Certain Results on Lorentzian Para-Kenmotsu Manifolds

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ABSTRACT: The object of the present paper is to study Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection. First, we study Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection satisfying the curvature conditions $\bar{R} \cdot \bar{S} = 0$ and $\bar{S} \cdot \bar{R} = 0$. Next, we study $\phi$-conformally flat, $\phi$-conharmonically flat, $\phi$-concircularly flat, $\phi$-projectively flat and conformally flat Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection and it is shown that in each of these cases the manifold is a generalized $\eta$-Einstein manifold.

Key Words: Lorentzian para-Kenmotsu manifold, $\eta$-Einstein manifold, Curvature tensor, Quarter-symmetric metric connection.

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2010 Mathematics Subject Classification: 53D15, 53C05, 53C25.
Conformally flat Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection

1. Introduction

Let \((M, g)\) be an \(n\)-dimensional connected semi-Riemannian manifold of class \(C^\infty\) and \(\nabla\) be its Levi-Civita connection. The Riemannian-Christoffel curvature tensor \(R\), the projective curvature tensor \(P\), the concircular curvature tensor \(V\), the conharmonic curvature tensor \(K\) and the conformal curvature tensor \(C\) of \((M, g)\) are defined by

\[
R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.1}
\]

\[
P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[g(Y,Z)QX - g(X,Z)QY], \tag{1.2}
\]

\[
V(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.3}
\]

\[
K(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y \tag{1.4}
\]

\[
+ g(Y,Z)QX - g(X,Z)QY],
\]

\[
C(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \tag{1.5}
\]

\[
- g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y],
\]

respectively, where \(r\) is the scalar curvature, \(S\) is the Ricci tensor and \(Q\) is the Ricci operator such that \(S(X,Y) = g(QX,Y)\).

A linear connection \(\bar{\nabla}\) defined on \((M, g)\) is said to be a quarter-symmetric connection \([8]\) if its torsion tensor \(T\)

\[
T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] \tag{1.6}
\]

satisfies

\[
T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y,
\]

where \(\eta\) is a 1-form and \(\phi\) is a (1,1)-tensor field. If moreover, a quarter-symmetric connection \(\bar{\nabla}\) satisfies the condition

\[
(\bar{\nabla}_X g)(Y,Z) = 0, \tag{1.7}
\]

where \(X,Y,Z \in \chi(M)\) and \(\chi(M)\) is the set of all differentiable vector fields on \(M\), then \(\bar{\nabla}\) is said to be a quarter-symmetric metric connection. If we change \(\phi X\) by \(X\), then the connection is known as semi-symmetric metric connection \([7]\). Thus the notion of quarter-symmetric connection generalizes the notion of semi-symmetric...
connection. A quarter-symmetric metric connection have been studied by many
geometers in several ways to a different extent such as ([1], [3], [5], [6], [9], [12], [15])
and many others.

A relation between the quarter-symmetric metric connection \( \bar{\nabla} \) and the Levi-
Civita connection \( \nabla \) in a Lorentzian para-Kenmotsu manifold \( M \) is given by
\[
\bar{\nabla} X Y = \nabla X Y + \eta(Y)\phi X - g(\phi X, Y)\xi.
\] (1.8)

The paper is organized as follows: In Section 2, we give a brief introduction of Lorentzian para-Kenmotsu manifolds. In Section 3, we establish the relation between the curvature tensors of the Riemannian connection and the quarter-
symmetric metric connection in a Lorentzian para-Kenmotsu manifold. Lorentzian
para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection
satisfying the curvature conditions \( \bar{R} \cdot \bar{S} = 0 \) and \( S \cdot \bar{R} = 0 \) have studied in Sections 4
and 5 respectively. Sections 6, 7, 8, 9 and 10 are devoted to study \( \phi \)-conformally flat, 
\( \phi \)-conharmonically flat, \( \phi \)-concircularly flat, \( \phi \)-projectively flat and conformally flat
Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric
connection, respectively.

2. Preliminaries

Let \( M \) be an \( n \)-dimensional Lorentzian metric manifold. If it is endowed with a
structure \( (\phi, \xi, \eta, g) \), where \( \phi \) is a \((1,1)\) tensor field, \( \xi \) is a vector field, \( \eta \) is a 1-form
on \( M \) and \( g \) is a Lorentz metric, satisfying [2]
\[
\phi^2 X = X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),
\]
(2.1)
\[
\eta(\xi) = -1, \quad g(X, \xi) = \eta(X)
\] (2.2)
for any \( X, Y \) on \( M \), then it is called Lorentzian almost paracontact manifold. In
the Lorentzian almost paracontact manifold, the following relations hold:
\[
\phi \xi = 0, \quad \eta(\phi X) = 0,
\] (2.3)
\[
\Phi(X, Y) = \Phi(Y, X),
\] (2.4)
where \( \Phi(X, Y) = g(X, \phi Y) \).
If \( \xi \) is a killing vector field, the (para) contact structure is called \( K \)-(para) contact.
In such a case, we have
\[
\nabla X \xi = \phi X.
\] (2.5)
A Lorentzian almost paracontact manifold \( M \) is called Lorentzian para-Sasakian
manifold if
\[
(\nabla X \phi) Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi
\] (2.6)
for any vector fields \( X, Y \) on \( M \).

Now, we define a new manifold called Lorentzian para-Kenmostu manifold:
Definition 2.1. A Lorentzian almost paracontact manifold \( M \) is called Lorentzian \( \text{para-Kenmotsu manifold} \) if
\[
(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X
\]
for any vector fields \( X, Y \) on \( M \).

In the Lorentzian \( \text{para-Kenmotsu manifold} \), we have
\[
\nabla_X \xi = -X - \eta(X)\xi, \quad (2.8)
\]
\[
(\nabla_X \eta)Y = -g(X, Y) - \eta(X)\eta(Y), \quad (2.9)
\]
where \( \nabla \) denotes the operator of covariant differentiation with respect to the Lorentzian metric \( g \).

Further, on a Lorentzian \( \text{para-Kenmotsu manifold} \) \( M \), the following relations hold:
\[
g(R(X,Y)Z, \xi) = \eta(R(X,Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \quad (2.10)
\]
\[
R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \quad (2.11)
\]
\[
R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad (2.12)
\]
\[
R(\xi, X)\xi = X + \eta(X)\xi, \quad (2.13)
\]
\[
S(X, \xi) = (n - 1)\eta(X), \quad S(\xi, \xi) = -(n - 1), \quad (2.14)
\]
\[
Q\xi = (n - 1)\xi, \quad (2.15)
\]
\[
S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y) \quad (2.16)
\]
for any vector fields \( X, Y \) and \( Z \) on \( M \).

Example 2.2. We consider the 3-dimensional manifold
\[
M^3 = \{(x, y, z) \in \mathbb{R}^3 : z > 0 \},
\]
where \( (x, y, z) \) are the standard coordinates in \( \mathbb{R}^3 \). Let \( e_1, e_2 \) and \( e_3 \) be the vector fields on \( M^3 \) given by
\[
e_1 = \frac{z}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = z \frac{\partial}{\partial z} = \xi,
\]
which are linearly independent at each point of \( M^3 \) and hence form a basis of \( T_p M^3 \).

Define a Lorentzian metric \( g \) on \( M^3 \) as
\[
g(e_1, e_1) = 1, \quad g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1, \quad g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.
\]
Let \( \eta \) be the 1-form on \( M^3 \) defined as \( \eta(X) = g(X, e_3) = g(X, \xi) \) for all \( X \in \chi(M) \), and let \( \phi \) be the \((1, 1)\)-tensor field on \( M^3 \) defined as
\[
\phi e_1 = -e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = 0.
\]
By applying linearity of $\phi$ and $g$, we have
\[
\eta(\xi) = g(\xi, \xi) = -1, \quad \phi^2 X = X + \eta(X)\xi, \quad \eta(\phi X) = 0,
\]
\[
g(X, \xi) = \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)
\]
for all $X, Y \in \chi(M)$.

Let $\nabla$ be the Levi-Civita connection with respect to the Lorentzian metric $g$. Then we have
\[
[e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_1, e_3] = -e_1, \quad [e_3, e_1] = e_1, \quad [e_2, e_3] = -e_2, \quad [e_3, e_2] = e_2.
\]

The Riemannian connection $\nabla$ of the Lorentzian metric $g$ is given by
\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]),
\]
which is known as Koszul’s formula. Using Koszul’s formula, we can easily calculate
\[
\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -e_1, \quad \nabla_{e_2} e_1 = 0,
\]
\[
\nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = -e_2, \quad \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.
\]

Let $X = \sum_{i=1}^{3} X^i e_i = X^1 e_1 + X^2 e_2 + X^3 e_3 \in \chi(M)$.

Also, one can easily verify that
\[
\nabla_X \xi = -X - \eta(X)\xi \quad \text{and} \quad (\nabla_X \phi) Y = -g(\phi X, Y)\xi - \eta(Y)\phi X.
\]

Now let
\[
X = \sum_{i=1}^{3} X^i e_i = X^1 e_1 + X^2 e_2 + X^3 e_3,
\]
\[
Y = \sum_{j=1}^{3} Y^j e_j = Y^1 e_1 + Y^2 e_2 + Y^3 e_3,
\]
\[
Z = \sum_{k=1}^{3} Z^k e_k = Z^1 e_1 + Z^2 e_2 + Z^3 e_3
\]
for all $X, Y, Z \in \chi(M)$. It is known that
\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z.
\]

From the equations (2.17) and (2.18), it can be easily verified that
\[
R(e_1, e_2)e_1 = -e_2, \quad R(e_1, e_3)e_1 = -e_3, \quad R(e_2, e_3)e_1 = 0,
\]
\[
R(e_1, e_2)e_2 = e_1, \quad R(e_1, e_3)e_2 = 0, \quad R(e_2, e_3)e_2 = -e_3,
\]
\[ R(e_1, e_2)e_3 = 0, \quad R(e_1, e_3)e_3 = -e_1, \quad R(e_2, e_3)e_3 = -e_2. \]

With the help of above expressions of the curvature tensors, it follows that

\[ R(X,Y)Z = g(Y,Z)X - g(X,Z)Y. \]

Hence, the manifold \((M^3, \phi, \xi, \eta, g)\) is a Lorentzian para-Kenmotsu manifold of constant curvature 1 and is locally isometric to the unit sphere \(S^3(1)\).

**Definition 2.3.** A Lorentzian para-Kenmotsu manifold \(M\) is said to be an \(\eta\)-Einstein manifold if its Ricci tensor \(S\) is of the form

\[ S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \]

where \(a\) and \(b\) are scalar functions on \(M\).

A Lorentzian para-Kenmotsu manifold \(M\) is said to be a generalized \(\eta\)-Einstein manifold if its Ricci tensor \(S\) is of the form

\[ S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + c\Phi(X,Y), \]

where \(a, b, c\) are scalar functions on \(M\) and \(\Phi(X,Y) = g(\phi X, Y)\). If \(c = 0\), then the manifold reduces to an \(\eta\)-Einstein manifold.

3. Curvature tensor of Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection

The curvature tensor \(\bar{R}\) of a Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection \(\bar{\nabla}\) is defined by

\[ \bar{R}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z. \tag{3.1} \]

From the equations (1.8), (2.1), (2.3), (2.7)-(2.9), we get

\[ \bar{R}(X,Y)Z = R(X,Y)Z + g(Y,Z)\phi X - g(X,Z)\phi Y + g(\phi Y,Z)X \]

\[ -g(\phi X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y, \tag{3.2} \]

where

\[ R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \]

is the Riemannian curvature tensor of the connection \(\nabla\). Taking inner product of (3.2) with \(W\), we have

\[ \bar{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(Y,Z)g(\phi X,W) - g(X,Z)g(\phi Y,W) \]

\[ +g(\phi Y,Z)g(X,W) - g(\phi X,Z)g(Y,W) + g(\phi Y,Z)g(\phi X,W) - g(\phi X,Z)g(\phi Y,W), \]

where \(\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W)\) and \(R(X,Y,Z,W) = g(R(X,Y)Z,W)\). Contracting (3.3) over \(X\) and \(W\), we get

\[ \bar{S}(Y,Z) = S(Y,Z) + (n - 2 + \psi)g(\phi Y,Z) + (\psi - 1)g(Y,Z) - \eta(Y)\eta(Z), \tag{3.4} \]
where $S$ and $\bar{S}$ are the Ricci tensors with respect to the connections $\nabla$ and $\bar{\nabla}$, respectively on $M$ and $\psi=\text{trace}\; \phi$.

From (3.4), we have

\[
\bar{Q}Y = QY + (n - 2 + \psi)\phi Y + (\psi - 1)Y - \eta(Y)\xi, \tag{3.5}
\]

where $Q$ and $\bar{Q}$ are the Ricci operators with respect to the connections $\nabla$ and $\bar{\nabla}$, respectively on $M$. Contracting (3.4) over $Y$ and $Z$, we get

\[
\bar{r} = r + (2\psi - 1)(n - 1) + \psi^2, \tag{3.6}
\]

where $r$ and $\bar{r}$ are the scalar curvatures with respect to the connections $\nabla$ and $\bar{\nabla}$, respectively on $M$.

Writing two more equations by the cyclic permutations of $X, Y$ and $Z$, we have

\[
\bar{R}(Y, Z)X = R(Y, Z)X + g(Z, X)\phi Y - g(Y, X)\phi Z + g(\phi Z, X)Y \tag{3.7}
\]

\[
- g(\phi Y, X)Z + g(\phi Z, X)\phi Y - g(\phi Y, X)\phi Z,
\]

\[
\bar{R}(Z, X)Y = R(Z, X)Y + g(X, Y)\phi Z - g(Z, Y)\phi X + g(\phi X, Y)Z \tag{3.8}
\]

\[
- g(\phi Z, Y)X + g(\phi X, Y)\phi Z - g(\phi Z, Y)\phi X.
\]

By adding (3.2), (3.7) and (3.8) and using the fact that $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$, we get

\[
\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = 0. \tag{3.9}
\]

Thus we can state that, if the manifold is a Lorentzian para-Kenmotsu, then the curvature tensor with respect to the quarter-symmetric metric connection satisfies the first Bianchi identity.

From (3.2), clearly

\[
\bar{R}(X, Y, Z, W) = -\bar{R}(Y, X, Z, W), \tag{3.10}
\]

\[
\bar{R}(X, Y, Z, W) = -\bar{R}(X, Y, W, Z). \tag{3.11}
\]

\[
\bar{R}(X, Y, Z, W) = \bar{R}(Z, W, X, Y). \tag{3.12}
\]

Combining equations (3.10)-(3.12), we have

\[
\bar{R}(X, Y, Z, W) = \bar{R}(Y, X, W, Z) = \bar{R}(W, Z, Y, X). \tag{3.13}
\]

Thus, in view of the equations (3.10)-(3.12), we can state the following theorem:

**Theorem 3.1.** The curvature tensor of type $(0, 4)$ of a Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is

(i) Skew-symmetric in first two slots,

(ii) Skew-symmetric in last two slots,

(iii) Symmetric in pair of slots.
Lemma 3.2. Let $M$ be an $n$-dimensional Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection. Then
\begin{align*}
\bar{R}(X,Y)\xi &= \eta(Y)X - \eta(X)Y + \eta(Y)\phi X - \eta(X)\phi Y, \quad (3.14) \\
\bar{R}(\xi,X)Y &= -\bar{R}(X,\xi)Y = g(X,Y)\xi - \eta(Y)X - \eta(Y)\phi X + g(\phi X,Y)\xi, \quad (3.15) \\
\bar{R}(\xi,X)\xi &= \eta(X)\xi + X + \phi X, \quad (3.16) \\
\bar{S}(X,\xi) &= (n + \psi - 1)\eta(X), \quad \bar{S}(\xi,\xi) = -(n + \psi - 1), \quad (3.17) \\
Q\xi &= (n + \psi - 1)\xi \quad (3.18)
\end{align*}
for all $X,Y$ on $M$.

4. Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection satisfying $\bar{R}\cdot\bar{S} = 0$

In this section we consider a Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection $\bar{\nabla}$ satisfying the condition $\bar{R}\cdot\bar{S} = 0$.

Then we have
\begin{equation}
\bar{S}(\bar{R}(X,Y)U,V) + \bar{S}(U,\bar{R}(X,Y)V) = 0 \quad (4.2)
\end{equation}
for any vector fields $X,Y,U,V \in \chi(M)$. Putting $X = \xi$ in (4.2), it follows that
\begin{equation}
\bar{S}(\bar{R}(\xi,Y)U,V) + \bar{S}(U,\bar{R}(\xi,Y)V) = 0. \quad (4.3)
\end{equation}

In view of (3.15) and (3.17), (4.3) yields
\begin{equation}
(n + \psi - 1)g(Y,U)\eta(V) - \eta(U)\bar{S}(Y,V) - \eta(U)\bar{S}(\phi Y,V) \quad (4.4)
\end{equation}
\begin{equation}
+ (n + \psi - 1)g(\phi Y,U)\eta(V) + (n + \psi - 1)g(Y,V)\eta(U) - \eta(V)\bar{S}(U,\phi Y) + (n + \psi - 1)g(\phi Y,V)\eta(U) = 0.
\end{equation}

By taking $U = \xi$ in (4.4) and using (2.2), we get
\begin{equation}
\bar{S}(Y,V) + \bar{S}(\phi Y,V) = (n + \psi - 1)g(Y,V) + (n + \psi - 1)g(\phi Y,V). \quad (4.5)
\end{equation}

In view of (3.4), (4.5) takes the form
\begin{equation}
S(Y,V) + S(\phi Y,V) = (2 - \psi)g(Y,V) + (3 - n - \psi)\eta(Y)\eta(V) + (2 - \psi)g(\phi Y,V). \quad (4.6)
\end{equation}

Thus we can state the following theorem:

Theorem 4.1. For a Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection satisfying the condition $\bar{R}\cdot\bar{S} = 0$, the Ricci tensor $S$ is given by (4.6).
5. Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection satisfying $\bar{S} \bar{R} = 0$

In this section we consider a Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection $\nabla$ satisfying the condition

$$(\bar{S}(X, Y), \bar{R})(U, V)Z = 0$$

(5.1)

for any vector fields $X, Y, Z, U, V \in \chi(M)$.

This implies that

$$(X \wedge_S Y)\bar{R}(U, V)Z + \bar{R}((X \wedge_S Y)U, V)Z + \bar{R}(U, (X \wedge_S Y)V)Z$$

$$+ \bar{R}(U, V)(X \wedge_S Y)Z = 0,$$

(5.2)

where the endomorphism $X \wedge_S Y$ is defined by

$$(X \wedge_S Y)Z = \bar{S}(Y, Z)X - \bar{S}(X, Z)Y.$$

(5.3)

Taking $Y = \xi$ in (5.2), we have

$$(X \wedge_S \xi)\bar{R}(U, V)Z + \bar{R}((X \wedge_S \xi)U, V)Z + \bar{R}(U, (X \wedge_S \xi)V)Z$$

$$+ \bar{R}(U, V)(X \wedge_S \xi)Z = 0.$$

(5.4)

From (3.17), (5.3) and (5.4), we have

$$(n + \psi - 1)[\eta(\bar{R}(U, V)Z)X + \eta(U)\bar{R}(X, V)Z + \eta(V)\bar{R}(U, X)Z$$

$$+ \eta(Z)\bar{R}(U, V)X] - \bar{S}(X, \bar{R}(U, V)Z)\xi - \bar{S}(X, U)\bar{R}(\xi, V)Z$$

$$- \bar{S}(X, V)\bar{R}(U, \xi)Z - \bar{S}(X, Z)\bar{R}(U, V)\xi = 0.$$

Taking inner product of (5.5) with $\xi$, we get

$$(n + \psi - 1)[\eta(\bar{R}(U, V)Z)\eta(X) + \eta(U)\eta(\bar{R}(X, V)Z)$$

$$+ \eta(V)\eta(\bar{R}(U, X)Z) + \eta(Z)\eta(\bar{R}(U, V)X)] + \bar{S}(X, \bar{R}(U, V)Z)$$

$$- \bar{S}(X, U)\eta(\bar{R}(\xi, V)Z) - \bar{S}(X, V)\eta(\bar{R}(U, \xi)Z) - \bar{S}(X, Z)\eta(\bar{R}(U, V)\xi) = 0.$$

By taking $U = Z = \xi$ in (5.6) and using (3.14)-(3.17) and (10), we get

$$\bar{S}(X, V) + \bar{S}(X, \phi V) = -(n + \psi - 1)g(X, V) - 2(n + \psi - 1)\eta(X)\eta(V)$$

$$- (n + \psi - 1)g(\phi V, X).$$

(5.7)

In view of (3.4), (5.7) takes the form

$$S(X, V) + S(X, \phi V) = -(2n + 3\psi - 4)g(X, V) - (3n + 3\psi - 5)\eta(X)\eta(V)$$

$$- (2n + \psi)g(\phi X, V).$$

(5.8)

Thus we can state the following theorem:

**Theorem 5.1.** For a Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection satisfying the condition $\bar{S} \bar{R} = 0$, the Ricci tensor $S$ is given by (5.8).
6. \(\phi\)-conformally flat Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Analogous to the equation (1.5), the conformal curvature tensor \(\bar{C}\) with respect to the quarter-symmetric metric connection is defined by

\[
\bar{C}(X,Y)\bar{Z} = \bar{R}(X,Y)\bar{Z} - \frac{1}{(n-2)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,\bar{Z})QX - g(X,\bar{Z})QY],
\]

where \(\bar{R}, \bar{S}\) and \(\bar{r}\) are the Riemannian curvature tensor, the Ricci tensor and the scalar curvature with respect to the connection \(\bar{\nabla}\), respectively on \(M\).

**Definition 6.1.** A Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is said to be \(\phi\)-conformally flat if [13,14]

\[
\phi^2\bar{C}(\phi X,\phi Y)\phi Z = 0
\]

for all \(X,Y,Z\) on \(M\).

Let \(M\) be an \(n\)-dimensional \(\phi\)-conformally flat Lorentzian para-Kenmotsu manifold with the quarter-symmetric metric connection. Then from (6.2), it follows that

\[
g(\bar{C}(\phi X,\phi Y)\phi Z,\phi W) = 0. \quad (6.3)
\]

From the equations (6.1) and (6.3), we have

\[
g[\bar{R}(\phi X,\phi Y)\phi Z,\phi W] = \frac{1}{(n-2)}[\bar{S}(\phi Y,\phi Z)g(\phi X,\phi W) - \bar{S}(\phi X,\phi Z)g(\phi Y,\phi W) - \bar{S}(\phi X,\phi W)g(\phi Y,\phi Z) + (\psi - 1)g(\phi X,\phi Z)g(\phi Y,\phi W)]. \quad (6.4)
\]

In view of (2.1), (2.3), (3.2) and (3.4), (6.4) takes the form

\[
g[\bar{R}(\phi X,\phi Y)\phi Z,\phi W] - g(\phi X,\phi Z)g(\phi Y,\phi W) + g(\phi Y,\phi Z)g(\phi X,\phi W)
\]

\[
+ g(\phi X,\phi W)g(\phi Y,\phi Z) - g(\phi X,\phi W)g(\phi Y,\phi Z)
\]

\[
+ g(Y,\phi Z)g(X,\phi W) - g(X,\phi Z)g(Y,\phi W) = \frac{1}{(n-2)}[(S(\phi Y,\phi Z) + (n-2 + \psi)g(Y,\phi Z) + (\psi - 1)g(\phi Y,\phi Z)]g(\phi X,\phi W)
\]

\[
- \{S(\phi X,\phi Z) + (n-2 + \psi)g(X,\phi Z) + (\psi - 1)g(\phi X,\phi Z)\}g(\phi Y,\phi W)
\]

\[
+ \{S(\phi X,\phi W) + (n-2 + \psi)g(X,\phi W) + (\psi - 1)g(\phi X,\phi W)\}g(\phi Y,\phi Z)
\]

\[
- \{S(\phi Y,\phi W) + (n-2 + \psi)g(Y,\phi W) + (\psi - 1)g(\phi Y,\phi W)\}g(\phi X,\phi Z)]
\]
\[ -\frac{\ddot{r}}{(n-1)(n-2)}[g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)g(\phi Y,\phi W)]. \]

Let \( \{e_1, e_2, \ldots, e_{n-1}, \xi\} \) be a local orthonormal basis of vector fields in \( M \). Using that \( \{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, \xi\} \) is also a local orthonormal basis in \( M \). If we put \( X = W = e_i \) in (6.5) and sum up with respect to \( i \), then we have

\[ \sum_{i=1}^{n-1} g[R(\phi e_i, \phi Y)\phi Z, \phi e_i] - \sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(Y, \phi e_i) + g(\phi Y, \phi Z) \sum_{i=1}^{n-1} g(e_i, \phi e_i) \quad (6.6) \]

\[ + g(Y, \phi Z) \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) - \sum_{i=1}^{n-1} g(\phi Y, \phi e_i)g(e_i, \phi Z) + \sum_{i=1}^{n-1} g(Y, \phi Z)g(e_i, \phi e_i) \]

\[ - \sum_{i=1}^{n-1} g(e_i, \phi Z)g(Y, \phi e_i) = \frac{1}{(n-2)}[S(\phi Y, \phi Z) + (n - 2 + \psi)g(Y, \phi Z) \]

\[ + (\psi - 1)g(\phi Y, \phi Z)] \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) - \sum_{i=1}^{n-1} S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) \]

\[ - (n - 2 + \psi) \sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) - (\psi - 1) \sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) \]

\[ + g(\phi Y, \phi Z) \sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) + (n - 2 + \psi)g(\phi Y, \phi Z) \sum_{i=1}^{n-1} g(e_i, \phi e_i) \]

\[ + (\psi - 1)g(\phi Y, \phi Z) \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) - \sum_{i=1}^{n-1} S(\phi Y, \phi e_i)g(\phi e_i, \phi Z) \]

\[ -(n - 2 + \psi) \sum_{i=1}^{n-1} g(Y, \phi e_i)g(\phi e_i, \phi Z) - (\psi - 1) \sum_{i=1}^{n-1} g(\phi Y, \phi e_i)g(\phi e_i, \phi Z) \]

\[ - \frac{\ddot{r}}{(n-1)(n-2)}[g(\phi Y, \phi Z) \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) - \sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)]. \]

It can be easily verified that

\[ \sum_{i=1}^{n-1} g[R(\phi e_i, \phi Y)\phi Z, \phi e_i] = S(\phi Y, \phi Z) - g(\phi Y, \phi Z), \quad (6.7) \]

\[ \sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = g(\phi Y, \phi Z), \quad (6.8) \]

\[ \sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(Y, \phi e_i) = g(Y, \phi Z), \quad (6.9) \]
\[ \sum_{i=1}^{n-1} g(e_i, \phi Z)g(Y, \phi e_i) = g(Y, Z) + \eta(Y)\eta(Z), \quad (6.10) \]
\[ \sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = r - (n - 1), \quad (6.11) \]
\[ \sum_{i=1}^{n-1} S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = S(\phi Y, \phi Z), \quad (6.12) \]
\[ \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n - 1, \quad (6.13) \]
\[ \sum_{i=1}^{n-1} g(e_i, \phi e_i) = \psi. \quad (6.14) \]

By virtue of (6.7)-(6.14), the equation (6.6) can be written as
\[ S(\phi Y, \phi Z) + (\psi - 2)g(\phi Y, \phi Z) + (n + \psi - 3)g(Y, \phi Z) \]
\[ = \frac{1}{(n - 2)}[(n - 3)S(\phi Y, \phi Z) + (n - 3)(n + \psi - 2)g(Y, \phi Z) \]
\[ + (2(n - 2)(\psi - 1) + r - n + 1 + (n - 2 + \psi)\psi}g(\phi Y, \phi Z)] - \frac{\bar{r}}{(n - 1)}g(\phi Y, \phi Z) \]
from which it follows that
\[ S(\phi Y, \phi Z) = \left[ r + 2n\psi + \psi^2 - 4\psi - n + 1 - \frac{(n - 2)\bar{r}}{n - 1}\right]g(\phi Y, \phi Z) - \psi g(Y, \phi Z). \quad (6.16) \]

In view of (2.1), (3.6) and (3.7), (6.16) yields
\[ S(Y, Z) = \left[ \frac{r + \psi^2}{n - 1} - 1 \right]g(Y, Z) + \left[ \frac{r + \psi^2}{n - 1} - n \right]\eta(Y)\eta(Z) - \psi g(Y, \phi Z). \]

Thus we can state the following theorem:

**Theorem 6.2.** An n-dimensional \( \phi \)-conformally flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is a generalized \( \eta \)-Einstein manifold with respect to the connection \( \nabla \).

7. \( \phi \)-conharmonically flat Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Analogous to the equation (1.4), the conharmonic curvature tensor \( \bar{K} \) with respect to the quarter-symmetric metric connection is defined by
\[ \bar{K}(X, Y)Z = R(X, Y)Z - \frac{1}{(n - 2)}[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y + g(Y, Z)\bar{Q}X - g(X, Z)\bar{Q}Y], \quad (7.1) \]
where \( \bar{R} \), \( \bar{S} \) and \( \bar{Q} \) are the Riemannian curvature tensor, the Ricci tensor and the Ricci operator with respect to the connection \( \nabla \), respectively on \( M \).
Definition 7.1. A Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is said to be $\phi$-conharmonically flat if

$$\phi^2 \bar{K}(\phi X, \phi Y)\phi Z = 0$$

(7.2)

for all $X, Y, Z$ on $M$.

Let $M$ be an $n$-dimensional $\phi$-conharmonically flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection. Then from (7.2), it follows that

$$g(\bar{K}(\phi X, \phi Y)\phi Z, \phi W) = 0.$$  

(7.3)

From the equations (7.1) and (7.3), we have

$$g[R(\phi X, \phi Y)\phi Z, \phi W] = \frac{1}{(n - 2)}[\bar{S}(\phi Y, \phi Z)g(\phi X, \phi W)$$

(7.4)

$$-\bar{S}(\phi X, \phi Z)g(\phi Y, \phi W) + \bar{S}(\phi X, \phi W)g(\phi Y, \phi Z) - \bar{S}(\phi Y, \phi W)g(\phi X, \phi Z)].$$

In view of (2.1), (2.3), (3.2) and (3.4), (7.4) takes the form

$$g[R(\phi X, \phi Y)\phi Z, \phi W] - g(\phi X, \phi Z)g(Y, \phi W) + g(\phi Y, \phi Z)g(X, \phi W)$$

$$+ g(\phi X, \phi W)g(Y, \phi Z) - g(\phi Y, \phi W)g(X, \phi Z)$$

$$+ g(Y, \phi Z)g(X, \phi W) - g(X, \phi Z)g(Y, \phi W)$$

$$= \frac{1}{(n - 2)}[(S(\phi Y, \phi Z) + (n - 2 + \psi)g(Y, \phi Z) + (\psi - 1)g(\phi Y, \phi Z))g(\phi X, \phi W)$$

$$-\{S(\phi X, \phi Z) + (n - 2 + \psi)g(X, \phi Z) + (\psi - 1)g(\phi X, \phi Z)\}g(\phi Y, \phi W)$$

$$+ \{S(\phi X, \phi W) + (n - 2 + \psi)g(X, \phi W) + (\psi - 1)g(\phi X, \phi W)\}g(\phi Y, \phi Z)$$

$$- \{S(\phi Y, \phi W) + (n - 2 + \psi)g(Y, \phi W) + (\psi - 1)g(\phi Y, \phi W)\}g(\phi X, \phi Z)].$$

Let $\{e_1, e_2, \ldots, e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in $M$. Using that $\{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, \xi\}$ is also a local orthonormal basis in $M$. If we put $X = W = e_i$ in (7.5) and sum up with respect to $i$, then we have

$$\sum_{i=1}^{n-1} g[R(\phi e_i, \phi Y)\phi Z, \phi e_i] - \sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(Y, \phi e_i) + g(\phi Y, \phi Z) \sum_{i=1}^{n-1} g(e_i, \phi e_i)$$

(7.6)

$$+ g(Y, \phi Z) \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) - \sum_{i=1}^{n-1} g(\phi Y, \phi e_i)g(e_i, \phi Z) + \sum_{i=1}^{n-1} g(Y, \phi Z)g(e_i, \phi e_i)$$

$$- \sum_{i=1}^{n-1} g(e_i, \phi Z)g(Y, \phi e_i) = \frac{1}{(n - 2)}[(S(\phi Y, \phi Z) + (n - 2 + \psi)g(Y, \phi Z)$$

$$+ (\psi - 1)g(\phi Y, \phi Z)] \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) - \sum_{i=1}^{n-1} S(\phi e_i, \phi Z)g(\phi Y, \phi e_i)$$
By virtue of (6.7)-(6.14), the equation (7.6) can be written as

\[ S(\phi Y, \phi Z) + (\psi - 2)g(\phi Y, \phi Z) + (n + \psi - 3)g(Y, \phi Z) \]

\[ = \frac{1}{(n - 2)}[(n - 3)S(\phi Y, \phi Z) + (n - 3)(n + \psi - 2)g(Y, \phi Z) \]

\[ + \left\{ 2(n - 2)(\psi - 1) + r - n + 1 + (n - 2 + \psi) \psi \right\} g(\phi Y, \phi Z)] \]

from which it follows that

\[ S(\phi Y, \phi Z) = [r + 2n\psi + \psi^2 - 4\psi - n + 1]g(\phi Y, \phi Z) - \psi g(Y, \phi Z). \] (7.8)

In view of (2.1) and (2.16), (7.8) yields

\[ S(Y, Z) = [r + 2n\psi + \psi^2 - 4\psi - n + 1]g(Y, Z) \] (7.9)

\[ + [r + 2n\psi + \psi^2 - 4\psi - 2n + 2]\eta(Y)\eta(Z) - \psi g(Y, \phi Z). \]

Contracting (7.9) over \( Y \) and \( Z \) gives

\[ r = \frac{(4\psi - \psi^2 - 2n\psi)(n - 1) + (n - 1)(n - 2) + \psi^2}{n - 2}. \] (7.10)

By using this value of \( r \) in (7.9), we get

\[ S(Y, Z) = -2\psi g(Y, Z) - (n + 2\psi - 1)\eta(Y)\eta(Z) - \psi g(Y, \phi Z). \]

Thus we can state the following theorem:

**Theorem 7.2.** An \( n \)-dimensional \( \phi \)-conhamonically flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is a generalized \( \eta \)-Einstein manifold with the scalar curvature \( r \) given by (7.10).
8. \(\phi\)-concircularly flat Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Analogous to the equation (1.3), the concircular curvature tensor \(\tilde{V}\) with respect to the quarter-symmetric metric connection is defined by
\[
\tilde{V}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{\tilde{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y],
\]
where \(\tilde{R}\) and \(\tilde{r}\) are the Riemannian curvature tensor and the scalar curvature with respect to the connection \(\tilde{\nabla}\), respectively on \(M\).

**Definition 8.1.** A Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is said to be \(\phi\)-concircularly flat if
\[
\phi^2 \tilde{V}(\phi X, \phi Y)\phi Z = 0
\]
for all \(X,Y,Z\) on \(M\).

Let \(M\) be an \(n\)-dimensional \(\phi\)-concircularly flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection. Then from (8.2), it follows that
\[
g(\tilde{V}(\phi X, \phi Y)\phi Z, \phi W) = 0.
\]
From the equations (8.1) and (8.3), we have
\[
g[R(\phi X, \phi Y)\phi Z, \phi W] = \frac{\tilde{r}}{n(n-1)}[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)].
\]

In view of (3.2), (8.4) takes the form
\[
g[R(\phi X, \phi Y)\phi Z, \phi W] - g(\phi X, \phi Z)g(Y, \phi W) + g(\phi Y, \phi Z)g(X, \phi W)
\]
\[
+ g(\phi X, \phi W)g(Y, \phi Z) - g(\phi Y, \phi W)g(X, \phi Z) + g(Y, \phi Z)g(X, \phi W)
\]
\[
- g(X, \phi Z)g(Y, \phi W) = \frac{\tilde{r}}{n(n-1)}[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)].
\]

Let \(\{e_1, e_2, \ldots, e_{n-1}, \xi\}\) be a local orthonormal basis of vector fields in \(M\). Using that \(\{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, \xi\}\) is also a local orthonormal basis in \(M\). If we put \(X = W = e_i\) in (8.5) and sum up with respect to \(i\), then we have
\[
\sum_{i=1}^{n-1} g[R(\phi e_i, \phi Y)\phi Z, \phi e_i] - \sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(Y, \phi e_i) + g(\phi Y, \phi Z)\sum_{i=1}^{n-1} g(e_i, \phi e_i)
\]
\[
+ g(Y, \phi Z)\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) - \sum_{i=1}^{n-1} g(\phi Y, \phi e_i)g(e_i, \phi Z) + \sum_{i=1}^{n-1} g(Y, \phi Z)g(e_i, \phi e_i)
\]
\[
- \sum_{i=1}^{n-1} g(e_i, \phi Z)g(Y, \phi e_i) = \frac{\tilde{r}}{n(n-1)}\sum_{i=1}^{n-1} [g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)].
\]
By virtue of (6.7)-(6.10), (6.13) and (6.14), the equation (8.6) becomes

\[ S(\phi Y, \phi Z) = \left[ \frac{\bar{r}(n-2)}{n(n-1)} - \psi + 2 \right] g(\phi Y, \phi Z) - (\psi + n - 3) g(Y, \phi Z). \]  

(8.7)

In view of (2.1) and (2.16), (8.7) takes the form

\[ S(Y, Z) = \frac{(n-2)[r + (2\psi - 1)(n-1) + \psi^2] - n(n-1)(\psi - 2)}{n(n-1)} g(Y, Z) \]  

(8.8)

\[ + \frac{(n-2)[r + (2\psi - 1)(n-1) + \psi^2] - n(n-1)(\psi + n - 3)}{n(n-1)} \eta(Y) \eta(Z) \]

\[ - (n + \psi - 3) g(Y, \phi Z). \]

Contracting (8.8) over \( Y \) and \( Z \) gives

\[ r = n(n - \psi) - (\psi - 1)^2. \]  

(8.9)

By using this value of \( r \) in (8.8), we get

\[ S(Y, Z) = \left( n - \frac{\psi}{n-1} \right) g(Y, Z) + \left( 1 - \frac{\psi}{n-1} \right) \eta(Y) \eta(Z) - (n + \psi - 3) g(Y, \phi Z). \]

Thus we can state the following theorem:

**Theorem 8.2.** An \( n \)-dimensional \( \phi \)-concircularly flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is a generalized \( \eta \)-Einstein manifold with the scalar curvature \( r \) given by (8.9).

9. \( \phi \)-projectively flat Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Analogous to the equation (1.2), the projective curvature tensor \( \bar{P} \) with respect to the quarter-symmetric metric connection is defined by

\[ \bar{P}(X, Y)Z = \bar{R}(X, Y)Z - \frac{1}{(n-1)}[g(Y, Z)\bar{Q}X - g(X, Z)\bar{Q}Y], \]  

(9.1)

where \( \bar{R} \) and \( \bar{Q} \) are the Riemannian curvature tensor and the Ricci operator with respect to the connection \( \bar{\nabla} \), respectively on \( M \).

**Definition 9.1.** A Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is said to be \( \phi \)-projectively flat if

\[ \phi^2 \bar{P}(\phi X, \phi Y)\phi Z = 0 \]  

(9.2)

for all \( X, Y, Z \) on \( M \).
Let $M$ be an $n$-dimensional $\phi$-projectively flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection. Then from (9.2), it follows that
\[ g(\tilde{\nabla}(\phi X, \phi Y)\phi Z, \phi W) = 0. \] (9.3)

From the equations (9.1) and (9.3), we have
\[ g[R(\phi X, \phi Y)\phi Z, \phi W] = \frac{1}{(n-1)}[S(\phi X, \phi W)g(\phi Y, \phi Z) - S(\phi Y, \phi W)g(\phi X, \phi Z)]. \] (9.4)

In view of (3.2) and (3.4), (9.4) takes the form
\[ g[R(\phi X, \phi Y)\phi Z, \phi W] = g(\phi X, \phi Y)g(\phi Z, \phi W) + g(\phi Y, \phi Z)g(\phi X, \phi W) \] (9.5)
\[ + g(\phi X, \phi W)g(\phi Y, \phi Z) - g(\phi Y, \phi W)g(\phi X, \phi Z) + g(Y, \phi Z)g(X, \phi W) \]
\[ - g(X, \phi Z)g(Y, \phi W) = \frac{1}{(n-1)}[(S(\phi X, \phi W) + (n-2+\psi)g(X, \phi W) \]
\[ + (\psi - 1)g(\phi X, \phi W)]g(\phi Y, \phi Z) - \{S(\phi Y, \phi W) + (n-2+\psi)g(Y, \phi W) \]
\[ + (\psi - 1)g(\phi Y, \phi W)]g(\phi X, \phi Z). \]

Let $\{e_1, e_2, \ldots, e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in $M$. Using that $\{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, \xi\}$ is also a local orthonormal basis in $M$. If we put $X = W = e_i$ in (9.5) and sum up with respect to $i$, then we have
\[ \sum_{i=1}^{n-1} g[R(\phi e_i, \phi Y)\phi Z, \phi e_i] - \sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) + g(\phi Y, \phi Z) \sum_{i=1}^{n-1} g(e_i, \phi e_i) \] (9.6)
\[ + g(Y, \phi Z) \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) - \sum_{i=1}^{n-1} g(\phi Y, \phi e_i)g(e_i, \phi Z) \]
\[ + \sum_{i=1}^{n-1} g(Y, \phi Z)g(e_i, \phi e_i) - \sum_{i=1}^{n-1} g(e_i, \phi Z)g(\phi Y, \phi e_i) \]
\[ = \frac{1}{(n-1)}[g(\phi Y, \phi Z) \sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) - \sum_{i=1}^{n-1} S(\phi Y, \phi e_i)g(\phi e_i, \phi Z) \]
\[ + (n-2+\psi)g(\phi Y, \phi Z) \sum_{i=1}^{n-1} g(e_i, \phi e_i) + (\psi - 1)g(\phi Y, \phi Z) \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) \]
\[ - (n-2+\psi) \sum_{i=1}^{n-1} g(Y, \phi e_i)g(\phi e_i, \phi Z) - (\psi - 1) \sum_{i=1}^{n-1} g(\phi Y, \phi e_i)g(\phi e_i, \phi Z)]. \]

By virtue of (6.7)-(6.14), the equation (9.6) turns to
\[ nS(\phi Y, \phi Z) = (r+n\psi - 3\psi +\psi^2 + 1)g(\phi Y, \phi Z) - (n^2 - 3n + n\psi + 1)g(Y, \phi Z). \] (9.7)
In view of (2.1) and (2.16), (9.7) becomes

\[ S(Y, Z) = \frac{r + (n - 3 + \psi)\psi + 1}{n} g(Y, Z) - \frac{(n^2 - 3n + n\psi + 1)}{n} g(Y, \phi Z) \]  

(9.8)

Contracting (9.8) over \( Y \) and \( Z \) gives

\[ r = n^2 - n\psi + 2\psi - \psi^2 - 1. \]  

(9.9)

By using this value of \( r \) in (9.8), we get

\[ S(Y, Z) = (n - \psi) g(Y, Z) + (1 - \psi) g(Y, Z) - \frac{(n^2 - 3n + n\psi + 1)}{n} g(Y, \phi Z). \]

Thus we can state the following theorem:

**Theorem 9.2.** An \( n \)-dimensional \( \phi \)-projectively flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is a generalized \( \eta \)-Einstein manifold with the scalar curvature \( r \) given by (9.9).

### 10. Conformally flat Lorentzian para-Kenmotsu manifolds with respect to the quarter-symmetric metric connection

Let \( M \) be an \( n \)-dimensional conformally flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection, that is, \( \bar{C} = 0 \). Then from (6.1), it follows that

\[ \bar{R}(X, Y)Z = \frac{1}{(n - 2)} \left[ \bar{S}(Y, Z)X - \bar{S}(X, Z)Y + g(Y, Z)\bar{Q}X - g(X, Z)\bar{Q}Y \right] \]  

(10.1)

\[ - \frac{\bar{r}}{(n - 1)(n - 2)} [g(Y, Z)X - g(X, Z)Y]. \]

By taking \( Y = \xi \) in (10.1) and using (2.2), (3.15) and (3.17), we have

\[ \eta(Y)X - \eta(X)Y + \eta(Y)\phi X - \eta(X)\phi Y = \frac{1}{(n - 2)} [(n + \psi - 1)\eta(Y)X \]  

(10.2)

\[-(n + \psi - 1)\eta(X)Y + \eta(Y)\bar{Q}X - \eta(X)\bar{Q}Y] - \frac{\bar{r}}{(n - 1)(n - 2)} [\eta(Y)X - \eta(X)Y]. \]

Now taking \( Z = \xi \) in (10.2) and then using (2.2), (2.3) and (3.18), we get

\[ \bar{Q}X = [n - 2 + \frac{\bar{r}}{n - 1}] (X + \eta(X)\xi) + (n - 2)\phi X - (n + \psi - 1)X - 2(n + \psi - 1)\eta(X)\xi \]

which by taking inner product with \( W \) and using the fact that \( g(\bar{Q}X, W) = \bar{S}(X, W) \) gives

\[ \bar{S}(X, W) = \frac{\bar{r}}{n - 1} - \psi - 1] g(X, W) + \left[ \frac{\bar{r}}{n - 1} - 2\psi - n\eta(X)\eta(W) + (n - 2)g(\phi X, W). \]  

(10.3)
In view of (3.4) and (3.6), (10.3) takes the form

\[ S(X, W) = \left[ \frac{r + \psi^2}{n - 1} - 1 \right] g(Y, Z) + \left[ \frac{r + \psi^2}{n - 1} - n \right] \eta(Y) \eta(Z) - \psi g(Y, \phi Z). \]

Thus we can state the following theorem:

**Theorem 10.1.** An n-dimensional conformally flat Lorentzian para-Kenmotsu manifold with respect to the quarter-symmetric metric connection is a generalized \( \eta \)-Einstein manifold with respect to the connection \( \nabla \).

**Acknowledgments**

The authors are thankful to the referees for their valuable suggestions towards the improvement of the paper.

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