ACCESSIBILITY OF DERIVED-FROM-ANOSOV SYSTEMS

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ABSTRACT. This paper shows any non-accessible partially hyperbolic diffeomorphism on the 3-torus which is homotopic to Anosov must in fact be Anosov.

1. INTRODUCTION

Let $M$ be a closed Riemannian manifold, and $\text{Diff}^r(M)$ be the space which consists of all $C^r$-diffeomorphisms ($r \geq 1$) of $M$ and is endowed with $C^r$-topology. We say $f \in \text{Diff}^r(M)$ is partially hyperbolic if there exists a continuous $Df$-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ and continuous functions $\sigma, \mu : M \to \mathbb{R}$, such that $0 < \sigma < 1 < \mu$ and

$$\|Df(v^\ast)\| < \sigma(p) < \|Df(v^c)\| < \mu(p) < \|Df(v^u)\|$$

for every $x \in M$ and unit vector $v^\ast \in E^\ast(x)$, for $\ast = s, c, u$. Let $\text{PH}^r(M)$ denote the set consisting of all partially hyperbolic diffeomorphisms on $M$. It is obvious that $\text{PH}^r(M)$ is an open set in $\text{Diff}^r(M)$ within the $C^r$-topology.

Since Pugh and Shub proposed the Stable Ergodicity Conjecture, the study of partially hyperbolic diffeomorphisms has been one of the main topics of research in dynamical systems. A key concept for studying the Stable Ergodicity Conjecture is accessibility.

For every $f \in \text{PH}^r(M)$, the stable bundle $E^s$ and unstable bundle $E^u$ are locally uniquely integrable. There exist two families of $f$-invariant foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ tangent to $E^s$ and $E^u$ everywhere. We say $f$ is accessible if for every $x, y \in M$, there exists a piecewise smooth curve from $x$ to $y$, such that each smooth piece is contained in a leaf of either $\mathcal{F}^s$ or $\mathcal{F}^u$.

Accessibility is an open dense property for partially hyperbolic diffeomorphisms with one-dimensional center bundle [Did03, RHRHU08a, BHH+08]. In this paper, we study the accessibility for certain 3-dimensional partially hyperbolic diffeomorphisms.

The simplest 3-manifold supporting partially hyperbolic diffeomorphisms is the 3-torus $\mathbb{T}^3$. For every $f \in \text{Diff}^r(\mathbb{T}^3)$, we call the induced action $f_\ast$ on $\pi_1(\mathbb{T}^3) = \mathbb{Z}^3$ the linear part of $f$. We have $f_\ast \in \text{GL}(3, \mathbb{Z})$. If $f \in \text{PH}^r(M)$, then there are two possibilities its linear part:

- either $f_\ast \in \text{GL}(3, \mathbb{Z})$ has an eigenvalue equal to -1 or 1;

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• or \( f_* \in \text{GL}(3, \mathbb{Z}) \) is Anosov, i.e. every eigenvalue of \( f_* \) has modulus not equal to 1.

In the second case, the moduli of three eigenvalues of \( f_* \) are distinct, and so \( f_* \in \text{GL}(3, \mathbb{Z}) \) is also a partially hyperbolic diffeomorphism on \( T^3 = \mathbb{R}^3/\mathbb{Z}^3 \).

**Definition.** We say \( f \in \text{PH}^r(T^3) \) is a derived-from-Anosov diffeomorphism or a DA-diffeomorphism, if its linear part \( f_* \in \text{GL}(3, \mathbb{Z}) \) is Anosov.

It is clear that the set of partially hyperbolic derived-from-Anosov diffeomorphisms is an open set in \( \text{Diff}^r(T^3) \). The study of derived-from-Anosov diffeomorphisms can be originated to R. Mañé, who constructed a robustly topological mixing derived-from-Anosov diffeomorphism on \( T^3 \), which is not Anosov [Mn78].

Recent years have seen many works studying various aspects of derived-from-Anosov systems. Every DA-diffeomorphism is dynamically coherent and leaf conjugate to its linear part [BB109, Ham11, Pot15]. There exists an open set of conservative DA-diffeomorphisms whose center Lyapunov exponents have a different sign than their linear part [PT14], and for such examples the center foliation is “minimal yet measurable” [PTVa14]. The disintegration of measure along the center foliation has been studied in detail [VY17]. Every conservative partially hyperbolic DA-diffeomorphism is ergodic [HU14, GS19]. In certain settings, it is further known to be Bernoulli [PTVa18].

In this paper, we study the accessibility of partially hyperbolic derived-from-Anosov diffeomorphisms.

**Theorem 1.1.** Let \( f : T^3 \to T^3 \) be a \( C^{1+\alpha} \) partially hyperbolic derived-from-Anosov diffeomorphism. If \( f \) is not accessible, then it is Anosov.

This theorem shows that every robustly mixing non-hyperbolic derived-from-Anosov diffeomorphism constructed in Mañé’s paper [Mn78] is accessible. Applying Theorem 5.1 of [GST19], we have the following corollary.

**Corollary 1.2.** A \( C^{1+\alpha} \) partially hyperbolic derived-from-Anosov diffeomorphism \( f : T^3 \to T^3 \) is not accessible, if and only if it is Anosov and every periodic point of \( f \) has the same center Lyapunov exponent with its linear part \( f_* \).

In particular, if a \( C^{1+\alpha} \) partially hyperbolic derived-from-Anosov diffeomorphism is not accessible, then it has an invariant minimal foliation tangent to \( E^s \oplus E^u \).

Hertz-Hertz-Ures studied the 3-dimensional conservative non-accessible partially hyperbolic diffeomorphisms in [RHRHU08b]. They showed such a diffeomorphism satisfies one of the following properties:

1. there is an \( f \)-periodic torus tangent to \( E^s \oplus E^u \);
2. there is an \( f \)-invariant lamination \( \phi \neq \Gamma(f) \neq M \) tangent to \( E^s \oplus E^u \) without compact leaves; or
3. there is a Reebless invariant foliation tangent to \( E^s \oplus E^u \).
The examples of Case 1 are defined on mapping tori whose maps commute with an Anosov automorphism on the 2-torus. These systems, called AB-systems in [Ham17], can be totally classified. The only known examples of Case 3 are either AB-systems or Anosov diffeomorphisms on the 3-torus $\mathbb{T}^3$. There are no known examples in Case 2, but it is an open question if such systems exist.

**Question 1.3.** Does there exist a partially hyperbolic diffeomorphism $f$ with one-dimensional center bundle, such that it has an $f$-invariant lamination $\mathcal{L} \neq \Gamma(f)$ not tangent to $E^s \oplus E^u$ without compact leaves?

Our result shows that, for a partial hyperbolic Anosov automorphism on $\mathbb{T}^3$, it is impossible to make a Hopf bifurcation in the center direction while preserving a lamination tangent to $E^s \oplus E^u$. This gives a strong evidence that this question has a negative answer in dimension 3.

Our work is also motivated by the following major open question for partially hyperbolicity on the 3-torus.

**Question 1.4.** Is every partially hyperbolic derived-from-Anosov diffeomorphism on $\mathbb{T}^3$ transitive?

We can show that if a partially hyperbolic derived-from-Anosov diffeomorphism $f$ has an invariant lamination $\mathcal{L} \neq \Gamma(f)$ not tangent to $E^s \oplus E^u$, then $f$ is chain-transitive but not transitive. However the results here show that it is impossible to construct a non-transitive partially hyperbolic derived-from-Anosov diffeomorphism in this way. We recommend [Pot14a] for more discussions about the transitivity of partially hyperbolic DA-diffeomorphisms.

**Organization of the paper:** In Section 2, we study the semi-conjugacy and the lift of invariant foliations in the universal cover $\mathbb{R}^3$. In Section 3, we prove a series of properties of the minimal invariant $su$-laminations on $\mathbb{T}^3$. In Section 4, we prove the main theorem at Proposition 4.8.

2. **Laminations on the Universal Cover**

This section analyzes the dynamics when lifted to the universal cover $\mathbb{R}^3$. Later sections will use the results proved here in order to analyse the original system on the 3-torus. Before looking at the partially hyperbolic system, we first establish a property for curves on $\mathbb{R}^2$ which will be of use later.

**Proposition 2.1.** Let $S$ be a collection of curves in $\mathbb{R}^2$ and $X$ be a dense subset of $\mathbb{R}^2$ with the following properties:

1. each curve in $S$ is the graph $\text{graph}(g) = \{(x, g(x)) : x \in \mathbb{R}\}$ of a continuous function $g : \mathbb{R} \to \mathbb{R}$;
2. no two curves topologically cross; and
3. if $\gamma$ is a curve in $S$ and $(x_0, y_0) \in X$, then the translation $\gamma + (x_0, y_0)$ is also a curve in $S$.

Then the curves in $S$ are straight lines and all have the same slope.
Remark. If graph\(g_1\), graph\(g_2\) are curves in \(S\), then the condition of no topological crossings implies that either \(g_1(x) \leq g_2(x)\) holds for all \(x \in \mathbb{R}\) or \(g_2(x) \leq g_1(x)\) holds for all \(x \in \mathbb{R}\).

Proof. Consider the closure of \(S\) in the compact-open topology. That is, the graph of a function \(g : \mathbb{R} \to \mathbb{R}\) belongs to \(\check{S}\) if and only if there is a sequence \{graph\(g_n\)\} in \(S\) such that \(g_n|_K\) converges uniformly to \(g|_K\) on every compact subset \(K\) of \(\mathbb{R}\). One can show that the no crossing condition on \(S\) implies no crossing condition on \(\check{S}\). If \(\gamma_n \to \gamma\) in the compact-open topology and \((x_0, y_0) \in X\), then the translates \(\gamma_n + (x_0, y_0)\) converge to \(\gamma + (x_0, y_0)\) in the compact-open topology. Hence, \(\check{S}\) is invariant under all translates \((x_0, y_0) \in X\). For \(\gamma \in \check{S}\) and any point \((x_0, y_0) \in \mathbb{R}^2\), let \((x_n, y_n)\) be a sequence in \(X\) converging to \((x_0, y_0)\). Then \(\gamma + (x_n, y_n)\) converges to \(\gamma + (x_0, y_0)\) in the compact-open topology and so \(\gamma + (x_0, y_0) \in \check{S}\).

Now knowing that \(\check{S}\) is invariant under all translations in \(\mathbb{R}^2\), we can show it is linear. Indeed, let graph\((g)\) be a curve in \(\check{S}\) passing through the origin so that \(g(0) = 0\). For \(a, b \in \mathbb{R}\), define functions \(g_n : \mathbb{R} \to \mathbb{R}\) by \(g_n(x) = g(x - a) + g(a) + \frac{1}{n}\). Then \(g_n(a) > g(a)\) and the no crossing condition imply that
\[
g(a) + g(b) + \frac{1}{n} = g_n(a + b) \geq g(a + b)
\]
for all \(n\). A similar argument shows that \(g(a) + g(b) - \frac{1}{n} \leq g(a + b)\) for all \(n\), and so \(g(a) + g(b) = g(a + b)\). As \(g\) is continuous and additive, it is linear. The translates of graph\((g)\) produce a linear foliation on all of \(\mathbb{R}^2\) that no other curve of \(\check{S}\) can cross. This implies that every curve of \(\check{S}\) is linear and of the same slope. \(\square\)

With proposition 2.1 established, we now consider the dynamics. Let \(f : \mathbb{T}^3 \to \mathbb{T}^3\) be partially hyperbolic and homotopic to an Anosov diffeomorphism. Lift \(f\) to a map on the universal cover. All of the analysis in this section will be on \(\mathbb{R}^3\), and so we again use \(f\) to denote the lifted map. This lift \(f : \mathbb{R}^3 \to \mathbb{R}^3\) is at finite distance from a hyperbolic linear map \(A : \mathbb{R}^3 \to \mathbb{R}^3\). Here \(A = f_s\) is the linear part of \(f\). Up to replacing \(f\) by its inverse, we may assume the center direction of \(A\) is contracting. That is, the logarithms of the eigenvalues of \(A\) satisfy
\[
\lambda^s(A) < \lambda^c(A) < 0 < \lambda^u(A).
\]
Many properties have been established for \(f\), first in the absolutely partially hyperbolic case [H08][BB09][Ham13] and then extended to the case of pointwise partial hyperbolicity [HP14].

There are unique invariant foliations tangent to the bundles \(E^u, E^s, E^{cs}, E^{cu}\), and \(E^c\) of \(f\). Denote these foliations by \(\mathcal{F}^u, \mathcal{F}^s, \mathcal{F}^{cs}, \mathcal{F}^{cu}\), and \(\mathcal{F}^c\). For the linear map \(A\), we adopt the notation used in [HP15][HP18] and write \(\mathcal{A}^u, \mathcal{A}^s, \mathcal{A}^{cs}, \mathcal{A}^{cu}, \mathcal{A}^c\), and \(\mathcal{A}^{us}\) for the invariant linear foliations of \(A\). For both \(f\) and \(A\), all of these foliations have quasi-isometrically embedded leaves [HP14 Theorem 3.5][Ham13 Proposition 2.6]. That is, there is \(Q > 1\) such that if \(x\) and \(y\) lie on the same leaf of the foliation, then \(d_{\mathcal{F}}(x, y) < Q\|x - y\| + Q\) where \(\|x - y\|\) is the
usual distance in $\mathbb{R}^3$ and $d_{\mathcal{F}}(x, y)$ is distance measured along the leaf. In this section, we use $d_u, d_c, d_s$ to denote distance measured along leaves associated to the non-linear system $f$.

The foliations have global product structure [Ham13, Proposition 2.15]. That is, for $x, y \in \mathbb{R}^3$, the following pairs of sets intersect in a unique point:

1. $\mathcal{F}^c u(x)$ with $\mathcal{F}^u(y)$,
2. $\mathcal{F}^c u(x)$ with $\mathcal{F}^c(y)$,
3. $\mathcal{F}^c(x)$ with $\mathcal{F}^u(y)$ if $x \in \mathcal{F}^c u(y)$, and
4. $\mathcal{F}^c(x)$ with $\mathcal{F}^s(y)$ if $x \in \mathcal{F}^c s(y)$.

There is a semiconjugacy [Fra70], a continuous surjective map $h : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies $h(f(x)) = A(h(x))$ and $h(x+z) = h(x) + z$ for all $x \in \mathbb{R}^3$ and $z \in \mathbb{Z}^3$. Further, $h$ is a finite distance from the identity map on $\mathbb{R}^3$.

For the center, center-stable, and center-unstable foliations, $h$ defines a bijection on the spaces of leaves [Ham13, §3]. That is, if $x \in \mathbb{R}^3$ and $v = h(x)$, then $h(\mathcal{F}^c(x)) = \mathcal{A}^c(v)$ and $h^{-1}(\mathcal{A}^c(v)) = \mathcal{F}^c(x)$. Similar equalities hold for $c_s$ and $c u$ in place of $c$. The restriction $h|_{\mathcal{F}^c(x)} : \mathcal{F}^c(x) \to \mathcal{A}^c(v)$ is a continuous surjective map, but in general it is not a homeomorphism. Throughout this section, we use the letter $v$ to denote a point in $\mathbb{R}^3$ associated to the linear dynamics of $A$ and $x$ and $y$ to denote points associated to the non-linear dynamics of $f$.

We assume all of the foliations have been given an orientation. For points $x$ and $y$ on a one-dimensional leaf, write $x = y$, $x < y$, or $x > y$ to denote their relative positions with respect to this orientation.

**Proposition 2.2.** The semiconjugacy is monotonic along center leaves. That is, the orientations may be chosen so that if $L \in \mathcal{F}^c$ and $h(L) \in \mathcal{A}^c$ is its image, then $x \leq y$ implies $h(x) \leq h(y)$ for all points $x, y \in L$.

This result is well known, but a proof does not appear to be given anywhere in the prior literature.

**Proof.** Suppose $h$ is not monotonic along a center leaf $L$. Then there are points $x < y < z$ along $L$ such that $h(x) = h(z) \neq h(y)$. As $A^{-1}$ expands the linear center direction, $\|A^{-n} h(x) - A^{-n} h(y)\| = \|hf^{-n}(x) - hf^{-n}(y)\| \to \infty$ as $n \to \infty$. As $h$ is a finite distance from the identity, it follows that $d_c(f^{-n}(x), f^{-n}(y)) \to \infty$. The same analysis shows $d_c(f^{-n}(y), f^{-n}(z)) \to \infty$ and since the points along the leaf have the ordering $f^{-n}(x) < f^{-n}(y) < f^{-n}(z)$, it follows that $d_c(f^{-n}(x), f^{-n}(z)) \to \infty$ as well. Using that center leaves are quasi-isometrically embedded, one can show that $\|A^{-n} h(x) - A^{-n} h(z)\| \to \infty$ which contradicts the fact that $h(x) = h(z)$.

**Corollary 2.3.** For each $v \in \mathbb{R}^3$, the preimage $h^{-1}(v)$ consists either of a single point or a compact segment inside a center leaf.

**Proof.** As $h$ is surjective, $h^{-1}(v)$ is non-empty. Let $L \in \mathcal{F}^c$ be such that $h(L)$ contains $v$. As $h$ is a bijection on the spaces of center leaves, it follows that
$h^{-1}(v) \subset L$. As $L$ is properly embedded and $h$ is a finite distance from the identity, $h^{-1}(v)$ is a compact subset of $L$. As $h|_L$ is monotonic, $h^{-1}(v)$ is connected. \hfill \Box

We now show $h$ also defines a bijection between the spaces of unstable leaves.

**Proposition 2.4.** For an unstable leaf $L \in \mathcal{F}_u$ of $f$, the image $h(L)$ is an unstable leaf of $A$, and $h|_L$ is a homeomorphism.

**Proof.** Suppose $x, y \in L$. Then $\|f^{-n}(x) - f^{-n}(y)\| \to 0$ as $n \to \infty$. As $h$ is uniformly continuous and a semiconjugacy, it follows that $\|A^{-n}h(x) - A^{-n}h(y)\| \to 0$ which is only possible if $h(x)$ and $h(y)$ are on the same linear unstable leaf. If $x \neq y$, then $\|f^n(x) - f^n(y)\| \to \infty$ from which one can show that $h(x) \neq h(y)$. As an injective proper map from one copy of $\mathbb{R}$ to another, $h|_L$ must be a homeomorphism. \hfill \Box

**Proposition 2.5.** If $L \in \mathcal{F}_c$ is a stable leaf of $f$, then $h(L)$ is a continuous curve embedded in a center stable leaf of $A$. In this linear center-stable leaf, each linear center leaf intersects $h(L)$ exactly once.

**Proof.** The stable leaf $L$ lies in a center-stable leaf of $f$ and by global product structure $L$ intersects every center subleaf exactly once. As $h$ is a bijection on the spaces of center and center-stable leaves, the result follows. \hfill \Box

We now assume that $f$ is not accessible. Then there is a non-empty lamination $\Gamma \subset \mathbb{R}^3$ consisting of the non-open accessibility classes [RHRHU08a]. Call these the $us$-leaves of $f$. By global product structure, each $us$-leaf is bifoliated by stable and unstable leaves and intersects every center leaf of $f$ exactly once.

Under this assumption, a stable analogue of proposition 2.4 holds.

**Proposition 2.6.** For a stable leaf $L^s$ of $f$, the image $h(L^s)$ is a (strong) stable leaf of $A$, and $h|_{L^s}$ is a homeomorphism.

**Proof.** This is an adaptation of the argument given in [HU14, §6]. The key idea is to intersect the $us$-leaves with one fixed center-stable leaf, apply $h$ to the resulting collection of stable leaves and show that the images satisfy the hypotheses of proposition 2.1. We now give the details.

Consider the linear center-stable leaf $A^{cs}(0)$ passing through the origin in $\mathbb{R}^3$. The pre-image $h^{-1}(A^{cs}(0))$ is a center-stable leaf of $f$. For any $us$-leaf $L \in \Gamma$, the intersection $L \cap h^{-1}(A^{cs}(0))$ is a stable leaf and so its image $h(L) \cap A^{cs}(0)$ satisfies the conclusions of proposition 2.5. Define a set of curves in $A^{cs}(0)$ by

$$S = \{ h(L) \cap A^{cs}(0) : L \in \Gamma \}.$$  

Since $h$ is monotonic along center leaves, the curves in $S$ do not topologically cross. For $z \in \mathbb{Z}^3$, define a translation $\tau_z : A^{cs}(0) \to A^{cs}(0)$ by setting $\tau_z(v)$ to the unique intersection of $A^u(v + z)$ with $A^{cs}(0)$. If $\gamma = h(L) \cap A^{cs}(0)$ is a curve in $S$, proposition 2.4 shows that $\tau_z(\gamma) = h(L + z) \cap A^{cs}(0)$ is also a curve in $S$. As the unstable foliation of a linear Anosov map on $\mathbb{T}^3$ is minimal [Fra70], the set of translations $\{ \tau_z : z \in \mathbb{Z}^3 \}$ is dense in the set of all rigid translations of $A^{cs}(0)$. 


The collection of curves $S$ satisfies the hypotheses of proposition 2.1, where $\mathcal{A}^c(0)$ is identified with $\mathbb{R}^2$ and the linear center foliation is identified with vertical lines on $\mathbb{R}^2$. All curves in $S$ are thus linear. Since the collection $S$ is invariant under $A$, these curves must be aligned with the linear stable direction.

We have shown that for a $us$-leaf $L$ of $f$, the image $h(L)$ is a $us$-leaf for the linear map $A$. On $\mathbb{T}^3$, such linear $us$-leaves are dense in $\mathbb{T}^3$ and so on the universal cover the image of the closed set $\Gamma$ must be $h(\Gamma) = \mathbb{R}^3$. Consider now any stable leaf $L^s \in \mathcal{F}^s$. Since $L^s$ does not cross through any $us$-leaf of $f$, the monotonicity of $h$ implies that the image $h(L^s)$ does not topologically cross any leaf of the linear $us$-foliation. Hence, $h(L^s)$ must lie in a single linear $us$-leaf. It also lies in a single linear $cs$-leaf and so it is a linear stable leaf. That $h|_{L^s}$ is a homeomorphism is proved similarly to proposition 2.4. □

We now consider the set $Y \subset \mathbb{R}^3$ consisting of all points where $h$ is injective. That is, $y \in Y$ if and only if $h^{-1}(h(y)) = \{y\}$.

**Proposition 2.7.** The set $Y$ is a union of $us$-leaves.

**Proof.** We first show that the complement is $us$-saturated. If $x \in \mathbb{R}^3 \setminus Y$, then $x$ lies on a compact interval $J = h^{-1}(h(x))$. Using stable and unstable holonomies, we can map $J$ to a compact interval on any other center leaf and propositions 2.4 and 2.6 show that this other interval is also mapped to a point by $h$.

If $U \subset \mathbb{R}^3$ is an open accessibility class, then $h(U)$ is a single linear $us$-leaf. Any center segment contained in $U$ maps to a single point under $h$ and so $U$ and $Y$ are disjoint. This shows that $Y$ is a subset of $\Gamma$. □

**Proposition 2.8.** For any linear center leaf $\mathcal{A}^c(v)$, the set $\mathcal{A}^c(v) \setminus h(Y)$ is countable.

**Proof.** Let $x$ be such that $h$ maps $\mathcal{F}^c(x)$ to $\mathcal{A}^c(v)$. Then any point of $\mathcal{A}^c(v) \setminus h(Y)$ is the image of an interval of positive length in $\mathcal{F}^c(x)$ and there can only be countably many disjoint intervals of this form. □

**Proposition 2.9.** For a point $x \in \mathbb{R}^3$, the following are equivalent:

1. $x$ is in the closure of $Y$;
2. $h|_{\mathcal{F}^c(x)}$ is not locally constant at $x$;
3. either $x$ lies in $Y$ or $x$ is an endpoint of the interval $h(h^{-1}(x))$.

**Proof.** Since $h|_{\mathcal{F}^c(x)} : \mathcal{F}^c(x) \to \mathcal{A}^c(h(x))$ is continuous, surjective, and monotonic, it is straightforward to show (2) $\iff$ (3). We show (1) $\iff$ (2). Suppose $x \in \bar{Y} \setminus Y$. As $Y$ is $us$-saturated, there are points $y_n \in \mathcal{F}^c(x) \cap Y$ converging to $x$. The images $h(y_n)$ are distinct from each other and so $h|_{\mathcal{F}^c(x)}$ is not locally constant at $x$. Conversely, if $h|_{\mathcal{F}^c(x)}$ is not locally constant at $x$, then for any neighbourhood $x \in J \subset \mathcal{F}^c(x)$, the image $h(J)$ has positive length and by proposition 2.8 there is $v \in h(J) \cap h(Y)$ and so $h^{-1}(v) \in J \cap Y$. □
Proposition 2.10. For any us-leaf $L \in \Gamma$, the closure of $\bigcup_{z \in \mathbb{Z}^3} (L + z)$ contains $\bar{Y}$.

Remark. This shows that $\bar{Y}$ when projected down to a subset of $\mathbb{T}^3$ yields a minimal us-lamination. Moreover, this is the unique minimal us-lamination.

Proof. For a point $x \in \bar{Y}$, consider a short center segment $x \in J \subset \mathcal{F}^c(x)$ such that $h(J)$ has positive length. As $\bigcup_{z \in \mathbb{Z}^3} h(L + z)$ is a dense union of linear us-leaves, there is $z \in \mathbb{Z}^3$ such that $h(L + z)$ intersects the interior of $h(f)$. Then $h^{-1}(v)$ is contained in $J$ and so $L + z$ intersects $J$. □

Proposition 2.11. For any open set $U$ which intersects $\bar{Y}$, there is $x \in \bar{Y} \cap U, k \geq 1$, and $z \in \mathbb{Z}^3$ such that $f^k(x) = x + z$. That is, $x$ projects down to a periodic point on $\mathbb{T}^3$.

Proof. Using proposition 2.9, one can show that $h(U)$ has non-empty interior. This interior contains a point $v$ which projects down a periodic point for the linear Anosov diffeomorphism on $\mathbb{T}^3$; that is, there are $k \geq 1$ and $z \in \mathbb{Z}^3$ such that $A^k(v) = v + z$. By the leaf conjugacy, $f^k(h^{-1}(v)) + z = h^{-1}(v)$ and as $v$ is in the interior of $h(U)$, it follows that $h^{-1}(v)$ is contained in $U$. If $h^{-1}(v)$ is a singleton set, it is the desired point $x$. Otherwise, we may take either endpoint of the interval $h^{-1}(v)$ to be $x$. □

3. Minimal lamination and semiconjugacy on $\mathbb{T}^3$

We have now finished working on the universal cover. All of the results from now until the end of the paper will be for the original partially hyperbolic diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$. This diffeomorphism is homotopic to an linear Anosov diffeomorphism $A : \mathbb{T}^3 \to \mathbb{T}^3$ and the logarithms of the eigenvalues of $A$ satisfy

$$\lambda^s(A) < \lambda^c(A) < 0 < \lambda^u(A).$$

As before, assume that $f$ is non-accessible. Then there is a lamination $\Gamma \subset \mathbb{T}^3$ consisting of the non-open accessibility classes of $f$. By the work of Potrie [Pot14b], the lamination $\Gamma$ contains a unique minimal sublamination. This can also be seen directly from proposition 2.10 above. Let $\mathcal{F}^u$, $\mathcal{F}^s$, $\mathcal{F}^{cs}$, $\mathcal{F}^{cu}$, and $\mathcal{F}^c$ denote the invariant foliations of $f$ considered as foliations defined on $\mathbb{T}^3$, and similarly let $\mathcal{A}^u$, $\mathcal{A}^s$, $\mathcal{A}^{cs}$, $\mathcal{A}^{cu}$, $\mathcal{A}^c$, and $\mathcal{A}^{us}$ denote the invariant linear foliations of the toral automorphism $A$. We fix an orientation of $\mathcal{F}^c$ and of $\mathcal{A}^c$. Then each center leaf $\mathcal{F}^c(q)$ splits into two half-leaves

$$\mathcal{F}^c(q) \setminus \{q\} = \mathcal{F}^c_+(q) \cup \mathcal{F}^c_-(q),$$

where $+$ and $-$ are determined by the orientation of $\mathcal{F}^c$ in $\mathbb{T}^3$. For every $y \in \mathcal{F}^c_+(x)$, we let $[x, y]^c$ and $(x, y)^c$ denote the closed and open segments contained in $\mathcal{F}^c(x)$ with endpoints $x$ and $y$ respectively.

Let $h : \mathbb{T}^3 \to \mathbb{T}^3$ be the Franks semiconjugacy, now considered as a homeomorphism on the 3-torus. That is, $h$ is isotopic to the identity and $h(f(x)) =
$A(h(x))$ for all $x \in \mathbb{T}^3$. Corollary 2.3 implies an analogous result for the semiconjugacy on the 3-torus: for each $\nu \in \mathbb{T}^3$, the preimage $h^{-1}(\nu)$ consists either of a single point or a compact segment inside a center leaf.

If $x \in \mathbb{T}^3$ is such that $h^{-1}(h(x)) = \{x\}$, we define $x_+ = x_- = x$. If instead $h^{-1}(h(x))$ is a positive length interval, we define $x_+$ and $x_-$ to be the two endpoints of $h^{-1}(h(x))$ with corresponding orientation. That is, $x_+ \in F_+^c(x_-)$ and $x_- \in F_+^c(x_+)$. Define

$$\Lambda = \bigcup_{x \in \mathbb{T}^3} \{x_+, x_-\}.$$

**Proposition 3.1.** The set $\Lambda$ and semiconjugacy $h$ satisfy the following properties:

1. The set $\Lambda$ is a $u$-saturated minimal set, i.e. if $x \in \Lambda$, then
   $$\mathcal{F}^{us}(x) \subset \Lambda \quad \text{and} \quad \mathcal{F}^{us}(x) = \Lambda.$$

2. The periodic points of $f|_{\Lambda}$ are dense in $\Lambda$.

3. For every $x \in \Lambda$, if $x = x_+$, then $y = y_+$ for every $y \in \mathcal{F}^{us}(x) \subset \Lambda$; if $x = x_-$, then $y = y_-$ for every $y \in \mathcal{F}^{us}(x) \subset \Lambda$.

4. For $\sigma = u, s, us$, the semiconjugacy $h|_{\Lambda}$ maps a leaf of the $\sigma$-foliation of $f$ to a leaf of the $\sigma$-foliation of $\Lambda$:
   $$h(\mathcal{F}^\sigma(x)) = \mathcal{A}^\sigma(h(x)), \quad \forall x \in \Lambda.$$

Moreover, $h : \mathcal{F}^\sigma(x) \to \mathcal{A}^\sigma(h(x))$ is a homeomorphism.

**Proof.** These are all consequences of the results in section 2. Item (1) follows from propositions 2.9 and 2.10. Item (2) follows from propositions 2.11. Item (3) follows from propositions 2.4 and 2.6 and global product structure. □

**Proposition 3.2.** There exist constants $C_1 > 1$ and $0 < \alpha < 1$, such that for every $\sigma = u, s, us$, the homeomorphism $h|_{\mathcal{F}^\sigma(x)} : \mathcal{F}^\sigma(x) \to \mathcal{A}^\sigma(h(x))$ is bi-Hölder continuous, i.e. for every $x_1, x_2 \in \mathcal{F}^\sigma(x)$, we have

$$d_{\mathcal{F}^\sigma}(x_1, x_2) \leq C_1 \cdot d_{\mathcal{A}^\sigma}(h(x_1), h(x_2))^\alpha, \quad d_{\mathcal{A}^\sigma}(h(x_1), h(x_2)) \leq C_1 \cdot d_{\mathcal{F}^\sigma}(x_1, x_2)^\alpha.$$

**Proof.** We first prove the case $\sigma = u$ and $d_{\mathcal{F}^u}(x_1, x_2) \leq C \cdot d_{\mathcal{A}^u}(h(x_1), h(x_2))^\alpha$. There exists $\delta_0 > 0$, such that for every $x_1, x_2 \in \mathcal{F}^u(x) \subset \Lambda$, if $d_{\mathcal{A}^u}(h(x_1), h(x_2)) < \delta_0$, then $d_{\mathcal{F}^u}(x_1, x_2) < 1$. Otherwise, there exist $x_1^n, x_2^n \in \mathcal{F}^u(x^n) \subset \Lambda$ such that $d_{\mathcal{F}^u}(x_1^n, x_2^n) = 1$ and $d_{\mathcal{A}^u}(h(x_1^n), h(x_2^n)) \to 0$. Taking a subsequence if necessary, we have $x_1^n \to y_1$ and $x_2^n \to y_2$ with $y_2 \in \mathcal{F}^u(y_1)$, $d_{\mathcal{A}^u}(y_1, y_2) = 1$, and $h(y_1) = h(y_2)$. This contradicts the fact that $h$ is a homeomorphism on $\mathcal{F}^u(y_1)$.

Now we assume that $d_{\mathcal{A}^u}(h(x_1), h(x_2)) \ll \delta_0$. Let $k$ be the largest positive number such that

$$d_{\mathcal{A}^u}(A^k \circ h(x_1), A^k \circ h(x_2)) \ll \delta_0.$$

Then we have

$$d_{\mathcal{F}^u}(h(x_1), h(x_2)) \geq \exp \{-(k + 1) \cdot \lambda^u(A)\} \cdot \delta_0.$$
On the other hand, from the semiconjugacy and \(d_{s_{\bar{f}}} (A^k \circ h(x_1), A^k \circ h(x_2)) < \delta_0\), we have \(d_{s_{\bar{f}}} (f^k(x_1), f^k(x_2)) < 1\). This implies

\[
d_{s_{\bar{f}}} (x_1, x_2) < \mu^{-k},
\]

where \(\mu = \inf_{z \in \mathbb{T}^3} m(Df|_{E^u(z)}) > 1\).

If \(\mu \geq \exp \lambda^u (A)\), then we have

\[
d_{s_{\bar{f}}} (x_1, x_2) < \frac{\exp \lambda^u (A)}{\delta_0} \cdot d_{s_{\bar{f}}} (h(x_1), h(x_2)).
\]

Otherwise, we take \(0 < \alpha < 1\) such that \(\exp (\alpha \lambda^u (A)) < \mu\). Then we have

\[
d_{s_{\bar{f}}} (x_1, x_2) < \mu^{-k} \exp \left( -\alpha k \lambda^u (A) \right) \cdot d_{s_{\bar{f}}} (h(x_1), h(x_2))^\alpha.
\]

The proof of the other inequality and the case \(\sigma = s\) are the same. If \(h|_\Lambda\) is bi-Hölder continuous on every leaf of \(\mathcal{F}^s\) and \(\mathcal{F}^u\), then it is bi-Hölder continuous on every leaf of \(\mathcal{F}^u\).

\(\Box\)

**Proposition 3.3.** Let \(x, y \in \Lambda\) and a sequence of points \(x_n \in \mathcal{F}^u (x) \cap \mathcal{F}^c (y)\) such that

\[
x_{n+1} \in (y, x_n) \quad \text{and} \quad \lim_{n \to \infty} d_{\mathcal{F}^c} (x_n, y) = 0.
\]

For every \(z \in \mathcal{F}^c (x) \cap \Lambda\) satisfying \(h(z) \neq h(x)\), there exists \(\delta_z > 0\), such that if we denote \(h^{-1}_{x,z} : \mathcal{F}^c (x) \to \mathcal{F}^c (z)\) the holonomy map induced by \(\mathcal{F}^u\) in \(\Lambda\) from \(x\) to \(x_n\) and \(z_n = h^{-1}_{x,z} (z)\), then

\[
d_{\mathcal{F}^c} (x_n, z_n) \geq \delta_z.
\]

The same conclusion holds for the sequence of points \(x_n \in \mathcal{F}^u (x) \cap \mathcal{F}^c (y)\).

**Proof.** We denote \(\delta_1 = d_{s_{\bar{f}}} (h(x), h(x)) > 0\) and assume \(x = x_+\). If \(z \in \mathcal{F}^c (x)\), then \(h(z) \in \mathcal{F}^c (h(x))\). Since \(\lim_{n \to \infty} d_{\mathcal{F}^c} (x_n, y) = 0\), we have \(y = y_+\). There exists \(w_1 \in \mathcal{F}^c (y)\), such that \(d_{s_{\bar{f}}} (h(y), h(w_1)) = \delta_1\). Moreover, there exists \(N_1 > 0\), such that \(d_{s_{\bar{f}}} (h(x_N), h(y)) \leq \delta_1 / 2\). This implies \(w_1 \in (x_n, z_n)\) and \([x_n, w_1] ^c \subset [z_n, y] ^c\) for every \(n \geq N_1\). So we define

\[
\delta_z = \min \{ d_{\mathcal{F}^c} (x_1, z_1), \ldots, d_{\mathcal{F}^c} (x_{N_1-1}, z_{N_1-1}), d_{\mathcal{F}^c} (x_{N_1}, w_1) \}.
\]

If \(z \in \mathcal{F}^c (x)\), then \(h(z) \in \mathcal{F}^c (h(x))\). Since \(\lim_{n \to \infty} d_{\mathcal{F}^c} (x_n, y) = 0\), there exists \(N_2 > 0\), such that \(d_{s_{\bar{f}}} (h(x_{N_2}), h(y)) \leq \delta_1 / 2\). This implies \(y \in (z_n, x_n) ^c\) and \([z_n, y] ^c \subset [z_n, x_n] ^c\) for every \(n \geq N_2\). So we define

\[
\delta_z = \min \{ d_{\mathcal{F}^c} (x_1, z_1), \ldots, d_{\mathcal{F}^c} (x_{N_2-1}, z_{N_2-1}), d_{\mathcal{F}^c} (x_{N_2}, z_{N_2}) \}.
\]

This proves the case \(x = x_+\). The proof for \(x = x_\_\) is the same.

Define the real number \(\lambda^- = \inf (\lambda^c (p) : p \in \text{Per} (f|_\Lambda))\).

**Lemma 3.4.** If \(p \in \text{Per} (f|_\Lambda)\), then \(\lambda^c (p) \leq 0\). This implies \(\lambda^- \leq 0\).

**Proof.** Let \(p \in \text{Per} (f|_\Lambda)\) with period \(\pi\). Since \(\Lambda = \mathcal{Z} = \bigcap_{x \in \mathbb{T}^3} \partial^c h^{-1} (h(x))\) and \(us\)-saturated, there exists a sequence of points \(x_n \in Z\), such that \(x_n \in \mathcal{F}^c (p)\) and \(x_n\) converge to \(p\) in \(\mathcal{F}^c (p)\). Moreover, for every \(n\), there exists \(k_n > 0\), such that
$A^k_n(h(x_1))$ is between $h(x_n)$ and $h(p)$ in $\mathcal{A}^c(h(p))$. From the semi-conjugacy and $x_n \to p$ in $\mathcal{F}^c(p)$, we have
\[
\lim_{k \to +\infty} d(f^{k\pi}(x_1), p) = 0.
\]
This implies $\lambda^c(p) \leq 0$ for every $p \in \text{Per}(f|_{\Lambda})$. □

**Proposition 3.5.** If $\mu$ is an ergodic measure supported on $\Lambda$ and $\lambda^c(\mu) < 0$, then there exists a sequence of periodic points $q_n \in \Lambda$, such that
\[
\lim_{n \to \infty} \lambda^c(q_n) = \lambda^c(\mu).
\]

In particular, if $\mu$ is an ergodic measure supported on $\Lambda$, then its central Lyapunov exponent satisfies $\lambda^c(\mu) \geq \lambda^-$. 

**Proof.** Assume $\mu$ is an ergodic measure supported on $\Lambda$ with $\lambda^c(\mu) < 0$. Take a $\mu$-typical point $x$. This point is recurrent and has a Pesin stable manifold. The shadowing lemma of Pesin theory leads to a periodic point $q_n$ of period $\pi(q_n)$ with center Lyapunov exponent satisfying
\[
|\lambda^c(q) - \lambda^c(\mu)| < \min \left\{ \frac{1}{n}, \frac{|\lambda^c(\mu)|}{2} \right\}.
\]

Moreover, the Pesin stable manifold of $q_n$ transversely intersects the unstable manifolds $\mathcal{F}^u(x)$ of $x$. If we denote $y_n$ this intersecting point, then we have
\[
y \in \mathcal{F}^u(x) \subset \mathcal{F}^{us}(x) \subset \Lambda.
\]
Since $\Lambda$ is a compact invariant set, and $f^{k\pi(q_n)}(y) \to q_n$ as $k \to +\infty$, we have $q_n \in \Lambda$. □

**Proposition 3.6.** The inequality $\lambda^- \leq \lambda^c(\Lambda) < 0$ holds.

**Proof.** We only need to show that there exists an ergodic measure $\mu$ supported on $\Lambda$, such that $\lambda^c(\mu) < 0$. Actually, we consider the measure $\mu_0$ of maximal entropy of $f$, then [Ure12] shows that its support $\text{supp}(\mu_0) \subset \Lambda$.

Again, [Ure12] shows that $\lambda^c(\mu_0) \leq \lambda^c(\Lambda) < 0$. This proves that $\lambda^- \leq \lambda^c(\Lambda) < 0$. □

4. **Rigidity of center Lyapunov exponents**

In this section, we prove Proposition 4.3 which states that all periodic points in $\Lambda$ have the same center Lyapunov exponent. This implies that $\Lambda$ is hyperbolic. We then use this hyperbolicity to show that $\Lambda = \mathbb{T}^3$.

**Lemma 4.1.** For every $\epsilon > 0$, up to changing the metric, there is a point $p \in \text{Per}(f|_{\Lambda})$ such that
\[
\log \|Df|_{E^c(x)}\| > \lambda^c(p) - \epsilon
\]
holds for all $x \in \Lambda$. 

Proof. We have proved that \( \lambda^- \leq \lambda^c(\mu) \) for every ergodic measure \( \mu \) supported on \( \Lambda \). From the definition of \( \lambda^- \), there exists a sequence of periodic points \( p_n \in \text{Per}(f|\Lambda) \) such that \( \lim_{n \to \infty} \lambda^c(p_n) = \lambda^- < 0 \).

For every \( \varepsilon > 0 \), up to changing the metric, we have

\[
\log \|Df|E^s(x)\| > \lambda^- - \frac{\varepsilon}{2}
\]

for all \( x \in \Lambda \). Then we take \( p = p_n \) for \( n \) large enough which proves the lemma.

\[\Box\]

Lemma 4.2. There exist two constants \( C_2 > 0 \) and \( 0 < \beta < 1 \), such that for every two periodic points \( p, q \in \Lambda \), there exist two sequence of points \( x_n \in \mathcal{F}^s(p) \), \( y_n \in \mathcal{F}^u(x_n) \) with \( y_n \in \mathcal{F}^c(q) \), such that

\[
\lim_{n \to \infty} d_{\mathcal{F}^s}(y_n, q) = 0, \quad \text{and} \quad d_{\mathcal{F}^u}(x_n, y_n) \leq \frac{C_2}{D_n^{\beta}}, \quad \text{where} \quad D_n = d_{\mathcal{F}^s}(p, x_n).
\]

Moreover, for every \( \eta > 0 \), there exists \( N_\eta > 0 \), such that for every \( n > N_\eta \),

\[
d_{\mathcal{F}^s}(f^k(y_n), f^k(q)) \leq \eta, \quad \forall k \geq 0.
\]

Proof. Since \( \Lambda = \bigcup_{x \in T^3} \partial^c h^{-1}(h(x)) \) is a us-minimal set, at least one branch \( \mathcal{F}^c_q(q) \) or \( \mathcal{F}^c_{p}(q) \) contains a sequence of points \( z_m \in \Lambda \), such that \( z_m \) converges to \( q \) in \( \mathcal{F}^c(q) \). We assume \( z_m \in \mathcal{F}^c_q(q) \cap \Lambda \) for every \( m \). Then we have \( h(z_m) \neq h(q) \) and \( h(z_m) \) converges to \( h(q) \) in \( \mathcal{A}^c_p(h(q)) \).

The points \( h(p) \) and \( h(q) \) are periodic points \( A \). Since \( \mathcal{A}^s \) is an algebraic foliation with algebraic irrational rotation vector on \( T^3 \), there exist \( C_2 > 0 \) and two sequences of points \( x'_n \in \mathcal{A}^s(h(p)), y'_n \in \mathcal{A}^u(x'_n) \) with \( y'_n \in \mathcal{A}^c_q(h(q)) \), such that

\[
d_{\mathcal{A}^s}(h(q), y'_n) \leq \frac{C_2}{\sqrt{D_n}} \quad \text{and} \quad d_{\mathcal{A}^u}(y'_n, x'_n) \leq \frac{C_2}{\sqrt{D_n}},
\]

where \( D'_n = d_{\mathcal{A}^s}(h(p), x'_n) \to +\infty \) as \( n \to +\infty \).

If we have \( p = h^{-1}(h(p)) \), then for every \( n, h^{-1}(x'_n) \) and \( h^{-1}(y'_n) \) are single points, we denote

\[
x_n = h^{-1}(x'_n) \in \mathcal{F}^s(p), \quad \text{and} \quad y_n = h^{-1}(y'_n) \in \mathcal{F}^u(x_n).
\]

Otherwise, the set \( \partial^c h^{-1}(h(p)) = \Lambda \cap h^{-1}(h(p)) \) consists of two periodic points, and \( p \) is one of them. This implies \( h^{-1}(h(\mathcal{F}^u(p))) \cap \Lambda \) contains two us-leaves, and one of them is \( \mathcal{F}^u(p) \). So there exist unique a pair of points

\[
x_n = h^{-1}(x'_n) \cap \mathcal{F}^s(p), \quad \text{and} \quad y_n = h^{-1}(y'_n) \cap \mathcal{F}^u(x_n).
\]

In both cases, we have \( y_n \in \mathcal{F}^c_q(q) \). Moreover, for every \( h(z_m) \in \mathcal{A}^c_q(h(q)) \), there exists \( N_m > 0 \), such that \( y'_n \) is contained in the open interval with endpoints \( h(q) \) and \( h(z_m) \) in \( \mathcal{A}^c_q(h(q)) \) for every \( n \geq N_m \). This implies \( y_n \) is between \( q \) and \( z_m \) in \( \mathcal{F}^c_q(q) \). Thus \( \lim_{n \to \infty} d_{\mathcal{F}^s}(z_m, q) = 0 \) implies \( \lim_{n \to \infty} d_{\mathcal{F}^s}(y_n, q) = 0 \).

Finally, let \( H : \mathbb{R}^3 \to \mathbb{R}^3 \) be the lifting map of the semi-conjugacy \( h : T^3 \to T^3 \).

Then there exists a constant \( K > 0 \), such that \( d(H, \text{Id}) < K \). For \( D'_n = d_{\mathcal{A}^s}(h(p), x'_n) \),
we have
\[ D_n = d_{\mathcal{F}}(p, x_n) > d_{\mathcal{F}}(h(p), x_n') - 2K = D_n' - 2K. \]
So for \( n \) large enough, we have \( D_n > D_n'/2 \). On the other hand, from \( d_{\mathcal{D}}(y_n', x_n') \leq C_{G}^{\prime}/\sqrt{D_n} \) and Proposition 3.2, we have
\[ d_{\mathcal{F}}(x_n, y_n) \leq C_1 \cdot d_{\mathcal{D}}(x_n, y_n)^{\alpha} \leq C_1 \cdot (2C_{G}^{\prime})^{\alpha} \cdot D_n^{-\frac{\alpha}{2}}. \]
Thus we set \( C_2 = C_1 \cdot (2C_{G}^{\prime})^{\alpha} \) and \( \beta = \alpha/2 \).

Finally, since \( y_n \in \mathcal{F}^{us}(p) \subset \Lambda \) converges to \( q \) in \( \mathcal{F}^c_+(q) \), this implies for every \( n \),
\[ \lim_{k \to \infty} d_{\mathcal{F}}(f^k(y_n), f^k(q)) = 0. \]
So for \( n = 1 \), there exists \( K_0 > 0 \), such that \( d_{\mathcal{F}}(f^k(y_n), f^k(q)) < \eta \) for every \( k \geq K_0 \). Let \( \pi(q) \) be the period of \( q \), and set \( N_\eta > 0 \), where \( y_{N_\eta} \) is contained in \( (q, f^{K_0 \pi(q)}(y_1)) \). Then for every \( n > N_\eta \),
\[ d_{\mathcal{F}}(f^k(y_n), f^k(q)) \leq \eta, \quad \forall k \geq 0. \]

\[ \square \]

**Remark.** Since \( \lim_{n \to -\infty} d_{\mathcal{F}}(y_n, q) = 0 \), we can assume \( y_{n+1} \in (q, y_n)^c \) by taking subsequence. This allows us to apply Proposition 3.3.

Now we fix the constant
\[ \theta = -\frac{1}{2} \cdot \frac{\beta \cdot \log(\sup_{x \in \mathbb{T}^3} \|Df^{\epsilon}|_{E^c(x)}\|)}{\log(\inf_{x \in \mathbb{T}^3} m(Df^{\epsilon}|_{E^c(x)}))} \in (0, 1). \]
We can state the main result of this section, which implies that \( \Lambda \) is a hyperbolic set and equal to \( \mathbb{T}^3 \). The proof of this proposition is similar to Proposition 4.1 of [GS19].

**Proposition 4.3.** All periodic points in \( \Lambda \) have the same center Lyapunov exponent and satisfies
\[ \lambda^c(p) \leq \lambda^c(A) < 0, \quad \forall p \in \text{Per}(f|_\Lambda). \]

**Proof.** Lemma 3.4 proved that \( \lambda^c(p) \leq 0 \) for every \( p \in \text{Per}(f|_\Lambda) \). Proposition 3.5 and 3.6 proved that there exists a sequence of periodic points \( q_n \in \text{Per}(f|_\Lambda) \), such that \( \lim_{n \to -\infty} \lambda^c(q_n) = \lambda^- \leq \lambda^c(A) < 0 \).

Assume there exist two periodic points \( r, q \in \text{Per}(f|_\Lambda) \) such that \( 0 < \lambda^c(r) < \lambda^c(q) \leq 0 \). Denote
\[ \delta_0 = \frac{\theta}{4} \left( \lambda^c(q) - \lambda^c(r) \right) > 0. \]
From Lemma 4.1, there exists a periodic point \( p \in \text{Per}(f|_\Lambda) \), such that
\[ \lambda^c(p) \leq \lambda^c(r), \quad \text{and} \quad \lambda^c(p) < \log \|Df^{\epsilon}|_{E^c(x)}\| + \delta_0, \quad \forall x \in \Lambda. \]
Denote \( n_0 \) be the minimal common period of \( p \) and \( q \).

Let \( \eta_0 > 0 \), such that for every \( z_1, z_2 \in \mathbb{T}^3 \) satisfying \( d(z_1, z_2) \leq 3\eta_0 \), we have
\[ \left| \log \|Df^{\epsilon}|_{E^c(z_1)}\| - \log \|Df^{\epsilon}|_{E^c(z_2)}\| \right| < \delta_0. \]
Since $p \in \text{Per}(f|_{\Lambda})$ and $\lambda^i(p) < 0$, there exists $z \in \mathcal{F}^c(p) \cap \Lambda$, such that

$$h(z) \neq h(p), \quad \text{and} \quad d_{\mathcal{F}^c}(f^k(p), f^k(z)) \leq \eta_0, \quad \forall k \geq 0.$$ 

We assume $x \in \mathcal{F}^c_+(p)$.

We apply Lemma 4.2 to $p$ and $q$, there exists $x_n \in \mathcal{F}^s(p)$, $y_n \in \mathcal{F}^u(x_n)$ with $y_n \in \mathcal{F}^c(q)$ such that

$$\lim_{n \to \infty} d_{\mathcal{F}^c}(y_n, q) = 0, \quad \text{and} \quad d_{\mathcal{F}^u}(x_n, y_n) \leq \frac{C_2}{D_n^\beta}, \quad \text{where} \quad D_n = d_{\mathcal{F}^s}(p, x_n).$$

By taking subsequence, we can assume $y_{n+1} \in (q, y_n)c$. Moreover, there exists $N_0 > 0$, such that for every $n > N_0$,

$$d_{\mathcal{F}^c}(f^k(y_n), f^k q) \leq \eta_0.$$ 

Denote $h_{p,y_n}^u : \mathcal{F}^c(p) \to \mathcal{F}^c(y_n)$ the holonomy map induced by $\mathcal{F}^u$ in $\Lambda$ from $p$ to $y_n$, and $z_n = h_{p,y_n}^u(z)$. Proposition 3.3 shows that there exists $\delta_z > 0$, such that

$$d_{\mathcal{F}^c}(y_n, z_n) \geq \delta_z.$$

**FIGURE 1. Holonomy maps of stable foliations**

**Claim 4.4.** If we denote $h_{p,x_n}^s : \mathcal{F}^c(p) \to \mathcal{F}^c(x_n)$ the holonomy map induced by $\mathcal{F}^s$ in $\mathcal{F}^c_+(p)$ from $p$ to $x_n$, and $w_n = h_{p,x_n}^s(z)$, then there exists $N_1 > 0$, such that for every $n > N_1$, it satisfies

$$d_{\mathcal{F}^c}(x_n, w_n) \geq \delta_z/2.$$ 

**Proof of the claim.** Consider the holonomy map $h_{y_n,x_n}^u : \mathcal{F}^c(y_n) \to \mathcal{F}^c(x_n)$ induced by $\mathcal{F}^u$ in $\mathcal{F}^c_+(q)$ for $y_n$ to $x_n$. We denote $w_n = h_{y_n,x_n}^u(z_n)$, $\mathcal{F}^c_+$ and $\mathcal{F}^c_-$ are $C^1$ continuous. Moreover, since $d_{\mathcal{F}^c}(x_n, y_n) \to 0$ as $n \to \infty$, the derivative of $h_{y_n,x_n}^u$ converges to 1 as $n \to \infty$. So there exists $N_1 > 0$, such that for every $n > N_1$, it satisfies

$$d_{\mathcal{F}^c}(x_n, w_n) \geq \delta_z/2.$$
For every $n$, we denote

- $m_n$ be the smallest positive integer where $d(x, f^{k_n n_0}(x_n)) \leq 1$.
- $k_n$ be the largest positive integer where $d(x, f^{k_n n_0}(x_n), f^{k_n_0}(y_n)) \leq \eta_0$.

**Claim 4.5.** There exists $N_2 > 0$, such that for every $n > N_2$, we have

$$\frac{k_n}{m_n} > \theta.$$  

**Proof of the claim.** From the definition of $m_n$ and $k_n$, they satisfy

- $m_n$ satisfies $D_n \cdot (\sup_{x \in T^3} \| Df_{E^c(x)} \|)(m_n - 1) n_0 > 1$, this implies

$$m_n < \frac{\log D_n}{n_0 \cdot \log (\sup_{x \in T^3} \| Df_{E^c(x)} \|)} + 1.$$  

- $k_n$ satisfies $(C_2/D_n^\beta) \cdot (\inf_{x \in T^3} m(Df_{E^c(x)})) k_n n_0 < \eta_0$, this implies

$$k_n > \frac{\beta \cdot \log D_n + \log \eta_0 - \log C_2}{n_0 \cdot \log (\inf_{x \in T^3} m(Df_{E^c(x)}))}.$$  

Since $D_n \to +\infty$ as $n \to \infty$, we also have $m_n \to +\infty$ as $n \to \infty$. Thus for

$$\theta = -\frac{1}{2} \cdot \frac{\beta \cdot \log (\sup_{x \in T^3} \| Df_{E^c(x)} \|)}{\log (\inf_{x \in T^3} m(Df_{E^c(x)}))},$$  

there exists $N_2 > 0$, such that if $n > N$, then $k_n/m_n > \theta$.  

**Claim 4.6.** For every $n > \max\{N_0, N_1, N_2\}$, we have

$$d(x, f^{m_n n_0}(z), p) \leq \exp \left[ m_n n_0 \cdot (\lambda^c(p) + \delta_0) \right],$$

and

$$d(x, f^{k_n n_0}(x_n), f^{k_n n_0}(w_n)) > \exp \left[ m_n n_0 \cdot (\lambda^c(p) + 2 \delta_0) \right] \cdot \min \left\{ \eta_0, \delta_z/2 \right\}.$$  

**Proof of the claim.** Now we let $n > \max\{N_0, N_1, N_2\}$. The length of $(m_n n_0)$-iteration of the segment $(p, z)^c$ satisfies

$$d(x, f^{m_n n_0}(z), p) \leq \exp \left[ m_n n_0 \cdot (\lambda^c(p) + \delta_0) \right].$$  

On the other hand, since $d(x_n, q) \leq 2\eta_0$, for every $1 \leq k \leq k_n n_0$, either

$$d(x, f^{k-1}(x_n), f^{k-1}(w_n)) > \eta_0;$$

or

$$d(x, f^{k}(x_n), f^{k}(w_n)) > \| Df_{E^c(f^{k-1}(q))} \| \cdot \exp(-\delta_0) \cdot d(x, f^{k-1}(x_n), f^{k-1}(w_n)).$$

Recall that $d(x_n, w_n) > \delta_z/2$, this implies that

$$d(x, f^{k_n n_0}(x_n), f^{k_n n_0}(w_n)) > \exp \left[ k_n n_0 \cdot (\lambda^c(q) - \delta_0) \right] \cdot \min \left\{ \eta_0, \delta_z/2 \right\}.$$  

Moreover, for every $k_n n_0 < k \leq m_n n_0$, either

$$d(x, f^{k-1}(x_n), f^{k-1}(w_n)) > \eta_0;$$
or
\[ \|Df|_{E^s(f^{-1}(x_n))} \cdot \exp(-\delta_0) \cdot d_{\mathcal{F}}(f^{k-1}(x_n), f^{k-1}(w_n)) > \exp\left(\lambda^c(p) - 2\delta_0\right) \cdot d_{\mathcal{F}}(f^{k-1}(x_n), f^{k-1}(w_n)). \]

This implies
\[ d_{\mathcal{F}}(f^{k_n n_0}(x_n), f^{k_n n_0}(w_n)) > \exp\left[k_n n_0 \cdot (\lambda^c(q) - \delta_0) + (m_n - k_n) n_0 \cdot (\lambda^c(p) - 2\delta_0)\right] \cdot \min\{\eta_0, \delta_z/2\} \]
\[ > \exp\left[m_n n_0 \cdot \left[\theta \lambda^c(q) + (1 - \theta)\lambda^c(p) - 2\delta_0\right]\right] \cdot \min\{\eta_0, \delta_z/2\}. \]

Since \( \delta_0 = \frac{\theta}{4} (\lambda^c(q) - \lambda^c(r)) \), we have
\[ d_{\mathcal{F}}(f^{k_n n_0}(x_n), f^{k_n n_0}(w_n)) > \exp\left[m_n n_0 \cdot \left[\lambda^c(p) + 2\delta_0\right]\right] \cdot \min\{\eta_0, \delta_z/2\}. \]

This proves the claim. \( \square \)

Since \( m_n \to +\infty \) as \( n \to \infty \), this claim implies
\[ \lim_{n \to \infty} \frac{d_{\mathcal{F}}(f^{k_n n_0}(x_n), f^{k_n n_0}(w_n))}{d_{\mathcal{F}}(f^{m n_0}(z), p)} \to +\infty \]
as \( n \to \infty \).

However, \( d_{\mathcal{F}}(p, f^{k_n n_0}(x_n)) \leq 1 \), this contradicts that the holonomy maps stable foliation in a center-stable leaf is \(C^1\), see Theorem B of [PSW97]. This proves
\[ \lambda^c(p) = \lambda^c(q), \quad \forall \ p, q \in \text{Per}(f|_\Lambda). \]

\( \square \)

**Corollary 4.7.** \( \Lambda \) is a uniformly hyperbolic attractor.

**Proof.** For every \( p \in \text{Per}(f|_\Lambda) \), we have \( p = p_+ \) or \( p = p_- \). If \( p = p_+ \), then \( \mathcal{F}^s(p) \) is contained in the stable manifold of \( p \). If \( p = p_- \), then \( \mathcal{F}^s(p) \) is contained in the stable manifold of \( p \). From the global product structure of \( \mathcal{F}^s \) and \( \mathcal{F}^u \), this implies every pair of periodic points \( p, q \in \Lambda \) are homoclinic related. Moreover, every periodic point \( r \notin \Lambda \) satisfies \( r \neq r_+ \) and \( r \neq r_- \). This implies its stable manifold
\[ W^s(r) \subset \bigcup_{x \in (r_-, r_+)} \mathcal{F}^s(x). \]

This implies
\[ W^s(r) \cap \mathcal{F}^{hs}(r_-) = \emptyset, \quad \text{and} \quad W^s(r) \cap \mathcal{F}^{hs}(r_+) = \emptyset. \]

Thus \( r \) is not homoclinic related to \( r_+ \) and \( r_- \). For every \( p \in \text{Per}(f|_\Lambda) \), its homoclinic class \( H(p) \subseteq \Lambda \). Proposition 3.1 shows that \( \Lambda = \text{Per}(f|_\Lambda) \), this implies
\[ \Lambda = H(p), \quad \forall \ p \in \text{Per}(f|_\Lambda). \]

Proposition 4.3 shows that \( \lambda^c(p) \leq \lambda^c(A) < 0 \) for every \( p \in \text{Per}(f|_\Lambda) \). We apply the Main Theorem of [BGY09], which shows that \( \Lambda \) is a hyperbolic set. Moreover,
the unstable manifold
\[ W^u(x) = \mathbb{F}^u(x) \subset A, \quad \forall x \in \Lambda. \]
This implies \( \Lambda \) is a hyperbolic attractor.

The following proposition finishes the proof of Theorem 1.1.

**Proposition 4.8.** The hyperbolic attractor \( \Lambda \) is the whole of \( \mathbb{T}^3 \) and so \( f \) is an Anosov diffeomorphism.

**Proof.** We have \( \Lambda = \mathbb{T}^3 \) if \( x = h^{-1}(h(x)) \) for every \( x \in \mathbb{T}^3 \). Assume there exists \( x \in \mathbb{T}^3 \) such that \( x \neq h^{-1}(h(x)) \). Then for every \( y \in \mathcal{A}^u(h(x)) \), \( h^{-1}(y) \) is a non-trivial center segment.

**Claim 4.9.** There exists \( z' \in \mathcal{A}^u(h(x)) \), such that the set
\[ B_1^{us}(z') = \{ y' \in \mathcal{A}^u(z') : d_{\mathcal{A}^u}(y', z') \leq 1 \} \]
satisfies
\[ A^{-k}(B_1^{us}(z')) \cap B_1^{us}(z') = \emptyset, \quad \forall k > 0. \]

**Proof of the claim.** There are two possibilities, either \( \mathcal{A}^u(h(x)) \) contains no periodic points of \( A \), or \( \mathcal{A}^u(h(x)) \) contains a periodic point.

If \( \mathcal{A}^u(h(x)) \) contains no periodic points of \( A \), then
\[ A^{-k}(\mathcal{A}^u(h(x))) \cap \mathcal{A}^u(h(x)) = \emptyset, \quad \forall k > 0 \]
Since \( B_1^{us}(h(x)) \subset \mathcal{A}^u(h(x)) \), we only need to choose \( z' = h(x) \).

If \( \mathcal{A}^u(h(x)) \) contains a periodic point \( p \) with period \( \pi \), then \( A^\pi : \mathcal{A}^u(h(x)) \to \mathcal{A}^u(h(x)) \) is a linear Anosov action on the plane. So there exists \( z' \in \mathcal{A}^s(p) \setminus \{ p \} \subset \mathcal{A}^u(h(x)) \) sufficiently far from \( p \) in \( \mathcal{A}^s(p) \), which satisfies
\[ A^{-k\pi}(B_1^{us}(z')) \cap B_1^{us}(z') = \emptyset, \quad \forall k > 0. \]
This implies \( A^{-k}(B_1^{us}(z')) \cap B_1^{us}(z') = \emptyset \) for every \( k > 0 \)

Let \( z \) be the center point of the non-trivial segment \( h^{-1}(z') \), and \( \delta > 0 \) satisfying
\[ B_3^\delta(z) \cap \Lambda = \emptyset. \]
The semi-conjugacy \( h : \mathbb{T}^3 \to \mathbb{T}^3 \) is a continuous map. There exists \( \varepsilon_0 > 0 \), such that if \( d(x_1, x_2) < \varepsilon_0 \) and \( h(x_2) \in \mathcal{A}^u(h(x_1)) \), then \( d_{\mathcal{A}^u}(h(x_1), h(x_2)) < 1 \). Here \( d_{\mathcal{A}^u}(\cdot, \cdot) \) is the distance in each leaf of \( \mathcal{A}^u \).

Since \( \Lambda \) is a hyperbolic attractor, there exists a constant \( 0 < \varepsilon < \min(\delta, \varepsilon_0) \), such that the \( \varepsilon \)-neighborhood \( B_\varepsilon(\Lambda) \) satisfies
\[ f^k(B_\varepsilon(\Lambda)) \subset B_\delta(\Lambda), \quad \forall k > 0. \]
This implies
\[ f^{-k}(B_\delta(z)) \subset B_\varepsilon(\Lambda), \quad \forall k > 0. \]
Otherwise, we have $B_δ(z) \cap B_δ(Λ) \neq \emptyset$, which contradicts to $B_{3δ}(z) \cap Λ = \emptyset$. Thus for every $k > 0$, we have

$$B_ε(f^{-k}(z)) \cap Λ = \emptyset, \quad \text{and} \quad h\left(B_ε(f^{-k}(z)) \right) \subset A^{us}(-k(z')).$$

Moreover, since $0 < ε < ε_0$, for every $y \in B_ε(f^{-k}(z))$, we have

$$d_{us}\left(h(y), A^{-k}(z')\right) < 1.$$ 

This implies

$$h\left(B_ε(f^{-k}(z)) \right) \subset B_1^{us}(A^{-k}(z')), \quad \forall k > 0.$$ 

Since $A^{-k}(B_1^{us}(z')) \cap B_1^{us}(z') = \emptyset$ for every $k > 0$, the semi-conjugacy property implies the sequence of balls

$$\left\{ B_ε(f^{-k}(z)) : k > 0 \right\}$$

are mutually disjoint. This is absurd since the volume of each $B_ε(f^{-k}(z))$ has a lower bound. Thus $h$ is injective everywhere and $Λ = \mathbb{T}^3$. □

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