The Turán Number for Spanning Linear Forests∗

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Abstract. For a set of graphs $F$, the extremal number $ex(n; F)$ is the maximum number of edges in a graph of order $n$ not containing any subgraph isomorphic to some graph in $F$. If $F$ contains a graph on $n$ vertices, then we often call the problem a spanning Turán problem.

A linear forest is a graph whose connected components are all paths and isolated vertices. In this paper, we let $L^k_n$ be the set of all linear forests of order $n$ with at least $n-k+1$ edges. We prove that when $n \geq 3k$ and $k \geq 2$,

$$ex(n; L^k_n) = \left(\frac{n-k+1}{2}\right) + O(k^2).$$

Clearly, the result is interesting when $k = o(n)$.

Keywords: spanning Turán problem, linear forests, Hamiltonian completion number

1 Introduction

For a set of graphs $F$, the extremal number $ex(n; F)$ is the maximum number of edges in a graph of order $n$ not containing any subgraph isomorphic to some graph in $F$. Turán introduced this problem in [11], and we recommend [8,10] for surveys on Turán problems for graphs and hypergraphs. Ore [9] proved that a non-Hamiltonian graph of order $n$ has at most $\left(\frac{n-1}{2}\right)+1$ edges. Ore’s theorem can also be expressed as $ex(n; \{C_n\}) = \left(\frac{n-1}{2}\right)+1$ where $C_n$ is the cycle of order $n$. Similarly, $ex(n; \{P_n\}) = \left(\frac{n-1}{2}\right)$ where $P_n$ is the path of order $n$.

Recently, many generalizations of Ore’s theorem have been studied. Alon and Yuster [1] extended Ore’s result to spanning structures other than just Hamilton cycles. They prove that if $H$ is a graph of order $n$ with minimum degree $\delta(H) > 0$, and maximum degree $\Delta(H) \leq \sqrt{n}/40$, then $ex(n; \{H\}) = \left(\frac{n-1}{2}\right) + \delta(H) - 1$ assuming $n$ is sufficiently large. Moreover, some researchers extend Ore’s theorem to the hypergraph setting [3,7,12].

Define the Hamiltonian completion number of a graph $G$, denoted by $h(G)$, to be the minimum number of edges that need be added to make $G$ Hamiltonian. The Hamiltonian completion problem was introduced in the 1970s by Goodman and Hedetniemi [4,5]. From the
viewpoint of Hamiltonian completion number of graphs, Ore’s Theorem can also be restated as a graph with \( h(G) > 0 \) of order \( n \) has at most \( \binom{n-1}{2} + 1 \) edges. Then, it’s natural to try to extend Ore’s result to graphs with \( h(G) \geq k \) for some \( k \geq 1 \). Let \( \mathcal{L}_n^k \) be the set of all linear forests of order \( n \) with at least \( n - k + 1 \) edges. Clearly, the problem is equivalent to determining the Turán number \( \text{ex}(n; \mathcal{L}_n^k) \). In this paper, we prove that when \( n \geq 3k \) and \( k \geq 2 \),

\[
\text{ex}(n; \mathcal{L}_n^k) = \left( \frac{n - k + 1}{2} \right) + O(k^2).
\]

It should be mentioned that Lidický et. al. determine the Turán number of linear forests of arbitrary order for \( n \) sufficiently large in [6]. However, the number of vertices in their forbidden linear forests does not depend on \( n \). The rest of this short paper is organized as follows. In Section 2, we give lower bounds on the Turán number \( \text{ex}(n; \mathcal{L}_n^k) \) and prove a useful lemma. In Section 3, we prove the main theorem. In Section 4, we give some concluding remarks.

## 2 Lower Bounds and A Useful Lemma

Let \( G_0 \) be the union graph of \( K_{n-k+1} \) and \( k - 1 \) isolated vertices. It is easy to see that \( G_0 \) does not contain any linear forest with more than \( n - k \) edges. Thus, we have the following lemma.

**Lemma 2.1.** The Turán number \( \text{ex}(n; \mathcal{L}_n^k) \) has the following lower bound.

\[
\text{ex}(n; \mathcal{L}_n^k) \geq e(G_0) = \left( \frac{n - k + 1}{2} \right).
\]

**Lemma 2.2** ([2]). Let \( G \) be a graph with \( n \) vertices and let \( P \) be a Hamiltonian path in \( G \) with endpoints \( u \) and \( v \). If \( d(u) + d(v) \geq n \), then \( G \) contains a Hamiltonian cycle.

For simplicity, we view isolated vertices as paths of length zero, whose end vertices are the same.

**Lemma 2.3.** Let \( k \geq 2 \) be an integer, and suppose that \( G \) is a graph that contains a spanning linear forest \( F \) with \( n - k \) edges. If \( u \) and \( v \) are vertices that are endpoints of different paths in \( F \) and \( d(u) + d(v) \geq n - k + 1 \), then \( G \) contains a spanning linear forest with \( n - k + 1 \) edges.

**Proof.** Let \( H \) be the join graph of \( G \) and \( k - 1 \) isolated vertices \( v_1, v_2, \ldots, v_{k-1} \). That means

\[
V(H) = V(G) \cup \{v_1, v_2, \ldots, v_{k-1}\}
\]

and

\[
E(H) = E(G) \cup \{v_1w: i = 1, 2, \ldots, k - 1 \text{ and } w \in V(G)\}.
\]

Let \( P_1, P_2, \ldots, P_k \) be \( k \) paths of \( F \), and assume that \( u \) is the first vertex of \( P_1 \), and \( v \) is the last vertex of \( P_k \). Then \( P_1v_1P_2v_2 \ldots, v_{k-1}P_k \) forms a Hamiltonian path of \( H \) with endpoints \( u \) and \( v \). Moreover, \( d_H(u) + d_H(v) \geq n - k + 1 + 2(k - 1) = n + k - 1 \). By Lemma 2.2, it follows that \( H \) contains a Hamiltonian cycle. Finally, by removing vertices \( v_1, v_2, \ldots, v_{k-1} \) from this Hamiltonian cycle, we obtain a linear forest of \( G \) with \( n + k - 1 - 2(k - 1) = n - k + 1 \) edges. Thus, the lemma holds.
3 The Main Result

Theorem 3.1. For \( n \geq 3k \) and \( k \geq 2 \), the Turán number \( \text{ex}(n; \mathcal{L}_n^k) \) has the following upper bound.

\[
\text{ex}(n; \mathcal{L}_n^k) \leq \left( \frac{n - k + 1}{2} \right) + \frac{k^2 - 3k + 4}{2}.
\]

Proof. Let \( G(V, E) \) be any graph with \( n \) vertices, and the maximum number of edges subject to \( h(G) \geq k \). We may assume that \( h(G) = k \) for if \( h(G) > k \), then one may add an edge to \( G \) to obtain a new graph \( G' \) with \( h(G') \geq k \), \( e(G') > e(G) \) and having the same number of vertices as \( G \). As \( h(G) = k \), we can choose a linear forest \( F \) with \( n - k \) edges and furthermore, among all such subgraphs, we may choose \( F \) so that it has the fewest number of isolated vertices.

Since \( F \) has \( n \) vertices and \( n - k \) edges, then \( F \) has \( k \) connected components consisting of paths and isolated vertices. By Lemma 2.3, for any two end vertices \( u, v \) in different connected components of \( F \), we have \( d(u) + d(v) \leq n - k \). Otherwise, \( G \) contains a linear forest with \( n - k + 1 \) edges, contradicting with \( h(G) \geq k \).

We claim that all but at most two end vertices of \( F \) have degree at most \( \frac{n - k}{2} \). Moreover, if two end vertices of \( F \) have degree greater than \( \frac{n - k}{2} \), then they are end vertices of the same path. If there are three vertices with degree greater than \( \frac{n - k}{2} \), then at least two of them fall into the different components, assume they are \( u \) and \( v \). Then \( d(u) + d(v) > n - k \). It follows that \( G \) contains a linear forest with \( n - k + 1 \) edges, contradicting with \( h(G) \geq k \). Thus, the claim holds.

The remainder of the proof splits into two cases, depending on whether or not \( F \) contains isolated vertices. For each case, we give the upper bound on the number of edges in \( G \).

**Case 1.** There are no isolated vertices in \( F \). Then we claim that any two end vertices in different paths of \( F \) are not adjacent. Otherwise, by using the edge between these two vertices we obtain a linear forest with \( n - k + 1 \) edges, contradicting with \( h(G) \geq k \). Thus, each end vertex of \( F \) has degree at most \( n - 1 - (2k - 2) = n - 2k + 1 \). Let \( X \) be the set of all the end vertices of the \( k \) paths in \( F \) and \( G' = G[V \setminus X] \). Since \( G' \) has \( n - 2k \) vertices, we have

\[
e(G') \leq \left( \frac{n - 2k}{2} \right).
\]

Now we bound the number of edges of \( G \) as follows:

\[
e(G) \leq \left( \frac{n - 2k}{2} \right) + 2(n - 2k + 1) + 2(k - 1) \cdot \frac{n - k}{2}
= \left( \frac{n - k + 1}{2} \right) + \frac{k^2 - 3k + 4}{2}.
\]

**Case 2.** There are \( i \) isolated vertices in \( F \) for some \( i \) with \( 0 < i < k \). Let \( X = \{ x_1, x_2, \ldots, x_i \} \) be the set of \( i \) isolated vertices in \( F \). We claim that any vertex in \( X \) has degree at most \( k - i - 1 \). Let \( x_s \in X \) and \( P = y_1y_2 \ldots y_m \) be any path in \( F \). If \( x_sy_t \) is an edge in \( G \) for some \( 1 < t < m \), then \( m \) has to be 3. Otherwise, if \( t = 2 \), then by replacing \( P \) and \( x_s \)
with \( y_1 y_2 x_s \) and \( y_3 \ldots y_m \) from \( F \), we obtain a linear forest with less isolated vertices, which contradicts with the selection of \( F \). If \( t \geq 3 \), then by replacing \( P \) and \( x_s \) with \( y_1 \ldots y_{t-1} \) and \( x_s y_t \ldots y_m \) from \( F \), we obtain a linear forest with less isolated vertices, a contradiction. Therefore, neighbors of \( x_s \) can only be internal vertices of paths of length two in \( F \). Since \( n \geq 3k \), \( F \) contains at most \( k - i - 1 \) paths of length two. Thus, any vertex in \( X \) has degree at most \( k - i - 1 \) in \( G \).

Let \( G' = G[V \setminus X] \) and \( F' = F[V \setminus X] \). We claim that \( F' \) is a spanning subgraph of \( G' \) that is a linear forest with maximum number of edges. Otherwise, if \( G' \) has a linear forest \( F^* \) with more edges, then by replacing edges in \( F' \) with those in \( F^* \), we obtain a spanning linear forest of \( G \) with more edges, a contradiction. Since \( F' \) has \( n - i \) vertices and \( k - i \) components and contains no isolated vertices, by inequality (3.1) we have

\[
e(G') \leq \binom{n - i}{2} + \frac{(k - i)^2 - 3(k - i) + 4}{2} + i(k - i - 1)
\]

Note that \( 1 \leq i \leq k - 1 \), we bound the number of edges of \( G \) as follows:

\[
e(G) \leq e(G') + i(k - i - 1)
\]

\[
\leq \binom{n - k + 1}{2} + \frac{(k - i)^2 - 3(k - i) + 4}{2} + i(k - i - 1)
\]

\[
= \binom{n - k + 1}{2} + \frac{k^2 - 3k}{2} - \frac{1}{2} \cdot \binom{i - 1}{2}^2 + \frac{17}{8}
\]

Combining the two cases, we conclude that

\[
ex(n; L^k_n) \leq \binom{n - k + 1}{2} + \frac{k^2 - 3k + 4}{2}.
\]

4 Concluding Remarks

In this paper, we prove that when \( n \geq 3k \) and \( k \geq 2 \),

\[
\binom{n - k + 1}{2} \leq ex(n; L^k_n) \leq \binom{n - k + 1}{2} + \frac{k^2 - 3k + 4}{2},
\]

the result is interesting when \( k = o(n) \). For \( k = 2 \), \( L^k_n \) denote the set of all linear forests of order \( n \) with at least \( n - 1 \) edges, which is exactly the set of Hamiltonian paths. Thus, the lower bound is reachable for \( k = 2 \). Furthermore, we guess that there exists a constant \( k_0 \) such that \( ex(n; L^k_n) = \binom{n - k + 1}{2} \) for \( k < k_0 \). And we end up this paper by proposing the following two problems.
Problem 4.1. Determine the exact value of $ex(n; L^k_n)$ for $k = o(n)$.

Problem 4.2. Let $c$ be a constant satisfying $0 < c < 1$. Determine the value of $ex(n; L^k_n)$ for $k = cn$.

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