Giant Gravitons as Fuzzy Manifolds

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Giant gravitons are described microscopically in terms of dielectric gravitational waves expanding into fuzzy manifolds. We review these constructions in $AdS_m \times S^n$ spacetimes, discussing the different fuzzy manifolds that appear in each case.\(^3\)

1 Introduction

Giant gravitons [1]-[3] are stable brane configurations with non-zero angular momentum, that are wrapped around $(n-2)$- or $(m-2)$-spheres in $AdS_m \times S^n$ spacetimes, and carry dipole moment with respect to the background gauge potential. They are stable because the contraction due to the tension of the brane is precisely cancelled by the expansion due to the coupling of the angular momentum to the background flux. These spherical brane configurations turn out to be massless, conserve the same number of supersymmetries and carry the same quantum numbers of a graviton. The fact that they are extended objects of finite size has lead to the name of giant gravitons.

These configurations were first proposed in [1] as a way to satisfy the stringy exclusion principle implied by the $AdS/CFT$ correspondence [4]. The spherical $(n-2)$-brane expands into the $S^n$ part of the geometry with a radius proportional to its angular momentum. Since this radius is bounded by the radius of the $S^n$, the configuration has associated a maximum angular momentum. There are as well [2, 3] “dual” giant graviton configurations, corresponding to spherical $(m-2)$-branes expanding in the $AdS_m$ part of the spacetime, and thus not satisfying the stringy exclusion principle. Possible ways of realizing this principle in the presence of these degenerate solutions have been proposed in for example [2, 5].

The appearance of giant gravitons as blown up massless particles hints to a connection with other examples of expanded brane configurations, more precisely to the dielectric effect [7], where multiple coinciding D$p$-branes can expand into higher dimensional D-brane configurations. There are two complementary descriptions of this effect. Consider the case in which the D$p$-branes expand into a spherical D$(p+2)$-brane. The first one is an Abelian, macroscopic description, describing the spherical D$(p+2)$-brane with a large number of D$p$-branes dissolved on its worldvolume [6]. The second one is a non-Abelian, microscopic formulation [8]-[7], describing how multiple coinciding D$p$-branes expand into a D$(p+2)$-brane with the topology of a fuzzy 2-sphere [7]. Both descriptions agree in the limit where the number $N$ of D$p$-branes is very large.

It has been suggested in the literature [10]-[12] that there should exist a microscopic description of the Abelian giant gravitons of [1]-[3] in terms of dielectric gravitational waves. Since massless particles, in particular gravitons, are the source terms for gravitational waves,

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it is natural to expect that a dielectric effect for gravitational waves will provide a microscopic picture for the giant graviton configurations. By analogy with the dielectric effect for D-branes, it is believed that in the limit when the number of gravitons is large, this microscopic description should match the macroscopical description of [1]-[3], where now the angular momentum of the giant graviton is interpreted microscopically as the total momentum of the multiple gravitational waves.

The actions describing coincident gravitational waves in Type IIA and M-theory were derived in [12, 13], using Matrix string theory and a chain of dualities. These actions contain the multiple gravitational waves.

In suitable coordinates the former here but refer to [13]-[16]. The giant gravitons in each case is very similar to the genuine ones, we will not deal with the AdS$_5$L direction (i.e. take the Ansatz \( \theta \) given by [1] for the metric and the volume form of a unit (\( \theta \sin L \)). Notice that since \( \sin \theta \) are in fact those of a massless particle [1, 2]. Notice that once the so called giant graviton, an expanded brane of finite size, whose quantum numbers give rise to the dielectric effect. With these actions, it is then possible to check explicitly the claims of [10]-[12] about the microscopical description of giant gravitons. This has been done in a series of papers [13]-[16] for the different AdS$_m \times S^n$ backgrounds. The aim of this paper is to review these constructions, emphasising the specific details of each case. In particular we will look at the AdS$_m \times S^n$ backgrounds relevant for string theory, namely AdS$_7 \times S^4$, AdS$_5 \times S^5$, AdS$_4 \times S^7$ and AdS$_3 \times S^3 \times T^4$. As the construction of the dual giant gravitons in each case is very similar to the genuine ones, we will not deal with the former here but refer to [13]-[16].

2 Macroscopic giant gravitons

In suitable coordinates the AdS$_m \times S^n$ backgrounds can be written as

\[
\begin{align*}
    ds^2 &= -(1 + \frac{r^2}{L^2})dt^2 + (1 + \frac{r^2}{L^2})^{-1}dr^2 + r^2 d\Omega_{m-2}^2 + L^2(d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\Omega_{n-2}^2), \\
    C^{(n-1)}_{\phi \chi_1, \ldots, \chi_{n-2}} &= \beta_n L^{n-1} \sin^{n-1} \theta \sqrt{\phi}, \\
    C^{(m-1)}_{\alpha_1, \ldots, \alpha_{m-2}} &= -r^{m-1} \frac{L}{\sqrt{\phi}}
\end{align*}
\]  

where \( L = \frac{\sqrt{2}}{2} \tilde{L} \) and \( \tilde{L} = -\beta_5 = -\beta_7 = 1 \), \( d\Omega_{m-2} \) and \( \sqrt{\phi} \) stand respectively for the metric and the volume form of a unit (\( \sqrt{\phi} \)) sphere (parametrized with the coordinates \( \{ \chi_i \} \)). The AdS$_5 \times S^3 \times T^4$ background is special. It arises as the near horizon geometry of the intersecting D1-D5 system:

\[
\begin{align*}
    ds^2 &= -(1 + \frac{r^2}{L^2})dt^2 + (1 + \frac{r^2}{L^2})^{-1}dr^2 + r^2 d\phi^2 + L^2(d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\chi^2) + R^2 dy_5^2, \\
    e^\phi &= R^2, \\
    C^{(2)}_{t\phi} &= -Q_5 L^{-3} r^2, \\
    C^{(2)}_{\phi \chi} &= Q_5 \sin^2 \theta.
\end{align*}
\]

where \( Q_1 \) and \( Q_5 \) are the total D1- and D5-brane charges, and \( y_5 \) (\( a = 1, \ldots, 4 \)) describe the relative transverse space. \( R, L \) and \( Q_5 \) are related via \( L^2 = Q_5 R^2 \).

Consider a test \((n-2)\)-brane wrapped around the \( S^{n-2} \) of (1) and moving along the \( \phi \) direction (i.e. take the Ansatz \( \theta = \text{constant}, r = 0, \phi = \phi(t) \)). Its energy is easily seen to be given by [1]

\[
E = \frac{P_\phi}{L} \sqrt{1 + \tan^2 \theta \left( 1 - \frac{\tilde{N}}{P_\phi} \sin^{n-3} \theta \right)^2}.
\]

Here \( \tilde{N} = T_{(n-2)}A_{(n-2)}L^{n-1} \) is an integer coming from the flux quantization on the \( S^{n-2} \) (with \( T_p \) the tension of the \( p \)-brane and \( A_p \) its area) and \( P_\phi \) is the angular momentum in \( \phi \). Minimizing with respect to \( \theta \) one finds two minima, both with energy \( E = P_\phi/L \), namely \( \sin \theta = 0 \) and \( \sin \theta = (P_\phi/\tilde{N})^{1/4} \). Since the radius of the \( (n-2) \)-sphere in (1) is given by \( L \sin \theta \), it is clear that the first solution corresponds to a point-like particle, while the second one is the so called giant graviton, an expanded brane of finite size, whose quantum numbers are in fact those of a massless particle [1, 2]. Notice that since \( \sin \theta \) is bounded, \( P_\phi \) is also bounded, thus leading to a realisation of the string exclusion principle [1].
In the $AdS_3 \times S^3 \times T^4$ background the most general giant graviton solution is in terms of a test brane consisting of a bound state of D1-branes and D5-branes wrapped on the 4-torus. Furthermore, since the cycles in $AdS_3$ and $S^3$ are both $S^1$, it is possible to construct mixed giant gravitons, a linear combination of genuine and dual ones. Here we will only deal with the simplest case of a D1-brane wrapped around the $S^1 \subset S^3$. For details of the general construction we refer to [17]. The energy of this brane is given by

$$E = \frac{P_\phi}{L} \sqrt{1 + \tan^2 \theta \left(1 - \frac{\tilde{N}}{R^2 P_\phi}\right)^2}.$$  (4)

The minimum energy is reached when $\theta = 0$ or $P_\phi = \tilde{N}/R^2$. Note that the giant graviton solution does not put constraints on the radius of the brane and that this solution only exists for specific values of the momentum. This poses a puzzle with the realisation of the stringy exclusion principle (see [3, 17]).

3 The action for M-theory gravitational waves

The action for $N$ 11-dim gravitational waves in an arbitrary background is given by [13]

$$S = -T_W \int d\tau \STr \left\{ k^{-1} \sqrt{-P[E_{00} + E_{0i}(Q^{-1} - \delta)^j_k E_{kj}] \det Q} \right\} + T_W \int d\tau \STr \left\{ -P[k^{-2}k^{(1)}] + iP[(iX_i X_i)C^{(3)}] + \frac{1}{2}P[(iX_i X_i)^2]C^{(6)} + \cdots \right\},$$  (5)

$$E_{\mu\nu} = g_{\mu\nu} - k^{-2}k_\mu k_\nu + k^{-1}(ikC^{(3)})_{\mu\nu}, \quad Q^j_i = \delta^j_i + ik[X^i, X^k]E_{kj}.$$

Here $T_W$ is the (momentum) charge of a single graviton. This action contains the direction of propagation of the waves as a special isometric direction, with Killing vector $k^\mu$. In the Abelian limit, a Legendre transformation restoring the dependence on this direction yields the usual action for massless particles. In turn, in the non-Abelian case, this action gives rise to Myers action for coincident D0-branes after dimensional reduction over $k^\mu$, and the action for coincident Type IIA gravitational waves, obtained in [12] via Matrix string theory in a weakly curved background, after dimensional reduction along a different direction. Consistently, the Matrix theory calculation yields as well an isometric action for coincident Type IIA waves. Notice that the waves are minimally coupled to the momentum operator $k^{(1)}_\mu/k^2 = g_{z\mu}/g_{zz}$, in coordinates adapted to the isometry in which $k^\mu = \delta^\mu_z$.

4 The microscopic description for giant gravitons

Using the action (5) we now provide a microscopical description for the giant gravitons in $AdS_m \times S^n$ spacetimes in terms of expanding gravitational waves.

4.1 Giant gravitons in $AdS_7 \times S^4$: the standard case

From the Abelian picture we know that the giant gravitons should come from waves propagating in the $\phi$-direction and expanding into a non-Abelian version of the $S^2$. We thus identify the isometry direction of the action (5) as $k^\mu = \delta^\mu_\phi$ and parametrise the $S^2$ in terms of cartesian coordinates $X^i = \frac{L \sin \theta}{\sqrt{N-1}} J^i$, with $J^i$ forming a $N$-dim representation of $SU(2)$. Since the $X^i$ satisfy $\sum_i (X^i)^2 = L^2 \sin^2 \theta \mathbb{1}$, through the quadratic Casimir of the group, they span a fuzzy two-sphere. Substituting this Ansatz in the action yields an energy [13]:

$$E = \frac{P_\phi}{L} \sqrt{1 + \tan^2 \theta \left(1 - \frac{\tilde{N}}{R^2 P_\phi}\right)^2}.$$  (4)


\[ E = \frac{T_W}{L \cos \theta} \text{STr} \left\{ 1 - \frac{4L \sin \theta}{\sqrt{N^2 - 1}} X^2 + \frac{4L^4 \sin^2 \theta \cos^2 \theta}{N^2 - 1} X^2 + \frac{4L^2 \sin^2 \theta}{N^2 - 1} X^2 X^2 \right\}. \]  

Given that we are interested in the large \( N \) limit in order to compare with the Abelian calculation, and taking into account that \( \text{STr}(\langle X^2 \rangle^n) = \text{Tr}(\langle X^2 \rangle^n) + \mathcal{O}(\frac{1}{N^{n-1}}) \), we can rewrite the action (6) for large \( N \) as

\[ E = \frac{NT_W}{L} \sqrt{1 + \tan^2 \theta \left( 1 - \frac{2L^3}{\sqrt{N^2 - 1}} \sin \theta \right)^2}. \]  

Identifying the macroscopic momentum \( P_\phi \) with the sum of the charges of the \( N \) gravitons and taking into account that \( \tilde{N} = T_2 A_2 L^3 \) we find perfect agreement up to order \( N^{-2} \) with the macroscopic computation (3). We also find agreement to the same order for the point-like and giant graviton solutions and the upper bound on the momentum.

### 4.2 Giant gravitons in \( AdS_5 \times S^5 \): the Hopf fibration

The action for coincident gravitational waves in Type IIB can be derived via reduction plus T-duality from the action (5). The BI part is of the same form as in (5), where now

\[ E_{\mu \nu} = g_{\mu \nu} - k^{-2} k_\mu k_\nu - l^{-2} l_\mu l_\nu - k^{-2} l^{-1} e^{\Phi(i_k i_l C(4))}_{\mu \nu}, \quad Q_j^i = \delta^i_j + i[X^i, X^k] e^{-\Phi} klE_{kj}. \]

Here \( l^\mu \) is a second Killing vector, pointing along the direction in which the T-duality is performed, which appears explicitly in the action for coincident Type IIB waves (see [14]). In the CS part, the coupling to the RR 4-form is of the form \( P[(i_X i_X) i_l C(4)] \).

The construction of non-Abelian giant gravitons in \( AdS_5 \times S^5 \) turns out to be more involved than the ones in \( AdS_7 \times S^4 \). Here the gravitational waves expand into a spherical D3-brane, and hence a fuzzy 3-sphere Ansatz is needed. Yet the extra isometry in the Type IIB action is of help. The presence of this compact isometry suggests the representation of the 3-sphere as a \( U(1) \) bundle over \( S^2 \), with the \( U(1) \) invariance associated to translations along this direction. The fuzzy \( S^3 \) is then constructed as an Abelian \( U(1) \) fibre over a fuzzy \( S^2 \). The details of the construction can be found in [14].

Parametrising the \( S^2 \) base manifold as in 4.1 by the cartesian coordinates \( X^i = \frac{\hat{R}}{\sqrt{N^2 - 1}} J_i \), with \( \tilde{R} \) the radius of the \( S^2 \) and \( \sum_i (X^i)^2 = \tilde{R}^2 \mathbb{1} \), we find that the energy is given by [14]

\[ E = \frac{T_W}{L \cos \theta} \text{STr} \left\{ 1 - \frac{L^4 \sin^4 \theta}{2R^2 \sqrt{N^2 - 1}} X^2 + \frac{L^8 \sin^6 \theta \cos^2 \theta}{16R^2 (N^2 - 1)} X^2 + \frac{L^8 \sin^4 \theta}{16R^4 (N^2 - 1)} (X^2)^2 \right\}, \]  

which for large \( N \) can again be rewritten to a form that agrees with the Abelian case (3):

\[ E = \frac{NT_0}{L} \sqrt{1 + \tan^2 \theta \left( 1 - \frac{L^4 \sin^2 \theta}{4\sqrt{N^2 - 1}} \right)^2}. \]  

### 4.3 Giant gravitons in \( AdS_4 \times S^7 \): fuzzy \( CP^2 \)

The giant graviton in this spacetime is an \( S^5 \), which can be described as a \( U(1) \)-bundle over the two dimensional complex projective plane, \( CP^2 \). It is again suggestive to identify the isometric direction in the action with the \( U(1) \) fibre. However, the action for M-theory gravitational waves contains only one isometric direction, which is typically identified with the direction of propagation of the waves. The relevant dielectric coupling in (5) to the 6-form potential is \( P[(i_X i_X) i_k C(6)] \), which is however vanishing for the 6-form potential of the
AdS\(_3 \times S^7\) background, given by (1). This is however not the case if we identify \(k^n\) with the coordinate along the \(U(1)\) fibre in the Hopf fibration of the \(S^5\), and the propagation direction \(\phi\) is in turn taken to be an isometry which is not explicit in the action (5). Analogously to the previous case, the fuzzy \(S^5\) onto which the gravitons expand is then the Hopf fibre of the fuzzy \(CP^2\), whose construction is known in the literature (see for instance [18]). Taking into account that \(CP^2\) can be defined as a submanifold of \(\mathbb{R}^8\) determined by certain constraints, its fuzzy version can be obtained imposing these conditions at the level of matrices. This is satisfied for \(X^i = \frac{1}{\sqrt{n^2+3n}} T^i\), where \(T^i\) are the \(SU(3)\) generators in the \((n,0)\) or \((0,n)\) irreducible representations. (For the technical details, we refer to [15].) Taking into account that the dimension of these representations is given by \(N = (n+1)(n+2)/2\), the Lagrangian of the system of waves is, up to order \(N^{-2}\),

\[
\mathcal{L} = -\frac{T_W}{L \sin \theta} \text{Str} \left\{ \sqrt{1 - L^2 \cos^2 \theta \dot{\phi}^2 \left(1 + \frac{3L^6 \sin^6 \theta}{8(N-1)} X^2 \right)} - \frac{9L^7 \sin^7 \theta}{8(N-1)} \dot{\phi}^2 (X^2)^2 \right\}.
\] (10)

Eliminating the velocity \(\dot{\phi}\) via a Legendre transformation, we obtain for the energy [15]

\[
E = \frac{P_\phi}{L} \sqrt{1 + \tan^2 \theta \left(1 - \frac{NT_0}{8(N-1)P_\phi} L^6 \sin^4 \theta \right)^2 + \frac{N^2 T_0^2}{P_\phi^2 \sin^2 \theta} \left(1 + \frac{L^6 \sin^6 \theta}{4(N-1)} \right)}. \tag{11}
\]

Here we have to recall that the gravitons described by the Lagrangian (10) carry, by construction, momentum along the \(U(1)\) fibre isometric direction, that we parametrize by \(\chi\), this momentum being given by \(P_\chi = NT_W\). Therefore, in order to make contact with the Abelian giant graviton of (3) we have to switch to zero this momentum, without setting to zero the number of expanding gravitons! The difference between \(P_\chi\) being zero or not is merely a coordinate transformation, a boost in \(\chi\). How to perform coordinate transformations in non-Abelian actions is however an open problem. Still, we can compare the gravitons described by the Lagrangian (10) with the corresponding macroscopical description, which is in terms of a spherical M5-brane moving in both the \(\phi\) and \(\chi\) directions. In this description the momentum along \(\chi\) can be neatly set to zero, since an M5-brane moving only along the \(\phi\) direction is perfectly well-defined. This shows that, at least for large \(N\), setting \(P_\chi\) to zero eliminates the last term in (11). We then obtain an expression that agrees with (3) for infinite number of gravitons:

\[
E = \frac{P_\phi}{L} \sqrt{1 + \tan^2 \theta \left(1 - \frac{NT_0}{8(N-1)P_\phi} L^6 \sin^4 \theta \right)^2}. \tag{12}
\]

### 4.4 Giant gravitons in \(AdS_3 \times S^3 \times T^4\): the worldvolume scalar

Giant gravitons in \(AdS_3 \times S^3 \times T^4\) are branes wrapped around an \(S^1\), which raises the immediate question of how to construct a non-Abelian realisation, as the \(S^1\) has associated an (Abelian) \(U(1)\) algebra. The solution is to embed the \(S^1\) in a cylinder, which can then be made fuzzy. In order to do this we have to take the worldvolume scalar associated by T-duality with the duality direction non-vanishing. This field will not have a geometrical meaning, since it is not a transverse scalar, but plays the role of the direction along the axis of the cylinder in the construction of the algebra.

The BI part is again of the form (5), but now we have to consider as well the contribution of the non-vanishing worldvolume scalar, \(\omega\), (see [16] for the explicit expression). The relevant CS coupling is now \(\text{Str} \{ [X^i, \omega] C^{(i)2} DX^j \}\).

Considering a fuzzy cylinder whose spatial section is a circle described by cartesian coordinates \(X^i (i = 1, 2)\) and whose axis is along the scalar field \(\omega\), the algebra is given by:
\[ [X^i, X^j] = 0, \quad [X^i, \omega] = i\epsilon^{ij} f X^j, \quad (13) \]

with \( f \) the non-commutative parameter. The quadratic Casimir gives \( \sum_i (X^i)^2 = L^2 \sin^2 \theta \). All the representations of this algebra are however infinite dimensional, so we are forced to deal with an infinite number of gravitons. Substituting in the action we find an energy [16]:

\[
E = \frac{T_W}{L \cos \theta} \text{Str} \left\{ \sqrt{\left( \mathbb{1} - \frac{fQ_5}{L^2} X^2 \right)^2 + \frac{f^2 Q_5^2}{L^2} \cos^2 \theta X^2} \right\}. \quad (14)
\]

The energy per unit length of the cylinder is then given by

\[
E = \frac{T_W}{fL} \sqrt{1 + \tan^2 \theta (1 - fQ_5)^2}, \quad (15)
\]

where we have taken into account that the length of the cylinder can be estimated as \( l = f \text{Tr} \mathbb{1} \). As in the macroscopic case, we find giant gravitons of arbitrary radius if \( f = Q_5^{-1} \).

Taking into account that the microscopical momentum per unit length is given by \( \text{Tr} \mathbb{1} T_W/l \), which is what should be compared with \( P_\phi \) in (4), we find perfect agreement between the two computations. This is in agreement with the fact that we need an infinite number of gravitons in order to span the fuzzy cylinder.

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