COUNTEREXAMPLES TO THE GAUSSIAN VS. MZ DERIVATIVES
CONJECTURE

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ABSTRACT. J. Marcinkiewicz and A. Zygmund proved in 1936 that the special
n-th generalized Riemann derivative \( \tilde{D}_n f(x) \) with nodes 0, 1, 2, 2², \( \ldots, 2^{n-1} \), is equivalent to
the n-th Peano derivative \( f^{(n)}(x) \), for all \( n - 1 \) times Peano differentiable functions \( f \)
at \( x \). Call every \( n \)-th generalized Riemann derivative with this property an MZ derivative.
A recent paper by the authors introduced the \( n \)-th Gaussian derivatives as the \( n \)-th gen-
eralized Riemann derivatives with nodes either 0, 1, \( q, q^2, \ldots, q^{n-1} \) or 1, \( q, q^2, \ldots, q^n \),
where \( q \neq 0, \pm 1 \), proved that the Gaussian derivatives are MZ derivatives, and conjec-
tured that these are all MZ derivatives. In this article, we invalidate the conjecture in three
different ways, by scales of generalized Riemann derivatives, by reference to the classi-
fication of generalized Riemann derivatives, and by means of an independent counterex-
ample. In addition, we show that Riemann differentiation is not equivalent to a Gaussian
differentiation in orders at least three, find an explicit description of the equivalence class
of a generalized Riemann derivative, and prove a more general quantum analogue of the
GGR Theorem. The symmetric versions of all these results are also included in the article.

For a positive integer \( n \), an \( n \)-th generalized Riemann difference of a function \( f \) at \( x \) is a
difference of the form
\[
\Delta_A(h, x; f) = \sum_{i=0}^{m} a_i f(x + b_i h),
\]
where the datum \( A = \{a_i; b_i\} \) has non-zero \( a_i \) and distinct \( b_i \), and these satisfy the Van-
dermonde linear system \( \sum a_i (b_j)^i = \delta_{j,n} \cdot n! \), for \( j = 0, 1, \ldots, n \). The numbers \( x + b_i h \)
are the base points, and the numbers \( b_i \) are the nodes of the given difference. The above
linear system is consistent when \( m \geq n \), and has a unique solution when \( m = n \), in which
case the difference is exact. The best known exact \( n \)-th generalized Riemann differences
are the \( n \)-th Riemann difference,
\[
\Delta_n(h, x; f) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + (n - i)h),
\]
the \( n \)-th symmetric Riemann difference,
\[
\Delta_n^s(h, x; f) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + (\frac{n}{2} - i)h),
\]

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and the $k$-shift of the $n$-th Riemann difference $\Delta_n$,

$$\Delta_{n,k}(h, x; f) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + (n + k - i)h),$$

where $k$ is a real number. When $k$ is a non-negative integer, the differences $\Delta_{n,k}$ and $\Delta_{n,-k}$ are respectively called the $k$-th forward shift and the $k$-th backward shift of the $n$-th Riemann difference. Note also that $\Delta_n^* = \Delta_{n,-n/2}$. A glossary of all exact $n$-th generalized Riemann differences used in this article, together with their sets of nodes, is given by the following table. The differences $\Delta_n, \Delta_n^*$ and $\Delta_n^1$ were computed in [10]. Their expressions involve Gaussian binomial coefficients and by this they are $q$-analogues of the Riemann differences. For this reason, they were called Gaussian Riemann differences. In this paper we will simply call them Gaussian differences. We do not have a notation for the exact differences whose sets of nodes are given at the bottom of the table. Instead, we refer to them by their associated generalized Riemann derivatives, which we define next.

| Difference | Nodes | Difference | Nodes |
|------------|-------|------------|-------|
| $\Delta_n$ | $0, 1, \ldots, n$ | $q\Delta_n$ | $0, 1, q, q^2, \ldots, q^{n-1}$ |
| $\Delta_{n,k}$ | $k, k+1, \ldots, k+n$ | $\Delta_n^*$ | $-\frac{q}{q-1} - \frac{n}{q-1} + 1, \ldots, -\frac{n}{q-1}$ |
| $q\Delta_n$ | $1, q, q^2, \ldots, q^n$ | $q\Delta_{n+1}$ | $0, \pm 1, \pm q, \pm q^2, \ldots, \pm q^{n-1}$ |
| $q\Delta_{n,k}$ | $q^k, q^{k+1}, \ldots, q^{k+n}$ | $q\Delta_{n+1}$ | $\pm 1, \pm q, \pm q^2, \ldots, \pm q^{n-1}$ |

An $n$-th generalized Riemann derivative is a limit of a difference quotient of the form

$$D_A f(x) = \lim_{h \to 0} \Delta_A(h, x; f)/h^n,$$

where $\Delta_A$ is an $n$-th generalized Riemann difference. In particular, the $n$-th Riemann derivative $D_n f(x)$ (resp. the $n$-th symmetric Riemann derivative $D_n^s f(x)$) correspond to $\Delta_n$ (resp. $\Delta_n^*$), the $n$-th Gaussian derivatives $q D_n f(x)$ and $q D_n^s f(x)$ (resp. the $n$-th symmetric Gaussian derivative $q D_n^s f(x)$) correspond to $q \Delta_n f(x)$ and $q \Delta_n^* f(x)$ (resp. to $q \Delta_n^1 f(x)$), and the $k$-th backward shift $D_{n-k} f(x)$ of the $n$-th Riemann derivative corresponds to $\Delta_{n-k}$. When the derivative $D_A f(x)$ exists, we say that $f$ is $n$ times generalized Riemann differentiable (of type $A$) at $x$, or $A$-differentiable at $x$.

A scale by a non-zero $r$ of an $n$-th generalized Riemann difference $\Delta_A(h, x; f)$ with data vector $A = \{a_1; b_1\}$ is the $n$-th generalized Riemann difference

$$\Delta_A(rh, x; f) := \frac{1}{r^n} \Delta_A(rh, x; f),$$

with data vector $A_r = \{a_i/r^n; rb_i\}$. For example, $\Delta_{n,k}$ is the scale by $q^k$ of $\Delta_n$. For all $A$ and $r$, a function $f$ is $A$-differentiable at $x$ if and only if $f$ is $A_r$-differentiable at $x$ and $D_A f(x) = D_A f(x)$.

By writing the general term of the most general arithmetic progression $\{a + bi \mid i = 0, 1, \ldots, n\}$ as $b(a/b + i)$, the exact generalized Riemann difference having this as its set of nodes is the scale by $b$ of the difference $\Delta_{n,a/b}$. And the exact generalized Riemann derivative whose set of nodes is the most general geometric progression $\{bq^i \mid i = 0, 1, \ldots, n\}$ is the scale by $b$ of the difference $q \Delta_{n,k}$.

The more widely used notion of an $n$-th difference is a non-zero scalar multiple of an $n$-th generalized Riemann difference. Vice-versa, an $n$-th generalized Riemann difference is a normalized $n$-th difference, where the normalization condition, or the $n$-th Vandermonde relation $\sum a_i(b_i)^n = n!$, is meant to make the value of the generalized derivative
agree with the value of the ordinary \( n \)-th derivative, for all \( n \) times ordinary differentiable functions at \( x \).

Another family of generalized derivatives is the Peano derivatives. A function \( f \) is \( n \) times Peano differentiable at \( x \) if it is approximated to order \( n \) about \( x \) by a polynomial, i.e., if there are constants \( c_0 = f(x), c_1, \ldots, c_n \) such that
\[
f(x + h) = c_0 + \frac{c_1}{1!} h + \frac{c_2}{2!} h^2 + \cdots + \frac{c_n}{n!} h^n + o(h^n).
\]
The numbers \( c_0, c_1, \ldots, c_n \) are usually denoted by \( f^{(0)}(x), f^{(1)}(x), \ldots, f^{(n)}(x) \) and called the first \( n \) Peano derivatives of \( f \) at \( x \). Note that the existence of the \( n \)-th Peano derivative \( f^{(n)}(x) \) implies the existence of all lower order Peano derivatives at \( x \). Moreover, \( f^{(0)}(x) = f(x) \) and \( f^{(1)}(x) = f'(x) \), while, in higher orders, the \( n \)-th Peano derivative is known to be strictly more general than the ordinary derivative.

Two generalized differentiations either imply or are equivalent to each other, if the existence of one implies or is equivalent to the existence of the other, for all functions \( f \) at \( x \). By Taylor or Peano expansion of the generalized Riemann difference, one can show that every \( n \) times Peano differentiable function \( f \) at \( x \) is \( n \) times generalized Riemann differentiable at \( x \) of any kind \( A \) and \( D_A f(x) = f^{(n)}(x) \). The converse of this, or the equivalence between Peano and (generalized) Riemann differentiations, has been a problem since 1927, initiated by Khintchine in [31]. Marcinkiewicz and Zygmund have shown in [34] that the \( n \)-th symmetric Riemann derivative \( R_n^s \) is equivalent to the \( n \)-th Peano derivative, for all functions \( f \) a.e. at \( x \) on a measurable set, and Ash has shown in [1] that the same result holds true for all \( n \)-th generalized Riemann derivatives. The measurability condition on the set was removed by Fejzić and Weil in [24].

Our first motivation comes from the following theorem, proved by Marcinkiewicz and Zygmund in 1936, that both of the above almost everywhere results relied on:

**Theorem MZ.** ([34], §10, Lemma 1). Let \( D_n, n \geq 1 \), be the \( n \)-th generalized Riemann derivative with nodes \( 0, 1, 2, 4, \ldots, 2^{n-1} \). Then, for all functions \( f \) and points \( x \),
\[
\text{both } f^{(n-1)}(x) \text{ and } D_n f(x) \text{ exist } \iff \ f^{(n)}(x) \text{ exists.}
\]

In other words, the \( n \)-th generalized Riemann derivative \( D_n f(x) \) is equivalent to the \( n \)-th Peano derivative \( f^{(n)}(x) \), for all \( n - 1 \) times Peano differentiable functions \( f \) at \( x \).

The goal of this article is to find all \( n \)-th generalized Riemann derivatives \( D_A f(x) \) that can play the role of \( D_n f(x) \) in Theorem MZ. More focused details about this goal are found in the next part of the introduction, where we introduce the MZ derivatives and two known open problems related to them, that both complete the explanation of the title and allow a brief discussion of some of the results.

0.1. **MZ derivatives and two conjectures related to them.** Theorem MZ prompts the first part of the following definition. The motivation for the second part will come later on in Section 0.2.

**Definition MZ.** (i) An \( n \)-th generalized Riemann differentiation \( D_A \) is an **MZ differentiation** at a point \( x \), if, for all \( n - 1 \) times Peano differentiable functions \( f \) at \( x \),
\[
f \text{ is } n \text{ times Peano differentiable at } x \iff f \text{ is } A\text{-differentiable at } x.
\]

(ii) A set \( \{ D_{A_1}, \ldots, D_{A_p} \} \) of \( n \)-th generalized Riemann differentiations is an **MZ-set** at a point \( x \), if, for all \( n - 1 \) times Peano differentiable functions \( f \) at \( x \),
\[
f \text{ is } n \text{ times Peano differentiable at } x \iff f \text{ is } A_j\text{-differentiable at } x, \text{ for all } j.
\]
In particular, when \( p = 1 \), the notion of an MZ differentiation and the one of an MZ-set coincide.

Our stated goal is then to solve the following problem:

**Problem MZ.** For each order \( n \) and point \( x \), determine all \( n \)-th MZ differentiations at \( x \).

A solution to Problem MZ was proposed in [10], where it was proved that, for all real numbers \( q \), with \( q \neq 0, \pm 1 \), the \( n \)-th Gaussian derivatives \( qD_n f(x) \) with nodes \( 0, 1, q, \ldots, q^{n-1} \) and \( qD_n f(x) \) with nodes \( 1, q, \ldots, q^n \) are MZ derivatives at \( x \). Furthermore, it was conjectured that these are all MZ derivatives of order \( n \) at \( x \), that is,

**Gaussian vs. MZ Derivatives Conjecture** ([10], Conjecture A). Each MZ differentiation is a Gaussian differentiation.

In other words, the Gaussian vs. MZ Derivatives Conjecture asserts that all MZ derivatives classify as Gaussian. Article [10] proved that the conjecture is true for \( n = 1, 2 \), and left it open for \( n \geq 3 \).

A particular case of the Gaussian vs. MZ Derivatives Conjecture is the following conjecture from [5] on the relation between Riemann differentiation and MZ differentiation:

**Riemann vs. MZ Derivatives Conjecture** ([5], Conjecture 4.2).

When \( n \geq 3 \), \( D_n \) is not an MZ differentiation, for all functions \( f \) at \( x \).

This conjecture was first formally stated in [5]. It was proved for \( n = 3 \) in [9] and for \( n = 7 \) in [10], and is now open for the remaining cases.

In this article we invalidate the Gaussian vs. MZ Derivatives Conjecture by three counterexamples. The order we chose to arrange them allowed the restatement of the conjecture in a sharper form after each of the first two counterexamples.

The first counterexample, given in Example 1.1, is a scale of a Gaussian derivative that is not a Gaussian. The conjecture is then updated to “every MZ derivative is a scale of a Gaussian”. Proving that the derivative is not a Gaussian is easily done by inspection. The motivation for this counterexample comes from [15], where all exact first order MZ derivatives are determined, and all such derivatives are scales of Gaussians, but not all of them are Gaussians.

The second counterexample, given in Example 1.3, is an MZ derivative that is equivalent to a Gaussian, but not a scale of a Gaussian. The conjecture is further updated to “every MZ derivative is equivalent to a Gaussian”. Proving that the derivative is not equivalent to a Gaussian is done by using a powerful recent result, the classification of generalized Riemann derivatives, [6, Theorem 2], explained in Section 0.2, and restated as Theorem A.

The third counterexample, given by Proposition 3.1, is an MZ derivative that is not equivalent to a Gaussian. Proving that it is not equivalent to a Gaussian is done by invoking another major recent result, Theorem GGR, proved in [5] and explained in Section 0.3. Since at the moment we do not have a third update of the statement of the conjecture, while Problem MZ remains an important problem, the Riemann vs. MZ Derivatives conjecture is elevated as the most outstanding open problem of this subject.
0.2. The Classification of Generalized Riemann Derivatives. Before getting into the
details of this subsection, consider the following three first order generalized Riemann
derivatives:

\[ D_A f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}, \quad D_B f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \]
and

\[ D_C f(x) = \lim_{h \to 0} \frac{3f(x+h) - 5f(x) + 2f(x-h)}{h}. \]
The first is the symmetric derivative \( f'_s(x) \), and the second is the ordinary derivative \( f'(x) \).

The example of \( f(x) = |x| \), for which \( f'_s(0) \) exists and is zero and \( f'(0) \) does not exist,
shows that \( D_A \) is not equivalent to \( D_B \) at \( x = 0 \). And by taking the limit as \( h \to 0 \) in the identity

\[ \frac{3}{5} \cdot \frac{3f(h) - 5f(0) + 2f(-h)}{h} + \frac{2}{5} \cdot \frac{3f(-h) - 5f(0) + 2f(h)}{h} = \frac{f(h) - f(0)}{h}, \]
the derivative \( D_C \) implies, hence is equivalent to, the derivative \( D_B \) at \( x = 0 \). We deduce
that in proving or disproving that two given generalized Riemann derivatives are equivalent
one needs either some inventive algebra or a counterexample.

Our second motivation comes from the classification of generalized Riemann derivatives,
due to Ash, Catoiu, and Chin in [6]. The symmetrizer and the skew-symmetrizer of an
\( n \)-th difference \( \Delta_A(h, x; f) \) are the differences

\[ \Delta_A^+(h, x; f) = \frac{1}{2}[\Delta_A(h, x; f) \pm (-1)^n \Delta_A(-h, x; f)]. \]
The first is an \( n \)-th symmetric generalized Riemann difference, while the second is either
zero or a symmetric difference whose order is higher than \( n \) and has a different parity
than \( n \). The identity \( \Delta_A(h, x; f) = \Delta_A^+(h, x; f) + \Delta_A^-(h, x; f) \) is the unique decomposi-
tion of \( \Delta_A(h, x; f) \) as a sum of even and odd differences. A difference is symmetric if and
only if it is its own symmetrizer, or if and only if its skew-symmetrizer is zero.

The following is a restatement of the main theorem in [6], which was extended to com-
plex functions in [7]. It characterizes all pairs of generalized Riemann differences whose
associated generalized Riemann derivatives are equivalent for all functions \( f \) at \( x \).

**Theorem A** (The Classification of Generalized Riemann Derivatives: [6], Theorem 2). Let \( A \) and \( B \) be the data sets for two generalized Riemann derivatives of orders \( n \) and \( N \). Then the following statements are equivalent for all functions \( f \) at \( x \):

(i) \( D_A f(x) \) is equivalent to \( D_B f(x) \);
(ii) \( N = n, \Delta_A^+(h, x; f) = r^{-n} \cdot \Delta_B^+(rh, x; f), \) and \( \Delta_A^{-}(h, x; f) = B \cdot \Delta_B^{-}(sh, x; f) \),
for some non-zero constants \( r, s, B \) and all \( h \).

Going back to the above example and with the notation \( \Delta(h) \) for a difference \( \Delta(h, 0; f) \),
the symmetrizers and skew-symmetrizers of the differences corresponding to the given
derivatives are

\[ \Delta_A^+(h) = \Delta_A(h), \quad \Delta_B^+(h) = \frac{1}{2}[f(h) - f(-h)], \quad \Delta_B^-(h) = \frac{1}{2}[f(h) + f(-h) - 2f(0)], \]
\[ \Delta_A^{-}(h) = 0, \quad \Delta_B^{-}(h) = \frac{1}{2}[f(h) - f(-h)], \quad \Delta_C^{-}(h) = \frac{1}{2}[f(h) + f(-h) - 2f(0)]. \]
By Theorem A, \( D_A \) is not equivalent to \( D_B \) at \( x = 0 \), since \( \Delta_A^{-}(h) = 0 \) and \( \Delta_B^{-}(h) \neq 0 \);
and \( D_C \) is equivalent to \( D_B \) at \( x = 0 \), since \( \Delta_C^{-}(h) = \Delta_B^{-}(h) \) and \( \Delta_C^{-}(h) = 5\Delta_B^{-}(h) \).
Thus Theorem A reduced the problem of deciding whether two generalized derivatives are equivalent
to the problem of inspecting the symmetrizers and skew-symmetrizers of the
two associated differences.

Theorem A follows a result by Ash, Catoiu, and Csörnyei in [8], which we restate here
in an equivalent form as the following theorem:
Theorem B ([8], Theorem 1).  
(i) All dilates $h \rightarrow sh$, for $s \neq 0$, of the first order generalized Riemann derivatives
\[
\lim_{h \to 0} \frac{f(x+h) - f(x-h) + B[f(x+rh) + f(x-rh) - 2f(x)]}{2h}, \quad Br \neq 0,
\]
are equivalent to the ordinary differentiation, for all functions $f$ at $x$.
(ii) Any other generalized Riemann derivative of any order $n$ is not equivalent to the $n$-th Peano derivative, for all $f$ at $x$.

Following the same example, and with the observation that $D_A f(0)$ is the limit in Theorem B(i) with $B = 0$, $D_B f(0) = f'(0)$, and $D_C f(0)$ is the limit in Theorem B(i) with $B = 5$ and $r = 1$, by the same theorem, $D_A$ is not equivalent to $D_B$ at 0, while $D_C$ is.

As we mentioned earlier, the first part of the chapter on the equivalence between Peano and generalized Riemann derivatives was an almost everywhere theory. The second part is a more exact, pointwise theory. This was initiated in [8], with Theorem B solving the century old problem, and continued with the subsequent papers. Ash’s result in [1], when translated as all $n$-th generalized Riemann derivatives form a single equivalence class of a.e. equivalent derivatives, can be interpreted as the a.e. solution to the problem of determining the equivalences between any two Generalized Riemann derivatives. Theorem A provided the pointwise solution to the same problem. Since then, the focus has been on a related problem, the equivalence between Peano and sets of generalized Riemann derivatives. Using either an inductive argument, as it was suggested in [9], or by means of a natural example, as shown in [16], this new problem was reduced to the equivalent problem of finding all sets of $n$-th generalized Riemann derivatives whose simultaneous existence at $x$ is equivalent to the existence at $x$ of the $n$-th Peano derivative, for all $n - 1$ times Peano differentiable functions at $x$.

This discussion both motivates the notion of an MZ-set, or part (ii) of Definition MZ, and brings us to Theorem GGR that will be explained in the next subsection.

0.3. MZ-Sets and Theorem GGR. Our third motivation comes from Theorem GGR, providing the first example of an MZ-set consisting of more than a single $n$-th generalized Riemann derivative. This is restated here in an equivalent form as the following theorem:

Theorem GGR ([5], Theorem 3.1). The set \{ $D_{n,-k}$ | $k = 1, \ldots, n$ \}, consisting of the first $n$ backward shifts of the $n$-th Riemann differentiation, is an MZ-set.

Theorem GGR has been a conjecture by Ginchev, Guerragio, and Rocca since 1998. The original statement of the GGR conjecture is an equivalent version of the theorem. It stated that the set of derivatives \{ $D_{j,-k} f(x)$ | $1 \leq k \leq j \leq n$ \} is equivalent to the $n$-th Peano derivative $f^{(n)}(x)$, for all $f$ at $x$. Ginchev, Guerragio, and Rocca proved the result for $n = 2, 3, 4$ by hand in [26], and, with the use of a computer, they proved it for $n = 5, 6, 7, 8$ in [28], leaving the rest as a conjecture. Theorem GGR was recently proved by Ash and Catoiu in [5], using a proof that is half-based on analysis and half-based on a combinatorial algorithm. Since then, there has been a second proof of it in [11] that replaced the combinatorial algorithm with an argument based on the linear algebra of Laurent polynomials, and a third proof in [16] that is entirely based on analysis. By replacing each $-k$ with $k$, a variant of Theorem GGR is obtained for forward shifts instead of backward shifts of the $n$-th Riemann derivative. The variant was proved in [9], before

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1The obvious fact that Theorem B completed the long lasting problem was first highlighted in [15], long after article [8] was published.
the original proof of the theorem, and the proof of the variant is a prerequisite for the original proof of the theorem.

Since every superset of an MZ-set consisting of \( n \)-th generalized Riemann derivatives is an MZ-set, whenever we have an MZ-set, it is important to eliminate redundancies, so that we are left with a minimal MZ-set, or one that does not contain a smaller MZ-set. The original proof of Theorem GGR did not require the shift \( D_{n,-n} \) as a hypothesis when \( n \) is odd. Moreover, it was observed in [16] Theorem 3.3 that only a half of the shifts are needed, regardless of the parity of \( n \), namely \( \{ D_{n,-k} \mid k = 1, \ldots, \lfloor n/2 \rfloor \} \). In particular, when \( n = 3 \), this is the singleton \( \{ D_{3,-1} \} \), for which Theorem GGR asserts that \( D_{3,-1} \) is an MZ differentiation. This is our third counterexample to the Gaussian vs. MZ Derivatives Conjecture, whose complete proof is given in Proposition 3.1.

### 0.4. The Remaining Results

Section [1] has two more results in addition to the first two counterexamples to the Gaussian vs. MZ Derivatives Conjecture, Examples [1.1] and [1.3] outlined in Section [0.1]. Proposition [1.2] shows that exactly one distinct Gaussian derivative is a scale of a given Gaussian derivative, namely, the one obtained from it by replacing \( q \) with either \(-q\) or \( \pm q^{-1} \). And Proposition [1.3] shows that the second update of the conjecture is no different than the first, for all exact MZ derivatives.

In Section 2 we observe that Theorem B explicitly describes the equivalence class of the ordinary first order derivative while Theorem A describes all equivalence classes of generalized Riemann derivatives in a more implicit way. The main result of the section is the following theorem, a restatement of Theorem 2.1, providing the desired, more explicit, equivalent version of Theorem A.

**Theorem A (Explicit).** Let \( \Delta_A(h, x; f) = \sum_i A_i f(x + a_i h) \) and \( \Delta_B(h, x; f) \) be two generalized Riemann differences. Then, the following statements are equivalent, for all functions \( f \) at \( x \):

1. \( D_B f(x) \) is equivalent to \( D_A f(x) \);
2. There exist non-zero numbers \( A, B, r, s \) such that
   \[
   \Delta_B(h, x; f) = \sum_{i=0}^{m} A_i \frac{A[f(x + a_i rh) - f(x - a_i rh)] + B[f(x + a_i sh) + f(x - a_i sh)]}{2}.
   \]

The result of the theorem is completed by Lemma 2.2 asserting that part (ii) of the theorem implies that the two differences have the same order \( n \), and that \( A = 1/r^n \) when \( n \) is odd and \( B = 1/s^n \) when \( n \) is even.

We are now applying the explicit version of Theorem A to the example in Section 0.2. The idea is that the terms with the highest absolute value nodes in the expression of one difference in part (ii) of the theorem should come from terms with the highest absolute value nodes in the other difference.

For the non-equivalence of \( D_A \) and \( D_B \), we apply the explicit Theorem A with \( A, B \) in place of \( B, A \). Since \( \Delta_B \) has only one term, \( 1 \cdot f(x + h) \), of highest absolute value node, the part of \( \Delta_A \) with the highest absolute value nodes, \( [f(x + h) - f(x - h)]/2 \), must be of the form \( 1 \cdot \{ A[f(x + h) - f(x - h)] + B[f(x + h) + f(x - h)] \}/2 \), leading to \( A = 1 \) and \( B = 0 \), the last of which implies, by the explicit Theorem A, that \( D_A \) is not equivalent to \( D_B \).

For the equivalence between \( D_B \) and \( D_C \), we apply the explicit Theorem A with \( C, B \) in place of \( B, A \). The part of \( \Delta_C \) with highest absolute value nodes is \( 3f(x + h) + 2f(x - h) \). This must equal the same generic part of \( \Delta_B \) with highest absolute value nodes. Coefficient identification for \( f(x + h) \) and \( f(x - h) \) leads to the linear system \((A + B)/2 = 3\) and...
\((-A + B)/2 = 2\), with solution \(A = 1\) and \(B = 5\). Since \(-1 \cdot \{1[f(x) - f(x)] + 5[f(x) + f(x)]\}/2 = -5f(x)\), the remaining part of \(\Delta_c\), or the term \(-5f(x)\), is consistent with these values of \(A\) and \(B\), and the explicit Theorem A assures that \(D_c\) is equivalent to \(D_B\).

In this example the base difference \(\Delta_A\), that played the role of \(\Delta_A\) in the explicit Theorem A, had one highest absolute value node. A case where the base difference has two highest absolute value nodes is shown in Example 2.3.

By the zeroth Vandermonde relation, \(\sum_i A_i = 0\), the difference \(\Delta_A\) in the explicit Theorem A can be written as \(\Delta_A(h, x; f) = \sum_{i=0}^m A_i[f(x + a_ih) - f(x)]\) and, similarly, the difference \(\Delta_B\) in the conclusion of the theorem is the difference

\[
\sum_{i=0}^m A_i \left[ f(x + a_i rh) - f(x - a_i rh) \right] + B \left[ f(x + a_i sh) + f(x - a_i sh) - 2f(x) \right].
\]

When \(\Delta_A(h, x; f) = f(x + h) - f(x)\), Theorem B(i) becomes the \(m = 0\) case in the explicit Theorem A, making the latter a more explicit generalization of Theorem B(i) than Theorem A is.

The power of the explicit Theorem A is validated by its numerous applications. Virtually all remaining results in this section and all in the next section are applications of it. Corollary 2.4 asserts that two generalized Riemann derivatives being equivalent is the same as being scales of each other when both derivatives have non-negative nodes. And Corollary 2.5 asserts the same result when the nodes of one of the derivatives have distinct absolute values. The most relevant of these applications to our subject of Gaussian and MZ derivatives is Theorem 2.7, or the classification of Gaussian derivatives, asserting that two Gaussian derivatives are equivalent if and only if they are scales of each other, which by Proposition 1.2 is equivalent to their being deduced from each other by replacing the parameter \(q\) with either \(-q\) or \(\pm q^{-1}\).

In Section 4, Proposition 3.1 provides the third counterexample to the Gaussian vs. MZ Derivatives Conjecture, namely, we show that the third generalized Riemann derivative,

\[D_{3, -1} f(x) := \lim_{h \to 0} \frac{f(x + 2h) - 3f(x + h) + 3f(x) - f(x - h)}{h^3},\]

1) is an MZ derivative; and 2) is not equivalent to a Gaussian derivative. The proof of 1) that uses Theorem GGR was discussed in Section 0.3. Although an earlier version of the paper had an independent proof of 2), the shorter proof we included here is based on Corollary 2.5 and on the explicit Theorem A. The section ends with an interplay between the Riemann vs. MZ Derivatives Conjecture and the above properties 1) and 2). The conjecture predicts that the \(n\)-th Riemann derivative \(D_n f(x)\) does not satisfy 1), and we improve the predicted result by showing in Theorem 3.3 that \(D_n\) satisfies 2).

Most of the results in Sections 1-3 have symmetric analogues. These are included in Section 4 and the similar proofs are omitted. One essential difference between the symmetric and non-symmetric cases is that the symmetric analogue of the third counterexample to the Gaussian vs. MZ Derivatives Conjecture is symmetric Gaussian, hence it is not a counterexample to the symmetric Gaussian vs. MZ Derivatives Conjecture. For this reason, unlike its non-symmetric version, the second update of the Symmetric Gaussian vs. MZ Derivatives Conjecture, that every symmetric MZ derivative is equivalent to a Gaussian derivative, is still standing.

In Section 5 we prove the quantum analogues of both Theorem GGR and its variant in a single, more general, theorem. Theorem 5.1 shows that the set consisting of the \(q\)-analogues of any \(n\) consecutive integer shifts of the \(n\)-th Riemann derivative \(D_n f(x)\) is
an MZ-set. Unlike the proofs of both Theorem GGR and its version, the proof of this theorem is quite simple, based on the fact that all elements of the set are scales of a single Gaussian derivative, and so, Theorem GGR and its version are among very few theorems in mathematics that both have quantum analogues and are much harder to prove than their quantum analogues. The section ends with a sketch of a potential $q$-analogue of the proof in [9] for the variant of Theorem GGR, leading to two new quantum derivatives, $qD_n$ and $q \overline{D}_n$, and the open question of their being MZ derivatives.

Section 6 starts with the observation that an $n$-th generalized Riemann derivative does not assume any lower order generalized Riemann derivative, while the existence of the $n$-th Peano derivative assumes the existence of every lower order Peano derivative, and proposes the $n$ times generalized Riemann differentiation to be a set \{\(D_{A_0}, D_{A_1}, \ldots, D_{A_n}\)\} of generalized Riemann differentiations, with each $A_i$ of order $i$ and $D_{A_i} f(x)$ being the continuity of $f$ at $x$. Example 6.2 shows that requiring one generalized Riemann differentiability in each order is necessary for every $n$ times generalized Riemann differentiability to imply, hence be equivalent, to the $n$ times Peano differentiability. Theorem 6.5 asserts that if an \{\(A_0, A_1, \ldots, A_n\)\}-differentiability is equivalent to $n$ times Peano differentiability, then the $A_n$-differentiation must be an MZ differentiation, while each $A_i$, for $i < n$, does not have to be an MZ differentiation.

* * * 

The Riemann derivatives were introduced by Riemann in the mid 1800s (see [37]). The Peano derivatives were invented by Peano in [36] in 1892 and developed by De la Vallée Poussin in [17]. The generalized Riemann derivatives were introduced by Denjoy in [18] in 1935. The symmetric generalized Riemann derivatives and the symmetric Peano derivatives that we will define in Section 4 were recently investigated in [5]. Other $q$-analogues of the Riemann derivatives were studied in [3][12].

The generalized Riemann derivatives were shown to satisfy properties similar to the ones satisfied by the ordinary derivatives, such as monotonicity [30][40][41], convexity [27][29][35], or the Mean Value Theorem [14][22]. They have many applications in the theory of trigonometric series [39][42] and in numerical analysis [13][33][38]. Multidimensional Riemann derivatives are studied in [4]. For more on generalized Riemann derivatives, see the expository article [2] by Ash. The Peano derivatives also have a long and rich history; see the expository article [19] by Evans and Weil. More recent developments on Peano derivatives are found in [20][21][23][25][32].

1. Two Counterexamples to the Gaussian vs. MZ Derivatives Conjecture

The first counterexample to the Gaussian vs. MZ Derivatives Conjecture as stated provides a scale of a Gaussian derivative that is not a Gaussian derivative.

Example 1.1. For all real $q$, $q \neq 0, \pm 1$, the scale by 2 of the second Gaussian difference

\[ q \Delta_2(h, x; f) = \frac{q^2}{(q^2-1)(q^2-q)} [f(x + q^2h) - (q+1)f(x + qh) + qf(x + h)] \]

is the non-Gaussian difference

\[ \frac{1}{4} \cdot \frac{q^2}{(q^2-1)(q^2-q)} [f(x + 2q^2h) - (q+1)f(x + 2qh) + qf(x + 2h)]. \]

This example shows that, in general, scales of Gaussian derivatives are not Gaussian, while, as we have seen in the introduction, they are equivalent to Gaussian. The first update of the Gaussian conjecture would either have to extend the notion of a Gaussian derivative to include all of its scales, or to reassert the same statement of the conjecture up to a scale.
We chose the second option as the first update of the Gaussian vs. MZ Derivatives Conjecture:

**Gaussian vs. MZ Derivatives Conjecture** (The first update). Each MZ differentiation is a scale of a Gaussian differentiation.

Before challenging this update of the Gaussian vs. MZ derivatives Conjecture, we are asking when two Gaussian derivatives are scales of each other. This questions and more are answered in the following proposition:

**Proposition 1.2.** (i) If two distinct Gaussian differentiations are scalar multiples of each other, then they must coincide.

(ii) Two distinct Gaussian differentiations are scales of each other if and only if they have the form

\[ qD_n \] and either one of \(-qD_n\) and \(\pm_q^{-1}D_n\), for \(n \geq 2\),

or

\[ q\bar{D}_n \] and either one of \(-q\bar{D}_n\) and \(\pm_q^{-1}\bar{D}_n\), for \(n \geq 1\).

**Proof.** (i) This follows from the same property for generalized Riemann derivatives, which is a consequence of the Vandermonde relations.

(ii) Suppose two Gaussian differentiations are scales of each other. Then either 0 is a node in both or is a node in neither. In the first case, the two differentiations are of the form \(qD_n\) and \(Q\bar{D}_n\) for some \(n\), \(q\) and \(Q\), with \(q, Q \neq 0, \pm 1\) and \(q \neq Q\). Their respective sets of nodes are \(\{0, 1, q, \ldots, q^{n-1}\}\) and \(\{0, 1, Q, \ldots, Q^{n-1}\}\). Let a scale be \(\pm r\) map the first differentiation into the other. This maps the first set of nodes, in order, into \(\{0, r, rq, \ldots, rq^{n-1}\}\). The absolute values of the non-zero elements of the same (differently ordered) set are two monotonic sequences, \(1, |Q|, \ldots, |Q^{n-1}|\) and \(|r|, |rq|, \ldots, |rq^{n-1}|\). This forces either \(|r| = 1\) and \(|rq| = |Q|\), or \(|r| = |Q^{n-1}|\) and \(|rq| = |Q^{n-2}|\), that is, either \(Q = -q\) or \(Q = \pm q^{-1}\). The second case has a similar proof. \(\square\)

The second counterexample to the Gaussian vs. MZ Derivatives Conjecture is a counterexample to the first update of the conjecture. This is based on using Theorem A to construct a generalized Riemann derivative that is both equivalent to a Gaussian derivative and not a scale of a Gaussian derivative.

Recall from Section 0.2 that the symmetrizer of an \(n\)th difference \(\Delta_A\) is the \(n\)th symmetric difference \(\Delta_A^n(h, x; f) = \frac{1}{2} [\Delta_A(h, x; f) + (-1)^n \Delta_A(-h, x; f)]\), and the skew-symmetrizer is \(\Delta_A^-(h, x; f) = \frac{1}{2} [\Delta_A(h, x; f) - (-1)^n \Delta_A(-h, x; f)]\). For simplicity, we denote each difference \(\Delta(h, 0; f)\) by \(\Delta(h)\).

**Example 1.3.** Let \(q\) be a real number, with \(q \neq 0, \pm 1\). Then the second Gaussian difference

\[ \Delta_A(h) := q\bar{\Delta}_A(h) = \bar{\lambda}_2[f(q^2h) - (q + 1)f(qh) + qf(h)], \]

where \(\bar{\lambda}_2 = 2!/(q^2 - 1)(q^2 - q)\), has the symmetrizer and skew-symmetrizer given by

\[ \Delta_A^+(h) = \frac{1}{2} \lambda_2 [f(q^2h) - (q + 1)f(qh) + qf(h) + qf(-h) - (q + 1)f(-qh) + f(-q^2h)], \]

\[ \Delta_A^-(h) = \frac{1}{2} \lambda_2 [f(q^2h) - (q + 1)f(qh) + qf(h) - qf(-h) + (q + 1)f(-qh) - f(-q^2h)]. \]

By taking \(\Delta_B(h) := \Delta_A^+(h) + 3\Delta_A^-(h)\), or

\[ \Delta_B(h) = \lambda_2 [2f(q^2h) - 2(q + 1)f(qh) + 2qf(h) - qf(-h) + (q + 1)f(-qh) - f(-q^2h)], \]

by Theorem A, the derivative \(D_Af(x)\) is equivalent to \(D_Bf(x)\), while, by inspection, the second order generalized Riemann difference \(\Delta_B(h)\) is not a scale of a Gaussian difference.
At this point, we can further update the Gaussian conjecture by either extending the notion of a Gaussian derivative to contain not only all of its scales, but also all generalized Riemann derivatives that are equivalent to it; or, by reasserting the conjecture as its previous statement up to the equivalence of generalized Riemann derivatives.

We chose the second option as the following higher update of the Gaussian conjecture:

**Gaussian vs. MZ Derivatives Conjecture (The second update).** Each MZ differentiation is equivalent to a Gaussian differentiation.

As the above example is a non-exact generalized Riemann derivative, one might think that by restricting the definition of an MZ derivative to only exact generalized Riemann derivatives, the second update of the Gaussian vs. MZ Derivatives Conjecture is no different than the first update. The next proposition shows that this is, indeed, the case.

**Proposition 1.4.** The second update of the Gaussian vs. MZ Derivatives Conjecture is the same as the first update, when all MZ derivatives are assumed exact.

**Proof.** This follows from Corollary 2.5.

The third counterexample to the Gaussian vs. MZ Derivatives Conjecture, given in Section 3, is based on more theory of Gaussian derivatives developed in Section 2.

2. **The Equivalence Class of a Generalized Riemann Derivative**

The second update of the Gaussian vs. MZ Derivatives Conjecture highlighted the need for an explicit description of the equivalence class of a Gaussian differentiation and, more generally, the need for an explicit description of the equivalence class of a generalized Riemann differentiation. This section is devoted specifically to serve this need.

The following theorem is a more explicit version of Theorem A, the classification of generalized Riemann derivatives, or a more explicit generalization of Theorem B(i). Recall that $\Delta = \Delta^+ + \Delta^-$ is the unique decomposition of a difference $\Delta$ as a sum of a symmetric and a skew-symmetric differences.

**Theorem 2.1.** Let $A(h, x; f) = \sum A_i f(x + a_i h)$ and $B(h, x; f)$ be two generalized Riemann differences. Then the following statements are equivalent for all functions $f$ at $x$:

(i) $D_A f(x)$ is equivalent to $D_B f(x)$;
(ii) There exist non-zero numbers $A, B, r, s$ such that

$$\Delta B(h, x; f) = \sum A_i \left[ f(x + a_i rh) - f(x - a_i rh) \right] + B \left[ f(x + a_i sh) + f(x - a_i sh) \right].$$

**Proof.** It suffices to show that Theorem A(ii) is equivalent to Theorem 2.1(ii), and we prove this by equivalence, first for $n$ odd. Indeed, Theorem A(ii) asserts that there exist non-zero numbers $B, r, s$ such that

$$\Delta^+ B(h) = r^{-n} \sum A_i [f(x + a_i rh) - f(x - a_i rh)]/2,$$

$$\Delta^- B(h) = B \sum A_i [f(x + a_i sh) + f(x - a_i sh)]/2.$$

The unique decomposition $\Delta B(h) = \Delta^+ B(h) + \Delta^- B(h)$ makes the system consisting of the equalities of the first and last terms in the above two chains equivalent to the weaker statement of Theorem 2.1(ii), Theorem 2.1(ii'), where, in addition, the orders $N$ and $n$ of
the two differences are required to be equal and \( A = 1/r^n \) when \( n \) is odd and \( B = 1/s^n \) when \( n \) is even. The equivalence between Theorem 2.1(ii) and Theorem 2.1(ii) follows from Lemma 2.2 below. The case when \( n \) is even is similar.

The following lemma is a prerequisite to the proof of Theorem 2.1(ii) and completes the statement of the theorem.

**Lemma 2.2.** Let \( \Delta_A(h, x; f) = \sum_{i=0}^{n} A_i f(x + a_i h) \) and \( \Delta_B(h, x; f) \) be two generalized Riemann differences of orders \( n \) and \( N \). If there exist non-zero numbers \( A, B, r, s \) such that

\[
\Delta_B(h, x; f) = \sum_{i=0}^{n} A_i \frac{A[f(x + a_i r h) - f(x - a_i r h)] + B[f(x + a_i s h) + f(x - a_i s h)]}{2},
\]

then \( N = n, A = 1/r^n \) when \( n \) is odd, and \( B = 1/s^n \) when \( n \) is even.

**Proof.** Suppose \( n \) is odd. Then the given expression of \( \Delta_B \) implies that \( \Delta_B(h, x; f) = A\Delta_A(rh, x; f) + B\Delta_A(rh, x; f) \). The first term here is a symmetric difference of order \( n \), and the second is a symmetric difference of order higher than \( n \) and of parity different than \( n \). It follows that \( N = n \) and \( \Delta_B(h, x; f) = A\Delta_A(rh, x; f) = Ar^n \cdot r^{-n} \Delta_A(rh, x; f) \).

Since both \( \Delta_A \) and the scale by \( r \) of \( \Delta_A \) are \( n \)-th generalized Riemann derivatives, the equality between the first and last terms in the above chain forces the scalar multiple \( Ar^n \) to be 1, or \( A = 1/r^n \). The case when \( n \) is even is similar. \( \square \)

When \( \Delta_A(h, x; f) = f(x + h) - f(x) \), Theorem B(i) becomes the case \( m = 0 \) in Theorem 2.1. In this way, Theorem 2.1 is an explicit generalization of Theorem B(i).

In Section 0.4 we had an example of how Theorem 2.1 is applied when the base difference that plays the role of \( \Delta_A \) in the theorem has only one node of the highest absolute value nodes. The following example applies Theorem 2.1 to two pairs of generalized Riemann derivatives, so that the base derivative each time has two nodes of highest absolute value.

**Example 2.3.** Consider the same first generalized Riemann derivatives, \( D_A f(x) = f'_s(x) \), \( D_B f(x) = f'(x) \), and \( D_C f(x) = \lim_{h \to 0} [3f(x + h) - 5f(x) + 2f(x - h)]/h \) as the ones used in Section 0.4.

We apply Theorem 2.1 to the first two derivatives using \( \Delta_A \) as the base difference. Its part consisting of the terms with the highest absolute value nodes is \( \frac{1}{2} f(x + h) - \frac{1}{2} f(x - h) \).

For equivalence, we need the part of \( \Delta_B \) consisting of the highest absolute value nodes, or \( f(x + h) \), to match the sum of the terms with the highest absolute value nodes in the expression listed in the theorem. This is

\[
\frac{1}{2} \{A[f(x + h) - f(x - h)] + B[f(x + h) + f(x - h)]\}/2
- \frac{1}{2} \{A[f(x - h) - f(x + h)] + B[f(x - h) + f(x + h)]\}/2
= A[f(x + h) - f(x - h)]/2,
\]

and the last expression cannot equal \( f(x + h) \) for any \( A \). Then \( D_A \) is not equivalent to \( D_B \).

We now apply Theorem 2.1 to \( D_B \) and \( D_C \), with \( \Delta_C \) as the base difference. Its part with highest absolute value nodes is \( 3f(x + h) + 2f(x - h) \). For equivalence, the part of \( \Delta_B \) with the highest absolute value nodes, the same \( f(x + h) \), has to equal the expression

\[
3\{A[f(x + h) - f(x - h)] + B[f(x + h) + f(x - h)]\}/2
+ 2\{A[f(x - h) - f(x + h)] + B[f(x - h) + f(x + h)]\}/2
= \frac{1}{2} (A + 5B)f(x + h) + \frac{1}{2} (-A + 5B)f(x - h).
\]
The last expression is \( f(x+h) \) when \( A+5B = 2 \) and \( -A+5B = 0 \), with solution \( A = 1 \) and \( B = 1/5 \). Moreover, the above work implicitly yields \( r = s = 1 \). We finally check that the remaining terms, those with node zero, are compatible in both \( \Delta_B \) and its desired form given by the theorem. Indeed, \(-5 \cdot \{1[f(x) - f(x)] + \frac{1}{2}[f(x) + f(x)]\}/2 = -f(x)\), as needed to conclude that, by Theorem 2.1, \( D_B \) is equivalent to \( D_C \).

The next corollary answers the question of when two generalized Riemann derivatives with non-negative nodes are equivalent to each other.

**Corollary 2.4.** Two generalized Riemann derivatives with non-negative nodes are equivalent if and only if they are scales of each other.

**Proof.** Let \( A = \{A_i; a_i\} \) and \( B = \{B_i; b_i\} \) be the data vectors of two equivalent generalized Riemann derivatives on orders \( n \) and \( N \). By Theorem 2.1, \( N = n \) and \( \Delta_B(h, x; f) \) has the expression in part (ii) of the theorem, for some non-zero \( A, B, r, s \). Without loss of generality, we may assume that \( n \) is odd and \( r, s \) are positive. Moreover, via a scale of \( \Delta_B \) by \( 1/r \), we may assume that \( r = 1 \), hence \( A = 1 \), making \( \Delta_B(h, x; f) \) have the expression

\[
\frac{1}{2} \sum_{i=0}^{m} A_i[f(x + a_i h) - f(x - a_i h)] + BA_i[f(x + a_i sh) + f(x - a_i sh)].
\]

Without loss of generality, we may assume that \( s > 0 \). Since \( \Delta_B \) has only non-negative nodes, the negative nodded terms cancel out, and hence

\[
\sum_{i=0}^{m} A_i f(x - a_i h) = B \sum_{i=0}^{m} A_i f(x - a_i sh).
\]

The equality of the sets of nodes in both sides, \(-\{a_0, \ldots, a_m\} = -s\{a_0, \ldots, a_m\}\) forces \( s = 1 \), which in turn makes \( B = 1 \), or \( \Delta_B = \Delta_A \), and the direct implication follows easily from here. The reverse implication is clear. \( \square \)

The second corollary of Theorem 2.1 addresses the situation of two exact equivalent generalized Riemann derivatives, when one of them has all distinct absolute value nodes.

**Corollary 2.5.** Two exact generalized Riemann derivatives, one of which having all nodes with distinct absolute values, are equivalent if and only if they are scales of each other.

**Proof.** Let \( A = \{A_i; a_i\} \) and \( B = \{B_i; b_i\} \) be the data vectors of two exact equivalent generalized Riemann derivatives \( D_A \) and \( D_B \) of orders \( n \) and \( N \). By Theorem 2.1, \( N = n \) and \( \Delta_B(h, x; f) \) is given by the formula in part (ii) of the theorem, for some non-zero \( A, B, r, s \). Working as in the proof of Corollary 2.4 without loss of generality, we may assume that \( n \) is odd, \( r = 1 = A \) and \( s \) is positive, and that \( \Delta_B(h, x; f) \) has the expression

\[
\sum_{i=0}^{n} A_i \frac{f(x + a_i h) - f(x - a_i h)}{2} + B \sum_{i=0}^{n} A_i \frac{f(x + a_i sh) + f(x - a_i sh)}{2}.
\]

Suppose now that \( |a_i| \neq |a_j| \) whenever \( i \neq j \). If there is an \( i \), with \( a_i \neq 0 \), such that none of \( \pm a_i \) is a node in \( \Delta_B \), then both terms \( \pm A_i f(x \pm a_i h)/2 \) in the first sum are canceled by terms of the second sum. Since \( \pm a_i \) have the same absolute values, we must have \( A_i [f(x + a_i h) - f(x - a_i h)] + BA_i [f(x + a_i sh) + f(x - a_i sh)] = 0 \), for some \( j \), making \( a_js = \pm a_i \) and \( BA_j = \pm A_i \). Since none of these choices leads to the cancellation of all four terms, this subcase is impossible. In the remaining case, for each \( i \), with \( a_i \neq 0 \), at least one of \( \pm a_i \) is a node in \( \Delta_B \). Since this difference is exact, for each \( i \), precisely one of the terms \( \pm A_i f(x \pm a_i h)/2 \) in the first sum is canceled by a term in the second...
sum. Accounting for the absolute values of the nodes of these canceled terms in each of the
two sums, we have \(|a_0|, \ldots, |a_n|\} = s\{|a_0|, \ldots, |a_n|\}, making s = 1 and B = \pm 1.
It follows that \(\Delta_B(h, x; f) = \pm \Delta_A(\pm h, x; f)\) and so \(D_B\) is a scale of \(D_A\), as needed. \(\square\)

The following example shows that the condition in Corollary \(2.5\) that one of the two
differences has all nodes of distinct absolute values is necessary. Here the equivalence of
two generalized Riemann differences is taken to be the same as the equivalence of their
associated generalized Riemann derivatives.

**Example 2.6.** (i) Let \(\Delta(h) := \Delta_{3,-1}(h, x; f) = f(x+2h) - 3f(x+h) + 3f(x) - f(x-h)\)
be the third generalized Riemann difference with nodes \(-1, 0, 1, 2\), that we considered in
Proposition \(1.2\). Then
\[
\begin{align*}
\Delta^+(h) &= \frac{1}{2}[\Delta(h) - \Delta(-h)] = \frac{1}{2}f(x+2h) - f(x+h) + f(x-h) - \frac{1}{2}f(x-2h) \\
\Delta^-(h) &= \frac{1}{2}[\Delta(h) + \Delta(-h)] = \frac{1}{2}f(x+2h) - 2f(x+h) + 3f(x) - 2f(x-h) + \frac{1}{2}f(x-2h),
\end{align*}
\]

hence the third generalized Riemann difference,
\[
\nabla(h) := \Delta^+(h) + \frac{1}{2}\Delta^-(h) = \frac{1}{2}[3f(x+2h) - 8f(x+h) + 6f(x) - f(x-2h)],
\]
by Theorem A, is equivalent to \(\Delta(h)\). Since the set of nodes of \(\nabla(h)\) is \([-2, 0, 1, 2]\), not a scale of the set of nodes of \(\Delta(h)\), \(\nabla(h)\) cannot be a scale of \(\Delta(h)\). Finally, by inspection, the
nodes in either difference do not have distinct absolute values.

(ii) More generally, for \(k \geq 2\), let \(\Delta(h)\) be the exact \(2k - 1\)-st generalized Riemann
difference with nodes \(-k+1, -k+2, \ldots, k\). The difference \(\nabla(h) := \Delta^+(h) + B\Delta^-(h),\) by
Theorem A, is equivalent to \(\Delta(h)\) for all non-zero values of \(B\). On the other hand, \(\Delta^+(h)\)
is the exact \(2k - 1\)-st generalized Riemann difference with nodes \(-k, \ldots, -1, 1, \ldots, k\)
and \(\Delta^-(h)\) is a difference of order \(2k\) and with nodes \(-k, -k+1, \ldots, k\). Then, for each \(i,\)
with \(i \neq 0, \pm k\), there exist a \(B\) such that \(\nabla(h)\) does not have \(i\) as a node, hence \(\nabla(h)\) is
the exact \(2k - 1\)-st generalized Riemann difference with nodes \(-k, \ldots, i-1, i+1, \ldots, k.\)
By inspection, \(\Delta(h)\) and \(\nabla(h)\) do not have all nodes of distinct absolute values, and they
are not scales of each other since their sets of nodes are not scales of each other.

The next theorem characterizes all pairs of equivalent Gaussian derivatives. In conjunction
with the result of Proposition \(1.2\), this is referred to as the classification of Gaussian
differentiations.

**Theorem 2.7 (The Classification of Gaussian Differentiations).** Two Gaussian differentiations are equivalent if and only if they are scales of each other.

**Proof.** The direct implication follows from Corollary \(2.5\). The reverse implication is due to the fact that scales of generalized Riemann derivatives are equivalent generalized Riemann
derivatives. \(\square\)

Following Proposition \(1.4\), Theorem \(2.7\) is the second application of Corollary \(2.5\)
hence of Theorem \(2.1\) to Gaussian derivatives. Two more such applications are given
in the next section.

3. **The Third Counterexample**

The third counterexample to the Gaussian vs. MZ Derivatives Conjecture is an example of an MZ derivative that is not equivalent to a Gaussian derivative. This is encoded in the following proposition, referring to the first backward shift \(D_{3,-1}\) of the third Riemann
differentiation that was mentioned in Section \(0.3\) and whose associated difference is
\[
\Delta_{3,-1}(h, x; f) = f(x+2h) - 3f(x+h) + 3f(x) - f(x-h).
\]
Proposition 3.1. The third generalized Riemann differentiation $D_{3,-1}$ has the following properties:

(i) $D_{3,-1}$ is an MZ differentiation;
(ii) $D_{3,-1}$ is not equivalent to a Gaussian differentiation.

Proof. (i) This follows from the simplified form of Theorem GGR, \[16\] Theorem 3.3], that we discussed in Section 0.3 of the introduction.

(ii) Suppose $D_{3,-1}$ is equivalent to a Gaussian differentiation and argue by contradiction. Then, by Theorem A, we must have $D_{3,-1}$ equivalent to one of $qD_3$ and $q\bar{D}_3$ for some $q$ with $q \neq 0, \pm 1$. And since 0 is a node in $D_{3,-1}$, this derivative is equivalent to $qD_3$. Then, by Corollary \[2.5\] $D_{3,-1}$ is a scale, say by a non-zero $a$, of $qD_3$. In particular, its set of nodes is $\{2, 1, 0, -1\} = \{0, a, aq, aq^2\}$. The contradiction comes from the set of absolute values on the left side having three elements, while on the right side has four. □

Remark 3.2. (i) Proposition 3.1 proved that the generalized Riemann differentiation corresponding to the difference $\Delta(h)$ in Example \[2.6\](i) is an MZ differentiation non-equivalent to a Gaussian differentiation, hence is a counterexample to the second update of the Gaussian vs. MZ Derivatives Conjecture. The same example proved that the difference $\nabla(h)$ is equivalent to $\Delta(h)$, and so its corresponding generalized Riemann derivative, or the limit

$$
\lim_{h \to 0} \frac{1}{2}(3f(x + 2h) - 8f(x + h) + 6f(x) - f(x - 2h))/h^3,
$$

will be a second counterexample to the same conjecture.

(ii) The linear combination $\Delta(2h) - 4\nabla(h)$ of dilates of the differences $\Delta(h)$ and $\nabla(h)$ from Example \[2.6\] is a third difference with nodes $0, 1, 2, 4$, hence is a scalar multiple of $\Delta_3(h)$, the third difference in Theorem MZ. By the same example, the differences $\Delta(h) := \Delta_{3,-1}(h)$ and $\nabla(h)$ are equivalent to each other, so $\Delta_{3,-1}(h)$ implies $\Delta_3(h)$. Then, by taking associated derivatives, $D_3$ (by Theorem MZ) is an MZ derivative implies that $D_{3,-1}$ is an MZ derivative. This was a second proof of Proposition 3.1(i) that did not require Theorem GGR.

Proposition 3.1 brings the Gaussian vs. MZ Derivatives Conjecture back to the drawing board, Problem MZ, where more ideas are needed. What we now know is that the set of all MZ differentiations contains all generalized Riemann differentiations equivalent to a Gaussian differentiation, and all equivalent to $D_{3,-1}$. Until new ideas become available, the Riemann vs. MZ Derivatives Conjecture will be left as the most important open question of this subject.

If the Riemann vs. MZ Derivatives Conjecture were true, then the Riemann differentiation will not be equivalent to a Gaussian differentiation in orders at least three.

The following theorem proves this weaker version of the conjecture and by this it provides more examples of generalized Riemann derivatives that are not equivalent to Gaussian derivatives.

Theorem 3.3. When $n \geq 3$, the Riemann differentiation $D_n$ is not equivalent to a Gaussian differentiation.

Proof. Suppose not. Then, for some $n$ at least 3, $D_n$ is equivalent to a Gaussian. By Theorem A, the Gaussian must be one of $qD_n$ or $q\bar{D}_n$, for some $q$ with $q \neq 0, \pm 1$. Since zero is a node in $D_n$, the Gaussian must be $qD_n$. Theorem \[2.7\] makes $D_n$ a scale of $qD_n$. In particular, the sequence of its non-zero nodes is on one hand the arithmetic sequence $1, 2, 3, \ldots, n$, and on the other hand is the geometric sequence $a, aq, aq^2, \ldots, aq^{n-1}$, for
some non-zero \( a \) and \( q \), with \( q \neq \pm 1 \). It follows that 2 is the geometric mean of 1 and 3, a contradiction. \( \square \)

4. SYMMETRIC GAUSSIAN AND MZ DERIVATIVES

Recall that an \( n \)-th general Riemann difference \( \Delta_A(h, x; f) \) is symmetric if it satisfies \( \Delta_A(-h, x; f) = (-1)^n \Delta_A(h, x; f) \). This translates in terms of symmetrizers and skew-symmetrizers as \( \Delta_A^s(h, x; f) = \Delta_A(h, x; f) \), or, equivalently, \( \Delta_A^s(h, x; f) = 0 \). For example, both \( \Delta_t^n \) and \( s \Delta_t^n \) are symmetric for all \( n \). A generalized Riemann derivative \( D_A \) is symmetric if its difference \( \Delta_A \) is symmetric.

A function \( f \) is \( n \) times symmetric Peano differentiable at \( x \) if there exist constants \( c_0, c_1, \ldots, c_n \) such that

\[
\frac{1}{2} [f(x + h) - f(x - h)] = c_0 + \frac{c_1}{1!} h + \frac{c_2}{2!} h^2 + \cdots + \frac{c_n}{n!} h^n + o(h^n),
\]

in which case the numbers \( c_i \), for \( i = 0, 1, \ldots, n \), are denoted by \( f^{(i)}(x) \) and called the first \( n \) symmetric Peano derivatives of \( f \) at \( x \). By replacing \( h \) with \( -h \), we have \( f^{(n-1)}_{(n-2)}(x) = f^{(n-3)}_{(n-2)}(x) = \cdots = f^{(1 \text{ or } 0)}(x) = 0 \), and the existence of the \( n \)-th symmetric Peano derivative \( f^{(n)}_{(n)}(x) \) implies the existence of all same parity lower order symmetric Peano derivatives of \( f \) at \( x \). The condition \( c_0 := f(x) \) when \( n \) is even is added to the above definition, in order to ensure that \( f^{(0)}_{(0)}(x) \) is the continuity of \( f \) at \( x \).

The symmetric version of Theorem MZ is the following theorem proved in [5]. For \( n \geq 2 \) we denote \( m = \lfloor (n - 1)/2 \rfloor \) and define \( D^s_n f(x) \) to be the \( n \)-th generalized Riemann derivative with nodes \((0), \pm 1, \pm 2, \pm 4, \ldots, \pm 2^m\), where \((0)\) means that 0 is taken only for \( n \) even.

Theorem MZ (Symmetric). [5] Theorem 2.2] For \( n \geq 2 \), let \( D^s_n f(x) \) be the above \( n \)-th symmetric generalized Riemann derivative. Then, for all functions \( f \) and points \( x \),

both \( f^{(n-2)}_{(n-2)}(x) \) and \( D^s_n f(x) \) exist \( \iff \) \( f^{(n)}_{(n)}(x) \) exists.

This theorem leads to the following definition and problem:

Definition MZ (Symmetric). (i) An \( n \)-th symmetric generalized Riemann differentiation \( D_A \) is a symmetric MZ differentiation at \( x \), if, for all functions \( f \),

both \( f^{(n-2)}_{(n-2)}(x) \) and \( D_A f(x) \) exist \( \iff \) \( f^{(n)}_{(n)}(x) \) exists.

(ii) A set \( \{D_{A_1}, \ldots, D_{A_p}\} \) of symmetric \( n \)-th generalized Riemann differentiations is a symmetric MZ-set at \( x \), if, for all functions \( f \),

\( f^{(n-2)}_{(n-2)}(x) \) and all of \( D_{A_1} f(x), \ldots, D_{A_p} f(x) \) exist \( \iff \) \( f^{(n)}_{(n)}(x) \) exists.

The following is the symmetric version of Problem MZ:

Problem MZ (Symmetric). For each order \( n \) and point \( x \), determine all \( n \)-th symmetric MZ differentiations at \( x \).

A solution to the symmetric Problem MZ was also proposed in [10], where it was proved that, for all real numbers \( q \), with \( q \neq 0, \pm 1 \), the \( n \)-th symmetric Gaussian derivatives \( qD^s_n f(x) \) are symmetric MZ derivatives at \( x \), and then it was conjectured that these are all symmetric MZ derivatives at \( x \) of order \( n \), that is,

Symmetric Gaussian vs. MZ Derivatives Conjecture ([10], Conjecture A). In orders at least 3, each symmetric MZ differentiation is a symmetric Gaussian differentiation.
In other words, all symmetric MZ derivatives classify as symmetric Gaussian. Article \[10\] proved it to be false for \( n = 1, 2 \), true for \( n = 3, 4 \), and left it open for \( n \geq 5 \).

A particular case of the symmetric Gaussian vs. MZ Derivatives Conjecture is the following conjecture from \[5\] on the relation between symmetric Riemann and symmetric MZ differentiations:

**Symmetric Riemann vs. MZ Derivatives Conjecture** (\[5\], Conjecture 4.1). For \( n \geq 5 \), the \( n \)-th symmetric Riemann differentiation \( D^n_{s} \) is not a symmetric MZ differentiation.

This conjecture was first stated in \[5\] for \( n \geq 3 \). It is obviously false for \( n = 2 \). Article \[10\] disproved it for \( n = 3, 4 \), updating the statement to the current \( n \geq 5 \), and proved the conjecture for \( n = 5, 6, 7, 8 \), leaving the result open for \( n \geq 9 \).

Similar to Example 1.1, the scale by 7 of the symmetric Gaussian difference \( q \Delta^n_{s}(h, x; f) \) is not symmetric Gaussian, and the symmetric Gaussian vs. MZ Derivatives Conjecture is updated to:

**Symmetric Gaussian vs. MZ Derivatives Conjecture** (The first update). Each symmetric MZ differentiation is a scale of a symmetric Gaussian differentiation.

The following proposition is the symmetric analogue of Proposition 1.2.

**Proposition 4.1.**

(i) No two distinct symmetric Gaussian differentiations are scalar multiples of each other.

(ii) Two distinct symmetric Gaussian differentiations are scales of each other if and only if they have the form

\[ qD^n_{s} \text{ and either one of } -qD^n_{s} \text{ and } \pm q^{-1}D^n_{s}, \text{ for } n \geq 3. \]

The following is the symmetric analogue of the second update of the Gaussian vs. MZ Derivatives Conjecture.

**Symmetric Gaussian vs. MZ Derivatives Conjecture** (The second update). Each symmetric MZ differentiation is equivalent to a symmetric Gaussian differentiation.

We were unable to provide a counterexample to the first update of the symmetric Gaussian vs. MZ Derivatives Conjecture, and the reason for this is that no such a counterexample exists. This in consequence of the symmetric analogue of Theorem A, or the classification of symmetric generalized Riemann derivatives of \[5\], Theorem 1.3], which we conveniently restate here in an equivalent form as the following theorem:

**Theorem C** (The Classification of Symmetric Generalized Riemann Derivatives: \[5\]).

Two symmetric generalized Riemann differentiations are equivalent if and only if they are scales of each other.

We highlight this consequence of Theorem C as the following symmetric analogue of Proposition 1.4:

**Proposition 4.2.** The second update of the symmetric Gaussian vs. MZ Derivatives Conjecture is the same as the first update, regardless of exactness.

Note that when the difference \( \Delta_A \) in Theorem A is symmetric, the expression of \( \Delta_B \) implied by Theorem 2.1 is the same as the one given by Theorem C. Thus, unlike Theorem 2.1 being a more explicit version of Theorem A, the symmetric analogue of Theorem 2.1 or the explicit version of Theorem C, is exactly Theorem C.
Another easy consequence of Theorem C is the following symmetric analogue of Theorem 2.7 which we refer to as the classification of symmetric Gaussian derivatives. Its proof is omitted.

**Corollary 4.3 (The Classification of Symmetric Gaussian Differentiations).**
Two symmetric Gaussian differentiations are equivalent if and only if they are scales of each other.

A natural third counterexample to the Symmetric Gaussian vs. MZ Derivatives Conjecture would be a symmetric analogue of the third counterexample to the Gaussian vs. MZ Derivatives Conjecture, which is the third derivative $D_{3,-1}$ of Proposition 3.1. And the only third symmetric derivative that we can think of as being naturally associated to it is its symmetrization, introduced in [5], whose expression is

$$\Delta_{3,-1}(h, x; f) = \frac{1}{2} \left[ \Delta_{3,-1}(h, x; f) + (-1)^3 \Delta_{3,-1}(-h, x; f) \right] = \frac{1}{2} \left[ f(x + 2h) - 3f(x + h) + 3f(x) - f(x - h) \right.$$

$$- f(x - 2h) + 3f(x - h) - 3f(x) + f(x + h) \left. \right] = \frac{1}{2} f(x + 2h) - f(x + h) + f(x - h) - \frac{1}{2} f(x - 2h).$$

However, this is the same as the third Gaussian difference $2\Delta^3_{0}$, and so it cannot be a counterexample to the conjecture. Then the second update of the Symmetric Gaussian vs. MZ Derivatives Conjecture is left standing.

We close this section on symmetric Peano and symmetric generalized Riemann derivatives with the symmetric analogue of Theorem 3.3. Its omitted proof is similar to the proof of the same result.

**Theorem 4.4.** When $n \geq 5$, the symmetric Riemann differentiation $D^n_s$ is not equivalent to a symmetric Gaussian differentiation.

5. **Quantum Analogues of Theorem GGR and Its Variant**

The most obvious $q$-analogue of the Riemann difference $\Delta_n$, with nodes 0, 1, \ldots, $n$, is the Gaussian difference $q\Delta_n$, with nodes $q^0, q^1, \ldots, q^n$. Then the $q$-analogue of the $k$-shift, $\Delta_{n,k}$, of the $n$-th Riemann difference, or the exact difference with nodes $k, k+1, \ldots, k+n$, is the exact difference with nodes $q^k, q^{k+1}, \ldots, q^{n+k}$, which we denote by $q\Delta_{n,k}$. This has the expression

$$q\Delta_{n,k}(h, x; f) = q^{-nk} \cdot q\lambda_n \sum_{i=0}^{n} \frac{q^{-i}}{i!} f(x + q^{n+k-i}h),$$

where $q\lambda_n = n! / \prod_{j=0}^{n-1} (q^n - q^j)$. The above expression was deduced from the fact that $q\Delta_{n,k}$ is the scale by $q^k$ of the difference $q\Delta_n = q\Delta_{n,0}$, whose formula was deduced in [10]. The same fact implies that, for each real number $k$, functions $f$, and points $x$, the generalized Riemann derivatives,

$$qD_{n,k}f(x) = \lim_{h \to 0} q\Delta_{n,k}(h, x; f)/h^n \quad \text{and} \quad q\bar{D}_n f(x) = \lim_{h \to 0} q\Delta_n(h, x; f)/h^n,$$

are equal to each other.

The next theorem when $\ell = -n$ is the $q$-analogue of Theorem GGR, and when $\ell = 0$ is the $q$-analogue of the variant of Theorem GGR. Recall that the reverse implication in the Gaussian vs. MZ Derivatives Conjecture is known to be true; see [10, Theorem B]. We will refer to it as the Gaussian lemma.
Theorem 5.1. For each integer \( \ell \), the set consisting of all \( n \)-th generalized Riemann derivatives,

\[
q \tilde{D}_{n,\ell} f(x), q \tilde{D}_{n,\ell+1} f(x), \ldots, q \tilde{D}_{n,\ell+n} f(x),
\]

is an MZ-set.

Proof. By the above observation, the set in the hypothesis is equivalent to the single derivative \( q \tilde{D}_n f(x) \), which by the Gaussian lemma, is an MZ differentiation. \( \square \)

In absence of the short argument we used above, the proof of Theorem 5.1 for \( \ell = 0 \) would have followed the proof of the variant of Theorem GGR, given in [9]. Note that one of the outer two derivatives in the hypotheses of both Theorem 5.1 and the GGR variant, those corresponding to \( k = 0, n \), is superfluous, since they are obtained from each other by respectively replacing \( q \) with \( q^{-1} \) and \( h \) with \( -h \) in their defining limits, so we may drop one of them from the hypothesis as we please.

The original proof of the GGR variant was done in two steps: first, by using a sophisticated combinatorial algorithm, it was shown that the set \( \{ D_{n,0}, \ldots, D_{n,n-1} \} \) implies the Gaussian \( _2D_n \), for all functions \( f \) at \( x \); then, by the Gaussian lemma, deduce that the same set is an MZ-set. Similarly, start with the set \( \{ D_{n,1}, \ldots, D_{n,n} \} \) and use the same combinatorial algorithm to first prove that this new set implies the Gaussian \( _2\tilde{D}_n \), and then, by the Gaussian lemma, deduce that the new set is an MZ-set.

In the quantum case, starting with the set \( \{ q \tilde{D}_{n,0}, \ldots, q \tilde{D}_{n,n-1} \} \) and using a similar combinatorial algorithm, one can first prove that this set implies the exact differentiation \( q \mathcal{D}_n \) with nodes \( 0, 1, q^2, q^2, \ldots, q^{2n-2} \), and then we need to employ a theorem for \( q \mathcal{D}_n \), similar to the Gaussian lemma for \( q \tilde{D}_n \), in order to deduce that the set is an MZ-set. Alternatively, one can start with the set \( \{ q \tilde{D}_{n,1}, \ldots, q \tilde{D}_{n,n} \} \) and, by a similar algorithm, show first that this implies the exact differentiation \( q \mathcal{D}_n \) with nodes \( 1, q^2, q^2, \ldots, q^{n-1} \), and then employ a theorem for \( q \mathcal{D}_n \), similar to the Gaussian lemma for \( q \tilde{D}_n \), in order to deduce that the set is an MZ-set.

Fixing the details of the above two suggested alternative proofs of Theorem 5.1 would amount to answering the following open question, our last highlighted item in the section.

Open Question 5.2. Are the above exact differentiations, \( q \mathcal{D}_n \) and \( q \mathcal{D}_n \), MZ differentiations?

6. The notions of \( n \) times generalized Riemann differentiation and \( n \) times MZ differentiation

In this section we are seated around the equivalence between \( n \) times Peano differentiation and sets of generalized Riemann differentiations, for all functions \( f \) at \( x \). We have seen that this is equivalent to the notions of MZ-sets and MZ derivatives.

One reason why, in orders at least two, by Theorem B, no \( n \)-th generalized Riemann derivative can possibly be equivalent to the \( n \)-th Peano derivative, for all functions at \( x \), is that \( n \) times Peano differentiation at \( x \) means the existence of all Peano derivatives of orders up to \( n \) at \( x \), while, up til now, \( n \) times generalized Riemann differentiation meant the existence of a single \( n \)-th generalized Riemann derivative at \( x \), without assuming the existence of lower orders generalized Riemann derivatives. From now on, this is about to change with the following new definition of \( n \) times generalized Riemann differentiation:
Definition 6.1. (i) A function \( f \) is \( n \) times generalized Riemann differentiable at \( x \), or \( f \) has \( n \) generalized Riemann derivatives at \( x \), if there exist generalized Riemann differentiations \( D_{A_0}, D_{A_1}, \ldots, D_{A_n} \) such that \( D_{A_j} \) has order \( j \) and \( D_{A_j} f(x) \) exists, for \( j = 0, 1, \ldots, n \). Here \( D_{A_0} f(x) := f_0(x) \) is the continuity of \( f \) at \( x \).

(ii) A function \( f \) is \( n \) times symmetric generalized Riemann differentiable at \( x \), or \( f \) has \( n \) symmetric generalized Riemann derivatives at \( x \), if there exist symmetric generalized Riemann differentiations \( D_{A_0}, D_{A_{n-2}}, \ldots \), such that \( D_{A_j} \) has order \( j \) and \( D_{A_j} f(x) \) exists, for \( j = n, n-2, \ldots \). Here \( D_{A_0} f(x) := f_0(x) \) is the continuity of \( f \) at \( x \), and \( D_{A_j} f(x) := f^{(j)}(x) \).

(iii) A function \( f \) is \( n \) times (symmetric) MZ differentiable at \( x \), if it is \( n \) times generalized Riemann differentiable and all differentiations \( D_{A_j} \) are (symmetric) MZ differentiations.

The next example provides the reason for why we require in Definition 6.1(i) that the set or sequence of generalized Riemann differentiations has an element in each order up to \( n \).

Example 6.2. Fix an index \( k \), with \( 1 \leq k \leq n \), and let \( A_0, A_1, \ldots, A_{k-1}, A_{k+1}, \ldots, A_n \) be the data vectors of generalized Riemann differentiations, with \( A_j \) of order \( j \) for all \( j \). Let \( S \) denote the set of all non-zero nodes of all these differentiations, and let \( G := \langle S \rangle \) be the multiplicative subgroup of non-zero real numbers generated by \( S \).

Let \( f \) be the function defined by \( f(h) = h^k \) for \( h \in G \) and \( f(h) = 0 \) for \( h \notin G \). Then \( f \) is \( k-1 \) times Peano differentiable at \( x = 0 \), hence is \( A_j \)-differentiable at \( 0 \) for \( j = 0, 1, \ldots, k-1 \). When \( j > k \), \( D_{A_j} f(0) = (h^k)^{(j)}(0) = 0 \) for \( h \in G \) and, by a similar reason, \( D_{A_j} f(0) = 0 \) for \( h \notin G \). Putting together, \( f \) is \( A_j \)-differentiable at \( 0 \) for \( j = k+1, \ldots, n \). On the other hand, \( f \) is not \( k \) times, hence not \( n \) times, Peano differentiable at \( 0 \), since \( \lim_{h \to 0} f(h)/h^k \) is both 1 and 0.

Summarizing, we have found an \( f \) that is \( \{A_0, A_1, \ldots, A_{k-1}, A_{k+1}, \ldots, A_n\} \)-differentiable at \( 0 \) but not \( n \) times Peano differentiable at \( 0 \).

Based on the above definition, the problem of finding all single \( n \)-th generalized Riemann derivatives that are equivalent to the \( n \)-th Peano derivative for all functions \( f \) at \( x \), which by Theorem B has a negative answer for \( n \geq 2 \), restricts to the problem of finding all \( n \) times Generalized Riemann differentiations \( \{A_0, A_1, \ldots, A_n\} \) that are equivalent to the \( n \) times Peano differentiation for all \( f \) at \( x \), that might have positive solutions in orders at least two. Clearly, every \( n \) times Peano differentiable function \( f \) at \( x \) is \( n \) times generalized Riemann differentiable of any kind \( \{A_0, A_1, \ldots, A_n\} \) and \( D_{A_j} f(x) = f^{(j)}(x) \), for all \( j \), so that the above equivalence problem relies on reversing this implication.

Note that each \( n \) times MZ differentiation, that is, each \( n \) times generalized Riemann differentiation \( \{A_0, A_1, \ldots, A_n\} \), where each \( D_{A_j} \) is an MZ differentiation of order \( j \), is a solution to the pointwise equivalence problem between the \( n \) times generalized Riemann and \( n \) times Peano differentiations. To illustrate this, we consider the following example:

Example 6.3. Let \( \{A_0, A_1, A_2, A_3\} \) be the three times generalized Riemann differentiation given by \( D_{A_0} \) is the continuity at \( x \), \( D_{A_1} = x D_1 \), \( D_{A_2} = x D_2 \), and \( D_{A_3} \) is the scale by 4.7 of \( D_{A_1} \). By our earlier work, each individual differentiation is an MZ differentiation, hence the \( \{A_0, A_1, A_2, A_3\} \)-differentiation is a three times MZ differentiation.

Let \( f \) be a continuous function at \( x \). Then \( f^{(0)}(x) \) exists. Since \( D_{A_1}, D_{A_2}, D_{A_3} \) are MZ differentiations of orders 1, 2, 3, we successfully deduce that \( f^{(1)}(x), f^{(2)}(x), f^{(3)}(x) \) exist.

The proof in the second paragraph of Example 6.3 shows how a three times MZ differentiation implies, hence is equivalent to, the three times Peano differentiation, for all functions.
at \( x \). The same argument, formalized by induction, can be used in the proof of the following, more general, result:

**Proposition 6.4.** (i) Each \( n \) times MZ differentiation is equivalent to the \( n \) times Peano differentiation, for all functions \( f \) at \( x \).

(ii) Each \( n \) times symmetric MZ differentiation is equivalent to the \( n \) times symmetric Peano differentiation, for all \( f \) at \( x \).

We close the section and the article with the following theorem on the pointwise equivalence between an \( n \) times generalized Riemann differentiation and the \( n \) times Peano differentiation.

**Theorem 6.5.** Let \( A_0, A_1, \ldots, A_n \) be generalized Riemann differentiations with the property that each \( A_j \) has degree \( j \) and, for all \( f \) and \( x \), \( f \) is \( A_k \)-differentiable at \( x \) for all \( k \) implies, hence is equivalent to, \( f \) is \( n \) times Peano differentiable at \( x \).

Then, the \( A_n \)-differentiation must be an MZ differentiation, while each \( A_k \)-differentiation, for \( 1 \leq k < n \), does not have to be an MZ differentiation.

**Proof.** If the \( A_n \)-differentiation is not an MZ differentiation, then there is an \( f \) that is \( n - 1 \) times Peano differentiable at \( x \), \( A_k \)-differentiable at \( x \) for all \( k \), and not \( n \) times Peano differentiable at \( x \). This contradicts the hypothesis that \( A_k \)-differentiability at \( x \) for all \( k \) implies \( n \) times Peano differentiability at \( x \).

For the second assertion, consider the \( \{A_0, A_1, A_2, A_3\} \)-differentiation, respectively corresponding to continuity at \( x \), \( D_1 \), \( D_2 \) and \( D_{3,-1} \), and make two claims about it:

(i) This is not a three times MZ differentiation.

(ii) This is equivalent to three times Peano differentiation, for all \( f \) at \( x \).

To prove claim (i), it suffices to show that \( D_2 \) is not an MZ differentiation. Indeed, the function \( f(x) = x^2 \cdot \text{sgn}(x) \) has: 1) \( f(h) = o(h) \) as \( h \to 0 \), hence \( f(1)(0) = 0 \); 2) \( f \) is an odd function, so \( D^2 f(0) = \lim_{h \to 0} [f(0+h)+f(0-h)-2f(0)]/h^2 = 0 \); and 3) \( f(2)(0) = \lim_{h \to 0} [f(0+h)-f(0)-f(1)(0)h]/h^2 = \pm 1 \), hence it does not exist.

For claim (ii), it suffices to show that the \( \{A_0, A_1, A_2, A_3\} \)-differentiation is equivalent to the three times MZ differentiation \( \{A_0, A_1, A_2, A_3\} \), where \( A_2 \) corresponds to \( D_2 = 2D_2 \).

Indeed, we assume that \( f \) is both \( A_2 \)- and \( A_3 \)-differentiable at \( x \). This means that both

\[
D_{A_2} f(x) = \lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}
\]

and

\[
D_{A_3} f(x) = \lim_{h \to 0} \frac{f(x+2h) - 3f(x+h) + 3f(x) - f(x-h)}{h^3}
\]

exist. Then the derivative

\[
D_{A_2}^2 f(x) = \lim_{h \to 0} \left( \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} \right)
\]

\[= \lim_{h \to 0} \left( \frac{f(x+2h) - 3f(x+h) + 3f(x) - f(x-h)}{h^3} \right) \cdot h
\]

\[+ \lim_{h \to 0} \left( \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \right)
\]

\[= D_{A_2} f(x) \cdot 0 + D_{A_2} f(x)
\]

exists. Similarly, one can prove that if \( f \) is both \( A_2 \)- and \( A_3 \)-differentiable at \( x \), then \( f \) is \( A_2 \)-differentiable at \( x \). \(\square\)
Proposition 3.1 has left Problem MZ under an elusive status. When this hard problem will be solved, then a new hard problem that could take its place is the following:

**Problem MZ (Extended).** Find all \( n \) times generalized Riemann differentiations that are equivalent to the \( n \) times Peano differentiation, for all functions \( f \) at \( x \).

Proposition 6.4 suggested that a first candidate for a solution to the extended Problem MZ could be the set of all \( n \) times MZ differentiations of functions at \( x \). The example in the second part of the proof of Theorem 6.5 has shown that the solution to the extended Problem MZ is much more complicated.

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