Adiabatic perturbation theory of nonequilibrium light-controlled superconductivity

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Recent experiments, in which Terahertz (THz) light has been used to induce nonequilibrium superconducting states, have raised a number of intriguing fundamental questions. Theoretically, these experiments are most often described within the Floquet formalism, which suffers a number of well-known limitations (e.g. Floquet heating). Alternative approaches rely on heavy numerical methods. In this Article we develop an analytical theory of nonequilibrium superconductivity that combines path integrals on the Kostantinov-Perel’ time contour with adiabatic perturbation theory [G. Rigolin, G. Ortiz, and V.H. Ponce, Phys. Rev. A 78, 052508 (2008)]. We consider a general system of electrons and Raman phonons coupled by the Fröhlich interaction, in the presence of a time-dependent external field which acts on the phonon subsystem. The latter is supposed to model the THz light-induced excitation of nonlinear interactions between infrared and Raman phonons. Assuming that the external field has a slow dependence on time, we derive equations for the dynamical Cooper gap, calculating the leading adiabatic term and the first non-adiabatic correction. Our nonequilibrium formulas can be solved numerically with a minimal increase of computational complexity with respect to that needed to calculate the Cooper gap at equilibrium.

I. INTRODUCTION

Discovering or engineering materials displaying superconductivity at room temperature represents an extraordinary challenge, with obvious disruptive technological implications. Since the equilibrium critical temperature $T_c$ for conventional superconducting states tends to be extremely low$^1$, large theoretical and experimental efforts are being devoted to the search for high-temperature superconductors$^{2,3}$ and generalizations of the equilibrium theory beyond the Eliashberg equations$^{4,5}$.

On the other hand, recent advances in the production and manipulation of intense Terahertz light sources have triggered a very interesting question. Is it possible to turn a normal material into a superconducting one, at least temporarily, by applying an appropriately designed time-dependent electromagnetic field? More precisely, recent experiments indicate that stimulation by light of a superconducting material at temperatures above $T_c$, even up to room temperature, may induce in the otherwise normal state at least some of the properties of the superconducting phase (e.g. coherent transport), avoiding the need to cool the material down to very low temperatures$^6$$^-$9. Of course this approach costs energy. For any technological application one should therefore assess whether the pros of operating a room-temperature nonequilibrium superconducting phase overcome the cons linked to sustaining the electromagnetic field over a certain time window. Ignoring such practical considerations, this fascinating question challenges our understanding of the mechanisms of interaction between THz light and matter degrees of freedom in the solid state.

From the point of view of theory, the main goal is to quantify how the superconducting gap ($\Delta$) changes in time due to the presence of an external time-dependent field. General integro-differential equations have been derived, e.g. within the formalism of Keldysh nonequilibrium Green’s functions$^{10,11}$, for the Bardeen-Cooper-Schrieffer (BCS) time-dependent gap. Despite their generality, their solutions rely on approximations or assumptions that limit their applicability (see e.g. the case of a dirty superconductor$^{10,12,13}$). One can group different theoretical approaches to nonequilibrium superconductivity on the basis of the time dependence of the external field. Three cases occur: (1) the field changes slowly, or (2) quickly, on the characteristic time scales set by the equilibrium parameters of the system (a condition that can be specified in different ways$^{11,14}$), and/or (3) the field is periodic in time. In our case, we say that the external field is slow if the transition amplitude between instantaneous eigenstates of the Hamiltonian $\mathcal{H}(t)$ induced by its time-derivative is much smaller than the ratio between the energy gap between those states and the time scale over which the system is observed ($T$). This requirement on the smallness of the time-derivative of the external field will remind the reader of the conditions of validity of the adiabatic theorem in quantum mechanics$^{15}$. Below we will see that we actually need a more powerful formalism. With this definition of the rapidity of variation of the external field, the approach that we pursue in this work deals with problems belonging to group (1). Let us briefly comment on other approaches first.

In relation to problems of type (2), many theoretical works have focussed on the non-adiabatic regime, which often requires a fully numerical treatment. Typical problems that have been investigated in the literature involve: i) an instantaneous switching-on of the field$^{16,17}$; ii) a quench of the attractive interaction between anti-parallel spin electrons$^{18}$$^{-22}$; iii) an ultrafast but non-instantaneous (e.g. Gaussian) external field acting on the electron subsystem$^{23,24}$; iv) simulations of ultrafast pump-probe experiments in superconductors$^{25}$; v) preparation of the system in a nonequilibrium state and study of its evolution under a time-independent Hamiltonian$^{26}$. The motivation to study this regime is given
by new experimental techniques for the ultrafast optical manipulation of superconductivity, including real-time tracking of the evolution of $\Delta$\textsuperscript{27,28}.

When the external field is \textit{periodic} in time, as in the case of problems of type (3), the Floquet formalism provides the simplest way to compute time-dependent observables. In Ref.\textsuperscript{29}, this approach was applied to the Hubbard model, showing that the super-exchange interaction can be modulated to become the dominant energy scale of the system, switching-on pair correlations that are responsible for superconductivity. In Ref.\textsuperscript{30}, Floquet theory was used to analyze Cooper-pair instabilities in nonequilibrium electron-phonon systems. The effective electron-electron interaction resulting from the electron-phonon coupling was treated in the Hubbard approximation and a quartic time-periodic phonon driving effectively modified the interaction parameter in time. During the transient, at low driving frequencies, a competition takes place between Cooper-pair enhancement due to the driving and Cooper-pair breaking due to the nonequilibrium distribution of phonons. The critical temperature $T_c$, defined with respect to the time-averaged Hamiltonian, was found to increase in a broad region of parameter space, with a complicated dependence on the driving frequency.

Despite its usefulness, the Floquet formalism can be strictly applied only when the driving is perfectly periodic, which is not consistent with realistic experimental situations. Establishing a field that can be modelled as periodic requires a switching-on procedure occurring on a long time scale, which may cost a significant amount of energy. For a rigorous application of Floquet theory, the field should then last forever. Moreover, a well-known problem with Floquet theory is the phenomenon of \textit{Floquet heating}, by which an interacting system heats up to an infinite temperature at infinite times. Although in some situations heating is slow enough in the time interval of interest\textsuperscript{30,31}, it is nevertheless a non-physical effect whose impact needs to be carefully addressed case by case.

The limitations of Floquet theory can be overcome via fully numerical approaches. For example, the authors of Ref.\textsuperscript{32} investigated the nonequilibrium dynamics of a phonon-mediated superconductor induced by a transiently modified electronic structure through non-linear phonon coupling. The system was modelled by a Fröhlich Hamiltonian with a dynamical electronic band structure (i.e. a two-dimensional square-lattice tight-binding model with time-dependent hopping). The time-dependent hopping amplitude was taken to evolve linearly with time from an initial value $J_0 = 0.25$ eV to a final value $J_f = 0.20$ eV (reached after a time $\tau$) and stayed equal to $J_f$ afterwards. For this model, the Kadanoff-Baym equations for the nonequilibrium Green’s functions were solved numerically. The authors demonstrated an enhancement of the superconducting gap and discussed mechanisms and time scales of relaxation through phononic channels. This approach, which represents the state-of-the-art of the level of numerical accuracy that can be currently reached, is computationally very demanding. Also, our understanding is that it allows little flexibility on the choice of the external time-dependent modulation.

A different approach was pursued in Ref.\textsuperscript{33}. Here the external field was taken care of through a time-dependent electronic band dispersion resulting from the direct action of the electromagnetic field on electronic subsystem. Two scenarios were discussed. In the first case (weak-field regime), the interaction was taken to be of the standard BCS form, and the equations of motion for the Anderson pseudospins\textsuperscript{34} were solved analytically up to second order in the vector potential describing the external field. In the second case, the interaction was taken to be of the Hubbard form and the dynamics of the superconducting order parameter was calculated numerically by using a dynamical mean-field theory approach in a one-dimensional system and in infinite dimensions, assuming a monochromatic oscillating time-dependent field, as in the Floquet formalism.

The purpose of this work is to lay down a nonequilibrium theory of superconductivity that allows us to bypass the aforementioned limitations. More precisely, we neither want to rely on a smallness assumption for the amplitude of the external field nor assume that the external field is periodic in time. The only assumption we want to make is that the field is slowly-varying in time, in the sense discussed above and as will be rigorously formalized below. In this adiabatic regime, we can employ the recently developed Adiabatic Perturbation Theory (APT)\textsuperscript{35}. This is a very general procedure that allows to deal with systems whose Hamiltonians have a slow dependence on time, while going systematically beyond the conventional adiabatic theorem of quantum mechanics, which represents the “zeroth order” of APT. Applying APT to our nonequilibrium superconducting problem, we are able to lay down a theory which falls into the category (1) of our previous list.

Of course, several theoretical treatments of superconductivity in the adiabatic regime are available. For example, in Ref.\textsuperscript{11} one can find a microscopic derivation of the time-dependent gap $\Delta(\omega)$ in the frequency representation and in a small-$\omega$ expansion. This derivation is heavily based on the strong assumption that the energy spectrum and quasiparticle distribution function remain the same as at equilibrium. More accurately, the phenomenological Ginzburg-Landau theory, which is applicable at equilibrium for temperatures $\approx T_c$, can be extended to nonequilibrium systems, yielding the time-dependent Ginzburg-Landau (TDGL) theory\textsuperscript{14,36,37}. The latter is designed to describe systems with temperature close to $T_c$ and subject to \textit{small} deviations from equilibrium. Approximate differential equations for the time-dependent gap are obtained from the general ones upon expanding in a Taylor series the time and space variations of $\Delta$ with respect to the equilibrium value\textsuperscript{14}. As such, this framework cannot describe large variations of the gap
Our main results are summarized into two equations, which determine the leading contributions the nonequilibrium gap parameter within the framework of APT [Eqs. (126) and (127)]. These can be easily solved by elementary numerical approaches. Such equations require the external field to be slowly varying in time (as specified above), but are neither restricted to small variations of the gap nor to periodic external drivings. The derivation of Eqs. (126) and (127) will be reported in great detail and can be summarized as follows. We start by describing our system by means of a Hamiltonian that includes electrons and phonons, a Fröhlich-type electron-phonon interaction, and a time-dependent external field acting on the phonon subsystem (Section II). We then apply the nonequilibrium path integral formalism on the Kostantinov-Perel’ time contour [38] to derive an effective electronic action, which is obtained after integrating out the phononic degrees of freedom (Section III). The equilibrium version of this approach is standard for stationary superconductivity [39,40]. Several notable differences, however, appear in the nonequilibrium case. In particular, one directly obtains the effective retarded electron-electron interaction in the real-time representation (Section III C) and, obviously, an action term accounting for the external field (Section III D). We then introduce sources that enable the calculation of the Cooper parameters by functional differentiation (Section III E). At this stage, we proceed by approximating the effective electron-phonon interaction à la Hubbard (Section IV). This allows us to perform a Hubbard-Stratonovich decoupling (Section IV A) and to integrate out the fermionic degrees of freedom (Section IV B). This procedure yields path-integral expressions for the time-dependent Cooper parameters and, in principle, other observables (Section IV C). In practice, these expressions should be evaluated under the nonequilibrium saddle-point approximation (Section IV D), which yields the nonequilibrium version of Gor’kov equations [42]. We finally proceed to determine the self-consistent equation for the nonequilibrium superconducting gap, in the case of a spatially-uniform external field (Section V). It is exactly at this step that we utilize APT. The main results of this work are presented in Section V G, in Eqs. (126) and (127), in a form that is easily tractable numerically. In Section VI we show that, at equilibrium, our nonequilibrium formulas reduce to the BCS result. In Section VI we discuss why the APT approach was necessary and how to assess its validity. A summary and a brief set of conclusions is reported in Section VII. A number of relevant technical details can be found in Appendices A and B.

II. HAMILTONIAN OF THE COUPLED ELECTRON-PHONON SYSTEM

A. Electronic representation

We consider a system of electrons and phonons described by the following time-dependent Hamiltonian:

\[ \hat{H}(t) = \hat{H}_e + \hat{H}_p + \hat{H}_{ep} + \hat{H}_{ext}(t). \]  (1)

Here,

\[ \hat{H}_e = \sum_{k,\sigma} \epsilon_{k,\sigma} \hat{c}_{k,\sigma}^\dagger \hat{c}_{k,\sigma} \]  (2)

is the free-electron Hamiltonian, where the dependence of the single-electron band energy \( \epsilon_{k,\sigma} \) on spin \( \sigma = \uparrow, \downarrow \) allows us to describe a Zeeman coupling to a static magnetic field as well as a spin-dependent chemical potential. The second term on the right-hand side of Eq. (1),

\[ \hat{H}_p = \sum_{q,\lambda} \omega_{q,\lambda} \hat{b}_{q,\lambda}^\dagger \hat{b}_{q,\lambda}, \]  (3)

is the free-phonon Hamiltonian,

\[ \hat{H}_{ep} = \sum_{q,\lambda} M_{q,\lambda} (\hat{b}_{q,\lambda}^\dagger \hat{b}_{q,\lambda} + \hat{b}_{q,\lambda}^\dagger \hat{b}_{q,\lambda}^\dagger) \sum_{k,\sigma} \epsilon_{q+k,\sigma}^\dagger \hat{c}_{k,\sigma} \]  (4)

is the electron-phonon interaction Hamiltonian [43], where \( \lambda \) labels the phonon branches, and

\[ \hat{H}_{ext}(t) = \sum_{q,\lambda} F_{q,\lambda}(t) (\hat{b}_{q,\lambda} + \hat{b}_{-q,\lambda}^\dagger), \]  (5)

describes a time-dependent external field displacing the ions from their equilibrium positions. For \( \hat{H} \) to be Hermitian, it must be

\[ M_{q,\lambda} = M_{-q,\lambda}^*, \quad F_{q,\lambda}(t) = F_{-q,\lambda}^*(t). \]  (6)

A mechanism that generates \( \hat{H}_{ext}(t) \) in the form of Eq. (5) could be a nonlinear coupling between infrared-active (IRA) and Raman-active (RA) phonons [30,32,44,45], the latter being responsible for conventional superconductivity via their interaction with the conduction electrons, while IRA phonons at zero momentum can be coherently excited by a laser.

Several types of nonlinearities have been recently discussed in great detail in Refs. 30 and 45. For example, the phonon Hamiltonians responsible for so-called type-I and type-II nonlinearities can be written, in first quantization, as

\[ \hat{H}_1 = \Lambda_1 (Q_0^{IRA})^2 Q_0^{RA} \]  (7)

and

\[ \hat{H}_1 = \sum_k \Lambda_{1k} (Q_0^{IRA})^2 Q_k^{RA} Q_{-k}^{RA}. \]  (8)
respectively, where $Q_\mathbf{q}^{\text{IRA(RA)}}$ is the IRA (RA) phonon displacement operator at wave vector $\mathbf{q}$ (we have neglected the band index for simplicity). Ref. 30 mostly focuses on type-II nonlinearities. Here, instead, we concentrate on a type-I phonon nonlinearity. If the IRA phonon field is treated classically and driven coherently by an external electromagnetic field\textsuperscript{44}, while the RA phonon field is treated quantum-mechanically, i.e. $Q_\mathbf{q}^{\text{RA}} \propto (\hat{b}_{\mathbf{0},\text{RA}} + \hat{b}_0^{\dagger,\text{RA}})$, Eq. (7) coincides with the $q = 0$ term in Eq. (5). Later in our derivation (Section V), we will take phonon modes at $q = 0$ (which is justified by the smallness of the photon momentum with respect to the reciprocal-lattice vector\textsuperscript{30}), although we develop the first part of the theory in full generality.

To establish a relationship with previous works, we note the following. As shown in Appendix A of Ref. 30, if the feedback of the electrons on the phonon subsystem is neglected, one can treat the nonlinear term given in Eq. (7) classically, i.e. by replacing $Q_0^{\text{RA}} \to Q_0^{\text{RA}}(t)$ where $Q_0^{\text{RA}}(t)$ is determined by a pumped oscillator equation of motion (EOM), pumping being provided by the coherently excited IRA mode. If the analytical solution of this EOM is inserted in our Eq. (4) in place of the second-quantized phonon operators, one obtains a replacement of $\mathcal{H}_{\text{ep}}$ with an effective time-dependent single-electron Hamiltonian. Leaving aside the specific choice of the time dependence of this term, this approach is equivalent to that of Ref. 32. Here, we do not fix \textit{a priori} the time dependence of the external field.

**B. Nambu representation**

We now apply the Nambu transformation on the fermionic fields\textsuperscript{39},

$$\hat{c}_{\mathbf{k},\uparrow} \equiv \hat{d}_{\mathbf{k},\uparrow}, \quad \hat{c}_{\mathbf{k},\downarrow} \equiv \hat{d}_{\mathbf{k},\downarrow}, \quad \hat{\epsilon}_{\mathbf{k},\downarrow} \equiv \hat{\epsilon}_{-\mathbf{k},\downarrow}, \quad \hat{\epsilon}_{\mathbf{k},\uparrow} \equiv \hat{\epsilon}_{-\mathbf{k},\uparrow}, \quad (9)$$

and we re-define the boson fields as\textsuperscript{41}

$$\hat{b}_{\mathbf{q},\lambda} \equiv \hat{a}_{\mathbf{q},\lambda} - \delta_{\mathbf{q},\mathbf{0},\lambda} \frac{N M_{0,\lambda}}{\omega_{0,\lambda}}, \quad (10)$$

where $N$ is the number of $\mathbf{k}$ points in the first Brillouin zone. After this substitution, the Hamiltonian retains the form of Eq. (1), with

$$\hat{\mathcal{H}}_e = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k},\sigma} \hat{d}_{\mathbf{k},\sigma}^{\dagger} \hat{d}_{\mathbf{k},\sigma} \quad (11)$$

(note that the first argument of $\epsilon_{\mathbf{k},\sigma}$ is $\sigma \mathbf{k}$, with $\sigma = \pm 1$, and not just $\mathbf{k}$, because the transformation (9) inverts the direction of the wave vector of spin-down particles),

$$\hat{\mathcal{H}}_p = \sum_{\mathbf{q},\lambda} \omega_{\mathbf{q},\lambda} \hat{a}_{\mathbf{q},\lambda}^{\dagger} \hat{a}_{\mathbf{q},\lambda}, \quad (12)$$

$$\hat{\mathcal{H}}_{\text{ext}}(t) = \sum_{\mathbf{q},\lambda} F_{\mathbf{q},\lambda}(t) (\hat{a}_{\mathbf{q},\lambda} + \hat{a}_{-\mathbf{q},\lambda}^{\dagger}), \quad (13)$$

and

$$\hat{\mathcal{H}}_{\text{ep}} = \sum_{\mathbf{q},\lambda} M_{\mathbf{q},\lambda} (\hat{a}_{\mathbf{q},\lambda} + \hat{a}_{-\mathbf{q},\lambda}^{\dagger}) \hat{\rho}_q. \quad (14)$$

In writing the previous equations, we have i) introduced

$$\hat{\rho}_q = \sum_{\mathbf{k},\sigma} \sigma \hat{d}_{\mathbf{k}+\mathbf{q},\sigma} \hat{d}_{\mathbf{k},\sigma}, \quad (15)$$

ii) renormalized the single-electron band energies by

$$\epsilon_{\sigma \mathbf{k}, \sigma} - 2N \sum_{\lambda} \frac{M_{0,\lambda}^2}{\omega_{0,\lambda}} \to \epsilon_{\sigma \mathbf{k}, \sigma}, \quad (16)$$

and iii) discarded a time-dependent quantity which involves no operators and, hence, can be gauged away via a common time-dependent phase factor for all wave functions, giving no contribution to the calculations of observables.

**III. NONEQUILIBRIUM SUPERCONDUCTIVITY IN THE PATH-INTEGRAL FORMALISM**

**A. Partition function and action**

Rather than solving numerically\textsuperscript{32} the EOMs for the nonequilibrium Green’s functions (GFs) for a chosen time-dependent external field, we here develop a semi-analytical approach that allows us to derive an easily-solvable equation for the time-dependent gap parameter.

In order to do so, we need to make some simplifying assumptions, without loosing certain essential nonequilibrium features. We use the nonequilibrium path-integral formalism on the Kostantinov-Perel’ time contour, which enables us to choose initial states of arbitrary nature, to integrate away the phononic degrees of freedom. While nonequilibrium path integrals on the Schwinger-Keldysh contour are thoroughly discussed in Ref. 13, their version on the Kostantinov-Perel’ time contour has not been studied with the same level of rigor. All necessary technical details can, however, be found in Ref. 38. We here use their results.

We assume that at the initial time $t = t_0$ the system is described by a known state or statistical mixture, specified by the density matrix operator $\hat{\rho}_0$. (This should not be confused with the operator introduced in Eq. (15).) The physical time domain is $t \in [t_0, \infty)$. The Kostantinov-Perel’ time contour $\gamma$ is then given by the union of three branches: $\gamma = \gamma_+ \cup \gamma_- \cup \gamma_M$. The forward ($\gamma_+$) and backward ($\gamma_-$) branches result from doubling the real time degrees of freedom along $[t_0, \infty)$. For a given physical time value $t$, we denote by the symbols $t_+$ and $t_-$ the two corresponding contour variables.
on $\gamma_+$ and $\gamma_-$, respectively. The Matsubara branch ($\gamma_M$) is the imaginary-time domain needed to describe the initial statistical mixture, $[t_0, t_0 - i\beta]$, where $k_B\beta$ is the initial inverse temperature ($h = 1$ throughout this Article). The density matrix at $t = t_0$ is written as 

$$\hat{\rho}_0 = \hat{U}_{\gamma_M} / \text{Tr} \left( \hat{U}_{\gamma_M} \right)$$

where $\hat{U}_{\gamma_M}$ is the evolution operator along $\gamma_M$. In the path-integral formalism, the nonequilibrium partition function is expressed in terms of an action on the contour $\gamma$, i.e.,

$$Z[V] \equiv \frac{\text{Tr} \left\{ U^{(V)}_\gamma \right\}}{\text{Tr} \left( \hat{U}_{\gamma_M} \right)}$$

$$\equiv \frac{1}{\text{Tr} \left( \hat{U}_{\gamma_M} \right)} \int \mathcal{D}(\bar{d}, d) \int \mathcal{D}(a^*, a) e^{iS^{(V)}[\bar{d}, d; a^*, a]} , \quad (17)$$

where we have included a fermionic source potential $\bar{V}(z)$, which depends on the contour variable $z$. If $\bar{V}(t_+) = \bar{V}(t_-)$, then $Z = 1$.

The nonequilibrium action is

$$S^{(V)}[\bar{d}, d; a^*, a] \equiv S^{(V)}_e[\bar{d}, d] + S_{\text{fp}}[\bar{d}, d; a^*, a] , \quad (18)$$

where

$$S^{(V)}_e[\bar{d}, d] = \int \gamma d\gamma' \sum_{k, \sigma} \bar{a}_{k, \sigma}(z) \hat{G}^{\text{fp}-1}_{k, \sigma}(z, z') d_{k, \sigma}(z')$$

$$- \int \gamma d\gamma' [\bar{M}_{q, \lambda} \rho_q(z) + F_{q, \lambda}(z)] a_{q, \lambda}(z)$$

$$- \int \gamma d\gamma' [\bar{M}_{q, \lambda} \rho_q(z) + F_{q, \lambda}(z)] a^*_q, \lambda(z) , \quad (19)$$

involves only electronic fields, while $S_{\text{fp}}[\bar{d}, d; a^*, a]$ involves the phonon fields and their coupling to the electronic fields, i.e., the contributions from the Hamiltonian terms given by Eqs. (12), (13) and (14). It can be separated as

$$S_{\text{fp}}[\bar{d}, d; a^*, a] = \sum_{q, \lambda} S_{\text{fp}, q, \lambda}[\bar{d}, d; a^*, a] , \quad (20)$$

where

$$S_{\text{fp}, q, \lambda}[\bar{d}, d; a^*, a]$$

$$= \int \gamma d\gamma' a^*_{q, \lambda}(z) \hat{G}^{\text{fp}-1}_{q, \lambda}(z, z') a_{q, \lambda}(z')$$

$$- \int \gamma d\gamma' [M_{q, \lambda} \rho_q(z) + F_{q, \lambda}(z)] a_{q, \lambda}(z)$$

$$- \int \gamma d\gamma' [M_{q, \lambda} \rho_q(z) + F_{q, \lambda}(z)] a^*_q, \lambda(z) . \quad (21)$$

The operators $\hat{G}^{\text{fp}-1}_{q, \lambda}(z, z')$ and $G^{\text{fp}-1}_{q, \lambda}(z, z')$ appearing in Eqs. (19) and (21) are the inverse free-electron (fe) and free-phonon (fp) GFs, respectively, defined on the contour $\gamma$. Their features are discussed in full generality in Ref. 38. Note that, in the case at hand, they are diagonal in the single-particle quantum labels. In Eq. (21) we have also used the symbol $\rho_q(z)$ to denote the Grassmann representation of the density operator given by Eq. (15), i.e.

$$\rho_q(z) = \sum_{k, \sigma} \bar{d}_{q+k, \sigma}(z) d_{k, \sigma}(z) . \quad (22)$$

### B. Effective electronic action

Since the action (21) is quadratic in the phonon fields, we can integrate them away. We can simplify Eq. (17) by carrying out explicitly the following integral

$$\int \mathcal{D}(a^*, a) e^{iS_{\text{ext}}[\bar{d}, d; a^*, a]}$$

$$= \prod_{q, \lambda} \int \mathcal{D}(a^*_{q, \lambda}, a_{q, \lambda}) e^{iS_{\text{ext}, q, \lambda}[\bar{d}, d; a^*, a]} . \quad (23)$$

The Gaussian functional integral required in Eq. (23) is reported in Appendix A. Using this result, Eq. (17) reduces to

$$Z[V] = \frac{1}{\text{Tr} \left( \hat{U}_{\gamma_M}^{\text{eff}} \right)} \int \mathcal{D}(\bar{d}, d) e^{iS_{\text{eff}}^{(V)}[\bar{d}, d]} , \quad (24)$$

where we have introduced the following effective electronic action:

$$S_{\text{eff}}^{(V)}[\bar{d}, d] = S^{(V)}_e[\bar{d}, d] + S_{\text{int}}[\bar{d}, d] + S_{\text{ext}}[\bar{d}, d] . \quad (25)$$

This consists of three terms: the electronic action $S^{(V)}_e$ given by Eq. (19) and two terms coming from the bosonic integration (as detailed in Appendix A), i.e. the effective electron-electron interaction $S_{\text{int}}[\bar{d}, d]$ and the phonon-mediated coupling between electrons and the external field $S_{\text{ext}}[\bar{d}, d]$. The last two contributions are expressed in terms of the direct free-phonon GF, which inverts the operator $\hat{G}^{\text{fp}-1}_{q, \lambda}(z, z')$ on the contour $\gamma$, and is given by

$$G^{\text{fp}}_{q, \lambda}(z, z') = -i e^{-i\omega_{q, \lambda}(t-t')} \left[ \Theta(z, z') + n^{(B)}_{q, \lambda} \right] . \quad (26)$$

Here, $t$ and $t'$ are complex time coordinates corresponding to the contour coordinates $z$ and $z'$, respectively, $\Theta(z, z')$ is the step function on $\gamma$, with $\Theta(z, z) = 1$, and

$$n^{(B)}_{q, \lambda} = \frac{1}{e^{\beta\omega_{q, \lambda}} - 1} \quad (27)$$

is the bosonic occupation number.

The two phonon-mediated contributions to the effective action in Eq. (25) are

$$S_{\text{int}}[\bar{d}, d] = - \sum_{q, \lambda} |M_{q, \lambda}|^2 \int d\gamma d\gamma' \rho_q(z) G^{\text{fp}}_{q, \lambda}(z, z')$$

$$\times \rho_{-q}(z') \quad (28)$$
and

\[ S_{\text{ext}}[\tilde{d},d] = -\sum_q \int_\gamma dz f_q(z) \rho_q(z) , \]  

(29)

where

\[ f_q(z) \equiv \sum_\lambda M_{q,\lambda} \int_\gamma dz' \left[ G_{q,\lambda}^{\text{fp}}(z, z') + G_{-q,\lambda}^{\text{fp}}(z', z) \right] \times F_{-q,\lambda}(z') . \]  

(30)

In Eq. (24), we have also introduced the effective evolution operator \( U_{\gamma_M}^{\text{eff}} \) along \( \gamma_M \) for the electrons only, which originates from the bosonic integration in the path-integral representation of the quantity \( \text{Tr}(U_{\gamma_M}) \). Details are given in Appendix A. In our main derivation, we will not need its explicit expression. It is enough to recall that it is a constant that ensures the normalization \( Z[V = 0] = 1 \).

The effective time-dependent (retarded) interaction between the electrons is given by the coefficient which couples the real-time densities \( \rho_q(t) \) and \( \rho_{-q}(t') \), i.e.

\[ V_q(t - t') \equiv -2 \sum_\lambda \left| M_{q,\lambda} \right|^2 \int_{t_0}^{\infty} dt dt' \Theta(t' - t) \sin \left[ \omega_{q,\lambda}(t - t') \right] \rho_{q}(t) \rho_{-q}(t') \]

\[ + i \sum_\lambda \left| M_{q,\lambda} \right|^2 (2n_{q,\lambda} + 1) \int_{t_0}^{\infty} dt e^{-i\omega_{q,\lambda} t} \rho_{q}(t) \int_{t_0}^{\infty} dt' e^{i\omega_{q,\lambda} t'} \rho_{-q}(t') \]

\[ + \sqrt{2} \sum_\lambda \left| M_{q,\lambda} \right|^2 \int_{t_0}^{\infty} d\tau \int_{t_0}^{\infty} dt \left[ \left( 1 + n_{q,\lambda} \right) e^{-i\omega_{q,\lambda}(t_0 - \tau) - t} + n_{q,\lambda} e^{-i\omega_{q,\lambda}(t_0 + \tau) - t} \right] \rho_{M}(\tau) \rho_{-q}(t) \]

\[ - i \sum_\lambda \left| M_{q,\lambda} \right|^2 \int_{t_0}^{\infty} d\tau d\tau' e^{i\omega_{q,\lambda}(\tau' - \tau)} \left[ \Theta(\tau - \tau') + n_{q,\lambda} \right] \rho_{M}(\tau) \rho_{-q}(\tau') . \]  

(32)

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\[ V_q(t - t') \equiv -2 \sum_\lambda \left| M_{q,\lambda} \right|^2 \Theta(t' - t) \sin \left[ \omega_{q,\lambda}(t - t') \right] . \]  

(33)

Its Fourier transform is

\[ V_q(\omega) = -2 \sum_\lambda \left| M_{q,\lambda} \right|^2 \lim_{\zeta \to 0^+} \frac{\omega_{q,\lambda}}{(\omega - i\zeta)^2 - \omega_{q,\lambda}^2} , \]  

(34)

which correctly reproduces the BCS retarded interaction (see e.g. Ref. 41, where \( g_{q,\lambda} \equiv \sqrt{2\omega_{q,\lambda} M_{q,\lambda}} \)).

### D. Effective action for the coupling with the external field

The action in Eq. (29) expresses the indirect effect of the external field on the electron subsystem, which acts directly on the phonons. We now discuss the additional terms of the effective electronic action.

#### C. Effective electron-electron interactions

The effective action given in Eq. (28) describes a phonon-mediated interaction between electrons. To show the correspondence with the well-known BCS retarded interaction, we use Eq. (26) and transform \( \rho_q(z) \) (as given in Eq. (22), specified to the cases \( z = t_+ \), \( z = t_- \) and \( z = t_0 - i\tau \) via the Keldysh rotation, i.e.

\[ \rho_q(t_\pm) \equiv \frac{\rho_q^C(t) \pm \rho_q^Q(t)}{\sqrt{2}} , \quad \rho_q(t_0 - i\tau) \equiv \rho_q^M(\tau) . \]  

(31)

Eq. (31) defines the classical [\( \rho_q^C(t) \)], quantum [\( \rho_q^Q(t) \)], and Matsubara [\( \rho_q^M(\tau) \)] components of the electronic density operator expressed in Grassmann variables. We obtain

\[ F_{q,\lambda}(t_+) = F_{q,\lambda}(t_-) = F_{q,\lambda}(t) , \quad F_{q,\lambda}(t - i\tau) = 0 , \]

(35)

we obtain

\[ f_q(z) = -i \int_\lambda M_{q,\lambda} \mu \int_{t_0}^{\infty} dt' \left\{ e^{-i\omega_{q,\lambda}(t - t') \Theta(z, t')} \right. \]

\[ + e^{i\omega_{q,\lambda}(t - t') \Theta(t'_+, z)} \left. \right\} F_{-q,\lambda}(t') . \]  

(36)

If \( z \) belongs to any of the real-time branches, i.e. \( z = t_+ \) or \( z = t_- \), we obtain

\[ f_q(t_+) = f_q(t_-) \equiv f_q(t) \]

\[ = -2 \sum_\lambda M_{q,\lambda} \int_{-\infty}^t \left( \sin \left[ \omega_{q,\lambda}(t - t') \right] F_{-q,\lambda}(t') \right) . \]  

(37)
On the other hand, if \( z \) belongs to the Matsubara branch,
\[
f_q(t_0 - i\tau) = 0 . \tag{38}
\]
The action term (29) then becomes
\[
S_{\text{ext}}[\overrightarrow{d}, \overrightarrow{d}] = -\sqrt{2} \sum_q \int_{t_0}^\infty dt f_q(t) \rho_q^2(t) . \tag{39}
\]
This action has the correct causal structure in the sense that the field \( F_{q,\lambda}(t') \) has an effect on \( \rho_q^2(t) \) only if \( t > t' \). This \textit{retarded} effect on the electron subsystem is qualitatively different from the instantaneous effect that would be obtained by coupling a field directly to the electron rather than the phonon subsystem.

E. Cooper parameters and fermionic sources

We now focus on the Cooper parameters, which are, in general, matrices in wave vector space:
\[
\begin{align*}
C(t)_{k,k'} &\equiv \langle \hat{c}_{-k,\uparrow}(t) \hat{c}_{k',\downarrow}(t) \rangle = \langle \hat{d}^\dagger_{k,\downarrow}(t) \hat{d}_{k',\uparrow}(t) \rangle , \\
C^\dagger(t)_{k,k'} &\equiv \langle \hat{c}_{k,\uparrow}(t) \hat{c}^\dagger_{-k',\downarrow}(t) \rangle = \langle \hat{d}_{k,\uparrow}(t) \hat{d}^\dagger_{k',\downarrow}(t) \rangle . \tag{40}
\end{align*}
\]
These quantities can be calculated by functional differentiating the partition function. To this end, we need to include in the action (19) the following source term:
\[
\int_\gamma dV [\overrightarrow{d}(z), \overrightarrow{d}(z); z]
\]
\[
= \int_\gamma dV \sum_{k,k'} \sum_{\sigma=\uparrow,\downarrow} V_{k,k'}^{\sigma,\sigma'}(z) \overrightarrow{d}_{k,\sigma}(z) \overrightarrow{d}_{k',-\sigma}(z) , \tag{41}
\]
and classical (C) and quantum (Q) components of the source potentials:
\[
V_{k,k'}^{\sigma,\sigma'}(t) = \frac{V_{k,k'}^{\sigma,\sigma}(t_+ + V_{k,k'}^{\sigma,\sigma}(t_-)}{2} . \tag{42}
\]
The Cooper parameters are obtained as
\[
C(t)_{k,k'} = i \left. \frac{\delta Z[V]}{\delta V_{k,k'}^{\uparrow,\downarrow}(t)} \right|_{V=0} = \frac{i}{2} \left\{ \left. \frac{\delta Z[V]}{\delta V_{k,k'}^{\uparrow,\downarrow}(t_+)} + \frac{\delta Z[V]}{\delta V_{k,k'}^{\uparrow,\downarrow}(t_-)} \right\} \right|_{V=0} , \tag{43}
\]
and \( C^\dagger(t)_{k,k'} \) has an analogous expression, except for the replacements \( \uparrow \leftrightarrow \downarrow \) and \( \downarrow \leftrightarrow \uparrow \). The fact that \( C(t)_{k,k'} \) is expressed in terms of sums of two terms in the last equality in Eq. (43) is understood by noticing that
\[
\delta X = \int_\gamma dV \frac{\delta X}{\delta V(z)} \delta V(z)
\]
\[
= \int_0^\infty dt \left[ \frac{\delta X}{\delta V(t_+)} \delta V(t_+) + \frac{\delta X}{\delta V(t_-)} \delta V(t_-) \right] + \int_\gamma dV \frac{\delta X}{\delta V(z)}
\]
\[
= \frac{1}{2} \int_0^\infty dt \left\{ \left[ \frac{\delta X}{\delta V(t_+)} - \frac{\delta X}{\delta V(t_-)} \right] \delta V(t) + \left[ \frac{\delta X}{\delta V(t_+)} + \frac{\delta X}{\delta V(t_-)} \right] \delta V(t) \right\} + \int_\gamma dV \frac{\delta X}{\delta V(z)}
\]
\[
= \frac{\delta X}{\delta V(t)} = \frac{1}{2} \left[ \frac{\delta X}{\delta V(t_+)} + \frac{\delta X}{\delta V(t_-)} \right] . \tag{44}
\]
Calculations of the Cooper parameters require an explicit expression for \( Z \), which we now proceed to derive within the well-known “Hubbard-BCS” approximation.

IV. HUBBARD-BCS APPROXIMATION

A. Hubbard-Stratonovich decoupling

To simplify the partition function, we need to perform the integral over the fermionic fields, which requires to decouple the interaction term by introducing suitable (complex) bosonic fields$^{13,40,43}$.$^4$ The simplest possibility is to adopt the local approximation$^{13}$ on the effective electron-electron interaction appearing in Eq. (28), i.e.
\[
- \sum_\lambda |M_{q,\lambda}|^2 G_{q,\lambda}^{\text{fp}}(z, z') \rightarrow - \frac{U}{2} \delta(z - z') , \tag{45}
\]
for all values of \( q \), where \( U \) is a Hubbard-like parameter expressing the effective strength of the interaction, which is local in space. The corresponding action term becomes
\[
S_{\text{int}}[\overrightarrow{d}, \overrightarrow{d}] \rightarrow S_{\text{U}}[\overrightarrow{d}, \overrightarrow{d}] = -U \sum_q \int_\gamma dV \overline{\Phi}_q(z) \Phi_q(z) , \tag{46}
\]
where we have introduced the Cooper pair fields
\[
\overline{\Phi}_q(z) \equiv \sum_k \overline{d}_{k,\uparrow}(z) d_{k+q,\downarrow}(z), \\
\Phi_q(z) \equiv \sum_k d_{k+q,\downarrow}(z) d_{k,\uparrow}(z) . \tag{47}
\]
We then perform the Hubbard-Stratonovich decoupling, based on the following exact identity:

$$
\exp \left\{ -i \int d\gamma \sum_q \bar{\Phi}_q(z) \Phi_q(z) \right\} = c \int \mathcal{D} \left[ \frac{\Delta}{U}, \frac{\Delta^*}{U} \right] \exp \left\{ i \sum_q \int d\gamma \left[ \Phi_q(z) \Delta_q(z) + \bar{\Phi}_q(z) \Delta_q^*(z) + \frac{1}{U} |\Delta_q(z)|^2 \right] \right\}, \tag{48}
$$

where the constant $c$ includes the integration measure. The partition function in the Hubbard-BCS approximation then becomes

$$
Z[V] \approx \frac{c}{\text{Tr} \left( \hat{U}_{\text{eff}}^\dagger \right)} \int \mathcal{D} \left[ \frac{\Delta}{U}, \frac{\Delta^*}{U} \right] \int \mathcal{D} \left[ \bar{d}, d \right] \times \exp \left\{ i \int d\gamma \sum_{kk'} \left[ \delta(z, z') \mathcal{G}^{-1}_{kk';zz'}(z) \right] \right\}, \tag{49}
$$

In Eq. (49) we have introduced the inverse BCS electronic GF on $\gamma$ (in the presence of sources):

$$
\mathcal{G}^{-1}_{kk';zz'}(z, z') = \begin{pmatrix}
\delta(z, z') - \delta(z, z') f_{k-k'}(z) & \delta(z, z') \left[ \Delta_{k-k'}(z) - V_{k-k'}^\dagger(z) \right] \\
\delta(z, z') \left[ \Delta_{k-k'}(z) - V_{k-k'}^\dagger(z) \right]^{-1} & \delta(z, z') + \delta(z, z') f_{k-k'}(z)
\end{pmatrix}. \tag{50}
$$

This is a matrix in the spaces of wave vectors, spins, and contour coordinates; the matrix on the right-hand side of Eq. (50) is written explicitly in spin space. In what follows, its individual elements will be denoted by $\mathcal{G}^{-1}_{kk';zz'}$.

The Dirac deltas in Eq. (50) should be interpreted as $\delta(z, z') \rightarrow \delta(z, z' + 0)$, i.e. one must account for an infinitesimal shift (along the contour) between the time arguments of the $\bar{d}$ and $d$ Grassmann fields appearing in Eq. (49).

### B. Fermionic Gaussian integration

We proceed by integrating away the Grassmann variables appearing in Eq. (49). To this aim, we use the general formula for a discrete-time action

$$
\int \mathcal{D} \left[ \bar{d}, d \right] \exp \left\{ -\frac{1}{2} \bar{d} \cdot X \cdot d \right\} = \det X = e^{\text{tr} \left( \ln \mathbf{X} \right)}. \tag{51}
$$

Performing the fermionic Gaussian integration, we obtain the following expression for the Hubbard-BCS partition function $Z[V]$:

$$
Z[V] = \frac{c}{\text{Tr} \left( \hat{U}_{\text{eff}}^\dagger \right)} \int \mathcal{D} \left[ \frac{\Delta}{U}, \frac{\Delta^*}{U} \right] \text{exp} \left\{ i S_{\text{BCS}}^{(V, \Delta)} \right\}, \tag{52}
$$

where we have introduced the effective action

$$
i S_{\text{BCS}}^{(V, \Delta)} = \text{Tr} \left[ \ln \left( -i G^{-1} \left( V, \Delta \right) \right) \right] + \frac{i}{U} \int d\gamma \sum_q |\Delta_q(z)|^2. \tag{53}
$$

Here, $G^{-1}$ is the discrete-time version of the operator $G^{-1}$. It is a matrix in the contour coordinates, whose elements we denote as $G^{-1}_{zz'}$. If $G_{zz'}$ is the direct GF on the contour, we have the following properties: i) if the time coordinates are taken on a discrete grid $\Gamma$, then

$$
\sum_{z'' \in \Gamma} G^{-1}_{zz''} G_{z', z''} = \delta_{zz'}, \tag{54}
$$

where $\delta_{zz''}$ is the Kronecker delta; ii) if the contour coordinates are taken on the continuous contour $\gamma$, then

$$
\int_{\gamma} dz' G^{-1}_{zz'} G_{z', z''} = \delta(z, z''), \tag{55}
$$

where $\delta(z, z'')$ is the Dirac delta on the contour.

Using functional differentiation, we now calculate 1) the Cooper parameters in Eq. (43) and 2) the values of the nonequilibrium gap parameters at the saddle point of the action. A general note about functional derivatives in this formalism is reported in Appendix B.

### C. Path-integral expressions for the Cooper parameters

We now derive explicit expressions for the nonequilibrium Cooper parameters. Applying Eq. (43) to the
partition function in the form of Eq. (52), we find:

$$C(t)_{k,k'} = -\frac{i}{2} \frac{c}{\text{Tr} \left( U^T \right)} \int D \left[ \frac{\Delta}{U} \right] \exp \left\{ i S_{\text{BCS}}^{(0, \Delta)} \right\} \times \left\{ G^{(0, \Delta)}_{k', \uparrow, t_+; k, \downarrow, (t+0)_+} + G^{(0, \Delta)}_{k', \uparrow, t_-; k, \downarrow, (t-0)_-} \right\},$$

where $t \pm 0$ denotes an instant of time infinitesimally after/before $t$. Therefore, the coordinate $(t+0)_+$ is reached infinitesimally later than $t_+$ while walking on the forward branch of the contour, and $(t-0)_-$ is reached infinitesimally later than $t_-$ while walking on the backward branch. Eq. (56) is exact within the Hubbard-BCS approximation, but it cannot be evaluated without resorting to further approximations.

D. Nonequilibrium saddle point

In order to compute Eq. (56) and its Hermitian conjugate, we use the saddle-point (SP) approximation, in full analogy to what is done in the framework of non-equilibrium many-body theory on the Schwinger-Keldysh contour and standard field-theoretical procedures at equilibrium.

The nonequilibrium saddle points of the action are obtained by finding extrema of Eq. (53) with respect to variations in the fields $\Delta_q(z)$ and $\Delta_q^*(z)$. Recalling that Eq. (56) requires $V = 0$, we write down the SP equations:

$$\frac{\delta i S_{\text{BCS}}^{(0, \Delta)}}{\delta \Delta_q^{(z)}} = 0, \quad \frac{\delta i S_{\text{BCS}}^{(0, \Delta)}}{\delta \Delta_q^{*(z)}} = 0. \quad (57)$$

Performing the functional derivatives (see Appendix B) we find

$$\Delta_q(z) = iU \sum_k \left\{ c^{(0, \Delta)}_{k, \uparrow, t_+; k+q, \downarrow, z+0} \right\}, \quad \Delta_q^*(z) = iU \sum_k \left\{ c^{(0, \Delta)}_{k, \uparrow, t_-; k-q, \downarrow, z+0} \right\}. \quad (58)$$

where $z + 0$ is a contour coordinate occurring infinitesimally later than $z$, for any $z \in \gamma$. In particular, on the Matsubara branch, i.e. for $z = t_0 - i\tau$,

$$\Delta_q^{(M)}(t_0 - i\tau) \overset{\text{SP}}{=} iU \sum_k \left\{ c^{(0, \Delta)}_{k, \uparrow, t_0 - i\tau; k+q, \downarrow, t_0 - i(\tau + 0)} \right\},$$

$$\Delta_q^{(M)}(t_0 - i\tau)^* \overset{\text{SP}}{=} iU \sum_k \left\{ c^{(0, \Delta)}_{k, \uparrow, t_0 - i\tau; k-q, \uparrow, t_0 - i(\tau + 0)} \right\}. \quad (59)$$

For $z$ belonging to one of the real-time branches ($\gamma_+$ or $\gamma_-$), we introduce the classical (C) and quantum (Q) combinations

$$\Delta_q^{C/Q}(t) \overset{\text{SP}}{=} \frac{\Delta_q(t_+)}{2} \pm \frac{\Delta_q(t_-)}{2},$$

$$\Delta_q^{C/Q}(t)^* \overset{\text{SP}}{=} iU \sum_k \left\{ c^{(0, \Delta)}_{k, \uparrow, t_0 - i\tau; k+q, \downarrow, (t+0)_+} \pm c^{(0, \Delta)}_{k, \downarrow, t_0 - i\tau; k+q, \uparrow, (t-0)_-} \right\}. \quad (60)$$

From Eq. (50) we see that, if $\Delta_q(t_+) = \Delta_q(t_-) \equiv \Delta_q^{(M)}(t)$ at all times $t$ and $\Delta_q^{(M)}(t_0 - i\tau) = \Delta_q^{(M)}(t_0)$ for all values of $\tau$, then $G^{(0, \Delta)}$ becomes a standard nonequilibrium GF corresponding to the following effective mean-field (MF) Hamiltonian:

$$\hat{H}_{\text{MF}}(t) \equiv \sum_{k,k'} \left\{ \delta_{k,k'} \hat{c}_{k,\uparrow}^\dagger \hat{c}_{k',\downarrow} \right\} \left( \delta_{k,k'} \epsilon_{k,\uparrow} + f_{k-k'}(t) - \Delta_{k-k'}^{(M)}(t) \right) \left( \delta_{k,k'} \epsilon_{k,\downarrow} - f_{k-k'}(t) \right) \left( \hat{c}_{k',\uparrow}^\dagger \right) \left( \hat{c}_{k,\downarrow} \right). \quad (62)$$

Here, $\Delta_q^{(M)}(t)$ acts as a classical time-dependent field (since the values of the field are the same, at a given $t$, on both branches of the real-time part of the KP contour). In this situation, one has

$$G^\ast_{t_+; (t+0)_+} = G^\ast_{t_-; (t-0)_-} \equiv G^\ast_{t,t}, \quad (63)$$
The solution of Eq. (53) functionally on $\Delta(\mathbf{q},t)$ where it is intended that the right-hand side depends on the saddle points of the BCS action, i.e.,

$$\Delta_q^{(M)}(t_0 - i\tau) \equiv \Delta_q^{(M)(t_0)}.$$  

(Eq. 61) should then be solved under the conditions (64) to determine the MF values $\Delta_q^{(M)}(t)$ corresponding to the saddle points of the BCS action, i.e.,

$$\Delta_q^{(M)}(t) \equiv iU \sum_k G_{k,q}^{<} \left| t, \uparrow, t, \downarrow, t \right|.$$  

(Eq. 65) The solution of Eq. (65) automatically satisfies the analogous equation for $\Delta_q^{(M)^*}(t)$.

In the SP approximation the Cooper-parameter matrix in Eq. (56) and its Hermitian conjugate are given by

$$C(t)_{k,k'}^{\text{SP}} \equiv -iG_{k,q}^{<} \left| t, \uparrow, t, \downarrow, t \right|,$$

$$C^\dagger(t)_{k,k'}^{\text{SP}} \equiv -iG_{k,q}^{<} \left| t, \uparrow, t, \downarrow, t \right|.$$  

(Eq. 66) Corrections to Eqs. (66) beyond the SP approximation can be included by considering Gaussian fluctuations of the action around the mean-field point(s). For the application of this procedure to a nonequilibrium problem, we refer the reader to Ref. 47. This is beyond the scope of the present Article.

V. THE CASE OF A SPATIALLY-UNIFORM TIME-DEPENDENT EXTERNAL FIELD

We now consider the case of a uniform time-dependent external field,

$$f_q(t) = \delta_q \alpha f(t),$$  

(Eq. 67) and look for spatially uniform solutions of the MF equations, i.e.,

$$\Delta_q^{(M)}(t) = \delta_q \alpha \Delta(t).$$  

(Eq. 68) We also assume no magnetic fields acting on the electron subsystem, as well as spatial inversion symmetry, so that $\epsilon_{+k,\uparrow} = \epsilon_k$ and $\epsilon_{-k,\downarrow} = \epsilon_k$.

In this case, the MF Hamiltonian (62) simplifies to (suppressing in what follows all “MF” labels)

$$\hat{H}(t) = \sum_k \left( \hat{d}_{k,\uparrow} \hat{d}_{k,\downarrow}^\dagger + \hat{d}_{k,\downarrow} \hat{d}_{k,\uparrow}^\dagger \right) \left( \epsilon_k + f(t) + \Delta(t) \right) - \left( \epsilon_k - f(t) + \Delta^*(t) \right) \left( \hat{d}_{k,\uparrow}^\dagger \right) \left( \hat{d}_{k,\downarrow} \right).$$

(Eq. 69) Eq. (65) becomes

$$\Delta(t) = -U \sum_k \sum_{\Psi_0} W_{\Psi_0} \langle \Psi(t) | \hat{d}_{k,\downarrow} \hat{d}_{k,\uparrow}^\dagger | \Psi(t) \rangle,$$  

(Eq. 70) where it is intended that the right-hand side depends functionally on $\Delta(t)$ and $\Delta^*(t)$. The quantities $W_{\Psi_0}$ in the second line of Eq. (70) represent the statistical weights of the states $|\Psi_0\rangle$, which are eigenstates of $\hat{H}(t = t_0)$. The time-dependent MF states satisfy the time-dependent Schrödinger equation,

$$i \partial_t |\Psi(t)\rangle = \hat{H}(t) |\Psi(t)\rangle, \quad |\Psi(t)\rangle = |\Psi_0\rangle.$$  

(Eq. 71) The explicit evaluation of $\Delta(t)$ requires an approximate solution of Eq. (71). To make analytical progress, we limit our attention to slowly-varying external fields. This suggests the application of the adiabatic theorem of quantum mechanics. However, as detailed in Sect. VI A, the strict application of the adiabatic theorem generates an inconsistency. On the one hand, it yields a self-consistent equation for $\Delta(t)$. On the other hand, the evaluation of the time derivative $\Delta(t)$ via another independent equation yields a vanishing result under the assumptions of the adiabatic theorem. This means that, for the problem at hand, the adiabatic theorem yields a meaningful result only at equilibrium and is therefore useless for our scope.

We now proceed to present a derivation of a time-dependent gap equation in the quasi-adiabatic limit, which bypasses the limitations of the adiabatic theorem.

A. Adiabatic Perturbation Theory

The appropriate tool to deal with systems close to the adiabatic regime is Adiabatic Perturbation Theory (APT). We here summarize the main results of Ref. 35.

We rescale the time coordinate $t$ according to $s = t/T$, where $T$ is the time scale over which the system is under observation. In the adiabatic regime, the external field (and, therefore, the full Hamiltonian) is assumed to vary slowly on the scale $T$. The time-dependent Schrödinger equation becomes

$$i \frac{\partial}{\partial s} |\Psi(s)\rangle = \hat{H}(s) |\Psi(s)\rangle.$$  

(Eq. 72) We introduce the instantaneous eigenstates $|n(s)\rangle$ of the Hamiltonian:

$$\hat{H}(s) |n(s)\rangle = \mathcal{E}_n(s) |n(s)\rangle,$$

$$\langle n(s)|n'(s)\rangle = \delta_{n,n'},$$  

(Eq. 73) for all $s$, where $n$ is the set of quantum numbers specifying $|n(s)\rangle$.

For every $s$, we expand the exact states $|\Psi(s)\rangle$ solving Eq. (72) on the basis of the complete set of states $|n(s)\rangle$. Without loss of generality, such expansion can be written as

$$|\Psi(s)\rangle \equiv \sum_n b_{\Psi,n}(s) e^{i[\gamma_n(s) - \omega_n(s)T]} |n(s)\rangle.$$  

(Eq. 74) In Eq. (74), we have introduced the geometrical phase factor

$$\gamma_n(s) = i \int_{s_0}^{s} ds' \langle n(s')|\partial_{s'}|n(s')\rangle.$$  

(Eq. 75)
and the dynamical phase factor
\[ \omega_n(s) = \int_{s_0}^{s} ds' \mathcal{E}_n(s') \, . \] (76)

The coefficients \( b_{\Psi, n}(s) \) can be determined by inserting Eq. (74) into the time-dependent Schrödinger equation. We obtain the following differential equation
\[ \partial_s b_{\Psi, n}(s) = - \sum_{m \neq n} e^{i[\gamma_{m, n}(s) - \omega_{m, n}(s) T]} M_{m, n}(s) b_{\Psi, m}(s) \, , \] (77)

where
\[ \gamma_{m, n}(s) \equiv \gamma_m(s) - \gamma_n(s), \quad \omega_{m, n}(s) \equiv \omega_m(s) - \omega_n(s), \]
\[ M_{m, n}(s) \equiv \langle m(s) | \partial_s \tilde{\mathcal{H}}(s) | n(s) \rangle = -i M^*_{m, n}(s). \] (78)

The initial condition is
\[ b_{\Psi, n}(s_0) = \delta_{\Psi, n}. \] (79)

Up to this point, no approximations were made. Note that the adiabatic theorem applies exactly if \( \partial_s b_{\Psi, n}(s) = 0 \). If the right-hand side of Eq. (77) is “small” (see below), but non-zero, we are in the regime of applicability of APT. This means that the quantities \( M_{m, n}(s) \) must be much smaller than unity. This is the requirement of “slowness” of the external field mentioned in Sect. I. A more rigorous assessment of the validity of APT in our case will be given in Sect. VI B.

Following Ref. 35, we make the following Ansatz:
\[ |\Psi(s)\rangle \equiv \sum_{p=0}^{\infty} T^{-p} |\Psi^{(p)}(s)\rangle \, , \] (80)

where
\[ |\Psi^{(p)}(s)\rangle \equiv \sum_n e^{i[\gamma_n(s) - \omega_n(s) T]} b^{(p)}_{\Psi, n}(s) |n(s)\rangle \] (81)

and
\[ b^{(p)}_{\Psi, n}(s) \equiv \sum_m e^{-i[\gamma_{m, n}(s) - \omega_{m, n}(s) T]} b^{(p)}_{\Psi, m}(s) \, . \] (82)

Note that this is equivalent to setting
\[ b_{\Psi, n}(s) \equiv \sum_{p=0}^{\infty} T^{-p} b^{(p)}_{\Psi, n}(s) \]

\[ = \sum_{p=0}^{\infty} T^{-p} \sum_m e^{-i[\gamma_{m, n}(s) - \omega_{m, n}(s) T]} b^{(p)}_{\Psi, m}(s) \, . \] (83)

After inserting this expansion into the time-dependent Schrödinger equation, one obtains equations for the coefficients \( b^{(p)}_{\Psi, m}(s) \), which can be solved order-by-order.

This is because the equation for \( b^{(p)}_{\Psi, m}(s) \) at any given \( p \) involves only derivatives of these coefficients corresponding to orders \( p' < p \). The initial condition is determined from
\[ |\Psi(s_0)\rangle = |\Psi^{(0)}(s_0)\rangle \, , \] (84)

which follows from Eq. (79), and by requiring that \( |\Psi^{(0)}(s)\rangle \) coincides with the strong adiabatic solution, which is obtained from Eq. (77) with the right-hand side \( = 0 \). One obtains
\[ |\Psi^{(0)}(s)\rangle = e^{i[\gamma_0(s) - \omega_0(s) T]} |\psi_0(s)\rangle \] (85)

and
\[ |\Psi^{(1)}(s)\rangle = i \sum_{n \neq 0} \left\{ e^{i[\gamma_0(s) - \omega_0(s) T]} \frac{M_{n, \Psi_0(s)}}{\mathcal{E}_n(s) - \mathcal{E}_{\Psi_0(s)}} \right\} |n(s)\rangle \]
\[ + i \sum_{n \neq 0} e^{i[\gamma_0(s) - \omega_0(s) T]} J_{n, \Psi_0(s)} |\Psi_0(s)\rangle \, , \] (86)

where \( |\Psi_0(s)\rangle \) is the instantaneous eigenstate of \( \tilde{\mathcal{H}}(s) \) that coincides with \( |\Psi_0(t)\rangle \) at \( s = s_0 \), and
\[ J_{m, n}(s) \equiv \int_{s_0}^{s} ds' \frac{|M_{m, n}(s')|^2}{\mathcal{E}_m(s') - \mathcal{E}_n(s')} \, . \] (87)

We now proceed to calculate Eqs. (85) and (86) for our problem.

B. Instantaneous eigenstates

The diagonalization of the uniform mean-field Hamiltonian \( \tilde{\mathcal{H}}(t) \) in Eq. (69) at each time \( t \) yields
\[ \tilde{\mathcal{H}}(t) = \sum_{k, \alpha=\pm 1} \alpha E_k(t) \hat{D}^\dagger_{k, \alpha}(t) \hat{D}_{k, \alpha}(t) \, , \] (88)

where
\[ E_k(t) \equiv \sqrt{[\epsilon_k + f(t)]^2 + |\Delta(t)|^2} \] (89)

is the gapped spectrum of the instantaneous quasiparticles (IQPs) corresponding to the time-dependent fermionic fields
\[ \hat{D}_{k, \alpha}(t) = a^\dagger_{k, \alpha}(t) \hat{a}_{k, \uparrow} + b^\dagger_{k, \alpha}(t) \hat{a}_{k, \downarrow}, \quad \alpha = \pm 1 \, . \] (90)

In Eq. (90)
\[ a_{k, \alpha}(t) = \frac{-\alpha \Delta(t)}{\sqrt{2E_k(t) [E_k(t) - \alpha |\epsilon_k + f(t)|]}} \, , \]
\[ b_{k, \alpha}(t) = \frac{E_k(t) - \alpha |\epsilon_k + f(t)|}{\sqrt{2E_k(t) [E_k(t) - \alpha |\epsilon_k + f(t)|]}} \, . \] (91)
At all times, it holds that
\[ a_{k,\alpha}(t) a_{k',\alpha'}(t) + b_{k',\alpha'}^*(t) b_{k,\alpha}(t) = \delta_{\alpha,\alpha'}, \tag{92} \]
and the inverse of Eq. (90) is
\[ \hat{d}_{k,\uparrow} = \sum_{\alpha} a_{k,\alpha}(t) \hat{D}_{k,\alpha}(t), \quad \hat{d}_{k,\downarrow} = \sum_{\alpha} b_{k,\alpha}(t) \hat{D}_{k,\alpha}(t). \tag{93} \]
The instantaneous eigenstates of \( \hat{H}(t) \) are then
\[ |n(t)\rangle = \prod_{k,\alpha} \left[ \hat{D}_{k,\alpha}(t) \right]^{n_{k,\alpha}} |0_D\rangle, \tag{94} \]
where the occupation numbers \( n_{k,\alpha} \) is 1 or 0, and \( |0_D\rangle \) is the vacuum of \( \hat{D} \) operators, i.e.
\[ \hat{D}_{k,\alpha}(t) |0_D\rangle = 0 \tag{95} \]
for all values of \( k, \alpha, \) and \( s \). Note that \( |0_D\rangle \) is also the vacuum of the \( \hat{d} \) operators: as such, it is independent of time. The instantaneous energy eigenvalues are
\[ E_{\alpha}(t) = \sum_{k,\alpha} \alpha n_{k,\alpha} E_k(t). \tag{96} \]

C. Time derivatives of the quasiparticle fields

The time derivatives of the IQP fields are expressed in terms of the IQP fields themselves as
\[ \partial_s \hat{D}_{k,\alpha}(s) = \sum_{\alpha'} A_{k}(s)_{\alpha,\alpha'} \hat{D}_{k,\alpha'}(s), \]
\[ \partial_s \hat{D}_{k,\alpha}^\dagger(s) = \sum_{\alpha'} A_{k}^\dagger(s)_{\alpha,\alpha'} \hat{D}_{k,\alpha'}^\dagger(s), \tag{97} \]
where
\[ A_{k}(s)_{\alpha,\alpha'} = \left[ \partial_s a_{k,\alpha}(s) a_{k,\alpha'}(s) + \partial_s b_{k,\alpha'}(s) b_{k,\alpha}(s) \right]. \tag{98} \]
Because of Eq. (92), one has
\[ A_{k}(s)_{\alpha,\alpha'} = -A_{k}^\dagger(s)_{\alpha',\alpha} . \tag{99} \]
Using Eqs. (91) and parameterizing the complex gap parameter as
\[ \Delta(s) = |\Delta(s)| e^{i\phi(s)} , \tag{100} \]
we obtain
\[ A_{k}(s)_{\alpha,\alpha} = \frac{-i |\Delta(s)|^2 \partial_s \phi(s)}{2E_k(s) \{ E_k(s) - \alpha [\epsilon_k + f(s)] \}} \tag{101} \]
and
\[ A_{k}(s)_{-\alpha,\alpha} = \frac{|\Delta(s)|}{2} \left[ \alpha V_{k}(s) + i W_{k}(s) \right] . \tag{102} \]
Here, the real quantities \( V_k(s) \) and \( W_k(s) \) are given by
\[ V_k(s) = \frac{1}{E_k^2(s)} \left[ \partial_s f(s) - \epsilon_k + f(s) \frac{\partial_s |\Delta(s)|}{|\Delta(s)|} \right] \]
\[ = \frac{\partial_s f(s) - [\epsilon_k + f(s)] \text{Re} \left[ \partial_s \Delta(s)/|\Delta(s)| \right]}{E_k^2(s)} \tag{103} \]
and
\[ W_k(s) = \frac{\partial_s \phi(s)}{E_k(s)} = \frac{\text{Im} \left[ \partial_s \Delta(s)/|\Delta(s)| \right]}{E_k(s)}. \tag{104} \]

D. Geometrical and dynamical phase factors

Using the results in the previous Sections, we can calculate the dynamical and geometrical factors. We find
\[ \omega_{n}(t) = -\sum_{k,\alpha} \alpha n_{k,\alpha} \int_{s_0}^{s} ds' E_k(s') \tag{105} \]
and
\[ \gamma_{n}(s) = i \sum_{k,\alpha} n_{k,\alpha} \int_{s_0}^{s} ds' \langle 0_D | \hat{D}_{k,\alpha}(s') \partial_{s'} \hat{D}_{k,\alpha}^\dagger(s') | 0_D \rangle \]
\[ = i \sum_{k,\alpha} n_{k,\alpha} \int_{s_0}^{s} ds' A_{k}(s')_{\alpha,\alpha} \]
\[ = \sum_{k,\alpha} n_{k,\alpha} \int_{s_0}^{s} ds' - \frac{|\Delta(s')|^2 \partial_{s'} \phi(s')}{2E_k(s') \{ E_k(s') - \alpha [\epsilon_k + f(s')]}]. \tag{106} \]

E. Components of the adiabatic parameter

From Eq. (78) at \( m \neq n \) and using Eq. (88) we have
\[ M_{m,n}(s) = \frac{1}{\bar{E}_m(s) - \bar{E}_n(s)} \sum_{k,\alpha} \alpha \langle n(s) | \left\{ \partial_s E_k(s) \hat{N}_{k,\alpha}(s) \right. \]
\[ + E_k(s) \left[ \partial_s \hat{D}_{k,\alpha}(s) \hat{D}_{k,\alpha}(s) \right] + \hat{D}_{k,\alpha}(s) \partial_s \hat{D}_{k,\alpha}(s) \right\} |m(s)\rangle , \tag{107} \]
where \( \hat{N}_{k,\alpha}(s) = \hat{D}_{k,\alpha}^\dagger(s) \hat{D}_{k,\alpha}(s) \). We now consider the quantity in curly brackets on the right-hand side of Eq. (107). The first term vanishes because
\[ \langle n(s) | \hat{N}_{k,\alpha}(s) |m(s)\rangle = n_{k,\alpha} \delta_{n,m} \tag{108} \]
and we are considering only \( m \neq n \). To calculate the second and third terms, we use Eqs. (97) and Eq. (108). We get
\[ M_{m,n}(s) = \frac{-2}{\bar{E}_m(s) - \bar{E}_n(s)} \sum_{k,\alpha} \alpha E_k(s) A_{k}(s)_{\alpha,\alpha} \]
\[ \times \langle n(s) | \hat{D}_{k,\alpha}^\dagger(s) \hat{D}_{k,\alpha}(s) |m(s)\rangle , \tag{109} \]
where we have used Eq. (99). The bra-ket appearing in the second line of Eq. (109) is zero unless the sets \( n \) and \( m \) are such that \( m_{k', \pm 1} = n_{k', \pm 1} \forall k' \neq k \), while \( m_{k, \alpha} = 1 = n_{k, -\alpha} \) and \( m_{k, -\alpha} = 0 = n_{k, \alpha} \). In this case, the bra-ket is equal to 1. To make the notation compact, when the set \( m \) satisfies these conditions with respect to the set \( n \), we will write that \( m = n[k, \alpha] \). We can then write

\[
\langle n(s)| \hat{D}_{k, -\alpha}(s) \hat{D}_{k, \alpha}(s)| m(s) \rangle = \delta_{n_{k, -\alpha}, 1} \delta_{n_{k, \alpha}, 0} \delta_{m, n[k, \alpha]} .
\]

(110)

Inserting Eq. (110) into Eq. (109), and observing, with the aid of Eq. (96), that \( \mathcal{E}_m(s) - \mathcal{E}_n(s) = 2\alpha E_k(s) \) for all the sets \( n \) and \( m \) such that Eq. (110) equals 1, we finally find:

\[
M_{n,m}(s) = - \sum_{k, \alpha} \delta_{n_{k, -\alpha}, 1} \delta_{n_{k, \alpha}, 0} \delta_{m, n[k, \alpha]} A_k(s) \epsilon_{k, \alpha} .
\]

(111)

Inserting this result into Eq. (87) and using Eq. (102), we get (for \( m \neq n \))

\[
J_{n,m}(s) = - \frac{1}{8} \sum_{k, \alpha} \alpha \delta_{n_{k, -\alpha}, 1} \delta_{n_{k, \alpha}, 0} \delta_{m, n[k, \alpha]} \sum_{s'} ds' \frac{|\Delta(s')|^2}{E_k(s')} \left[ V^2_k(s') + W^2_k(s') \right] .
\]

(112)

**F. APT expansion of the dynamical gap parameter**

After inserting the APT expansion (80) of the time-dependent states into Eq. (70), one directly obtains an expansion of \( \Delta(s) \) having the form

\[
\Delta(s) \equiv \sum_{p=0}^{\infty} T^{-p} \Gamma^{(p)}_{\Delta}(s) .
\]

(113)

with

\[
\Gamma^{(p)}_{\Delta}(s) = - U \sum_k \sum_{\psi_0} W_{\psi_0} \sum_q \langle \psi(q)(s)| \hat{d}^\dagger_{k, \alpha} \hat{d}_{k, \alpha}^\dagger | \psi(q-p)(s) \rangle .
\]

(114)

We observe that \( \Gamma^{(p)}_{\Delta}(s) \) is a functional of \( \Delta(s') \) through the dependence of the states \( |\psi(s)\rangle \) on such quantity. Therefore, \( \Gamma^{(p)}_{\Delta}(s) \) itself has a complicated dependence on all powers \( T^{-q} \), \( q \geq 0 \), as follows from Eq. (113). In turn, this means that truncating the sum in Eq. (113) with respect to \( p \) would be incorrect within the framework of APT. What we need is a perturbative expansion of the dynamical gap parameter of the form

\[
\Delta(s) \equiv \sum_{p=0}^{\infty} T^{-p} \Delta^{(p)}(s) ,
\]

(115)

where the coefficients \( \Delta^{(p)}(s) \) do not depend on powers of \( T^{-1} \). To determine them, we must 1) explicitly derive a sufficiently large set of quantities \( \Gamma^{(p)}_{\Delta}(s) \); 2) insert in these expressions the expansion (115); 3) in Eq. (113), insert the resulting expressions in the right-hand side, and replace the left-hand side with the expansion (115); 4) identify the terms with the same dependence on \( T^{-p} \) on both sides.

In this Article, we determine the coefficients \( \Delta^{(p)}(s) \) corresponding to the lowest values of \( p \), i.e. \( p = 0 \) (strictly adiabatic term) and \( p = 1 \) (first non-adiabatic correction). For this purpose, we only need the following quantities:

\[
\Gamma^{(0)}_{\Delta}(s) = - U \sum_k W_{\psi_0} \langle \psi(0)(s)| \hat{d}^\dagger_{k, \alpha} \hat{d}_{k, \alpha}^\dagger | \psi(0)(s) \rangle .
\]

(116)

and

\[
\Gamma^{(1)}_{\Delta}(s) = - U \sum_k \sum_{\psi_0} W_{\psi_0} \left\{ \langle \psi(1)(s)| \hat{d}^\dagger_{k, \alpha} \hat{d}_{k, \alpha}^\dagger | \psi(1)(s) \rangle + \langle \psi(0)(s)| \hat{d}^\dagger_{k, \alpha} \hat{d}_{k, \alpha}^\dagger | \psi(0)(s) \rangle \right\} .
\]

(117)

In order to obtain explicit expressions of \( \Gamma^{(0)}_{\Delta}(s) \) and \( \Gamma^{(1)}_{\Delta}(s) \), we use the results of the previous Sections, including Eqs. (85) and (86). We also note that

\[
\hat{d}^\dagger_{k, \alpha} \hat{d}_{k, \alpha}^\dagger = \sum_{\alpha, \alpha'} b^*_k(t)a_{k, \alpha'}(t)\hat{D}^\dagger_{k, \alpha}(t)\hat{D}_{k, \alpha'}^\dagger(t) ,
\]

(118)

where

\[
b^*_k(t)a_{k, \alpha}(t) = \frac{-\alpha \Delta(t)}{2E_k(t)} ,
\]

\[
b^*_k(t)a_{k, -\alpha}(t) = \frac{e^{i\omega(t)}}{2} \left[ \alpha - \frac{\epsilon_k + f(t)}{E_k(t)} \right] ,
\]

(119)

as a consequence of Eqs. (91).

After some algebra, we find

\[
\Gamma^{(0)}_{\Delta}(s) = \frac{U}{2} \sum_k \frac{w_k}{E_{\Delta,k}(s)} ,
\]

(120)

where

\[
\sum_n W_n (n_{k, -} - n_{k, +}) \equiv w_k
\]

(121)
\[ \Gamma_{\Delta}^{(1)}(s) = \frac{U}{4} \sum_{k} w_k \left\{ \frac{i \Delta(s) \partial_s f(s) - [\epsilon_k + f(s)] \partial_s \Delta(s)}{E_{\Delta,k}^{3}(s)} + \Delta(s) \left[ i \sin[\theta_{\Delta,k}(s)] - \frac{\epsilon_k + f(s)}{E_{\Delta,k}^{2}(s)} \cos[\theta_{\Delta,k}(s)] \right] \right. \]

\[ \left. - \Delta(s) \left[ i \cos[\theta_{\Delta,k}(s)] + \frac{\epsilon_k + f(s)}{E_{\Delta,k}^{2}(s)} \sin[\theta_{\Delta,k}(s)] \right] \right\} , \quad (122) \]

where we have used Eqs. (103) and (104), as well as the shorthand

\[ \theta_{\Delta,k}(t) = \int_{t_0}^{t} dt' \left[ \frac{\epsilon_k + f(t')}{E_{\Delta,k}(t')} \partial_{t'} \phi(t') - 2E_{\Delta,k}(t') \right] . \quad (123) \]

From Eq. (120) onwards, we have explicitly indicated which quantities depend on \( \Delta \), e.g. by writing \( E_k(s) \rightarrow E_{\Delta,k}(s) \). Because \( \theta_{\Delta,k}(s_0) = 0 \) (see Eq. (123)), one can verify that \( \Gamma_{\Delta}^{(1)}(s_0) = 0 \).

We now put in correspondence the two expansions given by Eqs. (115) and (113). In order to identify \( \Delta^{(0)}(s) \) and \( \Delta^{(1)}(s) \) we put \( \Delta(s) \approx \Delta^{(0)}(s) + T^{-1} \Delta^{(1)}(s) \) in Eq. (116) and we expand by assuming that the second term is small, obtaining

\[ \Gamma_{\Delta}^{(0)}(s) \approx -\Delta^{(0)}(s) \frac{U}{2} \sum_{k} \frac{w_k}{E_{\Delta^{(0)},k}(s)} - \frac{1}{T} \Delta^{(1)}(s) X_{\Delta^{(0)}}(s) , \quad (124) \]

\[ \Delta^{(1)}(t) = \frac{U}{4[1 + X_{\Delta^{(0)}}(t)]} \sum_{k} w_k \left\{ i \frac{\Delta^{(0)}(t) \dot{f}(t) - [\epsilon_k + f(t)] \dot{\Delta}^{(0)}(t)}{E_{\Delta^{(0)},k}(t)} + \Delta^{(0)}(t) \left[ i \sin[\theta_{\Delta^{(0)},k}(t)] - \frac{\epsilon_k + f(t)}{E_{\Delta^{(0)},k}(t)} \cos[\theta_{\Delta^{(0)},k}(t)] \right] \right. \]

\[ \left. - \Delta^{(0)}(t) \left[ i \cos[\theta_{\Delta^{(0)},k}(t)] + \frac{\epsilon_k + f(t)}{E_{\Delta^{(0)},k}(t)} \sin[\theta_{\Delta^{(0)},k}(t)] \right] \right\} \quad (127) \]

It should be noted that Eq. (127) exhibits a typical feature of APT: the corrections of higher order can be
calculated from the knowledge of terms of lower orders only. So, once \( \Delta^{(0)}(t) \) is known, the calculation of \( \Delta^{(1)}(t) \) is numerically trivial. Concerning the determination of \( \Delta^{(0)}(t) \), we see that the complexity of Eq. (126) is comparable to that of the equilibrium BCS gap equation, except that the calculation should be done at each instant of time (on a grid). This is a minimal increase of computational complexity, which was expected in going from an equilibrium problem to the corresponding nonequilibrium one. A simplification of the equations is obtained by noticing that Eq. (126) only determines the modulus of \( \Delta^{(0)}(t) \). This quantity can be therefore chosen to be real. This sets the second line of Eq. (127) to zero. However, \( \Delta^{(1)}(t) \) develops an imaginary part.

Because \( \theta^{(0)}_k(t_0) = 0 \), one can verify that \( \Delta^{(1)}(t_0) = 0 \). Therefore, \( \Delta^{(0)}(t_0) \) coincides with the total superconducting gap of the system at equilibrium, \( \Delta(t_0) \equiv \Delta_0 \).

Finally, we also notice that \( \Delta^{(0)}(t) \equiv 0 \) \( \forall t \) (which implies \( \Delta^{(1)}(t) \equiv 0 \) as well), corresponding to the normal state, is a possible solution.

### H. Constraints on the instantaneous range of variation of the gap

Discarding the normal-state solution, Eq. (126) can be written as

\[
- \frac{2}{U} = \sum_k \frac{w_k}{E_{\Delta^{(0)},k}(t)}.
\]

(128)

This condition must be satisfied at all times \( t \). Let us take Eq. (128) and subtract the same Equation taken at \( t = t_0 \), using the fact that \( f(t_0) = 0 \). We obtain

\[
0 = \sum_k w_k \frac{E_{\Delta^{(0)},k}(t_0) - E_{\Delta^{(0)},k}(t)}{E_{\Delta^{(0)},k}(t) E_{\Delta^{(0)},k}(t_0)}
+ \sum_k \frac{E^2_{\Delta^{(0)},k}(t_0) - E^2_{\Delta^{(0)},k}(t)}{E_{\Delta^{(0)},k}(t) E_{\Delta^{(0)},k}(t_0)}
+ \frac{2 \beta f(t) c(t)}{E_{\Delta^{(0)},k}(t_0) + E_{\Delta^{(0)},k}(t)}
= c(t) \left[ |\Delta_0|^2 - f^2(t) - |\Delta^{(0)}(t)|^2 \right] - 2 f(t) c(t),
\]

(129)

where we have defined

\[
c(t) \equiv \sum_k \frac{w_k}{E_{\Delta^{(0)},k}(t) E_{\Delta^{(0)},k}(t_0)}
\]

\[
c(t) \equiv \sum_k \frac{w_k}{E_{\Delta^{(0)},k}(t) E_{\Delta^{(0)},k}(t_0)} E_{\Delta^{(0)},k}(t_0)
E_{\Delta^{(0)},k}(t_0)
\]

(130)

Eq. (129) would look like a very simple relation between the quantities \( f(t) \), \( \Delta^{(0)}(t) \) and \( \Delta_0 \), if it were not for the fact that \( c(t) \) and \( e(t) \) depend on those quantities as well. However, their ratio is a weighted sum of the quantities \( \epsilon_k \), which satisfies

\[
\epsilon_{\min} \leq \frac{e(t)}{c(t)} \leq \epsilon_{\max} \forall t,
\]

(131)

where \( \epsilon_{\min} = \min_k \{ \epsilon_k \} \) and \( \epsilon_{\max} = \max_k \{ \epsilon_k \} \). It follows that, for a given field \( f(t) \),

\[
|\Delta_0|^2 + |f(t)| |x(t) - 2B| \leq \left| \Delta^{(0)}(t) \right|^2
\]

\[
\leq |\Delta_0|^2 + |f(t)| x(t),
\]

(132)

where

\[
x(t) \equiv B - |f(t)| - 2 \epsilon_c \text{sign} [f(t)],
\]

(133)

and we have introduced the central value of the quasi-particle energy band, \( \epsilon_c \equiv (\epsilon_{\max} + \epsilon_{\min})/2 \), and the band width \( B \equiv \epsilon_{\max} - \epsilon_{\min} \). Eq. (132) gives an exact (albeit not refined) constraint on how much \( \Delta^{(0)}(t) \) can vary from the equilibrium value \( \Delta_0 \) at each time \( t \).

Moreover, we can immediately see that Eq. (132) puts a restriction on the possibility to turn a normal material \( (\Delta_0 = 0) \) into a superconductor. In fact, for \( \Delta_0 = 0 \), it reduces to

\[
|f(t)| |x(t) - 2B| \leq \left| \Delta^{(0)}(t) \right|^2 \leq |f(t)| x(t),
\]

(134)

Now, if \( x(t) < 0 \), Eq. (134) admits no solutions, so the material stays in the normal state [the solution of Eq. (126) that was discarded in writing Eq. (128)]. If \( x(t) = 0 \), Eq. (134) is solved by \( \Delta^{(0)}(t) = 0 \), which is indistinguishable from the normal state. These two scenarios cannot be altered by considering the additional term \( \Delta^{(1)}(t) \) contributing to the dynamical gap: as discussed, \( \Delta^{(0)}(t) = 0 \) implies \( \Delta^{(1)}(t) = 0 \). So, in order to turn a normal material into a superconducting one, it is necessary (although not sufficient) that \( x(t) > 0 \).

### I. The particular cases of initial thermal equilibrium, and equilibrium

The present formulation allows for a certain flexibility in the initial conditions. Let us check that the gap equation, in the case of an initial thermal superposition, reduces to the usual BCS gap equation at equilibrium. An initial thermal superposition corresponds to

\[
W_n = Z^{-1} e^{-\beta E_n(t_0)} = Z^{-1} \prod_k e^{\frac{1}{|\Delta_0|^2 - f^2(t) - |\Delta^{(0)}(t)|^2} - 2 f(t) c(t)},
\]

(135)

\[
Z = \sum_n W_n = \prod_k \left[ 1 + e^{-\beta E_k(t_0)} + e^{\beta E_k(t_0)} \right],
\]

(136)

so that

\[
w_k = \sum_n W_n (n_{k, -} - n_{k, +}) = \tanh [\beta E_k(t_0)/2],
\]

(137)

which should then be used to compute Eqs. (126) and (127). In particular, Eq. (126), excluding the normal-state solution, reduces to

\[
1 = - \frac{U}{2} \sum_k \frac{\tanh \left( \frac{\beta \sqrt{\epsilon_k^2 + |\Delta_0|^2 / 2}}{2} \right)}{\sqrt{\epsilon_k^2 + f(t)^2 + |\Delta^{(0)}(t)|^2}},
\]

(138)
where we have assumed that \( f(t_0) = 0 \). If \( f(t) = 0 \), then \( \Delta^{(0)}(t) = \Delta_0 \) and \( \Delta^{(1)}(t) = 0 \), and Eq. (137) reduces to the standard BCS equation for the superconducting gap at equilibrium. In particular, it admits solution(s) only if \( U < 0 \) (attractive Hubbard model).

VI. NECESSITY AND VALIDITY OF ADIABATIC PERTURBATION THEORY

A. Inadequacy of the adiabatic theorem

We will now discuss the problem that we mentioned in Section V, namely the inconsistency that appears using the adiabatic theorem of quantum mechanics, rather than the APT approach that we have pursued here.

Consider, in all generality, the case of a spatially non-uniform modulation of the gap parameter. This changes in time if \( \Delta_q(t) \neq 0 \). From Eq. (65) and using the Schrödinger equation satisfied by the states \( |n(t)\rangle \), we obtain

\[
\Delta_q(t) = -iU \sum_n W_n \langle n(t)| \sum_k \left[ \hat{H}(t), \hat{d}_k^\dagger \right] n(t) \rangle \\
= iU \sum_n W_n \langle n(t)| \sum_{k,k'} \sigma \hat{d}_{k+q,\sigma} \hat{d}_{k',\sigma}^\dagger \\
+ 2 [f(t) + \epsilon_k] \hat{d}_k^\dagger \hat{d}_k^\dagger |n(t)\rangle , \quad (138)
\]

Let us restrict the analysis to the uniform case discussed above, with \( f_q(t) = \delta_{q,0} f(t) \) and \( \epsilon_{k,\sigma} = \epsilon_k \), and assume that \( \Delta_q(t) = \delta_{q,0} \Delta(t) \). We first need to check whether these two assumptions are consistent. If we assume so, the gap equation becomes

\[
\Delta(t) = iU \sum_n W_n \langle n(t)| \sum_k \left\{ \Delta(t) \sum_k \sigma \hat{d}_{k+q,\sigma} \hat{d}_{k,\sigma} \\
+ 2 [f(t) + \epsilon_k] \hat{d}_k^\dagger \hat{d}_k^\dagger \right\} |n(t)\rangle , \quad (139)
\]

while the condition

\[
0 = \sum_n W_n \langle n(t)| \sum_k \left\{ \Delta(t) \sum_{\sigma} \sigma \hat{d}_{k+q,\sigma} \hat{d}_{k,\sigma} \\
+ 2 [f(t) + \epsilon_k + \epsilon_{k+q}] \hat{d}_{k+q,\sigma} \hat{d}_{k,\sigma} \right\} |n(t)\rangle , \quad \forall q \neq 0 \quad (140)
\]

must be satisfied if the hypothesis that \( \Delta_q(t) = \delta_{q,0} \Delta(t) \) is valid. Using Eqs. (90) we write these relations in terms of the \( \hat{D} \) operators. Eq. (140) becomes

\[
0 = \sum_{k',\alpha,\alpha'} \left\{ \Delta(t) \left[ a_{k',\alpha}^*(t) a_{k',\alpha'}(t) - b_{k,\alpha}(t) b_{k',\alpha'}(t) \right] \\
+ 2 [f(t) + \epsilon_k + \epsilon_{k'}] b_{k,\alpha}(t) a_{k',\alpha'}(t) \right\} \\
\times \sum_n W_n \langle n(t)| \hat{D}_{k,\alpha}(t) \hat{D}_{k',\alpha'}(t) |n(t)\rangle . \quad (141)
\]

Under our assumptions, the total crystal momentum is a good quantum number, so

\[
\langle n(t)| \hat{D}_{k,\alpha}(t) \hat{D}_{k',\alpha'}(t) |n(t)\rangle = 0, \quad \text{if } k' \neq k , \quad (142)
\]

and therefore Eq. (140) is satisfied. Therefore, the assumptions that \( \Delta_q(t) = \delta_{q,0} \Delta(t) \) and \( f_q(t) = \delta_{q,0} f(t) \) are consistent. The same treatment applied to Eq. (139) yields

\[
\Delta(t) = iU \sum_k \sum_{\alpha} \left\{ \alpha [\epsilon_k + f(t)] - E_k \right\} \\
\times \sum_n W_n \langle n(t)| \hat{D}_{k,\alpha}(t) \hat{D}_{k,-\alpha}(t) |n(t)\rangle . \quad (143)
\]

In the case of the adiabatic theorem, the states \(|n(t)\rangle\) appearing in Eq. (143) would be approximated, apart from phase factors, with the instantaneous eigenstates of \( \hat{H}(t) \), see Eq. (94). Since these states have well-defined occupation numbers in the representation of the IQP fields, the bra-kets in the second line of Eq. (143) would vanish, yielding \( \Delta(t) = 0 \) \forall t. This is in contradiction with the time-dependent solution of the dynamical gap equation coming from the adiabatic theorem. This shows why we could not have applied the adiabatic theorem for the treatment of our nonequilibrium problem. First-order APT gives a more accurate approximation of \(|n(t)\rangle\), including terms that do not conserve the IQP occupation numbers. This yields \( \Delta(t) \neq 0 \), thereby removing the inconsistency.

B. Applicability of first-order APT

We now give a criterion to evaluate the accuracy of first-order APT, which we have used in this work. As mentioned in Sect. VA, in general one should require \(|M_{n,m}(s)| \ll 1\). However, the most accurate condition obviously depends on the order of truncation of the APT expressions. In our case, we have approximated the time-dependent state as

\[
|\Psi(s)\rangle \approx |\Psi^{(0)}(s)\rangle + T^{-1} |\Psi^{(1)}(s)\rangle . \quad (144)
\]

Let us compute the norm of this state using Eqs. (85) and (86). We observe that

\[
\langle \Psi^{(0)}(s)|\Psi^{(0)}(s)\rangle = 1 \quad (145)
\]

\[
\langle \Psi^{(0)}(s)|\Psi^{(1)}(s)\rangle = i \sum_{n \neq \Psi_0} J_{n,\Psi_0}(s) \quad (146)
\]

so that

\[
\langle \Psi(s)|\Psi(s)\rangle \approx 1 + T^{-2} \langle \Psi^{(1)}(s)|\Psi^{(1)}(s)\rangle . \quad (147)
\]
We then see that the condition
\[ T^{-2} \langle \Psi^{(1)}(s) | \Psi^{(1)}(s) \rangle \ll 1 \]  
(148)
is a good test of the validity of first-order APT. In fact, this ensures an (approximate) instantaneous normalization of the time-dependent state, as well as it states that quantities which are formally of order \( T^{-2} \) must be negligible. Before writing Eq. (148) explicitly, it is convenient to introduce the set \( S_n = \{(k, \alpha) : n_{k,\alpha} = 1, n_{k,-\alpha} = 0\} \). The left-hand side of Eq. (148) is then written as
\[ T^{-2} \langle \Psi^{(1)}(s) | \Psi^{(1)}(s) \rangle = \frac{1}{4T^2} \sum_{(k,\alpha) \in S_0} \left| \frac{A_k(s) - \alpha_\alpha e^{-i\alpha\theta_k(s)}}{\epsilon_k(s)} - \frac{A_k(s_0) - \alpha_\alpha}{\epsilon_k(s_0)} \right|^2 + \frac{1}{4T^2} \left( \sum_{(k,\alpha) \in S_0} \int_{s_0}^s ds' \left| A_k(s') - \alpha_\alpha \right|^2 \right)^2, \]  
(149)
where we have used Eqs. (111), (112), (123), and it is intended that the formula should be evaluated with \( \Delta(t) \to \Delta^{(0)}(t) \). The check of whether the right-hand side of Eq. (149) is much smaller than 1 should be done numerically, case by case.

To this end, Eq. (149) can be simplified as follows. First, we note that, since \( \Delta^{(0)}(t) \) can be chosen as real (as already discussed), Eq. (102) reduces to \( A_k(s) - \alpha_\alpha = \alpha \Delta^{(0)}(s) V_{\Delta^{(0)},k}(s)/2 \), which is the expression that should be inside Eq. (149). Then, Eq. (103) (see the first line) requires \( \partial_s |\Delta^{(0)}(s)| \). It is convenient to write this quantity in terms of \( \Delta^{(0)}(s) \), \( f(s) \), and \( \partial_s f(s) \). This can be done by taking the derivative with respect to \( s \) of Eq. (128), which yields
\[ |\Delta^{(0)}(s)| \partial_s |\Delta^{(0)}(s)| \equiv -[f(s) + J_{\Delta^{(0)}}(s)] \partial_s f(s), \]  
(150)
with
\[ J_{\Delta^{(0)}}(s) \equiv \left( \sum_k \frac{u_{k,k} \epsilon_k}{E_{\Delta^{(0)},k}(s)} \right) / \left( \sum_k \frac{u_k}{E_{\Delta^{(0)},k}(s)} \right). \]  
(151)
Note that Eq. (151) is a weighted sum, constrained by
\[ \epsilon_{\text{min}} < J_{\Delta^{(0)}}(s) < \epsilon_{\text{max}} \forall s. \]  
(152)
Eq. (150) can be used for manipulating Eq. (149) only if \( \Delta^{(0)}(s) \neq 0 \); otherwise it gives us information on that \( \Delta^{(0)}(s) \) can vanish instantaneously at \( s \) (while being allowed to be non-zero at other times) only if \( [f(s) + J_{\Delta^{(0)}}(s)] \partial_s f(s) = 0 \). For the sake of simplicity, we assume that we are in a situation in which \( \Delta^{(0)}(s) \neq 0 \). Then, after some straightforward algebraic manipulations, we obtain
\[ A_{\Delta^{(0)},k}(s) - \alpha_\alpha = \frac{\alpha \partial_s f(s)}{2 |\Delta^{(0)}(s)| E_{\Delta^{(0)},k}(s)} \left\{ |\Delta^{(0)}(s)|^2 + |J_{\Delta^{(0)}}(s) + f(s)| |\epsilon_k + f(s)| \right\}, \]  
(153)
which is the most simplified expression that should be used with Eq. (149) for the numerical assessment of the applicability of APT for the system of interest. Given the complexity of Eq. (149), further analytical estimates are difficult to perform without specifying the form of \( \epsilon_k \). Estimates for specific dispersion relations \( \epsilon_k \) are well beyond the scope of the general theoretical framework that we are formulating here. In any case, an accurate evaluation of Eq. (149) is best done numerically.

Importantly, since \( \partial_s = T \partial_t \), one can easily see, after plugging Eq. (153) into Eq. (149), that the quantity (149) is independent of \( T \).

VII. SUMMARY AND CONCLUSIONS

In this Article we have laid down a theoretical framework to compute the nonequilibrium superconducting gap for a coupled electron-phonon system subject to an external time-dependent electromagnetic field acting on the phonon subsystem. Since our main objective was to transcend the limitations of Floquet theory and/or heavy numerical methods, we had to make an assumption of slow time dependence of the external field.

As it happens with any approximate scheme, our formulation has both advantages and drawbacks with respect to previous works. From the point of view of the model, our approach has the advantage that it does not restrict the external field to be periodic (contrarily to Floquet theory), nor to have a small amplitude. The only restriction is that APT must be valid, which should be assessed case by case from the numerical evaluation of Eq. (149). One of the advantages of APT is that, in principle, further, higher-order corrections can be included, although the formulas get more involved. However, each additional correction can be computed from the knowledge of the lower-order terms contributing to \( \Delta(t) \). Therefore, the most numerically-demanding task is the solution of Eq. (116).

From the computational point of view, our formulas exhibit the minimal increase of computational complexity that can be expected in going from an equilibrium to a nonequilibrium problem. The nonequilibrium problem is mapped onto a set of equilibrium-like problems to be solved at every instant of time on a grid. The required computation, being analogous to an equilibrium BCS one, does not exhibit numerical difficulties such as the determination of huge two-times Green’s functions and self-energy matrices, which would require to adopt
simple expressions for the time-dependent field. The drawback of our approach is that it is based on mean-field theory, which can be transcended if a full numerical Keldysh calculation is done as e.g. in Ref. 32 (some effects that are not captured within mean-field theory are discussed there). It should be noted that the mean-field theory proposed in the Appendix of Ref. 32 is still to be intended as a numerical approximation, i.e. as a way to compare the full numerical calculation with a simpler one, but it is not equivalent to the semi-analytical formulas presented here [Eqs. (116) and (117)].

One of the earliest discussions on nonequilibrium superconductivity can be found in Ref. 14. It treats the case of a time- and space-dependent gap parameter, and it derives differential equations for it under several conditions, by means of second-order Taylor expansions in space and time gradients. Such an approach presents several difficulties, which are thoroughly discussed in Ref. 14. The resulting equations, depending on the various situations discussed, either assume smallness in the size variation of $\Delta$, or are valid on short time intervals due to use of a Taylor expansion. Moreover, the results are differential equations whose numerical solution is demanding. The advantage of our APT-based approach is that it reduces the problem to the solution of algebraic equations, whose validity is not restricted to small variations of $\Delta$ nor to small time intervals, provided that the external field is slow.

Our analytical theory can be used to answer several intriguing questions. For example, one may consider the problem of whether a non-superconducting material, with $\Delta(t_0) = 0$, can be driven into a non-equilibrium superconducting state, with $\Delta(t) \neq 0$, by applying an external time-dependent electromagnetic field. From Eq. (137) one sees that these two requirements are compatible, and with Eq. (134) we have provided a necessary (but not sufficient) condition for the transition to occur, which is aimed to guide computational studies. Since the normal state solution is always possible, it should be noted that the study of the normal-superconducting transition driven by the application of an external field requires a further stability analysis.

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**Appendix A: Bosonic Gaussian integration on the KP time contour**

We re-write Eq. (23) as

$$\prod_{q,\lambda} \int \mathcal{D} (a_{q,\lambda}^*, a_{q,\lambda}) \times \exp \left\{ i \int_\gamma dzdz' a_{q,\lambda}(z) \hat{G}_{q,\lambda}^{-1}(z, z') a_{q,\lambda}(z') \right\}$$

$$= \prod_{q,\lambda} \exp \left\{ -i \int_\gamma dzdz' \tilde{J}_{q,\lambda}(z) a_{q,\lambda}(z) + a_{q,\lambda}^*(z) J_{q,\lambda}(z) \right\}$$

$$= \prod_{q,\lambda} e^{-iG_{q,\lambda}^{-1}(z, z')} \times \det \left( -iG_{q,\lambda}^{-1} \right)$$

(A1)

where

$$J_{q,\lambda}(z) = M_{q,\lambda} \rho_{-q}(z) + F_{-q,\lambda}(z)$$

(A2)

$$\tilde{J}_{q,\lambda}(z) = M_{q,\lambda} \rho_{q}(z) + F_{q,\lambda}(z)$$

In the last step of Eq. (A1) we have applied to the $\gamma$ contour the standard rules of bosonic Gaussian integration on a continuous time domain, and we have used the direct free-phonon GF given by Eq. (26). The functional determinant appearing in Eq. (A1) is given by

$$\det \left( -iG_{q,\lambda}^{-1} \right) = 1 - e^{-\beta\omega_{q,\lambda}}$$

(A3)

Eq. (23) can be found by using Eqs. (A1) and (A3), and the definitions in (A2). We find

$$\int \mathcal{D} (a^*, a) e^{iS_{kp}} \prod_{d,d'} \left[ d^d a^*_{d'} \right]$$

$$= \text{Tr} \left( e^{-\beta\hat{H}_p} \right) \exp \left\{ -i \sum_{q,\lambda} \int_\gamma dzdz' \right.$$

$$\times \left[ F_{q,\lambda}(z) G_{q,\lambda}^{fp}(z, z') F_{-q,\lambda}(z') \right.$$

$$\left. + M_{q,\lambda} \left( G_{q,\lambda}^{fp}(z, z') + G_{-q,\lambda}^{fp}(z', z) \right) F_{-q,\lambda}(z') \rho_q(z) \right.$$

$$\left. + |M_{q,\lambda}|^2 G_{q,\lambda}^{fp}(z, z') \rho_q(z) \rho_{-q}(z') \right\}.$$ 

(A4)

Using Eq. (35), we see that the term appearing in Eq. (A4), which is independent of the Grassmann fields, vanishes:

$$\sum_{q,\lambda} \int_\gamma dzdz' F_{q,\lambda}(z) G_{q,\lambda}^{fp}(z, z') F_{-q,\lambda}(z') = 0.$$ 

(A5)

Eq. (A4) then reduces to

$$Z[V] = \frac{\text{Tr} \left( e^{-\beta\hat{H}_p} \right)}{\text{Tr} \left( \hat{U}_{\gamma M} \right)} \int \mathcal{D} [d, \bar{d}] e^{iS_{eff}^{(V)}} \prod_{d, \bar{d}} [d, \bar{d}],$$

(A6)
where $S^{(M)}_{\text{eff}}[\tilde{d}, d]$ is given by Eq. (25). In the right-hand side of Eq. (A6) we see the quantity

$$\text{Tr} \left( e^{-\beta \hat{H}_{\rho}} \right) = \prod_{q,\lambda} \frac{1}{1 - e^{-\beta \omega_{q,\lambda}}} .$$  \hspace{1cm} (A7)

This can be further simplified by performing the bosonic path integral in the denominator. Since the free-phonon and the phonon-electron interaction Hamiltonians have the same form on the Matsubara branch and on the real-time branches, we have

$$\text{Tr} \left( \hat{U}_{\gamma M} \right) = \text{Tr} \left( e^{-\beta \hat{H}_{\rho}} \right) \exp \left\{ i S^{(M)}_{\text{e}}[\tilde{d}, d] \right\}$$

$$- i \sum_{q,\lambda} \int_{\gamma M} \text{d}z \text{d}z' |M_{q,\lambda}|^2 G^{\text{tr}}_{q,\lambda}(z, z')$$

$$\times \rho_q(z) \rho_q(z')$$

$$\equiv \text{Tr} \left( e^{-\beta \hat{H}_{\rho}} \right) \text{Tr} \left( \hat{U}^{\text{eff}}_{\gamma M} \right) ,$$  \hspace{1cm} (A8)

where $S^{(M)}_{\text{e}}[\tilde{d}, d]$ is the quadratic electronic action on the Matsubara branch, and it should be noted that the time integrations run only on $\gamma M$. We then obtain Eq. (24).

**Appendix B: Functional derivatives**

In the main text, we have used the functional derivatives of Eq. (53) with respect to $V^{\uparrow\downarrow}, V^{\uparrow\downarrow}, \Delta, \Delta^*$. The first step can be done in general. If $x$ is the field with respect to which we differentiate, then

$$\frac{\delta}{\delta x} \text{tr} \left\{ \ln \left[ -i G^{-1}(V, \Delta) \right] \right\} = \text{tr} \left\{ G(V, \Delta) \frac{\delta}{\delta x} G^{-1}(V, \Delta) \right\}$$

$$= \sum_k \sum_{\sigma,\sigma'} \int \text{d}z \text{d}z' G^{(V, \Delta)}_{k, \sigma, z; k', \sigma', z'} \delta \frac{\delta}{\delta x} G^{-1}(V, \Delta)_{k', \sigma', z; k, \sigma, z}.$$  \hspace{1cm} (B1)

From Eq. (50), specifying the fields that we need, we get

$$\frac{\delta}{\delta \Delta_q(z)} \ln \left[ -i G^{-1}(V, \Delta) \right] = \sum_k G^{(V, \Delta)}_{k, \uparrow; z; k+q, \downarrow} ,$$  \hspace{1cm} (B2)

and

$$\frac{\delta}{\delta \Delta_q(z)} \ln \left[ -i G^{-1}(V, \Delta) \right] = \sum_k G^{(V, \Delta)}_{k, \downarrow; z; k-q, \uparrow} ,$$  \hspace{1cm} (B3)

$$\frac{\delta}{\delta V^{\sigma, -\sigma}_{k, k'}} \ln \left[ -i G^{-1}(V, \Delta) \right] = -G^{(V, \Delta)}_{k', -\sigma; z; k, \sigma, z} .$$  \hspace{1cm} (B4)

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1. J. Nagamatsu, N. Nakagawa, T. Muranaka, Y. Zenitani, and J. Akimitsu, Nature 410, 63 (2001).
2. P.W. Anderson, J. Phys.: Conf. Ser. 449, 012001 (2013).
3. B. Keimer, S.A. Kivelson, M.R. Norman, S. Uchida, and J. Zaanen, Nature 518, 179 (2015).
4. L. Pietronero, S. Strässler, and C. Grimaldi, Phys. Rev. B 52, 10516 (1995).
5. C. Grimaldi, L. Pietronero, and S. Strässler, Phys. Rev. B 52, 10530 (1995).
6. D. Fausti, R. Tobey, N. Dean, S. Kaiser, A. Dienst, M. Hoffmann, S. Pyon, T. Takayama, H. Takaqi, and A. Cavalleri, Science 331, 189 (2011).
7. W. Hu, S. Kaiser, D. Nicoletti, C.R. Hunt, I. Gierz, M.C. Hoffmann, M. Le Tacon, T. Loew, B. Keimer, and A. Cavalleri, Nature Mater. 13, 705 (2014).
8. S. Kaiser, C.R. Hunt, D. Nicoletti, W. Hu, I. Gierz, H. Y. Liu, M. Le Tacon, T. Loew, D. Haug, B. Keimer, and A. Cavalleri, Phys. Rev. B 89, 184516 (2014).
9. M. Mitrano, A. Cantaluppi, D. Nicoletti, S. Kaiser, A. Perucchi, S. Lupi, P. Di Pietro, D. Pontiroli, M. Riccò, S.R. Clark, D. Jaksh, and A. Cavalleri, Nature 530, 461 (2016).
10. J. Rammer and H. Smith, Rev. Mod. Phys. 58, 323 (1986).
11. N.B. Kopnin, Theory of Nonequilibrium Superconductivity (Clarendon Press, Oxford, 2001).
12. K. Usadel, Phys. Rev. Lett. 25, 507 (1970).
13. A. Kamenev, Field Theory of Non-Equilibrium Systems (Cambridge University Press, Cambridge, 2011).
14. E. Abrahams and T. Tsuneto, Phys. Rev. 152, 416 (1966).
15. A. Messiah, Quantum Mechanics, vol. 2, (John Wiley & Sons, New York, 1958).
16. R.A. Barankov, L.S. Levitov, and B.Z. Spivak, Phys. Rev. Lett. 93, 160401 (2004).
17. G.L. Warner and A.J. Leggett, Phys. Rev. B 71, 134514 (2005).
18. A.F. Volkov and S.M. Kogan, Sov. Phys. JETP 38, 1018 (1974).
19. R.A. Barankov and L.S. Levitov, Phys. Rev. Lett. 96, 230403 (2006).
20. E.A. Yuzbashyan and M. Dzero, Phys. Rev. Lett. 96, 230404 (2006).
21. T. Tomadin, M. Polini, M.P. Tosi, and R. Fazio, Phys. Rev. A 77, 033605 (2008).
22. F. Peronaci, M. Schirò, and M. Capone, Phys. Rev. Lett. 115, 257001 (2015).
23. T. Papenkort, V.M. Axt, and T. Kuhn, Phys. Rev. B 76, 224522 (2007).
