Generalized quantum measurements. Part II:
Partially-destructive quantum measurements in finite-dimensional Hilbert spaces

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Abstract

A concept of the generalized quantum measurement is introduced as the transformation, which establishes a correspondence between the initial states of the object system and final states of the object–measuring device (meter) system with the help of a classical informational index, unambiguously linked to the classically compatible set of states of the object–meter system. It is shown that the generalized measurement covers all the key known quantum measurement concepts—standard projective, entangling, fuzzy and the generalized measurement with the partial or complete destruction of the initial information contained in the object. A special class of partially-destructive measurements that map the continual set of the states in finite-dimensional quantum systems to that one of the infinite-dimensional quantum systems is considered. Their informational essence and some information characteristics are discussed in detail.

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I. INTRODUCTION

Recent progress in developing sophisticated modern methods for engineering of quantum information \(^1\) caused significant changes in the standard (quasiclassical) concept of the quantum measurement. From the early days of quantum mechanics, the procedure of quantum measurement is interpreted as setting a correspondence between the eigenstates of a measurable variable of the object quantum system (in the following, simply the object) and the indicator variable of the quasiclassical measuring device (in the following, simply the meter), which measures the state of the object system \(^2,3\). Nowadays, this concept of the quantum measurement has been changed towards more general concept that is setting the respected correspondence between the object and quantum meter in an essentially quantum form, which generally includes an entanglement in the object–meter system \(^4\) (the so called entangling measurement).

In the frame of this concept, it is natural to introduce the class of the “soft” measurements as the measurements accuracy of which is limited with the internal quantum uncertainty of the states of the indicator variable used in the measurement \(^5\). It presents the most basic class of a more general definition of a fuzzy measurement \(^6\): In the case of a continual-valued measurable variable, studying such class of measurements reveals the information essence of the internal quantum uncertainty of the nonorthogonal quantum states of the indicator, so that it is described not in the terms of the fluctuations of the physical variables (compare with the Heisenberg inequality) widely operated by physicists, but directly in the terms of the quantum states, i.e., irrespective of the values of the physical variables themselves.

For the classical standard measurement, one can easily contribute the requirement of the absolute accuracy of the measurement result with the requirement of the absolute absence of the perturbations in the measurable system during the measuring procedure. By contrast, in the generalized concept of the quantum measurement both these requirements cannot be fulfilled simply due to the specific properties of the set of quantum states forming a linear space. Any interaction with a quantum system inevitably changes at least a part of its possible quantum states. Being ever significant in terms of the Hilbert space, these changes are not take into account when we map onto the classical system the quantum algebra of the observable variables with the use of the classical limit \(\hbar \to 0\).

For a quantum system, the requirement for the absolute absence of the perturbations can
be fulfilled only with respect to the measurable variable selecting therefore a special class of the so called *nondemolition* measurements. Due to the uniqueness taken into account by the no-cloning principle \[7\]) of the entire quantum information the nondemolition measurement completely destroys the coherent, i.e., essentially quantum, information in the initial state of the object \[4\], which in the case of completely coherent measurement is distributed among the object and the meter and does not exist in the separate systems of the object and the meter.

Selecting the subclass of the soft nondemolition quantum measurements allows partial preserving of the coherent information in the object. On contrary, transmission of the major portion of information towards the meter can be realized only within the class of the so called *destructive* (demolition) measurement, when the state of the object is inevitably perturbed. A limiting case of maximally destructive measurement, which entirely destroys the initial information can be illustrated with the totally coherent transition of the excitation from one oscillator to another one that belong to a set of coupled oscillators. Another illustrating example can be given by considering a purely non-coherent measurement of the state of a two-level atom with the help of detection of the irradiated photon.

In this paper, we consider a more general class of measurements, which by contrast with the class of the soft measurements defined in Ref. \[5\] allows perturbation of the initial state along any variable. Respectively, we analyze the most interesting, in respect to the qualitative content of the mapped information, case of nonselected mapping of all states of the Hilbert space \(H\) of a quantum system onto the classically distinguishable eigen states of a continuous variable of another more complex quantum system with the Hilbert space \(L_2(H)\).

This class of quantum measurements, along with showing up the potential resources of the measuring transformation of this kind for developing new methods of engineering of quantum information, is also interested from the fundamental point of view of qualitative interpretation of quantum theory. It helps to exposure the most general relationships between the physical content of the transformations applied to a quantum system and the classical information contained in the values of the information index, which sets the unique correspondence between the initial quantum states and the quantum states after the measurement.
II. DEFINITION OF THE INDEXING DESTRUCTIVE MEASUREMENT AND ITS CORRESPONDENCE WITH THE GENERALIZED MEASUREMENT

We will start with the isometric transformation of the form

\[ V = \sum_{\alpha} \sqrt{\nu_{\alpha}} |\alpha\rangle_{AB} \langle \alpha |_{A} \]  

from Hilbert space \( H_{A} \) of the object \( A \) onto the space \( H_{A} \otimes H_{B} \) of the bipartite system object–meter \( A + B \). Here vectors \( |\alpha\rangle_{AB} \) define the orthogonal basis in the Hilbert space of the bipartite system object–meter, indexed by the values \( \alpha \) of an indicator variable. This basis allows new representation of the initial quantum information that can be measured in general case with the help of nonorthogonal “probe” states \( (\alpha|_{A} \); the set of positive numbers \( \nu_{\alpha} \) characterizes the repetition factor of the elementary maps \( |\alpha\rangle_{A} \rightarrow |\alpha\rangle_{AB} \), of which the resulted transformation \( V \) is constructed as the coherent (i.e., depending on the phases of the wave functions) superposition of the respective generalized projectors.

The relation

\[ V^{+}V = \sum_{\alpha} \nu_{\alpha} |\alpha\rangle_{A} \langle \alpha |_{A} = \hat{I}_{A}. \]  

assures the isometric property of the transformation. It admittedly can be fulfilled if the set of mapped states \( |\alpha\rangle_{A} \) is a set of orthogonal bases, randomly rotated with respect to each other. Specifically, for \( N \) equally represented bases in a \( D \)-dimensional space we have \( \nu_{\alpha} = 1/ND \).

The transformation \( \Box \) is a generalized modification of the canonical representation of the isometric mapping \( V = \sum |k\rangle_{C} \langle k |_{A} \) as the transformation of the entire orthogonal set in \( H_{A} \) into an orthogonal set in an arbitrary space \( H_{C} \). This transformation, first of all, concretizes the structure of the mapping space as the space of the states of the bipartite system object–meter, \( A + B \). Secondly, it uses in general case an overfull set of states \( |\alpha\rangle \) for the representation of the set of the initial states.

The physical meaning of consideration of the isometric mapping of \( A \) onto \( A + B \) is that it can always be redefined up to the unitary transformation \( U \) in the bipartite system \( A + B \), which corresponds to the transformation \( V \) at the fixed initial state \( |0\rangle_{B} \) of the meter:

\[ U |\psi\rangle_{A} |0\rangle_{B} = V |\psi\rangle_{A}, \quad \forall \psi \in H_{A}. \]

Therefore, the isometric property is the condition for the physical realizability of the transformation in the form of dynamically reversible evolution in the bipartite system object–meter.
Index \( \alpha \) in the transformation (1) accumulates in the classical form an information associated with the set of initial quantum states \( |\alpha\rangle_A \) of the object. Values of the index \( \alpha \) are mutually uniquely mapped with the set of classically distinguishable states \( |\alpha\rangle_{AB} \) of the bipartite system. This correspondence gives us the ground for the definition of the measurement transformation (1) as a sort of purely coherent measurement, which delivers the output information about the object in the form of entanglement of the state \( |\alpha\rangle_{AB} \). Such measurement on account of the dequantization effects, which are pronounced in the partial loss of coherency of the measurement results without loss of the classical information, can be represented with the following superoperator \( \mathcal{M} \)

\[
\mathcal{M} = \mathcal{D} (\mathcal{V} \circ \mathcal{V}^+) = \sum_{\alpha\beta} R_{\alpha\beta} \sqrt{\nu_{\alpha\beta}} |\alpha\rangle_{AB} (\alpha|A \odot |\beta\rangle_A \langle \beta|_{AB},
\]

where \( \mathcal{D} = \sum_{\alpha\beta} R_{\alpha\beta} |\alpha\rangle_{AB} \langle \alpha|AB \odot |\beta\rangle_A \langle \beta|_{AB} \) is the dephasing superoperator, \( R_{\alpha\beta} \) is an arbitrary positive-definite matrix with the only diagonal filled with the unit, which describes the dephasing of the states, and “\( \odot \)” is the substitution symbol to be replaced with the transformed superoperator (the density matrix). At \( R_{\alpha\beta} \equiv 1 \), i.e. without dephasing, this superoperator simply describes the transformation \( \mathcal{V} \) in terms of density matrix.

General representation of the measurement in the form (3) and its purely coherent modification (1) include:

- The standard projective and entangling measurements [4] at the choice of the mapped information in the form of a the complete set of classically compatible states, the orthogonal basis \( |k\rangle_A \), and as \( |\alpha\rangle_{AB} \)—the duplicated basis \( |k\rangle_A |k\rangle_B \).
- The soft measurement [5] when additionally replacing the orthogonal basis of the meter \( |k\rangle_B \) with the nonorthogonal set \( |k\rangle_B \).
- Considered in this work generalized measurement with partial destruction of the initial information at the choice \( |\alpha\rangle_{AB} = |\epsilon\alpha\rangle_A |\alpha\rangle_B \), where the set of states \( |\epsilon\alpha\rangle_A \) is arbitrary and \( |\alpha\rangle_B \) is formed of orthogonal states and unambiguously maps the values of the information index \( \alpha \), whereas the set \( (\alpha|A \) can count in nonorthogonal states, as well.

The transformation (1) corresponding to the generalized measurement takes the form:

\[
\mathcal{V} = \sum \sqrt{\nu_{\alpha}} |\epsilon\alpha\rangle_A |\alpha\rangle_B (\alpha|A,
\]

(4)
where in the case of nonorthogonal set \((\alpha|_A\) the information index \(\alpha\) is not unambiguously linked with the classically distinguishable states of \(A\) and its statistics includes the internal quantum uncertainty of the mapped states \(|\alpha\rangle_A\). It can be formally interpreted as the number of elementary coherent sub-channel \(|\alpha\rangle_A \rightarrow |e_\alpha\rangle_A |\alpha\rangle_B\), which links, in general, classically non-compatible input states of the object \(|\alpha\rangle_A\) with the states of the bipartite system object–meter \(|e_\alpha\rangle_A |\alpha\rangle_B\).

In case of the soft nondemolition measurement, the orthogonality of the set \(|e_\alpha\rangle_A = |k\rangle_A\) leads to the unique correspondence with the informational index \(\alpha = k\) and, respectively, to the nondemolition character of the measurement along the measurable variables of the form 
\[
\lambda = \sum \lambda_k |k\rangle_A \langle k|_A
\]
and to the complete vanishing of the coherent information of the meter in respect to the initial state of the object \([5]\).

Nonorthogonality of the set \(|e_\alpha\rangle_A\) leads, in its turn, to reduction of the information remaining in the object, i.e., to the destructive measurement. During this measurement some coherent information is transferred into the states of the meter indicator \(\alpha\) of which contains quantum uncertainty with respect to the object states \(|\alpha\rangle_A\) only if the latter have internal quantum uncertainty and are uniquely represented with the input states \(|\alpha\rangle_B\) of the meter. In the limiting case \(|e_\alpha\rangle_A \equiv |0\rangle_A\), the transformation \([4]\) corresponds to the complete transmission of the initial information from \(A\) into \(B\).

If one uses the orthogonal bases \(|k\rangle_A\) for the sets \(|e_\alpha\rangle_A, |\alpha\rangle_A\), the transformation \([4]\) corresponds to the entirely coherent entangling measurement \([4]\), which leads to the equitable probability distribution of the initial information between \(A\) and \(B\) and complete absence of the coherent information about initial states in the subcomponents of the bipartite system \(A + B\).

In general case, distribution of the initial information about the object among the object and meter is determined by the metric matrix 
\[
Q_{\alpha\beta} = (e_\alpha | e_\beta)_A |e_\alpha\rangle_A.
\]

In case of the overfull set \(|e_\alpha\rangle_A\) the representation of the operator \([4]\) as a sum over \(\alpha\) can be reduced into the superposition \(D^2\) of the projectors by shifting to the minimal orthogonal basis \(|k\rangle_A\). The respective representation has the form:

\[
\mathcal{V} = \sum_{kl} |k\rangle_A |k\rangle_B \langle l|_A, \tag{5}
\]
where $|kl\rangle_B = \sum_{\alpha} \sqrt{\nu_\alpha} (\alpha |l\rangle_A \langle k| e_\alpha) |\alpha\rangle_B$ with the scalar product
\[
(k\ell' \mid kl) = \sum_{\alpha} \nu_\alpha (\alpha |l\rangle_A \langle k| e_\alpha) (e_\alpha |k\ell'\rangle_A \langle \ell'| \alpha\rangle_B,
\]
which is determined only by the states in the Hilbert space of the object $H_A$. Therefore, the representation (4) clarifies the transformation of the form (5) as setting the correspondence between the input and output via the classical information index, which surely contains the internal quantum uncertainty.

III. RELATIONSHIP BETWEEN TRANSFORMATION OF THE GENERALIZED MEASUREMENT AND ITS REPRESENTATION IN THE FORM OF POVM

Let us consider the superoperator (3), which maps the generalized transformation (4) taking into account dephasing:
\[
\mathcal{M} = \sum_{\alpha\beta} R_{\alpha\beta} \sqrt{\nu_\alpha \nu_\beta} (e_\alpha |A\rangle \langle \alpha| \otimes |\beta\rangle_B \langle \beta|_A (e_\beta |A\rangle .
\]
(6)
It corresponds to the probability distribution $P(\alpha) = \langle \alpha|_B \hat{\rho}_B |\alpha\rangle_B$, where $\hat{\rho}_B = \text{Tr}_A \mathcal{M} \hat{\rho}_A$, for the results $\alpha$ of the measurement, which are physically realized in the form of quantum states of the meter. This distribution has the form
\[
P(\alpha) = \text{Tr}_A \hat{E}_\alpha \hat{\rho}_A
\]
(7)
with the positive operator valued measure (POVM) $\hat{E}_\alpha = \nu_\alpha |\alpha\rangle_A \langle \alpha|_A$.

This expression does not depend either on the coherency of the transformation or on the form of its representation in the output state of the object and corresponding entanglement in the bipartite system object–meter after the measurement because it describes only classically compatible information of the object about its initial state. However, the above expression does not describe the quantum result of the measurement, but the resulting nonselected information preserved in the object in quasiclassical form.

The complete resulting information, though displayed in the classically distinguishable form, is described by the contracted superoperator for the bipartite system object–meter
\[
\mathcal{M}_B = \sum_{\alpha\beta} R_{\alpha\beta} \sqrt{\nu_\alpha \nu_\beta} (e_\beta |A\rangle \langle \alpha| \otimes |\beta\rangle_B \langle \beta|_A (\beta|_B,
\]
(8)
which takes into account quantum correlations with the initial state. Even for the completely coherent measurement, it contains the decoherence factor \( R^{A\beta}_{\alpha\beta} = (e_{\beta} \mid e_{\alpha})_A \), which is due to the ignoring of the coherent information bundled in the form of entanglement in the bipartite system object–meter. This dequantization is not the complete one. Yet, considering the complete dequantization \( R_{\alpha\beta} = \delta_{\alpha\beta} \) of the output information leads, as one can easily see, to the transformation

\[
\mathcal{M}_B = \sum_{\alpha} \nu_{\alpha} |\alpha\rangle_B (\alpha |_A \otimes |\alpha\rangle_A \langle \alpha |_B).
\]

The respected probability distribution for this transformation is given by Eq. (7) on the algebra of classical events described with the set of compatible states \( |\alpha\rangle_B \) and its subsets.

It is worth to note here that the generalized measurements in terms of POVM have been widely discussed in the literature, particularly, in connection with the problem of optimal measurement of continual quantum variables, i.e., coordinates and momenta [9, 10, 11]. However, our consideration is qualitatively different because we consider the finite-dimensional Hilbert space \( H_A \) and, respectively, discuss potentially information about all quantum states in this space, whereas in the previous works the analysis has been done for the infinite-dimensional space, which cannot be applied to our case.

### IV. SELECTED GENERALIZED MEASUREMENT

A special case of the generalized measurement is the selected measurement, which has different from the entangling measurement generalized set of the output states of the corrected object: \( |k\rangle_A \rightarrow |e_{\alpha}\rangle_A = |k\rangle_A, k = 1, \ldots, D \). These states are different from the basis states \( |k\rangle_A \) of the measurable variable and, being in general case represented by the nonorthogonal set of states, prevent the object preserving the initial state of the measurable variable.

In case of purely coherent measurement, the meter attains a nonzero coherent information about the initial state of the object, which in the limiting case of \( |k\rangle_A \equiv |0\rangle_A \) is the complete information, i.e., the information equal numerically to the initial entropy of the object. In this case, the information relationships for the mapping object–meter reproduce obviously the same relationships for the mapping object–object for the case of the soft measurement, transformation for which is described with the transformation \( H_A \Leftrightarrow H_B \) of the resulting states of the object and meter. Therefore, the respective dependencies given in [5] for the coherent information object–object for the two-level system, retain their validity in the...
FIG. 1: Mapping of the elementary states in the process of nonselected generalized measurement.

New states $|e_\alpha\rangle_A$ of the object are, generally, different from $|\alpha\rangle_A$.

present case, as well. Quasiclassical information attained by the meter is due to the absolute accuracy of the measurement always complete, i.e., the amount of information coincides with the entropy of the measurable variable.

V. NONSELECTED GENERALIZED MEASUREMENT

The nonselected measurement gives us another special case of the measurements for which the set of mapped states $|\alpha\rangle_A$ includes all quantum states of the object. In this case, the information index $\alpha$ unambiguously maps all physically distinguishable elements of the Hilbert space $H_A$ and the appropriate representation of the set of its states is the unit $(2D - 2)$-mensional sphere of the real Euclidean space.

Then, the respected generalized measurement is the map $H_A \rightarrow H_A \otimes H_B$ with the states of the meter $H_B = L_2(H_A)$ with the wave functions of the continual argument $\psi_B(\alpha)$. Multiplicity of the states $d\nu = \sum d\nu_\alpha$, which corresponds to the element in the set $\alpha \in dV$, has in this case the form $d\nu = DdV/V$, where $V$ is the entire volume of the hypersphere of the physical states. Fig.[] illustrates an example of the two-level system.

A. Distribution of information between object and meter

The amount of information preserved in the object is determined by the information capacity of the overfull basis $|e_\alpha\rangle_A$, which duplicate information that is represented by the
states of the meter $|\alpha\rangle_B$, but, in general case, corresponds to a partial or complete loss of the initial information $|\alpha\rangle_A$ of the object. In case of completely coherent measurement, i.e. at $R_{\alpha\beta} \equiv 1$, the information capacity of the basis $|e_\alpha\rangle_A$ for the pure input state $\hat{\rho}_A = |\psi\rangle_A \langle \psi|_A$ is determined by the entanglement $E[|\psi\rangle_{AB}]$ of the resulted state $|\psi\rangle_{AB} = \mathcal{V} |\psi\rangle_A$ of the bipartite object–meter system. The meter in this case contains all accessible information about all Hilbert space of object states, which is represented, however, in quantum form including the entanglement with the object. This information is reduced into classical form either after additional projective measurement or after entirely dephasing transformation $D$ at $R_{\alpha\beta} = \delta_{\alpha\beta}$, which are equivalent from information point of view.

Let us illustrate the distribution of information among the object and meter on an example of a two-level system with $D = 2$ using as $|e_\alpha\rangle_A$ all states of a part of the Bloch sphere, which is formed by the mapping $\vartheta \rightarrow q \vartheta$, where $0 \leq q \leq 1$ is the compression coefficient of the initial Bloch sphere that is mapped into its part corresponding to $0 \leq \vartheta \leq \pi q$. With this choice of the mapping, at $q < 1$ there is some asymmetry with respect to the value of the polar angle $s$ of the initial state $\alpha_0 = (s, \varphi_0)$. This asymmetry reaches its maximum at $q = 0$ and vanishes at $q = 1$.

The entanglement in the object–meter system that arises after the measurement can be written as the entropy $S[\hat{\rho}'_A] = -\text{Tr} \hat{\rho}'_A \log_2 \hat{\rho}'_A$ of the partial density matrix $\hat{\rho}'_A = \text{Tr}_B |\psi\rangle_{AB} \langle \psi|_{AB}$ of the transformed object state. The corresponding dependence $E(s, q)$ is shown in Fig. 2.

The results of the analysis of the information distribution in the two-level system for the completely nonselected representation of the final state of the object at $q = 1$ and, respectively, $|e_\alpha\rangle_A = |\alpha\rangle_A$ is an obvious one, even without any calculations, because in this case $\hat{\rho}'_A$ corresponds to the entirely depolarized initial state (see, for example, Eq. (3.115) at $p = 1$ in Ref. 13)

$$\hat{\rho}'_A = (2/3) |\alpha_0\rangle \langle \alpha_0| + (1/3) |\alpha_0^*\rangle \langle \alpha_0^*|$$

($|\alpha_0^*\rangle$ is orthogonal to $|\alpha_0\rangle$) and, independently from $\alpha_0$, $E = E_0 = (2/3) \log_2(3/2) + (1/3) \log_2(3/1)$.

However, the result $E = 1$ bit, i.e., the complete entanglement between the object and the meter, which is achieved at the orientation of the initial state $s = \pi$, opposite to the Bloch sphere compression point $\vartheta = 0$, and at the intermediate value of the compression coefficient, is not trivial and requires a qualitative elucidation.
FIG. 2: The degree of entanglement $E$ (in bits) versus the compression coefficient $q$ of the Bloch sphere and the angle $\vartheta = s$ of the initial state. The maximum value $E = 1$ bit is achieved at $s = \pi$ and $q = 0.7978$. At $q = 1$ the degree of entanglement does not depend on $s$ and is equal to the entropy $E_0 = 0.918$ bit for the pure state of the qubit after its complete depolarization.

We can do that easily because at the chosen orientation the problem is symmetrical in respect to the axis of the Bloch sphere, thus the density matrix in the respective basis is a diagonal one and has the form: 

$$\hat{\rho}'_A = p_1 |1\rangle_A \langle 1|_A + p_2 |2\rangle_A \langle 2|_A.$$  

Also, the direction $\vartheta = 0$ is opposite to the direction of the initial state $s = \pi$ and, therefore, the probability $p_1$ in accordance with the given above equation for $q = 1$ is simply $p_1 = 1/3$. If one changes the compression coefficient up to the value of $q = 0$, which corresponds to the collapse of the Bloch sphere into the point $\vartheta = 0$, the probability of the opposite state $p_2$ reduces up to the zero and, respectively, the probability $p_1$ grows up to the unit in accordance with the following analytical formula:

$$p_1 = 1 - p_2 = \frac{3 - 2q^2 + \cos \pi q}{4(1 - q^2)} + \frac{1 - \cos \pi q}{4(4 - q^2)}.$$

This probability due to it continuity passes the value of $p_1 = 1/2$, which corresponds to the maximum possible entanglement between two systems, one of which if the two-level system (qubit).

Note also that the maximal degree of entanglement $E_0 = 0.918$ bit, achieved at the exact reproduction by the object after the measurement of all states of the Hilbert space, is very close to the maximal entanglement, which is achieved at the totally coherent nondemolition
entangling measurement. This, however, can be achieved only with the optimal choice of the initial wave function of the object. In case of the completely nonselected measurement, the degree of entanglement is invariant with regard to the initial state $|\alpha_0\rangle_A$ because all the states are due entirely equal.

VI. COMPETITION BETWEEN OBJECT AND METER IN SELECTION OF THE NONSELECTED QUANTUM INFORMATION

In case of nondemolition quantum measurement, there is no competition between the object and the meter because classically compatible information retrieved at such measurement can be duplicated without bound. However, with the choice of nonselected information, which is connected with the nonorthogonal overfull set $|\alpha\rangle_A$, typically, for instance for the quantum key distribution protocols [14], the competition arises. It is due to the impossibility of nondemolition duplication of the information about the nonorthogonal quantum states. Mathematically, such competition can be sufficiently treated with the Holevo information [11], which implicitly takes into account quantum nature of the information.

The Holevo information is defined for the semiclassical channel, which is characterized by the density matrix $\hat{\rho}(\alpha)$ depending on the classical messages $\alpha$ at the input of the channel, as

$$I_h = S[\hat{\rho}] - \int P(d\alpha) S[\hat{\rho}(\alpha)], \quad \hat{\rho} = \int \hat{\rho}(\alpha) P(d\alpha), \quad (9)$$

where $P(d\alpha)$ defines the probability distribution or the frequencies of the classical messages $\alpha$. One can easily see that in the considered transformation of the quantum measurement the classical parameter $\alpha$ corresponds to the informational index of the initial states of the object $|\alpha\rangle_A$ and two considered channels, object–object and object–meter, are described with the averaging of the wave function $\mathcal{V}|\alpha\rangle_A$ of the combined object+meter system (or, in general, of the density matrix, which results after the incoherent transformation (3)) over the competing system. For the uniform distribution $P(d\alpha)$, the density matrices for the
corresponding channels have the following form:

\[ \hat{\rho}_A(\alpha) = \frac{D}{V} \int dV_\beta \left( \beta | \alpha \right)_A (e_\beta |_A \langle e_\beta | ; \hat{\rho}_A = \frac{1}{V} \int dV_\beta |e_\beta \rangle \langle e_\beta | ; \right. \tag{10} \]

\[ \hat{\rho}_B(\alpha) = \sum_{\beta\beta'} \sqrt{\nu_\beta \nu_{\beta'}} (e_\beta | e_{\beta'} \rangle_A (\beta' | \alpha \rangle_A (\alpha | \beta)_{A} | \beta \rangle_{B} \langle \beta |_{B} ; \right. \tag{11} \]

\[ \hat{\rho}_B = \frac{1}{D} \sum_{\beta\beta'} \sqrt{\nu_\beta \nu_{\beta'}} (e_\beta | e_{\beta'} \rangle_A (\beta' | \beta \rangle_A | \beta' \rangle_{B} \langle \beta |_{B} \rightarrow \tag{12} \]

\[ \hat{\rho}_B = \frac{1}{V} \int dV_\beta |\beta \rangle_A (e^*_\beta | \langle e^*_\beta |_{A} . \tag{13} \]

The latter equation is the isometric display of the continual density matrix of the meter into the discrete space \( H_A \otimes H_A \), which realizes the active subspace of the states and that is used for the numerical calculations. Entropies of the density matrices \( \hat{\rho}_A(\alpha) \), \( \hat{\rho}_B(\alpha) \) coincide with each other, so that there is no need in using the continual representation. Respective dependencies for the information \( \mathbb{M} \) of the meter and the object that are calculated with the help of Eqs. (10) and (13) are shown in Fig. 3. They demonstrate the relatively weak competition character, for example, by contrast with the competition of the coherent information during the selective measurement, when preserving the entire information in the object corresponds to its complete absence in the meter \( \mathbb{M} \).

**VII. CONCLUSIONS**

In conclusion, the class of the partially-destructive quantum measurements discussed in this paper generalizes using the common mathematical formalism all the key classes of the quantum measurements discussed in the literature: standard projective, entangling, soft, destructive, coherent, and dequantized measurements.

The nonselective subclass of the partially-destructive quantum measurements has, as it was demonstrated on example of the two-level system, an interesting feature: the degree of entanglement in the bipartite object–meter system can reach its maximum value at the intermediate degree of preserving in the object its initial information. When the object preserves the maximum possible information about its initial state, the degree of entanglement nearly achieves its maximum for all possible pure initial states. It has also been shown that the nonselective quantum measurements realize the equitable measurement of all the dynamical variable of the measurable quantum system. Therefore, they are characterized
FIG. 3: The measured amount of Holevo information ($I_B$) and information preserved in the object ($I_A$) about the equally distributed ensemble of initial states $|\alpha\rangle_A$ of the object (qubit) versus the degree $q$ of the preserved information in the object. The maximum amount of information about the object $I_A = 0.081$ bit corresponds to the minimum of measured information $I_B = 0.874$ bit.

with the essentially low level of competition of quantum information in between the object and the meter, by contrast with the totally selective measurements.

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