On the existence of a Bose Metal at $T = 0$

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This paper aims to justify, at a microscopic level, the existence of a two-dimensional Bose metal, i.e. a metallic phase made out of Cooper pairs at $T = 0$. To this end, we consider the physics of quantum phase fluctuations in (granular) superconductors in the absence of disorder and emphasise the role of two order parameters in the problem, viz. phase order and charge order. We focus on the 2-d Bose Hubbard model in the limit of very large fillings, i.e. a 2-d array of Josephson junctions. We find that the algebra of phase fluctuations is that of the Euclidean group $E_2$ in this limit, and show that the model is equivalent to two coupled XY models in (2+1)-d, one corresponding to the phase degrees of freedom, and the other the charge degrees of freedom. The Bose metal, then, is the phase in which both these degrees of freedom are disordered (as a result of quantum frustration). We analyse the model in terms of its topological excitations and suggest that there is a strong indication that this state represents a surface of critical points, akin to the gapless spin liquid states. We find a remarkable consistency of this scenario with certain low-$T_c$ thin film experiments.

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I. INTRODUCTION

The superconductor-insulator (SI) transition in low-$T_c$ thin film systems [1] has drawn a lot of attention over the past couple of decades. These systems undergo transition from superconductor (SC) to insulator as the disorder, thickness or magnetic field is tuned. The problem has received a strong impetus after the experiment by Goldman et. al. [2] on homogeneous lead and bismuth films, which went from a superconducting phase to an insulating phase as a function of thickness, and which, right at the interface, was probably metallic. Since electrons usually do not form a metallic state in 2-d, it was argued [Ref.3b] this $T = 0$ transition is due to the localisation of preformed Cooper pairs. It was also claimed that the resistivity at the transition is universal[Ref.3b]. Similar SI transitions have been observed in granular superconductors and Josephson Junction arrays [3]. This scenario has been called into question after a recent magnetic field tuned experiment in the Mo-Ge sample [4], where the metal is no more a point in the phase diagram, but exists as a separate phase. We would like to point out that this is not the first observation of a metallic phase in a two-dimensional, otherwise superconducting, system. We came across, at least two separate instances of this phenomenon in granular superconductors — one in Ga films [5] and the other in granular Pb films [6], where a metallic phase is found to be sandwiched between the superconducting and the insulating phases. A similar observation has been reported in Josephson Junction arrays [7]. Each of these observations has probably a detailed explanation within the scope of the specific experimental system being measured. However, there is something common in these systems which is quite hard to overlook, viz. the preformed Cooper pairs are very much alive when the metallic phase is seen; and continue to exist till the insulating transition and beyond. Motivated by this fact, we would like to ask the question — is it possible for the charged bosons, i.e. Cooper pairs, to form an incoherent metallic phase at $T = 0$? The intriguing feature of this phase is that although the bosons are mobile, they do not bose condense at any temperature. Instead, they are dissipative and fail to drive a supercurrent even at $T = 0$, unlike a superconductor. We regard this phase as a Bose metal (BM). In this paper, we give arguments justifying the existence of a Bose metal in a physically realizable system.

The natural question, then is, what has been missing in the current theoretical models where a metallic phase was not obtained. Our thought on this issue is the following: most of the earlier theories [3,4,6] tried to attack this problem from the superconducting side of the phase diagram and projected it onto a basis diagonal in the phase states. In all these theories, there was a single order parameter, viz. $\psi = <e^{i\phi}>$, where $\phi$ refers to the phase of the charge boson. In the superconducting phase, this order parameter is well developed. At the SI transition, $\psi \to 0$ and the phases of the charge bosons get scrambled. The scrambling of the phases [3,4] had so far been taken as the indication of onset of the insulating phase. However, this is not enough to characterise the insulator. A Bose insulator (BI) phase is characterised by an extra order parameter, viz. the charge density. It is like a charge density wave but built out of Cooper pairs. This piece of physics has been missing in the existing theories [3,4,6]. The central point of this paper is that the phase fluctuation physics of superconductors should be viewed as a two order parameter problem, viz. there is a competition between phase order and charge order. It almost follows from this fact that the destruction of one order parameter does not necessitate the growth of the other order parameter. This implies a possible existence
of a disordered phase where both the order parameters are zero at \( T = 0 \). We consider this to be a Bose metal phase.

Secondly, we work in the limit when the average filling of the bosons\( (n_0) \) per site is very large. In this limit, the Bose Hubbard model [Ref.3b], which plays a central role in the SI transition, becomes equivalent to a Josephson junction array (JJA) model. Most of the existing work on the JJA model does not treat this large \( n_0 \) limit consistently. In many cases, the constraint due to the average filling, eqn.(2) below is neglected \([11]\); in certain other cases, there is a tendency to replace the \( E_2 \) algebra, which is the appropriate algebra for this limit (see sections II and V for more details), by a qualitatively different algebra, viz. SU(2) algebra, which holds in the hard core limit, i.e. for small fillings. Hence, we shall give a quite detailed description of the basic formulation of this limit in this paper.

Thus, we consider a pure model of Josephson-coupled Cooper pairs with extremely large fillings, interacting via on-site and near neighbour repulsive forces in two dimensions. This model captures the basic physics of granular superconductors [3] except that disorder is absent. It is applicable to a limited extent to the Josephson junction arrays as well. We are able to demonstrate that the physics of phase fluctuations, within such a model, can be described by two coupled anisotropic XY models in (2+1)-d, one corresponding to the charge degrees of freedom and the other the phase degrees of freedom, where the coupling is “XY-like”. The two mechanisms which drive the two transitions are as follows – disordering of the vortices and the Bose condensation of vortices. Phase order is destroyed when the vortex-antivortex pairs unbind (or, vortex loops blow up in (2+1)-d) as a result of the quantum fluctuations; the charge order grows when the vortices Bose condense \([11]\). In light of this, a search for a completely disordered phase in the charge picture translates into a search for a non superfluid(SF) liquid in the vortex picture. The presence of unbound, as yet uncondensed, dissipative vortices makes the disordered phase metallic. The two mechanisms mentioned above are separate processes owing to the presence of retardation, or equivalently, dissipative effects, coming from a gauge field mediated interaction in the vortex picture (for more details see sections IIId and IIIb). In the charge picture, the existence of a disordered phase like that of a Bose metal results from quantum frustration effects, i.e. the zero point motion is not strong enough to set up superconducting correlations and the long range interactions are not sufficient to set up a charge ordered state. This state seems to be intrinsically related to the (gapless) spin liquid states. Although disorder is not explicitly present in our model, our results on the nature of the Bose metal phase can be readily generalised to the case where disorder is present. We find a remarkable agreement of our predictions with the Gallium film experiment \([13]\). We believe this provides evidence of the existence of such a phase and supports the scenario described above. All of our considerations in this paper are restricted to \( T = 0 \) and zero magnetic field unless mentioned otherwise. This situation is relevant to the experiments in Refs. [3] and [14].

Before we go over to the main part of the paper, we would like to mention that the idea of two order parameters is not completely new and has appeared in the discussion of the hard-core limit of the Bose-Hubbard model \([12]\) and also in the gauge theory description of the Josephson Junction arrays \([13]\). The problem with the former is that superconductivity is never destroyed away from half-filling in their model (please see section IIb), while the latter invokes the idea of self-duality which implies logarithmic interaction among Cooper pairs and is not a good representation of the realistic thin film samples.

Apart from the existence of a novel disordered Bose liquid state, there is another important feature which emerges from our work. It is well known that \([4]\) non-interacting electrons cannot support a metallic state at low temperatures in 2-D in the absence of spin-orbit scattering. The situation is much less clear when both interactions and disorder are present. This fact has been brought into the limelight with the recent observation \([14]\) of metal-like states in two dimensional electron and hole systems in semiconductor based materials. In the wake of these observations, our investigations suggest that a metallic state is a possibility if the electrons bind themselves into Cooper pairs and behave as if they are bosons.

The plan of the paper is as follows. In section II, we introduce the model and straighten out some of the basic issues which relate to the model. Using a combination of qualitative and quantitative arguments, we explain why the BM phase is feasible. The mapping onto the coupled XY models is demonstrated here. We quantify the arguments in section III and sketch the phase diagram of the model. Section IV contains an estimate of the resistivity of the Bose metal phase and a comparison with the Gallium film experiment. The thermodynamics of this strange metallic state and its connection with the spin liquids are discussed in section V. We wind up with a discussion of certain relevant issues and conclusions in section VI. Some of the calculational details may be found in the appendices.

**II. THE FOUNDATIONS**

**A. The model**

We shall consider the following model Hamiltonian in this paper —

\[
H = -J \sum_{<i\alpha>} \cos(\phi_i - \phi_{i+\alpha}) + V_0 \sum_i (\delta n_i)^2 + V_1 \sum_{<i\alpha>} (\delta n_i + \delta n_{i+\alpha})^2 - \mu \sum_i \delta n_i - \mu Nn_0 \quad (1)
\]
where $\hat{\delta n}_i = \hat{n}_i - n_0$, with $\hat{n}_i =$ number density operator, $n_0 =$ neutralising background charge density (or, equivalently, the average density of Cooper pairs), $N =$ number of lattice sites, and $\tilde{\mu} = \mu + V_0$ the renormalised chemical potential in the problem. Also, $\alpha = \hat{x}, \hat{y}$ refers to the spatial unit vectors. The first term in eqn.(1) represents the kinetic energy of the bosons (Cooper pairs for our case), the second term the onsite repulsion among them (which should be non-zero to prevent any collapse of the bosons onto a single site), the next term the repulsion among the nearest neighbours which acts to set up a charge order in the system, and the last two terms are the chemical potential terms. Eqn.(1) has to be supplemented with the constraint $\frac{1}{V} \sum_i \langle \hat{\delta n}_i \rangle = n_0$, which in terms of the charge fluctuation operators convert into

$$\sum_i < \hat{\delta n}_i > = 0 \quad (2)$$

This equation implies $\tilde{\mu} = 0$ and, in the rest of the paper, we shall forget about the chemical potential terms completely.

On the experimental side, the three parameters of the Hamiltonian (1) can be determined in the following way [10].

$$J = \left(\frac{R_Q}{2R_n}\right)\Delta_0 \quad (3)$$

Here $R_Q = (\hbar/4e^2) = 6.45K\Omega$, $R_n =$ normal resistance of the film, and $\Delta_0 =$ pairing gap. The interaction constants $V_0$ and $V_1$ are related to the inverse of the capacitance matrix $C_{ij}$ of the grains.

B. The commutation relations

The model discussed in eqn(1) needs to be supplemented with the phase fluctuation algebra, which constitutes the appropriate commutation relations for this problem:

$$[\hat{\delta n}_i, \hat{\phi}_j] = i\delta_{ij} \quad (4)$$

Eqn.(4) implies an angular momentum representation[Ref.3a]

$$\hat{\delta n}_i = i \frac{\partial}{\partial \phi_i}.$$ 

Further, let us define the operators —

$$L = i \frac{\partial}{\partial \phi}, \text{ and } P = e^{i\phi} \quad (5)$$

Thus, eqn.(4) can be recast as —

$$[L, P] = -P \quad (6a)$$

$$[L, P^\dagger] = P^\dagger \quad (6b)$$

with $PP^\dagger = I$. It is well known in quantum optics that the phase operators being ladder operators [as is seen from eqns.(6a) and (6b)] usually do not commute [7]. However, in the large $n_0$—limit, they do (see appendix F). Thus, eqn.(6c) is strictly valid in the large $n_0$—limit of the problem and, hence, our discussion holds good in this limit only. Now, eqns.(6) constitute the algebra of the Euclidean group $E_2$ [18], the group of translations and rotations in 2-d, with the square of linear momentum($P$) restricted to unity for our case. It deserves to be mentioned here that the $SU(2)$ algebra used in the context of the hard-core limit of the Bose problem [2] is distinctly different from this algebra and is obtained in the opposite limit, i.e. the small $n_0$—limit of the Bose Hubbard model. The effects of the change in the group structure are quite significant. In the hardcore model, away from half filling, increased interactions changes the SF to a supersolid phase. Thus, superfluidity is never destroyed away from half filling. On the other hand, our model allows superfluidity to be quenched at arbitrary fillings. Further, the conserved quantities supported by the two algebras are different: the invariant of $E_2$ algebra is (square of linear momentum) $P_x^2 + P_y^2 = p^2$ (= 1 for our case), as compared to $S_x^2 + S_y^2 + S_z^2 = s(s + 1)$ of the $SU(2)$ algebra. Thus, unlike the $SU(2)$ case, (a) here the constraint on the $z$ component of the spin is much weaker and (b) the $z$ component of the spin enters very anisotropically in the algebra compared to the $x$ and $y$ components of the spin. (Here operator $L$ is referred to as the $z$ component of spin for $E_2$ algebra; see section V for more details.) Hence, in the disordered/symmetric phase, although the $x$ and $y$ components of $E_2$ spin might acquire a gap like the $SU(2)$ spin, the $z$ component of the former might remain gapless unlike the latter. Put in simple words, there may be a length scale determining the local superfluidity in this phase and as yet no length scale associated with the charge ordering in the system. A calculation of the charge charge correlation function indeed justifies this, as discussed in section V. Thus, given a disordered phase within a model built out of $E_2$ spin operators, it can support gapless excitations. We shall give detailed arguments in sections IID and III as to why the model (1) along with commutation rules (6) contain a completely disordered phase.

Now, if we rotate the charge fluctuations $L_i$ at each site, viz.

$$L_i \rightarrow e^{iQ_i \cdot R_i} L_i, \text{ where } Q = (\pi, \pi)$$

we obtain the Hamiltonian as

$$H = \sum_k J_k P_k^\dagger P_k + \sum_k V_k L_k^\dagger L_k, \quad (7)$$

where

$$J_k = - J(cosk_x + cosk_y)$$
\[ V_k = (V_0 + 4V_1) - 2V_1(\cos k_x + \cos k_y). \]

and \(P, L\) couple through the commutation relations \((6)\). In this form, it assumes the shape of a 2 order parameter problem. (Macroscopic occupation of \(k = 0\) mode of \(P\) reflects superconductivity and that of \(L\) a charge density wave.) However, because of commutation rules \((6)\), it is very hard to diagonalise this hamiltonian in this form. So, we seek alternative means.

C. The charge picture

In this section, we demonstrate that the model Hamiltonian \((1)\) is equivalent to two coupled XY models in (2+1)-d. To do this, we first write the model \((1)\) in terms of a path integral representation \([18,19]\) keeping the commutation relations \((6)\) in mind. The partition function \(Z\) reflects superconductivity and that of \(L\) is invariant under shifts of \(p_{i\alpha}\) by \(2\pi\). So, we break up the integral over \(p_{i\alpha}\) from \(-\infty\) to \(+\infty\) into that of periods of \(2\pi\). Further, we split up \(p_{i\alpha}\) into a curl and a gradient part. Since the divergence of \(p_{i\alpha}\) couples to \(m_i\), only the gradient part enters the dynamics. Thus, we obtain for this part,

\[
\sum_{\{m_i\}} \int_{0}^{2\pi} D\theta_{i} e^{-i \int \sum_{\alpha} (\cos(\nabla_\alpha \theta_i) e^{-i m_i})}
\]

Putting all these together, one obtains

\[
Z = \int_{0}^{2\pi} D\phi_{i} D\theta_i e^{S}
\]

with

\[
S = J \Delta \sum_{\alpha} \cos(\nabla_\alpha \phi_i)
\]

\[
+ \frac{1}{2V_0 \Delta \tau} \sum_{\alpha} \cos(e^{-iQ.r_i} \nabla_\tau \phi_i + \nabla^2 \theta_i)
\]

\[
+ \frac{1}{2V_1 \Delta \tau} \sum_{\alpha} \cos(\nabla_\alpha \theta_i)
\]

(10)

where the phase \(\phi\) is associated with superfluidity and the phase \(\theta\) with charge density wave. Eqn.(10) explicitly shows that the phase fluctuation physics of \((\text{granular})\) superconductors, in the absence of disorder, is equivalent to two coupled XY models in \((2+1)\)-d, where the coupling is XY-like. A few comments are in order. In the limit \(V_1 = 0\), we have \(\theta_i = 0\), the \(\theta\)-terms in action \((10)\) drop out, and we obtain a single XY model in \((2+1)\)-d, as has been discussed previously [13]. The coupling term \((\text{the second term})\) is highly anistropic in \(\phi\) and \(\theta\), explicitly breaking self-duality in this system, assumed in Ref. [13]. Eqn.(10) is one of the key results of this paper. Thus, we see that the destruction of superfluid state is driven by one XY model, whereas the other XY model characterises the growth of the charge ordered state. And, hence, as the parameters are tuned one transition does not necessarily accompany the other.
D. The vortex picture

Although the action (10) shows that the superfluid order and charge order are driven by different XY models, it is not clear whether the coupling between them, which is quite complicated, guarantees a completely disordered phase. In order to answer such a question, we now consider the model (1) in the dual picture, i.e. of the vortices. To do this, we invoke a duality transformation [22]. Starting from action(9) [21], one can show that

\[ S = \frac{\pi^2}{\ln(2/\Delta\tau)} \sum_{q,\omega} j_q^0 \frac{1}{q^2} j_{-q,-\omega} + \frac{\pi^2}{\ln(2/\Delta\tau)} \sum_{q,\omega} j_q^0 G(\omega, q) j_{-q,-\omega}, \]  

(11)

where \( j_q^\mu \) are Fourier transforms of the integer vortex variables \( j_i^\mu, j_i^0 = \) vortex density, \( j_i^\alpha = \) vortex current, and

\[ G^{-1}(\omega, q) = [\omega^2 + \frac{V_0 \Delta\tau}{\ln(2/\Delta\tau)} q^2 + \frac{V_1 \Delta\tau}{\ln(2/\Delta\tau)} q^2 (q - Q)] \]

(12)

The intervening steps are quite standard [22] and are discussed in Appendix A. \( \omega, q \) refer to lattice frequency and momentum respectively[Appendix A]. One may like to note here that there is no Magnus force term on the vortices, corresponding to the disappearance of \( n_0 \) from the problem in the large \( n_0 \) limit.

In the previous subsection, we observed that when \( V_1 \to 0 \), the model is equivalent to a single XY model rather than two coupled XY models. The question, obviously, is how this change is reflected in the dual picture. The basic answer lies in the appearance or disappearance of the zero point motion term for the vortices (more popularly known as the vortex mass term). To show this, we look at the long wavelength low frequency modes. In the limit \( \omega, q \to 0 \) and \( \omega \ll c_s q \) (where \( c_s \) is the plasmon velocity), the Green’s function \( G(\omega, q) \) splits up into a singular part \( G_s(\omega, q) \) and a constant part \( G_0 \), viz.

\[ G(\omega, q) \approx G_s(\omega, q) + G_0, \]

(13)

where

\[ G_s(\omega, q) = 1/(\omega^2 + c_s^2 q^2), \quad G_0 = b^2 / (8(b^2 + c^2)^2) \]

with

\[ c_s^2 = b^2 + c^2, \]

\[ b^2 = \frac{V_1 \Delta\tau}{\ln(2/\Delta\tau)}, \quad c^2 = \frac{V_0 \Delta\tau}{\ln(2/\Delta\tau)} \]

Thus the action is

\[ S = \frac{\pi^2}{\ln(2/\Delta\tau)} \sum_{q,\omega} j_q^0 \frac{1}{q^2} j_{-q,-\omega} + \frac{\pi^2}{\ln(2/\Delta\tau)} \sum_{q,\omega} j_q^0 [G_s(\omega, q) + G_0] j_{-q,-\omega}, \]

(14)

Since \( G_0 \) is a constant, it allows us to identify this term as the vortex kinetic energy, i.e. mass term. From (14) and (13), we notice that this term does not exist when \( V_1 = 0 \). Thus, we have shown that the change of the nature of the XY models is tied to the existence or non-existence of a vortex mass term.

We now look at the other terms in the action (14). The first term in the action is the usual logarithmic interaction term among the vortices, and the second term, because of the retardation effects, leads to a dissipative term for the vortices [16]. In the limit \( |r_i - r_j| \ll c_s |\tau - \tau'| \) (which is the same quasistatic limit discussed so far), this part of the action takes the form —

\[ \frac{\pi^2}{c_s^2 \ln(2/\Delta\tau)} \sum_{i,j} q_i q_j \sum_{\omega, q} \omega^2 \ln(\frac{1}{\omega}) r_{i,\tau} r_{j,-\omega} \]

where \( q_i = \pm 1 \) refers to the charge on the \( i \)th vortex. Physically, the aforesaid limit corresponds to the slow motion of vortices. The source of this heat bath (or, dissipation) is the gauge field (discussed in Appendix A), or more precisely, the transverse modes arising from the quantum fluctuations in the system. (There will be additional retardation effects in a real system from external heat bath mechanisms, e.g. electrons in the vortex core, etc.) These features of vortices in granular superconductors have been discussed previously by Eckern and Schmidt [14].

One may also undo some of the steps in Appendix A, and recast eqn.(14) in terms of a gauge field with appropriate action as —

\[ S = \ln(\frac{2}{\Delta\tau}) \sum_{q,\omega} [q^2 A_q^0 A_{q,-\omega} + (\omega^2 + c_s^2 q^2) A_q^\alpha A_{q,-\omega}^\alpha] + 2\pi i \sum_i j_i^\mu A_i^\mu + \frac{\pi^2}{\ln(2/\Delta\tau)} \sum_{q,\omega} j_q^\alpha j_{-q,-\omega}^\alpha \]

(15)

(with the gauge condition \( \nabla_\alpha A^\alpha = 0 \) imposed). A few comments are in order. Eqn. (15) shows that, considered in the vortex picture, the quantum phase fluctuations in a 2-d superconductor, as described by model (1), is equivalent to a two-component quantum plasma (bosons of two flavours, viz. vortices and antivortices) moving in a fluctuating gauge field \( A^\mu \) [23]. Secondly, the above scenario is a simple quantum mechanical extension of classical phase fluctuations in a 2-d superconductor, which is described by a two component classical plasma undergoing screening by a static electric field \( \vec{E} = -\nabla \vec{A}_0 \), as described by Kosterlitz and Thouless(KT) [24]. The effect of including quantum mechanics in the problem, apart from bringing up the importance of the quantum
statistics of vortices, is to make the electric field dynamical, viz. \( \vec{E} = -\nabla A_0 - (1/c_s) \partial \vec{A}/\partial \tau \) with a magnetic field \( \vec{B} = \nabla \times \vec{A} \). Whereas the importance of the statistics is to allow for the superfluidity of the vortices, an important consequence of the dynamical nature of the electromagnetic field is that there are retardation effects, viz.

\[
\sum_{i,j} \int d\tau d\tau' \frac{\vec{r}_i(\tau) \cdot \vec{r}_j(\tau')}{\sqrt{(\vec{r}_i(\tau) - \vec{r}_j(\tau'))^2 + c_s^2(\tau - \tau')^2}},
\]

which break Galilean invariance. (This feature is intensified in a real system by external heat bath mechanisms mentioned before [4].) This is not surprising because the action (15) has the structure of Maxwell’s action, which is reputed to have Lorentz invariance but lacks Galilean invariance. Now the absence of Galilean invariance will have a strong effect on our system, because it is bosonic. It is well known that all the delocalised bosons will condense into the superfluid state only if the system is Galilean invariant [26]. If this invariance is absent in a Bose system, then as the relevant parameter is tuned, the (vortex) condensate is gradually depleted and at one point the superfluidity will be completely lost. At this stage, it is important to recollect what important processes are going on in this system. There are two of them: destruction of vortex (and antivortex) superfluidity (owing to retardation effects) and the binding of vortex-antivortex pairs, corresponding respectively to the destruction of charge order and the growth of phase order. Now, these two processes are controlled effectively by two separate parameters, viz. \( c^2/g^2 \sim (V_0/8V_0)\Delta \tau \) (corresponding to the strength of the retardation effects) and \( g^2 \sim J\Delta \tau \) (corresponding to the strength of the logarithmic interaction) respectively, where \( \Delta \tau \sim 1/\Delta_0 \) [34] (please refer to appendix D and section III for the notation and appropriate details). As a result, the vortices (and antivortices) do not necessarily condense into the superfluid state as soon as they unbind. This leads to the possibility of a non-SF vortex liquid, or equivalently, a BM phase. In fact, that’s what we find when we quantitatively evaluate these processes in section III. There is a simpler way to see what is happening here. As we noted earlier, following eqn.(14), the kinetic energy of vortices originates from two sources — (a) quantum zero point motion (the \( G_0 \) term) and (b) action of an effective heat bath (the \( G_s \) term). In the delocalised state, if the source (a) dominates, the vortices move coherently, and since they are bosons, they form a superfluid. On the other hand, when the source (b) dominates, because of the random nature of the effective heat bath, the motion of the vortices is necessarily incoherent, and one is in a metallic phase. In this phase, no charge order is set up and one obtains a Bose metal phase.

These features of dual vortices are not special to a lattice model, but observable in the continuum formulation as well, as discussed in Appendix B.

### III. THE PHASE DIAGRAM

In the previous section, we argued why the Hamiltonian described by eqn.(1) may contain an incoherent metallic phase. In this section, we quantify these arguments by calculating the phase diagram of model (1): we shall locate the phase boundary where superconductivity is destroyed and the one where charge order is established.

#### A. Destruction of charge superfluidity

This is done non-perturbatively by estimating where the vortex loops blow up in (2+1)-d [27]. This happens when the entropy of the loops overcomes their interaction energy [28]. A good estimate of the interaction energy is the self energy of the loops, simply because dipole-dipole interactions fall off as \( 1/r^3 \) and the mutual interaction energy of the loops in a loop is much smaller than the self-energy of the loops when the loops are fairly large. Thus, the effective free energy of the loops is given by

\[
\mathcal{F} = \frac{\pi^2}{\ln(\Delta/r)} G(0) - \mu_l N \tag{16}
\]

where \( N \) is the number of the links in a vortex loop, \( G(0) = \text{diagonal part of the Green’s function} \), \( \mu_l = \text{entropy of the loops} = \ln 3 \) for our case [29]. An estimate of \( G(0) \) is given in the Appendix C. The loops blow up when \( \mathcal{F} < 0 \). Thus, superconductivity is destroyed when

\[
(V_0/J + 8V_1/J) > \tilde{b}_0, \tag{17}
\]

where \( \tilde{b}_0 = 2(b_0/\mu_l)^2 \) and \( b_0 = \text{a number of order unity} \approx 3.1725 \), defined by eqn. (C6) in appendix C. This phase boundary has the character of (2+1)-d XY model, at least when \( V_1 \ll V_0 \) (as seen from eqn.(10)), and hence, the superfluid density changes continuously across this transition line.

#### B. Destruction of vortex superfluidity

It is well established that the existence of charge order implies superfluidity of vortices and vice versa [11]. We shall follow this notion here and estimate the growth of charge order in terms of the superfluidity of vortices. We mentioned in section IIId that the vortices and antivortices move in the presence of a dynamical gauge field and argued that because of the latter there are retardation effects which deplete the (vortex) Bose condensate (due to broken Galilean invariance). As a result, the vortices and antivortices do not necessarily condense into a superfluid
state as soon as they unbind. This physics of suppression of Bose condensation as a result of gauge field fluctuations is not new but has been explored substantially in the context of spin charge separation theories in high $T_c$ superconductors \[24-25\]. The discussion here is very similar in spirit to that piece of work.

To estimate the strength of the parameters where vortex superfluidity is destroyed, we follow the self-consistent functional approach of Ioffe et. al. \[25\]. They did the calculation for a very similar piece of physics, i.e. how the coupling to a gauge field can kill superfluidity in a bosonic system (with logarithmic interaction) as a result of broken Galilean invariance stemming from retardation effects. We refer the reader to that paper for a full description of this technique. A short discussion of this is given in Appendix D. We can follow their approach here, simply because the lattice action and the continuum action have identical structure in the low frequency long wavelength limit, as discussed in Appendix B. To do this, we first replace the two component plasma by a one component plasma, i.e. charges/vortices of one flavour moving in the background of fixed neutralising charges of the other flavour. This approximation is very standard and captures the salient features of the problem, until and unless the plasma is extremely dense \[22\]. Then, the results of sec.V in Ref. \[24\] can be directly carried over here with the identification of the Coulomb interaction parameter $\alpha_c$ and transverse gauge field coupling constant $\alpha_g$ as \[30\] (Appendix D)

$$\begin{align*}
\alpha_c &= (\pi^2/2)v_1/(v_0 + 8v_1)^2 \\
\alpha_g &= (1/\pi n_v)[1 + (v_0/8v_1)]
\end{align*}$$

(18)

where $n_v = \text{average vortex density and } v_i = (V_i/J),(i = 0, 1)$. Since what counts in the destruction of vortex superfluidity are the free vortices (and antivortices), we take $n_v \sim n_f$, which goes inversely as the square of the correlation length $\xi$ which diverges at the SC-BM boundary from the BM side. Thus, from (17), $n_f \sim (\tilde{b}_0(V_0 + 8V_1)/J - 1)^{2\nu}, \ (\tilde{b}_0 = 1/b_0)$ with $\nu \sim 2/3$, since this phase boundary has the character of (2+1)-d XY model. Also, if we are above the phase boundary of eqn.(17) (see fig.1), we have $\alpha \ll 1$ as in Ref. \[24\], where $\alpha = \sqrt{\alpha_c}$. Since as we shall note below that the transition from SF to non-SF state takes place at a small value of $\alpha$, this assumption of small $\alpha$ is self-consistent.

Let us first consider the simple case when we are slightly above the phase boundary (16) and along the $V_1$ axis (see fig.1), so that $V_0 = 0, V_1 \sim 2.1J$. Then, we have, $\alpha \simeq 0.2$ and $\alpha_g \gg 1$; the calculation of Ref. \[24\], in that case, suggests we are in the disordered BM phase. Thus, there is at least a small region close to the phase boundary (17), where the system is metallic.

We now complete the calculation of phase boundary of BM-BI transition, following Ref. \[24\]. Since $0 < n_v < 1$, from (18), $\alpha_g > 0.32$. From fig.7 of Ref. \[24\], one can see that in this regime, the phase boundary tends to saturate at $\alpha = \alpha_{cr} \simeq 0.08$. Actually, for larger values of $\alpha$, $\alpha_{cr}$ is a little less; but, the point is that $\alpha_{cr}$ is always finite, however small it be, for large values of $\alpha_g$. That is, the vortices always form a superfluid for small enough $\alpha$. A physical way of seeing why there is always superfluidity for small enough Coulomb repulsion is as follows. The strength of the Coulomb repulsion is controlled by the effective charge $g$ (see appendix D), which also controls the strength of the coupling between the particles/vortices and the transverse gauge field. So, when the magnitude of Coulomb repulsion is small, $g$ is small, and hence, the coupling with the transverse gauge field is small as well. As a result, the particles do not feel the effect of retardation strongly enough in this limit and condense into a superfluid phase. Thus, the vortices are in a superfluid state when $\alpha < \alpha_{cr}$, which means

$$\frac{1}{(8V_1/J)[(V_0/J) + (8V_1/J)]^2} > c_0$$

(19)

where $c_0 = 4(\pi/8\alpha_{cr})^2 \sim 96$ for $\alpha_{cr} \simeq 0.08$. This calculation implies a jump in the (vortex) superfluid density at this phase boundary \[23\]. This means that the phase transition is either of first order or has a KT character. More calculations are necessary to resolve this point.

We display a schematic phase diagram determined by the eqns.(17) and (19) in fig.1. Eqn.(17) is represented by the curve LXMO and (19) by ZYX. They seem to meet at a tricritical point X. Since the retardation effects are very strong in this model (meaning that the value of $\alpha_{cr}$ is small), the constant $c_0$ in eqn.(19) is roughly an order of magnitude larger than the constant $\tilde{b}_0$ appearing in eqn.(17); and hence we expect that this crossing between the two curves will always occur, implying the existence of the Bose metal (see section VI as well). The various phases determined by these eqns. are as marked in fig.1 — when the zero point motion of the Cooper pairs is large, the system is superconducting (region LXMO); when the interactions dominate, the system is charge ordered and insulating as a result — we call it a Bose insulator(BI); and in the intermediate region, the system is disordered and, hence, metallic (please see section IV below) — the BM phase (region ZYXL).

What one sees on the phase diagram of fig.1 is that the metallic phase is more prominent towards the $V_1$ axis rather than the $V_0$ axis. Further, from eqn.(1) we note that the $V_1$-term contains both on-site and nearest neighbour repulsion energies. This shows that the Bose metallic phase is to be expected in cases where these energy scales are of comparable order of magnitude, a situation well represented by the granular superconductors \[26\].

We can provide a physical explanation in the charge picture as to why the metallic phase opens up along the $V_1$ axis: let us focus on eqn.(1) and say that superconductivity is already quenched. First consider the case $V_0$ large and $V_1$ small, so that we are along (or, close to) the $V_0$ axis. Naively, one would expect from the $V_1$ term that $\delta n_i + \delta n_{i+\sigma} \sim$ large. But, this costs a large energy from the $V_0$ term. As a result, what is favoured
is $\delta n_i \simeq 0$, i.e. $n_i \simeq n_0$, a (rather trivial) charge ordered state. Now, consider the opposite case: $V_1$ fairly large and $V_0$ small, so that we are close to (or, along) the $V_1$-axis. Since $V_1$ is appreciable, eqn.(1) suggests $\delta n_i + \delta n_{i+n} \simeq 0$. A non-trivial configuration may be $\delta n_i = -\delta n_{i+n} = 1$, for some $i$'s and zero otherwise. This state represents an RVB-like state of fluctuating charge (particle-hole) dipoles, the equivalent of spin singlets here (please see section V for more on this point). $V_0$ being small, the $V_0 \sum_i (\delta n_i)^2$ term does not cost much energy for this kind of state. This is the disordered BM phase. When $V_1$ is increased further, the dipoles freeze into a charge ordered solid. Thus, very crudely speaking, the smallness of $V_0$ is a source of frustration in this model for finite $V_1$.

Since $J$ is inversely proportional to the normal resistance $R_n$ [eqn.(3)], as the normal resistance of a thin film is tuned in an experiment we gradually cross from superconducting to non-superconducting regions according to the phase diagram of fig.1. Some such typical traces are shown in the figure — the dashed lines OA, OXB and OC. Traces OXB and OC represent a superconductor - insulator transition, a case which has been discussed very widely in the literature [3]. On the other hand, the trace OA represents a superconductor - metal - insulator transition. In this part of the phase diagram, the retardation effects are very strong and the system passes through an intermediate disordered phase. This is a new prediction made by our analysis. Model (1) which leads to this is good for s-i-s junctions.

IV. RESISTIVITY OF THE METALLIC PHASE

The discussion in the previous section suggests that the appropriate model for the Bose metal phase is that of uncondensed bosons (vortices and antivortices) in a transverse gauge field. In the frustrated BM phase, the vortices are unbound, but they fail to bose condense due to retardation, or equivalently dissipative, effects. As a result, the system exhibits metallic behaviour. This section is devoted to making a simplest possible estimate of the resistivity of this metallic phase. We shall evaluate the resistivity of the charge bosons $\rho_c$ in terms of the conductivity $\sigma_v$ of the double variables, viz. the vortices. The relation between the two is given by [3] (we mention the factor of $\hbar$ explicitly in this formula)

$$\rho_c = (\hbar/4e^2)\sigma_v \tag{20}$$

Also, $\sigma_v$ is given by the Drude formula $\sigma_v = n_{ef}\tau_{tr}/m_v$, where $\tau_{tr}$ refers to the transport time and $n_{ef}$ is the free vortex density. There are three contributions to the conductivity of the vortices — (i) dissipation due to the action of a heat bath (coming from the transverse modes), (ii) scattering from impurities, and (iii) Bardeen-Stephen processes. In this part, we shall focus on the contribution from process (i). We give here a short description of how this dissipation mechanism comes about in this BM phase. As one enters this phase from the SC phase, say (following curve A in fig.1, for example), the vortices and antivortices unbind in 2-d. These unbound charges screen each other leading to a finite screening length, i.e. a gap in the longitudinal part of the gauge field. However, the transverse part of the gauge field is gapless, because there is no spontaneous symmetry breaking. These transverse modes have their source in quantum fluctuations and mediate the dissipation process. They represent the plasmons. (Physically, this makes sense, since the plasmons are gapless in 2-d.) The physics is essentially that of damping which results when a charge (vortex) moves through a background of other charged particles. This lends a hydrodynamical character to this dissipation mechanism. In section IId, we saw that the most important modes are $\omega \ll c_s k$. So, we shall consider contributions from these modes only. The leading order contribution, then, comes from the $\omega = 0, k = \text{finite modes}$, and we make an estimate from this sector only. We think the higher order contributions will lead to simple renormalisation of coefficients, so long as the density of vortices (and antivortices) is low (please see below). Thus, we are led to evaluate the resistivity of a set of uncondensed bosons moving in a static random (but, annealed) gauge field, with a variance

$$< H(q)H(-q) > = 1/2c_s^2\ln(2/J\Delta\tau)$$

as seen from eqn.(15), where $H = \nabla \times \vec{A}$. Now, also in this delocalised phase, the interactions between the vortices are screened (by antivortices) and, in a lowest order approximation, we shall treat them as non-interacting bosons. Thus, the transport time is the same as that of a non-interacting particle scattering from a random magnetic field, which is given by [3]

$$\tau_{tr} = m_v/\pi^2.2(V_0 + 8V_1)\Delta\tau$$

where we have used the above field distribution and the parameters as enumerated following eqn.(13). The lattice constant in the time direction, $\Delta\tau \sim 1/\Delta_0$ [34]. Thus, we have —

$$\rho_c \sim 2/\pi^2 R_Q(n_{ef}\xi_0^2)(V_0 + 8V_1)/\Delta_0 \tag{21}$$

where $R_Q = h/4e^2 = 6.45K\Omega$. The quantity $n_{ef}\xi_0^2$ is inversely proportional to the square of the superconducting coherence length which diverges at the SC-BM phase boundary, i.e. $n_{ef}\xi_0^2 \sim (\xi_0/\xi_+)^2 \sim [(V_0 + 8V_1)/b_0 J - 1]^{2\nu}$, obtained from eqn.(17). Using eqn.(3), we can rewrite this as —

$$n_{ef}\xi_0^2 \sim (R_n/R_c - 1)^{2\nu}, \tag{22}$$

where $R_c = \tilde{b}_0 R_Q/[(V_0 + 8V_1)/\Delta_0]$ is the critical resistance of the film where the metallic phase sets in and
\( \nu \) = correlation length exponent. The exponent \( \nu \) is dependent on the particular universality class of SC-BM phase transition. For our case, the SC-BM phase boundary has the character of a \((2+1)\)-d XY model and, hence, \( \nu \approx 2/3 \). However, a real material, like the low-\( T \) thin film systems, is heavily disordered and \( \nu \) will certainly be very different from this, as we shall see below. So, from eqn.(21), we have

\[
R_0 / R_Q \sim b_0 ((R_n / R_c) - 1)^{2\nu} (R_Q / R_c)
\]

(23)

The case when disorder is present—The foregoing discussion can be readily generalised to the case when disorder is present. Owing to the Drude formula, the structure of eqn.(21) is not unique to model (1) but will come about when the bosons form a metallic phase as a result of phase fluctuations (zero magnetic field), i.e.

\[
R_0 \sim 2R_Q(n_{\nu f} \xi_0^2)\mu_D
\]

(24)

where \( \mu_D \) represents the vortex mobility, or the friction factor, arising from the dissipation of vortices and is model dependent [33]. For the pure case discussed above, the damping is due to the transverse modes and \( \mu_D \approx (V_0 + 8V_4) / \pi^2 \Delta_0 \). When the normal resistance is tuned through \( R_c \), as we saw above for the pure case, the most important dependence of BM resistivity in the formula (24) comes from the free vortex density \( n_{\nu f} \xi_0^2 \), i.e.

\[
n_{\nu f} \xi_0^2 \sim (R_a / R_c - 1)^{2\nu}
\]

where the critical exponent \( \nu \) characterising the SC-BM phase transition depends on the particular universality class being considered. As compared to this, the dissipation factor \( \mu_D \), in general being dependent on the morphological and normal properties of the film, is only weakly dependent on the normal resistivity about \( R_c \) [35]. Thus, in general,

\[
R_0 \sim R_Q[(R_n / R_c) - 1]^{2\nu}
\]

(25)

For the case of Ref. [3], \( R_c \approx R_Q \). As \( R_n \) is tuned in such an experiment, one executes a typical trajectory \( \Lambda \) on the phase diagram of fig.1. The phase which exists in a real material like Gallium is not captured by a pure model like model (1) is the Bose glass(BG) phase. So, instead of SC-BM-BI scenario along \( \Lambda \), one would probably have a SC-BM-BG(-BI) scenario. Either way, in the BM phase, because of formula (25) the resistivity of this metal increases continuously from zero through a wide spectrum of values as \( R_n \) is tuned through \( R_c \), as is seen in the experiments [33]. We display the low temperature metallic resistivity from Ref. [3][fig.2 of this reference] as a function of \( R_n \) in fig.2 and observe the remarkable fit to the formula (25) with \( \nu \approx 2 \). This clearly points out that the low temperature metallic phase observed in Ref. [3] was, in all probability, due to an incoherent motion of the Cooper pairs. The aforementioned value of the correlation exponent is consistent with the notion that Ga films were highly disordered samples [36] and the Chayes’ theorem [37] along with it.

V. HOW NEW IS THIS STATE OF MATTER?

The question now arises whether the Bose metal discussed in the previous sections is actually an adiabatic continuation of some known state of matter or if it is completely new. In this section, we argue that this state is actually a mathematical variant of a quantum disordered spin liquid with large spin and that it is a liquid of non fermionic variety.

To show this, we first note that the algebra which controls the quantum mechanics of our problem is \( E_2 \), as discussed in section II. Now, \( E_2 \) is a group contraction of \( SO(3) \) [18], i.e. as the radius \( R \) of the sphere on which \( SO(3) \) operations are defined tends to infinity, \( SO(3) \) contracts to \( E_2 \), viz. \( J_x / R \rightarrow -P_y, J_y / R \rightarrow -P_x, J_z \rightarrow L \), where the operators \( L \) and \( P \) are as defined in eqn.(5). And, in the same limit, i.e. as \( SO(3) \) angular momentum quantum number \( j = pR \rightarrow \infty \), where \( p \) refers to the linear momentum quantum number of \( E_2 \), the irreducible representations of \( SO(3) \) map onto those of \( E_2 \). Thus, the model (1) is a variation of a quantum spin model and the Bose metal obtained here, which is a quantum disordered phase, may be thought of as analogous to the frustrated spin liquids obtained within such a model in the limit of spin becoming large [35].

A more intriguing question is the issue of how consistent the existence of a disordered phase as that of a Bose metal at \( T = 0 \) is with the third law of thermodynamics. To this end, we calculate the low temperature specific heat of the metal under the same approximation as in section IV, i.e. non-interacting uncondensed bosons (vortices) moving in a transverse gauge field. The longitudinal part of the gauge field is gapped because of the screening effects and hence do not contribute significantly to the specific heat [39]. However, the transverse gauge field representing the quantum fluctuations associated with the transverse modes are gapless in this metallic phase [2] (because there is no spontaneous symmetry breaking) and, hence, make a dominant contribution to the specific heat. The calculation for uncondensed bosons interacting with a transverse gauge field [40,33] is analogous to the fermionic case for which an extensive literature exists [24]. Following Reizer [39], we find at low temperatures (Appendix E)

\[
C \sim A_p T^{2/3} \quad \text{(pure)}
\]

\[
\sim A_d T \ln(T_0/T) \quad \text{(with disorder)}
\]

(26)

This anomalous behaviour originates shearly from the fact that the transverse gauge field undergoes dynamical screening, which leads to the presence of diffusive modes \( \omega \sim -i q^n \), where \( n = 2 \)(disordered) or 3(pure).
The coefficient \( A_p \) scales as \( A_p \sim n_f^{2/3} \), \( A_d \sim n_f \), and \( T_0 \) as \( T_0 \sim 1/n_f \), where \( n_f \) refers to the free vortex density (Appendix E). This displays the non-fermi liquid behaviour of the Bose metal phase and verifies that there is no consistency problem with the third law. Clearly, the BM phase has more entropy than a normal fermi liquid at low temperatures.

The foregoing temperature variation of the specific heat shows that there are weak singularities at \( T = 0 \) in the metallic state. Thus, the BM state has the character of a critical point as \( T \to 0 \). This requires further study since the coefficients which enter the specific heat calculation would undergo strong renormalisation and the actual temperature dependence may be somewhat different. These arguments also suggest that there is no good length scale in the quantum liquid phase at \( T = 0 \). This can be seen by calculating the charge charge correlation function. Using equations (A4) (or, equivalently (B2)) and (E3), we obtained:

\[
<\delta n_{q,\omega} \delta n_{-q,-\omega} > \approx \frac{q^2}{-i\tilde{a}^2 + bq^2}
\]

where the coefficients \( \tilde{a} \) and \( \tilde{b} \) may be read off from eqn. (E3). Thus, the excitation spectrum of the charge fluctuations is gapless and diffusive. As we pointed out in section IIIB, because of the algebra of phase fluctuations, this does not contradict the fact that there is a length scale associated with superfluidity viz. the superfluid correlation length \( \xi_+ \) tied to the free vortex density \( n_f \). Thus, the BM phase is analogous to a gapless quantum spin liquid, very much like an RVB state.

VI. DISCUSSION AND CONCLUSIONS

In summary, we have argued that the superconductor-insulator transition in low-\( T_c \) superconductors should be viewed as a two order parameter problem and observed that it is described by two coupled XY models in \( (2+1)-d \) in the charge picture. This change from a single \( (2+1)-d \) XY model, as conventionally thought, is tied to the inclusion of an appropriate vortex mass term in the vortex picture or non-local interactions in the charge picture. This leads to the interesting possibility of a novel disordered Bose metal phase distinct from the traditional superconductor and Bose insulator phases. On the basis of our analysis of the model, we expect a superconductor - insulator transition when \( V_{on-site} \gg V_{n.n.} \) (\( n.n. = \) nearest neighbour), a case close to that of the Josephson Junction arrays (with \( B_{ext} = 0 \)); and a superconductor - metal - insulator transition when \( V_{on-site} \sim V_{n.n.} \), a situation close to that of the granular superconductors (it may be helpful to note that \( V_i \sim (2e)^2/C_i \), \( C_i = \) capacitance). The physics of the problem seems to be controlled by a multicritical point. We find the properties of the Bose metal to be critical. It is inherently connected to the quantum spin liquid states and is strongly reminiscent of the gapless RVB phase. The resistivity of this novel metallic phase predicted by our calculations finds an excellent match with the experiments on Gallium films. Dissipation in the bosonic system at low temperatures, within our model, is hydrodynamical: it comes from the fact that a moving charge (vortex) dissipates as it moves through a background of charged particles (vortices and antivortices). The heat is carried away by the gapless transverse modes representing the plasmons. This source of dissipation translates into quantum fluctuations in the charge picture.

To our knowledge, so far there has been no metallic phase proposed in the phase diagram of the Bose localisation problem, within the scope of any physically realisable model. This paper is the first attempt to propose this concept and establish some of the basic principles which underlie the existence of such a phase. As a result, many of the features which are tied to the current experiments have gone unaddressed. Some of these are: existence of disorder, long-range \( 1/r \) interactions, non-zero magnetic field, finite temperature effects, etc. Also, in a realistic sample, an important source of heat bath in addition to the emission of transverse excitations is the electrons in the vortex core. This needs to be incorporated into the calculation.

An important concern is whether the BM phase will survive in improved calculations, given that our calculations have been mean field like. As we saw, the experiments argue in favour of the existence of this phase. Further, as we argued in section IIIId, the retardation effects leading to the destruction of vortex superfluidity and the binding of vortex - antivortex pairs, the features corresponding to the loss of charge order and the growth of phase order respectively, are controlled effectively by two separate parameters. Plus, as we found in section III, the constants which determine the curves LXM and ZYX in fig.1 differ by about an order of magnitude because of the strong retardation effects felt by the vortices in this model. In view of these facts and the self consistent nature of the calculation, it is reasonable to think that the curve ZYX will lie above the curve LXM and intersect it at a multicritical point even if the fluctuations are taken into account, implying a separate BM phase. Either way, it will be very useful to check this via numerical simulations. An important issue which confronts us at the moment is — what is the universality class of the critical phenomena associated with model (1)? Although we have commented on this above, one needs to perform renormalisation group calculations or numerical simulations to answer this question completely. What is clear from our calculations, however, is that the critical behaviour, both at \( T = 0 \) and \( T = \) small but finite, implies a new universality class with a multicritical point which is very rich.

Given the considerable amount of work done on the Bose-Hubbard models, one might wonder why this phase was not observed in the other theoretical constructs, par-
ticularly in simulations. In view of this, we would like to mention a few points, in addition to the idea of two order parameters, which distinguishes our work from the previous ones. Firstly, we have worked in the large $n_0$-limit of the problem. Most of the current work on this model does not treat this limit consistently. Also, in this limit, the algebra which determines the quantum mechanics is qualitatively different from that of the hard core bosons, i.e. $E_2$, and the chemical potential $\mu$ does not play a significant role (as found in section Iib). In the small $n_0$-limit, $\mu$ plays a dominant role, especially in stabilising the commensurate Bose insulator (CBI) phase (mostly called Mott insulator in the Bose literature). The weakening effect of $\mu$ as $n_0$ increases is clearly observable in the shrinking of CBI lobes with $n_0$, evidenced by the perturbative and Quantum Monte Carlo calculations of Ref. [1]. This provides additional support to the results obtained here in this approximation. Further, we also observed that the nearest neighbour interaction plays a crucial role in opening up the metallic phase in the system. There is no such phase with just on-site repulsion.

*On the supersolid phase:* It is somewhat tempting to use eqn.(19) to the fullest extent and extend the curve ZYX beyond the point X along the dotted line XO. Eqn.(19) is really not good beyond the point X; but if we take it at face value, it seems to suggest a region OXM where both phase order and charge order exist, i.e. a supersolid phase. This has some support, if we look at the action (10). Here, as we noted earlier, when $V_1 \ll V_0$, $\theta_i \simeq 0$, implying the presence of charge order in the system. This, however, corresponds to trivial charge ordering only, viz. $\delta n_i \simeq 0$, i.e. $n_i \simeq n_0$. This phase needs to be verified by better calculations. If this phase exists in improved calculations, the point X would be tetracritical.

Before we close, we would like to make a short comment on the Mo-Ge system [1], a situation where magnetic field is present. As we argued in section IId, the physics of quantum phase fluctuations in 2-d superconductors is a quantum mechanical generalisation of the KT scenario. We expect this to happen when magnetic field is present as well. Let us recollect the classical case for completeness. Here, dislocation - antidislocation pairs are induced in the vortex lattice which unbind as a result of thermal fluctuations [12]. At low temperatures, quantum effects dominate and we expect the unbinding resulting from quantum fluctuations instead. But, as pointed out in this paper, their kinetic energy should receive two contributions — from the zero point motion and the action of a heat bath. The metallic state is realised when the dislocations in the vortex lattice (more precisely, vortex glass) phase move under the action of a heat bath. This is very much along the lines of what had been suggested in Ref. [1]. Following our discussion in section V, we suggest specific heat measurements be made to spot any non-fermionic behaviour of this metal-like phase. We plan to take up some of the unresolved issues mentioned in this section in a future publication.

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**APPENDIX A: THE DUALITY TRANSFORMATION**

We start from action (9), but with the transformation $m_i \to e^{-iQ \cdot r_i} m_i$ undone, so that

$$S = i \sum_i m_i (\nabla \tau \phi_i) + V_0 \Delta \tau \sum_i m_i^2$$

$$+ V_1 \Delta \tau \sum_{i \alpha} (m_i + m_{i+\alpha})^2 - J \Delta \tau \sum_{i \alpha} \cos(\nabla \alpha \phi_i). \quad (A1)$$

We now use Villain transformation on the $J$ term. For small values of $J \Delta \tau$, one has [3]

$$e^{J \Delta \tau \sum_{i \alpha} \cos(\nabla \alpha \phi_i)} \simeq e^{-\ln(\frac{2}{J \Delta \tau}) \sum_{i \alpha} n_{i+\alpha}^2 + \sum_i \phi_i (\nabla \alpha n_{i \alpha})}$$

Integrating out the phase degrees of freedom $\phi_i$, one obtains —

$$Z = \sum_{\{m_i\}} \sum_{\{n_{i \alpha}\}} e^{-S},$$

$$S = \ln(\frac{2}{J \Delta \tau}) \sum_{i \alpha} n_{i+\alpha}^2 + V_0 \Delta \tau \sum_i m_i^2$$

$$+ V_1 \Delta \tau \sum_{i \alpha} (m_i + m_{i+\alpha})^2 \quad (A2)$$

supplemented with the constraint

$$\nabla \tau m_i + \nabla \alpha n_{i \alpha} = 0, \quad (A3)$$

(with the summation over $\alpha$ implied). Defining $n_{i \alpha}^\mu = (m_i, n_{i \alpha}^\mu)$, $\mu = 0, 1, 2$, one obtains the constraint (A3) as

$$\nabla \mu n_{i \alpha}^\mu = 0.$$  

This allows us to define an integer gauge field $A_\mu^\mu$, via

$$n_{i \alpha}^\mu = \varepsilon_{\mu\nu\rho} \nabla \nu A_\rho^\mu.$$  

(A4)

A purist might object to our not using the shifted lattice operators $A_\mu^\rho_{\mu-\rho}$ in eqn.(A4) [13]. This does not matter for our case since the model involves a single gauge field. Now, transforming over to the Fourier space $(m_i = \sum_{\omega, q} e^{i q \cdot x_i} m_{q \omega})$ with $q = (\omega, \vec{q})$ and $x_i$ referring to the $i^{th}$ site on the space-time lattice, with $N_0$ the total no. of lattice sites, one obtains,
\[ Z = \sum_{\{A_{\mu}^\alpha\}} e^{-S}, \]

with

\[ S = \ln \left( \frac{2}{\Delta_T} \right) \sum_{q,\omega} \hat{q}^2 A_{q,\omega}^0 A_{-q,-\omega}^0 + G^{-1}(\omega,q) A_{q,\omega}^\alpha A_{-q,-\omega}^\alpha \]  \hspace{1cm} \text{(A5)}

and \( G^{-1}(\omega,q) \) given by eqn.(12) and \( A_{q,\omega}^\mu \) refer to the fourier transform of \( A_{\mu}^\alpha \). Here \( \hat{q} \) and \( \hat{\omega} \) refer to lattice momentum and frequency respectively, and \( \hat{q}^2 = \sum_{\alpha} \hat{K}_\alpha(q)K_\alpha(q), \) with \( \hat{K}_\alpha(q) = \frac{1}{4}(1 - e^{-i\hat{q}s}), K_\alpha(q) = \frac{1}{4}(e^{iq\alpha} - 1) \) \( [20] \). And, \( (\hat{q} - \hat{Q})^2 = \sum_{\alpha} \hat{K}_\alpha(q - \hat{Q})K_\alpha(q - \hat{Q}) \) in eqn.(12). Eqn.(14) is complete with the gauge condition,

\[ \nabla_\alpha A_{\mu}^\alpha = 0. \]  \hspace{1cm} \text{(A6)}

Now, we introduce the vortex variables \( j_{\mu}^\alpha \) via the Poisson summation formula \( [13] \)

\[ Z = \sum_{\{j_{\mu}^\alpha\}} \int D A_{\mu}^\alpha e^{-S - 2\pi i \sum \bar{j}_{\mu}^\alpha A_{\mu}^\alpha}. \]  \hspace{1cm} \text{(A7)}

Integrating out the gauge field degree of freedom, one obtains eqn.(11).

APPENDIX B: CONTINUUM VERSION

The features of the vortices discussed in section IIId are not special to the lattice case, but exists in the continuum case as well. This gives us a more general frame in which to study the model. The considerations in the continuum case start from the action \( [43,24] \)

\[ \hat{S} = \int d\tau d^2x [(m/\bar{\rho}) | \hat{J} |^2 + \frac{1}{2} \int d^2y V(x-y)\delta \rho(x)\delta \rho(y) + 2\pi i \bar{j}_{\mu} A_{\mu}] \]  \hspace{1cm} \text{(B1)}

where \( \hat{J} \) refers to the charge current and \( \delta \rho \) the charge fluctuations, with the gauge field \( A_{\mu} \) given by

\[ j_{\mu}^\mu(x) = (\delta \rho, \hat{J}) = \varepsilon_{\mu\nu\rho} \partial_\nu A_\rho \]  \hspace{1cm} \text{(B2)}

Also, \( \bar{j}_{\mu} = (\rho_v, \bar{J}_\nu) \) refers to the vortex three current density, \( m = \) mass of the Cooper pairs, and \( \bar{\rho} = \) average Cooper pair density. We have not written down a Magnus force term on vortices due to a background charge condensate, because this is not contained in the lattice model (1) discussed in the paper. However, this may be important in a uniform system like Mo-Ge \( [14] \). Working in the transverse gauge \( \hat{\nabla}A = 0 \), we obtain after integrating out the gauge field —

\[ \hat{S} = 2\pi^2 \bar{\rho} m \sum_{k,\omega} \bar{j}_k^0 \bar{j}_{-k,-\omega} + \frac{1}{k^2} \sum_{k,\omega} \bar{G}(k)\bar{G}(\omega) \bar{j}_k^0 \bar{j}_{-k,-\omega} \]  \hspace{1cm} \text{(B3)}

where the Green’s function \( \bar{G}(\omega,k) \) is given by —

\[ \bar{G}^{-1}(\omega, k) = \omega^2 + (\bar{\rho}/m)k^2 V_k \]  \hspace{1cm} \text{(B4)}

where \( V_k \) is the Fourier transform of the potential \( V(r) \). From here onwards we consider the screened Coulomb potential with

\[ V_k = V_0\alpha/\sqrt{k^2 + \alpha^2} \]

In the limit \( \omega, k \to 0 \) and \( \omega \ll \bar{c}_s k \) (with \( \bar{c}_s \) the plasmon velocity), the Green’s function splits up into a singular part \( \bar{G}_s(\omega,k) \) and \( \bar{G}_0 \) just like in the lattice model —

\[ \bar{G}(\omega,k) \approx \bar{G}_s(\omega,k) + \bar{G}_0, \]  \hspace{1cm} \text{(B5)}

where

\[ \bar{G}_s(\omega,k) = 1/(\omega^2 + \bar{c}_s^2 k^2), \bar{G}_0 = 1/(2\bar{\rho}V_0\alpha^2/m), \]

with

\[ \bar{c}_s^2 = \bar{\rho}V_0/m. \]

Thus, from (B3) and (B5), we obtain —

\[ \hat{S} = 2\pi^2 \bar{\rho} m \sum_{k,\omega} \bar{j}_k^0 \bar{j}_{-k,-\omega} + \frac{1}{k^2} \sum_{k,\omega} \bar{G}_s(\omega,k)\bar{G}_s(\omega,k) \bar{j}_k^0 \bar{j}_{-k,-\omega} \]  \hspace{1cm} \text{(B6)}

Thus, comparing eqns.(B6) and (14), we see that the lattice and continuum actions of phase fluctuations, within the scope of our model, have identical low frequency and long wavelength limits.

Correspondence between the lattice version and the continuum version is established via the following identifications —

\[ m/\bar{\rho} \leftrightarrow 2\ln(2/J\Delta\tau) \]  \hspace{1cm} \text{(B7)}

\[ \bar{c}_s^2 = \bar{\rho}V_0/m \leftrightarrow c_s^2 = (V_0 + 8V_1)\Delta\tau/\ln(2/J\Delta\tau) \]  \hspace{1cm} \text{(B8)}

\[ \bar{G}_0 = 1/(2\bar{\rho}V_0\alpha^2/m) \leftrightarrow G_0 = b^2/(8(b^2 + c_s^2)^2) \]  \hspace{1cm} \text{(B9)}

Following Ref. \( [13] \), one can easily show that eqn.(B3) is equivalent to the following Hamiltonian for the vortices —

\[ \hat{H}_v = \frac{1}{2m_v} \sum_i (\bar{p}_i^2 + 2\pi q_i \bar{A}(x_i))^2 + \frac{1}{2m_v} \sum_i q_i q_j \ln | r_i - r_j | \]  \hspace{1cm} \text{(B10)}

where the summation is over all vortices and antivortices and the gauge field \( \bar{A} \) has the spectrum determined by \( G^{-1}(\omega,k) \) as in eqn.(15). Here \( q_i = \pm 1 \) refers to the charges on the vortices. Here \( m_v \) refers to the vortex mass.
APPENDIX C: AN ESTIMATE OF \( G(0) \)

From section IIIa, we have —

\[
G(0) = \frac{1}{2} \int_{-\pi}^{\pi} \frac{d\omega}{2\pi} \int_{-\pi}^{\pi} \frac{d^2q}{(2\pi)^2} \frac{1}{1 - \cos \omega + A^2} \tag{C1}
\]

where

\[
A^2 = [2 - (\cos q_x + \cos q_y)][\kappa_0^2 + 2\kappa_1^2(2 + \cos q_x + \cos q_y)]
\tag{C2}
\]

with

\[
\kappa_0^2 = c^2 \text{ and } \kappa_1^2 = b^2/8,
\]

where \(b^2\) and \(c^2\) have been introduced in section IId.

From (C1), we obtain

\[
G(0) = \frac{1}{2} \int_{-\pi}^{\pi} \frac{d^2q}{(2\pi)^2} F(q), \tag{C3}
\]

with

\[
F(q) = \frac{1}{\sqrt{A^2(2 + A^2)}} \tag{C4}
\]

Important contributions to (C3) come from the low momentum region. Thus,

\[
F(q) \approx 1/\sqrt{2(\kappa_0^2 + 8\kappa_1^2)} \sqrt{2 - (\cos q_x + \cos q_y)} \tag{C5}
\]

Combining eqns.(C3) and (C5), we are led to the condition (17), with

\[
b_0 = \int_0^{\pi/2} \frac{d^2x}{\sqrt{\sin^2 x + \sin^2 y}} \tag{C6}
\]

APPENDIX D: SELF CONSISTENT FUNCTIONAL APPROACH

In this appendix, we give a brief review of the self-consistent functional approach discussed by Ioffe et. al. This is essentially a hydrodynamical kind of approach. We start with the following electromagnetic Lagrangian of bosons in imaginary time (in \(2 - d\)) —

\[
L_B = L_B^0 + \tilde{L}_B + L_B^\eta. \tag{D1}
\]

Here

\[
L_B^0 = \psi^* \frac{\partial \psi}{\partial \tau} + \frac{1}{2m} | \nabla \psi |^2 + \frac{g^2}{2} \rho(r) |n - r' | \rho(r') - \mu | \psi |^2 - (D2a)
\]

\[
\tilde{L}_B = -\frac{i}{2m} \vec{a} [\psi \nabla \psi^* - \psi^* \nabla \psi] + \frac{1}{2m} \vec{a}^2 | \psi |^2 - (D2b)
\]

\[
L_B^\eta = \frac{1}{2g^2} [ (\partial_r \vec{a})^2 + c^2 (\nabla \times \vec{a})^2 ] - (D2c)
\]

where \(\psi, \psi^*\) refer to the Bose fields and \(\rho = \psi \psi^*\). We work in the transverse gauge \(\nabla . \vec{a} = 0\). The dimensionless coupling constants determining the strength of the Coulomb repulsion and transverse gauge field, viz. \(\alpha_c\) and \(\alpha_g\) respectively, are defined as —

\[
\alpha_c = \frac{g^2 m}{16\pi^2 n}, \tag{D3}
\]

\[
\alpha_g = \frac{g^2}{8\pi mc^2}, \tag{D4}
\]

where \(n\) is the average density of bosons. Comparison of eqn.(D2) with eqn.(15) allows us to identify —

\[
g^2 \leftrightarrow 2\pi^2/ln(2/J\Delta \tau)
\]

\[
c^2 \leftrightarrow c_s^2
\]

and, \(m/2n \leftrightarrow \pi^2 G_0/ln(2/J\Delta \tau)\)

The last identification follows from the fact that the weight of the zero point motion term \(|\vec{f}|^2\) is \(m/2n\). Using eqns. (D3) and (D4), we are led to eqns.(18) [26].

The calculation of the depletion of the Bose condensate from the action (D2) proceeds in two steps. The first involves determining the effective functional of the superfluid Bose system in the absence of a transverse gauge field. And, the second involves determining the effect of the gauge field on this system, i.e. calculating the change in the superfluid density.

The effective functional of the superfluid Bose system (with \(\vec{a} = 0\) is obtained by expanding the Bose field in terms of the slow and fast modes, and integrating out the fast modes [34]. Galilean invariance of this system dictates \(n_s = n\) [26]. The leading terms in the effective action are (for small values of \(\alpha\)) —

\[
S_0 = \int d^3x d\tau \left[ \frac{n}{2m} | \nabla \phi |^2 - i \pi \partial_\tau \phi + \frac{1}{8mn} | \nabla \pi |^2 \right. \tag{D5}
\]

\[
+ \frac{g^2}{2} \sum_{q,\omega} \pi_{q,\omega} \frac{1}{q^2 - \pi_{q,\omega}^2} \left. \right]
\]

where \(\phi\) and \(\rho(x, \tau) = n + \pi(x, \tau)\) are the phase and the amplitude of the slow mode \(\psi_0 = \sqrt{\rho(x, \tau)} e^{i\phi(x, \tau)}\). We now work in the real time and integrate out the phase variables. This leads to an action solely in terms of the density fluctuation variables \(\pi\) —

\[
e^{-2} \int d^3x d^3x' \pi(x) C^{-1}(x-x') \pi(x')
\]

where

\[
C(\omega, q) = \frac{na^2/m}{\omega^2 - (q^2/2m)^2 - na^2/m + i\delta} \tag{D6}
\]

and \(d^3x = d^2xdt\).
To calculate the change of the superfluid density $n_s = n$ due to the action of the transverse gauge field, we focus on the diamagnetic term in (D2b)

$$-\frac{1}{2m} a^2 \psi_0^\dagger \psi_0 = -\frac{1}{2m} a^2 (n + \pi). \quad (D7)$$

The weight of the $a^2$ term determines the superfluid density. One now integrates out the $\pi$-degrees of freedom. The coupling term in (D7) leads to an $a^2 - a^2$ interaction term. One then (a) decouples this term in a mean field approximation and (b) rewrites the gauge field in terms of fast and slow modes, $a_1$ and $a_0$, and integrates out the fast modes. The weight of the $a_0^2$ term then gives the self-consistent equation for the superfluid density $n_s$,

$$n_s = n + \frac{i}{m} \int d^3x C(x) D(x) \quad (D8)$$

where $D(x)$ is the gauge field correlator

$$D(\omega, q) = \frac{g^2}{\omega^2 - (c^2 q^2 + n_s g^2/m) + i\delta} \quad (D9)$$

Eqn.(D8) leads to the phase diagram in $\alpha - \alpha_g$ plane, mentioned in section IIIb.

**APPENDIX E: SPECIFIC HEAT CALCULATION**

To calculate the specific heat, we consider the Lagrangian (D2) without the log interaction, which is screened in the metallic phase. Connection with the original model is established via the mapping (D4) [39]. We first rescale the transverse gauge field $\tilde{a} \rightarrow g \tilde{a}$ so that the effective charge $g$ appears in the eqn.(D2b) rather than (D2c). Thus, the unperturbed gauge propagator is

$$D^{(R)}_{\alpha \beta}(q, \omega_n) = K_{\alpha \beta}(q) \frac{1}{\omega_n^2 + c^2 q^2}, \quad (E1)$$

where $\omega_n = 2\pi n T$ ($n =$integer) are Matsubara frequencies and

$$K_{\alpha \beta}(q) = \delta_{\alpha \beta} - \frac{q_\alpha q_\beta}{q^2}.$$  

To obtain the contribution of specific heat from the transverse modes, one integrates out the particle degrees of freedom in a one-loop approximation [39]. This leads to the retarded polarization function —

$$\Pi^{(R)}_{\alpha \beta}(q, \Omega) = -K_{\alpha \beta}(q) g^2 \left[ -i \frac{2n \Omega}{m v_B q} + c^2 \chi D q^2 \right] \quad (E2)$$

where $\chi_D = n_B (-\mu)/24\pi mc^2$ and $mv_B^2/2 = |\mu_B|$, with $n_B(\xi) = 1/(e^{\beta \xi} - 1)$. The structure of eqn.(E2) is quite independent of the statistics because the integrals which enter the calculation are of the form $\int_{-\mu}^\infty d\xi n(\xi)$ and $\int_{-\infty}^{\infty} d\xi \bar{n}(\xi)$. The renormalised gauge propagator is then given by Dyson’s eqn. —

$$D = D_0 + D_0 \Pi D$$

In the quasistatic approximation $\Omega \ll q$, we have the retarded gauge propagator as

$$D^{(R)}_{\alpha \beta}(q, \Omega) = K_{\alpha \beta}(q) \left[ 1/( -i \frac{2ng^2}{m v_B q} + c^2 q^2) \right], \quad (E3)$$

where $c^2 = c^2(1 + g^2 \chi_D)$. Integrating out the gauge fields leads to the free energy —

$$\beta F = -\frac{1}{2} Tr ln D(q, \omega_n) \quad (E4)$$

The specific heat is obtained from $C = T \partial S/\partial T$. The details of the calculation may be found in Ref. [39]. For the disordered case, the Green’s function is written as $G(p, \omega_n) = 1/(i\omega_n - \xi_p + i/2\tau sgn(\omega_n))$, with $\tau$ the scattering time for the vortices from the impurities (which appear as a static gauge field in the vortex picture) [24]. In this factor, the $1/v_B q$ appearing in the frequency term in (E2) and (E3) gets replaced by $2\tau$. The coefficients $A_p$, $A_\varphi$ and $T_0$ in eqn.(26) are approximately given by —

$$A_p \approx (2ng^2/mv_B c^2)^{2/3}/\pi \quad A_\varphi \approx ng^2 \tau/\pi mc^2 \quad T_0 \approx \overline{q}_c^2 mc^2/4ng^2\tau \quad (E5)$$

where $\overline{q}_c$ denotes an upper momentum cutoff $\sim 1/\xi_0$, with $\xi_0$ the Cooper pair size.

**APPENDIX F: COMMUTATION RULE (6C)**

In this appendix, we discuss how the commutation rule (6c) comes about. From commutators (6a) and (6b), we observe that $P$ and $P^\dagger$ are ladder operators. Hence, in the angular momentum representation ($L \mid \Omega_m \rangle = m \mid \Omega_m \rangle$) [47,58].

$$P = i \sum_{m=-\infty}^{\infty} \mid \Omega_{m-1} \rangle < \mid \Omega_m \rangle \approx i \sum_{m=-\infty}^{\infty} \mid \Omega_{m+1} \rangle < \mid \Omega_m \rangle \quad (F1)$$

$$P^\dagger = -i \sum_{m=-\infty}^{\infty} \mid \Omega_{m+1} \rangle < \mid \Omega_m \rangle \approx -i \sum_{m=-\infty}^{\infty} \mid \Omega_{m+1} \rangle < \mid \Omega_m \rangle \quad (F2)$$

where $\{ n_0 \} = n_0$ for integer $n_0$, and the floor or ceiling of $n_0$ (appropriately taken) otherwise. The approximate expressions written on the right hold only when $n_0 \gg 1$.

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For this part, we have replaced $\ln$ by $\log$ for convenience. These two factors really come about at different limits of the magnitude of $J \Delta \tau$ while performing the familiar transformation. Please see Ref. [34] for more details. We believe that this replacement does not create any major problem, since $\Delta \tau$ simply sets the energy scale here.

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FIG. 1. A schematic phase diagram for model (1). The various phases are demarcated by solid boundaries. X denotes the multicritical point. The dashed lines (OA, OB, OC) denote typical trajectories executed as the normal resistance $R_n$ of a superconducting film is tuned. Please see the text for details.

FIG. 2. The resistivity of the metallic phase as a function of $R_n$ in Ga film experiment [5] (the diamonds are the datapoints from fig.2 of this reference) and the best fit of eqn.(25) (the dashed curve). (We have not displayed those points of fig.2 in Ref.5 which are in the insulating regime and, expectedly, deviate from the best fit shown here.)
Fig. 1
Fig. 2

\[ \log R_\Omega(K\Omega) \] vs. \[ \log \left( \frac{R}{R_c} - 1 \right) \]