POINCARÉ SERIES OF QUANTUM MATRIX BIALGEBRAS
DETERMINED BY A PAIR OF QUANTUM SPACES

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Abstract

The dimension of the third homogeneous component of a matrix quantum
bialgebra, determined by pair of quantum spaces, is calculated. The Poincaré
series of some deformations of $GL(n)$ is calculated. A new deformation of $GL(3)$
with the correct dimension is given.

1 Introduction

Quantum groups are the subcategory of the category of Hopf-algebras, which are
generally non-commutative and non-cocommutative and possess many properties
similar to those of classical universal enveloping algebras or, by duality, function
rings of affine algebraic groups. There are many approaches to quantum groups, the
"R-matrix" and Quantum Yang-Baxter equations approach, which was introduced
by Faddeev, Takhtajan, Reshetikhin and others, formal deformations of commutative
or cocommutative Hopf algebras especially of the universal enveloping algebra
$U(g)$ of a Lie algebra $g$, which was systematically studied by Drinfeld, Jimbo and
others. Quantum groups can be also considered as the symmetries of quadratic non-
commutative or quantum spaces. This construction was given by Yu.Manin [M2].
The methods in this paper are based on Manin’s construction.

A family of quantum spaces defines an algebra, which is universally coacting
upon the spaces of this family. This algebra because of its universality is a bialgebra.
We call it a quantum matrix bialgebra (QMB). To obtain a Hopf algebra envelope
of this bialgebra, one often has to invert a certain quantum determinant, which
is a group like element and in many examples a central element. In this paper we
restrict ourselves to an investigation of the quantum matrix bialgebra, which induces
deformations of $GL(n)$. More concretely we study a quantum matrix bialgebra,
which is determined by two quantum spaces, which generalize a pair of symmetric
and antisymmetric tensor algebras on a linear space. It is natural to consider the
above QMB as a deformation of the bialgebra $M(n)$ (a polynomial ring on the
semigroup of linear transformations of a vectorspace). Since our bialgebra is a
graded algebra, the question about its Poincaré series arises.

Note that in our construction we obtain an operator – “R-matrix”. We always
require it to satisfy the so called quantum Yang-Baxter equation. This operator
plays an important rôle in quantum theory. For deformations of $M(n)$ it should
satisfy the Hecke equation. In the undeformed case, for $M(n)$, it is just the twist
operator.

Recently several authors have considered the polynomiality and the Poincaré
series of a quantum matrix bialgebra by using the diamond lemma. The troubles
usually happen on cubic monomials. In general, the method of the diamond lemma
seems not to be effective because there is no sense of reduction system. In this work
we do the first step of calculating the Poincaré series without using the diamond
lemma. We use the method developed by Sudbery[S2]. We get a formula for calcul-
ating the dimension of the third component of the QMB via the dimension of the
components of the quantum spaces (Section 3). Nevertheless, we remark that the
obtained formula is independent of the matrix $R$. It seems to be complicated to do
the same for the higher components of the QMB. It should be interesting to show
that the same independence takes place for all higher homogeneous components.
If this would be true, we could say that if the quantum spaces have the correct
Poincaré series then the QMB has the correct Poincaré series, too.

On the other side, knowing the dimension of the second and third components
in some cases we can already obtain the Poincaré series of the QMB. Applying
Bondal’s theorem ([Bo], Theorem 3) to $M(n)$ we have, if $R = R(q), q \in \mathbb{R}$ is some
continuously parameterized deformation of $P$ – the twist operator, $R(1) = P$, and
if the Poincaré series of the quantum spaces are correct, then for every integer $n$
we can choose a neighborhood of 1, such that the dimensions of the homogeneous
components of the QMB obtained from $R(q)$ with $q$ in this neighborhood is correct
up to the $n$-th component.

In section 4 we investigate the polynomiality of all known QMBs which give
a deformation of $GL(n)$. The polynomiality of the multiparameter deformations
was shown in [ATS]. We show here the polynomiality of a deformation obtained by
Creme and Gervais [CG]. At the end of the section we give a new deformation of
$GL(3)$.

2 A Yang-Baxter operator on a universal bialgebra.

Let $k$ be a fixed algebraically closed field, $V$ be a $k$-vector space of finite dimension $n$.
A quadratic algebra $A$ is a factor algebra of the tensor algebra $T(V)$ by a two-sided
ideal generated by a subspace $R_A$ of $V \otimes V$. Hence $A$ is a graded algebra:

$$A = \bigoplus_{i=0}^{\infty} A_i, A_0 = k, A_1 = V$$

We call the pair $(A, V)$ a quantum space.
Let $B$ be a $k$-algebra. A left coaction of $B$ on the quantum space $(A, V)$ is an algebra homomorphism
\[ \delta_B : A \rightarrow B \otimes A \]
such that
\[ \delta_B(V) \subset B \otimes V \]
A coaction of $B$ on a family of quantum spaces $(A_\alpha, V_\alpha)$, $\alpha \in J$ is a family of coactions of $B$ on each $(A_\alpha, V_\alpha)$, $\alpha \in J$, such that their restrictions on $V$ coincide.

An algebra $E$ is said to universally coact upon $(A_\alpha, V_\alpha)$ if for every algebra $B$ coacting on $(A_\alpha, V_\alpha)$, $\alpha \in J$, there exists a unique algebra homomorphism $\phi : E \rightarrow B$ such that $(\phi \otimes \text{id})\delta_\alpha = \delta'_\alpha$, $\alpha \in J$, where $\delta_\alpha, \delta'_\alpha$ denote the coactions of $E$ and $B$ respectively. Mukhin [Mu] and Sudbery [S1] show that $E$ exists. The bialgebra structure on $E$ follows from the universality.

Let $\theta$ denote the isomorphism
\[ \theta : V^* \otimes V^* \otimes V \otimes V \rightarrow (V^* \otimes V)^{\otimes 2} \cong \text{End}(V) \otimes \text{End}(V) \]
which interchanges the 2nd and 3rd components. Analogously we have the isomorphism of the two spaces $(V^*)^{\otimes 3} \otimes V^{\otimes 3}$ and $\text{End}(V)^{\otimes 3}$, which is denoted by the same $\theta$.

In this paper we study certain deformations of $M(n)$. In this case there are only two quantum spaces $(A, V)$ and $(S, V)$ with $R_A \oplus R_S = V^{\otimes 2}$. $(A, V)$ and $(S, V)$ play the analogous roles to the antisymmetric and symmetric tensor algebras on the space $V$. Let $E$ be the universal matrix bialgebra coacting on $(A, V)$ and $(S, V)$. Then $E$ is a quadratic algebra over $E_1 = V^* \otimes V$. The relations on $E$ are given by $R_E = \theta(R_S^\perp \otimes R_S + R_A^\perp \otimes R_A)$.

Let $P_A$ and $P_S$ be the projections onto $R_A$ and $R_S$ respectively with respect to $R_A \oplus R_S = V^{\otimes 2}$. Let
\[ R = qP_S + P_A, \quad q \neq 1. \]
And $R^* : V^* \otimes V^* \rightarrow V^* \otimes V^*$. $R$ and $R^*$ are plainly diagonalizable. Let $t_i, i = 1, \ldots, n^2$ be the eigenvectors of $R$ which form a basis on $V \otimes V$, the dual basis to it consists of eigenvectors of $R^*$, denote them by $u_i, i = 1, \ldots, n^2$. The vectors $u_i \otimes t_j$ are eigenvectors of $R^* \otimes R^{-1}$ and form a basis on $V^* \otimes V^* \otimes V \otimes V$. Consequently
\[ R_S^\perp \otimes R_S + R_A^\perp \otimes R_A = \text{Im} (R^* \otimes R^{-1} - 1). \]
Thus
\[ R_E = \theta(\text{Im} \sum \alpha S_\alpha) = \text{Im} (\theta(R^* \otimes R^{-1})\theta^{-1} - 1). \]
We assume that our construction is Yang-Baxter, in other words, there exists a $q$ such that $R$ satisfies
\[ R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}, (R_{12} = R \otimes 1, R_{23} = 1 \otimes R). \]
\( R \) is called a Yang-Baxter operator. And we assume that \( q \) obeys the following condition
\[
q(q-1)(q^2-q+1) \neq 0. \tag{3}
\]
For a quadratic algebra \( A \) we have \( A = \bigoplus_{i=0}^{\infty} A_i \). Since the \( A_i \) are all finite dimensional over \( k \) we can form a formal series \( P_A(t) = \sum_{i=0}^{\infty} (\dim_k A_i) t^i \), which is called the Poincaré series of \( A \).

**Definition.**

a) A pair of quadratic algebras \((S, V)\) and \((\Lambda, V)\) is said to be compatible if their Poincaré series coincide with the ones of the symmetric and antisymmetric tensor algebras over \( V \) respectively.

b) The bialgebra \( E \) is said to be compatible if its Poincaré series coincides with the one of the polynomial ring of \( n^2 \) parameters.

### 3 Yang-Baxter operator with Hecke equation. The main theorem

The aim of this section is to calculate the dimension of the third homogeneous component of \( E \) via the dimensions of components of \( \Lambda \) and \( S \). Thus, let \( s_i = \dim S_i, \lambda_i = \dim A_i, i = 2, 3 \) be given. We want to find \( e_3 = \dim E_3 \).

Let \( R = qP_3 + P_A \) be Yang-Baxter and let \( q \) satisfy (3). Then \((R - q)(R - 1) = 0\).

Consider the following operators
\[
P_{\alpha} = P_{\alpha}(R) = (R_{12} - \alpha)(R_{23}R_{12} - \alpha R_{23} + \alpha^2)(\alpha = 1, q)
\]
\[
P_{\tau} = P_{\tau}(R) = (R_{12} - q)(R_{23} - 1)
\]
\[
P_{\L} = P_{\L}(R) = (R_{23} - q)(R_{12} - 1)
\]
Then \( R_{13} P_{\L} = P_{\tau} R_{13} \), where \( R_{13} = R_{12} R_{23} R_{12} = R_{23} R_{13} R_{23} \). By \( \Pi_{\alpha} \) we denote the subspaces \( \text{Im} P_{\alpha} \) of \( V^{\otimes 3} \) (\( \alpha = 1, q, \L, \tau \)). The notations \( \L \) and \( \tau \) are due to Sudbery, they are chosen for their resemblance to the Young tableau.

Analogously we define \( P_{\alpha}^+ = P_{\alpha}(R^*) \) and \( \Pi_{\alpha}^+ = \text{Im} P_{\alpha}^+ \), \( \alpha = 1, q, \L, \tau \). \( R^* \) is the adjoint operator of \( R \).

For \( \alpha = 1, q \) we have
\[
P_{\alpha} = (R_{12} - \alpha)(R_{23}R_{12} - \alpha R_{23} + \alpha^2)
\]
\[
= (R_{23} - \alpha)(R_{12}R_{23} - \alpha R_{12} + \alpha^2)
\]
\[
= (R_{23}R_{12} - \alpha R_{12} + \alpha^2)(R_{23} - \alpha)
\]
\[
= (R_{12}R_{23} - \alpha R_{23} + \alpha^2)(R_{12} - \alpha) \tag{4}
\]

The following lemma is due to Sudbery.

**Lemma 3.1** ([S2], Lemma 2) There exist constants \( c_{\alpha} \) such that \( c_{\alpha} P_{\alpha} \) are projections onto \( \Pi_{\alpha} \) and \( \bigoplus_{\alpha} \Pi_{\alpha} = V^{\otimes 3}, \alpha = 1, q, \L, \tau \).
From the definition we have
\[ s_2 = \dim \text{Im} (R - q), \lambda_2 = \dim \text{Im} (R - 1). \]  \tag{5}

We claim
\[ \ker P_q = \text{Im} (R_{12} - 1) + \text{Im} (R_{23} - 1). \]

Indeed if \( x \in \ker P_q : (R_{12} - q)(R_{23}R_{12} - qR_{23} + q^2)x = 0, \) then
\[ (R_{12} - 1)(R_{23}R_{12} - qR_{23} + q^2 - q + 1)x - (1 - q)(R_{23} - 1)(R_{12} - q)x = (q - 1)(q^2 - q + 1)x. \]

Hence, if \( (1 - q)(q^2 - q + 1) \neq 0, \) \( x \in \text{Im} (R_{12} - 1) + \text{Im} (R_{23} - 1). \) Conversely, if \( x \in \text{Im} (R_{12} - 1) + \text{Im} (R_{23} - 1), \) then \( x = x' + x'', \) where \( x' \in \text{Im} (R_{12} - 1), x'' \in \text{Im} (R_{23} - 1). \) And obviously \( P_qx = P_qx' + P_qx'' = 0. \) Consequently
\[ s_3 = \dim \text{Im} P_q, \lambda_3 = \dim \text{Im} P_1. \]  \tag{6}

We use the result obtained by Sudbery. For \( x \in \Pi_R \) put \( y = q^{-1}R_{13}x \) then \( y \in \Pi_L \) and
\[ R_{12}x = x \quad R_{23}x = qx + y \]
\[ R_{12}y = -qx + qy \quad R_{23}y = y. \]  \tag{7}

The first and last equations are obvious. According to (3), \( (R_{12}R_{23} - qR_{23} + q^2)P_r = 0, \) this shows the third equation \( R_{23}x = qx + y, \) the second one follows from it. Analogously we have, for \( x^+ \in \Pi_L^+ \) and \( y^+ = q^{-1}R_{13}x, \) then \( y^+ \in \Pi_L^+ \) and
\[ R_{12}^+x^+ = x^+ \quad R_{23}^+x^+ = qx^+ + y^+ \]
\[ R_{12}^+y^+ = -qx^+ + qy^+ \quad R_{23}^+y^+ = y^+. \]  \tag{8}

Now we consider the operator \( \overline{R} = \theta(R^* \otimes R^{-1}) \theta^{-1}. \) It this easy to check that
\[ \overline{R}_{12} = \theta(R_{12}^* \otimes R_{12}^{-1}) \theta^{-1} \]
\[ \overline{R}_{23} = \theta(R_{23}^* \otimes R_{23}^{-1}) \theta^{-1} \]

By the isomorphism \( \theta : (V^*)^\otimes 3 \otimes V^\otimes 3 \rightarrow (V^* \otimes V)^\otimes 3 \) we can regard \( \overline{R}_{12} \) as \( R_{12}^* \otimes R_{12}^{-1} \) acting upon \( (V^*)^\otimes 3 \otimes V^\otimes 3. \) Consequently \( \overline{R} \) is a Yang-Baxter operator.

By considering the eigenvectors of \( R \) which form a basis on \( V \otimes V \) we get \( (\overline{R} - 1)(\overline{R} - q)(\overline{R} - q^{-1}) = 0 \) and \( \dim \text{Im} (\overline{R} - 1) = 2\lambda_2 s_2. \) Since \( \lambda_2 + s_2 = n^2, \) by using (4) we have
\[ e_2 = \dim E_2 = n^4 - \dim \text{Im} (\overline{R} - 1) = \lambda_2^2 + s_2^2. \]  \tag{9}

**Lemma 3.2** Let \( \Phi = \overline{R}_{12} \overline{R}_{23} \overline{R}_{12} - \overline{R}_{12} \overline{R}_{23} \overline{R}_{23} \overline{R}_{12} + \overline{R}_{12} + \overline{R}_{23} - 1, \) then \( \text{Im} \Phi = \text{Im} (\overline{R}_{12} - 1) \cap \text{Im} (\overline{R}_{23} - 1). \)  \tag{10}
Proof. Since $\mathcal{R}$ is a Yang-Baxter operator

$$\Phi = (\mathcal{R}_{12} - 1)(\mathcal{R}_{23}\mathcal{R}_{12} - \mathcal{R}_{23} + 1) = (\mathcal{R}_{23} - 1)(\mathcal{R}_{12}\mathcal{R}_{23} - \mathcal{R}_{12} + 1).$$

Hence $\text{Im } \Phi \subset (\text{Im } (\mathcal{R}_{12} - 1) \cap \text{Im } (\mathcal{R}_{23} - 1)) = W$.

If we can show that $W \cap \text{Ker } \Phi = 0$, then $\dim W + \dim \text{Ker } \Phi \leq n^6$ (since they are subspaces of $(V^*)^3 \otimes V^3$). But $\dim \text{Im } \Phi + \dim \text{Ker } \Phi = n^6$, therefore $\dim W = \dim \text{Im } \Phi$, which means $W = \text{Im } \Phi$.

Assume $L := W \cap \text{Ker } \Phi \neq \{0\}$. Denote $\mathcal{R}_{13} = \mathcal{R}_{12}\mathcal{R}_{23}\mathcal{R}_{12} = \mathcal{R}_{23}\mathcal{R}_{12}\mathcal{R}_{23}$, then

$$\Phi \mathcal{R}_{13} = \mathcal{R}_{13} \Phi$$

since $\mathcal{R}_{13}\mathcal{R}_{12} = \mathcal{R}_{23}\mathcal{R}_{13}$, $\mathcal{R}_{12}\mathcal{R}_{13} = \mathcal{R}_{13}\mathcal{R}_{23}$. Hence $L$ is invariant under $\mathcal{R}_{13}$, therefore there exists a vector $x \neq 0$ in $L$ so that $\mathcal{R}_{13}x = \lambda x$.

If $\lambda = -1$ then

$$\Phi x = -2x - \mathcal{R}_{12}\mathcal{R}_{23}x - x + \mathcal{R}_{23}\mathcal{R}_{12}x + \mathcal{R}_{12}x + \mathcal{R}_{23}x = -2x + \mathcal{R}_{12}^{-1}x + \mathcal{R}_{23}^{-1}x + \mathcal{R}_{12}x + \mathcal{R}_{23}x.$$

Since $x \in W = \text{Im } (\mathcal{R}_{12} - 1) \cap \text{Im } (\mathcal{R}_{23} - 1)$, we get $(\mathcal{R}_{12}x - q^{-1})(\mathcal{R}_{12} - q) = 0$ and $(\mathcal{R}_{12} - (q + q^{-1})\mathcal{R}_{23} + 1) x = 0$. Thus

$$\begin{align*}
(\mathcal{R}_{12}^{-1} + \mathcal{R}_{12} - q - q^{-1})x & = 0 \\
(\mathcal{R}_{23}^{-1} + \mathcal{R}_{23} - q - q^{-1})x & = 0
\end{align*}$$

(11)

Hence $\Phi x = 2(q + q^{-1} - 1)x = 0$, $q + q^{-1} - 1 \neq 0$, therefore $x = 0$, which is a contradiction.

Let $\lambda \neq -1$. Since $\mathcal{R}_{13} = \mathcal{R}_{23}\mathcal{R}_{12}\mathcal{R}_{23}x = \lambda x$ we get

$$(\lambda \mathcal{R}_{23}^{-1} - x - \mathcal{R}_{23})x = (\mathcal{R}_{12}\mathcal{R}_{23} - \mathcal{R}_{23})x = (\mathcal{R}_{12} - 1)\mathcal{R}_{23}x \in \text{Im } (\mathcal{R}_{12} - 1)$$

(12)

According to (11), $(\lambda \mathcal{R}_{23}^{-1} - \mathcal{R}_{23})x = \lambda(q + q^{-1})x - (\lambda + 1)\mathcal{R}_{23}x$. And since $x \in W \subset \text{Im } (\mathcal{R}_{12} - 1)$ we get $\mathcal{R}_{23}x \in \text{Im } (\mathcal{R}_{12} - 1)$. Therefore we get

$$z := (\mathcal{R}_{23}\mathcal{R}_{12} - \mathcal{R}_{23} + 1)x = (\lambda \mathcal{R}_{12}^{-1} - \mathcal{R}_{23} + 1)x \in \text{Im } (\mathcal{R}_{12} - 1).$$

Then $(\mathcal{R}_{12} - q)(\mathcal{R}_{12} - q^{-1})z = 0$ and $(\mathcal{R}_{12} - 1)z = (\mathcal{R}_{12} - 1)(\mathcal{R}_{23}\mathcal{R}_{12} - \mathcal{R}_{23} + 1)x = \Phi x = 0$ hence $z = 0$, that means

$$(\mathcal{R}_{23}\mathcal{R}_{12} - \mathcal{R}_{23} + 1)x = 0.$$ 

(13)

Analogously we have

$$(\mathcal{R}_{12}\mathcal{R}_{23} - \mathcal{R}_{12} + 1)x = 0.$$ 

(14)

Multiplying (13) by $\mathcal{R}_{12}$ from the left and adding (14) we get $\mathcal{R}_{13}x = \mathcal{R}_{12}\mathcal{R}_{23}\mathcal{R}_{12}x = -x$ which is a contradiction. Thus we must have $L = \{0\}$, which means $\text{Im } \Phi = W$.

Q.E.D.
Corollary 3.3

$$\text{Im } \Phi \oplus \text{Ker } \Phi = (V^*)^\otimes 3 \otimes V^\otimes 3.$$ (15)

Indeed $\text{Im } \Phi \cap \text{Ker } \Phi = \{0\}$.

Proposition 3.4 Let $\Pi = \Pi_\mathbb{R} \oplus \Pi_\mathbb{L}$, $\Pi^+ = \Pi^+_{\mathbb{R}} \oplus \Pi^+_{\mathbb{L}}$ then

i) $\Pi^+_{\mathbb{R}} \oplus \Pi_q$, $\Pi^+_{\mathbb{L}} \oplus \Pi_1$ are subspaces of $\text{Im } \Phi$.

ii) $\Pi^+_{\mathbb{R}} \otimes \Pi_{\mathbb{R}}$, $\Pi^+_{\mathbb{L}} \otimes \Pi_{\mathbb{L}}$, $\Pi^+ \otimes \Pi_{\mathbb{R}}$, $\Pi^+ \otimes \Pi_{\mathbb{L}}$, $(\alpha = 1, q)$ are subspaces of $\text{Ker } \Phi$.

iii) The subspace $\Pi^+ \otimes \Pi$ is invariant under the action of $\Phi$ and decomposes into a direct sum of 4-dimensional $\Phi$-invariant subspaces. In each of them the operator $\Phi$ has rank 1.

Proof.

For $x \otimes y \in \Pi^+_{\mathbb{R}} \otimes \Pi_q$, $R_{12}x \otimes y = R_{23}x \otimes y = q^{-1}x \otimes y$, hence $\Phi x \otimes y = (q^{-1} - 1)(q^{-2} - q^{-1} + 1)x \otimes y$, therefore $x \otimes y \in \text{Im } \Phi$.

If $x \otimes y \in \Pi^+_{\mathbb{L}} \otimes \Pi_{\mathbb{L}}$, $(\alpha = 1, q)$ then $R_{12}x \otimes y = R_{23}x \otimes y = x \otimes y$ so that $\Phi x \otimes y = 0$. For $x \otimes y \in \Pi^+ \otimes \Pi$, $R_{12}x = R_{23}x = qx$ and $P_1(qR^{-1})y = -R_{13}^{-1}P_qy = 0$ hence $\Phi x \otimes y = x \otimes P_1(qR^{-1})y = 0$.

According to (3) and (8) if $x_j$ form a basis in $\Pi_{\mathbb{R}}$, $x_i^+$ form a basis in $\Pi^+_{\mathbb{R}}$ then $y_j = q^{-1}R_{13}x_j$ form a basis in $\Pi_{\mathbb{L}}$, $y_i^+ = q^{-1}R_{13}x_i^+$ form a basis in $\Pi^+_{\mathbb{L}}$, and the subspaces of $\Pi^+ \otimes \Pi$ spanned by the four vectors $x_i^+ \otimes x_j$, $x_i^+ \otimes y_j$, $y_i^+ \otimes x_j$, $y_i^+ \otimes y_j$, are $\Phi$-invariant. This shows the first part of iii). The second one can be proved by direct computation.

Q.E.D.

Theorem 3.5 If the operator $R$ satisfies the Yang-Baxter equation (3), with $q$ satisfying condition (1), then we have

$$e_3 = n^6 - 4n^2s_2\lambda_2 + 2s_3\lambda_3 + (n^3 - s_3 - \lambda_3)^2/4$$ (16)

Proof. From (1), the relations on $E_3$ are $\text{Im } (R_{12} - 1) + \text{Im } (R_{23} - 1)$, hence $\dim E_3 = n^6 - \dim \text{Im } (R_{12} - 1) + \dim \text{Im } (R_{23} - 1) = n^6 - \dim \text{Im } (R_{12} - 1) - \dim \text{Im } (R_{23} - 1) + \dim \text{Im } (R_{12} - 1) \cap \text{Im } (R_{23} - 1)) = n^6 - 4n^2s_2\lambda_2 + 4\dim \text{Im } \Phi$ (according to (3) and Lemma (3.2)). Since $\Pi \oplus \Pi_1 \oplus \Pi_q = V^\otimes 3$, we have

$$\bigoplus_{\alpha, \beta} (\Pi^+_{\alpha} \otimes \Pi_{\beta}) = (V^*)^\otimes 3 \otimes V^\otimes 3, \quad (\alpha, \beta = \emptyset, 1, q)$$ (17)

According to Corollary (3.3) and Proposition (3.4)

$$\dim \text{Im } \Phi = \dim (\Pi^+_{\mathbb{R}} \otimes \Pi_q) + \dim (\Pi^+_{\mathbb{L}} \otimes \Pi_1) + \dim (\Pi^+ \otimes \Pi)/4$$ (18)

Indeed, (17) is a decompositon of $(V^*)^\otimes 3 \otimes V^\otimes 3$, according to iii) in Proposition (3.4),

$$\Pi^+ \otimes \Pi = (\Pi^+ \otimes \Pi \cap \text{Im } \Phi) \oplus (\Pi^+ \otimes \Pi \cap \text{Ker } \Phi)$$
is a decomposition of $\Pi^+ \otimes \Pi$. And $\dim(\Pi^+ \otimes \Pi \cap \text{Im}\Phi) = \dim(\Pi^+ \otimes \Pi)/4$, which implies \textcolor{red}{[13]}. From Lemma \textcolor{red}{(3.1)} and \textcolor{red}{(3)} one follows

$$\dim \Pi_1 = \lambda_3, \dim \Pi_q = s_3, \dim \Pi = n^3 - s_3 - \lambda_3.$$ 

Thus $\dim \text{Im}\Phi = 2\lambda_3 s_3 + (n^3 - s_3 - \lambda_3)^2/4$ which gives \textcolor{red}{[14]}. Q.E.D.

Theorem \textcolor{red}{3.3} shows that if the construction is Yang-Baxter then the dimensions of $E_2$ and $E_3$ do not depend on the choice of the matrix $R$ and depend only on $\lambda_i$ and $s_i$. If the same independence would take place for all the higher components, we could directly calculate the Poincaré series of $E$ in many cases, avoiding the use of the diamond lemma, which is not always applicable.

In the next section we investigate the polynomiality of some deformations of $GL(n)$ however by using the diamond lemma. The following Corollary is useful.

Let $E_1$ be spanned by $\{z^i_j\}$, define an ordering of $z^i_j < z^k_l$ if either $i < k$ or $i = k, j < l$. Call a monomial $z$ in $z^i_j$ normally ordered if for any $z', z'' \in \{z^i_j\}$ in this monomial, $z' < z''$ iff $z'$ is to the left of $z''$. We consider the lexicographic order in the polynomials in $z^i_j$, which corresponds to this order.

**Corollary 3.6** If the dimensions of the second and third homogeneous components of $S$ and $\Lambda$ coincide with the ones of $S(V)$ and $\Lambda(V)$ respectively, then the dimensions of $E_2$ and $E_3$ coincide with the ones of a polynomial ring of $n^2$ parameters. Moreover, if every monomial, which is not normally ordered, can be represented as a sum of normally ordered monomials of strictly smaller order (in the lexicographic order), then the normally ordered monomials form a basis of $E$.

**Proof.** If

$$s_2 = n(n + 1)/2, \lambda_2 = n(n - 1)/2$$
$$s_3 = n(n + 1)(n + 2)/6, \lambda_3 = n(n - 1)(n - 2)/6$$

then, by using \textcolor{red}{(3),(14)}, we have

$$e_2 = \dim E_2 = n^2(n^2 + 1)/2$$
$$e_3 = \dim E_3 = n^2(n^2 + 1)(n^2 + 2)/6$$

(19)

If the last condition of the Corollary is verified then $z^i_j z^j_i, z^j_i \leq z^i_j$ generate $E_2$. According to \textcolor{red}{[19]} they must be independent and so form the basis on $E_2$. The reordering procedure of monomials of higher power is possible but not unique. Manin has shown \textcolor{red}{[M1]} that the different possible procedures will lead to the same result if they do so for cubic monomials, in other words if the cubic normally ordered monomials are independent. But they are so, according to \textcolor{red}{[19]}. So that we can apply the diamond lemma for the algebra $E$.

**Remark.** In general the normally ordered monomials will not span $E$, even if the subspace $E_2$ is spanned by normally ordered monomials. If they span $E_2$ they may even not span $E_3$. An example is the following quadratic algebra, $A = k < x, y > / xy = x^2 + y^2 + xy$. The normally ordered monomials span $A_2$ but do not span $A_3$. Note that, however its Poincaré series are correct \textcolor{red}{[EOW]}, p. 1. The next section considers this question for the deformations of $GL(n)$. 

\textcolor{red}{8}
4 Polynomiality of some deformations

There are mainly three series of deformation of $GL(n)$. Artin, Schelters and Tate [ATS] found the multiparameter deformation, which include Sudbery’s and Manin’s deformations [S], [DMMZ]. Cremer and Gervais [CG] found a deformation, which is associated with the non-linearly extended Virasoro algebras. The Jordan deformation of $GL(2)$ was found by Manin [DMMZ] and was systematically studied by Wess and others [EOW]. In this section we investigate the polynomiality of the deformation obtained by Cremer and Gervais. At the end of the section we form a new deformation for $GL(3)$.

4.1 The deformation of E.Cremer and J.-L.Gervais.

This deformation is given by the matrix

$$R = \sum_{k, i>j}(e_{kk} + q^{n+j-i}e_{ij} + q^{i-j}e_{ji} + (1-q^n)e_{ji} + \sum_{r=1}^{i+j}(q^n-1)q^{-r}\sum_{i+j}^{i+j+r} + (1-q^n)q^{i-r}e_{ij})$$  \hspace{1cm} (20)

Note that $R = P \cdot \rho$, where $\rho$ is the operator defined in [CG], p.625, $P$ is the usual twist operator. $R$ satisfies the Yang-Baxter equation and the equation

$$(R - 1)(R + q^n) = 0.$$  

We assume that $q \neq \sqrt{1}$. The subspace relations $R_S$ and $R_A$ can be spanned by the vectors

$$u_{ij} = x_i x_j - q^{i-j-n} x_j x_i - (q^n - q^{-n})q^{i-j-r-n}x_{i+r} x_{i-r} + p(1-q^n)x_{i+j} x_{i+j}$$  \hspace{1cm} (21)

$$(1 \leq i < j \leq n)$$

where

$$p = \begin{cases} 
0 & \text{if } i+j \text{ odd} \\
\frac{i-j}{2} + n - \frac{n(i-j)}{2} & \text{if } i+j \text{ even}
\end{cases}$$

and

$$\begin{cases} 
v_{ii} = x_i x_i \\
v_{ij} = x_i x_j + q^{i-j} x_j x_i, \; (1 \leq j < i \leq n)
\end{cases}$$  \hspace{1cm} (22)

respectively. We use the following lemma for investigating the compatibility of $S$ and $\Lambda$.

**Lemma 4.1** ([G]p.556) Let $P_S(t), P_A(t)$ be the Poincaré series of $S$ and $\Lambda$ respectively. If the matrix $R$ satisfies the Yang-Baxter equation [3], then

$$P_S(t)P_A(-t) = 1$$  \hspace{1cm} (23)

**Theorem 4.2** The quantum matrix bialgebra $E$ determined by $S$ and $\Lambda$, described in [21] and [22], is compatible.
Proof. Denote the Poincaré series of \( \Lambda \) and \( S \) by \( P_\Lambda(t) \) and \( P_S(t) \) respectively. According to lemma (4.1) we have \( P_\Lambda(t)P_S(-t) = 1 \). From (22) we have \( P_\Lambda = (1+t)^n \). Hence \( P_S(t) = \frac{1}{(1-t)^n} \) so that \( \Lambda \) and \( S \) are compatible.

As we have seen in section 2, the subspace relation on \( E \) are \( R^*_S \otimes R_S \oplus R^*_\Lambda \otimes R_\Lambda \) when \( E_1 \otimes E_1 \) and \( V^* \otimes V^* \otimes V \otimes V \) are identified.

Obviously the subspace \( R^*_S \) is spanned by
\[
v_{ij}^* = \xi_i^j - q^{n-i+j} \xi^i \xi^j, \quad 1 \leq j < i \leq n.
\] (24)

We show that \( R^*_S \) can be spanned by vectors of the type
\[
u_{ij} = \xi_i^j + q^{n-i+j} \xi^j \xi^i + \sum_{1 \leq r \leq j-1, r \leq n-i} c_{ij}^{rj} \xi^{j-r} \xi^{i+r}
\] (25)

\((1 \leq j \leq i \leq n)\).

In fact, we must find coefficients \( c_{ij}^{rj} \) so that \( \langle u_{ij} | u_{kl} \rangle = 0 \) if either \( k + l \neq i + j \) or \( l > j \). Hence we have to find \( c_{ij}^{rj} \) obeying only the following system of equations
\[
\langle u_{ij} | u_{kl} \rangle = 0 \quad (s < n - i + 1, s < j)
\] (26)

whose rank is \( \min(j-1,n-i) \).

Thus we get linear equations for \( c_{ij}^{rj} \), where the number of parameters equals the number of equation, and what is more, the coefficients \( a_r^s \) of \( c_{ij}^{rj} \) in the \( s \)-th equation is equal 1 if \( r = s \), and 0 if \( r < s \). Hence (25), (26) has a unique solution.

Using (21-25), we get the following relations in \( E \):
\[
z_i^l z_j^k + \sum_{m<n, m<i} c_{r}^{kl} z_i^l z_j^k + \sum_{m<n, m<i} p_{uv}^{mn} \overrightarrow{z_m^u \overrightarrow{z_n^v}} = 0, (1 \leq j < i \leq n, 1 \leq l \leq k \leq n)
\] (27)

\[
z_i^k z_j^l + \sum_{m<n, m<i} c_{r}^{kl} z_i^k z_j^l + \sum_{m<n, m<i} p_{uv}^{mn} \overrightarrow{z_m^u \overrightarrow{z_n^v}} = 0, (1 \leq j < i \leq n, 1 \leq l \leq k \leq n)
\] (28)

\[
z_i^l z_j^k - q^{kl} z_i^l z_i^k, (1 \leq i \leq n, 1 \leq l \leq k \leq n)
\] (29)

By induction (27), (28) can be written in the following form
\[
z_i^l z_j^k + \sum_{m<n, m<i} p_{uv}^{mn} \overrightarrow{z_m^u \overrightarrow{z_n^v}} = 0, (1 \leq j < i \leq n)
\] (30)

And so (27), (30) are generating relations for \( E \). According to the diamond lemma normally ordered monomials span \( E \). Finally, using corollary (3.6), one finds that \( E \) is compatible.

Q.E.D.
4.2 The Jordan deformation of $GL(3)$.

In [DMMZ] Manin formed the non-standard deformations of $GL(2)$. One of them is the Jordan deformation, which was systematically studied in [EOW]. Here we give its analogue for $GL(3)$. We show that the new bialgebra is still compatible. Let

$$S = k < x_1, x_2, x_3 > / \begin{pmatrix} x_2 x_1 - x_1 x_2 + p_{12} x_2 x_2 = 0 \\ x_3 x_2 - x_2 x_3 = 0 \\ x_3 x_1 - x_1 x_3 + p_{13} x_2 x_2 = 0 \end{pmatrix}$$

(31)

$$A^i = k < \xi^1, \xi^2, \xi^3 > / \begin{pmatrix} \xi^3 \xi^1 - \xi^1 \xi^3 + q_{31} \xi^1 \xi^1 = 0 \\ \xi^3 \xi^2 - \xi^2 \xi^3 + q_{32} \xi^1 \xi^1 = 0 \\ \xi^2 \xi^1 - \xi^1 \xi^2 = 0 \end{pmatrix}$$

(32)

where $A^i$ is the dual of $A_i$ (see [M]).

Let $E = E(S, A)$, then the matrix $R$ is given by $(q_{ij} = -q_{ji}, p_{ij} = -p_{ji})$

$$R = \begin{bmatrix} 1 & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . \\ q_{13} & . & . & . & . & 1 & . & . \\ . & . & 1 & . & . & . & . & . \\ q_{23} & . & . & . & . & . & 1 & . \\ q_{31} & . & 1 & . & . & . & . & . \\ q_{32} & . & . & . & . & 1 & . & . \\ p_{13} q_{31} & p_{12} & p_{13} & p_{21} & . & . & p_{31} & 1 \end{bmatrix}$$

(33)

is Yang-Baxter and $R^2 = 1$, it is easy to show that $S$ and $A$ are compatible.

Let us define the ordering in $E$ as follows, $z_i^j < z_k^l$ iff $i > k$ or $i = k, j > l$. Then we have

Theorem 4.3 $E$ is compatible and the normally ordered monomials with respect to the given order form a basis of $E$.

Proof. A direct computation shows that normally ordered monomials span $E$. By applying (3.6) we get the result. Q.E.D.

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