THE MAZUR-ULAM PROPERTY FOR UNIFORM ALGEBRAS

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ABSTRACT. We give a sufficient condition for a Banach space with which the homogeneous extension of a surjective isometry from the unit sphere of it onto another one is real-linear. The condition is satisfied by a uniform algebra and a certain extremely $C$-regular space of real-valued continuous functions.

1. INTRODUCTION

In 1987 Tingley [25] proposed a problem if a surjective isometry between the unit spheres of Banach spaces is extended to a surjective isometry between whole spaces. Wang [26] seems to be the first to solve Tingley’s problem between specific spaces. He dealt with $C_0(Y)$, the space of all real (resp. complex) valued continuous functions which vanish at infinity on a locally compact Hausdorff space $Y$. Although we do not exhibit each of the literatures, a considerable number of interesting results have shown that Tingley’s problem has an affirmative answer. No counterexample is known. Due to [29, p.730] Ding was the first to consider Tingley’s problem between different type of spaces [9]. Ding [10, Corollary 2] in fact proved that the real Banach space of all null sequences of real numbers satisfies now we call the Mazur-Ulam property. Later Cheng and Dong [5] introduced the concept of the Mazur-Ulam property. Following Cheng and Dong we say that a real Banach space $E$ satisfies the Mazur-Ulam property if a surjective isometry between the unit sphere of $E$ and that of any real Banach space is extended to a surjective real-linear isometry between the whole spaces. Tan [19, 20, 21] showed that the space $L^p(\mathbb{R})$ for $\sigma$-finite positive measure space satisfies the Mazur-Ulam property. In [22] Tan, Huang and Liu introduced the notion of generalized lush spaces and local GL spaces and proved that every local GL space satisfies the Mazur-Ulam property. New achievements by Mori and Ozawa [17] prove that the Mazur-Ulam property is satisfied by unital $C^*$-algebras and real von Neumann algebras. Cueto-Avellaneda and Peralta [7] proved that a complex (resp. real) Banach space of all continuous maps with the value in a complex (resp. real) Hilbert space satisfies the

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Mazur-Ulam property (cf. [8]). The result proving that all general JBW*-triples satisfy the Mazur-Ulam property is established by Becerra-Guerrero, Cueto-Avellaneda, Fernández-Polo and Peralta [2] and Kalenda and Peralta [14]. The study of the Mazur-Ulam property is nowadays a challenging subject of study (cf. [1, 4, 8, 28]).

In this paper we say that a complex Banach space $B$ satisfies the complex Mazur-Ulam property, emphasizing the term ‘complex’, if a surjective isometry between the unit spheres of $B$ and any complex Banach space is extended to a surjective real-linear isometry between the whole spaces. Jiménez-Vargas, Morales-Compoy, Peralta and Ramírez [13, Theorems 3.8, 3.9] probably provides the first examples of complex Banach spaces satisfying the complex Mazur-Ulam property (cf. [18]). Note that a complex Banach space satisfies the complex Mazur-Ulam property provided that it satisfies the Mazur-Ulam property as a real Banach space (a complex Banach space is a real Banach space simultaneously).

In [12] we proved that a surjective isometry between the unit spheres of uniform algebras is extended to a surjective real-linear isometry between whole of the uniform algebras. In this paper we show the complex Mazur-Ulam property for uniform algebras. Typical examples of a uniform algebra consist of analytic functions of one and several complex-variables such as the disk algebra, the polydisk algebra and the ball algebra. Through the Gelfand transform, the algebra of all bounded analytic functions on a certain domain is considered as a uniform algebra on the maximal ideal space. Hence the main result in this paper provides the first example of a Banach space of analytic functions which satisfies the complex Mazur-Ulam property. For further information about uniform algebras, see [3].

2. IS A HOMOGENEOUS EXTENSION LINEAR?

For a real or complex Banach space $B$, we denote $S(B)$ the unit sphere $\{a \in B : \|a\| = 1\}$ of $B$. A maximal convex subset of $S(B)$ is denoted by $\mathcal{F}_B$. Throughout the paper the map $T : S(B_1) \to S(B_2)$ always denotes a surjective isometry with respect to the metric induced by the norm, where $B_1$ and $B_2$ are both real Banach spaces or both complex Banach spaces. We define the homogeneous extension $\widetilde{T} : B_1 \to B_2$ of $T$ by

$$\widetilde{T}(a) = \begin{cases} \|a\| T \left( \frac{a}{\|a\|} \right), & 0 \neq a \in B_1 \\ 0, & a = 0. \end{cases}$$

By the definition $\widetilde{T}$ is a bijection which satisfies that $\|\widetilde{T}(a)\| = \|a\|$ for every $a \in B_1$. The Tingley’s problem asks if $\widetilde{T}$ is real-linear or not. In this paper we prove that $\widetilde{T}$ is real-linear for certain Banach spaces $B_1$ including uniform algebras.
It is well known that for every $F \in \mathfrak{F}_B$ of a real or complex Banach space $B$, there exists an extreme point $p$ in the closed unit ball $B(B^*)$ of the dual space $B^*$ of $B$ such that $F = p^{-1}(1) \cap S(B)$ (cf. [24, Lemma 3.3], [12, Lemma 3.1]). Let $Q$ be the set of all extreme points $p$ in $B(B^*)$ such that $p^{-1}(1) \cap S(B) \in \mathfrak{F}_B$. We define an equivalence relation $\sim$ in $Q$. We write $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ if $B$ is a complex Banach space, where $\mathbb{C}$ denotes the space of all complex numbers, and $\mathbb{T} = \{ \pm 1 \}$ if $B$ is a real Banach space.

From Definition 2.1 through Definition 2.6 is a uniqueness set for $\mathfrak{F}_B$ is a complex Banach space.

**Definition 2.1.** Let $p_1, p_2 \in Q$. We denote $p_1 \sim p_2$ if there exits $\gamma \in \mathbb{T}$ such that $p_1^{-1}(1) \cap S(B) = (\gamma p_2)^{-1}(1) \cap S(B)$.

Note that $\gamma p \in Q$ provided that $\gamma \in \mathbb{T}$ and $p \in Q$.

**Lemma 2.2.** The binary relation $\sim$ is an equivalence relation in $Q$.

A proof is by a routine argument and is omitted.

**Definition 2.3.** A set of all representatives with respect to the equivalence relation $\sim$ is simply called a set of representatives for $\mathfrak{F}_B$.

Note that a set of representatives exists due to the choice axiom. Note also that a set of representatives for $\mathfrak{F}_B$ is a norming family for $B$, hence it is a uniqueness set for $B$.

**Example 2.4.** Let $A$ be a uniform algebra. We assume a uniform algebra as a complex Banach space here and after. We denote the choquet boundary for $A$ by $\text{Ch}(A)$. By the Arens-Kelley theorem (cf. [11, p.29]) we have $Q = \{ \gamma \delta_x : x \in \text{Ch}(A), \gamma \in \mathbb{T} \}$, where $\delta_x$ denotes the point evaluation at $x$. In this case $\{ \delta_x : x \in \text{Ch}(A) \}$ is a set of representatives for $\mathfrak{F}_A$.

**Lemma 2.5.** Let $P$ be a set of representatives. For $F \in \mathfrak{F}_B$ there exists a unique $(p, \lambda) \in P \times \mathbb{T}$ such that $F = \{ a \in S(B) : p(a) = \lambda \}$. Conversely, for $(p, \lambda) \in P \times \mathbb{T}$ we have $\{ a \in S(B) : p(a) = \lambda \}$ is in $\mathfrak{F}_B$.

**Proof.** Let $F \in \mathfrak{F}_B$. We first prove the existence of $(p, \lambda) \in P \times \mathbb{T}$ satisfying the condition. There exists $p \in Q$ such that $F = p^{-1}(1) \cap S(B)$. By the definition, there exists $q \in P$ such that $p \sim q$. Hence there exists $\gamma \in \mathbb{T}$ such that $F = (\gamma q)^{-1}(1) \cap S(B)$. Letting $\lambda = q\gamma$ we have $F = \{ a \in S(B) : q(a) = \lambda \}$. We prove the uniqueness of $(q, \lambda)$. Suppose that $\{ a \in S(B) : q(a) = \lambda \} = \{ a \in S(B) : q'(a) = \lambda' \}$ for $(q', \lambda') \in P \times \mathbb{T}$. Then we have $q^{-1}(1) \cap S(B) = (\lambda \overline{\lambda'} q')^{-1} \cap S(B)$, that is $q \sim q'$. As $P$ is the set of all representatives with respect to the equivalence relation $\sim$ and $q, q' \in P$ we have $q = q'$. It follows that $\lambda = \lambda'$.

Conversely, let $(p, \lambda) \in P \times \mathbb{T}$. Then $\overline{\lambda} p \in Q$ and $\{ a \in S(B) : p(a) = \lambda \} = (\overline{\lambda} p)^{-1}(1) \cap S(B) \in \mathfrak{F}_B$. $\blacksquare$
Definition 2.6. For \((q, \lambda) \in Q \times T\), we denote \(F_{q, \lambda} = \{a \in S(B) : q(a) = \lambda\}\). A map

\[ I_B : \mathfrak{F}_B \rightarrow P \times T \]

is defined by \(I_B(F) = (p, \lambda)\) for \(F = F_{p, \lambda} \in \mathfrak{F}_B\).

By Lemma 2.5 the map \(I_B\) is well defined and bijective. An important theorem of Cheng, Dong and Tanaka states that a surjective isometry between the unit spheres of Banach spaces preserves maximal convex subsets of the unit spheres. This theorem was first exhibited by Cheng and Dong in [5, Lemma 5.1] and a crystal proof was given by Tanaka [23, Lemma 3.5]. In the following \(P_j\) is a set of representatives for \(\mathfrak{F}_B j\) for \(j = 1, 2\). Due to the theorem of Cheng, Dong and Tanaka a bijection \(T : \mathfrak{F}_{B_1} \rightarrow \mathfrak{F}_{B_2}\) is induced.

Definition 2.7. The map \(T : \mathfrak{F}_{B_1} \rightarrow \mathfrak{F}_{B_2}\) is defined by \(T(F) = T(F)\) for \(F \in \mathfrak{F}_{B_1}\). The map \(\Psi\) is well defined and bijective. Put

\[ \Psi = I_{B_2} \circ \mathfrak{F} \circ I_{B_1}^{-1} : P_1 \times T \rightarrow P_2 \times T. \]

Define two maps

\[ \phi : P_1 \times T \rightarrow P_2 \]

and

\[ \tau : P_1 \times T \rightarrow T \]

by

\[ \Psi(p, \lambda) = (\phi(p, \lambda), \tau(p, \lambda)), (p, \lambda) \in P_1 \times T. \]

If \(\phi(p, \lambda) = \phi(p, \lambda')\) for every \(p \in P_1\) and \(\lambda, \lambda' \in T\) we simply write \(\phi(p)\) instead of \(\phi(p, \lambda)\) by discarding the second term \(\lambda\).

Rewriting the equation (1) we have

\[ T(F_{p, \lambda}) = F_{\phi(p, \lambda), \tau(p, \lambda)}, (p, \lambda) \in P_1 \times T. \]

We point out that

\[ \phi(p, -\lambda) = \phi(p, \lambda), \tau(p, -\lambda) = -\tau(p, \lambda) \]

for every \((p, \lambda) \in P_1 \times T\). The reason is as follows. It is well known that \(T(-F) = -T(F)\) for every \(F \in \mathfrak{F}_{B_1}\) (cf. [16, Proposition 2.3]). Hence

\[ F_{\phi(p, -\lambda), \tau(p, -\lambda)} = T(F_{p, -\lambda}) = T(-F_{p, \lambda}) = -T(F_{p, \lambda}) = -F_{\phi(p, \lambda), \tau(p, \lambda)} = F_{\phi(p, \lambda), -\tau(p, \lambda)} \]

for every \(p \in P_1\). Since the map \(I_{B_2}\) is a bijection we have (3).

Rewriting (2) we get a basic equation in our argument:

\[ \phi(p, \lambda)(T(a)) = \tau(p, \lambda), a \in F_{p, \lambda}. \]
We will prove that under some condition on $B_1$, which we will exhibit explicitly later, we have that
\[ \phi(p, \lambda) = \phi(p, \lambda'), \quad p \in P_1 \]
for every $\lambda$ and $\lambda'$ in $T$, and
\[ \tau(p, \lambda) = \tau(p, 1) \times \begin{cases} \lambda, & \text{for some } p \in P_1 \\ \bar{\lambda}, & \text{for other } p \text{'s} \end{cases} \]
for $\lambda \in T$. We get, under some condition on $B_1$, that
\[ \phi(p)(T(a)) = \tau(p, 1) \times \begin{cases} p(a), & \text{for some } p \in P_1 \\ \bar{p}(a), & \text{for other } p \text{'s} \end{cases} \]
for $a \in F_{p,p(a)}$. If the equation (5) holds for any $a \in S(B_1)$, without the restriction that $a \in F_{p,p(a)}$, then applying the definition of $\tilde{T}$ we get
\[ \phi(p)(\tilde{T}(a)) = \tau(p, 1) \times \begin{cases} p(a), & \text{for some } p \in P_1 \\ \bar{p}(a), & \text{for other } p \text{'s} \end{cases} \]
for every $a \in B_1$, with which we infer that
\[ \phi(p)(\tilde{T}(a + rb)) = \phi(p)(\tilde{T}(a)) + \phi(p)(r\tilde{T}(b)) \]
for every pair $a, b \in B_1$ and every real number $r$. As $\phi(P_1) = P_2$ is a norming family, we conclude that $\tilde{T}$ is real-linear. It means that we arrive at the final positive solution for Tingley’s problem under some conditions on $B_1$.

3. Hausdorff Distance between the Maximal Convex Subsets

Recall that the Hausdorff distance $d_H(K,L)$ between non-empty closed subsets $K$ and $L$ of a metric space with metric $d(\cdot, \cdot)$ is defined by
\[ d_H(K,L) = \max \{ \sup_{a \in K} d(a,L), \sup_{b \in L} d(b,K) \}. \]

**Lemma 3.1.** Let $B$ be a complex Banach space and $P$ a set of representatives for $\overline{S}_B$. We consider $B$ as a metric space induced by the norm. For every $p \in P$ we have
\[ d_H(F_{p,\gamma}, F_{p,\gamma'}) = |\gamma - \gamma'| \]
for every $\lambda, \lambda' \in T$. Suppose further that $p' \in P$ is different from $p$. If $\overline{S}_{p,\gamma} \cap \overline{S}_{p',\gamma'} \neq \emptyset$ for some $\gamma, \gamma' \in T$, then we have
\[ d_H(F_{p,\gamma}, F_{p',\gamma'}) = 2. \]
Proof. Let \( a \in F_{p,\lambda} \). Then \( \overline{\lambda}a \in F_{p,\lambda'} \). Thus \( d(a,F_{p,\lambda'}) \leq \|a - \overline{\lambda}a\| = |\lambda - \lambda'| \). On the other hand, for every \( b \in F_{p,\lambda} \) we have \( |\lambda - \lambda'| = |p(a) - p(b)| \leq \|a - b\| \). Hence \( |\lambda - \lambda'| \leq d(a,F_{p,\lambda'}) \). Therefore \( d(a,F_{p,\lambda'}) = |\lambda - \lambda'| \) holds for every \( a \in F_{p,\lambda} \). In the similar way we obtain that \( d(b,F_{p,\lambda'}) = |\lambda - \lambda'| \) for every \( b \in F_{p,\lambda} \). We conclude that \( d_H(F_{p,\lambda},F_{p,\lambda'}) = |\lambda - \lambda'| \) for every \( \lambda,\lambda' \in \mathbb{T} \).

Suppose that \( P \ni p' \neq p \) and \( \mathcal{F}_{p,\gamma} \cap F_{p',\gamma'} \neq \emptyset \) for some \( \gamma,\gamma' \in \mathbb{T} \). Let \( a \in \mathcal{F}_{p,\gamma} \cap F_{p',\gamma'} \). Then for every \( b \in F_{p',\gamma'} \) we have
\[
2 = |\gamma - \gamma'| = |p'(a) - p'(b)| \leq \|a - b\| \leq 2.
\]
Hence \( d(a,F_{p',\gamma'}) = 2 \). It follows that \( d_H(F_{p,\gamma},F_{p',\gamma'}) = 2 \). \( \square \)

Note that the notion of the condition of the Hausdorff distance does not depend on the choice of \( P \). In fact, we can describe the condition applying the terms of \( Q \); the condition of the Hausdorff distance is satisfied by \( B \) if and only if
\[
d_H(F_{q,\lambda},F_{q',\lambda'}) = \begin{cases} |\lambda - \gamma\lambda'|, & q^{-1}(1) \cap S(B) = (\gamma q')^{-1}(1) \cap S(B) \\ 2, & q \neq q' \end{cases}
\]
for \( q,q' \in Q \).

Example 3.2. Let \( A \) be a uniform algebra and \( P = \{ \delta_x : x \in \text{Ch}(A) \} \), where \( \delta_x \) denotes the point evaluation at \( x \). Then \( \mathcal{F}_{\delta_x,\lambda} \cap \mathcal{F}_{\delta_x,\lambda'} \neq \emptyset \) for any pair of different points \( x \) and \( \lambda' \) in \( \text{Ch}(A) \) and any \( \lambda,\lambda' \in \mathbb{T} \) [12] Lemma 4.1. Thus a uniform algebra satisfies the condition of the Hausdorff distance.

Lemma 3.3. Suppose that \( B_1 \) satisfies the condition of the Hausdorff distance. Let \( P_1 \) be a set of representatives for \( \mathcal{F}_{B_1} \). Then we have \( \phi(p,\lambda) = \phi(p,\lambda') \) for every \( p \in P_1 \) and \( \lambda,\lambda' \in \mathbb{T} \). Put
\[
P_1^+ = \{ p \in P_1 : \tau(p,i) = i\tau(p,1) \}
\]
and
\[
P_1^- = \{ p \in P_1 : \tau(p,i) = i\tau(p,1) \}.
\]
Then \( P_1^+ \) and \( P_1^- \) are possibly empty disjoint subsets of \( P_1 \) such that \( P_1^+ \cup P_1^- = P_1 \). Furthermore we have
\[
\tau(p,\lambda) = \lambda \tau(p,1), \quad p \in P_1^+, \lambda \in \mathbb{T}
\]
and
\[
\tau(p,\lambda) = \lambda \tau(p,1), \quad p \in P_1^+, \lambda \in \mathbb{T}.
\]

Proof. First we prove that \( \phi(p,\lambda) = \phi(p,\lambda') \) for every \( p \in P_1 \) and \( \lambda,\lambda' \in \mathbb{T} \). Suppose not: there exist \( p \in P_1 \) and \( \lambda,\lambda' \in \mathbb{T} \) such that \( \phi(p,\lambda) \neq \phi(p,\lambda') \). We may assume that \( |\lambda - \lambda'| < 2 \). (If \( \phi(p,\lambda) \neq \phi(p,\lambda') \) with \( |\lambda - \lambda'| = \)
2, then \( \phi(p, \lambda) \neq \phi(p, i\lambda) \) or \( \phi(p, \lambda') \neq \phi(p, i\lambda) \). Replacing \( i\lambda \) by \( \lambda' \) in the first case, and \( i\lambda \) by \( \lambda \) in the later case we have \( \phi(p, \lambda) \neq \phi(p, \lambda') \) with \( |\lambda - \lambda'| = \sqrt{2} < 2 \). Letting \( \lambda'' = \lambda^2 \lambda' \) we have \( |\lambda - \lambda'| = |\lambda - \lambda''| \) and \( \lambda' \neq \lambda'' \). Since \( B_1 \) satisfies the condition of the Hausdorff distance, an element \((q, \alpha) \in P_1 \times \mathbb{T}\) which satisfies
\[
(7) \quad d_H(F_{p,\lambda}, F_{q,\alpha}) = |\lambda - \lambda'|
\]
is only two elements \((p, \lambda')\) and \((p, \lambda'')\). As \( \mathcal{F} \) preserves the Hausdorff distance we have
\[
d_H(F_{p,\lambda}, F_{p,\lambda'}) = d_H(F_{p,\lambda}, F_{p,\lambda'}) = |\lambda - \lambda'|.
\]
Applying Lemma 3.1 we have
\[
(8) \quad d_H(F_{p,\lambda}, F_{p,\lambda'}; F_{p,\lambda''}) = |\lambda - \lambda'|
\]
if \((t, \beta) = (\phi(p, \lambda), \lambda \lambda' \tau(p, \lambda)), (\phi(p, \lambda), \lambda \lambda'' \tau(p, \lambda)), (\phi(p, \lambda'), \tau(p, \lambda'))\). As we suppose that \( \phi(p, \lambda) \neq \phi(p, \lambda') \), the number of points \((t, \beta) \in P_1 \times \mathbb{T}\) which satisfy (8) is at least three, while the number of points \((q, \alpha) \in P_1 \times \mathbb{T}\) which satisfies (7) is two since \( B_1 \) satisfies the condition of the Hausdorff distance. On the other hand the numbers of \((q, \alpha)\) and \((t, \beta)\) which satisfy (7) and (8) respectively must coincide each other because \( \mathcal{F} \) preserves the Hausdorff distance between the maximal convex subset. We arrive at a contradiction proving that \( \phi(p, \lambda) = \phi(p, \lambda') \) for every \( p \in P_1 \) and \( \lambda, \lambda' \in \mathbb{T} \).

In the following we simply write \( \phi(p) \) instead of \( \phi(p, \lambda) \) by discarding the second term.

We prove that \( \tau(p, \lambda) = \lambda \tau(p, 1) \) or \( \tilde{\lambda} \tau(p, 1) \) for \( p \in P_1 \) and \( \lambda \in \mathbb{T} \). Since \( \mathcal{F} \) preserves the Hausdorff distance we have by Lemma 3.1 that
\[
|\lambda - 1| = d_H(F_{p,\lambda}, F_{p,1}) = d_H(F_{p,\lambda}, F_{p,1})
\]
\[
= |\tau(p, \lambda) - \tau(p, 1)| = |\tau(p, \lambda) \tau(p, 1) - 1|,
\]
hence
\[
(9) \quad \tau(p, \lambda) \tau(p, 1) = \lambda \text{ or } \tilde{\lambda}
\]
Thus we have \( \tau(p, \lambda) = \lambda \tau(p, 1) \) or \( \tilde{\lambda} \tau(p, 1) \). Letting \( \lambda = i \) we infer that \( P^+_1 \cup P^-_1 = P_1 \) and \( P^+_1 \cap P^-_1 = \emptyset \). We have
\[
2 = d_H(F_{p,i}, F_{p,-i}) = d_H(F_{p,i}, F_{p,-i}) = |\tau(p, i) - \tau(p, -i)|,
\]
hence \( \tau(p, -i) = -\tau(p, i) \) for every \( p \in P_1 \). We show that \( \tau(p, \lambda) = \lambda \tau(p, 1) \) for \( p \in P^+_1 \) and \( \lambda \in \mathbb{T} \). By (9) we may suppose that \( \text{Im } \lambda \neq 0 \). In the case of \( \text{Im } \lambda > 0 \) we have
\[
|i - \lambda| = d_H(F_{p,i}, F_{p,\lambda}) = d_H(F_{p,i}, F_{p,\lambda})
\]
\[
= |\tau(p, i) - \tau(p, \lambda)| = |i \tau(p, 1) - \tau(p, \lambda)| = |i - \tau(p, \lambda) \tau(p, 1)|.
\]
Applying (9) we infer that \( \tau(p, \lambda) \tau(p, 1) = \lambda \) as \( \text{Im} \lambda > 0 \), hence \( \tau(p, \lambda) = \lambda \tau(p, 1) \). Next we consider the case of \( \text{Im} \lambda < 0 \). We have
\[
| - i - \lambda | = d_H(F_p, -i, F_p, \lambda) = d_H(F_{\phi(p), \tau(p, -i)}, F_{\phi(p), \tau(p, \lambda)}) = | \tau(p, -i) - \tau(p, \lambda) |,
\]
as \( \tau(p, -i) = - \tau(p, i) \) we have
\[
= | - \tau(p, i) - \tau(p, \lambda) | = | - i \tau(p, 1) - \tau(p, \lambda) | = | - i - \tau(p, \lambda) \tau(p, 1) |.
\]
As \( \text{Im} \lambda < 0 \), we infer that \( \tau(p, \lambda) = \lambda \tau(p, 1) \) by (9).

Similarly, it can be proved that \( \tau(p, \lambda) = \lambda \tau(p, 1) \) if \( p \in P_1^- \) and \( \lambda \in \mathbb{T} \).

4. A SUFFICIENT CONDITION FOR THE COMPLEX MAZUR-ULAM PROPERTY

We define a set \( M_{p, \alpha} \) with which the map \( \Psi \) plays a crucial role to work out the complex Mazur-Ulam property for a uniform algebra. We exhibit the definition of \( M_{p, \alpha} \). We denote \( \mathbb{D} = \{ z \in \mathbb{F} : |z| \leq 1 \} \), where \( \mathbb{F} = \mathbb{R} \) if the corresponding Banach space is a real one and \( \mathbb{F} = \mathbb{C} \) if the corresponding Banach space is a complex one.

**Definition 4.1.** Let \( B \) be a real or complex Banach space and \( P \) a set of representatives for \( \mathfrak{B}_B \). For \( p \in P \) and \( \alpha \in \mathbb{D} \) we denote
\[
M_{p, \alpha} = \{ a \in S(B) : d(a, F_{p, \alpha/|\alpha|}) \leq 1 - |\alpha|, d(a, F_{p, -\alpha/|\alpha|}) \leq 1 + |\alpha| \},
\]
where we read \( \alpha/|\alpha| = 1 \) if \( \alpha = 0 \).

**Lemma 4.2.** If \( B_j \) is a real Banach space for \( j = 1, 2 \), then we have
\[
T(M_{p, \alpha}) = M_{\phi(p), \alpha \tau(p, 1)}
\]
for every \( (p, \alpha) \in P_1 \times \mathbb{T} \). If \( B_j \) is a complex Banach space which satisfy the condition of the Hausdorff distance for \( j = 1, 2 \), then we have
\[
T(M_{p, \alpha}) = \begin{cases} 
M_{\phi(p), \alpha \tau(p, 1)}, & p \in P_1^+ \\
M_{\phi(p), \alpha \tau(p, 1)}, & p \in P_1^- 
\end{cases}
\]
for every \( (p, \alpha) \in P_1 \times \mathbb{T} \).

**Proof.** Due to the definition of the map \( \Psi \) we have
\[
T(F_{p, \frac{a}{|a|}}) = F_{\phi(p), \frac{a}{|a|}, \tau(p, \frac{a}{|a|})}
\]
and
\[
T(F_{p, -\frac{a}{|a|}}) = F_{\phi(p, -\frac{a}{|a|}, \tau(p, -\frac{a}{|a|})}
\]
Suppose that $B_j$ is a real Banach space first. Then by the definition $T = \{\pm 1\}$. By (3) we have $\phi(p, 1) = \phi(p, -1)$ for every $p \in P_1$. Hence $\phi(p, \lambda)$ does not depend on the second term for a real Banach space. We also have $\tau(p, -1) = -\tau(p, 1)$ for every $p \in P_1$ by (3). It follows that $T(F_p, \frac{\alpha}{|\alpha|}) = F_p \phi(p), \frac{\alpha}{|\alpha|} \tau(p, 1)$ and $T(F_p, -\frac{\alpha}{|\alpha|}) = F_p \phi(p), -\frac{\alpha}{|\alpha|} \tau(p, 1)$. As $T$ is a surjective isometry we have

$$d(a, F_p, \frac{\alpha}{|\alpha|}) = d(T(a), F_p \phi(p), \frac{\alpha}{|\alpha|} \tau(p, 1))$$

and

$$d(a, F_p, -\frac{\alpha}{|\alpha|}) = d(T(a), F_p \phi(p), -\frac{\alpha}{|\alpha|} \tau(p, 1))$$

As $T$ is a bijection we conclude that $T(M_p, \alpha) = M_p \phi(p), \alpha \tau(p, 1)$ for every $p \in P_1$ and $\alpha \in \overline{D}$.

Suppose next that $B_j$ is a complex Banach space which satisfies that the condition of the Hausdorff distance. Let $p \in P_1^+$. Then by Lemma 3.3 we have

$$T(F_p, \frac{\alpha}{|\alpha|}) = F_p \phi(p), \frac{\alpha}{|\alpha|} \tau(p, 1)$$

and

$$T(F_p, -\frac{\alpha}{|\alpha|}) = F_p \phi(p), -\frac{\alpha}{|\alpha|} \tau(p, 1).$$

The left of the proof is similar to the case that $B_j$ is a real Banach space, hence we see that $T(M_p, \alpha) = M_p \phi(p), \alpha \tau(p, 1)$ for every $p \in P_1^+$ and $\alpha \in \overline{D}$. The proof for the case of $p \in P_1^-$ is similar and is omitted. □

**Lemma 4.3.** Suppose that $B$ is a real or complex Banach space and $P$ is a set of representatives for $\mathcal{S}_B$. For every $p \in P$ and $\alpha \in \mathbb{D}$ we have $M_{p, \alpha} \subset \{a \in S(B) : p(a) = \alpha\}$.

**Proof.** Let $a \in M_{p, \alpha}$. Then for arbitrary $b \in F_p, \frac{\alpha}{|\alpha|}$

$$|p(a) - \frac{\alpha}{|\alpha|}| = |p(a) - p(b)| \leq \|a - b\|.$$ 

As $d(a, F_p, \frac{\alpha}{|\alpha|}) \leq 1 - |\alpha|$

$$|p(a) - \frac{\alpha}{|\alpha|}| \leq d(a, F_p, \frac{\alpha}{|\alpha|}) \leq 1 - |\alpha|.$$
In the same way we have
\[
\left| p(a) - \left( -\frac{\alpha}{|\alpha|} \right) \right| \leq d(a, F_p, -\frac{\alpha}{|\alpha|}) \leq 1 + |\alpha|.
\]
Then by the two inequalities \( p(a) \) have to be \( \alpha \). \( \square \)

The following is an auxiliary result.

**Proposition 4.4.** Let \( B_1 \) be a complex Banach space and \( P_1 \) a set of representatives for \( \mathcal{S}_{B_1} \). Assume the following two conditions:

i) \( B_1 \) satisfies the condition of the Hausdorff distance,

ii) \( M_{p,\alpha} = \{ a \in S(B_1) : p(a) = \alpha \} \) for every \( p \in P_1 \) and \( \alpha \in \mathbb{D} \).

Then \( B_1 \) satisfies the complex Mazur-Ulam property.

**Proof.** Suppose that \( B_2 \) is a complex Banach space and \( T : B_1 \to B_2 \) a surjective isometry. Applying Lemma 3.3 for the equation (4) we get\[
\phi(p)(T(a)) = \tau(p, 1) \times \begin{cases} 
p(a), & p \in P_1^+ 
\frac{p(a)}{p(a)}, & p \in P_1^- 
\end{cases},
\]for \( a \in S(B_1) \) such that \( |p(a)| = 1 \). We prove that
\[
\phi(p)(T(a)) = \tau(p, 1) \times \begin{cases} 
p(a), & p \in P_1^+ 
\frac{p(a)}{p(a)}, & p \in P_1^- 
\end{cases}
\]for any \( a \in S(B_1) \). Let \( a \in S(B_1) \) and \( \alpha = p(a) \). Then \( |\alpha| \leq 1 \). By condition ii), \( a \in M_{p,\alpha} \) and \( T(a) \in T(M_{p,\alpha}) \). As i) is assumed, we have by Lemma 4.2 that
\[
T(M_{p,\alpha}) = \begin{cases} 
M_{\phi(p), \alpha \tau(p, 1)} : p \in P_1^+ 
M_{\phi(p), \alpha \tau(p, 1)} : p \in P_1^- 
\end{cases}
\]Therefore
\[
\phi(p)(T(a)) = \alpha \tau(p, 1) = p(a) \tau(p, 1)
\]if \( p \in P_1^+ \). In a similar way we have
\[
\phi(p)(T(a)) = \overline{p(a) \tau(p, 1)}
\]if \( p \in P_1^- \). We conclude that
\[
\phi(p)(T(a)) = \tau(p, 1) \times \begin{cases} 
p(a), & p \in P_1^+ 
\frac{p(a)}{p(a)}, & p \in P_1^- 
\end{cases}
\]
for every $a \in S(B_1)$. Let $c \in B_1$ and $c \neq 0$. Since $\frac{c}{\|c\|} \in S(B_1)$, and $\phi(p)$ and $p$ are real-linear we have

\[
\phi(p)(\tilde{T}(c)) = \phi(p)(\|c\|T\left(\frac{c}{\|c\|}\right)) = \|c\|\tau(p, 1) \times \begin{cases} p\left(\frac{c}{\|c\|}\right), & p \in P_1^+ \\ p\left(\frac{c}{\|c\|}\right), & p \in P_1^- \end{cases}
\]

We infer that

\[
\phi(p)(\tilde{T}(a + rb)) = \tau(p, 1) \times \begin{cases} p(a + rb), & p \in P_1^+ \\ p(a + rb), & p \in P_1^- \end{cases} + \tau(p, 1) \times \begin{cases} rp(b), & p \in P_1^+ \\ rp(b), & p \in P_1^- \end{cases}
\]

\[
= \phi(p)(\tilde{T}(a) + r\tilde{T}(b))
\]

for every pair $a, b \in B_1$ and every real number $r$. As $\phi(P_1) = P_2$ is a norming family we conclude that

\[
\tilde{T}(a + rb) = \tilde{T}(a) + r\tilde{T}(b)
\]

for every pair $a, b \in B_1$ and every real number $r$. By the definition of $\tilde{T}$ it is a bijection from $B_1$ onto $B_2$ and it satisfies the equality $\|\tilde{T}(a)\| = \|a\|$ for every $a \in B_1$. Thus $\tilde{T}$ is a surjective real-linear isometry from $B_1$ onto $B_2$ which extend $T$. $\square$

The following is the main result in this paper.

**Theorem 4.5.** A uniform algebra satisfies the complex Mazur-Ulam property.

*Proof.* Let $A$ be a uniform algebra. Put $P = \{\delta_x : x \in \text{Ch}(A)\}$, where $\delta_x$ is the point evaluation at $x \in \text{Ch}(A)$, the Choquet boundary. Then $P$ is a set of representatives for $\mathfrak{F}_A$. It is known that a uniform algebra satisfies the condition of the Hausdorff distance [12, Lemma 4.1]. By [12, Lemma 6.3] we have $M_{\delta_x, \alpha} = \{f \in S(A) : f(x) = \alpha\}$ for every $x \in \text{Ch}(A)$ and $\alpha \in \mathbb{D}$. Thus the conditions i) and ii) of Proposition 4.4 holds for $A$. Thus $A$ satisfies the complex Mazur-Ulam property by Proposition 4.4 $\square$

5. THE CASE OF A REAL BANACH SPACE

Throughout the section we denote $B_j$ a real Banach space, $P_j$ a set of representatives for $\mathfrak{F}_B$, and $T : S(B_1) \to S(B_2)$ is a surjective isometry. We
have by (3) that \( \phi(p, 1) = \phi(p, -1) \) for every \( p \in P_1 \). As \( \mathbb{T} = \{ \pm 1 \} \) for real Banach spaces we have that \( \phi(p, \lambda) \) does not depend the second term for real Banach spaces. We also have \( \tau(p, -1) = -\tau(p, 1) \) for every \( p \in P_1 \) by (3). The situation is rather simple than the case of complex Banach spaces, and by (4) we have the following equation (10) without further assumption on \( B_1 \), i.e.,

\[
\phi(p)(T(a)) = \tau(p, 1)p(a)
\]

for every \( p \in P_1 \) and \( a \in S(B_1) \) with \( |p(a)| = 1 \).

**Proposition 5.1.** Suppose that

\[
M_{p, \alpha} = \{ a \in S(B_1) : p(a) = \alpha \}
\]

for every \( p \in P \) and \( -1 \leq \alpha \leq 1 \). Then \( T \) is extended to a surjective real-linear isometry form \( B_1 \) onto \( B_2 \). Hence \( B_1 \) satisfies the Mazur-Ulam property.

**Proof.** We first prove the equation (10) for every \( p \in P_1 \) and \( a \in S(B_1) \) without assuming that \( |p(a)| = 1 \). Let \( p \in P_1 \) and \( a \in S(B_1) \). Put \( \alpha = p(a) \). Then by (11) \( a \in M_{p, \alpha} \). We have by Lemma [2.2] that

\[
\phi(p)(T(a)) = \alpha \tau(p, 1) = \tau(p, 1)p(a).
\]

It follows that for the homogeneous extension \( \tilde{T} \) of \( T \) we have

\[
\phi(p)(\tilde{T}(c)) = \phi(p) \left( \|c\|T \left( \frac{c}{\|c\|} \right) \right) = \|c\|\tau(p, 1)p \left( \frac{c}{\|c\|} \right) = \tau(p, 1)p(c)
\]

for every \( 0 \neq c \in B_1 \). As the equality \( \phi(p)(\tilde{T}(0)) = \tau(p, 1)p(0) \) holds, we obtain for \( a, b \in B_1 \) and a real number \( r \) that

\[
\phi(p)(\tilde{T}(a + rb)) = \tau(p, 1)p(a + rb) = \tau(p, 1)p(a) + r\tau(p, 1)p(b)
\]

and

\[
\phi(p)(\tilde{T}(a) + r\tilde{T}(b)) = \phi(p)(\tilde{T}(a)) + r\phi(p)(\tilde{T}(b)) = \tau(p, 1)p(a) + r\tau(p, 1)p(b).
\]

It follows that

\[
\phi(p)(\tilde{T}(a + rb)) = \phi(p)(\tilde{T}(a) + r\tilde{T}(b))
\]

for every \( p \in P_1 \), \( a, b \in B_1 \), and every real number \( r \). As \( \phi(P_1) = P_2 \) is a norming family we see that \( \tilde{T} \) is real-linear on \( B_1 \). As the homogeneous extension is a norm-preserving bijection as is described in the first parat of section [2] we complete the proof. \( \square \)
**Definition 5.2.** For a locally compact Hausdorff space, we denote by $C_0(Y, \mathbb{R})$ a real Banach space of all real-valued continuous functions which vanish at infinity on $Y$. Let $E$ be a closed subspace of $C_0(Y, \mathbb{R})$ which separates the points of $Y$, that is, for any pair $y_1$ and $y_2$ of different points in $Y$ there exists a function $e \in E$ such that $e(y_1) \neq e(y_2)$. In this paper we say that $E$ satisfies the condition $(r)$ if for any triple $y \in \text{Ch}(E)$, a neighborhood $V$, and $\varepsilon > 0$ there exists $u \in E$ such that $0 \leq u \leq 1 = u(y)$ on $Y$ and $0 \leq u \leq \varepsilon$ on $Y \setminus V$.

Note that if $E$ satisfies the condition $(r)$, then it is extremely $C$-regular (cf. [11, Definition 2.3.9]).

**Example 5.3.** The space $C_0(Y, \mathbb{R})$ satisfies the condition $(r)$ for any locally compact Hausdorff space $Y$. Let

$$E = \{ f \in C_0((0,2], \mathbb{R}) : f(t) = at \text{ on } (0,1] \text{ for some } a \in \mathbb{R} \}.$$ 

Then $\text{Ch}(E) = [1,2]$ and $E$ satisfies the condition $(r)$.

Liu [15, Corollary 6] established the Mazur-Ulam property for $C_0(Y, \mathbb{R})$ when $Y$ is compact. The following generalizes the result of Liu.

**Corollary 5.4.** Let $Y$ be a locally compact Hausdorff space. Suppose that $E$ is a closed subspace of $C_0(Y, \mathbb{R})$ which separates the points of $Y$. Suppose that $E$ satisfies the condition $(r)$. Then $E$ satisfies the Mazur-Ulam property. In particular, $C_0(Y, \mathbb{R})$ satisfies the Mazur-Ulam property.

**Proof.** Let $\text{Ch}(E)$ be the Choquet boundary for $E$. Then by the Arens-Kelly theorem $P = \{ \delta_x : x \in \text{Ch}(E) \}$ is a set of representatives for $\mathcal{E}_E$. Suppose that $p \in P$ and $-1 \leq \alpha \leq 1$. Proving the inclusion

$$M_{p,\alpha} \supset \{ a \in S(E) : p(a) = \alpha \},$$

we get [11] by Lemma 4.3. It will follow by Proposition 5.1 that $E$ satisfies the Mazur-Ulam property. Let $a \in S(B_1)$ such that $p(a) = \alpha$. First we consider the case of $\alpha = 1$. By definition $a \in F_{p,1}$, hence $d(a, F_{p,1}) = 0 = 1 - |\alpha|$. For every $b \in F_{p,-1}$ we have $2 = |p(a) - p(b)| \leq \|a - b\| \leq 2$. Hence $d(a, F_{p,-1}) = 2 = 1 + |\alpha|$. Thus $a \in M_{p,\alpha}$ if $\alpha = 1$. A proof for $\alpha = -1$ is similar and is omitted.

We consider the case that $-1 < \alpha < 1$. Let $a \in S(B_1)$ such that $p(a) = \alpha$. Let $\varepsilon$ be $0 < \varepsilon < 1 - |\alpha|$. Put $K = \{ q \in P : q(a) = \alpha \}$ and $F_0 = \{ q \in P : |q(a) - \alpha| \geq \varepsilon/4 \}$. For each positive integer $n$, put $F_n = \{ q \in P : \varepsilon/2^{n+2} \leq |q(a) - \alpha| \leq \varepsilon/2^{n+1} \}$. Then $P = K \cup (\bigcup_{n=0}^{\infty} F_n)$. For each positive integer $n$, choose $u_n \in S(B_1)$ such that $0 \leq u_n \leq 1 = p(u_n)$ on $Y$ and $u_n = 0$ on $F_0 \cup F_n$. Since $P = \{ \delta_x : x \in \text{Ch}(E) \}$ and $E$ satisfies the condition $(r)$ such $u_n \in E$ exists. Put $u = \sum_{n=1}^{\infty} u_n/2^n$. As the supremum norm of $u_n$ is dominated by
1 for every $n$, $u$ converges uniformly on $Y$ in $E$. Then put

$$g_+ = \left( \frac{\alpha}{|\alpha|} - \alpha \right) u + a,$$

and

$$g_- = \left( - \frac{\alpha}{|\alpha|} - \alpha \right) u + a.$$

We see that $g_+ \in F_{p, \frac{a}{|\alpha|}}$, $\|g_+ - a\| = 1 - |\alpha|$, and $d(a, F_{p, \frac{a}{|\alpha|}}) \leq 1 - |\alpha|$. A proof is as follows. All what is really needed is to prove that $|q(g_-)| \leq 1$ for every $q \in P$ as it is evident that $g_+ \in E$ and $p(g_+) = \frac{\alpha}{|\alpha|}$ by the definition of $g_+$. If $q \in F_0$, then $q(u) = 0$ asserts that $|q(g_+)| = |q(a)| \leq 1$. Suppose that $q \in F_n$ for some positive integer $n$. Then $q(u_n) = 0$ and $0 \leq u_k \leq 1$ on $Y$ we have $|q(u)| \leq \sum_{k \neq n} 1/2^k = 1 - 1/2^n$. Thus

$$|q(g_+)| \leq \left| \frac{\alpha}{|\alpha|} - \alpha \right| (1 - 1/2^n) + |q(a) - \alpha| + |\alpha|$$

$$\leq (1 - |\alpha|)(1 - 1/2^n) + \varepsilon/2^{n+1} + |\alpha|$$

$$\leq (1 - |\alpha|)(1 - 1/2^n) + (1 - |\alpha|)/2^{n+1} + |\alpha| \leq 1$$

Suppose that $q \in K$. Then

$$|q(g_+)| \leq 1 - |\alpha| + |\alpha| = 1.$$

We conclude that $g_+ \in F_{p, \frac{a}{|\alpha|}}$. By the definition of $g_+$, we infer that $\|g_+ - a\| = (1 - |\alpha|)\|u\| = 1 - |\alpha|$. It follows that

$$d(a, F_{p, \frac{a}{|\alpha|}}) \leq 1 - |\alpha|.$$  \hspace{1cm} (12)

We also see that $g_- \in F_{p, -\frac{a}{|\alpha|}}$, $\|g_- - a\| = 1 + |\alpha|$ and $d(a, F_{p, -\frac{a}{|\alpha|}}) \leq 1 + |\alpha|$. A proof is as follows. All what is really needed is to prove that $|q(g_-)| \leq 1$ for every $q \in P$ as it is evident that $g_- \in E$ and $p(g_-) = -\frac{\alpha}{|\alpha|}$. If $q \in F_0$, then $q(u) = 0$ asserts that $|q(g_-)| = |q(a)| \leq 1$. Suppose that $q \in F_n$ for some positive integer $n$. We have

$$q(g_-) = \left(-\frac{\alpha}{|\alpha|} + \alpha \right) q(u) - 2\alpha q(u) + \alpha + q(a) - \alpha,$$

hence

$$|q(g_-)| \leq (1 - |\alpha|)|q(u)| + |\alpha||1 - 2q(u)| + |q(a) - \alpha|.$$  

As $0 \leq u \leq 1$ on $Y$ by the definition of $u$, we have $|1 - 2q(u)| \leq 1$ for every $q \in P = \{ \delta_x : x \in Ch(E) \}$. As we have shown that $|q(u)| \leq 1 - 1/2^n$ for

\[ q \in F_n \]
\[ |q(g_-)| \leq (1 - |\alpha|)|q(u)| + |\alpha| + |q(a) - \alpha| \]
\[ \leq (1 - |\alpha|)(1 - 1/2^n) + |\alpha| + \epsilon/2^{n+1} \]
\[ \leq (1 - |\alpha|)(1 - 1/2^n) + |\alpha| + (1 - |\alpha|)/2^{n+1} \leq 1. \]

Suppose that \( q \in K \). Then we have
\[ q(g_-) = \left(-\frac{\alpha}{|\alpha|} + \alpha\right)q(u) - 2\alpha q(u) + \alpha. \]

As \( 0 \leq u \leq 1 \) on \( Y \) we have
\[ |q(g_-)| \leq 1 - |\alpha| + |\alpha||1 - 2q(u)| \leq 1. \]

We conclude that \( g_- \in F_{p, \frac{a}{|\alpha|}} \). By the definition of \( g_- \), we infer that \( \|g_- - a\| = (1 + |\alpha|)\|u\| = 1 + |\alpha| \). It follows that
\[ d(a, F_{p, \frac{a}{|\alpha|}}) \leq 1 + |\alpha|. \]

By (12) and (13) we conclude that \( a \in M_{p, \alpha} \).

Finally, as \( C_0(Y, \mathbb{R}) \) satisfies the condition \( (r) \) by Urysohn’s lemma, we see that \( C_0(Y, \mathbb{R}) \) satisfies the Mazur-Ulam property. \( \square \)

**Example 5.5.** Let \( \ell^\infty(\Gamma, \mathbb{R}) \) be the real Banach space of all real-valued bounded functions on a discrete space \( \Gamma \). Then there is a compact Hausdorff space \( X \) such that \( \ell^\infty(\Gamma, \mathbb{R}) \) is isometrically isomorphic to \( C_0(X, \mathbb{R}) \) as a real Banach space. Therefore \( \ell^\infty(\Gamma, \mathbb{R}) \) satisfies the Mazur-Ulam property (cf. [10, Corollary 2]). The Mazur-Ulam property of the space \( c_0(\Gamma, \mathbb{R}) \) is established by Ding [10, Corollary 2]. As \( c_0(\Gamma, \mathbb{R}) \) is isometrically isomorphic to \( C_0(Y, \mathbb{R}) \) for a locally compact Hausdorff space \( Y \), Corollary 5.4 gives an alternative proof of the result of Ding.

**Example 5.6.** Let \( \mu \) be a positive measure on a \( \sigma \)-algebra \( \Sigma \) of a subsets of a set \( \Omega \). Let \( L^\infty(\Omega, \Sigma, \mu, \mathbb{R}) \) be the usual real Banach space of all real-valued bounded measurable functions on \( (\Omega, \Sigma, \mu) \). Then there exists a compact Hausdorff space \( X \) such that \( L^\infty(\Omega, \Sigma, \mu, \mathbb{R}) \) is isometrically isomorphic to \( C_0(X, \mathbb{R}) \) as a real Banach space. Hence the space \( L^\infty(\Omega, \Sigma, \mu, \mathbb{R}) \) satisfies the Mazur-Ulam property (cf. [15, Corollary 6]). Note that Tan [19, Theorem 2.5] exhibits the result for the case of the measure being \( \sigma \)-finite by an alternative proof.

### 6. Remarks

We close the paper with a few remarks. In this paper we merely prove the complex Mazur-Ulam property for a uniform algebra. We conjecture that
a uniform algebra, generally a closed subalgebra of $C_0(Y, \mathbb{C})$, satisfies the Mazur-Ulam property.

The second remark concerns Tingley’s problem on a Banach space of analytic functions. The main result of this paper concerns with the complex Mazur-Ulam property. We expect that several Banach space of analytic functions including the Hardy spaces satisfy the complex Mazur-Ulam property and the Mazur-Ulam property.

As a final remark we encourage researches on Tingley’s problem on Banach algebras of continuous functions. Comparing with the theorem of Wang [27] on the Banach algebra of $C^{(n)}(X)$, it is interesting to study a surjective isometry on the unit sphere of a Banach space or algebra of Lipschitz functions, it has already been pointed out by Cueto-Avellaneda [6, Problem 4.0.8].

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