Cocycle and orbit superrigidity for lattices in \( \text{SL}(n, \mathbb{R}) \) acting on homogeneous spaces

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Abstract

We prove cocycle and orbit equivalence superrigidity for lattices in \( \text{SL}(n, \mathbb{R}) \) acting linearly on \( \mathbb{R}^n \), as well as acting projectively on certain flag manifolds, including the real projective space. The proof combines operator algebraic techniques with the property (T) in the sense of Zimmer for the action \( \text{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{R}^n \), \( n \geq 4 \). We also show that the restriction of the orbit equivalence relation \( \mathcal{R}(\text{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{R}^n) \) to a subset of finite Lebesgue measure, provides a \( \text{II}_1 \) equivalence relation with property (T) and yet fundamental group equal to \( \mathbb{R}_+ \).

1 Introduction and statement of main results

Over the last few years, operator algebraic methods were used to prove several orbit equivalence and cocycle superrigidity theorems: for Bernoulli actions of property (T) groups \([15]\) and of product groups \([14]\) and for profinite actions of property (T) groups \([10]\). In this paper, we extend the scope of these methods to a more geometric class of actions, like the natural actions of lattices \( \Gamma \subset \text{SL}(n, \mathbb{R}) \) on the vector space \( \mathbb{R}^n \), on the projective space \( \mathbb{P}^{n-1}(\mathbb{R}) \) and on certain flag manifolds, all of which can be viewed as \( \text{SL}(n, \mathbb{R}) \)-homogeneous spaces.

None of these actions is probability measure preserving. Hence, property (T) of the acting group, has to be replaced by Zimmer’s notion of property (T) for a non-singular action (see [20]), which plays a crucial role in this paper. It is shown that for any lattice \( \Gamma \subset \text{SL}(n, \mathbb{R}) \), the linear action \( \Gamma \curvearrowright \mathbb{R}^n \) has property (T) if and only if \( n \geq 4 \). We then deduce the following theorem.

**Theorem 1.1.** Let \( \Gamma < \text{SL}(n, \mathbb{R}) \) be a lattice and let \( \mathcal{R} \) be the \( \text{II}_1 \) equivalence relation obtained by restricting the orbit equivalence relation \( \mathcal{R}(\Gamma \curvearrowright \mathbb{R}^n) \) to a set of Lebesgue measure 1, for some \( n \geq 4 \). Then we have:

- \( \mathcal{R} \) has property (T), in the sense of Zimmer, yet the fundamental group of \( \mathcal{R} \) equals \( \mathbb{R}_+ \).
- \( \mathcal{R}_t \) cannot be implemented by a free action of a group, \( \forall t > 0 \). Also, \( \mathcal{R}_t \) cannot be implemented by an (not necessarily free) action of a discrete property (T) group, \( \forall t > 0 \).

We say that a Polish group is of finite type if it can be realized as the closed subgroup of the unitary group of some \( \text{II}_1 \) factor with separable predual. All countable and all second countable compact groups are Polish groups of finite type. In [15], the first author proved that every 1-cocycle for the Bernoulli action of a property (T) group with values in a Polish group of finite type, is cohomologous to a group morphism. We say that actions with this property are \( \text{U}_{\text{fin}} \)-cocycle superrigid. More precisely:

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**Definition 1.2.** The non-singular action $G \curvearrowright (X, \mu)$ of the locally compact second countable group $G$ on the standard measure space $(X, \mu)$ is called $U_{\text{fin}}$-cocycle superrigid if every 1-cocycle for the action $G \curvearrowright (X, \mu)$ with values in a Polish group of finite type $\mathcal{G}$ is cohomologous to a continuous group morphism $G \to \mathcal{G}$.

The following actions are known to be $U_{\text{fin}}$-cocycle superrigid: Bernoulli actions of property (T) groups [15] and of product groups [14], while in [10], virtual $U_{\text{fin}}$-cocycle superrigidity is proven for profinite actions of property (T) groups. We extend this to the following actions of geometric nature.

**Theorem 1.3.** The following actions are $U_{\text{fin}}$-cocycle superrigid.

1. For $n \geq 5$ and $\Gamma$ any lattice in $\text{SL}(n, \mathbb{R})$, the linear action $\Gamma \curvearrowright \mathbb{R}^n$.

2. For $n \geq 5$ and $\Gamma$ any finite index subgroup of $\text{SL}(n, \mathbb{Z})$, the affine action $\Gamma \ltimes \mathbb{Z}^n \curvearrowright \mathbb{R}^n$.

3. For $n \geq 4k + 1$, $\Gamma$ any lattice in $\text{SL}(n, \mathbb{R})$ and $H$ any closed subgroup of $\text{GL}(k, \mathbb{R})$, the action

$$G \curvearrowright M_{n,k}(\mathbb{R}) \text{ where } G := \begin{cases} \Gamma \times H & \text{if } (-1, -1) \in \Gamma \times H, \\ \Gamma \times H & \text{otherwise,} \end{cases}$$

by left-right multiplication on the space $M_{n,k}(\mathbb{R})$ of $n \times k$ matrices equipped with the Lebesgue measure.

In [15], the first author introduces the notion of malleability of a measure preserving action $\Gamma \curvearrowright (X, \mu)$, which roughly means that there is a flow on $X \times X$, commuting with the diagonal $\Gamma$-action and connecting the identity map to the flip map on $X \times X$. Theorem 0.1 in [15] says that every weakly mixing, malleable action of a property (T) group is $U_{\text{fin}}$-cocycle superrigid.

We generalize this cocycle superrigidity theorem to infinite measure preserving actions. But, property (T) of the group $\Gamma$ has to be replaced by property (T) for the diagonal action of $\Gamma \curvearrowright X \times X$. In the case of $\text{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{R}^n$, this forces $n \geq 5$. Finally, weak mixing has to be replaced by the ergodicity of the 4-fold diagonal action $\Gamma \curvearrowright X \times X \times X \times X$, which in the case of $\text{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{R}^n$ again holds exactly for $n \geq 5$.

Using the cocycle superrigidity of $\text{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{R}^n$, we give a full classification of all 1-cocycles for the action $\text{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, with values in a Polish group of finite type. As such, our Example 5.12 below, complements Zimmer’s celebrated cocycle superrigidity theorem [21]: Zimmer’s result treats arbitrary actions $\text{SL}(n, \mathbb{Z}) \curvearrowright (X, \mu)$, but specific target groups (simple linear algebraic groups), while our result treats a very specific action, but rather general target groups.

Given the cocycle superrigidity theorem [13], we can deduce several orbit equivalence (OE) superrigidity results. We are particularly interested in the following concrete actions of lattices $\Gamma$ in $\text{SL}(n, \mathbb{R})$ and $\text{PSL}(n, \mathbb{R})$.

1. The linear action $\Gamma \curvearrowright \mathbb{R}^n$.

2. If $\Gamma$ is a finite index subgroup of $\text{SL}(n, \mathbb{Z})$, the action $\Gamma \curvearrowright \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$.

3. The projective action $\Gamma \curvearrowright \mathbb{P}^{n-1}(\mathbb{R})$. 

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4. Let $X$ be the real flag manifold of signature $(d_1, \ldots, d_l, n)$. Recall that points in $X$ are flags

$$\{0\} \subset V_1 \subset \cdots \subset V_l \subset \mathbb{R}^n$$

where $V_i$ is a vector subspace of $\mathbb{R}^n$ with dimension $d_i$. We consider the natural action $\Gamma \curvearrowright X$ for any lattice $\Gamma$ in $\text{PSL}(n, \mathbb{R})$.

The action in 1 has the Lebesgue measure as infinite invariant measure, while the actions in 3 and 4 do not have finite or infinite invariant measures. All the actions in 1-4 are essentially free and ergodic, see Lemma 5.7 for details.

The natural invariant measure class on the flag manifold $X$ can be described as follows. Put $d_1 = k$ and consider the set $M_{n,k}(\mathbb{R})$ of $n \times k$ matrices of rank $k$, equipped with the Lebesgue measure. Denote by $E = (E_1, \ldots, E_l)$ the standard flag of signature $(d_1, \ldots, d_l, n)$, i.e. $E_i = \text{span}\{e_1, \ldots, e_{d_i}\}$, where $e_1, \ldots, e_n$ are the standard basis vectors in $\mathbb{R}^n$. The group $\text{GL}(k, \mathbb{R})$ acts on $M_{n,k}(\mathbb{R})$ by right multiplication. This action is free and proper and

$$M_{n,k}(\mathbb{R})/H \to X : A \mapsto (AE_1, \ldots, AE_l)$$

is an isomorphism. Here, $H = \{g \in \text{GL}(k, \mathbb{R}) \mid gE_i \subset E_i \text{ for all } i = 1, \ldots, l\}$. Writing $k_1 = d_1$ and $k_i = d_i - d_{i-1}$ for $i \geq 2$, the group $H$ can of course be written as

$$H = \begin{pmatrix}
\text{GL}(k_1, \mathbb{R}) & * & \cdots & * \\
0 & \text{GL}(k_2, \mathbb{R}) & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \text{GL}(k_l, \mathbb{R}) \\
\end{pmatrix}. \quad (1.1)$$

Before stating our OE superrigidity results, recall the following terminology.

**Definition 1.4.** Let $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$ and $\Lambda \overset{\beta}{\curvearrowright} (Y, \eta)$ be essentially free, ergodic, non-singular actions of countable groups on standard measure spaces.

- A stable orbit equivalence (SOE) between $\alpha$ and $\beta$ is a non-singular isomorphism $\Delta : X_0 \to Y_0$ between non-negligible subsets $X_0 \subset X, Y_0 \subset Y$, such that $\Delta$ is an isomorphism between the restricted orbit equivalence relations $\mathcal{R}(\Gamma \curvearrowright X)|_{X_0}$ and $\mathcal{R}(\Lambda \curvearrowright Y)|_{Y_0}$.

- We say that $\Gamma \curvearrowright X$ is induced from $\Gamma_1 \curvearrowright X_1$, if $\Gamma_1$ is a subgroup of $\Gamma$, $X_1$ is a non-negligible subset of $X$ and $g \cdot X_1 \cap X_1$ is negligible for all $g \in \Gamma - \Gamma_1$.

For the linear lattice actions and the quotient action $\text{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{T}^n$, we get the following.

**Theorem 1.5.** Let $n \geq 5$ and $\Gamma \subset \text{SL}(n, \mathbb{R})$ a lattice. Let $\Lambda \overset{\gamma}{\curvearrowright} (Y, \eta)$ be any essentially free, ergodic, non-singular action of the countable group $\Lambda$.

1. The actions $\Gamma \overset{\alpha}{\curvearrowright} \mathbb{R}^n$ and $\Lambda \overset{\beta}{\curvearrowright} Y$ are SOE if and only if $\Lambda \overset{\gamma}{\curvearrowright} Y$ is conjugate to an induction of one of the following actions:

   - $\Gamma \overset{\alpha}{\curvearrowright} \mathbb{R}^n$ itself,
   - (only in case $-1 \in \Gamma$) the quotient action $\Gamma/\{\pm 1\} \overset{\gamma}{\curvearrowright} \mathbb{R}^n/\{\pm 1\}$. 


Finally, combining the work of [6] and the above OE superrigidity results, we classify up to stable orbit equivalence, the actions $\text{SL}(n, \mathbb{Z}) \wedge \mathbb{T}^n$ and $\Lambda \wedge Y$ are SOE if and only if $\Lambda \wedge Y$ is conjugate to an induction of one of the following actions:

- $\text{SL}(n, \mathbb{Z}) \wedge \mathbb{T}^n$ itself,
- $\text{SL}(n, \mathbb{Z}) \wedge \mathbb{Z}^n \wedge \mathbb{R}^n$,
- $\text{SL}(n, \mathbb{Z}) \wedge (\mathbb{Z}/\Lambda \mathbb{Z})^n \wedge \mathbb{R}^n/(\mathbb{Z} \Lambda \mathbb{Z})^n$ for some $\lambda \in \mathbb{N} \setminus \{0, 1\}$,
- (only in case $n$ is even) one of the actions

$$\text{PSL}(n, \mathbb{Z}) \wedge \frac{\mathbb{T}^n}{\{\pm1\}} \propto \text{PSL}(n, \mathbb{Z}) \wedge \frac{(\mathbb{Z}/2\mathbb{Z})^n}{\{\pm1\}}.$$

To formulate easily the correct OE superrigidity statements for lattice actions on flag manifolds, make the following observations.

The real flag manifold of signature $(d_1, \ldots, d_l, n)$ has a natural $2^l$-fold covering $\tilde{X}$ consisting of oriented flags

$$\{0\} \subset (V_1, \omega_1) \subset \cdots \subset (V_l, \omega_l) \subset \mathbb{R}^n$$

where every $V_i$ is a vector subspace of $\mathbb{R}^n$ with an orientation $\omega_i$. Clearly, $\tilde{X} = M_{n,k}(\mathbb{R})/H_0$, where $H_0 = \{g \in H \mid \det(g|_{E_i}) > 0 \text{ for all } i = 1, \ldots, l\}$. In the expression (1.1) above, $H_0$ consists of those matrices $A$ that have on the diagonal $A_{ii} \in \text{GL}(k_i, \mathbb{R})$ with $\det A_{ii} > 0$ for all $i$.

Denote $\Sigma_l = H/H_0$ and observe that $\Sigma_l \cong (\mathbb{Z}/2\mathbb{Z})^\oplus l$. Then, $\Sigma_l$ acts on $\tilde{X}$ by reversing orientations, but keeping the flags. We denote by $-1 \in \Sigma_l$ the multiplication by $-1$ and observe that $-1 = 1$ in $\Sigma_l$ if all $d_i$ are even. Clearly, $X = \tilde{X}/\Sigma_l$.

**Theorem 1.6.** Let $X$ be the real flag manifold of signature $(d_1, \ldots, d_l, n)$. Let $\Gamma \subset \text{PSL}(n, \mathbb{R})$ be a lattice and assume that $n \geq 4d_l + 1$. Denote by $\tilde{X}$ the $2^l$-fold covering of $X$ consisting of oriented flags, as explained before the theorem. Let $\Lambda \wedge (Y, \eta)$ be any essentially free, ergodic, non-singular action of the countable group $\Lambda$.

The actions $\Gamma \wedge X$ and $\Lambda \wedge Y$ are SOE if and only if $\Lambda \wedge Y$ is conjugate to an induction of one of the actions

- $\Gamma \times \frac{\Sigma_l}{\Sigma} \wedge \tilde{X}/\Sigma$ for some subgroup $\Sigma < \Sigma_l$ with $-1 \in \Sigma$,
- $\tilde{\Gamma} \times \frac{\Sigma_l}{\Sigma\{\pm(1, 1)\}} \wedge \tilde{X}/\Sigma$ for some $\Sigma < \Sigma_l$ with $-1 \not\in \Sigma$ and with $\tilde{\Gamma} = \{\pm1\} \cdot \Gamma \subset \text{GL}(n, \mathbb{R})$.

**Example 1.7.** If $n \geq 5$ and $\Gamma \subset \text{PSL}(n, \mathbb{R})$ is a lattice, the action $\Gamma \wedge \mathbb{P}^{n-1}(\mathbb{R})$ is a special case of the flag manifold action treated in Theorem 1.6. Hence, $\Gamma \wedge \mathbb{R}^n$ and $\Lambda \wedge Y$ are SOE if and only if $\Lambda \wedge Y$ is conjugate to an induction of either $\Gamma \wedge \mathbb{P}^{n-1}(\mathbb{R})$ or its double cover $\tilde{\Gamma} \wedge \mathbb{R}^n/\mathbb{R}_+$, where $\tilde{\Gamma} = \{\pm1\} \cdot \Gamma \subset \text{GL}(n, \mathbb{R})$.

Finally, combining the work of [8] and the above OE superrigidity results, we classify up to stable orbit equivalence, the lattice actions on $\mathbb{R}^n$ and on flag manifolds, see Theorems [6.2] and [6.3]. At the same time, we compute the outer automorphism group of the associated orbit equivalence relation.

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(5) In fact, this statement is a slightly more detailed version, with very different proof, of [6, Corollary B], where it is shown that for all $n \geq 3$, the actions $\text{SL}(n, \mathbb{Z}) \wedge \mathbb{T}^n$ and $\Lambda \wedge Y$ are SOE if and only if they are virtually conjugate.
2 Preliminaries

We recall here Zimmer’s definition of property (T) for a II_1 equivalence relation \( R \) on a standard probability space \( (X, \mu) \).

To this end, first define \( \mathcal{R}^{(2)} = \{(x, y, z) \in X \times X \times X \mid x R y \text{ and } y R z\} \). Note that \( \mathcal{R} \), resp. \( \mathcal{R}^{(2)} \) come equipped with canonical \( \sigma \)-finite measures \( \mu^{(1)} \), resp. \( \mu^{(2)} \), given by

\[
\mu^{(1)}(Y) = \int_X \# \{y \in X \mid (x, y) \in Y\} \, d\mu(x),
\]

\[
\mu^{(2)}(Y) = \int_X \# \{(y, z) \in \mathcal{R} \mid (x, y, z) \in Y\} \, d\mu(x) = \int_X \# \{(x, z) \in \mathcal{R} \mid (x, y, z) \in Y\} \, d\mu(y)
\]

\[
= \int_X \# \{(x, y) \in \mathcal{R} \mid (x, y, z) \in Y\} \, d\mu(z).
\]

- A 1-cocycle of \( \mathcal{R} \) with values in the unitary group \( \mathcal{U}(H) \) of a Hilbert space \( K \) is a Borel map \( c : \mathcal{R} \to \mathcal{U}(K) \) satisfying \( c(x, z) = c(x, y)c(y, z) \) for almost all \( (x, y, z) \in \mathcal{R}^{(2)} \).

- Suppose that \( c : \mathcal{R} \to \mathcal{U}(K) \) is a 1-cocycle of \( \mathcal{R} \).
  - A unit invariant vector of \( c \) is a Borel map \( \xi : X \to K \) satisfying \( \xi(x) = c(x, y)\xi(y) \) for almost all \( (x, y) \in \mathcal{R} \) and \( \|\xi(x)\| = 1 \) for almost all \( x \in X \).
  - A sequence of almost invariant unit vectors of \( c \) is a sequence of Borel maps \( \xi_n : X \to K \) satisfying
    \[
    \|\xi_n(x) - c(x, y)\xi_n(y)\| \to 0 \quad \text{for almost all } (x, y) \in \mathcal{R}
    \]
    and \( \|\xi_n(x)\| = 1 \) for all \( n \in \mathbb{N} \) and almost all \( x \in X \).

**Definition 2.1.** A II_1 equivalence relation \( \mathcal{R} \) is said to have property (T) in the sense of Zimmer if the following holds: every 1-cocycle of \( \mathcal{R} \) with values in the unitary group of a Hilbert space and admitting a sequence of almost invariant unit vectors, admits a unit invariant vector.

3 Property (T) for actions of locally compact groups

We recalled above Zimmer’s definition of property (T) for a II_1 equivalence relation. In fact, one can define property (T) for measured groupoids in general, see [2]. We do not need this generality in this paper, but we do need the concept of property (T) for non-singular actions of locally compact second countable (l.c.s.c.) groups on measure spaces. For a groupoid approach to this definition, we refer to [2]. For the convenience of the reader, we gather in this section the necessary concepts and results and present them in an operator algebra framework.

All von Neumann algebras are supposed to have separable predual and all locally compact groups are supposed to be second countable.

If \( M \) is a von Neumann algebra, we equip \( \text{Aut}(M) \) with the Polish topology making the functions \( \text{Aut}(M) \to M_\omega : \alpha \mapsto \omega \circ \alpha \) continuous for all \( \omega \in M_\omega \). An action \( \alpha \) of a l.c.s.c. group \( G \) on a von Neumann algebra \( M \), denoted \( G \actson M \), is a continuous group morphism \( \alpha : G \to \text{Aut}(M) \).
• A 1-cocycle of an action $G \rtimes M$ with values in the unitary group $U(K)$ of a Hilbert space $K$, is a strongly continuous map $c : G \to U(M \otimes B(K))$ satisfying $c(gh) = c(g)(\alpha_g \otimes \text{id})(c(h))$ for all $g, h \in G$. Note that by Theorem 3 in [13], it makes no difference to assume only that $c$ is a measurable map, with the previous formula holding for almost all $(g, h) \in G \times G$.

• A unit invariant vector of the 1-cocycle $c$ of $G \rtimes M$, is an element $\xi$ in the $W^*$-module $M \otimes K$ satisfying $\xi^* \xi = 1$ and $c(g)(\alpha_g \otimes \text{id})(\xi) = \xi$ for all $g \in G$.

• A sequence of almost invariant unit vectors of the 1-cocycle $c$ of $G \rtimes M$, is a sequence $\xi_n \in M \otimes K$ satisfying $\xi_n^* \xi_n = 1$ for all $n$ and $c(g)(\alpha_g \otimes \text{id})(\xi_n) - \xi_n \to 0$ *-strongly, uniformly on compact subsets of $G$.

• The action $G \rtimes M$ is said to have property $(T)$ if every 1-cocycle with values in the unitary group of a Hilbert space and admitting a sequence of almost invariant unit vectors, admits a unit invariant vector.

The following result is proven for discrete groups in [20, Proposition 2.4], see also [2, Corollary 5.16]. These methods work as well in the locally compact case and for completeness, we give a proof in a von Neumann algebra setup.

**Proposition 3.1.** Let $G \rtimes M$ be an action of the l.c.s.c. group $G$ on the von Neumann algebra $M$. Suppose that $\tau$ is a faithful normal tracial state on $M$, invariant under $\alpha$. Then, $G \rtimes M$ has property $(T)$ if and only if the group $G$ has property $(T)$.

**Proof.** Suppose first that $G$ has property (T). Let $c : G \to U(M \otimes B(K))$ be a 1-cocycle of $G \rtimes M$ having $\xi_n \in M \otimes K$ as a sequence of almost invariant unit vectors. Define the unitaries $u_g$ on $L^2(M, \tau)$ by extending $\alpha_g$. Then,$$\pi : G \to U(L^2(M, \tau) \otimes K) : \pi(g) = c(g)(u_g \otimes 1)$$is a unitary representation of $G$ and we can view $\xi_n$ as a sequence of almost invariant unit vectors. Since $G$ has property (T), $\pi$ admits a unit invariant vector. Even more, we find a sequence $\eta_n \in L^2(M, \tau) \otimes K$ of $\pi$-invariant vectors satisfying $\eta_n - \xi_n \to 0$.

Since $\|\xi_n\| = 1$, it follows that $\|\xi_n^* \eta_n - 1\|_2 \to 0$. Hence, the right support projection of $\xi_n^* \eta_n$ converges strongly to 1. A fortiori, the right support projection of $\eta_n$ converges to 1. We view $\eta_n$ as a closed operator from $L^2(M, \tau)$ to $L^2(M, \tau) \otimes K$. Taking the polar decomposition of $\eta_n$, we find $v_n \in M \otimes K$ satisfying $c(g)(\alpha_g \otimes \text{id})(v_n) = v_n$ for all $g \in G, n \in \mathbb{N}$ and such that $v_n^* v_n$ is a sequence of projections in $M$ converging strongly to 1. Define the von Neumann algebra

$$N = \begin{pmatrix} M \otimes B(K) & M \otimes K \\ (M \otimes K)^* & M \end{pmatrix}.$$

Define the action $(\gamma_g)$ of $G$ on $N$ by

$$\gamma_g \begin{pmatrix} a & b \\ e & f \end{pmatrix} = \begin{pmatrix} c(g)(\alpha_g \otimes \text{id})(a)c(g)^* & c(g)(\alpha_g \otimes \text{id})(b) \\ (\alpha_g \otimes \text{id})(e)c(g)^* & \alpha_g(f) \end{pmatrix}.$$

Define $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $q = 1 - p$ and $w_n = \begin{pmatrix} 0 & v_n \\ 0 & 0 \end{pmatrix}$. Then, $w_n$ is a sequence of partial isometries in the fixed point algebra $N^G$, satisfying $w_n^* p N^G q w_n \to q$ strongly. It follows that $q < p$ in
the von Neumann algebra $N^G$. So, we find $v \in M \otimes K$ satisfying $c(g)(\alpha_g \otimes \text{id})(v) = v$ for all $g$ and $v^*v = 1$. Hence, $G \rtimes M$ has property (T).

Suppose conversely that $G \rtimes M$ has property (T). Let $\pi : G \to U(K)$ be a strongly continuous unitary representation of $G$ admitting $\xi_n$ as a sequence of almost invariant unit vectors. By [3 Theorem 2.12.9], it is sufficient to prove that $\pi$ has a non-zero finite dimensional $\pi(G)$-invariant subspace. Define $c : G \rightarrow U(M \otimes B(K)) : c(g) = 1 \otimes \pi(g)$. Obviously, $c$ is a 1-cocycle of $G \rtimes M$ having $1 \otimes \xi_n$ as a sequence of almost invariant unit vectors. By property (T) of $G \rtimes M$, we find $\xi \in M \otimes K$ satisfying $\xi^*\xi = 1$ and $c(g)(\alpha_g \otimes \text{id})(\xi) = \xi$ for all $g \in G$. Denoting again by $u : g \mapsto u_g$ the representation of $G$ on $L^2(M, \tau)$ obtained by extending $\alpha_g$, we find that $u \otimes \pi$ admits an invariant unit vector $\xi$. Identify $L^2(M, \tau) \otimes K$ with the Hilbert space of Hilbert-Schmidt operators from $L^2(M, \tau)$ to $K$. Then, $T := \xi \xi^*$ is a non-zero trace-class operator on $K$ satisfying $\pi(g)T\pi(g)^* = T$ for all $g \in G$. So, for $\varepsilon > 0$ sufficiently small, the spectral projection $\chi_{[\varepsilon, +\infty)}(T)$ projects onto a non-zero finite dimensional $\pi(G)$-invariant subspace of $K$. \hfill \Box

The following is a slight generalization of [2 Theorem 5.3]. When $H \rtimes M$ is an action, we denote by $M^H$ the von Neumann algebra of $H$-fixed points.

**Lemma 3.2.** Let $G \acts M$ be an action of the l.c.s.c. group $G$ on the von Neumann algebra $M$. Let $H \triangleleft G$ be a closed normal subgroup and assume that there exists a \*-(isomorphism $\theta : M \to L^\infty(H)\overline{\otimes} M^H$ satisfying $\theta \circ \alpha_h = (\rho_h \otimes \text{id}) \circ \theta$ for all $h \in H$, where $\rho_h$ denotes the right translation by $h$ on $L^\infty(H)$. Then, $G \rtimes M$ has property (T) if and only if $G/H \rtimes M^H$ has property (T).

**Proof.** We say that two 1-cocycles $c_1, c_2$ of $G \acts M$ with values in $U(K)$ are unitarily equivalent if there exists a unitary $v \in U(M \otimes B(K))$ satisfying $c_1(g) = vc_2(g)(\alpha_g \otimes \text{id})(v^*)$ for all $g \in G$. We denote by $H^1(G \acts M, U(K))$ the set of equivalence classes of 1-cocycles.

In the first part of the proof, we show that the obvious map

$$\Theta : H^1(G/H \rtimes M^H, U(K)) \to H^1(G \rtimes M, U(K)) : \Theta(c) = c \circ \pi \quad \text{with} \quad \pi : G \to G/H,$$

is a bijection. In the second part of the proof, we show that this map and its inverse preserve the property of having invariant, resp. almost invariant, vectors. Both parts together show that property (T) of $G \rtimes M$ is equivalent with property (T) of $G/H \rtimes M^H$.

It is straightforward to check that $\Theta$ is well defined and injective. Suppose that $c : G \to U(M \otimes B(K))$ is a 1-cocycle of $G \acts M$. In order to prove that $c$ is in the range of $\Theta$, it suffices to prove that $c$ is unitarily equivalent with $c'$ satisfying $c'(h) = 1$ for all $h \in H$. Identify throughout $U(M \otimes B(K))$ with $U(L^\infty(H)\overline{\otimes} M^H \otimes B(K))$ and view the latter as measurable functions $H \to U(M^H \otimes B(K))$, modulo equality almost everywhere. By [13 Theorem 1], take a measurable map $\varphi : H \times H \to U(M^H \otimes B(K))$ such that $c(h) = \varphi(\cdot, h)$ for all $h \in H$. Since $c$ is a cocycle, we find that

$$\varphi(k, hg) = \varphi(k, h)\varphi(kh, g) \quad \text{for almost all} \quad (k, h, g) \in H \times H \times H.$$

By the Fubini theorem, take $k_0 \in H$ such that for almost all $(h, g) \in H \times H$, the previous equality holds for $(k_0, h, g)$. Define the unitary $v \in U(M \otimes B(K))$ as $v = \varphi(k_0, k_0^{-1})^*$ and set $c'(g) = vc(g)(\alpha_g \otimes \text{id})(v^*)$. By construction, $c'(g) = 1$ for almost all $g \in H$ and hence for all $g \in H$ by continuity.

It is an exercise to check that the 1-cocycle $c \in H^1(G/H \rtimes M^H, U(K))$ has a unit invariant vector if and only if $\Theta(c)$ has. Also, a sequence of almost invariant unit vectors for $c$ defines a sequence
of almost invariant unit vectors for $\Theta(c)$. Finally, suppose that $\xi_n \in M\bar{\otimes}K$ is a sequence of almost invariant unit vectors for $\Theta(c)$. In order to conclude the proof of the lemma, it suffices to show that there exists a sequence $\eta_n \in M^H\bar{\otimes}K$ satisfying $\eta_n^*\eta_n = 1$ for all $n$ and $\xi_n - \eta_n \to 0$ *-strongly.

Define the sets $X = \{v \in M\bar{\otimes}K \mid v^*v = 1\}$ and $Y = \{v \in M^H\bar{\otimes}K \mid v^*v = 1\}$. Through the isomorphism $M \cong L^\infty(H)\bar{\otimes}M^H$, we identify $X$ with the set of measurable functions from $H$ to $Y$ (modulo equality almost everywhere). Take a bounded metric $d_0$ on $Y$ inducing the strong* topology. Let $\mu$ be a probability measure on $H$ in the same measure class as the Haar measure. Following [13, page 5], define the metric $d$ on $X$ by

$$d(v, w) = \int_H d_0(v(h), w(h))d\mu(h).$$

Then, $d$ induces the strong* topology on $X$. It is easy to check that when $v_n, w_n \in X$ such that $d(v_n, w_n)$ is summable, then $v_n(h) - w_n(h) \to 0$ *-strongly for almost every $h \in H$ (see [13, Proposition 6]).

View $Y \subset X$ as constant functions. We have to prove that $d(\xi_n, Y) \to 0$. Suppose the contrary. Write $H$ as an increasing union of compact subsets $H_n$. After passage to a subsequence, we find $\varepsilon > 0$ such that $d(\xi_n, Y) > \varepsilon$ for all $n$ and such that

$$d(\xi_n, (\alpha_g \otimes \text{id})(\xi_n)) < 2^{-n} \quad \text{for all } n \in \mathbb{N}, g \in H_n.$$

It follows that for all $g \in H$, we have

$$d_0(\xi_n(h), \xi_n(hg)) \to 0 \quad \text{for almost all } h \in H.$$

By the Fubini theorem, take $h_0 \in H$ such that $d_0(\xi_n(h_0), \xi_n(h_0g)) \to 0$ for almost all $g \in H$. It follows that $d(\xi_n, \xi_n(h_0)) \to 0$, contradicting the assumption that $d(\xi_n, Y) > \varepsilon$ for all $n$.

**Proposition 3.3.** Let $G$ be a l.c.s.c. group with closed subgroups $H_1, H_2$. Then, $H_1 \curvearrowright L^\infty(G/H_2)$ has property (T) if and only if $H_2 \curvearrowright L^\infty(G/H_1)$ has property (T).

**Proof.** Set $M = L^\infty(G)$ and $G = H_1 \times H_2$ acting by left-right translations on $M$:

$$(\alpha_{(g,h)}(F))(x) = F(g^{-1}xh) \quad \text{for all } F \in L^\infty(G), x \in G, g \in H_1, h \in H_2.$$

We apply Lemma 3.2 to $G \curvearrowright M$ and the closed normal subgroups $H = H_i$, $i = 1, 2$ of $G$. By the Effros-Mackey theorem (see e.g. [11, Theorem II.12.17]), the quotient map $G \to G/H$ admits a Borel lifting and hence, there exists an $H$-equivariant isomorphism $M \cong L^\infty(H)\bar{\otimes}M^H$. So, by Lemma 3.2 property (T) of $G \curvearrowright M$ is equivalent with property (T) of $H_1 \curvearrowright L^\infty(G/H_2)$ as well as with property (T) of $H_2 \curvearrowright L^\infty(G/H_1)$.

4 The lattice actions $\Gamma \curvearrowright \mathbb{R}^n$ have property (T)

Recall from [5] that if $\Gamma \curvearrowright (X, \mu)$ is a free ergodic p.m.p. action, then property (T) of $\mathcal{R}(\Gamma \curvearrowright X)$ in the sense of Zimmer, is equivalent with property (T) of the group $\Gamma$ (see also [20, Proposition 2.4] and Proposition 3.1 above).

If $\Gamma$ is a property (T) group, the fundamental group of $\mathcal{R}(\Gamma \curvearrowright X)$ is countable for any free ergodic p.m.p. action (see [9, Corollary 1.8] if $\Gamma$ is moreover ICC and see [10, Theorem 5.9] for the general case).
But more is true: we proved in [17, Theorem 6.1] that the fundamental group of a II\textsubscript{1} equivalence relation \( R \) on \((X, \mu)\) is countable whenever the full group \([R]\) contains a property (T) group that implements an ergodic action on \((X, \mu)\). As a result, the following theorem is rather surprising: we obtain a II\textsubscript{1} equivalence relation \( R \) with property (T) and fundamental group \([R]\); hence, none of the \( R^t \) can be implemented by a free action of a group and none of the \( R^t \) can be implemented by a possibly non-free action of a property (T) group.

**Theorem 4.1.** Let \( \Gamma < SL(n, \mathbb{R}) \) be a lattice and let \( R \) be the II\textsubscript{1} equivalence relation obtained by restricting the orbit equivalence relation \( R(\Gamma \curvearrowright \mathbb{R}^n) \) to a set of Lebesgue measure 1. If \( n \geq 4 \), the equivalence relation \( R \) has property (T) in the sense of Zimmer, but nevertheless \( \mathcal{F}(R) = \mathbb{R}_+ \). In particular,

- none of the equivalence relations \( R^t, t > 0 \), can be implemented by a free action of a group,
- none of the equivalence relations \( R^t, t > 0 \), can be implemented by a possibly non-free action of a property (T) group.

**Proof.** Proving property (T) of \( R \) amounts to proving property (T) for the action \( \Gamma \curvearrowright L^\infty(\mathbb{R}^n) \).

Define the l.c.s.c. group \( G = SL(n, \mathbb{R}) \) and set \( H_1 = \Gamma \). Consider the linear action \( G \curvearrowright \mathbb{R}^n \) and set \( H_2 = \{ A \in SL(n, \mathbb{R}) \mid Ae_1 = e_1 \} \), where \( e_1 \) denotes the first basis vector of \( \mathbb{R}^n \). By construction, the action \( \Gamma \curvearrowright L^\infty(\mathbb{R}^n) \) can be viewed as \( H_1 \curvearrowright L^\infty(G/H_2) \). Hence, by Proposition 3.3 property (T) for this last action is equivalent with property (T) of \( H_2 \curvearrowright L^\infty(G/H_1) \). This action admits a finite invariant measure, because \( H_1 \) is a lattice in \( G \). Moreover, \( H_2 \cong SL(n-1, \mathbb{R}) \rtimes \mathbb{R}^{n-1} \), which has property (T) for \( n \geq 4 \). So, it follows from Proposition 3.1 that \( H_2 \curvearrowright L^\infty(G/H_1) \) has property (T).

The action on \( \mathbb{R}^n \) by multiples of the identity matrix scales the Lebesgue measure and commutes with the action of \( \Gamma \). Hence, the fundamental group of \( R \) equals \( \mathbb{R}_+ \). The statements about implementing \( R^t \) by group actions, follow from the discussion preceding the theorem.

Note that in the case \( n = 2 \), a similar reasoning yields the following result of [1]: the action \( SL(2, \mathbb{Z}) \curvearrowright \mathbb{R}^2 \) is amenable and hence, \( L^\infty(\mathbb{R}^2) \rtimes SL(2, \mathbb{Z}) \) is isomorphic with the unique hyperfinite II\textsubscript{\infty} factor.

### 5 Cocycle and OE superrigidity theorems

#### 5.1 Proof of Theorem 1.3

We prove in this section the cocycle superrigidity theorem 1.3 as a consequence of the more general Theorem 5.3 below.

We do not know whether \( SL(n, \mathbb{R}) \curvearrowright \mathbb{R}^n \) is \( U_{\text{fin}} \)-cocycle superrigid for \( n = 3, 4 \). On the other hand, some condition is needed on the Polish group in which the 1-cocycle takes its values. Indeed, almost by construction, we have the following result, that we also prove at the end of this subsection.

**Proposition 5.1.** Let \( n \geq 3 \). The action \( SL(n, \mathbb{Z}) \curvearrowright \mathbb{R}^n \) admits a 1-cocycle with values in \( SL(n-1, \mathbb{R}) \times \mathbb{R}^{n-1} \) that is not cohomologous to a group morphism.

Recall from [15] the following definition of \( s \)-malleability of a measure preserving action.
Definition 5.2. Let $\Gamma$ be a locally compact second countable (l.c.s.c.) group and $\Gamma \curvearrowright (X, \mu)$ an action preserving the finite or infinite measure $\mu$. The action is called $s$-malleable if there exists

- a one-parameter group $(\alpha_t)_{t \in \mathbb{R}}$ of measure preserving transformations of $X \times X$,
- an involutive measure preserving transformation $\beta$ of $X \times X$,

such that

- $\alpha_t$ and $\beta$ commute with the diagonal action $\Gamma \curvearrowright X \times X$,
- $\alpha_1(x, y) \in \{y\} \times X$ for almost all $(x, y) \in X \times X$,
- $\beta(x, y) \in \{x\} \times X$ for almost all $(x, y) \in X \times X$,
- $\alpha_t \circ \beta = \beta \circ \alpha_{-t}$ for all $t \in \mathbb{R}$.

Theorem 0.1 in [15] says the following. Let $\Gamma \curvearrowright (X, \mu)$ be an $s$-malleable, probability measure preserving action and $\Lambda < \Gamma$ a normal subgroup with the relative property (T) such that the restriction of $\Gamma \curvearrowright (X, \mu)$ to $\Lambda$ is weakly mixing. Then every 1-cocycle of $\Gamma \curvearrowright (X, \mu)$ with values in a Polish group of finite type, is cohomologous to a group morphism.

Recall here that one of the equivalent formulations of weak mixing for a p.m.p. action $\Lambda \curvearrowright (X, \mu)$ is the ergodicity of the diagonal action $\Lambda \curvearrowright (X \times X, \mu \times \mu)$. If $\Lambda \curvearrowright (X, \mu)$ is a weakly mixing p.m.p. action and $\Gamma \curvearrowright (Y, \eta)$ is any ergodic p.m.p. action, then the diagonal action $\Gamma \curvearrowright X \times Y$ is ergodic. In particular, the diagonal action

$$\Gamma \curvearrowright \underbrace{X \times \cdots \times X}_{k \text{ times}}$$

is ergodic for every $k$, once it is ergodic for $k = 2$. For infinite measure preserving actions, things are more complicated and, for instance, the diagonal action

$$\text{SL}(n, \mathbb{Z}) \curvearrowright \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k \text{ times}}$$

is ergodic if and only if $k \leq n - 1$. This partially explains the formulation of the following result.

Theorem 5.3. Let $\Gamma \curvearrowright (X, \mu)$ be an infinite measure preserving, $s$-malleable action. Assume that

- the diagonal action $\Gamma \curvearrowright X \times X$ has property (T),
- the 4-fold diagonal action $\Gamma \curvearrowright X \times X \times X \times X$ is ergodic.

Then, $\Gamma \curvearrowright (X, \mu)$ is $U_{\text{fin}}$-cocycle superrigid.

The proof of Theorem 5.3 follows entirely the setup of the proof of [15 Theorem 0.1]. But, one has to be careful at those places in [15] where weak mixing is applied. To proper way to deal with these issues, lies in the following lemma distilled from the proof of [7 Lemmas 3.1].
Lemma 5.4. Let $(Z,d)$ be a Polish space with separable complete metric $d$ and $(\alpha_g)_{g \in \mathcal{G}}$ a continuous action of a Polish group $\mathcal{G}$ by homeomorphisms of $Z$. Assume that $d$ is $(\alpha_g)_{g \in \mathcal{G}}$-invariant.

Let $\Gamma$ be a l.c.s.c. group and $\Gamma \actson (X,\mu)$, $\Gamma \actson (Y,\eta)$ non-singular actions. Let $F : X \times Y \to Z$ be a measurable map satisfying

$$F(g \cdot x, g \cdot y) = \alpha_{\omega(g,x)}(F(x,y))$$

for almost all $(x,y) \in X \times Y$, $g \in \Gamma$, where $\omega : \Gamma \times X \to \mathcal{G}$ is some measurable map.

Consider the diagonal action $\Gamma \actson X \times Y \times Y$ and assume that $L^\infty(X \times Y \times Y)^\Gamma = L^\infty(X)^\Gamma \otimes 1 \otimes 1$ (which holds in particular if the diagonal action $\Gamma \actson X \times Y \times Y$ is ergodic). Then, there exists a measurable $H : X \to Z$ with $F(x,y) = H(x)$ for almost all $(x,y) \in X \times Y$.

Proof. Define the map $G : X \times Y \times Y \to \mathbb{R} : G(x,y,z) = d(F(x,y), F(x,z))$. Since $d$ is $(\alpha_g)_{g \in \mathcal{G}}$-invariant, the map $G$ is invariant under the diagonal $\Gamma$-action. By our assumption, $G(x,y,z) = G_0(x)$ for almost all $(x,y,z)$ and some measurable map $G_0 : X \to \mathbb{R}_+$. We claim that $G_0(x) = 0$ for almost all $x \in X$. Let $\delta > 0$ and assume that $G_0(x) \geq \delta$ for all $x$ in a non-negligible subset $U$ of $X$. Cover $Z$ by a sequence $(B_n)_{n \in \mathbb{N}}$ of balls of diameter strictly smaller than $\delta$. Write, for every $x \in X$, $F_x : y \mapsto F(x,y)$. By the Fubini theorem, for almost every $x \in U$, we have $G(x,y,z) \geq \delta$ for almost all $(y,z) \in Y \times Y$. Hence, for almost every $x \in U$, we have that $F_x^{-1}(B_n)$ is negligible for every $n$, which is absurd.

So, $G(x,y,z) = 0$ for almost all $(x,y,z) \in X \times Y \times Y$. Again by the Fubini theorem, take $z \in Y$ such that $d(F(x,y), F(x,z)) = 0$ for almost all $(x,y) \in X \times Y$. Putting $H(x) := F(x,z)$, we are done.

Proof of Theorem 5.3. Let $N$ be a $\Pi_1$ factor and $\mathcal{G} \subset \mathcal{U}(N)$ a closed subgroup. Let $\omega : \Gamma \times X \to \mathcal{G}$ be a 1-cocycle, meaning that for all $g,h \in \Gamma$, we have

$$\omega(gh,x) = \omega(g,h \cdot x) \omega(h,x) \quad \text{for almost all } x \in X.$$ 

Define the following 1-cocycles for the diagonal action $\Gamma \actson X \times X$.

$$\omega_0 : \Gamma \times X \times X \to \mathcal{G} : \omega_0(g,x,y) = \omega(g,x) \quad \text{and} \quad \omega_l : \Gamma \times X \times X \to \mathcal{G} : \omega_l(g,x,y) = \omega_0(g,\alpha_l(x,y)).$$

Define the action $(\rho_g)_{g \in \Gamma}$ of $\Gamma$ by automorphisms of $L^\infty(X) \overline{\otimes} N$ by the formula

$$(\rho_g^{-1}(F))(x) = \omega(g,x)^* F(g \cdot x) \omega(g,x).$$

Denote by $B$ the von Neumann subalgebra of $(\rho_g)_{g \in \Gamma}$-fixed points.

Claim. Whenever $p$ is a non-zero projection in $B$, there exists a measurable function $\varphi : X \times X \to N$ and a non-zero projection $q \in B$ such that $q \leq p$ and such that for all $g \in \Gamma$, we have

$$\omega(g,x) \varphi(x,y) = \varphi(g \cdot x, g \cdot y) \omega(g,y) \quad , \quad \varphi(x,y) \varphi(x,y)^* = q(x) \quad , \quad \varphi(x,y)^* \varphi(x,y) = q(y)$$

for almost all $(x,y) \in X \times X$.

Proof of the claim. Since $p$ is $(\rho_g)_{g \in \Gamma}$-invariant, the function $x \mapsto \tau(p(x))$ is $\Gamma$-invariant and hence constantly equal to $0 < \lambda \leq 1$. Let $p_0 \in P$ be a projection with $\tau(p_0) = \lambda$. It follows that, inside $L^\infty(X) \overline{\otimes} N$, the projections $p$ and $1 \otimes p_0$ are equivalent. Take a partial isometry $v \in L^\infty(X) \overline{\otimes} N$ such that $v^* v = p$ and $v v^* = 1 \otimes p_0$. Define $\eta(g,x) = v(g \cdot x) \omega(g,x) v(x)^*$ and note that $\eta$ is a 1-cocycle for $\Gamma \actson X$ with values in $\mathcal{U}(p_0 N p_0)$. Set $\eta_0(g,x,y) = \eta(g,x)$ and, for all $n \geq 1$, $\eta_n(g,x,y) = \eta_0(g,\alpha_{2^{-n}}(x,y))$. 

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Define the Hilbert space \( K = \bigoplus_{k=1}^{\infty} p_{0} L^{2}(N)p_{0} \). We define the following 1-cocycle of \( \Gamma \triangleleft X \times X \) with values in the unitary group \( \mathcal{U}(K) \) of \( K \).

\[
(c(g, x, y)\xi)_{k} = \eta(g, x) \xi_{k} \eta_{k}(g, x, y)^{*}.
\]

Define the map \( \xi_{n} : X \rightarrow (K)_{1} \) by the formula

\[
(\xi_{n}(x))_{k} = \begin{cases} 
\tau(p_{0})^{-1/2} & \text{if } k = n, \\
0 & \text{if } k \neq n.
\end{cases}
\]

One checks that \( \xi_{n} \) is a sequence of almost invariant unit vectors. Since \( \Gamma \triangleleft X \times X \) has property (T), we find a unit invariant vector, i.e. a measurable map \( \xi : X \times X \rightarrow K \) with \( \|\xi(x, y)\| = 1 \) for almost all \((x, y)\) and, for all \( g \in \Gamma \),

\[
\xi(g \cdot x, g \cdot y) = c(g, x, y) \xi(x, y)
\]

almost everywhere. It follows that

\[
\xi(g \cdot x, g \cdot y) = \eta(g, x) \xi_{k}(x, y) \eta_{k}(g, x, y)^{*}
\]

almost everywhere. In particular, for every \( k \), the function \((x, y) \mapsto \|\xi(x, y)\|\) is \( \Gamma \)-invariant and hence, constant. Since \( \|\xi(x, y)\| = 1 \) for almost all \((x, y)\), we can pick \( k \) such that \( \|\xi(x, y)\| \neq 0 \) for almost all \((x, y)\). Taking the polar decomposition of \( \xi(x, y)_{k} \), we find a non-zero partial isometry \( \psi \in L^{\infty}(X \times X) \otimes p_{0} N p_{0} \) satisfying

\[
\psi(g \cdot x, g \cdot y) \eta_{k}(g, x, y) = \eta(g, x) \psi(x, y)
\]

almost everywhere. We set \( \varphi_{0}(x, y) := v(x)^{*}\psi(x, y)v_{k}(x, y) \), where \( v_{0}(x, y) = v(x) \) and \( v_{k}(x, y) = v(\alpha_{2^{-k}}(x, y)) \). It follows that

\[
\omega(g, x) \varphi_{0}(x, y) = \varphi_{0}(g \cdot x, g \cdot y) \omega_{t_{0}}(g, x, y)
\]

where \( t_{0} = 2^{-k} \).

Set \( r(x, y) = \varphi_{0}(x, y)\varphi_{0}(x, y)^{*} \). It follows that

\[
r(g \cdot x, g \cdot y) = \omega(g, x) r(x, y) \omega(g, x)^{*}
\]

almost everywhere. By Lemma 5.4 we find a projection \( q \in L^{\infty}(X) \otimes N \) such that \( r(x, y) = q(x) \) almost everywhere. Then, \( q \) is a non-zero projection in \( B \) and \( q \leq p \). Also, \( \varphi_{0}(x, y)\varphi_{0}(x, y)^{*} = q(x) \) almost everywhere.

Set \( q_{0}(x, y) = q(x) \) and \( q_{k}(x, y) = q(\alpha_{k}(x, y)) \). We now construct \( \varphi_{1} : X \times X \rightarrow N \) such that \( \varphi_{1}(x, y)\varphi_{1}(x, y)^{*} = q(x) \), \( \varphi_{1}(x, y)^{*}\varphi_{1}(x, y) = q_{2t_{0}}(x, y) \) and

\[
\omega(g, x) \varphi_{1}(x, y) = \varphi_{1}(g \cdot x, g \cdot y) \omega_{2t_{0}}(g, x, y)
\]

almost everywhere. Continuing the same procedure \( k \) times (remember that \( t_{0} = 2^{-k} \)), we will have found \( \varphi = \varphi_{k} : X \times X \rightarrow N \) satisfying \( \varphi(x, y)\varphi(x, y)^{*} = q(x) \), \( \varphi(x, y)^{*}\varphi(x, y) = q_{1}(x, y) = q(y) \) and

\[
\omega(g, x) \varphi(x, y) = \varphi(g \cdot x, g \cdot y) \omega_{1}(g, x, y) = \varphi(g \cdot x, g \cdot y) \omega(g, y)
\]

hence proving the claim. In fact, it suffices to take

\[
\varphi_{1}(x, y) = \varphi_{0}(x, y)\varphi_{0}(\beta(\alpha_{2t_{0}}(x, y)))^{*}
\]

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and to use that \((\alpha_t)_{t\in \mathbb{R}}\) is a one-parameter group, \(\beta \circ \alpha_t = \alpha_{-t} \circ \beta\) and \(\beta(x, y) \in \{x\} \times Y\) for almost all \((x, y)\). So, the claim above has been proven.

Using the claim and a maximality argument, we find a measurable function \(\varphi : X \times X \to \mathcal{U}(N)\) such that

\[
\omega(g, x) \varphi(x, y) = \varphi(g \cdot x, g \cdot y) \omega(g, y)
\]

almost everywhere. Set \(H(x, y, z) = \varphi(x, y)\varphi(y, z)\). It follows that

\[
H(g \cdot x, g \cdot y, g \cdot z) = \omega(g, x) H(x, y, z) \omega(g, z)^*
\]

for almost all \((x, y, z)\). By Lemma 5.4 and because the 4-fold diagonal action \(\Gamma \acts X \times X \times X \times X\) is ergodic, \(H\) is essentially independent of its second variable. So, we find a measurable \(F : X \times X \to \mathcal{U}(N)\) such that \(\varphi(x, y) = F(x, z)\varphi(y, z)^*\) for almost every \((x, y, z)\). By the Fubini theorem, take \(z \in X\) such that the previous formula holds for almost all \((x, y)\). Set \(\psi(x) = F(x, z)^*\) and \(G(y) = \varphi(y, z)^*\). It follows that

\[
\psi(g \cdot x) \omega(g, x) \psi(x)^* = G(g \cdot y) \omega(g, y) G(y)^*
\]

almost everywhere. Hence, the left-hand side is independent of \(x\) and we have found a group morphism \(\delta : G \to \mathcal{U}(N)\) and a measurable map \(\psi : X \to \mathcal{U}(N)\) such that

\[
\omega(g, x) = \psi(g \cdot x)^* \delta(g) \psi(x)
\]

almost everywhere.

Consider the quotient Polish space \(\mathcal{U}(N)/\mathcal{G}\) with the induced metric, which is invariant under left multiplication by elements of \(\mathcal{U}(N)\). Write \(F : X \to \mathcal{U}(N)/\mathcal{G} : F(x) = \psi(x)\mathcal{G}\). It follows that \(F(g \cdot x) = \delta(g)F(x)\) almost everywhere. By Lemma 5.4 \(F\) is essentially constant. So, we find a unitary \(u \in \mathcal{U}(N)\) and a measurable map \(w : X \to \mathcal{G}\) such that \(\psi(x) = uw(x)\) almost everywhere. Replacing \(\delta\) by \(u^*\delta(\cdot)u\), it follows that \(\omega(g, x) = w(g \cdot x)^*\delta(g)w(x)\) almost everywhere. In particular, \(\delta(g) \in \mathcal{G}\) and we are done.

As a principle (cf. [15, Proposition 3.6]), once the restriction of a 1-cocycle \(\omega : G \times X \to \mathcal{G}\) to a closed subgroup \(H < G\), is cohomologous to a group morphism \(H \to \mathcal{G}\) and if \(H\) is sufficiently normal in \(G\) and acts sufficiently mixingly on \(X\), the entire 1-cocycle \(\omega\) is cohomologous to a group morphism \(G \to \mathcal{G}\). In our setting, we need the following.

**Lemma 5.5.** Let \(G \acts (X, \mu)\) be a non-singular action of the l.c.s.c. group \(G\) and \(\omega : G \times X \to \mathcal{G}\) a 1-cocycle with values in the Polish group of finite type \(\mathcal{G}\). Let \(H < G\) be a closed subgroup and assume that \(\omega|_H\) is cohomologous to a group morphism \(H \to \mathcal{G}\). If for every \(g \in G\), the diagonal action of the group \(H \acts gHg^{-1}\) on \(X \times X\) is ergodic, then \(\omega\) is cohomologous to a morphism \(G \to \mathcal{G}\).

**Proof.** We may assume that for every \(h \in H\), we have \(\omega(h, x) = \delta(h)\) for almost every \(x \in X\), where \(\delta : H \to \mathcal{G}\) is a continuous group morphism. Let \(g \in G\) and put \(F(x) = \omega(g, x)\). Using the cocycle equation, it follows that for all \(h \in H \acts g^{-1}Hg\), we have \(F(h \cdot x) = \delta(ghg^{-1})F(x)\delta(h)^{-1}\) almost everywhere. By Lemma 5.4 \(F\) is essentially constant. So, we have shown that for every \(g \in G\), the map \(x \mapsto \omega(g, x)\) is essentially constant. It follows that \(\omega(g, x) = \delta(g)\).

After showing the following lemma, we can prove Theorem [13] and Proposition 5.1.
Lemma 5.6. Let $\Gamma$ be a lattice in $\text{SL}(n, \mathbb{R})$ and consider the linear action of $\Gamma$ on $\mathbb{R}^n$. Let

$$\Gamma \acts X^{(k)} := \mathbb{R}^n \times \cdots \times \mathbb{R}^n \text{ \(k\) times}$$

be the $k$-fold diagonal action. Then, $\Gamma \acts X^{(k)}$

- is ergodic if and only if $k \leq n - 1$,
- has property (T) if and only if $k \leq n - 3$ or $k \geq n$ (the latter part being as interesting as the trivial group having property (T)).

Proof. Writing the elements of $\mathbb{R}^n$ as column vectors, identify, up to measure zero, $X^{(n)}$ with $\text{GL}(n, \mathbb{R})$, with the $\Gamma$-action given by left multiplication. The determinant function is invariant and not essentially constant, proving that $\Gamma \acts X^{(k)}$ is non-ergodic for $k \geq n$. It also follows that for $k \geq n$, the action $\Gamma \acts X^{(k)}$ is essentially free and proper. Hence, it has property (T) because the trivial group has property (T).

Let now $k \leq n - 1$. Denoting by $(e_i)_{i=1,\ldots,n}$ the standard basis vectors in $\mathbb{R}^n$, the orbit of $(e_1, \ldots, e_k) \in X^{(k)}$ under the diagonal $\text{SL}(n, \mathbb{R})$-action has complement of measure zero, so that we can identify $\Gamma \acts X^{(k)}$ with $\Gamma \acts \text{SL}(n, \mathbb{R})/H$, where $H = \{ A \in \text{SL}(n, \mathbb{R}) \mid Ae_i = e_i \text{ for all } i = 1, \ldots, k \}$. Observe that $H \cong \text{SL}(n-k, \mathbb{R}) \times \text{M}_{n-k}(\mathbb{R})$, where $\text{M}_{n-k}(\mathbb{R})$ denotes the additive group of $(n-k) \times (n-k)$ matrices on which $\text{SL}(n-k, \mathbb{R})$ acts by multiplication. By Moore’s ergodicity theorem (see e.g. [21, Theorem 2.2.6]), $H \acts \Gamma \text{SL}(n, \mathbb{R})$ is ergodic and hence, $\Gamma \acts \text{SL}(n, \mathbb{R})/H$ is ergodic.

By Proposition 5.3 property (T) of $\Gamma \acts \text{SL}(n, \mathbb{R})/H$ is equivalent with property (T) of $H \acts \Gamma \text{SL}(n, \mathbb{R})$. By Proposition 5.1, the latter is equivalent with the group $H$ having property (T), which is in turn equivalent with $n-k \geq 3$. \hfill \Box

Proof of Theorem 7.3 Observe that part 1 is a special case of part 3, by taking $k = 1$ and $H = \{1\}$. We start by proving part 3. Set $X = \text{M}_{n,k}(\mathbb{R})$. The action $\Gamma \acts X$ by left multiplication is $s$-malleable. It suffices to take

$$\alpha_t(A, B) = (\cos(\pi t/2)A + \sin(\pi t/2)A, -\sin(\pi t/2)A + \cos(\pi t/2)B) \text{ and } \beta(A, B) = (A, -B)$$

whenever $t \in \mathbb{R}$ and $A, B \in \text{M}_{n,k}(\mathbb{R})$.

Moreover, $\Gamma \acts X$ can be viewed as the $k$-fold diagonal action $\Gamma \acts \mathbb{R}^n \times \cdots \times \mathbb{R}^n$. By Lemma 5.6 and because $n \geq 4k + 1$, the diagonal action $\Gamma \acts X \times X \times X$ has property (T) and the 4-fold diagonal action $\Gamma \acts X \times X \times X \times X$ is ergodic. So, by Theorem 5.3 $\Gamma \acts X$ is $\mathcal{U}\text{fin}$-cocycle superrigid. Since the diagonal action $\Gamma \acts X \times X \times X$ is ergodic and $\Gamma$ is a normal subgroup of $G$, Lemma 5.5 implies that $G \acts X$ is $\mathcal{U}\text{fin}$-cocycle superrigid.

It remains to prove part 2 of the theorem. By part 1, we already know $\Gamma \acts \mathbb{R}^n$ is $\mathcal{U}\text{fin}$-cocycle superrigid. Define, for every $x \in \mathbb{Z}^n$, $\Gamma_x = \{ g \in \Gamma \mid gx = x \}$. We claim that the diagonal action $\Gamma_x \acts \mathbb{R}^n \times \mathbb{R}^n$ is ergodic for every $x \in \mathbb{Z}^n$. Once this claim is proven, Lemma 5.5 implies that $\Gamma \acts \mathbb{Z}^n \times \mathbb{R}^n$ is $\mathcal{U}\text{fin}$-cocycle superrigid.

For $x = 0$, the claim follows from Lemma 5.6. Let now $x \neq 0$. Define the closed subgroup $H = \{ g \in \text{SL}(n, \mathbb{R}) \mid ge_1 = e_1 \}$ of $\text{SL}(n, \mathbb{R})$. Exactly as in the proof of Lemma 5.6 when $n \geq 4$, the diagonal action $\Lambda \acts \mathbb{R}^n \times \mathbb{R}^n$ of any lattice $\Lambda \subset H$ is ergodic. Take $g_0 \in \text{SL}(n, \mathbb{Q})$ with
The following actions: The image of any group morphism $\text{SL}(n, xH)$ for an elementary argument). So, we have found a finite index subgroup $\Gamma$ of $\Lambda$. Hence, we can choose the lifting $\theta$ whenever $g \in \Gamma$ and $xH \in G/H$. Assume that $\omega$ is cohomologous to a group morphism $\delta : \Gamma \to \mathbb{R}$. This means that we can choose the lifting $\theta$ such that $g\theta(xH) = \theta(gxH)\omega(g, xH)$ whenever $g \in \Gamma$ and $xH \in G/H$.

The image of any group morphism $\text{SL}(n, Z) \to \text{SL}(n-1, R) \times R^{n-1}$ is finite (see [18, Theorem 6] for an elementary argument). So, we have found a finite index subgroup $\Gamma_0 \subset \text{SL}(n, Z)$ and a measurable map $\theta : R^n \to \text{SL}(n, R)$ such that $\theta(gx) = g\theta(x)$ for all $g \in \Gamma_0$ and almost all $x \in R^n$. It follows that the map $(x, y) \mapsto \theta(x)^{-1}\theta(y)$ is invariant under the diagonal $\Gamma_0$-action, which is ergodic by Lemma 5.6. Hence, $\theta$ is essentially constant, which is a contradiction with the formula $\theta(gx) = g\theta(x)$.

5.2 Proof of Theorems 1.5 and 1.6

We will deduce Theorems 1.5 and 1.6 from the general Theorem 5.8 below, dealing with arbitrary actions of the form $\Gamma \acts M_{n,k}(R)/H$, where $\Gamma$ is a lattice and $H < \text{GL}(k, R)$ a closed subgroup. First of all, observe that these actions are essentially free and ergodic.

Lemma 5.7. Let $n > k$ and $\Gamma < \text{SL}(n, R)$ any lattice. Let $H < \text{GL}(k, R)$ be a closed subgroup. If $(-1, -1) \in \Gamma \times H$, put $\Gamma_0 = \Gamma/\{\pm 1\}$, otherwise put $\Gamma = \Gamma_0$.

The action $\Gamma_0 \acts M_{n,k}(R)/H$ is essentially free and ergodic. It never admits an invariant probability measure. It admits an infinite invariant measure if and only if $H$ is unimodular and satisfies det $g = \pm 1$ for all $g \in H$.

Proof. By Lemma 5.6 $\Gamma \acts M_{n,k}(R)$ is ergodic, because $k < n$. A fortiori, $\Gamma \acts M_{n,k}(R)/H$ is ergodic.

Denote $V = \text{span}\{e_1, \ldots, e_k\} \subset R^n$ and define the closed subgroup $H_1 < \text{SL}(n, R)$ by

$$H_0 = \{g \in \text{SL}(n, R) \mid gV = V \text{ and } g|_V \in H\}.$$  

We can identify $\Gamma_0 \acts M_{n,k}(R)/H$ with $\Gamma \acts \text{SL}(n, R)/H_1$. From this description, essential freeness follows. Also the statement about invariant measures follows, because $H_1$ is unimodular if and only if $H$ is unimodular and satisfies det $g = \pm 1$ for all $g \in H$.

Theorem 5.8. Let $H < \text{GL}(k, R)$ be a closed subgroup and $\Lambda \acts (Y, \eta)$ any essentially free, ergodic, non-singular action of the countable group $\Lambda$. Suppose $n \geq 4k + 1$.

Case $-1 \in H$. Let $\Gamma \subset \text{PSL}(n, R)$ be a lattice and put $\Gamma := \{\pm 1\} \cdot \Gamma \subset \text{GL}(n, R)$. The actions $\Gamma \acts M_{n,k}(R)/H$ and $\Lambda \acts Y$ are SOE if and only if $\Lambda \acts Y$ is conjugate to an induction of one of the following actions:

$$x = g_0e_1. \text{ Since } \Gamma \text{ is a finite index subgroup of } \text{SL}(n, Z), \text{ it follows that } g_0^{-1}\Gamma xg_0 \text{ contains a finite index subgroup of } \{g \in \text{SL}(n, Z) \mid ge_1 = e_1\} \text{ and hence, is a lattice in } H. \text{ So, its diagonal action on } \mathbb{R}^n \times \mathbb{R}^n \text{ is ergodic. Then, the same is true for the diagonal action of } \Gamma_x \text{ on } \mathbb{R}^n \times \mathbb{R}^n.
1. $\Gamma \times H/N \acts M_{n,k}(\mathbb{R})/N$, where $N \triangleleft H$ is an open normal subgroup with $-1 \in N$,

2. $\tilde{\Gamma} \times H/N \acts M_{n,k}(\mathbb{R})/\{\pm(1,1)\}$, where $N \triangleleft H$ is an open normal subgroup with $-1 \notin N$.

Case $-1 \notin H$. Let $\Gamma \subset SL(n, \mathbb{R})$ be a lattice. The actions $\Gamma \acts M_{n,k}(\mathbb{R})/H$ and $\Lambda \acts Y$ are SOE if and only if $\Lambda \acts Y$ is conjugate to an induction of one of the following actions:

1. $\Gamma \times H/N \acts M_{n,k}(\mathbb{R})/N$, where $N \triangleleft H$ is an open normal subgroup,

2. (only when $-1 \in \Gamma$) $\Gamma \acts M_{n,k}(\mathbb{R})/\{\pm1\} \times H/N \acts M_{n,k}(\mathbb{R})/(\{\pm1\} \cdot N)$, where $N \triangleleft H$ is an open normal subgroup.

**Remark 5.9.** Given a stable orbit equivalence $\Delta : M_{n,k}(\mathbb{R})/H \to Y$ between the actions $\Gamma \acts M_{n,k}(\mathbb{R})/H$ and $\Lambda \acts Y$, Theorem 5.8 provides a conjugacy $\Psi$ between one of the listed actions and $\Lambda_1 \acts Y_1$ such that $\Lambda \acts Y$ is induced from $\Lambda_1 \acts Y_1$. In fact, one moreover has $\Psi(\overline{x}) \in \Lambda \cdot \Delta(\overline{x})$ for almost every $x \in M_{n,k}(\mathbb{R})$.

If $\Delta : X_0 \to Y_0$ is a SOE between $\Gamma \acts X$ and $\Lambda \acts Y$, we can (and will tacitly) extend $\Delta$ to a countable-to-one, measurable $\Delta : X \to Y$ satisfying $\Delta(g \cdot x) \in \Lambda \cdot \Delta(x)$ for all $g \in \Gamma$ and almost all $x \in X$.

If $\Delta : X \to Y$ is a SOE between the essentially free actions $\Gamma \acts X$ and $\Lambda \acts Y$, we get a *Zimmer 1-cocycle* $\omega$ for the action $\Gamma \acts X$ with values in $\Lambda$, determined by the formula

$$\Delta(g \cdot x) = \omega(g,x) \cdot \Delta(x)$$

almost everywhere. As a general principle, if the 1-cocycle $\omega$ is cohomologous to a group morphism $\Gamma \to \Lambda$, the stable orbit equivalence is ‘essentially’ given by a conjugacy of the actions: see [21 Proposition 4.2.11], [19 Lemma 4.7] and [7, Theorem 1.8]. In our framework of non p.m.p. actions, we again need such a principle: see Lemma 5.10 below.

The non-singular action $G \acts (X, \mu)$ of the l.c.s.c. group $G$ on the standard measure space $(X, \mu)$ is called *essentially free and proper* if there exists a measurable map $\pi : X \to G$ such that $\pi(g \cdot x) = g\pi(x)$ for almost all $(g,x) \in G \times X$. Equivalently, there exists a *-isomorphism $L^\infty(X, \mu) \to L^\infty(G \acts \otimes L^\infty(X, \mu)^G$ conjugating the natural $G$-actions.

**Lemma 5.10.** Let $G$ be a l.c.s.c. group and $G \acts (X, \mu)$ a non-singular, essentially free, ergodic action. Assume that $\sigma$ is $U_{\text{fin}}$-cocycle superrigid. Let $N \triangleleft G$ be an open normal subgroup such that the restricted action $\sigma|_N$ is proper. Let $\Lambda \acts (Y, \eta)$ be an essentially free, ergodic, non-singular action.

If $\Delta : X/N \to Y$ a SOE between the actions $G/N \acts X/N$ and $\Lambda \acts Y$, there exists

- a subgroup $\Lambda_1 \triangleleft \Lambda$ and a non-negligible $Y_1 \subset Y$ such that $\Lambda \acts Y$ is induced from $\Lambda_1 \acts Y_1$ ;

- an open normal subgroup $N_1 \triangleleft G$ such that $\sigma|_{N_1}$ is proper ;

such that the actions $G/N_1 \acts X/N_1$ and $\Lambda_1 \acts Y_1$ are conjugate through the non-singular isomorphism $\Psi : X/N_1 \to Y_1$ and the group isomorphism $\delta : G/N_1 \to \Lambda_1$. Furthermore, $\Delta(\overline{x}) \in \Lambda \cdot \Psi(\overline{x})$ for almost all $x \in X$. 

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Proof. Let Δ : X/N → Y be a SOE. By cocycle superrigidity of σ, take a measurable map ϕ : X → Λ such that, writing Θ(x) = ϕ(x)^−1 ∩ Δ(ϕ), we have Θ(g · x) = δ(g) · Θ(x) almost everywhere, where δ : G → Λ is a continuous group morphism. Put N₁ = Ker δ. So, N₁ is an open normal subgroup of G. Then, N ∩ N₁ is still open in G and we consider X/(N ∩ N₁) with the quotient map π : X/(N ∩ N₁) → X/N. It follows that we can view Θ as a measurable map Θ : X/(N ∩ N₁) → Y such that Δ(π(x)) ∈ Λ ∩ Θ(x) for almost all x ∈ X/(N ∩ N₁) and Θ(g · x) = δ(g) ∩ Θ(x) almost everywhere.

Using the facts that the countable group N/(N ∩ N₁) acts freely and properly on X/(N ∩ N₁), that Λ is countable and that Δ : X/N → Y is locally a non-singular isomorphism, it follows that X/(N ∩ N₁) can be partitioned in a sequence of non-negligible subsets {Uₙ}, such that for every n, Uₙ is a non-singular isomorphism between Uₙ and a non-negligible subset of Y. But then, for every non-trivial element g ∈ N/(N ∩ N₁) and every n, we conclude that g ∩ Uₙ ∩ Uₙ has measure zero. It follows that N₁/(N ∩ N₁) acts freely and properly on X/(N ∩ N₁). So, N₁ acts freely and properly on X. Hence, we can form the quotient space X/N₁ and view Θ as a measurable map Θ : X/N₁ → Y such that ∆(φ) ∈ Λ ∩ Θ(x) for almost all x ∈ X and Θ(φ ∩ π) = δ(φ) ∩ Θ(φ) almost everywhere. Now, δ : G/N₁ → Λ is an injective group morphism. Still, X/N₁ can be partitioned into a sequence of non-negligible subsets {Uₙ} such that Θ(Uₙ) is a non-singular isomorphism between Uₙ and Vₙ ⊂ Y.

We claim that Vₙ ∩ Vₘ has measure zero for every n ≠ m. If this is not the case, take W ⊂ Uₙ and W' ⊂ Uₙ non-negligible and a non-singular isomorphism ρ : W → W' such that Θ(ρ(x)) = Θ(x) for x ∈ W. Since Δ is a SOE, ρ(x) ∈ (G/N₁) · x for almost all x ∈ W. Hence, making W smaller but still non-negligible, we may assume that ρ(x) = φ · x for all x ∈ W and some φ ∈ G/N₁. Since W ∩ W' has measure zero and the action of G/N₁ on X/N₁ is essentially free, we get φ ≠ e. But also, δ(φ) ∩ Θ(x) = Θ(x) for almost all x ∈ W. This is a contradiction with the injectivity of δ and the essential freeness of Λ ∩ Y. This proves the claim and we have found that Θ is a non-singular isomorphism between X/N₁ and a non-negligible subset Y₁ ⊂ Y. Set Λ₁ = δ(G/N₁).

It remains to prove that Λ ∩ Y₁ is induced from Λ₁ ∩ Y₁. So, let h ∈ Λ and assume that h ∩ Y₁ ∩ Y₁ is non-negligible. We have to prove that h ∈ Λ₁. By our assumption, take W, W' ⊂ X/N₁ non-negligible and a non-singular isomorphism ρ : W → W' such that h ∩ Θ(x) = Θ(ρ(x)) for all x ∈ W. Since Δ was a SOE, we can make W smaller but still non-negligible and assume that ρ(x) = φ · x for all x ∈ W and some φ ∈ G/N₁. But then, h ∩ Θ(x) = δ(φ) ∩ Θ(x) for almost all x ∈ W. Since Λ ∩ Y₁ is essentially free, it follows that h = δ(φ) ∈ Λ₁. □

Proof of Theorem 5.8. We only prove the case -1 ∈ H, the case -1 ∉ H being analogous. By Theorem 1.3, the action of G := (Γ × H)/{±(1, 1)} on Mₙ,k(ℝ) is U₀₀-cocycle superrigid. By Lemma 5.10, we only have to prove that the following subgroups of Γ × H are the only open normal subgroups containing (-1, -1) and acting properly on Mₙ,k(ℝ).

- {±1} × N, where N < H is an open normal subgroup with -1 ∈ N.
- {(1) × N) ∪ (-1) × -N}, where N < H is an open normal subgroup with -1 ∉ N.

So, let N < (Γ × H) be a closed normal subgroup acting properly on Mₙ,k(ℝ). It is sufficient to prove that N ⊂ {±1} × H. Suppose that N ∉ {±1} × H and take (g, h) ∈ N with g ≠ ±1. Take k ∈ Γ such that the commutator t := kgg⁻¹g⁻¹ ≠ ±1. It follows that (t, 1) ∈ N. By Margulis’ normal subgroup theorem [12], we have Γ₀ × {1} ⊂ N for some finite index subgroup Γ₀ ⊂ Γ. By Lemma 5.6, Γ₀ acts ergodically on Mₙ,k(ℝ), contradicting the properness of N ∩ Mₙ,k(ℝ). □
Proof of Theorem 5.3. Statement 1 follows immediately from Theorem 5.8.

We now prove statement 2. We claim that the following are the only normal subgroups $N$ of $\text{SL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$ that act properly on $\mathbb{R}^n$:

- $N = \{e\}$,
- $N = \lambda \mathbb{Z}^n$ for some $\lambda \in \mathbb{N} \setminus \{0\}$,
- (only when $n$ is even) $N = \{\pm 1\} \rtimes \lambda \mathbb{Z}^n$ for $\lambda \in \{1, 2\}$.

Statement 2 of Theorem 5.3 then follows from Lemma 5.10 and Theorem 1.3, where the $U_{\text{fin}}$-cocycle superrigidity of the affine action $\text{SL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n \rtimes \mathbb{R}^n$ was established.

So, let $N \triangleleft \text{SL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$ be a normal subgroup acting properly on $\mathbb{R}^n$. Suppose first that $N \not\subseteq \mathbb{Z}^n$. Taking the commutator of $(g, x) \in N$ with $g \neq 1$ and an arbitrary $(1, y)$, $y \in \mathbb{Z}^n$, it follows that $H := N \cap \mathbb{Z}^n \neq \{0\}$. Hence, $H$ is a non-zero, globally $\text{SL}(n, \mathbb{Z})$-invariant subgroup of $\mathbb{Z}^n$. So, $H = \lambda \mathbb{Z}^n$ for some $\lambda \in \mathbb{N} \setminus \{0\}$. If $N \not\subseteq \{\pm 1\} \rtimes \mathbb{Z}^n$, it would follow that $N$ has finite index in $\text{SL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$, contradicting the properness of $N \rtimes \mathbb{R}^n$. So, we have shown that in all cases $N \subset \{\pm 1\} \rtimes \mathbb{Z}^n$. It is now straightforward to deduce the above list of possibilities for $N$. □

Proof of Theorem 1.6. This theorem is a special case of Theorem 5.8 □

5.3 Describing all 1-cocycles of quotient actions

Finally, cocycle superrigidity of $G \acts X, \mu$ allows to describe all 1-cocycles for $G/N \acts X/N$ when $N \triangleleft G$ is a closed normal subgroup of $G$ acting essentially freely and properly on $X$. We start with the following proposition, closely related to [16, Lemma 5.3], and illustrate it with two examples.

Proposition 5.11. Let $G$ be a l.c.s.c. group and $G \acts (X, \mu)$ a non-singular action. Let $N \triangleleft G$ be a closed, normal subgroup such that the restriction $\sigma|_N$ is essentially free and proper. Assume that $\sigma$ is $U_{\text{fin}}$-cocycle superrigid.

Choose a measurable map $\pi : X \to N$ satisfying $\pi(g \cdot x) = g\pi(x)$ for almost all $(g, x) \in N \times X$. Denote by $g \mapsto \overline{g}$ and $x \mapsto \overline{x}$ the quotient maps $G \to G/N$, resp. $X \to X/N$. Then,

$$\omega : \frac{G}{N} \times \frac{X}{N} \to G : \omega(\overline{g}, \overline{x}) = \pi(g \cdot x)^{-1}g\pi(x)$$

is a well-defined 1-cocycle.

Every 1-cocycle for the action $G/N \acts X/N$ with values in a Polish group of finite type $G$ is cohomologous to $\delta \circ \omega$ for a continuous morphism $\delta : G \to G$. If the diagonal action $G \acts X \times X$ is ergodic, $\delta$ is uniquely determined up to conjugacy by an element of $g \in G$.

Proof. Let $\Omega : G/N \times X/N \to G$ be a 1-cocycle with values in the Polish group of finite type $G$. From cocycle superrigidity of $\sigma$, let $\varphi : X \to G$ be a measurable map and $\delta : G \to G$ a continuous group morphism such that

$$\Omega(\overline{g}, \overline{x}) = \varphi(g \cdot x)^{-1}\delta(g)\varphi(x)$$

almost everywhere. Replacing $g$ by $hg$, $h \in N$, it follows that $\varphi(hg \cdot x)^{-1}\delta(h) = \varphi(g \cdot x)^{-1}$ and hence, $\varphi(h \cdot x) = \delta(h)\varphi(x)$ almost everywhere. So, we can define $\Psi(\overline{g}) = \delta(\pi(x))^{-1}\varphi(x)$. By construction, $\Psi$ makes $\Omega$ cohomologous to $\delta \circ \omega$. The uniqueness of $\delta$ follows directly from Lemma 5.3 □
Example 5.12. Let $\Gamma \subset \text{SL}(n, \mathbb{Z})$ be a finite index subgroup and consider the action $\Gamma \acts \mathbb{T}^n$.

- Choosing a measurable map $p : \mathbb{R}^n \to \mathbb{Z}^n$ such that $p(x+y) = x + p(y)$ for all $x \in \mathbb{Z}^n$ and almost all $y \in \mathbb{R}^n$, the formula
  \[ \omega : \Gamma \times \mathbb{R}^n / \mathbb{Z}^n \to \Gamma \acts \mathbb{Z}^n : \omega(g, \overline{x}) = p(g \cdot x)^{-1} gp(x) \]
defines a 1-cocycle for $\Gamma \acts \mathbb{T}^n$ with values in $\Gamma \acts \mathbb{Z}^n$.

- Every 1-cocycle with values in a Polish group of finite type $\mathcal{G}$ is cohomologous with $\delta \circ \omega$ for a group morphism $\delta : \Gamma \acts \mathbb{Z}^n \to \mathcal{G}$, uniquely determined up to conjugacy by an element in $\mathcal{G}$.

Note that by Zimmer’s cocycle superrigidity theorem [21 Theorem 5.2.5], any Zariski dense 1-cocycle for any ergodic p.m.p. action $\text{SL}(n, \mathbb{Z}) \acts (X, \mu)$, $n \geq 3$, taking values in a connected simple real algebraic non-compact center free group, is cohomologous to a group morphism. For the specific action $\text{SL}(n, \mathbb{Z}) \acts \mathbb{R}^n / \mathbb{Z}^n$, $n \geq 5$, the previous example provides an explicit description of all 1-cocycles of $\text{SL}(n, \mathbb{Z}) \acts \mathbb{R}^n / \mathbb{Z}^n$ with values in an arbitrary Polish group of finite type.

Example 5.13. Let $\Gamma \subset \text{PSL}(n, \mathbb{R})$ be a lattice and $X$ the real flag manifold of signature $(d_1, \ldots, d_l, n)$ with $n \geq 4d_l + 1$. We obtain as follows all 1-cocycles for the action $\Gamma \acts X$ with values in a Polish group of finite type.

- Identify $X = \text{M}_{n,k}(\mathbb{R}) / H$ and choose a measurable map $p : \text{M}_{n,k}(\mathbb{R}) \to H$ satisfying $p(Ag) = p(A)g$ for almost all $A \in \text{M}_{n,k}(\mathbb{R})$, $g \in H$. Let $\overline{\Gamma} = \{ \pm 1 \} \cdot \Gamma$ be the double cover of $\Gamma$ in $\text{GL}(n, \mathbb{R})$ and define $G := (\overline{\Gamma} \times H) / \{ \pm (1, 1) \}$. The formula
  \[ \omega : \overline{\Gamma} \times X \to G : \omega(\overline{g}, \overline{x}) = (g, p(gx)p(x)^{-1}) \mod \{ \pm (1, 1) \} \]
defines a 1-cocycle with values in $G$. Here $\overline{\Gamma} \to \Gamma : g \mapsto \overline{g}$ and $\text{M}_{n,k}(\mathbb{R}) \to X : x \mapsto \overline{x}$ are the quotient maps.

- Every 1-cocycle with values in a Polish group of finite type $\mathcal{G}$ is cohomologous with $\delta \circ \omega$ for a continuous group morphism $\delta : G \to \mathcal{G}$, uniquely determined up to conjugacy by an element in $\mathcal{G}$.

6 Classification up to orbit equivalence

Combining the results of [8] with Theorem 5.8, we classify up to stable orbit equivalence, the linear lattice actions $\Gamma \acts \mathbb{R}^n$, as well as the natural lattice actions on flag manifolds. At the same time, we compute the outer automorphism group of the associated orbit equivalence relations.

We start with the following elementary lemma.

**Lemma 6.1.** Let $\Gamma \acts (X, \mu)$ be a non-singular action of the countable group $\Gamma$ and assume that the diagonal action $\Gamma \acts X \times X$ is ergodic. Then, $\Gamma \acts X$ is not induced, i.e. if $\Gamma \acts X$ is induced from $\Gamma_1 \acts X_1$, then $\Gamma_1 = \Gamma$ and $\mu(X \setminus X_1) = 0$.

**Proof.** Assume that $\Gamma \acts X$ is induced from $\Gamma_1 \acts X_1$. So, we find a quotient map $\pi : X \to \Gamma / \Gamma_1$ satisfying $\pi(g \cdot x) = g \pi(x)$ almost everywhere and $X_1 = \pi^{-1}(e\Gamma_1)$. Hence, the subset $\{(x, y) \in X \times X \mid \pi(x) = \pi(y)\}$ is non-negligible and $\Gamma$-invariant. By the ergodicity of $\Gamma \acts X \times X$, it follows that $\pi(x) = \pi(y)$ for almost all $(x, y) \in X \times X$. This means that $\Gamma_1 = \Gamma$ and $\mu(X \setminus X_1) = 0$. \qed
If $R$ is a $\Pi_1$ equivalence relation on $(X, \mu)$, we denote by $[R]$ the \textit{full group} of $R$ consisting of non-singular automorphisms $\Delta : X \rightarrow X$ satisfying $(x, \Delta(x)) \in R$ for almost every $x \in X$. Then, $[R]$ is a normal subgroup of the automorphism group $\text{Aut}(R)$ of $R$. The quotient group is denoted by $\text{Out}(R)$ and called the \textit{outer automorphism group} of $R$. The \textit{full pseudogroup} of $R$ is denoted by $[[R]]$ and consists of non-singular partial isomorphisms $\phi : X_0 \subset X \rightarrow X_1 \subset X$ satisfying $(x, \phi(x)) \in R$ for almost every $x \in X_0$. We denote $X_0 = D(\phi)$ and $X_1 = R(\phi)$.

**Theorem 6.2.** Let $n \geq 5$ and $\Gamma \subset \text{SL}(n, \mathbb{R})$ a lattice. Let $n' \geq 2$ and $\Gamma' \subset \text{SL}(n', \mathbb{R})$ a lattice. If the non-singular isomorphism $\Delta : X_1 \subset \mathbb{R}^n \rightarrow X'_1 \subset \mathbb{R}^{n'}$ is a SOE between $\Gamma \rhd \mathbb{R}^n$ and $\Gamma' \rhd \mathbb{R}^{n'}$, then

- $n = n'$ and there exists $A \in \text{GL}(n, \mathbb{R})$ such that $\Gamma' = A\Gamma A^{-1}$,
- there exists $\phi \in [[R(\Gamma' \rhd \mathbb{R}^{n'})]]$ with $R(\phi) = X'_1$ such that $\Delta(x) = \phi(A(x))$ for almost every $x \in X_1$.

In particular, $\text{Out}(R(\Gamma \rhd \mathbb{R}^n)) = N_{\text{GL}(n, \mathbb{R})}(\Gamma)/\Gamma$.

**Proof.** Let $\Delta : X_1 \subset \mathbb{R}^n \rightarrow X'_1 \subset \mathbb{R}^{n'}$ be a SOE between $\Gamma \rhd \mathbb{R}^n$ and $\Gamma' \rhd \mathbb{R}^{n'}$. Since for $n' = 2$, the equivalence relation $R(\Gamma' \rhd \mathbb{R}^{n'})$ is hyperfinite, we have $n' \geq 3$ and hence the diagonal action $\Gamma' \rhd \mathbb{R}^{n'} \times \mathbb{R}^{n'}$ is ergodic. By Lemma 6.1, $\Gamma' \rhd \mathbb{R}^{n'}$ is not an induced action. By Theorem 1.5, the action $\Gamma' \rhd \mathbb{R}^{n'}$ is conjugate with either $\Gamma \rhd \mathbb{R}^n$ or, in case $n$ is even, $\Gamma/\{\pm 1\} \rhd \mathbb{R}^n/\{\pm 1\}$. By Mostow rigidity, the latter is impossible since $\Gamma' \not\cong \Gamma/\{\pm 1\}$. In the former case, we already conclude $n = n'$ and we have found a non-singular isomorphism $\Theta : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a group isomorphism $\delta : \Gamma \rightarrow \Gamma'$ satisfying

- $\Theta(g \cdot x) = \delta(g) \cdot \Theta(x)$,
- $\Delta(x) \in \Gamma' \cdot \Theta(x)$,

for all $g \in \Gamma$ and almost all $x \in \mathbb{R}^n$. Denoting by $B^\top$ the transpose of the matrix $B$, by Mostow rigidity, we find $A \in \text{GL}(n, \mathbb{R})$ such that a) $\delta(g) = AgA^{-1}$ for all $g \in \Gamma$, or b) $\delta(g) = A(g^\top)^{-1}A^{-1}$ for all $g \in \Gamma$.

Define the subgroup $H \subset \text{SL}(n, \mathbb{R})$ consisting of matrices $g$ with $ge_1 = e_1$ and identify $\mathbb{R}^n = \text{SL}(n, \mathbb{R})/H$. In case b), we would get a conjugacy between the $\Gamma$-actions on $\text{SL}(n, \mathbb{R})/H$ and $\text{SL}(n, \mathbb{R})/H^\top$, which is ruled out by [8, Theorem D]. In case a), multiplying $A$ by a non-zero scalar if necessary, [8, Theorem D] implies that $\Theta(x) = Ax$ for almost all $x \in \mathbb{R}^n$.

Defining the partial isomorphism $\phi := \Delta \circ A^{-1}$ with $R(\phi) = X'_1$, it follows that $\phi \in [[R(\Gamma' \rhd \mathbb{R}^{n'})]]$ and that $\Delta(x) = \phi(A(x))$ for almost every $x \in X_1$.

Let $X$ be the real flag manifold with signature $d := (d_1, \ldots, d_1, n)$. Denote

$$d^\top := (n - d_1, n - d_{l-1}, \ldots, n - d_1, n).$$

If $X'$ is the real flag manifold with signature $d^\top$, there is a natural diffeomorphism $X \rightarrow X' : x \mapsto \overline{x}$ satisfying $\overline{g \cdot x} = (g^\top)^{-1} \cdot \overline{x}$ for all $x \in X$, $g \in \text{SL}(n, \mathbb{R})$. 

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Theorem 6.3. Let $\Gamma \subset \text{PSL}(n, \mathbb{R})$ be a lattice and $X$ the real flag manifold with signature $d := (d_1, \ldots, d_l, n)$. Assume that $n \geq 4d_l + 1$. Let $\Gamma' \subset \text{PSL}(n', \mathbb{R})$ be a lattice and $X'$ the real flag manifold with signature $d' := (d'_1, \ldots, d'_l, n')$.

If the non-singular isomorphism $\Delta : X_1 \subset X \to X'_1 \subset X'$ is a SOE between $\Gamma \curvearrowright X$ and $\Gamma' \curvearrowright X'$, then $n = n'$ and there exists $A \in \text{PGL}(n, \mathbb{R})$, $\phi \in \text{Out}([\mathcal{R}(\Gamma' \curvearrowright X')])$ with $R(\phi) = X'_1$ such that

- either $d' = d$, $\Gamma' = \Delta A A^{-1}$, $\Delta(x) = \phi(A(x))$ for almost every $x \in X_1$,
- or $d' = d^T$, $\Gamma' = \Delta A^T A^{-1}$, $\Delta(x) = \phi(A(\mathcal{T}))$ for almost every $x \in X_1$.

In particular, $\text{Out}(\mathcal{R}(\Gamma \curvearrowright X)) = \mathcal{N}_{\text{PGL}(n, \mathbb{R})}(\Gamma)/\Gamma$.

Proof. Let $\Delta : X_1 \subset X \to X'_1 \subset X'$ be a SOE between $\Gamma \curvearrowright X$ and $\Gamma' \curvearrowright X'$.

We first prove that the diagonal action $\Gamma' \curvearrowright X' \times X'$ is ergodic. If $Y$ denotes the flag manifold of signature $(1, 2, \ldots, n')$, the action $\Gamma' \curvearrowright X'$ is a quotient of the action $\Gamma \curvearrowright Y$. So, it suffices to prove ergodicity of $\Gamma' \curvearrowright Y \times Y$. Denoting by $D \subset \text{SL}(n', \mathbb{R})$ the subgroup of diagonal matrices, $\Gamma' \curvearrowright Y \times Y$ can be identified with $\Gamma' \curvearrowright \text{SL}(n', \mathbb{R})/D$, which follows ergodic because $D \curvearrowright \text{SL}(n', \mathbb{R})/\Gamma'$ is ergodic by Moore's ergodicity theorem. So, by Lemma 6.1 $\Gamma' \curvearrowright X'$ is not an induced action.

Since $\Gamma'$ has trivial center, Theorem 1.6 yields a group isomorphism $\delta : \Gamma \to \Gamma'$ and a non-singular isomorphism $\Theta : X \to X'$ satisfying $\Theta(g \cdot x) = \delta(g) \cdot \Theta(x)$ and $\Delta(x) \in \Gamma' \cdot \Theta(x)$ almost everywhere.

By Mostow rigidity, $n = n'$ and there exists $A \in \text{PGL}(n, \mathbb{R})$ such that either $\delta(g) = A g A^{-1}$ or $\delta(g) = A (g^T)^{-1} A^{-1}$ for all $g \in \Gamma$.

Set $k_1 = d_1$ and $k_i = d_i - d_{i-1}$ for $2 \leq i \leq l$. Put $k = d_l$ and define the closed subgroups $H$ and $H_1$ of $\text{GL}(k, \mathbb{R})$ by

\[
H := \left\{ \begin{pmatrix} A_{11} & * & \cdots & * \\ 0 & A_{22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{ll} \end{pmatrix} \middle| A_{ii} \in \text{GL}(k_i, \mathbb{R}) \text{ and } \det(A_{ii}) \neq 0 \right\},
\]

\[
H_1 := \left\{ \begin{pmatrix} A_{11} & * & \cdots & * \\ 0 & A_{22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{ll} \end{pmatrix} \middle| A_{ii} \in \text{GL}(k_i, \mathbb{R}) \text{ and } \det(A_{ii}) = \pm 1 \right\}.
\]

We identify $H/H_1 = \mathbb{R}^k_+$. From now on, we write $X$ as $M_{n,k}(\mathbb{R})/H$. Define analogously the subgroups $H'$, $H'_1$ of $\text{GL}(k', \mathbb{R})$ and write $X'$ as $M_{n,k}(\mathbb{R})/H'$.

Whenever $N < H$ is a closed normal subgroup containing $-1$, the quotient morphism $H \to H/N$ gives rise, as in Example 5.13 to a 1-cocycle $\omega_N$ for the action $\Gamma \curvearrowright X$ with values in $H/N$ such that the action $\Gamma \curvearrowright M_{n,k}(\mathbb{R})/N$ can be identified with the action $\Gamma \curvearrowright X \times H/N$ given by

\[
g \cdot (x, h) = (g \cdot x, \omega_N(g, x) h).
\]

Since $\Gamma \curvearrowright M_{n,k}(\mathbb{R})/N$ is ergodic, the 1-cocycle $\omega_N$ cannot be cohomologous to a 1-cocycle taking values in a proper closed subgroup of $H/N$. We similarly define the 1-cocycles $\omega_{N'}$ for $\Gamma' \curvearrowright X'$.

Since $\delta, \Theta$ conjugate the actions $\Gamma \curvearrowright X$ and $\Gamma' \curvearrowright X'$, the map $\mu(g, x) = \omega_{N'}(\delta(g), \Theta(x))$ defines a 1-cocycle for $\Gamma \curvearrowright X$, with values in $\mathbb{R}^{k'}_+$ and with the property of not being cohomologous to a
1-cocycle taking values in a proper closed subgroup of $\mathbb{R}^k_\Gamma$. We now apply Example 5.13 describing all $U_{\text{fin}}$-valued 1-cocycles for $\Gamma \curvearrowright X$, and note that $\mathbb{R}^k_\Gamma$ belongs to $U_{\text{fin}}$. Every group morphism $\Gamma \to \mathbb{R}^k_\Gamma$ is trivial and every continuous group morphism $H \to \mathbb{R}^k_\Gamma$ is trivial on $H_1$. So, we find a continuous group morphism $\rho : \mathbb{R}^k_\Gamma \to \mathbb{R}^k_\Gamma$ such that $\mu$ is cohomologous to $\rho \circ \omega_{H_1}$. Since $\mu$ cannot be cohomologous to a 1-cocycle taking values in a proper closed subgroup of $\mathbb{R}^k_+, it follows that $\rho$ is onto.

Altogether, we find a closed normal subgroup $N < H$, containing $H_1$, and a continuous isomorphism $\rho : H/N \to H'/H'_1$ such that $\mu$ is cohomologous to $\rho \circ \omega_N$. It follows that there exists a non-singular isomorphism $\Theta_1: M_{n,k}(\mathbb{R})/N \to M_{n',k'}(\mathbb{R})/H'_1$ satisfying $\Theta_1(g \cdot x) = \delta(g) \cdot \Theta_1(x)$ and $\Theta_1(x)H' = \Theta(xH)$ almost everywhere.

Since the action of $\Gamma'$ on $M_{n',k'}(\mathbb{R})/H'_1$ is infinite measure preserving, Lemma 5.7 implies that $N = H_1$. We saw already that either $\delta(g) = AgA^{-1}$ or $\delta(g) = A(g^\top)^{-1}A^{-1}$ for all $g \in \Gamma$. In the former case, [8] Theorem D] implies that $d' = d$ and that there exists $B \in H$ such that $\Theta_1(x) = AxB$ for almost every $x \in M_{n,k}(\mathbb{R})/H_1$. It follows that $\Theta(x) = A(x)$ for almost every $x \in X$. In the latter case, we prove analogously that $d' = d^\top$ and $\Theta(x) = A(x)$. 

\begin{proof}

7 Implementation by group actions

In [1] page 292], the question is raised whether every $\Pi_1$ equivalence relation can be implemented by an essentially free action of a countable group. This question has been settled in the negative in [6] Theorem D]. In Proposition 7.1 below, we give examples of $\Pi_1$ equivalence relations $\mathcal{R}$ on $(X, \mu)$ with the following much stronger property: whenever $\Lambda \curvearrowright (Y, \eta)$ is an essentially free, non-singular action and $\Delta : X \to Y$ is a measurable map satisfying $\Delta(x) \in \Lambda \cdot \Delta(y)$ for almost all $(x,y) \in \mathcal{R}$, then there exists $y_0 \in Y$ such that $\Delta(x) \in \Lambda \cdot y_0$ for almost all $x \in X$.

Among other examples, [6] Theorem D] proves that the restriction of the orbit equivalence relation $\text{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$ to a subset of irrational measure, provides a $\Pi_1$ equivalence relation that cannot be implemented by a free action of a group. By [15] Theorem 0.3], if $\Gamma \curvearrowright [0,1]^{\Gamma}$ is the Bernoulli action of a property (T) group $\Gamma$ without finite normal subgroups, the restriction of its orbit equivalence relation to any subset of measure strictly between 0 and 1 is unimplementable by a free action.

But, [6] Theorem D] also provides examples of $\Pi_1$ equivalence relations $\mathcal{R}$ such that none of the amplifications $\mathcal{R}^t$, $t > 0$, can be implemented by a free action. These equivalence relations are constructed using the following method. Suppose that $G$ is a l.c.s.c. unimodular group and $G \curvearrowright (X, \mu)$ an essentially free, properly ergodic, p.m.p. action. There exists a Borel set $Y \subset X$, a probability measure $\eta$ on $Y$ and a neighborhood $\mathcal{U}$ of $e$ in $G$ that we equip with a multiple of the Haar measure, such that $\mathcal{U} \times Y \to X : (g,y) \mapsto g \cdot y$ provides a measure preserving isomorphism of $\mathcal{U} \times Y$ onto a non-negligible subset of $X$. The restriction of the orbit equivalence relation of $G \curvearrowright (X, \mu)$ to $Y$ is a $\Pi_1$ equivalence relation on $(Y, \eta)$. One calls $Y \subset X$ a measurable cross-section for $G \curvearrowright (X, \mu)$. A different choice of measurable cross-section yields a stably isomorphic $\Pi_1$ equivalence relation.

By Theorem 11 the restriction of the orbit equivalence relation of $\text{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{R}^n$ to a subset of finite measure, provides other examples of $\Pi_1$ equivalence relations $\mathcal{R}$ such that none of the finite amplifications can be implemented by a free action. In fact, one can show that $\mathcal{R}$ arises as the measurable cross-section for the action of $\text{SL}(n-1, \mathbb{R}) \ltimes \mathbb{R}^{n-1}$ on $\text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z})$. Nevertheless, this example is not covered by [6] Theorem D], since $\text{SL}(n-1, \mathbb{R}) \ltimes \mathbb{R}^{n-1}$ is not semi-simple.
Proposition 7.1. Let $G$ be a l.c.s.c. connected, unimodular group with normal closed subgroup $G_0$ having the relative property (T). Let $H_\mathbb{R}$ be a real Hilbert space and $\pi : G \to O(H_\mathbb{R})$ an orthogonal representation. Assume that $\pi$ is injective and that the restriction of $\pi$ to $G_0$ is weakly mixing (i.e. has no finite-dimensional invariant subspaces). Denote by $G \curvearrowright (X, \mu)$ the associated Gaussian action (see e.g. [7], Section 2.7). Choose a measurable cross-section $x \in G$ such that $\Delta(x) \in \Lambda$ · $\Delta(y)$ for almost all $(x,y) \in \mathcal{R}$, then there exists $y_0 \in Y$ such that $\Delta(x) \in \Lambda$ · $y_0$ for almost all $x \in X_1$.

Proof. Choose a measurable map $p : X \to X_1$ such that $p(x) \in G$·$x$ for almost all $x \in X$. Define the 1-cocycle $\omega : G \times X \to \Lambda$ such that $\Delta(p(g \cdot x)) = \omega(g, x) \cdot \Delta(p(x)) \omega(g, x)$ almost everywhere. As observed in [7], Theorem 0.1 in [15] applies to $G \curvearrowright (X, \mu)$. Since $G$ is connected, every group morphism from $G$ to $\Lambda$ is trivial and we find a measurable map $\varphi : X \to \Lambda$ such that $\omega(g, x) = \varphi(g \cdot x)^{-1} \varphi(x)$. So, the map $x \mapsto \varphi(x) \cdot \Delta(p(x))$ is $G$-invariant and hence, essentially constant. We therefore find $y_0 \in Y$ such that $\Delta(p(x)) \in \Lambda$ · $y_0$ for almost all $x \in X$. This concludes the proof of the proposition. \qed

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