COLORINGS WITH ONLY RAINBOW ARITHMETIC PROGRESSIONS

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(Received January 5, 2020; revised June 5, 2020; accepted June 20, 2020)

Dedicated to Endre Szemerédi on his 80th birthday

Abstract. If we want to color 1, 2, ..., n with the property that all 3-term arithmetic progressions are rainbow (that is, their elements receive 3 distinct colors), then, obviously, we need to use at least n/2 colors. Surprisingly, much fewer colors suffice if we are allowed to leave a negligible proportion of integers uncolored. Specifically, we prove that there exist $\alpha, \beta < 1$ such that for every $n$, there is a subset $A$ of $\{1, 2, \ldots, n\}$ of size at least $n - n^\alpha$, the elements of which can be colored with $n^\beta$ colors with the property that every 3-term arithmetic progression in $A$ is rainbow. Moreover, $\beta$ can be chosen to be arbitrarily small. Our result can be easily extended to $k$-term arithmetic progressions for any $k \geq 3$.

As a corollary, we obtain a simple proof of the following result of Alon, Moitra, and Sudakov, which can be used to design efficient communication protocols over shared directional multi-channels. There exist $\alpha', \beta' < 2$ such that for every $n$, there is a graph with $n$ vertices and at least $\binom{n}{2} - n^{\alpha'}$ edges, whose edge set can be partitioned into at most $n^{\beta'}$ induced matchings.

1. Introduction

Szemerédi’s regularity lemma [14] started a new chapter in extremal combinatorics and in additive number theory. In particular, it was instrumental...
in proving a famous conjecture of Erdős and Turán, according to which, for every real number \( \delta > 0 \) and every integer \( k > 0 \), there exists a positive integer \( n = n(\delta, k) \) such that every subset of \([n] = \{1, 2, \ldots, n\}\) that has at least \( \delta n \) elements contains an arithmetic progression of length \( k \) (in short, a \( k\text{-AP} \)); see [13]. The \( k = 3 \) special case of this theorem, originally proved by Roth [11], also follows from the celebrated triangle removal lemma [12], which is another direct consequence of the regularity lemma. It has several other closely related formulations and consequences:

1. If \( A \) is subset of \([n]\) with no 3-AP, then \( |A| = o(n) \).
2. If \( G \) is a graph on \( n \) vertices whose edge set can be partitioned into \( n \) induced matchings, then \( |E(G)| = o(n^2) \).
3. If \( G \) is a graph on \( n \) vertices which has \( o(n^3) \) triangles, then one can eliminate all triangles by removing \( o(n^2) \) edges of \( G \).
4. If \( H \) is a system of triples of \([n]\) such that every 6-element subset of \([n]\) contains at most 2 triples in \( H \), then \( |H| = o(n^2) \).

More precisely, the above statements apply to any infinite series of sets \( A \), graphs \( G \), and triple systems \( H \), resp., where \( n \to \infty \).

An old construction of Behrend [5] shows that there are 3-AP-free sets \( A \subset [n] \) of size at least \( ne^{-O(\sqrt{\log n})} \), so that 1 is not far from being tight. Ruzsa and Szemerédi [12] observed that Behrend’s construction can be used to show the existence of graphs \( G \) with \( n \) vertices and \( |E(G)| \geq n^2 e^{-O(\sqrt{\log n})} \) edges that can be partitioned into \( n \) induced matchings. Hence, 2 is also nearly tight, and the same is true for 3 and 4.

Szemerédi’s theorem on arithmetic progressions immediately implies van der Waerden’s theorem [15]: For any integer \( k \geq 3 \), let \( c_k(n) \) denote the minimum number of colors needed to color all elements of \([n]\) without creating a monochromatic \( k\text{-AP} \). Then we have \( \lim_{n \to \infty} c_k(n) = \infty \).

How many colors do we need if, instead of trying to avoid monochromatic \( k\text{-term arithmetic progressions} \), we want to make sure that every \( k\text{-term arithmetic progression} \) is \textit{rainbow}, that is, all of its elements receive distinct colors? For instance, it is easy to see that for \( k = 3 \), we need at least \( n/2 \) colors. Surprisingly, it turns out that much fewer colors suffice if we do not insist on coloring \textit{all} elements of \([n]\). In particular, there is a subset of \( A \subset [n] \) with \( |A| = (1 - o(1))n \) whose elements can be colored by \( n^{o(1)} \) colors with the property that all 3-term arithmetic progressions in \( A \) are rainbow.

More precisely, we prove the following result.

**Theorem 1.** There exist \( \alpha, \beta < 1 \) with the following property. For every sufficiently large positive integer \( n \), there is a set \( A \subset [n] \) with \( |A| \geq n - n^\alpha \) and a coloring of \( A \) with at most \( n^\beta \) colors such that every 3-term arithmetic progression in \( A \) is rainbow.

Moreover, for every \( \beta > 0 \), we can choose \( \alpha < 1 \) satisfying the above conditions.
Theorem 1 is used to construct graphs with \( n \) vertices and \( (1-o(1))\binom{n}{2} \) edges which can be partitioned into a small number of induced matchings. The first such constructions were found by Alon, Moitra, and Sudakov [2]. Theorem 1 easily implies the main result of [2], which is as follows.

**Corollary 2.** There exist \( \alpha', \beta' < 2 \) with the following property. For every sufficiently large positive integer \( n \), there is a graph with \( n \) vertices and at least \( \binom{n}{2} - n^{\alpha'} \) edges that can be partitioned into \( n^{\beta'} \) induced matchings.

Moreover, for every \( \beta' > 1 \), we can choose \( \alpha' < 2 \) satisfying the above condition.

Dense graphs that can be partitioned into few induced matchings have been extensively studied, partially due to their applications in graph testing [1,3,4,9] and testing monotonicity in posets [7]. The graphs satisfying the conditions in Corollary 2 can be used to design efficient communication protocols over shared directional multi-channels [2,6]. Some other interesting graphs decomposable into large matchings were constructed and studied in [8].

Our proof of Theorem 1 is inspired by the construction of Behrend [5], but it also has a lot in common with one of the two constructions given by Alon, Moitra, and Sudakov [2]. Roughly, the idea of Behrend is to identify the elements of \([n]\) with a high dimensional grid \([C]^d\), in which we find a sphere passing through many grid points. These points will correspond to a dense 3-AP-free set in \([n]\). We proceed similarly, but instead of taking a sphere, we take a small neighborhood \( S \) of a sphere. If we choose the radii properly, it follows by standard concentration laws that almost all points of the grid \([C]^d\) are contained in \( S \). On the other hand if 3 points form a 3-AP in \( S \), then they must be close to each other. This observation can be explored to give a coloring of \( S \cap [C]^d \) with the desired properties.

In Sections 2 and 3, we prove Theorem 1 and Corollary 2, respectively. In the last section, we indicate how to extend Theorem 1 to \( k \)-term arithmetic progressions for any \( k \geq 3 \); see Theorem 6.

### 2. Rainbow 3-AP’s—Proof of Theorem 1

We start by setting a few parameters. Let \( C \) be a sufficiently large integer. Suppose for simplicity that \( n = C^d \) for some integer \( d \). The general case can be treated in a similar manner. In the sequel, \( \log \) will stand for the natural logarithm.

Set \( \varepsilon = \frac{1}{C} \) and let \( B = \{0, 1, \ldots, C - 1\}^d \), so that \( |B| = C^d \). We view \( B \) as a subset of the vector space \( \mathbb{R}^d \) endowed with the Euclidean norm \( |.| \).
For \( x \in B \), let \( x(i) \in \{0, 1, \ldots, C - 1\} \) denote the \( i \)th coordinate of \( x \), where \( 1 \leq i \leq d \). Clearly, the map \( \phi : B \to [n] \) defined as

\[
\phi(x) = 1 + \sum_{i=1}^{d} x(i)C^{i-1}
\]

is a bijection.

Let \( z \) be an element chosen uniformly at random from the set \( B \), and let \( r = (\mathbb{E}[|z|^2])^{1/2} \). We have

\[
r^2 = \mathbb{E}[|z|^2] = \sum_{i=1}^{d} \mathbb{E}[z(i)^2] = \frac{d(C - 1)(2C - 1)}{6}.
\]

Therefore,

\[
\frac{dC^2}{6} < r^2 < \frac{dC^2}{3}.
\]

Let \( A' \) consist of the set of all points in \( B \) that lie in the spherical shell between the spheres of radii \( r(1 - \varepsilon) \) and \( r(1 + \varepsilon) \) about the origin. That is, let

\[
S = \{ x \in \mathbb{R}^d : r(1 - \varepsilon) \leq |x| \leq r(1 + \varepsilon) \},
\]

and let \( A' = B \cap S \). Finally, set \( A = \phi(A') \). Next we show, using standard concentration laws, that \( A' \) contains almost all elements of \( B \) and, hence, \( A \) contains almost all elements of \([n]\).

**Claim 3.** \( |A| = |A'| \geq C^d(1 - 2e^{-\frac{ \varepsilon^2}{18d^2}}) = n - 2n^{1 - \frac{\varepsilon^2}{18 \log C}} \).

**Proof.** Note that \( |z|^2 = \sum_{i=1}^{d} z(i)^2 \) is the sum of \( d \) independent random variables taking values in \( \{0, \ldots, (C - 1)^2\} \). We have \( r^2 = \mathbb{E}[|z|^2] \leq C^2d \). On the other hand, if \( x \not\in A' \), then \( |x|^2 - r^2 > \varepsilon r^2 > (1/6) \varepsilon dC^2 \). Thus, by Hoeffding’s inequality \([10]\), we obtain

\[
1 - \frac{|A'|}{C^d} \leq \mathbb{P}[|z|^2 - r^2 > (1/6) \varepsilon dC^2] \leq 2e^{-\frac{1}{18d^2}} = 2n^{-\frac{\varepsilon^2}{18 \log C}}.
\]

Therefore, with the choice \( \alpha = 1 - \frac{\varepsilon^2}{30 \log C} \), we have \( |A| \geq n - n^{\alpha} \), provided that \( n \) is sufficiently large.

It remains to define a coloring \( c \) of \( A \) with the desired properties. Using the bijection \( \phi \) between \( B \) and \([n]\), this corresponds to a coloring of \( A' \subset B \). We would like to guarantee that for every \( a, b \in A \) with \( a \neq b \) and \( c(a) = c(b) \), we have \( a + b \) and \( 2a - b \not\in A \). (By swapping \( a \) and \( b \), the latter condition

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also implies that \(2b - a \notin A\). Equivalently, we want that if \(c(a) = c(b)\) and \(\frac{a+b}{2}, 2a - b \in [n]\), then
\[
\phi^{-1}\left(\frac{a+b}{2}\right) \quad \text{and} \quad \phi^{-1}(2a-b) \notin A'.
\]
To achieve this, we would like to use the identities
\[
\phi^{-1}\left(\frac{a+b}{2}\right) = \frac{\phi^{-1}(a) + \phi^{-1}(b)}{2} \quad \text{and} \quad \phi^{-1}(2a-b) = 2\phi^{-1}(a) - \phi^{-1}(b).
\]
However, these equations hold if and only if
\[
\frac{\phi^{-1}(a) + \phi^{-1}(b)}{2} \in B \quad \text{and} \quad 2\phi^{-1}(a) - \phi^{-1}(b) \in B,
\]
respectively.

To overcome this problem, we first give an auxiliary coloring \(f\) of \(B\) such that if \(f(x) = f(y)\), then
\[
\frac{x+y}{2} \quad \text{and} \quad 2x - y \in B.
\]
We define \(f\) as follows. For any \(x \in B\), let \(f(x) = (a_1, \ldots, a_d, b_1, \ldots, b_d)\), where, for every \(i \in [d]\), we have
\[
a_i = \begin{cases} 0 & \text{if } x(i) \text{ is even,} \\ 1 & \text{if } x(i) \text{ is odd.} \end{cases}
\]
and
\[
b_i = \begin{cases} k & \text{if } x(i) \leq \frac{C}{2} \text{ and } 2^{k-1} - 1 \leq x(i) < 2^k - 1, \\ -k & \text{if } x(i) > \frac{C}{2} \text{ and } 2^{k-1} - 1 \leq C - 1 - x(i) < 2^k - 1. \end{cases}
\]
Then \(f\) uses at most \(2^d(2 \log_2 C)^d\) colors, and it is easy to verify that \(f\) satisfies the desired properties.

Next, we define a coloring \(g\) of \(A'\) such that any 3-AP in \(A'\) is rainbow. Then, the coloring \((f, g)\) induces a coloring on \(A\) for which every 3-AP is rainbow. In order to define \(g\), we need a simple geometric observation; see Fig. 1.

**Claim 4.** If \(x, y, z \in S\) such that \(y = \frac{x+z}{2}\), then \(|x - z| \leq 4\sqrt{\varepsilon} \cdot r\).

**Proof.** At least one of the angles \(0yx\) and \(0yz\) is at least \(\frac{\pi}{2}\), see Fig. 1. Assume without loss of generality that \(0yx\) is such an angle. Then we have \(|y|^2 + |y - x|^2 \leq |x|^2\). On the other hand,
\[
|x|^2 \leq (1 + \varepsilon)^2 r^2 \quad \text{and} \quad |y|^2 \geq (1 - \varepsilon)^2 r^2,
\]
so that we obtain

$$|x - z|^2 = 4|y - x|^2 \leq 4(|x|^2 - |y|^2) \leq 16\varepsilon r^2.$$  \[\Box\]

Define a graph $G$ on the vertex set $A'$, as follows. Join $x, y \in A'$ by an edge if at least one of the 3 vectors $2x - y$, $x + \frac{x+y}{2}$, $2y - x$ belongs to $A'$.

Claim 5. Let $\Delta$ denote the maximum degree of the vertices of $G$. Then we have

$$\Delta < 2^d C^{16\varepsilon d C^2}.$$

Proof. Fix any $x \in A'$. By Claim 4, every neighbor of $x$ is at distance at most $4\sqrt{\varepsilon}r < 4\sqrt{\varepsilon} \sqrt{d} C$ from $x$. If $|x - y| \leq 4\sqrt{\varepsilon} \sqrt{d} C$ for some $y \in A'$, then there are at most $16\varepsilon d C^2$ indices $i \in [d]$ such that $x(i) \neq y(i)$. The number of vertices $y$ with this property is smaller than $2^d C^{16\varepsilon d C^2}$. Indeed, there are fewer than $2^d$ ways to choose the indices $i$ for which $x(i) \neq y(i)$ and, for each such index $i$, there are fewer than $C$ different choices for $y(i)$. Therefore, we have

$$\Delta < 2^d C^{16\varepsilon d C^2}. \; \Box$$

It follows from Claim 5 that $G$ has a proper coloring with at most $\Delta + 1$ colors. By the definition of $G$, if in such a coloring two elements are colored with the same color, then this pair is not contained in any 3-AP in $A'$.

In the end, we obtain the coloring $(f, g)$ of $A'$ with at most

$$(\Delta + 1)2^d (2 \log_2 C)^d \leq (10C^{16\varepsilon C^2} \log_2 C)^d$$

colors such that every 3-AP in $A'$ is rainbow. The coloring $c$ on $A$ induced by $(f, g)$ has the same property.

Using that $\varepsilon = \frac{1}{C^3}$, we have $D = 10C^{16\varepsilon C^2} \log_2 C < C$, provided that $C$ is sufficiently large. Letting $\beta = \log_C D$, the number of colors used by $c$ is at most $n^{\beta}$.
Increasing $C$, $\beta$ tends to zero. Thus, in view of Claim 3, we obtain that for every $\beta > 0$, there is a suitable positive $\alpha < 1$ which satisfies the conditions of Theorem 1.

3. Induced matchings—Proof of Corollary 2

Let $\gamma > 0$, $s = n^\gamma$ and $m = n^{1-\gamma}$ (for simplicity, we omit the use of floors and ceilings). Let $V$ be a set of size $n$, and partition $V$ into $s$ sets $V_1, \ldots, V_s$ of size $m$. Let $\alpha, \beta < 1$ denote two constants meeting the requirements of Theorem 1. We will show that Corollary 2 is true with suitable constants $\alpha' = \max\{1+\alpha-\alpha\gamma, 2-\gamma\} + o(1)$ and $\beta' = 1 + \beta + \gamma - \beta\gamma + o(1)$, as $n \to \infty$. This illustrates that by choosing $\gamma$ sufficiently small, we can guarantee that $\beta'$ can be arbitrarily close to 1.

Let $A \subset [2m]$ be a set of size at least $2m - (2m)\alpha'$, and let $c$ be a coloring of $A$ with at most $(2m)^\beta$ colors such that every 3-AP in $A$ is rainbow.

Construct a graph $G$ on the vertex set $V$, as follows. Identify each $V_i$ with the set $[m]$ and, for every $1 \leq i < j \leq s$ and $x \in V_i$, $y \in V_j$, connect $x$ and $y$ by an edge of $G$ if and only if $x + y \in A$. If $xy$ is an edge, color it with the color $c'(xy) = (i, j, x - y, c(x + y))$.

Note that the same symbol $x$ denotes a different vertex in each $V_i$. Also, the third coordinate of the color $c'(xy)$ can be negative, zero, or positive.

First, we show that each color class is an induced matching. In other words, we show that if $xy \neq uv$ are distinct edges of $G$ such that $c'(xy) = c'(uv) = c'$, then $xy$ and $uv$ do not share a vertex and none of $xu, xv, yu, yv$ can be an edge of $G$ having color $c'$. The first two coordinates of the color $c'(xy) = c'(uv) = c'$ determine the pair of indices $(i, j)$, $i < j$, such that both $xy$ and $uv$ run between $V_i$ and $V_j$. Suppose without loss of generality that $x, u \in V_i$ and $y, v \in V_j$. If $x = u$, say, then $c'(xy) = c'(uv)$ implies that $x - y = u - v$, so that $y = v$, contradicting our assumption that $xy$ and $uv$ are distinct edges. Therefore, $xy$ and $uv$ cannot share a vertex. By definition, there is no edge between $x$ and $u$, and there is no edge between $y$ and $v$.

It remains to show that neither $xv$, nor $yu$ can be an edge of color $c'$. Let $d = x - y = u - v$. Suppose, for example, that $xv$ is an edge of color $c'$. Then $x + v \in A$, and we have

$$\frac{(x+y) + (u+v)}{2} = \frac{(2x-d) + (2v+d)}{2} = x + v.$$ 

Comparing the left-hand side and the right-hand side, it follows that $x + y$, $x + v$, $u + v$ are distinct numbers that form a 3-AP in $A$. However, the
fourth coordinate of the color \( c'(xy) = c'(uv) = c' \) guarantees that \( c(x + y) = c(u + v) \). Thus, we have found a non-rainbow 3-AP in \( A \), contradicting our assumptions. A symmetric argument shows that \( yu \) cannot be an edge of color \( c' \) either.

Let us count the number of edges of \( G \). For every pair \((i, j)\), \( 1 \leq i < j \leq s \), there are at least \( m^2 - m(2m)\alpha > m^2 - 2m^{1+\alpha} \) edges between \( V_i \) and \( V_j \). Indeed, for every \( t \in [2m] \setminus A \), there are at most \( m \) pairs \((x, y)\) \( \in [m]^2 \) such that \( x + y = t \), and the number of such elements \( t \) is at most \( (2m)^\alpha \). Hence, we have

\[
|E(G)| \geq \left( \frac{s}{2} \right) (m^2 - 2m^{1+\alpha}) \geq \left( \frac{n}{2} \right) - n^{2-\gamma} - n^{1+\alpha-\alpha\gamma}.
\]

The number of colors used by \( c' \) and, therefore, the number of induced matchings \( G \) can be partitioned into, is at most \( s^2(2m)(2m)^\beta \leq 4n^{\beta+\gamma-\beta\gamma} \). This completes the proof of Corollary 2.

4. Concluding remarks

Let us remark that in order to prove Corollary 2, it is enough to find a coloring of a large subset of \([n]\) such that in any 3-AP, the first and last elements have different colors. This can be achieved with slightly fewer colors: in the proof of Theorem 1, it is enough to define the coloring \( f \) as \( f = (a_1, \ldots, a_d) \) instead of \( f = (a_1, \ldots, a_d, b_1, \ldots, b_d) \).

Our proof of Theorem 1 can be easily extended to longer arithmetic progressions.

**Theorem 6.** For any positive integer \( k \), there exist \( \alpha, \beta < 1 \) with the following property. For every sufficiently large positive integer \( n \), there is a set \( A \subset [n] \) with \( |A| \geq n - n^\alpha \) and a coloring of \( A \) with at most \( n^{\beta} \) colors such that every arithmetic progression of length at most \( k \) in \( A \) is rainbow.

Moreover, for every \( \beta > 0 \), we can choose \( \alpha < 1 \) satisfying the above conditions.

In order to establish Theorem 6, we need to modify the proof of Theorem 1 at the following two points.

1. We should construct an auxiliary coloring \( f \) on \( B \) such that if \( f(x) = f(y) \), then \( \frac{p}{q}x + (1 - \frac{p}{q})y \in B \) for every \( p, q \in [k] \). Color \((x_1, \ldots, x_d)\) with the color \((a_1, \ldots, a_d, b_1, \ldots, b_d)\), where \( a_i \in \{0, \ldots, k! - 1\} \) such that \( a_i \equiv x_i \mod k! \), and \( b_i \equiv \frac{x(i)}{C} \mod \frac{k}{k-1} \). If \( x(i) \leq \frac{C}{2} \) then \( b_i = \left\lfloor \frac{x(i)}{C} \right\rfloor \), and if \( x(i) > \frac{C}{2} \) then \( b_i = \left\lfloor \frac{x(i)}{C} \right\rfloor \) and \( \left( \frac{k}{k-1} \right)^j \leq x(i) < \left( \frac{k}{k-1} \right)^{j+1} \).

Then \( f \) uses \( (k^k \log C)^{O(d)} \) colors.
2. Instead of Claim 4, we can show that if \( x_1, \ldots, x_k \) is a \( k \)-term arithmetic progression in \( S \), then \( |x_1 - x_k| \leq 10\sqrt{\varepsilon r} \).

After these changes, the proof can be completed by straightforward calculations, in the same way as in the case of Theorem 1.

Acknowledgements. We would like to thank Benny Sudakov for valuable discussions and the anonymous referee for useful comments and suggestions.

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