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Insights into the electromagnetic properties of the resonant slow-wave circuits from the resonant cavity perspective

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ABSTRACT
A multigap resonant cavity can simply be regarded as either a slow-wave structure (SWS) with shorted ends or as a resonator. However, two distinct physical images will be obtained when characterizing the electromagnetic properties of such circuits. In particular, dispersion is no longer a necessary concept from the resonant cavity perspective. The periodicity of the circuit and the synchronization characteristics with the beam are incorporated in the longitudinal electric field shape. Thus, we present insights from the resonant cavity perspective by taking the resonant rectangular grating SWS as an example. It is found that even if the fields are originally expressed as normal modes, they can be transformed into the space harmonic form. This is thought to be the result of the periodicity of the circuit, which provides a connection between the multigap resonant cavity and the SWS. We present the findings and discuss their physical significance.

I. INTRODUCTION
Extended-interaction devices are promising terahertz radiation sources due to their high power, moderate bandwidth, and compact size.1–4 These features are largely due to the special interaction circuitry of their multigap resonant cavity. These devices are a type of resonant slow-wave structure (SWS) that are shorted at both ends, thereby leading to discrete axial modes.5,6 From the SWS perspective, the electromagnetic properties of the finite length resonant SWS have been studied,7 where the formation of axial modes was interpreted as the transmission and reflection of the space harmonics. This occurs for a multigap resonant cavity when the reflection coefficients at both ends are equal to 1. However, in this case, the resonant periodic circuit is thought to satisfy the same dispersion relation as an ideal SWS, except for the discretization of the phase shifts and frequencies. We know that dispersion is important for the SWS because it plays a vital role in the synchronization characteristics of the circuit for interactions with electron beams. This poses the question of how the dispersion relations for the synchronization characteristics of the resonant periodic circuits are characterized, particularly for the multigap resonant cavity. That is, what is the relationship of the dispersion with the multigap resonant cavity.

It is difficult to directly answer this question. However, if we alter our perspective, a more explicit picture arises. Operating under a certain mode, the multigap resonant cavity acts as an ordinary resonant cavity. Thus, we can address this from the pure resonator perspective. In this case, the frequency and fields form a complete description of a mode while the dispersion is an unnecessary concept. Then, the synchronization is determined.
from the shape of the electric field shape along the beam path. In fact, it has been long realized that the field shape plays an essential role in the synchronization characteristics for multigap resonant cavities.\textsuperscript{5,6,8} More recent studies demonstrated this for an oversized multigap resonant cavity.\textsuperscript{9}

It is reasonable to regard a multigap resonant cavity as a resonant SWS only when it has a sufficient number of circuit periods. The behavior of the multigap resonant cavity is highly dependent on its circuit length (or number of periods). We can address this more generally from the resonant cavity perspective.\textsuperscript{10,11} In particular, the circuit periodicity is not strictly required because the fields are expressed from the normal modes rather than the space harmonics. However, neither dispersion nor periodicity are required. Therefore, physical insights into the multigap cavity as a periodic circuit appear to have been lost, especially the connection with the SWS. Therefore, we demonstrate that even if the modes were originally expressed using the normal mode expansion, they can still be transformed into an expression of spatial harmonics. Thus, we establish a connection between two different physical images.

This paper is organized as follows. In Section II, the theoretical model is explained by taking a resonant rectangular grating SWS as an example. In Section III, the axial modes and the dispersion are analyzed. In Section IV, the expansion coefficients of the axial modes are examined, revealing the distribution rules for the nonzero terms. In Section V, the physical meaning of the distribution rules are explored. Finally, the summary and comments are given in Section VI.

II. MODEL FOR THE RESONANT WAVEGUIDE-GRATING CIRCUIT

A structural diagram of the resonant waveguide-grating circuit is shown in Fig. 1. The width of the rectangular waveguide is $w$, the grating period is $p$, the groove depth is $D$, the groove width is $b$, and the height from the grating surface to the top is $H$. The term “resonant” means that the circuit is a finite-length structure with shorted boundaries at $z = 0$ and $z = L$, where $L = N \times p$ and is the overall length with $N$ being the number of periods. In this case, the continuous dispersion curves for the ideal SWS turn into the discrete axial modes of the corresponding resonant cavity.

The normal modes of the structure can be classified as either transverse electric (TE) or transverse magnetic (TM) with respect to the $x$ direction. The modes of interest belong to the TE$^{(x)}$ type, where $E_x = 0$, and $H_x$ satisfies the scalar wave equation

$$\nabla^2 H_x(x, y, z) + k_0^2 H_x(x, y, z) = 0. \quad (1)$$

where $k_0^2 = \omega^2/c^2$, $\omega = 2\pi f$, $f$ is the frequency of the mode, and $c$ is the speed of light in vacuum. The structure is divided into two regions

![Schematic diagram of the resonant rectangular grating circuit shorted at both ends: (a) 3D view and (b) y-z cross section.](image_url)
to solve the boundary-value problem: the groove region (-D < y < 0) and the common region (0 < y < H). According to the boundary conditions for each region, the normal mode expansion for $H_x$ in the common region is given by

$$H_{x,i} = \sum_{m=0}^{\infty} A_m \sin(k_m x) \cosh(r_m(H-y)) \cos(k_m z),$$

(2)

where $k_m = m \pi / L$, $m = 0, 1, 2, \ldots$, $r_m^2 = k_m^2 - k_x^2$, and $k_x^2 = (\omega/c)^2 - k_z^2$.

In the grooves, there is

$$H_{x,i} = \sum_{m=0}^{\infty} R_m \sin(k_m x) \cosh(r_m(D+y)) \cos(k_m z).$$

(3)

where $k_{2s} = sn/b$, $r_{2s}^2 = k_{2s}^2 - k_x^2$, $s = 0, 1, 2, \ldots$, and $z_0^{(s)}$ is the local origin of the $s$th gap given by $z_0^{(s)} = (p-d)/2 + (i-1)s + p$ with the index $i$ varied from 1 to $N$.

The other field components can be easily obtained from $H_x$ using Maxwell’s equations, where the longitudinal electric field $E_z$ is given by

$$E_z = \frac{j \omega \mu_0}{k_x^2} \frac{\partial H_x}{\partial y}.$$  

(4)

Hence, the interaction electric field is given by

$$E_{i,z} = -\frac{j \omega \mu_0}{k_x^2} \sum_{m=0}^{\infty} A_m \sin(k_m x) \tau_m \sinh(r_m (D+y)) \cos(k_m z).$$

(5)

The dispersion relation can be obtained through field matching at the interface $y = 0$. The procedure is detailed in Refs. 10 and 11. As a result, the algebraic equation system is given by

$$\hat{F} \bar{x} = \left( \sum_{i=1}^{N} \Psi_i \hat{K} \Psi_i^T \right) \bar{x},$$

(6)

where $\bar{x}$ is a coefficient vector to be solved with the elements defined by $x_m = A_m F_m(0)$. The $\hat{F}$ and $\hat{K}$ are diagonal matrices given by

$$\hat{F}_m(y = 0) = \frac{L}{2} \tau_m \tanh(\tau_m H),$$

$$\hat{K}_i(y = 0) = \frac{2}{b} \tau_i \tanh(\tau_i D).$$

(7)

The elements of $\Psi_i$ are given by

$$\psi_m^{(i)} = \int_{z_0}^{z_0^{(i)}} \cos(\tau_m(z - z_0^{(i)})) \cos(k_m z) dz.$$  

(8)

The analytical results of the integral can be obtained, which are

$$\psi_m^{(i)} = \begin{cases} b, & k_m = k_m^0 = 0, \\ \frac{b}{2} \cos(k_m^0 z_0^{(i)}), & k_m = k_m^0 \neq 0, \\ 2k_m \left((-1)^{i-1} \sin(k_m z_0^{(i)} + b) - \sin(k_m z_0^{(i)}) \right), & k_m \neq k_m^0 \end{cases}$$

(9)

The dispersion relation is given by

$$\det(\hat{F} - \sum_{i=1}^{N} \Psi_i \hat{K} \Psi_i^T) = 0.$$  

(10)

We see from Eq. (10) that all grooves are required to solve the equation, which differs from the ideal SWS situation where only one structural cell is required. Another major difference is that Eq. (10) has multiple roots, which correspond to the eigenfrequencies of the modes rather than the dispersion relations between the frequency ($\omega$) and axial phase shift ($\beta$). Thus, no axial phase shift is needed to be specified to solve the equation. With the obtained frequencies, the field coefficient $A_m$ can be solved from Eq. (6), which completes the modes.

III. THE AXIAL MODES AND DISPERSION

We use the following parameters as an illustrative example: $D = 0.5$, $b = 0.25$, $h = 1.0$, $p = 0.76$, and $w = 2.5$ in units of mm. Figure 2 shows the axial modes solved from Eq. (10) for $N = 8$. As a comparison, the dispersion curves of the ideal SWS with the same structural parameters are simulated using the Microwave Studio electromagnetic simulation software (CST), as shown in the continuous curves.

From Ref. 7, for an $N$-periodic structure, there are $N+1$ axial modes denoted as 0 (or 2n), $\pi/N$, $2\pi/N$, $3\pi/N$, $\ldots$, $\pi$. Therefore, nine modes are expected for each passband for the 8-periodic circuit. However, we see that some modes are missing that should have been located at the ends of the dispersion curves because their fields cannot satisfy the electric boundary conditions at $z = 0$ and $z = L$.

It can be seen from Fig. 2 that the phase shift sequence above the axial modes reconstructs the dispersion curve of the ideal SWS. As the number of periods increases, there is better agreement because the last axial mode gradually approaches the $\pi$-end of the dispersion curves. However, we cannot conclude that they share the same dispersion characteristics. It has been noted that the phase shifts are not required for Eq. (10) to solve these specified axial modes, suggesting they are tags for the modes. For example, in terms of the phase reversals of the electric field, we obtain a different sequence of $2n$ (or 0), $\pi/N$, $2\pi/N$, $3\pi/N$, $\ldots$, $\pi$. The last mode is the $\pi$ mode due to the opposite phase of the electric field in the two adjacent gaps. The resulting dispersion curves formed by the axial modes can no longer remain consistent with the ideal SWS. Therefore, unlike the behavior of the SWS, dispersion is no longer required in the multigap resonant cavity, and the electric field shape along the beam path plays...
the essential role in characterizing the synchronization. Therefore, we neglect the dispersion and consider the effects from the fields.

IV. FIELD COEFFICIENT $A_m$

In the form of the normal mode expansion, the electromagnetic fields are determined from the expansion coefficients $A_m$. It is known that the longitudinal field shapes are distinct from each other for the axial modes in the same passband. Thus, it is reasonable to believe that such differences are observed in the expansion coefficients.

Figure 3 shows the expansion coefficients $A_m$ for the $2\pi$ mode, which is the first axial mode of the second passband. The second passband was used because it includes the $2\pi$ end mode. The ordinate is expressed in decibels as $A_m$(dB)=$\log_{10}(A_m/A_{\max})$, where $A_{\max}$ is the term with the maximum value corresponding to 0-dB. We see that only the projected terms have significant values, which are called principal terms. All terms around the principal term are approximately ten orders of magnitude smaller, which can be regarded as zeros according to the numerical calculations. More importantly, we observe that the distributions of the principal terms follow a linear relationship. In Fig. 3, the first principal term corresponds to $m=0$, which is followed by $m=16, 32, 48, 60, ...$, that is, $m=2rN$, where $N$ is the number of periods and $r$ is an integer.

Figure 4 shows the coefficients $A_m$ for the second axial mode. In comparison to Fig. 3, the major differences are that the principal terms are paired after the first one. Specifically, the first principal term is $m=1$ followed by $m=(15, 17), (31, 33), (47, 49), ...$, with a general formula of $m=2rN\pm 1$ with $r=1, 2, 3,...$

Similarly, a regular distribution for the principal terms can be found for the other modes. The last axial mode is shown in Fig. 5. It can be seen that the first principal term is for $m=7$, and the general formula for the following principal terms is $m=2rN\pm 7$, where $r=1, 2, 3,...$. We summarized the distribution of the principal terms in Table I for all eight axial modes of the second passband to find the rules.

Rule-1: The first principal term is $A_l$, where $l=0, 1, 2, ..., N-1$. The subscript $l$ is the mode index that indicates the sequence of the axial modes.

Rule-2: Except for the $2\pi$ mode, the principal terms are paired after the first one, and the interval between each pair is exactly equal to the mode index $l$.

Rule-3: The interval between adjacent pairs of principal terms is equal to $2N$.

According to Rule-2, the exceptional case of the $2\pi$ mode can be interpreted as the degeneration of a pair of principal terms when $l=0$. Moreover, we deduce that the lost mode at the $\pi$ end has a mode index $l=N$ if it exists. Thus, there would also be a degeneration of the principal terms for the true $\pi$ mode. However, degeneration would occur between two adjacent pairs of principal terms rather than between two terms in one pair. In the case of the $8$-period circuit, the sequence of the principal terms for the lost $\pi$ mode are $8, 24, 40, ..., m=(2r-1)N$, where $N=8$ and $r=1, 2, 3,...$

Finally, we find that the principal terms for the eight existing axial modes and the missing $\pi$ mode together constitute a complete sequence for $A_m$, where $m=1, 2, 3,...$. Moreover, not every single principal term simultaneously belongs to two modes. We verified the above rules for different situations, such as the modes in the first passband, more period numbers, and different geometric parameters, and no violations have been found.

V. FROM EIGENMODES TO SPACE HARMONICS

Since only the principal terms are meaningful in the series, we only consider them in the summation. We rewrite Eq. (5) for the electric field $E_z$ in the following form

$$E_{\text{z,l}} = \sum_{m=0}^{\infty} A_m e_m(x,y) \cos(k_m z),$$  

(11)
where
\[ e_n(x, y) = -\frac{j\omega t_0}{k^2} \nu_m \sin(x) \sinh(\nu_m(H-y)). \] (12)

For the axial mode, which is denoted by mode index \( l \), the distributions of the principal terms satisfy:
\[ m = \begin{cases} \frac{1}{2rN} & r = 0 \\ \frac{1}{2rN} & r = 1, 2, 3, \ldots \end{cases} \] (13)

where \( r = 0, 1, 2, \ldots, N-1 \).

Each \( r \) for \( r \neq 0 \) corresponds to two terms of \( m \). Considering only the principal terms, Eq. (11) transforms to
\[ E_{z,l}^{(l)} = A_0 e_l(x, y) \cos(\beta_l z) \]
\[ + \sum_{r=1}^\infty A_{2N-r} e_{2N-r}(x, y) \cos\left(\frac{2\pi r}{N} - \beta_l \right) z \]
\[ + \sum_{r=1}^\infty A_{2N+r} e_{2N+r}(x, y) \cos\left(\frac{2\pi r}{N} + \beta_l \right) z, \] (14)

where \( \beta_l = \pi l / L \). It can be seen that the summation index changes from \( m \) to \( r \).

If we define
\[ c_0 = A_1 \quad c_r = A_{2N+l} \quad c_{-r} = A_{2N-l}, \] (15)

considering
\[ \cos\left(\frac{2|\pi r|}{N} + \beta_l \right) \right) z = \cos\left(\frac{2|\pi r|}{N} - \beta_l \right) \right) z, \]
the second term of Eq. (14) can be regarded as a sum over \( r = -1, -2, -3, \ldots \). Then, we have
\[ E_{z,l}^{(l)} = \sum_{r=\infty}^{\infty} c_r e_l(x, y) \cos(k_r z), \] (16)

where
\[ k_r = \beta_l + \frac{2\pi r}{N} \quad r = 0, \pm 1, \pm 2, \ldots \] (17)

For the \( 2\pi \) mode with \( l = 0 \), a pair of principal terms degenerate into one term. If we define
\[ c_0 = A_0 \quad c_r = c_{-r} = \frac{1}{2} A_{2N}, \] (18)

the form of Eq. (16) remains unchanged. On the other hand, the form of Eq. (16) can be obtained from the viewpoint of the transmission and reflection of the traveling waves. A wave propagating forward in the same circuit is given by
\[ E_{z,l} = \sum_{n=-\infty}^{\infty} a_n e_n(x, y) e^{j\beta_n z}, \] (19)

where \( a_n = \beta + 2\pi n / p \) is the propagation constant of the spatial harmonics and \( \omega(\beta) \) satisfies the dispersion relation of the ideal SWS. When reflection is introduced, the phase of the wave in the system must satisfy an additional boundary condition, of
\[ \rho \exp\left[ j(\beta_F - \beta_B)L \right] = 1, \] (20)

where \( \rho \) is the round-trip reflection coefficient, and \( \beta_F \) and \( \beta_B \) are the propagation constants of the forward and the backward waves, respectively. For a lossless circuit shorted at both ends, \( \rho = 1 \) and \( \beta_B = -\beta_F \), which leads to
\[ \beta_F = \frac{n\pi}{L} \quad n = 0, 1, 2, \ldots, N \]
\[ a_n = a_{-n}. \] (21)

Equation (19) then becomes
\[ E_{z,l}^{(l)} = \sum_{n=-\infty}^{\infty} 2a_n e_n(x, y) \cos(k_n z), \] (22)

which is the same as Eq. (16). Thus, we find that the principal terms are a set of spatial harmonic coefficients with propagation constants as defined in Eq. (17). By comparing Eqs. (22) and (16), we obtain the relationship between the traveling-wave coefficients and the standing-wave coefficients,
\[ a_0 = \frac{1}{2} A_0 \quad a_n = a_{-n} = \frac{1}{4} A_{2n} \] (23)

We interpret the special behaviors of the space harmonics coefficients at the passband ends from the perspective of the axial modes. It can be seen that \( an = a_n \) at the \( n \) end due to the degeneracy of the principal terms for the \( 2\pi \) mode. At the other end, while the true \( \pi \) mode does not exist, it is predicted that \( an = a_{n+1} \) when \( l \rightarrow N \) with \( N \rightarrow \infty \). These conclusions are consistent with the findings from Refs. 7 and 12.

Therefore, we better understand resonant periodic circuits. In this way, we see that the multigap cavity has no dispersion. The periodicity of the circuit means that we can express the fields in terms of the normal modes or, equivalently, in the form of spatial harmonics regardless of the number of periods for the circuit. For one axial mode with a frequency \( \omega_0 \), the resulting spatial harmonics have a fundamental propagation constant \( \beta \). The relationship between \( \omega_0 \) and \( \beta \) for all axial modes satisfies the dispersion of the ideal SWS. In other words, the dispersion curves for the ideal SWS dictate the phase constant.
to directly express the field in the form of a space harmonics series.

VI. CONCLUSION

The electromagnetic properties of the resonant periodic structure were studied from the perspective of a pure resonant cavity viewpoint taking a rectangular grating SWS as an example. The circuit was originally modeled using the normal mode expansion method. Therefore, we derived the eigenequation for the modes instead of the dispersion relation of the space harmonics. Particular attention was given to the expansion coefficients of the fields. It was found that there are clear regularities in the distributions of the nonzero terms, as summarized in Rules 1–3. Based on these rules, we transform the field expression from the normal mode expansion into the space harmonics series. Moreover, the same form was obtained from the SWS perspective based on the transmission and reflection of the space harmonics. The physical significance of the principal (nonzero) terms was found to be a set of space harmonics coefficients.

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