GLOBAL ASPECTS OF ABELIAN AND CENTER PROJECTIONS IN SU(2) GAUGE THEORY

by

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Abstract

We show that the global aspects of Abelian and center projection of a SU(2) gauge theory on an arbitrary manifold are naturally described in terms of smooth Deligne cohomology. This is achieved through the introduction of a novel type of differential topological structure, called Cho structure. Half integral monopole charges appear naturally in this framework.

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1. Introduction and conclusions

Since the early seventies, a considerable effort has been devoted to the theoretical understanding of confinement in non Abelian gauge theory. The two most accredited theories are the dual superconductor model [1–6] and the vortex condensation model [7–11].

According to the dual superconductor model picture, confinement is due to the condensation of chromomagnetic monopoles which forces the chromoelectric field into flux tubes, through a mechanism known as dual Meissner effect, leading to a linear rising confining chromoelectric potential.

Abelian gauge fixing and Abelian projection were proposed by ’t Hooft in 1980 as a means of explaining the emergence of monopoles in non Abelian gauge theory [6]. In an Abelian gauge, such as the maximal and the Laplacian Abelian gauges, the gauge symmetry associated with the coset $G/H$ of the maximal Abelian subgroup $H$ of the gauge group $G$ is fixed. The resulting gauge fixed field theory is an Abelian gauge theory with gauge group $H$, in which the $H$ and $G/H$ components of the original gauge field behave effectively as Abelian gauge fields and matter fields, respectively. The Abelian projection consists in keeping the $H$ component of the gauge field and neglecting the $G/H$ one, separating out in this way the Abelian sector of the theory relevant for confinement.

The gauge transformation required to transform a given gauge field into one satisfying the appropriate Abelian gauge fixing condition is not smooth in general. (This is just another manifestation of the well known Gribov problem.) The transformed gauge field has therefore defects. In a 4-dimensional space–time, the defects are located on closed loops. These are the chromomagnetic monopole world lines. The monopoles are remnants of the $G/H$ part of the gauge symmetry. Their condensation leads to confinement. Without the monopoles, Abelian projection would yield a rather trivial non confining Abelian gauge theory.

In the vortex condensation model picture, confinement is induced by the filling of the vacuum by closed chromomagnetic center vortices. Their condensation leads to an area law for the Wilson loop and thus to confinement.

The emergence of vortices is explained by means of center gauge fixing and center projection [12]. In a center gauge, such as the maximal and the Laplacian center gauges, the gauge symmetry associated with the coset $G/Z$ of the center $Z$ of the gauge group $G$ is fixed. Since $Z$ is typically a finite group, the resulting gauge fixed field theory is an exotic $Z$ gauge theory with no obvious continuum interpretation. The center projection consists
in keeping the $Z$ degrees of freedom of the gauge field and neglecting the $G/Z$ ones. The former are related to the global topological properties of the gauge field.

As for Abelian gauge fixing, the gauge transformation required to transform a given gauge field into one satisfying the appropriate center gauge fixing condition is not smooth in general. The transformed gauge field has therefore defects. In a 4–dimensional space–time, the defects are located on closed surfaces. These are the chromomagnetic center vortex world sheets. The center vortices are remnants of the $G/Z$ part of the gauge symmetry. Their condensation is responsible for confinement.

Since the dual superconductor model and the vortex condensation model describe the same physical phenomena, there should be a way of identifying vortices in the dual superconductor picture and monopoles in the vortex condensation picture. This is indeed possible provided the gauge fixing is carried out appropriately as follows.

Laplacian center gauge fixing can be implemented in two steps [13,14]. In the first step, the gauge symmetry is partially fixed from $G$ to $H$. In the second step, the gauge symmetry is further partially fixed from $H$ to $Z$. The first step is nothing but Abelian gauge fixing, which therefore becomes an integral part of center gauge fixing. Therefore, in this center gauge, the vortex defects of a gauge field include the monopole defects as a distinguished subset. Correspondingly, the vortex world sheets contain the associated monopole world lines. In this way, vortices appear as chromomagnetic flux tubes connecting monopoles [15].

To take into account the topologically non trivial gauge fields corresponding to monopoles, it is necessary to allow for twisted boundary conditions [16–18]. In pure gauge theory, gauge transformation proceeds via the adjoint representation of $G$. Since the center $Z$ of $G$ lies in the kernel of the latter, one may embed the original gauge theory in a broader one, by allowing the $G$ valued functions, in terms of which the boundary conditions obeyed by the gauge fields are expressed, to satisfy the appropriate 1–cocycle conditions only up to a $Z$ valued twist. There are several types of boundary conditions depending on the possible twist assignments. Correspondingly, the gauge fields fall in topological classes or twist sectors. The gauge fields of the original untwisted theory form the trivial twist sector. The gauge fields of the non trivial twist sectors are obtained from those of the trivial twist sector by applying suitable multivalued gauge transformations with $Z$ monodromy. The branching sheets of the latter are the vortex world sheets.
FORMULATION OF THE PROBLEM

One can view an Abelian gauge as map that assigns to each gauge field $A$ a non-vanishing Higgs field $\phi(A)$ transforming in the adjoint representation of the gauge group $G$, i.e. satisfying

$$\phi(A^g) = \text{Ad}g^{-1}\phi(A), \quad (1.1)$$

for any gauge transformation $g$, where $A^g$ is the gauge transform of $A$. By definition, a gauge transformation $g$ carries $A$ into the Abelian gauge if $\phi(A^g)$ is $\mathfrak{h}$ valued [6]. If $g$ has this property, then also $gh$ does, for any $H$ valued gauge transformation $h$. So, the Abelian gauge fixing leaves a residual unfixed $H$ gauge invariance. The defect manifold $N$ of the Abelian gauge is formed by those points $x$ of space time where $\phi(\bar{A})(x)$ is invariant under the adjoint action of a subgroup $K$ of $G$ properly containing $H$, where $\bar{A}$ is any gauge field belonging to the Abelian gauge, whose choice is immaterial [19,20]. If $g$ is a gauge transformation carrying $A$ into the Abelian gauge, then $g$ must be singular at a Dirac manifold $D_g$ bounded by $N$ depending on $g$. Consequently, $A^g$, also, is. $N$ consists of monopole world lines and $D_g$ of the associated Dirac strings. The Abelian projection consists in replacing $A$ with either one of the following

$$A_{\text{eff}} = \Pi_{\mathfrak{h}}A^g, \quad A_{\text{bff}} = (\Pi_{\mathfrak{h}}A^g)^g - 1, \quad (1.2)$$

where $\Pi_{\mathfrak{h}}$ is a suitable projector of $\mathfrak{g}$ onto $\mathfrak{h}$. The two choices corresponds to the so called space and body fixed frame [21]. The body fixed frame form $A_{\text{bff}}$, which we adopt in the following, has the advantage of being singular only at the defect manifold $N$.

The center gauge fixing works to some extent in similar fashion. It requires two linearly independent Higgs fields $\phi(A), \phi'(A)$ satisfying (1.1) rather than a single one. A gauge transformation $g$ carries $A$ into the center gauge if $\phi(A^g)$ is $\mathfrak{h}$ valued, as before, and $\phi'(A^g)$ is $\mathfrak{v}$ valued, where $\mathfrak{v}$ is a suitable proper subspace of $\mathfrak{g}$ not contained $\mathfrak{h}$ [13,14]. If $g$ has this property, then also $gz$ does, for any element $z$ of $Z$. So, the center gauge fixing leaves a residual unfixed $Z$ gauge invariance. The defect manifold $N^*$ of the center gauge is formed by those points $x$ of space time where $\phi(\bar{A})(x), \phi'(\bar{A})(x)$ are simultaneously invariant under the adjoint action of a subgroup $K$ of $G$ properly containing $Z$, where $\bar{A}$ is any gauge field belonging to the center gauge [19,20]. If $g$ is a gauge transformation carrying $A$ into the center gauge, then $g$ must be singular at a Dirac manifold $D^*_g$ bounded by $N^*$ depending on $g$ and, so, $A^g$ also is. $N^*$ consists of vortex world sheets and $D^*_g$ of the associated Dirac volumes. The singularity of $g$ appears as a non trivial $Z$ monodromy
around $N^*$, here working as a branching manifold. The center projection replaces $A^g$ with $g^{-1}dg$. The latter has non trivial $Z$ valued holonomy if $g$ has non trivial $Z$ monodromy [22]

By allowing gauge transformations with non trivial $Z$ monodromy in center gauge fixing, one is effectively trading the original $G$ gauge theory by a $G/Z$ one possessing a richer wealth of topological classes. These correspond to the twist sectors discussed earlier. These sectors should somehow appear also in Abelian gauge fixing, since one expects the various gauge fixing procedures to lead eventually to physically equivalent descriptions of gauge theory low energy dynamics.

Below, we restrict to SU(2) gauge theory to allow for a particularly simple and direct treatment. Here, as is usual in the physical gauge theory literature, we use the convenient isovector notation, in which the Lie algebra $su(2)$ and its Cartan and Lie brackets are identified with the 3–dimensional Euclidean space $E_3$ and its dot and cross product, respectively. Instead of the Higgs field $\phi$, it is customary to employ the normalized Higgs field

$$n = \phi/|\phi|, \quad n^2 = 1. \quad (1.3)$$

The body fixed frame gauge field $A_{bf}f$ is of the form

$$A_{bf}f = a n + d n \times n. \quad (1.4)$$

The gauge field $A_{bf}f$ was first written by Cho [23,24]. It is characterized by the fact that $n$ is covariantly constant with respect to it

$$D_{bf}f n = d n + A_{bf}f \times n = 0. \quad (1.5)$$

As a consequence, $A_{bf}f$ is a reducible gauge field. In this case, the defect manifold $N$ of an Abelian gauge is simply the vanishing locus of the associated Higgs field $\phi$. At the points of $N$, $n$ is singular, as $A_{bf}f$ is. The defects represent monopoles. The monopole charge is given by

$$m = -\frac{1}{4\pi} \int_\Sigma F_{bf}f \cdot n = \frac{1}{4\pi} \int_\Sigma \left( -da_{bf}f + \frac{1}{2} n \cdot d n \times d n \right), \quad (1.6)$$

where $F_{bf}f$ is the curvature of $A_{bf}f$ and $\Sigma$ is a surface surrounding the monopoles.

The center of SU(2) is the group $\{\pm \frac{1}{2}\} \cong Z_2$. For reasons discussed earlier, $A_{bf}f$ is an SU(2)/$Z_2 \cong SO(3)$ gauge field rather than an SU(2) one. In general, $A_{bf}f$ is not liftable to an SU(2) gauge field. The associated obstructions are related to center vortices.
Plan of the paper

The compatibility of Abelian and center projection with the global topology of the gauge background is not obvious a priori and requires a critical examination. In fact, the truncation of the gauge fields implicit in the projections, though doable locally, is not manifestly so globally, when the background topology is non trivial. We ask then the following question. Is there a natural global structure in which the local field theoretic data yielded by the projection can be fitted in? If so, which are its properties? The aim of this paper is to explore this matter.

The standard treatment of the defects associated with Abelian or center gauge fixing requires specific ad hoc choices of coordinates and trivializations making covariance obscure. Non trivial topology appears in the form of Dirac strings, sheets and the like emanating from the defects. We adopt an alternative approach avoiding this. It consists in defining all the fields locally and analyzing their gluing. Dirac strings, sheets etc. are then traded for cocycles specifying the gluing of the local fields. This is in the spirit of the seminal work of Wu and Yang [25]. Mathematically, it requires the apparatus of Čech cohomology [26–28]. It is an alternative approach to the geometry and topology of these gauge models which relies on cohomology rather than homotopy. It has the advantage of being very general allowing for the analysis of Abelian and center projections for a gauge theory on an arbitrary space time manifold.

To study the global features of the projections in SU(2) gauge theory, we shall work out a local formulation of Cho’s gauge theoretic framework à la Čech [23,24]. This will lead to a novel type of differential topological structures called Cho structures. In due course, we shall discover that the global aspects of a Cho structure $\mathcal{C}$ are encoded in a length 3 degree 3 Deligne cohomology class $D_{\mathcal{C}}$ [29–32], whose properties will be studied in detail. Next, we shall argue that a Cho structure $\mathcal{C}$ describes monopoles provided the associated Deligne class $D_{\mathcal{C}}$ vanishes and we shall show that, when this happens, the resulting monopole configurations are in one–to–one correspondence with certain differential topological structures subordinated to $\mathcal{C}$, called fine Cho structures. We shall also prove that these latter are classified by the length 2 degree 2 Deligne cohomology (up to equivalence). Next, we shall show that $D_{\mathcal{C}}$ does indeed always vanish. This will be the main result of the paper. Finally, we shall also interpret center vortices as $\mathbb{Z}_2$ topological obstructions to the lifting of the SO(3) bundles associated with the fine Cho structures to SU(2) ones. $\mathbb{Z}/2$ valued monopole charges and $\mathbb{Z}/4$ valued topological charges emerge naturally in this framework.
2. SU(2) principal bundles and associated adjoint vector bundles

In this paper, we consider exclusively principal SU(2) bundles and the associated vector bundles. SU(2) is the simplest non Abelian group and one can conveniently take advantage of that. We shall exploit throughout the following basic properties of SU(2). The material expounded below is well known and is collected only to define the notation and the conventions used in the sequel of the paper. The isovector notation used is standard in the physical literature.

**Generalities on SU(2)**

The 3-dimensional oriented Euclidean space \( \mathbb{E}_3 \) is a Lie algebra. The Lie brackets and the Cartan form of \( \mathbb{E}_3 \) are given by \([x, y] = x \times y\) and \((x, y) = -2x \cdot y\), for \(x, y \in \mathbb{E}_3\), respectively. The map \( h : su(2) \to \mathbb{E}_3 \) defined by \( \sigma(h(x))/2i = x \), for \(x \in su(2)\), where \(\sigma\) denotes the standard Pauli matrices, is a Lie algebra isomorphism. Hence, the Lie algebras \( su(2), \mathbb{E}_3 \) are isomorphic. The resulting identification is particularly convenient as the Lie brackets and the Cartan form of \( su(2) \) are represented by the ordinary cross and dot product of vectors of \( \mathbb{E}_3 \), respectively. Below, we shall use extensively the convenient shorthands \( x = h(x), \) for \(x \in su(2)\), and \( \star xy = -h(ad xy) = -x \times y \), for \(x, y \in su(2)\).

The center of SU(2) is \( \mathbb{Z}_2 \) realized as the sign group \( \{\pm 1\} \). The map \( H : SU(2)/\mathbb{Z}_2 \to SO(3) \) defined by \( \sigma(H(R)x) = AdR\sigma(x) \), for \(R \in SU(2)/\mathbb{Z}_2, x \in \mathbb{E}_3\), is a Lie group isomorphism. Hence, the groups \( SU(2)/\mathbb{Z}_2, SO(3) \) are isomorphic. Correspondingly, we have an isomorphism \( \tilde{H} : su(2) \to so(3) \). Hence, the Lie algebras \( su(2), so(3) \) are isomorphic. Below, we shall use the shorthand \( R = H(R) \), for \(R \in SU(2)\).

**Generalities on SU(2) principal bundles**

On a topologically non trivial manifold \( M \), fields are sections of vector bundles. These, in turn, are associated with principal bundles and linear representations of their structure group. Čech theory provides an advantageous description of these differential topological structures. It requires the introduction of a sufficiently fine open cover \( \{O_\alpha\} \) of \( M \) on whose sets the local representations of the fields are given. \(^1\) See for instance [33] for background material.

\(^1\) Below, we denote by \( \varphi_\alpha \) the local representation of a Čech 0–cochain \( \varphi \) on \( O_\alpha \). Similarly, we denote by \( \varpi_{\alpha \beta} \) the local representation of a Čech 1–cochain \( \varpi \) on \( O_\alpha \cap O_\beta \neq \emptyset \), etc. For convenience, we shall suppress the indexes \( \alpha, \beta, \ldots \) when confusion cannot arise.
Let $P$ be an SU(2) principal bundle on a manifold $M$. Then, $P$ is represented by some SU(2) Čech 1–cocycle $\{R_{\alpha\beta}\}$.

The adjoint bundle Ad$P$ of $P$ is the vector bundle

$$\text{Ad} \, P = P \times_{\text{SU}(2)} \mathfrak{su}(2), \quad (2.1)$$

where SU(2) acts on $\mathfrak{su}(2)$ by the adjoint representation. By the isomorphisms $\mathfrak{h} : \mathfrak{su}(2) \to \mathbb{E}_3$ and $H : \text{SU}(2)/\mathbb{Z}_2 \to \text{SO}(3)$, Ad$P$ is isomorphic to the oriented rank 3 vector bundle $E$

$$E = P \times_{\text{SU}(2)} \mathbb{E}_3, \quad (2.2)$$

where SU(2) acts on $\mathbb{E}_3$ via the representation induced by the natural map SU(2) → SU(2)/$\mathbb{Z}_2$ and the isomorphism $H$. $E$ is represented by the SO(3) Čech 1–cocycle $\{R_{\alpha\beta}\}$ corresponding to the SU(2) Čech 1–cocycle $\{R_{\alpha\beta}\}$ under the isomorphism $H$. The vector bundle $E$ is not generic, since its first and second Stiefel–Whitney classes both vanish, $w_1(E) = 0$, $w_2(E) = 0$. In the following, we find definitely more convenient to work with the vector bundle $E$ rather than the adjoint bundle Ad$P$ of $P$, since this allows us to exploit familiar techniques of vector calculus.

Let $A \in \text{Conn}(M,P)$ be a connection of the principal bundle $P$. $A$ is represented by an $\Omega^1 \otimes \mathfrak{su}(2)$ Čech 0–cochain $\{A_\alpha\}$ satisfying

$$A_\alpha = \text{Ad} \, R_{\alpha\beta} A_\beta - dR_{\alpha\beta} R_{\alpha\beta}^{-1}. \quad (2.3)$$

By the isomorphisms $\mathfrak{h} : \mathfrak{su}(2) \to \mathbb{E}_3$ and $H : \text{SU}(2)/\mathbb{Z}_2 \to \text{SO}(3)$, $A$ induces a connection $A \in \text{Conn}(M,E)$ of the vector bundle $E$. $A$ is represented by an $\Omega^1 \otimes \mathbb{E}_3$ Čech 0–cochain $\{A_\alpha\}$ satisfying

$$A_\alpha = R_{\alpha\beta} (A_\beta \omega_{\alpha\beta}), \quad (2.4)$$

where $\omega_{\alpha\beta}$ is the $\mathbb{E}_3$ valued 1–form defined by

$$\star \omega_{\alpha\beta} = -R_{\alpha\beta}^{-1} dR_{\alpha\beta}. \quad (2.5)$$

This correspondence establishes a canonical isomorphism of the affine spaces Conn$(M,P)$, Conn$(M,E)$ of connections of $P$, $E$.

Let $s \in \Omega^p(M,\text{Ad} \, P)$ be a $p$–form section of Ad$P$. Then, $s$ is represented by an $\Omega^p \otimes \mathfrak{su}(2)$ Čech 0–cochain $\{s_\alpha\}$ matching as

$$s_\alpha = \text{Ad} \, R_{\alpha\beta} s_\beta. \quad (2.6)$$
By the isomorphisms $h : \mathfrak{su}(2) \to \mathbb{E}_3$ and $H : \text{SU}(2)/\mathbb{Z}_2 \to \text{SO}(3)$ again, $s$ determines an element $\underline{s} \in \Omega^p(M, E)$. $\underline{s}$ is represented by a $\Omega^p \otimes \mathbb{E}_3$ Čech 0–cochain glueing as

$$\underline{s}_\alpha = R_{\alpha \beta} \underline{s}_\beta.$$  \hfill (2.7)

This correspondence establishes an isomorphism of the spaces $\Omega^p(M, \text{Ad} P)$, $\Omega^p(M, E)$ of $p$–form sections of $\text{Ad} P$, $E$.

Let $A \in \text{Conn}(M, P)$ be a connection of $P$. The covariant derivative $D_A s$ of $s \in \Omega^p(M, \text{Ad} P)$ is

$$D_A s = ds + [A, s].$$  \hfill (2.8)

$D_A s \in \Omega^{p+1}(M, \text{Ad} P)$. The isomorphisms $\text{Conn}(M, P) \cong \text{Conn}(M, E)$, $\Omega^p(M, \text{Ad} P) \cong \Omega^p(M, E)$, described above, map $D_A s$ into the covariant derivative $\underline{D}_A \underline{s}$ of $\underline{s} \in \Omega^p(M, E)$. Explicitly, $\underline{D}_A \underline{s}$ is

$$\underline{D}_A \underline{s} = d\underline{s} + A \times \underline{s}.$$  \hfill (2.9)

$\underline{D}_A \underline{s} \in \Omega^{p+1}(M, E)$ as expected.

The gauge curvature of a connection $A \in \text{Conn}(M, P)$ is

$$F_A = dA + \frac{1}{2} [A, A].$$  \hfill (2.10)

$F_A \in \Omega^2(M, \text{Ad} P)$. Under the isomorphism $\text{Conn}(M, P) \cong \text{Conn}(M, E)$, $F_A$ is mapped into the gauge curvature $\underline{F}_A$ of $A$. Explicitly, $\underline{F}_A$ is

$$\underline{F}_A = dA + \frac{1}{2} A \times A.$$  \hfill (2.11)

$\underline{F}_A \in \Omega^2(M, E)$.

Let $U \in \text{Gau}(M, P)$ be a gauge transformation of the principal bundle $P$. $U$ can be viewed as an $\text{SU}(2)$ Čech 0–cochain $\{U_\alpha\}$ satisfying

$$U_\alpha = R_{\alpha \beta} U_\beta R_{\alpha \beta}^{-1}.$$  \hfill (2.12)

By the isomorphism $H : \text{SU}(2)/\mathbb{Z}_2 \to \text{SO}(3)$, $U$ yields a $\text{SO}(3)$ valued endomorphism $\underline{U} \in \text{SO}(M, E)$ of the vector bundle $E$ (an $\text{SO}(3)$ endomorphism for short). $\underline{U}$ is represented by an $\text{SO}(3)$ Čech 0–cochain $\{\underline{U}_\alpha\}$ satisfying

$$\underline{U}_\alpha = R_{\alpha \beta} \underline{U}_\beta R_{\alpha \beta}^{-1}.$$  \hfill (2.13)
This establishes an injective homomorphism of $\text{Gau}(M, P)/\mathbb{Z}_2$ into $\text{SO}(M, E)$, where $\mathbb{Z}_2$ is realized as the sign group $\{\pm 1\}$. The homomorphism is not surjective in general.

A gauge transformation $U \in \text{Gau}(M, P)$ acts on a connection $A \in \text{Conn}(M, P)$ according to

$$A^U = \text{Ad}U^{-1}A + U^{-1}dU. \quad (2.14)$$

This action translates into one of the $\text{SO}(3)$ endomorphism $U \in \text{SO}(M, E)$ corresponding to $U$ under the homomorphism $\text{Gau}(M, P)/\mathbb{Z}_2 \rightarrow \text{SO}(M, E)$ on the connection $\underline{A} \in \text{Conn}(M, E)$ corresponding to $A$ under the isomorphism $\text{Conn}(M, P) \cong \text{Conn}(M, E)$ given by

$$\underline{A}^U = U^{-1}A + \lambda_U. \quad (2.15)$$

where $\lambda_U$ is the $\mathbb{E}_3$ valued 1–form defined by

$$\star \lambda_U = -U^{-1}dU. \quad (2.16)$$

A gauge transformation $U \in \text{Gau}(M, P)$ acts on a section $s \in \Omega^p(M, \text{Ad}P)$ as

$$s^U = \text{Ad}U^{-1}s. \quad (2.17)$$

The homomorphisms $\text{Gau}(M, P)/\mathbb{Z}_2 \rightarrow \text{SO}(M, E)$, $\Omega^p(M, \text{Ad}P) \cong \Omega^p(M, E)$, described above, map $s^U$ into the result of the action of the $\text{SO}(3)$ endomorphism $U$ on $s \in \Omega^p(M, E)$, $\underline{s}^U$. The latter is given by

$$\underline{s}^U = U^{-1}s. \quad (2.18)$$

All the isomorphisms established above will be repeatedly used in the following.

## 3. Cho structures

In this section, we shall introduce the basic notion of Cho structure. Cho structures provide a local formulation of Cho’s original gauge theoretic formalism [23,24].

### CHO STRUCTURES

Let $M$ be a manifold. Let $P$ be a principal $\text{SU}(2)$ bundle on $M$ and let $E$ be the vector bundle defined in (2.2).

**Definition 3.1** A Cho structure $\mathcal{C} = (\{n_i\}, \{A_i\}, \{T_{ij}\})$ of $E$ subordinated to the open cover $\{O_i\}$ of $M$ consists of the following elements.  

\[ \text{Note that the cover } \{O_i\} \text{ is logically distinct from the cover } \{O_\alpha\} \text{ of sect. 2.} \]
i) A collection \( \{ n_i \} \) of sections \( n_i \in \Omega^0(O_i, E) \) such that
\[
n_i^2 = 1. \tag{3.1}
\]

\( \text{(ii)} \) A collection \( \{ A_i \} \) of connections \( A_i \in \text{Conn}(O_i, E) \).

\( \text{(iii)} \) A collection \( \{ T_{ij} \} \) of sections \( T_{ij} \in \text{SO}(O_{ij}, E) \).

It is further assumed that these have the following properties.

\( a) \) \( n_i \) is covariantly constant with respect to \( A_i \),
\[
D_{A_i} n_i = 0. \tag{3.2}
\]

where \( D_{A_i} \) is the covariant derivative associated with \( A_i \) (cfr. eq. (2.9)).

\( b) \) \( T_{ij} \) matches \( n_i, n_j \),
\[
n_i = T_{ij} n_j. \tag{3.3}
\]

\( c) \) The \( T_{ij} \) are normalized so that
\[
T_{ii} = 1, \tag{3.4a}
\]
\[
T_{ij} T_{ji} = 1. \tag{3.4b}
\]

We denote by \( C_E \) the set of Cho structures \( \mathcal{C} \) of \( E \) subordinated to some open cover of \( M \).

**Remark 3.2** From (3.1), (3.2), it follows that a Cho structure \( \mathcal{C} \) of \( E \) describes a family of local SO(2) reduction of \( E \) with compatible SO(2) reduced connections subordinated to the associated covering [34]. We shall elaborate on this point in greater detail in sect. 5 below.

To compare Cho structures of \( E \) subordinated to different open covers, it is necessary to resort to refinement of the covers.

**Definition and Proposition 3.3** Let \( \mathcal{C} = (\{ n_i \}, \{ A_i \}, \{ T_{ij} \}) \) be a Cho structure of \( E \) subordinated to the open cover \( \{ O_i \} \). Let \( \{ O_i \} \) be an open cover which is a refinement of \( \{ O_i \} \) and let \( f \) be a refinement map (so that \( O_i \subseteq O_{f(i)} \)). We set
\[
\bar{n}_i = n_{f(i)}|\bar{O}_i, \tag{3.5a}
\]
\[
\bar{A}_i = A_{f(i)}|\bar{O}_i, \tag{3.5b}
\]
\[
\bar{T}_{ij} = T_{f(i)f(j)}|O_{ij}. \tag{3.5c}
\]
Then, \( \mathcal{C} = (\{\bar{n}_i\}, \{\bar{A}_i\}, \{\bar{T}_{ij}\}) \) is a Cho structure of \( E \) subordinated to the cover \( \{\bar{O}_i\} \). We call \( \mathcal{C} \) the refinement of \( \mathcal{C} \) associated with the refinement \( \{\bar{O}_i\} \) of \( \{O_i\} \) and the refinement map \( f \).

**Proof**  The verification of (3.1)–(3.4) is straightforward. \( \square \)

**Definition and Proposition 3.4** Two Cho structures
\( \mathcal{C} = (\{n_i\}, \{A_i\}, \{T_{ij}\}) \), \( \mathcal{C}' = (\{n'_i\}, \{A'_i\}, \{T'_{ij}\}) \) of \( E \) subordinated to the same open cover \( \{O_i\} \) are called equivalent if they are related as
\[
\begin{align*}
    n'_i &= U_{i}^{-1} n_i, \\
    A'_i &= U_{i}^{-1} (A_i - s_i n_i) + \lambda U_{i}, \\
    T'_{ij} &= U_{i}^{-1} T_{ij} \exp(-r_{ij} \star n_j) U_{j},
\end{align*}
\]
for some \( \text{SO}(E) \) Čech 0–cochain \( \{U_i\} \) and some \( \Omega^1 \) Čech 0–cochain \( \{s_i\} \) and \( \Omega^0 \) Čech 1–cochain \( \{r_{ij}\} \) independent from the trivialization of \( E \) used, where \( \lambda U_{i} \) is defined by (2.16) with \( U \) replaced by \( U_{i} \). More generally, two Cho structures
\( \mathcal{C} = (\{n_i\}, \{A_i\}, \{T_{ij}\}) \), \( \mathcal{C}' = (\{n'_i\}, \{A'_i\}, \{T'_{ij}\}) \) of \( E \) subordinated to the open covers \( \{O_i\}, \{O'_i\} \) are called equivalent if there is a common refinement \( \{\bar{O}_i\} \) of \( \{O_i\}, \{O'_i\} \) and refinement maps \( f, f' \) such that the associated refinements \( \mathcal{C} = (\{\bar{n}_i\}, \{\bar{A}_i\}, \{\bar{T}_{ij}\}) \), \( \mathcal{C}' = (\{\bar{n}'_i\}, \{\bar{A}'_i\}, \{\bar{T}'_{ij}\}) \) are equivalent in the restricted sense just defined. The above defines an equivalence relation on the set of Cho structures of \( E \). In the following, we denote by \( [\mathcal{C}] \) the equivalence class of the Cho structure \( \mathcal{C} \).

**Proof**  It is easy to see that (3.6) is consistent with (3.1)–(3.4). It is straightforward though tedious to verify the axioms of equivalence relation. \( \square \)

**Remark 3.5** The notion of equivalence defined in (3.6) is more general than gauge equivalence, to which it reduces when \( s_i = 0 \) and \( r_{ij} = 0 \).

**Remark 3.6** A Cho structure \( \mathcal{C} \) is equivalent to all its refinements \( \mathcal{C} \).

**Discussion**

The topological setting in which interesting Cho structures are defined has the following features. There are a manifold \( M_0 \), a principal SU(2) bundle \( P_0 \) on \( M_0 \) and a closed submanifold \( N_\varepsilon \) of \( M_0 \) depending on \( \mathcal{C} \) such that
\[
M = M_0 \setminus N_\varepsilon, \quad (3.7)
\]
and that \( n_i, A_i \) become singular on approaching \( N_\varepsilon \), when \( \partial O_i \) intersects \( N \). That is, \( N_\varepsilon \) is their defect manifold. \( N_\varepsilon \) has codimension 3, basically because the \( n_i \) are valued in the 3–dimensional vector space \( E_3 \). Further, there is a submanifold \( D \) of \( M_0 \) bounded by \( N_\varepsilon \),

\[
P = P_0|_M \tag{3.8}
\]

such that \( O_{ij} \cap D = \emptyset \) for all \( i, j \) and that the \( T_{ij} \) become singular when extended, if possible, beyond their domain \( O_{ij} \) to a larger one intersecting \( D \). \( D \) has codimension 2. \( D \) is not uniquely defined by \( C \) and is a generalization of the well known Dirac string.

The physical origin of the setting just described has been illustrated in the introduction (see also [19, 20]). In physical applications, \( M_0 \) is some 4–dimensional space–time manifold, such as \( \mathbb{R}^4, S^4, S^1 \times \mathbb{R}^3, T^4 \), \( P_0 \) is a principal \( SU(2) \) bundle defined by the \( SU(2) \) valued monodromy matrix functions of the gauge fields and \( N_\varepsilon \) is a set of closed monopole world lines forming a knot in \( M_0 \).

It is important to realize that the constructions worked out in this paper are fully general and do not require that \( M \) and \( P \) are of the form (3.7), (3.8) or that \( \dim M = 4 \).

**Examples**

We now illustrate the above analysis with a few examples.

**Example 3.7. A small monopole loop in a single instanton background**

The authors of ref. [35] found a solution of the differential maximal Abelian gauge in a single instanton background. Its defect manifold is a loop. When the radius of the loop is much smaller than the instanton size, an analytic expression is available.

In this case, \( M_0 \) is the 4–sphere \( S^4 \) and \( P_0 \) is the trivial principal bundle \( S^4 \times SU(2) \). Here, we view \( S^4 \) as the one point compactification of \( \mathbb{R}^4 \), \( S^4 = \mathbb{R}^4 \cup \{ \infty \} \). We define coordinates \( u, v \in [0, +\infty], \varphi, \psi \in [0, 2\pi] \) in \( \mathbb{R}^4 \cup \{ \infty \} \) by

\[
x^1 + \sqrt{-1}x^2 = u \exp(\sqrt{-1}\varphi), \quad x^3 + \sqrt{-1}x^4 = v \exp(\sqrt{-1}\psi). \tag{3.10}
\]

Note that these are ill defined at the planes \( x^1 = x^2 = 0, x^3 = x^4 = 0 \) and at infinity. Then, \( M = S^4 \setminus N, P = (S^4 \setminus N) \times SU(2) \) and \( E = (S^4 \setminus N) \times E_3 \), where

\[
N = \{ x | x \in \mathbb{R}^4, u = R_0, v = 0 \}, \tag{3.11}
\]

with \( R_0 > 0 \).
We set
\[
O_1 = \{x | x \in \mathbb{R}^4 \cup \{\infty\}, u < R_1 \text{ and } v < R_1 \} \setminus N,
\]
\[
O_2 = \{x | x \in \mathbb{R}^4 \cup \{\infty\}, u > R_2 \text{ or } v > R_2 \},
\]
where \(R_1 > R_2 > R_0\). Note that \(N \cap O_2 = \emptyset\). \(\{O_1, O_2\}\) is an open covering of \(M\).

Let \(\alpha : \mathbb{R}^4 \cup \{\infty\} \to [0, 2\pi], \beta : \mathbb{R}^4 \cup \{\infty\} \to [0, \pi], \gamma : \mathbb{R}^4 \cup \{\infty\} \to [0, 2\pi]\) be given by
\[
\alpha = \varphi - \psi \mod 2\pi, \tag{3.13a}
\]
\[
\beta = \tan^{-1}\left(\frac{2uv}{u^2 - v^2 - R_0^2}\right) \mod \pi, \tag{3.13b}
\]
\[
\gamma = \varphi + \psi \mod 2\pi. \tag{3.13c}
\]
\(\alpha, \gamma\) are ill defined at the planes \(u = 0, v = 0\), \(\beta\) is ill defined at the hypersurfaces \(u^2 - v^2 - R_0^2 = 0\). Let \(s_1, s_2, \exp(-\sqrt{-1}\chi_{12})\) be the local representatives of a connection \(s\) and the transition function of some \(\mathbb{T}\) principal bundle on \(M\), respectively, so that
\[
s_2 - s_1 = -d\chi_{12}. \tag{3.14}
\]

We set
\[
n_1 = \left[\sin \beta (\cos \alpha \epsilon_1 + \sin \alpha \epsilon_2) + \cos \beta \epsilon_3\right]|_{O_1}, \tag{3.15}
\]
\[
n_2 = \epsilon_3,
\]
\[
A_1 = \left[-d\alpha \epsilon_3 - d\beta (-\sin \alpha \epsilon_1 + \cos \alpha \epsilon_2) \right. \\
\left. + (\cos \beta d\alpha - s_1)(\sin \beta (\cos \alpha \epsilon_1 + \sin \alpha \epsilon_2) + \cos \beta \epsilon_3)\right]|_{O_1}, \tag{3.16}
\]
\[
A_2 = -s_2 \epsilon_3,
\]
\[
T_{12} = \exp(-\alpha \star \epsilon_3) \exp(-\beta \star \epsilon_2) \exp(-\gamma + \chi_{12} \star \epsilon_3)|_{O_{12}}, \tag{3.17}
\]
\[
T_{21} = \exp((\gamma + \chi_{12}) \star \epsilon_3) \exp(\beta \star \epsilon_2) \exp(\alpha \star \epsilon_3)|_{O_{12}},
\]
\[
T_{11} = T_{22} = 1.
\]

It is straightforward to check that (3.15)–(3.17) define a Cho structure \(\mathcal{C}\) of \(E\) subordinated to \(\{O_1, O_2\}\). Its defect manifold \(N_\mathcal{C}\) is precisely \(N\),
\[
N_\mathcal{C} = N. \tag{3.18}
\]
A Dirac sheet $D$ is the disk bounded by $N$ in the plane $v = 0$.

**Example 3.8 A Higgs field on a 4–torus**

In this case, $M_0$ is the 4–torus $T^4$ and $P_0$ is the trivial principal bundle $T^4 \times SU(2)$. The defect manifold $N$ is empty, $N = \emptyset$. Thus, $M = T^4$, $P = T^4 \times SU(2)$ and $E = T^4 \times E_3$.

We view $T^4$, as the quotient of $\mathbb{R}^4$ by a lattice $\Lambda \subseteq \mathbb{R}^4$, $T^4 = \mathbb{R}^4/\Lambda$. Let $\{O_i\}$ an open covering of $T^4$ such that, for each $i$, $O_i$ is a simply connected non empty open subset of $T^4$. As is well known, for each $i$, there is a local coordinate $x_i$ of $T^4$ defined on $O_i$ such that

$$\theta = x_i(\theta) + \Lambda,$$

for $\theta \in O_i$. Further, when $O_{ij} \neq \emptyset$, there is $\xi_{ij} \in \Lambda$ such that

$$x_i = x_j + \xi_{ij} \quad \text{on } O_{ij} \quad (3.20)$$

The collection $\{\xi_{ij}\}$ is a $\Lambda$ valued Čech 1–cocycle.

Let $c : T^4 \to \mathbb{R}^{4\hat{v}}$ be a smooth function and $k_0$, $e_0 \in E_3$ be unit vectors. Let $s_i$, $\exp (-\sqrt{-1}\chi_{ij})$ be the local representatives of a connection $s$ and the transition function of some $T$ principal bundle on $M_0$, respectively, so that

$$s_j - s_i = -d\chi_{ij}. \quad (3.21)$$

We set

$$\hat{n}_i = \exp (-\langle c, x_i \rangle \star k_0)e_0, \quad (3.22)$$

$$\hat{A}_j = -\exp (-\langle c, x_i \rangle \star k_0)[d\langle c, x_i \rangle e_0 \times (k_0 \times e_0) + s_i e_0], \quad (3.23)$$

$$\hat{T}_{ij} = \exp (-\langle c, x_i \rangle \star k_0) \exp (-\chi_{ij} \star e_0) \exp (\langle c, x_j \rangle \star k_0) |_{O_{ij}}. \quad (3.24)$$

It is straightforward to check that (3.20)–(3.23) define a Cho structure $\hat{\mathcal{C}}$ of $E$ subordinated to $\{O_i\}$. Its defect manifold $N_{\mathcal{C}}$ is empty

$$N_{\mathcal{C}} = \emptyset \quad (3.25)$$

**4. The Deligne cohomology class $D_{\mathcal{C}}$**

To any equivalence class of Cho structures, there is associated a flat degree 3 length 3 Deligne cohomology class. In this section, we describe its construction.
Construction of the Deligne cohomology class $D_\varphi$

Let $M$, $P$ and $E$ be as in sect. 3. Let $\mathcal{C} = \{\{n_i\}, \{A_i\}, \{T_{ij}\}\}$ be a Cho structure of $E$ subordinated to the open cover $\{O_i\}$ (cfr. def. 3.1).

Definition and Proposition 4.1 One has

$$A_i = A_{ii} - a_i n_i,$$  \hspace{1cm} (4.1)

where

$$A_{ii} = A_0 \cdot n_i + dn_i \times n_i,$$  \hspace{1cm} (4.2)

$A_0 \in \text{Conn}(M, E)$ is a fixed background connection of $E$ and $a_i \in \Omega^1(O_i)$. $A_{ii} \in \text{Conn}(O_i, E)$ is a connection of $E$ on $O_i$ and $a_i$ does not depend on the trivialization of $E$ used.

Proof This follows easily from (2.9), (3.2).

Remark 4.2 The connection $A_i$ has the body fixed form of ref. [21].

Definition and Proposition 4.3 Let $F_{Ai}$ be the curvature of $A_i$ (cfr. eq. (2.11)).

There is a 2–form $\eta_i \in \Omega^2(O_i)$ such that

$$F_{Ai} = -\eta_i n_i.$$  \hspace{1cm} (4.3)

$\eta_i$ is given explicitly by

$$\eta_i = \frac{1}{2} n_j \cdot dn_i \times dn_j - d(A_0 \cdot n_j) + da_i.$$  \hspace{1cm} (4.4)

$\eta_i$ does not depend on the trivialization of $E$ used.

Proof From (3.2) and the Ricci identity $D_{Ai}D_{Ai}n_i = F_{Ai} \times n_i$, one finds that

$$F_{Ai} \times n_i = 0.$$  \hspace{1cm} (4.5)

It follows that $F_{Ai}$ is of the form (4.3). The expression (4.4) of $\eta_i$ follows readily from (2.11), (3.1), (4.1), (4.2). The last statement is obvious from (4.3).

Definition and Proposition 4.4 There is a 1–form $\psi_{ij} \in \Omega^1(O_{ij})$ such that

$$A_i = T_{ij}(A_j - \zeta_{ij} + \psi_{ij} n_j),$$  \hspace{1cm} (4.6)
where $\zeta_{ij}$ is the $E_3$ valued 1-form defined by

$$
\ast \zeta_{ij} = -T_{ij}^{-1} dT_{ij}.
$$

(4.7)

$\psi_{ij}$ is given explicitly by

$$
\psi_{ij} = \zeta_{ij} \cdot n_j + A_0 \cdot (n_i - n_j) - a_i + a_j.
$$

(4.8)

$\psi_{ij}$ does not depend on the trivialization of $E$ used.

**Proof** Combining (3.2), (3.3), one finds that

$$
(A_i - T_{ij} (A_j - \zeta_{ij})) \times T_{ij} a_j = 0.
$$

(4.9)

It follows that $A_i$ is of the form (4.6) for some $\psi_{ij} \in \Omega^1(O_{ij})$. $\psi_{ij}$ can be computed explicitly from (3.1), (3.3), (4.1), (4.2). The last statement is easily verified. \qed

**Definition and Proposition 4.5** There is a 0-form $\phi_{ijk} \in \Omega^0(O_{ijk})$ such that

$$
T_{ij} T_{jk} T_{ki} = \exp(-\phi_{ijk} \ast n_i).
$$

(4.10)

$\phi_{ijk}$ does not depend on the trivialization of $E$ used.

**Proof** The consistency of (3.3) requires that

$$
n_i = T_{ij} T_{jk} T_{ki} n_i.
$$

(4.11)

From here it follows that there is $\phi_{ijk} \in \Omega^0(O_{ijk})$ satisfying (4.10). The last statement is obvious. \qed

**Definition 4.6** We set

$$
K_{ijkl} = \phi_{jkl} - \phi_{ikl} + \phi_{ijl} - \phi_{ijk}.
$$

(4.12)

**Remark 4.7** The $\phi_{ijk}$ are defined modulo $2\pi \mathbb{Z}$. So, we are allowed to redefine

$$
\phi_{ijk} \rightarrow \phi_{ijk} + m_{ijk},
$$

(4.13)

where $m_{ijk} \in 2\pi \mathbb{Z}$, if we wish so. To this, there corresponds an obvious redefinition of $K_{ijkl}$. 

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Definition and Proposition 4.8  \{\eta_i\} is an \Omega^2 \check{\text{Cech}} 0–cochain, \{\psi_{ij}\} is an \Omega^1 \check{\text{Cech}} 1–cochain, \{\phi_{ijk}\} is an \Omega^0 \check{\text{Cech}} 2–cochain and \{K_{ijkl}\} is a \pi 2\pi \check{\text{Cech}} 3–cochain (upon suitably fixing the indeterminacy (4.13)). Further, one has

\[
\begin{align*}
\delta \eta_{ij} &= d\psi_{ij}, \\
\delta \psi_{ijk} &= d\phi_{ijk}, \\
\delta \phi_{ijkl} &= K_{ijkl}, \\
\delta K_{ijklm} &= 0,
\end{align*}
\]  

(4.14a)  

(4.14b)  

(4.14c)  

(4.14d)

where \delta is the \check{\text{Cech}} coboundary operator defined in (A.7). As a consequence, the sequence \((\{\eta_i\}, \{\psi_{ij}\}, \{\phi_{ijk}\}, \{K_{ijkl}\})\) is a length 3 Deligne 3–cocycle (cfr. app. A, eq. (A.4)) and, thus, it defines a degree 3 length 3 Deligne cohomology class \(D_\varepsilon \in H^3(M, D(3)^\bullet)\). This class is unaffected by any redefinition of the form (4.13) and is thus associated with the Cho structure \(\mathcal{C}\).

Proof Using (3.4), (4.7), (4.10), it is a straightforward matter to verify that \(\{\eta_i\}, \{\psi_{ij}\}, \{\phi_{ijk}\}\) and \(\{K_{ijkl}\}\) are \check{\text{Cech}} cochains of the stated types. The \(2\pi \mathbb{Z}\) valuedness of \(\{K_{ijkl}\}\) follows from the relation

\[
\exp(-\delta \phi_{ijkl} \ast n_i) = 1,
\]

(4.15)

which can be shown by repeated application of (4.10). The proof of relations (4.14) is straightforward by observing that, from (4.3), (4.6),

\[
\begin{align*}
\delta \eta_{ij} &= F_{A_i} \cdot n_i - F_{A_j} \cdot n_j, \\
\delta \psi_{ijk} &= \zeta_{jk} \cdot n_k - \zeta_{ik} \cdot n_k + \zeta_{ij} \cdot n_j
\end{align*}
\]

(4.16)  

(4.17)

and upon using (3.1), (3.3), (3.4), (4.6), (4.7), (4.10) and the identities

\[
\exp(-\varphi \ast n) = 1 + (1 - \cos \varphi)(\ast n)^2 - \sin \varphi \ast n, \quad \exp(\varphi \ast n)d\exp(-\varphi \ast n) = -d\varphi \ast n - \sin \varphi \ast dn - (1 - \cos \varphi) \ast (dn \times n),
\]

(4.18)  

(4.19)

where \(\varphi\) is a local 0–form. The remaining statements are easily verified. \square

Next, we shall study the dependence of \(D_\varepsilon\) on \(\mathcal{C} \in C_E\).
Proposition 4.9  Let $\mathcal{C}$ be a Cho structure of $E$ subordinated to the open cover $\{O_i\}$. Let $\{\bar{O}_i\}$ be an open cover which is a refinement of $\{O_i\}$ and let $f$ be a refinement map. Let $\mathcal{C}$ be the associated refinement of $\mathcal{C}$ (cfr. def. 3.3). Then,

$$D_{\mathcal{C}} = D_{\mathcal{C}'} \quad (4.20)$$

Thus, refinement of Cho structures is compatible with Deligne cohomology.

Proof  It is readily verified that

$$\bar{\eta}_i = \eta_{f(i)}|_{\bar{O}_i}, \quad (4.21a)$$
$$\bar{\psi}_{ij} = \psi_{f(i)f(j)}|_{\bar{O}_{ij}}, \quad (4.21b)$$
$$\bar{\phi}_{ijk} = \phi_{f(i)f(j)f(k)}|_{\bar{O}_{ijk}}, \quad (4.21c)$$
$$\bar{K}_{ijkl} = K_{f(i)f(j)f(k)f(l)}|_{\bar{O}_{ijkl}}, \quad (4.21d)$$

with obvious notation. This means that the Deligne 3–cocycle corresponding to $\mathcal{C}$ is the refinement of the Deligne 3–cocycle corresponding to $\mathcal{C}$ associated with the refinement $\{\bar{O}_i\}$ of $\{O_i\}$ and the refinement map $f$. (4.20) follows immediately.

Proposition 4.10  Let $\mathcal{C}, \mathcal{C}'$ be equivalent Cho structures of $E$ (cfr. def. 3.4). Then,

$$D_{\mathcal{C}} = D_{\mathcal{C}’}. \quad (4.22)$$

Thus, the Deligne cohomology classes of equivalent Cho structures are equal. Since $D_{\mathcal{C}}$ depends on the Cho structure $\mathcal{C}$ only through its equivalence class $[\mathcal{C}]$, we shall use occasionally the notation $D_{[\mathcal{C}]}$.

Proof  Assume first that $\mathcal{C}, \mathcal{C}'$ are subordinated to the same open cover $\{O_i\}$. As $\mathcal{C}, \mathcal{C}'$ are equivalent, (3.6) holds. Using (4.1), (4.4), (4.7), (4.8) and (4.10) and exploiting (4.19), it is straightforward to show that $\{s_i\}$ is a $\Omega^1$ Čech 0–cochain, $\{r_{ij}\}$ is a $\Omega^0$ Čech 1–cochain and that, for some $2\pi Z$ Čech 2–cochain $\{k_{ijk}\}$,

$$\eta'_i = \eta_i + ds_i, \quad (4.23a)$$
$$\psi'_{ij} = \psi_{ij} + \delta s_{ij} + dr_{ij}, \quad (4.23b)$$
$$\phi'_{ijk} = \phi_{ijk} + \delta r_{ijk} + k_{ijk}, \quad (4.23c)$$
$$K'_{ijkl} = K_{ijkl} + \delta k_{ijkl}. \quad (4.23d)$$
Therefore, the Deligne 3–cocycles corresponding to \( \mathcal{C}, \mathcal{C}' \) differ by a length 3 Deligne 3–coboundary (cfr. eq. (A.5)). It follows immediately that \( D_\mathcal{C} = D_\mathcal{C}' \). Next, let \( \mathcal{C}, \mathcal{C}' \) be subordinated to distinct open covers \( \{O_i\}, \{O'_i\} \). Assume that \( \mathcal{C}, \mathcal{C}' \) are equivalent. Then, there is a common refinement \( \{\tilde{O}_i\} \) of \( \{O_i\}, \{O'_i\} \) and refinement maps \( f, f' \) such that the associated refinements \( \tilde{\mathcal{C}}, \tilde{\mathcal{C}}' \) are equivalent. By the result just shown, \( D_{\tilde{\mathcal{C}}} = D_{\tilde{\mathcal{C}}'} \).

On the other hand, by (4.20), \( D_\mathcal{C} = D_{\tilde{\mathcal{C}}}, D_\mathcal{C}' = D_{\tilde{\mathcal{C}}'} \). (4.22) follows.

\[ \text{Remark 4.11} \quad \text{Since a Cho structure} \; \mathcal{C} \; \text{is equivalent to anyone of its refinements} \; \tilde{\mathcal{C}}, \; \text{(4.20) is a particular case of (4.22).} \]

\[ \text{Proposition 4.12} \quad \text{For any Cho structure} \; \mathcal{C}, \; \text{the Deligne class} \; D_\mathcal{C} \; \text{is flat.} \]

\[ \text{Proof} \quad \text{From (3.2), (4.3) and the Bianchi identity} \; \begin{array}{c} \overline{D}_{\mathcal{A}}i\overline{F}_{\mathcal{A}}i = 0 \end{array}, \text{one verifies that} \]

\[ -d\eta_i = 0 \]  

(4.24)

and, so, the 2–form \( \eta_i \) is closed

\[ d\eta_i = 0. \]  

(4.25)

This shows that \( D_\mathcal{C} \) is flat as stated.

\[ \text{The main theorem on} \; D_\mathcal{C} \]

All the above results are summarized in the following theorem.

\[ \text{Theorem 4.13} \quad \text{There is a well defined natural map that associates with any equivalence class of Cho structures} \; [\mathcal{C}] \; \text{a flat degree 3 length 3 Deligne cohomology class} \; D_{[\mathcal{C}]} \in H^3(M, D(3)^*). \]

\[ \text{Remark 4.14} \quad \text{As discussed in app. A, with } D_\mathcal{C} \text{ there is associated an isomorphism class of Hermitian gerbes with Hermitian connective structure and curving. By prop. 4.12, this class is flat.} \]

This was to be expected in the present finite dimensional context (see ref. [32] for a discussion of this matter). Later, in sect. 6, we shall show that the class actually trivial.

\[ \text{Examples} \]

\[ \text{Example 4.15} \quad \text{A small monopole loop in a single instanton background} \]
This example was illustrated in sect. 3. It is straightforward to compute the Deligne 3–cocycle associated with the Cho structure $\mathcal{C}$ (3.15)–(3.17). Its only non vanishing components are

$$ \eta_1 = [-\sin \beta d\alpha d\beta] \big|_{O_1} + ds_1, \quad \eta_2 = ds_2, $$

$$ \psi_{12} = -\psi_{21} = [d\gamma + \cos \beta d\alpha] \big|_{O_{12}}. $$

**Example 4.16  A Higgs field on a 4–torus**

Also this example was illustrated in sect. 3. It is simple to compute the Deligne 3–cocycle associated with the Cho structure $\mathcal{C}$ defined (3.22)–(3.24)

$$ \eta_i = t \big|_{O_i}, $$

$$ \psi_{ij} = 0, $$

$$ \phi_{ijk} = 0, $$

$$ K_{ijkl} = 0, $$

where $t \in \Omega^2_{c2\pi\mathbb{Z}}(M)$ is a closed 2–form with periods in $2\pi\mathbb{Z}$ defined by $t \big|_{O_i} = ds_i$.

5. **Fine and almost fine Cho structures and the Deligne class $D_\mathcal{C}$**

In general, the local data of a Cho structure cannot be assembled in a global structure, but they do when the structure is fine.

**Fine Cho structures**

Let $M$, $P$ and $E$ be as in sect. 3.

**Definition 5.1**  A Cho structure $\mathcal{C} = (\{n_i\}, \{A_i\}, \{T_{ij}\})$ of $E$ subordinated to the open cover $\{O_i\}$ (cfr. def. 3.1) is said *fine* if

$$ T_{ij} T_{jk} T_{ki} = 1 $$

and

$$ A_i = T_{ij} (A_j - \zeta_{ij}) $$

(cfr. eqs. (4.6), (4.7). We denote by $\mathcal{C}_E^*$ the set of fine Cho structures of $E$ subordinated to some open cover of $M$. 

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Remark 5.2 Let \( \mathcal{C} \) be a fine Cho structure. If \( \mathcal{C} \) is a Cho structure refining \( \mathcal{C} \) (cfr. def. 3.3), then \( \mathcal{C} \), also, is fine. If \( \mathcal{C}' \) is a Cho structure equivalent to \( \mathcal{C} \) (cfr. def. 3.4), then \( \mathcal{C}' \) is not fine in general.

When dealing with fine Cho structures, one thus needs a sharper notion of equivalence compatible with fineness.

Definition and Proposition 5.3 Two fine Cho structures \( \mathcal{C} = (\{n_i\}, \{A_i\}, \{T_{ij}\}) \), \( \mathcal{C}' = (\{n'_i\}, \{A'_i\}, \{T'_{ij}\}) \) of \( E \) subordinated to the same open cover \( \{O_i\} \) are said finely equivalent, if

\[
\begin{align*}
n'_i & = U_i^{-1} n_i, & (5.3a) \\
A'_i & = U_i^{-1} A_i + \lambda U_i, & (5.3b) \\
T'_{ij} & = U_i^{-1} T_{ij} U_j, & (5.3c)
\end{align*}
\]

for some SO(\(E\)) Čech 0–cochain \( \{U_i\} \). It is readily checked that these relations are compatible with the fineness of \( \mathcal{C} \), \( \mathcal{C}' \). More generally, two fine Cho structures \( \mathcal{C} = (\{n_i\}, \{A_i\}, \{T_{ij}\}) \), \( \mathcal{C}' = (\{n'_i\}, \{A'_i\}, \{T'_{ij}\}) \) of \( E \) subordinated to the open covers \( \{O_i\}, \{O'_i\} \) are called finely equivalent if there is a common refinement \( \{\bar{O}_i\} \) of \( \{O_i\}, \{O'_i\} \) and refinement maps \( f, f' \) such that the associated fine refinements \( \mathcal{C} = (\{\bar{n}_i\}, \{\bar{A}_i\}, \{\bar{T}_{ij}\}) \), \( \mathcal{C}' = (\{\bar{n}'_i\}, \{\bar{A}'_i\}, \{\bar{T}'_{ij}\}) \) are finely equivalent in the restricted sense just defined. As suggested by the name, the above defines indeed an equivalence relation on the set of fine Cho structures. Below, we shall denote by \( \langle \mathcal{C} \rangle \) the fine equivalence class of a fine Cho structure \( \mathcal{C} \).

Proof These statements are straightforwardly verified. \( \square \)

Remark 5.4 Fine equivalence implies ordinary equivalence as defined in def. 3.4 above.

Definition and Proposition 5.5 A fine Cho structure \( \mathcal{C} \) defines an SO(3) vector bundle \( E_\mathcal{C} \), a connection \( A_\mathcal{C} \in \text{Conn}(M, E_\mathcal{C}) \) and a section \( n_\mathcal{C} \in \Omega^0(M, E_\mathcal{C}) \) such that

\[
\begin{align*}
n_\mathcal{C}^2 &= 1, & (5.4) \\
D_{A_\mathcal{C}} n_\mathcal{C} &= 0. & (5.5)
\end{align*}
\]

\( E_\mathcal{C} \) decomposes as

\[
E_\mathcal{C} = \dot{E}_\mathcal{C} \oplus C_\mathcal{C}. & (5.6)
\]
Here, $\hat{E}_\varphi, C_\varphi$ are the vector subbundles of $E_\varphi$ of local sections $s$ of $E_\varphi$ such that $s \cdot n_\varphi = 0$, $s \times n_\varphi = 0$, respectively. $\hat{E}_\varphi$ is an SO(2) vector bundle. $C_\varphi$ is isomorphic to the trivial bundle $\mathbb{R} \times M$. The complexification $\hat{E}_\varphi^C$ of $\hat{E}_\varphi$ decomposes as

$$\hat{E}_\varphi^C = e_\varphi \oplus e_\varphi^{-1}, \quad (5.7)$$

where $e_\varphi$ is a Hermitian line bundle. The connection $A_\varphi$ induces a Hermitian connection $a_\varphi \in \text{Conn}(M, e_\varphi)$ and the trivial connection on $C_\varphi$. The curvature $f_{a_\varphi} = da_\varphi$ of $a_\varphi$ is given by

$$f_{a_\varphi} = -F_{A_\varphi} \cdot n_\varphi. \quad (5.8)$$

Before proceeding to the mathematical study of fine Cho structures, it is necessary to realize the physical origin of this notion. Assume that $\dim M = 4$. In the body fixed frame of [21], the data $E_\varphi, A_\varphi, n_\varphi$ associated with a fine Cho structure $\mathcal{C}$ are precisely those describing a monopole configuration. The 1–dimensional defect manifold $N_\varphi$ of $\mathcal{C}$ (cfr. the discussion of sect. 3) is the set of the monopole world lines. The integral $\frac{-1}{4\pi} \int_{\Sigma} F_{A_\varphi} \cdot n_\varphi$, where $\Sigma$ is any 2–cycle of $M$, is the monopole charge enclosed by $\Sigma$. These aspects will be discussed in sect. 7 in greater detail.

**Definition and Proposition 5.6**  With any fine Cho structure $\mathcal{C}$, there is associated a degree 2 length 2 Deligne cohomology class $I_\varphi \in H^2(M, D(2)^\bullet)$. Its curvature $f_\varphi \in \Omega^2_{2\pi\mathbb{Z}}(M)$ is a closed 2–form with periods in $2\pi\mathbb{Z}$ called the Abelian curvature of $\mathcal{C}$. $f_\varphi$ is given by

$$f_\varphi = f_{a_\varphi}. \quad (5.9)$$

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Again, we shall sometimes use the notation \( I_{\langle \mathcal{C} \rangle}, f_{\langle \mathcal{C} \rangle} \).

**Proof** By prop. 5.5, the fine Cho structure \( \mathcal{C} \) determines a Hermitian line bundle with Hermitian connection \((e_\mathcal{C}, a_\mathcal{C})\). As is well known, the isomorphism class of the latter is represented by a degree 2 length 2 Deligne cohomology class \( I_\mathcal{C} \in H^2(M, D(2)\mathcal{C}) \) with curvature \( f_\mathcal{C} = f_{a_\mathcal{C}} \in \Omega^2_{c2\pi \mathbb{Z}}(M) \) (cfr. app. A). The remaining statement follows easily from the last part of prop. 5.5.

**Proposition 5.7** If \( \mathcal{C} \) is a fine Cho structure, then

\[
D_\mathcal{C} = 0
\]  
(5.10)

(cfr. def. 4.8)

**Proof** Let \( \mathcal{C} = (\{m_i\}, \{A_i\}, \{T_{ij}\}) \) be a fine Cho structure. We assume first that the open covering \( \{O_i\} \) underlying \( \mathcal{C} \) is good [33]. Combining (4.3), (4.6), (4.10) and (5.1), (5.2), (5.8), one finds that the associated degree 3 length 3 Deligne cocycle \((\{\eta_i\}, \{\psi_{ij}\}, \{\phi_{ijk}\}, \{K_{ijkl}\})\) (cfr. prop. 4.8) is given by

\[
\eta_i = f_{a_\mathcal{C}}|_{O_i},
\]  
(5.11a)

\[
\psi_{ij} = 0,
\]  
(5.11b)

\[
\phi_{ijk} = m_{ijk},
\]  
(5.11c)

\[
K_{ijkl} = \delta m_{ijkl},
\]  
(5.11d)

where \( \{m_{ijk}\} \) is some \( 2\pi \mathbb{Z} \) Čech 2–cochain. Here, \( f_{a_\mathcal{C}} \in \Omega^2_{c2\pi \mathbb{Z}}(M) \). Since the cover \( \{O_i\} \) is good, this suffices to show that our Deligne cocycle is a coboundary (cfr. eq. (A.5)) [32]. (5.10) follows. This result holds also when the cover \( \{O_i\} \) is not good. Indeed, every open cover \( \{O_i\} \) admits an open refinement \( \{\tilde{O}_i\} \) which is good [33]. Pick a refinement map \( f \). Let \( \tilde{\mathcal{C}} \) be the associated refinement. By rem. 5.2, \( \tilde{\mathcal{C}} \) is fine. Since the cover \( \{\tilde{O}_i\} \) is good, \( D_{\tilde{\mathcal{C}}} = 0 \), by the result just proved. By prop. 4.10, eq. (4.22), one has then \( D_\mathcal{C} = D_{\tilde{\mathcal{C}}} = 0 \).

Thus, (5.10) holds in general.

**Almost fine Cho structures**

The Cho structures which show up in physical applications are seldom fine. The following notion is therefore useful.
Definition 5.8 A Cho structure \( \mathcal{C} \) of \( E \) is said almost fine if it is equivalent to a fine Cho structure.

To anticipate, in sect. 6, by exploiting local diagonalizability, we shall see that, actually, every Cho structure is almost fine. However, this fact is not obvious \textit{a priori}.

Remark 5.9 A Cho structure equivalent to an almost fine Cho structure is almost fine. In particular, a fine Cho structure is also almost fine.

Proposition 5.10 A Cho structure \( \mathcal{C} \) is almost fine if and only if

\[ D_{\mathcal{C}} = 0. \tag{5.12} \]

Proof If \( \mathcal{C} \) is an almost fine Cho structure, then \( \mathcal{C} \) is equivalent to a fine Cho structure \( \mathcal{C}' \). By prop. 4.10, eq. (4.22), and prop. 5.7, eq. (5.10), \( D_{\mathcal{C}} = D_{\mathcal{C}'} = 0 \). Hence, (5.12) holds.

Conversely, let \( \mathcal{C} = (\{\underline{n}_i\}, \{A_i\}, \{T_{ij}\}) \) be a Cho structure satisfying (5.12). We assume first that the open covering \( \{O_i\} \) underlying \( \mathcal{C} \) is good. Then, the length 3 Deligne 3–cocycle \( (\{\eta_i\}, \{\psi_{ij}\}, \{\phi_{ijk}\}, \{K_{ijkl}\}) \) associated with \( \mathcal{C} \) is a coboundary. Therefore

\begin{align*}
\eta_i &= -ds_i, \tag{5.13a} \\
\psi_{ij} &= -\delta s_{ij} - dr_{ij}, \tag{5.13b} \\
\phi_{ijk} &= -\delta r_{ijk} - k_{ijk}, \tag{5.13c} \\
K_{ijkl} &= -\delta k_{ijkl}. \tag{5.13d}
\end{align*}

where \( \{s_i\} \) is an \( \Omega^1 \) ˇCech 0–cochain, \( \{r_{ij}\} \) is an \( \Omega^0 \) ˇCech 1–cochain and \( \{k_{ijk}\} \) is a \( 2\pi\mathbb{Z} \) ˇCech 2–cochain, by eq. (A.5). Then, the sequence \( \mathcal{C}' = (\{\underline{n}'_i\}, \{A'_i\}, \{T'_{ij}\}) \) defined by

\begin{align*}
\underline{n}'_i &= \underline{n}_i, \tag{5.14a} \\
A'_i &= A_i - s_i \underline{n}_j, \tag{5.14b} \\
T'_{ij} &= T_{ij} \exp(-r_{ij} \ast \underline{n}_j), \tag{5.14c}
\end{align*}

is a Cho structure of \( E \) subordinated to the cover \( \{O_i\} \) equivalent to \( \mathcal{C} \), by def. 3.4, eq. (3.6). By eq. (4.23), by construction, the associated length 3 Deligne 3–cocycle
\[(\{\eta'_i\}, \{\psi'_{ij}\}, \{\phi'_{ijk}\}, \{K'_{ijkl}\})\] is of the form
\[
\eta'_i = 0, \quad (5.15a)
\]
\[
\psi'_{ij} = 0, \quad (5.15b)
\]
\[
\phi'_{ijk} = m'_{ijk}, \quad (5.15c)
\]
\[
K'_{ijkl} = \delta m'_{ijkl}, \quad (5.15d)
\]
where \(\{m'_{ijk}\}\) is a \(2\pi\mathbb{Z}\) Čech 2–cochain. Thus, \(\mathcal{C}'\) is fine. Consequently, \(\mathcal{C}\) is almost fine.

This result holds also when the cover \(\{O_i\}\) is not good. Indeed, every open cover \(\{O_i\}\) admits an open refinement \(\{\bar{O}_i\}\) which is good. Pick a refinement map \(f\). Let \(\bar{\mathcal{C}}\) be the associated refinement. By rem. 3.6, prop. 4.10, eq. (4.22), and (5.12), \(D_\bar{e} = D_e = 0\). So, \(\bar{\mathcal{C}}\) is a Cho structure subordinated to the good open cover \(\{\bar{O}_i\}\) satisfying (5.12). Thus, \(\bar{\mathcal{C}}\) is almost fine, as just shown. On the other hand, \(\mathcal{C}\) is equivalent to \(\bar{\mathcal{C}}\). Thus, by rem. 5.9, \(\mathcal{C}\) is almost fine as well.

**Remark 5.11** When \(\mathcal{C}\) is almost fine, then \(\mathcal{C}\) is equivalent to infinitely many fine Cho structures \(\mathcal{C}'\). It is fairly obvious that equivalent almost fine Cho structures \(\mathcal{C}\) are characterized by the same set of fine Cho structures \(\mathcal{C}'\).

**Definition 5.12** We denote by \(F_\mathcal{C}\) the set of fine equivalence classes of fine Cho structures \(\mathcal{C}'\) equivalent to an almost fine Cho structure \(\mathcal{C}\).

By rem. 5.11, we shall also use the notation \(F_{[\mathcal{C}]}\).

**Proposition 5.13** Let \(\mathcal{C}\) be an almost fine Cho structure. Then, there exists a bijection \(H^2(M, D(2)^*) \cong F_\mathcal{C}\).

**Proof** Let \(\mathcal{C} = (\{\eta_i\}, \{A_i\}, \{T_{ij}\})\). By rem. 5.2 and rem. 5.11, we can assume without loss of generality that \(\mathcal{C}\) itself is fine and that the open cover \(\{O_i\}\) associated with \(\mathcal{C}\) is good. The degree 3 length 3 Deligne cocycle \((\{\eta_i\}, \{\psi_{ij}\}, \{\phi_{ijk}\}, \{K_{ijkl}\})\) associated with \(\mathcal{C}\) is thus of the form (5.11).

Let \(J \in H^2(M, D(2)^*)\). Let \((\{s_i\}, \{-r_{ij}\}, \{k_{ijk}\})\) be a degree 2 length 2 Deligne cocycle representing \(J\). Therefore, one has
\[
\delta s_{ij} + dr_{ij} = 0, \quad (5.16a)
\]
\[
\delta r_{ijk} + k_{ijk} = 0, \quad (5.16b)
\]
\[
\delta k_{ijkl} = 0 \quad (5.16c)
\]
by eq. (A.4). The sequence \( \mathcal{C}' = (\{\mu'_i\}, \{A'_i\}, \{T'_ij\}) \) defined by (5.14) is a Cho structure of \( E \) equivalent to \( \mathcal{C} \). By using (4.23), it is readily checked that the associated Deligne cocycle \( (\{\eta'_i\}, \{\psi'_ij\}, \{\phi'_ijkl\}) \) is also of the form (5.11). Hence, the Cho structure \( \mathcal{C}' \) is fine. If we replace the Deligne cocycle \( (\{s_i\}, \{-r_{ij}\}, \{k_{ijk}\}) \) by a cohomologous Deligne cocycle \( (\{s'_i\}, \{-r'_{ij}\}, \{k'_{ijk}\}) \), then the fine Cho structure \( \mathcal{C}' = (\{\mu'_i\}, \{A'_i\}, \{T'_ij\}) \) gets replaced by a finely equivalent fine Cho Structure \( \mathcal{C}'' = (\{\mu''_i\}, \{A''_i\}, \{T''ij\}) \). Indeed, by eq. (A.5), one has

\[
\begin{align*}
s'_i &= s_i + dp_i, \quad (5.17a) \\
r'_{ij} &= r_{ij} - \delta p_{ij} - l_{ij}, \quad (5.17b) \\
k'_{ijk} &= k_{ijk} + \delta l_{ijk}, \quad (5.17c)
\end{align*}
\]

where \( \{p_i\} \) is a \( \Omega^0 \) Čech 0–cochain and \( \{l_{ij}\} \) is a \( 2\pi\mathbb{Z} \) Čech 1–cochain. Using (4.18), (4.19), it is straightforward to check that \( \mathcal{C}' \), \( \mathcal{C}'' \) are related as in (5.3) with \( U_i = \exp(p_i \ast n_i) \) (after an obvious notational change). \( \mathcal{C}' \), \( \mathcal{C}'' \) are thus finely equivalent fine Cho structures. Therefore, we have a well defined map \( j_\varphi : H^2(M, D(2)\bullet) \to F_\varphi \) given by \( j_\varphi(J) = \langle \mathcal{C}' \rangle \). It is easy to see that \( j_\varphi \) does not depend on the good open cover \( \{O_i\} \).

Next, we shall show that \( j_\varphi \) is a bijection. We shall do so by proving that \( j_\varphi \) is injective and surjective. Let \( J^{(a)} \in H^2(M, D(2)\bullet), a = 1, 2 \), such that \( j_\varphi(J^{(1)}) = j_\varphi(J^{(2)}) \). Pick 2 length 2 Deligne cocycles \( (\{s^{(a)}_i\}, \{-r^{(a)}_{ij}\}, \{k^{(a)}_{ijk}\}) \) representing \( J^{(a)} \) and consider the representatives \( \mathcal{C}^{(a)} = (\{n^{(a)}_i\}, \{A^{(a)}_i\}, \{T^{(a)}ij\}) \) of \( j_\varphi(J^{(a)}) \) given by (5.14), as defined in the previous paragraph. By assumption, the fine Cho structures \( \mathcal{C}^{(1)}, \mathcal{C}^{(2)} \) are finely equivalent. So, they are related as in (5.3) for some \( \text{SO}(E) \) 0–cochain \( \{U_i\} \) (after an obvious notational change). On the other hand, for the representatives \( \mathcal{C}^{(a)}, n^{(a)}_i = n_i \) so that \( U_i = \exp(p_i \ast n_i) \) for an \( \Omega^0 \) Čech 0–cochain \( \{p_i\} \). Using (4.18), (4.19), it is straightforward to check that the Deligne cocycles \( (\{s^{(a)}_i\}, \{-r^{(a)}_{ij}\}, \{k^{(a)}_{ijk}\}) \) are related as in (5.17) for some \( 2\pi\mathbb{Z} \) Čech 1–cochain \( \{l_{ij}\} \) (after another obvious notational change), so that they are cohomologous. Hence, \( J^{(1)} = J^{(2)} \). This shows that \( j_\varphi \) is injective. Next, let \( \mathcal{C}' \) a fine Cho structure equivalent to \( \mathcal{C} \). Let us show that \( \langle \mathcal{C}' \rangle \) is contained in the range of \( j_\varphi \). By rem. 5.2 and for reasons explained in the previous paragraph, we can assume that \( \mathcal{C}, \mathcal{C}' \) are subordinated to the same open cover. Further, we are free to replace \( \mathcal{C}' \) be any other finely equivalent Cho structure subordinated to that cover. Let \( \mathcal{C}' = (\{n'_i\}, \{A'_i\}, \{T'ij\}) \). For the reasons just explained and the equivalence of \( \mathcal{C}, \mathcal{C}' \), we can assume that \( \mathcal{C}' \) is given by (5.14) for some \( \Omega^1 \) Čech 0–cochain \( \{s_i\} \) and \( \Omega^0 \) Čech 1–cochain \( \{r_{ij}\} \). On the other hand,
the Deligne cocycles \( \{\eta_i\}, \{\psi_{ij}\}, \{\phi_{ijk}\}, \{K_{ijkl}\} \) are of the form (5.11) and they are related as in (4.23) for some \( 2\pi Z \) Čech 2–cochain \( \{k_{ijk}\} \). It follows that, after perhaps a redefinition of \( \{k_{ijk}\} \), the sequence \( (\{s_i\}, \{-r_{ij}\}, \{k_{ijk}\}) \) is a degree 2 length 2 Deligne cocycle. This defines a class \( J \in H^2(M, D(2)\bullet) \). From the construction, it is obvious that \( j_{\mathfrak{F}}(J) = \langle \mathcal{C}' \rangle \). Hence, \( j_{\mathfrak{F}} \) is surjective. 

Remark 5.14 The bijection indicated in theorem (5.12) is not canonical, since it depends on a choice of a reference fine structure in the class [\( \mathcal{C} \)].

The relevance of the classification result of prop. 5.13 will become clear in sect. 7.

Examples

Example 5.15 A small monopole loop in a single instanton background

This example was illustrated in sect. 3. From (4.26), it appears that the Cho structure \( \mathcal{C} \) defined in (3.15)–(3.17) is not fine. Later, we shall see that \( \mathcal{C} \) is almost fine.

Example 5.16 A Higgs field on a 4–torus

Also this example was illustrated in sect. 3. From (4.27), it is evident that the Cho structure \( \hat{\mathcal{C}} \) defined (3.22)–(3.24) is fine and, thus, also almost fine.

6. Diagonalizability and almost fineness

In this section, we introduce a new type of Cho structures, the diagonalizable ones. In due course, we shall show that every diagonalizable Cho structure is almost fine (cfr. def. 5.8) and that every Cho structure is diagonalizable. In this way, we shall conclude that every Cho structure is almost fine.

Diagonalizable Cho structures

Let \( M, P \) and \( E \) be as in sect. 3.

Definition 6.1 A Cho structure \( \mathcal{C} = (\{n_i\}, \{A_i\}, \{T_{ij}\}) \) of \( E \) subordinated to the open cover \( \{O_i\} \) is said diagonalizable if it is of the form

\[
\begin{align*}
n_i &= S_i n_0, \\
A_i &= -S_i (\theta_i n_0 + \Lambda_i), \\
T_{ij} &= S_j \exp(-\omega_{ij} \star n_0) S_j^{-1},
\end{align*}
\]
where \( \mathbf{n}_0 \in \mathbb{E}_3 \) with \( \mathbf{n}_0^2 = 1 \), \( \{ \mathbf{S}_i \} \) is a Čech 0–cochain of the sheaf of SO(3) valued functions matching as

\[
\mathbf{S}_i = R_{\alpha\beta} \mathbf{S}_\beta
\]  

(6.2)

under changes of trivialization of \( E \), \( \{ \theta_i \} \) is an \( \Omega^1 \) Čech 0–cochain, \( \{ \omega_{ij} \} \) is an \( \Omega^0 \) Čech 1–cochain, both independent from the trivialization of \( E \) used, and \( \lambda_{\mathbf{S}_i} \) is defined as in (2.16) with \( \mathbf{U} \) replaced by \( \mathbf{S}_i \). More generally, \( C \) is called diagonalizable if it is equivalent to a Cho structure diagonalizable in the restricted sense just defined.

**Remark 6.2** It is easy to see that the restrictions imposed on \( \mathbf{S}_i, \theta_i, \omega_{ij} \) ensures that (3.1)–(3.4) are satisfied. We note that (6.1b), (6.1c) are implied by (6.1a), as is easy to see from (2.16), (3.3), (4.2).

The connection \( A_i \) above has the space fixed form of ref. [21].

**Proposition 6.3** A diagonalizable Cho structure \( C \) is almost fine.

**Proof** By rem. 5.9, one can assume that \( C \) satisfies (6.1) without loss of generality. Using (4.3), (4.6), (4.10), (6.1), one verifies directly that the associated length 3 Deligne 3–cocycle \( (\{ \eta_i \}, \{ \psi_{ij} \}, \{ \phi_{ijk} \}, \{ K_{ijkl} \}) \) is given by

\[
\eta_i = d\theta_i, \\
\psi_{ij} = \delta\theta_{ij} + d\omega_{ij}, \\
\phi_{ijk} = \delta\omega_{ijk} + l_{ijk}, \\
K_{ijkl} = \delta l_{ijkl},
\]

(6.3)

where \( \{ l_{ijk} \} \) is some \( 2\pi \mathbb{Z} \) Čech 2–cochain. From (A.5), it follows that \( (\{ \eta_i \}, \{ \psi_{ij} \}, \{ \phi_{ijk} \}, \{ K_{ijkl} \}) \) is a Deligne 3–coboundary. Hence, the Deligne class \( D_C \) vanishes. By prop. 5.10, eq. (5.12), one concludes that \( C \) is almost fine.

**The diagonalization theorem**

The natural question arises about under which conditions a Cho structure \( C \) is diagonalizable. The answer is that it always is.

**Theorem 6.4** Every Cho structure \( C \) is diagonalizable, thus almost fine, thus equivalent to a fine Cho structure. In particular, for every Cho structure \( C \), the Deligne class \( D_C \) is trivial

\[
D_C = 0.
\]  

(6.4)
Proof. To understand the essence of the matter, let us study the following closely related problem. Given an $E_3$ valued field $\mathfrak{n}$ defined on an open set $O$ of $\mathbb{R}^p$ such that $\mathfrak{n}^2 = 1$ and a vector $\mathfrak{n}_0 \in E_3$ with $\mathfrak{n}_0^2 = 1$, find an SO(3) valued function $R$ on $O$ such that
\[
\mathfrak{n} = R\mathfrak{n}_0. \tag{6.5}
\]
It is not difficult to find the general expression of such an $R$:
\[
R = R_{\mathfrak{n} \leftarrow \mathfrak{n}_0} \exp(-\varphi(\mathfrak{n}) \star \mathfrak{n}_0), \tag{6.6}
\]
where
\[
R_{\mathfrak{n} \leftarrow \mathfrak{n}_0} = 1 + \frac{1}{1 + \mathfrak{n} \cdot \mathfrak{n}_0} [(\mathfrak{n} \times \mathfrak{n}_0)]^2 + (\mathfrak{n} \times \mathfrak{n}_0). \tag{6.7}
\]
and $\varphi(\mathfrak{n})$ is an arbitrary function possibly depending on $\mathfrak{n}$. The point is that $R_{\mathfrak{n} \leftarrow \mathfrak{n}_0}$ is singular (though bounded) at those point $x \in O$ where $\mathfrak{n}(x) = -\mathfrak{n}_0$. These singularity cannot be compensated by a judicious choice of $\varphi(\mathfrak{n})$, as is easy to see from (4.18). So, when the field $\mathfrak{n}$ takes all possible values in the unit sphere $S^2(E_3)$ of $E_3$, a regular SO(3) valued function $R$ on $O$ fulfilling (6.4) cannot exist. However, it is easy to see that every point $x \in O$ has an open neighborhood $O_x \subseteq O$ such that the range of $\mathfrak{n}|O_x$ is a proper subset of $S^2(E_3)$. Indeed, if there were a point $x \in O$ such that for all open neighborhoods $O_x \subseteq O$ the range of $\mathfrak{n}|O_x$ were the whole $S^2(E_3)$, $\mathfrak{n}$ would be singular at $x$, while $\mathfrak{n}$ is regular on $O$. We conclude that $O$ has an open cover $\{O_a\}$, such that the problem posed has solution on each $O_a$ separately.

From the discussion of the previous paragraph, it follows that for every Cho structure $\mathcal{C} = (\{\mathfrak{n}_i\}, \{\mathcal{A}_i\}, \{\mathcal{T}_{ij}\})$ of $E$ subordinated to the open cover $\{O_i\}$, there is a refinement $\{\mathring{O}_i\}$ of $\{O_i\}$ and a refinement map $f$ such that the associated refinement $\mathring{\mathcal{C}} = (\{\mathfrak{n}_i\}, \{\mathcal{A}_i\}, \{\mathcal{T}_{ij}\})$ satisfies (6.1) (cfr. def. 3.3).}

Next we, explore the implications of this important result.

**Examples**

**Example 6.5** A small monopole loop in a single instanton background

This example was illustrated in sect. 3. The Cho structure $\mathcal{C}$ defined in (3.15)--(3.17) is not manifestly diagonalizable. But it actually is by theor. 6.4.

**Example 6.6** A Higgs field on a 4–torus

Also this example was illustrated in sect. 3. The Cho structure $\mathcal{C}$ defined (3.22)--(3.24) is evidently diagonalizable.

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7. Monopole and instanton charge, twist sectors and vortices

We want now find expressions for the monopole and instanton charge \( m_\cdot \), \( i_\cdot \) associated with a fine Cho structure \( \mathcal{C} \) and analyze their properties and relations.

**Monopole and instanton charge**

Let \( M \), \( P \) and \( E \) be as in sect. 3. Let \( \mathcal{C} \) be a fine Cho structure of \( E \) (cfr. def. 5.1). Let \( f_\cdot \in \Omega^2_{c2\pi\mathbb{Z}}(M) \) be the Abelian curvature of \( \mathcal{C} \) (cfr. def. 5.6).

**Definition 7.1** Let \( \Sigma \in Z^2_s(M) \) be a finite singular 2–cycle of \( M \) [33]. The monopole charge \( m_\epsilon(\Sigma) \) of \( \mathcal{C} \) in \( \Sigma \) is

\[
m_\epsilon(\Sigma) = \frac{1}{4\pi} \int_{\Sigma} f_\epsilon. \tag{7.1}
\]

Similarly, let \( \Omega \in Z^4_s(M) \) be a finite singular 4–cycle of \( M \). The instanton charge \( i_\epsilon(\Omega) \) of \( \mathcal{C} \) in \( \Omega \) is

\[
i_\epsilon(\Omega) = \frac{1}{16\pi^2} \int_{\Omega} f_\epsilon^2. \tag{7.2}
\]

It is important to realize that the objects just defined are indeed suitable generalizations of the customary physical objects carrying the same names. Indeed,

\[
\frac{1}{4\pi} f_\epsilon = -\frac{1}{4\pi} F_{A_\epsilon} \cdot n_\epsilon = \frac{1}{2\pi} \text{tr}(F_{A_\epsilon} n_\epsilon), \tag{7.3}
\]

\[
\frac{1}{16\pi^2} f_\epsilon^2 = \frac{1}{16\pi^2} F_{A_\epsilon} \cdot F_{A_\epsilon} = -\frac{1}{8\pi^2} \text{tr}(F_{A_\epsilon} F_{A_\epsilon}), \tag{7.4}
\]

by prop. 5.5, where the trace is over the fundamental representation of \( \mathfrak{su}(2) \) (cfr. sect. 2). This justifies the identification of \( m_\epsilon, i_\epsilon \) as monopole and instanton charge, respectively.

**Proof** From prop. 5.5, eq. (5.5), one has indeed that \( F_{A_\epsilon} = -f_\epsilon n_\epsilon \).

In the physical applications where \( M \) is a compact oriented 4–dimensional manifold, the 4–cycle \( \Omega \) is a representative of the fundamental class of \( M \). However, the following treatment applies to more general situations.

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\* Here and in the following, we denote by \( C^s_p(M) \), \( Z^s_p(M) \), \( B^s_p(M) \) the groups of \( p \)–dimensional singular chains, cycles and boundaries of \( M \), respectively, and by \( H^s_p(M) \) the degree \( p \) singular homology of \( M \). Further, we denote by \( C^s_p \) the presheaf of \( p \)–dimensional singular chains [36].
Remark 7.2 Since \( f_\epsilon \in \Omega^2_{c2\pi Z}(M) \), \( m_\epsilon(\Sigma) \in \mathbb{Z}/2 \), \( i_\epsilon(\Omega) \in \mathbb{Z}/4 \). However, if \( f_\epsilon/2 \in \Omega^2_{c2\pi Z}(M) \), then \( m_\epsilon(\Sigma) \), \( i_\epsilon(\Omega) \in \mathbb{Z} \).

Remark 7.3 By standard arguments [33], \( m_\epsilon \) induces a homomorphism \( m_\epsilon : H^2_2(M) \to \mathbb{Z}/2 \) on the degree 2 singular homology of \( M \). Similarly, \( i_\epsilon \) induces a homomorphism \( i_\epsilon : H^4_4(M) \to \mathbb{Z}/4 \) on the degree 4 singular homology of \( M \). To emphasize this, one may write \( m_\epsilon([\Sigma]), i_\epsilon([\Omega]) \) in (7.1), (7.2).

Remark 7.4 Since \( f_\epsilon \) depends on \( C \) only through its fine equivalence class \( \langle C \rangle \), by prop. 5.6, so do \( m_\epsilon, i_\epsilon \). For this reason, one occasionally writes \( m_\langle \epsilon \rangle, i_\langle \epsilon \rangle \) to emphasize this fact.

Discussion

Suppose that the Cho structure \( C = (\{\Sigma_i\}, \{A_i\}, \{T_{ij}\}) \) is subordinated to the open cover \( \{O_i\} \). From (4.4) and the relation \( F_{A_i} = -f_\epsilon|_{O_i} \), one has

\[
f_\epsilon|_{O_i} = \frac{1}{2}n_i \cdot d\Sigma_i \times d\Sigma_i - d(A_0 \cdot n_i) + da_i. \tag{7.5}
\]

We note that \( (\frac{1}{2}n_i \cdot d\Sigma_i \times d\Sigma_i)^2 = 0 \) identically, as is easy to see. From this fact and (7.5)

\[
f_\epsilon^2|_{O_i} = (n_i \cdot d\Sigma_i \times d\Sigma_i - d(A_0 \cdot n_i) + da_i)(- d(A_0 \cdot n_i) + da_i). \tag{7.6}
\]

This formula shows the importance of the terms \(-d(A_0 \cdot n_i) + da_i\) in (7.5) to yield a non vanishing instanton number. In the formulation of ref. [20], terms like these are associated with Dirac strings, sheets etc. and are distributional. In our formulation. Dirac strings, sheets etc. are traded for cocycles specifying glueing of locally defined fields. In turn, glueing requires these terms.

From (7.5), we see that \( w_\epsilon = m_\epsilon \) is a generalization of the customary winding number. Similarly, one can relate the instanton charge \( i_\epsilon \) to a suitable generalization of the Hopf invariant. To this end, we assume that the open sets of the cover \( \{O_i\} \) are contractible. As \( f_\epsilon \in \Omega^2_c(M) \) and \( O_i \) is contractible, one has

\[
f_\epsilon|_{O_i} = dv_i, \tag{7.7}
\]

for some \( v_i \in \Omega^1(O_i) \), by Poincaré’s lemma. Since \( f_\epsilon/4\pi \) is a winding number density, as explained above, \( v_i dv_i/16\pi^2 \) is a local Hopf invariant density on \( O_i \). Now, we note that

\[
f_\epsilon^2|_{O_i} = d(v_i dv_i). \tag{7.8}
\]
Let \( \Omega \in Z_4^s(M) \) be an \( \{O_i\} \)-small finite singular 4-cycle of \( M \) [33]. Then, there are a \( C_4^s \) Čech 0–chain \( \{U_i\} \) and a \( C_3^s \) Čech 1–chain \( \{V_{ij}\} \) such that

\[
\Omega = \sum_i U_i, \quad bU_i = \sum_j V_{ji},
\]

where \( b \) is the singular boundary operator [33]. Then, using (7.8), (7.9) and Stokes’ theorem, one easily shows that

\[
i_\varphi(\Omega) = \frac{1}{32\pi^2} \sum_{ij} \int_{V_{ij}} (v_j dv_j - v_i dv_i).
\]

Intuitively, \( i_\varphi(\Omega) \) is given by the net discontinuity of the local Hopf invariant densities \( v_i dv_i/16\pi^2 \) across the \( V_{ij} \). This is a cohomological interpretation of the calculations of ref. [37]. Exploiting barycentric subdivision, the above construction can be easily generalized to finite singular 4–cycles \( \Omega \in Z_4^s(M) \) which are not necessarily \( \{O_i\} \)-small [33].

**Liftability of \( \mathscr{C} \)**

Let \( \mathscr{C} \) be a fine Cho structure. The \( \text{SO}(3) \) vector bundle \( E_\varphi \) associated with \( \mathscr{C} \) (cfr. prop. 5.5) is not liftable to an \( \text{SU}(2) \) one in general. Thus, we expect the second Stiefel–Whitney class \( w_2(E_\varphi) \in H^2(M, \mathbb{Z}_2) \) of \( E_\varphi \) to play a role.

**Definition and Proposition 7.5** The Stiefel–Whitney class \( \varepsilon_\mathfrak{C} \in H^2(M, \mathbb{Z}_2) \) of \( \mathscr{C} \) is

\[
\varepsilon_\mathfrak{C} = w_2(E_\varphi).
\]

\( \varepsilon_\mathfrak{C} \) depends on \( \mathscr{C} \) only through its fine equivalence class \( \langle \mathscr{C} \rangle \).

For this reason, one may write \( \varepsilon_{\langle \mathfrak{C} \rangle} \) to stress this fact.

**Proof** The statement follows trivially from the last part of prop. 5.5.

**Definition and Proposition 7.6** The \( \text{SO}(3) \) bundle \( E_\varphi \) associated with \( \mathscr{C} \) is liftable to an \( \text{SU}(2) \) one precisely when \( \varepsilon_\mathfrak{C} \) vanishes. When this happens, the fine Cho structure \( \mathscr{C} \) is called *liftable*.

**Proof** The proof is trivial [34]

**Remark 7.7** If \( \mathscr{C} \) is liftable, there are in general several \( \text{SU}(2) \) lifts of \( E_\varphi \). As is well known, these are classified cohomologically by \( H^1(M, \mathbb{Z}_2) \) [34].
Proposition 7.8  One has
\[ \varepsilon_C = c(e_C) \mod 2 \]  
(7.12)
where \( c(l) \) denotes the 1st Chern class of a line bundle \( l \) and \( e_C \) is defined in prop. 5.5. Thus, \( C \) is liftable if and only if \( e_C \) is a square. In that case, the monopole charge \( m_C \) and the instanton charge \( i_C \) are \( \mathbb{Z} \) valued.

Proof  According to prop. 5.5, the SO(3) bundle \( E_C \) decomposes according to (5.6), (5.7). It is known that \( w_2(E_C) = c(e_C) \mod 2 \) (see ref. [34]). So (7.12) is clear. Under the natural homomorphism \( H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R}) \), \( c(e_C) \) is represented by \( [f_C/2\pi] \). So, when \( \varepsilon_C \) vanishes, \( f_C/2 \in \Omega^2_{c2\pi\mathbb{Z}}(M) \). Thus, by rem 7.2, \( m_C, i_C \) are integer valued.

Our framework naturally accommodates half integral monopole charges a quarter integral instanton charges. The reason why \( m_C, i_C \) are respectively \( \mathbb{Z}/2 \) and \( \mathbb{Z}/4 \) valued can be traced to the fact that \( C \) is generally non liftable.

Remark 7.9  View \( C \) as an almost fine Cho structure (cfr. def. 5.8). Then, the associated set of fine equivalence classes of fine Cho structures \( F_\langle C \rangle \) (cfr. def. 5.12) is partitioned in sectors depending on the value of \( \varepsilon_\langle C' \rangle \in H^2(M, \mathbb{Z}_2) \).

These sectors answer to the twisted sectors of refs. [16–18]. This is particularly clear from the analysis of sect. 2. Thus, they are also related to center vortices, as explained in the introduction.

Other topological features

Let \( C, C' \) be equivalent fine Cho structures. If their Abelian curvatures \( f_C, f_{C'} \) are equal, one cannot conclude that \( C, C' \) are finely equivalent in general.

Definition 7.10  Let \( C \) be an almost fine Cho structure. We denote by \( F_{\varepsilon_f} \) the subset of \( F_C \) of fine equivalence classes of fine Cho structures equivalent to \( C \) with assigned Abelian curvature \( f \in \Omega^2_{c2\pi\mathbb{Z}}(M) \).

Proposition 7.11  Let \( C \) be an almost fine Cho structure and \( f \in \Omega^2_{c2\pi\mathbb{Z}}(M) \). Then, there exists a bijection \( F_{\varepsilon_f} \cong H^1(M, \mathbb{T}) \)

Proof  There is an exact sequence
\[ 0 \to H^1(M, \mathbb{T}) \xrightarrow{\partial} H^2(M, D(2)^\bullet) \xrightarrow{\zeta} \Omega^2_{c2\pi\mathbb{Z}}(M) \to 0, \]  
(7.13)
where the map $\varsigma$ associates to any class in $H^2(M, D(2)\bullet)$ its curvature $\Omega^2_{c2\pi\mathbb{Z}}(M)$ \cite{32}.

Without loss of generality we can assume that $\mathcal{C}$ is fine. Then, we can identify $F_{\mathcal{C}'}$ with $H^2(M, D(2)\bullet)$ using the bijection $j^{-1}_{\mathcal{C}}$ defined in the proof of theor. 5.13. It is easy to see that

$$\varsigma \circ j^{-1}_{\mathcal{C}'}(\langle \mathcal{C}' \rangle) = f_{\mathcal{C}'} - f_{\mathcal{C}} \quad (7.14)$$

for a fine class $\langle \mathcal{C}' \rangle \in F_{\mathcal{C}'}$. Hence, $F_{\mathcal{C}'}$ is identified with $\ker \varsigma$. By the exactness of (7.13), $\ker \varsigma \cong \text{ran} \varrho \cong H^1(M, \mathbb{T})$. The statement follows. \hfill $\square$

**Remark 7.12** $H^1(M, \mathbb{T})$ has a well known interpretation. It is the group of flat $\mathbb{T}$ principal bundle on $M$. $H^1(M, \mathbb{T})$ is aptly described by the exact sequence

$$0 \to H^1_{dR}(M)/H^1_{dR\mathbb{Z}}(M) \xrightarrow{\chi} H^1(M, \mathbb{T}) \to \text{Tor} H^2(M, \mathbb{Z}) \to 0. \quad (7.15)$$

Here, $H^1_{dR}(M)$ is the degree 1 de Rham cohomology. $H^1_{dR\mathbb{Z}}(M)$ is the integral lattice in $H^1_{dR}(M)$. $\chi$ is essentially the map $\exp(2\pi \sqrt{-1} \cdot )$ in the Čech formulation of cohomology. $c$ is the Chern class homomorphism. The degree 2 torsion $\text{Tor} H^2(M, \mathbb{Z})$ is the kernel of the natural homomorphism $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$.

**Proof** (7.14) can be easily deduced from the long exact sequence of cohomology associated with the standard short exact sequence of sheaves $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$. \hfill $\square$

The above discussion shows that the monopole and instanton charges $m_{\mathcal{C}}, i_{\mathcal{C}}$ do not completely characterize the monopole configuration associated with a fine Cho structure $\mathcal{C}$. There are further topological features associated with the group $H^1(M, \mathbb{T})$. In the simple case where $M$ is compact and the torsion $\text{Tor} H^2(M, \mathbb{Z})$ vanishes, $H^1(M, \mathbb{T})$ is the torus $\mathbb{T}^{b_1}$, where $b_1$ is the 1st Betti number of $M$.

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A. Smooth Deligne cohomology

Our method relies heavily on the smooth version of Deligne cohomology. Since Deligne cohomology is defined in terms of hypercohomology, we review briefly below the basic facts about hypercohomology used in the paper. See ref. [32,33] for background material.

Abstractly, the computation of the hypercohomology of a complex of sheaves of Abelian groups $K^\bullet$

$$
K^0 \xrightarrow{d_K} K^1 \xrightarrow{d_K} \cdots \xrightarrow{d_K} K^p
$$

requires the choice of an appropriate resolution $R^{\bullet\bullet}$ of $K^\bullet$

\begin{align*}
K^p & \xrightarrow{r} R^{0,p} \xrightarrow{\delta} R^{1,p} \xrightarrow{\delta} R^{2,p} \xrightarrow{\delta} \cdots \\
R^{0,p} & \xrightarrow{d} R^{1,p} \xrightarrow{d} R^{2,p} \xrightarrow{d} \cdots \\
R^{0,0} & \xrightarrow{d} R^{1,0} \xrightarrow{d} R^{2,0} \xrightarrow{d} \cdots
\end{align*}

requires the choice of an appropriate resolution $R^{\bullet\bullet}$ of $K^\bullet$

\begin{align*}
K^p & \xrightarrow{r} R^{0,p} \xrightarrow{\delta} R^{1,p} \xrightarrow{\delta} R^{2,p} \xrightarrow{\delta} \cdots \\
R^{0,p} & \xrightarrow{d} R^{1,p} \xrightarrow{d} R^{2,p} \xrightarrow{d} \cdots \\
R^{0,0} & \xrightarrow{d} R^{1,0} \xrightarrow{d} R^{2,0} \xrightarrow{d} \cdots
\end{align*}

e. g. an injective resolution. $R^{\bullet\bullet}$ is a double complex of sheaves of Abelian groups. The hypercohomology of $K^\bullet$, $H^\bullet(M,K^\bullet)$, is the cohomology of the complex of Abelian groups $\text{Tot}R^\bullet(M)$

$$
\text{Tot}R^0(M) \xrightarrow{d_{\text{Tot}RM}} \text{Tot}R^1(M) \xrightarrow{d_{\text{Tot}RM}} \cdots \xrightarrow{d_{\text{Tot}RM}} \text{Tot}R^p(M),
$$

where $\text{Tot}R^k = \bigoplus_{l=0}^{\min(p,k)} R^{k-l,l}$ and $d_{\text{Tot}R} = \delta + (-1)^\partial d$ with $\partial$ denoting the horizontal degree. $H^\bullet(M,K^\bullet)$ is independent from the choice of the resolution up to isomorphism.

From the above definition, it follows that a hypercohomology class $c$ of $H^k(M,K^\bullet)$ is represented by a $k$–cocycle $\gamma$ of $\text{Tot}R^k(M)$ defined up to a $k$–coboundary. A $k$–cocycle $\gamma$ of $\text{Tot}R^k(M)$ is a sequence $(\gamma^{k-l,l})_{l=0,1,\ldots,\min(p,k)}$, where $\gamma^{k-l,l} \in R^{k-l,l}(M)$ satisfy

$$
\delta \gamma^{k-l,l} = d \gamma^{k-l+1,l-1}, \quad l = 0, 1, \ldots, \min(p,k) + 1,
$$

with $\gamma^{k+1,-1} = 0, \gamma^{k-\min(p,k)-1,\min(p,k)+1} = 0$. A $k$–coboundary $\beta$ of $\text{Tot}R^k(M)$ is a $k$–cocycle of $\text{Tot}R^k(M)$ of the form

$$
\beta^{k-l,l} = \delta \gamma^{k-1-l,l} + d \gamma^{k-l,l-1}, \quad l = 0, 1, \ldots, \min(p,k),
$$
where \( c_{k-1,l}^{l} \in R^{k-1-l,l}(M), l = 0, 1, \ldots, \min(p, k-1) \), and \( c_{k-1}^{k-1} = c^{-1}_k = 0 \) for \( k \leq p \).

In practical calculations, it is convenient to use a Čech resolution of \( K^\bullet \), which is constructed as follows. Let \( \{O_i\} \) be an open cover of \( M \). We set \( R^{n,l}(U) \) is the sheaf of Čech \( n \)-cochains of the sheaf \( K^l \). For an open set \( U \), an element \( \gamma^{n,l} \in R^{n,l}(U) \) is a collection \( \{\gamma_{n,i_0}^{l}\}_{i_0} \), where \( \gamma_{n,i_0}^{l} |_{O_{i_0} \cap U} \) is totally antisymmetric in the cover indices \( i_0, \ldots, i_n \). \( \boxed{41} \)

The inclusion \( r \) and the coboundary operators \( \delta \) and \( d \) are defined as

\[
(r \alpha^l)_i = \alpha^l |_{O_i \cap U}, \quad (A.6) \\
(\delta \gamma^{n,l})_{i_0 \cdots i_{n+1}} = \sum_{r=0}^{n+1} (-1)^r \gamma^{n,l}_{i_0 \cdots i_r \cdots i_{n+1}} |_{O_{i_0} \cap \cdots \cap O_{i_{n+1}} \cap U}, \quad (A.7) \\
(d \gamma^{n,l})_{i_0 \cdots i_n} = d_K \gamma^{n,l}_{i_0 \cdots i_n}. \quad (A.8)
\]

for \( \alpha^l \in K^l(U) \) and \( \gamma^{n,l} \in R^{n,l}(U) \). For a generic cover \( \{O_i\} \), the cohomology of the complex \( \text{Tot}R^\bullet(M) \) depends on \( \{O_i\} \) and does not directly compute the hypercohomology \( H^\bullet(M, K^\bullet) \). To that end, it is necessary to perform the direct limit of the cohomology of \( \text{Tot}R^\bullet(M) \) with respect to refinements of \( \{O_i\} \). In favorable conditions, there is a class of covers, called good covers, such that, when the cover \( \{O_i\} \) is good, the cohomology of \( \text{Tot}R^\bullet(M) \) is isomorphic to the hypercohomology \( H^\bullet(M, K^\bullet) \) and there is no need of the direct limit.

Let \( p \in \mathbb{N} \). The length \( p \) smooth Deligne complex \( D(p)^\bullet \) is the complex of sheaves

\[
2\pi\mathbb{Z} \rightarrow j^* \Omega^0 \rightarrow d \Omega^0 \rightarrow \Omega^1 \rightarrow \Omega^2 \rightarrow \cdots \rightarrow \Omega^{p-1}. \quad (A.9)
\]

Here, \( \mathbb{Z} \) is the sheaf of locally constant integer valued functions on \( M \). \( \Omega^k \) is the sheaf of real valued \( k \)-forms of \( M \). \( j \) is the natural injection and \( d \) is the customary de Rham differential. \( 2\pi\mathbb{Z} \) is put in degree 0. \( \boxed{52} \)

The length \( p \) smooth Deligne cohomology is by definition the hypercohomology of the smooth Deligne complex \( D(p)^\bullet \), \( H^\bullet(M, D(p)^\bullet) \).

Below, we shall deal with Deligne cohomology using the Čech resolution \( R^{\bullet,\bullet} \) of \( D(p)^\bullet \).

\( \boxed{41} \) Below, we denote by \( O_{ij}, O_{ijk}, \ldots \) the non empty intersections \( O_i \cap O_j, O_i \cap O_j \cap O_k, \ldots \), respectively.

\( \boxed{52} \) Our definition of the smooth Deligne complex differs from that of ref. [32], where one puts the sheaf \( \mathbb{Z}(p) = (2\pi \sqrt{-1})^p\mathbb{Z} \) in degree 0 and the sheaf \( \Omega^{k-1}_{\mathbb{C}} \) of complex valued \( k-1 \)-forms in degree \( k > 0 \).
associated with an open cover \( \{ O_i \} \), as outlined above. When necessary, we shall assume that \( \{ O_i \} \) is good. In the case considered here, an open cover \( \{ O_i \} \) is good if all the non empty finite intersections \( O_{i_0 \cdots i_k} \) are contractible. A cohomology class \( c \) of \( H^k(M, D(p)^\bullet) \) is then represented by a \( k \)-cocycle \( \gamma \) of \( \text{Tot} R^k(M) \) defined up to a \( k \)-coboundary (cfr. eqs. (A.4)–(A.5)). In the following, we shall call such cocycles, length \( p \) Deligne \( k \)-cocycles.

Some of the Deligne cohomology groups classify certain differential topological structures having the circle group \( \mathbb{T} \) as structure group [32]. Below, we describe the ones which are relevant in the paper.

\( H^2(M, D(2)^\bullet) \) is isomorphic to the group of isomorphism classes of Hermitian line bundles with Hermitian connection. Indeed, a class of \( H^2(M, D(2)^\bullet) \) is represented by a \( 2 \)-cocycle \((\{a_i\}, \{f_{ij}\}, \{m_{ijk}\})\), where \( \{a_i\} \) is an \( \Omega^1 \) Čech 0–cochain, \( \{f_{ij}\} \) is an \( \Omega^0 \) Čech 1–cochain and \( \{m_{ijk}\} \) is a \( 2\pi \mathbb{Z} \) Čech 2–cochain, satisfying the relations

\[
\delta a_{ij} = df_{ij}, \tag{A.10a}
\]
\[
\delta f_{ijk} = m_{ij}, \tag{A.10b}
\]
\[
\delta m_{ijkl} = 0. \tag{A.10c}
\]

These relations imply that \( \{\exp(\sqrt{-1} f_{ij})\} \) is a \( \mathbb{T} \) Čech 1–cocycle representing a Hermitian line bundle \( L \) and \( \{\sqrt{-1} a_{i}\} \) is a \( \sqrt{-1} \)Ω \( \Omega^1 \) Čech 0–cochain corresponding to a Hermitian connection \( A \) of \( L \). 6 If we replace the \( 2 \)-cocycle \((\{a_i\}, \{f_{ij}\}, \{m_{ijk}\})\) by a cohomologous one, we obtain an equivalent Hermitian line bundle with Hermitian connection. The \( \mathbb{Z} \) Čech 2–cocycle \( \{m_{ijk}/2\pi\} \) represents the Chern class \( c(L) \in H^2(M, \mathbb{Z}) \) of \( L \). The closed \( 2 \)-form with integer periods \( F \in \Omega^2_{\text{cZ}}(M) \), defined by \( F|_{O_i} = da_i/2\pi \), is the curvature of \( A. \) \( c(L) \) and \( F \) depend only on the isomorphism class of \( L, A \), hence on the corresponding Deligne cohomology class.

Similarly, \( H^3(M, D(3)^\bullet) \) is isomorphic to the group of isomorphism classes of Hermitian gerbes with Hermitian connective structure and curving. Indeed, a class of \( H^3(M, D(3)^\bullet) \) is represented by a \( 3 \)-cocycle \((\{h_i\}, \{g_{ij}\}, \{f_{ijk}\}, \{m_{ijkl}\})\), where \( \{h_i\} \) is an \( \Omega^2 \) Čech 0–cochain, \( \{g_{ij}\} \) is an \( \Omega^1 \) Čech 1–cochain, \( \{f_{ijk}\} \) is an \( \Omega^0 \) Čech 2–cochain and \( \{m_{ijkl}\} \) is a \( 2\pi \mathbb{Z} \) Čech 3–cochain satisfying the relations

\[
\delta h_{ij} = dg_{ij}, \tag{A.11a}
\]

\footnote{Below, for any Lie group \( G \), \( \underline{G} \) denotes the sheaf of \( G \) valued functions.}
\[ \delta g_{ijk} = df_{ijk}, \quad (A.11b) \]
\[ \delta f_{ijkl} = m_{ijkl}, \quad (A.11c) \]
\[ \delta m_{ijklm} = 0. \quad (A.11d) \]

These relations imply that \( \{ \exp(\sqrt{-1}f_{ijk}) \} \) is a 2–Čech cocycle representing a Hermitian gerbe \( H \) and \( \{ \sqrt{-1}g_{ij} \}, \{ \sqrt{-1}h_i \} \) are a 1–Čech cochain and a 2–Čech 0–cochain corresponding to a Hermitian connective structure and curving \( C \) of \( H \), respectively. If we replace the 3–cocycle \( \{ h_i \}, \{ g_{ij} \}, \{ f_{ijk} \}, \{ m_{ijkl} \} \) by a cohomologous one, we obtain an equivalent Hermitian gerbe with Hermitian connective structure and curving. The \( \mathbb{Z} \) Čech 3–cocycle \( m_{ijkl}/2\pi \) represents the Dixmier–Douady class \( d(H) \in H^3(M, \mathbb{Z}) \) of \( H \). The closed 3–form with integer periods \( G \in \Omega^3_{\mathbb{Z}}(M) \), defined by \( G|_{O_i} = dh_i/2\pi \), is the curvature of \( C \). \( d(H) \) and \( G \) depend only on the isomorphism class of \( H, C \), hence on the corresponding Deligne cohomology class.
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