Low-Temperature Transport Properties of Very Dilute Classical Solutions of $^3$He in Superfluid $^4$He

Gordon Baym · D. H. Beck · C. J. Pethick

Received: 7 August 2014 / Accepted: 22 September 2014 / Published online: 15 October 2014
© Springer Science+Business Media New York 2014

Abstract We report microscopic calculations of the thermal conductivity, diffusion constant, and thermal diffusion constant for classical solutions of $^3$He in superfluid $^4$He at temperatures $T \lesssim 0.6$ K, where phonons are the dominant excitations of the $^4$He. We focus on solutions with $^3$He concentrations $\lesssim 10^{-3}$, for which the main scattering mechanisms are phonon–phonon scattering via 3-phonon Landau and Beliaev processes, which maintain the phonons in a drifting equilibrium distribution, and the slower process of $^3$He–phonon scattering, which is crucial for determining the $^3$He distribution function in transport. We use the fact that the relative changes in the energy and momentum of a $^3$He atom in a collision with a phonon are small to derive a Fokker–Planck equation for the $^3$He distribution function, which we show has an analytical solution in terms of Sonine polynomials. We also calculate the corrections to the Fokker–Planck results for the transport coefficients.

Keywords Dilute solutions of $^3$He in $^4$He · Transport · Diffusion · Thermal conductivity · Boltzmann equation · Fokker–Planck equation · Neutron electric dipole moment experiment

G. Baym (✉) · D. H. Beck · C. J. Pethick
Department of Physics, University of Illinois, 1110 W. Green Street, Urbana, IL 61801, USA
e-mail: gbaym@illinois.edu

G. Baym · C. J. Pethick
The Niels Bohr International Academy, The Niels Bohr Institute, University of Copenhagen,
Blegdamsvej 17, 2100 Copenhagen, Denmark

C. J. Pethick
NORDITA, Roslagstullsbacken 23, 10691 Stockholm, Sweden
1 Introduction

*Problems Worthy of Attack, Prove Their Worth by Hitting Back* [1]

Historically, the study of equilibrium and transport processes in the helium liquids revealed important information about the basic physics of quantum liquids, e.g., [2]. A proposed experiment at Oak Ridge National Laboratory [3,4] to search sensitively for the electric dipole moment of the neutron (nEDM) has renewed interest in the transport properties of dilute solutions of $^3$He in liquid $^4$He in low concentration regimes never before studied in detail. The experiment will look for the effect of an electric field on the capture rate of polarized neutrons on polarized $^3$He atoms dissolved in $^4$He.

In a relative spin singlet state, the capture rate of a neutron on a $^3$He atom can proceed through a virtual $\alpha$ particle state, and is thus enormous compared with the rate for a relative triplet state. The experiment therefore aims to detect the precession of the neutron in an electric field from a change in the capture rate. With time, however, the $^3$He atoms depolarize through scattering against the container walls, and in the course of the experiment they will be driven out of the experimental volume by a phonon wind generated by a heater in the liquid $^4$He, and replaced with fresh polarized $^3$He.

A novel pilot experiment was performed by Lamoreaux et al. [5], who measured the $^3$He density gradient induced by a heat flow at temperatures $\lesssim 0.6$ K, at which the effects of rotons are negligible. In this experiment, the $^3$He number concentration, $x_3 = n_3/(n_3 + n_4)$, where $n_3$ and $n_4$ are the $^3$He and $^4$He number densities, lay in the range $7 \times 10^{-5}$–$1.5 \times 10^{-3}$ in the non-degenerate regime. The experiment was initially interpreted in terms of diffusion of $^3$He atoms with respect to a gas of phonons [5,6], but in Ref. [7] we showed that the relevant transport coefficient is the total thermal conductivity of the liquid, which consists of two contributions, one from the $^3$He and another from diffusive flow of phonons relative to $^3$He. This latter diffusion does not necessarily involve net relative motion of $^4$He atoms relative to $^3$He, as does diffusion in a normal system, since the net flux of $^4$He atoms has a contribution from the superfluid as well as that from the phonons.

In transport in this relatively high-$x_3$ regime, the $^3$He–$^3$He interactions are sufficiently strong that they keep the $^3$He in thermal equilibrium at rest at the local temperature $T(r)$, while phonon–phonon interactions keep the phonons in drifting local equilibrium. In thermal transport, phonons transfer momentum to the $^3$He atoms, via the $^3$He–phonon interactions. The calculations in Ref. [7] took into account a number of physical effects not included in earlier calculations of the thermal conductivity. Among these are phonon–phonon scattering, which rapidly establishes thermal equilibrium among phonons traveling in a given direction. $^3$He–$^3$He scattering, which is effective in maintaining thermal equilibrium of these atoms, and energy transfer in $^3$He–phonon collisions, which reduces the rate of these collisions. These calculations agree well with the data of Ref. [5]. By contrast, at lower concentrations, as will be encountered in the proposed search for the neutron electric dipole moment, the concentration of $^3$He will be in the much lower range $10^{-9}$–$10^{-11}$. This regime, never before studied, presents as we see here new and interesting physics. Thermal conduction is essentially limited by transfer of phonon momentum to the container walls by
viscous forces, with the \(^3\)He playing a negligible role. Furthermore, collisions of \(^3\)He with the phonons drive the \(^3\)He away from equilibrium.

In this paper, starting from microscopic scattering processes we calculate transport coefficients of dilute solutions in the regime where phonons dominate the \(^4\)He excitations and the \(^3\)He concentrations are low enough for classical statistics to be valid. This regime encompasses the range at relatively high \(^3\)He concentrations where measurements have been carried out \([5, 8]\), to the range of natural concentration, \(x_3 \sim 10^{-6}\) where future measurements are being prepared \([9]\), to the low concentrations anticipated in the Oak Ridge neutron electric dipole moment experiment \([3, 4]\). Our primary focus here is on the processes of thermal conduction and diffusion.

In the high-\(x_3\) regime, \(^3\)He–\(^3\)He collisions maintain the \(^3\)He quasiparticles in an equilibrium distribution. For this case, as we showed in Ref. \([7]\), in calculating the leading low-temperature contributions to the transport coefficients, one may neglect the energy transfer in \(^3\)He–phonon collisions. At low \(x_3\), however, the effects of \(^3\)He–\(^3\)He collisions are negligible and, as we show, it is necessary to take into account the energy transfer even in calculating the leading low-temperature properties. The reason for this is that while the fractional changes in the momentum and energy of a \(^3\)He quasiparticle in scattering by a phonon are both small, they are of the same order, \(\sim k/p \sim (T/m^*s^2)^{1/2}\), where \(k\) is the momentum transfer, \(p\) the momentum of a quasiparticle, \(m^*\) the \(^3\)He effective mass, and \(s\) the sound velocity in \(^4\)He. To calculate the leading low-temperature behavior we expand the collision term in the \(^3\)He Boltzmann equation to leading non-trivial (second) order in \(k/p\) and derive a Fokker–Planck equation for the distribution function. Quite remarkably, we find that the eigenfunctions of the collision integral may be found analytically and have the form of Sonine polynomials, which are familiar in the theory of transport coefficients in classical, single-component gases \([10, 11]\).

We begin in Sect. 2 by reviewing the conservation laws obeyed by the phonons and \(^3\)He, and define the particle and heat currents in detail. This section follows in large part the general approach to the dissipative hydrodynamics of mixtures in Ref. \([12]\), but with crucial differences. Then in Sect. 3, we review the microscopic \(^3\)He–\(^3\)He interactions, \(^3\)He–phonon interactions, and phonon–phonon interactions underlying the transport. Here we do not consider effects, such as phonon scattering from the walls, whose role depends on the specific geometry in question. The effects of walls, which are critical at ultralow \(x_3\) as will be encountered in the nEDM experiment, will be discussed in Ref. \([13]\). The scattering of phonons on \(^3\)He is characteristically that of a massless particle, as in Rayleigh scattering. We then develop the \(^3\)He and phonon Boltzmann equations in Sects. 4 and 5, and discuss recoil corrections in Sect. 6. The resulting transport coefficients are presented in Sect. 7. In general we work in the linear approximation, assuming that the deviations from equilibrium are small.

---

1 The expansion in the parameter, \(k/\sqrt{m^*T}\) is essentially one in small angle scattering. The \(^3\)He–phonon collision term is thus similar to a Fokker–Planck collision term, to which Sonine polynomials have previous been applied.
2 Hydrodynamics and Conservation Laws

In dilute solutions of $^3$He in low temperature $^4$He in bulk, mean free paths are generally sufficiently small that the system can be described in terms of dissipative hydrodynamics. In addition, at low temperatures the dominant bosonic excitations of solutions are phonons. We begin by laying out the general framework of the hydrodynamics, explicitly in terms of the $^3$He and $^4$He number densities, the superfluid velocity, and the normal fluid $^3$He and phonon velocities. In a later section we calculate the needed transport coefficients. The description we give here parallels in many ways the description of a normal two-component fluid in Ref. [12], as extended to dilute solutions by Khalatnikov and Zharkov [14,15], with the identification of the phonons and the $^3$He as the two fluids. An important difference from two component normal fluids is that here the excitations of the $^4$He are not conserved, and furthermore a phonon drift with respect to the $^3$He is a heat flow, whereas in a two-component normal fluid, heat flow is energy transport in the absence of relative motion of the two components.

2.1 Conservation of Particle Number and Momentum

The $^3$He excitation spectrum is effectively

$$
\epsilon_p = \epsilon_0 + \frac{p^2}{2m^*},
$$

where the $^3$He effective mass is $m^* = m_3 + \delta m \simeq 2.34m_3$ [16]; the $\delta m$ arises from backflow in the $^4$He as a $^3$He moves through it. The mean field shift $\epsilon_0$ depends on both the $^3$He and $^4$He densities; however at low concentrations the former dependence is negligible, and $\partial \epsilon_0/\partial n_4 = (m_4s^2/n_4)(1 + \alpha)$ where $\alpha \simeq 0.28$ is the fractional excess molar volume of the $^3$He. At the temperatures and concentrations of interest, the $^3$He atoms obey Boltzmann statistics, with the equilibrium density given by

$$
n_3 = \nu \int \frac{d^3p}{(2\pi)^3} e^{-\beta(p^2/2m^*+\epsilon_0-\mu_3)} = \nu e^{\beta(\mu_3-\epsilon_0)/\lambda_{th}^3},
$$

where $\mu_3$ is the $^3$He chemical potential, $\nu$ is the number of spin degrees of freedom: $\nu = 1$ in a fully spin-polarized gas and 2 in an unpolarized gas, $\beta = 1/T$, and $\lambda_{th} = \sqrt{2\pi/m^*T}$ is the $^3$He thermal wavelength. We generally work in units with $\hbar = k_B = 1$. At the low concentrations of interest, effects of $^3$He–$^3$He interactions on the thermodynamics are negligible.

The conservation law for $^3$He atoms is

$$
\frac{\partial n_3}{\partial t} + \nabla \cdot (n_3v_3) = 0,
$$
while the conservation law for $^4$He is

$$m_4 \frac{\partial n_4}{\partial t} + \nabla \cdot g_4 = 0,$$

(4)

where $v_3$ is the $^3$He flow velocity, and $g_4 = \rho_s v_s + \delta m n_3 v_3 + \rho_{ph} v_{ph}$ is the $^4$He mass current, with $v_s$ the superfluid velocity, $v_{ph}$ the phonon (normal) fluid velocity, and $m_4$ the $^4$He atomic mass. The $^4$He mass density is $\rho_4 = m_4 n_4$. The superfluid mass density is $\rho_s = \rho_3 + \rho_4 - m^* n_3 = \rho_4 - \rho_{ph} - \delta m n_3$, since the $^3$He effective mass correction does not participate in superfluid flow. The phonon fluid velocity is defined by writing the momentum density carried by phonons as $\rho_{ph} v_{ph}$, with the normal mass density of the phonons given by

$$\rho_{ph} = \beta \int \frac{d^3q}{(2\pi)^3} \frac{q^2}{3} n_q^0 (1 + n_q^0) = \frac{2\pi^2}{45} \frac{T^4}{s^5},$$

(5)

where $s$ is the first sound velocity, $q$ the phonon momentum, and $n_q^0 = (e^{\beta q} - 1)^{-1}$.

The mass current, or total momentum density, $g$, in the solution is given by

$$g = \rho_s v_s + \rho_{ph} v_{ph} + m^* n_3 v_3 = g_4 + g_3^0,$$

(6)

where $g_3^0 = m_3 n_3 v_3$ is the momentum density carried by $^3$He atoms, not the fully dressed quasiparticles. In the absence of $^4$He and $^3$He mass flow, $v_s = -(\rho_{ph}/\rho_s)v_{ph}$; at the temperatures and concentrations of interest, $v_s \ll v_{ph}$.

The linearized superfluid acceleration equation is

$$m_4 \frac{\partial v_s}{\partial t} + \nabla \mu_4 = 0,$$

(7)

where $\mu_4$ is the $^4$He chemical potential; note that in the absence of superfluid acceleration, $\mu_4$ is independent of position. In addition the hydrodynamic equation for conservation of total momentum in the dilute solutions, is, in linear order,

$$\frac{\partial g}{\partial t} + \nabla P = H,$$

(8)

where $P$ is the total pressure and $H$ is the gradient of the viscous stress tensor. In the phonon regime, the phonon first viscosity is dominant; indeed the phonon contributions to the dissipative second viscosity terms in Eqs. (7) and (8) vanish [17]. Similarly, the viscosity of the $^3$He gas can be neglected, as can small second viscosity effects associated with heat transfer between the $^3$He and phonons. Then

$$H \equiv \eta_{ph} \left( \nabla^2 v_{ph} + \frac{1}{3} \nabla (\nabla \cdot v_{ph}) \right),$$

(9)

---

2 Here, and in experiments determining heat transport in superfluid helium, the $v_{ph} - v_s$ achieved depends on the ambient temperature gradient.
where $\eta_{ph}$ is the first viscosity of the phonon fluid.

The $^3$He momentum density obeys [18]

$$\frac{\partial}{\partial t} (g_3 - \delta m n_3 v_s) + n_3 \nabla \mu_3 + S_3 \nabla T = \frac{\partial g_3}{\partial t} \bigg|_{coll},$$

(10)

where $g_3$ is the momentum density of the dressed $^3$He quasiparticles,

$$S_3 = \left( \frac{5}{2} - \beta (\mu_3 - \epsilon_0) \right) n_3 = \left[ \frac{5}{2} - \ln \left( \frac{n_3 \lambda_{th}^3}{\nu} \right) \right] n_3.$$  

(11)

is the $^3$He entropy density, and $(\partial g_3/\partial t) \big|_{coll}$ is the rate of change of the $^3$He momentum density resulting from collisions with phonons. This equation can also be written in terms of the $^3$He partial pressure, $P_3 = n_3 T$, if one uses the relation $dP_3 = S_3 dT + n_3 d(\mu_3 - \epsilon_0)$.

Similarly the phonon momentum density obeys

$$\frac{\partial}{\partial t} (\rho_{ph}(v_{ph} - v_s)) + S_{ph} \nabla T = -\frac{\partial g_3}{\partial t} \bigg|_{coll} + H,$$

(12)

where

$$S_{ph} = s^2 \rho_{ph}/T = \frac{2\pi^2}{45} \left( \frac{T}{s} \right)^3$$

(13)

is the $^4$He entropy density; this equation can similarly be written in terms of the phonon partial pressure, $P_{ph} = T S_{ph}/4$ if one uses the relation $dP_{ph} = S_{ph} dT - \frac{3}{4} s \rho_{ph} ds$.

With the Gibbs–Duhem relation,

$$\nabla P = n_4 \nabla \mu_4 + n_3 \nabla \mu_3 + S \nabla T,$$

(14)

conservation of momentum implies

$$\rho_{ph} \frac{\partial v_{ph}}{\partial t} + m^* n_3 \frac{\partial v_3}{\partial t} + n_3 \nabla \mu_3 + \frac{\rho_{ph} + \delta m n_3}{m_4} \nabla \mu_4 + S \nabla T = H;$$

(15)

here $S$ is the total entropy density, equal at low temperatures to $S_{ph} + S_3$. In a steady state,

$$n_3 \nabla \mu_3 + S \nabla T = H;$$

(16)

when the phonon viscosity contribution is negligible, a condition realized at $^3$He relative concentrations $\gtrsim 10^{-4}$, a temperature gradient is inevitably accompanied by a $^3$He chemical potential gradient and vice versa.
2.2 Energy Conservation

Using Eqs. (3)–(15), together with the relation for the internal energy \( dE_{\text{int}} = TdS + \mu_4dn_4 + \mu_3dn_3 \), we readily find the equation for conservation of energy,

\[
\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{j}_E = 0, 
\]

(17)

where the energy current is

\[
\mathbf{j}_E = g_4\mu_4/m_4 + n_3v_3\mu_3 + T\mathbf{v}_3S_3 + T\mathbf{v}_{ph}S_{ph} + Q_3. 
\]

(18)

(For the moment we omit the usual first viscosity terms in \( \mathbf{j}_E \).) For a purely linear phonon dispersion relation the total energy transported by the phonons is simply the drift term \( TS_{ph}\mathbf{v}_{ph} \), and \( Q_3 \) is the \(^3\text{He} \) heat current. Similarly the equation for entropy flow is

\[
\frac{\partial S}{\partial t} + \nabla \cdot \mathbf{j}_S = -\frac{1}{T}\mathbf{u} \cdot \frac{\partial g_3}{\partial t} \bigg|_{\text{coll}} - Q \cdot \frac{\nabla T}{T^2} \equiv \mathcal{R},
\]

(19)

where

\[
\mathbf{u} \equiv \mathbf{v}_3 - \mathbf{v}_{ph}; \]

(20)

here the entropy current is \( \mathbf{j}_S = \mathbf{v}_{ph}S_{ph} + \mathbf{v}_3S_3 + Q_3/T \). The quantity \( \mathcal{R} \) is the rate of entropy generation, which must be non-negative.

In general the change of \(^3\text{He} \) momentum density is driven by gradients in the chemical potential difference \( \mu \) and the temperature,\(^3\) as we see by subtracting \( y \equiv m^*n_3/(\rho_{ph} + m^*n_3) \) times Eq. (12) from \((1 - y)\) times Eq. (10):

\[
(1 - y)n_3 \left[ m^*\frac{\partial \mathbf{u}}{\partial t} + m_3\nabla \mu \right] - \sigma \nabla T = \frac{\partial g_3}{\partial t} \bigg|_{\text{coll}} - y\mathbf{H}. 
\]

(21)

Here

\[
\mu = \frac{\mu_3}{m_3} - \frac{\mu_4}{m_4},
\]

(22)

and

\[
\sigma = yS_{ph} - (1 - y)S_3. 
\]

(23)

Relative motion of the \(^3\text{He} \) and phonons can in general generate a \(^3\text{He} \) heat current, a type of “thermoelectric” effect; thus as a function of \( \nabla T \) and \( \mathbf{u} \), the \(^3\text{He} \) heat current to linear order has the form:

\footnote{Compare with Eq. (59.6) of Ref. [12], with the identifications \( i = m_3n_3(1 - y)\mathbf{u} \) and \( q - \mu i = Q_3 - T\sigma \mathbf{u} \).}
\( Q_3 = -K_3 \nabla T + T \chi \mathbf{u}. \)  \hfill (24)

In addition, the rate of momentum transfer in \(^3\text{He}\)–phonon collisions has the form:

\[
\frac{\partial g_3}{\partial t}_{\text{coll}} = -\frac{m^* n_3}{\tau} \mathbf{u} - \chi \nabla T,
\]  \hfill (25)

where the lifetime \( \tau \) determines the diffusion constant,

\[
D = T \tau / m^*
\]  \hfill (26)

for \(^3\text{He}\) against phonons. That the same off-diagonal thermoelectric coefficient \( \chi \) appears in both \( Q \) and \( (\partial g_3 / \partial T)_{\text{coll}} \) follows from general Onsager reciprocity relations for transport coefficients [19]. As we shall see, \( \chi \) vanishes for dilute solutions at very low temperature; the first non-vanishing result for \( \chi \) appears when we take into account phonon recoil corrections (Sect. 6).

With Eqs. (24) and (25), together with (21) in the static limit, we find the rate of entropy generation,

\[
\mathcal{R} = \frac{n_3}{D} \mathbf{u}^2 + K_3 \left( \frac{\nabla T}{T} \right)^2,
\]  \hfill (27)

plus the usual first viscosity term. The kinetic coefficients \( D, K_3, \chi \) and \( \eta_{ph} \), specify the transport properties of the solutions; the task we pursue in the following sections is to calculate these coefficients in terms of microscopic scattering processes.

Since in a steady state, \( n_3 \nabla \mu_3 + S_3 \nabla T = (\partial g_3 / \partial T)_{\text{coll}} \), we find from Eq. (25),

\[
\mathbf{u} = -\frac{D}{T} \left( \nabla \mu_3 + \frac{S_3 + \chi}{n_3} \nabla T \right).
\]  \hfill (28)

It will be more useful in later calculations of the transport properties to work in terms of the \(^3\text{He}\) density, \( n_3 \), rather than \( \mu_3 \). Using \( n_3 d \mu_3 = d(n_3 T) - S_3 dT + n_3 d\epsilon_0 \), we equivalently have

\[
\mathbf{u} = -D \left( \frac{\nabla n_3}{n_3} + \frac{\nabla \epsilon_0}{T} \right) - D_T \frac{\nabla T}{n_3},
\]  \hfill (29)

where \( D_T \), an effective thermoelectric diffusion constant, is given by

\[
D_T = \frac{D}{T} (n_3 + \chi).
\]  \hfill (30)

In the situation in which there is no net \(^3\text{He}\) or \(^4\text{He}\) particle flow, the total energy current, from Eq. (18), is

\[
Q = Q_3 - TS_{ph} \mathbf{u} = -K_3 \nabla T - T(S_{ph} - \chi) \mathbf{u};
\]  \hfill (31)
the $T S_{ph} u$ term is the heat carried by phonons with respect to the $^3$He. We define the total thermal conductivity, $K$, of the solutions by calculating, in the absence of $^3$He and $^4$He particle transport, the total energy flow, $Q = Q_3 - T S_{ph} u \equiv -K \nabla T$. In the absence of phonon viscous effects (which we include below), the Gibbs–Duhem relation implies in this situation that $n_3 \nabla \mu_3 = -S \nabla T$, so that in the static limit Eq. (28) yields

$$u = \frac{D}{n_3 T} (S_{ph} - \chi) \nabla T, \quad (32)$$

and thus

$$K = K_3 + \frac{D}{n_3} (S_{ph} - \chi)^2. \quad (33)$$

At constant pressure, the total entropy generation rate, $R$, is then $K (\nabla T / T)^2$.

2.3 Phonon Viscosity

We now extend the previous discussion of the thermal conductivity to include the phonon viscosity. We consider a steady state in which $v_3 = 0$, and assume that $v_{ph}$ is in the $z$-direction, but varying sinusoidally in the $x$-direction, i.e., $v_{ph} \sim e^{i k x} \hat{z}$. Equation (12) for conservation of momentum thus reads

$$S_{ph} \nabla T = -\frac{\partial g_3}{\partial t} \bigg|_{coll} - \eta_{ph} k_x^2 v_{ph}. \quad (34)$$

Using Eq. (25), we find

$$v_{ph} = -\frac{S_{ph} - \chi}{(T n_3 / D) + \eta_{ph} k_x^2} \nabla T. \quad (35)$$

Including the term $-T \chi v_{ph}$ in the total heat current $Q$, we then derive the total thermal conductivity of the solution,

$$K = K_3 + T \frac{(S_{ph} - \chi)^2}{(T n_3 / D) + \eta_{ph} k_x^2} \equiv K_3 + K_{ph}. \quad (36)$$

This equation defines the phonon thermal conductivity, $K_{ph}$.

Equation (36), which encompasses the range of $x_3$ from that in the Lamoreux experiment to that in the nEDM experiment, yields a number of physically interesting limits. Since $\eta_{ph} = \frac{1}{2} s \rho_{ph} \ell$, where $\ell$ is the phonon mean free path for viscosity (see Eq. (105)), the denominator in Eq. (36) shows the competition between the mean free path for phonon scattering against the $^3$He and the viscous diffusion length $\sim 1/k_x^2 \ell$. First, for non-zero $n_3$, as $k_x \to 0$, corresponding to a container infinite in all directions, we derive Eq. (33). On the other hand, in the absence of $^3$He,
\[ K \rightarrow K_{ph} = \frac{TS_{ph}^2}{\eta_{ph}k_x^2} = \frac{5sS_{ph}}{k_x^2 \ell}. \]  

This result can be understood in terms of a phonon undergoing a random walk of \( \sim (1/k_x \ell)^2 \) steps in going a wavelength \( \sim 1/k_x \).

For Poiseuille flow along the z-direction between parallel plates separated at \( x = \pm L_x/2 \), one has \( v_{ph} \sim L_x^2/4x^2 \), so that in terms of \( \tilde{v}_{ph} \), the average flow velocity, \( \nabla^2 \tilde{v}_{ph} = -12\tilde{v}_{ph}/L_x^2 \). Thus in linear order \( K_{ph} = 5sS_{ph}L_x^2/12\ell \), which agrees with Eq. (37) with the identification \( k_x^2 = 12/L_x^2 \). For Poiseuille flow in a cylinder of radius \( R \) one has rather \( \nabla^2 v_{ph} = -8\tilde{v}_{ph}/R^2 \) on average, which leads to the Benin-Maris result [20] \( K_{ph} = 5sS_{ph}R^2/8\ell \).

To see the regime in which phonon viscosity is important we note that the ratio of the viscous to diffusive terms in the denominator of Eq. (35) is

\[ \frac{\eta_{ph}k_x^2 D}{Tn_3} = \frac{\ell k_x^2 D S_{ph}}{5s n_3}. \]  

As extracted from viscosity measurements [21,22], the characteristic phonon mean free path for viscosity in pure \(^4\text{He}\) is

\[ \ell(T) \simeq 3.2 \times 10^{-3}/T^K \text{ cm} \]  

over a broad range of temperatures around \( T = 0.5 \text{ K} \); here \( T^K \) is the temperature measured in Kelvin. With Eq. (84) for \( D \) we find

\[ \frac{\eta_{ph}k_x^2 D}{Tn_3} \sim 0.5 \left( \frac{10^{-6}}{x_3} \right) \left( \frac{0.45K}{T} \right)^9 \left( \frac{1 \text{ cm}}{R} \right)^2, \]  

indicating that viscosity becomes important for \( x_3 \lesssim 10^{-6} \) at temperatures of order 0.5 K. At these low concentrations, effects of the \(^3\text{He}\) on the phonon viscosity are negligible.

As discussed in Sect. 7, \( K_3 \) reaches a maximum fraction of only about 1% of the overall thermal conductivity at the highest \(^3\text{He}\) concentrations considered here, \( x_3 = 10^{-3} \). Although the phonon thermal conductivity \( K_{ph} \) falls with increasing \( x_3 \) as \( 1/x_3 \) in this regime, \( K_3 \) is limited by \(^3\text{He}–^3\text{He}\) scattering and is never dominant.

2.4 Currents and Distribution Functions

At low temperatures, the \(^4\text{He}\) energy current is given in terms of the phonon distribution function, \( n_q \), by

\[ \mathbf{j}_{E,4} = \left( n_4 - \frac{\delta m}{m_4}n_3 \right) \mu_4 \mathbf{v}_s + \int \frac{d^3q}{(2\pi)^3}sq(s\hat{q})n_q, \]  

where the second term is the phonon energy current.

\[ \odot \text{ Springer} \]
\[ \mathbf{j}_{E,\text{ph}} = s^2 \int \frac{d^3 q}{(2\pi)^3} \mathbf{q} n_q = TS_{ph} \mathbf{v}_{ph}. \]

The integral is simply the momentum density carried by the phonons. The corresponding $^3$He energy current is similarly given in terms of the $^3$He distribution function, $f_p$, by

\[ j_{E,3} = \frac{\delta m n_3}{m_4} \mu_4 v_3 + \nu \int \frac{d^3 p}{(2\pi)^3} \frac{p}{m^*} \left( \frac{p^2}{2m^*} + \epsilon_0 \right) f_p, \]

with the $^3$He flow velocity, $v_3$, defined by

\[ \nu \int \frac{d^3 p}{(2\pi)^3} \frac{p}{m^*} f_p = n_3 v_3. \]

The $^3$He heat current is

\[ Q_3 = \nu \int \frac{d^3 p}{(2\pi)^3} \frac{p}{m^*} \left( \frac{p^2}{2m^*} - \frac{5}{2} T \right) f_p, \]

since for classical statistics, $(\mu_3 - \epsilon_0)n_3 + TS_3 = (5/2)n_3 T$.

### 3 Microscopic Scattering

The transport properties of the dilute solutions are determined microscopically by four scattering processes: $^3$He–phonon scattering, similar to the scattering of non-relativistic electrons and photons; phonon–phonon scattering, which tends to bring the phonons into equilibrium (although incompletely in the present situation); $^3$He–$^3$He scattering, and lastly; scattering of phonons from the walls (which we consider in Ref. [13]).

#### 3.1 $^3$He–$^3$He Scattering

The matrix element for $^3$He–$^3$He scattering for atoms of opposite spin for small momentum transfers is [16]

\[ V_0 = -0.064 \frac{m_4 s^2}{n_4}. \]

Thus the $^3$He–$^3$He scattering length is

\[ a = \frac{m^* V_0}{4\pi \hbar^2}, \]

the differential cross section is $d\sigma/d\Omega = a^2$, and the total cross section is
\[
\sigma_{33} = 4\pi a^2 = \frac{m^*}{4\pi\hbar^4}|V_0|^2
= \frac{9\pi^3}{k_D^2} (0.064)^2 \left( \frac{m^*}{m_4} \right)^2 \left( \frac{m_4 s}{\hbar k_D} \right)^4 \approx 10.5\text{Å}^2.
\]

(48)

where \( n_4 = k^2_D / 6\pi^2 \) and \( k_D \approx 1.089 \text{ Å} \) is the Debye wavevector. A \(^3\)He atom of low momentum scatters only from atoms of opposite spin, so that the mean free path of a \(^3\)He in a gas of unpolarized \(^3\)He is given by

\[
\ell_{33} = \frac{2}{n_3\sigma_{tot}} = \frac{8\pi n_4}{(0.064)^2 x_3} \left( \frac{m_4}{m^*} \right)^2 \left( \frac{\hbar}{m_4 s} \right)^4 \approx 8.66 \times 10^{-8} \text{ cm}.
\]

(49)

### 3.2 \(^3\)He–Phonon Scattering

For \(^3\)He thermal velocities, \( \sim \sqrt{3T/m^*} \), small compared to \( s \), the scattering is sufficiently elastic that one can, to a good approximation in calculating the \(^3\)He–phonon scattering matrix element, neglect the energy transfer in a collision. As shown in [16] [Eq. (24) there], the effective matrix element for elastic scattering of a phonon from momentum \( q \) to \( q' \) and a \(^3\)He from momentum \( p \) to \( p' \) is

\[
\langle p'q'|T|pq \rangle \equiv \langle T \rangle = \frac{s\sqrt{qq'}}{2n_4\Omega} (A + B \cos \theta_{qq'}),
\]

(50)

where \( \theta_{qq'} \) is the angle through which the phonon is scattered, the angle between \( q \) and \( q' \); the coefficients are parameters of a solitary \(^3\)He in \(^4\)He, deduced from experiment to be \( A = n_4 d\alpha/dn_4 = -1.2 \pm 0.2 \) [23,24], \( B = (1 + \alpha + \delta m/m_4)(m_4/m^*) \) \( (1 + \alpha - m_3/m_4) = 0.70 \pm 0.035 \) [23–25], and \( \Omega \) is the volume of the system. The largest uncertainty is in \( A \), owing to a systematic difference between the measurements [23,24] of the pressure dependence of the density of dilute solutions.

The differential rate of scattering of a phonon of momentum \( q \) by angle \( \theta_{qq'} \) is thus

\[
n_3 \frac{d\gamma_q(\theta)}{d\cos \theta_{qq'}} = n_3 \int \frac{q'^2dq'}{2\pi^2} 2\pi \delta(sq - s'q') |\langle p'q'|T|pq \rangle|^2 = \frac{x_3 s q^4}{4\pi n_4} (A + B \zeta)^2,
\]

(51)

where \( \zeta = \cos \theta_{qq'} \); the transport scattering rate for phonons colliding on \(^3\)He atoms is

\[
\gamma_q = \int \frac{d^3q'}{(2\pi)^3} 2\pi \delta(sq - s'q') |\langle p'q'|T|pq \rangle|^2 (1 - \cos \theta_{qq'})
= \int_{-1}^{1} d\zeta \frac{d\gamma_q(\theta)}{d\zeta} (1 - \zeta) = s \frac{q^4 J}{4\pi n_4^2}.
\]

(52)
where\(^4\) \(J = A^2 + (B^2 - 2AB)/3 \simeq 2.2 \pm 0.6\). The momentum dependent mean free path of a phonon scattering against the \(^3\)He is

\[
\ell_{ph3}(q) = \frac{s}{n_3 \gamma_q} = \frac{5.52}{3\pi x_3 k_d} \left(\frac{k_d}{q}\right)^4.
\] (53)

Similarly, as we shall see (Eq. (81) with \(n = 0\)), the effective \(^3\)He–phonon relaxation rate in diffusion is \(\Gamma/3m^*T\), where

\[
\Gamma = \int \frac{d^3q}{(2\pi)^3} q^2 \gamma_q n_q^0 (1 + n_q^0) = \frac{9!}{2} \pi \zeta(8) J \left(\frac{T}{\hbar k_D D}\right)^9 k_D^3 s,
\] (54)

with \(\zeta(8) = \pi^4/9450 \simeq 1.004\) the Riemann zeta function. The mean free path for scattering of a thermal \(^3\)He by phonons\(^5\) is

\[
\ell_{3ph} = \frac{2}{\sqrt{3}\pi 8! \zeta(8) J k_D} \left(\frac{m s}{k_D}\right)^{1/2} \left(\frac{\hbar k_D D}{T}\right)^{15/2} = 0.077 \left(\frac{0.45K}{T}\right)^{15/2}\; \text{cm}.
\] (55)

Comparing the mean free paths, Eqs. (49) and (55) to estimate the importance of 3-3 versus phonon scattering in bringing the \(^3\)He into equilibrium. we find

\[
\ell_{3ph}/\ell_{33} = 0.89 \times 10^6 x_3 \left(\frac{0.45K}{T}\right)^{15/2}
\] (56)

For \(T = 0.45\; \text{K}\) and \(x_3 = 10^{-6}\), \(\ell_{3ph} \approx \ell_{33}\), while for \(T = 0.65\; \text{K}\) and \(x = 3 \times 10^{-4}\), \(\ell_{3ph}/\ell_{33} \approx 16.9\).

### 3.3 Phonon–phonon Scattering

Phonon–phonon scattering conserves total phonon momentum, and thus does not contribute to the thermal conductivity directly. However, its effect on the phonon distribution is important. Because of the anomalous dispersion of phonons in liquid\(^4\)He at low pressure, three-phonon Beliaev and Landau-damping processes are allowed;

\(^4\) This uncertainty is directly reflected in an uncertainty in the calculated diffusion constant of 25%.

\(^5\) In Ref. [7] we wrote the effective \(^3\)He–phonon scattering rate as \(\Gamma/p^2\); we emphasize that when the energy transfer in \(^3\)He–phonon collisions is taken into account, the scattering rate is independent of \(p\) and has the value \(\Gamma/3m^*T\), which is the value of \(\Gamma/p^2\) for \(p^2\) replaced by its thermal average \(3m^*T\).
these processes rapidly equilibrate phonons propagating in a given direction, producing a phonon distribution,

\[ \tilde{n}_q = \frac{1}{e^{\beta(q, \tau) sq} - 1}, \]  

along rays in momentum space [21], in which the temperature is dependent on the momentum direction \( \hat{q} \). From energy conservation in these rapid scatterings,

\[ \int sq(q^2 dq) (n_q - \tilde{n}_q) = 0. \]  

For phonons with small momentum \( q \sim T/s \), the dominant three-phonon process is Landau damping, in which the phonon is absorbed or emitted by a thermal phonon, thereby producing another phonon with energy \( \sim T \). The rate of this process is

\[ \frac{1}{\tau_L} = \frac{3\pi}{2} (u + 1)^2 sq \frac{\rho_n}{m_4 n_4}, \]  

where \( u = \partial \ln s / \partial \ln n_4 \simeq 2.843 \) at SVP is the phonon Grüneisen parameter. The Beliaev process, the decay of one phonon into two, has a rate \( (u + 1)^2 q^5 / 240\pi m_4 n_4 \); this process\(^6\) dominates at large \( q \) (\( \gg T/s \)). The ratio of the rates of phonon scattering on \(^3\)He to Landau damping is

\[ \frac{15\pi J}{(u + 1)^2} \left( \frac{sq}{T} \right)^3 \frac{m_4 s^2}{T} \simeq 0.5x \left( \frac{sq}{T} \right)^3 \frac{m_4 s^2}{T} \ll 1. \]  

We will describe such rapid scatterings in the phonon Boltzmann equation in terms of a relaxation time, \( \tau_r \), with a collision rate,

\[ \left( \frac{\partial n_q}{\partial t} \right)_{\text{rapid}} = -\frac{1}{\tau_r} (n_q - \tilde{n}_q). \]  

Relaxation between rays, which conserves phonon momentum and energy, occurs on a longer time scale. We describe such processes, which determine the phonon viscosity, in a relaxation time approximation

\[ \left( \frac{\partial n_q}{\partial t} \right)_{\text{long}} = -\frac{1}{\tau_l} (n_q - n_{le}). \]  

\(^6\) Three-phonon Landau damping and the Beliaev process, although they involve phonons alone, can affect the rate at which momentum is transferred from phonons to \(^3\)He. These processes conserve the total momentum and energy flux of the phonons. However, the cross section for scattering of phonons by \(^3\)He atoms is strongly dependent on the phonon momentum and therefore the total rate at which momentum is transferred from phonons to \(^3\)He depends on the details of the phonon distribution, not just the total momentum of the phonons.
where \( \tau_l = \ell(T)/s \). Here

\[
n_{ql}^e = \frac{1}{e^{\beta(r)(sq - qv_{ph})} - 1}
\]  

(63)

is the local equilibrium phonon distribution function, in terms of the local temperature \( T(r) = 1/\beta(r) \) and mean flow velocity \( v_{ph} \) that the phonons would have by relaxing to local equilibrium through phonon–phonon collisions.

When phonon viscosity is important, the angular dependent temperature of the phonon distribution has the form

\[
\delta \beta(\hat{q}) = -\frac{\beta}{s} (\hat{q}_z v_{ph} + \hat{q}_z \hat{q}_x \lambda),
\]

(64)

where \( \lambda \) measures the quadrupolar distortion of the phonon distribution, proportional to the gradient in the transverse direction of the phonon velocity \( v_{ph} \), which we take along the x direction to be specific. The relaxation time, \( \tau_\ell \) depends strongly on the angular dependence of \( n_q \), which is a second spherical harmonic (the final term in Eq. (64)) for phonon viscosity. For \( \ell \gg 1 \) one expects \( 1/\tau_\ell \sim \ell^2 \), until \( 1/\ell \sim \theta \), where \( \theta \) is a typical scattering angle.

4 \( ^3 \text{He} \) Boltzmann Equation

We turn now to calculating the transport coefficients, \( D, K_3, \) and \( \chi \) from the \( ^3 \text{He} \) Boltzmann equation. We assume quite generally that the driving terms, \( \nabla \mu_3 \) and \( \nabla T \) and thus \( v_{ph} \) are in the z-direction. The \( ^3 \text{He} \) Boltzmann equation has the form

\[
\frac{\partial f_p}{\partial t} + \frac{p}{m^*} \cdot \nabla_r f_p = -\frac{f_p - f_p^{le}}{\tau_{33}} + \sum_{p',q,q'} |\langle T \rangle|^2 2\pi \delta(\Delta E) \delta_{p+q,p'+q'}
\]

\[
\times [f_{p',n_{q'}}(r)(1 + n_q(r)) - f_p n_q(r)(1 + n_{q'}(r))],
\]

(65)

where \( \tau_{33} \) is the \( ^3 \text{He} \)--\( ^3 \text{He} \) scattering time (appropriate for vector drivers of the \( ^3 \text{He} \) away from equilibrium); we take a mean thermal velocity to define \( \tau_{33} = \sqrt{m^*/3T\ell_{33}} \) in terms of the \( ^3 \text{He} \)--\( ^3 \text{He} \) mean free path. Also

\[
f_p^{le} = e^{-\beta(r)(p^2 / 2m^* - p \cdot v_3 - \mu_3(r))}
\]

(66)

is the distribution function toward which \( ^3 \text{He} \)--\( ^3 \text{He} \) collisions drive the \( ^3 \text{He} \), and \( \langle T \rangle \equiv \langle p'q' |T| pq \rangle \). In calculating \( D, K_3, \) and \( \chi \), phonon viscosity can be neglected. Then, as argued above, rapid phonon equilibration along rays in momentum space brings the
phonon distribution into the form (63). We first linearize the $^3\text{He}$–phonon collision term in terms of deviations of the phonon and $^3\text{He}$ distribution functions from equilibrium:

$$\delta \tilde{n}_q = \tilde{n}_q - n_q^{le} = \beta n_q^0 (1 + n_q^0) s q \cdot v_{ph},$$

(67)

and

$$\delta f_p = f_p - f_p^{le} = \beta f_p^0 p_z w_p;$$

(68)



here the local equilibrium $^3\text{He}$ distribution function is

$$f_p^{le} = e^{-\beta(r)(p^2/2m^* - \bar{\mu}_3(r))},$$

(69)

and the global equilibrium distribution function is

$$f_p^0 = e^{-\beta(p^2/2m^* - \bar{\mu}_3)},$$

(70)

where $\bar{\mu}_3 = \mu_3 - \epsilon_0$. With the detailed balance condition, $f_p^0 n_q^0 (1 + n_q^0) = f_p^{le} n_q^0 (1 + n_q^0)$, the term in square brackets in Eq. (65) becomes in linear order,

$$- f_p^0 n_q^0 (1 + n_q^0) \left[ \beta (w_p p_z - w_p' p_z' + (q - q') \cdot v_{ph}) \right].$$

(71)

It is most convenient to use momentum conservation to eliminate $p'$, and write $p' = p + k$, where $k = q - q'$. Then energy conservation implies that

$$\Delta E \equiv s q' - s q + \frac{p \cdot k}{2m^*} + \frac{k^2}{2m^*} = 0.$$

(72)

The momentum transfer $k$ is characteristically of order $T/s$, which is small compared with the momentum of a $^3\text{He}$ atom, $\sim (m^* T)^{1/2}$. The approach we take will be to expand the collision integrals in powers of $k/(m^* T)^{1/2}$; the leading terms are of second order. One might have imagined that to leading order one could neglect the energy transfer in collisions. However, this is not the case because, in a collision, the relative changes in the momentum and energy of a $^3\text{He}$ atom are both of order $(T/m^* s^2)^{1/2}$. As we show, the differential equation we derive for the distribution function to order $k^2/m^* T$, in the form of a Fokker–Planck equation, has an exact analytic solution.

Our calculations show that the energy transfer in a collision of $^3\text{He}$ with a phonon, although relatively small, has a large qualitative effect on the $^3\text{He}$ distribution function. If the energy transfer is neglected, the relaxation time for a quasiparticle varies as $p^2$ [7], whereas the exact solution above shows that when the energy transfer is taken into account, the relaxation time is independent of $p$.

The next higher-order terms vary as $k^4$ and we shall refer to them as “recoil corrections”. These give rise to contributions of order $T/m^* s^2$ times the leading term.
Despite the fact that $T/m^*s^2$ is $\sim 10^{-2}$ at temperatures of order 0.5 K, these corrections are not negligible because of the large numerical coefficients, as we demonstrated in the Appendix of Ref. [7]. We return to the recoil corrections in Sect. 6.

Expanding $w_p p_z - w_p' p_z'$ to order $k^2$ we find

$$w_p p_z - w_p' p_z' = -w_p k_z - (p_z + k_z) \mathbf{k} \cdot \nabla_p w_p - \frac{1}{2} p_z (\mathbf{k} \cdot \nabla_p)^2 w_p \quad = -w_p k_z - (p_z + k_z) \mathbf{k} \cdot \hat{p} \, w'_p - \frac{1}{2} p_z \left[ (\mathbf{k} \cdot \hat{p})^2 (w''_p - w'/p) + k^2 w'/p \right],$$

(73)

where primes on the $w$'s denote derivatives with respect to $p$.

The $^3\text{He}$–phonon collision term in Eq. (65), with the expansion (73), becomes

$$\left( \frac{\partial f_p}{\partial t} \right)_{3-\text{ph coll}} = \frac{1}{2} \sum_{q, q'} \left| (T) \right|^2 2\pi \delta(\Delta E) \delta_{p+q,p'+q'} f_0 n_0^0 (1 + n_q^0) \left( w_p - v_{ph} k_z + (p_z + k_z) \mathbf{k} \cdot \hat{p} \, w'_p + \frac{1}{2} p_z \left[ (\mathbf{k} \cdot \hat{p})^2 (w''_p - w'/p) + k^2 w'/p \right] \right).$$

(74)

The terms in the final parentheses are both linear and quadratic in $\mathbf{k}$. We expand the energy delta function to first order in $\mathbf{p} \cdot \mathbf{k}/m^*$, and average over the direction of $\mathbf{k}$, keeping the direction of $\mathbf{p}$ fixed. The terms in the final parentheses of order $k^2$ give

$$\frac{k^2 p_z}{6m^*} \left( 4w''_p / p + w''_p \right) n_q^0 (1 + n_q^0) \delta(\bar{s}q' - s q).$$

(75)

We symmetrize the terms of order $\mathbf{k}$ under the transformation $q \leftrightarrow q'$; the argument of the delta function remains fixed, while $n_q^0 (1 + n_q^0) \rightarrow n_q^0 (1 + n_q^0)$. Thus the terms of order $\mathbf{k}$ lead to

$$-\frac{\beta k^2 p_z}{6m^*} \left( w_p - v_{ph} + p w'_p \right) n_q^0 (1 + n_q^0) \delta(\bar{s}q' - s q).$$

(76)

When we integrate by parts in $q'$ this term becomes

$$-\frac{\beta k^2 p_z}{6m^*} \left( w_p - v_{ph} + p w'_p \right) n_q^0 (1 + n_q^0) \delta(s q' - \bar{s}q).$$

(77)

Altogether then, in terms of $\Gamma$ defined in Eq. (54), we have

$$\left( \frac{\partial f_p}{\partial t} \right)_{3-\text{ph coll}} = \frac{\beta \Gamma}{3} p_z f_0 \left( w''_p + 4 w''_p / p - \frac{\beta}{m^*} \left( p w'_p + w_p - v_{ph} \right) \right).$$

(78)

Remarkably, the derivatives of the distribution function in the parentheses on the right are just those entering into the equation for the polynomials, $g_n(y)$ [with $g_0 \sim 1$, $g_1 \sim y^2 - 5$, etc.], which obey
\[ g''_n + (4/y - y)g'_n + 2ng_n = 0. \] (79)

These polynomials are related to the Sonine polynomials familiar from the theory of dilute gases by the relation \( g_n(y) \propto S_{3/2}^n(y^2/2) \) [10,11]. Writing \( w_n(p) = g_n(p \sqrt{\beta/m^*}) \) we see that

\[ w''_n + 4w'_n/p - (\beta/m) pw'_n = -2n(\beta/m^*)w_n. \] (80)

so that the eigenfunctions of the differential operator in Eq. (78) are just the polynomials \( g_n(y) \). The collision term for a deviation proportional to \( g_n(y) \) is thus

\[ \left( \frac{\partial f_p}{\partial t} \right)_{3-\text{ph} \text{ coll}} = -\beta^2 \Gamma \frac{p_z f_p^0}{3m^*} \left[ (2n + 1)w_n - v_{ph} \right] \]
\[ = -\beta \Gamma \frac{p_z f_p^0}{3m^*} \left[ (2n + 1)\delta f_n - \beta f_p^0 p_z v_{ph} \right]. \] (81)

We first consider steady-state diffusion driven by a \(^3\text{He}\) chemical potential gradient at constant temperature. The driving term on the left side of the Boltzmann equation is \( (p_z/m^*) \beta f_p^0 \partial \tilde{\mu}_3/\partial z \), and the solution is the first Sonine polynomial, a constant. In other words, the \(^3\text{He}\) distribution is just an equilibrium drifting at velocity \( v_3 \), for which the \(^3\text{He}–^3\text{He}\) collision term vanishes. Then

\[ \frac{p_z}{m^*} \beta f_p^0 \frac{\partial \tilde{\mu}_3}{\partial z} = -\beta^2 \Gamma \frac{p_z f_p^0}{3m^*} u, \] (82)

where \( u = v_3 - v_{ph} \), so that \( u = -(3T/\Gamma) \partial \tilde{\mu}_3/\partial z \); the diffusion constant is thus

\[ D = \frac{3T^2}{\Gamma}, \] (83)

and the collision time in Eq. (25) is \( \tau = 3m^* T/\Gamma \). Using Eq. (54), with \( \hbar s k_D = 19.9 \text{K} \) and \( \hbar k_D/m_4s = 0.729 \) we find numerically that

\[ D = 173 \left( \frac{0.45K}{T_K} \right)^7 \text{cm}^2/\text{s} = 0.65 \frac{T_K^7}{T_K^7} \text{cm}^2/\text{s}. \] (84)

We note that there is a \( \sim 35\% \) increase in this lowest order calculation of \( D \) due to the recoil corrections (see Sects. 6 and 7).

We next compute the thermal conductivity of the \(^3\text{He}\), defined by the \(^3\text{He}\) heat current produced by a temperature gradient at constant \(^3\text{He}\) pressure, cf. Eq. (45). At constant \( P_3 \), the driving term on the left side of the Boltzmann equation is proportional to

\[ \left( \frac{\partial f_p^{\text{le}}}{\partial z} \right)_{P_3} = -f_p^0 \left( \frac{p^2}{2m^*} - \frac{5}{2} T \right) \frac{\partial \beta}{\partial z}. \] (85)
Thus the deviation of the $^3$He distribution is a first ($n = 1$) Sonine polynomial $\sim p^2 - 5m^*T$; we write

$$w_p = C \left( \frac{p^2}{2m^*} - \frac{5}{2} T \right).$$

(86)

With Eq. (81) the Boltzmann equation yields

$$C = -\frac{1}{\Gamma'} \frac{\partial T}{\partial z},$$

(87)

where $\Gamma' = \Gamma + m^*/\beta \tau_{33}$. Thus the $^3$He heat current is

$$Q_3 = -v \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{2m^*} \left( \frac{p^2}{2m} - \frac{5}{2} T \right)^2 \frac{\beta}{\Gamma'} \frac{\partial T}{\partial z}.$$  

(88)

Evaluating the integral we find the $^3$He thermal conductivity,

$$K_3 = \frac{\beta}{\Gamma' v} \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{2m^*} \left( \frac{p^2}{2m} - \frac{5}{2} T \right)^2 = \frac{5T^2}{2\Gamma'} n_3,$$

(89)

Numerically, the $^3$He–$^3$He scattering contribution $\Gamma'$ is of relative order $(x_3/10^{-6}) (0.45K/T)$, indicating that $^3$He–$^3$He scattering does not contribute importantly at concentrations well below $10^{-6}$ at temperatures of order 0.5 K.

At the present level of approximation the thermoelectric coefficient $\chi$ is zero. To see this we assume a uniform temperature; then the solution of the Boltzmann equation is simply a drifting local equilibrium [cf. Eq. (82)] for which, from the orthogonality of the Sonine polynomials, the $^3$He heat current vanishes. Thus from Eq. (24), $\chi = 0$. Similarly $(\partial g_3/\partial t)_{coll}$ also vanishes if the distribution function is proportional to the first Sonine polynomial, and thus from Eq. (25), we see again that $\chi$ vanishes. However, with terms of higher order in $k$ included in the collision term, the distribution functions are not given simply in terms of Sonine polynomials, and one finds $\chi \neq 0$ (see Sect. 6.2). We expect, from expanding the $^3$He–phonon collision term to higher order in $k$ that the corrections are $\sim k^2/p^2 \sim T/m^*s^2$. Thus in order of magnitude $\chi \sim n_3 T/m^*s^2$, which leads to negligible corrections to $D_T$, Eq. (30), as well as $K_{ph}$ in Eq. (33).

### 5 Phonon Boltzmann Equation

We turn to the details of the phonon Boltzmann equation. With phonon–phonon scatterings treated in a relaxation time approximation, the phonon Boltzmann equation in dilute solutions has the structure:

$$\frac{\partial n_q}{\partial t} + s \hat{q} \cdot \nabla_r n_q = \left( \frac{\partial n}{\partial t} \right)_{\text{ph–He}} - \frac{1}{\tau_r} \left( n_q - \tilde{n}_q \right) - \frac{1}{\tau_l} \left( n_q - n_{qe} \right).$$

(90)
where the phonon–$^3\text{He}$ collision term is, cf. Eq. (65),

$$
\left(\frac{\partial n_q}{\partial t}\right)_{\text{ph}^{-3}\text{He}} = \sum_{p, p', q} |\langle T \rangle|^2 2\pi \delta(\Delta E) \delta_{p+q, p'+q'} \\
\times \left[ f_{p'} n_q(r)(1 + n_q(r)) - f_{p} n_q(r)(1 + n_q'(r)) \right].
$$

(91)

We do not include explicit scattering of phonons with the container walls in this paper.

In the limit that the rate of phonon collisions with the $^3\text{He}$ is much smaller than phonon–phonon collisions along rays, cf. Eq. (60), and the spatial and temporal variations are slow, the deviation of the distribution function $n_q$ from $\tilde{n}_q$ is proportional to $\tau_r$; as we see from Eq. (90),

$$
n_q - \tilde{n}_q = \tau_r \left[ \frac{\partial n_q}{\partial t} + s\hat{q} \cdot \nabla_r n_q + \frac{1}{\tau_l} (n_q - n_{l e}^0) - \left(\frac{\partial n_q}{\partial t}\right)_{\text{ph}^{-3}\text{He}} \right].
$$

(92)

Thus all $n$ on the right can be replaced by $\tilde{n}$. The condition (58) then allows us to derive the effective Boltzmann equation for $\beta(\hat{q}, r)$, since the integral $\int dq q^3$ of the right side of Eq. (92) vanishes. We write

$$
\delta \tilde{n}_q = \tilde{n}_q - n_{l e}^0 = -n_q^0 (1 + n_q^0) s q \delta \beta(\hat{q}, r),
$$

(93)

where $\delta \beta(\hat{q}, r) = \beta(\hat{q}, r) - \beta(r)$, with $\beta(r)$ the local equilibrium temperature, and using $\int q^3 dq \delta \tilde{n}_q = -6\pi^2 s \rho_{ph} \delta \beta$. we arrive at the equation for $\beta(\hat{q}, r)$:

$$
\frac{\partial \beta(\hat{q}, r)}{\partial t} + s\hat{q} \cdot \nabla_r \beta(\hat{q}, r) = C - \frac{1}{\tau_l} (\beta(\hat{q}, r) + \beta q z v_{ph}/s).
$$

(94)

Here

$$
C = - \int \frac{q^3 dq}{6\pi^2 s \rho_{ph}} \left(\frac{\partial \tilde{n}_q}{\partial t}\right)_{\text{ph}^{-3}\text{He}};
$$

(95)

the tilde on the right side indicates that the phonon distribution functions $n_q$ in the collision terms are replaced by $\tilde{n}_q$.

The analysis of the $^3\text{He}$–phonon collision term in the phonon Boltzmann equation is similar to that in the $^3\text{He}$ Boltzmann equation. We assume $\beta(\hat{q})$ to have the form (64), and treat the $v_{ph}$ and $\lambda$ terms separately. In the $v_{ph}$ term, using Eq. (73), we see that when one averages over the direction of $^3\text{He}$ momenta $p$ only the terms first order in $k$ survive; then

$$
\left(\frac{\partial n}{\partial t}\right)_{\text{ph}^{-3}\text{He}} = \sum_q k_z |\langle T \rangle|^2 2\pi \delta(sq - sq') n_q^0 (1 + n_q^0) F_3
\\
= q \gamma q n_q^0 (1 + n_q^0) F_3,
$$

(96)
where

\[ F_3 = \sum_p f_p^0 \left( w_p - v_{ph} + \frac{1}{3} p w'_p \right). \]  

(97)

If the $^3$He are driven by $^3$He chemical potential gradients, then $F_3 = n_3 u$, while if the $^3$He are driven by a temperature gradient, then $w_p = -(1/\Gamma)(\partial T/\partial z) \times (p^2/2m^* - 5T/2)$, and $F_3 = -n_3 v_{ph}$.

The $\lambda$ terms in Eq. (64) lead to a similar term in the collision integral,

\[ -\gamma^{(2)}_q \left( \delta \tilde{n}_q - n_q^0 (1 + n_q^0) \beta \tilde{q}_z v_{ph} \right), \]  

(98)

where

\[ \gamma^{(2)}_q = \int \frac{d\cos \theta}{2} (1 - P_2(\cos \theta)) \frac{d\gamma_q(\theta)}{d\cos \theta} \]  

(99)

is the average of the scattering rate of phonons on $^3$He appropriate for viscosity, with $P_2$ the second Legendre polynomial. Thus

\[ C = \frac{n_3 \beta^2 \tilde{q}_z}{3s \rho_{ph}} \left( -\Gamma u + \Gamma^{(2)} \tilde{q}_z \lambda \right), \]  

(100)

where $\Gamma$ is given by Eq. (54), and

\[ \Gamma^{(2)} = \int \frac{d^3q}{(2\pi)^3} q^2 \left( \frac{\gamma^{(2)}_q(q)}{n_3} \right) n_q^0 (1 + n_q^0) = \frac{A^2 + B^2/5}{A^2 + (B^2 - 2AB)/3} \Gamma; \]  

(101)

the numerical coefficient is $\simeq 0.74$.

The linearized phonon Boltzmann equation, Eq. (94), separates into equations for the individual spherical harmonic components of $\beta(\tilde{q})$. We first calculate the phonon viscosity from the second spherical harmonic term; to do so we assume stationary flow and compute the off-diagonal component $T_{xz} = -\eta_{ph} \partial v_{ph}/\partial x$. Then

\[ \lambda = -s \tau_v \nabla x v_{ph}, \]  

(102)

where

\[ \frac{1}{\tau_v} = \frac{n_3 \beta}{3 \rho_{ph}} \Gamma^{(2)} + \frac{1}{\tau_l}. \]  

(103)

With Eq. (102), we find

\[ T_{zx} = -\int \frac{d^3q}{(2\pi)^3} s q_z \tilde{q}_z \delta n_q \]

\[ = \int \frac{d^3q}{(2\pi)^3} \beta s^2 \tau_v \tilde{q}_z^2 \tilde{q}_x^2 g^2 n_q^0 (1 + n_q^0) \frac{\partial v_{ph}}{\partial z}, \]  

(104)
and thus

\[ \eta_{ph} = \frac{1}{5} \rho_{ph} s^2 \tau_v. \]  

(105)

The contribution to the viscosity from phonon scattering on the \(^3\)He is important for \((n_3/S_{ph})(s \ell / D) \simeq 3 \times 10^4 x_3 / T_K \gtrsim 1\).

In extracting the first spherical harmonic component of Eq. (94), we encounter a term \( \hat{q}_z \hat{q}_x \hat{q} \cdot \nabla_r \), which from Eq. (102) equals \(-s \tau_v \hat{q}_z \hat{q}_x^2 \nabla_x^2 v_{ph}\). The first spherical harmonic component of \( \hat{q}_z \hat{q}_x^2 \) is \(3 \int d\Omega(q) \hat{q}_z^2 \hat{q}_x^2 / 4\pi = 1/5\). Thus the phonon momentum density obeys

\[ \rho_{ph} \frac{\partial v_{ph}}{\partial t} + S_{ph} \frac{\partial T}{\partial z} = \rho_{ph} s^2 \tau_v \frac{5}{3} \nabla_x^2 v_{ph} = \frac{n_3 \beta}{3} \Gamma u. \]  

(106)

which we recognize as the phonon momentum conservation equation (12).

### 6 Recoil Corrections

As we showed above, to leading order in an expansion of the \(^3\)He–phonon collision operator in powers of \(k^2 / m^* T \sim T / m^* s^2\), the eigenfunctions of the \(^3\)He–phonon collision operator in the \(^3\)He Boltzmann equation have the form of Sonine polynomials times \(p_i\) for driving terms proportional to \(p_i \partial f^{le}_{p} / \partial x_i\). Terms of order \(k^4 / m^* T^2\) in the expansion of the collision operator lead to contributions to the transport coefficients of nominal order \(T / m^* s^2\) relative to the leading term. The task in this section is to calculate these corrections explicitly. We have seen that in a temperature or \(^3\)He chemical potential gradient, the phonon distribution function is one of drifting local equilibrium, and therefore, it is convenient to work in the reference frame in which the phonon fluid is at rest \((v_{ph} = 0)\). We also assume that the system is in a steady state and, on the left hand of the Boltzmann equation (65), we may replace the distribution function by a local equilibrium one. At the low concentrations of interest, the \(^3\)He–\(^3\)He scattering may be neglected.

For temperature and chemical potential gradients in the \(z\)-direction, the left side of the Boltzmann equation (65) becomes

\[ \frac{p}{m^*} \cdot \nabla_r f_p = - \frac{f_0}{T} \frac{p_z}{m^*} \left( \frac{\partial (\mu_3 - \epsilon_0)}{\partial z} + \frac{p^2 / 2m^* + \epsilon_0 - \mu_3}{T} \frac{\partial T}{\partial z} \right) \]  

and

\[ \frac{p_z}{m^*} (\frac{\partial P_3}{\partial z} - \frac{\partial \epsilon_0}{\partial z} + (p^2 / 2m^* - 5T / 2) \frac{\partial T}{\partial z}). \]  

(107)

The latter form, in which we use the \(^3\)He pressure and the temperature as independent variables, is particularly convenient because it shows that the driving term due to the pressure gradient is proportional to the particle current, \(p_z / m^*\), carried by an atom, and that the driver due to the temperature gradient is proportional to the heat current, \((p^2 / 2m^* - 5T / 2) p_z / m^*\). Since the Boltzmann equation is linear, we write
\[ w(p) = W^c(p) \left( \frac{1}{n_3} \partial P_3 / \partial z - \partial \epsilon_0 / \partial z \right) + W^h(p) \partial T / \partial z, \]
and have the equations for the \( W \)'s:

\[
p^z_{m^*} = \sum_{p', q, q'} |\langle T \rangle|^2 2\pi \delta(\Delta E) \delta_{p+q, p'+q'} n^0_q (1 + n^0_{q'}) [p_z W^c(p) - p_z' W^c(p')].
\]

(108)

and

\[
\left( \frac{p^2}{2m^*} - \frac{5}{2} T \right) \frac{p_z}{m^*} = \sum_{p', q, q'} |\langle T \rangle|^2 2\pi \delta(\Delta E) \delta_{p+q, p'+q'} n^0_q (1 + n^0_{q'}) 
\times [p_z W^h(p) - p_z' W^h(p')].
\]

(109)

The response of the \( ^3 \)He particle number current density and heat current density to the chemical potential and temperature gradients may be written in terms of the correlation functions \( \Xi_{\eta \lambda} \) between the currents as

\[
\mathbf{j}_3 = - \Xi_{cc} \left( \frac{1}{n_3} \nabla P_3 - \nabla \epsilon_0 \right) - \Xi_{ch} \frac{\nabla T}{T},
\]

(110)

and

\[
\mathbf{Q}_3 = - \Xi_{hc} \left( \frac{\nabla P_3}{n_3} - \nabla \epsilon_0 \right) - \Xi_{hh} \frac{\nabla T}{T},
\]

(111)

where \( c \) denotes the current of particle number and \( h \) the heat current. The Onsager reciprocal relations imply that \( \Xi_{ch} = \Xi_{hc} \), a fact that can be confirmed explicitly using the results for the \( \Xi_{\eta \lambda} \) given below.

On eliminating \( (\nabla P_3) / n_3 - \nabla \epsilon_0 \) from Eqs. (110) and (111), we find

\[
\mathbf{Q}_3 = \frac{\Xi_{hc}}{\Xi_{cc}} n_3 \mathbf{u} - \left( \Xi_{hh} - \frac{\Xi_{hc}^2}{\Xi_{cc}} \right) \frac{\nabla T}{T},
\]

(112)

a result valid in an arbitrary frame. Comparison of this result with Eq. (24) shows that

\[
\chi = \frac{n_3 \Xi_{hc}}{T \Xi_{cc}}
\]

(113)

and the \( ^3 \)He thermal conductivity is \( K_3 = \Xi_{hh} - \Xi_{hc}^2 / \Xi_{cc} \).

Microscopically, the number and heat current densities are given by

\[
(j_3)_z = v \int \frac{d^3 p}{(2\pi)^3} \frac{p_z f^0_p}{m^* T} p_z w(p),
\]

(114)
\[ Q_{3z} = v \int \frac{d^3p}{(2\pi)^3} \left( \frac{p^2}{2m*} - \frac{5}{2}T \right) \frac{p_z f_0^p}{m*} p_z w(p), \]  

(115)

and therefore

\[ \Xi_{\eta,\lambda} = v \int \frac{d^3p}{(2\pi)^3} \frac{f_0^p}{T} X^\eta(p) \Phi^\lambda(p), \]  

(116)

where \( X^c(p) = p_z/m*, X^h(p) = (p^2/2m* - 5T/2)p_z/m*, \) and \( \Phi^\lambda(p) = W^{\lambda} p_z. \)

We may write Eqs. (108) and (109) in a compact matrix notation as

\[ |X^\lambda\rangle = I |\Phi^\lambda\rangle, \]  

(117)

in terms of which

\[ \Xi_{\eta,\lambda} = \langle X^\eta | \Phi^\lambda \rangle, \]  

(118)

where the inner product is defined by

\[ \langle C|D \rangle = v \int \frac{d^3p}{(2\pi)^3} \frac{f_0^p}{T} C^\eta(p) D^\lambda(p). \]  

(119)

We now expand the collision integral in powers of \( k^2/m*T, \)

\[ I = I_0 + I_1 + \cdots, \]  

(120)

where \( I_0 \) is the leading term, which is \( \sim T/m*s^2, \) and \( I_1 \) is the term of order \( (k^2/m*T)^2 \sim (T/m*s^2)^2. \) Similarly, we write the deviation function in the form

\[ |\Phi^\lambda\rangle = |\Phi^\lambda_0\rangle + |\Phi^\lambda_1\rangle + \cdots. \]  

(121)

Equating terms of the same order in \( T/m*s^2 \) in Eq. (117), one finds

\[ |X^\lambda\rangle = I_0 |\Phi^\lambda_0\rangle, \]  

(122)

and

\[ 0 = I_0 |\Phi^\lambda_1\rangle + I_1 |\Phi^\lambda_0\rangle, \]  

(123)

from which one sees that

\[ |\Phi^\lambda_1\rangle = -(I_0)^{-1} I_1 |\Phi^\lambda_0\rangle. \]  

(124)
The transport coefficient giving the response of a variable specified by $|X^n\rangle$ may be expressed in the form

$$\Xi^{n,\lambda} = \langle X^n|\Phi^{\lambda}\rangle = \langle X^n|\Phi^\lambda_0\rangle + \cdots$$

$$= \langle X^n|\Phi^\lambda_0\rangle - \langle X^n|(I_0^{-1})I_1|\Phi^\lambda_0\rangle + \cdots$$

$$= \langle \Phi^n|I_0|\Phi^\lambda_0\rangle - \langle \Phi^n|I_1|\Phi^\lambda_0\rangle + \cdots,$$  \hfill (125)

from which one sees that the corrections to the transport coefficient are given by

$$\Xi^{n,\lambda}_1 = -\langle \Phi^n_0|I_1|\Phi^\lambda_0\rangle.$$  \hfill (126)

6.1 Diffusion

In the absence of a temperature gradient, the flux density of $^3$He atoms in the frame moving with the phonons is given by

$$n_3u = -\Xi_{cc}\nabla(\mu_3 - \epsilon_0) = -\Xi_T\nabla n_3,$$  \hfill (127)

and therefore $\Xi_{cc}$ is related to the diffusion coefficient by

$$\Xi_{cc} = Dn_3/T.$$  \hfill (128)

One then sees from Eq. (125) that the fractional change in the diffusion coefficient due to recoil is

$$\frac{D_1(T)}{D_0(T)} \sim -\frac{\langle \Phi^n_0|I_1|\Phi^\lambda_0\rangle}{\langle \Phi^n_0|I_0|\Phi^\lambda_0\rangle}.$$  \hfill (129)

This ratio is precisely that evaluated in the Appendix of Ref. [7], and therefore

$$\frac{D(T)}{D_0(T)} \sim 1 + \frac{T}{m^*s^2}\left(\frac{100\pi^2 - 198}{33}\frac{\tilde{J}}{J} - 1\right),$$  \hfill (130)

where $\tilde{J} = 4A^2/3 - 4AB/3 + 8B^2/15 \approx 3.30$ and the coefficient of $T/m^*s^2$ is 35.5.

The reason that the calculation of recoil corrections is independent of $x_3$ is that the solution to the $^3$He Boltzmann equation is $u(p) = \text{constant}$: for the case of low $x_3$ considered in this paper, this is the exact solution for $^3$He–phonon scattering when $^3$He–$^3$He scattering is negligible. This is not altered at higher $x_3$ because $\Phi(p) \propto p_z$ is still a solution of the Boltzmann equation when $^3$He–$^3$He collisions are included, since they conserve the total $^3$He momentum.
An alternative approach to calculating recoil corrections is to start from the standard variational expression for transport coefficients [10], which in the case of diffusion is

$$\frac{Dn_3}{T} \geq \frac{\langle X^c | \Phi \rangle^2}{\langle \Phi | I | \Phi \rangle},$$

(131)

for an arbitrary form of the function $\Phi$. In particular, if one chooses $\Phi = \Phi^c_0$ one has

$$\frac{Dn_3}{T} \geq \frac{\langle X^c | \Phi^c_0 \rangle^2}{\langle \Phi^c_0 | I | \Phi^c_0 \rangle},$$

(132)

which recovers the exact results for $D_0$ and $D_1$ but also has terms of higher order in $k^2$.

6.2 Thermal Diffusion

We turn now to $\Xi^{hc}$, which describes the heat flow induced by a relative motion of phonons and $^3$He atoms (and also the relative motion of phonons and $^3$He induced by a temperature gradient). To first order in $T/m^*s^2$ this quantity vanishes since $\langle \Phi^h_0 | I_0 | \Phi^c_0 \rangle$ vanishes. This is because $\Phi^c$($p$) $\propto p_z$ is an eigenstate of $I_0$ and therefore $I_0 | \Phi^c_0 \rangle \propto | \Phi^c_0 \rangle$. As shown in Sect. 4, the solution for the Boltzmann equation for response to a temperature gradient is proportional to $(p^2/2m^* - 5T/2)p_z$ to leading order in $T/m^*s^2$ and consequently $\langle \Phi^h_0 | I_0 | \Phi^c_0 \rangle \propto \langle \Phi^h | \Phi^c \rangle = 0$ because of the orthogonality relation for Sonine polynomials for $n = 0$ and $n = 1$, $\int_0^\infty dx x^4 \exp(-x^2/2)(x^2 - 5) = 0$. The leading contributions to $\Xi^{hc}$ are therefore of second order in $T/m^*s^2$, while those to $\Xi^{cc}$ are of first order. In addition, $\Xi^{hc}$ has an extra factor of $T$ compared with $\Xi^{cc}$ because of the extra factor $p^2/2m^* - 5T/2$, and therefore as noted in Sect. 4, the magnitude of $\chi$ is $\sim n_3T/m^*s^2$.

In Sects. 2.2 and 2.3 we saw that $\chi$ usually enters in combinations such as $n_3 + \chi$, $S_3 + \chi$, or $S_{ph} - \chi$. Therefore, under the conditions of the nEDM proposed experiment, we expect the nonzero value of $\chi$ to have little effect and we shall not evaluate the prefactor.

7 Results

We now summarize the main results of this paper in graphical form. Over the full range of $^3$He concentrations in the non-degenerate regime, from that of the Lamoreaux et al. experiment [5] ($x_3 \sim 10^{-3}$) to that of the proposed nEDM experiment [3, 4] ($x_3 \lesssim 10^{-10}$), the diffusion constant, $D$, Eq. (84), is independent of $x_3$, even taking the recoil corrections of Sect. 6 into account. As Fig. 1 shows, its basic $T^{-7}$ temperature dependence is slightly modified by the recoil corrections. As expected, the phonon and $^3$He thermal conductivities (Eqs. (36) and (89), respectively) depend on both $T$ and $x_3$. As shown in Fig. 2, at low $^3$He concentrations, the phonon thermal conductivity is dominated by phonon viscosity; at the highest concentrations considered here, $K_{ph}$ is
Fig. 1 Diffusion constant multiplied by $T^2$, as a function of temperature. The results include recoil corrections and were obtained from numerical integration using Eq. (132). We note that the expression (130), which includes only the contributions $D_0$ and $D_1$, gives results that lie within 3% of this result over the range shown (Color figure online).

Fig. 2 Phonon and $^3$He thermal conductivities from Eqs. (36) and (89), respectively, at $T = 0.25, 0.35$ and 0.45 K. Note that the scale of phonon thermal conductivities is $10^5$ larger than the scale of $^3$He thermal conductivities. The phonon thermal conductivity is calculated using the phonon mean free path, Eq. (39), and $k_T^2 = 8/R_x^2$, with $R = 15$ cm (approximately 10 times the phonon mean free path at $T = 0.25$ K). Both $K_{ph}$ and $K_3$ are calculated using the recoil correction obtained from numerical integration as described in the caption of Fig. 1 (Color figure online).

simply proportional to $1/x_3$ because of phonon scattering from $^3$He. On the other hand, the $^3$He thermal conductivity grows linearly with increasing $x_3$ at low concentrations before becoming dominated by $^3$He–$^3$He scattering at concentrations of about $x_3 \sim 10^{-6}$. At all concentrations considered here, the $^3$He contribution to the transport of heat is $\lesssim 1\%$ that of the phonons. We note that we have not taken into account the effects of geometry on the mean free paths of phonons and $^3$He, especially important at low temperatures and low concentrations, respectively, in this treatment but will do so in Ref. [13].
8 Summary

In this paper we have calculated the transport properties of $^3$He in superfluid $^4$He in the non-degenerate regime $x_3 \lesssim 10^{-3}$ for $T \lesssim 0.6$ K where phonons are the dominant $^4$He excitations. These calculations are relevant to previous transport measurements at relatively high concentrations [5,8], to the range of natural concentration, $x_3 \sim 10^{-6}$ [9], as well as to the low concentrations expected in the neutron EDM experiment [3,4]. We began by considering particle number, momentum and energy conservation, including the effects of phonon viscosity. The time evolution of the relative velocity $u = v_3 - v_{ph}$ is given by Eq. (21); the relative velocity is driven by both gradients of the chemical potentials and temperature, and includes the dissipative effects of $^3$He–phonon scattering and phonon viscosity. We show explicitly that the entropy generation rate is positive definite. The energy currents lead us to identify the thermal conductivities of the $^3$He ($K_3$, Eq. (24)) and of the phonons ($K_{ph}$, Eq. (36)). In addition we also identify the ordinary and “thermoelectric” diffusion coefficients $D$ and $\chi$, Eqs. (25), as well as the Onsager reciprocity relation connecting the $^3$He heat current and the thermoelectric diffusion coefficient.

Before calculating the transport coefficients in a Boltzmann equation framework, we discuss in detail in Sect. 3 the various scattering rates in the problem. Whereas the $^3$He–$^3$He scattering rate is much larger than the $^3$He–phonon rate at the high concentrations of the Lamoreaux et al. [5] and Rosenbaum et al. [8] measurements, at $x_3 \sim 10^{-6}$, they are roughly equal, and at lower concentrations, the scattering of $^3$He by phonons (or walls, depending on the geometry [13]) dominates. In treating phonon–phonon scattering we use the fact that phonons propagating in a given direction in momentum are in local equilibrium, owing to the very large small angle scattering rate. The phonon viscosity arises from scattering of phonons through large angles, either in a single large angle event or in a sequence of small angle scatterings. When effects of phonon viscosity are important, it is necessary to consider the effects of phonons scattering from the walls, the only appearance of geometry in the present paper.

In Sects. 4 and 5 of the paper, we calculate the transport coefficients starting from the Boltzmann equations for the evolution of the $^3$He and phonon distribution functions. The $^3$He–phonon scattering amplitude is well established by measurements from the 1960s [23]-[25]. For the high $^3$He concentrations of Refs. [5,8], where $^3$He–$^3$He scattering keeps the $^3$He in equilibrium, we can ignore the energy transfer relative to the momentum transfer, $k$, in the $^3$He–phonon scattering. However, in general, it is necessary to consider both. The leading effects appear at order $k^2$, for which the $^3$He Boltzmann equation is effectively a Fokker–Planck equation, but there are important contributions at order $k^4$, which we describe as recoil corrections (see Sect. 6). We find that the solutions of the Fokker–Planck equation for the $^3$He distribution function are exactly Sonine polynomials, familiar from calculations of classical transport coefficients of one component gases. As we calculate, the diffusion constant varies approximately as $T^{-7}$, is independent of $x_3$, and, including the recoil corrections, has a leading coefficient approximately 70% as large as that found in Ref. [5]. The thermal transport is dominated by that of the phonons for the full range of $x_3$; for $x_3 \gtrsim 10^{-6}$ $^3$He–phonon scattering reduces $K_{ph}$ below its low $x_3$ value where it is determined.
by phonon viscosity. The $^3$He thermal conductivity is limited by $^3$He–$^3$He scattering for $x_3 \gtrsim 10^{-6}$ but is, in any case, much smaller than $K_{ph}$. Finally, the thermoelectric coefficient, $\chi$, because of its “off-diagonal” nature, vanishes to lowest order in $k^2$ due to the orthogonality of the Sonine polynomials; the $k^4$ corrections are furthermore negligible.

Acknowledgments  This research was supported in part by NSF Grants PHY08-55569, PHY09-69790, PHY-1205671, and PHY13-05891. Author GB is grateful to the Aspen Center for Physics, supported in part by NSF Grant PHY-1066292, and the Niels Bohr International Academy, where parts of this research were carried out.

References

1. P. Hein, Grooks (Doubleday, New York, 1969)
2. G. Baym, C.J. Pethick, Landau Fermi Liquid Theory: Concepts and Applications (Wiley, New York, 1991).
3. R. Golub, S.K. Lamoreaux, Phys. Rep. 237, 1 (1994)
4. S.K. Lamoreaux, R. Golub, J. Phys. G. 36, 104002 (2009).
5. S.K. Lamoreaux, G. Archibald, P.D. Barnes, W.T. Buttler, D.J. Clark, M.D. Cooper, M. Espy, G.L. Greene, R. Golub, M.E. Hayden, C. Lei, J.J. Marek, J.-C. Peng, S. Penttila, Europhys. Lett. 58, 718 (2002)
6. R.M. Bowley, Europhys. Lett. 58, 725–729 (2002)
7. G. Baym, D.H. Beck, C.J. Pethick, Phys. Rev B 88, 014512 (2013)
8. R.L. Rosenbaum, J. Landau, Y. Eckstein, J. Low Temp. Phys. 16, 131 (1974)
9. D.H. Beck et al., nEDM collaboration, to be published.
10. E.M. Lifshitz, L.P. Pitaevskii, Physics Kinetics (Pergamon Press, Oxford, 1981), (Sec. 10)
11. K. Abe, Y. Ushimi, Phys. Fluids 19, 2047 (1976)
12. L.D. Landau, E. M. Lifshitz, Fluid Mechanics (Pergamon Press, Oxford, 1959), Ch. VI
13. G. Baym, D.H. Beck, C.J. Pethick, to be published.
14. I.M. Khalatnikov, V.N. Zharkov, J. Exp. Theoret. Phys. (USSR) 32, 1108 (1957) [Engl. transl. Soviet Physics JETP 5, 95 (1957)].
15. I.M. Khalatnikov, Introduction to the Theory of Superfluidity, Chs. 24, 25 (W.A. Benjamin, New York, 1965).
16. G. Baym, C. Ebner, Phys. Rev. 164, 235 (1967)
17. I.M. Khalatnikov, Introduction to the Theory of Superfluidity (W.A. Benjamin, New York, 1965), pp. 65, 133.
18. G. Baym, W.F. Saam, Phys. Rev. 171, 172 (1968)
19. L.D. Landau, E.M. Lifshitz, Fluid Mechanics (Pergamon Press, Oxford, 1959), Ch. VI Eq. (59.6).
20. D. Benin, H.J. Maris, Phys. Rev. B 18, 3112 (1978)
21. H.J. Maris, Rev. Mod. Phys. 49, 341 (1977)
22. D. Greywall, Phys. Rev. B 23, 2152 (1981), p. 2164 (esp. first column near bottom).
23. C. Boghosian, H. Meyer, Phys. Lett. A25, 352 (1967)
24. G.E. Watson, J.D. Reppy, R.C. Richardson, Phys. Rev. 188, 384 (1969)
25. B.M. Abraham, C.G. Brandt, Y. Eckstein, J. Munarin, G. Baym, Phys. Rev. 188, 309 (1969)