SCHRÖDINGER OPERATORS AND ASSOCIATED HYPERBOLIC PENCILS

Sergey A. Denisov

Abstract. For a large class of Schrödinger operators, we introduce the hyperbolic quadratic pencils by making the coupling constant dependent on the energy in the very special way. For these pencils, many problems of scattering theory are significantly easier to study. Then, we give some applications to the original Schrödinger operators including one-dimensional Schrödinger operators with $L^2$–operator-valued potentials, multidimensional Schrödinger operators with slowly decaying potentials.

1. Introduction

In this paper, we consider two classes of Schrödinger operators: one-dimensional operator with operator-valued potential

$$L = -\frac{d^2}{dr^2} + V(r), r > 0$$

and the standard

$$H = -\Delta + V(x), x \in \mathbb{R}^3$$

Operator $L$ can be thought of as $L = L_0 + V$, where

$$L_0 = \begin{bmatrix} -\frac{d^2}{dr^2} & 0 & 0 & \cdots \\ 0 & -\frac{d^2}{dr^2} & 0 & \cdots \\ 0 & 0 & -\frac{d^2}{dr^2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

acts on $\bigoplus_{n=1}^{\infty} L^2(\mathbb{R}^+)\text{ with domain } \mathcal{D}(L_0) = \bigoplus_{n=1}^{\infty} \dot{H}^2(\mathbb{R}^+)$ (i.e. we consider the Dirichlet boundary conditions at zero). The selfadjoint $V(r)$ is given by

$$V(r) = \begin{bmatrix} v_{11}(r) & v_{12}(r) & \cdots \\ \overline{v_{12}(r)} & v_{22}(r) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

and $\|V(r)\| \in L^\infty(\mathbb{R}^+)$ where the norm is taken as an operator norm in $\ell^2$. By general spectral theory, $L$ is essentially selfadjoint with the same domain $\bigoplus_{n=1}^{\infty} \dot{H}^2(\mathbb{R}^+)$. For $H$, we assume $V(x) \in L^\infty(\mathbb{R}^3)$ and then, again, $\mathcal{D}(H) = H^2(\mathbb{R}^3)$. 

1
One of the basic questions of the scattering theory is under what decay assumptions on potential $V$ there is a nontrivial a.c. spectrum. The one-dimensional case suggests that some sort of $L^2(\mathbb{R}^+)$ condition should be sufficient. The one-dimensional case also has simple matrix-valued generalization [15]. In the meantime, the methods available now have not yet yielded the desired results for situations considered in this paper.

For both $L$ and $H$, we introduce and study the associated hyperbolic pencils. Then, we apply obtained estimates to the original Schrödinger operators. The structure of the paper is as follows. In the second section, we study $L$. The third one contains the discussion of three-dimensional case. The appendix contains the proof of Combes-Thomas estimate for Schrödinger pencils.

We will use the following notations: \( \langle \xi_1, \xi_2 \rangle \) denotes the inner product in $\mathbb{C}^n$, \[ \ln^- x = \begin{cases} \ln x, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases} \]
Also, \( \langle x \rangle = (|x|^2 + 1)^{1/2} \) for any vector $x$. For function $f(x)$, $f_r(x)$ denotes the radial component of the gradient and $f_t(x)$– the tangential component, $B$ will denote nonpositive Laplace-Beltrami operator. For operator $O$, $\sigma(O)$ will mean the spectrum of $O$.

Acknowledgements. This research was supported by Alfred P. Sloan Research Fellowship, and NSF Grant DMS-0500177.

2. ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH OPERATOR-VALUED POTENTIAL

Consider the family of operators $L(t)$, given by the coupling constant $t \in \mathbb{R}$:
\[ L(t) = -\frac{d^2}{dr^2} + tV(r), r > 0 \] (5)

Definition 2.1. We say that $\mathbb{R}^+ \subseteq \sigma_{ac}(L(t))$ generically if this property holds for all $t \in \Omega \subseteq \mathbb{R}$ where $\Omega$ is a full measure set in $\mathbb{R}$.

The main goal of this section is to prove the following

Theorem 2.1. Assume that self-adjoint $V(r)$ satisfies $|V(r)| \in L^2(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$. Then, $\mathbb{R}^+ \subseteq \sigma_{ac}(L(t))$ generically.

Notice carefully, that under the conditions of the Theorem, the essential spectrum of operator $L$ is not necessarily $\mathbb{R}^+$. For example, taking off-diagonal elements in $V(r)$ all equal to zero, one can arrange $v_{kk}(r), k = 1, 2, \ldots$ to be such that the spectrum of $L$ is purely a.c. on $\mathbb{R}^+$, and is, say, dense pure point on some negative interval.

Consider $F(r) = (f(r), 0, \ldots) \in \bigoplus_{n=1}^{\infty} L^2(\mathbb{R}^+)$ with $f(r)$– compactly supported function from $L^2(\mathbb{R}^+)$, $\|f\| = 1$. Assume that the support of $f$ is inside the interval $[0, \delta]$. Then, for each $t$, consider the spectral measure $d\sigma(\lambda, t)$ generated by $F$ and an operator $L(t)$. Take $\lambda \in [c, d] \subseteq \mathbb{R}^+$ and $|t| < T$. Under conditions of the Theorem, we will show that for generic $t \in [-T, T]$ the following is true: $d\sigma(\lambda, t)/d\lambda > 0$ for a.e. $\lambda \in [c, d]$. That would imply $[c, d] \subseteq \sigma_{ac}(H(t))$ for generic $t \in [-T, T]$. Since $c, d, T$ are arbitrary, the statement of the Theorem follows. But first we have to obtain some preliminary results.

Consider the family of measures $d\sigma(\lambda, t)$ restricted to $[c, d] \subseteq \mathbb{R}^+$. 

Lemma 2.1. The measure $d\sigma(\lambda, t)$ is weakly continuous in $t \in [-T, T]$.

Proof. Indeed, we obviously have $((L(t) - z)^{-1} F, F) \to ((L(t_0) - z)^{-1} F, F)$ for any $z \in \mathbb{C}^+$ as long as $t \to t_0$. Therefore, by the Spectral Theorem and Weierstrass approximation argument,

$$\int h(\lambda) d\sigma(\lambda, t) \to \int h(\lambda) d\sigma(\lambda, t_0)$$

for any compactly supported continuous $h(\lambda)$ and $t \to t_0$. \qed

The weak continuity allows us to use Riesz Representation Theorem to correctly define Radon measure $d\nu$ on the set $\Upsilon = (c, d) \times (-T, T)$ by letting

$$\int g(\lambda, t) d\nu(\lambda, t) = \int_{-T}^{T} dt \int_{c}^{d} g(\lambda, t) d\sigma(\lambda, t)$$

for any continuous $g(\lambda, t)$ supported inside $\Upsilon$. For each $t$, we have the decomposition $d\sigma(\lambda, t) = \sigma'(\lambda, t) d\lambda + d\sigma_s(\lambda, t)$. On the other hand, measure $d\nu$ allows decomposition with respect to two-dimensional Lebesgue measure $d\mu$ on $\Upsilon$:

$$d\nu(\lambda, t) = \nu'(\lambda, t) d\mu + d\nu_s(\lambda, t)$$

Lemma 2.2. We have

$$\nu'(\lambda, t) = \sigma'(\lambda, t)$$

for $\mu$–a.e. $\lambda, t \in \Upsilon$. Moreover

$$d\nu_s(\lambda, t) = dtd\sigma_s(\lambda, t)$$

Remark. The last equality is understood in the following sense

$$\int g(\lambda, t) d\nu_s(\lambda, t) = \int_{-T}^{T} dt \int_{c}^{d} g(\lambda, t) d\sigma_s(\lambda, t)$$

i.e. as equality of Radon measures generated by positive linear functionals on $C_c(\Upsilon)$.

Proof. Let us first show that $\sigma'(\lambda, t)$ is measurable with respect to $d\mu$. To do that, define the Herglotz function

$$M(z, t) = \int \frac{d\sigma(\lambda, t)}{\lambda - z}, z \in \mathbb{C}^+$$

By Spectral Theorem,

$$M(z, t) = ((H(t) - z)^{-1} F, F)$$

and this function is analytic in $z \in \mathbb{C}^+$ and continuous in $t \in [-T, T]$. Introduce the set $\Omega$ of $\lambda \in (c, d), t \in [-T, T]$ for which $\lim_{n \to \infty} \text{Im } M(\lambda + in^{-1}, t)$ exists and is finite. By Cauchy criteria,

$$\Omega = \bigcap_{j=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{m,k>N} \{ (\lambda, t) : |\text{Im } M(\lambda + im^{-1}, t) - \text{Im } M(\lambda + ik^{-1}, t)| < j^{-1} \}$$

and $\Omega$ is Borel. The boundary behavior of Herglotz functions implies that intersection of $\Omega$ with any line $t = t_0$ has a full one-dimensional Lebesgue measure.
Therefore, by Fubini, $\Omega$ has the full two-dimensional Lebesgue measure on $\Upsilon$. Also, for any $(\lambda, t) \in \Omega$, we have $\pi^{-1} \text{Im} M(\lambda + in^{-1}, t) \rightarrow \sigma'(\lambda, t)$ (that can be regarded as the definition of $\sigma'$, which is then equal to the corresponding maximal function $d\sigma/d\lambda$ Lebesgue a.e.). Therefore, $\sigma'(\lambda, t)$ is $d\mu$ measurable. Moreover, since

$$\int \sigma'(\lambda, t) d\lambda \leq \int d\sigma(\lambda, t) = 1$$

for any $t$, we have $\sigma'(\lambda, t) \in L^1(\Upsilon)$ by Fubini. Thus, we are left to show that $dtd\sigma_s(\Omega)$ is $d\mu$–singular. Besides $\Omega$, consider $\Omega_1 = \{(\lambda, t) \in \Upsilon : \text{Im} M(\lambda + in^{-1}, t) \rightarrow +\infty\}$ and $\Omega_2 = \{(\lambda, t) \in \Upsilon$ for which $\text{Im} M(\lambda + in^{-1}, t)$ has no limit, finite or infinite$\}$. In the same way, one can show that $\Omega_1(2)$ are Borel. For any $g(\lambda, t) \in C_c(\Upsilon)$, let

$$g_n(\lambda, t) = \frac{g(\lambda, t)}{1 + \pi^{-1} \text{Im} M(\lambda + in^{-1}, t)}$$

Clearly, $g_n(\lambda, t) \in C_c(\Upsilon)$ and by definition

$$\int g_n(\lambda, t) d\nu(\lambda, t) = \int_{-T}^{T} dt \int c g_n(\lambda, t) d\sigma(\lambda, t) \quad (6)$$

Consider the l.h.s. The functions $g_n(\lambda, t)$ are uniformly bounded and converge to $g(\lambda, t)(\sigma'(\lambda, t) + 1)^{-1}$ on $\Omega$ and to 0 on $\Omega_1$. By dominated convergence theorem,

$$\int_{\Omega} g_n d\nu \rightarrow \int_{\Omega} g(\sigma' + 1)^{-1} d\nu, \int_{\Omega_1} g_n d\nu \rightarrow 0$$

For the r.h.s. of $(6)$, apply dominated convergence theorem for the inner integral first. When doing that, we take into account that intersection of $\Omega_2$ with any line $t = t_0$ has zero measure with respect to $d\sigma(\lambda, t_0)$. Also, an intersection of $\Omega$ with any line $t = t_0$ has zero measure with respect to $d\sigma_s(\lambda, t_0)$. Therefore, the r.h.s. converges to

$$\int_{T} \frac{g\sigma'}{\sigma' + 1} d\mu$$

Comparing the limits, we have

$$\int_{T} \frac{g\sigma'}{\sigma' + 1} d\mu \geq \int_{\Omega} g(\sigma' + 1)^{-1} d\nu = \int_{\Omega} g(\sigma' + 1)^{-1} \sigma' d\mu + \int_{\Omega} g(\sigma' + 1)^{-1} dtd\sigma_s(\lambda, t)$$

Consequently,

$$\int_{\Omega} g(\sigma' + 1)^{-1} dtd\sigma_s(\lambda, t) = 0$$

for any $g \in C_c(\Upsilon)$. Therefore, $dtd\sigma_s(\Omega) = 0$ and $dtd\sigma_s$ is $d\mu$–singular since $\Omega$ is of the full Lebesgue measure. The statement of the Lemma now follows from the uniqueness of the $d\mu$–decomposition for the measure $d\nu$. □

The main idea of the proof of Theorem 2.1 is based on getting the entropy bound for the density of $\nu$, i.e., we will prove that

$$\int_{\Upsilon} \ln \sigma'(\lambda, t) d\mu > -\infty \quad (7)$$
Since \( \sigma'(\lambda, t) \in L^1(\Upsilon) \), an application of Fubini gives

\[
\int_{c}^{d} \ln \sigma'(\lambda, t) d\lambda > -\infty
\]

for Lebesgue a.e. \( t \in [-T, T] \). Clearly, summability of the logarithm ensures that the a.c. component of the measure is supported on \([c, d]\). That implies the statement of the Theorem.

Thus, we have to show (7).

The proof will be based on the approximation of operator-valued potential by the matrix-valued ones. Consider \( V_{n,R}(r) = \Pi_n V(r) \cdot \chi_{[0,R]}(r) \Pi_n \), where \( \chi_{[0,R]}(r) \) is the characteristic function of the interval \( \Delta \), and \( \Pi_n \) is the projection on first \( n \) coordinates in \( \ell^2 \). Thus, the non-zero part of \( V_{n,R}(r) \) is obtained by cutting \( n \times n \) matrix from the upper-left corner of the matrix representation for \( V(r) \) and restricting this matrix-function to the interval \([0, R]\). Notice that \( V_{n,R}(r) \) is self-adjoint, \( \|V_{n,R}(r)\|_{L^\infty(\mathbb{R}^+)} + \|V_{n,R}(r)\|_{L^2(\mathbb{R}^+)} < C \) uniformly in \( R \) and \( n \). Moreover, the new operator \( L_{n,R} \) is decoupled into the orthogonal sum of two operators: the first one, call it \( L_{1,n,R}^1 \), is one-dimensional Schrödinger operator with Dirichlet boundary conditions and \( n \times n \) matrix-valued potential \( V_{n,R}(r) \). The other operator is free one-dimensional Schrödinger operator acting in \( \bigoplus_{k=n+1} L^2(\mathbb{R}^+) \). Clearly, the spectral measure of \( F \) with respect to \( L_{n,R} \) coincides with the spectral measure of \( (f(r), 0, \ldots, 0) \) with respect to \( L_{1,n,R}^1 \). Thus, consider

\[
L_{1,n,R}^1(t) = -\frac{d^2}{dr^2} I_{n \times n} + tV_{n,R}(r)
\]

with Dirichlet boundary conditions at zero and

\[
V_{n,R}(r) = \begin{bmatrix}
v_{11}(r) & v_{12}(r) & \cdots & v_{1n}(r) \\
\vdots & \ddots & \ddots & \vdots \\
v_{1n}(r) & \cdots & \cdots & v_{nn}(r)
\end{bmatrix} \cdot \chi_{[0,R]}(r)
\]

Let \( d\sigma_{n,R}(\lambda, t) \) be the spectral measure of \( F \) with respect to \( L_{1,n,R}^1(t) \). Since \( V_{n,R}(r) \) is compactly supported, \( d\sigma(\lambda, t) = \sigma'(\lambda, t) d\mu \) with \( \sigma'(\lambda, t) \) smooth in \( (\lambda, t) \in \Upsilon \).

We will prove

\[
\int_{\Upsilon} \ln \sigma'_{n,R}(\lambda, t) d\mu > C
\]

uniformly in \( n, R \). Then, the standard argument with weak convergence of \( d\sigma_{n,R}(\lambda, t) \) to \( d\sigma(\lambda, t) \) and weak upper semicontinuity of the entropy ([11], Corollary 5.3) will imply (7).

We will need several simple and well-known facts (Lemmas 2.3 2.6). Consider

\[
L(t) = -\frac{d^2}{dr^2} + tQ(r)
\]

with Dirichlet boundary conditions and \( n \times n \) matrix-function \( Q(r) = Q^*(r) \in L^2(\mathbb{R}^+) \) compactly supported on \([0, R]\). Consider \( u(r, k, t) = (L(t) - k^2 - i(+0))^{-1} F, \)
the restriction of the solution to the real axis \((k \neq 0)\). The potential is compactly supported so this restriction clearly exists and has the following asymptotics

\[ u(r, k, t) = \exp(irk)A(k, t), r > R \]

where \(A(k, t)\) is a vector. Moreover, there is the unique \(u\) that both solves equation, satisfies boundary condition and asymptotics at infinity. Let \(\sigma(\lambda, t)\) be the spectral measure of \(F\) with respect to \(L(t)\).

**Lemma 2.3.** Let \(d\sigma(\lambda, t)\) be the spectral measure of \(F\) with respect to \(L(t)\). Then,

\[ \sigma'(\lambda, t) = k\pi^{-1}\|A(k, t)\|^2, \lambda = k^2, k > 0 \] (10)

**Proof.** Since the potential is finitely supported, we have

\[ \sigma'(\lambda, t) = \pi^{-1}\Im \int_0^\infty \langle u(r, k, t), F(r) \rangle dr \]

by the Spectral Theorem. On the other hand, from equation \(-u'' + tQu = k^2u + F\) we have

\[-\langle u, u'' \rangle + t\langle u, Qu \rangle = k^2\langle u, u \rangle + \langle u, F \rangle\]

Taking imaginary part, integrating over \(\mathbb{R}^+\), and using the boundary condition and asymptotics, we get

\[ \Im \int_0^\infty \langle u(r, k, t), F(r) \rangle dr = k\|A(k, t)\|^2 \]

\[ \square \]

We introduce now the standard object in the scattering theory, the Jost solution. Let \(k \in \mathbb{R}, k \neq 0\) and \(J(r, k, t)\) be the solution to

\[-J'' + tQJ = k^2J, \quad J(r, k, t) = \exp(ikr), r > R\]

One can easily show existence and uniqueness of \(J\). Also, let \(\alpha(r, k, t)\) be the solution to Cauchy problem

\[-\alpha'' + tQ\alpha = k^2\alpha, \quad \alpha(0, k, t) = 0, \quad \alpha'(0, k, t) = 1 \] (11)

**Lemma 2.4.** We have

\[ u(r, k, t) = -J(r, k, t) \int_0^r G_{12}(\rho, k, t)F(\rho)d\rho + \alpha(r, k, t) \int_r^\infty G_{22}(\rho, k, t)F(\rho)d\rho \] (12)

where

\[ \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} J & \alpha \\ J' & \alpha' \end{bmatrix}^{-1} \]

**Proof.** The proof is a simple calculation. \[ \square \]

The formula for the inverse is given by

**Lemma 2.5.** The following identity is true

\[ \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} J^{-1}(0, k, t) & 0 \\ -2ik|J^{-1}(0, k, t)|^2 & -[J^*(0, k, t)]^{-1} \end{bmatrix} \begin{bmatrix} [\alpha^*(r, k, t)]' & -\alpha^*(r, k, t) \\ [J^*(r, k, t)]' & -J^*(r, k, t) \end{bmatrix} \]
Proof. The proof is a simple calculation that uses identity 
\((Y^*_1)'Y_2-Y^*_1(Y^*_2)'=\text{const}\) 
which is true for any \(Y_{1(2)}\) that solve 
\(-Y''+tQY=k^2Y\). \(\square\)

In the previous Lemma, the invertibility of 
\(J(0, k, t)\) follows, for instance, from 
well-known formulas

**Lemma 2.6.** If 
\(\mathfrak{A}(k, t) = (J(0, k, t) + (ik)^{-1}J'(0, k, t))/2, \mathfrak{B}(k, t) = (J(0, k, t) - (ik)^{-1}J'(0, k, t))/2\) 
then 
\(J(0, k, t) = \mathfrak{A}(k, t) + \mathfrak{B}(k, t), |\mathfrak{A}(k, t)|^2 = I + |\mathfrak{B}(k, t)|^2\)

Proof. The second formula follows from the identity 
\(J^*(J)' - (J^*)'J = 2ik\). \(\square\)

Using previous Lemmas, we have 
\(A(k, t) = J^{-1}(0, k, t)\tilde{F}(k, t)\) 
where 
\(\tilde{F}(k, t) = \int_0^\delta \alpha^*(\rho, k, t)F(\rho)d\rho\) 
(14)

Recalling (10), 
\(\sigma'(\lambda, t) = k\pi^{-1}\|J^{-1}(0, k, t)\tilde{F}(k, t)\|^2, \lambda = k^2\) 
(15)

The function \(\tilde{F}(k, t)\) has analytic continuation in \(k\) to \(\mathbb{C}\) and depends on \(Q\) on \([0, \delta]\). Therefore, only \(J^{-1}(0, k, t)\) is responsible for scattering properties.

To study \(J(0, k, t)\), we will use the following argument. Instead of dealing with the standard Schrödinger equation 
\(-J''(r, k, t) + tQ(r)J(r, k, t) = k^2J(r, k, t)\) 
we will consider 
\(-D''(r, k, \xi) + k\xi QD(r, k, \xi) = k^2D(r, k, \xi), \xi \in \mathbb{R}\) 
(16)

Thus, we make the coupling constant energy-dependent. We can single out \(D(r, k, \xi)\) as the solution satisfying the same Jost asymptotics at infinity 
\(D(r, k, \xi) = \exp(ikr), r > R\)

Obviously, we have 
\(J(0, k, kt) = D(0, k, t)\) 
(17)

and, consequently, 
\(\sigma'(k^2, kt) = k\pi^{-1}\|D^{-1}(0, k, t)\tilde{F}(k, kt)\|^2\) 
(18)

Now, the main advantage of dealing with \(D(0, k, t)\) instead of \(J(0, k, t)\) is that it allows analytic continuation in \(k\) to the upper half-plane along with the nice uniform estimates. Indeed, consider equation 
\(-D'' + k\xi QD = k^2D\) 
(19)

for \(k \in \mathbb{C}^+\) and look for \(D(r, k, \xi)\) which satisfies Jost condition at infinity, i.e. \(D(r, k, \xi) = \exp(ikr), r > R\). This \(D\) can be easily obtained in the following fashion. Write (13) as a system.
\[ Y' = \begin{bmatrix} 0 & -k^2 \\ k\xi Q - k^2 & 0 \end{bmatrix} Y \]  

(20)

Introduce

\[ Y_0 = \begin{bmatrix} \exp(ikr) & \exp(-ikr) \\ ik\exp(ikr) & -ik\exp(-ikr) \end{bmatrix}, \quad U_{1(2)}(r_1, r_2, \xi) = \int_{r_1}^{r_2} \exp \left[ \pm \frac{i\xi}{2} Q(s)ds \right] \]

Then, for \( S = U^{-1}Y_0^{-1}Y \), we have

\[ S' = \begin{bmatrix} 0 & -A(r, \xi) \exp(-2ikr) \\ -A^*(r, \xi) \exp(2ikr) & 0 \end{bmatrix} S \]

where

\[ A(r, \xi) = -\frac{i\xi}{2}U_2^{-1}(0, r, \xi)Q(r)U_1(0, r, \xi) \]  

(21)

By letting \( S(\infty) = (I, 0) \), we have

\[ S_1(r, k, \xi) = 1 + \int_{s_1}^{\infty} \exp[2iks - s_1]A(s_2, \xi)S_1(s_2, k, \xi)ds_2 \]

Gronwall-Bellman’s Lemma yields

\[ \|S_1(r, k, \xi)\| \leq \exp \left[ \frac{\xi^2}{8\text{Im}k} \int_r^{\infty} \|Q(s)\|^2ds \right] \]

and for

\[ S_2(r, k, \xi) = \int_r^{\infty} A(s, \xi) \exp(2iks)S_1(s, k, \xi)ds \]

we have

\[ \|S_2(r, k, \xi)\| \leq \frac{|\xi|}{2} \exp \left[ \frac{\xi^2}{8\text{Im}k} \int_0^{\infty} \|Q(s)\|^2ds \right] \int_r^{\infty} \|Q(s)\| \exp(-2|\text{Im}ks|)ds \]

Clearly, we can express \( D(r, k, \xi) \) in the following way

\[ D(r, k, \xi) = [\exp(ikr)U_1(0, r, \xi)S_1(r, k, \xi) + \exp(-ikr)U_2(0, r, \xi)S_2(r, k, \xi)]U_1^{-1}(0, R, \xi) \]

Therefore, obviously, we have existence, analyticity, and continuity of \( D(r, k, \xi) \) for any \( k \in \mathbb{C}^+ \) (remember that \( Q \) is compactly supported). Moreover, the following uniform estimate holds

**Lemma 2.7.**

\[ \|D(0, k, \xi)\| \leq \exp \left[ \frac{\xi^2}{8\text{Im}k} \int_0^{\infty} \|Q(s)\|^2ds \right] \cdot \left[ 1 + \frac{|\xi|}{2\sqrt{2\text{Im}k}}\|Q\|_2 \right] \]  

(22)

holds true for any \( k \in \mathbb{C}^+ \) and any \( Q \in L^2(\mathbb{R}^+) \). Also,

\[ \|D^{-1}(0, k, \xi)\| < C, \quad \text{Im}k > \kappa \]

where \( C \) and \( \kappa \) depend on \( \xi \) and \( \|Q\|_2 \) only.
We have analogous estimate from above on $\|D'(0, k, \xi)\|$.

The next Lemma yields the quantitative version of invertibility of $D(0, k, \xi)$. Introduce $\mu(r, k, \xi) = D(r, k, \xi) \exp(-ikr)$. Then, we have

**Lemma 2.8.** The following identity is satisfied for any $k \in \mathbb{C}^+$

$$
|D^{-1}(0, k, \xi)|^2 + [D^*(0, k, \xi)]^{-1} \left[ \frac{\Im k}{|k|^2} \int_0^\infty |\mu'(s, k, \xi)|^2 ds \right] \cdot D^{-1}(0, k, \xi) = \Im \left[ \frac{D'(0, k, \xi) D^{-1}(0, k, \xi)}{k} \right]
$$

**Proof.** For $\mu$:

$$
\mu''(r, k, \xi) + 2ik\mu'(r, k, \xi) = k\xi Q(r)\mu(r, k, \xi), \quad \mu(r, k, \xi) = 1, r > R
$$

Divide the both sides by $2ik$, multiply from the left by $\mu^*(r, k, \xi)$ and integrate from 0 to $\infty$. Taking the real part, we have

$$
I - |\mu(0, k, \xi)|^2 + \frac{\Im k}{|k|^2} \int_0^\infty |\mu'(s, k, \xi)|^2 ds = \frac{1}{2ik} \mu^*(0, k, \xi) \mu'(0, k, \xi) - \frac{1}{2ik} \mu^*(0, k, \xi) \mu(0, k, \xi)
$$

Clearly, the last identity shows that $\mu(0, k, \xi)$ is invertible for any $k \in \mathbb{C}^\times \setminus \{0\}$.

Moreover,

$$
|\mu^{-1}(0, k, \xi)|^2 + [\mu^*(0, k, \xi)]^{-1} \left[ \frac{\Im k}{|k|^2} \int_0^\infty |\mu'(s, k, \xi)|^2 ds \right] \cdot \mu^{-1}(0, k, \xi) = I + \Re \left[ \frac{\mu'(0, k, \xi) \mu^{-1}(0, k, \xi)}{ik} \right] = \Im \left[ \frac{D'(0, k, \xi) D^{-1}(0, k, \xi)}{k} \right]
$$

since $\mu(r, k, \xi) = D(r, k, \xi) \exp(-ikr)$. $\square$

As a simple corollary, we get that the matrix-valued function

$$
G(k) = \frac{D'(0, k, \xi) D^{-1}(0, k, \xi)}{k} \tag{23}
$$

is Herglotz and its boundary value on the real line is factorized through $|D^{-1}(0, k, \xi)|^2$. Moreover, we have a uniform bound on $G(k)$ for large $\Im k$ due to Lemma 2.7 which yields $\|G(k)\| < C$ for large $\Im k$ (where $C$ depends only on $\xi$ and $\|Q\|_2$). As a corollary from the integral representation for Herglotz function, we have

**Lemma 2.9.** For any $k \in \mathbb{C}^+$ and $\|Q\| \in L^2(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$, we have

$$
\|D^{-1}(0, k, \xi)\| \leq C(\xi, \|Q\|) \|\Im k\|^{-1/2} (|\Re k| + 1) \tag{24}
$$

One may wonder why the function $D(r, k, \xi)$ possesses so many properties and may be there is some algebraic fact behind it. The partial answer to that question is contained in the following Lemma

**Lemma 2.10.** Let matrix $Y(r, k, \xi)$ solve (20) and

$$
E(r, k) = \begin{bmatrix} \exp(-2ikr) & 0 \\ 0 & 1 \end{bmatrix}
$$

If $X$ is defined by

$$
Y = Y_0 UEX
$$
Lemma 2.11. Fix any $C$ small enough so that there is a function $F$ for which $\sigma$ is specified later. Let $A(r, \xi)$ be the function $X(r, \tau, \xi) = \begin{bmatrix} i\tau & -A(r, \xi) \\ -A^*(r, \xi) & 0 \end{bmatrix} X(r, \tau, \xi)$ with $\tau = 2k$ and $A(r, \xi)$ defined by (27).

Proof. The proof is an elementary calculation. □

The matrix-valued Krein systems were studied (see, e.g., [2]). For the scalar case, see [12], [2], [9]. One can express $D(r, k, \xi)$ through the certain special solutions of the Krein systems that are know to have properties similar to those established in previous Lemmas.

Consider [13]. As was mentioned before, $\hat{F}(k, kt)$ has analytic continuation in $k$ to $\mathbb{C}$. The following Lemma is elementary.

Lemma 2.11. Fix any $k \in \mathbb{C}^+$ and $T_1 > 0$. Then, there is $\delta(k, T_1, \|Q\|_2) > 0$ small enough so that there is a function $F(r) = (f(r), 0, 0, \ldots)$, supported on $[0, \delta]$, for which

$$\|\hat{F}(k, kt)\| > C > 0, \forall t \in [-T_1, T_1]$$

where the constant $C$ depends on $k, T_1,$ and $\|Q\|_2$.

Proof. For $\alpha^*(r, k, kt)$ from (11), we have

$$\alpha^*(r, k, kt) = \frac{\sin(\rho k)}{k} + t \int_0^\rho \sin[k(r - \rho)]\alpha^*(\rho, k, kt)Q(\rho)d\rho$$ (26)

That integral equation can be used to define analytic continuation in $k$. Assume $k$ is fixed and $\delta \to 0$. Then,

$$\alpha^*(r, k, kt) = r(1 + o(1)), \quad 0 < r < \delta$$ (27)

uniformly in $|t| < T_1$. Therefore, to satisfy (25), it is sufficient to choose small $\delta$ and any nonnegative function $f(r)$ supported on $[0, \delta]$.

Proof of Theorem 2.1. Fix any $T > 0$ and $[a, b] \subset \mathbb{R}^+$. Let us show that $[a^2, b^2] \subset \sigma_{ac}(L(t))$ for generic $t \in [-T, T]$. For any $n, R$, consider $L_{n,R}^1$ given by (8).

Now, the potential $tV_{n,R}(r)$ in $L_{n,R}^1$ is an $n \times n$ matrix-function with compact support. Also, $\|V_{n,R}\|_2 \leq \|V\|_2$ for all $n, R$. Therefore, Lemmas 2.6, 2.10, 2.11 are applicable. Consider isosceles triangle in $\mathbb{C}^+$ with base $I = [a_1, b_1] \supset [a, b]$, sides $I_{I(2)}$, and the adjacent angles both equal to $\pi/\gamma$. Fix some $k_0 \in \mathbb{C}^+$ inside this triangle. Take the function $F(r)$ given by Lemma 2.11 applied to $k_0$ and some large $T_1(T)$ to be specified later. Let $d\sigma(\lambda, t)$ be the spectral measure of $F(r)$ corresponding to $L(t)$. We will show that for generic $t \in [-T, T]$ we have $\sigma'(\lambda, t) > 0$ for a.e. $\lambda \in [a^2, b^2]$. Let $d\sigma_{n,R}(\lambda)$ be the spectral measure of $F(r)$ with respect to $L_{n,R}^1$. By (18),

$$\sigma'_{n,R}(k^2, kt) = k\pi^{-1}\|D_{n,R}^{-1}(0, k, t)\hat{F}_{n,R}(k, kt)\|^2$$ (28)

For each $n, R$ and real $k$, we have factorization

$$\text{Im } G_{n,R}(k) = |D_{n,R}^{-1}(0, k, \xi)|^2$$
where $G_{n,R}(k)$ is Herglotz matrix-valued function given by \[ 23 \] with uniform in $n,R$ estimates for large $\text{Im} \, k$. Consequently,

$$\int \frac{\text{Im} \, G_{n,R}(k)}{k^2 + 1} dk < C$$

for all $n,R$. Since $\tilde{F}_{n,R}(k,kt)$ is entire in $k$ (with uniform estimates on Taylor coefficients), we have

$$\int \sigma_{n,R}'(k^2,kt) dk < C(J, T_1)$$

(29)

uniformly in $n,R, |t| < T_1$ for any interval $J \subset \mathbb{R}$.

Consider function

$$g_{n,R}(k) = \ln \| D_{n,R}^{-1}(0,k,t) \tilde{F}_{n,R}(k,kt) \|$$

Since $D_{n,R}^{-1}(0,k,t) \tilde{F}_{n,R}(k,kt)$ is analytic in $\mathbb{C}^+$ and continuous down to the real line, $g_{n,R}(k)$ is subharmonic. The mean value inequality applied to $g_{n,R}(k)$ at point $k_0$ yields

$$\int_I g_{n,R}(k) \omega(k,k_0) dk + \int_{I_{1(2)}} g_{n,R}(k) \omega(k,k_0) dk \geq g_{n,R}(k_0)$$

where $\omega(k,k_0)$ is the Green function for our triangle. It is well-known that $\omega(k,k_0)$ is smooth, positive inside $I,I_{1(2)}$, and vanishes at the vertices of triangle such that $\omega(k,k_0) \leq C|k-a_1(b_1)|^\gamma$. At $k_0$, we have

$$g_{n,R}(k_0) \geq \ln \left( \| D_{n,R}(0,k_0,t) \|^{-1} \| \tilde{F}_{n,R}(k_0,kt) \| \right) > C$$

uniformly in $n$ and $R$ due to \[ 22 \] and \[ 24 \]. By \[ 24 \] and trivial estimate on $\tilde{F}(k)$ from above, we have

$$\int_{I_{1(2)}} g_{n,R}(k) \omega(k,k_0) dk < C$$

uniformly in $n$ and $R$. In the last inequality, we also used the properties of the weight $\omega$.

Consequently, we have

$$\int_I g_{n,R}(k) \omega(k,k_0) dk > C > -\infty$$

(30)

uniformly in $n,R$. By \[ 28 \] and \[ 29 \],

$$\int_a^b \ln | \sigma_{n,R}'(k^2,kt) | dk > C, \quad \int_{a^2}^{b^2} \ln | \sigma_{n,R}'(\lambda,t\sqrt{\lambda}) | d\lambda > C$$

where the last inequality is satisfied uniformly in $n,R,t \in [T_1, T_1]$. Integration in $t$ yields

$$\int_{a^2}^{b^2} \int_{T_1}^T \ln | \sigma_{n,R}'(\lambda,t) | d\lambda dt > C$$

(31)

uniformly in $n,R$ by simple change of variables. Take $T_1 = a^{-1}T$.

Now, we consider the two-dimensional measures $d\sigma(\lambda,t)$ and $d\sigma_{n,R}(\lambda,t)$, both restricted to $[a^2, b^2] \times [-T, T]$. It is easy to show that $d\sigma_{n,R}(\lambda,t) \rightarrow d\sigma(\lambda,t)$ in the

\footnote{Analogous trick was used in \[ 10 \].}
weak-star sense. Therefore, the weak upper semicontinuity of the entropy (see [11], p. 293) and estimate (31) imply
\[
\int_{a^2}^{b^2} \int_{-T}^{T} \ln \left( \frac{d\sigma}{d\mu} \right) d\lambda dt > -\infty
\]
On the other hand, Lemma 2.2 implies that \( \frac{d\sigma}{d\mu} = \sigma'(\lambda, t) \). Consequently,
\[
\int_{a^2}^{b^2} \int_{-T}^{T} \ln -\sigma'(\lambda, t) d\lambda dt > -\infty
\]
Therefore, by Fubini theorem,
\[
\int_{a^2}^{b^2} \ln -\sigma'(\lambda, t) d\lambda dt > -\infty
\]
for a.e. \( t \in [-T, T] \). That, of course, implies \([a^2, b^2] \subset \sigma_{ac}(L(t))\) for generic \( t \in [-T, T] \). □

**Remark.** We proved that the function \( p(t) = \int_I \ln \sigma'(\lambda, t) d\lambda \) belongs to \( L^1_{loc}(\mathbb{R}) \) for any \( I \subset \mathbb{R}^+ \). Another simple property of \( p(t) \) is upper semicontinuity. It follows from the weak continuity of \( d\sigma(\lambda, t) \) with respect to \( t \) and weak upper semicontinuity of the entropy. Therefore, the set of “good” \( t \) for which \( p(t) \) is finite is necessarily \( F_\sigma \). We believe that the statement of the Theorem 2.1 holds for all \( t \). One can try to prove that by establishing the asymptotics of the Green functions as \( r \to \infty \). Let \( Y(r, k) \) be solution to
\[
-Y'' + QY = k^2 Y
\]
for \( k \in \mathbb{C}^+ \) that decays at infinity. If \( Y = \exp(ikr)\mu \), we have
\[
\mu'' + 2ik\mu' = Q\mu
\]
We try to find the solution in the form
\[
Z = \mu' \mu^{-1}
\]
Then
\[
Z' + 2ikZ = Q - Z^2
\]
and
\[
Z(r) = Z_0(r) + \int_r^\infty \exp(2ik(s - r))Z^2(s) ds, \quad Z_0(r) = -\int_r^\infty Q(s) \exp(2ik(s - r)) ds
\]
For \( \text{Im} k \) large enough, this integral equation can be solved by contraction argument and that gives us \( Z = Z_0 + Z_1 \), where \( \|Z_0\| \in L^2(\mathbb{R}^+) \) and \( \|Z_1\| \in L^1(\mathbb{R}^+) \). For \( \mu \), we have
\[
\mu' = (Z_0 + Z_1)\mu
\]
Unfortunately, the asymptotical analysis of this equation does not seem to be possible even in matrix-valued case although \( Z_0 \) is precise and \( \|Z_1\| \in L^1(\mathbb{R}^+) \). That
explains why we have to switch to different problem with energy dependent coupling constant. For this new problem, the semigroup generated by $Z_0$ happens to be bounded and the usual asymptotical analysis works.

We could have studied equation

$$-y'' + k\xi Q y = k^2 y$$

in the framework of the spectral theory for quadratic hyperbolic pencils.

Consider the following quadratic pencil [16]

$$P(k) = A_1 + kA_2 - k^2, k \in \mathbb{C}$$

where $A_1 = -d^2/dr^2 \cdot I_{n \times n}$ with Dirichlet boundary condition at zero, and $A_2 = \xi Q(r)$. Notice that $P(k)$ is hyperbolic [16], p. 169 since the quadratic polynomial

$$(P(k)G, G) = \int_0^\infty \|G'(r)^2 dr + k\xi \int_0^\infty \langle Q(r)G(r), G(r)\rangle dr - k^2$$

has two distinct real roots for any $G(r) \in \mathcal{D}(P(k)) = \bigoplus_{k=1}^n \hat{H}(\mathbb{R}^+), \|G\| = 1$

The general spectral theory of these pencils ensures invertibility of $P(k)$ for any $k \in \mathbb{C}, k \notin \mathbb{R}$. That is another explanation to the fact that function $D(r, k, \xi)$ is well-defined and invertible for all $k \in \mathbb{C}^+$. Notice that for Schrödinger operators, the Jost function $J(0, k, t)$ can be degenerate at some points $k_j = i\kappa_j$ that correspond to negative eigenvalues $-\kappa_j^2$. By Lemma 2.10, the study of $P(k)$ is essentially equivalent to analysis of the corresponding Krein systems and vice versa. Since Krein systems are understood rather well, we do not pursue any further analysis of $P(k)$. We just want to mention that matrix-valued Krein systems are essentially equivalent to matrix-valued Dirac operators. The $L^2$-conjecture for Dirac operators was resolved before [3] and the result obtained was much stronger than that of Theorem 2.1.

Pencils similar to $P(k)$ were studied before, especially for the purpose of solving the inverse problems (see, e.g. [17], and references there).

### 3. Multidimensional Schrödinger operator and corresponding pencils

In this section, we consider operator $H$ given by (2). For simplicity, we will work in the three-dimensional case. The $L^2$-conjecture for this case [22] reads

$$\int_{\mathbb{R}^3} \frac{V^2(x)}{|x|^2 + 1} dx < \infty$$

(32)

and one expects $\mathbb{R}^+ \subseteq \sigma_{ac}(H)$. This problem attracted a lot of attention recently and was resolved only for some special cases [14][1][3][6][7][8][13][14][18]. Basically, the main technical difficulty is absence of thorough asymptotical analysis for the Green function at complex energies. The operator-valued one-dimensional Schrödinger operator is a toy model for $H$ since it can be written as

$$-\frac{d^2}{dr^2} - \frac{B}{r^2} + V(r, \theta)$$

(33)

in spherical coordinates with $B$ being Laplace-Beltrami operator on the unit sphere $\Sigma$, $\theta \in \Sigma$. For general operator-valued case, the asymptotics at complex energies is not obtained (see discussion in the previous section). Of course, equation (33) is more complicated since $B$ is unbounded.
In this paper, we make another step toward understanding of the problem. Consider \( H(t) \) with potential \( V \) and the coupling constant \( t \). Our technique will allow us to easily prove the following results.

**Theorem 3.1.** Assume \( V(x) = \text{div} L(x) \) where smooth vector field \( L(x) \) satisfies

\[
L(x), |\nabla L(x)| \in L^\infty(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} \frac{|L(x)|^2}{|x|^2 + 1} dx < \infty
\]

Then for generic \( t, \mathbb{R}^+ \subseteq \sigma_{ac}(H(t)) \).

**Theorem 3.2.** Assume \( V(x) \) is bounded and

\[
\int_1^\infty r|v(r)|^2 < \infty
\]

for \( v(r) = \sup_{|x|=r} |V(x)| \). Then for generic \( t, \sigma_{ac}(H(t)) = \mathbb{R}^+ \).

Denote by \( H_R(t) \) the Schrödinger operator with potential \( tV_R(x) = tV(x) \cdot \chi_{|x|<R}(x) \). For fixed \( f(x) \in L^2(\mathbb{R}^3) \) with compact support inside the unit ball, introduce the spectral measures \( d\sigma(\lambda, t) \) and \( d\sigma_R(\lambda, t) \). For three-dimensional case, we have direct analogs of Lemmas proved in the last section. In particular (\cite{4}, p. 3974)

**Lemma 3.1.** Assume \( V(x) \) is real-valued compactly supported potential and \( u(x, k, t) = (-\Delta + tV - k^2 - (+0)i)^{-1} f \) is the restriction of the solution to real \( k \). Then, for \( u(x, k) \), the following asymptotics holds true

\[
u(x, k, t) = \frac{\exp(ik|x|)}{|x|} [A(k, \theta, t) + \bar{o}(1)], \quad \theta = x/|x|
\]

as \( |x| \to \infty \). Moreover,

\[
\sigma'(k^2, t) = k\pi^{-1}\|A(k, \theta, t)\|^2_{L^2(\Sigma)} \quad (34)
\]

where \( d\sigma \) is the spectral measure of \( f \) with respect to \( -\Delta + tV \).

Each \( A_R(k, \theta, t) \) has analytic continuation to \( \mathbb{C}^+ \) (besides points corresponding to negative discrete spectrum) but we are not able to prove any estimates uniform in \( R \) assuming only \( |V(x)| < C(|x| + 1)^{-1+} \). Therefore, instead of dealing with Schrödinger operator, we will consider the corresponding pencil given by

\[
P(k) = A_1 + kA_2 - k^2, k \in \mathbb{C}
\]

where \( A_1 = -\Delta, A_2 = \xi V(x), \xi \in \mathbb{R} \). Under the general assumption \( V(x) \in L^\infty(\mathbb{R}^3) \), \( P(k) \) is well-defined for any \( k \in \mathbb{C} \) with \( \mathcal{D}(P(k)) = H^2(\mathbb{R}^3) \). One can first define \( P(k) \) on the Schwarz space. Then it is an easy exercise to show that \( P(k) \) admits the closure which gives rise to the operator defined on \( H^2(\mathbb{R}^3) \). Moreover, one can show that \( P^*(k) = P(k) \) and that pencil \( P(k) \) is hyperbolic.

**Lemma 3.2.** Let \( V(x) \in L^\infty(\mathbb{R}^3) \). Then, for any \( k \not\in \mathbb{R} \), \( P(k) \) is invertible. If \( \psi(x, k, \xi) = P^{-1}(k)f, k \not\in \mathbb{R} \), then

\[
\|\psi\| \leq |\text{Im} k|^{-2} \|f\| \quad (35)
\]
Proof. This is a general fact of spectral theory for hyperbolic quadratic pencils. Let $k \not\in \mathbb{R}$. For any $f \in H^2(\mathbb{R}^3)$, consider
\[
\langle P(k)f, f \rangle = \int |\nabla f|^2 dx + k\xi \int V|f|^2 dx - k^2 \int |f|^2 dx = -(k - k_1)(k - k_2)||f||^2
\]
where $k_1(2) -$ real roots. Consequently,
\[
||P(k)f|| \cdot ||f|| \geq ||(P(k)f, f)|| \geq |\text{Im} k||f||^2
\]
which implies that $\text{Ker} P(k) = 0$ and $P^{-1}(k)$ is bounded. $\text{Ran} P(k)$ is dense in $L^2(\mathbb{R}^3)$ since $P^*(k) = P(\bar{k})$ and $\text{Ker} P(\bar{k}) = 0$. Then, (36) ensures that $\text{Ran} P(k) = L^2(\mathbb{R}^3)$ since $P(k)$ is closed.

Now, assume that $V(x)$ is compactly supported. Then, $\psi(x, k, \xi)$ can be continued in $k$ down to the real line by following, e.g., the proof of Agmon’s absorption principle (20), Chapter 13, sect. 8). Then, we have asymptotics
\[
\psi(x, k, \xi) = \exp(|k|x|J(k, \theta, \xi) + o(1)), \quad \theta = x/|x|
\]
as $|x| \to \infty$ for any $k \in \mathbb{C}^+$. From (36) and obvious identity $A(k, \theta, kt) = J(k, \theta, t)$ ($k -$ real), we have
\[
\sigma'(k^2, kt) = k\pi^{-1}||J(k, \theta, t)||^2_{L^2(\Sigma)}
\]
If $V(x)$ is only bounded, we can consider truncation $V_R(x)$ and the corresponding $\psi_R$ and $J_R(k, \theta, t)$. The last vector-function is analytic in $\mathbb{C}^+$ and is continuous down to the real line. For any bounded $V$, we can introduce
\[
\mu(x, k, \xi) = \psi(x, k, \xi) \exp(-ik|x|)|x|
\]
Since $\psi \in H^2(\mathbb{R}^3)$, the Sobolev embedding yields continuity of $\psi$ and $\mu$.

We start with

**Lemma 3.3.** For any compactly supported $V \in L^\infty(\mathbb{R}^3)$ and $f \in L^2(\mathbb{R}^3)$, we have
\[
||J(k, \theta, \xi)||_{L^2(\Sigma)} \leq \left[\sqrt{|k|} |\text{Im} k|^{-1} \left[ ||f(x)||_2 ||f(x)e^{2\text{Im} k|x||}_2 \right]^{1/2}
\]

Proof. For $\mu$,
\[
-\Delta \mu - 2\mu_r \left(ik - \frac{1}{|x|}\right) + k\xi V \mu = |x| \exp(-ik|x|) f(x)
\]
Divide the both sides by $2ik$, multiply by $\bar{\mu}(x)|x|^{-2}$ and integrate over the spherical layer $l < |x| < L$. Taking the real part, we have
\[
\frac{1}{L^2} \int_{|x|=L} |\mu(x, k, \xi)|^2 d\sigma - \frac{1}{L^2} \int_{|x|=l} |\mu(x, k, \xi)|^2 d\sigma + \frac{\text{Im} k}{|k|^2} \int_{l < |x| < L} \frac{|
abla \mu|^2}{|x|^2} dx
\]
\[
= -\text{Im} \left[ \frac{1}{k} \int_{l < |x| < L} f \bar{\mu} \exp(-ik|x|) \frac{dx}{|x|} \right] - \text{Re} \left[ \frac{1}{ik} \int_{|x|=l} \frac{\mu'(x) \bar{\mu}(x)}{|x|^2} d\sigma \bigg|_{s=L} \right]^{s=L}
\]
Then, take $l \to 0, L \to \infty$ and use asymptotics at infinity and regularity of $\mu$. We get
\[
||J(k, \theta, \xi)||^2 + \frac{\text{Im} k}{|k|^2} \int \frac{|
abla \mu|^2}{|x|^2} dx = -\text{Im} \left[ \frac{1}{k} \int f \bar{\mu} \exp(-ik|x|) \frac{dx}{|x|} \right]
\]
\[
\text{We have } \text{Proof.}
\]
Then (44) is an elementary corollary of (43) and equations (45) follows from (41). Also, for any compact \(V\), we have factorization (35) due to (44) and we have
\[
\int |\psi_R(k) - \psi(k)|^2_{L^2(\mathbb{R}^3)} \to 0
\]
if \(k \in \mathbb{C}^+\) is fixed.

**Remark.** Notice that the function \(g(k) = k\psi(k), f\) is Herglotz in \(\mathbb{C}^+\) and we have factorization
\[
\text{Im} [g(k)] = ||kJ(k, \theta, \xi)||^2_2
\]
for real \(k\). Also, for general \(V\), we have identity
\[
\frac{1}{|k|^2} \text{Im} [k\psi(k), f] = (\text{Im} k) \left[ \|\psi\|^2_2 + \frac{1}{|k|^2} \|\nabla \psi\|^2_2 \right]
\]
We will need the following auxiliary result.

**Lemma 3.4.** For any \(V \in L^\infty(\mathbb{R}^3)\) and \(f \in L^2(\mathbb{R}^3)\),
\[
\|\psi_R(k) - \psi(k)\|_{L^2(\mathbb{R}^3)} \to 0
\]
\[
\|\Delta \psi_R(k) - \Delta \psi(k)\|_{L^2(\mathbb{R}^3)} \to 0
\]
if \(k \in \mathbb{C}^+\) is fixed.

**Proof.** We have
\[
\psi_R = \psi + k\xi P_{R}^{-1}(k)(V - V_R)\psi
\]
and
\[
\|\psi_R - \psi\| \leq \frac{|k\xi|}{(\text{Im} k)^2} \|(V - V_R)\psi\| \to 0
\]
Then (44) is an elementary corollary of (13) and equations
\[-\Delta \psi + k\xi V\psi = k^2\psi + f, -\Delta \psi_R + k\xi V_R\psi_R = k^2\psi_R + f
\]
\[
\text{□}
\]
We will need some technical estimates

**Lemma 3.5.** For any \(k \in \mathbb{C}^+, V \in L^\infty(\mathbb{R}^3), \) and compactly supported \(f \in L^2(\mathbb{R}^3), \) we have
\[
\int |\nabla \mu|^2_{|x|^2} \, dx \leq \frac{|k|}{\text{Im} k} |f(x)||f(x)e^{2\text{Im} k|x|}||_2
\]
\[
\int_{R < |x| < R+1} \frac{|\mu|^2_{|x|^2}}{dx} < \frac{C}{|k|(|\text{Im} k|^2)^2} \left( 1 + \frac{1}{\text{Im} k} \right) \|f(x)||f(x)e^{2\text{Im} k|x|}||_2, \quad R > 1
\]
\[
\int_{\Sigma} |\mu(r\sigma)|^2 \, d\sigma < C \frac{1 + |k| \text{Im} k}{|\text{Im} k|^4} \|f(x)||f(x)e^{2\text{Im} k|x|}||_2
\]
where \(C\)'s are universal constants.

**Proof.** Consider \(V_R\) obtained from \(V\) by truncation. For the corresponding \(\mu_R,\)
(13) follows from (41). Also, for any compact \(K\) not containing zero, \(\nabla \mu_R \to \nabla \mu\)
in \(L^2(K)\) due to (44) and we have
\[
\int_K |\nabla \mu|^2_{|x|^2} \, dx \leq \frac{|k|}{\text{Im} k} \|f(x)||f(x)e^{2\text{Im} k|x|}||_2
\]
Since \(K\) is arbitrary, we have (15) for any bounded \(V.\)
To get (46), take \( l = 0, \ L = \rho \) in (40). We have
\[
\int_{\Sigma} |\mu(\rho \sigma)|^2 d\sigma + \frac{\text{Im} k}{|k|^2} \int_{|x| < \rho} |\nabla \mu|^2 dx =
\]
\[
= - \text{Re} \left[ \frac{1}{ik} \int_{\Sigma} \mu'(\rho \sigma) \bar{\mu}(\rho \sigma) d\sigma \right] + \text{Im} \left[ \bar{k}^{-1} \int_{|x| < \rho} \psi \bar{f} e^{2 \text{Im} k |x|} dx \right]
\]
Integrate in \( \rho \) from \( R \) to \( R + 1 \) and use
\[
|\mu \mu'| \leq \frac{1}{2} [\varepsilon |\mu|^2 + \varepsilon^{-1} |\mu'|^2]
\]
in the first term of the r.h.s. Then, taking \( \varepsilon = |k|/2 \) and using (45), we get (46). Since we have now the estimates on \( \mu(\rho \sigma) \) in \( H^1_{\text{loc}}(R^+) \), the standard Sobolev embedding argument yields (47).

The last Lemma essentially says that the average decay of Green’s function \( G(x, y, k) \) of \( P(k) \) is always at most \( \exp(-|\text{Im} k| \cdot |x - y|/|x - y|) \). That fact gives strong improvement of (72) and has no analogs in the spectral theory of Schrödinger operators.

We will need the following standard result later on

**Lemma 3.6.** Consider \( V(x) \in L^\infty(\mathbb{R}^3), V(m)(x) = V(x) \chi_{|x|>m}, m > 0 \) and the pencil \( P(m)(k) \) corresponds to potential \( V(m)(x) \). Then for fixed \( f \in L^2(\mathbb{R}^3) \) and \( k \in \mathbb{C}^+ \), we have
\[
\psi(m) = P^{-1}(m)(k)f \to \psi = (-\Delta - k^2)^{-1}f, \ m \to \infty
\]
uniformly over any compact in \( \mathbb{R}^3 \).

**Proof.** The second resolvent identity reads
\[
\psi(m) = \psi - k(-\Delta - k^2)^{-1}V(m)\psi(m)
\]
the last term can be written as
\[
k \int \frac{e^{ik|x-y|}}{|x-y|} V(m)(y) \psi(m)(y) dy = k \int \frac{e^{ik|x-y|}}{|x-y|} V(m)(y) \psi(m)(y) dy
\]
The application of Cauchy-Schwarz inequality and (33) finishes the proof.

The next Lemma controls the radial derivative of the solution in the case which is very close to condition (32).

**Lemma 3.7.** Let \( v(r) = \sup_{|x|=r} |V(x)| \in L^2(\mathbb{R}^+) \). Then, for any fixed \( k \in \mathbb{C}^+ \) and any \( f \in L^2(\mathbb{R}^3) \) supported within \( |x| < \rho \), we have
\[
4 \text{Im} k \int_{|x| > \rho} \frac{|\mu'(x,k)|^2}{|x|^2} dx < C(k) \int_{\rho} v^2(r) dr + \int_{\Sigma} |\nabla \cdot \mu(\rho \sigma)|^2 d\sigma \quad (48)
\]
Proof. In the spherical coordinates, the equation for \( \mu \) reads as follows

\[
- \mu'' - 2ik \mu' - \frac{B}{r^2} \mu + k \xi V \mu = r \exp(-ikr)f(r \sigma), \quad r > 0, \sigma \in \Sigma
\]  

(49)

Instead of \( V \), consider \( V_R \) and the corresponding \( \mu_R \). Multiply both sides by \( \mu_R' \) from the right and integrate from \( \rho \) to infinity. Taking the real part yields

\[
\|\mu_R'(\rho, \theta)\|^2 + 4 \text{Im} k \int_\rho^\infty \|\mu_R'(r, \theta)\|^2 dr + 2 \int_{|x|>\rho} \frac{\nabla \mu_R(x)\cdot \nabla \mu_R(x)}{|x|^3} =
\]

\[
= -\rho^{-2} \langle B \mu_R(\rho, \theta), \mu_R(\rho, \theta) \rangle - 2\xi \text{Re} \left[ k \int_\rho^\infty \langle V_R \mu_R, \mu_R' \rangle dr \right]
\]

(50)

The second integral in the r.h.s. can be bounded by \( C(k) \int_R^\infty v^2(r)dr \) due to (45) and (47). Thus, we have the statement of the Lemma for each \( R \). Take \( R \to \infty \). Lemma 3.4 and Sobolev embedding theorem, allows one to go to the limit and get (48).

□

Now, we have enough information to prove Theorems 3.1 and 3.2. The main idea is the same as in the proof of Theorem 2.1. That is to use subharmonicity in \( k \in \mathbb{C}^+ \) of the function \( \ln \|J_R(\theta, k, \xi)\| \) to obtain the lower bound on the entropy

\[
\int_a^b \int_{-T_1}^{T_1} \ln \sigma'(\lambda, t) d\lambda d\lambda
data \end{align*}

by using factorization (37). To do that, we have the uniform bound from above given by (38). This bound is true always, regardless of the behavior of potential at infinity. The only thing we need to do to make the argument work is to provide a bound from below for \( \ln \|J_R(\theta, k, \xi)\| \) which would be uniform in \( R \). Moreover, it is enough to prove this bound for at least some point \( k = k_0 \) inside a triangle considered in the proof of Theorem 2.1. Getting this bound will involve the information on decay of \( V \) and will be the core of the proofs for the next two Theorems.

Proof of Theorem 3.1.

Since the a.c. part of the measure is invariant under the trace-class perturbations (e.g., [19], Birman-Kuroda Theorem), it is enough to prove the statement for \( V_m = \text{div} L_m, L_m(x) = L(x) \cdot b_m(\|x\|) \) where \( m \) is arbitrary fixed number and \( b_m(t) = 0 \) on \( [0, m - 1] \), \( b_m(t) = 1 \) for \( t > m \) and is smooth on \([m, m + 1]\). For any \( f \in L^2(\mathbb{R}^3) \), \( d\sigma_m(\lambda, t) \) denotes the spectral measure of \( f \) corresponding to \( H_m(t) \). We will be taking \( m \) large later on.

Assume that the support of \( f \) is within \( |x| < 1 \) and consider compactly supported potentials \( V_{m, R} = \text{div} L_{m,R} \) with \( L_{m,R}(x) = L(x) \cdot b_{m,R}(\|x\|) \), where smooth \( b_{m,R} \) is such that \( b_{m,R}(t) = 0 \) for \( t \in (0, m - 1) \cup (R + 1, \infty) \), \( b_{m,R}(t) = 1 \) for \( t \in (m, R) \).

For that \( V_{m,R} \), multiply both sides of (39) by \( |x|^{-2} \), integrate over \( |x| > \rho > 1 \), and use asymptotics at infinity. We then have

\[
\int_{\Sigma} J_{m, R}(\theta, k, \xi) d\theta =
\]
\[ \frac{1}{\rho^2} \int_{|x|=\rho} \mu_{m,R}(x,k,\xi) d\sigma - \frac{i}{2k\rho^2} \int_{|x|=\rho} \mu'_{m,R}(x,k,\xi) dx + \frac{\xi}{2i} \int_{|x|>\rho} \frac{V_{m,R}\mu_{m,R}}{|x|^2} dx \]  

(51)

The last integral is equal to

\[- \int \frac{L_{m,R} \cdot \nabla \mu_{m,R}}{|x|^2} dx + 2 \int \frac{2\mu_{m,R}}{|x|^3} \left[ L_{m,R} \cdot \frac{x}{|x|} \right] dx\]

and its absolute value is not greater than

\[ C(k) \int \frac{|L_{m,R}(x)|^2}{|x|^2 + 1} dx \]

by Cauchy-Schwarz and \[\text{[3.6, 40]}\]. Notice that for fixed \(k\) the last quantity can be made arbitrarily small uniformly in \(R\) by choosing \(m\) large.

Now, fix any \(k_0 \in \mathbb{C}^+\). Take \(f\) to be spherically symmetric nonnegative and with unit norm. Then, by Lemma 3.6

\[ \mu_{m,R}(x,k,\xi) \to \mu^0(x,k) = \int \frac{|x|e^{-ik(|x|-|x-y|)}}{|x-y|} f(y) dy, \quad m \to \infty \]

for fixed \(k \in \mathbb{C}^+\) uniformly in \(R > R_0\) and in \(x \in K\) for any compact \(K\). By Lemma 3.6 and the theorem on trace of \(H^2\) functions, we have

\[ \mu'_{m,R} \to (\mu^0)', \quad m \to \infty \]

in \(L^2(S_r)\) on any fixed sphere \(S_r = \{|x| = r\}\).

This \(\mu^0\) is spherically symmetric since \(f\) is spherically symmetric. Moreover, for \(|x| \to \infty\),

\[ \mu^0(x,k) \to A^0(k) = \int e^{-ik\langle \theta,y \rangle} f(|y|) dy = \int_0^{\infty} tf(t) \frac{\sin(kt)}{k} dt \]

and

\[ (\mu^0)'(x,k) \to 0 \]

where \(A^0(k)\) is the amplitude of \(f\) with respect to unperturbed operator. This function is entire and therefore has only finite number of zeroes in any compact in \(\mathbb{C}\). For any \(k_0 \in \mathbb{C}^+\) which is not zero, we can arrange first \(\rho\) and then \(m\) such that the difference in the r.h.s. of 3.41 has absolute value greater than some \(2\delta\) and the last term in the r.h.s. of 51 has absolute value smaller than \(\delta\) (all that uniformly in \(R > R_0\) and \(\xi \in [-T_1, T_1]\), where \(T_1\) is any fixed constant). We get

\[ \left| \frac{1}{\rho^2} \int_{|x|=\rho} \mu_{m,R}(x,k_0,\xi) d\sigma - \frac{i}{2k_0\rho^2} \int_{|x|=\rho} \mu'_{m,R}(x,k_0,\xi) dx + \frac{\xi}{2i} \int_{|x|>\rho} \frac{V_{m,R}\mu_{m,R}}{|x|^2} dx \right| > \delta > 0 \]

for any \(|\xi| < T_1\) and any \(R > R_0\). Thus, for \(k = k_0\), we have

\[ ||J_{m,R}(\theta,k_0,\xi)|| \geq |\langle J_{m,R}(\theta,k_0,\xi), 1 \rangle| > \delta \]  

(52)

Then, since we also have the estimate 38 and factorization 37, the proof of absolute continuity for the measure repeats the argument for Theorem 2.1. The logic here is that we first choose an interval for the spectral parameter and for coupling constant, then take \(f\) and some \(k_0\) which is inside the triangle and is not a zero of \(A^0(k)\). Then we find large \(\rho\) and after that make a truncation by \(m\) so that the uniform in \(R\) estimates hold.
Now, we have two identities
\[
\sigma'_m(R)(k^2,kt) = k\pi^{-1}\|J_m(R(k,\theta,t))\|^2
\]
and
\[
\text{Im}(k\langle \psi_m(R(k),f) \rangle) = \|kJ_m(R(k,\theta,t))\|^2
\]
The functions \(k\langle \psi_m(R(k),f) \rangle\) are Herglotz in \(\mathbb{C}^+\) having uniform in \(m,R\) estimates. That yields the uniform bound on variations, i.e.
\[
\int_J \sigma'_m(R)(k^2,kt)dk < C(J,T_1)
\]
uniformly in \(m,R,|t| < T_1\). Here \(J\) is any interval in \(\mathbb{R}\).

Now the repetition of subharmonicity argument gives the uniform bounds on the entropy
\[
\int_{a^2}^{b^2} T \int_{-T}^{T} \ln^+ \sigma'_m(R)(\lambda,t)d\lambda dt > C
\]
The weak-star convergence of \(d\sigma_m(R(\lambda,t))\) to \(d\sigma_m(\lambda,t)\) as \(R \to \infty\) is a simple corollary of Lemma 3.4. It allows to conclude that
\[
\int_{a^2}^{b^2} T \int_{-T}^{T} \ln^+ \sigma'_m(R)(\lambda,t)d\lambda dt > -\infty
\]thus \((a^2,b^2) \subseteq \sigma_{ac}(H_m(t))\) for generic \(t \in [-T,T]\). Recall that parameter \(m\) corresponds to cutting \(L\) off inside the ball of radius \(m\). As mentioned earlier, this subscript \(m\) can be dropped due to trace-class type argument.

Remark. This result suggests that the method of [4] can probably be pushed forward to prove Theorem 2.1 for any coupling constant. Also, in the proof we have control over \(\langle J,1 \rangle\) and that implies that nontrivial energy is always present on low angular modes. We do not think that that is the case when \(V\) decays without substantial oscillation.

Proof of Theorem 3.3
By Weyl’s Theorem on essential spectrum [20], \(\sigma_{ess}(H(t)) = \mathbb{R}^+\). Consider \(V_m(R(x) = V(x)\chi_{m<|x|<R}, 1 < m < R\) and assume that \(f\) is spherically symmetric, has support within the unit ball and has the unit norm. Multiply (49) from the right by \(r\mu_R\) and integrate from \(\rho\) to infinity. Similarly to Lemma 3.7 we have
\[
\rho\|\mu'_{m,R}(\rho)\|^2 + \|\mu'_{m,R}(r)\|^2 dr + 4 \text{Im} k \int_{\rho} r\|\mu'_{m,R}(r)\|^2 dr + \int_{|x|>\rho} \frac{|\nabla^T \mu_{m,R}(x)|^2}{|x|^2} dx \leq
\]
\[
\frac{1}{\rho} \int_{|x|=\rho} |\nabla^T \mu_{m,R}(x)|^2 d\sigma + 2\xi \int_{|x|>\rho} \frac{V_m(x)\mu_{m,R}(x)}{|x|} dx
\]
The last integral is bounded by
\[
C(k) \left[ \int_{\rho} r\mu^2_{m,R}(r) dr \right]^{1/2} \left[ \int_{\rho} r\|\mu'_{m,R}(r)\|^2 dr \right]^{1/2}
\]
due to (45) and (47). Using inequality $2ab \leq \epsilon a^2 + \epsilon^{-1}b^2$ for the last product, we get

$$
\int_{|x|>\rho} \frac{|
abla \mu_{m,R}(x)|^2}{|x|^2} dx + 2 \text{Im} k \int_{\rho}^{\infty} r \|\mu'_{m,R}(r)\|^2 dr <
$$

$$
< C(k) \left[ \int_{\rho}^{\infty} r v_m^2(r) dr + \frac{1}{\rho} \int_{|x|=\rho} |\nabla \tau \mu_{m,R}(x)|^2 d\sigma \right]
$$

Notice that we infact show the weighted $L^2$ estimate for the full gradient

$$
\int_{|x|>\rho} \frac{|
abla \mu_{m,R}(x)|^2}{|x|^2} dx < C(k,\xi) \left[ \int_{\rho}^{\infty} \rho v_m(r) dr + \frac{1}{\rho} \int_{|x|=\rho} |\nabla \tau \mu_{m,R}(x)|^2 d\sigma \right]
$$

Now fix any positive interval for the spectral parameter and let the coupling constant $\xi \in [-T_1, T_1]$.

In (40), let $L \to \infty$. If $l = \rho > 1$, the integral with $f$ will be disappear and

$$
\|J_{m,R}(k,\theta,\xi)\|^2 = \int_{|x|=\rho} \frac{|\mu_{m,R}(x,k,\xi)|^2}{|x|^2} d\sigma - \text{Im} k \int_{\rho<|x|} |\nabla \tau \mu_{m,R}(x)|^2 d\sigma + \text{Re} \left[ \frac{1}{ik} \int_{|x|=\rho} \mu'_m(x) \bar{\mu}_{m,R}(x) \frac{1}{|x|^2} d\sigma \right]
$$

Now, as $m \to \infty$, the first term in the r.h.s. approaches

$$
\frac{1}{\rho^2} \int_{|x|=\rho} |\mu^0(x,k,\xi)|^2 d\sigma
$$

uniformly in $R > m$ and in $k \in D$ where $D$ is any domain in $\mathbb{C}^+$. Now, if $\rho \to \infty$, (55) will converge to $|A_0(k)|^2$. We take any $k = k_0 \in \mathbb{C}^+$ which is not a zero of $A_0(k)$. If $k_0$ is fixed,

$$
\frac{1}{\rho^2} \int_{|x|=\rho} |\mu_{m,R}(x,k_0,\xi)|^2 d\sigma > 2\delta
$$

if we first choose $\rho$ and then $m$ to be large. This inequality holds uniformly in $R$.

Consider

$$
- \text{Im} k_0 \int_{\rho<|x|} |\nabla \mu_{m,R}|^2 dx + \text{Re} \left[ \frac{1}{ik_0} \int_{|x|=\rho} \mu'_m(x) \bar{\mu}_{m,R}(x) \frac{1}{|x|^2} d\sigma \right]
$$

The function $\mu^0$ is spherically symmetric and therefore $\nabla \tau \mu^0 = 0$. Thus, by (53),

$$
\left| \frac{\text{Im} k_0}{|k_0|^2} \int_{\rho<|x|} \frac{|\nabla \mu_{m,R}|^2}{|x|^2} dx \right| < \delta
$$
uniformly in $R > R_0$, if $k_0$ and $\rho$ are fixed and $m$ is large. Moreover, we have

$$\left| \text{Re} \left[ \frac{1}{ik_0} \int_{|x| = \rho} \frac{\mu_m^\prime R(x) \bar{\mu}_m R(x)}{|x|^2} d\sigma \right] \right| < \delta$$

uniformly in $R > R_0$ if we first choose $\rho$ and then $m$ to be large. This is due to the fact that $\partial\mu^0(x, k_0, \xi) / \partial r \to 0$ as $|x| \to \infty$.

Thus, after arranging $\rho$ and $m$, we finally have

$$\|J_m R(k_0, \theta, \xi)\|_2^2 > \delta$$

uniformly in $R > R_0$. This uniform in $R$ estimate from below allows us to repeat the arguments from the Theorem 2.1 and finish the proof. □

Remark. Notice that even in one-dimensional case, the condition considered in the last Theorem leads to WKB correction

$$\exp \left[ \frac{1}{2ik} \int_0^r V(s) ds \right]$$

in the asymptotics of the Green function. The main point in the last proof is to show that the complete gradient of $\mu$ is small (not only its radial component). If so, the general identity (41) provides the bound from below for $\|\mu\|$. It is important to mention that the usual $L^2$ decay of potential guarantees that $\mu^\prime$ is small (due to Lemma 3.7) but we can say nothing about the size of $\nabla_r \mu$.

Now, we further study the Schrödinger pencil. As we saw before, the main equation to consider is (49), which can be rewritten as

$$\mu^\prime = -\frac{1}{2i} k \mu + \frac{\xi}{2i} V \mu + w_1 + w_2, \quad r > 0$$

(56)

with

$$\kappa = -\frac{1}{2i}, \quad w_1 = -\frac{1}{2i} \mu^\prime, \quad w_2 = -\frac{re^{-ikr}}{2ik} f$$

This is not quite an evolution equation on $L^2(\mathbb{R}^+, L^2(\Sigma))$ because of the second derivative present, but we can study the asymptotics of solution by writing Duhamel formula

$$\mu(r) = U(\rho, r, k) \mu(\rho) + \int_\rho^r U(s, r, k) w_1(s) ds$$

(57)

where $\rho > 1$ and $f$ is supported on $[0, 1]$ and

$$U'(\rho, r, k) = \kappa \frac{B}{r^2} U(\rho, r, k) + \frac{\xi}{2i} V U(\rho, r, k), \quad U(\rho, \rho, k) = I$$

(58)

Now, by considering $V(m) = V \cdot \chi_{(|x| > m)}$ instead of $V$ and taking $f$ spherically symmetric, we can always make sure that $\mu(\rho)$ in (57) is close to $\mu^0(\rho)$, a constant function in angles, in the uniform norm.

Now, we know that $\mu^\prime$ has small $L^2$ norm provided $V$ satisfies conditions of Lemma 3.7. Integration by parts and rather simple estimates on $\partial_s U(s, r, k)$ allow one to estimate the second term in (57). Therefore, to show that $\|\mu(r)\|$ is bounded away from zero, we need to concentrate mostly on the first term $U(\rho, r, k) \mu(\rho)$.

Notice that

$$\text{Re} \left[ \kappa \frac{B}{r^2} + \frac{\xi}{2i} V \right] = \frac{\text{Im} k}{2|k|^2} \cdot \frac{B}{r^2}$$
Since $B$ is nonpositive, the norm $\|U(\rho, r, k)\eta\|$ decreases in $r$. Moreover, it might be that high oscillation of $V$ kicks the Fourier spectrum of $u(r) = U(\rho, r, k)\eta$ to the higher and higher modes where the energy is dissipated due to the presence of $B$. In other words, we do not have any proof that $\|u(r)\|$ does not go to zero even for $V$: $|V(r, \theta)| < Cr^{-1+}$. Moreover, it might very well be that $\|u(r)\|$ does go to zero for some choice of $V$ satisfying this condition. Therefore, we have to use the following modification of the Hamiltonian $H$ itself. As we know, the operator $H$ is unitarily equivalent to the operator
\[-\frac{d^2}{dr^2} - \frac{B}{r^2} + V(r)\]
defined on $L^2(\mathbb{R}^+, L^2(\Sigma))$ with Dirichlet boundary condition at zero. In the previous part of the paper, we introduced the coupling constant in front of the potential.

Now, we consider a different family of operators. Let $\lambda_m = -m(m+1)$, $m = 0, 1, \ldots$ be distinct eigenvalues of $B$ and $Y^m_l$ the corresponding spherical harmonics ($|l| \leq m$). Let $\alpha \in (0, 1)$ be some positive parameter to be chosen later and $r_m$ be the points of intersection of the graph of $\omega(r) = r^\alpha$ with levels $|\lambda_m|^{1/2}$, $m = 0, 1, \ldots$.

On $r > 1$, we introduce the function $s(r)$ which is piecewise constant and equals to $|\lambda_m|^{1/2}$ on each $I_m = [r_m, r_{m+1})$.

We consider the function $s_1(\omega, r)$ defined for $\omega \geq 0, r > 1$ such that $s_1(\omega, r) = 0$ for $\omega > s(r)$ and $s_1(\omega, r) = 1$ for $\omega \leq s(r)$. The decomposition of unity on $r > 1, \omega \geq 0$ is defined through $s_2(\omega, r) = 1 - s_1(\omega, r)$. For each $r > 1$, these $s_{1(2)}$ define the multipliers and the corresponding operators
\[M_{1(2)}(r)f = \sum_{l,m} Y^m_l f^m_l s_{1(2)}(|\lambda_m|^{1/2}, r)\]
where $f \in L^2(\Sigma)$ and $f^m_l$ are Fourier coefficients with respect to spherical harmonics. The point here is that we want to separate frequencies along the level $\omega \sim r^\alpha$ and define
\[B_{1(2)}(r) = BM_{1(2)}(r), \quad \tilde{H}(t) = -\frac{d^2}{dr^2} + \frac{B_1(r)}{r^2} + V(r) - \frac{B_2(r)}{r^2}, \quad t \in \mathbb{R}\]
For $r \in [0, 1)$, we let $\tilde{H}(t) = H(t)$, this interval is not important. Of course, $\tilde{H}(1) = H(1)$. The operator $\tilde{H}(t)$ can be rewritten as
\[\tilde{H}(t) = H(t) + (1 - t) \frac{B_1(r)}{r^2}\]
Notice that $B_1(r)r^{-2}$ is bounded in the Hilbert space $L^2([1, \infty), L^2(\Sigma))$. Therefore for self-adjoint $\tilde{H}(t)$, we have $\mathcal{D}(\tilde{H}(t)) = H^2(\mathbb{R}^3)$ provided that $V \in L^\infty(\mathbb{R}^3)$. Essentially, in this approach we treat
\[\tilde{V}(r) = -\frac{B_1(r)}{r^2} + V(r)\]
as the perturbation of
\[\tilde{H}^0 = -\frac{d^2}{dr^2} - \frac{B_2(r)}{r^2}\]
The operator $\tilde{H}^0$ can be easily decoupled into the orthogonal sum of one-dimensional Schrödinger operators with explicit potentials. It is an easy exercise then to check that the spectrum of $\tilde{H}^0$ is $[0, \infty)$ and is purely a.c. One has to note thought that this perturbation $\tilde{V}$ is not a multiplication by a function any longer.
Now, we are ready to formulate our result.

**Theorem 3.3.** If \( \alpha = 2/3 - \) and \( |V(x)| < C|x|^{-\gamma} \), \( \gamma > 3/2 - \alpha \), then \( \sigma_{ac}(\tilde{H}(t)) = \mathbb{R}^+ \) for generic \( t \).

**Remark.** Since the a.c. spectrum covers the positive half-line for generic \( t \), it is true for some \( t \) accumulating to 1. That suggests (but does not prove) that the a.c. spectrum is likely to be preserved for \( t = 1 \) (i.e., the original Schrödinger operator), at least under the \( 5/6+ \) assumption on decay. In any case, this result is the first one when we are able to go below 1 in the decay assumption on the potential.

The proof follows the same lines. Consider truncations in space
\[
\tilde{V}_R(r) = \tilde{V}(r)\chi_{r < R}
\]
and damping of \( B_2 \) as
\[
\tilde{H}_{R,b} = -\frac{d^2}{dr^2} + \frac{B_{2,b}(r)}{r^2} + \tilde{V}_R
\]
where \( B_{2,b}(r) = B_b M_2(r) \),
\[
B_b f = \sum_{l,m,|m| < b} Y_l^m \lambda_m f_l^m - b(b + 1) \sum_{l,m,|m| \geq b} Y_l^m f_l^m
\]
Here \( b > R^3 \) and the damping is introduced to reduce the problem to one-dimensional Schrödinger operator with bounded operator-valued potential whose norm is in \( L^1[1, \infty) \). For these operators, we know absorption principle, absence of embedded positive eigenvalues, etc. The point, though, is to prove estimate on the entropy (e.g., \( \mathcal{F} \)) which is uniform in \( b \) and \( R \). Then, the following simple approximation result will do the job.

**Lemma 3.8.** For any \( f(r) \in L^2(\mathbb{R}^+, L^2(\Sigma)) \) and any \( z \in \mathbb{C}^+ \), we have
\[
\langle (\tilde{H}_{R,b} - z)^{-1} f, f \rangle \to \langle (\tilde{H} - z)^{-1} f, f \rangle
\]
as \( R \to \infty, b \to \infty \).

**Proof.** The second resolvent identity yields
\[
\langle R_{R,b}(z) f, f \rangle = \langle R(z) f, f \rangle - \left( \frac{B_{2,b}(r) - B_2(r)}{r^2} \right) R(z) f + (\tilde{V}_R - \tilde{V}) R(z) f, R^*_{R,b}(z) f \rangle
\]
Since
\[
\left\| \frac{B_{2,b}(r) - B_2(r)}{r^2} g \right\| \to 0, \left\| (\tilde{V}_R - \tilde{V}) g \right\| \to 0 \quad R \to \infty, b \to \infty
\]
for fixed \( g \in \mathcal{D}(\tilde{H}) = H^2(\mathbb{R}^3) \), we have the statement of a Lemma.

This Lemma yields the weak–star convergence of the spectral measures \( d\sigma_{R,b}(\lambda) \) to \( d\sigma(\lambda) \), where the spectral measures are calculated for fixed \( f \).

For \( \tilde{H}_{R,b} \), the analog of Lemma 2.3 (and Lemma 3.1) holds true.

**Lemma 3.9.** For any \( f(r) \in L^2(\mathbb{R}^+, L^2(\Sigma)) \) with compact support, we have
\[
\left[ (\tilde{H}_{R,b} - k^2 - i(+0))^{-1} f \right](r) \sim \exp(ikr) A_{R,b}
\]
as \( r \to \infty \). Moreover, for the spectral measure of \( f \), we have
\[
\sigma_{R,b}(k^2) = k\pi^{-1} \| A_{R,b}(k) \|^2, k > 0
\]
Lemma 3.12. Fix $J | \eta \rangle$ are introduced. The resulting estimates are uniform in $\tilde{\eta}$. Assume also that $\tilde{\eta} > 0$. For any compactly supported $f(r) \in L^2(\mathbb{R}^+, L^2(\Sigma))$, we introduce $\psi_{R,b} = \tilde{P}_{R,b}(k)f$, $\mu_{R,b} = \exp(-ikr)\psi_{R,b}$, $J_{R,b}(k, \xi) = \lim_{r \to \infty} \mu_{R,b}(r, k, \xi)$. We have

$$
\sigma'_{R,b}(k^2, kt) = k\pi^{-1}||J_{R,b}(k, t)||^2
$$

and the following analog of Lemma 3.3.

**Lemma 3.10.** For any compactly supported $f \in L^2(\mathbb{R}^+, L^2(\Sigma))$ and $k \in \mathbb{C}^+$, we have

$$
\|J_{R,b}(k, \xi)\|_{L^2(\Sigma)} \leq \left[ \sqrt{|k| \operatorname{Im} k} \right]^{-1} \left[ \|f(r)\|_2 \|f(r)e^{2\operatorname{Im} kr}\|_2 \right]^{1/2}
$$

uniformly in $R > R_0, b > 1$.

The estimate on the derivative of $\mu$ can be obtained in the same way.

**Lemma 3.11.** For any $k \in \mathbb{C}^+$, we have

$$
\int_0^\infty \|\mu'_{R,b}(r, k)\|^2 dr < C(k)
$$

What makes the situation different is the behavior of evolution $\tilde{U}(\rho, r)$

$$
\tilde{U}'(\rho, r, k) = \kappa \frac{B_2(r)}{r^2} \tilde{U}(\rho, r, k) + \frac{\xi}{2i} \tilde{V} \tilde{U}(\rho, r, k), \quad \tilde{U}(\rho, r, k) = I
$$

as $r \to \infty$. Recall that $\tilde{V}(d) = \tilde{V} \cdot \chi_{r > d}$. We have

**Lemma 3.12.** Fix $k \in \mathbb{C}^+$ and let $\alpha = 2/3-$, $\gamma > 3/2 - \alpha$, $|\xi| < T_1$. Assume that $|V(x)| \leq C|x|^{3/2}$ and consider the evolution

$$
\tilde{U}'(\rho, r, k) = \kappa \frac{B_2(r)}{r^2} \tilde{U}(\rho, r, k) + \frac{\xi}{2i} \tilde{V}(d) \tilde{U}(\rho, r, k), \quad \tilde{U}(\rho, r, k) = I
$$

where $d(k, V, T_1)$ is large enough. Then, we have

$$
\lim_{r \to \infty} \|\tilde{U}(1, r, k)\| > \delta(k, \gamma, T_1) > 0
$$

Assume also that $\tilde{V}$ has compact support in $[0, R]$. Then, for each $\eta \in L^2(\Sigma)$,

$$
\int_0^\infty \|\partial_\rho \tilde{U}^*(\rho, \infty)\eta\|^2 d\rho = \tilde{o}(1) \cdot \|\eta\|^2
$$

as $t \to \infty$ uniformly in $R$.

The same results hold true for the case when truncation by $R$ and damping by $b$ are introduced. The resulting estimates are uniform in $R > R_0, b > R^\alpha$.

**Proof.** For simplicity, we take $\xi = -2$ and $k = i/2$. Then, we have

$$
u' = \frac{B_2(r)}{r^2} u + i\tilde{V}(d)u, \quad u(1) = 1$$
Let us study this evolution. Obviously, \( \|u\| \) decreases. We split \( u(r) = M_1(r)u + M_2(r)u = u_1(r) + u_2(r) \). Operators \( M_1(2) = \text{const} \) on the intervals \( I_m = [r_m, r_{m+1}] \) and act as orthoprojectors, also \( r_m \sim m^{1/\alpha}, |I_m| \sim m^{1/\alpha-1} \). Let us control the variation of \( \|u_{1(2)}\| \) on each of \( I_m \). We have

\[
\frac{d}{dr} \begin{bmatrix} u_1(r) \\ u_2(r) \end{bmatrix} = \begin{bmatrix} iB_1(r)r^{-2} + iV^{11}(r) & iV^{12}(r) \\ iV^{21}(r) & B_2(r)r^{-2} + iV^{22}(r) \end{bmatrix} \begin{bmatrix} u_1(r) \\ u_2(r) \end{bmatrix}
\]

where

\[
V^{ij} = M_i V_d M_j, \quad \begin{bmatrix} u_1(r_m + 0) \\ u_2(r_m + 0) \end{bmatrix} = \begin{bmatrix} \alpha_m \\ \beta_m \end{bmatrix}
\]

Consider \( U_{1(2)} \) acting on \( \text{Ran} M_1(2)|I_m \) and defined as follows

\[
\begin{align*}
U_1^1(r, \rho) & = [iB_1(r)r^{-2} + iV^{11}(r)] U_1(\rho, r), U_1(\rho, \rho) = I \\
U_1^2(r, \rho) & = [B_2(r)r^{-2} + iV^{22}(r)] U_2(\rho, r), U_2(\rho, \rho) = I
\end{align*}
\]

where \( r_m < \rho < r < r_{m+1} \). \( U_1 \) is unitary and \( U_2 \) is a contraction satisfying

\[
\|U_2(\rho, r)\| \leq \exp \left[-\lambda_m \frac{r - \rho}{r \rho}\right], \quad r_m < \rho < r < r_{m+1}
\]

on \( \text{Ran} M_2 \). The dynamics under this evolution is as follows: \( U_1 \) doesn’t change the norm of \( u_1 \), \( U_2 \) suppresses \( u_2 \), and interaction between \( u_1 \) and \( u_2 \) is small due to decay of \( V^{12} \). This situation is standard in asymptotical analysis.

By Duhamel,

\[
\begin{align*}
u_1(r) & = U_1(r_m, r)\alpha_m + i \int_{r_m}^{r} U_1(\rho, r)V^{12}(\rho)u_2(\rho)d\rho \\
u_2(r) & = U_2(r_m, r)\beta_m + i \int_{r_m}^{r} U_2(\rho, r)V^{21}(\rho)u_1(\rho)d\rho
\end{align*}
\]

When moving from \( I_m \) to \( I_{m+1} \) the dimension of \( \text{Ran} M_{1(2)} \) increases/decreases by the geometric multiplicity of \( \lambda_{m+1} \). Therefore,

\[
\|u_2(\rho)\| \leq \|\beta_m\| \exp \left[-\lambda_m \frac{r - r_m}{r \rho_m}\right] + c \int_{r_m}^{r} \rho^{-\gamma} \exp \left[-\lambda_m \frac{r - \rho}{\rho \rho_m}\right]d\rho
\]

and

\[
\|\beta_{m+1}\| \leq \exp(-Cm^{1-\alpha^{-1}})\|\beta_m\| + c \int_{r_m}^{r_{m+1}} \rho^{-\gamma} \exp \left[-\lambda_m \frac{r_{m+1} - \rho}{\rho \rho_{m+1}}\right]d\rho < \exp(-Cm^{1-\alpha^{-1}})\|\beta_m\| + cm^{\alpha^{-1}(1-\gamma)-1}
\]

Let \( \kappa_1 = 1 - \alpha^{-1}, \kappa_2 = -\alpha^{-1}(1 - \gamma) + 1 \). The simple iteration gives

\[
\|\beta_m\| < C \sum_{j=1}^{m} \exp \left[-C(m^{\kappa_1+1} - j^{\kappa_1+1})\right] j^{-\kappa_2} \leq Cm^{-\kappa_1-\kappa_2}
\]
For $\alpha_{m+1}$,
\[
\|\alpha_{m+1}\| \geq \|\alpha_m\| - c\|\beta_m\| \int_{r_m}^{r_{m+1}} \rho^{-\gamma} \exp\left[-\lambda_m \frac{\rho - r_m}{\rho r_m}\right] d\rho
\]
\[-c \int_{r_m}^{r_{m+1}} \rho^{-\gamma} \int_{r_m}^{\rho} s^{-\gamma} \exp\left[-\lambda_m \frac{\rho - s}{\rho s}\right] ds d\rho
\]

For
\[
\zeta_m = \int_{r_m}^{r_{m+1}} \rho^{-\gamma} \exp\left[-\lambda_m \frac{\rho - r_m}{\rho r_m}\right] d\rho, \quad |\zeta_m| \leq C m^{\alpha - (1-\gamma) - 1}
\]
and
\[
\eta_m = \int_{r_m}^{r_{m+1}} \rho^{-\gamma} \int_{r_m}^{\rho} s^{-\gamma} \exp\left[-\lambda_m \frac{\rho - s}{\rho s}\right] ds d\rho < C m^{2(1-\gamma)/\alpha - 2}
\]
If $\alpha = 2/3 - \gamma > 3/2 - \alpha$, then
\[
\|\alpha_{m+1}\| \geq \|\alpha_m\| - c m^{-1-}
\]
Taking $d$ large and taking different $\gamma' \in (3/2 - \alpha, \gamma)$ in all estimates above, we can make sure that the constant $c$ is small with respect to $\|\alpha_1\| = 1$ and therefore the iteration of the last inequality yields (64).

To prove (65), we note that $\Psi(\rho, r) = \tilde{U}^*(\rho, r)$ solves
\[
\partial_\rho \Psi(\rho, r) = -\left[\frac{B_2(\rho)}{\rho^2} - i\tilde{V}(\rho)\right] \Psi(\rho, r), \quad \Psi(r, r) = I, \quad \rho < r
\]
Since $\tilde{V}$ is compactly supported, we can take $r \to \infty$ and consider $w(\rho) = \Psi(\rho, \infty)\eta$.
\[
w' = -\left[\frac{B_2(\rho)}{\rho^2} - i\tilde{V}(\rho)\right] w, \quad w(\infty) = \eta
\]
(67)
Multiply the both sides by $w$ take the real part and integrate. We have
\[
\|w(t)\|^2 + 2 \int_0^\infty \frac{\langle B_2(s)w, w\rangle}{s^2} ds = \|\eta\|^2, \quad 0 < t < \infty
\]
(68)
Then, multiplication of (67) by $w'$ and integration from $t$ to $\infty$ yields
\[
\int_t^\infty \|w'(s)\|^2 ds = -\int_t^\infty \frac{\langle B_2(s)w, w'\rangle}{s^2} ds + i \int_t^\infty \langle \tilde{V}(s)w, w'\rangle ds
\]
(69)
Denote the first term by $I$. Then, integration by parts gives
\[
\text{Re} I = \frac{1}{2} \left[ \frac{\langle B_2(t)w, w\rangle}{t^2} - 2 \int_t^\infty \frac{\langle B_2(s)w, w\rangle}{s^3} ds + i \int_t^\infty \langle \tilde{V}(s)w, w'\rangle ds \right]
\]
The first term is nonpositive. For the second one, (68) yields
\[
\int_t^\infty \frac{\langle B_2(s)w, w\rangle}{s^3} ds < t^{-1} \|\eta\|^2
\]
The third term can be bounded by
\[ \sum_{l,m,r_m \geq t} |w_l(r_m)|^2 = C\|\eta\|^2 \sum_{m \geq t} m^{2-2}\alpha = \bar{o}(1)\|\eta\|^2 \] (70)
since \( r_m \sim m^{1/\alpha} \) and we also used [68] once again to estimate the sum in \( l \) that corresponds to eigenspace of each \( \lambda_m \) for different values of \( r_m \).

The second term in [69] can be estimated by Cauchy-Schwarz since \( \|\tilde{V}\| \in L^2(1,\infty) \). Taking the real part of [69] yields
\[ \int_t^\infty \|w'(s)\|^2 ds < C\|\eta\|^2 \left[ \bar{o}(1) + \int_t^\infty \|\tilde{V}(s)\|^2 ds \right] \]
This inequality yields [65].

We will need the following statement later on. Recall that \( \tilde{V}_{(m)}(r) = \tilde{V}(r)\cdot \chi_{r\textgreater m} \).

**Lemma 3.13.** Let \( f(r) \in L^2(\mathbb{R}^+, L^2(\Sigma)) \) and \( k \in \mathbb{C}^+ \). Introduce \( \psi = [\tilde{P}(k)]^{-1}f \) and \( \psi_{(m),b} = [\tilde{P}_{(m),b}(k)]^{-1}f \). Then, for any \( \tau > 0 \),
\[ \|\psi_{(m),b}(\tau) - \psi(\tau)\|_{L^2(\Sigma)} \rightarrow 0, \|\psi'_{(m),b}(\tau) - \psi'(\tau)\|_{L^2(\Sigma)} \rightarrow 0 \]

**Proof.** From the second resolvent identity, we have
\[ \psi - \psi_{(m),b} = [\tilde{P}_{(m),b}(k)]^{-1} \left[ k\tilde{V}(d)\psi - \frac{B_{2,b}(r)}{r^2}\psi \right] \]
Since \( \psi \in H^2(\mathbb{R}^3) \), we have
\[ \frac{B}{r^2} \psi \in L^2(\mathbb{R}^+, L^2(\Sigma)) \]
and therefore
\[ \|\psi_{(m),b}\|_{L^2(\mathbb{R}^+, L^2(\Sigma))} \rightarrow 0 \]
as \( m, b \rightarrow \infty \). Compare two equations
\[ -\frac{d^2}{dr^2}\psi - \frac{B_2(r)}{r^2}\psi = k^2\psi + f \]
and
\[ -\frac{d^2}{dr^2}\psi_{(m),b} - \frac{B_{2,b}(r)}{r^2}\psi_{(m),b} + k\tilde{V}(d)\psi_{(m),b} = k^2\psi_{(m),b} + f \]
Now, the Theorem for traces of \( H^2(\mathbb{R}^3) \) functions written in spherical coordinates yields the statement of the Lemma. \( \square \)

Consider spherically symmetric function \( f \) having support on \( 0 < r < 1 \) such that \( \|f\|_2 = 1 \). Let \( \psi^0(r) = (\tilde{H}^0 - k^2)^{-1}f, \mu^0(r) = \exp(-ikr)\psi^0(r) \), and \( A^0(k) = \lim_{r \rightarrow \infty} \mu^0(r) \). Since \( f \) is spherically symmetric, \( \mu^0(r) \) is spherically symmetric as well and \( A^0(k) \) is nonzero function entire in \( k \).

**Lemma 3.14.** Let \( f \) be spherically symmetric with support in \([0,1]\), \( \alpha = 2/3- \) and \( \gamma \geq 3/2 - \alpha \). Then, for any \( k \in \mathbb{C}^+ \) which is not zero of \( A^0(k) \), there is \( d > 0 \) such that
\[ \|J_{\tilde{V}(d),R,b}(k,\xi)\| \geq \delta(k,d,V,T_1,f) > 0 \] (71)
uniformly in \( R > R_0, b > R^\alpha, \) and \( |\xi| < T_1 \).
Proof. Without loss of generality, we again assume that \( k = i/2 \), \( \xi = -2 \). Then, the equation for \( \mu \) can be rewritten as (we suppress the dependence of \( \mu \) on \( d \), \( b \), and \( R \))

\[
\mu' = \frac{B_{2,b}(r)}{r^2} u + i\tilde{V}_{(d),R}\mu + \mu'' + e^{r/2} f
\]

The support of \( f \) is within the interval \((0,1)\) and we therefore have

\[
\mu(r) = \tilde{U}_{(d),R,b}(\tau,r)\mu(\tau) + \int_{\tau}^{r} \tilde{U}_{(d),R,b}(\rho,r)\mu''(\rho)d\rho
\]

By making \( d \) and \( b \) large, we can make sure that \( \mu_{(d),R,b}(\tau) \) is close to \( \mu_0(\tau) \) uniformly in \( R > R_0 \) (by Lemma 3.13). On the other hand, \( \mu_0(\tau) \sim A^0(i/2) \neq 0 \) as \( \tau \) is large. Then, the absolute value of the first term can be controlled from below by Lemma 3.12. The second term can be made arbitrarily small if large \( \tau, d, b \) are fixed and \( r \to \infty \). Indeed, its limit as \( r \to \infty \) is equal to

\[
\int_{\tau}^{\infty} \tilde{U}_{(d),R,b}(\rho,\infty)\mu''(\rho)d\rho = I_1 + I_2
\]

where

\[
I_1 = -\tilde{U}_{(d),R,b}(\tau,\infty)\mu'(\tau)
\]

and

\[
I_2 = -\int_{\tau}^{\infty} \partial_\rho \tilde{U}_{(d),R,b}(\rho,\infty)\mu'(\rho)d\rho
\]

By fixing \( \tau, d, b \) large \((\tau < d)\), we can make \( I_1 \) arbitrarily small because \((\mu_0)'(\tau)\) tends to zero at infinity and \( \|\mu'(\tau) - (\mu_0)'(\tau)\|_2 \to 0 \) as \( d, b \to \infty \) (by Lemma 3.13). Thus, we are left only with \( I_2 \) to estimate. We have

\[
\|I_2\| \leq \max_{\|\eta\|_{L^2(\Sigma)} = 1} \left| \int_{\tau}^{\infty} \partial_\rho \tilde{U}_{(d),R,b}(\rho,\infty)\mu'(\rho)d\rho, \eta \right| \leq \sup_{\|\eta\|_{L^2(\Sigma)} = 1} \int_{\tau}^{\infty} \left| (\mu'(\rho), \partial_\rho \tilde{U}_{(d),R,b}(\rho,\infty)\eta) \right| d\rho
\]

By Cauchy-Schwarz and \((61)\), we have

\[
\|I_2\| \leq C \sup_{\|\eta\|_{L^2(\Sigma)} = 1} \left[ \int_{\tau}^{\infty} \|\partial_\rho \tilde{U}_{(d),R,b}(\rho,\infty)\eta\|^2 d\rho \right]^{1/2}
\]

and the last integral can be made arbitrarily small by choosing \( \tau \) large (see \((65)\)).

Proof of Theorem 3.3. The proof repeats the arguments given before. We have uniform control over \( \|J_{R,b}\| \) provided by \((60)\) and \((71)\). These estimates, \((59)\), and Lemma 3.8 allow to use the subharmonicity argument to get necessary bounds for the entropy.
4. Appendix: Combes-Thomas inequality.

Many properties of $P(k)$ are similar to those of general Schrödinger operators. For completeness of discussion on decay of Green’s function, we prove the analog of the so-called Combes-Thomas inequality (see, e.g. [9]). It gives a general uniform bound on Green’s function of $P(k)$. We do not have to use it to prove a.c. of the spectrum but we think it is interesting in itself. For the next Theorem, we assume $\xi = 1$.

**Theorem 4.1.** Let $V(x) \in L^\infty(\mathbb{R}^3)$ and $k \in \mathbb{C}^+$. Then,

$$\|\chi_{|x-x_1|<1}P^{-1}(k)\chi_{|x-x_1|<1}\|_{2,2} \leq C(k, \gamma) \exp(-\gamma|x_1-x_2|)$$

(72)

for any $x_1(2) \in \mathbb{R}^3$ and any $\gamma \in (0, \nu \Im k)$ (with $\nu$ – some universal constant).

**Proof.** We use the standard weight. Consider any $a \in \mathbb{R}^3$ and operator $P_a(k) = -\Delta - 2a\nabla - |a|^2 + kV - k^2 = e^{-ax}P(k)e^{ax}$

(73)

on $H^2(\mathbb{R}^3)$. It is easy to show that this operator is closed and $P_a(k) = P_{-a}(\bar{k})$. The last equality in (73) is justified for, e.g., $H^2(\mathbb{R}^3)$ functions with compact support.

Moreover, if $\|f\| = 1$, then

$$(P_a(k)f, f) = -(k - k_1)(k - k_2)$$

where

$$k_{1(2)} = c_1 \pm \sqrt{c_1^2 + 4c_2}$$

with

$$c_1 = \int V|f|^2dx, \quad c_2 = \int |
abla f|^2dx - |a|^2 - 2\int a\nabla \bar{f}f dx$$

Write

$$c_1^2 + 4c_2 = \alpha + i\beta$$

where

$$\beta = -8\text{ Im} \left[ \int a\nabla \bar{f}f dx \right]$$

and

$$\alpha = \left[ \int V|f|^2dx \right]^2 + 4\int |
abla f|^2dx - 4|a|^2$$

since

$$\text{Re} \int a\nabla \bar{f}f dx = 0$$

We are interested in the imaginary part of the square root of $\alpha + i\beta$. The inequality $\alpha \geq -4|a|^2$ is always true. If $\alpha \leq 0$, then

$$\int |
abla f|^2dx \leq |a|^2$$

and therefore

$$|\beta| \leq 8|a| \cdot \|\nabla f\|_2 \leq 8|a|^2$$

So,

$$|\text{Im} \sqrt{\alpha + i\beta}| \leq |\sqrt{\alpha + i\beta}| \leq (80)^{1/4}|a|$$

If, on the other hand, $\alpha > 0$, then there is $\kappa \in \mathbb{R}$, so that $\alpha = |a|^\kappa$ and

$$\|\nabla f\|^2_2 \leq |a|^2 + |a|^\kappa/4$$
For imaginary part of square root

\[ \text{Im} \sqrt{\alpha + i\beta} = \frac{2^{-1/2}|\beta|}{(\alpha + \sqrt{\alpha^2 + \beta^2})^{1/2}} \]

and this function increases in |\beta|. Moreover,

\[ |\beta| \leq 8|a| \left( |a|^2 + |a|^\kappa / 4 \right)^{1/2} \]

Thus,

\[ \text{Im} \sqrt{\alpha + i\beta} \leq C \frac{|a|(|a|^2 + |a|^\kappa)^{1/2}}{(|a|^2 + |a|^4 + |a|^2 + \kappa)^{1/4}} < C|a| \]

Consequently,

\[ |\text{Im} k_{1(2)}| \leq C_u |a| \]

where \( C_u \) is a universal constant (we believe more accurate analysis should yield \( C_u = 1 \)). That implies, of course, that \( \sigma(P_a(k)) \) lies inside the strip |Im\( k | \) < \( C_u |a| \).

**Lemma 4.1.** For any function \( f \in L^2(\mathbb{R}^3) \) with compact support and any \( k \) outside the strip \( |\text{Im} k| < C_u |a| \), we have

\[ \exp(ax)P_a^{-1}(k)\exp(-ax)f = P_a^{-1}(k)f \quad (74) \]

**Proof.** Consider \( \mathcal{R} = \text{Ran} P(k)|\mathcal{L} \), where \( \mathcal{L} \) denotes the linear manifold of \( H^2(\mathbb{R}^3) \) functions with compact support. For any \( f \in \mathcal{R} \), \( (74) \) is true just because the last equality of \( (73) \) is true for functions in \( \mathcal{L} \).

Take arbitrary open ball \( \Omega \) and those functions from \( \mathcal{L} \) that are supported inside \( \Omega \). Denote the linear manifold of these functions by \( \mathcal{L}_\Omega \). Operator \( P(k) \) defined on \( \mathcal{L}_\Omega \) can be closed to \( P_\Omega(k) = -\Delta_\Omega + kV - k^2 \) with \( \mathcal{D}[P_\Omega(k)] = H^2(\Omega) \), where \( -\Delta_\Omega \) is Laplace with Dirichlet b.c. on \( \partial\Omega \). Let \( \text{Ran} P(k)|\mathcal{L}_\Omega = \text{Ran} P_\Omega(k)|\mathcal{L}_\Omega \). Then, \( \mathcal{R}_\Omega = L^2(\Omega) \). All that can be justified in the standard way.

Now, consider any \( f \in L^2(\mathbb{R}^3) \) with support, say, within some ball \( \Omega \). One can find \( f_n \in \mathcal{R}_\Omega \) such that \( f_n \to f \) in \( L^2(\Omega) \). Since any function from \( \mathcal{R}_\Omega \) continued to \( \mathbb{R}^3 \) as zero is also from \( \mathcal{R} \), \( (74) \) is true for each \( f_n \). On the other hand, \( f_n \) is supported within \( \Omega \) and therefore \( \exp(-ax)f_n \to \exp(-ax)f \) in \( L^2(\mathbb{R}^3) \). So,

\[ P_a^{-1}(k)\exp(-ax)f_n \to P_a^{-1}(k)\exp(-ax)f, \quad P_a^{-1}(k)f_n \to P_a^{-1}(k)f \]

where the convergence is in \( L^2(\mathbb{R}^3) \). Thus, for arbitrary \( h \in L^2(\mathbb{R}^3) \) with compact support

\[ \langle \exp(ax)P_a^{-1}(k)\exp(-ax)f_n, h \rangle \to \langle \exp(ax)P_a^{-1}(k)\exp(-ax)f, h \rangle \]

and

\[ \langle \exp(ax)P_a^{-1}(k)\exp(-ax)f, h \rangle = \langle P_a^{-1}(k)f, h \rangle \]

Since \( h \) was arbitrary and \( \exp(ax)P_a^{-1}(k)\exp(-ax)f \) is in \( L^2_{\text{loc}} \) apriori, we have the statement of the Lemma. \( \square \)

To finish the proof of the Theorem, assume \( x_1 = 0 \) without loss of generality. Then, take \( a = -|a|x_2|/|x_2| \) with \( |a| < \nu \text{Im} k, \nu = C_u^{-1} \). Let \( f \) be any \( L^2 \) function supported within the unit ball around 0. One can then use Lemma and bound on \( ||P_a^{-1}(k)|| \) to get \( (72) \). \( \square \)
References

[1] P. Deift, R. Killip, On the absolutely continuous spectrum of one-dimensional Schrödinger operators with square summable potentials. Comm. Math. Phys. Vol. 203, 1999, no 2, 341–347.
[2] S.A. Denisov, Continuous analogs of polynomials orthogonal on the unit circle and Krein systems. IMRS Int. Math. Res. Surv. 2006, Art. ID 54517, 148 pp.
[3] S.A. Denisov, On the absolutely continuous spectrum of Dirac operator, Communications in PDE, Vol. 29, 2004, no. 9–10, 1403–1428.
[4] S.A. Denisov, Absolutely continuous spectrum for multidimensional Schrödinger operators, IMRN, 2004, no. 74, 3963–3982.
[5] S.A. Denisov, On the preservation of absolutely continuous spectrum for Schrödinger operators, J. Funct. Anal., Vol. 231, 2006, 143–156.
[6] S.A. Denisov, A. Kiselev, Spectral properties of Schrödinger operators with decaying potentials, to appear in Festschrift for B. Simon’s 60-th birthday, Proceedings of Symposia in Pure Mathematics.
[7] S.A. Denisov, An evolution equation as the WKB correction in long-time asymptotics of Schrödinger dynamics, (to appear in Comm. PDE).
[8] S.A. Denisov, Wave propagation through sparse potential barriers, (to appear in Comm. Pure Appl. Math.).
[9] F. Germinet, A. Klein, Operator kernel estimates for functions of generalized Schrödinger operators, Proc. Amer. Math. Soc. Vol. 131, 2003, no. 3, 911–920.
[10] R. Killip, Perturbations of one-dimensional Schrödinger operators preserving the absolutely continuous spectrum, IMRN, 2002, no. 38, 2029–2061.
[11] R. Killip, B. Simon, Sum rules for Jacobi matrices and their applications to spectral theory, Annals of Math., Vol. 158, 2003, 253–321.
[12] M.G. Krein, Continuous analogues of propositions on polynomials orthogonal on the unit circle, (Russian) Dokl. Akad. Nauk SSSR (N.S.), Vol. 105, 1955, 637–640.
[13] A. Laptev, S. Naboko, O. Saffronov, A Szegö condition for a multidimensional Schrödinger operator, J. Funct. Anal., Vol. 219, 2005, no.2, 285-305.
[14] A. Laptev, S. Naboko, O. Saffronov, Absolutely continuous spectrum of Schrödinger operators with slowly decaying and oscillating potentials, Comm. Math. Phys., Vol. 253, 2005, no.3, 611-631.
[15] A. Laptev, O. Saffronov, Absolutely continuous spectrum of matrix valued Schrödinger operators, Advances in differential equations and mathematical physics (Birmingham, AL, 2002), 215–221, Contemp. Math., 327.
[16] A.S. Markus, Introduction to the spectral theory of polynomial operator pencils, Translations of Mathematical Monographs, Vol. 71, AMS, 1988.
[17] A.A. Nabiev, Inverse scattering problem for the Schrödinger-type equation with a polynomial energy-dependent potential, Inverse Problems, Vol. 22, 2006, no. 6, 2055–2068.
[18] G. Perelman, Stability of the absolutely continuous spectrum for multidimensional Schrödinger operators, IMRN, 2005, no. 37, 2289–2313.
[19] M. Reed, B. Simon, Methods of modern mathematical physics. III. Scattering theory. Academic Press, New York-London, 1979
[20] M. Reed, B. Simon, Methods of modern mathematical physics. IV. Analysis of operators. Academic Press, New York-London, 1978.
[21] L. Sakhnovich, On the spectral theory of a class of canonical differential systems, Funktsional. Anal. i Prilozhen., Vol. 34, 2000, no. 2, 50–62, 96 (Russian); English transl. in: Funct. Anal. Appl., 34, 2000, no. 2, 119–128.
[22] B. Simon, Schrödinger operator in the 21-st century, Imp. Coll. Press, London, 2000, 283–288.

University of Wisconsin-Madison, Mathematics Department, 480 Lincoln Dr., Madison, WI, 53706-1388, USA, e-mail: denissov@math.wisc.edu