Uniform ergodicities and perturbation bounds
of Markov chains on ordered Banach spaces

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Abstract. In this paper, we consider uniformly mean ergodic and uniformly asymptotical stable Markov operators on ordered Banach spaces. In terms of the ergodicity coefficient, we show the equivalence of uniform and weak mean ergodicities of Markov operators. This result allowed us to establish a category theorem for uniformly mean ergodic Markov operators. Furthermore, using properties of the ergodicity coefficient, we develop the perturbation theory for uniformly asymptotical stable Markov chains in the abstract scheme.

1. Introduction

It is well-known [18] that the transition probabilities $P(x, A)$ (defined on a measurable space $(E, F)$) of Markov processes naturally define a linear operator by $Tf(x) = \int f(y)P(x, dy)$, which is called Markov operator and acts on the associated $L^1$-spaces. Quantum analogous of Markov processes naturally appear in various directions of quantum physics such as quantum statistical physics and quantum optics etc [1]. In these studies it is important to elaborate with associated quantum dynamical systems (time evolutions of the system) [20], which eventually converge to a set of stationary states. From the mathematical point of view, ergodic properties of quantum Markov operators were investigated by many authors. We refer a reader to [1, 7, 8] for further details relative to some differences between the classical and the quantum situations.

On the other hand, it is known [11, 14] that Dobrushin’s ergodicity coefficient is one of the effective tools in the investigations of limiting behavior of Markov processes. In [16, 17] we have defined such an ergodicity coefficient $\delta(T)$ of a positive mapping $T$ defined on ordered Banach space with a base, and studied its properties. In this paper, we consider uniformly mean ergodic and uniformly asymptotical stable Markov operators on ordered Banach spaces. In terms of the ergodicity coefficient, we prove the equivalence of uniform and weak mean ergodicities of Markov operators. This result allowed us to establish a category theorem for uniformly mean ergodic Markov operators. Furthermore, following some ideas of [11, 14] and using properties of $\delta(T)$, we develop the perturbation theory for uniformly asymptotical stable Markov chains.
in the abstract scheme. Our results open new perspectives in the perturbation theory for quantum Markov processes in more general von Neumann algebras setting, which have significant applications in quantum theory [20].

2. Preliminaries

In this section we firstly recall some necessary definitions and fact about ordered Banach spaces. Moreover Dobrushin’s ergodicity coefficient for Markov operator and Markov operator nets is also investigated.

Let $X$ be an ordered vector space with a cone $X_+ = \{ x \in X : x \geq 0 \}$. A subset $\mathcal{K}$ is called a base for $X$, if one has $\mathcal{K} = \{ x \in X_+ : f(x) = 1 \}$ for some strictly positive (i.e. $f(x) > 0$ for $x > 0$) linear functional $f$ on $X$. An ordered vector space $X$ with generating cone $X_+$ (i.e. $X = X_+ - X_+$) and a fixed base $\mathcal{K}$, defined by a functional $f$, is called an ordered vector space with a base [2]. In what follows, we denote it as $(X, X_+, \mathcal{K}, f)$. Let $U$ be the convex hull of the set $\mathcal{K} \cup (-\mathcal{K})$, and let

$$\|x\|_{\mathcal{K}} = \inf \{ \lambda \in \mathbb{R}_+ : x \in \lambda U \}.$$  

Then one can see that $\| \cdot \|_{\mathcal{K}}$ is a seminorm on $X$. Moreover, one has $\mathcal{K} = \{ x \in X_+ : \|x\|_{\mathcal{K}} = 1 \}$, $f(x) = \|x\|_{\mathcal{K}}$ for $x \in X_+$. If the set $U$ is linearly bounded (i.e. for any line $\ell$ the intersection $\ell \cap U$ is a bounded set), then $\| \cdot \|_{\mathcal{K}}$ is a norm, and in this case $(X, X_+, \mathcal{K}, f)$ is called an ordered normed space with a base. When $X$ is complete with respect to the norm $\| \cdot \|_{\mathcal{K}}$ and the cone $X_+$ is closed, then $(X, X_+, \mathcal{K}, f)$ is called an ordered Banach space with a base (OBSB). In the sequel, for the sake of simplicity instead of $\| \cdot \|_{\mathcal{K}}$ we will use usual notation $\| \cdot \|$.

Let us provide some examples of OBSB.

1. Let $M$ be a von Neumann algebra. Let $M_{h,s}$ be the Hermitian part of the predual space $M_s$ of $M$. As a base $\mathcal{K}$ we define the set of normal states of $M$. Then $(M_{h,s}, M_{h,s+}, \mathcal{K}, \mathbb{1})$ is a OBSB, where $M_{h,s+}$ is the set of all positive functionals taken from $M_{h,s}$, and $\mathbb{1}$ is the unit in $M$.

2. Let $X = \ell_p$, $1 < p < \infty$. Define

$$X_+ = \left\{ x = (x_0, x_1, \ldots, x_n, \ldots) \in \ell_p : x_0 \geq \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} \right\}$$

and $f_0(x) = x_0$. Then $f_0$ is a strictly positive linear functional. In this case, we define $\mathcal{K} = \{ x \in X_+ : f_0(x) = 1 \}$. Then one can see that $(X, X_+, \mathcal{K}, f_0)$ is a OBSB. Note that the norm $\| \cdot \|_{\mathcal{K}}$ is equivalent to the usual $\ell_p$-norm.

Let $(X, X_+, \mathcal{K}, f)$ be an OBSB. It is well-known (see [2, Proposition II.1.14]) that every element $x$ of OBSB admits a decomposition $x = y - z$, where $y, z \geq 0$ and $\|x\| = \|y\| + \|z\|$.

Let $(X, X_+, \mathcal{K}, f)$ be an OBSB. A linear operator $T : X \rightarrow X$ is called positive, if $Tx \geq 0$ whenever $x \geq 0$. A positive linear operator $T : X \rightarrow X$ is called Markov, if $T(\mathcal{K}) \subset \mathcal{K}$. It is clear that $\|T\| = 1$, and its adjoint mapping $T^* : X^* \rightarrow X^*$ acts in ordered Banach space $X^*$ with unit $f$, and moreover, one has $T^*f = f$. Now for each $y \in X$ we define a linear operator $T_y : X \rightarrow X$ by $T_y(x) = f(x)y$.

**Definition 2.1.** A Markov operator $T : X \rightarrow X$ is called
(i) uniformly asymptotically stable if there exist an element \( y_0 \in \mathcal{K} \) such that
\[
\lim_{n \to \infty} \| T^n - T_{y_0} \| = 0;
\]

(ii) weakly ergodic if one has
\[
\lim_{n \to \infty} \sup_{x, y \in \mathcal{K}} \| T^n x - T^n y \| = 0;
\]

Let \((X, X_+, \mathcal{K}, f)\) be an OBSB and \(T : X \to X\) be a linear bounded operator. Letting
\[
N = \{ x \in X : f(x) = 0 \},
\]
we define
\[
\delta(T) = \sup_{x \in N, x \neq 0} \frac{\|Tx\|}{\|x\|}.
\]

The quantity \(\delta(T)\) is called the Dobrushin’s ergodicity coefficient of \(T\) (see [16]).

**Remark 2.2.** We note that if \(X^*\) is a commutative algebra, the notion of the Dobrushin ergodicity coefficient was studied in [4]. In a non-commutative setting, i.e. when \(X^*\) is a von Neumann algebra, such a notion was introduced in [15]. We should stress that such a coefficient has been independently defined in [9]. Furthermore, for particular cases, i.e. in a non-commutative setting, such a coefficient explicitly has been calculated for quantum channels (i.e. completely positive maps).

The next result establishes several properties of the Dobrushin’s ergodicity coefficient.

**Theorem 2.3.** [16] Let \((X, X_+, \mathcal{K}, f)\) be an OBSB and \(T, S : X \to X\) be Markov operators. The following assertions hold:

(i) \(0 \leq \delta(T) \leq 1\);

(ii) \(|\delta(T) - \delta(S)| \leq \delta(T - S) \leq \|T - S\|\);

(iii) \(\delta(TS) \leq \delta(T)\delta(S)\);

(iv) if \(H : X \to X\) is a linear bounded operator such that \(H^*(f) = 0\), then \(\|TH\| \leq \delta(T)\|H\|\);

(v) one has
\[
\delta(T) = \frac{1}{2} \sup_{u, v \in \mathcal{K}} \|Tu - Tv\|;
\]

(vi) if \(\delta(T) = 0\), then there exists \(y_0 \in X_+\) such that \(T = T_{y_0}\).

**Remark 2.4.** Note that taking into account Theorem 2.3(v) we obtain that the weak ergodicity is equivalent to the condition \(\delta(T^n) \to 0\) as \(n \to \infty\).

The following theorem gives us the conditions that are equivalent to uniform asymptotical stability.

**Theorem 2.5.** [16] Let \((X, X_+, \mathcal{K}, f)\) be an OBSB and \(T : X \to X\) be a Markov operator. The following assertions are equivalent:

(i) \(T\) is weakly ergodic;

(ii) there exists \(\rho \in [0, 1)\) and \(n_0 \in \mathbb{N}\) such that \(\delta(T^{n_0}) \leq \rho\);

(iii) \(T\) is uniformly asymptotically stable. Moreover, there are positive constants \(C, \alpha, n_0 \in \mathbb{N}\) and \(x_0 \in \mathcal{K}\) such that
\[
\|T^n - T_{x_0}\| \leq Ce^{-\alpha n}, \ \forall n \geq n_0.
\]
3. Uniform LR-nets ergodicity

The special Markov operator net was firstly defined by Lotz [13] and Räbiger [19]. The terminology modified the recent paper [5] and they prefer to call Lotz-Räbiger net, LR-net. A family \( \Theta = (T_\lambda)_{\lambda \in \Lambda} \subseteq \mathcal{L}(X) \) indexed by a directed set \( \Lambda = (\Lambda, \prec) \) is called an operator net. A vector \( x \in X \) is called a fixed vector for the net \( \Theta \) if \( T_\lambda x = x \) for every \( \lambda \in \Lambda \). We denote by \( \text{Fix}(\Theta) \) the set of all fixed vectors of \( \Theta \). It is easy to see that \( \text{Fix}(\Theta) \) is a closed subspace of \( X \).

**Definition 3.1.** A net \( \Theta = (T_\lambda)_{\lambda \in \Lambda} \) is called a Lotz-Räbiger (LR)-net if

1. \( \Theta \) is uniformly bounded;
2. \( \lim_{\lambda \to \infty} \|T_\lambda(T_\mu - I)x\| = 0 \) for every \( \mu \in \Lambda \) and for every \( x \in X \);
3. \( \lim_{\lambda \to \infty} \|T_\lambda - T_\mu\| = 0 \) for every \( \mu \in \Lambda \) and for every \( x \in X \).

If the limit conditions hold in the uniform operator topology, then the operator net is called uniform LR-net, or ULR-net.

Elementary example of LR-net is the sequence \( \{A_n(T)\}_{n=1}^{\infty} \) of Cesàro averages
\[
A_n(T) := \frac{1}{n} \sum_{k=0}^{n-1} T^k
\]
of a power bounded operator \( T \in \mathcal{L}(X) \) with \( n^{-1}T^n \to 0 \) strongly.

In this section, we investigate the Dobrushin’s ergodicity coefficient to study a behavior of a Markov ULR-nets defined on OBSB \( X \).

**Definition 3.2.** Let \( \Theta = (T_\lambda)_{\lambda \in \Lambda} \) be a Markov ULR-net on OBSB \( X \). It is called

1. weakly ergodic if
   \[
   \lim_{\lambda \to \infty} \sup_{x,y \in \mathcal{K}} \|T_\lambda x - T_\lambda y\| = 0.
   \]
2. uniformly ergodic if there exist an element \( y_0 \in \mathcal{K} \) such that
   \[
   \lim_{\lambda \to \infty} \|T_\lambda - T_{y_0}\| = 0;
   \]

To formulate our main result of this section, we need a technical lemma.

**Lemma 3.3.** Let \( \Theta = (T_\lambda)_{\lambda \in \Lambda} \) be a Markov ULR-net on OBSB \( X \). The following statements hold:

1. There exist \( \lambda_0 \in \Lambda \) and \( 0 \leq \rho < 1 \) such that \( \delta(T_{\lambda_0}) < \rho \);
2. one has \( \lim_{\lambda \to \infty} \delta(T_{\lambda}) = 0 \);
3. weakly ergodic.

The following theorem shows that for strongly convergence of an operator net, it suffices to have one operator in the net which is uniformly ergodic to \( T_{y_0} \), where \( y_0 \in \mathcal{K} \) and the orbit at \( y_0 \) is convergent.

**Theorem 3.4.** Let \( \Theta = (T_\lambda)_{\lambda \in \Lambda} \) be a Markov ULR-net on OBSB \( X \). If there exist \( \lambda_0 \in \Lambda \) and \( y_0 \in \mathcal{K} \) such that \( T_{\lambda_0} \) is uniformly asymptotically stable to \( T_{y_0} \), and \( (T_{\lambda}y_0)_{\lambda \in \Lambda} \) is convergent in norm, then \( \Theta \) is strongly convergent.

Next theorem gives relations between uniform and weak ergodicities in terms of the Dobrushin’s ergodicity coefficient. This is an analogous of Theorem 2.5 for uniform ULR-nets.
Theorem 3.5. Let \((X, X_+, \mathcal{K}, f)\) be an OBSB and \(\Theta = (T_\lambda)_{\lambda \in \Lambda}\) be a Markov ULR-net on OBSB \(X\). Assume that there exists \(y_0 \in \mathcal{K}\) such that \((T_\lambda y_0)\) is convergent in norm. Then the following assumptions are equivalent:

(i) \(\Theta = (T_\lambda)_{\lambda \in \Lambda}\) is weakly ergodic;
(ii) There exists \(\rho \in [0, 1)\) and \(\lambda_0 \in \Lambda\) such that \(\delta(T_{\lambda_0}) \leq \rho\);
(iii) \(\Theta = (T_\lambda)_{\lambda \in \Lambda}\) is uniformly ergodic.

By \(\mathfrak{T}\) we denote the set of all \(\Theta = (T_\lambda)_{\lambda \in \Lambda}\) Markov ULR-nets on OBSB \(X\) which have non trivial fixed points belonging to \(\mathcal{K}\).

Corollary 3.6. Let \((X, X_+, \mathcal{K}, f)\) be an OBSB and \(\Theta = (T_\lambda)_{\lambda \in \Lambda} \in \mathfrak{T}\). Then the following assumptions are equivalent:

(i) \(\Theta = (T_\lambda)_{\lambda \in \Lambda}\) is weakly ergodic;
(ii) There exists \(\rho \in [0, 1)\) and \(\lambda_0 \in \Lambda\) such that \(\delta(T_{\lambda_0}) \leq \rho\);
(iii) \(\Theta = (T_\lambda)_{\lambda \in \Lambda}\) is uniformly ergodic.

As mentioned above, the most well-known example of ULR-net is the Cesaro averages of Markov operator \(T\). In this part, we restrict ourselves to ULR-net

\[
A_n(T) = \frac{1}{n} \sum_{k=0}^{n-1} T^k, \quad n \in \mathbb{N}.
\]

Definition 3.7. A Markov operator \(T : X \to X\) is called

(i) uniformly mean ergodic if there exist an element \(y_0 \in \mathcal{K}\) such that

\[
\lim_{n \to \infty} \|A_n(T) - T y_0\| = 0;
\]

(ii) weakly mean ergodic if one has

\[
\lim_{n \to \infty} \sup_{x, y \in \mathcal{K}} \|A_n(T)x - A_n(T)y\| = 0.
\]

Remark 3.8. We notice that uniform asymptotic stability implies uniform mean ergodicity. Moreover, if \(T\) is uniform mean ergodic, then \(y_0\), corresponding to \(T y_0\), is a fixed point of \(T\). Indeed, taking limit in the equality

\[
\left(1 + \frac{1}{n}\right) A_{n+1}(T) - \frac{1}{n} I = T A_n(T)
\]

we find \(T T y_0 = T y_0\), which yields \(T y_0 = y_0\). We stress that every uniformly mean ergodic Markov operator has a unique fixed point.

By \(\mathfrak{U}\) we denote the set of all Markov operators \(T\) from \(X\) to \(X\) such that it has an eigenvalue \(1\) and the corresponding eigenvector \(f\) belongs to \(\mathcal{K}\). From Theorem 3.5 one gets

Theorem 3.9. Let \((X, X_+, \mathcal{K}, f)\) be an OBSB and \(T \in \mathfrak{U}\). Then the following statements are equivalent:

(i) \(T\) is weakly mean ergodic;
(ii) There exist \(\rho \in [0, 1)\) and \(n_0 \in \mathbb{N}\) such that \(\delta(A_{n_0}(T)) \leq \rho\).
(iii) $T$ is uniformly mean ergodic.

By $\mathcal{U}_{ume}$ we denote the set of all uniformly mean ergodic Markov operators belonging to $\mathcal{U}$.

**Theorem 3.10.** Let $(X, X_+, \mathcal{K}, f)$ be an OBSB. Then the set $\mathcal{U}_{ume}$ is a norm dense and open subset of $\mathcal{U}$.

**Remark 3.11.** We point out that the question on the geometric structure of the set of uniformly ergodic operators its size and category was initiated in [10]. The proved theorem gives some information about the set of uniformly mean ergodic operators.

**Corollary 3.12.** Let $(X, X_+, \mathcal{K}, f)$ be an OBSB and $T \in \mathcal{U}$ with fixed point $x_0 \in \mathcal{K}$. If $\delta(A_m(T)) < 1$ for some $m \in \mathbb{N}$, then every Markov operator $S$ satisfying

$$\|S - T\| < \frac{2(1 - \delta(A_m(T))}{m + 1}$$

is uniformly mean ergodic and has a unique fixed point $z_0 \in \mathcal{K}$ such that

$$\|x_0 - z_0\| \leq \frac{\|A_m(S) - A_m(T)\|}{1 - \delta(A_m(T)) - \|A_m(S) - A_m(T)\|}.$$  

(3.1)

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**4. Perturbation Bounds of Markov operators**

In this section, we prove perturbation bounds in terms of $C$ and $e^\alpha$ under the condition $\|T^n - T_{z_0}\| \leq Ce^{-\alpha n}$. Moreover, we also give several bounds in terms of the Dobrushin ergodicity coefficient.

**Theorem 4.1.** Let $(X, X_+, \mathcal{K}, f)$ be an ordered Banach space with a base, and $S, T$ be Markov operators on $X$. If there exists $\rho \in [0, 1)$ and $n_0 \in \mathbb{N}$ such that $\delta(T^{n_0}) \leq \rho$, then one has

$$\|T^n x - S^n z\| \leq \begin{cases} \|x - z\| + n \|T - S\|, & \forall n \leq \tilde{n}, \\ Ce^{-\alpha n} \|x - z\| + \left(\tilde{n} + C\frac{e^{-\alpha \tilde{n}}}{1 - e^{-\alpha}}\right)\|T - S\|, & \forall n > \tilde{n} \end{cases}$$

where $\tilde{n} := \log\left[\log(1/C) \right] / e^{-\alpha}$, $C \in \mathbb{R}_+$, $\alpha \in \mathbb{R}_+$, $x, z \in \mathcal{K}$.

**Corollary 4.2.** Let $(X, X_+, \mathcal{K}, f)$ be an ordered Banach space with bases and $S, T$ be Markov operators on $X$. If there exists $\rho \in [0, 1)$ and $n_0 \in \mathbb{N}$ such that $\delta(T^{n_0}) \leq \rho$, then for every $x, y \in \mathcal{K}$ one has

$$\sup_{n \in \mathbb{N}} \|T^n x - S^n z\| \leq \|x - z\| + \left(\tilde{n} + C\frac{e^{-\alpha \tilde{n}}}{1 - e^{-\alpha}}\right)\|T - S\|.$$  

(4.2)

In addition, if $S$ is uniformly asymptotically stable to $S_{z_0}$ then

$$\|T_{z_0} - S_{z_0}\| \leq \left(\tilde{n} + C\frac{e^{-\alpha \tilde{n}}}{1 - e^{-\alpha}}\right)\|T - S\|.$$  

(4.3)

The inequality (4.1) allows us to obtain perturbation bounds in terms of the Dobrushin coefficient of $T$. Namely, we have the following result.
Theorem 4.3. Let \((X, X_+, \mathcal{K}, f)\) be an ordered Banach space with bases and \(S, T\) be Markov operators on \(X\). If \(T\) is uniformly asymptotically stable, then there exists a positive integer \(m\) such that \(\delta(T^m) < 1\) and for every \(x, z \in \mathcal{K}\) one has

\[
\sup_{k \in \mathbb{N}} \left\| T^{km}x - S^{km}z \right\| \leq \delta(T^m) \| x - z \| + \frac{\| T^m - S^m \|}{1 - \delta(T^m)} \quad (4.4)
\]

and

\[
\| T^n x - S^n z \| \leq \begin{cases} 
\| x - z \| + \max_{0 \leq i < m} \| T^i - S^i \|, & n \leq m, \\
\delta(T^m) (\| x - z \| + \max_{0 \leq i < m} \| T^i - S^i \|) + \frac{\| T^m - S^m \|}{1 - \delta(T^m)}, & n \geq m.
\end{cases} \quad (4.5)
\]

If, in addition, \(S\) is uniformly asymptotically stable, then

\[
\| T_{x_0} - S_{z_0} \| \leq \frac{\| T^m - S^m \|}{1 - \delta(T^m)}. \quad (4.6)
\]

The following theorem gives an alternative method of obtaining perturbation bounds in terms of \(\delta(T^m)\).

Theorem 4.4. Let \(\delta(T^m) < 1\) hold for some \(m \in \mathbb{N}\). Then for every \(x, z \in \mathcal{K}\) one has

\[
\| T^n x - S^n z \| \leq \delta(T^m)^{|n/m|} (\| x - z \| + \max_{0 \leq i < m} \| T^i - S^i \|) + \frac{1 - \delta(T^m)^{|n/m|}}{1 - \delta(T^m)} \| T^m - S^m \|, \quad n \in \mathbb{N}. \quad (4.7)
\]

Corollary 4.5. Let the condition of Theorem 4.4 be satisfied. Then for every \(x, z \in \mathcal{K}\) we have

\[
\sup_{n \in \mathbb{N}} \| T^n x - S^n z \| \leq \sup_{n \in \mathbb{N}} \delta(T^m)^{|n/m|} + \frac{m \| T - S \|}{1 - \delta(T^m)} \quad (4.8)
\]

Theorem 4.6. If \(\delta(T^m) < 1\) for some \(m \in \mathbb{N}\), then every Markov operator \(S\) satisfying \(\| S^m - T^m \| < 1 - \delta(T^m)\) is uniformly asymptotically stable and has a unique fixed point \(z_0 \in \mathcal{K}\) such that

\[
\| x_0 - z_0 \| \leq \frac{\| S^m - T^m \|}{1 - \delta(T^m) - \| S^m - T^m \|} \quad (4.9)
\]
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