SOME PRODUCTS IN FUSION SYSTEMS AND LOCALITIES

ELLEN HENKE

ABSTRACT. The theory of saturated fusion systems resembles in many parts the theory of finite groups. However, some concepts from finite group theory are difficult to translate to fusion systems. For example, products of normal subsystems with other subsystems are only defined in special cases. In this paper the theory of localities is used to prove the following result: Suppose $F$ is a saturated fusion system over a $p$-group $S$. If $E$ is a normal subsystem of $F$ over $T \leq S$, and $D$ is a subnormal subsystem of $N_F(T)$ over $R \leq S$, then there is a subnormal subsystem $ED$ of $F$ over $TR$, which plays the role of a product of $E$ and $D$ in $F$. If $D$ is normal in $N_F(T)$, then $ED$ is normal in $F$. It is shown along the way that the subsystem $ED$ is closely related to a naturally arising product in certain localities attached to $F$.

1. Introduction

The theory of localities was developed by Chermak [Che13, Che21a] and gives an alternative way of looking at linking systems and transporter systems. Here linking systems and transporter systems are categories associated to saturated fusion systems that were defined and studied before by various authors; see [BLO03, BCG+05, Oli10, OV07]. Linking systems are particularly important in the study of the homotopy theory of fusion systems. For an introduction to localities the reader is referred to [Che21a] or to the summary in [Hen21, Sections 3.1-3.5 and 3.7].

Many concepts and results from finite group theory can be translated both to saturated fusion systems and to a certain class of localities; for saturated fusion systems this is largely due to Aschbacher [Asc08, Asc11]. It is shown in [CH21] that the two theories are closely related. This can be used to reprove known results on fusion systems and, perhaps more importantly, to show some properties of fusion systems that seemed difficult to prove before. The present paper gives a further example of this.

To explain our approach and some results from [CH21] in more detail let us first introduce some definitions. If $(\mathcal{L}, \Delta, S)$ is a locality and $H$ is a partial subgroup of $\mathcal{L}$, then we can always naturally form the fusion system $F_{S \cap H}(H)$. By definition, this is the fusion system over $S \cap H$, which is generated by the maps between subgroups of $S \cap H$ that are conjugation maps by elements of $H$. The locality $(\mathcal{L}, \Delta, S)$ is said to be a locality over $F$ if $F = F_{S}(\mathcal{L})$.

As in [Hen19] we call a locality $(\mathcal{L}, \Delta, S)$ a linking locality if $F_{S}(\mathcal{L})$ is saturated, $F_{S}(\mathcal{L})^{cr} \subseteq \Delta$ and $N_{\mathcal{L}}(P)$ is a group of characteristic $p$ for every $P \in \Delta$. Here a finite group $G$ is of characteristic $p$ if $C_{G}(O_{p}(G)) \leq O_{p}(G)$. We caution the reader that Chermak [Che21b, Che17] calls such localities proper localities and this terminology is also used in [CH21]. If $F$ is a saturated fusion system over a $p$-group $S$, it is a consequence of the existence and uniqueness of centric linking systems that a linking locality $(\mathcal{L}, \Delta, S)$ over $F$ always exists (cf. [Che13, Oli13, GL16]). Indeed, for any “suitable” set $\Delta$ of subgroups of $S$, it is shown in [Hen19, Theorem A] that there is a linking locality $(\mathcal{L}, \Delta, S)$ over $F$ with object set $\Delta$. Examples for suitable object sets are the sets $F^{c}$, $F^{q}$ and $F^{s}$ of $F$-centric, $F$-quasicentric and $F$-subcentric subgroups of $S$ respectively; see [AKO11, Definition I.3.1, Definition III.4.5] and [Hen19, Definition 1]. The set of $F$-subcentric subgroups turns out to be the largest possible object set of a linking locality. Another particularly nice object set is the set $\delta(F)$ of regular objects, which was first defined by Chermak [Che17, p.36]. The reader is referred to Subsection 3.1 for details. A linking locality $(\mathcal{L}, \Delta, S)$ over $F$ is called regular if $\Delta = \delta(F)$. 

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Suppose now that $\mathcal{F}$ is a saturated fusion system and $(\mathcal{L}, \Delta, S)$ is a linking locality over $\mathcal{F}$. Write $\mathfrak{N}(\mathcal{F})$ for the set of normal subsystems of $\mathcal{F}$ and $\mathfrak{N}(\mathcal{L})$ for the set of partial normal subgroups of $\mathcal{L}$. Chermak and the author of this paper [CH21, Theorem A] proved that there is a bijection

$$\Psi_\mathcal{L}: \mathfrak{N}(\mathcal{L}) \to \mathfrak{N}(\mathcal{F})$$

which sends every partial normal subgroup $\mathcal{N} \in \mathfrak{N}(\mathcal{L})$ to a normal subsystem over $S \cap \mathcal{N}$, which is the smallest normal subsystem of $\mathcal{F}$ containing $\mathcal{F}_{S \cap \mathcal{N}}(\mathcal{N})$. If $\mathcal{F}_\mathcal{N} \subseteq \Delta$ or $\delta(\mathcal{F}) \subseteq \Delta$, then it even turns out that $\Psi_\mathcal{L}$ is given by $\Psi_\mathcal{L}(\mathcal{N}) = \mathcal{F}_{S \cap \mathcal{N}}(\mathcal{N})$ for all $\mathcal{N} \in \mathfrak{N}(\mathcal{L})$.

In fact, if $(\mathcal{L}, \Delta, S)$ is a regular locality over $\mathcal{F}$ then, writing $\mathcal{S}(\mathcal{F})$ for the set of subnormal subsystems of $\mathcal{F}$ and $\mathcal{S}(\mathcal{L})$ for the set of partial subnormal subgroups of $\mathcal{L}$, the map

$$\hat{\Psi}_\mathcal{L}: \mathcal{S}(\mathcal{L}) \to \mathcal{S}(\mathcal{F}), \mathcal{H} \mapsto \mathcal{F}_{S \cap \mathcal{H}}(\mathcal{H})$$

is well-defined and a bijection by [CH21, Theorem F]. Note that $\hat{\Psi}_\mathcal{L}$ restricts to $\Psi_\mathcal{L}$.

The correspondence between normal subsystems and partial normal subgroups is used in [CH21, Theorem C] to show that, for any two normal subsystems $\mathcal{E}_1, \mathcal{E}_2$ of $\mathcal{F}$, there is a meaningful notion of a product subsystem $\mathcal{E}_1 \mathcal{E}_2$. The proof uses that products in localities can be formed very naturally. Namely, if $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{L}$ and $\Pi: \mathcal{D} \to \mathcal{L}$ denotes the partial product on $\mathcal{L}$, we set

$$\mathcal{X}\mathcal{Y} := \Pi(\mathcal{X}, \mathcal{Y}) := \{\Pi(x, y) : x \in \mathcal{X}, y \in \mathcal{Y}, (x, y) \in \mathcal{D}\}.$$ 

It turns out that the product of any two partial normal subgroups of a locality is again a partial normal subgroup; see [Hen15]. Because of the existence of the bijection $\Psi_\mathcal{L}$ this makes it possible to define products of normal subsystems of fusion systems.

In the present paper we follow a relatively similar strategy to the one just described, but in a slightly different context. We are interested in the situation that $\mathcal{E}$ is a normal subsystem of $\mathcal{F}$ over a subgroup $T \leq S$ and that $\mathcal{D}$ is a subnormal subsystem of $N_\mathcal{F}(T)$. If $(\mathcal{L}, \Delta, S)$ is a linking locality over $\mathcal{F}$ as before, then by [Hen21, Lemma 3.35(b)], $(N_\mathcal{L}(T), \Delta, S)$ is a linking locality over $N_\mathcal{F}(T)$. Therefore, there is also a bijection

$$\Psi_{N_\mathcal{L}(T)}: \mathfrak{N}(N_\mathcal{L}(T)) \to \mathfrak{N}(N_\mathcal{F}(T))$$

with similar properties as mentioned before for $\Psi_\mathcal{L}$. Indeed, assuming that $(\mathcal{L}, \Delta, S)$ is regular, we show in Lemma 3.4 that $(N_\mathcal{L}(T), \delta(N_\mathcal{F}(T)), S)$ is a regular locality, and so the map

$$\hat{\Psi}_{N_\mathcal{L}(T)}: \mathcal{S}(N_\mathcal{L}(T)) \to \mathcal{S}(N_\mathcal{F}(T)), \mathcal{H} \mapsto \mathcal{F}_{S \cap \mathcal{H}}(\mathcal{H})$$

is well-defined and a bijection. Thus, in the situation described above, $\mathcal{E}$ corresponds to a partial normal subgroup $\mathcal{N}$ of $\mathcal{L}$ via $\Psi_\mathcal{L}$, and $\mathcal{D}$ corresponds to a partial subnormal subgroup $\mathcal{K}$ of $N_\mathcal{F}(T)$ via $\hat{\Psi}_{N_\mathcal{L}(T)}$. We use then Theorem 3.1 below to conclude that $\mathcal{N}\mathcal{K}$ is a partial subnormal subgroup of $\mathcal{L}$ and corresponds thus to a subnormal subsystem of $\mathcal{F}$ via $\hat{\Psi}_\mathcal{L}$. The proof of Theorem 3.1 builds on the following theorem, which holds for arbitrary localities.

**Theorem 1.** Let $(\mathcal{L}, \Delta, S)$ be a locality, $\mathcal{N} \subseteq \mathcal{L}$, $T := S \cap \mathcal{N}$ and $\mathcal{K} \subseteq N_\mathcal{F}(T)$. Then $\mathcal{N}\mathcal{K}$ is a partial normal subgroup of $\mathcal{L}$, $\mathcal{N}\mathcal{K} \cap S = T(\mathcal{K} \cap S)$ and $\mathcal{N}\mathcal{K} = \mathcal{K}\mathcal{N}$. Moreover, for every $g \in \mathcal{N}\mathcal{K}$ the following hold:

(a) There exist $n \in \mathcal{N}$ and $k \in \mathcal{K}$ with $(n, k) \in \mathcal{D}$, $g = nk$ and $S_g = S_{(n,k)}$.

(b) There exist $n \in \mathcal{N}$ and $k \in \mathcal{K}$ with $(k, n) \in \mathcal{D}$, $g = kn$ and $S_g = S_{(k,n)}$.

While the theorem above appears to be new, a somewhat similar (but slightly weaker) result was shown before by Chermak. Namely, if $(\mathcal{L}, \Delta, S)$ is a linking locality, Proposition 5.5 in [Che21a] states that, in the situation of Theorem 1, $(\mathcal{N}, \mathcal{K})$ is a partial normal subgroup with $(\mathcal{N}, \mathcal{K}) \cap S = (\mathcal{N} \cap S)(\mathcal{K} \cap S)$. Theorem 1 gives however some more precise information, which is actually needed to prove the theorems below.
Theorem 2. Let \((\mathcal{L}, \Delta, S)\) be a regular locality, \(\mathcal{N} \trianglelefteq \mathcal{L}, T := S \cap \mathcal{N}\) and \(\mathcal{K} \subseteq \mathcal{N}E(T)\). Then \(N\mathcal{K} = \mathcal{K}\mathcal{N}\) is a partial subnormal subgroup of \(\mathcal{L}\) and \(S \cap N\mathcal{K} = T(S \cap \mathcal{K})\).

The main results for fusion systems are summarized in the following theorem.

Theorem 3. Let \(\mathcal{F}\) be a saturated fusion system over \(S\), let \(\mathcal{E}\) be a normal subsystem of \(\mathcal{F}\) over \(T\), and let \(D\) be a subnormal subsystem of \(N_{\mathcal{F}}(T)\) over \(R\). Then there exists a subsystem \(\mathcal{E}D = (\mathcal{E}D)_{\mathcal{F}}\) of \(\mathcal{F}\) such that the following hold:

(a) \(\mathcal{E}D\) is a subnormal subsystem of \(\mathcal{F}\) over \(TR\) with \(\mathcal{E} \trianglelefteq \mathcal{E}D\) and \(D \trianglelefteq \mathcal{N}_{\mathcal{E}D}(T)\). Indeed, \(\mathcal{E}D\) is the smallest subnormal subsystem of \(\mathcal{F}\) with these properties.

(b) If \(D\) is normal in \(N_{\mathcal{F}}(T)\), then \(\mathcal{E}D\) is normal in \(\mathcal{F}\) and \(D\) is normal in \(\mathcal{N}_{\mathcal{E}D}(T)\).

(c) Let \(\mathcal{E}T\) be a subnormal subsystem of \(\mathcal{F}\) with \(\mathcal{E} \trianglelefteq \mathcal{E}T\) and \(D \trianglelefteq \mathcal{N}_{\mathcal{E}T}(T)\). Then \(\mathcal{E} \trianglelefteq \mathcal{E}T\), \(\mathcal{E}D \trianglelefteq \mathcal{E}T\) and \(\mathcal{D} \trianglelefteq \mathcal{N}_{\mathcal{E}T}(T)\).

(d) \(N_{\mathcal{E}D}(T) = (N_{\mathcal{E}}(T)D)_{N_{\mathcal{F}}(T)}\).

(e) Suppose \((\mathcal{L}, \Delta, S)\) is a regular locality over \(\mathcal{F}\). Let \(\Psi_{\mathcal{L}}: \mathcal{H}(\mathcal{L}) \to \mathcal{H}(\mathcal{F})\) and \(\hat{\Psi}_{N_{\mathcal{L}}(T)}: N_{\mathcal{L}}(T) \to \mathcal{H}(N_{\mathcal{F}}(T))\) be the maps from above given by Theorem A and Theorem F in [Che21a, Theorem 3.5]. Set \(\mathcal{N} := \Psi_{\mathcal{L}}^{-1}(S)\) and \(\mathcal{K} := \hat{\Psi}_{N_{\mathcal{L}}(T)}^{-1}(D)\). Then \(TR = (N\mathcal{K}) \cap S, N\mathcal{K} \trianglelefteq \mathcal{L}\) and \(\Psi_{\mathcal{L}}(N\mathcal{K}) = \mathcal{E}D\).

We write here \((\mathcal{E}D)_{\mathcal{F}}\) instead of \(\mathcal{E}D\) if we want to emphasize that the product subsystem depends not only on \(\mathcal{E}\) and \(\mathcal{D}\), but also on \(\mathcal{F}\) (cf. Remark 3.1). For the statement of (d) note that \(N_{\mathcal{F}}(T) \trianglelefteq N_{\mathcal{F}}(T)\) and \(D \trianglelefteq N_{\mathcal{F}}(T) = N_{\mathcal{F}}(T)\), so the product subsystem \((N_{\mathcal{F}}(T)D)_{N_{\mathcal{F}}(T)}\) is defined by the preceding claims in Theorem 3. As we state in the following corollary, the statement in (e) can be generalized to arbitrary linking localities if \(D\) is normal in \(N_{\mathcal{F}}(T)\).

Corollary 4. Assume the hypothesis of Theorem 3. Suppose \((\mathcal{L}, \Delta, S)\) is a linking locality over \(\mathcal{F}\). Let \(\Psi_{\mathcal{L}}: \mathcal{H}(\mathcal{L}) \to \mathcal{H}(\mathcal{F})\) and \(\hat{\Psi}_{N_{\mathcal{L}}(T)}: N_{\mathcal{L}}(T) \to \mathcal{H}(N_{\mathcal{F}}(T))\) be the maps from above given by [Che21a, Theorem A]. Set \(\mathcal{N} := \Psi_{\mathcal{L}}^{-1}(S)\) and \(\mathcal{K} := \hat{\Psi}_{N_{\mathcal{L}}(T)}^{-1}(D)\). Then \(TR = (N\mathcal{K}) \cap S, N\mathcal{K} \trianglelefteq \mathcal{L}\) and \(\Psi_{\mathcal{L}}(N\mathcal{K}) = \mathcal{E}D\) is the product subsystem from Theorem 3.

If \(D\) is normal in \(N_{\mathcal{F}}(T)\), then we can also give a more concrete description of the product subsystem \(\mathcal{E}D\). For this we refer the reader to Theorem 5.3(e).

2. THE PROOF OF THEOREM 1 AND FURTHER RESULTS ON LOCALITIES

2.1. Basic background. We adapt the terminology and notation of [Che21a] and of [Hen19], which is summarized in [Hen21] Sections 3.1-3.5]. The reader is referred to these sources for an introduction to partial groups and localities. However, we recall some of the notations and basic results now and along the way.

Suppose \(\mathcal{L}\) is a partial group with product \(\Pi: D \to L\). By \(W(\mathcal{L})\) we denote the set of words in \(\mathcal{L}\). Moreover, for every \(w = (f_1, f_2, \ldots, f_n) \in D\), the product \(\Pi(w)\) is also denoted by \(f_1 f_2 \cdots f_n\).

Assume now in addition that \((\mathcal{L}, \Delta, S)\) is a locality. By \(S_f\) we denote the set of all \(s \in S\) such that \(s^f\) is defined and an element of \(S\). More generally, if \(w = (f_1, \ldots, f_n) \in W(\mathcal{L})\), then \(S_w\) is the set of all \(s \in S\) for which there exist \(s = s_0, s_1, \ldots, s_n \in S\) such that \(s_{i-1}^f\) is defined and equal to \(s_i\) for all \(i = 1, \ldots, n\). We will frequently use that, by [Che21a, Corollary 2.6], for every \(w \in W(\mathcal{L})\), \(S_w\) is a subgroup of \(S\) and the following equivalence holds:

\[
S_w \in \Delta \text{ if and only if } w \in D. \tag{2.1}
\]

As \(f \in D\) for every \(f \in L\), it follows in particular that \(S_f \in \Delta\) for every \(f \in L\). If \(P \subseteq S_w\), then we say that \(w \in D\) via \(P\).

We will also use the following property which is a consequence of [Che21a, Lemma 2.3(c)]:

\[
S_w \leq S_{\Pi(w)} \text{ and } (\cdots (S_{w_1} f_2 \cdots f_n) = S_{w_{\Pi(w)}} \text{ for every } w = (f_1, \ldots, f_n) \in D. \tag{2.2}
\]
Using the axioms of a partial group, one sees moreover that

\[ v \circ (1) \in D \text{ and } \Pi(v) = \Pi((v \circ 0) = \Pi(v \circ 1)) \text{ for every } v \in D. \]  

(2.3)

2.2. The proof of Theorem I

Lemma 2.1. Let \((L, \Delta, S)\) be a locality with partial product \( \Pi: D \to L, N \triangleleft L, T := S \cap N \) and \( K \leq N_{S}(T) \). Then for every \( g \in N_{K} \), there exist \( n \in N \) and \( k \in K \) with \((n, k) \in D, g = nk \) and \( S_{g} = S_{(m, f)} \).

Proof. Let \( g \in N_{K} \). Then there exist \( n \in N \) and \( k \in K \) with \((n, k) \in D \) and \( g = nk \). By the Frattini Lemma and the Splitting Lemma [Che21a, Corollary 3.11, Lemma 3.12], there exist \( m \in N \) and \( f \in N_{S}(T) \) such that \( g = mf \) and \( S_{g} = S_{(m, f)} \). Then the first part of (2.2) yields

\[ P := S_{(n, k)} \leq S_{g} = S_{(m, f)}. \]

Notice that \( P \in \Delta \) by (2.1) since \((n, k) \in D \). Hence, using (2.1) and the second part of (2.2), we see that \( u := (f, m^{-1}, n^{-1}, k) \in D \) via \( P^{m} \). By [Che21a, Lemma 1.4(f)] we have \( g^{-1} = (mf)^{-1} = f^{-1}m^{-1} \). Hence, it follows from the axioms of a partial group and (2.3) that we can make the following computations, where the product is defined on all relevant words:

\[ \Pi(u) = \Pi((f, m^{-1}n^{-1}, nk)) = \Pi(f, g^{-1}, g) = \Pi(f, 1) = f \]

and

\[ \Pi(u) = \Pi(1, m^{-1}, n, k) = m^{-1}nk = (m^{-1}n)k. \]

As \( m, n \in N \), it follows \( m^{-1}n \in N_{X}(T) \) and thus \( f = (m^{-1}n)k \in N_{X}(T)K \). Notice \( N_{X}(T) \trianglelefteq N_{S}(T) \). As \((N_{S}(T), \Delta, S)\) is a locality by [Che21a, Lemma 2.12], it follows thus from [Hen15, Theorem 1] that there exist \( n' \in N_{X}(T) \) and \( k' \in K \) such that \((n', k') \in D, f = n'k' \) and \( S_{f} = S_{(n', k')} \). Now \( S_{g} = S_{(m, f)} = S_{(m, n', k')}, \) and so \((m, n', k') \in D \) by (2.1). Therefore \( g = mf \) \( \Pi((mn', k') = (mn', k') \in K \) where \( mn' \in N \) and \( k' \in K \). Moreover, using (2.2), one observes that \( S_{g} = S_{(m, n', k')} \leq S_{(mn', k')} \leq S_{(mn', k')} = S_{g} \) and thus \( S_{g} = S_{(mn', k')} \). So the assertion holds with \( mn' \) and \( k' \) in the roles of \( n \) and \( k \).

Given two partial groups \( L \) and \( L' \) with products \( \Pi: D \to L \) and \( \Pi': D' \to L' \) respectively, a projection from \( L \) to \( L' \) is a map \( \beta: L \to L' \) such that

\[ D' = \{(f_{1}\beta, \ldots, f_{n}\beta): (f_{1}, \ldots, f_{n}) \in D\} \]

and \( \Pi'(f_{1}\beta, \ldots, f_{n}\beta) = \Pi(f_{1}, \ldots, f_{n})\beta \) for every word \((f_{1}, \ldots, f_{n}) \in D \) (cf. [Che21a, Definition 4.4]). As \( L' \subseteq D' \), every projection \( \beta: L \to L' \) is surjective.

Lemma 2.2. Let \((L, \Delta, S)\) and \((\overline{L}, \overline{\Delta}, S)\) be localities and let \( \beta: L \to \overline{L} \) be a projection. Then \( M_{\beta} \leq \overline{L} \) for every partial normal subgroup \( M \) of \( L \).

Proof. Let \( M \leq L \). By [Che21a, Lemma 1.14], \( N := \ker(\beta) \) is a partial normal subgroup of \( L \). Hence, [Hen15, Theorem 1] yields \( MN \leq L \). It follows from the definition of the kernel and (2.3) that \( M_{\beta} = (MN)_{\beta} \). Hence, the partial subgroup correspondence [Che21a, Theorem 4.7] yields the assertion.

Proof of Theorem I Part (a) holds by Lemma 2.1. It follows from [Che21a, Lemma 3.2] that \( NK = K \cap N \) and that (a) implies (b). So it remains to show that \( NK \leq L \) and \( NK \cap S = T(K \cap S) \).

We show first that \( NK \) is a partial subgroup. If \( n \in N \) and \( k \in K \) with \((n, k) \in D \), then by [Che21a, Lemma 1.4(f)], we have \((nk)^{-1} = k^{-1}n^{-1} \in K \cap N \), so \( NK \) is closed under inversion. Let now \( v = (f_{1}, \ldots, f_{l}) \in W(NK) \). Then by (b), for each \( i = 1, \ldots, l \), there exist \( n_{i} \in N \) and
k_i \in \mathcal{K}_i \text{ such that } (k_i, n_i) \in \mathbf{D}, f_i = k_i n_i \text{ and } S_{f_i} = S_{(k_i, n_i)}. \text{ Then, setting } w := (k_1, n_1, \ldots, k_i, n_i), \text{ we have } S_w = S_k \in \Delta \text{ and thus } w \in \mathbf{D} \text{ by (2.1). Set } u := (k_1, \ldots, k_n). \text{ By [Che21a] Lemma 3.4, there exists } g \in \mathcal{N} \text{ such that } u \circ (g) \in \mathbf{D} \text{ and } \Pi(w) = \Pi(u \circ (g)). \text{ Observe that } \Pi(u) \in \mathcal{K} \text{ as } \mathcal{K} \text{ is a partial subgroup. Hence, by the axioms of a partial group, we have } \Pi(v) = \Pi(w) = \Pi(u \circ (g)) = \Pi(u) g \in \mathcal{K} \mathcal{N} = \mathcal{N} \mathcal{K}. \text{ This shows that } \mathcal{N} \mathcal{K} \text{ is a partial subgroup.}

Let } \alpha : \mathcal{L} \rightarrow \overline{\mathcal{L}} := \mathcal{L}/\mathcal{N} \text{ be the natural projection (which exists by [Che21a] Lemma 3.16, Corollary 4.5)]. Then } \alpha \text{ is a homomorphism of partial groups and so } \alpha|_{\mathcal{N}_L(T)} : \mathcal{N}_L(T) \rightarrow \overline{\mathcal{L}} \text{ is also a homomorphism of partial groups. Indeed, it follows from [Che21a] Theorem 4.3(b)] that } \alpha|_{\mathcal{N}_L(T)} \text{ is actually a projection of partial groups from } \mathcal{N}_L(T) \text{ to } \overline{\mathcal{L}}. \text{ Therefore, as } (\mathcal{N}_L(T), \Delta, S) \text{ is a locality by [Che21a] Lemma 2.12] and since } \mathcal{K} \subseteq \mathcal{N}_L(T), \text{ Lemma 2.2 yields } \mathcal{K} \alpha \subseteq \overline{\mathcal{L}}. \text{ As } \mathcal{N} \text{ is the kernel of } \alpha, \text{ it follows from (2.3) that } (\mathcal{K} \mathcal{N}) \alpha = \mathcal{K} \alpha \subseteq \overline{\mathcal{L}}. \text{ We have seen moreover that } \mathcal{N} \mathcal{K} = \mathcal{K} \mathcal{N} \text{ is a partial subgroup containing } \mathcal{N}. \text{ Using the partial subgroup correspondence [Che21a] Proposition 4.7], this implies } \mathcal{N} \mathcal{K} \subseteq \overline{\mathcal{L}}.

If } g \in \mathcal{N} \mathcal{K}(S), \text{ then it follows from (a) that } g = nk \text{ for some } n \in \mathcal{N}, \text{ } k \in \mathcal{K} \text{ with } (n, k) \in \mathbf{D} \text{ and } S_g = S_{(n, k)}. \text{ As } g \in \mathcal{N}_L(S), \text{ it follows } S = S_g = S_{(n, k)}, \text{ so } n \in \mathcal{N}_L(S) \text{ and } k \in \mathcal{N}_L(S). \text{ Thus } \mathcal{N} \mathcal{K}(S) \subseteq \mathcal{N}_L(S) \mathcal{N}_L(S). \text{ The converse inclusion holds as } \mathcal{N}_L(S) \text{ is a subgroup and thus closed under products. Hence, } \mathcal{N} \mathcal{K}(S) = \mathcal{N} \mathcal{K}, \text{ where } \mathcal{N} := \mathcal{N}_L(S) \text{ and } \mathcal{K} := \mathcal{N}_L(S) \text{ are normal subgroups of the group } \mathcal{N}_L(S); \text{ to see that } \mathcal{K} \subseteq \mathcal{N}_L(S) \text{ one uses here that } \mathcal{N}_L(S) \subseteq \mathcal{N}_L(T). \text{ Hence, } S \cap \mathcal{N} \mathcal{K} = S \cap \mathcal{N}_L(S) \cap \mathcal{N} \mathcal{K} = S \cap \mathcal{N}_L(S) = S \cap (\mathcal{N} \mathcal{K}) = (S \cap \mathcal{N})(S \cap \mathcal{K}) = (S \cap \mathcal{N})(S \cap \mathcal{K}) = T(S \cap \mathcal{K}). \tag{2.3}

2.3. Restrictions of localities. Let } (\mathcal{L}^+, \Delta^+, S) \text{ be a locality with product } \Pi^+ : \mathbf{D} \rightarrow \mathcal{L}^+. \text{ Suppose } \Delta \subseteq \Delta^+ \text{ is overgroup-closed in } S \text{ and closed under } \mathcal{F}_S(\mathcal{L}^+)-\text{conjugacy. We set then}

\[ \mathcal{L}^+|_{\Delta} := \{ f \in \mathcal{L}^+ : S_f \subseteq \Delta \} \text{ and } \mathbf{D} := \{ w \in \mathbf{D}^+ : S_w \subseteq \Delta \}. \]

It turns out that } \mathcal{L}^+|_{\Delta} \text{ together with } \Pi^+|_{\mathbf{D}} : \mathbf{D} \rightarrow \mathcal{L}^+|_{\Delta} \text{ is a partial group, and } (\mathcal{L}^+|_{\Delta}, \Delta, S) \text{ is a locality, called the restriction of } \mathcal{L}^+ \text{ to } \Delta.

Let now } \mathcal{F} \text{ be a saturated fusion system over } S. \text{ Recall that a locality } (\mathcal{L}, \Delta, S) \text{ over } \mathcal{F} \text{ is called a linking locality if } \mathcal{F}^{ct} \subseteq \Delta \text{ and, for every } P \subseteq \Delta, \text{ the normalizer } \mathcal{N}_L(P) \text{ is a group of characteristic } p. \text{ It is shown in [Hen19] Theorem A(b)] that there is always a linking locality } (\mathcal{L}^+, \mathcal{F}^{ct}, S) \text{ over } S \text{ whose object set is the set } \mathcal{F}^s \text{ of } \mathcal{F}-\text{subcentric subgroups of } S \text{ (cf. [Hen19] Definition 1). Moreover, if } (\mathcal{L}, \Delta, S) \text{ is any linking locality over } \mathcal{F}, \text{ then } \Delta \subseteq \mathcal{F}^s \text{ by [Hen19] Proposition 1(b)]}, \text{ and } \mathcal{L} \text{ is isomorphic to } \mathcal{L}^+|_{\Delta} \text{ via an isomorphism which restricts to the identity on } S \text{ by [Hen19] Theorem A(a)]. This gives us a way of moving between linking localities with different object sets. We will use this together with the following Lemma to deduce Corollary 4 from the statement in Theorem 3(e).}

Lemma 2.3. Let } (\mathcal{L}^+, \Delta^+, S) \text{ and } (\mathcal{L}, \Delta, S) \text{ be localities with } \Delta \subseteq \Delta^+ \text{ and } \mathcal{L}^+|_{\Delta} = \mathcal{L}. \text{ Write } \Pi^+ \text{ and } \Pi \text{ for the products on } \mathcal{L} \text{ and } \mathcal{L}^+ \text{ respectively. Let } \mathcal{N}^+ := \mathcal{L}^+ \text{ and } \mathcal{K}^+ := \mathcal{N}_L(T). \text{ Then } \mathcal{N} := \mathcal{N}^+ \cap \mathcal{L} \subseteq \mathcal{L} \text{ and } \mathcal{K} := \mathcal{K}^+ \cap \mathcal{L} \subseteq \mathcal{N}_L(T). \text{ Moreover, } \Pi^+(\mathcal{N}^+, \mathcal{K}^+) \cap \mathcal{L} = \Pi(\mathcal{N}, \mathcal{K}).

Proof. It is easy to observe that } \mathcal{N} \subseteq \mathcal{L}. \text{ It follows from [Hen20] Lemma 2.23(b)] that } \mathcal{N}_L(T) \cap \mathcal{L} = \mathcal{N}_L(T). \text{ Hence, } \mathcal{K} = \mathcal{K}^+ \cap \mathcal{N}_L(T) \subseteq \mathcal{N}_L(T). \text{ As } \Pi \text{ is the restriction of } \Pi^+, \text{ we have } \Pi(\mathcal{N}, \mathcal{K}) \subseteq \Pi^+(\mathcal{N}^+, \mathcal{K}^+) \cap \mathcal{L}. \text{ Let now } f \in \Pi^+(\mathcal{N}^+, \mathcal{K}^+) \cap \mathcal{L}. \text{ By Lemma 2.1, there is } n \in \mathcal{N}^+ \text{ and } k \in \mathcal{K}^+ \text{ such that } f = \Pi^+(n, k) \text{ and } S_f = S_{(n, k)}. \text{ As } f \in \mathcal{L}, \text{ we have } S_f \subseteq \Delta \text{ by definition of the restriction. Hence, by (2.1), } (n, k) \in \mathbf{D} \text{ and in particular } n, k \in \mathcal{L}. \text{ Hence, } n \in \mathcal{N}, k \in \mathcal{K} \text{ and } f = \Pi(n, k) \in \Pi(\mathcal{N}, \mathcal{K}). \tag{2.3} \]
3. Some results on linking localities and regular localities

3.1. Regular localities. The definition of regular localities and the most important results surrounding the concept are originally due to Chermak [Che17], but we refer to the treatment of the subject in [Hen21], as there appear to be some gaps in Chermak’s proofs.

As remarked before, if \( \mathcal{F} \) is a saturated fusion system, then by [Hen19] Theorem A] there exists a linking locality \((\mathcal{L}^s, \mathcal{F}^s, S)\) over \( \mathcal{F} \) whose object set is the set \( \mathcal{F}^s \) of subcentric subgroups. Building on earlier work of Chermak [Che17], we introduced in [Hen21] Definition 9.17 a certain partial normal subgroup \( \mathcal{F}^s(\mathcal{L}) \) of \( \mathcal{L} \), for every linking locality \((\mathcal{L}, \Delta, S)\) over \( \mathcal{F} \). Then the set \( \delta(\mathcal{F}) \) is defined as

\[
\delta(\mathcal{F}) := \{ P \leq S : P \cap \mathcal{F}^s(\mathcal{L}^s) \in \mathcal{F}^s \}.
\]

It is shown in [Hen21] Lemma 10.2 that the set \( \delta(\mathcal{F}) \) depends only on \( \mathcal{F} \) and not on the choice of \( \mathcal{L}^s \). Indeed, for every linking locality \((\mathcal{L}, \Delta, S)\), we have

\[
\delta(\mathcal{F}) := \{ P \leq S : P \cap \mathcal{F}^s(\mathcal{L}) \in \mathcal{F}^s \}.
\] (3.1)

A linking locality \((\mathcal{L}, \Delta, S)\) is called a regular locality, if \( \Delta = \delta(\mathcal{F}) \). For every saturated fusion system \( \mathcal{F} \), there exists a regular locality over \( \mathcal{F} \) (cf. [Hen21] Lemma 10.4).

The next theorem states one of the most important properties of regular localities.

**Theorem 3.1** ([Che17] Corollary 7.9, [Hen21] Corollary 10.19]). Let \((\mathcal{L}, \Delta, S)\) be a regular locality and \( \mathcal{H} \trianglelefteq \mathcal{L} \). Then \( \mathcal{F}_{\mathcal{S}\cap\mathcal{H}}(\mathcal{H}) \) is saturated and \( (\mathcal{H}, \delta(\mathcal{F}_{\mathcal{S}\cap\mathcal{H}}(\mathcal{H})), S \cap \mathcal{H}) \) is a regular locality.

Let \((\mathcal{L}, \Delta, S)\) be a regular locality. The theorem above leads to a natural definition of components of \( \mathcal{L} \) (cf. [Hen21] Definition 7.9, Definition 11.1]). If \( \mathcal{K}_1, \ldots, \mathcal{K}_r \) are components of \( \mathcal{L} \), then the product \( \prod_{i=1}^r \mathcal{K}_i \) does not depend on the order of the factors and is a partial normal subgroup of \( \mathcal{F}^s(\mathcal{L}) \) (cf. [Hen21] Proposition 11.7]). The product of all components of \( \mathcal{L} \) is denoted by \( E(\mathcal{L}) \) and turns out to be a partial normal subgroup of \( \mathcal{L} \) (cf. [Hen21] Lemma 11.13]). We have moreover \( \mathcal{F}^s(\mathcal{L}) = E(\mathcal{L})O_p(\mathcal{L}) \). Note that Theorem 3.1 makes it possible to form \( E(\mathcal{H}) \) for every partial subnormal subgroup \( \mathcal{H} \) of \( \mathcal{L} \).

3.2. Normalizers of strongly closed subgroups. The following lemma will be used frequently.

**Lemma 3.2** ([Hen21] Lemma 3.35(b)]). If \( T \) is a strongly closed subgroup of \( \mathcal{F} \), then \( N_{\mathcal{L}}(T), \mathcal{D}, S \) is a linking locality over \( N_{\mathcal{F}}(T) \).

The next goal is to show in Lemma 3.4 below that \( N_{\mathcal{L}}(T) \) can even be given the structure of a regular locality, provided \((\mathcal{L}, \mathcal{D}, S)\) is regular and there exists a normal subsystem of \( \mathcal{F}_S(\mathcal{L}) \) over \( T \). We first summarize some background that is needed in the proof.

Suppose \( S_1, \ldots, S_k \) are subgroups of a \( p \)-group \( S \) with \( [S_i, S_j] = 1 \) for \( i \neq j \). If, for all \( i = 1, 2, \ldots, k \), \( \mathcal{F}_i \) is a fusion system over \( S_i \) with \( S_i \cap \prod_{j \neq i} S_j \leq Z(\mathcal{F}_i) \), then we introduced in [CH21] Section 2.5 a fusion system \( \mathcal{F}_1 \ast \mathcal{F}_2 \ast \cdots \ast \mathcal{F}_k \) over \( S_1S_2 \cdots S_k \), which can be regarded as a central product of \( \mathcal{F}_1, \ldots, \mathcal{F}_k \). The notion is a priori slightly different from Aschbacher’s definition of a central product [Asc11] p.14], but of course closely related (cf. [CH21] Lemma 2.19(a),(b)]). Whenever we write \( \mathcal{F}_1 \ast \mathcal{F}_2 \ast \cdots \ast \mathcal{F}_k \) below, then we mean implicitly that this is well-defined, i.e. \( [S_i, S_j] = 1 \) for all \( i \neq j \) and \( S_i \cap \prod_{j \neq i} S_j \leq Z(\mathcal{F}_i) \) for all \( i = 1, \ldots, k \). We will use the following lemma.

**Lemma 3.3** ([Hen21] Lemma 2.14(g)]). Let \( S \) be a \( p \)-group, and let \( \mathcal{F}_i \) be a fusion system over \( S_i \leq S \) for \( i = 1, 2 \). Suppose \( [S_1, S_2] = 1 \) and \( S_1 \cap S_2 \leq Z(\mathcal{F}_i) \cap Z(\mathcal{F}_j) \). If \( P_i \leq S_i \) for \( i = 1, 2 \), then \( P_1P_2 \in (\mathcal{F}_1 \ast \mathcal{F}_2)^s \) if and only if \( P_i \in \mathcal{F}_i^s \) for each \( i = 1, 2 \).

If \( \mathcal{F} \) is a fusion system over \( S \), \( P \leq S \) and \( \mathcal{E} \) is a subsystem of \( \mathcal{F} \) over \( T \leq S \), then we will write \( P \cap \mathcal{E} \) for \( P \cap T \). In particular, \( S \cap \mathcal{E} = T \).
If \((\mathcal{L}, \Delta, S)\) is a linking locality over a saturated fusion system \(\mathcal{F}\), then it is stated in [CH21, Theorem E(d)] that \(F^*(\mathcal{L})\) corresponds to \(F^*(\mathcal{F})\) under the map \(\Psi_{\mathcal{L}}\) introduced in the introduction. As shown in [CH21, Lemma 7.21], this implies that
\[
\delta(\mathcal{F}) = \{ P \leq S : P \cap F^*(\mathcal{F}) \in F^*(\mathcal{F}) \} = \{ P \leq S : P \cap F^*(\mathcal{F}) \in F^*(\mathcal{F})^s \}. \tag{3.2}
\]

Lemma 3.4. Let \(\mathcal{F}\) be a saturated fusion system over \(S\), and let \(\mathcal{E}\) be a normal subsystem over \(T\). Then the following hold:
\[\begin{align*}
(a) & \quad \delta(N_{\mathcal{F}}(T)) = \{ Q \leq S : QO_{p}(N_{\mathcal{F}}(T)) \in \delta(\mathcal{F}) \}.
(b) & \quad If (\mathcal{L}, \Delta, S) is a regular locality over \(\mathcal{F}\), then \(N_{\mathcal{L}}(T), \delta(N_{\mathcal{F}}(T)), S\) is a regular locality over \(N_{\mathcal{F}}(T)\).
\end{align*}\]

Proof. (a) By [CH21, Lemma 7.13(c), Corollary 7.18], \(F^*(\mathcal{F}) = E(\mathcal{E}) \ast E(C_{\mathcal{F}}(\mathcal{E})) \ast F_{O_{p}(\mathcal{F})}(O_{p}(\mathcal{F}))\) and \(E(C_{\mathcal{F}}(\mathcal{E})) = E(N_{\mathcal{F}}(T))\). Hence, using [CH21, Lemma 2.16(c)], we can conclude that
\[
F^*(\mathcal{F}) = E(N_{\mathcal{F}}(T)) \ast (E(\mathcal{E}) \ast F_{O_{p}(\mathcal{F})}(O_{p}(\mathcal{F}))). \tag{3.3}
\]

By [CH21, Theorem 7.10(e)], we have also
\[
F^*(N_{\mathcal{F}}(T)) = E(N_{\mathcal{F}}(T)) \ast F_{O_{p}(N_{\mathcal{F}}(T))}(O_{p}(N_{\mathcal{F}}(T))). \tag{3.4}
\]

Fix now \(P \leq S\) with \(O_{p}(N_{\mathcal{F}}(T)) \leq P\). Then \(P \cap F^*(N_{\mathcal{F}}(T)) = (P \cap E(N_{\mathcal{F}}(T)))O_{p}(N_{\mathcal{F}}(T))\) by (3.4). As \((S \cap E(\mathcal{E}))O_{p}(\mathcal{F}) \leq TO_{p}(\mathcal{F}) \leq O_{p}(N_{\mathcal{F}}(T)) \leq P\), it follows moreover from (3.3) that
\[
P \cap F^*(\mathcal{F}) = (P \cap E(N_{\mathcal{F}}(T)))(S \cap E(\mathcal{E}))O_{p}(\mathcal{F}).
\]
Hence, we have the following equivalences:
\[\begin{align*}
P \in \delta(\mathcal{F}) \iff & \quad P \cap F^*(\mathcal{F}) \in F^*(\mathcal{F})^s \quad \text{(by (3.2))} \\
\iff & \quad (P \cap E(N_{\mathcal{F}}(T)))(S \cap E(\mathcal{E}))O_{p}(\mathcal{F}) \in F^*(\mathcal{F})^s \\
\iff & \quad P \cap E(N_{\mathcal{F}}(T)) \in E(N_{\mathcal{F}}(T))^s \quad \text{(by (3.3) and Lemma 3.3)} \\
\iff & \quad (P \cap E(N_{\mathcal{F}}(T)))O_{p}(N_{\mathcal{F}}(T)) \in F^*(N_{\mathcal{F}}(T))^s \quad \text{(by (3.4) and Lemma 3.3)} \\
\iff & \quad P \cap F^*(N_{\mathcal{F}}(T)) \in F^*(N_{\mathcal{F}}(T))^s \\
\iff & \quad P \in \delta(N_{\mathcal{F}}(T)) \quad \text{(by (3.2)).}
\end{align*}\]

It follows from [Hen21, Lemma 10.6] that \(\delta(N_{\mathcal{F}}(T)) = \{ Q \leq S : QO_{p}(N_{\mathcal{F}}(T)) \in \delta(N_{\mathcal{F}}(T)) \} \). So (a) follows since the above equivalences hold for every subgroup \(P \leq S\) with \(O_{p}(N_{\mathcal{F}}(T)) \leq P\).

(b) Let now \((\mathcal{L}, \Delta, S)\) be a regular locality over \(\mathcal{F}\). By Lemma 3.2, \((N_{\mathcal{L}}(T), \Delta, S)\) is a linking locality over \(N_{\mathcal{F}}(T)\). In particular, by [Hen19, Proposition 5], \(N_{N_{\mathcal{L}}(T)}(O_{p}(N_{\mathcal{F}}(T))) = N_{\mathcal{L}}(T)\). As \(\delta(\mathcal{F}) = \Delta\), part (a) gives \(\delta(N_{\mathcal{F}}(T)) = \{ P \leq S : PO_{p}(N_{\mathcal{F}}(T)) \in \Delta \} \). Now (b) follows from [CH21, Lemma 3.28].

4. THE PROOF OF THEOREM 2

The following lemma is needed in the proof of Theorem 2.

Lemma 4.1. Let \((\mathcal{L}, \Delta, S)\) be a regular locality, \(N \subseteq \mathcal{L}, T := N \cap S\) and \(T^* = F^*(\mathcal{L}) \cap S\). Then \(N_{\mathcal{N}}(T) \subseteq N_{\mathcal{L}}(T^*)\).

Proof. By [Hen21, Lemma 11.9], \(F^*(\mathcal{L}) = E(\mathcal{L})O_{p}(\mathcal{L})\) and \(T^* = (E(\mathcal{L}) \cap S)O_{p}(\mathcal{L})\). In [Hen21, Notation 5.12], a partial normal subgroup \(N^*\) of \(\mathcal{L}\) is defined, and it is shown in [Hen21, Lemma 11.16] that \(E(\mathcal{L}) = E(\mathcal{N})E(\mathcal{N}^*)\). Moreover, it follows from [Hen21, Lemma 11.13] that \(E(\mathcal{N})\) and \(E(\mathcal{N}^*)\) are normal in \(\mathcal{L}\). Hence, by [Hen13, Theorem 1], \(E(\mathcal{L}) \cap S = (E(\mathcal{N}) \cap S)(E(\mathcal{N}^*) \cap S)\) and so
\[
T^* = (E(\mathcal{N}) \cap S)(E(\mathcal{N}^*) \cap S)O_{p}(\mathcal{L}).
\]

As \(E(\mathcal{N}) \subseteq \mathcal{L}\) and \(E(\mathcal{N}) \cap S = E(\mathcal{N}) \cap T\), we have moreover
\(N_{\mathcal{N}}(T) \subseteq N_{\mathcal{L}}(T) \subseteq N_{\mathcal{L}}(E(\mathcal{N}) \cap S)\).
It is shown in [Hen21, Theorem 10.16(e)] that $N^\perp = C_L(N)$. Thus, [Hen21, Lemma 3.5] yields $N \subseteq C_L(N^\perp)$. In particular, 

$$N_N(T) \subseteq N \subseteq C_L(E(N^\perp) \cap S).$$

It is moreover shown in [Hen21, Lemma 3.13] that $\mathcal{L} = N_L(O_p(\mathcal{L}))$. This implies the assertion. □

Let now $(\mathcal{L}, \Delta, S)$ be a regular locality and set $T^* := S \cap F^*(\mathcal{L})$. As $\Delta = \delta(\mathcal{F})$ and $F^*$ is overgroup-closed in $S$ by [Hen19, Proposition 3.3], it follows from [3.1] that, for every subgroup $P$ of $S$, we have $P \in \Delta$ if and only if $P \cap T^* = P \cap F^*(\mathcal{L}) \in \Delta$. In particular, it follows from [2.1] that for every $w \in W(\mathcal{L})$ the following equivalence holds:

$$S_w \cap T^* \in \Delta \text{ if and only if } w \in D.$$

Proof of Theorem 3.3 Suppose the assertion is false, and let $(\mathcal{L}, \Delta, S, N, K)$ be a counterexample with $|\mathcal{L}| - |K|$ minimal. Set $\mathcal{F} := F_S(\mathcal{L})$ and $T := S \cap N$ so that $K \leq N_L(T)$. Write $\Pi: D \to \mathcal{L}$ for the partial product on $\mathcal{L}$. Fix a subnormal series

$$K = K_0 \leq K_1 \leq \cdots \leq K_m := N_L(T)$$

of minimal length $m$ and set $\tilde{K} := K_{m-1}$.

By follows from [Che21a, Lemma 3.2] that $NK = KN$. Hence, to obtain a contradiction, it is sufficient to prove that $NK$ is subnormal in $\mathcal{L}$ and $S \cap (NK) = T(S \cap K)$.

Since $\tilde{K} \leq N_L(T)$, it follows from Theorem 3.1 that $N\tilde{K} \leq \mathcal{L}$. As $(\mathcal{L}, \Delta, S, N, K)$ is a counterexample, it follows in particular that $K \neq \tilde{K}$ and thus $m \geq 2$. Moreover, by Theorem 3.1 $(N\tilde{K}, \Gamma, S \cap (N\tilde{K}))$ forms itself a regular locality for some appropriate set $\Gamma$ of subgroups of $S \cap (N\tilde{K})$.

Notice that $N \leq N\tilde{K}$. As $N_N(T) = N\tilde{K} \cap N_L(T)$ is a partial subgroup of $N_L(T)$ containing $K$, it follows moreover from [Hen21, Lemma 3.7(a)] that $K \unlhd N_N(T)$. Hence, if $N\tilde{K} \neq L$, then the minimality of $|\mathcal{L}| - |K|$ yields that $(N\tilde{K}, \Gamma, S \cap (N\tilde{K}), N, K)$ is not a counterexample. This means that $S \cap (N\tilde{K}) \cap (N\tilde{K}) = T(S \cap K)$ and $NK \leq N\tilde{K} \leq \mathcal{L}$. In particular, $N\tilde{K}$ is then subnormal in $\mathcal{L}$, so we obtain a contradiction to our assumption. Therefore we have shown that

$$\mathcal{L} = N\tilde{K}.$$  

(4.2)

By [Che21a, Theorem A], $F_T(N) \subseteq \mathcal{F}$. Thus, Lemma 3.4(b) gives that $(N_L(T), \delta(N_T(T)), S)$ is a regular locality. As $\tilde{K} \leq N_L(T)$, it follows thus from Theorem 3.1 that $(\tilde{K}, \tilde{\Gamma}, S \cap \tilde{K})$ forms a regular locality for some set $\tilde{\Gamma}$ of subgroups of $S \cap \tilde{K}$. Note that $N \cap K \leq \tilde{K}$. Moreover, $T_0 := S \cap N \cap \tilde{K} = T \cap K$. As $K \leq N_L(T)$, it follows $\tilde{K} = N_L(T_0)$. In particular, $K \unlhd N_L(T_0)$. The minimality of $m$ yields $\tilde{K} \neq N_L(T)$ and thus $|\tilde{K}| < |\mathcal{L}|$. Hence, it follows from the minimality of $|\mathcal{L}| - |K|$ that $(\tilde{K}, \tilde{\Gamma}, S \cap \tilde{K}, N \cap \tilde{K}, K)$ is not a counterexample. This means that

$$K' := (N \cap \tilde{K})K \unlhd \tilde{K} \text{ and } S \cap K' = (S \cap \tilde{K})K' = T_0(S \cap K).$$

As $\tilde{K} \leq N_L(T)$, we obtain therefore

$$K' \leq N_L(T) \text{ and } S \cap K' = T_0(S \cap K).$$  

(4.3)

We show next that

$$NK = NK'.$$  

(4.4)

Clearly, we have $K \subseteq K'$ and thus $NK \subseteq NK'$. Let now $n \in N$ and $f \in K'$ with $(n, f) \in D$. We need to show that $nf \in NK$. As $f \in K' = (N \cap \tilde{K})K$, there exist $\tilde{n} \in N \cap \tilde{K}$ and $k \in K$ with $(\tilde{n}, k) \in D$ and $f = \tilde{n}k$. It is indeed enough to show that $(n, \tilde{n}, k) \in D$, since then the “associativity axiom” of partial groups yields $nf = n(\tilde{n}k) = \Pi(n, \tilde{n}, k) = (n\tilde{n})k \in NK$ as required.

To prove $(n, \tilde{n}, k) \in D$, we note first that, by [Che21a, Lemma 1.4(d)] and the axioms of a partial group, $(\tilde{n}^{-1}, \tilde{n}, k) \in D$ and $k = \Pi(\tilde{n}^{-1}, \tilde{n}, k) = \tilde{n}^{-1}(\tilde{n}k) = \tilde{n}^{-1}f$. Moreover, by Lemma 4.1
we have $\tilde{n} \in \mathcal{N} \cap \tilde{K} \subseteq N_{\mathcal{N}}(T) \subseteq N_{\mathcal{L}}(T^*)$ and thus $S_f \cap T^* \subseteq S(\tilde{n}, \tilde{n}^{-1}, f)$. Using these properties along with (2.2), we see that

$$S(\tilde{n}, k) \cap T^* \leq S_f \cap T^* \leq S(\tilde{n}, \tilde{n}^{-1}, f) \cap T^* \leq S(\tilde{n}, \tilde{n}^{-1}, f) \cap T^* = S(\tilde{n}, k) \cap T^*$$

and thus $S(\tilde{n}, k) \cap T^* = S_f \cap T^*$. In particular, $S(n, f) \cap T^* = S(n, \tilde{n}, k) \cap T^*$. As $(n, f) \in D$, it follows from (4.1) first that $S(n, \tilde{n}, k) \cap T^* = S(n, f) \cap T^* \in \Delta$ and then $(n, \tilde{n}, k) \in D$. As argued before this shows (4.3).

Assume now that $\mathcal{K}$ is properly contained in $\mathcal{K}'$. The minimality of $|\mathcal{L}| - |\mathcal{K}|$ yields then that $(\mathcal{L}, \Delta, S, \mathcal{N}, \mathcal{K}')$ is not a counterexample. As $\mathcal{K}' \leq N_{\mathcal{L}}(T)$ and $S \cap \mathcal{K}' = T_0(S \cap \mathcal{K})$ by (4.3), it follows thus from (4.4) that $N \mathcal{K} = N \mathcal{K'} \leq \mathcal{L}$ and $S \cap (N \mathcal{K}) = S \cap (N \mathcal{K'}) = T(S \cap \mathcal{K'}) = T(T_0(S \cap \mathcal{K}) = T(S \cap \mathcal{K})$. This contradicts the assumption that $(\mathcal{L}, \Delta, S, \mathcal{N}, \mathcal{K})$ is a counterexample. Hence, we have shown that $\mathcal{K} = \mathcal{K}'$ and thus

$$\mathcal{N} \cap \tilde{K} \subseteq \mathcal{K}. \quad (4.5)$$

Recall that $m \geq 2$ and thus it makes sense to consider $\mathcal{K}_{m-2}$. We show next that

$$x^n \in \mathcal{K}_{m-2} \text{ for all } x \in \mathcal{K}_{m-2} \text{ and } n \in N_{\mathcal{N}}(T) \text{ with } (n^{-1}, x, n) \in D. \quad (4.6)$$

For the proof fix $x \in \mathcal{K}_{m-2}$ and $n \in N_{\mathcal{N}}(T)$ with $(n^{-1}, x, n) \in D$. Using [Che21a, Lemma 3.2(a)] (with $(n^{-1}, x)$ in place of $(x, f)$), we observe that $P := S(n^{-1}, x, n) \leq S_{x^{-1}} \leq S_{x}$ and thus $v := (x^{-1}, n^{-1}, x, n) \in D$ via $P^x$. As $\tilde{K} \leq N_{\mathcal{L}}(T)$ and $\mathcal{N} \leq \mathcal{L}$, we see then using (4.5) that

$$x^{-1}x^n = \Pi(v) = (n^{-1})^x n \in \tilde{K} \cap \mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{K}_{m-2}.$$

As $(x^{-1}, x^n) \in D$, [Che21a, Lemma 1.4(d)] allows us to conclude that $(x, x^{-1}, x^n) \in D$ and $x^n = x(x^{-1}, x^n) \in \mathcal{K}_{m-2}$. This proves (4.6).

We will now be able to obtain a contradiction to the minimality of $m$ by showing

$$\mathcal{K}_{m-2} \leq N_{\mathcal{L}}(T). \quad (4.7)$$

For the proof fix $x \in \mathcal{K}_{m-2}$ and $g \in N_{\mathcal{L}}(T)$ with $(g^{-1}, x, g) \in D$. By (4.2) and Theorem [1] there exist $n \in \mathcal{N}$ and $k \in \tilde{K}$ with $(n, k) \in D$, $g = nk$ and $S_g = S(n, k)$. It follows then from [Che21a, Lemma 1.4(f)] that $g^{-1} = k^{-1}n^{-1}$ and from [Che21a, Lemma 2.3(c), Proposition 2.5(b)] that $S_{g^{-1}} = S(n^{-1}, k^{-1})$. Hence, $w := (k^{-1}, n^{-1}, x, n, g) \in D$ via $S(g^{-1}, x, g)$ and so

$$x^g = \Pi(g^{-1}, x, g) = \Pi(w) = (x^n)^k.$$

As $T \leq S_g = S(n, k) \leq S_n$ and $T$ is strongly closed, we have $n \in N_{\mathcal{N}}(T)$. Hence, it follows from (4.6) that $x^n \in \mathcal{K}_{m-2}$. As $\mathcal{K}_{m-2} \leq \mathcal{K}_{m-1} = \tilde{K}$ and $k \in \tilde{K}$, we can conclude that $x^g = (x^n)^k \in \mathcal{K}_{m-2}$. This shows (4.7). As noticed before, this contradicts the minimality of $m$. \hfill $\Box$

5. Products in fusion systems

5.1. Reminder of some background. If $\mathcal{F}$ is a saturated fusion system, then as before, we write $\frak{N}(\mathcal{F})$ for the set of normal subsystems of $\mathcal{F}$, and $\frak{S}(\mathcal{F})$ for the set of subnormal subsystems of $\mathcal{F}$. Similarly, if $\mathcal{L}$ is a partial group, then $\frak{N}(\mathcal{L})$ denotes the set of partial normal subgroups of $\mathcal{L}$, and $\frak{S}(\mathcal{L})$ denotes the set of partial subnormal subgroups of $\mathcal{L}$.

Let now $(\mathcal{L}, \Delta, S)$ be a linking locality over $\mathcal{F}$. Then by [CH21, Theorem A], there is a bijection

$$\Psi_{\mathcal{L}}: \frak{N}(\mathcal{L}) \rightarrow \frak{N}(\mathcal{F})$$

which sends a partial normal subgroup $\mathcal{N}$ to the smallest normal subsystem of $\mathcal{F}$ over $S \cap \mathcal{N}$ containing $\mathcal{F}_{S \cap \mathcal{N}}(\mathcal{N})$. If $T \leq S$ is strongly $\mathcal{F}$-closed, then $(N_{\mathcal{L}}(T), \Delta, S)$ is a linking locality over $N_{\mathcal{F}}(T)$ by Lemma [3.2]. Thus, there exists a bijection

$$\Psi_{N_{\mathcal{L}}(T)}: N_{\mathcal{L}}(T) \rightarrow N_{\mathcal{F}}(T)$$
which sends a partial normal subgroup $K$ of $N_C(T)$ to the smallest normal subsystem over $S \cap K$
containing $F_{S\cap K}(K)$.

Suppose now that $(\mathcal{L}, \Delta, S)$ is a regular locality. Then the map $\Psi_{\mathcal{L}}$ from above is given by $\Psi_{\mathcal{L}}(N) = F_{S\cap N}(N)$. Moreover, by [CH21 Theorem F], the bijection $\Psi_{\mathcal{L}}$ extends to a bijection

$$\Psi_{\mathcal{L}}: \mathcal{S}(\mathcal{L}) \to \mathcal{S}(\mathcal{F}), H \mapsto F_{S\cap H}(H).$$

If $\mathcal{E}$ is a normal subsystem of $\mathcal{F}$ over $T$, we saw furthermore in Lemma 3.4 that $(N_C(T), \delta(N_F(T)), S)$ is a regular locality over $N_F(T)$. Hence, if a normal subsystem of $\mathcal{F}$ over $T$ exists, then the map $\Psi_{N_C(T)}$ from above extends to the bijection

$$\Psi_{N_C(T)}: \mathcal{S}(N_C(T)) \to \mathcal{S}(N_F(T)), K \mapsto F_{S\cap K}(K).$$

5.2. The proof of Theorem 3 and some related properties. To prove Theorem 3 we will use the existence of a regular locality over $\mathcal{F}$ and then apply Theorems 1 and 2. We need the following elementary lemma.

**Lemma 5.1.** Let $(\mathcal{L}, \Delta, S)$ be a locality, let $\mathcal{H}$ be a partial subgroup of $\mathcal{L}$, and set $\mathcal{E}_\mathcal{H} := F_{S\cap \mathcal{H}}(\mathcal{H})$.

Let $T \leq S \cap \mathcal{H}$ be such that $T$ is strongly $\mathcal{E}_\mathcal{H}$-closed. Then $S \cap \mathcal{H} = S \cap N_\mathcal{H}(T)$ and

$$N_\mathcal{H}(T) = F_{S\cap \mathcal{H}}(N_\mathcal{H}(T)).$$

**Proof.** Observe first that $T \leq S \cap \mathcal{H}$ as $T$ is strongly $\mathcal{E}_\mathcal{H}$-closed. Thus, $N_\mathcal{H}(T)$ is a fusion system over $S \cap \mathcal{H} = S \cap N_\mathcal{H}(T)$. Clearly $F_{S\cap \mathcal{H}}(N_\mathcal{H}(T)) \subseteq N_\mathcal{H}(T)$. On the other hand, if we consider a morphism $\varphi$ in $N_\mathcal{H}(T)$, we can assume that $\varphi$ is defined on $T$. By definition of $\mathcal{E}_\mathcal{H}$, $\varphi = (c_{f_1}[P_1] \circ (c_{f_2}[P_2] \circ \cdots \circ (c_{f_k}[P_k])$ for some $f_1, f_2, \ldots, f_k \in \mathcal{H}$ and $P_i \leq S_{f_i} \cap \mathcal{H}$ for $i = 1, \ldots, k$. Then $T \leq P_1$ and, as $T$ is strongly $\mathcal{E}_\mathcal{H}$-closed, it follows inductively that $T \leq P_i$ and $f_i \in N_\mathcal{H}(T)$ for $i = 1, \ldots, k$. So $\varphi$ is a morphism in $F_{S\cap \mathcal{H}}(N_\mathcal{H}(T))$ and the assertion holds.

We restate and prove Theorem 3 in Theorem 5.3 below. Apart from that, in Theorem 5.3(e), we will give a concrete description of $\mathcal{E}D$ in the case that $\mathcal{D}$ is normal in $N_F(T)$. We use that, for a normal subsystem $\mathcal{E}$ of $\mathcal{F}$ and a subgroup $R \leq S$, a product subsystem $\mathcal{E}R$ is defined as follows.

**Definition 5.2.** Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ over $T \leq S$. For every $P \leq S$, set

$$A^\varphi_{\mathcal{F}, \mathcal{E}}(P) := \langle \varphi \in \text{Aut}_\mathcal{F}(P) : \varphi \text{ p'}-element, [P, \varphi] \leq P \cap T \text{ and } \varphi|_{P \cap T} \in \text{Aut}_\mathcal{E}(P \cap T) \rangle.$$

For every subgroup $R$ of $S$, define

$$\mathcal{E}R := (\mathcal{E}R)_\mathcal{F} := (A^\varphi_{\mathcal{F}, \mathcal{E}}(P) : P \leq TR \text{ and } P \cap T \in \mathcal{E}^c)_{TR}$$

and call $\mathcal{E}R = (\mathcal{E}R)_\mathcal{F}$ the product of $\mathcal{E}$ with $R$ (formed inside of $\mathcal{F}$).

A product subsystem $\mathcal{E}R$ was first introduced by Aschbacher [Asc11 Chapter 8]. The construction above was given by the author of this paper [Hen13]. As shown in [Hen13 Example 7.4], the product $\mathcal{E}R = (\mathcal{E}R)_\mathcal{F}$ depends actually not only on $\mathcal{E}$, $R$ and $S$, but also on $\mathcal{F}$.

Note that Theorem 3 is implied by the following theorem.

**Theorem 5.3.** Let $\mathcal{F}$ be a saturated fusion system over $S$, let $\mathcal{E}$ be a normal subsystem of $\mathcal{F}$ over $T$, and let $\mathcal{D}$ be a subnormal subsystem of $N_F(T)$ over $R$. Then there exists a subsystem $\mathcal{E}D = (\mathcal{E}D)_\mathcal{F}$ of $\mathcal{F}$ such that the following hold:

(a) $\mathcal{E}D$ is a subnormal subsystem of $\mathcal{F}$ over $TR$ with $\mathcal{E} \trianglelefteq \mathcal{E}D$ and $\mathcal{D} \trianglelefteq N_{\mathcal{E}D}(T)$. If $\mathcal{D} \trianglelefteq N_F(T)$, then $\mathcal{E}D \leq \mathcal{F}$ and $\mathcal{D} \leq N_{\mathcal{E}D}(T)$.

(b) $\mathcal{E}D$ is the smallest subnormal subsystem $\tilde{\mathcal{E}}$ of $\mathcal{F}$ with $\mathcal{E} \trianglelefteq \tilde{\mathcal{E}}$ and $\mathcal{D} \trianglelefteq N_{\tilde{\mathcal{E}}}(T)$. Indeed, if $\tilde{\mathcal{E}} \trianglelefteq \mathcal{F}$ with $\mathcal{E} \trianglelefteq \tilde{\mathcal{E}}$ and $\mathcal{D} \trianglelefteq N_{\tilde{\mathcal{E}}}(T)$, then $\mathcal{E}D$ is a subnormal subsystem of $\tilde{\mathcal{E}}$, $\mathcal{E} \trianglelefteq \tilde{\mathcal{E}}$ and $\mathcal{D} \trianglelefteq N_{\tilde{\mathcal{E}}}(T)$. If $\mathcal{D} \leq N_F(T)$, then $\mathcal{E}D \leq \tilde{\mathcal{E}}$ and $\mathcal{D} \leq N_{\tilde{\mathcal{E}}}(T)$.

(c) $N_{\mathcal{E}D}(T)$ equals the product $(N_{\tilde{\mathcal{E}}}(T)\mathcal{D})_{N_F(T)}$. 

(d) Let \((L, \Delta, S)\) be a regular locality over \(F\), and suppose \(\Psi_L\) and \(\Psi_{N_L(T)}\) are the maps from above (given by Theorem A and Theorem F in [CH21]). Setting \(N := \Psi_L^{-1}(E)\) and \(K := \Psi_{N_L(T)}^{-1}(D)\), we have then \(TR = (NK) \cap S\) and \(\Psi_L(NK) = E\).

(e) If \(D \leq N_F(T)\), then
\[
E\mathcal{D} = \langle (E\mathcal{R})_F, (D\mathcal{T})_{N_F(T)} \rangle
\]
is the subsystem over \(TR\) which is generated by the automorphism groups \(A^0_{F,E}(P)\) where \(P \leq TR\) with \(P \cap T \in E\) and the automorphism groups \(A^0_{N_F(T),D}(Q)\) where \(Q \leq TR\) with \(Q \cap R \in D\).

As mentioned in the introduction, if \(E\) and \(D\) are as in Theorem 5.3, then \(N_F(T) \leq N_F(T)\) and \(D \leq N_F(T) = N_{N_F(T)}(T)\). Thus, by the first part of Theorem 5.3, the product subsystem \((E\mathcal{D})_{N_F(T)}\) is defined. Therefore, the statement in Theorem 5.3(c) makes sense.

**Proof of Theorem 5.3.** Let \((L, \Delta, S)\) be a regular locality over \(S\) (which exists by [Hen21 Lemma 10.4]). Then \((N_L(T), \delta_{N_F(T)}, S)\) is a regular locality over \(N_F(T)\) by Lemma 3.4(b). We will consider the maps \(\Psi_L, \Psi_E, \Psi_{N_L(T)}\) and \(\hat{\Psi}_{N_L(T)}\) given by [CH21 Theorem A, Theorem F] (which we introduced in Subsection 5.1). Set
\[
N := \Psi_L^{-1}(E) \leq L\text{ and } K := \hat{\Psi}_{N_L(T)}^{-1}(D) \leq N_L(T).
\]
Note that \(S \cap K = R\) by definition of \(\hat{\Psi}_{N_L(T)}\). By Theorem 2,
\[
NK = KN \leq L\text{ with } (NK) \cap S = T(S \cap K) = TR.
\]
Hence,
\[
E\mathcal{D} := \hat{\Psi}_{N_L(T)}(NK) = F_{TR}(NK)
\]
is well-defined and a subnormal subsystem over \(TR\).

(a) Note that \(N \leq NK\) as \(N \leq L\) and \(N \subseteq NK\). Hence, it follows from [CH21 Proposition 7.1(c)], that
\[
E = \Psi_L(N) = \Psi_E(N) \leq \hat{\Psi}_{N_L(T)}(NK) = E\mathcal{D}.
\]
Notice \(N_{NK}(T) = NK \cap N_L(T) \leq N_L(T)\) as \(NK \leq L\) and \(N_{NK}(T) \cap S = (NK) \cap S = TR\). Moreover, \(N_{E\mathcal{D}}(T) = F_{TR}(N_{NK}(T)) = \hat{\Psi}_{N_L(T)}^{-1}(N_{NK}(T))\), where the first equality uses Lemma 5.1 with \(NK\) in place of \(H\). Hence, \(K \subseteq N_{NK}(T)\) implies by [CH21 Proposition 7.1(c)] that \(D = \hat{\Psi}_{N_L(T)}(K) \leq \hat{\Psi}_{N_L(T)}^{-1}(N_{NK}(T)) = N_{N_F(T)}(T)\). Hence (a) holds.

(b) Let \(\tilde{E} \leq F\) with \(\tilde{E} \leq \tilde{E}\) and \(D \leq N_F(T)\). Recall that the bijections \(\hat{\Psi}_L\) and \(\hat{\Psi}_{N_L(T)}\) restrict to \(\Psi_L\) and \(\Psi_{N_L(T)}\) respectively. Set \(H := \hat{\Psi}_L^{-1}(\tilde{E})\). Then \(H \leq L\) and \(\tilde{E} = F_{S \cap H}(H)\).

Observe that \(N_H(T) \leq N_L(T)\), since a subnormal series \(H = H_0 \leq H_1 \leq \cdots \leq H_n = L\) of \(H\) in \(L\) leads to a subnormal series
\[
N_H(T) = H_0 \cap N_L(T) \leq H_1 \cap N_L(T) \leq \cdots \leq H_n \cap N_L(T) = N_L(T).
\]
Thus, by Lemma 5.1
\[
N_{\tilde{E}}(T) = F_{S \cap H}(N_H(T)) = \hat{\Psi}_{N_L(T)}(N_H(T)) \leq N_F(T).
\]
As \(\hat{\Psi}_L(N) = \Psi_L(N) = E \leq \tilde{E}\) and \(\hat{\Psi}_{N_L(T)}(K) = D \leq N_{\tilde{E}}(T)\) by assumption, it follow now from [CH21 Proposition 7.1(b)] that \(N \leq H\) and \(K \leq N_{\tilde{E}}(T)\). In particular, \(NK \leq H\). Observe that \(\hat{\Psi}_L(N) \leq H\) by [Hen21 Lemma 3.7(a)]. Hence, it follows now from [CH21 Proposition 7.1(c)] that \(E \leq \tilde{E}\), \(E\mathcal{D} \leq \tilde{E}\) and \(D \leq N_{\tilde{E}}(T)\).

If \(D \leq N_F(T)\), then \(K = \Psi_{N_L(T)}^{-1}(D) \leq N_L(T)\) and thus \(K \leq N_H(T)\). Moreover, \(NK \leq L\) by Theorem 1 and so \(NK \leq H\). Hence, [CH21 Proposition 7.1(c)] yields in this case that \(E\mathcal{D} \leq \tilde{E}\) and \(D \leq N_{\tilde{E}}(T)\).
(d) It follows from part (b) that the subsystem $\mathcal{E}D$ depends only on $\mathcal{F}$, $\mathcal{E}$ and $\mathcal{D}$ and not on the choice of the regular locality $(\mathcal{L}, \Delta, S)$. Hence, (d) follows from the definition of $\mathcal{E}D$ above.

(e) By the Dedekind Lemma for partial groups [Che21a, Lemma 1.10], $N_{\mathcal{H}}(T) = N_{\mathcal{H}}(T)K$. Notice moreover that $N_{\mathcal{H}}(T) \subseteq N_{\mathcal{L}}(T)$ and, by Lemma 5.1, applied with $N$ in place of $\mathcal{H}$, $N_{\mathcal{L}}(T) = F_T(N_{\mathcal{L}}(T)) = \Psi_{N_{\mathcal{L}}(T)}(N_{\mathcal{L}}(T)) \leq N_{\mathcal{F}}(T)$. As seen in the proof of (a), we have $N_{\mathcal{E}D}(T) = \Psi_{N_{\mathcal{E}D}(T)}(N_{\mathcal{E}D}(T))$. Hence, $N_{\mathcal{E}D}(T) = \Psi_{N_{\mathcal{E}D}(T)}(N_{\mathcal{E}D}(T))$. As $K \leq N_{\mathcal{E}D}(T) = N_{\mathcal{E}D}(T)$ and $(N_{\mathcal{L}}(T), \delta(N_{\mathcal{L}}(T)), S)$ is a regular locality over $N_{\mathcal{F}}(T)$, it follows thus from (d) that $N_{\mathcal{E}D}(T) = (N_{\mathcal{L}}(T)D)_{N_{\mathcal{E}D}(T)}$.

(e) By definition of $(\mathcal{E}R)_{\mathcal{F}}$ and $(DT)_{N_{\mathcal{F}}(T)}$ (cf. Definition 5.2), it is for the proof of (e) sufficient to show $\mathcal{E}D = ((\mathcal{E}R)_{\mathcal{F}}, (DT)_{N_{\mathcal{F}}(T)})$. As $(\mathcal{L}, \Delta, S)$ and $(N_{\mathcal{L}}(T), \delta(N_{\mathcal{L}}(T)), S)$ are regular localities over $\mathcal{F}$ and $N_{\mathcal{F}}(T)$ respectively, it follows from [CH21 Proposition 8.2] that $(\mathcal{E}R)_{\mathcal{F}} = F_{TR}(NR)$ and $(DT)_{N_{\mathcal{F}}(T)} = F_{TR}(KT)$. Hence, we only need to show that $\mathcal{E}D = (F_{TR}(NR), F_{TR}(KT))$. Recall that $\mathcal{E}D = F_{TR}(NK)$, $T = S \cap \mathcal{N}$ and $R = S \cap \mathcal{K}$. So it is clear that $(F_{TR}(NR), F_{TR}(KT)) \subseteq \mathcal{E}D$. Moreover, since $\mathcal{E}D$ is generated by morphisms of the form $c_f|_{S_f \cap (NK)}$ with $f \in NK$, we just need to argue that each such morphism can be written as the composite of a morphism in $F_{TR}(NK)$ and a morphism in $F_{TR}(KR)$. By Lemma 2.1, for each $f \in NK$ there are elements $n \in \mathcal{N}$ and $k \in \mathcal{K}$ such that $(n, k) \in \mathcal{D}$, $S_f = S(n, k)$ and $f = nk$, which implies that $c_f|_{S_f \cap (NK)} = (c_n|_{S_f \cap (NK)}) + (c_k|_{S_f \cap (NK)})$ is of the required form.

**Remark 5.1.** In [Hen13 Example 7.4] we constructed two saturated fusion systems $\mathcal{F}$ and $\mathcal{G}$ over the same $p$-group $S$ such that $\mathcal{F} \neq \mathcal{G}$, $E := O^p(\mathcal{F}) = O^p(\mathcal{G})$, and $S$ is normal in $\mathcal{F}$ and $\mathcal{G}$. It is a consequence of [Hen13 Theorem 1] that $(\mathcal{E}S)_{\mathcal{F}} = (O^p(\mathcal{F}))_{\mathcal{F}} = \mathcal{F}$ and similarly $(\mathcal{E}S)_{\mathcal{G}} = \mathcal{G}$.

Fix now $T \leq S$ such that $\mathcal{E}$ is a fusion system over $T$. As $S$ is normal in $\mathcal{F}$ and $\mathcal{G}$, it follows in particular that $D := F_D(S)$ is normal in $N_{\mathcal{F}}(T)$ and $N_{\mathcal{G}}(T)$. Hence, by Theorem 5.3(e), we have $(\mathcal{E}D)_{\mathcal{F}} = (\langle (\mathcal{E}S)_{\mathcal{F}}, F_{S}(S) \rangle) = (\mathcal{E}S)_{\mathcal{F}} = \mathcal{F}$ and similarly $(\mathcal{E}D)_{\mathcal{G}} = \mathcal{G}$. This shows that the product $(\mathcal{E}D)_{\mathcal{F}}$ depends actually not only on $\mathcal{E}$ and $\mathcal{D}$, but also on $\mathcal{F}$.

**5.3. The proof of Corollary 4.**

**Lemma 5.4.** Let $(\mathcal{L}, \Delta, S)$ and $(\mathcal{L}^+, \Delta^+, S)$ be linking localities over $\mathcal{F}$ with $\mathcal{L}^+|_{\Delta} = \mathcal{L}$. Let $\mathcal{E}$ be a normal subsystem of $\mathcal{F} \times T$ over $\mathcal{F}$, and let $D$ be a normal subsystem of $N_{\mathcal{F}}(T)$ over $R$. Set $\mathcal{N} := \Psi_{\mathcal{L}^+}(\mathcal{E})$, $\mathcal{K} := \Psi_{\mathcal{L}^+}(\mathcal{D})$, $\mathcal{N}^+ := \Psi_{\mathcal{L}^+}(\mathcal{E})$ and $\mathcal{K}^+ := \Psi_{\mathcal{L}^+}(\mathcal{D})$. Then

$$\Psi_{\mathcal{L}^+}(\mathcal{N}^+ \mathcal{K}^+) = \Psi_{\mathcal{L}^+}(\mathcal{N} \mathcal{K}).$$

**Proof.** It follows from [Hen20 Lemma 2.23(b)] that $N_{\mathcal{L}^+}(T) \cap \mathcal{L} = N_{\mathcal{L}^+}(T)|_{\Delta} = N_{\mathcal{L}}(T)$. Define

$$\Phi_{\mathcal{L}^+, \mathcal{L}} : \mathcal{M}(\mathcal{L}^+) \to \mathcal{M}(\mathcal{L}), \mathcal{M}^+ \mapsto \mathcal{M}^+ \cap \mathcal{L}$$

and similarly

$$\Phi_{\mathcal{L}^+, \mathcal{L}} : \mathcal{M}(\mathcal{L}^+) \to \mathcal{M}(\mathcal{L}^+), \mathcal{M}^+ \mapsto \mathcal{M}^+ \cap \mathcal{L}.$$

By [CH21 Theorem 5.14(c)], we have $\Psi_{\mathcal{L}^+} = \Psi_{\mathcal{L}^+} \circ \Phi_{\mathcal{L}^+, \mathcal{L}}$ and $\Psi_{\mathcal{L}^+}(T) = \Psi_{\mathcal{L}^+}(T) \circ \Phi_{\mathcal{L}^+, \mathcal{L}}(T)$. Hence, $\mathcal{N}^+ \cap \mathcal{L} = \Phi_{\mathcal{L}^+, \mathcal{L}}(\mathcal{N}^+) = \Psi_{\mathcal{L}^+}(\Psi_{\mathcal{L}^+}(\mathcal{N}^+)) \leq \Psi_{\mathcal{L}^+}(\mathcal{E}) = \mathcal{N}$ and $\mathcal{K}^+ \cap \mathcal{L} = \Phi_{\mathcal{L}^+, \mathcal{L}}(\mathcal{K}^+) = \Psi_{\mathcal{L}^+}(\Psi_{\mathcal{L}^+}(\mathcal{K})) = \Psi_{\mathcal{L}^+}(\mathcal{D}) = \mathcal{K}$. Hence, by Lemma 2.3 we have

$$\Phi_{\mathcal{L}^+, \mathcal{L}}(\mathcal{N}^+ \mathcal{K}^+) = \mathcal{N}^+ \mathcal{K}^+ \cap \mathcal{L} \subseteq \mathcal{N} \mathcal{K}$$

and thus $\Psi_{\mathcal{L}^+}(\mathcal{N} \mathcal{K}) = \Psi_{\mathcal{L}^+}(\mathcal{N}^+ \mathcal{K}^+)$. \hfill $\square$

**Proof of Corollary 4.** Let $(\mathcal{L}, \Delta, S)$ be a linking locality. By [Hen19 Proposition 3.3, Theorem 7.2], there exists a linking locality $(\mathcal{L}^*, \mathcal{F}^*, \mathcal{S})$ over $\mathcal{F}$ with $\mathcal{L}^*|_{\Delta} = \mathcal{L}$. By [Hen21 Lemma 10.4], $\mathcal{F}^* \subseteq \delta(\mathcal{F}) \subseteq \mathcal{F}$ and $\delta(\mathcal{F})$ is closed under $\mathcal{F}$-conjugacy and overgroup-closed in $\mathcal{S}$. Hence, $\mathcal{L}^* := \mathcal{L}^*|_{\delta(\mathcal{F})}$ is well-defined and a locality with object set $\delta(\mathcal{F})$. As $\mathcal{L}^*$ is a linking locality and $N_{\mathcal{L}^*}(P) = N_{\mathcal{L}^*}(P)$ for all $P \in \delta(\mathcal{F})$, it follows that $(\mathcal{L}^*, \delta(\mathcal{F}), \mathcal{S})$ is a linking locality and thus a regular locality.
Let now \( \mathcal{N} := \Psi^{-1}_L(\mathcal{E}) \), \( \mathcal{N}^s := \Psi^{-1}_{L^s}(\mathcal{E}) \), \( \mathcal{K} := \Psi^{-1}_{N_L(T)}(\mathcal{D}) \), \( \mathcal{K}^s := \Psi^{-1}_N(L^s(\mathcal{T})) \), \( \mathcal{K}_\delta := \Psi^{-1}_{N_L(\delta)}(\mathcal{T}) \). Then in particular \( \mathcal{N} \cap \mathcal{S} = \mathcal{T} \) and \( \mathcal{K} \cap \mathcal{S} = \mathcal{R} \). Note that \( \mathcal{N} \mathcal{K} \trianglelefteq \mathcal{L} \), \( (\mathcal{N} \mathcal{K}) \cap \mathcal{S} = \mathcal{T} \mathcal{R} \), \( \mathcal{N}^s \mathcal{K}^s \trianglelefteq \mathcal{L}^s \) and \( \mathcal{N} \mathcal{K}_\delta \trianglelefteq \mathcal{L} \delta \) by Theorem 1. By Theorem 5.3(d), we have \( \Psi_{\mathcal{L}^\delta}(\mathcal{N} \mathcal{K}_\delta) = \mathcal{E} \mathcal{D} \). Hence, applying Lemma 5.4 twice, we obtain \( \Psi_{\mathcal{L}}(\mathcal{N} \mathcal{K}) = \Psi_{\mathcal{L}^s}(\mathcal{N} \mathcal{K}_\delta) = \Psi_{\mathcal{L}^\delta}(\mathcal{N} \mathcal{K}_\delta) = \mathcal{E} \mathcal{D} \). This proves the assertion.

\[ \square \]

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Institut für Algebra, Fakultät Mathematik, Technische Universität Dresden, 01062 Dresden, Germany

Email address: ellen.henke@tu-dresden.de