A Limit Formula for Joint Spectral Radius with $p$-radius of Probability Distributions

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Abstract

In this paper we show a characterization of the joint spectral radius of a set of matrices as the limit of the $p$-radius of an associated probability distribution when $p$ tends to $\infty$. Allowing the set to have infinitely many matrices, the obtained formula extends the results in the literature. Based on the formula, we then present a novel characterization of the stability of switched linear systems for an arbitrary switching signal via the existence of stochastic Lyapunov functions of any higher degrees. Numerical examples are presented to illustrate the results.

Keywords: Joint spectral radius, $p$-radius, Lyapunov functions, absolute exponential stability

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1. Introduction

The joint spectral radius of a set of matrices, originally introduced in the short note [1], is a natural extension of the spectral radius of a single matrix and has found various applications in, for example, wavelet theory, functional analysis, and systems and control theory (see the monograph [2] for detail). This wide range of applications has motivated many authors to study the computation of joint spectral radius. Though even the approximation of joint spectral radius is in general an NP-hard problem [3], there are now a vast

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amount of efficient methods for the approximation of joint spectral radius [4, 5, 6] and also their implementations on mathematical softwares [7].

The result [4] by Blondel and Nesterov is of a particular theoretical interest because it characterizes joint spectral radius as the limit of another joint spectral characteristics called $L^p$-norm joint spectral radius when $p$ tends to $\infty$. Given a finite set $\mathcal{M} = \{A_1, \ldots, A_N\}$ of real and square matrices of a fixed dimension and a parameter $p \geq 1$, the $L^p$-norm joint spectral radius ($p$-radius for short) of $\mathcal{M}$ is defined by

$$
\rho_{p, \mathcal{M}} := \lim_{k \to \infty} \left( \sum_{i_1, \ldots, i_k \in \{1, \ldots, N\}} \|A_{i_k} \cdots A_{i_1}\|^p \right)^{1/kr},
$$

where $\|\cdot\|$ denotes any matrix norm. Firstly introduced [8, 9] for $p = 1$ and then extended [10] for a general $p$, $L^p$-norm joint spectral radius has found many applications in various areas of applied mathematics (see [11] and references therein). In particular $p$-radius has an application to the stability theory of stochastic switched systems [12, 13, 14], which is a dynamical system whose structure randomly experiences abrupt changes [15, 16].

Recently this “original” version of $L^p$-norm joint spectral radius was extended to probability distributions [13] (see Definition 3.3 for detail). Roughly speaking, the extension makes it possible to consider the $p$-radius of an infinite set of matrices and is useful when, for example, one wants to study the stability of a stochastic switched system with infinitely many subsystems that naturally arise as a result of uncertainty in modeling of dynamical systems. Being an extension, the $p$-radius of distributions inherits [13] from the $p$-radius of sets of matrices the characterization [10] as the spectral radius of a matrix. Though the characterization is valid only either when $p$ is an even integer or when matrices in $\mathcal{M}$ leave a common proper cone invariant, it still covers several interesting cases that appear in the stability analysis of stochastic switched linear systems. Then it is natural to expect that the other properties of the $p$-radius of sets of matrices can be extended to the $p$-radius of distributions.

In this paper we show that the characterization by Blondel and Nesterov [4] is still valid when we use the $p$-radius of probability distributions. This extension in particular circumvents the finiteness limitation of the original characterization. Since the proof for the original result relies on the finiteness of the number of matrices, it cannot be directly applied to the current setting. Instead, our proof utilizes nearly most unstable trajectories
of an associated switched linear system and is independent of the number of possible matrices.

As a theoretical application of the characterization of joint spectral radius, we will discuss the stability of switched linear systems. We will present a novel characterization of the stability of a switched linear system for an arbitrary switching signal with a so-called stochastic Lyapunov function [17, 18, 19], which is a positive definite functional whose value decreases along the trajectory of the switched linear system in expectation. The characterization in particular deduces the existence of a stochastic Lyapunov function from stability and hence is a variant of the converse Lyapunov theorems [20, 21] in systems and control theory. The construction of stochastic Lyapunov functions is also investigated.

This paper is organized as follows. After preparing necessary notations in Section 2, in Section 3 we give a brief overview of the joint spectral radius of sets of matrices and the \( L^p \)-norm joint spectral radius of probability distributions. Then Section 4 gives the characterization of joint spectral radius as the limit of \( L^p \)-norm joint spectral radius. In Section 5 we discuss the application of the characterization to the stability theory of switched linear systems.

2. Mathematical Preliminaries

For \( x \in \mathbb{R}^n \) its Euclidean norm is denoted by \( \|x\| \). The usual inner product in \( \mathbb{R}^n \) is denoted by \( \langle \cdot, \cdot \rangle \). For a real matrix \( A \) its maximal singular value is denoted by \( \|A\| \). If \( A \) is square then its spectral radius is denoted by \( \rho(A) \). When \( A \) is symmetric and negative semidefinite we write \( A \preceq 0 \). Let \( \mathcal{M} \subset \mathbb{R}^{n \times n} \). The interior and the boundary of \( \mathcal{M} \) are denoted by \( \text{int} \mathcal{M} \) and \( \partial \mathcal{M} \), respectively. The distance between \( A \) and \( \mathcal{M} \) is defined by \( d(A, \mathcal{M}) := \inf_{M \in \mathcal{M}} \|A - M\| \).

Let \( \Omega \) be a probability space with a probability measure \( \mu \). The support of \( \mu \), denoted by \( \text{supp} \mu \), is defined as the closed set such that \( \mu((\text{supp} \mu)^c) = 0 \) and, if \( G \) is open and \( G \cap (\text{supp} \mu) \neq \emptyset \), then \( \mu(G \cap \text{supp} \mu) > 0 \). Dirac’s delta distribution on \( x \in \Omega \) is denoted by \( \delta_x \). For an integrable random variable \( X \) on \( \Omega \) its expected value is denoted by \( E[X] \). The next proposition is fundamental.

**Lemma 2.1** ([22]). The boundary of a convex set in \( \mathbb{R}^{n \times n} \) is a null set with respect to the Lebesgue measure.
2.1. Proper Cones

A subset \( K \subset \mathbb{R}^n \) is called a cone if \( K \) is closed under multiplication by nonnegative numbers. The cone is said to be solid if it possesses a nonempty interior. We say that a cone is pointed if \( x, -x \in K \) implies \( x = 0 \). We say that \( K \) is proper if it is closed, convex, solid, and pointed. Let \( K \subset \mathbb{R}^n \) be a proper cone. A matrix \( A \in \mathbb{R}^{n \times n} \) is said to leave \( K \) invariant or to be \( K \)-nonnegative, written \( A \geq K 0 \), if \( AK \subset K \). The set of all matrices leaving \( K \) invariant is denoted by \( \pi(K) \) or simply by \( \pi \). Let \( B \in \mathbb{R}^{n \times n} \). By \( A \geq K B \) we mean \( A - B \geq K 0 \). A set \( M \subset \mathbb{R}^{n \times n} \) is said to leave \( K \) invariant if any \( A \in M \) leaves \( K \) invariant. \( A \) is said to be \( K \)-positive if \( A(K - \{0\}) \subset int K \) and we write \( A > K 0 \). It is known [23]

\[
\text{int } \pi = \{ A \in \mathbb{R}^{n \times n} : A > K 0 \}. \tag{2}
\]

The dual cone \( K^* \) is defined by \( K^* = \{ f \in \mathbb{R}^n : f^\top x \geq 0 \text{ for every } x \in K \} \).

A norm \( \| \cdot \| \) on \( \mathbb{R}^n \) is said to be cone absolute [24] with respect to a proper cone \( K \) if, for every \( x \in \mathbb{R}^n \),

\[
\| x \| = \inf_{v, w \in K, x = v - w} \| v + w \|. \tag{3}
\]

Also we say that \( \| \cdot \| \) is cone linear with respect to \( K \) if there exists \( f \in K^* \) such that

\[
\| x \| = f^\top x \tag{4}
\]

for every \( x \in K \). A norm that is cone linear and cone absolute with respect to a common proper cone is said to be cone linear absolute. It is known [24] that every \( f \in \text{int}(K^*) \) yields a cone linear absolute norm \( \| \cdot \|_f \) determined by (3) and (4). The norm \( \| \cdot \|_f \) induces a norm on \( \mathbb{R}^{n \times n} \) as

\[
\| A \|_f := \sup_{x \in \mathbb{R}^n} \frac{\| Ax \|_f}{\| x \|_f}. \tag{5}
\]

Some useful properties of this norm are collected in the next lemma.

**Lemma 2.2** ([24]). Let \( K \) be a proper cone and let \( \| \cdot \|_f \) denote the norm on \( \mathbb{R}^{n \times n} \) given by (5) for some \( f \in \text{int}(K^*) \).

1. If \( A \geq K 0 \) then

\[
\| A \|_f = \sup_{x \in K} \frac{\| Ax \|}{\| x \|}. \tag{6}
\]
2. If \( A_i \geq^K B_i \geq^K 0 \) for every \( i = 1, \ldots, k \) then

\[
\| A_k \cdots A_1 \|_f \geq \| B_k \cdots B_1 \|_f.
\] (7)

**Proof.** The first statement can be found in [24]. The second one is also proved in [24] when \( k = 1 \). Then the general case follows from the obvious relationship \( A_k \cdots A_1 \geq^K B_k \cdots B_1 \geq^K 0 \). \( \square \)

### 2.2. Lifts and Kronecker Products

Let \( p \geq 1 \) be an integer and let \( x \in \mathbb{R}^n \). The \( p \)-lift of \( x \), denoted by \( x^{[p]} \), is defined \([5]\) as the real vector of length \( n_p = \binom{n+p-1}{p} \) with its elements being the lexicographically ordered monomials \( \sqrt{\alpha!} x^\alpha \) indexed by all the possible exponents \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \{0,1,\ldots,p\}^n \) such that \( \alpha_1 + \cdots + \alpha_n = p \), where \( \alpha! := \frac{p!}{\alpha_1! \cdots \alpha_n!} \). For \( A \in \mathbb{R}^{n \times n} \) we define the matrix \( A^{[p]} \) as the unique matrix \([2]\) satisfying \((Ax)^{[p]} = A^{[p]}x^{[p]}\) for every \( x \in \mathbb{R}^n \). For a subset \( M \) of either \( \mathbb{R}^n \) or \( \mathbb{R}^{n \times n} \) we define \( M^{[p]} = \{ M^{[p]} : M \in M \} \). Also for real matrices \( A \) and \( B \), \( A \otimes B \) denotes the Kronecker product \([25]\) of \( A \) and \( B \). Define the Kronecker power \( A \otimes^p \) by \( A \otimes^1 := A \) and \( A \otimes^{(p-1)} \otimes A \) recursively for a general \( p \). We define \( M \otimes^p := \{ M \otimes^p : M \in M \} \). It is known that if \( AB \) is defined then

\[
(AB)^{\otimes^p} = A^{\otimes^p}B^{\otimes^p}.
\] (8)

The next lemma collects some properties of \( p \)-lifts and Kronecker products \([4]\).

**Lemma 2.3** \([4]\). Let \( M \subset \mathbb{R}^{n \times n} \).

1. If \( M \) leaves a proper cone invariant then \( M^{\otimes^p} \) also leaves a proper cone invariant for every \( p \geq 1 \).

2. \( M^{[2]} \) leaves a proper cone invariant.

For a probability distribution \( \mu \) on \( \mathbb{R}^{n \times n} \) we define the probability distribution \( \mu^{\otimes^p} \) on \( \mathbb{R}^{n^p \times n^p} \) as the image \([26]\) of \( \mu \) under the measurable mapping \( (\cdot)^{\otimes^p} : \mathbb{R}^{n \times n} \to \mathbb{R}^{n^p \times n^p} \). Let \( f \) be a measurable function on \( \mathbb{R}^{n^p \times n^p} \). If \( A \) and \( B \) are independent random variables following \( \mu \) and \( \mu^{\otimes^p} \), respectively, then

\[
E[f(B)] = E[f(A^{\otimes^p})].
\] (9)
3. Joint Spectral Characteristics

This section briefly overviews the notions of joint spectral radius and $L^p$-norm joint spectral radius. The joint spectral radius \([2]\) of a compact set $M \subset \mathbb{R}^{n \times n}$ is defined by

$$\hat{\rho}(M) := \limsup_{k \to \infty} \sup_{A_1, \ldots, A_k \in M} \|A_k \cdots A_1\|^{1/k}.$$  

One of the important applications of joint spectral radius is in the stability theory of switched linear systems \([15]\). Define the switched linear system $\Sigma_M$ by

$$\Sigma_M : x(k + 1) = A_k x(k), \quad A_k \in M$$

where $x(0) = x_0 \in \mathbb{R}^n$ is a constant vector. We say that $\Sigma_M$ is absolutely exponentially stable \([27]\) if there exist $C > 0$ and $\gamma \in [0, 1)$ such that $\|x(k)\| \leq C\gamma^k \|x_0\|$ for every $M$-valued sequence $\{A_k\}_{k=0}^\infty$ and $x_0$. This stability is characterized by joint spectral radius as follows (see, e.g., \([2]\)).

**Proposition 3.1.** $\Sigma_M$ is absolutely exponentially stable if and only if $\hat{\rho}(M) < 1$.

The following lemma lists some other properties of joint spectral radius. To state the lemma we recall that the set $\mathcal{K}(\mathbb{R}^{n \times n})$ of compact and nonempty subsets of $\mathbb{R}^{n \times n}$ becomes a complete metric space \([28]\) if it is endowed with the Hausdorff metric given by

$$H(M, N) := \max \left\{ \max_{A \in M} d(A, N), \max_{B \in N} d(B, M) \right\}. \quad (10)$$

**Lemma 3.2.**

1. The restriction of the mapping $\hat{\rho}$ to the metric space $\mathcal{K}(\mathbb{R}^{n \times n})$ is continuous \([28]\).

2. It holds \([2]\) that, for any $p \geq 1$,

$$\hat{\rho}(M[p]) = \hat{\rho}(M)^p. \quad (11)$$

Then we turn to the $L^p$-norm joint spectral radius of probability distributions. Let $\mu$ be a probability distribution on $\mathbb{R}^{n \times n}$ and let $A$ and $A_k$ ($k = 0, 1, 2, \ldots$) be random variables independently following $\mu$. Also let $p$ be a positive integer.
Definition 3.3 ([13]). The $L^p$-norm joint spectral radius ($p$-radius for short) of $\mu$ is defined by

$$\rho_{p,\mu} := \lim_{k \to \infty} \left( E[\|A_k \cdots A_1\|^p]\right)^{1/pk}. \quad (12)$$

This definition extends the $L^p$-norm joint spectral radius of a set of finitely many matrices shown in (1). One can check that if $\text{supp} \mu$ is compact then $\rho_{p,\mu}$ exists and is finite [13]. Thus, without being explicitly stated, we assume that probability distributions appearing in this paper have a bounded support. Though in general the computation of $p$-radius is a difficult problem [11], a simple formula for $p$-radius is available under certain assumptions [10, 13].

Proposition 3.4 ([10, 13]). Assume that one of the following conditions is true:

A$_1$. $p$ is even;

A$_2$. $\text{supp} \mu$ leaves a proper cone invariant.

Then

$$\rho_{p,\mu} = \rho(E[A^{\otimes p}])^{1/p}. \quad (13)$$

The next lemma collects some other properties of $p$-radius.

Lemma 3.5 ([13]). Let $p \geq 1$ be arbitrary.

1. If $p \leq q$ then $\rho_{p,\mu} \leq \rho_{q,\mu}$.

2. For any probability distribution $\mu$,

$$\rho_{p,\mu} \geq \rho(E[A^{\otimes p}])^{1/p}. \quad (14)$$

3. For every $m \geq 1$,

$$\rho_{p,\mu^{\otimes m}} = \rho_{p,\mu^{[m]}} = \rho_{mp,\mu}^m. \quad (15)$$

Proof. The first two statements can be found in [13]. The last statement can be proved in the same way as (11).

Remark 3.6. By the equivalence of the norms on a finite dimensional vector space, the value of $L^p$-norm joint spectral radius is independent of the norm used in (12).
4. Limit Formula for Joint Spectral Radius

This section presents a novel limit formula for joint spectral radius. We state the next assumption on a probability distribution \( \mu \) on \( \mathbb{R}^{n \times n} \).

A3. The singular part \( \mu_s \) of \( \mu \) is a linear combination of finitely many Dirac measures, i.e., either \( \mu_s = 0 \) or there exist positive numbers \( p_1, \ldots, p_N \) and matrices \( M_1, \ldots, M_N \) such that

\[
\mu_s = p_1 \delta_{M_1} + \cdots + p_N \delta_{M_N}.
\]

(16)

Notice that any of the assumptions from A1 to A3 does not require \( \mu \) to have a finite support.

The next theorem is the main result of this paper, which extends the characterization of the joint spectral radius of finite sets of matrices [4] to general compact sets.

**Theorem 4.1.** Let \( \mu \) be a probability distribution satisfying A2 and A3 and let \( M = \text{supp} \mu \). Then

\[
\hat{\rho}(M) = \lim_{p \to \infty} \rho \left( E[A^\otimes p] \right)^{1/p}.
\]

As a simple illustration of the theorem let us see the next example.

**Example 4.2.** Let \( \gamma > 0 \) and let \( \mu \) be the uniform distribution on \([0, \gamma]\). Clearly \( \mu \) is absolutely continuous and \( \mathcal{M} = \text{supp} \mu = [0, \gamma] \) leaves the proper cone \( \{ r : r \geq 0 \} \) of \( \mathbb{R} \) invariant. Now it is easy to observe \( \hat{\rho}(\mathcal{M}) = \gamma \) and \( \rho(E[A^\otimes p]) = \gamma^p / (p + 1) \). Therefore \( \lim_{p \to \infty} \rho(E[A^\otimes p]) = \gamma = \hat{\rho}(\mathcal{M}) \), as expected. The characterization in [4] cannot be applied to this simple example as \( \mu \) has an infinite support.

In the rest of this section we give the proof of Theorem 4.1. The proof relies on two propositions, each of which is preceded by a lemma. When \( \mu \) satisfies A3 we can uniquely decompose \( \mu \) as

\[
\mu = \mu_c + \mu_s,
\]

(17)

where \( \mu_c \) is an absolutely continuous measure and \( \mu_s \) is either the zero measure or is of the form (16). We write \( \mathcal{M} := \text{supp} \mu, \mathcal{M}_c := \text{supp} \mu_c, \) and
$${\mathcal{M}}_s := \text{supp}\mu_s = \{M_1, \ldots, M_N\}.$$ Clearly $${\mathcal{M}} = {\mathcal{M}}_c \cup {\mathcal{M}}_s.$$ For a proper cone $K$ of $\mathbb{R}^n$ and $r \geq 0$ we define

$$\pi_r := \{M \in \pi : d(M, \partial\pi) \geq r\}.$$ We notice that $\pi_r \subset \text{int}\pi$ if $r > 0$. Finally we let

$${\mathcal{M}}_r := ({\mathcal{M}}_c \cap \pi_r) \cup {\mathcal{M}}_s.$$ 

**Lemma 4.3.** Let $K$ be a proper cone. If $\mu$ satisfies $A_3$ then, for every $A \in \mathcal{M}$,

$$\lim_{r \to 0} d(A, {\mathcal{M}}_r) = 0. \quad (18)$$

*Proof.* Let $A \in \mathcal{M}$ be arbitrary. Notice that the set $\mathcal{M}_r$ is increasing with respect to $r$ so that the function $d(A, \mathcal{M}_r)$ of $r$ is increasing. Therefore the limit in $(18)$ is well defined.

First we suppose $\mu_s = 0$. Let $\epsilon > 0$ be arbitrary and let $B$ denote the open ball in $\mathbb{R}^{n \times n}$ with center $A$ and radius $\epsilon$. Since $\mu(B) > 0$ and the set $\{A\} \cup \partial\pi$ has the Lebesgue measure 0 by Lemma 2.1, $B \cap \mathcal{M}$ is not contained in $\{A\} \cup \partial\pi$. Therefore we can take $B \in B \cap \mathcal{M}$ that is distinct from $A$ and is not in $\partial\pi$. Now, if $r < d(B, \partial\pi)$ then we have $d(A, \mathcal{M}_r) \leq \|A - B\| < \epsilon$ because $B \in \mathcal{M}_r$. This shows $(18)$ since $\epsilon > 0$ was arbitrary.

Then let us consider the general case when $\mu_s$ is not necessarily 0. If $A \in \mathcal{M}_c$ then, since $\mathcal{M}_r \supset \mathcal{M}_c \cap \pi_r$,

$$\lim_{r \to 0} d(A, \mathcal{M}_r) \leq \lim_{r \to 0} d(A, \mathcal{M}_c \cap \pi_r) = 0,$$

where in the last equation we applied the above argument for the special case $\mu_s = 0$ to the normalized probability measure $\mu_c / (\mu_c(\mathbb{R}^{n \times n}))$. On the other hand, if $A \in \mathcal{M}_s$ then $(18)$ clearly holds because $A$ is also in $\mathcal{M}_r$ and hence $d(A, \mathcal{M}_r) = 0$ for every $r$. This completes the proof. \(\square\)

Using this lemma we can prove the following proposition on the Hausdorff distance $(10)$ between $\mathcal{M}$ and $\mathcal{M}_r$.

**Proposition 4.4.** If $\mu$ satisfies $A_3$ then

$$\lim_{r \to 0} H(\mathcal{M}, \mathcal{M}_r) = 0. \quad (19)$$
Proof. Let us first see that the limit in (19) is well defined. Clearly $M$ is in $\mathcal{K}(\mathbb{R}^{n \times n})$ because $\mu$ is assumed to have a bounded support. Then let us show that, if $r > 0$ is sufficiently small, the set $M_r$ also is in $\mathcal{K}(\mathbb{R}^{n \times n})$. Assume the contrary, i.e., $M_r = \emptyset$ for every $r > 0$. Then it must be that $M_c \subset \partial \pi$ and $\mu_s = 0$. The latter condition shows that $\mu_c$ is nonzero. Thus the former condition shows that the nonzero and absolutely continuous measure $\mu_c$ is concentrated on the null set $\partial \pi$, which is a contradiction. Therefore the distance $H(M, M_r)$ is well defined at least for a sufficiently small $r$. Finally, since the set $M_r$ is increasing with respect to $r$, the distance $H(M, M_r)$ is decreasing with respect to $r$ so that the limit $\lim_{r \to 0} H(M, M_r)$ does exist.

Now assume $\lim_{r \to 0} H(M, M_r) > 0$ to derive a contradiction. In this case there exists $\epsilon > 0$ such that $H(M, M_r) > \epsilon$ for every $r > 0$. By the definition of the Hausdorff metric (10) this implies $\max_{A \in \mathcal{M}} d(A, M_r) > \epsilon$ because $M_r \subset \mathcal{M}$. Therefore there exists $A_r \in \mathcal{M}$ such that $d(A_r, M_r) > \epsilon$ for each $r > 0$. Now let $\{r_i\}_{i=1}^{\infty}$ be a positive sequence decreasingly converging to 0. Since $\mathcal{M}$ is compact, there exists a subsequence $\{r_{i}'\}_{i=1}^{\infty}$ of $\{r_i\}_{i=1}^{\infty}$ such that $\{A_{r_i}'\}_{i=1}^{\infty}$ converges to some $A \in \mathcal{M}$. Using the triangle inequality we can show

$$d(A, M_{r_{i}'}) \geq d(A_{r_{i}'}, M_{r_{i}'}) - d(A_{r_{i}'}, A)$$

and hence $\liminf_{r \to 0} d(A, M_r) \geq \epsilon$ but this contradicts to (18). \hfill \Box

To prove another proposition we need the next lemma.

**Lemma 4.5.** Let $K$ be a proper cone in $\mathbb{R}^n$. For $M \in \mathbb{R}^{n \times n}$ and define

$$S_M := \{A \in \mathbb{R}^{n \times n} : A \geq^K M\}.$$

Let $\mu$ be a probability distribution on $\mathbb{R}^{n \times n}$ that is absolutely continuous with respect to the Lebesgue measure. If a sequence $\{M_k\}_{k=1}^{\infty} \subset \mathbb{R}^{n \times n}$ converges to $M$ then

$$\lim_{k \to \infty} \mu(S_M \setminus S_{M_k}) = 0,$$

$$\lim_{k \to \infty} \mu(S_{M_k} \setminus S_M) = 0.$$ (20)

**Proof.** By shifting the point $M$ to the origin and also $\mu$ accordingly, without loss of generality we can assume that $M = 0$. In this case $S_M = \pi$. We
only prove the first equation (20) because the second one can be proved in a similar way. Notice that the boundary \( \partial \pi \) is a null set with respect to \( \mu \) by Lemma 2.1. Therefore it is sufficient to show that, for every \( A \in M \setminus \partial \pi \),

\[
\lim_{k \to \infty} \chi_{\pi \setminus S_{M_k}}(A) = 0, \tag{21}
\]

where \( \chi_S \) denotes the characteristic function for a set \( S \). Since \( \pi \) is closed, \( A \) is either not in \( \pi \) or in \( \text{int} \pi \). In the former case (21) clearly holds. Next assume \( A \in \text{int} \pi \). Then \( A > K_0 \) by (2) so that the origin 0 is in the open set \( G := \{ B \in \mathbb{R}^{n \times n} : B <^K A \} \). Therefore, since \( \{ M_k \}_{k=1}^\infty \) converges to 0, if \( k \) is sufficiently large then \( M_k \) is in \( G \), i.e., \( A > K M_k \) and therefore \( A \in S_{M_k} \).

Thus (21) holds.

Then we can prove the next proposition.

**Proposition 4.6.** Assume that \( \mu \) satisfies \( A_2 \) and \( A_3 \). Let \( K \) be a proper cone left invariant by \( M \). Then, for every \( \epsilon > 0 \) and \( r > 0 \), there exists \( \delta > 0 \) such that \( \mu(S(1-\epsilon)M) \geq \delta \) for every \( M \in M_r \).

**Proof.** Fix \( \epsilon > 0 \) and \( r > 0 \). Define \( \phi : M_r \to \mathbb{R} \) by \( \phi(M) := \mu(S(1-\epsilon)M) \).

First assume that \( \mu \) is absolutely continuous with respect to the Lebesgue measure, i.e., \( \mu_s = 0 \). Since \( M_r \) is compact, it is sufficient to show that \( \phi \) is continuous and positive. In order to show that \( \phi \) is continuous at \( M \in M_r \), let \( \{ M_k \}_{k=1}^\infty \) be a sequence of \( M_r \) converging to \( M \). Then we can see that

\[
|\phi(M) - \phi(M_k)| \leq \mu(S(1-\epsilon)M \setminus S(1-\epsilon)M_k) + \mu(S(1-\epsilon)M_k \setminus S(1-\epsilon)M).
\]

Therefore \( \phi \) is continuous by Lemma 4.5. Then let us show that \( \phi \) is positive at \( M \). Since \( M_r \subset \text{int} \pi \) we have \( M >^K 0 \) and therefore \( M >^K (1-\epsilon)M \). This gives \( \text{int} S(1-\epsilon)M \cap \text{supp} \mu \ni M \) and thus \( \phi(M) \geq \mu(\text{int} S(1-\epsilon)M) > 0 \).

Then we consider the general case. Decompose \( \mu \) as (17). On the one hand, from the above argument, there exists a constant \( \delta_c > 0 \) such that \( \mu_c(S(1-\epsilon)M) \geq \delta_c \) for every \( M \in M_c \cap \pi_r \). On the other hand, if \( M \in M_s \) then \( M = M_i \geq^K 0 \) for some \( 1 \leq i \leq N \). Therefore \( M_i \in S(1-\epsilon)M \) because \( M_i \geq^K (1-\epsilon)M \). Hence \( \phi(M) \geq \mu(\{ M_i \}) = p_i > 0 \). Thus we can see that \( \delta := \min(\delta_c, p_1, \ldots, p_N) > 0 \) is a desired constant. This argument is valid even when \( \mu_c = 0 \) and therefore completes the proof.

Now we are at the position to prove the main result of Theorem 4.1.
Proof of Theorem 4.1. It is sufficient to show \( \hat{\rho}(\mathcal{M}) = \lim_{p \to \infty} \rho_{p,\mu} \) by A2 and (13). First notice that the definitions of \( p \)-radius and joint spectral radius immediately show \( \rho_{p,\mu} \leq \hat{\rho}(\mathcal{M}) \). Therefore \( \lim_{p \to \infty} \rho_{p,\mu} \leq \hat{\rho}(\mathcal{M}) \) because \( \rho_{p,\mu} \) is non-decreasing with respect to \( p \) by the first statement of Lemma 3.5.

To show the opposite direction let \( r > 0 \) be arbitrary and let \( \gamma_r := \hat{\rho}(\mathcal{M}_r) \). Let \( K \) be a proper cone left invariant by \( \mathcal{M} \) and take any cone linear absolute norm \( \| \cdot \| \) with respect to \( K \). By Proposition 3.1, there exist \( C > 0 \) and \( \{M_k\}_{k=1}^{\infty} \subset \mathcal{M}_r \) such that

\[
\|M_k \cdots M_1\| > C\gamma_r^k \tag{22}
\]

for infinitely many \( k \). Take an arbitrary \( \beta_r \) such that \( \gamma_r > \beta_r \) and define \( \epsilon := (\gamma_r - \beta_r)/\gamma_r \). Let us take the corresponding \( \delta > 0 \) given by Proposition 4.6. Observe that if \( A_i \in \mathcal{S}(1-\epsilon)_{M_i} \), then \( A_i \geq (1-\epsilon)M_i = (1-\epsilon)\gamma_r M_i \). Therefore, by (7) and (22), we have

\[
\|A_k \cdots A_1\| \geq (\beta_r/\gamma_r)^k \|M_k \cdots M_1\| > C\beta_r^k.
\]

Therefore

\[
E[\|A_k \cdots A_1\|^p] > (C\beta_r^k)^p \mu^{k} \{ (A_1, \ldots, A_k) : \|A_k \cdots A_1\| > C\beta_r^k \}
\]

\[
\geq C^p \beta_r^{pk} \prod_{i=1}^{k} \mu(S(1-\epsilon)_{M_i})
\]

\[
\geq C^p \beta_r^{pk} \delta^k
\]

and hence \( E[\|A_k \cdots A_1\|^p]^{1/kp} > C^{1/k} \delta^{1/p} \beta_r \) for infinitely many \( k \). Taking the limit \( k \to \infty \) in this inequality shows \( \rho_{p,\mu} \geq \delta^{1/p} \beta_r \) by Remark 3.6. Thus we obtain \( \lim_{p \to \infty} \rho_{p,\mu} \geq \beta_r \). Since \( \beta_r \) can be made arbitrarily close to \( \gamma_r \), we see \( \lim_{p \to \infty} \rho_{p,\mu} \geq \gamma_r \), which further shows \( \lim_{p \to \infty} \rho_{p,\mu} \geq \hat{\rho}(\mathcal{M}) \) since \( \gamma_r = \hat{\rho}(\mathcal{M}_r) \to \hat{\rho}(\mathcal{M}) \) as \( r \to 0 \) by Lemma 3.2 and Proposition 4.4. This completes the proof. \( \square \)

Finally the next corollary of Theorem 4.1 generalizes another characterization [29] of joint spectral radius.

Corollary 4.7. If \( \mu \) is of the form (16) then

\[
\hat{\rho}(\mathcal{M}) = \lim_{p \to \infty} \rho(E[A^{\otimes(2p)}])^{1/(2p)} \tag{23}
\]

\[
= \limsup_{p \to \infty} \rho(E[A^{\otimes p}])^{1/p}. \tag{24}
\]
Proof. Let $K$ be the convex hull of $(\mathbb{R}^n)^2$, which is a proper cone of $\mathbb{R}^{n^2}$ by Lemma 2.3. Then $\text{supp}(\mu^2) = (\text{supp } \mu)^2$ clearly leaves $K$ invariant. Also $\mu^2$ itself is a discrete measure consisting of finitely many point masses. Therefore $\mu^2$ satisfies $A_2$ and $A_3$.

Now, (11) shows
\[
\hat{\rho}(\mathcal{M}) = \hat{\rho}(\mathcal{M}^2)^{1/2}. \tag{25}
\]
Also (15) and Proposition 3.4 yield $\rho_{p,\mu^2}^{1/2} = \rho(E[A^\otimes(2p)])^{1/(2p)}$. Therefore Theorem 4.1 shows $\hat{\rho}(\mathcal{M}^2)^{1/2} = \lim_{p \to \infty} \rho(E[A^\otimes(2p)])^{1/(2p)}$. This identity and (25) prove (23). Then let us show (24). Using the inequality (14), the monotonicity of $p$-radius (Lemma 3.5), and Proposition 3.4, we can show
\[
\rho(E[A^\otimes(2p-1)])^{1/(2p-1)} \leq \rho_{2p-1,\mu} \leq \rho_{2p,\mu} = \rho(E[A^\otimes(2p)])^{1/(2p)}.
\]
This inequality and (23) prove the equation (24). \hfill \Box

5. Lyapunov Theorem for Switched Linear Systems

As a theoretical application of Theorem 4.1, in this section we show a novel characterization of the absolute exponential stability of the switched linear system $\Sigma_M$ via so-called stochastic Lyapunov functions. We will also investigate their construction. Define the stochastic switched linear system $\Sigma_\mu$ by
\[
\Sigma_\mu : x(k+1) = A_k x(k), \ A_k \text{ follows } \mu \text{ independently}
\]
where $x(0) = x_0 \in \mathbb{R}^n$ is a constant. We say that $\Sigma_\mu$ is \textit{exponentially stable in $p$th mean} ($p$th mean stable for short) if there exist $C > 0$ and $\gamma \in [0,1)$ such that $E[\|x(k)\|^p] \leq C \gamma^k \|x_0\|^p$ for every $x_0 \in \mathbb{R}^n$. We call $\gamma$ as a \textit{growth rate of the $p$th mean}. As is expected, $p$th mean stability is closely related to $p$-radius.

\textbf{Proposition 5.1 ([13])}. $\Sigma_\mu$ is $p$th mean stable if and only if $\rho_{p,\mu} < 1$. Moreover the infimum of the growth rate of the $p$th mean equals $\rho_{p,\mu}$.

Now we introduce the notion of stochastic Lyapunov functions [17, 18, 19] for $\Sigma_\mu$. 

Definition 5.2. We say that $V : \mathbb{R}^n \to \mathbb{R}$ is a stochastic Lyapunov function of degree $p$ for $\Sigma_\mu$ if there exist positive numbers $C_1, C_2$ and $\gamma \in [0, 1)$ such that

$$C_1 \|x\|^p \leq V(x) \leq C_2 \|x\|^p \quad (26)$$

and

$$E[V(Ax)] \leq \gamma^p V(x) \quad (27)$$

for every $x \in \mathbb{R}^n$. We say that $V$ has a growth rate $\gamma$.

The next theorem is the main result of this section and provides a connection between the stability of deterministic switched linear systems and that of stochastic switched linear systems.

Theorem 5.3. Let $\mu$ be a probability distribution satisfying $A_2$ and $A_3$. Let $M := \text{supp} \mu$. Then $\Sigma_M$ is absolutely exponentially stable if and only if there exists $\gamma < 1$ such that, for every $p \geq 1$, $\Sigma_\mu$ admits a stochastic Lyapunov function of degree $p$ with growth rate at most $\gamma$.

Remark 5.4. In contrast to the well known characterization of absolute exponential stability with the existence of a single Lyapunov function called a common Lyapunov function [21], Theorem 5.3 characterizes absolute exponential stability with the existence of infinitely many stochastic Lyapunov functions. Also we notice that the above theorem deduces the existence of Lyapunov functions from the absolute exponential stability and hence can be considered as a version of converse Lyapunov theorems [21, 20] in the systems and control theory literature.

For the proof of Theorem 5.3 we will need the next proposition.

Proposition 5.5. Let $\mu$ be a probability distribution on $\mathbb{R}^{n \times n}$.

1. If $\Sigma_\mu$ admits a stochastic Lyapunov function with degree $p$ and growth rate $\gamma < 1$ then $\Sigma_\mu$ is $p$th mean stable with growth rate $\gamma$.

2. If $\Sigma_\mu$ is $p$th mean stable then, for every $\gamma \in (\rho_{p,\mu}, 1)$, $\Sigma_\mu$ admits a stochastic Lyapunov function with degree $p$ and growth rate $\gamma$.

Proof. First assume that $\Sigma_\mu$ admits a stochastic Lyapunov function $V$ with degree $p$ and growth rate $\gamma < 1$. Let $x_0 \in \mathbb{R}^n$ be arbitrary. Using an induction we can show $E[V(x(k))] \leq \gamma^p V(x_0)$. Therefore (26) shows $E[\|x(k)\|^p] \leq (C_2/C_1)\gamma^p \|x_0\|^p$. Thus $\Sigma_\mu$ is $p$th mean stable with growth rate $\gamma$. 

14
On the other hand assume that $\Sigma_\mu$ is $p$th mean stable and let $\gamma \in (\rho_{p,\mu}, 1)$ be arbitrary. We follow the construction of Lyapunov functions in [30]. Define $h_k := E[\|A_k \cdots A_1\|^p]^{1/pk}$, where $A_1, A_2, \ldots$ are random variables following $\mu$ independently. Since $h_k \to \rho_{p,\mu}$ as $k \to \infty$, there exists $k_0$ such that $h_{k_0} \leq \gamma$. Define $V : \mathbb{R}^n \to \mathbb{R}$ by

$$V(x) := \sum_{k=0}^{k_0-1} \frac{E[\|A_k \cdots A_1\|^p]}{\gamma^{pk}},$$

where the product $A_k \cdots A_1$ is understood to be the identity matrix with probability one when $k = 0$. Let us show that $V$ is a stochastic Lyapunov function for $\Sigma_\mu$ with degree $p$ and growth rate $\gamma$. It is immediate to see that $\|x\|^p \leq V(x) \leq \left(\sum_{k=0}^{k_0-1} \frac{h_k}{\gamma^{pk}}\right)^p \|x\|^p$, where the first inequality can be obtained by truncating the series in (28) at $k = 0$. Moreover the independence of the random variables $A_k$ yields

$$E[V(Ax)] = \sum_{k=0}^{k_0-1} \frac{E[\|A_{k+1}A_k \cdots A_1\|^p]}{\gamma^{pk}} = \gamma^p \sum_{k=1}^{k_0} \frac{E[\|A_k \cdots A_1\|^p]}{\gamma^{pk}}.$$  

Since the last term of this sum can be bounded as

$$\frac{E[\|A_{k_0} \cdots A_1\|^p]}{\gamma^{p(k+1)}} \leq \frac{h_{k_0}^{pk}}{\gamma^{pk}} \leq \|x\|^p,$$

the equation (29) shows $E[V(Ax)] \leq \gamma^p V(x)$. This completes the proof of the proposition. \qed

Remark 5.6. When $\mu$ is the uniform distribution on a finite set, Proposition 5.5 can be seen to be true by just taking the ($\epsilon$-)extremal norms of a finite set of matrices studied in [31].

Now we prove Theorem 5.3.

Proof of Theorem 5.3. Assume that $\Sigma_\mathcal{M}$ is absolutely exponentially stable. Then there exist $C > 0$ and $0 \leq \gamma' < 1$ such that $\|x(k)\| < C\gamma'^k \|x_0\|$ for any choice of the sequence $\{A_k\}_{k=0}^\infty \subset \mathcal{M}$ and $x_0$. Now let $\gamma \in (\gamma', 1)$ be arbitrary and let us fix $p \geq 1$. Since $E[\|x(k)\|^p] < C^p \gamma^{pk}$, the system $\Sigma_\mu$
is $p$th mean stable with growth rate $\gamma'$. Therefore, by Proposition 5.5, $\Sigma_\mu$ admits a stochastic Lyapunov function with degree $p$ and growth rate $\gamma$.

On the other hand assume that there exists $\gamma < 1$ such that, for every $p \geq 1$, $\Sigma_\mu$ admits a stochastic Lyapunov function of degree $p$ with growth rate $\gamma$. By Proposition 5.5, $\Sigma_\mu$ is $p$th mean stable with growth rate $\gamma$, which further implies $\rho_{p,\mu} \leq \gamma$ by Proposition 5.1. Therefore $\lim_{p \to \infty} \rho_{p,\mu} \leq \gamma < 1$. Hence, by Theorem 4.1, we obtain $\hat{\rho}(\mathcal{M}) < 1$, which gives the absolute exponential stability of $\Sigma_{\mathcal{M}}$ by Proposition 3.1. \hfill \Box

5.1. Construction of Stochastic Lyapunov Functions

The realization (28) of a stochastic Lyapunov function as a sum involving several expected values of products of matrices is not useful in practice. In this section we will show that, if either the conditions $A_1$ or $A_2$ in Proposition 3.4 holds, then we can construct stochastic Lyapunov functions easily.

The next theorem covers the case when $A_1$ holds.

**Theorem 5.7.** Assume that $\Sigma_\mu$ is $p$th mean stable and $A_1$ holds. Let $q := p/2$ and let $\gamma \in (\rho_{p,\mu}, 1)$ be arbitrary. Then the function

$$V(x) = (Hx^{\otimes q}, x^{\otimes q}),$$

where the positive definite matrix $H \in \mathbb{R}^{nq \times nq}$ is a solution of the linear matrix inequality

$$E[(A^\top)^{\otimes q}HA^{\otimes q}] - \gamma^p H \preceq 0,$$

is a stochastic Lyapunov function for $\Sigma_\mu$ of degree $p$ with growth rate $\gamma$.

To prove this theorem we need its special case, which is proved in [13]. We here quote the result for ease of reference.

**Proposition 5.8.** Assume that $\Sigma_\mu$ is mean square stable. Let $\gamma \in (\rho_{2,\mu}, 1)$ be arbitrary. Then the function $V(x) = (Hx, x)$ on $\mathbb{R}^n$, where the positive definite matrix $H \in \mathbb{R}^{n \times n}$ is a solution of the linear matrix inequality $E[A^\top HA] - \gamma^2 H \preceq 0$, is a stochastic Lyapunov function for $\Sigma_\mu$ with degree $2$ and growth rate $\gamma$.

Let us prove Theorem 5.7.

**Proof of Theorem 5.7.** Assume that $\Sigma_\mu$ is $p$th mean stable and let $\gamma \in (\rho_{p,\mu}, 1)$ be arbitrary. Since (15) and Proposition 5.1 show $\rho_{2,\mu \otimes q} = \rho_{p,\mu}^q < 1$, the system $\Sigma_{\mu \otimes q}$ is mean square stable again by Proposition 5.1. Since $\gamma^q >$
Proposition 5.8 implies that $\Sigma_\mu$ admits a stochastic Lyapunov function $W(x) = (Hx, x)$ on $\mathbb{R}^n$ with growth rate $\gamma^q$. Moreover the matrix $H$ can be obtained as a solution of the matrix linear inequality $E[B^T H B] - (\gamma^q)^2 H \preceq 0$, where $B$ is a random variable following $\mu^\otimes q$. This linear matrix inequality is equivalent to (31) since taking a transpose and a Kronecker power commute. Now define $V: \mathbb{R}^n \to \mathbb{R}$ by (30) or, equivalently, by $V(x) := W(x^\otimes q)$. Let us show that $V$ is a stochastic Lyapunov function of degree $p$ with growth rate $\gamma$. Using (8) and (9) we can see that

$$
E[V(Ax)] = E[W(A^\otimes q x^\otimes q)] \\
= E[W(Bx^\otimes q)] \\
\leq (\gamma^q)^2 W(x^\otimes q) \\
= \gamma^p V(x).
$$
(32)

To show that an inequality of the form (26) holds for $V$, notice that there exist positive constants $C_1, C_2$ satisfying $C_1 \|x\|^2 \leq W(x) \leq C_2 \|x\|^2$ because $H$ is positive definite. From this inequality we can show (26) because $\|x^\otimes q\| = \|x\|^q$ for a general $q$ and $x \in \mathbb{R}^n$ provided $\|\cdot\|$ denotes the Euclidean norm. Hence $V$ is a stochastic Lyapunov function of degree $p$ with growth rate $\gamma$. □

Then we consider the condition $A_2$. In order to proceed we will need the next basic lemma.

**Lemma 5.9 ([32, 23]).** Let $K$ be a proper cone. If $A \geq^K B \geq^K 0$ then $\rho(A) \geq \rho(B)$. Moreover if $A >^K 0$ then the following statements are true.

1. $A$ has a simple eigenvalue $\rho(A)$, which is greater than the magnitude of any other eigenvalue of $A$;

2. The eigenvector corresponding to the eigenvalue $\rho(A)$ is in $\text{int}(K)$.

3. The dual cone $K^*$ is a proper cone and $A^\top$ is $K^*$-positive.

Then we prove the next proposition. Recall that, for a proper cone $K$ and $f \in \text{int}(K^*)$ the matrix norm $\|\cdot\|_f$ is defined by (5).

**Proposition 5.10.** Let $K \subset \mathbb{R}^n$ be a proper cone and assume that $M \geq^K 0$. Also let $\epsilon > 0$ be arbitrary. Then there exists $f \in \text{int}(K^*)$ such that $\|M\|_f < \rho(M) + \epsilon$. 

17
Proof. Let $\epsilon > 0$ be arbitrary. First assume $M >^K 0$. By Lemma 5.9 the matrix $M$ admits the Jordan canonical form $J = V^{-1}MV$ where $V \in \mathbb{R}^{n \times n}$ is an invertible matrix whose columns are the generalized eigenvectors of $M$ and $J \in \mathbb{R}^{n \times n}$ is of the form

$$J = \begin{bmatrix} J_0 & 0 \\ 0 & \rho(M) \end{bmatrix}$$

for some upper diagonal matrix $J_0 \in \mathbb{R}^{(n-1) \times (n-1)}$. Define $f \in \mathbb{R}^n$ by

$$V^{-1} = \begin{bmatrix} * \\ f^\top \end{bmatrix}.$$ 

Then we can easily see that $f$ is an eigenvector of $M^\top$ corresponding to the eigenvalue $\rho(M)$. Since $K^*$ is proper, Lemma 5.9 shows $f \in \text{int}(K^*)$. Also since $f^\top Mx = \rho(M)f^\top x$, the equation (6) shows $\|M\|_f = \rho(M)$.

Then we consider the general case of $M \geq^K 0$. Let $\epsilon > 0$ be arbitrary and take an arbitrary $P >^K 0$. Then there exists $\delta > 0$ such that $\rho(M + \delta P) < \rho(M) + \epsilon$ because $\rho(M + \delta P) \to \rho(M)$ as $\delta \to 0$ by the continuity of spectral radius. Since $M + \delta P >^K 0$, the above argument shows that there exists $f \in K^*$ satisfying $\|M + \delta P\|_f = \rho(M + \delta P) < \rho(M) + \epsilon$. Finally, since $0 \leq^K M \leq^K M + \delta P$, Lemma 2.2 shows $\|M\|_f \leq \|M + \delta P\|_f$ and thus we obtain the desired inequality. \(\square\)

The next theorem enables us to construct a stochastic Lyapunov function when $A_2$ holds.

Theorem 5.11. Assume that $\Sigma_\mu$ is $p$th mean stable and $A_2$ holds. Let $\gamma \in (\rho_{p,\mu}, 1)$ be arbitrary. Then there exists a cone linear absolute norm $\|\cdot\|_f$ on $\mathbb{R}^{n^p}$ such that $V(x) = \|x^{\otimes p}\|_f$ is a stochastic Lyapunov function for $\Sigma_\mu$ with degree $p$ and growth rate $\gamma$.

Proof. Assume that $\Sigma_\mu$ is $p$th mean stable and let $\gamma \in (\rho_{p,\mu}, 1)$ be arbitrary. Let $K^*$ be a proper cone left invariant by supp $\mu$.

First we consider the special case $p = 1$. Since $A$ leaves $K$ invariant with probability one we have $E[A] \geq^K 0$. Also Proposition 3.4 shows $\rho(E[A]) = \rho_{1,\mu} < \gamma$. Therefore, by Proposition 5.10, there exists a cone linear absolute norm $\|\cdot\|_f$ on $\mathbb{R}^n$ such that $\|E[A]\|_f < \gamma$. Let us show that $V(x) = \|x\|_f$ gives a stochastic Lyapunov function for $\Sigma_\mu$ with degree 1 and growth rate $\gamma$.
The inequality of the form (26) clearly holds for some positive constants \( C_1 \) and \( C_2 \) by the equivalence of the norms on a finite dimensional normed vector space. To show (27) let \( x \in \mathbb{R}^n \) and \( \delta > 0 \) be arbitrary. Since \( \| \cdot \|_f \) is cone linear absolute there exist \( x_1, x_2 \in K \) such that \( x = x_1 - x_2 \) and \( \| x_1 \|_f + \| x_2 \|_f = \| x_1 + x_2 \|_f \leq \| x \|_f + \delta \). Moreover we have \( Ax_i \in K \) and therefore \( \| Ax_i \|_f = (f, Ax_i) \) with probability one. Thus it follows that

\[
E[\| Ax_i \|_f] = (f, E[A]x_i) \\
= \| E[A]x_i \|_f \\
< \gamma \| x_i \|_f
\]

for each \( i \). Hence, since \( \| Ax \|_f = \| Ax_1 - Ax_2 \|_f \leq \| Ax_1 \|_f + \| Ax_2 \|_f \),

\[
E[\| Ax \|_f] < \gamma (\| x_1 \|_f + \| x_2 \|_f) \\
\leq \gamma (\| x \|_f + \delta).
\]

Since \( \delta > 0 \) was arbitrary we obtain \( E[\| Ax \|_f] \leq \gamma \| x \|_f \). This inequality shows that, since \( x \in \mathbb{R}^n \) was arbitrary, \( \| \cdot \|_f \) is a stochastic Lyapunov function for \( \Sigma_\mu \) with growth rate \( \gamma \) and degree 1.

Then let us give the proof for a general \( p \). Since \( \rho_{1,\mu \otimes p} = \rho_{p,\mu} < 1 \) by (15), \( \Sigma_{\mu \otimes p} \) is first mean stable by Proposition 5.1. Also notice that, by Lemma 2.3, \( \text{supp}(\mu \otimes p) = (\text{supp} \mu) \otimes p \) leaves a proper cone in \( \mathbb{R}^{np} \), say \( K_p \), invariant. Thus, by the above result for \( p = 1 \), since \( \gamma^p > \rho_{p,\mu} = \rho_{1,\mu \otimes p} \), the system \( \Sigma_{\mu \otimes p} \) admits a stochastic Lyapunov function \( \| \cdot \|_f \) with growth rate \( \gamma^p \) and degree 1, where \( \| \cdot \|_f \) is a cone linear absolute norm on \( \mathbb{R}^{np} \) with respect \( K_p \). Now we define \( V : \mathbb{R}^n \to \mathbb{R} \) by \( V(x) := \| x \otimes p \|_f \). Then, in the same way as (32), we can show that \( V \) is a stochastic Lyapunov function for \( \Sigma_\mu \) with degree \( p \) and growth rate \( \gamma \).

Example 5.12. Consider the probability distribution

\[
\mu = \begin{bmatrix} [0,1.5] & [0,1.8] \\ [0,0.15] & [0,1.2] \end{bmatrix},
\]

where each interval denotes the uniform distribution on it. Clearly \( \text{supp} \mu \) leaves the proper cone \( \mathbb{R}^2_+ \) invariant and moreover we can see \( \rho(E[A]) < 1 \). Therefore Propositions 5.1 and 3.4 show that \( \Sigma_\mu \) is first mean stable and therefore, by Proposition 5.5, \( \Sigma_\mu \) admits a stochastic Lyapunov function of degree 1. Following the proof of Theorem 5.11 we find a stochastic Lyapunov
function $\|x\|_f$ for $\Sigma_\mu$ where $f = [0.3838 \ 1]^\top$. We generate 200 sample paths of $\Sigma_\mu$ with the initial state $x_0 = [0 \ 1]^\top$. Figure 1 shows the sample means of the stochastic Lyapunov function $\|x(k)\|_f$ and the Euclidean norm $\|x(k)\|$. Figure 2 shows the average of the sample paths and the contour plot of the stochastic Lyapunov function and the Euclidean norm.

6. Conclusion

This paper presented a characterization of the joint spectral radius of a set of matrices as the limit of the $L^p$-norm joint spectral radius of a probability distribution when $p \to \infty$. The obtained characterization extends the ones in the literature by allowing the set to have infinitely many matrices. We made use of nearly most unstable trajectories of an associated switched linear system to derive the characterization. Based on the result, we also presented a novel characterization of the absolute exponential stability of switched linear systems via the existence of stochastic Lyapunov functions of any higher degrees. The construction of stochastic Lyapunov functions is also studied.
Figure 2: The averaged sample path (solid) and the level plots of the Lyapunov function (dashed) and the Euclidean norm (dotted)

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