Bridging Koopman Operator with Hilbert-Schmidt Operator Associated to Stationary Time Series

Yicun Zhen and Bertrand Chapron

Institut Français de Recherche pour l’Exploitation de la Mer, Plouzané 29280, France

zhenyicun@protonmail.com

Abstract. Given a stationary continuous-time process \( f(t) \), the Hilbert-Schmidt operator \( A_\tau \) can be defined for every finite \( \tau \). Let \( \lambda_{\tau,i} \) be the eigenvalues of \( A_\tau \) with descending order. In this article, a Hilbert space \( H_f \) and the (time-shift) continuous one-parameter semigroup of isometries \( K_s \) are defined. Let \( \{v_i, i \in \mathbb{N}\} \) be the eigenvectors of \( K_s \) for all \( s \geq 0 \). Let \( f = \sum_{i=1}^\infty a_i v_i + f^\perp \) be the orthogonal decomposition with descending \( |a_i| \). We prove that \( \lim_{\tau \to \infty} \lambda_{\tau,i} = |a_i|^2 \). The continuous one-parameter semigroup \( \{K_s: s \geq 0\} \) is equivalent, almost surely, to the classical Koopman one-parameter semigroup defined on \( L^2(X, \nu) \), if the dynamical system is ergodic and has invariant measure \( \nu \) on the phase space \( X \).

Keywords: Singular spectrum analysis, Koopman theory, Hilbert-Schmidt theory

1 Introduction

Let \( \{f(t) \in \mathbb{C}: t \geq 0\} \) be a continuous time process. We assume that \( f \) has zero temporal mean and the lagged moments exist for all \( s \geq 0 \):

\[
\rho(s) := \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t)\tilde{f}(t+s)dt.
\]

Define \( \rho_s = \tilde{\rho}_s \). In \cite{3} the self-adjoint operator \( A_\tau \) is defined to act on \( L^2([0, \tau]) \):

\[
(A_\tau g)(t) = \frac{1}{\tau} \int_0^\tau g(s)\rho(t-s)ds,
\]

for every \( g \in L^2([0, \tau]) \), and for all \( t \in [0, \tau] \). When \( \rho \in L^2_{\text{loc}}(\mathbb{R}) \) and \( \rho(s) \neq 0 \) for almost all \( s \in [0, \tau] \), \( A_\tau \) is a Hilbert-Schmidt operator. In particular, \( A_\tau \) is compact and always has purely point spectrum. As stated in \cite{3}, the SSA method is based on the spectral analysis of \( A_\tau \).

While in practice the SSA method has been applied successfully to a large variety of time series, it remains to ask, for theoretical interests, that what is the
relation between $A_{\tau_1}$ and $A_{\tau_2}$ for different $\tau_1$ and $\tau_2$? And what is the asymptotic behavior of $A_{\tau}$ as $\tau \to \infty$? In what way is the spectral property of $A_{\tau}$ associated to the intrinsic properties of the dynamical system? In this article we generalize the idea and tools developed in [4] and apply them to study $A_{\tau}$. We shall prove that

$$\lim_{\tau \to \infty} \lambda_{\tau,i} = |a_i|^2,$$

where $\lambda_{\tau,i}$ is the $i$–th largest eigenvalue of $A_{\tau}$ and $a_i$ is the $i$–th largest (in modulus) coefficient of some eigenvector $v_i$ (of unit length) of the time-shift operator $K^s$ (for all $s \geq 0$) in the orthogonal decomposition of $f$:

$$f = \sum_{i=1}^{\infty} a_i v_i + f^\perp.$$  

(4)

If there are only finitely many $i$ (say only $N$ terms in the summation) in Eq. (4), then we set $a_i = 0$ for $i > N$. The time-shift operator $K^s$ is closely related to the Koopman operator, which is defined to act, as a time-shift operator, on some function space the domain of the elements of which is the whole phase space of the dynamical system.

In section 2 we present the main result and a brief introduction of the mathematical tools used by the proof of the main result. All the quantities mentioned above are defined rigorously in section 2. The detailed proof of the main result is presented in section 3.

Notes and Comments. The main result and the techniques and ideas in the proof of the main result in this manuscript are completely parallel to those in [4]. But the Hilbert-Schmidt operator $A_{\tau}$ is defined for continuous time process and the theory developed in [4] does not cover the continuous-time case. The objective of this paper is to demonstrate that the asymptotic behavior of the Hilbert-Schmidt operator $A_{\tau}$ is related to the Koopman theory.

2 Preliminaries and the main result

Let $\{f(t) : t \geq 0\}$ be a continuous-time process.

Assumption 1 Assume that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(t) dt = 0.$$  

(5)

and that $\rho(s)$ is well-defined by Eq. (1) for all $s \geq 0$.

For any $s \geq 0$, we use $F_s$ to denote the time series $\{F_s(t) = f(t + s) : t \geq 0\}$. For any two time series $g = \{g(t) : t \geq 0\}$ and $h = \{h(t) : t \geq 0\}$, we define the new time series

$$ag + bh = \{ag(t) + bh(t) : t \geq 0\},$$  

(6)
where \( a, b \in \mathbb{C} \). We consider the following linear space:
\[
\tilde{H}_f = \text{Span}_{\mathbb{C}} \{ F_s : s \geq 0 \}.
\] (7)

Each element \( h \in \tilde{H}_s \) can be written as
\[
h = \sum_{i=1}^{n} c_i F_{s_i},
\] (8)
for any \( n \geq 1, c_i \in \mathbb{C}, s_i \geq 0 \). The existence of \( \rho(s) \) allows to define the following positive semi-definite Hermitian form:
\[
\langle h, g \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} h(t) \overline{g(t)} \, dt.
\] (9)

Let \( V = \{ v \in \tilde{H}_f : \langle v, v \rangle = 0 \} \). Since the Hermitian form is positive semi-definite, \( V \) is a linear subspace of \( \tilde{H}_f \). And the Hermitian form is strictly positive-definite on the quotient space \( \tilde{H}_f/V \). Hence it defines an inner product on \( \tilde{H}_f/V \). We define
\[
\mathcal{H}_f := \tilde{H}_f/V
\] (10)
where the closure is taken with respect to the inner product.

We define the operator \( \mathcal{K}^s \) on \( \tilde{H}_f \) for any \( s \geq 0 \):
\[
\mathcal{K}^s F_{s_1} = F_{s_1 + s}.
\] (11)

It is obvious that
\[
\langle \mathcal{K}^s h, \mathcal{K}^s g \rangle = \langle h, g \rangle
\] (12)
for any \( h, g \in \tilde{H}_f \) and any \( s \geq 0 \). Hence \( \mathcal{K}^s \) is well-defined on \( \tilde{H}_f/V \), and can be further extended to \( \mathcal{H}_f \) by continuity. Therefore we obtain a one parameter family of isometric operators \( \mathcal{K}^s \) that acts on the Hilbert space \( \mathcal{H}_f \). And obviously we have
\[
\mathcal{K}^{s_1} \mathcal{K}^{s_2} = \mathcal{K}^{s_1 + s_2}.
\] (13)

To simplify the notation, we use \( f \) to also denote the continuous-time process \( F_0 \). We further assume that

**Assumption 2**
\[
\lim_{s \to 0^+} \| \mathcal{K}^s f - f \| = 0.
\] (14)

In other words, assumption 2 assumes that the curve:
\[
\gamma : [0, \infty) \to \mathcal{H}_f
\]
\[
t \to \mathcal{K}^t f
\] (15)
is continuous. Since \( \mathcal{H}_f \) is generated by \( f \) and \( K^s \) are isometries for all \( s \geq 0 \), assumption 2 implies that \( K^s \to I \) in the strong operator topology as \( s \to 0^+ \). In other words, \( \{ K^s : s \geq 0 \} \) is a continuous one-parameter semigroup of isometries on \( \mathcal{H}_f \).

Under assumption 2 we have the following decomposition theorem (see Theorem 9.3 in [2]).

**Theorem 1.** Let \( \{ K^s : s \geq 0 \} \) be a continuous one-parameter semigroup of isometries on a Hilbert space \( \mathcal{H} \). Then \( \mathcal{H} \) has the orthogonal decomposition \( \mathcal{H} = \mathcal{H}_U \oplus \mathcal{H}_{NU} \), where \( \mathcal{H}_U = \bigcap_{s \geq 0} K^s \mathcal{H} \), and \( \mathcal{H}_{NU} \) is isomorphic to \( L^2([0, \infty), \mathcal{H}_0) \) for some Hilbert space \( \mathcal{H}_0 \). \( \mathcal{H}_U \) and \( \mathcal{H}_{NU} \) are invariant under \( K^s \) for all \( s \geq 0 \). The operator \( K^s \) restricted on \( \mathcal{H}_U \) is a one-parameter semigroup of unitary operators. And \( K^s \) restricted on \( \mathcal{H}_{NU} \) acts the unilateral shift operator, i.e. for any \( \gamma \in \mathcal{H}_{NU} = L^2([0, \infty], \mathcal{H}_0) \),

\[
(K^s \gamma)(t) = \gamma(t + s) \in \mathcal{H}_0.
\]  

Theorem 1 provides us with a useful tool to deal with the completely nonunitary component of \( K^s \). For the unitary component, we have the following spectral representation theorem.

**Theorem 2.** Let \( \{ U(s) : s \geq 0 \} \) be a continuous one-parameter semigroup of unitary operators on a Hilbert space \( \mathcal{H} \). Assume that \( \mathcal{H} \) can be generated by \( U \) and some \( f \in \mathcal{H} \). Then there exists a unitary map \( \phi : \mathcal{H} \to L^2(\mathbb{R}, d\mu) \) where \( \mu \) is some positive finite measure on \( \mathbb{R} \), such that

\[
(\phi(f))(x) = 1,
\]

\[
(\phi(K^s f))(x) = e^{isx}
\]

for all \( x \in \mathbb{R} \) and \( s \geq 0 \).

Theorems 1 and 2 suggest the orthogonal decomposition \( \mathcal{H}_f = \mathcal{H}_{f,U} \oplus \mathcal{H}_{f,NU} = L^2(\mathbb{R}, d\mu_f) \oplus L^2([0, \infty], \mathcal{H}_{f,0}) \). Further more, we can write \( \mu_f = \mu_{f,d} + \mu_{f,c} \), where \( \mu_{f,d} \) is a countable sum of dirac measures and \( \mu_{f,c} \) is continuous with respect to the Lebesgue measure. \( \mu_{f,c} \) can have both absolutely continuous part and singular continuous part. The decomposition of \( \mu_f \) suggests the orthogonal decomposition \( \mathcal{H}_{f,U} = L^2(\mathbb{R}, d\mu_{f,d}) \oplus L^2(\mathbb{R}, d\mu_{f,c}) \). In sum, we have

\[
f = f_{NU} + f_d + f_c,
\]

where \( f_{NU} \in L^2([0, \infty], \mathcal{H}_{f,0}) \), \( f_d \in L^2(\mathbb{R}, d\mu_{f,d}) \), and \( f_c \in L^2(\mathbb{R}, d\mu_{f,c}) \). Note that these subspace are pair-wise orthogonal and are all invariant under \( K^s \) for all \( s \geq 0 \). The support of \( \mu_{f,d} \) consists of countably many points. Each point \( x_i \) in the support of \( \mu_{f,d} \) corresponds to an eigenvector \( v_i \in \mathcal{H}_f \) of \( K^s \) for all \( s \geq 0 \), i.e.

\[
(\phi(u_i v_i))(x) = \begin{cases} 1 & \text{if } x = x_i, \\ 0 & \text{otherwise}, \end{cases}
\]
and \( \mu_{f,d}(\{x_i\}) = |a_i|^2 \). Therefore we can rewrite Eq. (19) as

\[
f = \sum_i a_i v_i + f_{NU} + f_c.
\]

We rearrange the index of \( v_i \) so that \( |a_1| \geq |a_2| \geq \cdots \geq 0 \). In order to make connection with \( A_\tau \), we need the following lemmas.

**Lemma 1.** For any \( \tau > 0 \) and any \( g \in L^2([0, \tau]) \), the following integral

\[
\int_0^\tau g(s)K_s f ds
\]

is well-defined and is an element of \( \mathcal{H}_f \).

The proof of this and the following lemma use standard argument from mathematical analysis and we leave the proof to the interested readers.

Let

\[
\tilde{\mathcal{H}}^\text{int}_f = \left\{ \int_0^\tau g(s)K_s f ds : \tau > 0, g \in L^2([0, \tau]) \right\}.
\]

\( \tilde{\mathcal{H}}^\text{int}_f \) is a linear subspace of \( \mathcal{H}_f \). We have

**Lemma 2.**

\[
\tilde{\mathcal{H}}^\text{int}_f = \mathcal{H}_f.
\]

Recall the following assumption are needed for \( A_\tau \) in [3]:

**Assumption 3** \( \rho(s) \) does not vanish for almost all \( s \geq 0 \). And \( \rho \in L^2_{\text{loc}}(\mathbb{R}) \).

Since \( |\rho(s)| \leq \rho(0) \) for all \( s \), assumption [3] assures that \( A_\tau g \in L^2([0, \tau]) \) for any \( g \in L^2([0, \tau]) \) and that \( A_\tau \) is a compact operator.

For simplicity, we use the notation \( L^2_\tau := L^2([0, \tau]) \). Given lemma [1] for any \( g \in L^2([0, \tau]) \) and \( t \in [0, \tau] \), we can rewrite \( A_\tau g \) in the following form:

\[
\langle A_\tau g \rangle(t) = \frac{1}{\tau} \int_0^\tau g(s)\rho(t-s)ds = \frac{1}{\tau} \int_0^\tau g(s)\langle K_s f, K_s f \rangle_{\mathcal{H}_f} ds
\]

\[
= \frac{1}{\tau} \left\langle K_t f, \int_0^\tau \tilde{g}(s) K_s f ds \right\rangle_{\mathcal{H}_f}.
\]

Similarly, for any \( g_1, g_2 \in L^2_\tau :

\[
\langle g_1, A_\tau g_2 \rangle_{L^2_\tau} = \int_0^\tau g_1(t) \langle A_\tau g_2 \rangle(t) dt
\]

\[
= \frac{1}{\tau} \int_0^\tau g_1(t) dt \left\langle \int_0^\tau \tilde{g}_2(s) K_s f ds, K_t f \right\rangle_{\mathcal{H}_f}
\]

\[
= \frac{1}{\tau} \left\langle \int_0^\tau \tilde{g}_2(s) K_s f ds, \int_0^\tau \tilde{g}_1(t) K_t f dt \right\rangle_{\mathcal{H}_f}
\]

where \( \langle \cdot, \cdot \rangle_{L^2_{\tau}} \) refers to the inner product in \( L^2_{\tau} \) and \( \langle \cdot, \cdot \rangle_{\mathcal{H}_f} \) refers to the inner product in \( \mathcal{H}_f \).

We shall use the following variational description of the eigenvalues.
Proposition 1 (The minimax principal). Let \( \mathcal{H} \) be a Hilbert space and \( A \) a Hermitian operator on \( \mathcal{H} \). \( \lambda_1 \geq \lambda_2 \geq \cdots \) be the eigenvalues of \( A \) in descending order. Then
\[
\lambda_i = \max_{\dim M = i} \min_{v \in M} \frac{\langle v, Av \rangle}{\|v\|^2}
\] (27)

Our main result states that,

Theorem 3 (Main result). Under assumptions 1, 2 and 3, we have
\[
\lim_{\tau \to \infty} \lambda_{\tau,i} = |a_i|^2
\] (28)

for all \( i \in \mathbb{N} \).

Remark 1. It is worth to point out that the one-parameter semigroup of isometries \( \{K^s : s \geq 0\} \) is equivalent to the classical Koopman one-parameter semigroup \( \{\tilde{K}^s : s \geq 0\} \) which acts on \( L^2(X, d\nu) \) almost surely, if the dynamical system is ergodic and has finite invariant measure \( \nu \) on the phase space \( X \). Because if \( f \in L^2(X, \nu) \), then \( f\tilde{K}^s f \in L^1(X, d\nu) \) and Birkhoff ergodic theorem states that \( \rho(s) = \nu(f\tilde{K}^s f) \) for almost every initial state \( x_0 \in X \). In other words, \( (f, \tilde{K}^s f)_{\mathcal{H}_f} = (f, \tilde{K}^s f)_{L^2(X, d\nu)} \). Note that \( f \) is interpreted as a given time series on the left of the equality and interpreted as a function on the right of the equality. This shows that under the assumption that the dynamical system is ergodic and (finite) measure-preserving, there is a natural isometric bijection from \( \mathcal{H}_f \) to \( L^2(X, d\nu) \).

3 Proof of the main result

For any fixed small \( \epsilon \geq 0 \), we choose \( k \), so that \( \sum_{i=k+1}^{\infty} |a_i|^2 \leq \epsilon \). We have the orthogonal decomposition
\[
f = f_d + f_{NU} + f_c = \sum_{i=1}^{k} a_i v_i + \sum_{i=k+1}^{\infty} a_i v_i + f_{d,k} + f_{NU} + f_c = f_{d,k} + f_{d,\epsilon} + f_{NU} + f_c
\] (29)

where \( f_{d,k} \in \mathcal{H}_{f,d,k} \) which is the subspace of \( \mathcal{H}_{f,d} \) spanned by \( \{v_1, \ldots, v_k\} \), and \( f_{d,\epsilon} \in \mathcal{H}_{f,d,\epsilon} \) the subspace spanned by the rest of the eigenvectors, \( f_{NU} \in \mathcal{H}_{f,NU} \), and \( f_c \in \mathcal{H}_{f,c} \). Note that \( \mathcal{H}_{f,d,k}, \mathcal{H}_{f,d,\epsilon}, \mathcal{H}_{f,NU}, \) and \( \mathcal{H}_{f,c} \) are pairwise orthogonal.
and invariant subspaces of $\mathcal{H}_f$. Hence following Eq. (20), for any $g_1, g_2 \in L^2_x,$

$$
\langle g_1, A_\tau g_2 \rangle_{L^2_x} = \frac{1}{\tau} \left( \int_0^\tau \langle \tilde{g}_2(s) \mathcal{K} s \mathcal{K} f \rangle ds, \int_0^\tau \langle \tilde{g}_1(t) \mathcal{K} f \rangle dt \right)_{\mathcal{H}_f} \\
= \frac{1}{\tau} \left( \int_0^\tau \tilde{g}_2(s) \mathcal{K} \mathcal{K}^* (f_{d,k} + f_{d,e} + f_{f + f_{NU}}) ds, \int_0^\tau \tilde{g}_1(t) \mathcal{K} \mathcal{K}^* (f_{d,k} + f_{d,e} + f_{f + f_{NU}}) dt \right)_{\mathcal{H}_f} \\
+ \frac{1}{\tau} \left( \int_0^\tau \tilde{g}_2(s) \mathcal{K} \mathcal{K}^* f_{d,e} ds, \int_0^\tau \tilde{g}_1(t) \mathcal{K} \mathcal{K}^* f_{d,e} dt \right)_{\mathcal{H}_f} \\
+ \frac{1}{\tau} \left( \int_0^\tau \tilde{g}_2(s) \mathcal{K} \mathcal{K}^* f_{f_{NU}} ds, \int_0^\tau \tilde{g}_1(t) \mathcal{K} \mathcal{K}^* f_{f_{NU}} dt \right)_{\mathcal{H}_f} \\
= \langle g_1, A_\tau d(k) g_2 \rangle_{L^2_x} + \langle g_1, A_\tau d(e) g_2 \rangle_{L^2_x} + \langle g_1, A_\tau e(g_2) \rangle_{L^2_x} + \langle g_1, A_\tau NU g_2 \rangle_{L^2_x},
$$

(30)
in which the definition of $A_\tau d(k), A_\tau d(e), A_\tau e$, and $A_\tau NU$ are obvious. Note that the cross product terms all vanish due to that $\mathcal{H}_{f,d,k}, \mathcal{H}_{f,d,e}, \mathcal{H}_{f,e}$ and $\mathcal{H}_{f,NU}$ are pairwise orthogonal and invariant under $\mathcal{K}^*$ for all $s \geq 0$.

Let $\lambda_{\tau,d,k,i}, \lambda_{\tau,d,e,i}, \lambda_{\tau,e,i},$ and $\lambda_{\tau,NU,i}$ be the $i$-th largest eigenvalue of $A_\tau d(k), A_\tau d(e), A_\tau e,$ and $A_\tau NU$ respectively. We will prove the following identities:

**Proposition 2.**

$$
\lim_{\tau \to \infty} \lambda_{\tau,d,k,i} = |a_i|^2 \text{ for } i = 1, ..., k, 
$$

(31)

$$
\lambda_{\tau,d,e,1} \leq \epsilon \text{ for any } \tau > 0, 
$$

(32)

$$
\lim_{\tau \to \infty} \lambda_{\tau,e,1} = 0 
$$

(33)

$$
\lim_{\tau \to \infty} \lambda_{\tau,NU,1} = 0 
$$

(34)

Before we start to prove Eqs. (31)–(34), it is not hard to see that proposition 1 and proposition 2 directly implies the main result. Indeed, for any fixed $n$ and any $\epsilon > 0$, we can find $k$ so that $n \leq k$ and $\sum_{i=k+1}^\infty |a_i|^2 \leq \epsilon$. Then we find $\tau$ large enough so that $\lambda_{\tau,e,1} \leq \epsilon$ and $\lambda_{\tau,NU,1} \leq \epsilon$. Note that $A_\tau d(k), A_\tau d(e), A_\tau e,$ and $A_\tau NU$ are all positive semi-definite. Applying the minimax principal we have

$$
\lambda_{\tau,n} = \max_{\mathcal{M} \subseteq L^2_x, \dim \mathcal{M} = n} \min_{v \in \mathcal{M}} \frac{\langle v, A_\tau v \rangle}{\|v\|^2} 
$$

(35)

$$
= \max_{\mathcal{M} \subseteq L^2_x, \dim \mathcal{M} = n} \min_{v \in \mathcal{M}} \frac{\langle v, A_\tau d(k) v \rangle + \langle v, A_\tau d(e) v \rangle + \langle v, A_\tau e v \rangle + \langle v, A_\tau NU v \rangle}{\|v\|^2} 
$$

(36)

$$
\geq \max_{\mathcal{M} \subseteq L^2_x, \dim \mathcal{M} = n} \min_{v \in \mathcal{M}} \frac{\langle v, A_\tau d(k) v \rangle}{\|v\|^2} = \lambda_{\tau,d,k,n}. 
$$

(37)
and that
\[
\lambda_{\tau,n} = \max_{M \subseteq L_2} \min_{v \in M} \frac{\langle v, A_v \rangle}{\|v\|^2}
\]
(38)
\[
\leq \max_{M \subseteq L_2} \min_{v \in M} \frac{\langle v, A_v \rangle}{\|v\|^2} + 2 \epsilon = \lambda_{\tau,d,k,n} + 2 \epsilon.
\]
(39)

Combining with Eq. (31), this implies Theorem 3.

**Proof (Eq. (31)).** Recall Eq. (20) that each eigenvector \(v_{i} \) corresponds to a point \(x_{i} \) in the support of \(\mu_{d} \). For any \(g \in L_{2} \), Theorem 2 states that \(\int_{0}^{\tau} \tilde{g}(s)K_{s}f_{d,k}ds \) has the following representation in \(L_{2}(\mathbb{R}, d\mu) \), for any \(x \in \mathbb{R} \),
\[
\left(\phi\left(\int_{0}^{\tau} \tilde{g}(s)K_{s}f_{d,k}ds\right)\right)(x) = \begin{cases} 
\int_{0}^{\tau} \tilde{g}(s)e^{isx}ds, & \text{if } x = x_{j} \text{ for some } j. \\
0, & \text{otherwise.}
\end{cases}
\]
(40)

And
\[
\langle g, A_{\tau,d,k}g \rangle_{L_{2}^{\tau}} = \frac{1}{\tau} \left(\int_{0}^{\tau} \tilde{g}(s)K_{s}f_{d,k}ds, \int_{0}^{\tau} \tilde{g}(t)K_{t}f_{d,k}dt\right)_{\mathcal{H}_{\tau}}
\]
(41)
\[
= \frac{1}{\tau} \sum_{j=1}^{k} \left| \int_{0}^{\tau} \tilde{g}(s)e^{isx_{j}}ds \right|_{L_{2}(\mathbb{R}, d\mu)}^{2}
\]
(42)
\[
= \frac{1}{\tau} \sum_{j=1}^{k} |a_{j}|^{2} \left| \int_{0}^{\tau} \tilde{g}(s)e^{isx_{j}}ds \right|^{2}.
\]
(43)

Let \(\xi_{j} \in L_{2}^{\tau} \) so that \(\xi_{j}(s) = e^{isx_{j}} \) for any \(s \in [0, \tau] \). Then \(\|\xi_{j}\|^{2}_{L_{2}^{\tau}} = \tau \) and
\[
\langle g, A_{\tau,d,k}g \rangle_{L_{2}^{\tau}} = \frac{1}{\tau} \sum_{j=1}^{k} |a_{j}|^{2} \langle \xi_{j}, g \rangle_{L_{2}^{\tau}}^{2} = \sum_{j=1}^{k} \left| \frac{a_{j}^{2}}{\sqrt{\tau}} \right|^{2}
\]
(44)

Let \(V_{\tau,k} = \text{Span} \{ \frac{a_{1}^{2}}{\sqrt{\tau}}, \frac{a_{2}^{2}}{\sqrt{\tau}}, \ldots, \frac{a_{k}^{2}}{\sqrt{\tau}} \} \). We write \(g = g_{\tau,k} + g^{\perp} \), where \(g_{\tau,k} \in V_{\tau,k} \), and \(g^{\perp} \in V_{\tau,k}^{\perp} \). Then
\[
\langle g, A_{\tau,k,d}g \rangle_{L_{2}^{\tau}} = \sum_{j=1}^{k} \left| \frac{a_{j}^{2}}{\sqrt{\tau}} \right|^{2}
\]
(45)

Note that \(\dim V_{\tau,k} = k \) for all \(\tau > 0 \). Direct calculation yields that, for \(j \neq l \),
\[
\langle \frac{a_{j}^{2}}{\sqrt{\tau}}, \frac{a_{l}^{2}}{\sqrt{\tau}} \rangle_{L_{2}^{\tau}} = \frac{\langle a_{j}^{2}, a_{l}^{2} \rangle}{\|a_{j}^{2} \|_{L_{2}^{\tau}}^{2}} \to 0 \text{ as } \tau \to \infty.
\]
Therefore the distribution of the eigenvalues of \(A_{\tau,k,d} \) shall approach to the distribution of the eigenvalues of
\[
\begin{pmatrix}
|a_{1}|^{2} & 0 & \cdots & 0 \\
0 & |a_{2}|^{2} & \cdots & 0 \\
\vdots & & \ddots & \vdots \\
0 & 0 & \cdots & |a_{k}|^{2}
\end{pmatrix}
\]
(46)
as \( \tau \to \infty \). This completes the proof of Eq. (31).

Proof (Eq. (32)). Similar to Eq. (44), for any \( g \in L^2_\tau \), \( \|g\|_{L^2_\tau} = 1 \), we have

\[
\langle g, A_{\tau,d,e}g \rangle_{L^2_\tau} = \frac{1}{\tau} \sum_{j=k+1}^{\infty} |a_j|^2 |\langle \xi_j, g \rangle_{L^2_\tau}|^2 = \sum_{j=k+1}^{\infty} \left( \frac{a_j}{\sqrt{\tau}} \right)^2 g \langle g \rangle_{L^2_\tau}^2 \leq \sum_{j=k+1}^{\infty} |a_j|^2 \leq \epsilon. \tag{47}
\]

Then the minimax principal implies that \( \lambda_{\tau,k,\epsilon,1} \leq \epsilon \).

Proof (Eq. (33)). Following [1] (page 39-41), we first show that

\[
\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau |\mu_{f,c}(e^{ix})| ds = 0, \tag{49}
\]

or equivalently

\[
\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau |\mu_{f,c}(e^{ix})|^2 ds = 0. \tag{50}
\]

Eq. (49) can be interpreted as that the large moments associated to the continuous spectral measure has density zero. We have

\[
\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau |\mu_{f,c}(e^{ix})|^2 ds = \frac{1}{\tau} \int_0^\tau e^{ixy} d\mu_{f,c}(y) dx
\]

\[
= \frac{1}{\tau} \int_0^\tau \int_{\mathbb{R}} e^{ix-y} d\mu_{f,c}(x) d\mu_{f,c}(y) \tag{51}
\]

\[
= \frac{1}{\tau} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{x-y} d\mu_{f,c}(x) d\mu_{f,c}(y) \int_0^\tau e^{ix} dx \tag{52}
\]

Note that

\[
\left| \frac{1}{\tau} \int_0^\tau e^{ix} dx \right| \leq 1 \tag{53}
\]

for any \( \tau > 0 \) and any \( x, y \in \mathbb{R} \). And when \( x \neq y \)

\[
\frac{1}{\tau} \int_0^\tau e^{ix-y} dx = \frac{e^{\tau(x-y)} - 1}{\tau i(x-y)} \to 0 \tag{54}
\]

as \( \tau \to \infty \). Since \( \mu_{f,c} \) is continuous, \( (\mu_{f,c} \times \mu_{f,c})(\{(x, y) \in \mathbb{R}^2 : x = y\}) = 0 \). Therefore Lebesgue’s dominated convergence theorem implies Eq. (50), hence Eq. (49).

For any \( g \in L^2(\mathbb{R}) \), Theorem 2 implies that

\[
\left( \phi \left( \int_0^\tau \tilde{g}(s) K^n f_{d,e} ds \right) \right)(x) = \int_0^\tau \tilde{g}(s) e^{i\tau x} ds. \tag{55}
\]
Therefore

$$
\langle g, A_{\tau,c}g \rangle_{L^2_T} = \frac{1}{\tau} \left\langle \int_0^\tau \tilde{g}(s)K^s f_{d,c} ds, \int_0^\tau \tilde{g}(t)K^t f_{d,c} dt \right\rangle_{\mathcal{H}_f}
$$

$$= \frac{1}{\tau} \left\langle \phi \left( \int_0^\tau \tilde{g}(s)K^s f_{d,c} ds \right), \phi \left( \int_0^\tau \tilde{g}(t)K^t f_{d,c} dt \right) \right\rangle_{L^2(\mathbb{R}, d\mu_c)}
$$

$$= \frac{1}{\tau} \int_{-\infty}^{\infty} d\mu_{f,c}(x) \int_0^\tau \int_0^\tau g(t)\tilde{g}(s)e^{is(t-s)x} ds dt
$$

$$= \frac{1}{\tau} \int_0^\tau \int_0^\tau g(t)\tilde{g}(s)\mu_{f,c}(e^{is(t-s)x}) ds dt
$$

Hence

$$\left| \langle g, A_{\tau,c}g \rangle \right| \leq \frac{1}{\tau} \int_0^\tau \int_0^\tau |g(t)| \cdot |\tilde{g}(s)| \cdot |\mu_{f,c}(e^{is(t-s)x})| ds dt
$$

$$= \frac{2}{\tau} \int_0^\tau |g(t)| \int_0^\tau |\tilde{g}(s)| \cdot |\mu_{f,c}(e^{is(t-s)x})| ds dt
$$

$$= \frac{2}{\tau} \int_0^\tau |g(t)| \int_0^{\tau-t} |g(t+s)| \cdot |\mu_{f,c}(e^{isx})| ds dt
$$

$$\leq \frac{2}{\tau} \int_0^\tau \int_0^{\tau-t} \frac{1}{2} (|g(t)|^2 + |g(t+s)|^2) |\mu_{f,c}(e^{isx})| ds dt
$$

$$= \frac{1}{\tau} \int_0^\tau |\mu_{f,c}(e^{isx})| \cdot \|g\|_{L^2_T}^2 ds
$$

Therefore

$$\lambda_{\tau,c,1} = \max_{g \in L^2_T} \frac{\langle g, A_{\tau,c}g \rangle}{\|g\|_{L^2_T}^2} \to 0,
$$

as $\tau \to \infty$. This completes the proof of Eq. \(\text{(53)}\).

Proof (Eq. \(\text{(54)}\)). Recall that $\mathcal{H}_{f,NU} \cong L^2([0, +\infty), \mathcal{H}_0)$. Hence $f_{NU}$ can be represented as a curve from $[0, \infty)$ to $\mathcal{H}_0$. We denote this curve by $\gamma$. Without ambiguity, we do not distinguish between $\gamma$ and $f_{NU}$. Hence for each $t \geq 0$, $\gamma(t) \in \mathcal{H}_0$. And $\|\gamma\|_{\mathcal{H}_{f,NU}}^2 = \int_0^\infty \|\gamma(t)\|_{\mathcal{H}_0}^2 dt$. Recall that $(K^t\gamma)(t) = \gamma(t+s)$. We set $\gamma(t) = 0$ for all $t < 0$. Hence for any $g \in L^2_T$,

$$\langle g, A_{\tau,NU}g \rangle_{L^2_T} = \frac{1}{\tau} \left\langle \int_0^\tau \tilde{g}(s)K^{s1}\gamma ds_1, \int_0^\tau \tilde{g}(s_2)K^{s2}\gamma ds_2 \right\rangle_{\mathcal{H}_{f,NU}}
$$

$$= \frac{1}{\tau} \int_0^\tau \int_0^\tau \tilde{g}(s_1)g(s_2)(\gamma(t+s_1), \gamma(t+s_2))_{\mathcal{H}_0} ds_1 ds_2 dt
$$

$$= \frac{1}{\tau} \int_0^\tau \int_0^\tau \tilde{g}(s_1)g(s_2) \int_0^\tau \langle \gamma(t+s_1), \gamma(t+s_2) \rangle_{\mathcal{H}_0} dt ds_1 ds_2
$$
We first show the following identity:
\[
\lim_{s \to \infty} \langle \gamma, K^s \gamma \rangle_{H_{f,NU}} = \lim_{s \to \infty} \int_0^\infty \langle \gamma(t), \gamma(t + s) \rangle_{H_0} \, dt = 0. \tag{70}
\]
To prove Eq. (70), without loss of generality we assume that \(\|\gamma\|_{H_{f,NU}} = 1\). For any \(\epsilon > 0\), there exists \(N_\epsilon\), so that \(\int_0^{N_\epsilon} \|\gamma(t)\|^2_{H_0} \, dt > 1 - \epsilon\). This means that \(\int_{N_\epsilon}^\infty \|\gamma(t)\|^2_{H_0} \, dt < \epsilon\). Therefore for any \(\tau > M\),
\[
\left| \int_0^\infty \langle \gamma(t), \gamma(t + s) \rangle_{H_0} \, dt \right|^2 \leq \left| \int_0^\infty \|\gamma(t)\|^2_{H_0} \, dt \right|^2 \cdot \left| \int_{N_\epsilon}^\infty \|\gamma(t)\|^2_{H_0} \, dt \right|^2 < \epsilon^2. \tag{71}
\]
This proves Eq. (70). Now we continue with Eq. (69):
\[
\langle g, A_{\tau,NU} g \rangle_{L^2} = \left| \frac{2}{\tau} \int_0^\tau \int_{s_1}^s \bar{g}(s_1)g(s_2)\langle (K^{s_1}g, K^{s_2}g)_{H_{f,NU}} \rangle \, ds_1 \, ds_2 \right| \tag{72}
\]
\[
= \frac{2}{\tau} \int_0^\tau \int_{M_\epsilon}^{s-\epsilon} \bar{g}(s_1)g(s_1 + s)\langle (\gamma, K^s \gamma)_{H_{f,NU}} \rangle \, ds_1 \, ds \tag{73}
\]
For any \(\epsilon > 0\), find \(M_\epsilon\), so that for any \(\|\gamma, K^s \gamma\| < \epsilon\) for any \(s > M_\epsilon\). Now for any \(\tau > M_\epsilon/\epsilon\) and any \(\|g\|_{L^2} = 1\), we have
\[
\langle g, A_{\tau,NU} g \rangle_{L^2} \leq \frac{2}{\tau} \int_0^\tau \int_0^{M_\epsilon} |g(s_1)| \cdot |g(s_1 + s)| \cdot \|\gamma, K^s \gamma\|_{H_{f,NU}} |ds_1| |ds|
\]
\[
\leq \frac{2}{\tau} \int_0^\tau \int_0^{M_\epsilon} |g(s_1)| \cdot |g(s_1 + s)| \cdot \|\gamma, K^s \gamma\|_{H_{f,NU}} |ds_1| |ds|
\]
\[
\leq \frac{1}{\tau} \int_0^\tau \int_0^{M_\epsilon} (|g(s_1)|^2 + |g(s_1 + s)|^2) \cdot \|\gamma, K^s \gamma\|_{H_{f,NU}} |ds_1| |ds|
\]
\[
\leq \frac{1}{\tau} \int_0^\tau \int_0^{M_\epsilon} (|g(s_1)|^2 + |g(s_1 + s)|^2) \cdot \|\gamma, K^s \gamma\|_{H_{f,NU}} |ds_1| |ds|
\]
\[
\leq \frac{1}{\tau} \int_0^\tau \int_0^{M_\epsilon} |g(s_1)|^2 |ds_1| |ds| + \frac{1}{\tau} \int_0^\tau \int_0^{M_\epsilon} |g(s_1 + s)|^2 \cdot \|\gamma, K^s \gamma\|_{H_{f,NU}} |ds_1| |ds|
\]
\[
\leq \frac{M_\epsilon}{\tau} + \frac{M_\epsilon}{\tau} + 2\frac{\epsilon}{\tau} (\tau - M_\epsilon) \leq 4\epsilon. \tag{81}
\]
Therefore for \(\tau > M_\epsilon/\epsilon\),
\[
\lambda_{\tau,NU,1} = \max_{g \in L^2 \atop \|g\| = 1} \langle g, A_{\tau,NU} g \rangle \leq 4\epsilon. \tag{82}
\]
This completes the proof of Eq. (33).
References

1. Paul R. Halmos. *Lectures on Ergodic Theory*. Chelsea Publishing Company, New York, N.Y, 1956.

2. Béla Szőkefalvi-Nagy, Ciprian Foias, Hari Bercovici, and László Kérchy. *Harmonic Analysis of Operators on Hilbert Space*. Springer Science & Business Media, 2010.

3. Robert Vautard and Michael Ghil. Singular spectrum analysis in nonlinear dynamics, with applications to paleoclimatic time series. *Physica D: Nonlinear Phenomena*, 35:395–424, 1989.

4. Yicun Zhen, Bertrand Chapron, Etienne Mémin, and Lin Peng. Eigenvalues of autocovariance matrix: A practical method to identify the koopman eigenfrequencies. *arXiv*, 2021.