Portfolio optimisation under non-linear drawdown constraints in a semimartingale financial model

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10 October 2011

Abstract
A drawdown constraint forces the current wealth to remain above a
given function of its maximum to date. We consider the portfolio optimi-
sation problem of maximising the long-term growth rate of the expected
utility of wealth subject to a drawdown constraint, as in the original setup
of Grossman and Zhou (1993). We work in an abstract semimartingale
financial market model with a general class of utility functions and draw-
down constraints. We solve the problem by showing that it is in fact
equivalent to an unconstrained problem but for a modified utility func-
tion. Both the value function and the optimal investment policy for the
drawdown problem are given explicitly in terms of their counterparts in
the unconstrained problem.

Our approach is very general but has an important limitation in that
we assume all admissible wealth processes have a continuous running max-
imum. This allows us to use Azéma–Yor processes. The proofs also rely
on convergence properties, in the utility function, of the unconstrained
problem which are of independent interest.

Keywords: Portfolio optimisation, Drawdown constraint, Asymptotic
growth rate, Azéma–Yor processes
MSC (2010): 91G10, 60G44, 60G17
JEL Classification: G11

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1 Introduction

We study portfolio optimisation subject to drawdown constraints. Such constraints specify that the investor’s wealth $V_t$ has to remain above a given function $w$ of its maximum to date: $V_t > w(\sup_{u \leq t} V_u)$. The motivating example is the case of a linear $w$, when the current wealth is always greater than a fixed fraction of its past maximum. It is a protection feature which is often promised by investment fund available in the financial markets.

This problem was originally introduced by Grossman and Zhou [12] who considered a power utility investor in a Black-Scholes market who faces a linear drawdown constraint and maximises the long-term (asymptotic) growth rate of the expected utility of her wealth. Grossman and Zhou [12] applied the forward approach and solved the problem using the dynamic programming principle. Later Cvitanic and Karatzas [5] generalised the setting in [12] to a complete $n$-dimensional market with deterministic coefficients. Using martingale theory they were able to link the solution to the optimisation problem with the drawdown constraint to an unconstraint problem which they could solve using the dual approach as in Karatzas, Lehoczky and Shreve [15].

In this paper we also consider maximisation of the asymptotic growth rate of the expected utility of wealth, as introduced in Grossman and Zhou [12]. We are inspired by ideas from [5] but we consider a very general setting. We work in an abstract semimartingale model and the investor is endowed with an arbitrary utility function $U$ and an arbitrary drawdown constraint $w$. We only assume that wealth processes are max-continuous (i.e. have a continuous running supremum) and that $U$ dominates a power function, and has a finite asymptotic elasticity as in Kramkov and Schachermayer [17]. The main contribution of the paper is an equivalence result: the drawdown constrained problem has the same value function as the unconstrained problem but with utility $U \circ F_w$, where $F_w$ is given explicitly in terms of $w$. Moreover, the optimal wealth process for the drawdown constrained problem are obtained as an explicit Azéma-Yor transformation of the optimal wealth process for unconstrained problem.

In the general setting of this paper there is usually no hope to solve explicitly the portfolio optimisation problem. Furthermore the drawdown constraint problem which features a path-dependent constraint on the admissible investment strategies appears significantly more complex at first sight. Rather surprisingly, our results show that in fact this problem is just as easy, or just as hard, as the analogue portfolio optimisation problem with no constraints.

This paper relies in an essential way on the so-called Azéma-Yor processes. They effectively provide us with a bijection between non-negative wealth processes and the wealth processes which satisfy a given drawdown constraint. Azéma-Yor martingales have initially appeared in [1] where they were used to solve the Skorokhod embedding problem. Carraro, El Karoui and Obłój [3] introduced a more general class of Azéma-Yor processes and studied them from an SDE perspective. In particular they investigated their properties in relation to drawdown constraints. These results provided crucial insights for our work. In fact, methods of Cvitanic and Karatzas [4] can be expressed using Azéma-Yor
processes simplifying greatly their proofs, see Section 7.2 below.

Finally, we mention that recently drawdown constraints have also been considered in setups with consumption. Roche [18] investigated maximisation of expected utility of consumption over infinite time horizon for a power utility and under a linear drawdown constraint. Elie and Touzi [9] generalised this to a general class of utility functions in the setting of zero interest rates obtaining explicit representation of the solution. Subsequently, Elie [8] analysed the problem of maximising the expected utility of consumption and terminal wealth on a finite time horizon. He did not have explicit formulae but rather represented the value function as the unique (discontinuous) viscosity solution to the Hamilton-Jacobi-Bellman equation. It is not clear at present if, and to what extent, our methods extend to such setups.

The paper is organised as follows. Firstly, in Section 2, we introduce the financial market, give definitions and formulate the main portfolio optimisation problems of interest. In Section 3, we recall the relevant results on Azéma-Yor processes. Section 4 presents the main result and its proof. It considers the problem with uniform units: the wealth in both the utility function and the drawdown constraint is discounted by the same numéraire. In Section 5, we provide our results for utility of “wealth in dollars” but subject to drawdown condition on the discounted wealth, as in [5, 12]. This requires stronger asymptotic assumptions on $U$ and $w$ as well as deterministic interest rates. Section 4 is devoted to the drawdown constrained optimisation problem with an asymptotically logarithmic utility. Finally, in Section 7, different examples are presented. We first consider a general market model which admits price deflators as in Karatzas and Kardaras [14] and give sufficient conditions for finiteness of the value function. Then in Section 7.2, we specialise to the complete market with deterministic coefficients and give explicit solutions, extending results in [5]. Finally, Section 7.3 provides an explicit solution for an incomplete market model.

The Appendix contains some technical lemmas needed in the proofs but which are of independent interest. In particular we show continuity of the value function – the long-term (asymptotic) growth rate of the expected utility of wealth – in the utility function $U$ and its invariance under perturbation of $U$ on some initial interval $[0, x_0]$.

2 Financial market model

We consider a general financial market model with no frictions. The dynamics of $d$ risky assets are represented by a vector $S_t = (S^1_t, \ldots, S^d_t)$ of semimartingales defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions. For simplicity we assume $\mathcal{F}_0$ is trivial. $S^0_t$ is the riskless asset (money market account) which is a non-decreasing adapted process with $S^0_0 = 1$. The discount factor is denoted $D_t = 1/S^0_t$. The only restriction we impose is that all assets are max-continuous, i.e. $(\sup_{0 \leq u \leq t} S^i_u)_{t \geq 0}$ is a continuous process, $i = 0, 1, \ldots, d$. 

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In this market agents are allowed to invest by trading in the usual self-financing way. However, we restrict their wealth processes to be max-continuous as well.

**Definition 2.1.** An adapted semimartingale \((V_t)\) is called a wealth process if it is strictly positive, max-continuous and there exists an \((F_t)\)-predictable process \(\pi_t = (\pi^1_t, \ldots, \pi^d_t)\) such that \(D_t V_t = V_0 + \sum_{i=1}^d \int_0^t \pi^i_u d(D_u S^i_u)\), where the integrals are assumed to be well-defined.

The set of wealth processes with \(V_0 = v_0\) is denoted \(A(v_0)\).

The choice of the bond \(S^0\) for units is customary. We say that \((N_t)\) is a numéraire if it is a continuous wealth process with \(N_0 = 1\). One can see that \((V_t)\) is a wealth process if and only if it is positive, max-continuous and there exists an \((F_t)\)-predictable process \(\pi_t = (\pi^1_t, \ldots, \pi^d_t)\) such that \(V_t/N_t = V_0 + \sum_{i=1}^d \int_0^t \pi^i_u d(S^i_u/N_u)\).

Note that so far we have not assumed no-arbitrage in the sense of existence of an equivalent martingale measure \(Q\). Neither have we made any strong assumptions on the integrability of \((\pi_t)\) which would make wealth processes \(Q\)-martingales. Instead we consider utility maximisation in a general setting. We are interested in maximising the long-term asymptotic growth rate of the expected utility of wealth. More precisely we consider the following

**Problem 2.2.** Given a numéraire \((N_t)\) and a function \(U\) compute

\[
\text{CER}_{U,N}(v_0) = \sup_{V \in A(v_0)} \mathcal{R}_U(V/N),
\]

where \(\mathcal{R}_U(V/N) = \limsup_{T \to \infty} \frac{1}{T} \log \mathbb{E}\left[U\left(\frac{V_T}{N_T}\right)\right]\),

along with the optimal wealth process which achieves the supremum.

In \(I\), and throughout the paper, we extend log to \(\mathbb{R} \setminus \{0\}\) via \(\log(x) = -\log(-x)\). Naturally, the problem only makes sense under some assumptions on \(U\). We say that \(U\) is a utility function if \(U : (0, \infty) \to \mathbb{R}\) is non-decreasing and concave. In Section \(I\) we consider the above problem for a utility function \(U\) which is either strictly positive or strictly negative. Utility functions with behaviour similar to logarithm are treated in Section \(I\) where \(I\) is modified into \(II\).

The idea to look at the growth rate of the expected utility goes back to Dumas and Luciano \(I\) and Grossman and Zhou \(II\). It is designed to capture the long-horizon optimality and is often a more tractable criterion than the fixed-horizon utility maximisation of terminal wealth. CER above stands for Certainty Equivalent Rate and is interpreted as the critical safe rate – if the investor was offered such (or higher) rate of growth via other investment opportunities she would be happy to abandon the market and move to the alternative investment opportunities. Note that \(V_t = N_t\) is an admissible wealth process so that \(\text{CER}_{U,N} \geq 0\).
Utility maximisation problem above can be solved in number of fairly general setups, see Section 7 below. The aim of this paper is to show a direct link between the solution to this problem and the solution to a seemingly much more complex problem with a pathwise drawdown constraint.

**Definition 2.3.** We say that \( w \) is a drawdown function if it is non-decreasing and
\[
\exists \alpha_1, \alpha_2 : 0 < \alpha_1 \leq w(x)/x \leq \alpha_2 < 1, \quad x \geq 0.
\]
(2)

We say that a wealth process \( (V_t) \) satisfies the \( w \)-drawdown \((w-DD)\) condition relative to a numéraire \( N \) if
\[
\min \{V_t/\sqrt{N_t}, V_t/N_t\} > w(\sup_{u \leq t} V_u/N_u), \quad t \geq 0.
\]

The set of such processes with \( V_0 = v_0 \) is denoted \( A^w_{\nu N}(v_0) \).

The classical example is the linear drawdown function: \( w(x) = \alpha x \). To the best of our knowledge this is the only example which has been considered in the literature, including [5, 8, 9, 12, 18]. We consider a general possibly non-linear drawdown constraint. In particular, Definition 2.3 allows for a piece-wise constant function \( w \). This seems like an interesting example for practical applications – drawdown constraint is updated discretely at times when the wealth process reaches a new threshold.

**Problem 2.4.** Given a numéraire \((N_t)\), a drawdown function \( w \) and a function \( U \) compute
\[
\text{CER}^w_{U,N}(v_0) = \sup_{V \in A^w_{\nu N}(v_0)} \mathcal{R}_U(V/N)
\]
(3)
along with the optimal wealth process which achieves the supremum.

This is a greatly generalised version of the problem introduced by Grossman and Zhou [12] and analysed later by Cvitanić and Karatzas [5]. Firstly, we allow for an almost arbitrary utility function \( U \) and not just the power utility. Secondly, we consider a general possibly non-linear drawdown constraint. Finally, we work in a general semimartingale financial market model and not a complete Black-Scholes-like model. Working in such a generality we can not hope for an explicit solution to Problem 2.4 as in [5, 12]. What we obtain is an explicit formula for the value function and the optimal investment strategy in Problem 2.4 in terms of the the value function and the optimal investment strategy in Problem 2.2 but with a suitably modified utility function.

Note that Problem 2.4 has unified units: both the drawdown and the utility are applied to wealth in units of \( N \). In [5, 12] the drawdown is relative to \( N = S^0 \) but the reward functional is taken of the wealth in dollars: \( \mathcal{R}_U(V) \). This introduces further inhomogeneity and is solved in Section 6 under additional assumptions.

Finally we note that we could allow a wealth process which become zero from some point onwards and likewise to allow the drawdown constraint to become
binding from some point onwards. Everything that follows extends to this setup, with \( U(0) := U(0^+) \), the results stay the same with some additional technical arguments in the proofs related to working on a stochastic interval.

### 3 Drawdown constraints and Azéma–Yor processes

To be able to formulate our results we need to introduce the so-called Azéma–Yor processes and recall their properties established in Carraro, El Karoui and Oblój [3]. This will equip us with the tools necessary to relate Problems 2.2 and their solutions in an explicit manner. We specialise here to the context of the present paper and refer to [3] for the general statements.

**Proposition 3.1** (Carraro, El Karoui and Oblój [3]). Let \( F_0 \) be a locally bounded function,  \( F(x) = F(x_0) + \int_{x_0}^x F'(u)du \), and \((X_t)\) a max-continuous \((\mathcal{F}_t)\)-semimartingale. The associated Azéma–Yor process is given via

\[
M_t^F(X) := F(X_t) - F'(X_t)(X_t - X_t) = F(X_0) + \int_0^t F'(X_u)dX_u, \tag{4}
\]

where \( X_t := \sup_{u \leq t} X_u \). Further

- if \( F' \geq 0 \) then \( M_t^F(X) = F(X_t) \),
- if \( F' > 0 \) then \( M_t^K(M_t^F(X)) = X_t \) with \( K = F^{-1} \) the inverse of \( F \),
- if \( F \) is concave then \( M_t^F(X) \geq F(X_t) \),

The crucial property of Azéma–Yor processes is that they automatically satisfy a drawdown property. In fact from (4), using that \( V \geq 0 \) it is not hard to see that \( X \) satisfies \( w-DD \) relative to \( N \) with \( w(x) = x - K(x)/K'(x) \), \( K := F^{-1} \). Crucially, we can start with \( w \) and find the suitable \( F \).

**Proposition 3.2** (Carraro, El Karoui and Oblój [3]). Let \((N_t)\) be a numéraire and \( w \) a function satisfying \((2)\). Define

\[
K(x) = v_0 \exp \left( \int_{v_0}^x \frac{1}{u - w(u)}du \right), \quad x \geq v_0 > 0, \tag{5}
\]

which is increasing and has a well defined inverse \( F = K^{-1} : [v_0, \infty) \to [v_0, \infty) \). If \((V_t)\in \mathcal{A}(v_0)\) then \( X_t := N_tM_t^F(V_t/N) \in \mathcal{A}_N(v_0) \) and

\[
d \left( \frac{X_t}{N_t} \right) = \left( \frac{X_t}{N_t} - w \left( \frac{X_t}{N_t} \right) \right) d(V_t/N_t). \tag{6}
\]

Conversely, if \( X_t \in \mathcal{A}_N(v_0) \) then \( V_t := N_tM_t^K(X_t/N) \in \mathcal{A}(v_0) \). Furthermore, if \( w \) is nondecreasing then \( K \) is convex and \( F \) is concave.
Proof. Properties of $K$ follow by a straightforward differentiation. Since $N$ is continuous, $X = N M^F(V/N)$ is well defined and max-continuous. Theorem 3.4 in [3] gives the required $w$–drawdown property and (6). In particular $X_t > 0$ and using (4) we see that $X$ satisfies Definition 2.1 with

$$\pi_t^X = \left( F(V_t/N_t) - F'(V_t/N_t)V_t/N_t \right) \pi_t^N + F'(V_t/N_t)\epsilon_t^V,$$

with the obvious (vector) notation. We conclude that $X \in A^w_N(v_0)$. The rest is easily derived from Theorem 3.4 in [3].

By (1), in the above it is sufficient to consider $F(v)$ for $v \geq v_0$ since $V_t/N_t \geq V_0 = v_0$. We are free to define $F$ on $[0, v_0)$ in any way without affecting $X_t$. As we will see later, any extension which preserves the sign, monotonicity and concavity will be allowed. For completeness we specify one such extension by extending $F$ for all positive $v$ as follows

$$F(v) := \begin{cases} K^{-1}(v) & \text{if } v \geq v_0 \\ F'(v_0^+)(v - v_0) + v_0 & \text{if } 0 \leq v < v_0 \end{cases}$$

(7)

so that $F(0) = w(F(v_0)) = w(v_0) > 0$ and $F$ is increasing and concave on $[0, \infty)$ if $w$ is nondecreasing. We write $K_w, F_w$ when we want to stress the dependence on the drawdown function $w$.

4 Main results

We are now ready to formulate our main results. The essence of the results is simple and explicit: the $w$–drawdown problem with a utility function $U$ has the same value as the unconstrained problem with the utility function $U \circ F_w$: $CER^w_{U,N} = CER_{U \circ F_w,N}$, where $w$ and $F_w$ are related by (5) and (7). Further, the optimal wealth process is given by $N_t M^F_w(V^*/N)$, where $V^*$ is the optimal wealth for the unconstrained problem. We impose

Assumption 4.1. Assume that, for some $\varepsilon > 0$, $U$ satisfies either

$$\frac{U(x)}{x^\varepsilon} \xrightarrow{x \to \infty} \infty, \quad \text{and } U \text{ is strictly positive on } (0, \infty),$$

or

$$\frac{U(x)x^\varepsilon}{t \to \infty} \to 0, \quad \text{and } U \text{ is strictly negative on } (0, \infty).$$

This insures that our utility functions are of constant sign and they dominate a power utility. We will further assume that they admit finite Asymptotic Elasticity in the sense of Kramkov and Schachermayer [17]. Recall that throughout the paper a utility function simply means a non-decreasing concave function.

Theorem 4.2. Let $w$ be a drawdown function, $N$ a numéraire, $v_0 > 0$ and $U$ a utility function satisfying Assumption 4.1 and with finite asymptotic elasticity

$$\limsup_{x \to \infty} \frac{xU'_-(x)}{|U(x)|} = \gamma \in (0, \infty).$$

(8)
Recall that $K_w$ is given by (4) and let $F_w$ be its inverse extended to $[0, \infty)$ as in (4) or in any other way which preserves monotonicity and concavity with $F_w(0) \geq 0$. Assume that, for some $\delta > 0$, $\text{CER}_{G,N}(v_0) < \infty$ where $G(x) = U \circ F_w(x)$ when $U < 0$ and $G(x) = (U \circ F_w(x))^{1+\delta}$ when $U > 0$. Then

$$\text{CER}_{U,N}^w(v_0) = \text{CER}_{U \circ F_w,N}(v_0) < \infty$$

and if $(V^*_w) \in A(v_0)$ achieves the maximum in the unconstrained problem then $N_i M_t^P v(V^*/N) \in A_N^w(v_0)$ achieves the maximum in the $w$-drawdown constrained problem.

**Remark 4.3.** It will be clear from the proof, and in particular from Lemma A.1, in the Appendix, that when $U < 0$ we have in fact $\text{CER}_{U,N}^w < \infty$ if and only if $\text{CER}_{U \circ F_w,N} < \infty$.

**Remark 4.4.** For $U \geq 0$ Assumption 4.4 implies $\delta$. It suffices to consider a sequence $x_k \to \infty$ such that $\left(\frac{U(x)}{x^\delta}\right)'|_{x=x_k} > 0$. The reverse is not true as is shown by $U_0$ with $U_0'(x)$ constant on $[x_{2k}, x_{2k+1}]$ and equal to $(x - x_{2k-1} + U_0'(x_{2k+1}))^{-1}$ on $[x_{2k-1}, x_{2k+2}]$ with

$$x_{2k+1} = \inf \left\{ x > x_{2k} : \frac{xU_0'(x)}{U_0(x)} \geq \frac{1}{2} \right\}, \quad x_{2k} = \inf \left\{ x > x_{2k-1} : U_0(x) \leq x^{-1/k} \right\}.$$ 

However for $U < 0$ we need both assumptions: $U(x) = -e^{-x}$ satisfies Assumption 4.4 but not $\delta$ and $-\frac{1}{U_0(x)}$ is a utility function which satisfies $\delta$ but not Assumption 4.4.

**Remark 4.5.** Observe that CER in Problems 2.3 and 2.4 are invariant under a multiplication of $U$ by a positive constant. Further, for a positive utility function $U$, they are invariant under a constant shift of $U$ which preserves the sign. More precisely, write $C = \text{CER}_{U,N}(v_0)$ and let $\kappa > 0$. For any $\delta > 0$, $V \in A(v_0)$, taking $T$ large enough we have

$$\log E[U(V_T/N_T) + \kappa] \leq \log(e^{T(C+\delta)} + \kappa) \leq \log(2e^{T(C+\delta)}).$$

This yields $R_{U+\kappa}(V/N) \leq C + \delta$ and letting $\delta \nrightarrow 0$ we have $\text{CER}_{U+\kappa,N}(v_0) = C$. In particular we may and will assume that a positive utility function has $U(0) > 0$.

**Proof of Theorem 4.2.** We start by observing that with no loss of generality we may take some $\gamma' \in \mathbb{R}, \ |\gamma'| > \gamma$ and assume that

$$0 < \frac{xU'(x)}{\gamma'U(x)} < 1, \quad x > 0.$$  \hfill (9)

To see this note that finite asymptotic elasticity of $U$ implies that there exists $x_0 > 0$ such that (9) holds but for $x \geq x_0$. Now let $\hat{U}(x)$ be equal to $U$ on $[x_0, \infty)$ and $\hat{U}(x) = U(x_0) + U'(x_0)(x-x_0)$ for $x \in [0, x_0]$ which is now a utility function $\hat{U} \geq U$ which satisfies Assumption 4.4 and 4.2 for all $x > 0$. 

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Observe that for any \((V_t) \in \mathcal{A}(v_0)\) and any \(c > 0\), \(\mathcal{R}_U(V/N) = \mathcal{R}_{cU}(V/N)\). Also if \(U_1 \leq U_2\) are two utility functions of the same sign then \(\mathcal{R}_{U_1}(V/N) \leq \mathcal{R}_{U_2}(V/N)\). It follows that if for some \(x_0 > 0\), \(U_1(x) = U_2(x)\) for all \(x \geq x_0\) then \(\mathcal{R}_{U_1}(X/N) = \mathcal{R}_{U_2}(X/N)\) for any \((X_t) \in \mathcal{A}_N^w(v_0)\) since \(X_t \geq w(v_0) > 0\). In consequence \(\text{CER}_{U_1,N}^w(v_0) = \text{CER}_{U_2,N}^w(v_0)\).

We conclude that \(\text{CER}_{U,N}^w(v_0) = \text{CER}_{U,N}^w(v_0)\) while \(\text{CER}_{U,N}^v(v_0)\) follows from Lemma \(\text{X.1}\). Similar reasoning shows that our problem is invariant under a change of \(F_w\) on \([0, v_0]\) which preserves the concavity and monotonicity of \(F_w\) and satisfies \(F_w(0) \geq 0\). Therefore we may, and will, assume that \(U\) satisfies \(\mathcal{B}\) and \(F = F_w\) is given by \(\mathcal{C}\).

Let \((V_t) \in \mathcal{A}(v_0)\) and \(X_t := N_t M^F_t(X/N)\) which is in \(\mathcal{A}_N^v(v_0)\) by Proposition \(3.2\). Now, by Proposition \(3.1\) we obtain directly

\[
U \left( \frac{X_t}{N} \right) = U \left( M^F_t \left( \frac{V}{N} \right) \right) \geq U \left( F \left( \frac{V_t}{N} \right) \right),
\]

which readily implies \(\mathcal{R}_U(X/N) \geq \mathcal{R}_{U \circ F}(V/N)\). Taking supremum over all \(V \in \mathcal{A}(v_0)\) we conclude

\[
\text{CER}_{U,N}^w(v_0) \geq \text{CER}_{U \circ F,N}^w(v_0).
\]

It follows also that if we had equality and the right hand side was attained by a wealth process \((V_t^*)\) then the left hand side is attained by \(N_t M^F_t(X^*/N)\), as required.

It remains to establish the reverse inequality to the one above. To this end we introduce a sequence of problems with modified utility functions. Let

\[
w_n(x) := x - \left(1 + \frac{1}{n}\right) \frac{K(x)}{K'(x)} = w(x) - \frac{1}{n} \frac{K(x)}{K'(x)} = (1 + \frac{1}{n})w(x) - \frac{1}{n}x,
\]

where the equalities follow from \(w(x) = x - K(x)/K'(x)\). Note that \(w_n(x)\) satisfies \(\mathcal{B}\) but may fail to be globally non-decreasing on \((0, \infty)\).

It follows by a direct computation that

\[
K_n(x) = K_{w_n}(x) = v_0 \exp \left( \int_{v_0}^x \frac{1}{u - w_n(u)} \, du \right) = v_0^\frac{1}{n} (K(x))^{\frac{1}{n}}, \quad x \geq v_0.
\]

Consider \((X_t) \in \mathcal{A}_N^w(v_0)\) and let \(Y^n_t := N_t M^{K^n}_t(X/N)\) which is an element of \(\mathcal{A}(v_0)\) by Proposition \(3.2\). Using \(\mathcal{D}\) and the drawdown property of \(X\) we obtain

\[
\frac{Y^n_t}{N_t} \geq K_n \left( \left( \frac{X}{N} \right)_t \right) - k_n \left( \left( \frac{X}{N} \right)_t - w \left( \left( \frac{X}{N} \right)_t \right) \right) = h_n \left( \left( \frac{X}{N} \right)_t \right),
\]

where

\[
h_n(x) := K_n(x) \left( 1 - \frac{K_n'(x)}{K(x)} \right) = K_n(x) \left( 1 - \frac{n}{n+1} \right) = \frac{1}{1+n} K_n(x).
\]
Let \( F_n(v) \) be the inverse of \( K_n(v) \), for \( v \geq v_0 \) and extended to \([0, \infty) \) via \( \text{E} \). Explicitly, we have \( F_n(v) = F(v^{1/n} v^{1/n}) \), \( v \geq v_0 \) and \( F_n(v) = F(v) - 1/n(F(v) - F(v)) \), \( v \in [0, v_0] \). \( F_n \) is increasing and we take \( n \) large enough so that \( F_n(0) > 0 \).

Observe that \( xF_n'(x)/F_n(x) \) is bounded by 1 for \( x > v_0 \) and by \((n+1)/n\) for \( 0 \leq x \leq v_0 \). It follows, using \( \text{E} \), that

\[
x(U \circ F_n(x))' = F_n(x)U'(F_n(x))\frac{xF_n'(x)}{F_n(x)} < \frac{\gamma'(n + 1)}{n} U \circ F_n(x).
\]

Applying Lemma A.3 to \( U \circ F_n \) we deduce that \( U \circ F_n(\lambda x) < \lambda^{2\gamma'} U \circ F_n(x) \). We have

\[
U \circ F_n \left( \frac{Y^n_t}{N_t} \right) \geq U \circ F_n \left( \frac{1}{1 + n} K_n \left( \frac{X}{N} \right)_t \right) \geq \left( \frac{1}{1 + n} \right)^{2\gamma'} U \left( \frac{X_t}{N_t} \right),
\]

The factor of \((1 + n)^{-2\gamma'}\) disappears when we apply \( \frac{1}{t} \log \) and let \( t \to \infty \):

\[
\mathcal{R}_{U \circ F_n}(Y^n_t/N) \geq \mathcal{R}_U(X_t/N).
\]

Taking supremum over \( X \in A^w_N(v_0) \) we conclude that

\[
\text{CER}_{U \circ F_n,N}(v_0) \geq \text{CER}_{U,N}^w(v_0).
\]

Finally, we verify the assumptions of Lemma A.2 in the Appendix. For \( v \geq v_0 \) we have \( F(v) \leq F_n(v) = F(v^{1/n} v^{1/n}) \) and for \( v \in [0, v_0] \) we have \( c_n F(v) \leq F_n(v) \), where \( c_n = 1 + \frac{1}{n} \frac{w(v_0) - v_0}{w(v_0)} \). Together with \( \text{CER}_{G,N}^w < \infty \), Lemma A.2 now yields

\[
\text{CER}_{U \circ F_n,N}(v_0) \xrightarrow{n \to \infty} \text{CER}_{U \circ F_n,N}(v_0) < \infty
\]

which concludes the proof. \( \square \)

From Lemmas A.1 and A.3 and the above proof, we immediately have

**Corollary 4.6.** Under the assumptions of Theorem 4.2 we have for any \( v > 0 \)

\[
\text{CER}_{U,N}(v) = \text{CER}_{U,N}^w(1) = \text{CER}_{U \circ F_n,N}(1) = \text{CER}_{U \circ F_n,N}(v) < \infty
\]

and if \( (V_t^*) \in A(1) \) achieves \( \text{CER}_{U \circ F_n,N}(1) \) then \( (vV_t^*) \) achieves \( \text{CER}_{U \circ F_n,N}(v) \) and \( N_t M_t^{F^w}(vV^*/N) \in A^w_N(v) \) achieves \( \text{CER}_{U,N}^w(v) \).

## 5 Utility of wealth in dollars

We turn now to the inhomogeneous problem as considered by Grossman and Zhou [12]. We seek to maximise the utility of wealth \( U(V_T) \), but the drawdown
constraint is imposed on the discounted wealth process $V_t/S_0$. Note that in the case of a linear constraint, $w(x) = \alpha x$, this is equivalent to saying that the drawdown constraint is growing at a hurdle rate equal to the riskless rate. In analogy to Problems 2.2, 2.4 we define

$$CER_U(v_0) = \sup_{V \in A(v_0)} \mathcal{R}_U(V), \quad CER_{w}^U(v_0) = \sup_{V \in A_{w}^0(v_0)} \mathcal{R}_U(V).$$

(12)

As previously, by considering $V = S_0$ we see that in both cases $CER \geq 0$. In order to be able to relate these problems we essentially need to go back to the homogenous case when $\mathcal{R}_U(V)$ is replaced by $\mathcal{R}_U(V/S_0)$ and for this we need to be able to factor the discounting in and out of the reward functional $\mathcal{R}_U$. This is possible when $U$ is a power utility, $w$ is linear and $S_0$ is deterministic as in [5] and [12]. Here we need to assume this holds asymptotically.

**Assumption 5.1.** Assume the following three conditions hold

(i) $U$ is either strictly positive or strictly negative on $(0, \infty)$ and the following limit exists $xU'(x)/U(x) \to \gamma \in (-\infty, 1) \setminus \{0\}$ as $x \to \infty$,

(ii) the following limit exists $w(x)/x \to \alpha \in (0, 1)$ as $x \to \infty$,

(iii) $S_0$ is deterministic and the following limit exists $r^* := \lim_{T \to \infty} \log S_T^0 / T$.

The first assumption is a strengthened version of the finite asymptotic elasticity of Kramkov and Schachermayer [17] which we assumed earlier in (8). It follows from Lemma A.4 in the Appendix that it implies Assumption 4.1 holds. The second condition above is in fact equivalent to saying that $K$ in [5] also has such (converging) finite asymptotic elasticity. This is immediate since $xK'(x)/K(x) = x/(x - w(x))$. We denote the CRRA (power) utility with $H(\gamma)(p) = x^p \gamma$, $p \leq 1$. We assume $p \neq 0$, which is the case of logarithmic utility treated below in Section 6. Finally, we denote $w_\alpha(x) = \alpha x$ the linear drawdown function.

**Theorem 5.2.** Let $U$ be a utility function and $w$ a drawdown function for which Assumption 5.1 holds. Assume further that $CER_{H(\gamma(1-\alpha)(1+\delta))}(v_0) < \infty$ for some $\delta > 0$. Then

$$CER_{w}^U(v_0) = CER_{w_\alpha}^H(\gamma)(v_0) = CER_{H(\gamma(1-\alpha))}(v_0) + |\gamma| \alpha r^* < \infty$$

and if $(V_t^*)$ achieves the maximum in the unconstrained problem then $S_0^0 M_t^Fw(V_t^*/S_0)$ achieves the maximum in the $w$-drawdown constrained problem, where $F_w$ is as in Theorem 4.2.

**Remark 5.3.** Theorem 1.1 of Grossman and Zhou [12] and Theorem 5.1 of Cvitanić and Karatzas [5] are consequences of the above statement. Namely, they specialise to $w = w_\alpha$, $U = H(\gamma)$ with $\gamma \in (0, 1)$ and a particular (deterministic, constant coefficients) market setup. Standard techniques allow then to compute
apply Lemma A.4 in the Appendix to obtain
\[ 0 < \gamma \]
Remark 5.4. Similarly to Theorem 4.2, it follows from Lemma A.2 that when
\( \gamma < 0 \) the equality \( \text{CER}_U^\gamma = \text{CER}_{H(\gamma(1-\alpha))} + |\gamma|ar^* \) holds without assuming that \( \text{CER}_{H(\gamma(1-\alpha)(1+\delta))} \) is finite.

**Proof.** Consider \((X_t) \in A_{\Sigma_0}^w(v_0)\) and a small \( \varepsilon > 0 \). As \( X_t \geq w(v_0) > 0 \), we can apply Lemma A.4 in the Appendix to obtain
\[
\mathcal{R}_U(X) \leq \mathcal{R}_{H(\gamma(1+\varepsilon))}(X) = \mathcal{R}_{H(\gamma(1+\varepsilon))}(X/S^0) + |\gamma|(1+\varepsilon)r^*.
\]
Recall \( K = K_w \) defined in \( \Theta \) and \( F \) its inverse defined on \([v_0, \infty)\). As noted before, \( xK'(x)/K(x) = x/(x - w(x)) \). This, together with Assumption 5.1, implies that \( K \) has asymptotic elasticity equal to \( 1/(1 - \alpha) \) and hence \( F \) has asymptotic elasticity equal to \( (1 - \alpha) \). Extend \( F \) to \([0, \infty)\) in such a way that it is increasing and concave and on some \([0, a)\) is equal to \( H((1-\alpha)(1+\varepsilon)) \). Lemma A.2 then implies that \( F(x) \leq c_2H((1-\alpha)(1+\varepsilon)) \) for some \( c_2 \geq 1 \) and all \( x \geq 0 \). In consequence, for any \( Y \in \mathcal{A}(v_0) \),
\[
\mathcal{R}_{H(\gamma(1+\varepsilon))}(Y/S^0) \leq \mathcal{R}_{H(\gamma(1-\alpha)(1+\varepsilon)^2)}(Y/S^0) = \mathcal{R}_{H(\gamma(1-\alpha)(1+\varepsilon)^2)}(Y) - |\gamma|(1-\alpha)(1+\varepsilon)^2r^*.
\]
By a similar reasoning and using the assumption of the theorem we conclude that \( \text{CER}_{G_0}(v_0) < \infty \) for \( G(x) = (H(\gamma(1-\alpha)(1+\varepsilon)) \circ F(x^{1+\delta}))^{1+\delta} \) with \( \delta = \varepsilon 1_{\gamma > 0} \) and for \( \varepsilon \) small enough. Applying Theorem 4.2 and combining [13] with [14] we obtain
\[
\text{CER}_U^\gamma(v_0) \leq \text{CER}_{H(\gamma(1+\varepsilon))}(v_0) + |\gamma|(1+\varepsilon)r^* \leq \text{CER}_{H(\gamma(1-\alpha)(1+\varepsilon)^2)}(v_0) + |\gamma|ar^* + \varepsilon |\gamma|r^*(1 - (2 + \varepsilon)(1 - \alpha)).
\]
Taking \( \varepsilon \to 0 \) and invoking Lemma A.2 yields \( \leq \) inequalities in the desired equality. The reverse inequalities are obtained in an analogous manner but exploiting the lower bound in Lemma A.2 and extending \( F \) so that it is equal to \( H((1-\alpha)(1-\varepsilon)) \) on \([0, a)\). Finally, a similar reasoning shows we can replace \( w \) by \( w_\alpha \).

**Remark 5.5.** Similarly to Theorem 4.2 and Corollary 4.6, \( \text{CER}_U^\gamma(v_0) \) above does not depend on \( v_0 \) and the optimal strategy for the unconstrained problem scales linearly in the initial wealth.

**Corollary 5.6.** In the setup of Theorem 5.2 we have
\[
\text{CER}_{H(\gamma(1-\alpha))}(v_0) = \text{CER}_{U\alpha F}(v_0).
\]

This Corollary follows from the proof of Theorem 5.2 or more directly from Lemma A.1 in the Appendix. Naturally, similar statements can be made relating in general CER for power utility and for \( U \) which satisfies (i) in Assumption
5.1 This is not surprising in the light of the results on the so-called turnpike theorems. In this stream of literature authors study the convergence of the value function and the optimal strategy for the Merton problem of maximising utility of terminal wealth as the horizon $T$ tends to infinity. In particular, Hubermann and Ross [13] argue that, in the case of a complete discrete market, the convergence of optimal strategies is equivalent to the convergence of the relative risk aversion, i.e. $-\frac{zU''(x)}{U'(x)} \to 1 - \gamma$, which is essentially (i) in Assumption 5.1. Huang and Zariphopoulou study the problem for a continuous time complete market model with deterministic coefficients, as in Section 7.2. They find sufficient conditions on $U$ for the optimal strategy to converge to the optimal strategy coming from the problem with a power utility. Note that our results apply in a much more general context but are also much weaker. Problem 2.2 looks at maximising the long-term asymptotic growth rate of the expected utility and the above Corollary shows that the resulting value function is the same when two utility functions have the same asymptotic behaviour. It does not say anything precise about finite horizon utility maximisation and its convergence.

6 Logarithmic Utility

So far we have only considered utility functions with constant sign and which dominated a power utility, as in Assumption 4.1. In this section, we consider utility functions akin to $U(x) = \log x$. The results are very close in spirit to ones in the previous two sections, but in fact require less technicalities in the proofs. First we need to introduce a modified version of maximisation criterion in (1).

**Problem 6.1.** Given a numéraire $(N_t)$, a drawdown function $w$ and function $U$ compute

$$\tilde{\text{CER}}_{U,N}(v_0) = \sup_{V \in A(v_0)} \tilde{R}_U(V/N),$$

$$\tilde{\text{CER}}^w_{U,N}(v_0) = \sup_{V \in A^w_N(v_0)} \tilde{R}_U(V/N),$$

$$\text{where } \tilde{R}_U(V/N) = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}[U(V_T / N_T)],$$

along with the optimal wealth processes which achieve the supremum.

**Theorem 6.2.** Let $N$ be a numéraire, $w$ a drawdown function and $U$ a utility function satisfying

$$\limsup_{x \to \infty} x U'(x) = \gamma < \infty.$$

Let $F_w$ be as in Theorem 4.2. If $\tilde{\text{CER}}_{\log^+,N}(v_0) < \infty$ then

$$\tilde{\text{CER}}^w_{U,N}(v_0) = \text{CER}_{U \circ F_w,N}(v_0) < \infty$$
and if \((V_t^*) \) achieves the maximum in the unconstrained problem then \(N_t M^{F^w}_t (V^*/N)\) achieves the maximum in the \(w\)-drawdown constrained problem, where \(F_w\) is as in Theorem 4.2.

**Proof.** The first part of the proof is identical to the first part of the proof of Theorem 4.2 and yields
\[
\tilde{\text{CER}}^w_{U,N}(v_0) \geq \tilde{\text{CER}}_{U \circ F,N}(v_0).
\]

We also obtain that \(N_t M^{F^w}_t (V^*/N)\) is optimal for constrained problem when \(V^*_t/N_t\) is optimal for unconstrained.

In analogy to the proof of Theorem 4.2, one argues that we can take \(F_w\) defined by (7) and replace \(U\) by another utility function which satisfies, for some \(\gamma' > \gamma\),
\[
xU'_t(x) < \gamma' < \infty, \quad x > 0.
\]

We leave the details to the reader. Notice also that \(\tilde{R}_{\lambda U} (V/N) = \lambda \tilde{R}_{U} (V/N)\) and, without loss of generality, we can take \(\gamma' = 1\). Finally, \(e^{U(x)}\) is non-decreasing and satisfies (9). Applying Lemma A.3, we deduce
\[
U(\lambda x) \leq U(x) + \log \lambda, \quad \lambda > 1, \quad x > 0.
\]

Define \(w_n, K_n, F_n, Y_n\) as in the proof of Theorem 4.2 above. Then
\[
Y^*_n \geq \frac{1}{1+n} K_n \left( \frac{X}{N} \right)_t,
\]
and therefore
\[
U \circ F_n \left( \frac{Y^*_n}{N_t} \right) \geq U \circ \left( \left( \frac{1}{1+n} \right)^{\frac{n+1}{n}} X_t \right) \geq U \left( \frac{X_t}{N_t} \right) - \frac{n+1}{n} \log(1+n).
\]

Now we apply \(\frac{1}{n} \log\) and let \(t \to \infty\) which yields:
\[
\tilde{R}_{U \circ F_n} (Y^*_N/N) \geq \tilde{R}_{U} (X/N).
\]

Taking supremum over \(X \in \mathcal{A}^v_N(v_0)\) we conclude that
\[
\tilde{\text{CER}}_{U \circ F_n, N}(v_0) \geq \tilde{\text{CER}}^w_{U, N}(v_0).
\]

It remains to establish the convergence in \(n\) on the LHS. Observe that
\[
x(U \circ F)'_\downarrow (x) = xU'_t(F(x)) F'(x) = x \frac{U'_t(F(x))}{K'(F(x))} = \frac{F(x) - w(F(x))}{F(x)} F(x) U'_t(F(x)) < 1.
\]

As above, applying Lemma A.3 to \(\exp(U \circ F)\), we see that (16) holds with \(U \circ F\) in place of \(U\). This yields
\[
U \circ F_n (x) - U \circ F(x) \leq U \circ F(\lambda_n(x)x) - U \circ F(x) \leq \log \lambda_n(x),
\]
where $\lambda_n(x) = (x^{1/n})^1 \lor 1$. We also have that $F_n(x) \geq F_n(0)/F(0) \cdot F(x)$ and, therefore,

$$U \circ F_n(x) - U \circ F(x) \geq \log F_n(0)/F(0).$$

(17)

Taking $x = \left(\frac{Y_t}{N_t}\right)_{t \geq 0}$, dividing by $t$ and letting $t \to \infty$ we obtain:

$$0 \leq \tilde{R}_{U\circ F_n,N}(Y/N) - \tilde{R}_{U\circ F,N}(Y/N) \leq \frac{1}{n}\tilde{CER}_{\log^+,N}(v_0),$$

Taking the limit in $n$ obtain convergence of $\tilde{CER}_{U\circ F_n,N}(v_0)$ to $\tilde{CER}_{U\circ F,N}(v_0)$ and thus the reverse inequality in the theorem. 

We close this section with a result similar to Theorem 5.2. The proof follows closely the arguments in Section 5 and we omit it for the sake of brevity.

**Theorem 6.3.** Let $U$ be a utility function with $xU'(x) \to \gamma \in (0, \infty)$ as $x \to \infty$ and $w$ a drawdown function such that (ii) and (iii) in Assumption 5.1 hold. Assume further that $\tilde{CER}_{\log^+}(v_0) < \infty$. Then

$$\tilde{CER}_w(v_0) = \gamma (1 - \alpha)\tilde{CER}_{\log}(v_0) + \gamma \alpha r^*$$

and $S_0^m M^F(V^*/S^0)$ achieves the maximum in the drawdown constrained problem if $(V^*_t)$ achieves the maximum in the unconstrained problem.

### 7 Examples

We discuss now some examples. Our aim is twofold. First, we want to give an example of a rather general setup in which sufficient conditions can be found which guarantee finiteness of CER for the unconstrained problem, as assumed in Theorem 4.2. Second, we want to discuss specific examples when the unconstrained, and hence also the drawdown constrained, portfolio optimisation problem is solved explicitly. In particular we relate our results and methods to the ones in [5].

#### 7.1 Market with price deflators

We start by assuming existence of a price deflator (or a state price density) process. In the setup of Section 2 we further assume that all $S_i^t$ are continuous and that there exists a $\mathbb{P}$-local martingale $(Z_t)$, $Z_t > 0$ for all $t \geq 0$, such that $(Z_tD_tS_i^t)$ are $\mathbb{P}$-local martingales, $i = 1, \ldots, d$. Note that we do not necessarily assume that $(Z_t)$ is a true martingale and hence that an equivalent martingale measure exists. Our setup is in fact analogous to the most general setup in which stochastic portfolio optimisation is considered, see Fernholz and Karatzas [10]. Note that if $(V_t) \in \mathcal{A}(v_0)$ then

$$d(Z_tD_tV_t) = D_t(V_t - \pi_t S_t)dZ_t + \pi_t d(Z_tD_tS_t),$$
so that \((Z_t D_t V_t)\) is a positive \(P\)-local martingale and hence a supermartingale. Karatzas and Kardaras [14] show that the existence of \((Z_t)\) is equivalent to the NUPBR condition (No Unbounded Profit With Bounded Risk). This condition is weaker than usual NFLVR condition from [6] and, therefore, some (weak) arbitrage opportunities may exist in such setup, see examples constructed in [14].

**Lemma 7.1.** Let \(N\) be a numéraire. The following implications hold for any \(p < 1\), \(p \neq 0\), and \(v_0 > 0\)

\[
\mathcal{R}_{H(-p/(1-p))}(DZ) < \infty \quad \implies \quad \text{CER}_{H(p)}(v_0) < \infty,
\]

\[
\mathcal{R}_{H(-p/(1-p))}(DZN) < \infty \quad \implies \quad \text{CER}_{H(p),N}(v_0) < \infty.
\]

**Proof.** Let \((V_t) \in A(v_0)\) so that \((Z_t D_t V_t)\) is a \(P\)-local martingale, as above. For \(p < 0\) we have

\[
E[V_T^p] = E[(D_T Z_T)^{-p}(Z_T D_T V_T)^p] 
\geq \left( E[(D_T Z_T)^{-p/(1-p)}] \right)^{(1-p)} \left( E[Z_T D_T V_T]^p \right) \geq v_0^p \left( E[(D_T Z_T)^{-p/(1-p)}] \right)^{(1-p)},
\]

where we used reversed Hölder’s inequality and the fact that a non-negative local martingale is a supermartingale. The inequalities above are reversed when we divide both sides by \(\frac{1}{p} < 0\) and the claim follows. The case \(p \in (0, 1)\) is even more straightforward – it suffices to reverse the inequalities in the above. The case with numéraire is entirely analogous.

The lemma above gives an example of sufficient conditions to apply Theorems 4.2 and 5.2 since they both require that \(\text{CER}_{G,N}\) or \(\text{CER}_{G}\) is finite. For the latter \(G\) is a power utility function and Lemma 7.1 applies directly. For the former we would need to bound \(G\) by a power utility.

Naturally, in the current very general setup there might be little hope to compute \(\text{CER}_U\) or find the optimal wealth process. However, one might expect this to be the simplest portfolio optimisation problem to solve. The strength of our results is to show that solving the seemingly much more complex problem with drawdown constraint on wealth paths is in fact equally simple (or hard).

Karatzas and Kardaras [14, Theorem 4.12] also show that the existence of \((Z_t)\) is equivalent to the existence of a benchmark numéraire \(\tilde{N}\) such that \(V/\tilde{N}\) is a supermartingale for any \(V \in A(v_0)\), see also Christensen and Larsen [4]. This readily implies that \(\widetilde{\text{CER}}_{\log(v_0)} = \mathcal{R}_{\log}(\tilde{N})\). Indeed, considering \(V \in A(v_0)\) and applying Jensen’s inequality gives

\[
\limsup_{T \to \infty} \frac{1}{T} E \log V_T \leq \limsup_{T \to \infty} \frac{1}{T} E \log \tilde{N}_T + \limsup_{T \to \infty} \frac{1}{T} E \log \frac{V_T}{\tilde{N}_T} \leq \limsup_{T \to \infty} \frac{1}{T} E \log \tilde{N}_T.
\]

This observation essentially goes back to Bansal and Lehmann [2]. In a no-arbitrage complete market model \((Z_t)\) is the density \(\frac{dQ}{dP}\) where \(Q\) is the equivalent martingale (risk-neutral) measure. Completeness means that \((D_t Z_t)^{-1}\) is an
admissible wealth process and thus the benchmark numeraire. In particular, in the setting of Theorem 6.3, we have
\[
\widehat{\text{CER}}_U(v_0) = \gamma r^* - \gamma (1 - \alpha) \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \log Z_T.
\]

7.2 Complete market model with deterministic coefficients

We consider now the classical complete financial market model with deterministic coefficients. \(W_t = (W^1_t, \ldots, W^d_t)'\) is a standard \(d\)-dimensional Brownian motion and \((\mathcal{F}_t)\) is the augmentation of its natural filtration. Here ‘ denotes vector transpose. \(S^0_t = \exp(\int_0^t r_u du)\) is deterministic and \(\frac{1}{T} \int_0^T r_u du \to r^* \geq 0\) as \(T \to \infty\). Each asset follows dynamics given by
\[
dS^i_t = \mu^i_t dt + \sigma^i_t dW^i_t, \quad S^i_0 = s^i_0 > 0
\]
where \(\mu^i_t\) and \(\sigma^i_t\) are bounded deterministic functions and \(\sigma_t\) is invertible. Recall Definition 2.1 of wealth process and let \(\tilde{\pi}^i_t := \pi^i_t S^i_t / V_t\) be the proportion of wealth invested in the \(i^{th}\) asset so that
\[
d(V_t) = \sum_{i=1}^d \tilde{\pi}^i_t V_t d(S^i_t).
\]
The market price of risk is given as \(\theta_t := \sigma^{-1} (\mu_t - r_t I)\), where \(I\) is a \(d\)-dimensional vector with all entries equal to one. We assume \(\theta_t\) is also bounded and that
\[
||\theta^*||^2 := \lim_{T \to \infty} \frac{1}{T} \log \int_0^T ||\theta_u||^2 du \quad \text{is well defined and finite.}
\]
The state price density
\[
Z_t := \exp \left\{ - \int_0^t \theta_u^* dW_u - \frac{1}{2} \int_0^t ||\theta_u||^2 du \right\}
\]
is a \(\mathbb{P}\)-martingale which defines the unique risk neutral measure \(Q\), \(\frac{dQ}{d\mathbb{P}} |_{\mathcal{F}_t} = Z_t\).

To solve \(\text{CER}_{H(p)}\) one first considers the problem of maximising the expected utility of wealth at a given horizon \(T\). The solution is obtained using, by now standard, convex duality arguments, see Karatzas, Lehoczky and Shreve [15] or Karatzas and Shreve [16, pp. 97–118]. The optimal wealth process \(V^*\) is described explicitly via
\[
\tilde{\pi}^*_t = \frac{1}{1 - p} \theta_t^* \sigma^{-1}_t
\]
and in particular it is independent of the time horizon \(T\). We conclude that it is also optimal for the long-term asymptotic growth rate optimisation. Taking limit of the value functions for the finite horizon problem we obtain
\[
\text{CER}_{H(p)}(v_0) = R_{H(p)}(V^*) = |p| r^* + \frac{|p|}{2(1 - p)} ||\theta^*||^2.
\]
Note that the difference of a factor $|p|$ when compared to [12, 5] is immediate since they consider $\frac{1}{|p|} R_{H(p)}(V)$ instead of $R_{H(p)}(V)$.

Applying Theorem 5.2 for a utility function $U$ and a drawdown function $w$, which satisfy Assumption 5.1, we obtain

$$\text{CER}^w_U(v_0) = \text{CER}_{H(\gamma(1-\alpha))}(v_0) + |\alpha r^*| = |\gamma| \left( r^* + \frac{(1-\alpha)}{2(1-\gamma(1-\alpha))} ||\theta^*||^2 \right)$$

which is achieved by the optimal wealth process $X_t = S^0_t M^F_t (V^*/S^0)$. Using (6) we see that

$$d(D_t X_t) = (D_t X_t - w(D_t X_t)) \sum_{i=1}^d \left( \frac{1}{1-\gamma(1-\alpha)} \theta'_i \sigma_i^{-1} \right) X_t d(D_t S^i_t) / D_t S^i_t$$

In particular, we recover Theorem 5.1 in [5] as the special case $U = H(\gamma)$, $\gamma \in (0,1)$ and $w(x) = \alpha x$. It is insightful to understand better how the objects in [5] relate to the tools of our paper. In fact the Auxiliary problem introduced and solved in [5] is nothing else but $\text{CER}_{U \circ F}(v_0) = \text{CER}_{H(\gamma(1-\alpha))}(v_0)$. Indeed, the process $N^\pi(t)$ defined in (4.1) therein is simply $S^0_t M^F_t (X/S^0) = Y_t$ and $\hat{\pi}_t = \frac{1}{1-\gamma(1-\alpha)} \theta'_t \sigma_t^{-1}$.

### 7.3 Incomplete market example

We present now an example of an incomplete market model where we can solve explicitly the drawdown constrained problem applying Theorem 5.2 and adapting methods of Fleming and Sheu [11] to solve the unconstrained problem.

Consider a market with constant interest rate $r$ and one risky asset $S(t)$ evolving according to

$$\frac{dS(t)}{S(t)} = (\mu_1 + \mu_2 x(t)) dt + \sigma dW^1_t + \rho dW^2_t,$$

$$d x(t) = b x(t) dt + dW^1_t,$$

where $W^1, W^2$ are two independent Brownian motions and $x(t)$ has an interpretation of an economical factor.

In Theorem 3.1 in [11] the authors provide a link between Problem 2.2 with a power utility function $H(\gamma)$ and a viscosity solution of the dynamic programming equation. In Theorem 4.1 the optimal investment policy is found. We refer the reader to [11] for further details of the method. In the last section of their paper Fleming and Sheu consider Vasicek interest rate model with a single stock and give an explicit solution to the utility maximisation problem. Our model above is slightly different but we are still able to use their solution.

The difference with Fleming and Sheu [11] example is that the interest rate is given by $r(t) = r$ in our work and by $r(t) = \lambda x(t) - \frac{b}{\rho}$ in theirs, which requires us to change some coefficients in final formulae in [11]. More precisely,
assume $\gamma < 0$ and $\mu^2 \geq \sigma^2 (K'(\gamma))^2$ where $K'(\gamma)$ is defined below. Then the value function is equal to

$$CER_{H}(v_0) = \frac{1}{2} K'(\gamma) + \frac{1}{2} |\eta|^2 + 1/2 \frac{\gamma}{1 - \gamma} \frac{(\mu_1 + \sigma \eta)^2}{\sigma^2 + \rho^2} + |\gamma|r,$$

where

$$\eta = -\frac{\gamma}{1 - \gamma} \frac{\mu_2 + K'(\gamma) \sigma_1}{(D(\gamma) + K'(\gamma) E(\gamma))(\sigma^2 + \rho^2)}$$

and

$$E(\gamma) = 1 + \frac{\gamma}{1 - \gamma} \frac{\sigma^2}{\sigma^2 + \rho^2},$$

$$K'(\gamma) = -\frac{b + \gamma}{1 - \gamma} \frac{1}{\sigma^2 + \rho^2} \frac{\mu_2 \sigma^2}{1 + \gamma \frac{\sigma^2}{\sigma^2 + \rho^2}} - \frac{1}{\sqrt{1 + \gamma \frac{\sigma^2}{\sigma^2 + \rho^2}}} \cdot \left( -\frac{\gamma}{1 - \gamma} \frac{\mu_2}{\sigma^2 + \rho^2} + \frac{b + \gamma}{1 - \gamma} \frac{1}{\sigma^2 + \rho^2} \right)^{1/2},$$

$$D(\gamma) = -\sqrt{1 + \gamma \frac{\sigma^2}{\sigma^2 + \rho^2}} \left( -\frac{\gamma}{1 - \gamma} \frac{\mu_2}{\sigma^2 + \rho^2} + \frac{b + \gamma}{1 - \gamma} \frac{1}{\sigma^2 + \rho^2} \right)^{1/2}. $$

And the optimal investment policy $\hat{\pi}_t$, which is fraction of wealth invested in risky asset at time $t$, is given by

$$\hat{\pi}_t = D(\gamma) x(t) + a(\gamma),$$

for some constant $a(\gamma)$.

In the setting of Theorem 5.2 we obtain that

$$CER_U(v_0) = \frac{1}{2} K'(\gamma(1-\alpha)) + \frac{1}{2} |\eta|^2 + 1/2 \frac{\gamma(1 - \alpha)}{1 - \gamma(1 - \alpha)} \frac{(\mu_1 + \sigma \eta)^2}{\sigma^2 + \rho^2} + |\gamma|(1 - \alpha)r,$$

where $\gamma < 0$ and $\alpha \in (0, 1)$ are defined in Assumption 5.1. Note that we could also consider $\gamma \in (0, 1)$ under the additional parameter restriction which makes appropriate $K'(\gamma)$ well defined.

## Appendix

We state and prove here lemmas used in the proofs in the main body of the paper. In fact the first two lemmas are of independent interests. Lemma A.1 shows that $CER_U$ is invariant under perturbation of $U$ on some initial interval $[0, x_0]$ and Lemma A.2 studies convergence of $CER_{U_n} \to CER_U$ as $U_n \to U$.

**Lemma A.1.** Let $U$ and $\tilde{U}$ be two continuous non-decreasing functions with well defined left derivatives. Assume $U, \tilde{U}$ are either both positive or both negative,
they satisfy Assumption 4.1, $U$ admits finite asymptotic elasticity and $\check{U}$ satisfies (9). Assume that $U \leq \check{U}$ and $\check{U}(x) = U(x)$, $x \geq x_0 > 0$ for some $x_0 > 0$. Then, with $N$ a numéraire,

$$CER_{\check{U},N}(v_0) = CER_{U,N}(v_0), \quad CER_{\check{U}}(v_0) = CER_{U}(v_0).$$

In particular, in both equations, the LHS is finite if and only if the RHS is finite.

**Proof.** For $U, \check{U} \geq 0$ we have $U(x) \leq \check{U}(x) \leq U(x) + U(x_0)$ and the claim follows by the argument in Remark 4.5.

Consider the case of negative $U, \check{U}$. We first argue that for $K$ large enough and for any positive process $(\xi_t)$ we have

$$\mathcal{R}_U(\xi) = \limsup_{T \to \infty} \frac{1}{T} \log E[\check{U}(\xi_T)] = \limsup_{T \to \infty} \frac{1}{T} \log E[\check{U}(\xi_T)1_{\xi_T \leq K^\tau}]. \quad (19)$$

Obviously $0 \geq U(\xi_T)1_{\xi_T \leq K^\tau} \geq U(\xi_T)$ so the above holds if $\zeta := \mathcal{R}_U(\xi) = \infty$.

Using Assumption 4.1 on $\check{U}$, for some $\varepsilon > 0$ and for $K$ large enough we have $E[\check{U}(\xi_T)1_{\xi_T \geq K^\tau}] \geq E[\check{U}(K^\tau)] \geq -K^{-T\varepsilon}$ which yields

$$E[\check{U}(\xi_T)] \leq E[\check{U}(\xi_T)1_{\xi_T \leq K^\tau}] \leq E[\check{U}(\xi_T)] + K^{-T\varepsilon} \quad (20)$$

If $\zeta = -\infty$ then for any constant $L > 0$, for $T$ large enough, $E[\check{U}(\xi_T)] \leq -\exp(TL)$ and it follows that (19) holds. Assume now that $\zeta$ is finite. Taking $K > 0$, possibly increasing $K$ so that $K > \exp((\zeta + \kappa)/\varepsilon)$, we have $E[\check{U}(\xi_T)] \geq e^{-T(\zeta + \kappa)}$ for $T$ large enough. It follows that

$$E[\check{U}(\xi_T)] \leq E[\check{U}(\xi_T)1_{\xi_T \leq K^\tau}] \leq E[\check{U}(\xi_T)] + K^{-T\varepsilon}$$

$$= E[\check{U}(\xi_T)] \left(1 - \frac{K^{-T\varepsilon}}{E[\check{U}(\xi_T)]}\right) \leq E[U_n(\xi_T)](1 - e^{-\varepsilon \ln K - \zeta - \kappa_T})$$

and taking $T \to \infty$ we see that (19) holds.

Consider now the process $X_t := S_t^0 F_t(Y/S^0)$, where $F_t(x) = v_0^\delta x^{1-\delta}$, $\delta \in (0,1)$. Explicitly we have $X_t = v_0^\delta S_t^0 \left(\delta(Y/S^0)_t + (1-\delta)Y_t/S^0_t\right)(Y/S^0)_t$ and $X_t \geq S_t^0 v_0^\delta (Y/S)^{1-\delta} \geq \delta v_0$. In consequence we have

$$E[\check{U}(X_T)1_{X_T \leq K^\tau}] \geq E\left(V_0^\delta Y_T(Y/S^0)_T^{-\delta}\right)1_{X_T \leq K^\tau} \geq E\left(cK^{-\frac{\delta}{1-\delta}}Y_T\right)1_{X_T \leq K^\tau}$$

for $c = (v_0^\delta)^{1-\delta/1-\delta} > 0$ and where in last inequality we used that $K^T \geq X_T \geq v_0^\delta (Y/S^0)_T$. Now, using Lemma 4.3 we conclude

$$E[\check{U}(X_T)1_{X_T \leq K^\tau}] \geq \left(cK^{-\frac{\delta}{1-\delta}}T\right)^{\gamma'} E[\check{U}(Y_T)1_{Y_T \leq K^\tau}] \geq \left(cK^{-\frac{\delta}{1-\delta}}T\right)^{\gamma'} E[\check{U}(Y_T)1_{Y_T \leq c_1 K^\tau/(1-\delta)}]$$

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Last inequality is valid as $1_{Y_T \leq \varepsilon, K_T^{(1-\varepsilon)}} \geq 1_{X_T \leq K_T}$ with $c_1 = (\delta c_0)^{-1/(1-\delta)}$. Taking $\frac{1}{\delta}$ log and passing to the limit in $T$, and using (19), gives

$$\mathcal{R}_G(X) \geq \mathcal{R}_G(Y) - \frac{|\gamma'|}{1-\delta} \log M.$$ 

Since $X$ satisfies the drawdown property, we have $X_t \geq \delta x_0$ which readily implies that $\mathcal{R}_U(X) = \mathcal{R}_G(X)$. Taking supremum over all $Y \in A$ we obtain

$$\text{CER}_U(v_0) \geq \text{CER}_G(v_0) - \frac{|\gamma'|}{1-\delta} \log K$$

Taking limit in $\delta \searrow 0$ we obtain one inequality, while the reverse inequality follows since $U \leq \bar{U}$. The case with numéraire is analogous but we take $X_t = N_t M_t^{\delta} (Y/N)$.}

\textbf{Lemma A.2.} Let $N$ be a numéraire and $U_n \xrightarrow{n \to \infty} U$, all nondecreasing functions of the same sign, continuous with a well defined left derivative, satisfying Assumption 4.1 and (9). Assume further that for some $c > 1$ and $c_n \to 1$

$$\forall \delta > 0 \exists n_0, \forall n \geq n_0 \quad c_n U(x) \leq U_n(x) \leq U(cx(x^\delta \vee 1)), \quad x \geq 0.$$  

If $U_n, U$ are negative we have, for any $v_0 > 0$,

$$\text{CER}_{U_n N}(v_0) \xrightarrow{n \to \infty} \text{CER}_{U N}(v_0),$$  

$$\text{CER}_{U_n}(v_0) \xrightarrow{n \to \infty} \text{CER}_{U}(v_0).$$  

If $U_n, U$ are positive the above holds assuming that $\text{CER}_{G, N}(1) < \infty$ and $\text{CER}_{G}(1) < \infty$ respectively, where $G(x) := U(x)^{1+\delta}$ for some $\delta > 0$. Consequently, we then have $\text{CER}_{U, N}(v_0) < \infty$ and $\text{CER}_{U}(v_0) < \infty$ respectively.

\textbf{Proof.} As in the proof of Theorem 4.3 by Lemma A.3 all the CER above are invariant under a change of $v_0$. Let $(V_t)$ be a wealth process. We prove both statements in (19) simultaneously. They follow respectively by taking $\xi_t = \frac{V_t}{V_0}$ and $\xi_t = V_t$ in what follows. First we argue that $K$ large enough, for all $n$, holds with $U_n$ in place of $U$. The case of negative $U_n, U$ follows from the proof of Lemma A.1 above. The only modification is that we use the assumption $U_n \geq c_n U$ and change $K^{-T \varepsilon}$ to $c_n K^{-T \varepsilon}$ on the RHS of (20), where $\varepsilon$ does not depend on $n$.

Consider $U_n, U \geq 0$. Using Remark 4.5 we may assume that $U$ and $U_n$ are bounded away from zero. Further, Assumption 4.1 implies that for some $x_0 > 0$ we have $x_0 < x < U(x)^{1/\varepsilon}$. Combining this with Lemma A.3 below gives, for $x > x_0$,

$$U(x^{1+\delta})^{1+\delta} \leq \left( x^{\gamma' \delta} U(x) \right)^{1+\delta} \leq U(x)^{1+\delta (1+\gamma'(1+\delta))}$$

and the last exponent is arbitrary close to 1 as $\delta \searrow 0$. Remark 4.5 allows us to discard the behaviour of $U$ on $[0, x_0]$ and we conclude that finiteness of $\text{CER}_{G, N}$
(reps. CER\(_G\)) for some \(\delta > 0\) is equivalent to finiteness of CER\(_{G,N}\) (reps. CER\(_G\)) for some \(\delta > 0\), where \(G(x) = U(x^{1+\delta})^{1+\delta}\).

Using Chebyshev’s inequality we obtain
\[
\mathbb{P}(\xi_T \geq K^T) \leq \frac{EU_n(\xi_T)}{U_n(K^T)} \leq \frac{EU_n(\xi_T)}{K^{\varepsilon T}}.
\]

In the last inequality we used again the fact that \(U_n \geq c_n U\) so that in Assumption \(\text{II}\) we may take the same \(\varepsilon\) for all \(U_n\). Combining this with Hölder’s inequality with \(p = 1 + \delta\) and \(1/p + 1/q = 1\) we have
\[
\mathbb{E}[U_n(\xi_T)1_{\xi_T \geq K^T}] \leq (EU_n(\xi_T)p)^{1/p} \mathbb{P}(\xi_T \geq K^T)^{1/q} \leq \frac{(EU_n(\xi_T)p)^{1/p}}{K^{\varepsilon T/q}} \frac{EU_n(\xi_T)^{1/q}}{K^{\varepsilon T}}.
\]

for some \(c_1 > 0\) and where in the last line we first took any \(\kappa > 0\), then \(T\) large enough and then \(K\) large enough, and where we used Lemma \(\text{A.3}\) to ignore the constant \(c\). It follows that for such \(T\) and \(K\),
\[
\mathbb{E}[U_n(\xi_T)] \geq \mathbb{E}[U_n(\xi_T)1_{\xi_T \leq K^T}] \geq \mathbb{E}[U_n(\xi_T)] - \kappa.
\]

To obtain \(\text{II}\) it suffices to observe that
\[
\log \mathbb{E}[U_n(\xi_T)] \leq \log (\mathbb{E}[U_n(\xi_T)] - \kappa) + \log(1 + \kappa/U(0)),
\]
where we recall that \(U(0) > 0\) by Remark \(\text{I}\).

The assumption \(U_n \geq c_n U\) instantly gives \(\mathcal{R}_{U_n}(\xi) \geq \mathcal{R}_U(\xi)\). For the reverse inequality, take \(\delta > 0\), large \(K\) as above and \(n\) large enough so that the assumptions yield
\[
U_n(\xi_T)1_{\xi_T \leq K^T} \leq U(c\xi_T(\xi_T^{1/\delta} \vee 1))1_{\xi_T \leq K^T} \leq U(cK^{\delta T}\xi_T)1_{\xi_T \leq K^T} \leq (cK^{\delta T})^{\gamma'} U(\xi_T)1_{\xi_T \leq K^T}.
\]

For \(U_n, U < 0\), the above and \(\text{I}\) imply that CER\(_{U_n,N}\) is finite if and only if CER\(_{U,N}\) is finite and likewise for CER\(_{U_n}\) and CER\(_U\). Assume therefore that all are finite. In the case of \(U_n, U > 0\) this is implied by the additional assumption. Combining the above and \(\text{II}\) gives
\[
0 \leq \mathcal{R}_{U_n}(\xi) - \mathcal{R}_U(\xi_T)
\]
\[
\leq \limsup_{T \to \infty} \frac{1}{T} \log \mathbb{E}[(cK^{\delta T})^{\gamma'} U(\xi_T)1_{\xi_T \leq K^T}] - \limsup_{T \to \infty} \frac{1}{T} \log \mathbb{E}[(U(\xi_T))^{1_{\xi_T \leq K^T}]}
\]
\[
= \limsup_{T \to \infty} \frac{\gamma'}{T} \log (cK^{\delta T}) = \gamma' \delta \log K.
\]
As this bound is independent of \((\xi_T)\) taking \(\delta \to 0\) gives the desired convergence of certainty equivalent rates.

The following Lemma is a slight extension of the first part of Lemma 6.3 in Kramkov and Schachermayer [17]. We omit the proof as it is identical to that in [17].

**Lemma A.3.** Let \(U\) be a continuous nondecreasing function with a well defined left derivative and which satisfies (9). Then \(U(x) \leq U(\lambda x) \leq \lambda^\gamma U(x)\) for \(\lambda > 1\).

**Lemma A.4.** Suppose \(U\) is a utility function which satisfies the first condition of Assumption 5.1. Then for any \(c > 0, \varepsilon > 0\), there exist \(c_1, c_2 > 0\) such that

\[
c_1 H^{(\gamma(1-\varepsilon))}(x) \leq U(x) \leq c_2 H^{(\gamma(1+\varepsilon))}(x), \quad x \geq c.
\]

**Proof.** The assumptions on \(U\) mean that there exists \(x_0 > 0\) such that

\[
\frac{yU'(y)}{U(y)} \in (\gamma(1-\varepsilon), \gamma(1+\varepsilon)) \text{ for } y \in [x_0, \infty).
\]

For \(x \geq x_0\) we express \(U(x)\) as

\[
U(x) = U(x_0) \exp \left\{ \int_{x_0}^x \frac{y U'(y)}{U(y)} dy \right\}. \quad (22)
\]

In the case of positive \(U\) the claim follows by taking \(c_1 := \min \left\{ \frac{U(x)}{H^{(\gamma(1-\varepsilon))}(x)}, x \in [c, y_1] \right\}\) and \(c_2 := \max \left\{ \frac{U(x)}{H^{(\gamma(1+\varepsilon))}(x)}, x \in [c, y_1] \right\}\). For a negative \(U\) we interchange max and min in definitions of \(c_1, c_2\).

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