Masslessness in \( n \)-dimensions

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ABSTRACT. We determine the representations of the “conformal” group \( \overline{SO}_0(2, n) \), the restriction of which on the “Poincaré” subgroup \( \overline{SO}_0(1, n - 1) \cdot T_n \) are unitary irreducible. We study their restrictions to the “De Sitter” subgroups \( \overline{SO}_0(1, n) \) and \( \overline{SO}_0(2, n - 1) \) (they remain irreducible or decompose into a sum of two) and the contraction of the latter to “Poincaré”. Then we discuss the notion of masslessness in \( n \) dimensions and compare the situation for general \( n \) with the well-known case of 4-dimensional space-time, showing the specificity of the latter.

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Introduction

The formulation of a unifying theory which would include all fundamental interactions of physics is still an open problem, though the need for it was realized early in this century, when modern physics (relativistic and quantum) was introduced and in spite of unsuccessful efforts of many of its founders. A major difficulty consists in unifying the so-called gauge interactions and gravitation. A number of approaches to this question, appearing in various models such as the Kaluza-Klein theories or supergravity \cite{1,12}, use an imbedding of the four-dimensional Minkowski space-time into a higher dimensional one (that is, $\mathbb{R}^n$ endowed with a $(1, n-1)$-Lorentz metric), then getting rid of the redundant spatial dimensions by various techniques, such as spontaneous compactification.

On the other hand, an essential feature of relativity is the boundary character of the speed of light, which implies qualitatively distinct behaviours for massless and for massive particles. Mathematically, this is expressed by the $(1,3)$ signature of Minkowski space and the distinction between the massless and massive case is kinematically expressed by distinct types of unitary irreducible representations (UIR) \cite{14} of the kinematic group, the Poincaré group $\mathcal{P}$.

Masslessness in four dimensions has been quite well studied from the group theoretical point of view. We shall start by recalling the relations between $\mathcal{P}$ and the De Sitter groups. Let $M_\rho$ be a four-dimensional manifold with constant curvature $\rho$. Its isometry group is the De Sitter group $G_\rho$, which is isomorphic to $SO_0(2,3)$ (resp. $SO_0(1,4)$) if $\rho > 0$ (resp. $\rho < 0$); a physical reason for the introduction of the curvature is that it provides efficient invariant infrared regularization in the limit of zero curvature \cite{4}. $M_\rho$ is isomorphic to the homogeneous space $G_\rho/L$ where $L$ is the Lorentz group $SO_0(1,3)$.

In the limit $\rho = 0$, $M_\rho$ becomes the (flat) Minkowski space and $G_\rho$ contracts to the Poincaré group $\mathcal{P}$. Concerning representations, it may however happen that two nonequivalent UIR of $G_\rho$ contract to the same massless representation of $\mathcal{P}$. Moreover, if $\rho < 0$, the representations of $\mathcal{P}$ one gets by contraction have an unbounded energy spectrum.

Now the conformal group $G = SO_0(2,4)$ acts on compactified Minkowski space. Massless UIR of $\mathcal{P}$ with discrete helicity extend uniquely to UIR of $G$ acting on the same Hilbert space, and are the only ones with this property, besides the trivial \cite{1}. It turns out that if such a UIR is extended to $G$, then restricted to the De Sitter subgroup $SO_0(2,3)$, and finally contracted to $\mathcal{P}$, the initial representation of $\mathcal{P}$ is recovered. Therefore, from a kinematical point of view, the representations of the De Sitter group $SO_0(2,3)$ thus obtained provide a satisfactory tool for the extension of masslessness on $M_\rho$. Furthermore one can define a gauge theory in the sense of Gupta-Bleuler and show that massless particles propagate on the light cone, and so on \cite{2,13}.

Since the geometry of the $n$-dimensional Minkowski space-time is determined by its Lorentzian metric $(1, n-1)$, its kinematic group is $\mathcal{P}_n = SO_0(1, n-1).T_n$, and all groups related to it (Lorentz, conformal, De Sitter) are similarly defined. It is then natural
to ask which properties of massless UIR of the Poincaré group extend to \( n \) dimensions, independently of other considerations: this is the purpose of this paper.

More precisely, we shall study here the following topics:

i) Which UIR \( U \) of \( \mathcal{P}_n \) extend to irreducible representations \( d \) of the corresponding conformal group \( G_n = SO_0(2,n) \)? To be precise we are dealing with projective representations but it turns out that all the interesting ones are representations of the twofold covering for all groups concerned.

ii) Is the extended representation \( d \) unitary, and is it unique?

iii) Is the restriction \( d' \) of \( d \) on the De Sitter subgroups \( SO_0(1,n) \) and \( SO_0(2,n-1) \) irreducible?

iv) Can \( d' \) be contracted to the initial one, \( U \)?

The answers to these questions for \( n = 4 \) were given in [2] (except for those concerning \( SO_0(1,4) \)), some of the results being anterior to that paper: only (and all) UIR with zero mass and discrete helicity do extend to \( G_n \) with uniqueness and unitarity; the restriction to \( SO_0(2,3) \) is irreducible unless the inducing representation of the little group is trivial (zero helicity) and it can be contracted back to the initial UIR.

For structural reasons (all groups concerned have similar structure and real rank at most 2), a straightforward generalization was expected, at least for \( n \) even. It turned out, however, that though most features do indeed generalize, the constraints on the existence of the extension increase significantly with \( n \). To be more precise, \( U \) must again be massless, that is induced by a UIR \( S \) of the little group \( SO_0(n-2).T_{n-2} \) (the Euclidean group in \( n-2 \) dimensions). Not only \( S \) has to be trivial on the translations (the analogue of discrete helicity), but it must also be a very degenerate representation of \( SO_0(n-2) \).

Acceptable \( S \) are characterized by a discrete parameter \( 2s \in \mathbb{Z} \) when \( n \) is even (\( 2s \) is the helicity for \( n = 4 \)), while for \( n \) odd \( S \) must be either trivial or spinorial. Also, the results for the De Sitter subgroup generalize, with the sole exception of the irreducibility of \( d' \) for odd \( n \): it reduces into the direct sum of two simple factors for spinorial \( S \) too. As far as \( SO_0(1,n) \) is concerned, \( d' \) is always irreducible.

We therefore see once more, in this simple (kinematical) group theoretical study, that the 4-dimensional space-time of special relativity and the related universes with constant curvature are really special. In higher dimensions the notion of masslessness becomes more involved and requires, in addition to zero mass, a degeneracy far greater than the requirement of discrete helicity in 4 dimensions.

The paper is organized as follows. In Section 1 we fix notations for \( G_n, \mathcal{P}_n \) and the normalizer \( \mathcal{W}_n \) of \( \mathcal{P}_n \) in \( G_n \). Since we are interested in projective UIR we also present their universal coverings; in fact, as we shall see later, only twofold coverings are needed, corresponding to the covering of \( SO(n) \) by \( Spin(n) \). We also identify the compactified \( n \)-Minkowski space with the quotient \( G_n/\mathcal{W}_n \) and describe the action of \( G_n \) on it. We then establish the unitary dual of \( \mathcal{P}_n/\mathcal{W}_n \), using the orbit-stabilizer method, and discuss the possibility of extending a UIR \( U \) to \( \mathcal{W}_n \). Propositions 1.1 and 1.2 give the (expected)
result: $U$ must be massless, and induced by a representation $S$ with trivial restriction on the Euclidean translation subgroup.

Section 2 is devoted to determine which among the representations $d$ of $G_n$ can be viewed as extensions of massless UIR of $\mathcal{P}_n$, using Lie algebraic methods. We begin by expressing the weight representations of the complexified $\mathfrak{so}(N)^C$, in a way which can be used both for $\mathfrak{so}(2, n)$ and for the compact real form $\mathfrak{so}(n - 2)$ of the little group. We next translate into enveloping algebra properties the fact that the squared $n$-mass operator $P^\mu P_\mu$ is mapped to 0 by $d$, calling such a $d$ a massless representation. After the study of low $N$, we determine the finite-dimensional ones, parameterizing them by a discrete parameter (Thm 2.3).

We next study infinite-dimensional ones, showing that on every $\mathfrak{so}(n)$-type the character of the $\mathfrak{so}(2)$ which commutes with $\mathfrak{so}(n)$ is fixed and increases in absolute value with the Casimir of $\mathfrak{so}(n)$, keeping a fixed sign. Moreover, the lowest $\ell$-type must be a massless representation of $\mathfrak{so}(n)$ itself: this constraint has no effect when $n = 4$, but does cut off a huge part in general (Thm 2.4). Massless representations are unitary and possess an extremal weight.

In the following paragraph we identify the UIR $U^S$ of $\mathcal{P}_n$ which extend to massless UIR of the conformal group $\mathcal{G}_n$. The inducing $S$ must be a finite-dimensional massless UIR of $SO(n - 2)$. The expression of the generators of $G_n$ as differential operators is uniquely determined by those of $\mathcal{P}_n$.

Section 3 discusses De Sitter subgroups. Irreducibility of the restriction is examined on $\ell$-types, which all have multiplicity one. As for the contraction to Poincaré, the proof of [2] extends easily to the general case. The paper ends with a few remarks, where in particular we briefly recall and present in the light of the present study the known results for the lower dimensional cases $n = 3$ and $n = 2$.

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1 Poincaré and Conformal Group in $n$-dimensions

a) The $n$-Poincaré group $\mathcal{P}_n$

Let $n \geq 3$ be a fixed integer. Let $\{e_\mu\}_{\mu \in J}$, with $J = \{0, 1, \ldots, n-1\}$ be a basis of $\mathbb{R}^n$; let $J' = \{1, \ldots, n-1\}$ and define a quadratic form $q$, such that:

$$(1.1) \quad q(e_0) = 1 = -q(e_{\mu'}) \quad \forall \mu' \in J'. $$

The associated symmetric bilinear form is denoted by $g$, with $g_{\mu\nu} = g(e_\mu, e_\nu)$, which is equal to $q(e_\mu)$ if $\mu = \nu$ and equal to zero otherwise. The quadratic space $(\mathbb{R}^n, q)$ will be denoted $\mathbb{R}_{1,n-1}^1$, and called the $n$-Minkowski space. It can be identified with its dual, the dual basis being $\{e^\mu\}_{\mu \in J}$, with

$$(1.2) \quad e^0 = e_0 \text{ and } e^{\mu'} = -e_{\mu'} \text{ for } \mu' \in J'$$

and we shall write $g^{\mu\nu} = g(e^\mu, e^\nu)$, with the same properties as $g_{\mu\nu}$. Any element $x \in \mathbb{R}_{1,n-1}^1$ has the form:

$$(1.3) \quad x = x_\mu e^\mu = x^\mu e_\mu, \text{ with } x_\mu = g_{\mu\nu} x^\nu \quad g_{\mu\nu} g^{\mu\lambda} = \delta^\lambda_\mu$$

where $\delta$ is the Kronecker symbol.

Remark: The Einstein summation convention over the set $J$ is used in (1.3). It will be used throughout this paper. The range of summation will not always be the same, and we shall use distinct index variables for distinct ranges of summation. For instance we shall write $x_{\mu'} x^{\mu''}$ instead of $-\sum_{1 \leq \mu' \leq n-1} (x_{\mu'})^2$, using the primed letter $\mu'$ instead of $\mu$ to avoid confusion. If the range of summation is $J$, greek letters $\lambda, \mu, \nu, \ldots$ will always be used.

The connected component of the Lie group of linear transformations of $\mathbb{R}^n$ which leave $g$ invariant, $SO_0(1, n-1)$, will be called the $n$-Lorentz group and denoted by $L_n$. Its maximal compact subgroup is $SO(n-1)$. The twofold covering of the latter (universal if $n > 3$) will be denoted by $Spin(n-1)$ and the corresponding covering of $L_n$ by $\overline{L}_n$.

The abelian group of translations of $\mathbb{R}^k$, homeomorphic to $\mathbb{R}^k$, will be denoted by $T_k$, for $k \in \mathbb{N}$. The semidirect product $L_n \cdot T_n$, where $L_n$ acts canonically on $T_n$, will be called the $n$-Poincaré group and denoted by $\mathcal{P}_n$. Its twofold covering (universal if $n > 3$) $\overline{L}_n \cdot T_n$ will be denoted by $\overline{\mathcal{P}}_n$.

The Lie algebra $\mathfrak{p}_n$ of $\mathcal{P}_n$ is spanned by generators $X_{\mu\nu} = -X_{\nu\mu} \in \mathfrak{t}_n = \text{Lie}(L_n)$ and $P_\mu \in \mathfrak{t}_n = \text{Lie}(T_n)$, with $\mu, \nu \in J$, satisfying the commutation relations

$$(1.4a) \quad [X_{\lambda\mu}, X_{\nu\rho}] = g_{\mu\nu}X_{\lambda\rho} - g_{\lambda\nu}X_{\mu\rho} - g_{\mu\rho}X_{\lambda\nu} + g_{\lambda\rho}X_{\mu\nu}$$
(1.4b) \[ [X_{\mu}, P_{\nu}] = g_{\mu\nu} P_{\lambda} - g_{\lambda\nu} P_{\mu} \]

(1.4c) \[ [P_{\lambda}, P_{\mu}] = 0 \]

The element \( P_{\mu} P_{\mu} = g^{\mu\nu} P_{\mu} P_{\nu} \) of the enveloping algebra \( \mathcal{U}(\mathfrak{p}_n) \) commutes with all generators. In the classical case \( n = 4 \), when used in theoretical physics, it gives the squared mass of a particle.

b) The \( n \)-conformal group \( G_n \)

Let \( I = \{-1, 0, 1, \ldots, n\} = J \cup \hat{J} \) with \( \hat{J} = \{-1, n\} \). Extend the basis \( \{e_\mu\}_{\mu \in J} \) of \( \mathbb{R}^n \) to the basis \( \{e_A\}_{A \in I} \) of \( \mathbb{R}^{n+2} \) by putting \( q(e_{-1}) = 1 = -q(e_n) \). The quadratic space thus obtained will be denoted \( \mathbb{R}^{2,n} \); the associated symmetric bilinear form will be again denoted by \( g \), with \( g_{AB} = g(e_A, e_B) \). The connected group \( SO_0(2, n) \) which conserves the bilinear form \( g \) will be called the \( n \)-conformal group and denoted by \( G_n \). Its maximal compact subgroup is \( SO(2) \times SO(n) \), with universal covering (for \( n \geq 3 \)) \( \mathbb{R} \times Spin(n) \), that is, infinite-fold times twofold. The universal covering of \( G_n \) will be denoted \( \overline{G_n} \).

The Lie algebra \( \mathfrak{g}_n \) of \( G_n \) is spanned by generators \( X_{AB} = -X_{BA}(A, B \in I) \), with commutation relations:

(1.5) \[ [X_{AB}, X_{CD}] = g_{BC} X_{AD} - g_{AC} X_{BD} - g_{BD} X_{AC} + g_{AD} X_{BC}. \]

We shall denote by \( C \) the Casimir element of the enveloping algebra \( \mathcal{U}(\mathfrak{g}_n) \), defined by:

(1.6) \[ C = \frac{1}{2} X_{AB} X^{BA} = \frac{1}{2} X_{AB} g^{BC} X_{CD} g^{DA}. \]

The stabilizer of the basis elements \( e_{-1}, e_n \) is obviously \( \mathcal{L}_n \). Moreover, the set of generators \( X_{\mu,-1} \pm X_{\mu,n} (\mu \in J) \) spans an \( n \)-dimensional abelian subalgebra isomorphic to \( \mathfrak{t}_n \), on which \( \mathfrak{t}_n \) acts like (1.4b), for either choice of the \( \pm \) sign. The corresponding group elements have the \( (n + 2) \times (n + 2) \) matrix form:

(1.7) \[
\exp t^\mu (X_{\mu,-1} \pm X_{\mu,n}) = \begin{bmatrix}
1 - q(t)/2 & t & \pm q(t)/2 \\
-t^\# & \mathbb{I}_n & \pm t^\# \\
\mp q(t)/2 & \pm t & 1 + q(t)/2
\end{bmatrix}
\]

where \( t^\# \) is the column vector \( (t_\mu) \) and \( t \) the line vector \( (t^\mu) \).
The two Poincaré subgroups thus obtained are conjugated in $G_n$, through the involutionary mapping:

$$\Theta = \text{Ad}\exp(\pi X_{n-1,n}).$$

We shall write hereafter

$$P_\mu = X_{\mu,-1} + X_{\mu,n} \quad ; \quad \hat{P}_\mu = X_{\mu,-1} - X_{\mu,n}$$

(1.8)

and we shall identify $T_n$ as the subgroup spanned by $\exp(t\mu P_\mu)$; the “other” translation subgroup will be denoted $\hat{T}_n$, and the corresponding $n$-Poincaré subgroups $P_n$ and $\hat{P}_n$.

The twofold covering $\overline{P}_n$ is a subgroup of $G_n$ (for $n = 3$ the universal covering of $P_n$ is not contained in $\overline{G}_n$).

The remaining generator $D = X_{n,-1}$ will be called the dilatation, it commutes with the Lorentz generators and its nonzero commutation relations are

$$[D, P_\mu] = P_\mu \quad ; \quad [D, \hat{P}_\mu] = -\hat{P}_\mu$$

(1.9)

One also has:

$$[P_\mu, \hat{P}_\nu] = -2(X_{\mu\nu} + g_{\mu\nu}D)$$

(1.10)

The normalizer $W_n$ of $P_n$ in $G_n$ is the semidirect product $Y_n \cdot T_n$, with $Y_n = A \times (W \cdot L_n)$, where $A = \{\exp tD\}_{t \in \mathbb{R}}$ and $W = \{1, w\}$ is a group of order two, with $w = \exp(\pi(X_{0,-1} + X_{n-1,n}))$. The action of $w$ on $P_\mu$ is given by:

$$\text{Ad}_w(P_\mu) = \epsilon_\mu P_\mu; \quad \epsilon_0 = \epsilon_{n-1} = 1; \quad \epsilon_j = -1 \text{ if } 1 \leq j \leq n - 2.$$  

(1.11)

$W \cdot L_n$ is the non-connected group $SO(1,n-1)$. The group $W_n$ is a maximal parabolic subgroup of $G_n$. The same holds for the normalizer $\hat{W}_n = Y_n \cdot \hat{T}_n$ of $\hat{P}_n$, and one has the Bruhat-type decomposition:

$$G_n \simeq \hat{T}_n Y_n T_n = \hat{W}_n T_n = \hat{T}_n W_n$$

(1.12)

that is, the set of elements which can be written in this form is a Zariski open in $G_n$. To be more precise, $P_n$ (resp. $\hat{P}_n$) stabilizes the point $e = e_{-1} + e_n$ (resp. $\hat{e} = e_{-1} - e_n$) of $\mathbb{R}^{n+2}$. The orbit of $e$ under $G_n$ is the isotropic cone minus the origin, that is $G_n/P_n = Q = \{y, y \in \mathbb{R}^{n+2}/y \neq 0 \text{ and } yAy^{-1} = 0\}$. The group $A \times W$ sends $e$ to $\lambda e$ (and $\hat{e}$ to $\lambda \hat{e}$), $\lambda \in \mathbb{R} - \{0\}$, so that $G_n/W_n = C_0 = Q/(\mathbb{R} - \{0\}) \cong (S_1 \times S_{n-1})/Z_2$ is the set of directions of $Q$. The translation group $T_n$ stabilizes the direction $\lambda e$ and acts transitively on the complementary subset of $C_0$. Thus the complementary subset of $\hat{W}_n T_n$ in $G_n$ is $\{g; ge \in e^{\perp}\}$. 


\( C_0 \) is thus diffeomorphic to the compactified \( T_n^c \) of \( T_n \), that is \( T_n^c = \mathbb{R}^{1,n-1} \cup C_\infty \) where \( C_\infty \) is the \((n-1)\)-dimensional compactified "light cone at infinity" \( \mathbb{R}^2_n \). Writing \( \mathbb{R}^{2,n} \) and \( \mathbb{R}^{1,n-1} \) as line vectors we have the imbedding \( \varphi \) from \( \mathbb{R}^{1,n-1} \) to \( Q \):

\[
\varphi(t) = e' \exp(t^\mu P_\mu)
\]

(1.13)

with \( e' = (1, 0, \ldots, 0, 1) \); using (1.7) one has:

\[
\varphi(t) = (1 - q(t), 2t, 1 + q(t)).
\]

(1.14)

One can thus define almost everywhere an action of \( G_n \) on \( \mathbb{R}^{1,n-1} \) by means of the decomposition (1.12), writing, for \( t \in T_n \) and \( g \in G_n \)

\[
tg = \gamma(t, g)t' \quad \text{; } \gamma(t, g) \in \hat{W}_n, e'g = \varphi(t')
\]

(1.15)

Clearly, if \( g = (\land, x) \in W_n \) with \( \land \in Y_n, x \in T_n \), one has \( t' = t + x \); if \( g = \hat{x} = \exp(\hat{x}^\mu \hat{P}_\mu) \in \hat{T}_n \) one gets

\[
t' = (t - (t^\mu \hat{x}_\mu)\hat{x})(1 - 2t^\mu \hat{x}_\mu + q(t)q(\hat{x}))^{-1}
\]

(1.16)

and \( t' \) is defined when the denominator does not vanish. Elements of \( \hat{T}_n \) acting on \( T_n \) are called \textit{special conformal transformations}.

c) Representations of \( \mathcal{P}_n \) and \( G_n \)

We are interested in determining which unitary irreducible representations of \( \mathcal{P}_n \) can be extended to \( \overline{\mathcal{G}}_n \), or, conversely, which ones of \( \overline{\mathcal{G}}_n \) remain irreducible when restricted to \( \mathcal{P}_n \). It will appear that they can all be realized as functional spaces over the \( n \)-Minkowski space. We shall here begin by studying the UIR of \( \mathcal{P}_n \) and operate a first selection among them; in the next chapter we shall study the representations of \( \overline{\mathcal{G}}_n \) which satisfy the necessary constraints, and give a complete description of the possible cases.

Since \( \mathcal{P}_n = \mathcal{L}_n \cdot T_n \) is a semidirect product with abelian normal subgroup \( T_n \), its UIR are determined by the theory of Mackey \cite{10}: let \( \mathcal{O} \) be an orbit of the dual of \( T_n \) under the action of \( \mathcal{L}_n \), and \( \Gamma \) the stabilizer of a point in \( \mathcal{O} \); every UIR of \( \mathcal{P}_n \) is equivalent to a representation \( U^S \) induced by a UIR \( S \) of \( \Gamma \). Different orbits, or non-equivalent representations of \( \Gamma \) for the same orbit, induce non-equivalent UIR of \( \mathcal{P}_n \).

To construct \( U^S \) one may proceed as follows: let \( V \) be the representation space of \( S \); denote by \( x \mapsto xh \) the action of \( \mathcal{L}_n \) on \( \mathcal{O} \), with \( x \in \mathcal{O} \) and \( h \in \mathcal{L}_n \); let \( \xi \) be the point of \( \mathcal{O} \) stabilized by \( \Gamma \), and let \( x \mapsto \tau_x \) be a smooth injective mapping from \( \mathcal{O} \) to \( \mathcal{L}_n \), so that \( x = \xi \tau_x \); denote by \( \gamma(x, h) \) the unique element of \( \Gamma \) satisfying \( \gamma(x, h)\tau_{(xh)} = \tau_x h \); let \( d\mu \) be a quasi-invariant measure on \( \mathcal{O} \) and let \( \alpha \) be the positive function of \( \mathcal{O} \times \mathcal{L}_n \) such that
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$$d\mu(xh) = \alpha(x, h)d\mu(x).$$

Let $\mathcal{H} = L^2(O, V, d\mu)$ be the Hilbert space of $V$-valued functions $f$ such that:

$$\int_O ||f(x)||^2_V d\mu(x) < \infty.$$  

For $h \in \mathcal{L}_n, t \in T_n$ and $f \in \mathcal{H}$ define $U^S$, acting on $\mathcal{H}$, by:

$$(1.17) \quad [U^S_{(h, t)} f](x) = [\alpha(x, h)]^{1/2} \exp[ig(x, t)]S_{\gamma(x, h)} f(xh)$$

where $i^2 = -1$.

One thus has to determine the partition of the $n$-Minkowski space into orbits under $\mathcal{L}_n$, and the corresponding stabilizers and their UIR. For $n = 4$ this was done by E.P. Wigner, who also established all the theoretical background needed for this purpose, in a famous paper [4], anterior to the formulation of the general theory by G. Mackey. For other $n \geq 3$, the resolution into orbits is a straightforward generalization of Wigner’s results; it is given in Table 1.

**Table 1: Orbits and little groups for the $n$-Poincaré group**

| Type | Orbit | Stabilized point | Stabilizer |
|------|-------|------------------|------------|
| 0    | $x = 0$ | 0                | $\mathcal{L}_n$ |
| $I^\pm_m$ | $q(x) = 0, \pm x_0 > 0$ | $\pm(e_0 + e_{n-1})$ | $E_{n-2}$ |
| $II^{\pm_m}_{|m|}$ | $q(x) = m^2 > 0, \pm x_0 > 0$ | $\pm|m|e_0$ | $Spin(n-1)$ |
| $III_{|m|}$ | $q(x) = -m^2 < 0$ | $|m|e_{n-1}$ | $Spin(1, n-2)$ |

In Table 1 the parameter $|m|$ runs over positive real numbers; $Spin(1, n-2)$ denotes the twofold covering of $SO_0(1, n-2)$ (for $n = 3$ this covering is merely $SO_0(1, 1) \times \mathbb{Z}_2$); $E_{n-2}$ is the twofold covering of the Euclidean group in $n-2$ dimensions, $Spin(n-2) \cdot T_{n-2}$ (for $n = 3$ this reduces to $\mathbb{Z}_2 \times T_1$).

One can immediately establish:

**Proposition 1.1:** The UIR of $\mathcal{P}_n$ corresponding to orbits of types II, III and 0 (with the exception of the trivial one) cannot be extended to UIR of $\mathcal{G}_n$.

Proof: UIR of type 0 have trivial restriction on $T_n$. Since $g_n = t_n \oplus [t_n, t_n] \oplus \bar{t}_n, [\bar{t}_n, t_n]$, the trivial representation of $\mathcal{P}_n$ is the only possibility.

Concerning types II and III, let $\tilde{U}$ be the UIR of $\mathcal{G}_n$ obtained by extending $U$. Since the parabolic subgroup $\mathcal{W}_n = Y_n \cdot T_n$ contains $\mathcal{P}_n$, the restriction of $\tilde{U}$ on $\mathcal{W}_n$ must also be irreducible. Since $\mathcal{W}_n$ is a semidirect product, its UIR are again obtained by the orbit-stabilizer method. It turns out that the orbits of $T_n$ under $\mathcal{W}_n$ are $x = 0, x_\mu x^\mu = 0, x_\mu x^\mu < 0, \text{ and } x_\mu x^\mu > 0$, with sign($x_0$) fixed if $x_\mu x^\mu \geq 0$; thus, if $x_\mu x^\mu \neq 0$, the restriction of $\tilde{U}$ to $\mathcal{P}_n$ is a direct integral, of representations over the parameter $|m|$, which is not irreducible.  

$\blacksquare$
So let us focus to UIR of type I, called massless hereafter, by reference to the $n = 4$ case. Since $\mathcal{L}_n$ is an invariant subgroup of $\mathcal{Y}_n$, both groups acting on the same homogeneous space $\mathcal{O}$, $E_{n-2}$ is an invariant subgroup of $\Gamma'$, the stabilizer of $\xi$ in $\mathcal{Y}_n$, and $\mathcal{Y}_n/\mathcal{L}_n = W \times A \cong \mathbb{Z}_2 \times \mathbb{R}^{++}$ is isomorphic to $\Gamma'/E_{n-2}$. More precisely one has

$$\Gamma' = (W \times A' \times \text{Spin}(n-2)).T_{n-2}$$

such that $\text{Lie} \,(T_{n-2})$ is generated by elements $L_j = X_{j,0} + X_{j,n-1}, 1 \leq j \leq n-2$; $\text{Lie} \,(\text{Spin}(n-2)) = so(n-2)$ is generated by $X_{j,k}, 1 \leq j, k \leq n-2; A' = \{ \exp(t(X_{0,n-1} + X_{n,-1})) \}_{t \in \mathbb{R}} \approx \mathbb{R}^{++}$ and $W = \{ 1, \exp(\pi(X_{0,-1} + X_{n-1,0})) \}; \Gamma'$ consists of the elements of $G_n$ which commute with $P_0 + P_{n-1} = X_{0,-1} + X_{0,n} + X_{n-1,-1} + X_{n,-1,0}$.

Let $S'$ be the inducing representation of $\Gamma'$ and $S$ its restriction to $E_{n-2}$, so that $U^S$ is the restriction to $\mathcal{P}_n$ of the representation $U^{S'}$ of $\mathcal{W}_n$. Since $U^S$ must be irreducible, $S$ must be irreducible too.

To determine the UIR of $\Gamma'$, one can again apply Mackey’s theory of resolution into orbits. Without entering into many details, one can see that $W \times \text{Spin}(n-2)$ stabilizes the “length” $x^2 = \sum_{j=1}^{n-2} (x_j)^2 = -x_j x^j$ of an element $x$ of $T_{n-2}$, acting transitively on the corresponding sphere.

On the other hand, $\lambda \in A'$ acts as a dilatation on $T_{n-2}$, sending $x$ to $\lambda x$. If $S'$ corresponds to a nonzero orbit, its restriction $S$ is a direct integral of representations and $U^S$ is reducible. This leaves us with:

**Proposition 1.2:** A necessary condition for a massless representation $U^S$ of $\mathcal{P}_n$ to extend to $\mathcal{G}_n$ is that the inducing representation $S$ is a (finite-dimensional) UIR of $\text{Spin}(n-2).T_{n-2}$ with trivial restriction to the normal subgroup $T_{n-2}$. \[\square\]

For every such choice of $S$ and for either choice of sign$(x_0)$, $U^S$ extends to $\mathcal{W}_n$, since $S$ always extends to $S'$: one can always do this by choosing a one-dimensional UIR of $A' \times W$ the choice being of course not unique. To see if the extension to $\mathcal{G}_n$ is possible, we shall use Lie algebraic methods. Before proceeding further, we shall give the expression of the infinitesimal operators of $\mathcal{P}_n$, acting on a dense subspace of analytic vectors of $\mathcal{H}$, the representation space of $U^S$.

To be more precise about $\mathcal{H}$, the orbit $\mathcal{O}$ can be parametrized by $\mathbb{R}^{n-1} - \{0\}$: if $(x_0, \vec{x}) \in T_n$ is in $\mathcal{O}$, let $||\vec{x}|| = (\sum_{\mu'} x_{\mu'}^2)^{1/2}$. Since the orbit is massless, one has $x_0^2 = ||\vec{x}||^2$, so that if $\vec{x} \in \mathbb{R}^{n-1} - \{0\}$ is given, $x_0$ is fixed, its sign being determined by the choice of $\mathcal{O}$. The quasi-invariant measure $d\mu$ is defined by

$$d\mu(x) = d^{n-1}\vec{x}/||x||.$$
In fact \(d\mu\) turns out to be invariant under the action of \(\mathcal{L}_n\) (but not under the action of dilatations), so that the factor \(\alpha\) in (1.17) equals 1. Putting \(S_{jk} = dS(X_{jk})\) acting on \(V\), one obtains the following expressions:

\[
\begin{aligned}
P_\mu &= \sqrt{1} x_\mu \\
X_{jk} &= L_{jk} + S_{jk}, \quad 1 \leq j, k \leq n - 2 \\
X_{jn-1} &= L_{j,n-1} + B_j, \quad 1 \leq j \leq n - 2 \\
X_{0j} &= x_0 \partial_j + B_j, \quad 1 \leq j \leq n - 2 \\
X_{0,n-1} &= x_0 \partial_{n-1}
\end{aligned}
\]

(1.20)

where

\[
L_{\mu'\nu'} = x_{\mu'} \partial_{\nu'} - x_{\nu'} \partial_{\mu'}, \quad B_j = (x_0 + x_{n-1})^{-1} \sum_{k=1}^{n-2} x^k S_{jk}.
\]

(1.21)

We recall that we use the standard notation

\[
\partial_{\mu'} = \partial/\partial x_{\mu'}, \quad (1 \leq \mu' \leq n - 1).
\]

(1.22)

This implies in particular:

\[
[\partial_{\mu'}, x_0] = -x_{\mu'}/x_0
\]

(1.23)

It is clear that \(U^S\) sends to zero the central element \(P_\mu P_\mu\) of \(\mathcal{U}(\mathfrak{p}_n)\). This feature will be the startpoint for the study of representations of \(\mathfrak{g}_n\), candidates to solve the problem.
2 Representations of $\mathfrak{so}(2,n)$ sending $P_\mu P^\mu$ to 0

a) Weight representations of $\mathfrak{so}(N)_C$ and the Casimir element

Let $g$ be a symmetric nondegenerate bilinear form on $\mathbb{R}^N$, $I$ a set of cardinality $N$, $\{e_A\}_{A \in I}$ a basis of $\mathbb{R}^N$ and $g_{AB} = g(e_A, e_B)$. The orthogonal Lie algebra $\mathfrak{g} = \mathfrak{so}(N, g)$ is spanned by generators $X_{AB} = -X_{BA}$ such that

$$[X_{AB}, X_{CD}] = g_{BC} X_{AD} - g_{BD} X_{AC} - g_{AC} X_{BD} + g_{AD} X_{BC}$$

(2.1)

their action on $\mathbb{R}^N$ being (with bracket notations)

$$[X_{AB}, e_C] = e_A g_{BC} - e_B g_{AC}$$

(2.2)

If $\{e^A\}$ is the dual basis, with $\langle e^A, e_B \rangle = \delta^A_B$, denoting by $g$ again the associated bilinear form on the dual, with $g(e^A, e^B) = g^{AB}$, nondegeneracy implies $g^{AB} g_{BC} = \delta^A_C$.

We shall use the tensor $g$ for raising and lowering indices, writing for instance $X_{AC} g^{CB}$.

The complexified Lie algebra $\mathfrak{g}^C$ is independent of the choice of $g$ (up to isomorphism), the various real forms being obtained by a suitable choice of the basis $\{e_A\}$, fixing $\mathbb{R}^N$ in $\mathbb{C}^N$.

We shall now introduce a Cartan subalgebra and a Borel-type decomposition in $\mathfrak{g}^C$ as follows:

**Proposition 2.1:** Let the indexing set $I$ be $\{1, \ldots, N\}$ and assume $(g_{AA})^2 = (g^{AA})^2 = 1$, for every $A \in I$. Fix the constant $\gamma$ by $\gamma = N/2 - \text{Rank}(\mathfrak{g})$, that is $\gamma = 0$ if $N$ is even and $\gamma = \frac{1}{2}$ if $N$ is odd. Let $\hat{I} = \{\gamma + 1, \gamma + 2, \ldots, N/2\}$ be an indexing set of cardinality $\text{Rank}(\mathfrak{g})$; let $q_A = g(e_A, e_A)$; for every $a \in \hat{I}$, fix the constant $\eta_a$ such that

$$\eta_a^2 = -q_{2a-1} q_{2a}$$

(2.3)

(Lence $\eta_a^4 = 1$ and $\eta_a^* = \eta_a^{-1} = \eta_a^3$) and define $H_a \in \mathfrak{g}^C$ by:

$$H_a = \eta_a X_{2a-1,2a}$$

(2.4)

The eigenvalues of $adH_a$ are $0, +1, -1$; for every index $A' \in I - \{2a-1, 2a\}$, the linear combinations

$$X \left( ^{\alpha} \right)_{A'} = X_{2a,A'} + \eta_a q_{2a} X_{2a-1,A'}$$

$$X \left( ^{\overline{\alpha}} \right)_{A'} = X_{2a-1,A'} + \eta_a q_{2a-1} X_{2a,A'}$$

(2.5)
are eigenvectors of \( \text{ad}H_a \), satisfying:

\[
\begin{align*}
[H_a, X(\pm \frac{a}{b})_{A'}] &= \pm X(\pm \frac{a}{b})_{A'} \\
\eta_a[X(\pm \frac{a}{b})_{A'}, X(\pm \frac{a}{b})_{B'}] &= 2 (X_{A'B'} + g_{A'B'} H_a) \\
[X(\pm \frac{a}{b})_{A'}, X(\pm \frac{a}{b})_{B'}] &= [X(\pm \frac{a}{b})_{A'}, X(\pm \frac{a}{b})_{B'}] = 0
\end{align*}
\]  

Similarly the linear combinations \( X\left(\frac{\varepsilon}{a} \frac{\varepsilon'}{b}\right) \) defined by:

\[
\begin{align*}
X\left(\frac{\varepsilon}{a} \frac{\varepsilon}{b}\right) &= X\left(\frac{\varepsilon}{a}\right)_{2b} + \eta_b \ q_{2b} \ X\left(\frac{\varepsilon}{a}\right)_{2b-1} \\
X\left(\frac{\varepsilon}{a} \ -\frac{\varepsilon}{b}\right) &= X\left(\frac{\varepsilon}{a}\right)_{2b-1} + \eta_b \ q_{2b-1} \ X\left(\frac{\varepsilon}{a}\right)_{2b}
\end{align*}
\]

are simultaneous eigenvectors for every \( \text{ad}H_c \), belonging to the eigenvalue \( \varepsilon 1 \) if \( c = a \), to \( \varepsilon' 1 \) if \( c = b \) and to 0 otherwise.

Then:

1) The elements \( H_a \) span a Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g}^C \).

2) The set \( \{X\left(\frac{\varepsilon}{a}\right)_{A'}, a \in \hat{I}, A' < 2a - 1\} \) span a nilpotent subalgebra \( \mathfrak{n}^\pm \) of \( \mathfrak{g}^C \), for either choice of the \( \pm \) sign, such that \( \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \) is a Borel-type decomposition of \( \mathfrak{g}^C \).

3) When \( N \) is an even integer, all elements \( X\left(\pm \frac{\varepsilon}{a} \frac{\varepsilon'}{b}\right) \) together with \( \mathfrak{h} \) span a subalgebra \( \mathfrak{l} \) isomorphic to \( \mathfrak{gl}(N/2) \), while elements \( X\left(\pm \frac{\varepsilon}{a} \frac{\varepsilon}{b}\right) \) span abelian subalgebras \( \mathfrak{n}^{\pm \pm} \), such that \( \mathfrak{l} \oplus (\mathfrak{n}^{++} \oplus \mathfrak{n}^{--}) \) is a Cartan decomposition of \( \mathfrak{g}^C \) corresponding to the real form \( \mathfrak{so}^r(N) \).

4) A Cartan-Weyl basis of \( \mathfrak{g}^C \) is

\[
\mathcal{B}_0 = \left\{ \frac{i}{2} \sqrt{\eta_a \eta_b} \ X\left(\frac{\varepsilon}{a} \frac{\varepsilon'}{b}\right), \varepsilon = \pm, \varepsilon' = \pm \right\}_{a<b}
\]

if \( N \) is even and

\[
\mathcal{B}_{1/2} = \mathcal{B}_0 \cup \left\{ \sqrt{\frac{\eta_a}{q_1}} X\left(\frac{\varepsilon}{a}\right), \varepsilon = \pm \right\}_a
\]

if \( N \) is odd. Indeed one has, if \( \{e_a\}_a \) is the dual basis of \( \{H_a\}_a \):

\[
\begin{align*}
\left[\frac{i}{2} \sqrt{\eta_a \eta_b} \ X\left(\frac{\pm}{a} \frac{\pm}{b}\right) + \frac{i}{2} \sqrt{\eta_a \eta_b} \ X\left(\frac{-}{a} \frac{\mp}{b}\right)\right] &= H_a + H_b \\
\left[H, \frac{i}{2} \sqrt{\eta_a \eta_b} \ X\left(\frac{\pm}{a} \frac{\pm}{b}\right)\right] &= \pm (e_a + e_b)(H) \frac{i}{2} \sqrt{\eta_a \eta_b} \ X\left(\frac{\pm}{a} \frac{\pm}{b}\right) \\
\left[H, \frac{i}{2} \sqrt{\eta_a \eta_b} \ X\left(\frac{\pm}{a} \frac{\mp}{b}\right)\right] &= \pm (e_a - e_b)(H) \frac{i}{2} \sqrt{\eta_a \eta_b} \ X\left(\frac{\pm}{a} \frac{\mp}{b}\right)
\end{align*}
\]
and, if \( N \) is odd:

\[
\left[ \sqrt{\frac{\eta a}{q}} X \left( \frac{\pm}{a} \right)_1 : \sqrt{\frac{\eta a}{q}} X \left( \frac{\mp}{a} \right)_1 \right] = 2H_a
\]

\[
\left[ H, \sqrt{\frac{\eta a}{q}} X \left( \frac{\pm}{a} \right)_1 \right] = \pm e_a(H) \sqrt{\frac{\eta a}{q}} X \left( \frac{\mp}{a} \right)_1.
\]

The root system is thus given by

\[
\Delta_0 = \{ \varepsilon e_a + \varepsilon' e_b, \varepsilon = \pm, \varepsilon' = \pm \}_{a < b}
\]

if \( N \) is even and

\[
\Delta_{1/2} = \Delta_0 \cup \{ \varepsilon e_a, \varepsilon = \pm \}_a
\]

if \( N \) is odd

**Remark:** If \( q_{2a} = -q_{2a-1} \) then the elements \( H_a \) and \( X \left( \frac{\pm}{a} \right)_{A'} \) belong to the real form \( g \) from which we started: with suitable modification of the indexing set \( I \), one sees that the dilatation operator or the Poincaré translations \( P_\mu \), imbedded in the conformal Lie algebra, are of this form. On the contrary, such elements do not belong to the real form when \( q_{2a} = q_{2a-1} \) since \( \eta_a \) is imaginary.

We are now interested in relating the eigenvalue of the Casimir element \( C \) of \( g \), defined by

\[
C = \frac{1}{2} X_{AB} g^{BC} X_{CD} g^{DA} = \frac{1}{2} X_{AB} X^{BA}
\]

(2.8)

to the extremal weight which defines a finite-dimensional irreducible representation \( D \) of \( g \). Though this result is classical, we shall give some details in view of further developments.

So let \( I = I' \cup I'' \), with \( I' = \{ 1, \ldots, N - 2 \} \) and \( I'' = \{ N - 1, N \} \). Splitting the summations one has:

\[
C = C' + C'' + B
\]

(2.9)

where \( C' \) is the Casimir element of \( so(N - 2) \) and \( C'' \) that of the complementary \( so(2) \), that is

\[
C'' = \frac{1}{2}(X_{N-1,N} X^{N,N-1} + X_{N,N-1} X^{N-1,N})
\]

(2.10)

\[
= -q^N q^{N-1}(X_{N-1,N})^2
\]

\[
= \eta_{N/2} X_{N-1,N}^2 = (H_{N/2})^2
\]

while

\[
B = -X_{A'A''} X^{A'A''}
\]

(2.11)
where primed and double-primed indices are summed over the sets $I'$ and $I''$ respectively.

Develop now the expression $B^{++}_{A'B'}$, symmetric in the indices $A', B' \in I'$, defined as follows:

\[
B^{++}_{A'B'} = \frac{1}{2} \eta_{N/2} (X^{(N/2)}_{A'} X^{(N/2)}_{B'} + X^{(N/2)}_{B'} X^{(N/2)}_{A'})
\]

\[
= \frac{1}{2} \eta_{N/2} (X_{N-1,A'} + \eta_{N/2} q_{N-1} X_{N,A'}) (X_{N,B'} + \eta_{N/2} q_{N} X_{N-1,B'}) +
\]

\[
\frac{1}{2} \eta_{N/2} (X_{N-1,B'} + \eta_{N/2} q_{N-1} X_{N,B'}) (X_{N,A'} + \eta_{N/2} q_{N} X_{N-1,A'})
\]

\[
= \frac{1}{2} (\eta_{N/2} (X_{N-1,A'} X_{N,B'} - X_{N,A'} X_{N-1,B'})) -
\]

\[
(q^{N-1} X_{N-1,A'} X_{N-1,B'} + q^{N} X_{N,A'} X_{N,B'}))
\]

\[
+ \frac{1}{2} (\eta_{N/2} (X_{N-1,B'} X_{N,A'} - X_{N,A'} X_{N-1,B'})) -
\]

\[
(q^{N-1} X_{N-1,B'} X_{N-1,A'} + q^{N} X_{N,B'} X_{N,A'}))
\]

\[
= -H_{N/2} g_{A'B'} - \frac{1}{2} g^{A''B''} (X_{A''A'} X_{B''B'} + X_{A''B'} X_{B''A'})
\]

Summing with $g^{A'B'}$ over the set $I'$ of cardinality $N - 2$ yields:

\[
(2.13) \quad B = (N - 2) H_{N/2} + B^{++}
\]

where $B^{++} = B^{++}_{A'B'} g^{A'B'}$ (notice that the permutation $N \leftrightarrow N - 1$ exchanges the + and − signs and transforms $H_{N/2}$ to $-H_{N/2}$, while $B$ is left unchanged). Thus one has

\[
(2.14) \quad C = H_{N/2} (H_{N/2} + N - 2) + C' + B^{++}
\]

Let now $V$ be a finite-dimensional irreducible $g$-module, corresponding to the representation $\mathcal{D}$. Let $s_{N/2}$ be the eigenvalue of $\mathcal{D}(H_{N/2})$ with maximal real part, and let $V'$ be the subspace

\[
(2.15) \quad V' = \{ \varphi \in V; \mathcal{D}(H_{N/2}) \varphi = s_{N/2} \varphi \}
\]

If $N = 2$, then $\dim V = \dim V' = 1$ since the Lie algebra is abelian, and $\mathcal{D}(C) = s_1^2 = \mathcal{D}(H_1^2)$. If $N > 2$, then $V' \subseteq \ker \mathcal{D}(X^{(N/2)}_{A'})$ for every $A' \in I'$. It follows that $V'$ is an irreducible $\mathfrak{g}'$-submodule, where $\mathfrak{g}' = \mathfrak{so}(N - 2)$ is the subalgebra generated by $X_{A'B'}$ with $A', B' \in I'$ and also $\mathcal{D}(B^{++}_{A'B'})$ vanishes on $V'$. Moreover, $\mathcal{D}$ is integrable to the compact real form since $V$ is finite dimensional, so that $\mathcal{D}(H_a)$ and $\pm \mathcal{D}(H_b)$ are conjugate for any choice of + or − and of $a, b$ in $I$: it can be proved that this implies that every eigenvalue of $\mathcal{D}(H_a \pm H_b)$ (hence $2 \mathcal{D}(H_a)$) is an integer; in particular, $2s_{N/2} \in \mathbb{N}$. 

Masslessness in $n$ Dimensions

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If $N = 3$, then $\mathfrak{g}^\prime = \{0\}$, $C^\prime = 0$, $\dim V^\prime = 1$ and one gets the well known formula

\begin{equation}
\mathcal{D}(C) = s(s + 1), \text{ with } s = s_{3/2}
\end{equation}

If $N > 3$, one may apply the same procedure to the $\mathfrak{g}^\prime$-module $V^\prime$, introducing the maximal eigenvalue $s_{N/2-1}$ of $\mathcal{D}(H_{N/2-1})$ restricted on $V^\prime$, and so on. Taking in account that $|s_a| \leq |s_{a+1}|$ because $\mathcal{D}(H_a)$ and $\mathcal{D}(H_b)$ are conjugate for every $a$ and $b$, one easily gets by induction:

**Theorem 2.1**: The extremal weight of an irreducible finite-dimensional representation $\mathcal{D}$ of $\mathfrak{so}(N)$, $N > 2$, is determined by a sequence of positive numbers $s_a, a \in \hat{I}$, satisfying $s_{a+1} - s_a \in \mathbb{N}, 2s_a \in \mathbb{N}$, and such that

\begin{equation}
\mathcal{D}(C) = \sum_{a=\gamma+1}^{N/2} s_a(s_a + 2a - 2).
\end{equation}

There is an extremal weight vector $\varphi \neq 0$, spanning a one-dimensional subspace invariant by the Borel subalgebra $\mathfrak{h} \oplus \mathfrak{n}$, such that

$\mathcal{D}(n^+)\varphi = \{0\}, \mathcal{D}(H_a)\varphi = s_a\varphi$ if $a > 1$

and, when $N$ is an even integer, $\mathcal{D}(H_1) = \pm s_1\varphi$ (representations with different choice of sign being inequivalent).

One can also show that $\mathcal{D}$ is determined by the extremal weight up to equivalence, and that the representation space is $\mathcal{D}(U(n^-))\varphi$. We shall denote such a representation here after by $\mathcal{D}(s_{N/2}, s_{N/2-1}, \ldots, s_{1+\gamma})$.

The corresponding Coxeter-Dynkin diagrams are:
Remark: Extremal weight representations of \( g \) with arbitrary range of the \( s_a \)'s can be defined, so that (2.17) still holds: \( I \) being the left ideal of \( \mathcal{U}(g) \) corresponding to a one-dimensional representation of \( \mathfrak{h} \oplus \mathfrak{n}^+ \), the left regular representation on \( \mathcal{U}(g)/I \) has the desired form. Integrability over some real form implies restrictions on the range of \( s_a \). In particular, for the real form \( \mathfrak{so}(2, N - 2) \), we shall denote by \( d_{(\alpha, \vec{s})}^{N-2,\epsilon} \) such a representation, where \( \epsilon \in \{-1, +1\} \) and \( 2\alpha \not\in \mathbb{N} \); the spectrum of \( d_{(\alpha, \vec{s})}^{N-2,\epsilon}(-\epsilon H_{N/2}) \) is \( \{\alpha-k, k \in \mathbb{N}\} \) and the eigenspace corresponding to the maximal eigenvalue \( \alpha \) is an irreducible \( \mathfrak{so}(N-2) \)-module corresponding to the weight \( \vec{s} = (s_{(N-2)/2}, \ldots, s_{1+\gamma}) \).

**b) Massless representations:**

Let us define the elements \( \bar{F}_{AB} \) of the enveloping algebra \( \mathcal{U} \) of \( g = \mathfrak{so}(N)^\mathbb{C} \) by:

\[
\bar{F}_{AB} = \frac{1}{2} (X_{AC}g^{CD}X_{DB} + X_{BC}g^{CD}X_{DA}) = X_A^C X_{CB} - \frac{1}{2} (N-1) X_{AB}
\]

and the elements \( F_{AB} \) as:

\[
F_{AB} = \bar{F}_{AB} - \frac{1}{N} g_{AB} \bar{F}_{CD} g^{CD} = \bar{F}_{AB} - \frac{2}{N} g^{AB} C
\]

The elements \( F_{AB} \) are symmetric in the indices \( A, B \) (as well as the \( \bar{F}_{AB} \)) and they span an irreducible \( g \)-submodule \( \mathcal{F} \) of \( \mathfrak{g} \otimes \mathfrak{g} \) under \( ad \otimes ad \). For \( N > 2 \) the dimension of \( \mathcal{F} \) is \( N(N+1)/2 - 1 = (N-1)(N+2)/2 \) (for \( N = 2, \mathcal{F} \) is \( \{0\} \)); \( \mathcal{F} \) is isomorphic, as a \( g \)-module, to the Cartan subspace \( \mathfrak{p} \) in the Cartan decomposition \( \mathfrak{sl}(N) = \mathfrak{so}(N) \oplus \mathfrak{p} \) of \( \mathfrak{sl}(N) \).

Since \( \mathcal{F} \) is irreducible, for every \( Y \in \mathcal{F} \) the two-sided ideal \( \mathcal{U} Y \mathcal{U} \) of \( \mathcal{U} \) contains \( \mathcal{F} \); it follows:

**Lemma 2.1:** Given a representation \( U \) of \( g \), if there is \( Y \in \mathcal{F} \) such that \( U(Y) = 0 \), then \( U(Y') = 0 \) for every \( Y' \in \mathcal{U} Y \mathcal{U} \), and, in particular, for every \( Y' \in \mathcal{F} \). \( \square \)

Split now the indexing set \( I \) into two disjoint sets \( I' \) and \( I'' = \{S, T\} \). Let, as in the preceding section, \( \eta \) be such that \( \eta^2 = -g^{SS}g^{TT} \) and let \( H = \eta X_{ST} \). The eigenvectors of \( ad_H \) are given by:

\[
X_{A'}^+ = X_{SA'} - \eta q_S X_{TA'}; \quad X_{A'}^- = X_{TA'} + \eta q_T X_{SA'}
\]

for every \( A' \) in \( I' \).

Summing over \( A' \in I' \) these expressions one gets

\[
\eta X_{A'}^\pm X_{B'}^\mp g^{AB'} = -\eta^2 (q_T F_S + q_S F_T) - 2H^2 \pm (N-2)H + \frac{4}{N} C
\]
and

\[ X^+_A X^+_B g^{AB'} = F_{TT} - 2\eta q_T F_{ST} - \eta^2 F_{SS} \]
\[ X^-_A X^-_B g^{AB'} = F_{SS} - 2\eta q_S F_{ST} - \eta^2 F_{TT} \]

(2.22)

One thus gets:

**Lemma 2.2:** For every generator \( X_{ST} \) with \( q^2_S = q^2_T = 1 \), the expressions \( X^+_A X^+_B g^{AB'} \), in which the summation runs over \( I - \{S, T\} \) and the \( X^+_A \) are the eigenvectors defined in (2.20), belong to \( F \). In particular, if \( N = n + 2, I = \{-1, 0, 1, \ldots, n\}, \{ST\} = \{-1, n\} \), the element \( P_\mu P^\mu \) of the Poincaré enveloping algebra, canonically imbedded in \( U(\mathfrak{so}(2, n)) \), belongs to \( F \). \( \square \)

From these two lemmas it follows:

**Proposition 2.2:** If a representation \( U \) of \( U(\mathfrak{so}(2, n)) \) satisfies \( U(P_\mu P^\mu) = 0 \), then \( U \) vanishes on \( F \).

Such a representation will be called **massless** hereafter.

We shall begin the study of massless representations by establishing:

**Proposition 2.3:** Let \( U \) be a representation of \( \mathfrak{g} \) which vanishes on \( F \). Let \( N = N' + N'' \) be any splitting of \( N \) into two positive integers, \( I = I' \cup I'' \) the corresponding splitting of the indexing set, \( \mathfrak{g}' = \mathfrak{so}(N') \) and \( \mathfrak{g}'' = \mathfrak{so}(N'') \) the corresponding subalgebras. Their Casimir elements \( C' \) and \( C'' \) are related to the Casimir element \( C \) of \( \mathfrak{g} \) by:

\[ U(C') - U(C'') = \frac{N' - N''}{N} U(C) \]

(2.23)

In particular, if \( N'' = 1 \) and \( I'' = \{1\} \) one has:

\[ U(C') = \frac{N - 2}{N} U(C) \]

(2.24)

\[ U(g^{AB} X_{1A} X_{B1}) = \frac{2q_1}{N} U(C) \]

(2.25)

**Proof:** Using distinct summations over \( I', I'' \) and using the definition of \( F_{AB'} \) one has

\[ g^{AB'} F_{AB'} = g^{AB'} (X_{A'C'} g^{C'D'} X_{D'B'} + X_{A'A''} g^{A'B''} X_{B'B''} - \frac{2g^{A'B'}}{N} C) \]

(2.26a)

\[ = 2C' + X_{A'A''} X_{B'B''} g^{AB'} g^{A'B''} - \frac{2N'}{N} C \]

(2.26b)

\[ g^{A'B''} F_{A'B''} = 2C'' + X_{A'A''} X_{B'B''} g^{A'B'} g^{A'B''} - \frac{2N''}{N} C \]

and by subtraction one gets the desired result, since \( U \) vanished on \( F \). \( \square \)
Let us now determine the irreducible massless representations. Starting from low values of $N$, one first establishes:

**Theorem 2.2:** For $N = 2$, every representation is massless, $\mathcal{F}$ being $\{0\}$. For $N = 3$ the only irreducible massless representations are the trivial and the spinorial (two-dimensional) one. For $N = 4$, if $g = g_1 \oplus g_2$ is the decomposition of $\mathfrak{so}(4)$ into two ideals, each isomorphic to $\mathfrak{so}(3)$, an irreducible representation is massless if and only if it vanishes on either $g_1$ or $g_2$.

Sketch of the proof: For $N = 3$, $g \otimes g = \mathcal{F} \oplus g \oplus \mathbb{C}C$, and one can show (we leave this to the reader) that $(C - \frac{3}{4})g$ belongs to the ideal $\mathcal{U}F\mathcal{U}$, so that the quotient is a five-dimensional complex algebra, which turns out to be $\text{End}_\mathbb{C}(\mathbb{C}^2) \oplus \mathbb{C}$.

For $N = 4$ one first sees that $\mathcal{F}$ is the span of all elements $X_1X_2$ with $X_i \in g_i$ so that $\mathcal{U}F = \mathcal{U}U$ is the intersection of the two maximal ideals $g_1\mathcal{U}$ and $g_2\mathcal{U}$, hence the result.$\square$

So, from now on we shall suppose $N \geq 5$.

Examining first the finite-dimensional case one gets:

**Theorem 2.3:** A representation $\mathcal{D}(s_{\frac{N}{2}}, \ldots, s_{\frac{N}{2}+\gamma})$ is massless if and only if $|s_a| = s$ for every $a \in \hat{I}$ where if $N$ is even (and $\gamma = 0$) then $2s \in \mathbb{N}$ while if $N$ is odd ($\gamma = \frac{1}{2}$) then $s \in \{0, \frac{1}{2}\}$. The corresponding value of the Casimir element is

$$C = \frac{1}{2}Ns\left(s + \frac{1}{2}N - 1\right).$$

Moreover, if $N$ is even, an extremal weight subspace carries a one dimensional representation of the parabolic subgroup $\mathfrak{gl}(N/2) \oplus \mathfrak{n}^{++}$, with trivial action of $\mathfrak{sl}(N/2)$ and $\mathfrak{n}^{++}$.

**Proof:** We shall calculate $\overline{F}_{AB}$ on an extremal vector $\varphi$. Using the notations of the preceding section and taking in account that $\mathfrak{n}^+$ vanished on $\varphi$, let $A, B < 2a - 1$ for some $a \in \hat{I}$; a calculation similar to (2.12) yields:

$$\sum_{i, j \in \{2a-1, 2a\}} \frac{1}{2}(X_{Ai}X_{jB} + X_{Bi}X_{jA})g^{ij}\varphi = H_ag_{AB}\varphi$$

(2.28)

On the other hand, let $I' = \{1, \ldots, 2b\}$ and $I''(b) = \{2b - 1, 2b\} \subset I'$. Using distinct summations on primed and double-primed indices, with $A', B' \in I'$ and $A'', B'' \in I''(b)$, one has, using inductively (2.12):

$$g^{A''B''}X_{A''A'}X_{B''B'}G^{AB'}\varphi = H_b(2H_b + 2b - 2)\varphi$$

(2.29)

hence

$$\sum_{A'', B'' \in \{2b-1, 2b\}} g^{A''B''} \overline{F}_{A''B''}\varphi = 2[H_b(H_b + b - 1) + \sum_{a>b} H_a]\varphi$$

(2.30)
Since $F_{A'B'}$ vanishes, one obtains:

\[(2.31) \quad \frac{2}{N} C \varphi = [H_b(H_b + b - 1) + \sum_{a > b} H_a] \varphi \]

Equalling the expressions obtained for $b$ and $b + 1$, one gets for consecutive eigenvalues $s_b$ and $s_{b+1}$:

\[(2.32) \quad 0 = s_b(s_b + b - 1) - s_{b+1}(s_{b+1} + b - 1) = (s_b - s_{b+1})(s_b + s_{b+1} + b - 1) \]

For $b \geq 1$ and $N$ odd or $b > 1$ and $N$ even one has $0 \leq s_b \leq s_{b+1}$ so that one must have $s_b = s_{b+1}$, and for $b = 1$, $N$ even, $(2.32)$ becomes $|s_1| = |s_2|$, so that $s = |s_a|$ is constant. For $b = N/2$, $(2.31)$ gives the values of the Casimir.

For $N$ odd one also has, by taking $A = B = 1$ in $(2.28)$ and summing all over $a \in \hat{I}$:

\[(2.33) \quad \sum_{a \in \hat{I}} s_a = \frac{1}{2}(N - 1)s = \frac{2}{N} C = s(s + \frac{1}{2}N - 1) \]

hence $s(s - \frac{1}{2}) = 0$.

Notice also that $X \left( \begin{array}{cc} \varepsilon & \varepsilon' \\ a & b \end{array} \right) \varphi = 0$ unless $\varepsilon = \varepsilon' = -$, because otherwise an eigenvalue equal to $s + 1$ would appear for some $H_a$, which is impossible. Since also $H_a - H_b$ vanishes on $\varphi$, $\varphi$ spans a one-dimensional representation of $\mathfrak{gl}(N/2) \oplus \mathfrak{n}^{++}$ for even $N$, as stated.

It remains to show every representation of this form is a massless one. If $s = 0$ we have the trivial one which is massless, and if $s = \frac{1}{2}$ we have a spinorial representation $\mathcal{D}$ and Ker $\mathcal{D}$ is a bilateral ideal of $\mathcal{U}$ containing $\mathcal{F}$; this ends the odd $N$ case. For even $N$ and $s \geq 1$ one has $\mathcal{D}(\mathfrak{g}, \mathfrak{g}) \varphi = (\mathcal{D}(\mathfrak{n}^{--}) \oplus \mathcal{D}(\mathfrak{n}^{--}) \oplus \mathcal{C}) \varphi$.

Diagonalizing the space $\mathcal{F}$ with respect to the Cartan subalgebra $\mathfrak{h}$ one gets, among others, elements $F_{a}^{++}$ and $F_{a}^{--}$ such that $[H_a, F_{b}^{\pm \pm}] = \pm 2\delta_{ab}F_{b}^{\pm \pm}$, and all these elements are in Ker $\mathcal{D}$, since no elements of $\mathfrak{n}^{--} \cdot \mathfrak{n}^{--}$ or $\mathfrak{n}^{--}$ have this property. Writing $\mathfrak{h} = \mathfrak{h}' \oplus \mathbb{C} H$ with $H = \sum_a H_a$ and $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{sl}(N/2)$, one can substitute $\mathfrak{h}'$ with any conjugated subalgebra, and this does not affect $\varphi$. The new elements $F_{b}^{\pm \pm}$ thus obtained are distinct from the original ones, and as $\mathfrak{h}$ varies the whole of $\mathcal{F}$ is spanned by such elements. It follows that $\mathcal{D}(\mathcal{F}) \varphi = \{0\}$, and since $\mathcal{FU} = \mathcal{UF}, \mathcal{D}(\mathcal{F})$ vanishes on $\mathcal{D}(\mathcal{U}) \varphi$, so the representation $\mathcal{D}$ is massless. \hfill \Box

Consider now infinite-dimensional massless representations integrable to the universal covering of the conformal group. Putting $n = N - 2$, the maximal compact subalgebra is $\mathfrak{k} = \mathfrak{so}(2) \oplus \mathfrak{so}(n)$, and the complexified Cartan subspace $\mathfrak{p}^\mathbb{C}$ is isomorphic to the $\mathfrak{k}$-module $\mathbb{C}^2 \otimes \mathbb{C}^n$. We shall again use the usual notations for the $n$-conformal algebra, that is the indexing set will be $\hat{I} = I' \cup I''$ with $I' = \{1, \ldots, n\}, I'' = \{-1, 0\}$ and the indexing set $\hat{I}$, for the Cartan subalgebra, $\{0, \frac{n}{2}, \frac{n}{2} + 1, \ldots\}$, we shall denote by $H_0$ the central element $\eta X_{-1,0}$ (with $\eta^2 = -1$ and $g_{-1,-1} = g_{00} = 1$) of $\mathfrak{k}^\mathbb{C}$. 
The space $\mathcal{H}$ of the representation $U$ is a direct sum of \mathfrak{k} submodules $W(s_0, \vec{s})$, where $s_0$ is the eigenvalue of $H_0$ and $\vec{s}$ the extremal weight of $\mathfrak{so}(n)$. $p^C$ acts on $W(s_0, \vec{s})$ like $(C^2 \otimes \mathbb{C}^n) \otimes W(s_0, \vec{s})$: this tensor product splits in general into $2n$ components $W(s_0, \vec{s})$: this tensor product splits in general into $2n$ components $W(s_0, \vec{s})$: this tensor product splits in general into $2n$ components $W(s_0 + \varepsilon, \vec{s} + \Delta \vec{s})$ with $\varepsilon = \pm 1$ and $\Delta s_a = (\Delta \vec{s})_a = \pm 1$ for one $a \in \hat{I} - \{0\}$ (at most if $n$ is odd, exactly if $n$ is even), all remaining coordinates of $\Delta \vec{s}$ being 0 (if $n$ is odd $\Delta \vec{s} = 0$ also exists in general). When $\Delta s_a = \pm 1$ and $s_{a+1} = s_a$ the corresponding component vanishes, since the resulting weight would not respect the ordering $s_{a+1} \geq s_a$. In particular, $\mathbb{C}^n \otimes W$ always contains a component $W^\uparrow$ for which $\Delta s_{n/2} = 1$ (the maximal eigenvalue increases) and a component $W^\downarrow$ for which $\Delta s_{n/2} = -1$; this latter is nonzero only if $s_{n/2} - 1 \geq s_{n/2-1}$.

Assume now $U$ irreducible and massless and take $\varphi$ in $W(s_0, \vec{s})$. Because of (2.23) $s_0$ is related to the Casimir $C'$ of $\mathfrak{so}(n)$ by

$$ (C' - s_0^2)\varphi = \frac{n - 2}{n + 2} C\varphi \quad (2.34) $$

Let $|s_0| = \varepsilon s_0$. For $\varphi$ in $W(s_0, \vec{s})$ one has:

$$ [H_0^2, X\left(\begin{array}{c} \pm \varepsilon \\ 0 \end{array}\right)_{A'}]\varphi = (\pm 2\varepsilon s_0 + 1)X\left(\begin{array}{c} \pm \varepsilon \\ 0 \end{array}\right)_{A'}\varphi \quad (2.35) $$

On the other hand, if $\varphi$ is an extremal vector then $X\left(\begin{array}{c} \mp n/2 \\ A'' \end{array}\right)\varphi$ belongs to $W(s_0, \vec{s})^\uparrow$ (with $A'' \in \{-1, 0\} = I''$) since the maximal eigenvalue increases, so that, by (2.17):

$$ [C', X\left(\begin{array}{c} + \\ n/2 \end{array}\right)_{A''}]\varphi = (2s_{n/2} + n - 1)X\left(\begin{array}{c} + \\ n/2 \end{array}\right)_{A''}\varphi \quad (2.36) $$

Since the difference $C' - H_0^2$ is constant, these two equations imply:

$$ (s_{n/2} + n/2 - 1 \mp |s_0|)X\left(\begin{array}{c} \pm \varepsilon \\ n/2 \end{array}\right)\varphi = 0 \quad (2.37) $$

hence $X\left(\begin{array}{c} -\varepsilon \\ 0 \pm n/2 \end{array}\right)$ vanishes on $\varphi$; $X\left(\begin{array}{c} +\varepsilon \\ 0 \pm n/2 \end{array}\right)\varphi$ is an extremal vector of $W(s_0, \vec{s})^\uparrow$ and the only non-vanishing component of $\sum_{A' \in I - \{n-1,n\}} \lambda^{A'} X\left(\begin{array}{c} + \\ n/2 \end{array}\right)_{A'}\varphi$, so it is nonzero (otherwise $s_{n/2}$, hence $C'$, would be bounded and $U$ would be finite dimensional), and we get:

$$ |s_0| = s_{n/2} + n/2 - 1 \quad (2.38) $$

for every $W(s_0, \vec{s})$. It also follows that
and one can transform (2.21) to:

(2.40a) \[ X \left( \frac{n}{2} - \epsilon, 0 \right) X \left( \frac{n}{2}, 0 \right) \varphi = 4\left( \frac{2}{n+2} C - \frac{n}{2} (s_n/2 + n/2) \right) \varphi \]

(2.40b) \[ X \left( \frac{n}{2} + \epsilon, 0 \right) X \left( \frac{n}{2} - \epsilon, 0 \right) \varphi = 4\left( \frac{2}{n+2} C - (s_n/2 - 1)(s_n/2 - 1 + n/2) \right) \varphi \]

One also checks that \( X \left( \frac{n}{2} - \epsilon, 0 \right) \varphi \) is the only nonvanishing component in

\[ \sum_{A' \in I'} \lambda_A' X \left( \frac{n}{2} - \epsilon, 0 \right) \varphi \] (otherwise an eigenvalue of \( H_{n/2} \) superior to \( s_n/2 - 1 \) would appear), and it is again an extremal vector. When \( s_n/2 \) reaches its minimal value, \( s \), every \( X \left( \frac{n}{2} - \epsilon, 0 \right) \varphi \) is zero and (2.40b) gives

(2.41) \[ 2C = (n + 2)(s - 1)(s - 1 + n/2) = (n + 2). \text{ Inf} |s_0|. (\text{ Inf}|s_0| - \frac{n}{2}). \]

It follows that \(-\epsilon H_0\) has a negative maximal value equal to \(-(s - 1 + n/2)\); an extremal vector \( \varphi \) for \( s_{n/2} = s \) is an extremal vector for the whole representation space and the nilpotent subalgebra \( n^+ \) vanishes on \( \varphi \). As for the remaining coordinates of \( \vec{s} \), one easily sees that they are all equal to \( s \) (or \( -s \) for the last one for even \( n \)), and that \( s = 0 \) or \( 1/2 \) when \( n \) is odd, the proof being exactly the same as in the finite-dimensional case.

Using the notations of Theorem 2.1 and the Remark following it, one can summarize:

**Theorem 2.4:** Every infinite-dimensional irreducible massless representation of \( \mathfrak{so}(2, n) \), for \( n \geq 3 \), integrable to \( \mathcal{G}_n \), is a weight representation \( d_{n,s}(s_{n/2}, \vec{s}) \), \( D(\vec{s}) \) being itself a massless representation of \( \text{Spin}(n) \), that is \( |s_a| = s \) for every \( a \). The eigenspace of \( \epsilon H_0 \) corresponding to the eigenvalue \( (s + n/2 - 1 + k), k \in \mathbb{N} \), is an irreducible \( \mathfrak{so}(n) \)-module corresponding to the representation \( D(\vec{s} + (k, 0, \ldots, 0)) \). The values of the Casimir element \( C \) is given by (2.41).

In addition one has:

**Proposition 2.4:** The massless representations \( d_{n,s}(s_{n/2} - k, \vec{s}) \) are integrable to unitary representations of \( \mathcal{G}_n \).

**Proof:** From what precedes, every \( \mathfrak{so}(n) \)-submodule \( W_k \) has multiplicity one, it carries the representation \( D(\vec{s} + (k, 0, \ldots, 0)) \) of \( \mathfrak{so}(n) \), and the unique eigenvalue of \( \epsilon H_0 \) on it is \( s + k + \frac{1}{2} n - 1 (k \in \mathbb{N}) \). Since there is a natural \( \mathfrak{g} \)-invariant scalar product on each \( W_k \) and since \( \mathfrak{p} W_k \subset W_{k-1} \oplus W_{k+1} \), it is sufficient to show that \( ||X\varphi||^2 = q(X)||\varphi||^2 \) for every
$X \in \mathfrak{p}$ such that $[[\varepsilon H_0, X] = \pm X$, with $q(X) \geq 0$; it is clear that $q(X)$ belongs to the spectrum of $X^*X$.

There is no loss of generality in assuming that $\varphi$ is an extremal vector of $W_k$; but then $X$ must be proportional to either $X_+ = X \begin{pmatrix} n/2 & 0 \\ 0 & -\delta \end{pmatrix}$ or $X_- = X \begin{pmatrix} -n/2 & 0 \\ 0 & -\delta \end{pmatrix}$, with $(X_\pm)^* = -X_\mp$ and where $X_\pm \varphi$ is an extremal vector of $W_{k\pm 1}$; from (2.40) one sees that $X^*X$ is scalar, with

$$X_+^*X_+ = (s + k)(s + k + \frac{n}{2}) - (s - 1)(s - 1 + \frac{n}{2}) = (k + 1)(2s + k - 1 + \frac{n}{2})$$

$$X_-^*X_- = (s + k - 1)(s + k - 1 + \frac{n}{2}) - (s - 1)(s - 1 + \frac{n}{2}) = k(2s + k - 2 + \frac{n}{2})$$

and these expressions are positive for $k \in \mathbb{N}, s \geq 0$ and $n \geq 3$. □

c) Conformal imbedding of Poincaré massless representations

Having determined all possible candidates, up to equivalence, we shall now examine whether a massless representation $U$ of Poincaré extends to one of them, and how.

We shall proceed by combining the expressions of the generators of $\mathcal{P}_n$ given in (1.20) to obtain elements of the ideal $\mathcal{UF}$. One first establishes:

**Proposition 2.5**: Given the expressions (1.20) of the Poincaré generators of $U$, if $P_\mu$ is identified with $X_\mu$ (with $\mu \in J = \{0, \ldots, n-1\}$), then the dilatation operator $D = X_{\mu,-1}$, satisfying $[D, P_\mu] = P_\mu$, is given by

$$D = x_\mu \partial^{\mu'} + (n - 2)/2, \quad \mu' \in \{1, \ldots, n-1\}. \tag{2.42}$$

**Proof**: Using a summation index $\lambda \in J$, and since $g_{-1,-1} = g_{n,n} = 1$, one has:

$$F_{-1 \mu} + F_{n \mu} = (X_{-1 \lambda} + X_{n \lambda})X_\mu^\lambda + (X_{n,-1}X_{-1,\mu} - X_{-1,n}X_{n,\mu}) - \frac{1}{2}n(X_{-1,\mu} + X_{n,\mu})$$

$$= P_\lambda(X_\mu^\lambda + (D + 1 - \frac{n}{2})\delta_\mu^\lambda)$$

substituting the expressions of the generators, and putting $F_{AB} = 0$, one gets, for every $\mu$ in $J$:

$$0 = x_\mu(D + 1 - \frac{n}{2} - x_{\mu'} \partial_{\mu'}) \tag{2.44}$$

hence the result announced. □

Now, one can rewrite (2.23) as:
\[ \frac{1}{2} X_{\lambda\mu} X^{\mu\lambda} = \frac{n-2}{n+2} C + D^2 \pmod{UF} \]

and one also has

\[ \frac{1}{2} (P_{\mu} \hat{P}_\nu + P_{\nu} \hat{P}_\mu) = X_\mu \lambda X_{\lambda\nu} - \frac{1}{2} (n-2) X_{\mu\nu} - g_{\mu\nu} (D + \frac{2}{n+2} C) - F_{\mu\nu} \]

where \( \hat{P}_\nu = X_{\nu-1} - X_{\nu n} \), satisfying \([D, \hat{P}_\nu] = -\hat{P}_\nu\).

Substituting the expressions of the generators in (2.45) and (2.46), one obtains, after some calculations which we do not reproduce:

\[ C = \frac{n+2}{n-2} C'' - \frac{1}{4} (n+2)(n-2) \]

where \( C'' = \frac{1}{2} S_{ij} S^{ij} (i, j \in \{1, \ldots, n-2\}) \) is the Casimir element of the inducing representation \( S \); from (2.46) one gets expressions of the form

\[ P_{\mu} \hat{P}_\nu + P_{\nu} \hat{P}_\mu = x_\nu G_\mu + x_\mu G_\nu + E_{\mu\nu} \]

with

\[
\begin{align*}
E_{ik} &= (S_{ij} S^{kj}_k + S_{kj} S^{ij}_k) - \frac{1}{n-2} g_{ik} C''; i, j, k \in \{1, \ldots, n-2\} \\
E_{ak} &= \sigma(\alpha) x^i E_{ik} (x_0 + x_{n-1})^{-1}; \alpha \in \{0, n-1\}, \sigma(0) = -\sigma(n-1) = -1 \\
E_{\alpha\beta} &= \sigma(\alpha) \sigma(\beta) x^i x^k E_{ik} (x_0 + x_{n-1})^{-2}; \alpha, \beta \in \{0, n-1\}
\end{align*}
\]

(2.49a) \quad E_{ik} = (S_{ij} S^{kj}_k + S_{kj} S^{ij}_k) - \frac{1}{n-2} g_{ik} C''; i, j, k \in \{1, \ldots, n-2\}

(2.49b) \quad E_{ak} = \sigma(\alpha) x^i E_{ik} (x_0 + x_{n-1})^{-1}; \alpha \in \{0, n-1\}, \sigma(0) = -\sigma(n-1) = -1

(2.49c) \quad E_{\alpha\beta} = \sigma(\alpha) \sigma(\beta) x^i x^k E_{ik} (x_0 + x_{n-1})^{-2}; \alpha, \beta \in \{0, n-1\}

Since \( P_{\mu} = \sqrt{-1} x_\mu \), the consistency of the \( \frac{n(n+1)}{2} \) equations (2.48) implies that \( E_{\mu\nu} = 0 \); in particular \( E_{\delta k} = 0 \), that is \( S \) is a massless representation of the little group \( \text{Spin}(n-2) \). Carrying out the calculations, one finally obtains:

**Theorem 2.5:** A massless representation of \( \bar{\mathcal{P}}_n (n \geq 3) \) induced by the representation \( S \) of \( \text{Spin}(n-2) \), \( T_{n-2} \) (trivial on \( T_{n-2} \)) extends to a massless UIR of \( G_n \) iff \( S \) itself is massless, that is of the form \( D(s, \ldots, s, \pm s), 2s \in \mathbb{N} \), if \( n \) is even and of the form \( D(s, \ldots, s), s = 0 \) or \( \frac{1}{2}, \) if \( n \) is odd. The extension is unique, the form of the remaining generator \( s \) (of \( g_n \)) being completely determined by those of \( p_n \) in (1.20): \( X_{n-1} = D \) is given by (2.42) and \( \hat{P}_\mu \) by:

\[ \sqrt{-1} \hat{P}_\mu = x_\mu \Delta + 2(x_0 + x_{n-1})^{-1} D_\mu + 2 D \partial_\mu \]
with $\partial_0 = 0, \Delta = \sum_{j=1}^{n-1} \partial_j^2$ and:

\[(2.50a) \quad D_j = (L_0 - L_{k,n-1})S^k_j \quad (j, k \in \{1, \ldots, n-2\})\]

\[(2.50b) \quad D_{n-1} = -D_0 = \frac{1}{2} L^j k S_{kj} + s(s + \frac{1}{2}n - 2)\]

and the values of the Casimir element for the inducing and the extended representations are:

\[(2.51) \quad C'' = \frac{1}{2} (n - 2) s(s + \frac{1}{2}n - 2); \quad C = \frac{1}{2} (n + 2)(s - 1)(s + \frac{1}{2}n - 1)\]

Remark: The constraints upon $S$ are relevant for $n > 4$. Indeed, for the classical case, $n = 4$, the little group is $SO(2).T_2$, and the elements $E_{ij}$ in (2.49) are identically zero: every such representation extends to the conformal group, as shown in [2]. For $n = 3, so(n-2) = \{0\}$ and all elements $S_{ij}$ vanish; notice that $C''$ vanishes in (2.51) for $n = 3$ and for either $s = 0$ or $s = \frac{1}{2}$. However, the choice of $S$ is relevant: it corresponds to the inducing representation of $Spin(1) = \{1, -1\}$ and determines whether the center of $Spin(3) = SU(2)$ is trivially represented ($s = 0$) or not ($s = \frac{1}{2}$), the lowest $so(3)$-module occurring in the representation space having dimension $2s + 1$.

Now, for given $s$, there are two possible choices for the extension, $d^{m,\varepsilon}_{(-s+1/2n-1),\delta}$, such that the spectrum of $\varepsilon\sqrt{-1} \ X_{1,0}$ is positive, so that it remains to identify which one is obtained. We shall show:

**Proposition 2.6:** For a given sign $\varepsilon$ of $x_0 = \varepsilon|x_0|$, the representation $U^S$ of $\bar{\mathcal{P}}_n$ extends to $d^{m,\varepsilon}_{(-s+1/2n-1),\delta}$

Proof: On every $so(3)$-submodule $W_k$ the absolute value of $s_0$ is $s + k + \frac{1}{2}n - 1$, while the eigenvalues of $H_{n/2}$ run from $-(s + k)$ to $s + k$, so that the spectrum of $E = 2\sqrt{-1}(X_{1,0} + X_{n-1,n})$ has the same sign as $H_0$ and a lowest element equal, in absolute value, to $n - 2$. Substituting with the differential operators obtained one gets:

\[E = \sqrt{-1}(-\hat{P}_0 - \hat{P}_{n-1} - P_0 + P_{n-1}) = -(x_0 + x_{n-1})\Delta - 2\partial_{n-1} + (x_0 - x_{n-1}), \quad D = x_{\mu'} \partial^{\mu'} + \frac{n-2}{2}\]

Take $f \in \mathcal{H}$ so that $f$ depends only on $x_0 = \varepsilon(-x_{\mu'}x_{\mu'})^{1/2}$, and denote by $d_0$ the differential operator $\frac{d}{dx_0}$. For such an $f$ one has:

\[Ef = ((x_0 - x_{n-1})(1 - d_0^2) - (n - 2)d_0)f.\]

If $d_0^2 f = f$, that is, for example, if $f(x_0) = e^{\pm \varepsilon x_0} v$, with $v \in V$, one gets:

\[Ef = \mp \varepsilon (n - 2)f\]
Since only $e^{-\varepsilon x_0 v}$ is a square-integrable function from $\mathbb{R}^{n-1}$ to $V$, $\varepsilon E$ has a positive spectrum, and so does $\varepsilon H_0$, hence the desired result.  

Remark: When $S$ is trivial, the Fourier transform on $\mathcal{H}$ sends it on the subspace $\hat{\mathcal{H}}$ of $L^2(\mathbb{R}^{1,n-1}, d\mu)$ which is the closure of all analytic functions satisfying $\partial_\mu \partial^\mu f = 0$. The action of $\bar{G}_n/\bar{P}_n$ on $\hat{\mathcal{H}}$ is obtained from the action of dilatations and special conformal transformations on the $n$-Minkowski space. What we have shown is that, for $S$ acting on $V$, the representation $U^S$ acting on $\mathcal{H} \otimes V$ can be extended iff $S$ is massless.
3 Massless representations and the De Sitter groups

a) Subgroups of \( G_n \)

Let \( x \in \mathbb{R}^{n+2} \). If its quadratic form \( q(x) \) is positive (negative), its stabilizer \( S_n(x) \) is isomorphic to \( SO_0(1, n)/SO_0(2, n-1) \). For distinct choices of \( x \), \( S_n(x) \) and \( S_n(x') \) are conjugated subgroups iff \( q(x), q(x') > 0 \), so we shall denote them by \( S_n^\pm \) (for \( q(x) = \pm q(x) \)) and call them the \( n \)-De Sitter subgroups of real rank 1 or 2 respectively, in analogy with the classical case \( n = 4 \). Clearly, \( S_n^- = G_{n-1} \). When \( q(x) = 0 \) the stabilizer is isomorphic to \( P_n \).

We shall examine here the restriction to the twofold covering \( \tilde{S}_n^\pm \) of a massless representation \( d \) of \( G_n \), establishing that it is either irreducible, or the direct sum of two factors. Also, since \( P_n \) is a Wigner-Inonü contraction of \( S_n^\pm \), we shall establish that the restriction of \( d \) on \( \tilde{S}_n^\pm \) can be contracted to its restriction on \( P_n \).

b) Restriction of \( d_{(m, \frac{N}{2} - 1), \tilde{S}_n^\pm} \) to \( S_n^\pm \)

We have already established that the Casimir element \( C' \) of \( S_n^\pm \) is scalar and equal to \( C(N - 1)/N \), in (2.27). We shall next continue with

**Lemma 3.1:** Let \( \mathfrak{g}' \) be the Lie algebra of \( S_n^\pm \), \( \mathcal{U}' \) its enveloping algebra, and let \( e_w, w \in I \), be a basis vector stabilized by \( \mathfrak{g}' \). Let \( d \) be a massless representation of \( \mathfrak{g} \) acting on a Hilbert space \( \mathcal{H} \) and \( W' \) be a \( \mathfrak{g}' \cap \mathfrak{k} \) invariant subspace of a \( \mathfrak{k} \)-type \( W \). Let \( V_0, V_1 \) be the prehilbert spaces

\[
V_0 = d(\mathcal{U}')W'; \quad V_1 = \sum_A d(\mathcal{U}'X_{Aw})W'
\]

and \( \mathcal{H}_0, \mathcal{H}_1 \) their closures. Then either \( \mathcal{H} = \mathcal{H}_0 = \mathcal{H}_1 \) or \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \).

**Proof:** Let \( \mathfrak{r} \) the \( \mathfrak{g}' \)-invariant subspace of \( \mathfrak{g} \) spanned by the generators \( X_{Aw} \), such that \( \mathfrak{g} = \mathfrak{r} \oplus \mathfrak{g}' \). Since \( \mathcal{U}' = \oplus_{k \in \mathbb{N}} \mathcal{U}'S^k(\mathfrak{r}) \), where \( S^k(\mathfrak{r}) \) contains the fully symmetrized polynomials of degree \( k \) in the generators of \( \mathfrak{r} \), it is sufficient to show that \( d \) sends \( S^2(\mathfrak{r}) \) to \( S^0(\mathfrak{r}) = \mathbb{C} \). But one has:

\[
X_{A\prime w}X_{B\prime w} + X_{B\prime w}X_{A\prime w} = g_{w\prime w}(X_{A\prime}D'B'X_{D'B} + X_{B\prime}D'X_{D'B'}) - 2\tilde{F}_{A'B'}
\]

and since \( d(\tilde{F}_{A'B'}) = g_{A'B'}2C/N \in \mathbb{C} \), \( d \) sends \( S^2(\mathfrak{r}) \) to \( \mathcal{U}' \). \( \Box \)

Now, if \( G' = S_n^+, \mathfrak{r} = \{ \lambda^A X_{-1, A} \} \), \( \mathfrak{g}' \cap \mathfrak{k} = \mathfrak{so}(n) \), and the \( \mathfrak{k} \)-type \( W(k) \) is irreducible under the action of \( \mathfrak{g}' \cap \mathfrak{k} \). The generator \( X_{A'0} \in \mathfrak{g} \) (for \( A' \in \{ 1, \ldots, n \} \)) sends \( W(k) \) to \( W(k) \oplus W(k) \) and so does \([C', X_{A'0}] \), \( C' \) being the Casimir of \( \mathfrak{so}(n) \), so that, for every \( k \in \mathbb{N} \), there is a shift operator \( X_{A'}^+ \in d(\mathcal{U}') \), linear combination of \( X_{A'0} \) and \([C', X_{A'0}] \), sending \( W(k) \) to \( W(k) \). Every \( W(k) \) being of multiplicity one, \( d(\mathcal{U}'), \mathfrak{r} \) contains every \( \mathfrak{k} \)-type of \( d \), so that the closure of \( V_0 \) is \( \mathcal{H} \) and the restriction to \( G' \) is irreducible.

If \( G' = S_n^- \), the situation is somewhat more complicated. Let \( e_1 \) be the stabilized vector, so that \( \mathfrak{r} \) is spanned by \( \{ X_{1A} \} \), \( \mathfrak{r} \cap \mathfrak{g}' \) being isomorphic to \( \mathfrak{so}(2) \oplus \mathfrak{so}(n-1) \).
Assume first $s = 0$, so that $\mathcal{H}$ contains a trivial $\mathfrak{so}(n)$-submodule $W(0)$. Let $\varphi \in W(0)$; clearly $X_{1A'}\varphi = 0$ if $A' \in \{2, \ldots, n\}$ and $X(\tilde{\varphi})_1 \varphi = 0$ too, so that $rW(0)$ is spanned by $X(\tilde{\varphi})_1 \varphi = \varphi^\perp$. Since $\mathfrak{so}(n-1)$ commutes with $X(\tilde{\varphi})_1$, it stills act trivially on $rW(0)$, while the eigenvalue of $H_0$ increases by 1 in absolute value, so that $\mathfrak{t} \cap \mathfrak{g}'$ stabilizes $rW(0)$. Moreover, for $A' \in \{2, \ldots, n\}$,

\begin{equation}
(3.3) \quad X(\tilde{\varphi})_{A'} \varphi^\perp = (X(\delta)_1 X(\tilde{\varphi})_{A'} - [X(\delta)_1, X(\tilde{\varphi})_{A'}]) \varphi = 2\varepsilon \sqrt{-1} X_{1A'} \varphi = 0
\end{equation}

so $\varphi^\perp$ is an extremal weight vector of $\mathfrak{g}'$, as well as $\varphi$, so that $V_0 \cap V_1 = \{0\}$.

Assume next $s \neq 0$ and $n$ even ($n \geq 4$). Since $d(H_1) = \pm d(H_a)$ on an extremal vector for every $W(k)$, one has $d(\sqrt{-1} X_{12}) \varphi = \pm s \varphi \neq 0$ on an extremal vector of $W(0)$, so that

\begin{equation}
(3.4) \quad \mathcal{U}(\mathfrak{so}(n-1))((\mathfrak{t} \cap \mathfrak{g}) W(0) = \mathcal{U}(\mathfrak{so}(n-1)) W(0) = \mathcal{U}(\mathfrak{so}(n)) W(0) = W(0)
\end{equation}

and $V_0 = V_1$.

Assume finally $n$ odd and $s = \frac{1}{2}$. The lowest $\mathfrak{so}(n)$-type is a spinorial representation, and it is well known that such a representation of $\mathfrak{so}(2r+1)(r \in \mathbb{N})$ splits into two inequivalent spinorial representations of $\mathfrak{so}(2r)$ of equal dimensions; they are labelled $D(\frac{1}{2}, \ldots, \frac{1}{2}, \pm \frac{1}{2})$ with the two different choices of sign.

Summarizing one has:

**Proposition 3.1:** The representation $d^{n,\varepsilon}_{-(s+\frac{n}{2}-1,\bar{s})}$ remains irreducible when restricted to $SO_0(1, n)$. Its restriction on $SO_0(2, n-1)$ when $s = 0$ is the direct sum

\[d^{n-1,\varepsilon}_{-(\frac{n}{2}-1,\bar{0})} \oplus d^{n-1,\varepsilon}_{-(\frac{n}{2},\bar{0})}\]

for $s = \frac{1}{2}$ and $n = 2r + 1$ odd, its restriction is

\[d^{2r,\varepsilon}_{-(r,\frac{1}{2},\ldots,\frac{1}{2},\frac{1}{2})} \oplus d^{2r,\varepsilon}_{-(r,\frac{1}{2},\ldots,\frac{1}{2},\frac{1}{2})}\]

for $s \neq 0$ and $n = 2r$ even, the restriction is irreducible and equal to

\[d^{2r-1,\varepsilon}_{-(s+r-1,\bar{s})}\]

where $\bar{s}$ comes from $\bar{s} = (s, \ldots, s, \pm s)$ by dropping the last coordinate $\pm s$.  

**c. Contraction of representations**

The Wigner-Inonü contraction of Lie algebras $[13]$ can be defined as follows: given a Lie algebra $\mathfrak{g}$ and a continuous family $\Phi_\alpha \in GL(\mathfrak{g})$ of linear transformations of the underlying vector space, with $0 < \alpha \leq 1$ and $\Phi_1 = 1$, a Lie algebra $\mathfrak{g}_\alpha$ isomorphic to $\mathfrak{g}$ is defined on the same underlying space by the Lie bracket:
(3.5) \[ [X,Y]_\alpha = \Phi_\alpha^{-1} [\Phi_\alpha X, \Phi_\alpha Y]. \]

If \( \lim_{\alpha \to 0} (\Phi_\alpha) \) is a non invertible mapping and \( [X,Y]_0 = \lim ([X,Y]_\alpha) \) exists when \( \alpha \to 0 \), the Lie algebra \( \mathfrak{g}_0 \) defined on the same underlying space is the contracted of \( \mathfrak{g} \) by the family \( \{ \Phi_\alpha \} \).

Contraction of representations \( U_\alpha \) of \( \mathfrak{g}_\alpha \) on \( \mathcal{H}_\alpha \) are defined in analogy. Here we shall limit ourselves to a fixed representation space \( \mathcal{H} \). Given a continuous family \( \{ Z_\alpha \} \) of closed invertible linear transformations of \( \mathcal{H} \) for \( 0 < \alpha \leq 1 \) with \( Z_1 = 1 \), and a representation \( U_1 = U \) of \( \mathfrak{g}_1 = \mathfrak{g} \), defined on a dense domain \( \mathcal{E} \) of analytic vectors, the map

(3.6) \[ X \mapsto U_\alpha(X) = Z_\alpha^{-1} U(\Phi_\alpha X) Z_\alpha \]

is a representation of \( \mathfrak{g}_\alpha \); indeed, one has:

(3.7) \[
[U_\alpha(X), U_\alpha(Y)] = Z_\alpha^{-1} [U(\Phi_\alpha X), U(\Phi_\alpha Y)] Z_\alpha = Z_\alpha^{-1} U([\Phi_\alpha X, \Phi_\alpha Y]) Z_\alpha = Z_\alpha^{-1} U(\Phi_\alpha [X,Y]_\alpha) Z_\alpha = U_\alpha([X,Y]_\alpha).
\]

If the limit of \( U_\alpha(X) \) exists for every \( X \in \mathfrak{g} \) when \( \alpha \to 0 \) (regardless to whether \( Z_\alpha \) has a limit), then \( U_0 = \lim_{\alpha \to 0} U_\alpha \) is a representation of the contracted Lie algebra \( \mathfrak{g}_0 \); we shall say that it is the contracted of \( U_1 \) through the family \( \{ Z_\alpha \} \).

Let us apply this to \( \mathfrak{g}_1 = \text{Lie}(S^+_{\pm n}) = \mathfrak{sl}_n \oplus \mathfrak{h} \) where \( \mathfrak{h} \) is spanned by the generators \( Y_\mu = X_{\mu \nu}, \mu \in J \), with \( w = n \) for \( S^+_{\pm n} \) and \( w = -1 \) for \( S^-_{\pm n} \); one has \( [\mathfrak{h}, \mathfrak{h}] = \mathfrak{sl}_n \). We shall define the familly \( \{ \Phi_\alpha \} \) by:

(3.8) \[ \Phi_\alpha(X_{\mu \nu}) = X_{\mu \nu}; \quad \Phi_\alpha(Y_\mu) = \alpha(2 - \alpha)Y_\mu. \]

Clearly, one has

(3.9) \[ [Y_\mu, Y_\nu]_\alpha = \alpha^2(2 - \alpha)^2 [Y_\mu, Y_\nu]; \quad [X_{\mu \nu}, Y_\lambda]_\alpha = [X_{\mu \nu}, Y_\lambda]. \]

so that the contacted algebra \( \mathfrak{g}_0 \) is isomorphic to \( \mathfrak{p}_n \).

Let now \( d \) be a massless representation of \( \bar{G}_n \) on \( \mathcal{H} \), with analytic domain \( \mathcal{E} \), on which all operators of the Lie algebra are defined, their expressions being given by (1.20) (in particular \( d(P_\mu) = \sqrt{-1} x_\mu \)) and (2.50).

Let \( U = U_1 \) be the restriction of \( d \) to \( \mathfrak{g}_1 \), so that one has
Define the family \{Z_\alpha\} by

\[(3.11)\quad (Z_\alpha \varphi)(x) = \alpha^{(n-2)/2} \varphi(\alpha x)\]

\(Z_\alpha\) is a unitary operator, equal to \(\exp(d(\text{Log} \alpha X_n, -1))\), which has no limit for \(\alpha \to 0\). It satisfies:

\[(3.12)\quad Z_\alpha^{-1}d(P_\mu)Z_\alpha = \alpha^{-1}d(P_\mu); \quad Z_\alpha^{-1}d(\hat{P}_\mu)Z_\alpha = \alpha d(\hat{P}_\mu)\]

so that:

\[(3.13)\quad U_\alpha(Y_\mu) = (1 - \frac{\alpha}{2})(d(P_\mu) \mp \alpha^2 d(\hat{P}_\mu))\]

while \(U_\alpha(X_{\mu\nu}) = U(X_{\mu\nu})\). It is clear that the limit of \(U_\alpha(Y_\mu)\) exists for \(\alpha \to 0\), and it is equal to \(d(\hat{P}_\mu)\). We have thus proved:

**Proposition 3.2:** The restriction \(U\) on \(\mathcal{S}_n^\pm\) of the massless representation \(d\) of \(\hat{G}_n\) contracts to its restriction on \(\mathcal{P}_n\) through the family of unitary operators \(\{Z_\alpha\}\). \(\Box\)
4 Conclusion

Comparing the results obtained here with the classical case $n = 4$, we first observe that the main features are conserved: only massless representations $U^S$ of $\mathcal{P}_n$ can be extended to ones of $\mathcal{G}_n$, and when this is possible the extension $d$ is unique: it is a unitary irreducible representation with extremal weight, vanishing on the two-sided ideal of the enveloping algebra generated by $P_\mu P^\mu$. The form of the remaining Lie algebra generators is completely determined when those of $\mathfrak{p}_n$ are given (that is, $d$ is not only fixed up to equivalence, but when $U^S$ is fixed inside its equivalence class so is $d$).

Moreover, $d$ is a representation of either $G_n$ itself (when $U^S$ is one of $\mathcal{P}_n$, that is $s$ integer) or of a twofold covering (when $s$ is half-integer). All representations $d$ are realizable on a functional space over the corresponding Minkowski space or over a half-cone of its Fourier dual (in fact, when $S$ is trivial $d$ is equivalent to the representation induced by the trivial representation of the parabolic subgroup $\mathcal{W}_n$).

The only feature which does not generalize concerns the restrictions imposed on the inducing representation $S$. For $n = 4$, the only restriction is that $S$ is trivial on the translation subgroup $T_2$ of the twofold covering of the Euclidean group $E_2$. This discards the so-called continuous spin representations and allows all helicities $\pm s \in \frac{1}{2} \mathbb{Z}$.

For $n > 4$, $S$ must still vanish on the translations, but there are additional constraints on $S$, depending on the parity of $n$: if $n = 2r$ is even every coordinate of the extremal weight must equal in absolute value to the last one (the minimal one), which is equal to $\pm s$. This constraint is automatically satisfied for $n = 4$, since $\mathfrak{so}(n - 2)$ has rank 1 and the last coordinate of the weight is also the only one: the study of the case $n = 4$ alone gives no hint about this new constraint.

For odd $n$ the constraints are quite drastic: $S$ may be either trivial or spinorial. This appears as a straightforward generalization of the case $n = 3$ [5], if one defines $\text{Spin}(1)$ as $\mathbb{Z}_2$.

If, instead of increasing $n$, one decreases it to $n = 2$, one finds again that the only UIR of the simply connected $\mathcal{P}_2 = SO_0(1, 1).T_2$ (besides the trivial one) which extend to $\mathcal{G}_2 = SO_0(2, 2)$ are the massless ones: the massless orbits are the connected components of the isotropic cone, that is, 4 half lines (instead of two half ones) and the stabilizer is just $\{1\}$, so that massless UIR vanish on the subgroup spanned by $P_0 + P_1$ or $P_0 - P_1$, the factor group on which they are faithful being here isomorphic to the connected affine group ($x \mapsto ax + b$) of the real line. By theorem 2.2, the extension to $\mathfrak{so}(2, 2) = \mathfrak{u}^+ \oplus \mathfrak{u}^-$ must vanish on one of the two factors $\mathfrak{u}^\pm$ (both isomorphic to $\mathfrak{so}(2, 1)$). The Casimir operator may take any value $C$ and by (2.46) one gets $\hat{P}_0 = (P_0)^{-1}(D^2 - D + \frac{1}{2} C)$. There is no uniqueness of the extension, not even unitarity ($C$ may be any complex number): lowering $n$ to 2 removes all constraints. One should however mention that in this case the full conformal group is infinite, as is well-known. We shall not discuss this case further here.

Concerning the Poincaré - De Sitter relations, the sequence “extension to $\mathcal{G}_n$, then
restriction to $S^+_n$, then contraction to $\overline{F}_n$ is cyclic for every $n \geq 3$; the demonstration is practically identical with the one for $n = 4$ [2]. As for the irreducibility of the restriction to the real rank two De Sitter subgroup $S^-_n$, the result for $n$ even is a straightforward generalization of the case $n = 4$: the restriction splits into two simple factors if the inducing representation is trivial, otherwise it is irreducible. When $n$ is odd, it splits into two simple factors for both $s = 0$ and $s = \frac{1}{2}$. The restriction on $\overline{S}^+_n = \overline{SO}_0(1, n)$ is always irreducible.

Finally we should mention that there are some interesting open problems involving massless representations, such as their tensor products with other representations (in particular their tensor squares) or their appearance as factors in indecomposable representations.
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