Shortened Linear Codes over Finite Fields

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Abstract

The puncturing and shortening technique are two important approaches to constructing new linear codes from old ones. In the past 70 years, a lot of progress on the puncturing technique has been made, and many works on punctured linear codes have been done. Many families of linear codes with interesting parameters have been obtained with the puncturing technique. However, little research on the shortening technique has been done and there are only a handful references on shortened linear codes. The first objective of this paper is to prove some general theory for shortened linear codes. The second objective is to study some shortened codes of the Hamming codes, Simplex codes, some Reed-Muller codes, and ovoid codes. Eleven families of optimal shortened codes with interesting parameters are presented in this paper. As a byproduct, five infinite families of 2-designs are also constructed from some of the shortened codes presented in this paper.

Index Terms

Linear code, cyclic code, punctured code, shortened code, $t$-design

I. INTRODUCTION

An $[n, \kappa, d]$ code over GF$(q)$ is a $\kappa$-dimensional linear subspace of GF$(q)^n$ with minimum Hamming distance $d$. By the parameters of a linear code, we refer to its length, dimension

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and minimum distance. An \([n, \kappa, d]\) code over \(GF(q)\) is called *distance-optimal* (respectively, *dimension-optimal* and *length-optimal*) if there is no \([n, \kappa, d'] \geq d + 1\) (respectively, \([n, \kappa' \geq \kappa + 1, d]\) and \([n' \leq n - 1, \kappa, d]\)) linear code over \(GF(q)\). An optimal code is a code that is length-optimal, or dimension-optimal, or distance-optimal, or meets a bound for linear codes.

An important problem in the theory and application of coding theory is the construction of optimal codes and codes with desirable parameters. To this end, one may construct a linear code with good or desirable parameters from a known linear code with optimal or good parameters. There are several standard ways of obtaining linear codes from a known one. The most famous are the puncturing and shortening methods. Since extending a linear code increases the code length by one and does not change the code dimension, the extending method is less interesting in constructing new codes.

Let \(C\) be an \([n, \kappa, d]\) code over \(GF(q)\), and let \(T\) be a set of \(t\) coordinates in \(C\). We puncture \(C\) by deleting all the coordinates in \(T\) in each codeword of \(C\). The resulting code is still linear and has length \(n - t\), where \(t = |T|\). We denote the punctured code by \(C^T\). Let \(C(T)\) be the set of codewords which are 0 on \(T\). Then \(C(T)\) is a subcode of \(C\). We now puncture \(C(T)\) on \(T\), and obtain a linear code over \(GF(q)\) with length \(n - t\), which is called a *shortened code* of \(C\), and is denoted by \(C_T\).

To explain the motivations of this paper, we recall a general construction of linear codes. Let \(D = \{d_1, d_2, \ldots, d_n\} \subseteq GF(r)\), where \(r = q^m\) and \(q = p^s\) with \(p\) a prime. Let \(Tr_{r/q}\) be the trace function from \(GF(r)\) to \(GF(q)\). Define a linear code of length \(n\) over \(GF(q)\) by

\[
C_D = \{(Tr_{r/q}(xd_1), Tr_{r/q}(xd_2), \ldots, Tr_{r/q}(xd_n)) : x \in GF(r)\}.
\]

The set \(D\) is called the defining set of \(C_D\). The next theorem says that the defining-set construction in (1) is fundamental (11).

**Theorem 1.** Any linear code of length \(n\) over \(GF(q)\) can be expressed as the code \(C_D\) in (1), where \(D = \{d_1, d_2, \ldots, d_n\} \subseteq GF(q^m)\) is a multiset and \(m\) is some positive integer.

Let \(\alpha\) be a primitive element of \(GF(q^m)\). The following code

\[
C(m, q, \alpha) = \{(Tr_{q^m/q}(a\alpha^0), Tr_{q^m/q}(a\alpha^1), \ldots, Tr_{q^m/q}(a\alpha^{q^m-2})) : a \in GF(q^m)\}
\]

is a \([q^m - 1, m, (q - 1)q^{m-1}]\) cyclic code with check polynomial \(M_{\alpha^{-1}}(x)\), which is the minimal polynomial of \(\alpha^{-1}\) over \(GF(q)\) and is irreducible over \(GF(q)\). The code \(C(m, q, \alpha)\) is said to be *irreducible*. It is easily seen that the weight enumerator of \(C(m, q, \alpha)\) is \(1 + (q^m - 1)z^{q^m-q^{m-1}}\) (6).
The next theorem then follows from Theorem 1 [10].

**Theorem 2.** Every linear code of length $n$ over $\text{GF}(q)$ with dual distance at least 2 is a punctured code of an irreducible cyclic code $C(m,q,\alpha)$ in $[2]$ for some integer $m$ and primitive element $\alpha \in \text{GF}(q^m)$.

While cyclic codes form a subclass of linear codes, every linear code with minimum distance at least two can be punctured from a special irreducible cyclic code $C(m,q,\alpha)$. This demonstrates the importance of the very special subclass of irreducible cyclic codes $C(m,q,\alpha)$ and the puncturing technique.

In the past ten years, a lot of research on the defining-set construction of linear codes has been done, and many families of linear codes with interesting parameters have been obtained. Recall that the defining-set construction is in fact to puncture a code $C(m,q,\alpha)$. Hence, the puncturing technique was extensively studied in the past ten years and proved to be an effective approach to obtaining linear codes with good or desirable parameters. However, every approach has limitations and there is no exception for the puncturing technique.

Motivated by the success of the puncturing technique, we ask the following questions on the shortening technique:

1) Is there a family of cyclic codes such that every linear code with minimum distance at least 2 is a shortened code of a cyclic code in this family?

2) Is every linear code with minimum distance at least 3 a shortened code of a Hamming code?

3) Can we obtain optimal linear codes or linear codes with desirable parameters by shortening certain known families of linear codes or cyclic codes? If the answer is positive, which known families of linear codes can be shortened for obtaining good linear codes, and which coordinate set $T$ leads to a shortened linear code $C_T$ with good or desirable parameters?

4) Can we develop some general theory for shortened linear codes?

The basic questions above are the major motivations of studying the shortening technique. In addition to obtaining new linear codes with desirable parameters from old ones, one may have to study some shortened codes of certain families of linear codes in particular applications. For instance, in order to obtain 2-designs and Steiner systems, certain shortened codes of a family of ternary codes were investigated in [13], where two families of shortened ternary codes with the best-known parameters were obtained. The study of shortened linear codes in [19] led
to a generalization of the Assmus-Mattson theorem. Shortened and punctured codes were used to prove a generalized MacWilliams Identity in [7]. These important applications of shortened linear codes are additional motivations of this paper.

The last motivation of this paper is that there are only several papers on shortened linear codes in the literature (see [3], [15], [9], [14], [17], [20]). This is quite amazing, as there are many works on the puncturing technique due to the defining-set construction. This fact shows that the shortening technique was overlooked by the coding theory community for 70 years.

It is known that $C_T = ((C^\perp)^T)^\perp$. This means that in theory shortening a linear code can be obtained by first performing the dual operation, then the puncturing operation, and finally the dual operation. But this involves the study of three codes in the sequence $(C^\perp, (C^\perp)^T, ((C^\perp)^T)^\perp)$, and is usually much more complicated. In addition, the puncturing technique also has a limitation in obtaining linear codes with good parameters in practice. Consequently, it is still necessary to study the shortening technique.

The objectives of this paper are the following:

1) Develop some general theory for shortened linear codes.
2) Construct linear codes over $\text{GF}(q)$ with various and good parameters by shortening some known families of linear codes over $\text{GF}(q)$.

In this paper, we present eleven families of optimal shortened codes and five families of 2-designs from some of the shortened codes.

II. GENERAL RESULTS ABOUT SHORTENED LINEAR CODES

In this section, we introduce known general results and prove new ones about shortened linear codes.

A. Known general results about shortened linear codes

Under certain conditions, the dimension of a shortened code $C_T$ is known and given in the next theorem [13, Theorem 1.5.7].

**Theorem 3.** Let $C$ be an $[n, \kappa, d]$ code over $\text{GF}(q)$ and let $T$ be any set of $t$ coordinates. Let $C^T$ denote the punctured code of $C$ in all coordinates in $T$. Then the following hold.

1) $(C^\perp)^T = (C^T)^\perp$ and $(C^\perp)^T = (C_T)^\perp$.
2) If $t < d$, then $C^T$ and $(C^\perp)^T$ have dimensions $\kappa$ and $n - t - \kappa$, respectively.
3) If \( t = d \) and \( T \) is the set of coordinates where a minimum weight codeword is nonzero, then \( C^T \) and \((C^\perp)_T\) have dimensions \( \kappa - 1 \) and \( n - t - \kappa + 1 \), respectively.

We will use this theorem to settle the dimension for the case \( t < d^\perp \), and will develop a case-specific method to determine the dimension of \( C_T \) for the case \( t > d^\perp \) according to the specific design of the original code \( C \). The most difficult task is to determine the minimum distance \( d(C_T) \) of a shortened code \( C_T \).

Let \( C \) be an \([n, \kappa, d]\) code over \( \text{GF}(q) \). Let \( M \) be the \( q^\kappa \times n \) matrix whose rows are all codewords in \( C \), and let \( M_i \) be the submatrix of \( M \) consisting of the codewords of weight \( i \). A code is homogeneous provided that for \( 0 \leq i \leq n \), each column of \( M_i \) has the same weight. Prange proved the following result [13, p. 271].

**Theorem 4.** Let \( C \) be an \([n, \kappa, d]\) code over \( \text{GF}(q) \) with \( d > 1 \), and let \( C^* \) and \( C_* \) be the code obtained from \( C \) by puncturing and shortening on some coordinate, respectively. Then for \( 0 \leq i \leq n - 1 \), we have

\[
A_i(C^*) = \frac{n-i}{n} A_i(C) + \frac{i+1}{n} A_{i+1}(C)
\]

and

\[
A_i(C_*) = \frac{n-i}{n} A_i(C),
\]

where \( A_i(C) \) denotes the number of codewords of weight \( i \) in \( C \).

It is known that \( C \) is homogeneous if \( C \) has a transitive automorphism group [13, p. 271]. This is about the only way to decide if a code is homogeneous. Hence, Theorem 4 has very limited applicability.

Recently, better results regarding the weight distribution of a shortened code \( C_T \) of special linear codes were developed in [19]. To introduce them, we need to introduce combinatorial \( t \)-designs.

Let \( \mathcal{P} \) be a set of \( n \) elements and \( \mathcal{B} \) a multiset of \( b \) \( k \)-subsets of \( \mathcal{P} \), where \( n \geq 1 \), \( b \geq 0 \) and \( 1 \leq k \leq n \). Let \( t \) be a positive integer satisfying \( 1 \leq t \leq n \). The pair \( \mathcal{D} = (\mathcal{P}, \mathcal{B}) \) is called a \( t-(n, k, \lambda) \) design, or simply \( t \)-design, if every \( t \)-subset of \( \mathcal{P} \) is contained in exactly \( \lambda \) elements of \( \mathcal{B} \). The elements of \( \mathcal{P} \) are called points, and those of \( \mathcal{B} \) are referred to as blocks.

When \( \mathcal{B} = \emptyset \), i.e., \( b = 0 \), we put \( \lambda = 0 \) and call \( (\mathcal{P}, \emptyset) \) a \( t-(n, k, 0) \) design for any \( t \) and \( k \) with \( 1 \leq t \leq n \) and \( 0 \leq k \leq n \). A \( t-(n, k, \lambda) \) design with \( t > k \) must have \( \lambda = 0 \) and must be the design...
(\mathcal{P}, \emptyset). These designs are called trivial designs. We will use the following conventions for the ease of description in the sequel. A \( t-(n,k,\lambda) \) design \((\mathcal{P}, \mathcal{B})\) is also said to be trivial if every \( k \)-subset of \( \mathcal{P} \) is a block.

A \( t \)-design is called \textit{simple} if \( \mathcal{B} \) does not contain repeated blocks. A \( t-(n,k,\lambda) \) design is called a \textit{Steiner system} and denoted by \( S(t,k,n) \) if \( t \geq 2 \) and \( \lambda = 1 \). The parameters of a \( t-(n,k,\lambda) \) design satisfy:

\[
\binom{n}{t} \lambda = \binom{k}{t} b.
\]

Let \( \text{GF}(q) \) denote the finite field with \( q \) elements, where \( q \) is a prime power. We assume that the reader is familiar with the basics of linear codes. Let \( C \) be an \([n,k,d]\) linear code over \( \text{GF}(q) \).

Let \( A_i := A_i(C) \), which denotes the number of codewords with Hamming weight \( i \) in \( C \), where \( 0 \leq i \leq n \). The sequence \((A_0, A_1, \ldots, A_n)\) is called the \textit{weight distribution} of \( C \), and \( \sum_{i=0}^{n} A_i z^i \) is referred to as the \textit{weight enumerator} of \( C \). Then the \( q \)-ary linear code \( C \) may induce a \( t \)-design under certain conditions, which is formed by the supports of codewords of a fixed Hamming weight in \( C \). Let \( \mathcal{P}(C) = \{0, 1, \ldots, n-1\} \) be the set of the coordinate positions of \( C \), where \( n \) is the length of \( C \). For a codeword \( \mathbf{c} = (c_0, \ldots, c_{n-1}) \) in \( C \), the \textit{support} of \( \mathbf{c} \) is defined by

\[
\text{Supp}(\mathbf{c}) = \{i : c_i \neq 0, i \in \mathcal{P}(C)\}.
\]

Let \( \mathcal{B}_w(C) = \frac{1}{q-1}\{\{\text{Supp}(\mathbf{c}) : \text{wt}(\mathbf{c}) = w \text{ and } \mathbf{c} \in C\}\} \), here and hereafter \( \{\{\}\} \) is the multiset notation and \( \frac{1}{q-1} S \) denotes the multiset obtained after dividing the multiplicity of each element in the multiset \( S \) by \( q-1 \). For some special \( C \), \((\mathcal{P}(C), \mathcal{B}_w(C))\) is a \( t-(n,w,\lambda) \) design with \( b \) blocks, where

\[
b = \frac{1}{q-1} A_w, \quad \lambda = \frac{\binom{w}{t}}{(q-1)\binom{n}{t}} A_w.
\]

If \((\mathcal{P}(C), \mathcal{B}_w(C))\) is a \( t \)-design for any \( 0 \leq w \leq n \), we say that the code \( C \) \textit{supports} \( t \)-designs. Notice that such a design \((\mathcal{P}(C), \mathcal{B}_w(C))\) may have repeated blocks or may be simple or trivial.

The following lemma provides a criterion for guaranteeing a simple block set \( \mathcal{B}_k(C) \) [4, Lemma 4.1].

**Lemma 5.** Let \( C \) be a linear code over \( \text{GF}(q) \) with length \( n \) and minimum weight \( d \). Let \( w \) be the largest integer with \( w \leq n \) satisfying

\[
w - \left\lfloor \frac{w+q-2}{q-1} \right\rfloor < d.
\]
Then there are no repeated blocks in $B_k(C)$ for any $d \leq k \leq w$. Such a block set is said to be simple.

The following theorem gives a characterization of codes supporting $t$-designs via the weight distributions of their shortened and punctured codes [19], and will be employed later in this paper.

**Theorem 6.** Let $C$ be an $[n,m,d]$ linear code over $GF(q)$ and $d^\perp$ the minimum distance of $C^\perp$. Let $t$ be a positive integer with $0 < t < \min\{d,d^\perp\}$. Then the following statements are equivalent.

1. $(P(C), B_k(C))$ is a $t$-design for any $0 \leq k \leq n$.
2. $(P(C^\perp), B_k(C^\perp))$ is a $t$-design for any $0 \leq k \leq n$.
3. For any $1 \leq t' \leq t$, the weight distribution $(A_k(C^T))_{k=0}^{n-t'}$ of the shortened code $C_T$ is independent of the specific choice of the elements in $T$, where $T$ is any set of $t'$ coordinate positions in $P(C)$.
4. For any $1 \leq t' \leq t$, the weight distribution $(A_k(C^T))_{k=0}^{n-t'}$ of the punctured code $C^T$ is independent of the specific choice of the elements in $T$, where $T$ is any set of $t'$ coordinate positions in $P(C)$.

Recall that the binomial coefficient $\binom{a}{b}$ equals 0 when $a < b$ or $b < 0$. We have the following useful result [19].

**Theorem 7.** Let $C$ be an $[n,m,d]$ linear code over $GF(q)$ and $d^\perp$ the minimum distance of $C^\perp$. Let $t$ be a positive integer with $0 < t < \min\{d,d^\perp\}$. Let $T$ be a set of $t$ coordinate positions in $P(C)$. Suppose that $(P(C), B_k(C))$ is a $t$-design for any $i$ with $d \leq i \leq n - t$. Then the shortened code $C_T$ is a linear code of length $n - t$ and dimension $m - t$. The weight distribution $(A_k(C_T))_{k=0}^{n-t'}$ of $C_T$ is independent of the specific choice of the elements in $T$. Specifically,

$$A_k(C_T) = \binom{k}{t} \binom{n-t}{k} \binom{n}{k} \binom{n-t}{k-t} A_k(C).$$

Theorem 7 is useful, and will be employed to determine the weight distributions of some shortened codes of several families of linear codes later. The following theorem will also be used later [19].

**Theorem 8.** Let $C$ be an $[n,m,d]$ linear code over $GF(q)$ and $d^\perp$ the minimum distance of $C^\perp$. Let $t$ be a positive integer with $0 < t < d^\perp$. Let $T$ be a set of $t$ coordinate positions in $P(C)$.
Suppose that \((P(C), B_i(C))\) is a t-design for any \(i\) with \(d \leq i \leq n\). Then the punctured code \(C^T\) is a linear code of length \(n - t\) and dimension \(m\). The weight distribution \((A_k(C^T))^{n-i}_{k=0}\) of \(C^T\) is independent of the specific choice of the elements in \(T\). Specifically,

\[
A_k(C^T) = \frac{\binom{n-i}{k} \binom{k+i}{i} \binom{n}{t}}{\binom{n-t+i}{t}} A_{k+i}(C).
\]

B. Some new general results

In this section, we prove two general results. The first one is the following.

**Theorem 9.** Every linear code \(C\) over \(\text{GF}(q)\) with minimum distance \(d \geq 2\) is a shortened code of \(C(m, q, \alpha)^\perp\) for some \(m\), \(q\) and \(\alpha\), where \(\alpha\) is a generator of \(\text{GF}(q)^*\) and \(C(m, q, \alpha)\) was defined in (2).

**Proof.** By assumption \(d \geq 2\). It follows from Theorem 2 that there are \(m\), \(q\) and a generator of \(\text{GF}(q)^*\) such that

\[
C^\perp = C(m, q, \alpha)^T,
\]

where \(T\) is a set of coordinates in \(C(m, q, \alpha)\). It then follows from Theorem 3 that

\[
C^\perp = C(m, q, \alpha)^T = ((C(m, q, \alpha)^\perp)^T = ((C(m, q, \alpha)^\perp)_T)^\perp.
\]

Hence, \(C = (C(m, q, \alpha)^\perp)_T\). This completes the proof.

The following is a corollary of Theorem 2.

**Corollary 10.** Every linear code over \(\text{GF}(q)\) with dual distance at least 3 is a punctured code of a Simplex code over \(\text{GF}(q)\).

Note that the dual of a Hamming code is called a Simplex code, which is a one-weight code.

**Theorem 11.** Every linear code with minimum distance at least 3 is a shortened code of a Hamming code over \(\text{GF}(q)\).

**Proof.** The desired conclusion follows from Corollary 10 and Theorem 3. The proof is similar to that of Theorem 9 and is omitted here.

We now prove the following result.
Theorem 12. Let $C$ be an $[n, k, d]$ code over $\text{GF}(q)$, and let $d^\perp$ denote the minimum distance of the dual code $C^\perp$. Let $t$ be an integer with $1 \leq t < \min\{d, d^\perp\}$. For any set $T = \{i_1, i_2, \ldots, i_t\}$ of $t$ coordinates, $C^T$ has length $n-t$, dimension $k$ and minimum distance at least $d-t$.

Furthermore, if $A_d(C) > q^k - q^{k-t}(q-1)^t - 1$ and $t \leq k$, then the minimum distance of $C^T$ equals $d-t$.

Proof. Since $t < d$, the punctured versions of any two distinct codewords in $C$ are distinct. It then follows that the dimension of $C^T$ is $k$. Clearly, the minimum distance of $C^T$ is at least $d-t$.

Let $M$ be the $q^k \times n$ matrix whose rows are all codewords of $C$. It is well known that $M$ is an orthogonal array of strength $t$. Let $S_1(T)$ denote the set of all codewords in $C$ whose coordinates in $T$ are all nonzero, and define $S_2(T) = C \setminus S_1(T)$. By definition, $S_1(T)$ and $S_2(T)$ partition $C$. Since $M$ is an orthogonal array of strength $t$, $|S_1(T)| = q^{k-t}(q-1)^t$. Consequently, $|S_2(T)| = q^k - q^{k-t}(q-1)^t$. Let $C_d$ denote the set of all codewords of weight $d$ in $C$. Then $S_1(T) \cap C_d$ and $S_2(T) \cap C_d$ partition $C_d$. Note that $d \geq 1$ and

$$|S_2(T) \cap C_d| \leq |S_2(T)| - 1 = q^k - q^{k-t}(q-1)^t - 1.$$

If $A_d(C) > q^k - q^{k-t}(q-1)^t - 1$ and $t \leq k$, then $|S_1(T) \cap C_d| \geq 1$. This means that there is at least one codeword in $C$ whose coordinates in $T$ are all nonzero. As a result, the punctured version of this codeword has Hamming weight $d-t$. Hence, the minimum distance of $C^T$ equals $d-t$. \qed

The only new result in Theorem 12 is the last conclusion on the minimum distance of the punctured code $C^T$. The following theorem shows that the minimum distance of $(C_T)^\perp$ can be determined in some cases.

Theorem 13. Let $C$ be an $[n, k, d]$ code over $\text{GF}(q)$, and let $d^\perp$ denote the minimum distance of the dual code $C^\perp$. Let $t$ be an integer with $1 \leq t < \min\{d, d^\perp\}$. For any set $T = \{i_1, i_2, \ldots, i_t\}$ of $t$ coordinates, $(C_T)^\perp$ has length $n-t$, dimension $n-k$ and minimum distance at least $d^\perp-t$.

Furthermore, if $A_{d^\perp}(C^\perp) > q^{n-k} - q^{n-k-t}(q-1)^t - 1$ and $t \leq n-k$, then the minimum distance of $(C_T)^\perp$ equals $d^\perp-t$.

Proof. By Theorem 3 we have $(C_T)^\perp = (C^\perp)^T$. The desired conclusions then follow from Theorem 12. \qed
Both Theorems 12 and 13 will be used to determine the parameters of some shortened codes and their duals later.

III. SOME SHORTENED CODES OF THE HAMMING CODES

A parity check matrix \( H_{(q,m)} \) of the Hamming code \( \mathcal{H}_{(q,m)} \) over GF\((q)\) is defined by choosing for its columns a nonzero vector from each one-dimensional subspace of GF\((q)^m\). In terms of finite geometry, the columns of \( H_{(q,m)} \) are the points of the projective geometry \( \text{PG}(m-1, \text{GF}(q)) \). Hence \( \mathcal{H}_{(q,m)} \) has length \( n = (q^m - 1)/(q - 1) \) and dimension \( n - m \). Note that no two columns of \( H_{(q,m)} \) are linearly dependent over GF\((q)\). The minimum weight of \( \mathcal{H}_{(q,m)} \) is at least 3. Adding two nonzero vectors from two different one-dimensional subspaces gives a nonzero vector from a third one-dimensional space. Therefore, \( \mathcal{H}_{(q,m)} \) has minimum weight 3. It is also well known that any \( [(q^m - 1)/(q - 1), (q^m - 1)/(q - 1) - m, 3] \) code over GF\((q)\) is monomially equivalent to the Hamming code \( \mathcal{H}_{(q,m)} \) \cite{13} Theorem 1.8.2]. The weight distribution of \( \mathcal{H}_{(q,m)} \) is given in the following lemma \cite{5}.

**Lemma 14.** The weight distribution of \( \mathcal{H}_{(q,m)} \) is given by

\[
q^m A_k(\mathcal{H}_{(q,m)}) \triangleq \sum_{0 \leq i \leq \frac{q^m - 1}{q - 1}} \left[ \binom{\frac{q^m - 1}{q - 1}}{i} \binom{q^m - 1}{q - 1} \left( (q - 1)^k + (-1)^j(q - 1)^i(q^m - 1) \right) \right]
\]

for \( 0 \leq k \leq (q^m - 1)/(q - 1) \).

The duals of the Hamming codes \( \mathcal{H}_{(q,m)} \) are called *Simplex codes*, which have parameters \( [(q^m - 1)/(q - 1), m, q^{m-1}] \). The nonzero codewords of the \( [(q^m - 1)/(q - 1), m, q^{m-1}] \) Simplex codes all have weight \( q^{m-1} \).

**Theorem 15.** Let \( n = (q^m - 1)/(q - 1) \geq 4 \), and let \( t_1 \) be any coordinator of codewords in \( \mathcal{H}_{(q,m)} \). Then the following hold:

- \( (\mathcal{H}_{(q,m)})_{\{t_1\}} \) is an \([n-1, n-m-1, 3]\) code over GF\((q)\) with
  \[
  A_k((\mathcal{H}_{(q,m)})_{\{t_1\}}) = \frac{n-k}{n} A_k(\mathcal{H}_{(q,m)})
  \]
  for \( 0 \leq k \leq n - 1 \), where \( A_k(\mathcal{H}_{(q,m)}) \) was given in Lemma 14.

- \( (\mathcal{H}_{(q,m)}^\perp)_{\{t_1\}} \) is an \([n-1, n-m-1, q^{m-1}]\) code over GF\((q)\) with weight enumerator
  \[
  1 + (q^{m-1} - 1)z^{q^{m-1}}.
  \]
• \((\mathcal{H}_{(q,m)})_{\{t\}}\) is an \([n - 1, m, q^{m-1} - 1]\) code over GF(q) with weight enumerator
\[
1 + (q - 1)q^{m-1}z^{q^{m-1} - 1} + (q^{m-1} - 1)z^{q^{m-1}}.
\]

• \((\mathcal{H}_{(q,m)}^{\perp})_{\{t\}}\) is an \([n - 1, n - m, 2]\) code over GF(q) with weight enumerator
\[
\frac{1}{q}[(1 + (q - 1)z)n - 1 + (q^{m-1} - 1)(1 - z)q^{m-1}(1 + (q - 1)z)n - 1 - q^{m-1}].
\]

**Proof.** By Lemma 5 \(B_{q^{m-1}}(\mathcal{H}_{(q,m)}^{\perp})\) does not have repeated blocks. It is known that the incidence structure \((\mathcal{P}(\mathcal{H}_{(q,m)}^{\perp}), B_{q^{m-1}}(\mathcal{H}_{(q,m)}^{\perp}))\) is a 2-design \([5]\). Since the Simplex code \(\mathcal{H}_{(q,m)}^{\perp}\) has weight enumerator \(1 + (q^{m} - 1)z^{q^{m-1}}\), \((\mathcal{P}(\mathcal{H}_{(q,m)}^{\perp}), B_{k}(\mathcal{H}_{(q,m)}^{\perp}))\) is the trivial 2-design \((\mathcal{P}(\mathcal{H}_{(q,m)}^{\perp}), \emptyset)\) or \((\mathcal{P}(\mathcal{H}_{(q,m)}^{\perp}), \emptyset)\) for each \(k\) with \(0 \leq k \leq n\) and \(k \neq q^{m-1}\). It then follows from Theorem 6 that \((\mathcal{P}(\mathcal{H}_{(q,m)}^{\perp}), B_{k}(\mathcal{H}_{(q,m)}^{\perp}))\) is a 2-design for each \(k\) with \(0 \leq k \leq n\). The desired conclusions on \((\mathcal{H}_{(q,m)})_{\{t\}}\) and \((\mathcal{H}_{(q,m)}^{\perp})_{\{t\}}\) then follow from Theorem 7 and Lemma 14.

We now prove the conclusions on the code \((\mathcal{H}_{(q,m)}^{\perp})_{\{t\}}\). It follows from Theorems 8 and 8 that
\[
A_{k}(\mathcal{H}_{(q,m)}^{\perp})_{\{t\}} = A_{k}(\mathcal{H}_{(q,m)}^{\perp})^{\{t\}}
\]
\[
= \sum_{i=0}^{\frac{n}{k}} \frac{(\frac{n}{k})}{(\frac{k+i}{1})} \frac{(\frac{k+i}{1})}{(\frac{n}{1})} A_{k+i}(\mathcal{H}_{(q,m)}^{\perp}).
\]

Notice that \((\mathcal{H}_{(q,m)}^{\perp})_{\{t\}}\) has weight enumerator \(1 + (q^{m} - 1)z^{q^{m-1}}\). Combining this with (5), we deduce that \(A_{k}(\mathcal{H}_{(q,m)}^{\perp})_{\{t\}} = 0\) for all \(k \notin \{0, q^{m-1} - 1, q^{m-1}\}\) and
\[
A_{q^{m-1}}(\mathcal{H}_{(q,m)}^{\perp})_{\{t\}} = (q - 1)q^{m-1}
\]
\[
A_{q^{m-1}}(\mathcal{H}_{(q,m)}^{\perp})_{\{t\}} = q^{m-1} - 1.
\]

This completes the proof of the desired conclusions on \((\mathcal{H}_{(q,m)}^{\perp})_{\{t\}}\).

Finally, we prove the conclusions on the code \((\mathcal{H}_{(q,m)})_{\{t\}}^{\perp}\). Note that the weight enumerator of \((\mathcal{H}_{(q,m)})_{\{t\}}^{\perp}\) is \(1 + (q^{m-1} - 1)z^{q^{m-1}}\). The desired weight enumerator in (4) then follows from the MacWilliams identity. It is easily verified that the coefficient of \(z\) in the polynomial in (4) equals 0, and the coefficient of \(z^2\) is
\[
\frac{(\frac{n}{2})}{(\frac{2}{2})} (q - 1)^2 + (q^{m-1} - 1)[(\frac{2}{2})] - (q - 1)q^{m-1}(n - 1 - q^{m-1})]
\]
\[
q^{m-1} = \frac{q(q - 1)(q^{m-1} - 1)}{2} > 0.
\]
TABLE I
EXAMPLES OF THE CODE $\mathcal{H}_{(q,m)}\{t_1\}$

| $q$ | $m$ | $[n,k,d]$ | Optimality |
|-----|-----|-----------|------------|
| 2   | 3   | [6, 3, 3] | Yes        |
| 2   | 4   | [14, 10, 3] | Yes |
| 2   | 5   | [30, 25, 3] | Yes |
| 2   | 6   | [62, 56, 3] | Yes |
| 2   | 7   | [126, 119, 3] | Yes |
| 3   | 2   | [3, 1, 3] | Yes |
| 3   | 3   | [12, 9, 3] | Yes |
| 3   | 4   | [39, 35, 3] | Yes |
| 3   | 5   | [120, 115, 3] | Yes |

Consequently, $((\mathcal{H}_{(q,m)}\{t_1\})^\perp)^\perp$ has minimum weight 2.

Theoretically, we have the following conclusions about the two shortened codes in Theorem 15 and their duals.

- Let $n \geq 7$ and $m \geq 2$. Then the shortened code $\mathcal{H}_{(q,m)}\{t_1\}$ is both length-optimal and dimension-optimal with respect to the sphere-packing bound.
- The code $\mathcal{H}_{(q,m)}\{t_1\}$ meets the Griesmer bound.
- The code $((\mathcal{H}_{(q,m)}\{t_1\})^\perp)^\perp$ is distance-optimal with respect to the sphere-packing bound, and is MDS when $m = 2$. Note that $((\mathcal{H}_{(q,m)}\{t_1\})^\perp)^\perp = (\mathcal{H}_{(q,m)}\{t_1\})$, which is a punctured Hamming code.
- The code $((\mathcal{H}_{(q,m)}\{t_1\})^\perp)^\perp$ meets the Griesmer bound.

Table I lists examples of the code $\mathcal{H}_{(q,m)}\{t_1\}$, which show that the code is distance-optimal in all these cases according to [8]. Table II lists examples of the code $\mathcal{H}_{(q,m)}\{t_1\}$, which show that the code is distance-optimal in all these cases according to [8].

**Theorem 16.** Let $n = (q^m - 1)/(q - 1) \geq 6$, and let $t_1$ and $t_2$ be any two distinct coordinators of codewords in $\mathcal{H}_{(q,m)}$. Then the following hold:

- $(\mathcal{H}_{(q,m)}\{t_1,t_2\})$ is an $[n-2,n-m-2,3]$ code over GF$(q)$ with

$$A_k((\mathcal{H}_{(q,m)}\{t_1,t_2\})) = \binom{n}{k} \binom{n-2}{k} A_k(\mathcal{H}_{(q,m)})$$

for $0 \leq k \leq n-2$, where $A_k(\mathcal{H}_{(q,m)})$ was given in Lemma 14.
\begin{table}[h]
\centering
\caption{Examples of the code $(\mathcal{H}_{(q,m)})\{(t_1,t_2)\}$}
\begin{tabular}{cccc}
\hline
$q$ & $m$ & $[n,k,d]$ & Optimality \\
\hline
2 & 3 & [6,2,4] & Yes \\
2 & 4 & [14,3,8] & Yes \\
2 & 5 & [30,4,16] & Yes \\
2 & 6 & [62,5,32] & Yes \\
2 & 7 & [126,6,64] & Yes \\
3 & 2 & [3,1,3] & Yes \\
3 & 3 & [12,2,9] & Yes \\
3 & 4 & [39,3,27] & Yes \\
3 & 5 & [120,4,81] & Yes \\
\hline
\end{tabular}
\end{table}

- $(\mathcal{H}_{(q,m)})\{(t_1,t_2)\}$ is an $[n-2,m-2,q^{m-1}]$ code over GF($q$) with weight enumerator
  \[1 + (q^{m-2} - 1) z^{q^{m-1}}.\]

- $(\mathcal{H}_{(q,m)})\{(t_1,t_2)\}^\perp$ is an $[n-2,m,q^{m-1}-2]$ code over GF($q$) with weight enumerator
  \[1 + (q-1)^2 q^{m-2} z^{q^{m-1}-2} + 2(q-1)q^{m-2} z^{q^{m-1}-1} + (q^{m-2} - 1) z^{q^{m-1}}.\]

\textbf{Proof.} The proof is similar to that of Theorem 15 and omitted here.

\begin{table}[h]
\centering
\caption{Examples of the code $(\mathcal{H}_{(q,m)})\{(t_1,t_2)\}$}
\begin{tabular}{cccc}
\hline
$q$ & $m$ & $[n,k,d]$ & Optimality \\
\hline
2 & 3 & [5,2,3] & Yes \\
2 & 4 & [13,9,3] & Yes \\
2 & 5 & [29,24,3] & Yes \\
2 & 6 & [61,55,3] & Yes \\
2 & 7 & [125,118,3] & Yes \\
3 & 3 & [11,8,3] & Yes \\
3 & 4 & [38,34,3] & Yes \\
3 & 5 & [119,114,3] & Yes \\
\hline
\end{tabular}
\end{table}

Let $n \geq 7$ and $m \geq 2$. Then the shortened code $(\mathcal{H}_{(q,m)})\{(t_1,t_2)\}$ is both length-optimal and dimension-optimal with respect to the sphere-packing bound. The code $(\mathcal{H}_{(q,m)})\{(t_1,t_2)\}^\perp$ is MDS.
when \( m = 2 \), and meets the Griesmer bound when \( q > 2 \). Table III lists examples of the code \((\mathcal{H}_{(q,m)}^{++})\{t_1,t_2\}\), which show that the code is distance-optimal in all these cases according to [8]. Table IV lists examples of the code \((\mathcal{H}_{(q,m)}^{-})\{t_1,t_2\}\), which show that the code is distance-optimal in most cases, and almost distance-optimal (i.e., the minimum distance is one less than the best possible value) in three cases according to [8].

| \( q \) | \( m \) | \([n,k,d]\) | Optimality |
|---|---|---|---|
| 2 | 3 | \([5,1,4]\) | Almost |
| 2 | 4 | \([13,2,8]\) | Yes |
| 2 | 5 | \([29,3,16]\) | Yes |
| 2 | 6 | \([61,4,32]\) | Yes |
| 2 | 7 | \([125,5,64]\) | Yes |
| 3 | 3 | \([11,1,9]\) | Almost |
| 3 | 4 | \([38,2,27]\) | Almost |
| 3 | 5 | \([119,3,81]\) | Yes |

**Theorem 17.** Let \( n = (q^m - 1)/(q - 1) \geq 6 \), and let \( T \) be the support of any codeword of weight 3 in \( \mathcal{H}_{(q,m)} \). Then \((\mathcal{H}_{(q,m)}^{+})T\) is an \([n-3,m-2,q^{m-1}]\) code over GF(q).

**Proof.** By Theorem 3 \((\mathcal{H}_{(q,m)}^{+})T\) has dimension \( m-2 \). Since \( m \geq 3 \), this code has \( q^{m-2} - 1 \geq q - 1 \) nonzero codewords. Consequently, \((\mathcal{H}_{(q,m)}^{+})T\) has minimum distance \( q^{m-1} \). \( \square \)

**Theorem 18.** Let \( n = (q^m - 1)/(q - 1) \), and let \( T = \{i_1,i_2,\ldots,i_t\} \) be any subset of \( t \) pairwise distinct coordinates in the codewords in \( \mathcal{H}_{(q,m)} \).

If \( t < q^{m-1} \), then \((\mathcal{H}_{(q,m)})_T\) is an \([n-t,n-m-t,d((\mathcal{H}_{(q,m)})_T)]\) code over GF(q), where \( d((\mathcal{H}_{(q,m)})_T) \geq 3 \).

If \( t = q^{m-1} \) and \( T \) is the support of a codeword of weight \( q^{m-1} \) in \( \mathcal{H}_{(q,m)} \), then \((\mathcal{H}_{(q,m)})_T\) is an \([n-t,n-m-t+1,d((\mathcal{H}_{(q,m)})_T)]\) code over GF(q), where \( d((\mathcal{H}_{(q,m)})_T) \geq 3 \).

**Proof.** The desired conclusions on the length and dimension of \((\mathcal{H}_{(q,m)})_T\) follow from Theorem 3. By definition, \( d((\mathcal{H}_{(q,m)})_T) \geq 3 \). \( \square \)
We inform the reader that the code \( (\mathcal{H}(q,m))_T \) in Theorem 18 is both length-optimal and dimension-optimal with respect to the sphere-packing bound when \( n \geq 7 \), \( m \geq 2 \) and \( t = |T| = 3 \). Hence, Theorem 18 does include a family of optimal shortened codes.

When \( t \in \{1,2\} \), the parameters and the weight distribution of the code \( (\mathcal{H}(q,m))_T \) were described in Theorems 15 and 16. In many other cases, \( d((\mathcal{H}(q,m))_T) = 3 \). Below we present a general result.

A partial \( k \)-spread \( S \) of the projective space \( \text{PG}(m-1,q) \) is a collection of pairwise disjoint \( k \)-dimensional subspaces. By definition, every point of \( \text{PG}(m-1,q) \) is contained in at most one element of \( S \).

**Lemma 19 ([2]).** Let \( 1 \leq k < m-1 \), where \( m \) and \( k \) are integers, and let \( r \) be the remainder of \( m \) divided by \( k+1 \). Then there exists a partial \( k \)-spread with cardinality \( \frac{q^m-q^r}{q^k-1} - q^r + 1 \).

**Theorem 20.** Let \( n = \frac{q^m-1}{q-1} \), and let \( T = \{i_1,i_2,\cdots,i_t\} \) be any subset of \( t \) pairwise distinct coordinates in the codewords in \( \mathcal{H}(q,m) \) with \( t < \frac{q-1}{q+1}n \) if \( m \) is even, and \( t < \frac{q-1}{q+1}(n-q^2) \) if \( m \) is odd. Then the minimum distance of the shortened Hamming code \( (\mathcal{H}(q,m))_T \) is equal to 3.

**Proof.** Let \( P_0, \cdots, P_{n-1} \) be the points of \( \text{PG}(m-1,q) \) and let \( D = \{P_i : 0 \leq i \leq n-1, i \notin T\} \). By Lemma 19 we can choose a 1-spread \( S = \{U_1,\cdots,U_L\} \) with \( L = \frac{q^m-q^r}{q^k-1} - q^r + 1 \), where

\[
    r = \begin{cases} 
      0, & \text{if } m \text{ is even,} \\
      1, & \text{otherwise.}
    \end{cases}
\]

It is obvious that

\[
(q-1)L = \begin{cases} 
  \frac{q-1}{q+1}n, & \text{if } m \text{ is even,} \\
  \frac{q-1}{q+1}(n-q^2), & \text{otherwise.}
\end{cases}
\]

By the pigeonhole principle, there must be a \( j \in \{1,\cdots,L\} \) such that \( |U_j \cap (\text{PG}(m-1,q) \setminus D)| < q-1 \). This clearly forces \( |U_j \cap D| \geq 3 \). Then we choose three distinct points \( P_{i_1}, P_{i_2}, P_{i_3} \) in \( U_j \cap D \). Thus \( P_{i_1}, P_{i_2} \) and \( P_{i_3} \) are collinear in \( \text{PG}(q,m-1) \). It is easily seen that any nontrivial linear relationship among \( P_{i_1}, P_{i_2}, P_{i_3} \) gives rise to a codeword of weight 3 in \( (\mathcal{H}(q,m))_T \). This completes the proof.

\[
\Box
\]

**IV. SOME SHORTENED CODES OF THE REED-MULLER CODES**

Reed-Muller codes can be defined by either the univariate or the multivariate approach. Each approach has advantages and disadvantages. We briefly recall the univariate definition below.
Let $m$ be a positive integer. Any function from $\GF(2^m)$ to $\GF(2)$ is called a (univariate) Boolean function. Let $B_m$ denote the set of all Boolean functions on $\GF(2^m)$. Every nonzero Boolean function $f$ on $\GF(2^m)$ can be uniquely expressed as

$$f(x) = \sum_{i=0}^{2^m-1} f_i x^i,$$

where $f_i \in \GF(2^m)$. Every integer $i$ with $0 \leq i \leq 2^m - 1$ has the unique 2-adic expansion $i = \sum_{j=0}^{m-1} i_j 2^j$, where $i_j \in \{0, 1\}$. The 2-weight of $i$ is defined to be the Hamming weight of $(i_0, i_1, \ldots, i_{m-1})$. The algebraic degree of a Boolean function $f$ on $\GF(2^m)$ defined in (6), denoted by $\deg(f)$, is defined to be the maximum 2-weight of all $i$ such that $f_i \neq 0$.

Let $\alpha$ be a generator of $\GF(2^m)^*$. Define $P_0 = 0$ and $P_i = \alpha^{i-1}$ for $1 \leq i \leq 2^m - 1$. The Reed-Muller code of length $2^m$ and order $r$ is defined by

$$R(r, m) = \{(f(P_0), f(P_1), \ldots, f(P_{2^m-1})) : f \in B_m, \deg(f) \leq r\}.$$

The reader is referred to [4, Chapter 5] and [16] for detailed information on the Reed-Muller code. The following are well known:

- $R(r, m)^\perp = R(m - 1 - r, m)$.
- $R(1, m)$ has parameters $[2^m, m+1, 2^{m-1}]$ and weight enumerator $1 + (2^{m+1} - 2)z^{2^{m-1}} + z^{2^m}$.
- $R(m - 2, m)$ has parameters $[2^m, 2^m - m - 1, 4]$.

Our objective in this section is to study some shortened codes of $R(1, m)$ and $R(m - 2, m)$ and their duals. As before, we are only interested in optimal codes or codes meeting a bound for linear codes. The first result of this section is the following.

**Theorem 21.** Let $m \geq 3$, and let $t_1$ be any coordinate. Then the following hold.

1) $R(1, m)_{\{t_1\}}$ is a $[2^m - 1, m, 2^{m-1}]$ binary code with weight enumerator $1 + (2^m - 1)z^{2^{m-1}}$, and is equivalent to the binary Simplex code. The code meets the Griesmer bound.

2) $(R(1, m)_{\{t_1\}})^\perp$ is a $[2^m - 1, 2^m - m - 1, 3]$ binary code, and is equivalent to the binary Hamming code. The code meets the sphere-packing bound and is perfect.

3) $R(m - 2, m)_{\{t_1\}}$ is a $[2^m - 1, 2^m - m - 2, 4]$ binary code. The code is distance-optimal with respect to the sphere-packing bound.

4) $(R(m - 2, m)_{\{t_1\}})^\perp$ is a $[2^m - 1, m+1, 2^{m-1} - 1]$ binary code with weight enumerator

$$1 + (2^m - 1)z^{2^{m-1}-1} + (2^m - 1)z^{2^{m-1}} + z^{2^{m-1}}.$$

This code meets the Griesmer bound.
**Proof.** We outline the proof as follows. Note that $\mathcal{B}_{2^m}(\mathcal{R}(1,m)) = \{\mathcal{P}(\mathcal{R}(1,m))\}$. It is then straightforward to see that $(\mathcal{P}(\mathcal{R}(1,m)), \mathcal{B}_{2^m}(\mathcal{R}(1,m)))$ is a $3-(2^m, 2^m, 1)$ simple design. It is known that $(\mathcal{P}(\mathcal{R}(1,m)), \mathcal{B}_{2^m-1}(\mathcal{R}(1,m)))$ is a $3-(2^m, 2^{m-1}, 2^{m-2} - 1)$ simple design [4, p. 143]. Clearly, $(\mathcal{P}(\mathcal{R}(1,m)), \mathcal{B}_k(\mathcal{R}(1,m)))$ is the trivial 3-design $(\mathcal{P}(\mathcal{R}(1,m)), \{\emptyset\})$ or $(\mathcal{P}(\mathcal{R}(1,m)), \emptyset)$ for all $k \notin \{2^{m-1}, 2^m\}$. It then follows from Theorem 6 that $(\mathcal{P}(\mathcal{R}(1,m)^\perp), \mathcal{B}_k(\mathcal{R}(1,m)^\perp))$ is a 3-design for all $k$ with $0 \leq k \leq 2^m$. We are now ready to apply Theorems 7 and 8. The desired conclusions on the codes can be similarly proved as the conclusions of Theorem 15. The details are omitted here. □

Similarly, one can prove the following result.

**Theorem 22.** Let $m \geq 3$, and let $t_1$ and $t_2$ be two distinct coordinates. Then the following hold.

1) $\mathcal{R}(1,m)_{\{t_1,t_2\}}$ is a $[2^m - 2, m - 1, 2^{m-1} - 1]$ binary code with weight enumerator

$$1 + (2^{m-1} - 1)z^{2^{m-1} - 1}.$$

This code almost meets the Griesmer bound.

2) $(\mathcal{R}(1,m)_{\{t_1,t_2\}})^\perp$ is a $[2^m - 2, 2^m - m - 1, 2]$ binary code, and is distance-optimal with respect to the sphere-packing bound.

3) $\mathcal{R}(m-2, m)_{\{t_1,t_2\}}$ is a $[2^m - 2, 2^m - m - 3, 4]$ binary code and is distance-optimal with respect to the sphere-packing bound.

4) $(\mathcal{R}(m-2, m)_{\{t_1,t_2\}})^\perp$ is a $[2^m - 2, m + 1, 2^{m-1} - 2]$ binary code with weight enumerator

$$1 + (2^{m-1} - 1)z^{2^{m-1} - 2} + 2^m z^{2^{m-1} - 1} + (2^{m-1} - 1)z^{2^{m-1}} + z^{2^m - 2}.$$

This code almost meets the Griesmer bound.

Note that all the codes in Theorems 21 and 22 are either optimal or almost optimal. We have also the following.

**Theorem 23.** Let $m \geq 3$, and let $t_1$, $t_2$ and $t_3$ be three pairwise distinct coordinates. Then the following hold.

1) $(\mathcal{R}(m-2, m)_{\{t_1,t_2,t_3\}})^\perp$ is a $[2^m - 3, m + 1, 2^{m-1} - 3]$ binary code with weight enumerator

$$1 + (2^{m-2} - 1)z^{2^{m-1} - 3} + 3 \times 2^{m-2} z^{2^{m-1} - 2} + 3 \times 2^{m-2} z^{2^{m-1} - 1} + (2^{m-2} - 1)z^{2^{m-1}} + z^{2^m - 3}.$$

This code almost meets the Griesmer bound.
2) $R(m - 2, m)_{\{1, 2, 3\}}$ is a $[2^m - 3, 2^m - m - 4, 4]$ binary code and is distance-optimal with respect to the sphere-packing bound.

Proof. The desired conclusions can be similarly proved with Theorems 6, 7, and 8. The parameters of the two codes can also be settled with Theorems 12 and 13. The details are omitted. \qed

V. SOME SHORTENED CODES OF THE OVOID CODES

In the projective space $\text{PG}(3, \text{GF}(q))$ with $q > 2$, an ovoid $\mathcal{O}$ is a set of $q^2 + 1$ points such that no three of them are collinear (i.e., on the same line). In other words, an ovoid is a $(q^2 + 1)$-cap (a cap with $q^2 + 1$ points) in $\text{PG}(3, \text{GF}(q))$, and thus a maximal cap. Two ovoids are said to be equivalent if there is a collineation (i.e., automorphism) of $\text{PG}(3, \text{GF}(q))$ that sends one to the other.

A classical ovoid $\mathcal{O}$ can be defined as the set of all points given by

$$\mathcal{O} = \{(0, 0, 1, 0)\} \cup \{(x, y, x^2 + xy + ay^2, 1) : x, y \in \text{GF}(q)\},$$

where $a \in \text{GF}(q)$ is such that the polynomial $x^2 + x + a$ has no root in $\text{GF}(q)$. Such ovoid is called an elliptic quadric, as the points come from a non-degenerate elliptic quadratic form.

For $q = 2^{2e+1}$ with $e \geq 1$, there is an ovoid which is not an elliptic quadric, and is called the Tits ovoid. It is defined by

$$\mathcal{T} = \{(0, 0, 1, 0)\} \cup \{(x, y, x^\sigma + xy + y^{\sigma+2}, 1) : x, y \in \text{GF}(q)\},$$

where $\sigma = 2^{e+1}$.

For odd $q$, any ovoid is an elliptic quadric. For even $q$, Tits ovoids are the only known ones which are not elliptic quadratics. In the case that $q$ is even, the elliptic quadrics and the Tits ovoid are not equivalent.

Let $\mathcal{O}$ be an ovoid in $\text{PG}(3, \text{GF}(q))$ with $q > 2$. Denote by

$$\mathcal{O} = \{v_1, v_2, \ldots, v_{q^2+1}\},$$

where each $v_i$ is a column vector in $\text{GF}(q)^4$. Let $C_{\mathcal{O}}$ be the linear code over $\text{GF}(q)$ with generator matrix

$$G_{\mathcal{O}} = [v_1v_2\cdots v_{q^2+1}].$$

It is known that $C_{\mathcal{O}}$ is a $[q^2 + 1, 4, q^2 - q]$ code over $\text{GF}(q)$ with weight enumerator

$$1 + (q^2 - q)(q^2 + 1)z^{q^2-q} + (q - 1)(q^2 + 1)z^{q^2}$$
and its dual $C_{q^r}^\perp$ is a $[q^2 + 1, q^2 - 3, 4]$ code over GF($q$) [4, Chapter 13]. Conversely, the set of column vectors of a generator matrix of any $[q^2 + 1, q^2 - 3, 4]$ code over GF($q$) is an ovoid in PG($3, GF(q)$). Hence, ovoids in PG($3, GF(q)$) and $[q^2 + 1, q^2 - 3, 4]$ codes over GF($q$) are the same, and a $[q^2 + 1, 4, q^2 - q]$ code over GF($q$) is called an ovoid code over GF($q$).

The weight distribution of an ovoid is given in the following lemma [4, p. 324], and will be employed later.

**Lemma 24.** Let $q \geq 4$, and let $C$ be a $[q^2 + 1, 4, q^2 - q]$ code over GF($q$). Then the weight distribution of $C^\perp$ is given by

$$q^4 A_\ell(C^\perp) = \binom{q^2 + 1}{\ell} (q-1)^\ell + u \sum_{i+j=\ell} \binom{q^2 - q}{i} (-1)^i \binom{q+1}{j} (q-1)^j +$$

$$v \left[ (-1)^\ell \binom{q^2}{\ell} + (-1)^{\ell-1} (q-1) \binom{q^2}{\ell-1} \right]$$

(11)

for all $4 \leq \ell \leq q^2$, and

$$q^4 A_{q^2+1} = (q-1)^{q^2+1} + u(q-1)^{q+1} + v(q-1),$$

where

$$u = (q^2 - q)(q^2 + 1), \quad v = (q-1)(q^2 + 1).$$

We are now ready to study some shortened codes of ovoid codes, and have the following results.

**Theorem 25.** Let $q \geq 4$, and let $C$ be a $[q^2 + 1, 4, q^2 - q]$ code over GF($q$). For any coordinate $t_1$, the following hold.

1) $C_{\{t_1\}}$ is a $[q^2, 3, q^2 - q]$ code over GF($q$) with weight enumerator

$$1 + q(q^2 - 1)z^{q^2 - q} + (q-1)z^{q^2}.$$  (13)

2) $(C_{\{t_1\}})^\perp$ is a $[q^2, q^2 - 3, 3]$ almost MDS code over GF($q$).

3) $((C^\perp)_{\{t_1\}})^\perp$ is a $[q^2, 4, q^2 - q - 1]$ code over GF($q$) with weight enumerator

$$1 + q^2(q-1)^2z^{q^2 - q - 1} + q(q^2 - 1)z^{q^2 - q} + q^2(q-1)z^{q^2 - 1} + (q-1)z^{q^2}.$$  

4) $(C^\perp)_{\{t_1\}}$ is a $[q^2, q^2 - 4, 4]$ almost MDS code over GF($q$) with weight distribution

$$A_k((C^\perp)_{\{t_1\}}) = \frac{\binom{k}{q^2}}{q^2+1} \frac{\binom{q^2}{k}}{q^2-1} A_k(C^\perp),$$  

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where $1 \leq k \leq q^2$, and $A_k(C^\perp)$ was given in Lemma 24.

Proof. It is straightforward to see that two codewords of weight $q^2$ in $C$ have the same support if and only if one is a nonzero multiple of the other. Hence, $B_{q^2}(C)$ has no repeated block and cardinality $q^2+1$, and every $q^2$-subset of $P(C)$ appears exactly once in $B_{q^2}(C)$. Consequently, $(P(C), B_{q^2}(C))$ is a $3$-$(q^2+1, q^2, q^2-2)$ design. It is known that $(P(C), B_{q^2-q}(C))$ is a $3$-$(q^2+1, q^2-q, (q-2)(q^2-q-1))$ simple design [4, p. 327]. Hence, $(P(C), B_k(C))$ is a $3$-design for all $k$ with $0 \leq k \leq q^2+1$. It then follows from Theorem 6 that $(P(C^\perp), B_k(C^\perp))$ is a $3$-design for all $k$ with $0 \leq k \leq q^2+1$.

The desired conclusions on $C_{\{t_1\}}$ then follow from Theorem 7 and the weight enumerator of $C$ given in (10). Using the weight enumerator of $C_{\{t_1\}}$ in (13) and the MacWilliams identity, one can prove that the minimum distance $d((C_{\{t_1\}})^\perp) = 3$. Thus, $(C_{\{t_1\}})^\perp$ is a $[q^2, q^2-3, 3]$ almost MDS code over GF($q$).

We now prove the desired conclusions on $((C^\perp)_{\{t_1\}})^\perp$. By Theorem 3, we have

$$((C^\perp)_{\{t_1\}})^\perp = C_{\{t_1\}}.$$

It then follows from Theorem 8 that

$$A_k(((C^\perp)_{\{t_1\}})^\perp) = A_k(C_{\{t_1\}}) = \sum_{i=0}^{k} \frac{\binom{q^2}{k} \binom{k+i}{1}}{\binom{q^2+1}{k+1}} A_{k+i}(C). \quad (14)$$

The desired conclusions then follow from the weight enumerator of $C$ in (10).

Since $d(C^\perp) = 4$, by definition $d(((C^\perp)_{\{t_1\}})^\perp) \geq 4$. If $d(((C^\perp)_{\{t_1\}})^\perp) = 5$, then $(C^\perp)_{\{t_1\}}$ would be a $[q^2, q^2-4, 5]$ MDS code, and $((C^\perp)_{\{t_1\}})^\perp$ would be $[q^2, 4, q^2-3]$ MDS code, which is a contradiction. Therefore, $d(((C^\perp)_{\{t_1\}})^\perp) = 4$. The desired conclusion on the weight distribution of $(C^\perp)_{\{t_1\}}$ then follows from (14) and the weight enumerator of $C$ given in (10).

Notice that all ovoid codes meet the Griemer bound. The shortened codes and their duals documented in Theorem 25 are very interesting due to the following.

- All the four classes of codes in Theorem 25 support $2$-designs.
- Both $C_{\{t_1\}}$ and $(C^\perp)_{\{t_1\}}$ meet the Griesmer bound.

Using the Assmus-Mattson theorem (see [1] or [4, Chapter 4]), Theorem 25 and Lemma 5, one can prove the following theorem. We omit the proofs here.

**Theorem 26.** Let $q \geq 4$, and let $C$ be a $[q^2+1, 4, q^2-q]$ code over GF($q$). For any coordinate $t_1$, the following hold.
1) The incidence structure \((\mathcal{P}(C_{t1}), \mathcal{B}_{q^2-q}(C_{t1}))\) is a 2-\((q^2, q^2-q, q^2-q-1)\) simple design.

2) The incidence structure \((\mathcal{P}(((C_{t1})^\perp), \mathcal{B}_{q^2-q-1}(((C_{t1})^\perp))))\) is a 2-\((q^2, 3, \lambda)\) simple design for some integer \(\lambda\).

3) The incidence structure \((\mathcal{P}(((C_{t1})^\perp), \mathcal{B}_{q^2-q-1}(((C_{t1})^\perp))))\) is a 2-\((q^2, q^2-q-1, (q-2)(q^2-q-1))\) simple design. The complement of this design is a 2-\((q^2, q+1, q)\) design.

4) The incidence structure \((\mathcal{P}(((C_{t1})^\perp), \mathcal{B}_{q^2-q}(((C_{t1})^\perp))))\) is a 2-\((q^2, q^2-q, q^2-q-1)\) simple design. The complement of this design is a Steiner system 2-\((q^2, q, 1)\), i.e., an affine plane.

5) The incidence structure \((\mathcal{P}(((C_{t1})^\perp), \mathcal{B}_4(((C_{t1})^\perp))))\) is a 2-\((q^2, 4, \lambda)\) simple design for some integer \(\lambda\).

Since \((\mathcal{P}(C^\perp), \mathcal{B}_k(C^\perp))\) is a 3-design for all \(k\) with \(0 \leq k \leq q^2+1\), we can similarly determine the parameters and weight distributions of the codes \((C^\perp)_{t1,t2}\) and \((C^\perp)_{t1,t2,t3}\). However, these shortened codes are less interesting.

VI. SUMMARY AND CONCLUDING REMARKS

The main contributions of this paper are the following.

- It was proved in Theorem 9 that every linear code \(C\) over GF\((q)\) with minimum distance \(d \geq 2\) is a shortened code of \(C(m, q, \alpha)^\perp\) for some \(m, q\) and \(\alpha\), where \(\alpha\) is a generator of GF\((q)^*\) and \(C(m, q, \alpha)\) was defined in \([2]\). This showed the importance of the shortening technique and the family of cyclic codes \(C(m, q, \alpha)^\perp\).

- It was proved in Theorem 11 that every linear code over GF\((q)\) with minimum distance at least 3 is a shortened code of a Hamming code over GF\((q)\). This showed the importance of the shortening technique and the family of Hamming codes.

- The parameters and weight distributions of two families of shortened codes of the Hamming codes and their duals were settled in Theorem 15. All of the shortened codes are optimal.

- The parameters and weight distributions of another two families of shortened codes of the Hamming codes and their duals were settled in Theorem 16. All of the shortened codes are optimal.

- The parameters of three families of shortened codes of the Reed-Muller codes \(R(1,m)\) and \(R(m-2, m)\) were settled in Theorems 21 and 23. All of the shortened codes are optimal.
• The parameters of another two families of shortened codes of the Reed-Muller codes $R(1,m)$ and $R(m-2,m)$ were settled in Theorem 22. All of the shortened codes are either optimal or almost optimal.

• The parameters of shortened codes of the ovoid code and its dual were settled in Theorem 25. The shortened codes are either optimal or almost optimal.

• Five families of 2-designs were obtained from the shortened codes of the ovoid codes and their duals and were documented in Theorem 26. Some of the 2-designs are interesting.

In summary, eleven infinite families of optimal shortened codes with new parameters were presented in this paper.

Since every linear code with minimum weight at least 3 is a shortened code of a Hamming code, some shortened codes must have bad parameters and some shortened codes must have good or optimal parameters. To obtain an optimal or good shortened code, a linear code $C$ and the coordinate set $T$ for shortening must be properly selected.

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