Nuclear polarization corrections in the $\mu^4{\text{He}}^+$ Lamb shift

C. Ji,$^1$ N. Nevo Dimur,$^2$ S. Bacca,$^1$ and N. Barnea$^2$

$^1$TRIUMF, 4004 Wesbrook Mall, Vancouver, BC V6T 2A3, Canada
$^2$Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel

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Stimulated by the proton radius conundrum, measurements of the Lamb shift in various light muonic atoms are planned at PSI. The aim is to extract the rms charge radius with high precision, limited by the uncertainty in the nuclear polarization corrections. We present an ab-initio calculation of the nuclear polarization for $\mu^4{\text{He}}^+$ leading to an energy correction in the $2S$-$2P$ transitions of $\delta_{pol}^{A} = -2.47$ meV $\pm 6\%$. We use two different state-of-the-art nuclear Hamiltonians and utilize the Lorentz integral transform with hyperspherical harmonics expansion as few-body methods. We take into account the leading multipole contributions, plus Coulomb, relativistic and finite-nucleon-size corrections. Our main source of uncertainty is the nuclear Hamiltonian, which currently limits the attainable accuracy. Our predictions considerably reduce the uncertainty with respect to previous estimates and should be instrumental to the $\mu^4{\text{He}}^+$ experiment planned for 2013.

Introduction — Recent laser spectroscopy measurements of the muonic Hydrogen Lamb shift $^1$ ($2S$-$2P$ transition) at PSI have tremendously improved the accuracy in determining the proton charge radius $\langle r_p^2 \rangle^{1/2}$. Besides experimental precision, the accurate deduction $\langle r_p^2 \rangle^{1/2}$ rely on theory. Theoretical estimates of quantum electro-dynamics (QED), recoil, and nuclear structure corrections are needed. The proton radius extracted at PSI $^2, ^3$ is 10 times more accurate than the value determined from electron Hydrogen, i.e., CODATA-2010 $^4$, but also deviates from it by 7 $\%$. This discrepancy, coined the “proton radius puzzle”, is challenging the understanding of experimental systematic errors and of theoretical calculations based on the standard model. Alternative explanations involving physics beyond the standard model (e.g., lepton flavor universality violations) have also been proposed (see $^5$ for a review). To understand this puzzle, one possible strategy is to investigate atoms with other nuclear charges $Z$ or mass numbers $A$, and track the persistence or variation of this discrepancy $^6$. Extending the Lamb shift measurements to other muonic atoms, e.g., $\mu D$, $\mu^3{\text{He}}^+$ and $\mu^4{\text{He}}^+$, must be complemented by corresponding theoretical calculations. Lamb shifts in light muonic atoms are very sensitive to nuclear structure effects since a muon is 206 times heavier than an electron and thus interacts more closely with the nucleus $^7, ^8$. The $2S$-$2P$ energy difference can be generally related to the nuclear charge radius $\langle R_n^2 \rangle^{1/2}$ (in $\hbar = c = 1$ units) by $^9$

$$\Delta E \equiv \delta_{QED} + \delta_{pol} + \delta_{Zem} + m_\mu^4(Z\alpha)^4(R_n^2)^{1/2},$$

in an expansion of $Z\alpha$ up to 5$^\text{th}$ order. Here $\alpha$ is the fine-structure constant and $m_\mu = m_\mu/M_A/(m_\mu + M_A)$ is the reduced mass related to the nuclear mass $M_A$ and the muon mass $m_\mu$. $\delta_{Zem}$ is the $3^\text{rd}$ Zemach moment defined via the nuclear charge density, $\rho_0(R)$, as

$$\delta_{Zem} = - \frac{m_\mu^4}{24}(Z\alpha)^5 \int dR dR' |R - R'|^3 \rho_0(R)\rho_0(R').$$

Contributions to $\delta_{QED}$ in Eq. $^1$ are from vacuum polarization, muon self energy and relativistic recoil; while $\delta_{pol} = \delta_{pol}^A + \delta_{pol}^N$ is the sum of the nuclear polarization $\delta_{pol}^A$ and the intrinsic nuclear polarizability $\delta_{pol}^N$. Since calculations of $\delta_{QED}$ and spectroscopy measurements of $\Delta E$ have both achieved high accuracy, the current bottleneck in accurately extracting $\langle R_n^2 \rangle^{1/2}$ from Eq. $^1$ lies in the polarization uncertainty. In muonic helium, to determine the nuclear radii with a relative accuracy of $3 \times 10^{-4}$, $\delta_{pol}$ needs to be known at the $\sim 5\%$ level $^6$. Here we focus on the nuclear polarization $\delta_{pol}^A, \delta_{pol}^N$ depends on the internal nucleon structure and can be evaluated separately $^7, ^8$ apart from nuclear dynamics.

FIG. 1. The lepton-nucleus two-photon exchange.

The nuclear polarization is induced by a two-photon exchange process (Fig. $^1$), where the nucleus in an atom is virtually excited by its Coulomb interaction with the lepton. Effects on the leptonic spectrum are evaluated in second-order perturbation theory with inputs from nuclear structure functions, also called response functions. In early calculations structure functions were either calculated using simple nuclear potentials (e.g., $\mu D$ $^14$ and $\mu^{-1}\text{C}$ $^15$) or extracted from measurements of photo-absorption cross sections (e.g., $\mu^4{\text{He}}^+$ $^16, ^18$). However, these approaches lack the desired accuracy. For example, Refs. $^14, ^18$ yielded $\delta_{pol}^A = -3.1$ meV $\pm 20\%$ for $\mu^4{\text{He}}^+$. Evaluations of the polarization effect in $\mu D$ using state-of-the-art potentials have significantly improved the accuracy $^12, ^20$. The purpose of this Letter is to extend these calculations to $\mu^4{\text{He}}^+$. We present the first ab-initio calculation of the nuclear polarization effects in
\( \mu^4 \text{He}^+ \) using modern nuclear potentials. We consider systematically all terms contributing to order \( (Z\alpha)^5 \) and estimate the theoretical error.

**Polarization Contributions** — Following works on \( \mu D \) by Pachucki [20] and Friar [21] we separate contributions to the \( \mu^4 \text{He}^+ \) polarization into non-relativistic, relativistic, Coulomb distortion and nucleon-size effects. The \( \mu^4 \text{He}^+ \) system is described as a muon interacting with the \( ^4\text{He} \) nucleus containing four point-like nucleons by

\[
H = H_{\text{nuc}} + H_\mu - \Delta H, \tag{3}
\]

where \( H_{\text{nuc}} \) denotes the nuclear Hamiltonian, and \( H_\mu \) is the non-relativistic Hamiltonian of a muon in the Coulomb potential of a point-like nucleus

\[
H_\mu = p^2/(2m_\mu) - Z\alpha/r . \tag{4}
\]

Here \( p = |p| \) \( (r = |r|) \) is the relative momentum (distance) of the muon from the center of mass (CM) of the nucleus. The last term in Eq. (3),

\[
\Delta H = \sum_a Z_a \Delta V(r,R_a) = \sum_a \frac{1}{r - R_a - 1} r , \tag{5}
\]

represents the difference between the muon interaction with the nucleus and the sum of Coulomb interactions between the muon and each proton, located at a distance \( R_a \) from the CM. Polarization effects are evaluated as corrections due to \( \Delta H \) in second-order perturbation theory. Utilizing the point-nucleon charge density operator

\[
\hat{\rho}(R) \equiv \frac{1}{Z} \sum_a Z_a \delta(R - R_a) , \tag{6}
\]

the nuclear polarization correction assumes the form

\[
\delta_{\text{pol}}^A = - \sum_{N \neq N_0} \int \! dR \bar{R}' P(R,R',\omega_N)\rho_N(R') , \tag{7}
\]

where \( \rho_N(R) = \langle N|\hat{\rho}(R)|N_0 \rangle \) is the charge density transition matrix element and

\[
P(R,R',\omega_N) = - Z^2 \int \! dV(r,R)\rho_{\mu}(r) \langle 0|\hat{\rho}(R)|N_0 \rangle \langle N|\hat{\rho}(R')|0 \rangle \langle 0|\hat{\rho}(R')|N_0 \rangle \langle N|\hat{\rho}(R')|0 \rangle \Delta V(r,R') \tag{8}
\]

is the muonic matrix element. Here \( \omega_N = E_N - E_{N_0} \) and \( E_{N_0}, E_N, |N_0 \rangle \) and \( |N \rangle \) are the nuclear ground- and excited-state energies and wave-functions, respectively. The symbol \( \langle \rangle \) indicates a sum over discrete plus an integration over continuum states. \( \epsilon_{\mu_0} \) and \( |\mu_0 \rangle \) are the unperturbed atomic energy and wave-function in either the 2S or 2P state. In Eq. (7) the nucleus is excited into all possible intermediate states, which represents the inelastic part of the two-photon exchange; while the elastic part is known as a finite-size effect [12].

The leading contribution to \( \delta_{\text{pol}}^A \) is obtained in the non-relativistic limit, neglecting in Eq. (8) the Coulomb potential part of \( H_\mu \). Only contributions to the 2S state are considered, as 2P-state effects enter only at order \( (Z\alpha)^6 \). In this limit, we have

\[
P(R,R',\omega) = - Z^2 \delta^2(0) \int \! \frac{dq}{(2\pi q^2)} \left( \frac{4\alpha}{q^2} \right)^2 (1 - e^{-i q R}) \times \frac{1}{q^2/(2m_\mu + \omega)} \left( 1 - e^{-i q R'} \right) , \tag{9}
\]

where \( \delta^2(0) = (m_\mu Z\alpha)^3/8\pi \) is the normalization coefficient of the muon 2S state. After integrating over \( q \) in Eq. (9), terms not depending on both \( R \) and \( R' \) drop out, due to the orthogonality of the nuclear eigenstates. The resulting muonic matrix element \( P \) is then a function of \( \sqrt{2m_\mu \omega} \), with \( \xi \equiv |\mathbf{R} - R'| \). Expanding \( P \) in powers of \( \sqrt{2m_\mu \omega} \) up to 4th order yields

\[
P(\xi,\omega) \simeq \frac{m_\mu^2(Z\alpha)^3}{12} \sqrt{\frac{2m_\mu \omega}{\omega^3}} \left[ \xi^2 - \frac{\sqrt{2m_\mu \omega}}{4} \xi + \frac{m_\mu \omega}{10} \xi^4 \right] . \tag{10}
\]

\( \xi \) indicates the “virtual” distance a proton travels during the two-photon exchange. According to the uncertainty principle it is related to \( \omega \) by \( \xi \sim 1/\sqrt{2M_A \omega} \). Therefore the expansion parameter \( \xi \sqrt{2m_\mu \omega} \) in Eq. (10) is of order \( \sqrt{m_\mu/M_A} \approx 0.17 \).

In the following we will relate the different \( \delta_{\text{pol}}^A \) terms coming from Eq. (10) to structure functions. Details will be given in a forthcoming paper [22]. The structure functions are defined as

\[
S_O(\omega) = \frac{1}{2J_0 + 1} \sum_{N \neq N_0} \langle |N_0 J_0|\hat{O}|N J \rangle^2 \delta(\omega - \omega_N) , \tag{11}
\]

where \( \hat{O} \) is a general operator. Here we use the reduced matrix elements by employing the Wigner-Eckart theorem [23]. \( J_0 \) \( (J) \) is the total angular momentum of the ground (excited) state of \( ^4\text{He} \).

The leading contribution to the nuclear polarization, denoted by the superscript \( (0) \), is the electric-dipole correction, which originates from the \( \xi^2 \) term in Eq. (10)

\[
\delta_{D1}^{(0)} = - \frac{2\pi m_\mu^3}{9} (Z\alpha)^5 \int_{\omega_{th}}^\infty \! d\omega \sqrt{\frac{2m_\mu \omega}{\omega}} S_{D1}(\omega) , \tag{12}
\]

where \( \hat{D}_1 = \frac{1}{2} \sum a R_a Y_1(\hat{R}_a) \) is the rank-1 spherical harmonics, and \( \omega_{th} \) indicates the threshold excitation energy of \( ^4\text{He} \), i.e., \( \omega_{th} = 19.8 \text{ MeV} \).

The sub-leading \( \xi^3 \) term is independent of \( \omega \). Replacing \( \sum_{N \neq N_0} |N \rangle \langle N | \) with \( 1 - |N_0 \rangle \langle N_0 | \), the contribution of this term, denoted by the superscript \( (1) \), is

\[
\delta^{(1)} = - \frac{m_\mu^4}{24} (Z\alpha)^5 \int \! dR dR' |R - R'|^3 \times \left[ |N_0 \rangle |\hat{\delta}(R)\hat{\rho}(R')|N_0 \rangle - \rho_0(R) \rho_0(R') \right] . \tag{13}
\]
where $\rho_0(R) \equiv \rho_{N_0}(R) = \langle N_0|\hat{\rho}(R)|N_0\rangle$ is the charge density, satisfying $\int dR \rho_0(R) = 1$. It is convenient to write Eq. (13) as $\delta^{(1)} = \delta_{R_{3pp}}^{(1)} + \delta_{Z_3}^{(1)}$. The first term $\delta_{R_{3pp}}^{(1)}$ is the ground-state expectation value of the proton-proton distance cubed. The second term $\delta_{Z_3}^{(1)}$ cancels exactly the 3rd Zemach moment $\delta_{Z_{em}}$ that appears in the finite-size corrections to the Lamb shift [11]. This cancellation was also found by Pachucki [21] and Friar [21, 24] in µD. Here we retain this term and calculate $\delta_{R_{3pp}}^{(1)}$ and $\delta_{Z_3}^{(1)}$ first in the point-nucleon limit and then add finite-nucleon-size corrections.

Contributions from the sub-sub-leading $\xi^4$ term, denoted with the superscript (2), are

$$\delta^{(2)} = \frac{m^5}{18} (Z \alpha)^5 \int_{\omega_{th}}^{\infty} dw \sum \frac{\omega}{2m_r} \left[ S_{R^2}(\omega) + \frac{16\pi}{25} S_Q(\omega) + \frac{16\pi}{5} S_{D_{1}D_{3}}(\omega) \right],$$

(14)

where $S_{R^2}$ and $S_Q$ are the respective structure functions of the monopole $R^2 = \frac{1}{2} \sum_{n} R_{n}^2$ and quadrupole $Q = \frac{1}{2} \sum_{n} R_{n}^2 Y_{2}(R_{n})$ operators. $S_{D_{1}D_{3}}$ indicates the interference between two multipolarity-1 operators $D_{1}$ and $D_{3} = \frac{1}{2} \sum_{n} R_{n}^2 Y_{1}(R_{n})$ is calculated as

$$S_{D_{1}D_{3}}(\omega) = \frac{1}{2} [S_{D_{1}D_{3}}(\omega) - S_{D_{1}}(\omega) - S_{D_{3}}(\omega)].$$

(15)

Effects from $\bar{R}^2$, $Q$ and the interference term in Eq. (14) are defined respectively as $\delta_{R_2}^{(2)}$, $\delta_{Q}^{(2)}$, and $\delta_{D_{1}D_{3}}^{(2)}$.

Since the electric-dipole contribution $\delta_{D_1}^{(0)}$ dominates in the non-relativistic approximation, we add relativistic corrections solely to this term. These corrections can be obtained from the longitudinal (L) and transverse (T) parts of the two-photon exchange amplitude [13, 10]. Replacing the non-relativistic Green’s function with relativistic expressions, we obtain relativistic corrections as

$$\delta_{L(T)}^{(0)} = \frac{2m^3}{9} (Z \alpha)^5 \int_{\omega_{th}}^{\infty} dw \Theta_{L(T)} \left( \frac{\omega}{m_r} \right) S_{D_{1}}(\omega),$$

(16)

where the two energy-dependent weights are

$$\Theta_{L}(\lambda) = \pi \sqrt{2/\lambda} + 2\mathcal{F}(\lambda),$$

(17)

$$\Theta_{T}(\lambda) = \lambda + \lambda \ln(2\lambda) + \lambda^2 \mathcal{F}(\lambda),$$

(18)

with

$$\mathcal{F}(\lambda) = \sqrt{(\lambda - 2)/\lambda} \arctanh \left( \sqrt{(\lambda - 2)/\lambda} \right) - \sqrt{(\lambda + 2)/\lambda} \arctanh \left( \sqrt{(\lambda + 2)/\lambda} \right),$$

(19)

and $\lambda \equiv \omega/m_r$ ranges from $\sim 0.2$ to infinity. Expressions similar to Eqs. (17) and (18) are also derived by Martorell et al. [23], whose transverse form is, however, valid only for $\lambda \geq 2$.

Even though $\delta_{C}^{(0)}$ is of order $(Z \alpha)^6$, its contribution is significantly enhanced by the 2S-logarithmic term in Eq. (20).

By including a Coulomb distorted muonic wavefunction in the intermediate state of the two-photon exchange in Fig. 1, $\delta_{D_{1}}^{(0)}$ is corrected in both the 2S and 2P states. We follow the derivation by Friar [13] and provide Coulomb-distortion corrections up to 2nd order in a $Z \alpha \sqrt{2m_r/\omega}$ expansion. The Coulomb-distortion correction is given as the difference between the 2S and 2P levels:

$$\delta_{C}^{(0)} = -\frac{2\pi m^3}{9} (Z \alpha)^6 \int_{\omega_{th}}^{\infty} dw \left[ \frac{m_r}{\omega} \left( \frac{1}{6} + \ln \frac{2m_r Z^2 \alpha^2}{\omega} \right) \right] + \frac{17}{16} Z \alpha \left( \frac{2m_r}{\omega} \right)^{3/2} S_{D_{1}}(\omega).$$

(20)

Even though $\delta_{C}^{(0)}$ is of order $(Z \alpha)^6$, its contribution is significantly enhanced by the 2S-logarithmic term in Eq. (20).

Considering the finite size of the nucleons, the proton position in Eq. (5) should be replaced by a convolution over the proton charge density, and a similar term should be added for the neutron. For the proton and neutron form factors [24], we use low-momentum approximations: $G_{p}^{E}(q^2) \simeq 1 - 2q^2/\beta^2$ and $G_{n}^{E}(q^2) \simeq \lambda q^2$. Following Ref. [24], we choose $\beta = 4.120$ fm$^{-1}$ and $\lambda = 0.01935$ fm$^{-2}$ which reproduce $\langle r^2_p \rangle^{1/2} = 0.8409$ fm and $\langle r^2_p \rangle = -0.1161$ fm$^2$ [24]. Since corrections to $\delta_{C}^{(0)}$ vanish, the leading nucleon-size (NS) correction enters in $\delta^{(1)}$ as

$$\delta_{N}^{(1)} = -\frac{m^4}{9} (Z \alpha)^5 \left[ \frac{2}{\beta^2} - \lambda \right] \int dR dR' |R - R'| \times \left( \langle N_0|\hat{\rho}_L(R)\hat{\rho}_L(R')|N_0\rangle - \rho_0(R)\rho_0(R') \right),$$

(21)

where we have used the isospin symmetry of the $^4$He ground-state [28, 29]. The prefactors $2/\beta^2$ and $-\lambda$ account for respective contributions from proton-proton and neutron-proton correlations, whereas neutron-neutron correlations are neglected. Similarly to $\delta^{(1)}$, contributions to $\delta_{N}^{(1)}$ from the two integrals in Eq. (21) are denoted as $\delta_{N}^{(1)} = \delta_{R_{3pp}}^{(1)} + \delta_{Z_3}^{(1)}$.

The sub-leading nucleon-size correction enters in $\delta^{(2)}$ as

$$\delta_{NS}^{(2)} = -\frac{16\pi}{9} m^5 (Z \alpha)^5 \left[ \frac{2}{\beta^2} - \lambda \right] \int_{\omega_{th}}^{\infty} dw \left( \frac{\omega}{2m_r} \right) S_{D_{1}}(\omega).$$

(22)

Summing up the nuclear polarization corrections to the Lamb shift we have

$$\delta_{pol}^{A} = \delta^{(0)} + \delta^{(1)} + \delta^{(2)} + \delta_{NS}$$

$$= \left[ \delta_{D_{1}}^{(0)} + \delta_{L}^{(0)} + \delta_{Q}^{(0)} + \delta_{C}^{(0)} \right] + \left[ \delta_{R_{3pp}}^{(1)} + \delta_{Z_3}^{(1)} \right]$$

$$+ \left[ \delta_{R_{2}}^{(2)} + \delta_{Q_2}^{(2)} + \delta_{D_{1}D_{3}}^{(2)} \right] + \left[ \delta_{R_{1}pp}^{(1)} + \delta_{Z_1}^{(1)} + \delta_{Z_3}^{(2)} \right].$$

(23)
Computational Tools — The $^4$He structure functions involve a sum over all the spectrum, including energies beyond the three-body disintegration threshold. Thus, we calculate them using the Lorentz integral transform (LIT) method \[39\] \[41\], which allows exact calculations in this energy range. We use the effective interaction hyperspherical harmonics (EIHH) \[32\] few-body technique to solve the $^4$He ground state and the LIT equations. The same methods were used, e.g., for the first realistic calculation of the $^4$He dipole structure function in Ref. \[33\].

For the nuclear Hamiltonian we use two state-of-the-art potential models that include three-nucleon (3N) forces: (i) the Argonne $v_{18}$ \[34\] nucleon-nucleon (NN) force supplemented by the Urbana IX \[35\] 3N force, denoted by AV18/UIX, and (ii) a chiral effective field theory potential \[36, 37\], denoted by $\chi_{EFT}$, where the NN and 3N forces are at $N^3$LO and $N^2$LO in the chiral expansion, respectively. For the chiral 3N force we use the parameterization of the low-energy constants obtained in \[38\] ($c_D = 1$ and $c_E = -0.029$). The calculated $^4$He binding energy, point-proton radius and electric-dipole polarizability $\alpha_r$ are respectively 28.422 MeV, 1.432 fm and 0.0651 fm$^3$ for the AV18/UIX potential. The corresponding numbers for the $\chi_{EFT}$ force are 28.343 MeV, 1.475 fm and 0.0694 fm$^3$. These numbers are in good agreement with previous calculations \[29\], \[39\]. \[40\]. The theoretical AV18/UIX ($\chi_{EFT}$) binding energy and radius are respectively within 0.3% (0.1%) and 3% (0.3%) of the experimental values. The uncertainty of $\alpha_r$, spanned by these two potentials, agrees with one in a recent study from variations of the $\chi_{EFT}$ low-energy constants \[41\], and is much smaller than the experimental error.

Results — We first check the formalism by comparing our $\mu D$ results with Pachucki \[20\]. In Table I we present all corrections related to the dipole structure function $S_{D1}(\omega)$ obtained from the AV18 potential \[41\]. We find a good agreement for $\delta_{D1}^{(0)}$ and $\delta_{C}^{(0)}$. A difference in the relativistic corrections appears because in Ref. \[20\] $\delta_{C}^{(0)}$ includes only the leading term of $\Theta_T$ \[17\] in an $\omega/m_r$ expansion and neglects $\Theta_T$ \[18\], since it is one-order higher in $\omega/m_r$. These higher-order terms, which we include, provide additional relativistic corrections to the Lamb shift in $\mu D$. Consequently, the $-1.680$ meV result of Ref. \[20\] changes to $-1.698$ meV, where in this case the cancellation of the $3^{rd}$ Zemach moment is implemented as in Ref. \[20\].

Now we turn to $\mu ^4$He$^+$ and discuss the first ab-initio calculations for $\delta_{pol}^{A}$. Numerical results for the AV18/UIX and $\chi_{EFT}$ potentials are presented in Table. II. Average of $\delta_{pol}^{A}$ to $-2.475$ meV. In the point-nucleon treatment, we observe that the leading contribution $\delta_{pol}^{(0)}$, amounting to $-3.743$ meV with AV18/UIX and $-3.981$ meV with $\chi_{EFT}$, strongly dominates in $\delta_{pol}^{A}$.

Regarding the sub-leading terms, each individual term in $\delta^{(1)}$ (or $\delta^{(2)}$) is not necessarily small. Only their complete combination at each order fulfills the expansion in $\xi \sqrt{2m_0 \omega} \sim \sqrt{m_r / M_A}$ as a consequence of the uncertainty principle, and yields $\delta^{(1)} = 0.775$ meV and $\delta^{(2)} = 0.089$ meV when averaging AV18/UIX and $\chi_{EFT}$ calculations. As expected, $\delta^{(1)}$ and $\delta^{(2)}$ are respectively one- and two-order smaller in $\sqrt{m_r / M_A}$ than $\delta^{(0)}$. The nucleon-size correction contributes an additional $\delta_{NS} = 0.523$ meV in average. The latter depends on the value of $(r_F^2)^{1/2}$; using 0.8775 fm \[4\] will increase $\delta_{NS}$ to 0.579 meV.

The numerical accuracy of $\delta_{pol}^{A}$ is also studied. The error in the EIHH method is controlled by the convergence with respect to the maximum grand-angular momentum, $K_{max}$, which determines the size of the model space \[32\]. This error, obtained by taking the difference between results with $K_{max} = 22$ \([20]\) to those with $K_{max} = 4$, is 0.4% (0.2%) for AV18/UIX ($\chi_{EFT}$). An additional 0.2% error is estimated by comparing the results from integrating the structure functions calculated using the LIT method, with those obtained by a Lanczos sum-rule method as in Ref. \[39\].

### Table I. Nuclear polarization contributions to the 2S-2P Lambda shift $\Delta E$ [meV] in $\mu D$, compared to Pachucki \[20\].

|          | AV18/UIX | $\chi_{EFT}$ |
|----------|----------|--------------|
| $\delta_{D1}^{(0)}$ | -4.418 | -4.701 |
| $\delta_{L}^{(0)}$ | 0.289 | 0.308 |
| $\delta_{T}^{(0)}$ | -0.126 | -0.134 |
| $\delta_{C}^{(0)}$ | 0.512 | 0.546 |
| $\delta_{R}^{(1)}$ | -3.442 | -3.717 |
| $\delta_{D1}^{(1)}$ | 4.183 | 4.526 |
| $\delta_{D2}^{(1)}$ | 0.259 | 0.324 |
| $\delta_{Q}^{(1)}$ | 0.484 | 0.561 |
| $\delta_{D1D2}^{(1)}$ | -0.666 | -0.784 |
| $\delta_{R}^{(2)}$ | -1.036 | -1.071 |
| $\delta_{D1}^{(2)}$ | 1.753 | 1.811 |
| $\delta_{NS}^{(2)}$ | -0.200 | -0.210 |
| $\delta_{pol}^{A}$ | -2.408 | -2.542 |
The difference in \( \delta_{\text{pol}}^A \) that comes from using the AV18/UIX or 3EFT potential amounts to 0.134 meV and represents the uncertainty in nuclear physics. Both potentials are tuned to fit the \(^3\)He binding energy, and they reproduce the \(^4\)He binding energy to few parts per mil. They differ, however, in their respective predictions for the nuclear charge radius. Given the relations between the structure functions and the charge radius, it is plausible that the uncertainty in \( \delta_{\text{pol}}^A \), can be reduced using the \(^4\)He charge radius to constrain the nuclear potential models. This systematic uncertainty dominates the errors in predicting \( \mu \) \(^4\)He\(^+\) polarization effects. The difference between the two models divided by \( \sqrt{2} \) gives a 4\% error, which can be interpreted as a 1\( \sigma \) deviation from the central value. The magnetic polarization is negligible in \(^4\)He \( \| \). Terms of order \((Z\alpha)^6\), relativistic corrections to polarizations other than dipole, and higher-order nucleon-size effects will be explored in the future. The sum of all these additional corrections is expected to be a few percent. In a quadratic sum of all the errors mentioned above we estimate the accuracy of our calculation to be \( \pm 6\% \). We did not include the contribution from the disputed intrinsic nucleon polarizability \( \eta \), because it can be estimated independently of the nuclear Hamiltonian (see e.g. \[20\), \[21\]).

Conclusions — We perform the first \textit{ab-initio} calculation for the \( \mu \) \(^4\)He\(^+\) polarization correction obtaining \( \delta_{\text{pol}}^A = -2.47 \) meV \( \pm 6\% \). This result significantly improves the accuracy and is close to the upper bound of previous predictions \( \delta_{\text{pol}}^A = -3.1 \) meV \( \pm 20\% \). \[16\), \[18\]. The theoretical accuracy is limited by the uncertainty in the nuclear Hamiltonian, which is probed by using two different state-of-the-art nuclear potentials. Exploring other choices for potential parameterizations and including higher-order \( \chi\text{EFT} \) forces can possibly narrow this uncertainty. Our result allows a significant improvement in the precision of \( \langle R_e^2 \rangle \) that will be extracted from the \( \mu \) \(^4\)He\(^+\) Lamb shift measurements planned for 2013.

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\[1\] W. E. Lamb and R. C. Rutherford, Phys. Rev. 72, 241 (1947).
\[2\] R. Pohl \textit{et al.}, Nature 466, 213 (2010).
\[3\] A. Antognini \textit{et al.}, Science 339, 417 (2013).
\[4\] P. J. Mohr, B. N. Taylor and D. B. Newell, Rev. Mod. Phys. 84, 1527 (2012).
\[5\] R. Pohl, R.Gilman, G.A. Miller and K. Pachucki, Annu. Rev. Nucl. Part. Sci 63 (2013) [arXiv:1301.0095].
\[6\] A. Antognini \textit{et al.}, Can. J. Phys. 89, 47 (2011).
\[7\] C. E. Carlson and M. Vanderhaeghen, Phys. Rev. A 84, 020102 (2011).
\[8\] M. C. Birse and J. A. McGovern, Eur. Phys. J. A 48 (2012) 120.
\[9\] G. A. Miller, Phys. Lett. B 718, 1078 (2013).
\[10\] E. Borie and G. A. Rinker, Rev. Mod. Phys. 54, 67 (1982).
\[11\] E. Borie, Annals of Phys. 327, 733 (2012).
\[12\] J. L. Friar, Annals of Phys. 122, 151 (1979).
\[13\] A. C. Zemach, Phys. Rev. 104, 1771 (1956).
\[14\] Y. Lu and R. Rosenfelder, Phys. Lett. B 319, 7 (1993) [Erratum-ibid. B 333, 564 (1994)].
\[15\] R. Rosenfelder, Nucl. Phys. A 393, 301 (1983).
\[16\] J. Bernabeu and C. Jarlskog, Nucl. Phys. B 75, 59 (1974).
\[17\] G. A. Rinker, Phys. Rev. A 14, 18 (1976).
\[18\] J. L. Friar, Phys. Rev. C 16, 1540 (1977).
\[19\] W. Leidemann and R. Rosenfelder, Phys. Rev. C 51, 427 (1995).
\[20\] K. Pachucki, Phys. Rev. Lett. 106, 193007 (2011).
\[21\] J. L. Friar, [arXiv:1306.3269 [nucl-th]].
\[22\] C. Ji, N. Nevo Dimur, S. Bacca, N. Barnea, in preparation.
\[23\] A. R. Edmonds, “Angular momentum in quantum mechanics”, Vol. 4, Princeton University Press (1996).
\[24\] J. L. Friar and G. L. Payne, Phys. Rev. C 56, 619 (1997); Phys. Rev. A 56, 5173 (1997).
\[25\] J. Martorell, D. W. L. Sprung and D. C. Zheng, Phys. Rev. C 51, 1127 (1995).
\[26\] J. L. Friar and G. L. Payne, Phys. Rev. C 72, 014002 (2005).
\[27\] J. Beringer \textit{et al.}, Phys. Rev. D 86, 010001 (2012).
\[28\] M. Viviani, A. Kievsyky, and S. Rosati, Phys. Rev. C 71, 024006 (2005).
\[29\] A. Kievsyky, S. Rosati, M. Viviani, L. E. Marcucci, and L. Girlanda, J. Phys. G 35, 063101 (2008).
\[30\] V. D. Efros, W. Leidemann, and G. Orlandini, Phys. Lett. B 338, 130 (1994).
\[31\] V. D. Efros, W. Leidemann, G. Orlandini, and N. Barnea, J. Phys. G: Nucl. Part. Phys. 34, R459 (2007).
\[32\] N. Barnea, W. Leidemann, and G. Orlandini, Phys. Rev. C 61, 054001 (2000); Nucl. Phys. A693, 565 (2001).
\[33\] D. Gazit, S. Bacca, N. Barnea, W. Leidemann, and G. Orlandini, Phys. Rev. Lett. 96, 112301 (2006).
\[34\] R. B. Wiringa, V. G. J. Stoks and R. Schiavilla, Phys. Rev. C 51, 38 (1995).
\[35\] B.S. Pudliner, V.R. Pandharipande, J. Carlson, and R.B. Wiringa, Phys. Rev. Lett. 74, 4396 (1995).
\[36\] E. Epelbaum, H.-W. Hammer and U.-G. Meißner, Rev. Mod. Phys. 81, 1773 (2009).
\[37\] R. Machleidt and D. R. Entem, Phys. Rept. 503, 1 (2011).
\[38\] P. Navrátil, Few Body Syst., 41, 117 (2007).
\[39\] D. Gazit, N. Barnea, S. Bacca, W. Leidemann and G. Orlandini, Phys. Rev. C (R) 74, 061001 (2006).
\[40\] I. Stetcu, S. Quaglioni, J. L. Friar, A. C. Hayes, and Petr Navrátil, Phys. Rev. C 79, 064001 (2009).
\[41\] G. Bampa, W. Leidemann and H. Arenhövel, Phys. Rev.
C 84, 034005 (2011); W. Leidemann, private communication.

[42] S. Bacca, H. Arenhövel, N. Barnea, W. Leidemann, and G. Orlandini, Phys. Rev. C 76, 014003 (2007). A numerical test based on this semirealistic description of $^4$He finds the magnetic contributions to be sub-percentage.