TWISTED MONODROMY HOMOMORPHISMS AND MASSEY PRODUCTS

ANDREI PAJITNOV

ABSTRACT. Let $\phi : M \to M$ be a diffeomorphism of a $C^\infty$ compact connected manifold, and $X$ its mapping torus. There is a natural fibration $p : X \to S^1$, denote by $\xi \in H^1(X, \mathbb{Z})$ the corresponding cohomology class. Let $\rho : \pi_1(X, x_0) \to \text{GL}(n, \mathbb{C})$ be a representation (here $x_0 \in M$); denote by $H^*(X, \rho)$ the corresponding twisted cohomology of $X$. Denote by $\rho_0$ the restriction of $\rho$ to $\pi_1(M, x_0)$, and by $\rho_\lambda^0$ the antirepresentation conjugate to $\rho_0$. We construct from these data the twisted monodromy homomorphism $\phi_\lambda$ of the group $H_\lambda(M, \rho_\lambda^0)$. This homomorphism is a generalization of the homomorphism induced by $\phi$ in the ordinary homology of $M$.

The aim of the present work is to establish a relation between Massey products in $H^*(X, \rho)$ and Jordan blocks of $\phi_\lambda$. We have a natural pairing $H^*(X, \mathbb{C}) \otimes H^*(X, \rho) \to H^*(X, \rho)$; one can define Massey products of the form $\langle \xi, \ldots, \xi, x \rangle$, where $x \in H^*(X, \rho)$. The Massey product containing $r$ terms $\xi$ will be denoted by $\langle \xi, \ldots, \xi, x \rangle_r$; we say that the length of this product is equal to $r$. Denote by $M_k(\rho)$ the maximal length of a non-zero Massey product $\langle \xi, x \rangle_r$ for $x \in H^k(X, \rho)$. Given a non-zero complex number $\lambda$ define a representation $\rho_\lambda : \pi_1(X, x_0) \to \text{GL}(n, \mathbb{C})$ as follows: $\rho_\lambda(g) = \lambda^{\xi(g)} \cdot \rho(g)$. Denote by $J_k(\phi_\lambda, \lambda)$ the maximal size of a Jordan block of eigenvalue $\lambda$ of the automorphism $\phi_\lambda$ in the homology of degree $k$.

The main result of the paper says that $M_k(\rho_\lambda) = J_k(\phi_\lambda, \lambda)$. In particular, $\phi_\lambda$ is diagonalizable, if a suitable formality condition holds for the manifold $X$. This is the case if $X$ a compact Kähler manifold and $\rho$ is a semisimple representation. The proof of the main theorem is based on the fact that the above Massey products can be identified with differentials in a Massey spectral sequence, which in turn can be explicitly computed in terms of the Jordan normal form of $\phi_\lambda$.

CONTENTS

1. Introduction 1
2. Overview of the article 2
3. Formal deformations and Massey spectral sequences 4
4. Twisted monodromy homomorphisms 8
5. Twisted monodromy maps and formal deformation spectral sequences 12
6. Proofs of the main results 14
References 16

1. INTRODUCTION

The relation between non-vanishing Massey products of length 2 in the cohomology of mapping tori and the Jordan blocks of size greater than 1 of the monodromy homomorphism was discovered in the work of M. Fernández, A. Gray, J.

2010 Mathematics Subject Classification. 55N25, 55T99, 32Q15.
Key words and phrases. mapping torus, Massey products, twisted cohomology, Kähler manifolds.
ANDREI PAJITNOV

Morgan [7]. This relation was used by these authors to prove that certain mapping tori do not admit a structure of a Kähler manifold. In the work of G. Bazzoni, M. Fernández, V. Muñoz [1] it was proved that the existence of Jordan blocks of size 2 implies the existence of a non-zero triple Massey product of the form $⟨ξ, ξ, a⟩$.

In the paper [14] we began a systematic treatment of this phenomena, relating the maximal length of non-zero Massey products to the maximal size of Jordan blocks of the monodromy homomorphism. The both numbers turn out to be less by a unit than the number of the sheet where the formal deformation spectral sequence degenerates. In that paper we dealt with the case of Massey products of the form $⟨ξ, . . . , ξ, x⟩$ where $ξ$ is the 1-dimensional cohomology class determined by the fibration of the mapping torus $X$ over the circle, and $x$ is an element in the cohomology of $X$ with coefficients in a 1-dimensional local system.

In the present paper we continue the study of this phenomena, and prove a theorem relating the Jordan blocks of the monodromy homomorphisms to the Massey products of the form $⟨ξ, . . . , ξ, x⟩$ where $x$ is an element in the twisted cohomology of $X$ corresponding to an arbitrary representation $π_1(M, x_0)$ of $M$. One technical problem here is that there is no immediate definition of the homomorphism induced in the twisted homology of $M$ by the diffeomorphism $φ : M → M$. We construct such homomorphism in the present paper (Section 4). The construction is based on the techniques developed by P. Kirk and C. Livingston [9] in the context of twisted Alexander polynomials.

One corollary is that if $X$ is a compact Kähler manifold, then all the Jordan blocks of this twisted monodromy homomorphism are of size 1, that is, the twisted monodromy homomorphism is diagonalizable. This result imposes new constraints on the homology of Kähler manifolds.

2. Overview of the article

Let $φ : M → M$ be a diffeomorphism of a $C^∞$ compact connected manifold, and $X$ its mapping torus. Choose a point $x_0 ∈ M$, and put $H = π_1(M, x_0)$, $G = π_1(X, x_0)$. We have an exact sequence

$$1 → H → G → P → Z → 1$$

(1)

Let $V$ be a finite dimensional vector space over $C$ and $ρ : G → GL(V)$ be a representation; denote by $ρ_0$ its restriction to $H$. Put $V^* = Hom(V, C)$ and let $ρ^* : G → GL(V^*)$ be the antirepresentation of $G$ conjugate to $ρ$. In Section 4 we construct from these data an automorphism $φ_τ$ of the group $H_s(M, ρ_0)$; we call it the twisted monodromy homomorphism. This homomorphism can be considered as a generalization of the map induced by $φ$ in the ordinary homology. Observe however, that the homomorphism $φ_τ$ is not entirely determined by $φ$ and $ρ_0$, but depends also on the values of the representation $ρ$ on the elements of $G \setminus H$.

In the particular case when $ρ_0$ is the trivial 1-dimensional representation, and the representation $ρ$ sends the positive generator $u$ of $G/H ≈ Z$ to $λ ∈ C^*$, the map $λφ_τ$ equals the homomorphism induced by $φ$ in the ordinary homology (see the details in Subsection 4.2).

The main result of the paper is the theorem A below. To state it we need some terminology. Denote by $H^*(X, ρ)$ the twisted cohomology of $X$ with respect to the representation $ρ$. We have a natural pairing $H^*(X, C) ⊗ H^*(X, ρ) → H^*(X, ρ)$; one can define Massey products of the form $⟨ξ, . . . , ξ, x⟩$, where $x ∈ H^*(X, ρ)$. 

The Massey product containing \( r \) terms \( \xi \) will be denoted by \( \langle \xi, x \rangle_r \); we say that the length of this product is equal to \( r \). Denote by \( M_k(\rho) \) the maximal length \( r \) of a non-zero Massey product \( \langle \xi, x \rangle_r \) for \( x \in H^k(X, \rho) \). For a number \( \lambda \in \mathbb{C}^* \) define a representation \( \rho_\lambda : \pi_1(X, x_0) \to \text{GL}(V) \) as follows: \( \rho_\lambda(g) = \lambda^{\xi(g)} \cdot \rho(g) \).

Denote by \( J_k(\phi_*, \lambda) \) the maximal size of a Jordan block of eigenvalue \( \lambda \) of the homomorphism \( \phi_* \) of \( H_k(M, \rho_0^s) \).

**Theorem A.** We have \( J_k(\phi_*, \lambda) = M_k(\rho_\lambda) \) for every \( k \) and \( \lambda \).

This theorem implies that the monodromy homomorphism \( \phi_* \) has only Jordan blocks of size 1 (that is, \( \phi_* \) is diagonalizable) provided that the space \( X \) satisfies a suitable formality condition. Such formality conditions are discussed in details in Subsection 3.3. The main application of these ideas is the following theorem.

**Theorem B.** Assume that the mapping torus \( X \) of a diffeomorphism \( \phi : M \to M \) is a compact Kähler manifold. Let \( \rho : \pi_1(X, x_0) \to \text{GL}(n, \mathbb{C}) \) be a semisimple representation. Then the twisted monodromy homomorphism \( \phi_* \) of \( H_*(M, \rho_0^s) \) is a diagonalizable linear map.

The proofs of these theorems are given in Section 6. They are based on the construction of the twisted monodromy homomorphism \( \phi_* \) (Section 4), and the computation of the Massey spectral sequences in terms of \( \phi_* \) (Section 5). In Subsection 4.2 we discuss a particular case of special interest. Let us say that \( \phi \) is \( \pi_1 \)-split if the exact sequence (1) splits. For this case we give a natural geometric construction of the twisted monodromy homomorphism \( \phi_* \) of \( H_*(M, \rho_0^s) \) which is entirely determined by \( \phi \) and \( \rho_0 \) (see the formula (14) and Remark 4.8 of the Subsection 4.2).

Theorem 6.2 states the version of Theorem A for this particular case.

Let us mention a generalization of Theorem A to the case of an arbitrary manifold \( Y \) endowed with a non-zero cohomology class \( \alpha \in H^1(Y, \mathbb{Z}) \) and a representation \( \theta : \pi_1(Y, y_0) \to \text{GL}(V) \) (the manifold \( Y \) is not assumed to be a mapping torus any more). Assume that \( \alpha \) is indivisible and consider the corresponding infinite cyclic covering \( \overline{Y} \). Let \( L = \mathbb{C}[t, t^{-1}] \). Similarly to the above, denote by \( \theta_0 \) the restriction of \( \theta \) to \( \pi_1(\overline{Y}) \), and by \( \theta_0^\ast \) its conjugate antirepresentation. The homology \( H_k(\overline{Y}, \theta_0^\ast) \) is a finitely generated \( L \)-module; denote by \( \mathcal{T}_k \) its torsion part. This is a finite dimensional vector space over \( \mathbb{C} \) endowed with an action of \( L \). In particular the element \( t \in L \) represents an automorphism of \( \mathcal{T}_k \).

**Theorem C.** Let \( Y \) be a connected compact manifold. For every \( k \) and \( \lambda \in \mathbb{C}^* \) the maximal length \( r \) of a Massey product of the form \( \langle \alpha, x \rangle_r \) (where \( x \in H^k(Y, \theta_\lambda) \)) equals the maximal size of a Jordan block of eigenvalue \( \lambda \) of the automorphism \( t : \mathcal{T}_k \to \mathcal{T}_k \).

**Theorem D.** Let \( Y \) be a connected compact Kähler manifold. \( \theta : \pi_1(Y, y_0) \to \text{GL}(V) \) a semisimple representation. Then the homomorphism \( t : \mathcal{T}_k \to \mathcal{T}_k \) is diagonalizable for every \( k \).

2.1. **About the terminology.** We will keep the notations from Section 2 throughout the paper. Namely, \( X \) will always denote the mapping torus of a diffeomorphism \( \phi \) of a compact connected manifold \( M \); the corresponding cohomology class in \( H^1(X, \mathbb{Z}) \) will be always denoted by \( \xi \), the ring \( \mathbb{C}[t, t^{-1}] \) is denoted by \( L \), and \( \mathbb{C}[[z]] \) by \( A \). There are two exceptions: in Subsection 3.3 \( G \) will denote any group.
and in Subsection 5.1 \( G \) will denote the fundamental group of a topological space \( Y \).

2.2. Relations with other works. The case of the trivial representation \( \rho_0 \) was settled in the author’s paper [14]. The diagonalizability of the monodromy homomorphism in the ordinary homology for Kähler manifolds was also proved by N. Budur, Y. Liu, B. Wang [2].

Another approach to the relation between the size of Jordan blocks and formality properties was developed by S. Papadima and A. Suciu [15], [16]. They prove in particular that if the monodromy homomorphism has Jordan blocks of size greater than 1, then the fundamental group of the mapping torus is not a formal group.

3. Formal deformations and Massey spectral sequences

The main aim of this section is to recall necessary definitions and results concerning the Massey spectral sequences. There are different versions of these spectral sequences in literature, see [5], [11], [12], [6], [10]. We will recall here the versions described in [10], referring to this article for details and proofs.

The only new material in the present section is the definition of \( \mathcal{F} \)-formal manifold, introduced in Subsection 3.3. This notion generalizes the classical notion of formality (D. Sullivan [18]) incorporating to it differential forms with coefficients in flat bundles.

3.1. Formal deformations of differential graded algebras. Let

\[
\mathcal{A}^* = \{A^k\}_{k \in \mathbb{N}} = \{A^0 \xrightarrow{d} A^1 \xrightarrow{d} \ldots\}
\]

be a graded-commutative differential algebra (DGA) over \( \mathbb{C} \). Let \( \mathcal{N}^* \) be a graded differential module (DGM) over \( \mathcal{A}^* \). We denote by \( \mathcal{A}^*[z] \) the algebra of formal power series over \( \mathcal{A}^* \) endowed with the differential extended from the differential of \( \mathcal{A}^* \). Let \( \theta \in A^1 \) be a cocycle. Consider the \( \mathcal{A}^*[z] \)-module \( \mathcal{N}^*[z] \) and endow it with the differential

\[
D_1 x = dx + z\theta x.
\]

Then \( \mathcal{N}^*[z] \) is a DGM over \( \mathcal{A}^*[z] \), and we have an exact sequence of DGMs:

\[
0 \longrightarrow \mathcal{N}^*[z] \xrightarrow{z} \mathcal{N}^*[z] \xrightarrow{\pi} \mathcal{N}^* \longrightarrow 0
\]

where \( \pi \) is the natural projection \( z \longrightarrow 0 \). The induced long exact sequence in cohomology can be considered as an exact couple

\[
H^*(\mathcal{N}^*[z]) \xrightarrow{z} H^*(\mathcal{N}^*[z]) \xrightarrow{\delta} H^*(\mathcal{N}^*)
\]

One can prove that the spectral sequence induced by the exact couple \( \mathcal{N}^* \) depends only on the cohomology class of \( \theta \) ([10], Prop. 2.1).
Definition 3.1. Put $\alpha = [\theta]$. The spectral sequence associated to the exact couple $E_r$ is called formal deformation spectral sequence and denoted by $E_r^*(\mathcal{N}^*, \alpha)$. If the couple $(\mathcal{N}^*, \alpha)$ is clear from the context, we suppress it in the notation and write just $E_r^*$.

Thus $E_1^* = H^*(\mathcal{N}^*)$, and it is easy to see that $E_2^* \cong \text{Ker} \ L_\alpha/\text{Im} \ L_\alpha$, where $L_\alpha$ is the homomorphism of multiplication by $\alpha$. The higher differentials in this spectral sequence can be computed in terms of special Massey products. Let $a \in H^*(\mathcal{N}^*)$.

An $r$-chain starting from $a$ is a sequence of elements $\omega_1, \ldots, \omega_r \in \mathcal{N}^*$ such that
\[
d \omega_1 = 0, \quad [\omega_1] = a, \quad d \omega_2 = \theta \omega_1, \quad \ldots, \quad d \omega_r = \theta \omega_{r-1}.
\]

Denote by $MZ_{(r)}^m$ the subspace of all $a \in H^m(\mathcal{N}^*)$ such that there exists an $r$-chain starting from $a$. Denote by $MB_{(r)}^m$ the subspace of all $\beta \in H^m(\mathcal{N}^*)$ such that there exists an $(r-1)$-chain $(\omega_1, \ldots, \omega_{r-1})$ with $\theta \omega_{r-1}$ belonging to $\beta$. It is clear that $MB_{(r)}^m \subset MZ_{(r)}^m$ for every $i, j$. Put
\[
MH_{(r)}^m = MZ_{(r)}^m/MB_{(r)}^m.
\]

In the next definition we omit the upper indices and write $MH_{(r)}$, $MZ_{(r)}$, etc.

Definition 3.2. Let $a \in H^*(\mathcal{N}^*)$, and $r \geq 1$. We say that the $r$-tuple Massey product $\langle \theta, \ldots, \theta, a \rangle$ is defined, if $a \in MZ_{(r)}$. In this case choose any $r$-chain $(\omega_1, \ldots, \omega_r)$ starting from $a$. The cohomology class of $\theta \omega_r$ is in $MZ_{(r)}$ (actually it is in $MZ_{(N)}$ for every $N$) and it is not difficult to show that it is well defined modulo $MB_{(r)}$. The image of $\theta \omega_r$ in $MZ_{(r)}/MB_{(r)}$ is called the $r$-tuple Massey product of $\theta$ and $a$:
\[
\langle \theta, \ldots, \theta, a \rangle_r = \langle \theta_1, \ldots, \theta_r \rangle \in MZ_{(r)}/MB_{(r)}.
\]

The correspondence $a \mapsto \langle \theta, a \rangle_r$ gives rise to a well-defined homomorphism of degree 1
\[
\Delta_r : MH_{(r)} \longrightarrow MH_{(r)}.
\]

The following result is proved in [10], Theorem 2.5.

Theorem 3.3. 1) For any $r$ we have $\Delta^2_r = 0$, and the cohomology group $H^*(MH_{(r)}, \Delta_r)$ is isomorphic to $MH_{(r+1)}^*$.

2) For any $r$ there is an isomorphism
\[
\phi : MH_{(r)}^* \cong E_r^*
\]

commuting with differentials.

Therefore the differentials in the spectral sequence $E_r^*$ are equal to the higher Massey products with the cohomology class of $\theta$. Observe that these Massey products, defined above, have smaller indeterminacy than the usual Massey products.

Now let us consider some cases when the spectral sequences constructed above, degenerate in their second term. Recall that a differential graded algebra $\mathcal{A}^*$ is called formal if it has the same minimal model as its cohomology algebra.

Theorem 3.4. ([10], Th. 3.14) Let $\mathcal{A}^*$ be a formal differential algebra, $\alpha \in H^1(\mathcal{A}^*)$. Then the spectral sequence $E_r^*(\mathcal{A}^*, \alpha)$ degenerates at its second term.
Definition 3.5. A differential graded module $\mathcal{N}^*$ over a differential graded algebra $\mathcal{A}^*$ will be called formal if it is a direct summand of a formal differential graded algebra $\mathcal{B}^*$ over $\mathcal{A}^*$, that is,

$$\mathcal{B}^* = \mathcal{N}^* \oplus \mathcal{K}^*,$$

where both $\mathcal{N}^*$ and $\mathcal{K}^*$ are differential graded $\mathcal{A}^*$-submodules of $\mathcal{B}^*$.

Proposition 3.6. [10] Th. 3.16 Let $\mathcal{N}^*$ be a formal DG-module over $\mathcal{A}^*$, and $\alpha \in H^1(\mathcal{A}^*)$. Then the spectral sequence $E_r(\mathcal{N}^*, \alpha)$ degenerates at its second term.

In our applications $\mathcal{N}^*$ will be a DGM of differential forms on a manifold. Let $Y$ be a connected $C^\infty$ manifold, and $\rho$ be a representation of the fundamental group of $Y$. Put $\mathcal{A}^* = \Omega^*(Y, \mathbb{C})$ and let $\mathcal{N}^* = \Omega^*(Y, \rho)$ be the DGM of differential forms with coefficients in the flat bundle $\mathcal{E}_\rho$, induced by $\rho$. Then $\mathcal{N}^*$ is a DGM over $\mathcal{A}^*$, so for any cohomology class $\alpha \in H^1(\mathcal{A}^*)$ we obtain a spectral sequence starting with the twisted cohomology $H^*(Y, \rho)$ (see the next subsection for recollections about the twisted cohomology). The differentials in this spectral sequence are the Massey products with the class $\alpha$. We denote this spectral sequence by $E_r(\mathcal{N}^*, Y, \rho, \alpha)$.

Corollary 3.7. In the above notations assume that $\mathcal{N}^* = \Omega^*(Y, \rho)$ is a formal differential graded module over $\mathcal{A}^* = \Omega^*(Y, \mathbb{C})$. Then the spectral sequence $E_r(\mathcal{N}^*, Y, \rho, \alpha)$ degenerates at its second term.

3.2. Homology with local coefficients and spectral sequences. Let us first recall the definition of homology and cohomology with local coefficients. Let $R$ be a commutative ring, and $K$ a free module over $R$; denote by $\text{GL}(K)$ the group of $R$-automorphisms of $K$. Let $Y$ be a connected topological space, and $\rho : \pi_1(Y, y_0) \to \text{GL}(K)$ a representation. Define the $R$-module of $\rho$-twisted cochains of $Y$ with coefficients in $\rho$ as follows:

$$C_*(Y, \rho) = \text{Hom}_\rho(C_*(\tilde{Y}), K).$$

Here $\tilde{Y}$ denotes the universal covering of $Y$; we endow it with the canonical free left action of $\pi_1(Y, y_0)$. We denote by $C_*(\tilde{Y})$ the group of singular chains of $\tilde{Y}$; if $Y$ is a CW-complex we can replace it by the group of cellular chains. The group $C_*(\tilde{Y})$ of $\tilde{Y}$ has a natural structure of a free left module over $\mathbb{Z}\pi_1(Y, y_0)$. The cohomology $H^*(C_*(Y, \rho))$ is called twisted cohomology of $Y$ with coefficients in $\rho$, or cohomology of $Y$ with local coefficients in $\rho$.

Let $\beta : \pi_1(Y, y_0) \to \text{GL}(K)$ be an antirepresentation (that is, $\beta(g_1 g_2) = \beta(g_2)\beta(g_1)$ for every $g_1, g_2 \in \pi_1(Y, y_0)$); it determines a right action of $\pi_1(Y, y_0)$ on $K$. Define the $R$-module of $\rho$-twisted chains of $Y$ with coefficients in $K$ as follows:

$$C_*(Y, \beta) = K \otimes_\beta C_*(\tilde{Y}).$$

The homology $H_*(C_*(Y, \beta))$ is called twisted homology of $Y$ with coefficients in $\beta$, or homology of $Y$ with local coefficients in $\beta$.

In this paper we will be dealing with the cases when $R$ is one of the following rings: $\mathbb{C}$, $L = \mathbb{C}[t, t^{-1}]$, $\Lambda = \mathbb{C}[[z]]$. Let $V$ be a finite-dimensional vector space over $\mathbb{C}$, put

$$V[t^\pm] = V \otimes_\mathbb{C} \mathbb{C}[t, t^{-1}], \quad V[[z]] = V \otimes_\mathbb{C} \mathbb{C}[[z]].$$
If \( \alpha \in H^1(Y, \mathbb{Z}) \) is a non-zero cohomology class, we can define two representations:

\[
[\alpha] : \pi_1(Y, y_0) \to L^*; \quad [\alpha](g) = e^{i \alpha \cdot g}, \quad \langle \alpha \rangle : \pi_1(Y, y_0) \to \Lambda^*; \quad \langle \alpha \rangle(g) = e^{i \alpha \cdot g}.
\]

For a representation \( \rho : \pi_1(Y, y_0) \to GL(V) \) put

\[
(7) \quad \mathfrak{p} : \pi_1(Y, y_0) \to GL(V[t]), \quad \mathfrak{p} = \rho \otimes [\alpha],
\]

\[
(8) \quad \hat{\rho} : \pi_1(Y, y_0) \to GL(V[[z]]), \quad \hat{\rho} = \rho \otimes \langle \alpha \rangle.
\]

The representation \( \mathfrak{p} \) is a basic tool in the theory of twisted Alexander polynomials (see [9], Section 2.1). Observe that \( \hat{\rho} = \text{Exp} \circ \mathfrak{p} \), where \( \text{Exp} : L \to \Lambda \) is the homomorphism sending \( t \) to \( e^z \).

Applying the functor Hom to the exact sequence

\[
0 \to \Lambda \xrightarrow{t} \Lambda \to \mathbb{C} \to 0
\]

we obtain an exact sequence of groups of twisted cochains:

\[
0 \to \mathbb{C}^*(Y, \hat{\rho}) \xrightarrow{z} \mathbb{C}^*(Y, \hat{\rho}) \to \mathbb{C}^*(Y, \rho) \to 0.
\]

The corresponding long exact sequence of twisted homomorphism modules can be considered as an exact couple

\[
(9) \quad H^*(Y, \hat{\rho}) \xrightarrow{z} \mathbb{C}^*(Y, \hat{\rho}) \xrightarrow{\mathbb{C}^*(Y, \hat{\rho})} \to H^*(Y, \rho)
\]

This exact couple induces a spectral sequence \( D^*_r(Y, \rho, \alpha) \) starting from the module \( H^*(Y, \rho) \). We have the following theorem

**Theorem 3.8.** [10, Th. 5.4] The spectral sequences \( E^*_r(Y, \rho, \alpha) \) and \( D^*_r(Y, \rho, \alpha) \) are isomorphic.

In particular the differentials in the spectral sequence \( D_r(Y, \rho, \alpha) \) are equal to the Massey products \( \langle \alpha, \cdot, \cdot \rangle_r \).

### 3.3. Formality with respect to a family of representations of the fundamental group.

Let us start with a definition.

**Definition 3.9.** Let \( Y \) be a manifold, \( y_0 \in Y \), denote \( \pi_1(Y, y_0) \) by \( G \). Let \( \mathcal{F} \) be a family of complex representations of \( G \), such that \( \mathcal{F} \) is closed under tensor products, that is, if \( \rho_1, \rho_2 \in \mathcal{F} \) then \( \rho_1 \otimes \rho_2 \in \mathcal{F} \). Put

\[
\overline{\Omega}^*(Y, \mathcal{F}) = \bigoplus_{\rho \in \mathcal{F}} \Omega^*(Y, \rho).
\]

Then \( \overline{\Omega}^*(Y, \mathcal{F}) \) has a natural structure of a DGA. We say that \( Y \) is \( \mathcal{F} \)-formal, if this DGA is formal.

**Examples.**

1) If \( \mathcal{F} \) is the trivial 1-dimensional representation, then the notion of \( \mathcal{F} \)-formality is the same as the classical notion of formality as introduced by D. Sullivan [18], [4].

2) Let \( \mathcal{F} \) be the family of all 1-dimensional representations of \( G \). Then the notion of \( \mathcal{F} \)-formality is the same as the notion of strong formality introduced in [10], see also [8]. All compact connected Kähler manifolds are strongly formal, as it follows from C. Simpson’s theorem [17].
3) Let \( G \) be a fundamental group of a compact connected Kähler manifold, let \( \rho \) be a semisimple representation of \( G \). Consider the family \( \mathcal{F} \) consisting of all tensor powers of \( \rho \) (including the trivial representation of the same dimension as \( \rho \)). It follows from theorem of K. Corlette \(^3\) that \( \mathcal{F} \) is closed under tensor products, see also \([17]\). C. Simpson’s theorem \([17]\) implies that \( Y \) is \( \mathcal{F} \)-formal.

**Theorem 3.10.** Assume that a manifold \( Y \) is \( \mathcal{F} \)-formal. Let \( \rho \in \mathcal{F} \) and \( \alpha \in H^1(Y, \mathbb{C}) \). Then the formal deformation spectral sequence \( D_r^\rho(Y, \rho, \alpha) \) degenerates at its second term. All Massey products \( \langle \alpha, x \rangle_r \) vanish for every \( x \in H^*(Y, \rho) \) and \( r \geq 2 \).

**Proof.** It suffices to apply Corollary \([3.7]\) to the module \( \Omega^*(Y, E_\rho) \).

The formality property of Example 3) above yields the following corollary.

**Corollary 3.11.** Assume that \( Y \) is a connected compact Kähler manifold, and \( \alpha \in H^1(Y, \mathbb{C}) \) a non-zero cohomology class. Let \( \rho : \pi_1(Y, y_0) \to \text{GL}(n, \mathbb{C}) \) be a semisimple representation. Then the spectral sequence \( D_r^\rho(Y, \rho, \alpha) \) degenerates at its second term. \( \square \)

### 4. Twisted monodromy homomorphisms

Let \( \phi : M \to M \) be a diffeomorphism of a \( C^\infty \) compact connected manifold, and \( X \) its mapping torus. Choose a point \( x_0 \in M \), and put \( H = \pi_1(M, x_0) \), \( G = \pi_1(X, x_0) \). Recall the exact sequence

\[
\begin{array}{cccccc}
1 & \longrightarrow & H & \overset{i}{\longrightarrow} & G & \overset{p}{\longrightarrow} & \mathbb{Z} & \longrightarrow & 1.
\end{array}
\]

Let \( W \) be a vector space of dimension \( n \) over \( \mathbb{C} \) and endowed with a right action of \( G \). Such action can be described as a map \( \beta : G \to \text{GL}(W) \approx \text{GL}(n, \mathbb{C}) \) satisfying \( \beta(g_1 g_2) = \beta(g_2) \beta(g_1) \), that is, an antirepresentation of \( G \). Set \( \beta_0 = \beta | H : H \to \text{GL}(W) \). In this section we associate to these data an isomorphism \( \phi_s : H_s(M, \beta_0) \to H_s(M, \beta_0) \) of vector spaces that we call twisted monodromy homomorphism induced by \( \phi \). This homomorphism can be considered as a generalization of the map induced by \( \phi \) in the ordinary homology. Observe however, that the homomorphism \( \phi_s \) is not entirely determined by \( \phi \) and \( \beta_0 \), but depends also on the values of \( \rho \) on the elements of \( G \backslash H \) (see the details in Subsection \([4.1]\)). The constructions of this section will be applied in Section \([5]\) to the map \( \beta : G \to \text{GL}(W) \) which is conjugate to the given representation \( \rho : G \to \text{GL}(W) \).

#### 4.1. Definition of the twisted monodromy homomorphism.

Choose any path \( \theta \) in \( M \) from \( x_0 \) to \( \phi(x_0) \). This choice determines three more geometric objects:

A) An element \( u \in G \) such that \( p(u) = 1 \). Namely let \( u \) be a composition of the path \( \theta \) with the image of the path \( \phi(x_0) \times [0, 1] \) in the mapping torus

\[
X = M \times [0, 1]/\sim (x, 0) \sim (\phi(x), 1).
\]

Observe that any element \( u \) with \( p(u) = 1 \) can be obtained this way with a suitable choice of \( \theta \).

B) A lift of the map \( \phi \) to a map \( \tilde{\phi} : \tilde{M} \to \tilde{M} \). Namely, represent a point \( x \in \tilde{M} \) by a path \( \gamma \) in \( M \) starting at \( x_0 \). The path \( \phi(\gamma) = \phi \circ \gamma \) joins the points
\[ \phi(x_0) \text{ and } \phi(x). \] The composition of paths \( \theta \cdot \phi(\gamma) \) joins \( x_0 \) and \( \phi(x) \). Now put \( \tilde{\phi}(\gamma) = \theta \cdot \phi(\gamma) \).

C) A homomorphism \( K_{\theta} : H \to H \) defined by \( K_{\theta}(\gamma) = \theta \phi(\gamma) \theta^{-1} \) where \( \gamma \) is a loop starting at \( x_0 \).

These objects satisfy the following easily checked properties:
\[ \tilde{\phi}(hx) = K_{\theta}(h) \tilde{\phi}(x) \text{ for every } h \in H \text{ and } x \in \bar{M}; \]
\[ uhu^{-1} = K_{\theta}(h) \text{ for every } h \in H. \]

Now we can define the homomorphism \( \phi_s \).

**Definition 4.1.** To simplify the notation, we shall abbreviate \( W \otimes C_s(\bar{M}) \) to \( W \otimes C_s(\bar{M}) \) up to the end of the present subsection.

Define a map
\[ \alpha : W \otimes C_s(\bar{M}) \to W \otimes C_s(\bar{M}); \quad \alpha(v \otimes \sigma) = vu^{-1} \otimes \tilde{\phi}(\sigma), \]
where \( v \in W \) and \( \sigma \) is a simplex in \( C_s(\bar{M}) \) (here and elsewhere we denote by \( vg \) the result of the action of \( g \in G \) on the vector \( v \in W \)).

**Lemma 4.2.** 1) The map \( \alpha \) factors to an endomorphism \( A \) of \( W \otimes C_s(\bar{M}) \).

2) The resulting map \( A \) is a chain map, and it does not depend on the choice of the path \( \theta \).

**Proof.** 1) We have to check that \( \alpha(vh \otimes \sigma) \) and \( \alpha(vh \sigma) \) give the same element in \( W \otimes C_s(\bar{M}) \) for every \( h \in H \). Observe that
\[ \alpha(vh \otimes \sigma) = (vh)u^{-1} \otimes \tilde{\phi}(\sigma) = vu^{-1} \cdot uhu^{-1} \otimes \tilde{\phi}(\sigma) = vu^{-1} \cdot \phi(\sigma) = vu^{-1} \otimes K_{\theta}(h) \otimes \phi(\sigma) \]
and this equals \( vu^{-1} \otimes K_{\theta}(h) \phi(\sigma) \) in \( W \otimes C_s(\bar{M}) \). Apply the formula [10] and the proof of the first part of Lemma is over.

2) Let \( \theta' \) be another path joining \( x_0 \) and \( \phi(x_0) \), so that \( \theta' = \gamma \theta \) where \( \gamma \) is a loop in \( M \) starting at \( x_0 \). The corresponding element \( u' \in G \) satisfies \( u' = \gamma u \), and \( \tilde{\phi}' = \gamma \phi \), so that \( vu'u^{-1} \otimes \phi'(\sigma) = vuu^{-1} \otimes \gamma \phi(\sigma) \) and the property 2) follows. \( \square \)

**Definition 4.3.** The map induced by \( A \) in the homology groups \( H_*(M, \beta) \) will be denoted by \( \phi_s[\beta] \) and called the twisted monodromy homomorphism associated to \( \phi \) and \( \beta \) (when the value of \( \beta \) is clear from the context we omit it in the notation).

**Definition 4.4.** For any antihomomorphism \( \beta : G \to GL(W) \) and \( \lambda \in \mathbb{C}^* \) define an antihomomorphism \( \beta_\lambda : G \to GL(W) \) as follows:
\[ \beta_\lambda(g) = \lambda^{\xi \cdot g} \cdot \beta(g). \]

The proof of the following proposition follows immediately from the definition of \( \phi_s \) (see the formula [12]).

**Proposition 4.5.** We have
\[ \phi_s[\beta_\lambda] = \frac{1}{\lambda} \cdot \phi_s[\beta]. \] \( \square \)
4.2. **The case** $G = H \times \mathbb{Z}$. The algebraically simplest case occurs when the exact sequence (1) splits. This case can be characterized by the following simple lemma (the proof will be omitted).

**Lemma 4.6.** The three following properties are equivalent:

1) For some path $\theta$ from $x_0$ to $\phi(x_0)$ the homomorphism $K_\theta : H \to H$ is an inner automorphism.

2) For every path $\theta$ from $x_0$ to $\phi(x_0)$ the homomorphism $K_\theta : H \to H$ is an inner automorphism.

3) The extension (1) splits. □

One can prove also that if the properties listed in the lemma hold for some choice of a base point $x_0$, then they hold for any other choice of the base point.

**Definition 4.7.** If the map $\phi$ satisfies the three equivalent properties of Lemma (4.6) we say that $\phi$ is $\pi_1$-split.

Assume that $\phi$ is $\pi_1$-split. Choose an element $u \in G$ commuting with $H$, and such that $p(u) = 1$. Let $\beta : H \to GL(W)$ be any antirepresentation, and let $\beta_0$ be its restriction to $H$. Put $B = \beta(u)$, then $B \in GL(W)$. (Observe that in the split case any antirepresentation of $H$ can be extended to an antirepresentation of $G$, sending $u$ to a scalar matrix.)

1. Consider first the case when $B$ is the identity map of $W$. The homomorphism $\phi_*$ in this case has an especially simple definition. Namely, choose a path $\theta$ from $x_0$ to $\phi(x_0)$ in such a way that for any $\gamma \in \pi_1(M, x_0)$ we have $\theta \phi(\gamma) \theta^{-1} = \gamma$. Then the corresponding lift $\tilde{\phi} : \tilde{M} \to \tilde{M}$ has the property

$$\tilde{\phi}(hx) = h\tilde{\phi}(x) \quad \text{for every} \ h \in \pi_1(M, x_0)$$

Denote such a lift by $\tilde{\phi}^\circ$. The automorphism of $H_\ast(M, \beta_0)$ corresponding to this choice will be denoted by $\phi_*^\circ$; it is defined by the following formula:

$$\phi_*^\circ(v \otimes \sigma) = v \otimes \tilde{\phi}^\circ \sigma$$

(where $\sigma$ is a singular simplex of $\tilde{M}$). This homomorphism $\phi_*^\circ$ is entirely determined by $\phi$ and $\beta_0$, and does not depend on the values of $\beta$ on $G \setminus H$. In the case when $\beta_0$ is the trivial representation the map $\phi_*^\circ$ is just the induced map in the ordinary homology.

2. Now let $B = \lambda \cdot Id$ where $\lambda \in \mathbb{C}^\ast$. Choosing for $\phi$ the same lift as in the previous case, we obtain the following formula for $\phi_*^\circ$:

$$\phi_*(v \otimes \sigma) = \frac{1}{\lambda} v \otimes \tilde{\phi}^\circ_\ast(\sigma) = \frac{1}{\lambda} \phi_*^\circ(v \otimes \sigma).$$

3. Now let $B$ be an arbitrary element of $GL(W)$. Since $B$ commutes with $H$, it induces a well-defined linear maps $C\ast_\ast(M, \beta_0) \to C\ast_\ast(M, \beta_0)$ and $H\ast_\ast(M, \beta_0) \to H\ast_\ast(M, \beta_0)$, that will be denoted by the same letter $B$. We have then

$$\alpha(v \otimes \sigma) = vu^{-1} \otimes \tilde{\phi}(\sigma) = B^{-1}(v \otimes \tilde{\phi}(\sigma))$$

and finally

$$\phi_* = B^{-1} \circ \phi_*^\circ.$$
Remark 4.8. In the case $\lambda = 1$ it is possible to reformulate our definition of $\phi^\circ_\ast$ in terms of induced representations of fundamental groups. To explain this, let us proceed to a slightly more general framework. Let $\phi : X \to Y$ be a map of connected topological spaces, $x_0 \in X$, $y_0 \in Y$. Let $\rho : \pi_1(Y, y_0) \to \text{GL}(W)$ be a representation. Choose a path $\mu$ from $y_0$ to $\phi(x_0)$. Define a representation $\rho' : \pi_1(X, x_0) \to \text{GL}(W)$ as follows $\rho'(g) = \rho(\mu \phi \mu^{-1})$. It is easy to check that $\phi$ induces a homomorphism

$$
\phi_\ast : H_\ast(X, \rho') \to H_\ast(Y, \rho),
$$

defined on the chain level as $v \otimes \sigma \mapsto v \otimes \bar{\phi} \circ \sigma$. This homomorphism depends obviously on $\mu$. In the case when $X = Y$ and $\phi$ is $\pi_1$-split choose a path $\mu$ in such a way that $\mu \phi(h) \mu^{-1} = h$ for every $h \in \pi_1(M, x_0)$. Then $\rho' = \rho$ and we return to the homomorphism $\phi^\circ_\ast$ of the above definition.

4.3. Relation with Kirk-Livingston’s setup. Let $\overline{X}$ be the infinite cyclic covering of $X$ corresponding to $\xi$. Observe that $\pi_1(\overline{X}) \cong H$, so that the twisted homology $H_\ast(\overline{X}, \beta_0)$ of $\overline{X}$ is defined. We have the following simple lemma; the proof follows from the observation that $\overline{X} \cong M \times \mathbb{R}$.

Lemma 4.9. The inclusion $i : M \hookrightarrow X$ induces an isomorphism

$$
i : H_\ast(M, \beta_0) \xrightarrow{\cong} H_\ast(\overline{X}, \beta_0).$$

In the work [9] P. Kirk and C. Livingston constructed an action of the group $\mathbb{Z}$ on the space $W \otimes C_\ast(\overline{X})$. Namely, choose any element $u \in G$ such that $p(u) = 1$, and let the generator $t$ of $\mathbb{Z}$ act on $W \otimes C_\ast(\overline{X})$ by the following formula

$$
t(v \otimes \sigma) = vu^{-1} \otimes u\sigma.
$$

It is shown in [9] that that this action does not depend on the particular choice of $u$. The next proposition follows readily from the definition of the monodromy homomorphism $\phi_\ast$.

Proposition 4.10. The following diagram is commutative

$$
\begin{array}{ccc}
H_\ast(\overline{X}, \beta_0) & \xrightarrow{t} & H_\ast(\overline{X}, \beta_0) \\
\downarrow I & & \downarrow I \\
H_\ast(M, \beta_0) & \xrightarrow{\phi_\ast} & H_\ast(M, \beta_0)
\end{array}
$$

Remark 4.11. We worked in this section in the assumption that $M$ is a compact $C^\infty$ manifold; the homology groups were in coefficients in $\mathbb{C}$, in view of the applications to Kähler manifolds.

However all the constructions and results of the section generalize without any changes to the case when $M$ is any CW-complex; the coefficient field $\mathbb{C}$ can be replaced by an arbitrary field $K$. 

5. Twisted monodromy maps and formal deformation spectral sequences

We begin with a discussion of a universal coefficient theorem for twisted cohomology (Subsection 5.1). The next subsection contains the computation of the formal deformation spectral sequences in terms of the monodromy maps. The proof of the main theorem in Section 6 is based on these computations.

5.1. Universal coefficient theorem for twisted cohomology. Let $Y$ be a connected topological space endowed with a non-zero cohomology class $\eta \in H^1(Y, \mathbb{Z})$. Denote by $G$ the fundamental group $\pi_1(Y, y_0)$. Let $V$ be a finite-dimensional vector space over $\mathbb{C}$ and $\rho : G \rightarrow \text{GL}(V)$ a representation. Let $L = \mathbb{Z}[t, t^{-1}]$, denote by $V[t^{\pm}]$ the free $L$-module $V \otimes L$. Recall from Subsection 3.2 the representation

$$\varphi : G \rightarrow \text{GL}(V[t^{\pm}]); \quad \varphi(g) = \rho(g)t^{\ell(g)}.$$  

The representation $\rho$ determines an action from the left of $G$ on $V$: put $V^* = \text{Hom}(V, \mathbb{C})$, and consider the corresponding right action $\rho^*$ of $G$ on $V^*$. Similarly we obtain a right action $\overline{\rho}^*$ of $G$ on $V[t^{\pm}]$. If we choose a basis in $V$, then $\rho^*$ is identified with an antihomomorphism $G \rightarrow \text{GL}(n, \mathbb{C})$ obtained from $\rho$ by transposition (similarly for $\overline{\rho}^*$). Associated to $\overline{\rho}^*$ there is the $L$-module of $\overline{\rho}^*$-twisted chains

$$V[t^{\pm}] \otimes C_\bullet(\tilde{Y}),$$

and its homology $H_\bullet(X, \overline{\rho}^*)$. Observe that for any right action $\chi$ of $G$ on a free $L$-module $W$ there is a natural isomorphism

$$\text{Hom}_L \left( W \otimes C_\bullet(\tilde{Y}), L \right) \xrightarrow{\Phi} \text{Hom}_\rho \left( C_\bullet(\tilde{Y}), W^* \right);$$

the value of $\Phi$ on an $L$-homomorphism $\alpha : W \otimes C_\bullet(\tilde{Y}) \rightarrow L$ is defined by the following formula:

$$\Phi(\alpha)(\sigma)(w) = \alpha(w \otimes \sigma)$$

(where $w \in W$ and $\sigma \in C_\bullet(\tilde{Y})$). Applying this to the right action $\rho^*$ on $W = V^*$ we obtain the following isomorphism (see [10], Lemma 4.3)

$$H^\bullet(Y, \overline{\rho}^*) \cong H^\bullet \left( \text{Hom}_L(C_\bullet(Y, V^*), L) \right).$$

The cohomology module in the right-hand side of [16] has the advantage that we can apply to it the universal coefficient theorem:

**Proposition 5.1.** For every $k$ we have an exact sequence

$$0 \rightarrow \text{Ext}_L^1 \left( H_{k-1}(Y, \overline{\rho}^*), L \right) \rightarrow H^k(Y, \overline{\rho}^*) \rightarrow \text{Hom}_L \left( H_k(Y, \overline{\rho}^*), L \right) \rightarrow 0. \tag{17}$$

We will now apply these results to mapping tori. In the rest of this subsection $X$ is the mapping torus of a homeomorphism $\phi : M \rightarrow M$ (see Subsection 2.1 for the notations). Endow the vector space $H_{k-1}(M, \rho_0^*)$ with the action of $L$ as follows: $ta = \phi_*(a)$, where $\phi_*$ is the twisted monodromy map from Subsection 4.1.

**Proposition 5.2.** We have an isomorphism of $L$-modules

$$H^k(X, \overline{\rho}^*) \cong H_{k-1}(M, \rho_0^*).$$
Proof. By Theorem 2.1 of [9] we have
\[ H^*(X, \overline{\rho}) \approx H_*(\overline{X}, \rho_0^*). \]

Lemma 4.9 says that \( H_*(\overline{X}, \rho_0^*) \approx H_*(M, \rho_0^*) \), therefore this \( L \)-module is a finite dimensional vector space over \( \mathbb{C} \), and hence a finitely generated torsion \( L \)-module. Thus we have \( \text{Hom}_L \left( H_k(X, \overline{\rho}), L \right) = 0 \) and
\[ \text{Ext}_L^1 \left( H_{k-1}(X, \overline{\rho}), L \right) \approx H_{k-1}(X, \overline{\rho}) \approx H_{k-1}(M, \rho_0^*). \]

The proposition follows. □

5.2. Computation of deformation spectral sequences in terms of monodromy maps. Our main aim here is to prove that the exact couples (18) and (19) induce isomorphic spectral sequences. We have \( \hat{\rho} = \text{Exp} \circ \overline{\rho} \), where \( \text{Exp} : L \to \Lambda \) is the ring homomorphism sending \( t \) to \( e^z \). The exact couple (20)
\[ H^*(X, \hat{\rho}) \xrightarrow{z} H^*(X, \hat{\rho}) \]

is obviously isomorphic to the following exact couple (21)
\[ H^*(X, \hat{\rho}) \xrightarrow{e^z - 1} H^*(X, \hat{\rho}) \]

The exact couple (22) below
\[ H^*(X, \rho) \xrightarrow{t-1} H^*(X, \rho) \]

maps to (21) via the homomorphism \( H_* (X, \rho) \to H_* (X, \hat{\rho}) \) induced by \( \text{Exp} \), therefore the spectral sequences induced by (22) and (20) are isomorphic. Applying Proposition 5.2 we deduce that (22) is isomorphic to the exact couple (23) of the form
\[ H_* (M, \rho_0^*) \xrightarrow{\phi_* - 1} H_* (M, \rho_0^*) \]

where the maps \( j \) and \( k \) have the degrees respectively 1 and 0. We obtain finally a homomorphism (23) \( \to (20) \) of exact couples, which equals the identity map on the term \( H^*(X, \rho) \). Therefore the exact sequences derived from these exact couples are isomorphic.
Remark 5.3. Similarly to the Section 4, all the constructions and results of the section generalize without any changes to the case when $M$ is any CW-complex.

6. PROOFS OF THE MAIN RESULTS

Now we can complete the proofs of the main results.

6.1. Theorem A. Let us first prove Theorem A for the case $\lambda = 1$, that is, $\rho\chi = \rho$. According to the previous section the exact couples $(\mathcal{D}_0)$ and $(\mathcal{D}_3)$ induce isomorphic spectral sequences. The differentials in the spectral sequence induced by $(\mathcal{D}_0)$ are equal to Massey products: $d_r(x) = \langle \xi, x \rangle_r$, therefore the spectral sequence degenerates in degree $k$ at the term number $M_k(\rho) + 1$. It suffices to prove that the spectral sequence induced by $(\mathcal{D}_3)$ degenerates at the term $J_k(\phi_u) + 1$ in degree $k$. Denote by $\phi^{(k)}$ the twisted monodromy homomorphism in degree $k$.

Let $A_0$ be the invariant linear subspace of eigenvalue 1 of $\phi^{(k)}$. Let $B_0$ be the sum of all invariant linear subspaces of $\phi^{(k)}$ corresponding to the eigenvalues different from 1. The restriction $(\phi^{(k)} - 1) | A_0$ is nilpotent of degree equal to $J_k(\phi_u, 1)$, and the restriction $(\phi^{(k)} - 1) | B_0$ is an isomorphism of $B_0$ onto itself. The assertion of the theorem follows now from the following lemma ([14], Lemma 3.3).

Lemma 6.1. Let $\varepsilon$ be a graded exact couple:

$$
\begin{array}{c}
D_i \rightarrow D_j \\
\downarrow \downarrow \\
E_i \leftarrow E_j
\end{array}
$$

Assume that the homomorphism $i : D_k \rightarrow D_k$ decomposes as follows:

$$
\delta \oplus \tau : A \oplus B \rightarrow A \oplus B
$$

where $\delta$ is nilpotent of degree $m$ and $\tau$ is injective.

1) Let $\deg i = \deg l = 0$, $\deg j = 1$. Then the spectral sequence induced by $\varepsilon$ degenerates at the step $m + 1$ in degree $k$.

2) Let $\deg i = 0$, $\deg l = 1$, $\deg j = 0$. Then the spectral sequence induced by $\varepsilon$ degenerates at the step $m + 1$ in degree $k - 1$.

□

Now let us consider the case when $\lambda$ is an arbitrary non-zero complex number. According to the Proposition 4.5 the monodromy homomorphism $\phi_u[\rho\chi]$ constructed from the representation $\rho\chi$ equals $\frac{1}{\lambda} \phi_u[\rho]$. Therefore the exact couple $(\mathcal{D}_3)$ for the case of the representation $\rho\chi$ has the following form

$$
\begin{array}{c}
H^\ast(M, \rho_0^\ast) \\
\downarrow \downarrow \\
H^\ast(X, \rho_\chi)
\end{array}
$$

(where $\phi_u$ denotes the monodromy homomorphism corresponding to $\rho$). It remains to observe that $J_k(\frac{1}{\lambda} \phi_u, 1) = J_k(\phi_u, \lambda)$. 

6.2. **Theorem B.** Let $X$ be a connected compact Kähler manifold, and $\rho$ a semisimple representation. In view of Theorem A it suffices to prove that $M_k(\rho) = 1$ for every $\lambda \in \mathbb{C}^*$, or, equivalently, that the spectral sequence associated to the exact couple

\[(24)\]

\[H^*(X, \hat{\rho}_\lambda) \xrightarrow{z} H^*(X, \hat{\rho}_\lambda) \]

\[H^*(X, \rho_\lambda) \]

degenerates at its second term. Observe that the representation $\rho_\lambda$ is also semisimple; apply to it Corollary (3.11) and the proof of Theorem B is over.

6.3. **The $\pi_1$-split case.** Recall from Subsection 4.2 that in this case we have an automorphism $\phi^\circ$ determined by $\phi$ and by the representation of $\pi_1$ of $M$. Choose an element $u \in G$ commuting with $H$ and such that $p(u) = 1$. Let $\chi : \pi_1(M, x_0) \to \text{GL}(V)$ be any representation of the fundamental group of $M$. Let $\lambda \in \mathbb{C}^*$. Extend $\chi$ to a representation $\chi_\lambda : \pi_1(X, x_0) \to \text{GL}(V)$ by $\chi(u) = \lambda$ (this is possible since $u$ commutes with $H$).

**Theorem 6.2.** We have

1) $J_k(\phi^\circ_\lambda, \lambda) = M_k(\chi_\lambda)$.

2) If moreover $X$ is a compact Kähler manifold, and the representation $\chi$ is semisimple, then $\phi^\circ_\lambda$ is diagonalizable.

**Proof.** Part 1) follows immediately from Theorem A. As for the part 2), observe that the representation $\chi_\lambda$ is also semisimple, so we can apply to it the Theorem B, and the proof is over. □

**Remark 6.3.** The particular case of the trivial representation $\chi$ corresponds to Theorems 3.1 and 5.1 of the paper [14].

6.4. **Theorem C.** We need some more terminology.

**Definition 6.4.** Let $R$ be a finitely generated $L$-module, and $a \in L$ a polynomial of degree 1. Denote by $R_a$ the $a$-primary part of $R$, that is, the submodule of all $x \in R$, such that $a^N x = 0$ for some $N$. Denote by $\text{Nil}(R, a)$ the minimal number $N$, such that $a^N R_a = 0$. The module $R_a$ is a finite-dimensional vector space, and $a$ determines a linear map of this space. The number $\text{Nil}(R, a)$ equals the maximal size of Jordan blocks of eigenvalue 0 of $a$.

Denote by $M_k(\alpha, \theta_\lambda)$ the maximal length of a non-zero Massey product of the form $\langle \alpha, x \rangle_r$ where $x \in H^k(Y, \theta_\lambda)$. Consider the spectral sequence associated to the exact couple

\[(25)\]

\[H^*(Y, \overline{\theta}_\lambda) \xrightarrow{t-1} H^*(Y, \overline{\theta}_\lambda) \]

\[H^*(Y, \theta_\lambda) \]

Applying the same argument as in the beginning of Subsection 6.1 we deduce that $M_k(\alpha, \theta_\lambda) + 1$ equals the the number $r$ of the sheet where this spectral sequence degenerates. By Lemma 6.1 this number $r$ equals $\text{Nil}(H^{k+1}(Y, \overline{\theta}_\lambda), t-1)$. Observe
that we have $\theta_\lambda = g_\lambda \circ \theta$, where $g_\lambda : L \to L$ is the isomorphism given by the formula $g_\lambda(t) = \lambda \cdot t$. Therefore the $L$-homomorphism $t - 1 : H^*(Y, \theta_\lambda) \to H^*(Y, \theta_\lambda)$ is isomorphic to the $L$-homomorphism $\lambda^{-1}t - 1 : H^*(Y, \theta_\lambda) \to H^*(Y, \theta_\lambda)$, and we have $\text{Nil}(H^{k+1}(Y, \theta_\lambda), t - 1) = \text{Nil}(H^{k+1}(Y, \theta), t - \lambda)$. The torsion submodule of $H^{k+1}(Y, \theta)$ is isomorphic to $\text{Ext}(H_k(Y, \theta), L)$ which is in turn isomorphic to the torsion submodule $\mathcal{T}_k$ of $H_k(Y, \theta)$. A theorem of P. Kirk and C. Livingston ([9], Th. 2.1) says that we have an isomorphism

$$H_*(Y, \theta) \approx H_*(Y, \theta_0),$$

Therefore the module $\mathcal{T}_k$ is isomorphic to the torsion submodule $\mathcal{T}_k$ of $H_*(Y, \theta_0)$, so, finally, $M_k(\alpha, \theta_\lambda) = \text{Nil}(\mathcal{T}_k, t - \lambda)$ and the proof of Theorem C is complete.

6.5. Theorem D. It follows readily from Theorem C; the proof is similar to the argument of Subsection 6.2.

6.6. Acknowledgments. I am grateful to Professor F. Bogomolov for valuable discussions and support.

References

[1] G. Bazzoni, M. Fernández, V. Muñoz, Non-formal co-symplectic manifolds, Trans. Amer. Math. Soc. 367, (2015), 4459 – 4481.
[2] N. Budur, Y. Liu, B. Wang, The monodromy theorem for compact Kähler manifolds and smooth quasi-projective varieties, arXiv:1609.06478.
[3] K. Corlette, Flat G-bundles with canonical metrics. J. Differential Geom. 28 (1988), no. 3, 361–382.
[4] P. Deligne, Ph. Griffiths, J. Morgan, D. Sullivan, Real homotopy theory of Kähler manifolds, Invent. Math. 29 (1975), 245 – 274.
[5] M. Farber, Exactness of Novikov inequalities, Functionalnyi Analiz i ego Prilozheniya 19, 1985 p. 49 – 59.
[6] M. Farber, Topology of closed 1-forms and their critical points, Topology. 40 (2001), p. 235 – 258.
[7] M. Fernández, A. Gray, J. Morgan, Compact symplectic manifolds with free circle actions and Massey products, Michigan Math. J. 36, 1990, 271 – 283.
[8] H. Kasuya, Minimal models, formality and hard Lefschetz properties of solvmanifolds with local systems. Journal of Differential Geometry. 93, (2013), 269 – 297.
[9] P. Kirk, C. Livingston, Twisted Alexander invariants, Reidemeister torsion, and Casson-Gordon invariants, Topology 38 (1999), 635 – 661.
[10] T. Kohno, A. Pajitnov, Novikov homology, jump loci and Massey products. Cent. Eur. J. Math. 12 (2014), 1285 – 1304.
[11] S.P. Novikov. Bloch homology, critical points of functions and closed 1-forms Soviet Math. Dokl. 287 (1986), 1321 – 1324.
[12] A. Pajitnov. Proof of a conjecture of Novikov on homology with local coefficients over a field of finite characteristic. Soviet Math. Dokl. 37 (1988), p. 824 – 828.
[13] A. Pajitnov. Novikov homology, twisted Alexander polynomials, and Thurston cones, Algebra i analiz, 18:5 (2006), 173 – 209.
[14] A. Pajitnov. Massey products in mapping tori. Eur. Journ. Math, published online November 16, 2016.
[15] S. Papadima, A. Suciu, Algebraic monodromy and obstructions to formality, Forum Math. 22 (2010), 973 – 983.
[16] S. Papadima, A. Suciu, Geometric and algebraic aspects of 1-formality, Bull. Math. Soc. Sci. Math. Roumaine 52 (100) (2009), 355 – 375.
[17] C. Simpson. Higgs bundles and local systems. Publ. I.H.E.S. 75 (1992), 5 – 95.
[18] D. Sullivan, Infinitesimal computations in topology, Publ. I.H.E.S. 47 (1977), p. 269 – 331.

Laboratoire Mathématiques Jean Leray UMR 6629, Université de Nantes, Faculté des Sciences, 2, rue de la Houssinière, 44072, Nantes, Cedex
E-mail address: andrei.pajitnov@univ-nantes.fr