ON CO-ORDINATED QUASI-CONVEX FUNCTIONS

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Abstract. In this paper, we give some definitions on quasi-convex functions and we prove inequalities contain J-quasi-convex and W-quasi-convex functions. We give also some inclusions.

1. INTRODUCTION

Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a convex function on the interval of \( I \) of real numbers and \( a, b \in I \) with \( a < b \). The following double inequality

\[
f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_{a}^{b} f(x)dx \leq \frac{f(a) + f(b)}{2}
\]

is well-known in the literature as Hadamard’s inequality. We recall some definitions;

In [25], Pecaric et al. defined quasi-convex functions as following

Definition 1. A function \( f : [a, b] \to \mathbb{R} \) is said quasi-convex on \([a, b]\) if

\[
f(\lambda x + (1-\lambda)y) \leq \max \{f(x), f(y)\}, \quad (QC)
\]

holds for all \( x, y \in [a, b] \) and \( \lambda \in [0, 1] \).

Clearly, any convex function is quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex.

Definition 2. (See [6], [12]) We say that \( f : I \to \mathbb{R} \) is a Wright-convex function or that \( f \) belongs to the class \( W(I) \), if for all \( x, y + \delta \in I \) with \( x < y \) and \( \delta > 0 \), we have

\[
f(x + \delta) + f(y) \leq f(y + \delta) + f(x)
\]

Definition 3. (See [6]) For \( I \subseteq \mathbb{R} \), the mapping \( f : I \to \mathbb{R} \) is wright-quasi-convex function if, for all \( x, y \in I \) and \( t \in [0, 1] \), one has the inequality

\[
\frac{1}{2} [f(tx + (1-t)y) + f((1-t)x + ty)] \leq \max \{ f(x), f(y) \}, \quad (WQC)
\]

or equivalently

\[
\frac{1}{2} [f(y) + f(x + \delta)] \leq \max \{ f(x), f(y + \delta) \}
\]

for every \( x, y + \delta \in I \), \( x < y \) and \( \delta > 0 \).
Definition 4. (See [6]) The mapping \( f : I \to \mathbb{R} \) is Jensen- or J-quasi-convex if
\[
f \left( \frac{x + y}{2} \right) \leq \max \{ f(x), f(y) \}, \quad (JQC)
\]
for all \( x, y \in I \).

Note that the class \( JQC(I) \) of J-quasi-convex functions on \( I \) contains the class \( J(I) \) of J-convex functions on \( I \), that is, functions satisfying the condition
\[
f \left( \frac{x + y}{2} \right) \leq \frac{f(x) + f(y)}{2}, \quad (J)
\]
for all \( x, y \in I \).

In [6], Dragomir and Pearce proved the following theorems containing J-quasi-convex and Wright-quasi-convex functions.

Theorem 1. Suppose \( a, b \in I \subseteq \mathbb{R} \) and \( a < b \). If \( f \in JQC(I) \cap L_1[a, b] \), then
\[
f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx + I(a, b)
\]
where
\[
I(a, b) = \frac{1}{2} \int_0^1 |f(ta + (1-t)b) - f((1-t)a + tb)| \, dt.
\]

Theorem 2. Let \( f : I \to \mathbb{R} \) be a Wright-quasi-convex map on \( I \) and suppose \( a, b \in I \subseteq \mathbb{R} \) with \( a < b \) and \( f \in L_1[a, b] \), one has the inequality
\[
\frac{1}{b - a} \int_a^b f(x) \, dx \leq \max \{ f(a), f(b) \}.
\]

In [6], Dragomir and Pearce also gave the following theorems involving some inclusions.

Theorem 3. Let \( WQC(I) \) denote the class of Wright-quasi-convex functions on \( I \subseteq \mathbb{R} \), then
\[
QC(I) \subset WQC(I) \subset JQC(I).
\]
Both inclusions are proper.

Theorem 4. We have the inclusions
\[
W(I) \subset WQC(I), \quad C(I) \subset QC(I), \quad J(I) \subset JQC(I).
\]
Each inclusion is proper.

For recent results related to quasi-convex functions see the papers [1]-[11] and books [23], [24]. In [19], Dragomir defined co-ordinated convex functions and proved following inequalities.

Let us consider the bidimensional interval \( \Delta = [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). A function \( f : \Delta \to \mathbb{R} \) will be called convex on the co-ordinates if the partial mappings
\[
f_y : [a, b] \to \mathbb{R}, \quad f_y(u) = f(u, y)
\]
and
\[
f_x : [c, d] \to \mathbb{R}, \quad f_x(v) = f(x, v)
\]
are convex where defined for all \( y \in [c, d] \) and \( x \in [a, b] \).
Recall that the mapping \( f : \Delta \to \mathbb{R} \) is convex on \( \Delta \), if the following inequality:

\[
(1.6) \quad f(\lambda x + (1 - \lambda) z, \lambda y + (1 - \lambda) w) \leq \lambda f(x, y) + (1 - \lambda) f(z, w)
\]

holds for all \((x, y), (z, w) \in \Delta \) and \( \lambda \in [0, 1] \).

**Theorem 5.** (see [19], Theorem 1) Suppose that \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) is convex on the co-ordinates on \( \Delta \). Then one has the inequalities:

\[
(1.7) \quad f\left(\frac{a + b}{2}, \frac{c + d}{2}\right) \leq \frac{1}{2} \left[ \frac{1}{b - a} \int_{a}^{b} f\left(x, \frac{c + d}{2}\right) \, dx + \frac{1}{d - c} \int_{c}^{d} f\left(\frac{a + b}{2}, y\right) \, dy \right]
\]

\[
\leq \frac{1}{(b - a)(d - c)} \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx
\]

\[
\leq \frac{1}{4} \left[ \frac{1}{b - a} \int_{a}^{b} f(x, c) \, dx + \frac{1}{b - a} \int_{a}^{b} f(x, d) \, dx
\]

\[
+ \frac{1}{d - c} \int_{c}^{d} f(a, y) \, dy + \frac{1}{d - c} \int_{c}^{d} f(b, y) \, dy \right]
\]

\[
\leq \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{4}
\]

The above inequalities are sharp.

Similar results can be found in [13]-[22].

This paper is arranged as follows. Firstly, we will give some definitions on quasi-convex functions and lemmas belong to this definitions. Secondly, we will prove several inequalities contain co-ordinated quasi-convex functions. Also, we will discuss the inclusions a connection with some different classes of co-ordinated convex functions.

### 2. DEFINITIONS AND MAIN RESULTS

We will start the following definitions and lemmas;

**Definition 5.** A function \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) is said quasi-convex function on the co-ordinates on \( \Delta \) if the following inequality

\[
f(\lambda x + (1 - \lambda) z, \lambda y + (1 - \lambda) w) \leq \max\{f(x, y), f(z, w)\}
\]

holds for all \((x, y), (z, w) \in \Delta \) and \( \lambda \in [0, 1] \)

\( f : \Delta \to \mathbb{R} \) will be called co-ordinated quasi-convex on the co-ordinates if the partial mappings

\[
f_{y} : [a, b] \to \mathbb{R}, \quad f_{y}(u) = f(u, y)
\]

and

\[
f_{x} : [c, d] \to \mathbb{R}, \quad f_{x}(v) = f(x, v)
\]

are convex where defined for all \( y \in [c, d] \) and \( x \in [a, b] \). We denote by \( QC(\Delta) \) the classes of quasi-convex functions on the co-ordinates on \( \Delta \). The following lemma holds.

**Lemma 1.** Every quasi-convex mapping \( f : \Delta \to \mathbb{R} \) is quasi-convex on the co-ordinates.
Proof. Suppose that \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) is quasi-convex on \( \Delta \). Then the partial mappings
\[
f_y : [a, b] \to \mathbb{R}, \quad f_y (u) = f (u, y), \quad y \in [c, d]
\]
and
\[
f_x : [c, d] \to \mathbb{R}, \quad f_x (v) = f (x, v), \quad x \in [a, b]
\]
are convex on \( \Delta \). For \( \lambda \in [0, 1] \) and \( v_1, v_2 \in [c, d] \), one has
\[
f_x (\lambda v_1 + (1 - \lambda) v_2) = f (x, \lambda v_1 + (1 - \lambda) v_2)
\]
\[
= f (\lambda x + (1 - \lambda) x, \lambda v_1 + (1 - \lambda) v_2)
\]
\[
\leq \max \{ f (x, v_1), f (x, v_2) \}
\]
which completes the proof of quasi-convexity of \( f_x \) on \([c, d]\). Therefore \( f_y : [a, b] \to \mathbb{R}, \quad f_y (u) = f (u, y) \) is also quasi-convex on \([a, b]\) for all \( y \in [c, d] \), goes likewise and we shall omit the details. \( \square \)

**Definition 6.** A function \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) is said \( J \)-convex function on the co-ordinates on \( \Delta \) if the following inequality
\[
f \left( \frac{x + z}{2}, \frac{y + w}{2} \right) \leq \frac{f (x, y) + f (z, w)}{2}
\]
holds for all \((x, y), (z, w) \in \Delta\). We denote by \( \mathcal{J}(\Delta) \) the classes of \( J \)-convex functions on the co-ordinates on \( \Delta \).

**Lemma 2.** Every \( J \)-convex mapping defined \( f : \Delta \to \mathbb{R} \) is \( J \)-convex on the co-ordinates.

Proof. By the partial mappings, we can write for \( v_1, v_2 \in [c, d] \),
\[
f_x \left( \frac{v_1 + v_2}{2} \right) = f \left( x, \frac{v_1 + v_2}{2} \right)
\]
\[
= f \left( \frac{x + z}{2}, \frac{v_1 + v_2}{2} \right)
\]
\[
\leq \frac{f (x, v_1) + f (x, v_2)}{2}
\]
\[
= f_x (v_1) + f_x (v_2)
\]
which completes the proof of \( J \)-convexity of \( f_x \) on \([c, d] \). Similarly, we can prove \( J \)-convexity of \( f_y \) on \([a, b]\). \( \square \)

**Definition 7.** A function \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) is said \( J \)-quasi-convex function on the co-ordinates on \( \Delta \) if the following inequality
\[
f \left( \frac{x + z}{2}, \frac{y + w}{2} \right) \leq \max \{ f (x, y), f (z, w) \}
\]
holds for all \((x, y), (z, w) \in \Delta\). We denote by \( \mathcal{JQC}(\Delta) \) the classes of \( J \)-quasi-convex functions on the co-ordinates on \( \Delta \).

**Lemma 3.** Every \( J \)-quasi-convex mapping defined \( f : \Delta \to \mathbb{R} \) is \( J \)-quasi-convex on the co-ordinates.
Proof. By a similar way to proof of Lemma 1, we can write for \( v_1, v_2 \in [c, d] \),

\[
\begin{align*}
  f_x \left( \frac{v_1 + v_2}{2} \right) & = f \left( x, \frac{v_1 + v_2}{2} \right) \\
  & = f \left( \frac{x + x}{2}, \frac{v_1 + v_2}{2} \right) \\
  & \leq \max \{ f(x, v_1), f(x, v_2) \} \\
  & = \max \{ f_x(v_1), f_x(v_2) \}
\end{align*}
\]

which completes the proof of J-quasi-convexity of \( f_x \) on \([c, d]\). We can also prove J-quasi-convexity of \( f_y \) on \([a, b]\). □

**Definition 8.** A function \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) is said Wright-convex function on the co-ordinates on \( \Delta \) if the following inequality

\[
f((1 - t)a + tb, (1 - s)c + sd) + f(ta + (1 - t)b, sc + (1 - s)d) \leq f(a, c) + f(b, d)
\]

holds for all \((a, c), (b, d) \in \Delta \) and \( t, s \in [0, 1] \). We denote by \( W(\Delta) \) the classes of Wright-convex functions on the co-ordinates on \( \Delta \).

**Lemma 4.** Every Wright-convex mapping defined \( f : \Delta \to \mathbb{R} \) is Wright-convex on the co-ordinates.

Proof. Suppose that \( f : \Delta \to \mathbb{R} \) is Wright-convex on \( \Delta \). Then by partial mapping, for \( v_1, v_2 \in [c, d], x \in [a, b], \)

\[
\begin{align*}
  f_x((1 - t)v_1 + tv_2) + f_x(tv_1 + (1 - t)v_2) & = f(x, (1 - t)v_1 + tv_2) + f(x, tv_1 + (1 - t)v_2) \\
  & = f((1 - t)x + tx, (1 - t)v_1 + tv_2) + f((1 - t)x, tv_1 + (1 - t)v_2) \\
  & \leq f(x, v_1) + f(x, v_2) \\
  & = f_x(v_1) + f_x(v_2)
\end{align*}
\]

which shows that \( f_x \) is Wright-convex on \([c, d]\). Similarly one can see that \( f_y \) is Wright-convex on \([a, b]\). □

**Definition 9.** A function \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) is said Wright-quasi-convex function on the co-ordinates on \( \Delta \) if the following inequality

\[
\frac{1}{2} \left[ f(tx + (1 - t)z, ty + (1 - t)w) + f((1 - t)x + tz, (1 - t)y + tw) \right] \leq \max \{ f(x, y), f(z, w) \}
\]

holds for all \((x, y), (z, w) \in \Delta \) and \( t \in [0, 1] \). We denote by \( WQC(\Delta) \) the classes of Wright-quasi-convex functions on the co-ordinates on \( \Delta \).

**Lemma 5.** Every Wright-quasi-convex mapping defined \( f : \Delta \to \mathbb{R} \) is Wright-quasi-convex on the co-ordinates.
Theorem 6. Suppose that $f : \Delta \to \mathbb{R}$ is Wright-quasi-convex on $\Delta$. Then by partial mapping, for $v_1, v_2 \in [c, d],$

\[
\frac{1}{2} [f_x (tv_1 + (1-t) v_2) + f_x ((1-t) v_1 + tv_2)]
\]
\[
= \frac{1}{2} [f (x, tv_1 + (1-t) v_2) + f (x, (1-t) v_1 + tv_2)]
\]
\[
= \frac{1}{2} [f (tx + (1-t) x, tv_1 + (1-t) v_2) + f ((1-t) x + tx, (1-t) v_1 + tv_2)]
\]
\[
\leq \max \{f (x, v_1), f (x, v_2)\}
\]
\[
= \max \{f_x (v_1), f_x (v_2)\}
\]

which shows that $f_x$ is Wright-quasi-convex on $[c, d]$. Similarly one can see that $f_y$ is Wright-quasi-convex on $[a, b]$. \qed

Theorem 6. Suppose that $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ is $J$-quasi-convex on the co-ordinates on $\Delta$. If $f_x \in L_1 [c, d]$ and $f_y \in L_1 [a, b]$, then we have the inequality:

\[
\frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f \left( x, \frac{c+d}{2} \right) dx + \frac{1}{d-c} \int_c^d f \left( \frac{a+b}{2}, y \right) dy \right]
\]
\[
\leq \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dxdy + H(x, y)
\]

where

\[
H(x, y) = \frac{1}{4(d-c)} \int_c^d \int_0^1 |f (ta + (1-t) b, y) - f ((1-t) a + tb, y)| dt dy
\]
\[
+ \frac{1}{4(b-a)} \int_a^b \int_0^1 |f (x, tc + (1-t) d) - f (x, (1-t) c + td)| dt dx.
\]

Proof. Since $f : \Delta \to \mathbb{R}$ is $J$-quasi-convex on the co-ordinates on $\Delta$. We can write the partial mappings

\[
f_y : [a, b] \to \mathbb{R}, \quad f_y (u) = f (u, y), \quad y \in [c, d]
\]

and

\[
f_x : [c, d] \to \mathbb{R}, \quad f_x (v) = f (x, v), \quad x \in [a, b]
\]

are $J$-quasi-convex on $\Delta$. Then by the inequality (1.2), we have

\[
f_y \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f_y(x) dx + \frac{1}{2} \int_0^1 |f_y (ta + (1-t) b) - f_y((1-t) a + tb)| dt.
\]

That is

\[
f \left( \frac{a+b}{2}, y \right) \leq \frac{1}{b-a} \int_a^b f(x, y) dx + \frac{1}{2} \int_0^1 |f (ta + (1-t) b, y) - f ((1-t) a + tb, y)| dt.
\]
Integrating the resulting inequality with respect to $y$ over $[c, d]$ and dividing both sides of inequality with $(d - c)$, we get

\begin{equation}
\frac{1}{d - c} \int_c^d f \left( \frac{a + b}{2}, y \right) dy 
\leq \frac{1}{(b - a) (d - c)} \int_c^d \int_a^b f(x, y) dxdy 
+ \frac{1}{2(d - c)} \int_c^d \int_0^1 |f \left( ta + (1 - t) b, y \right) - f \left( (1 - t) a + tb, y \right) | dt dy.
\end{equation}

By a similar argument, we have

\begin{equation}
\frac{1}{b - a} \int_a^b f \left( x, \frac{c + d}{2} \right) dx 
\leq \frac{1}{(b - a) (d - c)} \int_a^b \int_c^d f(x, y) dydx 
+ \frac{1}{2(b - a)} \int_a^b \int_0^1 |f \left( x, tc + (1 - t) d \right) - f \left( x, (1 - t) c + td \right) | dt dx.
\end{equation}

Summing (2.2) and (2.3), we get the required result. \hfill \Box

**Theorem 7.** Suppose that $f : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ is Wright-quasi-convex on the co-ordinates on $\Delta$. If $f_x \in L_1 [c, d]$ and $f_y \in L_1 [a, b]$, then we have the inequality;

\begin{equation}
\frac{1}{(b - a) (d - c)} \int_c^d \int_a^b f(x, y) dxdy 
\leq \frac{1}{2} \left[ \max \left\{ \frac{1}{(b - a)} \int_a^b f(x, c) dx, \frac{1}{(b - a)} \int_a^b f(x, d) dx \right\} 
+ \max \left\{ \frac{1}{(d - c)} \int_c^d f(a, y) dy, \frac{1}{(d - c)} \int_c^d f(b, y) dy \right\} \right].
\end{equation}

**Proof.** Since $f : \Delta \to \mathbb{R}$ is Wright-quasi-convex on the co-ordinates on $\Delta$. We can write the partial mappings

\[ f_y : [a, b] \to \mathbb{R}, \quad f_y (u) = f (u, y), \quad y \in [c, d] \]

and

\[ f_x : [c, d] \to \mathbb{R}, \quad f_x (v) = f (x, v), \quad x \in [a, b] \]

are Wright-quasi-convex on $\Delta$. Then by the inequality (1.3), we have

\[ \frac{1}{b - a} \int_a^b f_y (x) dx \leq \max \{ f_y (a), f_y (b) \}. \]

That is

\[ \frac{1}{b - a} \int_a^b f(x, y) dx \leq \max \{ f(a, y), f(b, y) \}. \]

Dividing both sides of inequality with $(d - c)$ and integrating with respect to $y$ over $[c, d]$, we get

\begin{equation}
\frac{1}{(b - a) (d - c)} \int_c^d \int_a^b f(x, y) dxdy \leq \max \left\{ \frac{1}{(d - c)} \int_c^d f(a, y) dy, \frac{1}{(d - c)} \int_c^d f(b, y) dy \right\}.
\end{equation}
By a similar argument, we can write
\[(2.6)\]
\[
\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y)dx\,dy \leq \max \left\{ \frac{1}{b-a} \int_a^b f(x,c)\,dx, \frac{1}{b-a} \int_a^b f(x,d)\,dx \right\}.
\]

By addition \((2.5)\) and \((2.6)\), we have
\[
\frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x,y)dx\,dy \leq \frac{1}{2} \left[ \max \left\{ \frac{1}{(b-a)} \int_a^b f(x,c)\,dx, \frac{1}{(b-a)} \int_a^b f(x,d)\,dx \right\} \right.
\]
\[+ \max \left\{ \frac{1}{(d-c)} \int_c^d f(a,y)\,dy, \frac{1}{(d-c)} \int_c^d f(b,y)\,dy \right\}\]

which completes the proof.

**Theorem 8.** Let \(C(\Delta), J(\Delta), W(\Delta), QC(\Delta), JQC(\Delta), WQC(\Delta)\) denote the classes of functions co-ordinated convex, co-ordinated J-convex, co-ordinated JQC, co-ordinated convex, co-ordinated J-convex, co-ordinated WQC functions on \(\Delta = [a,b] \times [c,d]\), respectively, we have the following inclusions.

\[(2.7)\]
\[QC(\Delta) \subset WQC(\Delta) \subset JQC(\Delta)\]

\[(2.8)\]
\[W(\Delta) \subset WQC(\Delta), \quad C(\Delta) \subset J(\Delta), \quad J(\Delta) \subset JQC(\Delta).\]

**Proof.** Let \(f \in QC(\Delta)\). Then for all \((x,y), (z,w) \in \Delta\) and \(t \in [0,1]\), we have
\[
f((1-t)x + tz, (1-t)y + tw) \leq \max \{f(x,y), f(z,w)\}.
\]

By addition, we obtain
\[(2.9)\]
\[
\frac{1}{2} \left[ f((1-t)x + tz, (1-t)y + tw) + f((1-t)x + tz, (1-t)y + tw) \right] \leq \max \{f(x,y), f(z,w)\}
\]

that is, \(f \in WQC(\Delta)\). In \((2.9)\), if we choose \(\lambda = \frac{1}{2}\), we obtain \(WQC(\Delta) \subset JQC(\Delta)\). Which completes the proof of \((2.7)\).

In order to prove \((2.8)\), taking \(f \in W(\Delta)\) and using the definition, we get
\[
\frac{1}{2} \left[ f\left( \left( (1-t)a + tb, (1-s)c + sd \right) + f\left( \left( (1-t)b, sc + (1-s)d \right) \right) \right \right] \leq \frac{f(a,c) + f(b,d)}{2}
\]

for all \((a,c), (b,d) \in \Delta\) and \(t \in [0,1]\). Using the fact that
\[
\frac{f(a,c) + f(b,d)}{2} + f(a,c) - f(b,d) = \max \{f(a,c), f(b,d)\}
\]

we can write
\[
\frac{f(a,c) + f(b,d)}{2} \leq \max \{f(a,c), f(b,d)\}
\]

for all \((a,c), (b,d) \in \Delta\), we obtain \(W(\Delta) \subset WQC(\Delta)\).

Taking \(f \in C(\Delta)\) and, if we choose \(t = \frac{1}{2}\) in \((1.0)\), we obtain
\[
\frac{f\left( \frac{x+y}{2}, \frac{z+w}{2} \right)}{2} \leq \frac{f(x,y) + f(z,w)}{2}
\]
for all \((x, y), (z, w) \in \Delta\). One can see that \(C(\Delta) \subset J(\Delta)\).

Taking \(f \in J(\Delta)\), we can write
\[
f\left(\frac{x + z}{2}, \frac{y + w}{2}\right) \leq \frac{f(x, y) + f(z, w)}{2}
\]
for all \((x, y), (z, w) \in \Delta\). Using the fact that
\[
f(x, y) + f(z, w) + \left|f(x, y) - f(z, w)\right| = \max\{f(x, y), f(z, w)\}
\]
we can write
\[
\frac{f(x, y) + f(z, w)}{2} \leq \max\{f(x, y), f(z, w)\}.
\]
Then obviously, we obtain
\[
f\left(\frac{x + z}{2}, \frac{y + w}{2}\right) \leq \max\{f(x, y), f(z, w)\}
\]
which shows that \(f \in JQ(\Delta)\). □

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