Milnor numbers for 2-surfaces in 4-manifolds

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ABSTRACT. In this paper \((\Sigma_n)\) is a sequence of surfaces immersed in a 4-manifold which converges to a branched surface \(\Sigma_0\). Up to sign, \(\mu^T_p\) (resp. \(\mu^N_p\)) will denote the amount of curvature of \(T\Sigma_n\) (resp. \(N\Sigma_n\)) which concentrates around a singular point \(p\) of \(\Sigma_0\) when \(n\) goes to infinity. By a slight abuse of notation, we call \(\mu^T_p\) (resp. \(\mu^N_p\)) the tangent (resp. normal) Milnor number of \((\Sigma_n)\) at \(p\). These numbers are not always well-defined; we discuss assumptions under which the existence of \(\mu^T\) implies that \(\mu^N\) also exists and that \(-\mu^T \geq \mu^N\).

When the second fundamental forms of the \(\Sigma_n\)'s have a common \(L^2\) bound, we relate \(\mu^T\) and \(\mu^N\) to a bubbling-off in the Grassmannian \(G^+_{2}(M)\).

KEYWORDS: surfaces in 4-manifolds, branch points, characteristic numbers, currents, braids, twistors, minimal surfaces

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1 Introduction - Motivation

1.1 Statement of the problem

If \(\Sigma_0\) is a complex curve in \(\mathbb{C}^2\) and \(p\) is a branch point of \(\Sigma_0\), one associates to the singularity at \(p\) an invariant, called the Milnor number ([Mi], see also [Ru 1]) which is computed from the Puiseux coefficients of \(\Sigma_0\) around \(p\). It gives us the following topological information. Let \(D\) be the unit disk in \(\mathbb{C}\), let \((\Sigma_s)_{s \in D}\) be a family of curves in a complex surface such that, for \(s \neq 0\), \(\Sigma_s\) is smooth and \(\Sigma_0\) has one branch point \(p\). Then the genus of \(\Sigma_0\) is smaller than the genus of \(\Sigma_s\): the difference of these genera is the Milnor number of the singularity. Very roughly speaking: what we lose in topology we gain in singularity.
QUESTION. What remains of this nice picture if \((\Sigma_n)\) is a sequence of 2-surfaces embedded in a 4-manifold which degenerates to a branched surface \(\Sigma_0\)? Can we define a Milnor number in this context? The question makes particular sense if the \(\Sigma_n\)'s are minimal: we remind the reader that complex curves in Kähler surfaces are a special case of minimal surfaces (Wirtinger’s theorem). However, when we try to generalize the Milnor number to arbitrary surfaces, we encounter the following two problems:

PROBLEM 1. The definition of a Milnor number will not depend only on the branched immersion \(\Sigma_0\). It might depend also on the sequence \((\Sigma_n)\) of smooth embedded surfaces converging to \(\Sigma_0\). In the complex analytic case, under mild assumptions, we can always find a complex 2-variable function \(F\) defined in a neighbourhood of the branch point in \(\mathbb{C}P^2\) such that the \(\Sigma_n\)'s are regular fibres of \(F\) and \(\Sigma_0\) is a singular fibre of \(F\).

PROBLEM 2. A key feature of the topology of complex curves in complex surfaces is the connection between the tangent and normal bundles (in other words, between the intrinsic and the extrinsic topology); this is reflected in the adjunction formula (see for example [B-P-V]).

This formula implies that if a sequence of embedded complex curves \((\Sigma_n)\) degenerates to a branched curve \(\Sigma_0\), we will have

\[
  c_1(T\Sigma_n) + c_1(N\Sigma_n) = c_1(N\Sigma_0) + c_1(N\Sigma_0)
\]

(we will recall below how to define the tangent and normal bundle for a branched surface \(\Sigma_0\)).

If the \(\Sigma_n\)'s are minimal surfaces, there is no adjunction formula so the limit behaviour of \(c_1(T\Sigma_n)\) and \(c_1(N\Sigma_n)\) will not be linked in the same strong fashion. It seems therefore reasonable to define two Milnor numbers, one for the tangent bundle, one for the normal bundle.

REMARK. The question of generalizing the Milnor number to the non-complex algebraic case has been around for some time. We would like to mention here the work of Rémi Langevin (see [La] for example) and of Lee Rudolph: in particular [Ru 2] which contains a construction closely related to ours.
1.2 Sketch of the paper

After some preliminaries, we consider the following situation: a 4-manifold $M$, a sequence $(\Sigma_n)$ of surfaces immersed in $M$ and a surface $\Sigma_0$ immersed in $M$ with branch points. We will be more precise below; for now we say that the $\Sigma_n$’s converge smoothly to $\Sigma_0$ on every compact set outside of the singular points of $\Sigma_0$.

For simplicity’s sake, let us assume that $\Sigma_0$ has only one singular point, $p$. We denote by $T\Sigma_n$ (resp. $N\Sigma_n$) the tangent (resp. normal bundle) of $\Sigma_n$. We focus on the case when the degree of $T\Sigma_n$ (resp. $N\Sigma_n$) has a well-defined limit as $n$ goes to infinity. Then we can define the tangent (resp. normal) Milnor number $\mu_p^T$ (resp. $\mu_p^N$). Its opposite $-\mu_p^T$ (resp. $\mu_p^N$) measures the amount of curvature of the bundle $T\Sigma_n$ (resp. $N\Sigma_n$) which gets concentrated around $p$ as $n$ goes to infinity. Although these numbers are not always well-defined, sometimes the topology of the situation ensures that one of them is. In particular if the $\Sigma_n$’s are closed without boundary, embedded and have bounded genus both Milnor numbers are well-defined.

If the $\Sigma_n$’s are complex curves in a Kähler surface $M$, then $|\mu^T| = |\mu^N|$. In a more general context, we have

**Theorem 1** Consider an oriented 4-manifold $M$ and a sequence $(\Sigma_n)$ of 2-surfaces immersed in $M$. Let $\Sigma_0$ be a 2-surface immersed in $M$ possibly with branch points and/or multiple components. Let $p$ be a singular point of $\Sigma_0$ and assume that the $\Sigma_n$’s converge to $\Sigma_0$ smoothly on compact subsets not containing $p$ (see Def. 2 below). Suppose moreover than either 1) or 2) below holds

1) i) the $\Sigma_n$’s are embedded
   ii) denoting by $S(p, \epsilon)$ the sphere centered at $p$ of radius $\epsilon$, $\Sigma_n \cap S(p, \epsilon)$ is connected for $\epsilon$ small enough and $n$ large enough

2) the $\Sigma_n$’s are minimal.

Then, if $\mu_p^T$ exists, so does $\mu_p^N$ and moreover

$$\mu_p^T \geq |\mu_p^N|.$$ 

**REMARK.** Let us comment on assumption 1) ii). It means that there is only one germ of disk (branched or not) of $\Sigma_0$ going through $p$. It does not
require \( \Sigma_0 \) to be a topologically embedded submanifold of \( M \) (although if this is the case, then 1) i) implies 1) ii)). For example, if \( \Sigma_0 \) is parametrized in a neighbourhood of \( p \) by
\[
z \mapsto (z^2, \text{Re}(z^3), 0),
\]
then it has self-intersections.

Before we state the corollary, we need to explain a notation we will use throughout the paper. If \( L \) is a \( U(1) \)-bundle above a connected oriented 2-surface \( \Sigma \) without boundary, we denote the degree of \( L \) by \( c_1(L) \): in other words, we identify the cohomology class \( c_1(L) \) with its integer representative in \( H^2(\Sigma, \mathbb{Z}) \).

**Corollary 1** If \( M, (\Sigma_n) \) and \( \Sigma_0 \) are as in Th. 1 and verify 1) or 2) of that theorem, then for \( n \) large enough,
\[
c_1(T\Sigma_n) + c_1(N\Sigma_n) \leq c_1(T\Sigma_0) + c_1(N\Sigma_0).
\]

**REMARK.** In the case of minimal surfaces, Corollary 1 was proved by ([Ch-T]).

If we assume that the second fundamental forms of the \( \Sigma_n \)'s have a common \( L^2 \) bound, we can derive a common upper bound on the areas of the lifts of the \( \Sigma_n \)'s in the Grassmannian \( G_2^+(M) \) of oriented 2-planes tangent to \( M \). A bubbling off phenomenon in \( G_2^+(M) \) ensues: a closed 2-current \( C \) in \( G_2^+(T_pM) \) appears (\( G_2^+(T_pM) \) denoting the fibre of \( G_2^+(M) \) above \( p \)). The numbers \( \mu^T_p \) and \( \mu^N_p \) can be computed on the homology class of \( C \).

Moreover, if the \( \Sigma_n \)'s are minimal surfaces, \( C \) is a complex curve.

**SKETCH OF THE PAPER**
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2 Preliminaries

Here and in the rest of the paper, $M$ is a $C^2$ 4-manifold. We endow it with an auxiliary Riemannian metric. We point out that except for the part on minimal surfaces, our results do not depend on the choice of metric.

2.1 Branched immersions

We recall the

Definition 1 ([G-O-R]) A map $f : D \to M$ from the disk $D$ to a 4-manifold $M$ is branched at $0$ if we have in a neighbourhood of $0$, $f^1(z) = \Re(z^N) + o_1(|z|^N)$
$f^2(z) = \Im(z^N) + o_1(|z|^N)$
$f^k(z) = o_1(|z|^N)$ (for $k = 3, 4$), for some $N$, $N \geq 2$.

In the formulae above, $z$ is a local complex coordinate on $D$ around the origin; the $f^k(z)$’s are the coordinates of $f(z)$ in some well chosen chart on $M$ around $f(0)$. A function is a $o_1(|z|^N)$ if it is a $o(|z|^N)$ and its first partial derivatives are $o_1(|z|^{N-1})$.

We will call $f(D)$ a branched disc.

For further use we now state the following fact (cf. [McD], [S-Vi], [Vi 4]); it follows from the implicit function theorem.
Lemma 1 Let $\epsilon$ be a small positive number, let $S(p, \epsilon)$ be the sphere centered at $p$ and of radius $\epsilon$ and put $\Gamma^\epsilon = S(p, \epsilon) \cap f(D)$. Then for $\epsilon$ small enough, there is a positive function $r_\epsilon$ such that $\Gamma^\epsilon$ is parametrized by

$$\theta \mapsto f(r_\epsilon(\theta)e^{i\theta}).$$

REMARK 1. For such a map $f$, one can check that the map from $D$ to the Grassmannian of oriented 2-planes $G_2^+(M)$ given by

$$p \mapsto T_p f(D)$$

($T_p f(D)$ denoting the tangent plane to $f(D)$ at $p$) extends continuously across the branch point.

We will say that a map $f : S \to M$ from a Riemann surface $S$ to a manifold $M$ is a branched immersion if it is an immersion everywhere except at a discrete set of points called branch points which are parametrized by branched discs as in Def. 1. It follows from Remark 1 above that we can define an oriented 2-plan bundle $Tf$ (of course, if $f$ is an immersion $Tf$ is isomorphic to the tangent bundle $T\Sigma$); and by taking orthogonal complements, an oriented 2-plane bundle $Nf$.

The bundles $Tf$ and $Nf$ have natural orientations. We remind the reader of the following orientation convention. Let $m$ be a point in $S$ and let $e_1, e_2$ be a positive basis of $T_m f(\Sigma)$, in other words, $T_m f$; a basis $e_3, e_4$ of $N_m f$ is positive if and only if $(e_1, e_2, e_3, e_4)$ is a positive basis of $T_m M$.

In the course of this paper, a surface immersed in $M$ with branch points means the following: a 2-dimensional CW-complex $\Sigma_0$ included in $M$ which is the image of some Riemann surface $S$ under a branched immersion $f : S \to M$. The bundles $T\Sigma_0$ and $N\Sigma_0$ are the bundles $Tf$ and $Nf$ which we have described above (so they are not bundles above $\Sigma_0$ but above the preimage $S$).

2.2 A lemma

Throughout the paper, we will rely on the following

Lemma 2 Let $\Sigma$ be a surface with boundary and let $F : L \to \Sigma$ be a $U(1)$-bundle. We denote by $<,>$ the scalar product on $L$ and by $J$ the complex
structure on $L$. We consider a section $s$ of $L$ which vanishes nowhere on the boundary of $\Sigma$. We let $\nabla$ be a $U(1)$-connection on $L$ and we define a connection 1-form $\omega$ by

$$\omega(u) = \frac{<\nabla_u s, Js>}{\|s\|^2} = <\nabla_u \left(\frac{s}{\|s\|}\right), J\left(\frac{s}{\|s\|}\right)>.$$ 

We let $\Omega = d\omega$ be the curvature form of $\nabla$. We have

$$\int_\Sigma \Omega = \int_{\partial \Sigma} \omega + \sum_{i=1}^{m} \text{index}(z_i)$$

where the $z_i$’s $i = 1, ..., m$ are the zeroes of $s$ inside $\Sigma$.

PROOF. For a small real number $\epsilon$, we consider the balls $B(z_i, \epsilon)$ centered at $z_i$ for $i = 1, ..., m$ and of radius $\epsilon$. We apply Stokes’ formula on $\Sigma - \cup B(z_i, \epsilon)$ to the form $\omega$. Letting $\epsilon$ tend to zero yields the desired result.

3 Definition of the Milnor numbers; statement of the results

3.1 Which convergence do we consider

Definition 2 Let $(\Sigma_n)$ be a sequence of surfaces immersed in $M$ (and embedded outside a set of codimension at least 1) and let $\Sigma_0$ be a surface which immersed in $M$ possibly with branch points. Let $p_1, ..., p_m$ be a finite number of points in $\Sigma_0$. The $\Sigma_n$’s converge to $\Sigma_0$ smoothly on compact sets outside of the $p_i$’s if the following is true:

for every small enough $\epsilon$, there exists a 2-surface with boundary $S_\epsilon$ and endowed with a metric $g_\epsilon$ such that:

there exists an integer $n_\epsilon$ such that for every $n > n_\epsilon$, there exists a smooth immersion $f_n^{(\epsilon)}$ which is almost everywhere 1 to 1 from $S_\epsilon$ into $M$ with

$$f_n^{(\epsilon)}(S_\epsilon) = \Sigma_n \cap (M - \cup_{i=1}^{m} B(p_i, \epsilon)).$$

Moreover the $f_n^{(\epsilon)}$’s converge $C^2$ to an immersion $f_0^{(\epsilon)} : S_\epsilon \rightarrow M$ with

$$f_0^{(\epsilon)}(S_\epsilon) = \Sigma_0 \cap (M - \cup_{i=1}^{m} B(p_i, \epsilon)).$$

We point out that we allow $\Sigma_0$ to have multiple components.
3.2 A definition of the Milnor numbers

We consider \( (\Sigma_n), \Sigma_0 \) and \( M \) as in Def. 2 above. We denote by \( \nabla \) the Levi-Civita connection on \( M \): it induces a connection \( \nabla^T_n \) (resp. \( \nabla^N_n \)) on \( T\Sigma_n \) (resp. \( N\Sigma_n \)): let \( \Omega^N_n \) (resp. \( \Omega^T_n \)) be the curvature 2-form of \( \nabla^T_n \) (resp. \( \nabla^N_n \)).

We are now ready to state

**Definition 3** The notations are as in Def. 2. Let \( p \) be a branch point of \( \Sigma_0 \). For a small number \( \epsilon \), we denote by \( B(p, \epsilon) \) the ball centered at \( p \) and of radius \( \epsilon \). If the following quantity exists

\[
\mu_p^T = -\frac{1}{2\pi} \lim_{\epsilon \to 0} \lim_{n \to \infty} \int_{B(p,\epsilon) \cap \Sigma_n} \Omega^T_n
\]

(resp \( \mu_p^N = -\frac{1}{2\pi} \lim_{\epsilon \to 0} \lim_{n \to \infty} \int_{B(p,\epsilon) \cap \Sigma_n} \Omega^N_n \))

we call it the tangent (resp. normal) Milnor number of the sequence \( (\Sigma_n) \) at the point \( p \).

From now on we will use the notation

\[ \Sigma^\epsilon_n = \Sigma_n \cap B(p, \epsilon). \]

**PLEASE NOTE.** In the Def. 3 as in the definition for \( \Sigma^\epsilon_n \) above, we committed a slight abuse of notation. When the \( \Sigma_n \)'s are not embedded but immersed, the \( \Sigma^\epsilon_n \)'s will not mean the subsets of \( M \) but their smooth preimages under an immersion into \( M \).

**REMARK.** We would like to say a word about the double limit. It means that the sequence \( \int_{\Sigma_n} \Omega_n \) converges for (almost) every \( \epsilon \) to a finite real number \( l(\epsilon) \); and that \( l(\epsilon) \) has a finite limit when \( \epsilon \) tends to 0. Double limits can be tricky; however in the present case, the situation is very much simplified by Lemma 3 below.

**NB.** Lemma 3 is valid both for \( \mu^T \) and \( \mu^N \); to lighten the writing, we have stated it using \( \Omega_n \) for \( \Omega^T_n \) (resp. \( \Omega^N_n \)) and \( \mu \) for \( \mu^T \) (resp. \( \mu^N \)).

**Lemma 3** Let \( (\Sigma_n), \Sigma_0, M \) and \( p \) be as in Def. 2. Then the following two assertions are equivalent
1) there is a sequence $(\epsilon_s)$ converging to 0 such that for every $s$, the quantities $\int_{\Sigma_n^s} \Omega_n$ have a finite limit $\mu^{(s)}$ when $n$ tends to infinity; and we have
\[
\lim_{s \to \infty} \mu^{(s)} = \mu
\]

2) there exists an $\epsilon_0$ such that, for every $\epsilon$ with $0 < \epsilon < \epsilon_0$,
\[
\lim_{n \to \infty} \int_{\Sigma_n^s} \Omega_n = \int_{\Sigma_0^s} \Omega_0 + \mu.
\]

PROOF. It is clear that 2) implies 1).

Assume now that 1) is true and fix a positive number $\eta$. For every $\epsilon$, $n$ and $\epsilon_s < \epsilon$, we have
\[
| \int_{\Sigma_n^s} \Omega_n - \int_{\Sigma_0^s} \Omega_0 - \mu | \quad (I)
\]
\[
\leq | \int_{\Sigma_n^s - \Sigma_0^s} \Omega_n - \int_{\Sigma_0^s - \Sigma_0^s} \Omega_0 | + | \int_{\Sigma_n^s} \Omega_n - \mu^{(s)} | \quad (II)
\]
\[
+ | \mu^{(s)} - \mu | + | \int_{\Sigma_0^s} \Omega_0 | \quad (III).
\]

We choose an $\epsilon_s$ such that $(III) \leq \frac{\eta}{2}$. Given this $\epsilon_s$, there exists an integer $N$ such that, for every $n > N$, $(II) < \frac{\eta}{2}$; thus $(I) < \eta$. This concludes the proof of the Lemma.

EXEMPLE 1. If the $\Sigma_n$’s are holomorphic curves in a Kähler surface,
\[
\mu_p^T = -\mu_p^N.
\]

In this case the Milnor number which algebraic geometers consider is not equal to the tangent or normal Milnor number we have just defined. It is equal to $\mu_p^T - (N - 1)$. We apologize to the readers for this risk of confusion. Nonetheless we chose to call our invariants Milnor numbers because, although neither is exactly equal to the traditional Milnor number, they both generalize it in a straightforward way.

EXEMPLE 2. Suppose now that $(\Sigma_n)$ is a sequence of surfaces in a 3-manifold $N$. If we embed $N$ in $N \times S^1$ and view $(\Sigma_n)$ as a sequence of surfaces in $N \times S^1$, then $\mu_p^N$ will be zero for every branch point $p$ of $\Sigma_0$.

We now state a topological criterion for the existence of $\mu^T$. 


We assume that there are $k$ disks, $D_1, D_2, \ldots, D_k$, of respective multiplicities $s_1, \ldots, s_k$, branched or not, going through $p$ in $\Sigma_0$. Each disk or branch is parametrized by a map $f_i : D \to M$ with $f_i(0) = p$. We can assume that each $f_i$ is of the form given by Def. 1; we denote by $N_i$ the integer appearing in that definition; however, unlike in Def. 1, we only assume $N_i \geq 1$. We put $m_i = N_i - 1$; it is zero if 0 is a smooth point of $f_i$; otherwise it is the branching order of $f_i$ at 0.

**Proposition 1** Suppose, as above, that there are $k$ branches going through the branch point $p$ of $\Sigma_0$ and the notations are as above.

If the left-hand side in the equality below is well-defined, then the tangent Milnor number is also well-defined and the following holds:

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \chi(\Sigma^n_\epsilon) = \sum_{i=1}^{k} s_i(m_i + 1) - \mu^T_p.$$  

**PROOF.** For $\epsilon$ and $n$, the Gauss-Bonnet formula with boundary writes

$$\int_{\Sigma^n_\epsilon} \Omega^n - 2\pi \chi(\Sigma^n_\epsilon) = -\int_{\partial \Sigma^n_\epsilon} k_g$$

where $k_g$ denotes the geodesic curvature of the curve $\partial \Sigma^n_\epsilon$ on the surface $\Sigma^n_\epsilon$. When $n$ tends to infinity, the right-hand side in the above formula tends to

$$-\int_{\partial \Sigma^0} k_g.$$

Here $k_g$ denotes the geodesic curvature of $\partial \Sigma^0$ inside $\Sigma^0$. To handle this last expression, we let $exp_p$ be the exponential map from a ball centered at the origin in $T_pM$ to a neighborhood of $p$ in $M$ and, we introduce, for small enough positive $\epsilon$’s, the surfaces

$$\hat{\Sigma}^\epsilon_0 = \frac{1}{\epsilon} exp^{-1}(\Sigma^\epsilon_0).$$

We let $P_1$ be the plane tangent to $D_1$ at $P$. When $\epsilon$ tends to 0, the $\hat{\Sigma}^\epsilon_0$’s tend to the union of the unit disks in the $P_1$’s, each disk counted $s_i(m_i + 1)$.
times, and endowed with the Euclidean metric of \( T_p M \). Likewise, the curves \( \partial (\tilde{\Sigma}_0) \) converge to the union of the unit circles \( C_i \)'s of the \( P_i \)'s, each counted \( s_i(m_i + 1) \) times. We let the reader derive from all this that

\[
\lim_{\epsilon \to 0} \int_{\partial \Sigma_0^\epsilon} k_g = \sum_{i=1}^k s_i(m_i + 1) \int_{C_i} k_g
\]

where the \( k_g \) in the right-hand side of this last expression is the geodesic curvature of \( C_i \) in the Euclidean plane \( P_i \). Hence

\[
\lim_{\epsilon \to 0} \int_{\partial \Sigma_0^\epsilon} k_g = [\sum_{i=1}^k s_i(m_i + 1)]2\pi.
\]

Prop. 1 follows.

4 Proof of Th. 1 1): embedded surfaces

We begin by a construction similar to [Vi 4]. We let \( P \) be the plane tangent to \( \Sigma_0 \) at \( p \) and we have

**Lemma 4** There exists an \( \epsilon_0 > 0 \) such that for a generic \( \epsilon \), with \( \epsilon < \epsilon_0 \), the following is true: for \( n \) large enough, the knot

\[
K_{n}^{\epsilon} = \partial \Sigma_{n}^{\epsilon}
\]

is a braid in \( S(p, \epsilon) \) with braid axis the great circle in \( P_{\perp} \).

**Proof.** We use the following characterization of a braid axis:

**Lemma 5** Let \( L \) be a link in \( S^3 \). Let \( P \) be a plane in \( \mathbb{R}^4 \) and let \( (e_1, e_2) \) be a (non necessary orthonormal) basis of the orthogonal complement \( P_{\perp} \). We denote the orthogonal projection of \( L \) to \( P_{\perp} \) by

\[
x(t) = x_1(t)e_1 + x_2(t)e_2.
\]

The following two assertions are equivalent:

1) the great circle \( \Gamma \) in \( P \) is a braid axis for \( L \)

2) the projection of \( L \) to \( P_{\perp} \) verifies

\[
x_1^2(t) + x_2^2(t) \neq 0, \quad x_1(t)x'_2(t) - x_2(t)x'_1(t) \neq 0.
\]
We let 
\[ \Gamma^\epsilon = S(p, \epsilon) \cap \Sigma_0 \] 
be a curve in \( M \), it is not necessarily a knot, just an immersion of the circle \( S^1 \) to \( M \). The expression of \( f \) given by Def. 1 together with Lemma 1 show us that for \( \epsilon \) small enough, \( \Gamma^\epsilon \) verifies assertion 2) of Lemma 5 above w.r.t the tangent plane \( P \). For a given \( \epsilon \), the \( K^\epsilon_n \)'s converge to \( \Gamma^\epsilon \). Thus for a small enough generic \( \epsilon \) and a large enough \( n \) the \( K^\epsilon_n \)'s also verify Lemma 5 2). This concludes the proof of Lemma 4.

We denote the algebraic crossing number of this braid by \( e(K^\epsilon_n) \). We take a non-zero vector \( X \) in \( Q \) and we denote by \( X^N_n \) its orthogonal projection to \( N\Sigma^\epsilon_n \). The Levi-Civita connection on \( N\Sigma^\epsilon_n \) yields a covariant derivative \( \nabla^{(n)} \) on \( N\Sigma^\epsilon_n \) and we derive a connection form \( \omega^N_n \) defined by

\[
\forall u \in T\Sigma_n, \quad \omega^N_n(u) = \frac{\langle \nabla^{(n)}_u X^N_n, J_n X^N_n \rangle}{\|X^N_n\|}
\]

where \( J_n \) denotes the complex structure on \( N\Sigma_n \) compatible with the \( SO(2) \)-structure. We denote by \( N(X^N_n, \Sigma^\epsilon_n) \) the number of zeroes of \( X^N_n \) in \( \Sigma^\epsilon_n \). We can write Lemma 2 above

\[
\int_{\Sigma^\epsilon_n} \Omega^N_n = \int_{\partial \Sigma^\epsilon_n} \omega^N_n + N(X^N_n, \Sigma^\epsilon_n) \quad (2)
\]

For a fixed \( \epsilon \), we have

\[
\lim_{n \to \infty} \int_{\partial \Sigma^\epsilon_n} \omega^N_n = \int_{\partial \Sigma^\epsilon_0} \omega^N_0.
\]

The tangent plane to \( \Sigma_0 \) at a point \( q \) near \( p \) tends to \( P \) as \( q \) tends to \( p \). On the other hand the vector \( X \) does not belong to \( P \), hence we derive the existence of a positive real number \( \alpha \) such such that,

\[
\alpha \leq \|X^N_0\| \quad (3)
\]

everywhere on \( \Sigma^\epsilon_0 \).

The form \( \omega^N_0 \) is defined everywhere on \( \Sigma^\epsilon_0 \), or rather on the disk in \( \mathbb{C} \) which parametrized \( \Sigma^\epsilon_0 \); it follows that, for some constant \( A \),and some positive number \( \epsilon_1 \), we have

\[
\forall \epsilon > 0, \ 0 < \epsilon \leq \epsilon_1, \ \forall x \in \Sigma^\epsilon_0, \ \forall u \in T_x \Sigma^\epsilon_0, \ |\omega^N_0(u)| \leq A\|u\| \quad (4).
\]
We derive from (3) and (4) that
\[ \lim_{\epsilon \to 0} \int_{\partial \Sigma_0^1} \omega_0^N = 0 \quad (5) \]

**Lemma 6** For \( \epsilon \) small enough and \( n \) large enough,
\[ N(X_n^N, \Sigma_n^\epsilon) = e(K_n^\epsilon). \]

**PROOF.** Inside \( \mathbb{R}^4 \), the vector \( X \) is never tangent to the knot \( K_n^\epsilon \); nor is it ever orthogonal to the sphere \( S(p, \epsilon) \) at a point in \( K_n^\epsilon \). It follows that \( X \), or rather its projection to \( S(p, \epsilon) \) along \( K_n^\epsilon \) defines a framing of the knot \( K_n^\epsilon \). We denote by \( \hat{K}_n^\epsilon \) a knot obtained by pushing \( K_n^\epsilon \) on \( S(p, \epsilon) \) slightly in the direction of \( X \). The linking number between \( K_n^\epsilon \) and \( \hat{K}_n^\epsilon \) is equal to the number of intersection points between two surfaces smoothly embedded in \( B(p, \epsilon) \) and bounded respectively by these two knots. In other words
\[ N(X_n^N, \Sigma_n^\epsilon) = \text{lk}(K_n^\epsilon, \hat{K}_n^\epsilon) \quad (6) \]

The right-hand side of the above identity is equal to the algebraic crossing number of the braid \( K_n^\epsilon \). For the reader who feels more at ease with braids in \( \mathbb{R}^3 \), we add the following. We complete \( X \) in an orthonormal basis \((X, Y)\) of \( Q \) and map the sphere \( S(p, \epsilon) \) to \( \mathbb{R}^3 \) by stereographic projection of pole \( Y \). The knot \( K_n^\epsilon \) becomes a closed braid of axis \( X \) and its linking number with \( \hat{K}_n^\epsilon \) is the algebraic crossing number of the braid.

At this juncture we need to recall the *slice Bennequin inequality* which was proved by Lee Rudolph.

**NOTATION.** If \( L \) is an oriented link in \( S^3 \) we let \( \chi_s(L) \) be the greatest Euler characteristic of a smooth 2-surface \( F \) in \( B^4 \) without closed components and smoothly embedded in \( B^4 \) with boundary \( L \).

**Theorem 2** ([Ru 3]) Let \( \beta \) be a closed braid with \( n \) strands and algebraic crossing number \( e(\beta) \). Then
\[ \chi_s(\beta) \leq n - e(\beta). \]

The braid index of \( L_n^\epsilon \) is equal to the quantity \( N \) appearing in Def. 1; in other words, it is \( m + 1 \), where \( m \) is the branching order of \( p \). So Th. 2 yields for \( \epsilon \) small enough, there exists an integer \( n_1 \) such that, for every \( n > n_1 \),
\[ \chi(\Sigma_n^\epsilon) - N \leq -\text{lk}(\hat{K}_n^\epsilon, K_n^\epsilon) \quad (7). \]
We can now reverse the orientation of \( M \): the quantities in the left hand-side of (7) will be unchanged and the right-hand side will be changed in its opposite, that is,

\[
\chi(S^+ - N) \leq lk(\hat{K}^\epsilon, K_n^\epsilon)
\]  

(8)

Putting (7) and (8) together we derive

\[
|lk(L_n^\epsilon, L_n^\epsilon)| \leq -\chi(S^+ + N)
\]  

(9)

The right-hand side of (9) can be interpreted in view of Prop. 1. We choose a small enough \( \epsilon \) for which the right-hand term of (9) converges as \( n \) goes to infinity; it follows that the left-hand side is bounded above independently of \( n \), hence there is a subsequence \( n_p \) for which the sequence

\[
|lk(\hat{K}^\epsilon, K_{n_p}^\epsilon)|
\]

converges. This last fact, coupled with (5) and (6) above ensures that

\[
\int_{S^+_{n_p}} \Omega^N
\]

has a finite limit when \( n_p \) tends to infinity.

We apply Lemma 3 above to derive that \( \mu^N \) exists for the subsequence \( n_p \). Moreover (9) ensures that for the subsequence \( n_p \),

\[
|\mu^N_{n_p}| \leq \mu^T_{n_p}.
\]

We can derive something else from this proof. If the limiting surface \( S_0 \) is topologically embedded, then \( \Gamma^\epsilon \) is also a braid for \( \epsilon \) small enough (see [Vi 4]); the quantities in (6) converges to the algebraic crossing number of \( \Gamma^\epsilon \). We derive

**Proposition 2** Let \( M \) an oriented 4-manifold and, \( (S_n) \) a sequence of surfaces converging to a branched surface \( S_0 \) as in Def. 2. Suppose moreover that \( S_0 \) is topologically embedded. We denote by \( \Gamma^\epsilon \) the braid defined on \( S_0 \) by the branch point \( p \) for \( \epsilon \) small enough and we let \( e(\Gamma^\epsilon) \) be its algebraic crossing number.

The quantity \( \mu^N_p \) is well-defined and for \( \epsilon \) small enough

\[
\mu^N_p = e(\Gamma^\epsilon)
\]
REMARK. If \( \Sigma_0 \) is not only embedded by closed without boundary, Prop. 2 immediately follows from

**Theorem 3 (\cite{Vi 4})** Let \( \Sigma \) be a closed surface without boundary immersed into an oriented 4-manifold \( M \) with one branch point \( p \). Suppose moreover that \( p \) is an isolated singularity of \( \Sigma \). Then for a small number \( \epsilon \), the link \( L^\epsilon = S(p, \epsilon) \cap \Sigma \) is a disjoint union of closed braids \( L^\epsilon_1, \ldots, L^\epsilon_s \) which all have, up to orientation, the same axis.

The degree of the normal bundle \( N\Sigma \) writes

\[
c_1(N\Sigma) = [\Sigma] \cdot [\Sigma] - \sum_{i=1}^s e(L^\epsilon_i) + 2 \sum_{1 \leq i < j \leq s} \text{lk}(L^\epsilon_i, L^\epsilon_j).
\]

where \([\Sigma] \cdot [\Sigma]\) denotes the self-intersection number of \( \Sigma \) in \( M \).

5 Proof of Th. 1: minimal surfaces

The specific form of the curvature for minimal surfaces (see Appendix 1) yields

**Proposition 3** Let \((M, g)\) be a Riemannian 4-manifold, let \((\Sigma_n)\) be a sequence of minimal surfaces converging to a branched minimal surface \( \Sigma_0 \) smoothly on compact subsets outside of the singular points of \( \Sigma_0 \). We denote by \( B_n \) the second fundamental form on \( \Sigma_n \) and by \( dA_n \) the area element of \( \Sigma_n \) for the metric induced on \( \Sigma_n \) by the metric \( g \). We let \((e_1, e_2)\) be a local positive orthonormal frame on \( \Sigma_n \). Then for a branch point \( p \) in \( \Sigma_0 \),

\[
\mu^T_p = \frac{1}{4\pi} \lim_{\epsilon \to 0} \lim_{n \to \infty} \int_{B(p, \epsilon) \cap \Sigma_n} \| B_n \|^2
\]

\[
\mu^N_p = \frac{1}{\pi} \lim_{\epsilon \to 0} \lim_{n \to \infty} \int_{B(p, \epsilon) \cap \Sigma_n} B(e_1, e_2) \wedge B(e_1, e_1) \, dA_n.
\]

Here we have identified the 2-vector \( B(e_1, e_2) \wedge B(e_1, e_1) \) to a number since it belongs to \( \Lambda^2(N\Sigma_n) \) which identifies with \( \mathbb{R} \) (via the orientation of \( N\Sigma_n \)). These formulae conclude the proof of Th. 1 for minimal surfaces.
6 Bounding the second fundamental form

If $\Sigma$ is a branched immersion in $M$, the data of its oriented tangent planes yield a lift in $G_2^+(M)$ which we denote by $\tilde{\Sigma}$.

We denote by $dA_n$ the area element of $\Sigma_n$ for the metric induced on $\Sigma_n$ by the metric on $M$ and we let $B_n$ be the second fundamental form of $\Sigma_n$. A straightforward computation yields

$$\text{area}(\tilde{\Sigma}_n) \leq \text{area}(\Sigma_n) + 2 \int_{\Sigma_n} \|B_n\|dA_n + 4 \int_{\Sigma_n} \|B_n\|^2dA_n.$$ 

The following ensues (the notations being as above):

**Proposition 4** Let $(\Sigma_n)$ be a sequence of surfaces immersed in a 4-manifold $M$. Suppose

1) $\exists C_1$ such that $\forall n \in \mathbb{N}$, $\text{area}(\Sigma_n) \leq C_1$

2) $\exists C_2$ such that $\forall n \in \mathbb{N}$, $\|B_n\|_2 \leq C_2$.

Then, $\exists C_3$ such that $\forall n \in \mathbb{N}$,

$$\text{area}(\tilde{\Sigma}_n) \leq C_3.$$ 

For our present purpose we do not need a global $L^2$ bound of the second fundamental forms of the $\Sigma_n$'s; a local bound as defined below will suffice:

**Definition 4** Let $M$ be a Riemannian manifold and let $(\Sigma_n)$ be a sequence of surfaces immersed in $M$.

The $\Sigma_n$'s have local common bounds for the area and for the $L^2$ norm of the second fundamental form if and only if for every point $p$ in $M$ there exists an $\epsilon_0$ and a constant $C(p)$ such that for every integer $n$

1) $\text{area}(\Sigma_n^\epsilon) < C(p)$

2) $\int_{\Sigma_n^\epsilon} \|B_n\|^2 < C(p)$

where $\Sigma_n^\epsilon = \Sigma_n \cap B(p, \epsilon)$.

We derive

**Theorem 4** Let $M$, $(\Sigma_n)$, $\Sigma_0$ be as in Def. 2. Let $p$ be a branch point of $\Sigma_0$. Suppose moreover that the $\Sigma_n$'s have local common bounds for the area and for the $L^2$ norm of the second fundamental form. For an $\epsilon > 0$, we denote by $\tilde{\Sigma}_n^\epsilon$ the lift in $G_2^+(M)$ of $\Sigma_n^\epsilon$.
There exists a closed 2-current $C$ in $G^+_2(T_pM)$ (the Grassmannian of oriented 2-planes tangent to $M$ at $p$) such that the following is true: for every $\epsilon > 0$, the sequence $(\tilde{\Sigma}^\epsilon_n)$ converges in the sense of currents and
\[
\lim_{n \to \infty} \tilde{\Sigma}^\epsilon_n = \tilde{\Sigma}^\epsilon_0 + C.
\]

### 6.1 Preliminaries about the Grassmann bundle

In order to make use of Th. 4 above, we need to recall some elementary facts on the Grassmann bundle.

Let $E$ be a 4-dimensional oriented Euclidean vector space, let $\Lambda^2(E)$ be the space of exterior 2-vectors and let
\[ *
\] be the Hodge star operator. $\Lambda^2E$ splits into the sum of its ±1-eigenspaces w.r.t. $*$, that is $\Lambda^2E = \Lambda^+E \oplus \Lambda^-E$.

We denote by $S(\Lambda^+E)$ and $S(\Lambda^-E)$ the unit spheres of these eigenspaces and by $G^+_2(E)$ the Grassmannian of oriented 2-planes in $E$. We recall the isomorphism ([Be])
\[
S(\Lambda^+E) \times S(\Lambda^-E) \to G^+_2(E)
\]
\[
(J, K) \mapsto \frac{J + K}{\sqrt{2}}.
\]
When we write this, we identify an oriented 2-plane $P$ with an element of $\Lambda^2E$: if $(e_1, e_2)$ is a positive orthonormal basis of $P$, $P$ is identified with $e_1 \wedge e_2$.

#### 6.1.1 The 2-homology of $G^+_2(\mathbb{R}^4)$

Fix any $J_0 \in S(\Lambda^+\mathbb{R}^4)$, $K_0 \in S(\Lambda^-\mathbb{R}^4)$. The 2-homology $H_2(G^+_2(\mathbb{R}^4), \mathbb{Z})$ is generated by the classes $[S_+]$ and $[S_-]$ where
\[
S_+ = \left\{ \frac{1}{\sqrt{2}}(h + K_0) \mid h \in S(\Lambda^+\mathbb{R}^4) \right\}
\]
\[
S_- = \left\{ \frac{1}{\sqrt{2}}(J_0 + k) \mid k \in S(\Lambda^-\mathbb{R}^4) \right\}
\]
We denote by $\omega_+$ (resp. $\omega_-$) the 2-cohomology class in $H^2(G_2^+(\mathbb{R}^4), \mathbb{Z})$ dual to $[S_+]$ (resp. $[S_-]$). Another way to define $\omega_\pm$ is to say: $\omega_+$ (resp. $\omega_-$) is the pull-back of the fundamental class of $S(\Lambda^+\mathbb{R}^4)$ (resp. $S(\Lambda^-\mathbb{R}^4)$) under the projection $G_2^+(\mathbb{R}^4) \rightarrow S(\Lambda^+\mathbb{R}^4)$ (resp. $G_2^+(\mathbb{R}^4) \rightarrow S(\Lambda^-\mathbb{R}^4)$).

6.1.2 The homology class of the lift of a branched immersion

If we consider now a Riemannian 4-manifold $M$ and the bundles $S(\Lambda^+M)$, $S(\Lambda^-M)$ and $G_2^+(M)$, the cohomology of the total spaces of these bundles can be described by the Leray-Hirsch theorem (see for example [Hi]) for $S(\Lambda^-M)$).

The classes $\omega_+$ and $\omega_-$ extend to two classes in $H^2(G_2^+(M), \mathbb{Z})$: we denote these classes by $\tilde{\omega}_+$ and $\tilde{\omega}_-$.

Consider now a surface $\Sigma$ immersed with branch points in $M$ and let $\tilde{\Sigma}$ be its lift in $G_2^+(M)$. The degrees of the tangent and normal bundle of $\Sigma$ can be computed via the homology class $[\tilde{\Sigma}]$ of $\tilde{\Sigma}$ in $G_2^+(M)$. Namely

**Proposition 5** ([E-S], [Vi 1]). Let $\Sigma$ be a surface immersed in $M$ with branch points; using the above notations,

\[
c_1(T\Sigma) = \langle \tilde{\omega}_+ + \tilde{\omega}_-, [\tilde{\Sigma}] \rangle
\]

\[
c_1(N\Sigma) = \langle \tilde{\omega}_+ - \tilde{\omega}_-, [\tilde{\Sigma}] \rangle
\]

6.2 The homology class of the current $C$

We can now write the homology class of $C$ in terms of the generators for the homology group $H_2(G_2^+(T_pM), \mathbb{Z})$ described above:

**Proposition 6** Consider $M$, $\Sigma_n$, $\Sigma_0$ and $C$ as in Th. 4 and let $[C] = a[S_+] + b[S_-]$, $a, b \in \mathbb{Z}$, be the homology class of $C$ in $H_2(G_2^+(T_pM), \mathbb{Z})$. Then

\[
\mu^T_p = a + b
\]

\[
\mu^N_p = a - b.
\]
6.3 Minimal surfaces

In the rest of this paper we will assume the $\Sigma_n$’s to be minimal surfaces. If $x$ is a point in $\Sigma_n$, we will denote by $K_{\Sigma_n}(x)$ (resp. $K_M(x)$) the sectional curvature of $\Sigma_n$ (resp. $M$) at $x$. The Gauss equation ([K-N]) yields

$$K_M(x) = K_{\Sigma_n}(x) - \frac{1}{2}\|B_n\|^2.$$

We derive

**Proposition 7** Let $(\Sigma_n)$ be a sequence of minimal surfaces which converges to a surface $\Sigma_0$ which is immersed with branch points. Let $p$ be a point in $M$ and let $\epsilon$ be a small enough number. Assume that:

1) there exists a positive number $C_1$ such that for every integer $n$,

$$\text{area}(\Sigma_n^\epsilon) \leq C_1$$

2) $\mu^T_p$ exists.

Then there exists a constant $C_2$ such that

$$\forall n \in \mathbb{N}, \int_{\Sigma_n^\epsilon} \|B_n\|^2 \leq C_2.$$

6.4 Preliminaries: Eells-Salamon’s result for the Grassmann bundle

A useful tool when dealing with minimal surfaces in 4-manifolds is the twistor space. Here we use the Grassmann bundle $G_2^+(M)$ as a twistor bundle and we endow it with an almost complex structure.

Consider first a Euclidean 4-vector space $E$; we construct a complex structure $\mathcal{I}$ on $G_2^+(E)$ as follows.

If $P$ is a 2-plane in $G_2^+(E)$, a vector tangent to $G_2^+(E)$ at $P$ will be a linear map from $P$ to its orthogonal complement $P^\perp$. In other words, $T_PG_2^+(E)$ identifies with $\text{Hom}_\mathbb{R}(P, P^\perp)$. We now let $J$ be the complex structure on $P$ compatible with the metric and orientation. The complex structure $\mathcal{I}$ on $T_PG_2^+(E)$ is defined by putting

$$\forall X \in P, \forall \Phi \in T_PG_2^+(E), \mathcal{I}(\Phi)(X) = \Phi(JX).$$
It turns out that $I$ is an integrable complex structure on $G^+_2(E)$. The $S_+$ and $S_-$ which we defined in §6.1.1 are complex lines w.r.t. $I$ and as a complex surface $G^+_2(E)$ is isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$.

In the spirit of Eells and Salamon, we endow the Grassmann bundle $G^+_2(M)$ of a Riemannian 4-manifold $M$ with an almost complex structure w.r.t. which the lifts of minimal surfaces in $M$ will be pseudo-holomorphic curves. Actually there are two equally good choices for this almost complex structure. We call them $J^+$ and $J^-$ and we proceed to describe them.

Let $p$ be a point in $M$ and let $P$ be an oriented 2-plane tangent to $M$ at $p$. We recall that $G^+_2(T_pM) = S(\Lambda^+(T_pM)) \times S(\Lambda^-(T_pM))$ and that $S(\Lambda^+(T_pM))$ (resp. $S(\Lambda^-(T_pM))$) denotes the set of all the complex structures on $T_pM$ compatible with the metric and preserving (resp. reversing) the orientation.

The tangent bundle $T_pG^+_2(M)$ splits into a horizontal and a vertical space,

$$T_pG^+_2(M) = H_pG^+_2(M) \oplus T_pG^+_2(M).$$

We define $J^\pm$ by restriction to these two spaces:

i) on $T_pG^+_2(M)$, both $J^+$ and $J^-$ are equal to $I$ defined above.

ii) to describe $J^\pm$ on $H_pG^+_2(M)$, we split $P$ into a sum of $\pm 1$-eigenvectors, namely

$$P = \frac{1}{\sqrt{2}}(J + K), \quad J \in S(\Lambda^+(T_pM)), K \in S(\Lambda^-(T_pM)).$$

The differential of the vector bundle projection from $G^+_2(M)$ to $M$ identifies $H_pG^+_2(M)$ with $T_pM$. Via this identification $J^+$ (resp. $J^-$) is given by the complex structure $J$ (resp. $K$) on $T_pM$. We can now rewrite Eells-Salamon’s result as follows:

**Theorem 5** Let $\Sigma$ be a Riemann surface, let $M$ be a Riemannian 4-manifold, and let $f : \Sigma \longrightarrow M$ be a conformal harmonic map. Then the lift

$$\tilde{f} : \Sigma \longrightarrow G^+_2(M)$$

is pseudo-holomorphic for both the almost complex structures $J^+$ and $J^-$. This theorem is neither new nor due to us; however to make the exposition clearer, we give a quick proof in Appendix 2.

**Remark.** We get the same complex structure (resp. almost complex structure) on $G^+_2(E)$ (resp. $G^+_2(M)$) if we put together the complex (resp. almost complex) structures Eells-Salamon consider on $S(\Lambda^+(M))$ and $S(\Lambda^-(M))$. 20
6.5 A complex curve in the Grassmannian

In view of this, we can restate Th. 4 for minimal surfaces:

**Theorem 6** Let $M$ be a Riemannian 4-manifold and let $(\Sigma_n)$ be a sequence of immersed minimal surfaces which converges to a minimal surface $\Sigma_0$ with a branch point $p$, smoothly on compact subsets outside of $p$. There exists a complex curve $S$ cohomologous to the current $C$ of Th. 4 such that, for every $\epsilon > 0$ small enough,

$$\lim_{n \to +\infty} \tilde{\Sigma}_n^\epsilon = \tilde{\Sigma}_0^\epsilon \cup S$$

where the limit means: convergence of pseudo-holomorphic curves with boundary in the sense of cusp-curves.

**Proof.** The $\tilde{\Sigma}_n$'s are pseudo-holomorphic curves with boundary; their areas and genera are bounded. Convergence of pseudo-holomorphic curves with bounded area and genus is described in [A-L]. It works much better for closed curves without boundary; however, in our present case, we know that in a neighbourhood of the boundary, the convergence of the $\Sigma_n^\epsilon$'s is a uniform $C^2$ convergence of immersions. It follows that, in a neighbourhood of the boundaries of $\Sigma_n^\epsilon$'s, the lifts $\tilde{\Sigma}_n^\epsilon$ also converge uniformly (in particular there cannot be any blowing up of a holomorphic disc or sphere at the boundary). That is, $(\tilde{\Sigma}_n^\epsilon)$ converges to a pseudo-holomorphic curve with boundary $C^\epsilon$.

For a given $\epsilon$, we put

$$\dot{S} = C^\epsilon - \tilde{\Sigma}_0^\epsilon;$$

it is clear that $\dot{S}$ does not depend on the $\epsilon$ we use.

We notice that $\dot{S}$ is contained in $G_2^+(T_pM)$ and that its closure, which we denote $S$, verifies

$$S = \dot{S} \cup \{q_0\}$$

where $q_0$ the point in $G_2^+(T_pM)$ corresponding to the plane tangent to $\Sigma_0$ at $p$; that is, $\{q_0\} = \tilde{\Sigma}_0 \cap G_2^+(T_pM)$.

$S$ represents the current $C$ hence it is not empty. Moreover $S - \{q_0\}$ is an analytic subvariety of $G_2^+(T_pM)$ so it follows from Remmert-Stein ([Si]) that $S$ is an analytic subvariety of $G_2^+(T_pM)$. QED

Fiberwise the Grassmann bundle is the product of the twistor bundles. Th.6 above yields
Corollary 2 We let \( M, (\Sigma_n), \Sigma_0 \) be as in Th. 6. We denote by \( Z_p^+ \) (resp. \( Z_p^- \)) the fibre of \( S(\Lambda^+(M)) \) (resp. \( S(\Lambda^-(M)) \)) above the point \( p \).

We let \( \tilde{\Sigma}_n^+ \) (resp. \( \tilde{\Sigma}_n^- \)) be the lift of \( \Sigma_n \) in \( S(\Lambda^+(M)) \) (resp. \( S(\Lambda^-(M)) \)). Then the sequence \( (\tilde{\Sigma}_n^+) \) (resp. \( (\tilde{\Sigma}_n^-) \)) converges to

\[
\tilde{\Sigma}_0^+ + (\mu_p^T + \mu_p^N)Z_p^+ \quad \text{(resp.} \quad \tilde{\Sigma}_0^- + (\mu_p^T - \mu_p^N)Z_p^-)\]

7 Superminimal surfaces

Superminimal surfaces are the closest Riemannian analogue to complex curves in Kähler surfaces (see [Gau] for details). Thus they are a good setting to apply the previous constructions.

A surface \( \Sigma \) immersed with possible branch points in an oriented Riemannian 4-manifold \( M \) is called right superminimal (resp. left superminimal) if its lift \( J \) (resp. \( K \)) in \( S(\Lambda^+(M)) \) (resp. \( S(\Lambda^-(M)) \)) is parallel w.r.t the connection induced by the Levi-Civita connection on \( M \).

Equivalently the second fundamental form \( B \) of \( \Sigma \) verifies for every two vectors \( X \) and \( Y \) tangent to \( \Sigma \):

\[
B(X, JY) = JB(X, Y) \quad \text{(resp.} \quad B(X, KY) = KB(X, Y))
\]

(10).

We plug (10) into the formulae of Prop. 3 and we derive

Proposition 8 Let \( M, (\Sigma_n), \Sigma_0 \) and \( p \) be as in Def. 2 and suppose moreover that the \( \Sigma_n \)'s are right (resp. left) superminimal. Assume that \( \mu^T \) exists. Then

\[
\mu_p^T = -\mu_p^N \quad \text{(resp.} \quad \mu_p^T = \mu_p^N).\]

Suppose moreover that the \( L^2 \)-norm of the second fundamental form of the \( \Sigma_n \)'s have a common bound - for example if the area of the \( \Sigma_n \)'s have a common bound. Then we can apply Cor. 2: if the \( \Sigma_n \)'s are right (resp. left) superminimal there is no bubbling off in \( S(\Lambda^+(M)) \) (resp. \( S(\Lambda^-(M)) \)). We derive

Proposition 9 Let \( \Sigma_0 \) be a surface which is immersed in \( M \) with branch points and let \( (\Sigma_n) \) be a sequence of right (resp. left) superminimal surfaces which converges to \( \Sigma_0 \) smoothly on compact sets outside of the branch points.
Suppose that the genera and areas of the $\Sigma_n$’s have local common bounds. Let $J_n$ (resp. $K_n$) be the lift of $\Sigma_n$ inside $S(\Lambda^+(M))$ (resp. $S(\Lambda^+(M))$). Let $p$ be a branch point of $\Sigma_0$ and assume that there is only one branched disk of $\Sigma_0$ going through $x_0$. We let $J_0$ (resp. $K_0$) be the complex structure on $T_p M$ compatible (resp. not compatible) with the orientation on $M$ for which the tangent plane to $\Sigma_0$ at $p$ is a complex line. Then the following is true:

let $(x_n)$ be a sequence of points in $M$, $x_n \in \Sigma_n$ and suppose that the sequence $(x_n)$ converges to $x_0$ in $\Sigma_0$. Then $(J_n)$ (resp. $(K_n)$) converges to $J_0$ (resp. $K_0$).

REMARK. Prop. 9 above would have significantly shortened the proof of [Vi 3].

REMARK. We conclude this section by recalling a result of [Vi 2] which has some relevance here: a branch point of a superminimal surface is $C^1$ diffeomorphic to the branch point of a holomorphic curve in a complex surface (see [Vi 2] for the exact formulation). This means that the braid $\beta$ of a branched point of a superminimal disk is the braid of an algebraic knot and thus verifies $n(\beta) < |e(\beta)|$.

8 Appendix 1: curvature computations

8.1 The curvature of the tangent and normal bundles

We denote by $\nabla^T$ (resp. $\nabla^N$) the connection on $T\Sigma$ (resp. $N\Sigma$) induced by the projection of the Levi-Civita connection on $M$. The goal of this paragraph is to compute its curvature $\Omega^T$ (resp. $\Omega^N$).

We choose $e_1, e_2$ (resp. $e_3, e_4$) a positive orthonormal basis of $T\Sigma$ (resp. $N\Sigma$) and we let $\omega^T$ (resp. $\omega^N$) be the connections 1-forms for $\nabla^T$ (resp. $\nabla^N$). We write

$$\omega^T(X) = \langle \nabla_X e_2, e_1 \rangle$$
$$\omega^N(X) = \langle \nabla_X e_4, e_3 \rangle,$$

where $X$ is a vector tangent to $\Sigma$; so the curvature forms are $\Omega^T = d\omega^T$ and $\Omega^N = d\omega^N$.

We recall Gauss’equation

**Proposition 10**

$$\Omega^T(e_1, e_2) = -\|B(e_1, e_2)\|^2 + \langle B(e_1, e_1), B(e_2, e_2) \rangle \quad (11)$$
\[ + < R^M(e_1, e_2)e_1, e_2 > \]

If \( \Sigma \) is a minimal surface, (11) = \(-\frac{1}{2} \|B\|^2 + < R^M(e_1, e_2)e_1, e_2 >\), where the norm \( \|B\| \) is taken w.r.t. the induced scalar product on \( T^* \Sigma \otimes N \Sigma \).

We now turn to \( \Omega^N \) and compute

\[ \Omega^N(e_1, e_2) = e_1 \omega(e_2) - e_2 \omega(e_1) - \omega([e_1, e_2]) \]

\[ = e_1 < \nabla e_2 e_4, e_3 > - e_2 < \nabla e_1 e_4, e_3 > - < \nabla [e_1, e_2] e_4, e_3 > \]

\[ = < \nabla e_1 \nabla e_2 e_4 - \nabla e_2 \nabla e_1 e_3 - \nabla [e_1, e_2] e_4, e_3 > \quad (12) \]

\[ + < \nabla e_2 e_4, \nabla e_1 e_3 > - < \nabla e_1 e_4, \nabla e_2 e_3 > \quad (13). \]

(12) is equal to \( < R^M(e_1, e_2)e_3, e_4 > \) where \( R^M \) is the curvature of the ambient manifold \( M \).

To estimate (13), we notice that only the components of \( \nabla e_3 \) and \( \nabla e_4 \) along the tangent vectors \( e_1, e_2 \) will contribute to \( < \nabla e_3, \nabla e_4 > \). So

\[ (13) = < \nabla e_2 e_4, e_1 > < \nabla e_1 e_3, e_1 > + < \nabla e_2 e_4, e_2 > < \nabla e_1 e_3, e_2 > \]

\[- < \nabla e_1 e_4, e_1 > < \nabla e_2 e_3, e_1 > - < \nabla e_1 e_4, e_2 > < \nabla e_2 e_3, e_2 > \]

\[ = < e_4, \nabla e_2 e_1 > < e_3, \nabla e_1 e_1 > + < e_4, \nabla e_2 e_2 > < e_3, \nabla e_1 e_2 > \]

\[- < e_4, \nabla e_1 e_1 > < \nabla e_2 e_1, e_3 > - < e_4, \nabla e_1 e_2 > < e_3, \nabla e_2 e_2 > . \]

We recall that we can identify the elements of \( N \Sigma \) to real numbers; hence we write

**Proposition 11**

\[ \Omega^N(e_1, e_2) = (B(e_1, e_1) - B(e_2, e_2)) \wedge B(e_1, e_2) + < R^M(e_1, e_2)e_3, e_4 > . \]

If \( \Sigma \) is minimal, then

\[ \Omega^N(e_1, e_2) = 2dB(e_1, e_2) \wedge B(e_1, e_1) + < R^M(e_1, e_2)e_3, e_4 > . \]
9 Appendix 2: a proof of Eells-Salamon’s result

We give here a quick proof of Eells-Salamon’s result (Th.5, cf.[E-S]). It relies on an explicit computation of the complex structure $\mathcal{I}$:

**Lemma 7** Let $(e_1, e_2, e_3, e_4)$ be a positive orthonormal basis of $\mathbb{R}^4$, and consider the 2-planes $P_i$, $i = 1, 2, 3$ given by

$$P_1 = e_1 \wedge e_2, \quad P_2 = e_1 \wedge e_3, \quad P_3 = e_1 \wedge e_4.$$ 

Note that $\ast P_1 = e_3 \wedge e_4, \ast P_2 = e_4 \wedge e_2, \ast P_3 = e_2 \wedge e_3$.

$(P_2, P_3, \ast P_2, \ast P_3)$ form a basis of the tangent space $T_p G^+_2(\mathbb{R}^4)$. In this basis the complex structure $\mathcal{I}$ writes

$$\mathcal{I} P_2 = - \ast P_3$$
$$\mathcal{I} \ast P_2 = - P_3$$
$$\mathcal{I} P_3 = \ast P_2$$
$$\mathcal{I} \ast P_3 = P_2$$

Let $z \in \Sigma$, $p \in M$, with $p = f(z)$, and let $(x, y)$ be an isothermal coordinate system around $z$. We put $\lambda = \|\frac{\partial f}{\partial x}\|$. Then

$$e_1 = \frac{1}{\lambda} \frac{\partial f}{\partial x}, \quad e_2 = \frac{1}{\lambda} \frac{\partial f}{\partial y}$$

constitute a positive orthonormal basis of $T_p f(\Sigma)$. We put

$$P = e_1 \wedge e_2 \in \tilde{f}(p).$$

We want to show that

$$\frac{\partial \tilde{f}}{\partial y} = \mathcal{J} \frac{\partial \tilde{f}}{\partial x} \quad (14).$$

Let $\tilde{f} = \frac{H+K}{\lambda}$, where $H \in Z^+(M), K \in Z^-(M)$. $\tilde{f}$ is an $H$-complex and $K$-complex line in $T_{\tilde{f}(p)} M$. So the horizontal part of the identity (14) follows from the definition of $\mathcal{J}$. 

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Proving the vertical part of (14) amounts to proving that
\[ \nabla e_2 P = \mathcal{I} \nabla e_1 P \quad (15). \]
To do this, we develop both sides of the equation (15) and plug in the identities from Lemma 8.

\[ \nabla e_2 P = \nabla e_2 e_1 \wedge e_2 + e_1 \wedge \nabla e_2 e_2. \] Since \( \nabla e_2 e_1 \) (resp. \( \nabla e_2 e_2 \)) is orthogonal to \( e_1 \) (resp. \( e_2 \)), we only need to take into account the components of \( \nabla e_2 e_1 \) and \( \nabla e_2 e_2 \) along \( e_3, e_4 \). We get

\[ \nabla e_2 P = -\langle \nabla e_2 e_1, e_3 \rangle P_3 + \langle \nabla e_2 e_1, e_4 \rangle *P_2 \]

\[ + \langle \nabla e_2 e_2, e_3 \rangle P_2 + \langle \nabla e_2 e_2, e_4 \rangle P_3. \]

For \( i = 3, 4 \),

\[ \langle \nabla e_2 e_2, e_i \rangle = -\langle \nabla e_1 e_1, e_i \rangle, \]

\[ \langle \nabla e_2 e_1, e_i \rangle = -\langle \nabla e_1 e_2, e_i \rangle \]

so we get

\[ \nabla e_2 P = -\langle \nabla e_1 e_1, e_3 \rangle P_3 + \langle \nabla e_2 e_2, e_3 \rangle P_2 \]

\[ - \langle \nabla e_1 e_1, e_3 \rangle P_3 - \langle \nabla e_1 e_1, e_4 \rangle P_2 \]

\[ = \mathcal{I}[\langle \nabla e_1 e_2, e_3 \rangle P_2 + \langle \nabla e_1 e_2, e_4 \rangle P_3 \]

\[ - \langle \nabla e_1 e_1, e_3 \rangle *P_3 + \langle \nabla e_1 e_1, e_4 \rangle *P_2] \]

\[ = \mathcal{I}[\langle \nabla e_1 e_2, e_3 \rangle e_1 \wedge e_3 + \langle \nabla e_1 e_2, e_4 \rangle e_1 \wedge e_4 \]

\[ - \langle \nabla e_1 e_1, e_3 \rangle e_2 \wedge e_3 + \langle \nabla e_1 e_1, e_4 \rangle e_4 \wedge e_2] \]

\[ = \mathcal{I} \nabla e_1 P. \]

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