End point gradient estimates for quasilinear parabolic equations with variable exponent growth on nonsmooth domains

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Abstract

In this paper, we study quasilinear parabolic equations with the nonlinearity structure modeled after the $p(x,t)$-Laplacian on nonsmooth domains. The main goal is to obtain end point Calderón-Zygmund type estimates in the variable exponent setting. In a recent work [1], the estimates obtained were strictly above the natural exponent $p(x,t)$ and hence there was a gap between the natural energy estimates and the estimates above $p(x,t)$ (see (1.3) and (1.2)). Here, we bridge this gap to obtain the end point case of the estimates obtained in [1]. To this end, we make use of the parabolic Lipschitz truncation developed in [2] and obtain significantly improved a priori estimates below the natural exponent with stability of the constants. An important feature of the techniques used here is that we make use of the unified intrinsic scaling introduced in [3], which enables us to handle both the singular and degenerate cases simultaneously.

Mathematics Subject Classification 35K59 · 35B65 · 35R05 · 46F30
1 Introduction

Calderón-Zygmund theory was first developed for the Poisson equation in [4], which related the integrability of the gradient of the solution for the Poisson equation with the associated data. This represented the starting point of obtaining a priori estimates in Sobolev spaces for
elliptic and parabolic equations. Since we are interested in Calderón-Zygmund theory for parabolic equations in this paper, we shall discuss the history of the problem only for parabolic equations and refer the reader to [5] and references therein for the elliptic counterpart.

All the estimates mentioned in this introduction are quantitative in nature, but to avoid being too technical, we only recall the qualitative nature of the bounds. This is sufficient to highlight the nature of the results that we will prove in this paper.

The starting point of Calderón-Zygmund theory for quasilinear parabolic equations was developed in [6], where they considered the following problem:

\[ u_t - \text{div}(a(x, t)|\nabla u|^{p-2}\nabla u) = -\text{div}(|f|^{p-2} f) \quad \text{in } \Omega \times (-T, T), \]

with \( a(x, t) \in \text{VMO} \) and \( p > \frac{2n}{n+2} \), proving

\[ |f| \in L^q_{\text{loc}}(\Omega \times (-T, T)) \implies |\nabla u| \in L^q_{\text{loc}}(\Omega \times (-T, T)) \quad \text{for all } q > p. \]

After this pioneering work, there have been numerous publications which extended these estimates to other quasilinear parabolic equations with constant \( p \)-growth. In [7], the authors improved the estimate in [6] to obtain global a priori estimates (with non homogeneous boundary data) and proved

\[ |f| \in L^q(\Omega \times (-T + \delta, T)) \implies |\nabla u| \in L^q(\Omega \times (-T + \delta, T)) \]

for all \( q > p \) and some \( \delta \in (0, 2T) \).

This was subsequently extended in [8] to prove global a priori estimates for more general nonlinear structures satisfying a small BMO condition and Reifenberg-flat domains (see Sect. 2 for the precise definitions).

In this paper, we are interested in obtaining Calderón-Zygmund type bounds for the problem

\[
\begin{cases}
    u_t - \text{div}A(x, t, \nabla u) = -\text{div}(|f|^{p(x, t)-2} f) & \text{in } \Omega \times (-T, T), \\
    u = 0 & \text{on } \partial_\Omega \Omega \times (-T, T).
\end{cases}
\] (1.1)

Here, the quasilinear operator \( A(x, t, \nabla u) \) is modeled after well known \( p(x, t) \)-Laplacian operator having the form \( |\nabla u|^{p(x, t)-2} \nabla u \) with \( p(\cdot) > \frac{2n}{n+2} \). For more on the importance of variable exponent problems, see [9–14] and the references therein.

In a recent paper [15], the authors were able to show

\[ |f|^{p(\cdot)} \in L^q_{\text{loc}}(\Omega \times (-T, T)) \implies |\nabla u|^{p(\cdot)} \in L^q_{\text{loc}}(\Omega \times (-T, T)) \quad \text{for all } 1 < q < \infty. \]

This was subsequently improved to a global estimate in [1], where they proved

\[ |f|^{p(\cdot)} \in L^q(\Omega \times (-T, T)) \implies |\nabla u|^{p(\cdot)} \in L^q(\Omega \times (-T, T)) \quad \text{for all } 1 < q^{-} \leq q(\cdot) \leq q^{+} < \infty. \] (1.2)

In particular, they could not take \( q^{-} = 1 \).

On the other hand, from the definition of weak solution, it is easy to see that the following energy-type estimate holds:

\[ |f|^{p(\cdot)} \in L^1(\Omega \times (-T, T)) \implies |\nabla u|^{p(\cdot)} \in L^1(\Omega \times (-T, T)). \] (1.3)

Comparing (1.2) and (1.3), it seems reasonable to expect that (1.2) should hold with \( 1 \leq q^{-} \leq q(\cdot) \leq q^{+} < \infty \), i.e., \textit{it should be possible to take } \( q^{-} = 1 \).

In this paper, we prove that we can indeed take \( q^{-} = 1 \) in (1.2). In order to do this, we will obtain improved estimates below the natural exponent \( p(\cdot) \) using the method of parabolic
Lipschitz truncation developed in the seminal paper [2], as well as the unified intrinsic scaling of [3].

In order to prove our results, we need to impose some restrictions on the variable exponent \( p(x, t) \), on the nonlinear structure \( A(x, t, \nabla u) \) as well as on the boundary of the domain \( \partial \Omega \). These restrictions will be described in detail in Sect. 2.

The plan of the paper is as follows: In Sect. 2, we collect all assumptions that will be needed on the structure of the nonlinearity \( A \), on the domain \( \Omega \) and on the variable exponent \( p(\cdot) \). In Sect. 3, we define the notion of weak solutions and collect some of their well known properties. In Sect. 4, we state the main results of this paper. In Sect. 5, we collect all the preliminary results and well known lemmas that will be needed in subsequent parts of the paper. In Sect. 6, we describe the approximations that will be made along the way. In Sects. 7 and 8, we prove crucial difference estimates below the natural exponent for energy solutions. In Sect. 9, we demonstrate some important covering arguments. In Sect. 10, the proof of the main theorems will be provided. Finally in Appendix A and Appendix B, we will describe the construction of test functions having Lipschitz regularity which will be needed to prove the estimates in Sects. 7 and 8, respectively.

2 Regularity assumptions and notation

In this section, we shall collect all the structure assumptions as well as recall several useful lemmas that are already available in existing literature.

2.1 Metrics needed

Let us first collect a few metrics on \( \mathbb{R}^{n+1} \) that will be used throughout the paper.

**Definition 2.1** We define the parabolic metric \( d_p \) on \( \mathbb{R}^{n+1} \) as follows: Let \( z_1 = (x_1, t_1) \) and \( z_2 = (x_2, t_2) \) be any two points on \( \mathbb{R}^{n+1} \), then

\[
d_p(z_1, z_2) := \max \left\{ |x_1 - x_2|, \sqrt{|t_1 - t_2|} \right\}.
\]

Since we will use intrinsically scaled cylinders where the scaling depends on the center of the cylinder, we will also need to consider the following localized parabolic metric:

**Definition 2.2** Given a function \( 1 < p(\cdot) < \infty \), some fixed point \( z = (x, t) \in \mathbb{R}^{n+1} \) and any \( \tau > 0, d > 0 \), we define the localized parabolic metric \( d^{\tau, d} \) as follows: Let \( z_1 = (x_1, t_1) \) and \( z_2 = (x_2, t_2) \) be any two points on \( \mathbb{R}^{n+1} \), then

\[
d^{\tau, d}(z_1, z_2) := \max \left\{ \tau^{\frac{1}{p(z)} - \frac{d}{2}} |x_1 - x_2|, \sqrt{\tau^{1-d} |t_1 - t_2|} \right\}.
\]

2.2 Structure of the variable exponent

**Definition 2.3** We say that, a bounded measurable function \( p(\cdot) : \mathbb{R}^{n+1} \to \mathbb{R} \) belongs to the log-Hölder class \( \log^\pm \), if the following conditions are satisfied:

- There exist constants \( p^- \) and \( p^+ \) such that \( 1 < p^- \leq p(z) \leq p^+ < \infty \) for every \( z \in \mathbb{R}^{n+1} \).
- \( |p(z_1) - p(z_2)| \leq \frac{L}{-\log |z_1 - z_2|} \) holds for every \( z_1, z_2 \in \mathbb{R}^{n+1} \) with \( d_p(z_1, z_2) \leq \frac{1}{2} \) and for some \( L > 0 \).

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We say that a bounded domain $\Omega$ is a Reifenberg flat domain if for every $x_0 \in \partial \Omega$ and every $r \in (0, S_0]$, there exists a system of coordinates $(y_1, y_2, \ldots, y_n)$ (possibly depending on $x_0$ and $r$) such that in this coordinate system, $x_0 = 0$ and

$$B_r(0) \cap \{y_n > \gamma r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y_n < -\gamma r\}.$$

The class of Reifenberg flat domains are standard in obtaining Calderón-Zygmund type estimates, in the elliptic case, see [16–19] and references therein, whereas for the parabolic case, see [8,20–22] and the references therein.

### 2.3 Structure of the domain

The domain that we consider may be nonsmooth but should satisfy some regularity condition. This condition would essentially say that at each boundary point and every scale, we require the boundary of the domain to be between two hyperplanes separated by a distance proportional to the scale.

#### Definition 2.5

Given any $\gamma \in (0, 1)$ and any $S_0 > 0$, we say that $\Omega$ is a $(\gamma, S_0)$-Reifenberg flat domain if for every $x_0 \in \partial \Omega$ and every $r \in (0, S_0]$, there exists a system of coordinates $(y_1, y_2, \ldots, y_n)$ (possibly depending on $x_0$ and $r$) such that in this coordinate system, $x_0 = 0$ and

$$B_r(0) \cap \{y_n > \gamma r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y_n < -\gamma r\}.$$

The class of Reifenberg flat domains are standard in obtaining Calderón-Zygmund type estimates, in the elliptic case, see [16–19] and references therein, whereas for the parabolic case, see [8,20–22] and the references therein.

#### Definition 2.6

We say that a bounded domain $\Omega$ is said to satisfy a uniform measure density condition with a constant $m_e > 0$ if for every $x \in \Omega$ and every $r > 0$, there holds

$$|\Omega^c \cap B_r(x)| \geq m_e |B_r(x)|.$$  

From the definition of $(\gamma, S_0)$-Reifenberg flat domains, it is easy to see that the following property holds:

#### Lemma 2.7

Let $\gamma \in (0, 1/8)$ and $S_0 > 0$ be given and suppose that $\Omega$ is a $(\gamma, S_0)$-Reifenberg flat domain. Then the following measure density conditions hold:

$$\sup_{y \in \Omega} \sup_{r \leq S_0} \frac{|B_r(y)|}{|B_r(y) \cap \Omega|} \leq \left(\frac{2}{1-\gamma}\right)^n \leq \left(\frac{16}{7}\right)^n,$$

$$\inf_{y \in \partial \Omega} \inf_{r \leq S_0} \frac{|\Omega^c \cap B_r(y)|}{|B_r(y)|} \geq \left(\frac{1-\gamma}{2}\right)^n \geq \left(\frac{7}{16}\right)^n.$$  

### 2.4 Structure of the nonlinearity $\mathcal{A}$

We first assume that $\mathcal{A}(-, -)$ is a Carathéodory function in the sense:

$$(x, t) \mapsto \mathcal{A}(x, t, \zeta) \text{ is measurable for every } \zeta \in \mathbb{R}^n,$$

$$\zeta \mapsto \mathcal{A}(x, t, \zeta) \text{ is continuous for almost every } (x, t) \in \mathbb{R}^n \times \mathbb{R}.$$

Let $\mu \in [0, 1]$ be given, then there exist two positive constants $\Lambda_0, \Lambda_1$ such that the following holds for almost every $x \in \Omega$ and every $\zeta, \eta \in \mathbb{R}^n$,

$$\left(\mu^2 + |\zeta|^2 \right)^{1/2} |D_\zeta \mathcal{A}(x, t, \zeta)| + |\mathcal{A}(x, t, \zeta)| \leq \Lambda_1 (\mu^2 + |\zeta|^2)^{p(x, t)-1},$$

where

$$p(x, t) := \frac{N}{2} \left( \frac{N}{2} + \frac{N}{p^*} \right).$$
$$\left(\mu^2 + |\zeta|^2\right)^{\frac{p(s,t)-2}{2}} \frac{1}{|\eta|^2} \Lambda_0 \leq \langle D_\xi A(x, t, \xi) \eta, \eta \rangle.$$  (2.3)

We point out that from (2.3), one can derive the following monotonicity bound:

$$\langle A(x, t, \xi) - A(x, t, \eta), \xi - \eta \rangle \geq \tilde{\Lambda}_0 \left(\mu^2 + |\zeta|^2 + |\eta|^2\right)^{\frac{p(s,t)-2}{2}} |\xi - \eta|^2,$$  (2.4)

where \(\tilde{\Lambda}_0 = \tilde{\Lambda}_0(\Lambda_0, n, p^+, p^-) > 0\). By inserting \(\eta = 0\) into (2.4), we also have the following coercivity bound:

$$\tilde{\Lambda}_2 |\zeta|^{p(s,t)} \leq \langle A(x, t, \xi), \xi \rangle + \tilde{\Lambda}_1,$$

where \(\tilde{\Lambda}_1 = \tilde{\Lambda}_1(\Lambda_1, \Lambda_0, p^+, p^-, n) > 0\) and \(\tilde{\Lambda}_2 = \tilde{\Lambda}_2(\Lambda_1, \Lambda_0, p^+, p^-, n) > 0\).

### 2.5 Smallness assumption

In order to prove the main results, we need to assume a smallness condition satisfied by \((p(\cdot), A, \Omega)\).

**Definition 2.8** Let \(\gamma \in (0, 1/8)\) and \(S_0 > 0\) be given, we then say \((p(\cdot), A, \Omega)\) is \((\gamma, S_0)\)-vanishing if the following three structure conditions are satisfied:

(i) Assumption on \(p(\cdot)\): The variable exponent \(p(\cdot)\) with modulus of continuity \(\omega_{p(\cdot)}\) as defined in Definition 2.3 with \(p^- > \frac{2n}{n+2}\), is further assumed to satisfy the smallness condition:

$$\sup_{0 < r \leq S_0} \omega_{p(\cdot)}(r) \log \left(\frac{1}{r}\right) \leq \gamma.$$  (2.5)

(ii) Assumption on \(A\): For a bounded open set \(U \subset \mathbb{R}^n\), let us denote

$$\Theta(A, U)(x, t) := \sup_{\zeta \in \mathbb{R}^n} \left| \frac{A(x, t, \zeta)}{\left(\mu^2 + |\zeta|^2\right)^{\frac{p(s,t)-2}{2}}} \right| - \left(\frac{A(\cdot, t, \zeta)}{\left(\mu^2 + |\zeta|^2\right)^{\frac{p(s,t)-2}{2}}} \right)_U,$$

where we have used the notation \((f)_U := \int_U f(y) \, dy\). Note that if \(\mu = 0\), then \(\zeta \in \mathbb{R}^n \setminus \{0\}\).

We assume that the nonlinearity \(A\) has small BMO with constant \(\gamma\) if there holds

$$\sup_{t_1, t_2 \in \mathbb{R}, 0 < r \leq S_0} \sup_{y \in \mathbb{R}^n} \int_{t_1}^{t_2} \int_{B_r(y)} \Theta(A, B_r(y))(x, t) \, dx \, dt \leq \gamma.$$  (2.6)

(iii) Assumption on \(\partial \Omega\): The domain \(\Omega\) is \((\gamma, S_0)\)-Reifenberg flat in the sense of Definition 2.5.
2.6 Notation

We shall use the following notations throughout the paper:

- We will use \( z, \tilde{z}, \dot{z}, \ldots \) to be points in \( \mathbb{R}^{n+1} \), symbols \( x, \tilde{x}, y, \tilde{y}, \ldots \) to denote space variables in \( \mathbb{R}^n \) and symbols \( t, \tilde{t}, s, \ldots \) to denote time variables. We will also specifically match symbols, i.e., \( z = (x, t) \) or \( \dot{z} = (x, t) \) and so on.
- In all subsequent sections, the subscript \( \cdot \) will always denote the usual Steklov average.
- In what follows, the function \( \omega_{p(\cdot)} \) denotes the modulus of continuity of \( p(\cdot) \) and we denote \( \omega_{q(\cdot)} \) for the modulus of continuity of \( q(\cdot) \).
- We shall write \( p(\cdot) \) as well as \( p(\cdot, \cdot) \) depending on the necessity and we will switch between the two notations without notice throughout the paper.
- For the variable exponent \( p(\cdot) \), we shall denote by \( p^\pm_{\log} \) to include the constants \( p^+, p^- \) and those that are part of the log-Hölder continuity structure of \( p(\cdot) \). Analogously, for variable exponents \( q(\cdot), r(\cdot) \) and \( s(\cdot) \), we shall use \( q^\pm_{\log}, r^\pm_{\log} \) and \( s^\pm_{\log} \) to denote corresponding constants.
- Capital alphabets with subscripts as in radii \( R, R^\cdot \), or bounding values \( M, M_0, M_1, \ldots \) will be fixed in subsequent sections once they are chosen.
- We shall use \( \lesssim, \gtrsim \) and \( \approx \) to suppress writing the constants that could possibly change from line to line as long as they depend only on the structure constants of the form \( n, p^\pm_{\log}, q^\pm_{\log}, \Lambda_0, \Lambda_1, S_0 \) and related quantities.
- We shall sometimes use \( \sim \) to denote variables (without subscripts) that occur only within the proof of the concerned result, for example \( r, \tilde{r}, \tilde{m}, \ldots \).
- Given a variable exponent \( p(\cdot) \), we shall use the following notation:
  \[
  p^+_E := \inf_{x \in E} p(x) \quad \text{and} \quad p^-_E := \sup_{x \in E} p(x).
  \]
  We will drop the set \( E \) and denote \( p^+ := \sup_{x \in \mathbb{R}^{n+1}} p(z) \) and \( p^- := \inf_{x \in \mathbb{R}^{n+1}} p(z) \).
- We will denote \( \Omega_T := \Omega \times (-T, T) \) which is the region on which (1.1) is considered. We will also use the notation \( \partial_p \) to denote the parabolic boundary, i.e,
  \[
  \partial_p Q_{\rho, s}(x, t) := B_\rho(x) \times \{t - s\} \bigcup B_\rho(x) \times \{t - s, t + s\}.
  \]

2.7 Unified intrinsic cylinders

We will describe the intrinsically scaled cylinders that will be used in this paper. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), and let \( \rho > 0, s > 0, \lambda > 0 \) and \( \dot{z} = (x, t) \in \mathbb{R}^{n+1} \) be given. Furthermore, let \( d \) be an exponent satisfying
\[
\min \left\{ \frac{2}{p(\dot{z})}, 1 \right\} > d > \frac{2n}{(n + 2)p(\dot{z})}.
\]  (2.7)

**Remark 2.9** Note that \( d \) depends on the point \( \dot{z} \) only through \( p(\dot{z}) \). Thus we can make a choice of \( d(\dot{z}) \) which is also log-Hölder continuous, for example take \( d \) to be the average of the upper and lower bounds in (2.7), which is possible since \( p(\dot{z}) \) is a bounded function away from \( \frac{2n}{n+2} \) and \( \infty \).

In terms of using \( d(\dot{z}) \), we locally obtain estimates on parabolic cylinders of the form \( Q_{\alpha - \frac{1}{p(\dot{z})} + d, \alpha - 1 + d} \) (3) and the exponent \( d = d(p(\dot{z})) \), where \( \dot{z} \) is the center of the cylinder. By
an abuse of notation, we shall ignore writing this dependence on the point \( z \) unless otherwise noted.

Since \( p(\cdot) \) is bounded away from zero and infinity, we see that the exponent \( d \) has upper and lower bounds depending only on \( n, p^+, p^- \).

We define the following cylinders that will be used throughout the paper:

\[
Q_{\rho, s}(\beta):=B_{\rho}(x) \times (t-s^2, t+s^2),
\]
\[
Q^{\lambda, +}_{\rho, s}(\beta):=B_{\rho}^{+}(x) \times \left( t-\lambda^{-1+d}s^2, t+\lambda^{-1+d}s^2 \right) :=B_{\rho}^{+}(x) \times I^{s}_{\rho}(t).
\]

We will also use the following short notation:

\[
\Omega_{\rho}(\beta):=B_{\rho}(x) \cap \Omega,
\]
\[
I_{\rho}(t):=\left( t-\rho^2, t+\rho^2 \right),
\]
\[
\Omega^{\rho}_{\rho}(\beta):=B_{\rho}^{+}(x) \cap \partial \Omega,
\]
\[
\partial w_{\rho}(\beta):=B_{\rho}(x) \cap \partial \Omega,
\]
\[
\partial w_{\rho}^{\rho}(\beta):=B_{\rho}^{+}(x) \cap \partial \Omega.
\]

\[
\partial w_{\rho, \rho}(\beta):=B_{\rho}(x) \cap \partial \Omega(x) \cap I_{\rho}(t),
\]
\[
\partial w_{\rho}^{\rho}(\beta):=B_{\rho}^{+}(x) \cap \partial \Omega \times (-T, T),
\]
\[
\partial w_{\rho}^{\rho}(\beta):=B_{\rho}^{+}(x) \cap \partial \Omega \times (-T, T).
\]

\[
\partial_{\beta}w_{\rho}(\beta):=B_{\rho}(x) \times \left( t-s^2 \right) \cup \partial B_{\rho}(x) \times I_{\rho}(t),
\]
\[
\partial_{\beta}w_{\rho}^{\rho}(\beta):=B_{\rho}^{+}(x) \times \left( t-\lambda^{-1+d}s^2 \right) \cup \partial B_{\rho}^{+}(x) \times I^{s}_{\rho}(t).
\]

\[
Q_{\rho}(\beta):=Q_{\rho, \rho}(\beta),
\]
\[
K_{\rho}(\beta):=K_{\rho, \rho}(\beta),
\]
\[
Q^{\rho}_{\rho}(\beta):=Q^{\rho, +}_{\rho, \rho}(\beta),
\]
\[
K^{\rho}_{\rho}(\beta):=K^{\rho, +}_{\rho, \rho}(\beta).
\]

We will also have to deal with half spaces, and use the following notation in that regard:

\[
B_{\rho}^{+}(x) :=B_{\rho}(x) \cap \{ x_n > 0 \},
\]
\[
B^{\lambda, +}_{\rho}(x) :=B^{+}_{\rho}(x) \cap \{ x_n > 0 \},
\]
\[
Q^{\lambda, +}_{\rho}(x) :=B^{+}_{\rho}^{\lambda}(x) \times \left( t-\lambda^{-1+d}\rho^2, t+\lambda^{-1+d}\rho^2 \right),
\]
\[
T^{\lambda}_{\rho}(x) :=B_{\rho}^{-1+d} \times \left( t-\lambda^{-1+d}\rho^2, t+\lambda^{-1+d}\rho^2 \right).
\]

An important thing to note is that the cylinders considered above are intrinsically scaled both in space and time simultaneously. This enables us to handle both the singular case \((p(\cdot) < 2)\) and degenerate case \((p(\cdot) > 2)\) simultaneously.

### 2.8 Restriction on radii

In this subsection, let us collect all the restrictions we will make on some universal constants. First, let us describe all the restriction on the radii \( \rho_{0} \):

(R1) Let \( \rho_{0} \leq \frac{1}{4} \) such that \(|Q_{\rho_{0}}| = (\rho_{0})^{n+2} |B_{1}| \leq 1 \).

(R2) Let \( \rho_{0} \) be such that \( \rho_{0} \leq \min \{ \beta_{1}, \beta_{2} \} \), where \( \beta_{1} \) is from Theorem 6.1 and \( \beta_{2} \) is from Theorem 6.2 applied with \( M_{\Gamma} = M_{0} \). Here \( M_{0} \) is given in (6.26).

(R3) Let \( \rho_{0} \leq \min \{ \hat{\beta}_{1}, \hat{\beta}_{2} \} \), where \( \hat{\beta}_{1} \) is from Theorem 6.1 and \( \hat{\beta}_{2} \) is from Theorem 6.2 applied with \( M_{\Gamma} = M_{0} \).

(R4) Let \( 1024 \rho_{0} \leq \min \left\{ \frac{1}{M_{0}}, \frac{1}{M_{u}}, \frac{1}{M_{w}} \right\} \), where \( M_{0}, M_{u}, \text{ and } M_{w} \) are from (6.26), (6.27), and (6.28), respectively.

(R5) Let \( \rho_{0} \) satisfy \( \rho_{0} \leq \rho_{0} \frac{12}{\rho_{0} - 1} \leq \beta_{0} \), where \( \beta_{1} \) is from Theorem 6.1 and \( \beta_{2} \) is from Theorem 6.2 applied with \( M_{\Gamma} = M_{0} \).

(R6) With \( M_{\rho} = \max \{ M_{0}, M_{u}, M_{w} \} \), we will apply Lemma 5.1 and Theorem 5.2 which will impose the restriction \( \rho_{0} \leq R_{\rho} \).

(R7) Let \( \rho_{0} \leq S_{0} \), where \( \Gamma \) is given in (9.4) and \( S_{0} \) is from Definition 2.8.
(R8) Let \( \omega_{p(\cdot)}(2\rho_0) \leq \min \left\{ \frac{p^+ - \sigma}{2}, \frac{\Lambda_0}{2\Lambda_1}, \frac{p^+ - (p^- - 1)\sigma}{4}, \frac{1}{4}, d_0 p^-, d_0 p^+(p^- - 1) \right\} \), where \( \sigma \) is given in Remark 2.11 and \( d_0 \) is defined in (6.4).

(R9) Let \( \omega_{q(\cdot)}(2\rho_0) \leq \min \left\{ \frac{q^+ - \sigma}{4}, \frac{1}{4}, q^-(q^- - 1) \right\} \), where \( \sigma \) is given in Remark 2.11 and \( q(\cdot) \) is the exponent appearing in Theorem 4.1.

(R10) We will also assume that 

\[
p_{128\rho_0}^+ - p_{128\rho_0}^- \leq \omega_{p(\cdot)}(512\rho_0) \leq \frac{1}{2} \min \left\{ \frac{1}{n + 2}, \frac{1}{4} \right\}.
\]  

(2.8)

**Remark 2.10** Note that all the restrictions on \( \rho_0 \) are such that \( \rho_0 = \rho_0(n, \Lambda_0, \Lambda_1, p_{\log}^+, M_0) \in (0, 1/4) \) and henceforth we will always take the radius \( \rho \) to satisfy \( 128\rho \leq \rho_0 \).

### 2.9 Fixing a few other exponents

We will first collect all the restrictions on the higher integrability exponent:

(B1) Let \( 0 < \beta_0 \leq \min \left\{ \frac{1}{p^+}, \beta_1, \beta_2, \beta_3, \beta_4 \right\} \), where \( \beta_1, \beta_2, \beta_3, \) and \( \beta_4 \) are given in Theorems 6.1, 6.2, 7.1, and 8.1, respectively.

(B2) Once \( \beta_0 \) is fixed, let \( \sigma_0 \) be a number chosen such that \( 0 < \sigma_0 \leq \min \left\{ \frac{\beta_0}{3(1 - \beta_0)}, \frac{q^- - 1}{3}, 1 \right\} \) holds.

(B3) Let \( \tilde{\sigma}_0 = \max\{\tilde{\beta}_1, \tilde{\beta}_2\} \), where \( \tilde{\beta}_1 \) and \( \tilde{\beta}_2 \) are from Theorems 6.1 and 6.2 with \( M_0 = M_0 \). Here \( M_0 \) is given in (6.26).

(B4) We will further assume \( \beta \leq \frac{p^- d_0}{2} \) and \( \beta \leq \frac{p^-(p^- - 1) d_0}{2} \) holds, where \( d_0 \) is as defined in (6.4).

**Remark 2.11** Henceforth, we will assume \( 0 < \sigma \leq \sigma_0 \) and \( 0 < \beta \leq \beta_0 \).

### 3 Weak solution

#### 3.1 Sobolev spaces with variable exponents

Let \( \tilde{\Omega} \) be a bounded domain in \( \mathbb{R}^N \) for some \( N \geq 1 \), and let \( s(\cdot) \) be an admissible variable exponent as in Section 2.2. Given a positive integer \( m \), the **variable exponent Lebesgue space** \( L^{s(\cdot)}(\tilde{\Omega}, \mathbb{R}^m) \) consists of all measurable functions \( f : \tilde{\Omega} \to \mathbb{R}^m \) satisfying 

\[
\int_{\tilde{\Omega}} |f(z)|^{s(z)} \, dz < \infty,
\]

endowed with the Luxemburg norm 

\[
\|f\|_{L^{s(\cdot)}(\tilde{\Omega}, \mathbb{R}^m)} := \inf \left\{ \lambda > 0 : \int_{\tilde{\Omega}} \left| \frac{f(z)}{\lambda} \right|^{s(z)} \, dz \leq 1 \right\}.
\]

Analogously, we can define the **variable exponent Sobolev space** as 

\[
W^{1,s(\cdot)}(\tilde{\Omega}, \mathbb{R}^m) := \{ f \in L^{s(\cdot)}(\tilde{\Omega}, \mathbb{R}^m) : \nabla f \in L^{s(\cdot)}(\tilde{\Omega}, \mathbb{R}^m) \}
\]

equipped with the norm 

\[
\|f\|_{W^{1,s(\cdot)}(\tilde{\Omega}, \mathbb{R}^m)} := \|f\|_{L^{s(\cdot)}(\tilde{\Omega}, \mathbb{R}^m)} + \|\nabla f\|_{L^{s(\cdot)}(\tilde{\Omega}, \mathbb{R}^m)}.
\]  

(3.1)
We shall denote $W_{0}^{1,s} (\Omega, \mathbb{R}^{m})$ to be the closure of $C_{c}^\infty(\Omega, \mathbb{R}^{m})$ under the norm from (3.1). Then all function spaces mentioned above are separable Banach spaces. For $m = 1$, we write $L^{s} (\Omega)$ and $W^{1,s} (\Omega)$ for simplicity. We will also use the following modular function:

$$
\varphi_{L^{s} (\Omega)} (f) := \int_{\Omega} |f(z)|^{\frac{s}{s'}} dz.
$$

We mention the following useful relation between the modular and the norm in variable exponent spaces (see [23, Lemma 3.2.5] for details):

**Lemma 3.1** For any $f \in L^{s} (\Omega)$, the following holds:

$$
\min \left\{ \varphi_{L^{s} (\Omega)} (f) \frac{1}{s'}, \varphi_{L^{s'} (\Omega)} (f) \frac{1}{s} \right\} \leq \| f \|_{L^{s} (\Omega)} \leq \max \left\{ \varphi_{L^{s} (\Omega)} (f) \frac{1}{s'}, \varphi_{L^{s'} (\Omega)} (f) \frac{1}{s} \right\}.
$$

Let us now define some function spaces involving time. Let $\Omega \subset \mathbb{R}^{n}$ be a bounded domain, and the space $L^{s} (-T, T; W^{1,s} (\Omega))$ is defined as

$$
L^{s} (-T, T; W^{1,s} (\Omega)) := \left\{ f \in L^{s} (\Omega_{T}) : \nabla \text{space } f \in L^{s} (\Omega_{T}, \mathbb{R}^{n}) \right\},
$$

equipped with the norm

$$
\| f \|_{L^{s} (-T, T; W^{1,s} (\Omega))} := \| f \|_{L^{s} (\Omega_{T})} + \| \nabla f \|_{L^{s} (\Omega_{T}, \mathbb{R}^{n})}.
$$

We shall define $L^{s} (-T, T; W_{0}^{1,s} (\Omega)) := L^{s} (-T, T; W^{1,s} (\Omega)) \cap L^{1} (-T, T; W_{0}^{1,1} (\Omega))$, and let us denote $L^{s} (-T, T; W^{1,s} (\Omega))'$ the dual space of $L^{s} (-T, T; W_{0}^{1,s} (\Omega))$. We remark that if $s(\cdot)$ is constant function, then all function spaces considered above become well known classical parabolic Sobolev spaces.

### 3.2 Definition of weak solution

There is a well known difficulty in defining the notion of solution for (1.1) due to a lack of time derivative of $u$. To overcome this, one can either use Steklov average or convolution in time. In this paper, we shall use the former approach (see also [24, Chapter 2] for further details).

Let us first define Steklov average as follows: let $h \in (0, 2T)$ be any positive number, then we define

$$
[u]_{h} (\cdot, t) := \left\{ \begin{array}{ll}
\frac{1}{h} \int_{t-h}^{t} u(\cdot, \tau) d\tau & t \in (-T, T-h), \\
\text{else.} & \end{array} \right.
$$

(3.2)

Let us now recall two equivalent notions of solutions (see [25, Section 3.4] and [24, Chapter 2, Remark 1.1]) that will be considered in this paper.

**Definition 3.2** Let $h \in (0, 2T)$ be given, then we say $u \in L^{2} (-T, T; L^{2} (\Omega)) \cap L^{p(\cdot)} (-T, T; W_{0}^{1,p(\cdot)} (\Omega))$ is a weak solution of (1.1) if for any $\phi \in W_{0}^{1,p(\cdot)} (\Omega)$, the following holds:

$$
\int_{\Omega \times \{t\}} \frac{d[u]_{h}}{dt} \phi + \langle [A(x, t, \nabla u)]_{h} , \nabla \phi \rangle dx = \int_{\Omega \times \{t\}} \langle [\mathcal{F}[p(x,t)-2]_{h} , \nabla \phi] \rangle dx
$$

for almost every $-T < t < T - h$. 
Definition 3.3 We say that a function \( u \in L^2(-T, T; L^2(\Omega)) \cap L^{p(\cdot)}(-T, T; W_0^{1,p(\cdot)}(\Omega)) \) is a weak solution of (1.1) provided
\[
\iint_{\Omega \times T} u \phi_t - \langle A(x, t, \nabla u), \nabla \phi \rangle \, dz = -\iint_{\Omega \times T} \langle |f|^{p(\cdot) - 2} \nabla \phi \rangle \, dz,
\]
holds for every \( \phi \in C_c^\infty(\Omega_T) \).

Further details regarding weak solutions can also be found in the introduction of [17].

3.3 Existence and uniqueness of weak solution

We begin with the following well known existence and uniqueness result:

Proposition 3.4 ([26,27]) Let \( \Omega \) be any bounded domain satisfying a uniform measure density condition (see Definition 2.6). Suppose that \( f \in L^p(\Omega_T) \), \( f \in L^{p(\cdot)}(-T, T; W_0^{1,p(\cdot)}(\Omega)) \) with \( \frac{df}{dt} \in L^p(-T, T; W_1^{1,p(\cdot)}(\Omega)^{\prime}) \) and \( f_0 \in L^2(\Omega) \) are given. Then there is a unique weak solution \( \phi \in C^0(-T, T; L^2(\Omega)) \cap L^{p(\cdot)}(-T, T; W_1^{1,p(\cdot)}(\Omega)) \) solving
\[
\begin{cases}
\phi_t - \text{div} D(z, \nabla \phi) = -\text{div}[f(\cdot, t)]^{p(\cdot)-2}f & \text{in } \Omega_T, \\
\phi = f & \text{on } \Omega \times (-T, T), \\
\phi(\cdot, -T) = f_0 & \text{on } \Omega.
\end{cases}
\]

where \( D \) is any operator satisfying all the assumptions in Sect. 2.4.

Moreover if \( f = 0 \), we then have the following energy estimate:
\[
\sup_{-T \leq t \leq T} \| \phi(\cdot, t) \|_{L^2(\Omega)}^2 + \iint_{\Omega_T} |\nabla \phi|^{p(\cdot)} \, dz \\
\lesssim \left( \int_{\Omega} \left[ |f|^{p(\cdot)} + 1 \right] \, dz + \| f_0 \|_{L^2(\Omega)}^2 \right).
\]

Returning to our problem (1.1), Proposition 3.4 yields the existence and uniqueness result as follows:

Corollary 3.5 There exists a unique weak solution \( u \in C^0(-T, T; L^2(\Omega)) \cap L^{p(\cdot)}(-T, T; W_1^{1,p(\cdot)}(\Omega)) \) solving (1.1) with the estimate
\[
\sup_{-T \leq t \leq T} \| u(\cdot, t) \|_{L^2(\Omega)}^2 + \iint_{\Omega_T} |\nabla u|^{p(\cdot)} \, dz \leq C_{n,p_0^\pm,\Lambda_0,\Lambda_1} \iint_{\Omega_T} \left[ |f|^{p(\cdot)} + 1 \right] \, dz. \quad (3.3)
\]

4 Main results

We now state the main results of this paper. Let us first set
\[
\vartheta(z) := \frac{1}{-\frac{n}{p(z)} + \frac{nd(z)}{2} + d(z)} \quad \text{and} \quad M := \iint_{\Omega_T} \left[ |f|^{p(z)\max\{1-\beta\eta^{-1},1\}} + 1 \right] \, dz + 1,
\]

where the exponent \( d \) is given in (2.7). From (2.7), it is easy to see that \( \vartheta(z) > 1 \).

The first theorem concerns the local estimate around small balls.
Theorem 4.1 Assume that $u$ is the weak solution of the problem (1.1) under the structure conditions (2.2) and (2.3). Let $0 < S_0 < 1$, and $q(\cdot)$ be log-Hölder continuous satisfying $1 < q^- \leq q(\cdot) \leq q^+ < \infty$. There exist constants $\gamma_0 \in (0, 1/8)$ and $\beta_0 \in (0, 1/4)$, both depending only on $\Lambda_0$, $\Lambda_1$, $p^\pm_{log}$, $q^\pm_{log}$, $n$, such that if $(p(\cdot), A, \Omega)$ is $(\gamma, S_0)$-vanishing for some $\gamma \in (0, \gamma_0)$, then there exists a constant $C_0 = C_0(\Lambda_0, \Lambda_1, p^\pm_{log}, q^\pm_{log}, n, S_0) \geq 1$ such that for any $\zeta \in \Omega_T$, $\beta \in (0, \beta_0)$ and $\rho \in (0, 1/(C_0M^2)]$, we have
\[
\iint_{K_{\rho}(\zeta)} |\nabla u|^{p(\zeta)(1-\beta)q(\zeta)} \, dz 
\leq C \left\{ \iint_{K_{\rho}(\zeta)} |\nabla u|^{p(\zeta)(1-\beta)} \, dz + \left( \iint_{K_{\rho}(\zeta)} |f|^{p(\zeta)(1-\beta)q(\zeta)} \, dz \right)^{\frac{1}{q(\zeta)}} + 1 \right\}^{\frac{1}{1-\beta}},
\]
for some constant $C = C(\Lambda_0, \Lambda_1, p^\pm_{log}, q^\pm_{log}, n) \geq 1$. Here $M$ and $\vartheta(\zeta)$ are given in (4.1).

In the above theorem, it is important to note that the exponent $q^- > 1$, on the other hand, the above estimate has $p(\zeta)(1-\beta)q(\zeta)$ as the exponent. In particular, the term $(1-\beta)$ in the exponent provides sufficient gap in order to prove the end point version of the result as highlighted in the introduction. To do this, we use a standard covering argument followed by uniformizing the exponents which enables us to remove the term $(1-\beta)$. Thus our main theorem now takes the following form:

Theorem 4.2 Let $M^+ > 1$ and let $r(\cdot)$ be log-Hölder continuous satisfying $1 \leq r^- \leq r(\cdot) \leq r^+ < M^+ < \infty$. Then under the assumptions in Theorem 4.1, there is a constant $\gamma_0 \in (0, 1/4)$ depending only on $\Lambda_0$, $\Lambda_1$, $p^\pm_{log}$, $r^\pm_{log}$, $M^+$, $n$, such that if $(p(\cdot), A, \Omega)$ is $(\gamma, S_0)$-vanishing for some $\gamma \in (0, \gamma_0)$, then we have the following global bound holds:
\[
\iint_{\Omega_T} |\nabla u|^{p(\zeta)r(\zeta)} \, dz \leq C \left\{ \left( \iint_{\Omega_T} |f|^{p(\zeta)r(\zeta)} \, dz \right)^{\frac{1}{r(\zeta)}} + 1 \right\},
\]
for some constants $C = C(\Lambda_0, \Lambda_1, p^\pm_{log}, r^\pm_{log}, M^+, n, \Omega_T, S_0) \geq 1$ and $\vartheta = \vartheta(p^\pm_{log}, r^\pm_{log}, M^+, n) \geq 1$.

Remark 4.3 When $p(\cdot) \equiv p$, if we were to make the choice $d = \min \left\{ \frac{2}{p}, 1 \right\}$ in (2.7) (see [3] for more details), then using this in (4.1), we see that
\[
\vartheta(\zeta) \equiv \vartheta = \begin{cases} \frac{p}{2} & \text{if } p \geq 2, \\ \frac{2p}{p(n+2)-2n} & \text{if } \frac{2n}{n+2} < p < 2. \end{cases}
\]
This is the standard scaling deficit coefficient introduced in [6]. In this sense, our scaling is an intermediate scaling between the standard singular and degenerate scalings used in existing literature.

5 Some useful inequalities

In this section, we shall collect and prove in some cases well known estimates that will be used in subsequent sections. We first recall an integral version of Poincaré’s inequality which was proved in [5, Lemma 4.12]:

\[\text{Springer}\]
Lemma 5.1 Let \( s(\cdot) \in \log^\pm \) and let \( M_p \geq 1 \) be given. Define \( R_p : = \min \left\{ \frac{1}{2M_p}, \frac{1}{\omega_n}, \frac{1}{2} \right\} \).

Then for any \( \phi \in W^{1,s(\cdot)}(B_{4r}) \) with \( 4r < R_p \) satisfying

\[
\int_{B_{4r}} |\nabla \phi(x)|^{s(x)} \, dx + 1 \leq M_p,
\]

the following estimate holds:

\[
\int_{B_r} \left( \frac{|\phi - (\phi)_{B_r}|}{\diam(B_r)} \right)^{s(x)} \, dx \lesssim_{(n,s,\log)} \int_{B_r} |\nabla \phi(x)|^{s(x)} \, dx + |B_r|,
\]

where we have used the notation \( (\phi)_{B_r} = \int_{B_r} \phi(y) \, dy \). Since \( \diam(B_r) = 2r \leq R_p < 1 \), we also obtain

\[
\int_{B_r} |\phi - (\phi)_{B_r}|^{s(x)} \, dx \lesssim_{(n,s,\log)} \int_{B_r} |\nabla \phi(x)|^{s(x)} \, dx + |B_r|.
\]

Another Poincaré’s inequality that will be needed is one where the function has a reasonable large zero set:

Theorem 5.2 Let \( s(\cdot) \in \log^\pm \) and let \( M_p \geq 1 \) and \( \varepsilon \in (0,1) \) be given. Define \( R_p : = \min \left\{ \frac{1}{2M_p}, \frac{1}{\omega_n}, \frac{1}{2} \right\} \). For any \( \phi \in W^{1,p(\cdot)}(B_{2r}) \) with \( 2r < R_p \) satisfying

\[
|(N(\phi))| : = |\{ x \in B_r : \phi(x) = 0 \} > \varepsilon |B_r| \quad \text{and} \quad \int_{B_{2r}} |\nabla \phi(x)|^{s(x)} \, dx + 1 \leq M_p,
\]

the following estimate holds:

\[
\int_{B_r} \left( \frac{|\phi|}{\diam(B_r)} \right)^{s(x)} \, dx \lesssim_{(s,\log,n,\varepsilon)} \int_{B_r} |\nabla \phi(x)|^{s(x)} \, dx + |B_r|.
\]

We note that Theorem 5.2 is slightly different than the one proved in [5, Theorem 4.13]. In order to obtain this improvement where the ball \( B_r \) is the same on both sides of the inequality, we can repeat the arguments in the proof of [5, Theorem 4.13] and combine them with the technical lemma from [28, Lemma 3.4].

The next lemma that we need is an estimate in \( L \log L \)-space which can be found in [29] and references therein:

Lemma 5.3 Let \( \beta > 0 \) and let \( s > 1 \). Then for any \( f \in L^\beta(\hat{\Omega}) \), we have

\[
\int_{\hat{\Omega}} |f| \left[ \log \left( e + \frac{|f|}{\langle |f| \rangle_{\hat{\Omega}}} \right) \right]^\beta \, dx \lesssim_{(n,s,\beta)} \left( \int_{\hat{\Omega}} |f|^\beta \, dx \right)^{\frac{1}{\beta}},
\]

where we have used the notation \( \langle |f| \rangle_{\hat{\Omega}} : = \int_{\hat{\Omega}} |f(y)| \, dy \).

We record some useful property as follows:

Lemma 5.4 Let \( \hat{\Omega} \) be an open set in \( \mathbb{R}^N \) and let \( q > s \geq 0 \). For \( g \in L^1(\hat{\Omega}) \), we have

\[
\int_{\hat{\Omega}} |g|^q \, dx = (q-s) \int_0^k a^{q-s-1} \int_{\{y \in \hat{\Omega} : |g(y)| > a\}} |g(x)| \, dx \, da,
\]

where the truncation function \( |g|_k : = \min \{|g|, k\} \) for some constant \( k > 0 \). If \( g \in L^{q-s+1}(U) \), then (5.1) also holds for \( k = \infty \).
Proof By Fubini’s theorem, it is easy to check that Lemma 5.4 holds.

We also use the following technical lemma which was proved in [28, Lemma 4.3]:

Lemma 5.5 Let g be a bounded nonnegative function in \([\tau_0, \tau_1]\) with \(\tau_0 \geq 0\). Suppose that for \(\tau_0 \leq s_1 < s_2 \leq \tau_1\), we have

\[
f(s_1) \leq \theta f(s_2) + \frac{P_1}{(s_2 - s_1)^k} + P_2,
\]

for some \(k, P_1, P_2 \geq 0\) and \(\theta \in [0, 1)\). Then for any \(\tau_0 \leq s_1 < s_2 \leq \tau_1\), there holds

\[
f(s_1) \lesssim_{(k, \theta)} \left\{ \frac{P_1}{(s_2 - s_1)^k} + P_2 \right\}.
\]

5.1 Maximal function

For any \(f \in L^1(\mathbb{R}^{n+1})\), let us now define the strong maximal function in \(\mathbb{R}^{n+1}\) as follows:

\[
\mathcal{M}(f)(x, t) := \sup_{\tilde{Q} \ni (x, t)} \iint_{\tilde{Q}} |f(y, s)| \, dy \, ds,
\]

(5.2)

where the supremum is taken over all parabolic cylinders \(\tilde{Q}_{a,b}\) with \(a, b \in \mathbb{R}^+\) such that \((x, t) \in \tilde{Q}_{a,b}\). An application of the Hardy-Littlewood maximal theorem in \(x-\) and \(t-\) directions shows that the Hardy-Littlewood maximal theorem still holds for this type of maximal function (see [30, Lemma 7.9] for details):

Lemma 5.6 If \(f \in L^{\vartheta}(\mathbb{R}^{n+1})\) for some \(1 < \vartheta \leq \infty\), then there holds

\[
\|\mathcal{M}(f)\|_{L^{\vartheta}(\mathbb{R}^{n+1})} \leq C(n, \vartheta) \|f\|_{L^{\vartheta}(\mathbb{R}^{n+1})}.
\]

6 Approximations

In this section, we describe gradient higher integrability type results and the approximations that will be made.

6.1 Gradient higher integrability estimates

In this subsection, let us collect a few important higher integrability results that will be used throughout the paper. In order to state the general theorems, let \(\phi \in L^{p(\cdot)}(-T, T; W^{1,p(\cdot)}(\Omega))\) be a weak solution of

\[
\begin{aligned}
\phi_t - \text{div} A(x, t, \nabla \phi) &= -\text{div}(|f|^{p(x, t)-2}f) \\
\phi &= 0
\end{aligned}
\quad \text{in } \Omega \times (-T, T),
\]

\[
\phi = 0
\quad \text{on } \partial_p \Omega \times (-T, T),
\]

(6.1)

where the nonlinearity is assumed to satisfy (2.2) and (2.4). Here the domain \(\Omega\) is assumed to satisfy a uniform measure density condition with constant \(m_\varepsilon\) as defined in Definition 2.6. Let us define

\[
M_f := \iint_{\Omega_T} [|f|^{p(\cdot)} + 1] \, dz + 1,
\]

(6.2)
which combined with (3.3) shows

$$M_\phi := \iint_{\Omega_T} |\nabla \phi|^{p(z)} + 1 \, dz + 1 \leq C_{(n,p^\pm_0,\Lambda_0,\Lambda_1)} M_f.$$  

The first result we recall is the higher integrability above the natural exponent. In the interior case, this was proved in [25,31] whereas in the boundary case, using the measure density condition satisfied by $\Omega$, the result was proved in [1, Lemma 3.5]. Combining the intrinsic geometry as considered in this paper with the covering arguments developed in [3, Section 9] applied verbatim to the estimates from [25] gives the following higher integrability above the natural exponent:

**Theorem 6.1** Let $\tilde{\sigma} > 0$ be given, then there exists $\tilde{\beta}_1 = \tilde{\beta}_1(n, \Lambda_0, \Lambda_1, p^\pm_0, \Omega) \in (0, \tilde{\sigma}]$ such that if $f \in L^{p(1+\tilde{\sigma})}(\Omega_T)$ and $\phi \in L^{p(\cdot)}(-T, T; W^{1,p(\cdot)}_0(\Omega))$ is a weak solution to (6.1), then $|\nabla \phi| \in L^{p(1+\tilde{\beta})}(\Omega_T)$ for all $\beta \in (0, \tilde{\beta}_1)$. Moreover, with $M_f$ defined as (6.2), there exists a radius $\tilde{\rho}_1 = \tilde{\rho}_1(n, p^\pm_0, \Lambda_0, \Lambda_1, M_f)$ such that for any $2\rho \in (0, \tilde{\rho}_1]$ and any $\bar{z} \in \bar{\Omega} \times (-T, T)$, there holds

$$\iint_{K_{2\rho}(\bar{z})} |\nabla \phi|^{p(\cdot)(1+\beta)} \, dz \lesssim_{(n,\Lambda_0,\Lambda_1,p^\pm_0,\Omega)} \left( \iint_{K_{2\rho}(\bar{z})} (|\nabla \phi| + |f|)^{p(\cdot)} \, dz \right)^{1+\beta \tilde{\rho}_1}
+ \iint_{K_{2\rho}(\bar{z})} (|f| + 1)^{p(\cdot)} \, dz,$$

where the constant $\tilde{\rho}_1 = \tilde{\rho}_1(p(\bar{z}), n) \geq 1$.

We will also need an improved higher integrability result below the natural exponent. The following theorem was proved for a weaker class of solutions called very weak solutions, but also holds true for weak solutions as considered in this paper. The interior regularity in the singular case, i.e., when $\frac{2n}{n+2} < p^+ \leq 2$, the result was proved in [32] and the interior regularity in the degenerate case, i.e., when $p^- \geq 2$, the result was proved in [33]. Subsequently, using the unified intrinsic scaling, this restriction can be removed and the full result up to the boundary with $\frac{2n}{n+2} < p^- \leq p(\cdot) \leq p^+ < \infty$ was proved in [3] for domains satisfying a uniform measure density condition as in Definition 2.6.

**Theorem 6.2** [[3]] Let $\tilde{\sigma} > 0$ be given and suppose $f \in L^{p(\cdot)(1+\tilde{\sigma})}(\Omega_T)$ and $\phi \in L^{p(\cdot)}(-T, T; W^{1,p(\cdot)}_0(\Omega))$ is a weak solution to (6.1). With $M_f$ defined as (6.2), there exist radius $\tilde{\rho}_2 = \tilde{\rho}_2(n, p^\pm_0, \Lambda_0, \Lambda_1, M_f)$ and $\tilde{\beta}_2 = \tilde{\beta}_2(n, \Lambda_0, \Lambda_1, p^\pm_0) \in (0, \tilde{\sigma}]$ with $\tilde{\beta}_2 \leq \frac{1}{4}$ such that for any $2\rho \in (0, \tilde{\rho}_2], \beta \in (0, \tilde{\beta}_2]$ and any $\bar{z} \in \bar{\Omega} \times (-T, T)$, there holds

$$\iint_{K_{2\rho}(\bar{z})} |\nabla \phi|^{p(\cdot)} \, dz \lesssim_{(n,\Lambda_0,\Lambda_1,p^\pm_0,\Omega)} \left( \iint_{K_{2\rho}(\bar{z})} (|\nabla \phi| + |f|)^{p(\cdot)(1-\beta)} \, dz \right)^{1+\beta \tilde{\rho}_2}
+ \iint_{K_{2\rho}(\bar{z})} (|f| + 1)^{p(\cdot)} \, dz,$$

where the constant $\tilde{\rho}_2 = \tilde{\rho}_2(n, p(\bar{z})) \geq 1$.  

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Remark 6.3 For weak solutions, from the papers \cite{1,25}, the exponent \( \tilde{\vartheta} \) in Theorem 6.1 was explicitly given by

\[
\tilde{\vartheta}_1 := \begin{cases} 
p(1) \over 2 & \text{if } p(3) \geq 2, \\
\frac{2p(3)}{p(3)(n+2)-2n} & \text{if } \frac{2n}{n+2} < p(3) < 2.
\end{cases}
\]

On the other hand, using the unified intrinsic scaling approach and recalculating the estimates from \cite{25}, we can obtain the following unified exponent \( \tilde{\vartheta}_1 = \frac{1}{n} - \frac{1}{p(3)(n+2)-2n} \) which holds in the full range \( \frac{2n}{n+2} < p(3) < \infty \). Here since the cylinders are chosen sufficiently small enough satisfying (R10), the choice of \( d \) is made such that

\[
\frac{2n}{(n+2)p_{loc}^-} < d < \min\{2/p_{loc}^+, 1\},
\]

where \( p_{loc}^- \) and \( p_{loc}^+ \) are infimum and supremum over the local cylinder.

For very weak solutions, in \cite{3}, the exponent \( \tilde{\vartheta}_2 = 2 \alpha K_3 \) is from Theorem 6.2. We further restrict \( \beta \leq \frac{1}{4} \), one can uniformize the exponent \( \tilde{\vartheta}_2 = \tilde{\vartheta}_2(n, p(3)) \) only, i.e., it does not depend on \( \beta \).

For the purposes of this paper, the explicitly computed exponents \( \tilde{\vartheta}_1 \) and \( \tilde{\vartheta}_2 \) will not be needed except for the following two properties: firstly, we need that \( \tilde{\vartheta}_1, \tilde{\vartheta}_2 \geq 1 \) and secondly, \( \tilde{\vartheta}_1 \) and \( \tilde{\vartheta}_2 \) can be made to depend only on \( n \) and \( p(3) \).

Before we end this subsection, let us prove the following important corollary:

Corollary 6.4 Let \( \tilde{z} \in \Omega_T \) be any fixed point, and let \( \alpha \geq 1 \) be given. Suppose that \( \phi \) and \( f \) solve

\[
\begin{align*}
\phi_t - \text{div}A(x, t, \nabla \phi) &= - \text{div}(|f|^p(x, t) - 2f) \quad \text{in } K_{3r}^\alpha(\tilde{z}), \\
\phi &= 0 \quad \text{on } \partial_u K_{3r}^\alpha(\tilde{z}).
\end{align*}
\]

Let \( \beta \leq \min\{\tilde{\vartheta}_1, \tilde{\vartheta}_2\} \) where \( \tilde{\vartheta}_1 \) is from Theorem 6.1 and \( \tilde{\vartheta}_2 \) is from Theorem 6.2. We further restrict \( \beta = \frac{p-1}{p+1} d_0 \) and \( \beta = \frac{p-1}{p+1} d_0 + 1 \), where \( d_0 \) is defined as

\[
d_0 := \inf_{\Omega_T} \left( \frac{d(z)(n+2)}{2} - \frac{n}{p(z)} \right) > 0.
\]

Assume the following are satisfied for some constants \( \tilde{M} \geq 1, c_*, c_p \) and \( \Gamma \):

\[
\iint_{K_{3r}^\alpha(\tilde{z})} |\nabla \phi|^{p(-)(1-\beta)} + |f|^{p(-)(1-\beta)} + 1 \, dz \leq \tilde{M},
\]

\[
\iint_{K_{3r}^\alpha(\tilde{z})} |\nabla \phi|^{p(-)(1-\beta)} \, dz + \left( \iint_{K_{3r}^\alpha(\tilde{z})} |f|^{p(-)(1-\beta)\kappa} \, dz \right)^\frac{1}{\kappa} \leq c_\kappa \alpha^{1-\beta} \quad \text{for some } \kappa \geq 1 + \beta.
\]

Let \( 3r \leq \min\{\tilde{\vartheta}_1, \tilde{\vartheta}_2\} \) where \( \tilde{\vartheta}_1 \) is from Theorem 6.1 and \( \tilde{\vartheta}_2 \) is from Theorem 6.2, furthermore assume the following assumptions hold:

\[
p_{K_{3r}^\alpha(\tilde{z})}^+ - p_{K_{3r}^\alpha(\tilde{z})}^- \leq \omega p(-) (32r) \leq \frac{d_0}{2} \min\{p^-, p^-(p^--1)\} \quad \text{and} \quad \alpha^{p_{K_{3r}^\alpha(\tilde{z})}^+ - p_{K_{3r}^\alpha(\tilde{z})}^-} \leq c_p.
\]
Then for any $0 < \sigma \leq \frac{2\beta}{1-\beta}$, the following estimate holds:

$$
\int_{K^p_1(3)} |\nabla \phi|^{p(\cdot)(1-\beta)(1+\sigma)} \, dz \lesssim_{(c_\phi, c_p, p^-, p(3))} \alpha^{(1-\beta)(1+\sigma)}.
$$

\textbf{Proof} From (6.3), we see that under the change of variables, $x : = \alpha^{-\frac{1}{p(3)\beta} + \frac{d}{d-2}} y$ and $t : = \alpha^{-1 + \frac{d}{d-2}} \tau$, with

$$
\tilde{p}(y, \tau) : = p(x, t), \quad \phi_1(y, \tau) : = \frac{\phi(x, t)}{d}, \quad f_1(y, \tau) : = \alpha^{\frac{1-p(3)}{p(3)(p(\cdot) - 1)}} f(x, t) \quad \text{and}
$$

$$
\tilde{a}(y, \tau, \zeta) : = \alpha^{\frac{1-p(3)}{p(3)}} A(x, t, 1/p(3) \zeta),
$$

the following equation is satisfied:

$$
\left\{ \frac{d\phi_1(y, \tau)}{d\tau} - \text{div}_y \tilde{a}(y, \tau, \nabla_y \phi_1(y, \tau)) = - \text{div}_y ([f_1(y, \tau)]^{\tilde{p}(y, \tau) - 2} f_1(y, \tau)) \right. \quad \phi_1 = 0 \quad \text{in} \quad K_{3r}(3),
$$

$$
\left. \quad \phi_1 = 0 \quad \text{on} \quad \partial_y K_{3r}(3). \right\}
$$

From the assumptions (6.4), (6.7) and (2.7), recalling the restrictions $\beta \frac{p^+}{p^-} \leq d_0$ and $\beta \leq \frac{p^-(p^- - 1) d_0}{p^+(p^+ - 1) 2}$, it is easy to see that the following bounds hold:

$$
- \frac{p(\cdot)(1 - \beta)}{p(3)} + \frac{n}{p(3)} - \frac{d(n + 2)}{2} + 1 \leq \frac{p(3) - p(\cdot)}{p(3)} + \frac{\beta}{p^-} + \frac{n}{p(3)} - \frac{d(n + 2)}{2} \leq 0,
$$

$$
\frac{1 - p(\cdot)(1 - \beta)}{p(3)(p(\cdot) - 1)} + \frac{n}{p(3)} - \frac{d(n + 2)}{2} + 1 \leq \frac{p(\cdot) - p(3)}{p(3)(p(\cdot) - 1)} - \frac{d_0}{2} \leq 0.
$$

From a simple change of variables and using the fact that $\alpha \geq 1$, we see that

$$
\int_{K_{3r}(3)} |\nabla_y \phi_1(y, \tau)|^{\tilde{p}(y, \tau)(1-\beta)} \, dy \, d\tau = \int_{K^p_1(3)} |\nabla_x \phi(x, t)| \phi(x, t)(1-\beta) \, dx \, dt \leq (6.9)
$$

$$
\int_{K^p_1(3)} |\nabla_x \phi(x, t)| \phi(x, t)(1-\beta) \, dx \, dt.
$$

Analogously, we get

$$
\int_{K_{3r}(3)} |f_1(y, \tau)|^{\tilde{p}(y, \tau)(1-\beta)} \, dy \, d\tau = \int_{K^p_1(3)} |f(x, t)| \phi(x, t)(1-\beta) \, dx \, dt \leq (6.10)
$$

$$
\int_{K^p_1(3)} |f(x, t)| \phi(x, t)(1-\beta) \, dx \, dt.
$$

Thus combining (6.11) and (6.12) and using the hypothesis (6.5), we get

$$
\int_{K_{3r}(3)} \left[ |\nabla \phi_1(y, \tau)|^{\tilde{p}(y, \tau)(1-\beta)} + |f_1(y, \tau)|^{\tilde{p}(y, \tau)(1-\beta) + 1} \right] dy \, ds \leq \tilde{M}.
$$
For the sake of simplicity, let us denote $\tilde{p}(y, \tau) = \tilde{p}(z)$ and $p(x, t) = p(z)$. We will now proceed with proving (6.8) as follows:

$$
\int K^\alpha_y(3) |\nabla \phi|^{p(z)(1-\beta)(1+\sigma)} dz \\
= \frac{\beta(z)(1-\beta)(1+\sigma)}{p(z)} |\nabla \phi|^{p(z)(1-\beta)(1+\sigma)} dz \\
\leq \int K^\alpha_y(3) |\nabla \phi|^{p(z)(1-\beta)(1+\beta)(1+\sigma)} dz \\
\leq c_p \int K^\alpha_y(3) |\nabla \phi|^{p(z)(1-\beta)(1+\sigma)} dz.
$$

(6.13)

To obtain (a), we made use of (6.7) and the fact $1 + \sigma \leq 2$.

Let us assume $(1 + \sigma)(1 - \beta) > 1$ and prove the following estimate. We can now apply Theorems 6.1 and 6.2 to obtain the higher integrability from $\tilde{p}(z)(1-\beta)$ to $\tilde{p}(z)(1-\beta)(1+\sigma)$. Thus the expression on the right of (6.13) can be estimated as

$$
\int K^\alpha_y(3) |\nabla \phi|^{p(z)(1-\beta)(1+\sigma)} dz \\
\leq \left[ \int K^\alpha_y(3) |\nabla \phi|^{p(z)(1-\beta)} dz \right]^{1+\beta \delta_2} + \int K^\alpha_y(3) |\nabla \phi|^{p(z)(1-\beta)(1+\sigma)} dz + 1.
$$

(6.14)

If $(1 + \sigma)(1 - \beta) < 1$, then we only need to apply Theorem 6.2 to obtain analogous estimate, the details of which are left to the reader.

In order to prove (6.8), it is sufficient to bound (6.14) by a constant from which the result will follow by using (6.13). In order to do this, we scale back to get

$$
\int K^\alpha_y(3) |\nabla \phi|^{p(z)(1-\beta)} dz = \left[ K^\alpha_y(3) \right]^{1+\beta \delta_2} \int K^\alpha_y(3) |\nabla \phi|^{p(z)(1-\beta)} dz \\
\leq c_p \int K^\alpha_y(3) |\nabla \phi|^{p(z)(1-\beta)(1+\sigma)} dz.
$$

(6.15)

To obtain (c), we used the fact that $\left[ K^\alpha_y(3) \right] = \alpha^{-n/p} + \frac{n}{p} \alpha^{-1+d}$ and to obtain (d), we made use of (6.6) and (6.7) along with the trivial bound $1 - \beta \leq 1$.

To estimate the terms containing $f_1$ in (6.14), let us denote $\sigma$ to be either $(1 - \beta)$, $1$ or $(1 - \beta)(1+\sigma)$ and estimate $\int K^\alpha_y(3) |f_1|^{p(z)\sigma} dz$ as follows:

$$
\int K^\alpha_y(3) |f_1|^{p(z)\sigma} dz = \int K^\alpha_y(3) \int K^\alpha_y(3) \int K^\alpha_y(3) |f_1|^{(1-p(z))p(z)(1-\beta)(1+\sigma)} dz \\
\leq c_p \int K^\alpha_y(3) |f_1|^{p(z)\sigma} dz.
$$

(6.16)

To obtain (e), we performed the usual change of variables, to obtain (f), we used the fact that $\left[ K^\alpha_y(3) \right] = \alpha^{-n/p} + \frac{n}{p} \alpha^{-1+d}$, to obtain (g), we used the fact that $\kappa(1 - \beta) \geq \sigma$ from (6.6) and finally to obtain (h), we made use of (6.6) and (6.7) along with the bound $\sigma \leq 2$. 

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Thus combining (6.15) and (6.16) into (6.14) and finally substituting the resulting expression into (6.13), we see that there holds
\[
\iint_{K^p_\varphi(\xi)} |\nabla \phi|^{p(\cdot)(1-\beta)(1+\sigma)} \, dz \leq C_{(c_\epsilon, c_p, p^- (\varphi))} \alpha^{(1-\beta)(1+\sigma)},
\]
which completes the proof.

### 6.2 Approximations

In this subsection, let $\alpha \geq 1$ be a given constant, let $\rho$ be as in Remark 2.10, and let $\xi = (x, t) \in \Omega_T$ be any fixed point. Also note that the existence of all the solutions considered below follows from Proposition 3.4.

First, let us consider the unique weak solution $w \in C^0 \left( I^{\alpha}_{4\rho}(t); L^2(\Omega^\alpha_{4\rho}(x)) \right) \cap L^p(1) \left( I^{\alpha}_{4\rho}(t); W^{1,p}(\Omega^\alpha_{4\rho}(x)) \right)$ solving
\[
\begin{cases}
    w_t - \text{div} A(x, t, \nabla w) = 0 & \text{in } K^\alpha_{4\rho}(t), \\
    w = u & \text{on } \partial_p K^\alpha_{4\rho}(t). 
\end{cases}
\tag{6.17}
\]

This is possible, since (1.1) shows $u \in L^p(1) \left( I^{\alpha}_{4\rho}(t); W^{1,p}(\Omega^\alpha_{4\rho}(x)) \right)$ and $\frac{du}{dt} \in L^p(1) \left( I^{\alpha}_{4\rho}(t); W^{1,p}(\Omega^\alpha_{4\rho}(x)) \right)$.

We can now compare the solutions of (1.1) and (6.17) to get the following lemma:

**Lemma 6.5** For any $\rho > 0$ and any weak solution $w$ to (6.17), the following estimate holds:
\[
\iint_{K^\alpha_{4\rho}(\xi)} |\nabla w - \nabla u|^{p(z)} \, dz \lesssim (n, p_0^z, \Lambda_0, \Lambda_1) \left( \iint_{K^\alpha_{4\rho}(\xi)} |\nabla u|^{p(z)} + |f|^{p(z)} + 1 \, dz \right),
\tag{6.18}
\]
\[
\iint_{K^\alpha_{4\rho}(\xi)} |\nabla w|^{p(z)} \, dz \lesssim (n, p_0^z, \Lambda_0, \Lambda_1) \left( \iint_{K^\alpha_{4\rho}(\xi)} |\nabla u|^{p(z)} + |f|^{p(z)} + 1 \, dz \right).
\tag{6.19}
\]

The proof of Lemma 6.5 follows by taking $u - w$ as a test function in (1.1) and (6.17) (see for example [1, (4.11)] for the proof of (6.18)). A simple application of triangle inequality to (6.18) implies (6.19).

**Lemma 6.6** Let $2\rho \leq \rho_0$ with $\rho_0$ as in Remark 2.10, then any weak solution $w \in L^p(1) \left( I^{\alpha}_{4\rho}(t); W^{1,p}(\Omega^\alpha_{4\rho}(x)) \right)$ has the improved regularity $\nabla w \in L^p(1) \left( K^\alpha_{3\rho}(\xi) \right)$.

**Proof** Since $\rho$ satisfies Remark 2.10, we can apply Theorem 6.1 to (6.17) which implies $\nabla w \in L^p(1+\beta) K^\alpha_{3\rho}(\xi)$ for any $\beta \in (0, \rho_0)$ with $\rho_0$ as in Remark 2.11. As a consequence, we have the following sequence of estimates
\[
\iint_{K^\alpha_{3\rho}(\xi)} |\nabla w|^{p(1)} \, dz = \iint_{K^\alpha_{3\rho}(\xi)} |\nabla w|^{p(1) + \rho_0} \frac{p(z)}{p(z) + \rho_0} \, dz \\
\lesssim \iint_{K^\alpha_{3\rho}(\xi)} (|\nabla w| + 1)^{p(z) + \rho_0} \, dz \\
\lesssim \iint_{K^\alpha_{3\rho}(\xi)} (|\nabla w| + 1)^{p(z) + \rho_0} \, dz \\
\lesssim \left( \iint_{K^\alpha_{3\rho}(\xi)} (|\nabla w| + 1)^{p(z)} \, dz \right)^{1+\rho_0 \delta_0}. 
\]
To obtain (a), we made use of (R5) which implies \( \frac{p_{K_{3p}(z)}^+}{p_{K_{3p}(z)}^-} (1 + \beta_0) \leq 1 \) and to obtain (b), we made use of Theorem 6.1 along (B3).

We will also need the following regularity with respect to the time derivative of the weak solution \( w \) to (6.17) which will enable us to use \( w \) as boundary data so that Proposition 3.4 can be applied.

**Lemma 6.7** We have \( \frac{dw}{dt} \in L^{p(3)} \left( I_{3p}(t) ; W^{1,p(3)}(\Omega_{3\rho}(x))^* \right) \) where the exponent * is used to denote the dual space.

**Proof** In order to prove the lemma, from (6.17), we see that it is sufficient to show \( \mathcal{A}(x, t, \nabla w) \in L^{p(3)-1} (K_{3\rho}^\alpha(3)) \). We show this as follows:

\[
\int_{K_{3\rho}(3)} |A(x, t, \nabla w)|^{p(3)-1} d\zeta \lesssim \int_{K_{3\rho}(3)} (|\nabla w| + 1)^{p(\cdot) - 1} \frac{p(3)}{p(3) - 1} d\zeta \\
\leq p(\cdot) \left( 1 + \frac{p_{K_{3\rho}(3)}^+ - p_{K_{3\rho}(3)}^-}{p_{K_{3\rho}(3)}^+ - 1} \right),
\]

To obtain (a), we used the following sequence of estimates on \( K_{3\rho}^\alpha(3) \):

\[
(p(\cdot) - 1) \frac{p(3)}{p(3) - 1} \leq (p_{K_{3\rho}(3)}^+ - 1) \frac{p(3)}{p(3) - 1} \leq (p_{K_{3\rho}(3)}^+ - 1) \frac{p_{K_{3\rho}(3)}^-}{p_{K_{3\rho}(3)}^- - 1} \leq p(\cdot) \left( 1 + \frac{p_{K_{3\rho}(3)}^+ - p_{K_{3\rho}(3)}^-}{p_{K_{3\rho}(3)}^- - 1} \right).
\]

Using Remark 2.4 with the observation \( \alpha \geq 1 \) which implies \( K_{3\rho}^\alpha(3) \subset K_{3\rho}(3) \), we see that

\[
\frac{p_{K_{3\rho}(3)}^+ - p_{K_{3\rho}(3)}^-}{p_{K_{3\rho}(3)}^- - 1} \leq \frac{\omega p(\cdot)(6\rho)}{p_{K_{3\rho}(3)}^- - 1} \leq \frac{\omega p(\cdot)(6\rho)}{p^- - 1} \leq \beta_0.
\]

Substituting (6.21) into (6.20) and making use of Theorem 6.1 (where \( \tilde{\beta}_1 \) is obtained), we get

\[
\int_{K_{3\rho}^\alpha(3)} |A(x, t, \nabla w)|^{p(3)-1} d\zeta \lesssim \int_{K_{3\rho}(3)} (|\nabla w| + 1)^{p(\cdot) - 1} \frac{p(3)}{p(3) - 1} d\zeta \\
\leq (b) \left| K_{3\rho}^\alpha(3) \right| \left( \frac{\int_{K_{3\rho}(3)} (1 + |\nabla w|)^{p(\cdot)(1-\beta_0)} d\zeta}{(1+\beta_0)(1+\beta_0) \vartheta_0} \right)^{(1+\beta_0)(1+\beta_0) \vartheta_0} \\
\leq (c) \left| K_{3\rho}^\alpha(3) \right| \left( \frac{\int_{K_{3\rho}(3)} (1 + |\nabla w|)^{p(\cdot)(1-\beta_0)} d\zeta}{(1+\beta_0) \vartheta_0} \right)^{(1+\beta_0) \vartheta_0}
\]

To obtain (b), we have used Theorems 6.1 and 6.2 along with (B3) and to obtain (c), we have used the fact that \( \beta_0 < 1 \) and \( \vartheta_0 \geq 1 \). This completes the proof of the lemma.

Let us now construct an averaged operator which will be needed. For any \( \alpha \geq 1 \) and any \( 4\rho \leq \rho_0 \), let us define the following vector valued function \( B : K_{3\rho}^\alpha(3) \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) by

\[
B(z, \xi) : = A(z, \xi) \left( \mu^2 + |\xi|^2 \right)^{\frac{p(3)-p(\cdot)}{2}}.
\]
From direct computations (see [1, (4.18)]), we see that the following bounds are satisfied:

\[
(\mu^2 + |\xi|^2) \frac{1}{2} |D_\xi \mathcal{B}(z, \xi)| + |\mathcal{B}(z, \xi)| \leq 3 \Lambda_1 (\mu^2 + |\xi|^2) \frac{\rho(z, \xi)}{2},
\]

\[
(\mu^2 + |\xi|^2) \frac{\rho(z, \xi)}{2} \leq (D \xi \mathcal{B}(z, \xi) \eta, \eta).
\]

(6.23)

In particular, the operator \( \mathcal{B} \) which admits a unique weak solution \((\mathcal{B}, \xi) = (\xi(\mathcal{B}, \xi), \xi(\mathcal{B}, \xi))\)

From Lemma 6.6 and Lemma 6.7, we can now define the following approximation:

\[
\mathcal{B}(t, \xi) = \int_B \frac{1}{\rho(z, \xi)} \mathcal{B}(y, t, \xi) dy.
\]

From (2.6), we see that

\[
\iint_{K_{3\rho}^2(\xi)} \sup_{z \in \mathbb{R}^n} \left| \mathcal{B}(t, \xi) - \mathcal{B}(z, \xi) \right| dz \\
\leq \iint_{Q_{3\rho}^2(\xi)} \Theta(A, B_{3\rho}^2(t))(z) dz \leq \gamma.
\]

In the above estimate, we have used the fact \( \alpha \geq 1 \) which implies \( \alpha^{-1} + \epsilon \leq 1 \).

Boundary case: Subsequently, in this case, we make use of the \((\gamma, \mathcal{S}_0)\)-Reifenberg flat condition, i.e., when \( K_{3\rho}^2(\xi) = B_{3\rho}^2(t) \cap \Omega \times I_{3\rho}^2(t) \) and

\[
B_{3\rho}^2(t) \subset \Omega \times I_{3\rho}^2(t) \subset B_{\rho}^2 \cap \{ x_n > -3\alpha^{-1} + \epsilon \gamma \rho \},
\]

we define another averaged operator \( \mathcal{B} : (t - \alpha^{-1} + \epsilon \gamma \rho, t + \alpha^{-1} + \epsilon \gamma \rho) \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) by

\[
\mathcal{B}(t, \xi) = \int_B \frac{1}{\rho(z, \xi)} \mathcal{B}(y, t, \xi) dy.
\]

From (2.6), we see that

\[
\iint_{Q_{3\rho}^2(\xi)} \sup_{z \in \mathbb{R}^n} \left| \mathcal{B}(t, \xi) - \mathcal{B}(z, \xi) \right| dz \\
\leq \iint_{Q_{3\rho}^2(\xi)} \Theta(A, B_{3\rho}^2(t))(z) dz \leq 4 \iint_{Q_{3\rho}^2(\xi)} \Theta(A, B_{3\rho}^2(t))(z) dz \leq 4 \gamma.
\]

In the above estimate, we have used the fact \( \alpha \geq 1 \) which implies \( \alpha^{-1} + \epsilon \gamma \rho \leq 1 \).

From Lemma 6.6 and Lemma 6.7, we can now define the following approximation:

\[
\begin{cases}
    v_t - \text{div} \mathcal{B}(t, \nabla v) = 0 & \text{in } K_{3\rho}^2(\xi), \\
    v = w & \text{on } \partial_{l} K_{3\rho}^2(\xi),
\end{cases}
\]

which admits a unique weak solution \( v \in C^0(I_{3\rho}^2(t); L^2(\Omega_{3\rho}^2(x))) \cap L^{p(\xi)}(I_{3\rho}^2(t); W^{1,p(\xi)}(\Omega_{3\rho}^2(x))) \) since Proposition 3.4 is applicable.
In the interior case, it is well known that the weak solution \( v \) has locally Lipschitz bounds (see [24] for details). On the other hand, in the boundary case, we need to make one further approximation in which we consider a weak solution \( \nabla \in C^0 \left( I^\alpha_{2p} (t); L^2 (\Omega^\alpha_{2p} (\gamma)) \right) \cap L^{p(3)} \left( I^\alpha_{2p} (t); W^{1,p(3)} (\Omega^\alpha_{2p} (\gamma)) \right) \) solving

\[
\begin{align*}
\nabla_t - \text{div} \mathcal{B}(t, \nabla) &= 0 \quad \text{in} \ Q^\alpha_{2p}, \\
\nabla &= 0 \quad \text{on} \ \partial w Q^\alpha_{2p}.
\end{align*}
\]  

(6.25)

Following the calculations from [1], we have the following important lemma:

**Lemma 6.8** For any \( \epsilon \in (0, 1) \), there exists \( \gamma = \gamma(n, \Lambda_0, \Lambda_1, p_{\log}^{\pm}, \epsilon) > 0 \) such that if \( v \) is the weak solution of (6.24) and the problem is \( (\gamma, S_0) \) vanishing, then there is a weak solution \( \nabla \in C^0 \left( I^\alpha_{2p} (t); L^2 (\Omega^\alpha_{2p} (\gamma)) \right) \cap L^{p(3)} \left( I^\alpha_{2p} (t); W^{1,p(3)} (\Omega^\alpha_{2p} (\gamma)) \right) \) solving (6.25) with

\[
\iint_{Q_{2p}^\alpha (\gamma)} |\nabla \nabla|^p dz \leq \overline{c} \alpha,
\]

such that the following holds:

\[
\iint_{Q_{p}^\alpha (\gamma)} |\nabla v - \nabla \nabla|^p dz \leq \epsilon \alpha.
\]

Moreover, we have the following qualitative Lipschitz bounds, see [1, Lemma 3.7] for the details:

**Lemma 6.9** For \( \nabla \) a weak solution of (6.25) with

\[
\iint_{Q_{2p}^\alpha (\gamma)} |\nabla \nabla|^p dz \leq \overline{C} \alpha,
\]

there holds \( |\nabla \nabla| \in L^\infty_{\text{loc}} \) satisfying

\[
\sup_{Q_{p}^\alpha (\gamma)} |\nabla \nabla|^p \leq C \alpha.
\]

### 6.3 Fixing the size of solutions

Let us define

\[
M_0 := \iint_{\Omega^\gamma} \left( |f|^p + 1 \right) dz + 1.
\]  

(6.26)

From (3.3), we see that

\[
M_w \leq C(n, p_{\log}^{\pm}, \Lambda_0, \Lambda_1) M_0 \quad \text{where we have set} \quad M_w := \iint_{\Omega^\gamma} \left( |\nabla u|^p + 1 \right) dz + 1.
\]  

(6.27)

From (6.19) (which holds for any \( \rho > 0 \)), we see that there holds

\[
M_w \leq C(n, p_{\log}^{\pm}, \Lambda_0, \Lambda_1) M_0 \quad \text{where we have set} \quad M_w := \iint_{K^\alpha_{2p} (\gamma)} \left( |\nabla w|^p + 1 \right) dz + 1.
\]  

(6.28)
7 First difference estimate below the natural exponent

In this section, we will prove a difference estimate between the weak solution of (1.1) and the weak solution of (6.17). To do this, we will use the method of Lipschitz truncation developed by [2] which is modified for use in the current setting in Appendix A.

**Theorem 7.1** Let $\alpha \geq 1$ be fixed, then there exists $\tilde{\rho}_3 = \tilde{\rho}_3(n, p_{\log}^\pm, \Lambda_0, \Lambda_1, M_0)$ such that for any $128 \rho \leq \tilde{\rho}_3$ and for any $\varepsilon \in (0, 1]$, there exists $\beta_3 = \beta_3(n, \Lambda_0, \Lambda_1, p_{\log}^\pm)$ such that for any $\beta \in (0, \beta_3]$, there holds the estimate

\[
\begin{align*}
\iint_{K_{a_0}^\rho(t)} |\nabla u - \nabla w|^{p_c(1-\beta)} \, dz & \leq \varepsilon \iint_{K_{a_0}^\rho(t)} |\nabla u|^{p_c(1-\beta)} \, dz + C_{(n, \Lambda_0, \Lambda_1, p_{\log}^\pm)}
\end{align*}
\]

\[
\begin{align*}
\iint_{K_{a_0}^\rho(t)} \left[ |f|^{p_c(1-\beta)} + 1 \right] \, dz.
\end{align*}
\]

Here $u$ is the weak solution of (1.1) and $w$ is the weak solution to (6.17).

**Proof** As needed in (A.1), let us denote

\[ s := \alpha^{-1+d(4\rho)}^2, \]

and we consider the following cut-off function $\zeta_\varepsilon \in C^\infty(\mathbb{R})$ such that $0 \leq \zeta_\varepsilon(t) \leq 1$ and

\[ \zeta_\varepsilon(t) = \begin{cases} 1 & \text{for } t \in (t-s+\varepsilon, t+s-\varepsilon), \\ 0 & \text{for } t \in (-\infty, t-s) \cup (t+s, \infty). \end{cases} \]

It is easy to see that

\[ \zeta_\varepsilon'(t) = \begin{cases} 0 & \text{for } t \in (-\infty, t-s) \cup (t-s+\varepsilon, t+s-\varepsilon) \cup (t+s, \infty), \\ \leq \frac{\zeta_\varepsilon(t)}{\varepsilon} & \text{for } t \in (t-s, t-s+\varepsilon) \cup (t-s+\varepsilon, t+s). \end{cases} \]

Without loss of generality, we shall always take $2h \leq \varepsilon$ since we will take limits in the following order $\lim_{\varepsilon \to 0} \lim_{h \to 0}$.

We shall use $v_{\lambda, h}(\cdot)\zeta_\varepsilon(t)$ as a test function in (1.1) and (6.17) where $v_{\lambda, h}$ is as constructed in Appendix A (more specifically in (A.5)). Thus we get

\[ L_1 + L_2 := \iint_{K_{a_0}^\rho(t)} \frac{d[u-w]_h}{dt} v_{\lambda, h} \zeta_\varepsilon \, dx \, dt 
\]

\[ + \iint_{K_{a_0}^\rho(t)} \langle A(x, t, \nabla u) - A(x, t, \nabla w), \nabla v_{\lambda, h} \rangle \zeta_\varepsilon \, dx \, dt 
\]

\[ = \iint_{K_{a_0}^\rho(t)} \langle |f|^{p_c(1-2\|f\|_{\log}^\pm)} + 1 \rangle \zeta_\varepsilon \, dx \, dt =: L_3. \]

Estimate for $L_1$: Setting $E_{\lambda} = \{(x, t) \in E_\lambda : t = \tau\}$ where $E_\lambda$ is as defined in (A.3), we get

\[ L_1 = \int_{t-s}^{t+s} \int_{\Omega_{d_0}^\rho(\tau)} \frac{d}{dx}(v_{\lambda, h} - w_{\lambda, h}) \zeta_\varepsilon(s) \, dy \, d\tau 
\]

\[ + \int_{t-s}^{t+s} \int_{\Omega_{d_0}^\rho(\tau)} \frac{d}{d\tau} \left( \frac{(v_{\lambda, h} - w_{\lambda, h})^2}{\tau} \right) \zeta_\varepsilon(\tau) \, dy \, d\tau 
\]

\[ - \int_{t-s}^{t+s} \int_{\Omega_{d_0}^\rho(\tau)} \frac{d}{ds} \left( v_{\lambda, h}^2 - (v_{\lambda, h} - w_{\lambda, h})^2 \right) \zeta_\varepsilon(s) \, dy \, d\tau 
\]

\[ := J_2 + J_1(t+s) - J_1(t-s) - J_3, \]
where we have set

\[ J_1(\tau) := \frac{1}{2} \int_{\omega_{4p}(\xi)} ((v_h)^2 - (v_{\lambda,h} - v_h)^2)(y, \tau) \xi \tau d\mathcal{M}(\xi). \]

Note that \( J_1(t - s) = J_1(t + s) = 0 \) since \( \xi = (t - s) = \xi(t + s) = 0 \).

Applying the bound from Lemma A.17, we have

\[
|J_2| \lesssim \int_{K_{4p}(\xi)} E_{\lambda} \left| \frac{d\nu_{\lambda,h}}{ds}(v_{\lambda,h} - v_h) \right| dy \ d\tau \lesssim \lambda |\mathbb{R}^{n+1} \setminus E_{\lambda}|.
\]

Estimate for \( L_2 \): We split \( L_2 \) and make use of the fact that \( v_{\lambda,h}(z) = v_h(z) \) for all \( z \in E_{\lambda} \cap K_{4p}(\xi) \).

\[
L_2 = \int_{K_{4p}(\xi)} [A(x, t, \nabla u) - A(x, t, \nabla w)]_{h} \nabla v_{\lambda,h} \xi \ dz + \int_{K_{4p}(\xi)} [A(x, t, \nabla u) - A(x, t, \nabla w)]_{h} \nabla v_{\lambda,h} \xi \ dz
\]

\[
= \int_{K_{4p}(\xi)} [A(x, t, \nabla u) - A(x, t, \nabla w)]_{h} \nabla [u - w]_{h} \xi \ dz + \int_{K_{4p}(\xi)} [A(x, t, \nabla u) - A(x, t, \nabla w)]_{h} \xi \ dz
\]

\[
= : L_2^1 + L_2^2.
\]

Estimate for \( L_2^1 \): Using (2.4), we get

\[
L_2^1 = \int_{K_{4p}(\xi)} [A(x, t, \nabla u) - A(x, t, \nabla w)]_{h} \nabla [u - w]_{h} \xi \ dz
\]

\[
\gtrsim \int_{K_{4p}(\xi)} |\nabla [u - w]_{h}|^2 (\mu^2 + |\nabla [u]_{h}|^2 + |\nabla [w]_{h}|^2)^{p-1 \over 2} \xi \ dz.
\]

Estimate for \( L_2^2 \): Using the bound from Lemma A.11, (2.2), we get

\[
L_2^2 \lesssim \int_{K_{4p}(\xi)} |[A(x, t, \nabla u) - A(x, t, \nabla w)]|_{h} |\nabla v_{\lambda,h}|| \ dz
\]

\[
\lesssim \sum_{i \in H} \lambda^{1 \over p_{2Q_i}} \int_{2Q_i} \left[ (\mu^2 + |\nabla u|^2 + |\nabla w|^2)^{p-1 \over 2} \right]_{h} dz \lesssim \sum_{i \in H} \lambda^{1 \over p_{2Q_i}} \frac{1}{p_{2Q_i}} |\mathcal{Q}_i| \lesssim \lambda |\mathbb{R}^{n+1} \setminus E_{\lambda}|.
\]

In the last inequality, we made use of \( \lambda^{1 \over p_{2Q_i}} + \frac{p_{2Q_i}}{p_{2Q_i}} - 1 \leq C(p_{\log}^{\pm}, n) \).

Estimate for \( L_3 \): Analogously to estimate \( L_2 \), we split \( L_3 \) as follows:

\[
L_3 = \int_{K_{4p}(\xi)} [A(x, t, \nabla u) - A(x, t, \nabla w)]_{h} \nabla v_{\lambda,h} \xi \ dz + \int_{K_{4p}(\xi)} [A(x, t, \nabla u) - A(x, t, \nabla w)]_{h} \nabla v_{\lambda,h} \xi \ dz
\]

\[
= \int_{K_{4p}(\xi)} [A(x, t, \nabla u) - A(x, t, \nabla w)]_{h} \nabla [u - w]_{h} \xi \ dz + \int_{K_{4p}(\xi)} [A(x, t, \nabla u) - A(x, t, \nabla w)]_{h} \xi \ dz
\]

\[
=: L_3^1 + L_3^2.
\]

Estimate for \( L_3^1 \): Using the fact that \( v_{\lambda,h}(z) = v_h(z) \) for all \( z \in E_{\lambda} \cap K_{4p}(\xi) \), we get

\[
L_3^1 = \int_{K_{4p}(\xi)} [A(x, t, \nabla u) - A(x, t, \nabla w)]_{h} \nabla [u - w]_{h} \xi \ dz
\]

\[
\lesssim \int_{K_{4p}(\xi)} [A(x, t, \nabla u) - A(x, t, \nabla w)]_{h} |\nabla [u - w]_{h}| \ dz.
\]

Estimate for \( L_3^2 \): Similar to the bound in (7.1), we get

\[
L_3^2 \lesssim \lambda |\mathbb{R}^{n+1} \setminus E_{\lambda}|.
\]
Combining all the above estimates, we get

\[-\int_{t-s}^{t+s} \int_{\Omega_{4p}^ε(\tau)} \frac{d\xi_ε}{ds} \left( v_h^2 - (v_{\lambda,h} - v_h)^2 \right) dy d\tau + \int_{K_{4p}^ε(\lambda) \cap E_\lambda} |\nabla[u - w]|^2 (\mu^2 + |\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}} \zeta_ε \, dz \\lesssim \int_{K_{4p}^ε(\lambda) \cap E_\lambda} \{ f \}^{p-1} |\nabla[u - w]| \, dz + \lambda |\mathbb{R}^{n+1} \setminus E_\lambda|.
\]

In order to estimate \(-\int_{t-s}^{t+s} \int_{\Omega_{4p}^ε(\tau)} \frac{d\xi_ε}{ds} \left( v_h^2 - (v_{\lambda,h} - v_h)^2 \right) dy d\tau\), we observe that on \(E_\lambda\), there holds \(v_{\lambda} = v\). Taking limits first in \(h \searrow 0\) followed by \(\epsilon \searrow 0\), we get

\[-\int_{t-s}^{t+s} \int_{\Omega_{4p}^ε(\tau)} \frac{d\xi_ε}{ds} \left( v_h^2 - (v_{\lambda,h} - v_h)^2 \right) dy d\tau \xrightarrow{\lim_{\epsilon \searrow 0 \ \lim h \searrow 0}} \int_{\Omega_{4p}^ε(\tau)} (v^2 - (v_{\lambda} - v)^2)(x, t + s) \, dx - \int_{\Omega_{4p}^ε(\tau)} (v^2 - (v_{\lambda} - v)^2)(x, t - s) \, dx.
\]

For the second term, we observe that on \(E_\lambda\), we have \(v_{\lambda} = v\); and on \(E_\lambda^c\), we have \(v_{\lambda} \cdot (t - s) = v(t - s) = 0\). Thus, the second term vanishes because on \(E_\lambda\), we can use the initial boundary condition; and on \(E_\lambda^c\), it is zero by construction. Thus we get

\[-\int_{t-s}^{t+s} \int_{\Omega_{4p}^ε(\tau)} \frac{d\xi_ε}{ds} \left( v_h^2 - (v_{\lambda,h} - v_h)^2 \right) dy d\tau \xrightarrow{\lim_{\epsilon \searrow 0 \ \lim h \searrow 0}} \int_{\Omega_{4p}^ε(\tau)} (v^2 - (v_{\lambda} - v)^2)(x, t + s) \, dx.
\]

In fact, if we consider a cut-off function \(\zeta_{\epsilon_0}(\tau)\) for some \(t_0 \in (t - s, t + s)\), where

\[\zeta_{\epsilon_0}(\tau) = \begin{cases} 1 & \text{for } \tau \in (-t_0 + \epsilon, t_0 - \epsilon), \\ 0 & \text{for } \tau \in (-\infty, -t_0) \cup (t_0, \infty), \end{cases}\]

we would have obtained the following estimate after taking limits:

\[\int_{\Omega_{4p}^ε(\tau)} (v^2 - (v_{\lambda} - v)^2)(x, t_0) \, dx + \int_{-t_0}^{t_0} \int_{\Omega_{4p}^ε(\tau) \cap E_\lambda} |\nabla(u - w)|^2 (\mu^2 + |\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}} \, dx \, dt \approx \int_{K_{4p}^ε(\lambda) \cap E_\lambda} \{ f \}^{p-1} |\nabla(u - w)| \, dz + \lambda |\mathbb{R}^{n+1} \setminus E_\lambda|.
\]

In particular, we get for any \(t_0 \in (t - s, t + s)\)

\[\int_{\Omega_{4p}^ε(\tau)} (v^2 - (v_{\lambda} - v)^2)(x, t_0) \, dx + \int_{K_{4p}^ε(\lambda) \cap E_\lambda} |\nabla(u - w)|^2 (\mu^2 + |\nabla u|^2 + |\nabla w|^2)^{\frac{p-2}{2}} \, dx \, dt \approx \int_{K_{4p}^ε(\lambda) \cap E_\lambda} |\nabla(u - w)| \, dz + \lambda |\mathbb{R}^{n+1} \setminus E_\lambda|.
\]
Using Lemma A.22, for any \( t \in (t - s, t + s) \), there holds
\[
\int_{\Omega_{2\rho}(y)} |(v)^2 - (v_h - v)^2(y, t)| \, dy \gtrsim -\lambda |\mathbb{R}^{n+1} \setminus E_\lambda|.
\]

Furthermore, using the above estimate in (7.2) gives
\[
\iint_{K_{4\rho}(3)\cap E_\lambda} |\nabla (u - w)|^2 \left( \mu^2 + |\nabla u|^2 + |\nabla w|^2 \right)^{\frac{p(\cdot) - 2}{2}} \, dx \, dt
\lesssim \iint_{K_{4\rho}(3)\cap E_\lambda} |f|^{p(\cdot)} |\nabla (u - w)| \, dz + \lambda |\mathbb{R}^{n+1} \setminus E_\lambda|.
\]

(7.3)

Let us now multiply (7.3) with \( \lambda^{-1 - \beta} \) and integrate over \((1, \infty)\) to get
\[
K_1 + K_2 \lesssim K_3,
\]
where we have set
\[
K_1 := \int_1^\infty \lambda^{-1 - \beta} \iint_{K_{4\rho}(3)\cap E_\lambda} |\nabla (u - w)|^2 \left( \mu^2 + |\nabla u|^2 + |\nabla w|^2 \right)^{\frac{p(\cdot) - 2}{2}} \, dz \, d\lambda,
\]
\[
K_2 := \int_1^\infty \lambda^{-1 - \beta} \iint_{K_{4\rho}(3)\cap E_\lambda} |f|^{p(\cdot)} |\nabla (u - w)| \, dz \, d\lambda,
\]
\[
K_3 := \int_1^\infty \lambda^{-1 - \beta} \lambda |\mathbb{R}^{n+1} \setminus E_\lambda| \, d\lambda.
\]

Let us define \( \bar{g} = \max\{g^{-1 - \beta}, 1\} \) where \( g \) is from (A.2), then we estimate each of the above terms as follows:

Estimate for \( K_1 \): Applying Fubini’s theorem, we get
\[
K_1 \gtrsim \frac{1}{\beta} \iint_{K_{4\rho}(3)} \bar{g}(z)^{-\beta} |\nabla (u - w)|^2 \left( \mu^2 + |\nabla u|^2 + |\nabla w|^2 \right)^{\frac{p(\cdot) - 2}{2}} \, dz.
\]

Let us define
\[
K_+ (3) := \{z \in K_{4\rho}(3) : p(z) \geq 2\} \quad \text{and} \quad K_- (3) := \{z \in K_{4\rho}(3) : p(z) \leq 2\},
\]
and consider the following two subcases:

Subcase \( K_- (3) \): We have the following simple decomposition:
\[
|\nabla u - \nabla w|^{p(z)(1 - \beta)} = \left( \left( \mu^2 + |\nabla u|^2 + |\nabla w|^2 \right)^{\frac{p(\cdot) - 2}{2}} |\nabla u - \nabla w|^2 \right)^{\frac{p(z)(1 - \beta)}{2}} \times
\frac{\left( \left( \mu^2 + |\nabla u|^2 + |\nabla w|^2 \right)^{\frac{p(\cdot)(1 - \beta)(2 - p(z))}{4}} \times \bar{g}(z)^{\frac{p(z)(1 - \beta)}{2}} \right)}{2 - p(z)}.
\]

Integrating (7.4) over \( K_- (3) \) and making use of Young’s inequality with exponents \( \frac{2}{p(z)(1 - \beta)} \), \( \frac{2}{2 - p(z)} \) and \( \frac{2}{p(z)\beta} \), we get
\[
\iint_{K_- (3)} |\nabla u - \nabla w|^{p(z)(1 - \beta)} \, dx
\lesssim \epsilon_1 \iint_{K_- (3)} \left( \mu^2 + |\nabla u|^2 + |\nabla w|^2 \right)^{\frac{p(\cdot)(1 - \beta)}{2}} \, dz + \epsilon_2 \iint_{K_- (3)} \bar{g}(z)^{1 - \beta} \, dz
\]
\[
+ C(\epsilon_1, \epsilon_2) \iint_{K_- (3)} \left( \mu^2 + |\nabla u|^2 + |\nabla w|^2 \right)^{\frac{p(\cdot) - 2}{2}} |\nabla u - \nabla w|^2 \bar{g}(z)^{-\beta} \, dz.
\]

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From the strong maximal function bound of Lemma 5.6, we see that
\[
\int K^{-(\beta)}_z \tilde{g}(z)^{1-\beta} \, dz \lesssim \int_{\mathbb{R}^{n+1}} g^z_1 \, dz \\
\lesssim \int_{K^z_{4p}(3)} \left( \frac{|u-w|}{p} + |\nabla u| + |\nabla w| + |f| + \mu + 1 \right)^{p(1-\beta)} \, dz \\
\lesssim \int_{K^z_{4p}(3)} (|\nabla u| + |\nabla w| + |f| + \mu + 1)^{p(1-\beta)} \, dz.
\]

Combining (7.5) and (7.6), we get
\[
\int K^{-(\beta)}_z |\nabla u - \nabla w|^{p(1-\beta)} \, dz \\
\lesssim \left( \epsilon_1 + \epsilon_2 \right) C_{(p_{\log}n, \Lambda_0, \Lambda_1)} \int_{K^z_{4p}(3)} |\nabla u|^{p(1-\beta)} + |\nabla w - \nabla u|^{p(1-\beta)} \, dz \\
+ C(\epsilon_1, \epsilon_2) \int K^{-(\beta)}_z (\mu^2 + |\nabla u|^2 + |\nabla w|^2)^{\frac{p(\beta-2)}{2} - \frac{p}{2}} |\nabla u - \nabla w|^2 \tilde{g}(z)^{-\beta} \, dz \\
+ \int_{K^z_{4p}(3)} |f|^{p(1-\beta)} + 1 \, dz.
\]

Subcase \( K^+(\beta) \): In this case, we proceed as follows:
\[
\int K^{+(\beta)}_z |\nabla u - \nabla w|^{p(1-\beta)} \, dz \\
\leq C(\epsilon_3) \int_{K^z_{4p}(3)} \tilde{g}(z)^{-\beta} |\nabla u - \nabla w|^{p(\beta)} \, dz + \epsilon_3 \int_{K^z_{4p}(3)} \tilde{g}(z)^{1-\beta} \, dz \\
\lesssim C(\epsilon_3) \int_{K^z_{4p}(3)} \tilde{g}(z)^{-\beta} (\mu^2 + |\nabla u|^2 + |\nabla w|^2)^{\frac{p(\beta-2)}{2} - \frac{p}{2}} |\nabla u - \nabla w|^2 \, dz \\
+ \epsilon_3 \int_{K^z_{4p}(3)} |\nabla u - \nabla w|^{p(1-\beta)} + |\nabla u|^{p(1-\beta)} + |f|^{p(1-\beta)} + 1 \, dz.
\]

Estimate for \( K_2 \): Again by Fubini’s theorem, we get
\[
K_2 = \frac{1}{\beta} \int_{K^z_{4p}(3)} \tilde{g}(z)^{-\beta} \langle |f|^{p(\beta-2)}, \nabla u - \nabla w \rangle \, dz.
\]

From the definition of \( g(z) \), we see that for \( z \in K^z_{4p}(3) \), we have \( \tilde{g}(z) \geq |\nabla u - \nabla w|^{p(z)} \) which implies \( \tilde{g}(z)^{-\beta} \leq |\nabla u - \nabla w|^{-\beta p(z)} \). We can now apply Young’s inequality with exponents \( \frac{p(\beta-1)}{p-1} \) and \( \frac{p(1-\beta)}{1-p(\beta)} \) to get:
\[
K_2 \lesssim C(\epsilon_3) \int_{K^z_{4p}(3)} |f|^{p(\beta-1)} \, dz + \frac{\epsilon_3}{\beta} \int_{K^z_{4p}(3)} |\nabla u - \nabla w|^{p(1-\beta)} \, dz.
\]

Estimate for \( K_3 \): Applying the layer cake representation followed by Lemma 5.6, we get
\[
K_3 = \frac{1}{\beta} \int_{\mathbb{R}^{n+1}} g^z_1 \, dz \\
\lesssim \int_{K^z_{4p}(3)} (|\nabla u - \nabla w| + |\nabla u| + |f| + \mu + 1)^{p(1-\beta)} \, dz.
\]
Combining everything, we get the following estimate:

\[
\int_{K_{4p}^2(3)} |\nabla u - \nabla w|^p \, dz \lesssim (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + C(\varepsilon_1, \varepsilon_2, \varepsilon_3) \beta) \int_{K_{4p}^2(3)} |\nabla u|^p(1-\beta) \, dz \\
+ (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + C(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \beta) \int_{K_{4p}^2(3)} |\nabla u - \nabla w|^p(1-\beta) \, dz \\
+ C(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \beta) \int_{K_{4p}^2(3)} \left[ |f|^p(1-\beta) + 1 \right] \, dz.
\]

Choosing \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) and \( \varepsilon_4 \) small followed by \( \beta \in (0, \tilde{\beta}_3] \), we get the proof of the estimate.

\section*{8 Second difference estimate below the natural exponent}

In this section, we will prove a difference estimate between the weak solution of (6.17) and the weak solution of (6.24). To do this, we will use the method of Lipschitz truncation from Appendix B.

For this section, as needed in (B.1), let us denote

\[ s := \alpha^{-1+d}(3\rho)^2. \]

From (2.7), recall that \( d_1 \geq -\frac{n}{p(\gamma)} + \frac{nd}{2} + d \geq d_0 \) where \( d_0 \) is the infimum as defined in (6.4) and \( d_1 \) is the supremum, both of which are finite and positive numbers.

\textbf{Theorem 8.1} Let \((\rho(\cdot), A, \Omega)\) be \((\gamma, S_0)\)-vanishing. Suppose that \( w \) and \( v \) are weak solutions of (6.17) and (6.24), respectively, and let \( \alpha \geq 1 \) be given such that the following assumptions hold:

\[
\int_{K_{4p}^2(3)} |\nabla w|^{p(\gamma)(1-\beta)} \, dz \leq c_s \alpha^{1-\beta} \quad \text{and} \quad \alpha p_{p_{4p}}^+ - p_{K_{4p}^2(3)} \leq c_p. \tag{8.1}
\]

Further assume that

\[ \alpha^{-\frac{n}{p(\gamma)} + \frac{nd}{2} + d-\beta} \leq \Gamma^2(4\rho)^{-(n+2)} \quad \text{and} \quad p_{K_{4p}^2(3)}^+ - p_{K_{4p}^2(3)}^- \leq \omega_{p(\cdot)}(4\rho \Gamma), \tag{8.2} \]

for some \( \Gamma, c_p, c_s > 1 \) to be selected as fixed constants in Sect. 9.

Then there exists \( \tilde{\beta}_4 = \tilde{\beta}_4(n, p_{\log}^\pm, \Lambda_0, \Lambda_1, M_0) \) such that for any \( 128 \rho \leq \tilde{\beta}_4 \) and for any \( \varepsilon \in (0, 1) \), there exist \( \tilde{\beta}_4 = \tilde{\beta}_4(\varepsilon, n, \Lambda_0, \Lambda_1, p_{\log}^\pm) \) and \( \gamma_0 = \gamma_0(\varepsilon, n, \Lambda_0, \Lambda_1, p_{\log}^\pm) \) such that for any \( \beta \in (0, \tilde{\beta}_4) \) and any \( \gamma \in (0, \gamma_0) \), the following estimate holds:

\[
\int_{K_{3p}^2(3)} |\nabla w - \nabla v|^{p(\gamma)(1-\beta)} \, dz \leq \varepsilon \alpha^{1-\beta} \quad \text{and} \quad \int_{K_{3p}^2(3)} |\nabla v|^{p(\gamma)(1-\beta)} \, dz \lesssim \alpha^{1-\beta}. \tag{8.3}
\]

\textbf{Proof} The first estimate in (8.3) and (8.1) directly implies the second estimate in (8.3) after making use of the triangle inequality. Thus we only prove the first estimate in (8.3).

Consider the following cut-off function \( \zeta_\varepsilon \in C^\infty(\mathbb{R}) \) such that \( 0 \leq \zeta_\varepsilon(t) \leq 1 \) and

\[
\zeta_\varepsilon(t) = \begin{cases} 
1 & \text{for } t \in (t-s+\varepsilon, t+s-\varepsilon), \\
0 & \text{for } t \in (-\infty, t-s) \cup (t+s, \infty).
\end{cases}
\]

It is easy to see that

\[
\zeta_\varepsilon(t) = 0 \quad \text{for} \quad t \in (-\infty, t-s) \cup (t-s+\varepsilon, t+s-\varepsilon) \cup (t+s, \infty), \\
|\zeta_\varepsilon'(t)| \leq \frac{\varepsilon}{\varepsilon} \quad \text{for} \quad t \in (t-s, t-s+\varepsilon) \cup (t+s-\varepsilon, t+s).
\]
Without loss of generality, we shall always take $2h \leq \varepsilon$ since we will take limits in the following order $\lim_{\varepsilon \to 0} \lim_{h \to 0}$.

We shall use $v_{\lambda,h}(z)\zeta_\varepsilon(t)$ as a test function where $v_{\lambda,h}$ is as constructed in Appendix B (more specifically in (B.5)). This is valid since $v_{\lambda,h} \in C^{0,1}(K_{3p}^{g}(\lambda))$. Using this, we get

$$
\begin{align*}
\iiint_{K_{3p}^{g}(\lambda)} & \frac{d|w - v|_{h}}{dt} v_{\lambda,h} \zeta_\varepsilon \ dx \ dt + \iiint_{K_{3p}^{g}(\lambda)} \langle [\overline{B}(t, \nabla v) - \overline{B}(t, \nabla w)]_{h} , \nabla v_{\lambda,h} \rangle \zeta_\varepsilon \ dx \ dt \\
= & \iiint_{K_{3p}^{g}(\lambda)} \langle [\overline{B}(t, \nabla w) - \mathcal{B}(x, t, \nabla w)]_{h} , \nabla v_{\lambda,h} \rangle \zeta_\varepsilon \ dx \ dt \\
& + \iiint_{K_{3p}^{g}(\lambda)} \langle [\mathcal{B}(x, t, \nabla w) - \mathcal{A}(x, t, \nabla w)]_{h} , \nabla v_{\lambda,h} \rangle \zeta_\varepsilon \ dx \ dt.
\end{align*}
$$

Proceeding as in Theorem 7.1, after taking limits, we get for any $t_0 \in (t - s, t + s)$, the estimate

$$
\int_{\Omega_{3p}^{g}(\lambda)} (v^2 - (v_{\lambda} - v)^2)(x, t_0) \ dx + \int_{\Omega_{3p}^{g}(\lambda) \cap E_{\lambda}} \langle [\overline{B}(t, \nabla v) - \mathcal{B}(t, \nabla w) , \nabla (v - w)] \rangle \zeta_\varepsilon \ dx \ dt
$$

$$
= - \int_{\Omega_{3p}^{g}(\lambda) \cap E_{\lambda}} \langle [\overline{B}(t, \nabla v) - \overline{B}(t, \nabla w)]_{h} , \nabla v_{\lambda,h} \rangle \zeta_\varepsilon \ dx \ dt \\
+ \int_{\Omega_{3p}^{g}(\lambda) \cap E_{\lambda}} \langle [\mathcal{B}(x, t, \nabla w) - \mathcal{A}(x, t, \nabla w)]_{h} , \nabla v_{\lambda,h} \rangle \zeta_\varepsilon \ dx \ dt \\
+ \int_{\Omega_{3p}^{g}(\lambda) \cap E_{\lambda}} \langle [\mathcal{B}(x, t, \nabla w) - \mathcal{A}(x, t, \nabla w)]_{h} , \nabla (v - w) \rangle \zeta_\varepsilon \ dx \ dt
$$

Let us multiply (8.4) by $\lambda^{-1-\beta}$ and integrate over $[1, \infty)$ to get

$$
K_1 + K_2 \leq K_3 + K_4 + K_5 + K_6 + K_7,
$$

where

$$
\begin{align*}
K_1 : = & \int_{1}^{\infty} \lambda^{-1-\beta} \int_{\Omega_{3p}^{g}(\lambda)} (v^2 - (v_{\lambda} - v)^2)(x, t_0) \ dx \ d\lambda, \\
K_2 : = & \int_{1}^{\infty} \lambda^{-1-\beta} \int_{\Omega_{3p}^{g}(\lambda) \cap E_{\lambda}} \langle [\overline{B}(t, \nabla v) - \overline{B}(t, \nabla w) , \nabla (v - w)] \rangle \zeta_\varepsilon \ dx \ d\lambda, \\
K_3 : = & - \int_{1}^{\infty} \lambda^{-1-\beta} \int_{\Omega_{3p}^{g}(\lambda) \cap E_{\lambda}} \langle [\overline{B}(t, \nabla v) - \overline{B}(t, \nabla w)]_{h} , \nabla v_{\lambda,h} \rangle \zeta_\varepsilon \ dx \ dt \ d\lambda, \\
K_4 : = & \int_{1}^{\infty} \lambda^{-1-\beta} \int_{\Omega_{3p}^{g}(\lambda) \cap E_{\lambda}} \langle [\mathcal{B}(x, t, \nabla w) - \mathcal{A}(x, t, \nabla w)]_{h} , \nabla v_{\lambda,h} \rangle \zeta_\varepsilon \ dx \ dt \ d\lambda, \\
K_5 : = & \int_{1}^{\infty} \lambda^{-1-\beta} \int_{\Omega_{3p}^{g}(\lambda) \cap E_{\lambda}} \langle [\mathcal{B}(x, t, \nabla w) - \mathcal{A}(x, t, \nabla w)]_{h} , \nabla (v - w) \rangle \zeta_\varepsilon \ dx \ dt \ d\lambda, \\
K_6 : = & \int_{1}^{\infty} \lambda^{-1-\beta} \int_{\Omega_{3p}^{g}(\lambda) \cap E_{\lambda}} \langle [\mathcal{B}(x, t, \nabla w) - \mathcal{A}(x, t, \nabla w)]_{h} , \nabla (v - w) \rangle \zeta_\varepsilon \ dx \ dt \ d\lambda, \\
K_7 : = & \int_{1}^{\infty} \lambda^{-1-\beta} \int_{\Omega_{3p}^{g}(\lambda) \cap E_{\lambda}} \langle [\mathcal{B}(x, t, \nabla w) - \mathcal{A}(x, t, \nabla w)]_{h} , \nabla (v - w) \rangle \zeta_\varepsilon \ dx \ dt \ d\lambda.
\end{align*}
$$

Let us set $\tilde{g}(z) := \max\{1, g(z)^{\frac{1}{m}}\}$ where $g(z)$ is defined in (B.3) and estimate each of the terms as follows:
Estimate for $K_1$: Using Lemma A.38, we see that

$$\int_{\Omega_{\psi}(t_0)} (v^2 - (v_\alpha - v)^2)(x, t_0) \, dx \geq -\lambda |\mathbb{R}^{n+1} \setminus E_\lambda|.$$  

Using this along with Fubini’s theorem, we see that

$$K_1 \gtrsim -\int_1^{\infty} \lambda^{-1-\beta} \lambda |z \in \mathbb{R}^{n+1} : \tilde{g}(z) \geq \lambda| \, d\lambda = -\frac{1}{1-\beta} \int_{\mathbb{R}^{n+1}} \tilde{g}(z)^{1-\beta} \, dz \gtrsim -\int_{K_{3p}(3)} (|\nabla v - \nabla w| + |\nabla w| + 1)^{p(3)(1-\beta)} \, dz.$$  

Estimate for $K_2$: Similar to the estimates in Theorem 7.1, we see that

$$\int_{K_{3p}(3)} |\nabla v - \nabla w|^{p(3)(1-\beta)} \, dz \lesssim C(\varepsilon_1) \beta K_2 + \varepsilon_1 \int_{K_{3p}(3)} |\nabla v|^{p(3)(1-\beta)} + 1 \, dz.$$  

Estimate for $K_3$: Using the bound from Lemma A.29, we get

$$\int_{K_{3p}(3)} \langle [\mathcal{B}(t, \nabla v) - \mathcal{B}(t, \nabla w)]_h, \nabla v_{\psi, h} \rangle \xi_\varepsilon \, dx \, dt \lesssim \sum_{i \in \mathbb{N}} \lambda_{n(\alpha)}^{-1} \int_{2Q_i} (\mu^2 + |\nabla w|^2 + |\nabla v|^2)^{\frac{p(1)}{2}} \, dz \lesssim \lambda_{n(\alpha)}^{-1} \int_{16Q_i} \lambda_{n(\alpha)}^{-1} \, dz \lesssim \lambda |\mathbb{R}^{n+1} \setminus E_\lambda|.$$  

Using the above bound in $K_3$ followed by applying Fubini’s theorem, we get

$$K_3 \lesssim \int_1^{\infty} \lambda^{-1-\beta} \lambda |z \in \mathbb{R}^{n+1} : \tilde{g}(z) \geq \lambda| \, d\lambda = -\frac{1}{1-\beta} \int_{\mathbb{R}^{n+1}} \tilde{g}(z)^{1-\beta} \, dz \lesssim \int_{K_{3p}(3)} (|\nabla v - \nabla w| + |\nabla w| + 1)^{p(3)(1-\beta)} \, dz. \quad (8.5)$$  

Estimate for $K_4$: Similar to the estimate for $K_3$, we get

$$K_4 \lesssim \int_{K_{3p}(3)} (|\nabla v - \nabla w| + |\nabla w| + 1)^{p(3)(1-\beta)} \, dz.$$  

Estimate for $K_5$: In this case, we proceed as follows:

$$\int_{K_{3p}(3) \setminus E_\lambda} \langle [\mathcal{B}(x, t, \nabla w) - \mathcal{A}(x, t, \nabla w)]_h, \nabla v_{\psi, h} \rangle \xi_\varepsilon \, dx \, dt \lesssim \sum_{i \in \mathbb{N}} \lambda_{n(\alpha)}^{-1} \int_{2Q_i} |\mathcal{B}(x, t, \nabla w) - \mathcal{A}(x, t, \nabla w)| \, dx \, dt \lesssim \sum_{i \in \mathbb{N}} |2Q_i| \lambda_{n(\alpha)}^{-1} \int_{2Q_i} (|\nabla w| + |\nabla v| + 1)^{p(3)-1} \, dx \, dt \lesssim \sum_{i \in \mathbb{N}} |2Q_i| \lambda_{n(\alpha)}^{-1} \lambda_{n(\alpha)}^{-1} \lambda_{n(\alpha)}^{-1} \lesssim \lambda |\mathbb{R}^{n+1} \setminus E_\lambda|.$$  

This is bounded exactly as in (8.5) to get

$$K_5 \lesssim \int_{K_{3p}(3)} (|\nabla v - \nabla w| + |\nabla w| + 1)^{p(3)(1-\beta)} \, dz.$$  

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Estimate for $K_6$: Applying Fubini’s theorem, we see that

$$K_6 := \frac{1}{p} \int \int K_{3^p}^\alpha (t) |\bar{B}(t, \nabla w) - B[x, t, \nabla w]| |\nabla (v-w)||\bar{g}^{-\beta}(z) \, dz$$

$$\lesssim \frac{1}{p} \int \int K_{3^p}^\alpha (t) \Theta(A, B_{3^p}^\alpha (t))(1 + |\nabla w|)^{p^2 - 1} |\nabla v - \nabla w||\bar{g}^{-\beta}(z) \, dz$$

We shall estimate the second term as follows: ($\sigma$ is to be chosen appropriately later on)

$$\frac{\tilde{K}_6}{|K_{3^p}^\alpha (t)|} \lesssim \left( \int \int K_{3^p}^\alpha (t) \Theta(A, B_{3^p}^\alpha (t))^{p^2 - 1} \beta \, dz \right)^{\frac{\sigma}{4 + \sigma}} \int \int (1 + |\nabla w|)^{p^2 - 1} \beta \, dz. \tag{8.7}$$

If we restrict $p_{K_{4^p}^\alpha (t)}^+ - p_{K_{4^p}^\alpha (t)}^- \leq \frac{(p^2 - 1)\alpha}{4}$, we see that the following two bounds hold:

$$p^2 \leq \frac{(p^2 - 1)\alpha}{4}$$

$$p^2 \left(1 + \alpha \right) \leq \frac{(p^2 - 1)\alpha}{4} \leq \frac{(p^2 - 1)\alpha}{4} \left(1 + \frac{\alpha}{4} \right) \leq p^2 \left(1 + \frac{\alpha}{4} \right) \leq p^2 \left(1 + \frac{\alpha}{4} \right) \leq p^2 \left(1 + \frac{\alpha}{4} \right). \tag{8.6}$$

Let us set $a = \frac{p_{K_{4^p}^\alpha (t)}^+ - p_{K_{4^p}^\alpha (t)}^-}{p^2 - 1} \left(1 + \frac{\sigma}{4} \right)$, then we get from Corollary 6.4 that

$$\int \int K_{3^p}^\alpha (t) \Theta(A, B_{3^p}^\alpha (t))^\alpha \beta \, dz \lesssim \int \int K_{3^p}^\alpha (t) \Theta(A, B_{3^p}^\alpha (t))^{p^2 - 1} \beta \, dz$$

$$\lesssim \alpha^{(p^2 - 1)\alpha} \leq \alpha^{(p^2 - 1)\alpha} \leq \alpha^{(p^2 - 1)\alpha} \leq \alpha^{(p^2 - 1)\alpha} \leq \alpha^{(p^2 - 1)\alpha}. \tag{8.7}$$

From (2.2) and (2.6), we see that

$$\left( \int \int K_{3^p}^\alpha (t) \Theta(A, B_{3^p}^\alpha (t))^{p^2 - 1} \beta \, dz \right)^{\frac{\sigma}{4 + \sigma}} \lesssim \gamma^{(p^2 - 1)\alpha}. \tag{8.7}$$
Combining everything, we see that

\[ K_6 \leq \frac{e^2}{\beta} \int_{K_{3\rho}^q(\beta)} |\nabla w - \nabla v|^{\rho(3)(1-\beta)} \, dz + \frac{C(\varepsilon_2)}{\beta^\gamma} \frac{1}{(1-\beta)} |K_{3\rho}^q(\beta)| c_p^{-\frac{(1-\beta)}{2}} \alpha^{1-\beta}. \]

Estimate for \( K_7 \): Applying Fubini’s theorem, we get

\[ K_7 = \frac{1}{\beta} \int_{K_{3\rho}^q(\beta)} |\mathcal{E}(x,t,\nabla w) - \mathcal{A}(x,t,\nabla w)||\nabla (v-w)|||\nabla (v-w)|^{1-\beta} p(3) \, dz \]

\[ \lesssim \frac{1}{\beta} \int_{K_{3\rho}^q(\beta)} |\mathcal{E}(x,t,\nabla w) - \mathcal{A}(x,t,\nabla w)||\nabla (v-w)|||\nabla (v-w)|^{1-\beta} p(3) \, dz \]

Applying Young’s inequality, we get for \( E := \{ z \in K_{3\rho}^q(\beta) : \mu^2 + |\nabla w(z)|^2 > 0 \} \), the bound

\[ \frac{K_7}{|K_{3\rho}^q(\beta)|} \leq \frac{1}{\beta} e^2 \int_{K_{3\rho}^q(\beta)} |\nabla w - \nabla v|^{\rho(3)(1-\beta)} \, dz \]

\[ + \frac{C(\varepsilon_2)}{\beta |K_{3\rho}^q(\beta)|} \int_{E} \left[ (\mu^2 + |\nabla w|^2)^{\frac{\rho(3)-p(c)}{2}} \right] \left[ 1 - (\mu^2 + |\nabla w|^2)^{\frac{\rho(3)-p(c)}{2}} \right]^\frac{p(3)(1-\beta)}{p(3)-p(c)} \, dz. \]

We shall now proceed with estimating the second term in (8.8) as follows: For each \( z \in E \), in view of the mean value theorem applied to \( (\mu^2 + |\nabla w|^2)^{\frac{\rho(3)-p(c)}{2}} \), there exists \( a_z \in [0, 1] \) such that we get

\[ (\mu^2 + |\nabla w|^2)^{\frac{\rho(3)-p(c)}{2}} - 1 = \frac{p(3)-p(z)}{2} (\mu^2 + |\nabla w|^2)^{\frac{\rho(3)-p(c)}{2}} a_z \log(\mu^2 + |\nabla w|^2). \]  

(8.9)

This implies

\[ (\mu^2 + |\nabla w|^2)^{\frac{\rho(3)-p(c)}{2}} \left[ 1 - (\mu^2 + |\nabla w|^2)^{\frac{\rho(3)-p(c)}{2}} \right] \]

\[ \lesssim \omega_{p(3)}(4\rho \Gamma)(\mu^2 + |\nabla w|^2)^{\frac{(\rho(3)-p(c))a_z + p(c)-1}{2}} \log(\mu^2 + |\nabla w|^2). \]

Let us now define the sets

\[ E^1 := \{ z \in K_{3\rho}^q(\beta) : |\nabla w(x)| \leq 1 \} \quad \text{and} \quad E^2 := \{ x \in K_{3\rho}^q(\beta) : |\nabla w(x)| > 1 \}. \]  

(8.10)

Recall that \( \mu \leq 1 \) and hence using the inequality \( t^\beta \log t \leq \max \left\{ \frac{1}{e^\beta}, 2^\beta \log 2 \right\} \) which holds for all \( t \in (0, 2) \) and any \( \beta > 0 \), we get for \( z \in E^1 \)

\[ (\mu^2 + |\nabla w|^2)^{\frac{\rho(3)-p(c)}{2}} \left[ 1 - (\mu^2 + |\nabla w|^2)^{\frac{\rho(3)-p(c)}{2}} \right] \lesssim \omega_{p(3)}(4\rho \Gamma) \max \left\{ \frac{1}{e^{\frac{a_z\beta-1}{2}}}, 2^{\frac{\rho(3)-p(c)}{2}} \log 2 \right\}. \]

(8.11)

To obtain the above estimate, with \( \beta(z) = \frac{a_z(\rho(3)-p(\beta^3)) + p(z)-1}{2} \), there holds

\[ \frac{p^--1}{2} \leq \beta(z) \leq \frac{p(3) - 1}{2} \leq \frac{p^+ - 1}{2}. \]
Hence using (8.10) and combining (8.11) into (8.9), we get
\[
|B(z, \nabla w) - A(z, \nabla w)| \lesssim \chi_{E^1} \omega \rho^\gamma(4 \rho \Gamma) \max \left\{ \frac{1}{e^{eta - 1}}, 2 \right\} \left( \frac{p - 1}{p} \right) \log 2 + \chi_{E^2} \omega \rho^\gamma(4 \rho \Gamma) |\nabla w|^{|p(\gamma) - p(\zeta)}|a + p(\zeta) - 1| \log(e + |\nabla w|).
\]
(8.12)

Combining (8.12) and (8.8), we get
\[
J \lesssim \omega \rho^\gamma(4 \rho \Gamma) \frac{p(\gamma)(1 - \beta)}{p(\gamma) - 1} |K_{3p}^\alpha(\gamma)| (1 + J_1),
\]
where \( J_1 := \iint_{K_{3p}^\alpha(\gamma)} |\nabla w|^b \log(e + |\nabla w|^b) \frac{p(\gamma)(1 - \beta)}{p(\gamma) - 1} dz \) with \( b := \frac{p(\gamma)(1 - \beta)}{p(\gamma) - 1} (p_{K_{3p}^\alpha(\gamma)}^+ - 1) \).

Using the inequality \( \log(e + ab) \leq \log(e + a) + \log(e + b) \) for \( a, b > 0 \) along with the simple bound \( \frac{p(\gamma)}{p(\gamma) - 1} \leq \frac{p}{p - 1} \), we get
\[
J_1 \lesssim \iint_{K_{3p}^\alpha(\gamma)} |\nabla w|^b \left[ \log \left( e + \frac{|\nabla w|^b}{(|\nabla w|^b)_{K_{3p}^\alpha(\gamma)}} \right) \right] \frac{p}{p - 1} \log \frac{p(\gamma)}{p(\gamma) - 1} \log \frac{p}{p - 1} dz
\]
\[
+ \iint_{K_{3p}^\alpha(\gamma)} |\nabla w|^b \left[ \log \left( e + \frac{|\nabla w|^b}{(|\nabla w|^b)_{K_{3p}^\alpha(\gamma)}} \right) \right] \frac{p(\gamma)}{p(\gamma) - 1} \log \frac{p}{p - 1} dz
\]
\[
= J_2 + J_3.
\]

Estimate for \( J_2 \): We now apply Lemma 5.3 with \( f = |\nabla w|^b \), \( \beta = \frac{p}{p - 1} \) and \( s = 1 + \frac{\sigma}{4} \) to get
\[
J_2 \lesssim \left( \iint_{K_{3p}^\alpha(\gamma)} |\nabla w|^b (1 + \frac{\sigma}{4}) dz \right)^{\frac{4}{1 + \frac{\sigma}{4}}} \lesssim \left( \iint_{K_{3p}^\alpha(\gamma)} (1 + |\nabla w|^p)(1 - \beta)(1 + a) dz \right)^{\frac{4}{1 + \frac{\sigma}{4}}},
\]
(8.13)

where \( a = \frac{p_{K_{3p}^\alpha(\gamma)}^+(p_{K_{3p}^\alpha(\gamma)}^-)}{p_{K_{3p}^\alpha(\gamma)}^- - 1} (1 + \frac{\sigma}{4}) + \frac{\sigma}{4} \) satisfying \( a \leq \sigma \).

Estimate for \( J_3 \): From (8.6) and (8.7), we see that
\[
\log \left( e + \frac{|\nabla w|^b}{(|\nabla w|^b)_{K_{3p}^\alpha(\gamma)}} \right) \leq \log(e + c_p \alpha^{1 - \beta}) = \log(e + c_1 \alpha^{1 - \beta}) \lesssim C_{(c_1)} (\log \alpha^{1 - \beta} + 1) \lesssim C_{(c_1)} \left\{ \log \left( \Gamma^2(4 \rho)^{-(n + 2)} \right) + 1 \right\},
\]
(8.14)

Here we have denoted \( c_1 = c_p^{-1} \) where \( c_p \) is from (8.7). Substituting (8.14) into \( J_3 \) and making use of the bound from (8.13), we get
\[
J_3 \lesssim C_{(c_1)} \left\{ \log \left( \Gamma^2(4 \rho)^{-(n + 2)} \right) + 1 \right\} \frac{p(\gamma)(1 - \beta)}{p(\gamma) - 1} \alpha^{1 - \beta}.
\]

Then we have
\[
|K_7| \lesssim \frac{c_1}{\beta} \iint_{K_{3p}^\alpha(\gamma)} |\nabla w - \nabla v|^p(\gamma)(1 - \beta) dz
\]
\[
+ \frac{C_{(c_1)}}{\beta} \omega \rho^\gamma(4 \rho \Gamma) \frac{p(\gamma)(1 - \beta)}{p(\gamma) - 1} |K_{3p}^\alpha(\gamma)| C_{(c_1)} \left\{ \log \left( \Gamma^2(4 \rho)^{-(n + 2)} \right) + 1 \right\} \frac{p(\gamma)(1 - \beta)}{p(\gamma) - 1} \alpha^{1 - \beta}.
\]
The restriction \( \rho \leq \frac{1}{4\epsilon n+5} \) implies
\[
\omega_{p(p)}(4\rho \Gamma) \left\{ \log \left( \Gamma^2(4\rho)^{-(n+2)} + 1 \right) \right\} = \omega_{p(p)}(4\rho \Gamma) \log \left( 4\rho e \Gamma^2(4\rho)^{-(n+3)} \right) \\
\leq \omega_{p(p)}(4\rho \Gamma) \log \left( \Gamma^{-(n+3)}(4\rho)^{-(n+3)} \right) \\
\leq (n+3)\omega_{p(p)}(4\rho \Gamma) \log \left( \frac{1}{4\rho \Gamma} \right) \\
\lesssim \gamma.
\]

Using this, we get
\[
|K_3| \lesssim \frac{\epsilon_3}{\beta} \int_{K_{3\rho}(3)} |\nabla w - \nabla v|^p(1-\beta) \, dz + \frac{C_{(e_3)}}{\beta^{\frac{p-1}{2}}} |K_{3\rho}(3)| C_{(e_1)} \alpha^{1-\beta}.
\]

Combining all the estimates, we get,
\[
\||\nabla w - \nabla v|^p(1-\beta) \|_z \lesssim (\epsilon_1 + \epsilon_2 + \epsilon_4 + \beta) \||\nabla w - \nabla v|^p(1-\beta) \|_z \\
+ 3\beta \||\nabla v|^p(1-\beta) \|_z \\
+ C_{(e_3)} \gamma^{\frac{p-1}{2}} |K_{3\rho}(3)| C_{(e_1)} \alpha^{1-\beta}.
\]

Now choosing \( \epsilon_1, \epsilon_2, \epsilon_4, \text{ and } \beta \) small, we get for any \( \epsilon > 0 \), the estimate
\[
\||\nabla w - \nabla v|^p(1-\beta) \|_z \lesssim (\epsilon_1 + \epsilon_2 + \epsilon_4 + \beta) \||\nabla w|^p(1-\beta) \|_z \\
+ \gamma^{\frac{p-1}{2}} C_{(e_1)} \alpha^{1-\beta} \\
\leq (8.1) \lesssim \epsilon \alpha^{1-\beta} + \gamma^{\frac{p-1}{2}} \left( 1 + c_\rho^{p-1} \right) \alpha^{1-\beta}.
\]

Now choose \( \gamma \) sufficiently small such that for any \( \epsilon \in (0, 1) \), there holds
\[
\||\nabla w - \nabla v|^p(1-\beta) \|_z \lesssim \epsilon \alpha^{1-\beta},
\]

which completes the proof.

## 9 Covering arguments

Let \( \beta \in (0, \beta_0) \), where \( \beta_0 \) is from Sect. 2.9. Assume that \( (p(\cdot), A, \Omega) \) is \( (\gamma, S_0) \)-vanishing for some \( S_0 \in (0, 1) \) in the sense of Definition 2.8. Let \( q(\cdot) \) be log-Hölder continuous in the sense of Definition 2.3. We fix any \( \rho \leq \frac{\epsilon_0}{4} \), where \( \rho_0 \) is given in Remark 2.10, and fix any \( \delta = (\gamma, \epsilon_0, 0) \in \Omega_T \) with \( (t - (4\rho)^2, t + (4\rho)^2) \subset (-T, T) \).

We observe from Theorems 6.1 and 6.2 that
\[
\||\nabla u|^p(1-\beta)(1+\sigma) \|_z \lesssim \left( \||\nabla u| + |f|\|^p(1-\beta) \|_z \right)^{1+\sigma} \\
+ \int_{K_{2\rho}} |f|^p(1-\beta)(1+\sigma) \, dz + 1,
\]

where \( \beta \) and \( \sigma \) are given in Remark 2.11 and for some \( \tilde{\sigma} = \tilde{\sigma}(n, p(\delta)) > 0 \).
It follows from Sect. 2.8 that for each \( z \in K_{4\rho}(\tilde{z}) \),

\[
\frac{p(z)(1 - \beta)q(z)}{q_{K_{4\rho}(\tilde{z})}} \leq p(z)(1 - \beta) \left( 1 + \frac{\omega_{q}(\lambda)}{q^{-}} \right) \leq p(z)(1 - \beta)(1 + \sigma), \tag{9.2}
\]

\[
\frac{p(z)(1 - \beta)q(z)(1 + \sigma)}{q_{K_{4\rho}(\tilde{z})}} \leq p(z)(1 - \beta)(1 + 3\sigma) \leq \min \left\{ p(z), p(z)(1 - \beta)q^{-} \right\}. \tag{9.3}
\]

We first verify some parabolic localization properties under our unified intrinsic cylinders.

**Lemma 9.1** Let \( c_{a} > 1 \) and let \( M_{0} \) be given in (6.26). Then there is a constant \( c_{1} = c_{1}(n, \Lambda_{0}, \Lambda_{1}, p_{\log}^{\pm}, q_{\log}^{\pm}) \geq 1 \) such that for any \( \lambda \geq 1, \) any \( \tilde{z} \in K_{2\rho}(\tilde{z}), \) and any \( \tilde{\rho} > 0, \)

\[
\tilde{\rho} \leq \Gamma^{-2} S_{0}, \quad \text{where} \quad \Gamma := 2c_{1} c_{a} M_{0} \nu^{-1} \geq 2, \tag{9.4}
\]

satisfying \( K_{\tilde{\rho}}^{a}(\tilde{z}) \subset K_{2\rho}(\tilde{z}), \) if

\[
a^{1 - \beta} \leq c_{a} \left\{ \iint_{K_{\tilde{\rho}}^{a}(\tilde{z})} |\nabla u|^{q_{K_{4\rho}(\tilde{z})}} dz + \frac{1}{\nu} \left( \iint_{K_{\tilde{\rho}}^{a}(\tilde{z})} |f|^{q_{K_{4\rho}(\tilde{z})}} dz \right) \right\}^{\frac{1}{1 + \sigma}}, \tag{9.5}
\]

then we have

\[
a^{- \frac{n}{3} + \frac{m}{2} + d - \beta} \leq \Gamma^{2} \tilde{\rho}^{-(n + 2)}, \quad p_{Q_{\tilde{\rho}}^{+}(\tilde{z})}^{+} - p_{Q_{\tilde{\rho}}^{+}(\tilde{z})}^{-} \leq \omega_{p}(\nu \tilde{\rho}),
\]

\[
p_{Q_{\tilde{\rho}}^{-}(\tilde{z})}^{+} - p_{Q_{\tilde{\rho}}^{-}(\tilde{z})}^{-} \leq e^{2n \nu / 5} =: c_{p}, \tag{9.6}
\]

\[
q_{Q_{\tilde{\rho}}^{+}(\tilde{z})}^{+} - q_{Q_{\tilde{\rho}}^{+}(\tilde{z})}^{-} \leq \omega_{q}(\nu \tilde{\rho}), \quad a^{- q_{Q_{\tilde{\rho}}^{+}(\tilde{z})}^{+} - q_{Q_{\tilde{\rho}}^{+}(\tilde{z})}^{-}} \leq e^{\frac{(2n + 4)\nu}{5}} =: c_{q}, \tag{9.7}
\]

where the positive constant \( d_{0} \) is given in (6.4).

**Proof** Fix \( K_{\tilde{\rho}}^{a}(\tilde{z}) \subset K_{2\rho}(\tilde{z}). \) We compute

\[
\frac{p(z)(1 - \beta)q(z)}{q_{K_{2\rho}(\tilde{z})}} \leq p(z)(1 - \beta) \left( 1 + \frac{\omega_{q}(\lambda)}{q^{-}} \right) \leq p(z)(1 - \beta)(1 + \sigma), \tag{9.2}
\]

\[
\frac{p(z)(1 - \beta)q(z)(1 + \sigma)}{q_{K_{2\rho}(\tilde{z})}} \leq p(z)(1 - \beta)(1 + 3\sigma) \leq \min \left\{ p(z), p(z)(1 - \beta)q^{-} \right\}. \tag{9.3}
\]

\[
\frac{M_{0}}{\nu |K_{4\rho}(\tilde{z})|} \left\{ \frac{M_{0}}{|K_{4\rho}(\tilde{z})|} \frac{\omega_{q}(\lambda)}{q^{-}} + 1 \right\} \text{Section } 2.8 \leq \frac{M_{0}}{\nu |K_{4\rho}(\tilde{z})|}. \tag{9.8}
\]
Then we see
\[\alpha^{-\frac{n}{p(z)} + \frac{nd}{2} + d - \beta} \leq \frac{c_d\alpha^{-\frac{n}{p(z)} + \frac{nd}{2} - 1 + d}|K_2\rho(z)|}{|\mathcal{K}_d\rho(z)|}\]

\[\left\{ \left( \int_{K_2\rho(z)} |\nabla u|^q K_4\rho(z) \, dz \right) + \frac{1}{\gamma} \left( \int_{K_2\rho(z)} |f|^q K_4\rho(z) \, dz \right) \right\}^{\frac{1}{1+\gamma}}\]

\[(9.8) c_1c_M \leq \Gamma \tilde{\rho}^{-(n+2)},\]

for some \(c_1 = c_1(n, \Lambda_0, \Lambda_1, p_{\log}, q_{\log}) \geq 1\).

On the other hand, it follows from Remark 2.4 that
\[p^+_{\rho(z)} - p^-_{\rho(z)} \leq \omega_{\rho(z)} \left( \max \left\{ \alpha^{-\frac{1}{p(z)} + \frac{d}{2}}, \alpha^{-\frac{1+d}{2}} \right\} 2\tilde{\rho} \right) \leq \omega_{\rho(z)}(2\tilde{\rho}) \leq \omega_{\rho(z)}(\Gamma \tilde{\rho}),\]

which implies
\[\Gamma^{p^+_{\rho(z)} - p^-_{\rho(z)}} \leq \Gamma^{\omega_{\rho(z)}(\Gamma \tilde{\rho})} \leq \left( \frac{\Gamma}{S_0} \right)^{\omega_{\rho(z)}(\frac{S_0}{\Gamma})} \leq \epsilon \leq e.\]

Combining the inequalities above and (2.5), we deduce
\[\alpha^{-\frac{n}{p(z)} + \frac{nd}{2} + d - \beta} \leq \frac{(n+1)\omega_{\rho(z)}(\Gamma \tilde{\rho})}{\omega_{\rho(z)}(\Gamma \tilde{\rho})} \leq \Gamma^{-\frac{n}{p(z)} + \frac{nd}{2} + d - \beta} \leq \Gamma^{-\frac{n}{p(z)} + \frac{nd}{2} + d - \beta} \leq e^{\frac{n+1}{n-1}}.\]

Note that \(d_0 - \beta > 0\) from Sect. 2.9. Similarly, we can also obtain the inequalities (9.7).

We now consider a Vitali type covering lemma for intrinsic parabolic cylinders as follow:

**Lemma 9.2** Let \(\alpha, c_p, c_q > 1\) and let \(\mathcal{F} := \left\{ Q_{\rho_j(z_j)}^{\alpha_j(z_j)} \right\}_{j \in J} \subset Q_{2r(z_j)}^{\alpha_j(z_j)} \) be any collection of intrinsic parabolic cylinders, where \(\alpha_j := \alpha^{-\frac{1}{p(\rho_j(z_j))}}\) and \(\rho_j > 0\), satisfying
\[p^+_{\alpha_j(z_j)} - p^-_{\alpha_j(z_j)} \leq c_p \quad \text{and} \quad q^+_{\alpha_j(z_j)} - q^-_{\alpha_j(z_j)} \leq c_q \quad \text{for every} \quad j \in J. \quad (9.9)\]

Then there exists a countable subcollection \(\mathcal{G} := \{ Q_{\rho_i(z_i)}^{\alpha_i(z_i)} \}_{i \in I}, I \subset J\), of mutually disjoint cylinders such that
\[\bigcup_{j \in J} Q_{\rho_j(z_j)}^{\alpha_j(z_j)} \subset \bigcup_{i \in I} Q_{\rho_i(z_i)}^{\alpha_i(z_i)},\]

for some constant \(\chi = \chi(n, c_p, c_q, p_{\log}, q_{\log}) \geq 1\).

**Proof** The proof is similar to that of the standard Vitali covering lemma except in the setting of the unified intrinsic cylinders. See [17, Lemma 5.3] and [33, Lemma 7.1] for other intrinsic cylinder cases. For completeness, we give the proof.

Write \(D := \sup_{j \in J} \rho_j\). Set
\[\mathcal{F}_k := \left\{ Q_{\rho_j(z_j)}^{\alpha_j(z_j)} \in \mathcal{F} : \frac{D}{2^k} < \rho_j \leq \frac{D}{2^{k-1}} \right\} \quad (k = 1, 2, \cdots).\]

\(\mathcal{F}_k\) contains the bounded open sets of \(\mathbb{R}^n\times(0, T)\) such that the \(\rho_j\) are all congruent to \(1\) modulo \(\frac{1}{2^k}\). Since \(\alpha > 1\) and \(\alpha > 1\), we have
\[\frac{D}{2^k} < \rho_j \leq \frac{D}{2^{k-1}} \leq \frac{D}{2^{k+1}}.\]

Thus, the \(\mathcal{F}_k\) cover all of \(\mathbb{R}^n\times(0, T)\) whenever \(D > 1\). Therefore, we can choose \(\mathcal{G}\) as desired.
We define $G_k \subset F_k$ as follows:

- Let $G_1$ be any maximal disjoint collection of intrinsic cylinders in $F_1$.
- Assuming that $G_1, \ldots, G_{k-1}$ have been selected, we choose $G_k$ to be any maximal disjoint subcollection of

\[
\left\{ Q \in F_k : Q \cap Q' = \emptyset \text{ for all } Q' \in \bigcup_{l=1}^{k-1} G_l \right\}.
\]

- Finally, we define

\[
G := \bigcup_{k=1}^\infty G_k.
\]

Clearly $G$ is a countable collection of disjoint intrinsic cylinders and $G \subset F$. Now it suffices to show that for each intrinsic cylinder $Q_{\rho_j}^{a_j}(z_j) \in F$, there exists an intrinsic cylinder $Q_{\rho_i}^{a_i}(z_i) \in G$ such that $Q_{\rho_j}^{a_j}(z_j) \cap Q_{\rho_i}^{a_i}(z_i) \neq \emptyset$ and $Q_{\rho_j}^{a_j}(z_j) \subset Q_{\rho_i}^{a_i}(z_i)$.

Fix $Q_{\rho_j}^{a_j}(z_j) \in F$. Then there is an index $k$ such that $Q_{\rho_j}^{a_j}(z_j) \in F_k$. By the maximality of $G_k$, there exists an intrinsic cylinder $Q_{\rho_i}^{a_i}(z_i) \in \bigcup_{l=1}^k G_l$ with $Q_{\rho_j}^{a_j}(z_j) \cap Q_{\rho_i}^{a_i}(z_i) \neq \emptyset$. Since $\rho_i > \frac{D}{2} \alpha$ and $\rho_j \leq \frac{D}{2} \alpha$, we know $\rho_j < 2 \rho_i$. Choose $z_0 \in Q_{\rho_j}^{a_j}(z_j) \cap Q_{\rho_i}^{a_i}(z_i)$. We compute

\[
\alpha_j^{-1+d(z_j)} = \alpha_i^{-1+d(z_i)} \frac{q_{\rho_j}(z_j)}{q_{\rho_i}(z_i)} \left(1-(1-d(z_j))(1-q_{\rho_j}(z_j)(1-d(z_j))(1-q_{\rho_i}(z_i))q_{\rho_i}(z_i)(1-q_{\rho_i}(z_i))q_{\rho_i}(z_i)\right)
\]

\[
\leq \alpha_i^{-1+d(z_i)} \alpha_j \frac{q_{\rho_i}(z_i)}{q_{\rho_i}(z_i)} \left(1-(1-d(z_j))(1-q_{\rho_i}(z_i))q_{\rho_i}(z_i)\right)
\]

\[
\leq \alpha_i^{-1+d(z_i)} \frac{2q_{\rho_i}^+}{q_{\rho_i}^-} \alpha_j^{-1+d(z_j)}.
\]

where $d$ is given in (2.7). Note that from (2.7), we see that the choice of $d(p(z))$ depends only on $p(z)$ up to a constant depending only on the dimension $n$ and hence, $d$ is also log-Hölder continuous. Thus, from (9.9), we see that $\alpha_j^{-1-(d(z_j)-(d(z_0)))} \leq c_d$ and $\alpha_i^{-1-(d(z_0)-(d(z_1)))} \leq c_d$ hold.

Similarly, it follows from (9.9) that

\[
\alpha_j \leq \frac{1}{p(z_j)^{1-(d(z_j))}} \leq c_{p(z_j)^{-1+(d(z_j))}} \alpha_i \leq \left(\frac{p(z_j)^{1-(d(z_j))}}{p(z_i)^{1-(d(z_i))}}\right)^{\alpha_i^{-1-(d(z_i))}}.
\]

Thus, from the definition of intrinsic cylinders in Sect. 2.7, there exists a constant $\chi = \chi_{n,c_p,c_q,p_{log}^+} \geq 1$ such that $Q_{\rho_j}^{a_j}(z_j) \subset Q_{\rho_i}^{a_i}(z_i)$, which completes the proof.

### 9.1 Stopping-time argument

We employ in this subsection a stopping-time argument from [34] to derive a covering of the upper-level set of $|\nabla u|_{q_{\rho_j}^{a_j}(z_j)}$ with respect to some parameter $\alpha$. bolts
Let us define $\tilde{\alpha}$ by
\[
\tilde{\alpha} := \int_{K_{2\rho}(\tilde{\zeta})} \frac{p(z)(1-\beta)q(z)}{(1-\beta)\vartheta_+ K_{4\rho}(\tilde{\zeta})} \left| \nabla u \right|^q K_{4\rho}(\tilde{\zeta}) \; dz + \frac{1}{\gamma} \left( \int_{K_{2\rho}(\tilde{\zeta})} \frac{p(z)(1-\beta)q(z)(1+\sigma)}{(1-\beta)\vartheta_+ K_{4\rho}(\tilde{\zeta})} \; dz \right)^{\frac{1}{1+\sigma}},
\]
(9.10)
where the constants $\beta$ and $\sigma$ are given in Remark 2.11 and
\[
\vartheta_+ K_{4\rho}(\tilde{\zeta}) := \sup_{z \in K_{4\rho}(\tilde{\zeta})} \vartheta(z),
\]
(9.11)
where $\vartheta(z)$ is given in (4.1). Note that $0 < \frac{1-\beta}{(1-\beta)\vartheta_+ K_{4\rho}(\tilde{\zeta})} < 1$ since $\vartheta_+ K_{4\rho}(\tilde{\zeta}) > 1$. For $\alpha \geq 1$ and $s \geq 1$, let $E(s, \alpha)$ denote the upper-level set of $|\nabla u(z)|^{q K_{4\rho}(\tilde{\zeta})}$, defined by
\[
E(s, \alpha) := \left\{ z \in K_{s\rho}(\tilde{\zeta}) : |\nabla u(z)|^{q K_{4\rho}(\tilde{\zeta})} > \alpha \right\}.
\]
(9.12)
Fix any $1 \leq s_1 < s_2 \leq 2$ and any $\alpha \geq 1$ satisfying
\[
\alpha > A\tilde{\alpha}, \quad \text{where} \quad A := \left[ \left( \frac{16}{7} \right)^n \left( \frac{120\chi}{s_2 - s_1} \right)^{n+2} \right]^{\frac{1-\beta}{1-\beta} K_{4\rho}(\tilde{\zeta})}.
\]
(9.13)
Here $\chi$ is given in Lemma 9.2. Fix any
\[
\tilde{\rho} \in \left( \frac{(s_2 - s_1)\rho}{60\chi}, (s_2 - s_1)\rho \right).
\]
(9.14)
Note that $K_{\tilde{\rho}^\alpha(\tilde{\zeta})}(\tilde{\zeta}) \subset K_{\tilde{\rho}(\tilde{\zeta})} \subset K_{s_2\rho}(\tilde{\zeta}) \subset K_{2\rho}(\tilde{\zeta})$, where $\alpha := \frac{q K_{4\rho}(\tilde{\zeta})}{\alpha_{\frac{n}{\alpha}} + 1}$. Then we compute that for all $\tilde{\zeta} \in K_{s_1\rho}(\tilde{\zeta})$,
\[
\begin{align*}
\int_{K_{\tilde{\rho}^\alpha(\tilde{\zeta})}(\tilde{\zeta})} \left| \nabla u \right|^q K_{4\rho}(\tilde{\zeta}) \; dz & \leq \frac{1}{\alpha_{\frac{n}{\alpha}} + 1 + d} \int_{K_{\tilde{\rho}(\tilde{\zeta})}} \left| \nabla u \right|^q K_{4\rho}(\tilde{\zeta}) \; dz \\
& \leq \frac{1}{\alpha_{\frac{n}{\alpha}} + \frac{d}{\alpha} + 1 + d} \int_{K_{2\rho}(\tilde{\zeta})} \left| \nabla u \right|^q K_{4\rho}(\tilde{\zeta}) \; dz \\
& \leq \left( \frac{16}{7} \right)^n \left( \frac{120\chi}{s_2 - s_1} \right)^{n+2} \frac{1}{\alpha_{\frac{n}{\alpha}} + \frac{d}{\alpha} + 1 + d} \left( \frac{1}{\vartheta_+ K_{4\rho}(\tilde{\zeta})} \right)^{\frac{1}{1+\sigma}}.
\end{align*}
\]
(9.10)
(9.13)
(9.14)
where the inequality (a) has used the fact that (4.1) and 1 ≤ α, β ≤ α.

On the other hand, in view of the Lebesgue differentiation theorem, for every Lebesgue point 3 of |∇u| qK4p(3) in E(s1, α), we have

\[
\lim_{\rho \to 0} \left\{ \iint_{K_{\rho}^3} |\nabla u|^{\frac{p(z)(1-\beta q(z)}{qK4p(3)} \, dz + \frac{1}{\gamma} \left( \iint_{K_{\rho}^3} |f|^{\frac{p(z)(1-\beta q(z)}{qK4p(3)} \, dz} \right)^{\frac{1}{p(z)-1}} \right\} > \alpha.
\]

Then for almost every such point, there exists ρ̄ ∈ (0, (s2-s1)ρ/60χ) such that

\[
\iint_{K_{\rho^3}^3} |\nabla u|^{\frac{p(z)(1-\beta q(z)}{qK4p(3)} \, dz + \frac{1}{\gamma} \left( \iint_{K_{\rho^3}^3} |f|^{\frac{p(z)(1-\beta q(z)}{qK4p(3)} \, dz} \right)^{\frac{1}{p(z)-1}} > \alpha \quad \forall \rho^3 \in \left(0, \frac{(s2-s1)\rho}{60\chi}\right).
\]

Applying Lemma 9.1 and Lemma 9.2 to the collection of intrinsic cylinders \( \{ Q_{\rho^3 i}^\alpha (3) \} \) with ρ̄ replacing ρ and ᾱ replacing α, there exist \( \{ 3_i \} \subset \subset E(s1, \alpha) \) and \( \rho_i \in (0, (s2-s1)\rho/60\chi) \) such that \( \{ \rho^3_i (3) \} \subset \subset K_{s2}\rho (3) \), where \( \alpha_i : = \alpha^{\frac{qK4p(3)}{p(z)(1-\beta q(z))}} \) for \( i = 1, 2, \ldots \), such that \( \{ Q_{\rho_i}^{\alpha_i} (3) \} \subset \subset \) is mutually disjoint,

\[
E(s1, \alpha) \setminus N \subset \bigcup_{i=1}^{\infty} K_{\rho_i}^{\alpha_i} (3) \subset K_{s2}\rho (3), \tag{9.15}
\]

for some Lebesgue measure zero set N, and for each i we have

\[
\iint_{K_{\rho_i}^{\alpha_i} (3)} |\nabla u|^{\frac{p(z)(1-\beta q(z))}{qK4p(3)} \, dz + \frac{1}{\gamma} \left( \iint_{K_{\rho_i}^{\alpha_i} (3)} |f|^{\frac{p(z)(1-\beta q(z))}{qK4p(3)} \, dz} \right)^{\frac{1}{p(z)-1}} = \alpha, \tag{9.16}
\]

and

\[
\iint_{K_{\rho_i}^{\alpha_i} (3)} |\nabla u|^{\frac{p(z)(1-\beta q(z))}{qK4p(3)} \, dz + \frac{1}{\gamma} \left( \iint_{K_{\rho_i}^{\alpha_i} (3)} |f|^{\frac{p(z)(1-\beta q(z))}{qK4p(3)} \, dz} \right)^{\frac{1}{p(z)-1}} < \alpha, \tag{9.17}
\]

for any \( \rho_i \in (\rho_i, (s2-s1)\rho] \).

### 9.2 Power decay estimates on unified intrinsic cylinders

Here we derive the power decay estimate (9.31) on the upper-level set of \( |\nabla u|^{\frac{p(z)(1-\beta q(z))}{qK4p(3)} } \), where \( \beta \) is given in Remark 2.11. For any \( 1 \leq s_1 < s_2 \leq 2 \) and any \( \alpha \geq 1 \) satisfying (9.13), we consider \( Q_{\rho_i}^{\alpha_i} (3) \), \( i = 1, 2, \ldots \), selected in the previous subsection, with

\[
\alpha_i : = \alpha^{\frac{qK4p(3)}{p(z)(1-\beta q(z))}} \quad \text{and} \quad 60\chi \rho_i \leq (s2-s1)\rho \leq \rho, \tag{9.18}
\]
where $\chi$ is given in Lemma 9.2.

We divide into the two cases: $Q_{4x,\rho_i}(\tilde{z}_i) \subset \Omega_T$ and $Q_{4x,\rho_i}(\tilde{z}_i) \not\subset \Omega_T$. We only consider the boundary case $Q_{4x,\rho_i}(\tilde{z}_i) \not\subset \Omega_T$. The interior case $Q_{4x,\rho_i}(\tilde{z}_i) \subset \Omega_T$ can be proved in a similar way.

Since $Q_{4x,\rho_i}(\tilde{z}_i) \not\subset \Omega_T$, there exists a boundary point $(\tilde{x}_i, t_i) \in (\partial\Omega \times (-T, T)) \cap Q_{4x,\rho_i}(\tilde{z}_i)$. Since $(\rho(\cdot), A, \Omega)$ is $(\gamma, S_0)$-vanishing, there exists a new coordinate system modulo rotation and translation, which we still denote by $\{x_1, \ldots, x_n, t\}$, with the origin is $(\tilde{x}_i, t_i) + 56\chi y \rho_i e_n$, where $e_n = (0, \ldots, 0, 1)$ and

$$B^+_{\rho}(0) \subset \Omega_{\rho}(0) \subset B_{\rho}(0) \cap \{(x, t) : x_n > -112\chi y \rho\} \text{ for any } 0 < \rho < 48\chi \rho_i.$$ 

Set $\tilde{z}_i = (0, t_i)$. Since $|\tilde{z}_i| \leq |\tilde{x}_i - \tilde{z}_i| + |\tilde{z}_i| \leq (4 + 56\gamma)\chi \rho_i \leq 11\chi \rho_i$, we have from (9.18) and (9.15) that

$$K_{4x,\rho_i}(\tilde{z}_i) \subset K_{12x,\rho_i}(\tilde{z}_i) \subset K_{48x,\rho_i}(\tilde{z}_i) \subset K_{60x,\rho_i}(\tilde{z}_i) \subset K_{52,\rho}(\tilde{z}) \subset K_{4\rho}(\tilde{z}),$$

and thus

$$\begin{align*}
&\quad p^+_{a,\alpha_i} - p^-_{a,\alpha_i} \\
&\leq \frac{p(\rho(1-\beta)\sigma(z))}{\int_{K_{48x,\rho_i}(\tilde{z}_i)} |\nabla u| q_{K_{4\rho}(\tilde{z})} \, dz + 1} \left( \frac{p(\rho(1-\beta)\sigma(z)(1+\sigma))}{\int_{K_{48x,\rho_i}(\tilde{z}_i)} |f| q_{K_{4\rho}(\tilde{z})} \, dz} \right)^{1+\sigma}.
\end{align*}$$

We employ (9.16) with taking $c_a = 2(48)^{n+2}$ to derive

$$\alpha_i^{1-\beta} \leq \alpha < c_a \left\{ \begin{array}{l}
\frac{p(\rho(1-\beta)\sigma(z))}{\int_{K_{48x,\rho_i}(\tilde{z}_i)} |\nabla u| q_{K_{4\rho}(\tilde{z})} \, dz + 1} \left( \frac{p(\rho(1-\beta)\sigma(z)(1+\sigma))}{\int_{K_{48x,\rho_i}(\tilde{z}_i)} |f| q_{K_{4\rho}(\tilde{z})} \, dz} \right)^{1+\sigma}.
\end{array} \right\},$$

where $\beta$ and $\sigma$ are given in Remark 2.11. Now applying Lemma 9.1 with $\alpha = \alpha_i$, $\tilde{\rho} = 48\chi \rho_i$ and $\tilde{z} = \tilde{z}_i$, we obtain

$$\begin{align*}
&\quad \alpha_i^{1-\beta} \leq \Gamma^2(48\chi \rho_i)^{-(n+2)}, \\
&\quad p^+_{a,\alpha_i} - p^-_{a,\alpha_i} \leq \omega_{p(\cdot)}(48\Gamma \chi \rho_i), \\
&\quad \alpha_i^{1-\beta} \leq c_p, \\
&\quad q^+_{a,\alpha_i} - q^-_{a,\alpha_i} \leq c_q,
\end{align*}$$

where $c_p$ and $c_q$ are given in (9.6) and (9.7), respectively. We can now directly compute to get

$$\begin{align*}
\frac{p(\rho(1-\beta)\sigma(z))}{\int_{K_{48x,\rho_i}(\tilde{z}_i)} |\nabla u| q_{K_{4\rho}(\tilde{z})} \, dz + 1} \left( \frac{p(\rho(1-\beta)\sigma(z)(1+\sigma))}{\int_{K_{48x,\rho_i}(\tilde{z}_i)} |f| q_{K_{4\rho}(\tilde{z})} \, dz} \right)^{1+\sigma}
\leq \alpha_i^{1-\beta} \leq \alpha_i^{1-\beta}. 
\end{align*}$$

(9.20)
Lemma 9.3 For any $\varepsilon \in (0, 1)$, there exist $\gamma = \gamma(n, \Lambda_0, \Lambda_1, p_{\text{log}}, q_{\text{log}}, \varepsilon) > 0$ and $\tilde{V}_i \in L^1(K_{24}^{\alpha_i} (\tilde{z}_i))$ with $\nabla \tilde{V}_i \in L^{\infty}(K_{12}^{\alpha_i} (\tilde{z}_i), \mathbb{R}^n)$ for each $i$ satisfying $Q_{36}^{\alpha_i} (\tilde{z}_i) \not\subset \Omega_T$ such that

\[
\iint_{K_{48}^{\alpha_i} (\tilde{z}_i)} |\nabla u - \nabla w_i|^p (z)(1-\beta) \, dz \leq \varepsilon \alpha_i^{-1-\beta}, \quad \iint_{K_{36}^{\alpha_i} (\tilde{z}_i)} |\nabla w_i|^p (z)(1-\beta) \, dz \leq \alpha_i^{-1-\beta}.
\]  

(9.22)

Under these settings, we have the following lemma:

Lemma 9.3 For any $\varepsilon \in (0, 1)$, there exist $\gamma = \gamma(n, \Lambda_0, \Lambda_1, p_{\text{log}}, q_{\text{log}}, \varepsilon) > 0$ and $\tilde{V}_i \in L^1(K_{24}^{\alpha_i} (\tilde{z}_i))$ with $\nabla \tilde{V}_i \in L^{\infty}(K_{12}^{\alpha_i} (\tilde{z}_i), \mathbb{R}^n)$ for each $i$ satisfying $Q_{36}^{\alpha_i} (\tilde{z}_i) \not\subset \Omega_T$ such that

\[
\iint_{K_{48}^{\alpha_i} (\tilde{z}_i)} |\nabla u - \nabla w_i|^p (z)(1-\beta) \, dz \leq \varepsilon \alpha_i^{-1-\beta}, \quad \iint_{K_{36}^{\alpha_i} (\tilde{z}_i)} |\nabla w_i|^p (z)(1-\beta) \, dz \leq \alpha_i^{-1-\beta}.
\]  

(9.23)

where $w_i$ is the weak solution of

\[
\begin{cases}
(w_i)_t - \text{div} A(x, t, \nabla w_i) = 0 & \text{in } K_{48}^{\alpha_i} (\tilde{z}_i), \\
w_i = u & \text{on } \partial K_{48}^{\alpha_i} (\tilde{z}_i).
\end{cases}
\]

Proof The first estimate of (9.22) follows from (9.20), (9.21) and Theorem 7.1. We see from Lemma 6.6 that $\nabla w_i \in L^p (\tilde{z}_i) (K_{56}^{\alpha_i} (\tilde{z}_i))$ and from Sect. 2.8 that

\[
p(\tilde{z}_i)(1-\beta) \leq \frac{p(\tilde{z}_i)(1-\beta) (p^{-1}_{K_{48}^{\alpha_i} (\tilde{z}_i)} - 1)}{p(\tilde{z}_i) - 1} \leq \frac{p^{-1}_{K_{48}^{\alpha_i} (\tilde{z}_i)} (1-\beta) (p^{-1}_{K_{48}^{\alpha_i} (\tilde{z}_i)} - 1)}{p^{-1}_{K_{48}^{\alpha_i} (\tilde{z}_i)} - 1}
\]

\[
\leq p(\cdot)(1-\beta) \left(1 + \frac{p^{+}_{K_{48}^{\alpha_i} (\tilde{z}_i)} - p^{-}_{K_{48}^{\alpha_i} (\tilde{z}_i)}}{p^{-1} - 1}\right) \leq p(\cdot)(1-\beta) \left(1 + \frac{\sigma}{4}\right).
\]

This inequality and Corollary 6.4 infer

\[
\iint_{K_{36}^{\alpha_i} (\tilde{z}_i)} |\nabla w_i|^p (\tilde{z}_i)(1-\beta) \, dz \leq \iint_{K_{36}^{\alpha_i} (\tilde{z}_i)} |\nabla w_i|^{p(z)(1-\beta)} \left(1 + \frac{p^{+}_{K_{48}^{\alpha_i} (\tilde{z}_i)} - p^{-}_{K_{48}^{\alpha_i} (\tilde{z}_i)}}{p^{-1} - 1}\right) \, dz \leq \alpha_i^{-1-\beta}.
\]

(9.24)

To prove the first estimate of (9.23), we split two parts as follows:

\[
\iint_{K_{12}^{\alpha_i} (\tilde{z}_i)} |\nabla w_i - \nabla \tilde{V}_i|^p (\tilde{z}_i)(1-\beta) \, dz \leq \iint_{K_{12}^{\alpha_i} (\tilde{z}_i)} |\nabla w_i - \nabla \tilde{V}_i|^p (\tilde{z}_i)(1-\beta) \, dz
\]

\[+ \iint_{K_{12}^{\alpha_i} (\tilde{z}_i)} |\nabla \tilde{V}_i|^p (\tilde{z}_i)(1-\beta) \, dz,
\]

(9.24)
where $v_i$ is the weak solution of
\[
\begin{aligned}
(v_i)_t - \text{div} B(t, \nabla v_i) &= 0 \quad \text{in } K_{36\chi_{\rho_i}}^a(\tilde{z}_i), \\
v_i &= w_i \quad \text{on } \partial_p K_{36\chi_{\rho_i}}^a(\tilde{z}_i).
\end{aligned}
\]

From the second estimate of (9.22), (9.19) and Theorem 8.1, we deduce
\[
\iint_{K_{12\chi_{\rho_i}}^a(\tilde{z}_i)} |\nabla v_i - \nabla \tilde{v}_i|^{p(\tilde{z}_i)(1-\beta)}\,dz \leq \frac{\varepsilon}{2} \alpha_i^{1-\beta}. \tag{9.25}
\]

Moreover, Lemma 6.8 implies that there exists a weak solution $\tilde{V}_i$ of
\[
\begin{aligned}
(\tilde{V}_i)_t - \text{div} B(t, \nabla \tilde{V}_i) &= 0 \quad \text{in } Q_{24\chi_{\rho_i}}^{a_i,+}(\tilde{z}_i), \\
\tilde{V}_i &= 0 \quad \text{on } \partial_{\omega} Q_{24\chi_{\rho_i}}^{a_i,+}(\tilde{z}_i)
\end{aligned}
\]
with
\[
\iint_{K_{24\chi_{\rho_i}}^a(\tilde{z}_i)} |\nabla \tilde{V}_i|^{p(\tilde{z}_i)}\,dz \lesssim \alpha_i,
\]
such that the following holds:
\[
\iint_{K_{12\chi_{\rho_i}}^a(\tilde{z}_i)} |\nabla v_i - \nabla \tilde{V}_i|^{p(\tilde{z}_i)(1-\beta)}\,dz \leq \left( \iint_{K_{12\chi_{\rho_i}}^a(\tilde{z}_i)} |\nabla v_i - \nabla \tilde{V}_i|^{p(\tilde{z}_i)}\,dz \right)^{1-\beta} \leq \frac{\varepsilon}{2} \alpha_i^{1-\beta}. \tag{9.26}
\]

Here $\tilde{V}_i$ is extended by zero from $Q_{24\chi_{\rho_i}}^{a_i,+}(\tilde{z}_i)$ to $K_{24\chi_{\rho_i}}^a(\tilde{z}_i)$. Inserting (9.25) and (9.26) into (9.24), we obtain the first estimate of (9.23).

Finally, Lemma 6.9 implies the second estimate of (9.23).

From a similar way in [1, Corollary 5.6], we can also obtain from Lemma 9.3 that the following estimates:

**Corollary 9.4** Under the assumptions as in Lemma 9.3, we have
\[
\iint_{K_{12\chi_{\rho_i}}^a(\tilde{z}_i)} |\nabla u - \nabla \tilde{V}_i|^{\frac{p(\tilde{z}_i)(1-\beta)q(\tilde{z}_i)}{2}}\,dz \leq \varepsilon \alpha \quad \text{and} \quad \sup_{z \in K_{12\chi_{\rho_i}}^a(\tilde{z}_i)} |\nabla \tilde{V}_i|^{\frac{p(\tilde{z}_i)(1-\beta)q(\tilde{z}_i)}{2}} \leq c_2 \alpha \tag{9.27}
\]
for some constant $c_2 = c_2(n, \Lambda_0, \Lambda_1, p_{\log}^\pm, q_{\log}^\pm) \geq 1$.

We now estimate the integration of $|\nabla u|^{q_{K_{\rho}}(\tilde{z})}$ on the upper-level set $E(s_1, B\alpha)$, where
\[
B : = 2 \frac{p^+(1-\beta)q^+}{q} c_2 \geq 1, \tag{9.28}
\]
and the constant $c_2$ is given in Corollary 9.4. Recalling (9.12), it follows from (9.15) that
\[
E(s_1, B\alpha) \setminus N \subset E(s_1, \alpha) \setminus N \subset \bigcup_{i=1}^{\infty} K_{\chi_{\rho_i}}^a(\tilde{z}_i) \subset K_{s_2\rho}(\tilde{z}),
\]
and
\[
\iint_{E(s_1, B\alpha)} |\nabla u|^{q_{K_{\rho}}(\tilde{z})}\,dz \leq \sum_{i=1}^{\infty} \iint_{E(s_1, B\alpha) \cap K_{\chi_{\rho_i}}^a(\tilde{z}_i)} |\nabla u|^{q_{K_{\rho}}(\tilde{z})}\,dz.
\]
Then this implies
\[\int_{E(s_1, B\alpha) \cap K_{\beta}^{q_i} (\tilde{z}_i)} |\nabla u| \, q_{K_{4p}^{q_i} (\tilde{z}_i)} \, dz \leq 2 \frac{p(1-\beta)q}{q} \int_{E(s_1, B\alpha) \cap K_{12\chi_{\rho_i} (\tilde{z}_i)}} |\nabla u - \nabla \tilde{V}_i| \, q_{K_{4p}^{q_i} (\tilde{z}_i)} \, dz\]
\[\leq 2 \frac{p(1-\beta)q}{q} \int_{E(s_1, B\alpha) \cap K_{12\chi_{\rho_i} (\tilde{z}_i)}} |\nabla u - \nabla \tilde{V}_i| \, q_{K_{4p}^{q_i} (\tilde{z}_i)} \, dz\]
\[\leq 2 \frac{p(1-\beta)q}{q} \int_{E(s_1, B\alpha) \cap K_{12\chi_{\rho_i} (\tilde{z}_i)}} |\nabla u - \nabla \tilde{V}_i| \, q_{K_{4p}^{q_i} (\tilde{z}_i)} \, dz\]
\[
\begin{cases}
\left| \nabla u \right|_{q_{K_{4p}^{q_i} (\tilde{z}_i)}} \leq 2 \frac{p(1-\beta)q}{q} \int_{E(s_1, B\alpha) \cap K_{12\chi_{\rho_i} (\tilde{z}_i)}} |\nabla u - \nabla \tilde{V}_i| \, q_{K_{4p}^{q_i} (\tilde{z}_i)} \, dz
\end{cases}
\]
that is,
\[
\int_{E(s_1, B\alpha)} |\nabla u| \, q_{K_{4p}^{q_i} (\tilde{z}_i)} \, dz \leq \varepsilon \alpha \sum_{i=1}^{\infty} |K_{\beta}^{q_i} (\tilde{z}_i)|. \quad (9.29)
\]

On the other hand, we know from (9.16) that either
\[
\alpha \leq \frac{1}{1+\sigma} \left( \int_{K_{\beta}^{q_i} (\tilde{z}_i)} |\nabla u| \, q_{K_{4p}^{q_i} (\tilde{z}_i)} \, dz \right)^{-1}
\]
and then we calculate
\[
\left| K_{\beta}^{q_i} (\tilde{z}_i) \right| \leq \frac{4}{\alpha} \left\{ \int_{K_{\beta}^{q_i} (\tilde{z}_i)} |\nabla u| \, q_{K_{4p}^{q_i} (\tilde{z}_i)} \, dz \right\}^\frac{\gamma}{q}
\]
\[
\left\{ \frac{p(1-\beta)q}{q} \int_{E(s_1, B\alpha) \cap K_{12\chi_{\rho_i} (\tilde{z}_i)}} |\nabla u - \nabla \tilde{V}_i| \, q_{K_{4p}^{q_i} (\tilde{z}_i)} \, dz \right\}
\]
\[
+ \left( \frac{4}{\gamma^\alpha} \right) \left\{ \frac{p(1-\beta)q}{q} \int_{E(s_1, B\alpha) \cap K_{12\chi_{\rho_i} (\tilde{z}_i)}} |\nabla u - \nabla \tilde{V}_i| \, q_{K_{4p}^{q_i} (\tilde{z}_i)} \, dz \right\}
\]
\[\left\{ \frac{p(1-\beta)q}{q} \int_{E(s_1, B\alpha) \cap K_{12\chi_{\rho_i} (\tilde{z}_i)}} |\nabla u - \nabla \tilde{V}_i| \, q_{K_{4p}^{q_i} (\tilde{z}_i)} \, dz \right\}
\]

Plugging (9.30) into (9.29) and using the fact that the family \( \left\{ K_{\beta}^{q_i} (\tilde{z}_i) \right\}_{i=1}^{\infty} \subset K_{s_2 \rho} (\tilde{z}) \) is pairwise disjoint, we conclude
\[
\int_{E(s_1, B\alpha)} |\nabla u| \, q_{K_{4p}^{q_i} (\tilde{z}_i)} \, dz \leq \varepsilon \int_{E(s_2, \frac{1}{2})} |\nabla u| \, q_{K_{4p}^{q_i} (\tilde{z}_i)} \, dz
\]
\[
+ \varepsilon \left\{ \frac{p(1-\beta)q}{q} \int_{E(s_1, B\alpha) \cap K_{12\chi_{\rho_i} (\tilde{z}_i)}} |\nabla u - \nabla \tilde{V}_i| \, q_{K_{4p}^{q_i} (\tilde{z}_i)} \, dz \right\}
\]
\[\left\{ \frac{p(1-\beta)q}{q} \int_{E(s_1, B\alpha) \cap K_{12\chi_{\rho_i} (\tilde{z}_i)}} |\nabla u - \nabla \tilde{V}_i| \, q_{K_{4p}^{q_i} (\tilde{z}_i)} \, dz \right\}
\]
10 Proof of the main results

10.1 Proof of Theorem 4.1

Fix any $z \in \Omega_T$, $\beta \in (0, \beta_0)$, and $\rho \in (0, \rho_0)$, where $\beta_0$ and $\rho_0$ are given in Sect. 2.9 and Remark 2.10, respectively. Recalling (4.1), we have $M \gtrsim M_0 \geq 1$, where $M_0$ is given in (6.26). In view of Lemma 9.1, putting $\rho_0 = \frac{1}{C_0 M_0}$ for some constant $C_0 = C_0(\Lambda_0, \Lambda_1, \frac{p^+}{p_-}, q_{log}, n, S_0) > 0$, we can apply all results in Sect. 9.

For $k > 0$, we define the truncation of $|\nabla u|^{q_{K^4_p(\rho)}}$ as

$$
\left(\frac{p(z(1-\beta)q(z))}{|\nabla u|^{q_{K^4_p(\rho)}}}\right)_{k} (z) := \min \left\{ \frac{p(z(1-\beta)q(z))}{|\nabla u|^{q_{K^4_p(\rho)}}}, k \right\}.
$$

Let $1 \leq s_1 < s_2 \leq 2$. Lemma 5.4 implies that for sufficiently large $k > 1$,

$$
\int_{K_{s_1, \rho}(3)} \left(\frac{p(z(1-\beta)q(z))}{|\nabla u|^{q_{K^4_p(\rho)}}}\right)_{k} d\alpha = \left( q_{K^4_p(\rho)} - 1 \right) \int_{0}^{k} \alpha \left( q_{K^4_p(\rho)} - 2 \right) \int_{E(s_1, \alpha)} |\nabla u|^{q_{K^4_p(\rho)}} d\alpha \int_{0}^{p(z(1-\beta)q(z))} d\alpha \int_{0}^{p(z(1-\beta)q(z))} d\alpha
$$

$$
\leq \left( q_{K^4_p(\rho)} - 1 \right) B q_{K^4_p(\rho)} - 1 \int_{0}^{\alpha \left( q_{K^4_p(\rho)} - 2 \right)} d\alpha \int_{K_{s_1, \rho}(3)} |\nabla u|^{q_{K^4_p(\rho)}} d\alpha \int_{0}^{p(z(1-\beta)q(z))} d\alpha \int_{0}^{p(z(1-\beta)q(z))} d\alpha
$$

$$
+ \left( q_{K^4_p(\rho)} - 1 \right) B q_{K^4_p(\rho)} - 1 \int_{0}^{\alpha \left( q_{K^4_p(\rho)} - 2 \right)} d\alpha \int_{E(s_1, \alpha)} |\nabla u|^{q_{K^4_p(\rho)}} d\alpha \int_{0}^{p(z(1-\beta)q(z))} d\alpha \int_{0}^{p(z(1-\beta)q(z))} d\alpha
$$

$$
=: I_1 + I_2,
$$

where $\tilde{\alpha}$, $A$, $B$, and $E(s_1, \alpha)$ are given in (9.10), (9.13), (9.28), and (9.12), respectively. For $I_1$, we compute directly that

$$
I_1 \leq (AB\tilde{\alpha}) q_{K^4_p(\rho)} - 1 \int_{K_{2, \rho}(3)} |\nabla u|^{q_{K^4_p(\rho)}} d\alpha \int_{0}^{p(z(1-\beta)q(z))} d\alpha \int_{0}^{p(z(1-\beta)q(z))} d\alpha
$$

$$
\lesssim \frac{\tilde{\alpha} q_{K^4_p(\rho)} - 1}{(s_2 - s_1) + 2 q_{K^4_p(\rho)} - 1} \int_{K_{2, \rho}(3)} |\nabla u|^{q_{K^4_p(\rho)}} d\alpha \int_{0}^{p(z(1-\beta)q(z))} d\alpha. \quad (10.2)
$$
For $I_2$, it follows from (9.31), Lemma 5.4 and Fubini’s theorem that

$$I_2 \leq \varepsilon \int_{K_{2^2}(\rho)} \left( \frac{p(1-\beta)\rho(z)}{q_{K_{4^2}(\rho)}} \right)^{q_{K_{4^2}(\rho)}-1} \frac{p(z)(1-\beta)q(z)}{q_{K_{4^2}(\rho)}} |\nabla u| \frac{p(z)(1-\beta)q(z)}{q_{K_{4^2}(\rho)}} dz + \varepsilon \gamma^{-q_{K_{4^2}(\rho)}}$$

(10.3)

Now we select $\varepsilon$ small enough which in turn determines $\gamma_0$. Plugging (10.2) and (10.3) into (10.1) and applying Lemma 5.5, we deduce

$$\int_{K_{2^2}(\rho)} |\nabla u|^p(z)(1-\beta)q(z) dz \leq C_{K_{2^2}(\rho)} q_{K_{4^2}(\rho)}^{-1} \int_{K_{2^2}(\rho)} |\nabla u|^{q_{K_{4^2}(\rho)}} dz + \int_{K_{2^2}(\rho)} |f|^p(z)(1-\beta)q(z) dz. $$

As $k \to \infty$, we have

$$\int_{K_{\rho}(\rho)} |\nabla u|^p(z)(1-\beta)q(z) dz \leq \alpha q_{K_{4^2}(\rho)}^{-1} \int_{K_{2^2}(\rho)} |\nabla u|^{q_{K_{4^2}(\rho)}} dz + \int_{K_{2^2}(\rho)} |f|^p(z)(1-\beta)q(z) dz. $$

(10.4)

On the other hand, we note that

$$\left( \int_{K_{2^2}(\rho)} |\nabla u|^p(z)(1-\beta) + |f|^p(z)(1-\beta)q(z) \right)^{\omega_{q(z)}(8\rho)} \leq \left( \frac{M}{|K_{2^2}(\rho)|} \right)^{\omega_{q(z)}(8\rho)} \leq \left( \frac{1}{8\rho} \right)^{(n+3)\omega_{q(z)}(8\rho)} \lesssim 1,$$

(10.5)

and similarly

$$\left( \int_{K_{2^2}(\rho)} |\nabla u|^p(z)(1-\beta) + |f|^p(z)(1-\beta)q(z) \right)^{\omega_{p(z)}(8\rho)} \lesssim 1.$$ 

(10.6)

Recalling (4.1), (9.11) and Remark 2.9, we discover

$$\vartheta^+_{K_{4^2}(\rho)} - \vartheta(\rho) \lesssim \omega_{p(z)}(8\rho).$$

(10.7)

Then we deduce

$$\int_{K_{2^2}(\rho)} |\nabla u|^p(z)(1-\beta)q(z) dz \leq \int_{K_{4^2}(\rho)} |\nabla u|^{p(z)(1-\beta)q(z)} dz + \int_{K_{4^2}(\rho)} |f|^p(z)(1-\beta)q(z) dz + 1$$

(10.8)
and
\[ g^{\Omega_{0}(z)} \lesssim \left\{ \int_{K_{\rho}(z)} |\nabla u|^{p(z)(1-\beta)q(z)} \, dz + \left( \int_{K_{\rho}(z)} |f|^{p(z)(1-\beta)q(z)} \, dz \right)^{\frac{1}{p(z)}} + 1 \right\}^{\frac{1}{(1-\beta)p(z)q(z)-1}} \] (10.9)

We finally obtain from (10.4), (10.8), and (10.9) that
\[ \int_{K_{\rho}(z)} |\nabla u|^{p(z)(1-\beta)q(z)} \, dz \lesssim \left\{ \int_{K_{\rho}(z)} |\nabla u|^{p(z)(1-\beta)} \, dz + \left( \int_{K_{\rho}(z)} |f|^{p(z)(1-\beta)q(z)} \, dz \right)^{\frac{1}{p(z)}} + 1 \right\}^{1 + \frac{1}{(1-\beta)p(z)q(z)-1}} \] (10.10)

which completes the proof.

### 10.2 Proof of Theorem 4.2

We extend the local estimate (10.10) up to the boundary. We first choose \( \rho = \frac{1}{C_{0}M^{2}} \), where \( C_{0} \) and \( M \) are given in Sect. 10.1. From the standard covering argument, we can find finitely many disjoint parabolic cylinders \( \{ Q_{\varphi}(\overline{3}k)\}_{k=1}^{m}, \overline{3}k \in \Omega_{T} \), such that \( \Omega_{T} \subset \bigcup_{k=1}^{m} Q_{\rho}(\overline{3}k) \).

Note that for an integrable function \( f \), we have \( \sum_{k=1}^{m} \int_{K_{\rho}(\overline{3}k)} f \, dz \lesssim \int_{\Omega_{T}} f \, dz \).

Then it follows from (10.10) that
\[ \int_{\Omega_{T}} |\nabla u|^{p(z)(1-\beta)q(z)} \, dz \leq \sum_{k=1}^{m} \int_{K_{\rho}(\overline{3}k)} |\nabla u|^{p(z)(1-\beta)q(z)} \, dz \lesssim \rho^{n+2} \left( \int_{\Omega_{T}} |\nabla u|^{p(z)(1-\beta)q(z)} \, dz \right)^{q^{+}} + \rho^{-(n+2)} \int_{\Omega_{T}} |f|^{p(z)(1-\beta)q(z)+1} \, dz \right)^{q(z)} \] (10.11)

where \( q(z) : = (1-\beta)q(z) \) for \( \beta \in (0, \beta_{0}) \) (it is important to note that we cannot take \( \beta = 0 \), then we trivially have \( r^{-} \geq (1-\beta)M^{-} \) and \( r^{+} \leq (1-\beta)M^{+} \). Note that \( r(\cdot) \) is clearly log-Hölder continuous with the log-Hölder constants equivalent to the ones satisfied by \( q(\cdot) \). Since all the estimates above are independent of \( M^{-} \) and \( \beta_{0} \) is independent of \( M^{-} \), we can choose \( M^{-} \) small enough with \( (1-\beta)M^{-} \leq 1 \). This in particular allows \( r^{-} = 1 \).
For this choice of the exponent $r(\cdot)$, we conclude from (10.11), (4.1), (6.27) and the definition of $\rho$ that
\[
\int_{\Omega_T} |\nabla u|^{p(z)r(\cdot)} \, dz \leq C \left\{ \left( \int_{\Omega_T} |f|^{p(z)r(\cdot)} \, dz \right)^{\theta_0(2n+3)q^+ - 2(n+2)} + 1 \right\},
\]
for some constants $C = C(\lambda_0, \lambda_1, p^{+}, \log^{-}, M^{+}, n, \Omega_T, s_0) \geq 1$ and $\tilde{\theta} = \tilde{\theta}(p^{+}, \log^{-}, M^{+}, n) \geq 1$, which completes the proof.

**Acknowledgements** The authors thank the anonymous referee for many helpful suggestions that improved the readability of the paper.

**A The method of Lipschitz truncation–first difference estimate**

In this appendix, following the techniques developed in [3] which were originally pioneered in [2], we will develop a modified version of Lipschitz truncation suited to our needs. Recall that $u$ is a weak solution of (1.1) and $w$ is a weak solution of (6.17). For this section, we only need to assume the following restrictions on the size of the region $K^p_{\mu}(\cdot)$: In particular, we will take $\tilde{\rho}_3$ small such that (R6) and (R4) are applicable.

To simplify the notation, we will define
\[
s : = \alpha^{-1+d}(4\rho)^2.
\]

Let us now collect some well known results that will be needed in the course of the proof. The first lemma is a time localised version of the parabolic Poincaré inequality (see [34, Lemma 4.2] for the proof):

**Lemma A.1** *Let $f \in L^\vartheta(-T, T; \mathcal{W}^{1, \vartheta}(\Omega))$ with $\vartheta \in (1, \infty)$ and suppose that $B_r \subset \subset \Omega$ be compactly contained ball of radius $r > 0$. Let $I \subset (-T, T)$ be a time interval and $\rho(x, t) \in L^1(B_r \times I)$ be any positive function such that*
\[
\|\rho\|_{L^\infty(B_r \times I)} \lesssim \|\rho\|_{L^1(B_r \times I)}
\]

*and $\mu(x) \in C^\infty_c(B_r)$ be such that $\int_{B_r} \mu(x) \, dx = 1$ with $|\mu| \leq \frac{C(n)}{\rho^d}$ and $|\nabla \mu| \leq \frac{C(n)}{\rho^{d+1}}$, then there holds:
\[
\int_{B_r \times I} \left[ f(z) \chi_{J} \left( \frac{f(x)}{\rho(z)} \right) \right]^\vartheta \, dz \lesssim_{\rho, \chi, \rho^d} \int_{B_r \times I} |\nabla f|^{\vartheta} \chi_{J} \, dz + \sup_{t_1, t_2 \in I} \left[ \left( f(x) \right)_{\mu}(t_2) - \left( f(x) \right)_{\mu}(t_1) \right]^\vartheta,
\]

*where $\left( f \chi_{J} \right)_{\rho} : = \int_{B_r \times I} f(z) \chi_{J} \rho(z) \, dz$, $\left( f \chi_{J} \right)_{\mu}(t) : = \int_{B_r} f(x, t) \mu(x) \chi_{J} \, dx$ and $J \subset (-\infty, \infty)$ is some fixed time-interval.*

**Lemma A.2** *For any $h \in (0, 2s)$ and let $\phi(x) \in C^\infty_c(\Omega^\vartheta_{\mu}(\cdot))$ and $\varphi(t) \in C^\infty(t-s, \infty)$ with $\varphi(t-s) = 0$ be a non-negative function and $[u]_h, [w]_h$ be the Steklov average as defined in*
With also note that \( \forall (t_1, t_2) \in [t - s, t + s] \):

\[
| (u - w)_h h (t_2) - (u - w)_h h (t_1) | 
\leq \| \nabla \phi \|_{L^\infty(\Omega^g (3))} \| \phi \|_{L^\infty(t_1, t_2)} \int_{\Omega^g (x) \times (t_1, t_2)} | A(x, \nabla w) - A(x, \nabla u) | \, dz
\]

\[ + \| \nabla \phi \|_{L^\infty(\Omega^g (3))} \| \phi \|_{L^\infty(t_1, t_2)} \int_{\Omega^g (x) \times (t_1, t_2)} \| f \|_{L^p(-1)} h \, dz
\]

\[ + \| \phi \|_{L^\infty(\Omega^g (3))} \| \phi \|_{L^\infty(t_1, t_2)} \int_{\Omega^g (x) \times (t_1, t_2)} | u - w | h \, dz.
\]

### A.1 Construction of test function

Let us denote the following functions:

\[ v(z) := u(z) - w(z) \quad \text{and} \quad v_h(z) := [u - w]_h (z), \]

where \( [u - w]_h (z) \) denotes the usual Steklov average. It is easy to see that \( v_h \xrightarrow{h \to 0} v \). We also note that \( v(z) = 0 \) for \( z \in \partial_p \Omega^g (3) \). For some fixed \( q \) such that \( 1 < q < \frac{p}{p + 1} \), with \( M \) as given in (5.2), let us now define

\[
g(z) := M \left( \frac{|v|}{\alpha - \frac{1}{p+1} + \frac{d}{2} \rho} + |\nabla u| + |\nabla w| + |f| + 1 \right)^{\frac{p-1}{q}} \chi_{\Omega^g (3)}(\rho). \quad (A.2)
\]

For a fixed \( \lambda \geq 1 \), let us define the good set by

\[ E_\lambda := \{ z \in \mathbb{R}^{n+1} : g(z) \leq \lambda^{1-\beta} \}. \quad (A.3) \]

For the rest of this section, we will always assume that the following bound holds:

**Lemma A.3** With \( \rho \leq \bar{\rho}_3 \), there holds

\[
\rho \geq \| p^+_{\Omega^g (3)} - p^-_{\Omega^g (3)} \| \leq C_{(p^\pm \log^n n)}.
\]

**Proof** Since \( p(\cdot) \in p^\pm \log \), we have from Remark 2.4,

\[
p^+_{\Omega^g (3)} - p^-_{\Omega^g (3)} \leq \omega_p(\cdot) \left( \max \left\{ 8\alpha - \frac{1}{p+1} + \frac{d}{2} \rho, \sqrt{\alpha - 1 + d} 32\rho^2 \right\} \right) \leq \omega_p(32\rho).
\]

Since \( \rho \leq 1 \), we only need to bound \( \rho^{-1}(p^+_{\Omega^g (3)} - p^-_{\Omega^g (3)}) \), which we do as follows:

\[
\rho^{-1}(p^+_{\Omega^g (3)} - p^-_{\Omega^g (3)}) \leq \rho^{-32\omega_p(\rho) \log \frac{1}{\rho}} \leq C_{(p^\pm \log^n n)}.
\]

This completes the proof of the lemma.

Following the ideas from [3, Lemma 5.10], we can obtain a Vitali-type covering lemma.

**Lemma A.4** Let \( \lambda \geq 1 \) be such that (A.3) is given, then for every \( z \in K_{\Omega^g (3)} \setminus E_\lambda \), consider the parabolic cylinders of the form

\[
Q^\lambda_{\rho, z}(z) := B_{\lambda^{\frac{1}{p(\cdot)} + \frac{d}{2} \rho_{\rho z}}} (x) \times (t - \lambda^{\frac{1-d}{2}} \rho_{\rho z}, t + \lambda^{\frac{1-d}{2}} \rho_{\rho z}^2)
\]
where $\rho_z := d_z^\lambda(z, E_\lambda) := \inf_{\tilde{z} \in E_\lambda} d_z^\lambda(z, \tilde{z})$. Let $\xi \in (0, 1]$ be a given constant and consider the open covering of $K_{4\rho}^\alpha(\lambda) \setminus E_\lambda$ given by
\[
\mathcal{F} := \left\{ Q_{\xi\rho_i}(z) \right\}_{z \in K_{4\rho}^\alpha(\lambda) \setminus E_\lambda}.
\]

Then there exists a universal constant $X = X(p_{\log}^+, n) \geq 9$ and a countable disjoint subcollection $G := \{ Q_{\rho_i}(z_i) \}_{i \in \mathbb{N}} \subset \mathcal{F}$ such that there holds
\[
\bigcup_{G} Q_{\rho_i}(z) \subset \bigcup_{\mathcal{F}} Q_{\xi\rho_i}(z_i).
\]

We now have the following Whitney type covering whose proof is very similar to [3, Lemma 5.11].

**Lemma A.5** There exists a universal constant $\delta \in (0, 1/4)$ such that for $\mathcal{F}$, a given covering of $K_{4\rho}^\alpha(\lambda) \setminus E_\lambda$ given by the cylinders: $\mathcal{F} := \left\{ Q_{\delta\rho_i}(z) \right\}_{z \in K_{4\rho}^\alpha(\lambda) \setminus E_\lambda}$, where $X$ is the constant from Lemma A.4, there exists a countable disjoint subcollection $G := \{ Q_{\delta\rho_i}(z_i) \}_{i \in \mathbb{N}} = \{ Q_{\rho_i}(z_i) \}_{i \in \mathbb{N}}$ subordinate to the covering $\mathcal{F}$ such that the following holds:

(W1) $K_{4\rho}^\alpha(\lambda) \setminus E_\lambda \subseteq \bigcup_{i \in \mathbb{N}} Q_i$.

(W2) Each point $z \in K_{4\rho}^\alpha(\lambda) \setminus E_\lambda$ belongs to utmost $C_{n, p_{\log}^+}$ cylinders of the form $2Q_i$.

(W3) There exists a constant $C = C_{n, p_{\log}^+}$ such that for any two cylinders $Q_i$ and $Q_j$ with $2Q_i \cap 2Q_j \neq \emptyset$, there holds
\[
|B_i| \leq C|B_j| \leq C|B_i| \quad \text{and} \quad |I_i| \leq C|I_j| \leq C|I_i|.
\]

In particular, there holds $|Q_i| \approx_{(p_{\log}^+, n)} |Q_j|$.

(W4) There exists a constant $\hat{c} = \hat{c}_{(p_{\log}^+, n)} \geq 9$ such that for all $i \in \mathbb{N}$, there holds:
\[
\hat{c}Q_i \subset \mathbb{R}^{n+1} \setminus E_\lambda \quad \text{and} \quad 8\hat{c}Q_i \cap E_\lambda \neq \emptyset.
\]

(W5) For the constant $\hat{c}$ from above, there holds $2Q_i \cap 2Q_j \neq \emptyset$ implies $2Q_i \subset \hat{c}Q_j$.

Once we have obtained the Whitney type covering lemma, we can now obtain the following standard partition of unity lemma:

**Lemma A.6** Subordinate to the covering $G$ obtained in Lemma A.5, we obtain a partition of unity $\{ \psi_i \}_{i=1}^\infty$ on $\mathbb{R}^{n+1} \setminus E_\lambda$ that satisfies the following properties:

- $\sum_{i=1}^\infty \psi_i(z) = 1$ for all $z \in K_{4\rho}^\alpha(\lambda) \setminus E_\lambda$.
- $\psi_i \in C_c^\infty(2Q_i)$.
- $\| \psi_i \|_\infty + \lambda^{-\frac{d}{p_{\log}^+} + \frac{d}{2}} r_i \| \nabla \psi_i \|_\infty + \lambda^{-1 + d/r_i} \| \partial_t \psi_i \|_\infty \leq C_{(p_{\log}^+, n)}$ where we have used the notation $r_i := \rho_{\partial_t}$ which is the parabolic radius of $Q_i$ with respect to the metric $d_z^\lambda$ (see Lemma A.5 for the notation).
- $\psi_i \geq C_{(p_{\log}^+, n)}$ on $Q_i$.

Before we end this subsection, let us recall the following useful bound that will be used throughout this section. For a proof, see the proof of [3, Lemma 5.10, (5.23)].
\[
\lambda^{p_{\log}^+ - p_{2Q_i}} \leq C_{(p_{\log}^+, n)} ,
\]
A.2 Construction of Lipschitz truncation function

Let us first clarify some of the notation that will subsequently be used in the rest of this section: for $\hat{c}$ from (W4), we denote

$$\hat{Q}_i := \hat{Q}_{\hat{r}_i}(z_i),$$

where $\hat{r}_i := \hat{c}r_i$.

We shall also use the notation

$$I(i) := \{ j \in \mathbb{N} : \text{spt}(\psi_i) \cap \text{spt}(\psi_j) \neq \emptyset \} \quad \text{and} \quad I_z := \{ j \in \mathbb{N} : z \in \text{spt}(\psi_j) \}.$$

We are now ready to construct the Lipschitz truncation function:

$$v_{\lambda,h}(z) := v_h(z) - \sum_i \psi_i(z) \left( v_h(z) - v_i^j \right),$$

(A.5)

where $v^j_i$ is defined as:

$$v^j_i := \left\{ \begin{array}{ll}
\int_{2Q_i} v_h(z) \chi_{[t-s,t+s]} \, dz & \text{if } 2Q_i \subset \Omega^\alpha_{4\rho}(x) \times (t-s, \infty), \\
0 & \text{else}.
\end{array} \right.$$

(A.6)

From construction in (A.5) and (A.6), we see that

$$\text{spt}(v_{\lambda,h}) \subset \Omega^\alpha_{4\rho}(x) \times (t-s, \infty).$$

We see that $v_{\lambda,h}$ has the right support for the test function and hence the rest of this section will be devoted to proving the Lipschitz regularity of $v_{\lambda,h}$ on $K^\alpha_{4\rho}(x)$ as well as some useful estimates.

A.3 Some estimates on the test function

In this subsection, we will collect some useful estimates on the test function. The proofs of these estimates follow similarly to those in [3] and hence we will only provide an outline of the proofs.

Lemma A.7 Let $z \in K^\alpha_{4\rho}(x) \setminus E_\lambda$, then from (W1), we have that $z \in 2Q_i$ for some $i \in I_z$. For any $1 \leq \theta \leq \frac{p}{q}$, there holds

$$|v^i_h|^\theta \leq \left\{ \begin{array}{ll}
\int_{2Q_i} |v_h(z)|^\theta \chi_{[t-s,t+s]} \, dz & \text{if } 2Q_i \subset \Omega^\alpha_{4\rho}(x) \times (t-s, \infty), \\
0 & \text{else}.
\end{array} \right.$$

(A.7)

$$\int_{2Q_i} |\nabla v_h(z)|^\theta \chi_{[t-s,t+s]} \, dz \lesssim \left( \int_{\log \rho} \frac{dz}{z^{\frac{4}{q}}} \right)^{\frac{1}{q}} \left( \int_{\log \rho} \frac{dz}{z^{\frac{4}{q}}} \right)^{\frac{4}{q}} \lesssim \left( \int_{\log \rho} \frac{dz}{z^{\frac{4}{q}}} \right)^{\frac{1}{q}} \left( \int_{\log \rho} \frac{dz}{z^{\frac{4}{q}}} \right)^{\frac{4}{q}}.$$

(A.8)

Proof Proof of (A.7): We prove this estimate as follows:

$$|v^i_h|^\theta \lesssim \left( \alpha - \frac{1}{\rho^{3^2}} \right)^\theta \left( \frac{\int_{2Q_i} |v(z)|^\theta \chi_{[t-s,t+s]} \, dz}{\int_{\log \rho} \frac{dz}{z^{\frac{4}{q}}} \left( \alpha - \frac{1}{\rho^{3^2}} \right)^\theta \left( \int_{\log \rho} \frac{dz}{z^{\frac{4}{q}}} \right)^{\frac{4}{q}} \right)^{\frac{1}{q}} \lesssim \left( \alpha - \frac{1}{\rho^{3^2}} \right)^\theta \left( \int_{\log \rho} \frac{dz}{z^{\frac{4}{q}}} \right)^{\frac{1}{q}} \left( \int_{\log \rho} \frac{dz}{z^{\frac{4}{q}}} \right)^{\frac{4}{q}}.$$
Proof of (A.8): From (A.3), we see that

$$
\int_{2Q_i} |\nabla v_h|^\theta \chi_{[t-s,t+s]} \, d\tilde{z} \lesssim \left( \int_{8\varepsilon Q_i} |\nabla v| + 1 \right)^{\theta p} d\tilde{z} \lesssim \lambda^\theta \frac{\rho}{r_{Q_i}^{\theta}} \lesssim \lambda^{\theta p/\gamma}.
$$

**Corollary A.8** For any $z \in K_{4\rho}^\alpha (z) \setminus E$, we have $z \in 2Q_i$ for some $i \in I$, then there holds

$$
|v_h(z)| \lesssim_{(\nu, p, \lambda_0, \Lambda_1)} (\alpha - \frac{1}{m_3} + \frac{d}{\rho}) \lambda \frac{1}{\rho_1},
$$

where $z_i$ is the centre of $Q_i$.

**Lemma A.9** Let $2Q_i$ be a parabolic Whitney type cylinder, then for any $1 \leq \theta \leq \frac{p}{q}$, there holds

$$
\int_{2Q_i} |v_h(z)\chi_{[t-s,t+s]} - v_h| \chi_{[t-s,t+s]} |^\theta \, d\tilde{z} \lesssim_{(p, \lambda_0, \Lambda_1, n)} \lambda \frac{1}{\rho_1}.
$$

**Proof** Let us consider the following two cases:

Case $\alpha - \frac{1}{m_3} + \frac{d}{\rho} \leq \lambda - \frac{1}{m_3} + \frac{d}{\rho_{Q_i}}$:

In this case, we can use triangle inequality along with (A.7) to get

$$
\int_{2Q_i} |v_h(z)\chi_{[t-s,t+s]} - v_h| \chi_{[t-s,t+s]} |^\theta \, d\tilde{z} \lesssim 2 \int_{2Q_i} |v_h(z)|^\theta \chi_{[t-s,t+s]} \, d\tilde{z} \lesssim (\alpha - \frac{1}{m_3} + \frac{d}{\rho}) \lambda \frac{1}{\rho_{Q_i}}.
$$

Case $\alpha - \frac{1}{m_3} + \frac{d}{\rho} \geq \lambda - \frac{1}{m_3} + \frac{d}{\rho_{Q_i}}$:

Applying Lemma A.2 with $\mu \in C_c(2B_i)$ such that $|\mu(x)| \lesssim \frac{1}{\lambda \frac{1}{m_3} + \frac{d}{\rho_{Q_i}}}$ and $|\nabla \mu(x)| \lesssim \frac{1}{\left(\frac{1}{m_3} + \frac{d}{\rho_{Q_i}}\right)^{\pi+\pi}}$, we get

$$
\int_{2Q_i} |v_h(z)\chi_{[t-s,t+s]} - v_h| \chi_{[t-s,t+s]} |^\theta \, d\tilde{z} \lesssim \left( \lambda - \frac{1}{m_3} + \frac{d}{\rho_{Q_i}} \right)^\theta \int_{2Q_i} |\nabla v_h|^\theta \chi_{[t-s,t+s]} \, d\tilde{z} + \sup_{t_1, t_2 \in 2I_i \cap [t-s,t+s]} |(v_h)_x (t_2) - (v_h)_x (t_1)|^\theta.
$$

The first term on the right of (A.10) can be estimated using (A.8) to get

$$
\left( \lambda - \frac{1}{m_3} + \frac{d}{\rho_{Q_i}} \right)^\theta \int_{2Q_i} |\nabla v_h|^\theta \chi_{[t-s,t+s]} \, d\tilde{z} \lesssim \left( \lambda - \frac{1}{m_3} + \frac{d}{\rho_{Q_i}} \right)^\theta \lambda \frac{1}{\rho_{Q_i}}.
$$

(A.11)
To estimate the second term on the right of (A.10), we make use of Lemma A.2 with \( \phi(x) = \mu(x) \) and \( \varphi(t) = 1 \), we get

\[
| (v_h)_\mu(t_2) - (v_h)_\mu(t_1) | \lesssim \frac{|2Q_i|}{\lambda^{-1 + d/2 - \delta} r_i} \ll \frac{|2Q_i|}{\lambda^{-1 + d/2 - \delta} r_i} \ll \frac{1}{(1 + |\nabla u| + |\nabla w| + |f|)(\frac{p(\tilde{z})}{\delta})^{-1}} \frac{q p_2^{-1} \ell_2^{-1}}{p_2 Q_i}.
\]

Now making use of (A.4) along with the fact that \( \lambda \geq 1 \) and \( p_2 Q_i \leq p(z_i) \), we get

\[
\lambda^{-1 + d/2 - \delta} r_i \ll \lambda^{-1 + d/2 - \delta} r_i \ll \lambda^{-1 + d/2 - \delta} r_i \ll C(p^{\log}_{\Lambda} n), \quad (A.13)
\]

Substituting (A.13) into (A.12), we get

\[
| (v_h)_\mu(t_2) - (v_h)_\mu(t_1) | \lesssim (\lambda^{-1 + d/2 - \delta} r_i \lambda^{-1 + d/2 - \delta} r_i) \ll (\lambda^{-1 + d/2 - \delta} r_i \lambda^{-1 + d/2 - \delta} r_i).
\]

Thus combining (A.11) and (A.14) into (A.10), we get

\[
\int_{2Q_i} |v_h(\tilde{z}) x_{[1-s, t+s]} - v_h|^{\theta} d\tilde{z} \lesssim (p^{\log}_{\Lambda} n, \lambda^{-1 + d/2 - \delta} r_i) \lambda^{-1 + d/2 - \delta} r_i, \quad (A.14)
\]

which proves the lemma.

**Corollary A.10** For any \( i \in \mathbb{N} \) and any \( j \in I_i \), there holds

\[
|v_j - v_i| \lesssim (p^{\log}_{\Lambda} n, \lambda^{-1 + d/2 - \delta} r_i) \min \left\{ \alpha^{-1 + d/2 - \delta} \rho, \lambda^{-1 + d/2 - \delta} r_i \right\} \lambda^{-1 + d/2 - \delta} r_i.
\]

**A.4 Bounds on** \( v_{\lambda,h} \) **and** \( \nabla v_{\lambda,h} \)

**Lemma A.11** Let \( Q_i \) be a parabolic Whitney type cylinder. Then for any \( z \in 2Q_i \), we have the following bound:

\[
\left( \frac{1}{\alpha^{-1 + d/2 - \delta} \rho} \right) |v_{\lambda,h}(z)| + |\nabla v_{\lambda,h}(z)| \lesssim (p^{\log}_{\Lambda} n, \lambda^{-1 + d/2 - \delta} r_i) \lambda^{-1 + d/2 - \delta} r_i. \quad (A.15)
\]

**Corollary A.12** Let \( z \in K^{\delta}_{4p}(\mathcal{I}) \setminus E_\delta \), then \( z \in 2Q_i \) for some \( i \in \mathbb{N} \). Then there holds for any \( \delta \in (0, 1] \), the estimates

\[
\frac{1}{\lambda^{-1 + d/2 - \delta} r_i} |v_{\lambda,h}(z)| \lesssim (p^{\log}_{\Lambda} n, \lambda^{-1 + d/2 - \delta} r_i) \lambda^{-1 + d/2 - \delta} r_i, \quad \lambda^{-1 + d/2 - \delta} r_i \frac{1}{\delta} |v_j|^2, \quad (A.16)
\]

\[
|\nabla v_{\lambda,h}(z)| \lesssim (p^{\log}_{\Lambda} n, \lambda^{-1 + d/2 - \delta} r_i) \lambda^{-1 + d/2 - \delta} r_i \frac{1}{\delta}. \quad (A.17)
\]
Lemma A.13 Let \( z \in K_{4 \rho}(\zeta) \setminus E_\lambda \), then \( z \in 2 Q_i \) for some \( i \in \mathbb{N} \). Then there holds for any \( \delta \in (0, 1] \), the estimates

\[
|v_{\lambda, h}(z)| \lesssim_{(p_{\log}^\pm, \Lambda_0, \Lambda_1, n)} \left( \frac{\lambda^{-\frac{1}{\rho(i)} + \frac{d}{2} r_i}}{\delta} \right) + \left( \frac{\delta}{\lambda^{-\frac{1}{\rho(i)} + \frac{d}{2} r_i}} \right) \mathcal{I}^h_{Q_i} |v_h(\tilde{z})|^2 d\tilde{z},
\]

\[
|\nabla v_{\lambda, h}(z)| \lesssim_{(p_{\log}^\pm, \Lambda_0, \Lambda_1, n)} \lambda^{\frac{1}{\rho(i)}} + \left( \frac{\delta}{\lambda^{-\frac{1}{\rho(i)} + \frac{d}{2} r_i}} \right) \mathcal{I}^h_{Q_i} |v_h(\tilde{z})|^2 d\tilde{z}.
\]

(A.18)

A.5 Estimates on the time derivative of \( v_{\lambda, h} \)

Lemma A.14 Let \( z \in K_{4 \rho}(\zeta) \), then \( z \in 2 Q_i \) for some \( i \in \mathbb{N} \). We then have the following estimates for the time derivative of \( v_{\lambda, h} \):

\[
|\partial_t v_{\lambda, h}(\tilde{z})| \lesssim_{(p_{\log}^\pm, \Lambda_0, \Lambda_1, n)} \left( \frac{1}{\lambda^{-1+d} r_i^2} \right) \mathcal{I}^h_{Q_i} |v_h(z)| \chi_{[s \to s]} dz.
\]

We also have the improved estimate

\[
|\partial_t v_{\lambda, h}(\tilde{z})| \lesssim_{(p_{\log}^\pm, \Lambda_0, \Lambda_1, n)} \left( \frac{1}{\lambda^{-1+d} r_i^2} \right) \min \left\{ \left( \frac{\lambda^{-\frac{1}{\rho(i)} + \frac{d}{2} r_i}}{\alpha^{-\frac{1}{\rho(i)} + \frac{d}{2} \rho} - \frac{1}{\rho(i)} + \frac{d}{2} r_i} \right) \right\}.
\]

Proof Let us prove each of the assertions as follows:

• Estimate (A.19): In this case, we proceed as follows

\[
|\partial_t v_{\lambda, h}(z)| \leq \sum_{j \in I_i} |v_j| |\partial_t \psi_j(z)| \lesssim \left( \frac{1}{\lambda^{-1+d} r_i^2} \right) \mathcal{I}^h_{Q_i} |v_h(z)| \chi_{[s \to s]} dz.
\]

• Estimate (A.20): From the fact that \( \sum_{j \in I_i} \partial_t \psi_j(z) = 1 \), we see that \( \sum_{j \in I_i} \partial_t \psi_j(z) = 0 \) which along with Lemma A.6 gives the following sequence of estimates

\[
|\partial_t v_{\lambda, h}(z)| \leq \left( \sum_{j \in I_i} \left( v_j - v_h \right) \partial_t \psi_j(z) \right) \lesssim \left( \frac{1}{\lambda^{-1+d} r_i^2} \right) \min \left\{ \left( \frac{\alpha^{-\frac{1}{\rho(i)} + \frac{d}{2} \rho} - \frac{1}{\rho(i)} + \frac{d}{2} \rho}{\alpha^{-\frac{1}{\rho(i)} + \frac{d}{2} \rho}} \right) \right\}.
\]

A.6 Some important estimates for the test function

Lemma A.15 Let \( Q_i \) be a Whitney-type parabolic cylinder for some \( i \in \mathbb{N} \). Then for any \( \vartheta \in [1, 2] \), there holds

\[
\int_{K_{4 \rho}(\zeta) \setminus E_\lambda} |v_{\lambda, h}(z)|^\vartheta dz \lesssim_{(p_{\log}^\pm, \Lambda_0, \Lambda_1, n)} \int_{K_{4 \rho}(\zeta) \setminus E_\lambda} |v_h(z)|^\vartheta dz.
\]

Lemma A.16 Let \( Q_i \) be a Whitney-type parabolic cylinder for some \( i \in \mathbb{N} \), then there holds

\[
\int_{Q_i} |v_{\lambda, h}(z) - uh(z)| dz \lesssim_{(p_{\log}^\pm, \Lambda_0, \Lambda_1, n)} \min \left\{ \left( \frac{\lambda^{-\frac{1}{\rho(i)} + \frac{d}{2} r_i}}{\alpha^{-\frac{1}{\rho(i)} + \frac{d}{2} \rho} - \frac{1}{\rho(i)} + \frac{d}{2} \rho} \right) \right\}.
\]
Lemma A.17 Let $Q_i$ be a Whitney-type parabolic cylinder for some $i \in \mathbb{N}$, then there holds
\[
\left\| \int_{K_{\rho_{i}}^{\alpha}(\beta)} \partial_{\nu} v_{\lambda, h}(z) \left( v_{\lambda, h}(z) - v_{h}(z) \right) \chi_{K_{\rho_{i}}^{\alpha}(\beta)} \, dz \right\|_{L^{\infty}(\mathbb{R}^{n+1} \setminus E_{\lambda})} \lesssim \left\langle \rho_{i} \right\rangle^{\theta} \lambda^{\theta} |E_{\lambda}|.
\]

Proof From (W2), we see that $K_{\rho_{i}}^{\alpha}(\beta) \setminus E_{\lambda} \subset \bigcup 2Q_i$, thus for a given $i \in \mathbb{N}$, let us define the following
\[
J_i := \int_{2Q_i} \left| \partial_{\nu} v_{\lambda, h}(z) \left( v_{\lambda, h}(z) - v_{h}(z) \right) \right|^{\theta} \chi_{K_{\rho_{i}}^{\alpha}(\beta)} \, dz.
\]
Making use of (A.20), we get
\[
J_i \lesssim \left( \frac{1}{\lambda^{1+d/r_{i}^{2}}} \min \left\{ \alpha - \frac{1}{\rho_{i}^{2}} + \frac{d}{\rho_{i}} , \lambda - \frac{1}{\rho_{i}^{2}} + \frac{d}{\rho_{i}} \right\} \right)^{\theta} \int_{2Q_i} \left| v_{\lambda, h}(z) \chi_{K_{\rho_{i}}^{\alpha}(\beta)} - v_{h}(z) \chi_{K_{\rho_{i}}^{\alpha}(\beta)} \right|^{\theta} \, dz.
\]
\[
\lesssim \left( \frac{1}{\lambda^{1+d/r_{i}^{2}}} \min \left\{ \alpha - \frac{1}{\rho_{i}^{2}} + \frac{d}{\rho_{i}} , \lambda - \frac{1}{\rho_{i}^{2}} + \frac{d}{\rho_{i}} \right\} \right)^{\theta} \sum_{j \in I_i} \int_{2Q_i} \left| v_{h}(z) \chi_{K_{\rho_{i}}^{\alpha}(\beta)} - v_{j} \right|^{\theta} \, dz.
\]
\[
\lesssim \left( \frac{1}{\lambda^{1+d/r_{i}^{2}}} \min \left\{ \alpha - \frac{1}{\rho_{i}^{2}} + \frac{d}{\rho_{i}} , \lambda - \frac{1}{\rho_{i}^{2}} + \frac{d}{\rho_{i}} \right\} \right)^{\theta} \left| \hat{Q}_i \right| = \left| \hat{Q}_i \right|^{\theta}.
\]
Summing over all $i \in \mathbb{N}$, we get the desired inequality.

A.7 Lipschitz continuity estimates

We will now show that the function $v_{\lambda}$ constructed in (A.5) is Lipschitz continuous on $B_{4\rho}(\bar{x}) \times (t-s, t+s)$ where $s$ is as defined in (A.1). To do this, we shall use the integral characterization of Lipschitz continuous functions obtained in [35, Theorem 3.1] which says the following:

Lemma A.18 (Lipschitz characterization) Let $\tilde{z} \in B_{4\rho}(\bar{x}) \times (t-s, t+s)$ and $r > 0$ be given. Define the parabolic cylinder $Q_r(\tilde{z}) := B_r(\bar{z}) \times (\tilde{t} - r^2, \tilde{t} + r^2)$, i.e., $Q_r(\tilde{z}) := \{ z \in \mathbb{R}^{n+1} : d_{\rho}(z, \tilde{z}) \leq r \}$ where $d_{\rho}$ is as defined in Definition 2.1. Furthermore suppose that the following expression is bounded independent of $\tilde{z} \in B_{4\rho}(\bar{x}) \times (t-s, t+s)$ and $r > 0$
\[
I_{r}(\tilde{z}) := \frac{1}{|B_{4\rho}(\bar{x}) \times (t-s, t+s) \cap Q_r(\tilde{z})|} \int_{B_{4\rho}(\bar{x}) \times (t-s, t+s) \cap Q_r(\tilde{z})} \left| v_{\lambda, h}(z) - \left( v_{\lambda, h} \right)_{B_{4\rho}(\bar{x}) \times (t-s, t+s) \cap Q_r(\tilde{z})} \right| \, dz < \infty,
\]
then $v_{\lambda} \in C^{0,1}(B_{4\rho}(\bar{x}) \times (t-s, t+s))$.

Remark A.19 From (2.7) and the fact that $\alpha \geq 1$, for any $\tilde{z}_1, \tilde{z}_2 \in \mathbb{R}^{n+1}$ and any $\tilde{z} \in \mathbb{R}^{n+1}$, we get
\[
d_{\rho}(\tilde{z}_1, \tilde{z}_2) \overset{\text{Definition 2.1}}{=} \max \left\{ |x_1 - x_2|, \sqrt{|t_1 - t_2|} \right\} \leq \max \left\{ \alpha^{-\frac{d}{2}} \frac{|x_1 - x_2|}{|t_1 - t_2|}, \sqrt{\alpha^{1-d}|t_1 - t_2|} \right\} \overset{\text{Definition 2.2}}{=} d_{\rho}(\tilde{z}_1, \tilde{z}_2) \leq \alpha^{-\frac{1}{2}} \alpha^{\frac{d}{2}} \max \left\{ |x_1 - x_2|, \sqrt{|t_1 - t_2|} \right\} \leq C(\alpha, p, \alpha) d_{\rho}(\tilde{z}_1, \tilde{z}_2),
\]
(A.21)
This shows that for any $\tilde{z} \in \mathbb{R}^{n+1}$, we have $d_p \approx_{\alpha} d_{\tilde{z}}$. 

In this subsection, we want to apply Lemma A.18, hence we only need to ensure the constants involved are independent of $r > 0$ and $\tilde{z}$ only. Only for this subsection, we will use the notation $o(1)$ to denote a constant which can depend on $\alpha, \alpha_0, p^\pm, \Lambda_0, \Lambda_1, n, \|uh\|_{L^1}, \|u\|_{L^1}$ but NOT on $r > 0$ and the point $\tilde{z}$.

Lemma A.20 Let $\alpha \geq 1$, then for any $\tilde{z} \in K_{4\rho_0}(3)$ and $r > 0$, there exists a constant $C > 0$ independent of $\tilde{z}$ and $r$ such that

$$I_r(\tilde{z}) := \frac{1}{|K_{4\rho_0}(3) \cap Q_r(\tilde{z})|} \int_{K_{4\rho_0}(3) \cap Q_r(\tilde{z})} \frac{|v_{\alpha,h}(z) - (v_{\alpha,h})_{K_{4\rho_0}(3) \cap Q_r(\tilde{z})}|}{r} \, dz \leq C < \infty.$$

In particular, this implies for any $\tilde{z}_1, \tilde{z}_2 \in B_{4\rho_0}(3) \times (t - s, t + s)$, there exists a constant $K > 0$ such that

$$|v_{\alpha,h}(\tilde{z}_1) - v_{\alpha,h}(\tilde{z}_2)| \leq K d_p(\tilde{z}_1, \tilde{z}_2).$$

Proof Let $r > 0$ and $\tilde{z} \in K_{4\rho_0}(3)$ and denote the cylinder $Q_r(\tilde{z}) = Q$. We will now proceed as follows: [Case 2 $Q \subseteq E^c_\lambda$:] From (A.5), it is easy to see that $v_{\alpha,h} \in C^\infty(E^c_\lambda)$. Thus, we can apply the mean value theorem to get

$$I_r(\tilde{z}) \lesssim \frac{1}{r} \int_{Q \cap (\mathbb{R}^n \times [t-s, t+s])} \int_{Q \cap (\mathbb{R}^n \times [t-s, t+s])} |v_{\alpha,h}(z_1) - v_{\alpha,h}(z_2)| \, dz_1 \, dz_2 \lesssim \sup_{z \in (\mathbb{R}^n \times [t-s, t+s])} \left( |\nabla v_{\alpha,h}(z)| + r|\partial_r v_{\alpha,h}(z)| \right).$$

(A.22)

Since $2Q \subseteq E^c_\lambda$, we can use (A.17) with $\delta = 1$ and (A.20) to bound (A.22) as follows:

$$I_r(\tilde{z}) \lesssim \sup_{z \in (\mathbb{R}^n \times [t-s, t+s])} \left( \lambda^{-\frac{1}{p-1}} + r \frac{\lambda^\frac{d}{p} \rho_l}{\lambda^{1+dr_i^2}} \right).$$

(A.23)

Here we recall that $z \in 2Q_i$ for some $i \in \mathbb{N}$ and $r_i$ is the radius of the cylinder $Q_i$.

Since $Q \subseteq E^c_\lambda$, we also have that $z \in 2Q_i$ for some $i \in \mathbb{N}$. Let $z_i$ be the centre of $Q_i$, then we have

$$r \leq d_p(z, E_\lambda) \leq d_p(z, z_i) + d_p(z_i, E_\lambda) \leq r_i + d_{z_i}(z_i, E_\lambda) \overset{(A.21)}{\leq} r_i + \hat{c}r_i = (1 + \hat{c})r_i.$$  \hspace{1cm} (A.24)

Substituting (A.24) into (A.23), we get

$$I_r(\tilde{z}) \lesssim \lambda^{-\frac{1}{p-1}} + (1 + \hat{c})\lambda^{\frac{d}{p} - \frac{d}{2}} = o(1).$$

[Case 2 $Q \not\subseteq E^c_\lambda$:] In this case, we split the proof into three subcases as follows: [Subcase 2 $Q \subseteq \mathbb{R}^n \times (-\infty, s]$ or $2Q \subseteq \mathbb{R}^n \times [-s, \infty)$:] In this situation, it is easy to see that the following holds:

$$|Q \cap (\mathbb{R}^n \times [t-s, t+s])| \gtrsim |Q|.$$  \hspace{1cm} (A.25)

We apply triangle inequality and estimate $I_r(\tilde{z})$ by

$$I_r(\tilde{z}) \leq 2J_1 + J_2,$$
where we have set
\[ J_1 := \iint_{Q \cap (\mathbb{R}^n \times [t-s, t+s])} \frac{v_{\lambda, d}(z) - v_b(z)}{r} \, dz, \]
\[ J_2 := \iint_{Q \cap (\mathbb{R}^n \times [t-s, t+s])} \frac{v_b(z) - (v_b)_Q \cap (\mathbb{R}^n \times [t-s, t+s])}{r} \, dz. \] (A.26)

We now estimate each of the terms of (A.26) as follows: Estimate for \( J_1 \): From (A.5), we get
\[ J_1 \lesssim \sum_{i \in \mathbb{N}} \left[ \frac{1}{|Q \cap (\mathbb{R}^n \times [t-s, t+s])|} \iint_{Q \cap (\mathbb{R}^n \times [t-s, t+s])} \frac{v_b(z) - (v_b)_Q \cap (\mathbb{R}^n \times [t-s, t+s])}{r} \, dz \right] \]
\[ = \sum_{i \in \mathbb{N}} \left[ \frac{1}{|Q \cap (\mathbb{R}^n \times [t-s, t+s])|} \iint_{Q \cap (\mathbb{R}^n \times [t-s, t+s])} \frac{v_b(z) - (v_b)_Q \cap (\mathbb{R}^n \times [t-s, t+s])}{r} \, dz \right]. \] (A.27)

Let us fix an \( i \in \mathbb{N} \) and take two points \( z_1 \in Q \cap 2Q_i \) and \( z_2 \in E_\lambda \cap 2Q_i \). Making use of (W5) along with the trivial bound \( d_p(z_1, z_2) \leq 4r \) and \( d_p(z_i, z_1) \leq 2r_i \), we get
\[ \hat{c}_r_i = d_p(z_i, E_\lambda) \leq d_p(z_i, z_1) + d_p(z_1, z_2) \leq 2r_i + 4r \implies r_i \lesssim (\hat{c}_r) r, \] (A.28)
where \( \hat{c}_r_i \) denotes the centre of \( Q_i \) as in (W2) and \( \hat{c} \) is from (W4).

Note that (A.25) holds and thus summing over all \( i \in \mathbb{N} \) such that \( Q \cap (\mathbb{R}^n \times [t-s, t+s]) \cap 2Q_i \neq \emptyset \) in (A.27) and making use of (A.28), we get
\[ J_1 \lesssim o(1). \]

Estimate for \( J_2 \): To estimate this term, we proceed as follows: Note that \( Q \cap (\mathbb{R}^n \times [t-s, t+s]) \) is another cylinder. If \( Q \subset B^{\epsilon}_{4\rho}(x) \times \mathbb{R} \), then choose a cut-off function \( \mu \in C^\infty_c(B) \) with \( |\nabla \mu| \leq \frac{C_{\alpha}}{\rho^{\alpha+1}} \) to get
\[ J_2 = \iint_{Q \cap (\mathbb{R}^n \times [t-s, t+s])} \frac{v_b(z)\chi_{[t-s, t+s]}}{r} \left( \frac{v_b\chi_{[t-s, t+s]}(Q \cap (\mathbb{R}^n \times [t-s, t+s]))}{r} \right) \, dz \]
\[ \lesssim \iint_{Q \cap (\mathbb{R}^n \times [t-s, t+s])} \frac{\nabla v_b \chi_{[t-s, t+s]}}{r} + \sup_{t_1, t_2 \in [t-s, t+s]} \nabla \frac{v_b\chi_{[t-s, t+s]}(Q \cap (\mathbb{R}^n \times [t-s, t+s]))}{r} \, dz. \]

Recall that we are in the case \( 2Q \cap E_\lambda \neq \emptyset \) and \( 2Q \cap E_\lambda^c \neq \emptyset \). Further applying Lemma A.2 and proceeding similarly to (A.12), we see that
\[ J_2 \lesssim o(1). \]

On the other hand, if \( Q \not\subset B^{\epsilon}_{4\rho}(x) \times \mathbb{R} \), then we can apply Poincaré’s inequality directly to get
\[ J_2 \lesssim \iint_{Q \cap (\mathbb{R}^n \times [t-s, t+s])} \frac{v_b(z)\chi_{[t-s, t+s]}}{r} \, dz \lesssim \iint_{Q \cap (\mathbb{R}^n \times [t-s, t+s])} \left| \frac{\nabla v_b(z)\chi_{[t-s, t+s]}}{r} \right| \, dz. \]
Recall that we are in the case $2Q \cap \mathcal{E}_\lambda \neq \emptyset$ and $2Q \cap E^c_\lambda \neq \emptyset$. Using (A.25), we thus get
\begin{equation}
J_2 \lesssim o(1). \tag{A.29}
\end{equation}

[Subcase 2] $2Q \cap \mathbb{R}^n \neq \emptyset$ and $2Q \cap \mathbb{R}^n \times (-s, \infty) \neq \emptyset$ AND $r^2 \leq s.$ In this case, we see that
\[ |Q \cap (\mathbb{R}^n \times [t-s, t+s])| = |B| \times s. \]

We apply triangle inequality and estimate $I_r(z)$ by
\[ I_r(z) \leq 2J_1 + J_2, \]
where we have set
\begin{align*}
J_1 &:= \mathcal{F}_Q(\mathbb{R}^n \times [t-s, t+s]) \left| \frac{v_{\lambda, h}(z) - v_h(z)}{r} \right| dz, \\
J_2 &:= \mathcal{F}_Q(\mathbb{R}^n \times [t-s, t+s]) \left| \frac{v_h(z) - (v_h)_Q(\mathbb{R}^n \times [t-s, t+s])}{r} \right| dz.
\end{align*}

Proceeding as before, we get
\begin{align*}
J_1 &\lesssim \sum_{i \in \mathbb{N}} [Q \cap (\mathbb{R}^n \times [t-s, t+s])]|2Q_i| \mathcal{F}_{2Q_i} \left| \frac{v_h(z) - (v_h)_Q(\mathbb{R}^n \times [t-s, t+s])}{r} \right| dz \\
&\lesssim \frac{r_{\lambda, h}^{n+2} - d - d}{r^d} \sum_{i \in \mathbb{N}} \mathcal{F}_{2Q_i} \left| \frac{v_h(z) - (v_h)_Q(\mathbb{R}^n \times [t-s, t+s])}{r_{i}} \right| dz \\
&\lesssim \frac{r_{\lambda, h}^{n+2} - d - d}{r^d} \sum_{i \in \mathbb{N}} \mathcal{F}_{2Q_i} \left| \frac{v_h(z) - (v_h)_Q(\mathbb{R}^n \times [t-s, t+s])}{r_{i}} \right| dz.
\end{align*}

Lemma A.9 \newline
To obtain the last inequality, we made use of the bound $r^2 \leq s.$

The estimate for $J_2$ is exactly as in (A.29) to get
\begin{equation}
J_2 \lesssim o(1). \tag{A.29}
\end{equation}

Subcase 2 $2Q \cap \mathbb{R}^n \times (-\infty, s) \neq \emptyset$ and $2Q \cap \mathbb{R}^n \times (-s, \infty) \neq \emptyset$ AND $r^2 \geq s.$ In this case, we proceed as follows. Using triangle inequality and the bound $|Q \cap (\mathbb{R}^n \times [t-s, t+s])| = |B| \times s$ where $s$ is from (A.1), we get
\begin{align*}
\mathcal{F}_Q(\mathbb{R}^n \times [t-s, t+s]) \left| \frac{v_{\lambda, h}(z) - v_h(z)}{r} \right| dz \\
\lesssim \frac{1}{|Q \cap (\mathbb{R}^n \times [t-s, t+s])|} \mathcal{F}_Q(\mathbb{R}^n \times [t-s, t+s]) \left| v_{\lambda, h}(z) \right| dz \\
+ \frac{1}{|Q \cap (\mathbb{R}^n \times [t-s, t+s])|} \mathcal{F}_Q(\mathbb{R}^n \times [t-s, t+s]) \left| (v_h)_Q(\mathbb{R}^n \times [t-s, t+s]) \right| dz.
\end{align*}

By construction of $v_{\lambda, h}$ in (A.5), we have $v_{\lambda, h} = v_h$ on $E_\lambda$. On $(\mathbb{R}^n \times [t-s, t+s]) \setminus E_\lambda$, we can apply Corollary A.8 to obtain the following bound:
\begin{equation}
\mathcal{F}_Q(\mathbb{R}^n \times [t-s, t+s]) \left| \frac{v_{\lambda, h}(z) - v_h(z)}{r} \right| dz \lesssim \frac{1}{r^d} \mathcal{F}(\mathbb{R}^n \times [t-s, t+s]) \left| v_h(z) \right| dz + o(1) \lesssim o(1). \tag{A.29}
\end{equation}

This completes the proof of the Lipschitz continuity.
A.8 Crucial estimates for the test function

In this subsection, we shall prove three crucial estimates that will be needed.

Lemma A.21 Let $\lambda \geq 1$, then for any $i \in \mathbb{N}$, $\delta \in (0, 1]$ and a.e. $t \in (t - s, t + s)$, there exists a constant $C = C(p^+_{\log}, \Lambda_0, A_1, n)$ such that there holds

$$
\left| \int_{\Omega^a_{4p}(\tau)} (v(x, t) - v') u_{\lambda,h}(x, t) \psi_i(x, t) \, dx \right| \leq C \left( \frac{1}{\delta^2} |Q_i| + \delta |\hat{B}_i| \iint_{\Omega_i} |v(z)|^2 \chi_{[t-s, t+s]} \, dz \right). \tag{A.30}
$$

**Proof** Let us fix any $t \in (-s, s)$, $t \in \mathbb{N}$ and take $\omega_i(y, \tau) u_{\lambda,h}(y, \tau)$ as a test function in (1.1) and (6.17). Further integrating the resulting expression over $(t_i - \lambda^{-1+d}4r_i^2, t)$ along with making use of the fact that $\psi_i(t_i - \lambda^{-1+d}4r_i^2) = 0$, we get for any $a \in \mathbb{R}$, the equality

$$
\int_{\Omega^a_{4p}(\tau)} (v_h - a) \psi_i u_{\lambda,h} \, dy = \int_{t_i - \lambda^{-1+d}4r_i^2}^{t_i} \int_{\Omega^a_{4p}(\tau)} \partial_t (v_h - a) \psi_i u_{\lambda,h} \, dy \, d\tau \\
= \int_{t_i - \lambda^{-1+d}4r_i^2}^{t_i} \int_{\Omega^a_{4p}(\tau)} (l - u) \partial_t \psi_i u_{\lambda,h} \psi_i u_{\lambda,h} \psi_i u_{\lambda,h} \, dy \, d\tau \\
+ \int_{t_i - \lambda^{-1+d}4r_i^2}^{t_i} \int_{\Omega^a_{4p}(\tau)} (\nabla |(\nabla u)|^2) \, d\tau.
$$

We can estimate $|\nabla (\psi_i u_{\lambda,h})|$ using the chain rule and Lemma A.6, to get

$$
|\nabla (\psi_i u_{\lambda,h})| \lesssim \frac{1}{\lambda - \rho_{(\tau)}} |v_{\lambda,h}| + |\nabla v_i|.
$$

Similarly, we can estimate $|\partial_t (\psi_i u_{\lambda,h})|$ using the chain rule, to get

$$
|\partial_t (\psi_i u_{\lambda,h})| \lesssim \frac{1}{\lambda^{-1+d}r_i^2} |v_i| + |\partial_t v_i|.
$$

Let us now prove each of the assertions of the lemma.

Proof of (A.30): Let us take $a = v_i^h$ in the (A.31) followed by letting $h \searrow 0$ and making use of (A.32), (2.2) and (6.23), we get

$$
\left| \int_{\Omega^a_{4p}(\tau)} (v - v') \psi_i u_{\lambda,h} \, dy \right| \leq J_1 + J_2 + J_3,
$$

where we have set

$$
J_1 := \frac{1}{\lambda - \rho_{(\tau)}} \int_{K^a_{4p}(\tau)} (|\nabla u| + |\nabla w| + |\nabla | + 1)^{p(z)-1} |v_i| \chi_{Q_i \cap K^a_{4p}(\tau)} \, dz,
$$

$$
J_2 := \int_{K^a_{4p}(\tau)} (|\nabla u| + |\nabla w| + |\nabla | + 1)^{p(z)-1} |v_i| \chi_{Q_i \cap K^a_{4p}(\tau)} \, dz,
$$

$$
J_3 := \int_{K^a_{4p}(\tau)} |v - v'| |\partial_t (\psi_i u_{\lambda})| \chi_{Q_i \cap K^a_{4p}(\tau)} \, dz.
$$

Let us now estimate each of the terms as follows: Bound for $J_1$: We split the estimate into two cases, the first is when $\alpha^{-\frac{1}{p+1} + \frac{d}{2}} \rho \leq \lambda^{-\frac{1}{p+1} + \frac{d}{2}} r_i$. In this case, we make use of (A.15)
along with (A.3) to get

\[
J_1 \lesssim \alpha - \frac{1 - \frac{1}{p} \lambda + \frac{d}{2} \rho \lambda}{\frac{1}{p} r_i} |Q_i| \bar{f}_{2Q_i} (|\nabla u| + |\nabla w| + |f| + 1)^{p(z) - 1} dz
\]

\[
\lesssim \alpha - \frac{1 - \frac{1}{p} \lambda + \frac{d}{2} \rho \lambda}{\frac{1}{p} r_i} |Q_i| \left( \bar{f}_{2Q_i} (|\nabla u| + |\nabla w| + |f| + 1)^{p(z) - 1} dz \right)^{\frac{r_{2Q_i}}{r_{2Q_i}}} - 1
\]

\[
\lesssim |Q_i| |\lambda| \lesssim \frac{1}{2} |2Q_i|.
\]

To obtain the last inequality, we have used \( \lambda \)

In the case \( \alpha - \frac{1}{p} \lambda + \frac{d}{2} \rho \geq \alpha - \frac{1}{p} \lambda + \frac{d}{2} r_i \), we get for any \( \delta \in (0, 1] \) using (A.18)

\[
J_1 \lesssim \left( \frac{1}{\delta} \frac{1}{\frac{1}{p} r_i} + \frac{\delta}{\frac{1}{p} \lambda + \frac{d}{2} \rho} \right)^{\frac{1}{p} r_{2Q_i}} \bar{f}_{Q_i} |v(z)|^2 \chi_{[t-s, t+s]} dz
\]

\[
\lesssim |Q_i| \left( \frac{1}{\delta} \frac{1}{\frac{1}{p} r_i} + \frac{\delta}{\frac{1}{p} \lambda + \frac{d}{2} \rho} \right)^{\frac{1}{p} r_{2Q_i}} \bar{f}_{Q_i} |v(z)|^2 \chi_{[t-s, t+s]} dz
\]

\[
\lesssim |Q_i| \frac{1}{\delta} |\hat{Q}_i| + \delta |\hat{B}_i| \int_{\hat{Q}_i} |v(z)|^2 \chi_{[t-s, t+s]} dz.
\]

To obtain the last inequality, we again made use of \( \lambda \)

\[
J_1 \lesssim \frac{1}{\delta} |\hat{Q}_i| + \delta |\hat{B}_i| \int_{\hat{Q}_i} |v(z)|^2 \chi_{[t-s, t+s]} dz.
\]

Bound for \( J_2 \): In this case, we can directly use (A.17) to get for any \( \delta \in (0, 1] \), the bound

\[
J_2 \lesssim \frac{1}{\delta} |Q_i| \bar{f}_{2Q_i} (|\nabla u| + |\nabla w| + |f| + 1)^{p(z) - 1} dz
\]

\[
\lesssim |Q_i| \frac{1}{\delta} \frac{1}{p} \lambda \frac{r_{2Q_i}}{r_{2Q_i}} \bar{f}_{Q_i} \lesssim |Q_i| \frac{1}{2}.
\]

To obtain the last inequality, we again made use of \( \lambda \)

\[
\frac{1}{\delta} |Q_i| \frac{r_{2Q_i}}{r_{2Q_i}} \bar{f}_{Q_i} \lesssim C(p_{\log}^{-1} n).
\]
We shall now estimate each of the terms as follows: Estimate of Lemma A.22

\[ \frac{1}{\lambda^{p(t)}} \left( \frac{4}{\lambda^{p(t)}} + \frac{2}{r_i} \frac{1}{\lambda^{p(t)}} \right) + \frac{1}{\lambda^{p(t)}} + \frac{2}{r_i} \frac{1}{\lambda^{p(t)}} \left( \frac{1}{\lambda^{p(t)}} + \frac{2}{r_i} \right) \]

Now making use of Lemma A.9, we see that

\[ \int_{K^{(3)}_{2\Omega}} |v - u| |\rho|^2 \chi_{|\Omega_1 \setminus \Omega_{t+s}|} \, dz \lesssim |\Omega| \left( \frac{1}{\lambda^{p(t)}} + \frac{2}{r_i} \right) |\rho|^2 \chi_{|\Omega_1 \setminus \Omega_{t+s}|} \, dz. \]  

Combining (A.33) and (A.34), we get

Now making use of Lemma A.9, we see that

Let us fix any \( x \in \Omega_{4\rho}(x) \setminus E_{\lambda}(t) \). Now define

\[ \Upsilon := \left\{ i \in \Theta : \text{spt}(\psi_i) \cap \Omega_{4\rho}(x) \times \{ t \} \neq \emptyset, |v| + |v_\lambda| \neq 0 \text{ on } \text{spt}(\psi_i) \cap (\Omega_{4\rho}(x) \times \{ t \}) \right\}. \]

If \( i \notin \Upsilon \), then \( v = v_\lambda = 0 \) on \( \text{spt}(\psi_i) \cap \Omega_{4\rho}(x) \times \{ t \} \), which implies

Hence we only need to consider \( i \in \Upsilon \). Noting that \( \sum_{i \in \Upsilon} \psi_i(\cdot, t) \equiv 1 \) on \( \mathbb{R}^n \cap E_{\lambda}(t) \), we can rewrite the left-hand side of (A.35) as

\[ \int_{\Omega_{4\rho}(x) \setminus E_{\lambda}(t)} (|v|^2 - |v - v_\lambda|^2) \, dx = \sum_{i \in \Upsilon} \int_{\Omega_{4\rho}(x)} \psi_i \left( |v|^2 - |v - v_\lambda|^2 \right) \, dx = J_1 - J_2. \]

where we have set

\[ J_1 := \sum_{i \in \Upsilon} \int_{\Omega_{4\rho}(x)} \psi_i \left( |v|^2 + 2v_\lambda(v - v') \right) \, dx, \quad J_2 := \sum_{i \in \Upsilon} \int_{\Omega_{4\rho}(x)} \psi_i |v_\lambda - v'|^2 \, dx. \]

We shall now estimate each of the terms as follows: Estimate of \( J_1 \): Using (A.30), we get

\[ J_1 \gtrsim \sum_{i \in \Upsilon} \int_{\Omega_{4\rho}(x)} \omega_i(z) |v'|^2 \, dz - \delta \sum_{i \in \Upsilon} |\tilde{\Omega}_i| |v'|^2 - \frac{\lambda}{\delta} \tilde{\Omega}_i. \]

\[ \square \] Springer
From (A.6), we have \( v^j = 0 \) whenever \( \text{spt} (\psi_j) \not\subseteq \Omega^{\alpha}_{t; \delta}(x) \times [-s, \infty) \). Hence we only have to sum over all those \( i \in \mathbb{T}_1 \) for which \( \text{spt} (\psi_i) \subseteq \Omega^{\alpha}_{t; \delta}(x) \times [-s, \infty) \). In this case, we make use of a suitable choice for \( \delta \in (0, 1] \), and use (W4) to estimate (A.37) from below. We get

\[
J_1 \gtrsim -\lambda |\mathbb{R}^{n+1} \setminus E_\lambda|.
\] (A.38)

Estimate of \( J_2 \): For any \( x \in K^{\alpha}_{t, \delta}(\delta) \setminus E_\lambda(t) \), we have from Lemma A.6 that \( \sum_j \psi_j (x, t) = 1 \), which gives

\[
\psi_t (z)|v_{\lambda, h} (z) - v^j|^2 \lesssim \sum_{j \in \mathbb{I}_i} |\psi_j (z)|^2 \left( v^j - v^j \right)^2 \lesssim \min \{ \rho, \lambda^{-\frac{1}{\rho} + \frac{d}{2} r_1} \} 2^j \lambda^{-\frac{2}{p(r_1)}}. \tag{A.39}
\]

To obtain (a) above, we made use of Corollary A.10 along with (W3). Substituting (A.39) into the expression for \( J_2 \), we get

\[
J_2 \lesssim \sum_{i \in \mathbb{T}} |\Omega^{\alpha}_{t; \delta}(x) \cap 2B_i| \left( \lambda^{-\frac{1}{\rho} + \frac{d}{2} r_1} \right) \left( \lambda^{-\frac{1}{\rho} + \frac{d}{2} r_1} \right)^2 \lambda^{-\frac{2}{p(r_1)}}.
\] (A.40)

Substituting (A.38) and (A.40) into (A.36), the proof of the lemma follows.

### B The method of Lipschitz truncation - second difference estimate

In Appendix A, we constructed a suitable test function which was used to obtain a difference estimate between the weak solutions of (1.1) and (6.17). In this appendix, we will obtain an analogous Lipschitz truncation method that will be used as a test function to obtain difference estimate between the weak solutions of (6.17) and (6.24). Most of the estimates follow exactly as in Appendix A and hence we will only highlight the modifications needed.

Let us first note that the Lipschitz truncation is now constructed over the constant exponent \( p(\mathfrak{z}) \) which actually simplifies a lot of the estimates from Appendix A. Let us denote

\[
s := \alpha^{-1 + d} (3\rho)^2. \tag{B.1}
\]

Firstly, let us recall the modified Lemma A.2:

**Lemma A.23** For any \( h \in (0, 2s) \) and let \( \phi (x) \in C^\infty_c (\Omega^{\alpha}_{t; \delta}(x)) \) and \( \varphi (t) \in C^\infty ([t - s, \infty)) \) with \( \varphi (t - s) = 0 \) be a non-negative function and \( [w]_h, [v]_h \) be the Steklov average as defined in (3.2). Then the following estimate holds for any time interval \( (t_1, t_2) \subset [t - s, t + s] \):

\[
| \{ (w - v)_h \psi \phi (t_2) \} - \{ (w - v)_h \psi \phi (t_1) \} | \leq \| \nabla \psi \|_{L^\infty(\Omega^{\alpha}_{t; \delta}(x))} \| \psi \|_{L^\infty([t_1, t_2])} \int_{\Omega^{\alpha}_{t; \delta}(x) \times (t_1, t_2)} |B(t, \nabla v) - A(z, \nabla w)| \text{ } dz + \| \phi \|_{L^\infty(\Omega^{\alpha}_{t; \delta}(x))} \| \psi \|_{L^\infty([t_1, t_2])} \int_{\Omega^{\alpha}_{t; \delta}(x) \times (t_1, t_2)} |[w - v]_h| \text{ } dz.
\] (B.2)

### B.1 Construction of test function

Let us denote the following functions:

\[
v_\lambda (z) := w(z) - v(z) \quad \text{and} \quad v_h (z) := [w - v]_h (z).
\]
where \([w − v]_h(z)\) denotes the usual Steklov average. It is easy to see that \(v_h \xrightarrow{h \searrow 0} v\). We also note that \(v(z) = 0\) for \(z \in \partial_{\mathcal{B}} N_{\beta, p}(3)\). For some fixed \(q\) such that \(1 < q < \frac{\beta}{p-1}\), with \(\mathcal{M}\) as defined in (5.2), let us now define

\[
g(z) := \mathcal{M}
\left(
\left|
\frac{v}{\alpha r_j^{\frac{1}{p(3) + \frac{d}{2}}}} + |\nabla w| + |\nabla v| + 1\right|
\right)^{\frac{p(1-\beta)}{q}} \chi_{N_{\beta, p}(3)}^{\mathcal{M}}(z)
\]  

(\text{B.3})

For a fixed \(\lambda \geq 1\), let us define the good set by

\[
E_\lambda := \{z \in \mathbb{R}^{n+1} : g(z) \leq \lambda^{1-\beta}\}.
\]  

(\text{B.4})

Since we are dealing with constant exponent \(p(z)\), we have the following Whitney-type covering lemma (see [36, Chapter 3] or [37, Lemma 3.1] for the proof):

**Lemma A.24** There exists a Whitney covering \(\{Q_i(z_i)\}\) of \(E^c_\lambda\) in the following sense:

(\text{W6}) \(Q_j(z_j) = B_j(x_j) \times I_j(t_j)\) where \(B_j(x_j) = B_{\lambda^{-\frac{1}{p(3) + \frac{d}{2}}}}(x_j)\) and \(I_j(t_j) = (t_j − \lambda^{-1 + d} r_j^2, t_j + \lambda^{-1 + d} r_j^2)\).

(\text{W7}) \(\bigcup_j Q_j(z_j) = E^c_\lambda\).

(\text{W8}) for all \(j \in \mathbb{N}\), we have \(8Q_j \subset E^c_\lambda\) and \(16Q_j \cap E_\lambda \neq \emptyset\).

(\text{W9}) if \(Q_j \cap Q_k \neq \emptyset\), then \(\frac{1}{3} r_k \leq r_j \leq c r_k\).

(\text{W10}) \(\sum_j \chi_{8Q_j}(z) \leq c(n)\) for all \(z \in E^c_\lambda\).

Subordinate to this Whitney covering, we have an associated partition of unity denoted by \(\{\psi_j\} \in C_\infty^c(\mathbb{R}^{n+1})\) such that the following holds:

(\text{W11}) \(\chi_{Q_j} \leq \psi_j \leq \chi_{2Q_j}\).

(\text{W12}) \(\|\psi\|_{\infty} + \lambda^{-\frac{1}{p(3) + \frac{d}{2}}} r_j \|\nabla \psi\|_{\infty} + \lambda^{-1 + d} r_j^2 \|\partial \psi\|_{\infty} \leq C\).

For a fixed \(k \in \mathbb{N}\), let us define

\[
A_k := \{j \in \mathbb{N} : \frac{3}{4} Q_k \cap \frac{3}{4} Q_j \neq \emptyset\},
\]

then we have

(\text{W13}) Let \(i \in \mathbb{N}\) be given, then \(\sum_{j \in A_i} \psi_j(z) = 1\) for all \(z \in 2Q_i\).

(\text{W14}) Let \(i \in \mathbb{N}\) be given and let \(j \in A_i\), then \(\max\{|Q_j|, |Q_i|\} \leq C(n)|Q_j \cap Q_i|\).

(\text{W15}) Let \(i \in \mathbb{N}\) be given and let \(j \in A_i\), then \(\max\{|Q_j|, |Q_i|\} \leq 2|Q_j \cap 2Q_i|\).

(\text{W16}) For any \(i \in \mathbb{N}\), we have \(#A_i \leq c(n)\).

(\text{W17}) Let \(i \in \mathbb{N}\) be given, then for any \(j \in A_i\), we have \(2Q_j \subset 8Q_i\).

**B.2 Construction of Lipschitz truncation function**

We shall also use the notation

\[
\mathcal{I}(i) := \{j \in \mathbb{N} : \text{spt}(\psi_i) \cap \text{spt}(\psi_j) \neq \emptyset\} \quad \text{and} \quad \mathcal{I}_z := \{j \in \mathbb{N} : z \in \text{spt}(\psi_j)\}.
\]

We are now ready to construct the Lipschitz truncation function:

\[
v_{\lambda, h}(z) := v_h(z) − \sum_i \psi_i(z) \left(v_h(z) − v^i_h\right),
\]

(\text{B.5})
where we have defined

\[ v^i_h := \begin{cases} \mathcal{J}_{2Q_i} v_h(z) \chi_{[t-s, t+s]} \, dz & \text{if } 2Q_i \subset \Omega_{3\rho}^q(\bar{x}) \times (t-s, \infty), \\ 0 & \text{else.} \end{cases} \]

From construction in (A.5) and (A.6), we see that

\[ \text{spt}(v^i_{\lambda, h}) \subset \Omega_{3\rho}^q(\bar{x}) \times (t-s, \infty). \]

We see that \( v^i_{\lambda, h} \) has the right support for the test function and hence the rest of this section will be devoted to proving the Lipschitz regularity of \( v^i_{\lambda, h} \) on \( K_{3\rho}^q(\bar{x}) \) as well as some useful estimates.

### B.3 Some estimates on the test function

In this subsection, we will collect some useful estimates on the test function. The proofs of these estimates are very similar to the corresponding ones from Appendix A (in fact simpler because we are dealing with the constant exponent \( p(\bar{z}) \)) and will be omitted. Let us first derive a useful estimate:

\[
\mathcal{B}(t, \nabla v) = A(z, \nabla w) = \mathcal{B}(t, \nabla v) - B(z, \nabla w) + |B(z, \nabla w) - A(z, \nabla w)| \leq (\mu^2 + |\nabla v|^2)^{p(\bar{z})-1} \left( \mu^2 + |\nabla w|^2 \right)^{p(\bar{z})-1} + |B(z, \nabla w) - A(z, \nabla w)|
\]

\[
\lesssim \left( \mu^2 + |\nabla v|^2 \right)^{p(\bar{z})-1} + \left( \mu^2 + |\nabla w|^2 \right)^{p(\bar{z})-1} + \left( \mu^2 + |\nabla w|^2 \right) \left( p(\bar{z}) - 1 \right)
\]

The primary use of (B.6) would be needed to estimate the first term on the right hand side of (B.2).

**Lemma A.25** Let \( \mathfrak{z} \in K_{3\rho}^q(\bar{z}) \setminus E_2 \), then from (W1), we have that \( \mathfrak{z} \in 2Q_i \) for some \( i \in \mathcal{I}_3 \). For any \( 1 \leq \theta \leq \frac{p_q-1}{q-1} \), there holds

\[
|v^i_h|_{\theta} \leq \mathcal{J}_{2Q_i} |v_h(z)|^{\theta} \chi_{[t-s, t+s]} \, dz \lesssim (p^q_{\mathfrak{z}, \mathfrak{z}})^A (\alpha^{-\frac{1}{m+1}} + \frac{d}{\rho})^{\theta} \lambda^{\frac{\theta}{m+1}},
\]

\[
\mathcal{J}_{2Q_i} |\nabla v_h(z)|^{\theta} \chi_{[t-s, t+s]} \, dz \lesssim (p^q_{\mathfrak{z}, \mathfrak{z}})^A \lambda^{\frac{\theta}{m+1}}.
\]

**Corollary A.26** For any \( \mathfrak{z} \in K_{3\rho}^q(\bar{z}) \setminus E_2 \), we have \( \mathfrak{z} \in 2Q_i \) for some \( i \in \mathcal{I}_3 \), then there holds

\[
|v_h(z)| \lesssim (p^q_{\mathfrak{z}, \mathfrak{z}})^A (\alpha^{-\frac{1}{m+1}} + \frac{d}{\rho}) \lambda^{\frac{1}{m+1}}.
\]

**Lemma A.27** Let \( 2Q_i \) be a parabolic Whitney type cylinder, then for any \( 1 \leq \theta \leq \frac{p_q-1}{q-1} \), there holds

\[
\mathcal{J}_{2Q_i} |v_h(z)|^{\theta} \chi_{[t-s, t+s]} \, dz \lesssim (p^q_{\mathfrak{z}, \mathfrak{z}})^A \min \left\{ \alpha^{-\frac{1}{m+1}} + \frac{d}{\rho}, \lambda^{-\frac{1}{m+1}} + \frac{d}{\rho} r_i \right\}^{\theta} \lambda^{\frac{\theta}{m+1}}.
\]

**Proof** Let us consider the following two cases: Case \( \alpha^{-\frac{1}{m+1}} + \frac{d}{\rho} \leq \lambda^{-\frac{1}{m+1}} + \frac{d}{\rho} r_i \): This is very similar to (A.9). Case \( \alpha^{-\frac{1}{m+1}} + \frac{d}{\rho} \geq \lambda^{-\frac{1}{m+1}} + \frac{d}{\rho} r_i \): Applying Lemma A.2 with \( \mu \in C^\infty_c(2B_i) \)
such that $|\mu(x)| \lesssim \left(\frac{1}{\lambda^{-\frac{1}{p+1} + \frac{d}{2} r_i}}\right)^n$ and $|\nabla \mu(x)| \lesssim \left(\frac{1}{\lambda^{-\frac{1}{p+1} + \frac{d}{2} r_i}}\right)^{n+1}$, we get

$$\int_{2Q_i} \left|v_h(z)\chi_{[t-s, t+s]} - v_h'\right|^\theta \, dz \leq \left(\frac{1}{\lambda^{-\frac{1}{p+1} + \frac{d}{2} r_i}}\right)^\theta \int_{2Q_i} |\nabla v_h|\theta \chi_{[t-s, t+s]} \, d\bar{z}$$

$$+ \sup_{t_1, t_2 \in 2Q_i \cap [t-s, t+s]} |(v_h) \mu (t_2) - (v_h) \mu (t_1)|^\theta. \tag{B.8}$$

The first term on the right of (B.8) can be estimated using (B.7) to get

$$\left(\frac{1}{\lambda^{-\frac{1}{p+1} + \frac{d}{2} r_i}}\right)^\theta \int_{2Q_i} |\nabla v_h|\theta \chi_{[t-s, t+s]} \, d\bar{z} \lesssim \left(\frac{1}{\lambda^{-\frac{1}{p+1} + \frac{d}{2} r_i}}\right)^\theta \lambda^{\frac{\theta}{p+\theta}} r_i^{\frac{p+1}{\theta}}.$$  

To estimate the second term on the right of (B.8), we make use of Lemma A.23 with $\phi(x) = \mu(x)$ and $\varphi(t) \equiv 1$, we get

$$\left|\langle v_h \rangle \mu (t_2) - \langle v_h \rangle \mu (t_1)\right| \lesssim \left(\frac{1}{\lambda^{-\frac{1}{p+1} + \frac{d}{2} r_i}}\right)^\theta \int_{2Q_i} |\nabla v_h| \chi_{[t-s, t+s]} \, d\bar{z} \lesssim \left(\frac{1}{\lambda^{-\frac{1}{p+1} + \frac{d}{2} r_i}}\right)^{\theta} \lambda^{\frac{\theta}{p+\theta}} r_i^{\frac{p+1}{\theta}}.$$  

Thus combining (A.11) and (A.14) into (A.10), we get

$$\int_{2Q_i} \left|v_h(z)\chi_{[t-s, t+s]} - v_h'\right|^\theta \, dz \lesssim \left\{\left(\frac{1}{\lambda^{-\frac{1}{p+1} + \frac{d}{2} r_i}}\right)^\theta \lambda^{\frac{\theta}{p+\theta}} r_i^{\frac{p+1}{\theta}}\right\} \lambda^{\frac{1}{1+\theta}}.$$

This proves the lemma.

**Corollary A.28** For any $i \in \mathbb{N}$ and any $j \in \mathcal{I}_i$, there holds

$$|v'_{h_i} - v_h| \lesssim (p_{\log, \Lambda_0, \Lambda_1, n}) \min \left\{\alpha^{-\frac{1}{p+1} + \frac{d}{2}} \rho, \lambda^{-\frac{1}{p+1} + \frac{d}{2} r_i} \right\} \lambda^{\frac{1}{1+\theta}}.$$  

**B.4 Bounds on $v_{\lambda, h}$ and $\nabla v_{\lambda, h}$**

**Lemma A.29** Let $Q_i$ be a parabolic Whitney type cylinder. Then for any $z \in 2Q_i$, we have the following bound:

$$\left(\frac{1}{\alpha^{-\frac{1}{p+1} + \frac{d}{2}} \rho} \right) |v_{\lambda, h}(z)| + |\nabla v_{\lambda, h}(z)| \chi_{[t-s, t+s]} \lesssim (p_{\log, \Lambda_0, \Lambda_1, n}) \lambda^{\frac{1}{1+\theta}}.$$  

**Corollary A.30** Let $z \in K_{\alpha, \rho}^d \setminus E_\lambda$, then $z \in 2Q_i$ for some $i \in \mathbb{N}$. Then there holds for any $\delta \in (0, 1]$, the estimates

$$\frac{1}{\lambda^{-\frac{1}{p+1} + \frac{d}{2} r_i}} |v_{\lambda, h}(z)| \lesssim \lambda^{\frac{1}{p+\theta}} \delta^{\frac{1}{\theta}} + \frac{\delta}{\lambda^{\frac{1}{p+1} + \frac{d}{2} r_i} \lambda^{\frac{1}{p+\theta}}} |v_h'|^2,$$

$$|\nabla v_{\lambda, h}(z)| \lesssim \frac{\lambda^{\frac{1}{p+\theta}}}{\delta}.$$  

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Lemma A.31 Let \( z \in K^\alpha_{3\rho}(3) \setminus E_\lambda \), then \( z \in 2\Omega_i \) for some \( i \in \mathbb{N} \). Then there holds for any \( \delta \in (0, 1] \), the estimates
\[
|v_{\lambda,h}(z)| \lesssim_{(p^\pm_{\log,\Lambda_0,\Lambda_1,1,n})} \frac{\lambda - \frac{1}{m^{\beta_j}} + \frac{d}{2} r_1 \lambda M_{\beta_j}}{\delta} + \frac{\delta}{\lambda - \frac{1}{m^{\beta_j}} + \frac{d}{2} r_1 \lambda M_{\beta_j}} \int_{\tilde{Q}_i} |v_h(\tilde{z})|^2 d\tilde{z},
\]
\[
|\nabla v_{\lambda,h}(z)| \lesssim_{(p^\pm_{\log,\Lambda_0,\Lambda_1,1,n})} \lambda M_{\beta_j} + \frac{\delta}{\lambda - \frac{1}{m^{\beta_j}} + \frac{d}{2} r_1 \lambda M_{\beta_j}} \int_{\tilde{Q}_i} |u_h(\tilde{z})|^2 d\tilde{z}.
\]

B.5 Estimates on the time derivative of \( v_{\lambda,h} \)

Lemma A.32 Let \( z \in K^\alpha_{3\rho}(3) \), then \( z \in 2\Omega_i \) for some \( i \in \mathbb{N} \). We then have the following estimates for the time derivative of \( v_{\lambda,h} \):
\[
|\partial_t v_{\lambda,h}(\tilde{z})| \lesssim_{(p^\pm_{\log,\Lambda_0,\Lambda_1,1,n})} \frac{1}{\lambda - 1 + d r_1^2} \int_{\tilde{Q}_i} |v_h(z)| |\chi_{[-s-s]}| dz.
\]
We also have the improved estimate
\[
|\partial_t v_{\lambda,h}(\tilde{z})| \lesssim_{(p^\pm_{\log,\Lambda_0,\Lambda_1,1,n})} \frac{1}{\lambda - 1 + d r_1^2} \lambda M_{\beta_j} \min \left\{ \lambda - \frac{1}{m^{\beta_j}} + \frac{d}{2} r_1, \frac{\alpha - \frac{1}{m^{\beta_j}} + \frac{d}{2} \rho}{\lambda M_{\beta_j}} \right\}.
\]

B.6 Some important estimates for the test function

Lemma A.33 Let \( Q_i \) be a Whitney-type parabolic cylinder for some \( i \in \mathbb{N} \). Then for any \( \theta \in [1, 2] \), there holds
\[
\int_{K_{3\rho}(3) \setminus E_\lambda} |v_{\lambda,h}(z)|^\theta dz \lesssim_{(p^\pm_{\log,\Lambda_0,\Lambda_1,1,n})} \int_{K_{3\rho}(3) \setminus E_\lambda} |v_h(z)|^\theta dz.
\]

Lemma A.34 Let \( Q_i \) be a Whitney-type parabolic cylinder for some \( i \in \mathbb{N} \), then there holds
\[
\int_{2\Omega_i} |v_{\lambda,h}(z) - \tilde{v}_h(z)| dz \lesssim_{(p^\pm_{\log,\Lambda_0,\Lambda_1,1,n})} \min \left\{ \lambda - \frac{1}{m^{\beta_j}} + \frac{d}{2} r_1, \frac{\alpha - \frac{1}{m^{\beta_j}} + \frac{d}{2} \rho}{\lambda M_{\beta_j}} \right\} \frac{1}{\lambda M_{\beta_j}}.
\]

Lemma A.35 Let \( Q_i \) be a Whitney-type parabolic cylinder for some \( i \in \mathbb{N} \), then there holds
\[
\int_{K_{3\rho}(3) \setminus E_\lambda} |\partial_t v_{\lambda,h}(z) (v_{\lambda,h}(z) - v_h(z))|^\theta dz \lesssim_{(p^\pm_{\log,\Lambda_0,\Lambda_1,1,n})} \lambda \theta |\mathbb{R}^{n+1} \setminus E_\lambda|.
\]

B.7 Lipschitz continuity

Lemma A.36 Let \( \lambda \geq 1 \), then for any \( \tilde{z} \in \Omega^\alpha_{3\rho}(3) \times [t - s, t + s] \) and \( r > 0 \), there exists a constant \( C > 0 \) independent of \( \tilde{z} \) and \( r \) such that
\[
I_r(\tilde{z}) := \frac{1}{\Omega^\alpha_{3\rho}(3) \times [t - s, t + s]} \int_{\Omega^\alpha_{3\rho}(3) \times [t - s, t + s]} \left| \frac{v_\lambda(z) - (v_\lambda)_{\Omega^\alpha_{3\rho}(3) \times [t - s, t + s]}}{r} \right| dz \leq C < \infty.
\]
In particular, this implies for any \( z_1, z_2 \in \Omega^\alpha_{3\rho}(3) \times [t - s, t + s] \), there exists a constant \( K > 0 \) such that
\[
|v_\lambda(z_1) - v_\lambda(z_2)| \leq Kd_{\rho}(z_1, z_2).
\]
B.8 Crucial estimates for the test function

In this subsection, we shall prove three crucial estimates that will be needed. Note that by the time these estimates are applied, we would have taken $h \downarrow 0$ in the Steklov average.

Lemma A.37 Let $\lambda \geq 1$, then for any $i \in \mathbb{N}$, $\delta \in (0, 1]$ and a.e. $t \in (t-s, t+s)$, there exists a constant $C = C_{\langle p^+_0, \Lambda_0, 1 \rangle}$ such that there holds

$$\left| \int_{\Omega_0^+(\delta)} \left( v(x, t) - v_i \right) v_{\lambda}(x, t) \psi_i(x, t) \, dx \right| \leq C \left( \frac{\lambda}{\delta} |Q_i| + 2|B_i| \int_{2Q_i} |v(z)|^2 \chi_{[t-s, t+s]} \, dz \right).$$

Lemma A.38 Let $\lambda \geq 1$, then for a.e. $t \in [t-s, t+s]$, there exists a constant $C = C_{\langle p^+_0, \Lambda_0, 1 \rangle}$ such that there holds

$$\int_{\Omega_0^+(\delta)) \setminus E_{\lambda}(t)} \left( |\tilde{v}|^2 - |v - v_{\lambda}|^2 \right) \, dx \geq -C\lambda |\mathbb{R}^{n+1} \setminus E_{\lambda}|.$$

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