Quantum criticality and universal scaling of strongly attractive spin-imbalanced Fermi gases in a one-dimensional harmonic trap

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We investigate quantum criticality and universal scaling of strongly attractive Fermi gases confined in a one-dimensional harmonic trap. We demonstrate from the power-law scaling of the thermodynamic properties that current experiments on this system are capable of measuring universal features at quantum criticality, such as universal scaling and Tomonaga-Luttinger liquid physics. The results also provide insights on recent measurements of key features of the phase diagram of a spin-imbalanced atomic Fermi gas [Y. Liao et al., Nature (London) 467, 567 (2010)] and point to further study of quantum critical phenomena in ultracold atomic Fermi gases.

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Very recently, the one-dimensional (1D) strongly attractive two-component Fermi gas has attracted much attention from theorists [1–7] and experimentalists [8] due to the existence of an exotic pairing mechanism. Investigation [9–11] shows that this novel pairing is closely related to the elusive Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) [12] state involving BCS pairs with nonzero center-of-mass momenta.

The 1D Fermi gas is one of the most important exactly soluble quantum many-body systems. It was solved long ago by Yang [13] and Gaudin [14] using the Bethe ansatz. Although the study of the attractive Fermi gas was initiated soon after [15,16], it was not until much later that this model began to receive more attention [17]. In terms of the polarization $p$, the model exhibits three quantum phases at zero temperature [1–3]: the fully paired phase which is a quasicondensate with $P = 0$, the fully polarized normal Fermi liquid with $P = 1$, and the partially polarized (1D FFLO-like) phase for $0 < P < 1$.

For a trapped imbalanced Fermi gas it is found [1,2] within the local density approximation (LDA) that a partially polarized 1D FFLO-like state sits in the trapping center surrounded by wings composed of either a fully paired state or a fully polarized Fermi gas. The phase boundaries of the zero-temperature phase diagram shown in Fig. 1 can be determined precisely from the exact solution by the vanishing of the axial density difference (solid line) and the minority state axial density (dashed line). The key features of this $T = 0$ phase diagram were experimentally confirmed using finite temperature density profiles of trapped fermionic $^6$Li atoms [8].

Most recently, quantum criticality and universal scaling behavior [18] are being explored in low-dimensional cold atomic matter, for example, in experiments on the two-dimensional Bose gas [19], following a theoretical scheme for mapping out quantum criticality [20,21]. From this viewpoint, the 1D imbalanced Fermi gas, exhibiting novel phase transitions at $T = 0$, is particularly valuable to test universal scaling through finite temperature density profiles of trapped ultracold atoms [22]. Here we illustrate that finite-temperature properties of the quasi-1D trapped Fermi gas allow the exploration of a wide range of physical phenomena, such as universal Tomonaga-Luttinger liquid (TLL) physics, scaling theory, and the nature of the FFLO state at quantum criticality.

We consider the 1D $\delta$-interacting attractive spin-1/2 Fermi gas with $N = N_{\uparrow} + N_{\downarrow}$ fermions of mass $m$ and external magnetic field $H$. The system is described by the Hamiltonian

$$
H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + g_{1D} \sum_{i=1}^{N_{\uparrow}} \sum_{j=1}^{N_{\downarrow}} \delta(x_i - x_j) - \frac{1}{2} \sum_{i=1}^{N_{\uparrow}} \sum_{j=1}^{N_{\downarrow}} \delta(x_i - x_j) \frac{1}{\sqrt{N_{\uparrow}}} \frac{1}{\sqrt{N_{\downarrow}}} \frac{H}{\sqrt{N}} \frac{N_{\uparrow} - N_{\downarrow}}{N}
$$

in which the three terms are kinetic energy, interaction energy, and Zeeman energy, respectively. Here the intercomponent interaction is related to an effective 1D scattering length $g_{1D} = -\frac{\hbar^2}{m\epsilon_{\text{F}}}$, which can be tuned from the weakly interacting regime ($g_{1D} \rightarrow 0^-$) to the strong-coupling regime ($g_{1D} \rightarrow -\infty$) via Feshbach resonances and optical confinement. For convenience, we define the interaction strength as $c = mg_{1D}/\hbar^2$ and dimensionless parameter $\gamma = c/n$ for physical analysis, where $n = N/L$ is the linear density and $L$ is the length of the system. We set the Boltzmann constant $k_B = 1$.

For the strongly attractive spin-1/2 Fermi gas at finite temperatures, the thermodynamics of the homogeneous system is described by two coupled Fermi gases of bound pairs and excess fermions in the charge sector and ferromagnetic spin-spin interaction in the spin sector. Spin fluctuations are suppressed by a strong effective magnetic field at low temperatures. For the physically interesting low-temperature and strong-coupling regime, i.e., $T \ll \epsilon_b, H$ and $\gamma \gg 1$, a high-precision equation of state [22–24] can be derived from the thermodynamic Bethe ansatz equations [25] in terms of the Yang-Yang grand canonical ensemble [26]. Using the
the equation of state (2) can be reformulated within the vacuum phase. Here FP denotes the fully paired phase, F the fully unpaired phase, and PP the FFLO-like phase. V is the vacuum phase.

binding energy \( \varepsilon_b = \hbar^2 c^2 / 4m \) as the unit of energy and defining \( \tilde{\mu} = \mu / \varepsilon_b \), \( \hbar = H / \varepsilon_b \), \( \tau = T / \varepsilon_b \), and \( \tilde{p} = p / |c\varepsilon_b| \), the pressure \( \tilde{p} = \tilde{p}^b + \tilde{p}^u \) of the system is found to be

\[
\begin{align*}
\tilde{p}^b &= -\frac{t^{3/2} f^{b}_{3/2}}{2\sqrt{\pi}} \left( 1 - t^{3/2} f^{b}_{3/2} \right) \frac{1}{\sqrt{2\pi}} 
\tilde{p}^u &= -\frac{t^{3/2} f^{u}_{3/2}}{2\sqrt{\pi}} \left( 1 - t^{3/2} f^{u}_{3/2} \right) \frac{1}{\sqrt{2\pi}},
\end{align*}
\]

where \( f^{b}_{3/2} = \text{Li}_{3/2}(-e^{\mu / \varepsilon_b}) \) and \( f^{u}_{3/2} = \text{Li}_{3/2}(-e^{\mu / \varepsilon_b}) \) in terms of the standard polylogarithm function \( \text{Li}_n(x) \), with

\[
\begin{align*}
A_b &= 2\tilde{\mu} + 1 - \tilde{p}^b - 4\tilde{p}^u - \frac{t^{5/2} f^{b}_{5/2}}{16\sqrt{\pi}} - \sqrt{\frac{2}{\pi}} t^{5/2} f^{u}_{5/2},
A_u &= \tilde{\mu} + \frac{\hbar}{2} - 2\tilde{p}^b - \frac{t^{5/2} f^{b}_{5/2}}{2\sqrt{\pi}} + f_1.
\end{align*}
\]

Here \( f_1 = t e^{-\tilde{\mu}/t} e^{-2\tilde{p}^b/t} \) and \( \text{Li}_n(x) = \sum_{n=0}^{\infty} x^{n+1} (1/n+1)^2 / (n+2)! \). The thermodynamic quantities such as the particle density \( n \), \( n^u \) for unpaired fermions, \( n^b \) for paired fermions, and compressibility follow from Eqs. (2) and the standard thermodynamic relations. The total pressure serves as the equation of state which provides high-precision thermodynamics over a wide temperature range \( T < 0.2\varepsilon_b \) (see [22,23]).

For spin-imbalanced attractive fermions in a harmonic trap, the equation of state (2) can be reformulated within the LDA by the replacement \( \mu(z) = \mu(0) - \frac{1}{2}m\omega_z^2 z^2 \) in which \( z \) is the position and \( \omega_z \) is the frequency within the trap. Using the dimensionless chemical potential, this becomes \( \tilde{\mu}(z) = \tilde{\mu}(0) - 2\tilde{z}^2 \), where \( \tilde{z} = z / (a_z^2 |c|) \) with the harmonic characteristic radius \( a_z = \sqrt{\hbar / (m\omega_z)} \). The total particle number \( N = \int_{-\infty}^{\infty} n(z)dz \) and polarization \( P = \int_{-\infty}^{\infty} n^u(z)dz / N \) become

\[
\begin{align*}
N\omega_{1D}^2 / a_z^2 &= 4 \int_{-\infty}^{\infty} \tilde{n}(z)dz/|c|, 
P &= 4 \int_{-\infty}^{\infty} \tilde{n}^u(z)dz a_z^2 / (N\omega_{1D}^2),
\end{align*}
\]

where \( \tilde{n}(z) = 1/|\gamma(z)| \) and \( \tilde{n}^u(z) = n^u(z)/|c| \).

At finite temperatures, the phase boundaries can be determined from the equation of state (2) within the LDA (4), where the boundaries of vanishing density difference (black solid line) and vanishing unpaired fermions (red dashed line) form three phases (see Fig. 1). As \( t \to 0 \), the phase boundaries determined via Eqs. (2) are consistent with the zero-temperature results. For strong attraction with polarization, the atoms with opposite spin states form hard-core bosons which are strongly correlated with excess fermions. The polarization can be changed by tuning the effective magnetic field. This results in two extreme phases: the fully paired and fully polarized phases. At zero temperature, these phases intersect at global polarization \( P = P_c \), where we find

\[
P_c = \frac{1}{5} \left( 1 - \frac{64}{75} \frac{n}{|c|} \right) + O(1/\gamma^2).
\]
For the 1D strongly attractive Fermi gas [22], the chemical potential $\mu_c$, the density $n$, and the compressibility $\kappa$ can be cast into a universal scaling form [18,20,21]. For example, the density and compressibility scale as

$$n(\mu,T) = n_0 + T^{\frac{1}{2}+1+\frac{1}{2}} \Gamma \left( \frac{\mu - \mu_c}{T^\pi} \right),$$

$$\kappa(\mu,T) = \kappa_0 + T^{\frac{1}{2}+1+\frac{1}{2}} \Phi \left( \frac{\mu - \mu_c}{T^\pi} \right).$$

Here the dimensionality $d = 1$, the dynamical critical exponent $z = 2$, and the correlation length exponent $\nu = 1/2$ for small polarization $P < P_c$, the chemical potential passes the lower critical point $\mu_{c2} \approx -1/2$ from the vacuum into the fully paired phase and then passes the upper critical point $\mu_{c4} \approx -1/2 + 1/(4\epsilon)(1-h)^2 + 1/(4\epsilon)(1-h)^2$ from the fully paired phase into the FFLO-like phase. At finite temperatures, contour plots of the entropy clearly indicate universal critical behavior near the critical points [see Fig. 3(a)]. The typical V-shaped crossover temperatures separate the quantum critical regimes of the hard-core bosonic TLL$_F$ phase and the two-component FFLO-like TLL$_{FF}$ phase near the critical points $\mu_{c2}$ and $\mu_{c4}$, respectively. Except for the leftmost line separating the vacuum from the critical regime, all of the lines can be determined by the breakdown of the TLL, i.e., when the entropy curves deviate from linear $T$-dependent relations for fixed values of $\mu$ and $h$. These boundaries indicate a crossover from linear dispersion into nonrelativistic dispersion rather than phase transitions at finite $T$.

For large polarization $P > P_c$, the chemical potential $\mu_{c1}$, the chemical potential $\mu_{c3}$, the density $n$, and the compressibility $\kappa$ can be cast into a universal scaling form [18,20,21]. For example, the density and compressibility scale as

$$n(\mu,T) = n_0 + T^{\frac{1}{2}+1+\frac{1}{2}} \Gamma \left( \frac{\mu - \mu_c}{T^\pi} \right),$$

$$\kappa(\mu,T) = \kappa_0 + T^{\frac{1}{2}+1+\frac{1}{2}} \Phi \left( \frac{\mu - \mu_c}{T^\pi} \right).$$

Here the dimensionality $d = 1$, the dynamical critical exponent $z = 2$, and the correlation length exponent $\nu = 1/2$.
We see that the density curves for temperatures below background near the critical points for practical purposes. Here the background compressibilities at low temperatures also well intersect at the compressibility near $\mu_c$ [see Fig. 3(b)]. Similarly the phase boundary separating the different temperatures intersects at the critical point $\mu_c$ [see Figs. 3(d) and 3(e)]. Near $\mu_c$ the background compressibility is $\kappa_{bF} = -\frac{|\mu|}{\sqrt{2\pi} |f_b|^2 f_u^2}$. In a similar fashion, the universal scaling for other thermodynamic quantities such as the specific heat and magnetization are testable through the quantum criticality of the trapped gas. This provides a reliable way to determine quantum phase diagrams and test universal scaling theory and TLL physics in 1D quantum gases of cold atoms.

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For the high-polarization case, the density profiles of unpaired and paired atoms can be used to map out the phase boundaries $\mu_{1V}$ ($V \to F$) and $\mu_{2F}$ ($F \to PP$), respectively [see Figs. 4(b) and 4(c)]. Moreover, we show that the corresponding compressibility curves of the trapped gas at low temperatures intersect at the critical points $\mu_{c1}$ and $\mu_{c2}$ [see Figs. 3(d) and 3(e)].

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