Symmetries of the similarity renormalization group for nuclear forces

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References

- V. S. Timoteo, S. Szpigel and E. Ruiz Arriola
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- E. Ruiz Arriola, nucl-th/1009.4161
Introduction

- How much do we need to know light nuclei to predict heavy nuclei?
- Nucleon size $a \sim 1\text{fm}$
- Nuclear Force $\sim 1/m_\pi = 1.4\text{fm}$
- Nuclear matter (interparticle distance)

$$\rho_{nm} = 0.17\text{fm}^{-3} = \frac{1}{(1.8\text{fm})^3}$$

- Fermi Momentum

$$k_F = 270\text{MeV} \quad \lambda_F = \pi/k_F = 2.3\text{fm} \gg 1/\sqrt{m_\pi M_N} = 0.5\text{fm}$$

Can we ignore explicit core and explicit pions?
Nuclear many body Hamiltonian $H$

$$H = \sum_i T_i + \sum_{i<j} V_{2,ij} + \sum_{i<j<k} V_{3,ijk} + \sum_{i<j<k<l} V_{4,ijkl} + \ldots$$

- NN: $V_{2,ij}$ (deuteron+NN scattering data)
- 3N: Triton+ N-deuteron scattering
- 4N: $\alpha$–particle, $dd, tp$ etc, scattering

Typical Range of multinucleon forces $e^{-m_\pi d} \sim 0.2$

$$V_{NN} \sim e^{-m_\pi d} \quad V_{NNN} \sim e^{-2m_\pi d} \quad V_{NNNN} \sim e^{-3m_\pi d}$$

Typical NN wavelengths $\geq 1/\sqrt{m_\pi M_N} \sim 0.5\text{fm}$

→ Few wavelengths within a range
(-Coarse grained Effective interactions) SYMMETRIES
WIGNER SYMMETRY
Long wavelength limit (Short range interactions)

- $^1S_0$ and $^3S_1$ channels (Square well potential)
  \[
  \alpha_{1S0} = -23.74 \text{fm} \quad r_{1S0} = 2.75 \text{fm} \quad \alpha_{3S1} = 5.4 \text{fm} \quad r_{3S1} = 1.75 \text{fm}
  \]

\[V_{l',l}^{JS}(p', p) = M_N \int_0^\infty j_{l'}(p'r)j_l(pr)V_{l'l}^{JS}(r)r^2\]

- Wigner symmetry: $V_{1S0}(p, p') \sim V_{3S1}(p, p')$
- Exact if $\int d^3x V_{1S0}(\vec{x}) = \int d^3x V_{3S1}(\vec{x})$ but NOT if $\alpha_{1S0}, \alpha_{3S1} \to \infty$
Long wavelength limit (EFT-approach)

- Effective NN-Lagrangian
  \[ \mathcal{L} = -\frac{1}{2} C_S (N^\dagger N)^2 - \frac{1}{2} C_T (N^\dagger \vec{\sigma} N)^2 \]

- Potential in momentum space
  \[ V_0(p', p) = C_0 \theta(\Lambda - p) \theta(\Lambda - p') \]

- Lippmann-Schwinger equation
  \[ \rightarrow C_0(\Lambda) = \frac{4\pi^2 \alpha_0}{M_N} \frac{1}{1 + \Lambda \alpha_0 / \pi} \quad \alpha_0 = \alpha_1 S_0, \alpha_3 S_1 \]

- Wigner symmetry exact if \( \alpha_0 = -\infty \) [Mehen, Steward, Wise, 1999]
  BUT there is Regularization dependence
SU(4) Wigner symmetry

- Generators

\[ T^a = \frac{1}{2} \sum_A \tau^a_A, \quad \text{ISOSPIN} \]
\[ S^i = \frac{1}{2} \sum_A \sigma^i_A, \quad \text{SPIN} \]
\[ G^{ia} = \frac{1}{2} \sum_A \sigma^i_A \tau^a_A, \quad \text{GAMOW – TELLER} \]

- Casimir operator (two body)

\[ C_{SU(4)} = T^a T_a + S^i S_i + G^{ia} G_{ia} , \]

- Irreducible representations \((\lambda, \mu, \nu)\)

\[ C_{SU(4)} = \mu(\mu + 4) + \nu(\nu + 2) + \lambda^2 \]
Selection rules in Gamow-Teller weak decays between supermultiplets

\[ \langle \lambda \mu \nu | G^{ia} | \lambda' \mu' \nu' \rangle = 0 \]

SU(4) mass formula [Franzini+Radicatti 63]

\[ E = c_1 A(A + 1) + c_2 \left[ \mu(\mu + 4) + \nu(\nu + 2) + \lambda^2 - \frac{15}{4} A \right] \]

Anomalously large double binding energy for even-even \( N = Z \) nuclei [Van Isaacker, Warnerr, Brenner, 1995].

SU(4) inequalities for nuclei on the lattice [Chen, Lee, Schaffer 2004]

SU(4) is crucial for Nuclear Lattice importance sampling [Lee, Epelbaum, 2008]
One nucleon state

\[ 4 = (p^\uparrow, p^\downarrow, n^\uparrow, n^\downarrow) = (S = 1/2, T = 1/2) \quad \text{Quartet} \]

Two nucleon states

\[ C_{SU(4)}^{ST} = \frac{1}{2} (\sigma + \tau + \sigma \tau) + \frac{15}{2} , \]

\[ \tau = \tau_1 \cdot \tau_2 = 2T(T+1) - 3 \]

\[ \sigma = \sigma_1 \cdot \sigma_2 = 2S(S+1) - 3 \]

Sextet and decuplet \((-1)^{S+L+T} = -1\)

\[ 6_A = (1,0) \oplus (1,0) \quad L = 0, 2, \ldots \quad (^1S_0, ^3S_1), (^1D_2, ^3D_{1,2,3}) \]

\[ 10_S = (0,0) \oplus (1,1) \quad L = 1, 3, \ldots \quad (^1P_1, ^3P_{0,1,2}) \]

At large \(N_c\) Wigner symmetry holds ONLY for even-L channels [Kaplan,Savage,Manohar,97]

This is confirmed by the data !! [Calle Cordon,Ruiz Arriola,08]
EFFECTIVE INTERACTIONS
Standard ways to coarse grain

- **Nuclear shell model (energy)**
  \[ \hbar \omega = 41 A^{-\frac{1}{3}} \text{MeV} \quad b = 1 A^{\frac{1}{6}} \text{fm} \]

- **\( V_{\text{low},k} \) (momentum)** [Kuo,Brown,Holt,Bogner,2000]
  \[ \Lambda_{\text{CM}} = 450 \text{MeV} \quad \pi / \Lambda = 1.5 \text{fm} \]

- **Nuclear lattice (space)** [D. Lee]
  \[ a = 2 \text{fm} \quad k \leq \pi / a = 300 \text{MeV} \]

- **Coarse grained delta-shells** [Navarro,Amaro,Ruiz Arriola,2011]
  \[ \Delta r \sim 1 / \sqrt{M_N m_\pi} \quad n \Delta r \sim 1 / m_\pi \]
Unitary transformation with generator $\eta_s \ H_s = e^{\eta_s H} e^{-\eta_s}$

Running Hamiltonian (SRG trajectory)

$$\frac{dH_s}{ds} = [\eta_s, H_s],$$

(1)

Initial condition $H_{s=0} = H$

Wegner generator

$$\eta_s = [D(H_s), H_s]$$

Glazek-Wilson

$$\eta_s = [T, H_s]$$
SRG long distance symmetries

- Operator evolution

\[
\frac{dO_s}{ds} = [[T, V_s], O_s]. \tag{2}
\]

- Symmetry group generator \(X\)

\[
\frac{dX_s}{ds} = [[T, V_s], X_s]. \tag{3}
\]

Jacobi’s identity \([[A, B], C] + [[C, A], B] + [[B, C], A] = 0 \]

\[
\frac{dX_s}{ds} = -[[X_s, T], V_s] - [[V_s, X_s], T] = 0. \tag{4}
\]

Long distance symmetry is a fixed-point of the SRG evolution
Momentum space representation

\[
\frac{dV_s(\vec{p}', \vec{p})}{ds} = -(E_p - E'_p)^2 V_s(\vec{p}', \vec{p}) \\
+ \int \frac{d^3 q}{(2\pi)^3} (E_p + E'_p - 2E_q) V_s(\vec{p}', \vec{q}) V_s(\vec{q}, \vec{p}). \tag{5}
\]

At large momenta (off-diagonal suppression)

\[
V_s(\vec{p}', \vec{p}) \approx V_{s=0}(\vec{p}', \vec{p}) e^{-(E_p' - E_p)^2 s} + \ldots \tag{6}
\]

Zero momentum states

\[
\frac{dV_s(\vec{0}, \vec{0})}{ds} = -2 \int \frac{d^3 q}{(2\pi)^3} E_q \langle \vec{q} | V_s V_s^\dagger | \vec{q} \rangle \leq 0, \tag{7}
\]

Momentum grid \( p_{\text{max}} = 5000 \text{MeV} \) and \( N = 250 \) points
General Structure of the potential

Momentum space

\[ V(\vec{p}', \vec{p}) = V_C + \vec{\tau}_1 \cdot \vec{\tau}_2 \, W_C + (V_S + \vec{\tau}_1 \cdot \vec{\tau}_2 \, W_S) \, \vec{\sigma}_1 \cdot \vec{\sigma}_2 \]
\[ + \left( V_{LS} + \vec{\tau}_1 \cdot \vec{\tau}_2 \, W_{LS} \right) i(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{p}' \times \vec{p}) \]
\[ + \left( V_T + \vec{\tau}_1 \cdot \vec{\tau}_2 \, W_T \right) S_{12}(\vec{p}' - \vec{p}) \]
\[ + \left( V_Q + \vec{\tau}_1 \cdot \vec{\tau}_2 \, W_Q \right) S_{12}(\vec{p}' \times \vec{p}) \]
\[ + \left( V_P + \vec{\tau}_1 \cdot \vec{\tau}_2 \, W_P \right) S_{12}(\vec{p}' + \vec{p}) \],

where the tensor operator is defined as

\[ S_{12}(\vec{q}) = \left[ \vec{\sigma}_1 \cdot \vec{q} \, \vec{\sigma}_2 \cdot \vec{q} - \frac{1}{3} q^2 \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right], \quad (8) \]

Pauli principle

\[ V_{1',2';1,2}(\vec{p}', \vec{p}) = -V_{2',1';1,2}(-\vec{p}', \vec{p}) \]
Partial wave decomposition

\[
\langle \vec{p}' | V^S_\lambda | \vec{p} \rangle = N \sum_{J'LL'} Y_{J'LS} (\hat{p}') V_{IL'}(p', p) Y_{L'S} (\hat{p}) \tag{9}
\]

, Coupled-channel equations,

\[
\frac{dV_s(p, p')}{ds} = -(p^2 - p'^2)^2 V_s(p, p') \\
+ \frac{2}{\pi} \int_0^\infty dq \ q^2 \ (p^2 + p'^2 - 2q^2) \ V_s(p, q) \ V_s(q, p'), \tag{10}
\]

where \( V_s(p, p') \) is used as a brief notation for the projected NN potential matrix elements,
\( V_{\text{low}k} \rightarrow \) Scattering reproduced until the cut-off.

\[
\delta_{\text{low}k}(k, \Lambda) = \delta(k)\theta(\Lambda - k)
\]

\( V_{\text{SRG}} \) Scattering reproduced at ALL energies.

\[
\delta_{\text{SRG}}(k, \lambda) = \delta(k)
\]
Infrared fixed points of SRG

- Fixed points for discretized equations
  \[
  \frac{dV_{ik}}{ds} = -(E_i - E_k)^2 V_{ik} + \sum_k (E_i + E_k - 2E_l) V_{il} V_{lk} = 0 \tag{11}
  \]

- Diagonal potential in momentum space basis
  \[
  V_{\alpha\beta} = v_\alpha \delta_{\alpha\beta}
  \]

- \( R \)-matrix
  \[
  R(p', p; k) = V(p', p) + \frac{2}{\pi} P \int_0^\infty q^2 dq \frac{V(p', q) R(q, p; k)}{k^2 - q^2}
  \]

- Excluding the mean value
  \[
  R(p, p; p) = -\frac{\tan \delta(p)}{p} = \lim_{\lambda \to 0} V_\lambda(p, p)
  \]
Sum rules

\[
V_{3P_C} = \frac{1}{9} \left( V_{3P_0} + 3 V_{3P_1} + 5 V_{3P_2} \right), \quad (12)
\]

\[
V_{3P_T} = -\frac{5}{72} \left( 2 V_{3P_0} - 3 V_{3P_1} + V_{3P_2} \right), \quad (13)
\]

\[
V_{3P_{LS}} = -\frac{1}{12} \left( 2 V_{3P_0} + 3 V_{3P_1} - 5 V_{3P_2} \right), \quad (14)
\]

for triplet D-waves

\[
V_{3D_C} = \frac{1}{15} \left( 3 V_{3D_1} + 5 V_{3D_2} + 7 V_{3D_3} \right), \quad (15)
\]

\[
V_{3D_T} = -\frac{7}{120} \left( 3 V_{3D_1} - 5 V_{3D_2} + 2 V_{3D_3} \right), \quad (16)
\]

\[
V_{3D_{LS}} = -\frac{1}{60} \left( 9 V_{3D_1} + 5 V_{3D_2} - 14 V_{3D_3} \right), \quad (17)
\]
for triplet F-waves

\[ V_{3F_C} = \frac{1}{21} \left( 5V_{3F_2} + 7V_{3F_3} + 9V_{3F_4} \right), \quad (18) \]
\[ V_{3F_T} = -\frac{5}{112} \left( 4V_{3F_2} - 7V_{3F_3} + 3V_{3F_4} \right), \quad (19) \]
\[ V_{3F_{LS}} = -\frac{1}{168} \left( 20V_{3F_2} + 7V_{3F_3} - 27V_{3F_4} \right), \quad (20) \]

and for triplet G-waves

\[ V_{3G_C} = \frac{1}{27} \left( 7V_{3G_3} + 9V_{3G_4} + 11V_{3G_5} \right), \quad (21) \]
\[ V_{3G_T} = -\frac{77}{2160} \left( 5V_{3G_3} - 9V_{3G_4} + 4V_{3G_5} \right), \quad (22) \]
\[ V_{3G_{LS}} = \frac{1}{360} \left( -35V_{3G_3} - 9V_{3G_4} + 44V_{3G_5} \right). \quad (23) \]
Wigner and Serber Symmetries

- Wigner symmetry
  
  \[ V_{1S_0} = V_{3S_C}, \quad V_{1D_2} = V_{3D_C}, \quad V_{1G_4} = V_{3G_C}. \]  
  \[ (24) \]

- Serber symmetry
  
  \[ 0 = V_{3P_C} = V_{3F_C} = V_{3H_C} = \ldots \]  
  \[ (25) \]

- The Wigner SRG scale:
  
  \[ \lambda_{\text{Wigner}} = 3 \text{fm}^{-1} \]

- Symmetry pattern consistent with large \( N_c \)
BINDING ENERGIES
Mean field Slater Determinant

\[ \psi(\vec{p}_1, \ldots, \vec{p}_A) = A \left[ \phi_{n_1, l_1, s, m_{s1}, t, m_{t1}}(\vec{p}_1) \cdots \phi_{n_A, l_A, s, m_{sA}, t, m_{tA}}(\vec{p}_A) \right] \quad (26) \]

Single particle states (Harmonic oscillator)

\[ P_{nl}(p) = N_{nl} e^{-\frac{1}{2} b^2 p^2} (bp)^l L_{n-1}^{l+\frac{1}{2}}(b^2 p^2) \quad (27) \]

Two body interaction (Talmi-Moshinsky)

\[ \langle V_2 \rangle_A = \sum_{nlJS} g_{nlJS} \langle nl|V^{JST}|nl\rangle, \quad (28) \]

Nuclei: Shell model (mean field)

\[ d : (1s)^2 \quad t : (1s)^3 \quad ^4\text{He} : (1s)^4, \]
\[ ^{16}\text{O} : (1s)^4(1p)^{12} \quad ^{40}\text{Ca} : (1s)^4(1p)^{12}(2s)^4(1d)^{20} \]
Binding Energies - Nijll

![Graphs showing binding energies for various nuclei](image)

- Binding Energies - Nijll
- AFDMC
- nuclear matter

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SRG Symmetries
Binding Energies - N3LO

- $B \text{(MeV)}$ vs. $b \text{(fm)}$
- $B \text{(MeV)}$ vs. $b_{\text{rms}} \text{(fm)}$
- $B / A \text{(MeV)}$ vs. $r_{\text{rms}} \text{(fm)}$
- $B / A \text{(MeV)}$ vs. $k_F \text{(fm^{-1})}$

For each nucleus: $d$, $^3\text{H}$, $^4\text{He}$, $^{16}\text{O}$, $^{40}\text{Ca}$, and nuclear matter.

- $\lambda = \infty$
- $\lambda = 3 \text{ fm}^{-1}$
- $\lambda = 2 \text{ fm}^{-1}$
- $\lambda = 1 \text{ fm}^{-1}$

Exp, CC, BHF, GFMC, UCDM.
From a variational point of view: GFM=SRG+Shell model

\[ B_{4\text{He}} = -24 \text{MeV} \]

\[
\min_\psi \langle \Psi | H_{\text{Av18}}^{\lambda=\infty} | \Psi \rangle = \min_b \langle (1s)^4 | H_{\text{Av18}}^{\lambda=\text{fm}^{-1}} | (1s)^4 \rangle = \min_b \langle (1s)^4 | H_{\text{Nijm}}^{\lambda=\text{fm}^{-1}} | (1s)^4 \rangle = \min_b \langle (1s)^4 | H_{\text{N3LO}}^{\lambda=\text{fm}^{-1}} | (1s)^4 \rangle
\]

Core + TPE are marginal for \(^4\text{He}\). Other nuclei?

SRG Wigner scale \( \lambda_{\text{Wigner}} = 3 \text{fm}^{-1} \) implies large 3-body and 4-body forces

Are the N-body forces SU(4)-invariant?
CONCLUSIONS
Wigner symmetry looks very different when finite range forces are considered.

In $V_{\text{lowk}}$ Wigner symmetry is saturated, i.e. it holds above a cut-off.

In SRG Wigner symmetry happens at one SRG cut-off.