Lie-Group Approach to Perturbative Renormalization-Group Method

Shin-itiro Goto, Yuji Masutomi and Kazuhiro Nozaki
Department of Physics, Nagoya University
Nagoya 464-8602, Japan

March 31, 2022

Abstract

The Lie-group approach to the perturbative renormalization group (RG) method is developed to obtain an asymptotic solutions of both autonomous and non-autonomous ordinary differential equations. Reduction of some partial differential equations to typical RG equations is also achieved by this approach and a simple recipe for providing RG equations is presented.
1 Introduction

A novel method based on the perturbative renormalization group (RG) theory has been developed as an asymptotic singular perturbation technique [1] and has been successively applied to ordinary differential equations (ODE) [2][3] and some partial differential equations (PDE) [2][4][5]. An approach from a geometrical point of view was also proposed, which is called an envelope method [6].

The renormalization group method removes secular or divergent terms from a naive perturbation series by renormalizing integral constants i.e., a renormalization transformation and reaches a slow-motion and/or large-scale equation as a renormalization group (RG) equation. In this reduction of an asymptotic equation, it is not necessary to introduce slow or large scales a priori in contrast to the other methods such as the reductive perturbation method [9] and various multiple-scale methods [10]. Instead, a renormalization transformation determines an asymptotic equation as a RG equation. Here, it is crucial to extract the structure of secular terms from a perturbation series in order to obtain a renormalization transformation. For ODE, the structure of secular terms are often relatively simple and, as shown in this paper, it is possible to calculate explicit secular solutions up to arbitrary order in principle and obtain an asymptotic form of the renormalization transformation of all order in some examples. However, for PDE, there are many divergent solutions in a perturbed equation and it is ambiguous which divergent solutions should be renormalized away [8]. Although an attempt to overcome this difficulty was presented [5], it is far from satisfactory.

The group structure of the renormalization transformation has not yet been necessarily clear in the previous applications of the RG method and it is desirable to make the renormalization procedure more transparent for its extended applications.

In this paper, we construct a representation of the Lie group from a renormalization transformation in several examples and identify the procedure of renormalization with one to derive an asymptotic expression to a generator of the Lie group. Although the view of the Lie group has been introduced to the RG method in systems with a translational symmetry with respect to an independent variable [1][4], there are few examples based on the Lie group theory and only the leading or low order RG equations are derived [12]. For the purpose of obtaining higher-order RG equations and applying the RG method to wider systems such as non-autonomous systems without a translational symmetry and PDE, much more investigations are still necessary to refine the Lie-group approach.

In Section 2, we present some clear examples, in which an asymptotic expression to a generator of the Lie group (a renormalization transformation) is obtained up to arbitrary order, in principle, by naive perturbation calcula-
tions. In the simplest example in Section 2.A, an asymptotic expression of a generator is shown to converge and we recover an exact solution. In Section 2.B, we extend the Lie-group approach to a non-autonomous system of ODE by introducing a simple shift operation. Through these simple examples, we present a recipe for obtaining an asymptotic representation to the Lie group underlying the original system. In the remaining parts of Section 2, we give some physically interesting examples. In an example in Section 2.C, a renormalization transformation has a translational symmetry though an original ODE does not have one. In Section 2.D, we present a peculiar example in the Einstein’s gravitational equation where an original ODE and its leading order RG equation are autonomous but a higher order RG equation becomes non-autonomous. Even in this peculiar case, our Lie-group approach works well and gives an asymptotic solution describing an expanding space. In Section 2.E, it is shown that secular solutions to perturbed equations for a weakly nonlinear oscillator can be removed asymptotically up to all order. We also derive adiabatic equations for parameters of the K-dV soliton under perturbations in Section 2.F.

In Section 3, an extension of the RG method in terms of the Lie group is presented for some nonlinear PDE. We derive two typical soliton equations such as the nonlinear Schrödinger equation and the Kadomtsev-Pitviashvili equation with higher order corrections as RG equations. As the final example, phase equations for interface and oscillating solutions are derived from a general reaction-diffusion system.

In Section 4, we give a simple recipe for providing RG equations as a summary of this paper.

2 Ordinary Differential Equations

2.A Linear Autonomous System

As an example in which a generator to the Lie group of a renormalization transformation is constructed exactly, we discuss a boundary layer type problem \[ \text{[10]} \].

\[ \left( \epsilon \frac{d^2}{dx^2} + \frac{d}{dx} + 1 \right) u = 0, \quad (2.1) \]

where \( u(0) = 0 \). Since a small parameter \( \epsilon \) multiplying the highest derivative causes a severe singularity at a boundary \( x = 0 \), we introduce a transformation
$x = et$. Then, (2.1) reads

$$\left( \frac{d^2}{dt^2} + \frac{d}{dt} + \epsilon \right) u = 0,$$  

(2.2)

which is called the inner equation in the boundary layer problem. Expanding $u$ as

$$u = A + Be^{-t} + \epsilon u_1 + \epsilon^2 u_2 + \cdots,$$  

(2.3)

we have

$$L(t)u_n \equiv \left( \frac{d^2}{dt^2} + \frac{d}{dt} \right) u_n = -u_{n-1},$$  

(2.4)

where $u_0 = A + Be^{-t}$ and $A, B$ are arbitrary constants. Secular solutions can be written as

$$u_n = AP_n(t) + BP_n(-t)e^{-t},$$  

(2.5)

where $P_0 = 1$ while $P_n(t)$ $(n \geq 1)$ is a polynomial solution of

$$L(t)P_n = -P_{n-1}, \quad P_1 = -t, \quad P_2 = t^2/2 - t, \quad P_3 = -t^3/6 + t^2 - 2t.$$

Here, we observe that a secular term $P_n(t)$ is a polynomial of degree $n$. In order to eliminate these secular terms by renormalizing integral constants $A, B$, the following naive renormalization transformations $A \rightarrow \tilde{A}(t)$ and $B \rightarrow \tilde{B}(t)$ are introduced as

$$\tilde{A}(t) = A(1 + \epsilon P_1(t) + \epsilon^2 P_2(t) + \cdots),$$  

(2.6)

$$\tilde{B}(t) = B(1 + \epsilon P_1(-t) + \epsilon^2 P_2(-t) + \cdots).$$  

(2.7)

Note that this definition of renormalization transformations is different from one in [2]. In order to obtain a representation to the Lie group from (2.6), we take advantage of a translational symmetry in (2.2). It is easy to see that $\tilde{A}(t), \tilde{B}(t)$ also enjoy a translational symmetry, that is

$$L(t)\tilde{A} + \epsilon \tilde{A} = 0, \quad L(-t)\tilde{B} + \epsilon \tilde{B} = 0.$$  

(2.8)

The renormalization transformation (2.6) is interpreted as a Taylor series for the solution $\tilde{A}$ of (2.8) around $t = 0$, while the origin of expansion can be shifted to arbitrary positions by virtue of a translational symmetry. Thus, we obtain an asymptotic representation to the Lie group $G_\tau$ immediately from (2.6).

$$\tilde{A}(t + \tau) = \tilde{A}(t)(1 + \epsilon P_1(\tau) + \epsilon^2 P_2(\tau) + \cdots),$$  

$$\equiv G_\tau \tilde{A}(t).$$  

(2.9)
In terms of a generator $\tau \partial_t$ of the Lie group, $\tilde{A}(t + \tau)$ is also expanded as

$$
\tilde{A}(t + \tau) = \exp(\tau \partial_t) \tilde{A}(t),
= (1 + \tau \partial_t + (\tau^2/2)\partial_t^2 + \cdots)\tilde{A}(t).
$$

Equating coefficients of equal powers of $\tau$ in (2.9) and (2.10), we have an asymptotic form of a generator of the Lie group called a RG equation

$$
d\tilde{A}/dt = \partial_\tau [G_\tau \tilde{A}(t)]_{\tau=0},
= \partial_\tau [\epsilon P_1(\tau) + \epsilon^2 P_2(\tau) + \cdots]_{\tau=0} \tilde{A},
= -\epsilon(1 + \epsilon + 2\epsilon^2 + \cdots)\tilde{A} \equiv \epsilon \delta \tilde{A},
$$

(2.11)

All other equations for higher order derivatives are guaranteed to be consistent with (2.11) by virtue of the theory of Lie group or arbitrariness of $\tau$. Since $\delta = -(1 + \epsilon + 2\epsilon^2 + \cdots)$ converges to $(-1 + \sqrt{1 - 4\epsilon})/(2\epsilon)$ for $\epsilon < 1/4$, (2.11) gives an exact form of the generator. Applying a similar analysis to (2.7), we obtain a RG equation for $\tilde{B}$:

$$
d\tilde{B}/dt = \partial_\tau [\epsilon P_1(-\tau) + \epsilon^2 P_2(-\tau) + \cdots]_{\tau=0} \tilde{B},
= \epsilon(1 + \epsilon + 2\epsilon^2 + \cdots)\tilde{B} = -\epsilon \delta \tilde{B}.
$$

Thus, renormalizing all secular terms in the perturbation series, we recover an exact solution of (2.1) with the boundary condition $u(0) = 0$.

$$
u = \tilde{A}(0) \exp(\delta x) + \tilde{B}(0) \exp(-x/\epsilon - \delta x),
$$

(2.12)

where $\tilde{A}(0) + \tilde{B}(0) = 0$.

### 2.B Oscillator with time-dependent spring constant

To extend the Lie-group approach described in the previous section to systems without a translational symmetry with respect to $t$, we consider an linear but nontrivial oscillator governed by

$$
\frac{d^2 u}{dt^2} + u = ctu,
$$

(2.13)
which was analyzed in [2] to the first order in $\epsilon$. Here, we try to extend the RG analysis to arbitrary order in terms of the Lie group. Expanding $u$ as

$$u = Ae^{-it} + \epsilon u_1 + \epsilon^2 u_2 + \cdots + \text{c.c.}, \quad (2.14)$$

we have

$$\left(\frac{d^2}{dt^2} + 1\right) u_n = tu_{n-1}, \quad (2.15)$$

where c.c. stands for the complex conjugate of the preceding expression, $n = 1, 2, \cdots$ and $u_0 = Ae^{-it}$. Secular solutions are given by

$$u_n = AP_t(t)e^{-it}, \quad L(t)P_n(t) \equiv \left(\frac{d^2}{dt^2} - 2i \frac{d}{dt}\right) P_n = tP_{n-1}, \quad (2.16)$$

where $n = 1, 2, \cdots$ and $P_0 = 1$, while an initial condition $P_n(0) = 0$ is set for $n \geq 1$. The important observation is that $P_n(t)$ is a polynomial of degree $2n$ which is given by, for example,

$$P_1 = (it^2 + t)/4, \quad P_2 = \sum_{j=1}^{4} p_j t^j, \quad p_4 = -1/32, \quad p_3 = 5i/48, \quad p_2 = 5/32, \quad p_1 = 5i/32.$$ 

Thus, we have

$$u = A(1 + \epsilon P_1(t) + \epsilon^2 P_2(t) + \cdots)e^{-it} + \text{c.c.}, \quad (2.17)$$

which yields a renormalization transformation $A \rightarrow \tilde{A}(t)$ defined by the same expression as the previous example (2.6), where secular terms $P_n$ are given by polynomial solutions of (2.16). Since $A(t)$ has no longer a translational symmetry in this case, it is necessary to introduce the following general procedure in order to derive an asymptotic representation to the Lie group from (2.6).

Let us shift $t \rightarrow t + \tau$ in (2.6):

$$\tilde{A}(t + \tau) = A(1 + \epsilon P_1(t + \tau) + \epsilon^2 P_2(t + \tau) + \cdots). \quad (2.18)$$

Noting that $P_n$ is a polynomial of degree $2n$, we have the following expansion of $\tilde{A}$ with respect to $\tau$ and $\epsilon$.

$$\tilde{A}(t + \tau) = A(1 + \sum_{n=1}^{\infty} \epsilon^n P_n(t)) \left[\tau \sum_{n=1}^{\infty} \epsilon^n P_{n,t} + \sum_{n=1}^{\infty} \epsilon^n P_{n,2t} + (\tau^3/3!) \sum_{n=2}^{\infty} \epsilon^n P_{n,3t} + \cdots \right]$$

$$+ \sum_{n=1}^{\infty} \epsilon^n P_{n,4t} + \cdots] A, \quad (2.19)$$
where $P_{n,kt}$ denotes the k-th derivative with respect to $t$. Replacing $A$ in (2.19) with $\tilde{A}$ by means of the renormalization transformation (2.6), we reach an asymptotic representation to the Lie group:

$$\tilde{A}(t+\tau) = \tilde{A}(t) + \left[ \tau \sum_{n=1}^{\infty} \epsilon^n P_{n,1t} + \frac{\tau^2}{2} \sum_{n=1}^{\infty} \epsilon^n P_{n,2t} \right]$$

$$+ \left[ \tau^3/3! \sum_{n=2}^{\infty} \epsilon^n P_{n,3t} + \frac{\tau^4}{4!} \sum_{n=2}^{\infty} \epsilon^n P_{n,4t} \right]$$

$$+ \cdots ]\tilde{A}(t)/(1 + \sum_{n=1}^{\infty} \epsilon^n P_n(t)).$$ (2.20)

Due to loss of the translational symmetry, the right-hand side of (2.20) explicitly depends on $t$ and so the representation (2.20) should read

$$\tilde{A}(t+\tau) = G_{\tau}\{\tilde{A}(t), T(t)\}, \quad T(t) = t.$$ (2.21)

From (2.20) and (2.21), we can derive an asymptotic expression of a generator of the Lie group, that is, a RG equation:

$$\frac{d\tilde{A}}{dt} = \partial_{\tau}[G_{\tau}\{\tilde{A}(t), T(t)\}]_{\tau=0}$$

$$= \frac{d}{dt}\{\ln(1 + \sum_{n=1}^{\infty} \epsilon^n P_n(t))\}\tilde{A}(t),$$ (2.22)

from which the following asymptotic solution $\tilde{A}(t)$ is easily obtained by integrating a truncated expression of (2.22).

$$\tilde{A}(t) = \tilde{A}(0) \exp\{\epsilon P_1(t) - \epsilon^2(P_2(t) + P_1(t)^2) + \cdots\},$$ (2.23)

where the argument of exp function in (2.23) is determined by integration of a truncated power series of

$$\frac{d}{dt} (\epsilon P_1 + \epsilon^2 P_2 + \cdots)/(1 + \epsilon P_1 + \epsilon^2 P_2 + \cdots).$$ (2.24)

The solution (2.23) is a generalization of the previous result [2]. Note that the RG equation (2.22) is formally integrated to recover the renormalization transformation (2.6).

Now, we summarize the above procedures for providing a RG equation as a generator of a renormalization group (the Lie group).

1) Calculate secular terms in a naive perturbation series.
(2) Find integral constants $A \in \mathbb{R}^n$, which are renormalizable in order to remove secular terms, and construct a renormalization transformation $\tilde{A}(t) = \tilde{R}(A, t)$ which is obtained as a power series of $t$.

(3) Rewrite the renormalization transformation in the form of the Lie group $\tilde{A}(t + \tau) = G_\tau\{\tilde{A}(t), T(t)\}$ by means of a shift operation $t \rightarrow t + \tau$, where $\tilde{A}$ and $T(t) = t$ constitute a differentiable manifold on which the Lie-group $G_\tau$ acts.

(4) A RG equation is obtained as a generator of the Lie group:

$$\frac{d\tilde{A}}{dt} = \partial_\tau[G_\tau\{\tilde{A}(t), T(t)\}]_{\tau=0}.$$ 

If a renormalization transformation $\tilde{A}(t) = \tilde{R}(A, t)$ has a translational symmetry with respect to $t$, a representation of the Lie group does not depend on $T(t) = t$ and takes the following simple form.

$$\tilde{A}(t + \tau) = G_\tau\tilde{A}(t) = \tilde{R}(\tilde{A}(t), \tau).$$

In the following subsections, we derive various RG equations following the above procedures.

## 2.C Mathieu Equation

As a second example of non-autonomous systems, we derive a RG equation near the transition curves that separate stable from unstable solutions of the Mathieu equation

$$\frac{d^2u}{dt^2} + u = -\epsilon\{\omega + 2\cos(2t)\}u,$$

where $\omega = \omega_1 + \epsilon\omega_2 + \epsilon^2\omega_3 + \cdots$. Expanding $u$ just as in the previous example

$$u = Ae^{-it} + \text{c.c.} + \epsilon u_1 + \epsilon^2 u_2 + \cdots,$$

we have, in the leading order,

$$\frac{d^2u_1}{dt^2} + u_1 = -(\omega_1A + A^*)e^{-it} - Ae^{-3it} + \text{c.c.},$$

of which general solution is

$$u_1 = -(\omega_1A + A^*)P_1(t)e^{-it} + A/8e^{-3it} + \text{c.c.},$$

where
where \( A^* \) denotes the complex conjugate of \( A \) and \( P_1(t) \) is a polynomial solution of
\[
\left( \frac{d^2}{dt^2} - 2i \frac{d}{dt} \right) P_1 \equiv L(t)P_1 = 1, \tag{2.30}
\]
where \( P_1(t) \) is uniquely determined for the initial condition \( P_1(0) = 0 \), i.e. \( P_1(t) = it/2 \). Similar perturbation calculations yield higher order solutions:
\[
\begin{align*}
\text{Performing a shift operation } t \to t + \tau \text{ on (2.34) and replacing } A \text{ in the right-hand side of (2.34) by } \hat{A}(t) + \epsilon(\omega_1 \hat{A} + \hat{A}^*)P_1(t) + O(\epsilon^2), \text{ we obtain a representation of the renormalization group}

\hat{A}(t + \tau) = G_\tau \hat{A}(t) = \hat{R}(\hat{A}(t), \tau). \tag{2.35}
\end{align*}
\]
This expression indicates that the renormalization group has a translational symmetry. In fact, (2.35) is also directly derived from (2.34) when we take into account of the fact that Fourier components of \( u \) obey a coupled but autonomous system of equations. This result shows an interesting example where the renormalization group has a translational symmetry even if the original
system does not have one. It is easy to see that non-resonant secular terms such as $Q_1$ in higher harmonics are automatically eliminated when resonant secular terms in the fundamental harmonic are renormalized by (2.34). Further discussions are presented about automatic elimination of non-resonant secular terms in higher harmonics in Section 2.E. By differentiating (2.35) with respect to $\tau$, we have a generator of the Lie group (2.35)

$$\frac{d\tilde{A}}{dt} = -(i\epsilon/2)(\omega_1\tilde{A} + \tilde{A}^*) - (i\epsilon^2/2)\Delta\tilde{A}$$

$$+ \frac{i\epsilon^3}{2}[(\omega_1/32 - \omega_3)\tilde{A} + (3/64)\tilde{A}^* + (\omega_1/2)\Delta\tilde{A}], \quad (2.36)$$

$$\Delta = \omega_2 + 1/8 + (1 - \omega_1^2)/4,$$

from which

$$\frac{d^2\tilde{A}}{dt^2} = [\epsilon^2(1 - \omega_1^2)/4 - (\epsilon^3\omega_1)\Delta$$

$$+ \epsilon^4\{\omega_1(\omega_1/32 - \omega_3) - 3/64 + \omega_1\Delta/2\}]\tilde{A}.$$ 

which gives the transition curves that separate stable from unstable solutions: $\omega_1 = \pm 1$, $\omega_2 = -1/8$, $\omega_3 = \mp 1/64$. This transition curves $\omega_1, \omega_2$ and a special RG equation for $\omega_1 = 1$ were obtained to $O(\epsilon^2)$ by means of the RG method [2]. The expression of a generator (2.36) is a generalization of the previous results.

### 2.D Large-scale expansion in Gravitational Equation

A uniformly and isotropically expanding space with dust is described by the following Einstein’s equation called the Friedman-Robertson-Walker equation

$$2a\ddot{a} + \dot{a}^2 = -(k - \Lambda a^2), \quad (2.37)$$

where $a$ is a time-dependent scale factor of space, a dot on $a$ denotes the derivative with respect to time $t$, $k(= \pm 1, 0)$ is a sign of curvature and $\Lambda$ is the cosmological constant. Although solutions of (2.37) with an initial condition $a(0) = 0$ are well-known and describe various kinds of expanding and/or contracting space, this example gives a peculiar asymptotic form of the representation of the Lie group for a large-scale solution as is shown later. That is, the higher order RG equation becomes non-autonomous though the original equation (2.37) and the leading order RG equation are autonomous. The leading order RG equation has been derived for a large-scale solution of (2.37) with $\Lambda = 0$ by Y.Nambu and Y.Y.Yamaguchi [12] and they show that
a contraction stage of the Friedman space \((k = 1)\) is qualitatively recovered by only the leading order RG equation. Here, we derive a higher order and non-autonomous RG equation by means of the present Lie-group approach for the purpose of quantitative comparison with the exact results.

If \(\Lambda = 0\), we can expand \(a\) around a large-scale solution \(a_0 = At^{2/3}\) as

\[
a(x) = x\left\{A + a_1(x)/A + a_2(x)/A^3 + a_3(x)/A^5 + \cdots\right\},
\]

(2.38)

where \(x = t^{2/3}\), \(A\) is an arbitrary constant and \(A \gg 1\). Substituting (2.38) into (2.37) and solving the resulting equation order by order with respect to \(A^{-1}\), we obtain the following secular terms

\[
a_j(x) = -c_j x^j, \quad j = 1, 2, 3, \ldots,
\]

(2.39)

\[
c_1 = \frac{9k^2}{20}, \quad c_2 = \frac{3c_1^2}{7}, \quad c_3 = \frac{23c_1c_2}{27}, \quad c_4 = \frac{9c_2^2 + 19c_1c_3}{22}.
\]

Renormalizing the integral constant \(A\) so that secular terms (2.39) are removed, we have a renormalization transformation

\[
\tilde{A}(x) = A + a_1(x)/A + a_2(x)/A^3 + a_3(x)/A^5 + \cdots.
\]

(2.40)

Replacing \(x\) in (2.40) by \(x + \xi\), we obtain a non-autonomous representation of the Lie group:

\[
\tilde{A}(x + \xi) = G_\xi\{\tilde{A}(x), x\},
\]

of which generator is explicitly given as

\[
\frac{d\tilde{A}}{dx} = \partial_\xi G_\xi(\tilde{A}(x), x)|_{\xi=0} = c_1 \frac{\tilde{A} + (c_1^2 - 2c_2)x}{\tilde{A}^3} - \frac{(2c_2^3 - 7c_1c_2 + 3c_3)x^2}{\tilde{A}^5} + \frac{(5c_1^2 - 24c_1^2c_2 + 6c_2^2 + 16c_1c_3 - 4c_4)x^3}{\tilde{A}^7} + \cdots.
\]

(2.41)

It is easy to see that this expression of the generator is asymptotic for \(x/\tilde{A}^2 = t/a(t) < 1\), that is, the scale factor \(a\) is greater than the cosmic horizon. Results of numerical integration of (2.41) for \(k = 1\) are compared with the exact solution in Fig.(1). Solutions of the RG equation with higher-order corrections (2.41) fits the exact solution closer in the region \(t/a(t) < 1\) but they separate away from the exact solution for \(t/a(t) > 1\) where the present asymptotic expansion is not valid. It may be interesting that the leading order RG solution gives the best fit for \(t/a(t) > 1\). This may come from fact that the leading order RG equation does not have the factor \(x/\tilde{A}^2 = t/a(t)\), which should be
small in the present asymptotic expansion.

When a value of $\Lambda$ is near the critical value where the Friedman space begins to expand again, $\Lambda$ should be scaled as

$$\Lambda = \frac{\lambda}{A_0^6},$$

(2.42)

where $A_0 = \tilde{A}(0)$ and the critical value of $\Lambda$ is given by $\lambda_c = 9/4$. Then, effects of this small $\Lambda$ enter into (2.41) as the following correction terms of $O(\tilde{A}^{-5})$ and $O(\tilde{A}^{-7})$

$$3\lambda_3\tilde{A}x^2 + 2(c_1\lambda_3 - 2\lambda_4)x^3,$$

(2.43)

where $\lambda_3 = \lambda/(12A_0^6)$, $\lambda_4 = 9\lambda c_1/(88A_0^6)$. Numerically integrating (2.41) with corrections (2.43), we obtain asymptotic solutions near the critical value $\Lambda_c$ as shown in Fig.(2), which demonstrates again that asymptotic solutions agree well with an exact solution for $t/a(t) < 1$. 
Figure 1: The scale factor $a$ vs. $t$ for $\Lambda = 0$ the line 0 is the cosmic horizon, i.e. $a = t$, the curve e is the exact solution, the curve 1 is a solution of the leading order RG equation, the curves 2, 3 and 4 are solutions of the second and the third and the fourth order RG equations respectively.
Figure 2: The scale factor $a$ vs. $t$ for $\Lambda = 1.1\Lambda_c$ the line 0 is the cosmic horizon ($a = t$), the curve e is the exact solution, the curve 1 is a solution of the RG equation.
2.E Weakly Nonlinear Oscillator

Let us apply the Lie-group approach to the motion of a weakly nonlinear oscillator.

\[ \frac{d^2 u}{dt^2} + u = \epsilon a_3 u^3 + \epsilon^2 a_5 u^5 + \epsilon^3 a_7 u^7 + \cdots. \quad (2.44) \]

Expanding \( u \) as

\[ u = A e^{-it} + \text{c.c.} + \epsilon u_1 + \epsilon^2 u_2 + \cdots, \quad (2.45) \]

we have

\[ \left( \frac{d^2}{dt^2} + 1 \right) u_1 = a_3 [3|A|^2 A e^{-it} + A^3 e^{-3it} + \text{c.c.}] \quad (2.46) \]

The general solution is given by

\[ u_1 = a_3 [3|A|^2 A P_1 e^{-it} + A^3 q_1 e^{-3it} + \text{c.c.}] \quad (2.47) \]

where \( q_1 = -1/8 \) and \( P_1(t) \) is a polynomial solution of (2.30). For \( O(\epsilon^2) \), we have

\[
\begin{align*}
\frac{d^2 u_2}{dt^2} & = \left\{ 3a_3^2(3P_2 + q_1 P_1) + 10a_5 P_1 \right\}|A|^4 A e^{-it} \\
& + \{ 9a_3^2 Q_1 + (6a_5^2 q_1 + 5a_2) q_1 \}|A|^2 A^3 e^{-3it} \\
& + (3a_3^2 q_1 + a_5) q_2 A^5 e^{-5it} + \text{c.c.,} \\
LP_2 & = P_1, \quad q_2 = -1/24, \quad L_3 Q_1 = P_1, \\
\end{align*}
(2.48)
\]

where \( L, \ L_3 \) are defined in (2.30) and (2.32). Explicit forms of secular terms are given in (2.33). Up to \( O(\epsilon^2) \), the secular solution is given by

\[
\begin{align*}
u & = \left( A + 3\epsilon a_3 |A|^2 A P_1 + \epsilon^2 \left\{ 3a_3^2 (3P_2 + q_1 P_1) + 10a_5 P_1 \right\}|A|^4 A \right) e^{-it} \\
& - (\epsilon/8) \{ a_3 + \epsilon (9a_3^2 P_1 + 21a_3^3 / 8 + 5a_5) |A|^2 \} A^3 e^{-3it} + \text{c.c.} + \cdots. (2.49) \\
\end{align*}
\]

In (2.49), coefficients of \( e^{-it} \) are summed up to give an asymptotic expansion to the renormalized amplitude \( \tilde{A} \).

\[
\begin{align*}
\tilde{A}(t) & = A + 3\epsilon a_3 |A|^2 A P_1(t) + \epsilon^2 \left\{ 3a_3^2 (3P_2(t) + q_1 P_1(t)) + 10a_5 P_1(t) \right\}|A|^4 A + O(\epsilon^3), \\
& \quad (2.50) \\
\end{align*}
\]

while coefficients of the third harmonic are summed up to give \( \epsilon \tilde{A}_3(t) \) as

\[
\begin{align*}
\epsilon \tilde{A}_3(t) & = -(\epsilon/8) \{ a_3 + 9\epsilon a_3^2 |A|^2 P_1(t) \\
& + \epsilon (21a_3^3 / 8 + 5a_5) |A|^2 + O(\epsilon^2) \} A^3. \\
& \quad (2.51) \\
\end{align*}
\]
Substituting an iterative expression of $A$ in terms of $\tilde{A}$ obtained from (2.50) into (2.51), we can eliminate a secular term in (2.51) and have

$$\tilde{A}_3(t) = -1(\alpha/8)\tilde{A}(t)^3\{a_3 + \epsilon(21a_3^2/8 + 5a_5)|\tilde{A}(t)|^2 + O(\epsilon^2)\}. \quad (2.52)$$

This elimination of secular terms in the higher harmonics is quite general as also seen in Section 2.C because secular terms in the higher harmonics originate directly from resonant secular terms in the fundamental harmonic and should disappear as soon as the amplitude $A$ of the fundamental harmonic is renormalized by the renormalization transformation (2.50). In this sense, we call secular terms in the higher harmonics as non-resonant secular terms, which disappear when resonant secular terms are removed by the renormalization transformation.

Performing a transformation $t \to t + \tau$ on (2.50) and using explicit forms of secular terms, we obtain

$$\tilde{A}(t + \tau) = \tilde{A}(t) + 3\epsilon a_3|\tilde{A}(t)|^2\tilde{A}(t)P_1(\tau) + \epsilon^2\{3a_3^2(3P_2(\tau) + q_1P_1(\tau))|\tilde{A}(t)|^4\tilde{A}(t) + O(\epsilon^3), \quad (2.53)$$

which is an explicit representation of the Lie group underlying the renormalization transformation (2.50). Since the original equation of motion (2.44) is translationally invariant, the amplitude of oscillation $\tilde{A}$ also enjoys a translational symmetry. Thus, (2.53) is also a direct result of (2.50) and the translational symmetry. Since we could proceed this perturbation calculations to arbitrarily higher order although calculations becomes tedious for the higher order, we can also extend (2.53) up to arbitrary order. Noting the important observation that secular terms appearing in the coefficient of the fundamental harmonic $\exp(-it)$ in $u_n$ are polynomials of degree $n$ and proportional to $|A|^{2n}A$, we have

$$\tilde{A}(t + \tau) = \tilde{A}(t) + \sum_{n=1}^{\infty} e^n \sum_{j=1}^{n} p_{n,j}\tau^j|\tilde{A}(t)|^{2n}\tilde{A}(t)$$

$$= \tilde{A}(t) + \epsilon p_{1,1}\tau|\tilde{A}(t)|^2\tilde{A}(t) + \epsilon^2(\tau^2p_{2,2} + \tau p_{2,1})|\tilde{A}(t)|^4\tilde{A}(t) + \cdots,$$

where $p_{n,j}$ is constant. Equating coefficients of equal powers of $\tau$ in (2.53) and (2.10), we have a set of equations

$$\frac{d^n\tilde{A}}{dt^n}/n! = e^n \sum_{j=0}^{\infty} e^j p_{n+j,n}|\tilde{A}(t)|^{2(n+j)}\tilde{A}(t). \quad (2.54)$$

Some explicit forms of $p_{n,j}$ are given as

$$p_{1,1} = i(3/2)a_3, \quad p_{2,1} = i\{(15/16)a_3^2 + 5a_5\},$$

16
\[ p_{3,1} = i\{(123/128)a_3^3 + 5a_3a_5 + (37/2)a_7\} \]
\[ p_{2,2} = -(9/8)a_3^2, \quad p_{3,2} = -a_3(45a_3^2/32 + 15a_5/2), \]
\[ p_{3,3} = -(9i/16)a_3^3, \]

The other \( p_{n,j} \) are obtained when higher order secular terms are calculated. Thus, for \( n = 1 \), eq.(2.54) gives an asymptotic representation for a generator of the Lie group.

\[
\frac{d\tilde{A}}{dt} = \sum_{j=0}^{\infty} \epsilon^j p_{1+j,1} |\tilde{A}(t)|^{2(1+j)} \tilde{A}(t) \\
= \epsilon i(3/2)a_3 |\tilde{A}|^2 \tilde{A} + \epsilon^2 i\{(15/16)a_3^2 + 5a_5\} |\tilde{A}|^4 \tilde{A} + \cdots. \quad (2.55)
\]

Since \( p_{1+j,1} \) is purely imaginary, it is easy to see that \( |\tilde{A}|^2 \) is a constant of motion and we formally integrate (2.55) to have an asymptotic expression to the amplitude of the fundamental harmonic by all order:

\[
\tilde{A}(t) = A e^{i \sum_{j=0}^{\infty} \epsilon^j p_{1+j,1} |A|^{2(1+j)} At},
\]

where \( A \) is an integral constant. Then, as noted earlier, secular terms in the higher harmonics are removed automatically, we finally obtain an asymptotic solution of (2.44)

\[
u = \tilde{A}(t)e^{-it} + \epsilon A_3(\tilde{A})e^{-3it} + \epsilon^2 A_5(\tilde{A})e^{-5it} + \cdots, \quad (2.57)
\]

where amplitudes of higher hamonics slave to \( \tilde{A} \) (e.g. see (2.52)). For the case that \( a_5 = a_7 = \cdots = 0 \), the asymptotic solution (2.57) may be interpreted as an asymptotic Fourier expansin of the Jacobi’s elliptic function, which is the exact solution of (2.44).

As far as the final RG equation (2.55) is concerned, the following convenient derivation is possible without exploiting explicit forms of secular terms. Let us operate the linear operator \( L(\tau) \) defined in (2.30) on the both sides of eq.(2.53) and set \( \tau = 0 \). Then, taking (2.30) and (2.48) into account, we have

\[
L(t)\tilde{A}(t) = 3\epsilon a_3 |\tilde{A}|^2 \tilde{A} + \epsilon^2 \{(3/2)a_3^2 + 10a_5\} |\tilde{A}|^4 \tilde{A}. \quad (2.58)
\]

After \( \partial_t^2 \) in \( L \) is eliminated iteratively , (2.58) reduces to (2.55). This procedure is valid for the secular solution (2.53) including secular terms higher order than \( O(\epsilon^2) \) in the present case. In fact, if this procedure is applied to the first order secular solution, we have

\[
L\tilde{A} = 3\epsilon a_3 |\tilde{A}|^2 \tilde{A}, \quad (2.59)
\]

which is not correct as the first order RG equation since \( \partial_t^2 \) in \( L \) is of \( O(\epsilon^2) \) (see (2.54)).
2.F Adiabatic Perturbation for a Soliton

Although a soliton is a special solution of PDE, adiabatic variations of soliton-parameters are described by a system of ODE and we will see that a corresponding renormalization group has only one parameter. Various perturbation methods for solitons have been developed based on the inverse scattering transform [13]. However, their applications are limited to nearly integrable systems. There are some attempts [15] to develop perturbation methods without utilizing the complete integrability of the unperturbed state, which are applicable to a pulse solution of non-integrable systems. Here, in line with these attempts, we derive the well-known adiabatic equation to soliton-parameters of the K-dV soliton by means of the RG method. Let us consider the K-dV equation with a perturbation

\[ \partial_t u - 6u \partial_x u + \partial_x^3 u = \epsilon f(u). \]  

(2.60)

Expanding \( u \) around a soliton solution

\[ u_0 = s(k, z) \equiv -2k^2 \text{sech}^2(kz), \quad z = x - v(k)t + \xi, \quad v(k) = 4k^2, \]  

(2.61)

we have, for the first order correction \( u_1 \),

\[ \{\partial_t + L(z)\} u_1 = f(s), \]  

(2.62)

\[ L = -v \partial_z - 6(s \partial_z + \partial_z s) + \partial_z^3. \]  

(2.63)

Since \( k \) and \( \xi \) in (2.61) are arbitrary parameters, zero-eigenfunctions of \( L(z) \) with a bounded boundary condition are given by \( \partial_z s(k, z) \equiv s', \quad \partial_k s(k, z) \equiv \dot{s}. \) In fact, we see that zero-eigenfunctions are degenerate [16] so that

\[ Ls' = 0, \quad L\dot{s} = \dot{v}s', \quad L^2 \dot{s} = 0, \]  

(2.64)

where \( \dot{v} = \partial_k v \). Here, let us introduce an adjoint zero-eigenfunction \( \hat{s} \) defined by the adjoint operator \( \hat{L} \) to \( L \) as

\[ \hat{L}\hat{s} = 0, \quad \hat{s} = \text{sech}^2(kz). \]  

(2.65)

From (2.64) and (2.65), we find

\[ <\hat{s} \cdot s'> = \int_{-\infty}^{\infty} \hat{s}s'dz = 0. \]  

(2.66)

In terms of these zero-eigenfunctions, we put a spatially bounded solution as

\[ u_1 = P_1(t)s' + Q_1(t)\hat{s} + \bar{u}_1(z). \]  

(2.67)

Substituting (2.67) into (2.62), we have an equation for \( \bar{u}_1(z) \)

\[ L(z)\bar{u}_1 + (P_{1,t} + \dot{v}Q_1)s' + Q_{1,t}\hat{s} = f(s), \]  

(2.68)
where the suffix \( t \) denotes the derivative with respect to \( t \). Since (2.68) is an equation for \( t \) independent and spatially bounded \( \bar{u}_1 \), the following consistency conditions must be satisfied:

\[
Q_{1,t} = \frac{\langle \hat{s} \cdot f(s) \rangle}{\langle \hat{s} \cdot \hat{s} \rangle} = \text{constant},
\]

\[
P_{1,t} + \dot{v}Q_1 = \text{constant},
\]

from which

\[
P_{1,2t} = -\dot{v}Q_{1,t}.
\]

From conditions (2.69) and (2.70), secular terms \( P_1(t) \) and \( Q_1(t) \) are found to be polynomials of degree 2 and 1 respectively.

\[P_1 = p_{1,2}t^2 + p_{1,1}t, \quad Q_1 = q_{1,1}t,\]

where coefficients \( p, q \) are

\[2p_{1,2} = -\dot{v}q_{1,1}, \quad q_{1,1} = \frac{\langle \hat{s} \cdot f(s) \rangle}{\langle \hat{s} \cdot \hat{s} \rangle},\]

while \( p_{1,1} \) is an arbitrary constant.

Thus, a secular solution to \( O(\epsilon) \) is found to be

\[u = s(k, x + \theta) + \epsilon(P_1(t)s' + Q_1(t)\hat{s}),\]

where \( \theta = -\nu t + \xi \). In order to eliminate secular terms \( P_1, Q_1 \) by renormalizing arbitrary soliton-parameters \( k \) and \( \xi \), we introduce a renormalized soliton solution as

\[s(\tilde{k}(t), x + \tilde{\theta}(t)) = s(k, x + \theta) + \epsilon \{ P_1(t)s'(k, x + \theta) + Q_1(t)\hat{s}(k, x + \theta) \},\]

where \( \tilde{\theta}(t) = -\nu(\tilde{k}(t))t + \tilde{\xi}(t) \). Setting \( \tilde{k}(t) = k + \epsilon k_1(t) \), \( \tilde{\theta}(t) = \theta + \epsilon \theta_1(t) \), the left-hand side of (2.74) is expanded as

\[s(\tilde{k}, x + \tilde{\theta}) = s(k, x + \theta) + \epsilon(\theta_1 s' + k_1 \hat{s}) + O(\epsilon^2).\]

From (2.74) and (2.75), we have a renormalization transformation

\[
\tilde{k}(t) = k + \epsilon Q_1(t; k), \quad \tilde{\theta}(t) = \theta + \epsilon P_1(t; k),
\]

where \( P_1, Q_1 \) depend on \( k \) through their coefficient \( p, q \) given in (2.72). Shifting \( t \) to \( t + \tau \) in (2.76) and replacing \( k \) by \( \tilde{k}(t) - \epsilon Q_1(t; k) \), we obtain a renormalization group

\[
\tilde{k}(t + \tau) = \tilde{k}(t) + \epsilon Q_1(\tau; \tilde{k}(t)),
\]

\[
\tilde{\theta}(t + \tau) = \tilde{\theta}(t) - \nu(\tilde{k}(t))\tau + \epsilon P_1(\tau; \tilde{k}(t)),
\]
from which we obtain a generator of the renormalization group

\[
\frac{d\tilde{k}}{dt} = \epsilon q_{1,1}(\tilde{k}), \quad (2.77)
\]

\[
\frac{d\tilde{\theta}}{dt} = -v(\tilde{k}) + \epsilon p_{1,1}(\tilde{k}), \quad (2.78)
\]

\[
\frac{d^2\tilde{\theta}}{dt^2} = 2\epsilon p_{1,2}(\tilde{k}), \quad (2.79)
\]

where consistency between (2.78) and (2.79) are assured by an arbitrary coefficient \(p_{1,1}\). Finally, from (2.72), (2.77) and (2.79), we reach the well-known adiabatic equations to soliton-parameters to \(O(\epsilon)\) \([14]\).

\[
\frac{d\tilde{k}}{dt} = -\frac{\epsilon}{4} \int_{-\infty}^{\infty} \text{sech}^2(\tilde{k}z) f[s(\tilde{k}, \tilde{k}z)]dz,
\]

\[
\frac{d^2\tilde{\theta}}{dt^2} = -\tilde{v}(\tilde{k}) \frac{d\tilde{k}}{dt} = -\frac{dv(k(t))}{dt}.
\]

3 Reduction of Partial Differential Equations

3.A General Discussion

If we try to extend the Lie-group approach discussed in the preceding sections to PDE, there is a difficulty to calculate secular terms so that infinitely many divergent terms such as exponentially divergent terms as well as polynomials appear even in a perturbed solution of the leading order. This is due to the fact that the dimension of "kernels" without any boundary conditions of a linearized partial differential operator is infinite in general. However, each divergent solution belonging to "kernels" is accompanied by an integral constant and so unsuitable divergent solutions such as exponentially divergent solutions are eliminated by nullifying corresponding integral constants. Thus, we restrict renormalizable secular terms to polynomials even in PDE. This restriction is quite natural in the Lie-group approach since a representation of the Lie group with multiple parameters should be expanded in terms of a generator as

\[
\tilde{A}(t + \tau, x + \xi) = \exp(\tau \partial_t + \xi \partial_x)\tilde{A}(t, x)
\]

\[
= \{1 + \tau \partial_t + \xi \partial_x + (\tau \partial_t + \xi \partial_x)^2/2 + \cdots \} \tilde{A}(t, x), \quad (3.1)
\]
which is a polynomial with respect to \( \tau, \xi \) when the expansion is truncated. Even if secular terms are restricted to polynomials, there are still infinitely many secular solutions in kernels of a linearized partial differential operator. For the purpose of classifying infinitely many polynomial terms, the following observation is crucial. In the expansion (3.1), each monomial is accompanied by a differential operator of the same order, for example, \( \xi \) by \( \partial_x \), \( \tau^2 \) by \( \partial_t^2 \) and \( \xi \tau \) by \( \partial_x \partial_t \) etc., while each differential operator, say, \( \partial_t \) maps \( \tilde{A} \) to the smaller value \( \partial_t \tilde{A} \ll |\tilde{A}| \) in the autonomous case, which will be discussed in the following sections. In this sense, each monomial in a representation of the Lie group is considered to have the same order of magnitude as the corresponding differential operator. On the other hand, for systems with a translational symmetry, each monomial with respect to \( t, x \) in secular terms in a perturbation series is replaced by the corresponding monomial with respect to \( \tau, \xi \) in a representation of the Lie group (see, (2.25)). Therefore, each monomial in secular terms has also the same order of magnitude as the corresponding differential operator. Thus, monomials in secular terms are ordered as

\[
x > x^2 > x^3, \quad t > tx > tx^2, \quad tx > t^2 x, \text{ etc.} \tag{3.2}
\]

With this kind of order for monomials in mind, we choose polynomial kernels from the lower order by nullifying integral constants accompanying higher-order kernels. In particular, the leading order secular term consists of the lowest order polynomial, i.e. a polynomial of degree one. As order of perturbed solutions increases, the higher order polynomial kernels are taken into secular solutions order by order so that order of differential operators accompanied by polynomial secular terms is kept consistent. Through this procedure, we can choose suitable kernels as necessary ingredients of a renormalization transformation.

On the other hand, there is inevitable flexibility appearing in the choice of initial setting of perturbation scheme, which brings about different RG equations from each other depending on the choice, although only one of them is often interesting as shown in the next example.

### 3.B Nonlinear Schrödinger Equation

As the first example, we consider a nonlinear wave equation, which is just a wave-equation version of the equation for a nonlinear oscillator (2.44).

\[
\partial_t^2 u - \nabla^2 u + u = \epsilon^m au^3, \tag{3.3}
\]

where \( m \) is a positive integer. Let us expand \( u \) around a plane-wave solution

\[
u_0 = A \exp \{i(kx - \omega t)\} + \text{c.c.}, \quad \omega^2 = 1 + k^2, \tag{3.4}\]
as \( u = u_0 + \epsilon u_1 + \epsilon u_2 + \cdots \).

For \( m = 1 \), we have

\[
\partial_t^2 u_1 - \nabla^2 u_1 + u_1 = au_0^3.
\]  

(3.5)

A secular solution of (3.5) is given by

\[
u_1 = 3a|A|^2 A P_1(t, x, r) e^{i\theta} - (a/8)A^3 e^{3i\theta} + c.c.,
\]  

(3.6)

\[
LP_1 = \{-2i\omega(\partial_t + \dot{\omega}\partial_x) + \partial_t^2 - \nabla^2\}P_1 = 1,
\]  

(3.7)

where \( \theta = kx - \omega t \), \( r_\perp = (y, z) \), \( \dot{\omega} = \partial_k \omega = k/\omega \). As discussed in Section 3.A, it is assumed that the leading order secular term is a polynomial of degree one, that is

\[
P_1 = p_{1,0,0} t + p_{0,1,0} x + p_{0,0,1} \cdot r_\perp.
\]  

(3.8)

Substituting (3.8) into (3.7), we have

\[
-2i\omega(p_{1,0,0} + \dot{\omega} p_{0,1,0}) = 1.
\]  

(3.9)

The secular term is eliminated by means of a renormalization transformation

\[
\tilde{A}(t, x, r_\perp) = A + 3\epsilon a|A|^2 Ap_{1,0,0} + p_{0,1,0} x + p_{0,0,1} \cdot r_\perp,
\]  

(3.10)

which is rewritten as, by executing arbitrary shifts on independent variables,

\[
\tilde{A}(t + \tau, x + \xi, r_\perp + \eta) = \tilde{A}(t, x, r_\perp) + 3\epsilon a|\tilde{A}|^2 \tilde{A}(t, x, r_\perp) P_1(\tau, \xi, \eta),
\]  

(3.11)

which gives a representation of the Lie group with multiple parameters to \( O(\epsilon) \). Here, it should be noted that the coefficients of a polynomial \( P_1 \) may depend on an "initial position", that is \( p_{j,k,l} \) is replaced by \( \tilde{p}_{j,k,l}(t, x, r_\perp) \) in (3.11). It is seen later in (3.13), (3.14) and (3.15) that this dependence comes through \( \tilde{A} \). In terms of a generator of the Lie group, (3.11) is expanded:

\[
\tilde{A}(t + \tau, x + \xi, r_\perp + \eta) = \exp(\tau \partial_t + \xi \partial_x + \eta \cdot \nabla_\perp) \tilde{A}(t, x, r_\perp)
\]  

(3.12)

where \( \nabla_\perp = (\partial_y, \partial_z) \). Equating coefficients of \( \tau, \xi, \eta \) in (3.11) and (3.12), we obtain

\[
\partial_\tau \tilde{A} = 3\epsilon a|A|^2 \tilde{A} p_{1,0,0},
\]  

(3.13)

\[
\partial_x \tilde{A} = 3\epsilon a|A|^2 \tilde{A} p_{0,1,0},
\]  

(3.14)

\[
\nabla_\perp \tilde{A} = 3\epsilon a|A|^2 \tilde{A} p_{0,0,1}.
\]  

(3.15)

From (3.9), (3.13) and (3.14), \( p_{j,k,l} \) or \( \tilde{p}_{j,k,l} \) are eliminated to yield a RG equation

\[
(\partial_t + \dot{\omega} \partial_x) \tilde{A} = 3i\epsilon a|A|^2 \tilde{A}/(2\omega),
\]  

(3.16)
which gives a steady propagation of the modulation of plane wave under effects of nonlinearity.

In order to include effects of dispersion, we set $m = 2$ in the initial setting of perturbation. Then, instead of (3.3), we have

$$\partial_t^2 u_1 - \nabla^2 u_1 + u_1 = 0,$$  

(3.17)

of which secular solution is

$$u_1 = P_0(t, x, r_\perp)e^{i\theta} + \text{c.c.},$$  

(3.18)

where $P_0$ is the lowest order secular solution of $L P_0 = 0$ given by the same form as $P_1$ in (3.8) with a different constraint

$$p_{1,0,0} + \dot{\omega} p_{0,1,0} = 0.$$  

(3.19)

If secular terms in (3.18) are removed by a similar renormalization transformation as above, the constraint (3.19) would yield a RG equation

$$(\partial_t + \dot{\omega} \partial_x) \tilde{A} = 0,$$  

(3.20)

of which general solution is written as $\tilde{A} = \tilde{A}(x', r_\perp), x' = x - \dot{\omega} t$. Therefore, it is convenient to introduce a Galilean transformation

$$x' = x - \dot{\omega} t, \quad t = t.$$  

(3.21)

Then, the operator $L$ in (3.7) is rewritten and $L P_0 = 0$ becomes

$$L P_0 = \{-2\dot{\omega} \partial_t + (\dot{\omega}^2 - 1) \partial_x^2 - \nabla_\perp^2 - 2\dot{\omega} \partial_t \partial_x' + \partial_t^2\} P_0 = 0.$$  

(3.22)

The lowest order secular solution of (3.22) is

$$P_0 = p_{0,1,0}^{(1)} x' + p_{0,0,1}^{(1)} \cdot r_\perp,$$  

(3.23)

where $p_{0,1,0}^{(1)}$ and $p_{0,0,1}^{(1)}$ are arbitrary constants. The second order secular solution $u_2$ is given by the same form as $u_1$ in (3.6), where $P_1$ is the second order polynomial solution of $L P_1(t, x', r_\perp) = 1$, that is

$$P_1 = p_{0,2,0}^{(2)} x'^2 + p_{0,1,0}^{(2)} x' + p_{0,0,2}^{(2)} \cdot r_\perp r_\perp + p_{0,0,1}^{(2)} \cdot r_\perp + p_{1,0,0}^{(2)} t,$$

$$-2i\dot{\omega} p_{1,0,0}^{(2)} + 2(\dot{\omega}^2 - 1) p_{0,2,0}^{(2)} - 2 \text{Tr}[p_{0,0,2}^{(2)}] = 1,$$  

(3.24)

(3.25)

where $\text{Tr}[p_{0,0,2}^{(2)}]$ denotes the trace of a matrix $p_{0,0,2}^{(2)}$. In terms of secular terms (3.23) and (3.24), a secular solution to $O(\epsilon^2)$ is obtained as

$$u = (A + \epsilon P_0 + 3\epsilon^2 a |A|^2 A P_1) e^{i\theta} - \epsilon^2 (a/8) A^3 e^{3i\theta} + \text{c.c.},$$  

(3.26)
which yields a renormalization transformation

$$\tilde{A}(t, x', r_\perp) = A + \epsilon P_0 + \epsilon^2 3a|A|^2 AP_1. \quad (3.27)$$

Performing the shift operation $t \rightarrow t + \tau, x' \rightarrow x' + \xi, r_\perp \rightarrow r_\perp + \eta$, we can rewrite (3.27) to obtain a representation of the Lie group

$$\tilde{A}(t + \tau, x' + \xi, r_\perp + \eta) = \tilde{A}(t, x', r_\perp) + \epsilon P_0(\tau, \xi, \eta; t, x', r_\perp) + \epsilon^2 3a|A|^2 \tilde{A}(t, x', r_\perp)P_1(\tau, \xi, \eta; t, x', r_\perp). \quad (3.28)$$

In deriving (3.28), we have used arbitrariness of the coefficients of $P_0$. Again, note that coefficients of secular terms $P_j$ depend on an "initial position" $(t, x', r_\perp)$. As in the preceding example, this dependence comes through $\tilde{A}$ and so the Lie group (3.28) also enjoys a translational symmetry which the original system (3.3) has. Expanding the right-hand side of (3.28) in terms of the generator in the same form as (3.12), we have

$$\partial_{x'} \tilde{A} = \epsilon \tilde{p}_{0,0,1} + \epsilon^2 \tilde{p}_{0,0,1}^{(2)}, \quad (3.29)$$
$$\nabla_\perp \tilde{A} = \epsilon \tilde{p}_{0,0,1}^{(1)} + \epsilon^2 \tilde{p}_{0,0,1}^{(2)}, \quad (3.30)$$
$$\partial_{x'}^2 \tilde{A} = 6\epsilon^2 a|A|^2 \tilde{A} \tilde{p}_{0,0,1}^{(2)}, \quad (3.31)$$
$$\nabla_\perp^2 \tilde{A} = 6\epsilon^2 a|A|^2 \tilde{A} \text{Tr}[\tilde{p}_{0,0,1}^{(2)}], \quad (3.32)$$
$$\partial_t \tilde{A} = 3\epsilon^2 a|A|^2 \tilde{A} \tilde{p}_{0,0,1}^{(2)}. \quad (3.33)$$

While (3.29) and (3.30) give order of operators $\partial_{x'} \sim \nabla_\perp \sim O(\epsilon)$, they do not put any constraints on $\tilde{A}$ because $\tilde{p}_{0,0,1}^{(1)}$ and $\tilde{p}_{0,0,1}^{(1)}$ are arbitrary coefficients of kernels of the linearized operator $L$. On the other hand, (3.31), (3.32), (3.33) and the constraint (3.25) yield the following nonlinear Schrödinger equation as a RG equation

$$i\partial_t \tilde{A} + (\dot{\omega}/2)\partial_{x'}^2 \tilde{A} + 1/(2\omega)\nabla_\perp^2 \tilde{A} + 3\epsilon^2/(2\omega)a|A|^2 \tilde{A} = 0, \quad (3.34)$$

where $\dot{\omega} = \partial_{x'}^2 \omega = (1 - \omega^2)/(2\omega)$.

It is possible to derive the final result (3.34) formally without exploiting explicit forms of secular terms by operating the linearized operator $L(\tau, \xi, \eta)$ defined in (3.22) on both sides of (3.28). Noting that $LP_0 = 0, LP_1 = 1$ and discarding $\partial_{x'}^2 \tilde{A}, \partial_t \partial_{x'}^2 \tilde{A}$ as higher order terms, we immediately obtain the nonlinear Schrödinger equation (3.34).

### 3.C Kadomtsev-Pitviashvili-Boussinesq Equation

Let us consider a weakly dispersive nonlinear wave equation:

$$\partial_t^2 u - \nabla \cdot \{a(u)\nabla u - c^2 b(u)\nabla (\nabla^2 u)\} = 0, \quad (3.35)$$
where the coefficient ($\epsilon^2 b(u)$) of a dispersive term is assumed to be as small as $\epsilon^2$. Expanding $u$ around constant $u_0$ as

\[
\begin{align*}
  u &= u_0 + \epsilon (\epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \epsilon^4 u_4 + \cdots), \\
a(u) &= a_0 + \epsilon [a'_0 (\epsilon u_1 + \epsilon^2 u_2) + \epsilon^3 (a''_0 u_3 + (a''_0/2) u_1^2) + \cdots], \\
b(u) &= b_0 + \epsilon b'_0 (\epsilon u_1 + \epsilon^2 u_2) + \cdots,
\end{align*}
\]

where a dash denotes the derivative with respect to $u$ and $a_0 = a(u_0) = \nu^2 > 0$ is assumed. The above setting of perturbation is justified a posteriori. We have the leading-order perturbed equation

\[
\partial_t^2 u_1 - \nu^2 \nabla^2 u_1 \equiv Lu_1 = 0.
\]

We may write a simple-wave solution of this linear wave equation in terms of an arbitrary function $h$:

\[
u^2 \nabla^2 u_1 \equiv Lu_1 = 0.
\]

\[
\begin{align*}
  u_1 &= h(\xi), \\
  L(t, r_\perp; \xi) &= \partial_t^2 - 2v\partial_t \partial_\xi - \nu^2 \nabla^2_\perp,
\end{align*}
\]

where $r_\perp = (y, z), \nabla^2_\perp = \partial_y^2 + \partial_z^2$. The next order correction obeys

\[
Lu_2 = 0,
\]

whose lowest order secular solution with respect to $t, r_\perp$ is given as

\[
\begin{align*}
  u_2(t, r_\perp; \xi) &= p^{(0)}_{0,1}(\xi) \cdot r_\perp, \\
  \nu^2 \nabla^2_\perp f \xi &= -2v\partial_t p^{(0)}_{0,1}(\xi) - \nu^2 \text{Tr}[p^{(0)}_{0,2}(\xi)],
\end{align*}
\]

where a monomial of $t$ is discarded because its coefficient is constant. From the next order equation, we have

\[
Lu_3 = f_\xi, \quad f = a'_{0} hh_\xi - b_0 h^2_\xi,
\]

where the subscript $\xi$ of $h, f$ denotes the partial derivative with respect to $\xi$. Since forcing terms in the right-hand side of (3.41) depend only on $\xi$, (3.41) has a non-trivial secular solution with respect to $t, r_\perp$:

\[
\begin{align*}
  u_3 &= p^{(1)}_{0,1}(\xi) \cdot r_\perp + p^{(1)}_{0,2}(\xi) \cdot r_\perp r_\perp + p^{(1)}_{1,0}(\xi) t, \\
f_\xi &= -2v\partial_\xi p^{(1)}_{0,1}(\xi) - \nu^2 \text{Tr}[p^{(1)}_{0,2}(\xi)],
\end{align*}
\]

where $p^{(1)}_{0,2}(\xi)$ is a $2 \times 2$ matrix function of $\xi$. Summing up secular terms up to $O(\epsilon^4)$, we obtain

\[
(u - u_0)/\epsilon^2 = h + \epsilon w_2 + \epsilon^2 u_3.
\]
In order to remove secular terms with respect to \( t, r_\perp \) in \( u_2, u_3 \) by renormalizing an arbitrary function \( h \), we introduce the following renormalization transformation

\[
\tilde{h}(t, r_\perp; \xi) = h(\xi) + \epsilon u_2(t, r_\perp; \xi) + \epsilon^2 u_3(t, r_\perp; \xi). \tag{3.44}
\]

Since \( \tilde{h}(t, r_\perp; \xi) \) has a translational symmetry with respect to \( t, r_\perp \) as well as \( u(t, r_\perp; \xi) \), (3.44) immediately leads to a representation of the Lie group with arbitrary parameters \( \tau, \eta \)

\[
\tilde{h}(t + \tau, r_\perp + \eta; \xi) = h(t, r_\perp; \xi) + \epsilon u_2(\tau, \eta; \xi) + \epsilon^2 u_3(\tau, \eta; \xi), \tag{3.45}
\]

where it is noted again that coefficients of secular terms \( p \) depend on an "initial position", e.g. \( p^{(1)}_{0,2}(\xi) \) is replaced by \( \tilde{p}^{(1)}_{0,2}(t, r_\perp; \xi) \). Expanding the right-hand side of (3.45) in terms of a generator \( \tau \partial_t + \eta \cdot \nabla_\perp \) and equating the same monomials, we have

\[
\nabla_\perp \tilde{h} = \epsilon \tilde{p}^{(0)}_{0,1}(t, r_\perp; \xi) + \epsilon^2 \tilde{p}^{(1)}_{0,1}(t, r_\perp; \xi), \tag{3.46}
\]

\[
\nabla^2 \tilde{h} = \epsilon^2 \text{Tr}[\tilde{p}^{(1)}_{0,2}(t, r_\perp; \xi)], \tag{3.47}
\]

\[
\partial_t \tilde{h} = \epsilon^2 \tilde{p}^{(1)}_{1,0}(t, r_\perp; \xi). \tag{3.48}
\]

Since \( \tilde{p}^{(j)}_{0,1} \) are arbitrary coefficients of kernels of the linearized operator \( L \), (3.46) does not put any constraints on \( \tilde{h} \) but provides order of the operator \( \nabla_\perp \sim O(\epsilon) \). From (3.47), (3.48) and the constraint (3.42), we obtain the Kadomtsev-Pitviashvili (K-P) equation in the three dimensional space

\[
2\nu \partial_\xi \tilde{h}_\xi + v^3 \nabla^2 \tilde{h} + \epsilon^2 (a_0^2 \tilde{h}_\xi - b_0 \tilde{h}_3 \xi) = 0. \tag{3.49}
\]

Next, we calculate higher-order corrections to the K-P equation. For \( u_4, u_5 \), we have

\[
Lu_4 = \partial_\xi^2 (a_0^2 h u_2 - b_0 \partial_\xi^2 u_2), \tag{3.50}
\]

\[
Lu_5 = (a_0^2/2)(h^2 h_\xi)_\xi + a_0^2 h \nabla^2 u_3 + a_0 h \nabla_\perp u_2 |^2
- b_0^2 (h h_3 \xi)_\xi - 2b_0 \partial_\xi^2 \nabla^2 u_3 + g(t, r_\perp; \xi), \tag{3.51}
\]

where \( g(0, 0; \xi) = 0 \). Explicit secular solutions of (3.50) and (3.51) are not necessary for later discussions but we note

\[
[L u_4]_{r_\perp=0} = 0. \tag{3.52}
\]

Then, we obtain a representation of the Lie group with higher order corrections

\[
\tilde{h}(t + \tau, r_\perp + \eta; \xi) = h(t, r_\perp; \xi) + \epsilon u_2(\tau, \eta; \xi) + \epsilon^2 u_3(\tau, \eta; \xi) + \epsilon^3 u_4(\tau, \eta; \xi) + \epsilon^4 u_5(\tau, \eta; \xi). \tag{3.53}
\]
Since it is straightforward but tedious to derive the K-P equation with higher order corrections from (3.53) by using explicit forms of secular solutions, we follow the simplified procedure mentioned in the last paragraph in the section 3.3. Operating $L(\tau, \eta; \xi)$ on both sides of (3.53) at $(\tau, \eta) = (0, 0)$ and noting (3.39), (3.41), (3.52) and (3.51), we obtain the following RG equation

\[
L(\tau, \eta; \xi) = \frac{\partial^2}{\partial t^2} - 2v \frac{\partial}{\partial \xi}(a_0 + \epsilon^2 a'_0 h) \nabla \perp h - \nabla \perp \left\{ (a_0 h + \epsilon^2 a'_0 h^2/2) h \right\} - (b_0 + \epsilon^2 b'_0 h) h_\perp - 2\epsilon^2 b_0 \nabla \perp^2 h \right\}. 
\]

(3.54)

If $\nabla \perp = 0$, (3.54) reduces to the Boussinesq equation with higher order corrections. Therefore, (3.54) may be considered as not only the K-P equation with higher order corrections but also the Boussinesq equation including multidimensional effects and higher order corrections.

### 3.D Phase Equations

First, we consider interface dynamics in the following general reaction-diffusion system.

\[
\partial_t U = F(U) + D \nabla^2 U, \tag{3.55}
\]

where $U$ is an $n$-dimensional vector and $D$ is a $n \times n$ constant matrix. Suppose (3.55) has an interface solution $U_0 = U_0(x - vt + \phi)$, where $\phi$ is an arbitrary constant, then

\[
vU_{0,\theta} + F(U_0) + DU_{0,\theta} = 0, \tag{3.56}
\]

where $\theta = x - vt + \phi$ and the suffix $\theta$ denotes the derivative with respect to $\theta$. Differentiating (3.56), we have

\[
L(\theta)U_{0,\theta} \equiv -v \partial_\theta + F'(u_0) \cdot + D \partial_\theta^2 U_{0,\theta} = 0, \tag{3.57}
\]

\[
LU_{0,\theta} = F'' \cdot U_{0,\theta}^2, \tag{3.58}
\]

where $F' \cdot V = (V \cdot \nabla U) F(U)|_{U=U_0}$, $F'' \cdot V^2 = (V \cdot \nabla U)^2 F(U)|_{U=U_0}$. It is convenient to introduce the following coordinate transformation

\[
\theta = x - vt + \phi, \quad x' = x, t' = t. \tag{3.59}
\]

Then, we have

\[
\partial_t = \partial_t - v \partial_\theta, \quad \partial_x = \partial_{x'} + \partial_\theta,
\]

\[
\nabla^2 = \partial_\theta^2 + 2 \partial_\theta \partial_{x'} + \nabla_{x'}^2,
\]

where $\nabla_{x'}^2 = \partial_{x'}^2 + \nabla_{x'}^2$, $\nabla_{x'} = (0, \partial_{y'}, \partial_{z'})$. We study the solution close to $U_0$ so that $U$ is expanded as

\[
U = U_0 + \epsilon U_1 + \epsilon^2 U_2 + \cdots. \tag{3.60}
\]
The first order term obeys
\[
\{ \partial_t + L(\theta) - (\nabla_{x'}^2 + 2\partial_\theta \partial_{x'})D \} U_1 \equiv \bar{L} U_1 = 0,
\]
(3.61)
The lowest order secular solution of (3.61) is supposed to be
\[
U_1 = P_1(t', r) U_{0,\theta}(\theta),
\]
(3.62)
where \(P_1\) is a scalar secular function of \(t'\) and \(r = (x', y, z)\). Then, we have
\[
\partial_t P_1 = 0, \quad \partial_{x'} P_1 = 0, \quad \nabla_{x'}^2 P_1 = 0,
\]
from which
\[
P_1 = \nabla_\perp P_1 \cdot r_\perp,
\]
(3.63)
where \(r_\perp = (0, y, z)\) and \(\nabla_\perp P_1\) is a constant three-dimensional vector.

The second order term \(U_2\) is governed by
\[
\bar{L} U_2 = (1/2) F'' \cdot U_1^2,
\]
(3.64)
of which secular solution is assumed to be
\[
U_2 = \bar{U}_2(\theta) + P_2(t', r) U_{0,\theta} + R_2(t', r) U_{0,2\theta},
\]
(3.65)
where \(\bar{U}_2\) is a bounded function of \(\theta\) and \(P_2, R_2\) are scalar secular terms. Substituting (3.63) into (3.64) and noting (3.57) and (3.58), we obtain
\[
L(\theta) \bar{U}_2 + \partial_{t'} P_2 U_{0,\theta} - \nabla_{x'}^2 P_2 D U_{0,\theta} - 2\partial_{x'} P_2 D U_{0,2\theta}
+ \partial_t R_2 U_{0,2\theta} - \nabla_{x'}^2 R_2 D U_{0,2\theta} - 2\partial_{x'} R_2 D U_{0,3\theta}
+ \{R_2 - (1/2) P_2^2\} F'' \cdot U_{1,0\theta}^2 = 0,
\]
which is a equation for \(\bar{U}_2\). Since \(\bar{U}_2\) is a function of \(\theta\) only, the following conditions are necessary.
\[
\partial_t P_2 = c_1, \quad \partial_{x'} P_2 = c_2, \quad \nabla_{x'}^2 P_2 = c_3, \quad R_2 = (1/2) P_1^2 + c_4,
\]
(3.66)
where \(c_j\) are constant. Since \(P_1\) is a linear function of \(r_\perp\) given in (3.63), (3.66) yields
\[
\partial_t R_2 = \partial_{x'} R_2 = 0, \quad \nabla_{x'}^2 R_2 = |\nabla_\perp P_1|^2.
\]
(3.67)
Then, a bounded solution \(\bar{U}_2\) is possible only if the following compatibility condition is satisfied.
\[
< \bar{U} \cdot U_{0,\theta} > \partial_t P_2 - < \bar{U} \cdot D U_{0,\theta} > \nabla_{x'}^2 P_2
- < \bar{U} \cdot D U_{0,2\theta} > (2\partial_{x'} P_2 + |\nabla_\perp P_1|^2) = 0,
\]
(3.68)
where $\hat{U}$ is an adjoint function of a null eigenfunction of $L$ and $<\hat{U} \cdot U> = \int_{-\infty}^{\infty} (\hat{U} \cdot U) d\theta$.

Now, a secular solution up to $O(\epsilon^2)$ is

$$U = U_0(x - vt + \phi) + \epsilon P_1(r_\perp)U_{0,\theta} + \epsilon^2 \{ P_2(t', x', r_\perp)U_{0,0} + R_2(r_\perp)U_{0,2\theta} \},$$

(3.69)

which should be equated with the renormalized $U_0(x - vt + \tilde{\phi})$, where $\tilde{\phi} = \tilde{\phi}(t', x', r_\perp)$ is a renormalized phase. Setting

$$\tilde{\phi}(t', x', r_\perp) = \phi + \delta(t', x', r_\perp), \quad |\delta| \ll |\phi|,$$

(3.70)

we expand $U_0(x - vt + \tilde{\phi})$ as

$$U_0(x - vt + \tilde{\phi}) = U_0(\theta) + \delta U_{0,\theta} + (\delta^2/2)U_{0,2\theta} + \cdots.$$

(3.71)

Equating (3.69) and (3.71), we have a renormalization transformation

$$\delta = \tilde{\phi}(t', x', r_\perp) - \phi = \epsilon P_1(r_\perp) + \epsilon^2 P_2(t', x', r_\perp),$$

$$\delta^2 = 2\epsilon^2 R_2(r_\perp),$$

(3.72)

(3.73)

where (3.73) and (3.66) are consistent with (3.72) up to $O(\epsilon^2)$ if $c_4 = 0$.

Following the same procedure as the previous examples, we rewrite (3.72) as

$$\tilde{\phi}(t' + \tau, x' + \xi, r_\perp + \eta) = \phi(t', x', r_\perp) + \epsilon P_1(\eta; t', x', r_\perp) + \epsilon^2 P_2(\tau, \xi; t', x', r_\perp),$$

(3.74)

from which we obtain

$$\nabla_\perp \tilde{\phi} = \epsilon [\partial \eta (P_1 + \epsilon P_2)]_0, \quad \nabla^2_\perp \tilde{\phi} = \epsilon^2 [\partial^2 \eta P_2]_0,$$

$$\partial_{x'} \tilde{\phi} = \epsilon^2 [\partial_\xi P_2]_0, \quad \partial_\theta \tilde{\phi} = \epsilon^2 [\partial_\eta P_2]_0,$$

(3.75)

where $[P(\tau, \xi, \eta)]_0 = P(0, 0, 0)$. Substitution of the relations (3.73) into (3.68), we obtain a phase equation for the interface by replacing $t', x'$ by $t, x$.

$$\begin{align*}
(\partial_t + 2V \partial_x) \tilde{\phi} &= D_\perp \nabla^2_\perp \tilde{\phi} + V |\nabla_\perp \tilde{\phi}|^2, \\
D_\perp &= <\hat{U} \cdot DU_{0,\theta}> / <\hat{U} \cdot U_{0,\theta}>, \\
V &= <\hat{U} \cdot DU_{0,2\theta}> / <\hat{U} \cdot U_{0,\theta}>.
\end{align*}$$

(3.76)

If (3.55) has a periodically oscillating solution $U_0(-\omega t + \phi) = U_0(\theta)$, the similar procedure as above leads to the isotropic Burgers equation for the phase $\phi$. In this case, the linealized operator $\hat{L}$ in (3.61) is replaced by
\[ L + D \nabla^2, \quad L = \partial_t - F' \cdot \] and secular solutions \( U_1, U_2 \) up to \( O(\epsilon^2) \) are obtained as

\[
U_1 = P_1(t, r)U_0(\theta), \quad U_2 = P_2(t, r)U_{0, \theta} + R_2(t, r)U_{0, 2\theta},
\]

where \( P_1 = \nabla P_1 \cdot r, \quad P_2 = \partial_t P_2 + [\nabla P_2]_0 \cdot r + p^{(2)} \cdot rr, \quad R_2 = P_2^2/2, \)

where \( p^{(2)} \) is a \( 2 \times 2 \) matrix. The constraint corresponding to (3.68) is given in terms of an adjoint null function \( \hat{U} \) of \( L = \partial_t - F' \cdot \) by

\[
< \hat{U} \cdot U_{0, \theta} > \partial_t P_2 - < \hat{U} \cdot DU_{0, \theta} > \nabla^2 P_2 - < \hat{U} \cdot DU_{0, 2\theta} > |\nabla P_1|^2 = 0, \quad (3.77)
\]

where \( < \hat{U} \cdot U > = \int_0^T (\hat{U} \cdot U)d\theta \) and \( T \) is a period of \( U_0 \) with respect to \( \theta \). A representation of a renormalization group and its generator have formally the same form as (3.74) and (3.75), where \( t', \nabla \perp \) are replaced by \( t, \nabla \) respectively. From (3.77) and (3.75), we obtain the three-dimensional Burgers equation

\[
\partial_t \tilde{\phi} = D_\perp \nabla^2 \tilde{\phi} + V|\nabla \tilde{\phi}|^2, \quad (3.78)
\]

which is known as the standard phase equation.

### 4 Summary

Through several examples, the perturbative renormalization group method is shown to be understood as the procedure to obtain an asymptotic expression of a generator of a renormalization transformation based on the Lie group. The present approach provides the following simple recipe for obtaining an asymptotic form of a RG equation from not only autonomous but also non-autonomous ODE.

1. Get a secular series solution of a perturbed equation by means of naive perturbation calculations.
2. Find integral constants, which are renormalized to eliminate all the secular terms in the perturbed solution and give a renormalization transformation.
3. Rewrite the renormalization transformation by executing an arbitrary shift operation on the independent variable: \( t \rightarrow t + \tau \) and derive a representation of the Lie group underlying the renormalization transformation.
4. By differentiating the representation of the Lie group with respect to arbitrary \( \tau \), we obtain an asymptotic expression of the generator, which yields an asymptotic RG equation.
This procedure is valid for general ODE regardless of a translational symmetry. When the renormalization transformation is known to have a translational symmetry in advance, the step (3) i.e. reduction to the Lie group from the renormalization transformation becomes trivially simple as implied in (2.23). The above recipe for ODE is also applicable to autonomous PDE if we choose suitable polynomial kernels of the linearized operator. First, we should take the lowest-order polynomial, of which degree is one, as the leading order secular term. As perturbation calculations proceed to higher order, polynomial kernels of higher degrees are included in the higher-order secular terms order by order. Through this procedure, we uniquely determine suitable polynomial kernels among infinite number of kernels of the linearized operator and the step (1) in the recipe is completed. There are no problems in the other steps. Thus, the present Lie-group approach is shown to be consistently applicable to PDE and some examples are presented. As more involved examples, we have succeeded in derivation of higher-order phase equations such as the non-isotropic Kuramoto-Sivashinsky equation and the K-dV-Burgers equation from the general reaction-diffusion system (3.53), which will be published elsewhere.

Acknowledgement

One of authors (K.N.) wishes to thank Prof. Y.Oono, University of Illinois, for valuable discussions about the RG method. We also appreciate Prof. Y. Nambu, Nagoya University, for his introduction to the Einstein’s gravitational theory.

References

[1] L.Y. Chen, N. Goldenfeld and Y. Oono, Phys. Rev. Lett. 73, 1311 (1994).
[2] L.Y. Chen, N. Goldenfeld and Y. Oono, Phys. Rev. E 54, 376 (1996).
[3] Y.Y. Yamaguchi and Y. Nambu, Prog. Theor. Phys. 100, 199 (1998).
[4] S. Sasa, Physica D 108 (1997), 45 (1997).
[5] K. Matsuba and K. Nozaki, Phys. Rev. E 56, R4926 (1997).
[6] T. Kunihiro, Prog. Theor. Phys. 94, 503 (1995).
[7] T. Maruo, K. Nozaki and A. Yosimori, Prog. Theor. Phys. 101, 243 (1999).
[8] K. Matsuba and K. Nozaki, J. Phys. Soc. Jpn. 66, 3315 (1997).
[9] T. Taniuti, Suppl. Prog. Theor. Phys. 55, 1 (1977).
[10] A. H. Nayfeh, Perturbation Methods (John Willy and Sons, New York, 1973).
[11] D. V. Shirkov, Intern. J. Modern Phys. A3, 1321 (1988).
[12] Y. Nambu and Y. Y. Yamaguchi, DPNU-99-09 (preprint).
[13] for example, D. J. Kaup and A. C. Newell, Proc. Roy. Soc. London A361, 413 (1978).
[14] for example, V. I. Karpman and E. M. Maslov, Sov. Phys. JETP 46, 281 (1977).
[15] for example, A. Bondeson, M. Lisak and D. Anderson, Physica Scripta 20, 479 (1979).
[16] T. Ohta, Mathematical Physics in Interface Dynamics (Nihon- Hyouronsya, Tokyo, 1997) in Japanese.