Microscopic shell-model counterpart of the Bohr–Mottelson model

H. G. Ganev\textsuperscript{1,2,a}

\textsuperscript{1} Joint Institute for Nuclear Research, Dubna, Russia
\textsuperscript{2} Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Sofia, Bulgaria

Abstract In the present paper we demonstrate that there exists a fully microscopic shell-model counterpart of the Bohr–Mottelson model by embedding the latter in the microscopic shell-model theory of atomic nucleus within the framework of the recently proposed fully microscopic proton-neutron symplectic model (PNSM). For this purpose, another shell-model coupling scheme of the PNSM is considered in which the basis states are classified by the algebraic structure \( \text{SU}(6) \otimes \text{SO}(5) \). It is shown that the configuration space of the PNSM contains a six-dimensional subspace that is closely related to the configuration space of the generalized quadrupole-monopole Bohr–Mottelson model and its dynamics splits into radial and orbital motions. The group \( \text{SO}(5) \) acting in this space, in contrast, e.g., to popular IBM, contains an \( \text{SU}(3) \) subgroup which allows to introduce microscopic shell-model counterparts of the exactly solvable limits of the Bohr–Mottelson model that closely parallel the relationship of the original Wilets-Jean and rotor models. The Wilets-Jean-type dynamics in the present approach, in contrast to the original collective model formulation, is governed by the microscopic shell-model intrinsic structure of the symplectic bandhead which defines the relevant Pauli allowed \( \text{SO}(6) \), and hence \( \text{SU}(3) \), subrepresentations. The original Wilets-Jean dynamics of the generalized Bohr–Mottelson model is recovered for the case of closed-shell nuclei, for which the symplectic bandhead structure is trivially reduced to the scalar or equivalent to it irreducible representation.

1 Introduction

Majority of atomic nuclei exhibit collective behavior in their spectra, which is primarily represented by the low-energy nuclear rotation and vibrations. One of the fundamental models of nuclear structure is the Bohr–Mottelson (BM) collective model [1], which has demonstrated that nuclear collective motion can be described by considering only few macroscopic collective degrees of freedom. Conceptually, this model has an invaluable impact on the understanding of nuclear collective motion and the development of nuclear structure models. Moreover, for heavy nuclei, the BM collective model provides the basic concepts and language in terms of which the nuclear collective phenomena are described.

It is well known that the BM model has three algebraically solvable limits [1]: the harmonic vibrator model, the \( \gamma \)-unstable Wilits-Jean (WJ) model [2], and the rigid-rotor model [3,4]. These solvable submodels provide approximate descriptions of a subset of nuclear collective states. The group-theory of the BM model was given in [5], where the conventional \( U(5) \supset \text{SO}(5) \) harmonic-vibrational basis is exploited. This basis is actually used in practical applications of the BM model by the Frankfurt school [6]. The physics and mathematics of the exactly solvable limits of the BM model, as well as their relationships, are presented in an exhaustive manner in a recent book [7].

The BM model has been formulated also in algebraic terms by means of different spectrum generating algebras (SGA) and dynamical groups. The position and momentum coordinates of the BM model, for example, close the Lie algebra of Heisenberg–Weyl group \( HW(5) = \{a_\mu, \pi^\nu, I\} \). It is too small to contain useful subgroup chains with which to classify basis states, but it provides the basic building blocks from which numerous dynamical groups and SGA can be constructed. Among them, the following two are important for our present considerations, namely \( [HW(5)]U(5) \) and \( SU(1, 1) \otimes \text{SO}(5) \) dynamical groups [7]. The latter leads to convergent solutions for a wide range of collective Hamiltonians and allowed the formulation of a powerful version of the BM model, called the algebraic collective model (ACM) [8,9]. For spherical nuclei, the \( SU(1, 1) \otimes \text{SO}(5) \) basis of the ACM reduces to that of the five-dimensional harmonic oscillator and is given by the harmonic series of \( SU(1, 1) \) irreps. For deformed nuclei, the modified oscillator series [10] of
SU(1, 1) irreps give much more rapidly convergent results. The three classical BM submodels have been expressed in terms of the \([HW(5)]U(5)\) dynamical group and its subgroups. Thus, the three dynamical groups \(U(5), [R^6]SO(5)\) and \([R^5]SO(3)\) have been shown to correspond to the spherical vibrator, Wilets-Jean \(\gamma\)-unstable and rigid-rotor limits of the BM model, respectively [7,8]. The irreducible representations of the \([R^5]SO(3)\) group, as first shown by Elliott et al. [11], are characterized by rigid value of \(\beta = \beta_0\), i.e. by a sharp value of the quadrupole moment. In turn, the rigid-rotor model irreps are characterized by both \(\beta\) and \(\gamma\) rigid values \(\beta_0\) and \(\gamma_0\), respectively.

There is a close correspondence of the physics of the BM model to that of the Interacting Boson Model (IBM) [12], between which many relationships have been established [7,13] using the known fact that two finite Hilbert spaces of equal dimension are isomorphic to each other. In this regard, the IBM achieves a reduction of the non-compact Hilbert state space to a finite dimensional space by compactifying the algebraic structure \([HW(5)]U(5)\) of the collective model to the \(U(6)\) group of six-dimensional harmonic oscillator. It was also pointed out that the IBM could be considered as an algebraic approximation to the Bohr–Mottelson collective model [14]. As will be shown further, although the WJ and rigid-rotor models have algebraic structures and their states are characterized by certain dynamical group chains, they are not particularly useful in the construction of square-integrable wave functions due to their delta function nature. This problem was circumvented in the ACM [7,8] by relaxing the rigidity of WJ model by replacing its dynamical group \([R^5]SO(5)\) with \(SU(1, 1) \otimes SO(5)\) one, which results in a more physical collective model. In fact, the ACM is a computationally tractable version of the BM collective model, which make use of the \(\beta\) wave functions given analytically by the softened-\(\beta\) version of the WJ model, initially considered by Elliott et al. [11,15]. This allows the \(\beta\)-rigid and \(\gamma\)-rigid limits to be approached in a continuous way with increasingly narrow but square-integrable \(\beta\) and \(\gamma\) wave functions.

The IBM also has three exactly soluble dynamical symmetry limits that correspond to similar dynamical symmetries of the BM model. A trivial relationship between the two models is obtained for the five-dimensional spherical vibrator submodel of BM model which corresponds to the \(U(5)\) limit of the IBM. The \(\beta\)-rigid but \(\gamma\)-unstable WJ model has firstly been shown by Meyer-ter-Vehn [16] to correspond to the \(SO(6) \supset SO(5) \supset SO(3)\) dynamical symmetry limit of the IBM. Further, it turns out that there is no analogue in the IBM of the \(\beta\)-rigid and \(\gamma\)-rigid rotor submodel of the BM model, which is obviously a submodel of the \(\beta\)-rigid but \(\gamma\)-unstable WJ model since \([R^5]SO(3) \subset [R^6]SO(5)\). This is because in IBM the \(SU(3)\) dynamical group, associated with the rotational states, is not a subgroup of \(SO(6)\). Hence, the rotor-like states in the \(SU(3)\) limit of the IBM are not related to those of its \(SO(6)\) limit in a way that parallels the relationship between the rigid-rotor and WJ states in the BM model. This fact was stressed in Ref. [13]. In this regard, it is the purpose of present work to demonstrate that there exists a microscopic many-particle counterpart of the Bohr–Mottelson model whose exactly solvable limits have a relationship that closely resembles the one between the original WJ and rigid-rotor submodels.

From another side, it is known that a limitation of the BM model is that it has irrotational-flow moments of inertia which are much smaller than those needed to describe the low-energy rotational states of deformed nuclei. Usually this problem is resolved by treating the moments of inertia as free parameters which are fitted to the experimental spectra of nuclei. Further, in its standard formulation, the BM model can not be naturally related to the microscopic shell-model theory of the nucleus. In particular, the vectors in the BM model which define the states of a quantum-mechanical liquid drop cannot be identified with the wave functions in the many-particle Hilbert space of \(A\) nucleon antisymmetric states. The problem of incorporating the BM model into the microscopic theory of the nucleus and its importance for nuclear structure physics have been realized long time ago. The solution of this problem was given through the algebraic approach. It was shown (see, e.g. [17,18]) that the collective model of Bohr and Mottelson admits a microscopic realization first by augmenting it by vorticity degrees of freedom, important for the appearance of low-lying collective states, and second by making it compatible with the composite many-fermion structure of the nucleus. The result is the one-component \(Sp(6, R)\) symplectic model [19] of nuclear collective motion, sometimes called a microscopic collective model, which is a submodel of the nuclear shell model. The \(Sp(6, R)\) model of nuclear rotations, among its submodels, contains the rigid-rotor model [4] and the Elliott’s \(SU(3)\) shell model of collective rotations [20]. The presence of vorticity in the \(Sp(6, R)\) model results in a complete range of possible collective flows from irrotational-flow (zero vorticity) to rigid rotations. This is of significant importance as well as the fact that the vortex-spin degrees of freedom are responsible for the appearance of low-lying collective states [17,18]. However, the microscopic collective model, which is just a microscopic version of the BM model augmented by the vortex spin degrees of freedom and compatible with the many-particle nucleon structure of nucleus, does not contain an \(SO(6)\) algebraic structure that could allow to establish a close relationship to the WJ model in a manner similar to that of IBM. In this way, the findings of the present work in embedding the generalized Bohr–Mottelson

---

1 Throughout the present work, we will use the notation \(Sp(2n,R)\) for the group of linear canonical transformations in \(2n\)-dimensional phase space. Some authors denote the \(Sp(2n,R)\) group by \(Sp(n,R)\).
model in the microscopic shell-model theory of the nucleus have a more natural interpretation of the underlying BM quadrupole-monopole collective dynamics than in the microscopic realization of the BM collective model provided by the one-component $Sp(6, R)$ symplectic model.

Recently, a fully microscopic proton-neutron symplectic model (PNSM) of nuclear collective motion with $Sp(12, R)$ dynamical algebra was proposed by considering the symplectic geometry and possible collective flows in the two-component many-particle nuclear system [21]. The PNSM naturally involves both the vertical and horizontal mixings of different $SU(3)$ multiplets. Thus, it contains rotations with vortex degrees of freedom, high-energy, and (in contrast to the $Sp(6, R)$ model) low-energy vibrations. Through its more general motion group $GL(6, R) \subset Sp(12, R)$, which allows for the separate treatment of the collective dynamics of proton and neutron subsystems as well as the combined proton-neutron collective excitations, the PNSM generalizes the $Sp(6, R)$ model for the case of two-component proton-neutron many-particle nuclear systems. This can be easily understood by the embedding $Sp(6, R) \subset Sp(12, R)$. The configuration space of the PNSM is isomorphic to the coset space $GL(6, R)/SO(6)$ and is spanned by the commuting quadrupole moment observables, i.e. $\mathbb{R}^{21} = \{Q_{ij}(\alpha, \beta)\}$ (cf. Eq. (29)). This configuration space contains a six-dimensional subspace of the combined proton-neutron collective dynamics $\mathbb{R}^6 \subset \mathbb{R}^{21}$ that is closely related to the configuration space of the generalized BM model, in which the monopole degrees of freedom are also included. The group $SO(6)$ acting in $\mathbb{R}^6 \subset \mathbb{R}^{21}$ contains an $SU(3)$ subgroup, as will be demonstrated further, and allows us to introduce a microscopic shell-model counterpart of the BM model, whose two exactly solvable limits closely parallel the relationship of the original WJ and rotor models. For this purpose, we first shortly consider the classical versions of the original BM submodels. Then we consider the reformulation of the BM limits in algebraic terms and consider their relation to the IBM, which will allow a better understanding of the close relationships between the WJ and rigid-rotor models and their parallel construction within the framework of the PNSM. This reveals further dynamical content of the latter, not considered before.

The present paper deals with another shell-model coupling scheme within the framework of the microscopic proton-neutron symplectic-based shell-model approach. In principle, this coupling scheme provides an alternative basis for the shell-model diagonalization of an arbitrary collective Hamiltonian, which could also be expressed as a polynomial in the many-particle position and momentum coordinates of the two-component proton-neutron nuclear systems. This will extend the applicability of the PNSM in describing the collective properties in various nuclei. It will be interesting also to compare the results obtained within the two shell-model coupling schemes of the PNSM introduced so far.

2 Bohr–Mottelson submodels

The configuration space of the BM model is $\mathbb{R}^2$, i.e. it has the geometry of Euclidean space. The volume element in spherical coordinates is given by

$$dV = \beta^4 d\beta \sin^3 \gamma d\gamma d\Omega,$$

where $d\Omega$ is the $SO(3)$ volume element. The Laplacian operator is

$$\nabla^2 = \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} - \frac{\Lambda^2}{\beta^2},$$

where the $SO(5)$ Casimir operator $\Lambda^2$ is expressed in terms of the intrinsic $SO(3)$ angular momentum operators $\{\mathcal{L}_k\}$ as

$$\Lambda^2 = -\frac{1}{\sin^3 \gamma} \frac{\partial}{\partial \gamma} \sin^3 \gamma \frac{\partial}{\partial \gamma} + \sum_{k=1}^{3} \frac{\mathcal{L}_k^2}{4 \sin^2(\gamma - 2\pi k / 3)}.$$  

Then the Bohr Hamiltonian takes the well-known form [1]:

$$H = -\frac{\hbar^2}{2\mathcal{B}} \nabla^2 + V(\beta, \gamma),$$

where $V(\beta, \gamma)$ is the potential energy and $\mathcal{B}$ is the mass parameter. An exact solution of the Schrödinger equation for the Bohr Hamiltonian (3) exists only for a few potentials. It is well known that there are three standard submodels of the BM collective model which correspond to the three cases of solvable $V(\beta, \gamma)$.

2.1 Harmonic spherical vibrator

The Hamiltonian for the spherical vibrator submodel of the BM model

$$H_{HV} = -\frac{\hbar^2}{2\mathcal{B}} \nabla^2 + \frac{1}{2} C \beta^2,$$

can be expressed, in terms of the creation and annihilation operators of quadrupole phonons as that of five-dimensional harmonic oscillator,

$$H_{HV} = \sum_{\mu} \left( d_{\mu}^\dagger d_{\mu} + \frac{5}{2} \right) = \left( \hat{N} + \frac{5}{2} \right),$$
where \( \hat{T} \) is the identity operator. The eigenvectors of this Hamiltonian define the wave functions, while its eigenvalues are the energies and give an equidistant harmonic spectrum with a characteristic two-phonon multiplet of degenerate 0\(^+\), 2\(^+\) and 4\(^+\) states.

### 2.2 Wilets–Jean model

A major simplification in solving the Schrödinger equation for the Bohr Hamiltonian (3) arises when the potential is \( \gamma \)-independent, i.e. \( V = V(\beta) \). The WJ model [2] is thus invariant under all \( SO(5) \) transformations. The WJ Hamiltonian is an \( SO(5) \) invariant and its eigenvectors occur in multiplets that span irreducible representations of the \( SO(5) \) group. The energies are labeled by the \( SO(5) \) quantum number \( \tau \). Recall that the generic \( SO(5) \) irreps are determined by two quantum numbers \( (\tau_1, \tau_2) \), but in the case of BM or IBM, they take one-rowed Young patterns \( (\tau_1, \tau_2) = (\tau, 0) \).

The wave functions of the BM model can be expressed as products of radial \( \beta \) functions and orbital \( SO(5) \) wave functions [7]:

\[
\Psi_{\alpha\tau\alpha\Lambda}\beta, \gamma, \Omega = R_\alpha(\beta)Y_{\tau\alpha\Lambda}(\gamma, \Omega),
\]

where \( \tau = 0, 1, 2, \ldots \) is an \( SO(5) \) angular momentum quantum number, which is often referred to as \( SO(5) \) seniority. The explicit expressions for the \( SO(5) \) spherical harmonics with \( L \leq 6 \) have been obtained by Bes [22] by solving the coupled system of differential equations that they obey.

The energies and \( \beta \) wave functions of the WJ model are solutions of the eigenvalue equation

\[
\left[ -\frac{\hbar^2}{2B} \left( \frac{\nabla^2}{\beta^2} - \frac{\tau(\tau + 3)}{\beta^2} \right) + V(\beta) \right] R_\alpha(\beta) = E_\tau R_\alpha(\beta).
\]

A \( \beta \)-rigid WJ model assumes, in addition, that \( \beta \) coordinate is frozen at some non-zero value \( \beta_0 \). Then, the \( \beta \) degree of freedom is suppressed and the Hamiltonian (3) reduces to

\[
H_{WJ} = \frac{\hbar^2}{2B\beta_0^2} A^2
\]

whose eigenvalues are the energies which are given by

\[
E_\tau = \frac{\hbar^2}{2B\beta_0^2} \tau(\tau + 3).
\]

The spectrum for the WJ model is given, for example, in Fig. 6.20 of Ref. [23]. For \( \tau > 1 \), each value of \( \tau \) corresponds to more than one level and the \( \tau \) values 2, 3, 4, \ldots include a low-lying set of levels analogous to the \( \gamma \)-vibrational band and to the anomalous levels of the Davydov–Filippov model [3] for large \( \gamma \). The yrast levels having \( L = 2\tau \), according to Eq. (8), produce a characteristic ratio \( E_{4^+} / E_{2^+} = 2.50 \) of the WJ \( \gamma \)-unstable model for the first 2\(^+\) and 4\(^+\) states.

### 2.3 Rigid rotor model

Further simplifications result when both the \( \beta \) and \( \gamma \) coordinates are frozen. The only remaining degrees of freedom are then rotations in the three-dimensional space. The \( SO(5) \) Casimir operator (2), with \( \beta \) and \( \gamma \) taking fixed values \( \beta_0 \) and \( \gamma_0 \), reduces to

\[
A^2 = \sum_{k=1}^{3} \frac{T_k^2}{4\sin^2(\gamma_0 - 2\pi k/3)}.
\]

The BM collective Hamiltonian in this limit correspondingly reduces to that of a rotor

\[
H_{rot} = \sum_{k=1}^{3} \frac{\hbar^2 T_k^2}{2J_k},
\]

with the irrotational-flow moments of inertia \( J_k = 4B\beta_0^2 \sin^2(\gamma_0 - 2\pi k/3) \). It is known that the experimental moments of inertia are much larger than these irrotational flow values. That is why, in numerical applications, the moments of inertia are treated as free parameters that are fitted to the experimental data.

### 3 Formulation of the BM submodels in algebraic terms. Relation with the IBM

It is known that almost all models of nuclear structure can be expressed in algebraic terms of some spectrum generating algebras and dynamical groups (see, e.g., [18]). As often happens, the BM model also has more than one dynamical group. The Heisenberg–Weyl group \( HW(5) = \{e^{\alpha_\mu}, \pi^\nu, I\} \), which is the simplest dynamical group spanned by the quadrupole collective variables and their conjugate momentum operators in \( \mathbb{R}^5 \), is too small to contain useful subgroup chains with which to classify the basis states. However, it provides the basic building blocks from which numerous dynamical groups and spectrum generating algebras can be constructed. Among them, we mention the following two dynamical groups: \( HW(5) U(5) \) and \( SU(1, 1) \otimes SO(5) \). The latter, as discussed in Sect. 3.4, turns out to be very efficient one on which the algebraic version of the BM collective model is based [8,9], while the former is relevant to the three submodels. The three BM submodels are therefore associated with dynamical group chains corresponding to different paths.
through the set of groups [7, 8]:

\[ [HW(5)]U(5) \supset [R^5]SO(5) \supset [R^5]SO(3) \]

\[ \bigcup \bigcup \bigcup \]

\[ U(5) \supset SO(5) \supset SO(3), \] \hspace{1cm} (10)

starting with \([HW(5)]U(5)\) and ending with \(SO(3)\), where \(R^5\) is the group with Abelian Lie algebra spanned by the quadrupole moments only, i.e. \(R^5 \equiv \{\alpha_\mu = \frac{1}{\sqrt{2}}(d_\mu^+ + d_\mu^-)\} \).

3.1 Harmonic spherical vibrator

The dynamical subgroup chain for the harmonic vibrator BM submodel is defined by [8]:

\[ [HW(5)]U(5) \supset U(5) \supset SO(5) \supset SO(3), \] \hspace{1cm} (11)

where \([HW(5)]U(5)\) is the semi-direct product group of Heisenberg-Weyl group \(HW(5) = \{\alpha_\mu, \pi^+, I\}\) and \(U(5) = \{\alpha_\mu \pi^+, \pi^+ \alpha_\mu, \ L_k = \sqrt{\frac{1}{15}}[\alpha \times \pi]_{1k}, k = 0, \pm 1, \ O_v = \sqrt{\frac{1}{15}}[\alpha \times \pi]_{10}, v = 0, \pm 1, \pm 2, \pm 3\}\). In terms of quadrupole phonon operators, one obtains the alternative realization \([HW(5)]U(5) = \{d_\mu^+, d_\mu^-, I, d_\mu^+ d_\mu^-\}\).

It was shown [13] that in the spherical vibrator \(U(5)\) limit, the IBM dynamical symmetry chain [12]

\[ U(6) \supset U(5) \supset SO(5) \supset SO(3) \] \hspace{1cm} (12)

contracts in the \(N \to \infty\) limit to the BM dynamical symmetry chain (11), which is actually based on the Holstein–Primakoff realization of the \(U(6)\):

\[ s^+ \rightarrow N, \]

\[ d_\mu^+ s \rightarrow \sqrt{N} d_\mu^+ s, \]

\[ s^+ d_\mu \rightarrow \sqrt{N} d_\mu, \]

\[ d_\mu^+ d_\nu \rightarrow d_\mu^+ d_\nu \] \hspace{1cm} (13)

This contraction of the IBM to BM spectrum generating algebra is valid only for low-energy IBM states close to \(U(5)\) limit. According to this contraction/compactification relation, any development in the \(U(5)\) limit of one model apply equally to the other.

3.2 Wilets–Jean model

The dynamical subgroup chain of the \(\gamma\)-unstable \(\beta\)-rigid WJ model is [8]:

\[ [HW(5)]U(5) \supset [R^5]SO(5) \supset [R^5]SO(3) \supset SO(3), \] \hspace{1cm} (14)

where the semi-direct product group \([R^5]SO(5)\) consists of an Abelian ideal \(R^5 \equiv \{\alpha_\mu; [\alpha_\mu, \alpha_\nu] = 0\}\) and the generators of the \(SO(5)\) group. Similarly to the \(U(5)\) limit, there is a close correspondence of the physics of the IBM in its \(O(6) \supset SO(5) \supset SO(3)\) limit with the BM model in its \(\beta\)-rigid \(\gamma\)-unstable WJ limit, as was first shown by Meyer-ter-Vehn [16]. This correspondence is precise in the limit in which the IBM dynamical symmetry subgroup chain [12]

\[ U(6) \supset O(6) \supset SO(5) \supset SO(3) \] \hspace{1cm} (15)

contracts in the \(N \to \infty\) limit to the chain (14) of the WJ model, based on the \(O(6) \to [R^5]SO(5)\) contraction [11, 13]:

\[ Q_\mu = d_\mu^+ s + s^+ d_\mu \rightarrow \sqrt{\nu} (v + 4)(d_\mu^+ + d_\mu^-), \]

\[ (d^\dagger \otimes d_\nu)_L \rightarrow (d^\dagger \otimes d_\nu)_L, \quad L = 1, 3 \] \hspace{1cm} (16)

where \(Q_\mu = d_\mu^+ s + s^+ d_\mu\) is the \(O(6)\) quadrupole operator and \(\nu\) is an \(O(6)\) irrep. The irreducible representations of \([R^5]SO(5)\) are characterized by the rigid values of \(\beta = \beta_0\). Hence, there is a problem with the delta-function nature of the \(\beta\) wave functions, which in turn don’t have a convergent expansion in terms of the harmonic oscillator \(U(5)\) states in the IBM. This problem, as will be shortly considered further, is circumvented in the ACM [8, 9] in which the \(\beta\)-rigid wave functions of the WJ model \([R^5]SO(5)\) algebra are replaced by the \(\beta\)-soft wave functions of the dynamical algebra \(SU(1, 1) \otimes SO(5)\).

3.3 Rigid rotor model

The dynamical subgroup chain of the \(\beta\)-rigid and \(\gamma\)-rigid rotor model is [8]:

\[ [HW(5)]U(5) \supset [R^5]SO(5) \supset [R^5]SO(3) \supset SO(3), \] \hspace{1cm} (17)

where \(ROT(3) \equiv [R^5]SO(3)\) is the rigid-rotor model group of Ui [4]. The irreducible representations of the \(ROT(3)\) group are characterized by both \(\beta\)-rigid and \(\gamma\)-rigid values. Thus, in the BM model \(\beta\)-rigid and \(\gamma\)-rigid subgroup chain (17) is a submodel of the \(\beta\)-rigid but \(\gamma\)-unstable subgroup chain (14) since the \([R^5]SO(3)\) is a subgroup of \([R^5]SO(5)\). Hence, in the BM rigid-rotor submodel again there is a problem with the wave functions which are delta functions in both \(\beta\) and \(\gamma\). Further, the IBM dynamical symmetry limit chain [12]

\[ U(6) \supset SU(3) \supset SO(3) \] \hspace{1cm} (18)

is not a submodel of the chain (15) since the \(SU(3)\) is not a subgroup of \(O(6)\). Hence, the rotor-like states in the \(SU(3)\) limit of the IBM are not related to those of its \(O(6)\) limit in
a way that parallels the relationship between the rigid rotor and \( \beta \)-rigid but \( \gamma \)-unstable states in the BM model. It is the purpose of the present paper to demonstrate that there are microscopic shell-model counterparts of the \( \beta \)-rigid or \( \beta \)-soft but \( \gamma \)-unstable WJ and the \( \beta \)-rigid and \( \gamma \)-rigid rotor limits of the BM model in the configuration space \( \mathbb{R}^6 \).

3.4 Algebraic collective model

Although the last two BM limiting cases just considered are characterized by dynamical subgroup chains, they are not particularly useful for the construction of basis states in which to diagonalize more general collective Hamiltonians, as this is done in the case of the five-dimensional oscillator. The resolution of this problem is the replacement of the surface shape as this is done in the case of the five-dimensional oscillator with the quantum mechanics. The resolution of this problem was given through the solution \([17,18]\) of this problem was given through the relation and also cause the non-square integrability of rigid rotor wave functions. It turns out that the \( \bar{\mathbf{Q}}_{ij} \) wave functions in the many-particle Hilbert space of \( A \) nucleon antisymmetry states. This is so, because its dynamical group \([H_W(5)]U(5)\) is not the most appropriate. In this respect, we recall briefly the embedding of the BM model into the one-component microscopic shell-model theory of nuclear collective motions obtained many years ago (see, e.g., \([17,18]\)).

The first step in the progression to a microscopic collective model \([7,17,18]\) is the replacement of the surface shape coordinates \( \{\alpha_i\} \), which don’t have a microscopic interpretation and also cause the non-square integrability of rigid rotor wave functions, by microscopic Cartesian components of the mass quadrupole tensor

\[
Q_{ij} = \sum_{s=1}^{A} x_{is} x_{js}, \quad i, j = 1, 2, 3; s = 1, \ldots, A. \tag{21}
\]

It immediately follows then that the time derivatives of the quadrupole moments and corresponding momentum observables are given by

\[
\dot{Q}_{ij} = \frac{dQ_{ij}}{dt} = \sum_{s=1}^{A} (\dot{x}_{is} x_{js} + x_{is} \dot{x}_{js}), \tag{22}
\]

\[
P_{ij} = M \dot{Q}_{ij} = \sum_{s=1}^{A} (p_{is} x_{js} + x_{is} p_{js}) \neq -i\hbar \frac{\partial}{\partial Q_{ij}}, \tag{23}
\]

where \( M \) is the nucleon mass. These moments and momenta are quantized by replacing the \( x_{is} \) and \( p_{is} \) coordinates by operators \( \hat{x}_{is} \) and \( \hat{p}_{is} \) with commutation relations \([\hat{x}_{is}, \hat{p}_{js}] = i\hbar \delta_{ij} \delta_{st}\) to obtain the quantum observables

\[
\hat{Q}_{ij} = \sum_{s=1}^{A} \hat{x}_{is} \hat{x}_{js}, \quad \hat{P}_{ij} = \sum_{s=1}^{A} (\hat{p}_{is} \hat{x}_{js} + \hat{x}_{is} \hat{p}_{js}). \tag{24}
\]

From (23) it becomes clear that the quantization of the BM model given by the standard Heisenberg-Weyl commutation relations

\[
[\alpha_{\mu}, \pi^\nu] = i\hbar \delta^\nu_{\mu}, \tag{25}
\]
where \( \pi^v = -i\hbar \frac{\partial}{\partial \alpha^v} \), was not correct. The new commutation relations emerge

\[
[\hat{Q}_{ij}, \hat{P}_{kl}] = i\hbar (\delta_{il} \hat{Q}_{jk} + \delta_{lk} \hat{Q}_{ij} + \delta_{jl} \hat{Q}_{ik} + \delta_{ik} \hat{Q}_{jl}).
\]

which together with the antisymmetric angular momentum operators \( \hbar \hat{L}_k = \hbar \epsilon_{ijk} \hat{L}_{ij} = \sum_{s=1}^A (\hat{x}_{si} \hat{p}_{sj} - \hat{x}_{sj} \hat{p}_{si}) \) span the Lie algebra of general collective motion group in three dimensions, i.e. \( GCM(3) = \{ \hat{L}_{ij}, \hat{Q}_{ij}, \hat{P}_{ij} \} \). For simplicity, further the hats in the notations of generators will be suppressed. The \( GCM(3) \) model is slightly extended version of the original \( CM(3) \) model of Weaver, Biedenharn and Cusson [24,25], which in addition includes the monopole degrees of freedom. The \( CM(3) \) model was obtained by extending the motion group (i.e. the group of transformations of the configuration space) from \( SO(3) \) to \( SL(3, R) \) which includes into considerations beyond the rotational also the vibrational degrees of freedom of shape change, the latter generated by the \( L = 2 \) components of \( \hat{P}_{ij} \). What is remarkable is that the new spectrum generating algebra of the \( GCM(3) \) model has irreps with different intrinsic angular momenta (vorticities).

It was shown by G. Rosensteel [26] that the invariant operator of the \( GCM(3) \) model is represented by the square of the conserved Kelvin circulation vector, i.e. \( V^2 \), and its eigenvalues \( V(V + 1) \) correspond to the quantized vorticity. The \( V = 0 \) representation has states that are in one-to-one correspondence with those of the BM model. In this way the \( GCM(3) \) model extends the irrotational-flows of the BM model to include an \( SO(3) \) intrinsic gauge (vorticity) degrees of freedom that are important for the appearance of low-lying collective states in nuclear spectra.

In addition of being a microscopic version of the BM model augmented by the intrinsic vortex degrees of freedom, the \( GCM(3) \) model has the desirable characteristic of containing all physical observables, i.e., quadrupole moments, angular momenta, vortex spin, and infinitesimal generators of deformation, that appear in the expression of the collective component \( T_{coll} \) [18] of the many-nucleon kinetic energy. As shown by Rosensteel, the \( GCM(3) \) is also related to the Riemann model of rotating ellipsoids [26] with linear combinations of rigid and irrotational flows and has a mathematical structure in terms of Yang-Mills theory [27]. The problem with the \( GCM(3) \) model is that it is not compatible with the shell model. The irreps of \( GCM(3) \) have no simple shell-model expression, except for the trivial case of vortex-free irreps. Additionally, the kinetic energy of the \( GCM(3) \) model has an exceedingly complicated expression and the model is difficult to use in the calculations of nuclear properties with a many-nucleon Hamiltonian. But the more serious concern is that the full many-particle kinetic energy does not conserve the vortex spin, i.e., it strongly mixes different \( GCM(3) \) irreps.

A resolution of the problem with the \( GCM(3) \) model was obtained by simply extending it to the one-component symplectic \( Sp(6, R) \) model [19] which includes the full many-particle kinetic energy \( T = \sum_{s=1}^A \hat{P}_{s}^2 / 2M \). The Lie algebra that emerges then contains all symmetric bilinear combinations of the many-nucleon position \( (x_{ij}) \) and momentum \( (p_{is}) \) coordinates. The SGA of the \( Sp(6, R) \) model thus becomes [7,17,18]:

\[
\hat{Q}_{ij} = \sum_{s=1}^A x_{is} x_{js}, \quad \hat{P}_{ij} = \sum_{s=1}^A (p_{is} x_{js} + x_{is} p_{js}),
\]

\[
\hbar \hat{L}_{ij} = \sum_{s=1}^A (x_{si} p_{sj} - x_{sj} p_{si}), \quad \hat{K}_{ij} = \sum_{s=1}^A p_{is} p_{js}.
\]

\( Sp(6, R) \) is also the smallest Lie algebra that contains the nuclear quadrupole moments and the many-nucleon kinetic energy, both of which are essential components of a complete microscopic model of nuclear collective states. In addition, the \( Sp(6, R) \) SGA contains the infinitesimal generators of both rigid- and irrotational-flow rotations. It contains also the \( U(3) \) Lie algebra of the Elliott model as a subalgebra and has the valuable property that it defines a coupling scheme in a \( U(3) \supset SU(3) \) basis for the many-nucleon Hilbert space in a straightforward way. The \( Sp(6, R) \) Lie algebra, like that of \( U(3) \), can be augmented to include the \( U(4) \) supermultiplet spin-isospin algebra with which it commutes. The one-component \( Sp(6, R) \) model then defines a complete coupling scheme for the many-nucleon shell-model Hilbert space in a spherical harmonic-oscillator basis. An important property of the \( Sp(6, R) \) model is that it also appears as a bridge between the shell model and the collective model. Thus, it is a microscopic unified model in every respect. Moreover, as we will see in the next subsection, the \( Sp(6, R) \) SGA can be embedded in the \( Sp(12, R) \) dynamical algebra, i.e. \( Sp(6, R) \subset Sp(12, R) \), which allows for the more complete description of the complex proton-neutron dynamics, as well as the separate treatment of the proton and neutron subdynamics. As a result, a new fully microscopic model of collective excitations in the two-component many-particle nuclear systems arises, namely the proton-neutron symplectic model [21,28]. The latter will allow us to give the BM collective model a microscopic foundation, admitting a more natural interpretation of the underlying BM quadrupole-monopole collective dynamics that is missing in the (one-component) \( Sp(6, R) \) symplectic model.

4.2 Embedding in the two-component proton–neutron microscopic shell-model theory

In the present subsection we will consider the embedding of the generalized BM collective model into the micro-
scopic shell-model theory of atomic nucleus in the framework of the proton-neutron symplectic model, which provides a more natural interpretation of the underlying BM quadrupole-monopole collective dynamics than the embedding given in the preceding subsection.

The PNSM with $Sp(12, R)$ SGA was formulated by considering the symplectic geometry and possible collective flows in the two-component many-particle nuclear system [21], generalizing in this way the $Sp(6, R)$ model. The PNSM collective observables are given by the following $O(A - 1)$-invariant one-body operators [21]:

$$Q_{ij}(\alpha, \beta) = \sum_{s=1}^{m} x_{is}(\alpha) x_{js}(\beta),$$

$$S_{ij}(\alpha, \beta) = \sum_{s=1}^{m} \left( x_{is}(\alpha) p_{js}(\beta) + p_{is}(\alpha) x_{js}(\beta) \right),$$

$$L_{ij}(\alpha, \beta) = \sum_{s=1}^{m} \left( x_{is}(\alpha) p_{js}(\beta) - x_{js}(\beta) p_{is}(\alpha) \right),$$

$$T_{ij}(\alpha, \beta) = \sum_{s=1}^{m} p_{is}(\alpha) p_{js}(\beta),$$

where $i, j = 1, 2, 3; \alpha, \beta = p, n$ and $s = 1, \ldots, m = A - 1$. In Eqs. (29)—(32), $x_{is}(\alpha)$ and $p_{is}(\alpha)$ denote the coordinates and corresponding momenta of the translationally-invariant Jacobi vectors of the $m$-quasiparticle two-component nuclear system and $A$ is the number of protons and neutrons. By considering the $m$ Jacobi quasiparticles instead of $A$ protons and neutrons, the problem of center-of-mass motion is avoided from the very beginning. Obviously, by summing over $\alpha$ in Eqs. (29)—(32) we obtain the one-component $Sp(6, R)$ symplectic model as a submodel.

The symplectic generators of the PNSM can be written in an alternative form in terms of all bilinear combinations of the raising and lowering operators of harmonic oscillator quanta

$$b_{i|s}^{\dagger} = \sqrt{\frac{m_{\alpha} \omega}{\hbar}} \left( x_{is}(\alpha) - \frac{i}{m_{\alpha} \omega} p_{is}(\alpha) \right),$$

$$b_{i|s} = \sqrt{\frac{m_{\alpha} \omega}{\hbar}} \left( x_{is}(\alpha) + \frac{i}{m_{\alpha} \omega} p_{is}(\alpha) \right)$$

that are $O(m)$ invariant [28]:

$$F_{ij}(\alpha, \beta) = \sum_{s=1}^{m} b_{i|s}^{\dagger} b_{j|s},$$

$$G_{ij}(\alpha, \beta) = \sum_{s=1}^{m} b_{i|s} b_{j|s},$$

$$A_{ij}(\alpha, \beta) = \frac{1}{2} \sum_{s=1}^{m} \left( b_{i|s}^{\dagger} b_{j|s} + b_{j|s}^{\dagger} b_{i|s} \right).$$

The operators $A_{ij}(\alpha, \beta)$ are the generators of the maximal compact subgroup $U(6) \subset Sp(12, R)$. We introduce also the operators [29]:

$$B_{ij}(\alpha) = \sum_{s} b_{i|s}^{\dagger},$$

and $B_{i}(\alpha) = (B_{ij}(\alpha))^{\dagger}$, which together with the identity operator close the six-dimensional Heisenberg-Weyl algebra $HW(6) = \{B_{ij}(\alpha), B_{i}(\alpha), I\}$.

An $Sp(12, R)$ unitary irreducible representation $|\sigma\rangle = (\sigma_1 + \frac{m}{2}, \ldots, \sigma_6 + \frac{m}{2})$ is obtained by acting on the $U(6)$ lowest-weight state $|\sigma\rangle$, defined by the $U(6)$ quantum numbers $\sigma = [\sigma_1, \ldots, \sigma_6]$, with the symmetric powers of symplectic raising operators $F_{ij}(\alpha, \beta)$ (34). The basis for a $Sp(12, R)$ irreducible representation can therefore be represented as [28]:

$$|\Psi(\sigma \rho \eta\rangle) = \left[ P^{(n)}(F) \times |\sigma\rangle \right]_{\eta} \rho E,$$

where $E = [E_1, \ldots, E_6]$ indicates the $U(6)$ quantum numbers of the coupled state, $\eta$ labels a basis of states for the coupled $U(6)$ irrep $E$ and $\rho$ is a multiplicity index. Thus we obtain a basis of $Sp(12, R)$ states that reduces the subgroup chain $Sp(12, R) \supset U(6)$. It will be shown further that $\eta = \upsilon qLM$ or equivalent to it set of quantum numbers $\eta = \upsilon (\lambda, \mu) qLM$.

The set of operators $\{L_{ij}(\alpha, \beta), S_{ij}(\alpha, \beta)\}$ form the Lie algebra of a more general motion group $GL(6, R) \subset Sp(12, R)$, which allows for the separate treatment of the collective dynamics of proton and neutron subsystems as well as the combined proton-neutron collective excitations. The configuration space of the PNSM is isomorphic to the coset space $GL(6, R)/SO(6)$ and is spanned by the commuting quadrupole moment observables, i.e. $\mathbb{R}^{21} = \{Q_{ij}(\alpha, \beta)\}$. Moreover, Eq. (29) can be considered as a map from the microscopic many-particle configuration space $\mathbb{R}^{6m}$ to the collective configuration space [21]:

$$Q : \mathbb{R}^{6m} \to \mathbb{Q}; \ x \to Q(x) = \tilde{x}x,$$

where $\tilde{x}$ denotes the transpose of the matrix $x \in \mathbb{R}^{6m}$. It follows that every path $x(t)$ in $\mathbb{R}^{6m}$ has an image $Q(x(t))$ in $\mathbb{Q}$. Thus, the collective motions in $\mathbb{R}^{6m}$ map to collective motions in $\mathbb{Q}$. In this way the components of the quadrupole moment (29) define the microscopic collective configuration space $\mathbb{R}^{21}$ of the PNSM.

Now, let us consider instead of the mapping $\{\alpha\} \to \{Q_{ij}\}$ the following one: $\{\alpha\} \to \{Q_{ij}(\rho, n)\}$. Then, in contrast to the $Sp(6, R)$ case, we obtain a six-dimensional microscopic configuration subspace $\mathbb{R}^{6} \subset \mathbb{R}^{21}$, spanned by the six commuting components $\{Q_{ij}(\rho, n) = Q_{ij}(n, \rho)\}$, in which the group of six-dimensional rigid rotations $SO(6)$...
acts. As will be demonstrated further, this configuration space \( \mathbb{R}^6 = \{Q_{ij}(p, n)\} \) is isomorphic to the configuration collective space of the generalized quadrupole-monopole Bohr–Mottelson dynamics. Further, the same kind of considerations are valid as those for the case of one-component nuclear systems given in the previous subsection.

The \( SO(6) \) group, spanned by the components of the six-dimensional angular-momentum operators \( \{L_{ij}(\alpha, \beta)\} \), can be expressed more conveniently in terms of the \( U(6) \) generators (36) as:

\[
A^{LM}(\alpha, \beta) = A^{LM}(\alpha, \beta) - (-1)^L A^{LM}(\beta, \alpha). \tag{40}
\]

It then allows us to consider the following reduction chain of the PNSM:

\[
Sp(12, R) \supset U(6) \supset SO(6) \supset G \supset SO(3). \tag{41}
\]

The reduction of \( SO(6) \) to \( SO(3) \) can be carried out in different ways, but for the present purposes we choose \( G = SU(3) \otimes SO(2) \) [30–33]. Thus, we obtain the subgroup chain

\[
Sp(12, R) \supset U(6) \supset SO(6) \supset SU_{pn}(3) \otimes SO(2) \supset SO(3), \tag{42}
\]

which defines a microscopic shell-model analogue of the BM \( \gamma \)-unstable reduction chain. We point out that the PNSM \( Sp(12, R) \) Lie algebra, realized in terms of many-particle position and momentum Jacobi coordinates, is a subalgebra of the shell-model algebra of one-body unitary transformations. Thus the problem of embedding of the generalized BM quadrupole-monopole collective dynamics of WJ-type into the microscopic shell-model theory of a nucleus is solved from the beginning.

The second-order invariant for the \( SO(6) \) group is

\[
\Lambda^2 = \sum_{\alpha, \beta, L, M} (-1)^M A^{LM}(\alpha, \beta) A^{L,-M}(\beta, \alpha). \tag{42}
\]

The generators of the \( SU_{pn}(3) \) group are defined by

\[
\tilde{q}^{2M} = \sqrt{3i}[A^{2M}(p, n) - A^{2M}(n, p)], \tag{43}
\]

\[
Y^{1M} = \sqrt{2}[A^{1M}(p, n) + A^{1M}(n, n)], \tag{44}
\]

whereas the single infinitesimal operator of \( SO(2) \) is proportional to the \( SO(3) \) scalar operator \( \Lambda^0(\alpha, \beta) \):

\[
M_{\alpha, \beta} = \Lambda^0(\alpha, \beta) = i[A^0(\alpha, \beta) - A^0(\beta, \alpha)]. \tag{45}
\]

Obviously, by construction the generator of \( SO(2) \) commute with the generators of \( SU_{pn}(3) \) which are rank-2 and rank-

1 tensors. The reduction of the \( SO(6) \) to \( SO(3) \) is therefore carried out by two mutually complementary groups \( SU_{pn}(3) \) and \( SO(2) \) [34], i.e. we have a direct-product group \( SU_{pn}(3) \otimes SO(2) \). Note also that in the present case the quadrupole moment operators \( \tilde{q} \) (43) are of proton-neutron nature. The second-order Casimir operators of the two groups \( SU_{pn}(3) \) and \( SO(2) \) are given by:

\[
C_2[SU_{pn}(3)] = \sum_M (-1)^M (\tilde{q}^{2M} \tilde{q}^{-1M} + Y^{1M} Y^{-1M}),
\]

\[
C_2[SO(2)] = M^2 = \sum_{\alpha, \beta} M_{\alpha \beta} \beta \alpha. \tag{46}
\]

From the above expressions, it follows that the second-order Casimir operator of \( SO(6) \) can be written in an alternative form as:

\[
C_2[SO(6)] = (\tilde{q} \cdot \tilde{q} + Y \cdot Y) + L_p^2 + L_n^2. \tag{47}
\]

and its eigenvalue \( \nu(\nu + 4) \) is determined by the quantum number \( \nu \) characterizing the \( SO(6) \) irreps.

For \( SO(6) \subset U(6) \), the symmetric representation \([E]_6\) of \( U(6) \) decomposes into fully symmetric \( (\nu, 0, 0)_6 \equiv (\nu)_6 \) irreps of \( SO(6) \) according to the rule [35]:

\[
[E]_6 = \bigoplus_{\nu=E, E-2, ..., 0(1)} (\nu, 0, 0)_6 = \bigoplus_{i=0}^{(E/2)} (E - 2i)_6, \tag{48}
\]

where \( (E/2) = E/2 \) if \( E \) is even and \( (E - 1)/2 \) if \( E \) is odd. Furthermore, the following relation between the quadratic Casimir operators \( C_2[SU_{pn}(3)] \) of \( SU_{pn}(3) \), \( M^2 \) of \( SO(2) \) and \( C_2[SO(6)] \) of \( SO(6) \) holds [31–33]:

\[
C_2[SO(6)] = 2C_2[SU_{pn}(3)] - \frac{1}{3} M^2. \tag{49}
\]

As a consequence, the irrep labels \([f_1, f_2, 0]_3\) of \( SU_{pn}(3) \) are determined by \( (\nu)_6 \) of \( SO(6) \) and by the integer label \((\nu)_2 \) of the associated irrep of \( SO(2) \) i.e.

\[
(\nu)_6 = \bigoplus[ f_1, f_2, 0]_3 \otimes (\nu)_2. \tag{50}
\]

Using the relation (49) of the Casimir operators, for their respective eigenvalues one obtains:

\[
\nu(\nu + 4) = \frac{4}{3} (f_1^2 + f_2^2 - f_1 f_2 + 3 f_1) - \frac{\nu^2}{3}. \tag{51}
\]
Thus (50) can be rewritten as

\[(v)_6 = \bigoplus_{i=0}^{v} [v, i, 0]_3 \otimes (v - 2i)_2\]

\[= \bigoplus_{v=\pm v, \pm (v-2), \ldots, 0(\pm 1)} [v, \frac{v - v}{2}, 0]_3 \otimes (v)_2,\]

or in terms of the Elliott’s notation \((\lambda, \mu)\):

\[(v)_6 = \bigoplus_{v=\pm v, \pm (v-2), \ldots, 0(\pm 1)} (\lambda = \frac{v + v}{2}, \mu = \frac{v - v}{2}) \otimes (v)_2.\]

(51)

Finally, the convenience of this reduction can be further enhanced through the use of the standard rules for the reduction of the \(SU_{pn}(3) \supset SO(3)\) chain in terms of a multiplicity index \(q\) which distinguishes the same \(L\) values in the \(SU_{pn}(3)\) multiplet \((\lambda, \mu)\) [20):

\[q = \min(\lambda, \mu), \min(\lambda, \mu) - 2, \ldots, 0(1)\]

\[L = \max(\lambda, \mu), \max(\lambda, \mu) - 2, \ldots, 0(1); q = 0\]

\[L = q, q + 1, \ldots, q + \max(\lambda, \mu); q \neq 0.\]

(52)

Using the above reduction rules we give in Table 1 the \(SU_{pn}(3)\) basis states for first few even \(SO(6)\) irreps, starting from zero, which correspond only to the lowest positive-parity states of the doubly-closed shell nuclei. Table 1 is given in order to show the structure of the \(SU_{pn}(3)\) irreps that appear within a given \(SO(6)\) seniority irreducible representation \(v\).

5 Microscopic counterparts of the exactly solvable limits of the BM model

We are now ready to define the microscopic many-particle counterparts of the generalized quadrupole-monopole BM submodels, which closely parallels the relationship between the original Wilets–Jean and rotor models given in Sect. 2.

5.1 Microscopic counterpart of the WJ model

Formally, we can form the six-dimensional analogue of BM dynamical group: \([-HW(6)] U(6) = \{B_i^\dagger(\alpha), B_i(\alpha), \lambda, A_i(\alpha, \beta)\}\) which is a semi-direct group of the \(HW(6)\) and \(U(6)\). Then, we can consider the following reduction chain

\[[-HW(6)] U(6) \supset [R^6] SO(6) \supset SO(6) \supset SU(3) \otimes SO(2) \supset SO(3),\]

which the semi-direct product group \([R^6] SO(6)\) consists of an Abelian ideal \(R^6 = \{x_1(\alpha); [x_j(\alpha), x_j(\beta)] = 0\}\) and the generators of the \(SO(6)\) \((40)\), i.e. \([R^6] SO(6) = \{x_1(\alpha) = \frac{1}{\sqrt{2}}[B_i^\dagger(\alpha) + B_i(\alpha)], A_iLM(\alpha, \beta)\}\) group. This construction parallels that of the \(\beta\)-rigid WM model and the irreps of \([R^6] SO(6)\) are characterized by a fixed value \(r_0\) of the radial coordinate (hyper-radius) \(r = \sqrt{r_p^2 + r_n^2}\), where \(r_n^2 = \sum x_i^2(\alpha)\) and \(\alpha = p, n\). The square of the hyper-radius \(r^2\) is invariant under \(SO(6)\) transformations.

5.2 Microscopic counterpart of the rigid-rotor model

An analogue of the rigid-rotor submodel of the BM model, defined by the dynamical group chain (17), can be defined within the framework of the PNSM by considering the following subgroup chain

\[[-HW(6)] U(6) \supset [R^6] SO(6) \supset SO(6) \supset SU_{pn}(3) \supset SO(3),\]

where for large dimensional \(SU(3)\) representations the \(SU_{pn}(3)\) group contracts to \([R^3] SO(3)\) of \(U_{36}\) and the rigid rotor model states are approached.

6 Many-particle quantum-mechanical shell-model counterpart of the BM model

Similarly to the case of \(\beta\)-rigid BM submodels, it is more convenient to relax the rigidity constrain and to consider the following reduction chain

\[SU(1, 1) \otimes SO(6) \supset U(1) \otimes SU_{pn}(3) \otimes SO(2) \supset SO(3),\]

which is naturally contained in the \(Sp_{12}(12, R)\) dynamical group of PNSM. To see this, one just needs to take into account that \(Sp(2, R)\) is locally isomorphic to \(SU(1, 1)\). We can then define the following dynamical symmetry limit of the PNSM:

\[Sp_{12}(12, R) \supset Sp(2, R) \supset SO(6)\]

\[\sigma \otimes SU(1, 1) \otimes SO(6)\]

\[\supset U(1) \otimes SU_{pn}(3) \otimes SO(2) \supset SO(3),\]

\[n \otimes (\lambda, \mu) \otimes v \otimes q \otimes L\]

(55)

which is well defined in the many-nucleon quantum mechanics and completely avoids the problem of non-normalizable wave functions of the WJ-type models in the zero-width limit in which they become proportional to a delta function. We
note that because of the dual pair relationships, dynamical symmetry chain (55) is equivalent to that defined by Eq. (41).

The \(Sp(2, R) \approx SU(1, 1)\) is a dynamical group for radial wave functions and \(SO(6)\) group determines the angular part (\(SO(6)\) spherical harmonics) that is characterized by the seniority quantum number \(\nu\). The infinitesimal generators of the \(SU(1, 1)\) group are expressed in terms of the symplectic generators (34)—(36) in the form:

\[
\begin{align*}
S^{(2, \nu)}_+ &= \frac{1}{2} \sum_{\alpha} F^0(\alpha, \alpha), \\
S^{(2, \nu)}_- &= \frac{1}{2} \sum_{\alpha} G^0(\alpha, \alpha), \\
S^{(2, \nu)}_0 &= \frac{1}{2} \sum_{\alpha} A^0(\alpha, \alpha),
\end{align*}
\]

(56) (57) (58)

for any value of \(\lambda_{\nu}\) (\(\lambda_{\nu} > 1\)) which, generally, define the so called modified oscillator \(SU(1, 1)\) irreps [10]. The wave functions can therefore be expressed as products of radial \(r\) functions and orbital \(SO(6)\) wave functions:

\[
\Psi_{\lambda_{\nu}; \nu, \nu, 0}^{q, \nu, \nu, 0} (r, \Omega_5) = R_{\nu}^{\lambda_{\nu}} (r) Y_{\nu, \nu, 0}^{q, \nu, \nu, 0} (\Omega_5),
\]

(59)

where \(Y_{\nu, \nu, 0}^{q, \nu, \nu, 0} (\Omega_5)\) are the \(SO(6)\) Dragt’s spherical harmonics [30,32].

The configuration space of the PNSM model in the present case is the six-dimensional Euclidean space \(\mathbb{R}^5\). The volume element in spherical coordinates is given by

\[
dV = r^5 dr d\Omega_5,
\]

where \(d\Omega_5\) is the volume element of the five-sphere. The Laplacian operator is

\[
\nabla^2 = \frac{1}{r^5} \frac{\partial}{\partial r} r^5 \frac{\partial}{\partial r} - \frac{\Lambda^2}{r^2},
\]

(60)

where the \(SO(6)\) Casimir operator \(\Lambda^2\) was given by Eq. (42). The energies and radial wave functions can then be found as solutions of the eigenvalue equation

\[
\left[-\frac{\hbar^2}{2m} \left(\nabla^2 - \frac{\nu(\nu + 4)}{r^2}\right) + V(r)\right] R_{\nu}^{\lambda_{\nu}} (r) = E_{\nu, \nu} R_{\nu}^{\lambda_{\nu}} (r),
\]

(61)

where we have assumed \(m = m_p = m_n\) (not to be confused with the number of Jacobi quasiparticles).

For strongly deformed rotational nuclei, the corresponding \(SU(3)\) structure is determined by the \(Sp(12, R)\) symplectic bandhead \(\langle \sigma \rangle\). For example, the Nilsson model ideas [37–40] and the shell-model considerations based on the pseudo-\(SU(3)\) scheme [41–43] give the \(Sp(12, R)\) irrep \(\langle \sigma \rangle = \{62 + \frac{157}{2}, 26 + \frac{157}{2}, 26 + \frac{157}{2}, 26 + \frac{157}{2}, 26 + \frac{157}{2}\}\), corresponding to the lowest-weight \(U(6)\) irrep \(\sigma = \{62, 26, 26, 26, 26\} \equiv \{36\}_0\), for \(^{158}\)Gd with a characteristic ratio \(E_{+}^{1+}/E_{+}^{1+} \simeq 3.26\) [44]. This symplectic irrep contains the lowest \(SU(3)\) irrep \(36(0)\) of the ground state (when there is no \(SU(3)\) mixing) that is embedded in the maximal \(SO(6)\) irreducible representation \(\nu_0 = 36\). We note that the relevant \(SU(3)\) structure in the present approach, in contrast to the IBM in which it is determined by the number of paired nucleons, is obtained by filling pairwise the three-dimensional (pseudo)harmonic oscillator with protons and neutrons, which \(SU(3)\) irreps are consequently strongly coupled to the common proton-neutron \(SU(3)\) irrep. If the same pseudo-\(SU(3)\) scheme is used in the one-component \(Sp(6, R)\) model, the Nilsson model ideas [37–40] and the shell-model considerations [41–43] will produce the \(Sp(6, R)\) irrep \(N_0(36, 0)\), where \(N_0\) is the minimal Pauli-allowed number of oscillator quanta. We notice that in contrast to the \(Sp(6, R)\) model, which symplectic bandhead does not contain other \(SU(3)\) multiplets, the PNSM \(Sp(12, R)\) symplectic bandhead con-
tains a plethora of \( SU(3) \) multiplets that are appropriate for the description of low-lying excited collective bands. For instance, the full set of \( SU_{pn}(3) \) multiplets that are contained in the \( SO(6) \) irreps \( \nu_0 = 36 \), according to Eq. (51), is: \( (36, 0), (35, 1), (34, 2), \ldots, (19, 17), (18, 18), (17, 19), \ldots, (2, 34), (1, 35), (0, 36) \). Thus, by mixing these \( SU(3) \) multiplets one will obtain a distribution over \( (\lambda, \mu) \) quantum numbers. Making use of the correspondence between the Bohr–Mottelson deformation parameters \( (\beta, \gamma) \) and the microscopic \( SU(3) \) quantum numbers \( (\lambda, \mu) \) [17, 37, 45], we obtain an equivalent distribution over different \( \beta \) and \( \gamma \). In other words, in contrast to the \( Sp(6, R) \) model, the PNSM naturally accounts for the low-energy vibrational degrees of freedom.

For harmonic oscillator potential \( V(r) = \frac{1}{2} Cr^2 \), the energy spectrum is that of six-dimensional oscillator \( E_N = \epsilon (N + \frac{1}{2}) \) with \( N = 0, 1, 2, \ldots \) and \( \epsilon = \sqrt{C/m} \). This case corresponds to the \( U(5) \) dynamical symmetry limit of the BM model or the IBM. The shell model consideration for vibrational-like nuclei within the present PNSM approach will restrict the number of oscillator quanta from bellow by \( N_0 \) that is allowed by the Pauli principle.

An \( r \)-rigid WJ-type model assumes, in addition, that radial coordinate \( r \) is frozen at some non-zero value \( r_0 \). Then, the radial degree of freedom can be suppressed and the Hamiltonian in (61) reduces to

\[
H_{6DWJ} = \frac{\hbar^2}{2mr_0^2} \Lambda^2
\]

and its eigenvalues determine the energies which now are not equidistant and are given by

\[
E_\nu = \frac{\hbar^2}{2mr_0^2} \nu(\nu + 4) \equiv A\nu(\nu + 4).
\]

Note that, in contrast to Eq. (8), this expression produces a characteristic ratio \( E_{4+} / E_{2+} \approx 2.67 \) of the ground state band energies, for which \( L = \nu \) (left diagonal of Table 1 with \( (\lambda, \mu) = (k, 0), k = 0, 2, 4, \ldots \) ). Notice also that the energies (63) are of kinetic origin and having increasing seniority \( \nu \).

For transitional nuclei, likewise, the shell model considerations will give a non-scalar \( \langle \sigma \rangle \neq 0 \) \( Sp(12, R) \) intrinsic bandhead structure, which will contain the dominant (maximal) seniority \( SO(6) \) configuration \( \nu_0 \). In phenomenological models, like the BM one, the relevant intrinsic structure is simply the vacuum state \( \langle \sigma \rangle \equiv \langle 0 \rangle \), corresponding to the scalar \( Sp(12, R) \) irreducible representation \( \langle \sigma \rangle = 0 \). In the microscopic shell-model theory, this corresponds to the physically unimportant case of doubly-closed shell nuclei (cf. Table 1). Thus, the main difference of the present symplectic-based shell-model approach from the phenomenological models is that the combined proton-neutron collective dynamics is governed by the non-scalar \( Sp(12, R) \) symplectic bandhead structure \( \langle \sigma \rangle \neq 0 \). The description of different kinds of collective dynamics along these lines will be given in detail elsewhere, where it will be shown that the low-lying quadrupole collectivity in different types of nuclei can be described without the introduction of an effective charge.

Usually, the potential energy \( V(r) \) is not invariant under six-dimensional rotations but only under rotations in three dimensions. The latter means that the potential energy breaks the \( SO(6) \) symmetry to \( SO(3) \). Then one could consider the following algebraic Hamiltonian

\[
H = AA^2 + BC_2[SU_{pn}(3)] + aL^2,
\]

which for \( B = a = 0 \) corresponds to the Hamiltonian (62). Note that, due to the mutual complementarity given by Eq. (49), we can equivalently make use of the \( SO(2) \) Casimir operator \( M^2 \) instead of \( C_2[SU_{pn}(3)] \). We want to point out that the states (59) defined by the irreps of the subgroup chain (55) diagonalize a more general \( r \)-soft Wilets-Jean-like model Hamiltonian of the form [46]:

\[
H = H \left( S_0^{(\lambda, \mu)}, S_+^{(\lambda, \mu)}, S_-^{(\lambda, \mu)} \right) + V(r)
\]

\[
+ AA^2 + BC_2[SU_{pn}(3)] + aC_2[SO(3)],
\]

which has an \( SU(1, 1) \) dynamical group spanned by the generators \( \{S_0^{(\lambda, \mu)}, S_+^{(\lambda, \mu)}, S_-^{(\lambda, \mu)} \} \). The latter largely simplifies the diagonalization.

Finally we note also that there is a prolate-oblate degeneracy related with the conjugate \( SU_{pn}(3) \) multiplets \( (\lambda, \mu) \) and \( (\mu, \lambda) \) contained within the corresponding \( SO(6) \) irreducible representations (cf. Table 1).

### 7 Conclusions

In the present paper, we consider another shell-model coupling scheme of the PNSM defined through the reduction of the direct-product group \( SU(1, 1) \otimes SO(6) \), which acts in the configuration subspace \( \mathbb{R}^6 \subset \mathbb{R}^{21} \) that is related to the combined proton-neutron excitations. It is demonstrated that this subspace is closely related to the configuration space of the generalized BM model, in which the monopole degrees of freedom are included together with the quadrupole ones. This in turn allows to formulate the shell-model many-particle counterparts of the two exactly solvable limits of the Bohr–Mottelson model, namely the \( \gamma \)-unstable and (soft-)rotor models because the group \( SO(6) \) acting in \( \mathbb{R}^6 \) contains a \( SU(3) \) subgroup which irreps could be readily mixed by a more realistic Hamiltonian. This is a significant result of the microscopic theory of collective motion in atomic nuclei.
In this respect we recall that a microscopic version of the BM model that is augmented by the intrinsic vortex-spin degrees of freedom and compatible with the composite many-nucleon structure of the nucleus is provided by the (one-component) symplectic model $Sp(6, R)$, which is sometimes called microscopic collective model. The $Sp(6, R)$ model, however, does not contain $O(5)$ or $O(6)$ structures, which could allow to associate it with the $\beta$-rigid or $\beta$-soft but $\gamma$-unstable type dynamics of the Wilets-Jean model. It contains the Elliott’s $SU(3)$ and Ui’s rigid-rotor $ROT(3)$ models, both of which can be associated only with the rotor-modal limit of the Bohr–Mottelson model. From the other side, in contrast to the original rigid rotor model, the rotational dynamics in the $Sp(6, R)$ model due to the intrinsic vortex-spin degrees of freedom span the continuous range from irrotational to rigid flows. As a result the rotational dynamics in the one-component symplectic model possesses a good $SU(3)$ or $ROT(3)$ dynamical or quasi-dynamical symmetry. The same type of rotational dynamics can be obtained for the proton, neutron or combined proton-neutron system in the framework of the PNSM with $Sp(12, R)$ dynamical group. The present paper shows further that when the combined proton-neutron dynamics is restricted to the $R^6 \subset R^{21}$ subspace, spanned by the six components $Q_{ij} (p, n)$, it can be associated with the quadrupole-monopole dynamics of the generalized Bohr–Mottelson model spanning another important class of $\gamma$-unstable collective models of Wilets-Jean type. The many-particle counterparts of the WJ and rotor models obtained in the present work are endowed with the microscopic shell-model potential (with the use of an effective charge) and closely parallel the relationship between the original BM submodels. In this way an embedding of the generalized Bohr–Mottelson model in the microscopic shell-model theory of the nucleus is obtained.

Equations (41) or (55) actually introduce another shell-model coupling scheme within the microscopic proton-neutron symplectic-based shell-model approach. In principle, this coupling scheme provides an alternative basis for shell-model diagonalization of an arbitrary collective Hamiltonian, which could also be expressed as a polynomial in the many-particle position and momentum Jacobi coordinates $x_{ij}(\alpha)$ and $p_{ij}(\alpha)$ in a manner similar to the ACM. Additionally, any Bohr–Mottelson Hamiltonian of the form $H = -\frac{\hbar^2}{2M} \nabla^2 + V(\beta, \gamma)$ (3) immediately defines a microscopic shell-model Hamiltonian in which the operator $-\frac{\hbar^2}{2M} \nabla^2$ is replaced by the many-particle kinetic energy (32), and $V(\beta, \gamma)$ can be expressed in terms of the microscopic quadrupole moment operators (29) since $[Q \times Q]^{(0)} \sim \beta^2$ and $[Q \times Q \times Q]^{(0)} \sim \beta^3 \cos 3\gamma$. The difference between the present approach and the ACM is in the irreducible collective subspaces in which the model Hamiltonians act. For the microscopic models, like the PNSM, the state space is defined by allowed $O(A - 1)$ (or complementary to it $Sp(12, R)$) irreducible representations $\omega$ that are consistent with the Pauli principle, whereas for the phenomenological models the state space in which the collective Hamiltonians act is defined by the $O(A - 1)$-scalar subspace of the many-particle Hilbert spaces with $\omega = (0)$. The specific structure of this violated permutational symmetry space $R^{12}_{\omega = (0)}$ is that it gives a “deep freezing” of the microscopic collective features of the used Hamiltonians and make them similar to those in the Bohr–Mottelson theory, associated with the irrotational-flow collective dynamics. In this way the results obtained in the present paper provide us with a fully microscopic proton-neutron symplectic-based shell-model approach to the generalization quadrupole-monopole Bohr–Mottelson dynamics that covers all the range from rigid to irrotational flows. The combined proton-neutron dynamics in $R^6 \subset R^{21}$ is governed by the microscopic shell-model intrinsic structure of the symplectic bandhead which defines the Pauli allowed $SO(6)$, and hence $SU(3)$, subrepresentations. The original Wilets-Jean-type dynamics of the BM model is recovered for the case of closed-shell nuclei, for which the symplectic bandhead structure is trivially reduced to the scalar or equivalent to it representation.

The present many-particle shell-model counterpart of the Bohr–Mottelson model, represented by the $SU(1, 1) \otimes SO(6)$ limit of the PNSM, could be used for a rough and fast evaluation of different collective observables in the exact limit of the present approach. This will allow to perform an everyday analysis of the experimental data with energies and transition probabilities (with the use of an effective charge) that are very close to the experimental values. Of course, then, the mixed representation symplectic-based shell-model calculations with no effective charge could be performed. All this, as well as the computational technique required for performing such detailed PNSM shell-model calculations, including some simple illustrative examples of the physics presented here will be given in Ref. [46].

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Author’s comment: The article describes entirely theoretical research and no datasets were generated or analysed during the current study.]

References

1. A. Bohr, B. R. Mottelson, *Nuclear Structure*, vol. II (W.A. Benjamin Inc., New York, 1975)
2. L. Wilets, M. Jean, Phys. Rev. **102**, 788 (1956)
3. A.S. Davydov, G.F. Filippov, Nucl. Phys. **8**, 237 (1958)
4. H. Ui, Prog. Theor. Phys. **44**, 153 (1970)
5. E. Chacon, M. Moshinsky, J. Math. Phys. **18**, 870 (1977)
6. D. Troltenier, J. A. Maruhn, P. O. Hess, in *Computational Nuclear Physics I*, ed. by K. Langanke, J. A. Maruhn, S. E. Koonin (Springer, Berlin, 1991)
7. D.J. Rowe, J.L. Wood, Fundamentals of Nuclear Models: Foundational Models (World Scientific Publisher Press, Singapore, 2010)
8. D.J. Rowe, Nucl. Phys. A 735, 372 (2004)
9. D.J. Rowe, P.S. Turner, Nucl. Phys. A 753, 94 (2005)
10. D.J. Rowe, J. Phys. A Math. Gen. 38, 10181 (2005)
11. J.P. Elliott, P. Park, J.A. Evance, Phys. Lett. B 171, 145 (1986)
12. F. Iachello, A. Arima, The Interacting Boson Model (Cambridge University Press, Cambridge, 1987)
13. D.J. Rowe, G. Thiamova, Nucl. Phys. A 760, 59 (2005)
14. J.P. Elliott, Rep. Prog. Phys. 48, 171 (1985)
15. J.P. Elliott, J.A. Evance, P. Park, Phys. Lett. B 169, 309 (1986)
16. J. Meyer-ter-Vehn, Phys. Lett. B 84, 10 (1979)
17. D.J. Rowe, Rep. Prog. Phys. 48, 1419 (1985)
18. D.J. Rowe, Prog. Part. Nucl. Phys. 37, 265 (1996)
19. D.J. Rowe, G. Rosensteel, Phys. Rev. Lett. 38, 10 (1977)
20. J.P. Elliott, Proc. R. Soc. A 245, 128 (1958). 245, 562 (1958)
21. H.G. Ganev, Eur. Phys. J. A 50, 183 (2014)
22. D.R. Bes, Nucl. Phys. 10, 373 (1959)
23. R.F. Casten, Nuclear Structure from a Simple Perspective (Oxford University, Oxford, 1990)
24. L. Weaver, R.Y. Cusson, L.C. Biedenharn, Ann. Phys. (NY) 77, 250 (1973)
25. O.L. Weaver, R.Y. Cusson, L.C. Biedenharn, Ann. Phys. (NY) 102, 493 (1976)
26. G. Rosensteel, Ann. Phys. (NY) 186, 230 (1988)
27. G. Rosensteel, N. Sparks, Eur. Phys. Lett. 119, 62001 (2017)
28. H.G. Ganev, Eur. Phys. J. A 51, 84 (2015)
29. H.G. Ganev, Phys. Rev. C 99, 054304 (2019)
30. A.I. Dragt, J. Math. Phys. 6, 533 (1965)
31. E. Chacon, G. German, Physica 114 A, 301 (1982)
32. E. Chacon, O. Castanos, A. Frank, J. Math. Phys. 25, 1442 (1984)
33. R. Le Blanc, D.J. Rowe, J. Phys. A Math. Gen. 19, 1111 (1986)
34. M. Moshinsky, C. Quesne, J. Math. Phys. 11, 1631 (1970)
35. V.V. Vanagas, Algebraic Methods in Nuclear Theory (Mintis, Vilnius, 1971). in Russian
36. R. Le Blanc, J. Carvalho, D.J. Rowe, Phys. Lett. B 140, 155 (1984)
37. J. Carvalho, D.J. Rowe, Nucl. Phys. A 548, 1 (1992)
38. J. Carvalho, P. Park, D.J. Rowe, G. Rosensteel, Phys. Lett. B 119, 249 (1982)
39. P. Park, J. Carvalho, M. Vassanji, D.J. Rowe, G. Rosensteel, Nucl. Phys. A 414, 93 (1984)
40. M. Jarrio, J.L. Wood, D.J. Rowe, Nucl. Phys. A 528, 409 (1991)
41. R.D. Ratnaraju, J.P. Draayer, K.T. Hecht, Nucl. Phys. A 202, 433 (1973)
42. J.P. Draayer, K.J. Weeks, Phys. Rev. Lett. 51, 1422 (1983)
43. J.P. Draayer, K.J. Weeks, Ann. Phys. 156, 41 (1984)
44. National Nuclear Data Center (NNDC). http://www.nndc.bnl.gov/
45. O. Castanos, J.P. Draayer, Y. Leschber, Z. Phys. A 329, 33 (1988)
46. H. G. Ganev, Matrix elements in the SU(1,1) ⊗ SO(6) limit of the proton–neutron symplectic model (to be published)