A Diophantine approximation problem with two primes and one $k$-th power of a prime

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Abstract

We refine a result of the last two Authors of [8] on a Diophantine approximation problem with two primes and a $k$-th power of a prime which was only proved to hold for $1 < k < 4/3$. We improve the $k$-range to $1 < k \leq 3$ by combining Harman’s technique on the minor arc with a suitable estimate for the $L^4$-norm of the relevant exponential sum over primes.

Keywords: Diophantine inequalities, Goldbach-type problems, Hardy-Littlewood method

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1. Introduction

This paper deals with an improvement of the result contained in [8], which is due to the last two Authors: we refer to its introduction for a more thorough description of the general Diophantine problem with prime variables. Here we just recall that the goal is to prove that the inequality

$$|\lambda_1 p_1^{k_1} + \cdots + \lambda_r p_r^{k_r} - \omega| \leq \eta,$$

where $k_1, \ldots, k_r$ are fixed positive numbers, $\lambda_1, \ldots, \lambda_r$ are fixed non-zero real numbers and $\eta > 0$ is arbitrary, has infinitely many solutions in prime variables $p_1, \ldots, p_r$ for any given real number $\omega$, under as mild Diophantine assumptions on $\lambda_1, \ldots, \lambda_r$ as possible. In some cases, it is even possible to prove that the above inequality holds when $\eta$ is a small negative power of the largest prime occurring in it, usually when $1/k_1 + \cdots + 1/k_r$ is large enough.

The problem tackled in [8] had $r = 3$, $k_1 = k_2 = 1$, $k_3 = k \in (1, 4/3)$. Assuming that $\lambda_1/\lambda_2$ is irrational and that the coefficients $\lambda_j$ are not all of the same sign, the last two Authors proved
that one can take \( \eta = (\max \{p_1, p_2, p_3^k\})^{-\phi(k)+\varepsilon} \) for any fixed \( \varepsilon > 0 \), where \( \phi(k) = (4 - 3k)/(10k) \).

Our purpose in this paper is to improve on this result both in the admissible range for \( k \) and in the exponent, replacing \( \phi(k) \) by a larger value in the common range. More precisely, we prove the following Theorem.

**Theorem 1.** Assume that \( 1 < k \leq 3 \), \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are non-zero real numbers, not all of the same sign, that \( \lambda_1 / \lambda_2 \) is irrational and let \( \omega \) be a real number. The inequality

\[
|\lambda_1p_1 + \lambda_2p_2 + \lambda_3p_3^k - \omega| \leq (\max \{p_1, p_2, p_3^k\})^{-\phi(k)+\varepsilon}
\]  

(1)

has infinitely many solutions in prime variables \( p_1, p_2, p_3 \) for any \( \varepsilon > 0 \), where

\[
\psi(k) = \begin{cases} 
(3 - 2k)/(6k) & \text{if } 1 < k \leq \frac{5}{3}, \\
1/12 & \text{if } \frac{5}{3} < k \leq 2, \\
(3 - k)/(6k) & \text{if } 2 < k < 3, \\
1/24 & \text{if } k = 3.
\end{cases}
\]  

(2)

We point out that in the common range \( 1 < k < 4/3 \) we have \( \psi(k) > \phi(k) \). We also remark that the strong bounds for the exponential sum \( S_k \), defined in (3) below, that recently became available for integral \( k \) (see Bourgain [1] and Bourgain, Demeter & Guth [2]) are not useful in our problem.

The technique used to tackle this problem is the variant of the circle method introduced in the 1940’s by Davenport & Heilbronn [4], where the integration on a circle, or equivalently on the interval \([0, 1]\), is replaced by integration on the whole real line. Our improvement is due to the use of the Harman technique on the minor arc and to the fourth-power average for the exponential sum \( S_k \) for \( k \geq 1 \).

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**2. Outline of the proof**

Throughout this paper \( p_i \) denotes a prime number, \( k \geq 1 \) is a real number, \( \varepsilon \) is an arbitrarily small positive number whose value may vary depending on the occurrences and \( \omega \) is a fixed real number. In order to prove that (1) has infinitely many solutions, it is sufficient to construct an increasing sequence \( X_n \) that tends to infinity such that (1) has at least one solution with \( \max \{p_1, p_2, p_3^k\} \in [\delta X_n, X_n] \), with a fixed \( \delta > 0 \) which depends only on the choice of \( \lambda_1, \lambda_2 \) and \( \lambda_3 \). Let \( q \) be a denominator of a convergent to \( \lambda_1 / \lambda_2 \) and let \( X_n = X \) (dropping the suffix \( n \)) run through the sequence \( X = q^3 \). The main quantities we will use are:

\[
S_k(\alpha) = \sum_{\delta X \leq p^k \leq X} \log p \ e(p^k \alpha), \quad U_k(\alpha) = \sum_{\delta X \leq n^k \leq X} e(n^k \alpha) \quad \text{and} \quad T_k(\alpha) = \int_{(\delta X)^{1/k}}^{X^{1/k}} e(t^k \alpha) \, dt,
\]  

(3)
where $e(\alpha) = e^{2\pi i \alpha}$. We will approximate $S_k$ with $T_k$ and $U_k$. By the Prime Number Theorem and first derivative estimates for trigonometric integrals we have
\begin{equation}
S_k(\alpha) \ll_{k, \delta} X^{1/k}, \quad T_k(\alpha) \ll_{k, \delta} X^{1/k-1} \min\{X, |\alpha|^{-1}\}.
\end{equation}

Moreover the Euler summation formula implies that
\begin{equation}
T_k(\alpha) - U_k(\alpha) \ll_{k, \delta} 1 + |\alpha|X.
\end{equation}

We also need a continuous function we will use to detect the solutions of (1), so we introduce
\[ \hat{K}_\eta(\alpha) := \max\{0, \eta - |\alpha|\}, \quad \text{where } \eta > 0, \]
which is the Fourier transform of the function $K_\eta$ defined by
\[ K_\eta(\alpha) = \left(\frac{\sin(\pi \alpha \eta)}{\pi \alpha}\right)^2 \]
for $\alpha \neq 0$ and, by continuity, $K_\eta(0) = \eta^2$. A well-known estimate is
\begin{equation}
K_\eta(\alpha) \ll \min\{\eta^2, |\alpha|^{-2}\}.
\end{equation}

Let now
\[ \mathcal{P}(X) = \{(p_1, p_2, p_3) : \delta X < p_1, p_2 \leq X, \delta X < p_3 \leq X\} \]
and
\[ \mathcal{J}(\eta, \omega, \mathcal{X}) = \int_{\mathcal{X}} S_1(\lambda_1 \alpha) S_1(\lambda_2 \alpha) S_k(\lambda_3 \alpha) K_\eta(\alpha)e(-\omega \alpha) \, d\alpha, \]
where $\mathcal{X}$ is a measurable subset of $\mathbb{R}$. From (3) and using the Fourier transform of $K_\eta(\alpha)$, we get
\[ \mathcal{J}(\eta, \omega, \mathbb{R}) = \sum_{(p_1, p_2, p_3) \in \mathcal{P}(X)} \log p_1 \log p_2 \log p_3 \max\{0, \eta - |\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^k - \omega|\} \leq \eta (\log X)^3 N(X), \]
where $N(X)$ actually denotes the number of solutions of the inequality (1) with $(p_1, p_2, p_3) \in \mathcal{P}(X)$. In other words $\mathcal{J}(\eta, \omega, \mathbb{R})$ provides a lower bound for the quantity we are interested in; therefore it is sufficient to prove that $\mathcal{J}(\eta, \omega, \mathbb{R}) > 0$.

We now decompose $\mathbb{R}$ into subsets such that $\mathbb{R} = \mathcal{M} \cup \mathcal{M}^* \cup \mathcal{M} \cup \mathcal{T}$ where $\mathcal{M}$ is the major arc, $\mathcal{M}^*$ is the intermediate arc (which is non-empty only for some values of $k$, see section 6), $\mathcal{M}$ is the minor arc and $\mathcal{T}$ is the trivial arc. The decomposition is the following: if $1 < k < 5/2$ we consider
\begin{align*}
\mathcal{M} &= [-P/X, P/X], & \mathcal{M}^* &= \emptyset, \\
\mathcal{M} &= [P/X, R] \cup [-R, -P/X], & \mathcal{T} &= \mathbb{R} \setminus (\mathcal{M} \cup \mathcal{M}^* \cup \mathcal{M}), \quad (7)
\end{align*}
while, for $5/2 \leq k \leq 3$, we set
\[
\mathfrak{M} = [-P/X, P/X], \quad \mathfrak{M}^* = [P/X, X^{-3/5}] \cup [-X^{-3/5}, -P/X],
\]
\[
m = [X^{-3/5}, R] \cup [-R, -X^{-3/5}], \quad t = \mathbb{R} \setminus (\mathfrak{M} \cup \mathfrak{M}^* \cup m),
\] (8)
where the parameters $P = P(X) > 1$ and $R = R(X) > 1/\eta$ are chosen later (see (15) and (16)) as well as $\eta = \eta(X)$, that, as we explained before, we would like to be a small negative power of $\max\{p_1, p_2, p_3^k\}$ (and so of $X$). We have to distinguish two cases in the previous decomposition of the real line in order to avoid a gap between the end of the major arc and the beginning of the minor arc, where we can prove Lemma 1 in the form that we need: see the comments at the beginning of section 6 and just before the statement of Lemma 1. As we will see later in section 6 we need to introduce intermediate arc only for $k \geq 5/2$.

The constraints on $\eta$ are in (18), (20) and (21), according to the value of $k$. In any case, we have $\mathcal{F}(\eta, \omega, \mathbb{R}) = \mathcal{F}(\eta, \omega, \mathfrak{M}) + \mathcal{F}(\eta, \omega, \mathfrak{M}^*) + \mathcal{F}(\eta, \omega, m) + \mathcal{F}(\eta, \omega, t)$. We expect that $\mathfrak{M}$ provides the main term with the right order of magnitude without any special hypothesis on the coefficients $\lambda_j$. It is necessary to prove that $\mathcal{F}(\eta, \omega, \mathfrak{M}^*)$, $\mathcal{F}(\eta, \omega, m)$ and $\mathcal{F}(\eta, \omega, t)$ are $o(\mathcal{F}(\eta, \omega, \mathfrak{M}^*))$ as $X \to +\infty$ on the particular sequence chosen: we show that the contribution from trivial arc is “tiny” with respect to the main term. The main difficulty is to estimate the minor arc contribution; in this case we will need the full force of the hypothesis on the coefficients $\lambda_j$ and the theory of continued fractions.

Remark: from now on, anytime we use the symbol $\ll$ or $\gg$ we drop the dependence of the approximation from the constants $\lambda_j, \delta$ and $k$. We use the notation $f = \infty(g)$ for $g = o(f)$.

3. Lemmas

In their original paper [4] Davenport and Heilbronn approximate directly the difference $|S_k(\alpha) - T_k(\alpha)|$ estimating it with $O(1)$. The $L^2$-norm estimation approach (see Brüdern, Cook & Perelli [3] and [8]) improves on this taking the $L^2$-norm of $|S_k(\alpha) - T_k(\alpha)|$: this leads to the possibility of having a wider major arc compared to the original approach. We introduce the generalized version of the Selberg integral
\[
\mathcal{F}_k(X, h) = \int_X^{2X} \left( \theta((x + h)^{1/k}) - \theta(x^{1/k}) - ((x + h)^{1/k} - x^{1/k}) \right)^2 \, dx,
\]
where $\theta(x) = \sum_{p \leq x} \log p$ is the usual Chebyshev function. We have the following lemmas.

Lemma 1 ([7], Theorem 3.1). Let $k \geq 1$ be a real number. For $0 < Y \leq 1/2$ we have
\[
\int_{-Y}^Y |S_k(\alpha) - U_k(\alpha)|^2 \, d\alpha \ll \frac{X^{2/k-2} \log^2 X}{Y} + Y^2 X + Y^2 \mathcal{F}_k(X, \frac{1}{2Y}).
\]
Lemma 2 ([2], Theorem 3.2). Let \( k \geq 1 \) be a real number and \( \varepsilon \) be an arbitrarily small positive constant. There exists a positive constant \( c_1(\varepsilon) \), which does not depend on \( k \), such that

\[
J_k(X, h) \ll h^2 X^{2/k - 1} \exp \left( -c_1 \left( \frac{\log X}{\log \log X} \right)^{1/3} \right)
\]

uniformly for \( X^{1-5/(6k+\varepsilon)} \leq h \leq X \).

In order to prove our crucial Lemma 4 on the \( L^4 \)-norm of \( S_k(\alpha) \), we need the following technical result.

Lemma 3. Let \( \varepsilon > 0 \), \( k > 1 \) and \( \gamma > 0 \). Let further \( \mathcal{B}(X^{1/k}; k; \gamma) \) denote the number of solutions of the inequalities

\[
|n_1^k + n_2^k - n_3^k - n_4^k| < \gamma, \quad X^{1/k} < n_1, n_2, n_3, n_4 \leq 2X^{1/k}.
\]

Then

\[
\mathcal{B}(X^{1/k}; k; \gamma) \ll (X^{2/k} + \gamma X^{4/k-1}) X^\varepsilon.
\]

Proof. This is an immediate consequence of Theorem 2 of Robert & Sargos [9]; we just need to choose \( M = X^{1/k} \), \( \alpha = k \) and \( \gamma = \delta M^k \) there. \( \Box \)

Lemma 4. Let \( \varepsilon > 0 \), \( \delta > 0 \), \( k > 1, n \in \mathbb{N} \) and \( \tau > 0 \). Then we have

\[
\int_{-\tau}^{\tau} |S_k(\alpha)|^4 \, d\alpha \ll (\tau X^{2/k} + X^{4/k-1}) X^\varepsilon \quad \text{and} \quad \int_{n}^{n+1} |S_k(\alpha)|^4 \, d\alpha \ll (X^{2/k} + X^{4/k-1}) X^\varepsilon.
\]

Proof. A direct computation gives

\[
\int_{-\tau}^{\tau} |S_k(\alpha)|^4 \, d\alpha = \sum_{\delta X < p_1^k, p_2^k, p_3^k, p_4^k \leq X} (\log p_1) \cdots (\log p_4) \int_{-\tau}^{\tau} e((p_1^k + p_2^k - p_3^k - p_4^k) \alpha) \, d\alpha
\]

\[
\ll (\log X)^4 \sum_{\delta X < p_1^k, p_2^k, p_3^k, p_4^k \leq X} \min \left\{ \tau, \frac{1}{|p_1^k + p_2^k - p_3^k - p_4^k|} \right\}
\]

\[
\ll (\log X)^4 \sum_{\delta X < n_1^k, n_2^k, n_3^k, n_4^k \leq X} \min \left\{ \tau, \frac{1}{|n_1^k + n_2^k - n_3^k - n_4^k|} \right\}
\]

\[
\ll U \tau (\log X)^4 + V(\log X)^4,
\]

(9)

where

\[
U := \sum_{\delta X < n_1^k, n_2^k, n_3^k, n_4^k \leq X} \frac{1}{|n_1^k + n_2^k - n_3^k - n_4^k|}, \quad V := \sum_{\delta X < n_1^k, n_2^k, n_3^k, n_4^k \leq X} \frac{1}{|n_1^k + n_2^k - n_3^k - n_4^k|}.
\]
say. Using Lemma 3 on \( U \) we get

\[
U \ll \mathcal{B}(X^{1/k}; k; 1/\tau) \ll \left( X^{2/k} + \frac{1}{\tau} X^{4/k-1} \right) X^\varepsilon. \tag{10}
\]

Concerning \( V \), by a dyadic argument we get

\[
V \ll \log X \left( \max_{1/\tau < W \ll X} \frac{1}{\delta X < n_1^k, n_3^k, n_4^k \leq X} \sum_{W < |n_1^k + n_2^k - n_3^k - n_4^k| \leq 2W} \frac{1}{|n_1^k + n_2^k - n_3^k - n_4^k|} \right)
\ll \log X \max_{1/\tau < W \ll X} \left( \frac{1}{W} \mathcal{B}(X^{1/k}; k; 2W) \right) \ll \max_{1/\tau < W \ll X} \left( X^{4/k-1} + \frac{X^{2/k}}{W} \right) X^\varepsilon
\ll (\tau X^{2/k} + X^{4/k-1}) X^\varepsilon. \tag{11}
\]

Combining (9)-(11), the first part of the lemma follows. The second part can be proved in a similar way. \( \square \)

We need the following result in the proof of Lemma 9 and also when dealing with \( \mathfrak{M}^* \); see section 6.

**Lemma 5.** Let \( \delta > 0, k > 1, n \in \mathbb{N} \) and \( \tau > 0 \). Then

\[
\int_{-\tau}^{\tau} |S_k(\alpha)|^2 \, d\alpha \ll (\tau X^{1/k} + X^{2/k-1})(\log X)^3 \quad \text{and} \quad \int_n^{n+1} |S_k(\alpha)|^2 \, d\alpha \ll X^{1/k}(\log X)^3.
\]

**Proof.** It follows directly from the proof of Lemma 7 of Tolev [10] by letting \( c = k \) and using \( X^{1/k} \) instead of \( X \) there. We explicitly remark that the condition \( c \in (1, 15/14) \) in Tolev’s original version of this lemma depends on other parts of his paper; in fact the proof of Lemma 7 of [10] holds for every \( c > 1 \). \( \square \)

We now state some other lemmas which will be mainly useful on the minor and trivial arcs.

**Lemma 6 (Vaughan [11], Theorem 3.1).** Let \( \alpha \) be a real number and \( a, q \) be positive integers satisfying \( (a, q) = 1 \) and \( |\alpha - a/q| < 1/q^2 \). Then

\[
S_1(\alpha) \ll \left( \frac{X}{\sqrt{q}} + \sqrt{Xq} + X^{4/5} \right)(\log X)^4.
\]

**Lemma 7.** Let \( X^{-1} \ll |\alpha| \ll X^{-3/5} \). Then \( S_1(\alpha) \ll X^{1/2}|\alpha|^{-1/2}(\log X)^4 \).

**Proof.** It follows immediately from Lemma 6 by choosing \( q = \lfloor 1/\alpha \rfloor \) and \( a = 1 \). \( \square \)
Lemma 8. Let \( \lambda \in \mathbb{R} \setminus \{0\}, X \geq Z \geq X^{4/5}(\log X)^5 \) and \( |S_1(\lambda \alpha)| > Z \). Then there are coprime integers \( (a, q) = 1 \) satisfying

\[
1 \leq q \ll \left( \frac{X(\log X)^4}{Z} \right)^2, \quad |q \lambda \alpha - a| \ll \frac{X(\log X)^{10}}{Z^2}.
\]

Proof. Let \( Q \) be a parameter that we will choose later. By Dirichlet’s theorem there exist coprime integers \( (a, q) = 1 \) such that \( 1 \leq q \leq Q \) and \( |q \lambda \alpha - a| \ll Q^{-1} \leq q^{-1} \). The choice

\[
Q = \frac{Z^2}{X(\log X)^{10}}
\]

allows us to prove the second part of the statement and to neglect some terms in the estimations of \( |S_1(\lambda \alpha)| \). Using Lemma \( \text{[6]} \) knowing that \( Z \geq X^{4/5}(\log X)^5 \) and \( |S_1(\lambda \alpha)| > Z \), we can rewrite the bound for \( |S_1(\lambda \alpha)| \) neglecting the term \( X^{4/5} \):

\[
Z < |S_1(\lambda \alpha)| \ll (Xq^{-1/2} + X^{1/2}q^{1/2})(\log X)^4.
\]

The condition \( q \leq Q \) allows us to neglect the term \( X^{1/2}q^{1/2} \) and deal with small values of \( q \); in fact, if \( q > X^{1/2} \) then we would have a contradiction

\[
Z < |S_1(\lambda \alpha)| \ll X^{1/2}q^{1/2}(\log X)^4 \leq X^{1/2} \frac{Z}{X^{1/2}(\log X)^5}(\log X)^4 = o(Z).
\]

Then \( q \leq \min\{X^{1/2}, Q\} = X^{1/2} \), since \( Z = X^{4/5}(\log X)^5 > X^{3/4}(\log X)^5 \). Moreover, we can rewrite the inequality on \( |S_1(\lambda \alpha)| \) as

\[
Z < |S_1(\lambda \alpha)| \ll Xq^{-1/2}(\log X)^4
\]

and finally we get \( q^{1/2}Z \ll X(\log X)^4 \), which completes the proof. \( \square \)

The optimizations in section \( \text{[7]} \) depend either on \( L^2 \) or on \( L^4 \) averages of \( S_k \), according to the value of \( k \); these are provided by the following Lemmas. For brevity, we skip the proof of the first one, remarking that it requires Lemma \( \text{[5]} \).

Lemma 9 (Lemma 5 of \( \text{[8]} \)). Let \( \lambda \in \mathbb{R} \setminus \{0\}, k > 1, 0 < \eta < 1, R > 1/\eta \) and \( 1 < P < X \). We have

\[
\int_{P/X}^{R} |S_1(\lambda \alpha)|^2 K_\eta(\alpha) \, d\alpha \ll \eta X \log X \quad \text{and} \quad \int_{P/X}^{R} |S_k(\lambda \alpha)|^2 K_\eta(\alpha) \, d\alpha \ll \eta X^{1/k}(\log X)^3.
\]

Lemma 10. Let \( \lambda \in \mathbb{R} \setminus \{0\}, \varepsilon > 0, k > 1, 0 < \eta < 1, R > 1/\eta \) and \( 1 < P < X \). Then

\[
\int_{P/X}^{R} |S_k(\lambda \alpha)|^4 K_\eta(\alpha) \, d\alpha \ll \eta \max\{X^{2/k}, X^{4/k-1}\} X^\varepsilon.
\]
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Proof. Using (6) we obtain
\[ \int_{P/X}^{R} |S_k(\lambda \alpha)|^4 K_\eta(\alpha) \, d\alpha \ll \eta^2 \int_{P/X}^{1/\eta} |S_k(\lambda \alpha)|^4 \, d\alpha + \int_{1/\eta}^{R} |S_k(\lambda \alpha)|^4 \frac{d\alpha}{\alpha^2} = A + B, \]

say. By Lemma 4, we immediately get
\[ A \ll \eta^2 \int_{-|\lambda|/\eta}^{|\lambda|/\eta} |S_k(\alpha)|^4 \, d\alpha \ll \eta \max\{X^{2/k}, \eta X^{4/k-1}\} X^\varepsilon. \]

Moreover, again by Lemma 4, we have that
\[ B \ll \int_{|\lambda|/\eta}^{+\infty} \frac{|S_k(\alpha)|^4 \, d\alpha}{\alpha^2} \ll \sum_{n \geq |\lambda|/\eta} \frac{1}{(n-1)^2} \int_{n-1}^{n} |S_k(\alpha)|^4 \, d\alpha \]
\[ \ll \eta \max\{X^{2/k}, X^{4/k-1}\} X^\varepsilon. \]

Combining (12)-(14) and using 0 < \eta < 1, the lemma follows. □

As we remarked in the introduction, stronger bounds are now available for larger integral \( k \), but they are not useful for our purpose. The next Lemma provides the additional information that enables us to give a non-trivial result also when \( k = 3 \).

Lemma 11. Let \( \lambda \in \mathbb{R} \setminus \{0\} \), \( \varepsilon > 0 \), 0 < \eta < 1, R > 1/\eta \) and 1 < \( P < X \). Then
\[ \int_{P/X}^{R} |S_3(\lambda \alpha)|^8 K_\eta(\alpha) \, d\alpha \ll \eta X^{5/3+\varepsilon}. \]

Proof. Inserting Hua’s estimate in (6), i.e. \( \int_{0}^{1} |S_3(\alpha)|^8 \, d\alpha \ll X^{5/3+\varepsilon} \), in the body of the proof of Lemma 10 and exploiting the periodicity of \( S_3(\alpha) \), the result follows immediately. □

Another lemma on the minor arc is inserted in the body of section 7

4 The major arc

We recall the definitions in (7) and (8). The major arc computation is the same as in (8):
\[ \mathcal{F}(\eta, \omega, \mathfrak{M}) = \int_{\mathfrak{M}} S_1(\lambda_1 \alpha) S_1(\lambda_2 \alpha) S_k(\lambda_3 \alpha) K_\eta(\alpha) e(-\omega \alpha) \, d\alpha \]
\[ = \int_{\mathfrak{M}} T_1(\lambda_1 \alpha) T_1(\lambda_2 \alpha) T_k(\lambda_3 \alpha) K_\eta(\alpha) e(-\omega \alpha) \, d\alpha \]
\[ + \int_{\mathfrak{M}} (S_1(\lambda_1 \alpha) - T_1(\lambda_1 \alpha)) T_1(\lambda_2 \alpha) T_k(\lambda_3 \alpha) K_\eta(\alpha) e(-\omega \alpha) \, d\alpha \]
As the reader might expect the main term is given by the summand $J_1$.

Let $H(\alpha) = T_1(\lambda_1 \alpha) T_2(\lambda_2 \alpha) T_3(\lambda_3 \alpha) K_\eta(\alpha) e(-\omega \alpha)$ so that

$$J_1 = \int_{\mathbb{R}} H(\alpha) \, d\alpha + \mathcal{O}\left(\int_{P/X} |H(\alpha)| \, d\alpha\right).$$

Using (6) and (4), we get

$$\int_{P/X} |H(\alpha)| \, d\alpha \ll \eta^2 X^{1/k-1} \int_{P/X} \frac{d\alpha}{\alpha^3} \ll \eta^2 \frac{X^{1+1/k}}{P^2} = o(\eta^2 X^{1+1/k}),$$

provided that $P \to +\infty$. Let now $D = [\delta X, X]^2 \times [(\delta X)^{1/k}, X^{1/k}]$. We obtain

$$\int_{\mathbb{R}} H(\alpha) \, d\alpha = \iint_{D} \int_{\mathbb{R}} e((\lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3^k - \omega)\alpha) K_\eta(\alpha) \, d\alpha \, dt_1 \, dt_2 \, dt_3$$

$$= \iint_{D} \max\{0, \eta - |\lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_3^k - \omega|\} \, dt_1 \, dt_2 \, dt_3.$$  

Apart from trivial changes of sign, there are essentially two cases:

1. $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 < 0$
2. $\lambda_1 > 0, \lambda_2 < 0, \lambda_3 < 0$.

We deal with the first one. We warn the reader that here it may be necessary to adjust the value of $\delta$ in order to guarantee the necessary set inclusions. After a suitable change of variables, letting $D' = [\delta X, (1 - \delta)X]^3$, we find that

$$\int_{\mathbb{R}} H(\alpha) \, d\alpha \gg \iiint_{D'} \max\{0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3|\} u_3^{1/k-1} \, du_1 \, du_2 \, du_3$$

$$\gg X^{1/k-1} \iiint_{D'} \max\{0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3|\} \, du_1 \, du_2 \, du_3.$$
Apart from sign, the computation is essentially symmetrical with respect to the coefficients $\lambda_j$: we assume, as we may, that $|\lambda_3| \geq \max\{|\lambda_1|, |\lambda_2|\}$, the other cases being similar. Now, for $j = 1, 2$ let

$$a_j = \frac{2|\lambda_3|}{|\lambda_j|}, \ b_j = \frac{3}{2}a_j$$

and $\mathcal{D}_j = [a_j X, b_j X]$; if $u_j \in \mathcal{D}_j$ for $j = 1, 2$ then

$$\lambda_1 u_1 + \lambda_2 u_2 \in [4|\lambda_3|\delta X, 6|\lambda_3|\delta X]$$

so that, for every choice of $(u_1, u_2)$ the interval $[a, b]$ with endpoints $\pm \eta / |\lambda_3| + (\lambda_1 u_1 + \lambda_2 u_2) / |\lambda_3|$ is contained in $[\delta X, (1 - \delta)X]$. In other words, for $u_3 \in [a, b]$ the values of $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$ cover the whole interval $[-\eta, \eta]$. Hence for any $(u_1, u_2) \in \mathcal{D}_1 \times \mathcal{D}_2$ we have

$$\int_{\delta X}^{(1 - \delta)X} \max\{0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3|\} \, du_3 = |\lambda_3|^{-1} \int_{-\eta}^\eta \max\{0, \eta - |u|\} \, du \gg \eta^2.$$ 

Summing up, we get

$$J_1 \gg \eta^2 X^{1/k-1} \int_{\mathcal{D}_1 \times \mathcal{D}_2} \, du_1 du_2 \gg \eta^2 X^{1/k-1} X^2 = \eta^2 X^{1+1/k},$$

which is the expected lower bound.

4.2. Bound for $J_2, J_3$ and $J_4$

The computations for $J_2$ and $J_3$ are similar to and simpler than the corresponding one for $J_4$; moreover the most restrictive condition on $P$ arises from $J_4$; hence we will skip the computation for both $J_2$ and $J_3$. Using the triangle inequality and (6),

$$J_4 \ll \eta^2 \int_{\mathfrak{N}} |S_1(\lambda_1 \alpha)||S_1(\lambda_2 \alpha)||S_k(\lambda_3 \alpha) - T_k(\lambda_3 \alpha)| \, d\alpha$$

$$\leq \eta^2 \int_{\mathfrak{N}} |S_1(\lambda_1 \alpha)||S_1(\lambda_2 \alpha)||S_k(\lambda_3 \alpha) - U_k(\lambda_3 \alpha)| \, d\alpha$$

$$+ \eta^2 \int_{\mathfrak{N}} |S_1(\lambda_1 \alpha)||S_1(\lambda_2 \alpha)||U_k(\lambda_3 \alpha) - T_k(\lambda_3 \alpha)| \, d\alpha$$

$$= \eta^2 (A_4 + B_4),$$

say, where $U_k(\lambda_3 \alpha)$ and $T_k(\lambda_3 \alpha)$ are defined in (3). Using the Cauchy-Schwarz inequality, Lemmas 15, 23 and trivial bounds yields, for any fixed $A > 0$,

$$A_4 \ll X \left( \int_{\mathfrak{N}} |S_1(\lambda_1 \alpha)|^2 \, d\alpha \right)^{1/2} \left( \int_{\mathfrak{N}} |S_k(\lambda_3 \alpha) - U_k(\lambda_3 \alpha)|^2 \, d\alpha \right)^{1/2}$$

$$\ll X^{1+1/k}(\log X)^{(1-A)/2} = o(X^{1+1/k})$$

say.
as long as \( A > 1 \), provided that \( P \leq X^{5/(6k) - \epsilon} \). Using again the Cauchy-Schwarz inequality, (5) and trivial bounds, we see that

\[
B_4 \ll \int_{0}^{1/X} |S_1(\lambda_1 \alpha)| |S_1(\lambda_2 \alpha)| \, d\alpha + X \int_{1/X}^{P/X} \alpha |S_1(\lambda_1 \alpha)| |S_1(\lambda_2 \alpha)| \, d\alpha \\
\ll X + P \left( \int_{1/X}^{P/X} |S_1(\lambda_1 \alpha)|^2 \, d\alpha \right)^{1/2} \left( \int_{1/X}^{P/X} |S_1(\lambda_2 \alpha)|^2 \, d\alpha \right)^{1/2} \ll PX \log X.
\]

Taking \( P = o(X^{1/k}(\log X)^{-1}) \) we get \( \eta^2 B_4 = o(\eta^2 X^{1+1/k}) \). We may therefore choose

\[
P = X^{5/(6k) - \epsilon}.
\]

5. The trivial arc

We recall that the trivial arc is defined in (7) and (8). Using the Cauchy-Schwarz inequality and \( (4) \), we see that

\[
|\mathcal{F}(\eta, \omega, t)| \ll \int_{R}^{+\infty} |S_1(\lambda_1 \alpha)S_1(\lambda_2 \alpha)S_k(\lambda_3 \alpha)| K_\eta(\alpha) \, d\alpha \\
\ll X^{1/k} \left( \int_{R}^{+\infty} |S_1(\lambda_1 \alpha)|^2 K_\eta(\alpha) \, d\alpha \right)^{1/2} \left( \int_{R}^{+\infty} |S_1(\lambda_2 \alpha)|^2 K_\eta(\alpha) \, d\alpha \right)^{1/2} \\
\ll X^{1/k} C_1^{1/2} C_2^{1/2},
\]

say. Using the PNT and the periodicity of \( S_1(\alpha) \), for every \( j = 1, 2 \) we have that

\[
C_j = \int_{R}^{+\infty} |S_1(\lambda_1 \alpha)|^2 \frac{d\alpha}{\alpha^2} \ll \int_{|\lambda_j|R}^{+\infty} |S_1(\alpha)|^2 \frac{d\alpha}{\alpha^2} \ll \sum_{n \geq |\lambda_j|} \frac{1}{(n-1)^2} \int_{n-1}^{n} |S_1(\alpha)|^2 \, d\alpha \ll \frac{X \log X}{|\lambda_j|R}.
\]

Hence, recalling that \( |\mathcal{F}(\eta, \omega, t)| \) has to be \( o(\eta^2 X^{1+1/k}) \), the choice

\[
R = \eta^{-2}(\log X)^{3/2}
\]

is admissible.

6. The intermediate arc: \( 5/2 \leq k \leq 3 \)

In section \( [7] \) we apply Harman’s technique to the minor arc, using Lemma \( [8] \) as the starting point. We remark that in the course of the proof of Lemma \( [12] \), it is crucial that both the integers \( a_1 \) and \( a_2 \) appearing in \( (22) \) below do not vanish; in fact, if \( a_1 = 0 \), say, then \( \alpha \) is very small (\( \alpha \ll X^{-2/3} \)) and, according to our definitions above, it belongs to \( \mathfrak{M} \cup \mathfrak{M}^* \).
For small $k$ we do not need an intermediate arc, because the major arc is wide enough to rule out the possibility that $a_1 a_2 = 0$ for $\alpha \in m$. For larger values of $k$, the constraint (15) implies that there is a gap between the major arc and the minor arc which we need to fill: see the definition in (8). Using the intermediate arc $\mathfrak{M}^*$, we are able to cover more than needed.

Let $5/2 \leq k \leq 3$: we now show that the contribution of $\mathfrak{M}^*$ is negligible. Using (6), Lemma 7, the Cauchy-Schwarz inequality and (15) we get

$$
\mathcal{I}(\eta, \omega, \mathfrak{M}^*) \ll \eta^2 \int_{P/X}^{X^{-3/5}} |S_1(\lambda_1 \alpha)| |S_1(\lambda_2 \alpha)| |S_k(\lambda_3 \alpha)| \, d\alpha
$$

$$
\ll \eta^2 X (\log X)^8 \int_{P/X}^{X^{-3/5}} |S_k(\lambda_3 \alpha)| \frac{d\alpha}{\alpha}
$$

$$
\ll \eta^2 X (\log X)^8 \left( \int_{X^{-3/5}}^{X^{-3/5}} |S_k(\lambda_3 \alpha)|^2 \, d\alpha \right)^{1/2} \left( \int_{P/X}^{X^{-3/5}} \frac{d\alpha}{\alpha^2} \right)^{1/2}
$$

$$
\ll \eta^2 X (X^{1/k-3/5})^{1/2} (X^{1-5/(6k)})^{1/2} X^2 \ll \eta^2 X^{6/5+1/(12k)+\epsilon},
$$

where we also used Lemma 5 with $\tau = X^{-3/5}$ and the fact that $k \geq 5/2$. The last estimate is $o(\eta^2 X^{1+1/k})$ for every $5/2 \leq k < 55/12$.

7. The minor arc

Here we use Harman’s technique as described in [5]. The minor arc $m$ is defined in (7) and (8), according to the value of $k$. In view of using Lemma 8, we now split $m$ into subsets $m_1$, $m_2$ and $m^* = m \setminus (m_1 \cup m_2)$, where

$$
m_i = \{ \alpha \in m : |S_1(\lambda_1 \alpha)| \leq X^{5/6}(\log X)^5 \} \quad \text{for } i = 1, 2.
$$

In order to obtain the optimization, we chose to split the range for $k$ into two intervals in which to take advantage of the $L^2$-norm of $S_k(\alpha)$ in one case (Lemma 9) and the $L^4$-norm of $S_k(\alpha)$ in the other one (Lemma 10). The same choice will be made later in the discussion of the arc $m^*$. We will see that it is not possible to split the minor arc in another way in order to get a better result, in the present state of knowledge on exponential sums.

7.1. Bounds on $m_1 \cup m_2$

Using Hölder’s inequality and Lemma 9, for $1 < k \leq 6/5$ we obtain

$$
|\mathcal{I}(\eta, \omega, m_i)| \ll \int_{m_i} |S_1(\lambda_1 \alpha)||S_2(\lambda_2 \alpha)||S_k(\lambda_3 \alpha)|K_\eta(\alpha) \, d\alpha
$$

$$
\ll \left( \max_{\alpha \in m_i} |S_1(\lambda_1 \alpha)| \right) \left( \int_{m_i} |S_2(\lambda_2 \alpha)|^2 K_\eta(\alpha) \, d\alpha \right)^{1/2}
$$
where \( f \) which justifies the last line of (12).

The estimate in (9) should be \( o(\eta^2 X^{1+1/k}) \); hence this leads to the constraint

\[
\eta = \infty(X^{1/3-1/(2k)+\epsilon}),
\]

where \( f = \infty(g) \) means \( g = o(f) \).

Using Hölder’s inequality and Lemmas [3] and [4] for \( 6/5 < k < 3 \) we obtain

\[
|\mathcal{F}(\eta, \omega, m_i)| \ll \int_{m_i} |S_1(\lambda_1 \alpha)||S_1(\lambda_2 \alpha)||S_3(\lambda_3 \alpha)|K_\eta(\alpha) \, d\alpha \\
\ll \left( \max_{\alpha \in m_i} |S_1(\lambda_1 \alpha)|^{1/2} \right) \left( \int_{m_i} |S_1(\lambda_1 \alpha)|^2 K_\eta(\alpha) \, d\alpha \right)^{1/4} \\
\times \left( \int_{m_i} |S_3(\lambda_3 \alpha)|^4 K_\eta(\alpha) \, d\alpha \right)^{1/4} \left( \int_{m_i} |S_1(\lambda_2 \alpha)|^2 K_\eta(\alpha) \, d\alpha \right)^{1/4} \\
\ll X^{5/12}(\log X)^{5/2}(\eta X \log X)^{1/4}(\eta \max\{X^{2/k}, X^{4/(k-1)}\})^{1/4}(\eta X \log X)^{1/2} \\
\ll \eta \max\{X^{7/6+1/(2k)}, X^{11/12+1/k}\}X^\epsilon.
\]

The estimate in (16) should be \( o(\eta^2 X^{1+1/k}) \); hence this leads to

\[
\eta = \infty\left( \max\{X^{1/6-1/(2k)+\epsilon}, X^{-1/12+\epsilon}\} \right).
\]

If \( k = 3 \) we use Lemmas [3] and [4] thus getting

\[
|\mathcal{F}(\eta, \omega, m_i)| \ll \int_{m_i} |S_1(\lambda_1 \alpha)||S_1(\lambda_2 \alpha)||S_3(\lambda_3 \alpha)|K_\eta(\alpha) \, d\alpha \\
\ll \left( \max_{\alpha \in m_i} |S_1(\lambda_1 \alpha)|^{1/4} \right) \left( \int_{m_i} |S_1(\lambda_1 \alpha)|^2 K_\eta(\alpha) \, d\alpha \right)^{3/8} \\
\times \left( \int_{m_i} |S_3(\lambda_3 \alpha)|^8 K_\eta(\alpha) \, d\alpha \right)^{1/8} \left( \int_{m_i} |S_1(\lambda_2 \alpha)|^2 K_\eta(\alpha) \, d\alpha \right)^{1/2} \\
\ll \eta X^{31/24+\epsilon}.
\]

This bound leads to the constraint

\[
\eta = \infty(X^{-1/24+\epsilon}),
\]

which justifies the last line of (12).
7.2. Bound on \( m^* \)

We recall our definitions in (7) and (8). It remains to discuss the set \( m^* \) where the following bounds hold simultaneously

\[
|S_1(\lambda_1 \alpha)| > X^{5/6}(\log X)^5, \quad |S_1(\lambda_2 \alpha)| > X^{5/6}(\log X)^5, \quad T \leq |\alpha| \leq \eta^{-2}(\log X)^{3/2} = R,
\]

where \( T = P/X = X^{5/(6k) - 1 - \varepsilon} \) by our choice in (15) if \( k < 5/2 \), and \( T = X^{-3/5} \) otherwise. Using a dyadic dissection, we split \( m^* \) into disjoint sets \( E(Z_1, Z_2, y) \) in which, for \( \alpha \in E(Z_1, Z_2, y) \), we have

\[
Z_i < |S_1(\lambda_i \alpha)| \leq 2Z_i, \quad y < |\alpha| \leq 2y,
\]

where \( Z_i = 2^k X^{5/6}(\log X)^5 \) and \( y = 2^k X^{5/(6k) - 1 - \varepsilon} \) for some non-negative integers \( k_1, k_2, k_3 \).

It follows that the number of disjoint sets is, at most, \( \ll (\log X)^3 \). Let us write \( \text{d} \) as a shorthand for the set \( E(Z_1, Z_2, y) \). We need an upper bound for the Lebesgue measure of \( \text{d} \). In the following Lemma, it is crucial that both the integers \( a_1 \) and \( a_2 \) appearing in (22) below do not vanish; in fact, if \( a_1 = 0 \), say, then \( q_1 = 1 \) and \( \alpha \) is so small that it cannot belong to \( m \). If \( k \) is large, we treat the range \( [P/X, X^{-3/5}] \) and its symmetrical by means of the argument in section 6; this is needed because, in this case, the inequalities (22) below do not rule out the possibility that \( a_1 a_2 = 0 \), unless \( |\alpha| \) is large enough.

**Lemma 12.** Let \( \varepsilon > 0 \). We have that \( \mu(\text{d}) \ll yX^{8/3+\varepsilon}Z_1^{-2}Z_2^{-2} \), where \( \mu(\cdot) \) denotes the Lebesgue measure.

**Proof.** If \( \alpha \in \text{d} \), by Lemma 3 there are coprime integers \((a_1, q_1)\) and \((a_2, q_2)\) such that

\[
1 \leq q_i \ll \left( \frac{X(\log X)^4}{Z_i} \right)^2, \quad |q_i \lambda_i \alpha - a_i| \ll \frac{X(\log X)^{10}}{Z_i^2}. \tag{22}
\]

We remark that \( a_1 a_2 \neq 0 \) otherwise we would have \( \alpha \in \mathcal{M} \cup \mathcal{M}^* \). In fact, if \( a_1 a_2 = 0 \), recalling the definitions of \( Z_i \) and (22), \( \alpha \ll q_i^{-1}X(\log X)^{10}Z_i^{-2} \ll X^{-2/3} \).

Now, we can further split \( m^* \) into sets \( I = I(Z_1, Z_2, y, Q_1, Q_2) \) where, on each set, \( Q_j \leq q_j \leq 2Q_j \). Note that \( a_i \) and \( q_i \) are uniquely determined by \( \alpha \); in the opposite direction, for a given quadruple \( a_1, q_1, a_2, q_2 \), the inequalities (22) define an interval of \( \alpha \) of length

\[
\ll \min \left\{ \frac{X(\log X)^{10}}{Q_1 Z_1^2}, \frac{X(\log X)^{10}}{Q_2 Z_2^2} \right\} \ll \frac{X(\log X)^{10}}{Q_1^{1/2} Q_2^{1/2} Z_1 Z_2},
\]

by taking the geometric mean.

Now we need a lower bound for \( Q_1 Q_2 \); by (22) we obtain

\[
\left| q_2 a_1 \lambda_1 - a_1 q_2 \right| = \left| \frac{a_2}{\lambda_2} (q_1 \lambda_1 \alpha - a_1) - \frac{a_1}{\lambda_2} (q_2 \lambda_2 \alpha - a_2) \right|
\]
This proves the lemma.

Recalling that $Q_i \ll (X\log X)^{10}/Z_1^2$ and that $Z_i \gg X^{5/6}\log X$, we have

$$\left|a_2q_1\frac{a_1}{a_2} - a_1q_2\right| \ll \left(\frac{X\log X}{Z_1^{5/6}\log X}\right)^2 \left(\frac{X^{1/2}\log X}{Z_1^{5/6}\log X}\right)^2 \ll X^{-1/3}(\log X)^{-2} < \frac{1}{4q}. \quad (23)$$

We recall that $q = X^{1/3}$ is a denominator of a convergent of $\lambda_1/\lambda_2$. Hence by (23), Legendre’s law of best approximation for continued fractions implies that $|a_2q_1| \geq q$ and by the same token, for any pair $\alpha, \alpha'$ having distinct associated products $a_2q_1$,

$$|a_2(\alpha)q_1(\alpha) - a_2(\alpha')q_1(\alpha')| \geq q,$$

thus, by the pigeon-hole principle, there is at most one value of $a_2q_1$ in the interval $[rq, (r + 1)q)$ for any positive integer $r$. Furthermore $a_2q_1$ determines $a_2$ and $q_1$ to within $X^{\epsilon/2}$ possibilities (from the bound for the divisor function) and consequently also $a_2q_1$ determines $a_1$ and $q_2$ to within $X^{\epsilon/2}$ possibilities from (23).

Hence we got a lower bound for $q_1q_2$, since, using $Q_j \leq q_j \leq 2Q_j$, we get

$$q_1q_2 = a_2q_1\frac{q_2}{a_2} \gg \frac{rq}{a} \gg rqa^{-1}.$$

for the quadruple under consideration.

As a consequence we obtain that the total length of the part of $I(Z_1, Z_2, y, Q_1, Q_2)$ with $a_2q_1 \in [rq, (r + 1)q)$ is

$$\ll X^{1+\epsilon/2}\log X)^{10}Z_1^{-1}Z_2^{-1}r^{-1/2}q^{-1/2}y^{1/2}.$$

Now we need a bound for $r$: since $a_2q_1 \in [rq, (r + 1), q)$, we have

$$rq \leq |a_2q_1| \ll q_1q_2|a| \ll y\left(\frac{X\log X}{Z_1}\right)^2\left(\frac{X\log X}{Z_2}\right)^2 \ll \frac{yX^4(\log X)^{16}}{Z_1^2Z_2^2}$$

and hence we get

$$r \ll q^{-1}yX^4(\log X)^{16}Z_1^{-2}Z_2^{-2}.$$

Next, we sum on every interval to get an upper bound for the measure of $\mathcal{A}$: we get

$$\mu(\mathcal{A}) \ll \frac{X^{1+\epsilon/2}y^{1/2}(\log X)^{10}}{Z_1Z_2q^{1/2}} \sum_{1 \leq r \leq q^{-1}yX^4(\log X)^{16}Z_1^{-2}Z_2^{-2}} r^{-1/2}.$$

Standard estimates imply that the sum on the right is $\ll (q^{-1}yX^4(\log X)^{16}Z_1^{-2}Z_2^{-2})^{1/2}$, and recalling that $q = X^{1/3}$ we can finally write

$$\mu(\mathcal{A}) \ll yX^{3+\epsilon/2}(\log X)^{18}Z_1^{-2}Z_2^{-2}q^{-1} \ll yX^{8/3+\epsilon}Z_1^{-2}Z_2^{-2}.$$

This proves the lemma. \qed
8. Conclusion

Here we finally justify the choice of the function η in the statement of the main Theorem. Using Lemmas 10 and 12 we are now able to estimate \( J(\eta, \omega, \mathcal{A}) \) for \( 1 < k \leq 3 \). For \( k \geq \frac{5}{2} \), we also need the result in section 6.

If \( 1 < k \leq 6/5 \) we proceed as follows:

\[
|J(\eta, \omega, \mathcal{A})| \ll \int_{\mathcal{A}} |S_1(\lambda_1 \alpha)||S_1(\lambda_2 \alpha)||S_3(\lambda_3 \alpha)|K_\eta(\alpha) \, d\alpha
\]

\[
\ll \left( \int_{\mathcal{A}} |S_1(\lambda_1 \alpha)S_1(\lambda_2 \alpha)|^2 K_\eta(\alpha) \, d\alpha \right)^{1/2} \left( \int_{\mathcal{A}} |S_3(\lambda_3 \alpha)|^2 K_\eta(\alpha) \, d\alpha \right)^{1/2}
\]

\[
\ll \left( \min\{\eta^2, y^{-2}\} \right)^{1/2} (Z_1 Z_2)^2 \mu(\mathcal{A})^{1/2} (\eta X^{1/k+\epsilon})^{1/2}
\]

\[
\ll (\min\{\eta^2, y^{-2}\})^{1/2} Z_1 Z_2 (y X^{8/3+\epsilon} Z_1^{-2} Z_2^{-2})^{1/2} \eta^{1/2} X^{1/(2k)+\epsilon/2}
\]

\[
\ll \eta X^{4/3+1/(2k)+\epsilon}.
\]

Hence we need η = \( \infty \{X^{1/3-1/(2k)+\epsilon}\} \), which is the same condition we got in (18).

If \( 6/5 < k < 3 \),

\[
|J(\eta, \omega, \mathcal{A})| \ll \int_{\mathcal{A}} |S_1(\lambda_1 \alpha)||S_1(\lambda_2 \alpha)||S_3(\lambda_3 \alpha)|K_\eta(\alpha) \, d\alpha
\]

\[
\ll \left( \int_{\mathcal{A}} |S_1(\lambda_1 \alpha)S_1(\lambda_2 \alpha)|^{4/3} K_\eta(\alpha) \, d\alpha \right)^{3/4} \left( \int_{\mathcal{A}} |S_3(\lambda_3 \alpha)|^4 K_\eta(\alpha) \, d\alpha \right)^{1/4}
\]

\[
\ll \left( \min\{\eta^2, y^{-2}\} \right)^{3/4} (Z_1 Z_2)^{4/3} \mu(\mathcal{A})^{3/4} (\eta \max\{X^{2/k}, X^{4/k-1}\} X^{\epsilon})^{1/4}
\]

\[
\ll (\min\{\eta^2, y^{-2}\})^{3/4} Z_1 Z_2 (y X^{8/3+\epsilon} Z_1^{-2} Z_2^{-2})^{3/4} \eta^{1/4} \max\{X^{1/(2k)}, X^{1/(k-1/4)}\} X^{\epsilon/4}
\]

\[
\ll \eta Z_1^{-1/2} Z_2^{-1/2} X^{2+\epsilon} \max\{X^{1/(2k)}, X^{1/k-1/4}\}
\]

\[
\ll \eta \max\{X^{7/6+1/(2k)}, X^{11/12+1/k}\} X^{\epsilon}.
\]

Hence we need η = \( \infty \{\max\{X^{1/6-1/(2k)+\epsilon}, X^{-1/12+\epsilon}\}\} \), which is the same condition we got in (20).

If \( k = 3 \), using Lemmas 11 and 12 we obtain

\[
|J(\eta, \omega, \mathcal{A})| \ll \int_{\mathcal{A}} |S_1(\lambda_1 \alpha)||S_1(\lambda_2 \alpha)||S_3(\lambda_3 \alpha)|K_\eta(\alpha) \, d\alpha
\]

\[
\ll \left( \int_{\mathcal{A}} |S_1(\lambda_1 \alpha)S_1(\lambda_2 \alpha)|^{8/7} K_\eta(\alpha) \, d\alpha \right)^{7/8} \left( \int_{\mathcal{A}} |S_3(\lambda_3 \alpha)|^8 K_\eta(\alpha) \, d\alpha \right)^{1/8}
\]

\[
\ll \eta Z_1^{-3/4} Z_2^{-3/4} X^{7/3+5/24+\epsilon} \ll \eta X^{31/24+\epsilon}.
\]

This leads to the same constraint for η that we had in (21).
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