The three-loop $\beta$-function of QCD with the clover action.

A. Bode$^a$ and H. Panagopoulos$^b$

$^a$CSIT, Tallahassee, USA
email: bode@scri.fsu.edu

$^b$Department of Physics, University of Cyprus,
P.O.Box 20537, Nicosia CY-1678, Cyprus
email: haris@ucy.ac.cy

Abstract

We calculate, to 3 loops in perturbation theory, the bare $\beta$-function of QCD, formulated on the lattice with the clover fermionic action. The dependence of our result on the number of colors $N$, the number of fermionic flavors $N_f$, as well as the clover parameter $c_{SW}$, is shown explicitly.

A direct outcome of our calculation is the two-loop relation between the bare coupling constant $g_0$ and the one renormalized in the $\overline{\text{MS}}$ scheme.

Further, we can immediately derive the three-loop correction to the relation between the lattice $\Lambda$-parameter and $g_0$, which is important in checks of asymptotic scaling. For typical values of $c_{SW}$, this correction is found to be very pronounced.

Keywords: Lattice QCD, Lattice gauge theory, Beta function, Asymptotic scaling, Lattice perturbation theory, Running coupling constant, Clover action.

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I. INTRODUCTION

The clover action for lattice fermions was introduced a number of years ago [1], as a means of reducing finite lattice spacing effects, leading to a faster approach to the continuum. It is widely used nowadays in Monte Carlo simulations of dynamical fermions.

To monitor the onset of the continuum limit, tests of scaling must be performed on measured quantities. In particular, asymptotic scaling is governed by the bare $\beta$-function, defined in the standard way:

$$\beta_L(g_0) = -a \frac{dg_0}{da} \bigg|_{g, \mu},$$

where $a$ is the lattice spacing, $g_0 = g_0(1)$ the renormalized (bare) coupling constant, $\mu$ the renormalization scale. In the asymptotic region $g_0 \to 0$ one may write $\beta_L$ as:

$$\beta_L(g_0) = -b_0 g_0^3 - b_1 g_0^5 - b_2^L g_0^7 + ..., \quad (2)$$

where, in $SU(N)$ gauge theory with $N_f$ fermion species,

$$b_0 = \frac{1}{(4\pi)^2} \left( \frac{11}{3} N - \frac{2}{3} N_f \right), \quad (3)$$

$$b_1 = \frac{1}{(4\pi)^4} \left[ \frac{34}{3} N^2 - N_f \left( \frac{13}{3} N - \frac{1}{N} \right) \right]. \quad (4)$$

It is well known that the first two coefficients, $b_0$ and $b_1$, are universal; the 3-loop coefficient $b_2^L$, on the other hand, is regularization dependent. In the case at hand, $b_2^L$ is thus expected to depend not only on $N$ and $N_f$, but also on the free parameter $c_{SW}$ which appears in the clover action (see next Section).

In the present work we calculate $b_2^L$ for arbitrary $N$, $N_f$ and $c_{SW}$. The analogous calculation for the case of pure gauge theory without fermions, as well as for the case with Wilson fermions, was done a few years ago (Refs. [3] and [4], respectively). We follow the general setup and notation of those publications, to which we refer the reader for further details.

The $\beta$-function enters directly into the relation defining the renormalization group invariant parameter $\Lambda_L$:

$$a\Lambda_L = \exp \left( -\frac{1}{2 b_0 g_0^2} \right) (b_0 g_0^2)^{-b_1/2b_0^2} \left[ 1 + q g_0^2 + O \left( g_0^4 \right) \right], \quad q = \frac{b_1^2 - b_0 b_2^L}{2 b_0^3}. \quad (5)$$

The "correction" factor $q$ is, as we shall see, very pronounced for typical values of $c_{SW}$ and $g_0$.

A direct outcome of our calculation is the two-loop relation between the $\overline{MS}$ coupling $\alpha \equiv g^2/(4\pi)$ and $\alpha_0 \equiv g_0^2/(4\pi)$:

$$\alpha = \alpha_0 + d_1(a\mu) \alpha_0^3 + \frac{d_2(a\mu)}{2} \alpha_0^3 + O \left( \alpha_0^4 \right),$$

(6)

This relation is useful in studies involving running couplings or renormalized quark masses (see e.g. Refs. [5–8]). In fact, we will first compute this two-loop relation, which in turn will allow us to derive $b_2^L$. 

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In Sec. II we describe the method we employed to calculate $b_2^L$. We present our results for $b_2^L$, $d_1(a\mu)$ and $d_2(a\mu)$, as functions of $N$, $N_f$ and $c_{SW}$. Discussion of further technical details and checks of our calculations is relegated to Sec. III.

Given the difficulty of this computation, two different strategies have been adopted by each of the authors independently. H.P. performed a standard perturbation theory calculation, as described in Section II. A.B. used the approach of ref. [7], in which $d_2(a\mu) - d_2(a\mu)|_{c_{SW}=0}$ is obtained through a two-loop computation of the lattice Schrödinger functional. A detailed comparison of our results is presented at the end of Section III, showing very good agreement.

II. SETUP OF THE CALCULATION AND RESULTS.

Our starting point is the Wilson formulation of the QCD action on the lattice, with the addition of the clover (SW) fermion term. In standard notation:

$$S_L = \frac{1}{g_0^2} \sum_{x, \mu, \nu} \text{Tr} [1 - U_{\mu, \nu}(x)] + \sum_f \sum_x (4r + m_{f,0}) \bar{\psi}_f(x) \psi_f(x)$$

$$- \frac{1}{2} \sum_f \sum_{x, \mu} [\bar{\psi}_f(x)(r - \gamma_\mu) U_\mu(x) \psi_f(x + \hat{\mu}) + \bar{\psi}_f(x + \hat{\mu})(r + \gamma_\mu) U_\mu(x) \psi_f(x)]$$

$$+ \frac{i}{4} c_{SW} \sum_f \sum_{x, \mu, \nu} \bar{\psi}_f(x) \sigma_{\mu\nu} \tilde{F}_{\mu\nu}(x) \psi_f(x),$$

where : $\tilde{F}_{\mu\nu} \equiv \frac{1}{8} (Q_{\mu\nu} - Q_{\nu\mu})$, $Q_{\mu\nu} = U_{\mu, \nu} + U_{\nu, -\mu} + U_{-\mu, -\nu} + U_{-\nu, \mu}$ (8)

Here $U_{\mu, \nu}(x)$ is the usual product of link variables $U_\mu(x)$ along the perimeter of a plaquette in the $\mu$-$\nu$ directions, originating at $x$; $r$ is the Wilson parameter; $f$ is a flavor index; $m_{f,0}$ are the bare fermionic masses; $\sigma_{\mu\nu} = (i/2)[\gamma_\mu, \gamma_\nu]$; powers of $a$ have been omitted and may be directly reinserted by dimensional counting. The quantity $c_{SW}$ is a free parameter in the present work; it is normally tuned in a way as to minimize $O(a)$ effects.

The procedure we have followed here is analogous to the one employed in Refs. [2–4]; we shall only briefly describe it here, in the interest of a self-contained exposition.

We set out to compute the relation between the bare lattice coupling $g_0$ and the renormalized coupling $g$ defined in the $\overline{\text{MS}}$ renormalization scheme:

$$g_0 = Z_g(g_0, a\mu) g,$$

where $\mu$ indicates the renormalization scale. $Z_g$ is related to the lattice $\beta$-function $\beta^L(g_0)$, and to its $\overline{\text{MS}}$-renormalized counterpart $\beta(g)$, through:

$$\beta^L(g_0) = -g_0 a \frac{d}{da} \ln Z_g(g_0, a\mu) |_{\mu, g}$$

$$\beta(g) = -g \mu \frac{d}{d\mu} \ln Z_g(g_0, a\mu) |_{\mu, g_0} = -b_0 g^3 - b_1 g^5 - b_2 g^7 + ...$$ (10)
The lattice β-function, as defined above, is independent of the renormalized fermionic masses; this follows from the renormalizability of the theory \[9\] combined with dimensional arguments (see e.g. Ref. \[4\]). Thus, we will consider fermions with zero renormalized mass.

From the above one can derive the relation \[4\]

\[
\beta^L(g_0) = \left( 1 - g_0^2 \frac{\partial}{\partial g_0} \ln Z_g^2 \right)^{-1} Z_g \beta(g_0 Z_g^{-1}),
\]

valid to all orders of perturbation theory.

The one- and two-loop coefficients of $Z_g$ have the form:

\[
Z_g(g_0, a\mu)^2 = 1 + L_0(a\mu) g_0^2 + L_1(a\mu) g_0^4 + O(g_0^6)
\]

\[
L_0(x) = 2b_0 \ln x + l_0, \quad L_1(x) = 2b_1 \ln x + l_1.
\]

The constant $l_0$ is related to the ratio of the Λ parameters associated with the particular lattice regularization and the $\overline{\text{MS}}$ renormalization scheme:

\[
l_0 = 2b_0 \ln (\Lambda_L/\Lambda_{\overline{\text{MS}}}).
\]

Its value as a function of $N$, $N_f$, $c_{\text{SW}}$, is known (see e.g. Ref. \[7\] and references therein) and is presented here with increased accuracy for the $c_{\text{SW}}$-dependent coefficients:

\[
l_0 = \frac{1}{8N} - 0.1699559991998031(2) N + N_f l_{01}
\]

where:

\[
l_{01}(r=1) = 0.006696001(5) - c_{\text{SW}} 0.00504671402(1) + c_{\text{SW}}^2 0.02984346720(1)
\]

We have performed all our calculations for a range of $r$-values. In what follows we will be presenting only our results for $r = 1$, which is the value used in most Monte Carlo simulations; nevertheless, just for comparison, we give here $l_{01}$ for $r \neq 1$:

\[
l_{01}(r=0.5) = 0.020173519(2) - c_{\text{SW}} 0.0144735322(1) + c_{\text{SW}}^2 0.0545958518(1)
\]

\[
l_{01}(r=2.0) = -0.00181358(2) - c_{\text{SW}} 0.0011966285(1) + c_{\text{SW}}^2 0.0119267696(1)
\]

We see that this range of $r$-values induces significant variations on $\Lambda_L$; the dependence on $c_{\text{SW}}$ is also quite pronounced, as can be seen from Eq. (14), leading to changes in $\Lambda_L$ of up to a factor of 2.

Expanding Eq. (11) in powers of $g_0^2$ leads to a tower of relations between the coefficients of $\beta^L(g_0)$ and $\beta(g)$; namely, the difference between the $(i+1)$-loop coefficients, $b_i^{L} - b_i$, can be written in terms of $i$-loop contributions to $Z_g$. In particular, as is well known, $b_0^L - b_0 = 0$, $b_1^L - b_1 = 0$, and, what will be important below:

\[
b_2^L = b_2 - b_1 l_0 + b_0 l_1.
\]

Thus, since $b_2$ is known from the continuum \[10\],

\[
b_2 = \frac{1}{(4\pi)^6} \left[ \frac{2857}{54} N^3 + N_f \left( \frac{-1709N^2}{54} + \frac{187}{36} + \frac{1}{4N^2} \right) + N_f^2 \left( \frac{56N}{27} - \frac{11}{18N} \right) \right],
\]
the evaluation of $b_2^q$ requires only a two-loop calculation on the lattice, of the quantity $l_1$.

We will compute $Z_g$ in the background field gauge \[11\] \[15\], where it is simply related to the background field renormalization constant $Z_A$,

$$Z_A(g_0, a\mu) Z_g(g_0, a\mu)^2 = 1.$$  \hspace{1cm} (18)

In the background field formulation the links are written as

$$U_\mu(x) = e^{ig_0 Q_\mu(x)} e^{ig_0 A_\mu(x)},$$ \hspace{1cm} (19)

where $Q_\mu(x) = T^c Q^c_\mu(x)$ and $A_\mu(x) = T^c A^c_\mu(x)$ are the quantum and background fields respectively; $T^a$ are Hermitian $su(N)$ generators. We use the standard gauge-fixing term for this formulation, along with the corresponding Faddeev-Popov action for ghost fields, and the contribution from the integration measure (in our notation, these terms are written explicitly in \[11\]).

As in Ref. \[2\], we write the renormalized one-particle irreducible two-point functions of the background and quantum fields as

$$\Gamma_R^{AA}(p)_{\mu\nu} = \delta^{ab} (\delta_{\mu\nu} p^2 - p_\mu p_\nu) (1 - \nu_R(p))/g^2, \quad \nu_R(p) = \sum_{l=1}^\infty g^{2l} \nu_R^{(l)}(p),$$ \hspace{1cm} (20)

$$\Gamma_R^{QQ}(p)_{\mu\nu} = \delta^{ab} [(\delta_{\mu\nu} p^2 - p_\mu p_\nu) (1 - \omega_R(p)) + \lambda p_\mu p_\nu], \quad \omega_R(p) = \sum_{l=1}^\infty g^{2l} \omega_R^{(l)}(p).$$ \hspace{1cm} (21)

Correspondingly on the lattice

$$\sum_\mu \Gamma_L^{AA}(p)_{\mu\nu} = \delta^{ab} 3\bar{p}^2 [1 - \nu(p)]/g_0^2, \quad \nu(p) = \sum_{l=1}^\infty g_0^{2l} \nu^{(l)}(p),$$ \hspace{1cm} (22)

$$\sum_\mu \Gamma_L^{QQ}(p)_{\mu\nu} = \delta^{ab} 3\bar{p}^2 [3 (1 - \omega(p)) + \lambda_0], \quad \omega(p) = \sum_{l=1}^\infty g_0^{2l} \omega^{(l)}(p).$$ \hspace{1cm} (23)

There follows

$$Z_g^2 = Z_A^{-1} = \frac{1 - \nu(p)}{1 - \nu_R(p)}.$$ \hspace{1cm} (24)

The bare and renormalized gauge parameters appearing above, $\lambda_0$ and $\lambda$, are related by $\lambda = Z_Q \lambda_0$, where $Z_Q$ is the renormalization constant of the quantum field:

$$[1 - \omega_R(p)] = Z_Q [1 - \omega(p)].$$ \hspace{1cm} (25)

Both $\nu(p)$ and $\omega(p)$ depend on $(a\mu)$, $g_0$ and $\lambda_0$, whereas $\nu_R(p)$ and $\omega_R(p)$ depend on $(p/\mu)$, $g$ and $\lambda$. One can fix $\lambda_0 = 1$, and we have done so; this entails computing $\lambda = Z_Q$, which is only needed here to one loop.

The $\overline{\text{MS}}$ renormalized functions entering into $Z_g$ to two loops: $\nu_R^{(1)}(p)$, $\omega_R^{(1)}(p)$ and $\nu_R^{(2)}(p) |_{\lambda = 1}$ were calculated and presented in Ref. \[4\] (and Ref. \[13\] for $N_f = 0$).

The lattice two-point functions, $\nu^{(1)}(p)$, $\omega^{(1)}(p)$ and $\nu^{(2)}(p)$, were the main object of the present work, for $c_{SW} \neq 0$. We will present here directly our results; details on our
calculations will be provided in the next section. For easier reference, we also include below results pertaining to pure gauge theory without fermions (from Refs. [2,3]) and with fermions at $c_{SW} = 0$ (from Ref. [4]).

$$\nu^{(1)}(p) |_{\lambda_0=1} = \nu^{(1)}(p) |_{\lambda_0=1, N_f=0} + N_f \left[ \frac{1}{24 \pi^2} \ln(a^2 p^2) + k_{1f} \right],$$

$$\omega^{(1)}(p) |_{\lambda_0=1} = \omega^{(1)}(p) |_{\lambda_0=1, N_f=0} + N_f \left[ \frac{1}{24 \pi^2} \ln(a^2 p^2) + k_{1f} \right],$$

$$\nu^{(2)}(p) |_{\lambda_0=1} = \nu^{(2)}(p) |_{\lambda_0=1, N_f=0} + N_f \left[ \frac{1}{16 \pi^2} \left( 3N - \frac{1}{N} \right) \ln(a^2 p^2) + k_{2f} \frac{1}{N} + k_{3f}N \right].$$

The $N_f=0$ quantities are [4]:

$$\nu^{(1)}(p) |_{\lambda_0=1, N_f=0} = -\frac{11N}{48 \pi^2} \ln(a^2 p^2) - \frac{1}{8N} + N \ 0.217098494366724...$$

$$\omega^{(1)}(p) |_{\lambda_0=1, N_f=0} = -\frac{5N}{48 \pi^2} \ln(a^2 p^2) - \frac{1}{8N} + N \ 0.13728678290915...$$

$$\nu^{(2)}(p) |_{\lambda_0=1, N_f=0} = -\frac{N^2}{32 \pi^4} \ln(a^2 p^2) + \frac{3}{128 N^2} - 0.016544619540(4) + N^2 \ 0.0074438722(2).$$

For $r = 1$ we found

$$k_{1f} = -0.013732194(5) + c_{SW} \ 0.00504671402(1) - c_{SW}^2 \ 0.02984346720(1),$$

$$k_{2f} = 0.0011877(14) - c_{SW} \ 0.0001578(3) - c_{SW}^2 \ 0.0052931(2) - c_{SW}^3 \ 0.00050624(3) - c_{SW}^4 \ 0.00008199(1),$$

$$k_{3f} = -0.0013617(16) - c_{SW} \ 0.0000981(4) + c_{SW}^2 \ 0.0052440(6) + c_{SW}^3 \ 0.00021431(3) + c_{SW}^4 \ 0.00004382(1).$$

The origin and meaning of the errors in the above results will be explained in the next section. Contributions which vanish in the continuum limit, $a \to 0$, do not contribute to $Z_g$ and are neglected as in Refs. [2,4].

We have now what is needed to calculate $Z_g$ to two loops using Eq. (24). The resulting two-loop constant $l_1$ is given by

$$l_1 = -\frac{3}{128 N^2} + 0.018127763034(4) - N^2 \ 0.0079101185(2) + N_f \left[ l_{11} \frac{1}{N} + l_{12} N \right].$$

$$l_{11} = -0.0011967(14) + c_{SW} \ 0.0001578(3) + c_{SW}^2 \ 0.0052931(2) + c_{SW}^3 \ 0.00050624(3) + c_{SW}^4 \ 0.00008199(1),$$

$$l_{12} = 0.0009998(16) + c_{SW} \ 0.0000342(4) - c_{SW}^2 \ 0.0048660(6) - c_{SW}^3 \ 0.00021431(3) - c_{SW}^4 \ 0.00004382(1).$$

At this point, it is a trivial exercise to recover $b_2^f$, for any value of $N$, $N_f$, $c_{SW}$, by substitution in Eq. (28). We list here some particular cases of interest, setting $N = 3$:

$$b_2^f(c_{SW}=0) = -0.00159983232(13) + N_f \ 0.0000799(4) - N^2 \ 0.00000605(2),$$

$$b_2^f(c_{SW}=1) = -0.00159983232(13) - N_f \ 0.0009449(4) + N^2 \ 0.00006251(2),$$

$$b_2^f(N_f=3) = -0.0014144(10) + c_{SW} \ 0.0000654(2) - c_{SW}^2 \ 0.0024241(4) - c_{SW}^3 \ 0.0008108(2) - c_{SW}^4 \ 0.0001781(1).$$
The correction coefficient $q$ of Eq. (5) also follows immediately. In the particular cases considered above we obtain ($N = 3$):

$c_{SW} = 0$:

- $q = 0.18960350(1)$ ($N_f = 0$),
- $q = 0.2160(1)$ ($N_f = 2$),
- $q = 0.2355(2)$ ($N_f = 3$)

$c_{SW} = 1$:

- $q = 0.18960350(1)$ ($N_f = 0$),
- $q = 0.4529(1)$ ($N_f = 2$),
- $q = 0.6138(2)$ ($N_f = 3$)

$q(N_f = 3) = 0.2355(2) - c_{SW} 0.01007(4) + c_{SW}^2 0.3731(1) + c_{SW}^3 0.01248(1) + c_{SW}^4 0.002741(1)$.  

Clearly, these values of $q$ bring about a substantial correction to asymptotic scaling, with a pronounced $c_{SW}$ dependence.

Finally, the coefficients $d_1(a\mu)$ and $d_2(a\mu)$, in the relation (6) between the $\overline{MS}$ coupling $\alpha$ and $\alpha_0$, can be read off $L_0(x)$ and $L_1(x)$, defined in Eq. (12):

$$d_1(x) = -4\pi L_0(x), \quad d_2(x) = (4\pi)^2 \left[ L_0(x)^2 - L_1(x) \right]$$

### III. THE CALCULATION IN LATTICE PERTURBATION THEORY

We describe here briefly some technical features and checks of our calculation of the quantities $\nu^{(1)}(p)$, $\nu^{(2)}(p)$. Fermionic contributions to $\omega^{(1)}(p)$ are identical to those in $\nu^{(1)}(p)$. Many aspects are as in Ref. [4], and are discussed there in more detail.

Two diagrams containing fermions contribute to $\nu^{(1)}(p)$, shown in Figure 1. There are 18 two-loop fermionic diagrams contributing to $\nu^{(2)}(p)$, as well as two diagrams containing an insertion of the one-loop fermion mass counterterm; these are shown in Figure 2.

![FIG. 1. Fermion contributions to the one-loop function $\nu^{(1)}(p)$. Dashed lines ending on a cross represent background gluons. Solid lines represent fermions.](image)

The algebra involving lattice quantities was performed using a symbolic manipulation package in Mathematica, developed by one of the authors (H.P.) and collaborators over the past years (see [4] and references therein). In order to carry out the present work, this package was augmented to include vertices of the clover action. These vertices are more complicated than those of Wilson fermions, as can be seen by the form of the action (7); as a result, the algebraic expression for each diagram is considerably longer in the present case. On the other hand, all new contributions to fermionic vertices are accompanied by extra powers of momenta, and consequently lead to expressions with improved infrared behaviour.

The two-loop amplitude $\nu^{(2)}(p)$ can be written as:
\[ \nu^{(2)}(p) = \nu^{(2)}(p)|_{c_{SW}=0} + \sum_i \nu_i(p) \]  

(the index \(i\) runs over the diagrams shown in Fig. 2). Since two-loop diagrams can have a maximum of 4 fermionic vertices, \(\nu_i(p)\) is a polynomial in \(c_{SW}\) of degree up to 4, starting from \(c_{SW}^1\).

On general grounds we expect:

\[ \hat{a}p^2 \nu_i(p) = c_{0,i} + c_{1,i} a^2 \sum_{\mu} \frac{p_{\mu}^4}{p^2} + a^2 p^2 \left\{ c_{2,i} \left( \frac{\ln a^2 p^2}{(4\pi)^2} \right)^2 + c_{3,i} \frac{\ln a^2 p^2}{(4\pi)^2} + c_{4,i} \right\} + O((ap)^4) \]  

\( (\hat{p}^2 = 4 \sum_{\mu} \sin^2(p_{\mu}/2)) \), where the most general dependence of \(c_{n,i}\) on \(N, N_f, c_{SW}\) is:

\[ c_{n,i} = \sum_{j=1}^{4} c_{SW}^j \left[ \frac{c_{n,i}^{(-1,j)}}{N} + c_{n,i}^{(1,j)} N \right] N_f \]  

FIG. 2. Fermion contributions to the two-loop function \(\nu^{(2)}(p)\). Dashed lines represent gluonic fields; those ending on a cross stand for background gluons. Solid lines represent fermions. The filled circle is a one-loop fermion mass counterterm.
As a byproduct of improved infrared convergence, the coefficients $c_{1,i}$, $c_{2,i}$ vanish for each diagram separately in this case; all other coefficients are also separately infrared convergent.

To extract the $p$-dependence, we apply a procedure of successive subtractions on terms containing (sub-)divergences; our expressions then split into a relatively small part made of factorized one-loop terms, whose (possibly logarithmic) $p$-dependence is easily extracted in terms of tabulated lattice integrals, and a much larger part which can be evaluated by naive Taylor expansion in $ap$.

Once the $p$-dependence has been made explicit, the resulting expressions for $c_{n,i}$ must be numerically integrated over internal momenta. For each expression, our package creates highly optimized numerical code for integration in momentum space, over finite lattices of size $L$. An extrapolation to infinite size is then performed. This operation is of course a source of systematic error. In a nutshell, we estimate this error as follows: First, different extrapolations are performed using a broad spectrum of functional forms of the type:

$$
\sum_{i,j} e_{i,j} L^{-i} (\ln L)^j
$$

For the $k^{th}$ such extrapolation, a deviation $d_k$ is calculated applying three different quantitative criteria for quality of fit. Finally, these deviations are used to assign weights $d_k^{-2}/(\sum_k d_k^{-2})$ to each extrapolation, which are then combined to produce a final value together with the error estimate. We have checked the validity of these estimates in cases where the exact answer was known, finding the estimate to be always correct.

The improved infrared behaviour of each diagram led to a rather stable extrapolation; indeed, for the accuracy reported in this work, lattices up to $L \leq 28$ were sufficient in all but a few cases.

The results for each diagram are presented in Tables I through VI. Diagrams not appearing in these Tables give vanishing contributions.

For the one-loop amplitude $\hat{a}p^2 \nu^{(1)}(p)$, the individual contributions from the corresponding two diagrams in Figure 1 are:

$$
\nu_1(p) = c_{SW} 0.00504671402(1) - c_{SW^2} 0.02984346720(1), \quad \nu_2(p) = 0
$$

There exist several constraints on the coefficients $c_{n,i}$, which we have used as verification both on the algebraic expressions and on the numerical results:

- $c_{0,i}$: Gauge invariance requires $\sum_i c_{0,i} = 0$. This relation amounts to 4 independent nontrivial checks, for the coefficients of $c_{SW} N$, $c_{SW^2} N$, $c_{SW}/N$, $c_{SW^2}/N$. Our algebraic expressions pass these checks. Our numerical results also give zero, well within the systematic errors; this effectively controls the consistency of our error estimation.

- $c_{3,i}$: The total contribution for single logarithms must give zero: $\sum_i c_{3,i} = 0$. Again, this relation amounts to 5 independent nontrivial checks, which were performed both algebraically and numerically.

- There are several other relations among the coefficients appearing in individual diagrams (e.g. $c_{3,10}^{(1,1)} = 4 c_{3,13}^{(1,1)}$, $c_{3,17}^{(1,2)} = -5 c_{3,13}^{(1,2)}$, etc.); these can be understood in terms of the vertices entering these diagrams, and have been used as further consistency controls.
The most stringent check is provided by Ref. [7], in which the Schrödinger functional in lattice QCD is computed to two loops, using the clover action with $c_{SW} = 0, 1$. Eq. (5.6) of that reference reads: $d_{21}(N=3, c_{SW}^{(0)}=1) = 1.685(9) - 8.6286(2) c_{SW}^{(1)}$. Calculating the same quantity from our results, we deduce: 1.6828(8) − 8.62843775(1) $c_{SW}^{(1)}$. Both sets of numbers are clearly in very good agreement.

Since the dependence of $d_2$ on $c_{SW}$ is polynomial of degree 4, all coefficients of the polynomial (except for the 0th degree, which is gotten from [4]) can be checked by repeating the calculations of Ref. [7] for a few more values of $c_{SW}$. We have done so for all values of $c_{SW}$ listed in Table VII; the results obtained in this way have a larger error than those stemming from Eqs. (33, 12, 29, 30), but they provide a very solid and thorough confirmation of our main result, Eq.(30).

The entire procedure employed in this paper can be applied without modifications to a number of other cases of interest, among them the higher order renormalization of $O(a)$-improved operators. We hope to study these cases in a future work.

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1In the notation of Ref. [7]: $c_{SW} = c_{SW}^{(0)} + c_{SW}^{(1)} g_0^2 + \ldots$, $d_2(a\mu) = d_1(a\mu)^2 + \sum_{r=0}^1 N_f \{-32\pi^2 b_1, \ln(a\mu) + d_{2r}\}$
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### TABLE I. Coefficients $c_{0,i}^{(-1,j)}$, $r = 1$.

| $i$  | $c_{0,i}^{(-1,1)}$ | $c_{0,i}^{(-1,2)}$ |
|------|--------------------|--------------------|
| 4    | 0.000040200(3)     | 0                  |
| 12   | 0.00114029(4)      | 0.00114886(4)      |
| 14   | -0.00008041(1)     | 0                  |
| 15   | -0.00013096(2)     | -0.00007625(1)     |
| 18   | -0.0009691(2)      | -0.0010726(1)      |
| 19   | 0.002862565(1)     | 0.001191276(1)     |
| 20   | -0.0028625645(1)   | -0.0011912762(1)   |

### TABLE II. Coefficients $c_{0,i}^{(1,j)}$, $r = 1$.

| $i$  | $c_{0,i}^{(1,1)}$ | $c_{0,i}^{(1,2)}$ |
|------|--------------------|--------------------|
| 4    | -0.00016080(1)    | 0.0071556(1)       |
| 5    | 0.000040202(2)    | -0.0077678(3)      |
| 11   | -0.00021503(4)    | 0.0023875(1)       |
| 12   | -0.00114029(4)    | -0.00114886(4)     |
| 13   | -0.0000575(1)     | 0.000956(1)        |
| 14   | 0.00029874(1)     | 0.00026811(2)      |
| 16   | 0.00004358(1)     | -0.00011562(2)     |
| 17   | 0.0002220(2)      | -0.002807(1)       |
| 18   | 0.0009691(2)      | 0.0010726(1)       |
| 19   | -0.002862565(1)   | -0.001191276(1)    |
| 20   | 0.0028625645(1)   | 0.0011912762(1)    |

### TABLE III. Coefficients $c_{3,i}^{(-1,j)}$, $r = 1$.

| $i$  | $c_{3,i}^{(-1,1)}$ | $c_{3,i}^{(-1,2)}$ | $c_{3,i}^{(-1,3)}$ |
|------|--------------------|--------------------|--------------------|
| 8    | -0.30986678046212  | 0                  | 0                  |
| 14   | -0.01449434463(1)  | 0                  | 0                  |
| 15   | 0.00500026042(1)   | -0.00589886486(1)  | 0                  |
| 18   | -0.0933193205(1)   | 0.0566733951(1)    | 0.03619153756(1)   |
| 19   | 0.41268018519(1)   | -0.05077453020(1)  | -0.03619153756(1)  |
### TABLE IV. Coefficients $c_{3,i}^{(1,1)}$. $r = 1.$

| $i$  | $c_{3,i}^{(1,1)}$ | $c_{3,i}^{(1,2)}$ | $c_{3,i}^{(1,3)}$ |
|------|------------------|------------------|------------------|
| 8    | 0.30986678046212 | 0                | 0                |
| 10   | 0.01009342804(1)  | -0.11937386878(1)| 0                |
| 13   | 0.00252335701(1)  | -0.02984346720(1)| 0                |
| 14   | -0.00500026042(1) | 0.01052896544(1) | 0                |
| 16   | 0.0271111297(1)   | -0.00463010057(1)| 0                |
| 17   | -0.02523357012(1) | 0.14921733598(1) | 0                |
| 18   | 0.0933193205(1)   | -0.0566733951(1) | -0.0361915376(1) |
| 19   | -0.41268018519(1) | 0.05077453020(1) | 0.0361915376(1)  |

### TABLE V. Coefficients $c_{4,i}^{(-1,1)}$. $r = 1.$

| $i$  | $c_{4,i}^{(-1,1)}$ | $c_{4,i}^{(-1,2)}$ | $c_{4,i}^{(-1,3)}$ | $c_{4,i}^{(-1,4)}$ |
|------|------------------|------------------|------------------|------------------|
| 3    | 0.00082631538(1)  | -0.007460866799(1)| 0                | 0                |
| 4    | 0.000020100(1)    | -0.00119261(2)    | 0                | 0                |
| 8    | 0.003071859(3)    | 0.00121700384(1)  | 0                | 0                |
| 11   | -0.000020100(1)   | 0.0059631(2)      | 0                | 0                |
| 14   | 0.00018663(2)     | 0.00005901(1)     | 0                | 0                |
| 15   | 0.0000931(2)      | 0.0001160(1)      | -0.00000655(2)   | -0.000038170(1)  |
| 18   | 0.0004905(4)      | -0.0002038(3)     | -0.00059685(2)   | 0.00008405(1)    |
| 19   | -0.004826287(1)   | 0.0016290054(5)   | 0.0000971639(1)  | -0.0001278777(1) |

### TABLE VI. Coefficients $c_{4,i}^{(1,1)}$. $r = 1.$

| $i$  | $c_{4,i}^{(1,1)}$ | $c_{4,i}^{(1,2)}$ | $c_{4,i}^{(1,3)}$ | $c_{4,i}^{(1,4)}$ |
|------|------------------|------------------|------------------|------------------|
| 3    | -0.00093961433(1)| 0.008800841023(1)| 0                | 0                |
| 4    | -0.0000050250(3) | 0.000298152(4)   | 0                | 0                |
| 5    | 0.0000004358(3)  | -0.0001706(1)    | 0                | 0                |
| 8    | -0.003071859(3)  | -0.00121700384(1)| 0                | 0                |
| 10   | -0.000169546987(1)| 0.002005213655(1)| 0                | 0                |
| 11   | 0.00003775(1)    | -0.00074560(4)   | 0                | 0                |
| 13   | 0.00004011(2)    | -0.0005579(2)    | 0                | 0                |
| 14   | -0.0002223(1)    | -0.0001602(1)    | -0.00001405(1)   | 0                |
| 16   | -0.0001813(4)    | -0.0000980(5)    | -0.00027132(2)   | 0                |
| 17   | 0.0000775(3)     | -0.001486(1)     | 0                | 0                |
| 18   | -0.0004905(4)    | 0.0002038(3)     | 0.00059685(2)    | -0.00008405(1)   |
| 19   | 0.004826287(1)   | -0.0016290054(5) | -0.0000971639(1) | 0.0001278777(1)  |
TABLE VII. Comparison of our results for \((d_{21} \mid c_{SW}^{(0)} = d_{21} \mid c_{SW}^{(0)} = 0) / (16\pi^2)\) coming from standard perturbation theory (PT) (Eqs. (33, 12, 29, 30)), to those we obtain using the Schrödinger functional (SF) methods of Ref. [7]. \((N = 3, c_{SW}^{(1)} = 0)\)

| \(^{(0)}c_{SW}\) | PT         | SF         |
|----------------|------------|------------|
| -4.0           | 0.202270(29) | 0.2029(21) |
| -2.0           | 0.049518(10) | 0.04968(53) |
| -1.0           | 0.012619(7)  | 0.013041(60) |
| -0.5           | 0.003233(2)  | 0.003258(96) |
| 0.5            | 0.003197(2)  | 0.003191(18) |
| 1.0            | 0.013257(7)  | 0.01327(4)  |
| 2.0            | 0.056484(10) | 0.05659(45) |
| 4.0            | 0.261725(29) | 0.2627(33)  |