Cosmic Inhomogeneities and the Average Cosmological Dynamics

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If general relativity (GR) describes the expansion of the Universe, the observed cosmic acceleration implies the existence of a ‘dark energy’. However, while the Universe is on average homogeneous on large scales, it is inhomogeneous on smaller scales. While GR governs the dynamics of the inhomogeneous Universe, the averaged homogeneous Universe obeys modified Einstein equations. Can such modifications alone explain the acceleration? For a simple generic model with realistic initial conditions, we show the answer to be ‘no’. Averaging effects negligibly influence the cosmological dynamics.

Cosmological observations have established that we live in an accelerating Universe [1]. Assuming the dynamics of the expanding Universe to be described by general relativity (GR), the observed acceleration compels us to conclude the existence of a ‘substance’ with negative pressure named ‘dark energy’ [2]. The standard model of particle physics offers no candidate for dark energy, nor does it provide any convincing clues as to its origin. While this situation might suggest new physics, it is still worthwhile to ask if a conventional explanation for the acceleration might yet be possible.

It is true that the standard hot Big Bang model for a homogeneous and isotropic Universe described by the Friedmann-Lemaître-Robertson-Walker (FLRW) solution of Einstein’s equations agrees well with observations. However, the actual Universe is highly inhomogeneous and homogeneity is recovered only in an average sense on sufficiently large scales [3]. Strictly speaking, only the inhomogeneous Universe obeys Einstein’s equations, and recovering the homogeneous limit by averaging leads to corrections in the equations [4]. Could these correction terms (backreaction) be significant enough to account for dark energy? This question has attracted considerable recent attention [4, 5, 6, 7].

One might simply take the view that since the perturbed FLRW model works well throughout the evolution of the Universe, then the backreaction must be evaluated for this specific perturbative case. This has been done in [10], by assuming the metric to have the form

$$ds^2 = a^2(\eta) \left[ -(1 + 2\varphi) d\eta^2 + (1 - 2\psi) \gamma_{AB} dx^A dx^B \right],$$

(1)

with \( \eta \) the conformal time and \( \gamma_{AB} \) the (flat) 3-space metric. In the present paper we work with the cosmic time \( \tau \) related to the conformal time by \( d\tau = a(\eta) d\eta \), with appropriate conversions applied. The modified FLRW equations are given by

$$H^2 = \left( \frac{1}{a} \frac{da}{d\tau} \right)^2 = \frac{8\pi G_N}{3} \bar{\rho} - \frac{1}{6} \left[ P^{(1)} + S^{(1)} \right],$$

$$\frac{1}{a} \frac{d^2 a}{d\tau^2} = \frac{4\pi G_N}{3} (\bar{\rho} + 3\bar{p}) + \frac{1}{3} \left[ P^{(1)} + P^{(2)} + S^{(2)} \right],$$

(2)

where \( \bar{\rho} \) and \( \bar{p} \) are the background energy density and pressure respectively and the combinations \( (P^{(1)} + S^{(1)}) \)
and \((P^{(1)} + P^{(2)} + S^{(2)})\) are covariant scalars given by

\[
P^{(1)} = \left[2((\partial_\tau \psi)^2 + ((\partial_\tau \varphi - \partial_\tau \psi)^2) - \langle \nabla_A \nabla_B \partial_\tau \beta \rangle (\nabla^A \nabla^B \partial_\tau \beta) \right],
\]

\[
S^{(1)} = -\frac{1}{a^2} \left[6\langle \partial_\psi \partial_A^3 \rangle + \langle \partial_A^3 (\varphi - \psi) \partial_\psi \rangle \right] - \langle \nabla_A \nabla_B \nabla^C \partial_\tau \beta \rangle (\nabla^A \nabla^B \nabla^C \partial_\tau \beta) \right],
\]

\[
P^{(1)} + P^{(2)} = \left[\langle \partial_\tau \varphi (\partial_\tau \varphi - \partial_\tau \psi) \rangle - 2H \left(\langle \varphi \partial_\tau \varphi \rangle - \langle \psi \partial_\tau \psi \rangle + \langle (\varphi - \psi) \partial_\tau \psi \rangle + \langle \nabla_A \nabla_B \partial_\tau \beta \rangle (\nabla^A \nabla^B \partial_\tau \beta) \right) \right],
\]

\[
S^{(2)} = -\left[\partial_\tau^2 (\partial_\tau^2 \beta) + H \partial_\tau \beta \partial_\tau (\varphi - a^2 H \partial_\tau \beta) \right].
\]

Here \(\nabla_A (A, B = 1, 2, 3)\) is the 3-space covariant derivative and \(\beta\) solves \(\nabla^2 \beta = \varphi - 3\psi\) with \(\nabla^2 = \gamma^{ab} \nabla_A \nabla_B\), with the condition that when \(\varphi = 0, \beta = 0\). The angular brackets in \((3)\) denote a spatial averaging defined as

\[
\langle f \rangle (\tau, \vec{x}) = \frac{1}{V_L} \int_{\mathcal{V}(\vec{x})} d^3 y f(\tau, \vec{y}),
\]

for any function \(f(\tau, \vec{x})\) where \(\mathcal{V}(\vec{x})\) is the 3-dimensional averaging domain of “Eulerian” length scale \(L\) (comoving with the background) and volume \(V_L\). This averaging operation is derived by considering a spatially averaging limit of the full MG averaging technique and working in a specific “volume preserving” gauge in which the metric determinant depends only on cosmic time with the form

\[
ds^2 = -(1 + \tilde{\psi}) d\tau^2 + a^2(\tau) (1 - 2\tilde{\psi}) d\vec{s}^2,
\]

where \(\tilde{\psi} = \varphi/3\) (see \((10)\)). This procedure is thus different from the spatial averaging of Buchert \((6, 9)\). Having defined the averaging in this gauge, one rewrites all quantities in terms of the conformal Newtonian gauge potentials \(\varphi\) and \(\psi\), and \((3)\) show the final result. In this paper we work with the background metric in spherical coordinates,

\[
ds^{2}_{bg} = -d\bar{r}^2 + a^2(\tau) [d\vec{r}^2 + r^2 d\Omega^2],
\]

where \(\bar{r}\) is the Eulerian radial coordinate. It is then most convenient to consider a single domain which is a sphere of radius \(L\) centered at the origin. The expressions for the backreaction \((3)\) will then correspond to this single domain.

When nonlinear structures such as galaxy clusters and voids form during late stages of evolution of the Universe, is the spacetime metric still perturbed FLRW? The answer is yes, provided matter peculiar velocities remain small. We showed this by considering a simple but generic model of spherically symmetric pressureless dust collapse \((11)\) (see also \((12)\). We now apply the backreaction results \((3)\) to our collapse model and show that the corrections are extremely small. This is the first example of a fully covariant calculation of the backreaction valid even in the nonlinear regime of structure formation.

The spacetime of the spherically collapsing dust is described by the Lemaitre-Tolman-Bondi (LTB) metric given by

\[
ds^2 = -dt^2 + \frac{R^2 d\bar{r}^2}{1 - k(r)} + R^2 d\Omega^2.
\]

Here \(t\) is the proper time measured by observers with fixed coordinate \(r\), which comoves with the dust. \(R(t, r)\) is the area radius of the dust shell labelled by \(r\), and satisfies the equation \(\dot{R}^2 = 2GM/R^3 - kr^2\). Here \(M(r)\) is the mass inside each comoving shell and a dot denotes a derivative with respect to \(t\). The energy density of dust measured by a comoving observer satisfies \(\rho(t, r) = M'/4\pi R^2 R'\) where the prime denotes a derivative with respect to \(r\). Initial conditions are completely specified by choosing an initial density \(\rho(t_i, r)\), velocity \(\dot{R}(t_i, r)\) and an initial scaling function \(R(t_i, r)\). In \((11)\) we made the following choices: To set the initial conditions as being a perturbation around the Einstein-deSitter (EdS) solution characterised by the scale factor \(a(t) \equiv (t/t_0)^{2/3}\) with \(t_0 = 2/(3H_0)\) where \(H_0\) is the standard Hubble constant, we chose the functions

\[
R(t_i, r) = a_ir; \quad \dot{R}(t_i, r) = a_iH_ir, \quad \rho(t_i, r) = \bar{\rho}_i \begin{cases} (1 + \delta_\star), & r < r_\star \\ (1 - \delta_v), & r_\star < r < r_v \\ 1, & r > r_v \end{cases},
\]

where \(\bar{\rho}_i = \dot{\bar{\rho}}(t_i), a_i = (t_i/t_0)^{2/3}, H_i = 2/(3t_i), \) and \(\delta_\star, \delta_v, r_\star, \) and \(r_v\) are constants whose values were chosen so that the system being described is initially a small overdensity of extent \(0 < r < r_\star\), surrounded by a small underdensity out to radius \(r_\star\). In particular, the following values were chosen for the various parameters (cf. Table 1 of \((11)\))

\[
a_i = 0.001; \quad H_0 = 72\text{ km s}^{-1}\text{Mpc}^{-1},
\]

\[
\delta_\star = 2.21 \times 10^{-3}; \quad \delta_v = 5 \times 10^{-3},
\]

\[
r_\star = 16.7\text{ Mpc}; \quad r_v = 23.5\text{ Mpc}.
\]

The exact solution for \(R(t, r)\) can be written in parametric form in terms of trigonometric or hyperbolic functions depending on the sign of \(k(r)\). One then numerically obtains the function \(R(t, r)\) in the region of interest in the \((t, r)\) plane.

It was further shown in \((11)\) that for the chosen model, for all times \(t_i < t < t_0\), the LTB metric \((7)\) can be
brought to the form
\[ ds^2 = -(1+2\varphi)dr^2 + a^2(\tau)(1-2\psi)(d\theta^2 + \nu^2d\Omega^2), \]
where \( \varphi \) and \( \psi \) satisfy \(|\varphi|, |\psi| \ll 1 \), and \( a(\tau) = (3H_0^2/2)^{2/3} \). This is achieved by the coordinate transformation \((t, r) \rightarrow (\tau, \tilde{r})\) given by
\[ \tau = t + \xi^0(t,r) \quad ; \quad \tilde{r} = (R(t,r)/a(t))(1 + \xi(t,r)). \]
Here \( \xi^0 \) and \( \xi \) are assumed to satisfy \(|\xi^0|, |\xi| \ll 1 \) and are determined by integrating the equations
\[ \xi^0 = (1/2) (k(r)r^2 + (a\tilde{v})^2) (R'/R) \quad ; \quad \xi^0 = a\tilde{v}R' \]
where \( \tilde{v} \equiv (\partial \tilde{r}/\partial t) \approx \partial_t(R/a) \) is the comoving peculiar velocity, also assumed to remain small \([13]\). This is a self-consistent calculation, in that one assumes such functions to exist to set up the equations, and then solves for them showing that they satisfy the required properties. In particular, one finds that the peculiar velocity \( \tilde{v} \) does remain small throughout the evolution. Of course, models with large peculiar velocities can be studied (e.g. the \( \sim 1\) Gpc void studied in \([2]\)) and such a case would indicate a breakdown of the weak field approximation. What is important however, is whether such inhomogeneities are generic. The parameter choices in our model reflect the nature of typical inhomogeneities at the Last Scattering epoch and do not lead to large underdense regions and/or large peculiar velocities. With these choices, we find that the metric potentials \( \varphi \) and \( \psi \) can be obtained self-consistently as \( \varphi = -\xi^0 + (a\tilde{v})^2/2 \) and \( \psi = \xi^0H + \xi \), where, at the leading order we have \( H \equiv 2/3t \approx 2/3\tau \).

It can be analytically shown by working in the Newtonian gauge \([12]\) that the metric potentials \( \varphi, \psi \) are in fact equal at leading order, and this can also be checked explicitly. We emphasize that the metric potentials remain small in magnitude compared to unity, even though the density contrast \( \rho/\bar{\rho} - 1 \) becomes completely nonlinear at late times \([11]\). While a more careful second order calculation can in principle be performed, it is not difficult to show that the higher order terms will contribute insignificantly to the backreaction, so long as peculiar velocities are small. [This stems from the basic structure of the nonlinear metric potentials which depends on an expansion in \( (HR) \ll 1 \).] Hereon when considering time derivatives of small quantities we shall not distinguish between \( t \) and \( \tau \). Terms quadratic in \( \tilde{v} \) are retained since on dimensional grounds one expects \( a\tilde{v} \sim HR \) whereas \( \varphi, \psi \sim (HR)^2 \), which is confirmed by our numerics (see also \([12]\)).

We now compute the backreaction given in \([3]\). These expressions were derived in \([10]\) under the requirement that the averaging operation be free of gauge related ambiguities, in linear perturbation theory. However, the actual conditions used to derive \([3]\) only depended on considering leading order effects in the metric perturbations. A key step was the transformation between the metric \([11]\) (in Cartesian spatial coordinates) and the volume preserving form \([5]\), which was achieved by the transformation \( \tau \rightarrow \tau, \ x^A \rightarrow x^A + \partial^A \beta \ (A = 1, 2, 3) \) where \( \beta \) is the function appearing in \([3]\). In the present context, the same transformation remains valid at the leading order and hence the backreaction in \([3]\) is physically relevant here as well. We emphasize that this truncated averaging operation remains valid even at late times since the weak field approximation for gravity works well during nonlinear structure formation.

Since our numerical results involve \((t, r)\) where \( r \) comoves with the matter, we must reexpress the averaging operation \([4]\) in terms of these variables. It is easy to show that, at the leading order, the average of a scalar \( s(t,r) \) defined in \([4]\) can be written as
\[ \langle s \rangle = \frac{3}{(a(t)L)^3} \int_0^{r_L(t)} sR^2R'dr, \]
where \( r_L(t) \) solves \( R(t, r_L(t)) = a(t)L \). Recall that \([12]\) gives the average of \( s \) over a single domain centered at the origin, which is what we restrict ourselves to in this paper. We are constrained to consider values \( r < r_v \), due to unphysical shell crossing singularities in the region beyond (see \([11]\)), so the largest value of \( L \) we can choose is \( L = r_v \), which then ensures \( r_L(t) < r_v \) since \( r_L(t) \) is a decreasing function for this choice. This gives us an Eulerian averaging scale of \( L = 23.5\) Mpc which is smaller than the more realistic expected value of \( \sim 100h^{-1}\) Mpc. Our model potentials \( \varphi, \psi \) and their derivatives hence do not strictly average to zero as needed \([10, 11]\). One can check that actual average values \( \langle (\partial_\lambda \varphi)^2 \rangle \) are small \((\lesssim 10\%) \) for all times compared to terms like \( \langle (\partial_\lambda \varphi)^2 \rangle \) which are needed in the backreaction calculations, although it turns out that the time derivatives satisfy \( \langle \dot{\varphi}^2 \rangle \sim \langle (\dot{\varphi})^2 \rangle \). However, since the averaging scale chosen here is large enough to encompass all the inhomogeneity of this system, we expect that our estimates for the backreaction are fairly representative.

Consider now the function \( \beta \) which satisfies the Poisson equation on a flat 3-space background, such that \( \beta = 0 \) if \( \varphi = 0 = \psi \). We can directly write the solution for \( \beta \) in terms of the coordinate \( \tilde{r} \) as
\[ \beta(\tau, \tilde{r}) = -(1/4\pi) \int d^3yq(\tau, \tilde{g})||\tilde{r} - \tilde{g}|| \\
= -\frac{1}{\tau} \int_0^{\tilde{r}} q \sqrt{\tilde{y}^2dy} - \int_{\tilde{r}}^{\infty} q \sqrt{\tilde{y}^2dy}, \]
where \( q \equiv -2\varphi \) (we have set \( \varphi = \psi \) at leading order) and the integration is over Eulerian spatial coordinates. The
The following relations are useful in the calculations,
\[
\partial_t \beta = \frac{1}{r} \int_0^r q y^2 dy, \quad (14a)
\]
\[
\partial_r^2 \beta = q - \frac{2}{r} \partial_r \beta, \quad (14b)
\]
\[
\partial_t^2 \beta = \frac{6}{r^2} \partial_t \beta - \frac{2}{r} q + \partial_t q. \quad (14c)
\]

We will need \(\partial_t \beta\), \(\partial_r^2 \beta\) and \(\partial_t^2 \beta\) as functions of \((t,r)\), which is done by replacing \(\tau\) and \(\hat{r}\) at leading order by \(t\) and \(R/a\) respectively. This gives us
\[
(\partial_t \beta)(t,r) = \frac{1}{a R^2} \int_0^r q R^2 R' dr, \quad (15a)
\]
\[
(\partial_r^2 \beta)(t,r) = q - \frac{2}{R^3} \int_0^r q R^2 R' dr, \quad (15b)
\]
\[
(\partial_t^2 \beta)(t,r) = \frac{6a}{R^4} \int_0^r q R^2 R' dr - \frac{2a}{R} q + \frac{a}{R^2} q', \quad (15c)
\]

where in the last equation we have used \(\partial_t q = (a/R') q'\) at leading order.

Also, noting that the time derivatives in (3) are taken keeping \(\hat{r}\) fixed, we have
\[
(\partial_{\hat{r}} \beta)(t,r) = \frac{1}{a R^2} \int_0^r \hat{q} R^2 R' dr, \quad (16a)
\]
\[
(\partial_t \hat{\beta})(t,r) = \frac{1}{a R^2} \int_0^r \hat{q} R^2 R' dr, \quad (16b)
\]
\[
(\partial_t^2 \hat{\beta})(t,r) = \hat{q} - \frac{2}{R^3} \int_0^r \hat{q} R^2 R' dr, \quad (16c)
\]

which follow from [14]. The expressions (3), rewritten in terms of \(t\) and valid at leading order, reduce to
\[
\mathcal{P}(1) = \frac{2((\dot{\varphi})^2) - 2((\partial_t \hat{\beta})^2) - 2((1/r^2)(\partial_t \hat{\beta})^2)}{(1/r^2)} ,
\]
\[
\mathcal{S}(1) = \frac{1}{a r} \left[ 6((\partial_t \varphi)^2) - 6((\partial_t \hat{\beta})^2) - 6((\partial_t \beta - \hat{r} \partial_t^2 \beta)^2/r^4) \right] ,
\]
\[
\mathcal{P}(1) + \mathcal{P}(2) = -2H \left[ ((\partial_t \hat{\beta})(\partial_t \dot{\beta})) + 2((1/r^2)(\partial_t \beta)(\partial_t \hat{\beta})) \right] ,
\]
\[
\mathcal{S}(2) = \langle (\partial_t \hat{\beta} + H \partial_t \beta)(a^2 H \partial_t \hat{\beta} - \partial_t^2 \varphi) \rangle , \quad (17)
\]

where the angular brackets are now defined by (12) and the various integrands can be read off using (15), (16) and the results \(\partial_t \varphi \approx (a/R') q'\) and \(\hat{r} \approx (R/a)\).

Figs. 1 and 2 show results of numerical calculations performed with Mathematica. Fig. 1 shows the evolution of the dominant correction \(-\mathcal{S}(1)/6H^2\), as a function of the scale factor. The dashed line shows a hypothetical curvature-like correction. The actual backreaction evolves differently due to significant evolution of \(\varphi\). Note that the largest value of \(|\mathcal{S}(1)/H^2|\) computed here is \(\sim 10^{-6}\), whereas estimates using linear theory suggest this value should be \(\sim 10^{-4}\). This discrepancy highlights an issue noted in [10], namely that nonlinear inhomogeneities on small scales do not contribute significantly to the backreaction. Our model has no large scale inhomogeneities and underestimates the backreaction. Reassuringly, accounting for the deficit only requires a calculation in linear theory, such as the one in [10]. Fig. 2 shows the evolution of the remaining (normalised) integrals. An initial rapid decay of \(\mathcal{P}(1)/H^2\) starting from \(\sim 10^{-8}\) has not been shown, in order to enhance the contrast in the late time behaviour of the three functions. The other functions remain subdominant compared to \(\mathcal{P}(1)\) at the early times not shown.

Our covariant and self-consistent calculation of the backreaction in this spherical collapse model establishes...
that inhomogeneities have an insignificant impact on the average cosmological dynamics. In particular, the observed cosmic acceleration cannot be explained by the averaging of inhomogeneities. Our nonlinear dust model can be regarded as representing a realistic situation, because it has a overdensity-void structure, and departure from sphericity, tidal interactions, and second order corrections are not expected to introduce any significant change in the results. What appears true in general is that as long as peculiar velocities remain small, as seems to be the case in the real Universe, a description as a perturbed FLRW model is valid, and this keeps the back-reaction small.

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[13] This corrects an error in Eq.35 of [11]. We thank Karel Van Acoleyen for pointing this out to us.