On some algebraic aspects of $\eta$-intuitionistic fuzzy subgroups

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ABSTRACT

In this study, we present the idea of $\eta$-intuitionistic fuzzy subgroup (IFSG) defined on $\eta$-intuitionistic fuzzy set (IFS). Furthermore, we prove that every IFSG is an $\eta$-IFSG. Also, we extend the study of this notion to define $\eta$-intuitionistic fuzzy cosets and $\eta$-intuitionistic fuzzy normal subgroups of a given group and investigate some of their fundamental algebraic features. Besides, we define the $\eta$-intuitionistic fuzzy homomorphism between two $\eta$-IFSGs and show that an $\eta$-intuitionistic fuzzy homomorphic image (inverse image) of the $\eta$-IFSG is an $\eta$-IFSG.

1. Introduction

The thought of a set and set hypothesis are compelling ideas in mathematics. However, the predominant idea of the fundamental set hypothesis, that a component may belong to a set or not be in a set makes it roughly difficult to speak to quite a bit of human correspondence. In a crisp set, we have a clear idea of whether an element exists in a set or not. Fuzzy sets enable components to be modestly in a set. Every component gives a level of enrolment in a set. This enrolment esteem can go from 0 to 1. If it just permits the outrageous participations estimations of 0 and 1, it would be equal to crisp sets. Fuzzy logic has been utilized as a part of different applications. Specifically, facial example: acknowledgment, ventilation systems, clothes washers and vacuum cleaners. The idea of intuitionistic fuzziness began by happenstance. The intuitionistic fuzzy set theory serves significantly in modern mathematics as it generalizes the fuzzy set. This particular theory is being applied in many disciplines, including medical diagnosis, vulnerability assessment of gas pipeline networks, travel time and neural network models. Zadeh [1] proposed the idea of fuzzy sets in 1965. The most important motivation for studying the theory of intuitionistic fuzzy sets is the ability to deal with the uncertainty and vagueness of a physical problem much more effectively than with the theory of the classic fuzzy set, especially in the area of logic programming and decision-making, Financial services, psychological examinations, medical diagnosis, career determination and artificial intelligence. For instance, This logic enables to identify the type of course to be taken to obtain a certain type of job by developing suitable skills. In addition, this theory is also used to find the relationship between the different types of academic courses and the different types of skills that can be developed through the courses. The traffic problem is a wide human-oriented area with diverse and challenging tasks that need to be solved. Characteristics and performance of transport system services, costs, infrastructure, vehicles and control systems are usually determined on the basis of a quantitative assessment of their main effects. Most transportation decisions take place under inaccuracy, uncertainty and partial truth. Some lenses and boundary conditions are often difficult to measure against crisp values and in the context of classic fuzzy logic. The concept of the intuitionist fuzzy subgroup offers a useful technique for real life transportation problems. This special phenomenon is used to model the structure of the controlling an intersection of two one-way streets. Fuzzy subgroups on fuzzy sets and their elemental consequences were studied by Rosenfeld [2] in 1971. Das [3] remodelled these concepts and proposed the definition of level subgroups in 1981. Mukherjee and Bhattacharya [4] explored the normality of a fuzzy subgroup and defined fuzzy cosets in 1984. The idea of fuzzy homomorphism and the related consequences of fuzzy subgroups was initiated by Choudhury et al. [5] in 1988. Gupta and Qi [6] extended these ideas and worked on $t$-norms accompanied by the fuzzy inference method in 1991. Ajmal and Prajapati [7] anal...
vised the concept of fuzzy cosets along with fuzzy normal subgroups in 1992. In 1998, Yaun and Zou [8] investigated the equivalence relation on fuzzy subgroups. Mordeson et al. [9] conversed about the nilpotency of fuzzy subgroups and many valuable algebraic features of fuzzy subgroups in 2005. Atanassov [10] augmented the fuzzification of sets to commence the idea of Intuitionistic fuzzy sets in 1986. Biswas [11] enhanced this notion by studying the intuitionistic fuzzification of subgroups and proposed some new definitions. Georgiev and Atanassov [12] extended this idea of intuitionistic fuzzification and defined their logic operations in 1995. In 1998, Coker [13] presented the idea of intuitionistic point. Atanassov [14] discussed many important properties of the intuitionistic fuzzy set in 1999. Hur et al. [15] introduced intuitionistic M-fuzzy groups in 2004. This idea was extended by Zhan and Tan [16] to initiate the idea of multi M-fuzzy groups and some associated results in 2004. Palaniappan [17] initiated the concept of intuitionistic L-fuzzy subgroup in 2009. In 2010, Marashdesh and Salleh [18] utilized the idea of intuitionistic fuzzy space and commenced the study of intuitionistic fuzzy normal subgroups. Li and Wang [19] investigated the $(\lambda, \alpha)$ homomorphism of fuzzy subgroups in 2011. Sharma [20] defined $(\alpha, \beta)$-cut of intuitionistic fuzzy groups in 2011. He also defined the notions of t-intuitionistic fuzzy set and t-intuitionistic fuzzy subgroup in [21]. Doda and Sharma [22] studied the finite groups of different orders and gave the idea of recording the count of intuitionistic fuzzy subgroups in 2013. In [23] the authors used interval-valued intuitionistic fuzzy values to set up a new multiple attribute decision making approach. Zeng et al. [24] established a new technique of induced aggregation for intuitionistic fuzzy set. For more study on intuitionistic fuzzy sets, we recommend reading of [25–34].

The major contributions of this paper are;

1. The notion of $\eta$-IFSG over $\eta$-IFS has been introduced.
2. A study to obtain a class of IFSG that correspond to a given IFSG has been proposed.
3. The notions of $\eta$-intuitionistic fuzzy cosets and $\eta$-intuitionistic fuzzy normal subgroups have been defined and their various fundamental algebraic attributes have been established.
4. The study of this phenomenon has been extended by presenting the concept of $\eta$-intuitionistic fuzzy homomorphism between any two $\eta$-IFSG's.
5. The behaviour of an $\eta$-intuitionistic fuzzy homomorphic image of this particular homomorphism has been investigated.

The rest of the paper is organized as follows: The basic definitions of the intuitionistic fuzzy subgroup and the associated results have been presented in Section 2. In Section 3, we define $\eta$-intuitionistic fuzzy subgroups based on $\eta$-intuitionistic fuzzy sets and establishes many basic algebraic properties of this notion. Moreover, we use this idea to define $\eta$-intuitionistic fuzzy cosets and $\eta$-intuitionistic fuzzy normal subgroups and investigate many algebraic properties for these particular groups. The Section 4 extends the study of this phenomenon to introduce $\eta$-intuitionistic fuzzy homomorphism between two given $\eta$-IFSG's and describes the effect of $\eta$-intuitionistic fuzzy homomorphism on these groups.

2. Preliminaries

This section is devoted to review some productive concepts of intuitionistic fuzzy group and related results that are mandatory to understand the subsequent study of this article.

Definition 2.1: [10] An intuitionistic fuzzy set (IFS) $\alpha$ is an augmentation of the classical fuzzy set which defines the degree of membership and non-membership of an element in the universe $S$ to the close unit interval. Each ordinary intuitionistic fuzzy set is given by $\alpha = \{ (s_1, \tau_{\alpha}(s_1), \xi_{\alpha}(s_1)) | s_1 \in S >\}, 0 \leq \tau_{\alpha}(s_1) + \xi_{\alpha}(s_1) \leq 1$.

Definition 2.2: [11] An IFS $\alpha$ is called intuitionistic fuzzy subgroup (IFSG) if it admits the following conditions:

1. $\tau_{\alpha}(s_1 s_2) \geq \min\{\tau_{\alpha}(s_1), \tau_{\alpha}(s_2)\}$,
2. $\xi_{\alpha}(s_1 s_2) \leq \max\{\xi_{\alpha}(s_1), \xi_{\alpha}(s_2)\}$,
3. $\tau_{\alpha}(s_1^{-1}) = \tau_{\alpha}(s_1)$,
4. $\xi_{\alpha}(s_1^{-1}) = \xi_{\alpha}(s_1)$, for all $s_1, s_2 \in G$.

Definition 2.3: [20] Let $\gamma$ and $\delta$ be positive real numbers lie in closed unit interval such that $0 \leq \gamma + \delta \leq 1$. The $(\gamma, \delta)$- cut set of an IFS $\alpha$ of the universe $S$ is a crisp set consisting of all those elements of $S$ for which $\tau_{\alpha}(s_1) \geq \gamma$ and $\xi_{\alpha}(s_1) \leq \delta$ for all $s_1 \in S$.

Remark 2.4: [20] An IFS $\alpha$ of a group $G$ is IFSG if each of its $(\gamma, \delta)$-cut set is a subgroup of $G$.

Definition 2.5: [15] Let $\alpha$ be an IFSG of $G$ and $s_1 \in S$. An IFS $\alpha s_1$ of $G$ is called intuitionistic fuzzy right coset of $\alpha$ in $G$ if $(\alpha s_1)(g) = (\tau_{\alpha s_1}(g), \xi_{\alpha s_1}(g))$, where $\tau_{\alpha s_1}(g) = \tau_{\alpha}(g s_1^{-1})$ and $\xi_{\alpha s_1}(g) = \xi_{\alpha}(g s_1^{-1})$ for all $g \in G$. The intuitionistic fuzzy left coset of $\alpha$ can be defined in the similar manner.

Definition 2.6: [15] An IFS $\alpha$ is said to be intuitionistic fuzzy normal subgroup (IFNSG) if it meets the following conditions:

1. $\tau_{\alpha}(s_1 s_2) = \tau_{\alpha}(s_2 s_1)$
2. $\xi_{\alpha}(s_1 s_2) = \xi_{\alpha}(s_2 s_1)$, for all $s_1, s_2 \in G$. 

\[ \begin{align*}
\tau_{\alpha}(s_1 s_2) &\geq \min\{\tau_{\alpha}(s_1), \tau_{\alpha}(s_2)\}, \\
\xi_{\alpha}(s_1 s_2) &\leq \max\{\xi_{\alpha}(s_1), \xi_{\alpha}(s_2)\}, \\
\tau_{\alpha}(s_1^{-1}) &\tau_{\alpha}(s_1), \\
\xi_{\alpha}(s_1^{-1}) &\xi_{\alpha}(s_1), \\
\end{align*} \]
Definition 2.7: The averaging operator of two IFS’s $\alpha$ and $\beta$ of the universe $S$, denoted by $\alpha \ast \beta$, is defined as
\[
\alpha \ast \beta = \left[ s \in S \right], \sqrt{\tau_{\alpha}(s) \tau_{\beta}(s)}, \sqrt{\xi_{\alpha}(s) \xi_{\beta}(s)} = s \in S
\]

3. Algebraic aspects of $\eta$-intuitionistic fuzzy subgroups

In this section, we study $\eta$-intuitionistic fuzzy subgroup. Moreover, numerous useful results and algebraic properties are introduced.

Definition 3.1: Suppose $\alpha$ is an IFS of a universe $S$ and $\eta \in [0, 1]$. Then IFS $\alpha^\eta = (\tau_{\alpha^\eta}, \xi_{\alpha^\eta})$ is called $\eta$-IFS of universe $S$ with respect to IFS $\alpha$, where, $\tau_{\alpha^\eta}(s) = \psi[\min(\tau_{\alpha}(s), \eta)], \xi_{\alpha^\eta}(s) = \psi[\min(\tau_{\alpha}(s), 1 - \eta)]$ and $\psi$, $\psi'$ denote the averaging operator defined in (2.7).

Next result demonstrates the essential attribute about the intersection of any two $\eta$-IFS’s of universe $S$.

Proposition 3.2: The intersection of any two $\eta$-IFS’s is an $\eta$-IFS.

Proof: Let $\alpha^\eta$ and $\beta^\eta$ be two $\eta$-IFS’s of the universe $S$, then
\[
\tau_{(\alpha \cap \beta)^\eta}(s) = \psi[\min(\tau_{\alpha}(s), \tau_{\beta}(s)), \eta] = \psi[\min(\tau_{\alpha}(s), \eta), \min(\tau_{\beta}(s), \eta)] = \min(\tau_{\alpha^\eta}(s), \tau_{\beta^\eta}(s)) = \tau_{(\alpha \cap \beta)^\eta}(s)
\]
for all $s \in S$.

Similarly, it can be proved that $\xi_{(\alpha \cap \beta)^\eta}(s) = \xi_{\alpha^\eta}(s) \cap \xi_{\beta^\eta}(s)$ for all $s \in S$.

Hence $(\alpha \cap \beta)^\eta = \alpha^\eta \cap \beta^\eta$.

Remark 3.3: The union of any two $\eta$-IFS’s is also $\eta$-IFS.

Definition 3.4: An $\eta$-IFSG of group $G$ is an $\eta$-IFS $\alpha^\eta$ satisfying the following conditions:

1. $\tau_{\alpha^\eta}(s_1 s_2) = \min(\tau_{\alpha^\eta}(s_1), \tau_{\alpha^\eta}(s_2))$.
2. $\xi_{\alpha^\eta}(s_1 s_2) = \max(\xi_{\alpha^\eta}(s_1), \xi_{\alpha^\eta}(s_2))$.
3. $\tau_{\alpha^\eta}(s_1^{-1}) = \tau_{\alpha^\eta}(s_1)$.
4. $\xi_{\alpha^\eta}(s_1^{-1}) = \xi_{\alpha^\eta}(s_1)$, for all $s_1, s_2 \in G$.

Remark 3.5: Let $e$ be an identity element of $G$, then $\alpha^\eta(s_1) \leq \alpha^\eta(e), \forall s_1 \in G$.

Also, $\alpha^\eta(s_1 s_2^{-1}) = \alpha^\eta(e)$ implying that $\alpha^\eta(s_1) = \alpha^\eta(s_2)$, for all $s_2 \in G$.

Proposition 3.6: Every IFSG of $G$ is an $\eta$-IFSG of $G$.

Proof: Let $s_1, s_2 \in G$, then by using the fact that $\alpha$ is an IFSG, we have
\[
\tau_{\alpha^\eta}(s_1 s_2) = \psi[\min(\tau_{\alpha}(s_1), \tau_{\alpha}(s_2)), \eta] = \psi[\min(\tau_{\alpha}(s_1), \tau_{\alpha}(s_2)), \eta] = \min(\tau_{\alpha^\eta}(s_1), \tau_{\alpha^\eta}(s_2)).
\]

Similarly, it can be proved that $\xi_{\alpha^\eta}(s_1 s_2) = \max(\xi_{\alpha^\eta}(s_1), \xi_{\alpha^\eta}(s_2)).$

Moreover,
\[
\tau_{\alpha^\eta}(s_1^{-1}) = \psi[\min(\tau_{\alpha}(s_1), \eta)] = \psi[\tau_{\alpha}(s_1), \eta] = \tau_{\alpha^\eta}(s_1).
\]

Similarly, $\xi_{\alpha^\eta}(s_1^{-1}) = \xi_{\alpha^\eta}(s_1)$.

Consquently, $\alpha^\eta$ is an $\eta$-IFSG of a group $G$.

Remark 3.7: An $\eta$-IFSG need not be an IFSG, that is, the converse of Proposition 3.6 does not hold.

The above algebraic fact can be viewed in the following example.

Example 3.8: Let $G = \{\pm 1, \pm i\}$ be the fourth root of unity. We define IFS $\alpha$ of $G$ as
\[
\alpha = \{< 1, 0.5, 0.4 >, < 1, 0.2, 0.7 >, < 1, 0.3, 0.7 >, < 1, 0, 2, 0.7 > \}
\]

Note that, $\alpha$ is not IFSG of $G$. Let $\eta = 0.4$, then
\[
\alpha^\eta = \{< 1, 0.5, 0.4 >, < 1, 0.2, 0.7 >, < 1, 0.3, 0.7 >, < 1, 0, 2, 0.7 > \}
\]

It is clear that $(0.5, 0.4)$ – cut set of 0.4-IFS is $\alpha^{0.4} = \{1\}$ and $\alpha^{0.6} = \{1\}$, whereas $(0.3, 0.7)$ – cut set of 0.4-IFS is given by $\alpha^{0.4} = \{1, i\}$ and $\alpha^{0.6} = \{\pm 1, \pm i\}$.

Note that, each of the above cut set of $\eta$-IFS is a subgroup of $G$. Hence it is an $\eta$-IFS.

The following result presents the condition under which a given $\eta$-IFS is an $\eta$-IFSG.

Proposition 3.9: Let $\alpha$ be any IFS of a group $G$ such that $\tau_{\alpha^\eta}(s_1^{-1}) = \tau_{\alpha^\eta}(s_1)$ and $\xi_{\alpha^\eta}(s_1^{-1}) = \xi_{\alpha^\eta}(s_1)$ for all $s_1 \in G$.

Moreover, $\eta < \min(1, 1 - m)$, where, $l = \min(\tau_{\alpha^\eta}(s_1) : s_1 \in G)$ and $m = \max(\xi_{\alpha^\eta}(s_1) : s_1 \in G)$, then $\alpha$ is an $\eta$-IFSG of $G$.

Proof: In view of given conditions, we have $l > \eta$ and $m < 1 - \eta$. It follows that $\tau_{\alpha^\eta}(s_1) > \eta$ and $\xi_{\alpha^\eta}(s_1) < 1 - \eta$, for all $s_1 \in G$. Therefore, $\tau_{\alpha^\eta}(s_1 s_2) \geq \min(\tau_{\alpha^\eta}(s_1), \tau_{\alpha^\eta}(s_2))$ and $\xi_{\alpha^\eta}(s_1 s_2) \leq \max(\xi_{\alpha^\eta}(s_1), \xi_{\alpha^\eta}(s_2))$ for all $s_1, s_2 \in G$. Moreover, for any $s_1 \in G$, we obtain
Definition 3.13: Let \( \alpha^n \) be an \( \eta \)-IFSG of \( G \) and \( s_1 \in G \). An \( \eta \)-intuitionistic fuzzy left coset of \( \alpha^n \) is defined as \( \alpha^n \cdot s_1 = \{ \tau \cdot s_1 : \tau \in \alpha^n \} \), for all \( s_1 \in G \).

The following result shows that every IFNSG of \( G \) is also \( \eta \)-IFNSG of \( G \).

**Proposition 3.15:** If \( \alpha \) is an IFNSG of a group \( G \), then \( \alpha^n \) is also an \( \eta \)-IFNSG of \( G \).

**Proof:** Let \( \alpha \) be an IFNSG of \( G \), then for all \( s_1, g \in G \), \( \tau_\alpha (g s_1^{-1}) = \tau_\alpha (s_1^{-1})g \) and \( \xi_\alpha (g s_1^{-1}) = \xi_\alpha (s_1^{-1})g \), implying that 

\[
\tau_\alpha (s_1^{-1}) = \psi \{ \tau_\alpha (s_1^{-1}), \eta \} = \psi \{ \tau_\alpha (g s_1^{-1}), \eta \} = \tau_\alpha (g s_1^{-1}).
\]

Similarly, it can be proved that \( \xi_\alpha (s_1^{-1}) = \eta \).

Corollary 3.11: The intersection of any number of \( \eta \)-IFSG’s of \( G \) is an \( \eta \)-IFSG of \( G \).

Remark 3.12: The union of two \( \eta \)-IFSG’s of \( G \) may not be an \( \eta \)-IFSG of \( G \).

**Example 3.16:** Consider the symmetric group of order 6, that is, \( S_3 = \{ e, p, q : p^3 = q^3 = e, qp = p^2q \} \). Define IFNSG \( \alpha \) of \( S_3 \) by

\[
\tau_\alpha (s_1) = \begin{cases} 0.80 & \text{if } s_1 \in \{ e, q \} \\ 0.70 & \text{otherwise} \end{cases}
\]

and

\[
\xi_\alpha (s_1) = \begin{cases} 0.10 & \text{if } s_1 \in \{ e, q \} \\ 0.13 & \text{otherwise} \end{cases}
\]

then

\[
\alpha = \begin{cases} < e, 0.08, 0.10 >, < e, 0.70, 0.13 >, & \text{if } s_1 \in \{ e, q \} \\ < pq, 0.70, 0.13 >, < q, 0.80, 0.10 >, & \text{if } s_1 \in \{ pq, q \} \end{cases}
\]

Since \( \tau_\alpha (s_1 s_2) = 0.70 \neq 0.80 = \tau_\alpha (s_2 s_1) \), therefore \( \alpha \) is not an IFNSG.

Next, let \( \eta = 0.1 \), then

\[
\alpha^{0.1} = \begin{cases} < e, 0.3, 0.3 >, < e, 0.3, 0.3 >, & \text{if } s_1 \in \{ e, q \} \\ < pq, 0.3, 0.3 >, < q, 0.3, 0.3 >, & \text{if } s_1 \in \{ pq, q \} \end{cases}
\]

One can see that \( \alpha^{0.1} \) is an \( \eta \)-IFNSG.

**Definition 3.13:** Let \( \alpha^n \) be an \( \eta \)-IFSG of \( G \) and \( s_1 \in G \). An \( \eta \)-intuitionistic fuzzy left coset of \( \alpha^n \) is defined as \( \alpha^n \cdot s_1 = \{ \tau \cdot s_1 : \tau \in \alpha^n \} \), for all \( s_1 \in G \).

**Definition 3.14:** An \( \eta \)-IFSG \( \alpha^n \) of \( G \) is called an \( \eta \)-intuitionistic fuzzy normal subgroup (IFNSG) of \( G \) if \( s_1 \alpha^n = \alpha^n s_1 \), for all \( s_1 \in G \).
Also, ξ_{α}(s_{1}) < 1/η, so \( \tau_{α}(s_{1}) = \psi\{\tau_{α}(g_{s_{1}}^{-1}), η\} = θ \) and \( \xi_{α}(s_{1}) = \psi\{\xi_{α}(g_{s_{1}}^{-1}), 1 - η\} = φ \) for all \( g \in G \).

Similarly, \( λ_{α}(g) = \psi\{λ_{α}(s^{-1}g), η\} = \theta \) and \( \xi_{α}(g) = \psi\{ξ_{α}(s^{-1}g), 1 - η\} = φ \).

This concludes the proof.

4. Some characterizations of \( η \)-intuitionistic fuzzy homomorphism

In this section, we define \( η \)-intuitionistic fuzzy homomorphism between any two \( η \)-IFSG's and establish some important characterizations of this phenomenon.

**Definition 4.1:** Let \( α^{η} \) and \( β^{η} \) be two \( η \)-IFSG's of the groups \( G_{1} \) and \( G_{2} \) respectively and \( φ : G_{1} \rightarrow G_{2} \) be a group homomorphism. Then \( φ \) is called \( η \)-intuitionistic fuzzy homomorphism from \( α^{η} \) to \( β^{η} \) if \( φ(α^{η}) = β^{η} \).

The following result indicates that an \( η \)-intuitionistic fuzzy homomorphic image of the \( η \)-IFSG is an \( η \)-IFSG.

**Theorem 4.2:** Let \( α^{η} \) be an \( η \)-IFSG of group \( G_{1} \) and \( φ : G_{1} \rightarrow G_{2} \) be a surjective homomorphism, then \( φ(α^{η}) \) is an \( η \)-IFSG of group \( G_{2} \).

**Proof:** In view of the given condition, for any two elements \( t_{1}, t_{2} \in G_{2} \), there exits \( s_{1}, s_{2} \in G_{1} \), such that \( φ(s_{1}) = t_{1} \) and \( φ(s_{2}) = t_{2} \).

Consider \( φ(α^{η})(t_{1}, t_{2}) = (τ_{φ}(α^{η})(t_{1}, t_{2}), ξ_{φ}(α^{η})(t_{1}, t_{2})) \). Then

\[
τ_{φ}(α^{η})(t_{1}, t_{2}) = τ_{φ}(α^{η})(t_{1}, t_{2})
\]

\[
= \psi\{τ_{φ}(α^{η})(φ(s_{1}), φ(s_{2}))\}, η\}
\]

\[
= \psi\{τ_{φ}(α^{η})(φ(s_{1}s_{2}))\}, η\}
\]

\[
≥ \psi\{τ_{α^{η}}(s_{1}s_{2}), η\}
\]

\[
= τ_{α^{η}}(s_{1}s_{2})
\]

\[
≥ \min\{τ_{α^{η}}(s_{1}), τ_{α^{η}}(s_{2})\}
\]

\[
= \min\{τ_{φ}(α^{η})(t_{1}), τ_{φ}(α^{η})(t_{2})\}.
\]

Thus, \( τ_{φ}(α^{η})(t_{1}, t_{2}) ≥ \min\{τ_{φ}(α^{η})(t_{1}), τ_{φ}(α^{η})(t_{2})\} \).

Similarly, it can be proved that

\[
ξ_{φ}(α^{η})(t_{1}, t_{2}) \leq \max\{ξ_{φ}(α^{η})(t_{1}), ξ_{φ}(α^{η})(t_{2})\}.
\]

Further,

\[
τ_{φ}(α^{η})(t_{1}) = \psi\{τ_{φ}(α^{η})(φ(s_{1})) = τ_{1}\}
\]

\[
= \psi\{τ_{φ}(α^{η})(φ(s_{1})) = τ_{1}\}
\]

\[
= \psi\{τ_{φ}(α^{η})(t_{1})\}.
\]

Similarly,

\[
ξ_{φ}(α^{η})(t_{1}) = \xi_{φ}(α^{η})(t_{1})\]

The following result depicts that every \( η \)-intuitionistic fuzzy homomorphic image of the \( η \)-IFNSG is an \( η \)-IFNSG.

**Theorem 4.3:** Let \( α^{η} \) be an \( η \)-IFNSG of \( G_{1} \) and \( φ : G_{1} \rightarrow G_{2} \) be a bijective homomorphism, then \( φ(α^{η}) \) is an \( η \)-IFNSG of \( G_{2} \).

**Proof:** In view of the given condition, for \( t_{1}, t_{2} \in G_{2} \), there exists a unique pair of elements \( s_{1}, s_{2} \in G_{1} \) such that \( φ(s_{1}) = t_{1} \) and \( φ(s_{2}) = t_{2} \).

Consider,

\[
(φ(α^{η}))^{η}(t_{1}, t_{2}) = (τ_{φ}(α^{η})(t_{1}, t_{2}), ξ_{φ}(α^{η})(t_{1}, t_{2}))
\]

which implies that

\[
τ_{φ}(α^{η})(t_{1}, t_{2}) = \psi\{τ_{φ}(α)(φ(s_{1}), φ(s_{2})), η\}
\]

\[
= \psi\{τ_{φ}(α)(φ(s_{1}s_{2})), η\}
\]

\[
= \psi\{τ_{φ}(α)(φ(s_{1}s_{2})), η\}
\]

which follows that

\[
τ_{φ}(α^{η})(t_{1}, t_{2}) = \tau_{φ}(φ(s_{1}s_{2}))
\]

\[
= \psi\{τ_{φ}(φ(s_{1}s_{2})), η\}
\]

\[
= \psi\{τ_{φ}(φ(s_{1}s_{2})), η\}
\]

implies that

\[
φ(τ_{φ}(α^{η})(φ(s_{1})), η) = \psi\{τ_{φ}(φ(t_{2}t_{1})), η\}
\]

\[
= τ_{φ}(α^{η})(t_{2}t_{1}).
\]

Similarly, one can prove that

\[
ξ_{φ}(α^{η})(t_{1}, t_{2}) = \xi_{φ}(α^{η})(t_{2}t_{1}).
\]

The following result depicts that every \( η \)-intuitionistic fuzzy inverse homomorphic image of \( η \)-IFSG is always \( η \)-IFSG.

**Theorem 4.4:** Let \( β^{η} \) be an \( η \)-IFSG of a group \( G_{2} \) and \( φ \) be a group homomorphism from groups \( G_{1} \) to \( G_{2} \), then \( φ^{-1}(β^{η}) \) is also an \( η \)-IFSG of \( G_{1} \).

**Proof:** Suppose \( β^{η} \) is an \( η \)-IFSG of \( G_{2} \), then there exists a unique pair of elements \( s_{1}, s_{2} \in G_{1} \) such that \( φ^{-1}(β^{η})(s_{1}s_{2}) = (τ_{φ^{-1}(β^{η})}(s_{1}s_{2}), ξ_{φ^{-1}(β^{η})}(s_{1}s_{2})) \). Also,

\[
τ_{φ^{-1}(β^{η})}(s_{1}s_{2}) = τ_{φ^{-1}(β^{η})}(φ(s_{1}s_{2}))
\]

\[
= τ_{φ^{-1}(β^{η})}(φ(s_{1}))
\]

\[
= \min\{τ_{φ^{-1}(β^{η})}(φ(s_{1})), τ_{φ^{-1}(β^{η})}(φ(s_{2}))\}
\]

\[
= \min\{τ_{φ^{-1}(β^{η})}(s_{1}), τ_{φ^{-1}(β^{η})}(s_{2})\}.
\]

Thus, \( τ_{φ^{-1}(β^{η})}(s_{1}s_{2}) ≥ \min\{τ_{φ^{-1}(β^{η})}(s_{1}), τ_{φ^{-1}(β^{η})}(s_{2})\} \).

Similarly, one can prove that \( ξ_{φ^{-1}(β^{η})}(s_{1}s_{2}) \leq \max\{ξ_{φ^{-1}(β^{η})}(s_{1}), ξ_{φ^{-1}(β^{η})}(s_{2})\} \).
Moreover,
\[
\tau_{\phi^{-1}(\beta \eta)}(t_1^{-1}) = \tau_{\beta \eta}(\phi(t_1^{-1}))
\]
\[
= \tau_{\beta \eta}(\phi(t_1))^{-1}
\]
\[
= \tau_{\beta \eta}(\phi(t_1))
\]
\[
= \tau_{\phi^{-1}(\beta \eta)}(t_1).
\]
Similarly, \(\xi_{\phi^{-1}(\beta \eta)}(t_1^{-1}) = \xi_{\phi^{-1}(\beta \eta)}(t_1)\). Consequently, \(\phi^{-1}(\beta \eta)\) is an \(\eta\)-IFNSG of \(G_1\).

The following theorem, we prove that every \(\eta\)-intuitionistic fuzzy homomorphic inverse image of \(\eta\)-IFNSG is an \(\eta\)-IFNSG.

**Theorem 4.5:** Let \(\beta \eta\) be an \(\eta\)-IFNSG of a group \(G_2\) and \(\phi : G_1 \rightarrow G_2\) be a group homomorphism, then \(\phi^{-1}(\beta \eta)\) is an \(\eta\)-IFNSG of \(G_1\).

**Proof:** Suppose \(\beta \eta\) is an \(\eta\)-IFNSG of a group \(G_2\) then there exists a unique pair of elements \(s_1, s_2 \in G_1\) such that \(\phi^{-1}(\beta \eta)(s_1 s_2) = (\tau_{\phi^{-1}(\beta \eta)}(s_1 s_2), \xi_{\phi^{-1}(\beta \eta)}(s_1 s_2))\), where
\[
\tau_{\phi^{-1}(\beta \eta)}(s_1 s_2) = \tau_{\beta \eta}(\phi(s_1 s_2))
\]
\[
= \tau_{\beta \eta}(\phi(s_1) \phi(s_2))
\]
\[
= \tau_{\beta \eta}(\phi(s_2) \phi(s_1))
\]

implying that
\[
\tau_{\beta \eta}(\phi(s_2 s_1)) = \tau_{\phi^{-1}(\beta \eta)}(s_2 s_1).
\]
Similarly, one can prove that \(\xi_{\phi^{-1}(\beta \eta)}(s_1 s_2) = \xi_{\phi^{-1}(\beta \eta)}(s_2 s_1)\). Thus, \(\phi^{-1}(\beta \eta)\) is an \(\eta\)-IFNSG of \(G_1\).

**5. Conclusion**

The \(\eta\)-intuitionistic fuzzy set generalizes the concept of classical fuzzy set intending to assess the ambiguity level of a fuzzy situation. In this research, we have presented cosets and subgroups of \(\eta\)-intuitionistic fuzzy sets and then used these concepts to construct the \(\eta\)-intuitionistic fuzzy normal subgroup. After discussing some important features of these concepts, we demonstrated the effectiveness of the image and inverse image of \(\eta\)-intuitionistic fuzzy normal subgroup followed by \(\eta\)-intuitionistic fuzzy homomorphism.

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**References**

[1] Zadeh LA. Fuzzy sets. Inf Control. 1965 Jun 1;8(3):338–353.

[2] Rosenfeld A. Fuzzy groups. J Math Anal Appl. 1971 Sep 1;35(3):512–517.

[3] Das PS. Fuzzy groups and level subgroups. J Math Anal Appl. 1981 Nov 1;84(1):264–269.

[4] Mukherjee NP, Bhattacharya P. Fuzzy normal subgroups and fuzzy cosets. Inf Sci (NY). 1981 Aug 1;13(1):225–239.

[5] Choudhury FP, Chakraborty AB, Khare SS. A note on fuzzy subgroups and fuzzy homomorphism. J Math Anal Appl. 1988 May 1;131(2):537–553.

[6] Gupta MM, Qi J. Theory of T-norms and fuzzy inference methods. Fuzzy Sets Syst. 1991 Apr 15;40(3):431–450.

[7] Ajmal N, Prajapati AS. Fuzzy cosets and fuzzy normal subgroups. Inf Sci (NY). 1992 Oct 1;64(1-2):17–25.

[8] Zhang Y, Zou K. A note on an equivalence relation on fuzzy subgroups. Fuzzy Sets Syst. 1998/95(2):243–247.

[9] Mordeson JN, Bhutani KR, Rosenfeld A. Fuzzy subsets and fuzzy subgroups. In Fuzzy group theory. Berlin, Heidelberg: Springer; 2005. p. 1–39.

[10] Atanassov KT. Intuitionistic fuzzy sets. Fuzzy Sets Syst. 1986;20(1):87–96.

[11] Biswas R. Intuitionistic fuzzy subgroups. Math Forum. 1989;10:37–46.

[12] Gargov G, Atanassov K. On the intuitionistic fuzzy logic operations. Notes on IFS. 1995(11):1–4.

[13] Çoker D. Fuzzy rough sets are intuitionistic L-fuzzy sets. Fuzzy Sets Syst. 1998;96(3):381–383.

[14] Atanassov KT. Intuitionistic fuzzy sets. In Intuitionistic fuzzy sets. Heidelberg: Physica; 1999. p. 1–137.

[15] Hur K, Jang SY, Kang HW. Intuitionistic fuzzy normal subgroups and intuitionistic fuzzy cosets. Int J Fuzzy Logic Intell Syst. 2010;10(1):82–88.

[16] Zhan JIANMING, Tan Z. Intuitionistic M-fuzzy groups. Soochow J Math. 2004;30(1):85–90.

[17] Palaniappan N, Naganathan S, Arjunan K. A study on intuitionistic L-fuzzy subgroups. Appl Math Sci. 2009;3(53):2619–2624.

[18] Marasheh MF, Salleh AR. The intuitionistic fuzzy normal subgroup. Int J Fuzzy Logic Intell Syst. 2010;10(1):82–88.

[19] Li XP, Wang GJ. Intuitionistic fuzzy groups. Hacettepe J Math Stat. 2011;40(5):663–672.

[20] Sharma PK. Cut of intuitionistic fuzzy groups. Int Math Forum. 2011;6(53):2605–2614.

[21] Sharma PK. t-Intuitionistic fuzzy subgroups. Int J Fuzzy Math Syst. 20123:233–243.

[22] Doda N, Sharma PK. Counting the number of intuitionistic fuzzy subgroups of finite Abelian groups of different order. Notes on Intuitionistic Fuzzy Sets. 2013;19:42–47.
[23] Zeng S, Chen SM, Fan KY. Interval-valued intuitionistic fuzzy multiple attribute decision making based on non-linear programming methodology and TOPSIS method. Inf Sci (NY). 2020;506:424–442.

[24] Zeng S, Llopis-Albert C, Zhang Y. A novel induced aggregation method for intuitionistic fuzzy set and its application in multiple attribute group decision making. Int J Intell Syst. 2018;33(11):2175–2188.

[25] Zeng S, Chen SM, Kuo LW. Multiattribute decision making based on novel score function of intuitionistic fuzzy values and modified VIKOR method. Inf Sci (Ny). 2019;488:76–92.

[26] Adamu IM, Tella Y, Alkali AJ. On normal Sub-intuitionistic fuzzy Multigroups. Ann Pure Appl Math. 2019;19(2):127–139.

[27] Sahoo S, Pal M. Different types of products on intuitionistic fuzzy graphs. Pac Sci Rev A: Nat Sci Eng. 2015;17(3):87–96.

[28] Kozae AM, Elshenawy A, Omran M. Intuitionistic fuzzy set and its application in selecting specialization: a case study for engineering students. Int J Math Anal Appl. 2015;2(6):74–78.

[29] Kang KT, Song SZ, Jun YB. Multipolar intuitionistic fuzzy Set with finite degree and Its Application in BCK/BCI-Algebras. Mathematics. 2020;8(2):177.

[30] Akram M, Akmal R. Operations on intuitionistic fuzzy graph structures. Fuzzy Inf Eng. 2016;8(4):389–410.

[31] Wan S, Dong J. A Selection method based on MAGDM with interval-valued intuitionistic fuzzy sets. In Decision making theories and methods based on interval-valued intuitionistic fuzzy sets. Singapore: Springer; 2020. p. 115–137.

[32] Al-Husbana R, Sallehb AR, Ahmadb AGB. Complex-intuitionistic fuzzy group. Global J Pure Appl Math. 2016;12(6):4929–4949.

[33] Ejegwa PA, Tyoakaa GU, Ayenge AM. Application of intuitionistic fuzzy sets in electoral system. Int J Fuzzy Math Arch. 2016;10(1):35–41.

[34] Lee JG, Lim PK, Kim JH, et al. Intuitionistic continuous, closed and open mappings. Infin Study. 2018.