1 Introduction and main statements

Coadjoint orbits play an important role in representation theory, harmonic analysis and theory of symplectic varieties \[1, 6\]. For semisimple Lie groups coadjoint orbits coincide with with adjoint orbits and their classification is well known. The classical example of nilpotent Lie group is the unitriangular group. The problem of classification of coadjoint orbits for this group is still far from its solution \[1, 2\]. Description of regular orbits (i.e. orbits of maximal dimension) for UT\((n, K)\) was achieved by A.A.Kirillov in the his origin paper on the orbit method \[3\]. Subregular orbits for an arbitrary \(n\) and all orbits for \(n \leq 7\) are classified in \[4\].

The goal of this paper is to enlarge the class of orbits that have complete description. To each involution \(\sigma\) in the symmetric group \(S_n\) we correspond a family of coadjoint orbits \(\{\Omega(f)\mid f \in X_\sigma\}\). We construct a polarization of the canonical form of any such orbit (Theorem 1.1). We have got the formula for dimension of these orbits (Theorem 1.2).

Orbits of a regular action of an algebraic unipotent group on an affine algebraic variety are closed \[6\ 11.2.4\]. In the paper we find generators in the defining ideal of the orbit \(\Omega(f)\) (Theorem 1.4). Note that each of constructed generators has the form \(P - c\), where \(P\) is a coefficient of minor of characteristic matrix and \(c \in K\). The class of orbit associated with involutions contains all regular and some subregular orbits( see Examples 2,3 in §1 and \[4\]). Remark also that one can set up a correspondence between involutions and families of adjoint orbits satisfying the equation \(X^2 = 0\) \[5\].

Now go on to formulation of the main statements of this paper. An unitriangular group is a group \(N := \text{UT}(n, K)\) which consists of all lower triangular matrices over a field \(K\) with units on the diagonal. We suppose that the field \(K\)
has zero characteristic. The Lie algebra \( n = \text{ut}(n, K) \) of group \( N \) consists of the lower triangular matrices with zeros on the diagonal. With the help of Killing form \( (\cdot, \cdot) \) we identify the conjugate space \( n^* \) with the subspace \( n_+ \) that consists of the upper triangular matrices with zeros on the diagonal. We also identify the symmetric algebra \( S(n) \) with the algebra \( K[n^*] \) of regular functions on \( n^* \). The group \( N \) acts on \( n^* \) by the formula \( \text{Ad}_g^* f(x) = f(\text{Ad}_g^{-1} x), \ g \in N, \ f \in n^* \).

A subalgebra \( p \) in \( n \) is a polarization of \( f \in n^* \) if \( p \) is maximal isotropic subspace with respect to the skew symmetric form \( f([x, y]) \). According to the orbit method for any linear form on a nilpotent Lie algebra there exists a polarization. Existence of polarization gives the possibility to construct a primitive ideal in the universal enveloping algebra \( U(n) \) [6] and in the case \( K = \mathbb{R} \) the irreducible representation of the group \( N \) [1].

Denote by \( \Delta^+ \) the system of positive roots of the Lie algebra \( \text{gl}(n, K) \) [7]. the set of positive roots. The relation \( > \) in \( \Delta^+ \) is a lexicographical order. Any positive root \( \xi \) has the form \( \alpha_{ji} = \varepsilon_j - \varepsilon_i \), where \( j < i \). We denote \( j = j(\xi) \) and \( i = i(\xi) \). Any positive \( \xi \) root defines the reflection \( r_\xi \in S_n \).

Let \( \sigma \) be the involution in \( S_n \) (i.e. \( \sigma^2 = \text{id} \)). The involution \( \sigma \) uniquely decomposes into a product of commuting reflections

\[
\sigma = r_1 r_2 \cdots r_s, \tag{1}
\]

where each \( r_m \) is the reflection with respect to the positive root \( \xi_m \). We assume that \( \xi_1 > \xi_2 > \ldots > \xi_s \). Since reflections mutually commute, then \( j(\xi_1) < j(\xi_2) < \ldots < j(\xi_s) \). Form the subset

\[ S = \{\xi_1, \xi_2, \ldots, \xi_s\}. \]

Let \( \{y_{ij}\}_{1 \leq j < i \leq n-1} \) be the standard basis \( n \). In what follows we shall also use the notation \( y_\gamma \) for \( y_{ij} \) if \( \gamma = \varepsilon_j - \varepsilon_i \).

Consider the subset \( X_\sigma \subset n^* \), which consists of \( f \in n^* \) that \( f(y_{\xi_m}) \neq 0, \ 1 \leq m \leq s, \) and \( f \) annihilates on the other vectors of standard basis.

**Main definition.** We call the orbit \( \Omega(f) \) of \( f \in X_\sigma \) a coadjoint orbit associated with the involution \( \sigma \).

Our first goal is to construct a polarization of any \( f \in X_\sigma \). One can decompose the set of positive roots:

\[ \Delta^+ = \bigcup_{1 \leq t \leq n-1} \Delta^{(t)}, \text{ where } \Delta^{(t)} = \{\gamma \in \Delta^+ | j(\gamma) = t\}. \]

For any \( 1 \leq t \leq n - 1 \) we denote

\[ \sigma_t = \prod_{j(\xi_m) \leq t} r_m. \]
Easy to see that $\sigma_{n-1} = \sigma$. We put $\sigma_0 = \text{id}$. Denote by $\Pi_\sigma$ the following subset of positive roots

$$
\Pi := \Pi_\sigma := \bigsqcup_{1 \leq t \leq n-1} \Pi^{(t)},
$$

where $\Pi^{(t)} = \{ \eta \in \Delta^{(t)} \mid \sigma_{t-1}(\eta) > 0 \}$. Denote by $p_\sigma$ the linear subspace spanned by $\{ y_{it} : \varepsilon_t - \varepsilon_i \in \Pi \}$.

**Theorem 1.1.** $p_\sigma$ is a polarization for any $f \in X_\sigma$.

As usual $l(\sigma)$ (resp. $s(\sigma)$) is a number of simple (resp. arbitrary) reflections in reduced decomposition of $\sigma$ into a product of simple(resp. arbitrary) reflections. Easy to see that $s(\sigma) = s$ (see (1)).

**Theorem 1.2.** For any $f \in X_\sigma$ the dimension of the orbit $\Omega(f)$ equals to $l(\sigma) - s(\sigma)$.

Our next goal is to find system of generators in the defining ideal $\mathcal{I}(\Omega(f))$ of the orbit $\Omega(f)$ (i.e. in the ideal that consists of all polynomials that annihilate on $\Omega(f)$). Consider the following formal matrix $\Phi$ and its characteristic matrix $\Phi(\tau)$:

$$
\Phi = \begin{pmatrix}
0 & 0 & \ldots & 0 \\
y_{21} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
y_{n1} & y_{n2} & \ldots & 0
\end{pmatrix}, \quad \Phi(\tau) = \tau \Phi + E = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
\tau y_{21} & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\tau y_{n1} & \tau y_{n2} & \ldots & 1
\end{pmatrix}.
$$

For any subsets $J, I \subset \{1, \ldots, n\}$, $|I| = |J|$, we denote by $M_I^J$ the minor of matrix $\Phi$ with ordered increasing systems of columns $\text{ord}(J)$ and rows $\text{ord}(I)$.

We denote by $M_I^J(\tau)$ the corresponding minor of matrix $\Phi(\tau)$. The minor $M_I^J(\tau)$ is a polynomial in $\tau$ with coefficients in the symmetric algebra $S(n)$:

$$
M_I^J(\tau) = \tau^m(P_{I,0}^J + P_{I,1}^J \tau + P_{I,2}^J \tau^2 + \ldots),
$$

(2) where $m$ equals to $|J \setminus I| = |I \setminus J|$. The coefficient $P_{I,0}^J$ is up to a sign equals to the minor $M_{I \setminus J}^J$ of the matrix $\Phi$ (in what follows we frequently drop this sign).

**Remark 1.** If $I \supset J$ (in the sense Definition 3.4), then $M_I^J(\tau) \neq 0$ and its first coefficient $P_{I,0}^J$ is also nonzero. If $I \not\supset J$, then $M_I^J(\tau) = 0$.

Note that for any $f \in n_+ = n^*$ the value of $M_I^J$ at the point $f$ coincides with the transposed minor of $f$ presented as an upper triangular matrix. Similarly, the values of $P_{I,0}^J, P_{I,1}^J, P_{I,2}^J, \ldots$ at the point $f$ coincides with values of coefficients of the corresponding minor of the characteristic matrix $\tau f + E$.
For any pair \(1 \leq k, t \leq n\) we consider the systems
\[
J'(k, t) = \{1 \leq j < t : \sigma(j) > k\}, \quad I'(k, t) = \sigma J'(k, t),
\]
\[
J(k, t) = J'(k, t) \sqcup \{t\}, \quad I(k, t) = I'(k, t) \sqcup \{k\}.
\]

Denote by \(D_{k,t}\) (resp. \(D_{k,t}(\tau)\)) the minor of the matrix \(\Phi\) (resp. \(\Phi(\tau)\)) with the systems of columns \(J(k, t)\) and rows \(I(k, t)\).

As in (2) we decompose
\[
D_{k,t}(\tau) = \tau^m(P_{k,t,0} + P_{k,t,1}\tau + P_{k,t,2}\tau^2 + \ldots),
\]
where the coefficients \(P_{k,t,i}\) are elements of \(S(n) = K[n^*]\) and \(P_{k,t,0} \neq 0\). Note that if \(k > t\) then \(D_{k,t}(\tau) = \tau^mP_{k,t,0}\). Denote
\[
\mathcal{M} = \{ \gamma \in \Delta^+ | \sigma_t(\gamma) > 0, \text{ where } t = j(\gamma) \}.
\]

To each root of \(\mathcal{M}\) we shall assign a polynomial on \(n^*\).

**Definition 1.3.** Let \(i > t\) and \(\eta = \varepsilon_t - \varepsilon_i \in \mathcal{M}\). We say that \(\eta\) has type 0, if there is no \(\xi \in S\) such that \(j(\xi) \leq t - 1\) and \(i(\xi) = i\). Otherwise, we say that a \(\eta\) of \(\mathcal{M}\) has type 1.

Put \(k = \sigma_{t-1}(i)\). We denote
\[
Q_{i,t} = \begin{cases} 
P_{k,t,0}, & \text{if } \eta \text{ has type 0}, \\
P_{k,t,1}, & \text{if } \eta \text{ has type 1}. 
\end{cases}
\]

Note that, if \(\eta\) has type 0, then \(k = i\) and \(Q_{i,t} = P_{i,t,0}\) coincides (up to a sign) with the minor \(D_{i,t}\) of matrix \(\Phi\). If \(\eta\) has type 1, then \(k < t\) and \(k = \sigma(t)\). For each \(1 \leq m \leq s\) we denote
\[
D_m = D_{i(\xi_m), j(\xi_m)}.
\]

Easy to see that \(D_m(f) \neq 0\) for any \(f \in X_\sigma\).

**Theorem 1.4.** For any \(f \in X_\sigma\) the defining ideal of coadjoint orbit \(\Omega(f)\) is generated by \(Q_{i,t}\), where \(\varepsilon_t - \varepsilon_i \in \mathcal{M}\), and \(D_m - D_m(f)\), where \(1 \leq m \leq s\).

**Corollary 1.5.** For any \(f \in X_\sigma\) the intersection \(\Omega(f) \cap X_\sigma\) contains the only element \(f\).

For each \(1 \leq t \leq n - 1\) decompose the involution \(\sigma = \sigma_t \sigma'_t\) where
\[
\sigma'_t = \prod_{j(\xi_m) > t} r_m.
\]

Denote
\[
\Delta^{\dagger}_\sigma = \{ \zeta \in \Delta^+ : \sigma(\eta) > 0 \}.
\]
By definition, \(|\Delta^+_\sigma| = |\Delta^+| - l(\sigma)\).

**Remark 2.** The positive root \(\zeta\) lies in \(\Delta^+_\sigma = \Delta^+ \cap \Delta^{(t)}\) if and only if \(\sigma_t(\eta) > 0\) where \(\eta = \sigma'_t(\zeta)\). This shows that \(\sigma_t\) bijectively maps \(\Delta^{(t)}\) onto \(\mathcal{M}^{(t)} = \mathcal{M} \cap \Delta^{(t)}\). This implies \(|\mathcal{M}^{(t)}| = |\Delta^{(t)}|\) and, therefore, \(|\mathcal{M}| = |\Delta^+_\sigma| = |\Delta^+| - l(\sigma)\). From Remark 2 and Theorems 1.2 and 1.4 we obtain

**Corollary 1.6.** Each coadjoint orbit \(\Omega(f), f \in X_\sigma\) is a complete intersection.

Theorems 1.1 and 1.2 are proved in §2. The next §3 is completely devoted to the proof of Theorem 1.4. In the sequel of this section we introduce the language of admissible diagrams and reformulate Theorems 1.1 in new terms. This approach has its own advantages. We present examples of involutions, construct their admissible diagrams and generators of defining ideals of their associated orbits.

As above \(\sigma\) be an involution in \(S_n\). Consider the following subsets in the set of positive roots \(\Delta^+\):

\[
\mathcal{C}_+ = \{\gamma \in \Delta^+ \setminus \mathcal{S} | \sigma_{t-1}(\gamma) > 0, \sigma_t(\gamma) < 0, \text{ where } t = j(\gamma)\},
\]

\[
\mathcal{C}_- = \{\gamma \in \Delta^+ | \sigma_{t-1}(\gamma) < 0, \text{ where } t = j(\gamma)\}
\]

The set of positive roots \(\Delta^+\) and its subset \(\Pi\) decomposes: \(\Delta^+ = \Pi \sqcup \mathcal{C}_-\) and \(\Pi = \mathcal{S} \sqcup \mathcal{M} \sqcup \mathcal{C}_+\).

We begin construction of an admissible diagram from the empty \(n \times n\)-matrix. We shall not fill the squares \((i, j), i \leq j\). To each positive root \(\gamma = \alpha_{ji}, i > j\), we correspond the square \((i, j)\). We put one of the symbols \(\otimes, +, -, \bullet\) into square \((i, j)\) according to these rules:

\[
\begin{cases}
\otimes, & \text{if } \gamma \in \mathcal{S}, \\
+ , & \text{if } \gamma \in \mathcal{C}_+, \\
- , & \text{if } \gamma \in \mathcal{C}_-, \\
\bullet , & \text{if } \gamma \in \mathcal{M}.
\end{cases}
\]

One can construct the same admissible diagram using the method of paper [4]. Consider the first root \(\xi_1\) in \(\mathcal{S}\). Put the symbol \(\otimes\) into square \((i(\xi_1), j(\xi_1))\). Next we put the symbol + into all squares that lie in the column \(j(\xi_1)\) lower the diagonal and higher than the square \((i(\xi_1), j(\xi_1))\). We put the symbol – into all squares that lie in the row \(i(\xi_1)\) lower the diagonal and righter than the square \((i(\xi_1), j(\xi_1))\).

After this procedure, we take the other root \(\xi_2\) of \(\mathcal{S}\). We put the symbol \(\otimes\) into the square \((i(\xi_2), j(\xi_2))\) that as it easy to see are not filled after the first step. Similar to the case of root \(\xi_1\) we put the symbols + and – into
the matrix using the following additional rule: if for certain \( k \) one of this two squares \((k, j(\xi_2)), (i(\xi_2), k)\) are already filled we do not fill the other square.

We continue this procedure for all other roots \( \xi \in S \). We put the symbol \( \bullet \) into all unfilled squares lower the diagonal. Note that the roots of \( \Pi \) correspond to the squares filled by symbols \( \otimes, + \) and \( \bullet \).

**Example 1 (begin).** Involution \( \sigma = (1, 4)(2, 7)(3, 6) \) in the symmetric group \( S_7 \) corresponds to the admissible diagram

\[
\begin{array}{ccccccc}
+ & + & \otimes & - & - & \bullet & + \\
+ & \otimes & - & - & \bullet & + & \bullet \\
\otimes & - & - & \bullet & + & \otimes & - \\
+ & + & \otimes & - & - & \bullet & + \\
+ & \otimes & - & - & \bullet & + & \bullet \\
\otimes & - & - & \bullet & + & \otimes & - \\
\end{array}
\]

The diagram is constructed in 4 steps:

\[
\begin{array}{ccccccc}
+ & + & \otimes & - & - & \bullet & + \\
+ & \otimes & - & - & \bullet & + & \bullet \\
\otimes & - & - & \bullet & + & \otimes & - \\
+ & + & \otimes & - & - & \bullet & + \\
+ & \otimes & - & - & \bullet & + & \bullet \\
\otimes & - & - & \bullet & + & \otimes & - \\
\end{array} \quad \Rightarrow \quad \begin{array}{ccccccc}
+ & + & \otimes & - & - & \bullet & + \\
+ & \otimes & - & - & \bullet & + & \bullet \\
\otimes & - & - & \bullet & + & \otimes & - \\
+ & + & \otimes & - & - & \bullet & + \\
+ & \otimes & - & - & \bullet & + & \bullet \\
\otimes & - & - & \bullet & + & \otimes & - \\
\end{array} \quad \Rightarrow \quad \begin{array}{ccccccc}
+ & + & \otimes & - & - & \bullet & + \\
+ & \otimes & - & - & \bullet & + & \bullet \\
\otimes & - & - & \bullet & + & \otimes & - \\
+ & + & \otimes & - & - & \bullet & + \\
+ & \otimes & - & - & \bullet & + & \bullet \\
\otimes & - & - & \bullet & + & \otimes & - \\
\end{array} \quad \Rightarrow \quad \begin{array}{ccccccc}
+ & + & \otimes & - & - & \bullet & + \\
+ & \otimes & - & - & \bullet & + & \bullet \\
\otimes & - & - & \bullet & + & \otimes & - \\
+ & + & \otimes & - & - & \bullet & + \\
+ & \otimes & - & - & \bullet & + & \bullet \\
\otimes & - & - & \bullet & + & \otimes & - \\
\end{array}
\]

In this example \( S = \{\alpha_{14}, \alpha_{27}, \alpha_{36}\} \), \( M = \{\alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{45}, \alpha_{46}, \alpha_{47}\} \), \( \Pi = \Delta^+ \setminus \{\alpha_{24}, \alpha_{34}, \alpha_{37}, \alpha_{56}, \alpha_{57}, \alpha_{67}\} \). In the following Remarks 3-5 we reformulate Theorems of this section in terms of admissible diagrams.

**Remark 3.** According to Theorem 1.1. the polarization \( p_\sigma \) is spanned by \( \{y_{it}\} \) such that the square \((i, t)\) is filled by one of the symbols \( \otimes, \bullet \) or \( + \) in the corresponding diagram.

**Remark 4.** By Theorem 1.2 and Remark 2, \( \dim \Omega(f) = l(\sigma) - s = |\Delta^+| - |M| - s \). We see that \( \dim \Omega(f) \) is equal to the number of \( \pm \)’s in the admissible diagram.

**Remark 5.** Let the square \((i, t), t < i\), be filled by \( \bullet \) (i.e. \( \varepsilon_t - \varepsilon_i \in M \)). We say that \((i, t)\) has type 0 (resp. type 1), if the root \( \varepsilon_t - \varepsilon_i \) of \( M \) has type 0 (resp. type 1). Easy to see that \((i, t)\) has type 0, if there is no symbol \( \otimes \) in the \( i \)th row and on the left side of \((i, t)\). If the above symbol \( \otimes \) exists, then \((i, t)\) has type 1. In the Example 1, \((5, 4)\) is type 0, but \((6, 4)\) and \((7, 4)\) have type 1.

Note that, if \((i, t)\) is filled by symbol \( \bullet \) and has type 1, then \( a < k < t < i \) where \( a = \sigma(t) \) and as above \( k = \sigma_{t-1}(i) = \sigma(i) \). The squares \((t, a)\) and \((i, k)\) are filled by symbol \( \otimes \). In Example 1, the square \((7, 4)\) is filled by symbol \( \bullet \) and
has type 1, \( a = 1 \) and \( k = 2 \). The squares \((4,1)\) and \((7,2)\) are filled by symbol \( \otimes \). End of Remark.

**Example 1 (end).** The defining ideal \( I(\Omega(f)) \) of Example 1 is generated by the elements, \( D_m - c_m \), where \( 1 \leq m \leq 3 \) and \( c_m \neq 0 \), and \( Q_{i,t} \) corresponding to pairs \( \{(i,t)\} \) filled by symbol \( \bullet \). Here are the generating polynomials of Example 1:

\[
D_1 = y_{41}, \quad Q_{5,1} = y_{51}, \quad Q_{6,1} = y_{61}, \quad Q_{7,1} = y_{71},
\]
\[
D_2 = y_{72}, \quad D_3 = \begin{vmatrix} y_{62} & y_{63} \\ y_{72} & y_{73} \end{vmatrix}, \quad Q_{5,4} = \begin{vmatrix} y_{52} & y_{53} & y_{54} \\ y_{62} & y_{63} & y_{64} \\ y_{72} & y_{73} & y_{74} \end{vmatrix},
\]
\[
Q_{6,4} = \begin{vmatrix} y_{62} & y_{64} \\ y_{72} & y_{74} \end{vmatrix} + \begin{vmatrix} y_{62} & y_{63} \\ y_{72} & y_{73} \end{vmatrix} y_{31}, \quad Q_{7,4} = y_{74}y_{41} + y_{73}y_{31} + y_{72}y_{21}.
\]

**Example 2.** \( \sigma \) is an substitution of maximal length, i.e \( \sigma = (1,n)(2,n-1) \cdots (k,k+1) \) where \( k = \lceil \frac{n}{2} \rceil \). The associated coadjoint orbit is a regular orbit (i.e. orbit of maximal dimension). The defining ideal \( I(\Omega(f)) \) is generated by \( D_{i,n-i+1} - D_{i,n-i+1}(f), 1 \leq i \leq k \).

**Example 3.** \( \sigma = (1,n-1)(2,n)(3,n-3) \cdots (k,k+1) \). The associated coadjoint orbit of this involution is a subregular orbit (i.e. orbit of dimension \( \dim(\text{reg.orbit}) - 2 \)) (see [4]). The defining ideal \( I(\Omega(f)) \) is generated by

\[
D_{i,n-i+1} - D_{i,n-i+1}(f), \quad 3 \leq i \leq k,
\]
\[
y_{n1}, \quad y_{n-1,1} - f(y_{n-1,1}), \quad y_{n2} - f(y_{n2}), \quad z = y_{n,n-1}y_{n-1,1} + y_{n,n-2}y_{n-2,1} + \cdots + y_{n2}y_{21}.
\]

Here are the admissible diagrams for Examples 2 and 3 for \( n = 6 \):

![Example 2 Diagram](image1.png)

![Example 3 Diagram](image2.png)

2 Polarization and dimension of orbit

The goal of this section is to prove Theorems 1.1. and 1.2. With the help of the Killing form we identify the Cartan subalgebra \( \mathfrak{h} \) with its conjugate space. This identification allows to define the Killing form \((\cdot, \cdot)\) on \( \mathfrak{h}^* \). Note that the condition \( \sigma^2 = \text{id} \) implies \( (\xi_i, \xi_j) = 0 \) for different elements \( \xi_i, \xi_j \) in \( \mathcal{S} \).

**Lemma 2.1.** Let \( \eta \in \Delta^+ \) and \( t = j(\eta) \). We claim:

1) if \( \sigma_{i-1}(\eta) > 0 \) (i.e \( \eta \in \Pi^{(t)} \)), then \( \sigma_k(\eta) > 0 \) for all \( 0 \leq k \leq t - 1 \);
2) if $\sigma_t(\eta) > 0$, then $\sigma_{t-1}(\eta) > 0$.

**Proof.** We shall prove statement 1). The statement 2) is proved similar. If $\eta \in \mathcal{S}$, then $r_m(\eta) = \eta$ for all $m$. This implies 1).

Suppose that $\eta \in \Delta^+ \setminus \mathcal{S}$. Then the only one of the following cases will take place:

i) $(\eta, \xi) = 0$ for all $\xi \in \mathcal{S}$;

ii) there exists a unique $\xi' \in \mathcal{S}$ such that $(\eta, \xi') \neq 0$ and $(\eta, \xi) = 0$ for all $\xi \in \mathcal{S} \setminus \{\xi'\}$;

iii) there exist exactly two $\xi', \xi'' \in \mathcal{S}$ such that $(\eta, \xi') \neq 0$, $(\eta, \xi'') \neq 0$ and $(\eta, \xi) = 0$ for all $\xi \in \mathcal{S} \setminus \{\xi', \xi''\}$.

The statement 1) is trivial for the case i). In the case ii) we embed $y_\eta$ in to the subalgebra span\{ $y_\gamma | \gamma \in \Delta^+, \gamma \in \mathbb{Z}\eta + \mathbb{Z}\xi'$\} that is isomorphic to $\mathfrak{ut}(3, K)$. The proof is reduces to the special case $n = \mathfrak{ut}(3, K)$ and $\mathcal{S} = \{\alpha\}$ where $\alpha$ is one of positive roots $\alpha_{12}, \alpha_{13}, \alpha_{23}$.

In the case iii) embed $y_\eta$ in to subalgebra span\{ $y_\gamma | \gamma \in \Delta^+, \gamma \in \mathbb{Z}\eta + \mathbb{Z}\xi' + \mathbb{Z}\xi''$\} that is isomorphic to $\mathfrak{ut}(4, K)$. The proof is reduces to the special case $n = \mathfrak{ut}(4, K)$ and $\mathcal{S}$ coincides with one of the following subsets $\mathcal{S}_1 = \{\alpha_{14}, \alpha_{23}\}$, $\mathcal{S}_2 = \{\alpha_{13}, \alpha_{24}\}$, $\mathcal{S}_3 = \{\alpha_{12}, \alpha_{34}\}$. □

**Lemma 2.2.** For any $\gamma' \in \mathcal{C}_-$ there exists a unique $\gamma \in \mathcal{C}_+$ such that $\gamma + \gamma' \in \mathcal{S}$. The correspondence $\gamma' \mapsto \gamma$ is bijective.

**Proof** is proceeded similar to Lemma 1 reducing to the unitriangular Lie subalgebra on lower size. □

**Proof of the Theorem 1.1.**

1) Let us show that $p_\sigma$ is a subalgebra. It suffices to prove that, if $\gamma, \gamma' \in \Pi$ and $\gamma + \gamma' \in \Delta^+$, then $\gamma + \gamma' \in \Pi$. Let $j(\gamma) = t$, $j(\gamma') = t'$ and $t < t'$. Then $j(\gamma + \gamma') = t'$. Since $\gamma \in \Pi$, then $\sigma_{t-1}(\gamma) > 0$. From Lemma 2.1 we conclude $\sigma_{t-1}(\gamma') > 0$. Therefore, $\sigma_{t-1}(\gamma + \gamma') > 0$ and $\gamma + \gamma' \in \Pi$.

2) Let us show that $p_\sigma$ is an isotropic subspace. It suffices to show that $(P + P) \cap \mathcal{S} = \emptyset$. Assume the contrary: $\xi = \gamma + \gamma'$ where $\xi \in \mathcal{S}$, $\gamma, \gamma' \in \Pi$. Let $t = j(\gamma) < t' = j(\gamma')$. Lemma 2.1 implies $\sigma_t(\gamma') > 0$. On the other hand, since $\gamma \in \Pi$, then

$$\sigma_t(\gamma') = \sigma_{t-1}r_\xi(\gamma') = -\sigma_{t-1}(\gamma) < 0.$$ A contradiction.

3) Let us show that $p_\sigma$ is maximal isotropic subspace. Assume the contrary. Let $p_\sigma + Ka$ be an isotropic subspace where

$$a = \sum_{\gamma' \in \mathcal{C}_-} c_{\gamma'}y_{\gamma'} \neq 0.$$
Suppose that \( c_{\gamma_0} \neq 0 \) for some \( \gamma_0 \in C_- \). There exists a root \( \gamma_0 \in C_+ \subset \Pi \) such that \( \gamma_0 + \gamma_0' \in S \) and \( \gamma_0 + \gamma' \notin S \) for all \( \gamma' \in C_- \setminus \{ \gamma_0' \} \). Then

\[
0 = f([y_{\gamma_0}, a]) = f\left( \sum_{\gamma' \in C_-} c_{\gamma'} y_{\gamma_0 + \gamma'} \right) = c_{\gamma_0} f(y_{\gamma_0 + \gamma_0}).
\]

This contradicts to \( f(y_{\gamma_0 + \gamma_0}) \neq 0 \). Finally, the points 1, 2, 3 conclude that \( p_\sigma \) is a polarization for all \( f \in X_\sigma \).

**Proof of the Theorem 1.2.** The dimension of any maximal isotropic subspace of the skew symmetric form \( \langle x, y \rangle = f([x, y]) \) equals to \( \frac{1}{2}(\dim n + \dim n^f) \) where \( n^f \) is a Lie algebra of stabilizer of \( f \). Hence

\[
\dim \Omega(f) = \dim n - \dim n^f = 2 \text{ codim } p_\sigma = 2|C_-| = |C_+| + |C_-| = |\Delta^+| - |M| - |S|.
\]

By Remark 4, \( |M| = |\Delta^+_\sigma| \). Therefore, \( |M| = |\Delta^+| - l(\sigma) \). Substituting in (4) we have got \( \dim \Omega(f) = l(\sigma) - |S| = l(\sigma) - s(\sigma) \).

**3 Defining ideal of orbit**

In this section we shall prove Theorem 1.4 on the defining ideal of an orbit associated with involution. Let \( f \in X_\sigma \). Denote by \( \mathcal{I} \) the ideal in the symmetric algebra \( S(n) \) generated by \( Q_{i,t}, \varepsilon_t - \varepsilon_i \in M, \) and \( D_m - D_m(f), 1 \leq m \leq s \).

Our goal is to prove that \( \mathcal{I} \) coincides with the defining ideal \( \mathcal{I}(\Omega(f)) \) of the orbit \( \Omega(f) \). The proof is divided into several steps:

1) the ideal \( \mathcal{I} \) annihilated at \( f \) (Proposition 3.1);
2) the ideal \( \mathcal{I} \) is prime and \( \dim(\text{Ann}\mathcal{I}) = \dim \Omega \). (Proposition 3.7)
3) the ideal \( \mathcal{I} \) in \( S(n) \) is stable with respect to the adjoint action of the group \( N \) (Proposition 3.8).

**Proposition 3.1** The ideal \( \mathcal{I} \) annihilates at \( f \).

**Proof.** Easy to see that the elements \( D_m - D_m(f) \) annihilate at \( f \). If the square \((i, t), i > t,\) is filled by symbol \( \bullet \) and has type 0, then \( Q_{i,t} = D_{i,t}. \) All entries of \( i \)th row of \( D_{i,t} \) annihilate at \( f \). Hence \( Q_{i,t}(f) = 0. \)

Suppose that the \((i, t), i > t\) is filled by symbol \( \bullet \) and has type 1. As in Remark 5 \( k = \sigma_{t-1}(i) = \sigma(i), a = \sigma(t) \) and \( a < k < t < i. \) Put \( J = J(k, t), I = I(k, t) \) (see §1). Decompose

\[
J = \{a, t\} \sqcup \Lambda_1 \sqcup \Lambda_2, \quad \text{where}
\]

\[
\Lambda_1 = \{j \in J : k < \sigma(j) < t\}
\]

\[
\Lambda_2 = \{j \in J : \sigma(j) > t\}.
\]
Present $\Lambda_1$ as a union $\Lambda_1 = \Lambda_{1,0} \cup \Lambda_{1,1}$, where

$$\Lambda_{1,0} = \{ j \in \Lambda_1 : 1 \leq j < k \};$$
$$\Lambda_{1,1} = \{ j \in \Lambda_1 : k < j < t \}.$$

The subset $\Lambda_{1,1}$ is stable with respect to action of $\sigma$.

Note that since $\sigma(i) = t$, then $i = \sigma(t)$ and $k \in \Lambda_2$. We have

$$J = J(k, t) = \{ a, t \} \cup \Lambda_{1,0} \cup \Lambda_{1,1} \cup \Lambda_2,$$
$$I = I(k, t) = \{ k, t \} \cup \sigma\Lambda_{1,0} \cup \Lambda_{1,1} \cup \sigma\Lambda_2,$$
$$I \cap J = \{ k, t \} \cup \Lambda_{1,1} \quad (6)$$

As in (3) we decompose $D_{k,t}(\tau)$ in powers of $\tau$. Denote

$$J_0 = J \setminus I = \{ a \} \cup \Lambda_{1,0} \cup \Lambda_2 \setminus \{ k \};$$
$$I_0 = I \setminus J = \sigma\Lambda_{1,0} \cup \sigma\Lambda_2. \quad (7)$$

We obtain

$$Q_{i,t} = P_{k,t,1} = \sum_{j \in I \cap J} \epsilon(j) M^J_{I_0 \cup \{ j \}} = \epsilon(t) M^J_{I_0 \cup \{ t \}} + \sum_{j \in I \cap J, j \neq t} \epsilon(j) M^J_{I_0 \cup \{ j \}}, \quad (8)$$

where $\epsilon(j) = \pm 1$. The minor $M^J_{I_0 \cup \{ t \}}$ annihilates at $f$ since all its $t$th column annihilates are equal to zero at $f$. Each minor $M^J_{I_0 \cup \{ j \}}, j \in I \cap J$ and $j \neq t$, annihilates at $f$ since its $a$th column annihilates at $f$. We have got $Q_{i,t}(f) = 0$.

Remark 6. If the square $(i, t)$ is filled by symbol $\bullet$ and has type 1, then the polynomial $P_{k,t,0}$ coincides with $M^J_{I_0}$. This minor annihilates at $f$ since all entries of its $a$th column annihilates at $f$.

Let $A = (a_{ij})$ be a $n \times n$-matrix where entries $a_{ij}$ are elements of some commutative domain $\mathcal{C}$.

Lemma 3.2. Let $I, J$ be two subsystems of $\{1, \ldots, n\}$ and $|I| = |J|$. Suppose that $J = J_1 \cup J_2$, $I = I_1 \cup I_2$, where $|I_1| = |J_1|$.

1) Let $S \subset J_2$. The following identity holds for minors of the matrix $A$:

$$M^J_{I_1} M^J_{I} = \sum_{T \subset I_2, |T| = |S|} \epsilon(T, S) M^J_{I_0 \cup \emptyset} M^J_{I \setminus T}. \quad (9)$$

2) Let $T \subset I_2$. The following identity holds for minors of the matrix $A$:

$$M^J_{I_1} M^J_{I} = \sum_{S \subset J_2, |S| = |T|} \epsilon(T, S) M^J_{I_0 \cup \emptyset} M^J_{I \setminus T}. \quad (10)$$
Here $\epsilon(T, S) = \pm 1$. The sign depends on the choice of $T, S, I_1, I_2, J_1, J_2$.

**Proof.** We may assume that $a_{ij}$ are variables. Let $M^J_I$ be the minor of the matrix $A$ with the system of columns $J$ and the system of rows $I$. Consider the determinate

$$\tilde{M}^J_{I_2} = \det(\tilde{a}_{ij})$$

of size $|I_2| = |J_2|$ with entries

$$\tilde{a}_{ij} = M^J_{I_i \cup \{i\}}, \text{ where } j \in J_2, i \in I_2.$$

Apply the Laplace formula:

$$\tilde{M}^J_{I_2} = \sum_{T \subset I_2, |T| = |S|} \pm \tilde{M}^S_T \tilde{M}^{J_2 \setminus S}_{I_2 \setminus T}$$

(11)

The following formula holds

$$\left( M^J_{I_1} \right)_{|J_2|-1} M^J_I = \pm \tilde{M}^J_{I_2}$$

(12)

Let us prove (12). By elementary transformations of $M^J_I$ one can obtain zeroes in $I_1 \times J_2$ and $I_2 \times J_1$ preserving entries of $I_1 \times J_1$. Then

$$M^J_I = \pm M^J_{I_1} \overline{M}^J_{I_2}$$

where $\overline{M}^J_{I_2} = \det(\overline{a}_{ij})$, $\overline{a}_{ij} \in \text{Fract}(C)$ and $\overline{a}_{ij} = \pm a_{ij} M^J_{I_1}$. We have got

$$\tilde{M}^J_{I_2} = \pm \left( M^J_{I_1} \right)_{|J_2|} \overline{M}^J_{I_2} = \pm \left( M^J_{I_1} \right)_{|J_2|-1} M^J_{I_1} \overline{M}^J_{I_2} = \left( M^J_{I_1} \right)_{|J_2|-1} M^J_I.$$

This proves (12). Substituting (12) for (11) we have got:

$$\left( M^J_{I_1} \right)_{|J_2|-1} M^J_I = \sum_{T \subset I_2, |T| = |S|} \epsilon(T, S) \left( M^J_{I_1} \right)_{|S|-1} M^J_{I_1 \cup T} \left( M^J_{I_1} \right)_{|J_2 \setminus S|-1} M^J_{I \setminus T}$$

Cutting $\left( M^J_{I_1} \right)_{|J_2|-2}$ in the left and right sides of this equality, we obtain (9).

The statement 2) is proved similarly. $\square$

Let $C = \tau^m (c_0 + c_1 \tau + c_2 \tau^2 + \ldots)$, $c_0 \neq 0$, be a polynomial in $\tau$ with coefficients in some commutative domain $C$. We call $m$ the lower degree of $C$. Easy to see that $m(AB) = m(A)m(B)$.

**Definition 3.3.** We call the presentation $C$ as a sum $C = A_1 + \ldots + A_k$ of polynomials $A_i \in C[\tau]$ an admissible presentation, if $m(C) \leq m(A_i)$ for all nonzero $A_i$.
Remark 7. Let $D = \tau^m(d_0 + d_1 \tau + d_2 \tau^2 + \ldots)$, $A_i = \tau^m(A_i)(a_{i,0} + a_{i,1} \tau + \ldots)$, $B_i = \tau^m(B_i)(b_{i,0} + b_{i,1} \tau + \ldots)$ and $C$ as above. Let $D, C \neq 0$. If $DC = \sum A_i B_i$ is an admissible presentation, then

$$d_0c_0 = \sum_{i} a_{i,0}b_{i,0},$$

$$d_0c_1 = -d_1c_0 + \sum_{i}(a_{i,1}b_{i,0} + a_{i,0}b_{i,1}) + \sum_{i} a_{i,0}b_{i,0},$$

where the sum $\sum'$ means the sum over those $i$ that obeys $m(D) + m(C) = m(A_i) + m(B_i)$; the sum $\sum''$ means the sum over those $i$ that obeys $m(D) + m(C) + 1 = m(A_i) + m(B_i)$.

Remark 8. Let $A, B, C, D$ as above. If all $a_{i,0}$, $a_{i,1}$ lie in some ideal $I$ of $C$ and if $d_0$ is invertible modulo $I$, then $c_0$, $c_1$ lie in $I$.

Definition 3.4. Let $v = (v_1 < \ldots < v_n)$ and $u = (u_1 < \ldots < u_n)$ be two increasing $n$-dimensional vectors. We say $v \leq u$, if $v_i \leq u_i$ for all $i$. Respectively, $v < u$, if $v \leq u$ and $v \neq u$. For $I, J \subset \{1, \ldots, n\}$, $|I| = |J|$, we say $I \triangleright J$ if $\text{ord}(I) \geq \text{ord}(J)$ in the above sense.

Lemma 3.5. Let $I_1$, $J_1$, $I$, $J$ be as in Lemma 3.2. Suppose that $I \triangleright J$ and $I \triangleright J_1$. We claim that the decompositions (9) and (10) are admissible over $S(n)$.

Proof is given for (9). The case of formula (10) is similar. Under the assumptions the left side of (9) is nonzero (see Remark 1). Let $m$ (resp. $m_T$) be the lower degree of the left part of (9) (resp. of the nonzero $T$th summand in the right side) Let us show that $m \leq m_T$. By definition,

$$m = |J_1| + |J| - |I_1 \cap J_1| - |I \cap J|$$

$$m_T = |J_1 \cup S| + |J \setminus S| - |(I_1 \cup T) \cap (J_1 \cup S)| - |(I \setminus T) \cap (J \setminus S)|.$$

Since

$$|J_1| + |J| = |J_1 \cup S| + |J \setminus S|,$$

$$|(I_1 \cup T) \cap (J_1 \cup S)| = |I_1 \cap J_1| + |I_1 \cap S| + |T \cap J_1| + |T \cap S|,$$

$$(I \setminus T) \cap (J \setminus S) = |I \cap J| - |I \cap S| - |T \cap J| + |T \cap S|;$$

then $m_T - m = |I \cap S| + |T \cap J| - |I_1 \cap S| - |T \cap J_1| - 2|T \cap S| = |I_2 \cap S| + |T \cap J_2| - 2|T \cap S| = |T \cap (J_2 \setminus S)| + |(I_2 \setminus T) \cap S| \geq 0$. □

Lemma 3.6. Let $J \subset \{1, \ldots, n\}$. Suppose that $J$ have the following property: if $j \in J$ and $j' < j$, $\sigma(j') > \sigma(j)$, then $j' \in J$. Denote $P_{\mu} = P_{I,J, \mu}$, $\mu \in \{0, 1\}$ (see (2)). Let $I \subset \{1, \ldots, n\}$ and $|I| = |J|$. We claim,
1) if \( I = \sigma(J) \), then \( P_0 - P_0(f) \in \mathcal{I} \), \( P_0(f) \neq 0 \) for any \( f \in X_\sigma \);
2) if \( I > \sigma(J) \), then \( P_\mu \in \mathcal{I} \) for \( \mu \in \{0, 1\} \).

**Proof.** First, note that the assumption on \( J \), imply that \( \sigma(J) \geq J \) in the sense of Definition 3.4. By Remark 1, \( M_I^J(\tau) \neq 0 \) and its lower degree equals to \(|J \setminus I| = |I \setminus J|\).

We prove the statement using the method of induction on \(|I|\). Easy to see that the claim is true for \(|I| = 1\). Assume that the claim is true for \(|I| < m\). We are going to prove the claim for \(|I| = m\).

**Case 1.** Let \( I = \sigma(J) \). Easy to see that \( P_0(f) \neq 0 \). Put \( t = \max J, \sigma(t) = a \). Denote

\[
J_1 = \{j \in J : \sigma(j) > a\}, \quad I_1 = \sigma(J_1).
\]

**Point 1a.** \(|J_1| < |J| - 1\). Denote

\[
M^{(1)}(\tau) = M_{I_1}^J(\tau), \quad M(\tau) = M_I^J(\tau), \quad M^{(\beta)}(\tau) = M_{I_1 \cup \{t\}}^{J_1 \cup \{t\}}(\tau), \quad M^{(*\beta)}(\tau) = M_{I_1 \cup \{\beta\}}^{J_1 \cup \{\beta\}}(\tau).
\]

If \( J_1 = \emptyset \), then we put \( M^{(1)}(\tau) = 1 \). Write down the admissible decomposition (9) with \( S = \{t\} \) for matrix \( \Phi(\tau) \):

\[
M^{(1)}(\tau)M(\tau) = \sum_{\beta \in I(\tau), \beta \leq a} \epsilon(\beta)M^{(\beta)}(\tau)M^{(*\beta)}(\tau) = \pm M^{(a)}(\tau)M^{(*a)}(\tau) + \sum_{\beta \in I(\tau) \setminus \{a\}, \beta > a} \epsilon(\beta)M^{(\beta)}(\tau)M^{(*\beta)}(\tau). \tag{15}
\]

Denote by \( P_0^{(1)}, P_0^{(\beta)}, P_0^{(*\beta)} \) the lowest coefficients of \( M^{(1)}(\tau), M^{(\beta)}(\tau), M^{(*\beta)}(\tau) \). Recall \( P_0 \) is the lowest coefficient of \( M(\tau) \). By (13), we have got

\[
P_0^{(1)}P_0 = \sum_{\beta \in I(\tau), \beta \leq a} P_0^{(\beta)}P_0^{(*\beta)}. \tag{16}
\]

The subset \( J_1 \) also obey assumption of Lemma. By induction assumption, \( P_0^{(1)} - P_0^{(1)}(f) \in \mathcal{I}, P_0^{(1)}(f) \neq 0 \). The left side of (16) is nonzero at \( f \).

The subset \( I \setminus \{t\} \) also obey the assumption of Lemma. If \( \beta < a \), then \( I \setminus \{a\} < I \setminus \{\beta\} \) (in the sense of Definition 3.4). By the induction assumption, \( P_0^{(*\beta)} \in \mathcal{I} \) for \( \beta < a \). By Lemma 3.1, \( P_0^{(*\beta)}(f) = 0 \). Considering values of right and left parts of (16) at \( f \in X_\sigma \), we conclude that the term \( \beta = a \) occurs in \( \sum' \):

\[
P_0^{(1)}P_0 = \pm P_0^{(a)}P_0^{(*a)} + \sum_{\beta \in I, \beta < a} P_0^{(\beta)}P_0^{(*\beta)}. \tag{17}
\]
By the induction assumption, $\pm P_0^{(a)}$, $\pm P_0^{(*)}$ satisfy 1). That is $P - P(f) \in \mathcal{I}$, $P(f) \neq 0$ for any of this two polynomials. This proves $P_0 - P_0(f) \in \mathcal{I}$ and $P_0(f) \neq 0$ in Point 1a.

**Point 1b.** $|J_1| = |J| - 1$. As above $t = \max J$, $a = \sigma(t) = \min I$. In Point 1b $J = J(a, t)$, $I = I(a, t)$ (see §1). If $a > t$, then $P_0$ is one of the minors $D_p$, $1 \leq p \leq s$. This proves statement 1).

Let $a \leq t$. Define $I_1$ and $J_1$ as in Point 1a. In Point 1b $J_1 = J - \{t\}$, $I_1 = I - \{a\}$ and $P_0 = P_0^{(1)}$. Subsets $I_1$, $J_1$ obey induction assumption. The statement 1) is true for $P_0^{(1)}$. Hence 1) is true for $P_0$.

**Case 2.** $I > \sigma(J)$. There exists $k \in I$ and $b \in J$ such that $k \notin \sigma(J)$ and $k > \sigma(b)$. Similar to Case 1, we denote

$$J_1 = \{j \in J : \sigma(j) > k\}, \quad I_1 = \sigma(J_1), \quad P^{(1)} = P_{I_1, 0}^{J_1}.$$ 

**Point 2a.** $|J_1| < |J| - 1$. Write down the admissible decomposition (10) with $T = \{k\}$ for matrix $\Phi(\tau)$:

$$M_{I_1}^{J_1}(\tau)M_I^J(\tau) = \sum_{\alpha \in J \setminus J_1} \epsilon(k)M_{I_1 \cup \{k\}}^{J_1 \cup \{\alpha\}}(\tau)M_{I \setminus \{k\}}^J(\tau)$$

(18)

The pairs of subsets $I_1$, $J_1$ and $I_1 \cup \{k\}$, $J_1 \cup \{\alpha\}$ obey the conditions of Lemma 1) and 2) respectively. Hence, $P_0^{(1)} - P_0^{(1)}(f) \in \mathcal{I}$, $P_0^{(1)}(f) \neq 0$ and

$$P_{I_1 \cup \{k\}, \mu}^{J_1 \cup \{\alpha\}} \in \mathcal{I}$$

for $\mu \in \{0, 1\}$. By (18) and Remark 8, $P_\mu \in \mathcal{I}$.

**Point 2b.** $|J_1| = |J| - 1$. In this case $I = I_1 \cup \{k\}$ and $k = \min I$. As above $t = \max J$.

If $b < t$, then we consider $J_2 = \{j \in J : 1 \leq j < b\}$ and $I_2 = \sigma(J_2)$. We write down (9) for subsets $I_2 \subset I$, $J_2 \subset J$ and $S = \{b\}$. We conclude the proof similar to Point 2a.

Let $b = t$. It follows $J = J(k, t)$, $I = I(k, t)$. If $k > t$, then the square $(k, t)$ is filled by symbol $\bullet$ in the admissible diagram; the square $(k, t)$ has type 0. The minor $M_I^J(\tau)$ is a homogeneous polynomial in $\tau$. We have $P_0 = Q_{k, t} \in \mathcal{I}$ and $P_1 = 0$. If $k \leq t$, then the square $(i, t)$, where $i = \sigma(k)$, is filled by symbol $\bullet$ in the admissible diagram; the square $(i, t)$ has type 1. This proves $P_1 = Q_{i, t} \in \mathcal{I}$.

To prove that $P_0 \in \mathcal{I}$ we recall that the pair $I_1, J_1$ obeys the induction assumption. Denote $(I_* = I_1 \setminus \{t\}) \sqcup \{k\}$. Note $I_* > I_1$ and $P_0 = P_{I_*, 0}^{J_1}$. The induction assumption concludes the proof. □

**Proposition 3.7.**
\( \mathcal{I} \) is a prime ideal and \( \dim(\text{Ann}\mathcal{I}) = \dim \Omega(f) \).

**Proof.** Consider the linear order in the set of squares \( \{(i, t) : t < i\} \) of admissible diagram such that that

1) if \( t_1 < t \), then \( (i_1, t_1) < (i, t) \);
2) if the square \( (i, t) \) is filled by symbol \( \bullet \) or symbol \( \otimes \), then \( (i, t) > (i_1, t) \) for any square \( (i_1, t) \) filled by symbol \( + \) or \( - \).
3) \( (i_1, t) < (i, t) \) if \( i_1 > i \) in any other case that do not mentioned in 1) and 2).

We shall construct a system of elements \( \{\tilde{y}_{it} \mid i > t\} \) such that

i) any \( \tilde{y}_{it} \) has the form \( \tilde{y}_{it} = y_{it} + v_{it} \) where \( v_{it} \) lies in subalgebra in \( S(n) \) generated by \( y_{i_1,t_1} \), \( (i_1, t_1) < (i, t) \);
ii) the ideal \( \mathcal{I} \) is generated by elements \( \tilde{y}_{it} - \tilde{y}_{it}(f) \), where the square \( (i, t) \) filled by symbol \( \otimes \), and \( \tilde{y}_{it} \), where he square \( (i, t) \) is filled by symbol \( \bullet \).

The condition i) implies that the system of elements \( \{\tilde{y}_{it} \mid i > t\} \) generate the symmetric algebra \( S(n) \) and ii) implies the factor algebra of \( S(n) \) over the ideal \( \mathcal{I} \) is isomorphic to the algebra of polynomials generated by \( \{y_{it}\} \), where the square \( (i, t) \) is filled by + or – in the admissible diagram. This proves that the ideal \( \mathcal{I} \) is prime and \( \dim(\text{Ann}\mathcal{I}) \) is equal to the number of ±’s in the admissible diagram. By Remark 4, \( \dim(\text{Ann}\mathcal{I}) = \dim \Omega(f) \).

So it suffices to construct the required system of elements \( \{\tilde{y}_{it} \mid i > t\} \) that obey i) and ii). First, we put \( \tilde{y}_{it} = y_{it} \) if the square \( (i, j) \) is filled by symbol \( + \) or \( - \).

Suppose that the square \( (i, t) \) is filled by symbol \( \otimes \) or symbol \( \bullet \) of type 0. Decomposing \( D_{i,t} \) by the column \( t \), we obtain \( D_{k,t} = Ay_{it} + u_{it} \), where \( u_{it} \) as i) and \( A \) equals to some nonzero constant \( a \) modulo \( \text{mod}\mathcal{I} \) (see Lemma 3.6(1)). Put \( \tilde{y}_{it} = y_{it} + a^{-1}u_{it} \).

Suppose that the square \( (i, t) \) is filled by symbol \( \bullet \) and has type 1. As in Remark 5 we denote \( k = \sigma(i) < t \). We preserve the notations (5). Using (6) and (7) we have got (8). Hence,

\[
Q_{i,t} = \pm \mathcal{M}^{J_0 \sqcup \{t\}}_{I_0 \sqcup \{t\}} + u_{it} \tag{19}
\]

where \( u_{it} \) lies in the subalgebra generated by \( y_{cd}, \ d < t \). Let as in Lemma 3.1 \( a = \sigma(t) \). Recall \( J_0 = \{a\} \sqcup \Lambda_1 \sqcup (\Lambda_2 \setminus k) \), \( I_0 = \sigma \Lambda_1 \sqcup \sigma \Lambda_2 \). Decompose \( \Lambda_2 = \Lambda_{2,0} \sqcup \Lambda_{2,1} \), where

\[
\Lambda_{2,0} = \{j \in \Lambda_2 : 1 \leq j < a\}, \quad \Lambda_{2,1} = \{j \in \Lambda_2 : a < j < t\}.
\]

Let \( \Lambda_{2,1} = \{k_1, \ldots, k_p\} \) and \( \sigma \Lambda_{2,1} = \{i_1, \ldots, i_p\} \) where each \( i_b = \sigma(k_b) \). By definition, \( k \in \Lambda_{2,1} \) and \( i \in \sigma \Lambda_{2,1} \). Note that all squares \( \{i_m, t\} \) are filled by symbol \( \bullet \) and have type 1. For convenience of reader we put just here
Example 4. The diagram below is the $10 \times 10$ admissible diagram of involution \( \sigma = (1,10)(2,5)(3,7)(4,9)(6,8) \). We put number of columns in the squares of first row and the numbers of of rows in the first column. Here we write decompositions for \( t = 7, \ i = 9 \). Hence, \( a = 3 \) and \( k = 4 \).

\[ \begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & & & & & & & & & \\
2 & + & & & & & & & & \\
3 & + & + & & & & & & & \\
4 & + & + & + & & & & & & \\
5 & + & \otimes & - & - & & & & & \\
6 & + & \bullet & + & + & \bullet & & & & \\
7 & + & \bullet & \otimes & - & \bullet & - & & & \\
8 & + & \bullet & \bullet & + & \bullet & \otimes & \bullet & & \\
9 & + & \bullet & \bullet & \otimes & \bullet & - & \bullet & - & \\
10 & \otimes & - & - & - & - & - & - & - & - \\
\end{array} \]

\( \Lambda_1 = \Lambda_{10} = \{2\}, \ \sigma \Lambda_{10}, = \{5\} \)

\( \Lambda_{20} = \{1\}, \ \sigma \Lambda_{20} = \{10\} \),

\( \Lambda_{21} = \{6\}, \ \sigma \Lambda_{20} = \{8\} \),

\( J_0 = \{1,2,3,7\}, \ I_0 = \{5,8,9,10\} \),

\( M_{J_0}^I = M_{5,8,9,10}^{1,2,3,6} \).

\( Q_{9,7} = -M_{5,7,8,9,10}^{1,2,3,6,7} - M_{4,5,8,9,10}^{1,2,3,4,6} \)

The last formula for \( Q_{9,7} \) is the decomposition (19) for \( i = 9 \) and \( t = 7 \). End of Example.

We continue the proof. Applying the formula (9) with \( S = \Lambda_{1,0} \sqcup \{a\} \) for the matrix \( \Phi \) we obtain

\[
M_{\sigma \Lambda_{2,0}}^\Lambda A_{2,0} M_{I_0 \sqcup \{t\}}^J = \sum_{T \subset \sigma \Lambda_{1,0} \sqcup \sigma \Lambda_{2,1} \sqcup \{t\}} \pm M_{\sigma \Lambda_{2,0} \sqcup T}^\Lambda A_{2,0} M_{I_0 \sqcup \{t\} \setminus T}^J (20)
\]

Since \( t \) is greater than any number of \( \sigma \Lambda_{1,0} \) and smaller than any number of \( \sigma \Lambda_{2,1} \), then all minors

\( M_T := M_{\sigma \Lambda_{2,0} \sqcup \{a\}}^\Lambda A_{2,0} \)

for \( T \subset \sigma \Lambda_{1,0} \sqcup \sigma \Lambda_{2,1} \sqcup \{t\} \) and \( T \neq \sigma \Lambda_{1,0} \sqcup \{t\} \) lie in the ideal \( \mathcal{I} \). We obtain

\[
M_{\sigma \Lambda_{2,0}}^\Lambda A_{2,0} M_{I_0 \sqcup \{t\}}^J = \pm M_{\sigma \Lambda_{2,0} \sqcup \sigma \Lambda_{1,0} \sqcup \{t\} \sqcup \{a\}}^\Lambda A_{2,0} M_{\sigma \Lambda_{2}}^{(A_2 \setminus \{k\}) \sqcup \{t\}} \text{ mod } \mathcal{I}. \quad (21)
\]

The first minors in left and right sides in (21) are nonzero constants modulo the ideal \( \mathcal{I} \) (see Lemma 3.6(1)). We have

\[
M_{I_0 \sqcup \{t\}}^J = \text{const} \cdot M_i, \quad \text{where } M_i = M_{\sigma \Lambda_2}^{(A_2 \setminus \{k\}) \sqcup \{t\}} \text{ mod } \mathcal{I}. \quad (22)
\]

Substituting (22) for (19) we have got

\[
Q_{it} = c_{i,t} M_i + u_{i,t} \text{ mod } \mathcal{I} \quad \text{where } c_{i,t} \in K^*.
\]
Then
\[ c_{i,t}^{-1}Q_{i,t} = M_i + c_{i,t}^{-1}u_{i,t}. \] (23)

Decomposing \( M_i \) by last \( t \)th column we obtain
\[ M_i = \sum_{r=1}^{p} A_i^{(r)} y_{i,r,t} + v_{i,t} \] (24)

where \( v_{i,t} \) lies in subalgebra generated by \( y_{c,d}, \ d < t, \) and \( y_{c,t}, \) where the square \( (c,t) \) is filled by symbol \(-\). Therefore, \( v_{i,t} \) lies in subalgebra generated by \( y_{c,d}, \) where \( (c,d) \) is smaller in the sense of order defined in the beginning of this proof.

Note that the matrix 
\( A = \det(A_{im}^{(r)})_{r,m=1}^{p} \)

is invertible modulo ideal \( \mathcal{I} \) (see Lemma 3.6 and formula (12)). Denote \( B = (B_m^{(r)}) \) the inverse matrix for \( A. \) After, we write down (23) and (24) for \( i = i_m, \ 1 \leq m \leq p, \) substitute (24) in (23). Further, multiply (23) by \( B_{s}^{(q)} \) and summarize:
\[ \sum_{q=1}^{p} B_m^{(q)} c_{i,q,t}^{-1} Q_{i,q,t} = y_{i,m,t} + w_{i,m,t} \in \mathcal{I}. \]

Finally, put \( \tilde{y}_{i,t} = y_{i,t} + w_{i,t} \) for any \( i \in \sigma \Lambda_{21} = \{ i_1, \ldots, i_p \}. \) The system \( \{ \tilde{y}_{i,t} \} \) obey i), ii). \( \Box \)

**Proposition 3.8** The ideal \( \mathcal{I} \) in \( S(\mathfrak{n}) \) is stable with respect to the adjoint action of group \( N. \)

**Proof.** The statement of Proposition if equivalent to claim that \( \mathcal{I} \) is a Poisson ideal in \( S(\mathfrak{n}) \) (with respect to natural Poisson bracket). Let \( y = y_{p+1,p} \) where \( 1 \leq p \leq n - 1. \) It suffices to prove that \( \{ y, P \} \in \mathcal{I} \) for any standard generator \( P. \) Recall that, by definition, \( \mathcal{I} \) is generated by \( Q_{i,t}, \) \( \varepsilon_t - \varepsilon_i \in \mathcal{M}, \) and \( D_m - D_m(f), \) \( 1 \leq m \leq s. \)

**Case 1.** \( P \) is equal to \( D_m \) or \( Q_{i,t}, \) where square \( (i,t) \) is filled by \( \bullet \) and has type 0. Then \( P \) is a minor \( M^J_I \) of the matrix \( \Phi; \) squares \( I \times J \) are placed under the diagonal and \( I \supseteq \sigma(J). \) Easy to see that
\[ \{ y, M^J_I \} = \epsilon_1 M^J_{I*} - \epsilon_2 M^J_{I'}, \] where
\[ \epsilon_1 = \begin{cases} 1, & \text{if } p \in I, \ p + 1 \notin I, \\ 0, & \text{otherwise.} \end{cases} \quad \epsilon_2 = \begin{cases} 1, & \text{if } p + 1 \in J, \ p \notin J, \\ 0, & \text{otherwise.} \end{cases} \]
\[ I_* = (I \setminus \{ p \}) \cup \{ p + 1 \}, \quad J_* = (J \setminus \{ p + 1 \}) \cup \{ p \}. \]
One can see that $I_*>I\geq \sigma(J)$ and $I>\sigma(J_*)$. By Lemma 3.2, minors $M^J_{I_*}, M^J_I$ lie in $\mathcal{I}$. Hence, $\{y, M^J_I\} \in \mathcal{I}$.

**Case 2.** Suppose that $P = Q_{it}$ and the square $(i, t)$ is filled by $\bullet$ and has type 1.

**Point 2a.** Denote $[t] = \{1, \ldots, t\}$ and $P_\mu = P_{I_{\mu}}^{[t]}$ where $\mu \in \{0, 1\}$. By Lemma 3.6, $P_\mu, \mu \in \{0, 1\}$, lies in the ideal $\mathcal{I}$. We shall show in this point that

$$\{y, P_\mu\} \in \mathcal{I}, \mu \in \{0, 1\}.$$  

More precisely,

$$\begin{align*}
\{y, P_0\} &= c_0(P_*)_0, \\
\{y, P_1\} &= c_1(P_*)_1 + c_2yP_0,
\end{align*}$$  

(25)

(26)

where $c_0, c_1, c_2 \in \{0, 1, -1\}$, $P_\mu = P_{I_{\mu}}^{[t]}$, $I_* = (I \setminus \{p\}) \cup \{p + 1\}$. In (25) and (26) $c_0 = c_1 = 0$ if the condition $p \in I$, $p + 1 \notin I$ is not true.

The proof is proceeded separately in each of 12 cases, concerning entry of $p$, $p + 1$ in $I$, $[t]$ (note that the case $p \notin [t]$, $p + 1 \in [t]$ is not possible). In other cases proof is similar.

Here we present the complete proof of (25) and (26) in the case $p \in I$, $p + 1 \notin I$, $p, p + 1 \in [t]$.

Suppose that $I$ is an ordered increasing subset. Denote $\Lambda = [t]$, $\Lambda_0 = \Lambda \setminus I$, $I_0 = I \setminus \Lambda$.

The polynomial $P_0$ coincides with the minor $M^{\Lambda_0}_{I_0}$. Since $p, p + 1 \notin I_0$ and $p + 1 \in \Lambda_0$, $p \notin \Lambda_0$, then

$$\{y, P_0\} = -M^{(\Lambda_0)*}_{I_0}, \text{ where } (\Lambda_0)* = (\Lambda_0 \setminus \{p + 1\}) \cup \{p\}.$$  

The minor $M^{(\Lambda_0)*}_{I_0}$ coincides with $(P_*)_0$.

Denote $F_0 = \Lambda \cap I$. By assumptions, $p \in F_0$, $p + 1 \notin F_0$. We have

$$P_1 = \sum_{\alpha \in F_0} \epsilon(\alpha)M_{I_0 \uplus \{\alpha\}}^{\Lambda_0 \uplus \{\alpha\}} = \epsilon(p)M_{I_0 \uplus \{p\}}^{\Lambda_0 \uplus \{p\}} + \sum_{\alpha \in F_0, \alpha \neq p} \epsilon(\alpha)M_{I_0 \uplus \{\alpha\}}^{\Lambda_0 \uplus \{\alpha\}}.$$  

Here $\epsilon(\alpha) = (-1)^{i+j}$ where $i$ equals to the index of $\alpha$ in $\Lambda$ and $j$ equals to the index of $\alpha$ in $I$. By direct calculations we have got

$$\begin{align*}
\{y, P_1\} &= \epsilon(p) \left(M^{\Lambda_0 \uplus \{p\}}_{I_0 \uplus \{p+1\}} \pm yM^{J_0}_{I_0} \right) - \sum_{\alpha \in F_0, \alpha \neq p} \epsilon(\alpha)M^{(\Lambda_0 \setminus \{p+1\}) \uplus \{p, \alpha\}}_{I_0 \uplus \{\alpha\}}. \\
&= \epsilon(p + 1)M^{\Lambda_0 \uplus \{p\}}_{I_0 \uplus \{p+1\}} + \sum_{\alpha \in F_0, \alpha \neq p} \epsilon'(\alpha)M^{(\Lambda_0 \setminus \{p+1\}) \uplus \{p, \alpha\}}_{I_0 \uplus \{\alpha\}}.
\end{align*}$$  

(27)

On the other hand,

$$\begin{align*}
(P_*)_1 &= \epsilon'(p + 1)M^{\Lambda_0 \uplus \{p\}}_{I_0 \uplus \{p+1\}} + \sum_{\alpha \in F_0, \alpha \neq p} \epsilon'(\alpha)M^{(\Lambda_0 \setminus \{p+1\}) \uplus \{p, \alpha\}}_{I_0 \uplus \{\alpha\}}.
\end{align*}$$  

(28)
Here \( \epsilon'(\alpha) = (-1)^{i'+j'} \) where \( i' \) equals to the index of \( \alpha \) in \( \Lambda \) and \( j' \) equals to the index of \( \alpha \) in \( I_\ast = (I \setminus \{p\}) \cup \{p + 1\} \). Similarly, \( \epsilon'(p + 1) = (-1)^{i'+j'} \) where \( i' \) equals to the index of \( p + 1 \) in \( \Lambda \) and \( j' \) equals to the index of \( p + 1 \) in \( I_\ast \). Comparing (27) and (28), we conclude that \( \epsilon(p) = -\epsilon'(p + 1) \) and \( \epsilon(\alpha) = \epsilon'(\alpha) \) for \( \alpha \neq p, \alpha \in F_0 \). We obtain

\[
\{y, P_1\} = -(P_\ast)_1 \pm yP_0.
\]

This proves (26).

**Point 2b.** We finish the proof of Case 2. Denote as above \( k = \sigma(i) \). Denote \( J_1 = J'(k, t), \quad I_1 = I'(k, t), \quad J(k, t) = J_1 \cup \{t\}, \quad I(k, t) = I_1 \cup \{k\} \) (see §1). Recall that \( Q_{i,t} \) is the second coefficient of minor with columns \( J(k, t) \) and rows \( I(k, t) \) of the matrix \( \Phi(\tau) \). That is

\[
Q_{i,t} = P^J_{i,t}. 
\]

Apply (9) for \( J_1 \subset [t], \quad I_1 \subset I = \sigma[t] \cup \{k\} \) and \( S = \{t\} \):

\[
M_{I_1}^{J_1}(\tau)M_{I}^{[t]}(\tau) = \sum_{\beta \supset k, \beta \in I} \epsilon(\beta)M_{I_1 \cup \{\beta\}}^{J_1 \cup \{t\}}(\tau)M_{I_1 \setminus \{\beta\}}^{[t-1]}(\tau) = \\
\epsilon(k)M_{I_1 \cup \{k\}}^{J_1 \cup \{t\}}(\tau)M_{\sigma[t-1]}^{[t-1]}(\tau) + \sum_{\beta > k, \beta \in I} \epsilon(\beta)M_{I_1 \cup \{\beta\}}^{J_1 \cup \{t\}}(\tau)M_{I_1 \setminus \{\beta\}}^{[t-1]}(\tau).
\]

Denote \( m_1 = |I_1| = |J_1|, \quad m = |I| = |J|, \quad q_1 = |I_1 \cap J_1|, \quad q = |I \cap J| \).

According Lemma 3.5, decomposition (29) is an admissible decomposition. This means that the lower degree of each summand in the right side of (29) is greater than the lower degree of the left side.

We show that the lower degree of the first term of the right side coincides with the lower degree of the left side. Indeed

\[
\text{ldeg} \left( M_{I_1}^{J_1}(\tau)M_{I}^{[t]}(\tau) \right) = \text{ldeg} \left( M_{I_1}^{J_1}(\tau) \right) + \text{ldeg} \left( M_{I}^{[t]}(\tau) \right) = (m_1 - q_1) + (m - q), \]

\[
\text{ldeg} \left( M_{I_1 \cup \{k\}}^{J_1 \cup \{t\}}(\tau)M_{\sigma[t-1]}^{[t-1]}(\tau) \right) = \text{ldeg} \left( M_{I_1 \cup \{k\}}^{J_1 \cup \{t\}}(\tau) \right) + \text{ldeg} \left( M_{\sigma[t-1]}^{[t-1]}(\tau) \right) = \\
[(m_1 + 1) - (q_1 + 2)] + [(m - 1) - (q - 2)] = (m_1 - q_1) + (m - q).
\]

This prove coincidence of lower degree. By Lemma 3.6, the first coefficients of \( M_{I_1}^{J_1}(\tau) \) and \( M_{\sigma[t-1]}^{[t-1]}(\tau) \) are invertible modulo \( \mathcal{I} \). Comparing the coefficients of (29) we have got formulas of type (13) and (14). Formula (14) provides an expression of \( Q_{i,t} \) as a sum of polynomials of type \( P_{I_\ast \mu}^{[t]} \) and \( P_{I_\ast \mu}^{[t-1]} \), where \( I > \sigma[t] \) and \( I' > \sigma[t - 1] \), with coefficients in \( S(n) \). Applying Point 1a), we conclude \( \{y, Q_{i,t}\} \in \mathcal{I} \). □
The proof of Theorem 1.4 follows directly from Propositions 3.1, 3.7 and 3.8.

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