NUMERICAL METHODS FOR THE DETERMINISTIC SECOND MOMENT EQUATION OF PARABOLIC STOCHASTIC PDES

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Abstract. Numerical methods for stochastic partial differential equations typically estimate moments of the solution from sampled paths. Instead, we shall directly target the deterministic equations satisfied by the first and second moments, as well as the covariance.

In the first part, we focus on stochastic ordinary differential equations. For the canonical examples with additive noise (Ornstein–Uhlenbeck process) or multiplicative noise (geometric Brownian motion) we derive these deterministic equations in variational form and discuss their well-posedness in detail. Notably, the second moment equation in the multiplicative case is naturally posed on projective–injective tensor product spaces as trial–test spaces. We construct Petrov–Galerkin discretizations based on tensor product piecewise polynomials and analyze their stability and convergence in these natural norms.

In the second part, we proceed with parabolic stochastic partial differential equations with affine multiplicative noise. We prove well-posedness of the deterministic variational problem for the second moment, improving an earlier result. We then propose conforming space-time Petrov–Galerkin discretizations, which we show to be stable and quasi-optimal.

In both parts, the outcomes are illustrated by numerical examples.

1. Introduction

1.1. Introduction. Ordinary and partial differential equations are pervasive in financial, biological, engineering and social sciences, to name a few. Often, randomness is introduced in order to model uncertainties in the coefficients, in the geometry of the physical domain, in the boundary or initial conditions, or in the sources (right-hand sides). In this work we aim at the latter scenario, specifically ordinary or partial differential evolution equations driven by additive or multiplicative noise. The random solution is then a stochastic process with values in a certain state space. If the noise is a Wiener process, the solution paths are continuous in time. When the state space is of finite dimension (≤ 3, say), it may be possible to approximate numerically the temporal evolution of the probability density function of the stochastic process. For stochastic PDEs, this is in general computationally too expensive. One therefore estimates the mean and possibly the covariance of the solution process, also given by its first two statistical moments.

To estimate moments of the random solution one can resort to sampling methods such as Monte Carlo (MC). For every sample path, viz. a realization of the random input, a deterministic ordinary or partial differential evolution equation is solved. The vanilla MC exhibits the notorious convergence rate 1/2 in the number of samples. On the upside, sampling methods are usually trivial to parallelize across samples. Recent developments include multilevel MC [9, 11, 12, 16, 17, 41], quasi-MC [14, 19, 20, 25], and combinations thereof [18, 26]. More on solving random and parametric equations can be found in [8, 10, 13, 21, 37].

For the covariance of the solution to a parabolic stochastic PDE driven by additive Wiener noise, an alternative to sampling was proposed in [27]. It is based on the insight that the second moment solves a well-posed linear deterministic space-time variational problem on Hilbert tensor products of Bochner spaces. The main promise of space-time variational formulations is in potential computing time and memory savings through space-time compressive schemes, e.g., using
adaptive wavelet methods [40] or low-rank tensor approximations [8, 21, 22]. In principle, it is straightforward to construct numerical methods for the formulation from [27] by tensorizing existing discretizations of deterministic parabolic evolution equations (space-time or not), the main practical issue being the high dimensionality of the resulting equations.

The space-time variational formulation from [27] was extended in [23] to include multiplicative Lévy noise. This required a more careful analysis because firstly, an extra term in the space-time variational formulation constrains it to non-Hilbert tensor product spaces for the trial and test spaces; secondly, the well-posedness is self-evident only as long as the volatility of the multiplicative noise is sufficiently small. Consequently, contrary to the additive case, a dedicated design and analysis of numerical schemes is required for stochastic PDEs with multiplicative noise. To fully explain and address these issues, in this work we first focus on canonical examples of stochastic ODEs driven by additive or multiplicative Wiener noise. To facilitate the transition and the comparison to parabolic stochastic PDEs, our estimates are explicit and sharp in the relevant parameters. We then proceed with parabolic stochastic PDEs driven by multiplicative Lévy noise as in [23]. The transition from convolutions of real-valued functions to semigroup theory on tensor product spaces allows us to prove well-posedness of the deterministic second moment equation also in the vector-valued situation even beyond the smallness assumption on the multiplicative noise term made in [23, Eq. (5.5)].

This article is structured as follows. In §2 we introduce the model stochastic ODEs and the necessary definitions, derive the deterministic equations for the first and second moments and discuss their well-posedness. In §3 we present conforming Petrov–Galerkin discretizations of these equations and discuss their stability, concluding with a numerical example. In §4 we generalize the results of §§2–3 to stochastic PDEs with affine multiplicative noise and, again, verify these by numerical experiments. The outcomes of this work are summarized in §5.

1.2. Notation. We briefly comment on notation. If $X$ is a Banach space then $S(X)$ denotes its unit sphere and $X'$ its dual, i.e., all linear continuous mappings from $X$ to $\mathbb{R}$. We write $s \land t := \min\{s, t\}$. The symbol $\delta$ denotes the Dirac measure (at $s$). The closure of an interval $J$ is $\bar{J}$. We mark equations which hold almost everywhere or $\mathbb{P}$-almost surely with a.e. and $\mathbb{P}$-a.s., respectively. The space of bounded linear operators $X \to Y$ is denoted by $\mathcal{L}(X; Y)$; those on $X$ by $\mathcal{L}(X)$.

Depending on the context, the symbol $\otimes$ denotes the tensor product of two functions or operators, the algebraic tensor product of function spaces, or the Kronecker product of matrices.

If $H$ is a Hilbert space then the Hilbert tensor product space $H_2 := H \otimes_2 H$ is obtained as the closure of the algebraic tensor product $H \otimes H$ under the norm $\| \cdot \|_2$ induced by the “tensorized” inner product $(a \otimes b, c \otimes d)_2 := (a, c)_H (b, d)_H$.

A function $w \in L_2(J \times J)$ is called symmetric positive semi-definite (SPSD) if

$$w(s, t) = w(t, s) \quad \text{a.e. in } J \times J \quad \text{and} \quad \int_J \int_J w(s, t) \varphi(s) \varphi(t) \, ds \, dt \geq 0 \quad \forall \varphi \in L_2(J). \quad (1)$$

More generally, if $H$ is a Hilbert space (we have $H = L_2(J)$ in (1), cf. (21)), then an element $w$ of the Hilbert tensor product space $H \otimes_2 H$ is symmetric and positive semi-definite, abbreviated as $H$-SPSD, if

$$(w, \varphi \otimes \tilde{\varphi})_2 = (w, \tilde{\varphi} \otimes \varphi)_2 \quad \text{and} \quad (w, \varphi \otimes \varphi)_2 \geq 0 \quad \forall \varphi, \tilde{\varphi} \in H. \quad (2)$$

It is called symmetric if the equality in (2) holds, and antisymmetric if $(w, \varphi \otimes \tilde{\varphi})_2 = -(w, \tilde{\varphi} \otimes \varphi)_2$.

A functional $\ell$ defined on some closure of $H \otimes H$ is called symmetric positive semi-definite (SPSD) if

$$\ell(\psi \otimes \tilde{\psi}) = \ell(\tilde{\psi} \otimes \psi) \quad \text{and} \quad \ell(\psi \otimes \psi) \geq 0 \quad \forall \psi, \tilde{\psi} \in H. \quad (3)$$

It is called antisymmetric if $\ell(\psi \otimes \tilde{\psi}) = -\ell(\tilde{\psi} \otimes \psi)$. If (3) holds only on a subset $\psi, \tilde{\psi} \in V \subset H$, we say that $\ell$ is SPSD on $V \otimes V$ for short.
2. Derivation of the Deterministic Moment Equations

2.1. Model stochastic ODEs. Let $T > 0$, set $J := (0, T)$. The first part of this article focusses on the model real-valued stochastic ODEs with additive noise

$$dX(t) + \lambda X(t)\, dt = \mu \, dW(t), \quad t \in J, \quad \text{with} \quad X(0) = X_0, \quad (4)$$

or with multiplicative noise

$$dX(t) + \lambda X(t)\, dt = \rho X(t) \, dW(t), \quad t \in J, \quad \text{with} \quad X(0) = X_0. \quad (5)$$

Here,

- $\lambda > 0$ is a fixed positive number that models the action of an elliptic operator,
- $W$ is a real-valued Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
- $\mu, \rho \geq 0$ are parameters specifying the volatility of the noise,
- the initial value $X_0 \in L^2(\Omega)$ is a random variable, independent of the Wiener process, with known first and second moments (but not necessarily with a known distribution).

We call $\mathcal{F}_t$ the $\sigma$-algebra generated by the Wiener process $\{W(s) : 0 \leq s \leq t\}$ and the initial value $X_0$, and $\mathcal{F}$ the resulting filtration. The expectation operator is denoted by $\mathbb{E}$. We refer to [24, 31] for basic notions of stochastic integration and Itô calculus.

A real-valued stochastic process $X$ is said to be a (continuous strong) solution of the stochastic differential equation “$dX + \lambda X = \sigma(X)\, dW$ on $J$ with $X(0) = X_0$” if a) $X$ is progressively measurable with respect to $\mathcal{F}$, b) the expectation of $||\lambda X||_{L^1(J)} + ||\sigma(X)||^2_{L^2(J)}$ is finite, c) the integral equation

$$X(t) = X_0 - \lambda \int_0^t X(s)\, ds + \int_0^t \sigma(X(s)) \, dW(s) \quad \forall t \in J$$

holds $(\mathbb{P}$-a.s.), and d) $t \mapsto X(t)$ is continuous $(\mathbb{P}$-a.s.). By standard theory ([24, Thm. 4.5.3] or [31, Thm. 5.2.1]) a Lipschitz condition on $\sigma$ implies existence and uniqueness of such a solution. Moreover, it has finite second moments. For future reference, we state here the integral equations for (4)–(5):

$$X(t) = X_0 - \lambda \int_0^t X(s)\, ds + \mu \int_0^t \, dW(s) \quad \forall t \in J \quad (\mathbb{P}$$-a.s.), \quad (6)

$$X(t) = X_0 - \lambda \int_0^t X(s)\, ds + \rho \int_0^t X(s) \, dW(s) \quad \forall t \in J \quad (\mathbb{P}$$-a.s.). \quad (7)

The solution processes and their first/second moments are known explicitly (e.g. [24, §4.4]):

| $(8a)$ | $X(t)$ | Additive (4)/(6) Ornstein–Uhlenbeck process | $e^{-\lambda t}X_0 + \mu \int_0^t e^{-\lambda(t-s)} \, dW(s)$ | $X_0e^{-(\lambda + \rho^2/2)t + \rho W(t)}$ |
| $(8b)$ | $\mathbb{E}[X(t)]$ | | $e^{-\lambda t}\mathbb{E}[X_0]$ | $e^{-\lambda t}\mathbb{E}[X_0]$ |
| $(8c)$ | $\mathbb{E}[X(t^2)]$ | | $e^{-\lambda t}\mathbb{E}[X_0^2] + \frac{\mu^2}{2} (e^{-\lambda t} - e^{-\lambda t})$ | $e^{-\lambda (t+s) + \rho^2(s+t)}\mathbb{E}[X_0^2]$ |
| $(8d)$ | $\mathbb{E}[||X||^2_{L^2(J)}]$ | | $\frac{1-e^{-2\lambda t}}{2\lambda} \mathbb{E}[X_0^2] + \frac{\mu^2}{4\lambda^2} (e^{-2\lambda t} + 2\lambda T - 1)$ | $\frac{e^{(\rho^2-2\lambda)t} - 1}{\rho^2 - 2\lambda} \mathbb{E}[X_0^2]$ |

The square integrability (8d) in conjunction with Fubini’s theorem will be used to interchange the order of integration over $J$ and $\Omega$ without further mention. Square integrability also implies the useful martingale property ([24, Thm. 3.2.5] or [31, Cor. 3.2.6 & Def. 3.1.4])

$$\mathbb{E}\left[ \int_0^t X(r) \, dW(r) \bigg| \mathcal{F}_s \right] = \int_0^s X(r) \, dW(r), \quad 0 \leq s \leq t. \quad (9)$$

Choosing $s = 0$ shows that the stochastic integral $\int_0^t X(r) \, dW(r)$ has expectation zero. If $Y_1$ and $Y_2$ are two square integrable processes adapted to $\mathcal{F}$, the Itô isometry ([24, Thm. 3.2.3] or [31,
Cor. 3.1.7]) along with (9) and the polarization identity yield the equality
\[
\mathbb{E}\left[\int_0^s Y_1(r) \, dW(r) \int_t^s Y_2(r) \, dW(r)\right] = \int_0^s \mathbb{E}[Y_1(r)Y_2(r)] \, dr. \tag{10}
\]
These are the main tools in the derivation of (8). We will write \( X \otimes X \) for the real-valued stochastic process \((s, t) \mapsto X(s)X(t)\) on \((\Omega, \mathcal{F}, \mathbb{P})\) indexed by the parameter space \(J \times J\).

Our first aim will be to derive deterministic equations for the first and the second moments
\[
m(t) := \mathbb{E}[X(t)] \quad \text{and} \quad M(s, t) := \mathbb{E}[X(s)X(t)], \quad s, t \in J,
\]
as well as for the covariance function
\[
\text{Cov}(X) := \mathbb{E}[(X - m) \otimes (X - m)] = M - (m \otimes m) \tag{11}
\]
of the stochastic process \(X\).

In showing well-posedness of the deterministic equations, the notions (1)–(3) of positive semi-definiteness will be important. Indeed, the second moment and the covariance of a real-valued stochastic process are SPSD. Importantly, the SPSD functions form a cone, so that sums (and integrals) thereof remain SPSD.

### 2.2. Deterministic First Moment Equations

We first introduce the spaces
\[
E := L_2(J) \quad \text{and} \quad F := H_0^1(J),
\]
where the latter denotes the closed subspace of the Sobolev space \(H^1(J)\) of functions vanishing at \(t = T\). Thanks to the embedding \(F \hookrightarrow C^0(J)\), elements of \(F\) will be identified by their continuous representative. These spaces are equipped with the \(\lambda\)-dependent norms
\[
\|w\|_E^2 := \lambda \|w\|_{L_2(J)}^2 \quad \text{and} \quad \|v\|_F^2 := \lambda^{-1}\|v\|_{H_0^1(J)}^2 + \lambda \|v\|_{L_2(J)}^2 + |v(0)|^2, \tag{12}
\]
and the obvious corresponding inner products \((\cdot, \cdot)_E\) and \((\cdot, \cdot)_F\). The norm on \(F\) is motivated by the fact that
\[
\|v\|_F^2 = \lambda^{-1}|v(T) - v(0)|^2 + \lambda \|v\|_{L_2(J)}^2 \quad \forall v \in F. \tag{13}
\]

#### Lemma 2.1.

Let \(v \in F\). Then
\[
|v(t)| \leq \frac{1}{\sqrt{2}} \|v\|_F \quad \forall t \in J. \tag{14}
\]

**Proof.** Suppose that the supremum of \(|v(t)|\) is attained at some \(0 \leq t \leq T\). Integrating \((v^2)' = 2vv'\) over \((0, t)\), applying the Cauchy–Schwarz and the Young inequalities leads to the estimate
\[|v(t)|^2 \leq \frac{\lambda^{-1}}{2} |v(0)|^2 + \lambda \|v\|_{L_2(J)}^2 + |v(0)|^2\]
in terms of the \(L_2(0, t)\) norms. In a similar way, observing that \(v(T) = 0\), we obtain \(|v(t)|^2 \leq \lambda^{-1}|v(0)|^2 + \lambda \|v\|_{L_2(J)}^2\) in terms of the \(L_2(0, T)\) norms. Adding the two inequalities gives (14). \(\square\)

The inequality (14) is sharp as the function \(\psi(t) := \sinh(\lambda(T - t))/\sinh(\lambda T)\) attests:
\[
1 = \psi(0) = \sup_{t \in J} |\psi(t)| \quad \text{and} \quad \|\psi\|_F = \sqrt{\coth(\lambda T) + 1} \to \sqrt{2} \quad \text{as} \quad \lambda T \to \infty. \tag{15}
\]

The deterministic moment equations will be expressed in terms of the continuous bilinear form
\[
b : E \times F \to \mathbb{R}, \quad b(w, v) := \int_J w(t)(-v'(t) + \lambda v(t)) \, dt. \tag{16}
\]

We employ the same notation for the induced bounded linear operator
\[
b : E \to F', \quad (bw, v) := b(w, v),
\]
and use whichever is more convenient, as should be evident from the context. The operator \(b\) arises in the weak formulation of the ordinary differential equation \(u' + \lambda u = f\). With the definition of the norms (12), it is an isometric isomorphism,
\[
\|bw\|_{F'} = \|w\|_E \quad \forall w \in E. \tag{17}
\]
Indeed, \( \|bw\|_{F'} \leq \|w\|_E \) is obvious from (12)–(13). To verify \( \|bw\|_{F'} \geq \|w\|_E \), let \( w \in E \) be arbitrary. Taking \( v \) as the solution to the ODE \(-v' + \lambda v = \lambda w \) with \( v(T) = 0 \), it follows using (12)–(13) that \( \langle bw, v \rangle = \|w\|_E^2 = \|v\|_F^2 \). Therefore, \( \langle bw, v \rangle = \|w\|_E \|v\|_F \), and in particular \( \|bw\|_{F'} \geq \|w\|_E \). This shows the isometry property. By a similar argument, \( \sup_{w} \langle bw, v \rangle \neq 0 \) for all nonzero \( v \in F \). By [5, Thm. 2.1], \( b \) is an isomorphism.

If a functional \( \ell \in F' \) can be expressed as \( \ell(v) = \int_J g v \) for some \( g \in L_1(J) \), then \( u = b^{-1} \ell \) enjoys the representation

\[
u(t) = (b^{-1} \ell)(t) = \int_0^t e^{-\lambda(t-s)} g(s) \, ds.
\]

(18)

Despite this integral representation, \( b^{-1} \) is not a compact operator (it is an isomorphism).

Applying the expectation operator to (6)–(7) shows that the first moment \( m \) of the solution satisfies the integral equation

\[
m(t) = E[X_0] - \lambda \int_0^t m(s) \, ds.
\]

Testing this equation with the derivative of an arbitrary \( v \in F \) and integrating by parts in time shows that the first moment of (4)–(5) solves the deterministic variational problem

Find \( m \in E \) s.t. \( b(m, v) = E[X_0]v(0) \) \( \forall v \in F \).

(19)

2.3. Second moment equations: additive noise. For the deterministic equations for the second moment and the covariance we need the Hilbert tensor product spaces

\[E_2 := E \otimes_2 E \quad \text{and} \quad F_2 := F \otimes_2 F,
\]

(20)

with \( \| \cdot \|_2 \) denoting the norms on both spaces. We further write \( \| \cdot \|_{-2} \) for the norm of the dual space \( F'_2 \) of \( F_2 \). We recall the canonical isometry (see [34, Thm. II.10] or [4, Thm. 12.6.1])

\[E_2 = L_2(J) \otimes_2 L_2(J) \cong L_2(J \times J).
\]

(21)

By virtue of square integrability (8d), the second moment \( M \) is an element of \( E_2 \). We define the bilinear form

\[B : E_2 \times F_2 \to \mathbb{R}, \quad B := b \otimes b,
\]

or explicitly as

\[B(w, v) := \int_J \int_J w(s, t)(-\partial_s + \lambda)(-\partial_t + \lambda)v(s, t) \, ds \, dt.
\]

(22)

More precisely, \( B \) is the unique continuous extension of \( b \otimes b \) by bilinearity from the algebraic tensor products to \( E_2 \times F_2 \). Boundedness and injectivity of the operator \( B : E_2 \to F_2' \) induced by the bilinear form \( B \) follow readily from the corresponding properties of \( b \), so that the operator \( B \) is an isometry and its inverse is the due continuous extension of \( b^{-1} \otimes b^{-1} \). A representation of the inverse analogous to (18) also holds. For example, the integral kernel of the functional \( \ell(v) := v(0) \) is \( \delta_0 \otimes \delta_0 \), which gives \( (B^{-1} \ell)(t, t') = e^{-\lambda(t+t')} \).

Recall the definitions of SPSD-ness from (1)–(3).

**Lemma 2.2.** The function \( U := B^{-1} \ell \in E_2 \) is SPSD if and only if the functional \( \ell \in F'_2 \) is.

**Proof.** Identifying \( \varphi \in L_2(J) \) with \( \psi \in F \) via \( (w, \varphi)_{L_2(J)} = b(w, \psi) \) for all \( w \in E \), we observe that \( (U, \varphi \otimes \tilde{\varphi})_{L_2(J \times J)} = B(U, \psi \otimes \tilde{\psi}) = \ell(\psi \otimes \tilde{\psi}) \). Thus \( U \) is SPSD iff \( \ell \) is. \( \square \)

Finally, we introduce the bounded linear functional

\[\delta : F_2 \to \mathbb{R}, \quad \delta(v) := \int_J v(t, t) \, dt.
\]

(23)

As in [27, Lem. 4.1], one could use [42, Lem. 5.1] to show boundedness of \( \delta \). We give here an elementary and quantitative argument. Writing \( \delta(v) \) as the integral of \( \delta(s-s')v(s, s') \) over \( J \times J \)
and exploiting the representation (18) of $b^{-1}$ we find $(B^{-1} \delta)(t, t') = (e^{-\lambda|t-t'|} - e^{-\lambda(t+t')})/(2\lambda)$. Since $B$ is an isometry, the operator norm of $\delta$ is

$$\|\delta\|_{-2} = \lambda \|B^{-1} \delta\|_{L_2(J \times J)} = \frac{1}{4\lambda^2} (4\lambda T - 5 + (8\lambda T + 4)e^{-2\lambda T} + e^{-4\lambda T})^{1/2}.$$  

In particular, this yields the asymptotics $\|\delta\|_{-2} \sim T^2 \lambda/\sqrt{\lambda}$ for small $\lambda$ and $\|\delta\|_{-2} \sim \sqrt{T/(4\lambda)}$ for large $\lambda$. In addition, the uniform bound $\|\delta\|_{-2} \leq 1/2 T$ holds, see Remark 2.9.

We are now ready to state the deterministic equation for the second moment (derived for stochastic PDEs in [27]).

**Proposition 2.3.** The second moment $M = \mathbb{E}[X \otimes X]$ of the solution $X$ to the stochastic ODE (4) with additive noise solves the deterministic variational problem

$$\text{Find } M \in E_2 \text{ s.t. } B(M, v) = \mathbb{E}[X_0^2] \nu(0) + \nu^2 \delta(\nu) \quad \forall \nu \in F_2. \quad (24)$$

**Proof.** Inserting the solution (6) in the first argument of $b(X, \nu) = \int_{J} (-X \nu' + \lambda X \nu)$ and integrating by parts one finds

$$b(X, \nu) = X_0 \nu(0) - \mu \int_{J} W(t) \nu'(t) dt = X_0 \nu(0) + \mu \int_{J} \nu(t) dW(t) \quad \forall \nu \in F \quad (\mathbb{P}\text{-a.s.}),$$

where the stochastic integration by parts formula [31, Thm. 4.1.5] was used in the second equality. Employing this in $B(M, \nu_1 \otimes \nu_2) = \mathbb{E}[b(X, \nu_1) b(X, \nu_2)]$ with (10) for the $\mu^2$ term leads to the desired conclusion. $\square$

From the equations for the first and second moments, an equation for the covariance function $\text{Cov}(X) \in E_2$ follows:

$$B(\text{Cov}(X), \nu) = \text{Cov}(X_0) \nu(0) + \nu^2 \delta(\nu) \quad \forall \nu \in F_2.$$  

The proof is straightforward and is therefore omitted.

### 2.4. Second moment equations: multiplicative noise

Before proceeding with the second moment equation for the case of multiplicative noise, we formulate a lemma which repeats the derivation of the first moment equation (19) without taking the expectation first.

**Lemma 2.4.** Let $X$ be the solution (7) to the stochastic ODE (5). Then

$$b(X, \nu) = X_0 \nu(0) - \rho \int_{J} \left( \int_{0}^{t} X(r) \, dW(r) \right) \nu'(t) dt \quad \forall \nu \in F \quad (\mathbb{P}\text{-a.s.}). \quad (25)$$

**Proof.** Let $\nu \in F$. We employ the definition (7) of the solution in the first term of $b(X, \nu)$ and integration by parts on the first two summands of the integrand to obtain (observing that the terms at $t = T$ vanish due to $\nu(T) = 0$)

$$\int_{J} X(t) \nu'(t) dt = -X_0 \nu(0) + \lambda \int_{J} X(t) \nu(t) dt + \rho \int_{J} \left( \int_{0}^{t} X(r) \, dW(r) \right) \nu'(t) dt$$

$$= -X_0 \nu(0) + \lambda \int_{J} X(t) \nu(t) dt + \rho \int_{J} \left( \int_{0}^{t} X(r) \, dW(r) \right) \nu'(t) dt \quad (\mathbb{P}\text{-a.s.}).$$

Inserting this expression in the definition (16) of $b(X, \nu)$ yields the claimed formula. $\square$

The next ingredient in the second moment equation for the case of multiplicative noise, which appears due to the integral term in (25), is the bilinear form

$$\Delta(w, v) := \int_{J} w(t, t) \nu(t, t) dt, \quad w \in E \otimes E, \quad v \in F \otimes F, \quad (26)$$

referred to as the trace product. Again, we use the same symbol for the induced operator, where convenient. Here, $\otimes$ denotes the algebraic tensor product. The expression (26) is meaningful.
because functions in $F \subset H^1(U)$ are bounded. As we will see in Lemma 2.8, this bilinear form extends continuously to a form
\begin{equation}
\Delta: E_\pi \times F_\varepsilon \to \mathbb{R}
\end{equation}
on the projective and the injective tensor product spaces
\begin{equation}
E_\pi := E \otimes \pi E \quad \text{and} \quad F_\varepsilon := F \otimes \varepsilon F.
\end{equation}
These spaces are defined as the closure of the algebraic tensor product under the projective norm
\begin{equation}
\|w\|_\pi := \inf \left\{ \sum_i \|w_i\|_E \|w_i^2\|_E : w = \sum_i w_i^n \otimes w_i^n \right\},
\end{equation}
and the injective norm
\begin{equation}
\|v\|_\varepsilon := \sup \left\{ \|(g_1 \otimes g_2)(v)\) : g_1, g_2 \in S(F') \right\},
\end{equation}
respectively. Note that, initially, these norms are defined on the algebraic tensor product space. In particular, the sums in (29) are finite and the action of $g_1 \otimes g_2$ in (30) is well-defined. The spaces in (28) are separable Banach spaces. They are reflexive if and only if their dimension is finite [35, Thm. 4.21]. By [35, Prop. 6.1(a)], these tensor norms satisfy
\begin{equation}
\|w_1 \otimes w_2\|_\pi = \|w_1\|_E \|w_2\|_E \quad \text{and} \quad \|v_1 \otimes v_2\|_\varepsilon = \|v_1\|_F \|v_2\|_F,
\end{equation}
as well as
\begin{equation}
\| \cdot \|_2 \leq \| \cdot \|_\pi \quad \text{on} \quad E \otimes E \quad \text{and} \quad \| \cdot \|_\varepsilon \leq \| \cdot \|_2 \quad \text{on} \quad F \otimes F.
\end{equation}
We write $\| \cdot \|_{-\varepsilon}$ for the norm of the continuous dual $F'_\varepsilon := (F_\varepsilon)'$.

**Example 2.5.** Consider $V := \mathbb{R}^N$ with the Euclidean norm. Elements $A \in V \otimes V$ can be identified with $N \times N$ real matrices. Let $\sigma(A)$ denote the singular values of $A$. The projective, the Hilbert, and the injective norms on $V \otimes V$ are the nuclear norm $\|A\|_\pi = \sum_{s \in \sigma(A)} s$, the Frobenius norm $\|A\|_2 = (\sum_{s \in \sigma(A)} s^2)^{1/2}$, and the operator norm $\|A\|_F = \max \sigma(A)$, respectively. They are also known as the Schatten $p$-norms with $p = 1$, 2, and $\infty$. Evidently, $\| \cdot \|_\pi \geq \| \cdot \|_2 \geq \| \cdot \|_F$.

If a function $w \in E_2$ is SPSD (1), the operator $S_w: E \to E$ defined by $S_w \varphi := \int_J w(s, \cdot) \varphi(s) \, ds$ is self-adjoint and positive semi-definite. Let $(s_n) \subset [0, \infty)$ denote its eigenvalues. If their sum is finite then the operator is trace-class and $\|w\|_\pi = \sum_n s_n$, see [32, Thm. 9.1.38 and comments]. We note that the correspondence between symmetric positive semi-definite kernels, covariances, and trace-class operators was already observed in [30], [29, Thm. XI.37.1.A] and [33, Thm. A.8 and p. 363] and extended to case of Hilbert space valued kernels in [38, Thm. 2.3 and Cor. 2.4]. For our purposes, the following specialization will be particularly useful.

**Lemma 2.6.** If $w \in E_\pi$ is SPSD then $\|w\|_\pi = \lambda \delta(w)$ with $\delta$ from (23).

**Proof.** Let $\{e_n\}_n$ be an orthonormal basis of $E$ consisting of eigenvectors of $S_w$ with the eigenvalues $(s_n)_n$. By symmetry, $w = \sum_n s_n e_n \otimes e_n$. Since $\lambda \delta(e_n \otimes e_n) = 1$, we have $\lambda \delta(w) = \sum_n s_n = \|w\|_\pi$. \hfill \Box

An arbitrary $w \in E_\pi$ can be decomposed (via the corresponding integral operator) as $w = w^+ - w^- + w^a$ with SPSD $w^\pm \in E_\pi$ and an antisymmetric $w^a \in E_\pi$. This decomposition is stable in the sense that
\begin{equation}
\|w^a\|_\pi \leq \|w\|_\pi \quad \text{and} \quad \|w^+ - w^-\|_\pi = \|w^+\|_\pi + \|w^-\|_\pi \leq \|w\|_\pi.
\end{equation}
The tensor product spaces $E_\pi$ and $F_\varepsilon$ seem necessary because the trace product $\Delta$ is not continuous on the Hilbert tensor product spaces $E_2 \times F_2$ as the following example illustrates.

**Example 2.7.** To simplify the notation, suppose $T = 1$, so that $J = (0, 1)$. Define $v \in F_2$ by $v(s, t) := (1-s)(1-t)$ for $s, t \in J$. Consider the sequence $u_1, u_2, \ldots$ of indicator functions
\begin{equation}
u_n(s, t) := \chi_{A_n}(s, t), \quad \text{where} \quad A_n := \left(0, \frac{1}{n}\right)^2 \cup \left(\frac{1}{n}, \frac{2}{n}\right)^2 \cup \cdots \cup \left(\frac{n-1}{n}, 1\right)^2 \subset J \times J.
\end{equation}
In view of the canonical isometry (21), this sequence is a null sequence in $E_2$. However, $\Delta(u_n, v) = \int_J u_n(t, t) v(t, t) \, dt = \frac{1}{3}$ for all $n \geq 1$. Therefore, $\Delta(\cdot, v)$ is not continuous on $E_2$. 

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The example additionally shows that $\Delta$ is not continuous on $E_x \times F_x$ either, since by (31)–(32) we have $\|v\|_\pi = \|v\|_2$, while $\|u_n\|_e \leq \|u_n\|_2 \to 0$ as $n \to \infty$.

By contrast, $\{u_n\}_{n \geq 1}$ is not a null sequence in $E_\pi$. Indeed, Lemma 2.6 gives $\|u_n\|_\pi = \lambda$ for all $n \geq 1$.

**Lemma 2.8.** The trace product $\Delta$ in (27) is continuous on $E_\pi \times F_x$ with $\|\Delta\| \leq 1/(2\lambda)$.

**Proof.** By density it suffices to bound $\Delta(w, v)$ for arbitrary $w \in E \otimes E$ and $v \in F \otimes F$. By [36, Thm. 2.4] we may assume that $w = w^1 \otimes w^2$. We note first that the point evaluation functionals $\delta_i : v \mapsto v(t)$ have norm $1/\sqrt{2}$ on $F$ by (14). Therefore, if $v = \sum_j v_j^1 \otimes v_j^2$ then
\[
|v(s, t)| = \left| \sum_j \delta_i(v_j^1) \delta_i(v_j^2) \right| \leq \sup \left\{ \frac{1}{2} \sum_j g_1(v_j^1) g_2(v_j^2) : g_1, g_2 \in S(F') \right\} = \frac{1}{2} \|v\|_e
\]
and the continuity of $\Delta$ follows:
\[
|\Delta(w, v)| = \left| \int \int w(t, s) v(t, s) \, ds \, dt \right| \leq \frac{1}{2} \|v\|_e \int \int |w(t, s)| \, ds \, dt \leq \frac{1}{2\lambda} \|v\|_e \|w\|_\pi,
\]
where the integral Cauchy–Schwarz inequality on $w(t, s) = w^1(t) w^2(s)$ was used in the last step, together with the fact that $\lambda\|w^1\|_{L_2(E)}\|w^2\|_{L_2(E)} = \|w^1\|_E\|w^2\|_E = \|w^1 \otimes w^2\|_\pi$.

We note that the bound $\|\Delta\| \leq 1/(2\lambda)$ is sharp: For $\eta > 0$ take $w = \varphi \otimes \varphi$ with $\varphi = \chi_{(0, \eta)} / \sqrt{\eta}$ and $v = \psi \otimes \psi$ with $\psi(t) := \sinh(\lambda(T-t))/\sinh(\lambda T)$ as in (15). Then $\lim_{\eta \to 0} \Delta(w, v) = 1$ and $\lim_{T \to \infty} \|v\|_e \|w\|_\pi = 2\lambda$, and the bound is tight when applying both limits.

**Remark 2.9.** Consider the functional $\delta$ from (23). Since $\delta = \Delta(1 \otimes 1)$ and $\|1 \otimes 1\|_\pi = \lambda T$, we have $\|\delta : F_2 \to \mathbb{R}\|_{-\infty} \leq T/2$. In view of $\|v\|_e \leq \|v\|_2$ from (32), we find $\|\delta : F_2 \to \mathbb{R}\|_{-2} \leq T/2$. Finally, $\|\delta : E_\pi \to \mathbb{R}\|_{-\infty} = 1/\lambda$ by the integral Cauchy–Schwarz inequality and Lemma 2.6.

A crucial observation is that the second moment $M$ lies not only in the Hilbert tensor product space $E_2$ but in the smaller projective tensor product space, $M \in E_\pi$. This follows by passing the norm under the expectation $\mathbb{E}[X \otimes X] \|_\pi \leq \mathbb{E}[\|X \otimes X\|_\pi]$, then using (31) and the square integrability (8d) of $X$.

We recall here from [36, Thm. 2.5 and Thm. 5.13] the fact that
\[
F'_\epsilon = (F \otimes F') \cong F' \otimes \pi F' \quad \text{isometrically},
\]
(wheras the space $(F')'$ is isometric to a proper subspace of $(F_\pi)'$, see [35, pp. 23/46]). A corollary of this representation is that
\[
b \otimes b : E_\pi \to F'_\epsilon \quad \text{defines an isometric isomorphism},
\]
(35)

because $b \otimes b$ extends to an isometric isomorphism from $E \otimes \pi E$ onto $F' \otimes \pi F'$. We call it also $B$. This isometry property (35), Lemma 2.2 and Lemma 2.6 produce the useful identity
\[
\|B^{-1}\|_{-\epsilon} = \lambda \delta(B^{-1} \ell)
\]
for any $\ell \in F'_\epsilon$ which is SPSD (3). Here and below, Lemma 2.2 applies to functionals in $F'_\epsilon$ mutatis mutandis. Using the decomposition from (33) we can decompose any $\ell = \ell^+ - \ell^- + \ell^0$ into SPSD and antisymmetric parts with
\[
\|\ell^0\|_{-\epsilon} \leq \|\ell\|_{-\epsilon} \quad \text{and} \quad \|\ell^+ - \ell^-\|_{-\epsilon} = \|\ell^+\|_{-\epsilon} + \|\ell^-\|_{-\epsilon} \leq \|\ell\|_{-\epsilon}.
\]

Now we are in position to introduce the bilinear form
\[
\mathcal{B} : E_\pi \times F_\epsilon \to \mathbb{R}, \quad \mathcal{B} := B - \rho^2 \Delta,
\]
(38)
or more explicitly,
\[
\mathcal{B}(w, v) = \int \int w(s, t)(-\partial_t + \lambda)(-\partial_t + \lambda) v(s, t) \, ds \, dt - \rho^2 \int w(t, t) v(t, t) \, dt.
\]

The reason for this definition is the following result from [23, Thm. 4.2] derived there for stochastic PDEs. The simplified proof is given here for completeness.
Proposition 2.10. The second moment $M = \mathbb{E}[X \otimes X]$ of the solution $X$ to the stochastic ODE (5) with multiplicative noise solves the deterministic variational problem

$$
\text{Find } M \in E_\pi \text{ s.t. } \mathcal{B}(M, \nu) = \mathbb{E}[X_0^2 \nu(0)] \forall \nu \in F_\varepsilon. 
$$

Proof. It suffices to verify the claim for $\nu$ of the form $\nu = \nu_1 \otimes \nu_2$ with $\nu_1, \nu_2 \in F$. The more general statement follows by linearity and continuity of both sides in $\nu \in F_\varepsilon$. We first observe with Fubini’s theorem on $\Omega \times J$ that $B(M, \nu_1 \otimes \nu_2) = B(\mathbb{E}[X \otimes X], \nu_1 \otimes \nu_2) = \mathbb{E}[b(X, \nu_1) b(X, \nu_2)]$. Next, we insert the expression (25) for both $b(X, \nu_1)$ and expand the product. The cross-terms vanish because the terms of the form $X_0 \int_0^t X(r) \, dW(r)$ vanish in expectation; this is seen by conditioning this term on $\mathcal{F}_0$ and employing the martingale property (9). With the identity (10) and $\mathbb{E}[X(r)^2] = M(r, r)$ we arrive at

$$
B(M, \nu_1 \otimes \nu_2) = \mathbb{E}[X_0^2 \nu(0)] + \rho^2 \iint J J' v_1'(s) v_2'(t) \int_0^{s \wedge t} M(r, r) \, dr \, ds \, dt. 
$$

It remains to verify that $\rho^2 \Delta(M, \nu)$ coincides with the last term on the right-hand side. Let us distinguish the two cases $s = s \wedge t$ and $t = s \wedge t$ and write that triple integral as

$$
\int_s^t v_1'(s) \int_s^t v_2'(t) \, ds \int_0^t M(r, r) \, dr \, dt + \int_t^s v_2'(t) \int_t^s v_1'(s) \, ds \int_0^t M(r, r) \, dr \, dt. 
$$

Evaluating the $dt$ integral in the first summand and the $ds$ integral in the second summand, we see that $(40 - \Delta(M, \nu)) = \int_0^t \frac{d}{dt} (-v_1(t) v_2(t)) \int_0^t M(r, r) \, dr \, dt = 0$. Hence, (40) $\Delta(M, \nu)$. This completes the proof.

Using the equations for the first and second moments we obtain an equation for the covariance function $\text{Cov}(X) \in E_\pi$ from (11):

$$
\mathcal{B}(\text{Cov}(X), \nu) = \text{Cov}(X_0) \nu(0) + \rho^2 \Delta(m \otimes m, \nu) \forall \nu \in F_\varepsilon. 
$$

Identity (36) yields $||\nu \mapsto \nu(0)||_{-\varepsilon} = ||\nu \otimes \nu||_{-\varepsilon} = \frac{1}{2} (1 - e^{-2\lambda T})$ for the functional appearing on the right-hand side of (39) and (41). Similarly, $||\Delta(m \otimes m)||_{-\varepsilon} = \frac{1}{2} \int_0^{\infty} \int_0^T (1 - e^{-2\lambda (t - \tau)}) |m(t)|^2 \, d\tau \leq \frac{1}{2\lambda} ||m||_{-\varepsilon}^2$, in agreement with Lemma 2.8.

We emphasize that it is not possible to replace in the present case of multiplicative noise the pair of trial and test spaces $E_\pi \times F_\varepsilon$ by either pair $E_2 \times F_2$ or $E_\varepsilon \times E_\pi$, because by Example 2.7 the operator $\Delta$ is not continuous there. We note, however, that in the case of additive noise ($\S 2.3$) the pair $E_\pi \times F_\varepsilon$ could be used instead of $E_2 \times F_2$. Then $|||\delta|||_{-\varepsilon} = \lambda \delta(B^{-1} \delta) = \frac{1}{\lambda} (e^{-2T \lambda} - 1 + 2T \lambda)$ with the asymptotics $\frac{1}{2} T^2 \lambda$ (small $\lambda$) and $\frac{1}{2} T$ (large $\lambda$).

In order to discuss the well-posedness of the variational problem (39), given a functional $\ell \in F'_\varepsilon$, we consider the more general problem:

$$
\text{Find } U \in E_\pi \text{ s.t. } \mathcal{B}(U, \nu) = \ell(\nu) \forall \nu \in F_\varepsilon. 
$$

Owing to $||Bw||_{-\varepsilon} = ||w||_\pi$ and $||\Delta|| \leq 1/(2\lambda)$ we have $||Bw||_{-\varepsilon} \geq (1 - \rho^2/(2\lambda)) ||w||_\pi$. Thus, injectivity of $\mathcal{B}$ holds under the condition $\rho^2 < 2\lambda$ of small “volatility”. A similar condition was imposed in [23, Thm. 5.5]. This is exactly the threshold for the second moment (8c) to diverge as $s = t \to \infty$, but it stays nevertheless finite for all finite $s = t$. We discuss here what happens in the variational formulation (42) for larger volatilities $\rho$, and summarize in Theorem 2.11 below.

Since $B$ is an isomorphism, problem (42) is equivalent to $U = \rho^2 B^{-1} \Delta U + B^{-1} \ell$. Using the representation of $\Delta(U, \nu)$ as the double integral of $\delta(s - s') U(s, s') \nu(s, s')$, and the integral representation of $B^{-1}$ through (18), we obtain the integral equation

$$
U(t, t') = \rho^2 \int_0^{t \wedge t'} e^{-\lambda (t' - 2s)} U(s, s) \, ds + (B^{-1} \ell)(t, t').
$$
Defining \( f(t) := (B^{-1}\Delta U)(t, t) = \int_0^t e^{-2\lambda(t-s)}U(s, s)\,ds \) and \( g(t) := (B^{-1}f)(t, t) \) we find from (43) the ODE \( f'(t) + 2\lambda f(t) = \rho^2 f(t) + g(t) \) with the initial condition \( f(0) = 0 \). The solution is

\[
 f(t) = (B^{-1}\Delta U)(t, t) = \int_0^t e^{-(2\lambda-\rho^2)(t-s)}g(r)\,dr. \tag{44}
\]

Inserting

\[
 U(s,s) = \rho^2 f(s) + g(s) = \rho^2 \int_0^s e^{-(2\lambda-\rho^2)(s-r)}g(r)\,dr + g(s) \tag{45}
\]

under the integral of (43) provides a unique candidate for \( U \). Moreover, \( U \in E_2 \). We now estimate \( \|U\|_\pi \) in terms of the norm of \( \ell \).

Clearly, not all functionals \( \ell \) lead to solutions that are potential second moments. Let us therefore assume first that \( \ell \) is SPSD. Then \( B^{-1}\ell \) is SPSD by Lemma 2.2. In particular, \( f \geq 0 \) and \( g \geq 0 \). Thus the functional \( \nu \mapsto \Delta(U, \nu) = \int_0^T (\rho^2 f(t) + g(t))\nu(t, t)\,dt \) is SPSD. Now \( U = \rho^2 B^{-1}\Delta U + B^{-1}\ell \) is the sum of two SPSD functions (Lemma 2.2) and is therefore SPD. Under these assumptions, Lemma 2.6 gives

\[
 \|U\|_\pi = \lambda \delta(U) = \rho^2 \lambda \delta(B^{-1}\Delta U) + \lambda \delta(B^{-1}\ell). \tag{46}
\]

For the first term on the right-hand side of (46) we employ (44) as follows:

\[
 \delta(B^{-1}\Delta U) = \int_0^T g(r) \int_r^T e^{-(2\lambda-\rho^2)(s-r)}\,ds\,dr \leq \delta(B^{-1}\ell) \frac{\rho^2 \rho^2 - 2\lambda}{\rho^2 - 2\lambda}, \tag{47}
\]

where we have exchanged the order of integration in the first step, evaluated the inner integral and used \( g \geq 0 \) with \( \|g\|_{L_2(t)} = \delta(B^{-1}\ell) \) in the last step. The fraction evaluates to \( T \) in the limit \( \rho^2 = 2\lambda \). Combining (46)–(47) and (36), we arrive at the following theorem.

**Theorem 2.11.** Suppose that \( \ell \in F'_e \) is SPSD. Then, for any \( \rho \geq 0 \) and \( \lambda > 0 \), the variational problem (42) has a unique solution \( U \in E_\pi \). This solution is SPSD and admits the bound

\[
 \|U\|_\pi \leq C\|\ell\|_{-\pi} \quad \text{with} \quad C := \frac{\eta}{\rho^2 - 2\lambda} \tag{48}
\]

where \( C = \rho^2 T + 1 \) for \( \rho^2 = 2\lambda \).

The bound in (48) is sharp: for \( \eta > 0 \) and \( \ell := \eta^{-1}B(\chi_{(0,\eta)} \otimes \chi_{(0,\eta)}) \) we have \( g = \eta^{-1}\chi_{(0,\eta)} \) in (47), and the inequality in (47) approaches an equality as \( \eta \to 0 \).

For a general functional \( \ell \in F'_e \), we decompose \( \ell = \ell^+ - \ell^- + \ell^a \) as in (37). The corresponding solutions \( U^\pm := B^{-1}\ell^\pm \) and \( U^a := B^{-1}\ell^a = B^{-1} \ell^a \) (noting that \( \Delta U^a = 0 \) by antisymmetry) satisfy the bounds \( \|U^\pm\|_\pi \leq C\|\ell^\pm\|_{-\pi} \) and \( \|U^a\|_\pi = \|\ell^a\|_{-\pi} \). By linearity, \( U := U^+ - U^- + U^a \) is the solution to (42), and the estimate \( \|U\|_\pi \leq C(\|\ell^+\|_{-\pi} + \|\ell^-\|_{-\pi} + \|\ell^a\|_{-\pi}) \leq (C + 1)\|\ell\|_{-\pi} \) follows by triangle inequality in the first step and by (37) in the last step.

In contrast to Lemma 2.2, the solution \( U \to (42) \) may be SPSD even though the right-hand side \( \ell \) is not. Indeed, for any \( (w, v) \in E \times F \) with \( \Delta(w \otimes w, v \otimes v) = \int_0^T [w(t)v(t)]^2\,dt \neq 0 \), the expression

\[
 B(w \otimes w, v \otimes v) = |b(w, v)|^2 - \rho^2 \Delta(w \otimes w, v \otimes v) \tag{39}
\]

is negative for sufficiently large \( \rho \).

The variational formulations (39), (41) for the second moment and the covariance function are of the form (42) for the functionals \( \ell := \mathbb{E}[X^2](\delta_0 \otimes \delta_0) \) and \( \ell := \text{Cov}(X_0)(\delta_0 \otimes \delta_0) + \rho^2 \Delta(m \otimes m) \).

The proof of the above theorem highlights the special status of the diagonal \( t \mapsto U(t, t) \). First, it is uniquely defined as the solution of an integral equation. Second, it determines all other off-diagonal values of \( U \). Finally, the projective norm (46) only “looks” at the diagonal (when \( U \) is SPSD). These insights will guide **a** the development of the numerical methods in §3 and **b** the proof of well-posedness of the deterministic second moment equation also for the vector-valued case in §4.
3. CONFORMING DISCRETIZATIONS OF THE DETERMINISTIC EQUATIONS

3.1. Orientation. In §2 we have derived deterministic variational formulations for the first and second moments of the stochastic processes (6) and (7). In particular, the first moment satisfies a known “weak” variational formulation of an ODE. To our knowledge, [6, 7] were the first to discuss the numerical analysis of conforming finite element discretizations of a space-time variational formulation for linear parabolic PDEs. The problem was first reduced to the underlying family of ODEs parameterized by the spectral parameter $\lambda$. With the notation from §2.2 for the bilinear form $b$ and the spaces $E$ and $F$, the solution $u$ to such an ODE is characterized by a well-posed variational problem of the above form (19), with a general right-hand side $\ell$. The temporal discretization analyzed in [7] was of the conforming type, employing discontinuous piecewise polynomials as the discrete trial space for $u$ and continuous piecewise polynomials of one degree higher as the discrete test space for $v$. The analysis in essence revealed that the discretization is not uniformly stable (in the Petrov–Galerkin sense, as discussed below) in the choice of the discretization parameters such as the polynomial degree and the location of the temporal nodes [7, Thm. 2.2.1].

The same question of stability of was taken up in [2] for a “strong” space-time variational formulation of linear parabolic PDEs and for the two classes of discretizations, of Gauss–Legendre (e.g., Crank–Nicolson, CN) or Gauss–Radau (e.g., implicit Euler, iE) type. It was confirmed that both types are in general only conditionally space-time stable, but the Gauss–Radau type can be made unconditionally stable under mild restrictions on the temporal mesh. We will first revisit the simplest representative of each group adapted to the present variational formulation. The adaptation consists in switching the roles of the discrete trial and test spaces and by reversing the temporal direction, the latter due to the integration by parts that was used in the derivation of the variational formulation (19). The resulting adjoint discretizations will therefore be denoted by CN$^*$ and iE$^*$, respectively. The CN$^*$ discretization is thus a special case of the discretizations analyzed in [7].

In summary, in §3.2 we will discuss two conforming discretizations for the deterministic first moment equation (19): CN$^*$ which is only conditionally stable (depending on the spectral parameter $\lambda$) and iE$^*$ which is stable under a mild condition on the temporal mesh (comparable size of neighboring temporal elements). Both employ discontinuous trial spaces but iE$^*$ requires additional discussion due to the somewhat unusual shape functions, whereby the discrete trial spaces are not nested and therefore do not generate a dense subspace in the usual sense. The situation transfers with no surprises to the second moment equations with additive noise (24) by tensorizing the discrete trial/test spaces. The case of multiplicative noise (39), however, presents a significant twist due to:

1. the presence of the $\Delta$ term in the definition (38) of the bilinear form $B$. We will see that CN$^*$ interacts naturally with the $\Delta$ operator while iE$^*$ requires a modification to restore the expected convergence order.
2. the non-Hilbertian nature of the trial and test spaces in (39).

We will then provide a common framework for both discretizations, generalizing to arbitrary polynomial degrees. This will allow us to use the unconditionally stable Gauss–Radau discretization family without resorting to the modification of the lowest-order iE$^*$ discretization because the discrete trial spaces with higher polynomial degree do generate a dense subspace.

Since the trial and test spaces in (39) are not Hilbert spaces, we briefly state results on Petrov–Galerkin discretizations of variational problems on normed spaces in §3.3. In §3.4 we construct discretizations on tensor product spaces and comment on their stability. These are applied to the variational problem (24) for the second moment in the additive case in §3.5.

In the multiplicative case we obtained existence and stability of the exact solution for arbitrary $\rho \geq 0$ in Theorem 2.11, even beyond the trivial range $0 \leq \rho^2 < 2\lambda$. The situation is similar in the discrete setting, where this trivial range is reduced by the discrete inf-sup constant $\gamma_k$ to $0 \leq \rho^2 < 2\lambda_k \gamma_k^2$. In §3.6 we will therefore investigate, for the low order CN$^*$ and iE$^*$ schemes
and some of their variants, whether stability holds for all $\rho \geq 0$. The behavior of the high order discretizations beyond the trivial stability range remains an open question.

3.2. First moment discretization. We are using the notation from §2.2. Let us consider the general formulation of (19) as the variational problem

$$\text{Find } u \in E \text{ s.t. } b(u, v) = \ell(v) \quad \forall v \in F$$

(49)

with some bounded linear functional $\ell \in F'$. Recall that the spaces $E$ and $F$ carry the $\lambda$-dependent norms (12) that render $b : E \to F'$ an isometric isomorphism. This variational problem is formally obtained by testing the real-valued ODE

$$u'(t) + \lambda u(t) = f \quad \text{on } J = (0, T), \quad u(0) = g,$$

(50)

with a test function $v$, integrating over $J$, moving the derivative from $u'$ to $v$ via integration by parts and then replacing the exposed $u(0)$ by the given initial datum $g$. The corresponding right-hand side then reads as $\ell(v) := \int_J (f, v) \, dt + (g, v(0))$. We write $\langle \cdot, \cdot \rangle$ for the simple multiplication to emphasize the structure of the problem and to facilitate the transition to vector-valued ODEs.

For the discretization of the variational problem (49) we need to define subspaces

$$E^k \subset E \quad \text{and} \quad F^k \subset F$$

of the same (nontrivial) finite dimension. We then consider the discrete variational problem

$$\text{Find } u^k \in E^k \text{ s.t. } b(u^k, v) = \ell(v) \quad \forall v \in F^k.$$  

(51)

The well-posedness of this discrete problem is quantified by the discrete inf-sup constant

$$\gamma_k := \inf_{w \in S(E^k)} \sup_{v \in S(F^k)} b(w, v) > 0,$$

(52)

since the norm of the discrete data-to-solution mapping $\ell|_{F^k} \mapsto u_k$ equals $1/\gamma_k$. Moreover, the quasi-optimality estimate

$$\|u - u^k\|_E \leq (\|b\|/\gamma_k) \inf_{w \in E^k} \|u - w\|_E$$

(53)

holds [44, Thm. 2], where in fact $\|b\| = 1$ by (17). We call a family $\{E^k \times F^k\}_{k>0}$ of discretization pairs uniformly stable if $\inf_{k>0} \gamma_k > 0$. To construct $E^k \times F^k$ we introduce a temporal mesh

$$\mathcal{T} := \{0 =: t_0 < t_1 < \ldots < t_N := T\}$$

(54)

subdividing $J = (0, T)$ into $N$ temporal elements. Below, the dependence on $\mathcal{T}$ is implicit in the notation. We write

$$J_n := (t_{n-1}, t_n) \quad \text{and} \quad k_n := |t_n - t_{n-1}|, \quad n = 1, \ldots, N.$$

As announced above, we first discuss the simplest representatives of the Gauss–Legendre and Gauss–Radau discretizations in §3.2.1–§3.2.2, which are the CN$^*$ and the iE$^*$ schemes. For both methods, the discrete test space $F^k \subset F$ is defined as the spline space of continuous piecewise affine functions $v$ with respect to the temporal mesh $\mathcal{T}$ such that $v(T) = 0$. A common framework is the subject of §3.2.3.

3.2.1. The CN$^*$ discretization. For the discrete trial space $E^k \subset E$, the space of piecewise constant functions with respect to $\mathcal{T}$ seems a natural choice. We call this discretization CN$^*$ in reference to the reversal of the roles of the trial and test spaces compared to the usual Crank–Nicolson time-stepping scheme. Unfortunately, if we keep the temporal mesh $\mathcal{T}$ fixed, the discrete inf-sup constant (52) of the couple $E^k \times F^k$ depends on the spectral parameter $\lambda$, see Figure 1. This was already observed in [7, Eqn. (2.3.10)]. It can be shown along the lines of [2] that $\gamma_k \gtrsim (1 + \min\{\sqrt{\lambda T}, \text{CFL}\})^{-1}$, where $\text{CFL} := \max_n k_n \lambda$ is the parabolic CFL number. The three-phase behavior of the CN$^*$ scheme in Figure 1 can be intuitively understood as follows: Consider $b(w, v) = \int_J (-v' + \lambda v) \, dw$ from (52). For any $w \in E^k$ we can find a $v \in F^k$ such that $-v' = w$, so that at sufficiently low spectral numbers $\lambda$, the estimate $\gamma_k \geq 1 - \varepsilon$ is evident. For large $\lambda$, the
Figure 1. The inf-sup constant (52) for the CN* and the iE* discretizations on the same “random” temporal mesh of the interval (0, 1) with 210 nodes and backward successive temporal element ratio \( \sigma \leq 3 \) in (58). The bound shown is the estimate from (59).

Function \(-v' + \lambda v\) is, up to negligible jumps, a piecewise linear continuous one. Such functions approximate a general piecewise constant \( w \) poorly, see [7, Eqn. (2.3.10)].

This behavior renders the method less useful for parabolic PDEs because following a spatial semi-discretization, a low parabolic CFL number has to be maintained for uniform stability.

3.2.2. The iE* discretization. To obtain stability under only mild restrictions we recur to an observation of [2, §3.4]; for the sake of a self-contained exposition and sharp results we confine the discussion first to the lowest order case. We take \( E_k \) as the space of functions \( w \in L_2(J) \) for which each \( w|_{J_n} \) is a dilated translate of the shape function \( \phi : s \mapsto (4 - 6s) \) from the reference temporal element \((0, 1)\) to the temporal element \( J_n = (t_{n-1}, t_n) \). We refer to this combination of \( E_k \times F_k \) as iE* (adjoint implicit Euler). The motivation for this definition is as follows. Consider the adjoint (backward) ODE

\[-v' + \lambda v = f, \quad v(T) = 0, \tag{55}\]

with a given \( f \) that for the sake of argument is piecewise affine with respect to \( \mathcal{T} \). Define the approximate continuous piecewise affine solution \( v \in F_k \) (hence, \( v(T) = 0 \)) through the implicit Euler time-stepping scheme backward in time:

\[-\frac{1}{k_n}(v(t_n) - v(t_{n-1})) + \lambda v(t_{n-1}) = f(t_{n-1}^+), \quad n = N, \ldots, 1, \tag{56}\]

where \( t_{n-1}^+ \) denotes the limit from above. We shall use the obvious abbreviations \( v_n \) and \( f_{n-1}^+ \) when referring to (56). The definition of the discrete trial space \( E_k \) implies that the time-step condition (56) is equivalent to the variational requirement

\[\int_{J_n} \left( w, -v' + \lambda v - f \right) \, dt = 0 \quad \forall w \in E_k \quad \forall n = N, \ldots, 1. \tag{57}\]

The equivalence is due to the identity \( \int_0^1 \phi(s)(as + b) \, ds = b \) for all real \( a \) and \( b \), which implies that the integral in (57) is a multiple of \((-v' + \lambda v - f)(t_{n-1}^+)\).

The role of the adjoint ODE (55) is elucidated in the proof of the following proposition that bounds the inf-sup constant (52) for the iE* discretization. The result is formulated in terms of the backward successive temporal element ratio

\[\sigma := \max_{n=1, \ldots, N-1} \frac{k_n}{k_{n+1}}. \tag{58}\]

Proposition 3.1. The inf-sup condition (52) holds for the iE* discretization with

\[\gamma_k \geq \gamma_\sigma := \frac{1}{\sqrt{2(1 + \max\{1, \sigma\})}}, \tag{59}\]
uniformly in $\lambda > 0$.

Thus, in order to obtain uniform stability of the iE* discretization it suffices to ensure that the backward successive temporal element ratio (58) stays bounded. This is verified numerically in Figure 1. We generated an initial temporal mesh for $T = 1$ with 129 nodes by distributing the inner nodes in interval $(0, 1)$ uniformly at random. New nodes were inserted by subdividing large temporal elements into two equal ones until $\sigma \leq 3$, leading to a temporal mesh with 210 nodes. On this new temporal mesh, we observe that the inf-sup constant of the iE* discretization is controlled as in (59), while that of CN* depends strongly on the spectral parameter $\lambda$, as already explained in §3.2.1.

Proof of Proposition 3.1. Let $w \in E^k$ be arbitrary nonzero. We will find a discrete $v \in F^k$ such that $b(w, v) \geq \gamma_\sigma \|w\|_E \|v\|_F$. To this end, consider the adjoint ODE (55) with $f := \lambda w$. If we took $v$ as the exact solution we would obtain $b(w, v) = \|w\|_E^2 = \lambda^{-1}\|v\|_F^2$. However, the exact solution is not necessarily an element of the discrete test space $F^k$, so we take $v \in F^k$ according to the implicit Euler scheme (56) instead. By the equivalence of (56)–(57) we see that $b(w, v) = \int_J \langle w, -v' + \lambda v \rangle \, dt = \int_J \langle w, \lambda w \rangle \, dt = \|w\|_E^2$ still holds.

To conclude, it is enough to establish $\|w\|_E \geq \gamma_\sigma \|v\|_F$. To that end, we square (56) with $f := \lambda w$ on both sides and rearrange to obtain

$$\lambda^{-1}k^{-1}_n |v_n - v_{n-1}|^2 + \lambda k_n |v_{n-1}|^2 + |v_n - v_{n-1}|^2 - |v_n|^2 = \lambda k_n |w^+_n|^2.$$  \hfill (60)

Let $i_k v$ denote the piecewise constant function with $i_k v(t_{n-1}^+) = v(t_{n-1})$ for all $n = 1, \ldots, N$. The dual of $E$ is identified via the (unweighted) $L_2(J)$ inner product. We introduce the mesh-dependent norm

$$\|v\|_F^2 := \|v\|_E^2 + \|i_k v\|_E^2 + |v(0)|^2 + \sum_{n=1}^N |v_n - v_{n-1}|^2$$

and sum up (60) over $n$. This yields the equality $\|w\|_E = \frac{1}{2} \|v\|_F$, since $\int_0^1 |\phi(s)|^2 \, ds = 4 = \frac{1}{4} |\phi(0)|^2$. With $\sigma$ from (58) we obtain the estimate (the last term is omitted for $n = N$)

$$\|v\|_{L_2(J_n)}^2 \leq \frac{1}{2} k_n (|v_{n-1}|^2 + |v_n|^2) \leq \frac{1}{2} \|i_k v\|_{L_2(J_n)}^2 + \frac{1}{2} \sigma \|i_k v\|_{L_2(J_{n+1})}^2.$$  \hfill (62)

Summation over $n$ yields $\|v\|_F^2 \leq 2(1 + \max(1, \sigma)) \frac{1}{2} \|v\|_F^2$. In concatenation, $\|w\|_E = \frac{1}{2} \|v\|_F \geq \gamma_\sigma \|v\|_F$, as anticipated. $\square$

The choice of the shape function $\phi : s \mapsto (4 - 6s)$ in the trial space $E^k$ defining the iE* discretization leads to uniform stability as discussed above. In view of the quasi-optimality estimate (53) we need to address the approximation properties of this trial space $E^k$. Unfortunately, we do not have nestedness $E^k \subset E^{k+1}$. Moreover, no matter how fine the temporal mesh, $E^k$ does not approximate the constant function. To be precise, let $P_d$ denote the $L_2$-orthonormal Legendre polynomial (normalized to $P_d(1) = \sqrt{1 + 2d}$) of degree $d \geq 0$ on the reference interval $(0, 1)$. For real $a, b$, set $u := a P_0 + b P_1 + r$, where $r$ is E-orthogonal to $P_0$ and $P_1$. The E-orthogonal projection of $u$ onto the span of the shape function $\phi = P_0 - \sqrt{3} P_1$ is $w := c \phi$ with $c = \frac{1}{4} (a - \sqrt{3} b)$. The error $\|u - w\|_E^2 = \lambda^{-1} |\sqrt{3} a + b|^2 + \|r\|_E^2$ may be large, for example, if $u$ is constant.

3.2.3. Common framework. On the $n$-th element of the temporal mesh $\mathcal{T}$ in (54), let $\mathcal{N}_n \subset [t_{n-1}, t_n)$ be a set of $p \geq 1$ collocation nodes (we choose the same $p$ for all $n$ for simplicity). The compound element-wise interpolation operator based on these collocation nodes $\mathcal{I}_n$ is denoted by $i_k$. As the discrete test space $F^k \subset F$, we take the subspace of continuous piecewise polynomials of degree $p$ with respect to $\mathcal{T}$. We introduce $i_k : i_k F^k \to F^k$ by $(i_k \cdot, \cdot)_{L_2(J)} = (\cdot, i_k \cdot)_{L_2(J)}$ on $F^k \times i_k F^k$. The discrete trial space is then defined as $E^k := i_k^* F^k$. Note that the dimensions $\dim E^k = \dim F^k$ match.

We are interested in two types of nodes: Gauss–Legendre nodes and (left) Gauss–Radau nodes, to which we refer as GL$^+$ and GR$^-$, respectively. All temporal elements host the same type of nodes. The lowest-order examples are $\mathcal{N}_n = \{ \frac{1}{2}(t_{n-1} + t_n) \}$ for GL$^+$ and $\mathcal{N}_n = \{ t_{n-1} \}$ for GR$^-$.
corresponding to the CN* and iE' schemes. The shape functions on the reference element \((0, 1)\) for the space \(E^k = i_k^p F^k\) are (cf. [2, §2.3])

1. the Legendre polynomials \(P_0, \ldots, P_{p-1}\) for GL\(_p\), and
2. the Legendre polynomials \(P_0, \ldots, P_{p-2}\) together with \(P_{p-1} - \frac{p_{p-1}(1)}{p_p} P_p\) for GR\(_p^-\).

In particular, for \(p \geq 2\), the GR\(_p^-\) family contains the piecewise constant functions, which means that any function in \(E\) can be approximated to arbitrary accuracy upon mesh refinement.

Define the mesh-dependent norm \(\| \cdot \|_F\) by

\[
\|v\|_F^2 := \|v\|_{E,x}^2 + \|i_k v\|_{E,x}^2 + \|v(0)\|^2 + \left\{ \begin{array}{ll}
0 & \text{for GL\(_p\),} \\
\sum_{n=1}^N [v - i_k v]_{-n}^2 & \text{for GR\(_p^-\),}
\end{array} \right.
\]

where \([f]_{-n}\) denotes \(\lim_{t \to -n} f(t)\). This is the generalization of (61).

Following [2, Proof of Thm. 3.3], we can now show:

**Lemma 3.2.** For any \(v \in F^k\) there exists a nonzero \(w \in E^k = i_k^p F^k\) such that

\[
b(w, v) \geq \|(i_k^*)^{-1}w\|_E \|v\|_F. \tag{63}
\]

**Proof.** The space \(i_k^p F^k \subset E\) carries the norm of \(E\). Let \(v \in F^k\). We first show that \(\|\Gamma v\|_E = \|v\|_F\), where \(\Gamma : F^k \to i_k^p F^k\) is defined by

\[
(\Gamma v, \hat{w})_E = b(i_k^* \hat{w}, v) \quad \forall (v, \hat{w}) \in F^k \times i_k^p F^k.
\]

To this end, we expand \(\|\Gamma v\|_E^2 = \|\Gamma v - i_k v\|_E^2 + 2(\Gamma v, i_k v) - \|i_k v\|_E^2\). For the first term we have

\[
\|\Gamma v - i_k v\|_E = \sup_{\hat{w} \in S(i_k^p F^k)} (\Gamma v - i_k v, \hat{w}) = \sup_{\hat{w} \in S(i_k^p F^k)} \{b(i_k^* \hat{w}, v) - (i_k v, \hat{w})_E\} = \|v\|_F^2.
\]

For the second term, we use the definition of \(\Gamma\), followed by [2, Lem. 3.1]:

\[
(\Gamma v, i_k v)_E = \|i_k v\|_E^2 - (i_k v, v)_L^2 = \left\{ \begin{array}{ll}
\|i_k v\|_E^2 + \|v(0)\|^2 & \text{(GL\(_p\))}, \\
\frac{1}{2} \sum_{n=1}^N [v - i_k v]_{-n}^2 & \text{(GR\(_p^-\)).}
\end{array} \right.
\]

Hence, \(\|\Gamma v\|_E = \|v\|_F\). Now take \(\hat{w} := \Gamma v\). Then \(b(i_k^* \hat{w}, v) = (\Gamma v, \hat{w})_E = \|\Gamma v\|_E^2 = \|\hat{w}\|_E \|v\|_F\). The claim (63) follows for \(w := i_k^* \hat{w}\).

In order to convert (63) to a statement with the original norms, we need to compare these norms. First, it can be shown as in [2, §3.2.2] that \(\|w\|_E \leq \|i_k^*\| \|w\|_{L^2(GR_p^-)} \leq 2\|i_k^*\|^{-1} \|w\|_E\).

Second, we need to quantify \(\|v\|_F \leq \|v\|_E\). For the Gauss–Radau family GR\(_p^-\) we can, for example, use the estimate (akin to (62)); see [2, §3.4])

\[
\|v - i_k v\|_{L^2(t_{n-1}, t_n)}^2 \leq \frac{2p^2}{4p^{2-1/p}} \left( \|i_k v\|_{L^2(t_{n-1}, t_n)}^2 + \frac{\gamma_n}{\gamma_{n+1}} \|i_k v\|_{L^2(t_{n}, t_{n+1})}^2 \right)
\]

to derive \(\|v\|_F \leq C \sqrt{p(1 + \sigma)} \|v\|_E\) with the backward successive temporal element ratio \(\sigma\) from (58) and a universal constant \(C > 0\). Therefore, the discrete inf-sup condition (52) holds for the GR\(_p^-\) family with

\[
\gamma_k := \frac{\gamma_0}{\sqrt{p(1 + \sigma)}}, \tag{64}
\]

where \(\gamma_0 > 0\) is a constant independent of all parameters. The Gauss–Legendre family GL\(_p\) suffers from the same potential instability as the CN* scheme, see §3.2.1.

Consider now the solution \(u^k\) to (51). From the ODE (50), the reconstruction

\[
\hat{u}^k := g + \int_0^t \{f(s) - \lambda u^k(s)\} \, ds
\]

can be expected to provide a better approximation of the exact solution. With (51) we find the orthogonality property \((\hat{u}^k - u^k, v')_E = 0\) for all \(v \in F^k\). Let

\[
q_k : E \to \partial_i F^k
\]

(65)
be the orthogonal projection (in $E$ or in $L_2(J)$). The orthogonality property gives $q_k \tilde{u}^k = q_k u^k$. Hence, the postprocessed solution $\tilde{u}^k := q_k u^k$ is an approximation of the reconstruction $\bar{u}^k$. In the case of Gauss–Legendre collocation nodes, $i_\ast^k$ is the identity, so that $E^k = i_k F^k$, and therefore $q_k u^k = u^k$ has no effect. In the Gauss–Radau case, however, the projection is useful to improve the convergence rate upon mesh refinement, as will be seen in §3.6.4.

Note that $q_k$ is injective on $E^k$ in both cases. In the Gauss–Radau case, $q_k^{-1}$ sends the shape function $P_{p-1}$ to $P_{p-1} - \frac{p_{1,1}(1)}{p_{p,1}(1)} P_p$. Since $p_{p,1}(1) = \sqrt{2d+1}$, this gives

$$||q_k^{-1}||^2 = 1 + \frac{2p+1}{2(p-1)+1}. \tag{66}$$

### 3.3. Petrov–Galerkin approximations

In this subsection we comment on Petrov–Galerkin discretizations of the generic linear variational problem

$$\text{Find } u \in X \text{ s.t. } \langle Bu, v \rangle = \langle \ell, v \rangle \quad \forall v \in Y,$$

where $X$ and $Y$ are normed vector spaces. This generalization away from Hilbert spaces (that can also be found e.g. in [39]) will allow us to address the variational problem (39).

We assume that $X_h \times Y_h \subset X \times Y$ are finite-dimensional subspaces with nonzero dim $X_h = \dim Y_h$. Here, $h$ refers to the “discrete” nature of these subspaces, and the pair $X_h \times Y_h$ is fixed. We write $\| : \|_{\gamma'_k} := \sup_{v \in \mathcal{S}(Y'_h)} \{ , v \}$. (70)

In order to admit variational crimes we suppose that we have access to an operator $\bar{B}: X \rightarrow Y' \,$ that approximates $B$ (although $\bar{B}: X \rightarrow Y'_h \,$ suffices). For this approximation we assume the discrete inf-sup condition in the form of a constant $\gamma_h > 0$ such that $\|\bar{B}w_h\|_{Y'_h} \geq \gamma_h \|w_h\|_X$ for all $w_h \in X_h$. The proof of the following Proposition is obtained by standard arguments (for the discussion of the constant “1+” see [1, 39, 44]).

**Proposition 3.3.** Fix $u \in X$. Under the above assumptions there exists a unique $u_h \in X_h$ such that

$$\langle \bar{B}u_h, v_h \rangle = \langle Bu, v_h \rangle \quad \forall v_h \in Y_h.$$

The mapping $u \mapsto u_h$ is linear with $\|u_h\|_X \leq \gamma^{-1}_h \|Bu\|_{Y'_h}$, and satisfies the quasi-optimality estimate

$$\|u - u_h\|_X \leq (1 + \gamma^{-1}_h \|\bar{B}\|) \inf_{w_h \in X_h} \|u - w_h\|_X + \gamma^{-1}_h \|\bar{B}\| \|u\|_{Y'_h}.$$

### 3.4. Tensorized discretizations

Recall the definition of the tensor product spaces $E_{2/\pi}$ and $F_{2/\epsilon}$ from (20) and (28). Recall also that we can extend $B := (b \otimes b)$ to an isometric isomorphism $B: E_2 \rightarrow F'_2$ or $B: E_{\pi} \rightarrow F'_{\epsilon}$. We discuss here these two viewpoints in parallel. Consider the variational formulation

$$\text{Find } U \in E_{2/\pi} \text{ s.t. } B(U, v) = \ell(v) \quad \forall v \in F_{2/\epsilon}, \tag{67}$$

where $\ell \in F'_{2/\epsilon}$. If $F^k \times F^k$ is a discretization for (49) then the pair of tensorized subspaces

$$E_{2/\pi}^k \times F_{2/\epsilon}^k := (E^k \otimes E^k) \times (F^k \otimes F^k) \subset E_{2/\pi} \times F_{2/\epsilon} \tag{68}$$

is a natural choice for the discretization for (67). The subscript 2 or $\pi$ (and 2 or $\epsilon$) indicates which norm the algebraic tensor product $E^k \otimes E^k$ (and $F^k \otimes F^k$) is equipped with; since these spaces are finite-dimensional, no norm-closure is necessary.

We now turn to the discrete variational formulation

$$\text{Find } U^k \in E_{2/\pi}^k \text{ s.t. } B(U^k, v) = \ell(v) \quad \forall v \in F_{2/\epsilon}^k. \tag{69}$$

The inf-sup constant required in the analysis is the square $\gamma^2_k$ of the discrete inf-sup constant $\gamma_k$ from (52) in both cases:

$$\inf_{w \in \mathcal{S}(E_{2/\pi}^k)} \sup_{v \in \mathcal{S}(F_{2/\epsilon}^k)} B(w, v) = \gamma^2_k = \inf_{w \in \mathcal{S}(E_{2/\pi}^k)} \sup_{v \in \mathcal{S}(F_{2/\epsilon}^k)} B(w, v) \tag{70}$$

Indeed, consider the $\pi/\epsilon$ situation. For $w \in E^k$ let $b^w$ denote the restriction of $bw$ to $F^k$. The discrete inf-sup condition (52) says that $b_k: E^k \rightarrow (F^k)'$ is an isomorphism with $\|b_k^{-1}\| = \gamma_k^{-1}$. 


The mapping $B_k := b_k \otimes b_k : E^k \otimes_{\pi} E^k \to (F_k)^\prime \otimes_{\pi} (F_k)^\prime$ has the inverse $b_k^{-1} \otimes b_k^{-1}$. It is therefore an isomorphism with $\|B_k^{-1}\| = \gamma_k^{-2}$. The identification $(F_k)^\prime \otimes_{\pi} (F_k)^\prime \cong (F_{\varepsilon}^k)^\prime$ shows that for any $w \in E_{\varepsilon}^k$, the functional $B_k w$ is the restriction of $B w$ to $F_{\varepsilon}^k$. This gives (70).

Proposition 3.3 (with $B := B$) provides a unique solution $U_k \in E^k \otimes E^k$ to the discrete variational problem (69) that approximates the solution $U$ of (67) as soon as $\gamma_k > 0$ in (52). The solution is, moreover, quasi-optimal (recall that $\|B\| = 1$):

$$\|U - U_k\|_{2/\pi} \leq (1 + \gamma_k^{-2}) \inf_{w \in E^k \otimes E^k} \|U - w\|_{2/\pi}. \quad (71)$$

We will also be interested in the postprocessed solution $\tilde{U}_k := (q_k \otimes q_k) U_k$, where $q_k : E \to \partial_t F^k$ is the orthogonal projection in (65).

Analogously to Lemma 2.2 one proves:

**Lemma 3.4.** The discrete solution $U_k$ to (69) is SPSD if and only if $\ell$ is SPSD on $F_k \otimes F^k$. The same is true for the postprocessed solution.

### 3.5. Second moment discretization: additive noise.

In view of the previous section, any discretization pair $E^k \times F^k$ satisfying the discrete inf-sup condition (52) induces a valid discretization of the variational problem (24) for the second moment of the solution process to the stochastic ODE with additive noise (4) if we choose the trial space as $E^k \otimes E^k$ and the test space as $F^k \otimes F^k$. The functional on the right-hand side of (67) is then $\ell := \mathbb{E}[X_0^2](\delta_0 \otimes \delta_0) + \mu^2 \delta$. Moreover, the discrete solution satisfies the quasi-optimality estimates in (71) simultaneously with respect to $\|\cdot\|_2$ and $\|\cdot\|_\pi$, because $\ell \in F_{\varepsilon}^k \subset F_{\varepsilon}^k$.

### 3.6. Second moment discretization: multiplicative noise.

As in the continuous case for sufficiently small values of the volatility $\rho$, namely in the range

$$0 \leq \rho^2 < 2\lambda \gamma_k^2, \quad (72)$$

we immediately obtain a discrete inf-sup condition for the operator $B - \rho^2 \Delta$. The purpose of this section is to address the whole range $\rho \geq 0$.

We will focus on the CN and iE’ discretizations discussed in §§3.2.1–3.2.2, although with some work, our methods may be adapted to higher-order schemes from §3.2.3. Throughout, we assume that the discretization pair $E^k \times F^k \subset E \times F$ satisfies the discrete inf-sup condition (52). The discrete trial and test spaces $E_{\varepsilon}^k \times F_{\varepsilon}^k \subset E_{\pi}^k \times F_{\pi}^k$ are defined as in (68).

We introduce some more notation. In what follows, the default range of the indices (we use $m$ as an index, since the first moment does not appear anymore) is

$$0 \leq i, j \leq N - 1 \quad \text{and} \quad 1 \leq m, n \leq N.$$

Recall that the discrete test space $F_{\varepsilon}^k \subset F$ consists of continuous piecewise affine functions with respect to the temporal mesh $\mathcal{T}$ in (54) that vanish at the terminal time $T$. It is equipped with the hat function basis $\{v_i\}_i$, determined by $v_i(t_j) = \delta_{ij}$. The basis functions $\{e_n\}_n$ of the discrete trial space $E_{\varepsilon}^k \subset E$ are supported on supp($e_n$) = $[t_{n-1}, t_n]$ in both schemes. Specifically, $e_n$ is a constant for CN and is a dilated translate of the shape function $\phi : s \mapsto (4 - 6s)$ for iE’.

The following statements do not depend on the scaling of the basis functions, if not specified otherwise.

#### 3.6.1. The discrete problem.

In the multiplicative case, the trace product $\Delta$ from (26) appears in the variational problem (39) for the second moment. The basis functions $\{e_n\}_n \subset E_{\pi}^k$ for the iE’ discretization lead to an inconsistency in the $\Delta$ term, see §3.6.5. For this reason, we introduce the approximate trace product

$$\Delta^k : E_{\pi}^k \times F_{\varepsilon} \to \mathbb{R}, \quad (73)$$

to be specified below. We require that $\Delta^k$ reproduces the following properties of the exact trace product $\Delta$: 
(i) **Symmetry and definiteness:** for every SPSD \( w \in E^k_{\pi} \), the functional \( \Delta^k w \) is SPSD on \( F^k \otimes F^k \), i.e.,

\[
\Delta^k(w, \psi \otimes \tilde{\psi}) = \Delta^k(w, \tilde{\psi} \otimes \psi) \quad \text{and} \quad \Delta^k(w, \psi \otimes \tilde{\psi}) \geq 0 \quad \forall \psi, \tilde{\psi} \in F^k.
\]

(ii) **Locality:**

\[
\Delta^k(e_m \otimes e_n, \nu_i \otimes \nu_j) \neq 0 \quad \text{only if} \quad m = n \quad \text{and} \quad i, j \in \{n-1, n\}.
\]

(iii) **Bilinearity and continuity on \( E_\pi \times F_\epsilon \).**

The corresponding approximation of the operator \( \mathcal{B} \) is defined as \( \mathcal{B}^k := B - \rho^2 \Delta^k \). We are now interested in the solution of the discrete variational problem

\[
\text{Find} \quad U^k \in E^k_{\pi} \quad \text{s.t.} \quad \mathcal{B}^k(U^k, \nu) = \ell(\nu) \quad \forall \nu \in F^k \quad (74)
\]

which approximates (42).

### 3.6.2. Well-posedness of the discrete problem

The solution \( U^k \) to (74) can be expanded in terms of the basis \( \{e_m \otimes e_n\}_{mn} \) of \( E^k_{\pi} \) as

\[
U^k = \sum_{mn} U_{mn}(e_m \otimes e_n) \quad \text{with} \quad U_{mn} = \frac{(U^k, e_m \otimes e_n)}{\|e_m\|_E^2 \|e_n\|_E^2} \quad (75)
\]

We combine its coefficients in the \( N \times N \) matrix \( U := (U_{mn})_{mn} \). Furthermore, we define the values

\[
b_{in} := b(e_n, \nu_i) \quad \text{and} \quad \ell_{ij} := \ell(\nu_i \otimes \nu_j).
\]

If the discrete inf-sup condition (52) is satisfied then \( b_{n-1,n} \neq 0 \) follows.

The sparsity assumption on \( \Delta^k \) together with the fact that the discretization pair \( E^k_{\pi} \times F^k_\epsilon \) is a tensor product discretization allows for an explicit formula for the diagonal entries of \( U \). This is presented in the lemma below.

For future purpose, we note that \( w \in E^k \otimes E^k \) is SPSD if and only if the matrix of coefficients \( w := (w_{mn})_{mn} \) with respect to \( \{e_m \otimes e_n\}_{mn} \) is. Indeed, if \( \varphi \in L_2(J) \) and \( \varphi = ((e_n, \varphi)_{L_2(J)})_n \in \mathbb{R}^N \) then \( \varphi^T w \varphi = \sum_{mn} w_{mn}(e_m, \varphi)_{L_2(J)}(e_n, \varphi)_{L_2(J)} = (w, \varphi \otimes \varphi)_{L_2(J \times J)} \).

According to the locality assumption (ii), the nonzero values of \( \Delta^k \) (as acting on the basis functions) can be combined in the \( 2 \times 2 \) matrices

\[
\Delta^n := \begin{pmatrix}
\Delta^k(e_n \otimes e_n, \nu_{n-1} \otimes \nu_{n-1}) & \Delta^k(e_n \otimes e_n, \nu_{n-1} \otimes \nu_n) \\
\Delta^k(e_n \otimes e_n, \nu_n \otimes \nu_{n-1}) & \Delta^k(e_n \otimes e_n, \nu_n \otimes \nu_n)
\end{pmatrix}, \quad 1 \leq n \leq N - 1,
\]

and in \( \Delta^N := \Delta^k(e_N \otimes e_N, \nu_{N-1} \otimes \nu_{N-1}) \). The foregoing remark and Assumption (i) on \( \Delta^k \) imply that each \( \Delta^n \) is SPSD.

We define

\[
\beta_n := (1 - \rho^2 b_{n-1,n}^{-2} \Delta^n_{11})^{-1}, \quad n = 1, \ldots, N, \quad (77)
\]

where \( \Delta^n_{pq} \) denotes the \((p, q)\)-th entry in the matrix \( \Delta^n \), and for \( n \geq 2 \):

\[
\theta_n := b_{n-1,n} b_{n-1,n-1}^{-1}, \quad (78)
\]

\[
\alpha_n := \beta_n \left[ \theta_n^2 + \rho^2 b_{n-1,n}^{-2} \left( \Delta_{22} - 2 b_{n-2,n-1}^{-1} b_{n-1,n-1} \Delta_{11}^{-1} \right) \right] \quad (79)
\]

We note that

\[
\frac{\|e_n\|_E^2}{\|e_n\|_E^2} \alpha_n, \quad \frac{\|e_n\|_E^2}{\|e_n\|_E^2} \theta_n, \quad \text{and} \quad \beta_n \quad (80)
\]

do not depend on the scaling of the basis \( \{e_n\}_n \).

For technical reasons we also introduce the function \( G^k \in E^k_{\pi} \) as the solution (which is well-defined under the inf-sup condition (52)/(70)) to

\[
\text{Find} \quad G^k \in E^k_{\pi} \quad \text{s.t.} \quad B(G^k, \nu) = \ell(\nu) \quad \forall \nu \in F^k_\epsilon \quad (81)
\]

Let \( G_{mn} \) denote its coefficients with respect to \( \{e_m \otimes e_n\}_{mn} \).
Lemma 3.5. Let $\ell \in F'_e$. Assume that $\beta_n$ is finite for all $n$. Then there exists a unique solution $U^k \in E^k_n$ to the discrete variational problem (74). Its diagonal coefficients in (75) are

$$U_{nn} = \beta_n G_{nn} + \sum_{m=1}^{n-1} G_{mn}(\beta_m \alpha_{m+1} - \beta_{m+1} \theta^2_{m+1}) \prod_{v=m+2}^{n} \alpha_v. \quad (82)$$

Proof. By locality of the support of $e_n$ and $v_i$, the values $b_{in} = b(e_n, v_i)$ are non-zero at most for $i \in \{n-1, n\}$. Therefore, the coefficients $\{w_{in}\}_n$ of the solution $w \in E^k$ to the problem “$b(w, v) = f(v)$ for all $v \in E^k_n$” are obtained by recursion,

$$b_{m-1,n} w_{n} = f(v_{n-1}) - b_{n-1,n-1} w_{n-1} = \prod_{j=0}^{n-1} f(v_j), \quad \text{where} \quad \Pi_j = \prod_{i=j+1}^{n} -b_{ii}. \quad (83)$$

Hence, the coefficients of the solution $G^k$ to the tensorized problem (81) satisfy

$$b_{m-1,n} b_{n-1,n} G_{mn} = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \Pi_j^{m-1} \Pi_j^{n-1} \ell_{ij}. \quad (84)$$

Applying this formula to $BU = \ell + \rho^2 \Delta^k U$ instead of $BG = \ell$ gives

$$b_{m-1,n} b_{n-1,n} U_{mn} = b_{m-1,n} b_{n-1,n} G_{mn} + \rho^2 \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \Pi_j^{m-1} \Pi_j^{n-1} [\Delta^k U]_{ij}. \quad (84)$$

Due to the locality (ii) of $\Delta^k$, the double sum contains only the diagonal coefficients $U_{rr}$ with $r \leq \min\{m, n\}$ and no off-diagonal ones; specifically, only the entries

$$[\Delta^k U^k]_{r-1, r-1} = U_{r-1, r-1} \Delta^r_{22} + U_{rr} \Delta^r_{11}, \quad (85a)$$

$$[\Delta^k U^k]_{r-2, r-1} = U_{r-1, r-1} \Delta^r_{12}, \quad (85b)$$

$$[\Delta^k U^k]_{r-1, r-2} = U_{r-1, r-1} \Delta^r_{21}, \quad (85c)$$

occur. In particular, if $m = n$ then the formula gives a recursion for $U_{nn}$ with $\rho^2 \Delta^n_{11} U_{nn}$ on the right-hand side. Therefore, we can solve for $U_{nn}$ if $b_{n-1,n}^2 \neq \rho^2 \Delta^n_{11}$ (which is equivalent to $\beta_n$ being finite). The formula then provides the remaining off-diagonal coefficients $U_{mn}$. With this, the existence of the discrete solution is established.

To obtain the representation (82), we subtract from formula (84) for $U_{nn}$ that for $U_{n-1,n-1}$. After some manipulation, this leads to the iteration

$$U_{nn} = \beta_n G_{nn} - \beta_n \theta^2_n G_{n-1,n-1} + \alpha_n U_{n-1,n-1}, \quad 2 \leq n \leq N,$$

and by induction to the claim (82).

Equation (82) is the discrete version of the identity in (45), which was used to prove (see Theorem 2.11) that an SPSD right-hand side $\ell$ entails the same property for the solution $U$. The following lemma characterizes the conditions on the discretization parameters for which this is true in the discrete.

**Lemma 3.6.** The following are equivalent:

(i) $\beta_n > 0$ in (77) for all $n$;

(ii) For every SPSD $\ell \in F'_e$ the discrete variational problem (74) has a unique solution $U^k \in E^k_n$, and it is SPSD.

Proof. Assume (i). Let $\ell \in F'_e$ be SPSD. Then $G^k \in E^k_n$ defined in (81) is also SPSD by Lemma 3.4. As remarked above, its matrix of coefficients is therefore also SPSD, in particular $G_{nn} \geq 0$. From this and (82), it follows that also $U_{nn} \geq 0$. Indeed, with (i) $\beta_n > 0$, we obtain the equivalence

$$\beta_n^{-1} \alpha_{n+1} \geq \beta_n^{-1} \theta^2_{n+1} \iff (-b_{n-1,n}^{-1} b_{nn}, 1) \Delta^n (-b_{n-1,n}^{-1} b_{nn}, 1)^T \geq 0. \quad (86)$$
Since the matrices $\Delta^n$ are positive semi-definite, $\beta_n \alpha_{n+1} \geq \beta_{n+1} \theta_{m+1}^2$ holds and, thus, $\alpha_{n+1} \geq 0$ and $U_{nn} \geq 0$ for all $n$. Set now $\hat{U}^k := \sum_{n=1}^N U_{nn}(e_n \otimes e_n)$. Since the discrete inf-sup condition (52) is assumed, there exists a unique $U^k \in E_{n}^k$ satisfying $B(U^k, \nu) = \hat{\ell}(\nu)$ for all $\nu \in F_{\ell}^k$, where $\hat{\ell} := \rho^2 \Delta^k \hat{U}^k + \ell$. By Assumption (i) on $\Delta^k$, the functional $\hat{\ell}$ is SPSD on $F^k \otimes F^k$. By Lemma 3.4, $U^k$ is also SPSD. Moreover, the identity (83) applied to the right-hand side $\hat{\ell}$ yields $b_{n-1,n}^2 U_{nn} = \sum_{i,j=1}^n \Pi_i^{n-1} \Pi_j^{n-1} [\rho^2 \Delta^k \hat{U}^k + \ell]_{ij} = b_{n-1,n}^2 \hat{U}_{nn}$, where the last equality follows from the definition of the coefficients $\hat{U}_{nn} = U_{nn}$ and the locality properties (85). Consequently, $\Delta^k \hat{U}^k = \Delta^k U^k$ on $F^k$, and $U^k$ is the desired solution.

Conversely, assume (ii). For any $g_1, \ldots, g_N \geq 0$, the function $G^k := \sum_{n} g_n (e_n \otimes e_n) \in E_{n}^k \subset E_{n}$ is SPSD. By Lemma 2.2, the functional $\ell := BG^k \in F_{\ell}^k$ inherits this property and, moreover, by assumption also the solution $U^k$ to (74) is positive semi-definite. In particular, $U_{nn} \geq 0$. Fix $n \in \{1, \ldots, N\}$ and choose $g_n = 1$ and $g_m = 0$ for all $m \neq n$. With this choice, the nonnegativity of $U_{nn}$ along with its representation in (82) imply that $\beta_n \geq 0$. Since $\beta_n$ is a fraction (77), we conclude that (i) $\beta_1, \ldots, \beta_N$ are positive.

3.6.3. Discrete stability and inf-sup. The representation of $U_{nn}$ in (82) in combination with the Lemmas 2.6 and 3.6 allow for an explicit representation of the $E_{n}$-norm of the discrete solution:

**Corollary 3.7.** Suppose $\beta_n > 0$ in (77) for all $n$. Let $\ell \in F_{\ell}^k$ be SPSD. Then the discrete variational problem (74) admits a unique solution $U^k \in E_{n}^k$. It is SPSD with norm

$$||U^k||_n = \sum_{n=1}^N \left( \beta_n G_{nn} + \sum_{m=1}^{n-1} G_{mn}(\beta_m \alpha_{m+1} - \beta_{m+1} \theta_{m+1}^2) \prod_{v=m+2}^{n} \alpha_v \right) ||e_n||_{E_{n}}^2. \quad (87)$$

**Proof.** Lemmas 2.6, 3.5 and 3.6 give $||U^k||_n = \lambda \hat{\ell}(U^k) = \sum_{n=1}^N U_{nn} ||e_n||_{E_{n}}^2$. Inserting the expression (82) for $U_{nn}$ yields (87).

From Corollary 3.7, the norm of the discrete solution $U^k$ can be estimated in terms of the norm of the right-hand side $\ell$. We shall do this under the additional assumption of a uniform temporal mesh. For convenience of notation, we rescale the basis $\{e_n\}$ to $||e_n||_{E_{n}} = 1$, so that in view of (80), the numbers $(\alpha, \beta, \theta) := (\alpha_n, \beta_n, \theta_n)$ do not depend on $n$ (cf. Table 1).

**Theorem 3.8.** In addition to the conditions posed in Corollary 3.7, assume that the temporal mesh is uniform. Then the discrete solution $U^k$ to (74) satisfies the stability bound

$$||U^k||_n \leq C_k ||\ell||_{-\epsilon} \quad \text{with} \quad C_k := \gamma_k^2 \beta (1 + (\alpha - \theta^2) \frac{\alpha^{N-1}}{\alpha - 1}), \quad (88)$$

where $\gamma_k$ is the discrete inf-sup constant from (52). If $\alpha = 1$ then $C_k = \gamma_k^2 \beta (\theta^2 + N(1 - \theta^2))$.

**Proof.** Corollary 3.7 yields

$$||U^k||_n = \beta \sum_{n=1}^N G_{nn} + \beta (\alpha - \theta^2) \sum_{m=1}^{N-1} G_{mn} \sum_{n=0}^{N-m-1} \alpha^n, \quad (89)$$

where we have changed the order of summation.

It follows from the observations in (86) that $\alpha \geq \theta^2 \geq 0$. Thus, if $\alpha \neq 1$ we have $\frac{1 - \alpha^{N-n}}{1 - \alpha} \leq \frac{1 - \alpha^{N-1}}{1 - \alpha}$ and evaluating the geometric sum in (89) yields

$$||U^k||_n = \beta \sum_{n=1}^N (1 + (\alpha - \theta^2) \frac{1 - \alpha^{N-n}}{1 - \alpha}) G_{nn} \leq \beta (1 + (\alpha - \theta^2) \frac{1 - \alpha^{N-1}}{1 - \alpha}) ||G^k||_n \leq C_k ||\ell||_{-\epsilon}.$$

For $\alpha = 1$, the claim follows directly from (89).

As a consequence of the the stability bound in the previous theorem we obtain an inf-sup condition for $\mathcal{A}^k = B - \rho^2 \Delta^k$. It is convenient to formulate it on the subspaces $E_{n}^{k} \subset E_{n}$ and $F_{\ell}^{k} \subset F_{\ell}$ of symmetric functions.
Suppose the temporal mesh is uniform with $\beta > 0$. Then $\mathcal{B}_k$ in (74) satisfies the discrete inf-sup condition (note the symmetrization)

$$\inf_{w\in S(F^k)} \sup_{v\in S(F^k)} \mathcal{B}_k(w, v) \geq C_k^{-1}, \quad (90)$$

where $C_k$ is the discrete stability constant in (88).

**Proof.** Fix a symmetric $w \in \hat{E}_k$. On $\hat{E}_k$ define the functional $\ell := \mathcal{B}_k w$, extending it via Hahn–Banach with equal norm to $F_k$. Decompose it as $\ell =: \ell^+ - \ell^- + \ell^s$ as in (37). Then $\ell^s = 0$ by symmetry of $w$. Let $w^+ \in \hat{E}_k$ be the solution to (74) with the right-hand side $\ell^+$. Clearly, $w = w^+ - w^-$. Therefore,

$$\|w\|_{\pi} \leq \|w^+\|_{\pi} + \|w^-\|_{\pi} \leq C_k(\|\ell^+\|_\pi + \|\ell^-\|_\pi) \overset{(37)}{=} C_k \|\ell\|_\pi.$$

Since $w \in \hat{E}_k$ was arbitrary and $\|\ell\|_\pi = \sup_{v\in S(F^k)} \mathcal{B}_k(w, v)$, the conclusion (90) follows. \hfill $\square$

Now we introduce some approximations $\Delta^k$ of the trace product $\Delta$. This is of interest primarily for the iE$^*$ discretization. The schemes we consider are

- CN$^*$: The CN$^*$ discretization discussed in §3.2.1 with the exact trace product $\Delta^k := \Delta$.
- iE$^*$: The iE$^*$ discretization introduced in §3.2.2 with the exact trace product $\Delta^k := \Delta$.
- iE$^*_Q$: iE$^*$ with pre-processing: $\Delta^k := \Delta \circ (q_k \otimes q_k)$ with $q_k$ from (65).
- iE$^*_K$: iE$^*$ with the “box rule”

$$\Delta^k(w, v) := \sum_{n=1}^{N} \left( \int_{J_n \times J_n} w(s, t) v(s, t) \, ds \, dt \right), \quad (w, v) \in E_k \times F_k. \quad (91)$$

This definition is motivated by observing that $\Delta(w, v)$ is the double integral of $\delta(s - t)w(s, t)v(s, t)$ over all “boxes” $J_n \times J_n$ and approximating $\delta(s - t)$ by $k_n^{-1}$ on $J_n \times J_n$.

All these candidates for the approximate trace product $\Delta^k$ satisfy the assumptions (i)–(iii) made above. In particular, they are bilinear and continuous, as quantified in the following lemma.

**Lemma 3.10.** Each of the above $\Delta^k$ is bounded on $E_k \times F_k$ with

$$\Delta^k(w, v) \leq \frac{1}{2\pi} \|w\|_{\pi} \|v\|_{\pi}, \quad \forall (w, v) \in E_k \times F_k.$$

**Proof.** Boundedness of the exact trace product is the subject of Lemma 2.8. For the approximation with pre-processing $\Delta^k := \Delta \circ (q_k \otimes q_k)$ we have the same bound, because $\|q_k : E \rightarrow E^k\| = 1$ and therefore $\|q_k \otimes q_k : E_k \rightarrow E^k\| = 1$.

Now consider the “box rule” $\Delta^k$ as in (91). Let $(w, v) \in E_k \times F_k$. By [36, Thm. 2.4] we may assume that $w = w^1 \otimes w^2$. Employing $\|v(s, t)\| \leq \frac{1}{2} \|v\|_{\pi}$ from (34) in (91) results in the estimate

$$\Delta^k(w, v) \leq \frac{1}{2} \|w\|_{\pi} \sum_n \|w_1\|_{L^2(J_n)} \|w_2\|_{L^2(J_n)} \leq \frac{1}{2\pi} \|w\|_{\pi} \|w_1\|_{\pi} \|w_2\|_{\pi}. \quad \square$$

The values of $\Delta^k, \alpha$ and $\beta$ for each scheme are given in Table 1 below in terms of the time-step size $k > 0$ (assumed uniform) and the dimensionless numbers $z := \lambda k$ and $q := \rho^2/(2\lambda)$. Recall that the basis $\{e_n\}_n \subset E^k$ is normalized to $\|e_n\|_E = 1$ to define these values. The denominator of $\beta_n = \lambda k_n b_n^{-1, n} / D_n$ is $D_n = \lambda k_n (b_n^{-1, n} - \rho^2 \Delta^k_{11})$. Thus, $D_n > 0$ necessary and sufficient for $\beta_n > 0$ in Lemma 3.6. On a uniform mesh we write $D := D_n$. We remark that $D > 0$ holds for all our schemes if the temporal mesh width $k$ is sufficiently small, namely when $k \rho^2 \lesssim 1$.

With Theorem 3.8 we find that $\lim_{k \rightarrow 0} C_k = C$ for the schemes CN$^2$, iE$^*_Q$, and iE$^*_K$ (but not for iE$^*_2$), where $C$ is the stability constant in (48) of the continuous problem (42).

3.6.4. Error analysis and convergence. In this subsection we estimate the difference between the exact solution $U$ to (42) and the discrete solution $U^k$ to (74). We first remark that by Lemma 3.10, the norm of $\mathcal{B}_k = B - \rho^2 \Delta^k$ is bounded by

$$\|\mathcal{B}_k\| \leq 1 + \frac{\rho^2}{2\pi}.$$
for each $\Delta^k \in \{\Delta, \Delta \circ (q_k \otimes q_k), (q_1)^t\}$. Moreover, $\mathcal{B}^k$ satisfies the inf-sup condition (90) on $\tilde{E}^k \times \tilde{F}^k \subset E_{\pi} \times F_{\pi}$, and the dimensions of these subspaces coincide. Hence, Proposition 3.3 on quasi-optimality of the discrete solution applies. This quasi-optimality is formulated in terms of the symmetric subspace $\tilde{E}^k$, but we can improve this to $E^k_{\pi}$ for symmetric solutions $U$. Indeed, if $U \in E_{\pi}$ is symmetric then $\|U - \frac{1}{2}(w + w')\|_{\pi} \leq \frac{1}{2}(\|U - w\|_{\pi} + \|(U - w')^\ast\|_{\pi}) = \|U - w\|_{\pi}$ for any $w \in E_{\pi}$, where $(\cdot)^\ast(s, t) := (\cdot)(t, s)$. Furthermore, the appearing residual $(\mathcal{B} - \mathcal{B}^k)U = (\Delta - \Delta^k)U$ is a symmetric functional, whether $U$ is symmetric or not, and therefore vanishes on anti-symmetric elements of $F_{\pi}$. This leads to the estimate

$$\|U - U^k\|_{\pi} \leq (1 + C_k\|\mathcal{B}^k\|) \inf_{w \in E^k_{\pi}} \|U - w\|_{\pi} + C_k\|\Delta - \Delta^k\|U\|_{(F^2, \gamma)}$$

for symmetric $\ell$. Replacing $C_k$ by $\gamma^{-1}_k + C_k$, the assumption of symmetry may be dropped.

This result shows convergence for the CN$_2^*$ scheme, where $\Delta^k = \Delta$. Unfortunately, it is not useful for the iE$_2^*$ scheme and its variants, because the best approximation from the discrete space $E^k_{\pi}$ does not converge to $U$ as we refine the temporal mesh, see the discussion at the end of §3.2.2. This motivates looking at the postprocessed solution

$$\bar{U}^k := Q_kU^k \quad \text{with} \quad Q_k := (q_k \otimes q_k) \quad (92)$$

for these schemes, where $q_k$ is the projection from (65). Recall that $q_k$ is injective on $E^k$. By $Q_k^{-1}$ we will mean the inverse of $Q_k : E^k_{\pi} \to Q_kE^k_{\pi}$. In the case of the iE$_2^*$ discretization, (66) implies

$$\|Q_kw\|_{\pi} = \frac{1}{4}\|w\|_{\pi} \quad \forall w \in E^k_{\pi}. \quad (93)$$

The convergence of the postprocessed solution will again be obtained via Proposition 3.3. To this end, we define $\bar{\mathcal{B}} := \mathcal{B} \circ Q_k^{-1}Q_k : E_{\pi} \to F_{\pi}'$ with the motivation that the postprocessed solution solves the modified discrete problem

Find $\bar{U}^k \in Q_kE^k_{\pi}$ s.t. $\bar{\mathcal{B}}(\bar{U}^k, v) = \ell(v) \quad \forall v \in F_{\pi}'. \quad (94)$

The operator $\bar{\mathcal{B}}^k$ is bounded with $\|\bar{\mathcal{B}}^k\| \leq 4\|\mathcal{B}^k\|$. Moreover, it follows from (93) that if $\mathcal{B}^k$ satisfies the discrete inf-sup condition (90) on $E^k_{\pi} \times F^k_{\pi}$ with the constant $C_k^{-1}$ then so does $\bar{\mathcal{B}}^k$ on $Q_kE^k_{\pi} \times \tilde{F}^k_{\pi}$ with the constant $4C_k^{-1}$. The following is our main result of this section.

**Proposition 3.11.** Let $\ell \in F_{\pi}'$ be symmetric. Assume the discrete inf-sup condition (90). Then the exact solution $U \in E_{\pi}$ to (42) and the postprocessed discrete solution $\bar{U}^k \in Q_kE^k_{\pi}$ to (94) differ by

$$\|U - \bar{U}^k\|_{\pi} \leq (1 + C_k\|\mathcal{B}^k\|) \inf_{w \in Q_kE^k_{\pi}} \|U - w\|_{\pi},$$

for the CN$_2^*$ scheme, and by

$$\|U - \bar{U}^k\|_{\pi} \leq (1 + C_k\|\mathcal{B}^k\|) \inf_{w \in Q_kE^k_{\pi}} \|U - w\|_{\pi} + \frac{1}{4} C_k\|\mathcal{B} - \mathcal{B}^k\|U\|_{(F^2, \gamma)} \quad (95)$$

for any of the iE$_2^*$ schemes.
To complete the analysis we need to estimate the residual term in (95). Hence, from now on we focus entirely on the \( iE^*_2 \) schemes. Recalling that \( \mathcal{B} = B - \rho^2 \Delta \) and \( \mathcal{B}^k = (B - \rho^2 \Delta^k)Q_k^{-1}Q_k \) we split the residual according to

\[
\mathcal{B} - \mathcal{B}^k = \mathcal{B}(\text{Id} - Q_k) - B(\text{Id} - Q_k)Q_k^{-1}Q_k - \rho^2(\Delta Q_k - \Delta^k)Q_k^{-1}Q_k \tag{96}
\]

and address it term by term.

- The first term \( T_1 := \|\mathcal{B}(\text{Id} - Q_k)U\|_{(F_{\pi}^r)} \) in (95)/(96) goes to zero upon mesh refinement by density of the subspaces \( Q_k E_n^\pi \subset E_n^\pi \).

- To bound the second term \( T_2 := \|B(\text{Id} - Q_k)Q_k^{-1}Q_k U\|_{(F_{\pi}^r)} \) in (95)/(96) we proceed in two steps. First, we observe that \( b((\text{Id} - Q_k)w, v) = (\text{Id} - Q_k)w, v)_E = ((\text{Id} - Q_k)w, (\text{Id} - Q_k)v)_E \leq \|w\|_E\|(\text{Id} - Q_k)v\|_E \) for any \((w, v) \in E \times F\). The Poincaré–Wirtinger inequality on each temporal element yields \(\|(\text{Id} - Q_k)v\|_E \leq \frac{1}{\sqrt{12}} \lambda \max k_n \|v\|_F \) for all \( v \in F^k \). Second, we write

\[
\text{Id} - Q_k = \frac{1}{2}[(\text{Id} - q_k) \otimes (\text{Id} + q_k) + (\text{Id} + q_k) \otimes (\text{Id} - q_k)],
\]

and use this identity in \( B(\text{Id} - Q_k) \). Recalling \(\|Q_k^{-1}Q_k U\|_{\pi} = 4\|Q_k U\|_{\pi} \leq 4\|U\|_{\pi} \) from (93), this gives \( T_2 \leq \frac{4}{\sqrt{3}} \lambda \max k_n \|U\|_{\pi} \).

- Consider now the third term \( T_3 := \rho^2 \|\Delta Q_k - \Delta^k\|Q_k^{-1}Q_k U\|_{(F_{\pi}^r)} \) in (95)/(96). For the \( iE^*_2 \) scheme where \( \Delta^k = \Delta \), this term does not converge to zero upon mesh refinement, see §3.6.5. For the \( iE^*_2/Q \) scheme where \( \Delta^k = \Delta Q_k \), this term vanishes identically.

It remains to discuss the “box rule” where \( \Delta^k = (91) \). To this end, we first note that for \( v \in F^k \)

\[
(\Delta Q_k - \Delta^k)(Q_k^{-1}Q_k U, v) = \Delta(Q_k U, (\text{Id} - I_k)v),
\]

where \( I_k := i_k \otimes i_k \) with the interpolation operator \( i_k \) onto the space of piecewise constants from (61). To estimate the last expression, we expand \( \text{Id} - I_k \) as in (97). Recalling from [35, §3.2] that \( C^0(J \times J) = C^0(J) \otimes \epsilon C^0(J) \), the estimates \( \|\psi - i_k \psi\|_{C^0(J)} \leq \sqrt{\lambda \max k_n} \|\psi\|_F \) and \( \|\psi + i_k \psi\|_{C^0(J)} \leq \sqrt{2} \|\psi\|_F \) for \( \psi \in F^k \), then imply

\[
\| (98) \| \leq \delta (\|Q_k U\|)(\|\text{Id} - I_k\|v)_{C^0(J \times J)} \leq \sqrt{2} \max k_n \sqrt{\frac{1}{2}} \|U\|_\pi \|v\|_e.
\]

### 3.6.5. Non-convergence of \( iE^*_2 \) with postprocessing.

We introduced the approximate trace product (73) because even with postprocessing, the \( iE^*_2 \) scheme with the exact trace product does not converge upon temporal mesh refinement. In fact, it is consistent with the value \( 2 \rho \) for the volatility instead of \( \rho \), as we will indicate here. First, as in (66), we have \( \Delta(w, Q_k v) = \Delta(4Q_k w, Q_k v) \) for all \((w, v) \in E_{\pi}^r \times F^k \). Therefore, invoking \(\delta((w - 4Q_k w)\|_{\pi} \leq \frac{1}{\lambda} \|w - 4Q_k w\|_{\pi} \) and the identity (93),

\[
|\Delta(w, v) - 4\Delta(Q_k w, v) | = |\Delta(w - 4Q_k w, v - Q_k v) | \leq \frac{2}{\lambda} \|w\|_{\pi} \|v\|_{C^0(J \times J)}.
\]

To bound the last term, we use the estimates \( \|\psi - q_k \psi\|_{C^0(J)} \leq \frac{1}{2} \sqrt{\lambda \max k_n} \|\psi\|_F \) and \( \|\psi + q_k \psi\|_{C^0(J)} \leq \sqrt{2} \|\psi\|_F \) for \( \psi \in F^k \), which yield \(\|v - Q_k v\|_{C^0(J \times J)} \leq \sqrt{2} \max k_n \|v\|_e \). By the preceding subsection, the \( iE^*_2/Q \) scheme with \( \Delta Q_k \) does provide a consistent approximation, so (99) shows that \( iE^*_2 \) does not.

### 3.7. Numerical example.

In the following numerical experiment we implement the schemes \( CN^*_2, iE^*_2, iE^*_2/Q, \) and \( iE^*_2/\boxplus \) proposed in §3.6 to solve the discrete variational problem (74). In addition, we apply the discretizations of polynomial degree \( p = 2 \) from §3.2.3 with the exact trace product \( \Delta \), denoted by \( CN^*_2(2) \) and \( iE^*_2(2) \). We choose \( T = 2, \lambda = 3, \rho^2 = \lambda/2, \) and for the right-hand side \( \ell(v) := v(0) \), motivated by (39). The error against the exact solution from (8c) is measured as the \( L_1 \) error on the diagonal, \( E(U^\text{num}) := \delta(|U - U^\text{num}|) \) for \( \text{num} = U^k \) (without postprocessing) and \( \text{num} = U^k \) (with postprocessing). Note that only the inequality \( \delta(|w|) \leq \frac{1}{\lambda} \|w\|_e \) holds (with equality when \( w \) is SPSD). Nevertheless, we use this measure for simplicity and for easier comparison with Monte Carlo below. The results are shown in Figure 2.
Convergence of the schemes is summarized in the following table (along with the number of conjugate gradients iterations as discussed below). The convergence, where present, is of first order in the temporal mesh width.

|       | $\mathbf{CN}_1^*$ | $\mathbf{CN}_2^*(2)$ | $\mathbf{iE}_2^*$ | $\mathbf{iE}_2^*(2)$ | $\mathbf{iE}_2^*/Q$ | $\mathbf{iE}_2^*/\Box$ |
|-------|-------------------|----------------------|-------------------|----------------------|-------------------|-------------------|
| $U^k$ | ✓                 | ✓                    | ✗                 | ✓                    | ✓                 | ✓                 |
| $U^k$ | ✓                 | ✓                    | ✗                 | ✓                    | ✓                 | ✗                 |
| $n_{CG}$ | 50               | 71                   | 294               | 113                  | 50               | 49               |

These results are in line with the convergence results established in §3.6. The schemes of polynomial degree $p = 2$ exhibit only first order convergence, presumably due to the limited smoothness of the second moment across the diagonal. However, they do not require pre- or postprocessing for convergence. The stability of the $\mathbf{iE}_2^*(2)$ scheme, in particular, does not depend on the temporal mesh width as long as it is equidistant, see (64), but our analysis does not cover this statement beyond the trivial range (72).

The discrete variational problem (74), with the choice of bases described at the beginning of §3.6, leads to the linear algebraic problem $\mathbf{B Vec(U)} = \mathbf{F}$. Here, $\mathbf{Vec}$ stacks the columns of the matrix $\mathbf{U}$ into one long vector. Let $\mathbf{M} := \mathbf{m} \otimes \mathbf{m}$ and $\mathbf{N} := \mathbf{n} \otimes \mathbf{n}$, where $\mathbf{m}/\mathbf{n}$ are the mass matrices for $E/F$. We symmetrize the problem as $\mathbf{B}^T \mathbf{N}^{-1} \mathbf{B Vec(U)} = \mathbf{B}^T \mathbf{N}^{-1} \mathbf{F}$ and solve this with the conjugate gradients method preconditioned with $\mathbf{M}$. The matrix-vector products are implemented in a matrix-free fashion with linear complexity in the size of $\mathbf{U}$, which is of order $k^{-2}$. The symmetrization is motivated by operator preconditioning that was shown to be effective for space-time discretizations of parabolic evolution equations [3], but the correct adaptation to the present setting of Banach spaces that are not strictly convex is an open issue. We use the MATLAB pcg solver with tolerance $10^{-10}$, resulting in a number of iterations $n_{CG}$ that increases with increasing temporal resolution. Thus the computational effort is of order $n_{CG} k^{-2}$. The number of iterations $n_{CG}$ for $k = 2^{-9} T$ is shown in Table (100).

Another possibility to solve the discrete problem is indicated by Lemma 3.5, where first only the diagonal of the discrete second moment is determined from the data. More fundamentally, one could directly target numerically the ordinary differential / integral equation satisfied by the diagonal of the continuous second moment, see the proof of Theorem 2.11.

We comment briefly on the error of the Monte Carlo empirical estimate of the second moment. Let $X_1, \ldots, X_R$ be i.i.d. copies of the solution process $X$. The empirical estimate of the second moment $M$ in $s, t \in J$ with $R$ samples is the random variable $M_R(s, t) := \frac{1}{R} \sum_{r=1}^R X(r, s) X(r, t)$. Then we have $\mathbb{E}[(M(s, t) - M_R(s, t))^2] = \text{Var}(M_R(s, t)) = \frac{1}{R} \text{Var}(X(s) X(t))$. Setting $s = t$ and integrating over $J$ leads to the strong error estimate $\mathbb{E}[(M - M_R)^2] \leq \frac{T}{R} \int_J \text{Var}(X(t)^2) \, dt$. We expect a similar estimate to hold for the $\mathbb{E}[(M - M_R)^2]$ norm. Balancing the Monte Carlo error $1/\sqrt{R}$ with the
temporal discretization error $k$ requires $R \sim k^{-2}$ samples; since adding one summand to $M_R$ is on the order of $k^{-2}$ operations, this leads to an overall effort of $O(k^{-4})$. The effort could be reduced with parallelization and other techniques mentioned in the introduction.

4. Stochastic PDEs with affine multiplicative noise

In this section we generalize the preceding discussion of scalar stochastic ODEs to vector-valued stochastic PDEs

$$dX(t) + AX(t)\,dt = G[X(t)]\,dL(t), \quad t \in \bar{J}, \quad \text{with } X(0) = X_0. \quad (101)$$

Here, $A: \mathcal{D}(A) \subset H \to H$ is a self-adjoint, positive definite operator, densely defined on a real Hilbert space $H$, with a compact inverse $A^{-1} : H \to H$. Furthermore, $L := (L(t), t \geq 0)$ is a square-integrable zero-mean Lévy process taking values in a Hilbert space $\mathcal{U}$ with a self-adjoint positive semi-definite trace-class covariance operator $Q$, i.e., $\mathbb{E}[(L(s), x)_{\mathcal{U}}(L(t), y)_{\mathcal{U}}] = \langle s \wedge t \rangle Q(x, y)_{\mathcal{U}}$ for all $s, t \geq 0$ and $x, y \in \mathcal{U}$. For each $\varphi \in H$, $G[\varphi] : \mathcal{U} \to H$ is a bounded linear operator and $G$ is affine: $G[\varphi] = G_1[\varphi] + G_2$, for certain $G_1 \in \mathcal{L}(H; \mathcal{L}(\mathcal{U}; H))$ and $G_2 \in \mathcal{L}(\mathcal{U}; H)$. Further technical assumptions on $G$, $L$ and $X_0$ are those of [23, §2].

We define the space $V \subset H$ with the norm $\| \cdot \|_{V} := \|A^{1/2} \cdot \|_H$. Identifying $H$ with its dual $H'$, we obtain the Gelfand triple

$$V \hookrightarrow H \cong H' \hookrightarrow V'$$

with continuous and dense embeddings, and the $H$ inner product has a unique extension by continuity to the duality pairing on $V' \times V$, denoted by $\langle \cdot, \cdot \rangle$. Moreover, akin to (32), we find

$$V_\pi \hookrightarrow H_\pi \hookrightarrow H_2 \cong H'_2 \hookrightarrow V'_2 \hookrightarrow (V')_e, \quad (102)$$

and the $H_2$ inner product extends continuously to the duality pairing $\langle \cdot, \cdot \rangle_{\pi, e}$ on $V_\pi \times (V')_e$. The functional framework for the deterministic PDE of the second moment is based on the Bochner spaces

$$\mathcal{X} := L_2(J; V) \quad \text{and} \quad \mathcal{Y} := \{ v \in H^1(J; V') \cap L_2(J; V) : v(T) = 0 \} \quad (103)$$

which are equipped with the norms $\| w \|_{\mathcal{X}}^2 := \int_J \| w(t) \|_{V'}^2 \, dt$ and $\| v \|_{\mathcal{Y}}^2 := \| - \partial_t v + Av \|_{L_2(J; V')}^2 = \int_J \| \partial_t v(t) \|_{V'}^2 \, dt + \int_J \| v(t) \|_{V}^2 \, dt + \| v(0) \|_{V}^2$ and the obvious corresponding inner products. The norm on $\mathcal{Y}$ is equivalent to the one used in [23]. Analogously to (13), these norms render the operator $b : \mathcal{X} \to \mathcal{Y}'$, stemming from the bilinear form

$$b : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}, \quad b(w, v) := \int_J \langle w(t), -\partial_t v(t) + Av(t) \rangle \, dt, \quad (104)$$

an isometric isomorphism. Consequently, on the tensor product spaces $\mathcal{X}_\pi := \mathcal{X} \otimes_\pi \mathcal{X}$ and $\mathcal{Y}_\pi := (\mathcal{Y} \otimes_\epsilon \mathcal{Y}') \cong \mathcal{Y}' \otimes_\pi \mathcal{Y}'$, the (properly extended) operator

$$B := (b \otimes b) : \mathcal{X}_\pi \to \mathcal{Y}_\pi' \quad \text{is an isometric isomorphism}. \quad (105)$$

As in (38), the multiplicative noise in (101) causes an additional term acting on the temporal diagonals of elements in $\mathcal{X}_\pi$ and $\mathcal{Y}_\pi$ in the bilinear form $\mathcal{B} : \mathcal{X}_\pi \times \mathcal{Y}_\pi \to \mathbb{R}$ for the variational formulation of the second moment equation. The continuity of this diagonal term is a consequence of the following two properties of the tensor spaces $\mathcal{X}_\pi$ and $\mathcal{Y}_\pi$: a) the boundedness of point evaluation functionals on $\mathcal{Y}$ and $\mathcal{Y}_\pi$ addressed in Lemma 4.1, and b) the role of the diagonal for elements in $\mathcal{X}_\pi$ emphasized in Lemma 4.2. Being simple extensions of Lemmas 2.1 and 2.6, respectively, the proofs are omitted.

**Lemma 4.1.** Let $\tilde{v} \in \mathcal{Y}$, $v \in \mathcal{Y}_e$. Then

$$\| \tilde{v}(t) \|_H \leq \frac{1}{\sqrt{2}} \| \tilde{v} \|_{\mathcal{Y}} \quad \text{and} \quad \| v(s, t) \|_{H_\pi} \leq \frac{1}{\sqrt{2}} \| v \|_{\mathcal{Y}_\pi} \quad \forall s, t \in \bar{J}.$$  

In particular, the temporal diagonal $\tilde{v} : t \mapsto v(t, t)$ belongs to $L_1(J; V_\pi)$ and satisfies $\| \tilde{v} \|_{L_1(J; V_\pi)} \leq \| w \|_{\mathcal{X}_\pi}$. If $w \in \mathcal{X}_\pi$ is $\mathcal{X}$-SPSD (2) then equality holds.

**Lemma 4.2.** If $w \in \mathcal{X}_\pi$, then its temporal diagonal $\tilde{w} : t \mapsto w(t, t)$ belongs to $L_1(J; V_\pi)$ and satisfies $\| \tilde{w} \|_{L_1(J; V_\pi)} \leq \| w \|_{\mathcal{X}_\pi}$. If $w \in \mathcal{X}_\pi$ is $\mathcal{X}$-SPSD (2) then equality holds.
Together with (102), the two lemmas imply that the vector analogue of the trace product (26),
\[
\Delta(w, v) := \int J \langle w(t, t), v(t, t) \rangle_{\pi, \ell} \, dt. \tag{106}
\]
is well-defined on $\mathcal{X}_\pi \times \mathcal{Y}_\epsilon$.

The covariance of the Lévy process will enter the deterministic PDE for the second moment through the linear operator $G_1: V_\pi \to V_\pi$ defined by
\[
(G_1[\psi \otimes \tilde{\psi}], \varphi \otimes \tilde{\varphi})_{H_2} = (Q^{1/2} G_1[\psi'] \varphi, Q^{1/2} G_1[\tilde{\psi}'] \tilde{\varphi}) \quad \forall \varphi, \tilde{\varphi} \in H, \tag{107}
\]
where $G_1[\psi'] : H \to \mathcal{Y}$ is the adjoint of $G_1[\psi]$. In the scalar case, $G_1 = \rho^2$. This operator is well-defined under suitable boundedness assumptions on $G_1$. For example, if $\psi \mapsto G_1[\psi]Q^{1/2}$ is a bounded map from $V$ into the Hilbert–Schmidt operator space $\mathcal{L}_2(\mathcal{Y}; V)$, then
\[
C_G := \|G_1[\cdot]\|_\mathcal{L}(V, \mathcal{Y}) = \|G_1[\cdot]Q^{1/2}\|_\mathcal{L}(V, \mathcal{L}(\mathcal{Y}; V)). \tag{108}
\]
Henceforth, we assume that $C_G$ is indeed finite.

We write $\Delta G_1 : \mathcal{X}_\pi \to \mathcal{Y}_\epsilon'$ for the operator corresponding to $\Delta(G_1[\cdot, \cdot])$. This composition is also well-defined because the temporal diagonal of $G_1 w$ belongs to $L_1(J; V_\pi)$ if $w \in \mathcal{X}_\pi$, cf. [23, §3].

Finally, we define the operator for the second moment equation in the vector-valued case,
\[
\mathcal{B} := B - \Delta G_1.
\]

Given a functional $\ell \in \mathcal{Y}_\epsilon'$, we are now interested in the variational problem
\[
\text{Find } U \in \mathcal{X}_\pi \text{ s.t. } \mathcal{B}(U, v) = \ell(v) \quad \forall v \in \mathcal{Y}_\epsilon. \tag{109}
\]

The second moment $M$ and the covariance $C$ of the solution process $X$ to the SPDE (101) satisfy the deterministic variational problem (109) with suitable right-hand sides $\ell_M$ and $\ell_C$, see [23, Thms. 4.2 & 6.1]. These are given by $\ell_M(v) := \langle [EX_0 \otimes X_0], v(0) \rangle_{\pi, \ell} + \Delta((G_1 - G_1)[\mathbb{E}X \otimes \mathbb{E}X], v)$ and $\ell_C(v) := \langle \mathbb{C}(X_0, v(0))_{\pi, \ell} + \Delta(G_1[EX \otimes EX], v) \rangle$. Here, $G_1[\cdot]$ is defined as in (107) with $G_1$ replaced by $G$. Note that $\ell_M, \ell_C \in \mathcal{Y}_\epsilon'$ are both SPD.

4.1. Well-posedness of the second moment PDE. Well-posedness of (109) was deduced in [23, Thm. 5.5] under the smallness condition $\|G_1[\cdot]Q^{1/2}\|_\mathcal{L}(V, \mathcal{L}(\mathcal{Y}; H)) < 1$. The following theorem disposes of this assumption by exploiting semigroup theory on the Banach space $V_\pi$. It is the vector analogue of Theorem 2.11. As in the scalar case, the solution $U$ of (109) inherits symmetry and definiteness from an SPD right-hand side $\ell \in \mathcal{Y}_\epsilon'$. This crucial structural property allows us to derive a stability bound in the natural tensor norm.

**Theorem 4.3.** Suppose $G_1 \in \mathcal{L}(V_\pi)$ with norm $C_G = (108)$. Then, for every functional $\ell \in \mathcal{Y}_\epsilon'$ there exists a unique solution $U \in \mathcal{X}_\pi$ to the variational problem (109). If $\ell$ is SPD then $U$ is $\mathcal{X}$-SPSD and satisfies the stability bound
\[
\|U\|_{\mathcal{X}_\pi} \leq C \|\ell\|_{\mathcal{Y}_\epsilon'} \quad \text{with} \quad C := \frac{C_G \epsilon(C_0 - 2\lambda_1)^{\gamma - 2\lambda_1}}{C_0 - 2\lambda_1}, \tag{110}
\]
where $\lambda_1 > 0$ is the smallest eigenvalue of $A$. In the limit $C_0 = 2\lambda_1$ we have $C = C_0 T + 1$.

**Proof.** Recall that $B : \mathcal{X}_\pi \to \mathcal{Y}_\epsilon'$ is an isometric isomorphism. Thus, the variational problem (109) is equivalent to the following equality in $\mathcal{X}_\pi$,
\[
U = B^{-1} \ell + B^{-1} \Delta G_1 U. \tag{111}
\]
Let $\tilde{U}$, $g$ and $f$ denote the diagonals of $U$, $B^{-1} \ell$ and $B^{-1} \Delta G_1 U$. These are functions $J \to V_\pi$. By the assumptions at the beginning §4, a $C_0$-semigroup of contractions $(S(t))_{t \geq 0}$ is generated by $-A$ on $H$ and also on $V$. Owing to $\Delta(w, v \otimes \tilde{v}) = \int J \int J \delta(s - s')(w(s, s'), v(s) \otimes \tilde{v}(s'))_{H_2} ds ds'$, for $w \in \mathcal{X}_\pi$ and $v, \tilde{v} \in \mathcal{Y}$, we can represent $B^{-1} \Delta w \in \mathcal{X}_\pi$ explicitly in terms of the semigroup by
\[
(B^{-1} \Delta w)(t, t') = \int_0^{\min\{t, t'\}} (S(t - s) \otimes S(t' - s))w(s, s) \, ds, \quad t, t' \in J. \tag{112}
\]
Set \( \mathcal{S}(r) := S(r) \otimes S(r) \). Then \((\mathcal{S}(t))_{t \geq 0}\) forms a \( C_0 \)-semigroup on \( V_r \) generated by \( \mathcal{A} := -\text{Id} \otimes \mathcal{A} - \mathcal{A} \otimes \text{Id} \). If \((-\mathcal{A})\) is the Laplacian in \( d \) dimensions, then \( \mathcal{A} \) is the \( 2d \)-Laplacian. By the perturbation theorem [15, Thm. 1.3], also \( \mathcal{A} := \mathcal{A} + \mathcal{G}_1 \) is a generator of a \( C_0 \)-semigroup \((\mathcal{S}(t))_{t \geq 0}\) on \( V_r \), and \( \|\mathcal{S}(t)\|_{\mathcal{L}(V_r)} \leq e^{(C_g-2\lambda)t} \). With these definitions, we find that \( f(s) = \int_s^t \mathcal{S}(s-r) \mathcal{G}_1 \mathcal{U}(r) \, dr \). Thus, the derivative of \( f \) satisfies \( f' = \mathcal{A} f + \mathcal{G}_1 \mathcal{U} = \mathcal{A} f + \mathcal{G}_1 g \) and \( f \in L_1(J; V_r) \) can be identified uniquely with \( f(s) = \int_s^T \mathcal{S}(s-r) \mathcal{G}_1 g(r) \, dr \). It follows that \( \mathcal{U} = g + f \) is well-defined in \( L_1(J; V_r) \). By (111)--(112) then, \( \mathcal{U} \in \mathcal{X} \) is uniquely determined via

\[
U(t, t') = (B^{-1} \ell)(t, t') + \int_0^{t' - t} (S(t-s) \otimes S(t'-s)) \mathcal{G}_1 [g(s) + f(s)] \, ds. \tag{113}
\]

Assume now that \( \ell \in \mathcal{W} \) is SPSD. Then, as in Lemma 2.2, one can show that \( B^{-1} \ell \) is \( \mathcal{X} \)-SPSD and symmetry of \( U \) is evident from the representation (113). The operators \( \mathcal{G}_1 \) and \( \mathcal{A} \) both preserve \( V \)-SPSD-ness; therefore the semigroup \( \mathcal{G} \) generated by \( \mathcal{A} + \mathcal{G}_1 \) does, too: \( \mathcal{G}(t)w \) is \( V \)-SPSD, \( t \geq 0 \), if \( w \in V_r \) is \( V \)-SPSD. Therefore, for (a.e.) \( s \in J \), we have semi-definiteness on \( V \) for quantities appearing under the integral in (113):

\[
(\mathcal{G}_1, g(s), \theta \otimes \theta)_{V_r} \geq 0, \quad (\mathcal{G}_1, f(s), \theta \otimes \theta)_{V_r} = \int_0^s (\mathcal{G}_1 \mathcal{S}(s-r) \mathcal{G}_1 g(r), \theta \otimes \theta)_{V_r} \, dr \geq 0 \quad \forall \theta \in V.
\]

Setting \( z_\varphi(s) := \int_s^T S(t-s) \varphi(t) \, dt \) with the \( V \)-adjoint \( S(t) \varphi \)' of \( S(t) \varphi \), we find for all \( \varphi \in \mathcal{X} \)

\[
(U, \varphi \otimes \varphi)_{\mathcal{X}^2} = (B^{-1} \ell, \varphi \otimes \varphi)_{\mathcal{X}^2} + \int_0^T \mathcal{G}_1 [g(s) + f(s), \varphi(s) \otimes \varphi(s)]_{V_r} \, ds \geq 0.
\]

This proves that \( U \) is \( \mathcal{X} \)-SPSD. By Lemma 4.2 above, we have \( \|U\|_{\mathcal{X}_n} = \|\tilde{U}\|_{L_1(J; V_n)} \) and with \( \tilde{U} = g + f \) we conclude that

\[
\|U\|_{\mathcal{X}_n} \leq \|\ell\|_{\mathcal{W}_d} + \int_0^T \|\mathcal{S}(t-s) \mathcal{G}_1 g(s)\|_{V_n} \, ds \, dt \leq \|\ell\|_{\mathcal{W}_d} + C_g \int_0^T e^{(C_g-2\lambda)(t-s)} \|g(s)\|_{V_n} \, ds,
\]

where we have used (108) and the bound \( \|\mathcal{S}(t)\|_{\mathcal{L}(V_n)} \leq e^{(C_g-2\lambda)t} \). In this way, the stability estimate (110) follows as in the scalar case (47). \( \square \)

### 4.2. Second moment discretization.

In order to introduce conforming discretizations of the second moment equation in the vector case (109), let \( (V^h)_{h>0} \) be a family of finite-dimensional subspaces of \( V \), whose members carry the same norm as on \( V \). In addition, let \( E^k \times F^k \subset E \times F \) be a discretization pair as considered in §3 with basis functions \( \{e_n\} \subset E^k \) and \( \{v_i\} \subset F^k \). The family \( \{v_i\} \) is normalized in \( L_2(J) \). As before, \( N := \dim E^k = \dim F^k \) is the dimension of the temporal discretization. If not specified otherwise, the range of the indices is

\[
0 \leq i, j \leq N - 1, \quad 1 \leq m, n \leq N, \quad 1 \leq p, q, r, s \leq \dim V^h. \tag{114}
\]

By choosing \( \mathcal{X}^{k,h} := E^k \otimes V^h \) and \( \mathcal{Y}^{k,h} := F^k \otimes V^h \) we obtain finite-dimensional subspaces of the Bochner spaces (103). The discrete spaces \( \mathcal{X}^{k,h} := \mathcal{X}^{k,h} \otimes \mathcal{X}^{k,h} \) and \( \mathcal{Y}^{k,h} := \mathcal{Y}^{k,h} \otimes \mathcal{Y}^{k,h} \) then form a conforming discretization pair of the trial and test spaces in (109). As in §§3.4--3.6, the subscript indicates the norm.

The discrete operator \( A^h \) on \( V^h \) is defined by \( (A^h \varphi^h, \psi^h)_H = (A \varphi^h, \psi^h)_H \) for \( \varphi^h, \psi^h \in V^h \). Its eigenvalues and the corresponding \( H \)-orthonormal eigenvectors are denoted by \( \{\lambda_p^h\} \) and \( \{\varphi_p^h\}^h \), respectively. We define the bilinear form \( b_p \) as in (16), replacing \( \lambda \) by \( \lambda_p^h \).

Let the discretization pair \( E^k \times F^k \) satisfy

\[
Y_{k,p} := \inf_{w \in \mathcal{X}^{k,h}} \sup_{v \in \mathcal{Y}^{k,h}} b_p(w, v) > 0, \quad 1 \leq p \leq \dim V^h.
\]

Then the inf-sup constant of

- \( b \) from (104) on \( \mathcal{X}^{k,h} \times \mathcal{Y}^{k,h} \) equals \( \min_p Y_{k,p} > 0 \). \( \tag{115} \)

where

\[
\Delta \text{ following definition of the approximate trace product in the vector-valued case,}
\]

\[
w_{pq} := (w, \varphi_p^h \otimes \varphi_q^h)_{H_2} \quad \text{and} \quad v_{pq} := (v, \varphi_p^h \otimes \varphi_q^h)_{H_2}, \quad 1 \leq p, q \leq \dim V^h,
\]

can be identified with elements in \( E_x \) and \( F_x \), respectively. Furthermore, if we let \( P_h \) denote the \( H \)-orthonormal projection onto \( V^h \), we obtain \( (P_h \otimes P_h)w = \sum_{pq} w_{pq}(\varphi_p^h \otimes \varphi_q^h) \in \mathcal{X}_\pi \) and, similarly, for \( (P_h \otimes P_h)v \in \mathcal{Y}_e \). We can then approximate the vector trace product as follows,

\[
\Delta(w, v) \approx \Delta((P_h \otimes P_h)w, (P_h \otimes P_h)v) = \sum_{pq} \Delta(w_{pq}, v_{pq}) \approx \sum_{pq} \Delta^k(w_{pq}, v_{pq}),
\]

where \( \Delta^k : E_x \times F_x \to \mathbb{R} \) is the scalar approximate trace product from (73). This motivates the following definition of the approximate trace product in the vector-valued case,

\[
\Delta^{k,h} : \mathcal{X}_\pi \times \mathcal{Y}_e \to \mathbb{R}, \quad \Delta^{k,h}(w, v) := \sum_{pq} \Delta^k(w_{pq}, v_{pq}),
\]

with \( w_{pq} \in E_x \) and \( v_{pq} \in F_x \) from (117). We note that the identities \( \Delta^{k,h} = \Delta \) and \( \Delta^{k,h} \mathcal{G}_1 = \Delta \mathcal{G}_1 \) hold on the discrete subspaces \( \mathcal{X}_\pi^{k,h} \times \mathcal{Y}_e^{k,h} \) if \( \Delta^k := \Delta \) is the exact scalar trace product. We furthermore point out that the definition of \( \Delta^{k,h} \) in (118) depends on the subspace \( V^h \subset V \), but it is independent of the choice of the \( H \)-orthonormal basis \( \{\varphi_p^h\}_p \subset V^h \).

Setting

\[
\mathcal{B}^{k,h} := B - \Delta^{k,h} \mathcal{G}_1,
\]

we introduce the discrete variational problem

\[
\text{Find} \quad U^{k,h} \in \mathcal{X}_\pi^{k,h} \quad \text{s.t.} \quad \mathcal{B}^{k,h}(U^{k,h}, v) = \ell(v) \quad \forall v \in \mathcal{Y}_e^{k,h}.
\]

In the following, we suppose that the temporal mesh is uniform. Then \( \Delta^a \) in (76) does not depend on \( n \) and, for all \( n \),

\[
b_p(e_n, v_{n-1}) = b_{p0} := b_p(e_1, v_0) \quad \text{and} \quad b_p(e_n, v_n) = b_{p1} := b_p(e_1, v_1).
\]

Furthermore, \( b_{p0} \neq 0 \) for all \( p \) by (115). Under these assumptions, we derive existence, uniqueness and stability of a solution \( U^{k,h} \) to the discrete problem (120) for any SPSD functional \( \ell \). As in §3.6, the formula is formulated using the following constants:

\[
\beta := (1 - \hat{\beta})^{-1}, \quad \hat{\beta} := \|P_n G[\cdot]Q^{1/2}\|^2_{\mathcal{L}(V^h; \mathcal{L}(V; V^h))} \Delta_{11} \max_p b_{p0}^{-2} \quad \text{and} \quad \theta_p^+ := \max_p |\theta_p|, \quad \theta_p^- := \min_p |\theta_p|, \quad \theta_p := \theta_p^+ \theta_p^-,
\]

\[
\alpha := \beta \max_p \left\{ \theta_p^2 + b_{p0}^{-2} \right\} \Delta_{22} - 2\Delta_{12} \theta_p \|P_n G[\cdot]Q^{1/2}\|^2_{\mathcal{L}(V^h; \mathcal{L}(V; V^h))} \right\}.
\]

These quantities should be compared to (77)–(79) from the scalar case. The following result is the analogue of Theorem 3.8.

**Theorem 4.4.** Suppose the temporal mesh is uniform and that \( \beta > 0 \). Then, if \( \ell \in \mathcal{Y}_e^{k,h} \) is SPSD, the discrete variational problem (120) has a unique solution \( U^{k,h} \in \mathcal{X}_\pi^{k,h} \), it is \( \mathcal{X} \)-SPSD, and

\[
\|U^{k,h}\|_{\mathcal{X}_\pi} \leq C_{k,h}\|\ell\|_{\mathcal{Y}_e}, \quad \text{where} \quad C_{k,h} := \max_p \gamma_{k,p}^{-2} \beta \left( 1 + (\alpha - \theta_p^2) a^{n-1} / a^{-1} \right).
\]

Let \( \mathbb{M}_k^h \) denote the set of SPSD matrices of size \( \dim V^h \times \dim V^h \). Define the matrix-valued operator

\[
(\mathcal{T}(\mathbf{W}))_{pq} := \Delta_{11} b_{p0}^{-1} b_{q0}^{-1} \sum_{rs} W_{rs}(\mathcal{G}_1[\varphi_r^h \otimes \varphi_s^h], \varphi_p^h \otimes \varphi_q^h)_{H_2} = \Delta_{11} b_{p0}^{-1} b_{q0}^{-1} (\mathcal{G}_1 W, \varphi_p^h \otimes \varphi_q^h)_{H_2},
\]

(123)
where the second equality holds whenever
\[ W = (W_{rs})_{rs} \in \mathbb{R}^{\dim V \times \dim V} \quad \text{and} \quad W := \sum_{rs} W_{rs} \varphi_r^h \otimes \varphi_s^h \in V^h \otimes V^h. \] (124)

The following lemma is the key ingredient for the proof of Theorem 4.4. For \( \Lambda := \text{diag}(\lambda_p^h) \), we introduce the weighted trace \( \text{tr}_\Lambda(W) := \text{tr}(\Lambda^{1/2}W\Lambda^{1/2}) \). Note that \( \text{tr}_\Lambda(W) = \|W\|_{V^h} \) for \( W \in \mathbb{M}_+^h \).

**Lemma 4.5.** Recall \( \tilde{\beta} \) and \( \beta \) from (121). For \( W \in \mathbb{M}_+^h \) the following hold:

(i) \( \mathbb{M}_+^h \) is invariant under \( \mathcal{T} \), i.e., \( \mathcal{T}W \in \mathbb{M}_+^h \).

(ii) \( \text{tr}_\Lambda(\mathcal{T}W) \leq \tilde{\beta} \text{tr}_\Lambda(W) \).

(iii) If \( \tilde{\beta} < 1 \) then \((\text{Id} - \mathcal{T})^{-1}W \) exists in \( \mathbb{M}_+^h \) and \( \text{tr}_\Lambda((\text{Id} - \mathcal{T})^{-1}W) \leq \beta \text{tr}_\Lambda(W) \).

**Proof.** With (124), we have \( W \in \mathbb{M}_+^h \) if and only if \( W \in V^h \otimes V^h \) is \( V \)-SPSD. Since \( \mathcal{G}_1 \) preserves this property, see (107), and \( \Delta_{11} > 0 \), the claim (i) follows. For (ii), let \( W = \sum_q s_q \psi_q \otimes \psi_q \) be an expansion with \( s_q \geq 0 \) and \( V \)-orthonormal \( \psi_q \in V^h \) for which \( \text{tr}_\Lambda(W) = \|W\|_{V^h} = \sum_q s_q \). The assertion (ii) follows:

\[ \text{tr}_\Lambda(\mathcal{T}W) = \sum_p (\mathcal{T}W)_{pp}^h \lambda_p^h \leq \Delta_{11} \max_p b_{p0}^2 \max_q \|P_h G_1[\psi_q]Q^{1/2}\|_{\mathcal{L}(V^h; V^h)}^2 \text{tr}_\Lambda(W) \leq \tilde{\beta} \text{tr}_\Lambda(W), \]

since, letting \( P'_h \) denote the \( H \)-adjoint of the projection \( P_h \), we find

\[ \sum_p (\mathcal{G}_1[\psi_q \otimes \psi_q], \varphi_p^h \otimes \varphi_p^h) = \sum_p \|Q^{1/2}G_1[\psi_q]P'_h \varphi_p^h\|_h^2 \lambda_p^h = \|P_h G_1[\psi_q]Q^{1/2}\|_{\mathcal{L}(V^h; V^h)}^2 \]

and \( \psi_q \) has unit \( V \)-norm. Finally, if \( \tilde{\beta} < 1 \) then the Neumann series \( \sum_{n \geq 0} \mathcal{T}^nW \) consists of terms in \( \mathbb{M}_+^h \) and converges, which gives (iii). \( \square \)

**Proof of Theorem 4.4.** Consider the expansion of \( U^{k,h} \) in terms of the basis \( \{e_m \otimes \varphi_p^h\}_{m,p} \in \mathcal{X}^{k,h} \)

\[ U^{k,h} = \sum_{mn,pq} U_{mn,pq} (e_m \otimes \varphi_p^h) \otimes (e_n \otimes \varphi_q^h) \in \mathcal{X}^{k,h} \otimes \mathcal{X}^{k,h}. \]

Similarly, let \( G_{nn,pq} \) denote the corresponding coefficients of the solution \( c^{k,h} \) to the problem

Find \( G^{k,h} \in \mathcal{X}^{k,h} \) s.t. \( B(G^{k,h}, v) = \ell(v) \quad \forall v \in \mathcal{X}^{k,h}. \)

Define the matrices \( U_n := (U_{nn,pq})_{pq} \) and \( G_n := (G_{nn,pq})_{pq} \). Since \( \ell \) is \( V \)-SPSD, we have \( G_n \in \mathbb{M}_+^h \). By testing (120) with \( v_{ij,pq} := (v_i \otimes \varphi_j^h) \otimes (v_j \otimes \varphi_p^h) \) for fixed \( p, q \), we find as in (84) that

\[ b_{p0}b_{q0}U_{nn,pq} = b_{p0}b_{q0}G_{nn,pq} + \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \det(\Pi_p^n) \det(\Pi_q^n) \Delta^{n} \Theta \Theta U_{ij,pq}, \]

where \( \det(\Pi_p^n) := (-b_{p0}/b_{p0})^{-n} \). After rearranging, this gives \( (U_n - \mathcal{T}U_n)_{pq} = (G_1)_{pq} \) when \( n = 1 \), and, for \( n \geq 2 \),

\[ (U_n - \mathcal{T}U_n)_{pq} = \theta_p \theta_q (U_{n-1} - G_{n-1})_{pq} + (G_{n})_{pq} + \Delta_{11}^{-1} (\Delta_{22} - \Delta_{12} \theta_p - \Delta_{21} \theta_q) (\mathcal{T}U_{n-1})_{pq}, \]

where \( \mathcal{T} \) is the operator from (123). In terms of \( \Theta := (-\theta_p, 1)^T \) and \( \Delta \) we thus have

\[ (U_n - \mathcal{T}U_n)_{pq} = \theta_p \theta_q (U_{n-1} - \mathcal{T}U_{n-1} - G_{n-1})_{pq} + (G_{n})_{pq} + \Delta_{11}^{-1} (\theta_p^T \Delta \theta_q) (\mathcal{T}U_{n-1})_{pq}. \]

We define the diagonal matrices \( \Theta = \text{diag}(\theta_p) \) and \( D_q := \Delta_{11}^{-1/2} \text{diag}(L_1 \theta_p - L_2 \eta)^T \), for \( \eta = 1, 2 \) and a Cholesky factor \( L \) of \( \Delta = LL^T \), and can then express (126) in matrix form as

\[ (\text{Id} - \mathcal{T})U_n = \Theta(\text{Id} - \mathcal{T})U_{n-1} + G_n - \Theta G_{n-1} \Theta + \sum_{\eta=1}^{2} D_q \mathcal{T}U_{n-1} D_q, \quad n \geq 2. \] (127)
For $n = 1$, we have $(\text{Id} - \mathcal{T})U_1 = G_1$. By induction, it follows from (127) that

$$(\text{Id} - \mathcal{T})U_n = G_n + \sum_{n=1}^{N} \sum_{n=1}^{N} \Theta^{n-1} D_\eta \mathcal{T} U_n D_\theta \Theta^{n-1}. \quad (128)$$

Since $\beta > 0$ by assumption, $(\text{Id} - \mathcal{T})$ is invertible on $M_h^2$ by Lemma 4.5, so that (128) defines $U_1, \ldots, U_N$ in $M_h^2$. Let $\hat{U}^{k,h} := \sum_{n,p} U_{n,p}(e_n \otimes \varphi_p^h) \otimes (e_n \otimes \varphi_q^h)$ and $\ell := \Delta^{k,h} \mathcal{T} \hat{U}^{k,h} + \ell$. The fact that $\mathcal{T}$ preserves $H$-SPSD-ness and $U_1, \ldots, U_N \in M_h^2$ imply that $\ell$ is SPSD on $B^{k,h}$. Owing to (116), there exists a unique $\mathcal{X}^{k,h}$ with $B(U^{k,h}, v) = \ell(v)$ for all $v \in B^{k,h}$. As in the scalar case, we conclude from the construction of $\hat{U}^{k,h}$ and $U^{k,h}$ that $U_{n,p} = \hat{U}_{n,p}$ so that $U^{k,h}$ is the unique solution to (120).

The $\mathcal{X}$-SPSD-ness of $U^{k,h}$ and the $L_2(J; H)$-orthonormality of $(e_m \otimes \varphi^h_p)_{m,p} \subset \mathcal{X}^{k,h}$ imply that

$$\|U^{k,h}\|_{\mathcal{X}} = \sum_{n,p} \lambda^h_{\mathcal{X}}(U_{n,p}) = \sum_n \lambda_h(A_n). \quad (129)$$

A similar equality holds also for $G^{k,h}$. For $n \geq 2$, we estimate with Lemma 4.5 (i)–(ii)

$$0 \leq \beta^{-1} \text{tr}_A(U_n) = (1 - \beta) \text{tr}_A(U_n) - \text{tr}_A(U_{n-1}) = \text{tr}_A((\text{Id} - \mathcal{T}U_{n-1})U_n), \quad (130)$$

and use the identity (125) as well as $\Delta_{12} \text{tr}_A(\Theta \mathcal{T} U_{n-1}) = \Delta_{21} \text{tr}_A(\mathcal{T} U_{n-1} \Theta)$ to derive the bound

$$\text{tr}_A((\text{Id} - \mathcal{T})U_{n}) = \text{tr}_A(\Theta \mathcal{T} U_{n-1} \Theta + \Delta_{12} - 2 \Delta_{12} \Theta) \mathcal{T} U_{n-1} + G_n - \Theta G_{n-1} \Theta) \leq \beta^{-1} \alpha \text{tr}_A(U_{n-1}) + \text{tr}_A(G_n) - \theta^2 \text{tr}_A(G_{n-1}). \quad (131)$$

Furthermore, by Lemma 4.5 (iii) above we find $\text{tr}_A(U_1) = \text{tr}_A((\text{Id} - \mathcal{T})^{-1}G_1) \leq \beta \text{tr}_A(G_1)$. By combining this with (130)–(131) we obtain by induction, for all $n$, $n \geq 1$,

$$\text{tr}_A(U_n) \leq \beta \text{tr}_A(G_n) + \beta(\alpha - \theta^2) \sum_{n=1}^{N} \alpha^{n-1} \text{tr}_A(G_n).$$

Inserting this estimate in (129) and changing the order of summation in the second term gives

$$\|U^{k,h}\|_{\mathcal{X}} \leq \beta \|G^{k,h}\|_{\mathcal{X}} + \beta(\alpha - \theta^2) \sum_{n=1}^{N} \text{tr}_A(G_n) \sum_{n=1}^{N} \alpha^{n-1} \leq \beta(1 + (\alpha - \theta^2) \sum_{n=0}^{N-1} \alpha^{n-1}) \|G^{k,h}\|_{\mathcal{X}}.$$ 

The application of the discrete stability estimate $\|G^{k,h}\|_{\mathcal{X}} \leq C \|\nu\|_{\mathcal{X}'}$ from (116) completes the proof of the stability bound (122).

As for the scalar case, the discrete stability estimate (122) implies an inf-sup condition for $\mathcal{B}^{k,h}$ in (119) on the subspaces $\mathcal{X}^{k,h} \subset \mathcal{X}^{k,h}$ and $\mathcal{B}^{k,h} \subset \mathcal{B}^{k,h}$ of symmetric elements. Subsequently, Proposition 3.3 is applicable, which gives a quasi-optimality estimate for the CN scheme. These observations are summarized in the following theorem.

**Theorem 4.6.** Suppose that the temporal mesh is uniform with $\beta > 0$, and let $\mathcal{B}_1 \in \mathcal{L}(V_\pi)$ with norm $C_G = (108)$. Then $\mathcal{B}^{k,h}$ in (119) satisfies the discrete inf-sup condition

$$\inf_{w \in \mathcal{B}^{k,h}} \sup_{v \in \mathcal{B}^{k,h}} \mathcal{B}^{k,h}(w, v) \geq C_{k,h}^{-1}, \quad (132)$$

where $C_{k,h}$ is the discrete stability constant in (122). If $\ell \in \mathcal{B}^{k,h}$ is symmetric, the error between the exact solution $U \in \mathcal{X}^{k,h}$ to (109) and the discrete solution $U^{k,h} \in \mathcal{X}^{k,h}$ to (120) for the CN scheme admits the bound

$$\|U - U^{k,h}\|_{\mathcal{X}} \leq (1 + C_{k,h}) \|\mathcal{B}^{k,h}\| \inf_{w \in \mathcal{B}^{k,h}} \|U - w\|_{\mathcal{X}}, \quad (133)$$

where $\|\mathcal{B}^{k,h}\| \leq 1 + \frac{C_G}{2}$ is the operator norm of $\mathcal{B}^{k,h} : \mathcal{X}^{\pi} \to \mathcal{B}^{k,h}$ induced by (119).
Proof. Since the inf-sup estimate (132) follows by exactly the same arguments as in the scalar case, Corollary 3.9, we focus on the derivation of the quasi-optimality estimate (133). By Proposition 3.3 we have
\[ \| U - U_h \|_{X_h} \leq (1 + C_h) \| \mathcal{B}^{k,h} \| \inf_{w \in \mathcal{B}^{k,h}} \| U - w \|_{X_h} + C_h \| (\Delta - \Delta^{k,h}) \mathcal{G}_h U \|_{(\mathcal{B}^{k,h})^*}. \]
For the exact scalar trace product \( \Delta^k := \Delta \), the definition of \( \Delta^{k,h} \) in (118) gives
\[ \Delta^{k,h}(w, \nu) = \Delta(w, (P_h \otimes P_h)\nu) \quad \forall (w, \nu) \in \mathcal{X}_h \times \mathcal{Y}_h. \]
This shows that the residual term \( \| (\Delta - \Delta^{k,h}) \mathcal{G}_h U \|_{(\mathcal{B}^{k,h})^*} \) vanishes for the CN scheme. Furthermore, \( \mathcal{B}^{k,h} \) is continuous on \( \mathcal{X}_h \times \mathcal{Y}_h \) with \( \| \mathcal{B}^{k,h} \| \leq 1 + C_h \), and it is symmetric, taking the symbolization \( w^{\ast} = w \ast \nu \). The quasi-optimality estimate (133) for the exact scalar trace product \( \Delta^k := \Delta \), the definition of \( \Delta^{k,h} \) in (118) gives
\[ \Delta^{k,h}(w, \nu) = \Delta(w, (P_h \otimes P_h)\nu) \quad \forall (w, \nu) \in \mathcal{X}_h \times \mathcal{Y}_h. \]
This shows that the residual term \( \| (\Delta - \Delta^{k,h}) \mathcal{G}_h U \|_{(\mathcal{B}^{k,h})^*} \) vanishes for the CN scheme. Furthermore, \( \mathcal{B}^{k,h} \) is continuous on \( \mathcal{X}_h \times \mathcal{Y}_h \) with \( \| \mathcal{B}^{k,h} \| \leq 1 + C_h \), and it is symmetric, taking the symbolization \( w^{\ast} = w \ast \nu \). The quasi-optimality estimate (133) for the exact scalar trace product \( \Delta^k := \Delta \), the definition of \( \Delta^{k,h} \) in (118) gives
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This shows that the residual term \( \| (\Delta - \Delta^{k,h}) \mathcal{G}_h U \|_{(\mathcal{B}^{k,h})^*} \) vanishes for the CN scheme. Furthermore, \( \mathcal{B}^{k,h} \) is continuous on \( \mathcal{X}_h \times \mathcal{Y}_h \) with \( \| \mathcal{B}^{k,h} \| \leq 1 + C_h \), and it is symmetric, taking the symbolization \( w^{\ast} = w \ast \nu \). The quasi-optimality estimate (133) for the exact scalar trace product \( \Delta^k := \Delta \), the definition of \( \Delta^{k,h} \) in (118) gives
\[ \Delta^{k,h}(w, \nu) = \Delta(w, (P_h \otimes P_h)\nu) \quad \forall (w, \nu) \in \mathcal{X}_h \times \mathcal{Y}_h. \]
This shows that the residual term \( \| (\Delta - \Delta^{k,h}) \mathcal{G}_h U \|_{(\mathcal{B}^{k,h})^*} \) vanishes for the CN scheme. Furthermore, \( \mathcal{B}^{k,h} \) is continuous on \( \mathcal{X}_h \times \mathcal{Y}_h \) with \( \| \mathcal{B}^{k,h} \| \leq 1 + C_h \), and it is symmetric, taking the symbolization \( w^{\ast} = w \ast \nu \). The quasi-optimality estimate (133) for the exact scalar trace product \( \Delta^k := \Delta \), the definition of \( \Delta^{k,h} \) in (118) gives
\[ \Delta^{k,h}(w, \nu) = \Delta(w, (P_h \otimes P_h)\nu) \quad \forall (w, \nu) \in \mathcal{X}_h \times \mathcal{Y}_h. \]
This shows that the residual term \( \| (\Delta - \Delta^{k,h}) \mathcal{G}_h U \|_{(\mathcal{B}^{k,h})^*} \) vanishes for the CN scheme. Furthermore, \( \mathcal{B}^{k,h} \) is continuous on \( \mathcal{X}_h \times \mathcal{Y}_h \) with \( \| \mathcal{B}^{k,h} \| \leq 1 + C_h \), and it is symmetric, taking the symbolization \( w^{\ast} = w \ast \nu \). The quasi-optimality estimate (133) for the exact scalar trace product \( \Delta^k := \Delta \), the definition of \( \Delta^{k,h} \) in (118) gives
\[ \Delta^{k,h}(w, \nu) = \Delta(w, (P_h \otimes P_h)\nu) \quad \forall (w, \nu) \in \mathcal{X}_h \times \mathcal{Y}_h. \]
This shows that the residual term \( \| (\Delta - \Delta^{k,h}) \mathcal{G}_h U \|_{(\mathcal{B}^{k,h})^*} \) vanishes for the CN scheme. Furthermore, \( \mathcal{B}^{k,h} \) is continuous on \( \mathcal{X}_h \times \mathcal{Y}_h \) with \( \| \mathcal{B}^{k,h} \| \leq 1 + C_h \), and it is symmetric, taking the symbolization \( w^{\ast} = w \ast \nu \). The quasi-optimality estimate (133) for the exact scalar trace product \( \Delta^k := \Delta \), the definition of \( \Delta^{k,h} \) in (118) gives
\[ \Delta^{k,h}(w, \nu) = \Delta(w, (P_h \otimes P_h)\nu) \quad \forall (w, \nu) \in \mathcal{X}_h \times \mathcal{Y}_h. \]
$X_0(x) := \sqrt{30}(x - x^2)$ from [28, §4], which is normalized to $\|X_0\|_H = 1$. We furthermore let $\mu_\nu := 32 \nu^{-5}$ be the eigenvalues of the covariance operator $Q$, i.e., $\mu_\nu = C \lambda_\nu^{-\tau}$ for $C = 32 \pi^5$, $r = 5/2$, and, thus, $Q$ is a constant multiple of the inverse fractional 1d-Laplacian. Stochastic processes with covariance operators of this type are sometimes called Riesz fields, see [43, §4].

As a reference solution $U^{ref}$ for the second moment $U = E[X \otimes X] \in \mathcal{X}_\pi$ of the solution $X$ to (101), we take the Monte Carlo estimator from $2^{12} = h_{ref}^{-2}$ sample paths generated with the Euler–Maruyama method with a constant time step size $k_{ref} = 2^{-15}$ and continuous piecewise affine basis functions on spatial grid with uniform mesh width $h_{ref} = 2^{-6}$. The sample paths of the Q-Wiener process are simulated from a truncation of the representation $\mathcal{W}(t) = \sum_\nu \mu_\nu W_\nu(t) \psi_\nu$, where $\{W_\nu\}_\nu$ is a sequence of mutually independent real-valued Wiener processes. Since the decay of the eigenvalues of $Q$ is given by $\mu_\nu \lesssim \nu^{-\eta}$ for $\eta = 5$, we truncate this series after $\kappa := (h_{ref})^{-\frac{2}{\nu+5}} = 8$ terms, see [28, Thm. 3.2].

We let $\dim V^h = 5$ and discretize the system (135) with the CN$_\pi$ scheme proposed in §3.6. To this end, we use the representation $\rho_{pq,rs} = \sum_\nu \mu_\nu \sigma_{p,r}^\nu \sigma_{q,s}^\nu$, where $\sigma_{p,r}^\nu := \langle G_1(\varphi_r) \psi_\nu, \varphi_p \rangle_H$, and use the same truncation for this series as for the Q-Wiener process in the simulation of the reference solution, i.e., we truncate after $\kappa = 8$ terms. By evaluating the integrals we find

$$\sigma_{p,r}^\nu = \int_0^1 \varphi_r(x) \psi_\nu(x) \varphi_p(x) \, dx = \begin{cases} \frac{\lambda_\nu^{-1/2} \sqrt{\pi}}{(v+p+r)(v-p+r)(v+p-r)(v+p-r)} & \text{if } (v+p+r) \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases}$$

In this way, we obtain approximations $U^k_{pq} \in E^k_\pi$ of the coefficients $U_{pq} \in E_\pi$, and an overall approximation $U^{k,h} = \sum_{pq} U^k_{pq} (\varphi_p \otimes \varphi_q) \in \mathcal{X}^{k,h}_\pi$ of the solution $U \in \mathcal{X}_\pi$ to (109).

We use the symmetrization and preconditioning from §3.7 and solve the discretized system with the conjugate gradients method by applying the MATLAB pcg solver with tolerance $10^{-10}$. For $1 \leq p \leq \dim V^h = 5$, we measure the error of $U^{num}_{pp} := U^k_{pp}$ against the coefficient $U^{ref}_{pp}$ of the reference solution as the $L_1$ error on the diagonal $E_p(U^{num}) := \delta((U^{ref}_{pp} - U^{num}_{pp}))$, as in §3.7. Finally, we approximate the total error $\|U - U^{k,h}\|_{\mathcal{X}_\pi}$ by the weighted sum $E(U^{num}) := \sum_p \lambda_p E_p(U^{num})$, motivated by Lemma 4.2 which gives $|w|_{\mathcal{X}_\pi} = ||\tilde{w}||_{L_1(\mathcal{X}_\pi)} = \sum_{\nu \in \mathbb{H}} \lambda_\nu \delta(w_\nu)$ for every $\mathcal{X}$-SPSD $w \in \mathcal{X}_\pi$. The results are presented in Figure 3, showing first order convergence with respect to the temporal discretization parameter $k$ for every coefficient $U^{num}_{pp}$ as well as for the measure of the total error, which is in accordance with Theorem 4.6.

5. CONCLUSIONS

We have considered the model stochastic ODEs (4), (5) with additive and multiplicative Wiener noise and have derived the deterministic equations in variational form satisfied by the first (19)
and second moment (24), (39) of the solution. The equations for the second moment are posed on tensor products of function spaces, which can be taken as Hilbert tensor products (20) in the additive case, whereas projective–injective tensor product spaces (28) as trial–test spaces are required in the multiplicative case. The well-posedness of these equations is evident in the additive case (24) by the isometry property of the operator (22), but the multiplicative case, analyzed in Theorem 2.11, requires more work due to the presence of the trace product (27) in the operator.

We have discussed Petrov–Galerkin discretizations of two basic kinds for the first moment: CN$^*$ (§3.2.1) and iE$^*$ (§3.2.2). The main difference is in the stability behavior documented in Figure 1, wherein CN$^*$ requires the CFL number to be small, as opposed to iE$^*$ which can be made stable (59) under mild restrictions on the temporal mesh. Higher order generalizations followed in §3.2.3. From these, tensor product Petrov–Galerkin discretizations are constructed in §3.4. We have addressed the additive case briefly in §3.5 in order to focus the multiplicative case in §3.6. We have found the condition $C_0 = (108) < \infty$ for well-posedness of the deterministic second moment equation (109) in the vector-valued case (see Theorem 4.3), which is less restrictive than the smallness assumption on the multiplicative noise term made in [23, Eq. (5.5)]. Furthermore, we have discussed stability of numerical approximations based on the tensor product Petrov–Galerkin discretizations from §3.6 in time, and standard Galerkin discretizations in space, see Theorem 4.4. From this, the quasi-optimality estimate (133) for approximations generated with the CN$^*_2$ scheme has followed. Since no postprocessing is necessary for the CN$^*_2$ discretization (see §3.6.4), for the sake of brevity, we have focussed on this method for the quasi-optimality analysis in §4.2, and for the numerical experiment in §4.3, see Figure 3. However, we point out that the definition (118) of the vector approximate trace product decouples the discretizations in space and in time. Thus, the convergence results of the (postprocessed) scalar iE$^*_2$ schemes from §3.6.4 should also readily transfer to the vector-valued situation.

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**ACKNOWLEDGMENT**

The author thanks Roman Andreev for his contributions and support during the writing of this article as well as Stig Larsson and Sonja Cox for their valuable input.