Some Properties of Coideal over Coalgebras

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Abstract. A coalgebra is a module over commutative ring with comultiplication and counit. In ring theory, we know ideal over a ring and in coalgebra we also have a coideal over coalgebras. There are two definitions of coideal over coalgebra and we show that both of them are equivalence by some proof. Corings is the generalization of coalgebra by change commutative ring R with any ring (not need commutative). Coideal of Corings is defined analog with definition coideal of coalgebras. In this paper, we have some properties of coideal over coalgebras have a similar patterns with properties of ideals over rings. Intersections and sums two coideals are also coideals and this result can be generalized for the finite family of coideals.

1. Introduction

This paper present theory of coalgebra structure. A commutative ring with unity denoted by $R$ and $A$ is notation for a unity ring (not need commutative). In 1960's [1] introduced coalgebras over a field as the dualization of algebras over fields. The generalization of coalgebra over a field is a coalgebra over a commutative ring $R$ and it generalization is a coring. Corings defined by replacing a commutative ring with any ring [2]. On ring theory, a subset I of ring R where for any $r \in R$, then $rI = Ir = I$ is called an ideal over $R$. Intersections and sums of some finite family ideals is also ideals. For study this paper we need some basic concepts on modules and ring theory we can see on [3, 4, 5] and also coalgebras concept that we can get on [1, 2, 6]. In his paper [7] explained algebras and coalgebras concept with categories approach. A coideal is a submodule of coalgebras with some unusual conditions. As well known there are two kinds of definitions coideal. We will show that both meanings are isomorphic and furthermore we showed that the properties of the ideal are also fulfilled on coideal of a coalgebra. In this paper, an identity map of $C$ denoted by $IC$. Coalgebra over coring defined as below.

Definition 1.1 [2] Given ring $R$ with multiplicative identity. An $R$-coalgebra is an $R$-module $C$ with $R$-homomorphism module $\Delta: C \rightarrow C \otimes R C$ and $\varepsilon: C \rightarrow R$ called (coassosiative) comultiplication and counit and satisfied commutativity of the diagrams

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes R C \\
\downarrow & & \downarrow \\
C \otimes R C & \xrightarrow{\Delta \otimes I_C} & C \otimes R C \\
\end{array}
\]

Figure 1. Coassosiative Diagram
Sweedler was introducing the Sweedlers $\Sigma$ -notation [1] i.e. for any $c \in C$, 
$\Delta(c) = \sum c_1 \otimes c_2$. Figure 1 and 2 is commutative it means that for any $c \in C$, then 
$\sum c_{11} \otimes c_{12} \otimes c_{13} = \sum c_1 \otimes c_2 \otimes c_3$ and $\varepsilon(c_1) c_2 = c = c_1 \varepsilon(c_2)$. A Coalgebra $C$ over $R$ with comultiplication $\Delta$ and counit $\varepsilon$ denoted by $(C, \Delta, \varepsilon)$. A Coalgebra $C$ is cocommutative if $
abla = \tau \circ \Delta$ where 
$\tau : C \otimes R C \rightarrow C \otimes C R,$ 
$a \otimes b \mapsto b \otimes a$

$\tau$ is called a twist map. As an $R$-modules, two $R$-coalgebra have some relation to each other by a function, i.e., $R$-module homomorphism, and as a coalgebra structure, we can define a coalgebra morphism as a linear map between coalgebra that respects the coalgebra structure. A Morphism of a coalgebra is given on the following definition.

**Definition 1.2** [2] Given an $R$-coalgebra $(C_1, \Delta_1, \varepsilon_1)$ and $(C_2, \Delta_2, \varepsilon_2)$. An $R$-module homomorphism $f : C_1 \rightarrow C_2$ is a coalgebra morphism if the following diagrams is commutative.

On the ring $R$, non-empty set $S \subseteq R$ where $S$ is also a ring over the same addition and multiplication operation with $R$, is called subring. In a coalgebra, a subring like that we called as a coalgebra.

**Definition 1.3**[2] Given an $R$-coalgebra $(C, \Delta, \varepsilon)$. An $R$-submodul $K$ of $C$ is called a subcoalgebra if 
$\Delta(K) \subseteq K \otimes R K$.

Corings is a generalization of coalgebras over $R$. By replace $R$ with any ring $A$. Various general properties of corings can be proved by using the same technique as for coalgebras over commutative rings. Here we give definitions of corings and coring morphism that have the same meaning as coalgebra over a commutative ring. Extending the notion of an $R$-coalgebra to noncommutative base rings leads to corings over an associative algebra $A$. Some properties and notions of corings have similarity with coalgebra, but we must pay attention to the left and right sided of the modules.
**Definition 1.4** [2] A-coring is an \((A,A)\)-bimodule \(C\) with bilinear function \(\Delta : C \to C \otimes_A C\) and \(\varepsilon : C \to A\) called (coassociative) comultiplication and counit, i.e \((I_c \otimes \Delta) \circ \Delta = (\Delta \otimes I_c) \circ \Delta\) and \((I_c \otimes \varepsilon) \circ \Delta = I_c = (\varepsilon \otimes I_c) \circ \Delta\).

Given two \(A\)-corings \((C_1,\Delta_1,\varepsilon_1),(C_2,\Delta_2,\varepsilon_2)\). An \((A,A)\)-bilinear map \(f : C_1 \to C_2\) is said to be a corings morphism provided \(\Delta_2 \circ f = (f \otimes f) \circ \Delta_1\) and \(\varepsilon_1 \circ f = \varepsilon_2\). An images set and kernel of morphism coalgebra \(f\) similar with image and kernel when \(f\) considered as an \(R\)-homomorphism. On the next section we study about coideal over coalgebra and coring. A Coideal defined as ideal in ring theory.

**2. Coideals over Coalgebras**

In this section, we will study about coideal and properties of coideal. We start by given two definitions of coideal and this section aims are to see that the definition are an equivalence. The first definition was introduced in 2003 [2] and the second definition proposed in 2012 [8].

**Definition 2.1** [2] Let \(\left(C,\Delta,\varepsilon\right)\) be an \(R\)-coalgebra. We called \(\text{Ker}(f)\) is a coideal of \(C\) if \(f : C \to C^*\) is a surjective coalgebra morphism.

**Definition 2.2** [8] Let \(\left(C,\Delta,\varepsilon\right)\) be an \(R\)-coalgebra. A \(R\)-submodule \(K\) of \(C\) is called left (right) coideal of \(K\) if \(K \subseteq \text{Ker}(\varepsilon)\) and \(\Delta(K) \subseteq C \otimes_R K\) \((\Delta(K) \subseteq K \otimes_R C)\). A \(R\)-submodule \(K\) of \(C\) is called coideal of \(C\) if \(K \subseteq \text{Ker}(\varepsilon)\) and \(\Delta(K) \subseteq C \otimes_R K + K \otimes_R C\).

Based on two Definition of coideal we will prove that Definition 2.1 and Definition 2.2 are an equivalence.

**Theorem 2.3** Given an \(R\)-coalgebra \(\left(C,\Delta_c,\varepsilon_c\right)\). For any \(R\)-submodule \(K\), \(K\) satisfies condition in Definition 2.1 if and only if \(K\) is also fullfiled condition in Definition 2.2.

**Proof.**

\((\Rightarrow)\) First, we will show that \(\text{Ker}(f)\) that is fulfilled condition in Definition 2.1 is also satisfied condition on Definition 2.2. \(R\)-coalgebra \(\left(C,\Delta_c,\varepsilon_c\right)\) and surjective coalgebra morphism \(f : C \to C^*\). Let \(K = \text{Ker}(f)\). We will show that \(K \subseteq \text{Ker}(\varepsilon)\) and \(\Delta(K) \subseteq C \otimes_R K + K \otimes_R C\).

We know that \(f\) is coalgebra morphism, then so

1. \(\varepsilon_c \circ f = \varepsilon_c\). It implies if \(I = \text{Ker}(f)\),

   \[\varepsilon_c \circ f(I) = \varepsilon_c(I) \Leftrightarrow \varepsilon_c(0) = \varepsilon_c(I) \Leftrightarrow 0 = \varepsilon_c(I)\]

   then \(I \subseteq \text{Ker}(\varepsilon_c)\).

2. From axioms of coalgebra morphism we know that \(\Delta_c \circ f = (f \otimes f) \circ \Delta_c\), then for any \(x \in \text{Ker}(f)\),

   \[\Delta_c \circ f(x) = (f \otimes f) \circ \Delta_c(x)\]

   \[\Rightarrow 0 = (f \otimes f) \circ \Delta_c(x)\).

It mean that \(\Delta_c(x) \in \text{Ker}(f \otimes f)\), then
\[ \Delta_c \left( \text{Ker}(f) \right) \subseteq \text{Ker}(f \otimes f) \subseteq C \otimes \text{Ker}(f) + \text{Ker}(f) \otimes C. \]

From point (1) \( K = \text{Ker}(f) \) satisfied the condition on Definition 2.2.

(\( \subseteq \)) Now we would to proof the converse, i.e. every submodule \( K \) of coalgebra \( C \) with \( K \subseteq \text{Ker}(f) \) and \( \Delta_c(K) \subseteq K \otimes C + C \otimes K \) then there is \( f \) a morphism coalgebra surjective with domain \( C \) such as \( K = \text{Ker}(f) \).

For any \( R \)-submodules \( K \) of \( C \) that satisfied condition on Definition 2.2 we can construct an \( R \)-modul factor \( C/K \) and then we can define natural \( R \)-module homomorphism \( f : C \rightarrow C/K \).

Natural \( R \)-module homomorphism \( f \) is surjective and \( f \) is coalgebra morphism [8]. Furthermore, we always have that \( \text{Ker}(f) = K \). So \( K \) is a coideal on Definition 2.1. 

**Example 2.4** Let \( R \)-coalgebra \((R[x], \Delta, \varepsilon)\) with coassociative comultiplication

\[ \Delta : R[x] \rightarrow R[x] \otimes R[x], x' \mapsto (x \otimes 1) + (x' \otimes 1) \text{ and } 1 \mapsto 1 \otimes 1. \]

Counit defined by \( \varepsilon : R[x] \rightarrow R, x' \mapsto 0, 1 \mapsto 1 \). Prove that \( \text{Ker}(\varepsilon) \) is coideal of \( R[x] \).

Let \( D = \text{Ker}(\varepsilon) \).

1. Clearly \( D = \text{Ker}(\varepsilon) \subseteq \text{Ker}(\varepsilon) \).
2. prove that \( \Delta(D) \subseteq D \otimes R[x] + R[x] \otimes D \). Let \( p(x) \in \Delta(D) \), i.e. \( p(x) = \Delta(r(x)) \) for some \( r(x) \in D \). Based on definition of counit \( \varepsilon \), \( r(x) \neq 1 \). If \( r(x) = 1 \) then \( \varepsilon(r(x)) = 1 \neq 0 \). Consequently \( \Delta(r(x)) \) must be form as sums of \( (x \otimes 1) + (x' \otimes 1) \) and implies that \( p(x) = \Delta(r(x)) \subseteq D \otimes R[x] + R[x] \otimes D \)

Form 1 and 2 we have that \( D = \text{Ker}(\varepsilon) \) is coideal of \((R[x], \Delta, \varepsilon)\). On the other hand, \( D = \text{Ker}(\varepsilon) \) is also satisfied the condition on Definition 2.1 because we can construct a coalgebra morphism surjective \( f : C \rightarrow C/K \).

**Theorem 2.5** [8] Every coideal over coalgebras is a subcoalgebra.

**Proof.**

Suppose \( I \) is acoideal of coalgebra \((C, \Delta, \varepsilon)\). Then \( \Delta(I) \subseteq I \otimes C + C \otimes I \). We will show that \( \Delta(I) \subseteq I \otimes I \). Since \( I \subseteq C \) then \( I \otimes C \subseteq C \otimes C \) and \( C \otimes I \subseteq C \otimes C \). Consequently \( \Delta(I) \subseteq I \otimes C + C \otimes I \subseteq C \otimes C + C \otimes C = C \otimes C \) or \( I \) is a subcoalgebra of \( C \).

Converse of Theorem 2.5 is not true. We give the counter example as following example.

**Example 2.6** Let a \( Z \)-module \( R \) and \( Z \)-coalgebra \((R, \Delta, \varepsilon)\) with \( \Delta : R \rightarrow R \otimes Z, r \mapsto r \otimes 1 \) and counit \( \varepsilon : R \rightarrow Z, 1 \neq r \mapsto 0 \) and \( 1 \mapsto 1 \). By \( \Delta \) and \( \varepsilon \) we have \( \Delta(Z) \subseteq Z \otimes Z \). Then \( Z \) is subcoalgebra of \((R, \Delta, \varepsilon)\) with counit \( \varepsilon |_Z \). Subcoalgebra \( Z \) is not a coideal of \( R \) because \( Z \) is not contain at \( \text{Ker}(\varepsilon) \).

Now we have similar pattern between morphism of coalgebra and homomorphism module properties as below.
Theorem 2.7 [8] Given two $R$-coalgebra $(C, \Delta, \epsilon)$ and $(C', \Delta', \epsilon')$. If $f : C \to C'$ is a coalgebra morphism, then $\text{Ker}(f)$ is a coideal of $C$.

Proof.
Suppose $f$ is a coalgebra morphism from $C$ to $C'$. We will show that $\text{Ker}(f)$ is acoideal of $C$. Let $x \in \text{Ker}(f)$. Since $f$ is a coalgebra morphism then $\epsilon \circ f = \epsilon$ such that $\epsilon(x) = \epsilon \circ f(x) = \epsilon'(f(x)) = \epsilon'(0) = 0$, then $x \in \text{Ker}(\epsilon)$. Furthermore let $y \in \Delta(\text{Ker}(f))$, i.e $y = \Delta(x)$ for some $x \in \text{Ker}(f) \subseteq C$. Then $y = \Delta(x) = \sum x_i \otimes x_2$, for some $x_i, x_2 \in C$.

$\Rightarrow y = \Delta(x) = \sum 0 + x_i \otimes x_2 + 0$, for some $x_i, x_2 \in C$.

$\Rightarrow y \in \text{Ker}(f) \otimes C + C \otimes \text{Ker}(f)$

$\Rightarrow \Delta(\text{Ker}(f)) \subseteq \text{Ker}(f) \otimes C + C \otimes \text{Ker}(f)$.

Consequently $\text{Ker}(f)$ is a coideal of $C$. ■

Theorem 2.8 Given two $R$-coalgebra $(C, \Delta, \epsilon)$ and $(C', \Delta', \epsilon')$. If $f : C \to C'$ is a coalgebra morphism, then $\text{Image}(f)$ is a subcoalgebra of $C'$.

Proof.
Let $f$ is a coalgebra morphism between $C$ to $C'$. We want to show that $\text{Image}(f)$ is a subcoalgebra of $C$. From the module theory for every module homomorphism $f$, $\text{Image}(f)$ is a submodule $C'$, then we only need to prove that $\Delta'(\text{Image}(f)) \subseteq \text{Image}(f) \otimes \text{Image}(f)$. Suppose $\alpha \in \Delta'(\text{Image}(f))$ i.e $\alpha = \Delta'(y)$ for some $y \in \text{Image}(f)$ or $\alpha = \Delta'(y') = \Delta'(f(x))$. Since $f$ is a coalgebra morphism then $\Delta \circ f = (f \otimes f) \circ \Delta$ then we have $\Delta'(f(x)) = f \otimes f \circ \Delta(x)$

$\Rightarrow \Delta'(f(x)) = f \otimes f(\Delta(x))$

$\Rightarrow \Delta'(f(x)) = f \otimes f(\sum x_i \otimes x_2)$ for some $x_i, x_2 \in C$

$\Rightarrow \Delta'(f(x)) = \sum f(x_i) \otimes f(x_2)$ for some $x_i, x_2 \in C$

So $\Delta'(f(x)) \subseteq \text{Image}(f) \otimes \text{Image}(f)$ or $\text{Image}(f)$ is a subcoalgebra of $C'$. ■

3. Some properties of Coideals
For any left $R$-module $N$ and right $R$-module $M$ we always have tensor product from $N$ and $M$ denoted by $N \otimes_R M$. We can study properties of tensor product in [4, 5]. We use tensor product properties to prove that intersection and sum of a finite family of subcoalgebra (coideal) is subcoalgebra (coideal). The following Lemma is used to determine proof of the next theorem.

Lemma 3.1 Given $U, V$ are $R$-modules. If $\{U_i\}_{i\in I}$ is family of submodules $U$ and $\{V_i\}_{i\in I}$ is family of submodules $V$, then $\bigcap_{i\in I} U_i \otimes V_i = \bigcap_{i\in I} U_i \otimes \bigcap_{i\in I} V_i$.

Proof.
First we show that \( \bigcap_{i \in I} U_i \otimes V_i \subseteq \bigcap_{i \in I} U_i \otimes \bigcap_{i \in I} V_i \). Let \( a \otimes b \in \bigcap_{i \in I} U_i \otimes V_i \) then \( a \otimes b \in U_i \otimes V_i \) for every \( i \in I \). It is implies that \( a \in U_i \) and \( b \in V_i \) for every \( i \in I \). Consequently \( a \in \bigcap_{i \in I} U_i \) and \( b \in \bigcap_{i \in I} V_i \) or \( a \otimes b \in \bigcap_{i \in I} U_i \otimes \bigcap_{i \in I} V_i \).

Second we show that \( \bigcap_{i \in I} U_i \otimes V_i \supseteq \bigcap_{i \in I} U_i \otimes \bigcap_{i \in I} V_i \). Let \( a \otimes b \in \bigcap_{i \in I} U_i \otimes \bigcap_{i \in I} V_i \) then \( a \in \bigcap_{i \in I} U_i \) and \( b \in \bigcap_{i \in I} V_i \). It is means that \( a \in U_i \) and \( b \in V_i \) for every \( i \in I \). It is implies that \( a \otimes b \in U_i \otimes V_i \) for any \( i \in I \) then \( a \otimes b \in \bigcap_{i \in I} U_i \otimes V_i \). From the first and second step we have that
\[
\bigcap_{i \in I} U_i \otimes V_i = \bigcap_{i \in I} U_i \otimes \bigcap_{i \in I} V_i .
\]

**Lemma 3.2** If \( I, J \) is a \( R \)-submodule \( M \), then \( I \otimes I + J \otimes J = I + J \otimes I + J \)

**Proof.**
1. We will prove that \( I \otimes I + J \otimes J \subseteq I + J \otimes I + J \). For any \( a_i \otimes a_j + b_i \otimes b_j \in I \otimes I + J \otimes J \).

   By existence of tensor product
   \[
a_i \otimes a_j + b_i \otimes b_j = (a_i, a_j) + (b_i, b_j) = (a_i + b_i, a_j + b_j) = a_i + b_i \otimes a_j + b_j \quad \text{and} \quad a_i + b_i \otimes a_j + b_j \in I + J \otimes I + J \,
   \]

   or \( I \otimes I + J \otimes J \subseteq I + J \otimes I + J \)

2. We want to show that \( I \otimes I + J \otimes J \supseteq I + J \otimes I + J \). For every \( \alpha \in I + J \otimes I + J \) means that \( \alpha = a \otimes b \) for some \( a, b \in I + J \) and then

   \[
   \alpha = a \otimes b = (a, b) \quad \text{for some} \quad a, b \in I + J
   \]

   \[
   \Rightarrow \alpha = a \otimes b = (a_i + a_j, b_i + b_j) \quad \text{for some} \quad a_i, b_i \in I \quad \text{and} \quad a_j, b_j \in J
   \]

   \[
   \Rightarrow \alpha = (a_i, b_i) + (a_j, b_j) \quad \text{for some} \quad a_i, b_i \in I \quad \text{and} \quad a_j, b_j \in J
   \]

   \[
   \Rightarrow \alpha = a_i \otimes b_i + a_j \otimes b_j \quad \text{for some} \quad a_i, b_i \in I \quad \text{and} \quad a_j, b_j \in J
   \]

   \[
   \Rightarrow \alpha = a_i \otimes b_i + a_j \otimes b_j \in I \otimes I + J \otimes J
   \]

From 1 and 2 \( I \otimes I + J \otimes J = I + J \otimes I + J \). ■

Lemma 3.2 can be generalize for the set of finite family of submodules.

This paper aims to prove some properties of coideal over coalgebra. The following theorem describes that intersections of two subcoalgebra (coideal) are also subcoalgebra (coideal).

**Theorem 3.3** If \( I, J \) are subcoalgebra of coalgebra \( (C, \Delta, \varepsilon) \), then intersection of I and J is a subcoalgebra of C.

**Proof.**
Suppose \( I, J \) are subcoalgebra of \( C \), then \( \Delta(I) \subseteq I \otimes I \) and \( \Delta(J) \subseteq J \otimes J \). We will see that \( \Delta(I \cap J) \subseteq I \cap J \otimes I \cap J \). Let \( x \in \Delta(I \cap J) \) then \( x = \Delta(y) \) for some \( y \in I \) and \( y \in J \). So we have \( x = \Delta(y) \) for some \( y \in I \) and \( x = \Delta(y) \) for some \( y \in J \) or \( x \in \Delta(I) \cap \Delta(J) \)

\[
\Rightarrow x \in \Delta(I) \cap \Delta(J) \subseteq I \otimes I \cap J \otimes J
\]
\( \Rightarrow x \in I \otimes I \cap J \otimes J = I \cap J \otimes I \cap J \) (from Lemma 3.1)

Consequently \( \Delta(I \cap J) \subseteq I \cap J \otimes I \cap J \) and implies that \( I \cap J \) is a subcoalgebra of \( C \). \( \blacksquare \)

Theorem 3.3 can be generalized for the family of subcoalgebra \( \{U_i\}_{i \in I} \) of \( C \) and we have \( \bigcap_{i \in I} U_i \) is also a subcoalgebra of \( C \). In the next theorem, we will prove that the sum of two subcoalgebra is also subcoalgebra.

**Theorem 3.4** If \( I, J \) are subcoalgebra of coalgebra \( (C, \Delta, \varepsilon) \), then \( I + J \) is a subcoalgebra of \( C \).

**Proof.**

Suppose \( I, J \) are subcoalgebra of coalgebra \( (C, \Delta, \varepsilon) \), i.e \( \Delta(I) \subseteq I \otimes I \) and \( \Delta(J) \subseteq J \otimes J \). For any \( x \in \Delta(I + J) \) we prove that \( x \in I + J \otimes I + J \). Since \( x \in \Delta(I + J) \), then \( x = \Delta(a + b) \) for some \( a \in I \) and \( b \in J \). Comultiplication \( \Delta \) is module homomorphism then so \( x = \Delta(a + b) = \Delta(a) + \Delta(b) \) for some \( a \in I \) and \( b \in J \)

\[ \Rightarrow x \in \Delta(I) + \Delta(J) \subseteq I \otimes I + J \otimes J \]

\[ \Rightarrow x \in I \otimes I + J \otimes J = I + J \otimes I + J \] (Lemma 3.2)

It is mean that \( I + J \) is a subcoalgebra of \( C \). \( \blacksquare \)

A subcoalgebra not always be a coideal, but properties of subcoalgebra which are given on the Theorem above can be satisfied on coideal over coalgebra.

**Theorem 3.5** If \( I, J \) are two left (right) coideal of \( R \)-coalgebra \( (C, \Delta, \varepsilon) \), then

1. \( I \cap J \) is a left (right) coideal of \( C \);
2. \( I + J \) is a left (right) coideal of \( C \).

**Proof.**

1. we only show for left coideal case, because for the right side it is analog by moving \( C \) on the right side. Suppose \( I, J \) are left coideal of \( C \) i.e \( I \subseteq Ker(\varepsilon) \), \( J \subseteq Ker(\varepsilon) \), \( \Delta(I) \subseteq C \otimes I \) and \( \Delta(J) \subseteq C \otimes J \). Let \( x \in \Delta(I \cap J) \) then \( x = \Delta(y) \) for some \( y \in I \cap J \) then

\[ \Rightarrow x = \Delta(y) \) for some \( y \in I \) and \( y \in J \)

\[ \Rightarrow x = \Delta(y) \) for some \( y \in I \) and \( x = \Delta(y) \) for some \( y \in J \)

\[ \Rightarrow x \in \Delta(I) \) and \( x \in \Delta(J) \)

\[ \Rightarrow x \in \Delta(I \cap J) \)

\[ \Rightarrow x \in \Delta(I \cap J) \subseteq C \otimes I + C \otimes J \]

\[ \Rightarrow x \in \Delta(I \cap J) \subseteq C \otimes I + C \otimes J = C + C \otimes I + J \]

\[ \Rightarrow x \in C \otimes I + J \)

Then we prove that \( \Delta(I \cap J) \subseteq C \otimes I + J \) or \( I \cap J \) is left coideal of \( C \).

2. It is easy to prove the second statement by apply Theorem 3.4 for coideal case. Suppose \( I, J \) are left coideal of \( C \) i.e \( I \subseteq Ker(\varepsilon) \), \( J \subseteq Ker(\varepsilon) \), \( \Delta(I) \subseteq C \otimes I \) and \( \Delta(J) \subseteq C \otimes J \). Since \( I \subseteq Ker(\varepsilon) \) and \( J \subseteq Ker(\varepsilon) \), then \( I + J \subseteq Ker(\varepsilon) + Ker(\varepsilon) \subseteq Ker(\varepsilon) \). For second axioms let \( x \in \Delta(I + J) \), then
\[ x \in \Delta(I + J) = \Delta(I) + \Delta(J) \]
\[ \Rightarrow x \in \Delta(I) + \Delta(J) \subseteq C \otimes I + C \otimes J \]
\[ \Rightarrow x \in C \otimes I + C \otimes J = C \otimes I + J \]

then \( x \in C \otimes I + J \) or implies that \( I + J \) is left coideal of \( C \). ■

Corollary 3.6 If \( I, J \) are two coideal of coalgebra \( C \), then \( I + J, I \cap J \) is coideal of coalgebra \( C \).

Proof.
Let \( I, J \) are coideal of \( C \) i.e. \( I, J \) are left and right coideal. By Theorem 3.6, \( I + J, I \cap J \) are left and right coideal then so \( I + J, I \cap J \) are coideal of the coalgebra \( C \). ■

We can generalize Corollary 3.6 for the infinite family of coideal of \( C \) \( \{U_i\}_{i \in I} \). It means that \( \bigcap_{i \in I} U_i \) and \( \sum_{i \in I} U_i \) is coideal of \( C \). A coring is generalization of coalgebra. Morphism and Kernel of corings define by a similar pattern with coalgebra. Coideals of an \( A \)-coring \( C \) defined as the kernels of surjective \( A \)-corings morphism \( C \rightarrow C' \). It implies that all of the properties of coideal over corings in this paper is also proved for corings by a similar technique of proof.

4. Conclusions
Let \( C \) be an \( R \)-coalgebra. We show that for a surjective coalgebra morphism \( f \) with domain \( C \), \( D = \text{Ker}(f) \) if only and if \( \Delta(D) \subseteq D \otimes C + C \otimes D \). Coideal on coalgebras has the similar properties with ideal on rings. Using the tensor product properties, we can prove that intersection and sum of finite coideal over \( R \)-coalgebra \( C \) is also coideal. A Coring is a generalization of coalgebra. Properties of coideal over a coalgebra can preserve on the structure of coideal over corings, but we should be carrefuly with left and right scalar multiplication on modules.

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