Critical rotation of a harmonically trapped Bose gas

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We study experimentally and theoretically a cold trapped Bose gas under critical rotation, \textit{i.e.} with a rotation frequency close to the frequency of the radial confinement. We identify two regimes: the regime of explosion where the cloud expands to infinity in one direction, and the regime where the condensate spirals out of the trap as a rigid body. The former is realized for a dilute cloud, and the latter for a Bose-Einstein condensate with the interparticle interaction exceeding a critical value. This constitutes a novel system in which repulsive interactions help in maintaining particles together.

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The rotation of a macroscopic quantum object is a source of spectacular and counter-intuitive phenomena. In superfluid liquid helium contained in a cylindrical bucket rotating around its axis \( z \), one observes the nucleation of quantized vortices for a sufficiently large rotation frequency \( \Omega \). A similar phenomenon occurs in a Bose-Einstein condensate confined in a rotating harmonic trap \( \text{\[2, 3, 4, 5\]} \). In particular, vortices are nucleated when the condensate spirals out of the trap as a rigid body. The former is realized for a dilute cloud, and the latter for a Bose-Einstein condensate with the interparticle interaction exceeding a critical value. This constitutes a novel system in which repulsive interactions help in maintaining particles together.

Several theoretical studies have recently considered the critical rotation of the gas, \textit{i.e.} \( \Omega \sim \omega_\perp \), which presents remarkable features \( \text{\[6, 7, 8, 9, 10, 11, 12, 13, 14\]} \). From a classical point of view, for \( \Omega = \omega_\perp \), the centrifugal force compensates the harmonic trapping force in the \( xy \) plane. Hence the motion of a single particle of mass \( m \) in the frame rotating at frequency \( \Omega \) is simply due to the Coriolis force \( 2m \ddot{r} \times \Omega \). This force is identical to the Lorentz force acting on a particle of charge \( q \) in the magnetic field \( \mathbf{B} = 2(m/q) \Omega \). The analogy between the motion of charged particles in a magnetic field and neutral particles in a rotating frame also holds in quantum mechanics. In this respect, a quantum gas of atoms confined in a harmonic trap rotating at the critical frequency is analogous to an electron gas in a uniform magnetic field. One can then expect \( \text{\[\Box\]} \) to observe phenomena related to the Quantum Hall Effect.

This paper presents an experimental and theoretical study of the dynamics of a magnetically trapped rubidium (\(^{87}\text{Rb}\)) gas stirred at a frequency close to \( \omega_\perp \). We show that the single particle motion is dynamically unstable for a window of frequencies \( \Omega \) centered around \( \omega_\perp \). This result entails that the center-of-mass of the atom cloud (without or with interatomic interactions) is destabilized, since its motion is decoupled from any other degree of freedom for a harmonic confinement. This also implies that a gas of non-interacting particles “explodes”, which we indeed check experimentally. When one takes into account the repulsive interactions between particles, which play an important role in a \(^{87}\text{Rb}\) condensate, one would expect naively that this explosion is enhanced. However, we show experimentally that this is not the case, and repulsive interactions can “maintain the atoms together”. This has been predicted for a Bose-Einstein condensate in the strongly interacting – Thomas-Fermi (TF) – regime \( \text{\[\[\]]\} \). Here we derive the minimal interaction strength which is necessary to prevent the explosion. This should help studies of the Quantum Hall related physics in the region of critical rotation.

Consider a gas of particles confined in an axisymmetric harmonic potential \( V_0(r) \), with frequency \( \omega_z \) along the trap axis \( z \), and \( \omega_\perp \) in the \( xy \) plane. To set this gas into rotation, one superimposes a rotating asymmetric potential in the \( xy \) plane. In the reference frame rotating at an angular frequency \( \Omega \) around the \( z \) axis, this potential reads \( V_1(r) = \epsilon m \omega^2 (Y^2 - X^2)/2 \), where \( \epsilon > 0 \). The rotating frame coordinates \( X, Y \) are deduced from the lab frame coordinates \( x, y \) by a rotation at an angle \( \Omega t \).

For a non-interacting gas, the equation of motion for each particle reads:

\begin{align}
\dot{X} - 2\Omega \dot{Y} + (\omega_\perp^2 (1 - \epsilon) - \Omega^2) X &= 0 \\
\dot{Y} + 2\Omega \dot{X} + (\omega_\perp^2 (1 + \epsilon) - \Omega^2) Y &= 0,
\end{align}

while the motion along \( z \) is not affected by the rotation. One deduces from this set of equations that the motion in the \( xy \) plane is dynamically unstable if the stirring frequency \( \Omega \) is in the interval \( [\omega_\perp \sqrt{1 - \epsilon}, \omega_\perp \sqrt{1 + \epsilon}] \). In particular, for \( \Omega = \omega_\perp \) and \( \epsilon \ll 1 \), one finds that the quantity \( X + Y \) diverges as \( \exp(\omega_\perp t/2) \), whereas \( X - Y \) remains finite.

To test this prediction we use a \(^{87}\text{Rb}\) cold gas in an Ioffe-Pritchard magnetic trap, with frequencies \( \omega_x = \omega_y = 2\pi \times 180 \text{ Hz} \), and \( \omega_z = 2\pi \times 11.7 \text{ Hz} \). The initial tem-
constant with the measured displacement increases exponentially, with an exponent the quasi-pure condensate (Fig. 1b). The center-of-mass motion both for the non-condensed cloud (Fig. 1a) and for the condensate with an exponent consistent with the measured \( \epsilon \).

After evaporative cooling, the atomic cloud is stirred during an adjustable period \( t \) by a focused laser beam of wavelength 852 nm and waist \( w_0 = 20 \mu m \), whose position is controlled using acousto-optic modulators. The beam is switched on abruptly and it creates a rotating optical-dipole potential which is nearly harmonic over the extension of the cloud. We measure the transverse density profile of the condensate after a period of free expansion. In this pursuit, we suddenly switch off the magnetic field and the stirrer, allow for a 25 ms free-fall, and image the absorption of a resonant laser by the expanded cloud. The imaging beam propagates along the \( z \) axis. We fit the density profile of the sample assuming a Gaussian shape for the non-condensed cloud, and a parabolic TF shape for the quasi-pure condensate. We extract from the fit the long and short diameters in the \( xy \) plane. The long radius increases with time, while the short one remains approximately constant (Fig. 2a). On the opposite, we find that the condensate remains circular (Fig. 2b), with no systematic increase in size. We then obtain the following counter-intuitive result: for a significant repulsive interaction the atoms remain in a compact cloud, while they fly apart if the interaction is negligible. We observe this stability of the shape of the condensate rotating at the critical velocity for \( \epsilon \leq 0.2 \). Above this value of \( \epsilon \) we find that the atomic cloud rapidly disintegrates. For \( \epsilon \approx 0.3 \), after a stirring time of 50 ms, we observe several fragments in the time-of-flight picture.

We now perform a theoretical analysis of how the interparticle interaction stabilizes a rotating condensate. To this end we use the 2D \( (x, y) \) time-dependent Gross-Pitaevskii (GP) equation for an idealized cylindrical trap \( (\omega_z = 0) \). In the rotating frame the GP equation reads:

\[
\frac{i}{2} \partial_t \psi = \frac{1}{2} \left[-\Delta + (1 - \epsilon) X^2 + (1 + \epsilon) Y^2 + 2g|\psi|^2 - 2\Omega \hat{L}_z \right] \psi, \tag{3}
\]

where \( \hat{L}_z \) is the \( z \) component of the angular momentum operator. In Eq. (3) the coordinates are given in units of the initial harmonic oscillator length \( \sqrt{\hbar/m \omega_z} \), and the frequencies in units of \( \omega_z \). The condensate wave function \( \psi(X, Y, t) \) is normalized to unity, and the effective coupling constant is \( g = 4\pi a \tilde{N} \), with \( a \) being the positive scattering length, and \( \tilde{N} \) the number of particles per unit axial length. The effective coupling \( g \) depends on density and characterizes the ratio of the mean-field interparticle interaction to the radial frequency \( \omega_z \).

Since the trapping potential is harmonic, the average center of mass motion of the condensate is described by the classical equations (1) and is decoupled from the evolution of the condensate wave function in the center of mass reference frame. We shall therefore restrict to wave functions \( \psi \) centered at \( x = y = 0 \) for all times.

We start with a variational analysis of the steady state of the condensate in the rotating frame, using a Gaussian...
ansatz for the condensate wave function [15]:

$$\psi(X,Y) \propto \exp(i\alpha XY - \beta X^2/2 - \gamma Y^2/2). \quad (4)$$

We use the symmetry properties of the Hamiltonian and assume that the condensate wave function remains invariant under the combination of a time reversal ($\psi \rightarrow \psi^*$) and a reflection with respect to the $XZ$ plane. This implies that the parameters $\alpha$, $\beta$, $\gamma$ are real numbers. We extremize the GP energy functional with respect to these parameters. Extremizing with respect to the phase parameter $\alpha$ gives $\alpha = \Omega(\gamma - \beta)/(\gamma + \beta)$. As $\beta$ and $\gamma$ should be finite and positive this sets the constraint $\alpha^2 < \Omega^2$. Extremizing over $\beta$ and $\gamma$ and expressing $\beta/\gamma$ in terms of $\alpha$, we obtain a closed equation for $\alpha$:

$$\left(\frac{\epsilon}{\Omega}\right) \left[ \alpha^2 + 2\alpha \Omega(\Omega^2 - 1)/\epsilon + \Omega^2 \right] - \left(\frac{g}{2\pi}\right) \sqrt{1 - \alpha^2/\Omega^2} \left[ \alpha^3 + (1 - 2\Omega^2)\alpha - \Omega\epsilon \right] = 0. \quad (5)$$

In the non-interacting case ($g = 0$) the Gaussian ansatz is exact, and $\alpha$ is the root of a quadratic equation. This ansatz also captures the scaling properties of the rotating condensate in the regime of strong interactions. In the TF limit ($g \rightarrow \infty$) the first line of (5) can be neglected and we recover the cubic equation for $\alpha$ derived in [1].

For $g = 0$ the parameter $\alpha$ is complex in the interval of rotation frequencies $\sqrt{1 - \epsilon} < \Omega < \sqrt{1 + \epsilon}$, and there is no steady state solution for the condensate wave function at these $\Omega$ [6]. For a finite $g$ the lower border $\Omega_-$ of this frequency interval remains equal to $\sqrt{1 - \epsilon}$ irrespective of the value of $g$. The upper border, dashed curves in Fig.3, decreases with increasing $g$ at a given $\epsilon$. For small anisotropy $\epsilon < 1/5$ it reaches the lower border at a critical coupling strength. For larger $g$ the steady states exist at any $\Omega$. If $\epsilon > 1/5$, the upper border never reaches $\Omega_+ = \sqrt{1 + \epsilon}$ and for any $g$ one has an interval of $\Omega$ where steady state solutions are absent. The $\epsilon = 1/5$ threshold was derived for the TF limit in [6].

We now analyze the time evolution of the condensate after the stirring potential has been switched on. For this purpose we use an approximate scaling approach to the solution of Eq.(5). We assume (and later on check) that the evolution of the condensate shape is well described by dilations with factors $b_u(t)$ and $b_v(t)$ along the axes $X$ and $Y$, rotating at an angular frequency $\phi(t)$ with respect to the laboratory frame $x, y$. To determine $b_u(t), b_v(t)$, and $\phi(t)$, we write the wave function as

$$\psi(\tilde{X},\tilde{Y},t) = (b_u b_v)^{-1/2} \chi(u,v,t) \exp\left(i\Phi(\tilde{X},\tilde{Y},t)\right), \quad (6)$$

where we have set $u = \tilde{X}/b_u$, $v = \tilde{Y}/b_v$, and

$$\Phi(\tilde{X},\tilde{Y},t) = \tilde{\alpha}(t) \tilde{X}\tilde{Y} + \left(b_u/2b_v\right)\tilde{X}^2 + \left(b_v/2b_u\right)\tilde{Y}^2, \quad (7)$$

with $\tilde{\alpha} = -\dot{\phi} \tanh \xi$ and $\xi(t) = \ln(b_v/b_u)$. Then the GP equation reduces to the following equation for $\chi(u,v,t)$:

$$i \left( \partial_t - \frac{\phi}{\cosh \xi} (u\partial_u - v\partial_v) \right) \chi = \left[ -\frac{\partial_u^2}{2b_u^2} - \frac{\partial_v^2}{2b_v^2} + \frac{1}{2} (\nu^2 u^2 + \nu^2 v^2) \right] \chi \quad (8)$$

The “frequencies” $\nu_u$ and $\nu_v$ are given by

$$\nu^2 = 1 + \epsilon \cos(2\Omega t - 2\phi) + \alpha^2 + 2\phi\partial_t + b_{u,v}/b_{u,v}. \quad (9)$$

In Eq.(8) we took into account that $u, v$ are eigenaxes of the condensate, which requires the absence of terms proportional to $uv$ and is provided by the relation

$$\epsilon \sin(2\Omega t - 2\phi) - \dot{\alpha} + \tilde{\alpha}(b_u/b_v + b_v/b_u) + \dot{\phi} \xi = 0. \quad (10)$$

We now replace the lhs of Eq.(8) by $\tilde{\mu} \chi$, where $\tilde{\mu}$ follows from the normalization condition for $\chi$. The solution of the resulting equation is a function of $b_{u,v}, \phi$ and $\nu_{u,v}$. We then require that (8) is a relevant scaling transform, i.e., that the function $\chi(u,v,t)$ is most similar to the initial function $\chi(u,v,0)$. More precisely we set the averages $\langle u^2 \rangle_t, \langle v^2 \rangle_t$ equal to their values at $t = 0$. This fixes $\nu_u$ and $\nu_v$ in terms of $b_u, b_v, \phi$. The solution of Eqs.(8) and (10) then gives the desired scaling parameters.

The omitted lhs of Eq.(8) only insignificantly changes $\nu_u$ and $\nu_v$. It vanishes in both the TF regime and for an ideal-gas condensate. For the TF limit our procedure gives the same results as the scaling approach of [1].

For an abrupt switching of the rotating potential we use the initial conditions $b_{u,v}(0) = 1$, $b_{u,v}(0) = \phi(0) = 0$. We find two types of solution: (i) oscillating functions $b_{u,v}(t)$, (ii) one of the scaling parameters eventually grows exponentially. Case (ii) describes an infinite expansion of the condensate in one direction, similarly to the expansion of the ideal gas under rotation.

For a given $\epsilon$ we obtain the upper $(\Omega_+)$ and lower $(\Omega_-)$ instability borders in the $\Omega - g$ space (see Fig.3). The lower border is always equal to $\sqrt{1 - \epsilon}$. The upper border
Ω+(g) decreases with increasing g and for ϵ < 0.17 it reaches Ω− at a critical value of the coupling strength. For ϵ > 0.17 we have Ω+ > Ω− at any g.

The obtained results can be understood by comparing the frequency Ωq of the rotating quadrupole mode of the condensate with the rotation frequency Ω ~ 1 of the perturbation $V_1(r)$. In the absence of interaction one has Ωq = 1, and the corresponding resonance leads to the condensate explosion. The interactions reduce the frequency of the rotating quadrupole mode ($Ω_q = 1/\sqrt{2}$ in the TF limit), suppressing the resonance at $Ω \approx 1$: the deformation of the condensate induced by $V_1$ remains small, at least for ϵ smaller than the detuning from the resonance. For larger ϵ the condensate explodes.

In the presence of interactions ($1/\sqrt{2} < Ω_q < 1$), one could expect naively that the explosion occurs for a resonant drive with Ω ∼ Ωq, even for small ϵ. This is not the case because of a non-linear character of the dynamics. As the system starts to elongate under the action of the resonant excitation, it becomes closer to an ideal gas, for which Ωq = 1. The gas is then driven away from the resonance and its deformation stops. This explains why the lower instability border Ω− is independent of g.

The scaling method can also be used to identify stationary solutions, by setting $\dot{\phi} = Ω$ and constant $b_{u,v}$ in Eqs. (3,4). The results nearly coincide with the ones from the Gaussian ansatz. The existence of these solutions can also be explored using an adiabatic switching of the rotating potential. As shown in Fig. 3 the domain of instability for an abrupt switching of the rotation includes the domain for the absence of stationary solutions.

In Fig. 4 we present the minimum coupling strength $g_c(ε)$ required for the stability of the shape of the condensate, for both abrupt and adiabatic switching of the potential rotating at $Ω = 1$.

To summarize, our analysis shows that the condensate can preserve its shape and size for rotation frequencies in the instability window for the center of mass motion, $\sqrt{1-ε} < Ω < \sqrt{1+ε}$. This means that the condensate will spiral out of the trap as a rigid body after the rotation is switched on. For TF condensates this is the case if $ε \lesssim 0.28$, which explains why the repulsive inter-particle interaction maintains particles together in our experiment. On the other hand, for larger $ε$ there are always frequencies at which even TF condensates are unstable. This gives an account for the destruction of the condensate in our experiment at $ε \approx 0.3$. One can think of observing related effects in rotating ion clouds in electromagnetic traps (see [17] and refs. therein).

Even though our picture properly describes the experiment, we are likely not to deal with the ground state of the system. On a very long time scale, the rotating gas can evolve to a more complex state, for instance to a multi-vortex state (with possible quantum melting) discussed in [17, 18, 19, 20, 21, 22], or to a Quantum-Hall-like state [23, 24, 25]. However these states have been discussed for the axially symmetric case and a natural development will be to include a finite rotating anisotropy $ε$, which is a necessary ingredient for most present experiments. In particular, for $Ω = ω_1/\sqrt{1-ε}$, one reaches a 1-body Hamiltonian corresponding to an unbound motion (with a gauge field) in the X direction, similar to a Quantum Hall fluid in a narrow channel. We believe that the study of many body aspects of this regime will bring in new features of quantum mesoscopic physics.

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