Quantum random walks and vanishing of the second Hochschild cohomology
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Abstract
Given a conditionally completely positive map $L$ on a unital $*$-algebra $A$, we find an interesting connection between the second Hochschild cohomology of $A$ with coefficients in the bimodule $E_L = B^a(A \oplus M)$ of adjointable maps, where $M$ is the GNS bimodule of $L$, and the possibility of constructing a quantum random walk (in the sense of [2, 11, 13, 16]) corresponding to $L$.

1 Introduction
Quantum dynamical semigroups (QDS for short), which are $C_0$- semigroups of completely positive, contractive maps on $C^*$ or von Neumann algebras (with appropriate continuity assumptions), are interesting and important objects of study both from physical as well as mathematical viewpoints. A very useful tool for understanding such semigroups is Evans-Hudson dilation (EH dilation for short). By an E-H dilation of a QDS $(T_t)_{t \geq 0}$ on a von Neumann algebra $A \subseteq B(h)$, we mean a family $j_t$ of normal $*$-homomorphism from $A$ into $A \otimes B(h \otimes \Gamma(L^2(R_+, k)))$, where $k$ is a Hilbert space, and $j_t$ satisfies a quantum stochastic differential equation of the form

$$dj_t(x) = j_t(\theta_t^a(x)), \quad j_0(x) = x \otimes I,$$

for $x$ belonging to a suitable dense $*$-subalgebra on which a family of linear maps $\theta_t^a$ are defined, and $\theta_0^a$ coincides with the generator of $T_t$. For more details of this concept, we refer the reader to the books [12, 7] and references therein. While there is a complete theory of such dilations for semigroups with norm-bounded generator (i.e. uniformly continuous semigroups), there is hardly any hope for a general theory for an arbitrary QDS. Nevertheless, there have been several attempts to construct EH dilation for different classes of QDS with unbounded generator. Moreover, there are more than one constructions of the family $j_t$ for a QDS with bounded generator. In addition to the traditional approach by iteration, there is a very interesting

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construction (see [13, 14, 4]) of EH dilation as a strong limit of a sequence of homomorphism which can be thought of as ‘quantum random walk’. It should be mentioned that for building a satisfactory general theory of EH dilation covering a reasonably large class of QDS with unbounded generator, it is absolutely crucial to deeply look into all the different approaches available in the bounded generator case, and to see whether some of them, or a suitable combination of them, can be generalized to cover QDS with unbounded generator. Indeed, the approach through quantum random walk seems to have a great promise in this context. However, there are two issues involved in this approach: first, to construct a quantum random walk for a given QDS (possibly with unbounded generator), and then to see whether it converges strongly. In the present article we study some algebraic conditions for the possibility of constructing a quantum random walk in the general situation. We work in a purely algebraic setting, and are able to discover a very interesting connection between the algebraic relations satisfied by components of a quantum random walk (if it exists) and the second Hochschild cohomology of the algebra with coefficient in a module naturally associated with the CCP generator of the given QDS. We leave the study of analytic aspects of our results for later work. It may be remarked here that the first and second Hochschild cohomologies did appear in several other works on QDS and quantum probabilistic dilation, for example the celebrated work of Christensen and Evans ([5]), and also in work of Hudson ([10]). However, none of those works are concerned with the quantum random walks and do not have any overlap with the results obtained in the present article.

2 Notations and Preliminaries

Quantum Random Walk

Let $\mathcal{K} = L^2(\mathbb{R}_+, k)$ where $k$ is a Hilbert space and let $\Gamma$ be the symmetric Fock space $\Gamma(\mathcal{K})$ over $\mathcal{K}$. For any partition $S \equiv (0 = t_0 < t_1 < t_2 \cdots)$ of $\mathbb{R}_+$, $\mathcal{K} = \bigoplus_{n \geq 1} \mathcal{K}_n$, where $\mathcal{K}_n$ is the range of projection $1_{(t_{n-1}, t_n]}$ and the Fock space $\Gamma$ can be viewed as the infinite tensor product $\bigotimes_{n \geq 1} \Gamma_n$ of symmetric Fock spaces $\{\Gamma_n = \Gamma(\mathcal{K}_n)\}_{n \geq 1}$ with respect to the stabilizing sequence $\Omega = \{\Omega_n : n \geq 1\}$, where $\Omega_n = \Omega(t_{n-1}, t_n]$ is the vacuum vector in $\Gamma_n$. Let denote the interval $(t_{n-1}, t_n]$ by $[n]$ and the orthogonal projection of $\Gamma_n$ onto the $m$-particle space by $P_m[n]$.

For $n \geq 1$, consider the subspace $\mathbf{k}_n = \mathbb{C} \Omega_n \oplus \mathbf{k}_n$ of $\Gamma$, where $\mathbf{k}_n = \{1_{[n]} \phi : \phi \in k\}$. The spaces $\mathbf{k}_n$, are isomorphic with $\mathbf{k} := \mathbb{C} \oplus k$.

**Definition 2.1.** The toy Fock space associated with the partition $S$ of $\mathbb{R}_+$ is defined to be the subspace $\Gamma(S) := \bigotimes_{n \geq 1} \mathbf{k}_n$ with respect to the stabilizing sequence $(\Omega_n)_{n \geq 1}$.

Let $P(S)$ be the orthogonal projection of $\Gamma$ onto the toy Fock space $\Gamma(S)$. Now onwards let us consider toy Fock space $\Gamma(S_h)$ associated with regular partition $S_h \equiv$
($0, h, \cdots$) for some $h > 0$ and denote the orthogonal projection by $P_h$. Denoting the restriction of orthogonal projection $P_h$ to $\Gamma_n$ by $P_h[n], P_h = \bigotimes_{n \geq 1} P_h[n]$.

Now we define basic operators associated with a toy Fock space $\Gamma(\mathfrak{h})$ as follows, for $0 \leq h$, the annihilation process $P_h$ define basic operators associated with toy Fock space $\Gamma(\mathfrak{h})$.

For $S \in \mathcal{B}(\mathfrak{h}), R \in \mathcal{B}(\mathfrak{h} \otimes \mathfrak{k}), Q \in \mathcal{B}(\mathfrak{h} \otimes \mathfrak{k}, \mathfrak{h})$ and $T \in \mathcal{B}(\mathfrak{h} \otimes \mathfrak{k})$ let us define four basic operators on $\Gamma$ as follows, for $n \geq 1,

\begin{align*}
N^0_S[n] &= SP_0[n], \\
N^0_Q[n] &= \frac{a_Q[n]}{\sqrt{h}} P_1[n], \\
N^1_R[n] &= P_1[n] \frac{a_R[n]}{\sqrt{h}}, \\
N^1_T[n] &= P_1[n] (\lambda_T[n] P_1[n] P_h[n].
\end{align*}

(2.1)

Here all these operators act nontrivially only on $\Gamma_n$. For definition of coordinate-free fundamental processes $\Lambda$’s we refer to [8]. Here, we note that in the notation of [8], the annihilation process $P_h$ appear above is $a_Q[n]$. All these maps $\mathcal{B}(\mathfrak{h}) \ni S \mapsto N^0_S[n], \mathcal{B}(\mathfrak{h} \otimes \mathfrak{k}, \mathfrak{h}) \ni Q \mapsto N^0_Q[n], \mathcal{B}(\mathfrak{h} \otimes \mathfrak{k}) \ni R \mapsto N^1_R[n]$ and $\mathcal{B}(\mathfrak{h} \otimes \mathfrak{k}) \ni T \mapsto N^1_T[n]$ are linear. It is clear that these operators $N$’s are bounded and leave the subspace $\Gamma(S_h)$ invariant. It can be shown that (for detail see [14]):

- $(N^\mu_X[n])* = N^\nu_X[n], \forall \mu, \nu \in \{0, 1\}$
- $N^0_S[n] + N^1_{S \otimes 1_k}[n] = S \otimes P_h[n]$
- $N^\mu_X[n] N^\nu_Y[n] = \delta^\mu_\nu N^0_XY[n]$, where $\delta^\mu_\nu$ is Dirac delta function of $\eta$ and $\nu$.

Let $\mathcal{A}$ be a unital *-subalgebra of $\mathcal{B}(\mathfrak{h})$. Suppose we are given with a family of *-homomorphisms $\{\beta(h)\}_{h > 0}$ from $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{B}(\mathfrak{k})$. It can be written that $\beta(h) = \begin{pmatrix} \beta_{00}(h) & \beta_{01}(h) \\ \beta_{10}(h) & \beta_{11}(h) \end{pmatrix}$, where the components $\beta_{00}(h) \in \mathcal{B}(\mathcal{A}), \beta_{11}(h) \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes \mathcal{B}(\mathfrak{k}))$ and $\beta_{10}(h) \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes \mathfrak{k})$ such that

$$
\beta_{\mu\nu}(h)(x) = (\beta_{\mu\nu}(h)(x))^*,
$$

$$
\beta_{\mu\nu}(h)(xy) = \sum_{\eta=0}^{1} \beta_{\mu\eta}(h)(x) \beta_{\eta\nu}(h)(y).
$$

Let us define a family of maps $\{\mathcal{P}_{t}^{(h)} : \mathcal{A} \otimes \mathcal{E}(\mathcal{K}) \to \mathcal{A} \otimes \Gamma\}_{t \geq 0}$ as follows. First subdivide the interval $[0, t]$ into $[k] \equiv ((k - 1)h, kh], 1 \leq k \leq n$ so that $t \in ((n - 1)h, nh]$.
and set for \( x \in \mathcal{A}, \ f \in \mathcal{K} \)
\[
\begin{align*}
\mathcal{P}_0^{(h)}(xe(f)) &= xe(f) \\
\mathcal{P}_k^{(h)}(xe(f)) &= \sum_{\mu, \nu=0}^{1} \beta_{\mu, \nu}(h, x) [k]e(f)
\end{align*}
\]
and \( \mathcal{P}_k^{(h)} = \mathcal{P}_{nh}^{(h)} \).

Setting \( p_t^{(h)}(x)ue(f) := \mathcal{P}_t^{(h)}(xe(f))u, \forall u \in \mathcal{h} \), by the properties of the family \( \{\beta_{\mu, \nu}(h)\} \) and \( \{N^{\mu \nu}[k]\} \), \( p_t^{(h)} \) are \(*\)-homomorphism from \( \mathcal{A} \) into \( \mathcal{A} \otimes \mathcal{B}(\Gamma) \).

**Definition 2.2.** This family of \(*\)-homomorphisms \( \{p_t^{(h)}: t \geq 0\} \) is called a quantum random walk (QRW) associated with \( \beta(h) \).

**Hochschild cohomology**

Let us recall the definition of the Hochschild cohomology \( H^n(\mathcal{A}, N) \) for \( \mathcal{A} \) with coefficients in an \( \mathcal{A} \)-\( \mathcal{A} \) bimodule \( N \) (for detail we refer to [15]). It is the cohomology of the cochain complex \( \mathcal{C}^n \equiv \mathcal{C}^{n}(\mathcal{A}, N, b) \), \( n \geq 0 \), where \( \mathcal{C}^0 = N \), and for \( n \geq 1 \), \( \mathcal{C}^n \) consists of all multi-\( \mathcal{C} \)-linear maps \( f : \mathcal{A} \times \cdots \mathcal{A} \ (n \text{ copies}) \rightarrow N \), with the coboundary map \( b \) given by
\[
b f(a_0, a_1, \cdots, a_n) := a_0 f(a_1, \cdots, a_n)
+ \sum_{i=0}^{n-1} (-1)^{i+1} f(a_0, \cdots, a_{i-1}, a_i a_{i+1}, \cdots, a_n) + (-1)^{n+1} a_n f(a_0, \cdots, a_n).
\]

Let us introduce one more notation. Let \( \mathcal{C}[[t]] \) denote the ring of formal power series in one indeterminate \( t \) with coefficients in a ring \( \mathcal{C} \). If \( \mathcal{C} \) is a \(*\)-algebra, so is \( \mathcal{C}[[t]] \).

### 3 Main results

Let \( \mathcal{A} \) be a unital \(*\)-subalgebra of \( \mathcal{B}(\mathcal{h}) \) and \( \mathcal{L} \) be a conditionally completely positive (CCP) map from \( \mathcal{A} \) into itself, satisfying \( \mathcal{L}(1) = 0 \). Then there exist a canonical (unique upto isomorphism) pre-Hilbert \( \mathcal{A} \)-\( \mathcal{A} \) bimodule \( M \), with the left action denoted by \( \pi \) (can also be viewed as a \(*\)-representation of \( \mathcal{A} \) into the algebra \( \mathcal{B}^a(M) \) of adjointable maps on \( M \)), and a bimodule-derivation \( \delta : \mathcal{A} \rightarrow M \), such that \( M \) coincides with the right \( \mathcal{A} \)-linear span of \( \delta(\mathcal{A}) \). Note that we can identify an element \( \xi \in M \) with the rank-one map \( \xi^* \equiv <\xi, \cdot > : M \rightarrow \mathcal{A} \) given by \( M \ni \eta \mapsto <\xi, \eta> \in \mathcal{A} \), where \( <\cdot, \cdot> \) denotes the \( \mathcal{A} \)-valued inner product on \( M \). We have
\[
\mathcal{L}(xy) - x\mathcal{L}(y) - \mathcal{L}(x)y = \delta^\dagger(x)\delta(y), \ \forall x, y \in \mathcal{A},
\]
where \( \psi^\dagger \) for a linear map \( \psi \) on \( \mathcal{A} \) is defined as \( \psi^\dagger(x) := (\psi(x^*))^* \).
When $A$ is a von Neumann algebra and $L$ is norm-bounded then one can imbed $M$ in a Hilbert von Neumann module of the form $A \otimes k \subseteq (B(h, h \otimes k))$ for some Hilbert space $k$, and show that $\delta, \delta^1, \pi$ are all bounded maps. Furthermore, using the explicit structure of $L$ as obtained from the Christensen-Evans Theorem (ref. [5]) one can construct (see [4, 14]) a family of $*$-homomorphism $\{\beta(h) : A \rightarrow A \otimes B(k) : h > 0\}$ such that $\beta(h) = \begin{pmatrix} \beta_{00}(h) & \beta_{01}(h) \\ \beta_{10}(h) & \beta_{11}(h) \end{pmatrix}$, where

- $\beta_{00}(h) = \sum_{n \geq 0} h^n \theta_{00}^{(n)}$ with $\theta_{00}^{(0)}(x) = x, \theta_{00}^{(1)}(x) = \theta_{00}(x)$
- $\beta_{10}(h) = \sum_{n \geq 1} h^{2n-1} \theta_{10}^{(n)}$ with $\theta_{10}^{(1)}(x) = \delta(x)$,
- $\beta_{11}(h) = \sum_{n \geq 1} h^n \theta_{11}^{(n)}$ with $\theta_{11}^{(1)}(x) = \pi(x)$.

Using this, an EH flow for the QDS generated by $L$ can be constructed (see [4, 14]) as strong limit of quantum random walks discussed in the previous section.

However, in this paper we concentrate on the purely algebraic aspect of such construction only and make the interesting observation that this is related intimately to the vanishing of second Hochschild cohomology of $A$.

Now, for a purely algebraic treatment, let us fix a $*$-algebra $A$, CCP map $L$ as in the beginning, and the bimodule $M$ and the derivation $\delta$ as mentioned before (not assumed to be bounded in any sense). Let us also consider the pre-Hilbert $A$-$A$ bimodule $E_L := B^a(M)$ $\equiv B^a(A \oplus M)$, with the bimodule actions given by $x.R = \tilde{\pi}(x)R$ and $R.x = R\tilde{\pi}(x)$ where $\tilde{\pi}(x) = x \oplus \pi(x)$. Let us denote by $M^*$ the submodule of $E_L$ consisting of $\xi^*$, $\xi \in M$. It is clear that $M, M^*, A$ and $B^a(M)$ are canonically imbedded as complemented submodules of $E_L$ and in fact, $E_L$ is the direct sum of these four submodules. Any element $X$ of $E_L$ can be written as a $2 \times 2$ matrix form

\[ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \]

where $X_{11} \in A$, $X_{12} \in M^*$, $X_{21} \in M$ and $X_{22} \in B^a(M)$.

**Theorem 3.1.** If $H^2(A, E_L) = 0$ then there exists a $*$-homomorphism $\beta : A \rightarrow E_L[[t]]$ such that $\beta(t) = \begin{pmatrix} \beta_{00}(h) & \beta_{01}(h) \\ \beta_{10}(h) & \beta_{11}(h) \end{pmatrix}$, where $h = t^2$ and

- $\beta_{00}(h) = \sum_{n \geq 0} h^n \theta_{00}^{(n)}$ with $\theta_{00}^{(0)}(x) = x, \theta_{00}^{(1)}(x) = \theta_{00}(x)$
- $\beta_{10}(h) = \sum_{n \geq 1} h^{2n-1} \theta_{10}^{(n)}$ with $\theta_{10}^{(1)}(x) = \delta(x)$,
• \( \beta_{10}(h) = \sum_{n \geq 1} h^{\frac{2n+1}{2}} \theta_{01}^{(n)} \) with \( \theta_{01}^{(1)}(x) = \delta(x) \),

• \( \beta_{11}(h) = \sum_{n \geq 1} h^{n-1} \theta_{11}^{(n)} \) with \( \theta_{11}^{(1)}(x) = \pi(x) \).

**Proof.** First of all we note that \( H^2(A, N) = 0 \) for any complemented submodule \( N \) of \( E_L \), for example, for \( N = M, M^*, A, B^0(M) \). Moreover, we shall view any map from some module to any such submodule \( N \) of \( E_L \) as a map into \( E_L \). Also, it is easy to verify that the \( * \)-homomorphic property of \( \beta \) is equivalent to

\[
\beta_{\mu \nu}(h)(x^*) = (\beta_{\nu \mu}(h)(x))^*,
\]

\[
\beta_{\mu \nu}(h)(xy) = \sum_{n=0}^{1} \beta_{\mu \nu}(h)(x) \beta_{\nu \mu}(h)(y).
\]

To prove existence and \( * \)-homomorphic properties of \( \beta \), by induction, we shall show the existence of maps \( \theta_{\mu \nu}^{(n)} \in C^1(A, E_L) \) satisfying

\[
\theta_{11}^{(n)}(xy) = \sum_{k=1}^{n-1} \theta_{01}^{(k)}(x) \theta_{01}^{(n-k)}(y) + \sum_{k=1}^{n} \theta_{11}^{(k)}(x) \theta_{11}^{(n-k+1)}(y)
\]

\[
\theta_{10}^{(n)}(xy) = \sum_{k=1}^{n} \theta_{00}^{(k)}(x) \theta_{00}^{(n-k)}(y) + \sum_{k=1}^{n} \theta_{10}^{(k)}(x) \theta_{01}^{(n-k+1)}(y)
\]

\[
\theta_{01}^{(n)}(xy) = \sum_{k=0}^{n-1} \theta_{00}^{(k)}(x) \theta_{00}^{(n-k)}(y) + \sum_{k=1}^{n} \theta_{01}^{(k)}(x) \theta_{01}^{(n-k+1)}(y)
\]

\[
\theta_{00}^{(n)}(xy) = \sum_{k=0}^{n} \theta_{00}^{(k)}(x) \theta_{00}^{(n-k)}(y) + \sum_{k=1}^{n} \theta_{01}^{(k)}(x) \theta_{01}^{(n-k+1)}(y)
\]

\[
\theta_{\mu \nu}^{(n)}(x^*) = (\theta_{\nu \mu}^{(n)}(x))^*.
\]

First, let us consider the following elements of \( C^2(A, E_L) \) and \( C^1(A, E_L) \)

• \( \phi_{11}^{(2)}(x, y) := \theta_{10}^{(1)}(x) \theta_{01}^{(1)}(y) \).

As \( \partial \theta_{10}^{(1)}(x, y) = 0, \partial \theta_{01}^{(1)}(x, y) = 0 \) we have \( \partial \phi_{11}^{(2)}(x, y, z) = 0 \). Now since \( H^2(A, E_L) = 0 \), there exists a map, say \( \phi_{11}^{(2)} \in C^1(A, E_L) \) such that \( \partial \phi_{11}^{(2)} = \phi_{11}^{(2)} \).

Since we have \( \phi_{01}^{(1)} = \phi_{10}^{(1)} \), it is easy to see that \( (\phi_{11}^{(2)}(x^*, x^*))^* = \phi_{11}^{(2)}(x, y) \), so \( \partial \phi_{11}^{(2)} = \phi_{11}^{(2)} \). Thus, taking \( \gamma = \frac{1}{2}(\phi_{11}^{(2)} + \phi_{11}^{(2)}) \), we have \( \partial \gamma = \phi_{11}^{(2)} \) and \( \gamma^\dagger = \gamma \). By replacing \( \theta_{11}^{(2)} \) by \( \gamma \), we can assume without loss of generality that \( \theta_{11}^{(2)}(x^*)^* = \theta_{11}^{(2)}(x) \).

• \( \phi_{10}^{(2)}(x, y) := \theta_{10}^{(1)}(x) \theta_{00}^{(1)}(y) + \theta_{11}^{(1)}(x) \theta_{01}^{(1)}(y) \) Now

\[
\partial \phi_{10}^{(2)}(x, y, z)
\]

\[
\begin{align*}
= & \theta_{10}^{(1)}(x) \partial \theta_{00}^{(1)}(y, z) - \partial \theta_{10}^{(1)}(x, y) \theta_{00}^{(1)}(z) + \theta_{11}^{(1)}(x) \partial \theta_{01}^{(1)}(y, z) - \partial \theta_{11}^{(1)}(x, y) \theta_{10}^{(1)}(z) \\
= & \theta_{10}^{(1)}(x) \theta_{01}^{(1)}(y) \theta_{10}^{(1)}(z) - 0 + 0 - \theta_{10}^{(1)}(x) \theta_{01}^{(1)}(y) \theta_{10}^{(1)}(z) = 0.
\end{align*}
\]
Since \( H^2(A, E_L) = 0 \), there exists a map, say \( \theta^{(2)}_{10} \in C^1(A, E_L) \) such that 
\[ \partial \theta^{(2)}_{10} = \phi^{(2)}_{10}. \]

Now define \( \theta^{(2)}_{01}(x) := (\theta^{(2)}_{10}(x^*))^* \). Then

\[
\phi^{(2)}_{01}(x, y) := \partial \theta^{(2)}_{01}(x, y) = \theta^{(2)}_{01}(xy) - x\theta^{(2)}_{01}(y) - \theta^{(2)}_{01}(x)\pi(y) \\
= \{\theta^{(2)}_{10}(y^*x^*) - \theta^{(2)}_{10}(y^*)x^* - \pi(y^*)\theta^{(2)}_{10}(x^*)\pi(y)\}^* = \{\partial \theta^{(2)}_{10}(y^*, x^*)\}^* \\
= \{\theta^{(1)}_{10}(y^*)\theta^{(1)}_{00}(x^*) + \theta^{(2)}_{11}(y^*)\theta^{(1)}_{11}(x^*)\}^* \\
= \theta^{(1)}_{00}(x)\theta^{(1)}_{00}(y) + \theta^{(1)}_{01}(x)\theta^{(1)}_{10}(y).
\]

\[ \phi^{(2)}_{00}(x, y) := \theta^{(1)}_{00}(x)\theta^{(1)}_{00}(y) + \theta^{(1)}_{01}(x)\theta^{(1)}_{10}(y) + \theta^{(2)}_{01}(x)\theta^{(1)}_{10}(y). \]

\[
\begin{align*}
\partial \phi^{(2)}_{00}(x, y, z) \\
&= \theta^{(1)}_{00}(x)\partial \theta^{(1)}_{00}(y, z) - \partial \theta^{(1)}_{00}(x, y)\theta^{(1)}_{00}(z) \\
&+ \theta^{(1)}_{01}(x)\partial \theta^{(1)}_{10}(y, z) - \partial \theta^{(1)}_{01}(x, y)\theta^{(1)}_{10}(z) \\
&+ \theta^{(2)}_{01}(x)\partial \theta^{(1)}_{10}(y, z) - \partial \theta^{(2)}_{01}(x, y)\theta^{(1)}_{10}(z) \\
&= \theta^{(1)}_{00}(x)\theta^{(1)}_{00}(y)\theta^{(1)}_{10}(z) - \theta^{(1)}_{01}(x)\theta^{(1)}_{10}(y)\theta^{(1)}_{10}(z) \\
&+ \theta^{(1)}_{01}(x)\theta^{(1)}_{00}(y)\theta^{(1)}_{00}(z) + \theta^{(1)}_{11}(y)\theta^{(1)}_{10}(z) \\
&+ 0 - \theta^{(1)}_{00}(x)\theta^{(1)}_{01}(y) + \theta^{(1)}_{01}(x)\theta^{(1)}_{10}(y)\theta^{(1)}_{10}(z) \\
&= 0.
\end{align*}
\]

Since \( H^2(A, E_L) = 0 \), there exists a map, say \( \theta^{(2)}_{00} \in C^1(A, E_L) \) such that 
\[ \partial \theta^{(2)}_{00} = \phi^{(2)}_{00}. \]

As seen before, it can be arranged, by replacing \( \theta^{(2)}_{00} \) by \( \frac{1}{2}(\theta^{(2)}_{00} + \theta^{(2)}_{00}^\dagger) \) if necessary, that \( \theta^{(2)}_{00}(x^*) = (\theta^{(2)}_{00}(x))^* \).

Now we prove by induction that there exists a family of maps \( \{\theta^{(n)}_{\mu\nu}\} \in C^1(A, E_L) : \mu, \nu \in \{0, 1\}, n \geq 1 \) such that

1. \[ \partial \theta^{(n)}_{11}(x, y) = \theta^{(n)}_{11}(xy) - \pi(x)\theta^{(n)}_{11}(y) - \theta^{(n)}_{11}(x)\pi(y) \\
= \sum_{k=1}^{n-1} \theta^{(k)}_{10}(x)\theta^{(n-k)}_{01}(y) + \sum_{k=2}^{n-1} \theta^{(k)}_{11}(x)\theta^{(n-k)}_{11}(y) \]
   \[ \theta^{(n)}_{11}(x^*) = \theta^{(n)}_{11}(x)^* \]

2. \[ \partial \theta^{(n)}_{10}(x, y) = \theta^{(n)}_{10}(xy) - \pi(x)\theta^{(n)}_{10}(y) - \theta^{(n)}_{10}(x)y \\
= \sum_{k=1}^{n-1} \theta^{(k)}_{10}(x)\theta^{(n-k)}_{00}(y) + \sum_{k=2}^{n-1} \theta^{(k)}_{11}(x)\theta^{(n-k)}_{10}(y) \]

3. \[ \theta^{(n)}_{01}(x) = \theta^{(n)}_{01}(x^*)^*, \]
   \[ \partial \theta^{(n)}_{01}(x, y) = \theta^{(n)}_{01}(xy) - x\theta^{(n)}_{01}(y) - \theta^{(n)}_{01}(x)\pi(y) \\
= \sum_{k=1}^{n-1} \theta^{(k)}_{00}(x)\theta^{(n-k)}_{01}(y) + \sum_{k=1}^{n-1} \theta^{(k)}_{01}(x)\theta^{(n-k)}_{11}(y) \]
4. \( \partial \theta_{00}^{(n)}(x, y) = \theta_{00}^{(n)}(xy) - x \theta_{00}^{(n)}(y) - \theta_{00}^{(n)}(x) y \)
\[= \sum_{k=1}^{n-1} \theta_{00}^{(k)}(x) \theta_{00}^{(n-k)}(y) + \sum_{k=1}^{n} \theta_{01}^{(k)}(x) \theta_{10}^{(n-k+1)}(y). \]
\(\theta_{00}^{(n)}(x^*) = (\theta_{00}^{(n)}(x))^*\).

Let us assume that for some \( m \geq 2 \), there exist a family of maps \( \{\theta_{\mu \nu}^{(m)} \in C^1(\mathcal{A}, E_{\mathcal{L}}) \} \) satisfying above relations.

Consider the map
\[
\phi_{11}^{(m)}(x, y) = \theta_{11}^{(m)}(xy) - \pi(x) \theta_{11}^{(m)}(y) - \theta_{11}^{(m)}(x) \pi(y)
\]
\[= \sum_{k=1}^{m-1} \theta_{10}^{(k)}(x) \theta_{01}^{(m-k)}(y) + \sum_{k=2}^{m-1} \theta_{11}^{(k)}(x) \theta_{11}^{(m-k+1)}(y). \]

Then we have
\[
\partial \phi_{11}^{(m)}(x, y, z)
\]
\[= \sum_{k=1}^{m-1} \{ \theta_{10}^{(k)}(x) \partial \theta_{01}^{(m-k)}(y, z) - \partial \theta_{10}^{(k)}(x, y) \theta_{01}^{(m-k)}(z) \}
\]
\[+ \sum_{k=2}^{m-1} \{ \theta_{11}^{(k)}(x) \partial \theta_{11}^{(m-k+1)}(y, z) - \partial \theta_{11}^{(k)}(x, y) \theta_{11}^{(m-k+1)}(z) \}
\]
\[= \sum_{k=1}^{m-1} \theta_{10}^{(k)}(x) \{ \sum_{l=1}^{m-k-1} \theta_{00}^{(l)}(y) \theta_{01}^{(m-k-l)}(z) + \sum_{l=1}^{m-k-1} \theta_{01}^{(l)}(y) \theta_{01}^{(m-k-l+1)}(z) \}
\]
\[+ \sum_{l=1}^{m-1} \{ \sum_{k=1}^{l-1} \theta_{10}^{(k)}(x) \theta_{00}^{(l-k)}(y) + \sum_{k=1}^{l-1} \theta_{11}^{(k)}(x) \theta_{01}^{(l-k+1)}(y) \theta_{01}^{(m-k)}(z) \}
\]
\[+ \sum_{k=2}^{m-1} \{ \sum_{l=1}^{m-k-1} \theta_{10}^{(k)}(x) \theta_{01}^{(l)}(y) \theta_{01}^{(m-k-l+1)}(z) + \sum_{l=2}^{m-k} \theta_{11}^{(k)}(y) \theta_{11}^{(m-k-l+2)}(z) \}
\]
\[+ \sum_{k=2}^{m-1} \{ \sum_{l=1}^{m-k-1} \theta_{10}^{(k)}(x) \theta_{01}^{(l-k)}(y) + \sum_{l=2}^{k-1} \theta_{11}^{(l)}(x) \theta_{11}^{(l-k+1)}(y) \theta_{11}^{(m-k+1)}(z) \}
\]
\[= 0\]

Since \( H^2(\mathcal{A}, E_{\mathcal{L}}) = 0 \), there exists a map, say \( \phi_{11}^{(m)} \in C^1(\mathcal{A}, E_{\mathcal{L}}) \) such that \( \partial \phi_{11}^{(m)} = \phi_{11}^{(m)} \). Moreover, it is easily seen that \( \partial \phi_{11}^{(m)} = \partial \phi_{11}^{(m)} \), and so without loss of generality we can assume that \( \theta_{11}^{(m)}(x^*) = (\theta_{11}^{(m)}(x))^* \). Proceeding similarly it can be shown the existence of maps \( \theta_{10}^{(m)}(x), \theta_{01}^{(m)}(x) \) and \( \theta_{00}^{(m)}(x) \) with required relations.

From this the algebraic relations \( (3.2) \) follow. Now it is easy to get \( (3.1) \), which completes the proof.

It is interesting to investigate whether the converse of the above result also holds; i.e. whether vanishing of \( H^2(\mathcal{A}, E) \) is necessary for the existence of a ‘quantum random walk’ in the formal algebraic sense as in the above theorem. If the converse
to Theorem 3.1 holds, then it will give a ‘quantum probabilistic’ interpretation of $H^2(\mathcal{A}, \mathcal{E})$ as the obstruction to construction of a quantum random walk. However, in order to meaningfully apply Theorem 3.1 to the theory of EH dilation, one must obtain an appropriate analytic version of it, giving conditions for the formal power series in the statement of Theorem 3.1 to converge. We hope to take up these questions in a future work.

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