Bloch oscillations of spin-orbit-coupled cold atoms in an optical lattice and spin current generation

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We study the Bloch oscillation dynamics of a spin-orbit-coupled cold atomic gas trapped inside a one-dimensional optical lattice. The eigenspectra of the system is identified as two interpenetrating Wannier-Stark ladder. Based on that, we carefully analyzed the Bloch oscillation dynamics and found out that intraladder coupling between neighboring rungs of Wannier-Stark ladder give rise to ordinary Bloch oscillation while interladder coupling lead to small amplitude high frequency oscillation superimposed on it. Specifically spin-orbit interaction breaks Galilean invariance, which can be reflected by out-of-phase oscillation of the two spin components in the accelerated frame. The possibility of generating spin current in this system are also explored.

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I. INTRODUCTION

Bloch oscillation describe that inside a lattice potential a particle will perform periodic oscillation instead of constant acceleration when subject to a constant external force. It was first proposed in electronic system \cite{1}, however have not been observed until the use of semiconductor superlattice \cite{2} due to the small lattice constant and imperfections in conventional crystal. The frequency of Bloch oscillation is proportional to the applied force $F$, which can have potential application in precision measurement. Besides that, the dynamics concerning particles moving in periodic structures is itself important due to that it is a pure quantum effect and reflects the properties of energy band such as the topology \cite{3}. These extends people’s interest in Bloch oscillation beyond the electronic system. Bloch oscillation have been experimentally observed in optical system \cite{4} and ultracold atoms trapped in an optical lattice \cite{5,6}. Recently it was demonstrated that impurity moving in quantum liquids can also display the behavior of Bloch oscillation \cite{3,7}. Theoretically Bloch oscillation can be well-understood within adiabatical approximation in which the particles move in Bloch energy band under the action of the force \cite{5}. The eigenstate of Bloch oscillation is also well-known as Wannier-Stark ladder (WSL) \cite{10}.

On the other hand besides the external centre-of-mass motion, particles possess internal degree-of-freedom such as the electronic spin. Pseudospin can also be constructed from the atomic internal energy level structure. Through the mechanism of spin-orbit (SO) coupling particle’s orbital motion can be connected to its spin dynamics and lead to rich physics. Recently SO coupling have been successfully implemented in neutral atom \cite{11,13}. Along with that, interesting physics have been predicted in SO-coupled atomic system such as dipole oscillation \cite{11,12}, Zitterbewegung \cite{15,16}, spin-dependent pairing \cite{17}, SO-modulated Anderson localization \cite{18,20}, SO-modulated atom optics \cite{21} and exotic dynamics \cite{22,26}.

Then it is natural to ask how Bloch oscillation will be affected by SO interaction. In the present work we will investigate the Bloch oscillation dynamics of SO-coupled cold atoms in a one-dimensional optical lattice. An important motivation lies in the recent achievement of SO-coupled Bose-Einstein condendates (BEC) in a one-dimensional optical lattice \cite{22}, which guarantee that the results obtained here can be readily observed in experiment. In previous theoretical works, Larson and co-workers investigated Bloch oscillation of SO-coupled BEC in a two-dimensional optical lattice, in which transverse spin current and atomic Zitterbewegung are predicted \cite{27}. Bloch oscillation of a SO-coupled helicoidal molecule was studied by Caeteno in \cite{28}. Kartashov et al. studied Bloch oscillation in one-dimensional optical and Zeeman lattices in the presence of SO coupling, in which they give a detailed discussion on the amplitude and wavepacket width of Bloch oscillation \cite{29}. Although the WSL eigen-spectra have been given in \cite{29}, its relation with the oscillation dynamics was not clarified yet. Here we will solve the dynamics using the theory of WSL. We show that one can understand the properties of Bloch oscillation dynamics in the presence of SO coupling via analyzing the coupling of WSLs. Especially in the case with finite Zeeman detuning which was not considered in \cite{29}, the two spin components will display unusual out-of-phase oscillation. In addition we show how this can serve as an unambiguous proof of broken Galilean invariance caused by SO interaction. Since SO interaction can play a crucial role in generating and manipulating spin current \cite{30}, we’ll also look into the possibility of generating

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spin current in the present one-dimensional system.

The article is organized as follows: In Sec. II we present our model and the dynamics are solved with WSL. Section III is devoted to the detailed discussion of Bloch oscillation. The possibility of generating spin current in the present system is explored in Sec. IV. Finally we conclude in Sec. V.

II. MODEL

As shown in Fig. 1 our model is based on the recent experiment \[22\] with a \(^{87}\)Rb BEC prepared in a one-dimensional optical lattice along the z-direction, inside which the effective SO interaction is induced via coupling the \(|1, -1\rangle\) (\(|↓↓\rangle\)) and \(|1, 0\rangle\) (\(|↑↑\rangle\)) hyperfine states with Raman lasers. In addition to that, here we consider that a constant external force \(F\) is exerted on the atoms via tilting the optical lattice. The effective single-particle Hamiltonian reads

\[
\hat{H} = \hat{H}_{SO} + U_0 \sin^2(2kz) - Fz,
\]

\[
\hat{H}_{SO} = \frac{(p_z - A)^2}{2m} + \frac{\hbar \Omega}{2} \hat{\sigma}_x + \frac{\hbar \delta}{2} \hat{\sigma}_z,
\]

in which the SO coupling is embodied in the effective vector potential \(A = -ma\hat{\sigma}_z\) (\(a = \hbar k_R/m\) characterizes SO coupling strength with \(k_R\) the Raman beam wavevector), \(\Omega\) is the Raman coupling strength with \(\delta\) the two-photon detuning. The periodic potential is characterized by the depth \(U_0\) and period \(d = \pi/k_l\).

By performing lowest energy band truncation and assuming tight binding approximation, Hamiltonian (1) can be expanded in the \(s\)-Wannier basis \(|j, \sigma\rangle\) (with \(j\) the lattice site index) as

\[
\hat{H} = \sum_j \left\{ -\frac{J}{2} \cos(\pi \gamma) \sum_\sigma |j, \sigma\rangle \langle j + 1, \sigma| + \frac{\hbar \Omega}{2} |j, \uparrow\rangle \langle j, \downarrow| - |j, \downarrow\rangle \langle j, \uparrow| + \frac{\hbar \delta}{2} \langle j, \uparrow| j, \downarrow| - |j, \downarrow| \langle j, \uparrow| \right\} + F d \sum_j |j, \sigma\rangle \langle j, \sigma| + \frac{\hbar \delta}{2} \langle j, \uparrow| j, \downarrow| - |j, \downarrow| \langle j, \uparrow| \right\}
\]

in which the spin-dependent hopping matrix element \(\hat{T} = J \exp\left(-i/\hbar \int \hat{A} \right)/2\) is obtained through Peierls substitution \[31\]. \(J\) is the tunneling amplitude without SO coupling, \(\gamma = k_R/k_l\). \(J\) can be calculated as

\[
J = -2 \int dz U_{j+1}(z) \left[-\frac{d^2}{dz^2} + U_0 \sin^2(2kz)\right] w_j(z),
\]

with \(w_j(z) = w(z - z_j)\) is the Wannier state of the lowest energy band at the \(j\)-th site which can be obtained numerically \[32\]. Here we consider the case of \(U_0 > 0\) with \(z_j = jd\).

In order to find out the eigenstates of Hamiltonian \[2\], it will be more convenient to transform it into the Bloch basis via the Fourier transformation \[33\]

\[
|q, \sigma\rangle = \sqrt{\frac{d}{2\pi}} \sum_{j=-\infty}^{\infty} |j, \sigma\rangle e^{iqjd}.
\]

One can then obtain

\[
\hat{H}(q) = \langle q | \hat{H} | q \rangle = \left( \frac{H_j^+}{\hbar \Omega/2} \right),
\]

with \(H_j^+ = -J \cos(qd + \pi \gamma) \pm \hbar \delta/2 - iF \partial/\partial q\). The eigenvalue problem then resort to

\[
-iF \frac{\partial \psi_T(q)}{\partial q} - J \cos(qd - \pi \gamma) \psi_T(q) + \frac{\hbar \delta}{2} \psi_T(q) + \frac{\hbar \Omega}{2} \psi_T(q) = E \psi_T(q),
\]

\[
-iF \frac{\partial \psi_T(q)}{\partial q} - J \cos(qd + \pi \gamma) \psi_T(q) - \frac{\hbar \delta}{2} \psi_T(q) + \frac{\hbar \Omega}{2} \psi_T(q) = E \psi_T(q),
\]

where \(\psi(q) = [\psi_T(q), \psi_T(q)]^T\) is the eigenvector.

Consider that \(\psi_\nu(q) = [\psi_\nu(q), \psi_\nu(q)]^T\) to be the \(\nu\)-th eigensolution of Eqs. (3) with the corresponding eigenvalue \(E_\nu\), it can be solved via performing the Fourier expansion

\[
\psi_T(q) = \sqrt{\frac{d}{2\pi}} \sum_{m=-\infty}^{\infty} A_\nu \exp\left[iqmd + iJ \frac{\hbar \delta}{d} \sin(qd - \pi \gamma)\right],
\]

\[
\psi_T(q) = \sqrt{d} \sum_{m=-M}^{M} B_\nu \exp\left[iqmd + iJ \frac{\hbar \delta}{d} \sin(qd + \pi \gamma)\right],
\]

where \(A_\nu\) and \(B_\nu\) are expansion coefficients with the truncation number \(M\). Through numerical calculation we found that \(M = 50\) to be a good approximation for the parameters considered in the present work. Substitute
Eqs. (6) is known as WSL [34], which consists of quantized energy levels with equal energy spacing $F_d$. In the presence of SO coupling WSL still exists, as can be seen from the Hamiltonian (1) with $\hat{A}' = E_\nu A'_\nu$, 

$$
\begin{align*}
\frac{\hbar \Omega}{2} \sum_{m'} \left( \frac{2J}{F_d} \sin \left( \pi \gamma \right) \right) B_{m'}^\nu & + \left( mF_d + \frac{\hbar \delta}{2} \right) A_m^\nu = E_\nu A'_\nu, \\
\frac{\hbar \Omega}{2} \sum_{m'} \left( -i \right)^{m-m'} J_{m-m'} \left( \frac{2J}{F_d} \sin \left( \pi \gamma \right) \right) A_{m'}^\nu & + \left( mF_d - \frac{\hbar \delta}{2} \right) B_{m'}^\nu = E_\nu B'_m, \\
\end{align*}
$$

with $J_n(z)$ the $n$th-order Bessel functions of the first kind. One can then numerically solve Eqs. (3) and obtain the coefficients $A'_\nu$, $B'_m$ and the corresponding eigenenergy $E_\nu$. The Wannier amplitudes of the corresponding eigenvector read

$$
\begin{align*}
W_{j,\uparrow}^\nu & = \sum_{m} A_m^\nu J_{j-m} \left( \frac{J}{F_d} \right) e^{i(j+m)\pi\gamma}, \\
W_{j,\downarrow}^\nu & = \sum_{m} B_m^\nu J_{j-m} \left( \frac{J}{F_d} \right) e^{-i(j+m)\pi\gamma}.
\end{align*}
$$

In the case without SO coupling the eigenenergies of Eqs. (6) is known as WSL [34], which consists of quantized energy levels with equal energy spacing $F_d$. In the presence of SO coupling WSL still exists, as can be seen from the Hamiltonian (1) with $\hat{A}' = E_\nu (z \psi (z + d) + (E + F_d) \psi (z + d))$. However the coupling between two pseudo-spin states will lead to two interpenetrating WSL which positioned symmetrically around $0$ [29], with an intra-ladder separation $s$, as shown in Fig. 2(a). The inter-ladder spacing within the two WSLs is still $F_d$. By considering that, we can label the WSL eigenenergies with $\nu_1(2)$ and $E_{\nu_1(2)} = \nu_1(2) F_d \mp s/2$. The intra-ladder spacing $s$ is a composite function of $\gamma$, $\Omega$, and $\delta$. As shown in Fig. 2(b), $s$ is a periodic function of $\gamma$. When $\delta = 0$, $s = 0$ for integer values of $\gamma$, the two WSL overlaps. This can be seen from that Eqs. (6) (a) and (b) are the same by replacing $\psi_{\uparrow} (q) \rightarrow \psi_{\downarrow} (q)$ at $\delta = 0$ and integer $\gamma$, signaling identical dynamics for the two spin components. Interestingly in addition to that, at some specific values of $\gamma$ marked by asterisks in Fig. 2(b) $s = F_d$, also indicating overlapping WSL. A nonzero $\delta$ separates the two ladder even at $\gamma = 0$.

The relation between the WSL spectrum and dynamics can be understood from the mean velocity. The velocity operator can be defined as $d\hat{z}/dt = i [\hat{H}, \hat{z}]/\hbar$, using the Hamiltonian (2) and assume the atomic wavefunction $|\psi (t)\rangle = \sum_{j,\sigma} \psi_{j,\sigma} (t) |j, \sigma\rangle$, one can calculate the mean velocity as

$$
\frac{dz}{dt} = \frac{Jd}{\hbar} \sum_{j} \text{Im} \left\{ a^*_\nu a_{\nu'} \sum_{j'} W_{j,\uparrow}^{\nu}\ast W_{j',\downarrow}^{\nu'} e^{-i\pi\gamma} + \sum_{j} W_{j,\uparrow}^{\nu}\ast W_{j',\downarrow}^{\nu'} e^{i\pi\gamma} \right\} \left[ 1 - e^{(E_{\nu} - E_{\nu'})t/\hbar} \right] + z_{\gamma(\downarrow)} (0)
$$

symbol the mean position of spin-$\sigma$ component. Eq. (12) predict that the oscillation frequencies are ruled by the energy difference between two Wannier-Stark levels with the amplitude of each frequency inversely proportional to the energy distance of those Wannier-Stark states and proportional to the overlap of their wavefunctions.

In the absence of SO coupling it is well-known that the Wannier-Stark eigenstate $W_j^{\nu}$ have the form of Bessel function of the first kind $(J_{\nu+j}(z))$ with $W_{j+1}^{\nu} = W_{j}^{\nu+1}$ [34], then $\sum_{\nu} W_{j}^{\nu}\ast W_{j+1}^{\nu} = \sum_{\nu} W_{j}^{\nu}\ast W_{j+1}^{\nu}$ take the value 1 for $\nu = \nu' + 1$ and 0 otherwise. It indicates that in the oscillation dynamics each rung of the WSL is only coupled to its neighboring rung with the Bloch frequency $\omega_B = (E_{\nu} - E_{\nu'})/\hbar = F_d/\hbar = 2\pi/T_B$. One can notice that in the presence of SO coupling the coupled equations [8] indicate two WSL in which any rung of the ladder is coupled to all the rungs of the other ladder, which will substantially modify the Bloch oscillation dynamics. This will be discussed in detail in the subsequent section.

FIG. 2: (Color online) (a) Eigenenergy spectra of the system under consideration. The spectra consists of two interpenetrating WSL, the intraladder spacing of both ladder is $F_d$ while the interladder spacing is $s$. (b) $s$ versus $\gamma$ at $\delta = 0$ (black solid line), $\delta = 0.2\Omega$ (red dashed line) and $\delta = 0.5\Omega$ (blue dotted line). The asterisks mark the values of $\gamma$ at which $s = F_d$. The other parameters are set as $J = 10 F_d$ and $\hbar \Omega = 80 F_d$. Then one can take advantage of Wannier-Stark eigenstates by considering that $\psi_{j,\sigma} (t) = \sum_{\nu} a_{\nu} W_{j,\sigma}^{\nu} \exp (-i E_{\nu} t/\hbar)$ with $a_{\nu} = \sum_{j,\sigma} W_{j,\sigma}^{\nu} \psi_{j,\sigma} (0)$, and the mean velocity can be expressed as
III. BLOCH OSCILLATION DYNAMICS

The Bloch oscillation dynamics have been studied in \[29\] for the case of \(\delta = 0\). The results predicted there can be well understood under adiabatical theory. When \(F\) is weak enough not to induce intraladder transitions the adiabatic approximation can be applied, under which the atoms move adiabatically along the energy band with the quasimomentum \(q(t) = q(0) + Ft / \hbar\). One can predict that the frequency of Bloch oscillation is proportional to \(Fd\) with the amplitude proportional to the bandwidth. The properties of Bloch oscillation can then be captured via further looking into the energy band structure, which can be obtained through diagonalizing the Hamiltonian \[5\] without the force \((F = 0)\). This result in a two-band structure with \(\varepsilon_\pm (q) = -J \cos qd \cos \pi \gamma \pm \sqrt{J^2 \sin^2 qd \sin^2 \pi \gamma - \hbar \delta J \sin qd \sin \pi \gamma + \hbar^2 \delta^2 / 4 + \hbar^2 \Omega^2 / 4}\).

Two major results are predicted in \[29\]: (i) In analogue to increasing the potential depth \(U_0\) of the optical lattice, SO interaction can take the same effect of band flattening \[29\]. In this case the Bloch oscillation amplitude will be suppressed and thus make it difficult to measure. An example for this is given at \(\gamma = 0.5\) with the energy band shown in Fig. 3(a). (ii) Since that in the adiabatic approximation the mean velocity of the atom \(v(q) = dq \langle \psi | q / \hbar dq\rangle\), the change in the band structure indicate that the atomic dynamics will subject to strong modification. As an example, for the band profile at \(\gamma = 0.8\) shown in Fig. 3(b), the initial atomic moving direction will be reversed.

These phenomena can also be explained using the theory of WSL. By considering that the eigenstate of the system consists of two interpenetrating WSL, one can group their contribution to the dynamics into two terms. Similar to the case without SO coupling, start from Eqs. \(5\) and \(6\), one can prove that within each ladder \(W_{j+1,\sigma} = W_{j,\sigma}^{\nu,\nu} (i = 1, 2\) label the two ladders) still hold true, then according to Eq. \(12\) one can conclude that in the presence of SO interaction the Bloch oscillation dynamics in general are still dominated by intraladder coupling between neighboring rungs within each ladder, indicating the oscillation frequency \(T_B\). At \(\delta = 0\), due to the symmetry between spin-up and \(\downarrow\) components, we have \(\sum_j |W_{j,\uparrow}^\nu|^2 = \sum_j |W_{j,\downarrow}^\nu|^2 = 1 / 2\), then according to Eqs. \(11\) and \(12\) one can predict that \(z(t) = 0\) at \(\gamma = 0.5\) and \(dz/dt < 0\) at \(\gamma = 0.8\) for initial small \(t\), indicating that Bloch oscillation dynamics are substantially modified by SO interaction.

We assume that initially the atomic wavefunction

\[
\psi_j (t = 0) = (a \sqrt{\pi})^{-1/2} e^{-(j - j_0)^2 / 2a^2} + i q_0 (1) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{13}
\]

to be a spin-polarized Gaussian wave-packet with width \(a\), where \(j_0\) is the center of the wave-packet while \(q_0\) denotes the initial quasimomentum. In our calculations the parameters are chosen as \(j_0 = 0\) and \(q_0 = 0\). The dynamics are simulated using the method of eigenstate expansion and the results are demonstrated in Figs. 3(c)-(f), from which one can see that the results of numerical simulation are consistent with the above theoretical analysis.

Besides intraladder coupling, interladder coupling also contribute to the oscillation dynamics. We calculate the value of \(\sum_j W_{j,\nu}^\nu W_{j+1,\nu}^\nu\) and found out that for relatively large \(|\nu_1 - \nu_2|\) (approaching 100) it really matters. This can be traced to the symmetry within WSL. Eq. \(8\) indicate that if \((A_m, B_m)\) are eigensolutions with eigenvalue \(E_{\nu}\), then \((-B_m, A_m^*)\) are eigensolutions with eigenvalue \(-E_{\nu}\). Due to the large energy difference of interladder coupling, it will superimpose small amplitude high frequency oscillation on the dynamics dominated by intraladder coupling.

An interesting case is that at \(\gamma = 0.5\), since the intraladder coupling are canceled out, then the dynamics deviating from \(z = 0\) is the result of interladder coupling, which is shown in Fig. 3(g). One can observe small amplitude high frequency oscillations, which become prominent around \(t = nT_B / 2\). Similar behavior
can also be observed for $\gamma = 0.8$ in Fig. 3(h), in which the small oscillations are superimposed on the traditional Bloch oscillation.

The Klein four-group [29] or CPT symmetry [33] is conserved by the Hamiltonian $\hat{H}_{SO} + U_0 \sin^2 (k_l z)$ at $\delta = 0$, then in the corresponding energy band the eigenfunctions are symmetric for spin-$\uparrow$ and $\downarrow$ ($\psi_\uparrow (q) = \psi_\downarrow (-q)$) at the centre and edge of Brillouin zone, which can also be seen from Eqs. (9). Then within adiabatical theory one can predict that $\langle \hat{\sigma}_z \rangle = 0$ when the atoms pass through the centre and edge of Brillouin zone. However this symmetry is broken at finite $\delta$. At finite $\delta$ the upper energy band and the lower one are shifted to opposite directions with respect to $q = 0$, as shown in Fig. 4(a). Physically this band asymmetry can be captured through Bloch oscillation via exerting force in opposite directions. The numerical results are shown in Figs. 4(b) and (c), in which a force $F$ is considered to be exerted along the $+z$ and $-z$ direction, respectively. At $\delta = 0$ one would expect that these two dynamics are identical, here the different dynamics signal the energy band asymmetry. Since the atomic initial state can be viewed as the superposition of the upper and lower eigenstate of the two bands, then in adiabatic limit they will subject to different dispersion under the action of the force. This cannot take place at $\delta = 0$ where the energy band are always symmetric and the two bands possess almost identical dispersion. The combined effect will lead to different oscillation dynamics for the two spin components as we illustrated in Fig. 4(c), the dynamics become out-of-phase for the two spin components. One can also notice that in Fig. 4(d) the high frequency oscillations for the two components are out-of-phase, this is because $W_{j\uparrow}^\dagger W_{j+1\uparrow}^\dagger = -W_{j\downarrow}^\dagger W_{j+1\downarrow}^\dagger$ for interladder couplings. In the meanwhile, $\langle \hat{\sigma}_z \rangle$ deviate from 0 when the wavepacket passes through the centre and edge of the Brillouin zone, as shown in Figs. 4(f) and (g).

![Fig. 4](image_url)

**FIG. 4:** (Color online) (a) Asymmetric energy band at $\delta = 0.5\Omega$ with the color indicating spin polarization $\langle \hat{\sigma}_z \rangle$. (b) Dynamics of mean position $z$ with the exerting force $F$ along the $+z$ direction. Same dynamics of $z_\uparrow$ (blue line) and $z_\downarrow$ (red line) are shown in (d). (c) and (e) Same as (b) and (d) except that the force $F$ is exerted along the $-z$ direction. (f) and (g) Mean value of pseudospin $\langle \hat{\sigma}_z \rangle$ versus time for the force $F$ exerted along $+z$ and $-z$ direction, respectively. The other parameters are set as $\gamma = 0.2$, $J = 10Fd$ and $h\Omega = 80Fd$.

![Fig. 5](image_url)

**FIG. 5:** (Color online) Oscillation dynamics in the lab frame (left column) and the accelerated frame (right column). The dynamics of $z \equiv z_\uparrow + z_\downarrow$ (first row), $z_\uparrow$ (middle row) and $z_\downarrow$ (bottom row) are shown in black, red and blue lines respectively. The parameters are set as $\gamma = 0.2$, $J = 10Fd$, $\delta = 0$, $h\Omega = 80Fd$ and $E_c / J = 8.55$.

In the case without SO coupling, one can introduce a linearly time-dependent frequency difference $\Delta \nu (t) = -Ft / md$ between the two lattice beams [5], the lattice potential becomes $U_0 \sin^2 \left[ k_l z - \pi \int_0^t d\tau \Delta \nu (\tau) \right]$ and in an accelerated frame it is equivalent to exerting a constant inertial force $F$ on the atoms trapped in a station-
ary lattice. However this equivalence cannot be established in the presence of SO coupling. This is due to that the SO Hamiltonian $\hat{H}_{\text{SO}}$ breaks Galilean invariance as the physical momentum $p_z - \hat{A}$ does not commute with $\hat{H}_{\text{SO}}$. In this case going into a moving inertial frame will result in an additional time-dependent term $-\alpha F t \hat{\sigma}_z$ in Hamiltonian (1), which play the role as a time-dependent effective detuning.

We calculate the oscillation dynamics in the stationary frame (lab frame) with the exerting force $F$ and that in the accelerated frame within which the atoms are subject to an effective force $F$ as well as an effective time-dependent detuning $-\alpha F t \hat{\sigma}_z$, the results are shown in Fig. 5. The dynamics in the lab frame are simulated with eigenstate expansion while that in the accelerated frame are calculated by means of the Fourth-order Runge-Kutta method. Both the initial state are given by Eq. (13). In the numerical simulation we consider the recoil energy $E_r = \hbar^2 k_z^2 / 2m = 8.55 J$ for a typical experimental value of $U_0 = 4 E_r$. As one can expect, in the lab frame the oscillation dynamics for spin-$\uparrow$ and $\downarrow$ components are in phase, as shown in Figs. 5(c) and (e). However the dynamics shown in Figs. 5(d) and (f) indicate that they are out-of-phase (phase separated in the time domain) in the accelerated frame. This interesting dynamics can be readily captured in experiment and serve as a clear proof of broken Galilean invariance, which is also the mechanism underlying other unusual behaviors such as the deviation of dipole oscillation frequency in a harmonically trapped system [11, 14], the ambiguity in defining Landau critical velocity in SO coupled condensates [36], finite-momentum dimer bound state in a SO coupled Fermi gas [17] and asymmetric expansion of SO coupled atomic Bose gas [25]. The effect of broken Galilean invariance can be signified via introducing a frequency difference between the two laser beams forming the optical lattice [22].

IV. SPIN CURRENT GENERATION

An interesting question is how to create a spin current with SO coupling [30]. Spin current have been experimentally generated in a SO-coupled BEC via spin Hall effect [27] and quenching [38]. In theory, Larson et al. studied bloch oscillations of atomic BEC in a tilted two-dimensional (2D) optical lattice [27], in which the atoms are subject to a 2D SO interaction $\hat{H}_{\text{SO}} \propto \hat{p}_x \hat{\sigma}_z + \hat{p}_y \hat{\sigma}_y$ and in turn give rise to a spin-dependent effective force proportional to $\hat{\sigma}_z$. As a result an oscillating transverse spin current can be generated. For the present 1D system we have

$$\hat{F}_z = \left[ \hat{H}_{\text{SO}}, \hat{z} \right] = \frac{\hbar^3 k_R \Omega}{m} \hat{\sigma}_y \mathbf{e}_z,$$

indicating an SO aroused effective force along $\mathbf{e}_z$-direction and proportional to $\hat{\sigma}_y$.

Here we would like to explore the possibility of generating spin current in the present 1D system with this effective force. As suggested by Shi et al. [39], the spin current operator along the $z$-direction can be defined as

$$\hat{J}_z (t) = \frac{d}{dt} (\hat{\sigma}_z \hat{z}).$$

Follow the very similar procedure as deducing Eqs. (10) and (11), make use of the WSL eigenstate, the mean-value of $\sigma_z$-component of spin current can be calculated as

$$\langle J_z \rangle (t) = \sum_j \text{Im} \left[ \frac{J_d}{\hbar} e^{-i \pi \gamma} \psi_j^{\dagger} (t) \psi_{j+1} (t) \right] \langle \hat{\sigma}_z \rangle,$$

which predicts that in addition to the coupling between different rungs, the last term in Eq. (10) indicate that the coupling between spin-$\uparrow$ and $\downarrow$ components also contribute to the spin current, resulting from that the effective force $\hat{F}_z$ is proportional to $\hat{\sigma}_y$.

FIG. 6: (Color online) (a) Energy band at $\gamma = 1$ and $\delta = 0$. (b) Dynamics of mean position $z$. (c) Dynamics of the spin current $J_z$. The parameters are set as $J = 10 F d$ and $\hbar \Omega = 80 F d$ with the atoms initially prepared in a Gaussian wavepacket [13].

In order to illustrate the contribution of this term, one can choose $\gamma = 1$ at which the major intraladder contribution from first two terms in Eq. (16) canceled out at $\delta = 0$ due to the symmetry. Physically it is equivalent to that the two spin components are performing identical Bloch oscillation and in the meanwhile subject to on-site Raman coupling, as one can see from the Hamiltonian (2). In the case $\langle \hat{\sigma}_z \rangle = 0$ for the Bloch eigenstate without the force. We then numerically calculate $J_z (t)$ and
the results are shown in Fig. 6. One can expect that in the absence of Raman coupling no spin current can be generated since that spin-\(\uparrow\) and spin-\(\downarrow\) components both move adiabatically along the energy band and exhibit typical properties of Bloch oscillation, as can be seen from Fig. 6(a) and (b). The small amplitude high frequency oscillation is aroused by interladder coupling as we discussed in Sec. 11. The time evolution of spin current \(J(z)\) exhibit the behavior of collapse and revival shown in Fig. 6(c), reminiscent of the Jaynes-Cummings model in quantum optics [40]. This collapse and revival behavior can be understood as a result of the complex interplay between the external force \(F\) and the intrinsic force \(F_{\text{SO}} = \hbar k R \Omega \sigma_z / 2\) aroused by SO interaction. One can also understand this collapse and revival behavior the same as Zitterbewegung [27].

Zitterbewegung results from coherent coupling between eigenstates of Dirac cone with different helicity [41] and have been successfully observed in experiment with cold atoms [15, 16], while here the trembling oscillation is aroused by spin swapping.

We also examined the case with finite Zeeman detuning. As one can expect, although the spin-\(\uparrow\) and spin-\(\downarrow\) components are performing different oscillation, it will be immersed in the dynamics aroused by Raman coupling and in general the spin current will exhibit the dynamics of collapse and revival. In order to achieve constant directional spin current, one can either adapt time-dependent SO coupling [12] or unbiased external force [43].

V. SUMMARY AND OUTLOOK

Before concluding the paper, we need to note that in the presence of SO interaction one should be very careful while using the above lowest energy band truncation. As was pointed out by Zhou and Cui [44], in this case tight-binding models have limitations in predicting the correct single-particle physics due to the missed high-band contributions. Physically the Raman lasers inducing SO interaction also inevitably couple atoms to high-lying bands which will significantly affect the single-particle physics [24]. Experimentally atomic BEC can also be prepared in excited bands of an optical lattice [45]. Ao and Rammer also pointed out that high-band contributions can substantially affect the Bloch oscillation dynamics [46]. Contributions from higher Bloch bands will be important and interesting in orbital optical lattices [17]. By considering that, we compare the results presented in this work with those obtained through numerical simulation of the corresponding Schrodinger equation and found good agreement in the case of large energy gap and small external force.

In summary, we have studied the Bloch oscillation dynamics of a SO-coupled cold atomic gas trapped inside a 1D optical lattice. The eigen-spectra of the system have been identified as two interpenetrating WSL. The Bloch oscillation dynamics in this system can be well-understood via analyzing the coupling between different rungs of the WSL. In the presence of finite Zeeman detuning, we show that the two spin components can display out-of-phase oscillation. This can also serve as an unambiguous proof of broken Galilean invariance aroused by SO coupling. In addition to that, we numerically explored the possibility of generating spin current in the present system. Since SO interaction have been implemented in BEC in a 1D optical lattice [22], our findings of the interesting dynamical phenomena should be within reach of present-day experiments. For BEC it will be interesting to study the impact of interparticle collisions on Bloch oscillation [48] and spin current generation, which can be investigated by the Gaussian variational approach [19, 50]. It will also be interesting to investigate Landau-Zener tunneling [51, 52]. These will be left for further investigation.

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