Remarks on the Current-Carrying State of Hall Superfluids

FRANK WILCZEK

School of Natural Sciences
Institute for Advanced Study
Olden Lane
Princeton, N.J. 08540

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† Research supported in part by DOE grant DE-FG02-90ER40542.  wilczek@sns.ias.edu
ABSTRACT

I discuss in an elementary and self-contained way the nature of the current-carrying state in Hall superfluids. The explicit connection of fractional charge and rationality of the filling fraction $\nu$ with quantization of angular momentum, and the discussion of the preferred velocity profile for bulk current flows may be new; the latter might provide an interesting experimental handle on a fundamental aspect of quantum superfluidity.
The quantum Hall effect exhibits many analogies to the more venerable superfluidities, epitomized by the \( \lambda \) phase of \(^4\text{He}\) and BCS superconductivity. The most obvious macroscopic commonality is of course the possibility of (approximately) dissipationless flow. At a more microscopic and theoretical level, there are striking analogies between the Laughlin quasiparticles and vortices, and effective Landau-Ginzburg like theories using interesting concepts including fictitious gauge fields and Chern-Simons interactions which conveniently summarize important aspects of the phenomenology [1, 2].

Here I would like to focus on one particular question which lies at the heart of every form of superfluidity, and can be discussed in a self-contained and (I hope) instructive fashion: that is, the nature of the ideal current-carrying state and particularly its realization in a non-simply connected geometry, where fundamental questions of quantization come into play. I have not seen a discussion of quite this type for the Hall superfluid, but that may well just reflect ignorance on my part.

1. Approach Through Wave Functions

A remarkable feature of the theory of the quantum Hall effect is how much insight can be gained by consideration of simple trial wave functions. I will review, partially explain, and exploit this feature in the following.

1.1. Quick March Through Droplets and Annuli

Let me remind you of some basic results about electrons in a strong magnetic field. Here and throughout I will assume that the motion of the electrons is confined to a plane. I will also ignore the spin degree of freedom. The energy levels are highly degenerate Landau levels, with a density of states \((2\pi l_B^2)^{-1}\) per unit area per Landau level, where the magnetic length \(l\) is defined through \(l_B^2 \equiv eB/\hbar c\). The splitting between levels is \(\hbar\) times the cyclotron frequency, \(\text{viz. } \Delta E = \hbar (eB/mc)\). At low temperatures and for densities small compared to \((2\pi l_B^2)^{-1}\) it ought to be a good approximation to restrict attention to states formed from single-particle states.
taken from the lowest Landau level, unless there is some very special energetic advantage to admixing higher levels. Within the lowest Landau level, the single particle wave functions take a particularly attractive form if one employs the so-called symmetric gauge, defined by the vector potentials $A_x = By/2, A_y = -Bx/2$. With this gauge choice, the wave functions in the lowest Landau level take the form

$$\psi = f(z)e^{-\frac{1}{4}|z|^2},$$

where $f(z)$ is an arbitrary holomorphic (i.e., complex analytic) function of $z \equiv x + iy$, subject to reasonable growth conditions insuring that the wave function is normalizable. Distances are measured in units of the magnetic length. A basis of orthogonal vectors in this Hilbert space is provided by the functions $f_l(z) = z^l$. $l$ is the canonical angular momentum around the origin, which here is intrinsically non-negative. For reasonably large $l$, the corresponding wave function is concentrated in a circular ring of radius $\sqrt{2l}$ and width $\sqrt{2\pi}$ centered at the origin. It follows, by comparing the size of the region where the wavefunction is large to the inverse density, or by direct calculation, that the supports of these wave functions are highly overlapping.

Now let us consider an assembly of (non-interacting) electrons. Let us suppose that they are subject to a very small potential which draws them toward the origin, but does not appreciably change the form of the wave functions. Then the ground state will be formed by occupying the states with the smallest values of $l$, consistent with Fermi statistics. It will be the Slater determinant

$$\psi_1 = \det\{z_r^{c-1}\}e^{-\frac{1}{4}\sum|z_k|^2},$$

where the row variable $r$ (a particle identity index), the column variable $c$, and $k$ all run from 1 to $N$, the number of electrons. Given the spatial character of the wave functions as discussed above, one easily sees that $\psi_1$ represents, for large values of $N$, a droplet of uniform density $1/2\pi$ and radius $\sqrt{2N}$, with some fuzziness in an
ring of width unity near the edge. The uniformity of the density is far from obvious upon inspecting (1.2), but follows from the fact that we have occupied all the lowest Landau level states with substantial support in the specified geometric region, since the completely occupied Landau level is certainly uniform. The deep reason why this uniformity is not more obvious is that to maintain any fixed gauge condition, such as the symmetric gauge we have chosen, a translation must be accompanied by a non-trivial gauge transformation. One can also re-write the droplet wave function in a suggestive way using the Vandermonde identity\[
\det\{z_r^{c-1}\} = \prod_{k<l}^N (z_k - z_l).
\]

Part of Laughlin’s inspiration in understanding the fractional quantized Hall effect was to notice that the cube of this wave function has remarkable qualities, that make it a particularly attractive trial wave function for an assembly of interacting electrons. After taking the cube the Gaussian factor is no longer appropriate to the lowest Landau level, but that defect can be compensated by a trivial re-definition of the length scale, which we suppose done. Then clearly one has a wave function again describing a uniform droplet centered at the origin, now with radius $\sqrt{6N}$, density $1/6\pi$ (that is, filling factor $1/3$) and fuzziness in an ring of width $\sqrt{6\pi}$ at the edge. From the alternative Vandermonde representation of the determinantal wave function as a product, we see that there is a strong vanishing as two particles approach one another. Thus we might expect – and one does find, through numerical work – that this is a particularly favorable wave function for short-range repulsive interactions among the electrons.

Now consider, in either the integer or fractional case, the process of adiabatically turning on a localized point source of flux. (Flux comes in tubes in three space dimensions, and therefore as localized points in two space dimensions.) Let us implement the potential as a branch cut. In this situation the equations which forced $f(z)$ to be holomorphic (in the lowest Landau level) still hold in the bulk, but a singularity is allowed along the cut. For adiabatic evolution, since there is a gap, one will not leave the lowest Landau level, and the wave functions will be multiplied by factors $z^\alpha$, with $\alpha$ proportional to the flux. The effect of adiabatically
adding a quantum unit $h/e$ of flux at the origin is to multiply the single particle wave functions by a factor $z$, an operation which maps us between proper lowest Landau level wave functions. From our previous discussion of the one-particle wave functions, we see that the effect of this factor is to push them out from the origin. The effect on the many-particle droplet wave function is to produce a localized hole in the charge distribution. Proceeding similarly and adding many units of flux, we will carve out a large geometrical hole in the charge droplet, producing an annulus of uniform charge density.

Thus far I have been speaking of free electrons. It would impede my narrative flow to give an exposition of the fundamentals of the quantum Hall effect, but I would like to say just enough to make it plausible why the sort of “wave functionology” discussed here has some relevance for describing the real phenomena. The two most profound features of the observed phenomena are the existence of incompressible states of the electron fluid in a strong magnetic field, and the fact that these incompressible states occur when the ratio of the density to the density required to fill the Landau level is a simple rational fraction. For example, the most robust and prominent such states occur at $\nu \equiv \rho2\pi l_B^2 = 1, 1/3$. The key physical point for incompressibility is the absence of low-energy excitations – “sound waves” – characterized by small deviations from the favored density over a large spatial region. The rigidity of the ground-state wave function against density variations arise because correlations fix a preferred length scale. In general this is not a single-particle phenomenon. That is clear for $\nu = 1/3$: in the absence of interactions a one-third filled Landau level would remain hugely degenerate, since one could pick and choose among many states to occupy, and in this case is easy to construct “sound wave” configurations. Exactly at $\nu = 1$ all the available states are filled, and there is indeed a finite energy gap to density fluctuations, since these require promotion of some electron to the second Landau level. However this argument fails at any infinitesimal distance from $\nu = 1$, and does not fully explain the observed phenomenon, which cannot and does not require infinite precision. The full explanation requires careful consideration of impurities, and is well beyond my
Despite these essential complications the simple (Laughlin) wave functions we have been discussing are profoundly correct and relevant. The reason lies in the nature of the quantized Hall state. Since the ground state is characterized by an energy gap, a set of trial wave function with the correct short-range correlations and a sensible thermodynamic limit can have significant overlap with the true ground state and capture its universal features. There is ample numerical work indicating that the Laughlin wave functions, with their built-in feature of uniformity and strong short-range repulsion, make an excellent approximation to the true ground states.

Likewise adiabatic flux insertion, which as we have seen creates a localized hole in the charge distribution, produced a good wave function for the low-energy charged excitations – quasiholes. Adiabatically inserting many flux units at the origin produces a macroscopic hole, and provides a good trial wave function for annular geometry.

2. Adiabatic Charge Insertion and Angular Momentum

Now I would like to reconsider in the context of the Hall effect the analysis of flow in an annular geometry, which plays a classic role in the discussion of quantization for the other superfluidities. I will do this first in the style of the preceding section, using explicit wave functions.

2.1. Adiabatic Charge Insertion

Let us now consider adiabatic charge insertion – or, more precisely, response to an adiabatic change in the electric potential. This proves to be subtly but crucially different from adiabatic flux insertion. We can consider adiabatic charge insertion as arising from the slow lowering of charge down toward the sample plane. For simplicity let us consider lowering a negative charge toward the origin.
One’s first reaction might be that the droplet or annulus responds to such a perturbation by carving out a bigger hole, as we discussed for flux insertion. This is not quite adequate, however. During the charge lowering process the degree of the wave function, regarded as a polynomial in the angular variables $e^{i\theta_j}$, cannot change. Indeed, since this degree is necessarily an integer it can only vary adiabatically if it does not change at all. This defect can be remedied in a simple manner by applying together with the old factor

\[ \text{Annulizing Factor} = \prod z_k^p \quad (2.1) \]

a new factor

\[ \text{Rotation Factor} = \prod e^{-ip\theta_k}. \quad (2.2) \]

At the moment this probably appears to be nothing but an ad hoc fix; but I hope to convince you there is more to it.

\[ \text{Angular Momentum} \]

Before proceeding further let me remind you of some very elementary but sometimes confusing facts about angular momentum, which justify the terminology above and will be of use below. The canonical angular momentum is associated with the generator of spatial rotations or the operator $-i\frac{\partial}{\partial \phi}$; its value is $l$ on a wave function whose angular dependence is $e^{il\phi}$. The canonical angular momentum of a particle is conserved in its response to azimuthally symmetric external fields. The kinetic angular momentum operator is defined with covariant derivatives, and is gauge invariant. It is what appears in energy expressions, is what gravitons couple to, ... and in general is what we think of as a proper measure of rate of rotation. The sum of the kinetic angular momentum of particles and the field angular momentum of the electromagnetic field is conserved. In symmetric gauge in the lowest Landau level all the wave functions $z^l e^{-\frac{1}{2}|z|^2}$ have unit kinetic angular momentum, corresponding to cyclotron motion in a small orbit, independent of $l$. 

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The factor \((2.2)\) implements a dynamical change in the kinetic angular momentum, by a fixed amount \(\hbar p\) for each particle.

It may appear paradoxical that a procedure (lowering the charge source) which can be carried out in a completely rotation-invariant way could cause rotational motion of the electron fluid. However this “paradox” is closely related to a classic one in electromagnetic theory [3] and it is resolved in a similar way, very instructive for our purpose. Radial expansion of the annulus involves transport of charge and thus a change in the electric field and eventually a change in the field angular momentum, as we shall now compute.

For a constant magnetic field, \(B = B\hat{z}\) and an electric potential, \(V\), with azimuthal symmetry, the total field angular momentum is in the \(z\) direction and has magnitude

\[
L_{EM} = \frac{1}{4\pi c} 2\pi B \int dz r dr r \frac{\partial V}{\partial r}. \tag{2.3}
\]

Notice that this diverges for a point charge at the origin.

Consider then a point charge \(+Q\) at the origin and a ring of charge \(-Q'\) at radius \(r_0\). Using the expansion of \(1/|\mathbf{x} - \mathbf{x}'|\) appropriate to cylindrical coordinates we obtain

\[
L_{EM} = -\frac{2BQ}{c} \int_0^\infty dk \ k \delta(k) \int_0^\infty dr r^2 I_0(0) K_0'(kr) = -\frac{2BQ'}{c} \int_0^\infty dk \ k \delta(k) \left\{ \int_0^{r_0} dr r^2 I_0'(kr) K_0(kr_0) + \int_{r_0}^\infty dr r^2 I_0(kr_0) K_0'(kr) \right\}, \tag{2.4}
\]

where \(f'(x)\) denotes \(\frac{df}{dx}\). Doing the \(r\) integrals by parts and making use of the expansions \(I_0(x) = 1 + \frac{x^2}{4} + O(x^4), K_0(x) = -\ln x + x(x^2 + \frac{2}{x}) + O(x^4 \ln x)\) (where \(\alpha = 0.5772\ldots\)) to evaluate the resulting \(k\)-integrand for small \(k\), we find

\[
L_{EM} = -\frac{2BQ}{c} \int_0^\infty dk \ k \delta(k) \left\{ \frac{2(Q - Q')}{k^2} + \frac{Q' r_0^2}{2} \right\}.
\]
When $Q = Q'$ the divergent pieces at $k = 0$ cancel and we find the simple result

$$L_{EM} = -\frac{BQr_0^2}{2c}. \quad (2.5)$$

It is important that this arises as the finite residual of an infrared divergence, since this indicates the insensitivity of the result to details, including notably the precise shape of the edge.

Of course, if we move the annulus so as to move charge $Q$ from the inner radius $r_1$ to the outer radius $r_2$, the change in electromagnetic field angular momentum is $BQ(r_2^2 - r_1^2)/2c$. Since the number of particles in the annulus is given by $N = 2\pi(r_2^2 - r_1^2)\rho$ with $\rho = \frac{veB}{2\pi\hbar c}$, we find the relation

$$\frac{\Delta L}{N} = \frac{\hbar Q}{e\nu} \quad (2.6)$$

for the angular momentum per particle.

Note that according to (2.6) a unit change in angular momentum corresponds to a generally fractional change $e\nu$ in the ring charge $Q$. Thus we see how fractional charge can be transported from one edge of the sample to the other. Also note that for transport of an integer number of electrons across the ring to be possible, $\nu$ must be rational. Evidently elementary considerations on angular momentum and its quantization are tied up with the most profound quantum aspects of the Hall states.
3. Analogies and Identification

In this section I will argue, both by analogy and by energetics, that one should identify the product of the standard annulus wave function with a rotation factor of type (2.2) as the preferred current-carrying Hall state.

3.1. The London-Feynman-Girvin Description

The Hall fluid at rest is supposed to be described by a unique featureless ground state wave function $\Phi(x_k)$, that has a finite energy gap against localized density inhomogeneities. Overall translational motion without change of density is then described by wave functions of the type

$$\Psi_q(x_k) = \prod e^{iqx_k} \Phi(x_k).$$

(3.1)

One may also consider configurations in which $q$ becomes a slowly varying function of position, $q \to q(x)$. These again do not violate the incompressibility constraint, and ought to represent bulk motions of the fluid. Following Feynman’s approach to superfluid helium, one can work to extract dynamics directly from these wave-functions.

An alternative approach involves the concept of a macroscopic wave function or order parameter, as pioneered by London and by Landau and Ginzburg. In this approach one postulates the existence of a macroscopic complex wave function $\Upsilon(x)$ which describes the ordered state and its low-energy excitations. One expresses the dynamics directly in terms of $\Upsilon(x)$. The ground state is characterized by a uniform constant value of $\Upsilon$, and the energy functional for the low-energy excitations is supposed to be expanded in gradients, $\mathcal{H} \approx \kappa |\partial \Upsilon|^2$. In London’s work the magnitude of $\Upsilon$ was taken constant, and this will be good enough for my immediate purpose here.

Girvin and his collaborators adapted this approach to the Hall fluids in a very fruitful way [1], and here I am just filling out a particular corner of their picture.
We have two alternative descriptions of the current-carrying states, which is potentially one too many. But it is easy to see the relation: with $\Upsilon(x) = \text{const. } e^{i q(x)}$ we are describing (3.1), and vice versa. To firm up the connection we should calculate the energy associated with (3.1) and check that it takes the assumed form, in the process determining $\kappa$.

With this perspective we come to view (2.2) in a different way. If there is an underlying London theory of the Hall superfluids, then this must be the way to describe the ideal current-carrying state for annular geometry. To check this identification, let us consider the energetics. Having appended a factor $r^p$ to each single-particle wave function (combining the annular and rotation factors), the resulting single-particle state is no longer within the lowest Landau level. However for large $l$ one can compute the relevant overlap to be

$$\frac{\int_0^\infty r^{2l+p} e^{-r^2/2} r dr}{\sqrt{\int_0^\infty r^{2l+2} e^{-r^2/2} r dr \int_0^\infty r^{2l+2p} e^{-r^2/2} r dr}} \approx 1 - \frac{p^2}{8l} \quad (3.2)$$

The energy will thus be of order the cyclotron energy times the sum of $\frac{p^2}{4l}$ taken over the relevant $l$-values for the annulus, that is for between $l_i \approx r_i^2$ at the inner radius and $l_o \approx r_o^2$ at the outer. We see that this sum goes quadratically with the overall current strength and logarithmically with the radius. But this precisely matches what we expect from the macroscopic wave function approach. Indeed the current density is of the form $\Upsilon^*(\frac{i}{r} \frac{\partial}{\partial \theta}) \Upsilon \sim p/r$ and the energy density is proportional to the square of this, with $\Upsilon \sim e^{ip\theta}$.

### 3.2. Profile

From this analysis we conclude that the ideal current density will exhibit a $1/r$ profile. Indeed, this is an immediate consequence of the form of the rotation factor (2.2). From the point of view of the macroscopic wave function, it is a signature of condensation in a definite partial wave. From the single-particle point of view, it expresses the conservation of circulation.
There is an absolutely essential aspect of the problem which has not been mentioned so far. That is, when I discussed the bulk translational motion and considered the effect on the wave function (3.1), I failed to take into account that the whole process is going on in a background magnetic field. Because of this background, a proper implementation of the boost will require the existence of an appropriate electric field. We really only reproduce a configuration related locally by boosts to the ground state when the famous Hall relation \( j/E = \nu e^2/h \) between the current and perpendicular electric field is obeyed.

If the ideal current profile we discussed above is achieved, then the Hall relation gives a simple definite result for \( E(r) \) – a conveniently measured quantity. We were led to infer its form by assuming that the bulk flow occurred with uniform density, and (implicitly) that the voltage took care of itself. Neither of these assumptions is automatically fulfilled. The presence of Laughlin quasiholes, which cost only a finite energy, will generally both spoil the density uniformity and help in reaching a self-consistent potential. However one should note that the \( j(r) \propto 1/r \) profile besides maintaining strict uniformity in correlated flow also minimizes the kinetic energy for a given total current. This is more favorable than the situation in superfluid helium, where the natural constraint is given total angular momentum. With that constraint, the most favorable profile is instead \( j(r) = \text{const} \).

We are actively investigating the question under what practical conditions, if any, the ideal flow is a good approximation [4].

Acknowledgment: A brief indication of these ideas occurs at the end of [5]. The talk as actually given was closely based on that paper; here I have given a more self-contained and extended discussion of one aspect. Anything original in this talk represents joint work with R. Levien and C. Nayak, from whom I have learned much regarding these matters.
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