Multivortex Solutions of the Weierstrass Representation

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Abstract

The connection between the complex Sine and Sinh-Gordon equations on the complex plane associated with a Weierstrass type system and the possibility of construction of several classes of multivortex solutions is discussed in detail. We perform the Painlevé test and analyse the possibility of deriving the Bäcklund transformation from the singularity analysis of the complex Sine-Gordon equation. We make use of the analysis using the known relations for the Painlevé equations to construct explicit formulae in terms of the Umemura polynomials which are \( \tau \)-functions for rational solutions of the third Painlevé equation. New classes of multivortex solutions of a Weierstrass system are obtained through the use of this proposed procedure. Some physical applications are mentioned in the area of the vortex Higgs model when the complex Sine-Gordon equation is reduced to coupled Riccati equations.

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I. INTRODUCTION.

The complex Sine and Sinh-Gordon equations have been of considerable interest recently in many areas of mathematical physics. They originally appeared in the reduction of the $O(4)$ nonlinear sigma model\textsuperscript{1,2}. They have also appeared in a number of other physical contexts, for example, in the study of a massless fermion model with chiral symmetry, and also in the study of the motion of a vortex filament in an inviscid incompressible fluid\textsuperscript{3}. The equations have been found to be completely integrable, and some work on the construction of multi-soliton solutions has been carried out\textsuperscript{1-5}. It has also been shown that the complex Sine-Gordon theory may be reformulated in terms of the Wess-Zumino-Witten action and interpreted as the integrably deformed $SU(2)/U(1)$-coset model, (as in\textsuperscript{6} and references therein). These studies are based on the complex Sine and Sinh-Gordon theory in the $(1+1)$-dimensional Minkowski space-time. These models have energy functionals which are related to that of the Ginsburg-Landau energy

$$E_{GL} = \int [||\nabla \psi||^2 + \frac{1}{2}(1 \pm |\psi|^2)^2] \, d^2x. \quad (1.1)$$

From the physical point of view, this expression constitutes the basis of the phenomenological theory of superfluidity\textsuperscript{7,8} and has appeared in particle physics as well\textsuperscript{1}. The Ginsburg-Landau functional is minimized by the Gross-Pitaevski vortices\textsuperscript{7}. These are topological solitons of the form

$$\psi(x,y) = \Phi_n(r)e^{in\theta}, \quad \Phi_n \to 1, \quad r \to \infty.$$ 

It is the purpose here to study vortex solutions for the equations of interest discussed in this paper.

Another important physical application is to the area of superconductivity\textsuperscript{3}, where vortex solutions play an essential role. The Lagrangian for the superconducting system usually takes the form

$$L = -\frac{1}{4}F_{\mu \nu}^2 + |\nabla_{\mu} \psi|^2 - V(\psi). \quad (1.2)$$

Here, $\psi$ plays the role of the Higgs field, $F_{\mu \nu}$ pertains to the electrodynamic term and $V(\psi)$ is the scalar or Higgs potential function which is responsible for mass generation in the system.

In particular, the classical Sine-Gordon equation has relevance to models which are of interest to particle theory. The complete integrability of the Gross-Neveu model, in the large $N$ approx-
imation, is quite analogous to the complete integrability of the classical Sine-Gordon equation, where $1/N$ plays the role of the coupling constant. For $N = 1$, a large coupling case, the model reduces to the massless Thirring model, which is scale invariant with anomalous dimensions. For $N > 1$, the theory exhibits non-trivial renormalization group behavior and mass generation through dimensional transmutation.

Another area of recent importance with regard to applications is to the area of liquid crystals and membranes. Fluid membranes may be idealized as two-dimensional surfaces in an aqueous solution with each membrane being made up of a double layer of long molecules. The curved fluid membrane may be treated as a bending liquid crystal cell with uniaxial molecular order. Various physical properties of interest can be calculated in terms of quantities which are directly related to the geometry of the surface. For example, in one model for uniaxial liquid crystals, with the normal to the membrane $\vec{n}$ of the liquid crystal, it has been shown that the elastic energy of curvature per unit area of the membrane is

$$F = \int g \, dA, \quad g = \frac{1}{2} k (c_1 + c_2 - c_0)^2 + \bar{k} c_1 c_2.$$  

Here, $c_1, c_2$ are the two principal curvatures of the surface of the membrane, and $c_0$ is called the spontaneous curvature of the surface. The quantity, $F$ is referred to as the total bending energy of the membrane, the constant $c_0$ is related to the asymmetry of the layers. The positive constant $k$ is the bending rigidity and $\bar{k}$, which could have either sign, is the elastic modulus of the Gaussian curvature. The curvature elastic free energy per unit area of the membrane can also be formulated rigorously in terms of two-dimensional differential invariants of the surface.

This paper is organized as follows. A generalization of a Weierstrass system for inducing two-dimensional surfaces in $\mathbb{R}^4$ is presented. It is shown that this system is related to the complex Sine and Sinh-Gordon equations. Several properties namely the Lax pair, the Painlevé property and the Bäcklund transformation following from the singularity analysis, are investigated for the complex Sine-Gordon equation and then extended to the Weierstrass system in Sections 3 and 4. Multivortex solutions are constructed and the Painlevé structure of the associated radial equations is studied. The $\tau$ functions for the rational class of solutions of the Painlevé equation P5 are written
in terms of Umemura polynomials, and explicit forms of such solutions are given in Section 5. We also present a particular class of solutions of the Weierstrass system via the solution of the complex Sine-Gordon equation. Section 6 contains examples of solving the Weierstrass system by means of solutions of the complex Sine-Gordon equation.

II. THE GENERALIZED WEIERSTRASS SYSTEM AND ASSOCIATED COMPLEX SINE AND SINH-GORDON EQUATIONS.

The Gauss-Codazzi equations describing a two-dimensional surface immersed in a three-dimensional sphere which is itself again immersed into a four-dimensional Euclidean space has been studied by Darboux. He investigated the non-linear Dirac-type system for four complex valued functions $\psi_i$ and $\varphi_i$, $i = 1, 2$ satisfying the following system of equations

$$ \partial \psi_1 = Q_1(\psi_1 + \frac{\varphi_1}{2\psi_2\varphi_2}), \quad \bar{\partial} \psi_2 = Q_1\psi_2, $$

$$ \bar{\partial} \varphi_1 = Q_2(\varphi_1 - \frac{\psi_1}{2\varphi_2\psi_2}), \quad \partial \varphi_2 = Q_2\varphi_2, $$

$$ Q_1 = |\psi_2|^2 \pm |\psi_1|^2, \quad Q_2 = |\varphi_2|^2 \pm |\varphi_1|^2, $$

and its respective complex conjugate equations. The bar denotes complex conjugate, $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial \bar{z}$. The above system can be considered as a variant of the Weierstrass representation for surfaces immersed in $\mathbb{R}^4$ and will refer to it as such. System (2.1) is a nonlinear first order system of eight equations, for which eight of sixteen first order derivatives with respect to $z$ or $\bar{z}$ are known in terms of functions $\psi_i$ and $\varphi_i$. System (2.1) admits several conservation laws such as

$$ \partial(\ln \bar{\psi}_2) = \bar{\partial}(\ln \psi_2), \quad \partial(\ln \varphi_2) = \bar{\partial}(\ln \bar{\varphi}_2), \quad \partial(\frac{\psi_1\varphi_1}{\psi_2\varphi_2}) = \bar{\partial}(\frac{\bar{\psi}_1\bar{\varphi}_1}{\bar{\psi}_2\bar{\varphi}_2}). $$

As a consequence of conservation laws (2.2), there exist four real-valued functions $X^i(z, \bar{z})$ which are defined by

$$ X^1 = \int_\Gamma \ln \psi_2 \, dz + \ln \bar{\psi}_2 \, d\bar{z}, \quad X^2 = \int_\Gamma \ln \varphi_2 \, dz + \ln \bar{\varphi}_2 \, d\bar{z}, $$

$$ X^3 = \int_\Gamma \ln \psi_2\varphi_2 \, dz + \ln \bar{\psi}_2\bar{\varphi}_2 \, d\bar{z}, \quad X^4 = \int_\Gamma \frac{\bar{\psi}_1\varphi_1}{\psi_2\varphi_2} \, dz + \frac{\psi_1\bar{\varphi}_1}{\psi_2\bar{\varphi}_2} \, d\bar{z}, $$

for any contour $\Gamma$ in the complex plane which begins at a fixed $z_0$ and ends at variable point $z$. The right hand side of (2.3) does not depend on the choice of the curve $\Gamma$, since the differential of
equations (2.3) are exact ones. Thus, equations (2.1) and (2.2) allow us to identify the real-valued functions $X^i(z, \bar{z}), i = 1, \cdots, 4$, as the coordinates of a surface immersed in four-dimensional Euclidean space.

In the present paper, we propose a procedure for constructing explicit multivortex solutions of system (2.1). These solutions are obtained through the use of a link between the complex Sine or Sinh-Gordon equations on the plane and Weierstrass system (2.1). To our knowledge, this connection between these two systems has been observed for the first time here. At this point, we want to underline that, for the Weierstrass system (2.1), few explicit solutions have been found up to now, and this link with complex Sine-Gordon and complex Sinh-Gordon equations allows us to construct new classes of solutions explicitly. We subject system (2.1) to several transformations in order to simplify its structure. We start by defining two new complex valued functions

$$u = \frac{\psi_1}{\psi_2}, \quad v = \frac{\varphi_1}{\bar{\varphi}_2}. \quad (2.4)$$

It is easy to show that if the complex functions $\psi_i$ and $\varphi_i,$ are solutions of the first order system (2.1), then the complex-valued functions $u$ and $v$ defined by (2.4) are solutions of the first order system of two equations

$$\partial u = \frac{1}{2}(1 \pm |u|^2)v, \quad \bar{\partial}v = -\frac{1}{2}(1 \pm |v|^2)u, \quad (2.5)$$

and their respective complex conjugate equations. The elimination of one of the functions $u$ or $v$ in system (2.5) leads to the complex Sinh-Gordon (CShG) equation when the sign is positive in (2.5), and Sine-Gordon (CSG) equation when the sign is negative in (2.5). Thus we get for both cases

$$\partial \bar{\partial}u = \frac{\bar{u}}{1 \pm |u|^2} \partial u \bar{\partial}u + \frac{1}{4}u(1 \pm |u|^2) = 0. \quad (2.6)$$

As it was shown in$^1$, equation (2.6) was derived in the context of the reduction of the $O(4)$ nonlinear sigma model and as well, the reduction of the self-dual Yang-Mills equations and relativistic equations$^{2,12,13}$.

Note that if $v$ tends to one in CSG equations (2.5), then $Q_2$ vanishes and system (2.1) takes
the form
\[ \partial \psi_1 = Q_1(\psi_1 + \frac{1}{2\psi_2}), \quad \bar{\partial} \psi_2 = Q_1 \psi_2. \] (2.7)

Conversely, if \( u \) tends to one in CSG equations (2.5), then \( Q_1 \) vanishes and system (2.1) becomes
\[ \bar{\partial} \varphi_1 = Q_2(\varphi_1 - \frac{1}{2\varphi_2}), \quad \partial \varphi_2 = Q_2 \varphi_2. \] (2.8)

These limits characterize the properties of the solutions of system (2.1).

Now, we discuss a set of conditions which allow the system (2.1) to become a decoupled system of equations.

**Proposition 1.** Let the complex functions \( u \) and \( v \) be solutions of system (2.5). Let the functions \( \psi_i \) and \( \varphi_i \) be defined in terms of \( u \) and \( v \) by
\[ \psi_1 = \epsilon u(1 \pm |u|^2)^{-1/2}, \quad \varphi_1 = \epsilon v(1 \pm |v|^2)^{-1/2}, \]
\[ \psi_2 = \epsilon(1 \pm |u|^2)^{-1/2}, \quad \varphi_2 = \epsilon(1 \pm |v|^2)^{-1/2}, \quad \epsilon = \pm 1. \] (2.9)

Then the general integrals of system (2.1) are given by
\[ \psi_1 = uA(z) e^z, \quad \varphi_1 = vB(z) e^z, \]
\[ \psi_2 = A(z) e^z, \quad \varphi_2 = \bar{B}(\bar{z}) e^z, \] (2.10)

where the complex functions \( A \) and \( B \) satisfy the following conditions,
\[ |A|^2 = e^{-(z+\bar{z})(1 \pm |u|^2)^{-1}}, \quad |B|^2 = e^{-(z+\bar{z})(1 \pm |v|^2)^{-1}}. \] (2.11)

**Proof:** Substituting (2.4) into system (2.1), we obtain an overdetermined system for the functions \( \psi_2 \) and \( \varphi_2 \) of the following form
\[ \partial(u\bar{\psi}_2) = (1 \pm |u|^2)|\psi_2|^2(u\bar{\psi}_2 + \frac{v}{2\psi_2}), \quad \bar{\partial} \psi_2 = (1 \pm |u|^2)|\psi_2|^2 \psi_2, \]
\[ \bar{\partial}(v\varphi_2) = (1 \pm |v|^2)|\varphi_2|^2(v\varphi_2 - \frac{u}{2\varphi_2}), \quad \partial \varphi_2 = (1 \pm |v|^2)|\varphi_2|^2 \varphi_2. \] (2.12)

Consider the first equation in the first line of (2.12). Expanding the derivative term \( \partial(u\bar{\psi}_2) \) and using (2.5), this equation reduces to the form
\[ \partial \bar{\psi}_2 = (1 \pm |u|^2)|\psi_2|^2 \bar{\psi}_2. \]
Using (2.9) to replace $|\psi_2|^2$ in this result, it reduces to $\partial \bar{\psi}_2 = \psi_2$. Making use of (2.9), the remaining three equations in (2.12) can be treated in a similar way. Thus the initial system (2.1) becomes a linear system of the form

$$\bar{\partial} \psi_2 = \psi_2, \quad \partial \varphi_2 = \varphi_2. \quad (2.13)$$

These two equations can be easily integrated to give $\psi_2$ and $\varphi_2$ as given in (2.10), then (2.4) can be used to obtain $\psi_1$ and $\varphi_1$. The results in (2.10) must be consistent with those in (2.9). If we calculate the modulus of $\psi_2$ and $\varphi_2$ from (2.10) and equate to the modulus calculated from (2.9), the conditions (2.11) are obtained. In fact, equating $\psi_i$, $\varphi_i$ in (2.10) to their corresponding forms in (2.9), we must also have that

$$\bar{A}(\bar{z})e^z = \epsilon(1 \pm |u|^2)^{-1/2} = A(z)e^\bar{z}, \quad B(z)e^\bar{z} = \epsilon(1 \pm |v|^2)^{-1/2} = \bar{B}(\bar{z})e^z.$$

A set of differential constraints which must be satisfied can be obtained by substituting (2.9) into (2.13), and we find

$$(\bar{\partial}u\bar{u} + \frac{1}{2}uv(1 \pm |u|^2)) = \mp 2(1 \pm |u|^2), \quad (\partial v\bar{v} - \frac{1}{2}uv(1 \pm |v|^2)) = \mp 2(1 \pm |v|^2).$$

QED

From the computational point of view, it is more convenient to deal with the CShG or CSG equations (2.5) than with the original system (2.1). From every solution of CShG or CSG equations (2.5), we can integrate a linear system (2.13), and consequently, a very large class of solutions of system (2.1) can be found explicitly by making use of formulae (2.10) and (2.11).

Using the connection between the CSG equation (2.6) and Weierstrass system (2.1), we discuss in the next section in detail the Painlevé analysis of the CSG equation which allows us to extend this analysis to the Weierstrass system.

**III. PAINLEVÉ ANALYSIS OF THE COMPLEX SINE-GORDON EQUATION.**

Integrability of the CSG equation (2.6) is confirmed by tests for the Painlevé property. We perform the classical test of extended to partial differential equations in, assuming a solution in the form of a Laurent series about an arbitrary singularity manifold $F(z, \bar{z}) = 0$ and checking
compatibility of the resulting recurrence formulae. Detailed discussion of the meaning and validity of this test may be found in\textsuperscript{20}. The test is carried out for the CSG equation (2.6) and for the system (2.5). Both versions of the CSG pass the test. For the system (2.5), we discuss the possibilities and limitations of the approach to the Bäcklund transformation\textsuperscript{15}.

Both those forms of the CSG contain non-analytic expressions, therefore $z$ and $\bar{z}$ should be treated as two independent variables and extended to two separate complex planes. Similarly, the functions $u$ and $\bar{u}$ need not be complex conjugates of each other, when their independent variables are separately extended. We assume also the same holds for $v$ and $\bar{v}$. To avoid the misleading complex conjugate symbol, we denote $\bar{z}$ by $t$, $\bar{u}$ by $w$ and $\bar{v}$ by $s$, while symbols of the derivatives $\partial$ and $\bar{\partial}$ will be replaced by alphabetic subscripts $z$ and $t$, respectively.

In this notation, the original CSG equation (2.6), is cast into polynomial form. It has the form

\begin{align*}
(1 - uw)u_{zt} + wu_zu_t + \frac{1}{4}u(1 - uw)^2 = 0, \\
(1 - uw)w_{zt} + uw_zw_t + \frac{1}{4}w(1 - uw)^2 = 0.
\end{align*}

(3.1)

Both dependent variables, $u$ and $w$, begin their series with $F$ inverse. The coefficients at this power of $F$ may be written as

\begin{equation}
\begin{aligned}
u_0 &= 2(F_zF_t)^{1/2}/Q_0, \\
w_0 &= 2(F_zF_t)^{1/2}Q_0,
\end{aligned}
\end{equation}

(3.2)

where $Q_0$ is an arbitrary function of $z$ and $t$.

The indices, at which the determinant of the linear recurrence formulae turns zero, are $-1$ (corresponding to arbitrariness of $F$), 0 (due to arbitrariness of $Q_0$), 1 and 2. Both compatibility conditions at the last two indices are satisfied. As the $2 \times 2$ matrices of the respective linear systems have rank one at each of these indices, each of the recurrence formulae yields one arbitrary function (say, $Q_1$ and $Q_2$), to complete the set of four first integrals ($F, Q_0, Q_1, Q_2$) in the general solution of (2.6). To conclude that the CSG equation (2.6) has the Painlevé property, we should also examine the singularity $uw \to 1$. This task is straightforward when working with system (2.5).

The system (2.5) is just a form of the CSG equation (2.6), so it should pass the same test. We describe it in more detail because of the close connection of (2.5) with the Weierstrass system.
For the purpose of the “Painlevé test”, equation (2.5) is a $4 \times 4$ system (we choose the minus sign for further analysis) given by

\[
\begin{align*}
 u_z - (1 - wu)v/2 &= 0, \\
 w_t - (1 - wu)s/2 &= 0, \\
 v_t + (1 - sv)u/2 &= 0, \\
 s_z + (1 - sv)w/2 &= 0.
\end{align*}
\]

(3.3)

We may choose $-1$ as the initial exponents in the Laurent series of either $u$ and $w$ or $v$ and $s$. We perform the calculations for the former choice, which corresponds to our previous analysis of the CSG equation (2.6); the latter choice is equivalent to it by exchange of $u$ with $v$, $w$ with $s$ and $z$ with $-t$. The above-mentioned singularity $uw \rightarrow 1$ corresponds to the latter choice. The resulting initial exponents of $v$ and $s$ are both equal to 0 and their Laurent expansions turn into the Taylor ones. Coefficients $u_0$ and $w_0$ are the same as those in (3.2), while

\[
v_0 = (F_z/F_t)^{1/2}/Q_0, \quad s_0 = (F_t/F_z)^{1/2}Q_0.
\]

(3.4)

At this point, we would like to mention the following fact. Usually, the classical “Painlevé test” is not possible when terms of lowest order in $F$ have nonnegative exponents. Such terms may lack the property (necessary for the algorithm of\textsuperscript{14} and\textsuperscript{15}) that differentiation decreases their order of magnitude by one. However, here the derivatives of $v$ and $s$ do not contribute to the dominant terms in the last two equations of (3.3) and that deficiency is meaningless.

The remaining terms are derived from the recurrence formulae

\[
\begin{align*}
 nF_zu_n + (F_z/Q_0^2)w_n + 2F_zF_tv_n &= -(u_{n-1})_z + \frac{1}{2}v_{n-2} \\
 -\frac{1}{2}u_0 \sum_{k=1}^{n-1} w_kv_{n-k} - \frac{1}{2} \sum_{k=0}^{n-1} \sum_{l=1}^{k-1} w_ku_lv_{n-k-l}, \\
 (F_tQ_0^2)u_n + nF_tw_n + 2F_tF_vw_n &= -(w_{n-1})_t + \frac{1}{2}s_{n-2} \\
 -\frac{1}{2}w_0 \sum_{k=1}^{n-1} u_kw_{n-k} - \frac{1}{2} \sum_{k=0}^{n-1} \sum_{l=1}^{k-1} u_kw_lv_{n-k-l}, \\
 (n-1)FTv_n - (F_z/Q_0^2)s_n &= -(v_{n-1})_t \\
 +\frac{1}{2}u_0 \sum_{k=1}^{n-1} s_kv_{n-k} + \frac{1}{2} \sum_{k=0}^{n-1} \sum_{l=1}^{k-1} s_kw_lv_{n-k-l}, \\
 -(F_tQ_0^2)v_n + (n-1)Fzv_n &= -(s_{n-1})_z \\
 +\frac{1}{2}w_0 \sum_{k=1}^{n-1} v_kw_{n-k} + \frac{1}{2} \sum_{k=0}^{n-1} \sum_{l=1}^{k-1} v_kw_lv_{n-k-l},
\end{align*}
\]

(3.5a, 3.5b, 3.5c, 3.5d)

where $v$ and $s$ with negative subscripts are both set equal to zero, and have been admitted for compact notation.
The determinant of the system (3.5) reads

\[(F_z F_t)^2 (n + 1)n(n - 1)(n - 2),\]  \hspace{1cm} (3.6)

Hence, the indices at which it becomes zero, are $-1, 0, 1$ and $2$ as in the previous case of the CSG equation (2.6). Similarly, all the compatibility conditions are satisfied, whence we conclude that equation (2.5) has the Painlevé property.

This Painlevé integrability obviously extends to the Weierstrass system (2.1) as the Painlevé property is invariant under the homographic transformation which converts the equations (2.4) to (2.1).

The method of the Laurent expansion may often be extended to deriving the Bäcklund transformation and further an explicit integration scheme\textsuperscript{15}. The usual approach relies on truncation of the Laurent series at the term of order $F^0$, which is assumed to satisfy the original equation. The truncation usually is possible through appropriate choice of the arbitrary functions (first integrals). Some extra assumptions on the coefficients and expansion variable may also be necessary. A systematic approach to that problem may be found in \textsuperscript{16,20}.

However, the usual method contains assumptions which are too restrictive for application to equations (3.3). Namely, the Laurent series of $v$ and $s$, which begin with the $F^0$ terms, reduce to a single term each. This means that $v$ and $s$ would not be transformed at all. Moreover, the truncation at $F^0$ implies vanishing of terms proportional to $F^1$. This imposes further constraints on these variables: from their recurrence equations (3.5) at $n = 1$

\[-(F_z/Q_0^2)s_1 = -(v_0)_t = 0,\] \hspace{1cm} (3.7a)

\[-(F_t Q_0^2)v_1 = -(s_0)_z = 0.\] \hspace{1cm} (3.7b)

It follows that $v_0$ should be independent of $t$, while $s_0$ should be independent of $z$. However, these coefficients are reciprocals of each other (3.4), whence neither of them may depend on $z$ or $t$. This, together with the truncation of the series, reduces $u$ and $v$ to constants. If we denote

\[s = k, \hspace{1cm} v = 1/k, \hspace{1cm} k = constant,\] \hspace{1cm} (3.8)
then the original equations (3.3) reduce to a system of coupled Riccati equations

\[ u_z - (1 - wu)/(2k) = 0, \quad w_t - (1 - wu)k/2 = 0, \quad (3.9) \]

which may immediately be linearized by substitution

\[ u = (2/k)(\ln \Psi)_t, \quad w = 2k(\ln \Psi)_z, \quad (3.10) \]

to the Helmholtz equation

\[ \Psi_{zt} = (1/4)\Psi. \quad (3.11) \]

Obviously, this linearization also yields an (almost trivial) transformation of \( u \) and \( v \), a superposition principle, and other properties.

The method of the Laurent expansion starting from eqs. (3.1) is free of those shortcomings. However the condition of vanishing of the higher order coefficients results in cumbersome equations, which makes the procedure hardly worthwhile in comparison with other methods of integration (see next section).

**IV. ON EQUIVALENCE OF TWO FORMS OF THE COMPLEX SINE-GORDON EQUATION.**

Throughout this paper we investigate the CSG equation in the form of equations (2.6). Another form of the CSG equation was given in\textsuperscript{17,18} as follows

\[
\left( \frac{q \xi}{\sqrt{1 + qp}} \right)_\eta = 4q, \quad \left( \frac{p \xi}{\sqrt{1 + qp}} \right)_\eta = 4p.
\quad (4.1)
\]

Let

\[ u = \sin(\Phi/2) \exp(i\alpha). \quad (4.2) \]

The polar coordinates \( \Phi \) and \( \alpha \) satisfy a system of equations similar to that given by Lund\textsuperscript{4}

\[
\partial \bar{\partial} \Phi - 2 \frac{\sin(\Phi/2)}{\cos^3(\Phi/2)} \partial \alpha \bar{\partial} \alpha + \frac{1}{4} \sin \Phi = 0, \quad (4.3)
\]

\[
\partial(\tan^2(\Phi/2) \bar{\partial} \alpha) + \bar{\partial}(\tan^2(\Phi/2) \partial \alpha) = 0. \quad (4.4)
\]

We formulate the following statement for systems (2.6) and (4.1).
**Proposition 2.** Equation (4.1) is transformed into CSG equation (2.6) through the following relations

\[ \xi = \frac{1}{4}\bar{z}, \quad \eta = -\frac{1}{4}z, \quad q = -\sin\Phi \exp(-i\beta), \quad p = \sin\Phi \exp(i\beta), \quad (4.5) \]

where the phase is given by

\[ \beta = \int_{z_0}^{z} \left\{ [1 + \tan^2(\Phi(z', \bar{z})/2)]\partial'\alpha(z', \bar{z}) dz' + [1 - \tan^2(\Phi(z, \bar{z}')/2)]\bar{\partial}\alpha(z, \bar{z}') d\bar{z}' \right\}. \quad (4.6) \]

The lower limit of integration, \( z_0 \), is fixed and depends on the initial conditions.

**Proof.** Substitution of (4.5) and (4.6) into (4.1) yields the system (4.3) and (4.4).

Note that both equations, (2.6) and (4.1) depend on their phases \( \alpha \) and \( \beta \) respectively, through their derivatives only, except for linear dependence on factors \( \exp(i\alpha) \) and \( \exp(i\beta) \). Therefore, any change of \( \beta \), which leaves its derivatives unchanged, for example, one that arises from deformation of the integration contour in (4.6), does not affect equivalence of those equations. Moreover, for those \( u \), which satisfy the CSG equation (2.6), the integrand is an exact differential and the path of integration does not even affect the value of the phase.

Note also that the form of the equation determining evolution of the phase (4.4) suggests integration in terms of an arbitrary potential \( \psi(z, \bar{z}) \)

\[ \tan^2(\Phi/2)\partial\alpha = \partial\psi, \quad \tan^2(\Phi/2)\bar{\partial}\alpha = -\bar{\partial}\psi. \quad (4.7) \]

However, the potential is not arbitrary since the compatibility condition \( \partial\bar{\partial}\alpha = \bar{\partial}\partial\alpha \) imposes a constraint of the form similar to the original phase equation (4.4)

\[ \partial(cot^2(\Phi/2)\bar{\partial}\psi) + \bar{\partial}(cot^2(\Phi/2)\partial\psi) = 0. \quad (4.8) \]

Obviously, if \( \alpha \) is a solution of (4.4) for a given \( \Phi \) then \( \psi \) solves the same equation for \( \Phi \) shifted by an odd multiple of \( \pi \) or subtracted from such a multiple. However, this is not a symmetry of the CSG equation (2.6), as the signs in the amplitude equation (4.3) are changed by such a transformation. Finally, repetition of the transformation brings us back to the original equation. A similar property holds for the CSG equation in the form (4.1), for which the phase equation may be written as

\[ \partial(\cos^{-1}\Phi\bar{\partial}\beta) - \bar{\partial}(\cos\Phi\partial\beta) = 0, \quad (4.9) \]
where the change of independent variables has already been performed.

This symmetry has an additional consequence. The transformations (4.5) and (4.6) from equations (2.6) to (4.1) have some features of a Bäcklund transformation. Namely, the definition of $\beta$ by means of the contour integral (4.6) is a solution of a coupled pair of differential equations

$$
\partial \beta = [1 + \tan^2(\Phi/2)] \partial \alpha, \quad \bar{\partial} \beta = [1 - \tan^2(\Phi/2)] \bar{\partial} \alpha. \quad (4.10)
$$

System (4.10) is overdetermined, as the right hand sides of the above equations must satisfy the compatibility condition $\partial \bar{\partial} \beta = \bar{\partial} \partial \beta$. This condition is indeed satisfied as it proves to be equivalent to equation (4.4). Thus we obtain (4.4) in two ways: either through direct substitution of (4.10) to (4.9), or from the above compatibility condition.

The transformation defined by (4.5) and (4.6) may be extended to the inverse scattering method. The inverse scattering technique for (4.1) was given in\textsuperscript{17}. Using (4.5) and (4.6), we obtain the Lax pair for (2.6)

$$
\partial X = -\frac{1}{4\lambda} [Y_1, X], \quad \bar{\partial} X = \frac{1}{4} [Y_2 + \lambda Y, X], \quad (4.11)
$$

where

$$
Y = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad Y_1 = \begin{pmatrix} i \cos \Phi & \sin \Phi e^{i\beta} \\ -\sin \Phi e^{-i\beta} & -i \cos \Phi \end{pmatrix},
$$

$$
Y_2 = 2i \begin{pmatrix} 0 & \bar{\partial}(\sin \Phi e^{i\beta}) \\ \frac{\bar{\partial}(\sin \Phi e^{-i\beta})}{\cos \Phi} & 0 \end{pmatrix}. \quad (4.12)
$$

Further extension to the Weierstrass system (2.1) is also possible, by means of the transformation (2.4). The matrices $Y_1$ and $Y_2$ expressed in terms of $u$ become

$$
Y_1 = \begin{pmatrix} i(1 - 2u^2e^{-2i\alpha}) & \sqrt{1 - (1 - 2u^2e^{-2i\alpha})^2} e^{i\beta} \\ -\sqrt{1 - (1 - 2u^2e^{-2i\alpha})^2} e^{-i\beta} & -i(1 - 2u^2e^{-2i\alpha}) \end{pmatrix},
$$

$$
Y_2 = \begin{pmatrix} 0 & \bar{\partial}(\sqrt{1 - (1 - 2u^2e^{-2i\alpha})^2} e^{i\beta}) \\ \bar{\partial}(\sqrt{1 - (1 - 2u^2e^{-2i\alpha})^2} e^{-i\beta}) & 1 - 2u^2e^{-2i\alpha} \end{pmatrix}. \quad (4.13)
$$
From (2.4), complex functions $u$ and $v$ are given in terms of $\psi_1$, $\bar{\psi}_2$ and $\varphi_1$, $\bar{\varphi}_2$, respectively. Taking into account that functions $u$, $v$ satisfy the same equation (2.6), the resulting Lax pair (4.11) for Weierstrass system (2.1) can be described by a system of five two by two matrices $Y_1$, $Y_2$ in terms of $(\psi_1, \bar{\psi}_2)$ and $(\varphi_1, \bar{\varphi}_2)$ respectively, and the constant matrix $Y$.

V. MULTIVORTEX SOLUTIONS

At this point, we would like to derive, through the link between first order system (2.1) and equations (2.6), a procedure for constructing multivortex solutions in explicit form. Now, we concentrate on certain class of multivortex solutions of equations (2.6) in polar coordinates $(r, \theta)$ on the plane determined by

$$u = A_n(r)e^{in\theta}, \quad n \in \mathbb{Z}. \quad (5.1)$$

Equation (2.6), under the assumption (5.1) is reducible to a second order ODE of the form

$$\frac{d^2 A_n}{dr^2} + \frac{1}{r} \frac{dA_n}{dr} = \frac{A_n}{1 \pm A_n^2} \left[ \left( \frac{dA_n}{dr} \right)^2 \pm \frac{n^2}{r^2} \right] + (1 \pm A_n^2)A_n = 0. \quad (5.2)$$

By a homographic transformation of the dependent variable

$$A_n = c \frac{1 + w(z)}{1 - w(z)}, \quad z = r, \quad (5.3)$$

where $c = -i$ for the CShG system, and $c = 1$ for the CSG system, respectively, equation (5.2) has the structure of the fifth Painlevé (P5) equation

$$w'' = \frac{3w - 1}{2w(w - 1)} w'^2 - \frac{w'}{z} + \frac{(w - 1)^2}{z} (\alpha w + \beta w) + \gamma w + \delta \frac{w(w + 1)}{w - 1}, \quad (5.4)$$

with the coefficients $\alpha$ and $\beta$ parametrized by a number $n \in \mathbb{Z}$ and $\gamma, \delta$ fixed as follows

$$\alpha = -\beta = \frac{n^2}{8}, \quad \gamma = 0, \quad \delta = -2. \quad (5.5)$$

Such reduction to P5 has recently been performed\textsuperscript{19}. In general, equation P5 is not integrable in terms of known classical transcendental functions. However, for specific values of the parameters, solutions of equation (5.4) can be reduced to two types of nontranscendental functions, that is, to solutions of a Riccati equation with one arbitrary parameter or to three types of rational
solutions of equation $P5^{20,21}$. According to $^{21}$, equation (5.4) with coefficients (5.5) can be written in equivalent form as a first order system of ODEs

$$
z \frac{dp}{dz} = -\frac{\epsilon n}{2} - \epsilon np - pq - p^2 q, \quad \epsilon = \pm 1,
$$

$$
z \frac{dq}{dz} = -2z^2 + \epsilon nq - 4z^2 p + \frac{q^2}{2} + pq^2,
$$

where $p = w/(1 - w)$. The function $q(z)$ satisfies a Painlevé type equation of the form

$$q'' = \frac{q}{q^2 - 4z^2} q'^2 - \frac{q^2 + 4z^2}{q^2 - 4z^2} \cdot \frac{q'}{z} + \frac{q}{4z^2(q^2 - 4z^2)} [16n z^2 (2\epsilon - n) - (q^2 - 4z^2)^2].$$

The function $q^2 - 4z^2$ has two roots at $q = 2z$. Using the transformation

$$y(z) = \frac{q + 2z}{q - 2z}, \quad q \neq 2z,$

we get that $y(z)$ is also a solution of equation $P5$ with parameters

$$\tilde{\alpha} = -\tilde{\beta} = \frac{(1 - \epsilon n)^2}{8}, \quad \tilde{\gamma} = 0, \quad \tilde{\delta} = -2. \quad (5.7)$$

Propositions 3 to 5 are special cases studied by V. Gromak$^{21}$ (Chapter 12, section 14) concerning the fifth Painlevé equations with specific parameters. This analysis is used to construct solutions to Weierstrass system (2.1).

**Proposition 3.** Let $w = w(z)$ be a solution of the fifth Painlevé equation $P5$ (5.4) with parameters given by (5.5), such that the function

$$\Phi_1(w) \equiv zw' - \frac{\epsilon n}{2} w^2 + 2zw + \frac{\epsilon n}{2} \neq 0, \quad (5.8)$$

does not vanish for any $n \in \mathbb{Z}$, then the function

$$w_1 = 1 - \frac{4z}{\Phi_1(w)}, \quad (5.9)$$

is a solution of the fifth Painlevé equation (5.4) with parameters given by (5.7).

Proposition 3 establishes the Auto-Bäcklund transformation (Auto-BT) for equation $P5$ when $\gamma = 0, \delta = -2$ and $\alpha = -\beta$ are parametrized by $n \in \mathbb{Z}$. 

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Now, let us discuss the link between equations P5 with different values of the parameter \( \delta \), namely, \( \delta \neq 0 \) and \( \delta = 0 \). Note that for \( \delta \neq 0 \), the solutions of equation (5.4) are expressible in terms of Bessel functions, whereas for \( \delta = 0 \), they can be expressed in terms of Umemura polynomials, which is presented below.

**Proposition 4.** Let \( u(z) \neq 0 \) be a solution of equation P5 with parameters given by (5.5). Then the function

\[
\tilde{u}(z) = \frac{f^2(\sqrt{z})}{f^2(\sqrt{z}) - 1},
\]

where \( f(z) \) is defined by

\[
f(z) = \frac{d}{dz} \ln u(z) - \frac{n}{4z} (u(z) - \frac{1}{u(z)}), \quad n \in \mathbb{Z},
\]

is a solution of equation P5 with parameters

\[
\tilde{\alpha} = \frac{(1 + n^2)^2}{2}, \quad \tilde{\beta} = \tilde{\delta} = 0, \quad \tilde{\gamma} = -\frac{1}{2}.
\]

Based on reference 21 and using the result of Proposition 6, we can find in our case the relation between equations P3 with \( \gamma \delta \neq 0 \)

\[
w'' = \frac{w'^2}{w} - \frac{w'}{w} + \frac{1}{z} (\alpha \gamma w^2 + \beta) + \gamma^2 w^3 + \frac{\delta}{w},
\]

and P5 with coefficients given by (5.11). Indeed, the third Painlevé equation (5.12) can be written as a first order system of ODEs

\[
zw' = (\epsilon \alpha - 1)w + \epsilon \gamma zw^2 + zw, \quad \epsilon = \pm 1,
\]

\[
zwv' = \beta w + \delta z + (\epsilon \alpha - 2)wv + zv^2.
\]

From system (5.13), the elimination of \( w \) gives

\[
v'' - \frac{v}{v^2 + \delta} v' + \frac{v'}{z} + \frac{\beta^2 - (2 - \epsilon \alpha)^2 \delta}{z^2(v^2 + \delta)} v + \epsilon \gamma (v^2 + \delta) - \frac{2 \delta \beta (\epsilon \alpha - 2)}{z^2 v^2 + \delta} - \frac{\beta}{z^2(\epsilon \alpha - 2)} = 0.
\]

By a homographic transformation of the dependent variable and a change of the independent variable

\[
v = -i \sqrt{\delta} \frac{y + 1}{y - 1}, \quad z = \sqrt{2} \tau,
\]

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we obtain from (5.14) equation P5

\[ y'' + \frac{3y - 1}{2(y - 1)} y' + \frac{y'}{\tau} + \frac{1}{32\delta \tau^2} [(y^2 - 1)(Ay + B)] + \frac{\epsilon}{\tau} \gamma (-\delta)^{1/2} y = 0, \quad (5.16) \]

where \(A\) and \(B\) are defined as follows

\[
A = \beta^2 + 4(-\delta)^{1/2} \beta - \delta \alpha^2 - 4\delta - 2(-\delta)^{1/2} \epsilon \alpha \beta + 4\epsilon \delta \alpha,
\]

\[
B = \delta \alpha^2 - 2(-\delta)^{1/2} \epsilon \alpha \beta + 4(-\delta)^{1/2} \beta + 4\delta - \beta^2 - 4\epsilon \delta \alpha.
\]

**Proposition 5.** Let \(y = y(z)\) be a solution of the fifth Painlevé equation (5.16) with parameters given by (5.17), such that the function

\[ r(z) = w' - (\epsilon \alpha - 1) \frac{w}{z} - \epsilon \gamma w^2 - 1 \neq 0, \]

does not vanish. Then the function

\[ S(\tau) = 1 - 2r^{-1}(\sqrt{2\tau}), \quad (5.18) \]

is a solution of the third Painlevé equation (5.12) with parameters \(\gamma \neq 0\) and \(\delta = -2\).

The \(\tau\)-functions for the rational class of solutions of the Painlevé equation P3 can be constructed\(^{22,23}\) in terms of the Umemura polynomials \(T_n = T_n(z, l)\) which are determined by a sequence of polynomials in \(z\) and defined through the recurrence relation

\[
T_{n+1}T_{n-1} = \left(\frac{z}{8} - l + \frac{3}{4}n\right)T_n^2 + \frac{\partial T_n}{\partial z} T_n + z[\frac{\partial^2 T_n}{\partial z^2} T_n - (\frac{\partial T_n}{\partial z})^2], \quad (5.19)
\]

with initial conditions \(T_0 = T_1 = 1\). Based on reference \(^{22}\), we have

**Proposition 6.** For the existence of rational solutions of equation P3 of the form

\[ w(z) = \frac{T_{n+1}(z, l - 1)T_n(z, l)}{T_{n+1}(z, l)T_n(z, l - 1)}, \quad (5.20) \]

where the Umemura polynomials \(T_n = T_n(z, l)\) satisfy recurrence relation (5.19), it is necessary and sufficient that the parameters of equation P5 satisfy

\[
\alpha = 4(n + l), \quad \beta = 4(n - l), \quad \gamma = -\delta = 4.
\]
Note that from system (5.13) and the transformation (5.15), there is a connection between solutions \( w \) of the third Painlevé equation (5.12) and the solutions \( y \) of the fifth Painlevé equation (5.16),
\[
y = \frac{zw' - [(4\epsilon(n + l) - 1) + 4\epsilon zw]w + 2z}{z w' - [(4\epsilon(n + l) - 1) + 4\epsilon zw]w - 2z}.
\] (5.21)
Substituting the rational solutions (5.20) into formula (5.21) and next replacing the \( w \) which appears in (5.3) by the function \( u \) so obtained, we get multivortex solutions of equations (2.6). Consequently, by applying Proposition 2 to the so obtained multivortex solution of (2.6), certain classes of solutions of system (2.1) can be found.

Another class of vortex solutions to CSG equations (2.5) can be provided if we define functions \( u \) and \( v \) in the polar form following
\[
u = A_n(r)e^{i\theta}, \quad v = A_{n-1}(r)e^{i(n-1)\theta}, \quad n \in \mathbb{Z}.
\] (5.22)
which transforms the CSG system (2.5) into
\[
(i) \quad \frac{dA_n}{dr} + \frac{n}{r}A_n = (1 - A_n^2)A_{n-1},
\]
\[
(ii) \quad -\frac{dA_{n-1}}{dr} + \frac{(n-1)}{r}A_{n-1} = (1 - A_{n-1}^2)A_n.
\] (5.23)
When \( n = 1 \), the second equation (5.23) is solved by taking \( A_0 = \pm 1 \) and then the first equation (5.23) becomes a Riccati equation which can be linearized by a Cole-Hopf transformation and solved in terms of Bessel functions. The vortex solution (5.22) takes the form
\[
u = \frac{I_1(r)}{I_0(r)}e^{i\theta}, \quad v = \epsilon = \pm 1,
\] (5.24)
where \( I_1 \) is the Bessel function of the first order, that is, \( I_1 = I_1'(r) \), and the prime denotes differentiation with respect to \( r \). Such a reduction of (5.23) has been recently obtained\(^\text{19}\). Consequently, from transformation (2.4), we get
\[
\psi_1 = \frac{I_1(r)}{I_0(r)}e^{i\theta} \bar{\psi}_2, \quad \varphi_1 = \epsilon \bar{\varphi}_2.
\] (5.25)
Substituting (5.25) into Weierstrass system (2.1) and solving the resulting equations, we obtain
\[
\varphi_2 = F(re^{-i\theta}),
\] (5.26)
where $F$ is an arbitrary function of one variable $re^{-i\theta}$ and the function $\psi_2$ satisfies the PDE

$$\frac{\partial \psi_2}{\partial r} + \frac{ie^{-i\theta}}{r} \frac{\partial \psi_2}{\partial \theta} = 2e^{-i\theta} R(r) |\psi_2|^2 \psi_2, \quad R(r) = 1 - \frac{I_0^2(r)}{I_0^2(\tau)}. \quad (5.27)$$

Equation (5.27) has a solution of the form

$$\psi_2 = g(v(r)e^{-i\theta}), \quad (5.28)$$

where the functions $g$ and $v$ satisfy the following ODEs,

$$v' + \frac{v}{r} - R(r) \lambda = 0, \quad \dot{g} \lambda + |g|^2 g = 0, \quad \lambda \in \mathbb{C},$$

and $g$ denotes the derivative of $g$ with respect to $s = v(r)e^{-i\theta}$. These two equations can be integrated to give the following expressions

$$v(r) = \frac{\lambda}{r} \int_0^r \tau R(\tau) \, d\tau, \quad \frac{\dot{g}}{g^\lambda} = c. \quad (5.29)$$

From equations (5.25), (5.26) and (5.28), we can summarize the results as follows,

$$\psi_1 = e^{i\theta} \frac{I_0(r)}{I_0(\tau)} g \left( \frac{\lambda e^{i\theta}}{r} \int_0^r \tau R(\tau) \, d\tau \right), \quad \psi_2 = g \left( \frac{\lambda e^{-i\theta}}{r} \int_0^r \tau R(\tau) \, d\tau \right). \quad (5.30)$$

$$\varphi_1 = \epsilon F(r e^{-i\theta}), \quad \varphi_2 = F(r e^{-i\theta}),$$

where $g$ is a function of one variable, which is restricted by relation (5.29). Note that the solution for the function $u$ in (5.24) has the form of a scalar field which has appeared in the study of the vortex solutions of superconductivity with asymptotic behavior of the radial part of the solution going to zero as $r$ goes to zero and constant as $r$ goes to infinity. Consequently, solutions (5.30) of the Weierstrass system (2.1) possess similar asymptotic behavior when functions $F$ and $g$ are bounded.

VI. EXAMPLES OF SOLVING THE WEIERSTRASS SYSTEM VIA THE COMPLEX SINE-GORDON EQUATION

Now we investigate the possibility of generating new multi-soliton solutions by taking products of known solutions of the CSG system (2.6). Thus we can formulate the following:
Proposition 7. Suppose $u$ is a solution of equation (2.6) with constant modulus $|u|^2 = |c|^2 \neq 1$. Suppose also that a complex function $w$ exists which satisfies $|w|^2 = 1$, and the differential constraint equation

$$u(\partial \bar{w} w + \bar{u}w) + \bar{u}\bar{w}w + \bar{u}\bar{w}w = 0. \quad (6.1)$$

Then the product function $U = uw$ is a solution of system (2.6).

Proof: Differentiating the function $U = uw$, we get the following expressions

$$\partial(wu) = (\partial u)w + u(\partial w), \quad \partial(uw) = (\partial u)w + (\partial w)u,$$

$$\partial\bar{w} = (\partial\bar{u})w + (\partial u)(\bar{w}) + (\partial u)(\bar{w}) + u(\partial\bar{w}).$$

Substituting $U$ into equation (2.6), using $|u|^2 = |c|^2$ and $|w|^2 = 1$, we obtain that

$$(\partial\bar{u})w + (\partial u)(\bar{w}) + (\partial u)(\bar{w}) + u(\partial\bar{w})$$

$$= \frac{\bar{w}}{1 \pm |c|^2}(\partial u)w + u(\partial w)) + \frac{uw}{4} (1 \pm |c|^2)$$

$$= (\partial\bar{u})w + (\partial u)(\bar{w}) + (\partial u)(\bar{w}) + u(\partial\bar{w})$$

$$= \frac{\bar{u}w}{1 \pm |c|^2}(\partial u)(\bar{w}) + \frac{|c|^2 u\bar{w}}{1 \pm |c|^2}(\bar{u})(\partial w) + \frac{uw}{4} (1 \pm |c|^2).$$

Substituting the second derivative $\partial\bar{u}$ from equation (2.6) into equation (6.2) and next collecting terms with respect to first derivatives of $u$ and $w$ and simplifying, we obtain the differential constraint (6.1). QED

In the case of the CShG equation, the constant $c$ need not necessarily have modulus different from one, since there is no singularity in the $(1 + |c|^2)^{-1}$ term in this case.

At this point, we would like to illustrate Proposition 10 for constructing a solution to system (2.6) with an elementary example. The simplest solution of analytic type, a vacuum solution, is given by

$$u = ce^{(Az - A\bar{z})}, \quad (6.3)$$

where $c$ and $A$ are complex constants. By substituting the solution $u$ into (2.6), it is easy to show that this is a solution provided that the following constraint holds between $c$ and $A$,

$$2c|A| = 1 \pm |c|^2, \quad \epsilon = \pm 1.$$
Suppose that \( f \) is a complex valued function of one complex variable \( z \) and define the function \( w \) as follows,

\[
w = \frac{f(z)}{\bar{f}(\bar{z})},
\]
such that \( f(z) \) satisfies the constraint (6.1), namely,

\[
\partial f(z) \bar{\partial} f(\bar{z}) + A \partial f(z) \bar{f}(\bar{z}) + \bar{A} f(z) \bar{\partial} f(\bar{z}) = 0.
\]

Then Proposition 7 implies that the function

\[
U = ce^{(\bar{A}z - A\bar{z})} \frac{f(z)}{\bar{f}(\bar{z})},
\]
is also a solution to system (2.6) and represents a one-soliton solution.

Let us illustrate our considerations by an example of the exponential solution (6.3) for \( A = -ia \). We show that we can use this to obtain a solution to system (2.1). First of all, since eliminating \( u \) from the pair of equations in (2.5) results in an equation which is identical to (2.6) but with \( u \) replaced by \( v \), we can assign the solution obtained from the second order equation to either the \( u \) or the \( v \) variable. Let us take \( u = ce^{ia(z+\bar{z})} \). The second function \( v \) is obtained from (2.5), that is

\[
u = \frac{2\partial u}{1 \pm |u|^2} = ice^{ia(z+\bar{z})}.
\]
These satisfy (2.5) provided that \( 2a = 1 \pm |c|^2 \). From (2.4) we can write \( \psi_1 = u\bar{\psi}_2 \) and \( \varphi_1 = v\bar{\varphi}_2 \), and calculate the quantities \( Q_{1,2} \) from (2.1)

\[
Q_1 = |\psi_2|^2(1 \pm |c|^2), \quad Q_2 = |\varphi_2|^2(1 \pm |c|^2).
\]

Now, we will show that we can find a particular class of solutions which satisfy (2.1). First from equations (2.1) with (6.5), the equation

\[
\bar{\partial} \psi_2 = |\psi_2|^2(1 \pm |c|^2)
\]
is satisfied by a function of the form \( \psi_2 = e^{b(z-\bar{z})} \) provided that \( b = -2a \) to be consistent with the condition \( 2a = 1 \pm |c|^2 \). Similarly, a function of the form \( \varphi_2 = e^{2a(z-\bar{z})} \) satisfies the equations
\[ \partial \varphi_2 = Q_2 \varphi_2. \] The equations for \( \psi_1 \) and \( \varphi_1 \) in (2.1) can also be integrated, and the set of functions below
\[
\psi_2 = ce^{ia(z + \bar{z})} e^{2a(z - \bar{z})}, \quad \varphi_1 = ice^{ia(z + \bar{z})} e^{-2a(z - \bar{z})}, \quad \psi_2 = e^{-2a(z - \bar{z})}, \quad \varphi_2 = e^{2a(z - \bar{z})}
\] constitutes a solution to system (2.1). Substituting (6.6) into (2.3) and integrating, we obtain the parametric form of a surface. One obtains that
\[
X^1 = -a(z - \bar{z})^2 + 2a|z|^2 = ay^2 + 2ar^2, \quad X^2 = a(z - \bar{z})^2 - 2a|z|^2 = -ay^2 - 2ar^2,
\]
\[
X^3 = 0, \quad X^4 = i|c|^2(z - \bar{z}) = -|c|^2 y.
\] (6.7)
where we set \( z - \bar{z} = iy \) and \( |\bar{z}|^2 = r^2 \). Treating \( y \) and \( r > 0 \) as parameters, one can plot \( X^1, X^2 \) and \( X^4 \) to give the surface given in Figure 1, which has the form of a parabolic cylinder. Moreover, from (6.12), we can calculate the components of the induced metric
\[
g_{zz} = \sum_{i=1}^{4} (X^i_z)^2 = 8a^2(z^2 - 4|z|^2 + 4\bar{z}^2) - |c|^4 = \bar{g}_{zz},
\]
\[
g_{z\bar{z}} = \sum_{i=1}^{4} X_i^z X_i^\bar{z} = 8a^2(2z + \bar{z})(z - 2\bar{z}) + |c|^4.
\]
A procedure for obtaining solutions to (2.1) as a result of using Proposition 10 can be developed from the above example. The new product solution can be called either \( u \) or \( v \). One substitutes this in the corresponding equation in (2.5) to obtain the remaining unknown solution \( v \) or \( u \). Using these results in (2.4) to eliminate \( (\bar{\psi}_2, \bar{\varphi}_2) \), one tries to integrate the nonlinear system (2.1) to obtain the required complex functions \( \psi_1 \) and \( \varphi_1 \), which are solutions of the Weierstrass system (2.1).

IV. FINAL REMARKS.

Equations (2.1) and (2.5) under investigation here have long been of interest in field theory. In particular, there has been the extensive use of applying soliton solutions to construct models of extended particles \(^6\,^{24}\). The Sine-Gordon equation, it seems is the only Lorentz-invariant, nonlinear equation whose initial value problem has been solved \(^4\). This equation also describes a completely integrable Hamiltonian system. It would certainly be of great interest to find other
Lorentz invariant integrable systems. Of more recent interest is the study of vortex tubes. The motion of vortex tubes in an inviscid incompressible fluid is described by the Biot-Savard law. The recently proposed localized induction equation is the simplest model to capture the leading order behavior of the three-dimensional self-induced motion of a vortex filament. This type of equation is in fact related to the cubic nonlinear Schrödinger equation for a complex variable, and implies that the localized induction equation is completely integrable. Note that from solutions (3.36), when the functions $F$ and $g$ are real polynomials in a single variable, the vortex structure of the solutions is preserved at the level of the functions $\psi_\alpha$ and $\varphi_\beta$. These functions are applied to generate surfaces. Consequently, by plotting such results in two or three dimensions, these surfaces could model a vortex filament in such a fluid.

We have presented a new approach to the study of the Weierstrass system (2.1) in connection with CShG and CSG equations (2.6). It proved to be particularly effective in constructing multi-vortex solutions of (2.1) in terms of $\tau$-functions based on rational solutions of the third and fifth Painlevé equations. It is worth noting that the approach to the Weierstrass system (2.1) proposed here can be applied with some necessary modifications, to more general cases of Weierstrass type systems describing more diverse surfaces immersed in multi-dimensional Minkowski and pseudo-Riemannian spaces. The task of obtaining new types of minimal surfaces described by system (2.1) will be undertaken in our future work.

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Figure Captions.

Figure 1. The surface which corresponds to the solution (6.6) of the CShG equation and coordinates (6.7) for the choice of constants $a = 1$ and $c = 1$. 
