Complete addition laws on abelian varieties

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Abstract

We prove that under any projective embedding of an abelian variety $A$ of dimension $g$, a complete set of addition laws has cardinality at least $g + 1$, generalizing of a result of Bosma and Lenstra for the Weierstrass model of an elliptic curve in $\mathbb{P}^2$. In contrast, we moreover prove that if $k$ is any field with infinite absolute Galois group, then there exists, for every abelian variety $A/k$, a projective embedding and an addition law defined for every pair of $k$-rational points. For an abelian variety of dimension 1 or 2, we show that this embedding can be the classical Weierstrass model or the embedding in $\mathbb{P}^{15}$, respectively, up to a finite number of counterexamples for $|k| \leq 5$.

1. Introduction

The notion of completeness of a set of addition laws for an abelian variety $A$ in $\mathbb{P}^r$ was introduced by Lange and Ruppert [LR85]. We recall that an addition law is an $(r + 1)$-tuple of bihomogeneous polynomials $(p_0, \ldots, p_r)$ such that the map

$$(x, y) \mapsto (p_0(x, y), \ldots, p_r(x, y)),$$

determines the group law $\mu : A \times A \to A$ on an open subset of $A \times A$, and a set of addition laws is complete if these open sets cover $A \times A$ (see Definition 2.1). The bidegree $(m, n)$ of an addition law is the bidegree of the polynomials $p_i$ in $x$ and $y$. Lange and Ruppert prove that the minimal bidegree of any addition law is $(2, 2)$ and determine exact dimensions for the spaces of all addition laws of given bidegree. For an elliptic curve $E$ in $\mathbb{P}^2$ in Weierstrass form, the space of addition laws has dimension 3, and Bosma and Lenstra [BL95] proved that two suffice for a complete set, determining $\mu$ on all of $E \times E$.

In 2007, Edwards introduced a new normal form for elliptic curves

$$x_1^2 + x_2^2 = a^2(1 + x_1^2x_2^2),$$

with particularly simple rational expression for the group law. After a coordinate scaling, Bernstein and Lange [BL07] descend this model to

$$x_1^2 + x_2^2 = 1 + dx_1^2x_2^2,$$

for $d = a^4$, which admits the group law $x + y = z$ where

$$z = \left(\frac{x_3y_2 + x_2y_3}{1 + dx_3y_3}, \frac{y_3 - x_3}{1 - dx_3y_3}\right),$$

for $x_3 = x_1x_2$ and $y_3 = y_1y_2$. In addition to giving a precise analysis of the efficiency of this group law, Bernstein and Lange observe that the addition law is $k$-complete over any field $k$ in which $d$ is a nonsquare (i.e. the addition law is well-defined on all pairs of $k$-rational points of $E$). To interpret these rational expressions in terms of projective addition laws as analyzed

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by Lange and Ruppert, we note that \( \{1, x_1, x_2, x_3\} \) forms a basis of global sections for the Riemann–Roch space of the divisor at infinity for the pair of coordinate functions \((x_1, x_2)\), and that this basis determines a projective embedding
\[
(x_1, x_2, x_3) \mapsto (1 : x_1 : x_2 : x_3)
\]
in \(\mathbb{P}^3\) which is projectively normal (see Section 2 for precise definitions). Namely the image curve is of the form
\[
X_1^2 + X_2^2 = X_0^2 + dX_3^2, \quad X_0X_3 = X_1X_2.
\]
The Edwards addition law can be interpreted as the bidegree \((2, 2)\) addition law
\[
( (X_0Y_0 + dX_3Y_3)(X_0Y_0 - dX_3Y_3), (X_0Y_0 - dX_3Y_3)(X_1Y_2 + X_2Y_1),
(X_0Y_0 + dX_3Y_3)(X_0Y_3 - X_3Y_0), (X_0Y_3 - X_3Y_0)(X_1Y_2 + X_2Y_1)).
\]
Any elliptic curve specified by an affine model has a canonical embedding associated to the complete linear system. Consequently, we refer only to such abelian varieties with projective embeddings.

In terms of degree 3 models, Bernstein, Kohel and Lange [BKL09] construct a \(k\)-complete addition law on the family of twisted Hessian curves
\[
aX_0^3 + X_1^3 + X_2^3 = dX_0X_1X_2,
\]
which admit the \(k\)-complete addition laws
\[
(X_0X_1Y_1^2 - X_2^2Y_0Y_2, \quad aX_0X_2Y_0^2 - X_1^2Y_1Y_2, \quad -aX_0^2Y_0Y_1 + X_1X_2Y_2^2),
\]
and
\[
(X_0X_2Y_2^2 - X_1^2Y_0Y_1, \quad -aX_0^2Y_0Y_2 + X_1X_2Y_2, \quad aX_0X_1Y_0^2 - X_2^2Y_1Y_2),
\]
over any field \(k\) in which \(a\) is not a cube. Any such model is equivalent to a Weierstrass model by a linear change of variables, which shows that the property of \(k\)-completeness is not special to quartic models in \(\mathbb{P}^3\).

Both the Edwards and twisted Hessian models share the property that they require a level structure of rational torsion. In analogy with the quartic Edwards model, Bernstein and Lange [BL09] demonstrate by example that a general elliptic curve admits a quartic model with \(k\)-complete addition law (subject to some coefficient being a nonsquare), while resorting to a rational expression for an addition law of high bidegree. The second author of the present article gives an elementary characterization of \(k\)-completeness of addition laws of bidegree \((2, 2)\) in terms of the Galois action on an associated divisor on the curve [Koh11, Corollary 12]. In particular, the property of \(k\)-completeness on elliptic curves is not special.

In this paper, we generalize the above results to abelian varieties. We determine new, tight bounds on the size of a complete set of addition laws under any embedding, a generalization of the result of Bosma and Lenstra [BL95] for elliptic curves. Moreover we prove that if \(k\) is any field with infinite absolute Galois group, then there exists, for every abelian variety \(A/k\), a projective embedding and an addition law defined for every pair of \(k\)-rational points (see Theorem 3.1).

Our work builds on the elegant paper of Lange and Ruppert [LR85], in which the authors interpret addition laws on an abelian variety \(A/k\) in terms of sections of a certain line bundle \(\mathcal{M}\) on \(A \times A\). Our key idea is to observe that an addition law associated to a section \(s\) of \(\mathcal{H}^0(\mathcal{A} \times A, \mathcal{M})\) with zero divisor \(D_s := (s)_0\) is defined on \(A \times A \setminus D_s\). We obtain a \(k\)-complete addition law by constructing a \(k\)-rational divisor \(D_s\) without any \(k\)-rational point. This gives an exact analog of the elliptic curve case studied by the second author [Koh11].

In Section 2, we recall some definitions and concepts of [LR85], explain more explicitly the link between addition laws on a projective embedding of \(A/k\) and sections of \(\mathcal{H}^0(\mathcal{A} \times A, \mathcal{M})\), and also deal with the geometric case \(k = \bar{k}\). For any principally polarized abelian variety
of dimension \( q \), we give bounds on the cardinality of any complete set of addition laws. In particular we show that its cardinality is at least \( q + 1 \).

In Section 3, we consider the case of a field \( k \) with infinite absolute Galois group, and prove the aforementioned result on existence of a pair consisting of a projective embedding and a \( k \)-complete addition law.

In Section 4, we specialize to elliptic curves and Jacobians of genus 2 curves over a finite field \( k \), noting that the results also extend to other fields (see Remarks 4.4 and 4.9). We prove that there exists a \( k \)-complete addition law for their classical embeddings in \( \mathbb{P}^2 \) and \( \mathbb{P}^{15} \), respectively, as soon as \( |k| \geq 5 \) for elliptic curves and \( |k| \geq 7 \) for Jacobian surfaces. In particular, we exhibit an explicit \( k \)-complete addition law on a Weierstrass model of an elliptic curve \( E \) over \( k \) when \( E \) has no nontrivial rational 2-torsion point.

2. Addition laws and completeness

Let \( k \) be a field and \( A/k \) be an abelian variety of dimension \( q \). We assume that \( A \) is embedded in some projective space \( \mathbb{P}^r \) over \( k \), by a very ample line bundle \( \mathcal{L} = \mathcal{L}(D) \) for \( D \) an effective divisor, and we denote by \( \iota : A \hookrightarrow \mathbb{P}^r \) the corresponding morphism. We also assume in the sequel that the embedding is projectively normal. Recall that \( A \) is said to be projectively normal in \( \mathbb{P}^r \) if for every \( n \geq 1 \) the restriction map \( \Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to \Gamma(A, \mathcal{L}^n) \) is surjective. This is the case in the classical settings where \( \mathcal{L} = \mathcal{L}_0^0 \) with \( \mathcal{L}_0 \) an ample line bundle and \( a \geq 3 \) [BL04, p.187].

Let \( I_1 \) and \( I_2 \) be the homogeneous defining ideal for \( A \) in \( k[X_0, \ldots, X_r] \) and \( k[Y_0, \ldots, Y_r] \), respectively. The group law

\[
\mu : A \times A \to A,
\]

defined by \( (x, y) \mapsto x + y \), can be locally described by bihomogenous polynomials. More precisely, an addition law \( p \) of bidegree \( (m, n) \) on \( \iota(A) \subset \mathbb{P}^r \) is an \((r+1)\)-tuple \((p_0, \ldots, p_r)\) of elements

\[
p_i \in k[X_0, \ldots, X_r]/I_1 \otimes k[Y_0, \ldots, Y_r]/I_2,
\]

which are bihomogeneous of degree \( m \) and \( n \) in \( X_0, \ldots, X_r \) and \( Y_0, \ldots, Y_r \), respectively, and for which there exists a nonempty open subset \( U \) of \( A \times A \) such that, for all \((x, y) \in U(\bar{k})\),

\[
\iota \circ \mu (x, y) = (p_0(\iota(x), \iota(y)) : \ldots : p_r(\iota(x), \iota(y))).
\]

When \( A \) is given with a fixed embedding in \( \mathbb{P}^r \) we may suppress the reference to the embedding \( \iota \) and speak of addition laws on \( A \).

**Definition 2.1** A set \( S \) of addition laws is said to be \( k \)-complete if for any \( k \)-rational point \((x, y) \in (A \times A)(k)\) there is an addition law in \( S \) defined on an open set \( U \) containing \((x, y)\). This set is said to be complete if the previous property is true over \( \bar{k} \). If \( S = \{p\} \) is a singleton, we say the addition law \( p \) is \( k \)-complete and complete when \( k = \bar{k} \).

In [LR85, Lem.2.1], Lange and Ruppert give the interpretation of the possible addition laws in terms of the sections of certain line bundles.

**Proposition 2.2** Let \( \pi_1, \pi_2 : A \times A \to A \) be the projection maps on the first and second factor. There is an addition law (respectively a complete set of addition laws) of bidegree \((m, n)\) on \( A \) with respect to the embedding in \( \mathbb{P}^r \) determined by \( \mathcal{L} \) if and only if

\[
H^0(A \times A, \mathcal{L}_{m,n}) \neq 0
\]
(respectively the linear system $|\mathcal{M}_{m,n}|$ is basepoint-free), where
\[ \mathcal{M}_{m,n} = \mu^* L^{-1} \otimes \pi_1^* L^m \otimes \pi_2^* L^n. \]

We explain how one associates an addition law to a nonzero section $w$ in $H^0(A \times A, \mathcal{M}_{m,n})$. For $0 \leq j \leq n$, let $t_j \in H^0(A, L)$ be the basis given by $t_j = \iota^* X_j$ where $X_j$ are the coordinate functions on $\mathbb{P}^r$. As shown in [LR85, p.607], $H^0(A \times A, \mu^* L) = \mu^* H^0(A, L)$, so $s_j = \mu^* t_j$ is a basis of $H^0(A \times A, \mu^* L)$. For each $j$ and $(x, y) \in A \times A$, we have
\[ s_j(x, y) = t_j \circ \mu(x, y) = X_j(\iota \circ \mu(x, y)). \]

Now $w \otimes s_j \in H^0(A \times A, \pi_1^* L^m \otimes \pi_2^* L^n)$. As the embedding is projectively normal we have
\[ \pi_1^* L^m \otimes \pi_2^* L^n = (\iota \otimes \iota)^* \mathcal{O}_{\mathbb{P}^r}(m) \otimes \mathcal{O}_{\mathbb{P}^r}(n), \]
then there exists a bihomogeneous polynomial $p_j$ of bidegree $(m, n)$ such that for all points $(x, y) \in A \times A$
\[ (w \otimes s_j)(x, y) = p_j(\iota(x), \iota(y)). \]

Therefore, if $U = A \times A \setminus (w)_0$, we have
\[ (p_0(\iota(x), \iota(y)) : \ldots : p_r(\iota(x), \iota(y))) = ((w \otimes s_0)(x, y) : \ldots : (w \otimes s_r)(x, y)) = (s_0(x, y) : \ldots : s_r(x, y)) = (X_0(\iota \circ \mu(x, y)) : \ldots : X_r(\iota \circ \mu(x, y))) = \iota(\mu(x, y)). \]

Another natural requirement to ask is that $L = L(D)$ be symmetric, i.e., $[-1]^* L \cong L$, or equivalently $D \sim [-1]$, as we can see in the following lemmas.

**Lemma 2.3** If $A/k$ is embedded in $\mathbb{P}^r$ by a very ample symmetric line bundle $L$ (projectively normal), then the inversion map $[-1]$ on $A$ is induced by a linear automorphism of $\mathbb{P}^r$. Moreover if $\text{char}(k) \neq 2$ there is a choice of coordinates such that the inversion acts by $\pm 1$ on each coordinate.

**Proof.** The first statement is a direct consequence of the symmetry of $L$. Now fix a basis $(t_i)$ of $H^0(A, L)$ and let $M$ be the matrix of the coordinates of $[-1]^* t_i$ in the basis $(t_i)$. The morphism $[-1]$ is induced by an involution of $\mathbb{P}^r$ so there exists $\varepsilon \in k$ such that $M^2 - \varepsilon \text{Id} = 0$.

The neutral element $O = (a_0 : \ldots : a_r)$ of $\mathbb{A} \rightarrow \mathbb{P}^r$ is a fixed point for $[-1]$. Hence, the vector $(a_0, \ldots, a_r)$ is an eigenvector of the matrix $M$ with eigenvalue $\varepsilon_0 \in k$. This implies that $\varepsilon = \varepsilon_0^2$ and if $\text{char}(k) \neq 2$ then $M^2 - \varepsilon \text{Id}$ factors as $(M - \varepsilon_0 \text{Id})(M + \varepsilon_0 \text{Id})$. This proves that $M$ can be diagonalized over $k$ with eigenvalues in $\{\pm \varepsilon_0\}$ and the conclusion holds.

Before considering non-algebraically closed fields, it is natural to consider what happens over $k$. We start by giving an upper bound on the cardinality of a complete set of addition laws. In what follows we define the difference map $\delta : A \times A \rightarrow A$ by $(x, y) \mapsto x - y$, and use the product partial order on bidegree given by $(k, l) \leq (m, n)$ if and only if $k \leq m$ and $l \leq n$. For bidegree $(m, n) = (2, 2)$ we denote the line bundle $\mathcal{M}_{m,n}$ of Proposition 2.2 by $\mathcal{M}$. We begin by recalling a fundamental lemma of Lange and Ruppert [LR85, Prop.2.2, Prop.2.3].

**Lemma 2.4** Let $L$ be an ample line bundle on $A$.

1. If $L$ is not symmetric then $H^0(A \times A, \mathcal{M}) = 0$, and
2. If $L$ is symmetric then $\mathcal{M}$ is isomorphic to $\delta^* L$ and is basepoint-free, and consequently $h^0(\mathcal{M}) = h^0(\mathcal{L})$.

If $(m, n) > (2, 2)$ then $h^0(\mathcal{M}_{m,n}) = h^0(\mathcal{L})^2(mn - m - n)^9$. 


Proof. For \((m, n) > (2, 2)\), the proof follows the case \((m, n) = (2, 3)\) treated in [LR85] Prop.2.3. For \((m, n) = (2, 2)\), Lange and Ruppert prove in [LR85] Proposition 2.2] that \(\mathcal{M} \cong \delta^* \mathcal{L}\) and that \(\mathcal{M}\) is basepoint-free. The equality \(h^0(\mathcal{M}) = h^0(\mathcal{L})\) is an easy consequence of the fact proved in loc. cit. that \(\mathcal{M}|_{K(\mathcal{M})}\) is trivial and of the fact that, as \(\mathcal{L}\) is ample, its index is zero. Indeed, according to [Mum70] Theorem 1(ii) p.95, one then has the isomorphism \(H^0(A \times A, \mathcal{M}) \cong H^0(A, \mathcal{L})\). 

The isomorphism of \(\mathcal{M}\) with \(\delta^* \mathcal{L}\) allows us to consider line bundles on \(A\) instead of \(A \times A\). The following well-known lemma shows that we can always find a symmetric embedding of \(A/\bar{k}\).

**Lemma 2.5** Let \((A, \lambda)\) be a principally polarized abelian variety over \(\bar{k}\). There exists a symmetric line bundle which induces the polarization \(\lambda\) on \(A\).

**Proof.** Suppose that \(\mathcal{L}'\) is a line bundle attached to the polarization \(\lambda\). We construct a symmetric line bundle \(\mathcal{L}\) algebraically equivalent to \(\mathcal{L}'\). Since \(\mathcal{L}'\) is algebraically equivalent to \([-1]^* \mathcal{L}'\) (see [Lan83a] p.93]), there exists \(x \in A(\bar{k})\) such that the translation \(\tau_x^* \mathcal{L}'\) is algebraically equivalent to \([-1]^* \mathcal{L}'\). Let \(y\) be an element of \(A(\bar{k})\) such that \(2y = x\), and set \(\mathcal{L} = \tau_y^* \mathcal{L}'\). Then \(\mathcal{L}\) is algebraically equivalent to \(\mathcal{L}'\) and

\[
\mathcal{L} = \tau_y^* \mathcal{L}' = \tau_x^* \tau_y^* \mathcal{L}' = \tau_y^* [-1]^* \mathcal{L}' = [-1]^* \mathcal{L},
\]

hence symmetric. 

Suppose that \(\mathcal{L}\) is a symmetric line bundle as in the preceding lemma. By Lemma [2.3] the embedding defined by \(\mathcal{L}^3\) has a complete set of biquadratic addition laws of cardinality equal to \(h^0(A, \mathcal{L}^3) = 3^g\). This gives an upper bound on the minimal size of a complete set of addition laws. We now determine a lower bound.

**Theorem 2.6** Assume \(A\) is embedded in \(\mathbb{P}^r\) by a symmetric line bundle. If \(S\) is a complete set of addition laws on \(A\) then \(|S| \geq g + 1\).

**Proof.** Suppose that \(S\) is a complete set of addition laws of bidegree \((m, n)\) on \(A\), and let \(\nabla = \ker(\mu) \subset A \times A\). By Lemma [2.3] the isomorphism

\[
[1] \times [-1] : A \rightarrow \nabla
\]

is linear, and so \(((1] \times [-1])^* S\) is a set of polynomial (rational) maps for \(A \rightarrow \{O\} \subset A\). It follows that there exists a set \(I\) of polynomials of degree \(m + n\) such that

\[
([1] \times [-1])^* S = \{(a_0q(X_0, \ldots, X_r), \ldots, a_rq(X_0, \ldots, X_r)) : q \in I\},
\]

where \(O = (a_0 : \cdots : a_r)\). Since \(S\) is complete, the subvariety \(V(I) \cap A\) is empty. On the other hand, its dimension is at least \(\dim(A) - |I| \geq g - |S|\), hence the cardinality of \(S\) must be at least \(g + 1\). 

Although the interval \([g + 1, 3^g]\) is quite large, the lower bound shows that there is no complete addition law on any abelian variety of any dimension. For \(g = 1\), these bounds show that the minimal size of a complete set of addition laws is either 2 or 3. An explicit set of cardinality 3 was already given by Lange and Ruppert [LR85] Sec.3 if \(\text{char}(k) \neq 2, 3\), and in [LR87] for any characteristic, and Bosma and Lenstra [BL95] proved that a set of minimal cardinality 2 is in fact sufficient.
3. $k$-complete addition laws

Let $\mathcal{L}$ be a very ample symmetric line bundle defined by an effective $k$-rational divisor $D$ on $A/k$.

Since $\delta^* \mathcal{L} \cong \mathcal{M} = \mu^* \mathcal{L}^{-1} \otimes \pi_1^* \mathcal{L}^2 \otimes \pi_2^* \mathcal{L}^2$ there exists $w$ in $H^0(A \times A, \mathcal{M})$ such that $(w)_0 = \delta^*(D)$. As we have seen in Section 2, $w$ defines a biquadratic addition law on the complement of $(w)_0 = \delta^*D$. Hence it is sufficient that $D$ has no $k$-rational point for the group law to be $k$-complete. Note that this is also a necessary condition since a $k$-rational point $x$ on $D$ gives the $k$-rational point $(x,0)$ on $\delta^*D$.

**Theorem 3.1** Let $A/k$ be an abelian variety and $\iota_0 : A \hookrightarrow \mathbb{P}^{r_0}$ be an embedding for some $r_0 > 1$. Assume that $k$ has infinite absolute Galois group and let $d > r_0$ be such that there exists a separable extension $K/k$ of degree $d$ over $k$. Then there exists an embedding $\iota : A \hookrightarrow \mathbb{P}^r$ and a $k$-complete biquadratic addition law on $\iota(A)$, with $r = (2d)^9(r_0 + 1) - 1$.

**Proof.** Let $K = k(\alpha_0)/k$ be a separable extension, and denote by $\alpha_0, \ldots, \alpha_{d-1}$ its distinct Galois conjugates in the normal closure of $K/k$. For $i = 0, \ldots, d-1$, let $H_i$ be the hyperplane in $\mathbb{P}^{r_0}$

$$H_i : X_0 + \alpha_i X_1 + \ldots + \alpha_i^{r_0} X_{r_0} = 0.$$ 

Since $d > r_0$, the sets $\{1, \alpha_i, \ldots, \alpha_i^{r_0}\}$ are linearly independent over $k$ for every $i$ and hence $H_i(k)$ is empty. Now $\sum H_i$ is a $k$-rational divisor, so let $D_0 = \iota_0(\sum H_i)$ and define the divisor $D = D_0 + [-1]^D_0$. Then $D$ is a symmetric, effective, $k$-rational divisor without $k$-rational points. Denote by $\mathcal{L}_0$ the line bundle associated to the embedding $\iota_0$. The line bundle $\mathcal{L} = \mathcal{L}(D)$ is isomorphic to $\mathcal{L}_0^{2d}$, so $\mathcal{L}$ is very ample and provides a projectively normal embedding $A \hookrightarrow \mathbb{P}^{r}$ with a $k$-complete biquadratic addition law. By the Riemann-Roch theorem, the dimension $r$ is equal to $(2d)^9(r_0 + 1) - 1$. 

4. The genus one and two cases

In the previous section, a $k$-complete (biquadratic) addition law is proved to exist, for an embedding of the abelian variety in a projective space of high dimension. When $k = \mathbb{F}_q$ is a finite field and the abelian variety $A/k$ has dimension 1 or 2, we will show that we can take the embedding to be the classical ones. In what follows, we let $\sigma$ denote the Frobenius automorphism of $\bar{k}/k$.

4.1. Elliptic curves

Let $A = E$ be an elliptic curve defined over $k = \mathbb{F}_q$.

**Lemma 4.1** If $q \geq 5$, there exists $P_0 \in E(\bar{k})$ whose Galois orbit is given by three distinct points whose sum is $O$.

**Proof.** Consider the group homomorphism $N : E(\mathbb{F}_q^2) \to E(\mathbb{F}_q)$ given by

$$P \mapsto P + P^\sigma + P^{\sigma^2}.$$ 

We are looking for a point $P_0 \in \ker(N) \setminus E(\mathbb{F}_q)$, hence we want

$$|\ker(N)| > |\ker(N) \cap E(\mathbb{F}_q)|.$$
The intersection of ker(N) with $E(F_q)$ is the group of $F_q$-rational 3-torsion points of $E$ so $|\ker(N) \cap E(F_q)| \leq 9$. On the other hand, for all $q \geq 5$, we have

$$|\ker(N)| \geq \frac{|E(F_{\overline{q}})|}{|E(F_q)|} \geq \frac{q^3 + 1 - 2\sqrt{q^3}}{q + 1 + 2\sqrt{q}} > 9,$$

so such a point $P_0$ exists in $E(F_{\overline{q}})$. □

Remark 4.2 For each of $q = 2, 3$ and 4, there exists at least one elliptic curve over $F_q$ for which $|\ker(N)| = |\ker(N) \cap E(F_q)|$.

Theorem 4.3 Let $k$ be the finite field $F_q$ with $q \geq 5$ and $E/k$ be an elliptic curve. There exists a $k$-complete biquadratic addition law on the Weierstrass model of $E \subset \mathbb{P}^2$.

Proof. Let $P_0$ be a point as in Lemma 4.1 and $D$ be the divisor given by the sum of the Galois conjugates of $P_0$. It is a $k$-rational divisor without $k$-rational points. It is not a symmetric divisor but $L = L(D)$ is a symmetric line bundle as $D = 3(O) \sim [-1]^*D$. Another consequence of the relation $D \sim 3(O)$ is that the embedding associated to $L(D)$ is projectively equivalent to the Weierstrass model of $E$. □

Remark 4.4 We use the fact that $k$ is a finite field only to prove the existence of the point $P_0$. It is easy to see that when $k$ is a number field, such a point always exists and so the conclusion of Theorem 4.3 still holds. Indeed, if $E$ is defined by $y^2 + h(x)y = f(x)$, then, since $k$ is Hilbertian (see [Lan83b, p.225]), there exists $y_0 \in k$ such that $y_0^2 + h(x)y_0 - f(x)$ is irreducible. We can take $P_0 = (x_0, y_0)$ where $x_0$ is any root of $y_0^2 + b(x)y_0 - f(x) = 0$ in $k$.

In particular, for $\text{char}(k) \neq 2$ or 3, by means of a change of variables we may assume $E$ is of the form $y^2 = x^3 + ax + b$. Moreover, if $E$ has no non trivial $k$-rational 2-torsion point, then the polynomial $f(x) = x^3 + ax + b$ is irreducible over $k$ and the sum $(X_1 : Y_1 : Z_1) + (X_2 : Y_2 : Z_2)$ is given by the addition law $(X^{(2)}_3, Y^{(2)}_3, Z^{(2)}_3)$ of Bosma and Lenstra [BL95].

\[
\begin{align*}
(X_1Y_2 + Y_1X_2)(Y_1Y_2 - 6bZ_1Z_2) - a(Y_1Z_2 + Z_1Y_2)(2X_1X_2 - aZ_1Z_2) \\
- X_1Z_2(aX_1Y_2 + 3bY_1Z_2) - Z_1X_2(aY_1X_2 + 3bZ_1Y_2), \\
Y_1^2Y_2^2 + aX_1X_2(3X_1X_2 - 2aZ_1Z_2) - a^2(X_1Z_2 + Z_1X_2)^2 \\
+ 3b(X_1Z_2 + Z_1X_2)(3X_1X_2 - aZ_1Z_2) - (a^3 + 9b^2)Z_1^2Z_2, \\
Y_1Y_2(Y_1Z_2 + Z_1Y_2) + (3X_1X_2 + 2aZ_1Z_2)(X_1Y_2 + Y_1X_2) \\
+ (aX_1 + 3bZ_1)Y_1Z_2^2 + Z_1^2(aX_2 + 3bZ_2)Y_2,
\end{align*}
\]

specialized to $(a_1, a_2, a_3, a_4, a_6) = (0, 0, 0, a, b)$. Under the hypothesis on the 2-torsion, the exceptional divisor $\delta^*\{Y = 0\}$ is irreducible, hence the addition law is $k$-complete.

4.2. Genus 2 curves

Let $C$ be a genus 2 curve over a finite field $k = F_q$, with hyperelliptic involution $P \mapsto \overline{P}$. By [TV91] Proposition 2.3.21, p.180], there exists a (not necessarily effective) $k$-rational divisor $P_\infty$ of degree 1, such that $2P_\infty$ is equivalent to the canonical divisor $\kappa$ of $C$. The divisor $\Theta$, defined as the image of $C$ in $\text{Jac}(C)$ under the map $P \mapsto (P) - P_\infty$, is then a $k$-rational, ample, symmetric divisor which defines the canonical principal polarization on $\text{Jac}(C)$. For any $z \in \text{Jac}(C)(\bar{k})$, we denote by $\Theta_z$ its translation $(\tau^z_\infty)^{-1}\Theta = \Theta + z$.

The following result can be found for instance in [Mum75, p.275].

Proposition 4.5 Let $P$ and $Q$ be points in $C(\bar{k})$, and set $z = (P)-(Q) \in \text{Jac}(C)(\bar{k})$. Then we have

$$\Theta \cap \Theta_z = \{(P) - P_\infty, (\overline{Q}) - P_\infty\}.$$
As in the previous section, we will need the existence of a Galois orbit of points for the construction of a divisor on Jac(C).

**Lemma 4.6** If \( q \geq 7 \), there exists a point \( P_0 \in C(\overline{k}) \) whose Galois orbit has cardinality four and \( P_0^g = \overline{P}_0 \).

**Proof.** Let \( \phi : C \to \mathbb{P}^1 \) be the quotient by the hyperelliptic involution. Note that \( P_0 \) is a point in \( C(\mathbb{F}_{q^2}) \) such that \( \phi(P_0) \) is in \( \mathbb{P}^1(\mathbb{F}_{q^2}) \). Moreover, no such point exists if and only if \( \phi(C(\mathbb{F}_{q^2})) = \mathbb{P}^1(\mathbb{F}_{q^2}) \), or equivalently if
\[
|C(\mathbb{F}_{q^2})| = 2(q^2 + 1) - e_2,
\]
where \( e_2 \leq 6 \) is the number of ramification points of \( \phi \) in \( C(\mathbb{F}_{q^2}) \). For \( q \geq 7 \), this equality contradicts the Weil bound \( |C(\mathbb{F}_{q^2})| \leq q^2 + 4q + 1 \), and such a point exists.

**Remark 4.7** For each \( q = 2, 3, 4 \) and 5, there exists at least one genus 2 curve over \( \mathbb{F}_q \) with no such point \( P_0 \). In particular, for \( q = 5 \), the bound is tight (for \( e_2 = 6 \)):
\[
|C(\mathbb{F}_{q^2})| = 2(q^2 + 1) - 6 = q^2 + 4q + 1 = 46,
\]
and is satisfied for the curve \( y^2 = x^6 + 1 \) over \( \mathbb{F}_5 \).

**Theorem 4.8** Let \( C \) be a genus 2 curve over \( \mathbb{F}_q \) with \( q \geq 7 \). There exists a \( k \)-complete biquadratic addition law for the classical embedding of Jac(C) in \( \mathbb{P}^{15} \) determined by \( \mathcal{L}(4\Theta) \).

**Proof.** For the canonical divisor \( \kappa \) and a point \( P_0 \) as in Lemma 4.6, we define
\[
\alpha_0 = (P_0) + (P_0^g) - \kappa, \quad \alpha_1 = (P_0^g) + (P_0) - \kappa, \quad \alpha_2 = (P_0) + (P_0^g) - \kappa, \quad \alpha_3 = (P_0^g) + (P_0) - \kappa.
\]
Using Proposition 4.5 we find
\[
\Theta_{\alpha_0} \cap \Theta_{\alpha_1} = (\tau_{\alpha_0}^*)^{-1} \left( \Theta \cap \Theta_{(P_0^g) - (P_0)} \right) = \left\{ (P_0^g) - P_\infty + \alpha_0 \right\},
\]
\[
\Theta_{\alpha_0} \cap \Theta_{\alpha_3} = (\tau_{\alpha_0}^*)^{-1} \left( \Theta \cap \Theta_{(P_0) - (P_0^g)} \right) = \left\{ (P_0^g) - P_\infty + \alpha_0 \right\}.
\]
By construction, the divisor \( D = \sum \Theta_{\alpha_i} \) is ample, symmetric and \( k \)-rational. Moreover, since there exists a transitive action on the components \( \Theta_{\alpha_i} \), any \( k \)-rational point of \( D \) must be a point of the intersection
\[
\Theta_{\alpha_0} \cap \Theta_{\alpha_1} \cap \Theta_{\alpha_2} \cap \Theta_{\alpha_3},
\]
which is empty. Finally, we have \( \sum \alpha_i = 0 \) by construction, so \( D \sim 4\Theta \) and \( D \) determines a \( k \)-complete addition law for the classical embedding of Jac(C) in \( \mathbb{P}^{15} \) determined by \( \mathcal{L}(4\Theta) \).

**Remark 4.9** This construction can be generalized to other fields. For instance, following the same lines as Remark 4.4, Lemma 4.6 has an analogue over number fields \( k \). However, a \( k \)-rational divisor \( P_\infty \) of degree 1 may no longer exist, but for the family of curves \( C \) such as \( y^2 = f(x) \) with \( \deg f = 5 \), we can take \( P_\infty \) to be the divisor with support the point at infinity. In this case, the analogue of Theorem 4.8 holds over a number field. Arene and Costet have developed an algorithm to construct such an addition law [AC12].

**Remark 4.10** The construction of Theorem 4.8 uses differences of effective divisors of degree \( g = 2 \). In general such degree \( g \) divisors are necessary, since if \( C \) is a curve of genus \( g \) and if we define \( W_i = \text{im}(\text{Sym}^i C \to \text{Jac}(C)) \), then by [FK80, p. 146] the intersection
\[
\bigcap \{ W_{g-1} - a : a \in W_r + b \}
\]
is nonempty for any $0 \leq r \leq g - 1$ and any $b \in \text{Jac}(C)$.

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