On the rank of the distance matrix of graphs

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Abstract

Let $G$ be a connected graph with $V(G) = \{v_1, \ldots, v_n\}$. The $(i, j)$-entry of the distance matrix $D(G)$ of $G$ is the distance between $v_i$ and $v_j$. In this article, using the well-known Ramsey’s theorem, we prove that for each integer $k \geq 2$, there is a finite amount of graphs whose distance matrices have rank $k$. We exhibit the list of graphs with distance matrices of rank 2 and 3. Besides, we study the rank of the distance matrices of graphs belonging to a family of graphs with their diameters at most two, the trivially perfect graphs. We show that for each $\eta \geq 1$ there exists a trivially perfect graph with nullity $\eta$. We also show that for threshold graphs, which are a subfamily of the family of trivially perfect graphs, the nullity is bounded by one.

Keywords— Distance Matrix, Distance Rank, Threshold Graph, Trivially Perfect Graph.

1 Introduction

All graphs mentioned in this article are finite and have neither loops nor multiple edges. Let $G$ be a connected graph on $n$ vertices with vertex set $V = \{v_1, \ldots, v_n\}$. The distance in $G$ between vertices $v_i$ and $v_j$, denoted $d_G(v_i, v_j)$, is the number of edges of a shortest path linking $v_i$ and $v_j$. When the graph $G$ is clear from the context we write $d(v_i, v_j)$. The distance matrix of $G$, denoted $D(G)$, is the $n \times n$ symmetric matrix having its $(i, j)$-entry equal to $d(v_i, v_j)$. The distance matrix has attracted the attention of many researchers. The interest in this matrix was motivated by the connection with a communication problem (see [9, 10] for more details). In an early article, Graham and Pollack [10] presented a remarkable result, proving that the determinant of the distance matrix of a tree $T$ on $n$ vertices only depends on $n$, being

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equal to \((-1)^{n-1}(n-1)2^{n-2}\). More recently, formulas for the determinant of connected graphs on \(n\) vertices with \(n\) edges \([1]\) (unicyclic graphs) and \(n+1\) edges \([3]\) (bicyclic graphs) have been computed.

Determining the family of graphs with a given nullity for some associated matrix is a problem of interest for the graph-theoretic community. For instance, it is well-known that the nullity of the Laplacian matrix \(L(G)\) of a given graph \(G\) coincides with the number of connected components of \(G\) (see \([11]\)). Bo and Liu considered graphs whose adjacency matrix has rank two or three \([2]\); i.e., graphs with nullity \(n-2\) and \(n-3\), where \(n\) is the number of vertices of the graph. Later, Cang et al. characterized graphs whose adjacency matrix has rank four \([4]\) and five \([4]\).

The remainder of this article is organized as follows. In Section 2 we present some definitions and preliminary results. Section 3 is devoted to proving that for any integer \(k\ge 2\), there exists a finite number of graphs with distance rank \(k\). Section 4 presents a collection of results in connection with the distance rank of a graph and a partition of its vertex set into sets of twins. In Section 5 we prove that the nullity of any threshold graph is at most one, and we also present an infinite family of threshold graphs with nullity one. Finally, Section 6 contains a sufficient condition for a trivially perfect graph to have a nonsingular distance matrix and a result that guarantees an example of a trivially perfect graph with nullity \(\eta\), for each positive integer \(\eta\ge 2\). In Section 7 we close the article with some conclusions and open questions.

2 General concepts

Let \(G\) be a graph. We use \(V(G)\) and \(E(G)\) to denote the set of vertices of \(G\) and the set of edges of \(G\), respectively. We use \(N_G(v)\) to denote the set of neighbors of a vertex \(v\in V(G)\) and \(N_G[v] = N_G(v) \cup \{v\}\), we omit the subscript in case the context is clear enough. A vertex \(v\) is a universal vertex if \(N_G[v] = V(G)\). Let \(S \subseteq V(G)\). We use \(N_G(S)\) to denote the set of those vertices with at least one neighbor in \(S\) and \(N_G[S] = N_G(S) \cup S\), omitting the subscript in case the context is clear enough. Two vertices \(u\) and \(v\) are true twins (resp. false twins) if \(N[u] = N[v]\) (resp. \(N(u) = N(v)\)). A vertex \(v\) is universal if \(N[v] = V(G)\). Let \(X \subseteq V(G)\). We use \(G[X]\) to denote the subgraph of \(G\) induced by \(X\). A stable set (or independent set) of a graph is a set of pairwise nonadjacent vertices. By \(\overline{G}\), we denote the complement graph of \(G\). The maximum independent number, denoted \(\alpha(G)\), is the cardinality of an independent set with the maximum number of vertices. A clique is a set of pairwise adjacent vertices. A split graph is a graph whose vertices can be partitioned into an independent set and a clique. A complete graph is a graph such that all its vertices are pairwise adjacent. We use \(C_n, K_n, K_{t_n-1}\) and \(P_n\) to denote the isomorphism classes of cycles, complete graphs, stars and paths, all of them on \(n\) vertices, respectively. Let \(\mathcal{H}\) be a set of graphs. A graph is said to be \(\mathcal{H}\)-free if it does not contain any graph in \(\mathcal{H}\) as an induced subgraph. In the case in which \(\mathcal{H} = \{H\}\), we use \(H\)-free for short. Let \(G\) and \(H\) be two graphs. We use \(G + H\) (resp. \(G \vee H\)) to denote the disjoint union of \(G\) and \(H\) (resp. the joint between \(G\) and \(H\)); i.e., \(G + H\) plus all edges having an endpoint in \(V(G)\) and the other one in \(V(H)\).

A cograph is a \(P_2\)-free graph. If \(G\) is a cograph, then \(G\) or \(\overline{G}\) is connected \([5]\). Thus, if \(G\) is a connected cograph, then \(G = H \vee J\), for two cographs \(H\) and \(J\). A graph is trivially perfect if, for each induced subgraph, the maximum cardinality of an independent set agrees with the number of maximal cliques. Indeed, trivially perfect graphs are precisely the \(\{P_1, C_4\}\)-free graphs \([7]\). In addition, a graph is trivially perfect if and only if every connected induced subgraph has a universal vertex (see \([13]\)). A graph is threshold if it is \(\{2K_2, P_4, C_4\}\)-free. Observe that threshold graphs are precisely the split cographs. For more details about the graph classes described above, we refer the reader to \([8]\).
3 Distance rank of general graphs

The rank of a graph $G$, denoted $\text{rank}(G)$, is the rank of its adjacency matrix. For each integer $k \geq 2$ there exists an infinite family of graphs having rank $k$ (see [4]). The rank of $D(G)$, denoted $\text{rank}_d(G)$, is called the distance rank of $G$. Unlike what happens with the rank of a graph, as a consequence of Ramsey’s Theorem, for every integer $k \geq 2$ there exists a finite family of graphs having distance rank equal to $k$.

Recall that given two integers $r,t \geq 2$ there exists a positive integer $R(r,t)$, such that for every graph $G$ with $|V(G)| \geq R(r,t)$, $G$ contains either a clique with at least $r$ vertices or an independent set with at least $t$ vertices [12]. When $r = t$, $R(t)$ stands for $R(t,t)$. For bounds of $R(r,t)$ see for instance [13].

3.1 General characteristic

Let $n \geq 2$. If $G = K_n$, clearly $n = \text{rank}(G) = \text{rank}_d(G)$. Besides, if $G$ is a tree on $n$ vertices, then $\text{rank}_d(G) = n$ [10], and thus $\text{rank}_d(G)(K_{1,n-1}) = \text{rank}_d(G)(P_n) = n$. Let $G$ and $H$ be two graphs. The graph $H$ is said to be an isometric subgraph of $G$ if $H$ is a subgraph of $G$ such that $d_H(u,v) = d_G(u,v)$ for every $u,v \in V(H)$. We state the following immediate lemma without proof.

Lemma 1. If $H$ is an isometric subgraph of $G$, then $\text{rank}_d(H) \leq \text{rank}_d(G)$.

The diameter of a graph $G$, denoted $\text{diam}(G)$, is the maximum distance between two vertices. An induced path $P$ of $G$ on $\text{diam}(G) + 1$ vertices is called a diameter path. By Lemma 1 and [10], since every graph contains a diameter path as an isometric subgraph, the lemma below follows.

Lemma 2. If $G$ is a connected graph, then $\text{diam}(G) + 1 \leq \text{rank}_d(G)$.

It is well-known that the number of vertices of a graph $G$ is upper-bounded by a function on its maximum degree $\Delta(G)$ and $\text{diam}(G)$.

Lemma 3. [14] Exercise 2.1.60] Let $G$ be a graph. If $\text{diam}(G) = d$ and $\Delta(G) = r$, then

$$|V(G)| \leq \frac{1 + [(r - 1)^d - 1]r}{r - 2} = f(d,r)$$

As a consequence of Ramsey’s theorem we prove the main result of this section.

Theorem 1. If $k$ is an integer with $k \geq 2$, then there is a finite number of connected graphs $G$ such that $\text{rank}_d(G) = k$.

Proof. Consider a connected graph $G$ such that $\text{rank}_d(G) = k$. On the one hand if $\text{diam}(G) \geq k$, by Lemma 2 $\text{rank}_d(G) > k$. On the other hand, if $\Delta(G) \geq R(k)$, by Ramsey’s Theorem, $G$ contains either a complete subgraph $K_{k+1}$ or a star $K_{1,k}$ as an isometric subgraph. Thus, Lemma 1 $\text{rank}_d(G) > k$. Hence, if $\text{rank}_d(G) = k$, then $\text{diam}(G) < k$ and $\Delta(G) < R(k)$. Therefore, by Lemma 3 $|V(G)| \leq f(k,R(k))$ and the result holds.

3.2 Graphs with distance rank $k \in \{2,3\}$

A connected graph $G$ with at least three vertices contains either $P_3$ or $K_3$ as isometric subgraphs and thus $\text{rank}_d(G) \geq 3$. For graphs used throughout this section, see Figure 1. In particular, it is easy to check that $\text{rank}_d(P_3) = \text{rank}_d(D_4) = \text{rank}_d(H_4) = 4$.

Remark 1. A connected graph $G$ has $\text{rank}_d(G) = 2$ if and only if $G = K_2$.

The following lemma is a consequence of the isometric subgraph definition.
Lemma 4. If $H$ is a connected induced subgraph of a connected graph $G$ such that $d_H(u, v) \leq 2$ for every $u, v \in V(H)$, then $H$ is an isometric subgraph of $G$.

As a consequence of the above lemma the graphs with distance rank equals three are cographs.

Lemma 5. If $G$ is a connected graph with $\text{rank}_d(G) = 3$, then $G$ is a cograph.

Proof. We prove the contrapositive statement. Assume that $G$ contains a path with four vertices $P : a, b, c, d$ as an induced subgraph. If $P$ was an isometric subgraph, then $\text{rank}_d(G) \geq 4$ by Lemma 4. Assume that $d_G(a, d) = 2$. Consequently, there exists a vertex $v$ in $G$ that is adjacent to $a$ and $d$. Thus $G'[[a, b, c, d, v]]$ contains a diamond as an induced subgraph or is isomorphic to $C_5$ or the house. Since the diamond and the house have distance rank 4 and the $C_5$ has distance rank 5, it follows from Lemma 4 that $\text{rank}_d(G) \geq 4$. Thus, if $G$ is not a cograph, then $\text{rank}_d(G) \geq 4$. Therefore, the result follows.

Theorem 2. If $G$ is a connected graph with $\text{rank}_d(G) = 3$, then $G$ is one of the following graphs: $K_3$, $P_3$, or $C_4$.

4 Twins and null space

Let $G$ be a graph with vertices $v_1, v_2, \ldots, v_n$, and assume that $v_1$ and $v_2$ are either true twins or false twins. Notice that if $j \notin \{1, 2\}$, then $d_G(v_1, v_j) = d_G(v_2, v_j)$. Let $D$ be the distance matrix of $G$ and $\vec{x}$ a vector in the null space of $D$. We denote the coordinate of $\vec{x}$ that corresponds to vertex $v_i$ as $\vec{x}_{v_i}$. Notice that the coordinate

\begin{center}
\begin{tabular}{ccc}
\text{Pa} & \text{Di} & \text{Hou} \\
\begin{tikzpicture}
  \node[vertex] (a) at (0,0) {};
  \node[vertex] (b) at (1,0) {};
  \node[vertex] (c) at (1,1) {};
  \node[vertex] (d) at (0,1) {};
  \draw (a) -- (b) -- (c) -- (d) -- (a);
  \end{tikzpicture} & \begin{tikzpicture}
  \node[vertex] (a) at (0,0) {};
  \node[vertex] (b) at (1,0) {};
  \node[vertex] (c) at (1,1) {};
  \node[vertex] (d) at (0,1) {};
  \draw (a) -- (b) -- (c) -- (d) -- (a);
  \end{tikzpicture} & \begin{tikzpicture}
  \node[vertex] (a) at (0,0) {};
  \node[vertex] (b) at (1,0) {};
  \node[vertex] (c) at (1,1) {};
  \node[vertex] (d) at (0,1) {};
  \node[vertex] (e) at (1,2) {};
  \draw (a) -- (b) -- (c) -- (d) -- (e) -- (a);
  \end{tikzpicture}
\end{tabular}
\end{center}

Figure 1: Pa, the paw graph; Di, the diamond graph; and Hou, the house graph.
Lemma 7. Let $G$ be a graph with distance matrix $D$. If $v_i$ and $v_j$ are either true twins or false twins and $\vec{x}$ is in the null space of $D$, then $\vec{x}_{v_i} = \vec{x}_{v_j}$.

Lemma 6 allows to use a smaller matrix to study the null space of $D$. To do that, we introduce some notation. We say that a partition $W = \{W_1, W_2, \ldots, W_k\}$ of the set of vertices is a twin partition of a graph $G$ if $W_i$ is either a set of true twins or a set of false twins for every $i$. Notice that we allow $|W_i| = 1$. If $W_i$ is a set of true (false) twins for every $i$, then we say that $W$ is a true (false) twin partition of $G$.

Let $W = \{W_1, W_2, \ldots, W_k\}$ be a twin partition of $G$ and $w_1, \ldots, w_k$ a set of vertices with $w_i \in W_i$ for each $1 \leq i \leq k$. We define the quotient matrix $D/W$ by

\[
(D/W)_{i,j} = \begin{cases} 
|W_j|d_{G}(w_i, w_j) & \text{if } i \neq j, \\
(|W_i| - 1) & \text{if } i = j \text{ and } W_i \text{ is a set of true twins}, \\
2(|W_i| - 1) & \text{if } i = j \text{ and } W_i \text{ is a set of false twins}.
\end{cases}
\]

Let $\vec{x} \in \mathbb{R}^n$ be a vector such that $\vec{x}_{v_i} = \vec{x}_{v_j}$ if $v_i$ and $v_j$ are twin vertices and let $\vec{y} \in \mathbb{R}^k$ such that $\vec{y}_{w_i} = \vec{x}_{w_i}$. We have

\[
[D/W]\vec{y}_{w_i} = \sum_{j=1, j \neq i}^{k} d_{G}(w_i, w_j)|W_j|\vec{x}_{w_j} + c_i(|W_i| - 1)\vec{x}_{w_i},
\]

where $c_i = 1$ if $W_i$ consists of true twins and $c_i = 2$ if $W_i$ consists of false twins. On the other hand

\[
[D]\vec{x}_{w_i} = \sum_{v_j \in V} d_{G}(v_i, v_j)\vec{x}_{v_j}
= \sum_{\ell=1, \ell \neq i}^{k} \sum_{v_j \in W_\ell} d_{G}(v_i, v_j)\vec{x}_{v_j} + \sum_{v_j \in W_i, v_j \neq w_i} d_{G}(v_i, v_j)\vec{x}_{v_j}
= \sum_{\ell=1, \ell \neq i}^{k} |W_\ell|d_{G}(v_i, w_\ell)\vec{x}_{w_\ell} + c_i(|W_i| - 1)\vec{x}_{w_i}
= [D/W]\vec{y}_{w_i}.
\]

Thus, $\vec{x}$ is in the null space of $D$ if and only if $\vec{y}$ is in the null space of $D/W$. Combined with Lemma 6 this implies that the nullity of $D$ equals the nullity of $D/W$.

Lemma 7. Let $G$ be a graph, $D$ the distance matrix of $G$ and $W = \{W_1, \ldots, W_k\}$ be a partition of the vertices of $G$ into sets of twins, each of them consisting of either
true twins or false twins. For each \( i \), let \( w_i \) be a vertex in \( W_i \). If \( D/W \) is the
matrix defined as

\[
D/W_{i,j} = \begin{cases} 
|W_i|d_G(w_i, w_j) & \text{if } i \neq j, \\
|W_i| - 1 & \text{if } i = j \text{ and } W_i \text{ consists of true twins, and} \\
2(|W_i| - 1) & \text{if } i = j \text{ and } W_i \text{ consists of false twins,}
\end{cases}
\]

then the nullity of \( D \) is equal to the nullity of \( D/W \).

5 Threshold graphs

It is well-known that we can obtain any threshold graph by repeatedly adding either isolated vertices or universal vertices. Thus, a threshold graph can be represented by a
finite sequence \((a_i)_{i=1}^n\), with \( a_i \in \{0, 1\} \), with edges of the form \( \{v_i, v_j\} \) if \( a_i = 1 \) and \( i > j \). We are going to assume
\( a_1 = 1 \) as otherwise the graph is not connected. Notice that
\( a_1 \) can be assumed to be 0 since otherwise would give place to the same graph. Since the sequence \((a_i)\) consists of some consecutive zeros, followed by consecutive ones and so
on, we can write it as \([0^n, 1^n, 0^n, \ldots, 1^n, 0^n, 1^n] \), where \( a^k \) represents \( b \) consecutive copies of the number \( a \). Notice that in \([0^n, 1^n, 0^n, \ldots, 1^n, 0^n, 1^n]\) the
number 0 appears in every odd position and 1 in every even position, thus the
only values providing information are \((n_i)\). We can represent \((a_i)\) with the sequence
\([n_1, n_2, n_3, \ldots, n_{2k-2}, n_{2k-1}, n_{2k}]\), called the power sequence of the
threshold graph \( G \).

As every 0 vertex is at distance 2 of all previous vertices and every 1 vertex is
at distance 1 of all previous vertices, if \([n_1, n_2, n_3, \ldots, n_{2k-2}, n_{2k-1}, n_{2k}]\) is the power sequence of a threshold graph \( G \), then the distance matrix \( D \) of \( G \) is

\[
\begin{pmatrix}
2(J - I) & J & 2J & J & \ldots & J & 2J & J \\
J & J - I & 2J & J & \ldots & J & 2J & J \\
2J & 2J & J - I & J & \ldots & J & 2J & J \\
J & J & J & J - I & \ldots & J & 2J & J \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
J & J & J & J & \ldots & J - I & 2J & J \\
2J & 2J & 2J & 2J & \ldots & 2J & 2(J - I) & J \\
J & J & J & J & \ldots & J & J & (J - I)
\end{pmatrix},
\]

where each \( J \) in position \( i, j \) stands for a block of \( n_i \times n_j \) ones, and each \( I \) in position
\( i, i \) an \( n_i \times n_i \) identity matrix. Notice that consecutive zeros produce false twins, whereas consecutive ones produce true twins. We can partition the vertices of \( G \) into
\( W = \{W_1, \ldots, W_{2k}\} \), where \( W_i \) consists of \( n_i \) false twins if \( i \) is odd and \( n_i \) true twins if \( i \) is even. Consequently \( D/W \) equals

\[
\begin{pmatrix}
2n_1 - 2 & n_2 & n_3 & n_4 & \ldots & n_{2k-2} & 2n_{2k-1} & n_{2k} \\
n_1 & n_2 - 1 & 2n_3 & n_4 & \ldots & n_{2k-2} & 2n_{2k-1} & n_{2k} \\
2n_1 & 2n_2 & n_3 - 2 & n_4 & \ldots & n_{2k-2} & 2n_{2k-1} & n_{2k} \\
n_1 & n_2 & n_3 & n_4 - 1 & \ldots & n_{2k-2} & 2n_{2k-1} & n_{2k} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
n_1 & n_2 & n_3 & n_4 & \ldots & n_{2k-2} - 1 & 2n_{2k-1} & n_{2k} \\
2n_1 & 2n_2 & 2n_3 & 2n_4 & \ldots & 2n_{2k-2} & 2n_{2k-1} - 2 & 2n_{2k} \\
n_1 & n_2 & n_3 & n_4 & \ldots & n_{2k-2} & n_{2k-1} - 2 & n_{2k} - 1
\end{pmatrix}
\]

Lemma [4] allows us to use \( D/W \) instead of \( D \) to study its nullity. Given a matrix \( A \)
having \( m \) rows, we denote by \( r_i(A) \) the \( i \)-th row of \( A \) for each \( 1 \leq i \leq m \). When the
context is clear enough, we use \( r_i \) for shortness. We proceed to apply row operations to \( D/W \). We begin by doing \( r_i - r_{i+1} \rightarrow r_i \) for \( i \) moving from 1 to \( 2k - 1 \)

\[
\begin{pmatrix}
n_1 - 2 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
-1 & -n_2 - 1 & 2 & 0 & \ldots & 0 & 0 & 0 \\
n_1 & n_2 & n_3 - 2 & 1 & \ldots & 0 & 0 & 0 \\
-1 & -n_2 & -n_3 & -n_4 - 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-1 & -n_2 & -n_3 & -n_4 & \ldots & -n_{2k-2} - 1 & 2 & 0 \\
n_1 & n_2 & n_3 & n_4 & \ldots & n_{2k-2} & n_{2k-1} - 2 & 1 \\
n_1 & n_2 & n_3 & n_4 & \ldots & n_{2k-2} & n_{2k-1} & n_{2k} - 1
\end{pmatrix},
\]

we multiply every even row by \(-1\), but the last one

\[
\begin{pmatrix}
n_1 - 2 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
n_1 & n_2 + 1 & -2 & 0 & \ldots & 0 & 0 & 0 \\
n_1 & n_2 & n_3 - 2 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
n_1 & n_2 & n_3 & n_4 & \ldots & n_{2k-2} + 1 & -2 & 0 \\
n_1 & n_2 & n_3 & n_4 & \ldots & n_{2k-2} & n_{2k-1} - 2 & 1 \\
n_1 & n_2 & n_3 & n_4 & \ldots & n_{2k-2} & n_{2k-1} & n_{2k} - 1
\end{pmatrix}
\]

Finally, we do \( r_{2k-i} - r_{2k-i-1} \rightarrow r_{2k-i} \) for \( i \) moving from 0 to \( 2k - 2 \),

\[
\begin{pmatrix}
n_1 - 2 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
2 & n_2 & -2 & 0 & \ldots & 0 & 0 & 0 \\
0 & -1 & n_3 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 2 & n_4 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & n_{2k-2} - 2 & 0 \\
0 & 0 & 0 & 0 & \ldots & -1 & n_{2k-1} & 1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 2 & n_{2k} - 2
\end{pmatrix}
\]

The first \( 2k - 1 \) rows are linearly independent. Thus the nullity of \( D/W \) is at most 1. Lemma 7 yields the following.

**Theorem 3.** If \( D \) is the distance matrix of a connected threshold graph, then the nullity of \( D \) is at most 1.

We now want to find precisely which threshold graphs have nullity 1. Dividing even rows of the last matrix by \(-2\), we obtain

\[
\begin{pmatrix}
n_1 - 2 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
-1 & -n_2/2 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & -1 & n_3 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & -1 & -n_4/2 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & -n_{2k-2}/2 & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & -1 & n_{2k-1} & 1 \\
0 & 0 & 0 & 0 & \ldots & 0 & -1 & (2 - n_{2k})/2
\end{pmatrix}
\]

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that has the same nullity as $D/W$. Notice that if we let

$$\alpha_i = \begin{cases} 
& n_1 - 2 \quad \text{if } i = 1 \\
& n_i \quad \text{if } i > 1 \text{ is odd} \\
& -n_i/2 \quad \text{if } i < 2k \text{ is even} \\
& (2 - n_{2k})/2 \quad \text{if } i = 2k
\end{cases}$$

the last matrix is of the form

$$\begin{pmatrix}
\alpha_1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
-1 & \alpha_2 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & -1 & \alpha_3 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & -1 & \alpha_4 & \ldots & 0 & 0 & 0 \\
& \vdots & & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & \alpha_{2k-2} & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & -1 & \alpha_{2k-1} & 1 \\
0 & 0 & 0 & 0 & \ldots & 0 & -1 & \alpha_{2k}
\end{pmatrix}$$

We can obtain the determinant of this last matrix inductively. Let $D_i$ be the main minor of $D$ obtained by deleting each row $k$ greater than $i$ and its corresponding columns, and let $d_i$ be the determinant of $D_i$. It is not hard to prove that $d_1 = \alpha_1$ and $d_2 = 1 + \alpha_1\alpha_2$ and

$$d_i = \alpha_id_{i-1} + d_{i-2},$$

for each integer $3 \leq i \leq 2k$.

Thanks to the recursion, we can find some infinite families of threshold graphs with nullity 1. For example, if $\alpha_1, \ldots, \alpha_{2k-2}$ are such that $d_{2k-2} = 0$, then $\alpha_{2k} = 0$ implies $d_{2k} = 0$ regardless of the value of $\alpha_{2k-1}$. As a way to apply this, notice that both $(\alpha_1, \alpha_2) = (2, -1/2)$ and $(\alpha_1, \alpha_2) = (1, -1)$ imply $d_2 = 0$. In addition, if $\alpha_4 = 0$, then $[4, 1, n_3, 2]$ and $[3, 2, n_3, 2]$ are power sequences of threshold graphs with distance nullity 1 for every $n_3$, meaning that $K_2 \lor (n_3K_1 + (K_1 \lor 4K_1))$ and $K_2 \lor (n_3K_1 + (K_2 \lor 3K_1))$ are threshold graph whose distance matrices have nullity one.

Unfortunately if we wanted to keep applying this construction as is to yield a power sequence of length 6 we would need to do $[4, 1, n_3, 0, n_5, 2] = [4, 1, n_3 + n_5, 2]$ because of the difference between $\alpha_i$ when $i < 2k$ and $\alpha_{2k}$. What we can do instead is use the fact that, when $d_{i-2} = 0$, we have

$$d_i = \alpha_id_{i-1}$$

$$d_{i+1} = \alpha_{i+1}d_i + d_{i-1} = (\alpha_{i+1}\alpha_i + 1)d_{i-1}$$

which is similar to how the recursion begins, multiplying by $d_{i-1}$ and replacing $(\alpha_1, \alpha_2)$ with $(\alpha_1, \alpha_{i+1})$. Thus, if $\alpha_1, \ldots, \alpha_i$ yield $d_i = 0$ and $\alpha_1, \ldots, \alpha_j$ imply $d_j = 0$, setting $\alpha_{i+k} = \alpha_k$ implies $d_{i+k} = 0$. As a way to apply this, we can use $(\alpha_1, \alpha_2) = (1, -1)$ together with $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, -1, \epsilon, 0)$, with $\epsilon$ being any value we want to choose. This yields that threshold graphs with power sequences $[3, 2, 1, 2, \epsilon, 2]$ have nullity 1. And repeatedly applying this construction, we get that threshold graphs with power sequences of the form

$$[3, 2, 1, 2, 1, 2, 1, 2, 1, 2, \ldots, 1, 2, \epsilon, 2]$$

have nullity 1.
6 Trivially perfect graphs

In this section, we give sufficient conditions for a trivially perfect graph to have a nonsingular distance matrix. Let $G$ be a trivially perfect graph and let $\mathcal{W}$ be a true twin partition of $G$. There exists a tree $T = (\mathcal{W}, E)$, called rooted clique tree of $G$, such that if $W, W' \in \mathcal{W}$, $w \in W$ and $w' \in W'$, then $w$ and $w'$ are adjacent if and only if $W = W'$, or $W'$ is descendant of $W$ in $T$ or vice versa. By $T_W$, we denote the subtree of $T$ rooted at $W$ containing all descendants of $W$. The arrow matrix of $T$ is recursively defined as follows. If $\mathcal{W} = \{R\}$, $A_T = |R| - 1$. Assume that $|\mathcal{W}| \geq 3$.

Let the elements of $\mathcal{W}$ be numbered as follows:

- if $i < j$ then $W_i$ is not a descendant of $W_j$;
- if $i < j < k$ and $W_k$ is a descendant of $W_i$, then $W_j$ is a descendant of $W_k$.

See Fig. 6.1 Further, let $W_{i_1}, W_{i_2}, \ldots, W_{i_k}$ be the children of $R = W_1$, renumbered so that if $i < j$, $W_i = W_{h_m}$ and $W_j = W_{h_n}$, then $h_m < h_n$. We define the arrow matrix of $T$ as

$$A_T = \begin{bmatrix} |R| + 1 & |W_2| & |W_3| & \cdots & |W_i| \\ |R| & |R| & & & \\ \vdots & \vdots & \ddots & \vdots \\ |R| & & & |R| & \end{bmatrix} \cdot B_T,$$

where

$$B_T = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_i \end{bmatrix}.$$

where $A_i$ is the arrow matrix of $T(V_i)$. The ordering of $\mathcal{W}$ induced by the rows of $A_T$ is called an arrow ordering.

**Theorem 4.** Let $G$ be a trivially perfect graph, having a true twin partition $\mathcal{W} = \{W_1, W_2, \ldots, W_k\}$ such that $|W_i| \geq 6$ for each $i = 1, \ldots, k$, then $D(G)$ has an inverse, i.e., $\eta(G) = 0$.

As the proof of Theorem 4 is a bit technical, we give an illustration of how it works before proceeding with the actual proof.

6.1 Illustration of Theorem 4

Consider the trivially perfect graph $G = K_6 \lor ((K_7 \lor (K_9 + K_9)) \lor (K_9 \lor ((K_9 + K_7) + K_6)))$ with the vertex set partition $\mathcal{W} = \{W_1, W_2, W_3, W_4, W_5, W_6\}$ (see Fig. 6.1), whose rooted clique tree appears on Figure 2. Notice that the quotient matrix $D/\mathcal{W}$ is

$$\begin{pmatrix} 6 - 1 & 7 & 9 & 8 & 9 & 8 & 6 & 7 & 6 \\ 6 & 7 & 9 - 1 & 2 - 8 & 2 - 9 & 2 - 8 & 2 - 6 & 2 - 7 & 2 - 6 \\ 6 & 7 & 2 - 9 & 8 - 1 & 2 - 9 & 2 - 8 & 2 - 6 & 2 - 7 & 2 - 6 \\ 6 & 2 - 7 & 2 - 9 & 2 - 8 & 9 - 1 & 8 & 6 & 7 & 6 \\ 6 & 2 - 7 & 2 - 9 & 2 - 8 & 9 & 8 & 5 & 2 - 7 & 2 - 6 \\ 6 & 2 - 7 & 2 - 9 & 2 - 8 & 9 & 8 & 2 - 6 & 7 - 1 & 2 - 6 \\ 6 & 2 - 7 & 2 - 9 & 2 - 8 & 9 & 2 - 8 & 2 - 6 & 2 - 7 & 6 - 1 \end{pmatrix}.$$
Figure 2: Rooted tree of $G = K_6 \vee ((K_6 \vee (K_6 + K_6)) + (K_6 \vee ((K_6 \vee (K_6 + K_6)) + K_6)))$; i.e., each $W_i$ is a clique with six vertices. Rooted tree of $G = K_6 \vee ((K_7 \vee (K_9 + K_8)) + (K_9 \vee ((K_8 \vee (K_6 + K_7)) + K_6)))$.

where the $i$-th row represents $W_i$. We denote such a row by $r_{W_i}$.

Now we apply on $D/W$ the following elementary operations, first $r_{W_1} - 2r_{W_2} \rightarrow r_{W_1}$ and then $-r_{W_5} \rightarrow r_{W_5}$, for each $i \geq 2$, obtaining the following matrix

$$M_1 = \begin{pmatrix}
5 & 7 & 9 & 8 & 9 & 8 & 6 & 7 & 6 \\
4 & 8 & 9 & 8 & 0 & 0 & 0 & 0 & 0 \\
4 & 7 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 7 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 10 & 8 & 6 & 7 & 6 \\
4 & 0 & 0 & 0 & 9 & 9 & 6 & 7 & 0 \\
4 & 0 & 0 & 0 & 9 & 8 & 7 & 0 & 0 \\
4 & 0 & 0 & 0 & 9 & 8 & 0 & 8 & 0 \\
4 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 7
\end{pmatrix}. $$

To make more $0$'s we do the following row operations. First we subtract from the row corresponding to the root the rows corresponding to its children, i.e., $r_{W_1} - r_{W_2} - r_{W_5} \rightarrow r_{W_1}$. Do the same for $r_{W_5}$, $r_{W_5} - r_{W_6} - r_{W_9} \rightarrow r_{W_5}$. This was done because $W_5$ has grandchildren (i.e. it has child who has children of its own). This yields the matrix

$$M_2' = \begin{pmatrix}
-3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
4 & 8 & 9 & 8 & 0 & 0 & 0 & 0 & 0 \\
4 & 7 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 7 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & -8 & -1 & 0 & 0 & -1 \\
4 & 0 & 0 & 0 & 9 & 9 & 6 & 7 & 0 \\
4 & 0 & 0 & 0 & 9 & 8 & 7 & 0 & 0 \\
4 & 0 & 0 & 0 & 9 & 8 & 0 & 8 & 0 \\
4 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 7
\end{pmatrix}. $$

We keep making $0$'s appear as follows. We take every vertex that has children, but not grandchildren, and use them to make $0$'s. This means we do $r_{W_2} - \frac{2}{10}r_{W_3} - \frac{8}{7}r_{W_4} \rightarrow r_{W_2}$ and $r_{W_6} - \frac{6}{7}r_{W_7} - \frac{5}{8}r_{W_8} \rightarrow r_{W_6}$. This gives the matrix
$$M_2 = \begin{pmatrix} -3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 45 & 407 & 90 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 7 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 7 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & -3 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 96 & 34 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 8 & 7 & 0 & 0 \\ 4 & 0 & 0 & 0 & 9 & 8 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 7 \end{pmatrix}.$$ 

We can do now something similar for \( r_{W_5} \), although we need to multiply \( r_{W_6} \) by a different value. We do \( r_{W_5} - \frac{7}{34} r_{W_6} \rightarrow r_{W_5} \) and then \( r_{W_5} + \frac{1}{17} r_{W_6} \rightarrow r_{W_5} \). Notice that in this case we have \( \frac{7}{34} = \frac{34}{-1} = M_{2,6,6} \), and \( \frac{1}{17} = N_{5,5} \). This yields the matrix

$$N = \begin{pmatrix} -3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 45 & 407 & 90 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 7 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 7 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \\ -192 & 0 & 0 & 0 & -10201 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -399 & -34 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 8 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 7 \end{pmatrix}.$$ 

Finally, we can do the same process for \( r_{W_1} \), using \( r_{W_2} \) and \( r_{W_5} \). Thus we do \( r_{W_1} - \frac{90}{4151807} r_{W_2} \rightarrow r_{W_1} \) and then \( r_{W_1} - \frac{1904}{1904} r_{W_5} \rightarrow r_{W_1} \). In this case, as neither \( W_2 \) nor \( W_5 \) were leaves, we are just using \( \frac{90}{4151807} = N_{2,6} \) and \( \frac{1904}{1904} = N_{5,5} \). Thus, we obtain the following lower triangular matrix, which is non-singular because it does not have any zeros in the main diagonal.

$$\begin{pmatrix} -7368677 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4151807 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -142 & -407 & 90 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 7 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 7 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \\ -1345 & 0 & 0 & 0 & -10201 & 0 & 0 & 0 & 0 \\ -14 & 0 & 0 & 0 & -399 & -34 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 8 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 7 \end{pmatrix}.$$ 

### 6.2 Proof of Theorem 4

Before proceeding with the proof, we need to define the height of the vertices of a rooted tree. This definition is done inductively. If \( v \) has no children we define the height of \( v \) as \( h(v) = 0 \). If \( v \) has children, and the height of every child of \( v \) has been defined, we define the height of \( v \) as

$$h(v) = 1 + \max_{w | w \text{ is a child of } v} h(w).$$
Thus, for the vertices of the rooted tree in Figure 2 we have

\[ h(W_3) = h(W_4) = h(W_7) = h(W_8) = h(W_9) = 0, \]
\[ h(W_2) = h(W_6) = 1, \]
\[ h(W_5) = 2, \]
\[ h(W_1) = 3. \]

We are ready now to present the prove Theorem 4.

**Proof of Theorem 4.** Let \( T = (\mathcal{W}, E) \) be a rooted clique tree of \( G \). Consider now an arrow ordering \( W_1, \ldots, W_{|\mathcal{W}|} \). The quotient matrix of \( G \), under this ordering, has the following structure.

\[
D/\mathcal{W} = \begin{pmatrix}
|R| - 1 & \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_k \\
|R| - 1 & B_1 & 2 \cdot \vec{x}_2 & \cdots & 2 \cdot \vec{x}_k \\
|R| - 1 & 1 & B_2 & \cdots & 2 \cdot \vec{x}_k \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
|R| - 1 & 2 \cdot \vec{x}_1 & 2 \cdot \vec{x}_2 & \cdots & B_k
\end{pmatrix},
\]

where \( R \) is the root of \( T \), \( B_i \) is the quotient matrix of the distance matrix, induced by those vertices in \( G \) belonging to some vertex of \( T_{W_i} \), where \( W_i \) is the \( i \)-th child of \( R \) under the considered ordering of \( \mathcal{W} \), and the vector \( \vec{x}_i \) has \(|\mathcal{W}| \) in each entry corresponding to \( W \in V(T_{W_i}) \) for each \( 1 \leq i \leq k \). Now we apply on \( D/\mathcal{W} \) the following elementary operations, first \( r_W - 2r_r \to r_W \) and then \(-r_W \to r_W \), for each \( W \in \mathcal{W} \setminus \{R\} \), obtaining the following matrix

\[
M_1 = \begin{pmatrix}
|R| - 1 & \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_k \\
(|R| - 2) \cdot 1 & A_1 & 0 & \cdots & 0 \\
(|R| - 2) \cdot 1 & 0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(|R| - 2) \cdot 1 & 0 & 0 & \cdots & A_k
\end{pmatrix},
\]

where the \( A_i \)'s are the arrow matrices of the subtrees \( T_{W_i} \)'s of \( T \). We can transform \( M_1 \) into

\[
M_2 = \begin{pmatrix}
|R| - 1 & -\vec{a}_1 & -\vec{a}_2 & \cdots & -\vec{a}_k \\
\vec{b}_1 & C_1 & 0 & \cdots & 0 \\
\vec{b}_2 & 0 & C_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vec{b}_k & 0 & 0 & \cdots & C_k
\end{pmatrix},
\]

such that, for each \( i \), the first entry of \( \vec{b}_i \) is \( k_i(|R| - 2) \) with \( k_i \leq \frac{1}{|R|} \),

\[
C_i = \begin{pmatrix}
1 + |W_i| & -\vec{d}_1 & -\vec{d}_2 & \cdots & -\vec{d}_k \\
\vec{c}_1 & C_1 & 0 & \cdots & 0 \\
\vec{c}_2 & 0 & C_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vec{c}_k & 0 & 0 & \cdots & C_k
\end{pmatrix},
\]

and \( \vec{a}_i \) stands for the vector having as many rows as \( B_i \), a 1 in the first column and 0’s in the rest of its entries. The vector \( \vec{d}_i \) has as many rows as \( C_j \), a 1 in the first columns and 0’s in the rest of its entries. The block \( C_j \) is a lower matrix, \( (C_j)_{11} = (1 + m_j)|S_j| \) with \( m_j \leq \frac{1}{|R|} \) and \( S_j \) is the child of \( W_i \) corresponding to the first row of \( C_j \), and the first entry of \( C_j \) is \( m_j|S_j| \).
We will prove that there exists a sequence of elementary row operations leading $M_1$ to $M_2$. First, we do $r_R \sum_W r_W \rightarrow r_R$, the sum is taken among all vertices $W \in V(T)$ such that $W$ is a child of $R$. We repeat this procedure on each $T_W$ such that $h(T_W) \geq 2$ and $W$ is a child of $R$. Then we proceed with every child of the $T_W$'s and so on as long as possible. Let us call this new matrix $M_2'$. Notice that entries of $M_2'$ have been modified according to $M_2$ as follow $(M_2')_{WV} = |W| + 1 - \alpha_W |W|$ for each $W = V$ or $W$ ancestor of $V$, where $\alpha_W$ is the number of children of $W$ on $T_W$; and $(M_2')_{RR} = |R| - 1 - \alpha_R (|R| - 2)$, where $\alpha_R$ is the number of children of $R$ on $T$.

We proceed by applying induction on $h(T_W)$, the height of $T_W$.

Base case: $h(T_W) = 1$. We do $r_W \sum_V \frac{V}{V + 1} \cdot r_V \rightarrow r_W$, the sum is taken over all children $V$ of $W$. Under this row operation we obtain a matrix $N$ such that $N_{WV} = 0$ for every descendant $V$ of $W$, $N_{WW} = |W| + 1 - \alpha_W |W|$, $N_{WV} = |V| - \alpha_W |V|$ for each $V$ ancestor of $W$ distinct of $R$, and $N_{WR} = |R| - 2 - \alpha_W (|R| - 2)$. Thus $N_{SV} = 1 + m_W |V|$, $N_{WV} = m_W |V|$ for each ancestor $W$ distinct of $R$ and $N_{WR} = m_W (|R| - 2)$, where $m_W = 1 - s_W + \sum_V \frac{V}{V + 1}$ and $s_W$ is the number of children of $W$. Hence, since $|V| \geq 6 > 3$, $m_W < 1 - s_W + \frac{4}{3}$. Therefore, $s_W \geq 2$ implies $m_W < \frac{1}{4}$.

Assume now, by inductive hypothesis, that we can obtain a matrix $N$ from $M_2'$, by means of elementary rows operations such that if $1 \leq h(T_W) < k < h(T_R)$ with $W \neq R$, $N_{WV} = 0$ for each descendant $V$ of $W$, $N_{WW} = 1 + m_W |W|$ and $N_{WV} = m_W |V|$ with $m_W \leq \frac{1}{4}$ for each ancestor $V$ of $W$ distinct of $R$, and $N_{WR} = m_W (|R| - 2)$ with $m_R \leq \frac{1}{4}$. These are the only entries modified concerning to $M_2'$. Let $W' \in V(T)$ such that $1 < h(T_{W'}) = k$. We modify row $W'$ according to $r_W \sum_V \frac{V}{V + 1} \cdot r_V \rightarrow r_{W'}$, where the sum is taken over all children $V$ of $W'$ such that $h(T_V) \geq 1$; and then we do $r_W \sum_V \frac{V}{V + 1} \cdot r_V \rightarrow r_{W'}$, the sum is taken over all children $V'$ of $W'$ such that $h(T_{V'}) = 0$. Hence the new matrix $N'$ satisfies

$$N'_{W'W'} = \left(|W'| + 1 - s_{W'} |W'| + \sum_{h(T_V) \geq 1} \frac{m_V |W'|}{|V| + 1} + \sum_{h(T_V) = 0} \frac{|W'|}{|V| + 1} \right) \leq 1 + |W'| \left(1 - s_{W'} + \sum_{h(T_V) \geq 1} \frac{1}{|V| - 2} + \sum_{h(T_V) = 0} \frac{1}{|V| + 1}\right) \leq 1 + |W'| \left(1 - \frac{3}{4} s_{W'}\right).$$

By the inductive hypothesis, $m_V \leq \frac{1}{4}$ for each $V$ child of $W'$ such that $h(T) < k$ and thus the first inequality holds. The last one follows from $|V| \geq 6$ for each vertex $V$ of $T$. We conclude that $N_{W'W'} < 0$. Using the inductive hypothesis and reasoning as in the base case, it follows that $N_{W'V} = 0$ for each descendant $V$ of $W$ and $N_{W'V} = m_V |V|$ with $m_V \leq \frac{1}{4}$ for each ancestor $V$ of $W$ distinct of $R$, and $M_{WR} = (|R| - 2)m_R$. In particular, the result holds for each child $W$ of $R$. Hence $M_2$ can be obtained from $D/W$ through elementary row operations.

Finally, use the same strategy as in the inductive hypothesis to prove our result.

We can prove that, if we do $r_R + \sum_V \frac{V}{V + 1} \cdot r_V \rightarrow r_R$, where the sum is taken over all children $V$ of $R$ such that $h(T_V) \geq 1$; and then we do $r_R + \sum_V \frac{V}{V + 1} \cdot r_V \rightarrow r_R$, where the sum is taken over all children $V'$ of $R$, we obtain a lower matrix whose main diagonal has no zero entry.

\section{Nullity}

Trivially perfect graphs are a superclass of threshold graphs, but, unlike threshold graphs, for every $k \geq 2$ there exists a trivially perfect graph with nullity $k$. 

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Theorem 5. Let \( N = 3k + r \), where \( r \in \{1, 2, 3\} \) and \( k \geq 2 \) is an integer. Let \( n \in \mathbb{N} \) with \( n \geq 7k + r \). Then, there exists a trivially perfect graph \( G \) that has a true twin partition into \( N \) sets and \( |V(G)| = n \) such that the distance matrix of \( G \) has nullity \( \ell \), where

- \( \ell = k - 1 \) if \( n = 7k + r \) or if \( r = 1 \) and \( n \geq 7k + 3 \),
- \( \ell = k \) if \( r = 1 \) and \( n = 7k + 2 \),
- \( \ell \in \{k-1,k\} \) otherwise.

Proof. Let \( N \) be an integer number such that \( N = 3k + r \) with \( r = 1, 2 \) or \( 3 \) and \( k \geq 2 \).

Let \( G \) be a trivially perfect graph with \( |V(G)| = n \geq 7k + r \), having a true twin partition \( \mathcal{W} = \{R_1, \ldots, R_r, W_{1,1}, W_{1,2}, W_{2,1}, W_{2,2}, \ldots, W_k, W_{k,1}, W_{k,2}\} \) such that

- \( |W_i| = 3 \) for \( 1 \leq i \leq k \),
- \( |W_{i,1}| = |W_{i,2}| = 2 \) for \( 1 \leq i \leq k \),
- \( |R_i| \geq 1 \) for \( 1 \leq i \leq r \),
- \( |R_1| + \cdots + |R_r| = n - 7k \).

Let \( T = (\mathcal{W}, E) \) a rooted clique tree of \( G \), where

- \( R_1 \) is the root,
- \( R_i \) is descendant of \( R_1 \) for \( 2 \leq i \leq r \), if \( r \geq 2 \),
- \( W_i \) is descendant of \( R_1 \) for \( 1 \leq i \leq k \),
- \( W_{i,1} \) and \( W_{i,2} \) are descendants of \( W_i \) for \( 1 \leq i \leq k \).

Consider the distance matrix \( D \) of \( G \), Lemma \[7\] allows us to use \( D/\mathcal{W} \) instead of \( D \) to study its nullity. Using the same transformations as in the proof of Theorem \[3\], we obtain the matrix

\[
M = \begin{pmatrix}
|R_1| - 1 & x_{11}^r & x_{12}^r & \cdots & x_{1k}^r \\
(|R_1| - 2) \cdot 0^r & A_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(|R_1| - 2) \cdot 0^r & 0 & A_2 & \cdots & 0 \\
& & & & A_k 
\end{pmatrix}
\]

if \( r = 1 \), or

\[
M = \begin{pmatrix}
|R_1| - 1 & |R_2| & \cdots & |R_r| & x_{11}^r & x_{12}^r & \cdots & x_{1k}^r \\
|R_1| - 2 & |R_2| + 1 & 0^r & 0 & 0^r & 0^r & \cdots & 0^r \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(|R_1| - 2) \cdot 1 & 0 & \cdots & 0 & A_1 & 0 & \cdots & 0 \\
(|R_1| - 2) \cdot 1 & 0 & \cdots & 0 & 0 & A_2 & \cdots & 0 \\
& & & & & & & A_k 
\end{pmatrix}
\]

if \( r \geq 2 \), where

\[
A_i = \begin{pmatrix}
|W_i| + 1 & |W_{i,1}| & |W_{i,2}| \\
|W_i| & |W_{i,1}| + 1 & 0 \\
|W_i| & 0 & |W_{i,2}| + 1 
\end{pmatrix} = \begin{pmatrix}
4 & 2 & 2 \\
3 & 3 & 0 \\
3 & 0 & 3 
\end{pmatrix},
\]

and

\[
x_i^r = (|W_i|, |W_{i,1}|, |W_{i,2}|) = (3, 2, 2),
\]
for \(1 \leq i \leq k\).

By elementary row operations, we obtain
\[
\tilde{M} = \begin{pmatrix}
\hat{R} & \hat{\alpha}^t & \hat{\alpha}^t & \ldots & \hat{\alpha}^t \\
(|R_1| - 2) \cdot \hat{y} & 0 & \cdots & 0 \\
(|R_1| - 2) \cdot \hat{y} & 0 & A & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(|R_1| - 2) \cdot \hat{y} & 0 & \cdots & 0 & A \\
\end{pmatrix},
\]
if \(r = 1\), or
\[
\tilde{M} = \begin{pmatrix}
\hat{R} & 0 & \cdots & 0 & \hat{\alpha}^t & \hat{\alpha}^t & \cdots & \hat{\alpha}^t \\
|R_1| - 2 & |R_2| + 1 & 0^t & 0^t & 0^t & \cdots & 0^t \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
|R_1| - 2 & 0 & 0^t & |R_r| + 1 & 0^t & 0^t & \cdots & 0^t \\
(|R_1| - 2) \cdot \hat{y} & 0 & \cdots & 0 & A & 0 & \cdots & 0 \\
(|R_1| - 2) \cdot \hat{y} & 0 & \cdots & 0 & 0 & A & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(|R_1| - 2) \cdot \hat{y} & 0 & \cdots & 0 & 0 & 0 & \cdots & A \\
\end{pmatrix}
\]
if \(r \geq 2\), where
\[
\hat{R} = \begin{pmatrix}
(|R_1| - 1) - \frac{|R_1|}{M_1} \cdot (|R_1| - 2) & \cdots & \frac{|R_1|}{M_1} \cdot (|R_1| - 2) \\
\end{pmatrix} \quad \text{if } r = 1
\]
\[
\hat{R} = \begin{pmatrix}
(|R_1| - 1) - \frac{|R_1|}{M_1} \cdot (|R_1| - 2) & \cdots & \frac{|R_1|}{M_1} \cdot (|R_1| - 2) \\
\end{pmatrix} \quad \text{if } r \geq 2,
\]
\[
\hat{A} = \begin{pmatrix}
0 & 0 & 0 \\
3 & 3 & 0 \\
3 & 0 & 3
\end{pmatrix}, \quad \hat{y}^t = (-1, 0, 0), \quad \text{and} \quad \hat{y} = \begin{pmatrix}
-\frac{1}{3} & 1, 1
\end{pmatrix}.
\]

Notice that the \((r + 1 + 3i)\)-th row corresponds to the first row of \(\hat{y}^t\) and \(\hat{A}\), and thus it equals
\[
\begin{pmatrix}
-\frac{|R_1|}{3} & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]

Thus, the nullity of \(\tilde{M}\) is at least \(k\) if \(|R_1| = 2\), and at least \(k - 1\) otherwise. Furthermore, it is easy to check that the rest of the rows form a linearly independent set, and that this set does not generate the \((r + 1 + 3i)\)-th row if \(|R_1| \neq 2\). The result now follows from the fact that \(1 \leq |R_1| \leq n - 7k - r + 1\), and that if \(r = 1\), then \(|R_1| = n - 7k - r + 1\).

In the proof of Theorem 5, we assign values to \((|W_i|, |W_{i,1}|, |W_{i,2}|)\) so that \(A_i\) has nullity 1, for all \(1 \leq i \leq k\). We can see that the matrix \(A_i\) has nullity 1 if and only if \((|W_i|, |W_{i,1}|, |W_{i,2}|)\) is one of \((2, 3, 3), (2, 2, 5), (2, 5, 2), (3, 2, 2), (3, 1, 5), (3, 5, 1), (4, 1, 3), (4, 3, 1), (6, 1, 2), (6, 2, 1)\), for \(1 \leq i \leq k\). In particular we use \((3, 2, 2)\) because with this choice we obtain the minimum lower bound for the number of vertices.

### 7 Conclusion and further research

The proof of Theorem 1 presents an upper bound for the number of graphs with distance rank equals \(k\) in terms of the Ramsey number \(R(k)\). Nevertheless, this upper bound seems to be far from being tight. Indeed, \([f(3, R(3))] = 186\), and the number of connected graphs with distance rank 3 is equal to three. It would be interesting to find a tighter upper bound for the number of connected graphs with distance rank \(k\).
In Theorem 3, we prove that a connected threshold graph has nullity at most one. We also present a family of infinite power sequences giving place to an infinite family of connected threshold graphs with nullity one. A challenging problem is characterizing those connected threshold graphs with nullity equal to zero or one. Unlike threshold graphs, for each integer \( k \geq 2 \) there exists a trivially perfect graph with nullity equal to \( k \), see Theorem 5. Notice that, Theorem \( 4 \) guarantees that if each set of the twin partition of a trivially perfect graph is big enough, then its distance matrix is nonsingular. Consequently, connected threshold graphs with nullity one have a small set in their twin partition, as they are a subclass of trivially perfect graphs.

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