Abstract: In this short note we define a Poissonian model of directed random graphs which generalises the undirected Poissonian random graph process introduced by Norros and Reittu in [12]. We discuss the relation of our model to the Norros-Reittu model, characterise the limiting distribution of the degree of a typical vertex and discuss the component structure of the model in some special cases which are relevant to the theory of infection processes.

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1 Motivation

Our motivation comes from the study of random processes on sparse, inhomogeneous random graphs of large but finite size. ‘Sparseness’ refers to the number of directed edges (or arcs), which should be of the same order as the graph size and ‘inhomogeneous’ means that arc probabilities are non-constant, i.e. may depend on the starting- and endpoint of an arc. In particular we are interested in scale free random graphs, i.e. graphs in which the asymptotic proportion of vertices with (in- and/or out-)degree $k$ decays as $k^{-\tau}$ for some $\tau > 2$.

A plenitude of different undirected graph models can be found in the literature and a lot of work has been done over the last decades concerning dynamics of and on undirected random graphs, see e.g. the monographs [8, 10]. Some processes, e.g. random walks, several types of infection dynamics and bond percolation are by now rather well understood, at least for some underlying graphs models.

In the undirected setting, the mathematical literature is not as vast and mathematically rigorous results seem to be harder to obtain. However, recently there has been
considerable interest in dynamics on inhomogeneous directed graph models, examples include random walks [3, 6], page rank algorithms [5] and the threshold contact process [4].

The canonical choice for an inhomogeneous directed graph model is to specify a (random or deterministic) degree sequence (say of length $N \in \mathbb{N}$) and then choose one graph among all graphs with this degree sequence uniformly at random. This model is usually called directed configuration model and is the underlying graph model in the studies [3, 6, 5, 4] cited above. Structural results concerning this model have been obtained in [7]. However, arcs in the configuration are not independent and thus often considerable technical effort is needed to turn rather intuitive heuristic derivations into rigorous proofs.

To avoid these technical difficulties one may rely on a random graph model with independent arcs. A very general model for inhomogeneous random digraphs was proposed in [1]. This model is an adaptation of the model for undirected random graphs introduced in [2]. These models are parametrised by vertex weights and a bivariate kernel function which together essentially correspond to the limiting edge probabilities. Unfortunately, both models are not very suited to model power law graphs with $τ \leq 3$, since in this case the corresponding kernels are unbounded. One therefore either needs to rely on approximation arguments or slightly change the proposed framework to include such graphs, cf. [2, Sections 16.4 and 18]. Since power law degree sequences with this tail behaviour are relevant for some applications, one of our motivations is to specify a model which is tailored to include this range of $τ$. The approach we propose is based on the ‘Conditionally Poissonian random graph process’ or Norros-Reittu model defined in [12]. We recall the explicit definition of this model in Section 3.

2 Model definition

The random graph we propose is in fact a multigraph with a Poissonian number of arcs between two given vertices. It is parametrised by a bivariate probability law $λ_W$ on $(0, \infty) \times (0, \infty)$, which we call the weight distribution. The notation $W = (W^{(\text{in})}, W^{(\text{out})})$ is throughout reserved for a generic random variable with distribution $λ_W$. The weight $W$ may be thought of as the limit law of joint in- and outdegree of a typical vertex as the graph size tends to infinity. To form a graph we therefore require

$$\mathbb{E}W^{(\text{in})} = \mathbb{E}W^{(\text{out})} =: \mu < \infty,$$

i.e. on average the weight of arcs pointing towards a vertex should match the weight of arcs pointing away from it and it should be finite in line with our assumption of sparseness. We further denote by

$$ν := (ν^{(\text{in})}, ν^{(\text{out})}) = (\mathbb{E}(W^{(\text{in})})^2, \mathbb{E}(W^{(\text{out})})^2) \in [0, \infty] \times [0, \infty]$$
the second moment of the weight and by

$$\rho := \mathbb{E}(W^{(\text{in})} W^{(\text{out})}) \leq \sqrt{\nu^{(\text{in})} \nu^{(\text{out})}}$$

the correlation between the marginals.

Let $W = (W_n)_{n \in \mathbb{N}}$ denote a sequence of i.i.d. $\lambda_W$-distributed random variables, i.e. $W_n = (W^{(\text{in})}_n, W^{(\text{out})}_n) \sim W$. Here and throughout the article ‘~’ denotes equality in distribution. Given $W$ and $N \in \mathbb{N}$, we now define a random directed (multi-)graph $G_N(W)$ with vertex set $[N] := \{1, \ldots, N\}$ by inserting for each pair $(v, w) \in [N] \times [N]$ independently a random number $E(v, w)$ of directed edges where

$$E(v, w) \sim \text{Poiss}\left(\frac{W^{(\text{out})}_v W^{(\text{in})}_w}{L_N}\right)$$

and $L_N$ is an a.s. non-decreasing normalisation satisfying

$$\frac{1}{L_N} \sum_{v, w \in [N]} W^{(\text{out})}_v W^{(\text{in})}_w = (\mu + o_p(1)) N \text{ as } N \to \infty,$$

which guarantees the sparseness of the graph. We use the common asymptotic notations, $o(\cdot), O(\cdot)$ and $\Theta(\cdot)$ and the subscript indicates that the implicit limit statements holds $\mathbb{P}$-asymptotically almost surely, i.e. with probability $\mathbb{P}$ tending to one as $N \to \infty$.

Remark 1. In the undirected Norros-Reittu model, each vertex just gets one weight and the normalisation $L_N$ equals the sum of all weights. In the directed setting, there is no straightforward equivalent to this set up. If $|v| < \infty$ then the deterministic choice $L_N = \mu N$ already yields a satisfactory model, in fact this is a special case of the model in [1]. With view towards applications and to include ‘randomly directed’ Norros-Reittu graphs as a special case of our model, see Section 3, we make no explicit choice at this point and only ask for (2) to hold.

Note that the graph is defined conditional on $W$. We assume that $W$ and the family $(G_N)_{N \in \mathbb{N}}$ are defined on the same probability space. The corresponding measure is denoted by $\mathbb{P}$ and blackboard font quantities generally refer to this measure. Some of our results concern the conditional distribution of $(G_N)_{N \in \mathbb{N}}$ given $W$. We indicate this in our notation by using the bold face letters $P_W, E_W$, etc.

In the following sections we will

- discuss the relation to the undirected Norros-Reittu model;
- sketch the derivation of some results concerning the connectivity of the graph in special cases;

1 Another common terminology for this type of statement is ‘with high probability’ which sounds a little less technical. In this article the reference measure sometimes changes, thus should be included in the notation and we feel that ‘$\mathbb{P}$-a.a.s.’ sounds marginally better than ‘w.h.p. $\mathbb{P}$’.
• calculate the limiting distribution of a uniformly chosen vertex in $G_N$.

Limiting results for the empirical degree sequence, the general component structure of $G_N$ and graph distances will be addressed in detail in a forthcoming article [11].

3 Relation to the Norros-Reittu graph process with i.i.d. weights

We now discuss the relation with the original (undirected) Norros-Reittu graph process $(NR_N)_{N \in \mathbb{N}}$ and discuss how its fundamental properties carry over to the directed model. The most useful feature of $NR_N$ is the representation of the local neighbourhood of a typical vertex by truncated Poisson branching processes. This property greatly simplifies the analysis of the component structure and graph distances in $NR_N$. In this and the following section, we show that $NR_N$ is essentially a special case of our model, discuss some implication of the connection and also give a formulation of our model in terms of an evolving graph process.

To begin with, we give the original definition of $NR_N$ given in [12]. Conditionally on a sequence $\Lambda = (\Lambda_1, \Lambda_2, \ldots)$ of i.i.d. ‘capacities’, define a random multigraph $NR_N(\Lambda)$ by inserting independently for each unordered pair $\{v, w\}$ of vertices in $[N]$ a random number $E\{v, w\}$ of edges with $E\{v, w\} = \text{Poisson}\left(\frac{\Lambda_v \Lambda_w}{\bar{L}_N}\right)$, where $\bar{L}_N = \sum_{v \in [N]} \Lambda_v$. Clearly, the capacities in $NR_N$ play the role of the weights in our model.

To be able to compare the two models, we first turn our model into an undirected model by letting $W_\Lambda, p$ be an i.i.d. sequence of copies of $(B\Lambda^{(1)}, (1 - B)\Lambda^{(2)})$, where $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are distributed (not necessarily independent) as the capacities in $NR_N$ and $B \sim \text{Bernoulli}(p)$ is an independent Bernoulli random variable which equals one with probability $p \in [0, 1]$. Ignoring the direction of the edges we obtain an undirected multigraph. This construction corresponds to discarding either the out-weight or the in-weight of each vertex with a fixed probability before sampling the arcs. Consequently, if we set $L_N = \sum_{v \in [N]} W_v^{(m)} + W_v^{(w)}$, we obtain $NR_N(\Lambda)$ from $G_N(W_\Lambda, p)$ for any $p \in [0, 1]$.

As a byproduct of this construction, we obtain the following. Let $W = W_\Lambda, p + \tilde{W}_\Lambda, p$ where $\tilde{W}_\Lambda, p$ is obtained from $W_\Lambda, p$ via the transformation $B_v \mapsto 1 - B_v$, i.e. $\tilde{W}_\Lambda, p$ contains precisely the weights which were removed in $W_\Lambda, p$. Then conditional on $W$, the random graph $G_N(W)$ may be viewed as a mixture of $G_N(W_\Lambda, p)$ and $G_N(\tilde{W}_\Lambda, p)$. In particular, we obtain a characterisation of one instance of our model as a randomly oriented Norros-Reittu graph.

**Proposition 1** (Sum of oriented Norros-Reittu graphs). Choose $p \in \{0, 1\}$ and $\Lambda^{(1)} = \Lambda^{(2)} = \Lambda_v$. Then, conditional on $W = W_\Lambda, p + \tilde{W}_\Lambda, p$, $G_N$ has the same distribution as both of the following models.
(i) the sum of two independently generated $\text{NR}_N(\Lambda^{(i)})$ models $G^1_N, G^2_N$ with their edges oriented in such a way that they always point in opposing directions;

(ii) a random graph $\text{NR}_N(2\Lambda)$ in which for each edge an orientation is chosen independently with probability $\frac{1}{2}$.

When working with the model mentioned in Proposition 1(i), we make the following convention in the choice of orientation: in $G^1_N$, arcs always point toward the vertices with higher index and in $G^2_N$ arcs always point towards the vertex with lower index.

Before we conclude this section, we return to the general setting and observe that due to Poisson thinning, our model has a dynamical representation as a graph process just like the Norros-Reittu model.

**Proposition 2.** Conditional on $W$, $G_{N+1}$ may be obtained from $G_N$ by removing each existing edge independently with probability $1 - L_N/L_{N+1}$ and adding only vertex $N + 1$ together with its incoming and outgoing edges as specified in (1).

**Proof.** This follows immediately upon comparing the edge numbers in both graphs and using the fact that independent thinning with removal probability $p$ of a Poisson($\lambda$)-distribution yields a Poisson($(1 - p)\lambda$)-distribution. \qed

### 4 Cluster sizes in special cases

The particular setting of Proposition 1 plays a role in applications in which dependence between in- and outdegree are required. Since here $W^{(in)} = W^{(out)}$, there is a rather strong positive correlation between (unconditioned) in- and outdegrees in the graph. As a consequence of the above construction, we can infer a result about the connectivity of $G_N(W)$ in this situation. If there exists a directed path from $u$ to $v$ in $G_N$ we write $u \to v$ and if there is also a directed path from $v$ to $u$ we write $v \leftrightarrow u$. Clearly, ‘$\leftrightarrow$’ induces an equivalence relation on $[N]$ and the equivalence classes (or rather the subgraphs induced by them) are called strongly connected components of $G_N$. Fix $v \in [N]$, the subgraph induced by $F[v] = \{ u \in [N] : v \to u \}$ is called the forward component or forward cluster of $v$. The set $B[v] = \{ u \in [N] : u \to v \}$ is called the backward component or backward cluster of $v$. $C[v] = F[v] \cap B[v]$ denotes the strongly connected component/cluster of $[v]$. It is clear, that by taking the sum of two graphs which are directed, the sum graph inherits the weak components from its constituent graphs.

If there is a connected component $C_N \subset G_N$ such that $\lim_{N \to \infty} \#C_N/N = \pi > 0$, then we call this component the giant component. As a corollary of Proposition 1 we immediately obtain that a sum of $\text{NR}_N$-models has a giant component, if and only if the constituent models have a giant (undirected) component.

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2By an abuse of notation, we will not distinguish between the vertex sets and the subgraphs they induce in $G_N(W)$
Proposition 3 (Giant clusters in sum graphs). Let \( G_N(W) = G^1 + G^2 \) be as in Proposition 1. If the constituent NR\(_N\) models possess a giant (undirected) component of asymptotic size \( \zeta_N \), then \( G_N \) possess giant weak components of the same size and strong component of asymptotic size \( \pi N \), where \( \pi > 0 \).

Proof. The weak components are inherited from the constituent graphs. Furthermore, it is easy to see that giant components are unique and well known, see e.g. [norros], that \( \zeta \) corresponds to the survival probability of a weighted Poisson branching process which approximates the neighbourhood of a uniformly chosen vertex \( V \). Conditioning on the weights we may thus write
\[
\zeta = \int \zeta(W) \, dP_W,
\]
where \( \zeta(W) \) corresponds to the survival probability of the branching process given the weights \( W \). Now, by construction, the asymptotic fraction \( \pi \) of vertices in the strongly connected component equals the joint survival probability of two branching processes approximating the neighbourhoods of \( U \) in \( G_1, G_2 \) respectively. Conditional on \( W \), these processes are independent and we can infer
\[
\pi = \int \zeta(W)^2 \, dP_W > 0.
\]

Remark 2. With a more careful analysis of the possible dependencies between the two exploration processes, the proof of Proposition 3 remains valid without the assumptions on \( G_N(W) \) to be a sum of independent graphs. This argument is carried out in [1].

For the Norros-Reittu model NR(\( \Lambda \)), much is known about the size of the largest components for critical weight sequences, i.e. if \( E\Lambda^{(1)} = 1 \), see [9]. This case is particularly interesting, therefore we mention the induced result for oriented sum graphs as well.

Corollary 4 (Critical weak clusters sum graphs). Let \( G_N(W) \) be as in Proposition 1 with power law weights s.t. \( \tau > 3 \) and \( v^{(\text{out})} / \mu = v^{(\text{in})} / \mu = 1 \), i.e. the constituent NR\(_N\) models possess an (undirected) component of asymptotic size \( \theta P(N^\alpha) \) where \( \alpha = \frac{\tau - 1}{\tau - 2} \wedge \frac{2}{3} \). Then \( G_N(W) \) has forward and backward components of asymptotic size \( \theta P(N^\alpha) \).

On the other end of the spectrum of in- and out-degree dependencies are instances of \( G_N(W) \) for which \( W^{(\text{in})} \) and \( W^{(\text{out})} \) are independent. These are relatively easy to treat, but differ from Norros-Reittu graphs in the following way: If we follow a random edge in the Norros-Reittu model, the weight on its end does not follow the distribution \( \lambda_W \), but rather its size biased version (which is also the reason why the critical regime in Corollary 4 is at \( v^{(\text{in})}/\mu = 1 \) and not at \( \mu = 1 \)). This is not the case in an oriented exploration in \( G_N(W) \), since incoming and outgoing arcs are only correlated through the dependence of \( W^{(\text{in})} \) and \( W^{(\text{out})} \). For deterministic \( L_N \), we may again interpret \( G_N(W) \) as
sum of two independent undirected graphs which have been given a fixed orientation, but this time also the weight sequence is sampled independently for each graph. The following result about component sizes can be inferred from the undirected situation, see [2] or by applying the result of [1].

**Proposition 5** (Giant clusters in independent sum graphs). Let $L_N = \mu N$ and $G_N(W) = G^1(W_1) \oplus G^2(W_2)$, using the above orientation convention. If the constituent graphs possess a giant (undirected) component of asymptotic proportions $\zeta_1, \zeta_2$, then $G_N$ possesses giant weak components of the same size and strong component of asymptotic size $\pi N$, where $\pi = \zeta_1 \zeta_2$.

**Proof.** The argument is exactly the same as for the dependent situation. However, the branching processes which need to be considered are now simply mixed Poisson with mixing distributions given by $W_1$ and $W_2$. \hfill \Box

### 5 Vertex degrees in $G_N(W)$

From the definition of the model it is straightforward to derive that under $P_W$ each vertex degree is a Poisson-distributed random variable, since the family of Poisson-distributions is stable under convolution. We use the following conventions for indegree and outdegree of a vertex $v \in [N]$

$$D_v = (D_v^{(i)}, D_v^{(o)}), \quad D_v^{(i)} := \sum_{u \in [N] \setminus \{v\}} E(u, v) \quad \text{and} \quad D_v^{(o)} := \sum_{u \in [N] \setminus \{v\}} E(v, u),$$

i.e. we do not count loops towards the in- or outdegree but include them in the total degree $\deg(v) := D_v^{(i)} + D_v^{(o)} + E(v, v)$. If $v > N$, we set $D_v = (0, 0)$. The following proposition summarises our observation of the conditional degree distribution.

**Proposition 6** (Conditional degrees). For $v \in [N]$, let $D_v$ denote its degree in $G_N$. Under $P_W$, the random variables $D_v^{(i)}$ and $D_v^{(o)}$ are independent and $D_v^{(i)} \sim \text{Poisson}(\Lambda_i)$, $D_v^{(o)} \sim \text{Poisson}(\Lambda_o)$ and $\deg(v) \sim \text{Poisson}(\Lambda_t)$, where

$$\Lambda_i = \frac{1}{L_N} \sum_{u \in [N] \setminus \{v\}} W_u^{\text{out}}, \quad \Lambda_o = \frac{1}{L_N} \sum_{u \in [N] \setminus \{v\}} W_u^{\text{out}} \quad \text{and} \quad \Lambda_t = \Lambda_i + \Lambda_o + \frac{W_v^{\text{out}} W_v^{\text{in}}}{L_N}.$$

From the independence of the edge variable array $E(\cdot, \cdot)$ it is evident that the indegrees of different vertices are independent under $P_W$ and the same holds for the outdegrees. For $u \neq v \in [N]$, we have that $D_u^{(o)}$ depends on $D_v^{(i)}$ only through $E(u, v)$. As $N \to \infty$ this dependence is negligible, as long as the vertex weights do not get too large. To see this we need the following lemma, which is frequently used in the remainder of the article.
Lemma 7 (Convergence of mixed Poisson random variables). Let \( M_1, M_2, \ldots \) be nonnegative real random variables and \( X_i \sim \text{Poisson}(M_i) \), i.e. \((X_i)_{i=1}^{\infty}\) are mixed Poisson with mixing measures \( L(M_i), i \in \mathbb{N} \). If \( M_i \to \lambda \) in distribution for some constant \( \lambda \in [0, \infty) \), then
\[
\text{d}_{TV}(L(X_i), \text{Poisson}(\lambda)) \to 0 \quad \text{as } i \to \infty,
\]
where \( \text{d}_{TV} \) denotes total variation distance.

Proof. For fixed \( \lambda \), the function
\[
g_{\lambda}(u) = \text{d}_{TV}(\text{Poisson}(u), \text{Poisson}(\lambda)) = \frac{1}{2} \sum_{j=0}^{\infty} \left| e^{-\lambda} \frac{\lambda^j}{j!} - e^{-u} \frac{u^j}{j!} \right|
\]
is continuous in \( u \). Thus
\[
\text{d}_{TV}(L(X_i), \text{Poisson}(\lambda)) = \int g_{\lambda}(M_i) \, dM_i \to 0,
\]
since convergence of distributions is equivalent to convergence of all integrals against continuous functions and \( g_{\lambda}(\lambda) = 0 \).

Proposition 8 (Asymptotic independence of conditional degrees). Fix \( \mathcal{W} \) and a finite set \( v(1), \ldots, v(k) \in \mathbb{N} \) of vertices. Then, as \( N \to \infty \),
\[
(D_{v(1)}, \ldots, D_{v(k)}) \xrightarrow{\mathcal{P}_N} (\tilde{D}_{v(1)}, \ldots, \tilde{D}_{v(k)})
\]
where the limiting vector has independent components with entries
\[
\tilde{D}_{v(i)} \sim \text{Poisson}(W_{v(i)}^{\text{in}}) \otimes \text{Poisson}(W_{v(i)}^{\text{out}}).
\]

Proof. Conditional on \( \mathcal{W} \), we define two random multigraphs \( \mathcal{G}_N^l, \mathcal{G}_N^u \) on \( [N] \), such that \( \mathcal{G}_N^l, \mathcal{G}_N^u \) and \( \mathcal{G}_N \) can be coupled to satisfy
\[
\mathcal{G}_N^l \leq \mathcal{G}_N \leq \mathcal{G}_N^u,
\]
where ‘ \( \leq \)’ denotes dominations of (multi-)graphs in the usual sense. If now both \( (\mathcal{G}_N^l)_{N \in \mathbb{N}} \) and \( (\mathcal{G}_N^u)_{N \in \mathbb{N}} \) obey the stated limiting behaviour, the same must be true for \( \mathcal{G}_N \). Let \( V = \{v(1), \ldots, v(k)\} \) and define \( \mathcal{G}_N^l \) by letting the edge numbers independently be given by
\[
E'(v, w) \sim \text{Poisson} \left( \mathbb{I}\{v \notin V \text{ or } w \notin V\} \frac{W_{v}^{\text{out}} W_{w}^{\text{out}}}{L_N} \right).
\]
Thus, by Poisson-thinning, coupled versions of \( \mathcal{G}_N^l, \mathcal{G}_N \) can be obtained by removing all arcs in \( \mathcal{G}_N \) with both endpoints in \( V \). Clearly, the degrees of \( v(1), \ldots, v(k) \) in \( \mathcal{G}_N^l \) are independent. Now, since \( \sum_{v \in [N] \setminus V} W_{v}^{\text{out}} / L_N \to 1 \) and \( \sum_{v \in [N] \setminus V} W_{v}^{\text{out}} / L_N \to 1 \) by (2), the limiting degree distribution in \( \mathcal{G}_N^l \) is as stated. To define \( \mathcal{G}_N^u \), let \( M = \)
max\{W_v^{(in)}, W_v^{(out)}; v \in V\} and sample \(k(k-1)\) i.i.d. Poisson\((M^2/L_N)\)-distributed random variables \(F(v(i), v(j)), i \neq j = 1, \ldots, k\). Let \(G_N^u\) be given by the edge variables

\[ E^u(v, w) = E^l(v, w) + \mathbb{1}\{v, w \in V\}F(v, w). \]

Then clearly, \(G_N^u\) and \(G_N\) can be coupled, such that \(G_N^u \geq G_N\). But \(G_N^u\) and \(G_N^l\) only differ by a Poisson\((k(k-1)M^2/L_N)\)-distributed number of edges and this difference converges to 0 for almost every \(W\). Hence the degrees in \(G_N^u\) are asymptotically equal to the degrees in \(G_N^l\). \(\square\)

If \(\rho\) is finite then the model produces at most a finite number of loops.

**Proposition 9** (Finite number of loops). If \(\rho < \infty\), then total number of loops in \(G_N\) is asymptotically distributed as Poisson\((\rho/\mu)\).

**Proof.** By definition of the model, it is clear that under \(P_W\), the loop numbers

\[ E(1,1), \ldots, E(N,N) \]

are independent Poisson\((W_v^{(in)}W_v^{(out)}/L_N)\)-distributed random variables. Thus their sum is Poisson with parameter

\[ \frac{\sum_{v=1}^{N} W_v^{(in)}W_v^{(out)}}{L_N} = \frac{N}{L_N} \left( \sum_{v=1}^{N} \frac{W_v^{(in)}W_v^{(out)}}{N} \right). \]

The term in brackets \(P_W\)-almost surely converges to \(\rho\) by the strong law of large numbers. The factor \(N/L_N\) satisfies

\[ \frac{N}{L_N} = (\mu + o_P(1)) \frac{N^2}{\sum_{v=1}^{N} W_v^{(in)}W_v^{(out)}} = (1 + o_P(1))\mu^{-1} \]

by (2) and the strong law of large numbers. By Lemma 7, it follows that the limiting distributions is Poisson\((\rho/\mu)\). \(\square\)

Finally, as a corollary of the conditional results, we obtain a characterisation of the limiting distribution of the in- and outdegree of a typical vertex under \(P\).

**Corollary 10** (Limiting distribution of vertex degrees). For \(U(N) \in [N]\) uniformly chosen denote by \(D(N) = (D^{(in)}(N), D^{(out)}(N))\) its joint in- and outdegree in \(G_N(W)\). Then

\[ D(N) \to D \text{ in distribution as } N \to \infty, \]

where \(D\) is bivariate mixed Poisson with mixing distribution \(W\), i.e. conditionally on the mixing variable \(W = (W^{(in)}, W^{(out)})\),

\[ D \sim \text{Poisson}(W^{(in)}) \otimes \text{Poisson}(W^{(out)}). \]
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