Instantons: thick-wall approximation

V.F. Mukhanov\textsuperscript{a,b} and A.S. Sorin\textsuperscript{c}

\textsuperscript{a}Ludwig Maxmillian University, Theresienstr. 37, 80333 Munich, Germany
\textsuperscript{b}Korea Institute for Advanced Study, Seoul 02455, Korea
\textsuperscript{c}Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Moscow Region, Russia

E-mail: mukhanov@physik.lmu.de, sorin@theor.jinr.ru

ABSTRACT: We develop a new method for estimating the decay probability of the false vacuum via regularized instantons. Namely, we consider the case where the potential is either unbounded from below or the second minimum corresponding to the true vacuum has a depth exceeding the height of the potential barrier. In this case, the materialized bubbles dominating the vacuum decay naturally have a thick wall and the thin-wall approximation is not applicable. We prove that in such a case the main contribution to the action determining the decay probability comes from the part of the solution for which the potential term in the equation for instantons can be neglected compared to the friction term. We show that the developed approximation exactly reproduces the leading order results for the few known exactly solvable potentials. The proposed method is applied to generic scalar field potentials in an arbitrary number of dimensions.

KEYWORDS: Nonperturbative Effects, Phase Transitions, Solitons Monopoles and Instantons

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1 Introduction

In this work, we consider the problem of false vacuum decay for a scalar field with potential $V(\varphi)$, which has a local minimum normalized to zero (false vacuum) at $\varphi_f < 0$, separated from the unbounded part of the potential or from the deep true vacuum by the barrier of the height $V_{\text{bar}}$ at $\varphi = 0$ (see figure 1).

The original approach to the general problem of false vacuum decay in 4-dimensional scalar field theory was elaborated in [1, 2]. This approach proved to be inapplicable for a wide class of the potentials and was recently modified in [3–7] for a very general situation. The key element of our modified approach is the correct treatment of the role of quantum fluctuations, which regularize the Euclidean classical $O(4)$-invariant singular solutions and lead to the appearance of a whole class of new instantons, all contributing to the vacuum decay rate. These new instantons exist even for those potentials for which the instantons with Coleman boundary conditions do not exist, although the false vacuum must obviously be unstable.

The aim of this work is to generalise our approach [3–7] to thick-wall instantons for arbitrary potentials in $D$-dimensional spacetime. Our purpose is to derive the generic asymptotic formulas valid in the leading order for these thick-wall bubbles.

2 Instantons with quantum core in D dimensions

The generalization of the basic formulas in [3, 4, 6] for the new instantons with quantum core in $D$-dimensional spacetime is quite straightforward. In particular, the decay rate of the false vacuum per unit time per unit volume must be given by

$$\Gamma \simeq g_0^{-D} \exp(-S_I),$$

(2.1)
Figure 1. The potential with a metastable vacuum at $\phi_f$, which must decay via instantons, regardless of whether this potential is unbounded from below or has the second true minimum represented by the dot-dashed line. For each concrete potential, there is a full spectrum of instantons. The instantons corresponding to the deep subbarrier transition with $|V_b| \gg V_{\text{bar}}$ (as shown, for example, by the lower blue line) are dominated by “friction” and cannot be described in the thin-wall approximation.

where $\rho_0$ is the typical size, i.e., “radius” of the bubble and the finite on-shell instanton action is equal to

$$S_I = \frac{2\pi D}{\Gamma\left(\frac{D}{2}\right)} \left( \int_{\rho_{uv}}^{\rho_{ir}} d\rho \rho^{D-1} \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right) + \frac{\rho_{uv}^D}{D} V_{uv} \right). \quad (2.2)$$

The total energy of the regularized instantons,

$$V = \frac{2\pi D}{\Gamma\left(\frac{D-1}{2}\right)} \left( \int_{\rho_{uv}}^{\rho_{ir}} d\rho \rho^{D-2} \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right) + \frac{\rho_{uv}^{D-1}}{D-1} V_{uv} \right), \quad (2.3)$$

vanish at the moment of their exit from under the barrier with the precision allowed by the time-energy uncertainty relation

$$\rho_{uv} V \approx O(1). \quad (2.4)$$

The second terms in equations (2.2) and (2.3) are the contributions from the homogeneously filled quantum core with radius $\rho_{uv}$ and energy-density $V_{uv} \equiv V(\phi(\rho_{uv}))$, where $\rho_{uv}$ and $\rho_{ir}$ are ultraviolet and infrared cutoff scales determined by parameters of the corresponding instanton solutions.

The instants are the $O(D)$-invariant solutions of the Euclidean scalar field equation

$$\frac{\partial^2 \phi}{\partial \tau^2} + \Delta \phi - \frac{dV(\phi)}{d\phi} = 0, \quad (2.5)$$

where $\phi$ is the field and $\Delta$ is the Laplacian.
which in this case reduces to the ordinary differential equation
\[ \ddot{\varphi} + \frac{D-1}{\varrho} \dot{\varphi} - V' = 0 \] (2.6)
with the boundary conditions:
\[ \varphi (\varrho \to \infty) = \varphi_f , \] (2.7)
\[ \dot{\varphi} (\varrho = \varrho_b) = 0 . \] (2.8)

Here, \( \varphi (\tau, x) = \varphi (\varrho) \), \( \varrho \equiv \sqrt{\tau^2 + x^2} \) with Euclidean time \( \tau \), \( V' \equiv dV/d\varphi \) and \( \dot{\varphi} \equiv d\varphi/d\varrho \), \( \varrho_b \) is a bouncing parameter characterizing the corresponding instanton solution. For a given potential \( V(\varphi) \), both the ultraviolet and infrared cutoffs \( \varrho_{uv} \) and \( \varrho_{ir} \) can be fully expressed in terms of \( \varrho_b \) and parameters of the potential \( V(\varphi) \), by equating the derivative of the instanton solution \( \dot{\varphi} (\varrho) \) to the derivative of the typical amplitude of quantum fluctuations \( \delta \dot{\varphi} (\varrho) \) in corresponding scales (see [3, 4, 6]), that is
\[ |\dot{\varphi}_{uv}| \equiv |\dot{\varphi} (\varrho_{uv})| = |\delta \dot{\varphi} (\varrho_{uv})| , \quad |\dot{\varphi}_{ir}| \equiv |\dot{\varphi} (\varrho_{ir})| = |\delta \dot{\varphi} (\varrho_{ir})| . \] (2.9)

In the special case \( \varrho_b = 0 \) the boundary condition (2.8) coincides with those of Coleman [1] imposed to avoid a singularity. The scaling dimension of the massless scalar field \( \varphi \) in \( D \) dimensional spacetime is \( cm^{\frac{2-D}{2}} \), consequently the typical amplitude of its quantum fluctuations in scales \( \varrho \) is approximately
\[ |\delta \varphi_q (\varrho)| \simeq \sigma \varrho^{\frac{D-2}{2}} , \quad |\delta \dot{\varphi}_q (\varrho)| \simeq \frac{\sigma (D-2)}{2} \varrho^{\frac{D-2}{2}} , \] (2.10)
where the parameter \( \sigma \) is of order one, and the relations (2.9) become:
\[ |\dot{\varphi}_{uv}| = \frac{\sigma (D-2)}{2} \varrho_{uv}^{\frac{D-2}{2}} , \quad |\dot{\varphi}_{ir}| = \frac{\sigma (D-2)}{2} \varrho_{ir}^{\frac{D-2}{2}} . \] (2.11)

The classical solutions with \( \varrho_b \neq 0 \) are singular at \( \varrho \to 0 \) and have a divergent action. However, as shown in [3, 4, 6], these solutions for \( \varrho < \varrho_{uv} \) and \( \varrho > \varrho_{ir} \) are completely saturated by quantum fluctuations and therefore are trustworthy only for \( \varrho_{uv} < \varrho < \varrho_{ir} \). In particular, the region \( 0 < \varrho < \varrho_{uv} \), which makes an infinite contribution to the action, must be replaced by the quantum core with radius \( \varrho_{uv} \). For these regularised instantons, the action is finite and they provide a finite contribution to the decay rate. So instead of a single instanton with Coleman boundary conditions (which, for example does not exist at all for the steep unbounded potentials [5, 7]), we obtain the entire spectrum of regularized instantons, all of which contribute to the false vacuum decay.

Using the identities:
\[ \varrho^{D-2} \left( \frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right) \equiv -\frac{d}{d\varrho} \left( \frac{\varrho^{D-1}}{D-1} \left( \frac{1}{2} \varphi^2 - V(\varphi) \right) \right) \]
\[ + \frac{\varrho^{D-1}}{D-1} \dot{\varphi} \left( \dot{\varphi}(\varrho) + \frac{D-1}{\varrho} \dot{\varphi}(\varrho) - V'(\varphi) \right) \] (2.12)
\[ \varrho^{D-1} \left( \frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right) = \frac{\varrho^{D-1}}{D} \varphi^2 - \frac{d}{d\varrho} \left( \frac{\varrho^D}{D} \left( \frac{1}{2} \dot{\varphi}^2 - V(\varphi) \right) \right) \\
+ \frac{\varrho^D}{D} \dot{\varphi} \left( \ddot{\varphi}(\varrho) + \frac{D-1}{\varrho} \dot{\varphi}(\varrho) - V'(\varphi) \right), \] (2.13)

where the last terms on the right-hand sides vanish on-shell (see (2.6)), we can simplify the expressions for the total energy (2.3) and for the action (2.2) as:

\[ V = \frac{\pi^{D-1}}{\Gamma \left( \frac{D+1}{2} \right)} \left( \frac{\varrho_{uv}^{D-1}}{2} \dot{\varphi}_{uv}^2 - \varrho_{ir}^{D-1} \left( \frac{1}{2} \dot{\varphi}_{ir}^2 - V(\varphi_{ir}) \right) \right) \approx \frac{\sigma^2 \pi^{D-1}}{2 \Gamma \left( \frac{D+1}{2} \right)} \left( 1 - \frac{\varrho_{uv}}{\varrho_{ir}} \right) \frac{1}{\varrho_{uv}}, \] (2.14)

and

\[ S_I = \frac{\pi^D}{\Gamma \left( \frac{D+2}{2} \right)} \left( \int_{\varrho_{uv}}^{\varrho_{ir}} d\varrho \varrho^{D-1} \dot{\varphi}^2 + \frac{\varrho_{uv}^D}{2} \dot{\varphi}_{uv}^2 - \varrho_{ir}^D \left( \frac{1}{2} \dot{\varphi}_{ir}^2 - V(\varphi_{ir}) \right) \right) \approx \frac{\pi^D}{\Gamma \left( \frac{D+2}{2} \right)} \int_{\varrho_{uv}}^{\varrho_{ir}} d\varrho \varrho^{D-1} \dot{\varphi}^2, \] (2.15)

where we have also used relations (2.11) and assumed that \( \varrho_{ir}^{D-1} V(\varphi_{ir}) \approx 0 \) and \( \varrho_{ir}^D V(\varphi_{ir}) \approx O(1) \) as \( \varrho_{ir} \to \infty \), as well as we have neglected the term \( \varrho_{ir}^D V(\varphi_{ir}) \) of order one in the action \( S_I \), because the semiclassical approximation is valid anyway only when \( S_I \gg 1 \). Moreover, if we consider that

\[ \int_{\varrho_{uv}}^{\varrho_{ir}} d\varrho \varrho^{D-1} \dot{\varphi}^2 < \frac{1}{D} \varphi_{uv}^2 \varrho_{uv}^D \approx O(1), \quad \int_{\varrho_{uv}}^{\infty} d\varrho \varrho^{D-1} \dot{\varphi}^2 \approx O(1), \] (2.16)

then finally the instanton action can be rewritten in a very simple way

\[ S_I \approx \frac{\pi^D}{\Gamma \left( \frac{D+2}{2} \right)} \int_{\varrho_{uv}}^{\varrho_{ir}} d\varrho \varrho^{D-1} \dot{\varphi}^2 = \frac{\pi^D}{\Gamma \left( \frac{D+2}{2} \right)} \int_{\varrho_{b}}^{\varrho_{f}} d\varphi \varrho^{D-1} \dot{\varphi}, \] (2.17)

where \( \varphi_b \equiv \varphi(\varrho_b) \) and both the ultraviolet and infrared cutoff scales do not enter at all. Since \( \varrho_{uv} < \varrho_{ir} \), it follows from (2.14) that the total energy \( V \) actually vanishes with the required precision allowed by the time-energy uncertainty relation (2.4). This satisfy the necessary condition for the bubble with quantum core to emerge from under the barrier at \( \varrho = \varrho_{uv} \). The bubble materializes and expands, filling the Minkowski space with a new phase.

3 Friction dominated instantons

To calculate the false vacuum decay rate, we need first to solve the nonlinear equation (2.6) with boundary conditions (2.7)–(2.8) and then determine the value of the instanton action \( S_I \) (2.17) for this solution. This equation can be solved exactly only for some very special potentials.
Coleman [1] has developed the thin-wall approximation method, which allows us to estimate the contribution of leading order to the action when the depth of the tunnelling \( |V_b| \equiv |V(\varphi_b)| \) (the magnitude of the potential at the center of the emergent bubble) is much smaller than the height of the potential barrier \( V_{\text{bar}} \), i.e., \( |V_b| \ll V_{\text{bar}} \). In this case, the friction term in equation (2.6) can be initially neglected and then considered perturbatively.

In the opposite case of deep tunnelling, for \( |V_b| \gg V_{\text{bar}} \), as shown in [6], the friction term dominates over major part of the instanton, which gives the main contribution to the action. Such instantons determine the decay rate for the unbounded potentials or for the potentials with the deep true vacuum. The main aim of this work is to develop a general method for calculating the decay rate in the case when it is determined by the friction dominated instantons, such as for the unbounded potentials shown in figure 1.

When the friction term in equation (2.6) dominates over the potential term, i.e.,

\[
\left| \frac{D-1}{\varrho} \dot{\varphi} \right| \gg V',
\]

this equation simplifies to

\[
\dot{\varphi} + \frac{D-1}{\varrho} \varphi \simeq 0
\]

and has an obvious solution

\[
\varphi(\varrho) = \varphi_f + \frac{E}{(D-2)^2 |\varphi_f| \varrho^{D-2}}
\]

which satisfies the boundary condition (2.7). We have introduced here the constant of integration \( E \) which in principle can be expressed in terms of \( \varphi_b \equiv \varphi(\varrho_b) \), where \( \varrho_b \) enters the second boundary condition (2.8). Therefore, \( E \) can be used instead of \( \varphi_b \) to parametrize the regularized friction dominated instantons. One can easily verify that the friction dominated solution (3.3) satisfies the following useful relation

\[
E = \varrho^D \left( \dot{\varphi}^2 + \frac{D-2}{\varrho} \varphi \dot{\varphi} \right).
\]

Estimating the expression on the right-hand side of this equality for quantum fluctuations (2.10) we get

\[
\varrho^D \left( |\delta \dot{\varphi}_q(\varrho)|^2 + \frac{D-2}{\varrho} |\delta \varphi_q(\varrho)||\delta \dot{\varphi}_q(\varrho)| \right) \simeq \frac{3(D-1)^2 \sigma^2}{4} \sim O(1),
\]

which is independent on the scale \( \varrho \). From this we conclude that \( E \) must be much larger than unity, i.e.,

\[
E \gg 1,
\]

otherwise the quantum fluctuations would dominate everywhere over the friction part of the classical solution (mean field) and therefore its contribution to the action cannot be trusted at all.
An important property of the solution (3.3) is that it can be inverted and one can express $\varrho$ in terms of $\varphi$:

$$\varrho(\varphi) = \left( \frac{E}{(D-2)^2 |\varphi_f| (\varphi - \varphi_f)} \right)^{1/(D-2)}.$$  \hfill (3.7)

As a result, the equation (3.2) can be rewritten in the following autonomous (no explicit $\varrho$ dependence) form

$$\ddot{\varphi}(\varrho) + V_{fr}'(\varphi) = 0,$$  \hfill (3.8)

where

$$V_{fr}(\varphi) \equiv -\frac{1}{2} (D-2) \frac{|\varphi_f|}{E} \left( \frac{\varphi - \varphi_f}{E} \right)^{2(D-1)/(D-2)} \leq 0$$ \hfill (3.9)

is the “friction potential”. The equation (3.8) must be supplemented by the boundary condition at $\varrho \to \infty$ (2.7) and the condition that its first integral vanishes, i.e.,

$$\frac{1}{2} \dot{\varphi}(\varrho)^2 + V_{fr}(\varphi) = 0.$$ \hfill (3.10)

In this case, its solution corresponds exactly to (3.3).

This consideration suggests that for the cases where the friction dominated part of the instanton is the main contributor to the action, one can replace the exact equation (2.6) with autonomous equation

$$\ddot{\varphi} + U'_{eff}(\varphi) = 0,$$ \hfill (3.11)

where

$$U_{eff} = V_{fr} - V.$$ \hfill (3.12)

The first boundary condition for this equation must be as before (2.7) and the second one follows from equating the first integral of this equation to zero by analogy with (3.10), that is

$$\frac{1}{2} \dot{\varphi}^2 + U_{eff} = 0.$$ \hfill (3.13)

Let us now determine under which conditions the non-autonomous equation (2.6) can be well approximated by its nontrivial substitution by the autonomous equation (3.11). As follows from (3.13), the value of the scalar field at which its velocity vanishes must satisfy

$$U_{eff}(\varphi) = 0.$$ \hfill (3.14)

One of the solutions of this equation is obvious, namely, the location of the false vacuum, i.e., $\varphi = \varphi_f$. The other solution determines the bouncing value $\varphi_b$ for the case when the main part of the instanton is dominated by friction, i.e., satisfies the equation (3.2). It is clear that $\varphi_b$ must lie to the right of the maximum of the potential, i.e., $\varphi_b > 0$. If we assume that $\dot{\varphi}$ at the location of the maximum of the potential $V(\varphi = 0) = V_{bar}$ is determined by the friction term, then it follows from (3.13) that the friction dominated instanton must satisfy the following necessary condition

$$|V_{fr}(\varphi = 0)| \gg V_{bar},$$ \hfill (3.15)
and using (3.9) we find that for these instantons
\[ 1 \ll E \ll (D - 2)^D |\varphi_f| D (2V_{\text{bar}})^{\frac{2-D}{2}}. \] (3.16)

As follows from (3.14), at the bounce point \( \varphi_b \) the depth of the tunnelling is
\[ V_b \equiv V(\varphi_b) \simeq V_{\text{fr}}(\varphi_b). \] (3.17)

Since \( |V_{\text{fr}}(\varphi_b)| \geq |V_{\text{fr}}(0)| \), the condition (3.15) can be rewritten in a slightly stronger form, namely,
\[ |V_b| \gg V_{\text{bar}}, \] (3.18)
i.e., when the tunnelling depth is much larger than the height of the potential barrier, the instantons are mainly dominated by the friction term. This is converse to the condition for the applicability of thin-wall approximation, that is why we alternatively refer to these instantons as thick-wall instantons. It is interesting to note that in equation (2.6) the friction term becomes comparable to the potential term at \( \varphi_m \), which satisfies
\[ |V'(\varphi_m)| \simeq |V_{\text{fr}}'(\varphi_m)|. \] (3.19)

Comparing this with (3.17), we see that for the thick-wall instantons the potential term \( V' \) becomes important only near the bounce point, i.e., at \( (\varphi_b - \varphi_m)/\varphi_m \sim O(1) \).

4 The decay rate

To calculate the false vacuum decay rate, we need to find the typical size of the bubble and the action (2.17) for the corresponding thick-wall instantons. To calculate the action, we first rewrite the equation (2.6) into the following form
\[ \frac{d}{d\varphi} \left( \varrho^{D-1} \dot{\varphi} \right) = \frac{V' \varrho^{D-1}}{\dot{\varphi}}. \] (4.1)

After integration with taking into account the boundary condition at \( \varphi \to \varphi_f \) we get
\[ \varrho^{D-1} \dot{\varphi} = -\frac{E}{(D - 2)|\varphi_f|} + \int_{\varphi_f}^{\varphi} d\varphi' \frac{V' \varrho^{D-1}}{\dot{\varphi}}. \] (4.2)

Before we use this expression in the action (2.17), we estimate the integral on the right-hand side of this equation. For \( \varphi < \varphi_m \) the friction term dominates and therefore we can substitute into the integral
\[ \frac{D - 1}{q} \dot{\varphi} \simeq V_{\text{fr}}', \quad \varrho^{D-2} \simeq \frac{E}{(D - 2)^2 |\varphi_f| (\varphi - \varphi_f)} \] (4.3)

and obtain
\[ \int_{\varphi_f}^{\varphi} d\varphi' \frac{V' \varrho^{D-1}}{\dot{\varphi}} \simeq \frac{E}{2(D - 2)|\varphi_f|} \int_{\varphi_f}^{\varphi} d\varphi' \frac{V' \varrho_{\text{fr}}'}{V_{\text{fr}}} \simeq \frac{E}{2(D - 2)|\varphi_f|} \left( \frac{V(\varphi)}{V_{\text{fr}}} + \ldots \right), \] (4.4)
where we have used the explicit expression (3.9) for $V_{\text{fr}}$. It is clear that this integral term is small compared to the first term on the right-hand side of (4.2) up to almost the bounce and thus

$$\theta^{D-1}\dot{\varphi} \simeq -\frac{E}{(D-2)|\varphi_f|}$$

in the interval $\varphi_f < \varphi < \varphi_m$.

To find the behavior of $\theta^{D-1}\dot{\varphi}$ near the bounce, where the potential dominates over the friction potential, we consider that this happens in the interval $\varphi_m < \varphi < \varphi_b$, where $(\varphi_b - \varphi_m)/\varphi_m \sim O(1)$. In this case, $V'(\varphi)$ can be approximated by its value at the bounce $V_b' \equiv V'_{\varphi_b}$, and taking into account that $\dot{\varphi}(\theta_b) = 0$, we obtain from (4.1)

$$\theta^{D-1}\dot{\varphi} \simeq \frac{V_b'}{D} \left( \theta^D - \theta_b^D \right).$$

(4.6)

This equation can be further integrated further to give

$$\varphi = \varphi_b + \frac{V_b'}{2D\theta^D} \left[ 1 - \frac{D}{D-2} \left( \frac{\theta_b}{\theta} \right)^2 + \frac{2}{D-2} \left( \frac{\theta_b}{\theta} \right)^{D} \right].$$

(4.7)

For $\theta > \theta_b$ we can neglect the last two terms inside the parentheses and express $\theta$ in terms of $(\varphi_b - \varphi)$. Then equation (4.6) can be rewritten as

$$\theta^{D-1}\dot{\varphi} \simeq -2 \left( \frac{2D}{V_b'} \right)^{\frac{D+2}{2}} (\varphi_b - \varphi)^{\frac{D}{2}}$$

(4.8)

for the interval $\varphi_m < \varphi < \varphi_b$. The contribution of this term to the action is about

$$S_{\text{pot}} \sim \int_{\varphi_m}^{\varphi_b} d\varphi \theta^{D-1}\dot{\varphi} \sim |V_b'| \frac{2-D}{2} (\varphi_b - \varphi_m)^{\frac{2+D}{2}} \sim |V_b| \frac{2-D}{2} \varphi_b^D,$$

(4.9)

where we skipped all numerical coefficients of order one and made the following rough estimates $\varphi_m \sim \varphi_b$ and $V_b' \sim V_b/\varphi_b$. This contribution never exceeds the contribution of the friction dominated part of the instanton, which, as can be seen using (4.5) is of order

$$S_{\text{fr}} \sim \int_{\varphi_f}^{\varphi_m} d\varphi \theta^{D-1}\dot{\varphi} \sim \frac{E}{|\varphi_f|} (\varphi_m - \varphi_f) \sim \frac{E}{|\varphi_f|} (\varphi_b - \varphi_f).$$

(4.10)

Then from (4.9) and (4.10) it follows

$$\frac{S_{\text{pot}}}{S_{\text{fr}}} \sim \left( \frac{\varphi_b}{\varphi_b - \varphi_f} \right)^{\frac{D}{2}},$$

(4.11)

where we have taken into account that $V_b = V_{\text{fr}}(\varphi_b)$ and used the expression (3.9) for the friction potential. Thus, if $\varphi_b$ determined from the equation

$$V(\varphi_b) = V_{\text{fr}}(\varphi_b)$$

(4.12)

1To simplify the consideration, we assume that near the false vacuum $V \propto (\varphi - \varphi_f)^\alpha$, where $\alpha > 2(D-1)/(D-2)$. 

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is much smaller than $|\varphi_f|$, then $S_{\text{pot}} \ll S_{\text{fr}}$, while for $\varphi_b \gg |\varphi_f|$ both terms give the contribution of same order, i.e., $S_{\text{pot}} \sim S_{\text{fr}}$. Therefore

$$S = \frac{\alpha \pi D E (\varphi_b - \varphi_f)}{\Gamma \left( \frac{D+2}{2} \right) (D-2) |\varphi_f|} = \frac{\alpha (D-2)^{D-1} \pi D}{\Gamma \left( \frac{D+2}{2} \right)} \frac{(\varphi_b + |\varphi_f|)^D}{|2V_b|^{D-2}},$$

(4.13)

where the numerical coefficient $\alpha \to 1$ for $\varphi_b \ll |\varphi_f|$ and it is of order one, i.e., $\alpha \sim O(1)$ if $\varphi_b \gg |\varphi_f|$. Note that in the case $\varphi_b \ll |\varphi_f|$ the numerical coefficient in the action is exact in the leading order, while for $\varphi_b \gg |\varphi_f|$ the action is estimated even in the leading order with an accuracy up to a numerical factor of order one, the concrete value of which depends on the potential $V(\varphi)$.

To obtain the expression for the typical “size” of the thick-wall bubble, we note that the friction term still dominates when the field crosses $\varphi = 0$ and if we define the “size of the bubble” as $\varrho_0 = \varrho(\varphi = 0)$ we obtain from (3.7)

$$\varrho_0 = \left( \frac{E}{(D-2)^2 \varphi_f^2} \right)^{\frac{1}{D+2}} = \sqrt{|2V_b|} \left( 1 + \frac{\varphi_b}{|\varphi_f|} \right)^{\frac{D-1}{D+2}} |\varphi_f|.$$

(4.14)

To express $E$ in the formulas (4.13) and (4.14) in terms of $\varphi_b$ and $V_b$, we considered that $V_b = V_{\text{fr}}(\varphi_b)$ and used the expression (3.9) for $V_{\text{fr}}$. Thus, one can use the value of the scalar field at the bounce (in the “center of the bubble”) $\varphi_f$ to parametrize the regularized instantons. The condition (3.16) under which the thick-wall approximation is applicable can be rewritten as

$$\left( 1 + \frac{\varphi_b}{|\varphi_f|} \right)^{\frac{2(D-1)}{D+2}} V_{\text{bar}} \ll |V_b| \ll \frac{1}{2} (D-2)^{\frac{2D}{D+2}} \left( 1 + \frac{\varphi_b}{|\varphi_f|} \right)^{\frac{2(D-1)}{D+2}} |\varphi_f|^{\frac{2D}{D+2}}.$$

(4.15)

For the case when the potential has the second true minimum of depth $|V_{\text{fr}}|$ in the interval (4.15), the largest contribution to the decay rate comes from the instantons with $|V_b| \simeq |V_{\text{fr}}|$. Substituting (4.13) and (4.14) into (2.1), we obtain the contribution of the instantons parametrized by $\varphi_b$ to the total decay rate of the false vacuum. Note that even if the potential is unbounded the results are only credible if $E \gg 1$. Therefore, to estimate the maximal depth of the potential to which the field can tunnel, we must first determine the maximum value of $\varphi_b^{\text{max}}$ by equating $V(\varphi_b)$ with $V_{\text{fr}}(\varphi_b)$ for $E \simeq 1$, i.e.,

$$V(\varphi_b^{\text{max}}) \simeq -\frac{1}{2} (D-2)^{\frac{2D}{D+2}} \left( 1 + \frac{\varphi_b^{\text{max}}}{|\varphi_f|} \right)^{\frac{2(D-1)}{D+2}} |\varphi_f|^{\frac{2D}{D+2}},$$

(4.16)

solve this equation for $\varphi_b^{\text{max}}$ and then determine the maximal possible penetration depth. For example, if the potential has a second very deep true minimum at $\varphi_{\text{fr}}$, at which $|V(\varphi_{\text{fr}})| > V(\varphi_b^{\text{max}})$ and the Coleman instanton that connects $\varphi_f$ with the neighbourhood of $\varphi_{\text{fr}}$ exists, this instanton is nevertheless not trustworthy because it is below the level of quantum fluctuations. In this case, the main contribution to the decay rate comes from the instantons describing tunnelling from $\varphi_f$ to $\varphi_b^{\text{max}}$. A particular example of such a situation was presented in [3] for an exactly solvable linear potential.
5 Example

To illustrate how the above approximation works, and to compare our approximate formulas with the exact results, we consider the case of an exactly solvable quartic potential in $D = 4$ dimensions, which we studied in [4]:

\begin{equation}
V(\varphi) = \begin{cases} 
\frac{\lambda_+}{4} (\varphi - \varphi_f)^4 & \text{for } \varphi \leq \beta \varphi_f, \\
-\frac{\lambda_-}{4} \varphi^4 + V_{\text{bar}} & \text{for } \varphi \geq \beta \varphi_f,
\end{cases}
\end{equation}

(5.1)

where $0 < \beta < 1$ and the height of the barrier $V_{\text{bar}}$ are expressed in terms of the positive coupling constants $\lambda_+$ and $\lambda_-$ as

\begin{equation}
\beta \equiv \frac{\lambda_+}{\lambda_+ + \lambda_-}, \quad V_{\text{bar}} \equiv \frac{\lambda_-}{4} \beta^4 \varphi_f^4.
\end{equation}

(5.2)

This potential is composed of two power-law potentials glued at $\varphi = \beta \varphi_f < 0$, so that both the potential and its first derivative are continuous at this point. Taking into account that the thick-wall approximation is valid only when $|V_{\text{b}}| \gg V_{\text{bar}}$, we can neglect $V_{\text{bar}}$ in (5.1) and use

\begin{equation}
V_b \simeq -\frac{\lambda_-}{4} \varphi_b^4
\end{equation}

(5.3)
in (4.13) and (4.14) for $D = 4$:

\begin{equation}
S = \frac{8\pi^2 \alpha}{\lambda_-} \left(1 + \frac{|\varphi_f|}{\varphi_b}\right)^4,
\end{equation}

(5.4)

\begin{equation}
\rho_0 = \left[\frac{8}{\lambda_-} \left(1 + \frac{\varphi_b}{|\varphi_f|}\right)^{3/2} \frac{|\varphi_f|}{\varphi_b^2}\right].
\end{equation}

(5.5)

According to (4.15) these expressions are valid only when $\varphi_b$ satisfies inequality

\begin{equation}
\left(1 + \frac{\varphi_b}{|\varphi_f|}\right)^3 \frac{V_{\text{bar}}}{4} \ll \frac{1}{4} \lambda_- \varphi_b^4 \ll 8 \left(1 + \frac{\varphi_b}{|\varphi_f|}\right)^3 |\varphi_f|^4,
\end{equation}

(5.6)

and therefore the corresponding instanton is friction dominated. To compare the above formulas with those in [4], presented in terms of $E$, we need to express $\varphi_b$ in terms of $E$. The corresponding relation follows directly from the equation $V_b = V_{\text{fr}}(\varphi_b)$ and in our particular example becomes

\begin{equation}
\left(1 + \frac{|\varphi_f|}{\varphi_b}\right)^3 \frac{|\varphi_f|}{\varphi_b} = \frac{\lambda_- E}{32}.
\end{equation}

(5.7)

We consider separately two cases where the coupling constant $\lambda_-$ is either much smaller or much larger than $\lambda_+$.

For the potential that is very flat near its maximum, i.e., $\lambda_- \ll \lambda_+$, the height of the potential barrier is $V_{\text{bar}} \simeq \frac{1}{4} \lambda_- \varphi_f^4$ and as follows from (5.6)

\begin{equation}
|\varphi_f| \ll \varphi_b \ll \frac{32}{\lambda_-} |\varphi_f|.
\end{equation}

(5.8)
In this case $\varphi_b$ is always larger than $|\varphi_f|$ and it follows from (5.7) that

$$\varphi_b \simeq \frac{32}{\lambda_+ E} |\varphi_f|$$

(5.9)

for $1 \ll E \ll 32/\lambda_-$. The formulas (5.4) and (5.5) simplify to

$$S \simeq \frac{8\pi^2 \alpha}{\lambda_-}, \quad \varrho_0 \simeq \sqrt{\frac{E}{4}} \frac{1}{|\varphi_f|}.$$ 

(5.10)

Comparing these results to those from [4], we see that $\alpha = 1/3$. All other numerical coefficients in the expressions for $\varphi_b$ and $\varrho_0$ are exact in the leading order.

For the potential that drops very rapidly after reaching its maximum, i.e., $\lambda_- \gg \lambda_+$ the interval of $\varphi_b$ for which our thick-wall approximation holds is larger. Considering that in this case $V_{\text{bar}} \simeq \frac{1}{4} \lambda_+ \varphi_f^4$, we find that the inequality (5.6) in this case reduces to

$$\left(\frac{\lambda_-}{\lambda_+}\right)^{1/4} |\varphi_f| \ll \varphi_b \ll \frac{32}{\lambda_-} |\varphi_f|.$$ 

(5.11)

In the range (5.8), the expressions (5.9) and (5.10) remain unchanged. For

$$\left(\frac{\lambda_+}{\lambda_-}\right)^{1/4} |\varphi_f| \ll \varphi_b \ll |\varphi_f|,$$

(5.12)

the solution of the equation (5.7) is

$$\varphi_b \simeq \left(\frac{32}{\lambda_- E}\right)^{1/4} |\varphi_f|,$$

(5.13)

and it holds for $32/\lambda_- \ll E \ll 32/\lambda_+$. Since for $\varphi_b \ll |\varphi_f|$ the value of $\alpha$ tends to one, the expressions (5.4) and (5.5) simplify to

$$S = \frac{\pi^2}{4} E, \quad \varrho_0 \simeq \sqrt{\frac{E}{4}} \frac{1}{|\varphi_f|}.$$ 

(5.14)

in full agreement with the results derived in [4] from exact solutions. Note, that the maximum depth of tunnelling via classical instantons in both cases is approximately

$$V(\varphi_b^{\text{max}}) = -\frac{1}{4} \lambda_- \left(\frac{32}{\lambda_-} |\varphi_f|\right)^4 = \left(\frac{64}{\lambda_-}\right)^3 \varphi_f^4$$

(5.15)

in agreement with [4]. As we have checked, the results obtained above reproduce not only the parametric dependence but also the numerical coefficients for the corresponding asymptotic formulas in case of an exactly solvable potential, as considered in [3]. Moreover, we have verified that this is also true for the deep tunnelling in the case of exactly solvable potentials which are constructed piecewise from quartic and quadratic potentials in three and four dimensions.
6 Conclusions

The only known case in which the probability of decay of the false vacuum can be estimated for general potentials is the case in which the depth of the true vacuum is much smaller than the height of the potential barrier separating the false and true vacuums. In such a case, the thin-wall approximation for instantons applies [1]. In this work, we have considered the reverse case, i.e., we have assumed that the depth of the true vacuum (or tunnelling at unbounded potential) significantly exceeds the height of the barrier. We found that in this case the corresponding instantons describing the tunnelling are dominated mainly by the friction term in equation (2.6) and the resulting bubbles of a true vacuum always have thick walls. This simplifies the problem considerably and allows us to replace the non-autonomous equation (2.6) by the autonomous completely integrable equation (3.11), which is a good approximation for the original exact equation when the depth of tunnelling $|V_b|$ significantly exceeds the height of the barrier $V_{bar}$. As a result, we were able to derive the general simple formulas (4.13) and (4.14) for arbitrary potentials in any number of dimensions. We applied the general results to some exactly solvable potentials and showed that our simple thick-wall approximation exactly reproduces the leading order asymptotic, including the numerical coefficients, for the deep tunnellings.

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