Reconstruction of the scalar–tensor Lagrangian from a ΛCDM background and Noether symmetry

S Capozziello1, S Nesseris2 and L Perivolaropoulos2

1 Dipartimento di Scienze Fisiche, Università di Napoli ‘Federico II’, and INFN, Sez. di Napoli, Compl. Univ. di Monte S. Angelo, Edificio G, Via Cinthia, I-80126—Napoli, Italy
2 Department of Physics, University of Ioannina, Greece
E-mail: capozzie@na.infn.it, snesseris@grads.uoi.gr, leandros@cc.uoi.gr and leandros@uoi.gr

Received 30 May 2007
Accepted 26 November 2007
Published 14 December 2007

Online at stacks.iop.org/JCAP/2007/i=12/a=009
doi:10.1088/1475-7516/2007/12/009

Abstract. We consider scalar–tensor theories and reconstruct their potential $U(\Phi)$ and coupling $F(\Phi)$ by demanding a background ΛCDM cosmology. In particular we impose a background cosmic history $H(z)$ provided by the usual flat ΛCDM parameterization through the radiation ($w_{\text{eff}} = 1/3$), matter ($w_{\text{eff}} = 0$) and de Sitter ($w_{\text{eff}} = -1$) eras. The cosmological dynamical system which is constrained to obey the ΛCDM cosmic history presents five critical points in each era, one of which corresponding to the standard General Relativity (GR). In the cases that differ from GR, the reconstructed coupling and potential are of the form $F(\Phi) \sim \Phi^2$ and $U(\Phi) \sim F(\Phi)^m$, where $m$ is a constant relating the two functions $F(\Phi)$ and $U(\Phi)$: this assumption is necessary to find out the fixed points of the dynamical system. This class of scalar tensor theories is also theoretically motivated by a completely independent approach: imposing maximal Noether symmetry on the scalar–tensor Lagrangian. This approach independently (i) provides the form of the coupling and the potential as $F(\Phi) \sim \Phi^2$ and $U(\Phi) \sim F(\Phi)^m$, (ii) provides a conserved charge related to the potential and the coupling, and (iii) allows the derivation of exact solutions by first integrals of motion.

Keywords: dark energy theory, gravity, cosmology of theories beyond the SM

ArXiv ePrint: 0705.3586
1. Introduction

There is accumulating observational evidence based mainly on type Ia supernovae standard candles [1] and also on standard rulers [2, 3] that the universe has entered a phase of accelerating expansion at a recent cosmological timescale. Such an expansion implies the existence of a repulsive pressure on cosmological scales which counterbalances the gravitational attraction of matter on these scales giving rise to an overall accelerating behavior. There have been several theoretical approaches (see [4, 5] for a review) to understanding this phenomenon. The simplest of such approaches assumes the existence of a positive cosmological constant which is small enough to have started dominating the universe at recent times. The predicted cosmic expansion history, in this case (assuming flatness), is

$$\frac{H(z)^2}{H_0^2} = \Omega_{\text{0r}} (1 + z)^3 + \Omega_{\text{0m}} (1 + z)^4 + \Omega_{\Lambda} ,$$

where $\Omega_{\text{0r}} = \rho_r/\rho_{\text{crit}} \simeq 10^{-4}$ is the energy density of radiation today normalized over the critical density for flatness $\rho_{\text{crit}}$. Also $\Omega_{\text{0m}} = \rho_m/\rho_{\text{crit}} \simeq 0.3$ is the normalized present matter density and $\Omega_{\Lambda} = 1 - \Omega_{\text{0m}} - \Omega_{\text{0r}}$ is the normalized energy density due to the cosmological constant. This model provides an excellent fit to the cosmological observational data [2] and has the additional bonus of simplicity due to a single free parameter. Despite its simplicity and good fit to the data, such a model fails to explain why the cosmological constant, so unnaturally small, dominates the universe at recent cosmological times. This problem is known as the coincidence problem. Furthermore, there is no urgent theoretical reason implying $\Omega_{\Lambda} \simeq 0.7$ and $\Omega_{\text{0m}} \simeq 0.3$ at the present time so also a fine tuning problem has to be taken into account.

In an effort to address these problems, several models, which essentially can be grouped in two classes, have been proposed: The first class assumes that General
Relativity (GR) is valid at cosmological scales, and attributes the accelerating expansion to a dark energy component which has repulsive gravitational properties due to its negative pressure. The role of dark energy is usually played by a scalar field minimally coupled to gravity called quintessence [6]. Alternatively, the role of dark energy can be played by various perfect fluids (e.g. Chaplygin gas [7]) topological defects [8], holographic dark energy [9], etc.

The second class of models attributes the accelerating expansion to modifications and extensions of GR which convert gravity to a repulsive interaction at late times and on cosmological scales. Examples of this class of models include scalar–tensor theories [10,11], $f(R)$ extended gravity theories [12], braneworld models [13], etc. An advantage of models in this class is that they naturally allow [14,15] a super-accelerating expansion of the universe where the effective dark energy equation of state $w = p/\rho$ crosses the phantom divide line $w = -1$. Such a crossing is consistent with current cosmological data [16,17].

A representative model of the second class is provided by scalar–tensor theories of gravity. In these theories, the Newton constant is obtained by dynamical properties expressed through the coupling $F(\Phi)$. The dynamics is given by the Lagrangian density [18]–[21]

$$\mathcal{L} = \frac{F(\Phi)}{2} R - \frac{1}{2} \epsilon g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - U(\Phi) + \mathcal{L}_m[\psi_m; g_{\mu\nu}],$$

where $\mathcal{L}_m[\psi_m; g_{\mu\nu}]$ represents matter fields approximated by a pressureless perfect fluid in the dust dominated regime. We have set $8\pi G = 1$ and $\epsilon = \pm 1$ for standard scalar and phantom fields respectively, i.e., we have negative kinetic energy for the scalar degree of freedom. Note, however, that negative energy needs $\epsilon = -1$, but there also exist positive energy configurations with $\epsilon = -1$. This happens due to the fact that the kinetic term of the actual spin-0 degree of freedom does not come only from the obvious $(\partial_\mu \Phi)^2$, but also from the cross term $F(\Phi) R$, since $R$ involves second derivatives (see [21] for details). As discussed below, for $\epsilon = 0$ and $\Phi \to R$ the Lagrangian (1.2) can also describe $f(R)$ generalizations of GR.

In the present study, we reconstruct the potential $U(\Phi)$ and the coupling $F(\Phi)$. Instead of specifying various forms of $U(\Phi)$ and $F(\Phi)$ and finding the corresponding cosmological dynamics, we specify the cosmological dynamics to that of the $\Lambda$CDM cosmology and search for possible corresponding forms of $U(\Phi)$ and $F(\Phi)$. The original method for the reconstruction of scalar–tensor theories from a given cosmic history $H(z)$ was introduced in [15] and applied to specific cases of late cosmic history in [14,21]. Our reconstruction approach is different in two aspects:

- We use a dynamical system formalism and find the critical points that determine the generic evolution of the system.
- We start the reconstruction from the radiation era rather than focusing only at late times through the acceleration epoch.

In particular, we consider the general dynamical system for scalar–tensor theories and study the dynamics of $U(\Phi)$ and $F(\Phi)$ using as input a $\Lambda$CDM cosmic expansion history. Our study is performed both analytically (using the critical points and their stability) and numerically by explicitly solving the dynamical system.
Reconstruction of the scalar–tensor Lagrangian from a ΛCDM background and Noether symmetry

The structure of the paper is the following. In the next section we derive the dynamical system for the cosmological dynamics of scalar–tensor theories. Using as input a particular cosmic history $H(z)$ (e.g. ΛCDM), we show how this system can be transformed so that its solution provides the dynamics and the functional form of $U(\Phi)$ and $F(\Phi)$. We also study the dynamics of this transformed system analytically by deriving its critical points during the three eras of the cosmic background history (radiation, matter and de Sitter). We find that the cosmological dynamical system constrained to obey the ΛCDM cosmic history has five critical points in each era, one of which corresponds to GR. In section 3, we use the solution of the above system to reconstruct the cosmological evolution and functional form of the coupling and potential, which are of the form $F(\Phi) \sim \Phi^4$ and $U(\Phi) \sim F(\Phi)^m$, where $m$ is an arbitrary constant. We show that such forms are also motivated by a completely independent approach, i.e., by imposing Noether symmetry on the scalar–tensor Lagrangian [22]. We also demonstrate the agreement between the analytical and numerical results of our reconstruction scheme. Finally, in section 4, we conclude, summarize and refer to future prospects of this work.

2. Dynamics of scalar–tensor cosmologies

Let us consider the action (1.2) describing the dynamics of scalar–tensor theories in the Jordan frame. In the context of flat Friedman–Robertson–Walker (FRW) universes, the metric is homogeneous and isotropic, i.e.,

$$ds^2 = -dt^2 + a^2(t) dx^2.$$  \hspace{1cm} (2.1)

Variation of the action (1.2) with respect to the metric leads to the following dynamical equations which are the generalized Friedman equations [20, 21, 23]:

$$3FH^2 = \rho_m + \rho_r + \frac{1}{2} \epsilon \dot{\Phi}^2 - 3H \dot{F} + U, \hspace{1cm} (2.2)$$

$$-2F\dot{H} = \rho_m + \frac{4}{3} \rho_r + \epsilon \dot{F}^2 + \ddot{F} - H \dot{F}, \hspace{1cm} (2.3)$$

and variation with respect to $\Phi$ gives the Klein–Gordon equation:

$$\epsilon(\ddot{\Phi} + 3H \dot{\Phi}) = 3F(\Phi, \phi) (\dot{H} + 2H^2) - U(\Phi, \phi), \hspace{1cm} (2.4)$$

where we have assumed the presence of perfect fluids $\rho_m$, $\rho_r$ representing the matter and radiation energy densities which are conserved according to

$$\dot{\rho}_m + 3H \rho_m = 0, \hspace{1cm} (2.5)$$

$$\dot{\rho}_r + 4H \rho_r = 0. \hspace{1cm} (2.6)$$

The background equations in equations (2.2) and (2.3) can be rewritten in a more convenient form, which is easier to confront with observations; see for example [24]:

$$3F_0H^2 = \rho_{DE} + \rho_m + \rho_r, \hspace{1cm} (2.7)$$

$$-2F_0\dot{H} = \rho_{DE} + p_{DE} + \rho_m + \frac{4}{3} \rho_r. \hspace{1cm} (2.8)$$
Reconstruction of the scalar–tensor Lagrangian from a $\Lambda$CDM background and Noether symmetry

where we have set
\[ \rho_{DE} = \frac{1}{2} \epsilon \dot{\Phi}^2 - 3 H^2 (F - F_0) - 3 H \dot{F} + U, \]
\[ p_{DE} = \frac{3}{2} \dot{\Phi}^2 + \dot{F} + 2 H \dot{F} - U - (2 \dot{H} + 3 H^2)(F_0 - F), \]
and the subscript ‘0’ denotes present day values. The function $\rho_{DE}$ defined in this way can be shown to satisfy the usual energy conservation equation:
\[ \dot{\rho}_{DE} + 3 H (\rho_{DE} + p_{DE}) = 0, \]
where the dark energy (DE) equation of state is defined as
\[ w_{DE} \equiv \frac{p_{DE}}{\rho_{DE}} = -1 + \frac{\epsilon \dot{\Phi}^2 - 3 H^2 (F - F_0) - 3 H \dot{F} + U}{3 F H^2 - \Phi'}. \]
By using equations (2.7) and (2.8) we can express the equation of state $w_{DE}$ as
\[ w_{DE} = -\frac{3 E(z) - (1 + z)(dE(z)/dz)}{3 E(z) - 3 \Omega_m^0 (1 + z)^3}, \]
where we have set $E(z) \equiv H^2(z)/H_0^2$. This relation is exactly the same as in standard Einstein gravity [5], and therefore we can constrain $w_{DE}$ from type Ia supernovae observations in the same way.

In order to study the cosmological dynamics implied by equations (2.2), (2.3) and (2.4) we express them as a dynamical system of first-order differential equations. To achieve this, let us first write (2.2) in dimensionless form as
\[ 1 = \frac{\rho_m}{3 F H^2} + \frac{\rho_r}{3 F H^2} + \frac{\epsilon \dot{\Phi}^2}{6 F} + \frac{U}{3 F H^2} - \frac{F'}{F}, \]
where
\[ \frac{d}{dN} \equiv \frac{d}{H dt}. \]
Let us now define the dimensionless variables $x_1, \ldots, x_4$ as
\[ x_1 = -\frac{F'}{F}, \]
\[ x_2 = \frac{U}{3 F H^2}, \]
\[ x_3 = \frac{\dot{\Phi}}{6 F}, \]
\[ x_4 = \frac{\rho_r}{3 F H^2} = \Omega_r, \]
where we can associate $x_4$ with $\Omega_r$ and $x_1 + x_2 + \epsilon x_3 \equiv \Omega_{DE}$ with curvature dark energy (dark gravity). Defining also $\Omega_m \equiv \rho_m/3 F H^2$ we can write equation (2.14) as
\[ \Omega_m = 1 - x_1 - x_2 - \epsilon x_3 - x_4. \]
Let us now use (2.15) to express (2.3) as
\[
\frac{H'}{H} = -\frac{\rho_m}{2FH^2} - \frac{2}{3} \frac{\rho_r}{FH^2} - \phi \frac{\Phi'^2}{2F} - \frac{F''}{2FH} + \frac{F'}{2F},
\]  
(2.21)
or
\[
x_1' = 3 - 2x_1 - 3x_2 + x_4 + 3\epsilon x_3^2 + x_1^2 + 2\frac{H'}{H} - x_1 \frac{H'}{H}.
\]  
(2.22)
Differentiating \( x_4 \) of (2.19) with respect to \( N \), we have
\[
x_4' = \frac{\rho_r'}{3FH^2} - \frac{\rho_r}{3FH^2} \frac{F'}{F} - \frac{\rho_r}{3FH^2} \frac{H'}{H},
\]  
(2.23)
or
\[
x_4' = -4x_4 + x_4 x_1 - 2x_4 \frac{H'}{H},
\]  
(2.24)
where we have used (2.6). Similarly, differentiating (2.17) with respect to \( N \), we find
\[
x_2' = x_2 \left[ x_1 (1 - m) - 2\frac{H'}{H} \right],
\]  
(2.25)
where
\[
m = \frac{U_{,\Phi}/U}{F_{,\Phi}/F},
\]  
(2.26)
and \( \phi \) implies derivative with respect to \( \Phi \). Finally, differentiating (2.18) with respect to \( N \) and using (2.4), one finds
\[
\epsilon(x_3')' = \epsilon x_3^2 x_1 - 6\epsilon x_3^2 - 2x_1 + mx_2 x_1 - 2\epsilon x_3^2 \frac{H'}{H} - x_1 \frac{H'}{H}.
\]  
(2.27)
The dynamical system (2.22), (2.24), (2.25), (2.27) describes the cosmological dynamics of scalar–tensor theories. We have to stress the fact that we are looking for fixed points and a necessary condition for that is the constancy of \( m \). As we shall discuss in section 3.2, this situation corresponds to the presence of a Noether symmetry for the system.

In two special limits, it can be transformed into the dynamical systems obtained in \( f(R) \) theories and quintessence respectively [25]. However, we have to say that there is one more degree of freedom in scalar–tensor theories, compared to \( f(R) \) theories, since here we have two arbitrary functions, i.e., \( F(\Phi) \) and \( U(\Phi) \). In general, we should add a further parameter \( n \) which could be related to \( F_{,\Phi} \) and \( F_{,\Phi\Phi} \), beside the above \( m \).

In fact, if we do not try to reconstruct the function \( F(\Phi) \), such a function can be fixed \textit{a priori} and the corresponding parameter \( n \) would be, for example, \( F_{,\Phi}/F \), or some similar condition. In such a case \( H(N) \) would not be fixed as in our reconstruction approach but would have to be determined by the autonomous system. This is the approach followed in [26], where \( F_{,\Phi}/F \) is not present as a variable in the autonomous system since \( F(\Phi) \) is fixed \textit{a priori}. The parameters in [26] are the power law of the fixed potential (there called \( n \)) and \( \xi \), the coupling \( F(\Phi) = \xi \Phi^2 \).

However, in a reconstruction approach where \( H(N) \) is fixed \textit{a priori}, while \( F_{,\Phi}/F \) are allowed to vary, \( F(\Phi) \) can be reconstructed from the autonomous system. Thus, in
Reconstruction of the scalar–tensor Lagrangian from a ΛCDM background and Noether symmetry

our reconstruction approach where \( H(N) \) is fixed, \( n \) is no longer a parameter and it is allowed to evolve. Finally, as we shall see below, the reconstructed function \( F(\Phi) \) and \( U(\Phi) \) can be independently obtained by looking for a sort of ‘first principle’, the existence of a Noether symmetry. Since the results coincide (the form of \( F \) and \( U \) are the same in both the approaches), we are confident that the presented method is consistent.

Considering again the \( f(R) \)-theories, by setting \( U = FR - f/2 \) [27] and using the transformations

\[
x_1 \rightarrow \tilde{x}_1, \tag{2.28}
\]
\[
x_2 \rightarrow \tilde{x}_2 + \tilde{x}_3, \tag{2.29}
\]
\[
x_4 \rightarrow \tilde{x}_4, \tag{2.30}
\]
\[
\frac{H'}{H} \rightarrow \tilde{x}_3 - 2, \tag{2.31}
\]
\[
\Phi \rightarrow R, \tag{2.32}
\]
\[
\epsilon \rightarrow 0, \tag{2.33}
\]

one recovers the dynamical system of \( f(R) \) theories (equations (2.15), (2.17) and (2.21) in [28]), where the \textit{tilded} (\( \tilde{\cdot} \)) quantities are the ones defined for \( f(R) \) theories:

\[
\tilde{x}_1' = -1 - \tilde{x}_3 - 3\tilde{x}_2 + \tilde{x}_1^2 + \tilde{x}_4, \tag{2.34}
\]
\[
\tilde{x}_2' = -\tilde{x}_3' - 2\tilde{x}_3(\tilde{x}_3 - 2) - \tilde{x}_2(2\tilde{x}_3 - \tilde{x}_1 - 4), \tag{2.35}
\]
\[
\tilde{x}_4' = -2\tilde{x}_3 \tilde{x}_4 + \tilde{x}_1 \tilde{x}_4. \tag{2.36}
\]

On the other hand, the following set of transformations gives the autonomous system for quintessence (see equations (175) and (176) in [5]):

\[
x_1 \rightarrow 0, \tag{2.37}
\]
\[
x_2 \rightarrow y^2, \tag{2.38}
\]
\[
x_3 \rightarrow x^2, \tag{2.39}
\]
\[
x_4 \rightarrow 0, \tag{2.40}
\]

with dynamical equations

\[
x' = -3x + \frac{\sqrt{6}}{2} \epsilon y^2 + \frac{3}{2} x(\epsilon x^2 + 1 - y^2), \tag{2.41}
\]
\[
y' = -\frac{\sqrt{6}}{2} \lambda xy + \frac{3}{2} y(\epsilon x^2 + 1 - y^2), \tag{2.42}
\]

and \( \lambda = -U,\Phi/U \).

It is worth noting that the results of our analysis do not rely on the use of any particular form of \( H(z) \). They only require that the universe goes through the radiation era (high redshifts), matter era (intermediate redshifts) and acceleration era (low redshifts).
The corresponding total effective equation of state

\[ w_{\text{eff}} = -1 - \frac{2}{3} \frac{H'(N)}{H(N)} \]  

is

\[ w_{\text{eff}} = \frac{1}{3} \text{ Radiation era,} \]
\[ w_{\text{eff}} = 0 \text{ Matter era,} \]
\[ w_{\text{eff}} = -1 \text{ deSitter era.} \]

For the sake of definiteness, however, we will assume a specific form for \( H(N) \) corresponding to a ΛCDM cosmology which, in terms of \( N \), takes the form

\[ H(N)^2 = H_0^2 \left[ \Omega_0\text{m}e^{-3N} + \Omega_0\text{r}e^{-4N} + \Omega_\Lambda \right], \]  

where \( N \equiv \ln(a) = \ln(1 + z) \) and \( \Omega_\Lambda = 1 - \Omega_0\text{m} - \Omega_0\text{r} \). Also, we can use (2.45) to find \( H'(N)/H(N) \), a quantity needed in the dynamical system, as

\[ \frac{H'(N)}{H(N)} = \frac{-3 \Omega_0\text{m}e^{-3N} - 4 \Omega_0\text{r}e^{-4N}}{2 (1 - \Omega_0\text{m} - \Omega_0\text{r} + \Omega_0\text{m}e^{-3N} + \Omega_0\text{r}e^{-4N})}. \]  

The crucial generic properties of \( H'(N)/H(N) \) are their values at the radiation, matter and de Sitter eras:

\[ \frac{H'(N)}{H(N)} = -2 \quad N < N_{\text{rm}}, \]  
\[ \frac{H'(N)}{H(N)} = -\frac{3}{2} \quad N_{\text{rm}} < N < N_{m\Lambda}, \]  
\[ \frac{H'(N)}{H(N)} = 0 \quad N > N_{m\Lambda}, \]

where \( N_{\text{rm}} \approx -\ln(1 + z) \) and \( N_{m\Lambda} \approx -1/3\ln(1 + z) \) are the \( N \) values for the radiation–matter and matter–de Sitter transitions. For \( \Omega_0\text{m} = 0.3, \Omega_0\text{r} = 10^{-4} \) we have \( N_{\text{rm}} \approx -8, N_{m\Lambda} \approx -0.3 \). The transition between these eras is model dependent but rapid, and it will not play an important role in our analysis.

It is straightforward to study the dynamics of the system (2.22), (2.24), (2.25), (2.27) by finding the critical points and their stability in each one of the three eras. Notice that even though this dynamical system is not autonomous at all times, it can be approximated as such during the radiation, matter and de Sitter eras when \( H'(N)/H(N) \) is approximately constant. The critical points and their eigenvalues are shown in table 1. An interesting feature to observe in table 1 is that in each era there are five critical points, but only one of them is a stable ‘attractor’ for a given value of \( m \). We have to stress, however, that the system gets fixed points (i.e., \( x_i \) constant) if and only if \( m \) is a constant. This fact restricts the set of theories looked at to those which verify the following condition (4.2). Also, remembering that the positivity of the energy of the (helicity zero) scalar partner of the graviton, i.e., the positivity of the kinetic energy of the scalar field (see [11]) is expressed by

\[ \phi'^2 = \frac{3}{4} \left( \frac{F'}{F} \right)^2 + \frac{\epsilon \phi'^2}{2F} > 0, \]  

\( \epsilon \phi'^2 \)
Table 1. The critical points of the system (2.22), (2.24), (2.25), (2.27) and their eigenvalues in each one of the three eras (radiation era $N < -\ln \Omega_{0\text{m}}/\Omega_0$, matter era $-\ln \Omega_{0\text{m}}/\Omega_0 < N < -\frac{1}{3}\ln \Omega_\Lambda/\Omega_{0\text{m}}$, de Sitter era $N > -\frac{1}{3}\ln \Omega_\Lambda/\Omega_{0\text{m}}$).

| Era    | CP $x_1$ | $x_2$ | $x_3^2$ | $x_4$ | $\Omega_m$ | $\Omega_{\text{DE}}$ | Eigenvalues |
|--------|----------|-------|---------|-------|------------|----------------|-------------|
| Radiation $w_{\text{eff}} = \frac{1}{3}$ | $R_1$ 2 | 0 | -1 | 0 | 0 | 0 | 1 | $(2,3,1,6 - 2m)$ |
| $R_2$ 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | $(1,2,-1,5 - m)$ |
| $R_3-1$ | 0 | 0 | 0 | 2 | -1 | -1 | -2 | $(-1,-2,-3,3 + m)$ |
| $w_{\text{eff}} = 0$ | $M_4$ | 3 | -1 | 1/2 | 0 | 0 | 1 | $(1/2,3/2,-1/2,-3/2(m - 3))$ |
| $M_5$ | 1 | 0 | -1/4 | 1/4 | 0 | 3/4 | | $(1,-1/2,-1,1 - m)$ |
| de Sitter $w_{\text{eff}} = -1$ | $\Lambda_4$ | 0 | 1 | 0 | 0 | 0 | 1 | $(-4,-3,-\sqrt{24m+1/2},\sqrt{24m+1/2})$ |
| $\Lambda_5$ | 4 | 0 | -4 | 1 | 0 | 0 | | $(1,1,2,4 - 4m)$ |

The other two eigenvalues are $-\frac{3m+\sqrt{6}m^2+63m^4+118m^6+111}{2(m-1)}$, $-\frac{3m+\sqrt{6}m^2+63m^4+118m^6+111}{2(m-2)}$.

The other two eigenvalues are $-\frac{7m+\sqrt{48}m^2-26m^4+35m^6+35}{4(m-1)}$, $-\frac{7m+\sqrt{48}m^2-26m^4+35m^6+35}{4m-2}$.

Notice that $\Lambda_2$ and $\Lambda_3$ are degenerate.

Then we have in dimensionless variables

$$\frac{x_1^2}{4} + x_3^2 > 0.$$ (2.51)

All points of table 1 satisfy (2.51) except $R_4$ and $M_4$. In order for $R_4$ to satisfy (2.51), we must have $m < 2$ or $m > 3$, while for $M_4$ it is $m < 2$ or $m > 5/2$. Therefore the allowed region for $m$, so that we have critical points that are physical ($(x_1^2/4) + x_3^2 > 0$), is

$$m \leq 2 \quad \text{or} \quad m \geq 3.$$ (2.52)

Now, regarding the ‘attractor’ behavior of the system in each era separately, we see that

- in the radiation era, $R_3$ is an ‘attractor’ for $m < -3$ while $R_4$ is an ‘attractor’ for $-3 < m < 1$;
- in the matter era, $M_4$ is an ‘attractor’ for $m < 1$;
- in the de Sitter era, $\Lambda_1$ is an ‘attractor’ for $m > 1$ and $\Lambda_4$ is an ‘attractor’ for $m < 1$.

The significance of the ‘attracting’ nature of the critical points is limited because we have not found all possible theories that reconstruct the $\Lambda$CDM expansion history.
Reconstruction of the scalar–tensor Lagrangian from a ΛCDM background and Noether symmetry

but only a subclass of them that corresponds to a fixed $m$. The assumption of a constant $m$ then directly leads to the quest for fixed points. Moreover, it should be noted that the search for fixed points is just a convenient way for finding one solution for the system that reconstructs the ΛCDM expansion history on the radiation, matter and de Sitter eras. However, we can use the ‘attractor’ critical points as a prediction for the numerical evolution of our reconstruction dynamical system.

A particularly interesting feature of the critical points of table 1 is that in all the cases that differ from GR, the expansion rate in each era is induced by dark energy ($\Omega_{DE} = 1$), which implies that the scalar field $\Phi$ could also play the role of dark matter if it is found to have the proper perturbation properties at early times. We postpone the analysis of such perturbations for a future study.

To confirm the dynamical evolution implied by the ‘attractors’ of table 1, we have performed a numerical analysis of the dynamical system (2.22), (2.24), (2.25) and (2.27) using the ansatz (2.46) for $H'(N)/H(N)$ with $\Omega_{0m} = 0.3$ and $\Omega_{0r} = 10^{-4}$. This ansatz, for $H'(N)/H(N)$, leads to the $w_{\text{eff}}(N)$ shown in figure 1. We have set up the system initially, on $R_4$, with $m = -0.5$. As seen in figure 2 and figure 3, the system follows the evolution of $R_4$ from the radiation era, to the matter era ($M_4$) and finally to the de Sitter era ($\Lambda_4$). We have checked that if we choose initial conditions not exactly coinciding with any of the other critical points then the system is captured by the $R_4$ ‘attractor’ and follows the trajectory mentioned above, i.e., (init) $\rightarrow$ $R_4 \rightarrow M_4 \rightarrow \Lambda_4$.

Finally, if the initial conditions of the system are set to be exactly on the top of other critical points then the system will end up on the $\Lambda_1$ critical point.

The choice of a constant $m$ is justified by the fact that one looks for fixed points (i.e., all the $x_i$ constants). Also, the potential $U(\Phi)$ and coupling $F(\Phi)$ that are mostly used in the literature are power-laws or exponentials, which also give a constant $m$ (see...
3. Reconstruction of $F(\Phi)$ and $U(\Phi)$

3.1. Analytical results

Our goal is now the reconstruction of the form of the potential $U(\Phi)$ and coupling $F(\Phi)$ corresponding to each one of the critical points of the system shown in table 1. Let us consider a critical point of the form $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$. Using (2.16), we find

$$F = F_0 e^{-\bar{x}_1 N},$$

(3.1)
where $F_0 = F(N = 0)$ is the present value of $F$. Using now equation (2.18) we find

$$
\Phi(N) = -2 \sqrt{6} \frac{\bar{x}_3}{\bar{x}_1} F_0^{1/2} e^{-\bar{x}_1 N/2} + C
$$

$$
= 2 \sqrt{6} \frac{\bar{x}_3}{\bar{x}_1} F_0^{1/2} (1 - e^{-\bar{x}_1 N/2}) + \Phi_0,
$$

(3.2)

where

$$
C = \Phi_0 + 2 \sqrt{6} \frac{\bar{x}_3}{\bar{x}_1} F_0^{1/2},
$$

(3.3)

and $\Phi_0 \equiv \Phi(N = 0)$. Equations (3.1) and (3.2) allow us to eliminate $N$ in favour of $\Phi$:

$$
F(\Phi) = \frac{1}{24} \frac{\bar{x}_2}{\bar{x}_3} (\Phi - C)^2 \equiv \xi (\Phi - C)^2,
$$

(3.4)

where $\xi \equiv (1/24)(\bar{x}_2^2/\bar{x}_3^2)$. The quadratic form of $F(\Phi)$ can be achieved in a completely different approach imposing Noether symmetry in the scalar–tensor Lagrangian.

From equation (2.17), we have

$$
U(N) = \bar{x}_2 \cdot 3F(N)H(N)^2.
$$

(3.5)

Using now the input form of $H(N)$ (equation (2.45)), we find the dominant term of $H(N)$ in each era, that is,

$$
H(N)^2/H_0^2 = \begin{cases} 
\Omega_{0r} e^{-4N}, & \text{Rad. era} \\
\Omega_{0m} e^{-3N}, & \text{Mat. era} \\
1 - \Omega_{0r} - \Omega_{0m}, & \text{dS era}
\end{cases}
$$

Thus using equation (3.2) we have $H(\Phi)$:

$$
H(\Phi)^2/H_0^2 = \begin{cases} 
\frac{\Omega_{0r}}{F_0^{1/2}} [\xi (\Phi - C)^2]^{4/2}, & \text{Rad. era} \\
\frac{\Omega_{0m}}{F_0^{1/2}} [\xi (\Phi - C)^2]^{3/2}, & \text{Mat. era} \\
1 - \Omega_{0r} - \Omega_{0m}, & \text{dS era}
\end{cases}
$$

and then we can express (3.5) in terms of $\Phi$ to get the relevant form

$$
U(\Phi) = \lambda (\Phi - C)^{2+\alpha},
$$

(3.6)

where

$$
\lambda = \begin{cases} 
3 \bar{x}_2 \frac{\Omega_{0r}}{F_0^{1/2}} \xi^{1+4/2}, & \text{Rad. era} \\
3 \bar{x}_2 \frac{\Omega_{0m}}{F_0^{1/2}} \xi^{1+3/2}, & \text{Mat. era} \\
3 \bar{x}_2 \xi (1 - \Omega_{0r} - \Omega_{0m}), & \text{dS era}
\end{cases}
$$

and

$$
\alpha = \begin{cases} 
8/\bar{x}_1, & \text{Rad. era} \\
6/\bar{x}_1, & \text{Mat. era} \\
0, & \text{dS era}
\end{cases}
$$
By using equations (2.26), (3.4) and (3.6), it is straightforward to find
\[ 2 + \alpha = 2m, \]  
(3.7)
so that equation (3.6) can be written as
\[ U(\Phi) = \lambda(\Phi - C)^{2m}. \]  
(3.8)
Notice that even though the eigenvalue \( \bar{x}_1 \) changes in the sequence \( R_4 \rightarrow M_4 \), the exponent \( 2 + \alpha \) remains constant.

Furthermore, since \( m \) is constant (see equation (3.7)) equation (2.26) allows to write \( U \) in terms of \( F \), i.e.,
\[ U = cF^m. \]  
(3.9)
The above analysis is only valid when the parameters \( \bar{x}_1, \bar{x}_2 \) and \( \bar{x}_3 \) are not equal to zero, as for example is the case for \( R_4 \) or when we have perturbed the initial conditions around a critical point. Otherwise, we have the following cases:

- \( \bar{x}_2 = \bar{x}_3 = 0 \), then \( \Phi = \Phi_0 = \text{const}, U = 0 \) and \( F \) is undefined, as is the case for \( R_2 \) and \( R_3 \);
- \( \bar{x}_1 = \bar{x}_3 = 0 \) and we are in the de Sitter era, then \( \Phi = \Phi_0 = \text{const}, F = F_0 \) and \( U = U_0 \), as is the case for \( \Lambda_4 \);
- \( \bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 0 \), then \( \Phi = \Phi_0 = \text{const}, F = F_0 \) and \( U = 0 \), as is the case for \( R_5 \).

Note that if we were considering other (non-critical) points, not only would the reconstruction involve a time dependence, but we would not be able to reconstruct simultaneously \( F(\Phi) \) and \( U(\Phi) \) from the single function (1.1). Two arbitrary functions obviously need two observed functions to be reconstructed in the general case (see [14,20,21]). In our case, however, the fact that the reconstruction occurs on the critical points means that \( m \) is fixed to a constant, and this closes the system of equations (2.22), (2.24), (2.25), (2.27), allowing us to proceed with the reconstruction numerically and analytically.

Also, it is easy to see that the reconstructed theories are merely Brans–Dicke theories with an additional potential \( U \). If we define \( F = \beta \phi_{BD} = \xi \Phi^2 \) then \( \Phi = \sqrt{(\beta/\xi)}\phi_{BD} \) and the Lagrangian (1.2) becomes
\[ \mathcal{L} = \frac{\beta \phi_{BD}}{2} R - \frac{1}{2} \frac{\omega_{BD}}{\phi_{BD}} g^{\mu\nu} \partial_\mu \phi_{BD} \partial_\nu \phi_{BD} - U(\phi_{BD}) + \mathcal{L}_m, \]  
(3.10)
where \( \omega_{BD} \equiv \epsilon{\alpha \over \xi} = \text{constant} \). The Einstein frame version of the Brans–Dicke theory can also be easily obtained via the usual conformal transformation
\[ g_{\mu\nu} \equiv A^2(\varphi)\tilde{g}_{\mu\nu}, \]  
(3.11)
\[ \phi_{BD} \equiv A^{-2}(\varphi), \]  
(3.12)
\[ \tilde{V}(\varphi) \equiv A^4(\varphi)U(\phi_{BD}), \]  
(3.13)
\[ \alpha(\varphi)^2 \equiv \left( \frac{d\ln(A)}{d\phi} \right)^2 = \frac{1}{2\omega_{BD} + 3}, \]  
(3.14)
which implies that the potential $\tilde{V}(\varphi)$ in the Einstein frame is of the form

$$\tilde{V}(\varphi) \sim e^{\lambda(\omega_{\text{BD}}, m)\varphi},$$

where $\lambda(\omega_{\text{BD}}, m)$ is a constant that depends on $\omega_{\text{BD}}$ and $m$.

A noteworthy feature of the above reconstruction scheme is that equation (3.9) and equation (3.4) are exactly the conditions for the existence of a Noether symmetry in scalar–tensor theories, as we discuss in what follows.

3.2. Noether symmetries in scalar–tensor gravity

3.2.1. Generalities on the method. Solutions for the Lagrangian (1.2) can be sought by the so-called Noether symmetry approach [22]. This approach allows one, in principle, to find cyclic variables related to conserved quantities and then to reduce the dynamics. Besides, the existence of symmetries fixes the forms of the coupling $F(\Phi)$ and of the potential $U(\Phi)$, and gives the relation between them.

Let us give a quick summary of the approach for finite-dimensional dynamical systems before the application to our specific problem.

Let $L(q_i, \dot{q}_i)$ be a canonical, non-degenerate point-like Lagrangian in the configuration coordinates $q_i$ (the ‘positions’), where

$$\frac{\partial L}{\partial \lambda} = 0; \quad \det H_{ij} \equiv \det \left| \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right| \neq 0. \quad (3.16)$$

$H_{ij}$ is the Hessian matrix related to $L$. The dot indicates derivatives with respect to the affine parameter $\lambda$ which, in general, corresponds to the time $t$. We are going to consider only transformations which are point-transformations. Any invertible and smooth transformation of the ‘positions’ $Q^i = Q^i(q)$ induces a transformation on the ‘velocities’ such that

$$\dot{Q}^i(q) = \frac{\partial Q^i}{\partial q^j} \dot{q}^j. \quad (3.17)$$

The matrix $J = |\partial Q^i / \partial q^j|$ is the Jacobian of the transformation on the positions, and it is assumed to be non-zero. The Jacobian $\tilde{J}$ of the ‘induced’ transformation is easily derived, and it has to be $\tilde{J} \neq 0 \rightarrow J \neq 0$. Usually, this condition is not satisfied in the whole space but only in the neighbor of a point. It is a local transformation. A point transformation $Q^i = Q^i(q)$ can depend on one (or more than one) parameter. In general, an infinitesimal point transformation is represented by a generic vector field acting on the space $\{q^i, \dot{q}^i\}$. The transformation induced by (3.17) is then represented by

$$X = \alpha^i(q) \frac{\partial}{\partial q^i} + \left( \frac{d}{d\lambda} \alpha^i(q) \right) \frac{\partial}{\partial \dot{q}^i}. \quad (3.18)$$

$X$ is called the ‘complete lift’ of $X$ [29]. A function $f(q, \dot{q})$ is invariant under the transformation $X$ if

$$L_X f \equiv \alpha^i(q) \frac{\partial f}{\partial q^i} + \left( \frac{d}{d\lambda} \alpha^i(q) \right) \frac{\partial f}{\partial \dot{q}^i} = 0, \quad (3.19)$$

where $L_X f$ is the Lie derivative of $f$ with respect to $X$.
where \( L_X f \) is the Lie derivative of \( f \). In particular, if
\[
L_X \mathcal{L} = 0,
\] (3.20)
\( X \) is said to be a symmetry for the dynamics derived from the Lagrangian \( \mathcal{L} \). To see how Noether’s theorem and cyclic variables are related, let us consider a Lagrangian \( \mathcal{L} \) and the related Euler–Lagrange equations
\[
\frac{d}{d\lambda} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) - \frac{\partial \mathcal{L}}{\partial q^i} = 0.
\] (3.21)
Let us consider also the vector field (3.18). By contracting (3.21) with the \( \alpha^i \), one obtains
\[
\alpha^j \left( \frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{q}^j} - \frac{\partial \mathcal{L}}{\partial q^j} \right) = 0.
\] (3.22)
With
\[
\alpha^j \frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{q}^j} = \frac{d}{d\lambda} \left( \alpha^j \frac{\partial \mathcal{L}}{\partial \dot{q}^j} \right) - \left( \frac{d\alpha^j}{d\lambda} \right) \frac{\partial \mathcal{L}}{\partial \dot{q}^j},
\] (3.23)
from (3.22), we have
\[
\frac{d}{d\lambda} \left( \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) = L_X \mathcal{L}.
\] (3.24)
As a consequence, the Noether Theorem enunciates:
If \( L_X \mathcal{L} = 0 \), the function
\[
\Sigma_0 = \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i}
\] (3.25)
is a constant of motion.
It is worth noting that equation (3.25) can be expressed independently of coordinates as a contraction of \( X \) by a Cartan one-form
\[
\theta_\mathcal{L} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^i} dq^i.
\] (3.26)
Thus equation (3.25) can be written as
\[
i_X \theta_\mathcal{L} = \Sigma_0,
\] (3.27)
where \( i_X \) is defined through the relation
\[
i_X dq^i = \alpha^i.
\] (3.28)
By a point-transformation, the vector field \( X \) becomes
\[
\tilde{X} = (i_X dQ^k) \frac{\partial}{\partial Q^k} + \left( \frac{d}{d\lambda} (i_X dQ^k) \right) \frac{\partial}{\partial \dot{Q}^k}.
\] (3.29)
\( \tilde{X} \) is still the lift of a vector field defined on the ‘space of positions’. If \( X \) is a symmetry and we choose a point transformation such that
\[
i_X dQ^1 = 1 \quad i_X dQ^i = 0 \quad i \neq 1,
\] (3.30)
Reconstruction of the scalar–tensor Lagrangian from a ΛCDM background and Noether symmetry

we get

\[ \tilde{X} = \frac{\partial}{\partial Q^1}, \quad \frac{\partial \mathcal{L}}{\partial Q^1} = 0. \]  

(3.31)

Thus \( Q^1 \) is a cyclic coordinate and the dynamics can be reduced [30,31]. Clearly the change of coordinates defined by (3.30) is not unique. Usually a clever choice is very important. It is possible that more than one vector field \( X \) is found. In this case, more than one symmetry exists.

3.2.2. The case of scalar–tensor gravity. The above method can be used to seek for solutions in the dynamics given by Lagrangian (1.2). In particular, for a flat FRW metric, the field Lagrangian (1.2) reduces to the point-like Lagrangian

\[ \mathcal{L} = -3a\dot{a}^2F - 3F_\Phi \dot{a}^2 + a^3 \left( \frac{1}{2} \dot{\Phi}^2 - U(\Phi) \right) - Da^{-3(\gamma - 1)}, \]  

(3.32)

where, for the sake of simplicity, we are considering only the scalar field case (the generalization to the phantom field case is obvious). The constant \( D \) is related to the perfect-fluid matter density, with \( \rho_m = D(a_0/a)^{3\gamma} \), where \( 1 \leq \gamma \leq 2 \) defines the Zel’dovich range for the equation of state of standard matter. The above dynamical system (2.2), (2.3), (2.4) is immediately deduced considering the energy condition and the Euler–Lagrange equations for (3.32). In the case of standard dust matter, \( \gamma = 1 \), the last term in (3.32) reduces to an additive constant. With \( \{ a, \Phi \} \) being the configuration space of the system, the problem is two dimensional, and then the infinitesimal generator of the Noether symmetry is

\[ X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \Phi} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{\Phi}}, \]  

(3.33)

where \( \alpha \) and \( \beta \) are functions depending on \( a \) and \( \Phi \), and

\[ \dot{\alpha} \equiv \frac{\partial \alpha}{\partial a} \dot{a} + \frac{\partial \alpha}{\partial \Phi} \dot{\Phi}; \quad \dot{\beta} \equiv \frac{\partial \beta}{\partial a} \dot{a} + \frac{\partial \beta}{\partial \Phi} \dot{\Phi}. \]  

(3.34)

The condition for the existence of a Noether symmetry is \( L_X \mathcal{L} = 0 \). It, explicitly, gives an expression of second degree in \( \dot{a} \) and \( \dot{\Phi} \), whose coefficients are zero due to the fact they are considered to be linearly independent. Then this set of coefficients gives rise to the following system of partial differential equations [22]:

\[ \alpha + 2a \frac{\partial \alpha}{\partial a} + a^2 \frac{\partial \beta}{\partial a} F_\Phi + a\beta \frac{F_\Phi}{F} = 0, \]  

(3.35)

\[ \left( 2\alpha + a \frac{\partial \alpha}{\partial a} + a \frac{\partial \beta}{\partial \Phi} \right) F_\Phi + a F_\Phi \beta + 2F \frac{\partial \alpha}{\partial \Phi} - \frac{a^2 \partial \beta}{3 \partial a} = 0, \]  

(3.36)

\[ 3\alpha - 6F_\Phi \frac{\partial \alpha}{\partial \Phi} + 2a \frac{\partial \beta}{\partial \Phi} = 0, \]  

(3.37)

\[ \frac{U_\Phi}{U} = \frac{3\alpha}{a\beta}. \]  

(3.38)
Equation (3.38) can be rewritten in the form
\[
\frac{U_\Phi}{U} = m \cdot \frac{F_\Phi}{F},
\] (3.39)
where
\[
m \equiv -\frac{3\alpha F}{a\beta F\Phi}.
\] (3.40)

It is worth noting that equation (3.39) is a relation between the potential and the coupling, and it exactly coincides with (2.26). Solving the above system means finding out the explicit form of the set of functions \(\{\alpha, \beta, F, U\}\). For this purpose, we consider the separation of variables,
\[
\alpha = A_1(a)A_2(\Phi),
\] (3.41)
and
\[
\beta = B_1(a)B_2(\Phi).
\] (3.42)
Then from equation (3.37) we get
\[
\frac{B_1 a}{6A_1} = -\frac{A_2}{4B_2} + \frac{A_2'F'}{2B_2} = C,
\] (3.43)
where \(C\) is a separation constant. The solution of (3.43) is simple, and we get
\[
A_1 = \frac{B_1 a}{6C},
\] (3.44)
and
\[
B_2' = -\frac{A_2 - 2A_2'F'}{4C}.
\] (3.45)
Using equations (3.41) and (3.42) on (3.35), we get
\[
-\frac{a}{B_1} \frac{dB_1}{da} = \frac{3(A_2F + 2CB_2F')}{2(A_2F + 3CB_2F')} = -s,
\] (3.46)
where \(s\) is a separation constant. Hence, from equations (3.44) and (3.46) we get
\[
B_1 = Ba^s,
\] (3.47)
and
\[
A_1 = \frac{B}{6C} a^{s+1},
\] (3.48)
where \(B\) is a constant of integration. Also, from (3.46) we have
\[
\frac{F'}{F} = -\frac{2s + 3}{6C(s + 1)} \frac{A_2}{B_2}.
\] (3.49)
Using equations (3.47), (3.48) and (3.37) yields
\[
2FA_2' + (A_2(s + 3) + 6CB_2')F' - 2B_2C(s - 3F'') = 0.
\] (3.50)
Now we are left with three equations (3.45), (3.49) and (3.50) to solve for the three unknown functions $A_2$, $B_2$ and $F$. Using equations (3.45) and (3.49) for $B_2$ and $B'_2$ on equation (3.50), we get

$$FA'_2 + \frac{F'}{4}(A_2(2s + 3) + 6A'_2 F') + \frac{A_2 F(2s + 3)(s - 3F'')}{6(s + 1)F'} = 0. \quad (3.51)$$

Also, using (3.49) in (3.45) yields

$$2A'_2 F' \left[3(s + 1)(F')^2 + F(2s + 3)\right] + A_2 \left[(s + 3)(F')^2 - 2F(2s + 3)F''\right] = 0. \quad (3.52)$$

Now, we can use equations (3.51) and (3.52) to eliminate $A_2$ and $A'_2$ in favor of $F$:

$$F'' = \frac{3s(s + 1)(s + 2)F'^4}{(2s + 3)F^2} + \frac{(s + 1)(8s^2 + 16s + 3)F'^2}{2(2s + 3)F} + \frac{s(2s + 3)}{3}. \quad (3.53)$$

Equation (3.53) is nonlinear and its complete solution is an elliptical integral of the second kind which is not simple to handle. However, an exact solution can be found to be of the form

$$F = \xi(\Phi - \Phi_0)^2. \quad (3.54)$$

Using the ansatz (3.54) in (3.53) we get that $\xi = -(2s + 3)^2/24(s + 1)(s + 2)$ or $\xi = -1/6$, with the latter corresponding to the conformal coupling [32]. Note also that the conformal coupling $\xi = -1/6$ corresponds to a model where there is actually no scalar degree of freedom despite the fact that there seems to exist one in the parameterization (1.2) (see [33]). The free parameter $s$ has a physical meaning since it is connected to the ratio of critical points $\bar{x}_1$ and $\bar{x}_3$ and to the coupling. For this form of $F(\Phi)$ we can now determine $A_2$ from (3.51) or (3.52) and $B_2$ from (3.49), and finally arrive at a solution for $\alpha$ and $\beta$. For the two values of $\xi$ and equations (3.51) and (3.52) we get three degenerate solutions for $\alpha$ and $\beta$, i.e., they correspond to the same form of the potential $U$ (see equation (3.38)). The solutions are

$$\alpha_1 = \frac{a^{s+1}AB(\Phi - \Phi_0)^{2s(s+2)/(2s+3)}}{6C},$$

$$\beta_1 = \frac{a^sAB(2s + 3)(\Phi - \Phi_0)^{2s(s+2)/(2s+3)+1}}{12C(s + 1)},$$

$$\alpha_2 = \frac{a^{s+1}AB(\Phi - \Phi_0)^s(2s+3)/2(s+1)}{12C(s + 1)},$$

$$\beta_2 = \frac{a^sAB(2s + 3)(\Phi - \Phi_0)^{s(2s+3)/(2(s+1))+1}}{12C(s + 1)},$$

$$\alpha_3 = \frac{a^{s+1}AB(\Phi_0 - \Phi)^s}{6C},$$

$$\beta_3 = \frac{a^sAB(2s + 3)(\Phi_0 - \Phi)^{s+1}}{12C(s + 1)}.$$

It is easy to show that for all three cases we have

$$m(s) = \frac{3(s + 1)}{2s + 3}. \quad (3.55)$$
Reconstruction of the scalar–tensor Lagrangian from a $\Lambda$CDM background and Noether symmetry

and that from equation (3.39) we get

$$U(\Phi) = U_0(\Phi - \Phi_0)^{6(\gamma+1)/(2\gamma+3)} = U_0(\Phi - \Phi_0)^{2m(s)},$$

(3.56)

where $U_0$ is a constant determining the scale of the potential and is not directly measurable, but can be rewritten in terms of observable parameters like $H_0$, $q_0$, $\Omega_m$. A noteworthy feature of equation (3.56) is that it exactly coincides with (3.8), thus hinting towards a non-trivial physical content in this class of scalar–tensor Lagrangians.

In order to find the solution to the field equations we need to find the value of the constant of motion $\Sigma_0$ from equation (3.25). This is

$$\Sigma_0 = \frac{b(2s + 3)^2}{(s + 1)^2(s + 2)(s + 3)} \frac{d(a^{s+3}(\Phi - \Phi_0)^{(2s+2)/(2s+3)+2})}{dt},$$

(3.57)

where $b = -AB/48C$. Integration of (3.57) yields

$$\Phi - \Phi_0 = a^{-(2s+3)/(2s+2)}c^{1+3/2s}t(2s+3)/(2s^2+8s+6),$$

(3.58)

where $c = (((s+1)^2(s^2+5s+6)\Sigma_0)/(b(2s+3)^2))^{s/(s+1)(s+3)}$.

Plugging (3.56) and (3.58) into (2.2) we can get $a(t)$ and $\Phi(t)$:

$$a(t) = t^{(s+2)/(s+3)} \left( \frac{8cs(s+1)(s+2)(s+3)^2U_0(t^{(s+6)/(s+3)} + a_0)^2}{(s+6)(2s+3)^2} \right)^{1+1/s},$$

(3.59)

and

$$(\Phi(t) - \Phi_0)^2 = c^{2+(3/s)}t^{-(2s+3)/(s+3)} \left( \frac{8cs(s+1)(s+2)(s+3)^2U_0t^{(s+6)/(s+3)} + a_0}{(s+6)(2s+3)^2} \right)^{-2-(3/s)},$$

(3.60)

where $a_0$ is an integration constant.

Due to the structure of the above general solution, the cases $s = 0$, $-1$, $-3/2$, $-2$ and $-3$ have to be considered separately. The solutions for $s = 0$ and $-3$ correspond to the minimal coupling where $F = F_0$ and $U = \Lambda$ and to the quartic potential case where $F \sim \Phi^2$ and $U \sim \Phi^4$. In these situations, the solutions assume oscillating or exponential behavior (for a discussion see [22, 23, 34]).

3.3. Numerical results

As a final step, in order to confirm the validity of our analysis, we perform a numerical evolution of the dynamical system to compare the form of the potential $U$ and coupling $F$ of equations (3.4) and (3.8) (found also by the Noether symmetry approach), with the corresponding form obtained from the numerical analysis. The steps involved in the comparison are the following:

- Numerically solve the dynamical system (2.22), (2.24), (2.25), (2.27) and obtain $x_1(N)$, $x_2(N)$, $x_3(N)$ and $x_4(N)$. Integrate equation (2.16) to get

$$F(N) = F_0e^{-\int_{N_{\text{min}}}^{N} x_1(N')dN'},$$

(3.61)
Reconstruction of the scalar–tensor Lagrangian from a ΛCDM background and Noether symmetry

Figure 4. The form of \( \log(F(\Phi)) \) in the numerical reconstruction (red continuous line), its analytical approximation (blue dotted line) and a fit of the numerical reconstruction using equation (3.61) (green long-dashed line). The agreement between the three approaches is very good. The reason for the existence of the small plateau, see the enlarged region, is that as the system evolves towards the de Sitter era the potential \( F(\Phi) \) ‘freezes’ much faster than the field \( \Phi \).

and use equation (2.18) in the form

\[
\Phi'(N)^2 = 6x_3^2(N)F(N)
\]  
(3.62)

to obtain \( \Phi(N) \). The resulting form of \( F(\Phi) \) in both the numerical (red continuous line) equation (3.61) and its analytical approximation (blue dotted line) equation (3.4) is shown in figure 4.

• Use equation (2.17) to obtain \( U(N) \) numerically (red continuous line in figure 5)

\[
U(N) = 3x_2(N)F(N)H(N)^2
\]  
(3.63)

and compare with the analytical form (blue dotted line in figure 5) of equation (3.8).

In the methodology mentioned above we used the exact coefficients \( \bar{x}_1, \bar{x}_2, \bar{x}_3 \) and \( \bar{x}_4 \) from table 1 in each era for the analytic forms (3.4) and (3.8). Also, the initial conditions used in the numerical evolution were \( F(N = -30) = \Phi(N = -30) = 1 \). As another test, we fitted the numerically obtained \( F(N) \) and \( U(N) \) from equations (3.61) and (3.63) respectively to obtain the coefficients of the analytic forms (3.4) and (3.8) and found that the results were in good agreement (see figures 4 and 5: the green long-dashed lines). A noteworthy feature of figures 4 and 5 is a small plateau that appears during the de Sitter era (see the enlarged region in figure 4). The reason for the existence of the plateau is that as the system evolves towards the de Sitter era the coupling \( F(\Phi) \) ‘freezes’, since \( x_1(N_{ds}) \to 0 \) (see equation (2.16) and figure 2(a)), much faster than the field \( \Phi \).
4. Conclusions and outlook

We have reconstructed the form of the gravitational coupling $F(\Phi)$ and the potential $U(\Phi)$ of scalar–tensor quintessence by demanding that it reproduces a $\Lambda$CDM cosmic history through the radiation ($w_{\text{eff}} = \frac{1}{3}$), matter ($w_{\text{eff}} = 0$), and de Sitter ($w_{\text{eff}} = -1$) eras. We have found that apart from the usual general relativistic solution with a constant coupling $F(\Phi) = F_0$ and potential $U(\Phi) = U_0$ (corresponding to Newton and cosmological constants), there is another consistent solution which reproduces the same cosmic history. According to this solution

$$ F(\Phi) = \xi(\Phi - C)^2, $$

$$ U(\Phi) \sim F(\Phi)^m, $$

where $m$ and $C$ are arbitrary constants ($m$, however, is negative). The fact that the functional forms (4.1) and (4.2) can indeed reproduce the full $\Lambda$CDM cosmic history has been demonstrated not only through the critical points analysis but also through the numerical solution shown in figure 2 for a fixed $m = -0.5$. This numerical solution assumes a $\Lambda$CDM background and shows that functional forms (4.1) and (4.2) (with $m = -0.5$) are reproduced from this background and are therefore consistent with it.

The direct comparison of our results with the corresponding results of [15] and [21] would be particularly interesting. Unfortunately, such a comparison can not be made for the following reasons. First, [21] had set up the general framework for the reconstruction but did not attempt an explicit reconstruction of $F(\Phi)$ and $U(\Phi)$ from $H(z)$ because they found that there are more unknown functions than equations (one equation with two unknown functions $F(\Phi)$ and $U(\Phi)$) and therefore it was necessary to make an additional...
Reconstruction of the scalar–tensor Lagrangian from a $\Lambda$CDM background and Noether symmetry assumption. In our case, this assumption is a constant $m$ which is necessary on the critical points. The corresponding assumption of [15] was a very small value for the kinetic term $\Phi'$. This assumption was used to reproduce not $\Lambda$CDM but an $H(z)$ that crossed the phantom divide at late times. Also, this reconstruction was made at late times only and not throughout the cosmic history. For these reasons, it is not possible to make a meaningful connection with the results of [15] and [21].

In this new solution the ‘radiation’, ‘matter’ and ‘de Sitter’ expansion rates, for the ‘attractor’ trajectory shown in figures 2 and 3, is dominated by dark gravity through all epochs. This is indeed a potential problem for this type of trajectory but it could also be a potential blessing since this type of solution has the correct expansion rate at all epochs without the use of dark matter or dark energy. A proper test of these models for detailed comparison with observations would require analysis of large scale structure formation (analysis of evolution of perturbations). Such an analysis is beyond the aims of our present analysis but it is an interesting extension of this project.

We should also stress that we have not found all possible theories that reconstruct the $\Lambda$CDM expansion history but only a subclass of them that corresponds to a fixed $m$. The assumption of a constant $m$ then directly leads to the quest for fixed points, which is just a convenient way for finding one solution for the system that reconstructs the $\Lambda$CDM expansion history on the radiation, matter and de Sitter eras. On the other hand, going back to the considerations developed in section 3 regarding the number of parameters beside $m$, we have to stress again that if we had used the parameter $n \sim F_{\Phi}/F$ in our analysis (thus fixing $F(\Phi)$ and allowing $H(N)$ to vary), the autonomous system would be very different and, depending on the values of $n$, the stability and the form of $H(N)$ would correspondingly vary. An alternative approach could be to fix $F$ and $U$, as in equations (4.1) and (4.2), and allow $H(N)$ to vary about a $\Lambda$CDM background, along the lines of [26]. Very likely, this approach could provide a comprehensive stability analysis of the $\Lambda$CDM model since it involves also the proper fluctuation modes of $H(N)$. This extension is out of the scope of this work and will be faced in a forthcoming paper.

Another point is that phantom behavior can be easily realized in scalar tensor theories, see [11,35] for a discussion, and this is an attractive feature of these theories. In fact we could have chosen to reconstruct different forms of $H(N)$ giving late time phantom behavior, thus deriving different forms of $F(N)$ at late times with different fixed points. This late time reconstruction has been undertaken in [14,17] and [36]. However, clearly the early times behavior of the reconstructed $F(N)$ would be unchanged even in the phantom case. Fixing $F(\Phi)$ could also lead to phantom behavior but we would have to guess a proper form of $F$.

A completely independent way that can also lead to the form of the new solution presented here is obtained by imposing maximal Noether symmetry on the scalar–tensor Lagrangian. We have demonstrated that imposing such a symmetry leads uniquely to exactly the same form of potentials as (4.1) and (4.2). It also leads to a conserved charge $\Sigma_0$ which allows the derivation of exact solutions for the evolution of the scale factor $a(t)$ and the scalar field $\Phi(t)$.

This intriguing coincidence of the two approaches hints towards a non-trivial physical content in this class of scalar–tensor Lagrangians, and this seems to be a peculiar and interesting coincidence which could have physical roots. It is therefore important to study
Reconstruction of the scalar–tensor Lagrangian from a ΛCDM background and Noether symmetry

the evolution of cosmological perturbations in this class of models in order to test them using large scale structure and CMB observations.

Acknowledgments

The authors would like to thank Gilles Esposito-Farese and David Polarski for helpful suggestions and discussions. This work was supported by the European Research and Training Network MRTN-CT-2006 035863-1 (UniverseNet). SN acknowledges support from the Greek State Scholarships Foundation (IKY).

References

[1] Perlmutter S et al., 1999 Astrophys. J. 517 565 [SPIRES]
Riess A G et al., 1998 Astron. J. 116 1009
Riess A G et al., 1999 Astron. J. 117 707
Tonry J L et al., 2003 Astrophys. J. 594 1 [SPIRES]
Knop R A et al., 2003 Astrophys. J. 598 102 [SPIRES]
Astier P et al., 2006 Astron. Astrophys. 447 31 [SPIRES]
Miknaitis G et al., 2007 Preprint astro-ph/0701043
Riess A G et al., 2006 Preprint astro-ph/0611572

[2] Spergel D N et al., 2003 Astrophys. J. Suppl. 148 175
Spergel D N et al., 2006 Preprint astro-ph/0603449

[3] Eisenstein D J et al., 2005 Astrophys. J. 633 560 [SPIRES]
Blake C, Parkinson D, Bassett B, Glazebrook K, Kunz M and Nichol R C, 2006 Mon. Not. R. Astron. Soc. 365 255

[4] Sahni V and Starobinsky A A, 2000 Int. J. Mod. Phys. D 9 373 [SPIRES]
Carroll S M, 2001 Living Rev. Rel. 4 1

[5] Copeland E J, Sami M and Tsujikawa S, 2006 Int. J. Mod. Phys. D 15 1753 [SPIRES] [hep-th/0603057]

[6] Fujii Y, 1982 Phys. Rev. D 26 2580 [SPIRES]
Ford L H, 1987 Phys. Rev. D 35 2339 [SPIRES]

[7] Bilic N, Tupper G B and Viollier R D, 2002 Phys. Lett. B 535 17 [SPIRES] [astro-ph/0111325]

[8] Friedland A, Murayama H and Perelstein M, 2003 Phys. Rev. D 67 043519 [SPIRES] [gr-qc/0205520]

[9] Li M, 2004 Phys. Lett. B 603 1 [SPIRES] [hep-th/0403127]
Huang Q G and Gong Y G, 2004 J. Cosmol. Astropart. Phys. JCAP08(2004)006 [SPIRES] [astro-ph/0403590]

[10] Fujii Y, 2000 Phys. Rev. D 62 044011 [SPIRES]
Bartolo N and Pietroni M, 2000 Phys. Rev. D 61 023518 [SPIRES]
Perrotta F, Baccigalupi C and Matarrese S, 2000 Phys. Rev. D 61 023507 [SPIRES]
Esposito-Farese G and Polarski D, 2001 Phys. Rev. D 63 063504 [SPIRES]
Torres D F, 2002 Phys. Rev. D 66 043522 [SPIRES]

Journal of Cosmology and Astroparticle Physics 12 (2007) 009 (stacks.iop.org/JCAP/2007/i=12/a=009) 23
Reconstruction of the scalar–tensor Lagrangian from a $\Lambda$CDM background and Noether symmetry

[11] Gannouji R, Polarski D, Ranquet A and Starobinsky A A, 2006 J. Cosmol. Astropart. Phys. JCAP09(2006)016 [SPIRES]

[12] Amendola L, Polarski D and Tsujikawa S, 2007 Phys. Rev. Lett. 98 131302 [SPIRES] [astro-ph/0603703]

Capozziello S, 2002 Int. J. Mod. Phys. D 11 883 [SPIRES]

Capozziello S, Carloni S and Troisi A, 2003 Astron. Astrophys. 11 [SPIRES] [astro-ph/0303041]

Nojirí S and Odintsov S D, 2004 Gen. Rel. Grav. 36 1765 [SPIRES]

Soussa M E and Woodard R P, 2004 Gen. Rel. Grav. 36 855 [SPIRES]

Allemandi G, Borowiec A and Francaviglia M, 2004 Phys. Rev. D 70 103503

Easson D A, 2004 Int. J. Mod. Phys. A 19 5343 [SPIRES]

Carroll S M, De Felice A, Duvvuri V, Easson D A, Trodden M and Turner M S, 2005 Phys. Rev. D 71 063513 [SPIRES]

Capozziello S, Dunsby P K S, Carloni S and Troisi A, 2005 Class. Quantum Grav. 22 4839 [SPIRES]

Capozziello S, Cardone V F and Troisi A, 2005 Phys. Rev. D 71 043503 [SPIRES]

Nojirí S and Odintsov S D, 2004 Gen. Rel. Grav. 36 1765 [SPIRES]

Soussa M E and Woodard R P, 2004 Gen. Rel. Grav. 36 855 [SPIRES]

Allemandi G, Borowiec A and Francaviglia M, 2004 Phys. Rev. D 70 103503

Easson D A, 2004 Int. J. Mod. Phys. A 19 5343 [SPIRES]

Carroll S M, De Felice A, Duvvuri V, Easson D A, Trodden M and Turner M S, 2005 Phys. Rev. D 71 063513 [SPIRES]

Capozziello S, Dunsby P K S, Carloni S and Troisi A, 2005 Class. Quantum Grav. 22 4839 [SPIRES]

Capozziello S, Cardone V F and Troisi A, 2005 Phys. Rev. D 71 043503 [SPIRES]

Cognola G, Elizalde E, Nojirí S, Odintsov S D and Zerbini S, 2005 J. Cosmol. Astropart. Phys. JCAP02(2005)010 [SPIRES]

Nojirí S, Odintsov S D and Tsujikawa S, 2005 Phys. Rev. D 71 063004 [SPIRES]

Abdalla M C B, Nojirí S and Odintsov S D, 2006 Preprint hep-th/0601213

Woodard R P, 2006 Preprint astro-ph/0601672

Das S, Banerjee N and Dadhich N, 2006 Class. Quantum Grav. 23 4159 [SPIRES]

Capozziello S, Cardone V F, Elizalde E, Nojirí S and Odintsov S D, 2006 Phys. Rev. D 73 043512 [SPIRES]

Srivastava S K, 2006 Preprint astro-ph/0602116

Sotiriou T P, 2006 Class. Quantum Grav. 23 5117 [SPIRES]

Sotiriou T P, 2006 Preprint gr-qc/0601107

Sotiriou T P, 2006 Preprint gr-qc/0611158

Sotiriou T P and Liberati S, 2006 Preprint gr-qc/0604006

De Felice A, Hindmarsh M and Trodden M, 2006 J. Cosmol. Astropart. Phys. JCAP08(2006)005 [SPIRES]

Bludman S, 2006 Preprint astro-ph/0605198

Carroll S M, Sawicki I, Silvestri A and Trodden M, 2006 Preprint astro-ph/0607458

Huterer D and Linder E V, 2006 Preprint astro-ph/0608681

Jin X h, Liu D j and Li X z, 2006 Preprint astro-ph/0610854

Poplawski N J, 2006 Phys. Rev. D 74 084032 [SPIRES]

Poplawski N J, 2006 Preprint gr-qc/0610133

Poplawski N J, 2006 Preprint astro-ph/0610734

Chiha T, Smith T L and Erickcek A L, 2006 Preprint astro-ph/0611867

Faraoni V and Nadeau S, 2006 Preprint astro-ph/0612075

Li B and Barrow J D, 2007 Preprint gr-qc/0701111

Navarro I and Van Acoleyen K, 2006 Preprint gr-qc/0611127

Brookfield A W, van de Bruck C and Hall L M H, 2006 Phys. Rev. D 74 064028 [SPIRES]

Fairbairn M and Rydbeck S, 2007 Preprint astro-ph/0701900

Dick R, 2004 Gen. Rel. Grav. 36 217 [SPIRES] [gr-qc/0307052]

Capozziello S, Carloni S and Troisi A, 2003 Preprint astro-ph/0303041

Sawicki I and Hu W, 2007 Preprint astro-ph/0702278

Hu W and Sawicki I, 2007 Preprint 0705.1158 [astro-ph]

Amendola L and Tsujikawa S, 2007 Preprint 0705.0396 [astro-ph]

Nojirí S, Odintsov S D and Tretyakov P V, 2007 Preprint 0704.2520 [hep-th]

[13] Maartens R, 2004 Living Rev. Rel. 7 7 [gr-qc/0312059]

Sahni V and Shtanov Y, 2003 J. Cosmol. Astropart. Phys. JCAP11(2003)014 [SPIRES] [astro-ph/0202346]

Kofinas G, Panotopoulos G and Tomaras T N, 2006 J. High Energy Phys. JHEP01(2006)107 [SPIRES] [hep-th/0510207]

Apostolopoulos P S and Tetradis N, 2006 Phys. Rev. D 74 064021 [SPIRES] [hep-th/0604014]

Bogdanos C, Dimitriadis A and Tamvakis K, 2006 Preprint hep-th/0611094

Perivolaropoulos L, 2005 J. Cosmol. Astropart. Phys. JCAP10(2005)001 [SPIRES] [astro-ph/0504582]

Nesseris S and Perivolaropoulos L, 2007 Phys. Rev. D 75 023517 [SPIRES] [astro-ph/0611235]

Boisseau B, Esposito-Far`ese G, Polarski D and Starobinsky A A, 2000 Phys. Rev. Lett. 85 2236 [SPIRES]

Alam U, Sahni V, Saini T D and Starobinsky A A, 2004 Mon. Not. R. Astron. Soc. 354 275 [astro-ph/0311364]

Nesseris S and Perivolaropoulos L, 2005 Phys. Rev. D 72 123519 [SPIRES] [astro-ph/0511040]
Reconstruction of the scalar–tensor Lagrangian from a ΛCDM background and Noether symmetry

Lazkoz R, Nesseris S and Perivolaropoulos L, 2005 *J. Cosmol. Astropart. Phys.* JCAP11(2005)010 [SPIRES] [astro-ph/0503230]

Nesseris S and Perivolaropoulos L, 2004 *Phys. Rev.* D 70 043531 [SPIRES] [astro-ph/0401556]

[17] Nesseris S and Perivolaropoulos L, 2007 *J. Cosmol. Astropart. Phys.* JCAP01(2007)018 [SPIRES] [astro-ph/0610092]

[18] Capozziello S and de Ritis R, 1993 *Phys. Lett.* A 177 1 [SPIRES]

[19] Capozziello S and de Ritis R, 1994 *Class. Quantum Grav.* 11 107 [SPIRES]

[20] Boisseau B, Esposito-Farese G, Polarski D and Starobinsky A A, 2000 *Phys. Rev. Lett.* 85 2236 [SPIRES] [gr-qc/0001066]

[21] Esposito-Farese G and Polarski D, 2001 *Phys. Rev.* D 63 063504 [SPIRES] [gr-qc/0009034]

[22] Capozziello S, de Ritis R, Rubano C and Scudellaro S, 1996 *Riv. Nuovo Cimento* 19 1–114

Kamyla S, Modak B and Biswas S, 2004 *Gen. Rel. Grav.* 36 661 [SPIRES]

[23] Capozziello S, Demianski M, de Ritis R and Rubano C, 1995 *Phys. Rev.* D 52 3288 [SPIRES]

[24] Tsujikawa S, 2007 *Preprint* 0705.1032 [astro-ph]

[25] Capozziello S, Nojiri S and Odintsov S D, 2006 *Phys. Lett.* B 634 93 [SPIRES]

Capozziello S, Nojiri S, Odintsov S D and Troisi A, 2006 *Phys. Lett.* B 639 135 [SPIRES]

[26] Carloni S, Leach J A, Capozziello S and Dunsby P K S, 2007 *Preprint* gr-qc/0701009

[27] Amendola L, Gannouji R, Polarski D and Tsujikawa S, 2007 *Phys. Rev.* D 75 083504 [SPIRES] [gr-qc/0612180]

[28] Fay S, Nesseris S and Perivolaropoulos L, 2007 *Preprint* gr-qc/0703006

[29] Morandi G, Ferrario C, Lo Vecchio G, Marmo G and Rubano C, 1990 *Phys. Rep.* 188 149 [SPIRES]

[30] Arnold V I, 1978 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)

[31] Marmo G, Saletan E J, Simoni A and Vitale B, 1985 *Dynamical Systems. A Differential Geometric Approach to Symmetry and Reduction* (New York: Wiley)

[32] Faraoni V, Gunzig E and Nardone P, 1999 *Fundam. Cosm. Phys.* 20 121 [gr-qc/9811047]

[33] Deser S, 1970 *Ann. Phys., NY* 59 248 [SPIRES]

[34] Demianski M, Piedipalumbo E, Rubano C and Tortora C, 2006 *Astron. Astrophys.* 454 55 [SPIRES]

[35] Martin J, Schimd C and Uzan J P, 2006 *Phys. Rev. Lett.* 96 061303 [SPIRES] [astro-ph/0510208]

[36] Tsujikawa S, 2005 *Phys. Rev.* D 72 083512 [SPIRES] [astro-ph/0508542]