Anomalous Lagrangians and the radiative muon capture in hydrogen

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Abstract

The structure of an anomalous Lagrangian of the $\pi\rho\omega$ system is investigated within the hidden local $SU(2)_R \times SU(2)_L$ symmetry approach. The interaction of the external electromagnetic and weak vector and axial–vector fields with the above hadron system is included.

The Lagrangian of interest contains the anomalous Wess–Zumino term following from the well known Wess–Zumino–Witten action and six independent homogenous terms. It is characterized by four constants that are to be determined from a fit to the data on various elementary reactions. Present data allows one to extract the constants with a good accuracy.

The homogenous part of the Lagrangian has been applied in the study of anomalous processes that could enhance the high energy tail of the spectrum of photons, produced in the radiative muon capture in hydrogen. It should be noted that recently, an intensive search for such enhancement processes has been carried out in the literature, in an attempt to resolve the so called "$g_P$ puzzle": an $\approx 50\%$ difference between the theoretical prediction of the value of the induced pseudoscalar constant $g_P$ and its value extracted from the high energy tail of the photon spectrum, measured in the precision TRIUMF experiment.

Here, more details on the studied material are presented and new results, obtained by using the Wess–Zumino term, are provided.
1 Introduction

The theory of the weak nuclear interaction aims to describe nuclear phenomena induced by the external interaction, that is mediated by the intermediate bosons $W^\pm$ and $Z^0$ of the Standard Model [1-2]. At low and intermediate energies, the strangeness conserving semileptonic nuclear interaction Hamiltonian is of the current-current form [3-6]

\[
\mathcal{H}_W = -\frac{G_F}{\sqrt{2}} \cos \theta_C \left[ \bar{J}^\alpha_{W,\mu}(q_1) l^\mu + \text{h.c.} \right].
\] (1)

Here the weak interaction constant $G_F/(\hbar c)^3 \approx 1.16637(1) \times 10^{-5} \text{GeV}^{-2}$ [7], the Cabibbo angle \cite{8} $\cos \theta_C = 0.9738(5)$ [7], $l_\mu$ is the lepton current and the operator of the weak nucleon current is

\[
\bar{J}^a_{W,\mu}(q_1) = \bar{J}^a_{V,\mu}(q_1) + \bar{J}^a_{A,\mu}(q_1)
\]

\[
= i \left( g_V(q_1^2) \gamma_\mu - \frac{g_M(q_1^2)}{2M} \sigma_{\mu\nu} q_{1\nu} - g_A(q_1^2) \gamma_\mu \gamma_5 + i \frac{g_P(q_1^2)}{m_l} q_{1\mu} \gamma_5 \right) \frac{\tau^a}{2},
\] (2)

where $M(m_l)$ is the nucleon (lepton) mass and $q_{1\mu} = p'_\mu - p_\mu$, where $p'_\mu (p_\mu)$ is the 4-momentum of the final (initial) nucleon.

The least known of the form factors entering Eq. (2) is the induced pseudoscalar form factor $g_P(q_1^2)$. The presence of this form factor in the weak nucleon current is a consequence of the intimate relation between the strong and weak interaction processes. The contribution of the pion pole to $g_P(q_1^2)$ is

\[
g_P(q_1^2) = -2g_{\pi NN}f_\pi m_l \Delta^\pi_F(q_1^2),
\] (3)

where $g_{\pi NN}$ is the pseudoscalar $\pi NN$ coupling constant \footnote{Modern phenomenological nucleon-nucleon potentials use $g_{\pi NN} \approx 13.0$.}, $f_\pi$ is the pion decay constant \footnote{According to Ref. [7], $\sqrt{2}f_\pi^+ = 130.7 \pm 0.1 \pm 0.36 \text{MeV}$, thus providing $f_\pi^+ = 92.7 \pm 0.3 \text{MeV}$; let us note, however, that $\sqrt{2}f_\pi^0 = 130 \pm 5 \text{MeV}$ [8].} and $\Delta^\pi_F(q_1^2) = 1/(m_\pi^2 + q_1^2)$ is the pion propagator.

The matrix element of the axial current $\bar{J}^a_{A,\mu}$ should satisfy the partial conservation of the axial current (PCAC)

\[
\bar{u}(p')q_{1\mu} J^a_{A,\mu} u(p) = \bar{u}(p') \left[ 2M g_A F_A(q_1^2) - \frac{g_P(q_1^2)}{m_l} q_{1\mu} \gamma_5 \right] \frac{\tau^a}{2} u(p) = if_\pi m_\pi^2 \Delta^\pi_F(q_1^2) M^a_\pi,
\] (4)

where $M^a_\pi$ is the pion absorption amplitude. We also put

\[
g_A(q_1^2) = g_A F_A(q_1^2), \quad g_A \equiv g_A(0) = -1.2695 \pm 0.0029.
\] (5)

The value of the constant $g_A$ is taken from Ref. [7]. In order to fulfil Eq. (4) one subtracts from the induced pseudoscalar form factor a piece

\[
\Delta g_P(q_1^2) = -2g_{\pi NN}f_\pi m_l \frac{1}{q_1^2} \left[ 1 + \frac{M g_A}{g_{\pi NN}f_\pi} F_A(q_1^2) \right]
\]

\[
\approx 2M g_A m_l \frac{1}{q_1^2} \left[ 1 - F_A(q_1^2) \right] \approx \frac{1}{3} M g_A m_l r_A^2.
\] (6)
Here $r_A$ is a nucleon axial radius measured independently in the quasi-elastic neutrino scattering, $r_A^2 = 0.42 \pm 0.04 \text{ fm}^2$ [9], and in the charged pion electroproduction, $r_A^2 = 0.403 \pm 0.030 \text{ fm}^2$ [10].

In deriving Eq. (6), we use the Goldberger–Treiman relation

$$M |g_A| = g_{\pi NN}(0)f_\pi,$$

as being satisfied exactly and we assume a weak dependence of the couplings on the momentum transfer. Using the constants $g_A$, $f_{\pi^+}$ and $g_{\pi NN}$ as given above and the mean nucleon mass $M = 938.92 \text{ MeV}$, one observes that the left- and right hand sides of Eq. (7) differ by 1.1%.

The prediction for the form factor $g_P$, following from the above discussed material, is given for the ordinary muon capture (OMC) in the hydrogen,

$$\mu^- + p \rightarrow \nu_\mu + n,$$

as [11]

$$g_P(q_1^2 = 0.877m_\mu^2) = \frac{2Mm_\mu}{0.877m_\mu^2 + m_\pi^2} g_A = 6.87 g_A = -8.72,$$

whereas the correction, demanded by the PCAC is [11]

$$\Delta g_P = 0.34 g_A = -0.43.$$

This correction is obtained by using the dipole form factor for $F_A(q_1^2)$ and the axial mass $m_A = 1.077 \text{ GeV}$, extracted from the data in Ref. [10], which is equivalent to taking the nucleon axial radius $\langle r_A^2 \rangle^{1/2} = 0.635 \text{ fm}$ [10]. Then the resulting value of the induced pseudoscalar constant, as predicted by the PCAC, is

$$g_P^{\text{PCAC}} = 6.44 g_A = -8.29.$$  

Let us note that this value of $g_P$, obtained from the PCAC constraint, is of fundamental importance. It is confirmed by the calculations, performed within the framework of the effective field theory [12, 13, 14] that incorporates the chiral symmetry of the quantum chromodynamics.

The influence of the induced pseudoscalar form factor $g_P$ on observables and the related extraction of this quantity from experiments was studied intensively in the OMC and radiative muon capture (RMC) in nuclei and in the electroproduction of charged soft pions. This activity has recently been reviewed in Refs. [15, 16, 17, 18].

Let us note here first that in our opinion, the electroproduction of charged soft pions cannot provide any information on the induced pseudoscalar form factor $g_P$. Attempts to study it [16, 20] in the reaction

$$e + p \rightarrow e' + \pi^+ + n,$$

at the threshold stem from the soft pion electroproduction amplitude derived from the low energy theorem. As usually derived, this amplitude contains the current–current amplitude and the weak axial nucleon current. In the next step, one calculates the contributions to the current–current amplitude. If one restricts oneself only to the contributions due to the nucleon Born terms, one really has an
electroproduction amplitude containing the $g_P$ form factor. However, as shown in detail [21], the correct calculation of the contribution to the current–current amplitude due to the pion and the heavy axial meson emitted in the t–channel, leads to a complete cancellation of the weak axial nucleon current in the electroproduction amplitude. Simultaneously, a contact term, containing $g_A$ form factor, and the pion pole production amplitude appear, as expected from physical intuition.

Since the extraction of the information on $g_P$ from the data on the OMC and RMC in nuclei is extensively reviewed [15, 16, 17, 18], we restrict ourselves only to necessary comments on the RMC in hydrogen, allowing us to proceed to the material that is the subject of this review.

For the OMC in the hydrogen, the induced pseudoscalar form factor $g_P$ only for one value of the momentum transfer needs to be considered. In contrast to it, the RMC in hydrogen

$$\mu^- + p \longrightarrow \nu + \gamma + n,$$

(13)

allows one to study the momentum dependence of $g_P$ in a certain interval of the values of the momentum transfer.

The RMC amplitude generally consists of two parts. One part is due to the lepton radiation, another one is due to the hadron radiation. In the lepton radiation amplitude, the weak form factors depend on the four–momentum transfer $q^L = p_1 - p'_1 = \nu + k - \mu = - q_1$, whereas the momentum dependence of the weak form factors entering the hadron radiation amplitude is given by $q^N = \nu - \mu = q^L - k$. At the end of the photon spectrum, we have $(q^L)^2 \approx + m^2_\mu$ and $(q^N)^2 \approx - m^2_\mu$. Thus one obtains the value of $g_P$ in the hadron radiation amplitude larger by a factor of about 3 in comparison with the value of $g_P$ in the lepton radiation amplitude. This enhancement factor makes the reaction (13) attractive for the study of the sensitivity of the high energy tail of the photon spectrum to the form factor $g_P$.

The photon spectrum was measured in the reaction (13) in a TRIUMF experiment [22, 23]. The comparison of the measured spectrum with the calculations [24, 25] provided the value of the pseudoscalar coupling constant that is by about 50% smaller than its value [11] predicted by the PCAC. In these calculations, a relativistic RMC amplitude was used, derived from Feynman tree graphs. This amplitude includes also the contribution from the $\Delta(1232)$ excitation process. Besides, it satisfies approximately the Ward–Takahashi identities, generally derived in Ref. [26]. Subsequent studies, aiming to find an additional enhancement mechanism of the tail of the photon spectrum, were performed by several authors [11, 19, 27, 28, 29, 30]. Only the model taking into account the off–shell degrees of freedom of the $\Delta(1232)$ isobar is able to provide enough enhancement of the photon spectra [11].

One of the possible contributions to the enhancement mechanism has been studied in detail in Refs. [11, 31, 32]. This is the contribution of the processes described by anomalous Lagrangians. We next provide a detailed account of the construction of such a Lagrangian for the $\pi-\rho-\omega-a_1$ system, then we present the structure of the RMC amplitude arising from it. We show that this part of the full RMC amplitude satisfies the PCAC constraint by itself.

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3The notations are obvious: $\nu, \mu, k$ is the four–momentum of the neutrino, muon and photon, respectively.
2 Anomalous Lagrangian of the $\pi\rho\omega a_1$ system

As in the OMC, in order to calculate the capture rate, one needs to know the effective Hamiltonian. The velocity independent part of it is

$$H_{\text{eff}}^{(0)} = \frac{1}{\sqrt{2}m_\mu} (1 - \vec{\sigma}_l \cdot \hat{\nu}) \left[ g_1 (\vec{\sigma}_l \cdot \vec{\varepsilon}) + g_2 (\vec{\sigma} \cdot \vec{\varepsilon}) + g_3 i (\vec{\sigma} \cdot \vec{\varepsilon} \times \vec{\sigma}_l) + g'_4 (\vec{\sigma}_l \cdot \vec{\varepsilon}) \left( \vec{\sigma} \cdot \hat{k} \right) + g''_4 (\vec{\sigma} \cdot \vec{\varepsilon}) \left( \vec{\sigma} \cdot \hat{k} \right) \cdot (\vec{\varepsilon} \cdot \hat{\nu}) \right] + g'_5 i \left( \vec{\sigma} \cdot \hat{k} \times \vec{\varepsilon} \right) + g''_5 i \left( \vec{\sigma} \cdot \vec{\varepsilon} \times \vec{\varepsilon} \right) + g'_6 \left( \vec{\sigma}_l \cdot \vec{\varepsilon} \right) \left( \vec{\sigma} \cdot \vec{\varepsilon} \right) + g''_6 \left( \vec{\sigma}_l \cdot \vec{\varepsilon} \right) \left( \vec{\sigma} \cdot \vec{\varepsilon} \right) + g'_7 \left( \vec{\sigma}_l \cdot \vec{\varepsilon} \right) \left( \vec{\sigma} \cdot \vec{\varepsilon} \right) + g''_7 \left( \vec{\sigma}_l \cdot \vec{\varepsilon} \right) \left( \vec{\sigma} \cdot \vec{\varepsilon} \right) + g'_8 \left( \vec{\sigma}_l \cdot \vec{\varepsilon} \right) \left( \vec{\sigma} \cdot \vec{\varepsilon} \right) + g''_8 \left( \vec{\sigma}_l \cdot \vec{\varepsilon} \right) \left( \vec{\sigma} \cdot \vec{\varepsilon} \right) + g'_9 \left( \vec{\sigma}_l \cdot \vec{\varepsilon} \right) + g''_9 \left( \vec{\sigma}_l \cdot \vec{\varepsilon} \right) + g'_{10} \left( \vec{\sigma}_l \cdot \vec{\varepsilon} \right) \left( \vec{\sigma} \cdot \vec{\varepsilon} \right) + g''_{10} \left( \vec{\sigma}_l \cdot \vec{\varepsilon} \right) \left( \vec{\sigma} \cdot \vec{\varepsilon} \right) + g'_{11} \left( \vec{\sigma}_l \cdot \vec{\varepsilon} \right) \left( \vec{\sigma} \cdot \vec{\varepsilon} \right) + g''_{11} \left( \vec{\sigma}_l \cdot \vec{\varepsilon} \right) \left( \vec{\sigma} \cdot \vec{\varepsilon} \right) \right].$$ (14)

Here $\vec{\sigma}_l$ ($\vec{\sigma}$) are the lepton (nucleon) spin Pauli matrices, $\hat{\nu}$ ($\hat{k}$) is the unit vector in the direction of the neutrino (photon) momentum vector $\vec{\nu}$ ($\vec{k}$) and $\vec{\varepsilon}_\lambda$ is the photon polarization. The most important form factors are $g_1$, $g_2$ and $g_3$. All other $g_i$ contain at least one damping factor $1/M$.

The aim of any model Lagrangian of the interaction of a hadron system with an external electroweak fields is to provide the form factors $g_i$ entering Eq. (14). Usually, one considers processes, described by the normal Lagrangians [19, 25, 27, 28, 34], where a natural parity of the in- and outcoming channels does not change. The natural parity of a particle is defined for bosons only and it is $P_n = P (-1)^J$, where $P$ is the intrinsic parity and $J$ is the spin of the particle. The natural parity of the channel is defined as the product of the natural parities of the channel particles.

The RMC amplitude, presented in Ref. [34], is derived from a non–anomalous Lagrangian of the $\pi\rho\omega a_1$ system that reflects the $SU(2)_L \times SU(2)_R$ hidden local symmetry [35, 36, 37, 38]. This amplitude extends the amplitude obtained from the low energy theorem to higher values of the photon and weak current momenta.

Here our goal is to construct the RMC amplitude which contributes to the anomalous processes. Let first discuss the generalities related to the construction of the necessary anomalous Lagrangian.

In meson physics, anomalous processes are defined as processes in which the natural parity is not preserved. The value of $P_n$ for some bosons is given in Table 1.

| boson | $P$ | $J$ | $P_n$ |
|-------|-----|-----|-------|
| $\sigma$ | +1 | 0 | +1 |
| $\pi$ | -1 | 0 | -1 |
| $\rho$ | -1 | 1 | +1 |
| $\omega$ | -1 | 1 | +1 |
| $a_1$ | +1 | 1 | -1 |
| $f_1(1285)$ | +1 | 1 | -1 |
| $\gamma$ | -1 | 1 | +1 |
Then the vertices $\rho \to \pi\pi$ and $a_1 \to \rho\pi$ are described by the non–anomalous Lagrangian, whereas the vertices $\rho \to \gamma\pi$ and $\omega \to \gamma\pi$ are anomalous and play an important role in describing the deuteron electromagnetic form factors. As we shall see soon, the anomalous vertices $a_1 \to \rho\omega$ and $\rho \to \omega\pi$ enter into the amplitudes for the process (13).

There exists a unique way to construct the interaction Lagrangian that would violate the natural parity and would simultaneously conserve the intrinsic parity and would be Lorentz invariant: the Levi–Civita pseudotensor should be used. So anomalous processes are defined as processes, described by a Lagrangian containing the pseudotensor $\varepsilon_{\alpha\beta\gamma\lambda}$. Let us note that this definition of the anomalous processes is more general than the definition using the natural parity, as it can be extended from pure meson processes also to processes, where mesons interact also with the gauge bosons of the electroweak interaction and subsequently, with fermions.

Let us note that the term ”anomalous” refers to the axial Abelian as well as non–Abelian anomaly, yielding an anomalous breaking of the chiral symmetry in the theory of quantum fields [2]. The first such chiral anomaly was observed in the width of the decay

$$\pi^0 \to \gamma\gamma. \tag{15}$$

The observed decay width turned out to be enhanced by three orders of magnitude in comparison with the one derived from the models incorporating spontaneously and explicitly broken chiral symmetry [1, 2]. Anomalous behavior of the decay width for the process (15) indicates the existence of a new mechanism of the chiral symmetry breaking. It was found [40, 41] that it is the regularization of the Feynman one–loop diagrams that produces the chiral symmetry breaking.

An elegant study of the chiral anomaly has later been performed within the path integral method in Ref. [42]. Non–invariance under the chiral transformations of the measure in the path integral over the fermion fields means that the functional integral is not invariant and the chiral symmetry is broken. This formally provides Jacobi determinant different from identity. It follows that the logarithm of such a determinant leads to an anomaly function [1, 2] that can be given in terms of the gauge fields of the electroweak interaction, but it does not depend on the quark fields. Finally, its presence produces in the effective Lagrangian the term

$$L_{\pi^0\gamma\gamma} = -\frac{e^2}{32\pi^2 f_\pi} \varepsilon_{\alpha\beta\gamma\lambda} F_{\alpha\beta}(x) F_{\gamma\lambda}(x) \pi^0(x), \quad F_{\alpha\beta}(x) = \partial_\alpha A_\beta(x) - \partial_\beta A_\alpha(x). \tag{16}$$

This Lagrangian can be qualified as anomalous in accord with both definitions of anomalous Lagrangians discussed above and it provides the correct value of the decay width for the process (15).

It is important to note that the form of the anomaly function is rather independent of the detailed features of the theory [2]. However, it should satisfy the Wess–Zumino consistency conditions [43] that depend only on the properties of the anomalously broken symmetry group. These conditions strongly restrict the form of the anomaly function, but they do not fix it uniquely.

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4The axial–vector anomaly is frequently called the chiral anomaly [39].
The independence of the anomaly function on the details of the theory is useful if one would like to pass from the quantum chromodynamics (QCD) to the low energy effective field theories given in terms of the Goldstone bosons and the gauge fields of the electroweak interaction. The consistency conditions will be the same as in QCD and it can be shown that the anomaly function will be the same, too \[44, 2\].

At the level of effective low energy Lagrangians, the term, breaking the chiral symmetry, should be present in the Lagrangian from the very beginning and it is called the anomalous Lagrangian. It follows from what we said above that its change under the infinitesimal chiral transformation is fixed by the anomaly function.

The form of the anomalous Lagrangian for the Goldstone bosons of the spontaneously broken global chiral group $SU(3)_L \times SU(3)_R$ was derived in Ref. \[43\]. However, this Lagrangian cannot be given in a simple closed form \[1, 2\], but it is possible for the anomalous action \[45\]. Without gauge fields, this action is invariant under the global transformations of the chiral group

$$G_g = [SU(3)_L \times SU(3)_R]_g,$$

(17)

whereas the related anomalous Lagrangian is not invariant. It is possible to extend this global symmetry to the local one by introducing the gauge fields of the electroweak interaction into the anomalous action, the fields of the photon and of the intermediate bosons $W^{\pm}$ and $Z^0$ \[15\]. Then the gauged Wess–Zumino–Witten action will contain terms, describing anomalous processes taking place between the $\pi^-$, $K$- and $\eta$ mesons in the presence of the external electroweak fields, including the Lagrangian \[16\], responsible for the decay \[15\].

Anomalous processes are important not only in the case of the Goldstone bosons, but are abundantly observed in the reactions with vector mesons, such as $\omega$ and $\rho$ mesons, e.g. the radiative decay of these mesons,

$$B \rightarrow \pi + \gamma, \quad B = \omega, \rho,$$

(18)

is the anomalous process, as it follows from table 1.

For an extension of the theory to incorporate the vector mesons, it is important to recognize that any such theory should be free of axial anomaly, as it follows from the gauge invariance \[2\]. In other words, the anomaly functions, arising in different sectors of the theory, should compensate among themselves. It turns out that this restriction does not allow one to incorporate the vector mesons into the anomalous Lagrangians as Yang–Mills gauge fields. This follows from the fact that in this case, the anomaly function, arising in the vector meson sector of the anomalous Lagrangian, differs from the one, derived from the QCD. Then such a theory is in contradiction with the fundamental postulate \[16\] that any effective low energy theory of hadrons should strictly respect symmetries imbedded in the QCD and the consequences following from it, including the mode in which these symmetries are broken. Since the anomaly function is a direct consequence of the anomalously broken chiral symmetry of the QCD, the same anomaly function should be immanent also to any anomalous hadron Lagrangian.

On the contrary, introducing the vector mesons within the framework of the hidden local symmetries (HLS) provides a theory of the interaction of the mesons with the electroweak fields that is free of the
above mentioned defects \[35, 36, 37, 47\]. Methodologically, the construction of the anomalous HLS
Lagrangian does not differ from that of the non–anomalous one. The extension of the symmetry \[17\]
by the group \(G_t \equiv [SU(3)_L \times SU(3)_R]_L\) of the local transformations needs introduction both of the
related massless gauge fields and non–physical compensators. In the next step, after the spontaneously
breaking of the extended symmetry, the compensator fields disappear, whereas the gauge fields acquire
the mass. These new gauge fields are identified with the physical vector and axial–vector mesons. Then
the procedure results in the appearance of the anomalous invariants, consisting of the fields of particles,
which anomalous processes one would like to describe of: of the Goldstone-, vector- and axial–vector
meson fields and of the fields of the electroweak interaction. In general, the anomalous Lagrangian
can be chosen as the sum of the Wess–Zumino–Witten Lagrangian and of the linear combination of
the anomalous invariants \[35, 36, 37\]. Thus, such a Lagrangian contains free parameters that can be
fixed by analyzing anomalous processes, or by imposing additional semi–phenomenological conditions.

The anomalous Lagrangian, containing both the vector- (\(\rho, \omega\)) and axial–vector (\(a_1, f_1\)) mesons,
was first constructed in Ref. \[37\]. Such a Lagrangian contains 14 invariants. In Ref. \[37\], only the
electromagnetic anomalous processes were studied. The extension of this model to the weak anomalous
processes was accomplished in Refs. \[31, 32\]. To carry out this step, it is advantageous to use the
anomalous invariants, constructed in Ref. \[37\] and to express the external gauge fields using both the
photon field and the boson fields \(W^\pm\) and \(Z^0\).

The most general anomalous action of the \(\pi\rho\omega a_1 f_1\) system reads \[37\]
\[
\Gamma_{an}[\xi_L, \xi_R, \xi_M, L, R, L, R] = \Gamma_{cov}^{WZW}[U, L, R] + \sum_{i=1}^{14} \int_{M^4} c_i L_i[\xi_L, \xi_R, \xi_M, L, R, L, R].
\] (19)

Here \(\Gamma_{cov}^{WZW}[U, L, R]\) is the covariant Wess-Zumino-Witten action containing pseudoscalars and the
electroweak fields. It already satisfies the anomaly constraints. Generally, the 14 independent (ho-
monicous) terms in the r. h. s. of Eq. (19) are given in Eqs. (3.8) and (3.9) of Ref. \[37\]. As the terms
\(L_1- L_8\) contain at least 4 particles in each vertex, only the terms \(L_9- L_{14}\) are of interest for our purpose.

The covariant Wess-Zumino-Witten anomalous action of pseudoscalars
reads \[35, 36, 45\]
\[
\Gamma_{cov}^{WZW}[U, L, R] = -i \frac{N_c}{240 \pi^2} \int_{M^5} Tr[\alpha^5]_{covariantized},
\] (20)
where \(N_c\) is the number of colors and \(\alpha\) is a differential one-form
\[
\alpha = (\partial_\mu U) U^\dagger dx_\mu, \quad U(x) = exp[-i\Pi^a(x)\tau^a/f_\pi] \equiv \xi^2, \quad (21)
\]
and \(L_\mu, R_\mu\) are the external gauge fields.

The contribution from the action (20) to the 3-point Lagrangian of interest is \[31\]
\[
\mathcal{L}_{WZW} = i \frac{e^2}{8\pi^2 f_\pi} \varepsilon_{\kappa\lambda\mu\nu}(\partial_\kappa \vec{B}_\lambda)(\partial_\mu \vec{V}_\nu \cdot \vec{\Pi}),
\] (22)
where \(\vec{B}_\lambda\) is an electroweak neutral field and \(\vec{V}_\nu\) is a weak vector field.
In the homogenous terms, we include both the electromagnetic and weak interactions, but we omit the field of the $f_1$ meson. Keeping only the 3–particle terms, the anomalous Lagrangian of the $\pi \rho \omega a_1$ system \cite{31,32} is obtained

$$\tilde{\mathcal{L}}_{an} = \sum_{i=7}^{10} \tilde{c}_i \tilde{L}_i , \quad (23)$$

where the $\tilde{L}_i$ terms are

$$\tilde{L}_7 = 2i g_\rho \varepsilon_{\kappa \lambda \mu \nu} \left\{ \partial_\kappa \omega_\lambda \left[ (g_\rho \bar{\rho}_\mu - e \bar{V}_\mu) \cdot \left( \frac{1}{f_\pi} \partial_\lambda \bar{\pi} + e \bar{A}_\nu \right) \right] 
+ (g_\rho \omega_\kappa - \frac{1}{3} e \mathcal{B}_\kappa) \left[ (\partial_\lambda \bar{\rho}_\mu) \cdot \left( \frac{1}{f_\pi} \partial_\nu \bar{\pi} + e \bar{A}_\nu \right) \right] \right\} , \quad (24)$$

$$\tilde{L}_8 = -2i g_\rho \varepsilon_{\kappa \lambda \mu \nu} \left\{ \partial_\kappa \omega_\lambda \left[ (g_\rho \bar{\rho}_\mu - e \bar{V}_\mu) \cdot (g_\rho \bar{A}_\nu + \frac{1}{2 f_\pi} \partial_\nu \bar{\pi}) \right] 
+ (g_\rho \omega_\kappa - \frac{1}{3} e \mathcal{B}_\kappa) \left[ (\partial_\lambda \bar{\rho}_\mu) \left( \frac{1}{f_\pi} \partial_\nu \bar{\pi} + e \bar{A}_\nu \right) \right] \right\} , \quad (25)$$

$$\tilde{L}_9 = 2i e \varepsilon_{\kappa \lambda \mu \nu} \left\{ \frac{1}{3} \partial_\kappa \mathcal{B}_\lambda \left[ (g_\rho \bar{\rho}_\mu - e \bar{V}_\mu) \cdot \left( \frac{1}{f_\pi} \partial_\nu \bar{\pi} + e \bar{A}_\nu \right) \right] 
+ (g_\rho \omega_\kappa - \frac{1}{3} e \mathcal{B}_\kappa) \left[ (\partial_\lambda \bar{V}_\mu) \left( \frac{1}{f_\pi} \partial_\nu \bar{\pi} + e \bar{A}_\nu \right) \right] \right\} , \quad (26)$$

$$\tilde{L}_{10} = -2i e \varepsilon_{\kappa \lambda \mu \nu} \left\{ \frac{1}{3} \partial_\kappa \mathcal{B}_\lambda \left[ (g_\rho \bar{\rho}_\mu - e \bar{V}_\mu) \cdot (g_\rho \bar{A}_\nu + \frac{1}{2 f_\pi} \partial_\nu \bar{\pi}) \right] 
+ (g_\rho \omega_\kappa - \frac{1}{3} e \mathcal{B}_\kappa) \left[ (\partial_\lambda \bar{V}_\mu)(g_\rho \bar{A}_\nu + \frac{1}{2 f_\pi} \partial_\nu \bar{\pi}) \right] \right\} . \quad (27)$$

External fields $\bar{V}$ and $\bar{A}$ correspond to the gauge fields of the Standard Model \cite{35}

$$\nu_\mu^\pm = -\mathcal{A}_\mu^\pm = \frac{1}{\sin \Theta_w} \mathcal{W}^\pm \mu \cos \Theta_c , \quad (28)$$

$$\nu_\mu^3 = \bar{B}_\mu + \cot (2 \Theta_w) Z_\mu = \mathcal{B}_\mu + \frac{1}{\sin (2 \Theta_w)} Z_\mu , \quad (29)$$

$$\mathcal{A}_\mu^3 = -\frac{1}{\sin (2 \Theta_w)} Z_\mu . \quad (30)$$

The constants $\tilde{c}_i$ are

$$\tilde{c}_7 = \tilde{c}_7 + \frac{1}{2} \tilde{c}_8 = c_9 , \quad \tilde{c}_8 = \tilde{c}_8 = c_9 - 2c_{10} ,$$

$$\tilde{c}_9 = \tilde{c}_9 + \frac{1}{2} \tilde{c}_{10} = c_{12} , \quad \tilde{c}_{10} = \tilde{c}_{10} = c_{12} - 2c_{13} . \quad (31)$$

The constants $\tilde{c}_i$ were first determined in Ref. \cite{37}. The progress in acquiring the data on several reactions \cite{48} allowed in Ref. \cite{32} to improve the analysis considerably. The difference between the new data \cite{7} and the data \cite{48} is not so dramatic and only the constants $\tilde{c}_7$ and $\tilde{c}_9$ slightly changed in comparison with \cite{32}

$$\tilde{c}_7 = 8.72 \times 10^{-3} , \quad \tilde{c}_8 = -1.07 \times 10^{-1} , \quad \tilde{c}_9 = 9.76 \times 10^{-3} , \quad \tilde{c}_{10} = 7.59 \times 10^{-2} . \quad (32)$$

$^5$In the proceedings version of the Ref. \cite{31}, the factor 2g is lacking at the r.h.s. of Eqs. (50) and (51), whereas the factor g in the first term in the braces is superfluous; the same is true for Eqs. (52) and (53), but with $g \to e$. 


Let us note that we prefer to choose $\tilde{c}_9$ as in [32], by averaging the data on the processes $\rho^\pm \to \pi^\pm \gamma$ and $\omega \to \pi^0 \gamma$, since the data on the reaction $\rho^0 \to \pi^0 \gamma$ show clear tendency to move to the data on the charged $\rho$ meson radiative decay [4].

Having the homogenous part of the anomalous Lagrangian at our disposal, we pass to the construction of the anomalous RMC amplitude.

### 3 Structure of the anomalous RMC amplitude

The contribution to the RMC amplitude due to the Wess–Zumino term (22) is presented in Fig. 1. It reads

$$
J^{a,WZ}_{\mu\nu} = -\frac{g_{\pi NN}}{8\pi^2 f_\pi} \varepsilon_{\mu\nu\eta\alpha} k_\eta q_\alpha \Delta_T^a(q_1^2) \Gamma_5^a, \quad (33)
$$

where

$$
\Gamma_5^a = \bar{u}(p')\gamma_5 \tau^a u(p). \quad (34)
$$

This amplitude satisfies the continuity equations

$$
k_\nu J^{a,WZ}_{\mu\nu} = q_\mu J^{a,WZ}_{\mu\nu} = 0. \quad (35)
$$

Since this amplitude contains the pion propagator, it contributes to the form factor $g_P$. However, $q_1 = q - k = \mu - \nu - k = -q^L$ and this contribution has no enhancement factor. The influence of this amplitude on the photon spectrum was earlier calculated in Ref. [49] and it was found to be negligible.

---

**Fig. 1.** The Wess–Zumino anomalous amplitude $J^{a,WZ}_{\mu\nu}$.
Using particular terms, the contributions to the anomalous RMC amplitude are calculated. From the vertex $\tilde{L}_{7,1}$, one generates 3 Feynman amplitudes, presented in Fig. 2a, Fig. 2b and Fig. 2c. They are of the form

\begin{align}
J_{\mu\nu}^a(7, 1a) &= -2m^2 \frac{g_\rho g_{\pi NN}}{f_\pi} \varepsilon_{\alpha\beta\gamma\kappa} k_{\kappa} q_{1\lambda} \Delta^\kappa_F(q_1^2) \Delta_{\alpha\nu}^\omega(k) \Delta_{\beta\mu}^\rho(q) \Gamma_5^a, \\
J_{\mu\nu}^a(7, 1b) &= -im^2 \frac{g_\rho^3}{g_\omega f_\pi} \varepsilon_{\alpha\beta\gamma\kappa} k_{\kappa} q_{\beta} q_{\mu} \Delta^\kappa_F(q_1^2) \Delta_{\alpha\nu}^\omega(k) \Delta_{\beta\mu}^\rho(q_1) \Gamma_5^a, \\
M_{\nu}^a(7, 1c) &= m^2 \frac{g_\rho^3}{g_\omega f_\pi} \varepsilon_{\alpha\beta\gamma\kappa} k_{\kappa} q_{\lambda} \Delta_{\alpha\nu}^\omega(k) \Delta_{\beta\mu}^\rho(q_1) \Gamma_5^a. 
\end{align}

Here

$$\Gamma_5^a = \bar{u}(p') (\gamma_\eta - \frac{k_V}{2M} \sigma_{\eta\lambda} q_{1\lambda}) u(p).$$

Both amplitudes, $J_{\mu\nu}^a(7, 1a)$ and $J_{\mu\nu}^a(7, 1b)$, contain the pion propagator and contribute to $g_P$. However, only the pion propagator of the amplitude $J_{\mu\nu}^a(7, 1b)$ provides the enhancement factor $\approx 3$.

Calculating the divergence yields

\begin{align}
k_{\nu} J_{\mu\nu}^a(7, 1a) &= k_{\nu} J_{\mu\nu}^a(7, 1b) = 0, \\
q_{\mu} J_{\mu\nu}^a(7, 1a) &= -2m^2 \frac{g_\rho g_{\pi NN}}{g_\omega f_\pi} \varepsilon_{\alpha\beta\gamma\kappa} k_{\kappa} q_{1\lambda} \Delta^\kappa_F(q_1^2) \Delta_{\alpha\nu}^\omega(k) \Gamma_5^a, \\
q_{\mu} J_{\mu\nu}^a(7, 1b) &= -if_\pi q^2 \Delta^\kappa_F(q^2) M_{\nu}^a(7, 1c).
\end{align}

Relative to the index $\nu$, attached to the photon line, the amplitudes are transverse separately.

The vertex $\tilde{L}_{7,2}$ generates the amplitude presented in Fig. 2d. Explicitly we have

\begin{align}
J_{\mu\nu}^a(7, 2) &= 2m^2 \frac{g_\rho g_{\pi NN}}{g_\omega f_\pi} \varepsilon_{\alpha\beta\gamma\kappa} k_{\kappa} q_{1\lambda} \Delta^\kappa_F(q_1^2) \Delta_{\alpha\nu}^\omega(k) \Gamma_5^a, \\
k_{\nu} J_{\mu\nu}^a(7, 2) &= 0, \quad q_{\mu} J_{\mu\nu}^a(7, 2) = -q_{\mu} J_{\mu\nu}^a(7, 1a).
\end{align}

Again, the amplitude is transverse in the electromagnetic sector. As follows from the second part of Eq. (45), the weak vector amplitudes $J_{\mu\nu}^a(7, 1a)$ and $J_{\mu\nu}^a(7, 2)$ satisfy the CVC hypothesis.

From the vertex $\tilde{L}_{7,3}$ we have also only one amplitude of Fig. 2e

\begin{align}
J_{\mu\nu}^a(7, 3) &= im^2 \frac{g_\rho^3}{g_\omega} \varepsilon_{\mu\alpha\beta\kappa} k_{\kappa} \Delta_{\alpha\nu}^\omega(k) \Delta_{\beta\mu}^\rho(q_1) \Gamma_5^a, \\
k_{\nu} J_{\mu\nu}^a(7, 3) &= 0, \quad q_{\mu} J_{\mu\nu}^a(7, 3) = if_\pi M_{\nu}^a(7, 1c).
\end{align}

It is seen from Eq. (43) and Eq. (47) that the axial amplitudes $J_{\mu\nu}^a(7, 1b)$ and $J_{\mu\nu}^a(7, 3)$ satisfy the PCAC

$$q_{\mu} [J_{\mu\nu}^a(7, 1b) + J_{\mu\nu}^a(7, 3)] = if_\pi m^2 \Delta^\omega_F(q^2) M_{\nu}^a(7, 1c).$$
The vertex $\bar{\mathcal{L}}_{7,4}$ does not contribute to processes triggered by the charged current.

In its turn, the vertex $\bar{\mathcal{L}}_{7,5}$ generates 3 amplitudes, again represented by the graphs Fig. 2a, Fig. 2b and Fig. 2c

\begin{align}
J_{\mu\nu}^{a}(7,5a) &= 2m^{2}_{\omega}m^{2}_{\rho} \frac{g_{\rho}g_{\pi NN}}{g_{\omega}f_{\pi}} \varepsilon_{\alpha\beta\lambda\kappa} q_{\alpha} q_{\lambda} \Delta_{\rho}^{\pi}(q_{1}^{2}) \Delta_{\lambda\kappa}^{\omega}(k) \Delta_{\beta\mu}^{\rho}(q) \Gamma_{5}^{a}, \quad (49) \\
J_{\mu\nu}^{a}(7,5b) &= -im^{2}_{\omega} \frac{g_{3}^{2}}{g_{\omega}} \varepsilon_{\alpha\beta\lambda\kappa} q_{\alpha} q_{\lambda} q_{\mu} \Delta_{\rho}^{\pi}(q_{1}^{2}) \Delta_{\lambda\kappa}^{\omega}(k) \Delta_{\beta\eta}^{\rho}(q_{1}) \Gamma_{\eta}^{a} \\
&= -if_{\pi} q_{\mu} \Delta_{\rho}^{\pi}(q^{2}) M_{\nu}^{a}(7,5c), \quad (50) \\
M_{\nu}^{a}(7,5c) &= m^{2}_{\omega} \frac{g_{3}^{2}}{g_{\omega} f_{\pi}} \varepsilon_{\alpha\beta\lambda\kappa} q_{\lambda} q_{\mu} \Delta_{\kappa\mu}^{\omega}(k) \Delta_{\beta\eta}^{\rho}(q_{1}) \Gamma_{\eta}^{a}. \quad (51)
\end{align}

Fig. 2. The processes contributing to the anomalous amplitude from the Lagrangian $\bar{\mathcal{L}}_{7}$. 


Again, both amplitudes, \( J_{\mu\nu}^a(7, 5a) \) and \( J_{\mu\nu}^a(7, 5b) \), contain the pion propagator and contribute to \( g_P \). However, only the pion propagator of the amplitude \( J_{\mu\nu}^a(7, 5b) \) provides the enhancement factor \( \approx 3 \). The divergence of these amplitudes reads

\[
k_{\nu}J_{\mu\nu}^a(7, 5a) = 2m_\rho^2 g_\rho g_{\pi NN} g_\omega \varepsilon_{\alpha\beta\lambda\kappa} k_{\kappa} q_{\alpha} q_{1\lambda} \Delta_\pi^\nu(q_1^2) \Delta_\rho^\mu(q) \Gamma_\kappa^a, \tag{52}
\]

\[
q_{\mu}J_{\mu\nu}^a(7, 5a) = 0, \tag{53}
\]

\[
k_{\nu}J_{\mu\nu}^a(7, 5b) = -ig_\rho^3 g_\omega \varepsilon_{\alpha\beta\lambda\kappa} k_{\kappa} q_{1\alpha} q_{\lambda} q_{\mu} \Delta_\pi^\nu(q_1^2) \Delta_\rho^\eta(q_1) \Gamma_\eta^a, \tag{54}
\]

\[
q_{\mu}J_{\mu\nu}^a(7, 5b) = -if_{\pi}q^2 \Delta_\pi(2^2) M_{\nu}^a(7, 6h). \tag{55}
\]

Now the amplitudes are not separately transverse in the electromagnetic sector. On the contrary, the weak vector amplitude \( J_{\mu\nu}^a(7, 5a) \) satisfies the CVC by itself.

From the vertex \( \tilde{L}_{7, 6} \), one again obtains 3 amplitudes, given in Fig. 2f, Fig. 2g and Fig. 2h

\[
J_{\mu\nu}^a(7, 6a) = 2m_\rho^2 g_\rho g_{\pi NN} g_\omega \varepsilon_{\nu\kappa\beta\lambda} q_{\lambda} q_{1\alpha} \Delta_\pi^\nu(q_1^2) \Delta_\rho^\mu(q) \Gamma_\kappa^a, \tag{56}
\]

\[
J_{\mu\nu}^a(7, 6b) = -ig_\rho^3 g_\omega \varepsilon_{\nu\kappa\beta\lambda} q_{\lambda} q_{1\alpha} \Delta_\pi^\nu(q_1^2) \Delta_\rho^\eta(q_1) \Gamma_\eta^a \]

\[-i f_{\pi}q^2 \Delta_\pi(2^2) M_{\nu}^a(7, 6h), \tag{57}
\]

\[
M_{\nu}^a(7, 6h) = \frac{g_\rho^3}{g_\omega} \varepsilon_{\nu\kappa\beta\lambda} q_{\lambda} q_{1\alpha} \Delta_\rho^\eta \Gamma_\eta^a. \tag{58}
\]

In deriving these equations, we used the relation \( g_\omega = 3 g_\rho \). It is the amplitude \( J_{\mu\nu}^a(7, 6b) \) that contains the enhancement factor. The divergence of the amplitudes \( J_{\mu\nu}^a(7, 6a) \) and \( J_{\mu\nu}^a(7, 6b) \) is

\[
k_{\nu}J_{\mu\nu}^a(7, 6a) = 2m_\rho^2 g_\rho g_{\pi NN} g_\omega \varepsilon_{\nu\kappa\beta\lambda} k_{\kappa} q_{\nu} q_{1\lambda} \Delta_\pi^\nu(q_1^2) \Delta_\rho^\mu(q) \Gamma_\kappa^a, \tag{59}
\]

\[
q_{\mu}J_{\mu\nu}^a(7, 6a) = 0, \tag{60}
\]

\[
k_{\nu}J_{\mu\nu}^a(7, 6b) = ig_\rho^3 g_\omega \varepsilon_{\alpha\beta\lambda\kappa} k_{\kappa} q_{1\alpha} q_{\lambda} q_{\mu} \Delta_\pi^\nu(q_1^2) \Delta_\rho^\eta(q_1) \Gamma_\eta^a, \tag{61}
\]

\[
q_{\mu}J_{\mu\nu}^a(7, 6b) = -if_{\pi}q^2 \Delta_\pi(2^2) M_{\nu}^a(7, 6h). \tag{62}
\]

It follows from Eq. (54) and Eq. (61) that

\[
k_{\nu}[J_{\mu\nu}^a(7, 5b) + J_{\mu\nu}^a(7, 6b)] = 0. \tag{63}
\]

The vertex \( \tilde{L}_{7, 7} \) yields the amplitude of Fig. 2e. Together with the divergence, this amplitude is

\[
J_{\mu\nu}^a(7, 7) = im_\rho^2 g_\rho^3 g_\omega \varepsilon_{\alpha\beta\mu\kappa} q_{1\alpha} \Delta_{\pi\nu}(k) \Delta_\rho^\eta(q_1) \Gamma_\eta^a, \tag{64}
\]

\[
k_{\nu}J_{\mu\nu}^a(7, 7) = ig_\rho^3 g_\omega \varepsilon_{\alpha\beta\mu\kappa} k_{\kappa} q_{1\alpha} \Delta_\rho^\eta(q_1) \Gamma_\eta^a. \tag{65}
\]
\[ q_\mu J_{\mu\nu}^a(7, 7) = if_\pi M^a_\nu(7, 5c) . \]  

Combining Eq. (55) and Eq. (66) we obtain another PCAC constraint

\[ q_\mu [ J_{\mu\nu}^a(7, 5b) + J_{\mu\nu}^a(7, 7) ] = if_\pi m^2_\pi \Delta^a_F(q^2)M^a_\nu(7, 5c) . \]  

From the last vertex \( \bar{L}_7, 8 \), one obtains the amplitude of Fig. 2i. It reads, together with the divergence, as follows

\[ J_{\mu\nu}^a(7, 8) = -i g^3_\rho g_\omega \epsilon_{\alpha\beta\mu\nu} q_1^\alpha \Delta^a_\beta \eta(q_1)\Gamma^a_\eta , \]  

\[ k_\nu J_{\mu\nu}^a(7, 8) = -i g^3_\rho g_\omega \epsilon_{\alpha\beta\mu\nu} k_\nu q_1^\alpha \Delta^a_\beta \eta(q_1)\Gamma^a_\eta , \]  

\[ q_\mu J_{\mu\nu}^a(7, 8) = if_\pi M^a_\nu(7, 6h) . \]  

It follows from Eq. (65) and Eq. (69) that

\[ k_\nu [ J_{\mu\nu}^a(7, 7) + J_{\mu\nu}^a(7, 8) ] = 0 . \]  

From Eq. (62) and Eq. (70) we have the PCAC constraint

\[ q_\mu [ J_{\mu\nu}^a(7, 6b) + J_{\mu\nu}^a(7, 8) ] = if_\pi m^2_\pi \Delta^a_\beta \eta F(q^2)M^a_\nu(7, 6h) . \]  

Let us note that the amplitudes that contain the \( \omega \) meson propagator can be simplified at once, since it holds for the real photon with a good accuracy

\[ \Delta^a_{\nu\kappa}(k) \approx \delta_{\nu\kappa}/m^2_\omega . \]  

Summing up, the vector–axial-vector (VA) amplitudes are

\[ J^a_\nu(7, 1b), J^a_\nu(7, 3), J^a_\nu(7, 5b), J^a_\nu(7, 6b), J^a_\mu(7, 7) \text{ and } J^a_\mu(7, 8). \]  

Using Eq. (73), we have from Eq. (51) and Eq. (58)

\[ M^a_\nu(7, 5c) \approx -M^a_\nu(7, 6h) , \]  

from which it follows that

\[ J^a_\mu(7, 5b) + J^a_\mu(7, 6b) \approx 0 . \]  

It also follows that in this approximation

\[ J^a_\mu(7, 7) + J^a_\mu(7, 8) \approx 0 . \]  

Then from the VA amplitudes, only \( J^a_\mu(7, 1b) \) and \( J^a_\mu(7, 3) \) survive. Since \( \Gamma^a_\eta \) depends on the momentum transfer \( q_1 \), we can write

\[ J^a_\mu(7, 1b) = -i g^3_\rho g_\omega \epsilon_{\nu\eta\lambda\kappa} k_\kappa q_\lambda q_\mu \Delta^a_F(q^2)\Delta^a_F(q_1^2)\Gamma^a_\eta , \]  

\[ J^a_\mu(7, 3) = i g^3_\rho g_\omega \epsilon_{\nu\eta\mu\kappa} k_\kappa \Delta^a_F(q^2)\Gamma^a_\eta . \]  

The vector–vector (VV) amplitudes are
For the weak momentum transfer \( q_\mu \) of interest, the following approximation is quite acceptable

\[
\Delta^0_\beta\mu(q) \approx \delta_\beta\mu/m_\rho^2. 
\]  

(79)

This approximation simplifies the analysis of the VV amplitudes to a great extent, since it is clear from Eq. (38) and Eq. (45) that

\[
J^a_{\mu\nu}(7, 1a) + J^a_{\mu\nu}(7, 2) \approx 0, 
\]  

(80)

and from Eq. (50) and Eq. (56) we have

\[
J^a_{\mu\nu}(7, 5a) + J^a_{\mu\nu}(7, 6a) \approx 0. 
\]  

(81)

It means that the VV amplitudes do not contribute to the anomalous amplitude in the approximation (79).

In the next step, analogous construction of the amplitudes arising from the Lagrangians \( \bar{\mathcal{L}}_i \), \( i = 8, 9, 10 \) is carried out. It turns out that only the same amplitudes, \( J^a_{\mu\nu}(7, 1b) \) and \( J^a_{\mu\nu}(7, 3) \), result. The full anomalous amplitude can be expressed, using Eq. (23), as

\[
\begin{align*}
J^a_{\mu\nu} &= (\bar{c}_7 - 1/2 \bar{c}_8 + \bar{c}_9 - 1/2 \bar{c}_{10})[J^a_{\mu\nu}(7, 1b) + J^a_{\mu\nu}(7, 3)] \\
&= \bar{c}[J^a_{\mu\nu}(7, 1b) + J^a_{\mu\nu}(7, 3)],
\end{align*}
\]  

(82)

where

\[
\bar{c} = \bar{c}_7 + \bar{c}_9 = 1.85 \times 10^{-2}.
\]  

(83)

Here we have used Eqs. (31) and Eqs. (32). The amplitude \( J^a_{\mu\nu,an} \) satisfies the continuity equations

\[
\begin{align*}
k_\nu J^a_{\mu\nu,an} &= 0, \\
g_\mu J^a_{\mu\nu,an} &= igF^\pi_\mu \lambda \bar{\rho} \bar{\rho} n \,(\bar{q}^2)\bar{c}M^a_\nu(7, 1c).
\end{align*}
\]  

(84)

We now proceed to calculate the contributions to the form factors \( g_i \) entering the effective Hamiltonian (14). For this purpose, we multiply the amplitudes \( J^a_{\mu\nu,wz} \), Eq. (83), and \( J^a_{\mu\nu,an} \), Eq. (82), by \( \varepsilon_\nu l_\mu \), where the weak lepton current is \( l_\mu = -i\bar{u}_\nu \gamma_\mu (1 + \gamma_5)u_\mu \). After the non–relativistic reduction, we obtain from the Wess–Zumino amplitude \( J^a_{\mu\nu,wz} \)

\[
\begin{align*}
g_{4,wz} &= -\frac{E_k^2}{8\pi^2 f^2_\pi} \frac{\lambda \bar{E}_k + y \bar{E}_\nu}{2m_\mu} g_P, \\
g_{4',wz} &= -\frac{E_k \bar{E}_\mu}{8\pi^2 f^2_\pi} \frac{\lambda \bar{E}_k + y \bar{E}_\nu}{2m_\mu} g_P, \\
g_{7,wz} &= g_{11,wz} = \frac{E_k}{8\pi^2 f^2_\pi} \frac{\lambda \bar{E}_\nu}{2m_\mu} g_L, \\
g_{10,wz} &= g_{11,wz} = \frac{E_k}{8\pi^2 f^2_\pi} \frac{\lambda \bar{E}_\nu^2}{2m_\mu} g_L.
\end{align*}
\]  

(86)

(87)

(88)

(89)
Here $\eta = m_\mu/2M$, $E_k(E_\nu)$ is the photon (neutrino) energy, $y = \hat{\nu} \cdot \hat{k}$ and the induced pseudoscalar form factor $g_P^N = 2Mm_\mu g_A\Delta_F^\rho((q^N)^2)$ depends on the momentum $q^I = -q_1$. As it has already been noted, this momentum dependence does not exhibit the enhancement factor for the photon energies at the high energy tail of the photon spectrum.

Let us calculate next the contributions from the amplitude $J^a_{\mu\nu}(7,1b)$. We obtain

$$
g_{2,1} = -(1 + \kappa_V)\eta \frac{g_\rho^3}{g_A g_\omega} \frac{E_k}{2M} \frac{|\vec{k} + \vec{\nu}|^2}{m_\rho^2} g_P^N, \tag{90}
$$

$$
g^\prime_{10,1} = -(1 + \kappa_V)\eta \frac{g_\rho^3}{g_A g_\omega} \frac{E_\nu}{2M} g_P^N, \tag{91}
$$

$$
g^\prime_{10,1} = -(1 + \kappa_V)\eta \frac{g_\rho^3}{g_A g_\omega} \frac{E_\nu}{2M} g_P^N. \tag{92}
$$

Here $g_P^N = 2Mm_\mu g_A\Delta_F^\rho((q^N)^2)$ depends on the momentum $q^N = \nu - \mu$ and it provides the enhancement factor $\approx 3$ in the amplitude at the high energy tail of the photon spectrum. We have also used the approximation

$$
\Delta_F^\rho(q_1^2) \approx m_\rho^{-2}. \tag{93}
$$

The last contributions to calculate are from the amplitude $J^a_{\mu\nu}(7,3)$

$$
g_{2,3} = (1 + \kappa_V)\eta \frac{g_\rho^3}{g_\omega} \frac{E_k(E_k + yE_\nu)}{m_\rho^2}, \tag{94}
$$

$$
g^\prime_{3,3} = -g^\prime_{3,3} = -(1 + \kappa_V)\eta \frac{g_\rho^3}{g_\omega} \frac{E_k E_\nu}{m_\rho^2}. \tag{95}
$$

These form factors do not contribute to $g_P$. However, the form factor $g_{2,3}$ is large at the high energy tail of the photon spectrum. Together with the form factor $g_{2,1}$ of Eq. (90), they are by a factor $\approx 1/2M$ larger than other form factors $g_{i,1}$ and $g_{j,3}$. It can be seen that due to the enhancement factor in the form factor $g_{2,1}$, these two form factors are about the same size at the end of the photon spectrum. Indeed, taking $|g_P^N| \approx 30$, we have $E_k g_P^N / 2M \approx 1.5$. Besides, $|\vec{k} + \vec{\nu}|^2 \approx E_k(E_k + yE_\nu) \approx E_k^2$. On the other hand, comparing the form factors $g_{2,1}$ and $g^\prime_{4,1WZ}$ we have at the high energy tail of the photon spectrum

$$
\frac{g_{2,1}}{g^\prime_{4,1WZ}} \approx -4\pi^2 \frac{1 + \kappa_V}{3g_A} \frac{m_\mu g_P^N}{M g_P^2} \approx 40. \tag{97}
$$

In deriving this ratio, we used the Kawarabayashi–Suzuki–Fayazuddin–Riazuddin relation $2f^2_\pi \rho^2 = m_\rho^2$. Nevertheless, the form factors $g_{2,1}$ and $g_{2,3}$ do not enhance sensibly the photon spectrum, because of the small factor $\tilde{c} = 1.85 \times 10^{-2}$, Eq. (83). The presence of this small factor in the homogenous part of the anomalous Lagrangian makes its influence approximately equal to the Wess–Zumino term.

The numerical analysis shows that the contribution of the anomalous processes to the singlet and triplet capture rates for the RMC in hydrogen is $\approx 0.2\,$%. 

16
Let us briefly discuss the double radiative muon capture (DRMC) in hydrogen

\[ \mu^- + p \rightarrow \nu_\mu + \gamma \gamma + n. \]  

One of the possible processes contributing to this reaction is presented in Fig. 3.

![Fig. 3. The amplitude of the double radiative muon capture.](image)

The charged weak vector interaction converts \( \pi^+ \) into \( \pi^0 \) that decays into two photons. This decay is possible due to the chiral anomaly and is described by the anomalous Lagrangian \([16]\). Since in the experiment on the RMC only one photon is detected, the double differential capture rate for the process \([98]\), integrated over one photon, can contribute to the measured photon spectrum. Calculations, analogous to those performed above, provide the following anomalous amplitude for the DRMC process

\[ J_{\mu \nu \eta}^a = \frac{g_{\pi NN} m^2}{8 \pi^2 f_{\pi}} \varepsilon_{\nu \alpha \eta} k_{1\zeta} k_{2\alpha} \Delta_{\lambda \mu}(q) \Delta_F^\tau(q_1^2) \Delta_F^\tau(q_2^2)(q_{1\lambda} + q_{2\lambda}) \varepsilon^{3ab} \Gamma_5^b. \]  

This amplitude is gauge invariant

\[ k_\nu J_{\mu \nu \eta}^a = k_\eta J_{\mu \nu \eta}^a = 0. \]  

In the weak sector it satisfies the following Ward–Takahashi identity

\[ q_\mu J_{\mu \nu \eta}^a = \frac{g_{\pi NN}}{8 \pi^2 f_{\pi}} \varepsilon_{\nu \alpha \eta} k_{1\zeta} k_{2\alpha} \left[ \Delta_F^\tau(q_1^2) - \Delta_F^\tau(q_2^2) \right] \varepsilon^{3ab} \Gamma_5^b. \]  

At the right hand side of Eq. \((101)\), the amplitude of more simple double radiative process enter. This amplitude is given at Fig. 1 with the wavy lines corresponding to outgoing photons.

The probability of the DRMC per unit volume is

\[ dw_{fi} = (e^2 G_F \cos \theta_C)^2 \frac{1}{2(2\pi)^8} \frac{1}{4} \sum_{s, p} |M_{fi}|^2 \delta^4(P_f - P_i) d^3p d^3\nu \frac{d^3k_1}{2E_1} \frac{d^3k_2}{2E_2}. \]  

Here

\[ M_{fi} = \tilde{\lambda}_\mu(0) \varepsilon^*_\nu(k_1) \varepsilon^*_\eta J_{\mu \nu \eta}^a. \]
The form of the DRMC rate resembles that of the single RMC rate \(^6\)

\[
\Lambda_{\text{DRMC}} = \frac{\alpha^5}{28\pi^9} (G_F \cos \theta_C \frac{g_{\pi NN}}{2Mf_\pi})^2 m^3 M_n \int \frac{v_0^2}{W + E_1(\cos \theta_1 - 1) + E_2(\cos \theta_2 - 1)} \times (\Delta_F(q_1^2) \Delta_F(q_2^2))(k_1 \cdot k_2)^2 (q_1^2)^2 F E_1 d(E_1) E_2 d(E_2) \\
\times \sin \theta_1 d\theta_1 \sin \theta_2 d\theta_2 d\phi_2. \tag{104}
\]

Here

\[
F = [m_\mu - 2(E_1 + E_2)]^2 + 4(\vec{k}_1 + \vec{k}_2)^2 - 4[m_\mu + 2(E_1 + E_2)][\vec{\nu} \cdot (\vec{k}_1 + \vec{k}_2)], \tag{105}
\]

\[
v_0 = \frac{W^2 - M_n^2 - 2W(E_1 + E_2) + 2E_1 E_2 (1 - \cos \theta_{12})}{2[W + E_1(\cos \theta_1 - 1) + E_2(\cos \theta_2 - 1)]}, \tag{106}
\]

\(\alpha\) is the fine structure constant, \(W = M_p + m_\mu\), \(M_p\) (\(M_n\)) is the proton (neutron) mass, \(m\) is the reduced proton-muon mass and \(\cos \theta_{12} = \hat{k}_1 \cdot \hat{k}_2\).

In comparison with the single RMC rates \([11]\), \(\Lambda_{\text{DRMC}}\) is suppressed by the factor \(\alpha/2^6 \pi^6\). This clearly indicates that \(\Lambda_{\text{DRMC}}\) is much smaller. The numerical calculations yield for the triplet capture rate

\[
\Lambda_{\text{DRMC}, t} \times 10^3 = 106.7 \times 10^{-13} \text{s}^{-1}, \tag{107}
\]

that is damped by the factor \(10^{-13}\) in comparison with the radiative muon capture rate \(\Lambda_t\) for the reaction \([13] [11]\).

Besides the weak vector amplitude \([9]\), one can also construct weak axial amplitudes for the DRMC. However, they are expected to provide an effect of the same order of magnitude.

### 4 Conclusions

Starting from the most general anomalous action of the \(\pi\rho\omega a_1\) system, we arrive at the 3–point anomalous Lagrangian, that includes the Wess–Zumino term and four homogenous terms. Application to radiative muon capture in hydrogen is considered in order to establish possible contributions that may enhance the photon spectrum at its high energy tail. It is important to find and understand the possible sources that may explain the discrepancy between the experimental results and the conventional methods to calculate this spectrum.

The constructed anomalous amplitudes are gauge invariant and in the weak sector, the vector amplitudes satisfy the CVC hypothesis and the axial amplitudes satisfy the PCAC constraint. Using reasonable approximations, only two axial amplitudes result, besides the Wess–Zumino one. All three terms yield corrections that turn out to be of the same order of magnitude for the high photon energies. The numerical estimates show \([11]\) that they cannot provide the necessary enhancement.

We have also made here an estimate of the capture rate for the double radiative muon capture in hydrogen. This reaction is triggered by the anomalous decay \(\pi^0 \rightarrow \gamma\gamma\). The calculations show that

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For the comparison see, e.g., Eq. (4.1) of Ref. \([11]\).
its capture rate turns out to be strongly suppressed in comparison with the capture rate for the single radiative muon capture.

Let us note finally that we have recently suggested [11] that off–shell effects due to the isobar intermediate state can, in principle, provide an enhancement that brings the calculated results in reasonable agreement with the experiment.

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