Gauged String Actions and $O(\tilde{d}, \tilde{\bar{d}})$ Transformation

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Abstract

An $O(\tilde{d}, \tilde{\bar{d}})$ transformation is given which relates the background configuration of the ungauged string action to the gauged ones for a large class of models discussed recently by Giveon and Rocek. Interestingly, the transformation is background independent and has a unique matrix representation in a given space-time dimension.

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Last year, in a series of papers, it was shown by Meissener and Veneziano[1], Sen and Hassan[2, 3] that the string effective action in any space-time dimension is invariant under an $O(\tilde{d}, \tilde{d})$ group of symmetry transformations when the background fields are independent of $\tilde{d}$ of the space-time coordinates. Such transformations generate new classical solutions from a given one, and have been conjectured to be a generalization of the Narain construction[4] to the curved background. Indeed, they have been a useful tool for generating several interesting nontrivial solutions, both the cosmological[5] as well as the static ones with curvature singularity, such as rotating black holes[6], charged branes[2, 3] etc.

On the other hand, the space-time geometry of the gauged WZW models has also been analyzed intensively since last year and gives an exact conformal field theory representation of many interesting target space structures such as two dimensional black holes[7, 8, 9, 10], three dimensional charged black string[11, 12] and many other solutions[13]. In this context, in order to formulate duality in curved backgrounds [14], a large class of string actions and their abelian gauging was recently studied by Giveon and Rocek[15]. The gauged string action of ref.[15] corresponds to the most general background configuration in a given space-time dimension D which is independent of d of the coordinates. The corresponding ungauged string action is the (D+d)-dimensional one with its background parameterized by the elements of the backgrounds of the gauged action. It has been pointed out in ref.[15] that
the two dimensional black hole as well as some other interesting classical solutions belong to this class of backgrounds. We believe that, being a large class, many more interesting classical solutions are likely to fall into this category of models.

In this paper, we show an intimate connection between the two approaches, namely the $O(\tilde{d}, \tilde{d})$ transformation [1,2,3] and the abelian gauging [15], used for the construction of string backgrounds. We give an $O(\tilde{d}, \tilde{d})$ transformation, with $\tilde{d} = 2d$ in ref.[15], which relates the ungauged backgrounds to the gauged ones for the complete class of models studied in ref.[15]. An interesting aspect of the result is that the corresponding $O(\tilde{d}, \tilde{d})$ transformation is represented by a unique (D+d)-dimensional matrix for the whole class of models. Under this transformation, the background configuration of the ungauged string action transforms to a direct product of the one corresponding to the gauged action and a set of free scalar fields whose number is same as that of the gauged U(1) currents. These transformations can therefore be interpreted as a deformation from a conformal field theory $[\mathcal{F}]$, described by the ungauged action of ref.[15], to another one $[\mathcal{F}/U(1)^{d}] \times U(1)^{d}$. The background independence of the transformation may have relation with the characteristic features of the operator responsible for such a deformation in the corresponding CFT.

We start by reviewing the relevant aspects of ref.[15]. It was shown in ref.[15] that the string action in a space-time dimension D, with most general
background independent of $d$ of the coordinates, can be constructed by gauging a class of $(D+d)$-dimensional action with respect to its $U(1)^d_L \times U(1)^d_R$ conserved currents. The ungauged action is written as

$$S_{D+d} = \frac{1}{2\pi} \int d^2 z \left[ (E_{D+d})_{IJ} \partial X^I \partial X^J - \frac{1}{4} \phi_{D+d} R^{(2)} \right]$$

where $X^I \equiv (x^a, \theta^i_1, \theta^i_2)$, $a = 1...(D-d)$ and $i= 1...d$. The background $E_{IJ}(x)$ depends only on coordinates $x^a$ and is written as,

$$E_{D+d} = (G_{D+d} - B_{D+d}) = \begin{pmatrix} \Gamma & \Gamma_1^T & 0 \\ 0 & I_d & -I_d \\ \Gamma_2 & 2\Sigma + I_d & I_d \end{pmatrix},$$

where $\Gamma$, $\Sigma$ and $\Gamma_{1,2}$ are $(D-d) \times (D-d)$, $d \times d$ and $d \times (D-d)$ dimensional $x^a$ dependent matrices, respectively. Subscript of the Identity matrix in eqn.(2) and in the rest of the paper denotes its dimension. The $(D+d)$-dimensional background is, therefore, parameterized by four matrices $\Gamma$, $\Gamma_{1,2}$, and $\Sigma$. The $G_{D+d}, B_{D+d}$ and $\phi_{D+d}$ are the background metric, antisymmetric tensor and dilaton fields such that the action (1) is conformal invariant. Therefore, these fields satisfy the equations of motion of the string effective action in $(D+d)$-dimensions.

It has already been observed in ref.[15] that a constant shift in the antisymmetric tensor in $(\theta_1, \theta_2)$ space corresponds to the addition of a total derivative in the action (1). We will later on also show that such shifts can be obtained by $O(\tilde{d}, \tilde{d})$ transformation. Taking these facts into account, we
ignore a shift of this type in the expression for $E_{D+d}$ in eqn.(2) and write $G_{D+d}$ and $B_{D+d}$ as,

$$G_{D+d} = \begin{pmatrix} \Gamma_s & \frac{1}{2} \Gamma_1^T & \frac{1}{2} \Gamma_2^T \\ \frac{1}{2} \Gamma_1 & I_d & \Sigma^T \\ \frac{1}{2} \Gamma_2 & \Sigma & I_d \end{pmatrix},$$  \hspace{1cm} (3)$$

and

$$B_{D+d} = \begin{pmatrix} \Gamma_a & -\frac{1}{2} \Gamma_1^T & \frac{1}{2} \Gamma_2^T \\ \frac{1}{2} \Gamma_1 & 0 & \Sigma^T \\ -\frac{1}{2} \Gamma_2 & -\Sigma & 0 \end{pmatrix},$$  \hspace{1cm} (4)$$

where $\Gamma_{s,a} = \pm \frac{1}{2}(\Gamma \pm \Gamma^T)$. The action (1) is invariant under $U(1)^d_L \times U(1)^d_R$ transformations. By gauging these symmetries and integrating out the gauge fields one obtains

$$S_D = \frac{1}{2\pi} \int d^2z \left[ (E_D)_{AB} \partial X^A \partial X^B - \frac{1}{4} \phi_D R(2) \right]$$  \hspace{1cm} (5)$$

where $E_D \equiv (G_D - B_D)$, $X^A \equiv (x^a, \theta^i)$ and $\theta^i = (\theta_1^i - \theta_2^i)$. The resulting D-dimensional background metric ($G_D$), antisymmetric tensor ($B_D$) and dilaton ($\phi_D$) are [15]:

$$G_D = \begin{pmatrix} \Gamma_s - \frac{1}{4} \Gamma_1^T (I_d + \Sigma)^{-1} \Gamma_2 & \frac{1}{2} \Gamma_1^T (I_d + \Sigma)^{-1} \\ -\frac{1}{4} \Gamma_2^T (I_d + \Sigma)^{-1} \Gamma_1 & -\Gamma_2^T (I_d + \Sigma)^{-1} \end{pmatrix}$$  \hspace{1cm} (6)$$

\begin{pmatrix} \frac{1}{2} [(I_d + \Sigma)^{-1} \Gamma_1 & \frac{1}{2} [(I_d - \Sigma)(I_d + \Sigma)^{-1} \\ -\Gamma_2^T (I_d + \Sigma)^{-1} \Gamma_1 & + (I_d + \Sigma)^{-1} (I_d - \Sigma^T) \end{pmatrix}$$
\[ B_D = \begin{pmatrix} \frac{1}{2} \Gamma_1^T (I_d + \Sigma)^{-1} \Gamma_2 \\ -\frac{1}{2} \Gamma_2^T (I_d + \Sigma^T)^{-1} \Gamma_1 \\ \frac{1}{2} [(I_d + \Sigma^T)^{-1} \Gamma_1 \\ +(I_d + \Sigma)^{-1} \Gamma_2 ] \end{pmatrix}, \tag{7} \]

and

\[ \phi_D = \phi_{D+d} + \log \det (I_d + \Sigma). \tag{8} \]

It has been pointed out\(^{[15]}\) that, \(G_D, B_D\) and \(\phi_D\) obtained in this manner are consistent D-dimensional string backgrounds.

We now come to the main point of the paper. In this paper, we show that there is an \(O(\tilde{d}, \tilde{d})\) transformation \(^{[1,2,3]}\), represented by the \(2(D+d) \times 2(D+d)\) dimensional matrix:

\[ \Omega \equiv \begin{pmatrix} I_{D-d} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_d & -I_d & 0 & -I_d & -I_d \\ 0 & I_d & I_d & 0 & I_d & -I_d \\ 0 & 0 & 0 & I_{D-d} & 0 & 0 \\ 0 & -I_d & -I_d & 0 & I_d & -I_d \\ 0 & I_d & -I_d & 0 & I_d & I_d \end{pmatrix}, \tag{9} \]

which is the symmetry of the \((D+d)\)-dimensional string effective action and transforms the background fields, \((G_{D+d}, B_{D+d})\) to \((\tilde{G}_{D+d}, \tilde{B}_{D+d})\) respectively, where

\[ \tilde{G}_{D+d} = \begin{pmatrix} G_D & 0 \\ 0 & I_d \end{pmatrix}, \tag{10} \]

\[ \tilde{B}_{D+d} = \begin{pmatrix} B_D & 0 \\ 0 & 0 \end{pmatrix}. \tag{11} \]

We also show that the dilaton \(\phi_{D+d}\) transforms under the above \(O(\tilde{d}, \tilde{d})\) transformation as

\[ \phi_{D+d} \to \phi_D. \tag{12} \]
Therefore, as stated earlier, under the above $O(d, d)$ transformation the backgrounds $(G_{D+d}, B_{D+d}, \phi_{D+d})$ for the ungauged string action transform to the gauged one $(G, B, \phi)$ together with a set of free scalars. It is remarkable to note that a background independent $\Omega$ in eqn.(9) transforms the nontrivial expressions of eqns.(3-4) and (6-7) into each other, especially since $O(d, d)$ does not act linearly on the background graviton and antisymmetric tensor fields.

$\Omega$ in eqn.(9) satisfies the condition, $\Omega^T \eta \Omega = \eta$, where

$$\eta = \begin{pmatrix} 0 & I_{D+d} \\ I_{D+d} & 0 \end{pmatrix}.$$  \hspace{1cm} (13)$$

Its action on $(G_{D+d}, B_{D+d})$ is given by a linear transformation on a matrix $M$:

$$\Omega M_{D+d} \Omega^T = \tilde{M}_{D+d},$$  \hspace{1cm} (14)$$

where

$$M_{D+d} = \begin{pmatrix} G^{-1}_{D+d} & -G^{-1}_{D+d}B_{D+d} \\ B_{D+d}G^{-1}_{D+d} & G_{D+d} - B_{D+d}G^{-1}_{D+d}B_{D+d} \end{pmatrix},$$  \hspace{1cm} (15)$$

and $\tilde{M}_{D+d}$ in eqn.(14) is a matrix of the same form as $M_{D+d}$, with $(G_{D+d}, B_{D+d})$ replaced by $(\tilde{G}_{D+d}, \tilde{B}_{D+d})$. The dilaton field $\phi_{D+d}$ transforms under this transformation to [1,2,3]

$$\tilde{\phi}_{D+d} = \phi_{D+d} + \frac{1}{2} \log \left( \frac{\det G_{D+d}}{\det G} \right).$$  \hspace{1cm} (16)$$

and it will be shown later on that $\tilde{\phi}_{D+d} = \phi_D$.

The proof that the transformation $\Omega$ in eqn.(9) is a symmetry of the $(D + d)$- dimensional string effective action follows directly from the results
of ref.[3]. By setting the gauge field backgrounds in ref.[3] to zero, Ω in eqn.(9) is easily seen to belong to the set of symmetry transformations given in eq.(3.21) of ref.[3], when one takes into account that η in ref.[3] has a different form than ours.

Our major task now is to verify eqn.(14). Algebraic complexity in this calculation arises mainly due to the complicated forms of $G^{-1}_{D+d}$ and $G^{-1}_D$ as well as other elements of the matrices $M_{D+d}$, and $\tilde{M}_{D+d}$ in terms of $\Gamma$, $\Gamma_1$, $\Gamma_2$ and $\Sigma$. For example,

$$G^{-1}_{D+d} = \begin{pmatrix} x & y^T & z^T \\ y & A & B^T \\ z & B & C \end{pmatrix},$$

(17)

where

\begin{align*}
    x &= x^T = \left[ \Gamma_s - \frac{1}{4}\Gamma^T_{2}\Gamma_2 - \frac{1}{4}\Gamma^T_{1}(I_d - \Sigma^T\Sigma)^{-1}\Gamma_1 - \frac{1}{4}\Gamma^T_{2}\Sigma(I_d - \Sigma^T\Sigma)^{-1}\Sigma^T\Gamma_2 \\
    &+ \frac{1}{4}\Gamma^T_{1}(I_d - \Sigma^T\Sigma)^{-1}\Sigma^T\Gamma_2 + \frac{1}{4}\Gamma^T_{2}\Sigma(I_d - \Sigma^T\Sigma)^{-1}\Gamma_1 \right]^{-1} \\
    y &= \frac{1}{2}\left[ -(I_d - \Sigma^T\Sigma)^{-1}\Gamma_1 + (I_d - \Sigma^T\Sigma)^{-1}\Sigma^T\Gamma_2 \right]x \\
    z &= \frac{1}{2}\left[ \Sigma(I_d - \Sigma^T\Sigma)^{-1}\Gamma_1 - \Gamma_2 - \Sigma(I_d - \Sigma^T\Sigma)^{-1}\Sigma^T\Gamma_2 \right]x \\
    A &= (I_d - \Sigma^T\Sigma)^{-1} + yx^{-1}y^T \\
    B &= -\Sigma(I_d - \Sigma^T\Sigma)^{-1} + zx^{-1}y^T \\
    C &= I_d + \Sigma(I_d - \Sigma^T\Sigma)^{-1}\Sigma^T + zx^{-1}z^T
\end{align*}

(18)

$G^{-1}_{D+d}$ in eqn.(17) has been obtained by using a general inversion formula for a matrix of the type:

$$D = \begin{pmatrix} P & R^T \\ R & Q \end{pmatrix},$$

(19)
then,

\[
D^{-1} = \begin{pmatrix} (P - R^T Q^{-1} R)^{-1} & -(P - R^T Q^{-1} R)^{-1} R^T Q^{-1} \\ -Q^{-1} R(P - R^T Q^{-1} R)^{-1} & Q^{-1} + Q^{-1} R(P - R^T Q^{-1} R)^{-1} R^T Q^{-1} \end{pmatrix}
\]

(20)

Using the expressions for \( G_{D+d} \), \( B_{D+d} \) and \( G_{D+d}^{-1} \) in equations (3), (4) and (17), the matrix \( M_{D+d} \) in eqn.(15) can be written explicitly in terms of matrices \( \Gamma_{s,a}, \Gamma_{1,2}, \Sigma, x, y, z, A, B, \) and \( C \). The expression for \((G_{D+d}^{-1} B_{D+d})\) is given as,

\[
-(G_{D+d}^{-1} B_{D+d}) = \begin{pmatrix} -x \Gamma_a - \frac{1}{2} y^T \Gamma_1 + \frac{1}{2} z^T \Gamma_2 & \frac{1}{2} x \Gamma_1^T + z^T \Sigma & -\frac{1}{2} x \Gamma_2^T - y^T \Sigma^T \\ -y \Gamma_a - \frac{1}{2} A \Gamma_1 + \frac{1}{2} B^T \Gamma_2 & \frac{1}{2} y \Gamma_1^T + B^T \Sigma & -\frac{1}{2} y \Gamma_2^T - A \Sigma^T \\ -z \Gamma_a - \frac{1}{2} B \Gamma_1 + \frac{1}{2} C \Gamma_2 & \frac{1}{2} z \Gamma_1^T + C \Sigma & -\frac{1}{2} z \Gamma_2^T - B \Sigma^T \end{pmatrix}
\]

(21)

An explicit expression for the matrix \((G_{D+d} - B_{D+d} G_{D+d}^{-1} B_{D+d})\) can be obtained in a similar manner. Since it is straightforward to obtain and not convenient to write in a compact form, we do not give it here.

For further calculations, we find it convenient to use a set of identities which are obtained from the requirements,

\[
G_{D+d} G_{D+d}^{-1} = G_{D+d}^{-1} G_{D+d} = I_{D+d},
\]

(22)

where \( G_{D+d} \) and \( G_{D+d}^{-1} \) are given in eqns.(3) and (17) respectively.

Then, the left hand side of eqn. (14) is computed by using the expression of \( \Omega \) from eqn.(9) and \( M_{D+d} \). The calculation is rather long and tedious. For simplifications we use the identities obtained from eqn.(22). Finally, for the
left hand side of eqn.(14) one finds:

\[
\Omega M_{D+d} \Omega^T \equiv 
\begin{pmatrix}
(\Omega M \Omega^T)_{11} & (\Omega M \Omega^T)_{12} \\
((\Omega M \Omega^T)_{12})^T & (\Omega M \Omega^T)_{22}
\end{pmatrix},
\]

(23)

where

\[
(\Omega M \Omega^T)_{11} =
\begin{pmatrix}
x & y^T - z^T & 0 \\
y - z & -I_d + A + C & 0 \\
0 & 0 & I_d
\end{pmatrix},
\]

(24)

and

\[
(\Omega M \Omega^T)_{12} =
\begin{pmatrix}
[-x \Gamma_a - \frac{1}{2} y^T \Gamma_1 + \frac{1}{2} z^T \Gamma_2] & -(y^T + z^T) & 0 \\
[-y + z] \Gamma_a + \frac{1}{2} (-A \Gamma_1 + [-A - B^T & 0 \\
+ B^T \Gamma_2 + B \Gamma_1 - C \Gamma_2)] & + B + C \\
0 & 0 & 0
\end{pmatrix},
\]

(25)

The expression for \((\Omega M \Omega^T)_{22}\) is again large and we do not give it here. One can then verify, using the expressions for \(\tilde{G}_{D+d}\) and \(\tilde{B}_{D+d}\) in eqns.(10-11) and (6-7), and the inversion formula (20), that,

\[
(\Omega M \Omega^T)_{11} = \tilde{G}_{D+d}^{-1}.
\]

(26)

In proving eqn.(26), we also used the following simple identities,

\[
\frac{1}{2} \left[(I_d - \Sigma)(I_d + \Sigma)^{-1} + (I_d + \Sigma^T)^{-1}(I_d - \Sigma^T \Sigma)(I_d + \Sigma)^{-1}\right] = (I_d + \Sigma^T)^{-1}(I_d - \Sigma^T \Sigma)(I_d + \Sigma)^{-1},
\]

(27)

and

\[
-I_d + A + C - B - B^T = (I_d + \Sigma)(I_d - \Sigma^T \Sigma)^{-1}(I_d + \Sigma^T) + (y - z)x^{-1}(y^T - z^T).
\]

(28)
Similarly one gets,

\[
(\Omega M \Omega^T)_{12} = -\tilde{G}_{D+d}^{-1} \tilde{B}_{D+d} \\
(\Omega M \Omega^T)_{22} = \tilde{G}_{D+d} - \tilde{B}_{D+d} \tilde{G}_{D+d}^{-1} \tilde{B}_{D+d}.
\]  (29)

After verifying eqn.(14), we now evaluate \(\tilde{\phi}_{D+d}\) in eqn.(16). We note that the determinant of the matrix \(D\) in eqn.(19) is given as

\[
Det(D) = Det(Q).Det(P - R^T Q^{-1} R) \quad (30)
\]

Using this expression one gets,

\[
Det(G_{D+d}) = \frac{Det(I_d - \Sigma^T \Sigma)}{Det(x)} \quad (31)
\]

and

\[
Det(\tilde{G}_{D+d}) = \frac{Det[(I_d + \Sigma^T)^{-1}(I_d - \Sigma^T \Sigma)(I_d + \Sigma)^{-1}]}{Det(x)}. \quad (32)
\]

Then, for \(\tilde{\phi}_{D+d}\), we have

\[
\tilde{\phi}_{D+d} = \phi_{D+d} + \frac{1}{2} \log \left( \frac{detG_{D+d}}{detG_D} \right) \\
= \phi_{D+d} + \log[det(I_d + \Sigma)] \\
= \phi_D, \quad (33)
\]

which is the desired result in eqn.(12).

Finally, we like to mention that a constant shift in the antisymmetric tensor \(B_{D+d}\) in \((\theta_1, \theta_2)\) space can be obtained by an \(O(\tilde{d}, \tilde{d})\) transformation,

\[
\Omega_s = \begin{pmatrix}
I_{D-d} & I_{2d} \\
I_{2d} & I_{D-d} \\
b & 0
\end{pmatrix},
\]  (34)
where b is the constant shift in the antisymmetric tensor in $(\theta_1, \theta_2)$ space. This justifies such shift in the expression for $B_{D+d}$ in eqn.(4).

To conclude, we have established that, for the whole class of models in ref.[15], background field configuration for the unaguged string action is related to the gauged ones by a background independent $O(\tilde{d}, \tilde{d})$ transformation. A simple application of the result is that the $SL(2, R)$ WZW model can be transformed by an $O(2, 2)$ transformation of the above type to the uncharged black string[11]. We expect that these results being general, can be applied in various other circumstances. It will be interesting to investigate whether the results of this paper can be generalized even further, such as to heterotic strings, nonabelian gaugings etc.

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