AREA AND HOLONOMY ON THE PRINCIPAL $U(n)$ BUNDLES OVER THE DUAL OF GRASSMANNIAN MANIFOLDS

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Abstract. Consider the principal $U(n)$ bundles over the dual of Grassmann manifolds $U(n) \to U(n, m)/U(m) \to D_{n,m}$. Given a 2-dimensional subspace $m' \subset m \subset u(n,m)$, assume either $m'$ is induced by $X, Y \in U_{m,n}(\mathbb{C})$ with $X^*Y = \mu I_n$ for some $\mu \in \mathbb{R}$ or by $X, iX \in U_{m,n}(\mathbb{C})$. Then $m'$ gives rise to a complete totally geodesic surface $S$ in the base space. Furthermore, let $\gamma$ be a piecewise smooth, simple closed curve on $S$ parametrized by $0 \leq t \leq 1$, and $\tilde{\gamma}$ its horizontal lift on the bundle $U(n) \to \pi^{-1}(S) \to S$, which is immersed in $U(n) \to U(n, m)/U(m) \to D_{n,m}$. Then

$$\tilde{\gamma}(1) = \tilde{\gamma}(0) \cdot (e^{i\theta} I_n) \quad \text{or} \quad \tilde{\gamma}(1) = \tilde{\gamma}(0),$$

depending on whether $S$ is a complex submanifold or not, where $A(\gamma)$ is the area of the region on the surface $S$ surrounded by $\gamma$ and $\theta = 2 \frac{1}{n} A(\gamma)$.

1. Introduction

For two natural numbers $n, m \in \mathbb{N}$, define an Hermitian form $F : \mathbb{C}^{n+m} \to \mathbb{C}$ by

$$F(v, w) = v^* \Lambda^*_m w$$

$$= - \sum_{k=1}^{n} \bar{v}_k w_k + \sum_{s=n+1}^{n+m} \bar{v}_s w_s,$$

where $v$ and $w$ are regarded as column vectors, and

$$\Lambda^*_m = \begin{pmatrix} -I_n & O_{n \times m} \\ O_{m \times n} & I_m \end{pmatrix}.$$

Consider the Lie group

$$U(n,m) = \{ \Phi \in GL_{n+m}(\mathbb{C}) | \Phi^* \Lambda^*_m \Phi = \Lambda^*_m \}.$$

Then,

$$U(n,m) = \{ \Phi \in GL_{n+m}(\mathbb{C}) | F(\Phi v, \Phi w) = F(v, w), v, w \in \mathbb{C}^{n+m} \},$$

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and \( D_{n,m} := U(n,m)/(U(n) \times U(m)) \) can be identified with the set of all \( n \)-dimensional subspaces \( V \) of \( \mathbb{C}^{n+m} \) such that \( F(v,v) \leq 0 \) for every \( v \in V \) by considering the first \( n \) columns of an element in \( U(n,m) \).

Let

\[ U_{m,n}(\mathbb{C}) := \{ X \in M_{m,n}(\mathbb{C}) \mid X^*X = \lambda I_n \text{ for some } \lambda \in \mathbb{C} \setminus \{0\} \}, \]

which may be regarded as generalizations of a unitary group. As it did for the principal \( U(n) \) bundles

\[ U(n) \to U(n + m)/U(m) \to G_{n,m} \]

over a Grassmannian manifold \( G_{n,m} \) in \([1]\), it will also play an important role in studying the principal \( U(n) \) bundles

\[ U(n) \to U(n, m)/U(m) \to D_{n,m} \]

over \( D_{n,m} \) such that \( U(n,m) \) has a left invariant metric, related to the Killing-Cartan form, given by

\[ \langle A, B \rangle = \frac{1}{2} \text{Re} \left( \text{Tr} (A^*B) \right), \quad A, B \in \mathfrak{u}(n,m), \]

and that \( D_{n,m} = U(n,m)/(U(n) \times U(m)) \) has the induced metric which makes the projection a Riemannian submersion.

Consider the Hopf fibration \( S^1 \to S^3 \to S^2 \). Let \( \gamma \) be a simple closed curve on \( S^2 \). Pick a point in \( S^3 \) over \( \gamma(0) \), and take the unique horizontal lift \( \tilde{\gamma} \) of \( \gamma \). Since \( \gamma(1) = \gamma(0) \), \( \tilde{\gamma}(1) \) lies in the same fiber as \( \tilde{\gamma}(0) \) does. We are interested in understanding the difference between \( \tilde{\gamma}(0) \) and \( \tilde{\gamma}(1) \). The following equality was already known \([4]\):

\[ V(\gamma) = e^{\frac{1}{2}A(\gamma)i}, \]

where \( V(\gamma) \) is the holonomy displacement along \( \gamma \), and \( A(\gamma) \) is the area of the region surrounded by \( \gamma \).

In this paper, we generalize this fact to the \( U(n) \) bundle over \( D_{n,m} \) through

\[ U(n) \to U(n, m)/U(m) \to D_{n,m}. \]

The main results are stated as follows: For \( \hat{X} \in \mathfrak{u}(n,m) \) which is induced by \( X \in U_{m,n}(\mathbb{C}) \), consider a 2-dimensional subspace \( \mathfrak{m}' \subset \mathfrak{m} \subset \mathfrak{u}(n,m) \) with \( \hat{X} \in \mathfrak{m}' \). Assume either

\[ \mathfrak{m}' = \text{Span}_\mathbb{R} \{ \hat{X}, \hat{Y} \} \]

for some \( Y \in U_{m,n}(\mathbb{C}) \) with \( X^*Y = \mu I_n \) for some \( \mu \in \mathbb{R} \) or

\[ \mathfrak{m}' = \text{Span}_\mathbb{R} \{ \hat{X}, i\hat{X} \}. \]

Then \( \mathfrak{m}' \) gives rise to a complete totally geodesic surface \( S \) in the base space. Furthermore, let \( \gamma \) be a piecewise smooth, simple closed curve on \( S \) parametrized by \( 0 \leq t \leq 1 \), and \( \tilde{\gamma} \) its horizontal lift on the bundle \( U(n) \to \pi^{-1}(S) \to S \), which is immersed in \( U(n) \to U(n, m)/U(m) \to D_{n,m}. \) Then

\[ \tilde{\gamma}(1) = \tilde{\gamma}(0) \cdot (e^{it}I_n) \quad \text{or} \quad \tilde{\gamma}(1) = \tilde{\gamma}(0), \]
depending on whether $S$ is a complex submanifold or not, where $A(\gamma)$ is the area of the region on the surface $S$ surrounded by $\gamma$ and $\theta = 2 \cdot \frac{1}{n} A(\gamma)$. See Theorem 3.12.

2. Preliminaries

To begin with, we introduce the results of \cite{2}: for the circle group

$$S^1 \cong S(U(1) \times U(1)) = \left\{ \begin{pmatrix} e^{-iz} & 0 \\ 0 & e^{iz} \end{pmatrix} : 0 \leq z \leq 2\pi \right\},$$

consider the principal bundle

$$S^1 \to SU(1,1) \to \mathbb{C}H^1$$

under the representation of $SU(1,1)$ into $GL(4,\mathbb{R})$ such that

$$w = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \\ -w_2 & w_1 & -w_4 & w_3 \\ w_3 & -w_4 & w_1 & -w_2 \\ w_4 & w_3 & w_2 & w_1 \end{pmatrix}$$

with the condition $-w_1^2 - w_2^2 + w_3^2 + w_4^2 = -1$, which induces the following identifications

$$\mathbb{C}H^1 = \left\{ \begin{pmatrix} x & 0 & y & z \\ 0 & x & -z & y \\ y & -z & x & 0 \\ z & y & 0 & x \end{pmatrix} : -x^2 + y^2 + z^2 = -1, x > 0 \right\}$$

$$= \{ (x, y, z) \in \mathbb{R}^3 : -x^2 + y^2 + z^2 = -1, x > 0 \}$$

$$=: \mathbb{H}^2,$$

where

$$p : SU(1,1) \to \mathbb{C}H^1$$

is defined by $p(w) = \bar{w}$ with

$$\bar{w} = \begin{pmatrix} w_1 & -w_2 & w_3 & w_4 \\ w_2 & w_1 & -w_4 & w_3 \\ w_3 & -w_4 & w_1 & w_2 \\ w_4 & w_3 & -w_2 & w_1 \end{pmatrix}.$$ 

Note that $p$ has the following properties:

$$p(wv) = wp(v)\bar{w} \quad \text{for all } w, v \in SU(2)$$

$$p(wv) = p(w) \quad \text{if and only if } \quad v \in S(U(1) \times U(1)) \cong S^1,$$

which shows that it is the orbit map of the principal bundle. But we have to be careful that the inclusion map $\mathbb{C}H^1 \hookrightarrow SU(1,1)$ is not a cross-section in this bundle. In fact, $p(v) = v^2 \in \mathbb{C}H^1$ for any $v \in \mathbb{C}H^1$. 
Theorem 2.1 ([2]). Let \( S^1 \to SU(1, 1) \xrightarrow{p} (\mathbb{C}H^1, \langle \cdot, \cdot \rangle_{\mathbb{C}H^1}) \) be the natural fibration. Let \( \gamma \) be a piecewise smooth, simple closed curve on \( \mathbb{C}H^1 \). Then the holonomy displacement along \( \gamma \) is given by

\[
V(\gamma) = e^{\frac{1}{2} A(\gamma)i}
\]

where \( A(\gamma) \) is the area of the region on \( \mathbb{C}H^1 \) enclosed by \( \gamma \).

To apply the result of Theorem 2.1, we will study the isomorphic equivalence of the principal bundle

\[
S^1 \to SU(1, 1) \xrightarrow{p} \mathbb{C}H^1
\]

to the one

\[
S(U(1) \times U(1)) \to SU(1, 1) \to SU(1, 1)/S(U(1) \times U(1)),
\]

but not the isometric equivalence. In fact, a conformal map \( h : SU(1, 1)/S(U(1) \times U(1)) \to \mathbb{C}H^1 \) will be constructed such that the identity map on \( SU(1, 1) \) is the bundle map covering it.

Define a map \( h : SU(1, 1)/S(U(1) \times U(1)) \to \mathbb{C}H^1 \) by

\[
h(vH) = v^2 = p(v) \quad v \in \mathbb{C}H^1,
\]

where \( H = S(U(1) \times U(1)) \). Then, the identity map of \( SU(1, 1) \) is a trivially isomorphic bundle map which covers the map \( h \).

The Lie group \( SU(1, 1) \) will have a left-invariant Riemannian metric given by the following orthonormal basis on the Lie algebra \( \mathfrak{su}(1, 1) \)

\[
E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},
\]

which correspond to

\[
e_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},
\]

in \( \mathfrak{gl}(2k, \mathbb{R}) \), respectively. Notice that \([e_1, e_2] = -2e_3\).

In order to understand the map \( h \) between base spaces and the projection map \( p \) better, refer to [2] and consider the subset of \( SU(1, 1) \):

\[
T = \left\{ \begin{bmatrix} \cosh x & (\sinh x)e^{-iy} \\ (\sinh x)e^{iy} & \cosh x \end{bmatrix} : x \geq 0, \ 0 \leq y \leq 2\pi \right\}
\]

\[
= \left\{ \begin{pmatrix} \cosh x & 0 & (\sinh x)(\cos y) & (\sinh x)(\sin y) \\ 0 & \cosh x & -(\sinh x)(\sin y) & (\sinh x)(\cos y) \\ (\sinh x)(\cos y) & -(\sinh x)(\sin y) & \cosh x & 0 \\ (\sinh x)(\sin y) & (\sinh x)(\cos y) & 0 & \cosh x \end{pmatrix} \right\}
\]
which is the exponential image of
\[ m = \left\{ \left[ \begin{array}{cc} 0 & \xi^2 \\ \xi & 0 \end{array} \right] : \xi \in \mathbb{C} \right\}. \]
Furthermore, it is exactly same as $\mathbb{C}H^1$, so the map $p$ restricted to $T$ is just the squaring map; that is,
\[ p(w) = w^2, \quad w \in T. \]

To check $h$ is a conformal map: given
\[ w = (\cosh x, (\sinh x)(\cos y), (\sinh x)(\sin y)) \in T = \mathbb{C}H^1, \]
\[ |D_1(wH)| = |(D_1w)| \]
\[ = |((\cos y)L_{w*}e_1 + (\sin y)L_{w*}e_2)| \]
\[ = 1 \]
and
\[ |D_2(wH)| = |(D_2w)| \]
\[ = \left| -\frac{1}{2}(\sinh 2x)(\sin y)L_{w*}e_1 + \frac{1}{2}(\sinh 2x)(\cos y)L_{w*}e_2 + (\sinh^2 x)L_{w*}e_3 \right| \]
\[ = \frac{1}{2} |\sinh 2x|, \]

while
\[ |D_1 h(wH)| = |D_1 w^2| \]
\[ = |(2 \sinh 2x, 2(\cosh 2x)(\cos y), 2(\cosh 2x)(\sin y))| \]
\[ = 2 \]
and
\[ |D_2 h(wH)| = |D_2 w^2| \]
\[ = |(0, -(\sinh 2x)(\sin y), (\sinh 2x)(\cos y))| \]
\[ = |\sinh 2x|. \]

Thus $h$ is a conformal map.

From Theorem 2.1 we get the following result:

**Theorem 2.2.** Consider
\[ S(U(1) \times U(1)) \rightarrow SU(1,1) \rightarrow SU(1,1)/S(U(1) \times U(1)). \]

Let $\gamma$ be a piecewise smooth, simple closed curve on $SU(1,1)/S(U(1) \times U(1))$. Then the holonomy displacement along $\gamma$ is given by
\[ V(\gamma) = e^{2A(\gamma)} = e^{\frac{i}{2} A(h\circ\gamma)} i \in S(U(1) \times U(1)) \cong S^1. \]
where $A(\gamma)$ is the area of the region on $SU(1, 1)/S(U(1) \times U(1))$ enclosed by $\gamma$ and

$$
\Phi = \begin{bmatrix}
i & 0 \\
0 & -i
\end{bmatrix}.
$$

3. **The bundle $U(n) \to U(n, m)/U(m) \to D_{n,m}$**

To deal with the bundle $U(n) \to U(n, m)/U(m) \to D_{n,m}$, we investigate $U(n) \to U(n, m) \to D_{n,m}$.

The Lie algebra $u(n, m)$ of $U(n, m)$ has the following canonical decomposition:

$$
u(n, m) = h + m,$$

where

$$
h = u(n) + u(m) = \left\{ \left( \begin{array}{cc} A & O_{n \times m} \\ O_{m \times n} & B \end{array} \right) : A \in u(n), B \in u(m) \right\}$$

and

$$
m = \left\{ \hat{X} := \left( \begin{array}{cc} O_n & X^* \\ X & O_m \end{array} \right) : X \in M_{m,n}(\mathbb{C}) \right\}.$$

Define an Hermitian inner product $h : \mathbb{C}^m \to \mathbb{C}$ by

$$
h(v, w) = v^* w,$$

where $v$ and $w$ are regarded as column vectors. Then the following lemma is obvious:

**Lemma 3.1.** If a matrix $X \in M_{m,n}(\mathbb{C})$ satisfies $X^*X = \lambda I_n$ for some $\lambda \in \mathbb{C}$, then $\lambda$ will be a nonnegative real number and $\lambda = 0$ only if $X$ is trivial.

From the lemma 3.1, we obtain that

$$U_{m,n}(\mathbb{C}) = \{ X \in M_{m,n}(\mathbb{C}) | X^*X = \lambda I_n \text{ for some } \lambda \in \mathbb{C} - \{0\} \}$$

$$= \{ X \in M_{m,n}(\mathbb{C}) | X^*X = \lambda I_n \text{ for some } \lambda > 0 \}.$$

**Lemma 3.2.** Let

$$X = \left( a_k^r + ib_k^r \right), Y = \left( c_k^r + id_k^r \right) \in M_{m,n}(\mathbb{C})$$

for $r = 1, \cdots, m$, and $k = 1, \cdots, n$. Suppose that for their induced $\hat{X}, \hat{Y} \in m$,

$$[[\hat{X}, \hat{Y}], \hat{X}] = \hat{Z} \in m$$
for some \( Z = \left( \alpha_k^r \right) \in M_{m,n}(\mathbb{C}) \) for \( r = 1, \cdots, m \), and \( k = 1, \cdots, n \). Then we have
\[
\alpha_k^r = \sum_{j=1}^n (a_j^r + ib_j^r) (2h(Y_j, X_k) - h(X_j, Y_k)) - \sum_{j=1}^n (c_j^r + id_j^r) h(X_j, X_k),
\]
where \( X_k \) and \( Y_k \) are \( k \)-column vectors of \( X \) and \( Y \) for \( k = 1, \cdots, n \), respectively.

**Proof.** It is easily obtained from
\[
[[\hat{X}, \hat{Y}], \hat{X}] = \hat{X} (2\hat{Y} \hat{X} - \hat{X} \hat{Y}) - \hat{Y} \hat{X} \hat{X}.
\]
\( \square \)

Recall the following proposition, which gives the clue for the holonomy displacement in the principal \( U(n) \) bundles over \( D_{n,m} \).

**Proposition 3.3.** [3] Let \((G, H, \sigma)\) be a symmetric space and \( \mathfrak{g} = \mathfrak{h} + \mathfrak{m} \) the canonical decomposition. Then there is a natural one-to-one correspondence between the set of linear subspaces \( \mathfrak{m}' \) of \( \mathfrak{m} \), such that \([[\mathfrak{m}', \mathfrak{m}'], \mathfrak{m}'] \subset \mathfrak{m}' \) and the set of complete totally geodesic submanifolds \( \mathcal{M}' \) through the origin \( 0 \) of the affine symmetric space \( \mathcal{M} = G/H \), the correspondence being given by \( \mathfrak{m}' = T_0(M') \).

Note that \( \mathfrak{m}' \) in the Proposition 3.3 will make a bunch of complete totally geodesic submanifolds, each of which is obtained from another one by a translation, in the affine symmetric space \( G/H \).

The role of \( U_{m,n}(\mathbb{C}) \) in this paper will be seen from now on.

**Theorem 3.4.** Given \( X \in U_{m,n}(\mathbb{C}) \) and the natural fibration
\[
U(n) \times U(m) \rightarrow U(n, m) \rightarrow D_{n,m},
\]
assume a 2-dimensional subspace \( \mathfrak{m}' = \text{Span}_\mathbb{R} \{ \hat{X}, \hat{Y} \} \) of \( \mathfrak{m} \subset u(n, m) \) satisfies
\[
(3-1) \quad X^* X = \lambda I_n, \quad X^* Y = \mu I_n, \quad \mu \in \mathbb{C}
\]
for \( Y \in M_{m,n}(\mathbb{C}) \). Then \( \mathfrak{m}' \) gives rise to a complete totally geodesic surface \( S \) in \( D_{n,m} \) if and only if either \( \text{Im} \mu \neq 0 \) and \( iX \in \text{Span}_\mathbb{R} \{ X, Y \} \) or \( (\text{Im} \mu = 0 \) and \( Y' \in U_{m,n}(\mathbb{C})) \) holds.

**Proof.** To begin with, note that \( \lambda > 0 \). Assume that \( \mathfrak{m}' \) gives rise to a complete totally geodesic surface \( S \) in \( D_{n,m} \). By a translation, without loss of generality, we can assume that \( S \) passes through the origin of the affine symmetric space \( D_{n,m} = U(n, m)/(U(n) \times U(m)) \).
To show the necessary condition: let \( e_k \in \mathbb{C}^m, k = 1, \cdots, m \), be an elementary vector which has all components 0 except for the \( k \)-component with 1. Then
\[
h(X_k, Y_j) = h(X e_k, Y e_j) = e_k^*(X^* Y) e_j,
\]
so the condition (3–1) is equivalent to
\[
h(X_k, Y_k) = \mu, \quad h(X_k, X_k) = \lambda, \quad h(X_k, X_j) = 0, \quad h(X_k, Y_j) = 0
\]
for \( k \neq j \) in \( \{1, \cdots, n\} \). From \( h(X_k, Y_k) = \mu \), we obtain
\[
2h(Y_k, X_k) = h(X_k, Y_k) = 2\mu - 3iIm \mu.
\]
The hypothesis of totally geodesic and Proposition 3.3 say that
\[
a \hat{X} + b \hat{Y} = [\hat{X}, \hat{Y}],
\]
for some \( a, b \in \mathbb{R} \). So, from Lemma 3.2,
\[
aX + bY = (Re \mu - 3iIm \mu)X - \lambda Y = -3iIm(iX) + (Re \mu X - \lambda Y),
\]
and then \( Im \mu \neq 0 \) implies \( iX \) will lie in \( \text{Span}_\mathbb{R} \{X, Y\} \) and that \( i \hat{X} \) will lie in \( \text{Span}_\mathbb{R} \{\hat{X}, \hat{Y}\} = m' \subset u(n, m) \).

If \( Im \mu = 0 \), then
\[
X^* Y - Y^* X = X^* Y - (X^* Y)^* = 2iIm \mu I_n = O_n;
\]
so
\[
(3–2) \quad [\hat{X}, \hat{Y}] = \begin{bmatrix} O_n & 0 \\ 0 & XY^* - YX^* \end{bmatrix} \in u(m) \subset u(n, m).
\]
Let \( M = XY^* - YX^* \). Then
\[
[\hat{X}, \hat{Y}] = \begin{bmatrix} O_n & 0 \\ 0 & M \end{bmatrix}
\]
and \([[\hat{Y}, \hat{X}], \hat{Y}] = -\hat{M} Y \in m' \) from the hypothesis of the condition of totally geodesic and from Proposition 3.3. Note that
\[
MY = XY^* Y - Y X^* Y = XY^* Y - Y \mu I_n = XY^* Y - (Re \mu) Y.
\]
Thus \( XY^* Y = aX + bY \) for some \( a, b \in \mathbb{R} \). Then
\[
\lambda Y^* Y = X^*(XY^* Y) = X^*(aX + bY) = (a \lambda + bRe \mu) I_n
\]
and so
\[
Y^* Y = \frac{a \lambda + bRe \mu}{\lambda} I_n, \quad \frac{a \lambda + bRe \mu}{\lambda} \in \mathbb{R}.
\]
Since \( m' = \text{Span}_\mathbb{R} \{\hat{X}, \hat{Y}\} \) is 2-dimensional, \( Y \) is not a zero matrix and so from Lemma 3.1 \( Y \in U_{m,n}(\mathbb{C}) \).

Conversely, assume the necessary part holds.

Assume \( Im \mu = 0 \) and \( Y^* Y = \eta I_n \) for some \( \eta > 0 \). Then,
\[
[\hat{X}, \hat{Y}] = \begin{bmatrix} O_n & 0 \\ 0 & M \end{bmatrix}, \quad [[[\hat{X}, \hat{Y}], \hat{X}] = \hat{M} \hat{X} \quad \text{and} \quad [[[\hat{Y}, \hat{X}], \hat{Y}] = -\hat{M} \hat{Y},
\]

Since \( m' = \text{Span}_\mathbb{R} \{\hat{X}, \hat{Y}\} \) is 2-dimensional, \( Y \) is not a zero matrix and so from Lemma 3.1 \( Y \in U_{m,n}(\mathbb{C}) \).

Conversely, assume the necessary part holds.

Assume \( Im \mu = 0 \) and \( Y^* Y = \eta I_n \) for some \( \eta > 0 \). Then,
where $M = XY^* - YX^*$. Note it suffices to show that $[[\hat{X}, \hat{Y}], \hat{X}] \in m'$ and $[[\hat{Y}, \hat{X}], \hat{Y}] \in m'$. Since
\[
MX = XY^*X - YX^*X = X\mu I_n - Y\lambda I_n = \text{Re} \mu X - \lambda Y,
\]
we get $[[\hat{X}, \hat{Y}], \hat{X}] \in m'$. We also get $[[\hat{Y}, \hat{X}], \hat{Y}] \in m'$ since
\[
MY = XY^*Y - YX^*Y = X\eta I_n - Y\mu I_n = \eta X - \text{Re} \mu Y.
\]
Assume $\text{Im} \mu \neq 0$ and $iX \in \text{Span}_R \{X, Y\}$. Since
\[
\text{Span}_R \{X, Y\} = \text{Span}_R \{X, iX\}
\]
and
\[
m' = \text{Span}_R \{\hat{X}, \hat{Y}\} = \text{Span}_R \{\hat{X}, i\hat{X}\},
\]
it suffices to show that $[[\hat{X}, i\hat{X}], \hat{X}] \in m'$ and $[[i\hat{X}, \hat{X}], i\hat{X}] \in m'$. From
\[
h(-iv, w) = iv^*w = h(v, iw)
\]
and from Lemma 3.2 it is easily obtained that
\[
[[\hat{X}, i\hat{X}], \hat{X}] = -4\lambda i\hat{X}
\]
and that
\[
[[i\hat{X}, \hat{X}], i\hat{X}] = -4\lambda \hat{X}.
\]

Corollary 3.5. Given $X \in U_{m,n}(\mathbb{C})$ and $Y \in M_{m,n}(\mathbb{C})$ with $X^*Y = \mu I_n$ for some $\mu \in \mathbb{C}$, and given the natural fibration $U(n) \times U(m) \to U(n, m) \to D_{n,m}$, assume $m' = \text{Span}_R \{\hat{X}, \hat{Y}\}$ produce a 2-dimensional subspace of $m \subset u(n, m)$. If $m'$ gives rise to a complete totally geodesic surface $S$ in $D_{n,m}$, then $Y \in U_{m,n}(\mathbb{C})$ and $\mu$ determines whether $S$ is a complex submanifold or not.

Corollary 3.6. Given $X, Y \in U_{m,n}(\mathbb{C})$ and given the natural fibration $U(n) \times U(m) \to U(n, m) \to D_{n,m}$, assume $m' = \text{Span}_R \{\hat{X}, \hat{Y}\}$ produce a 2-dimensional subspace of $m \subset u(n, m)$. If $X^*Y = \mu I_n$ for some $\mu \in \mathbb{R}$, then $m'$ will give rise to a complete totally geodesic surface $S$ in $D_{n,m}$.

Remark 3.7. Note that $\mathbb{C}^n \to \mathbb{C} \times \{0\} \subset \mathbb{C}^{n+m}$ is an $n$-dimensional subspace such that $F(v, v) \leq 0$ for every $v \in \mathbb{C}^n$ and that $\mathbb{C}^m \to \{0\} \times \mathbb{C}^m \subset \mathbb{C}^{n+m}$ is an $m$-dimensional subspace such that $F(w, w) \geq 0$ for every $w \in \mathbb{C}^m$. Given $X \in U_{m,n}(\mathbb{C})$, if $n \leq m$, then $X : (\mathbb{C}^n, h_{\mathbb{C}^n}) \to (\mathbb{C}^m, h_{\mathbb{C}^m})$ is a conformal one-one linear map, where $h_{\mathbb{C}^k}$ is an Hermitian on $\mathbb{C}^k$, $k = 1, 2, \cdots$, given by
\[
h_{\mathbb{C}^k}(u_1, u_2) = u_1^*u_2 \quad \text{for } u_1, u_2 \in \mathbb{C}^k.$
In view of \( \hat{X} \in \mathfrak{u}(n, m) \subset \text{End}(\mathbb{C}^{n+m}) \), \( \hat{X} \) sends the subspace \( \mathbb{C}^n \) to the subspace \( \mathbb{C}^m \) and satisfies
\[
F(\hat{X}v_1, \hat{X}v_2) = -\lambda F(v_1, v_2)
\]
for \( v_1, v_2 \in \mathbb{C}^n \). And the condition of the relation between \( X \) and \( Y \) in Theorem 3.4 says that
\[
F(\hat{X}v_1, \hat{Y}v_2) = h_{\mathbb{C}^m}(Xv_1, Yv_2) = \mu h_{\mathbb{C}^n}(v_1, v_2) = -\mu F(v_1, v_2)
\]
for \( v_1, v_2 \in \mathbb{C}^n \).

When \( n = 1 \), the condition (3–1) is satisfied automatically for any two vectors in \( \mathbb{C}^m \) by identifying \( M_{m,1}(\mathbb{C}) \) with \( \mathbb{C}^m \). So we get

**Corollary 3.8.** A 2-dimensional subspace \( \mathfrak{m}' \) of \( \mathfrak{m} \subset \mathfrak{u}(1, m) \) gives rise to a complete totally geodesic submanifold in the affine symmetric space \( \mathbb{C}H^m = U(1, m)/(U(1) \times U(m)) \) if and only if \( \mathfrak{m}' \) has two linearly independent tangent vectors \( \hat{w}_1 \) and \( \hat{w}_2 \) such that either \( w_2 = iw_1 \) or \( \text{Im}h_{\mathbb{C}^m}(w_1, w_2) = 0 \) holds.

We return to the bundle \( U(n) \to U(n, m)/U(m) \xrightarrow{\pi} D_{n,m} \). Any submanifold \( A \subset D_{n,m} \) induces a bundle \( U(n) \to \pi^{-1}(A) \to A \), which is immersed in the original bundle and diffeomorphic to the pullback bundle with respect to the inclusion of \( A \) into \( D_{n,m} \). In fact, in the bundle \( U(n) \times U(m) \to U(n, m) \xrightarrow{\tilde{\pi}} D_{n,m} \), the induced distribution in \( \tilde{\pi}^{-1}(A) \) from \( \mathfrak{u}(m) \) in \( U(n, m) \) is integrable and preserved by the right multiplication of \( U(n) \), and produces the bundle \( U(n) \to \pi^{-1}(A) \to A \).

**Theorem 3.9.** Assume that a complete totally geodesic surface \( S \) in \( D_{n,m} \) is induced by a 2-dimensional subspace \( \mathfrak{m}' \subset \mathfrak{m} \) with the necessary condition in Theorem 3.4 satisfied. In case of \( \text{Im} \mu = 0 \), the bundle \( U(n) \to \pi^{-1}(S) \to S \), which is immersed in the original bundle \( U(n) \to U(n, m)/U(m) \xrightarrow{\pi} D_{n,m} \), is flat.

**Proof.** By a left translation, without loss of generality, assume that \( S \) passes through the origin of the affine symmetric space \( D_{n,m} \).

Consider the bundle \( U(n) \times U(m) \to U(n, m) \xrightarrow{\hat{\pi}} D_{n,m} \). Then \( S \) induces a bundle \( U(n) \times U(m) \to \hat{\pi}^{-1}(S) \to S \). Totally geodesic condition says that the distribution induced from \( \text{Span}_{\mathbb{R}}\{\hat{X}, \hat{Y}, [\hat{X}, \hat{Y}]\} \) is integrable. Since \( \text{Im} \mu = 0 \) implies that \( [\hat{X}, \hat{Y}] \) is contained in the Lie algebra \( \mathfrak{u}(m) \) of \( U(m) \) from the equation (3–2) in the proof of Theorem 3.4, the conclusion is obtained. \( \square \)

**Theorem 3.10.** Given \( X \in U_{m,n}(\mathbb{C}) \) and the natural fibration \( U(n) \times U(m) \to U(n, m) \xrightarrow{\hat{\pi}} D_{n,m} \), consider the 2-dimensional subspace \( \mathfrak{m}' = \text{Span}_{\mathbb{R}}\{\hat{X}, i\hat{X}\} \). Then,
(1) \( \mathfrak{m}' \) gives rise to a complete totally geodesic surface \( S \) in \( D_{n,m} \).
(2) \( \mathfrak{m}' \) induces a \( U(1) \)-subbundle of a bundle

\[
U(n) \times U(m) \to \tilde{\pi}^{-1}(S) \to S,
\]

which is an immersion of the bundle

\[
S(U(1) \times U(1)) \to SU(1,1) \to SU(1,1)/S(U(1) \times U(1))
\]

into

\[
U(n) \times U(m) \to U(n, m) \xrightarrow{\tilde{\pi}} D_{n,m},
\]

such that it is isomorphic to the Hopf bundle \( S^1 \to S^2 \to CH^1 \).

(3) the immersion is conformal. In fact,

\[
|\tilde{\pi}_*v| = \sqrt{n}|v|
\]

under the expression \( \tilde{\pi} : SU(1,1) \to U(n, m) \) for the immersion.

Proof. From Lemma 3.1, let \( X^*X = \lambda I_n \) for some \( \lambda > 0 \).

By a left translation, without loss of generality, assume that \( S \) passes through the origin of the affine symmetric space \( D_{n,m} \).

Note that, for \( K = \begin{bmatrix} -i\lambda I_n & 0 \\ 0 & iXX^* \end{bmatrix} \in \mathfrak{u}(n) \times \mathfrak{u}(m), \)

\[
[\hat{X}, i\hat{X}] = -2K, \quad [K, \hat{X}] = 2\lambda i\hat{X}, \quad [K, i\hat{X}] = -2\lambda \hat{X},
\]

which implies \([\mathfrak{m}', \mathfrak{m}'], \mathfrak{m}' \subset \mathfrak{m}' \) and the conclusion (1).

Consider an orthonormal basis of \( \mathfrak{su}(1,1): \)

\[
E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},
\]

and a Lie algebra monomorphism \( f : \mathfrak{su}(1,1) \to \mathfrak{u}(n, m) \), given by

\[
f(aE_1 + bE_2 + cE_3) = \frac{a}{\sqrt{\lambda}} \hat{X} + \frac{b}{\sqrt{\lambda}} i\hat{X} + \frac{c}{\lambda} K
\]

for \( a, b, c \in \mathbb{R} \), from

\[
[E_1, E_2] = -2E_3, \quad [E_3, E_1] = 2E_2, \quad [E_3, E_2] = -2E_1.
\]

For any \( \theta \in \mathbb{R} \),

\[
e^{\theta E_3} = \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \in S(U(1) \times U(1)).
\]

Thus \( f \) will induce a Lie group monomorphism \( \hat{f} : SU(1,1) \to U(n, m) \) with \( \hat{f}(S(U(1) \times U(1))) \subset U(n) \times U(m) \) since \( SU(1,1) \) is simply connected and \( S(U(1) \times U(1)) \) is connected. Furthermore, it is the bundle map from

\[
S(U(1) \times U(1)) \to SU(1,1) \to SU(1,1)/S(U(1) \times U(1))
\]

to

\[
U(n) \times U(m) \to U(n, m) \xrightarrow{\tilde{\pi}} D_{n,m},
\]
so the connected component of the integral manifold of the distribution induced by Span_{\mathbb{R}}\{K, \hat{X}, \hat{i}X\}, which is the image of \( \hat{f} \), shows (2).

Note that \( \{ \frac{1}{\sqrt{\lambda}}\hat{X}, \frac{1}{\sqrt{\lambda}}\hat{i}X, \frac{1}{\lambda}K \} \) is an orthogonal basis of the image of \( \hat{f} \) such that

\[
\sqrt{\lambda} = \frac{1}{\sqrt{\lambda}}\hat{X} = \frac{1}{\sqrt{\lambda}}\hat{i}X = \frac{1}{\lambda}K,
\]

which shows (3).

\[ \square \]

**Remark 3.11.** Let \( \theta = \frac{\theta}{\lambda} \). Then, for \( \Phi = -E_3 \),

\[
\tilde{f}(e^{\theta \Phi}) = \tilde{f}(e^{-\theta E_3}) = e^{-\theta K} = \left[ \begin{array}{cc} e^{i\theta}I_n & 0 \\ 0 & I_m + \frac{e^{-i\theta - 1}XX^*}{\lambda} \end{array} \right]
\]

from

\[
(-i\theta XX^*)^j = \left( \frac{\theta}{\lambda} \right)^j X(X^*X)^{-1}X^* = \left( \frac{(-i\theta)^j}{\lambda} \right)XX^*
\]

for \( j = 1, 2, \cdots \). Furthermore,

\[
\left( I_m + \frac{e^{-i\theta - 1}XX^*}{\lambda} \right) \left( I_m + \frac{e^{-i\phi - 1}XX^*}{\lambda} \right)
= I_m + \frac{e^{-i(\theta + \phi)} - e^{-i\theta} - e^{-i\phi} - 1}{\lambda}X(X^*X)X^*
= I_m + \frac{e^{-(i(\theta + \phi) - 1)}XX^*}{\lambda},
\]

from which it is also obtained that

\[
I_m = \left( I_m + \frac{e^{-i\theta - 1}XX^*}{\lambda} \right) \left( I_m + \frac{e^{-i\phi - 1}XX^*}{\lambda} \right)^*.
\]

We return to the bundle \( U(n) \to U(n,m)/U(m) \to D_{n,m} \). In fact, Remark 3.11 implies that the immersed \( U(1) \)-subbundle, which is the image of \( \tilde{f} \), gives two \( U(1) \)-bundles, one of which is an immersed \( U(1) \)-subbundle in the bundle \( U(n) \to U(n,m)/U(m) \to D_{n,m} \) and the other one is an immersed \( U(1) \)-subbundle in the bundle \( U(m) \to U(n,m)/U(n) \to \tilde{\pi} D_{n,m} \).

**Theorem 3.12.** Assume the same condition for a complete totally geodesic surface \( S \) of Theorem 3.4 and consider the immersed bundle \( U(n) \to \pi^{-1}(S) \to S \) in the bundle \( U(n) \to U(n,m)/U(m) \to D_{n,m} \). Let \( \gamma \) be a piecewise smooth, simple closed curve on \( S \). Then the holonomy displacement along \( \gamma \),

\[
\tilde{\gamma}(1) = \tilde{\gamma}(0) \cdot V(\gamma),
\]

is given by the right action of

\[
V(\gamma) = e^{i\theta}I_n \text{ or } e^{i\phi}I_n \in U(n),
\]

depending on whether \( S \) is a complex submanifold or not, where \( A(\gamma) \) is the area of the region on the surface \( S \) surrounded by \( \gamma \) and \( \theta = 2 \cdot \frac{1}{n}A(\gamma) \). Especially, \( \theta = 2 \cdot A(\gamma) \) in case of \( n = 1 \).
Proof. If $S$ is not a complex manifold, then, from Theorem 3.9, the immersed bundle is flat, and so it is obvious that the holonomy displacement is trivial.

If $S$ is a complex manifold, then assume the condition of Theorem 3.10 for the immersed $U(1)$-subbundle, which is the image of $\tilde{f}$. Consider the induced map $\tilde{\phi} : B \to S \subset D_{n,m}$ between base spaces from the bundle map $\tilde{f} : SU(1,1) \to \text{Im}(\tilde{f}) \subset U(n,m)$, which is a monomorphism, where $B = SU(1,1)/S(U(1) \times U(1))$. Let $\theta = 2 \cdot \frac{1}{n} A(\gamma)$. Without loss of generality, assume that the origin of $D_{n,m}$ lies on $S$ and is the initial point of $\gamma$. The Theorem 3.10, Theorem 2.2 and Remark 3.11 say that the holonomy displacement of $\gamma$ in the bundle $U(n) \times U(m) \to \pi^{-1}(S) \to S$, which is immersed in the bundle $U(n)m \to U(n,m) \to D_{n,m}$, is given by the right action of

$$V(\gamma) = \tilde{\phi}(V(\tilde{f}^{-1} \circ \gamma)) = \tilde{\phi}(e^{i \theta} A(\tilde{f}^{-1} \circ \gamma) \Phi) = \tilde{\phi}(e^{i \theta} \Phi) = \begin{bmatrix} e^{i \theta} I_n & 0 \\ 0 & I_m + e^{-i \theta} - 1 \lambda_{XX^*} \end{bmatrix}.$$

Thus in the bundle $U(n) \to \pi^{-1}(S) \to S$, which is immersed in the bundle $U(n) \times U(m) \to U(n,m) \to D_{n,m}$, the holonomy displacement is given by the right action of

$$V(\gamma) = e^{i \theta} I_n.$$ 

□

Remark 3.13. For $n = 1$, we have the following Hopf bundle $S^1 \to S^{2m,1} \to \mathbb{C}H^m$ under the identification $S^1 = U(1), \ \mathbb{C}H^m = D_{1,m}$

and

$$S^{2m,1} = U(1,m)/U(m) \cong H^{1,2m} = \{(z_0, \cdots, z_m) \in \mathbb{C}^{m+1} : -|z_0|^2 + \sum_{k=1}^{m} |z_k|^2 = -1\},$$

where $\mathbb{C}H^m = U(1,m)/U(1) \times U(m)$ is given by the quotient metric, so the projection is a Riemannian submersion. Let $S$ be a complete totally geodesic surface in $\mathbb{C}H^m$ and $\gamma$ be a piecewise smooth, simple closed curve on $S$. Identify $\mathbb{C}^{m} \cong M_{m,1}(\mathbb{C})$. If $S$ is induced by $\text{Span}\{v, w\} \subset \mathbb{C}^m$ with $\text{Im} h_{\mathbb{C}^m}(v, w) = 0$, then the holonomy displacement along $\gamma$ is trivial. See Corollary 3.8 and Theorem 3.9. If $S$ is induced by a two dimensional subspace with complex structure in $\mathbb{C}^m$, then the holonomy displacement depends
only on the area of the region surrounded by $\gamma$. In case of $m = 1$, $\mathbb{C}H^m$ with the quotient metric is isometric to $\mathbb{H}^2\left(\frac{1}{2}\right)$, where

$$\mathbb{H}^m(r) := \{(x_0, \cdots, x_m) \in \mathbb{R}^{m+1} : -x_0^2 + x_1^2 + \cdots + x_m^2 = -r^2\}$$

for $m \in \mathbb{N}$ and for $r > 0$. Refer to the map $h$ defined in Section 2 and notice that the immersion $\tilde{f}$ of the Theorem 3.10 is isometric if $n = 1$.

**Remark 3.14.** Let $U(m) \to U(n, m)/U(n) \xrightarrow{\pi} D_{n,m}$ be the natural fibration. Assume the same condition for a complete totally geodesic surface $S$ of Theorem 3.10 and consider the bundle $U(m) \to \hat{\pi}^{-1}(S) \xrightarrow{\hat{\pi}} S$. Let $\gamma$ be a piecewise smooth, simple closed curve on $S$. Then the holonomy displacement along $\gamma$ is given by the right action of

$$V(\gamma) = I_m + \frac{e^{-i\theta} - 1}{\lambda} XX^* \in U(m),$$

which depends on $X$, not only on $n$, where $\theta = 2 \cdot \frac{1}{n} A(\gamma)$.

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