The formulation of $f(R)$-gravity on singular semi-Riemannian manifolds

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Abstract. In singular semi-Riemannian manifolds, the metric is allowed to be degenerate, i.e. it is not necessarily non-degenerate. The formulation of $f(R)$-gravity on this kind of manifold has been studied. The field equation is obtained via variational principle approach. The quantities which is involved in the formulation is a smooth quantity even when the metric becomes degenerate. These quantities are the densitized energy-momentum tensor, the Ricci curvature tensor, and the arbitrary function of scalar curvature. The resulting filed equation is a more general field equation. It reduces to the standard field equation when the metric is non-degenerate. The extension of the modified Friedmann equations has also been obtained. The generalized mass (energy) density and the generalized pressure density can be constructed based on the definition of the densitized energy-momentum tensor. These quantities remain smooth even when the metric is degenerate. The modified-$f(R)$ Schwarzschild black holes has also been considered. By coordinate transformation, the metric which is semi-regular has been obtained. The consequences of this formulation to the spacetime singularity has also been discussed.

1. Introduction

General Relativity by its cosmological model known as Λ-CDM (Lambda-Cold Dark Matter) still cannot gives an adequate explanation concerning accelerated expansion of the universe. In the framework of Λ-CDM model, this acceleration is caused by an unknown form of energy that exist abundantly in the universe. The term ”dark energy” is assigned to describe this kind of energy. Until now, the origin and the nature of dark energy is still become one of mysteries in our universe.

An alternative way to understand that phenomena is by considering the modification of geometric term of the field equation. It is done by replacing Langrangian density of Einstein-Hilbert action formerly Ricci scalar, $R$ becomes an arbitrary function that depends on $R$. This idea is known as $f(R)$-gravity. Based on this theory, the accelerated expansion of the universe is understood as a nature of spacetime, i.e it is purely geometrical property of spacetime.

It is well known that the mathematical framework of general relativity is semi-Riemannian manifold. That is a differentiable manifold endowed with a symmetric, non-degenerate bilinear form on its tangent bundle. In 1996, Kupeli introduced singular semi-Riemannian manifold as a generalization of semi-Riemannian manifold[1]. That is a differentiable manifold having on its tangent bundle a symmetric bilinear form which is not necessarily non-degenerate, i.e. it is allowed degenerate.
The application of singular semi-Riemannian geometry to general relativity is studied extensively by Stoica [2]. Stoica generalized the work of Kupeli by dropping the assumption of constant signature of the metric and then formulated general relativity on it [3]. Moreover, Stoica considered the spacetime singularity [4],[5]. It is defined as a point in spacetime where the metric is degenerate.

In this paper, we are interested in the formulation of \( f(R) \)-gravity on singular semi-Riemannian manifolds. By extending field equation that includes degenerate point, we can consider spacetime singularity in the theory of \( f(R) \)-gravity.

2. A brief review on \( f(R) \)-gravity

Action for \( f(R) \)-gravity in vacuum is given as follows

\[
S = \int d^4x \sqrt{-g} f(R). \tag{1}
\]

By applying the variational principle, \( \delta S = 0 \), we can finally obtain modified field equations in vacuum,

\[
f'(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} + g_{\mu\nu} \nabla^\alpha \nabla_\alpha f'(R) - \nabla_\mu \nabla_\nu f'(R) = 0. \tag{2}
\]

When we consider the present of matter, the full field equations is written as

\[
f'(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} + g_{\mu\nu} \Box f'(R) - \nabla_\mu \nabla_\nu f'(R) = \kappa T_{\mu\nu}, \tag{3}
\]

where \( \Box := g^{\alpha\beta} \nabla_\beta \nabla_\alpha \) is Laplace-Beltrami operator and \( T_{\mu\nu} \) is energy-momentum tensor.

The first modified Friedmann equation is

\[
3f'(R)H^2 = \frac{f'(R)R - f(R)}{2} - 3H \dot{f}'(R) + \kappa \rho, \tag{4}
\]

and the second modified Friedmann equation is

\[
-2\dot{H} f'(R) = \ddot{f}'(R) - H \dot{f}'(R) + \kappa(\rho + P). \tag{5}
\]

In the framework of standard model of cosmology (Λ-CDM), the accelerated expansion of the universe is caused by an unknown form of energy whose parameter of state is \( \omega < -\frac{1}{3} \). The term "dark energy" is assigned to describe this kind of energy. On the other hand, in \( f(R) \)-gravity, that parameter of state can be obtained naturally from field equations.

\( f(R) \)-gravity theory is not only succeed to give a satisfied explanation on the accelerated expansion of the universe, but also can gives an explanation about the other phenomena that has been predicted by Einstein general relativity. One of them is collapsing star that become black holes. The metric of modified Schwarzschild black holes for the case \( R = R_0 \) is written as follows.

\[
ds^2 = -\left(1 - \frac{2m}{r} - \frac{\lambda}{3} r^2\right) dt^2 + \left(1 - \frac{2m}{r} - \frac{\lambda}{3} r^2\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{6}
\]

where \( m \) is mass of the object, \( \lambda = \frac{f_0}{f'} \) is the ratio between value of \( f(R) \) and its derivative when evaluated at \( R = R_0 \) and \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \), is the metric of the sphere \( S^2 \).
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3. Singular semi-Riemannian manifolds

In this section, we will give a bird’s eye view on singular semi-Riemannian manifold. We shall be closely following Stoica [2]. More detailed and rigorous exposition can be found in [2].

A singular semi-Riemannian manifold \((M, g)\) is a differentiable manifold \(M\) endowed with a symmetric bilinear form \(g\) named metric. Note that the metric is not necessarily non-degenerate. This condition, in fact, bring some obstacles. We cannot define an isomorphism between the tangent space \(T_p M\) and the dual \(T^*_p M\). It is induce the absence of the inverse metric. Thus, the geometric objects which is defined with the help of inverse metric cannot be constructed.

The concept of radical in the vector space can be extended to manifold by considering the tangent space which posses the property of vector space. This concept is needed to construct the covariant contraction which is normally defined with the help of inverse metric.

The radical of \(T_p M\) is denoted by \((T_p M)_0 = (T_p M)^\perp\). It is the set of tangent vectors in \(T_p M\) which its inner product to any tangent vector in \(T_p M\) is zero. An inner product \(g_p\) on \(T_p M\) is non-degenerate if and only if \((T_p M)_0 = \{0\}\). The radical of \(TM\) is defined as \(T_p M = \bigcup_{p \in M} (T_p M)_0\).

The radical-annihilator of \((T_p M)_0\) is then defined as \(T^*_p M = \bigcup_{p \in M} (T_p M)^\perp\), where \((T_p M)^\perp \subseteq T^*_p M\) is the space of covectors at \(p\) which can be expressed as \(\omega_p(X_p) = \langle Y_p, X_p \rangle\) for some \(Y_p \in T_p M\) and any \(X_p \in T_p M\). The sections of \(T^* M\) can be defined, in the general case, by

\[
A^\bullet(M) := \{ \omega \in A^1(M) | \omega_p \in (T_p M)^\perp \ \text{for any} \ p \in M \} \tag{7}
\]

A unique non-degenerate inner product \(g_p\) on \(T^* M\) is defined by \(g_\bullet(\omega, \tau) := \langle X, Y \rangle\) where \(X^\bullet = \omega, Y^\bullet = \tau, X, Y \in \mathcal{X}(M)\).

Let \(T\) be a tensor type \((r, s)\). Tensor \(T\) is called radical in the \(k\)-th contravariant slot if \(T \in T^{k-1}_0 M \otimes_M T^s_0 M \otimes_M T^r_0 M\). It is called radical-annihilator in the \(l\)-th covariant slot if \(T \in T^r_{l-1} M \otimes_M T^* M \otimes_M T^s_{l-1} M\). The covariant contraction between two covariant indices in which a tensor is radical-annihilator can be defined by the inner product \(g_\bullet\). The contraction \(C_{kl}\) of a tensor field \(T\) is defined by \(T(\omega_1, ..., \omega_r, v_1, ..., s_1, ..., s_k)\).

The Koszul form is defined as \(\mathcal{K} : \mathcal{X}(M)^3 \rightarrow C^\infty(M)\),

\[
\mathcal{K}(X, Y, Z) := \frac{1}{2} \left( \langle X, Y \rangle \langle Z, X \rangle + \langle Y, Z \rangle - \langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle \right) \tag{8}
\]

The Koszul formula becomes \(\langle \nabla_X Y, Z \rangle = \mathcal{K}(X, Y, Z)\). For non-degenerate metric, the unique Levi-Civita connection is obtained by raising the 1-form, \(\mathcal{K}(X, Y, \cdot)\). It is \(\nabla_X Y = \mathcal{K}(X, Y, \cdot)\).

In a coordinate basis, the Koszul form reduce to,

\[
\mathcal{K}_{\mu\nu\alpha} = \mathcal{K}(\partial_\mu, \partial_\nu, \partial_\alpha) = \frac{1}{2} (\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) = \Gamma_{\mu\nu\alpha}, \tag{9}
\]

which are Christoffels symbols of the first kind.

The properties of the Koszul form is correspond to the Levi-Civita connection which is only defined when the metric is non-degenerate. Thus, the construction of the Koszul form is then used to define covariant derivative in singular semi-Riemannian manifolds.

The lower covariant derivative of a vector field \(Y\) in the direction of a vector field \(X\) is the differential 1-form \(\nabla_X^p Y \in A^1(M)\) defined as \(\langle \nabla_X^p Y \rangle(Z) := \mathcal{K}(X, Y, Z)\) for any \(Z \in \mathcal{X}(M)\).

The lower covariant derivative operator is the operator \(\nabla^\bullet : \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow A^1(M)\) which associates to each \(X, Y \in \mathcal{X}(M)\) the differential 1-form \(\nabla_X^p Y\).

Singular semi-Riemannian manifold \((M, g)\) is radical-stationary if it satisfies the condition \(\mathcal{K}(X, Y, \cdot) \in A^\bullet(M)\) for any vector fields \(X, Y \in \mathcal{X}(M)\). The covariant derivative of a radical-annihilator 1-form \(\omega \in A^\bullet(M)\) in the direction of a vector field \(X \in \mathcal{X}(M)\) defined as

\[
(\nabla_X \omega)(Y) := X(\omega(Y)) - g_\bullet(\nabla_X^p Y, \omega), \tag{10}
\]
where \( \mathcal{A}^1_\omega(M) \) denotes the set of 1-forms which are smooth on the regions of constant signature.

For a radical-stationary semi-Riemannian manifold \((M, g)\), then can be defined the vector spaces of differential forms having smooth covariant derivatives:

\[
\mathcal{A}^\bullet(M) = \{ \omega \in \mathcal{A}^\bullet(M) | (\forall X \in \mathcal{X}(M)) \nabla_X \omega \in \mathcal{A}^\bullet(M) \}
\]

A semi-regular semi-Riemannian manifold is a singular semi-Riemannian manifold \((M, g)\) which satisfies \(\nabla^\omega_X Y \in \mathcal{A}^\bullet(M)\) for any vector fields \(X, Y \in \mathcal{X}(M)\). A radical-stationary semi-Riemannian manifold \((M, g)\) is semi-regular if and only if for any \(X, Y, Z, T \in \mathcal{X}(M)\) satisfies \(\mathcal{K}(X, Y, \mathcal{Z}, T) \in \mathcal{F}(M)\).

The Riemann curvature tensor of a singular semi-Riemannian manifold \((M, g)\) is defined as \(R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{C}^\infty(M)\),

\[
R(X, Y, Z, T) := \nabla_X \nabla^\omega_Y Z - \nabla_Y \nabla^\omega_X Z - \nabla^\omega_{[X,Y]}Z,
\]

for any \(X, Y, Z, T \in \mathcal{X}(M)\).

The Riemann curvature of a semi-regular semi-Riemannian manifold \((M, g)\) is a smooth tensor field \(R \in T^4_4M\). In a coordinate basis, the components of the Riemann curvature tensor are given by

\[
R^\mu_{\rho\nu\alpha} = \partial_\alpha R^\mu_{\rho\nu} - \partial_\rho R^\mu_{\alpha\nu} + g^\beta_{\nu} (K_{\beta\mu\rho} - K_{\beta\mu\rho}K_{\alpha\lambda} - K_{\beta\mu\rho}K_{\alpha\lambda}).
\]

The Ricci curvature tensor is defined as the covariant contraction of the Riemann curvature tensor, \(\text{Ric}(X, Y) := R(X, Y, \mathcal{Z}, T)\) and the Ricci scalar is defined as the covariant contraction of the Ricci curvature tensor, \(R := \text{Ric}(\mathcal{Z}, \mathcal{Z})\).

4. The formulation and its relation to spacetime singularity

4.1. Extended field equation

Let us reconsider action for \(f(R)\)-gravity in vacuum as follows

\[
S = \int d^4x \sqrt{-g} f(R).
\]

By considering the construction of Ricci scalar on singular semi-Riemannian manifold, the variation of action becomes

\[
\delta S = \int d^4x \sqrt{-g} [f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu}] \delta g^{\mu\nu} + \int d^4x \sqrt{-g} f'(R) g^{\mu\nu} \delta R_{\mu\nu}.
\]

The last term in the equation \([15]\) above must be formed as variation of \(\delta g^{\mu\nu}\). By calculating the expression of \(g^{\mu\nu} \delta R_{\mu\nu}\), the last term of \([15]\) is written as follows

\[
\int d^4x \sqrt{-g} f'(R) g^{\mu\nu} \delta R_{\mu\nu} = \int d^4x \sqrt{-g} g_{\mu\nu} \delta g^{\beta}_{\alpha} \partial_\beta \partial_\alpha f'(R) \delta g^{\mu\nu} - \int d^4x \sqrt{-g} \partial_\mu \partial_\nu f'(R) \delta g^{\mu\nu}.
\]

By joining the above term to the total action and then applying variational principle, \(\delta S = 0\), we arrive to the extended field equation as follows

\[
f'(R) R_{\mu\nu} \sqrt{-g} - \frac{1}{2} f(R) g_{\mu\nu} \sqrt{-g} + g_{\mu\nu} \Box f'(R) \sqrt{-g} - \nabla_{\mu} \nabla_{\nu} f'(R) \sqrt{-g} = 0,
\]

where \(\Box := g^{\beta}_{\alpha} \nabla_\beta \nabla_\alpha\) is Laplace-Beltrami operator in singular semi-Riemannian manifolds.

When the present of matter is considered, the Lagrangean density is also contain the component of volume form, \(\sqrt{-g}\). Therefore, the full field equations is written as

\[
f'(R) R_{\mu\nu} \sqrt{-g} - \frac{1}{2} f(R) g_{\mu\nu} \sqrt{-g} + g_{\mu\nu} \Box f'(R) \sqrt{-g} - \nabla_{\mu} \nabla_{\nu} f'(R) \sqrt{-g} = \kappa T_{\mu\nu} \sqrt{-g}.
\]

In obtaining the field equations above, it is not just refrain to divide by \(\sqrt{-g}\). But, we have to ensure that at every step in the formulation we have only smooth quantities.
4.2. Extension of Modified Friedmann equation
The construction below is obtained by following the method developed by Stoica [5]. We extend the result to the $f(R)$-gravity.

Consider the first modified Friedmann equation which is expressed in term mass (energy) density and pressure density as follows.

$$\rho = \frac{3}{\kappa} \left[ \left( \frac{\dot{a}}{a} \right)^2 f'(R) - \frac{f'(R)R - f(R)}{6} + \frac{\dot{a}}{a} f'(R) \right].$$

and the second modified Friedmann equation

$$\rho + 3P = -\frac{3}{\kappa} \left[ 2 \frac{\ddot{a}}{a} f'(R) + \frac{\dot{a}}{a} f'(R) - \frac{2}{6} \left( f'(R)R - f(R) \right) - \ddot{f}'(R) \right].$$

Two equations above shows that when $a \to 0$, mass (energy) density and pressure density appears to tend to infinity. In other words, a finite amount of matter occupies a very small volume. At that situation, mass (energy) density becomes infinite and then "bang!" spanned spacetime. That event is called by bigbang.

The construction of field equations in the previous section shows that the more fundamental object is in fact, energy-momentum tensor weight $+1$, that is $T_{\mu\nu} \sqrt{-g}$. This implies to introduce the definition of generalized mass (energy) density $\rho \sqrt{-g}$ and generalized pressure density $P \sqrt{-g}$ as a natural quantity in fluid equation of the model of the universe. Both quantities is expressed as $\rho$ and $P$ only in an orthonormal frame, where the determinant of the metric equals $-1$ so that component $\sqrt{-g}$ is negligible. But, when $a = 0$, an orthonormal frame would become singular, because $g = 0$. Therefore, we avoid the usage of $\rho$ and $P$ because they cannot be define there.

This idea is in accordance with the study of differential form. Both quantities $\rho$ and $P$ are scalar fields as seen by an observer moving in an orthonormal frame. The invariant quantities is in fact must involve the volume form which its component in a coordinate system are $\rho \sqrt{-g}$ and $P \sqrt{-g}$.

In a non-singular coordinate system, $g$ has to be variable, as it is in the comoving coordinate system of the flat FLRW model which is expressed by $\sqrt{-g} = a^3$. Based on those idea, we can define densitized energy-momentum tensor as

$$T_{\mu\nu} \sqrt{-g} = (\dot{\rho} + \ddot{P})u_\mu u_\nu + \dot{P}g_{\mu\nu},$$

where $\dot{\rho} = \rho \sqrt{-g} = \rho a^3$ and $\dot{P} = P \sqrt{-g} = Pa^3$. By substituting the expression and to the modified Friedmann equation, we finally obtain the modified Friedmann equation which is smooth even at $a = 0$ as follows

$$\dot{\rho} = \frac{3}{\kappa} \left[ \ddot{a} \alpha f'(R) - \frac{a^3}{6} \left( f'(R)R - f(R) \right) + \dot{a} \dot{f}'(R) \right]$$

and

$$\dot{\rho} + 3\dot{P} = -\frac{3}{\kappa} \left[ 2 \ddot{a} a f'(R) + \dot{a} \dot{f}'(R) - \frac{2a^3}{6} \left( f'(R)R - f(R) \right) - a^3 \ddot{f}'(R) \right].$$

4.3. Semi-regularity of modified Schwarzschild black holes
Following Stoica [6], we develop here the coordinate transformation for modified Schwarzschild black holes. By coordinate change $r = \tau^2$ and $t = \xi \tau^2$, we can obtain a new metric which is written as follows

$$ds^2 = -\frac{12\tau^4}{6m - 3\tau^2 + \lambda \tau^6}d\tau^2 + \left( \frac{6m - 3\tau^2 + \lambda \tau^6}{3} \right) \tau^{2\tau - 4}(T\xi d\tau + \tau d\xi)^2 + \tau^4 d\Omega^2.$$
By finding the value of $T$ which satisfy the semi-regularity condition, we obtain $T = 4$ as a unique solution. Finally, we obtain a metric which is an analytic extension of the modified Schwarzschild metric. That is a metric which is semi-regular and analytic including at the point $r = 0$ by a coordinate transformation $r = \tau^2$ and $t = \xi \tau^4$ as follows

$$ds^2 = -\left(\frac{12\tau^4}{6m - 3\tau^2 + \lambda\tau^6}\right)d\tau^2 + \left(\frac{6m - 3\tau^2 + \lambda\tau^6}{3}\right)\tau^4(4\xi d\tau + \tau d\xi)^2 + \tau^4d\Omega^2. \tag{25}$$

5. Conclusion and discussion

In the framework of the metric, there are two types of singularities[6]:

(i) **malign singularity**: some of the components of the metric are divergent, $g_{\mu\nu} \to 0$

(ii) **benign singularity**: the components of the metric $g_{\mu\nu}$, are smooth and finite, but its determinant, $g \to 0$

An obvious definition of singularity would seem to be that it is a point where the metric is singular i.e fails to be suitably differentiable[8]. Based on this idea, such spacetime singularity refers to **malign singularity**. That is because the usual spacetime is modeled as a differentiable manifold which is equipped by the metric that must be non-degenerate. On the other hand, the other type, **benign singularity** is not used to define spacetime singularity.

In the previous section, we have shown the work of Stoica [2] for the mathematical construction of manifold which is equipped by the metric that can be degenerate. The concept of **benign singularity** is then used to consider spacetime singularity. Spacetime singularity is defined as a point where the metric becomes degenerate. By this point of view, the field equation that has been obtained above is more general because it includes singularities. The extension of the modified Friedmann equation that includes degenerate point can be viewed as the extension to the bigbang singularity. We also obtain a new metric for modified Schwarzschild black holes. It can be viewed as a transformation from malign singularity to benign singularity. In the new metric, the singularity at $r = 0$ is benign. The consideration of singularity problem in $f(R)$-gravity as in [9] is left for future work.

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