MHD SIMULATIONS OF SMALL AND LARGE SCALE DYNAMOS

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Abstract. Isotropic homogeneous hydromagnetic turbulence is studied using numerical simulations at resolutions of up to $1024^3$ meshpoints. It is argued that, in contrast to the kinematic regime, the nonlinear regime is characterized by a spectral magnetic power that is decreasing with increasing wavenumber, regardless of whether or not the turbulence has helicity. This means that the root-mean-square field strength converges to a limiting value at large magnetic Reynolds numbers. The total (magnetic and kinetic) energy spectrum tends to be somewhat shallower than $k^{-5/3}$, in agreement with the findings of other groups. In the presence of helicity, an inverse cascade develops, provided the scale separation between the size of the computational box and the scale of the energy carrying eddies exceeds a ratio of at least two. Finally, the constraints imposed by magnetic helicity conservation on mean-field theory are reviewed and new results of simulations are presented.

1. Introduction

MHD turbulence is an important and ubiquitous phenomenon in astrophysics. Stellar convection zones are known to harbor magnetic fields that can be very strong locally, as evidenced by sunspots. In the solar wind fully developed MHD turbulence has been measured in situ. Accretion discs and galaxies are further examples where MHD turbulence must be present. Much of our current understanding of MHD turbulence has come from analytical theories using a number of simplifications, and from simulations at quite limited numerical resolution. The time has now come to try and confirm various aspects of MHD turbulence such as the power spectra of velocity and magnetic field and the coupling between the two via dynamo
action in numerical simulations at higher resolution, with $1024^3$ meshpoints being feasible now.

One important question is how intermittent the magnetic field is and whether it is dominated by large or small scale structures. This question is directly related to observable quantities in the interstellar medium where the contributions from large and small scale field are found to be comparable; see Beck et al. (1996) for a review. The question of small scale fields is also related to theoretical approaches to understand the amplification of the large scale magnetic fields on scales ranging from stars to galaxies in general, as well as the cyclic modulation of such fields on late type stars such as the sun.

2. Isotropic MHD turbulence

We are particularly interested in the case where the magnetic field is not externally maintained, but constantly regenerated by dynamo action. Dynamo action is nothing special, but a rather general phenomenon that is common to virtually all turbulent and also many non-turbulent flows. One important requirement is that the magnetic Reynolds number is large, i.e. that resistive effects are weak and that the field is nearly frozen into the flow at large and moderate scales. Only under rather special conditions can dynamo action be suppressed, so for example if the magnetic field is forced to be two-dimensional.

2.1. THE KINEMATIC GROWTH PHASE

There are normally always two phases to dynamo action: the kinematic phase where the field is still weak and the dynamical phase where the velocity is being affected by the Lorentz force which then leads to the saturation of the magnetic field. It is worth recalling that there can be exceptions to this rather simple rule: there are self-killing and self-generating dynamos (Fuchs et al. 1999). The self-generating dynamos are relevant to accretion discs where the turbulence driving the dynamo is only generated by the magnetic field itself via the magneto-rotational instability (Balbus and Hawley 1991). This requires a doubly positive feedback from both the magneto-rotational and the dynamo instability. Simulations of this phenomenon (Brandenburg et al. 1995) have not only confirmed that this process works, but they have also provided one of the clearest examples of large scale dynamo action in general, and migratory cyclic behavior in particular. The geodynamo is another example where a finite amplitude magnetic field is essential for the operation of the dynamo on the so-called strong-field branch that is thought to be relevant for the Earth (Glatzmaier and Roberts 1995).
Figure 1. Magnetic power spectra during the early growth phase of a forced MHD turbulence simulation. The straight lines denote the $k^{3/2}$ Kazantsev slope.

In the following we return to the more standard situation where the turbulence exists already in the absence of magnetic fields. In this section we restrict ourselves to the case where the turbulence is non-helical, so there is no well established mechanism that could generate large scale fields (the $\alpha$-effect, which is discussed in the following sections, is absent). The field grows then predominantly at small scales. This process is well described by the theory of Kazantsev (1968) which predicts that the magnetic energy spectrum, $E_M(k)$, increases with $k$ like

$$E_M(k) \sim k^{3/2} \quad \text{(kinematic regime)}. \quad (1)$$

In simulations of forced MHD turbulence this behavior has been demonstrated in the case of large magnetic Prandtl numbers (Maron and Cowley 2001). In this case the kinetic energy cascade terminates much before the magnetic energy cascade. Recent simulations of Haugen et al. (2003) have shown that the same behavior is also found in the case of unit magnetic Prandtl number, $\Pr_M \equiv \nu/\eta = 1$; see Fig. 1.
Figure 2. Kinetic and magnetic energy spectra compared with the total energy spectrum (dashed, dotted, and solid lines, respectively) for the saturated state of a forced MHD turbulence simulation. At small wavenumbers, the spectral magnetic energy increases approximately like $k^{-1/3}$ (at higher resolution, however, the spectrum is flatter; see Haugen et al. 2003). The forcing is nonhelical with amplitude $f_0 = 0.02$, $\nu = \eta = 2 \times 10^{-4}$ and 512\(^3\) meshpoints.

2.2. THE SATURATED PHASE

The rapid build-up of small scale fields, as seen in linear (kinematic) theory, has often been viewed as an indication that also in the nonlinear, saturated phase the field is dominated by small scale fields (e.g. Kulsrud and Andersen 1992); see Widrow (2002) for a review. This picture might be in conflict with the ordinary picture of a turbulent cascade where an approximately scale-independent energy flux $\epsilon$ gives rise to energy transfer from large to small scales (e.g. Frisch 1995). This question has indeed been matter of recent debate. For decades the leading theory of MHD turbulence has been that of Iroshnikov (1963) and Kraichnan (1965). They suggested that in the inertial range the kinetic and magnetic power spectra are given by

$$E_M(k) = E_K(k) = C_{IK} \epsilon^{1/2} v_A^{1/2} k^{-3/2} \quad \text{(nonlinear regime)},$$

where $v_A$ is the Alfvén speed and $C_{IK}$ is a constant. Until recently this spectrum seemed to be verified by simulations (Biskamp 1993, 1995, Biskamp and Bremer 1994), but solar wind data always suggested a $k^{-5/3}$ spectrum.
Figure 3. Comparison of total energy spectra for runs with different Reynolds numbers. Stronger forcing: $f_0 = 0.05$. The vertical arrows at the top give the visco-resistive cutoff wavenumber, $(c/\nu^3)^{1/4}$.

(Matthaeus and Goldstein 1982) as in Kolmogorov theory. This issue has been resolved by Goldreich and Sridhar (1995, 1997) who showed that the magnetic field makes the flow strongly anisotropic (especially on small scales) and that an energy cascade occurs only for the components perpendicular to the field lines. This picture was also supported by simulations with imposed fields (Maron and Goldreich 2001, Cho and Vishniac 2000a, Cho et al. 2002a; see also Oughton et al. 1998) and with decaying fields (Biskamp and Müller 2000).

The existence of the Goldreich–Sridhar cascade requires the presence of reasonably strong large scale fields, and it is not clear whether dynamo-generated fields satisfy this condition. Simulations of nonhelical dynamos by Maron and Cowley (2001) and Cho and Vishniac (2000b) do not seem to show power law behavior for the magnetic power spectrum, but rather a peak in the spectrum near the resistive scale that would move to larger wave numbers (smaller scales) as the magnetic Reynolds number is increased.

A different suggestion was made by Kida et al. (1991) who found from a dynamo simulation that the total energy spectrum, $E_T(k) \equiv E_M(k) + E_K(k)$, is of the form

$$E_T(k) = C_{KYM} \epsilon^{2/3} k^{-5/3}, \quad (3)$$
Figure 4. Comparison of total energy spectra for runs with different Reynolds numbers. Weaker forcing: $f_0 = 0.02$. The ‘hook’ to the right of the high resolution spectrum indicates that the run has not yet sufficiently relaxed yet and is typical of similar runs shortly after remeshing from a lower resolution run.

where $\epsilon = \epsilon_M + \epsilon_K$ is the sum of magnetic and kinetic energy dissipation rates, and $C_{KYM} \approx 2.1$ is the ‘Kolmogorov’ constant found by Kida et al. (1991).

Simulations by Haugen et al. (2003) at resolutions of up to $1024^3$ meshpoints seem to confirm that the magnetic energy spectrum does not show power law behavior, but only the kinetic energy spectrum and the total energy spectrum. The precise form of the power spectrum is not fully settled however; cf. Figs 2–4 for results with different forcing amplitudes and Reynolds numbers.

Already at a resolution of $512^3$ (Fig. 3) it becomes evident that the $k^{-5/3}$ spectrum is followed by a shallower spectrum just before entering the dissipation range. At a resolution of $1024^3$ meshpoints (Fig. 4), the emergence of a shallower spectrum ($k^{-3/2}$ or perhaps even $k^{-1}$) becomes even more pronounced. This result is very similar to the “bottleneck effect” seen quite strongly in simulations using hyperviscosity and hyperresistivity (Cho and Vishniac 2000b, Biskamp and Müller 2000). Here the ordinary $\nabla^2$ operator is replaced by an operator $(-1)^{n-1}\nabla^{2n}$, where $n$ is an integer usually taken between 2 and 8. The bottleneck effect exists also in two-dimensional turbulence even with ordinary viscosity and resistivity (Biskamp et al. 1998).
It is worthwhile recalling that a $k^{-1}$ bottleneck subrange is also observed in simulations where the magnetic Prandtl number is large (Cho et al. 2002b). The emergence of a shallower $k^{-1}$ subrange has also been found in hydrodynamic simulations with numerical dissipation via the piecewise parabolic method (PPM); see Porter et al. (1992). Again, this subrange has become ever more pronounced as the resolution was increased to $1024^3$ meshpoints (Porter et al. 1998). PPM is believed to act similarly to hyper-viscosity, so the emergence of a $k^{-1}$ bottleneck subrange seems therefore not too surprising. In the present simulations with magnetic fields, however, this explanation does not apply, because ordinary viscosity and magnetic diffusion operators are used. The nature of a shallower $k^{-3/2}$ or even $k^{-1}$ subrange in our MHD simulations is therefore unclear. One wonders what will happen at much larger Reynolds numbers. Will the spectrum then be dominated by a long $k^{-1}$ subrange, or is this just an artifact of still too small Reynolds numbers? Or is this a subtle feature of meshpoint schemes being run too closely to the maximum permissible value of the Reynolds number? On the other hand, the bottleneck effect may well be physical, and the reason why this possibility is emerging only now is that it is much weaker in the one-dimensional longitudinal or transversal spectra accessible from experiments as compared to the fully three-dimensional spectra available in simulations (Dobler et al. 2003). In the following section we attempt a preliminary convergence study by comparing runs at different numerical resolution and different values of the Reynolds number.

3. Reynolds number dependence and convergence

In turbulence research it is customary to display compensated power spectra, $E(k)/(\varepsilon^{2/3}k^{-5/3})$. If the spectrum obeys the anticipated scaling one expects the compensated spectrum to be flat over a certain range. The value of the spectrum gives the Kolmogorov constant or, in the present case, $C_{KYM}$, because, following Kida et al. (1991), we are considering here the total (kinetic plus magnetic) energy spectrum and use the rate of total energy dissipation, $\varepsilon = \varepsilon_M + \varepsilon_K$. The result is shown in Fig. 5 for three different values of $Re = R_m$ between 270 and 960. In the first case with $Re = R_m = 270$ the compensated spectrum shows a reasonably flat range with $C_{KYM} \approx 1.3$, which is less than the value found by Kida et al. (1991). In the second case with $Re = R_m = 440$ the compensated spectrum shows clear indications of excess power just before turning into the dissipative subrange. This effect becomes even more dramatic in the third case with $Re = R_m = 960$.

In order to assess the reliability of the results we have carried out a convergence study using the same value of $\nu$, but different mesh resolution;
Figure 5. Comparison of compensated total energy spectra for runs with different Reynolds numbers. In all runs the horizontal dash-dotted line goes through the value 1.3. In the second panel two runs with the same Reynolds numbers, but different resolution are compared. $f_0 = 0.02$.

see the second panel of Fig. 5. In both cases, $\nu = \eta = 2 \times 10^{-4}$, while $u_{\text{rms}} = 0.13$ for $512^3$ meshpoints and 0.12 for $256^3$. The energy dissipation is also similar, $\epsilon = 2.8 \times 10^{-4}$ for $512^3$ meshpoints and $2.3 \times 10^{-4}$ for $256^3$. The compensated energy spectra agree quite well for the two different reso-
Figure 6. Comparison of runs with and without helicity (left and right hand columns, respectively) and for the different forcing wavenumbers: $k_f = 1$ (top) 2 (middle), 5 (bottom).

lutions, and both show excess power just before turning into the dissipative subrange. This supports the result that the excess power may be real.

4. When helical and nonhelical turbulence are similar

It is important to point out that the simulations of Kida et al. (1991) are not strictly comparable to our present case, because they considered the case of helical forcing. In the following we shall discuss the difference between helical and nonhelical turbulence.

In the case of helical forcing one expects an inverse cascade to smaller wavenumbers rather than a direct cascade to larger wavenumbers. This
is not really seen in the simulations of Kida et al. (1991). There are two reasons for this. On the one hand the inverse cascade takes a resistive time to develop (Brandenburg 2001, hereafter B01), and this time tends to be too long if magnetic hyperdiffusivity is used (Brandenburg and Sarson 2002). Kida et al. (1991) did use magnetic hyperdiffusivity and their resistive times where at least a hundred times longer than the duration of their runs, so no inverse cascade should be expected. But there is another, perhaps more important reason. In order for the inverse cascade to develop, scale separation is required, i.e. the magnetic field must be allowed to grow on scales larger than the forcing scale (which corresponds to the energy carrying scale) of the turbulence. This was not the case in the simulations of Kida et al. (1991), and thus, there is no inverse cascade present in their simulations. Therefore their results should be close to those without helicity. This is shown more clearly in Fig. 6 where we compare the power spectra of helical and nonhelical simulations with forcing at wavenumber $k_f = 1.5$ (no scale separation), $k_f = 2.3$ (weak scale separation), and $k_f = 5$ (considerable scale separation). For $k_f = 1.5$ the spectra of the helical and nonhelical simulations are indeed quite similar to each other.

5. Inverse cascade and nonlinear $\alpha$-effect

The magnetic field generated by the inverse cascade tends to have helicity of opposite signs at large and small scales. This is a consequence of magnetic helicity conservation which, in the limit of large magnetic Reynolds numbers, prevents the build-up of any net magnetic helicity. Forcing with positive helicity, for example, results in positive magnetic helicity at small scales (in particular the forcing scale). To conserve magnetic helicity, this positive magnetic helicity must be compensated by negative magnetic helicity at large scales, which in turn is a direct consequence of the inverse cascade, through which magnetic energy flows from small to large scales.

Another way of seeing this is by noting that helical turbulence leads to an $\alpha$-effect. This means that the large scale magnetic field, $\overrightarrow{B}$, can be described by the mean-field dynamo equation (Moffatt 1978)

$$\frac{\partial \overrightarrow{B}}{\partial t} = \nabla \times (\alpha \overrightarrow{B} - \eta T \nabla \times \overrightarrow{B}),$$

where $\eta T = \eta_t + \eta$ is the total (sum of turbulent and microscopic) magnetic diffusivity. In periodic domains, this equation has solutions where $\alpha \overrightarrow{B} = \eta T \nabla \times \overrightarrow{B}$, so

$$\mu_0 \overrightarrow{J} \cdot \overrightarrow{B} = (\alpha/\eta T) \overrightarrow{B}^2,$$

where $\overrightarrow{J} = \nabla \times \overrightarrow{E}/\mu_0$ is the mean current. Thus, for these solutions the sign of the current helicity density of the large scale field, $\overrightarrow{J} \cdot \overrightarrow{B}$, is the same as the sign of $\alpha$. 

Late saturation phase of fully helical turbulent dynamos for three different values of the magnetic Reynolds number: $R_m \equiv u_{rms}/\eta k_\ell = 2.4, 6, \text{ and } 18$ for Runs 1, 2, and 3 respectively; see B01. The mean magnetic field, $B$, is normalized with respect to the equipartition value, $B_{eq} = \sqrt{\mu_0 \rho_0 u_{rms}}$, and time is normalized with respect to the kinematic growth rate, $\lambda$. The dotted lines represent the fit formula (6) which tracks the simulation results rather well.

The problem with mean-field theory is that not only the magnitude (and sometimes even the sign) of $\alpha$ are not well understood, but in particular that the feedback from the magnetic field has completely ignored the effects of the small scale fields (Vainshtein and Rosner 1991, Vainshtein and Cattaneo 1992).

5.1. THE HELICITY CONSTRAINT

In this section we briefly explain how the question of feedback is directly related to magnetic helicity conservation; see Eq. (7) below. Even without thinking about mean-field theory, the magnetic helicity balance poses a problem when it comes to building up a large scale field fast. In simulations with helicity, the build-up of large scale magnetic field is always seen to proceed on a resistive timescale. Ideally, one would like to see such built-up to proceed on a dynamical timescale, but so far there have been no
simulations that are actually able to generate large scale fields faster than on the resistive timescale. A resistively slow saturation phase means that the resistivity, or magnetic diffusivity $\eta$, enters the timescale of the saturation. This is demonstrated in Fig. 7 where we show the saturation phase of a turbulent dynamo together with the fit formula

$$\frac{B^2}{B_{eq}^2} = \frac{\epsilon_1 k_1}{\epsilon_1 k_1} \left( 1 - e^{-2\eta k_1^2(t-t_{sat})} \right)$$

(6)

that was derived in B01, see also Brandenburg et al. (2002) using the assumptions that the large and small scale fields are nearly fully helical, i.e. that the helicity fractions $\epsilon_1$ and $\epsilon_\ell$ of the mean and fluctuating fields, respectively, are nearly 100%.

The resistively slow saturation behavior, which is also seen in oscillatory dynamos with shear ($\alpha \Omega$-type dynamos; see Brandenburg et al. 2001) is not reproduced by standard mean-field dynamo theory. Only in the special case of an $\alpha^2$-type dynamo (no shear) it has been possible to describe the saturation behavior correctly by simultaneously invoking $R_m$-dependent $\alpha$ and $\eta$ quenchings (B01). For an extensive review including a discussion of the effects of shear, helicity cancellation across the equator, and the effect on the cycle frequency, see Brandenburg et al. (2002). Here we just want to emphasize that magnetic helicity conservation is a real issue and not just a theoretical complication.

5.2. SMALL-SCALE MAGNETIC HELICITY EVOLUTION

The next question is how to deal with magnetic helicity conservation and how to incorporate it into mean-field models. The answer is relatively simple if the system is homogeneous and there are no boundaries, i.e. if the boundary conditions are periodic. In that case we have to satisfy

$$d\langle A \cdot B \rangle / dt = -2\eta \mu_0 (J \cdot B).$$

(7)

Here, $A$ is the magnetic vector potential, and $\langle A \cdot B \rangle$ is the magnetic helicity, with angular brackets denoting volume averages.\(^1\) The mean-field dynamo equation (4) may also be written in the form

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{E} + \mathbf{u} \times \mathbf{B} - \eta \mu_0 \mathbf{J}),$$

(8)

where $\mathbf{E} = \alpha \mathbf{B} - \eta \mu_0 \mathbf{J}$ is the mean electromotive force expressed in its simplest form in terms of the mean magnetic field and its curl via $\alpha$-effect and turbulent diffusion. Mean fields are here defined as averages over one

\(^1\) In a periodic domain $\langle A \cdot B \rangle$ is gauge invariant, because $\langle \nabla \phi \cdot B \rangle = \langle \Phi \nabla \cdot B \rangle = 0$
or at most two coordinate directions. In the case of the $\alpha^2$ dynamo the wavevector of the mean field, $k_m$, may point in any of the three coordinate directions, so the averages should be taken over the two directions perpendicular to $k_m$.

Equation (8) implies its own evolution equation for the magnetic helicity of the mean field, $\langle A \cdot B \rangle$,

$$\frac{d\langle A \cdot B \rangle}{dt} = 2\langle \mathbf{E} \cdot \mathbf{B} \rangle - 2\eta \mu_0 \langle J \cdot \mathbf{B} \rangle.$$  \hfill (9)

Note that any field-aligned component of $\mathbf{E}$ produces excess magnetic helicity at large scales relative to the magnetic helicity of the total field. This can only be consistent with equation (7) if magnetic helicity of opposite sign is absorbed by the small scale (or fluctuating) field. We shall now quantify this.

The departures from the mean field are referred to as fluctuations, i.e. $\mathbf{b} = \mathbf{B} - \overline{\mathbf{B}}$ and $\mathbf{a} = \mathbf{A} - \overline{\mathbf{A}}$. This implies

$$\langle \mathbf{a} \cdot \mathbf{b} \rangle = \langle \mathbf{A} \cdot \mathbf{B} \rangle - \langle \overline{\mathbf{A}} \cdot \overline{\mathbf{B}} \rangle$$ \hfill (10)

and

$$\langle \mathbf{j} \cdot \mathbf{b} \rangle = \langle \mathbf{J} \cdot \mathbf{B} \rangle - \langle \overline{\mathbf{J}} \cdot \overline{\mathbf{B}} \rangle$$ \hfill (11)

and hence the evolution equation for the magnetic helicity of the fluctuations is

$$\frac{d\langle \mathbf{a} \cdot \mathbf{b} \rangle}{dt} = -2\langle \mathbf{E} \cdot \mathbf{B} \rangle - 2\eta \mu_0 \langle \mathbf{j} \cdot \mathbf{b} \rangle.$$ \hfill (12)

This equation plays a key role in the dynamical $\alpha$-quenching approach that was first developed by Kleeorin and Ruzmaikin (1982) and turns now out to be the only theoretically acceptable theory that reproduces the helicity constraint seen in simulations (see Field and Blackman 2002, Blackman and Brandenburg 2002, Subramanian 2002). We shall show now that this equation can be coupled back into the mean-field equation by noting that the quantities $\langle \mathbf{a} \cdot \mathbf{b} \rangle$ and $\langle \mathbf{j} \cdot \mathbf{b} \rangle$ are proportional to magnetic contributions to the $\alpha$-effect.

5.3. CALCULATING THE $\alpha$-EFFECT

A new way of deriving the expression coupling the mean electromotive force with the mean field has recently been put forward by Blackman and Field (2002) who calculated the time derivative of $\mathbf{E}$,

$$\frac{d\mathbf{E}}{dt} = \varepsilon_{ijk} \left( u_j \mathbf{b}_k + \mathbf{u}_j \mathbf{b}_k \right),$$ \hfill (13)

where

$$\mathbf{b} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \ldots,$$ 

$$\nabla \cdot \mathbf{b} = 0,$$ \hfill (14)
\[ \dot{u} = -\nabla p + b \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla b +..., \quad \nabla \cdot u = 0, \tag{15} \]

and diffusive and nonlinear terms have been ignored (first order smoothing approximation). Inserting this into equation (16) yields

\[ \frac{d\xi}{dt} = \bar{\alpha}_{ij}B_j - \bar{\beta}_{ijk}B_{j,k} - \frac{\xi}{\tau}, \tag{16} \]

where all terms beyond the (linear) first order smoothing approach have been subsumed into the last term, which acts as a damping term provided \( \tau \) is positive. The positivity is indeed confirmed using simulations (Brandenburg and Blackman, unpublished). The calculation of \( \bar{\alpha}_{ij} \) is relatively straightforward and yields (cf. Blackman and Field 2002)

\[ \bar{\alpha}_{ip} = \epsilon_{jnp}u_ju_{n,i} - \epsilon_{inp}u_ju_{n,j} + \epsilon_{ijk}b_kb_{j,p}. \tag{17} \]

If the assumption of isotropy is made one has \( \bar{\alpha}_{ip} = \bar{\alpha}\bar{\delta}_{ij} \) with

\[ \bar{\alpha} = -\frac{1}{3}\mathbf{\omega} \cdot \mathbf{u} + \frac{1}{3j} \cdot \mathbf{B}/\rho_0. \tag{18} \]

One usually ignores the explicit time derivative of \( \xi \) in equation (16), in which case \( \alpha = \tau \bar{\alpha} \) is the well-known \( \alpha \)-effect (e.g. Krause and Rädler 1980). The magnetic contribution, \( \tau \mathbf{j} \cdot \mathbf{b}/(3\rho_0) \), was first derived by Pouquet et al. 1976, and used heavily in all recent approaches to \( \alpha \)-quenching (Gruzinov and Diamond 1994, Bhattacharjee and Yuan 1995, Field et al. 1999, Blackman and Field 2000).

5.4. THE DYNAMICAL QUENCHING MODEL

We have seen that the evolution equation for the small scale magnetic helicity involves the term \( \langle j \cdot b \rangle \), which is also part of the \( \alpha \)-effect; see equation (18). Here we have also made use of the assumption of isotropy. This assumption, together with the assumption of scale separation, allows us to write

\[ \langle a \cdot b \rangle \approx \langle j \cdot b \rangle/k_t^2. \tag{19} \]

This equation can be used to write equation (12) fully in terms of \( \alpha \),

\[ \frac{d\alpha}{dt} = -2\eta k_t^2 \left( \frac{\langle \mathbf{E} \cdot \mathbf{B} \rangle}{B_{eq}^2} + \alpha - \alpha_K \right), \tag{20} \]

where \( \alpha_K = \frac{1}{3}\tau \langle \mathbf{\omega} \cdot \mathbf{u} \rangle \) is the kinematic contribution to the \( \alpha \)-effect. Equation (20) has first been obtained by Kleeorin and Ruzmaikin (1982), and it has been generalized by Kleeorin et al. (2000, 2002) to incorporate the flux of small scale magnetic helicity through the boundaries. In the special
context of homogeneous systems this model was rederived and analyzed by Field and Blackman (2002) and Blackman and Brandenburg (2002).

The most important property of equation (20) is that it reproduces the late, resistively slow saturation phase in a physically motivated fashion. It is for this reason that an explicitly time-dependent $\alpha$-effect appears to be mandatory for all models of large scale dynamos based on the helicity (or $\alpha$) effect.

5.5. CATASTROPHIC QUENCHING AS A LIMITING CASE

The time-dependent equations (16) and (20) for $\alpha$ and $\mathcal{E}$ may seem rather unfamiliar. In order to make contact with earlier more familiar formulations we now ignore the explicitly time dependence in equation (20) and obtain

$$
\alpha = \alpha_K - R_m \frac{(\mathcal{E} \cdot \mathbf{B})}{B_{eq}^2} = \alpha_K - \alpha R_m \frac{B^2}{B_{eq}^2} + \eta R_m \frac{\mathcal{J} \cdot \mathbf{B}}{B_{eq}^2},
$$

(21)

where $\mathcal{E} = \alpha \mathbf{B} - \eta \mathcal{J}$ itself depends still on $\alpha$. Solving for $\alpha$ yields

$$
\alpha = \frac{\alpha_K + \eta R_m \langle \mathcal{J} \cdot \mathbf{B} \rangle/B_{eq}^2}{1 + R_m \langle B^2 \rangle/B_{eq}^2}. 
$$

(22)

This equation was already obtained by Gruzinov and Diamond (1994) and Kleedorin et al. (1995). In numerical experiments with an imposed large scale

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure8}
\caption{Comparison of the magnetic field evolution in the dynamical and algebraic quenching models for different values of the magnetic Reynolds number. Both models have in common that $\eta$ is assumed to be proportional to $\alpha$ at all times. While both models agree in their late saturation behavior, the onset of saturation is markedly more abrupt in the dynamical quenching model.}
\end{figure}
field over the scale of the box, $\mathbf{B}$ is spatially uniform and hence $J = 0$. In that special case we have

$$\alpha = \frac{\alpha_K}{1 + R_m \langle \mathbf{B}^2 \rangle / B_{eq}^2} \quad (J = 0),$$

which implies that $\alpha$ becomes quenched when $\langle \mathbf{B}^2 \rangle / B_{eq}^2 = R_m^{-1} \approx 10^{-8}$ for the sun. This would seem to preclude any involvement of $\alpha$ in the sun (Vainshtein and Cattaneo 1992). However, if the field is not imposed but maintained by dynamo action, $\mathbf{B}$ is not spatially uniform and then $J$ is finite. In the case of a Beltrami field (see B01), $\langle \mathbf{J} \cdot \mathbf{B} \rangle / \langle \mathbf{B}^2 \rangle = \tilde{k}_m$ is some effective wavenumber of the large scale field. Since $R_m$ enters both the nominator and the denominator, $\alpha$ tends to $\eta \tilde{k}_m$. Thus, the question of how strongly $\alpha$ is quenched in the sun has been diverted to the question of how strongly $\eta$ is quenched (Blackman and Brandenburg 2002). One way to determine this quantity is by looking at cyclic dynamos with shear ($\alpha \Omega$-type dynamos), because here the cycle frequency is equal to $\eta \tilde{k}_m^2$. Blackman and Brandenburg (2002) came to the conclusion that $\eta$ is quenched when $\langle \mathbf{B}^2 \rangle / B_{eq}^2$ becomes of order unity, i.e. it is noncatastrophically quenched. However, more work at larger magnetic Reynolds numbers needs to be done.

5.6. COMPARISON WITH SIMULATIONS

At magnetic Reynolds numbers up to a few tens the comparison between simulations and the dynamical quenching model turned out to be quite favorable. However, as we increase the magnetic Reynolds number to one hundred and above a new difficulty arises that was already visible in Run 5 of B01. The point is that both during the entire kinematic phase and during much of the saturation phase several modes of the large scale field with the same or very similar values of $|k|$ are excited and contribute to the evolution of the mean field. Which one survives in the end can remain undecided for quite a long time when the magnetic Reynolds number is large. This is also the main reason why in Fig. 7 the runs with smaller magnetic Reynolds number reached equipartition earlier than those with larger magnetic Reynolds number. This can be seen from Fig. 6 of B01 which shows the interchange of energy in different modes during the intermediate and late saturation phase.

One can artificially try to favor a particular mode by initializing the simulation with a strong Beltrami field; see Fig. 9. This was done in a series of runs shown in Fig. 10, where we show the evolution of the mean field for magnetic Reynolds numbers up to about $R_m \equiv u_{rms} / (\eta k_t) = 200$. (Here, $k_t = 5$, so $R_m$ is, in this definition, $\sim 3$ times smaller than with $k_t = 1.5$ and the same resolution.) Although the late saturation behavior
agrees fairly well with the dynamical quenching model, at intermediate times the agreement is less favorable. An obvious possibility that needs to be considered is $R_m$-independent (non-catastrophic) quenching of $\alpha$ and $\eta_t$. In Blackman and Brandenburg (2002) such quenching for $\eta_t$ was already found to be vital for explaining simulations with shear.

5.7. COMMENTS ON MODELS WITHOUT HELICITY

Given the severe constraints encountered in connection with the helicity effect, one has to ask the question whether or not astrophysical large scale dynamos do operate with the helicity effect, or whether there are other effects that produce large scale fields without producing net magnetic helicity. Several alternative effects have been mentioned in recent years. Vishniac and Cho (2001) suggested a new effect that might operate in accretion discs where shear is very strong. Simulations by Arlt and Brandenburg (2001) have so far not been able to show that such an effect would be powerful enough to generate large scale fields. A similar effect is Rädler’s (1969) $\Omega \times \mathbf{J}$ effect (see also Krause and Rädler 1980), and Kitchatinov (these proceed-
Figure 10. Evolution of the mean magnetic field in simulations with different magnetic Reynolds numbers starting from an initial Beltrami field. Runs for the same $R_m$, but with different initial field strengths result in overlapping curves. In these runs $u_{\text{rms}} \approx 0.2$ and $k_f = 5$.

ings) has given reasons why this effect may indeed be important in the sun. Finally, we mention negative magnetic diffusivity effects (Zheligovsky et al. 2001), but again it is not clear that this actually works in astrophysical turbulence. We emphasize, however, that these are possibilities that should not be overlooked.

6. Concluding remarks

The present investigations have demonstrated that a lot can be learned from numerical simulations at different resolutions going up to the highest
resolution possible today, which is around $1024^3$ meshpoints. An important issue is whether these simulations have all run for long enough to give reliable statistics. The spectra vary a lot in time and averaging the spectra over long enough time intervals is crucial. In addition, there is the worry that the entire spectrum may evolve slowly in time after remeshing from a lower resolution run. It seems, however that at least the simulations with $512^3$ meshpoints, which have generally run for between 50 to 80 (large-scale) turnover times, have reached a statistically steady state. The simulation with $1024^3$ meshpoints, however, has run for only 5 turnover times, but even that may be long enough, although one can not be certain. We also recall that, as we have shown in this paper, simulations at different resolution and with different initial conditions give the same result.

Another question is what can be learned from simulations at intermediate Reynolds numbers when one is really interested in asymptotically large Reynolds numbers like those for the sun and many other astrophysical bodies. It is possible that the shallower subrange just before the dissipative subrange would disappear as the Reynolds number becomes larger. This possibility would be reminiscent of what happened in the mid-eighties, when hydrodynamic simulations began to show that turbulence is permeated by many thin and long vortex tubes (Kerr 1985, She et al. 1990, Vincent and Meneguzzi 1991). These tubes have a diameter comparable to the dissipative scale and their length was thought to be equal to the integral scale, which in turn is comparable to the size of the computational domain. Only recently, simulations at resolutions up to $1024^3$ meshpoints (Porter et al. 1998) have begun to show that the length of the tubes is significantly less than the integral scale. Instead, visualizations of vorticity show that clusters of vorticity are now a more dominant feature of high Reynolds number turbulence. Thus, there may well be surprises ahead of us in our quest for a deeper understanding of astrophysical MHD turbulence.

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