Improvements on some partial trace inequalities for positive semidefinite block matrices

Yongtao Li
School of Mathematics, Hunan University, Changsha, Hunan, People’s Republic of China

ABSTRACT
We study matrix inequalities involving partial traces for positive semidefinite block matrices. First of all, we present a new method to prove a celebrated result of Choi [Linear Algebra Appl. 516 (2017)]. The method also allows us to prove a generalization of another result of Choi [Linear Multilinear Algebra 66 (2018)]. Furthermore, we shall give an improvement on a recent result of Li, Liu and Huang [Operators and Matrices 15 (2021)]. In addition, we include with some majorization inequalities involving partial traces for two by two block matrices, and also provide inequalities related to the unitarily invariant norms as well as the singular values, which can be viewed as slight extensions of two results of Lin [Linear Algebra Appl. 459 (2014)] and [Electronic J Linear Algebra 31 (2016)].

ARTICLE HISTORY
Received 21 November 2021
Accepted 3 August 2022

COMMUNICATED BY
Y.-T. Poon

KEYWORDS
Partial transpose; partial traces; Cauchy and Khinchin; positive semidefinite

MATHEMATICS SUBJECT CLASSIFICATIONS
15A45; 15A60; 47B65

1. Introduction
The space of \( m \times n \) complex matrices is denoted by \( \mathbb{M}_{m \times n} \); if \( m = n \), we write \( \mathbb{M}_n \) for \( \mathbb{M}_{n \times n} \) and if \( n = 1 \), we use \( \mathbb{C}^m \) for \( \mathbb{M}_{m \times 1} \). By convention, if \( A \in \mathbb{M}_n \) is positive semidefinite, we write \( A \geq 0 \). For Hermitian matrices \( A \) and \( B \) with the same size, \( A \geq B \) means that \( A - B \) is positive semidefinite, i.e. \( A - B \geq 0 \). We denote by \( \mathbb{M}_m(\mathbb{M}_n) \) the set of \( m \times m \) block matrices with each block in \( \mathbb{M}_n \). Each element of \( \mathbb{M}_m(\mathbb{M}_n) \) is also viewed as an \( mn \times mn \) matrix with numerical entries and usually written as \( A = [A_{ij}]_{i,j=1}^m \), where \( A_{ij} \in \mathbb{M}_n \). We denote by \( A \otimes B \) the Kronecker product of \( A \) with \( B \), that is, if \( A = [a_{ij}] \in \mathbb{M}_m \) and \( B \in \mathbb{M}_n \), then \( A \otimes B \in \mathbb{M}_m(\mathbb{M}_n) \) whose \((i,j)\) block is \( a_{ij}B \).

In the present paper, we are mainly concentrated on the block positive semidefinite matrices. Given \( A = [A_{ij}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n) \), the partial transpose of \( A \) is defined as

\[
A^\tau = [A_{ji}]_{i,j=1}^m = \begin{bmatrix}
A_{1,1} & \cdots & A_{m,1} \\
\vdots & \ddots & \vdots \\
A_{1,m} & \cdots & A_{m,m}
\end{bmatrix}.
\]
This is different from the usual transpose, which is defined as

\[ A^T = [A^T_{j,i}]_{i=1}^m = \begin{bmatrix}
A^T_{1,1} & \cdots & A^T_{1,m} \\
\vdots & \ddots & \vdots \\
A^T_{m,1} & \cdots & A^T_{m,m}
\end{bmatrix}. \]

The primary application of partial transpose is materialized in quantum information theory [1–3]. It is easy to see that \( A \geq 0 \) does not necessarily imply \( A^T \geq 0 \). If both \( A \) and \( A^T \) are positive semidefinite, then \( A \) is said to be positive partial transpose (or PPT for short). For more explanations and applications of the PPT matrices, we recommend a comprehensive monograph [4], and see, e.g. [5–9] for recent results. Next we introduce the definition of two partial traces \( \text{tr}_1 A \) and \( \text{tr}_2 A \) of \( A \)

\[ \text{tr}_1 A = \sum_{i=1}^m A_{i,i} \quad \text{and} \quad \text{tr}_2 A = [\text{tr}A_{i,j}]_{i,j=1}^m, \]

where \( \text{tr}(\cdot) \) stands for the usual trace. Clearly, we have \( \text{tr}_1 A \in \mathbb{M}_n \) and \( \text{tr}_2 A \in \mathbb{M}_m \).

It is believed that there are many elegant matrix inequalities that have arisen from the probability theory and quantum information theory in the literature. As we all know, these two partial trace maps are linear and trace-preserving. Furthermore, if \( A = [A_{i,j}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n) \) is positive semidefinite, it is easy to see that both \( \text{tr}_1 A \) and \( \text{tr}_2 A \) are also positive semidefinite; see, e.g. [10, p.237] or [11, Theorem 2.1]. Over the years, various results involving partial transpose and partial traces have been obtained in the literature, e.g. [12–16]. We introduce the background and recent progress briefly. In the following results, we always assume that \( A \in \mathbb{M}_m(\mathbb{M}_n) \) is a positive semidefinite block matrix.

(R1) Choi [13] presented by using induction on \( m \) that \( I_m \otimes \text{tr}_1 A^T \geq A^T \).
(R2) Zhang [9] revisited Choi’s result and proved \( (\text{tr}_2 A^T) \otimes I_n \geq A^T \).
(R3) Choi [14] further extended the result of Zhang and proved

\[ I_m \otimes \text{tr}_1 A^T \geq \pm A^T \] \hspace{1cm} (1)

and

\[ (\text{tr}_2 A^T) \otimes I_n \geq \pm A^T. \] \hspace{1cm} (2)

(R4) Ando [12] (or see [17] for an alternative proof) revealed a nice connection between the first partial trace and the second partial trace, and established

\[ (\text{tr}A)I_{mn} - (\text{tr}_2 A) \otimes I_n \geq I_m \otimes (\text{tr}_1 A) - A. \] \hspace{1cm} (3)

(R5) Motivated by (1) and (3), Li et al. [16] (or see [18] for a unified treatment) proved recently an analogous complement, which states that

\[ (\text{tr}A)I_{mn} - (\text{tr}_2 A) \otimes I_n \geq \pm (I_m \otimes (\text{tr}_1 A) - A), \] \hspace{1cm} (4)

and Li et al. also obtained

\[ (\text{tr}A)I_{mn} \pm (\text{tr}_2 A) \otimes I_n \geq A \pm I_m \otimes (\text{tr}_1 A). \] \hspace{1cm} (5)
We will present some partial trace inequalities for positive semidefinite block matrices which improve the abovementioned results. The paper is organized as follows. In Section 2, we shall give a new method to prove Choi’s result (R1). This method can allow us to give an improvement on (1); see Theorem 2.1. In Section 3, we present inequalities for the second partial trace and show an improvement on (2); see Theorem 3.2. In Section 4, we shall prove some inequalities involving both the first and second partial trace. Our results can be viewed as improvements on (4) and (5); see Theorems 4.1, 4.2 and 4.3. In Section 5, we give an application on Cauchy–Khinchin’s inequality by using Theorems 4.1 and 4.2; see Corollary 5.1. In Section 6, we study the majorization inequalities related to partial traces for two by two block matrices; see Theorem 6.2. In addition, we also prove some inequalities about the unitarily invariant norms as well as the singular values, which extend slightly two recent results of Lin [8] and [19]; see Theorems 6.3 and 6.6.

2. Inequalities about the first partial trace

We shall give a short proof of Choi’s result (R1). The original proof stated in [13, Theorem 2] used a standard decomposition of positive semidefinite matrices and then applied inductive techniques. Our method is quite different and transparent. As an application of this method, we shall present an improvement on (1) of Choi’s result (R3).

**Alternative proof of (R1).** Denote \( D_A := A_{1,1} \oplus A_{2,2} \oplus \cdots \oplus A_{m,m} \). We are going to prove

\[
I_m \otimes \text{tr}_1 A^\tau + (m - 2)D_A \geq A^\tau + (m - 2)D_A,
\]

which is the same as

\[
\begin{bmatrix}
(m-1)A_{1,1} + \sum_{i \neq 1} A_{i,i} & 0 & \cdots & 0 \\
0 & (m-1)A_{2,2} + \sum_{i \neq 2} A_{i,i} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & (m-1)A_{m,m} + \sum_{i \neq m} A_{i,i}
\end{bmatrix} \geq
\begin{bmatrix}
(m-1)A_{1,1} & A_{2,1} & \cdots & A_{m,1} \\
A_{1,2} & (m-1)A_{2,2} & \cdots & A_{m,2} \\
\vdots & \vdots & \ddots & \vdots \\
A_{1,m} & A_{2,m} & \cdots & (m-1)A_{m,m}
\end{bmatrix}. \tag{6}
\]

It suffices to show that for every pair \((i, j)\) with \(1 \leq i < j \leq m\),

\[
\begin{bmatrix}
\text{i-th} & \text{j-th} \\
\vdots & \vdots \\
\cdots & \cdots \\
n \cdot \cdot \cdot & n \cdot \cdot \cdot \\
0 & \cdots \\
\cdots & \cdots \\
A_{i,i} + A_{j,j} & \cdots & 0 & \cdots \\
\cdots & \vdots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \cdots \\
\end{bmatrix} \geq
\begin{bmatrix}
\vdots & \vdots & \vdots \\
\cdot \cdot \cdot \cdot & \cdot \cdot \cdot \cdot & \cdot \cdot \cdot \cdot \\
\cdot \cdot \cdot \cdot & \cdot \cdot \cdot \cdot & \cdot \cdot \cdot \cdot \\
A_{i,i} & \cdots & A_{j,j} & \cdots \\
\cdots & \vdots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}. \tag{7}
\]
Indeed, we can sum all inequalities (7) over all \( 1 \leq i < j \leq n \), which leads to the desired (6).

Note that the omitted blocks in (7) are zero matrices, so we need to prove

\[
\begin{bmatrix}
A_{ij} + A_{ji} & 0 \\
0 & A_{ii} + A_{jj}
\end{bmatrix} \geq \begin{bmatrix}
A_{ii} & A_{ji} \\
A_{ij} & A_{jj}
\end{bmatrix}.
\]

This inequality immediately holds by observing that

\[
\begin{bmatrix}
A_{ij} - A_{ji} \\
-A_{ii}
\end{bmatrix} = \begin{bmatrix} 0 & -I_n \\
I_n & 0 \end{bmatrix} \begin{bmatrix}
A_{ii} & A_{ij} \\
A_{ji} & A_{jj}
\end{bmatrix} \begin{bmatrix} 0 & I_n \\
-I_n & 0 \end{bmatrix}.
\]

Hence, we complete the proof.

As promised, we shall provide an improvement on (1) by using the above method.

**Theorem 2.1:** Let \( A = [A_{ij}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n) \) be positive semidefinite. Then

\[
I_m \otimes \text{tr}_1 A^\tau \geq -A^\tau + 2D_A,
\]

where \( D_A = A_{1,1} \oplus A_{2,2} \oplus \cdots \oplus A_{m,m} \).

**Proof:** We intend to prove

\[
I_m \otimes \text{tr}_1 A^\tau - mD_A \geq -A^\tau - (m - 2)D_A.
\]

This inequality can be written as

\[
\begin{bmatrix}
-(m-1)A_{1,1} + \sum_{i \neq 1} A_{ii} & 0 & \cdots \\
0 & -(m-1)A_{2,2} + \sum_{i \neq 2} A_{ii} & \cdots \\
\vdots & \vdots & \ddots \\
0 & 0 & \cdots \\
n & \cdots \\
-(m-1)A_{m,m} + \sum_{i \neq m} A_{ii}
\end{bmatrix} \geq \begin{bmatrix}
-(m-1)A_{1,1} & -A_{2,1} & \cdots & -A_{m,1} \\
-A_{1,2} & -(m-1)A_{2,2} & \cdots & -A_{m,2} \\
\vdots & \vdots & \ddots & \vdots \\
-A_{1,m} & -A_{2,m} & \cdots & -(m-1)A_{m,m}
\end{bmatrix}.
\]

Using a similar treatment as to the above proof of (R1), it is sufficient to prove

\[
\begin{bmatrix}
-A_{ii} + A_{jj} & 0 \\
0 & -A_{jj} + A_{ii}
\end{bmatrix} \geq \begin{bmatrix}
-A_{ii} & -A_{ji} \\
-A_{ij} & -A_{jj}
\end{bmatrix}
\]

for every pair \((i,j)\) with \( 1 \leq i < j \leq m \), which follows by noting that

\[
\begin{bmatrix}
A_{ij} & A_{ji} \\
A_{ij} & A_{ii}
\end{bmatrix} = \begin{bmatrix} 0 & I_n \\
I_n & 0 \end{bmatrix} \begin{bmatrix}
A_{ii} & A_{ij} \\
A_{ji} & A_{jj}
\end{bmatrix} \begin{bmatrix} 0 & I_n \\
I_n & 0 \end{bmatrix}.
\]

Thus, this completes the proof. ■
Although Theorem 2.1 is an improvement of Choi’s result (1), as pointed out by a referee, a concise proof of Theorem 2.1 can also be given from Choi’s proof in [14].

**Corollary 2.2:** Let $A = [A_{i,j}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$ be Hermitian. Then,

$$(m - 1)\lambda_{\max}(A)I_{mn} \geq I_m \otimes \text{tr}_1 A^\tau - A^\tau \geq (m - 1)\lambda_{\min}(A)I_{mn}$$

and

$$(m - 1)\lambda_{\max}(A)I_{mn} + 2D_A \geq I_m \otimes \text{tr}_1 A^\tau + A^\tau \geq (m - 1)\lambda_{\min}(A)I_{mn} + 2D_A,$$

where $D_A = A_{1,1} \oplus A_{2,2} \oplus \cdots \oplus A_{m,m}$.

**Proof:** The required result holds from Choi’s result (R1) and Theorem 2.1 by replacing $A$ with $A - \lambda_{\min}(A)I_{mn}$ and $\lambda_{\max}(A)I_{mn} - A$. We leave the detailed proof to the readers. □

### 3. Inequalities about the second partial trace

We present some inequalities of the second partial trace in this section. First of all, we shall give an improvement on (2). To illustrate the relations between the first and second partial traces, we shall apply a useful technique, which was recently introduced by Choi [14]. Assume that $A = [A_{i,j}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$, where $A_{i,j} = [a_{i,j}]_{r,s=1}^n$. We define $B_{r,s} := [a_{r,s}]_{i,j=1}^m \in \mathbb{M}_m$ and

$$\tilde{A} := [B_{r,s}]_{r,s=1}^m \in \mathbb{M}_n(\mathbb{M}_m).$$

Clearly, both $A$ and $\tilde{A}$ are matrices of order $mn \times mn$. Furthermore, $A$ and $\tilde{A}$ are unitarily similar; see, e.g. [20] or [21]. The similarity implies that $\tilde{A}$ is positive semidefinite whenever $A$ is positive semidefinite. Next, we make a brief review of some useful properties.

**Lemma 3.1 ([14]):** Each of the following holds.

(a) For $A, B \in \mathbb{M}_m(\mathbb{M}_n)$, $A \leq B$ implies $\tilde{A} \leq \tilde{B}$.
(b) For $X \in \mathbb{M}_m$ and $Y \in \mathbb{M}_n$, $\tilde{X} \otimes \tilde{Y} = \tilde{Y} \otimes \tilde{X}$.
(c) $\tilde{A}^\tau = (\tilde{A^\tau})^T = (\tilde{A})^T$.
(d) For $A \in \mathbb{M}_m(\mathbb{M}_n)$, $\text{tr}_2 A = \text{tr}_1 \tilde{A}$ and $\text{tr}_2 \tilde{A}^\tau = \text{tr}_1 A^T$.

In the sequel, we write $A \circ B$ for the Hadamard (Schur) product of $A$ and $B$.

**Theorem 3.2:** Let $A \in \mathbb{M}_m(\mathbb{M}_n)$ be positive semidefinite. Then,

$$(\text{tr}_2 A^\tau) \otimes I_n \geq -A^\tau + 2A^\tau \circ J,$$

where $J$ is the $m \times m$ block matrix with each block $I_n$. 


**Proof:** Replacing $A$ with $\tilde{A}$ in Theorem 2.1, we have

$$I_n \otimes \text{tr}_1 \tilde{A}^T \geq -\tilde{A}^T + 2\tilde{D}_A.$$  

Applying Lemma 3.1, we get

$$I_n \otimes \text{tr}_1 (A^T)^T \geq -(A^T)^T + 2\tilde{D}_A.$$

Observe that $X \geq Y$ implies $\tilde{X} \geq \tilde{Y}$. Invoking Lemma 3.1 again, we obtain

$$\left( \text{tr}_2(A^T)^T \right) \otimes I_n = \text{tr}_1 (A^T)^T \otimes I_n \geq -(A^T)^T + 2\tilde{D}_A.$$  

(9)

Recall in (8) that $B_{rs} = [a_{r,s}]_{i,j=1}^m \in \mathbb{M}_m$. A direct computation reveals that

$$\tilde{D}_A = \text{diag}(B_{1,1}, B_{2,2}, \ldots, B_{n,n}) = [A_{ij} \circ I_n]_{i,j=1}^m.$$

Note that $(X \otimes Y)^T = X^T \otimes Y^T$ and

$$(\tilde{D}_A)^T = \left[ (A_{ij} \circ I_n)^T \right]_{i,j=1}^m = [A_{ij} \circ I_n]_{i,j=1}^m = A^T \circ J.$$

Taking transpose in both sides of (9) yields the required result.  

We replace $A$ with $A - \lambda_{\min}(A)I_{mn}$ and $\lambda_{\max}(A)I_{mn} - A$ in Theorem 3.2, which yields the following corollary.

**Corollary 3.3:** Let $A \in \mathbb{M}_m(\mathbb{M}_n)$ be Hermitian. Then

$$(n - 1)\lambda_{\max}(A)I_{mn} + 2A^T \circ J \geq (\text{tr}_2 A^T) \otimes I_n + A^T \geq (n - 1)\lambda_{\min}(A)I_{mn} + 2A^T \circ J,$$

where $J$ is the $m \times m$ block matrix with each block $I_n$.

4. **Inequalities about two partial traces**

In this section, we shall present inequalities involving both the first and second partial traces, which are improvements on Li–Liu–Huang’s result (R5). Recall that a map $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_k$ is called positive if it maps positive semidefinite matrices to positive semidefinite matrices. A map $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_k$ is said to be $m$-positive if for $[A_{ij}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$,

$$[A_{ij}]_{i,j=1}^m \geq 0 \Rightarrow [\Phi(A_{ij})]_{i,j=1}^m \geq 0.$$  

We say that $\Phi$ is completely positive if it is $m$-positive for every integer $m \geq 1$. On the other hand, a map $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_k$ is called $m$-co-positive if

$$[A_{ij}]_{i,j=1}^m \geq 0 \Rightarrow [\Phi(A_{ij})]_{i,j=1}^m \geq 0.$$  

Similarly, the map $\Phi$ is completely co-positive if it is $m$-co-positive for all $m \geq 1$. Furthermore, $\Phi$ is called completely PPT if the block matrix $[\Phi(A_{ij})]_{i,j=1}^m$ is PPT for all $m \geq 1$.  


whenever \([A_{ij}]_{i,j=1}^m \geq 0\), i.e. \(\Phi\) is both completely positive and completely co-positive. It is well known that both the determinant map and the trace map are completely PPT; see [10, p.221,237] and see [4, Chapter 3] for more standard results of completely positive maps.

In 2014, Lin [8] proved that \(\Phi_1(X) = (\text{tr}X)I + X\) is a completely PPT map. Two years later, Lin [17] obtained that \(\Psi_1(X) = (\text{tr}X)I - X\) is a completely co-positive map; see [22] for related applications. It is easy to see that the completely co-positivity of these two maps can also be deduced from Choi’s result (2). In this section, we shall obtain some new matrix inequalities involving both the first and the second partial trace. Our results are improvements on (4) and (5).

It was proved in [23, Example 1] that if \(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\) is positive semidefinite, then
\[
\text{tr}A\text{tr}C - |\text{tr}B|^2 \geq \text{tr}(AC) - \text{tr}(B^*B). \tag{10}
\]

We remark that (10) was also formulated in [8, 24]. Note that the positivity of the block matrix entails \(\text{tr}A\text{tr}C - |\text{tr}B|^2 \geq 0\); see [10, p.237] or [25, Theorem IX.5.10]. However, the right-hand side of (10) might be negative. It has been shown that the PPT condition on the block matrix can ensure \(\text{tr}AC \geq \text{tr}B^*B\); see [26, Theorem 2.1]. Motivated by this observation, Kittaneh and Lin [27] further generalized (10) by an elegant self-improved technique:
\[
\text{tr}A\text{tr}C - |\text{tr}B|^2 \geq \text{tr}(B^*B) - \text{tr}(AC). \tag{11}
\]

By applying the 2-co-positivity of \(\Psi_1(X) = (\text{tr}X)I - X\), they proved also that
\[
\text{tr}A\text{tr}C + |\text{tr}B|^2 \geq \text{tr}(AC) + \text{tr}(B^*B). \tag{12}
\]

It is worth noting that the 2-co-positivity of \(\Psi\) could also lead to the inequality (11); see [22] for more details. Let \(A \in M_m(M_n)\) be positive semidefinite. By using the 2-co-positivity of \(\Psi(X) = (\text{tr}X)I - X\), one can obtain Ando’s result (3). By employing the 2-co-positivity of \(\Phi(X) = (\text{tr}X)I + X\), Li et al. [16] established (4) and (5). Last but not least, one may observe an interesting phenomenon that (4) is similar to (10) and (11) in mathematical writing form. Correspondingly, (5) is also similar to (11) and (12).

In what follows, we will give an improvement on (5).

**Theorem 4.1:** Let \(A = [A_{ij}]_{i,j=1}^m \in M_m(M_n)\) be positive semidefinite. Then
\[
(\text{tr}A)I_{mn} + (\text{tr}2A) \otimes I_n \geq A + I_m \otimes (\text{tr}1A) + 2(\text{tr}2D_A) \otimes I_n - 2D_A,
\]
where \(D_A = A_{1,1} \oplus A_{2,2} \oplus \cdots \oplus A_{m,m}\).

**Proof:** The desired inequality can be written as
\[
\begin{bmatrix}
\sum_{i \neq 1} (\text{tr}A_{i,i})I_n & (\text{tr}A_{1,2})I_n & \cdots & (\text{tr}A_{1,m})I_n \\
(\text{tr}A_{2,1})I_n & \sum_{i \neq 2} (\text{tr}A_{i,i})I_n & \cdots & (\text{tr}A_{2,m})I_n \\
\vdots & \vdots & \ddots & \vdots \\
(\text{tr}A_{m,1})I_n & (\text{tr}A_{m,2})I_n & \cdots & \sum_{i \neq m} (\text{tr}A_{i,i})I_n
\end{bmatrix}
\]
\[ \begin{bmatrix}
\sum_{i \neq 1} A_{i,i} & A_{1,2} & \cdots & A_{1,m} \\
A_{2,1} & \sum_{i \neq 2} A_{i,i} & \cdots & A_{2,m} \\
\vdots & \vdots & & \vdots \\
A_{m,1} & A_{m,2} & \cdots & \sum_{i \neq m} A_{i,i}
\end{bmatrix}.\]

It is easy to see from Theorem 2.1 that
\[ \begin{bmatrix}
\sum_{i \neq 1} A_{i,i} & A_{1,2} & \cdots & A_{1,m} \\
A_{2,1} & \sum_{i \neq 2} A_{i,i} & \cdots & A_{2,m} \\
\vdots & \vdots & & \vdots \\
A_{m,1} & A_{m,2} & \cdots & \sum_{i \neq m} A_{i,i}
\end{bmatrix} \geq 0.\]

Since \( \Psi(X) = (\text{tr}X)I - X \) is completely co-positive. Then applying \( \Psi \) to the above positive semidefinite block matrix yields the required inequality. \( \blacksquare \)

Note that \( (\text{tr}A_{i,i})I_n \geq A_{i,i} \) for each integer \( i \), then \( (\text{tr}_2 D_A) \otimes I_n \geq D_A \), i.e.
\[ 2(\text{tr}_2 D_A) \otimes I_n - 2D_A \geq 0.\]

So Theorem 4.1 is indeed a generalization of (5). In view of symmetry of definitions of \( \text{tr}_1 \) and \( \text{tr}_2 \), one can easily obtain the following equivalent theorem.

**Theorem 4.2:** Let \( A \in \mathbb{M}_m(\mathbb{M}_n) \) be positive semidefinite. Then
\[ (\text{tr}A)I_{mn} - (\text{tr}_2 A) \otimes I_n \geq A - I_m \otimes (\text{tr}_1 A) + 2(I_m \otimes \text{tr}_1 A - A) \circ J, \]
where \( J \) is the \( m \times m \) block matrix with each block \( I_n \).

The next result is an analogue of Theorem 4.1.

**Theorem 4.3:** Let \( A = [A_{i,j}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n) \) be positive semidefinite. Then
\[ (\text{tr}A)I_{mn} + (\text{tr}_2 A) \otimes I_n + I_m \otimes (\text{tr}_1 A) + A \geq 2(\text{tr}_2 D_A) \otimes I_n + 2D_A, \]
where \( D_A = A_{1,1} \oplus A_{2,2} \oplus \cdots \oplus A_{m,m} \).

We remark that Theorems 4.1, 4.2 and 4.3 all require the positivity of \( A \), by replacing \( A \) by \( A - \lambda_{\text{min}}(A)I_{mn} \) and \( \lambda_{\text{max}}(A)I_{mn} - A \), one can extend these theorems to Hermitian matrices. Motivated by Theorems 4.1 and 4.2, we believe highly that it is possible to improve Ando’s result (3). In other words, for every positive semidefinite matrix \( A \in \mathbb{M}_m(\mathbb{M}_n) \), is there a positive semidefinite matrix \( T \) satisfying the following inequality?
\[ (\text{tr}A)I_{mn} + A \geq I_m \otimes (\text{tr}_1 A) + (\text{tr}_2 A) \otimes I_n + T.\]
5. Extensions on Cauchy–Khinchin’s inequality

In this section, we shall provide some applications of Theorems 4.1 and 4.2 in the field of numerical inequalities. The Cauchy–Khinchin inequality (see [28, Theorem 1]) states that if \( X = [x_{ij}] \) is an \( m \times n \) real matrix, then

\[
\left( \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \right)^2 + mn \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^2 \geq m \sum_{i=1}^{m} \left( \sum_{j=1}^{n} x_{ij} \right)^2 + n \sum_{j=1}^{n} \left( \sum_{i=1}^{m} x_{ij} \right)^2.
\]

(13)

It was proved that this inequality has many applications on problems related to the directed graph in combinatorics; see, e.g., [29, 30] for more details. In 2016, Lin [17] provided a simple proof using Ando’s inequality (3). Under the similar line, Li et al. [16] proved some generalizations by applying (4) and (5). In this section, we shall prove the following corollary, which is an easy consequence of Theorem 4.1.

**Corollary 5.1:** Let \( X = [x_{ij}] \) be an \( m \times n \) real matrix. Then

\[
(m - 2)n \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^2 + n \sum_{j=1}^{n} \left( \sum_{i=1}^{m} x_{ij} \right)^2 \geq \left( \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \right)^2 + (m - 2) \sum_{i=1}^{m} \left( \sum_{j=1}^{n} x_{ij} \right)^2
\]

and

\[
m(n - 2) \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^2 + m \sum_{i=1}^{m} \left( \sum_{j=1}^{n} x_{ij} \right)^2 \geq \left( \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \right)^2 + (n - 2) \sum_{j=1}^{n} \left( \sum_{i=1}^{m} x_{ij} \right)^2.
\]

**Proof:** We only prove the first inequality since the second is similar. Let \( J_n \) be an \( n \)-square matrix with all entries 1. Setting \( A = J_m \otimes J_n \) in Theorem 4.1. Clearly, \( \text{tr}_1 A = m J_n \) and \( \text{tr}_2 A = n J_m \). Additionally, we have \( D_A = J_n \oplus \cdots \oplus J_n \), which leads to \( (\text{tr}_2 D_A) \otimes J_n = (n I_m) \otimes J_n \). Therefore,

\[
(m - 2)n I_{mn} + n I_m \otimes J_n \geq J_m \otimes J_n + (m - 2) I_m \otimes J_n.
\]

(14)

Let \( \text{vec} X = [x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{mn}]^T \in \mathbb{R}^{mn} \) be the column vector determined by matrix \( X \). Then some calculations give the following equalities:

\[
(\text{vec} X)^T I_{mn} (\text{vec} X) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^2,
\]

\[
(\text{vec} X)^T (J_m \otimes J_n) (\text{vec} X) = \sum_{j=1}^{n} \left( \sum_{i=1}^{m} x_{ij} \right)^2,
\]

\[
(\text{vec} X)^T (J_m \otimes J_n) (\text{vec} X) = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \right)^2.
\]
\[(\text{vec } X)^T (I_m \otimes J_n)(\text{vec } X) = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} x_{ij} \right)^2.\]

Thus, the desired inequality is equivalent to
\[
(\text{vec } X)^T ((m - 2) n I_{mn} + n J_m \otimes I_n)(\text{vec } X) \geq (\text{vec } X)^T (J_m \otimes J_n + (m - 2) I_m \otimes J_n)(\text{vec } X).
\]

This inequality follows immediately from the matrix inequality (14).

**Remark 5.2:** The inequality (14) can also be proved by a standard argument on eigenvalues by noting that $I_{mn}, I_{mn}, J_m \otimes J_n$ and $J_m \otimes I_n$ mutually commute. In addition, setting $A = J_m \otimes J_n$, we see that $A = A^\tau$ and $A$ is PPT. Unfortunately, our results in Sections 2 and 3, e.g. Theorems 2.1 and 3.2, will lead to some trivial inequalities.

### 6. More inequalities for two by two block matrices

In this section, we will investigate the $2 \times 2$ block positive semidefinite matrices. First of all, we introduce the notion of majorization. For a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we arrange the coordinates of $x$ in non-increasing order $x_{i_1}^\downarrow \geq \cdots \geq x_{i_n}^\downarrow$ and denote $x^\downarrow = (x_{i_1}^\downarrow, \ldots, x_{i_n}^\downarrow)$. Given $x, y \in \mathbb{R}^n$, we say that $x$ is weakly majorized by $y$, written as $x \asymp_W y$, if
\[
\sum_{i=1}^{k} x_{i}^\downarrow \leq \sum_{i=1}^{k} y_{i}^\downarrow \quad \text{for } k = 1, 2, \ldots, n.
\]

We say that $x$ is majorized by $y$, denoted by $x < y$, if $x \asymp_W y$ and the sum of all entries of $x$ equal to the sum of all entries of $y$. More generally, if the dimension of $x$ is larger than $y$, the inequality $x < y$ really means that $x < (y, 0)$, where the zero vector is added to make the length of $(y, 0)$ the same as that of $x$. The majorization theory has become a rich research field with far-reaching applications to a wide number of areas, we refer to the recent monograph [31] or [10, Chapter 10] for comprehensive surveys on this subject.

Let $H$ be a Hermitian matrix. We denote by $\lambda(H)$ the vector of eigenvalues of $H$ in which the components are sorted in decreasing order. Historically, the first result of majorization arising in matrix theory is usually attributed to Issai Schur, who proved that the diagonal elements of $H$ are majorized by its eigenvalues, i.e. $d(H) < \lambda(H)$. This majorization provided a new and profound understanding on Hadamard’s determinant inequality; see, e.g. [32, p.514]. Due to Schur’s discovery, a large number of majorization inequalities have been found in the context of matrix analysis. For instance, if $A = [A_{ij}]_{i,j=1}^{m} \in M_m(M_n)$ is Hermitian, then Schur’s inequality implies $d(A) < \lambda(A_{1,1} \oplus \cdots \oplus A_{m,m})$. Furthermore, if $A$ is positive semidefinite, it is well-known (see [32, p.259] and [31, p.308]) that
\[
\lambda(A_{1,1} \oplus \cdots \oplus A_{m,m}) < \lambda(A) < \lambda(A_{1,1}) + \cdots + \lambda(A_{m,m}).
\]

Moreover, Rotfeld and Thompson [31, p. 330] obtained an analogous complement,
\[
\lambda(A_{1,1} \oplus \cdots \oplus A_{m,m}) < \lambda(\text{tr}_1 A) < \lambda(A_{1,1}) + \cdots + \lambda(A_{m,m}).
\]
In the sequel, we shall provide a comparison between (15) and (16) under the PPT condition. First of all, we recall the following lemma, which was proved by Hiroshima [1] in the language of quantum information theory; see [33] for an alternative proof.

**Lemma 6.1 ([1, 33]):** Let \( A \in \mathbb{M}_m(\mathbb{M}_n) \) be positive semidefinite.

1. If \( I_m \otimes \text{tr}_1 A \geq A \), then \( \lambda(A) \prec \lambda(\text{tr}_1 A) \).
2. If \( (\text{tr}_2 A) \otimes I_n \geq A \), then \( \lambda(A) \prec \lambda(\text{tr}_2 A) \).

Our first result in this section is the following majorization inequalities, which is a direct consequence of Lemma 6.1 by applying inequalities (1) and (2).

**Theorem 6.2:** Let \( A \in \mathbb{M}_m(\mathbb{M}_n) \) be PPT. Then

\[
\lambda(A) \prec \lambda(\text{tr}_1 A) \quad \text{and} \quad \lambda(A) \prec \lambda(\text{tr}_2 A).
\]

Similarly, we have

\[
\lambda(A^\tau) \prec \lambda(\text{tr}_1 A) \quad \text{and} \quad \lambda(A^\tau) \prec \lambda(\text{tr}_2 A).
\]

In particular, for the case of \( 2 \times 2 \) block matrices, we remark here that inequalities involving \( \text{tr}_1 \) in Theorem 6.2 was partially proved by Bourin et al. [34, 35] by making use of a simple but useful decomposition lemma for positive semidefinite matrices. Moreover, other special cases can be found in [36] and [37, Corollary 5].

Over the past few years, \( 2 \times 2 \) block positive partial transpose matrices play an important role in matrix analysis and quantum information, such as the separability of mixed states and the subadditivity of \( q \)-entropies; see [23, 26] for related topics and references to the physics literature. It is extremely meaningful and significant to finding \( 2 \times 2 \) block PPT matrices. To the author’s best knowledge, the most famous example is commonly regarded as the Hua matrix; see [26, 38–41] for more details.

In 2014, Lin [8, Proposition 2.2] proved that if \( \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n) \) is positive semidefinite, then

\[
\begin{bmatrix}
(\text{tr}A)I + A & (\text{tr}B)I + B \\
(\text{tr}B^*)I + B^* & (\text{tr}C)I + C
\end{bmatrix}
\]

is PPT. In 2017, Choi [13, Theorem 4] also showed an extremely similar result, which states that

\[
\begin{bmatrix}
(\text{tr}A)I + C & (\text{tr}B)I - B \\
(\text{tr}B^*)I - B^* & (\text{tr}C)I + A
\end{bmatrix}
\]

is positive semidefinite. In the next theorem, we shall prove that the \( 2 \times 2 \) block matrix in (17) is further PPT.

**Theorem 6.3:** Let \( \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n) \) be positive semidefinite. Then

\[
\begin{bmatrix}
(\text{tr}A)I + C & (\text{tr}B)I - B \\
(\text{tr}B^*)I - B^* & (\text{tr}C)I + A
\end{bmatrix}
\]

is PPT.
Proof: In view of the positivity of (17), we only need to prove

\[
\begin{bmatrix}
(trA)I + C & (trB^*)I - B^*\\
(trB)I - B & (trC)I + A
\end{bmatrix} \succeq 0.
\]

Note that

\[
\begin{bmatrix}
C & -B^* \\
-B & A
\end{bmatrix} = \begin{bmatrix}
0 & -I \\
I & 0
\end{bmatrix} \begin{bmatrix}
A & B \\
B^* & C
\end{bmatrix} \begin{bmatrix}
0 & I \\
-I & 0
\end{bmatrix} \succeq 0.
\]

It suffices to show

\[
\begin{bmatrix}
(trA)I & (trB^*)I \\
(trB)I & (trC)I
\end{bmatrix} \succeq 0,
\]

which follows from the complete positivity of trace map.

A norm \(\|\cdot\|\) on \(\mathbb{M}_n\) is called unitarily invariant if \(\|UAV\| = \|A\|\) for any \(A \in \mathbb{M}_n\) and any unitary matrices \(U, V \in \mathbb{M}_n\). The unitarily invariant norm of a matrix is closely related to its singular values; see, e.g. [10, p.372–376] and [25, p.91–98]. Recall in Section 4 that \(\Phi(X) = (trX)I + X\) and \(\Psi(X) = (trX)I - X\). In the sequel, we shall present some inequalities involving unitarily invariant norms and singular values for these two maps.

**Corollary 6.4:** Let \(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)\) be positive semidefinite. Then

\[
2\|\Phi(B)\| \leq \|\Phi(A) + \Phi(C)\|
\]

and

\[
2\|\Psi(B)\| \leq \|\Phi(A) + \Phi(C)\|
\]

for any unitarily invariant norm.

**Proof:** Invoking a theorem of von Neumann (see [10, p. 375]), it is sufficient to prove

\[
2s((trB)I \pm B) \prec_w s((tr(A + C))I + A + C),
\]

where \(s(X)\) is the vector consisting of singular values of \(X\). We next prove the second inequality only. There is a well-known fact that if \(X = \begin{bmatrix} Y & Z \\ Z^* & W \end{bmatrix}\) is positive semidefinite, then \(2s_i(Z) \leq s_i(X)\) for every \(i\). Thus

\[
2s((trB)I - B) \prec_w s\left[ (trA)I + C & (trB^*)I - B^* \\
(trB)I - B & (trC)I + A
\right] \prec_w s((tr(A + C))I + A + C),
\]

where we used Theorems 6.3 and 6.2.

The second partial trace inequality in Theorem 6.2 yields the following corollary.

**Corollary 6.5:** Let \(H = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)\) be positive semidefinite. Then

\[
2\|\Phi(B)\| \leq (n + 1) \|\text{tr}_2 H\|
\]

for any unitarily invariant norm.
At the end of this paper, we conclude with the following inequality involving singular values, which is a stronger inequality than Corollary 6.4.

**Theorem 6.6:** Let $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ be positive semidefinite. Then for $j = 1, 2, \ldots, n$,

$$2s_j(\Phi(B)) \leq s_j(\Phi(A) + \Phi(C))$$

and

$$2s_j(\Psi(B)) \leq s_j(\Phi(A) + \Phi(C)).$$

Note that the first inequality in Theorem 6.6 was proved by Lin as a main result in [19]. To some extent, our proof of Theorem 6.6 grows out from [19] with some simplifications. To proceed the proof, we need to present the following two lemmas.

**Lemma 6.7 ([25, p. 262]):** For any $M, N \in \mathbb{M}_{n \times m}$ and $j = 1, 2, \ldots, n$, we have

$$2s_j(MN^*) \leq s_j(M^*M + N^*N).$$

**Lemma 6.8:** For any $M, N \in \mathbb{M}_{n \times m}$ and $j = 1, 2, \ldots, n$, we have

$$\lambda_j(M^*M + N^*N) \leq \lambda_j(MM^* + NN^*) + \frac{1}{2} \text{tr}(M^*M + N^*N - M^*N - N^*M).$$

**Proof:** By a direct computation, we can get

$$2\lambda_j(MM^* + NN^*) = \lambda_j((M + N)(M + N)^* + (M - N)(M - N)^*)$$

$$\geq \lambda_j((M + N)(M + N)^*) + \lambda_n((M - N)(M - N)^*)$$

$$\geq \lambda_j((M + N)^*(M + N))$$

$$= \lambda_j(2(M^*M + N^*N) - (M - N)^*(M - N))$$

$$\geq 2\lambda_j((M^*M + N^*N) - \lambda_1((M - N)^*(M - N)),$$

where the first and last inequality hold by Weyl’s eigenvalue inequality (see [25, p. 63]). Moreover, the positivity of $(M - N)^*(M - N)$ leads to

$$\lambda_1((M - N)^*(M - N)) \leq \text{tr}((M - N)^*(M - N))$$

$$= \text{tr}(M^*M + N^*N - M^*N - N^*M).$$

This completes the proof. □

**Proof of Theorem 6.6:** Since $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ is positive semidefinite, we may write

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} = \begin{bmatrix} MM^* & MN^* \\ NM^* & NN^* \end{bmatrix}$$

for some $M, N \in \mathbb{M}_{n \times 2n}$. Then the desired inequality is the same as

$$2s_j(\text{tr}(MN^*)I \pm MN^*) \leq s_j(\text{tr}(MM^* + NN^*)I + MM^* + NN^*).$$

By the Weyl’s inequality of singular value and Lemma 6.7, we have

$$2s_j(\text{tr}(MN^*)I \pm MN^*) \leq 2s_j(\pm MN^*) + 2s_1((\text{tr}MN^*)I).$$
\[ = 2s_j(MN^*) + 2|\text{tr}(MN^*)| \]
\[ \leq s_j(M^*M + N^*N) + 2|\text{tr}(MN^*)|, \]

On the other hand, we observe that

\[ s_j(\text{tr}(MM^* + NN^*)I + MM^* + NN^*) = \lambda_j(MM^* + NN^*) + \text{tr}(MM^* + NN^*). \]

It is sufficient to prove that for every \( M, N \in \mathbb{M}_{n \times m} \),

\[ s_j(M^*M + N^*N) + 2|\text{tr}(MN^*)| \leq \lambda_j(MM^* + NN^*) + \text{tr}(MM^* + NN^*). \quad (18) \]

We may assume without loss of generality that \( \text{tr}(MN^*) \geq 0 \) in (18), since it is clear that (18) is invariant when we replace \( M \) with \( e^{i\theta}M \) for every \( \theta \in [0, 2\pi] \). Therefore,

\[ s_j(M^*M + N^*N) + |\text{tr}(MN^*)| \]
\[ = \lambda_j(M^*M + N^*N) + \frac{1}{2}\text{tr}(M^*N + N^*M) \]
\[ \leq \lambda_j(MM^* + NN^*) + \frac{1}{2}\text{tr}(M^*M + N^*N), \]

where the last inequality holds by Lemma 6.8. Note that

\[
\begin{pmatrix}
\text{tr}MM^* & \text{tr}MN^*\\
\text{tr}NM^* & \text{tr}NN^*
\end{pmatrix}
\]

is a positive semidefinite matrix, we have

\[ |\text{tr}(MN^*)| = \frac{1}{2}\text{tr}(MN^* + NM^*) \leq \frac{1}{2}\text{tr}(MM^* + NN^*). \quad (20) \]

The desired inequality (18) follows by combining (19) and (20).

\[ \square \]

**Acknowledgments**

The paper is dedicated to Prof. Weijun Liu, my teacher on the occasion of his 60th birthday, October 22 of the lunar calendar in 2021. The author would like to thank Prof. Minghua Lin for bringing the topic on partial traces to his attention. Thanks also go to Prof. Fuzhen Zhang and Prof. Yuejian Peng for reading an earlier draft of the paper, Prof. Xiaohui Fu for the inspiring discussions over the years, and the anonymous referee for helpful suggestions on improving the presentation of this paper.

**Disclosure statement**

No potential conflict of interest was reported by the author.

**Funding**

This work was supported by NSFC [grant no. 11931002].
References

[1] Hiroshima T. Majorization criterion for distillability of a bipartite quantum state. Phys Rev Lett. 2003;91:057902.
[2] Horodecki M, Horodecki P, Horodecki R. Separability of mixed states: necessary and sufficient conditions. Phys Lett A. 1996;223:1–8.
[3] Petz D. Quantum information theory and quantum statistics. Berlin: Springer; 2008. (Theoretical and mathematical physics).
[4] Bhatia R. Positive definite matrices. Princeton: Princeton University Press; 2007.
[5] Fu X, Lau P-S, Tam T-Y. Inequalities on partial traces of positive semidefinite block matrices. Canad Math Bull. 2021;64(4):964–969.
[6] Fu X, Lau P-S, Tam T-Y. Inequalities on $2 \times 2$ block positive semidefinite matrices. Linear Multilinear Algebra. 2020; DOI: 10.1080/03081087.2020.1969327:
[7] Lee E-Y. The off-diagonal block of a PPT matrix. Linear Algebra Appl. 2015;486:449–453.
[8] Lin M. A completely PPT map. Linear Algebra Appl. 2014;459:404–410.
[9] Zhang P. On some inequalities related to positive block matrices. Linear Algebra Appl. 2019;576:258–267.
[10] Zhang F. Matrix theory: basic results and techniques. 2nd ed. New York: Springer; 2011.
[11] Zhang F. Positivity of matrices with generalized matrix functions. Acta Math Sin (Engl Ser). 2012;28:1779–1786.
[12] Ando T. Matrix inequalities involving partial traces. ILAS Conference; 2014.
[13] Choi D. Inequalities related to partial transpose and partial trace. Linear Algebra Appl. 2017;516:1–7.
[14] Choi D. Inequalities about partial transpose and partial traces. Linear Multilinear Algebra. 2018;66:1619–1625.
[15] Fu X, Lau P-S, Tam T-Y. Linear maps of positive partial transpose matrices and singular value inequalities. Math Inequal Appl. 2020;23(4):1459–1468.
[16] Li Y, Liu W, Huang Y. A new matrix inequality involving partial traces. Operators Matrices. 2021;15(3):1189–1199.
[17] Lin M. A determinantal inequality involving partial traces. Canad Math Bull. 2016;59:585–591.
[18] Li Y. Extensions of some matrix inequalities related to trace and partial traces. Linear Algebra Appl. 2022;639:205–224.
[19] Lin M. A singular value inequality related to a linear map. Electron J Linear Algebra. 2016;31:120–124.
[20] Choi D. Inequalities related to trace and determinant of positive semidefinite block matrices. Linear Algebra Appl. 2017;532:1–7.
[21] Li Y, Feng L, Huang Z, et al. Inequalities regarding partial trace and partial determinant. Math Inequal Appl. 2020;23(2):477–485.
[22] Li Y, Huang Y, Feng L, et al. Some applications of two completely copositive maps. Linear Algebra Appl. 2020;590:124–132.
[23] Desenyei A, Petz D. Partial subadditivity of entropies. Linear Algebra Appl. 2013;439:3297–3305.
[24] Zhang F. Trace inequality for positive block matrices. IMAGE Solution 50-3.1, Bull Internat Linear Algebra Soc. (Fall 2013).
[25] Bhatia R. Matrix analysis. New York: Springer-Verlag; 1997. (GTM; 169).
[26] Lin M. Inequalities related to $2 \times 2$ block PPT matrices. Operators Matrices. 2015;9(4):917–924.
[27] Kittaneh F, Lin M. Trace inequalities for positive semidefinite block matrices. Linear Algebra Appl. 2017;524:153–158.
[28] van Dam ER. A Cauchy–Khinchin matrix inequality. Linear Algebra Appl. 1998;280:163–172.
[29] de Caen D. An upper bound on the sum of squares of degrees in a graph. Discrete Math. 1998;185:245–248.
[30] Li J-S, Pan Y-L. de Caen’s inequality and bounds on the largest Laplacian eigenvalue of a graph. Linear Algebra Appl. 2001;328:153–160.
[31] Marshall AW, Olkin I, Arnold BC. Inequalities: theory of majorization and its applications. 2nd ed. New York: Springer; 2011.
[32] Horn RA, Johnson CR. Matrix analysis. 2nd ed. New York: Cambridge University Press; 2013.
[33] Lin M. Some applications of a majorization inequality due to Bapat and Sunder. Linear Algebra Appl. 2015;469:510–517.
[34] Bourin J-C, Lee E-Y, Lin M. On a decomposition lemma for positive semi-definite block-matrices. Linear Algebra Appl. 2012;437:1906–1912.
[35] Bourin J-C, Lee E-Y, Lin M. Positive matrices partitioned into a small number of Hermitian blocks. Linear Algebra Appl. 2013;438:2591–2598.
[36] Lin M, Wolkowicz H. An eigenvalue majorization inequality for positive semidefinite block matrices. Linear Multilinear Algebra. 2012;60:1365–1368.
[37] Turkmen R, Paksoy VE, Zhang F. Some inequalities of majorization type. Linear Algebra Appl. 2012;437:1305–1316.
[38] Ando T. Positivity of operator-matrices of Hua-type. Banach J Math Anal. 2008;2:1–8.
[39] Hua L-K. Inequalities involving determinants (in Chinese). Acta Math Sin. 1955;5:463–470. See also Trans. Am. Math. Soc. Ser. II 32 (1963) 265–272.
[40] Lin M. The Hua matrix and inequalities related to contractive matrices. Linear Algebra Appl. 2016;511:22–30.
[41] Xu G, Xu C, Zhang F. Contractive matrices of Hua type. Linear Multilinear Algebra. 2011;59:159–172.