Wright functions of the second kind and Whittaker functions

Francesco Mainardi\textsuperscript{1} · Richard B. Paris\textsuperscript{2} · Armando Consiglio\textsuperscript{3}

Received: 21 December 2021 / Revised: 1 May 2022 / Accepted: 2 May 2022 / Published online: 27 May 2022
© Diogenes Co. Ltd 2022

Abstract
In the framework of higher transcendental functions, the Wright functions of the second kind have increased their relevance resulting from their applications in probability theory and, in particular, in fractional diffusion processes. Here, these functions are compared with the well-known Whittaker functions in some special cases of fractional order. In addition, we point out two erroneous representations in the literature.

Keywords Fractional calculus · Wright functions · Whittaker functions · Hypergeometric functions · Laplace transform

Mathematics Subject Classification 26A33 (primary) · 30B10 · 30E15 · 33C20 · 34E05 · 41A60

Francesco Mainardi
francesco.mainardi@bo.infn.it

Richard B. Paris
r.paris@abertay.ac.uk

Armando Consiglio
armando.consiglio@physik.uni-wuerzburg.de

\textsuperscript{1} Department of Physics and Astronomy, University of Bologna, & INFN, Via Irnerio 46, 40126 Bologna, Italy
\textsuperscript{2} Division of Computing and Mathematics, University of Abertay, Dundee DD1 1HG, UK
\textsuperscript{3} Institut für Theoretische Physik und Astrophysik, and Würzburg-Dresden Cluster of Excellence ct.qmat, Universität Würzburg, 97074 Würzburg, Germany
1 Introduction

The Wright function under consideration (also known as a generalised Bessel function) is defined by

\[ \phi(\lambda, \mu, z) := W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n + \mu)}, \quad z \in \mathbb{C}, \quad (1.1) \]

where \( \lambda \) is supposed real and \( \mu \) is, in general, an arbitrary complex parameter. The series converges for all finite \( z \) provided \( \lambda > -1 \) and, when \( \lambda = 1 \), it reduces to the modified Bessel function \( z^{(1-\mu)/2}I_{\mu-1}(2\sqrt{z}) \). We point out that the \( \phi \) notation is originally due to Wright while the \( W \) notation was introduced by Mainardi in the 1990’s, see [13, 14].

For the Wright function corresponding to a negative \( \lambda \) it is convenient to denote by \( \nu \) the positive parameter \( \nu = -\lambda \), with \( 0 < \nu < 1 \). The function with negative \( \lambda \) has been termed a Wright function of the second kind by Mainardi [15], with the function with \( \lambda > 0 \) being referred to as a Wright function of the first kind.

In order to avoid confusion with the identical notation for the Whittaker function, we shall denote in the following this latter function by \( \mathcal{W} \). Hereafter we recall the definition of the Whittaker functions, which are confluent hypergeometric functions as found in the NIST Handbook [20,(13.2.42), (13.14.5)]. Indeed, the Whittaker function \( \mathcal{W}_{\kappa,\mu}(z) \) can be expressed in terms of the confluent hypergeometric function \( _1F_1(z) \) by

\[ \mathcal{W}_{\kappa,\mu}(z) = e^{-z/2}z^{\frac{1}{2}+\mu} \left\{ \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2} - \mu - \kappa\right)} _1F_1\left(\frac{1}{2} + \mu - \kappa; 1 + 2\mu; z\right) \right. \\
+ \left. \frac{z^{-2\mu}}{\Gamma\left(\frac{1}{2} + \mu - \kappa\right)} _1F_1\left(\frac{1}{2} - \mu - \kappa; 1 - 2\mu; z\right) \right\}, \quad (1.2) \]

where

\[ _1F_1(a; b, z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(b)_k k!}, \quad (|z| < \infty). \]

For more details the reader is referred, for example, to the NIST Handbook [20,p. 334].

The plan of our paper is as follows. In Section 2 we give the definition of the auxiliary functions \( F_{\nu}(z) \) and \( M_{\nu}(z) \) as special cases of the Wright function \( W_{-\nu,\mu}(z) \) with \( 0 < \nu < 1 \) and \( 0 \leq \mu \leq 1 \). Various plots of these functions for real argument are shown to illustrate their behaviour. The relation of these function to the Whittaker function is indicated. In Section 3 we present a list of special evaluations of \( W_{-\nu,\mu}(\pm x) \) for certain rational values of \( \nu \) in the range \( 0 < \nu < 1 \) and \( 0 \leq \mu \leq 1 \) expressed in terms of Whittaker, Airy and Bessel functions. Section 4 deals with a Laplace transform pair arising in time fractional diffusion processes and its relation to the
so-called “four sisters”. The proof of a typical result stated in Section 3 is given in Appendix A. The proof of the Laplace transform pair discussed in Section 4 is given in Appendix B. Section 5 is devoted to concluding remarks.

2 The Wright function of the second kind versus the Whittaker function

The Wright functions arise in probability theory related to the analysis of some Levy-stable distributions and, more specifically, in processes governed by time-fractional diffusion and diffusion-wave equations. Indeed, partial differential equations of non-integer order in time

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{\partial^2 u}{\partial x^2}, \quad u = u(x, t), \quad \beta = 2\nu \in (0, 2),$$

were outlined by Mainardi in the early 1990’s; see, for example, [18]. For more details, see the 2010 book by Mainardi [15], the recent survey by Consiglio and Mainardi [5], and references therein.

In the above context, the following auxiliary functions were introduced:

$$F_\nu(z) = W_{-\nu, 0}(-z) = \sum_{n=1}^{\infty} \frac{(-z)^n}{n!\Gamma(-n\nu)}, \quad 0 < \nu < 1,$$

$$M_\nu(z) = W_{-\nu, 1-\nu}(-z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-n\nu + 1 - \nu)}, \quad 0 < \nu < 1.$$ (2.2)

These functions are interrelated by the following relation:

$$F_\nu(z) = \nu z M_\nu(z).$$ (2.4)

For the asymptotic expressions of the Mainardi auxiliary functions we refer the reader to [18, 19, 25]. For further information about the general Wright functions we refer to the papers by Luchko [11, 12] and by Paris [21–23] and references therein.

For particular rational values of the parameter $\nu$ the Wright functions of the second kind are expected to be represented in terms of known special functions of the hypergeometric class. For instance, referring to the $M$-Wright functions with positive variable $x$ the following particular representations are nowadays well known in terms of some simpler functions:

$$M_0(x) = e^{-x}, \quad M_{1/3}(x) = 3^{2/3}\text{Ai}(x/3^{1/3}), \quad M_{1/2}(x) = \frac{1}{\sqrt{\pi}} e^{-x^2/4},$$

$$M_{2/3}(x) = 3^{-2/3} \left[ 3^{1/3} x \text{Ai}(x^2/3^{4/3}) - 3\text{Ai}'(x^2/3^{4/3}) \right] e^{-2x^3/27},$$ (2.5)

where $\text{Ai}$ is the Airy function and $\text{Ai}'$ its derivative. As $\nu \to 1^-$ the function $M_\nu(x)$ tends to the Dirac delta generalized function $\delta(x - 1)$.

\[ \text{Springer} \]
Wright functions of the second kind... 861

Plots of $M_\nu(\nu)$ for real $x$ and varying $\nu$ are presented in [15,Appendix F] and [17] in order to illustrate the transition between the special values $\nu = 0, 1/2, 1$, and the physical transition between diffusion and wave propagation. As an example we include in Fig. 1 the plot of the function $M_{2/3}(x) = W_{-2/3,1/3}(-x)$ for $0 \leq x \leq 5$.

For details on the $M$-Wright function in probability theory, see the paper by Cahoy [3]. The case $\mu = 0$ in (1.1) also finds application in probability theory and is discussed extensively by Paris and Vinogradov in [26] where it is referred to as a ‘reduced’ Wright function. In both representations we write

$$
\phi(\lambda, 0; z) = W_{\lambda,0}(z).
$$

The simple representations (2.5) have motivated us to explore the possibility that for rational values of $\nu = 1/4, 1/3, 1/2, 2/3, 3/4$ the Wright functions of the second kind can be represented in terms of other special functions (of hypergeometric type) including the Whittaker functions. In particular, we concentrate our attention on the representations with $\nu = 2/3$, where we are able to correct two erroneous results existing in the literature.

We note that another method for representation of the Wright function in terms of the hypergeometric functions for the rational values of its parameters was introduced in the paper by Gorenflo et al. [9]. This method is based on the representation of the Wright function as a particular case of the Fox $H$-function and on using the Gauss-Legendre formula for the gamma function (see Section 2.2 of [9]).

We recall that the Whittaker functions are so named after the fundamental 1903 paper by Whittaker [28]. They are particular confluent hypergeometric functions that are the solutions of the following differential equation

$$
\frac{d^2}{dx^2} W_{\mu,\nu}(x) + \left( -\frac{1}{4} + \frac{\mu}{x} + \frac{1/4 - \nu^2}{x^2} \right) W_{\mu,\nu}(x) = 0.
$$

In [28] Whittaker noted (using our notation) that the differential equation is unchanged if $\nu$ is replaced by $-\nu$ and if $\mu$ is replaced by $-\mu$, provided $x$ is replaced by $-x$ at the
same time. Hence the four functions $W_{\mu,v}(x)$, $W_{\mu,-v}(x)$, $W_{-\mu,v}(-x)$, $W_{-\mu,-v}(-x)$ are solutions of the differential equation (2.7). Then we have two linearly independent solutions of the Whittaker equation (2.7) $W_{\mu,v}(x)$, $W_{-\mu,v}(-x)$, $x \geq 0$; see [7,p. 6] and the NIST Handbook [20,p. 335]. We note that the Whittaker functions exhibit a branch cut on the negative real axis, so that they assume complex values on this semi-axis. This fact is clarified in Fig. 2 concerning the above Whittaker functions for $0 \leq x \leq 5$ in the special cases $\mu = \pm 1/2$ and $\nu = 1/6$ of most interest in the following.

In [27], Stanković, in addition to wrongly reporting the Whittaker differential equation (2.7), which was presumably a misprint, he obtained the following representation of the Wright function (also reported in the treatise on Mittag-Leffler functions by Gorenflo et al. [8] in Eq. (7.2.12), p. 214)

$$W_{-\frac{2}{3},0}(-x^{-\frac{2}{3}}) = -\frac{1}{2\sqrt{3}\pi} \exp\left(-\frac{2}{27x^2}\right) W_{-\frac{1}{2},\frac{1}{6}}\left(-\frac{4}{27x}\right).$$

We note that Stanković’s representation appears to be wrong because the corresponding Whittaker function is expected to be complex valued.

In order to derive the correct result we take advantage of the Whittaker function representations for the reduced Wright functions $W_{-2/3,0}(\pm x)$ given by Paris and Vinogradov in [26,Appendix C] for $x \geq 0$ and checked in the plots in Fig. 3,

$$\phi(-2/3, 0, x) = W_{-2/3,0}(x) = -\frac{1}{2\sqrt{3} \pi} e^{2x^3/27} W_{-1/2,1/6}\left(\frac{4x^3}{27}\right), (2.8)$$

$$\phi(-2/3, 0, -x) = W_{-2/3,0}(-x) = \sqrt{\frac{3}{\pi}} e^{-2x^3/27} W_{1/2,1/6}\left(\frac{4x^3}{27}\right). (2.9)$$
Indeed, replacing $x$ by $x^{-2/3}$ in (2.9), we obtain the correct result for the Stanković representation checked as usual for the identity between the Wright and Whittaker representations, see Fig. 4,

$$W_{-\frac{2}{3}, 0}(-x^{-\frac{2}{3}}) = \sqrt{\frac{3}{\pi}} \exp\left(-\frac{2}{27x^2}\right) W_{\frac{1}{2}, 1, \frac{1}{2}} \left(\frac{4}{27x^2}\right).$$  

(2.10)

### 3 A table of special evaluations of the Wright function

In this section we present a list of special evaluations of the Wright function $W_{-\nu, \mu}(\pm x)$ for certain rational values of $\nu$ satisfying $0 < \nu < 1$ and $0 \leq \mu \leq 1$,
with \( x > 0 \). The functions \( \mathcal{M}_{\kappa, \mu}(x) \) and \( \mathcal{W}_{\kappa, \mu}(x) \) denote the Whittaker functions, \( J_{\nu}(x) \) and \( K_{\nu}(x) \) are the usual Bessel functions and \( p F_q(x) \) is the generalised hypergeometric function.

The method of proof of these results is the same in each case. An example of the proof when \( \nu = 2/3 \) is supplied in Appendix A.

### 3.1. The case \( \nu = 1/2, X = x^2/4 \):

\[
W_{-1/2,0}(\pm x) = \frac{X^{1/2}e^{-x}}{\sqrt{\pi}},
\]
\[
W_{-1/2,1/4}(+x) = \frac{1}{\sqrt{\pi}} X^{-1/4}e^{-x/2} \left\{ W_{1/2,1/4}(X) - \frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)} \mathcal{M}_{1/2,1/4}(X) \right\},
\]
\[
W_{-1/2,1/4}(-x) = \frac{1}{\sqrt{\pi}} X^{-1/4}e^{-x/2} W_{1/2,1/4}(X),
\]
\[
W_{-1/2,1/2}(\pm x) = e^{-x/\sqrt{\pi}},
\]
\[
W_{-1/2,3/4}(+x) = \frac{1}{\sqrt{\pi}} X^{-1/4}e^{-x/2} \left\{ W_{0,1/4}(X) + \frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)} \mathcal{M}_{0,1/4}(X) \right\},
\]
\[
W_{-1/2,3/4}(-x) = \frac{1}{\sqrt{\pi}} X^{-1/4}e^{-x/2} W_{0,1/4}(X),
\]
\[
W_{-1/2,1}(\pm x) = \frac{1}{\sqrt{\pi}} X^{-1/4}e^{-x/2} W_{-1/4,1/4}(X) + 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

The last entry can also be expressed more simply as an error function, namely

\[
W_{-1/2,1}(\pm x) = 1 \pm \text{erf} \sqrt{X}.
\]

### 3.2. The case \( \nu = 1/3, X = 2(x/3)^{3/2} \):

\[
W_{-1/3,0}(+x) = -X/2 J_{-1/3}(X) + J_{1/3}(X) = -3^{-1/3} x \text{Ai}(-3^{-1/3} x),
\]
\[
W_{-1/3,0}(-x) = \frac{\sqrt{3}}{2 \pi} X K_{1/3}(X) = 3^{-1/3} x \text{Ai}(3^{-1/3} x),
\]
\[
W_{-1/3,1/3}(+x) = (X/2)^{2/3} J_{1/3}(X) - J_{2/3}(X) = -3^{1/3} \text{Ai}'(-3^{-1/3} x),
\]
\[
W_{-1/3,1/3}(-x) = \frac{\sqrt{3}}{\pi} (X/2)^{2/3} K_{2/3}(X) = -3^{1/3} \text{Ai}'(3^{-1/3} x),
\]
\[
W_{-1/3,2/3}(+x) = (X/2)^{2/3} J_{2/3}(X) + J_{2/3}(X) = 3^{2/3} \text{Ai}(-3^{-1/3} x),
\]
\[
W_{-1/3,2/3}(-x) = \frac{\sqrt{3}}{\pi} (X/2)^{1/3} K_{1/3}(X) = 3^{2/3} \text{Ai}(3^{-1/3} x),
\]
Wright functions of the second kind
is involved
In the fractional processes defined by the partial differential equation of type \((2.1)\) the
Laplace transform pair occurring in time fractional diffusion
functions of the type \(0\) special function representations. They are found to involve generalised hypergeometric
functions of the type \(0\) with \(x\) are not included here.

The cases \(\nu = 1/4, X = 4x^3/27\):

\[
W_{-1/3,1}(\pm x) = 1 \pm \frac{x}{\Gamma\left(\frac{1}{3}\right)} F_2\left(\frac{1}{3}; \frac{2}{3}, \frac{4}{3}; \mp x^2/4\right) + \frac{x^2}{2\Gamma\left(\frac{1}{3}\right)} F_2\left(\frac{2}{3}; \frac{4}{3}, \frac{5}{3}; \mp x^2/4\right).
\]

3.3. The case \(\nu = 2/3, X = 4x^3/27\):

\[
W_{-2/3,0}(+x) = -\frac{1}{2\sqrt{3\pi}} e^{x^2/2} \mathcal{W}_{-1/2,1/6}(X),
\]

\[
W_{-2/3,0}(-x) = \sqrt{\frac{3}{\pi}} e^{-x^2/2} \mathcal{W}_{1/2,1/6}(X),
\]

\[
W_{-2/3,1/3}(+x) = \frac{2-4/3}{\sqrt{3\pi}} e^{x^2/2} X^{-1/3} \mathcal{W}_{-1/2,1/6}(X),
\]

\[
W_{-2/3,1/3}(-x) = 2^{-1/3} \sqrt{\frac{3}{\pi}} e^{-x^2/2} X^{-1/3} \mathcal{W}_{1/2,1/6}(X),
\]

\[
W_{-2/3,2/3}(+x) = 2^{-2/3} \sqrt{3/\pi} e^{x^2/2} X^{-1/6} \mathcal{W}_{0,1/3}(X),
\]

\[
W_{-2/3,2/3}(-x) = 2^{-2/3} \sqrt{3/\pi} e^{-x^2/2} X^{-1/6} \mathcal{W}_{0,1/3}(X),
\]

\[
W_{-2/3,1}(\pm x) = 1 + \frac{2^{-1/3} x}{\sqrt{\pi}} \left\{ \pm \frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{2}{3}\right)} F_2\left(\frac{1}{3}; \frac{5}{6}, \frac{2}{3}, \frac{4}{3}; \pm X\right) - X^{1/3} \frac{\Gamma\left(\frac{5}{6}\right)}{4\Gamma\left(\frac{1}{3}\right)} F_2\left(\frac{2}{3}; \frac{7}{6}, \frac{4}{3}, \frac{5}{3}; \pm X\right) \right\}.
\]

The cases \(\nu = 1/4\) and \(\nu = 3/4\), with \(\mu = 0, 1/4, 1/2, 3/4, 1\) do not yield any special function representations. They are found to involve generalised hypergeometric functions of the type \(0\) \(F_2(-X), 1 F_3(-X)\) and \(2 F_3(-X)\), where \(X = (x/4)^4\), and so are not included here.

4 The Laplace transform pair occurring in time fractional diffusion processes and the four sisters

In the fractional processes defined by the partial differential equation of type \((2.1)\) the Wright function of the second kind is involved in the following Laplace transform
pair with \(x, t > 0\) and \(0 < \nu < 1, \mu \geq 0:\n
\[
L^{-1}\left[s^{-\mu} e^{-sx^\nu}\right] := \frac{1}{2\pi i} \int_{Br} e^{st-x^\nu} \frac{ds}{s^\mu} = t^{\mu-1} \sum_{n=0}^{\infty} \frac{(-x/t^\nu)^n}{n! \Gamma(-\nu n + \mu)} \quad (4.1)
\]

where \(\text{Re}(s) > 0\) and \(Br\) stands for the Bromwich path in the complex plane, namely an infinite line parallel to the imaginary axis cutting the positive real axis to the right of the branch cut on the negative real axis. For the reader’s convenience, we prove

\(\square\) Springer
the related inversion of the Laplace transform pair (4.1) in Appendix B, where both heuristic and rigorous demonstrations are given. We note that this pair can also be derived from the 1970 paper by Stanković [27]. Other Laplace transform pairs related to $s^{-\mu} \exp(-xs^\nu)$ with $x = 1$ can be found in the recent article by Apelblat and Mainardi [1].

Mainardi and Consiglio [17] have utilized the Laplace transform pair (4.1) to define the so-called ‘four sisters’ relevant in time-fractional diffusion and related to the Mainardi auxiliary functions. These four functions are obtained for any $\nu \in (0, 1)$, with $\mu = 0$, $\mu = 1 - \nu$, $\mu = \nu$ and $\mu = 1$, and are the natural generalization of the three sisters functions obtained for $\nu = 1/2$ that henceforth we recall for the reader’s convenience (for more detail, see Appendix A of [17]). The character of the sisters, because of their inter-relations, was put forward by one of us (F.M.) in his lecture notes on Mathematical Physics [16]. The three sisters, being related to the fundamental solutions of the standard diffusion equation obtained from (2.1) with $\beta = 1$, read

\begin{align*}
\phi(x, t) &= \text{erfc} \left( \frac{x}{2\sqrt{t}} \right), \\
\psi(x, t) &= \frac{x}{2\sqrt{\pi}} t^{-3/2} e^{-x^2/4t}, \\
\chi(x, t) &= \frac{1}{\sqrt{\pi}} t^{-1/2} e^{-x^2/4t},
\end{align*}

(4.2)

Then, based on the Laplace transform pair (4.1) we can get for $\nu = 1/2$ and $\mu = 1, 0, 1/2$ the representations and plots of the three sisters for $x, t > 0$ in terms of the Wright functions:

\begin{align*}
\phi(x, t) &= W_{-1/2,1}(-x/t^{1/2}), \\
\psi(x, t) &= t^{-1} W_{-1/2,0}(-x/t^{1/2}), \\
\chi(x, t) &= t^{-1/2} W_{-1/2,1/2}(-x/t^{1/2}).
\end{align*}

(4.3)

We easily recognize from Section 3.1 the representations of the three sisters in terms of the Whittaker functions by putting $X = x^2/(4t)$ and multiplying by $t^{-\mu}$ accordingly.

In [17], the four sisters were obtained for $\nu = 1/4, 1/2, 3/4$ and were plotted versus $x$ at fixed time ($t = 1$) and versus $t$ at fixed space ($x = 1$). Then, in Figs. 6, 7, we present the new plots of the four sisters corresponding to $\nu = 1/3$ and $\nu = 2/3$. We can get their representations in terms of the Whittaker functions by using the results of Section 2.2 and 2.3, respectively.

One sister, namely that corresponding to the left-hand display of Fig. 7 with $\nu = 2/3$ and $\mu = 1 - \nu = 1/3$, concerns the case considered by Humbert in 1945 [10], which
Wright functions of the second kind... 867

Fig. 5 The three sisters versus \( t \) at \( x = 1 \) (left) and versus \( x \) at \( t = 1 \) (right)

Fig. 6 The four sisters for \( \nu = 1/3 \), versus \( t \) at \( x = 1 \) (left) and versus \( x \) at \( t=1 \) (right)

according to (4.1) is given by the Laplace transform pair

\[
L^{-1} \left[ s^{-1/3} \exp(-s^{2/3}) \right] = t^{-2/3} \ W_{-2/3,1/3}(-1/t^{2/3}) .
\] (4.5)

However, Humbert \(^1\), ignoring the Wright function, provided without proof the inverse of the Laplace transform in terms of the Whittaker function as follows, see p.124 in [10],

\[
L^{-1} \left[ s^{-1/3} \exp(-s^{2/3}) \right] = -\frac{1}{4} \sqrt{\frac{3}{\pi}} e^{-2/(27t^2)} \ Y_{-1/2,-1/6} \left( -\frac{4}{27t^2} \right) .
\]

\(^1\) Humbert used the so-called Carson transform that is the standard Laplace transform multiplied for \( p \), where \( p \) stands for our \( s \).
Fig. 7  The four sisters for $\nu = 2/3$, versus $t$ at $x = 1$ (left) and versus $x$ at $t = 1$ (right)

Fig. 8  The correct result from the Laplace transform by Humbert for $0 < t \leq 5$

which is surely wrong.

Indeed, from Section 3.3 we have with

$$x \rightarrow 1/t^{2/3}, \quad X = 4x^3/27 \rightarrow 4/(27t^2),$$

the expression

$$t^{-2/3} W_{-2/3,1/3}(-1/t^{2/3}) = \frac{3}{2} \sqrt{\frac{3}{\pi}} e^{-2/(27t^2)} W_{1/2,1/6} \left( \frac{4}{27t^2} \right). \quad (4.6)$$

The plot in Fig. 8 for $0 < t \leq 5$ demonstrates the correctness of the expression (4.6) in terms of Wright and Whittaker functions. Furthermore, this plot is seen to be equivalent to that of the sister for $\nu = 2/3$ and $\mu = 1 - \nu = 1/3$ in the left-hand display of Fig. 7, as expected.
5 Conclusions

A list of evaluations of the Wright function $\mathcal{W}_{-\nu,\mu}(\pm x)$ of real argument has been given for certain rational values of $\nu$ satisfying $0 < \nu < 1$ and $0 \leq \mu \leq 1$ in terms of familiar special functions, such as the Whittaker, Airy and Bessel functions. Two erroneous representations existing in the literature have been pointed out and corrected.

The Laplace transform pair occurring in time fractional diffusion processes is established and is employed to define the so-called “four sisters”. Plots of these latter functions and other examples of the Wright function are presented to give a graphical indication of their behaviour.

All the plots presented in the paper have been realised using MATLAB. When evaluating power series, it is necessary to employ a sufficiently large number of terms, checking that with fewer terms the results lie within the margin of error. The Whittaker functions have been plotted using the routines provided by MATLAB itself whereas the Wright functions have been plotted using their series representation. It should be noted that only recently has it been recognised to be advantageous to produce efficient numerical methods for the Wright functions of the second kind, as outlined by Aceto and Durastante [2].

Acknowledgements The research activity of A.C. and F.M. has been carried out in the framework of the activities of the National Group of Mathematical Physics (GNFM, INdAM). The activity of A.C., a PhD student at the University of Wuerzburg, is carried out also in the Wuerzburg-Dresden Cluster of Excellence - Complexity and Topology in Quantum Matter (ct.qmat).

We wish to thank the anonymous referees for their helpful suggestions that have improved the presentation of the paper.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix A: Proof of the case $\nu = 2/3$ in §3.3

In this appendix we prove the Whittaker function evaluations given in Section 3.3. The proofs of the other results in Section 3 follow the same procedure and are consequently omitted. We consider the case $\nu = 2/3$ for $0 \leq \mu \leq 1$ and write $W_{-2/3,\mu}(\pm x)$ in the form

$$W_{-2/3,\mu}(\pm x) = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(\pm x)^n}{n!} \Gamma\left(\frac{2}{3}n + 1 - \mu\right) \sin\pi\left(\frac{2}{3}n + 1 - \mu\right).$$

Replacement of $n$ by $3m + j$, $j = 0, 1, 2$ then produces

$$W_{-2/3,\mu}(\pm x) = \sin\pi(1 - \mu) \sum_{m=0}^{\infty} \frac{(\pm x)^{3m}}{(3m)!} \Gamma(2m + 1 - \mu).$$
Using the reflection and triplication formulas for the gamma function

\[
\Gamma(2a) = \frac{2^{2a-1}}{\sqrt{\pi}} \Gamma(a) \Gamma(a + \frac{1}{2}), \quad \Gamma(3a) = \frac{3^{3a-\frac{1}{2}}}{2\pi} \Gamma(a) \Gamma(a + \frac{1}{3}) \Gamma(a + \frac{2}{3})
\]

and introducing of the new variable \( X = 4x^3/27 \), we find

\[
W_{-2/3,\mu}(\pm x) = \frac{2^{1-\mu}}{\sqrt{3\pi}} \left\{ \sin(1 - \mu) \sum_{m=0}^{\infty} (\pm X)^m \Gamma(m + \frac{1}{3} - \frac{1}{2}\mu) \Gamma(m + \frac{1}{3}) \Gamma(m + \frac{2}{3}) m! \Gamma(m + \frac{1}{3} - \frac{1}{2}\mu) \Gamma(m + \frac{1}{3}) \Gamma(m + \frac{2}{3}) m! \right\}
\]

\[
\pm X^{1/3} \sin \left( \frac{5}{3} - \mu \right) \sum_{m=0}^{\infty} (\pm X)^m \frac{\Gamma(m + \frac{5}{6} - \frac{1}{2}\mu) \Gamma(m + \frac{4}{3} - \frac{1}{2}\mu) \Gamma(m + \frac{5}{6}) \Gamma(m + \frac{4}{3}) m!}{\Gamma(m + \frac{1}{3} - \frac{1}{2}\mu) \Gamma(m + \frac{1}{3}) \Gamma(m + \frac{2}{3}) m!}
\]

\[
+ X^{2/3} \sin \left( \frac{7}{3} - \mu \right) \sum_{m=0}^{\infty} (\pm X)^m \frac{\Gamma(m + \frac{7}{6} - \frac{1}{2}\mu) \Gamma(m + \frac{5}{3} - \frac{1}{2}\mu) \Gamma(m + \frac{7}{6}) \Gamma(m + \frac{5}{3}) m!}{\Gamma(m + \frac{1}{3} - \frac{1}{2}\mu) \Gamma(m + \frac{1}{3}) \Gamma(m + \frac{2}{3}) m!}
\]

(A.1)

**The case \( \mu = 1/3 \).**

When \( \mu = 1/3 \), we have from (A.1)

\[
W_{-2/3,1/2}(\pm x) = \frac{2^{-1/3}}{\sqrt{\pi}} \left\{ \sum_{m=0}^{\infty} \frac{(\pm X)^m \Gamma(m + \frac{5}{6}) \Gamma(m + \frac{4}{3}) m!}{\Gamma(m + \frac{1}{3}) m!} \mp X^{1/3} \sum_{m=0}^{\infty} \frac{(\pm X)^m \Gamma(m + \frac{7}{6}) \Gamma(m + \frac{5}{3}) m!}{\Gamma(m + \frac{1}{3}) m!} \right\}
\]

\[
= \frac{2^{-1/3}}{\sqrt{\pi}} \left\{ \frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{7}{6}\right)} F_1\left(\frac{5}{6}; \frac{2}{3}; \pm X\right) \mp X^{1/3} \frac{\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{5}{6}\right)} F_1\left(\frac{7}{6}; \frac{4}{3}; \pm X\right) \right\}
\]

\[
= \frac{2^{-4/3}}{\sqrt{5\pi}} \left\{ \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} F_1\left(\frac{5}{6}; \frac{2}{3}; \pm X\right) \mp X^{1/3} \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} F_1\left(\frac{7}{6}; \frac{4}{3}; \pm X\right) \right\}.
\]

(A.2)

where \( F_1(a; b; z) \) is the confluent hypergeometric function. From (1.2) we have the following combination of confluent hypergeometric functions expressed in terms of the Whittaker function \( W_{k,\mu}(z) \):

\[
\frac{\Gamma(1 - b)}{\Gamma(a - b + 1)} F_1(a; b; z) + x^{1-b} \frac{\Gamma(b - 1)}{\Gamma(a)} F_1(a - b + 1; 2 - b; z)
\]

\[
= e^{x/z} x^{-b/2} W_{-1/2 - b/2, a-b/2}(z) \quad (z \in \mathbb{C}).
\]

(A.3)
Inserting the values \( a = 5/6 \) and \( b = 2/3 \) into (A.3) we find upon taking the upper signs in (A.2) that

\[
W_{-2/3,1/3}(x) = \frac{2^{-4/3}}{\sqrt{3\pi}} e^{X/2} X^{-1/3} W_{-1/2,1/6}(X),
\]

where we have made use of the fact that \( W_{κ,-μ}(z) = W_{κ,μ}(z) \).

Taking the lower sign in (A.2), we have upon application of the well-known Kummer transformation \( _1F_1(a; b; z) = e^z _1F_1(b - a; b; -z) \)

\[
W_{-2/3,1/3}(-x) = \frac{2^{-1/3}}{\sqrt{3\pi}} e^{-X} \left\{ \frac{\Gamma(1/3)}{\Gamma(1/6)} _1F_1(-1/6; 2/3; X) \right. \\
\left. + X^{1/3} \frac{\Gamma(-1/3)}{\Gamma(-1/6)} _1F_1(1/6; 4/3; X) \right\}.
\]

Insertion of the values \( a = -1/6 \) and \( b = 2/3 \) into (A.3) then yields

\[
W_{-2/3,1/3}(-x) = \frac{2^{-1/3}}{\sqrt{3\pi}} e^{-X} X^{-1/3} W_{1/2,1/6}(X). \tag{A.5}
\]

**The case \( μ = 2/3 \).**

Proceeding in the same manner when \( μ = 2/3 \), we have from (A.1)

\[
W_{-2/3,2/3}(±x) = \frac{2^{-2/3}}{\sqrt{π}} \left\{ \sum_{m=0}^{∞} (±X)^m \frac{\Gamma(m + 1/6)}{\Gamma(m + 5/6) m!} - X^{2/3} \sum_{m=0}^{∞} (±X)^m \frac{\Gamma(m + 5/6)}{\Gamma(m + 5/3) m!} \right\}
\]

\[
= 2^{-2/3} \frac{3}{\pi} \left\{ \frac{\Gamma(2/3)}{\Gamma(5/6)} _1F_1(1/6; 1/3; ±X) \right. \\
\left. + X^{2/3} \frac{\Gamma(-2/3)}{\Gamma(5/6)} _1F_1(5/6; 5/3; ±X) \right\}.
\]

With the values \( a = 1/6, b = 1/3 \) in (A.3) we find upon taking the upper signs that

\[
W_{-2/3,2/3}(x) = 2^{-2/3} \sqrt{3/π} e^{X/2} X^{-1/6} W_{0,1/3}(X). \tag{A.6}
\]

Application of Kummer’s transformation to the expression with the lower signs and the values \( a = 1/6, b = 1/3 \) in (A.3) then yields

\[
W_{-2/3,2/3}(-x) = 2^{-2/3} \sqrt{3/π} e^{-X/2} X^{-1/6} W_{0,1/3}(X). \tag{A.7}
\]

**The case \( μ = 0 \).**
Finally, when \( \mu = 0 \) we have from (A.1)

\[
W_{-2/3,0}(\pm x) = \mp \frac{X^{1/3}}{\sqrt{\pi}} \left\{ \sum_{m=0}^{\infty} \frac{(-X)^m \Gamma(m + \frac{5}{6})}{\Gamma(m + \frac{2}{3}) m!} \pm X^{1/3} \sum_{m=0}^{\infty} \frac{(-X)^m \Gamma(m + \frac{5}{6})}{\Gamma(m + \frac{2}{3}) m!} \right\}
\]

\[
= \mp \frac{X^{1/3}}{2\sqrt{3\pi}} \left\{ \frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{5}{6}\right)} F_1\left(\frac{5}{6}; \frac{2}{3}; \pm X\right) \pm X^{1/3} \frac{\Gamma\left(-\frac{1}{3}\right)}{\Gamma\left(-\frac{1}{3}\right)} F_1\left(\frac{7}{6}; \frac{4}{3}; \pm X\right) \right\}.
\]

Taking the upper signs and the values \( a = 5/6, b = 2/3 \) in (A.3), we find

\[
W_{-2/3,0}(x) = -\frac{1}{2\sqrt{3\pi}} e^{x/2} W_{-1/2,1/6}(X),
\]

(A.8)

and with the lower signs, after application of Kummer’s transformation, we have with \( a = -1/6, b = 2/3 \) in (A.3)

\[
W_{-2/3,0}(-x) = \sqrt{\frac{3}{\pi}} X^{1/3} e^{-x} \left\{ \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{6}\right)} F_1\left(-\frac{1}{6}; \frac{2}{3}; X\right) + X^{1/3} \frac{\Gamma\left(-\frac{1}{3}\right)}{\Gamma\left(-\frac{1}{3}\right)} F_1\left(\frac{1}{6}; \frac{4}{3}; X\right) \right\}
\]

\[
= \sqrt{\frac{3}{\pi}} e^{-x/2} W_{1/2,1/6}(X).
\]

(A.9)

The case \( \mu = 1 \) in (A.1) results in no cancellations of the gamma functions in the second and third series (the first series vanishing identically). As a consequence, these series are expressible in terms of \( 2F_2(\pm X) \) functions as stated in Section 2.3, with the result that they will not reduce to Bessel or Whittaker functions.

**Appendix B: On inversion of a relevant Laplace transform**

In this appendix we evaluate the inverse Laplace transform (LT) (4.1) whose expression is repeated here for the reader’s convenience,

\[
L^{-1} \left[ s^{-\mu} \exp(-sx^{\nu}) \right] = t^{\mu-1} W_{-\nu,\mu}(-x/t^{\nu}), \quad 0 < \nu < 1, \quad \mu > 0.
\]

(B.1)

We first provide a formal justification of this result starting from the following LT pair reported by Doetsch for the (extended) Laplace transforms of

\[
L \left[ Pf \frac{t^{-\alpha-1}}{\Gamma(-\alpha)} \right] = s^\alpha, \quad \alpha > 0, \quad \alpha \neq 0, 1, 2, \ldots,
\]

(B.2)

where \( Pf \) denotes the Hadamard finite part, see [6, pp. 63, 318]. For \( \alpha = n \), with \( n = 0, 1, 2 \), we obtain the Laplace transforms of the generalised functions \( \delta^{(n)}(t) \), that is of the derivatives of the Dirac ‘delta function’. Indeed it is known that the standard Laplace transforms of locally integrable functions are required to vanish at infinity.
We note that (B.2) can also be derived from the standard LT pair
\[ L^\left[t^\alpha\right] = \frac{\Gamma(\alpha + 1)}{s^{\alpha + 1}}, \quad \alpha > -1. \] (B.3)

Now we proceed formally expanding the exponential series term by term as follows taking into account the LT pair (B.2) for \( t > 0 \):
\[ L^{-1}\left[s^{-\mu}\exp(-xs^\nu)\right] = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} L^{-1}\left[s^{\nu n - \mu}\right] \]
\[ = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \frac{t^{-\nu n + \mu - 1}}{\Gamma(-\nu n + \mu)} := t^{\mu - 1} W_{-\nu, \mu} (-x/t^\nu). \] (B.4)

We note that this result is considered valid for \( t > 0 \); that is excluding the point \( t = 0 \) because when the exponent of the Laplace parameter turns out to be a non-negative integer the contributions of the corresponding generalised functions vanish.

We now provide a rigorous justification of the result (B.1) using the Bromwich integral formula for the inverse LT followed by use of the Hankel formula for the reciprocal of the gamma function given by
\[ \frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{Ha} \frac{e^{\tau}}{\tau^z} d\tau, \quad z \in \mathbb{C}, \] (B.5)
where \( Ha \) denotes the Hankel contour defined as a path that begins at \( \tau = -\infty - ia \) \((a > 0)\), encircles the branch cut that lies along the negative real axis, and terminates at \( \tau = -\infty + ib \) \((b > 0)\). The branch cut is necessary when \( z \) is non-integer because \( \tau^{-z} \) is a multivalued function. Then, provided \( 0 < \nu < 1 \), the Bromwich path may be deformed into the Hankel path to yield
\[ L^{-1}\left[s^{-\mu} e^{-xs^\nu}\right] = \frac{1}{2\pi i} \int_{Ha} e^{st} \frac{e^{-xs^\nu}}{s^{\mu}} ds \]
\[ = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \frac{1}{2\pi i} \int_{Ha} \frac{e^{st}}{s^{\nu n + \mu}} ds \] (B.6)
\[ = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} t^{\mu - 1} W_{-\nu, \mu} (-x/t^\nu). \]

References
1. Apelblat, A., Mainardi, F.: Applications of the Efros theorem to the function represented by the inverse Laplace transform of \( s^{-\mu} \exp(-s^\nu) \). Symmetry 13, Art. 354, 1–15 (2021). https://doi.org/10.3390/sym13020354; E-print arXiv:2012.07068 [math.CA]
2. Aceto, L., Durastante, F.: Efficient computation of the Wright function and its applications to fractional diffusion-wave equations. E-print arXiv:2202.00397v2
3. Cahoy, D.O.: Estimation and simulation for the $M$-Wright function. Communications in Statistics - Theory and Methods 41(8), 1466–1477 (2012). https://doi.org/10.1080/03610926.2010.543299
4. Consiglio, A., Luchko, Yu., Mainardi, F.: Some notes on the Wright functions in probability theory. WSEAS Transactions on Mathematics 18, 389–393 (2019)
5. Consiglio, A., Mainardi, F.: Fractional diffusive waves in the Cauchy and signalling problems. In: Beghin, L., Mainardi, F., Garrappa, R. (Eds), Nonlocal and Fractional Operators, SEMA-SIMAI Springer Ser. No 26, pp. 133–153, Springer Nature Switzerland (2021)
6. Doetsch, G.: Introduction to the Theory and Application of the Laplace Transformation. Springer, Berlin (1974)
7. Erdélyi, A., Swanson, C.A.: Asymptotic forms of Whittaker’s confluent hypergeometric functions. Memoirs of the American Mathematical Society 1(25), 1–50 (1957)
8. Gorenflo, R., Kilbas, A.A., Mainardi, F., Rogosin, S.V.: Mittag-Leffler Functions. Related Topics and Applications. 2nd Ed., Monographs in Mathematics, Springer Verlag, Berlin (2020). [1st Ed. (2014)]
9. Gorenflo, R., Luchko, Yu., Mainardi, F.: Analytical properties and applications of the Wright function. Fract. Calc. Appl. Anal. 2(4), 383–414 (1999). E-print http://arxiv.org/abs/math-ph/0701069
10. Humbert, P.: Nouvelles correspondances symboliques. Bull. Sci. Math. (Paris, II Ser.) 69, 121–129 (1945)
11. Lipnevich, V., Luchko, Yu.: The Wright function: its properties, applications, and numerical evaluation. AIP Conference Proceedings 1301, 614–622 (2010). https://doi.org/10.1063/1.3526663
12. Luchko, Yu.: The Wright functions and its applications. In: Kochubei, A., Luchko, Yu. (eds.) Handbook of Fractional Calculus with Applications, vol. 1, pp. 241–268. De Gruyter, Berlin (2019)
13. Mainardi, F.: On the initial value problem for the fractional diffusion-wave equation. In: Rionero, S., Ruggeri (Eds), 7th Conference on Waves and Stability in Continuous Media (WASCOM 1993), pp. 246–251, World Scientific, Singapore (1994)
14. Mainardi, F.: The fundamental solutions for the fractional diffusion-wave equation. Applied Mathematics Letters 9(6), 23–28 (1996)
15. Mainardi, F.: Fractional Calculus and Waves in Linear Viscoelasticity. World Scientific, Singapore (2010) [2nd Ed. in press (2022)]
16. Mainardi, F.: The Linear Diffusion Equation. Lecture Notes in Mathematical Physics, University of Bologna, Department of Physics, 19 pp. (1996–2006)
17. Mainardi, F., Consiglio, A.: The Wright function of the second kind in mathematical physics. Mathematics 8(6) (SI on Special Functions with Applications in Mathematical Physics), Art. 884, 1–26 (2021). https://doi.org/10.3390/math8060884; [E-print arXiv:2007.02098]
18. Mainardi, F., Tomirotti, M.: On a special function arising in the time fractional diffusion-wave equation. In: Rusev, P., Dimovski, I., Kiryakova, V. (Eds), Transform Methods and Special Functions, 1994 (Proc. Int. Workshop, Sofia 12–17 August 1994), 171–183, Science Culture Technology, Singapore (1995)
19. Mainardi, F., Tomirotti M.: Seismic pulse propagation with constant $Q$ and stable probability distributions. Annali di Geofisica 40, 1311–1328 (1997). [E-print http://arxiv.org/abs/1008.1341]
20. Olver, F.W., Lozier, D.W.: Boisvert, R.F. and Clark, C.W. (Eds.) NIST Handbook of Mathematical Functions. Cambridge University Press, Cambridge (2010)
21. Paris, R.B.: Exponentially small expansions of the Wright function on the Stokes lines. Lithuanian Math. J. 54, 82–105 (2014)
22. Paris, R.B.: The asymptotics of the generalised Bessel function. Math. Aeterna 7, 381–406 (2017)
23. Paris, R.B.: Asymptotics of the special functions of fractional calculus. In: Kochubei, A., Luchko, Yu. (eds.) Handbook of Fractional Calculus with Applications, vol. 1, pp. 297–325. De Gruyter, Berlin (2019)
24. Paris, R.B., Kaminski, D.: Asymptotics and Mellin-Barnes Integrals. Cambridge University Press, Cambridge (2001)
25. Paris, R.B., Consiglio, A., Mainardi, F.: On the asymptotics of Wright functions of the second kind, Fract. Calc. Appl. Anal. 24(1), 54–72 (2021). https://doi.org/10.1515/fca-2021-0003; [E-print arXiv:2103.04284]
26. Paris, R.B., Vinogradov, V.: Asymptotic and structural properties of the Wright function arising in probability theory. Lithuanian Math. J. 56, 377–409 (2016)
27. Stanković, B.: On the function of E.M. Wright. Publ. Inst. Math. (Beograd, Nouv. Sér.) 10(24), 113–124 (1970)
28. Whittaker, E.T.: An expression of certain known functions as generalised hypergeometric functions. Bull. Amer. Math. Soc. 10(3), 125–134 (1903)
29. Wright, E.M.: The asymptotic expansion of the generalised Bessel function. Proc. Lond. Math. Soc. (Ser. 2), 38, 286–293 (1934)
30. Wright, E.M.: The generalised Bessel function of order greater than one. Quart. J. Math. 11, 36–48 (1940)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.