LOCAL WELL-POSEDNESS AND BLOW-UP CRITERIA OF MAGNETO-VISCOELASTIC FLOWS

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(Communicated by Fanghua Lin)

Abstract. In this paper, we investigate a hydrodynamic system that models the dynamics of incompressible magneto-viscoelastic flows. First, we prove the local well-posedness of the initial boundary value problem in the periodic domain. Then we establish a blow-up criterion in terms of the temporal integral of the maximum norm of the velocity gradient. Finally, an analog of the Beale-Kato-Majda criterion is derived.

1. Introduction. In the past years, the study of magnetoelastic materials has attracted attention of scientists not only from the point of view of mathematical modeling but also for applications. A general magneto-viscoelastic model describing magnetoelastic materials was derived in Forster [8] based on an energetic variational approach (see, e.g., [13]). The resulting system of partial differential equations consists of the incompressible Navier-Stokes equations coupled with balance equations for the deformation gradient and the magnetization field. In the general magneto-viscoelastic model, the dissipative dynamical behavior of the magnetization \( M \) satisfies a Landau-Lifshitz-Gilbert (LLG) system with convection, which is mathematically involved due to its highly nonlinear structure (see [2, 3, 8] and the references therein). In this paper, we consider the following simplified magneto-viscoelastic system:

\[
\begin{align*}
    v_t + (v \cdot \nabla)v &= -\mu \Delta v + \nabla p \\
    &= \nabla \cdot (FF^T) - \nabla \cdot (\nabla^T M \nabla M) + \nabla^T H_{ext} M, \\
    \nabla \cdot v &= 0, \\
    F_t + (v \cdot \nabla)F &= \nabla v F, \\
    M_t + (v \cdot \nabla)M &= \Delta M - \frac{1}{\alpha^2}(|M|^2 - 1)M + H_{ext},
\end{align*}
\]

in \( \Omega \times (0, T) \), where \( T > 0 \) and \( \Omega = \mathbb{T}^d \) for \( d = 2 \) or \( 3 \). In the system (1) – (4), \( v(x, t) : \Omega \times (0, T) \rightarrow \mathbb{R}^d \) is the velocity field, \( p(x, t) : \Omega \times (0, T) \rightarrow \mathbb{R} \) is the pressure, \( F : \Omega \times (0, T) \rightarrow \mathbb{R}^{d \times d} \) denotes the deformation gradient and \( M : \Omega \times (0, T) \rightarrow \mathbb{R}^3 \) describes the magnetization vector. The magneto-viscoelastic fluid is sometimes exposed to an external effective magnetic field \( H_{ext}(x, t) : \Omega \times (0, T) \rightarrow \mathbb{R}^3 \). The

2010 Mathematics Subject Classification. Primary: 35A01, 35B44; Secondary: 76A10.

Key words and phrases. Local well-posedness, blow-up criteria, magneto-viscoelastic flows, initial boundary value problem, hydrodynamic system.

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fluid viscosity $\mu$ is assumed to be a positive constant and $\alpha > 0$ stands for the parameter that controls the strength of penalization on the deviation of $|M|$ from 1.

The magneto-viscoelastic system (1) – (4) describes the motion of an incompressible fluid that responds mechanically to applied magnetic field and changes its magnetic property in response to mechanical stress. It can be viewed as a simplification of the general model derived in [8] such that the magnetization $M$ now satisfies a gradient flow type equation with convection (see [27, Appendix A] for a sketch of the derivation for system (1) – (4)). In [8], the author proved the existence of global weak solutions to the magneto-viscoelastic system (1) – (4) with an additional regularizing term $\kappa \Delta F$ ($\kappa > 0$) in the deformation equation (3). Later, Schlömerkemper and Žabensky [27] investigated the uniqueness of global weak solutions for the same problem with slightly different boundary conditions. It is worth mentioning that the artificial regularizing term $\kappa \Delta F$ plays an essential role in their analysis.

The magneto-viscoelastic system (1) – (4) can be rewritten into the following form:

\[
\begin{align*}
 v_t + (v \cdot \nabla)v - \mu \Delta v + \nabla p &= (E^T \cdot \nabla) \cdot E + \nabla \cdot E - \nabla \cdot (\nabla^T M \nabla M), \\
 \nabla \cdot v &= 0, \\
 E_t + (v \cdot \nabla)E &= \nabla v E + \nabla v, \\
 M_t + (v \cdot \nabla)M &= \Delta M - \frac{1}{\alpha^2} (|M|^2 - 1)M.
\end{align*}
\]

Here, we use the following notations

\[
(\nabla v)_{ij} = \frac{\partial v_i}{\partial x_j}, \quad (\nabla v E)_{ij} = (\nabla v)_{ik} E_{kj}, \quad (\nabla \cdot E)_i = \partial_j E_{ij},
\]

\[
(E^T \cdot \nabla) \cdot E)_i = E_{ik} \partial_j E_{kj}, \quad (\nabla \cdot (\nabla^T M \nabla M))_i = \partial_j (\partial_i M_k \partial_j M_k).
\]

The system (5) – (8) is subject to the following initial conditions:

\[ v(x, 0) = v_0(x), \quad F(x, 0) = I + E_0(x), \quad M(x, 0) = M_0(x), \quad \forall x \in \Omega = \mathbb{T}^d. \]

In this paper, we aim to establish the local well-posedness and some blow-up criteria of the magneto-viscoelastic system (1) – (4), in particular, without the regularizing term $\kappa \Delta F$ as in [8, 27]. Following the notation in [22], we define the usual strain tensor in the form of

\[ E = F - I, \]

where $I$ is the $d \times d$ identity matrix ($d = 2, 3$). Beside, in the remaining part of the paper, we always take $H_{ext} = 0$ in (1) – (4) for the sake of simplicity. Then the magneto-viscoelastic system (1) – (4) can be rewritten into the following form:
Now we are in a position to state the main results of this paper.

**Theorem 1.1.** Let $d = 2, 3$. Suppose that the initial data satisfy $v_0, E_0 \in H^2(\Omega)$, $M_0 \in H^3(\Omega)$ and the following constraints

$$
\begin{align*}
\nabla \cdot v_0 &= 0, \quad \det(I + E_0) = 1, \quad \nabla \cdot E_0 = 0, \\
\nabla_m E_{0ij} - \nabla_j E_{0im} &= E_{0lj} \nabla_k E_{0lm} - E_{0lm} \nabla_l E_{0ij}.
\end{align*}
$$

Then there exists a positive time $T$ depending on $\|v_0\|_{H^2}$, $\|E_0\|_{H^2}$ and $\|M_0\|_{H^3}$ such that the periodic initial-boundary value problem (5)−(9) has a unique classical solution $(v, E, M)$ on $[0, T]$, which satisfies

$$
\begin{align*}
v &\in C([0, T]; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega)) \cap W^{1, \infty}(0, T; L^2(\Omega)), \\
E &\in C([0, T]; H^2(\Omega)) \cap W^{1, \infty}(0, T; L^2(\Omega)), \\
M &\in C([0, T]; H^3(\Omega)) \cap L^2(0, T; H^4(\Omega)) \cap W^{1, \infty}(0, T; L^2(\Omega)).
\end{align*}
$$

**Remark 1.** (1) As indicated in [15], the first three relations in (10) are consequences of the incompressibility condition, while the last one can be understood as a consistency condition for changing of variables between the Lagrangian and Eulerian coordinates.

(2) Similar to [15, 20, 22], if the initial data satisfy $v_0, E_0 \in H^s(\Omega)$ and $M_0 \in H^{s+1}(\Omega)$, for any integer $s \geq 3$, then by an inductive argument, the local classical solution $(v, E, M)$ obtained in Theorem 1.1 satisfies

$$
\begin{align*}
\partial_t^j \nabla^2 v &\in L^\infty(0, T; H^{s-2j-|\beta|}(\Omega)) \cap L^2(0, T; H^{s-2j-|\beta|+1}(\Omega)), \\
\partial_t^j \nabla^2 E &\in L^\infty(0, T; H^{s-2j-|\beta|}(\Omega)), \\
\partial_t^j \nabla^2 \nabla M &\in L^\infty(0, T; H^{s-2j-|\beta|}(\Omega)) \cap L^2(0, T; H^{s-2j-|\beta|+1}(\Omega)),
\end{align*}
$$

for all integer $j$, and multi-index $\beta$ satisfying $2j + |\beta| \leq s$.

**Theorem 1.2.** Suppose that the assumptions of Theorem 1.1 are satisfied. Let $(v, E, M)$ be the local classical solution to problem (5)−(9) and $T^*$ be the maximal existence time. If $T^* < +\infty$, then it holds

$$
\int_0^{T^*} \|\nabla v(t)\|_{L^\infty} \, dt = +\infty.
$$

**Theorem 1.3.** Suppose that the initial data satisfy $v_0, E_0 \in H^s(\Omega)$, $M_0 \in H^{s+1}(\Omega)$ and the constraint (10), for some integer $s \geq 3$. Let $(v, E, M)$ be the local classical solution to problem (5)−(9) and $T^*$ be the maximal existence time. If $T^* < +\infty$, then it holds

$$
\int_0^{T^*} \left(\|\nabla \times v(t)\|_{L^\infty} + \sum_{k=1}^d \|\nabla \times E_k(t)\|_{L^\infty}\right) \, dt = +\infty,
$$

where $E_k = Ee_k$ stands for the $k$-th column of the matrix $E$.

The magneto-viscoelastic system (1)−(4) has a highly nonlinear coupling structure due to the interconnection of viscoelasticity with magnetism. Nevertheless, it consists of two subsystems that have been extensively studied in the literature: one is the incompressible viscoelastic system (i.e., taking $M \equiv 0$), while the other one is the simplified Ericksen-Leslie (E-L) system for incompressible nematic liquid crystal flows with Ginzburg-Landau approximation (i.e., taking $F \equiv 0$). We recall that existence of local classical solutions as well as global classical solutions
near-equilibrium of the incompressible viscoelastic system in the two-dimensional case was first proved in Lin et al. [20] (see also Lei and Zhou [16] for the same result via incompressible limit). Corresponding well-posedness results in the three dimensional case and a blow-up criterion can be found in Lei et al. [15] (see also Chen and Zhang [6]). We refer to Lin and Zhang [22] for the case of bounded domain, and to [5, 7, 10, 29, 30] for various types of blow-up criteria. On the other hand, concerning the simplified E-L system for nematic liquid crystal flow, Lin and Liu [19] first proved the existence of global weak solutions as well as local classical solutions and they also studied its long-time behavior (see Wu [28], Grasselli and Wu [9] for improved results on the uniqueness of asymptotic limit as $t \to +\infty$). Besides, some regularity criteria were derived in Liu and Cui [24] and Liu et al. [25] for the simplified E-L system in 3D (see Cavaterra et al. [4] for the generalized E-L system). For more detailed information on the mathematical analysis of these two subsystems, we refer to the recent review papers [18, 21].

The rest of this paper is organized as follows. In Section 2, we provide some lemmas that will be used in the subsequent proofs. The local well-posedness of problem (5)–(9) (i.e., Theorem 1.1) is proved in Section 3. The last Section 4 is devoted to the proof of two blow-up criteria (i.e., Theorem 1.2 and 1.3).

2. Preliminaries. We denote by $L^p(\Omega)$, $W^{m,p}(\Omega)$ the usual Lebesgue and Sobolev spaces on $\Omega$, with norms $\| \cdot \|_p$, $\| \cdot \|_{W^{m,p}}$ respectively. For $p = 2$, we simply denote $H^m(\Omega) = W^{m,2}(\Omega)$ with norm $\| \cdot \|_{H^m}$. The norm and inner product on $L^2(\Omega)$ will be denoted by $\| \cdot \|$ and $(\cdot, \cdot)$, respectively. For simplicity, we do not distinguish functional spaces when scalar-valued, vector-valued or matrix-valued functions are involved. We denote by $C$ a generic positive constant throughout this paper, which may vary at different places. Its special dependence will be indicated explicitly if necessary.

First, we recall the following important properties of the strain tensor $E$ (see [15, 20, 23]).

**Lemma 2.1.** Assume that $(v, E, M)$ is the solution of problem (5)–(9), with the initial data (9) satisfying the constraint (10). Then the following identities hold

\begin{align}
\det(I + E) &= 1, \\
\nabla \cdot E^T &= 0, \\
\nabla_m E_{ij} - \nabla_j E_{im} &= E_{lj} \nabla_l E_{im} - E_{lm} \nabla_l E_{ij},
\end{align}

for all time $t \geq 0$.

The following interpolation inequalities are consequences of the well-known Gagliard-Nirenberg inequality (see, e.g., [31]). They will be frequently used in the higher-order energy estimates.

**Lemma 2.2.** Assume that $v \in H^k(\Omega)$, $k \geq 3$, $\Omega \subset \mathbb{R}^d$ is bounded, and $0 \leq j < s \leq k$, the following interpolation inequalities hold:

\begin{align}
\| \nabla^j v \|_{L^\infty} &\leq C \| \nabla^s v \|^{\frac{d+2j}{d+2s}} \| v \|^{1 - \frac{d+2j}{d+2s}} + C \| v \|, \\
\| \nabla^j v \|_{L^4} &\leq C \| \nabla^s v \|^{\frac{d+2j}{d+4s}} \| v \|^{1 - \frac{d+2j}{d+4s}} + C \| v \|,
\end{align}

for some constant $C$ that is independent of $v$.

Besides, the following inequalities can be found in [14].
Lemma 2.3. Assume that \( f, g \in H^k(\Omega) \), \( k \geq 0 \) being an integer. Then for any multi-index \( \beta, |\beta| \leq k \), we have
\[
\| \nabla^\beta (fg) \| \leq C(\|f\|_{L^\infty} \|g\|_{H^k} + \|g\|_{L^\infty} \|f\|_{H^k}),
\]
(19)
\[
\| \nabla^\beta (fg) - f \nabla^\beta g \| \leq C(\|f\|_{H^k} \|g\|_{L^\infty} + \|\nabla f\|_{L^\infty} \|g\|_{H^{k-1}}).
\]
(20)

Finally, we derive the following basic energy law that reflects the energy dissipation property of magneto-viscoelastic flows.

Proposition 1 (Basic energy law). Let \((v, E, M)\) be a classical solution of problem (5) - (9) on \( \Omega \times [0,T] \). Then we have
\[
\frac{1}{2} \frac{d}{dt} \left[ \|v\|^2 + \|E\|^2 + \|\nabla M\|^2 + \int_\Omega \frac{1}{2\alpha^2} (M^2 - 1)^2 \, dx \right]
+ \mu \int_\Omega |\nabla v|^2 \, dx + \int_\Omega |\Delta M - \frac{1}{\alpha^2} (M^2 - 1)M|^2 \, dx = 0,
\]
(21)
for all \( t \in (0,T) \).

Proof. We multiply (5), (7) and (8) by \( v \), \( E \) and \( \Delta M - \frac{1}{\alpha^2} (|M|^2 - 1)M \) respectively, and integrate over \( \Omega \). Adding the three resultants together, we obtain that
\[
0 = (v_i, v) + (v \cdot \nabla v, v) + (\nabla p, v) - \mu(\Delta v, v) - (E_{jk} \nabla_j E_{ik}, v_i) - (\nabla_j E_{ij}, v_i)
+ (\nabla_j (\nabla_i M_i \nabla_j M_i), v_i) + (E_i, E) + (v \cdot \nabla E, E) - (v \nabla E, E) - (\nabla v, E)
+ \|\Delta M - \frac{1}{\alpha^2} (|M|^2 - 1)M\|^2 - (M_i, \Delta M - \frac{1}{\alpha^2} (|M|^2 - 1)M)
- (v \cdot \nabla M, \Delta M - \frac{1}{\alpha^2} (|M|^2 - 1)M).
\]
(22)

Applying (6) we see that
\[
(v \cdot \nabla v, v) = (v, \nabla p) = (v \cdot \nabla E, E) = 0,
\]
\[
(v \cdot \nabla M, \frac{1}{\alpha^2} (|M|^2 - 1)M) = -(\nabla_i v_i, \frac{1}{4\alpha^2} (|M|^2 - 1)^2) = 0.
\]
Besides, recalling that \( E \) satisfies (15) and (16), then using integration by parts we obtain
\[
-(\nabla_j E_{ij}, v_i) - (\nabla_j v_i, E_{ij}) = 0,
\]
\[
-(E_{jk} \nabla_j E_{ik}, v_i) - (\nabla_j v_i E_{jk}, E_{ik})
= -(\nabla_j (E_{jk} E_{ik}), v_i) - (\nabla_j v_i E_{jk}, E_{ik})
= (E_{jk} E_{ik}, \nabla_j v_i) - (\nabla_j v_i, E_{jk}, E_{ik})
= 0,
\]
\[
(\nabla_j (\nabla_i M_i \nabla_j M_i), v_i) - (v \cdot \nabla M, \Delta M)
= -(\nabla_i M_i \nabla_j M_i, \nabla_j v_i) + (\nabla_j (v_i \nabla_i M_i), \nabla_j M_i)
= -(\nabla_i M_i \nabla_j M_i, \nabla_j v_i) + (\nabla_j v_i \nabla_i M_i, \nabla_j M_i) + (v, \nabla (\frac{|\nabla M|^2}{2}))
= 0.
\]
Inserting the above identities into (22), we arrive our conclusion (21).
Remark 2. Proposition 1 implies an important feature of problem (5) – (9) such that it only has a partially dissipative structure. Finding the hidden dissipative mechanism for $E$ (or $F$) will play an essential role in the study of global well-posedness of problem (5) – (9).

3. Local Well-posedness: Proof of Theorem 1.1. As in [15, 19], one can use the Galerkin method to construct approximating solutions to the momentum equation (5), then use this approximating solution $v$ and the equations for the deformation gradient (7) and the magnetization (8) to obtain approximating solutions for $E$ and $M$. Thus, to prove the convergence of the approximating solutions, we need only a priori estimates. For the sake of simplicity, below we derive a priori estimates for smooth solutions of problem (5) – (9) in $\mathbb{T}^3$.

First estimate. Integrating (21) over $(0, t)$, we infer that
\[
\|v(t)\|^2 + \|E(t)\|^2 + \|\nabla M(t)\|^2 + \int_{\Omega} \frac{1}{2\alpha^2} (M(t)^2 - 1)^2 dx + 2\mu \int_0^t \|\nabla v\|^2 d\tau \\
+ 2 \int_0^t \int_{\Omega} |\Delta M - \frac{1}{\alpha^2}(M^2 - 1)M|^2 dx d\tau \\
= \|v_0\|^2 + \|E_0\|^2 + \|\nabla M_0\|^2 + \int_{\Omega} \frac{1}{2\alpha^2}(M_0^2 - 1)^2 dx, \forall t \geq 0.
\]

Second estimate. Taking $L^2$ inner product of the momentum equation (5) with $v_t$, using (6), (15) and integration by parts, we obtain
\[
\mu \frac{d}{dt} \|\nabla v\|^2 + \|v_t\|^2 = -(v \cdot \nabla v, v_t) - (\nabla p, v_t) + ((E^T \cdot \nabla) \cdot E, v_t) + (\nabla \cdot E, v_t) \\
- (\nabla \cdot (\nabla^T M \nabla M), v_t) \\
= (v_j v_t, \nabla_j v_t) - (E_{jl} E_{lt}, \nabla_j v_t) - (E_{ij}, \nabla_j v_t) \\
+ (\nabla_i M_l \nabla_j M_l, \nabla_j v_t) \\
\leq C \|\nabla v_t\| (\|v\|_{L^2}^2 + \|E\|_{L^4}^2 + \|E\| + \|\nabla M\|_{L^4}^2).
\]

This together with the interpolation inequality (18) and Young’s inequality implies that
\[
\mu \frac{d}{dt} \|\nabla v\|^2 + \|v_t\|^2 \\
\leq C \|\nabla v_t\| (\|v\|_{L^4}^2 |\nabla v|_{L^4}^2 + \|E\|_{L^4}^2 |\Delta E|_{L^4}^2 + \|\nabla M\|_{L^4}^2 |\nabla \Delta M|_{L^4}^2 + 1) \\
\leq \frac{\mu}{8} \|\nabla v_t\|^2 + g(\|\nabla v\|, \|\Delta E\|, \|\nabla \Delta M\|),
\]
where $g(\cdot, \cdot, \cdot)$ stands for a generic nonnegative and increasing function of its variables.

Taking time derivative of the momentum equation (5) and taking $L^2$ inner product of the resulting equation with $v_t$, using (6), (15), (17) and integration by parts, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|v_t\|^2 + \mu \|\nabla v_t\|^2 \\
= - (\partial_t (v \cdot \nabla v), v_t) - (\nabla p_t, v_t) + (\partial_t (E_{jl} \nabla_j E_{lt}), v_t) + (\partial_t \nabla_j E_{ij}, v_{lt}) \\
- (\partial_t \nabla_j (\nabla_i M_l \nabla_j M_l), v_{lt})
\]
Similarly, taking 
Substituting (27) into (28), we find
\[
\begin{align*}
L_{j}\|
\nabla \mu _{j}\| & \leq C\|
\nabla v_{i}\|\|v_{i}\|\|v_{i}\|_{L^{\infty }} + \|E_{i}\|\|E\|_{L^{\infty }} + \|E_{i}\| + \|\nabla M_{i}\|\|\nabla M\|_{L^{\infty }} \\
& \leq C\|
\nabla v_{i}\|\|v_{i}\|\|v_{i}\|_{L^{\infty }} + \|E_{i}\|\|E\|_{L^{\infty }} + \|E_{i}\| + \|\nabla M_{i}\|\|\nabla M\|_{L^{\infty }} \\
& + \|\nabla M_{i}\|\|\nabla M\|_{L^{\infty }} + 1)\\
& \leq \frac{H}{8}\|
\nabla v_{i}\|^{2} + g(\|v_{i}\|, \|\Delta v\|, \|E_{i}\|, \|\Delta E\|, \|\nabla M_{i}\|, \|\nabla M\|),
\end{align*}
\]
where in the last step of the above derivation, we used the \(L^{2}\) estimate (23).

**Third estimate.** Taking the \(L^{2}\) inner product of the momentum equation (5) with \(\Delta v\), using (17), (18) and integration by parts, we have
\[
\begin{align*}
\mu \|\Delta v\|^{2} = (v_{i}, \Delta v) + (v \cdot \nabla v, \Delta v) + (\nabla p, \Delta v) \\
- ((\nabla \cdot v) \cdot E, \Delta v) - (\nabla \cdot E, \Delta v) + (\nabla \cdot (\nabla T M\nabla M), \Delta v) \\
\leq C\|\Delta v\|\|v_{i}\| + \|v_{i}\|_{L^{\infty }}\|\nabla v\| + \|E\|_{L^{\infty }}\|\nabla E\| + \|\nabla E\| \\
+ \|\nabla M\|\|\nabla M\|_{L^{\infty }} \\
\leq C\|\Delta v\|\|v_{i}\| + (\|v_{i}\|_{L^{\infty }}\|\Delta v\|_{L^{2}}^{2} + 1)\|\nabla v\| + (\|E\|_{L^{\infty }}\|\Delta E\|_{L^{2}}^{2} + 1)(\|E\| \\
+ \|\Delta E\| + (\|\nabla M\|_{L^{2}}\|\nabla \Delta M\|_{L^{2}}^{2} + 1)(\|\nabla M\|_{L^{2}}\|\nabla \Delta M\|_{L^{2}}^{2} + 1) \\
\leq \frac{H}{2}\|\Delta v\|^{2} + g(\|v_{i}\|, \|v_{i}\|, \|\Delta v\|, \|\nabla M\|),
\end{align*}
\]
and hence
\[
\|\Delta v\|^{2} \leq g(\|v_{i}\|, \|v_{i}\|, \|\Delta v\|, \|\nabla M\|).
\]
(27)
From the transport equation (7), We can get the following estimate with the help of (17),
\[
\begin{align*}
\|E_{i}\| & \leq \|E\|_{L^{\infty }}\|\nabla v\| + \|v_{i}\|_{L^{2}}\|\nabla v\| + \|\nabla v\| \\
& \leq C\left((\|E\|_{L^{\infty }}\|\Delta E\|_{L^{2}}^{2} + 1)\|\nabla v\| + (\|v_{i}\|_{L^{2}}\|\Delta v\|_{L^{2}}^{2} + 1)(\|E\| + \|\Delta E\|) + \|\nabla v\| \right) \\
& \leq \|E_{i}\| \\
& \leq g(\|v_{i}\|, \|v_{i}\|, \|\Delta E\|, \|\nabla M\|).
\end{align*}
\]
(28)
Similarly, taking \(\nabla v\) to the magnetization equation (8), we have
\[
\nabla v_{i}M_{i} + \nabla v_{i}(v_{i}\nabla v_{i}) = \nabla v_{i}M_{i} - \frac{1}{\alpha^{2}}\nabla v_{i}[(|M|^{2} - 1)M_{i}].
\]
Using (17) and (18), we arrive at
\[
\begin{align*}
\|\nabla M_{i}\| & \leq \|\nabla v_{i}\|_{L^{2}}\|\nabla M\|_{L^{2}} + \|v_{i}\|_{L^{2}}\|\nabla v_{i}\|_{L^{2}} + \|\nabla \Delta M\|_{L^{2}} \\
& + \|\nabla M\| + \frac{1}{\alpha^{2}}(\|\nabla M\| + 3\|\nabla M\|_{L^{2}}\|\nabla M\|_{L^{2}} + 1) \\
& \leq C\left((\|v_{i}\|_{L^{2}}\|\nabla v\|_{L^{2}}^{2} + 1)(\|\nabla M\|_{L^{2}}\|\nabla \Delta M\|_{L^{2}}^{2} + 1) \\
& + (\|v_{i}\|_{L^{2}}\|\nabla v\|_{L^{2}}^{2} + 1)(\|\nabla M\|_{L^{2}}\|\nabla M\|_{L^{2}}^{2} + 1) + \|\nabla \Delta M\| \\
& + \frac{1}{\alpha^{2}}(\|\nabla M\| + 3\|\nabla M\|_{L^{2}}\|\nabla M\|_{L^{2}}^{2} + 1)(\|\nabla M\|_{L^{2}}\|\nabla \Delta M\|_{L^{2}}^{2} + 1)) \\
& \leq \|\nabla M_{i}\| \\
& \leq g(\|v_{i}\|, \|v_{i}\|, \|\Delta E\|, \|\nabla M\|).
\end{align*}
\]
So we find the following estimate with the help of (27) such that
\[ || \nabla M || \leq g(|| \nabla v ||, || v_t ||, || \Delta E ||, || \nabla \Delta M ||). \] (31)

Plugging (27), (29) and (31) into (26), one has
\[ \frac{1}{2} \frac{d}{dt} || v_t ||^2 + \frac{7\mu}{8} || \nabla v_t ||^2 \leq g(|| \nabla v ||, || v_t ||, || \Delta E ||, || \nabla \Delta M ||). \] (32)

**Fourth estimate.** Noting (24) and (32), it is clear that we need to gain the estimates for \( || \Delta E || \) and \( || \nabla \Delta M || \). Applying \( \Delta \) to the transport equation (7), multiplying \( \Delta \) to the momentum equation (5), and then take the inner product of the resulting equation with \( \nabla \Delta v \), we find that
\[ \mu || \nabla \Delta v ||^2 = (\nabla_j v_{it}, \nabla_j \Delta v_i) + (\nabla_j (v_i \nabla v_t), \nabla_j \Delta v_i) + (\nabla_j \nabla_i v_t, \nabla_j \Delta v_i) \]
\[ - (\nabla_j (\nabla_i E_{it} E_{st}), \nabla_j \Delta v_i) - (\nabla_j \nabla_i E_{it}, \nabla_j \Delta v_i) \]
\[ + (\nabla_j \nabla s (\nabla_i M_t \nabla_s M_t), \nabla_j \Delta v_i) \]
\[ \leq C || \nabla \Delta v || \left( || \nabla v_t || + || \nabla v || \| \Delta v || + || \nabla E || \| \Delta E || + || \nabla \Delta M || \| \Delta M || \right). \]

Applying (17) and (18), we obtain
\[ \frac{1}{2} \frac{d}{dt} || \Delta E ||^2 \leq C || \Delta E || \left( || v || \| \Delta v || + || \nabla \Delta v || \| \Delta E || + || \nabla \Delta v || \| \Delta M || \right). \] (33)

On the other hand, we take \( \nabla \) to the momentum equation (5), and then take the \( L^2 \) inner product of the resulting equation with \( \nabla \Delta v \), we can get
\[ \mu || \nabla \Delta v ||^2 \leq C || \nabla \Delta v || \left( || \nabla v_t || + || \nabla v || \| \Delta v || + || \nabla v || \| \Delta E || + || \nabla M || \| \Delta M || \right). \]

Applying (17) and (18), we find that
\[ \mu || \nabla \Delta v ||^2 \leq C || \nabla \Delta v || \left( || \nabla v_t || + || \nabla v || \| \Delta v || + || \nabla v || \| \Delta E || + || \nabla M || \| \Delta M || \right). \] (34)

Inserting (34) into (33), we get
\[ \frac{1}{2} \frac{d}{dt} || \Delta E ||^2 \leq \frac{\mu}{8} || \nabla v_t ||^2 + g(|| \nabla v ||, || v_t ||, || \Delta E ||, || \nabla \Delta M ||). \] (35)
Fifth estimate. Now we take triple derivatives to the magnetization equation (8), multiply $\nabla \Delta M$ both sides, integrate over $\Omega$, we get
\begin{align}
\frac{1}{2} \frac{d}{dt} \| \nabla \Delta M \|^2 + \| \nabla^2 \Delta M \|^2 &= -(\nabla \Delta (v \cdot \nabla M), \nabla \Delta M) - (\nabla \Delta (\frac{1}{\alpha} (|M|^2 - 1) M), \nabla \Delta M) \\
&= -(\nabla \Delta (v \cdot \nabla M) - v \cdot (\nabla \Delta M), \nabla \Delta M) - (\nabla \Delta (\frac{1}{\alpha} (|M|^2 - 1) M), \nabla \Delta M) \\
&\leq C \| \nabla \Delta M \| \left( \left\| \nabla \Delta v \right\| \| \nabla M \| \| \nabla \Delta M \| + \left\| \nabla \Delta M \right\| + \| M \| \| \nabla \Delta M \| + \| M \| \| \nabla \Delta M \| \right).
\end{align}

Using again the interpolation inequalities (17) and (18), we have
\begin{align}
\left\| \nabla^2 \Delta M \right\|^2 + \frac{1}{2} \frac{d}{dt} \| \nabla \Delta M \|^2 &\leq C \| \nabla \Delta M \| \left( \left\| \nabla \Delta v \right\| \left( \| \nabla M \|^{\frac{4}{3}} \| \nabla \Delta M \|^{\frac{2}{3}} + 1 \right) \\
&+ \left( \left\| \nabla \Delta v \right\|^{\frac{8}{3}} + 1 \right) \left( \| \nabla M \| \left\| \nabla \Delta M \right\|^2 + 1 \right) \\
&+ \left( \left\| \nabla \Delta v \right\|^{\frac{4}{3}} \| \nabla \Delta M \| + \left( \| M \| \| \nabla \Delta M \| + 1 \right) \| M \| \| \nabla \Delta M \| \right).
\end{align}

Noticing the orders of $\| \nabla \Delta v \|$ are all smaller than two, using the Young inequality and substituting (21), and (34) into (36), we find
\begin{align}
\left\| \nabla^2 \Delta M \right\|^2 + \frac{1}{2} \frac{d}{dt} \| \nabla \Delta M \|^2 &\leq \frac{\mu}{8} \| \nabla v_t \|^2 + g \left( \| \nabla \Delta v \|, \| \nabla v \|, \| \Delta E \|, \| \nabla \Delta M \| \right). \tag{37}
\end{align}

Sixth estimate. Combining (24), (32), (35) and (37), we finally arrive at
\begin{align}
\frac{d}{dt} (\mu \| \nabla v \|^2 + \| v_t \|^2 + \| \Delta E \|^2 + \| \nabla \Delta M \|^2) + \| \nabla v_t \|^2 + \mu \| v_t \|^2 + \| \nabla^2 \Delta M \|^2 &\leq g \left( \| \nabla \Delta v \|, \| v_t \|, \| \Delta E \|, \| \nabla \Delta M \| \right). \tag{38}
\end{align}

It follows from the momentum equation (5) that
\begin{align}
\| v_t (0, x) \| &\leq C (\| v_0 \|_{H^2}, \| E_0 \|_{H^2}, \| M_0 \|_{H^3}). \tag{39}
\end{align}

This last estimate together with (38) yields that there exist positive constants $T$, $C_0$, depending on $\| v_0 \|_{H^2}$, $\| E_0 \|_{H^2}$, $\| M_0 \|_{H^3}$ such that
\begin{align}
&\mu \| \nabla v(t) \|^2 + \| v_t(t) \|^2 + \| \Delta E(t) \|^2 + \| \nabla \Delta M(t) \|^2 \\
&+ \int_0^t \left( \| \nabla v \|^2 + \mu \| \nabla v_t \|^2 + \| \nabla^2 \Delta M \|^2 \right) \, dt \leq C_0, \quad \forall 0 \leq t \leq T.
\end{align}

Besides, we deduce from (27), (29), (31) and (34) that
\begin{align}
\| \Delta v(t) \|^2 + \| E_t(t) \|^2 + \| \nabla M_t(t) \|^2 + \int_0^t \| \nabla \Delta v \|^2 \, dt \leq C_0, \quad \forall 0 \leq t \leq T. \tag{40}
\end{align}

Keeping the above estimates in mind, by a standard argument, we can obtain the existence of local classical solution to problem (5) – (9). Uniqueness of local classical solutions can be easily derived by using the energy method and Gronwall’s lemma.

The proof of Theorem 1.1 is complete. \hfill \Box
4. Blow-up Criteria.

4.1. Proof of Theorem 1.2. Theorem 1.2 can be proved by using a contradiction argument. Assume that \([0, T^*)\) is the maximal existence interval of the local classical solution and

\[
\int_0^{T^*} \|\nabla v(t)\|_{L^\infty} \, dt := C_1 < +\infty. \tag{41}
\]

Keeping in mind that we have the global \(L^2\)-estimate (23), thanks to the basic energy law, then we proceed to get the \(H^1\)-estimate. Taking \(\nabla\) to the equations (5) and (7), multiplying with \(\nabla v\) and \(\nabla E\) respectively, and then integrating over \(\Omega\), we get

\[
\frac{1}{2} \frac{d}{dt} \|\nabla v\|^2 + \mu \|
abla v\|^2 = -\langle \nabla (v \cdot \nabla v), \nabla v \rangle - \langle \nabla_k \nabla_i p, \nabla_k v_i \rangle \\
+ \langle \nabla_k (E_{jl} \nabla_j E_{il}), \nabla_k v_i \rangle + \langle \nabla_k \nabla_i E_{il}, \nabla_k v_i \rangle \\
- \langle \nabla_k (\nabla_j (\nabla_i M_l \nabla_j M_l)), \nabla_k v_i \rangle \\
:= H_1^{(1)} + H_2^{(1)} + H_3^{(1)} + H_4^{(1)} + H_5^{(1)},
\]

\[
\frac{1}{2} \frac{d}{dt} \|\nabla E\|^2 = -\langle \nabla_k (v \cdot \nabla E), \nabla_k E \rangle + \langle \nabla_k (\nabla_i E_{ij}), \nabla_k E_{ij} \rangle \\
+ \langle \nabla_k \nabla_i v_i, \nabla_k E_{ij} \rangle \\
:= H_6^{(1)} + H_7^{(1)} + H_8^{(1)}.
\]

Applying \(\Delta\) to the magnetization equation (8), taking \(L^2\) inner product with \(\Delta M\), we obtain that

\[
\frac{1}{2} \frac{d}{dt} \|\Delta M\|^2 + \|\nabla \Delta M\|^2 \\
= -(\Delta (v \cdot \nabla M), \Delta M) - \frac{1}{\alpha^2} (\Delta M^3, \Delta M) + \frac{1}{\alpha^2} \|\Delta M\|^2 \\
:= H_9^{(1)} + H_{10}^{(1)} + H_{11}^{(1)}.
\]

Using the incompressibility of \(v\), we have

\[
H_1^{(1)} = -\langle \nabla (v_j \cdot \nabla v_i) - v_j \cdot \nabla_j \nabla v_i, \nabla v_i \rangle \\
= -\langle \nabla (v \cdot \nabla v), \nabla v \rangle \leq C \|\nabla v\|_{L^\infty} \|\nabla v\|^2,
\]

\[
H_2^{(1)} = -\langle \nabla \nabla_i p, \nabla v_i \rangle = (\nabla p, \nabla_i v_i) = 0,
\]

and

\[
H_6^{(1)} = -\langle \nabla_k (v_i \cdot \nabla_i E), \nabla_k E \rangle \\
= -\langle \nabla_k v_i \nabla_i E, \nabla_k E \rangle - \langle v_i \nabla \nabla_k E, \nabla_k E \rangle \\
= -\langle \nabla_k v_i \nabla_i E, \nabla_k E \rangle \\
\leq C \|\nabla v\|_{L^\infty} \|\nabla E\|^2.
\]

Next, using integration by parts, we get

\[
H_3^{(1)} + H_7^{(1)} = (\nabla_k (E_{jl} \nabla_j E_{il}), \nabla_k v_i) + (\nabla_k (\nabla_j v_i E_{jl}), \nabla E_{il}) \\
= (\nabla_k E_{jl} \nabla_j E_{il}, \nabla_k v_i) + (E_{jl} \nabla_k \nabla_j E_{il}, \nabla_k v_i) \\
+ (\nabla_k \nabla_j v_i E_{jl}, \nabla_k E_{il}) + (\nabla_j v_i \nabla_k E_{jl}, \nabla_k E_{il})
\]
get equality, the Cauchy–Schwarz inequality and the Sobolev imbedding theorem, we deduce that

\[ \nabla E_i - \nabla E_i, \nabla v_i - (E_i \nabla E_i, \nabla v_i) + (\nabla \nabla E_i, \nabla v_i) = (\nabla E_i \nabla v_i, \nabla v_i) + (\nabla \nabla E_i, \nabla v_i) \leq C\|\nabla v\|_{L^\infty}\|\nabla E\|^2, \]

and

\[ H_4^{(1)} + H_8^{(1)} = (\nabla_k \nabla_j E_{ij}, \nabla_k v_i) + (\nabla_k \nabla j v_i, \nabla_k E_{ij}) = -(\nabla_k E_{ij}, \nabla_j \nabla v_i) + (\nabla_k \nabla j v_i, \nabla_k E_{ij}) = 0. \]

Similarly, we obtain

\[ H_5^{(1)} + H_9^{(1)} = -(\nabla_k \nabla_j (\nabla_i M_i \nabla_j M_i), \nabla_k v_i) - (\nabla (v_i \nabla_i M_i), \nabla M_i) = (\nabla_j (\nabla_i M_i \nabla_j M_i), \Delta v_i) - (\nabla (v_i \nabla_i M_i), \Delta M_i) = (\nabla \nabla M_i \nabla_j M_i, \Delta v_i) - 2(\nabla_j v_i \nabla_j \nabla_i M_i, \Delta M_i) = -2(\nabla_j v_i \nabla_j \nabla_i M_i, \Delta M_i) \leq C\|\nabla v\|_{L^\infty}\|\Delta M\|^2, \]

as \((\nabla_j \nabla M_i \nabla_j M_i, \Delta v_i) = -(\nabla_j \nabla M_i \nabla_j M_i, \Delta v_i) = 0\). Moreover, using H"older’s inequality, the Cauchy–Schwarz inequality and the Sobolev imbedding theorem, we get

\[ H_{10}^{(1)} = -\frac{1}{\alpha^2}(\Delta(M^3), \Delta M) = \frac{1}{\alpha^2}(\nabla(M^3), \nabla \Delta M) \leq C\|\nabla \Delta M\|\|\nabla M\|_{L^6}\|M\|^2_{L^6} \leq \frac{1}{2}\|\nabla \Delta M\|^2 + C\|\nabla M\|^2_{L^6}\|M\|^2_{L^6} \leq \frac{1}{2}\|\nabla \Delta M\|^2 + C(\|\Delta M\|^2 + 1). \]

Summing up (42) – (44), we infer from the above estimates that

\[ \frac{d}{dt}(\|\nabla v\|^2 + \|\nabla E\|^2 + \|\Delta M\|^2) + \mu\|\Delta v\|^2 + \|\nabla \Delta M\|^2 \leq C(1 + \|\nabla v\|_{L^\infty})^2(\|\nabla v\|^2 + \|\nabla E\|^2 + \|\Delta M\|^2 + 1). \]  

Then by Gronwall’s inequality, we deduce that

\[ \|\nabla v(t)\|^2 + \|\nabla E(t)\|^2 + \|\Delta M(t)\|^2 + \int_0^t (\mu\|\Delta v\|^2 + \|\nabla \Delta M\|^2) d\tau \leq (\|\nabla v_0\|^2 + \|\nabla E_0\|^2 + \|\Delta M_0\|^2) \exp\left(C \int_0^t (1 + \|\nabla v\|_{L^\infty}) d\tau\right), \]  

for all \(0 \leq t \leq T^*\), and some universal constant \(C\).

We proceed to obtain \(H^2\)-estimate. Taking \(\Delta\) to (5) and (7), \(\nabla \Delta\) to (8), multiplying the resultants with \(\Delta v\), \(\Delta E\) and \(\nabla \Delta M\) respectively, and integrating over \(\Omega\), we get

\[ \frac{1}{2} \frac{d}{dt}\|\Delta v\|^2 + \mu\|\nabla \Delta v\|^2 = -(\Delta(v \cdot \nabla v), \Delta v) - (\Delta \nabla p, \Delta v) + (\Delta [(E^T \cdot \nabla) \cdot E], \Delta v) + (\Delta \nabla \cdot E, \Delta v) - (\Delta \nabla \cdot (\nabla^T M \nabla M), \Delta v) \]
Using the incompressibility of \( v \), we have

\[
H_1^{(2)} = -\langle \Delta (v_j \cdot \nabla v_i) - v_j \cdot \nabla_j \Delta v_i, \Delta v_i \rangle \leq C\|\nabla v\|_{L^\infty} \|\Delta v\|^2,
\]

and

\[
H_2^{(2)} = -\langle \Delta \nabla_i p, \Delta v_i \rangle = -\langle \Delta p, \Delta \nabla_i v_i \rangle = 0.
\]

On account of (23) and integration by parts, we deduce

\[
H_6^{(2)} = -\langle \Delta (v_i \cdot \nabla_i E), \Delta E \rangle
= -\langle \Delta \nabla_i (v_i E) - v_i \nabla_i \Delta E, \Delta E \rangle
\leq C\|\nabla v\|_{H^3} \|E\|_{L^\infty} + \|\nabla v\|_{L^\infty} \|E\|_{H^2} \|\Delta E\|
\leq \frac{\mu}{8} \|\nabla \Delta v\|^2 + C(\|\nabla v\|_{L^\infty} + \|E\|_{L^\infty}^2 + 1) \|\Delta E\|^2 + 1).
\]

\[
H_3^{(2)} + H_7^{(2)} = \langle \Delta \{ (E^T \cdot \nabla) \cdot E \}, \Delta v \rangle + \langle \Delta (\nabla v E), \Delta E \rangle
= \langle \nabla_j \nabla_j v_i, \Delta v_i \rangle + \langle \nabla_j (v_i E_{ji}), \Delta E_{ii} \rangle
\leq C(\|E\|_{L^\infty} \|E\|_{H^2} \|\nabla \Delta v\|^2 + \|\nabla v\|_{L^\infty} \|E\|_{L^\infty}^2 + \|E\|_{H^2} \|\Delta E\|)
\leq \frac{\mu}{8} \|\nabla \Delta v\|^2 + C(\|\nabla v\|_{L^\infty} + \|E\|_{L^\infty}^2 + 1) \|\Delta v\|^2 + \|\Delta E\|^2 + 1),
\]

and

\[
H_4^{(2)} + H_8^{(2)} = -\langle \Delta \nabla_j E_{ij}, \Delta v_i \rangle + \langle \Delta \nabla_j v_i, \Delta E_{ij} \rangle
= -\langle \Delta E_{ij}, \nabla_j \Delta v_i \rangle + \langle \nabla_j v_i, \Delta E_{ij} \rangle
= 0.
\]

Similarly, we obtain

\[
H_5^{(2)} = \langle \Delta (\nabla_i M_i \nabla_j M_j), \nabla_j \Delta v_i \rangle
\leq C \|\nabla \Delta v\| \|\nabla M\|_{H^2} \|\nabla M\|_{L^\infty}
\leq \frac{\mu}{8} \|\nabla \Delta v\|^2 + \|\nabla M\|_{L^\infty}^2 \|\nabla \Delta M\|^2 + 1),
\]

\[
H_9^{(2)} = -\langle \nabla_j \Delta (v_i \nabla_i M_i), \nabla_j \Delta M_j \rangle
= -\langle \nabla_j \Delta (v_i \nabla_i M_i) - v_i \nabla_i \nabla_j \Delta M, \nabla_j \Delta M_j \rangle
\leq C(\|v\|_{H^3} \|\nabla M\|_{L^\infty} + \|\nabla v\|_{L^\infty} \|\nabla M\|_{H^2} \|\nabla \Delta M\|
\leq \frac{\mu}{8} \|\nabla \Delta v\|^2 + C(\|\nabla v\|_{L^\infty} + \|\nabla M\|_{L^\infty}^2 + 1) \|\Delta v\|^2 + \|\nabla \Delta M\|^2 + 1),
\]
and
\[ H_{10}^{(2)} = \frac{1}{\alpha^2}(\nabla \Delta M^3, \nabla \Delta M) = \frac{1}{\alpha^2} (\Delta (M^3), \Delta^2 M) \]
\[ \leq C \| \Delta^2 M \| \| M \|_{H^2}^3 \leq \frac{1}{2} \| \Delta^2 M \|^2 + C \| M \|_{H^2}^6. \]

Collecting the above estimates and using the bound (46), we obtain the \( H^2 \)-estimate
\[ \frac{d}{dt}(\| \Delta v \|^2 + \| \Delta E \|^2 + \| \nabla \Delta M \|^2) + \mu \| \nabla \Delta \|^2 + \| \Delta^2 M \|^2 \]
\[ \leq C(\| \nabla v \|_{L^\infty} + \| E \|_{L^\infty} + \| \nabla M \|_{L^\infty}) (\| \Delta v \|^2 + \| \Delta E \|^2 + \| \nabla \Delta M \|^2 + 1). \]
(47)

Using the equation for \( E \)
\[ E_t + (v \cdot \nabla) E = \nabla v E + \nabla v, \]
by integration along the characteristic passing [17] one can get
\[ E(t) = \int_0^t (\nabla v E + \nabla v) d\tau + E_0, \]
and
\[ \| E(t) \|_{L^\infty} \leq \int_0^t (\| \nabla v \|_{L^\infty} \| E \|_{L^\infty} + \| \nabla v \|_{L^\infty}) d\tau + \| E_0 \|_{L^\infty}. \]

With the help of Gronwall’s lemma, we obtain
\[ \| E(t) \|_{L^\infty} \leq (\| E_0 \|_{L^\infty} + C_1) \exp(\int_0^t \| \nabla v \|_{L^\infty} d\tau). \]
(48)

On the other hand, applying the Sobolev imbedding theorem and (46), we have
\[ \int_0^t \| \nabla M \|_{L^\infty}^2 d\tau \leq C \int_0^t \| \nabla M \|_{H^2}^2 d\tau \]
\[ \leq C(\| \nabla v_0 \|^2 + \| \nabla E_0 \|^2 + \| \Delta M_0 \|^2) \exp(\int_0^t \| \nabla v \|_{L^\infty} d\tau). \]
(49)

Then going back to (47), by Gronwall’s lemma we infer that
\[ \| \Delta v(t) \|^2 + \| \Delta E(t) \|^2 + \| \nabla \Delta M(t) \|^2 + \int_0^t (\mu \| \nabla v(t) \|^2 + \| \Delta^2 M(t) \|^2) d\tau \]
\[ \leq C, \quad 0 \leq t \leq T^*, \]
(50)
where \( C \) is a constant depending on \( C_1, \| v_0 \|_{H^2}, \| E_0 \|_{H^2} \) and \( \| M_0 \|_{H^3} \). Hence, the local solution \( (v, E, M) \) can be extended beyond \( t = T^* \). This leads to a contradiction of the definition of maximal existence time \( T^* \).

The proof of Theorem 1.2 is complete.

4.2. Proof of Theorem 1.3. The proof is again based on a contradiction argument. Assume now
\[ \int_0^{T^*} (\| \nabla \times v \|_{L^\infty} + \sum_{k=1}^d \| \nabla \times E_k \|_{L^\infty}) dt < +\infty. \]
(51)
From the assumption (51), we see that for any small constant \(0 < \varepsilon \ll 1\), there exists a corresponding \(T = T(\varepsilon) \in (0, T^*)\) such that

\[
\int_T^{T^*} \left( \| \nabla \times v \|_{L^\infty} + \sum_{k=1}^d \| \nabla \times E_k \|_{L^\infty} \right) dt < \varepsilon. \tag{52}
\]

Recalling the well-known Kato’s inequality (see [1])

\[
\| \nabla v \|_{L^\infty} \leq C \left[ 1 + \| \nabla \times v \| + \| \nabla \times v \|_{L^\infty} \ln(e + \| v \|_{H^1}) \right],
\]

from the simple facts \( \| \nabla \times v \| \leq C \| \nabla \times v \|_{L^\infty} \), \( \| \nabla \times E \| \leq C \| \nabla \times E \|_{L^\infty} \) that are valid on \( T^d \) and the lower-order estimate (23), we infer that

\[
\| \nabla v \|_{L^\infty} + \| \nabla E \|_{L^\infty} \leq C_2 \left[ 1 + (\| \nabla \times v \|_{L^\infty} + \| \nabla \times E \|_{L^\infty}) \ln(C_3 + \| \nabla^3 v \| + \| \nabla^3 E \|) \right], \tag{53}
\]

where \( C_2 \) depends on \( T^d \) and \( C_3 > \varepsilon \) depends on \( \| v_0 \|, \| E_0 \|, \| M_0 \|_{H^1}, \alpha \) and \( T^d \).

For all \( t \in (T, T^*) \), we define

\[
H(t) := \sup_{T \leq \tau \leq t} \left\{ \| \nabla^3 u(\tau) \|^2 + \| \nabla^3 E(\tau) \|^2 + \| \nabla^4 M(\tau) \|^2 \right\}. \tag{54}
\]

Recalling (45), by Gronwall’s lemma and (53), we find that

\[
\sup_{T \leq \tau \leq t} \left\{ \| \nabla v(\tau) \|^2 + \| \nabla E(\tau) \|^2 + \| \Delta M(\tau) \|^2 \right\} \\
\leq (\| \nabla v(T) \|^2 + \| \nabla E(T) \|^2 + \| \Delta M(T) \|^2) \exp \left( \int_T^t (1 + \| \nabla v \|_{L^\infty}) d\tau \right) \\
\leq C(\varepsilon) \exp \left( C_2 \int_T^t (1 + (\| \nabla \times v \|_{L^\infty} + \| \nabla \times E \|_{L^\infty}) \ln(C_3 + \| \nabla^3 v \| + \| \nabla^3 E \|)) d\tau \right) \\
\leq C(\varepsilon) \exp \left( C_2 \int_T^t (\| \nabla \times v \|_{L^\infty} + \| \nabla \times E \|_{L^\infty}) d\tau \ln(C_3 + H(t)) \right) \\
\leq C(\varepsilon) \exp(C_2 \varepsilon \ln(C_3 + H(t))) \\
= C(\varepsilon)(C_3 + H(t))^C_1 \varepsilon,
\]

where \( C(\varepsilon) \) is a positive constant depending on \( \varepsilon, T^*, \) and \( \Omega \), which may change from line to line.

Next, applying \( \nabla^3 \) to the equations (5) and (7), \( \nabla^4 \) to (8), taking \( L^2 \) inner product with \( \nabla^3 v, \nabla^3 E \) and \( \nabla^2 \Delta M \) respectively, and adding the resultants together, we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla^3 v \|^2 + \| \nabla^3 E \|^2 + \| \nabla^4 M \|^2 \right) + \mu \| \nabla^4 v \|^2 + \| \nabla^5 M \|^2 \\
= - (\nabla^3 (v \cdot \nabla v), \nabla^3 v) - (\nabla^3 \nabla p, \nabla^3 v) + (\nabla^3 (E^T \cdot \nabla) \cdot E), \nabla^3 v) \\
+ (\nabla^3 \nabla \cdot E, \nabla^3 v) - (\nabla^3 \nabla \cdot (E^T \nabla M), \nabla^3 v) \tag{55} \\
+ (\nabla^3 (\nabla v E), \nabla^3 E) + (\nabla^3 \nabla v, \nabla^3 E) \tag{55} \\
- \frac{1}{\alpha^2} (\nabla^4 (M^3), \nabla^2 \Delta M) + \frac{1}{\alpha^2} \| \nabla^4 M \|^2 \\
:= \sum_{i=1}^{11} H_i^{(3)}, \quad t \in [T, T^*].
\]
We estimate the right-hand side of (55) term by term. Using the incompressibility of \( v \) and Lemma 2.3, we have

\[
H_1^{(3)} = -(\nabla^3 (v \cdot \nabla v), \nabla^3 v) \\
= (\nabla^3 (v \cdot \nabla v) - v \cdot \nabla v^3, \nabla^3 v) \\
\leq \|\nabla v\|_{L^\infty} \|v\|_{H^3}^2 \\
\leq C \|\nabla v\|_{L^\infty} (H(t) + C_3),
\]

and

\[
H_2^{(3)} = -(\nabla^3 \nabla p, \nabla^3 v) = 0.
\]

Similarly, using the property (15) and applying the estimate (20), we deduce that

\[
H_3^{(3)} + H_7^{(3)} = (\nabla^3 [(E^T \cdot \nabla) E], \nabla^3 v) + (\nabla^3 (\nabla v E), \nabla^3 E) \\
= (\nabla^3 (E_{ji} \nabla_j E_{ik}) - E_{ji} \nabla_j \nabla^3 E_{ik}, \nabla^3 v) \\
\quad + (\nabla^3 (\nabla_j v_{Ej}) - E_{jk} \nabla_j \nabla^3 v_{ik}, \nabla^3 E_{ik}) \\
\leq C \|\nabla E\|_{L^\infty} \|E\|_{H^3} \|\nabla^3 v\| \\
\quad + (\|v\|_{H^3} \|\nabla E\|_{L^\infty} + \|\nabla v\|_{L^\infty} \|E\|_{H^3}) \|\nabla^3 E\| \\
\leq C (\|\nabla v\|_{L^\infty} + \|\nabla E\|_{L^\infty}) (H(t) + 1),
\]

\[
H_4^{(3)} + H_8^{(3)} = (\nabla^3 \nabla \cdot E, \nabla^3 v) + (\nabla^3 \nabla v, \nabla^3 E) = 0,
\]

and

\[
H_6^{(3)} = -(\nabla^3 (v \cdot \nabla E), \nabla^3 E) \\
= -(\nabla^3 (v_i \nabla_i E) - v_i \nabla_i \nabla^3 E, \nabla^3 E) \\
\leq C (\|v\|_{H^2} \|\nabla E\|_{L^\infty} + \|\nabla v\|_{L^\infty} \|E\|_{H^2}) \|\nabla^3 E\| \\
\leq C (\|\nabla v\|_{L^\infty} + \|\nabla E\|_{L^\infty}) (H(t) + 1).
\]

Concerning the terms involving \( M \), we see that

\[
H_5^{(3)} + H_9^{(3)} \\
= -(\nabla^3 \nabla \cdot (\nabla^T M \nabla M), \nabla^3 v) - (\nabla^4 (v \cdot \nabla M), \nabla^4 \Delta M) \\
= -(\nabla^3 (\nabla_i M \Delta M) - \nabla_i \nabla_i \nabla^3 \Delta M, \nabla^3 v) \\
\quad + (\nabla^3 (v \cdot \nabla M) - \nabla^3 v_i \nabla_i M, \nabla^3 \Delta M) \\
= -(\nabla^3 \nabla_i M \Delta M + 3 \nabla^2 \nabla_i M \Delta \nabla M + \nabla \nabla_i M \Delta \nabla^2 M, \nabla^3 v) \\
\quad + (3 \nabla v_i \nabla_i \nabla M + 3 \nabla v_i \nabla_i \nabla^2 M + \nabla v_i \nabla \nabla^3 M, \nabla^3 \Delta M) \\
\quad = (\nabla^4 \nabla_i M \Delta M, \nabla^2 v_i) + (\nabla^3 \nabla_i M \Delta \nabla M, \nabla^2 v_i) \\
\quad - 3(\nabla^4 \nabla_i M \Delta \nabla M + 2 \nabla^3 \nabla_i M \Delta \nabla^2 M + \nabla^2 \nabla_i M \Delta \nabla^3 M, \nabla v_i) \\
\quad + 3(\nabla^2 \nabla_i M \Delta \nabla^2 M + \nabla \nabla_i M \Delta \nabla^3 M, \nabla v) \\
\quad + 3(\nabla^2 v_i \nabla_i \nabla M, \nabla^3 \Delta M) + 3(\nabla v_i \nabla \nabla^2 M, \nabla^3 \Delta M) \\
\quad - (\nabla v_i \nabla \nabla^3 M, \nabla^3 \nabla j M) - (v_i \nabla_i \nabla^3 \nabla j M, \nabla^3 \nabla M) \\
\quad = (\nabla^4 \nabla_i M \Delta M, \nabla^2 v_i) - (\nabla^2 \nabla_i M \Delta \nabla M, \nabla v) + (\nabla^3 \nabla_i M \Delta \nabla^2 M, \nabla v_i) \\
\quad - 3(\nabla^4 \nabla_i M \Delta \nabla M, \nabla v_i) - 6(\nabla^3 \nabla_i M \Delta \nabla^2 M, \nabla v) - 3(\nabla^2 \nabla_i M \Delta \nabla^3 M, \nabla v_i)
\[ -3(\nabla^3 J, \Delta^2 M, \nabla v) - 3(\nabla^2 J, \Delta^3 M, \nabla v) + 3(\nabla J, \Delta^3 M, \nabla^2 v) \\
+ 3(\nabla^2 v, \Delta^3 M, \nabla J) + 3(\nabla v, \Delta^3 M, \nabla^3 J) - (\nabla_j v_i \nabla_i \Delta^3 M, \nabla^3 J) \]

\[ \leq C \int_{\Omega} |\nabla^2 v|^{5} M |\Delta M| \, dx + C \int_{\Omega} |\nabla v|^{5} M |\nabla^3 M| \, dx + C \int_{\Omega} |\nabla v|^{4} M |\nabla^4 M| \, dx \]

\[ := J_1 + J_2 + J_3. \]

Using Hölder’s inequality, Young’s inequality, the Gagliardo-Nirenberg inequalities (17), (18) and the following inequalities

\[ ||\nabla v||_{L^\infty} \leq C ||\nabla v||^{\frac{1}{5}} ||\nabla^3 v||^{\frac{2}{5}} + C ||\nabla v||, \]
\[ ||\nabla^2 v||_{L^4} \leq C ||\nabla v||^{\frac{1}{2}} ||\nabla^3 v||^{\frac{1}{2}} + ||\nabla v||_{L^\infty}, \]

we can estimate \(J_1, J_2\) and \(J_3\) as follows

\[ J_1 \leq C ||\nabla^5 M|| ||\nabla^2 v||_{L^4} ||\Delta M||_{L^4} \]
\[ \leq \frac{1}{8} ||\nabla^5 M||^2 + C ||\nabla^2 v||_{L^4}^2 ||\Delta M||_{L^4}^2 \]
\[ \leq \frac{1}{8} ||\nabla^5 M||^2 + C (||\nabla v||_{L^\infty} ||\nabla^3 v|| + ||\nabla v||_{L^\infty}^{\frac{1}{2}}) (||\Delta M||^{\frac{1}{2}} ||\nabla^4 M||^{\frac{1}{2}} + ||\Delta M||^2) \]
\[ \leq \frac{1}{8} ||\nabla^5 M||^2 \]
\[ + C ||\nabla v||_{L^\infty} (||\nabla^3 v|| + ||\nabla v||^{\frac{1}{2}} ||\nabla^3 v||^{\frac{1}{2}} + ||\nabla v||) (H(t)^{\frac{1}{2}} + \frac{1}{2} C_2 \epsilon + H(t)^C_2 \epsilon + 1) \]
\[ \leq \frac{1}{8} ||\nabla^5 M||^2 + C ||\nabla v||_{L^\infty} (H(t)^{\frac{1}{2}} + \frac{1}{2} C_2 \epsilon + 1) (H(t)^{\frac{1}{2}} + C_2 \epsilon + 1) \]
\[ \leq \frac{1}{8} ||\nabla^5 M||^2 + C ||\nabla v||_{L^\infty} (H(t)^{\frac{1}{2}} + \frac{1}{2} C_2 \epsilon + 1), \]

\[ J_2 \leq ||\nabla^5 M|| ||\nabla v||_{L^\infty} ||\nabla^3 M|| \]
\[ \leq \frac{1}{8} ||\nabla^5 M||^2 + C ||\nabla v||_{L^\infty} (||\nabla v||^{\frac{1}{2}} ||\nabla^3 v||^{\frac{1}{2}} + ||\nabla v||) (||\Delta M|| ||\nabla^4 M|| + ||\Delta M||^2) \]
\[ \leq \frac{1}{8} ||\nabla^5 M||^2 \]
\[ + C ||\nabla v||_{L^\infty} (H(t)^{\frac{1}{2}} + \frac{1}{2} C_2 \epsilon + H(t)^{\frac{1}{2}} C_2 \epsilon + 1) (H(t)^{\frac{1}{2}} + \frac{1}{2} C_2 \epsilon + 1) \]
\[ \leq \frac{1}{8} ||\nabla^5 M||^2 + C ||\nabla v||_{L^\infty} (H(t)^{\frac{1}{2}} + \frac{1}{2} C_2 \epsilon + 1), \]

and

\[ J_3 \leq C ||\nabla v||_{L^\infty} ||\nabla^4 M||^2 \leq C ||\nabla v||_{L^\infty} H(t). \]

For \(H^{(3)}_{10}\), we have

\[ H^{(3)}_{10} = -\frac{1}{\alpha^2} (\nabla^4 (M^3), \nabla^2 \Delta M) = \frac{1}{\alpha^2} (\nabla^3 (M^3), \nabla^3 \Delta M) \]
\[ \leq C ||\nabla^5 M|| ||M|| \nabla^5 M || ||\nabla^5 M||_{L^\infty} \]
\[ \leq \frac{1}{8} ||\nabla^5 M||^2 \]
\[ + C (||M|| + ||\Delta M||^{\frac{1}{2}} ||\nabla^4 M||^{\frac{1}{2}} + ||\Delta M||) (||\nabla M||^{\frac{1}{2}} ||M||^{\frac{1}{2}} + ||M||)^4 \]
\[ \leq \frac{1}{8} ||\nabla^5 M||^2 + C (H(t)^{\frac{1}{2}} + \frac{1}{2} C_2 \epsilon + H(t)^C_2 \epsilon + 1) (H(t)^{\frac{1}{2}} C_2 \epsilon + 1) \]
\[ \leq \frac{1}{8} ||\nabla^5 M||^2 + C (H(t)^{\frac{1}{2}} + \frac{1}{2} C_2 \epsilon + 1). \]
Inserting the above estimates into (55), and choosing \( \varepsilon > 0 \) small enough such that 
\[ 12C_2 \varepsilon \leq 1, \]
we obtain
\[
\frac{d}{dt}(\|\nabla^3 v(t)\|^2 + \|\nabla^3 E(t)\|^2 + \|\nabla^4 M(t)\|^2 + \|\nabla^5 M(t)\|^2) 
\leq C(\varepsilon)(1 + \|\nabla v\|_{L^\infty} + \|\nabla E\|_{L^\infty})(C_3 + H(t)), \quad \forall t \in (T, T^*). \tag{56}
\]
Then integrating the above inequality with respect to time from \( T \) to \( t \in (T, T^*) \),
by the inequality (53), we have
\[
C_3 + H(t) 
\leq C_3 + \|\nabla^3 v(T)\|^2 + \|\nabla^3 E(T)\|^2 + \|\nabla^4 M(T)\|^2 
+ C(\varepsilon)C_2 \int_T^t (1 + \|\nabla \times v\|_{L^\infty} + \|\nabla \times E\|_{L^\infty}) \ln(C_3 + H(\tau))(C_3 + H(\tau))d\tau,
\]
for all \( T \leq t < T^* \). From Gronwall’s lemma, we deduce that
\[
C_3 + H(t) \leq (C_3 + \|\nabla^3 v(T)\|^2 + \|\nabla^3 E(T)\|^2 + \|\nabla^4 M(T)\|^2) 
\times \exp \left( C(\varepsilon)C_2 \int_T^t (1 + \|\nabla \times v\|_{L^\infty} + \|\nabla \times E\|_{L^\infty}) \ln(C_3 + H(\tau))d\tau \right),
\]
which implies
\[
\ln(C_3 + H(t)) \leq \ln(C_3 + \|\nabla^3 v(T)\|^2 + \|\nabla^3 E(T)\|^2 + \|\nabla^4 M(T)\|^2) 
\quad + C(\varepsilon)C_2 \int_T^t (1 + \|\nabla \times v\|_{L^\infty} + \|\nabla \times E\|_{L^\infty}) \ln(C_3 + H(\tau))d\tau.
\]
Applying Gronwall’s lemma again, we have for all \( t \in [T, T^*] \) (recalling (52))
\[
\ln(C_3 + H(t)) 
\leq \ln(C_3 + \|\nabla^3 v(T)\|^2 + \|\nabla^3 E(T)\|^2 + \|\nabla^4 M(T)\|^2) 
\times \exp \left( C(\varepsilon)C_2 \int_T^t (1 + \|\nabla \times v\|_{L^\infty} + \|\nabla \times E\|_{L^\infty})d\tau \right) 
\leq \ln(C_3 + \|\nabla^3 v(T)\|^2 + \|\nabla^3 E(T)\|^2 + \|\nabla^4 M(T)\|^2) \exp \left( C(\varepsilon)C_2(T^* + \varepsilon) \right),
\]
and thus
\[
\lim_{t \to (T^*)^-} (\|u(t)\|_{H^3} + \|E(t)\|_{H^3} + \|M(t)\|_{H^1}) \leq C.
\]
Hence, we can extend the local classical solution \((v, E, M)\) beyond \( T^* \), which leads to a contradiction with the definition of \( T^* \).

The proof of Theorem 1.3 is complete. \( \square \)

**Acknowledgments.** The author would like to express her sincere gratitude to Prof. JiaXing Hong for his longtime encouragements and supports. She would also like to thank Prof. Hao Wu for helpful discussions and comments on a previous version of the manuscript. Part of the work was done during the author’s visit to School of Mathematical Sciences at Fudan University, whose hospitality is gratefully acknowledged.
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Received for publication December 2017.

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