THE STRUCTURE OF COMPLETELY MEET IRREDUCIBLE
CONGRUENCES IN STRONGLY FREGEAN ALGEBRAS

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Abstract. A strongly Fregean algebra is an algebra such that the class of its homomorphic images is Fregean and the variety generated by this algebra is congruence modular. To understand the structure of these algebras we study the prime intervals projectivity relation in the posets of their completely meet irreducible congruences and show that its cosets have natural structure of Boolean group. In particular, this approach allows us to represent congruences and elements of such algebras as the subsets of upward closed subsets of these posets with some special properties.

1. Introduction

The word Fregean comes from Frege’s idea that sentences should denote their logical values. This idea was an inspiration for Pigozzi [11], who transferred it to the field of universal algebra, see [8], [4], [3] and [9] for thorough historical discussion. A Fregean algebra is an algebra \( A \) with a distinguished constant term \( 1 \) satisfying two axioms:

\[
\Theta_A(1,a) = \Theta_A(1,b) \implies a = b \quad \text{for} \quad a, b \in A \quad (\text{congruence orderability}),
\]

and

\[
1/\alpha = 1/\beta \implies \alpha = \beta \quad \text{for} \quad \alpha, \beta \in \text{Con}(A) \quad (1\text{-regularity}).
\]

A class of algebras of the same type is called Fregean if all algebras in this class are Fregean with respect to the common constant term 1. Many natural examples of Fregean varieties, like Boolean algebras, Heyting algebras, Brouwerian semilattices, Boolean groups, equivalential algebras (i.e., the equivalential subreducts of Heyting algebras), equivalential algebras with negation, Hilbert algebras or Hilbert algebras with supremum, come from the algebraization of fragments of classical or intuitionistic logics. Fregean varieties are congruence modular, but not necessarily congruence permutable: Hilbert algebras can serve as an example here. To see that this distinction is important, note that the structure of congruence permutable Fregean varieties is quite well understood, in particular, it was proved in [9] that every congruence permutable Fregean variety consists of algebras that are expansions of equivalential algebras. On the other hand, the structure of Fregean algebras in general remains elusive.

The main technical tool used in analyzing the structure of Fregean algebras was prime intervals projectivity (PIP) relation. (Note, however, that in the congruence distributive case this relation is trivial.) In [13] and [9] we used this relation in the set of join irreducible elements of congruence lattices of finite algebras from congruence permutable Fregean varieties to understand their structure. However, in a series of papers [14], [15] and [16] we apply the same relation, but this time in the set of completely meet irreducible elements of congruence lattice, in particular, to construct free algebras in the varieties under consideration. This perspective is, in a sense, dual to the former, though substantially different, since the posets of join and meet irreducible elements of the congruence lattice need not to be isomorphic,
and the latter approach can be applied also in the infinite case, in contrast to the former one.

In the present paper we show that one can apply this technique to a broad class of Fregean algebras also out of the congruence permutability realm. This is surprising, since we do not know almost nothing about the language of algebra in this situation, in contrast to the congruence permutable case, when we are at least sure that it contains equivalence operation. Namely, we shall consider strongly Fregean algebras, i.e., the algebras such that the classes of their homomorphic images are Fregean and the varieties generated by these algebras are congruence modular. This class lies in between Fregean algebras and Fregean varieties, as every algebra from a Fregean variety must be strongly Fregean. We know from [9] that strongly Fregean algebras fulfill the (SC1) condition isolated in [7]. This condition simplifies studying the PIP relation in the poset of completely meet irreducible congruences significantly, in particular, it allows us to introduce the structure of Boolean group into the PIP equivalence classes (Theorem 13) and to comprehend their location in the congruence lattice (Theorem 16). Moreover, for finite strongly Fregean algebras we get a formula for the length of the congruence lattice as the sum of dimensions of these classes treated as vector spaces over the field \( \mathbb{Z}_2 \) (Theorem 19).

In the set of upward closed subsets of the poset of completely meet irreducible congruences of a strongly Fregean algebra \( A \) we can distinguish two subsets \( \mathcal{H}(A) \) and \( \mathcal{S}(A) \). The larger subset \( \mathcal{S}(A) \) consists of sets that intersected with any PIP equivalence class form its subgroup, whereas the smaller subset \( \mathcal{H}(A) \) comprises the sets \( Z \) such that for each PIP equivalence class such that all the elements larger than the elements of this class are contained in \( Z \), the intersection of \( Z \) with this class is either a maximal subgroup of this class or is equal to this class. The natural map that sends a congruence into the set of all larger completely meet irreducible congruences gives us an embedding of \( \text{Con}(A) \) into \( \mathcal{S}(A) \) that is surjective for finite algebras (Theorem 23). The same map sends principle congruences into \( \mathcal{H}(A) \) (Proposition 27), closed under the natural equivalence operation (Theorem 28). Much more can be said if \( A \) is additionally endowed with a principle congruence term (or, which is here the same, a Malcev term) and, in consequence, its reduct is an equivalential algebra. Then the natural embedding preserves equivalence operation (Proposition 31). Moreover, we show that a finite congruence orderable algebra with a Malcev term generates a Fregean variety, and in this case there is a one-to-one correspondence between \( A \) and \( \mathcal{H}(A) \) preserving equivalence operation (Theorem 33).

2. Prime Intervals Projectivity (PIP) Relation

Let \( A \) be an algebra such that its congruence lattice \( \text{Con}(A) \) is modular. We denote by \( \text{Cm}(A) \) the set of all completely meet irreducible congruences of \( A \). For each \( \eta \) of \( \text{Cm}(A) \) there is a unique congruence \( \eta^+ \in \text{Con}(A) \) such that \( \alpha \geq \eta^+ \) whenever \( \alpha > \eta \) for \( \alpha \in \text{Con}(A) \). In [7] we considered an equivalence relation PIP (prime intervals projectivity, denoted by \( \sim \)) on \( \text{Cm}(A) \) by putting for \( \varphi, \psi \in \text{Cm}(A) \):

\[ \varphi \sim \psi \text{ if and only if the prime intervals } I[\varphi, \varphi^+] \text{ and } I[\psi, \psi^+] \text{ are projective}, \]

which has been studied for many years in the theory of modular lattices [1].
Observe that whenever $I[\alpha_1, \beta_1] \not\triangleleft I[\alpha_2, \beta_2] \setminus I[\alpha_3, \beta_3]$ holds for three prime intervals in $\text{Con}(A)$, then there is $\mu \in \text{Cm}(A)$ with $I[\alpha_1, \beta_1] \not\triangleleft I[\mu, \mu^+] \setminus I[\alpha_3, \beta_3]$. Indeed, it is enough to pick a completely meet irreducible congruence $\mu \geq \alpha_2$ with $\mu \not\geq \beta_2$. Hence for $\varphi, \psi \in \text{Cm}(A)$, $\varphi \sim \psi$ there exists $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \text{Con}(A)$ and $\mu_1, \ldots, \mu_n \in \text{Cm}(A)$ such that

$$I[\varphi, \varphi^+] \setminus I[\alpha_1, \beta_1] \not\triangleleft I[\mu_1, \mu_1^+] \setminus I[\psi, \psi^+].$$

(*)

(in particular, for $n = 1$, we put $I[\varphi, \varphi^+] \setminus I[\alpha_1, \beta_1] \not\triangleleft I[\psi, \psi^+]$).

In the sequel the following theorem from [7, Lemma 22] will be useful.

**Theorem 1.** Let $A$ be an algebra with modular congruence lattice. Then for all $\alpha, \beta \in \text{Con}(A)$ and $\eta \in \text{Cm}(A)$ with $\alpha \land \beta \leq \eta$ there are $\mu_1, \mu_2 \in \eta/\sim \cup \{1_A\}$ such that $\alpha \leq \mu_1, \beta \leq \mu_2$ and $\mu_1 \land \mu_2 \leq \eta$, where $1_A$ denotes the largest element in $\text{Con}(A)$.

Applying this theorem, we see that the relation PIP can be used to characterize distributivity property of the congruence lattice of $A$.

**Proposition 2.** The following conditions are equivalent for an algebra $A$ with modular congruence lattice:

(i) $\text{Con}(A)$ is distributive;

(ii) $\varphi/\sim = \{\varphi\}$ for every $\varphi \in \text{Cm}(A)$.

**Proof.** (i) $\Rightarrow$ (ii) Let $\psi \in \varphi/\sim$. Observe that for $\nu, \mu \in \text{Cm}(A)$, $\alpha, \beta \in \text{Con}(A)$ from $I[\nu, \nu^+] \setminus I[\alpha, \beta] \not\triangleleft I[\mu, \mu^+]$ it follows that $\nu = \mu$. Indeed, we have $\beta \land \mu = \alpha \leq \nu$, and, by (i), $\nu = (\beta \lor \nu) \land (\mu \lor \nu) = \nu^+ \land (\mu \lor \nu)$. Hence $\mu \lor \nu = \nu$ and, analogously, $\mu \lor \nu = \mu$, as desired. Taking this observation into account, we get from (*) the equalities $\varphi = \mu_1 = \ldots = \mu_n = \psi$.

(ii) $\Rightarrow$ (i) Let $\alpha, \beta, \gamma \in \text{Con}(A)$. To show that $\alpha \land (\beta \lor \gamma) = (\alpha \land \beta) \lor (\alpha \land \gamma)$ it suffices to prove that $(\alpha \land \beta) \lor (\alpha \land \gamma) \leq \mu$ implies $\alpha \land (\beta \lor \gamma) \leq \mu$ for every $\mu \in \text{Cm}(A)$. If $\mu \in \text{Cm}(A)$ and $(\alpha \land \beta) \lor (\alpha \land \gamma) \leq \mu$, then $\alpha \land \beta, \alpha \land \gamma \leq \mu$. From $\mu/\sim = \{\mu\}$ and from Theorem 1 we deduce that either $\alpha \leq \mu$ or $\beta \leq \mu$ and either $\alpha \leq \mu$ or $\gamma \leq \mu$. Hence either $\alpha \leq \mu$ or $\beta \lor \gamma \leq \mu$, and so $\alpha \land (\beta \lor \gamma) \leq \mu$. \hfill $\Box$

From now on in this section we strengthen our assumption on $A$ by requiring that $A$ belongs to a congruence modular variety. In particular, this means that we can apply the commutator theory of Freese and McKenzie [5] to study the properties of $A$. The definition of the relation PIP simplifies considerably in the case of algebras satisfying the condition (SC1) that was isolated and studied by Idziak and Słomczyński in [7].

**Definition 3.** An algebra $A$ from a congruence modular variety fulfills the condition (SC1) if and only if for every $\varphi \in \text{Cm}(A)$ the centralizer $(\varphi : \varphi^+) \leq \varphi^+$.

Note that this condition implies the condition (C1) from [5] described by a commutator identity: $[\alpha, \beta] = ([\alpha, \alpha] \land \beta) \lor ([\beta, \beta] \land \alpha)$ for every $\alpha, \beta \in \text{Con}(A)$, see [7, Theorem 15]. We will see further that this class covers many interesting cases, in particular Fregean algebras that play an important role in the algebraization of intuitionistic logic and its fragments [9]. From the definition of (SC1) we get the following lemma that will be frequently used throughout the paper, see also [7, Lemma 21].
Lemma 4. Let $A$ be an algebra from a congruence modular variety and $\varphi \in \operatorname{Cm}(A)$. Then

1. if $\varphi^+$ is not Abelian over $\varphi$, then $(\varphi : \varphi^+) = \varphi$, and $\varphi/\sim = \{\varphi\}$.
2. if $A$ fulfills (SC1) and $\varphi^+$ is Abelian over $\varphi$, then
   (a) $(\varphi : \varphi^+) = \varphi^+$, and
   (b) $\psi^+$ is Abelian over $\psi$ and $\psi^+ = \varphi^+$ for every $\psi \in \varphi/\sim$.

Proof. We have $\varphi \leq (\varphi : \varphi^+)$. Note that $\varphi^+$ is Abelian over $\varphi$ if and only if $(\varphi : \varphi^+) \geq \varphi^+$. Since projectivity preserves centralizers and abelianity \cite{5}, for $\psi \in \operatorname{Cm}(A)$, $\psi \sim \varphi$, we deduce that $(\psi : \psi^+) = (\varphi : \varphi^+)$ and $\psi^+$ is Abelian over $\psi$ if and only if $\varphi^+$ is Abelian over $\varphi$. This implies (1). If $A$ fulfills (SC1) and $\varphi^+$ is Abelian over $\varphi$, we get $\psi^+ = (\psi : \psi^+) = (\varphi : \varphi^+) = \varphi^+$, and (2) follows. \hfill \square

Assume that an algebra $A$ from a congruence modular variety satisfies (SC1).

Definition 5. For $U \in \operatorname{Cm}(A)/\sim$ we put $0_U := \bigwedge U$. For $S \subseteq U$, set $\overline{S} := S \cup \{\eta^+\}$, where $\eta \in U$. (It follows from Lemma 4.2 that the choice of $\eta$ is irrelevant.)

In this situation, we can easily characterized these congruences $\alpha$ for which $[\alpha, \alpha] \leq 0_U$.

Proposition 6. Suppose that an algebra $A$ from a congruence modular variety satisfies (SC1). Let $U = \eta/\sim \in \operatorname{Cm}(A)/\sim$. For $\alpha \in \operatorname{Con}(A)$ we have:

1. if $[\alpha, \alpha] \leq 0_U$, then $\alpha \leq \eta^+$;
2. if $\eta^+$ is Abelian over $\eta$, then $[\alpha, \alpha] \leq 0_U$ iff $\alpha \leq \eta^+$.

Proof. (1) Let $[\alpha, \alpha] \leq 0_U$. Applying the definition of (SC1), we get $(\eta : \eta^+) \leq \eta^+$. Suppose that $\alpha \nsubseteq \eta^+$. Hence $[\eta^+, \alpha \vee \eta] \nsubseteq \eta$. Then $\eta^+ = \eta \vee [\eta^+, \alpha \vee \eta] \leq \eta \vee [\alpha \vee \eta, \alpha \vee \eta] = \eta \vee [\alpha, \alpha] = \eta$, a contradiction.

(2) Now, let $\eta^+$ is Abelian over $\eta$ and $\alpha \leq \eta^+$. For $\psi \in \eta/\sim$, it follows from Lemma 4.2, that $\psi^+ = \eta^+$ is Abelian over $\psi$. In consequence, $[\eta^+, \eta^+] \leq \psi$. Thus $[\alpha, \alpha] \leq [\eta^+, \eta^+] \leq 0_U$. \hfill \square

Observe that from Lemma 4.2 it follows immediately that two different congruences $\varphi, \psi \in \operatorname{Cm}(A)$ such that $\varphi \sim \psi$ fulfills $\varphi^+ = \psi^+$ and so are incomparable. This observation allows us to obtain the projectivity of completely meet irreducible congruences in (*) in short chains.

Lemma 7. Suppose that an algebra $A$ from a congruence modular variety satisfies (SC1). Then for all $\varphi, \psi \in \operatorname{Cm}(A)$ such that $\varphi \sim \psi$ there exist $\alpha, \beta \in \operatorname{Con}(A)$ with $I[\varphi, \varphi^+] \searrow I[\alpha, \beta] \nearrow I[\psi, \psi^+]$.

Proof. Let $\varphi, \psi \in \operatorname{Cm}(A)$, $\varphi \sim \psi$. We assume that $\varphi \neq \psi$, since otherwise the statement is obvious. Then, there exist $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \operatorname{Con}(A)$ and $\mu_1, \ldots, \mu_{n-1} \in \operatorname{Cm}(A)$ fulfilling (*). To reduce this chain to a short one we may induct on $n$, the number of up-arrows. Actually, it suffices to consider the case $n = 2$. Therefore suppose that

$$I[\varphi, \varphi^+] \searrow I[\alpha_1, \beta_1] \nearrow I[\mu, \mu^+] \searrow I[\alpha_2, \beta_2] \nearrow I[\psi, \psi^+]$$

holds for some $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \operatorname{Con}(A)$ and $\mu \in \operatorname{Cm}(A)$.

In consequence, it follows from Lemma 4 that (1) $\varphi^+ = \mu^+ = \psi^+$. Then we may assume that (2) $\beta_2 \leq \varphi \vee \alpha_2$ and (3) $\beta_1 \leq \psi \vee \alpha_1$, since otherwise we would...
get either $I[\varphi, \varphi^+] \setminus I[\alpha_2, \beta_2]$ or $I[\alpha_1, \beta_1] \nrightarrow I[\psi, \psi^+]$, respectively. Now, we will show that the congruences

$$\alpha := (\varphi \land \alpha_2) \lor (\psi \land \alpha_1)$$

and

$$\beta := \mu \land ((\varphi \land \beta_2) \lor (\psi \land \beta_1))$$

witness our Lemma. Put $\gamma := (\varphi \land \beta_2) \lor (\psi \land \beta_1)$. Using modularity (m) we get

$$\varphi \lor \beta = \varphi \lor ((\varphi \land \beta_2) \lor (\mu \land \gamma)) = \varphi \lor (((\varphi \land \beta_2) \lor (\mu \land \gamma)) \land (\mu \land \gamma)) = \varphi \lor (((\varphi \land \beta_2) \lor (\mu \land \gamma)) \land (\mu \land \gamma))$$

(2) $$= \varphi \lor ((\beta_2 \lor \mu) \land \gamma) = \varphi \lor [\mu^+ \land \gamma] = \varphi \lor [\varphi^+ \land \gamma]$$

(1) $$= \varphi \lor \gamma = \varphi \lor (\psi \land \beta_1) = \varphi \lor \alpha_1 \lor (\psi \land \beta_1)$$

(3) $$= \varphi \lor (((\alpha_1 \lor \psi) \land \beta_1) = \varphi \lor \beta_1 = \varphi^+,$$

and in the very same way $\psi \lor \beta = \psi^+$. Moreover

$$\varphi \land \beta = \mu \land \varphi \land ((\varphi \land \beta_2) \lor (\psi \land \beta_1))$$

(2) $$= \mu \land ((\varphi \land \beta_2) \lor (\varphi \land \psi \land \beta_1))$$

(1) $$= \mu \land (((\varphi \land \beta_2) \lor (\psi \land \alpha_1))$$

(m) $$= (\mu \land \varphi \land \beta_2) \lor (\psi \land \alpha_1)$$

(3) $$= (\varphi \land \alpha_2) \lor (\psi \land \alpha_1) = \alpha,$$

and analogously $\psi \land \beta = \alpha$, which completes the proof.

The main focus of this work is to understand the behaviour of the relation PIP for an algebra $A$ from a congruence modular variety such that $H(A)$ is Fregean, see Sec. 3. In particular, we show that in this case each equivalence class $\eta/\sim$ in $\text{Cm}(A)$, supplemented with a unit element $\eta^+$, forms a Boolean group.

3. PIP in Fregean algebras

In the sequel we assume that the language of an algebra (or class of algebras of the same type) we consider contains a distinguished constant term 1.

**Definition 8.** We call an algebra $A$ congruence orderable (with respect to 1) if $\Theta_A(1,a) = \Theta_A(1,b)$ implies $a = b$ for $a, b \in A$, where $\Theta_A(c,d)$ denotes the smallest congruence containing $(c,d)$. We call it Fregean if, additionally, $1/\alpha = 1/\beta$ implies $\alpha = \beta$ for $\alpha, \beta \in \text{Con}(A)$ (i.e., $A$ is 1-regular).

Thus, if an algebra is congruence orderable, then the formula $a \leq b$ iff $\Theta_A(1,a) \subset \Theta_A(1,b)$ for $a, b \in A$ introduces a partial order in $A$, and, if it is Fregean, then the congruences are uniquely determined by their 1–equivalence classes. A class of algebras of the same type is called Fregean (congruence orderable) if all algebras in this class are Fregean (congruence orderable) with respect to the common constant term 1.

The following property of subdirectly irreducible congruence orderable algebras is crucial for understanding the structure of Fregean algebras.
Proposition 9. [9] Lemma 2.1] If μ is the monolith of a subdirectly irreducible congruence orderable (with respect to 1) algebra A, then |1/μ| = 2 and all other μ–cosets are one element.

Hence, if A is a subdirectly irreducible congruence orderable algebra, then there is the unique non-unit element in the unique non trivial coset of the monolith of A. Clearly, this element is the largest non-unit element in A. If A is Fregean, then the reverse is true: if there is the largest non-unit element in A, then A is subdirectly irreducible.

Observe that for an algebra A, it follows directly from the definition of orderability that H(A), the class of all homomorphic images (quotient algebras) of A, is congruence orderable if and only if for every a, b ∈ A and α ∈ Con(A) we have α ∨ ΘA(1, a) = α ∨ ΘA(1, b) iff (a, b) ∈ α [7, Lemma 3.3]. What is more, we can characterize this property in a simple way.

Proposition 10. The following conditions are equivalent:

1. H(A) is congruence orderable with respect to 1;
2. For all φ ∈ Cm(A) and a, b ∈ A
   \[(a, b) ∈ φ^+ \text{ implies } \{(a, b), (1, a), (1, b)\} ∩ φ ≠ ∅.\]

Proof. (1) ⇒ (2). Let φ ∈ Cm(A), a, b ∈ A and (a, b) ∈ φ^+. Then A/φ is subdirectly irreducible with the monolith φ^+/φ. Assuming that (a, b) /∈ φ, we get \{a/φ, b/φ\} is a two-element and (a/φ, b/φ) ∈ φ^+/φ. By Proposition [9] 1/φ ∈ \{a/φ, b/φ\}, and so (1, a) ∈ φ or (1, b) ∈ φ.

(2) ⇒ (1). Choose α ∈ Con(A) and a, b ∈ A such that α ∨ ΘA(1, a) = α ∨ ΘA(1, b). Suppose that (a, b) /∈ α. Then we can find φ ∈ Cm(A) such that α ≤ φ and (a, b) ∈ φ^+ \φ. Thus φ ∨ ΘA(1, a) = φ ∨ ΘA(1, b), and either (1, a) ∈ φ or (1, b) ∈ φ. Hence (a, b) ∈ φ, a contradiction.

Moreover, observe that if A is 1-regular, then it is easy to check that H(A) is 1–regular as well. In consequence, H(A) is Fregean iff H(A) is congruence orderable and A is 1-regular. It follows from [9] that 1-regular varieties (and so Fregean varieties) are congruence modular. While we need congruence modularity in our applications, the assumption that the whole variety generated by A is 1-regular seems to be too strong. Thus, in the sequel, we shall assume that H(A) is Fregean, but the variety generated by A is just congruence modular. From [9 Theorem 2.3] it follows that all such algebras satisfy the condition (SC1):

Theorem 11. Let A be an algebra from a congruence modular variety and let H(A) is Fregean. Then A satisfies (SC1).

Thus, it seems natural to distinguish this class of algebras.

Definition 12. We call an algebra A strongly Fregean if:

- H(A) is Fregean;
- V(A), the variety generated by A, is congruence modular.

Note that if A is from Fregean variety, then A is strongly Fregean. However, the reverse implication is not true. To show this, it is enough to consider two-element lattice with the greatest element 1. From now on, unless otherwise stated, we shall assume that A is strongly Fregean.
Now let us go back to the study of the relation PIP. We will show that each equivalence class \( \eta/\sim \) in \( \text{Cm}(A) \), supplemented with a unit element \( \eta^+ \), forms a Boolean group with the complement of symmetric difference restricted to \( \eta^+ \).

Let \( \varphi, \psi \in \text{Cm}(A) \) and \( \varphi^+ = \psi^+ \). Define

\[
\varphi \cdot \psi := (\varphi \div \psi)' \cap \varphi^+.
\]

**Theorem 13.** Let \( A \) be strongly Fregean and let \( U \in \text{Cm}(A) / \sim \). Then \( (U, \bullet) \) is a Boolean group.

**Proof.** Let \( \eta \in \text{Cm}(A) \) such that \( U = \eta/\sim, \) and \( \varphi, \psi \in U \). It follows from Lemma 42 that \( \varphi^+ = \psi^+ = \eta^+ \). It suffices to show that \( \varphi \cdot \psi \in U \) for \( \varphi \neq \psi \), as \( \varphi \cdot \varphi = \eta^+ \). By Theorem 11 and Lemma 1 there exist \( \alpha, \beta \in \text{Con}(A) \) such that \( I[\varphi, \eta^+] \setminus I[\alpha, \beta] \nearrow I[\psi, \eta^+] \). Put \( \gamma := (\varphi \land \psi) \lor \beta \). It is enough to show that \( \varphi \cdot \psi = \gamma \) and \( \gamma \in U \).

Step 1. Clearly \( \alpha \leq \varphi \land \psi \), and so, by modularity, \( \varphi \land \gamma = (\varphi \land \psi) \lor (\varphi \land \beta) = (\varphi \land \psi) \lor \alpha = \varphi \land \psi \). Analogously, \( \psi \land \gamma = \varphi \land \psi \). Hence \( \gamma \subseteq \varphi \cdot \psi \), since \( \varphi \land \psi = \varphi \land \gamma = \psi \land \gamma \). Moreover \( \varphi \lor \gamma = \eta^+ = \psi \lor \gamma \). Consequently, \( I[\varphi, \eta^+] \setminus I[\varphi \land \psi, \gamma] \nearrow I[\psi, \eta^+] \) and \( I[\varphi, \eta^+] \setminus I[\varphi \land \psi, \psi] \nearrow I[\gamma, \eta^+] \). Now, we show that \( \gamma \supseteq \varphi \cdot \psi \). On the contrary, suppose that there exists \( (x, y) \in (\varphi \cdot \psi) \setminus \gamma \). Then \( (x, y) \in \eta^+ \setminus (\varphi \lor \psi) \). Applying Proposition 10 we deduce that \( \text{card}(\{(1, x), (1, y)\} \cap \varphi) = \text{card}(\{(1, x), (1, y)\} \cap \psi) = 1 \), and so \( (1, x), (1, y) \in \eta^+ \). In consequence, \( \gamma \lor \Theta_A(1, x), \gamma \lor \Theta_A(1, y) \in I[\gamma, \eta^+] \). From projectivity, we have \( \text{card}(I[\gamma, \eta^+]) = \text{card}(I[\varphi, \eta^+]) = 2 \). From orderability of \( \text{A}/\gamma \in \text{H}(A) \) we get \( \gamma \lor \Theta_A(1, x) \neq \gamma \lor \Theta_A(1, y) \), since otherwise \( (x, y) \in \gamma, \) a contradiction. Thus either \( \gamma \lor \Theta_A(1, x) = \gamma \) or \( \gamma \lor \Theta_A(1, y) = \gamma \). Without loss of generality we can assume that \( \gamma \lor \Theta_A(1, x) = \gamma \), and so \( (1, x) \in \gamma \). Then \( (1, y) \notin \gamma \). Hence \( (1, x) \in \gamma \lor \varphi \land \psi \), and so \( (1, y) \notin \gamma \lor \varphi \lor \psi \). From 1-regularity there exists \( s \in A \) such that \( (1, s) \in \gamma \setminus (\varphi \land \psi) \). Then \( (1, s) \notin \varphi \lor \psi \), and we can apply again Proposition 10 obtaining \( (s, y) \in \varphi \land \psi \leq \gamma \). Hence \( (1, y) \in \gamma \), a contradiction.

Step 2. We show that \( \gamma \in U \). Let \( \gamma = \bigwedge \Gamma \), where \( \Gamma \subseteq \text{Cm}(A) \). Since \( \text{card}(I[\gamma, \eta^+]) = 2 \), there exists \( \mu \in \Gamma \) such that \( \eta^+ \not\subseteq \mu \), and so \( \gamma = \eta^+ \land \mu \). Hence \( I[\gamma, \eta^+] \setminus I[\mu, \eta^+ \lor \mu] \nearrow I[\gamma, \eta^+] \). From \( \text{card}(I[\mu, \eta^+ \lor \mu]) = 2 \) we get \( \mu^+ = \eta^+ \lor \mu \). As the intervals \( I[\varphi, \eta^+] \), \( I[\gamma, \eta^+] \) are projective, we deduce that \( \mu \sim \varphi \). Lemma 41 gives us \( \eta^+ = \mu^+ \). Thus \( \gamma = \mu^+ \land \mu = \mu \in U \), as desired. \( \square \)

**Corollary 14.** Let \( A \) be strongly Fregean, \( \varphi, \psi \in \text{Cm}(A) \), \( \varphi \neq \psi \) and \( \varphi^+ = \psi^+ \). Then the following conditions are equivalent:

1. \( \varphi \sim \psi \);  
2. \( \varphi \cdot \psi \in \text{Cm}(A) \);  
3. \( \varphi \cdot \psi \in \text{Con}(A) \).

**Proof.** (1) \( \Rightarrow \) (2) follows form the theorem above, and (2) \( \Rightarrow \) (3) is obvious. To prove (3) \( \Rightarrow \) (1), observe that there exist \( a, b \in A \) such that \( (1, a) \in \varphi \setminus \psi \) and \( (1, b) \in \psi \setminus \varphi \). Hence \( (a, b) \notin \varphi \lor \psi \) and \( (a, b) \in \varphi^+ \). Consequently, \( (a, b) \in \varphi \lor \psi \), and so \( \varphi \lor \psi = \varphi^+ = (\varphi \cdot \psi) \lor \psi \). Thus \( I[\varphi, \varphi^+] \setminus I[\varphi \land \psi, \varphi \cdot \psi] \nearrow I[\psi, \psi^+] \), as desired. \( \square \)

The following lemma and theorem describe the location of the equivalence classes of the relation PIP in the congruence lattice.
Lemma 15. Let $\mathbf{A}$ be strongly Fregean, $\mu \in \text{Cm}(\mathbf{A})$, $U = \mu \sim$, and $a \in A$. If $(1, a) \not\in \mu^+ \backslash 0_U$, then $\Theta_{\mathbf{A}}(1, a) \cap 0_U$ is an atom in $[0_U, \mu^+]$.

Proof. Since $(1, a) \not\in 0_U$, it follows that there exists $\varphi \in U$ such that $(1, a) \not\in \varphi$. Clearly, from Lemma 12 we have $\varphi^+ = \mu^+$. We show that $\varphi \wedge \Theta_{\mathbf{A}}(1, a) \subseteq 0_U$. On the contrary, suppose that there exists $(x, y) \in \varphi \wedge \Theta_{\mathbf{A}}(1, a)$ and $(x, y) \not\in 0_U$. Hence there exists $\psi \in U$ such that $(x, y) \not\in \psi$. In consequence, $\psi \neq \varphi$ and $(1, a) \not\in \psi$. From Theorem 13 we get $(1, a) \in \varphi \bullet \psi$, and so $(x, y) \in \varphi \wedge (\varphi \bullet \psi) = \varphi \wedge \psi \subseteq \psi$, a contradiction. Thus, from modularity, $\varphi \wedge (\Theta_{\mathbf{A}}(1, a) \cap 0_U) = 0_U$. In consequence, $I[0_U, \Theta_{\mathbf{A}}(1, a) \cap 0_U] = I[\varphi, \varphi^+]$. Hence $\Theta_{\mathbf{A}}(1, a) \cap 0_U$ covers $0_U$, as desired. □

Theorem 16. Let $\mathbf{A}$ be strongly Fregean, $\mu \in \text{Cm}(\mathbf{A})$ and $U = \mu \sim$. Then

1. $U = \{\beta \in \text{Con}(\mathbf{A}) : 0_U \leq \beta < \mu^+\}$
2. $U = \text{Cm}(\mathbf{A}) \cap I[0_U, \mu^+]$
3. If $0_U \leq \alpha \in \text{Con}(\mathbf{A})$, then either $\alpha \leq \mu^+$ or $\mu^+ \leq \alpha$.

Proof. 1. Clearly, if $\beta \in U$, then $0_U \leq \beta < \mu^+$. Let $\beta \in \text{Con}(\mathbf{A})$ be such that $0_U \leq \beta < \mu^+$. If we pick $\varphi \in \text{Cm}(\mathbf{A})$ over $\beta$ but not over $\mu^+$, then $\mu^+ \wedge \varphi = \beta$, and so $I[\beta, \mu^+] \not\succ I[\varphi, \varphi^+]$. From 1-regularity we deduce that there is $x \in A$ such that $(1, x) \in \mu^+ \setminus \beta$. Clearly, $(1, x) \not\in 0_U$, and so there exists $\nu \in U$ such that $(1, x) \not\in \nu$. Put $\gamma = \Theta_{\mathbf{A}}(1, x) \setminus 0_U$. It follows from Lemma 15 that $0_U < \gamma$. Then $I[\nu, \nu^+] \setminus I[0_U, \gamma] \not\succ I[\beta, \mu^+] \not\succ I[\varphi, \varphi^+]$. Hence $\varphi \sim \nu$ and $\varphi^+ = \nu^+ = \mu^+$. Thus $\beta = \mu^+ \wedge \varphi = \varphi \in U$.

2. If $\text{card} U = 1$, then the equality is obvious. Let us assume that $\text{card} U \geq 2$. Take $\gamma \in \text{Cm}(\mathbf{A}) \cap I[0_U, \mu^+]$. Clearly, $\gamma^+ \leq \mu^+$. From Lemma 4 we deduce that $\mu^+$ is Abelian over $\mu$. Consequently, by Proposition 6 $\mu^+$ is Abelian over $0_U$. We get $[\gamma^+, \mu^+] \leq [\mu^+, \mu^+] \leq 0_U \leq \gamma$, and so $\mu^+ \leq (\gamma : \gamma^+)$. On the other hand, it follows from Theorem 10 that $(\gamma : \gamma^+) \leq \gamma^+$. Thus $\mu^+ \leq \gamma^+$, and so $\mu^+ = \gamma^+$. In consequence, $0_U \leq \gamma \prec \mu^+$ and from (1) we get $\gamma \in U$.

3. Let $\alpha \in \text{Con}(\mathbf{A})$, $0_U \leq \alpha$. As in the proof of (2) we consider two cases: $\text{card} U = 1$ and $\text{card} U \geq 2$. In the former case, we have $U = \{\mu\}$, and so $\mu = 0_U \leq \alpha$. Hence, either $\alpha = \mu < \mu^+$, or $\mu < \alpha$, and so $\mu^+ \leq \alpha$. In the latter case, we see from Lemma 4 that $\mu^+$ is Abelian over $\mu$. Assume that $\mu^+ \not\geq \alpha$. Then $0_U \leq \alpha \wedge \mu^+ < \mu^+$. We choose $\eta \in \text{Cm}(\mathbf{A})$ such that $\alpha \wedge \mu^+ \leq \eta$ and $\mu^+ \not\leq \eta$. Since $[\mu^+, \mu^+] \leq 0_U \leq \eta$, we have $[\mu^+ \vee \eta, \mu^+] \leq [\mu^+ \vee \eta, \mu^+ \vee \eta] \leq \eta$. Thus $\mu^+ \vee \eta \leq \eta$. Using (SC1) condition, we get $(\eta : \eta^+) \leq \eta^+$, and so $\mu^+ \vee \eta \leq \eta^+$. Consequently, $\mu^+ \vee \eta = \eta^+$, and so $I[\mu^+ \vee \eta, \mu^+] \not\succ I[\eta, \eta^+]$. Then $0_U \leq \mu^+ \vee \eta < \mu^+$, and from (1) we deduce that $\mu^+ \wedge \eta \in U \in \text{Cm}(\mathbf{A})$. Hence $\eta = \mu^+ \wedge \eta \in U$ and $\eta^+ = \mu^+$. Then, from modularity, $\eta = \eta \vee (\alpha \wedge \mu^+) = \eta \vee (\alpha \wedge \eta^+) = (\eta \vee \alpha) \wedge \eta^+$, and so $\alpha \leq \eta < \eta^+ = \mu^+$.

Applying Theorem 11 we can significantly strengthen Theorem 10. We start from the observation describing the relation between the Boolean operation and meet operation in a given coset of the relation $\text{PIP}$.

Lemma 17. Let $\mathbf{A}$ be strongly Fregean, $U \in \text{Cm}(\mathbf{A}) / \sim$ and $\eta, \nu_1, \nu_2 \in U$, $\eta \neq \nu_1 \neq \nu_2 \neq \eta$. Then $\nu_1 \bullet \nu_2 = \eta$ if and only if $\nu_1 \wedge \nu_2 \leq \eta$.

Proof. Clearly $\nu_1 \wedge \nu_2 \leq \nu_1 \bullet \nu_2$. Assume that $\nu_1 \wedge \nu_2 \leq \eta$. By Theorem 13 we know that $\nu_1 \bullet \nu_2 \in U$. It is enough to show that $\eta \leq \nu_1 \bullet \nu_2$, since all elements in $U$ are incomparable. On the contrary, suppose that there exists $(a, b) \in \eta \setminus (\nu_1 \bullet \nu_2)$. 

□
Then either \((a, b) \in v_1 \setminus v_2\) or \((a, b) \in v_2 \setminus v_1\). There is no loss of generality in assuming that \((a, b) \in v_1 \setminus v_2\). Using modularity of \(\text{Con}(A)\) we get \(v_1 = v_1 \land \eta^+ = v_1 \land (v_1 \cdot v_2) \lor \Theta_A(a, b)) = (v_1 \land (v_1 \cdot v_2)) \lor \Theta_A(a, b) = (v_1 \land v_2) \lor \Theta_A(a, b) \leq \eta\). Hence \(v_1 = \eta\), a contradiction. \(\square\)

**Proposition 18.** Let \(A\) be strongly Fregean, \(\eta \in \text{Con}(A)\), and \(\alpha_1, \ldots, \alpha_n \in \text{Con}(A)\) be such that \(\alpha_1 \land \cdots \land \alpha_n \leq \eta\). Put \(U := \eta/\sim\). Then there exist \(\mu_1, \ldots, \mu_n \in U \cup \{1_A\}\) such that \(\alpha_i \leq \mu_i\) for \(i = 1, \ldots, n\), \(\mu_1 \land \cdots \land \mu_n \leq \eta\), and \(\eta\) belongs to a subuniverse of \((U, \bullet)\) generated by \(\{\mu_i : 1, \ldots, n\} \setminus \{1_A\}\), i.e., \(\eta = \mu_i \cdot \cdots \cdot \mu_i\) for some \(1 \leq i_1 < \ldots < i_k \leq n\).

**Proof.** The proof is by induction on \(n\). For \(n = 1\) the assertion is obvious. Suppose that \(n \geq 2\) and the assertion is true for every \(k < n\). We can assume that \(\alpha_i \not\leq \eta\) \((i = 1, \ldots, n)\), since otherwise we put \(\mu_i = \eta\) and \(\mu_j = 1_A\) for \(j \neq i\). It follows from Lemma \(\text{[1]}\) that there exist \(v_1, v_2 \in U \cup \{1_A\}\) such that \(\alpha_1 \land \cdots \land \alpha_{n-1} \leq v_1\) and \(\alpha_n \leq v_2\) and \(v_1 \land v_2 \leq \eta\). If \(v_1 = 1_A\) or \(v_1 = v_2\), or \(v_2 = \eta\), then \(\alpha_n \leq \eta\), a contradiction. From \(v_1 \neq 1_A\), by the induction hypothesis, it follows that there exist \(\mu_1, \ldots, \mu_{n-1} \in U \cup \{1_A\}\) such that \(\alpha_i \leq \mu_i\) for \(i = 1, \ldots, n-1\), \(\mu_1 \land \cdots \land \mu_{n-1} \leq v_1\), and \(v_1\) belongs to a subuniverse of \((U, \bullet)\) generated by \(\{\mu_i : 1, \ldots, n-1\} \setminus \{1_A\}\). If \(v_1 = \eta\), which covers also the case \(v_2 = 1_A\), we can put \(\mu_n = 1_A\), and the assertion holds. If \(\eta \not\in \{v_1, v_2\}\), we get from Lemma \(\text{[17]}\) that \(\eta = v_1 \cdot v_2\). Putting \(\mu_n = v_2 \neq 1_A\), we obtain the assertion. \(\square\)

By the length \(\delta(L)\) of a finite modular lattice \(L\) we mean the length of an arbitrary maximal chain in \(L\) (where the length of a finite chain with \(n + 1\) element is defined to be \(n\)). This quantity can be characterized with the help of the PIP relation.

**Theorem 19.** Let \(A\) be finite and strongly Fregean. Then

\[
\delta(\text{Con}(A)) = \sum \{\dim U : U \in \text{Con}(A) \setminus \sim\},
\]

where \(\dim U\) denotes the dimension of \((U, \bullet)\) treated as a vector space over the field \(\mathbb{Z}_2\).

**Proof.** The proof is by induction on \(\delta(\text{Con}(A))\). If \(\delta(\text{Con}(A)) = \emptyset\), then the theorem is trivial. Let now \(\delta(\text{Con}(A)) > 0\). From 1-regularity, we can find \(a \in A\) such that \(\Theta_A(1, a)\) is an atom of \(\text{Con}(A)\). Take \(\mu \in \text{Con}(A)\) such that \(\Theta_A(1, a) \not\leq \mu\). As \(\mu \land \Theta_A(1, a) = 0_A\), we have \(\mu^+ = \mu \lor \Theta_A(1, a)\). Put \(V := \mu/\sim\). It is easy to observe that, \(\Theta_A(1, a) \leq \varphi\) for every \(\varphi \in \text{Con}(A) \setminus V\), since otherwise \(I[\varphi, \varphi^+] \land I[0_A, \Theta_A(1, a)] \land I[\mu, \mu^+]\), a contradiction. Then \(M(a) := \{ \eta \in \text{Con}(A) : (1, a) \in \eta\} = (M(a) \cap V) \cup (\text{Con}(A) \setminus V)\). Clearly, \(\text{Con}(A/\Theta_A(1, a))\) is isomorphic to the interval \(I[\Theta_A(1, a), 1_A]\) in \(\text{Con}(A)\), and so \(\text{Con}(A/\Theta_A(1, a)) = \{\eta/\Theta_A(1, a) : \eta \in M(a)\}\). We use Corollary \(\text{[13]}\) to deduce that \(\varphi \sim \psi\) in \(\text{Con}(A)\) if and only if \(\varphi/\Theta_A(1, a) \sim \psi/\Theta_A(1, a)\) in \(\text{Con}(A/\Theta_A(1, a))\) for \(\varphi, \psi \in M(a)\). Thus, putting the quotient set \(\text{Con}(A/\Theta_A(1, a))/\sim\) we get, in fact, the same equivalence classes (up to the natural isomorphism) as in \(M(a) \setminus \sim\). What is more, for any \(W \in \text{Con}(A) \setminus \sim\), \(W \neq V\), we have \(W \cap M(a) = W\).

Moreover, in the next section we show (Proposition \(\text{[28]}\)) that \(\overline{V} \cap \overline{M(a)}\) is a hyperplane in \((V, \bullet)\), and so \(\dim \overline{V} \cap \overline{M(a)} + 1 = \dim \overline{V}\). Combining all these facts,
and using induction assumptions for $A/\Theta_A(1,a)$, we get:

\[
\delta(\operatorname{Con}(A)) = \delta(\operatorname{Con}(A/\Theta_A(1,a))) + 1
\]
\[
= \sum \{ \dim U : U \in \text{Cm}(A/\Theta_A(1,a)) / \sim \} + 1
\]
\[
= \dim V \cap M(a) + 1 + \sum \{ \dim W : W \in \text{Cm}(A) / \sim, W \neq V \}
\]
\[
= \sum \{ \dim W : W \in \text{Cm}(A) / \sim \}.
\]

\[
\square
\]

4. Representations

Let us start by recalling the well-known construction. Let $\mathcal{P} = (P, \leq)$ be a poset. For $S \subseteq P$ we write $S^\uparrow := \{ \mu \in P : \mu \geq \varphi \text{ for some } \varphi \in S \}$ and $S^\downarrow := \{ \mu \in P : \mu \leq \varphi \text{ for some } \varphi \in S \}$. The set $Up(P) = \{ S \subseteq P : S = S^\uparrow \}$ has the natural structure of Heyting algebra with the operations: $\cup, \cap, 0 := \emptyset, 1 := P$, and $\rightarrow$ defined by $S \rightarrow T := ((S \cap T) \downarrow)^\uparrow$ for $S, T \in Up(P)$. Then the equivalence operation $\leftrightarrow$ in $Up(P)$ is given by

\[
S \leftrightarrow T := (S \rightarrow T) \cap (T \rightarrow S) = ((S \cap T) \downarrow)^\uparrow
\]

for $S, T \in Up(P)$. Alternatively we can also define $S \leftrightarrow T$ as the largest $C \in Up(P)$ fulfilling $C \cap S = C \cap T$.

In this section we apply the above construction to the poset $\text{Cm}(A)$. Now, let $A$ be an arbitrary algebra. Then, it is well known that the map $M : \operatorname{Con}(A) \ni \varphi \rightarrow M(\varphi) := \{ \mu \in \text{Cm}(A) : \varphi \leq \mu \} \subseteq Up(\text{Cm}(A))$ is one-to-one, as $\varphi = \bigwedge M(\varphi)$ holds for every $\varphi \in \operatorname{Con}(A)$ (Birkhoff’s theorem).

In the next proposition that shows how the principal congruences behave under the map $M$, we merely assume that $H(A)$ is congruence orderable.

**Proposition 20.** Let $H(A)$ be congruence orderable (with respect to a constant term 1). Then

\[
M(\Theta_A(a,b)) = M(a) \leftrightarrow M(b),
\]

for $a, b \in A$, where $M(c) := M(\Theta_A(1,c)) = \{ \mu \in \text{Cm}(A) : (1,c) \in \mu \}$ for $c \in A$.

**Proof.** Let $a, b \in A$. It is enough to show that $M(\Theta_A(a,b))$ is the largest $C \in Up(\text{Cm}(A))$ such that $C \cap M(a) = C \cap M(b)$. Clearly, $M(\Theta_A(a,b)) \cap M(a) = M(\Theta_A(a,b)) \cap M(b)$. Let $C \in Up(\text{Cm}(A))$ fulfill $C \cap M(a) = C \cap M(b)$. Assume that there exists $\mu \in C \setminus M(\Theta_A(a,b))$. Then $(a,b) \notin \mu$. From the fact that $H(A)$ is congruence orderable we get $\mu \vee \Theta_A(1,a) \notin \mu \vee \Theta_A(1,b)$. Without loss of generality we can assume that $\mu \vee \Theta_A(1,a) \notin \mu \vee \Theta_A(1,b)$. Then there exists $v \in \text{Cm}(A)$ such that $\mu \vee \Theta_A(1,a) \leq v$ and $(1,b) \notin v$. Hence $v \in C \cap M(a)$ and $v \notin M(b)$, a contradiction.

From now on, as in the preceding section, we assume that $A$ is strongly Fregean. Let us consider the following subfamily of $Up(\text{Cm}(A))$, which for finite algebras can be identified with $\operatorname{Con}(A)$.

**Definition 21.** $S(A) := \{ S \in Up(\text{Cm}(A)) : S \cap U \text{ is a subalgebra of } (U, \bullet) \text{ for every } U \in \text{Cm}(A) / \sim \}$.

From Proposition 13 it is easy to deduce the following lemma.
**Lemma 22.** Let $A$ be strongly Fregean, $S$ be a finite set from $S(A)$ and $\mu \in \text{Cm}(A)$ be such that $\bigwedge S \leq \mu$. Then $\mu \in S$.

**Proof.** Let $S = \{\alpha_1, \ldots, \alpha_n\} \subset \text{Cm}(A)$ and $\alpha_1 \wedge \cdots \wedge \alpha_n \leq \mu$. According to Proposition 18 there exist $\mu_1, \ldots, \mu_n \in \mu/\sim \cup \{1\}$ such that $\mu_i \leq \mu_i$ for $i = 1, \ldots, n$, $\mu_1 \wedge \cdots \wedge \mu_n \leq \mu$, and $\mu = \mu_1 \cdot \cdots \cdot \mu_{ik}$ for some $1 \leq i_1 < \ldots < i_k \leq n$. Put $U = \mu/\sim$. From $S = S \uparrow$ we get $\{\mu_1, \ldots, \mu_n\} \setminus \{1\} \subset S \cap U$. Since $S \cap U$ is a subgroup of $(\overline{U}, \cdot)$ we deduce that $\mu \in S \cap U$. Hence $\mu \in S$. \hfill $\Box$

As a consequence, we get

**Theorem 23.** Let $A$ be strongly Fregean. Then

1. $M(\text{Con}(A)) \subset S(A)$.
2. If $A$ is finite, then $M$ establishes a one-to-one correspondence between $\text{Con}(A)$ and $S(A)$.

**Proof.** It follows immediately from the definition of the operation ‘$\cdot$’ that the image of $M$ is contained in $S(A)$. To prove (2) assume that $S \in S(A)$. Clearly, $S \subset M(\bigwedge S)$. From Lemma 22 we deduce that $S = M(\bigwedge S)$, which completes the proof. \hfill $\Box$

Our ultimate aim here is to characterize, for $A$ with a Malcev term, those sets from $Up(\text{Cm}(A))$ that correspond to congruences $\{\Theta_A(1, a) : a \in A\}$, or in other words, due to 1-regularity of $A$, to elements of $A$. We start from the definition of the family of hereditary sets in $Up(\text{Cm}(A))$ representing the principal congruences.

Let $\eta \in \text{Cm}(A)$ and let $U = \eta/\sim$. We put $U^+ := U \upharpoonright U$. Then

$$U^+ = \{\vartheta \in \text{Cm}(A) : \vartheta > \eta\}$$

$$= \{\vartheta \in \text{Cm}(A) : \vartheta > \varphi \text{ for all } \varphi \in U\}$$

$$= \{\vartheta \in \text{Cm}(A) : \vartheta \geq \eta^+\}$$

$$= M(\eta^+).$$

**Definition 24.** Let $Z \subset \text{Cm}(A)$. We say that $Z$ is hereditary, if:

(1) $Z = Z_\uparrow$;

(II) for all $U \in \text{Cm}(A)/\sim$, if $U^+ \subset Z$, then $\overline{Z \cap U} = U$ or $Z \cap U$ is a hyperplane in $(\overline{U}, \cdot)$.

(We use the word ‘hyperplane’ because we can interpret a Boolean group as a vector space over the field $\mathbb{Z}_2$.)

We denote the set of all hereditary subsets of $\text{Cm}(A)$ by $\mathcal{H}(A)$. Clearly, $\mathcal{H}(A) \subset S(A)$.

**Remark 25.** Observe that if $U \in \text{Cm}(A)/\sim$, $Z \in \mathcal{H}(A)$ and $U^+ \not\subset Z$, then $Z \cap U = \emptyset$. Converse implication is true, if $|U| > 1$. Moreover, if $U = \eta/\sim$, $\eta \in \text{Cm}(A)$, and $U^+ \subset Z$, then $\bigwedge Z \leq \eta^+$.

**Remark 26.** It follows from Proposition 2 that if we assume additionally that $\text{Con}(A)$ is distributive, then $\mathcal{H}(A) = S(A) = Up(\text{Cm}(A))$.

**Proposition 27.** Let $A$ be strongly Fregean and let $a, b \in A$. Then $M(\Theta_A(a, b)) \in \mathcal{H}(A)$.

**Proof.** Let $a, b \in A$. From Theorem 23 we deduce that, $M(\Theta_A(a, b)) \in S(A)$. Let $U = \eta/\sim$ for some $\eta \in \text{Cm}(A)$. Assume that $U^+ \subset M(\Theta_A(a, b))$. Clearly,
Then is no loss of generality in assuming that \( \varphi \) preserves equivalence operation, that is \( \varphi \models \leftrightarrow \). In consequence, \( \overline{M(\Theta_A(a,b)) \cap U} \) is a hyperplane in \( \overline{(U, \cdot)} \), as desired. \[ \square \]

It turns out that \( \mathcal{H}(A) \) inherits a natural equivalence operation from the Heyting algebra \( (Up(Cm(A)), \cup, \cap, \emptyset, Cm(A), \rightarrow) \).

**Theorem 28.** Let \( A \) be strongly Fregean. Then \( \mathcal{H}(A) \) is closed under the equivalence operation \( \leftrightarrow \) in \( Up(Cm(A)) \).

**Proof.** Let \( S, T \in \mathcal{H}(A) \). Then \( S \leftrightarrow T = ((S \div T) \downarrow) \in Up(Cm(A)) \). To prove that \( S \leftrightarrow T \) is hereditary it is enough to show for all \( U \in Cm(A) / \sim \), if \( U^{+} \subset S \leftrightarrow T \), then \( (S \leftrightarrow T) \cap U = U \) or \( (S \leftrightarrow T) \cap U \) is a hyperplane in \( \overline{(U, \cdot)} \). Let \( U \in Cm(A) / \sim \) and \( U^{+} \subset S \leftrightarrow T \). We know that \( U^{+} \cap S = U^{+} \cap T \). Let us consider two possibilities: 1) \( U^{+} \nsubseteq S \) and \( U^{+} \nsubseteq T \) (and then \( U \cap S = U \cap T = \emptyset \)); 2) \( U^{+} \subset S \cap T \).

1) In this situation we show that \( U \subset S \leftrightarrow T \). Assume that \( \mu \in U \setminus (S \leftrightarrow T) \). Then \( \mu \in (S \div T) \downarrow \), and so there is \( \varphi \in Cm(A) \) such that \( \mu \leq \varphi \in S \div T \). There is no loss of generality in assuming that \( \varphi \in S \setminus T \). Clearly \( \mu < \varphi \), since otherwise \( \mu \in U \cap S = \emptyset \), a contradiction. Hence \( \varphi \in U^{+} \cap S \) and \( \varphi \notin U^{+} \cap T \), which contradicts our assumption.

2) In this case \( S \cap U \) either equals \( U \) or is a hyperplane in \( \overline{(U, \cdot)} \). The same is true for \( T \cap U \). It is straightforward to prove that for \( \mu \neq \nu \) and \( \mu, \nu \in (S \div T) \downarrow \cap U \) we have \( \mu \cdot \nu \in (S \div T) \downarrow \cap U \). Hence \( (S \div T) \cap U \) is either \( U \) or a hyperplane in \( \overline{(U, \cdot)} \). Now, observe that \( S \div T \subset (S \div T) \downarrow \). Then \( (S \leftrightarrow T) \cap U \subset (S \div T) \downarrow \cap U \). It is enough to show that these sets are equal. Assume that \( \mu \in (S \div T) \downarrow \cap U \) and \( \mu \notin S \leftrightarrow T \). Then \( \mu \in (S \div T) \downarrow \). Thus, there is \( \varphi \in Cm(A) \) such that \( \mu \leq \varphi \in S \div T \). Clearly, \( \mu \neq \varphi \), and so \( \mu < \varphi \). In consequence, \( \varphi \in U^{+} \subset S \cap T \), a contradiction. \[ \square \]

In [3] Theorem 3.8] the following result, showing that the equivalential algebras form a paradigm of congruence permut able Fregean varieties, was proven. Recall that equivalential algebras are defined as equivalential subreducts of Heyting algebras [10]. For more information on equivalential algebras, see [15].

**Theorem 29.** Let \( H(A) \) be congruence orderable with respect to a constant term 1. Then the following conditions are equivalent:

1. \( A \) has a Malcev term;
2. \( A \) has a binary term \( e \), satisfying one of two equivalent conditions:
   a. \( e \) is a principle congruence term, i.e., \( \Theta_A(a,b) = \Theta_A(1,e(a,b)) \) for all \( a, b \in A \);
   b. \( e \)-reduct of \( A \) is an equivalential algebra.

Note that if these conditions are satisfied, then \( A \) is strongly Fregean.

Now, from Proposition [20] and Theorem [29] we get

**Corollary 30.** Let \( A \) be strongly Fregean with a principle congruence term \( e \). Then \( M \) preserves equivalence operation, that is

\[ M(e(a,b)) = M(a) \leftrightarrow M(b) \]

for \( a, b \in A \).
In particular, the assumptions of Theorem 29 are true if $\mathbf{A}$ is finite, congruence orderable and congruence permutable. Namely, the following proposition holds:

**Proposition 31.** Let $\mathbf{A}$ be a finite, congruence orderable (with respect to a constant term 1) and congruence permutable algebra. Then $H(\mathbf{A})$ is congruence orderable.

**Proof.** Let $\alpha \in \text{Con}(\mathbf{A})$, $a, b \in A$ fulfill $a \vee \Theta_{\mathbf{A}}(1, a) = a \vee \Theta_{\mathbf{A}}(1, b)$. It is enough to show that $(a, b) \in \alpha$. On the contrary, suppose that $(a, b) \notin \alpha$. There is no loss of generality in assuming that $a \not\equiv b$. Then, from congruence permutability of $\mathbf{A}$, there exists $c \in A$ such that $b \equiv c \Theta_{\mathbf{A}}(1, a) 1$. Thus, $c > a$ and $(b, c) \in \alpha$. Let $c_0 \in A$ be a maximal element with this property. Hence, $(1, b) \in \alpha \vee \Theta_{\mathbf{A}}(1, c_0)$, and so $(1, a) \in \alpha \vee \Theta_{\mathbf{A}}(1, c_0)$. Consequently, there is $d \in A$ such that $a \equiv d \Theta_{\mathbf{A}}(1, c_0) 1$. Clearly, we have $d \geq c_0 > a$ and $(1, b) \in \alpha \vee \Theta_{\mathbf{A}}(1, d)$, and so there exists $t \in A$ such that $c_0 \equiv t \Theta_{\mathbf{A}}(1, d) 1$. Hence, $t \geq d \geq c_0 > a$ and $(b, t) \in \alpha$. Finally, from maximality of $c_0$ we get $t = c_0 = d$, and so $a \equiv d = c_0 \equiv b$, a contradiction. \hfill \Box

**Remark 32.** In fact, the congruence permutability condition in the above proposition can be weakened. We see from the proof that it suffices to assume that $\mathbf{A}$ is congruence 1-permutable, i.e., $(1, a) \in \alpha \vee \beta \iff (1, a) \in \alpha \circ \beta$ for $\alpha, \beta \in \text{Con}(\mathbf{A})$.

The following theorem extends the result proven for equivalential algebras in [14, Theorem 5].

**Theorem 33.** Let $\mathbf{A}$ be finite, congruence orderable (with respect to a constant term 1) with a Malcev term. Then

1. $\mathcal{V}(\mathbf{A})$ is Fregean with a principle congruence term $e$ that turns every algebra in $\mathcal{V}(\mathbf{A})$ into an equivalential algebra;
2. the map

$$A \ni a \rightarrow M(a) := \{ \mu \in \text{Con}(\mathbf{A}) : (1, a) \in \mu \} \in \mathcal{H}(\mathbf{A})$$

establishes a one-to-one correspondence between $A$ and $\mathcal{H}(\mathbf{A})$. Moreover, $(\mathcal{H}(\mathbf{A}), \leftrightarrow)$ is an equivalential algebra isomorphic with $(A, e)$.

**Proof.** (1) From [9, Theorem 2.10] to show that $\mathcal{V}(\mathbf{A})$ is Fregean for finite $\mathbf{A}$, it is enough to prove that $H(\mathbf{A})$ is congruence orderable and $\mathcal{V}(\mathbf{A})$ is 1-regular. Let $B \in S(\mathbf{A})$ and $\Theta_{\mathbf{B}}(1, a) = \Theta_{\mathbf{B}}(1, b)$ for $a, b \in B$. Clearly, $\Theta_{\mathbf{B}}(1, a) \leq \Theta_{\mathbf{A}}(1, a) \cap B^2$, and so $(1, b) \in \Theta_{\mathbf{A}}(1, a)$. Consequently, $\Theta_{\mathbf{A}}(1, b) \leq \Theta_{\mathbf{A}}(1, a)$, and, analogously, $\Theta_{\mathbf{A}}(1, a) \leq \Theta_{\mathbf{A}}(1, b)$. From congruence orderability of $\mathbf{A}$ we get $a = b$. Thus $S(\mathbf{A})$ is congruence orderable. By Proposition 31 $H(\mathbf{A})$ is congruence orderable. To show that $\mathcal{V}(\mathbf{A})$ is 1-regular, we use Theorem 29 Since $H(\mathbf{A})$ is congruence orderable and $\mathbf{A}$ has a Malcev term, we deduce that there exists a binary term $e$ such that $e$ is a principle congruence term in $\mathbf{A}$, i.e., $\Theta_{\mathbf{A}}(a, b) = \Theta_{\mathbf{A}}(1, e(a, b))$ for all $a, b \in A$, and $(A, e)$ belongs to the variety of equivalential algebras [14]. Hence, $(C, e)$ must be an equivalential algebra for every $C \in \mathcal{V}(\mathbf{A})$. Let $\alpha \in \text{Con}(C) \subset \text{Con}((C, e))$. Thus $(c, d) \in \alpha \iff (1, e(c, d)) \in \alpha$. In consequence, $\mathcal{V}(\mathbf{A})$ is 1-regular with a principle congruence term $e$.

(2) By (1) $\mathbf{A}$ is strongly Fregean with a principle congruence term. Now, from Proposition 27 we know that the image of $M$ is contained in $H(\mathbf{A})$. On the other hand, the injectivity of $M$ follows from Theorem 28. Moreover, from Corollary...
we deduce that $M$ preserved equivalence operation. Hence it is enough to prove that $M$ is surjective.

Let $Z \in \mathcal{H}(A)$. Assume that $a$ is a minimal (with respect to the natural order `$\leq$' in $A$) element in $\{c \in A : (1, c) \in \wedge Z\}$. We show that $Z = M(a)$. Clearly, $Z \subseteq M(a)$. Suppose that $Z \not\subseteq M(a)$. Let $\mu$ be a maximal element in $\{\eta \in \text{Cl}(A) : \eta \in M(a) \setminus Z\}$. Put $U = \mu/\sim$. Then $Z \cap U \neq U$ and $U^+ = \{\vartheta \in \text{Cl}(A) : \vartheta > \mu\} \subset Z$. Hence, as $Z \in \mathcal{H}(A)$, we know that $Z \cap U$ is a hyperplane in $(\overline{U}, \bullet)$. Moreover, $Z \cap U \subseteq M(a) \cap U$. Then $M(a) \cap U$ is not a hyperplane, and so $M(a) \cap U = U$. Thus $(1, a) \in \mathcal{V}$, it follows from Lemma 22 that $\mu \notin Z$ implies $\wedge Z \leq \mu$. Since $\wedge Z \leq \wedge U^+ = \mu^+$, we get $I[\mu, \mu^+] \sim I[\wedge Z \wedge \mu, \wedge Z]$ and consequently $\text{card} I[\wedge Z \wedge \mu, \wedge Z] = 2$. Take a join irreducible $\beta \in \text{Con}(A)$ such that $\beta \leq \wedge Z$ and $\beta \notin \wedge Z \wedge \mu$. Let $\beta^-$ denote the unique subcover of $\beta$ in $\text{Con}(A)$. Then $I[\wedge Z \wedge \mu, \wedge Z] \sim I[\wedge Z \wedge \mu, \beta, \beta] = I[\beta^- \beta, \beta]$. Consequently, $I[\mu, \mu^+] \sim I[\beta^- \beta, \beta]$. From the fact that $A$ is 1-regular and $\beta$ is join irreducible we deduce that there exists $b \in A$ such that $\beta = \Theta_A(1, b)$, and so $(1, b) \notin \mu$ and $(1, b) \in \wedge Z$. In consequence, $(a, b) \notin \mu$ and $(a, b) \in \wedge Z$. From the fact that $e$ is a binary principle congruence term in $A$ we get $\Theta_A(a, b) = \Theta_A(1, e(a, b))$. Thus $(1, e(a, b)) \notin \mu$ and $(1, e(a, b)) \in \wedge Z$. Hence, $M(a) \neq M(e(a, b))$, and so from the minimality of $a$ in $\{c \in A : (1, c) \in \wedge Z\}$, we get $M(e(a, b)) \notin M(a)$. Let $\gamma$ be a maximal element in $\{\varphi \in \text{Cl}(A) : \varphi \in M(e(a, b)) \setminus M(a)\}$ and $W := \gamma/\sim$. Then for $v \in \text{Cl}(A)$ we know that $v > \gamma$ implies $v \in M(a)$, which gives $(1, a) \in \gamma^+$, and, since $(1, e(a, b)) \in \gamma^+$, we get $(1, b) \in \gamma^+$. Note that $(1, a) \notin 0_W$ as $(1, a) \notin \gamma$, and so $W \neq U$. Let us consider two cases: $(1, b) \in 0_W$ and $(1, b) \notin 0_W$. If $(1, b) \in 0_W$, then $(1, b) \in \gamma$, and we obtain $(1, a) \in \gamma$, a contradiction. If $(1, b) \notin 0_W$, then $(1, b) \in \gamma^+ \setminus 0_W$, and so there exists $\psi \in \text{Con}(A)$ such that $(1, b) \notin \psi \in W$. Then $I[\beta^- \beta] \sim I[\psi, \psi^+]$ which gives $\mu \sim \psi$, and consequently, $U = W$, a contradiction.

The assumption that $A$ has a Malcev term cannot be dropped. If $A$ does not come from congruence permutative variety, then we can represent congruences of the form $\Theta_A(1, a)$ as elements of $\mathcal{H}(A)$, however, the latter set can be larger than the image of $M$. To show this it is enough to consider the following simple example from [12].

**Example 34.** Take $A = (A, \rightarrow)$, where $A := \{a, b, 1\}$, and $\rightarrow$ is given by $x \rightarrow y := 1$, where $x = y$, and $x \rightarrow y := y$ where $x \neq y$. In fact, $A$ is a Hilbert algebra as the implicational subreduct of the Boolean algebra $B := 2 \times 2$, where $2$ denotes the two-element Boolean algebra, and so $V(A)$ is Fregean and congruence distributive. Then $\text{Cl}(A)$ is consisted of two incomparable congruences $\alpha := \Theta_A(1, a)$ and $\beta := \Theta_A(1, b)$. Hence $\mathcal{H}(A) = U \cap \text{Cl}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ has four elements, whereas $|A| = 3$.

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