Yang-Mills Inspired Solutions for General Relativity

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Abstract

Several exact, cylindrically symmetric solutions to Einstein’s vacuum equations are given. These solutions were found using the connection between Yang-Mills theory and general relativity. Taking known solutions of the Yang-Mills equations (e.g. the topological BPS monopole solutions) it is possible to construct exact solutions to the general relativistic field equations. Although the general relativistic solutions were found starting from known solutions of Yang-Mills theory they have different physical characteristics.

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I. THE ERNST EQUATIONS

Recently several exact solutions for the Yang-Mills field equations were found \cite{1} \cite{2} by using the correspondence between Yang-Mills theory and general relativity. The idea was to use known general relativistic solutions, such as the Schwarzschild and Kerr solutions, to find analogous Yang-Mills solutions. It was conjectured that these solutions may be connected with the confinement mechanism in QCD, since just as the Schwarzschild solution of general relativity will confine any particle which carries the gravitational “charge” behind its event horizon, so the corresponding Yang-Mills solution might confine particles behind its “event horizon”. Although the Yang-Mills solutions were functionally similar to the analogous general relativistic solutions there were significant physical differences between the two. For example, the spherical singularity at \( r = 2GM \) of the Schwarzschild solution is a coordinate singularity whereas, in the Yang-Mills case the spherical singularity is a true singularity in the gauge fields and in the energy density. Thus, although the solutions look functionally similar, they nevertheless have some different physical characteristics (e.g. the \( r = 2GM \) singularity in the Schwarzschild solution acts as a one way membrane while the equivalent singularity of the Yang-Mills solution appears to act as a two way barrier).

In the present paper we want to invert the above process and use known solutions of the Yang-Mills field equations to find solutions to Einstein’s vacuum equations. The best known exact solutions of the Yang-Mills field equations are the monopole solutions (e.g. the Prasad-Sommerfield-Bogomolnyi \cite{3} solution). These Yang-Mills monopole solutions can be viewed as topological solitons, whose fields are non-singular. The standard interpretation of this Yang-Mills solution is as a localized particle which carries magnetic (and possibly electric) charge. In contrast the corresponding general relativistic BPS solutions, which are presented here, do not seem to have an interpretation as arising from a localized distribution of gravitational charge (mass-energy). In particular the general relativistic version of the BPS monopoles are not asymptotically flat making their physical meaning unclear. It is also found that two different forms of the BPS monopole solution, give physically different
solutions when carried over to general relativity.

To examine the connection between solutions to Einstein’s equations and solutions to the Yang-Mills equations we use the Ernst equations \( \text{[4]} \). The Ernst equations were originally formulated to simplify the general relativistic field equations for axially symmetric solutions (particularly solutions which were parameterized via the Papapetrou metric). Later it was shown \( \text{[5]} \) that using the axially symmetric ansatz of Manton \( \text{[6]} \) one could write the field equations of an SU(2) Yang-Mills-Higgs system in the form of the Ernst equations. For axially symmetric solutions to Einstein’s field equations one can write the down the metric using the Papapetrou ansatz

\[
\begin{align*}
    ds^2 &= f(\rho, z)[dt - \omega(\rho, z)d\phi]^2 - \frac{1}{f(\rho, z)}[e^{2\gamma(\rho, z)}(d\rho^2 + dz^2) + \rho^2 d\phi^2] \\
    &= \left(\begin{array}{cc}
        f & e^{2\gamma} \\
        0 & 1
    \end{array}\right)^2 - \frac{1}{f} [\partial^2_{\rho\rho} + \partial^2_{\rho z} + \rho^2 \partial^2_{\phi\phi}]
\end{align*}
\]

Plugging this ansatz into the vacuum Einstein field equations yields four coupled differential equations for the ansatz functions \( f(\rho, z) \), \( \omega(\rho, z) \) and \( \gamma(\rho, z) \) \( \text{[4]} \)

\[
\begin{align*}
    f\nabla^2 f &= \nabla f \cdot \nabla f - \frac{f^4}{\rho^2} \nabla \omega \cdot \nabla \omega \\
    \nabla \cdot \left( \frac{f^2}{\rho^2} \nabla \omega \right) &= 0 \\
    \frac{\partial \gamma}{\partial \rho} &= \frac{\rho}{4f^2} \left[ \left( \frac{\partial f}{\partial \rho} \right)^2 - \left( \frac{\partial f}{\partial z} \right)^2 \right] - \frac{f^2}{4\rho} \left[ \left( \frac{\partial \omega}{\partial \rho} \right)^2 - \left( \frac{\partial \omega}{\partial z} \right)^2 \right] \\
    \frac{\partial \gamma}{\partial z} &= \frac{\rho}{2f^2} \frac{\partial f}{\partial \rho} \frac{\partial f}{\partial \rho} - \frac{f^2}{2\rho} \frac{\partial \omega}{\partial \rho} \frac{\partial \omega}{\partial \rho} \\
\end{align*}
\]

Ernst was able to re-write these field equations through the introduction of a complex potential

\[\epsilon = f + i\Psi\]

in terms of which some of the field equations became

\[Re(\epsilon) \nabla^2 \epsilon = \nabla \epsilon \cdot \nabla \epsilon \]

or more explicitly

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\[ f \nabla^2 f = \nabla f \cdot \nabla f - \nabla \Psi \cdot \nabla \Psi \]
\[ f \nabla^2 \Psi = 2 \nabla f \cdot \nabla \Psi \quad (5) \]

The function \( \omega \) is determined from \( \Psi \) via \[ \nabla \omega = \frac{\rho}{f^2} \hat{n} \times \nabla \Psi \quad (6) \]

where \( \hat{n} \) is the unit vector in the azimuthal direction. Since our ansatz functions only depend on \( \rho \) and \( z \) we can re-write the above condition as \[ \frac{\partial \omega}{\partial z} = -\frac{\rho}{f^2} \frac{\partial \Psi}{\partial \rho} \quad \frac{\partial \omega}{\partial \rho} = \frac{\rho}{f^2} \frac{\partial \Psi}{\partial z} \quad (7) \]

Thus if a solution is found to the Ernst equation in terms of the \( f \) and \( \Psi \) functions, then one can use Eq. (7) to determine the original function \( \omega \). Then the final ansatz function \( \gamma \) can be determined using the last two equations of Eq. (2). An important point to emphasize is that once \( f \) and \( \Psi \) are found (or alternatively \( f \) and \( \omega \)) then the solution is found up to an integration, since the last two equations of Eq. (2) automatically imply that the integrability condition, \( \partial^2 \gamma/\partial z \partial \rho = \partial^2 \gamma/\partial \rho \partial z \), is satisfied. Thus a unique solution to \( \gamma \) can be given via a line integration \[ \ref{7} \]. It may not be possible, however, to obtain a closed form solution for \( \gamma \).

### II. Monopole Solutions of Einstein’s Equations

Several authors have looked for exact solutions for an SU(2) system using Ernst or modified Ernst equations \[ \ref{5} \] \[ \ref{10} \]. Using this technique it is possible to construct monopole and multi-monopole solutions for the Yang-Mills field equations. Since the Ernst equation is also used to examine exact solutions in general relativity, it should be possible to take the Yang-Mills monopole solutions written in the formulation of Refs. \[ \ref{5} \] \[ \ref{10} \] and arrive at corresponding general relativistic solutions. The BPS monopole solutions have several good features such as having non-singular fields and finite energy. It was originally hoped that the general relativistic versions would inherit these good features, however, the general
relativistic solutions discussed here either have metrics whose components are singular at some point or they do not become asymptotically flat.

In Refs. [10] Chakrabarti and Koukiou obtain various solutions to the Yang-Mills equations by starting with a seed solution to a modified version of the Ernst equations. Then by applying Harrison-Neugebauer type transformations [11] [12] on this seed solution they obtain various monopole solutions of the Yang-Mills equations. (Forgács et. al. [5] work directly with the Ernst equation rather than the Ernst-like equation used in Ref. [10]. However their version of the BPS monopole solution is much more complicated than in Ref. [10]. Thus it is easier to obtain a closed form general relativistic solution starting with the monopole solution in the form given by Chakrabarti and Koukiou). Before trying to map over the monopole solution into a general relativistic solution we will examine the easier example of how the seed solution of Ref. [10] can be used to give a solution to Einstein’s vacuum equations. The general seed solution used in Ref. [10] is

\[ f(r) = \exp\left(-br - ar^2 \cos(\theta)\right) \quad \Psi(r) = 0 \quad (8) \]

where \( a \) and \( b \) are arbitrary constants, and we have used spherical coordinates in writing out the solution as in Ref. [10]. This seed solution satisfied a modified Ernst equation, which is related to the Ernst equation, Eq. (3), by the transformation \( r \to 1/r \). Thus to turn the solution of Eq. (8) into a solution of Eq. (5) we apply the same transformation to the solution. This yields

\[ f(\rho, z) = \exp\left(-\frac{b}{\sqrt{\rho^2 + z^2}} - \frac{az}{(\rho^2 + z^2)^{3/2}}\right) \quad \Psi(\rho, z) = 0 \quad (9) \]

where we have also changed from spherical to cylindrical coordinates, since these are the coordinates in which the original ansatz functions in Eq. (2) are formulated. It is easily checked by direct substitution that Eq. (2) solves Eq. (5). Since \( \Psi(\rho, z) = 0 \) in the above solution \( \omega(\rho, z) = 0 \) by Eq. (7). This is a static solution with no angular momentum. Using \( f \) from Eq. (2) the function \( \gamma(\rho, z) \) can be found by integrating the last two equations of Eq. (2) to yield a closed form result.
\[ \gamma(\rho, z) = \frac{9a^2 \rho^4}{16(\rho^2 + z^2)^4} - \frac{a \rho^2(a + bz)}{2(\rho^2 + z^2)^3} - \frac{b^2 \rho^2}{8(\rho^2 + z^2)^2} \]  (10)

It is easy to see that this solution becomes asymptotically flat \((i.e. \ f \to 1, \ \omega = \gamma \to 0\) as \(\sqrt{\rho^2 + z^2} = r \to \infty\)). By looking at the far field behaviour of this solution we find that the Newtonian potential is

\[ \Phi = (g_{00} - 1)/2 = (f - 1)/2 \approx -\frac{b}{2r} + \frac{1}{2r^2} \left( \frac{b^2}{2} - a \cos(\theta) \right) \]  (11)

Thus the constant \(b\) is related to the mass of the solution \((i.e. \ b = 2GM)\) and the \(a\) term looks like a dipole term. If \(a = 0\) we just recover the Curzon metric \([13]\). If \(b = 0\) the Newtonian far field potential looks like the dipole potential of electromagnetism. This could be taken to indicate that this special case of the solution is not physical. However recent work \([14]\) on massless black holes also finds a Newtonian potential whose leading term falls off like \(1/r^2\) rather than \(1/r\). The physical interpretation of these massless objects was as bound states of positive and negative mass. For the \(b = 0\) case of the above solution it may be even more appropriate to consider the possibility that the solution represents some kind of positive-negative mass bound state since the far field has exactly the kind of angular dependence one would expect of a dipole field, while the solution in Ref. \([14]\) only has the \(1/r^2\) behaviour, but not the dipole angular dependence. In a certain sense the solution given by Eqs. \((9)\) and \((10)\) is not mathematically very interesting since it is just a specific example of a Weyl solution. However it is an asymptotically free, closed form solution, and in light of Ref. \([14]\) it may be of some physical interest. Although this solution becomes asymptotically flat it has the undesirable feature that some of the components of its metric become singular at \(r = 0\) \((e.g. \ g_{33} \) diverges as \(\sqrt{\rho^2 + z^2} = r \to 0\) since \(f \to 0\)).

A more interesting solution, which is not just a particular example of a Weyl solution is the BPS monopole solution. In Ref. \([10]\) it is found that the BPS monopole can be expressed in terms of the ansatz functions of the Ernst-like equation as

\[ f(r) = csch(r) \quad \Psi(r) = i \coth(r) \]  (12)
Where we have again written the solution in spherical coordinates. In this form the connection to the BPS solution is very apparent since these are exactly the hyperbolic functions used in the BPS monopole solution. Since \( \Psi(r) \) is imaginary, \( \omega \) will also be imaginary, which makes the solution unphysical. However taking the complementary hyperbolic functions of the above solution we find that we can get a completely real valued solution. Making the transformation \( r \to 1/r \) (so that our solution satisfies the Ernst equation rather than the modified Ernst equation of Ref. [10]), and switching to cylindrical coordinates (so that the solutions can be checked in Eqs. (2)) we find the following real solution to the Ernst equation

\[
f(\rho, z) = D \tanh \left( a + \frac{b}{\sqrt{\rho^2 + z^2}} \right) \quad \Psi(\rho, z) = D \tanh \left( a + \frac{b}{\sqrt{\rho^2 + z^2}} \right)
\]

where \( D, a, \) and \( b \) are constants, and we have generalized the solution somewhat by introducing the constant \( a \). Using Eq. (7) we can determine the ansatz function \( \omega \) from \( \Psi \).

Integrating the equations gives

\[
\omega(\rho, z) = \frac{bz}{D\sqrt{\rho^2 + z^2}} = \frac{b\cos(\theta)}{D}
\]

where in the last step we have written the result in spherical coordinates. Since \( \omega \neq 0 \) this is a stationary solution as opposed to the first solution which was static. A non-zero \( \omega \) usually indicates a source with some angular momentum. However, for a body with angular momentum \( S \) one would expect \( \omega \to -2S \rho^2/(\rho^2 + z^2)^{3/2} \) as \( r = \sqrt{\rho^2 + z^2} \to \infty \) [9] which is not the case here. This behaviour of \( \omega \) indicates that this solution does not become asymptotically flat, and makes a physical interpretation difficult. There are other known stationary solutions, such as the NUT-Taub metric [15], the Lewis metric [16] and the Van Stockum metric [17], which also do not become asymptotically flat. The ansatz function \( \omega \) can be made small by letting \( D \) become large and/or allowing \( b \) to become small. Finally using Eq. (2) we can determine the last ansatz function

\[
\gamma(\rho, z) = \frac{-b^2 \rho^2}{8(\rho^2 + z^2)^2} = \frac{-b^2 \sin^2(\theta)}{8r^2}
\]

where in the last step we have again used spherical coordinates. Thus this form of the BPS monopole solution gives an exact, closed form solution to Einstein’s vacuum equations, and
the components of the metric tensor are non-singular except at the origin (in particular $g_{33} \to \infty$ as $r \to 0$). At large distances (i.e. $r \to \infty$) $\gamma \to 0$ as one would want for an asymptotically flat solution. Also if one requires that $D = \cosh(a)$ then $f \to 1$ as $r \to \infty$ also as expected for an asymptotically flat solution. However $\omega$ does not go to zero at large distances (except in the $x - y$ plane where $z = 0$ or $\theta = \pi/2$), thus this solution can not be viewed as some finite, localized distribution of rotating mass. To determine the physical meaning of the arbitrary constants of this solution we can again examine the Newtonian potential at large distances

$$\Phi = (f - 1)/2 \approx -\frac{1 + D\text{sech}(a)}{2} - \frac{bD\text{sech}(a) \tanh(a)}{2r} + \frac{D\text{sech}(a)(-b^2/2 + b^2 \tanh^2(a))}{2r^2} + \mathcal{O}(1/r^3) \quad (16)$$

First we can choose $D = \cosh(a)$ so that the leading term of the potential goes as $1/r$. Then we can set $b \tanh(a) = 2GM$ so that $a$ and $b$ appear to be related to the mass of the solution. Finally as a special case we could take $\tanh(a) = 0$. In this case we would obtain a Newtonian potential similar to that of the massless Reissner solution or to the far field found in Ref. [14]. It is not clear what physical use if any this closed form solution to the vacuum equations may have. Since this solution is not asymptotically flat, it can not represent the exterior field of some localized distribution of rotating matter. Since only $\omega$ does not approach its correct asymptotic value, this solution appears to have a source with an infinite angular momentum. The similarity between this solution and the NUT-Taub solution [15] should be pointed out. The Newtonian potential of both the present metric and the NUT-Taub metric fall off as $1/r$ at large distances. More importantly the $g_{03}$ term of both solutions have exactly the same form, and this keeps both solutions from being asymptotically flat.

Finally it was shown in Ref. [10] that the BPS monopole could also be obtained from the following alternative form of the solution to the modified Ernst equations

$$f(\rho, z) = \frac{\sinh(r) \sin(\theta)}{r} \quad \Psi(\rho, z) = \cos(\theta) \quad (17)$$
Going through the usually transformation to turn this into a solution of the regular Ernst equation \((i.e. \ r \to 1/r)\), scaling the variable \(r \ (i.e. \ r \to b r)\), and finally converting to cylindrical coordinates we arrive at the following solution to the Ernst equations

\[
\begin{align*}
f(\rho, z) &= b \rho \sinh \left( \frac{1}{b \sqrt{\rho^2 + z^2}} \right) \\
\Psi(\rho, z) &= \frac{z}{\sqrt{\rho^2 + z^2}}
\end{align*}
\] (18)

Using the above function \(\Psi\) in Eq. (7) we find the ansatz function \(\omega\)

\[
\omega(\rho, z) = \frac{1}{b} \coth \left( \frac{1}{b \sqrt{\rho^2 + z^2}} \right)
\] (19)

Already one can see that this solution will not give an asymptotically flat solution, since as \(\sqrt{\rho^2 + z^2} = r \to \infty\), \(\omega \to \infty\) rather than 0. The last ansatz function, \(\gamma\), can not be obtained in closed form in this case. As \(r \to \infty\) one can obtain the following approximate form for the equations that determine \(\gamma\)

\[
\begin{align*}
\frac{\partial \gamma}{\partial \rho} &\approx \frac{z^2 - \rho^2}{4 \rho (\rho^2 + z^2)} \\
\frac{\partial \gamma}{\partial z} &\approx -\frac{z}{2 (\rho^2 + z^2)}
\end{align*}
\] (20)

These can be integrated to give

\[
\gamma(\rho, z) \approx \frac{1}{4} \ln \left( \frac{\rho}{\rho^2 + z^2} \right)
\] (21)

which is valid as \(\sqrt{\rho^2 + z^2} = r \to \infty\). Thus the ansatz function, \(\gamma\), does not indicate an asymptotically flat solution. The asymptotic behaviour here is worse than for the previous solution since both \(\omega\) and \(\gamma\) diverge as \(r \to \infty\). In addition \(f\) is divergent as \(r \to 0\). Even though the solutions of Eq. (12) and Eq. (17) yield the same field configuration for the Yang-Mills equations \((i.e. \ they \ both \ give \ the \ BPS \ monopole)\) they give apparently different solutions when carried over to general relativity. Also the physical characteristics of the Yang-Mills solution do not necessarily carry over into the general relativistic solution \((e.g. \ the \ BPS \ monopole \ is \ well \ behaved \ over \ all \ space, \ while \ the \ three \ general \ relativistic \ solutions \ presented \ here \ have \ some \ undesired \ features: \ they \ are \ not \ asymptotically \ flat \ or \ the \ components \ of \ the \ metric \ become \ singular \ at \ certain \ points)\). At this point it is not
known whether the singularities are real features of these solutions or whether they might not be coordinate singularities as is the case for the event horizon of the Schwarzschild solution.

III. DISCUSSION AND CONCLUSIONS

We have presented three solutions to Einstein’s vacuum field equations which were found by exploiting the connection between Yang-Mills theory and general relativity. In Ref. [3] it was shown that the Yang-Mills field equations could be put in the form of the Ernst equations of general relativity through the use of Manton’s ansatz [3]. Previously this framework has been used by the author to map the Kerr solution of general relativity into a Yang-Mills counterpart [2]. Here we have carried out this procedure in reverse: by starting with the known monopole solutions to the Yang-Mills field equations we obtained new solutions to the general relativistic field equations. In doing this we used the BPS monopole solution in the forms given in Ref. [10] where the monopole solution was derived from a modified Ernst equation. This made it straightforward to covert the Yang-Mills solutions to general relativistic counterparts. The first solution studied in this paper was the general seed solution used in Ref. [10] to obtain the monopole solutions via Harrison-Neugebauer transformations. This Yang-Mills solution gave a general relativistic solution, which was a generalization of the Curzon metric. In the special case of this solution where the constant $b$ was set to zero (i.e. the mass of the solution became zero) we found that the far field Newtonian potential behaved like a dipole field. Thus this solution may have some connection with some recent work on black diholes [14]. This solution had a singularity at the origin in some of the components of its metric and it became asymptotically flat.

The second solution which we examined - Eq. (12) - was one form of the BPS monopole. The general relativistic version of this solution - Eqs. (13) (14) (15) - did not become asymptotically flat as $r \to \infty$ (in particular $\omega$ did not go to zero). This may be related to the fact that in the BPS solution the Higgs field does not go to zero as $r \to \infty$. Since $\omega$
did not go to zero asymptotically one could interpret this solution as having a source with infinite angular momentum. The final solution which we examined was a different version of the BPS monopole. This alternative form of the BPS monopole gave a general relativistic solution with a different asymptotic behaviour and different physical characteristics from the general relativistic solution obtained from the version of the BPS monopole given by Eq. (12). This highlights the fact that although solutions of one theory can be used to find solutions in the other, the physical characteristics of the original solution are not all necessarily inherited by the new solution. For example, in the Yang-Mills version of the Schwarzschild solution the spherical singularity of the solution is a true singularity, while for the general relativistic Schwarzschild solution the spherical singularity is a coordinate singularity. Nevertheless, both Yang-Mills and general relativity do seem to share some degree of mathematical similarity at the level of the classical field equations, which allows one to use the solutions of one theory to obtain solutions in the other.

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