Analytical Results for the Grand Canonical Partition Function for Unidimensional Hubbard Model, Up to Order $\beta^5$.

I.C. Charret\textsuperscript{1,}
Departamento de Física
Instituto de Ciências Exatas
Universidade Federal de Minas Gerais
Campus da Pampulha
Belo Horizonte, M.G., 31270–901 BRAZIL

E.V. Corrêa Silva\textsuperscript{2}
Centro Brasileiro de Pesquisas Físicas
R. Dr. Xavier Sigaud n.º 150
Rio de Janeiro, R.J., 22290-180 BRAZIL

S.M. de Souza\textsuperscript{3},
Departamento de Ciências Exatas
Universidade Federal de Lavras
C.P.: 37
Lavras, M.G., 37200–000 BRAZIL

M.T. Thomaz\textsuperscript{4}
Instituto de Física
Universidade Federal Fluminense
Av. Gal. Milton Tavares de Souza s/nº
Campus da Praia Vermelha
Niterói, R.J., 24210–310 BRAZIL

Abstract

We calculate the exact analytical coefficients of the $\beta$–expansion of the grand canonical partition function of the unidimensional Hubbard model up to order $\beta^5$, using an alternative method, based on properties of the Grassmann algebra. The results derived are non–perturbative and no restrictions on the set of parameters that characterize the model are required. By applying this method we obtain analytical results for the thermodynamical quantities, in the high–temperature limits, for arbitrary density of electrons in the unidimensional chain.

\textsuperscript{1} E–mail: iraziet@if.uff.br
\textsuperscript{2} E–mail: ecorrea@cbpfsl1.cat.cbpf.br
\textsuperscript{3} E–mail: smartins@if.uff.br
\textsuperscript{4} Corresponding author: Dr. Maria Teresa Thomaz; R. Domingos Sávio Nogueira Saad n.º 120 apto 404, Niterói, R.J., 24210–340, BRAZIL –Phone/Fax: (21) 620–6735;
E–mail: mtt@if.uff.br
1. Introduction

Working with fermionic variables seems discouraging, for their non-commuting nature, in a certain way, compells one to a higher degree of care than that required by commutative variables. Nevertheless, Grassmann algebra properties justify their use in many circumstances [1]. With that in mind, we have recently described an alternative method to calculate the terms of the $\beta$-expansion of the grand canonical partition function of periodic unidimensional self-interacting fermionic models, in which Grassmann algebra properties play a central role. No auxiliary fields are needed, and we express our results in terms of matrices with commuting elements. The general approach for a periodic unidimensional fermionic model has been applied to the unidimensional Hubbard model up to order $\beta^3$ [2]. An important point about the method developed in reference [2] is that even though the unidimensional Hubbard model has exact solutions [3], the analytical expressions are only known in the half-filled case. This drawback hinders the analytical evaluation of the partition function for the model from the knowledge of its energy spectrum when we are not considering the half-filling case. Takahashi[4] derived a closed expression for the grand canonical partition function of the unidimensional Hubbard model, but besides requiring the formulation of some additional hypothesis, a simple closed expression for the grand canonical partition function is only obtained in the strong coupling limit. The literature offers many examples of high temperature expansions of the grand canonical partition function for the Hubbard model in different space dimensions, some of them up to order $\beta^9$ [5]. However, all these works referer to either some approximation in which one of the characteristic constants of the model has to be much bigger than the other, or some consideration based on important numerical analysis.

Our results are *analytical* and do not rest upon any additional hypothesis on the constants that characterize the model. We should point out that we do *not* perform a perturbative expansion valid in the high temperature limit [5], but a $\beta$-expansion of the grand canonical partition function [6] where we calculate the exact analytical coefficients of the terms up to order $\beta^5$ for any density of electrons in the unidimensional chain.

In the present paper, we will apply the approach developed in reference [2] to get the exact terms at orders $\beta^4$ and $\beta^5$ of the grand canonical partition function of the unidimensional Hubbard model. In section 2 we present a review of the results of reference [2]. In section 3,
we write down the unidimensional Hubbard model and the Grassmann functions necessary in the calculations that follow. In section 4 we obtain the coefficients at orders $\beta^4$ and $\beta^5$ of the $\beta$-expansion of the grand canonical partition function of the unidimensional Hubbard model. We use the grand potential derived up to order $\beta^4$ to calculate some physical quantities. A certain property of the multivariable Grassmann integrals — namely, its factorization — opens the way to extending the method of reference [2] to orders higher than $\beta^3$. This factorization property is presented in Appendix A through one example. In Appendix B we introduce a graphical notation that greatly simplifies the calculations. In Appendix C, we present a table of the necessary multivariable Grassmann integrals. Section 5 is dedicated to our conclusions.

2. Review of Previous Results [2]

The grand canonical partition function of any system is given by:

$$
Z(\beta; \mu) = \text{Tr}(e^{-\beta K}), \quad (2.1)
$$

where $\beta = \frac{1}{kT}$, $k$ is the Boltzmann constant, $T$ is the absolute temperature, and

$$
K = H - \mu N, \quad (2.1a)
$$

$H$ is the hamiltonian of the system, $\mu$ is the chemical potential and $N$ is the total number of particles operator.

In the high temperature limit, $\beta \ll 1$, $Z(\beta; \mu)$ has the expansion

$$
Z(\beta, \mu) = \text{Tr}[\mathbb{1} - \beta K] + \sum_{n=2}^{\infty} \frac{(-\beta)^n}{n!} \text{Tr}[K^n], \quad (2.2)
$$

that we call the $\beta$-expansion of the grand canonical partition function.

For any self-interacting fermionic quantum system, in reference [2] we used the Grassmann algebra to show that

$$
\text{Tr}[K^n] = \int \prod_{I=1}^{2nN} d\eta_I d\bar{\eta}_I \sum_{I,J=1}^{2nN} \bar{\eta}_I A_{IJ} \eta_J \times 
\times K^{\otimes}(\bar{\eta}, \eta; \nu = 0) K^{\otimes}(\bar{\eta}, \eta; \nu = 1) \cdots K^{\otimes}(\bar{\eta}, \eta; \nu = n-1), \quad (2.3)
$$
where the matrix $A$ is given by,

$$ A = \begin{pmatrix} A^{\uparrow\uparrow} & \emptyset \\ \emptyset & A^{\downarrow\downarrow} \end{pmatrix}, \quad (2.3a) $$

and

$$ A^{\uparrow\uparrow} = A^{\downarrow\downarrow} = \begin{pmatrix} \mathbf{1}_{N \times N} & -\mathbf{1}_{N \times N} & \mathbf{0}_{N \times N} & \cdots & \mathbf{0}_{N \times N} \\ \mathbf{0}_{N \times N} & \mathbf{1}_{N \times N} & -\mathbf{1}_{N \times N} & \cdots & \mathbf{0}_{N \times N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{1}_{N \times N} & \mathbf{0}_{N \times N} & \mathbf{0}_{N \times N} & \cdots & \mathbf{1}_{N \times N} \end{pmatrix}. \quad (2.3b) $$

Each matrix $A^{\sigma\sigma}$ has dimension $nN \times nN$, $\mathbf{1}_{N \times N}$ been the identity matrix in dimension $N \times N$ and $\mathbf{0}_{N \times N}$ the null matrix in this dimension. $N$ is the number of space sites and $n$ is the power of the $\beta$ term. The non–null elements of $A^{\sigma\sigma}$, $\sigma = \uparrow$ and $\sigma = \downarrow$, are

$$ A^{\sigma\sigma}_{ij} = \begin{cases} a_{II} = 1, & I = 1, 2, \cdots, nN \\
-1, & I = 1, 2, \cdots, (n-2)N \\
1, & I = 1, 2, \cdots, N. \end{cases} \quad (2.3c) $$

The Grassmann function $K^{\otimes}(\bar{\eta}, \eta)$ is the kernel of the fermionic operator $K$ in the normal order [7]. In writing down eq.(2.3), we have used a particular mapping for the Grassmann generators [2], that greatly simplify our calculations:

$$ \eta^{\uparrow}(x_I, \tau_\nu) \equiv \eta_{\nu N+l} \quad (2.4a) $$

and

$$ \eta^{\downarrow}(x_I, \tau_\nu) \equiv \eta_{(n+\nu)N+l}, \quad (2.4b) $$

where $l = 1, 2, \cdots, N$, and $\nu = 0, 1, \cdots, n-1$. The mappings (2.4a–b) can be summarized as [8]:

$$ \eta^{\sigma}(x_I, \tau_\nu) \equiv \eta_{(\nu(n+\sigma)-1)N+l}. \quad (2.4c) $$

The generators $\bar{\eta}^{\sigma}(x_I, \tau_\nu)$ have an equivalent mapping.
In reference [9] we showed that the Grassmann integrals (2.3) can be written as co–factors of the matrix $A^{\sigma\sigma}$. As we have stated before, the calculation of the co–factors of matrix $A^{\sigma\sigma}$ gets simpler when we diagonalize it through a similarity transformation

$$P^{-1}A^{\sigma\sigma}P = D,$$

where the matrix $D$ is,

$$D = \begin{pmatrix}
\lambda_1 I_{N \times N} & 0 & \cdots & 0 \\
0 & \lambda_2 I_{N \times N} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n I_{N \times N}
\end{pmatrix},$$

(2.5a)

$\lambda_i$, $i = 1, 2, \cdots, n$, are the eigenvalues of matrices $A^{\sigma\sigma}$, $\sigma = \uparrow, \downarrow$. At the same time, we transform of the anti–commuting variables,

$$\eta' = P^{-1}\eta \quad \text{and} \quad \bar{\eta}' = \bar{\eta}P.$$ 

(2.5b)

The matrices $P$ and $P^{-1}$ have a block–structure, and in reference [2] we got the elements of matrix $P$ and its inverse for the particular cases $n = 2$ and $n = 3$, besides the eigenvalues of matrices $A^{\sigma\sigma}$, $\sigma = \uparrow, \downarrow$. However, for any value of $n$, we have that

$$p_{\nu\nu'}^{(n)} = \frac{1}{\sqrt{n}} e^{\frac{i\pi}{n}(2\nu'+1)(\nu+1)},$$

(2.6a)

and

$$q_{\nu'\nu}^{(n)} = \frac{1}{\sqrt{n}} e^{-\frac{i\pi}{n}(2\nu'+1)(\nu+1)},$$

(2.6b)

with $\nu, \nu' = 0, 1, \cdots, n - 1$, and

$$P = \begin{pmatrix}
p_{00}^{(n)} I_{N \times N} & \cdots & p_{0,n-1}^{(n)} I_{N \times N} \\
\vdots & \ddots & \vdots \\
p_{n-1,0}^{(n)} I_{N \times N} & \cdots & p_{n-1,n-1}^{(n)} I_{N \times N}
\end{pmatrix},$$

(2.6c)

and

$$P^{-1} = \begin{pmatrix}
q_{00}^{(n)} I_{N \times N} & \cdots & q_{0,n-1}^{(n)} I_{N \times N} \\
\vdots & \ddots & \vdots \\
q_{n-1,0}^{(n)} I_{N \times N} & \cdots & q_{n-1,n-1}^{(n)} I_{N \times N}
\end{pmatrix}.$$ 

(2.6d)
The diagonal elements of matrix $D$ are:

$$\lambda^{(n)}_\nu = 1 - e^{\frac{i\pi}{n} (2\nu + 1)}, \quad \nu = 0, 1, \cdots, n - 1. \quad (2.6e)$$

where the eigenvalues are $N$-fold degenerated, $N$ being the number of space sites. Due to space translation symmetry, we should note that the elements $p^{(n)}_{\nu\nu'}$ and $q^{(n)}_{\nu\nu'}$ do not carry any space site index.

Once the matrix $A$ has a block-structure as depicted in eq.(2.3), the integrals (2.3) are equal to the product of the integral in the sector $\sigma\sigma = \uparrow\uparrow$ times the integral in the sector $\sigma\sigma = \downarrow\downarrow$. The Grassmann integrals in the sector $\uparrow\uparrow$ have the form:

$$M(L, K) = \int d\eta_1 d\bar{\eta}_1 \eta_{k_1} \cdots \bar{\eta}_{k_m} \eta_1 \cdots \bar{\eta}_{k_m} e^{\frac{i\pi}{n} \sum_{i,j=1}^{nN} \bar{\eta}_i A_{ij} \eta_j}, \quad (2.7)$$

with $L = \{l_1, \cdots, l_m\}$ and $K = \{k_1, \cdots, k_m\}$. The products $\bar{\eta} \eta$ are ordered in such a way that $l_1 < l_2 < \cdots < l_m$ and $k_1 < k_2 < \cdots < k_m$. From reference [9], the result of this type of integrals is equal to:

$$M(L, K) = (-1)^{(l_1 + l_2 + \cdots + l_m) + (k_1 + k_2 + \cdots + k_m)} A(L, K), \quad (2.7a)$$

where $A(L, K)$ is the determinant of the matrix obtained from matrix $A$ by deleting the lines $\{l_1, \cdots, l_m\}$ and the columns: $\{k_1, \cdots, k_m\}$. The Grassmann integrals to be calculated in sector $\downarrow\downarrow$ are of the same type as that of eq.(2.7).

After the similarity transformation (2.5) and the change of variables (2.5b), in a schematic way, the integral (2.7) becomes

$$M(L, K) = \int d\eta_1 d\bar{\eta}_1 (\bar{\eta} P^{-1})_{l_1} (\eta P)_{k_1} \cdots (\eta P^{-1})_{l_m} (\bar{\eta} P^*)_{k_m} e^{\frac{i\pi}{n} \sum_{i,j=1}^{nN} \bar{\eta}_i D_{ij} \eta_j}, \quad (2.8)$$

where $D$ is a diagonal matrix whose entries are given by eq.(2.6e).

We should point out that the relations (2.6a)–(2.6e) are valid for any unidimensional self-interacting fermionic model with space translation symmetry.
We have a large number of integrals that contribute to eq.(2.3). For a discussion on some useful symmetries and their use in the reduction of the number of contributing integrals, the reader is referred to [2].

3. Unidimensional Hubbard Model

The hamiltonian that describes the Hubbard model in one space dimension is [10]:

$$H = \sum_{i=1}^{N} \sum_{\sigma=-1,1} t_{ij} a_{i\sigma}^\dagger a_{j\sigma} + U \sum_{i=1}^{N} a_{i\uparrow}^\dagger a_{i\uparrow} a_{i\downarrow}^\dagger a_{i\downarrow} + \lambda_B \sum_{i=1}^{N} \sum_{\sigma=-1,1} \sigma a_{i\sigma}^\dagger a_{i\sigma}$$  \hspace{1cm} (3.1)

where $a_{i\sigma}^\dagger$ is the creation operator of an electron in site $i$ with spin $\sigma$ and $a_{i\sigma}$ is the destruction operator of an electron in site $i$ with spin $\sigma$. The first term on the r.h.s. of eq.(3.1) is the kinetic energy operator. All diagonal elements of $t_{ij}$ are equal, $t_{ii} = E_0$, the only non–null off–diagonal terms are $t_{i,i-1} = t_{i,i+1} = t$, where $i = 1, 2, \ldots, N$, and they contribute to the hopping term. $U$ is the strength of the interaction between the electrons in the same site but with different spins. We have defined $\lambda_B = -\frac{1}{2} g \mu_B B$, where $g$ is the Landé’s factor, $\mu_B$ is the Bohr’s magneton and $B$ is the constant external magnetic field in the $\hat{z}$ direction.

The periodic boundary condition in space is implemented by imposing that $a_{0\sigma} = a_{N\sigma}$ and $a_{N+1,\sigma} = a_{1\sigma}$. Therefore, the hopping terms $t_{10} a_{1\sigma}^\dagger a_{0\sigma}$ and $t_{N,N+1} a_{N\sigma}^\dagger a_{N+1,\sigma}$ become $t_{1N} a_{1\sigma}^\dagger a_{N\sigma}$ and $t_{N,1} a_{N\sigma}^\dagger a_{1\sigma}$ respectively. We point out that the hamiltonian (3.1) is already in normal order.

The kernel of the operator $K$ (eq.(2.1a)) for the unidimensional Hubbard model on a lattice with $N$ space sites, written in terms of the generators $\bar{\eta}_I$ and $\eta_J$, is equal to

$$K(\bar{\eta}, \eta; \nu) = \sum_{l=1}^{N} \sum_{\sigma=\pm 1} (E_0 + \sigma \lambda_B - \mu) \bar{\eta}_{\frac{1-\sigma}{2} n+\nu} |N_l+1\rangle \eta_{\frac{1-\sigma}{2} n+\nu} |N_l+1\rangle^+$$

$$+ \sum_{l=1}^{N} \sum_{\sigma=\pm 1} t [\bar{\eta}_{\frac{1-\sigma}{2} n+\nu} |N_l\rangle \eta_{\frac{1-\sigma}{2} n+\nu} |N_{l+1}\rangle + \bar{\eta}_{\frac{1-\sigma}{2} n+\nu} |N_l\rangle \eta_{\frac{1-\sigma}{2} n+\nu} |N_{l+1}\rangle^+]$$

$$+ \sum_{l=1}^{N} U \bar{\eta}_{(n+\nu)N+l} \eta_{(n+\nu)N+l} \bar{\eta}_{\nu N+l} \eta_{\nu N+l},$$  \hspace{1cm} (3.2a)

with the periodic spatial boundary condition:
\[ t \tilde{\eta}[(1-\sigma) n + \nu] N + N \eta[(1-\sigma) n + \nu] N + N + 1 \equiv t \tilde{\eta}[(1-\sigma) n + \nu] N + N \eta[(1-\sigma) n + \nu] N + 1 \]  

(3.2b) and

\[ t \tilde{\eta}[(1-\sigma) n + \nu] N + 1 \eta[(1-\sigma) n + \nu] N \equiv t \tilde{\eta}[(1-\sigma) n + \nu] N + 1 \eta[(1-\sigma) n + \nu] N + N \]  

(3.2c)

and the anti-periodic boundary condition in \( \nu \):

\[ \eta[(1-\sigma) n + \nu] N + l = -\eta[(1-\sigma) n + \nu] N + l, \]  

(3.2d)

for \( l = 1, 2, \cdots, N \), and \( \sigma = \uparrow, \downarrow \). We have used the mapping (2.4c) to write the previous expressions.

In order to write down the terms that contribute to \( Tr[K^4] \) and \( Tr[K^5] \) in a simplified way, we define:

\[ \mathcal{E}(\tilde{\eta}, \eta; \nu; \sigma) \equiv \sum_{l=1}^{N} \tilde{\eta}[(1-\sigma) n + \nu] N + l \eta[(1-\sigma) n + \nu] N + l \]  

(3.3a)

\[ \mathcal{T}^{-}(\tilde{\eta}, \eta; \nu; \sigma) \equiv \sum_{l=1}^{N} \tilde{\eta}[(1-\sigma) n + \nu] N + l \eta[(1-\sigma) n + \nu] N + l + 1 \]  

(3.3b)

and

\[ \mathcal{T}^{+}(\tilde{\eta}, \eta; \nu; \sigma) \equiv \sum_{l=1}^{N} \tilde{\eta}[(1-\sigma) n + \nu] N + l \eta[(1-\sigma) n + \nu] N + l - 1 \]  

(3.3c)

We also define

\[ \mathcal{E}(\tilde{\eta}, \eta; \nu) \equiv \sum_{\sigma=\pm 1} E(\sigma) \mathcal{E}(\tilde{\eta}, \eta; \nu; \sigma), \]  

(3.4a)

\[ \mathcal{T}^{-}(\tilde{\eta}, \eta; \nu) \equiv \sum_{\sigma=\pm 1} t \mathcal{T}^{-}(\tilde{\eta}, \eta; \nu; \sigma), \]  

(3.4b)

\[ \mathcal{T}^{+}(\tilde{\eta}, \eta; \nu) \equiv \sum_{\sigma=\pm 1} t \mathcal{T}^{+}(\tilde{\eta}, \eta; \nu; \sigma), \]  

(3.4c)

and

\[ \mathcal{U}(\tilde{\eta}, \eta; \nu) \equiv \sum_{l=1}^{N} \tilde{\eta}(n + \nu) N + l \eta(n + \nu) N + l \tilde{\eta} N + l \eta N + l \]  

(3.4d)
where $E(\sigma) \equiv E_0 - \sigma \lambda_B - \mu$. The term $E(\bar{\eta}, \eta; \nu)$ represents the diagonal part of the kinetic energy, $T^-(\bar{\eta}, \eta; \nu)$ and $T^+(\bar{\eta}, \eta; \nu)$ are the hopping terms and $U(\bar{\eta}, \eta; \nu)$ is the fermionic interaction term.

For the unidimensional Hubbard model, the Grassmannian function $K^\oplus(\bar{\eta}, \eta; \nu)$ is written as

$$K^\oplus(\bar{\eta}, \eta; \nu) = E(\bar{\eta}, \eta; \nu) + T^-(\bar{\eta}, \eta; \nu) + T^+(\bar{\eta}, \eta; \nu) + U(\bar{\eta}, \eta; \nu). \quad (3.5)$$

4. The Exact Coefficients of the $\beta$–Expansion of the Grand Canonical Partition Function for the Unidimensional Hubbard Model

In eq.(2.2) we have the $\beta$–expansion of the grand canonical partition function for any quantum system. For the particular case of self–interacting unidimensional fermionic models, the expression of $Tr[K^n]$ is given by eq.(2.3).

In reference [2], we calculated the exact coefficients of the terms $\beta^2$ and $\beta^3$ of the expression (2.2) for the unidimensional Hubbard model for arbitrary values of the constants $E_0$, $t$, $U$ and $\mu$, that characterize the model, and for any value of the constant external magnetic field $B$.

The evaluation of integrals has been performed by a number of procedures (computer programs) developed by the authors in the symbolic language Maple V.3, that consist in the computational implementation of the method described in reference [2]. We have called this package of procedures GINT.

The method applied in [2] greatly simplifies the calculations made in reference [11], but memory utilization problems have appeared as we tried to go beyond $n > 3$. Luckily, the factorization property of multivariable integrals of type (2.8) allowed us to optimize the performance of the package. In Appendix A we consider one typical Grassmann integral to exemplify the factorization of the sub–graphs.

The procedure perm is one of the procedures contained in the package, being a useful tool to calculate the independent non–null terms [12] that contribute to $Tr[K^4]$ and $Tr[K^5]$. In this procedure are implemented the symmetries discussed in reference [2]. The procedure
gint has been used to calculate the multivariable Grassmann integrals, taking into account the factorization into sub–graphs. The package can be downloaded through ftp from the site http://www.if.uff.br.

4.1. Calculation of Tr[K⁴]

For the case \( n = 4 \), we get from eq.(2.3) that

\[
\text{Tr}[K^4] = \int \prod_{I=1}^{8N} d\eta_I d\bar{\eta}_I \sum_{I,J=1}^{8N} \bar{\eta}_I A_{IJ} \eta_J \times \]

\[
\times K^\oplus(\bar{\eta}, \eta; \nu = 0) K^\ominus(\bar{\eta}, \eta; \nu = 1) K^\circ(\bar{\eta}, \eta; \nu = 2) K^\circ(\bar{\eta}, \eta; \nu = 3). \quad (4.1.1)
\]

The expressions of \( A_{\sigma \sigma}, p^{(4)}_{\nu \nu'}, q^{(4)}_{\nu \nu'} \) and \( \lambda^{(4)}_{\nu} \) are obtained from eqs.(2.3b), (2.6a), (2.6b) and (2.6e).

From eq.(3.5), for the unidimensional Hubbard model, we have that the Grassmann function \( K^\ominus(\bar{\eta}, \eta; \nu) \) is equal to

\[
K^\ominus(\bar{\eta}, \eta; \nu) = \mathcal{E}(\bar{\eta}, \eta; \nu) + \mathcal{T}^-(\bar{\eta}, \eta; \nu) + \mathcal{T}^+(\bar{\eta}, \eta; \nu) + \mathcal{U}(\bar{\eta}, \eta; \nu). \quad (4.1.1a)
\]

We defined a simplified notation,

\[
< \mathcal{O}_1(\nu_1) \cdots \mathcal{O}_m(\nu_m) > \equiv \int \prod_{I=1}^{2nN} d\eta_I d\bar{\eta}_I \sum_{I,J=1}^{2nN} \bar{\eta}_I A_{IJ} \eta_J \times \]

\[
\times \mathcal{O}_1(\bar{\eta}, \eta; \nu_1) \cdots \mathcal{O}_m(\bar{\eta}, \eta; \nu_m) \quad (4.1.2a)
\]

and

\[
< \mathcal{O}_1(\sigma, \nu_1) \cdots \mathcal{O}_m(\sigma, \nu_m) > \equiv \int \prod_{I=(1-\sigma)nN+1}^{(3-\sigma)nN} d\eta_I d\bar{\eta}_I \sum_{I,J=(1-\sigma)nN+1}^{2nN} \bar{\eta}_{(1-\sigma)nN+1} A_{IJ} \eta_{(1-\sigma)nN+J} \times \]

\[
\times \mathcal{O}_1(\bar{\eta}, \eta; \nu_1) \cdots \mathcal{O}_m(\bar{\eta}, \eta; \nu_m), \quad (4.1.2b)
\]

where \( \mathcal{O}_j(\bar{\eta}, \eta; \nu_j) \) are Grassmann functions.

The independent terms that contribute to \( Tr[K^4] \) are:
\[ Tr[K^4] = - \epsilon_0, \epsilon_0, \epsilon_0, \epsilon_0 > + 4 < U, \epsilon_0, \epsilon_0, \epsilon_0 > + 2 < U, \epsilon_0, U, \epsilon_0 > + 8 < U, T^-, T^+, \epsilon_0 > + 4 < U, U, \epsilon_0 > + 4 < U, U, U, \epsilon_0 > + 4 < T^-, \epsilon_0, T^+, \epsilon_0 > + 8 < T^-, U, T^+, \epsilon_0 > + 4 < T^-, U, T^+, U > + 8 < T^-, T^+, \epsilon_0 > + 4 < T^-, T^-, \epsilon_0 > + 4 < T^-, T^- > . \] (4.1.3)

In order to calculate the terms on the r.h.s. of (4.1.3), we need the result of a set of Grassmann multivariable integrals that are presented on Appendix C. We used the procedure \textit{gint} to calculate them. Before using the procedure \textit{gint} to calculate the terms in eq.(4.1.3) that include \( \epsilon_0, T^- \) and \( T^+ \), we need to explicit the contributions coming from the sectors \( \sigma = \uparrow \) and \( \sigma = \downarrow \). For example, in the term \( < \epsilon(0), \epsilon(1), \epsilon(2), \epsilon(3) > \) we have 16 terms when we explicitly write down the spin–sectors. However, the number of terms is diminished when we use the symmetries discussed in reference [2] and the fact that \( A^{\uparrow \uparrow} = A^{\downarrow \downarrow} \). The application of these symmetries is simplified by using the graphic notation explained in Appendix B.

By taking into account the results of integrals in Appendix C and their contributions to the sum over the space indices in each term of \( Tr[K^4] \) (eq.(4.1.3)), we finally obtain

\[ Tr[K^4] = N^4 \left( U \Delta E^3 + \frac{1}{256} U^4 + \frac{3}{8} U^2 \Delta E^2 + \Delta E^4 + \frac{1}{16} U^3 \Delta E \right) + N^3 \left( 3 \Delta E^2 \lambda_B^2 + 3 \Delta E^4 + 3 t^2 U \Delta E + 9 \frac{1}{2} U \Delta E^3 + 3 \frac{9}{16} U^2 \lambda_B^2 + 6 t^2 \Delta E^2 + \frac{9}{128} U^4 + \frac{45}{16} U^2 \Delta E^2 + \frac{3}{8} t^2 U^2 + \frac{3}{4} U^3 \Delta E + \frac{3}{2} U \Delta E \lambda_B^2 \right) + N^2 \left( \frac{3}{4} \Delta E^4 + \frac{3}{16} U^2 \lambda_B^2 + 3 U \Delta E^3 + \frac{51}{16} U^2 \Delta E^2 + 3 t^2 U \Delta E + \frac{3}{2} \Delta E^2 \lambda_B^2 + 3 t^2 \lambda_B^2 + 3 \frac{9}{4} \lambda_B^4 + \frac{9}{8} t^2 U^2 + 3 t^4 + 3 t^2 \Delta E^2 + \frac{21}{16} U^3 \Delta E + \frac{51}{256} U^4 \right) \]
\[ + N \left( 3 t^2 \lambda_B^2 + \frac{3}{2} t^4 + 3 t^2 \Delta E^2 + \frac{1}{2} U \Delta E^3 + \frac{5}{4} t^2 U^2 + \frac{3}{8} U^2 \lambda_B^2 + \frac{1}{4} \Delta E^4 + 3 t^2 U \Delta E + \frac{3}{2} \Delta E^2 \lambda_B^2 + \frac{3}{8} U^2 \Delta E^2 + \frac{1}{4} \lambda_B^4 + \frac{1}{8} U^3 \Delta E + \frac{3}{2} U \Delta E \lambda_B^2 + \frac{3}{128} U^4 \right) . \] (4.1.4)

We use the short notation: \( \Delta E \equiv E_0 - \mu \).
4.2. Calculation of $Tr[K^5]$

For $n=5$, the expression of $Tr[K^5]$ obtained from the procedure perm, where each term is multiplied by the respective constant, is:

\[
Tr[K^5] = 5 < U, \varepsilon_0, \varepsilon_0, \varepsilon_0, \varepsilon_0 > + 5 < U, \varepsilon_0, U, \varepsilon_0, \varepsilon_0 > + 5 < U, U, \varepsilon_0, \varepsilon_0, \varepsilon_0 > + \\
+ 5 < U, U, \varepsilon_0, U, \varepsilon_0 > + 10 < U, U, T^-, T^+, \varepsilon_0 > + 5 < U, U, \varepsilon_0, \varepsilon_0, \varepsilon_0 > + \\
+ 5 < U, U, U, \varepsilon_0, \varepsilon_0 > + 10 < T^-, T^+, \varepsilon_0 > + 10 < U, U, T^-, T^+, \varepsilon_0 > + \\
+ 10 < T^-, T^+, \varepsilon_0, \varepsilon_0 > + 10 < T^-, T^+, \varepsilon_0, \varepsilon_0 > + 10 < T^-, U, U, T^+, \varepsilon_0 > + \\
+ 10 < U, U, \varepsilon_0, \varepsilon_0, \varepsilon_0 > + 10 < T^-, U, T^+, \varepsilon_0, \varepsilon_0 > + 10 < T^-, \varepsilon_0, T^+, U, \varepsilon_0 > + \\
+ 10 < T^-, U, T^+, U, \varepsilon_0 > + 10 < T^-, U, T^+, U, U, \varepsilon_0 > + \\
+ 10 < U, \varepsilon_0, T^+, \varepsilon_0, \varepsilon_0 > + 10 < U, T^-, \varepsilon_0, T^+, \varepsilon_0 > + 10 < U, T^-, \varepsilon_0, T^+, \varepsilon_0 > + \\
+ 10 < U, T^-, U, T^+, \varepsilon_0 > + 10 < U, T^-, U, T^+, \varepsilon_0 > + 10 < U, T^-, U, T^+, U, \varepsilon_0 > + \\
+ 10 < U, T^-, T^+, \varepsilon_0 > + 10 < T^-, T^-, T^+, \varepsilon_0 > + 10 < T^-, T^-, T^+, \varepsilon_0 > + \\
+ 10 < T^-, T^-, U, T^+, \varepsilon_0 > + 10 < T^-, T^-, U, T^+, \varepsilon_0 > + 10 < T^-, T^-, U, T^+, U, \varepsilon_0 >
\]

(4.2.1)

where we are using the convention (4.1.2a) to write down each term.

In $n=5$ we have seven new type of integrals that do not have an equivalent one for $n<5$. We table those integrals in section C.2 of Appendix C.

After a long but convergent calculation, we get

\[
Tr[K^5] = N^5 \left( \frac{1}{1024} U^5 + \frac{5}{256} U^4 \Delta E + \frac{5}{4} U \Delta E^4 + \frac{5}{32} U^3 \Delta E^2 + \Delta E^5 + \frac{5}{8} U^2 \Delta E^3 \right) + \\
+ N^4 \left( \frac{105}{16} U^2 \Delta E^3 + \frac{15}{16} U^2 \Delta E \lambda^2_B + \frac{5}{32} U^3 t^2 + 10 t^2 \Delta E^3 + 5 \Delta E^3 \lambda^2_B + 5 \Delta E^5 + \frac{35}{4} U \Delta E^4 + \\
+ \frac{15}{4} U \Delta E^2 \lambda^2_B + \frac{15}{2} U t^2 \Delta E^2 + \frac{15}{512} U^5 + \frac{5}{64} U^3 \lambda^2_B + \frac{155}{64} U^3 \Delta E^2 + \frac{55}{128} U^4 \Delta E + \\
+ \frac{15}{8} U^2 t^2 \Delta E \right) + N^3 \left( 15 t^2 \Delta E^3 + 15 t^2 \Delta E \lambda^2_B + \frac{195}{16} U \Delta E^4 + \frac{45}{8} U \Delta E^2 \lambda^2_B + \\
+ \frac{15}{16} U \lambda^4_B + \frac{15}{2} U \Delta E^3 \lambda^2_B + \frac{15}{4} U \Delta E \lambda^2_B + \frac{15}{4} U \lambda^4_B + \frac{15}{4} U t^4 + \frac{225}{16} U^2 \Delta E^3 + \frac{45}{16} U^2 \Delta E \lambda^2_B + \\
+ \frac{15}{4} U t^2 \lambda^2_B + \frac{75}{4} U t^2 \Delta E^2 + 15 t^4 \Delta E + \frac{495}{256} U^4 \Delta E + \frac{15}{32} U^3 \lambda^2_B + \frac{15}{2} U^3 \Delta E^2 + \\
+ \frac{75}{8} U^2 t^2 \Delta E + \frac{195}{1024} U^5 + \frac{45}{32} U^3 t^2 \right) + N^2 \left( -\frac{25}{4} U^2 t^2 \Delta E + \frac{5}{4} U^2 \Delta E^3 - \right)
\]

11
\[- \frac{15}{4} U^2 \Delta E \lambda_B^2 + \frac{75}{512} U^5 - \frac{15}{16} U \Delta E^4 - \frac{75}{8} U \Delta E^2 \lambda_B^2 - \frac{35}{16} U \lambda_B^4 - \frac{15}{2} t^4 \Delta E - \]
\[- 15 U t^2 \Delta E^2 - \frac{15}{2} U t^2 \lambda_B^2 + \frac{115}{128} U^4 \Delta E - \frac{15}{2} \Delta E^3 \lambda_B^2 - \frac{5}{4} \Delta E^5 - \frac{5}{4} \Delta E \lambda_B^4 - 15 t^2 \Delta E^3 - \]
\[- 15 t^2 \Delta E \lambda_B^2 - \frac{45}{64} U^3 \lambda_B^2 + \frac{125}{64} U^3 \Delta E^2 - \frac{15}{8} U t^4 - \frac{5}{8} U^3 t^2 \right) + \]
\[+ N \left( - \frac{5}{2} U^2 \Delta E^3 - \frac{15}{2} U t^2 \Delta E^2 + \frac{15}{2} U t^2 \lambda_B^2 - \frac{15}{2} U^2 t^2 \Delta E - \frac{25}{32} U^4 \Delta E - \frac{65}{32} U^3 \Delta E^2 + \right. \]
\[+ \frac{5}{32} U^3 \lambda_B^2 - \frac{15}{128} U^5 - \frac{15}{8} U^3 t^2 - \frac{5}{4} U \Delta E^4 + \frac{5}{4} U \lambda_B^4 \right) \right].
\]

(4.2.2)

We continue to use the notation: \( \Delta E \equiv E_0 - \mu \).

Certainly, the most subtle part in calculating expression (4.2.2) comes from the product of Grassmann integrals for different \( \sigma \)–sectors when we have Grassmann generators at the same space indice. In this case, we have to suit the conditions satisfied by both integrals and calculate the contribution of the product to the sum over space indices.

**4.3. The \( \beta \)–Expansion of the Grand Potential Up to Order \( \beta^4 \) and Physical Quantities**

The relation between the grand potential \( W(\beta; \mu) \) and the grand canonical partition function \( Z(\beta; \mu) \) is

\[ W(\beta; \mu) = - \frac{1}{\beta} \ln Z(\beta; \mu). \]  

(4.3.1)

The \( \beta \)–expansion of \( Z(\beta; \mu) \) up to order \( \beta^5 \) is (see eq.(2.2)):

\[ Z(\beta, \mu) \approx \text{Tr}[\mathbb{I} - \beta \mathbf{K}] + \frac{\beta^2}{2!} \text{Tr}[\mathbf{K}^2] - \frac{\beta^3}{3!} \text{Tr}[\mathbf{K}^3] + \frac{\beta^4}{4!} \text{Tr}[\mathbf{K}^4] - \frac{\beta^5}{5!} \text{Tr}[\mathbf{K}^5]. \]  

(4.3.2)

The first term on the r.h.s. of (4.3.2) was calculated in reference[11], the second and third terms were calculated in reference [2] and in its last two terms we substitute the results of eqs.(4.1.4) and (4.2.2).

From eqs.(4.3.1) and (4.3.2), we get the grand potential up to order \( \beta^4 \), that is,
\[ W(\beta; \mu) = -N \left\{ \frac{2}{\beta} \ln 2 + \left( -\frac{1}{16} Ut^2 \lambda_B^2 + \frac{1}{16} U^2 t^2 \Delta E + \frac{1}{1024} U^5 + \frac{1}{16} Ut^2 \Delta E^2 + \frac{13}{768} U^3 \Delta E^2 - \frac{1}{768} U^3 \lambda_B^2 - \frac{1}{96} U \lambda_B^4 + \frac{1}{96} U \Delta E^4 + \frac{5}{768} U^4 \Delta E + \frac{1}{48} U^2 \Delta E^3 + \frac{1}{64} U^3 t^2 \right) \beta^4 - \left( \frac{1}{8} t^2 U \Delta E + \frac{1}{16} U \Delta E \lambda_B^2 + \frac{1}{96} U \Delta E^4 + \frac{1}{16} t^4 + \frac{1}{96} \lambda_B^4 + \frac{1}{1024} U^4 + \frac{1}{48} U \Delta E^3 + \frac{1}{8} t^2 \Delta E^2 + \frac{1}{192} U^3 \Delta E + \frac{5}{96} t^2 U^2 + \frac{1}{64} U^2 \lambda_B^2 + \frac{1}{64} U^2 \Delta E^2 + \frac{1}{16} \Delta E^2 \lambda_B^2 + \frac{1}{8} t^2 \lambda_B^2 \right) \beta^3 + \left( \frac{1}{64} - \frac{1}{16} U \lambda_B^2 - \frac{1}{16} \Delta E^2 U - \frac{1}{16} \Delta E U^2 \right) \beta^2 + \left( \frac{1}{4} \Delta E^2 + \frac{1}{4} \lambda_B^2 + \frac{1}{4} \Delta E U + \frac{t^2}{2} + \frac{3}{32} U^2 \right) \beta - (\Delta E + \frac{U}{4}) + \mathcal{O}(\beta^5) \right\}. \]

\[ (4.3.3) \]

It is important to stress out that the coefficients of the \( \beta \)-expansion of function \( W(\beta; \mu) \) are exact for any set of constants: \( E_0, t, U, \mu \) and \( B \) for the unidimensional Hubbard model. From expression (4.3.3) we can get the strong limit approximation by taking \( U \gg t \), as well the atomic limit approximation when \( U \ll t \).

From expression (4.3.3), we can derive any physical quantity for the model at thermal equilibrium at high temperature. As examples, we consider the following quantities:

i) specific heat at constant length and constant number of fermions: \( C_L(\beta) \).

\[ C_L(\beta) = -k \beta \beta \left[ \frac{\partial^2 W(\beta; \mu)}{\partial \beta^2} \right], \]

where \( k \) is the Boltzmann constant. From eq.(4.3.3), we get,

\[ C_L(\beta) = N k \left\{ \left( \frac{5}{256} U^5 - \frac{5}{24} U \lambda_B^4 + \frac{5}{24} U \Delta E^4 + \frac{65}{192} U^3 \Delta E^2 + \frac{5}{12} U^2 \Delta E^3 - \frac{5}{192} U^3 \lambda_B^2 - \frac{5}{4} U t^2 \lambda_B^2 + \frac{5}{4} U t^2 \Delta E^2 + \frac{5}{4} U^2 t^2 \Delta E + \frac{25}{192} U^4 \Delta E + \frac{5}{16} U^3 t^2 \right) \beta^5 + \left( \frac{3}{256} U^4 - \frac{3}{2} t^2 \Delta E^2 - \frac{3}{8} \lambda_B^4 - \frac{3}{4} t^4 - \frac{3}{8} \Delta E^4 - \frac{5}{8} t^2 U^2 - \frac{1}{16} U^3 \Delta E - \frac{3}{4} U \Delta E \lambda_B^2 - \frac{3}{4} \Delta E^2 \lambda_B^2 - \frac{3}{2} t^2 \lambda_B^2 - \frac{3}{2} t^2 U \Delta E - \frac{3}{16} U^2 \lambda_B^2 - \frac{3}{16} U^2 \Delta E^2 - \frac{1}{4} U \Delta E^3 \right) \beta^4 + \right\}. \]
\[ + \left( -\frac{3}{8} U^2 \Delta E - \frac{3}{32} U^3 + \frac{3}{8} U \lambda_B^2 - \frac{3}{8} U \Delta E^2 \right) \beta^3 + \]
\[ + \left( \frac{1}{2} U \Delta E + t^2 + \frac{3}{16} U^2 + \frac{1}{2} \lambda_B^2 + \frac{1}{2} \Delta E^2 \right) \beta^2 + \mathcal{O}(\beta^6) \right). \tag{4.3.4a} \]

\[ ii) \text{ average energy per site: } <h> = \frac{1}{N} \frac{\partial \mathcal{W}(\beta; \mu; \alpha)}{\partial \alpha} \bigg|_{\alpha = 1}. \tag{4.3.5} \]

From eq.(4.3.3), we get that

\[
<h>(\beta) = E_0 + \frac{U}{4} + \left( -t^2 - \frac{3}{16} U^2 - \frac{1}{2} E_0 U + \frac{1}{4} U \mu - \frac{1}{2} E_0^2 + \frac{1}{2} E_0 \mu - \frac{1}{2} \lambda_B^2 \right) \beta + \\
+ \left( \frac{3}{16} E_0 U^2 - \frac{1}{8} U^2 \mu - \frac{3}{16} U \lambda_B^2 + \frac{3}{16} U E_0^2 - \frac{1}{4} U E_0 \mu + \frac{1}{16} U \mu^2 + \frac{3}{64} U^3 \right) \beta^2 + \\
+ \left( -\frac{3}{16} U E_0^2 \mu + \frac{1}{256} U^4 + \frac{1}{4} t^4 + \frac{1}{24} \lambda_B^2 + \frac{1}{16} U^2 \lambda_B^2 + \frac{5}{24} t^2 U^2 + \frac{1}{2} t^2 \lambda_B^2 + \\
+ \frac{1}{16} U^2 E_0^2 + \frac{1}{32} U^2 \mu^2 + \frac{1}{12} U E_0^3 - \frac{1}{48} U \mu^3 + \frac{1}{48} U E_0^3 E_0 - \frac{1}{64} U^3 \mu + \frac{1}{24} E_0^4 - \\
- \frac{3}{32} U^2 E_0 \mu + \frac{1}{8} U E_0 \mu^2 + \frac{1}{2} U t^2 E_0 - \frac{3}{8} U t^2 \mu - \frac{1}{8} E_0^3 \mu + \frac{1}{8} E_0^2 \mu^2 - \\
- \frac{1}{24} E_0 \mu^3 + \frac{1}{2} t^2 E_0^2 + \frac{1}{4} t^2 \mu^2 - \frac{3}{4} t^2 E_0 \mu + \frac{1}{4} U E_0 \lambda_B^2 - \frac{3}{8} \lambda_B^2 E_0 \mu - \\
- \frac{3}{16} U \lambda_B^2 \mu + \frac{1}{4} \lambda_B^2 E_0^2 + \frac{1}{8} \lambda_B^2 \mu^2 \right) \beta^3 + \\
+ \left( -\frac{5}{1024} U^5 - \frac{25}{768} U^4 E_0 - \frac{5}{64} U^3 t^2 + \frac{5}{96} U \lambda_B^4 + \frac{5}{192} U^4 \mu - \frac{5}{48} U^2 E_0^3 + \\
+ \frac{5}{768} U^3 \lambda_B^2 - \frac{5}{96} U E_0^4 - \frac{1}{96} U \mu^4 - \frac{65}{768} U^3 E_0^2 + \frac{1}{24} U^2 \mu^3 + \frac{1}{6} U E_0^3 \mu - \\
+ \frac{13}{96} U^3 E_0 \mu - \frac{3}{16} U E_0^2 \mu^2 + \frac{1}{4} U^2 E_0^2 \mu - \frac{3}{8} U^2 E_0 \mu^2 + \frac{1}{12} U E_0 \mu^3 - \frac{5}{16} U^2 t^2 E_0 + \\
+ \frac{5}{16} U t^2 \lambda_B^2 - \frac{5}{16} U t^2 E_0^2 - \frac{3}{16} U t^2 \mu^2 + \frac{1}{4} t^2 U^2 \mu + \frac{1}{2} U t^2 E_0 \mu - \frac{13}{256} U^3 \mu^2 \right) \beta^4 + \\
+ \mathcal{O}(\beta^5). \tag{4.3.5a} \]
iii) difference between average numbers of spin up and spin down particles per site:

\[ < n^\uparrow > - < n^\downarrow >. \]

From the definition of the grand potential (eq.(4.3.1)), we have that

\[ < n^\uparrow > (\beta) - < n^\downarrow > (\beta) = \frac{1}{N} \frac{\partial W(\beta; \mu)}{\partial \lambda_B}. \] (4.3.6)

Up to order \( \beta^4 \), we get from eq.(4.3.3) that,

\[ < n^\uparrow > (\beta) - < n^\downarrow > (\beta) = -\frac{\lambda_B}{8} \left\{ -(U\beta + 4) + \left( \Delta E^2 + U\Delta E + \frac{1}{3}\lambda_B^2 + \frac{1}{4}U^2 + 2t^2 \right) \beta^3 + \right. \]

\[ \left. + \left( \frac{1}{3}U\lambda_B^2 + Ut^2 + \frac{1}{48}U^3 \right) \beta^4 + O(\beta^5) \right\}. \] (4.3.6a)

iv) average of the square of the magnetization per site: \( < m_z^2 > (\beta) \).

\[ < m_z^2 > (\beta) = \frac{\lambda_B}{B^2} < (n^\uparrow_i - n^\downarrow_i)^2 > \]

\[ = -\left( \frac{1}{2}g\mu_B \right)^2 \frac{1}{N} \left[ \frac{\partial W(\beta; \mu)}{\partial \mu} + 2 \frac{\partial W(\beta; \mu)}{\partial U} \right], \] (4.3.7)

where \( B \) is the external magnetic field. From eq.(4.3.3), we obtain that

\[ < m_z^2 > (\beta) = \frac{1}{4} g^2 \mu_B^2 \left\{ \frac{1}{2} + \frac{1}{8}U\beta + \left( -\frac{1}{8}UE_0 + \frac{1}{8}\mu U - \frac{1}{32}U^2 + \frac{1}{8}\lambda_B^2 - \frac{1}{8}E_0^2 + \right. \right. \]

\[ + \frac{1}{4}E_0\mu - \frac{1}{8}\mu^2 \right\} \beta^2 - \left( \frac{1}{12}Ut^2 + \frac{1}{384}U^3 \right) \beta^3 + \]

\[ + \left( \frac{5}{1536}U^4 + \frac{15}{384}U^2E_0^2 + \frac{15}{384}U^2\mu^2 + \frac{1}{24}UE_0^3 - \frac{1}{24}U\mu^3 + \right. \]

\[ + \frac{7}{384}U^3E_0 - \frac{7}{384}U^3\mu + \frac{1}{32}t^2U^2 - \frac{3}{384}U^2\lambda_B^2 - \frac{1}{8}t^2\lambda_B^2 - \frac{1}{48}\lambda_B^2 - \frac{1}{8}UE_0^2\mu + \]

\[ + \frac{1}{8}UE_0\mu^2 + \frac{1}{8}Ut^2E_0 - \frac{1}{8}Ut^2\mu + \frac{1}{48}E_4^4 + \frac{1}{48}\mu^4 - \frac{1}{12}E_0^3\mu + \frac{1}{8}E_0^2\mu^2 - \]

\[ - \frac{1}{12}E_0\mu^3 + \frac{1}{8}t^2E_0^2 + \frac{1}{8}t^2\mu^2 - \frac{1}{4}t^2E_0\mu - \frac{15}{192}U^2E_0\mu \right\} \beta^4 + O(\beta^5) \}, \] (4.3.7a)

where \( g \) is the Landé's factor and \( \mu_B \) is the Bohr’s magneton.

v) magnetic susceptibility: \( \chi(\beta) \).
\[ \chi(\beta) = -\left(\frac{1}{2}g\mu_B\right)^2 \frac{1}{N} \frac{\partial^2 W(\beta; \mu)}{\partial \lambda_B^2}. \]  

(4.3.8)

From eq. (4.3.3), we obtain that

\[ \chi(\beta) = -\left(\frac{1}{2}g\mu_B\right)^2 \left[ \left(\frac{1}{8}Ut^2 + \frac{1}{8}U\lambda_B^2 + \frac{1}{384}U^3\right)\beta^4 + \left(\frac{1}{4}t^2 + \frac{1}{8}\lambda_B^2 + \frac{1}{8}U\Delta E\right) \beta^3 + \left(\frac{1}{8}\Delta E^2 + \frac{1}{32}U^2\right)\beta^3 - \frac{1}{8}U\beta^2 - \frac{1}{2}\beta + O(\beta^5) \right]. \]  

(4.3.8a)

5. Conclusions

With the implementation of the factorization into sub-graphs of the Grassmann multi-variable integrals, we can certainly go beyond the calculation of the term $\beta^4$ of the $\beta$-expansion of the grand potential of the unidimensional Hubbard model. Even though the physics for $U > 0$ and $U < 0$ are different, the results of section 4.3 apply equally well for both cases. Recently, dos Santos and Thomaz have applied the results of reference [2] to calculate the $\beta$-expansion of the grand canonical partition function of the extended unidimensional Hubbard model up to order $\beta^3$ [16]. But the important point is that the present approach opens the possibility to calculate the first terms of the $\beta$-expansion of the grand canonical partition function of the Hubbard model in two space dimensions, as well as of unidimensional models with impurities. We believe that improvements on the present approach will render a valuable tool for tackling with such problems.
Appendix A

Factorization of Grassmannian Sub–Graphs

For calculating the co–factors of matrices $A^{\sigma \sigma}$, $\sigma = \uparrow, \downarrow$, it helps to have the value of their determinant. For arbitrary $n$, the determinant of these matrices is equal to

$$\det A^{\sigma \sigma} = \left[ \prod_{\nu=0}^{n-1} \lambda^{(n)}_{\nu} \right]^N,$$

(A.1)

where $N$ is the number of space sites and $\lambda^{(n)}_{\nu}$ are the $N$–fold degenerated eigenvalue of $A^{\sigma \sigma}$. From eq.(2.6e) we have that

$$\lambda^{(n)}_{\nu} = 1 - e^{\frac{i\pi}{n}(2\nu+1)}.$$

(A.2)

We should notice that $\lambda^{(n)}_{(n-1)-\nu} = \lambda^{(n)*}_{\nu}$.

We define:

$$P^{(n)} \equiv \prod_{\nu=0}^{n-1} \lambda^{(n)}_{\nu}.$$

(A.3)

For calculating $P^{(n)}$ we need to consider the cases $n$ even and $n$ odd separately.

For $n$ even, expression (A.3) can be rewritten as,

$$P^{(n)} = \prod_{\nu=0}^{n-2} \left[ 2 - 2 \cos\left(\frac{\pi}{n}(2\nu+1)\right) \right] = 2,$$

(A.4)

where the last equality is already known [13].

For $n$ odd, expression (A.3) can be rewritten as,

$$P^{(n)} = 2 \times 2^{n-1} \prod_{\nu=0}^{n-1} \sin\left(\frac{\pi}{2n}(2\nu+1)\right).$$

(A.5)

From reference [13], we have that [14]:

$$2^{n-1} \prod_{\nu=0}^{n-1} \sin\left(\frac{\pi}{2n}(2\nu+1)\right) = 1,$$

(A.6)

that substituted in eq. (A.5) gives
\[ P^{(n)} = 2. \quad (A.7) \]

From the results (A.4) and (A.7), for any \( n \), we get that

\[
\text{det} A^\sigma \sigma = \left[ \prod_{\nu=0}^{n-1} \lambda^{(n)}_{\nu} \right]^N = 2^N. \quad (A.8)
\]

To present the factorization of the Grassmannian integrals, we consider an example and use the graphic notation explained in Appendix B.

Let us consider the integral for fixed space indices \( l_1 \) and \( l_3 \), that contributes to \( \langle \mathcal{E}_0(\uparrow), \mathcal{E}_1(\uparrow), \mathcal{E}_2(\uparrow) > \),

\[
\mathcal{I}(l_1, l_1, l_3) = \int \prod_{I=1}^{4N} d\eta_I d\bar{\eta}_I e^{I, J=1} \eta_{l_1} \eta_{l_1} \eta_{N+l_1} \eta_{N+l_1} \eta_{2N+l_3} \eta_{2N+l_3}
\]

where \( l_1 \neq l_3 \). Under the similarity transformation (2.5), eq. (A.9) becomes

\[
\mathcal{I}(l_1, l_1, l_3) = \int \prod_{I=1}^{4N} d\bar{\eta}_I d\eta_I \left[ \sum_{\nu_1, \nu_2=0}^{3} q_{\nu_1 0} p_{0 \tau_1} q_{\nu_2 1} p_{1 \tau_2} \bar{\eta}_{\nu_1 \tau_1, N+l_1} \eta_{\nu_1 \tau_1, N+l_1} \bar{\eta}_{\nu_2 \tau_2, N+l_1} \eta_{\nu_2 \tau_2, N+l_1} \right] \times
\]

\[
\times \left[ \sum_{\nu_3=0}^{3} q_{\nu_3 2} p_{2 \nu_3} \bar{\eta}_{\nu_3 N+l_3} \eta_{\nu_3 N+l_3} \right] e^{I, J=1} \bar{\eta}_I D_{IJ} \eta_J
\]

\[
= \left[ 2^N \sum_{\nu_1, \nu_2=0}^{3} \sum_{\tau_1, \tau_2=0}^{3} q_{\nu_1 0} p_{0 \tau_1} q_{\nu_2 1} p_{1 \tau_2} \lambda^{(4)}_{\nu_1} \lambda^{(4)}_{\nu_2} \right] \times \left[ \sum_{\nu_3=0}^{3} q_{\nu_3 2} p_{2 \nu_3} \lambda^{(4)}_{\nu_3} \right] \quad (A.10)
\]

where to write the second equality on the r.h.s. of eq. (A.10), we used the result (A.8).

By explicitly writing down the expressions, we see that,

\[
2^N \sum_{\nu_1, \nu_2=0}^{3} \sum_{\tau_1, \tau_2=0}^{3} q_{\nu_1 0} p_{0 \tau_1} q_{\nu_2 1} p_{1 \tau_2} \lambda^{(4)}_{\nu_1} \lambda^{(4)}_{\nu_2} = \int \prod_{I=1}^{4N} d\eta_I d\bar{\eta}_I e^{I, J=1} \bar{\eta}_I \eta_I \eta_{N+l_1} \eta_{N+l_1},
\]

\[
(A.10a)
\]

and
\[ \sum_{\nu_3=0}^{3} \frac{q_{\nu_3} p_{2\nu_3}}{\lambda^{(4)}_{\nu_3}} = \frac{1}{2^N} \int \prod_{I=1}^{4N} d\eta_I d\bar{\eta}_I \sum_{I,J=1}^{4N} \bar{\eta}_I A^{I\bar{J}}_{IJ} \eta_J \bar{\eta}_{2N+l_3} \eta_{2N+l_3}. \] (A.10b)

Using the graphic representation of Appendix B, we write result (A.10) as,

\[ \text{(A.11)} \]

The factorization (A.11) comes directly from the fact that the matrix \( D \) is diagonal (see eq.(2.5a)) and the result (2.7a). Once the presence of Grassmann generators in the integrand of integrals (2.8) correspond to cutting lines and columns of matrix \( D \), then only for cuts at the same space index and any \( \nu \)-indices we get co–factors of matrix \( D \) that are non–zero. In summary, the factorization of the type (A.11) always happens when two or more space indices are different.

In a similar way and by the reasons discussed before, it is simple to show that \( I(l_1, l_2, l_3) \), where all the space indices \( l_i, i = 1, 2, 3 \), are distinct, is easily written as:
Appendix B
Graphic Notation of the Multivariable Grassmann Integrals

To exemplify our graphic notation for the graphs that contribute to eq. (2.3), for fixed value of \( n \), we consider some terms of \( Tr[K^4] \). This graphic notation is very helpful when we apply the symmetries discussed in reference [2] to identify equivalent terms in \( Tr[K^n] \).

We present the graphic notation through examples and its application to the identification of equivalent integrals.

1) \[
< \mathcal{E}(\uparrow,0),\mathcal{E}(\downarrow,1),\mathcal{E}(\uparrow,2),\mathcal{E}(\uparrow,3) >= E(\uparrow)^3 E(\downarrow) \times
\]

2) \[
The constants \( E(\uparrow) \) and \( E(\downarrow) \) are defined just below eq.(3.4d). In order to show that terms (B.1) and (B.2) are equal, we use the invariance of the integrals under a cyclic translation in the temperature parameter \( \nu \) in each sector \( \sigma = \uparrow \) and \( \sigma = \downarrow \), separately. Therefore,
\[
< \mathcal{E}(\uparrow,0)\mathcal{E}(\downarrow,1)\mathcal{E}(\uparrow,2)\mathcal{E}(\uparrow,3) >= < \mathcal{E}(\downarrow,0),\mathcal{E}(\uparrow,1),\mathcal{E}(\uparrow,2),\mathcal{E}(\uparrow,3) > \quad (B.3)
\]
3) Due to the presence of the term $U(0)$, the integrals in the two $\sigma$–sectors have one space index $l$ in common.

$$< U(0) T^- (\uparrow, 1) T^+ (\uparrow, 2) E (\uparrow, 3) > = U t^2 E (\uparrow) \times$$

4) The terms (B.4) and (B.5) are equal, up to a multiplicative factor, due to the fact that $A^{\uparrow \downarrow} = A^{\downarrow \downarrow}$. The graphic notation was used along all the calculations and permitted us to considerably reduce the number of terms that contribute to the expressions of $Tr[K^4]$ and $Tr[K^5]$. 

21
Appendix C
Useful Multivariable Grassmann Integrals at \( n = 4 \) and \( n = 5 \)

C.1. Useful integrals for \( n=4 \)

We need the result of twelve integrals only, to calculate the terms that contribute to (4.1.3). In this Appendix, we present the value of these integrals according to the conditions satisfied by the space indices [15].

1) \( I^{(4)}_1(l) \equiv \int \prod_{I=1}^{4N} d\eta_I d\bar{\eta}_I \ e^{\sum_{I,J=1}^{4N} \bar{\eta}_I A^{\uparrow I\uparrow}_I \eta_J} \bar{\eta}_l \ \eta_l = 2^{N-1} \), \( (C.1.1) \)

for \( l = 1, 2, \ldots, N \).

2) \( I^{(4)}_2(l_1, l_2) \equiv \int \prod_{I=1}^{4N} d\eta_I d\bar{\eta}_I \ e^{\sum_{I,J=1}^{4N} \bar{\eta}_I A^{\uparrow I\uparrow}_I \eta_J} \bar{\eta}_{l_1} \ \eta_{l_1} \ \bar{\eta}_{N+l_2} \ \eta_{N+l_2} \)

\[ = \begin{cases} 2^{N-1}, & l_1 = l_2, \quad l_1 = 1, 2, \ldots, N \\ 2^{N-2}, & l_2 \neq l_1, \quad l_1, l_2 = 1, 2, \ldots, N \end{cases} \) \( (C.1.2) \)

3) \( I^{(4)}_3(l_1, l_2) \equiv \int \prod_{I=1}^{4N} d\eta_I d\bar{\eta}_I \ e^{\sum_{I,J=1}^{4N} \bar{\eta}_I A^{\uparrow I\uparrow}_I \eta_J} \bar{\eta}_{l_1+1} \ \eta_{l_1+1} \ \bar{\eta}_{N+l_2} \ \eta_{N+l_2} \)

\[ = 2^{N-2}, l_2 = l_1 + 1, \quad l_1 = 1, 2, \ldots, N. \) \( (C.1.3) \)

4) \( I^{(4)}_4(l_1, l_2, l_3) \equiv \int \prod_{I=1}^{4N} d\eta_I d\bar{\eta}_I \ e^{\sum_{I,J=1}^{4N} \bar{\eta}_I A^{\uparrow I\uparrow}_I \eta_J} \bar{\eta}_{l_1} \ \eta_{l_1} \ \bar{\eta}_{N+l_2} \ \eta_{N+l_2+1} \ \bar{\eta}_{2N+l_3} \ \eta_{2N+l_3-1} \)

\[ = \begin{cases} 2^{N-3}, & l_3 = l_2 + 1 \\ 2^{N-2}, & l_1 = l_3 = l_2 - 1 \end{cases} \) \( (C.1.4) \)
5) \[
\mathcal{I}_5^{(4)}(l_1, l_2, l_3) \equiv \int \prod_{I=1}^{4N} d\eta_I d\bar{\eta}_I \quad e^{I,J=1} \quad \bar{\eta}_I \quad \bar{\eta}_I+1 \quad \bar{\eta}_{N+l_2} \quad \bar{\eta}_{N+l_2} \quad \bar{\eta}_{2N+l_3} \quad \bar{\eta}_{2N+l_3}
\]
\[
= \begin{cases} 
2^{N-3}, & l_1 \neq l_2 \neq l_3 \\
2^{N-2}, & l_1 = l_2, \text{ or } l_1 = l_3, \text{ or } l_2 = l_3 \\
2^{N-1}, & l_1 = l_2 = l_3.
\end{cases}
\] (C.1.5)

6) \[
\mathcal{I}_6^{(4)}(l_1, l_2, l_3) \equiv \int \prod_{I=1}^{4N} d\eta_I d\bar{\eta}_I \quad e^{I,J=1} \quad \bar{\eta}_I \quad \bar{\eta}_{I+1} \quad \bar{\eta}_{N+l_2} \quad \bar{\eta}_{N+l_2} \quad \bar{\eta}_{2N+l_3} \quad \bar{\eta}_{2N+l_3-1}
\]
\[
= \begin{cases} 
2^{N-3}, & l_3 = l_1 + 1 \\
2^{N-2}, & l_2 = l_1 \text{ and } l_3 = l_1 + 1.
\end{cases}
\] (C.1.6)

7) \[
\mathcal{I}_7^{(4)}(l_1, l_2, l_3) \equiv \int \prod_{I=1}^{4N} d\eta_I d\bar{\eta}_I \quad e^{I,J=1} \quad \bar{\eta}_I \quad \bar{\eta}_I+1 \quad \bar{\eta}_{N+l_2} \quad \bar{\eta}_{N+l_2}+1 \quad \bar{\eta}_{2N+l_3} \quad \bar{\eta}_{2N+l_3-1}
\]
\[
= \begin{cases} 
2^{N-3}, & l_2 = l_3 - 1 \\
2^{N-2}, & l_1 = l_3 = l_2 + 1.
\end{cases}
\] (C.1.7)

8) \[
\mathcal{I}_8^{(4)}(l_1, l_2, l_3, l_4) \equiv \int \prod_{I=1}^{4N} d\eta_I d\bar{\eta}_I \quad e^{I,J=1} \quad \bar{\eta}_I \quad \bar{\eta}_I+1 \quad \bar{\eta}_{N+l_2} \quad \bar{\eta}_{N+l_2} \quad \bar{\eta}_{2N+l_3} \quad \bar{\eta}_{2N+l_3} \quad \bar{\eta}_{3N+l_4} \quad \bar{\eta}_{3N+l_4}
\]
\[
= \begin{cases} 
2^{N-4}, & l_1 \neq l_2 \neq l_3 \neq l_4 \\
2^{N-3}, & l_1 = l_2, \text{ or } l_1 = l_3, \text{ or } l_3 = l_4 \\
2^{N-2}, & l_1 = l_2 \text{ and } l_3 = l_4, \text{ or } l_1 = l_2 = l_3 = l_4 \text{ or } \text{all permutations } 2 \text{ by } 2 \\
2^{N-1}, & l_1 = l_2 = l_3 = l_4.
\end{cases}
\] (C.1.8)
\[ I_9^{(4)}(l_1, l_2, l_3, l_4) \equiv \prod_{I=1}^{4N} d\eta_I d\bar{\eta}_I e^{I', J=1} \sum_\bar{\eta}_I A_{I,j}^{\dagger} \eta_J \bar{\eta}_{I_1} \eta_{I_1+1} \bar{\eta}_{l_2} \eta_{l_2+1} \bar{\eta}_{l_3} \eta_{l_3+1} \times \eta_{3N+l_4} \eta_{3N+l_4-1} \]

\[ = \begin{cases} 2^{N-4}, & l_2 = l_1 + 1 \text{ and } l_4 = l_3 + 1, \text{ or, } l_2 = l_3 + 1 \text{ and } l_4 = l_1 + 1 \\ 2^{N-2}, & l_2 = l_3 + 1 \text{ and } l_1 = l_3 \text{ and } l_4 = l_3 + 1. \end{cases} \quad (C.1.9) \]

\[ I_{10}^{(4)}(l_1, l_2, l_3, l_4) \equiv \prod_{I=1}^{4N} d\eta_I d\bar{\eta}_I e^{I', J=1} \sum_\bar{\eta}_I A_{I,j}^{\dagger} \eta_J \bar{\eta}_{I_1} \eta_{I_1-1} \bar{\eta}_{l_2} \eta_{l_2+1} \bar{\eta}_{l_3} \eta_{l_3+1} \times \eta_{3N+l_4} \eta_{3N+l_4+1} \]

\[ = \begin{cases} 2^{N-4}, & l_2 = l_3 + 1 \text{ and } l_1 = l_4 + 1, \text{ or, } l_4 = l_2 - 1 \text{ and } l_1 = l_3 + 1 \\ 2^{N-3}, & l_2 = l_3 + 1 \text{ and } l_1 = l_3 + 2 \text{ and } l_4 = l_3 + 1 \\ 2^{N-2}, & l_2 = l_3 + 1 \text{ and } l_1 = l_3 \text{ and } l_4 = l_3 - 1. \end{cases} \quad (C.1.10) \]

\[ I_{11}^{(4)}(l_1, l_2, l_3, l_4) \equiv \prod_{I=1}^{4N} d\eta_I d\bar{\eta}_I e^{I', J=1} \sum_\bar{\eta}_I A_{I,j}^{\dagger} \eta_J \bar{\eta}_{I_1} \eta_{I_1-1} \bar{\eta}_{l_2} \eta_{l_2+1} \bar{\eta}_{l_3} \eta_{l_3+1} \times \eta_{3N+l_4} \eta_{3N+l_4-1} \]

\[ = \begin{cases} 2^{N-4}, & l_4 = l_3 + 1 \\ 2^{N-3}, & l_2 = l_3 + 1 \text{ and } l_4 = l_3 + 1, \text{ or, } l_1 = l_2 \text{ and } l_4 = l_3 + 1, \text{ or, } l_1 = l_3 + 1 \text{ and } l_4 = l_3 + 1 \\ 2^{N-2}, & l_1 = l_3 + 1 \text{ and } l_2 = l_3 + 1 \text{ and } l_4 = l_3 + 1. \end{cases} \quad (C.1.11) \]

\[ I_{12}^{(4)}(l_1, l_2, l_3, l_4) \equiv \prod_{I=1}^{4N} d\eta_I d\bar{\eta}_I e^{I', J=1} \sum_\bar{\eta}_I A_{I,j}^{\dagger} \eta_J \bar{\eta}_{I_1} \eta_{I_1-1} \bar{\eta}_{l_2} \eta_{l_2+1} \bar{\eta}_{l_3} \eta_{l_3+1} \times \eta_{3N+l_4} \eta_{3N+l_4-1} \]

\[ = \begin{cases} 2^{N-4}, & l_4 = l_2 + 1 \\ 2^{N-3}, & l_1 = l_3 \text{ and } l_4 = l_2 + 1, \text{ or, } l_1 = l_2 + 1 \text{ and } l_4 = l_2 + 1, \text{ or, } l_2 = l_3 \text{ and } l_4 = l_2 + 1 \text{ and } l_4 = l_2 + 1 \\ 2^{N-2}, & l_1 = l_2 + 1 \text{ and } l_3 = l_2 \text{ and } l_4 = l_2 + 1. \end{cases} \quad (C.1.12) \]
C.2. Useful integrals for n=5

We present here the seven integrals that have no equivalent ones for \( n < 5 \); i.e., integrals that cannot be factorized into any of the integrals for \( n < 5 \) for all the conditions satisfied by the space indices. In some graphs, we do not have generators in the integrand of integrals of type (2.7) at a given value of \( \nu \), as we can see for example in the graphs presented in Appendix B. Those rings in the integrals of type (2.7) that have no associated Grassmann generators in the integrand, we call \textit{empty rings}. For example, in (B.1) we have one empty ring at \( \sigma = \uparrow \) (\( \nu = 1 \)), and three empty rings at \( \sigma = \downarrow \) (\( \nu = 0, 1, \text{and} \ 3 \)). The integrals for \( n = 5 \) with empty rings give the same results to the equivalent integrals for \( n = 4 \). We have not demonstrated this property in general form for any \( n \), but we have detected it by evaluating these integrals through the procedure \texttt{gint}.

The seven integrals for \( n = 5 \) and the conditions satisfied by the space indices \([15]\) are:

1) 
\[
\mathcal{G}_1^{(5)}(l_1,l_2,l_3,l_4,l_5) \equiv \int \prod_{I=1}^{5N} d\bar{\eta}_I d\bar{\eta}_I \quad e_{I,J=1}^{\bar{I}} \sum_{I,J=1}^{5N} \bar{\eta}_I A_{I,J}^{\bar{I}} \eta_J \eta_{1I} \eta_{1J} \eta_{N+l_2} \eta_{N+l_2} \eta_{N+l_3} \eta_{2N+l_3} \times \eta_{3N+l_4} \eta_{3N+l_4} \eta_{4N+l_5} \eta_{4N+l_5} 
\]

\[
= \begin{cases} 
2^{N-5}, & l_1 \neq l_2 \neq l_3 \neq l_4 \neq l_5 \\
2^{N-4}, & l_1 = l_2, \text{or, all permutations with 2 equal space indices} \\
2^{N-3}, & l_1 = l_2 = l_3, \text{or, all permutations with 3 equal space indices} \\
 & l_1 = l_2 \text{ and } l_3 = l_4, \\
 & \text{or, all permutations 2 by 2} \\
2^{N-2}, & l_1 = l_2 = l_3 = l_5, \text{or,} \\
 & \text{all permutations with 4 equal space indices} \\
 & l_1 = l_2 = l_3 \text{ and } l_4 = l_5, \text{or,} \\
 & \text{all permutations with 2 or 3 equal space indices} \\
2^{N-1}, & l_1 = l_2 = l_3 = l_4 = l_5.
\end{cases} 
\]

(C.2.1)

2) 
\[
\mathcal{G}_2^{(5)}(l_1,l_2,l_3,l_4,l_5) \equiv \int \prod_{I=1}^{5N} d\bar{\eta}_I d\bar{\eta}_I \quad e_{I,J=1}^{\bar{I}} \sum_{I,J=1}^{5N} \bar{\eta}_I A_{I,J}^{\bar{I}} \eta_J \eta_{1I} \eta_{1J} \eta_{N+l_2} \eta_{N+l_2+1} \eta_{2N+l_3} \eta_{2N+l_3} \times \eta_{3N+l_4} \eta_{3N+l_4} \eta_{4N+l_5} \eta_{4N+l_5} 
\]

\[
= \begin{cases} 
2^{N-5}, & l_1 \neq l_2 \neq l_3 \neq l_4 \neq l_5 \\
2^{N-4}, & l_1 = l_2, \text{or, all permutations with 2 equal space indices} \\
2^{N-3}, & l_1 = l_2 = l_3, \text{or, all permutations with 3 equal space indices} \\
 & l_1 = l_2 \text{ and } l_3 = l_4, \\
 & \text{or, all permutations 2 by 2} \\
2^{N-2}, & l_1 = l_2 = l_3 = l_5, \text{or,} \\
 & \text{all permutations with 4 equal space indices} \\
 & l_1 = l_2 = l_3 \text{ and } l_4 = l_5, \text{or,} \\
 & \text{all permutations with 2 or 3 equal space indices} \\
2^{N-1}, & l_1 = l_2 = l_3 = l_4 = l_5.
\end{cases} 
\]
\[ G_3^{(5)}(l_1, l_2, l_3, l_4, l_5) \equiv \int \prod_{I=1}^{5N} d\eta_I d\bar{\eta}_I \sum_{I,J=1}^{5N} \bar{\eta}_I A_{Ij}^{\dagger} \eta_J \bar{\eta}_1 \eta_{l_1+1} \bar{\eta}_{N+l_2} \eta_{N+l_2+1} \bar{\eta}_{2N+l_3} \eta_{2N+l_3+1} \times \bar{\eta}_{3N+l_4} \eta_{3N+l_4} \bar{\eta}_{4N+l_5} \eta_{4N+l_5} \]

\[ = \begin{cases} 
2^{N-5}, & l_3 = l_2 + 1 \\
2^{N-4}, & l_3 = l_2 + 1 \text{ and two other space indices are equal} \\
2^{N-3}, & l_3 = l_2 + 1 \text{ and } l_1 = l_4 \text{ and } l_3 = l_5, \text{ or, } \\
& \text{all permutations } 2 \times 2 \\
& l_3 = l_2 + 1 \text{ and } l_1 = l_3 = l_4, \text{ or, } \\
& \text{all permutations with } 3 \text{ equal space indices} \\
2^{N-2}, & l_3 = l_2 + 1 \text{ and } l_1 = l_3 = l_4 = l_5. 
\end{cases} \tag{C.2.3} \]

\[ G_4^{(5)}(l_1, l_2, l_3, l_4, l_5) \equiv \int \prod_{I=1}^{5N} d\eta_I d\bar{\eta}_I \sum_{I,J=1}^{5N} \bar{\eta}_I A_{Ij}^{\dagger} \eta_J \bar{\eta}_1 \eta_{l_1+1} \bar{\eta}_{N+l_2} \eta_{N+l_2+1} \bar{\eta}_{2N+l_3} \eta_{2N+l_3+1} \times \bar{\eta}_{3N+l_4} \eta_{3N+l_4} \bar{\eta}_{4N+l_5} \eta_{4N+l_5} \]

\[ = \begin{cases} 
2^{N-5}, & l_1 = l_2 + 1 \text{ and } l_4 = l_3 + 1, \text{ or, } \\
& l_1 = l_3 + 1 \text{ and } l_4 = l_2 + 1 \\
2^{N-4}, & l_1 = l_2 + 1 \text{ and } l_5 = l_3 + 1 \text{ and } l_4 = l_5, \text{ or, } \\
& l_2 = l_4 = l_1 - 1 = l_3 + 1, \text{ or, } \\
& l_2 = l_5 = l_1 - 1 \text{ and } l_3 = l_4 - 1, \text{ or, } \\
& l_1 = l_3 + 1 \text{ and } l_4 = l_5 = l_2 + 1, \text{ or, } \\
& l_3 = l_5 = l_1 - 1 \text{ and } l_2 = l_4 - 1, \text{ or, } \\
& l_1 = l_3 = l_4 - 1 \text{ and } l_4 = l_2 + 2 \\
2^{N-3}, & l_1 = l_3 = l_5 + 1 \text{ and } l_2 = l_5 \text{ and } l_4 = l_2 + 2, \text{ or, } \\
& l_1 = l_5 + 1 \text{ and } l_2 = l_5 \text{ and } l_3 = l_5 - 1 \text{ and } l_4 = l_5, \text{ or, } \\
& l_1 = l_3 = l_5 - 1 \text{ and } l_2 = l_5 - 2 \text{ and } l_4 = l_5. 
\end{cases} \tag{C.2.4} \]
5) 

\[ G_5^{(5)}(l_1, l_2, l_3, l_4, l_5) = \int \prod_{l=1}^{5N} d\eta_l d\bar{\eta}_l \ e^{I_{J=1}^{5N} \bar{\eta}_l A_{I J}^{\uparrow \uparrow} \eta_J} \bar{\eta}_{l_1} \eta_{l_1+1} \eta_{N+l_2} \eta_{N+l_2} \bar{\eta}_{2N+l_3} \eta_{2N+l_3} \times \]

\[ \times \bar{\eta}_{3N+l_4} \eta_{3N+l_4-1} \eta_{4N+l_5} \eta_{4N+l_5} \]

\[ = \begin{cases} 
2^{N-5}, & l_4 = l_1 + 1 \\
2^{N-4}, & l_4 = l_1 + 1 \text{ and two other space indices are equal} \\
2^{N-3}, & l_4 = l_1 + 1 \text{ and } l_1 = l_2 = l_3, \text{ or}, \\
& \text{all permutations with 3 equal space indices} \\
2^{N-2}, & l_4 = l_1 + 1 \text{ and } l_1 = l_2 = l_3 = l_5 - 1.
\end{cases} \]

(C.2.5)

6) 

\[ G_6^{(5)}(l_1, l_2, l_3, l_4, l_5) = \int \prod_{l=1}^{5N} d\eta_l d\bar{\eta}_l \ e^{I_{J=1}^{5N} \bar{\eta}_l A_{I J}^{\uparrow \uparrow} \eta_J} \bar{\eta}_{l_1} \eta_{l_1+1} \eta_{N+l_2} \eta_{N+l_2-1} \bar{\eta}_{2N+l_3} \eta_{2N+l_3+1} \times \]

\[ \times \bar{\eta}_{3N+l_4} \eta_{3N+l_4-1} \eta_{4N+l_5} \eta_{4N+l_5} \]

\[ = \begin{cases} 
2^{N-5}, & l_2 = l_1 + 1 \text{ and } l_4 = l_3 + 1, \text{ or}, \\
& l_4 = l_1 + 1 \text{ and } l_2 = l_3 + 1 \\
2^{N-4}, & l_2 = l_1 + 1 \text{ and } l_4 = l_5 = l_3 + 1 \\
& l_2 = l_5 = l_1 + 1 \text{ and } l_4 = l_3 + 1 \\
& l_4 = l_5 = l_1 + 1 \text{ and } l_2 = l_3 + 1 \\
& l_1 = l_4 - 1 \text{ and } l_3 = l_5 = l_2 - 1 \\
2^{N-3}, & l_1 = l_3 = l_4 - 1 \text{ and } l_2 = l_4 \\
2^{N-2}, & l_1 = l_3 = l_5 - 1 \text{ and } l_2 = l_4 = l_5
\end{cases}. \]

(C.2.6)

7) 

\[ G_7^{(5)}(l_1, l_2, l_3, l_4, l_5) = \int \prod_{l=1}^{5N} d\eta_l d\bar{\eta}_l \ e^{I_{J=1}^{5N} \bar{\eta}_l A_{I J}^{\uparrow \uparrow} \eta_J} \bar{\eta}_{l_1} \eta_{l_1+1} \eta_{N+l_2} \eta_{N+l_2+1} \bar{\eta}_{2N+l_3} \eta_{2N+l_3-1} \times \]

\[ \times \bar{\eta}_{3N+l_4} \eta_{3N+l_4-1} \eta_{4N+l_5} \eta_{4N+l_5} \]

27
\[
\begin{cases}
2^{N-5}, & l_3 = l_1 + 1 \text{ and } l_4 = l_2 + 1, \text{ or,} \\
2^{N-4}, & l_3 = l_1 + 1 \text{ and } l_4 = l_5 = l_2 + 1, \text{ or,} \\
2^{N-3}, & l_2 = l_4 = l_5 = l_3 - 1 = l_1 + 1, \text{ or } l_1 = l_3 = l_4 - 1 = l_2 + 1, \\
& l_4 = l_5 = l_1 + 1 \text{ and } l_2 = l_3 - 1, \text{ or,} \\
& l_2 = l_4 = l_5 = l_3 - 1 = l_1 + 1, \text{ or } l_1 = l_3 = l_4 - 1 = l_2 + 1, \\
& l_1 = l_3 = l_5 - 1 \text{ and } l_4 = l_5 \text{ and } l_2 = l_5 - 2, \text{ or,} \\
& l_2 = l_4 = l_5 - 1 \text{ and } l_3 = l_5 \text{ and } l_1 = l_5 - 2.
\end{cases}
\] (C.2.7)

Acknowledgements

The authors thank J. Florencio Jr. for interesting discussions and A.T. Costa Jr. for making the figures. I.C.C thanks FAPMG and E.V.C.S. thanks CNPq for financial support. M.T.T. thanks CNPq and FINEP for partial financial support.

REFERENCES

1. S. Samuel, J. Math. Phys. 21 (1980) 2806–2833; C. Itzykson, Nucl. Phys. B210 [FS6] (1982) 448; V.N. Plechko, Physica A 152 (1988) 51; V.N. Plechko and I.K. Sobolev, Physica A 197 (1993) 323;
2. I.C. Charret, S.M. de Souza, E.V. Corrêa Silva and M.T. Thomaz, Grand Canonical Partition Function for the Unidimensional Systems: Application to Hubbard Model Up to Order $\beta^3$, submitted to Jour. of Phys. A,(pre-print cond/mat/ 9607171);
3. E.H. Lieb and F.Y. Wu, Phys. Rev. Letters 20 (1968) 1445; A.A. Ovchinnikov, Sov. Phys. JETP 30 (1970) 1160;
4. M. Takahashi, Prog. Theoret. Phys. (Kyoto) 43 (1970) 1619;
5. K. Kubo and M. Tada, Prog. Theoret. Phys. 69 (1983) 1345; C.J. Thompson, Y.S. Young, A.J. Guttmann and M.F. Sykes, J. Phys. A: Math. Gen. 24 (1991) 1261; J.A. Henderson, J. Oitmaa and M.C.B. Ashley, Phys. Rev. B46 (1992) 6328;
6. H.E. Stanley, Introduction to Phase Transitions and Critical Phenomena, Oxford Univ. Press (1971);
7. C. Itzykson and J.–B. Zuber, *Quantum Field Theory*, McGraw–Hill (1980);
U. Wolf, Nucl. Phys. **B225** [FS9] (1983) 391;
8. We are using the convention: \( \sigma = 1 = \uparrow \) and \( \sigma = -1 = \downarrow \);
9. I.C. Charret, S.M. de Souza and M.T. Thomaz, Braz. Jour. of Phys. **26** (1996) 720;
10. J. Hubbard, Proc. Roy. Soc. **A277** (1963) 237; **A281** (1964) 401;
    M. Gutzwiller, Phys. Rev. Lett. **10** (1963) 159; Phys. Rev. **A137** (1965) 1726;
11. I.C. Charret, E.V. Corrêa Silva, S.M. de Souza and M.T. Thomaz, J. Math. Phys. **36** (1995) 4100;
12. See section 4.1 of reference [2] for a discussion on the vanishing Grassmann integrals;
13. I.S. Gradshteyn and I.M. Ryzhik; *Table of Integrals, Series and Products*, 4th edition,
    Academic Press (1965); expression: 1.396.4;
14. Reference [11], expression: 1.392.2;
15. When the space index does not appear among the conditions, we mean that it is distinct
    from any other space index in the graph;
16. O.R. dos Santos and M.T. Thomaz, private communication.