A cellular approach to the Hecke–Clifford superalgebra

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Abstract
The Hecke–Clifford superalgebra is a super-analogue of the Iwahori–Hecke algebra of type $A$. The classification of its simple modules is done by Brundan, Kleshchev and Tsuchioka using a method of categorification of affine Lie algebras. In this paper, we introduce another way to produce its simple modules with a generalized theory of cellular algebras which is originally developed by Graham and Lehrer. In our construction the key is that there is a right action of the Clifford superalgebra on the super-analogue of the Specht module. With the help of the notion of the Morita context, a simple module of the Hecke–Clifford superalgebra is made from that of the Clifford superalgebra.

Keywords: Cellular algebras, the Iwahori–Hecke algebra, the Hecke–Clifford superalgebra

Introduction
The purpose of this paper is to classify the simple modules over the Hecke–Clifford superalgebra by use of an extended theory of cellular algebras. The original theory of cellular algebras is developed by Graham and Lehrer [13] as an axiomatization of various algebras arising as endomorphism algebra on natural representation of classical groups and quantum groups: the symmetric group algebra, the Brauer algebra, the partition algebra, the Iwahori–Hecke algebra, the Birman–Murakami–Wenzl algebra and so on so forth. First recall the notion of cellular algebra with more general one introduced by Du and Rui [11]. The definition below is based on that given by König and Xi [31], which is equivalent but slightly ring-theoretic than the original one. Let $(\Lambda, \preceq)$ be a partially ordered set. In this introduction we assume that the set $\Lambda$ is finite for simplicity.

Definition 0.1. Let $A$ be an algebra over a commutative ring $k$. $A$ is called a standardly based algebra on $\Lambda$ if it is equipped with a particular basis over $k$

$$\{a_{ij}^\lambda \in A | \lambda \in \Lambda, i \in I(\lambda), j \in J(\lambda)\}$$

parametrized by families of finite sets $I(\lambda)$ and $J(\lambda)$ for each $\lambda$ which satisfies the following properties.

1. For each $\lambda \in \Lambda$, the $k$-submodule $A^{\lambda} \subset A$ spanned by

$$\{a_{ij}^\mu \in A | \mu < \lambda, i \in I(\mu), j \in J(\mu)\}$$

is a 2-sided ideal of $A$. 
For each \( \lambda \in \Lambda \), there exist a left \( A \)-module \( M_\lambda = k\{m^i_\lambda \mid i \in I(\lambda)\} \) and a right \( A \)-module \( N_\lambda = k\{n^j_\lambda \mid j \in J(\lambda)\} \), which also have parametrized bases, such that
\[
M_\lambda \otimes_k N_\lambda \to A/A^{\lambda,\lambda},
\]
\[
m^i_\lambda \otimes n^j_\lambda \mapsto a^i^j_{\lambda}
\]
is a homomorphism between \((A, A)\)-bimodules.

\( A \) is also called a cellular algebra if \( I(\lambda) = J(\lambda) \) for all \( \lambda \) and the map \( a^i^j_{\lambda} \mapsto a^j^i_{\lambda} \) defines an anti-involution on the algebra \( A \).

We here do not pay much attention to anti-involutions, so standardly based algebras are fundamental for us. Intuitively the cell \( k\{a^i^j_{\lambda} \mid i \in I(\lambda), j \in J(\lambda)\} \) for each \( \lambda \in \Lambda \) is made to imitate the structure of matrix algebra, so that the modules \( M_\lambda \) and \( N_\lambda \) respectively correspond to column and row vector spaces. As a semisimple algebra decompose into a direct sum of matrix algebras, a cellular algebra has a filtration whose successive quotients are such cells.

One of the most striking result of the theory is the classification of simple modules performed as follows. First we can show that there is a canonical \( A \)-bilinear form \((\cdot, \cdot) : N_\lambda \times M_\lambda \to k\) between \( M_\lambda \) and \( N_\lambda \) for each \( \lambda \). Now suppose \( k \) is a field and let
\[
L_\lambda := M_\lambda / \{x \in M_\lambda \mid (y, x) = 0 \text{ for all } y \in N_\lambda\}
\]
for each \( \lambda \). Graham and Lehrer [13, Theorem 3.4] prove that an \( A \)-module \( L_\lambda \) is either zero or simple, and the set \( \{L_\lambda \mid \lambda \in \Lambda, L_\lambda \neq 0\} \) consists of pairwise distinct all simple \( A \)-modules. This is an analogue of the fact that each matrix component of a semisimple algebra produces its simple module.

However, this strategy does not work well in representation theory of superalgebras; there are no known non-trivial cellular superalgebras in the original definition. This is essentially because there is another kind of simple superalgebras in addition to matrix algebras, namely matrix algebras over the Clifford superalgebra. The key idea is that we allow a generalized cellular algebra to have such a new kind of cells.

The construction above of simple modules, though the developers of the theory might have not noticed, implicitly use the notion of Morita context which connect the two algebras \( A/A^{\lambda,\lambda} \) and \( k \).

**Definition 0.2.** A Morita context between algebras \( A \) and \( B \) is a pair of an \((A, B)\)-bimodule \( M \) and a \((B, A)\)-bimodule \( N \) equipped with bimodule homomorphisms \( \eta : M \otimes_B N \to A \) and \( \rho : N \otimes_A M \to B \) which satisfy the associativity laws
\[
\eta(x \otimes y) \cdot x' = x \cdot \rho(y \otimes x'), \quad \rho(y \otimes x) \cdot y' = y \cdot \eta(x \otimes y')
\]
for every \( x, x' \in M \) and \( y, y' \in N \).

This is a weaker version of the Morita equivalence [38] and studied in detail by Nicholson and Watters [41]. Morita’s original theorem says that if both \( \eta \) and \( \rho \) are surjective then we have a category equivalence between the module categories of these algebras, and every category equivalence is obtained in this form. For such data, we can prove the following statement
**Theorem 0.3.** Let $I \subset A$, $J \subset B$ be the images of $\eta$ and $\rho$ respectively. Let $\text{Irr}(A)$ be the set of isomorphism class of simple $A$-modules, and let $\text{Irr}'(A)$ be its subset consisting of simple modules $V$ such that $IV = V$. We similarly define $\text{Irr}'(B)$. For a $B$-module $W$, let $DW$ be the image of the $A$-homomorphism

$$M \otimes_B W \rightarrow \text{Hom}_B(N, W)$$

$$m \otimes w \mapsto (n \mapsto \rho(n \otimes m)w).$$

Then $W \mapsto DW$ induces a one-to-one correspondence $\text{Irr}'(A) \xrightarrow{1:1} \text{Irr}'(B)$.

We will prove it in Theorem 3.15 in a more general setting: we also treat a Morita context between two abelian categories instead of that between two algebras $A$ and $B$, so that it is redefined as that between their module categories $A$-$\text{Mod}$ and $B$-$\text{Mod}$. We do this process for two reasons. First since our purpose is a classification of simple objects in the module category, it is more essential to deal directly with the module category $A$-$\text{Mod}$ rather than the algebra $A$ itself. Second we expect that our strategy works in more general settings outside representation theory of algebras.

Anyway, note that for a standardly based algebra, for each $\lambda \in \Lambda$ the embedding $M_\lambda \otimes N_\lambda \hookrightarrow A/A^{-\lambda}$ and the bilinear form $N_\lambda \otimes_A M_\lambda \rightarrow k$ make pair $(M_\lambda, N_\lambda)$ into a Morita context between the algebras $A/A^{-\lambda}$ and $k$, and $L_\lambda$ above is just $Dk$ where $k$ is viewed as a trivial $k$-module. The classification of simple modules of a cellular algebra is a consequence of this theorem.

For a general Morita context we do not need that one algebra is a base ring $k$. Hence by replacing $k$ with a more general one, such as the Clifford superalgebra, we can define generalized cellular algebras in order to obtain a similar method of classification which we can apply to more various things. In this paper we introduce the notion of *standardly filtered* algebra over a family of algebras $\{B_\lambda\}_{\lambda \in \Lambda}$; see Definition 4.1. A standardly filtered algebra $A$ also consists of a Morita context $(M_\lambda, N_\lambda)$ for each $\lambda \in \Lambda$, between quotient algebras $A/A^{-\lambda}$ of $A$ and $B_\lambda/B'_\lambda$ of $B_\lambda$. Let $B''_\lambda/B'_\lambda \subset B_\lambda/B'_\lambda$ the image of the Morita context map $N_\lambda \otimes_A M_\lambda \rightarrow B_\lambda/B'_\lambda$, and write $\text{Irr}'(B_\lambda) := \text{Irr}'(B''_\lambda/B'_\lambda)$. These data induce the following classification which generalizes [15, Theorem 3.4].

**Theorem 0.4.** The Morita contexts induce a one-to-one correspondence

$$\text{Irr}(A) \xrightarrow{1:1} \bigsqcup_{\lambda \in \Lambda} \text{Irr}'(B_\lambda).$$

In the classical case, each $B_\lambda$ is taken to be a base field $k$ so that $\text{Irr}'(B_\lambda)$ is either $\{k\}$ or $\varnothing$. Thus in this case $\text{Irr}(A)$ is in bijection with some subset of $\Lambda$. If so, we simply say that $A$ is a standardly filtered algebra over $k$ on the set $\Lambda$, similarly as before. In any case the classification of simple modules of $A$ can be reduced to those of $B_\lambda$’s via this correspondence. In this paper we also introduce the notion of generalized standardly based algebra and that of generalized cellular algebra over the family $\{B_\lambda\}$, not over the single base ring $k$. It seems better to list several examples rather than to introduce its detailed definition. In many cases a standardly filtered algebra is produced from a category as follows.

**Lemma 0.5.** Let $\mathcal{A}$ be a $k$-linear (super)category. For each $\lambda \in \Lambda$, let us take an object $X_\lambda \in \mathcal{A}$ and a subalgebra $B_\lambda \subset \text{End}_\mathcal{A}(X_\lambda)$. Let $\mathcal{A'} \subset \mathcal{A}$ be a 2-sided ideal of $\mathcal{A}$ generated by $X_\lambda$; that is,

$$\mathcal{A'}(X, Y) := \text{Hom}_\mathcal{A}(X_\lambda, Y) \circ \text{Hom}_\mathcal{A}(X, X_\lambda) \subset \text{Hom}_\mathcal{A}(X, Y)$$

and $\text{Irr}'(\mathcal{A'})$ is in bijection with some subset of $\text{Irr}'(\mathcal{A})$. Then $\mathcal{A'}$ is a standardly filtered algebra of $\mathcal{A}$.

This is just a special case of the above theorem.
where $\lambda \leq \mu$. Then for every $\omega \in \mathcal{B}$, $A$ is a standardly filtered algebra over $\mathcal{B}$. Here, for each pair of objects $X, Y \in \mathcal{A}$, we have

$$\text{End}(X, Y) = \mathcal{B} + \sum_{\omega} \mathcal{B}(X, Y)$$

$$M_i = \text{End}_{\mathcal{A}}(X, Y)$$

$$N_i = \text{End}_{\mathcal{A}}(X, Y)$$

As we listed in the examples above, the Iwahori-Hecke algebra $H_n(q)$ is one of the most important examples of a cellular algebra whose cellular basis is given by Kazhdan and Lusztig's canonical basis $[26]$ or Murphy's basis $[39, 40]$. It first comes from a study of flag varieties important example of a cellular algebra whose cellular basis is given by Kazhdan and Lusztig's.

Example 0.8. The Temperley-Lieb algebra $\mathcal{T}(V)$ is standardly based over $k$. We can prove similar results for the orthogonal group $O_n$ and for $k$ the general linear group $GL_n$ respectively.

Example 0.9. The partition algebra $P(n)$ is standardly based over $k$. We can prove similar results for the symmetric group $S_n$ and for $k$ the general linear group $GL_n$. A canonical basis for $k$ and $GL_n$ is given by Kazhdan and Lusztig’s. It first comes from a study of flag varieties.

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Note the natural inclusion $\mathcal{B} \subset \mathcal{A}$ and the index set can be restricted to partitions of $k$.

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over the finite fields, and also appears as an endomorphism algebra of a certain representation
of the quantum general linear group via an analogue of the Schur–Weyl duality, then considered
as a q-analogue of the symmetric group algebra. Now suppose \( k \) is a field and \( q \in \k \) be a non-
zero element. When \( q \) is not a root of unity, its representations are very similar to those of the
symmetric group in characteristic zero, and a concrete construction of the simple modules called
Young’s seminormal form is given by Hoefsmit [16]. For modular representations \( q = \sqrt{\k} \), its
simple modules are studied by Dipper and James [8, 9] in a cellular way. In addition to cellular
representation theory there is a beautiful approach on the classification of simple modules made
by Lascoux, Leclerc, Thibon [33], Ariki [1], Grojnowski [14], Brundan [3], Kleshchev [28]
and others, called the categorification. Based on their works it is proven that the union set
\( \bigsqcup_{n \in \mathbb{N}} \text{Irr}(H_n) \) of simple modules of \( H_n \) for all \( n \in \mathbb{N} \) has a structure of Kashiwara crystal [25]
over the quantum affine algebra \( U_q(\mathfrak{s}\mathfrak{l}_n) \) of type \( A^{(1)}_{n-1} \), and is isomorphic to the crystal basis \( B(\Lambda_0) \)
of the irreducible representation \( V(\Lambda_0) \) whose highest weight is the fundamental weight \( \Lambda_0 \) (Theorem 14.2, 14.3) for \( \lambda = \Lambda_0 \)). One can obtain each simple module by applying Kashiwara
operators \( \tilde{f}_i \) on the trivial module of \( H_0 = k \). However this construction is too abstract and hard
to compute in practical use. Compared with this Lie theory, the cellular theory has advantages
that we can construct simple modules in a concrete way, and that we can apply it even when \( k \) is
a more general commutative ring: we only require that \( q \in \k \) is invertible.

**Theorem 0.10.** Suppose \( q \in \k \) is invertible. Then there is a one-to-one correspondence
\[
\text{Irr}(H_n) \leftrightarrow \bigcup_{\lambda: \text{partition}} \text{Irr}^{k\lambda}(\k).
\]
Here \( f_3 \equiv [\lambda_1 - \lambda_3][\lambda_2 - \lambda_3] \cdots [\lambda_r]! \in \k \) for each partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) where \([k]!\)
denotes the \( q \)-factorial.

Our main target in this paper is the Hecke–Clifford superalgebra \( H_n^c = H_n^c(a; q) \) for \( a, q \in \k \),
which is a super version of the Iwahori–Hecke algebra. It is introduced by Olshanski [42] as a
partner of the quantum Queer superalgebra via the Schur–Weyl duality and is considered as a
\( q \)-analogue of the wreath product \( W_n = C_n \rtimes \mathfrak{S}_n \) of the Clifford superalgebra, which is called the
Sergeev superalgebra. It is known that the spin representation theory of the symmetric group \( \mathfrak{S}_n \)
is controlled by \( W_n \); see [5] or [29]. Young’s seminormal form of \( H^c_a \) for characteristic zero case
is independently founded by Hill, Kujawa and Sussan [15] and Wan [45], and the categorification
method for \( q = \sqrt{\k} \) is developed by Brundan and Kleshchev [4] for odd \( e \) and by Tsuchioka [44]
for even \( e \) using the quantum affine algebra of type \( A^{(2)}_{n-1} \) and of type \( D^{(2)}_{n-1} \) respectively. Hence
this paper fills the missing one: the cellular representation theory of \( H^c_a \). In our cellular method
the classification can be done in a very weak assumption same as the case of the Iwahori–Hecke
algebra above. This is our main theorem.

**Theorem 0.11.** Suppose \( q \in \k \) is invertible. Then there is a one-to-one correspondence
\[
\text{Irr}(H_n^c) \leftrightarrow \bigcup_{\lambda: \text{partition}} \text{Irr}^{\Lambda_0+\Theta_\lambda}(\Gamma_\lambda).
\]
Here \( \Gamma_\lambda \) is the Clifford superalgebra defined on the quadratic form with respect to the scalars
\( a[l_1] \equiv a[l_2] \equiv \cdots \equiv a[l_r] \) where \([k]\) denotes the \( q^2 \)-integer, and \( \Delta_\lambda, \Theta_\lambda \subset \Gamma_\lambda \) are its 2-sided ideals
defined in Section 6.
This paper is organized as follows. There are roughly two parts. First we extend the theory of cellular algebras so that we can apply it to our target, the Hecke–Clifford superalgebra. We start by reviewing the enriched category theory in Section 1 in order to treat representation category of superalgebra. Next in Section 2 and 3 we define Morita contexts between two categories, and develop the theory of classification of simple objects. We then in Section 4 introduce the notion of standardly filtered algebra by use of Morita contexts.

Using these results we apply our generalized cellular theory to the two algebras. In Section 5 we reconstruct the cellular representation theory of the Iwahori–Hecke algebra from a generalized viewpoint to make it suitable for our purpose. Finally in Section 6 we proceed to the study of the cellular representation theory of the Hecke–Clifford superalgebra and completes the classification of its simple modules.

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1. Basics on enriched categories

Throughout in this paper, we fix a commutative ring $k$. Tensor products over $k$ are simply denoted by $\otimes$. We denote by $k^\times$ the set of invertible elements in $k$. We here recall the basic notions of enriched categories in a special case. For details we refer the reader to the textbook [27].

1.1. Enriched categories

We denote by $\mathcal{M}$ the symmetric tensor category of $k$-modules, by $\mathcal{S}$ that of $k$-supermodules and by $\mathcal{G}$ that of graded $k$-modules. So they consist of $k$-modules $V = \bigoplus_{i \in I} V_i$ graded by the abelian group $I = \{1\}, \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}$ respectively, and $k$-homomorphisms which respect these gradings. When we take an element $x \in V$, we always assume that $x$ is a homogeneous element. For such $x$, we denote by $|x| \in I$ the degree of $x$. The symmetries on them are defined as

$$V \otimes W \to W \otimes V$$

$$x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$$

using the Koszul sign $(-1)^{|x||y|}$. If you want to use the naïve symmetry in the graded case, you should concentrate on evenly graded spaces for convention (actually we can take an arbitrary abelian group $I$ with a homomorphism $I \to \{\pm 1\}$).

Now let $\mathcal{V}$ be one of $\mathcal{M}, \mathcal{S}$ or $\mathcal{G}$. In each case, a $\mathcal{V}$-category is called a $k$-linear category, a $k$-linear supercategory or a $k$-linear graded category. Shortly, a $\mathcal{V}$-category $\mathcal{C}$ consists of a hom object $\text{Hom}_\mathcal{C}(X, Y) \in \mathcal{V}$ for each pair of objects $X, Y \in \mathcal{C}$ instead of an ordinary hom set. By taking the degree-zero part $\text{Hom}_\mathcal{C}(X, Y) = \text{Hom}_\mathcal{C}(X, Y)_0$ we obtain the underlying $\mathcal{M}$-category $\mathcal{C}_0$. When we write $f : X \to Y$ we mean that $f$ is a homogeneous element of $\text{Hom}_\mathcal{C}(X, Y)$. For $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$, the tensor product $\mathcal{V}$-category $\mathcal{C} \boxtimes \mathcal{D}$ and the opposite $\mathcal{V}$-category $\mathcal{C}^{op}$ are
defined through the symmetry on \( \mathcal{V} \). Their morphisms are in the form \( f \boxtimes g \) and \( f^\circ \) respectively and the compositions are given by

\[
(f_1 \boxtimes g_1) \circ (f_2 \boxtimes g_2) := (-1)^{|\text{id}|_f} (f_1 \circ f_2) \boxtimes (g_1 \circ g_2),
\]

\[
f_1^\circ \circ f_2^\circ := (-1)^{|\text{id}|_f} (f_2 \circ f_1)^\circ.
\]

Similarly a \( \mathcal{V} \)-functor \( F : C \to D \) is a collection of a degree preserving homomorphism \( \text{Hom}_C(X,Y) \to \text{Hom}_D(FX, FY) \) for each pair of \( X,Y \in C \). By taking its degree-zero part we obtain its underlying usual functor \( F_0 : C_0 \to D_0 \). A \( \mathcal{V} \)-natural transformation \( F \to G \) is defined as a usual natural transformation between the underlying functors \( F_0 \to G_0 \) which satisfies an additional condition.

We denote by \( \mathcal{H} \text{om}(C,D)_0 \) the usual category consisting of \( \mathcal{V} \)-functors and \( \mathcal{V} \)-natural transformations between them. The set (or the class) of natural transformations between \( \mathcal{V} \)-functors \( F, G : C \to D \) is denoted by \( \text{Hom}_{C,D}(F,G)_0 \) for short. We can also complete this category to a \( \mathcal{V} \)-category \( \mathcal{H} \text{om}(C,D) \) (except that it may not be locally small) by letting its hom object \( \text{Hom}_{C,D}(F,G) \) as a equalizer of the parallel morphisms

\[
\prod_{X \in C} \text{Hom}_C(FX, GX) \rightrightarrows \prod_{Y \in C} \text{Hom}_D(Y, \text{Hom}_C(Y,Z), \text{Hom}_D(FY, GZ)).
\]

We represent its homogeneous element as \( h : F \to G \). The reader should pay attention to that its naturality means

\[
(hY) \circ (Ff) = (-1)^{|\text{id}|_f} (Gf) \circ (hX)
\]

for every \( f : X \to Y \). Note that a \( \mathcal{V} \)-algebra \( A \) (a monoid object in \( \mathcal{V} \)) is nothing but a \( \mathcal{V} \)-category \( C \) with a single object \( * \in C \) such that \( A \cong \text{End}_C(*) \). Then the category of left \( A \)-modules, which we denote by \( A\text{-Mod} \), is just the functor category \( \mathcal{H} \text{om}(C,\mathcal{V}) \). The tensor product \( \mathcal{V} \)-algebra \( A \otimes B \) and the opposite \( \mathcal{V} \)-algebra \( A^\circ \) are special cases of the operations for \( \mathcal{V} \)-categories above. We also denote by \( \text{Mod}-A = A^\circ \text{-Mod} \) the category of right \( A \)-modules.

### 1.2. Alternative definitions

Now for a while consider the super case \( \mathcal{V} = S \). \( S \) has the parity change functor \( \Pi : S \to S \) which exchanges the grading

\[
(\Pi IV)_0 := V_1, \quad (\Pi IV)_1 := V_0
\]

for supermodule \( V = V_0 \oplus V_1 \). In a general \( S \)-category \( C \), for an object \( Y \in C \) its parity change \( \Pi Y \in C \), if exists, is defined as a representation of the \( S \)-functor

\[
\text{Hom}_C(X, \Pi Y) \cong \Pi \text{Hom}_C(X, Y).
\]

If every object in \( C \) has its parity change, \( \Pi \) can be defined as an \( S \)-functor \( \Pi : C \to C \) and we also have an \( S \)-natural isomorphism

\[
\Pi \text{Hom}_C(X, Y) \cong \text{Hom}_C(\Pi X, Y).
\]

We say that such \( C \) is \( \Pi \)-closed. Instead of to treat the theory of enriched categories directly, we can view a \( \Pi \)-closed \( S \)-category as a usual category with additional informations as follows.
Lemma 1.1. Giving a $\Pi$-closed $\mathcal{S}$-category $\mathcal{C}$ is equivalent to giving an $\mathcal{M}$-category $\mathcal{C}_0$ with an $\mathcal{M}$-functor $\Pi : \mathcal{C}_0 \rightarrow \mathcal{C}_0$ and an isomorphism $\xi : \Pi^2 \simeq \text{Id}_{\mathcal{C}_0}$ such that $\xi \Pi = \Pi \xi$ as $\mathcal{S}$-natural isomorphisms $\Pi^3 \rightarrow \Pi$.

Proof. Clearly a $\Pi$-closed $\mathcal{S}$-category induces such a datum. Conversely let $\mathcal{C}_0$ be an $\mathcal{M}$-category equipped with $\Pi : \mathcal{C}_0 \rightarrow \mathcal{C}_0$ and $\xi : \Pi^2 \simeq \text{Id}_{\mathcal{C}_0}$. For each $X, Y \in \mathcal{C}_0$, let $\text{Hom}_C(X, Y)$ be a supermodule defined by

$$\text{Hom}_C(X, Y)_0 := \text{Hom}_{\mathcal{C}_0}(X, Y), \quad \text{Hom}_C(X, Y)_1 := \text{Hom}_{\mathcal{C}_0}(X, \Pi Y).$$

Then we can define the composition $\text{Hom}_C(Y, Z) \otimes \text{Hom}_C(X, Y) \rightarrow \text{Hom}_C(X, Z)$ by

$$\text{Hom}_{\mathcal{C}_0}(Y, Z) \otimes \text{Hom}_{\mathcal{C}_0}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}_0}(X, Z),$$
$$\text{Hom}_{\mathcal{C}_0}(Y, \Pi Z) \otimes \text{Hom}_{\mathcal{C}_0}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}_0}(X, \Pi Z),$$
$$\text{Hom}_{\mathcal{C}_0}(Y, Z) \otimes \text{Hom}_{\mathcal{C}_0}(X, \Pi Y) \simeq \text{Hom}_{\mathcal{C}_0}(Y, \Pi Z) \otimes \text{Hom}_{\mathcal{C}_0}(X, \Pi Y)$$
$$\rightarrow \text{Hom}_{\mathcal{C}_0}(X, \Pi Z),$$
$$\text{Hom}_{\mathcal{C}_0}(Y, \Pi Z) \otimes \text{Hom}_{\mathcal{C}_0}(X, \Pi Y) \simeq \text{Hom}_{\mathcal{C}_0}(Y, \Pi^2 Z) \otimes \text{Hom}_{\mathcal{C}_0}(X, \Pi Y)$$
$$\rightarrow \text{Hom}_{\mathcal{C}_0}(X, \Pi Z).$$

The condition $\xi \Pi = \Pi \xi$ is needed for that composition of three odd morphisms is associative.

Lemma 1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be $\Pi$-closed $\mathcal{S}$-categories. Then giving an $\mathcal{S}$-functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is equivalent to giving an $\mathcal{M}$-functor $F_0 : \mathcal{C}_0 \rightarrow \mathcal{D}_0$ between their underlying $\mathcal{M}$-categories equipped with an isomorphism $\alpha : F_0 \Pi \simeq \Pi F_0$ which makes the diagram below commutes:

$$\begin{array}{ccc}
F_0 \Pi^2 & \xrightarrow{\alpha \Pi} & \Pi F_0 \Pi \\
\downarrow{F_0 \xi} & & \downarrow{\xi F_0} \\
F_0 \Pi & \xrightarrow{\Pi \alpha} & \Pi^2 F_0
\end{array}$$

Proof. The one direction is clear. So let $F_0 : \mathcal{C}_0 \rightarrow \mathcal{D}_0$ be an $\mathcal{M}$-functor equipped with an isomorphism $\alpha : F_0 \Pi \simeq \Pi F_0$. On objects simply let $FX := F_0X$. Then we can define the degree preserving map $F : \text{Hom}_C(X, Y) \rightarrow \text{Hom}_{\mathcal{C}_0}(FX, FY)$ by

$$\text{Hom}_{\mathcal{C}_0}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}_0}(FX, FY),$$

$$\text{Hom}_{\mathcal{C}_0}(X, \Pi Y) \rightarrow \text{Hom}_{\mathcal{C}_0}(FX, F\Pi Y) \simeq \text{Hom}_{\mathcal{C}_0}(FX, \Pi FY).$$

The commutativity of the diagram above is used to ensure that $F$ preserves composition of two odd morphisms.

Lemma 1.3. Let $F$ and $G$ be $\mathcal{S}$-functors $\mathcal{C} \rightarrow \mathcal{D}$ between $\Pi$-closed $\mathcal{S}$-categories. Then a natural transformation $h : F_0 \rightarrow G_0$ is $\mathcal{S}$-natural if and only if the square

$$\begin{array}{ccc}
F_0 \Pi & \xrightarrow{h \Pi} & G_0 \Pi \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
\Pi F_0 & \xrightarrow{\Pi h} & \Pi G_0
\end{array}$$

commutes.
Proof. The S-naturality just says that the two parallel maps $\text{Hom}_C(X, Y) \Rightarrow \text{Hom}_C(FX, GY)$, which are induced by $h$, coincide. By the usual naturality it is satisfied for the even parts. The condition above is equivalent to that it also holds for the odd parts.

Remark 1.4. This characterization of $\Pi$-closed $S$-category is appeared in [24, 22, 23] as their definition of supercategory. In [23], our $S$-category is called a 1-supercategory.

Next consider the graded case $V = G$. Analogously we have the $k$-th degree shift functor $\Sigma^k$ defined by $(\Sigma^k V)_i := V_{i+k}$ for $V \in G$. We say that a $G$-category is $\Sigma$-closed if each $Y \in C$ has its degree shift $\Sigma^k Y$ defined by

$$\Sigma^k \text{Hom}_C(X, Y) = \text{Hom}_C(X, \Sigma^k Y).$$

$\Sigma$-closed $G$-category can be also characterized as follows. The proofs are similar as before so we omit them.

**Lemma 1.5.** Giving a $\Sigma$-closed $G$-category $C$ is equivalent to giving a $M$-category $C_0$ with a $M$-functor $\Sigma : C_0 \to C_0$ which is an equivalence.

**Lemma 1.6.** Let $C$ and $D$ be $\Sigma$-closed $G$-categories. Then giving a $G$-functor $F : C \to D$ is equivalent to giving a $M$-functor $F_0 : C_0 \to D_0$ equipped with an isomorphism $\alpha : F_0 \Sigma \cong \Sigma F_0$.

**Lemma 1.7.** Let $F$ and $G$ be $G$-functors $C \to D$ between $\Sigma$-closed $G$-categories. Then a natural transformation $f : F_0 \to G_0$ is $G$-natural if and only if the square

\[
\begin{array}{ccc}
F_0 \Sigma & \xrightarrow{\Sigma f} & G_0 \Sigma \\
\downarrow \alpha & & \downarrow \alpha \\
\Sigma F_0 & \xrightarrow{\Sigma f} & \Sigma G_0
\end{array}
\]

commutes.

1.3. Limits in enriched category

Let $C$ be a $\mathcal{V}$-category. We say a usual functor $I \to C_0$ from a small category $I$ to the underlying category of $C$ a diagram in $C$. So a diagram consists of $Y_i \in C$ for each $i \in I$ and a degree-zero morphism $Y_i \to Y_j$ for each arrow $i \to j$ in $I$. For a diagram $\{Y_i\}_{i \in I}$, its (conical) $\mathcal{V}$-limit is an object $\lim_{\leftarrow I} Y_i \in C$ equipped with a family of degree-zero morphisms $\lim_{\leftarrow I} Y_i \to Y_i$ for each $i$ which satisfies the $\mathcal{V}$-natural isomorphism

$$\text{Hom}_{C_0}(X, \lim_{\leftarrow I} Y_i) \cong \lim_{\leftarrow I} \text{Hom}_C(X, Y_i).$$

Note that the usual limit only implies

$$\text{Hom}_{C_0}(X, \lim_{\leftarrow I} Y_i) \Rightarrow \lim_{\leftarrow I} \text{Hom}_C(X, Y_i).$$

Since taking the degree-zero part $\mathcal{V} \to M; V \mapsto V_0$ preserves limits, the $\mathcal{V}$-limit of an diagram is also its usual limit. The converse does not hold in general but there are no differences between them in a suitable condition.
Lemma 1.8. Suppose $C$ is an $M$-category (resp. a $\Pi$-closed $S$-category, a $\Sigma$-closed $G$-category). Then for any diagram its limit in $C_0$ is automatically its $\mathcal{V}$-limit in $C$.

Proof. It is trivial for the $\mathcal{V} = M$ case. When $\mathcal{V} = S$, we have

$$
\text{Hom}_C(X, \lim_i Y_i) = \lim_i \text{Hom}_{C_0}(X, Y_i) \oplus \text{Hom}_{C_0}(\Pi X, \lim_i Y_i) = \lim_i \text{Hom}_{C_0}(X, Y_i) \oplus \lim_i \text{Hom}_{C_0}(\Pi X, Y_i) = \lim_i \text{Hom}_{C}(X, Y_i).
$$

The similar proof works for $\mathcal{V} = G$ since a limit commutes with direct sums in a Grothendieck category $G$.

The dual notion (conical) $\mathcal{V}$-colimit of a diagram is introduced similarly. Next we introduce the notion of abelian $\mathcal{V}$-category.

Definition 1.9. We say that a $S$-category (resp. a $G$-category) $C$ is abelian if it is $\Pi$-closed (resp. $\Sigma$-closed) and its underlying $M$-category $C_0$ is abelian in an ordinary sense.

By the lemma above, in an abelian $\mathcal{V}$-category we can use several categorical notions such as zero object, direct sum, kernel, cokernel, monomorphism, epimorphism and exactness defined via enriched Hom functors without any modifications. We are also allowed to operate homological computations as follows.

Lemma 1.10. Let $C$ be an abelian $\mathcal{V}$-category. Then $P \in C$ is projective in $C_0$ if and only if the functor $\text{Hom}_C(P, \bullet): C \to \mathcal{V}$ is exact.

Proof. The “if” part follows from that taking the degree-zero part $\mathcal{V} \mapsto \mathcal{V}_0$ is exact. The “only if” part can be proven similarly as the lemma above.

We here study limits and colimits in a functor category. Let $\{F_i: C \to D\}$ be a diagram in $\text{Hom}(C, D)$. If the $\mathcal{V}$-limit $\lim_i F_i X$ exists for each $X \in C$, then the $\mathcal{V}$-limit $\lim_i F_i$ also exists and is defined as

$$
(\lim_i F_i)X := \lim_i F_i X.
$$

Dually $\mathcal{V}$-colimits of this diagram is also computed value-wise. We remark that the composition of $\mathcal{V}$-functors

$$
\text{Hom}(\mathcal{D}, \mathcal{E}) \boxtimes \text{Hom}(C, \mathcal{D}) \to \text{Hom}(C, \mathcal{E}),
$$

$$
F \boxtimes G \mapsto FG
$$

is also defined as a $\mathcal{V}$-functor. In particular, if $\mathcal{D}$ is abelian then so is $\text{Hom}(C, \mathcal{D})$. By definition, the right multiplication

$$
\bullet G: \text{Hom}(\mathcal{D}, \mathcal{E}) \to \text{Hom}(C, \mathcal{E})
$$

is both $\mathcal{V}$-continuous (i.e. preserves $\mathcal{V}$-limits) and $\mathcal{V}$-cocontinuous (i.e. does $\mathcal{V}$-colimits). In contrast, the left multiplication

$$
F \bullet: \text{Hom}(C, \mathcal{D}) \to \text{Hom}(C, \mathcal{E})
$$

preserves certain \( \mathcal{V} \)-limits or \( \mathcal{V} \)-colimits for all \( C \) if and only if \( F \) does.

Suppose that \( \mathcal{V} \)-categories \( C \) and \( D \) are both abelian, and \( C \) has enough projectives. Then for each \( F: C \to D \), its \( i \)-th left derived \( \mathcal{V} \)-functor \( L_i F: C \to D \) makes sense as usual. \( L_i \) can be viewed as a \( \mathcal{V} \)-endofunctor on \( \mathcal{V} \text{Hom}(C, D) \), which is also abelian, and from a short exact sequence 

\[
0 \to F \to G \to H \to 0
\]

of \( \mathcal{V} \)-functors they yield a long exact sequence

\[
\cdots \to L_2 H \to L_1 F \to L_1 G \to L_1 H \to L_0 F \to L_0 G \to L_0 H \to 0.
\]

Dually, when \( C \) has enough injectives we define the \( i \)-th right derivation \( R_i \).

We are most interested in the zeroth derivation \( L_0 \). By definition there is a canonical \( \mathcal{V} \)-natural transformation \( L_0 F \to F \). \( L_0 F: C \to D \) is right exact and of course \( L_0 F \cong F \) when \( F \) is already right exact. For another right exact \( \mathcal{V} \)-functor \( G: C \to D \), the map

\[
\text{Hom}_{C, D}(G, L_0 F) \to \text{Hom}_{C, D}(G, F)
\]
is an isomorphism since its inverse map is given by

\[
\text{Hom}_{C, D}(G, F) \to \text{Hom}_{C, D}(L_0 G, L_0 F) \cong \text{Hom}_{C, D}(G, L_0 F).
\]

Thus \( L_0 F \) is considered as the most applicable right exact approximation of \( F \).

1.4. Adjunctions

An adjunction from \( C \) to \( D \) is a pair of adjoint \( \mathcal{V} \)-functors \( F: C \to D \) and \( F^\vee: D \to C \) where \( F \) is left adjoint to \( F^\vee \). That is, it is called so if there is a \( \mathcal{V} \)-natural isomorphism

\[
\text{Hom}_{D}(FX, Y) \cong \text{Hom}_{C}(X, F^\vee Y).
\]

Then \( F \) is \( \mathcal{V} \)-cocontinuous and \( F^\vee \) is \( \mathcal{V} \)-continuous. It is also characterized by degree-zero natural transformations \( \delta: \text{Id}_C \to F^\vee F \) and \( \epsilon: FF^\vee \to \text{Id}_D \) which satisfy the zig-zag identities

\[
\text{id}_F = (F \xrightarrow{F\delta} FF^\vee F \xrightarrow{id_{FF^\vee}} F), \quad \text{id}_{F^\vee} = (F^\vee \xrightarrow{\delta F^\vee} F^\vee FF^\vee F^\vee \xrightarrow{id_{FF^\vee}} F^\vee).
\]

\( \delta \) and \( \epsilon \) are called the unit and the counit of the adjunction respectively. For a \( \mathcal{V} \)-functor \( F: C \to D \), the rest datum \( (F^\vee, \delta, \epsilon) \) which makes an adjunction is uniquely determined up to unique isomorphism if it exists. In order to keep notations simple we say that \( \mathcal{F} \) is an adjunction from \( C \) to \( D \) \( \mathcal{V} \)-functors between \( \mathcal{V} \) categories \( C \) and \( D \) has a fixed right adjoint \( \mathcal{V} \)-functor \( F^\vee \). We denote by \( \mathcal{A} \text{dj}(C, D) \) the full subcategory of \( \mathcal{V} \text{Hom}(C, D) \) consisting of adjunctions.

For an adjunction \( F: C \to D \), the left multiplication \( F \bullet \) is left adjoint to \( F^\vee \bullet \) while the right multiplication \( \bullet F \) is right adjoint to \( \bullet F^\vee \). Thus for two parallel adjunction \( F, G: C \to D \), we have a canonical isomorphism

\[
\text{Hom}_{C, D}(F, G) \cong \text{Hom}_{D, C}(G^\vee, F^\vee).
\]

We here list few examples of adjunction category.

**Example 1.11.** Let \( A \) and \( B \) be \( \mathcal{V} \)-algebras. For an \((A, B)\) bimodule \( M \), the \( \mathcal{V} \)-functors between their module categories

\[
F: B\text{-Mod} \to A\text{-Mod}, \quad F^\vee: A\text{-Mod} \to B\text{-Mod},
\]

\[
W \mapsto M \otimes_B W, \quad V \mapsto \text{Hom}_A(M, V)
\]
form an adjunction from $B$-$\text{Mod}$ to $A$-$\text{Mod}$. Conversely let $F: B$-$\text{Mod} \to A$-$\text{Mod}$ be an arbitrary adjunction. When we put $M := FB$ the multiplication on $B$ from right makes $M$ a right $B$-module. We have

$$F^\vee V \cong \text{Hom}_B(B, F^\vee V) \cong \text{Hom}_A(FB, V) = \text{Hom}_A(M, V)$$

so every adjunction between the module categories can be obtained in this way. In addition, $\text{Adj}(B$-$\text{Mod}, A$-$\text{Mod})$ is equivalent to $A$-$\text{Mod}$-$B$, the category of $(A, B)$-bimodules. In particular it is abelian and the embedding to $\text{Hom}(B$-$\text{Mod}, A$-$\text{Mod})$ is right exact, but not left exact in general.

Example 1.12. Suppose $k$ is a field, and for $A$ and $B$ above let $A$-$\text{Mod}^l$, $B$-$\text{Mod}^l$ be the categories of their finite dimensional left modules. Suppose that $N$ is a $(B, A)$-bimodule which satisfies these finiteness conditions:

1. if a right $A$-module $V$ is finite dimensional then so is $\text{Hom}_A(V, N)$,
2. if a left $B$-module $W$ is finite dimensional then so is $\text{Hom}_B(W, N)$,
3. $N$ is locally finite dimensional, that is, it is the union of its finite dimensional $(B, A)$-submodules.

We denote by $V^\vee := \text{Hom}_V(V, k)$ the dual space of a finite dimensional vector space. Then the functors

$$F: B$-$\text{Mod}^l \to A$-$\text{Mod}^l, \quad W \mapsto \text{Hom}_B(W, N)^\vee,$$

$$F^\vee: A$-$\text{Mod}^l \to B$-$\text{Mod}^l, \quad V \mapsto \text{Hom}_A(V^\vee, N)$$

form an adjunction via the natural isomorphism

$$\text{Hom}_A(\text{Hom}_B(W, N)^\vee, V) \cong \text{Hom}_A(V^\vee, \text{Hom}_B(W, N))$$

$$\cong \text{Hom}_B(W, \text{Hom}_A(V^\vee, N)).$$

In this case $N$ is recovered from $F$ by the formula $N \cong \lim_{B \to B'} (FB')^\vee$ where $B'$ runs over all finite dimensional quotient algebras of $B$. Every adjunctions are obtained in this way and $\text{Adj}(B$-$\text{Mod}^l, A$-$\text{Mod}^l)$ is equivalent to the opposite of the category of such $(B, A)$-bimodules. This category does not necessarily have kernels.

Example 1.13. For a small $\mathcal{V}$-category $\mathcal{A}$, we call a $\mathcal{V}$-functor $\mathcal{A} \to \mathcal{V}$ "a left $\mathcal{A}$-module", and denote by $\mathcal{A}$-$\text{Mod} := \text{Hom}(\mathcal{A}, \mathcal{V})$. Similarly, for another $\mathcal{V}$-category $\mathcal{B}$, a right $\mathcal{B}$-module (resp. an $(\mathcal{A}, \mathcal{B})$-bimodule) is just a $\mathcal{V}$-functor $\mathcal{B}^{op} \to \mathcal{V}$ (resp. $\mathcal{B}^{op} \boxtimes \mathcal{A} \to \mathcal{V}$). Since a $\mathcal{Z}$-linear category is a "ring with several objects" as Mitchell [36] noticed, this terminology is a generalization for usual algebras.

For an $(\mathcal{A}, \mathcal{B})$-bimodule $M$ and an $(\mathcal{A}, \mathcal{C})$-bimodule $N$, we can form an $(\mathcal{B}, \mathcal{C})$-bimodule $\text{Hom}_{\mathcal{A}}(M, N)$ defined as

$$C^{op} \boxtimes \mathcal{B} \to \mathcal{V},$$

$$Z \boxtimes Y \mapsto \text{Hom}_{\mathcal{A}, \mathcal{C}}(M(Y, \bullet), N(Z, \bullet)).$$

On the other hand, if $P$ is an $(\mathcal{B}, \mathcal{C})$-bimodule, there is an $(\mathcal{A}, \mathcal{C})$-bimodule denoted by $M \otimes_{\mathcal{B}} P$, which sends $Z \otimes X \in C^{op} \boxtimes \mathcal{A}$ to the coequalizer of the parallel maps

$$\bigoplus_{Y', Y'' \in \mathcal{B}} M(Y', X) \otimes \text{Hom}_{\mathcal{B}}(Y'', Y') \otimes P(Z, Y'') \rightrightarrows \bigoplus_{Y \in \mathcal{B}} M(Y, X) \otimes P(Z, Y).$$
Similarly as above, every adjunction $F : \mathcal{B}\text{-Mod} \to \mathcal{A}\text{-Mod}$ is represented by some $(\mathcal{A}, \mathcal{B})$-bimodule $M$ using $\otimes$ and Hom. The identity functor on $\mathcal{A}\text{-Mod}$ corresponds to the $(\mathcal{A}, \mathcal{A})$-bimodule $\text{Hom}_\mathcal{A}(\bullet, \bullet)$.

1.5. The category of adjunctions

First of all, we make sure a well-known fact that adjunctions are closed under colimits, especially cokernels. Let $\{F_i\}$ be a diagram in $\mathcal{A}\text{adj}(\mathcal{C}, \mathcal{D})$. For each arrow $i \to j$ between indices, the $\mathcal{V}$-natural transformation $F_i \to F_j$ induces the corresponding $F_i^\vee \to F_j^\vee$. Thus they form the diagram $\{F_i^\vee\}$ in $\mathcal{Hom}(\mathcal{D}, \mathcal{C})$ whose arrows are reversed.

Proposition 1.14. Let $\{F_i\}$ be as above, and suppose that the $\mathcal{V}$-colimit $\lim_i F_i$ and the $\mathcal{V}$-limit $\lim_i F_i^\vee$ are both exist. Then $F$ is left adjoint to $F^\vee$. Moreover the canonical morphisms $F_i \to F$ and $F^\vee \to F_i^\vee$ are mapped to each other by the isomorphism

$$\text{Hom}_{\mathcal{C}, \mathcal{D}}(F_i, F) \simeq \text{Hom}_{\mathcal{D}, \mathcal{C}}(F_i^\vee, F^\vee).$$

We here give two proofs for this easy but important result.

First proof. First we prove the lemma by studying the functors value-wise. The statements are obvious by the $\mathcal{V}$-natural isomorphism

$$\text{Hom}_{\mathcal{D}}(FX, Y) \simeq \lim_i \text{Hom}_{\mathcal{D}}(F_iX, Y) \simeq \lim_i \text{Hom}_{\mathcal{C}}(X, F_i^\vee Y) \simeq \text{Hom}_{\mathcal{C}}(X, F^\vee Y).$$

Second proof. The second is a “2-categorical” proof. For each arrow $i \to j$, we have a commutative diagram

$$\begin{array}{ccc}
\text{Id}_\mathcal{C} & \to & F_i^\vee F_j \\
\downarrow & & \downarrow \\
F_i^\vee F_i & \to & F_i^\vee F_j
\end{array}$$

so it induces the unit $\text{Id}_\mathcal{C} \to \lim_i F_i^\vee F \simeq F^\vee F$. The counit $\text{FF}^\vee \to \text{Id}_\mathcal{D}$ is defined analogously. To prove that $F \to \text{FF}^\vee F \to F$ is equal to $\text{Id}_F$, it suffices to show that its pullback $F_i \to F \to \text{FF}^\vee F \to F$ is equal to $F_i \to F$ for each $i$. It follows from the commutativity of the diagram below:

$$\begin{array}{ccc}
F & \leftarrow & F_i \\
\downarrow & & \downarrow \\
\text{FF}^\vee F & \leftarrow & F_i^\vee F_i
\end{array}$$

Similarly $F^\vee \to \text{FF}^\vee F \to F^\vee$ is also equal to the identity so $F$ and $F^\vee$ are adjoint to each other. The diagram above also shows us that $F_i \to F$ also coincides with the composite $F_i \to F_i^\vee F \to F_i^\vee F_i F \to F$ which is induced by $F^\vee \to F_i^\vee F_i^\vee$. 

$\blacksquare$
Thus $\mathcal{L}p\text{hism}$ also has a similar representation. For any $H \in \mathcal{D}$ we have a natural isomorphism $\text{Hom}_{\mathcal{D}}(D, H)$ if and only if so is the corresponding sequence $0 \to H^0 \to G^0 \to F^0$ in $\text{Hom}(D, C)$. In particular, $F \to G$ is epic if and only if the corresponding $G^0 \to F^0$ is monic.

As we have seen in the examples, the $\mathcal{V}$-category $\mathcal{A}dj(C, D)$ may have limits which differ from those taken in $\text{Hom}(C, D)$. We here give a simple sufficient condition for $\mathcal{A}dj(C, D)$ to be abelian.

Lemma 1.17. Let $C$ and $D$ be abelian $\mathcal{V}$-categories, and suppose that $C$ has enough projectives and $D$ has enough injectives. Let $F, G : C \to D$ be parallel adjunctions with a degree-zero natural transformation $F \to G$. Let $K := \text{Ker}(F \to G)$ and $C := \text{Coker}(G^0 \to F^0)$. Then $L_0K$ is left adjoint to $R^0C$, so it is an adjunction. Moreover, $L_0K$ is the kernel of $F \to G$ in $\mathcal{A}dj(C, D)$.

Proof. Let $X \in C$, $Y \in D$ and take a projective resolution $P' \to P \to X \to 0$ and an injective resolution $0 \to Y \to Q \to Q'$ respectively. By definition $\text{Hom}_D((L_0K)X, Y)$ is the kernel of the map

$$\text{Hom}_D(KP, Q) \to \text{Hom}_D(KP', Q) \otimes \text{Hom}_D(KP, Q').$$

Since $P$ is projective and $Q$ is injective, each term can be represented as

$$\text{Hom}_D(KP, Q) \cong \text{Coker}(\text{Hom}_D(GP, Q) \to \text{Hom}_D(FP, Q))$$

$$\cong \text{Coker}(\text{Hom}_C(P, G^0Q) \to \text{Hom}_C(P, F^0Q))$$

$$\cong \text{Hom}_C(P, CQ).$$

Hence we have a natural isomorphism $\text{Hom}_D((L_0K)X, Y) \cong \text{Hom}_C(X, (R^0C)Y)$ since its right-hand side also has a similar representation. For any $H \in \mathcal{A}dj(C, D)$ there is a natural isomorphism

$$\text{Hom}_D((L_0K)H, L_0K) \cong \text{Hom}_C(H, K) \cong \text{Ker}(\text{Hom}_C(H, F) \to \text{Hom}_C(H, G)).$$

Thus $L_0K$ is the kernel of $F \to G$ taken in $\mathcal{A}dj(C, D)$. \hfill $\square$

Proposition 1.18. Let $C$ and $D$ be as above. Then $\mathcal{A}dj(C, D)$ is abelian and the embedding $\mathcal{A}dj(C, D) \hookrightarrow \text{Hom}(C, D)$ is right exact.

Proof. Clearly it is closed under $\Pi$ or $\Sigma$. By Proposition 1.14 and Lemma 1.17 it also has finite direct sums, kernels and cokernels. In $\mathcal{A}dj(C, D)$ the image of the morphism $F \to G$ is isomorphic to its coimage, since they give same value $\text{Image}(FP \to GP)$ on enough projectives $P \in C$. The cokernel of a morphism in $\mathcal{A}dj(C, D)$ is equal to that taken in $\text{Hom}(C, D)$ so the embedding is right exact. \hfill $\square$
2. Ideal functors in abelian categories

From now on, we omit all prefixes “-V-” in order to avoid redundant descriptions, so “a category” means a V-category, “a functor” a V-functor, etc.

We here study some kind of endofunctors which we call ideal functors. These are analogues of 2-sided ideals in a ring. Later it is used to divide the category into two parts by a Morita context.

2.1. Ideal functors

Let C be an abelian category (i.e. an abelian V-category) and consider the category \( \text{End}(C) := \mathcal{H}om(C, C) \) which is also abelian and has the specific object \( \text{Id}_C \), the identity functor.

**Definition 2.1.** A subfunctor \( I \subset \text{Id}_C \) is called an ideal functor on \( C \) if its cokernel \( \text{Coker}(I \hookrightarrow \text{Id}_C) \) is an adjunction. The cokernel of the corresponding morphism \( \text{Id}_C / I \hookrightarrow \text{Id}_C \) (monic by Corollary 1.16) is denoted by \( I' \).

**Example 2.2.** Consider the case that \( C \) is a module category \( A\text{-Mod} \). Then quotient adjunctions of \( \text{Id}_{A\text{-Mod}} \) are in bijection with quotient \( (A, A) \)-bimodules of \( A \), that is, \( A/I \) for a 2-sided ideal \( I \subset A \). Thus the corresponding ideal functor maps an \( A \)-module \( V \) to the kernel of the map

\[
V \twoheadrightarrow T_I V := A/I \otimes_A V \cong V/I_V,
\]

namely \( IV \). This is why we call such kind of functor an “ideal functor”. In this case the right adjoint \( T'_I \) can be represented as

\[
T'_I V := \text{Hom}_A(A/I, V) = \{v \in V | Iv = 0\}.
\]

**Example 2.3.** When \( C = \mathcal{A}\text{-Mod} \) is a module category of a category \( \mathcal{A} \), an ideal functor on \( C \) is also corresponds to a 2-sided ideal \( I \subset \mathcal{A} \). Here a 2-sided ideal in a category is a collection of spaces of morphisms

\[
I(V, W) \subset \text{Hom}_\mathcal{A}(V, W)
\]

for each pair of \( V, W \in \mathcal{A} \), which is closed under compositions with all morphisms in \( \mathcal{A} \). From such an ideal we can form a quotient category \( \mathcal{A}/I \), whose hom sets are defined by pairwise quotient.

**Example 2.4.** The socle \( \text{Soc}(X) \) of an object \( X \in C \) is the sum of all simple subobjects of \( X \). Dually, its top \( \text{Top}(X) \) is defined as \( X/\text{Rad}(X) \) where the radical \( \text{Rad}(X) \) is the intersection of all its maximal subobjects of \( X \). If these objects exist for every \( X \in C \), then \( \text{Soc}, \text{Top} \) and \( \text{Rad} \) can be defined as endofunctors on \( C \).

Suppose that for all object \( X \in C \), \( \text{Top}(X) \) and \( \text{Soc}(X) \) are both semisimple. Then the functor \( \text{Top} \) is left adjoint to \( \text{Soc} \), thus Rad = Ker(\( \text{Id}_C \rightarrow \text{Top} \)) is an ideal functor.

**Example 2.5.** If \( I \) is an ideal functor on \( C \), \( (I')^\# \) is an ideal functor on \( C^\# \).

On a general abelian category, a typical example is the image of an adjunction.

**Proposition 2.6.** Let \( F : C \rightarrow C \) be an adjunction with a degree-zero natural transformation \( F \rightarrow \text{Id}_C \). Then \( I := \text{Image}(F \rightarrow \text{Id}_C) \) is an ideal functor. \( I' \) is also the image of the corresponding natural transformation \( \text{Id}_C \rightarrow F' \).
Proof. By Proposition 1.14 the cokernel \( T_I = \text{Coker}(F \to \text{Id}_C) \) has the right adjoint functor \( T'_I := \text{Ker}(\text{Id}_C \to F') \) so it is an adjunction. Thus by definition \( I \) is an ideal functor. Since \( 0 \to T'_I \to \text{Id}_C \to F' \) is exact by Corollary 1.16 \( F' \) contains \( I = \text{Coker}(T'_I \to \text{Id}_C) \) as a subfunctor.

**Remark 2.7.** Suppose \( C \) has enough projectives and injectives. Then for an ideal functor \( I \) on \( C \), \( L_0I \) is an adjunction and \( L_0I \to I \) is epic by Proposition 1.18. Thus in this case every ideal functor is obtained as the image of an adjunction. When \( C = A-\text{Mod} \), \( L_0I \) for a 2-sided ideal \( I \subset A \) is just the tensor functor \( I \otimes_A \bullet \) where \( I \) is viewed as an \((A, A)\)-bimodule.

On the other hand, let \( A \) be a polynomial algebra \( k[x_1, x_2, \ldots] \) in infinitely many variables over a field \( k \) and consider the case \( C = A-\text{Mod}^f \), the category of its finite dimensional modules. Then \( I: V \mapsto \sum x_i V \) is clearly an ideal functor on \( A-\text{Mod}^f \), but there are no adjunctions which cover \( I \).

We here list basic properties of ideal functors. For simplicity if there is a canonical isomorphism \( F \to G \) between objects which is clear from the context, we write \( F = G \) for short.

**Lemma 2.8.** Let \( I \subset \text{Id}_C \) be an ideal functor. Then

1. the morphisms \( T_I = T_I \cdot \text{Id}_C \to T^2_I \) and \( T_I = \text{Id}_C \cdot T_I \to T^2_I \) coincide,
2. \( T'_I T_I = T_I T'_I = T^2_I \),
3. \( IT_I = 0 = I'T_I \).

**Proof.** (1) follows from that the epimorphism \( \text{Id}_C \to T_I \) equalize these two morphisms. Since \( \text{Id}_C \to T_I \) factors through \( \text{Id}_C \to T'_I T_I \to T_I \), the monomorphism \( T'_I T_I \to T_I \) is also epic. So it must be an isomorphism since the functor category is abelian. Similarly \( T'_I = T_I T'_I \) holds and we obtain

\[
T_I = T'_I T_I = T_I T'_I T_I = T^2_I
\]

so (2) holds. (3) is just a rephrasing of (2).

In general an ideal functor is neither left exact nor right exact. Still, we can prove the following useful properties.

**Proposition 2.9.** An ideal functor preserves all images.

**Proof.** Let \( I \) be an ideal functor on \( C \). A functor is called mono (resp. epi) if it preserves all monomorphisms (resp. epimorphisms). Since \( \text{Id}_C \) is clearly a mono functor, so is its subfunctor \( I \). \( I \) is also epi because \( \text{Id}_C \) is epi and the cokernel \( T_I \) is right exact; apply the nine lemma. Thus \( I \) preserves all epi-mono factorizations.

Recall that for a possibly infinite family of subobjects \( \{X_i \subset X\} \), their sum, if exists, is the minimum subobject \( \sum X_i \subset X \) which contains all \( X_i \).

**Lemma 2.10.** Let \( X \in C \) be an object and \( Y, Z \subset X \) its subobjects. Then \( IZ \subset Y \) if and only if \( Z \subset \text{Ker}(X \to X/Y \to I'(X/Y)) \) (in other words, \( X/Z \) is a quotient of \( I'(X/Y) \)).

**Proof.** Suppose \( IZ \subset Y \), or equivalently, the composite \( IZ \hookrightarrow Z \to X/Y \) is zero. Then \( Z \to X/Y \) factors through \( Z \to T_I Z \). Hence \( Z \to X/Y \to I'(X/Y) \) factors through \( I'T_I Z = 0 \) so it must be zero. Taking the opposite category we can dually prove the other implication.

**Proposition 2.11.** An ideal functor commutes with summation.
Proof. Let $I$ be an ideal functor on $C$. Take an object $X \in C$ and a family of subobjects $\{X_i \subset X\}$ whose sum $\sum_i X_i$ exists. Note that $X_i \hookrightarrow \sum_i X_i$ induces $IX_i \hookrightarrow I\sum_i X_i$. Thus if $\sum_i IX_i$ exists, it is contained in $I\sum_i X_i$.

Conversely, let $Y \subset X$ be a subobject which contains every $IX_i$. Let

$$Y' := \text{Ker}(X \twoheadrightarrow X/Y \twoheadrightarrow I'(X/Y)).$$

Then $X_i \subset Y'$ for all $i$ by the “only if” part of Lemma 2.10, so $\sum_i X_i \subset Y'$. On the other hand, its “if” part says that $II' \subset Y$. Thus $I\sum_i X_i \subset Y$, so $\sum_i IX_i$ exists and actually

$$\sum_i IX_i = I\sum_i X_i.$$ 

2.2. Subcategories defined by ideal functors

In this subsection we fix an ideal functor $I$ on $C$. Using an ideal functor, we define two full subcategory of $C$ in the following manner.

Lemma 2.12. For an object $X \in C$, the following conditions are equivalent.

1. $IX = 0$ ($\iff X = T_i X$),
2. $I'X = 0$ ($\iff T'_i X = X$).

Proof. Similar to the proof of $T_i = T'_i$ in Lemma 2.8.

Definition 2.13. An object $X \in C$ is called $I$-annihilated if it satisfies the conditions above. We denote by $C_I$ the full subcategory of $C$ consisting of $I$-annihilated objects.

The other subcategory is defined analogously.

Definition 2.14. An object $X \in C$ is called

1. $I$-accessible if $IX = X$ ($\iff T_i X = 0$),
2. $I$-torsion-free if $X = I'X$ ($\iff T'_i X = 0$).

We denote by $C^I$ the full subcategory of $C$ consisting of $I$-accessible $I$-torsion-free objects.

By definition it is clear that these subcategories are closed under the parity change $\Pi$ or the degree shift $\Sigma$.

Example 2.15. Let $A$ be a ring and $I \subset A$ a 2-sided ideal. Then an $A$-module $V$ is $I$-annihilated if and only if it can be defined over the quotient ring $A/I$. Thus there is a canonical category equivalence $(A\text{-Mod})_I \simeq (A/I)\text{-Mod}.$

Example 2.16. Suppose that every object in $C$ is of finite length. Then the assumption in Example 2.4 is satisfied. For an ideal functor $I = \text{Rad}$, we have that $C_{\text{Rad}} = \{0\}$ (Nakayama’s lemma) and $C_{\text{Rad}}$ consists of all semisimple objects in $C$.

Clearly the intersection of $C^I$ and $C_I$ is $\{0\}$. In addition, there are no non-zero morphisms between objects in these categories.

Lemma 2.17. Let $X, Y, Z \in C$ and suppose that $X$ is $I$-accessible, $Y$ is $I$-torsion-free and $Z$ is $I$-annihilated. Then $\text{Hom}_C(X, Z) = 0$ and $\text{Hom}_C(Z, Y) = 0$. 

17
Proof. Follows from

\[
\begin{align*}
\text{Hom}_C(X, Z) &= \text{Hom}_C(X, T^\vee_I Z) \cong \text{Hom}_C(T_I X, Z) = 0, \\
\text{Hom}_C(Z, Y) &= \text{Hom}_C(T_I Z, Y) \cong \text{Hom}_C(Z, T^\vee_I Y) = 0.
\end{align*}
\]

The important property is that simple objects in \( C \) are divided into these subcategories. The proof of the lemma below is obvious.

Lemma 2.18. When \( X \in C \) is simple, these three conditions are all equivalent.

(1) \( X \) is \( I \)-accessible,
(2) \( X \) is \( I \)-torsion-free,
(3) \( X \) is not \( I \)-annihilated.

Notation 2.19. We denote by \( \text{Irr} C \) the isomorphism class of simple objects in \( C \). For an ideal functor \( I \) on \( C \), we also denote by \( \text{Irr} C^I \) and \( \text{Irr} C_I \) the subsets of \( \text{Irr} C \) whose members are simple objects contained in respective subcategories.

By the lemma, we have a decomposition \( \text{Irr} C = \text{Irr} C^I \sqcup \text{Irr} C_I \). Note that the definitions of \( \text{Irr} C^I \) and \( \text{Irr} C_I \) need both the category \( C \) and the ideal functor \( I \), not only the subcategories themselves.

Proposition 2.20.

(1) \( I \)-accessible objects are closed under quotients, extensions and direct sums,
(2) \( I \)-torsion-free objects are closed under subobjects, extensions and direct products,
(3) \( I \)-annihilated objects are closed under subobjects, quotients, direct products and direct sums.

In particular, \( C^I \) is an exact subcategory of \( C \) in Quillen’s sense, and \( C_I \) is an abelian subcategory.

Proof. Follow from that \( T_I \) is cocontinuous and that \( T^\vee_I \) is continuous.

Obviously \( \text{Irr} C_I \) is equal to the isomorphism class of simple objects in an abelian category \( C_I \), so this notation makes no confusions. Let us denote by the embedding \( C_I \hookrightarrow C \) of abelian categories by \( \Phi_I \). Namely, for \( X \in C_I \), we explicitly write \( \Phi_I X \in C \) when we need to emphasis the categories in which these objects belong.

Lemma 2.21. \( \Phi_I \) has both the left adjoint functor \( \Phi_I^\vee \) and the right adjoint functor \( \Phi_I^\wedge \) such that \( \Phi_I \Phi_I^\vee = T_I, \Phi_I \Phi_I^\wedge = T^\vee_I \) and \( \Phi_I \Phi_I^\vee \Phi_I = \Phi_I \Phi_I^\wedge = \text{Id}_{C_I} \).

Proof. Since \( T_I X \) and \( T^\vee_I X \) for any \( X \in C \) are \( I \)-annihilated, the functors \( \Phi_I^\vee \) and \( \Phi_I^\wedge \) can be defined as the restriction of \( T_I \) and \( T^\vee_I \) respectively. For any \( Y \in C_I \), we have naturally

\[
\begin{align*}
\text{Hom}_C(X, \Phi_I Y) &= \text{Hom}_C(X, T^\vee_I \Phi_I Y) = \text{Hom}_C(T_I X, \Phi_I Y) = \text{Hom}_C(\Phi_I^\vee X, Y), \\
\text{Hom}_C(\Phi_I Y, X) &= \text{Hom}_C(T_I \Phi_I Y, X) = \text{Hom}_C(\Phi_I^\wedge Y, T^\vee_I X) = \text{Hom}_C(Y, \Phi_I^\wedge X).
\end{align*}
\]

Thus these two functors are respective adjoints of the embedding.
Corollary 2.22. For an ideal functor \( J \) on another abelian category \( \mathcal{D} \), the functor
\[
\text{Hom}(C_1, \mathcal{D}) \to \text{Hom}(C, \mathcal{D}), \\
F \mapsto \Phi_J F \Phi_J^\wedge
\]
is fully faithful. Its image is equivalent to the full subcategory
\[
\{ G : C \to \mathcal{D} | G = T_J GT_I \} \subset \text{Hom}(C, \mathcal{D})
\]
and the inverse is induced by
\[
\text{Hom}(C, \mathcal{D}) \to \text{Hom}(C_1, \mathcal{D}), \\
G \mapsto \Phi_J^\wedge G \Phi_I.
\]
In particular, \( F \) is an adjunction if and only if so is \( \Phi_J F \Phi_J^\wedge \).

Remark 2.23. \( C_I \) is characterized up to equivalence by these data: let \( \mathcal{E} \) be an abelian category with an adjoint \( \Phi : \mathcal{E} \to C \) which also has a left adjoint functor \( \Phi^\wedge \), and suppose that the counit \( \Phi^\wedge \Phi \to \text{Id}_C \) is an isomorphism and the unit \( \text{Id}_C \to \Phi \Phi^\wedge \) is epic. Then \( \mathcal{E} \) is canonically equivalent to \( C_I \) where \( I := \text{Ker}(\text{Id}_C \to \Phi \Phi^\wedge) \).

2.3. Ideal operations

In this subsection we consider operations against ideal functors: summation, product and quotient. These are analogues of those against usual 2-sided ideals in rings. Firstly we introduce summation of ideal functors.

Proposition 2.24. Let \( \{I_i\} \) be a family of ideal functors on \( C \), and suppose that their sum \( \sum_i I_i \subset \text{Id}_C \) and the intersection \( \bigcap_i T_{I_i} \subset \text{Id}_C \) exist. Then \( \sum_i I_i \) is again an ideal functor.

Proof. The cokernel of \( \sum_i I_i \to \text{Id}_C \) is the pushout of adjunctions \( \text{Id}_C \to T_{I_i} \) under \( \text{Id}_C \), which is left adjoint to the pullback \( \bigcap_i T_{I_i}' \) by Proposition 1.14.

Remark 2.25. In contrast, the intersection of ideal functors is not an ideal functor. This is because even in the module category of a ring, the equation \((I \cap J)V = IV \cap JV\) does not hold in general.

In particular, a finite sum always exists. We can represent finite sum in another way as follows.

Proposition 2.26. Let \( I \) and \( J \) be ideal functors on \( C \). Then \( T_I T_J = T_{I+J} \).

Proof. We have a commutative diagram
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & \text{Id}_C & \longrightarrow & T_I & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & IT_J & \longrightarrow & T_J & \longrightarrow & T_I T_J & \longrightarrow & 0
\end{array}
\]
where the rows are exact since the right multiplication of \( T_J \) is exact, and \( I \to IT_J \) is epic since \( I \) preserves images. These properties imply that the right square is cocartesian. In other words, \( T_I T_J \) is the pushout of \( T_I \) and \( T_J \) under \( \text{Id}_C \).
Secondly we consider the product of two ideal functors defined by composition.

**Lemma 2.27.** For ideal functors $I$ and $J$ on $C$, we have $T_1 J = JT_1 J$.

Here we let $T_1 J := \text{Coker}(IJ \hookrightarrow \text{Id}_C)$ though we have not yet proved that $IJ$ is an ideal functor.

**Proof.** Let us consider the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & IIJ \\
\downarrow \text{id} & & \downarrow \text{id} \\
0 & \rightarrow & J \\
\downarrow & & \downarrow \\
J & \rightarrow & JI J \\
\downarrow & & \downarrow \\
\text{Id}_C & \rightarrow & T_1 J \\
\end{array}
\]

whose rows are exact. The commutative square induces $T_1 J \rightarrow T_1 J$ which is monic by the four lemma. On the other hand, we have another commutative diagram

\[
\begin{array}{ccc}
J & \rightarrow & JI J \\
\downarrow & & \downarrow \\
\text{Id}_C & \rightarrow & T_1 J \\
\end{array}
\]

where $J \rightarrow JI J$ is epic and $JI J \rightarrow T_1 J$ is monic. Thus we have two epi-mono factorization of the diagonal morphism so that its images must be equal.

**Lemma 2.28.** For $I$ and $J$ as above, we have $I \circ J = J \circ I$.

**Proof.** Follows in a similar way from that both are the image of $J \hookrightarrow \text{Id}_C \twoheadrightarrow I \circ I \circ$.

**Proposition 2.29.** Let $I$ and $J$ be ideal functors on $C$. Then $IJ$ is also an ideal functor such that $(IJ)^\vee = J^\vee I^\vee$.

**Proof.** Let $T_1 J := \text{Coker}(IJ \hookrightarrow \text{Id}_C)$ and $T_1^\vee J := \text{Ker}(\text{Id}_C \twoheadrightarrow J^\vee I^\vee)$. It suffices to prove that these functors actually form an adjunction. By the lemma above, we have $IJ T_1 J = I T_1 J = 0$. So $J T_1 J$ is $I$-annihilated and this implies $J^\vee T_1 J = J^\vee J T_1 J = 0$. Thus $T_1 J$ is $J$-annihilated so $J^\vee J T_1 J = 0$. Equivalently, we have $T_1^\vee J T_1 J = T_1^\vee J$. We can prove $T_1^\vee J T_1^\vee J = T_1^\vee J$ in a similar way. Using these isomorphisms, we can define the unit $\text{Id}_C \hookrightarrow T_1 J = T_1^\vee J T_1 J$ and the counit $T_1 J T_1^\vee J = T_1^\vee J \hookrightarrow \text{Id}_C$. Now it is obvious that these morphisms satisfy the zig-zag identities.

Lastly we study about quotient of ideal functors.

**Definition 2.30.** Let $I$ and $J$ be two ideal functors on $C$ such that $J \subset I \subset \text{Id}_C$. Since $C_J$ is closed under subobjects, $X \in C_J$ implies $IX \in C_J$. We denote this restricted functor of $I$ by $I_J : C_J \rightarrow C_J$.

Recall that we denote by $\Phi_J : C_J \rightarrow C$ the embedding of abelian category, and the endofunctor category can be also exactly embedded as

\[
\text{End}(C_J) \rightarrow \text{End}(C), \\
F \mapsto \Phi_J^* F \Phi_J^\vee.
\]
Lemma 2.31. Let I and J be as above. Then

\[ \Phi_J J \Phi_J^* = IT_J = \text{Ker}(T_J \to T_I) \cong I/J. \]

Proof. By definition, \( \Phi_J J \Phi_J^* = I \Phi_J \) so \( \Phi_J J \Phi_J^* = IT_J \). Since \( T_J T_J = T_{1_J} = T_J \), it is equal to \( IT_J = \text{Ker}(T_J \to T_{1_J}) = \text{Ker}(T_J \to T_I) \).

The last isomorphism is obvious.

Proposition 2.32. For I and J as above, \( IJ \) is an ideal functor on \( C_J \). Moreover every ideal functors on \( C_J \) is obtained in this way, and ideal functors on \( C_J \) are in one-to-one correspondence with those on \( C \) which contain \( J \).

Proof. Via the above embedding, \( \text{Coker}(I_J \to \Phi_I \Phi_J) \) is mapped to an adjunction \( \text{Coker}(IT_J \to T_{1_J}) = T_{T_I} \). Thus \( IJ \) is an ideal functor and \( I \) is recovered from \( IJ \) in this way.

Conversely, suppose that \( K \) is an ideal functor on \( C_J \). Then \( T_K = \text{Ker}(T_J \to T_K) \) is an ideal functor which contains \( J \). Moreover we have \( \Phi J = \text{Ker}(\Phi_J \to T_{T_{1_J}}) = \text{Ker}(\Phi_J \to T_J) = \Phi_J K \),

which means that \( \Phi J = K \). Thus these operations gives a one-to-one correspondence up to unique isomorphism.

Now the next statements are clear.

Proposition 2.33. Let \( I, J \subset \Phi I \) be ideal functors. Then

1. \( C_{I+J} = C_I \cap C_J \),
2. \( C_{IJ} = C_I \cap C_J \),
3. if \( J \subset I \) then \( (C_J)_I = C_I \) and \( (C_J)^I = C_I \cap C_J \).}

2.4. Compatibility with extension

In this subsection we fix an abelian category \( C \) and an ideal functor \( I \subset \Phi I \). For each pair of \( X, Y \in C \), let us denote \( \text{Ext}^i_C(X, Y)_0 := \text{Ext}^i_C(X, Y) \) the Ext group taken in \( C_0 \). It is defined as the set of equivalence classes of exact sequences

\[ 0 \to Y \to E_i \to \cdots \to E_1 \to A \to 0. \]

We also define the graded Ext group \( \text{Ext}^i_C(X, Y) \) as

\[ \text{Ext}^i_C(X, Y) := \text{Ext}^i_C(X, Y) \oplus \text{Ext}^i_C(X, Y) \]

for the super case \( V = S \) and

\[ \text{Ext}^i_C(X, Y) := \bigoplus_k \text{Ext}^ik_C(X, \Sigma^k Y) \]

for the graded case \( V = G \).

Every exact sequences in \( C_I \) are still exact in \( C \), so we have a canonical map

\[ \text{Ext}^i_C(X, Y) \to \text{Ext}^i_C(\Phi I X, \Phi I Y). \]
However this map is rarely an isomorphism because when we take an exact sequence in \( C \) whose both ends are in \( C_I \), the rest terms do not need to belong in \( C_I \). In this subsection we give some characterizations of that \( \Phi_I \) preserves Ext functors. These results are a reformulation of those in [2].

In a module category case, the condition for \( \text{Ext}^1 \) is well-known: it is equivalent to that the ideal is idempotent. We can easily generalize this fact as follows.

**Proposition 2.34.** The followings are equivalent.

1. \( \text{Ext}^1_I(X, Y) = \text{Ext}^1_C(X, Y) \) for every \( X, Y \in C_I \).
2. \( C_I \) is closed under extensions,
3. \( I^2 = I \).

**Proof.** (1) \( \Leftrightarrow \) (2) is immediate by definition. Now let \( 0 \to X \to Y \to Z \to 0 \) be a short exact sequence in \( C \) and suppose \( X, Z \in C_I \). Then clearly \( I^2Y = 0 \) so (3) implies (2). Conversely assume that (3) fails so that \( I^2 \nsubseteq I \). Then there exists \( X \in C \) such that \( I^2X \nsubseteq IX \). By \( ITI = IT^2 \) the sequence
\[
0 \to ITIX \to ITX \to TX \to 0
\]
is exact. Both the left and the right term is in \( C_I \) but the middle is not since \( I^2X \nsubseteq IX \) implies \( ITX = ITIX \neq 0 \). Hence (2) does not hold, so we have (2) \( \Leftrightarrow \) (3).

**Lemma 2.35.** Suppose that \( C \) has enough projectives.

1. If \( P \in C \) is projective, then so is \( \Phi_I^iP \in C_I \).
2. \( C_I \) also has enough projectives.
3. Each projective object in \( C_I \) is a direct summand of \( \Phi_I^iP \) for some projective object \( P \in C \).

**Proof.** (1) follows from that its right adjoint \( \Phi_I \) is exact. For any \( X \in C_I \), there is a projective object \( P \in C \) and an epimorphism \( P \to \Phi_I X \) which induces \( \Phi_I^iP \to X \) so (2) and (3) follow from (1).

In the rest of this subsection, we require that \( C \) has enough projectives and injectives in order to use its homological properties. Then we have that \( \text{Ext}^i_C(X, \bullet) \) and \( \text{Ext}^i_C(\bullet, Y) \) are the \( i \)-th left derived functors of \( \text{Hom}_C(X, \bullet) \) and \( \text{Hom}_C(\bullet, Y) \) respectively.

**Lemma 2.36.** Let \( 2 \leq k \leq \infty \). For \( X \in C \), the followings are equivalent.

1. \( \text{Ext}^i_C(\Phi_I^iX, Y) = \text{Ext}^i_C(X, \Phi_I^iY) \) for any \( Y \in C_I \) and \( 0 \leq i < k \),
2. \( \text{Ext}^i_C(X, \Phi_I^iQ) = 0 \) for any injective \( Q \in C_I \) and \( 1 \leq i < k \),
3. \( (L_i\Phi_I^i)X = 0 \) for any \( 1 \leq i < k \),
4. \( (L_0)X \cong IX \) and \( (L_1)X = 0 \) for any \( 1 \leq i < k - 1 \).

**Proof.** Let \( P_k \to \cdots \to P_1 \to P_0 \to X \to 0 \) be a projective resolution of \( X \). First (1) \( \Rightarrow \) (2) is trivial. Assume (2) then it implies
\[
0 \to \text{Hom}_{C_I}(\Phi_I^iX, Q) \to \text{Hom}_{C_I}(\Phi_I^iP_0, Q) \to \cdots \to \text{Hom}_{C_I}(\Phi_I^iP_k, Q)
\]
is exact for any injective \( Q \in C_I \). Since \( C_I \) has enough injectives by the dual of Lemma 2.35, the sequence
\[
\Phi_I^iP_k \to \cdots \to \Phi_I^iP_1 \to \Phi_I^iP_0 \to \Phi_I^iX \to 0
\]
must be exact so (3) holds. Conversely, if (3) is satisfied, the sequence above is a projective resolution of \( \Phi_i X \). Thus

\[
\text{Ext}_{C_i}^i(\Phi_i X, Y) \approx H_i(\text{Hom}_{C_i}(\Phi_i P, Y)) \approx H_i(\text{Hom}_{C}(P, \Phi_i Y)) \approx \text{Ext}_{C}^i(X, \Phi_i Y)
\]

so (1) holds. Finally we have \( \Phi_i(L_i \Phi_i^\circ) = L_i T_I \) since \( \Phi_i \) is exact. Hence (3) and (4) are equivalent by that the sequence

\[
0 \to L_i T_I \to L_0 I \to \text{Id}_C \to T_I \to 0
\]

is exact and that \( L_i T_I \cong L_{i-1} I \).

One can easily check that if every \( X \in C \) satisfies the above conditions for \( k = 2 \), then it is also true for \( k = \infty \). In this situation, we can rewrite the conditions as follows.

**Corollary 2.37.** The followings are equivalent.

1. \( \text{Ext}_{C_i}^i(\Phi_i X, Y) = \text{Ext}_{C}^i(X, \Phi_i Y) \) for any \( X \in C \), \( Y \in C_I \) and \( i \geq 0 \).
2. \( \Phi_i \) sends injectives to injectives,
3. \( \Phi_i^\circ \) is exact,
4. \( I \) is an adjunction.

However this condition is too strong for our purpose because we only need objects of the form \( \Phi_i X \in C \) for \( X \in C_I \). Now we state a criteria of Ext preserving property for ideal functors.

**Proposition 2.38.** Let \( 2 \leq k \leq \infty \). Then the followings are equivalent.

1. \( \text{Ext}_{C_i}^i(\Phi_i X, \Phi_i Y) = \text{Ext}_{C}^i(X, \Phi_i Y) \) for any \( X, Y \in C_I \) and \( 0 \leq i < k \),
2. \( \text{Ext}_{C_i}^i(\Phi_i X, \Phi_i Y) = 0 \) for any projective \( P \in C_I \), injective \( Q \in C_I \) and \( 1 \leq i < k \),
3. \( (L_i \Phi_i^\circ)^{\circ} \Phi_i = 0 \) for any \( 1 \leq i < k \),
4. \( (L_i \Phi_i^\circ)^{\circ} \Phi_i P = 0 \) for any projective \( P \in C_I \) and \( 1 \leq i < k \),
5. \( (L_i I) \Phi_i = 0 \) for any \( 0 \leq i < k - 1 \),
6. \( (L_i I) \Phi_i P = 0 \) for any projective \( P \in C_I \) and \( 0 \leq i < k - 1 \).

When \( k \geq 3 \), it is also equivalent to that:

7. \( (L_0 I)^2 \cong L_0 I \), and \( (L_i I) P = 0 \) for any projective \( P \in C \) and \( 1 \leq i < k - 2 \).

Since the conditions (1) and (2) are self-dual, we can replace the rest conditions by their dual statements.

**Proof.** (1) \( \iff \) (3) \( \iff \) (5) and (2) \( \iff \) (4) \( \iff \) (6) follow from the previous lemma. (3) \( \implies \) (4) is obvious. Now suppose (4). Let \( X \in C \) and take an exact sequence \( 0 \to K \to P \to X \to 0 \) with \( P \) projective. By applying \( (L_i \Phi_i^\circ)^{\circ} \Phi_i \) it yields the long exact sequence

\[
\cdots \to (L_i \Phi_i^\circ)^{\circ} \Phi_i K \to (L_i \Phi_i^\circ)^{\circ} \Phi_i P \to (L_i \Phi_i^\circ)^{\circ} \Phi_i X \to K \to P \to X \to 0.
\]

Then the assumption implies \( (L_i \Phi_i^\circ)^{\circ} \Phi_i X = 0 \). Moreover we have \( (L_i \Phi_i^\circ)^{\circ} \Phi_i X \cong (L_{i-1} \Phi_i^\circ)^{\circ} \Phi_i K \) for each \( 2 \leq i < k \) so by induction all of them must be zero. Hence (4) implies (3).

Now suppose \( k \geq 3 \) and we prove that (6) is also equivalent to (7). First by Lemma 2.35 (6) can be replaced by

(6') \( (L_i I) T_I P = 0 \) for any projective \( P \in C \) and \( 0 \leq i < k - 1 \).
Applying \(L_iI\)'s to the exact sequence \(0 \to IP \to P \to TI \to 0\), we obtain that it is equivalent to that \((L_0I)IP \simeq IP\) and \((L_iI)IP = 0\) for \(1 \leq i < k - 2\). Moreover, the first condition is equivalent to that \((L_0I)^2 \simeq L_0I\) since a right exact functor is determined by the values on projectives which generate whole \(C\).

Hence it also yields a well-known statement as a special case: for an algebra \(A\) and its 2-sided ideal \(I \subset A\), \(\text{Ext}^2_A/I = \text{Ext}^2_A\) if and only if the multiplication \(I \otimes A I \to I\) is an isomorphism.

**Example 2.39.** Suppose \(I^2 = I\) and \(I\) sends projectives to projectives. Then for \(k = \infty\) the condition (7) above is easily verified.

### 2.5. Ideal filters

In this subsection we consider a family of ideal functors indexed by a partially ordered set \((\Lambda, \leq)\). Such situation occurs mainly in the study of cellular algebras or quasi-hereditary algebras. In this subsection we fix an abelian category \(C\) which is closed under sums and intersections of subobjects with cardinality \(#\Lambda\).

**Definition 2.40.** An ideal filter on \(C\) indexed by \((\Lambda, \leq)\) is a family of ideal functors \(\{I_{\leq \lambda}\}_{\lambda \in \Lambda}\) which satisfies these three conditions:

\[
I_{\leq \lambda} \subset I_{\leq \mu} \quad \text{if} \quad \lambda \leq \mu,
\]

\[
\text{Id}_C = \sum_{\lambda} I_{\leq \lambda},
\]

\[
I_{\leq \lambda}I_{\leq \mu} \subset \sum_{\nu \leq \lambda, \mu} I_{\leq \nu}.
\]

From now on \(\{I_{\leq \lambda}\}\) denotes an ideal filter on \(C\) indexed by \((\Lambda, \leq)\).

**Notation 2.41.** For each \(\lambda \in \Lambda\), we define

\[
I^{\leq \lambda} := \sum_{\mu \leq \lambda} I_{\leq \mu}.
\]

When an ideal filter on \(C\) is fixed, we write \(C^{\leq \lambda} := C^{I_{\leq \lambda}}\) and \(C_{\leq \lambda} := C_{I_{\leq \lambda}}\) for short. We also denote \(C[\lambda] := C^{\leq \lambda} \cap C_{\leq \lambda}\) and \(\text{Irr} C[\lambda] := \{V \in \text{Irr} C \mid V \in C[\lambda]\}\).

The purpose of introducing an ideal filter is to divide the category into a small subcategories as follows.

**Lemma 2.42.** If \(\lambda \neq \mu\) we have \(C[\lambda] \cap C[\mu] = \{0\}\), so that \(\text{Irr} C[\lambda] \cap \text{Irr} C[\mu] = \emptyset\).

**Proof.** Suppose that \(X \in C[\lambda] \cap C[\mu]\). If \(\lambda < \mu\), we have

\[
X = I_{\leq \lambda}X \subset I_{\leq \mu}X = 0.
\]

Otherwise \(\lambda \not< \mu\). Then

\[
X = I_{\leq \lambda}I_{\leq \mu}X \subset \sum_{\nu \leq \lambda, \mu} I_{\leq \nu}X \subset I_{\leq \lambda}X = 0.
\]

In either case, we have that \(X = 0\).

A partially ordered set \((\Lambda, \leq)\) is said to be well-founded if its every non-empty subset has a minimal element. Under the axiom of choice, this property is equivalent to the conditions below:

1. there are no infinite descending chains \(\lambda_1 > \lambda_2 > \cdots\),
24
there is a well-ordering extension of $\leq$.

**Proposition 2.43.** Suppose that $\Lambda$ is well-founded. Then

$$\text{Irr } \mathcal{C} = \bigcup_{\lambda \in \Lambda} \text{Irr } \mathcal{C}[\lambda].$$

**Proof.** Suppose that $X \in \mathcal{C}$ is simple. Since we have $0 \neq X = \sum_{\lambda} I^{\leq \lambda} X$, the set $\{ \lambda \in \Lambda \mid I^{\leq \lambda} X \neq 0 \}$ is non-empty. By the assumption it has a minimal element $\lambda \in \Lambda$, so that $X \in \mathcal{C}[\lambda]$. □

In practice we can choose a partially ordered set from various choices to obtain a same result.

**Lemma 2.44.** Suppose that there is a subset $\Lambda_0 \subset \Lambda$ such that $\text{Id}_{\mathcal{C}} = \sum_{\mu \in \Lambda_0} I^{\leq \mu}$. Let $\Lambda' := \{ \mu \in \Lambda \mid \exists \lambda \in \Lambda_0 \text{ s.t. } \mu \leq \lambda \}$ be the order ideal generated by $\Lambda_0$. Then

1. $\{ I^{\leq \lambda} \}_{\lambda \in \Lambda'}$ is also an ideal filter indexed by $(\Lambda', \leq)$,
2. $I^{\leq \lambda} = I^{\leq \mu}$ unless $\lambda \in \Lambda'$, so that $\mathcal{C}[\lambda] = \{ 0 \}$.

**Proof.** (1) is obvious. Suppose $\mu \notin \Lambda'$. Then

$$I^{\leq \lambda} \subset \sum_{\mu \in \Lambda'} I^{\leq \mu} \subset \sum_{\mu \in \Lambda'} \sum_{\nu \leq \mu} I^{\leq \nu} \subset I^{\leq \lambda}$$

so (2) holds. □

Next we consider how these subcategories will be affected when we strengthen the order on $\Lambda$.

**Lemma 2.45.** Let $\preceq$ be an extension of $\leq$, that is, another partial ordering on $\Lambda$ such that $\lambda \preceq \mu$ implies $\lambda \leq \mu$. For each $\lambda \in \Lambda$, define

$$I^{\leq \lambda} := \sum_{\mu \preceq \lambda} I^{\leq \mu}.$$

Then

1. $\{ I^{\leq \lambda} \}$ is also an ideal filter indexed by $(\Lambda, \preceq)$,
2. $C^{\preceq \lambda} \cap C_{\preceq \lambda} = C^{\leq \lambda} \cap C_{\leq \lambda}$,
3. there is a surjective morphism $I^{\leq \lambda} / I^{\preceq \lambda} \to I^{\leq \lambda} / I^{\preceq \lambda}$.

**Proof.** Let us check the conditions in the definition for $\{ I^{\leq \lambda} \}$. The first two clearly hold. Since ideal functors commute with summation, we also have the third one

$$I^{\leq \lambda} I^{\leq \mu} = \sum_{\nu \preceq \lambda, \pi \preceq \mu} I^{\leq \nu} I^{\leq \pi} \subset \sum_{\nu \preceq \lambda, \pi \preceq \mu} I^{\leq \pi} \subset \sum_{\nu \preceq \lambda, \pi \preceq \mu} \sum_{\rho \preceq \mu, \pi} I^{\leq \rho} = \sum_{\rho \preceq \mu} I^{\leq \rho}. $$

Thus $\{ I^{\leq \lambda} \}$ is an ideal filter.

In order to prove (2), first note that

$$I^{\leq \lambda} \equiv \sum_{\mu \preceq \lambda} I^{\leq \mu} \equiv \sum_{\rho \preceq \lambda} \sum_{\mu \preceq \rho} I^{\leq \mu} = \sum_{\rho \preceq \lambda} I^{\leq \rho}. $$
Suppose that \( X \in C_{\leq 1} \cap C_{<1}. \) Since \( I_{=1} \subset I_{\leq 1}, \) clearly \( X \in C_{\leq 1}. \) Moreover,

\[
I_{\leq 1}^X = I_{=1}^X I_{\leq 1}^X = \sum_{\mu \leq 1} I_{=1}^\mu I_{\leq 1}^X = \sum_{\mu \leq 1} \sum_{\lambda \leq 1, \lambda \geq \mu} I_\lambda^X \subset I_{\leq 1}^X = 0
\]

so \( X \in C_{\leq 1}. \) Conversely, suppose that \( X \in C_{=1} \cap C_{<1}. \) Then \( I_{=1} \subset I_{<1} \) immediately implies that \( X \in C_{=1}. \) We also have

\[
X = I_{\leq 1}^X = I_{=1}^X + I_{\leq 1}^X = I_{\leq 1}^X,
\]

that is, \( X \) is \( \leq 1 \)-accessible. We can prove that \( X \) is \( \leq 1 \)-torsion-free in a similar manner so \( X \in C_{=1}. \) Thus \( C_{=1} \cap C_{<1} = C_{=1} \cap C_{<1}. \)

Finally (3) follows from \( I_{=1} \subset I_{\leq 1}, I_{<1} \subset I_{\leq 1} \) and \( I_{\leq 1} = I_{=1} + I_{<1}. \)

Hence the notation \( C[l] := C_{\leq 1} \cap C_{<1} \) does not depend on taking extension of ordering. Unfortunately, the functor \( I_{\leq 1}/I_{<1} \) does change by extension of ordering. The condition for stability of this functor is described as follows.

**Definition 2.46.** For each \( \lambda, \) let

\[
I_{=1}^\lambda = \sum_{\mu \leq 1} I_{=1}^\mu.
\]

An ideal filter \( \{I_{=1}\} \) is said to be **rigid** if it satisfies \( I_{=1} \cap I_{\leq 1} = I_{\leq 1} \) for every \( \lambda. \)

Note that the condition \( I_{=1} \cap I_{\leq 1} \supset I_{<1} \) is always satisfied. Clearly if \( \leq \) is a total order then every ideal filter is rigid.

**Proposition 2.47.** Suppose \( \{I_{=1}\} \) is rigid. Then for any extension \( \leq \) of \( \leq, \)

1. \( \{I_{=1}\} \) is also rigid,
2. the canonical morphism \( I_{\leq 1}/I_{<1} \rightarrow I_{\leq 1}/I_{<1} \) is an isomorphism.

**Proof.** Note that \( I_{=1} \subset I_{\leq 1} \subset I_{\leq 1}. \) These inclusions imply

\[
I_{\leq 1} \cap I_{\leq 1} = (I_{\leq 1} + I_{=1}) \cap I_{\leq 1} = (I_{\leq 1} \cap I_{\leq 1}) + I_{=1} \subset I_{\leq 1} \cap I_{\leq 1} + I_{=1}.
\]

By the assumption, the right hand side is equal to \( I_{<1} + I_{<1} = I_{<1} \), so \( \{I_{=1}\} \) is rigid. Moreover

\[
I_{=1} \subset I_{\leq 1} \subset I_{\leq 1} \cap I_{\leq 1} = I_{<1}.
\]

so \( I_{=1} \cap I_{<1} = I_{<1}. \) Thus the morphism \( I_{\leq 1}/I_{<1} \rightarrow I_{\leq 1}/I_{<1} \) is an isomorphism since its kernel is \( (I_{\leq 1} \cap I_{<1})/I_{<1} = 0. \)

When an ideal filter is rigid we have an additional result on simple objects.

**Proposition 2.48.** Suppose \( \{I_{=1}\} \) is rigid and \( \Lambda \) is well-founded. Then in the Grothendieck group of finite length objects in \( C \), we have the equation

\[
[X] = \sum_{\lambda \in \Lambda} [I_{=1}^X/ I_{<1}^X].
\]
Proof. By the proposition above, by taking its extension we may assume that \( \leq \) is a well-ordering. Suppose \( X \in C \) is of finite length. Let us denote by \( l(Y) \) the length of \( Y \in C \). Since \( X = \sum \lambda I^{\leq k} X \), the set \( \{ \lambda \in \Lambda | l(I^{\leq k} X) \geq k \} \) is not empty for each \( 0 \leq k \leq l(X) \). Let \( \lambda_k \) be its minimum element. Then we have
\[
I^{\leq k} X = \begin{cases} 0 & \text{if } k = 0, \\ I^{\leq k-1} X & \text{if } \lambda_k \neq \lambda_{k-1}. \end{cases}
\]
Hence by taking the composition series
\[
0 = I^{\leq 0} X \subset I^{\leq 1} X \subset \cdots \subset I^{\leq l(X)} X = X
\]
we have
\[
[X] = [I^{\leq k} X] + \sum_{1 \leq k \leq l(X)} [I^{\leq k} X/I^{\leq k+1} X] = \sum_{\lambda \in \Lambda_0} [I^{\leq k} X/I^{\leq l(X)} X]
\]
where \( \Lambda_0 = \{ \lambda_0, \lambda_1, \ldots, \lambda_{l(X)} \} \) (overlapping elements are excluded). It is also clear that \( I^{\leq k} X = I^{\leq k} X \) when \( \lambda \notin \Lambda_0 \), so the statement holds. \( \square \)

3. Morita context between abelian categories

The classical Morita theory [35] treats a category equivalence between respective module categories of two rings \( A \) and \( B \). It is performed as a tensor functor \( P \otimes_B \bullet : B-\text{Mod} \to A-\text{Mod} \) and a hom functor \( \text{Hom}_A(P, \bullet) : A-\text{Mod} \to B-\text{Mod} \) by use of a progenerator \( P \), which is an \((A, B)\)-bimodule such that finitely generated and projective as both left and right modules. To make this correspondence symmetric, we can take a \((B, A)\)-bimodule \( P^\vee := \text{Hom}_A(P, A) \) and rewrite \( \text{Hom}_A(P, \bullet) \simeq P^\vee \otimes_A \bullet \). A Morita context between rings is a weaker notion of Morita equivalence consists of such pair \((P, P^\vee)\), which still provides an equivalence between certain full subcategories of the module categories. We here introduce a more generalized notion, a Morita context between two abelian categories.

3.1. Morita context and its trace ideals

Definition 3.1. Let \( C \) and \( D \) be abelian categories. A Morita context between \( C \) and \( D \) is a pair of adjunctions \( F : D \to C \) and \( G : C \to D \) equipped with two degree-zero natural transformations \( \eta : FG \to \text{Id}_C \) and \( \rho : GF \to \text{Id}_D \) such that \( F \rho \simeq \eta F \) as morphisms \( FG \simeq F \) and \( \rho G \simeq G \eta \) as \( GFG \simeq G \). These equations are called the associativity laws.

When \( C = A-\text{Mod} \) and \( D = B-\text{Mod} \) are respectively the module categories of algebras \( A \) and \( B \), the above definition of Morita context between \( C \) and \( D \) coincides with Definition [12] of that between \( A \) and \( B \) we introduced in the introduction.

Remark 3.2. Iglesias and Torrecillas [19, 20] has defined a more general notion called wide (right) Morita context. They only required that \( F \) and \( G \) are right exact.

For a while we fix a Morita context \((F, G)\) between \( C \) and \( D \) as above.

Notation 3.3. We denote by
\[
\bar{\eta} : G \to F^\vee, \\
\bar{\rho} : F \to G^\vee,
\]
the morphisms induced by adjunctions. Let \( D \) and \( D' \) be the images of \( \bar{\eta} : G \to F^\vee \) and \( \bar{\rho} : F \to G^\vee \) respectively.
The functors $D: C \to D$ and $D': D \to C$ are called Morita context functors. Similarly as ideal functors, a Morita context functor is not left nor right exact. However it has a following property again.

**Lemma 3.4.** Morita context functors preserve all images.

**Proof.** $D$ is both mono and epi since it is a subobject of a left exact functor $F^\vee$ as well as a quotient of a right exact functor $G$. \hfill \Box

Let $I \subset \text{Id}_C$ and $J \subset \text{Id}_D$ be the images of $\eta: FG \to \text{Id}_C$ and $\rho: GF \to \text{Id}_D$ respectively. These are ideal functors on the respective categories by Proposition \ref{prop:ideal-functors} which we call trace ideals. First we study how these functors act on the subcategories defined by these ideal functors $I$ and $J$.

**Lemma 3.5.** Suppose $X \in C$ and consider three morphisms $\eta X: FGX \to X$, $\bar{\eta} X: GX \to F^\vee X$ and $\eta^\vee X: X \to G^\vee F^\vee X$.

1. $X$ is $I$-accessible if and only if $\eta X$ is epic.
2. $X$ is $I$-torsion-free if and only if $\eta^\vee X$ is monic.
3. $X$ is $I$-annihilated if and only if $\eta X = 0$ (equivalently, $\bar{\eta} X = 0$ or $\eta^\vee X = 0$). In particular, it is also equivalent to that $DX = \text{Image}(\bar{\eta} X) = 0$.

**Proof.** Obvious by definition. \hfill \Box

**Lemma 3.6.** Suppose $X \in C$.

1. If $X$ is $I$-accessible then $GX$ is $J$-accessible.
2. If $X$ is $I$-torsion-free then $F^\vee X$ is $J$-torsion-free.
3. If $X$ is $I$-annihilated then both $GX$ and $F^\vee X$ are $J$-annihilated.

In particular, so is $DX$ in each cases.

**Proof.** Since $\rho GX: GFGX \to GX$ is equal to $G\eta X$ and $G$ is right exact, if $\eta X$ is epic then so is $\rho GX$. This means that if $X$ is $I$-accessible then $GX$ is $J$-accessible by the lemma above. (2) and (3) can be proven in a similar manner. The last statement follows from that these properties are inherited to subobjects or quotients. \hfill \Box

**Lemma 3.7.** $\text{Coker}(DX \hookrightarrow F^\vee X)$ is $J$-annihilated for any $X \in C$.

**Proof.** Let $C := \text{Coker}(D \hookrightarrow F^\vee) = \text{Coker}(G \to F^\vee)$ and consider the commutative diagram

$$
\begin{array}{ccc}
GFG & \to & GFF^\vee \\
\downarrow & & \downarrow \rho F^\vee \\
G & \xrightarrow{\eta} & F^\vee \\
& \downarrow \rho C & \downarrow \\
& C & \to 0 \\
\end{array}
$$

Its rows are exact since $GF$ is right exact. Since $\rho F^\vee: GFF^\vee \to F^\vee$ factors through $G$ by the associativity on $FGF$, the induced morphism $\rho C: GFC \to C$ is zero. In other words, $JC = 0$. \hfill \Box

**Proposition 3.8.** If $X \in C$ is $I$-accessible, $DX$ is the unique largest $J$-accessible subobject of $F^\vee X$.

**Proof.** Let $Y \subset F^\vee X$ be $J$-accessible. Then by Proposition \ref{prop:ideal-functors} $Y' := DX + Y$ and $Y'/DX$ are also $J$-accessible. On the other hand, by the lemma above $Y'/DX \subset F^\vee X/DX$ must be $J$-annihilated too. Hence we conclude that $Y' = DX$, that is, $Y \subset DX$. \hfill \Box
3.2. Category equivalence

The first remarkable result which Morita context brings is the equivalence of categories between respective subcategories defined by ideal functors.

**Theorem 3.9.** D and D‘ induce a category equivalence C" ≃ D'.

To prove this theorem, first we list several endofunctors on C into a diagram.

**Lemma 3.10.** Consider the following epi-mono factorizations

\[ \eta: FG \rightarrow I \hookrightarrow \text{Id}_C, \quad \bar{\eta}G: FG \rightarrow D'G \hookrightarrow G^\vee G, \quad \bar{\rho}D: FD \rightarrow D'D \hookrightarrow G^\vee D. \]

These epimorphisms factor through

\[ FG \rightarrow FD \rightarrow I \rightarrow D'G \rightarrow D'D. \]

Dually, monomorphisms \( \text{Id}_C \hookrightarrow G^\vee F^\vee, D'F \hookrightarrow G^\vee F^\vee \) and \( D'D \hookrightarrow G^\vee D \) factor through

\[ D'D \hookrightarrow D'F^\vee \hookrightarrow I^\vee \hookrightarrow G^\vee D \hookrightarrow G^\vee F^\vee. \]

These chains of morphisms fit into the commutative diagram

\[
\begin{array}{cccccc}
FG & \rightarrow & FD & \rightarrow & I & \rightarrow & D'G & \rightarrow & D'D \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
FF^\vee & \rightarrow & \text{Id}_C & \rightarrow & G^\vee G & \rightarrow & \rightarrow & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
D'D & \rightarrow & D'F^\vee & \rightarrow & I^\vee & \rightarrow & G^\vee D & \rightarrow & G^\vee F^\vee.
\end{array}
\]

**Proof.** First the morphisms \( FG \rightarrow FD \) and \( D'G \rightarrow D'D \) at both ends are induced by \( \bar{\eta}: G \rightarrow D \hookrightarrow F^\vee \). Since the functors F and D' are both epi, these morphisms are epic. Now consider the diagram

\[
\begin{array}{c}
FG \\
\downarrow & \downarrow \downarrow \\
FD & I & D'G \\
\downarrow & \downarrow & \downarrow \\
FF^\vee & \rightarrow & \text{Id}_C & \rightarrow & G^\vee G.
\end{array}
\]

The right pentagon is commutative by the help of the associativity on \( GFG \) while the commutativity of the left one is trivial. Since \( I \) and \( D'G \) are the images of the respective pentagons, there exist unique morphisms \( FD \rightarrow I \rightarrow D'G \) which make the diagram commutes. Now it is left us to check the commutativity for \( \bar{\rho}D \). We have the diagram

\[
\begin{array}{c}
FG \\
\downarrow & \downarrow \\
FD & \rightarrow & D'D
\end{array}
\]
where the outer square trivially commutes. Since $FG \to FD$ is epic and the upper triangle commutes, the lower also does. The dual statement goes similarly and the last commutativity has been already proven. 

**Corollary 3.11.** $D'C'$ is equal to the image of the composite $I \to \text{Id}_C \to I$. In particular, if $X \in C'$ then canonically $X \cong D'DX$. 

Putting this corollary and Lemma 3.6 together, we obtain Theorem 3.9.

**Remark 3.12.** Though the categories $C'$ and $D'$ are equivalent, their exact structures may differ. For example, let $A$ and $B$ be the upper triangle matrix algebras

\[
A := \begin{pmatrix}
* & * & *
\end{pmatrix}
\begin{pmatrix}
0 & * & *

0 & 0 & *
\end{pmatrix},
B := \begin{pmatrix}
* & *
0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & *
\end{pmatrix}
\]

over $k$. Let $M$ and $N$ be the bimodules

\[
M := \begin{pmatrix}
* & *
0 & *
\end{pmatrix},
N := \begin{pmatrix}
* & *
0 & *
\end{pmatrix}
\begin{pmatrix}
0 & *
\end{pmatrix}
\]

and define $\eta: M \otimes_B N \to A$ and $\rho: N \otimes_A M \to B$ by matrix multiplication. These data define a Morita context between $A$-$\text{Mod}$ and $B$-$\text{Mod}$. $\rho$ is surjective and the image of $\eta$ is

\[
I := \begin{pmatrix}
* & * & *
0 & 0 & *
0 & 0 & *
\end{pmatrix} \subset A,
\]

so the Morita context functors induce a category equivalence $(A$-$\text{Mod})' \cong B$-$\text{Mod}$. However it sends a short exact sequence

\[
0 \to \begin{pmatrix}
\ast
\end{pmatrix} \to \begin{pmatrix}
\ast
\end{pmatrix} \to \begin{pmatrix}
\ast
\end{pmatrix} \to 0
\]

in $B$-$\text{Mod}$ to the sequence

\[
0 \to \begin{pmatrix}
\ast
\end{pmatrix} \to \begin{pmatrix}
\ast
\end{pmatrix} \to \begin{pmatrix}
\ast
\end{pmatrix} \to 0
\]

in $A$-$\text{Mod}$, which is obviously not exact at the middle term.

As we have seen in this remark, the category equivalence does not preserve extensions in general. However, it is true if one of the categories is semisimple.

**Lemma 3.13.** If $\mathcal{D}$ is semisimple, then $\text{Ext}_C^1(X, Y) = 0$ for any $X, Y \in C'$.

**Proof.** Let $0 \to Y \to E \to X \to 0$ be a short exact sequence in $C$. Since $C'$ is closed under extensions, $E$ is also in $C'$. Now $C' \cong D'$ is semisimple, so that this sequence splits.

On the other hand, the Ext preserving property for the other category is induced from the following condition.
Lemma 3.14. Suppose that $\rho : GF \to \text{Id}_D$ is surjective. Then $\text{Ext}_{C^I}^1(X,Y) \cong \text{Ext}_{C^I}^1(X,Y)$ for any $X,Y \in C^I$.

Proof. By the assumption $FGFG \to FG$ is also surjective. This implies $f^2 = I$, so that we can use Proposition 2.34. $\square$

3.3. Correspondence on simple objects

Next we prove the correspondence between the simple objects in the respective subcategories.

Theorem 3.15. $D$ and $D'$ induce a one-to-one correspondence $\text{Irr} C^I \cong \text{Irr} D^J$.

Though we have proven the category equivalence $C^I \cong D^J$, we have to prove this theorem independently since we do not know how to characterize the set $\text{Irr} C^I$ from the category $C^I$ itself. Actually, using Lemma 3.6 again, this theorem is obtained as an immediate corollary of the next theorem.

Theorem 3.16. Let $X \in \text{Irr} C^I$. Then $DX$ is the simple socle of $F^\vee X$ as well as the simple top of $GX$.

Proof. By the assumption $X \notin C_I$, we have $DX \neq 0$. Take any non-zero subobject $Y \hookrightarrow F^\vee X$. Then the corresponding morphism $FY \to X$ is also non-zero, so it must be epic since $X$ is simple. Now consider the commutative diagram

\[
\begin{array}{ccc}
GFY & \longrightarrow & GFF^\vee X \\
\downarrow & & \downarrow \\
JY & \longrightarrow & JF^\vee X
\end{array}
\]

Since $G$ is right exact $GFY \to GX$ is also epic. So we have

$JY = \text{Image}(GFY \to F^\vee X) = \text{Image}(GX \to F^\vee X) = DX.$

This implies that $DX$ is contained in an arbitrary non-zero subobject $Y \hookrightarrow F^\vee X$, so it must be a simple socle of $F^\vee X$. Dually it is also a simple top of $GX$. $\square$

Putting it together with Lemma 3.6 and Lemma 3.7 we obtain the next corollary.

Corollary 3.17. Let $X \in \text{Irr} C$ and $Y \in \text{Irr} D^J$. Then

$[GX : Y] = [F^\vee X : Y] = \begin{cases} 1 & Y \cong DX, \\ 0 & \text{otherwise.} \end{cases}$

Here $[M : S]$ is the multiplicity of a simple object $S$ in the composition factors of $M$. If $M$ is not of finite length this symbol does not make sense in general, but the formula above can be always read in an appropriate manner.

When we replace $X$ above to its injective hull or its projective cover, we obtain similar statements.

Proposition 3.18. Let $X \in \text{Irr} C^I$ and suppose that it has a projective cover $P \twoheadrightarrow X$. Then $GP \twoheadrightarrow GX \twoheadrightarrow DX$ is the top of $GP$. 31
**Proof.** Take \( Y \in \text{Irr } \mathcal{D} \) and let \( C := \text{Coker}(D'Y \hookrightarrow G'Y) \). By the projectiveness of \( P \), the sequence
\[
0 \to \text{Hom}_{\mathcal{C}}(P, D'Y) \to \text{Hom}_{\mathcal{C}}(P, G'Y) \to \text{Hom}_{\mathcal{C}}(P, C) \to 0
\]
is exact. By Lemma [5.7] \( C \) is \( I \)-annihilated. Hence it has no subquotients isomorphic to \( X \), so \( \text{Hom}_{\mathcal{C}}(P, C) = 0 \) by a property of projective cover. Thus
\[
\text{Hom}_{\mathcal{D}}(GP, Y) \cong \text{Hom}_{\mathcal{C}}(P, G'Y) \cong \text{Hom}_{\mathcal{C}}(P, D'Y).
\]
Now \( D'Y \) is simple or zero, so there is a non-zero morphism \( GP \to Y \) if and only if \( D'Y \cong X \), or equivalently, \( Y \cong DX \). Moreover
\[
\text{Hom}_{\mathcal{D}}(GP, DX) \cong \text{Hom}_{\mathcal{C}}(P, X) \cong \text{End}_{\mathcal{C}}(X) \cong \text{End}_{\mathcal{D}}(DX).
\]
Thus \( GP \to DX \) is the unique its simple quotient.

**Remark** that \( \text{Ker}(GP \to DP) \) is \( J \)-annihilated by the dual of Lemma [5.7] but in general \( \text{Ker}(DP \to DX) \) is not, so may contains a composition factor in \( \mathcal{C} \).

### 3.4. Morita context among multiple categories

We here generalize the notion of Morita context, from that between two categories to that among more than two categories. Let us take an index set \( \Lambda \) which is not necessarily finite. We assume that every category appears in this subsection is closed under sums and intersections with cardinality \( \# \Lambda \).

**Definition 3.19.** Let \( \{C_\lambda\}_{\lambda \in \Lambda} \) be a family of abelian categories indexed by a set \( \Lambda \). A Morita context among \( \{C_\lambda\} \) is a family of adjunctions \( F_{\lambda \mu} : C_\mu \to C_\lambda \) indexed by a pair of \( \lambda, \mu \in \Lambda \), equipped with a family of degree-zero natural transformations \( \eta_{\lambda \mu \nu} : F_{\lambda \mu}F_{\mu \nu} \to F_{\lambda \nu} \) indexed by a triple of \( \lambda, \mu, \nu \in \Lambda \) which satisfies the following conditions.

1. (The associativity law) For each \( \lambda, \mu, \nu, \pi \in \Lambda \), the square
\[
\begin{array}{ccc}
F_{\lambda \mu}F_{\mu \nu} & \xrightarrow{\eta_{\lambda \mu \nu}} & F_{\lambda \nu} \\
F_{\lambda \pi} & \downarrow{\eta_{\lambda \pi \nu}} & F_{\lambda \pi} \\
F_{\lambda \mu}F_{\mu \pi} & \xrightarrow{\eta_{\lambda \mu \pi}} & F_{\lambda \pi}
\end{array}
\]
commutes.

2. (The unit law) For each \( \lambda \), there is a fixed isomorphism \( F_{\lambda \lambda} \cong \text{Id}_{C_\lambda} \) such that \( \eta_{\lambda \lambda \mu} \) and \( \eta_{\lambda \mu \lambda} \) are respectively equal to
\[
F_{\lambda \mu}F_{\mu \lambda} \cong \text{Id}_{C_\lambda}F_{\mu \lambda} \cong F_{\mu \lambda}, \quad F_{\lambda \mu}F_{\mu \lambda} \cong F_{\mu \lambda}F_{\mu \lambda} \cong \text{Id}_{C_\lambda} \approx F_{\lambda \mu}.
\]

One can easily verify that when \( \# \Lambda = 2 \) this definition is equivalent to the previous one.

**Remark 3.20.** Let \( \mathcal{A} \) be a 2-category which consists of abelian categories as 0-cells, adjunctions as 1-cells and natural transformation as 2-cells. Consider \( \Lambda \) as a codiscrete category, that is, we regard that there exists a unique morphism \( \mu \to \lambda \) for each \( \lambda, \mu \in \Lambda \). Then a Morita context is just a lax functor \( \mathcal{F} : \Lambda \to \mathcal{A} \) (where \( C_\lambda = \mathcal{F}(\lambda), F_{\lambda \mu} = \mathcal{F}(\mu \to \lambda) \)) such that the unit \( \text{Id}_{\mathcal{F}(\lambda)} \to \mathcal{F}(\lambda \to \lambda) \) is an isomorphism for every \( \lambda \in \Lambda \).
In particular, $\alpha, \lambda, \mu \in \{A, \lambda, \mu\}$ also defines a Morita context among the collection of an $(\mathcal{A}_\lambda, \mathcal{A}_\mu)$-bimodule $\Lambda$. Moreover the composition of morphisms

$$\text{Hom}_\mathcal{A}(X_\mu, X_\lambda) \otimes \text{Hom}_\mathcal{A}(X_\lambda, X_\mu) \to \text{Hom}_\mathcal{A}(X_\mu, X_\mu)$$

gives these natural transformations between these adjunctions which is associative and unital. Hence these define a Morita context among the categories $[\mathcal{A}_\lambda, \text{Mod}]$. Conversely all Morita contexts among module categories are obtained in this way.

**Example 3.21.** Let $\mathcal{A}$ be a category, and take an object $X_\lambda \in \mathcal{A}$ for each $\lambda$. Let $\mathcal{A}_\lambda := \text{End}_{\mathcal{A}}(X_\lambda)$ be its endomorphism algebra. Then for each pair of $\lambda, \mu$, $\text{Hom}_{\mathcal{A}}(X_\mu, X_\lambda)$ is a $(\mathcal{A}_\lambda, \mathcal{A}_\mu)$-bimodule so it induces an adjunction $\mathcal{A}_\mu \rightarrow \mathcal{A}_\lambda$. Moreover the composition of morphisms

$$\text{Hom}_\mathcal{A}(X_\mu, X_\lambda) \otimes \text{Hom}_\mathcal{A}(X_\lambda, X_\mu) \to \text{Hom}_\mathcal{A}(X_\mu, X_\mu)$$

gives a natural transformations between these adjunctions which is associative and unital. Hence these define a Morita context among the categories $[\mathcal{A}_\lambda, \text{Mod}]$. Conversely all Morita contexts among module categories are obtained in this way.

**Example 3.22.** More generally, take a small full subcategory $\mathcal{A}_\lambda \subset \mathcal{A}$ for each $\lambda$. Then a collection of an $(\mathcal{A}_\lambda, \mathcal{A}_\mu)$-module

$$\mathcal{A}_\mu^{op} \otimes \mathcal{A}_\lambda \to \mathcal{V}$$

$X \otimes Y \mapsto \text{Hom}_\mathcal{A}(X, Y)$

also defines a Morita context among $[\mathcal{A}_\lambda, \text{Mod}]$.

Suppose that $\{F_{\lambda\mu}\}_{\lambda, \mu \in \Lambda}$ is a Morita context among categories $[\mathcal{C}_\lambda]_{\lambda \in \Lambda}$. For each triple of $\alpha, \lambda, \mu \in \Lambda$, let $\Gamma_{\lambda\mu}$ be a subfunctor of $F_{\lambda\mu}$ defined by

$$\Gamma_{\lambda\mu} := \text{Image}(\eta_{\lambda\mu} : F_{\mu\lambda} \to F_{\lambda\mu}).$$

In particular, $\Gamma_{\lambda\lambda} \subset F_{\lambda\lambda} = \text{Id}_{\mathcal{C}_\lambda}$ is an ideal functor on $\mathcal{C}_\lambda$. The unit law implies that $I_{\lambda\mu} = I_{\mu\lambda} = F_{\lambda\nu}$. By the associativity law there are natural transformations

$$\Gamma_{\lambda\mu} F_{\mu\nu} \to \Gamma_{\lambda\nu} \quad \text{and} \quad F_{\lambda\mu} \Gamma_{\mu\nu} \to \Gamma_{\lambda\nu}$$

induced by $\eta_{\mu\nu}$. Since $F_{\lambda\mu}$ is right exact, they induce

$$(F_{\lambda\mu} / \Gamma_{\lambda\mu}) (F_{\mu\nu} / \Gamma_{\mu\nu}) \to (F_{\lambda\nu} / \Gamma_{\lambda\nu})$$

Now take a subset $\Lambda' \subset \Lambda$. Then clearly the restriction $\{F_{\lambda\mu}\}_{\lambda, \mu \in \Lambda'}$ gives a Morita context among the subcollection $(\mathcal{C}_\lambda)_{\lambda \in \Lambda'}$. In contrast, we can also take a "quotient" of this Morita context with respect to $\Lambda'$ as follows.

**Proposition 3.23.** For each $\lambda, \mu \in \Lambda$, let

$$I_{\lambda\mu}' := \sum_{\alpha \in \Lambda'} T_{\lambda\mu}^{\alpha}$$

and $\mathcal{C}'_{\lambda} := (\mathcal{C}_\lambda)_{T_{\lambda\mu}^{\alpha}}$. Then there exists a Morita context $\{F_{\lambda\mu}'\}$ among the abelian categories $[\mathcal{C}_\lambda']$ defined by

$$F_{\lambda\mu}' := \Phi_{\lambda\mu}^\Lambda (F_{\lambda\mu} / \Gamma_{\lambda\mu}) \Phi_{\lambda\mu}.$$ 

**Proof.** Since $F_{\lambda\mu}$ is cocontinuous, by taking colimits of natural transformations above we obtain

$$(F_{\lambda\mu} / \Gamma_{\lambda\mu}) (F_{\mu\nu} / \Gamma_{\mu\nu}) \to (F_{\lambda\nu} / \Gamma_{\lambda\nu})$$

Moreover, by the unit law we have $T_{\lambda\mu} (F_{\lambda\mu} / I_{\lambda\mu}') T_{\mu\nu} = F_{\lambda\mu} / I_{\lambda\mu}'$. Thus there are a natural transformation

$$\eta_{\lambda\mu}' : F_{\lambda\mu}' F_{\mu\nu}' = \Phi_{\lambda\mu}^\Lambda (F_{\lambda\mu} / \Gamma_{\lambda\mu}) (F_{\mu\nu} / \Gamma_{\mu\nu}) \Phi_{\lambda\mu} \to \Phi_{\lambda\mu}^\Lambda (F_{\lambda\mu} / \Gamma_{\lambda\mu}) \Phi_{\lambda\mu} = F_{\lambda\nu}'$$

33
and an isomorphism
\[ F'_{\lambda\lambda} \cong \Phi_{\lambda\lambda}^* T_{\lambda\lambda} \Phi_{\lambda\lambda} = \text{Id}_{C_\lambda} \]
which form a Morita context.

Note that \( C'_\alpha = \{0\} \) for every \( \alpha \in \Lambda' \), so the quotient Morita context above should be considered as parameterized by the complement set \( \Lambda \setminus \Lambda' \). When \( \Lambda' \) has a decomposition \( \Lambda' = \Lambda'_1 \sqcup \Lambda'_2 \), taking the quotient by \( \Lambda' \) is equal to first taking by \( \Lambda'_1 \), then by \( \Lambda'_2 \).

### 3.5. Morita context with a partial order

As a special case of quotient, let us consider the case that \( \Lambda' \) in the previous subsection consists of a single element \( \alpha \). For each \( \lambda \in \Lambda \setminus \{\alpha\} \), the pair \((F_{\lambda\alpha}, F_{\alpha\lambda})\) is a Morita context between two categories \( C_\lambda \) and \( C_\alpha \). Hence we have a category equivalence
\[ (C_\lambda)^{F_{\lambda\alpha}} \cong (C_\alpha)^{F_{\alpha\lambda}} \]
and a one-to-one correspondence
\[ \text{Irr}(C_\lambda)^{F_{\lambda\alpha}} \cong \text{Irr}(C_\alpha)^{F_{\alpha\lambda}}. \]

In practice we should choose \( \alpha \) such that the structure of \( C_\alpha \) is very simple so that we can describe a part of \( C_\lambda \), which may be hard to study, by terms of \( C_\alpha \). Now on the collection of the rest part \( C'_\lambda = (C_\lambda)^{F_{\lambda\alpha}} \) we have a new Morita context, so we can recursively continue this process for \( C'_\lambda \) by choosing another \( \beta \in \Lambda \) to decompose \( C_\lambda \) into small parts. In order to perform this strategy at one time, we introduce a partial order on the set \( \Lambda \) as we do before in the previous subsection. Intuitively it indicates the order of \( \alpha, \beta, \ldots \) we pick up from \( \Lambda \).

**Definition 3.24.** Let \( [F_{\lambda\mu}] \) be a Morita context among the categories \( \{C_\lambda\} \). A partial order \( \leq \) on the set \( \Lambda \) is said to be compatible with \( [F_{\lambda\mu}] \) if it satisfies
\[ F_{\lambda\mu} = \sum_{\nu \leq \lambda, \mu} I_{\nu\lambda\mu}, \text{ where } I_{\nu\lambda\mu} := \text{Image}(F_{\lambda\nu} F_{\nu\mu} \to F_{\lambda\mu}) \]
for each pair of \( \lambda, \mu \in \Lambda \).

When \( \lambda \) and \( \mu \) are comparable then the condition above is trivially satisfied. Hence every total order on \( \Lambda \) is compatible.

**Lemma 3.25.** If \( \Lambda \) is well-founded, then the condition above is equivalent to that
\[ F_{\lambda\mu} = \sum_{\nu < \lambda} I'_{\nu\lambda\mu} \]
is satisfied for each pair of \( \lambda, \mu \in \Lambda \) such that \( \lambda \nleq \mu \).

**Proof.** Clearly the first condition implies the second. Suppose the second one. We prove the first condition for a fixed \( \mu \) by transfinite induction on \( \lambda \). So assume that for every \( \nu < \lambda \) we have \( F_{\nu\mu} = \sum_{\rho \leq \nu, \mu} I'_{\rho\nu\mu} \). If \( \lambda \leq \mu \) then the condition is trivially satisfied so assume \( \lambda \nleq \mu \). Then by the assumption we have \( F_{\lambda\mu} = \sum_{\nu < \lambda} I'_{\nu\lambda\mu} \). Each \( I'_{\nu\lambda\mu} \) is contained in
\[ \sum_{\pi \leq \lambda, \mu} \text{Image}(F_{\lambda\nu} F_{\nu\mu} F_{\mu\rho} \to F_{\lambda\rho}) \subset \sum_{\pi \leq \lambda, \mu} I'_{\pi\lambda\mu} \]
since \( F_{\lambda\rho} \) is cocontinuous. Thus the condition is also satisfied for \( \lambda \). \qed
Now assume that a partial order \( \preceq \) is compatible with \( \{F_{\mu \nu}\} \). Let us denote
\[
I_{\preceq \lambda \alpha \beta} := \sum_{\mu \preceq \lambda} I_{\mu \alpha \beta} \quad \text{and} \quad I_{\prec \lambda \alpha \beta} := \sum_{\mu \prec \lambda} I_{\mu \alpha \beta}.
\]

**Proposition 3.26.** For each \( \omega \in \Lambda \) the family \( \{I_{\preceq \lambda \omega \omega}\}_{\lambda \in \Lambda} \) is an ideal filter on \( C_{\omega} \).

**Proof.** For simplicity let us write \( I_{\preceq \lambda \omega \omega} \) and \( I_{\prec \lambda \omega \omega} \). The first two conditions in Definition 2.40 are obvious. So we prove \( I_{\preceq \lambda \omega \omega} \subset \sum_{\rho \leq \lambda, \mu} I_{\preceq \rho \omega \omega} \) for each \( \lambda, \mu \in \Lambda \). Let us take \( \nu \leq \lambda \) and \( \pi \leq \mu \). Since \( F_{\omega \nu} \) is cocontinuous, we have
\[
I_{\nu \omega \omega} \subset \text{Image}(F_{\omega \nu}F_{\nu \pi}F_{\pi \omega} \to \text{Id}_{C_{\omega}}) \subset \sum_{\rho \leq \nu, \pi} I_{\rho \omega \omega}.
\]
Hence by taking sum we obtain the inclusion as desired. \( \square \)

Using this ideal filter the category \( C_{\omega} \) is divided into \( C_{\omega}[\lambda] = (C_{\omega})_{\preceq \lambda} \cap (C_{\omega})_{\prec \lambda} \). For each \( \lambda \), by taking the quotient with respect to the subset \( \Lambda' = \{\mu \in \Lambda | \mu < \lambda\} \) we have a Morita context between \( (C_{\omega})_{\prec \lambda} \) and \( (C_{\omega})_{\preceq \lambda} \). The corresponding trace ideal in \( (C_{\omega})_{\preceq \lambda} \) is \( (I_{\preceq \lambda \omega \omega})_{\preceq \lambda} \). Thus by letting
\[
C_{\lambda}(\omega) := (C_{\lambda})_{\preceq \lambda} \cap (C_{\lambda})_{\prec \lambda}
\]
and \( \text{Irr} C_{\lambda}(\omega) := \{V \in \text{Irr} C | V \in C_{\lambda}(\omega)\} \) we obtain the following theorem.

**Theorem 3.27.** For each \( \lambda \leq \omega \), there is a Morita context between \( (C_{\omega})_{\prec \lambda} \) and \( (C_{\lambda})_{\preceq \lambda} \) which induces a category equivalence \( C_{\omega}[\lambda] \equiv C_{\lambda}(\omega) \) and a one-to-one correspondence \( \text{Irr} C_{\omega}[\lambda] \leftrightarrow \text{Irr} C_{\lambda}(\omega) \). If \( \lambda \not\preceq \omega \), then \( C_{\omega}[\lambda] = 0 \). \( \square \)

**Corollary 3.28.** If \( \Lambda \) is well-founded, we have
\[
\text{Irr} C_{\omega} = \bigsqcup_{\lambda \leq \omega} \text{Irr} C_{\omega}[\lambda] \leftrightarrow \bigsqcup_{\lambda \leq \omega} \text{Irr} C_{\lambda}(\omega). \quad \square
\]

4. Generalized cellular algebras

Now we concentrate on representation theory of algebras. Here continuously the term “an algebra” means a \( V \)-algebra. With the help of the category equivalence \( \text{Adj}(B-\text{Mod}, A-\text{Mod}) \simeq A-\text{Mod}-B \), we can interpret all the notions we have introduced in the previous sections into the language of modules. For example, ideal functors on the category are replaced by 2-sided ideals in an algebra. So an ideal filter is just a collection of 2-sided ideals which satisfies the similar conditions.

In this section we fix a partially ordered set \( (\Lambda, \preceq) \) and an indexed family \( \{B_{\lambda}\}_{\lambda \in \Lambda} \) of algebras. We introduce a generalized notion of standardly based algebra and that of cellular algebra over the family \( \{B_{\lambda}\} \), not over the single base algebra \( k \). We also study its Morita invariance motivated by the work of König and Xi [32].

35
4.1. Standard filter

We start from a very general setting. In the last of previous section we decompose a category into small parts in order to study them one by one. A standardly filtered algebra is defined so that we can perform similar strategy for its module category.

**Definition 4.1.** Let $A$ be an algebra. A *prestandard filter* of $A$ over $\{B_i\}$ is a datum consisting of:

- an ideal filter $\{A^{\leq i}\}_{i \in \Lambda}$ on $A$,
- for each $i \in \Lambda$, a 2-sided ideal $B'_i \subset B_i$,
- for each $i \in \Lambda$, a Morita context $(M_i, N_i)$ between $A/A^{\leq i}$ and $B_i/B'_i$ whose trace ideal in $A$ is $A^{\leq i}/A^{\leq i}$.

Moreover if it satisfies $A^{\leq i}M_i = 0$ and $N_iA^{\leq i} = 0$ for each pair of $i, \mu$ such that $i \neq \mu$, we call it a *standard filter*. An algebra equipped with a standard filter is called a *standardly filtered algebra*. If each $B_i$ is just the base ring $k$, we simply say it is a standard filter over $k$ instead of over the family $[k]$.

Now Lemma 0.5 in the introduction is just a reformulation of Theorem 3.27 on Example 3.21.

**Remark 4.2.** By Lemma 3.25 if $\Lambda$ is well-founded the first assumption of Lemma 0.5 can be weakened to

$$\text{Hom}_A(X_\mu, X_i) = \mathcal{A}^{\leq 1}(X_\mu, X_i).$$

**Remark 4.3.** In the settings of the lemma, for $\omega_1, \ldots, \omega_n \in \Lambda$, the algebra

$$\text{End}_A(\bigoplus_i X_{\omega_i}) = \bigoplus_{i,j} \text{Hom}_A(X_{\omega_i}, X_{\omega_j})$$

is also standardly filtered. It can be proven by adding a new index $\infty$ which is greater than any element of $\Lambda$ so that $X_{\infty} = \bigoplus_i X_{\omega_i}$, then remove it since it is needless by that $\mathcal{A}^{\leq \infty} = \mathcal{A}^{< \infty}$.

Actually the condition for being a standardly filtered algebra can be weakened as follows.

**Lemma 4.4.** Suppose that $(M, N)$ is a Morita context between algebras $A$ and $B$, and let us write its equipped maps as $\eta: M \otimes_B N \to A$ and $\rho: N \otimes_A M \to B$. Let

$$B' := \{b \in B | \eta(mb \otimes n) = 0 \text{ for all } m \in M, n \in N\},$$

$$M' := \{m \in M | \eta(m \otimes n) = 0 \text{ for all } n \in N\},$$

$$N' := \{n \in N | \eta(m \otimes n) = 0 \text{ for all } m \in M\}.$$ 

Then $(M/M', N/N')$ is a Morita context between algebras $A$ and $B/B'$ with the same trace ideal in $A$.

**Proof.** First by definition $\eta: M/M' \otimes_B N/N' \to A$ is well-defined. In addition we have $MB' \subset M'$ and $B'N \subset N'$ so that $M/M'$ and $N/N'$ can be considered as modules over $B/B'$. Moreover $\rho(M' \otimes_A N), \rho(M \otimes_A N') \subset B'$ by the associativity, so that $\rho: N/N' \otimes_B M/M' \to B/B'$ is also well-defined. Now it is clear that these data form a Morita context between $A$ and $B/B'$. \qed

Note that $B'$ above is the common annihilator of $M/M'$ and $N/N'$, so that these are faithful modules over $B/B'$.  

36
Proposition 4.5. If an algebra $A$ has a prestandard filter, it also has a standard filter.

Proof. Take a prestandard filter of $A$ as above. For each $\lambda$, let $A^{\leq \lambda} := \sum_{\mu \leq \lambda} A^{= \mu}$. Then
\[
\begin{align*}
\eta(A^{\leq \lambda} M_{\lambda} \otimes B_{\lambda} N_{\lambda}) &= A^{\leq \lambda} A^{= \lambda} + A^{< \lambda} = A^{\leq \lambda}, \\
\eta(M_{\lambda} \otimes B_{\lambda} A^{\leq \lambda}) &= A^{\leq \lambda} A^{\leq \lambda} + A^{< \lambda} = A^{\leq \lambda}.
\end{align*}
\]
Hence taking $M'_{\lambda} \subset M_{\lambda}$ and $N'_{\lambda} \subset N_{\lambda}$ as in the lemma above, we have inclusions $A^{\leq \lambda} M_{\lambda} \subset M'_{\lambda}$ and $N_{\lambda} A^{\leq \lambda} \subset N'_{\lambda}$, Thus by replacing $(M_{\lambda}, N_{\lambda})$ with the quotients $(M_{\lambda} / M'_{\lambda}, N_{\lambda} / N'_{\lambda})$ we obtain a standard filter.

The notion of standardly filtered algebra is a Morita invariant and inherited by Peirce decomposition.

Proposition 4.6. Let $A$ be a standardly filtered algebra over $\{B_{\lambda}\}$.

1. For any idempotent $e \in A$, the algebra $eAe$ is also standardly filtered.
2. If an algebra $A'$ is Morita equivalent to $A$ (i.e. $A\text{-Mod} \simeq A'\text{-Mod}$), $A'$ is also standardly filtered.

Proof. (1) follows from that the pair $(eM_{\lambda}, N_{\lambda}e)$ forms a Morita context between $eAe / eAe < \lambda e$ and $B_{\lambda} / B_{\lambda}'$. (2) is a consequence of that the definition of standard filter on an algebra can be translated into the language of its module category.

For an algebra $A$ and a 2-sided ideal $I \subset A$, let us write
\[
\text{Irr}(A) := \text{Irr}(A\text{-Mod}) \quad \text{and} \quad \text{Irr}^J(A) := \text{Irr}(A\text{-Mod}^J) = \text{Irr}(A) \setminus \text{Irr}(A/I)
\]
for short. More generally, for $J \subset I \subset A$ let
\[
\text{Irr}^J_I(A) := \text{Irr}^{I/J}(A/J) = \text{Irr}(A/J) \setminus \text{Irr}(A/I).
\]

Then Proposition 4.3 and Theorem 3.15 immediately bring us the following classification of simple $A$-modules. This is a generalization of [13, Theorem 3.4].

Theorem 4.7. Suppose that $\Lambda$ is well-founded. Let $A$ be a standardly filtered algebra over $\{B_{\lambda}\}$ and take its prestandard filter as above. For each $\lambda$, let $B''_{\lambda} / B'_{\lambda} \subset B_{\lambda} / B'_{\lambda}$ be the trace ideal of the Morita context. Then there is a one-to-one correspondence
\[
\text{Irr}(A) \cong \bigsqcup_{\lambda \in \Lambda} \text{Irr}_{B_{\lambda}}^{B''_{\lambda}}(B_{\lambda})
\]
induced by Morita contexts.

Let us write $[M : S]$ the multiplicity of a simple module $S$ in the composition factors of $M$. The analogue of the decomposition matrix of cellular algebra can be defined as follows. It also satisfies the unitriangular property.

Lemma 4.8. Take a standard filter of $A$ as above. Let $\lambda, \mu \in \Lambda$ and take $S \in \text{Irr}^{A^{\leq \mu}}_{A^{\leq \lambda}}(A)$, $T \in \text{Irr}(B_{\lambda})$. Then unless $\lambda \leq \mu$
\[
[M_{\lambda} \otimes B_{\lambda} T : S] = [\text{Hom}_{B_{\lambda}}(N_{\lambda}, T) : S] = 0.
\]
Moreover, if $A = \mu$,

$$[M_\lambda \otimes_{B_\lambda} T : S] = [\text{Hom}_{B_\lambda}(N_\lambda, T) : S] = \begin{cases} 1 & \text{if } T \cong DS, \\ 0 & \text{otherwise}. \end{cases}$$

Here $D$ is the Morita context functor which induces $\text{Irr}^{A_{\lambda}}_{A_{\lambda}}(A) \xrightarrow{1:1} \text{Irr}^{B'_{\lambda'}}_{B'_{\lambda'}}(B_\lambda)$.

**Proof.** The first equation follows from that $M_\lambda \otimes_{B_\lambda} T$ and $\text{Hom}_{B_\lambda}(N_\lambda, T)$ are $A_{\lambda}$-annihilated. The second follows from Corollary [3.17] if $T \in \text{Irr}(B/B'_\lambda)$; otherwise $M_\lambda \otimes_{B_\lambda} T = \text{Hom}_{B_\lambda}(N_\lambda, T) = 0$ so the formula also holds trivially. \(\square\)

### 4.2. Well-based standard filter

Graham and Lehrer [13, Theorem 3.7] also proved that for a cellular algebra we can compute its Cartan matrix by its decomposition matrix. A general standardly filtered algebra does not have this property, so we strengthen its conditions to prove an analogue of the theorem.

**Definition 4.9.** Let $(M, N)$ be a Morita context between algebras $A$ and $B$. We say that $(M, N)$ is well-based over $B$ if $M$ and $N$ are both finitely generated and projective over $B$ and the map $M \otimes_B N \to A$ is injective.

**Definition 4.10.** A prestandard filter of $A$ is said to be well-based if

1. the ideal filter $\{A^{\perp}\}$ is rigid,
2. the Morita context $(M_\lambda, N_\lambda)$ is well-based over $B_\lambda/B'_\lambda$ for every $\lambda \in \Lambda$.

An algebra equipped with a well-based standard filter is called a weakly standardly based algebra.

We can prove a statement similar to Proposition [4.5] for weakly standardly based algebras. The proof is clear from the lemmas below.

**Lemma 4.11.** Let $B$ be an algebra. Let $M$ be a finitely generated projective right $B$-module and $N$ be a left $B$-module. Then $x \in M$ satisfies $0 = x \otimes y \in M \otimes_B N$ for all $y \in N$ if and only if $x \in M \cdot \text{Ann}_B(N)$. Here $\text{Ann}_B(N) := \{b \in B | bN = 0\}$ denotes the (left) annihilator of $N$.

**Proof.** The “if” part is obvious so we prove the “only if” part. We may assume that there is an $m \times m$ idempotent matrix $e = (e_{ij})$ such that $M = eB^m$. Since $M \subseteq B^m$ is an direct summand, we can regard $M \otimes_B N \subseteq B^m \otimes_B N = N^m$. Suppose $x = (x_1, \ldots, x_m) \in M$ satisfies $x \otimes N = 0$. This means that $0 = x \otimes n = \sum_i (x_1, \ldots, x_m, n) \in N^m$ for all $n \in N$, that is, $x_1, \ldots, x_m \in \text{Ann}_B(N)$. Thus

$$x = ex = (e_{11}, \ldots, e_{mm})x_1 + \cdots + (e_{1m}, \ldots, e_{mm})x_m \in M \cdot \text{Ann}_B(N).$$

\(\square\)

**Lemma 4.12.** Let $A$, $B$, $M$ and $N$ as in Lemma [4.4]. If $(M, N)$ is well-based over $B$, then so is $(M/M', N/N')$ over $B/B'$.

**Proof.** By the lemma above, we have $M' = M \cdot \text{Ann}_B(N) \subseteq MB'$. We already has the other inclusion so $M' = MB'$, hence $M/M' \cong M \otimes_B (B/B')$ is finitely generated and projective over $B/B'$. The same holds for $N/N'$. It is clear that $\eta: M/M' \otimes_B N/N' \to A$ is also injective. \(\square\)

**Proposition 4.13.** If an algebra $A$ has a well-based prestandard filter, it also has a well-based standard filter.
We also prove the statements similar to Proposition 4.6.

**Proposition 4.14.** Let $A$ be a weakly standardly based algebra over $\{B_i\}$.

1. For any idempotent $e \in A$, the algebra $e Ae$ is also weakly standardly based.
2. If an algebra $A'$ is Morita equivalent to $A$, $A'$ is also weakly standardly based.

**Proof.** (1) follows from that $eM_1$ and $N_1e$ are also finitely generated and projective, and that $eM_1 \otimes_{B_i} N_1e \simeq e(M_1 \otimes_{B_i} N_1)e$. (2) follows from the following categorical characterization of being finitely generated and projective: a right (resp. left) $B$-module $M$ is finitely generated projective if and only if the functor $M \otimes \bullet$ (resp. $\text{Hom}_B(M, \bullet)$) also have its left (resp. right) adjoint functor. \hfill $\Box$

Now [13, Theorem 3.7] can be generalized as follows.

**Theorem 4.15.** Suppose that $k$ is a field, $A$ is well-founded and each $B_i$ is finite dimensional and semisimple. Let $A$ be a weakly standardly based algebra over $\{B_i\}$ and take its well-based prestandard filter. Let $S_1, S_2 \in \text{Irr}(A)$ and suppose that they have projective covers $P_i \twoheadrightarrow S_i$. Then

$$[P_2 : S_1] = \dim_k \text{End}_A(S_2) \sum_{\lambda \text{dim}(B_i)} \frac{[M_\lambda \otimes_{B_i} T : S_1][\text{Hom}_B(N_\lambda, T) : S_2]}{\dim_k \text{End}_{B_i}(T)}.$$  

Note that we can take $\mu \in A$ such that $S_2 \in \text{Irr}^{\text{wr}}(A)$ then by the Morita context we have an isomorphism $\text{End}_A(S_2) \simeq \text{End}_{B_i}(DS_2)$, so that its dimension is also easy to compute.

**Proof.** Since the ideal filter is rigid, we have $[P_2 : S_1] = \sum \{[A^{\leq i}P_2/A^{< i}P_2 : S_1] \}$ by Proposition 4.14. Then for each $\lambda$, we have

$$[A^{\leq i}P_2/A^{< i}P_2 : S_1] = \dim_k \text{Hom}_A(P_1, A^{\leq i}P_2/A^{< i}P_2)/\dim_k \text{End}_A(S_1)$$

and by using $A^{\leq i}/A^{< i} \simeq M_\lambda \otimes_{B_i} N_\lambda$ and that $P_2$ is flat,

$$\text{Hom}_A(P_1, A^{\leq i}P_2/A^{< i}P_2) \simeq \text{Hom}_A(P_1, M_\lambda \otimes_{B_i} N_\lambda \otimes_A P_2) \simeq \text{Hom}_{B_i}(M_\lambda^i \otimes_A P_1, N_\lambda \otimes_A P_2)$$

where $M_\lambda^i := \text{Hom}_{B_i}(M_\lambda, B_i)$. Since $B_i$ is semisimple,

$$\dim_k \text{Hom}_{B_i}(M_\lambda^i \otimes_A P_1, N_\lambda \otimes_A P_2) = \sum_{T \in \text{Irr}(B_i)} [M_\lambda^i \otimes_A P_1 : T][N_\lambda \otimes_A P_2 : T] \dim_k \text{End}_{B_i}(T).$$

Moreover we have

$$\text{Hom}_{B_i}(M_\lambda^i \otimes_A P_1, T) = \text{Hom}_A(P_1, M_\lambda \otimes_{B_i} T)$$

which implies

$$[M_\lambda^i \otimes_A P_1 : T] \dim_k \text{End}_{B_i}(T) = [M_\lambda \otimes_{B_i} T : S_1] \dim_k \text{End}_A(S_1).$$

Similarly we have

$$[N_\lambda \otimes_A P_2 : T] \dim_k \text{End}_{B_i}(T) = [\text{Hom}_{B_i}(N_\lambda, T) : S_2] \dim_k \text{End}_A(S_2).$$

Putting them all together, we obtain the equation. \hfill $\Box$
Quasi-hereditary algebra is an important class of algebra introduced by Cline, Parshall and Scott [6]. The condition for a non-generalized standardly based algebra to be quasi-hereditary is given by Graham and Lehrer [13], and Du and Rui [11]. We can prove an analogous partial result for our generalized standardly based algebra.

**Lemma 4.16.** Let \((M, N)\) be a well-based Morita context between \(A\) and \(B\), and suppose that the algebra \(B\) is semisimple. If \(\rho: N \otimes_A M \to B\) is surjective, then the trace ideal \(I := \eta(M \otimes_B N) \subset A\) is generated by an idempotent, and finitely generated and projective as both a left and a right \(A\)-module.

**Proof.** By the Artin–Wedderburn theorem and the Morita equivalence, we may assume that \(B\) is a product of finitely many division algebras:

\[B = D_1 \times D_2 \times \cdots \times D_l\]

(here we mean that every non-zero homogeneous element \(x \in D_i\) is invertible). Let us write \(1_i = (0, \ldots, 1, \ldots, 0) \in B\) the identity element of each \(D_i\). Since \(\rho\) is surjective, for each \(i\) we can find \(m_i \in M\) and \(n_i \in N\) such that \(\rho(n_i \otimes m_i) 1_i \neq 0\). By multiplying elements in \(D_i\) we may assume that \(\rho(n_i \otimes m_i) = 1_i, 1_i n_i = n_i\) and \(m_i 1_i = m_i\). Thus \(\rho(n_i \otimes m_i) = 0\) for \(i \neq j\). Let \(e := \sum_i \eta(m_i \otimes n_i) \in A\). Then by the associativity \(e\) is an idempotent. Moreover the maps

\[
M \to Ae, \quad Ae \to M, \quad m \mapsto \sum_i \eta(m \otimes n_i), \quad a \mapsto \sum_i am_i
\]

are inverses of each other. Hence \(M \simeq Ae\) is a finitely generated and projective left \(A\)-module.

Similarly \(N \simeq eA\) as right \(A\)-modules so that \(I \simeq M \otimes_B N\) is finitely generated and projective from both sides. By these isomorphisms we also have \(I \simeq AeA\).

Hence in this case we have \(\text{Ext}^{i}_{\mathcal{A}/\mathcal{I}} \simeq \text{Ext}^{i}_{\mathcal{A}}\) for any \(i\) by Proposition 2.38. We also have \(\text{Ext}^{i}_{\mathcal{A}}(V, W) = 0\) for \(V, W \in \text{Irr}^{i}(A)\) by Lemma 3.13.

4.3. Standard basis

We here give the definition of class of algebras which is more closely related to the original one of cellular algebra.

**Definition 4.17.** A (generalized) standard basis of an algebra \(A\) is a direct sum decomposition

\[A = \bigoplus_{\lambda \in \Lambda} A^\lambda\]

as a \(k\)-module (not as a left or right \(A\)-module) such that for each \(\lambda\)

\[A^{\geq \lambda} := \bigoplus_{\mu \geq \lambda} A^\mu \quad \text{and} \quad A^{< \lambda} := \bigoplus_{\mu < \lambda} A^\mu\]

are both \(2\)-sided ideals of \(A\), equipped with for each \(\lambda\) an isomorphism of \((A, A)\)-bimodules

\[M_{\lambda} \otimes_{B_{\lambda}} N_{\lambda} \simeq A^{\geq \lambda}/A^{< \lambda}\]

for a pair of an \((A, B_{\lambda})\)-bimodule \(M_{\lambda}\) and a \((B_{\lambda}, A)\)-bimodule \(N_{\lambda}\) which are both finitely generated and free over \(B_{\lambda}\). An algebra equipped with a standard basis is called a (generalized) standardly based algebra.
When every $B_1$ is the base algebra $k$, this definition coincides with the original we given at the beginning.

**Proposition 4.18.** A standardly based algebra is a weakly standardly based algebra.

*Proof.* Since $A^{\geq 1} A^{\mu} \subset A^{\geq 1} \cap A^{\mu} = \bigoplus_{\nu \geq \mu} A^\nu$, the collection $\{A^{\geq 1}\}_\lambda$ is an ideal filter on $A$. As proved in [13] Proposition 2.4, we can construct a suitable $(B_1, B_2)$-homomorphism $\rho: M \otimes A N_i \to B_2$ for each $\lambda$ which completes a Morita context between $A/A^{\geq 1}$ and $B_1$. \hfill $\square$

The converse is also holds when the following assumptions are satisfied.

**Proposition 4.19.** Suppose that $\Lambda$ is well-founded and every $B_1$ is projective over $k$. Then a weakly standardly based algebra is a standardly based algebra if $M_1$ and $N_1$ are free over $B_1$ for every $\lambda$.

*Proof.* Let $A$ be a weakly standardly based algebra. Since its ideal filter $\{A^{\geq 1}\}$ is rigid, by taking a well-ordering extension, we obtain a well-ordered filtration of $A$ whose successive quotients are $A^{\geq 1}/A^{\geq 1}$.

Each of them is isomorphic to $M_1 \otimes_{B_1} N_1$ which is projective over $k$, so we can lift $M_1 \otimes_{B_1} N_1 \hookrightarrow A/A^{\geq 1}$ to some $k$-linear map $\iota_j: M_1 \otimes_{B_1} N_1 \hookrightarrow A$. Then as a $k$-module $A$ decompose into a direct sum of $k$-modules $A^j \coloneqq \iota_j(M_1 \otimes_{B_1} N_1)$ as desired. \hfill $\square$

It is a natural question to ask whether the freeness condition of the definition of standardly based algebra can be weakened to the projectiveness. So suppose that we are given an $(A, B)$-bimodule $M$ and a $(B, A)$-bimodule $N$ which are both finitely generated and projective over $B$, equipped with an injective $(A, A)$-homomorphism $\eta: M \otimes B N \hookrightarrow A$. By replacing them with their quotients, we may assume that $B', M'$ and $N'$ taken as in lemma 4.1 are all zero. The existence of $\rho: N \otimes B M \to A$ fails in this general situation: consider the following counterexample that

$$A = \mathbb{k}, \quad B = \begin{bmatrix} 0 & \ast \\ \ast & 0 \end{bmatrix}, \quad M = \begin{bmatrix} \ast \\ \ast \end{bmatrix}, \quad N = \begin{bmatrix} 0 \\ \ast \end{bmatrix}$$

with natural isomorphism $\eta: M \otimes B N \cong A$. We state a sufficient condition for its existence as follows. This is a generalization of [13] Proposition 2.4 we quoted above.

**Lemma 4.20.** Let $A$, $B$, $M$ and $N$ as above. Let $\text{Tr}_{B^\vee}(M), \text{Tr}_{B}(N) \subset B$ be the trace ideals of $M$ and $N$ in $B$, that is,

$$\text{Tr}_{B^\vee}(M) \coloneqq \text{Image}(M^\vee \otimes_A M \to B), \quad \text{Tr}_{B}(N) \coloneqq \text{Image}(N \otimes_A N^\vee \to B)$$

where $M^\vee \coloneqq \text{Hom}_{B}(M, B)$ and $N^\vee \coloneqq \text{Hom}_{B}(N, B)$. If $M$ is $\text{Tr}_{B}(N)$-accessible and $N$ is $\text{Tr}_{B^\vee}(M)$-accessible, then there exists a unique $(B, A)$-homomorphism $\rho: N \otimes_A M \to B$ which makes $(M, N)$ into a Morita context between $A$ and $B$.

*Proof.* The uniqueness of $\rho$ is clear from that $B' = 0$, so we prove its existence. First consider the sequence

$$M \otimes_B N \otimes_A M \otimes_B N \xrightarrow{\eta \otimes_A (M \otimes_B N)} M \otimes_B N \xrightarrow{\eta} A.$$  

Since the two parallel homomorphisms above are equalized by $\eta$ which is injective, these are equal. This implies that the diagram below is commutative:

$$\begin{array}{c}
N \otimes_A M \xrightarrow{N \otimes_A \eta \otimes_A N^\vee} N \otimes_A N^\vee \xrightarrow{\eta} \text{End}_A(M)^\vee.
\end{array}$$
Here the map at the bottom is given by $n \otimes m \mapsto (m' \mapsto \eta(m' \otimes n)m)$. By the assumption that $M$ is $\text{Tr}_B(N)$-accessible, the left vertical arrow is surjective. On the other hand, since $M$ is faithful over $B$, the right vertical arrow is injective. Hence the diagram induces a $(B, B)$-homomorphism $\rho: N \otimes_A M \to B$ which satisfies $\eta(m' \otimes n)m = m' \rho(n \otimes m)$ for all $m, m' \in M$ and $n \in N$. Dually we can prove the existence of another $\rho': N \otimes_A M \to B$ such that $m\eta(n \otimes n') = \rho'(n \otimes m)n'$, but we have $\rho = \rho'$ by the uniqueness.

When $M$ and $N$ are free over $B$ the accessibility condition is trivially satisfied, so that this proof is essentially the same as the original one by Graham and Lehrer. Note that this condition is not necessary: consider the example above with replacing $A$ with $k \oplus k\epsilon$, $\epsilon^2 = 0$ so that $\epsilon M = 0, N\epsilon = 0$ and $\eta: M \otimes_B N \cong A\epsilon$, which clearly has $\rho = 0$. One necessary condition is that $\eta(M \otimes_B N)M \subset M \text{Tr}_B(N)$ and $N\eta(M \otimes_B N) \subset \text{Tr}_B(M)N$, but the author does not know it is sufficient for the existence of $\rho$ or not.

### 4.4. Involution on algebras

For an algebra $A$, we call an algebra homomorphism $A \to A^{op}$ whose square is equal to the identity an anti-involution on $A$. That is, it is a degree-zero map $\bullet^* : A \to A$ satisfying

$$1^* = 1, \quad (ab)^* = (-1)^{|a||b|}b^*a^* \quad \text{and} \quad a^{**} = a$$

(beware the Koszul sign). If $A$ has an anti-involution, for each left $A$-module $M$ there is a corresponding right $A$-module $M^*$ whose underlying set is equal to $M$ and action is defined by $x^* \cdot a^* := (-1)^{|a||x|}(ax)^*$, where we write $x^* \in M^*$ the element corresponds to $x \in M$. Similarly for a right $A$-module $N$ we denote by $N^*$ the corresponding left $A$-module, so that $M^{**} \cong M$.

**Definition 4.21.** Let $A$ and $B$ be algebras with anti-involution. A Morita context $(M, N)$ between $A$ and $B$ is said to be involutive if there is an isomorphism $M \cong N^*$ of $(A, B)$-bimodules (so $M^* \cong N$) which satisfies

$$\eta(x \otimes y)^* = (-1)^{|a||b|}\eta(y^* \otimes x^*), \quad \rho(y \otimes x)^* = (-1)^{|a||b|}\rho(x^* \otimes y^*)$$

for every $x \in M, y \in N$.

Now we assume that each $B_A$ has a fixed anti-involution.

**Definition 4.22.** A standardly filter on an algebra $A$ with anti-involution is said to be involutive if for each $\lambda, A^{-\lambda}$ and $B^{\lambda}_f$ are closed under anti-involution and the Morita context $(M_\lambda, N_\lambda)$ between $A/A^{-\lambda}$ and $B^{\lambda}_f/B^{\lambda}_f$ is involutive. An algebra equipped with an involutive well-based standard filter is called a weakly cellular algebra.

Note that in the settings of Lemma 4.3 when $\mathcal{A}$ has an anti-involution $\mathcal{A} \to \mathcal{A}^{op}$ which fixes all $X_\lambda$ and $B_\lambda$, it produces an involutive standard filter. The statements below are clear from the definition.

**Lemma 4.23.** Let $A, B, M$ and $N$ as in Lemma 4.2. If $A$ and $B$ have their anti-involutions and $(M, N)$ is involutive, then $(B')^* = B'$ and $(M/M', N/N')$ is also involutive.

**Proposition 4.24.** If an algebra $A$ with anti-involution has an involutive (well-based) prestandard filter, it also has an involutive (well-based) standard filter.
To deal its Morita invariant property we should be careful with the compatibility between Morita equivalence and anti-involution. See the hypotheses (* and †) in [32]. Recall that the Morita equivalence \( A-\text{Mod} \cong A'-\text{Mod} \) also induces \( \text{Mod}-A \cong \text{Mod}-A' \). We here say that algebras \( A \) and \( A' \) with anti-involution are involutively Morita equivalent if the category equivalence makes the diagram

\[
\begin{array}{ccc}
A-\text{Mod} & \cong & \text{Mod}-A \\
\downarrow & & \downarrow \\
A'-\text{Mod} & \cong & \text{Mod}-A'
\end{array}
\]

commutes up to natural isomorphism. Note that if \( A' \) is equivalent to \( A \) we can find an idempotent \( m \times m \) matrix \( e = (e_{ij}) \) over \( A \) such that \( A' = e \cdot \text{Mat}_m(A) \cdot e \). The condition above is equivalent to that we can also take \( e \) so that \( e'_{ij} = e_{ij} \).

The proofs of the statements below are obvious by Proposition 4.14.

**Proposition 4.25.** Let \( A \) be a weakly cellular algebra over \( \{B_\lambda\} \).

1. For any idempotent \( e \in A \) such that \( e^* = e \), the algebra \( eAe \) with the same anti-involution is also weakly cellular.
2. If an algebra \( A' \) with anti-involution is involutively Morita equivalent to \( A \), \( A' \) is also weakly standardly filtered. \( \square \)

In [32] it is also proved that even if we are not given a such anti-involution on \( A' \) we can construct it from that on \( A \). Thus their result is stronger than above.

Finally we give the definition of cellular algebra in terms of basis. Note the next lemma which follows by the uniqueness of \( \rho \).

**Lemma 4.26.** Suppose that \( M, N \) and \( \eta \) in Lemma 4.20 satisfies \( M \cong N^* \) and \( \eta(x \otimes y)^* = (-1)^{|x||y|}\eta(y^* \otimes x^*) \). Then the induced map \( \rho: N \otimes_A M \rightarrow B \) also satisfies \( \rho(y \otimes x)^* = (-1)^{|x||y|}\rho(x^* \otimes y^*) \) so that the Morita context \( (M, N) \) between \( A \) and \( B \) is involutive. \( \square \)

**Definition 4.27.** A standardly based algebra \( A \) over \( \{B_\lambda\} \) with anti-involution is called a (generalized) cellular algebra if each component \( A^\lambda \) is closed under anti-involution and the isomorphism \( M_\lambda \otimes_{B_\lambda} N_\lambda \cong A^\lambda \) satisfies the involutive property similar as above.

Again, this definition is same as the original one when every \( B_\lambda \) is just the base ring \( k \). Then the next statement is obvious from the lemma above.

**Proposition 4.28.** A cellular algebra is a weakly cellular algebra. \( \square \)

We prove the converse in suitable conditions.

**Proposition 4.29.** Suppose that the assumptions in Proposition 4.19 are satisfied, in addition to that \( 2 \in k \) is invertible. Then a weakly cellular algebra is a cellular algebra if \( M_\lambda \) and \( N_\lambda \) are free over \( B_\lambda \) for every \( \lambda \).

**Proof.** The problem is that \( \iota_\lambda \) we chose in the proof of Proposition 4.19 does not preserve anti-involution. So we retake a new map \( \iota'_\lambda: M_\lambda \otimes_{B_\lambda} N_\lambda \leftarrow A \) defined by

\[
\iota'_\lambda(x \otimes y) := \frac{\iota_\lambda(x \otimes y) + (-1)^{|x||y|}\iota_\lambda(y^* \otimes x^*)}{2}.
\]

Then \( \iota'_\lambda \) is also a lift of \( M_\lambda \otimes_{B_\lambda} N_\lambda \leftarrow A/A^\lambda \) which satisfies \( \iota'_\lambda(x \otimes y)^* = (-1)^{|x||y|}\iota'_\lambda(y^* \otimes x^*) \). Thus by putting \( A^\lambda := \iota'_\lambda(M_\lambda \otimes_{B_\lambda} N_\lambda) \) we obtain a desired direct sum decomposition. \( \square \)

\[43\]
5. Cellular structure on the Iwahori–Hecke algebra

In this section we review the definition and the representation theory of the Iwahori–Hecke algebra. We give a new proof for that the Iwahori–Hecke algebra and the associated $q$-Schur algebra are cellular with respect to Murphy’s basis [39, 40] in a more simple and sophisticated way than his original one or given in [34]. We give a generalized theorem that classify its simple modules on a very few assumptions.

5.1. The symmetric groups

We here introduce standard notions on Young tableaux used in representation theory of the symmetric group and the Iwahori–Hecke algebra, and we briefly recall some of their basic facts. We refer the standard textbooks [18], [12] and [34] for details.

We write $\mathbb{N} = \{0, 1, 2, \ldots \}$ the set of natural numbers. We denote by $\mathfrak{S}_n$ the symmetric group of rank $n \in \mathbb{N}$ acting on the set $\{1, 2, \ldots , n\}$ from left. For $1 \leq i \leq n - 1$, let $s_i$ be the basic transposition $(i, i + 1)$. As a Coxeter group, $\mathfrak{S}_n$ is generated by the elements $s_1, s_2, \ldots , s_{n-1}$. With respect to this generator set, the length

$$\ell (w) = \# \{(i, j) \mid 1 \leq i < j \leq n \text{ and } w(j) < w(i)\}.$$

A composition of $n \in \mathbb{N}$ is an infinite sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of natural numbers whose sum, written as $|\lambda| := \sum \lambda_i$, is equal to $n$. Alternatively we often represent a composition $\lambda$ as a finite tuple $\lambda = (\lambda_1, \lambda_2, \ldots , \lambda_r)$ if it satisfies $\lambda_i = 0$ for all $r > i$. For such $\lambda$, the corresponding parabolic subgroup (also called the Young subgroup)

$$\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_r} \subset \mathfrak{S}_n$$

is defined. It is known that the quotient set $\mathfrak{S}_n / \mathfrak{S}_\lambda$ has the minimal length coset representatives

$$\mathfrak{D}_\lambda := \{ w \in \mathfrak{S}_n \mid \ell (w s_i) > \ell (w) \text{ for every } s_i \in \mathfrak{S}_\lambda \}.$$

With respect to this set, every $w \in \mathfrak{S}_n$ is uniquely decomposed as $w = uv$ to a pair of $u \in \mathfrak{D}_\lambda$ and $v \in \mathfrak{S}_\lambda$ which satisfies $\ell (w) = \ell (u) + \ell (v)$. For another composition $\mu$, the sets

$$\mathfrak{D}_\mu := \{ w \in \mathfrak{S}_n \mid \ell (s_i w) > \ell (w) \text{ for every } s_i \in \mathfrak{S}_\mu \}$$

and $\mathfrak{D}_\lambda \cap \mathfrak{D}_\mu^{-1}$ are the minimal length representatives of the left cosets $\mathfrak{S}_n / \mathfrak{S}_\mu$ and the double cosets $\mathfrak{S}_n \backslash \mathfrak{S}_\mu / \mathfrak{S}_\lambda$ respectively.

The Poincaré polynomial of a subset $S \subset \mathfrak{S}_n$ is defined by $P_S (q) := \sum_{w \in S} q^{\ell (w)} \in \mathbb{Z}[q]$. If $S$ has a decomposition $S = S_1 \cdot S_2$ which preserves lengths, it follows by definition that $P_S (q) = P_{S_1} (q) P_{S_2} (q)$. We have a $q$-factorial as the Poincaré polynomial of whole $\mathfrak{S}_n$,

$$P_{\mathfrak{S}_n} (q) = [n]! := [1] [2] \cdots [n],$$

where $[k]$ is a $q$-integer $[k] = 1 + q + \cdots + q^{k-1}$, which follows inductively from the decomposition $\mathfrak{S}_n = \bigsqcup_{1 \leq i < n} S_i S_{i+1} \cdots S_{n-1} \mathfrak{S}_{n-1}$. Then for a composition $\lambda = (\lambda_1, \lambda_2, \ldots , \lambda_r)$ of $n$, we obtain

$$P_{\mathfrak{D}_\lambda} (q) = P_{\mathfrak{S}_{\lambda_1}} (q) P_{\mathfrak{S}_{\lambda_2}} (q) \cdots P_{\mathfrak{S}_{\lambda_r}} (q) = [\lambda_1] [\lambda_2] ! \cdots [\lambda_r] !$$

and

$$P_{\mathfrak{D}_\mu^{-1}} (q) = \frac{P_{\mathfrak{S}_n} (q)}{P_{\mathfrak{D}_\lambda} (q)} = \frac{[n]!}{[\lambda_1] [\lambda_2] ! \cdots [\lambda_r] !}.$$
This polynomial
\[
\begin{bmatrix} n \\ \lambda \end{bmatrix} = [n_{\lambda_1, \lambda_2, \ldots, \lambda_r}] := P_D(n)
\]
is called the \textit{q-multinomial coefficient}. In particular, when \( \lambda = (n - k, k) \) is of length 2, the Poincaré polynomial of \( D_{(n-k,k)} \) is given by a \textit{q-binomial coefficient}
\[
\begin{bmatrix} n \\ k \end{bmatrix} := \begin{bmatrix} n \\ n - k, k \end{bmatrix} = \frac{[n][n-1] \cdots [n-k+1]}{[k]!}.
\]

5.2. Combinatorics on tableaux

The \textit{Young diagram} of a composition \( \lambda \) is defined by
\[
Y(\lambda) := \{(i,j) \mid 1 \leq i, 1 \leq j \leq \lambda_i\}.
\]
We represent it by boxes placed in the fourth quadrant arranged as matrix indices (the English notation):

\[
(3,2) = \begin{array}{cc}
\hline \\
\hline \\
\hline \\
\end{array}, \quad (2,4,1) = \begin{array}{ccc}
\hline \\
1 \\
\hline \\
1 \\
\hline \\
\end{array}, \quad (2,0,3) = \begin{array}{ccc}
\hline \\
\hline \\
\hline \\
\end{array}.
\]

A \textit{tableau} of shape \( \lambda \) is a function \( T : Y(\lambda) \rightarrow \{1,2,\ldots\} \). The \textit{weight} of a tableau \( T \) is a composition \( \mu = (\mu_1, \mu_2, \ldots) \) whose \( i \)-th component is \( \mu_i := \#T^{-1}(i) \). \( T \) is said to be \textit{row-semistandard} if it satisfies \( T(i,j) \leq T(i,j+1) \) for each pair of adjacent boxes \((i,j), (i,j+1) \in Y(\lambda)\), that is, entries in each row of \( T \) are weakly increasing. We denote by Tab_{\lambda,\mu} the set of row-semistandard tableaux of shape \( \lambda \) and weight \( \mu \). For example,

\[
\text{Tab}_{(3,3),(3,1,0,1)} = \{ \begin{array}{ccc}
1 & 4 & 5 \\
3 & 7 & 8 \\
6 & &
\end{array}, \begin{array}{ccc}
1 & 2 & 4 \\
1 & 1 & 3 \\
1 & 1 & 2
\end{array}, \begin{array}{ccc}
1 & 2 & 4 \\
1 & 1 & 2 \\
1 & 1 & 1 \\
\end{array}, \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{array}\}.
\]

A row-semistandard tableau is called a \textit{row-standard tableau} if it satisfies \( T(i,j) < T(i,j+1) \) for each pair of adjacent boxes \((i,j), (i,j+1) \in Y(\lambda)\), that is, entries in each row of \( T \) are strictly increasing. For example,

\[
T = \begin{array}{cccc}
1 & 2 & 4 & 5 \\
3 & 7 & 8 & 6
\end{array}
\]
corresponds to \( d(T) = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 2 & 4 & 5 \\
1 & 2 & 4 & 5 \\
1 & 2 & 4 & 5
\end{array} = s_3 s_4 s_5 s_7 \).

Actually any tableau of weight \((1^n)\) provides a permutation in this manner, and the increasing condition on the rows just say that this permutation is in \( S_\lambda \). We denote by \( \pi_\lambda \) the largest element in \( S_\lambda \). Its corresponding tableau is obtained by putting numbers on \( Y(\lambda) \) from bottom to top, conversely as before. For each \( T \in \text{Tab}_\lambda \), let us write \( \ell(T) := \ell(d(T)) \) for short which we also call the \textit{length} of \( T \). \( \ell(T) \) can be also expressed as the inversion number

\[
\ell(T) = \{ (i,j),(k,l) \mid i < k \text{ and } T(k,l) < T(i,j) \}.
\]

Next let us take another composition \( \mu \) and consider the action \( S_\mu \sim S_\mu / S_\lambda \). For \( S \in \text{Tab}_\mu \), we denote by Tab_{\mu} the set \( \{ T \in \text{Tab}_{\lambda} \mid T|_\mu = S \} \) where \( T|_\mu \) is a row-semistandard tableau of weight
\( \mu \) obtained from \( T \) by replacing its entries 1, 2, \ldots, \mu_1 by 1, \mu_1 + 1, \ldots, \mu_1 + \mu_2 by 2, and so forth. For example, for

\[
S = \begin{bmatrix}
1 & 4 & 3 \\
1 & 4 & 4 \\
3 & 3 & 3
\end{bmatrix},
\]

we have

\[
\text{Tabs} = \left\{ \begin{bmatrix}
1 & 2 & 4 & 5 \\
3 & 7 & 8 & 6 \\
5 & 0 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 3 & 4 & 5 \\
2 & 7 & 8 & 6 \\
5 & 0 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 2 & 4 & 6 \\
1 & 7 & 8 & 5 \\
5 & 0 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 3 & 4 & 6 \\
1 & 7 & 8 & 5 \\
5 & 0 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
2 & 3 & 4 & 6 \\
1 & 7 & 8 & 5 \\
5 & 0 & 0 & 0
\end{bmatrix} \right\}.
\]

Then via the one-to-one correspondence \( \text{Tab}_d \overset{1:1}{\longleftrightarrow} \mathcal{S}/\mathcal{S}_d \), each subset \( \text{Tabs} \subset \text{Tab}_d \) clearly corresponds to each orbit of the action above. Hence the set \( \text{Tab}_{\lambda, \mu} \) is in bijection with the set \( \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1} \). Namely, for each \( S \in \text{Tab}_{\lambda, \mu} \), there is a unique tableau \( S_1 \in \text{Tabs} \) which has the minimal length, so that \( d(S_1) \in \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1} \). We can construct \( S_1 \) from \( S \) in the following manner: first we mark subscripts 1, 2, \ldots, \mu_k to all \( k \)'s in \( S \) for each number \( k \) along with the above reading order. Then \( S_1 \) is obtained by replacing the entries of \( S \) by 1, 2, \ldots, \( n \) with respect to the total order

\[ 1_1 < 1_2 < \cdots < 1_{\mu_1} < 2_1 < 2_2 < \cdots < 2_{\mu_2} < \cdots. \]

For example, the row-semistandard tableau \( S \) above is marked as

\[
\begin{bmatrix}
1 & 1 & 2 & 5 \\
1 & 4 & 4 & 3 \\
5 & 3 & 3 & 3
\end{bmatrix},
\]

and gives the corresponding row-standard tableau \( S_1 = T \) in the previous example; so \( d(S_1) = d(T) = s_3 s_4 s_5 s_6 s_7 \). Other elements in \( \text{Tabs} \) can be constructed from \( S_1 \) as follows: let \( \#_i(S) \) be the number of \( j \)'s in the \( i \)-th row of \( S \), and \( S[j] \) be the composition of \( \mu_j \) defined by \( S[j] := \#_j(S) \). We define \( \mathcal{D}_S \subset \mathcal{S}_n \) by

\[ \mathcal{D}_S := \mathcal{D}_{S[1]} \times \mathcal{D}_{S[2]} \times \cdots \times \mathcal{D}_{S[r]} \subset \mathcal{S}_\mu \subset \mathcal{S}_n. \]

Then we have a one-to-one correspondence \( \mathcal{D}_S \rightarrow \text{Tabs} : w \mapsto wS_1 \) which preserves lengths, that is, \( \ell(wS_1) = \ell(w) + \ell(S_1) \). Let \( \sigma_S \in \mathcal{D}_S \) be its longest element \( \sigma_S := (\sigma_{S[1]}, \sigma_{S[2]}, \ldots, \sigma_{S[r]}) \). The tableau \( S^\dagger := \sigma_S S_1 \in \text{Tabs} \) which has maximal length is obtained by replacing the entries of \( S \) from bottom to top, contrary to \( S_1 \).

The matrix \( (\#_j(S))_{i,j=1}^{r} \) uniquely determines a row-semistandard tableau \( S \), and its shape \( \lambda \) and its weight \( \mu \) are recovered from this matrix as

\[ \lambda_i = \sum_j \#_j(S) \quad \text{and} \quad \mu_j = \sum_i \#_j(S). \]

So for each \( S \), there exists a unique tableau \( S^* \) of shape \( \mu \) and of weight \( \lambda \), which satisfies \( \#_j(S^*) = \#_j(S) \). We call it the dual tableau of \( S \). For example, for \( S \) above, its dual is

\[
S^* = \begin{bmatrix}
1 & 1 & 2 \\
1 & 3 & 3 \\
2 & 2
\end{bmatrix}.
\]

It easily follows that \( d((S^*)_1) = d(S_1)^{-1} \). So taking dual \( \text{Tab}_{\lambda, \mu} \rightarrow \text{Tab}_{\mu, \lambda} : S \mapsto S^* \) corresponds to the inversion \( \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1} \rightarrow \mathcal{D}_\mu \cap \mathcal{D}_\lambda^{-1} : w \mapsto w^{-1} \) via the bijection \( d \).
5.3. The Iwahori–Hecke algebra

Hereafter we fix a parameter $q \in \mathbb{k}$. For each $n \in \mathbb{N}$, the Iwahori–Hecke algebra $H_n = H_n(q)$ of rank $n$ (or of type $A_{n-1}$) is an algebra generated by elements $T_1, T_2, \ldots, T_{n-1}$ with defining relations

$$T_i T_j = T_j T_i \quad \text{if } |i - j| \geq 2, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad (T_i - q)(T_i + 1) = 0.$$  

Here for $n = 0$ or $1$, it is defined as $H_0 = H_1 = \mathbb{k}$. For each $w \in \mathfrak{S}_n$, we take a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ and define an element $T_w := T_{i_1} T_{i_2} \cdots T_{i_k}$ of $H_n$. Then it is known that it does not depend on choice of expression, and that $H_n$ is a free $\mathbb{k}$-module with basis $\{T_w \mid w \in \mathfrak{S}_n\}$. Thus $H_n$ can be considered as a $q$-deformation of $\mathbb{k}\mathfrak{S}_n$, the group ring of the symmetric group. The element $T_w$ is invertible if and only if $q \in \mathbb{k}^*$; in such a case, we have $T_w^{-1} = q^{-1}(T_w - q + 1)$ and $T_w^{-1} = T_{i_1}^{-1} \cdots T_{i_{k}}^{-1}$. By definition, if $u, v \in \mathfrak{S}_n$ satisfy $\ell(uv) = \ell(u) + \ell(v)$ then $T_{uv} = T_u T_v$. The algebra $H_n$ has an anti-involution defined by $(T_w)^* := T_{w^{-1}}$. Thus the category of left $H_n$-modules is equivalent to that of right modules.

For a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of $n$, let $H_\lambda$ be a subalgebra of $H_n$ spanned by $\{T_w \mid w \in \mathfrak{S}_\lambda\}$. Then $H_\lambda$ is free as a right $H_n$-module with basis $\{T_w \mid w \in \mathfrak{S}_\lambda\}$ by the decomposition $\mathbb{V}_n = \mathbb{D}_I \mathfrak{S}_\lambda$. As an abstract algebra, we have an isomorphism

$$H_\lambda \cong H_{\lambda_1} \otimes H_{\lambda_2} \otimes \cdots \otimes H_{\lambda_r}.$$  

It is called a parabolic subalgebra of $H_n$.

In representation theory of the symmetric groups and the Iwahori–Hecke algebra, it is important to treat modules over these algebras for all ranks at once. So it is better to consider the direct sum of all their module categories. Convolution product of modules is defined as a binary operation on this category.

**Definition 5.1.** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a composition of $n$. For each $i = 1, 2, \ldots, r$, let $V_i$ be an $H_{\lambda_i}$-module. We define an $H_\lambda$-module

$$V_1 \ast V_2 \ast \cdots \ast V_r := H_n \otimes_{H_\lambda} (V_1 \boxtimes V_2 \boxtimes \cdots \boxtimes V_r)$$  

where $\boxtimes$ denotes the outer tensor product of modules. It is called the convolution product of $V_1, V_2, \ldots, V_r$.

Obviously this product is associative up to natural isomorphism. By the basis theorem, we have a direct sum decomposition

$$V_1 \ast V_2 \ast \cdots \ast V_r = \bigoplus_{w \in \mathbb{D}_\lambda} T_w (V_1 \boxtimes V_2 \boxtimes \cdots \boxtimes V_r)$$  

as a $\mathbb{k}$-module. The convolution product $\ast$ defines a structure of tensor category on the direct sum of the module categories $\bigoplus_{\lambda}(H_n\text{-Mod})$. This tensor category also admits a braiding

$$\sigma(V,W): V \ast W \rightarrow W \ast V$$  

$$x \boxtimes y \mapsto T_{\sigma_{n,m}}(y \boxtimes x)$$  

in a weak sense; it satisfies the hexagon axioms of braiding but is not invertible unless $q \in \mathbb{k}^\times$. Here $\sigma_{(n,m)}$ is the longest element in $\mathbb{D}_{(n,m)}$ defined by

$$\sigma_{(n,m)}(i) := \begin{cases} i + m & \text{if } 1 \leq i \leq n, \\ i - n & \text{if } n + 1 \leq i \leq m + n. \end{cases}$$  

47
The hexagon axioms follow from the decompositions
\[ \varpi_{(n+p,m)} = (\varpi_{(n,m)}, 1_p) \cdot (1_n, \varpi_{(p,m)}), \quad \varpi_{(p,m+n)} = (1_m, \varpi_{(p,m)}) \cdot (\varpi_{(p,m)}, 1_n) \]
which preserve lengths. Here we denote by $1_n$ the unit element of $\mathfrak{S}_n$.

5.4. Parabolic modules and the $q$-Schur algebra

Let $\lambda$ be a composition. We define an element $m_\lambda \in H_\lambda$ by
\[ m_\lambda := \sum_{w \in \mathfrak{S}_\lambda} T_w. \]
Note that $T_\lambda(1 + T_\mu) = (1 + T_\lambda)T_\mu = q(1 + T_\mu)$. Hence $m_\lambda$ satisfies $T_w m_\lambda = m_\lambda T_w = q^{\lambda(w)} m_\lambda$ for all $w \in \mathfrak{S}_\lambda$ since it can be also written as
\[ m_\lambda = \sum_{w \in \mathfrak{S}_\lambda, \ell(w) = \ell(w)} (1 + T_\lambda)T_w = \sum_{w \in \mathfrak{S}_\lambda, \ell(w) = \ell(w)} T_w(1 + T_\lambda) \]
for each $\lambda \in \mathfrak{S}_\lambda$. In particular, $km_\lambda$ is a 2-sided ideal of $H_\lambda$.

Let $M_\lambda := H_\lambda m_\lambda$ be a left ideal of $H_\lambda$ generated by $m_\lambda$, which we call a parabolic module. In particular, the trivial module $\mathbb{I}_n := M_{(n)}$ is a free $k$-module of rank one spanned by $m_n := m_{(n)}$, on which every $T_w$ acts by a scalar $q^{\lambda(w)}$. Since the action $H_n \subset H_\lambda$ is free, $M_\lambda$ is isomorphic to $H_n \otimes H_\lambda [km_\lambda]$ as an $H_\lambda$-module; so it has a basis $\{ T_w m_j | w \in \mathfrak{S}_\lambda \}$ over $k$. Or equivalently, by using convolution product, we can also represent it as $M_\lambda = \mathbb{I}_{\lambda_1} \ast \mathbb{I}_{\lambda_2} \ast \cdots \ast \mathbb{I}_\lambda$. Elements of $M_\lambda \subset H_\lambda$ are characterized as
\[ M_\lambda = \{ x \in H_\lambda | xT_w = q^{\lambda(w)}x \text{ for all } w \in \mathfrak{S}_\lambda \} \]
because for $x = \sum_{w \in \mathfrak{S}_\lambda} x_w T_w$ ($x_w \in k$), $xT_i = qx$ is equivalent to that $x_w = x_{w_i}$ for all $w \in \mathfrak{S}_\lambda$. For each $w \in \mathfrak{S}_\lambda$, we take the corresponding row-standard tableau $T_i$ such that $w = d(T)$ and write $m_T := T_w m_\lambda$. The action of $H_n$ on it is described as follows: suppose each number $i$ is contained in the $r(i)$-th row of $T$. Then
\[ T_i \cdot m_T = \begin{cases} \lambda q \tau & \text{if } r(i) = r(i + 1), \\ m_{\lambda, \tau} & \text{if } r(i) < r(i + 1), \\ \lambda q \tau + (q - 1)m_{\lambda, \tau} & \text{if } r(i) > r(i + 1). \end{cases} \]
We similarly define right ideals $M^*_\lambda := m_\lambda H_\lambda$ and $\mathbb{I}^*_\lambda := M^*_\lambda$. Then we have
\[ M^*_\lambda = \{ x \in H_\lambda | T_w x = q^{\lambda(w)}x \text{ for all } w \in \mathfrak{S}_\lambda \}. \]

Now take two compositions $\lambda, \mu$ of $n$. Since $M_\mu$ is a cyclic module generated by $m_\mu$ with the relations $T_w m_\mu = q^{\lambda(w)}m_\mu$ for every $w \in \mathfrak{S}_\mu$, by taking the image of the generator $m_\mu$ we have an isomorphism
\[ \text{Hom}_{H_n}(M_\mu, M_\lambda) = \{ x \in M_\lambda | T_w x = q^{\lambda(w)}x \text{ for all } w \in \mathfrak{S}_\mu \} = M_\lambda \cap M^*_\mu. \]
Let us write $M_{\lambda, \mu} := M_\lambda \cap M^*_\mu$. The collection of these $k$-modules has a natural product
\[ \circ \mu : M_{\lambda, \mu} \otimes M_{\lambda, \mu} \rightarrow M_{\lambda, \mu}, \]
\[ xm_\mu \otimes m_{\mu,y} \mapsto xm_{\mu,y}. \]
48
According to the isomorphism above, this product corresponds to the opposite of the composition of homomorphisms. Note that $M_{\mu,\mu}$ naturally acts on the parabolic module $M_\mu$ from right, so that the composition is given by the reversed product. On the other hand, $M_{\lambda,\mu}$ is also isomorphic to $\text{Hom}_H^\text{op}(M_\lambda', M_\mu')$, the set of homomorphisms between right modules. In this view, the product is just the same as the composition of such homomorphisms. Anyway, the algebra with this product

$$\mathcal{A}_{\mu,\lambda} := \bigoplus_{\lambda,\mu} M_{\lambda,\mu} \cong \text{End}_H\left( \bigoplus_{\lambda} M_\lambda \right)^{\text{op}} \cong \text{End}_H(\bigoplus_{\lambda} M_\lambda)$$

is called the $q$-Schur algebra introduced by Dipper and James \cite{10}. Here $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_r)$ runs over all compositions of $n$ whose components are zero except for the first $r$ ones. Note that the Iwahori–Hecke algebra itself can be obtained similarly:

$$H_n = M_{(1^{r})} \cong \text{End}_H(M_{(1^{r})})^{\text{op}} \cong \text{End}_H(M_{(1^{r})}^*)$$

Since we can write

$$M_{\lambda,\mu} = \{ x \in H_n | T_wxT_\lambda = q^{\ell(w)}x \text{ for all } v \in G_\mu, w \in G_\lambda \},$$

it has a basis $\{ \sum_{v \in G_\mu} \theta_v T_w | w \in D_{\lambda} \cong G_\lambda \}$ which corresponds to the double cosets $G_\mu \backslash G_\lambda / G_\lambda$. Similarly as before, for $w \in D_{\lambda} \cap D_{\mu}^{-1}$ we take the corresponding row-semistandard tableau $S \in \text{Tab}_{\lambda,\mu}$ such that $w = d(S_\mu)$ and write $m_S := \sum_{v \in G_\mu} \theta_v T_v$. As an element of $M_\lambda$, we can decompose it as $m_S = \sum_{T \in \text{Tab}_\mu} m_T$. The anti-involution on $H_n$ induces a map

$$\bullet : M_{\lambda,\mu} \rightarrow M_{\mu,\lambda}$$

which induces that on $\mathcal{A}_{\mu,\lambda}$. By definition we have $(m_S)^* = m_{S^*}$.

5.5. Decomposing a tableau

In this subsection we observe that for each $S \in \text{Tab}_{\lambda,\mu}$, $m_S \in M_{\lambda,\mu}$ has a canonical decomposition

$$m_S = m_{\mu} \circ_v m_{\mu_\mu} \circ_w m_{\lambda}$$

into three tableaux. We first explain each of these terms.

Let $\mu$ and $\nu$ be compositions of $n$. We say that $\nu$ is a refinement of $\mu$ when there is an increasing sequence of indices $1 \leq a_1 \leq a_2 \leq \cdots$ such that $\mu_i = \sum_{a_j \leq i} \nu_j$. Clearly it is equivalent to that $G_{\nu} \subset G_{\mu}$. Hence $m_{\nu}$ is contained in both $M_{\mu,\nu}$ and $M_{\nu,\mu}$. As elements of these sets, $m_{\mu}$ is respectively represented by tableaux $S$ and its dual $S^*$ defined by $S^*(j,k) := i$ for $a_i \leq j < a_{i+1}$, such as

$$S = \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
4 & 3 & 2 & 1 & 5 & 6
\end{array}$$

and

$$S^* = \begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2
\end{array}$$

for $\mu = (4,5)$ and $\nu = (1,2,1,3,2)$. For $T \in \text{Tab}_{\lambda,\nu}$ of weight $\nu$, let $T_{\mu} \in \text{Tab}_{\lambda,\mu}$ be a row-standard tableau of weight $\mu$ obtained by replacing each entry $j$ in $T$ such that $a_i \leq j < a_{i+1}$ with $i$, similarly as before. Since $(1^n)$ is a refinement of every composition, this notation coincides with the previous one.

Lemma 5.2. Let $\nu$ be a refinement of $\mu$, and take $a_1 \leq a_2 \leq \cdots$ as above.
Lemma 5.4. Let $\nu$ be a permutation tableau, there is a permutation $v \in \mathfrak{S}_n$ such that $m_{v,\nu} = T \cdot m_{w,\nu}$. The formula follows from that every $R \in \text{Tab}_T$ satisfies $v R \in \text{Tab}_{vT}$ and $\ell(vR) = \ell(v) + \ell(R)$.

Proof. The tableau $wT$ is also row-standard by the assumption. By the definition of permutation tableau, there is a permutation $v \in \mathfrak{S}_n$ such that $m_{v,\nu} = T \cdot m_{w,\nu}$. The formula follows from that every $R \in \text{Tab}_T$ satisfies $v R \in \text{Tab}_{vT}$ and $\ell(vR) = \ell(v) + \ell(R)$.

50
Proposition 5.5. For each $S \in \text{Tab}_{\lambda \mu}$, there exists a unique pair $(\nu, w)$ of a composition $\nu = (\nu_1, \nu_2, \ldots, \nu_r)$ with $\nu_1, \nu_2, \ldots, \nu_r > 0$ and a permutation $w \in S_r$ such that $w\nu$ and $\nu$ are respectively refinements of $\lambda$ and $\mu$, and

$$m_S = m_\mu \circ \nu \circ m_{P_{w\nu}} \circ w \circ m_\lambda.$$ 

Proof. For such $S$, it suffices to put

$$\nu := (\#_{11}(S), \#_{21}(S), \ldots, \#_{12}(S), \#_{22}(S), \ldots, \#_{13}(S), \#_{23}(S), \ldots),$$

$$w\nu := (\#_{11}(S), \#_{12}(S), \ldots, \#_{21}(S), \#_{22}(S), \ldots, \#_{31}(S), \#_{32}(S), \ldots)$$

with removing zero entries $\#_{ij}(S) = 0$, and take the corresponding permutation $w$. Then by the two lemmas above we have a desired decomposition. For example,

$$\begin{align*}
\begin{array}{cccccc}
2 & 2 & 3 & 3 & 3 & 3 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 3 & 3 & 3 & 3 & 3 \\
1 & 3 & 3 & 3 & 3 & 3
\end{array}
& \quad \circ_{(4,1,2,3,2)}
\begin{array}{cccccc}
3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 \\
2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2
\end{array}
& \quad \circ_{(2,3,4,1,2)}
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3
\end{array}
\end{align*}$$

where we represent an element $m_T$ by the tableau $T$ itself for short. The uniqueness is obvious from this construction.

5.6. Good tableaux

We introduce a partial order $\leq$ on the set of compositions of $n \in \mathbb{N}$ called the dominance order. Here for two compositions $\lambda$ and $\mu$, they are defined to be $\lambda \leq \mu$ if and only if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \leq \mu_1 + \mu_2 + \cdots + \mu_k$$

is satisfied for each $k \in \mathbb{N}$. It is not a total order; for example, the compositions $(3, 3)$ and $(4, 1, 1)$ are incomparable. According to the reversed dominance order, we make a filtration on the module category as we did in the previous part. For each composition $\lambda$, the set $\{\mu | \mu > \lambda\}$ is finite. Hence the set of all compositions with the reversed dominance order is a well-founded partially ordered set.

Notation 5.6. Let $X, Y \in H_n \text{-Mod}$. For a composition $\lambda$, let

$$\mathcal{H}^\lambda(X, Y) := \text{Hom}_{H_\mu}(M_\lambda, Y) \circ \text{Hom}_{H_\mu}(X, M_\lambda)$$

be the set of homomorphisms which factor through $M_\lambda$. In other words, $\mathcal{H}^\lambda$ is a 2-sided ideal of $H_n \text{-Mod}$ generated by $M_\lambda$. By using the dominance order we define

$$\mathcal{H}^{\geq \lambda}(X, Y) := \sum_{\mu \geq \lambda} \mathcal{H}^\mu(X, Y), \quad \mathcal{H}^{> \lambda}(X, Y) := \sum_{\mu > \lambda} \mathcal{H}^\mu(X, Y)$$

and

$$\text{Hom}_{H_n}^{(\lambda)}(X, Y) := \text{Hom}_{H_n}(X, Y) / \mathcal{H}^{> \lambda}(X, Y).$$

The last one is a hom set in the quotient category $(H_n \text{-Mod})/\mathcal{H}^{> \lambda}$. 

51
When $X$ and $Y$ above are parabolic modules, we write these submodules or quotient modules of $M_{\lambda \mu}$ as $M_{\lambda \mu}^+$, $M_{\lambda \mu}^{\odd}$, $M_{\lambda \mu}^{\even}$, and $M_{\lambda \mu}^{(v)}$ respectively. In particular,

$$M_{\lambda \mu}^{(v)} = \text{Hom}^{(v)}_{H_n}(M_{\mu}, M_{\lambda})$$

is the $k$-module equipped with the reversed composition as product. As its special case we let $S_{\lambda \mu} := M_{\lambda \mu}^{(0)}$. Then $S_{\lambda \mu}$ is a quotient algebra of $M_{\lambda \mu}$ and $S_{\lambda \mu}$ is a right module over this algebra.

When $\mu = (1^\nu)$ we simply write $S_{\lambda} := S_{\lambda(1^\nu)}$. $S_{\lambda}$ is also a left module over $H_n \cong M_{(1^\nu)(1^\nu)}$ and called the Specht module. We denote equalities in the quotient set $S_{\lambda \mu}$ by the symbol $\equiv$.

Note that if a composition $\lambda = (\lambda_1, \lambda_2, \ldots)$ has $\lambda_i = 0$ such that $\lambda_{i+1} \neq 0$, letting $\tilde{\lambda} := (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots)$ we have $\lambda < \tilde{\lambda}$ and $M_{\lambda} \equiv M_{\tilde{\lambda}}$. Hence for such $\lambda$, $M_{\lambda}$ is zero in the quotient category $(H_n\text{-}\text{Mod})/\mathcal{H}^{\lambda}$; in particular we have $S_{\lambda \mu} = 0$ for all $\mu$. We can remove such needless compositions from the index set. Then the rest is now a finite set.

For a while we fix $n \in \mathbb{N}$ and $\lambda, \mu$ denote compositions of $n$. In order to study this quotient category, we introduce a combinatorial notion on tableau as follows.

**Definition 5.7.** Let $T \in \text{Tab}_{\lambda \mu}$ be a row-semistandard tableau. We say that a box $(i, j) \in Y(\lambda)$ in the Young diagram is **good** if it satisfies $T(i, j) \geq i$, and $T$ is said to be **good** if all boxes in $Y(\lambda)$ are good.

**Lemma 5.8.** $S_{\lambda \mu}$ is spanned by $\{m_T \mid T \in \text{Tab}_{\lambda \mu} \text{ which is good}\}$.

**Proof.** Suppose that $T$ is not good. For such $T$, let us define a tableau $T_1$ of shape $\lambda$ by

$$T_1(i, j) := \min[i, T(i, j)].$$

Next let $T_2$ be a tableau obtained by moving up all ungood boxes of $T$ to its $i$-th row, so that $T_2$ is good. For example, when

$$T = \begin{array}{cccc}
| & | & | & |\\
2 & 2 & 1 & 3 \\
1 & 2 & 3 & \\
\end{array}$$

which has ungood 1 and 2 in the third row, we let

$$T_1 = \begin{array}{cccc}
| & | & | & |\\
1 & 1 & 1 & 1 \\
2 & 2 & 2 & \\
1 & 2 & 3 & \\
\end{array} \quad \text{and} \quad T_2 = \begin{array}{cccc}
| & | & | & |\\
1 & 1 & 1 & 1 & 2 & 3 \\
2 & 2 & 3 & 3 & \\
3 & \\
\end{array}.$$ 

Let $\nu$ be the weight of $T_1$, which is equal to the shape of $T_2$. For each $k$ we have

$$\nu_1 + \nu_2 + \cdots + \nu_k = \lambda_1 + \lambda_2 + \cdots + \lambda_k + \#((i, j) \in Y(\lambda) \mid i > k \text{ and } T(i, j) \leq k).$$

Since $T$ is not good, we have $\nu > \lambda$ so that $m_{T_2} \circ \nu, m_{T_1} \equiv 0$ in $S_{\lambda \mu}$.

On the other hand, observe that the $i$-th row of $T_2$ is obtained by reading entries of $T$ at boxes $(k, l)$ such that $T_1(k, l) = i$ from bottom to top. So taking $w := d(T_1) \in \mathcal{D}$, we have $T^+ = wT_1$ and $\ell(T^+) = \ell(w) + \ell(T_1^+)$. This induces the following decomposition in $M_{\lambda \mu}$:

$$m_{T_2} \circ \nu, m_{T_1} = m_T + \sum_{S \in \text{Tab}_{\lambda \mu}, \ell(S^+) < \ell(T^+)} c_S m_S \quad (c_S \in k).$$

Hence in $S_{\lambda \mu}$ we can replace ungood $m_T$ by a linear combination of elements $m_S$ which has smaller lengths. Consequently it inductively follows that any tableau can be written as a linear combination of good ones.  

$\square$

52
Lemma 5.9. 
(1) Tab,λ has only one good tableau.
(2) There are no good tableau in Tab,µ unless λ ⩾ µ.

Proof. If T ∈ Tab,µ is good, then for each k, all i’s in T less than or equal to k are placed in its k-th row or upper. The number of such numbers (= µ1 + ··· + µk) must be equal to or less than that of such boxes (= λ1 + ··· + λk) so we have λ ⩾ µ. Moreover if λ = µ, all i’s in T must be in its i-th row.

By these two lemmas, the statements below are obvious.

Corollary 5.10. 
(1) S,λ,µ is spanned by m,λ. Hence it is isomorphic to a quotient ring of k.
(2) S,λ,µ = 0 unless λ ⩾ µ.

Hence the category H_n-Mod with objects {M,λ} and algebras End_{k,λ} satisfies the assumptions in Lemma 0.5 (see Remark 4.2), so it produces several standardly filtered algebras.

Theorem 5.11. The Iwahori–Hecke algebra H_n = M_{[1,1) and the q-Schur algebra M_{[1,1),p} = ⊕_{λ,µ} M_{λ,µ} are standardly filtered algebras over k on the set of compositions. Here for each composition ν, their ideal filter and the attached Morita contexts is given by

\[ H^ν_n := M^ν_{(1,1)} \quad \text{with} \quad \langle S_ν, S^*_ν \rangle \]

and

\[ M^ν_{(1,1),p} := \bigoplus_{λ,µ} M^ν_{λ,µ} \quad \text{with} \quad \bigoplus_{λ,µ} S^ν_{κ,λ} \bigoplus_{λ,µ} S^ν_{λ,µ} \]

where S^ν_{κ,λ} := M^ν_{(1,1),p} and S^ν_{λ,µ} := S^ν_{κ,λ}. These standard filters are involutive.

It seems to be an interesting problem to determine the k-module structure of S,λ,µ (or more general M^ν_{(1,1),p}) in detail. For the case that q is invertible we can completely determine its structure by taking its free basis as we will study in later subsections. In the other case the situation is more complicated so that these modules even need not to be free. The author conjectures that each S,λ,µ is isomorphic to k or k/(h(λ))k for some h(λ) ∈ N determined by the shape of λ, but a general one is still unable to describe.

5.7. Local transformations in Specht modules

In this subsection we prove useful formulas for computation on Specht modules.

Lemma 5.12. Suppose we have an equation \( \sum T c_T t_T \equiv 0 \) in S,λ,µ for some c_T ∈ k. Take an arbitrary sequence a_1 ⩽ a_2 ⩽ ··· ⩽ a_k. For each T ∈ Tab,λ,µ let T^+ ∈ Tab,k,µ^+ be the tableau obtained by adding a new row \( [a_0, a_1, \ldots, a_k] \) at the top of T; here \( (k, µ) := (k, λ_1, λ_2, \ldots) \) and µ^+_j = µ_j + \#(i | a_i = j). Then we have an equation \( \sum T^+ c_T t_{T^+} \equiv 0 \) in S,(k,µ)^+.

Proof. First note that the convolution functor with trivial module

\[ \otimes_k : (H_n-Mod)^{H^n} \to (H_{k-λ}-Mod)^{H^n} \]

is still well-defined, because for any V → W which factors through some M, for ν > λ, corresponding \( \otimes_k V \to \otimes_k W \) factors through M_(k,ν) with \( (k, ν) > (k, λ) \). For each T, let us define \( T^0 ∈ Tab,λ,µ^+ \) by

\[ T^0(i, j) = \begin{cases} 1 & \text{if } i = 1, \\
\frac{T(i - 1, j) + 1}{53} & \text{otherwise,}
\end{cases} \]

so that $m_{T^*} = \mathbb{1} + m_T$. On the other hand, let $R \in \text{Tab}_{k,\mu}$ be the tableau defined by

$$R(i,j) = \begin{cases} a_j & \text{if } i = 1, \\ i - 1 & \text{otherwise.} \end{cases}$$

Then we have $m_R \circ_{(k,\mu)} m_{T^*} = m_T$, by the decomposition of $m_R$ according to Proposition 5.5 and the formulas in Lemma 5.2 and Lemma 5.3. Hence

$$\sum_T c_T m_{T^*} = m_R \circ_{(k,\mu)} \sum_T c_T m_{T^*} = m_R \circ_{(k,\mu)} (\mathbb{1} + \sum_T c_T m_T) \equiv 0. \quad \square$$

By the same argument, we can also add a new row to the bottom of tableaux. For the bottom row of a tableau we have another kind of formula.

**Lemma 5.13.** Let $\sum_T c_T m_{T^*} = 0 \in S_{k,\mu}$ as above. Take a number $a$ which is greater than or equal to any entries of $T$ (so $\mu_i = 0$ for $i > a$). For each $T \in \text{Tab}_{k,\mu}$, let $T^* \in \text{Tab}_{k',\mu}$ be the tableau obtained by joining a bar $\hline a \hline$ of length $l$ at the right of the bottom row of $T$; here $\lambda^* := (\lambda_1, \ldots, \lambda_{l-1}, \lambda_l + l)$ and $\mu^* := (\mu_1, \ldots, \mu_{a-1}, \mu_a + l)$. Then we also have $\sum_T c_T m_{T^*} = 0 \in S_{k',\mu^*}$.

**Proof.** For a composition $\nu = (\nu_1, \ldots, \nu_r)$, we write $(\nu, l) := (\nu_1, \ldots, \nu_r, l)$. We define $T^* \in \text{Tab}_{k,\mu}$ for each $T \in \text{Tab}_{k,\mu}$ by

$$T^*(i,j) := \begin{cases} T(i,j) & \text{if } i \leq r, \\ a & \text{if } i = r + 1 \end{cases}$$

and $R \in \text{Tab}_{k',\mu}$ by

$$R(i,j) := \begin{cases} i & \text{if } i < r \text{ or } (i = r, j \leq \lambda_i), \\ r + 1 & \text{if } i = r, j > \lambda_i, \end{cases}$$

so that $m_{T^*} \circ_{(k,\mu)} m_R = \left[\nu_a(T) + l\right] m_{T^*}$ similarly to the previous proof. By the similar argument we can prove $\sum_T c_T m_{T^*} = 0$, and more strongly, this element can be written as a linear combination of elements which factor through $M_{\nu,l}$ for $\nu > \lambda$. This implies $(\nu, l) \notin \lambda^*$; thus by Corollary 5.11 in $S_{k',\mu^*}$ we have

$$\sum_T c_T \left[\nu_a(T) + l\right] m_{T^*} = \sum_T c_T m_{T^*} \circ_{v} m_R \equiv 0. \quad \square$$

The formula below will be needed for a later computation.

**Lemma 5.14.** Let $k, l, n \in \mathbb{N}$ such that $k \leq l \leq n$ and let $\lambda := (n-k, k)$ and $\mu := (n-l, l)$. For each $i$, let $T_i \in \text{Tab}_{k,\mu}$ be the tableau determined by $\#_{21}(T_i) = i$, that is, it is in the form

$$T_i = \begin{array}{cccccc} i & 1 & 1 & 1 & 2 & 2 \\ \hline 1 & 2 & 2 & 2 & 2 \end{array}.$$  

Then we have $m_{T_i} \equiv (-1)^i q^{\binom{i}{2}} [i] m_{T_0}$ in $S_{k,\mu}$.  

54
Proof. We prove it by an induction on \(k\). The case \(i = 0\) is trivial so assume that \(0 < i \leq k\). For \(i < k\), using the assumption of induction, the formula is implied by the lemma above. On the other hand, by Lemma 5.2(1) we have

\[
0 \equiv \sum_{0 \leq j \leq k} m_T = \sum_{0 \leq j \leq k} n_T,
\]

so that the statement also holds for \(i = k\) by the formula

\[
\sum_{0 \leq j \leq k} (-1)^j q^j \left[ \begin{array}{c}
\lambda_1 \\
\vdots \\
\lambda_k
\end{array} \right] = 0 \quad \text{implied by} \quad \prod_{0 \leq j \leq k} (1 + q^j t) = \sum_{0 \leq j \leq k} q^j \left[ \begin{array}{c}
\lambda_1 \\
\vdots \\
\lambda_k
\end{array} \right]. \quad \Box
\]

Multiplying an element to the both-hand sides of this formula for \(i = k = l\), we obtain the following corollary by Lemma 5.2(1).

Corollary 5.15. Let \(\lambda = (n-k, k)\) as above. For arbitrary entries \(a_1 \leq \cdots \leq a_k\), we have

\[
\begin{array}{c}
1 \\
\vdots \\
1
\end{array} \quad \equiv \quad (-1)^k q^k \left[ \begin{array}{c}
\lambda_1 \\
\vdots \\
\lambda_k
\end{array} \right]. \quad \Box
\]

5.8. Semistandard tableaux

Hereafter in this section we assume \(q \in k^\times\). Then the braiding \(\sigma\) of the convolution \(*\) is now invertible so we have \(M_\lambda \cong M_w\) for any \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)\) and \(w \in \Theta_r\). Recall that a composition \(\lambda\) is called a partition if it is a descending sequence: \(\lambda_1 \geq \lambda_2 \geq \ldots\). So in this case, unless \(\lambda\) is a partition, we can take some \(w\) such that \(\lambda < w\lambda\), so that \(M_\lambda\) is zero in the quotient category \((H_r \text{-Mod})/H_r^\sim\) again.

A row-semistandard tableau \(T \in \text{STab}_{\lambda, \rho}\) is called a semistandard tableau if its shape \(\lambda\) is a partition and for all vertically adjacent boxes \((i, j), (i+1, j) \in Y(\lambda)\) it satisfies \(T(i, j) < T(i+1, j)\); or equivalently, all its columns are strictly increasing. We denote by \(\text{Stab}_{\lambda, \rho}\) the set of all semistandard tableaux of shape \(\lambda\) of weight \(\rho\). Note that the strictly increasing condition clearly implies that every semistandard tableau is good. Now we can improve a lemma in the previous subsection.

Lemma 5.16. \(S_{\lambda, \rho}\) is spanned by \(\{m_T \mid T \in \text{STab}_{\lambda, \rho}\}\).

Proof. The statement is clear if \(\lambda\) is not a partition, so we may assume so. Suppose \(T\) is not semistandard and take its box \((k, l) \in Y(\lambda)\) such that \(T(k, l) \geq T(k+1, l)\). Let \(\nu\) be a composition

\[
\nu := (\lambda_1, \ldots, \lambda_{k-1}, l-1, \lambda_k + 1, \lambda_{k+1} - l, \lambda_{k+2}, \lambda_{k+3}, \ldots).
\]

We define tableaux \(T_1 \in \text{Tab}_{\lambda, \nu}\) and \(T_2 \in \text{Tab}_{\nu, \rho}\) by

\[
T_1(i, j) = \begin{cases} 
1 & \text{if} \ i < k \text{ or } (i = k, j < l) \text{ or } (i = k+1, j \leq l), \\
i + 1 & \text{otherwise}.
\end{cases}
\]

\[
T_2(i, j) = \begin{cases} 
T(i, j) & \text{if} \ i \leq k \text{ or } (i = k+1, j \leq l), \\
T(k, j-1) & \text{if} \ i = k+1, j > l, \\
T(k+1, j+l) & \text{if} \ i = k+2, \\
T(i-1, j) & \text{if} \ i > k+2.
\end{cases}
\]

We define tableaux \(T_1 \in \text{Tab}_{\lambda, \nu}\) and \(T_2 \in \text{Tab}_{\nu, \rho}\) by

\[
T_1(i, j) = \begin{cases} 
i & \text{if} \ i < k \text{ or } (i = k, j < l) \text{ or } (i = k+1, j \leq l), \\
i + 1 & \text{otherwise}.
\end{cases}
\]

\[
T_2(i, j) = \begin{cases} 
T(i, j) & \text{if} \ i \leq k \text{ or } (i = k+1, j \leq l), \\
T(k, j-1) & \text{if} \ i = k+1, j > l, \\
T(k+1, j+l) & \text{if} \ i = k+2, \\
T(i-1, j) & \text{if} \ i > k+2.
\end{cases}
\]
For example, when 
\[ T = \begin{array}{cccccc}
1 & 1 & 1 & 2 & 2 & 3 \\
1 & 2 & 2 & 2 & 3 & 2 \\
\end{array} \]
and \((k, l) = (1, 4)\), the corresponding tableaux are 
\[ T_1 = \begin{array}{cccccc}
1 & 1 & 1 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 3 & 2 \\
\end{array} \quad \text{and} \quad T_2 = \begin{array}{cccccc}
1 & 1 & 1 & 2 & 2 & 3 \\
2 & 2 & 2 & 2 & 3 & 2 \\
\end{array} \].

So intuitively \( T_2 \) is obtained by picking up entries of \( T \) in the polygonal chain 
\[ \bullet \bullet \bullet \bullet \bullet \bullet \bullet \]
which turns at \((k, l)\) and \((k + 1, l)\) as a new row. Then by the same argument in the proof of Lemma 5.8, \( m_{T_2} \circ m_{T_1} \in M_{\lambda \mu} \) has the leading term \( m_T \). Now let \( \nu' + \) be another composition
\[ \nu' := (\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + 1, l - 1, \lambda_{k+1} - l, \lambda_{k+2}, \ldots, \lambda_r) \]
which is obtained by swapping middle two entries of \( \nu \). By the assumption \( q \in k^{\times} \), we have \( M_{\nu'} = M_{\nu''} \). Hence \( m_{T_2} \circ m_{T_1} \) also factors through \( M_{\nu''} \). On the other hand, we have clearly \( \nu' \not\leq \lambda \). Hence by Corollary 5.10 (2), \( m_{T_2} \circ m_{T_1} \equiv 0 \) in \( S_{\lambda \mu} \). By induction on length as before we obtain the statement.

**Theorem 5.17.** Recall the assumption \( q \in k^{\times} \). Then \( M_{\lambda \mu} \) has a basis
\[ \bigsqcup \{ m_S \circ m_T \mid S \in S\text{Tab}_{\nu \mu}, T \in S\text{Tab}_{\nu' \lambda} \}. \]

**Proof.** First we prove that the set above spans the hom space. Take an appropriate total order on the set of all compositions \( \{ \nu_1, \nu_2, \ldots, \nu_p = \lambda, \ldots \} \) which is stronger than the reversed dominance order, so that \( i \leq j \) whenever \( \nu_i \geq \nu_j \). We take a filtration on \( M_{\lambda \mu} \) by letting \( M_{\lambda \mu}^{\leq k} := \sum_{i \leq k} M_{\nu_i \mu}^{\nu_i} \) for each \( k \) so that \( M_{\lambda \mu} = M_{\lambda \mu}^{\leq p} \). Then, on each composition factor, by inclusion \( M_{\nu_i \mu}^{\nu_i} \subset M_{\nu_i \mu}^{\nu_i - 1} \) there is a natural surjective map
\[ \circ_{\nu_i} : S_{\nu_i \mu} \otimes S_{\nu_i \lambda}^{*} \to M_{\nu_i \mu}^{\nu_i} / M_{\nu_i \mu}^{\nu_i} \to M_{\nu_i \mu}^{\nu_i - 1} / M_{\nu_i \mu}^{\nu_i - 1} \]
here recall that \( S_{\nu_i \mu} = M_{\nu_i \mu}^{\nu_i} \) and we define \( S_{\nu_i \lambda}^{*} := M_{\lambda \nu_i}^{\nu_i} \). Hence by Lemma 5.16, the right-hand side is spanned by \( \{ m_S \circ m_T \} \) above.

Now remember the Robinson–Schensted–Knuth correspondence \( \text{Tab}_{\lambda \mu} \leftrightarrow \bigsqcup \bigsqcup \text{STab}_{\nu \mu} \times \text{STab}_{\nu' \lambda} \).

Hence the rank of the free \( k \)-module \( M_{\lambda \mu} \) is equal to the number of elements in the generating set above. Consequently this set is also linearly independent, so that it forms a basis. \( \square \)
Corollary 5.18. (1) $S_{\lambda,\mu}$ has a basis $\{m_T | T \in \text{STab}_{\lambda,\mu}\}$. In particular,

$$S_{\lambda,\mu} \cong \begin{cases} \mathbb{k} & \text{if } \lambda \text{ is a partition}, \\ 0 & \text{otherwise}. \end{cases}$$

(2) The product

$$\alpha_v : S_{\nu,\mu} \otimes S^*_{\nu,\lambda \mu} \to M^{(\nu)}$$

is injective.

(3) $H_n$ and $\mathcal{A}_n$ are cellular algebras. \qed

Now for the $q$-Schur algebra $\mathcal{S}_{r,n} = \bigoplus_{\lambda,\mu} M_{\lambda,\mu}$ its simple modules are easily classified. Let $\nu$ be a partition of $n$. If $\nu$ is of at most length $r$ then the trace ideal of the Morita context $(\bigoplus_{\lambda} S_{\lambda,\nu}, \bigoplus_{\lambda} S^*_{\lambda,\nu})$ in $S_{\nu,\nu} \cong \mathbb{k}$ is clearly $\mathbb{k}$. Otherwise the Morita context is zero since $\lambda, \mu \not\subseteq \nu$ for all such $\lambda, \mu$. Hence we obtain the following classification.

Theorem 5.19. When $q \in \mathbb{k}^\times$, there is a one-to-one correspondence

$$\text{Irr}(\mathcal{S}_{r,n}) \longleftrightarrow \{\nu = (\nu_1, \ldots, \nu_r) : \text{partition of } n\} \times \text{Irr}(\mathbb{k})$$

induced by the Morita context functors. Here for a pair of $\nu$ and $V \in \text{Irr}(\mathbb{k})$, the corresponding simple module is given by

$$\text{Image}\left(\bigoplus_{\lambda} S_{\lambda,\nu} \otimes V \to \text{Hom}_{\mathbb{k}}\left(\bigoplus_{\lambda} S^*_{\lambda,\nu}, V\right)\right).$$

5.9. Identification of the ideals

Recall the assumption $q \in \mathbb{k}^\times$. We then proceed to the classification of simple modules of the Iwahori–Hecke algebra $H_n = M_{[\nu_{(\nu)}]}$. For each partition $\lambda$, let $J_\lambda = S^*_\lambda : S_\lambda$ be the trace ideal of the Morita context $(S_\lambda, S^*_\lambda)$ in $S_{\nu,\nu} \cong \mathbb{k}$; here note that the product $\alpha_\nu$ is just the ordinary multiplication. Since $S^*_\lambda$ is generated by $m_{\lambda,1}$, we have $J_\lambda = m_{\lambda,1} : S_\lambda$. In order to classify simple modules, we have to determine it.

Lemma 5.20. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition. For such $\lambda$, let

$$f_\lambda := [\lambda_1 - \lambda_2]![\lambda_2 - \lambda_3]![\lambda_3 - \lambda_4]! \cdots [\lambda_r]!.$$

Then we have inclusions $\mathbb{k} f_\lambda^r \subset J_\lambda \subset \mathbb{k} f_\lambda$. In particular, $\text{Irr}^{(\lambda)}(\mathbb{k}) = \text{Irr}^{\mathbb{k} f_\lambda}(\mathbb{k})$.

Proof. First we prove $J_\lambda \subset \mathbb{k} f_\lambda$. So it suffices to prove that for an arbitrary $T \in \text{Tab}_\lambda$ we have $m_{\lambda,1} : m_T \in \mathbb{k} f_\lambda m_{\lambda,1}$ as an element of $S_{\lambda,\lambda}$. Note that taking a refinement $\mu := (\lambda_1, 1^{n-\lambda_1})$ of $\lambda$ we can decompose $m_{\lambda,1} \in M_\lambda^+ = m_{\lambda_1} \circ_{\mu} m_{\mu}$. So let $S := T|_{\mu}$. Explicitly, $S$ is a row-semistandard tableau of shape $\lambda$ of weight $\mu$ defined by

$$S(i,j) := \begin{cases} 1 & \text{if } 1 \leq T(i,j) \leq \lambda_1, \\ T(i,j) - \lambda_1 + 1 & \text{otherwise}. \end{cases}$$

Let $\nu := S[1]$ be the composition of $\lambda_1$ where $\nu_i$ is the number of entries $1, 2, \ldots, \lambda_1$ in the $i$-th row of $T$. Then by Lemma 5.2 (2) we obtain that

$$m_{\mu} : m_T = q^\nu \nu_1 ![\nu_2]! \cdots ![\nu_r]! m_S$$

57
for some \( \ell \in \mathbb{N} \). In particular, the coefficient can be divided by \( [\nu_1]! \). Let \( \lambda \setminus \nu \) be the composition of \( n - \lambda_1 \) defined by \((\lambda \setminus \nu)_i := \lambda_i - \nu_i \). Since \( m_S \) factors through \( M_{(\lambda \setminus \nu), (\nu')} \) as before, if \( \nu_1 < \lambda_1 - \lambda_2 \) then \( \lambda \notin (\lambda_1, \lambda \setminus \nu) \), which implies \( m_S \equiv 0 \) in \( S_{\lambda', \mu} \). Thus the statement trivially holds in this case. Otherwise \( [\nu_1]! \) can be divided by \( [\lambda_1 - \lambda_2]! \). By induction, for \( \lambda' = (\lambda_2, \ldots, \lambda_\ell) \) we may assume that \( m_{\lambda'} \cdot S_{\lambda', \mu} \subseteq k f_{\lambda'} m_{\lambda'} \). Note that \( S_{\lambda', \mu} = \mathbb{1}_{\lambda'} + S_{\lambda', \mu} \). Therefore

\[
m_{\lambda} \cdot m_{\tau} \in [\lambda_1 - \lambda_2]! m_{\lambda} \circ_{\mu} S_{\lambda, \mu} \subseteq [\lambda_1 - \lambda_2]! ([\mathbb{1}_{\lambda'} + k f_{\lambda'} m_{\lambda'}) = k f_{\lambda'} m_{\lambda'} .
\]

Next we prove the other inclusion \( k f_{\lambda'} \subseteq J_{\lambda} \). Let \( R \in \text{Tab}_{\lambda, \mu} \) be the row-semistandard tableau determined by \( \#_{\lambda}(R) = \lambda_{\mu + 1} - \lambda_{\mu} \). For example, when \( \lambda = (6, 4, 1) \),

\[
R = \begin{array}{cccc}
1 & 1 & 1 & 2 \\
2 & 3 & 4 & 5 \\
6 & & & \\
\end{array}
\]

Then by taking its underlying row-standard tableau \( R \in \operatorname{Tab}_{\lambda} \), by Lemma \( \text{5.2} \) (2) again we obtain

\[
m_{\lambda} \cdot m_{R} = [\lambda_1 - \lambda_2]! [\lambda_2 - \lambda_3]! \cdots [\lambda_\ell]! m_{R} .
\]

On the other hand, by using Corollary \( \text{5.15} \) repeatedly, we also obtain that \( m_{R} \in k^* m_{\lambda} \) in \( S_{\lambda, \mu} \). For example,

\[
\begin{array}{cccc}
1 & 1 & 2 & 2 \\
1 & 1 & 1 & 2 \\
2 & & & \\
\end{array}
\equiv
\begin{array}{cccc}
1 & 1 & 2 & 2 \\
1 & 1 & 1 & 2 \\
1 & 1 & 1 & 1 \\
2 & & & \\
\end{array}
\equiv
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & \\
2 & 2 & 2 & \\
3 & & & \\
\end{array}
\equiv
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & \\
2 & 2 & 2 & 2 \\
3 & 3 & & \\
\end{array}
\]

This implies \( J_{0} m_{\lambda} \supseteq k m_{\lambda} \cdot m_{R} \supseteq k f_{\lambda'} m_{\lambda} \) as desired.

This completes the classification we noted in the introduction.

**Theorem 5.21.** When \( q \in k^* \), there is a one-to-one correspondence

\[
\text{Irr}(H_n) \leftrightarrow \begin{array}{c}
\lambda: \text{partition of } n \\
\end{array}
\implies \text{Irr}^{k_{\lambda}/k} (k)
\]

induced by the Morita context functors. For a partition \( \lambda \) and \( V \in \text{Irr}^{\lambda} (k) \), the corresponding simple module is

\[
\text{Image}(S_{\lambda} \otimes V \to \text{Hom}_k (S_{\lambda}, V)) .
\]

Finally let us consider the case that \( k \) is a field. Let \( e \in \mathbb{N} \cup \{\infty\} \) be the \( q \)-characteristic of \( k \), namely \( e := \min \{ k | [k] = 0 \} \) (the case \( e = \infty \) is usually written as \( e = 0 \), but we use this definition for simplicity). A partition \( \lambda \) is called \( e \)-restricted if \( \lambda_i - \lambda_{i+1} < e \) holds for every \( i \). Then clearly we have that \( \lambda \) is \( e \)-restricted if and only if \( f_{\lambda} = 0 \). Thus as a corollary of the theorem we obtain the well-known classification.

**Corollary 5.22.** If \( k \) is a field whose \( q \)-characteristic is \( e \) (we still assume that \( q \in k^* \)), there is a one-to-one correspondence

\[
\text{Irr}(H_n) \leftrightarrow \{ \text{\( e \)-restricted partition of } n \} .
\]

The right-hand side is actually the crystal \( B(\Lambda_0) \) of type \( A_{e-1}^{(1)} \) under the description of Misra and Miwa [35].
6. Cellular structure on the Hecke–Clifford superalgebra, I

We are now ready to introduce the main topic of this paper, the Hecke–Clifford superalgebra. In this section we introduce analogues of the Murphy basis, the \( q \)-Schur algebra and the Specht modules for this superalgebra and develop the cellular representation theory parallel to the Iwahori–Hecke algebra.

6.1. The Clifford superalgebra

First we define the most basic superalgebra, the Clifford superalgebra.

**Definition 6.1.** Let \( n \in \mathbb{N} \) and take \( a_1, a_2, \ldots, a_n \in \mathbb{k} \). The **Clifford superalgebra** (or the Clifford–Grassman superalgebra) \( C_n(a_1, a_2, \ldots, a_n) \) is generated by the odd elements \( c_1, c_2, \ldots, c_n \) with relations

\[
c_i^2 = a_i, \quad c_ic_j = -c_jc_i \quad \text{for } i \neq j.
\]

We have a canonical isomorphism of superalgebras

\[
C_n(a_1, a_2, \ldots, a_n) \cong C_1(a_1) \otimes C_1(a_2) \otimes \cdots \otimes C_1(a_n)
\]

(note that by the help of Koszul sign \( c_i \) and \( c_j \) for \( i \neq j \) (anti-)commutes). Clearly \( C_1(a) = \mathbb{k} \oplus k c_1 \) so \( c_1^2 \cdots c_n^2 | p_k \in [0, 1] \) is a basis of \( C_n(a_1, a_2, \ldots, a_n) \).

**Remark 6.2.** More generally, for a free \( \mathbb{k} \)-module \( V \) equipped with a quadratic form \( Q : V \to \mathbb{k} \), we have the corresponding Clifford superalgebra \( C_Q \) generated by \( V \) with the relation \( v^2 = Q(v) \).

When \( \mathbb{k} \) is a field whose characteristic is different from 2, we can always take an orthogonal basis with respect to \( Q \), so that \( C_Q \) is isomorphic to the above form.

The classification of simple modules of \( C_n(a_1, a_2, \ldots, a_n) \) is well-known for special cases (see [29 §12]). We here state a more general result.

**Proposition 6.3.** Suppose \( \mathbb{k} \) is a field. Then \( C_n(a_1, a_2, \ldots, a_n) \) has a unique maximal 2-sided ideal. In particular, it has a unique simple module up to isomorphism and parity change \( \Pi \).

**Proof.** First we prove the case that \( \mathbb{k} \) is algebraically closed. By replacing \( c_i \) with \( c_i/\sqrt{a_i} \) for \( a_i \neq 0 \) and permuting the generators, we may assume that it is in the form \( C_n(1, 1, 0, 0, 0, 0) \).

If the characteristic of \( \mathbb{k} \) is 2, \( C_2(1, 1) \) is isomorphic to \( C_2(1, 0) \) since \( c_1 + c_2 \) (anti-)commutes with \( c_1 \) and its square is zero. Otherwise \( C_2(1, 1) \) is isomorphic to the matrix algebra \( \text{Mat}_{11}(\mathbb{k}) \) via the isomorphism using the Pauli matrices below:

\[
1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c_2 \mapsto \begin{pmatrix} 0 & -1 \\ \sqrt{-1} & 0 \end{pmatrix},
\]

\[
c_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad c_1c_2 \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.
\]

So in both cases, \( C_n(a_1, a_2, \ldots, a_n) \) is isomorphic to \( A \otimes \text{Mat}_{pq}(\mathbb{k}) \otimes \mathbb{C}_0(0, 0, 0) \) for some \( p, q \in \mathbb{N} \) and \( A = \mathbb{k} \) or \( C_1(1) \). Then central idempotent elements \( c_1, \ldots, c_q \) in \( C_0(0, 0, 0) \) are contained in its Jacobson radical, and the quotient superalgebra \( A \otimes \text{Mat}_{pq}(\mathbb{k}) \) with respect to these elements is Morita equivalent to \( A \), which is clearly simple (note that \( (1 + c_1)C_1(1) \) is not considered as an ideal because it is not homogeneous).

Now let \( \mathbb{k} \) be an arbitrary field and \( \overline{\mathbb{k}} \) its algebraic closure. Let us write \( C_n = C_n(a_1, a_2, \ldots, a_n) \) for short. Take a proper 2-sided ideal \( I \subset C_n \). Then \( \overline{I} \subset \overline{C}_n \) is also a proper 2-sided ideal so contained in the Jacobson radical of \( C_n \otimes \overline{\mathbb{k}} \) by the previous case. Since the Jacobson radical is nilpotent, so is \( I \). Hence \( I \) is contained in the Jacobson radical of \( C_n \). \( \square \)
6.2. The Hecke–Clifford superalgebra

Henceforth we fix elements $a, q \in \mathbb{k}$. Let us write $C_n = C_n(a) := C_n(a, \ldots, a)$ for short.

**Definition 6.4.** The Hecke-Clifford superalgebra $H_n^c = H_n^c(a; q)$ is generated by $C_n(a)$ and $H_n(q)$ with relations

$$T_i c_j = c_j T_i, \quad \text{for } j \neq i, i+1, \quad T_i c_i = c_{i+1} T_i, \quad (T_i - q + 1)c_{i+1} = c_i (T_i - q + 1).$$

Note that if $q \in \mathbb{k}^*$, the second relation implies the third.

Here in order to make it compatible with the notions in the previous part we slightly modified the original definition by Olshanski [42] so that our $q$ is isomorphic to the wreath product of the Clifford superalgebra

$$W_n(a) := C_1(a) \wedge \mathfrak{S}_n = C_n(a) \otimes \mathfrak{S}_n$$

which is called the Sergeev superalgebra. In this case there is a natural anti-homomorphism $\ast : W_n(a)^{op} \to W_n(-a)$ between superalgebras defined by $s_i^\ast := s_i$ and $c_i^\ast := c_i$ (note that due to the Koszul sign we have $(c_i^{op})^2 = -(c_i^2)^{op} = -a$), but unfortunately this involution does not have its $q$-analogue.

The next basis theorem is well-known, but we make it our proof by itself since we modified the definition.

**Proposition 6.5.** The multiplication maps $C_n \otimes H_n \to H_n^c$ and $H_n \otimes C_n \to H_n^c$ are isomorphisms of supermodules.

**Proof.** We prove the first isomorphism. By the defining relations this map is surjective. In order to show that it is also injective, we construct an action of $H_n^c$ on $C_n \otimes H_n$ by

$$T_i (x \otimes y) := s_i (x) \otimes T_i y + (q - 1)t_i (x) \otimes y$$

$$c_i (x \otimes y) := c_i x \otimes y$$

for $x \in C_n$ and $y \in H_n$. Here $s_i$ is the automorphism of superalgebra $C_n$ which exchanges $c_i$ and $c_{i+1}$, and $t_i$ is the $\mathbb{k}$-linear map $C_n \to C_n$ defined by

$$t_i (1) := 0, \quad t_i (c_i) := -c_i + c_{i+1},$$

$$t_i (c_i) := 0, \quad t_i (c_i c_{i+1}) := a + c_i c_{i+1}$$

and $t_i (zc_j) = t_i (c_j) z$ for $j \neq i, i+1$. It is a routine work to verify that the action is well-defined. This action satisfies $xy \cdot (1 \otimes 1) = x \otimes y$ for $x \in C_n$ and $y \in H_n$ so it defines the inverse map $H_n^c \to C_n \otimes H_n$.

Now we have $H_n^c \cong C_n \otimes H_n$ so that $H_n^c$ is a free supermodule over $\mathbb{k}$ of rank $2^n n!$ with a basis $\{c_i^n \cdots c_{i+1}^n T_n\}$. By the commutation relation $[T_n, c_i^n \cdots c_{i+1}^n]$ also forms a basis of $H_n^c$. This implies the second isomorphism. \qed

In particular, $C_n$ and $H_n$ can be identified with subsuperalgebras of $H_n^c$. For each left $H_n$-module $V$, $C_n \otimes V \cong H_n^c \otimes H_n V$ is naturally a left $H_n^c$-module.
Remark 6.6. By the commutation relation, for \( n \geq 2 \), \( I_n := \sum_{1 \leq i < j \leq n} (c_i - c_j)H_n^c \) is a 2-sided ideal of \( H_n \) whose quotient superalgebra is

\[
H_n^c/I_n \cong C_1 \otimes H_n \otimes (k/2\mathfrak{k}c).
\]

Now suppose that \( 2a = 0 \). Since \((c_i - c_j)^2 = 2a = 0 \) and \((c_i - c_j)(c_i - c_k) = -(c_j - c_k)(c_i - c_j)\) for mutual different \( i, j \) and \( k \), the ideal \( I_n \) is nilpotent. Thus it is contained in the Jacobson ideal of \( H_n^c \), so that

\[
\text{Irr}(H_n^c) = \text{Irr}(H_n^c/I_n) \cong \text{Irr}(C_1 \otimes H_n) = \{ V, PV | V \in \text{Irr}(H_n), aV = 0 \} \cup \{ V \otimes cV | V \in \text{Irr}(H_n), aV = V \}.
\]

Hence the classification of simple module of \( H_n^c \) is reduced to that of \( H_n \).

The next computation is a key of our theory. Recall that \( m_n = \sum_{w \in \mathfrak{S}_n} T_w \in H_n \).

**Lemma 6.7.** Let \( \gamma_n^c := c_1 + qc_2 + \cdots + q^{r-1}c_n \) and \( \gamma_n^R := q^{r-1}c_1 + q^{r-2}c_2 + \cdots + c_n \). Then for \( 1 \leq i_1 < i_2 < \cdots < i_r \leq n \),

\[
m_{i_1}c_{i_1}c_{i_2} \cdots c_{i_r}m_n = \begin{cases} \left( \frac{q^{r-1}}{q} \right)^{i_r}m_n & \text{if } r = 2s, \\ \left( \frac{q^{r-1}}{q} \right)^{i_r-1}m_n & \text{if } r = 2s + 1 \end{cases}
\]

(note that even \( q \)-integers \([2], [4], [6], \ldots\) can be divided by \([2]\), so that the coefficients are polynomials in \( \mathbb{Z}[a, q] \)). Moreover we have \( \gamma_n^R m_n = m_n \gamma_n^R \).

**Proof.** Since \( H_n^c \) is free over \( k \), it suffices to prove for the field of rational functions \( k = \mathbb{Q}(a, q) \) in variables \( a \) and \( q \), which contains the universal ring \( \mathbb{Z}[a, q] \). If \( i_{j-1} < i_j - 1 \) holds for some \( j \), we have

\[
m_n \cdots c_{i_j} \cdots m_n = q^{r-1}m_n \cdots c_{i_j} \cdots T_{i_{j-1}}m_n
\]

\[
= q^{r-1}m_n T_{i_{j-1}} \cdots c_{i_j-1} \cdots m_n = m_n \cdots c_{i_{j-1}} \cdots m_n.
\]

Hence we may assume \( i_j = j \). Then for \( r = 0 \) or \( 1 \), we have \( m_n^2 = [n!]m_n \) and

\[
m_{n}c_{i_j}m_n = (c_1 + c_2 T_1 + \cdots + c_{n} T_{n-1} \cdots T_2 T_1)m_{n-1}m_n = [n-1]! \gamma_n^R m_n
\]

where \( m_{n-1} = m_{(1,n-1)} \). Moreover we have

\[
q m_n c_1 c_2 c_3 \cdots m_n = m_n c_1 c_2 c_3 \cdots T_1 m_n
\]

\[
= m_n c_1 T_1 c_1 c_3 \cdots m_n = m_n (T_1 c_2 + (q-1)(c_1 - c_2)) c_1 c_3 \cdots m_n
\]

\[
= a(q-1)m_n c_1 c_2 c_3 \cdots m_n
\]

so that

\[
m_n c_1 c_2 c_3 \cdots m_n = \frac{a(q-1)}{2} m_n c_1 c_2 \cdots m_n.
\]

Hence inductively we obtain the equation. Similarly as above we have

\[
m_n c_{i_j}m_n = m_n m_{n-1}(c_n + T_{n-1}c_{n-1} + \cdots + T_{n-1} \cdots T_2 T_1 c_1) = [n-1]! m_n \gamma_n^R
\]

which implies \( \gamma_n^R m_n = m_n \gamma_n^R \). \[\square\]
6.3. Parabolic supermodules

Analogously to the Iwahori–Hecke algebra, for each composition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) we introduce the parabolic subalgebra

\[
H^*_\lambda = \bigoplus_{w \in \mathcal{O}_\lambda} C_n T_w = \bigoplus_{w \in \mathcal{O}_\lambda} T_n C_n = H_{\lambda_1}^* \otimes H_{\lambda_2}^* \otimes \cdots \otimes H_{\lambda_r}^*.
\]

Then \( H^*_\lambda \) is again a free right \( H^*_\lambda \)-module with a basis \( \{T_w | w \in \mathcal{O}_\lambda \} \). For \( m_1 = \sum_{w \in \mathcal{O}_\lambda} T_w \cdot C_n \) is a left (but not right) ideal of \( H^*_\lambda \) and the parabolic module \( M^*_\lambda := H^*_\lambda m_1 = H^*_\lambda \otimes_{H^*_\lambda} C_n m_1 \) is defined as its induced module. Then by the basis theorem \( H^*_\lambda \cong C_n \otimes H^*_\lambda \) we have \( M^*_\lambda = C_n \otimes M_\lambda \), and

\[
M^*_\lambda = \{ x \in H^*_\lambda | x T_w = q^{(w)} x \text{ for all } w \in \mathcal{O}_\lambda \}.\]

In particular we define the trivial module \( \mathbb{I}^*_\lambda := M^*_0 \cong C_n \). Similarly right modules \( M^*_\lambda \) are defined. Then we have

\[
\text{Hom}_{H^*_\lambda}(M^*_\lambda, M^*_\lambda) = M^*_\lambda \cap M^*_\lambda
\]
equipped with the reversed product

\[
\circ: (M^*_\lambda \cap M^*_\lambda) \otimes (M^*_\lambda \cap M^*_\lambda) \rightarrow M^*_\lambda \cap M^*_\lambda \text{,}
\]

\[
x m \cdot y \mapsto x m y.
\]

With Lemma 6.7 in mind, for each composition \( \lambda \) we define elements

\[
\gamma^L_{2,1} := c_{1,2,\ldots,\lambda_1}, \quad \gamma^R_{2,1} := c_{1,1,2,\ldots,\lambda_1},
\]

\[
\gamma^L_{2,2} := c_{1,1,2,\ldots,\lambda_1}, \quad \gamma^R_{2,2} := c_{1,1,1,2,\ldots,\lambda_1+1},
\]

\[
\ldots
\]

where \( c_{i_1,i_2,\ldots,i_k} := c_{i_1} q c_{i_2} + \cdots + q^{i_k-1} c_{i_k} \). Then we have \( \gamma^L_{2,1} m_{1,2} = m_{1,2} \gamma^R_{2,1} \). So let us define the endomorphism \( \gamma_{2,1} \) acts on \( M^*_\lambda \) and \( M^*_\lambda \) as

\[
x m_{1,2} \cdot \gamma_{2,1} = x m_{1,2} \gamma^R_{2,1} = \gamma^L_{2,1} m_{1,2},
\]

\[
\gamma_{2,1} \cdot m_{1,2} \gamma_{2,1} = \gamma^L_{2,1} m_{1,2} \gamma^R_{2,1} = m_{1,2} \gamma^R_{2,1}.
\]

Note that these endomorphisms anti-commute and we have

\[
(\gamma^L_{2,1})^2 = (\gamma^R_{2,1})^2 = a \llk \lambda \rrk
\]

where \( \llk k \rrk \) is a \( q \)-integer \( 1 + q^2 + \cdots + q^{2(k-1)} \). We can abstractly define a superalgebra consisting of these actions as follows.

**Definition 6.8.** Let \( \lambda \) be a composition. Let \( \Gamma_\lambda \) be a superalgebra generated by odd elements \( \gamma_{2,1}, \gamma_{2,2}, \ldots \) with relations

\[
(\gamma_{2,1})^2 = a \llk \lambda \rrk, \quad \gamma_{2,1} \gamma_{2,1} = - \gamma_{2,1} \gamma_{2,1} \quad \text{for } i \neq j, \quad \gamma_{2,1} \cdot m_{1,2} = m_{1,2} \gamma_{2,1} \text{ if } \lambda_i = 0.
\]

By definition, it is just isomorphic to the Clifford superalgebra

\[
\Gamma_\lambda \cong C_n(\llk \lambda_1 \rrk, \llk \lambda_2 \rrk, \ldots, \llk \lambda_r \rrk)
\]

where \( \{i_1, i_2, \ldots, i_r\} \) are indices such that \( \lambda_i \neq 0 \). By the action above \( M^*_\lambda \) (resp. \( M^*_\lambda \)) is now an \((H^*_\lambda, \Gamma_\lambda)\)-bimodule (resp. a \((\Gamma_\lambda, H^*_\lambda)\)-bimodule). Since the set

\[
\{(\gamma^L_{2,1})^{p_1} (\gamma^L_{2,2})^{p_2} \cdots (\gamma^L_{2,r})^{p_r} | p_k \in \{0, 1\}\}
\]
is linearly independent in \( C_n \), the map \( \Gamma_\lambda \rightarrow \Gamma_\lambda m_1 \subset M^*_\lambda \cap M^*_\lambda \) is an inclusion of superalgebra.
6.4. Circled tableaux

In order to denote elements of the parabolic module $M^r$ graphically we introduce the notion of *circled tableau* [33]. Here a circled tableau of shape $\lambda$ is a map $Y(\lambda) \rightarrow \{1, 2, \ldots, \} \sqcup \{(1), (2), \ldots\}$. From a circled tableau $T$ we obtain its underlying ordinary tableau $T^\circ$ by removing circles from numbers. The weight of a circled tableau is defined as that of underlying tableau. We say that a circled tableau is *row-standard* if its underlying tableau is row-standard. Let $\mathcal{T}_\lambda^r$ be the set of row-standard circled tableau of shape $\lambda$. For $T \in \mathcal{T}_\lambda^r$ we define the corresponding element $m_T := T_\circ c_i \cdot \cdots \cdot c_i m_i$ where $w = d(T^\circ)$ and $i_1, \ldots, i_w$ are indices of positions of circled entries in $T$ according to the top-to-bottom reading order. For example,

$$
\begin{array}{c|c|c|c|c}
 & 1 & 2 & 4 & 5 \\
\hline
3 & 7 & 8 & 5 & 3 \\
\end{array}, \\
\begin{array}{c|c|c|c|c}
 & 1 & 2 & 4 & 5 \\
\hline
3 & 7 & 8 & 5 & 3 \\
\end{array}
$$

For such $T$, we define its length as $\ell(T) := \ell(d(T^\circ))$. If we focus only on leading terms with respect to this length, we have

$$
T_n c_i \cdot \cdots \cdot c_i \cdot m_i \cdot \cdots \cdot m_i + \text{(lower terms)}
$$

so by $M^r \cong C_n \otimes M_1$ the set $\{m_T | T \in \mathcal{T}_\lambda^r\}$ forms a basis of $M^r$. The action of $T_i$ is described as

$$ T_i \cdot m_T = \begin{cases} 
 m_{iT} & \text{if } r(i) < r(i+1), \\
 qm_T + (q-1)m_{iT} & \text{if } r(i) > r(i+1)
\end{cases} $$

where $r(i)$ is the index of the row which contains $i$ or $\bigcirc$ similarly as before, and $s_iT$ is the circled tableau whose underlying tableau is $(s_iT)^\circ = s_i(T^\circ)$ and which has circles at the same boxes as $T$. If $r(i) = r(i+1)$, putting $j = i + 1$ it acts by

$$ T_i \cdot \begin{array}{c|c|c|c|c}
 & 1 & 2 \\
\hline
1 & 2 & 1 & 2 \\
\end{array} = q \begin{array}{c|c|c|c|c}
 & 1 & 2 \\
\hline
1 & 2 & 1 & 2 \\
\end{array}, \\
T_i \cdot \begin{array}{c|c|c|c|c}
 & 1 & 2 \\
\hline
2 & 3 & 2 & 1 \\
\end{array} = q \begin{array}{c|c|c|c|c}
 & 1 & 2 \\
\hline
2 & 3 & 2 & 1 \\
\end{array} - \begin{array}{c|c|c|c|c}
 & 1 & 2 \\
\hline
2 & 3 & 2 & 1 \\
\end{array}.
$$

In contrast the action of $c_i$ is hard to describe due to the commutation relation of $C_n$ and $H_n$, but on leading terms we have

$$ c_i \cdot \begin{array}{c|c|c}
 & 1 \\
\hline
1 & 2 \\
\end{array} = \pm \begin{array}{c|c|c}
 & 2 \\
\hline
1 & 2 \\
\end{array} + \cdots, \\
T_i \cdot \begin{array}{c|c|c}
 & 1 \\
\hline
1 & 2 \\
\end{array} = \pm \begin{array}{c|c|c}
 & 2 \\
\hline
1 & 2 \\
\end{array},
$$

as desired. Here the signs above are taken to be + if the tableau has even number of circles before this box with respect to the reading order, and otherwise −. The right action of $\Gamma_4$ is easy: for example,

$$
\begin{array}{c|c|c|c|c}
 & 1 & 2 \bigcirc & 2 \\
\hline
3 & 7 & 8 & 5 & 3 \\
\end{array} \cdot \gamma_{\{4,3,1\},2} = \begin{array}{c|c|c|c|c}
 & 1 & 2 & 4 \bigcirc \\
\hline
3 & 7 & 8 & 5 & 3 \\
\end{array} - \begin{array}{c|c|c|c|c}
 & 1 & 2 & 4 \bigcirc \\
\hline
3 & 7 & 8 & 5 & 3 \\
\end{array}.
$$

Beware the signs due to the exchange of $c_i$ and $c_j$.

**Remark 6.9.** Usually the shifted form

$$
\begin{array}{c|c|c|c|c}
 & 1 & 2 & 4 & 5 \\
\hline
3 & 7 & 8 & 5 & 3 \\
\end{array}
$$

is used in literatures for circled tableaux. We continue to use the non-shifted form since it seems to be troublesome to change the notations from the previous section.

63
Furthermore we introduce the set of row-semistandard circled tableau Tab$_{\mu}$ to denote elements of $M^r_\mu \cap M^\ast_\mu$. We call a circled tableau of shape $\lambda$ and of weight $\mu$ is row-semistandard if its underlying tableau is row-semistandard and it does not contain patterns $[2 1]$ or $[1 2]$. In other words, circled numbers must be placed at the rightmost of a bar in a row. It is also equivalent to say that each row is weakly increasing with respect to the order $1 < (1) < 2 < (2) < 3 < (3) < \cdots$ and a circled number can not be adjacent to itself. For such $S \in \text{Tab}^c_{\mu}$, we define an element $m_S \in M^r_\mu \cap M^\ast_\mu$ as follows: first we make a formal linear combination of tableaux from $S$ by distributing
\[
\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} + q \begin{bmatrix} 1 & 3 & 1 \end{bmatrix} + q^2 \begin{bmatrix} 4 & 4 & 3 \end{bmatrix}
\]
for each circled bar of length $r$, then by replacing each term $q^i \mathcal{R}$ with the sum of $q^im_T \in M^r_\mu$ for all $T$ such that $T^\ast \in \text{Tab}^c_{\mu}$ and its positions of circles are same as that of $\mathcal{R}$. For example, for
\[
\text{S} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 5 & 0 & 3 \end{bmatrix}
\]
we have
\[
m_S = \begin{bmatrix} 6 & 2 & 4 & 3 \\ 3 & 8 & 0 & 0 \\ 2 & 0 & 8 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 7 & 3 & 3 \end{bmatrix} + q \begin{bmatrix} 6 & 2 & 4 & 3 \\ 3 & 8 & 0 & 0 \\ 2 & 0 & 8 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 7 & 3 & 3 \end{bmatrix} + q^2 \begin{bmatrix} 6 & 2 & 4 & 3 \\ 3 & 8 & 0 & 0 \\ 2 & 0 & 8 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 7 & 3 & 3 \end{bmatrix} + q^3 \begin{bmatrix} 6 & 2 & 4 & 3 \\ 3 & 8 & 0 & 0 \\ 2 & 0 & 8 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 7 & 3 & 3 \end{bmatrix} + q^4 \begin{bmatrix} 6 & 2 & 4 & 3 \\ 3 & 8 & 0 & 0 \\ 2 & 0 & 8 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 7 & 3 & 3 \end{bmatrix} + q^5 \begin{bmatrix} 6 & 2 & 4 & 3 \\ 3 & 8 & 0 & 0 \\ 2 & 0 & 8 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 7 & 3 & 3 \end{bmatrix} + q^6 \begin{bmatrix} 6 & 2 & 4 & 3 \\ 3 & 8 & 0 & 0 \\ 2 & 0 & 8 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 7 & 3 & 3 \end{bmatrix} + q^7 \begin{bmatrix} 6 & 2 & 4 & 3 \\ 3 & 8 & 0 & 0 \\ 2 & 0 & 8 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 7 & 3 & 3 \end{bmatrix} + q^8 \begin{bmatrix} 6 & 2 & 4 & 3 \\ 3 & 8 & 0 & 0 \\ 2 & 0 & 8 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 7 & 3 & 3 \end{bmatrix}.
\]

**Proposition 6.10.** The set $\{m_S | S \in \text{Tab}^c_{\mu}\}$ is linearly independent in $M^r_\mu \cap M^\ast_\mu$. Moreover if $[2] \in k$ is not a zero-divisor, it is also a basis of $M^r_\mu \cap M^\ast_\mu$.

**Proof.** For each $S \in \text{Tab}^c_{\mu}$, take $T \in \text{Tab}^c_\mu$ so that $T^\ast = (S^\ast)_1$ and $T$ has a circle at each box whose position is the leftmost of circled bars $[1 1 \cdots 1]$. Then the coefficient of $m_S$ at the basis element $m_T$ is 1. Since this map $S \mapsto T$ is injective, the set $\{m_S | S \in \text{Tab}^c_{\mu}\}$ is linearly independent in $M^r_\mu$.

On the other hand, let us take $x \in M^r_\mu$ and write $x = \sum_{T \in \text{Tab}^c_\mu} c_T^r m_T$. Suppose that $x \in M^\ast_\mu$ and let $s_1 \in \mathcal{S}_\mu$. Then by the above description of the action of $T_1$, for $T \in \text{Tab}^c_\mu$ such that $r(i) \not= r(i + 1)$, $T_1 x = q x$ implies $c_T = c_{s_1 T}$. On the other hand, suppose $r(i) = r(i + 1)$. Let $T_1$, $T_2$, $T_3$ and $T_4$ be circled tableaux obtained by replacing $i$ and $j = i + 1$ in $T$ by $[1 1]$ and $[1 1]$ and $[0 0]$ and $[0 0]$ respectively. Then by $(1 + T_1)x = [2]x$, we have $[2]e_1 c_{T_2} - [2]c_{T_1} = [2]c_{T_4} = 0$. Using the assumption that $[2]$ is not a zero-divisor, we obtain $c_{T_4} = q c_{T_1}$ and $c_{T_2} = 0$. Hence $x$ can be written as a linear combination of $m_S$. It is also clear that this condition is sufficient for that $x \in M^\ast_\mu$.

Unfortunately, if the assumption is not satisfied then this statement may fail. For example, when $q = 1$ and $2 = 0$ in $k$, the element $e_1 c_2 m_3$ is incidentally contained in $L^\ast_k \cap L^\ast_k$. The set $M^r_\mu \cap M^\ast_\mu$ is not suitable for our use, so instead we use a well-behaved set
\[
M^r_{\mu, \mu} := k \{m_S | S \in \text{Tab}^c_{\mu, \mu}\} \subset M^r_\mu \cap M^\ast_\mu.
\]
This free $k$-module is preserved by an extension of scalars. Since the universal ring $k = \mathbb{Z}[a, q]$ satisfies the assumption, it is closed under product

$$o_{\mu} : M_{\mu^\vee}^\epsilon \otimes M_{\lambda^\mu}^\epsilon \to M_{\lambda^\epsilon}^\epsilon.$$ 

By definition we can represent $\gamma_{ij}^\mu \mu_3 = m_{ij}^\mu \mu_3 \in M_{ij}^\epsilon \cap M_{ij}^\mu$ by a circled tableau, so $\Gamma_\mu \mu_3$ is contained in $M_{ij}^\epsilon$. Hence $\Gamma_\mu$ also acts on $M_{ij}^\epsilon$ from left (resp. $M_{ij}^\mu$ from right) and the product $o_{\mu}$ above is $\Gamma_\mu$-bilinear, so that we can define it as

$$o_{\mu} : M_{\mu^\vee}^\epsilon \otimes_{\Gamma_\mu} M_{\lambda^\mu}^\epsilon \to M_{\lambda^\epsilon}^\epsilon.$$ 

Let $\mathcal{E}_{\rho} := \bigoplus_{\lambda} M_{ij}^\epsilon$, where $\alpha = (\lambda_1, \ldots, \lambda_i)$ and $\mu = (\mu_1, \ldots, \mu_i)$ run over compositions of at most $r$ components as before. We call it the queer $q$-Schur superalgebra. When $q = 1$ and $2 \neq 0$, it is equal to the Schur superalgebra of type $Q$ introduced in [5].

We finish this subsection with a remark on involution. For $S \in \text{Tab}^\epsilon_3$, we can similarly define an element $m^\gamma_\mathcal{E} \in M^\gamma_\mathcal{E}$ in the dual manner by multiplying elements on $m_{ij}^\gamma$ from right. When $q = 1$, it is actually the dual element of $m_{ij}^\gamma \in W(-a)^\text{op}$ mapped via the anti-homomorphism $\ast : W_\phi(-a)^\text{op} \to W(\phi(a))$. For such $S$ we define its dual tableau $S^\ast \in \text{Tab}^\epsilon_\mathcal{E}$ so that $(S^\ast)^\ast = (S^\ast)^\ast$ and $S$ has $\gamma$ in its $i$-th row if and only if $S^\ast$ has $\gamma$ in its $i$-th row. Then by the commutation relation on Lemma 6.7 $m^\gamma_\mathcal{E}$ has the leading term $m_{ij}^\gamma$, but they are not equal unless $q = 1$. The map $m_{ij}^\gamma \mapsto m_{ij}^\gamma$ does not either preserve the reversed product in general.

6.5. Good circled tableaux

Analogously to the non-super case, we introduce a filtration into our subcategory of $H^\epsilon_\mathcal{E}$-$\text{Mod}$. According to this filtration we decompose the set of simple modules of $H^\epsilon_\mathcal{E}$ into small parts.

**Definition 6.11.** For each compositions $\lambda, \mu$, and $\nu$, let

$$M_{\lambda^\mu}^{\epsilon^\nu} := M_{\lambda^\mu}^\epsilon \circ_{\nu} M_{\nu^\mu}^\epsilon \subset M_{\lambda^\nu}^\epsilon.$$ 

Then we define

$$M_{\lambda^\mu}^{\epsilon^\nu} := \sum_{\pi < \nu} M_{\pi^\mu}^{\epsilon^\nu}, \quad M_{\lambda^\mu}^{\epsilon^\nu} := \sum_{\pi > \nu} M_{\pi^\mu}^{\epsilon^\nu}$$

and finally

$$M_{\lambda^\mu}^{\epsilon^\nu} := M_{\lambda^\mu}^\epsilon / M_{\lambda^\mu}^{\epsilon^\nu}.$$ 

In particular we let $S_{\lambda^\mu}^{\epsilon^\nu} := M_{\lambda^\mu}^{\epsilon^\nu}$ and $S_{\lambda^\mu}^{\epsilon^\nu} := S_{\lambda^\mu}^{\epsilon^\nu}$.

We say that a circled tableau $T \in \text{Tab}^\epsilon_{\lambda^\mu}$ is good if its underlying tableau $T^\gamma$ is good.

**Lemma 6.12.** $S_{\lambda^\mu}^{\epsilon^\nu}$ is spanned by $\{m_T \mid T \in \text{Tab}^\epsilon_{\lambda^\mu} \}$ which is good.

**Proof.** Similarly to the proof of Lemma 5.8 we prove it inductively by replacing each $m_T$ for ungood $T \in \text{Tab}^\epsilon_{\lambda^\mu}$ with tableaux which have smaller lengths. However in this case we can not perform this method at a time, so we do for each number one by one. Suppose that $T$ has ungood $\square_i$ which we choose so that $i$ is minimum. In particular, $i$’s in its $i$-th row are at the leftmost of $T$ if exist. If it has circled $\bigcirc$ in its $i$-th row, multiplying $y_{ij}$, from right we can represent $m_T$ by tableaux without this circle; for example,

$$\begin{array}{cccc} 1 & 1 & 1 & 2 \ 2 & 3 & 1 \ 3 & 3 \ \end{array} \cdot \gamma_{h, l} = \begin{array}{cccc} 1 & 1 & 1 & 3 \ 2 & 2 & 3 \ 3 \ 3 \ \end{array} - aq^2 \begin{array}{cccc} 1 & 1 & 1 & 3 \ 2 & 3 & 1 \ 3 & 3 \ \end{array} - q^2 \begin{array}{cccc} 1 & 1 & 1 & 3 \ 2 & 3 & 1 \ 3 & 3 \ \end{array}.$$
Hence we may assume that $i$ in the $i$-th row of $T$ is not circled. We define tableaux $T_1$, $T_2$ from $T$ by moving up ungood $\boxed{1}$ and $\boxed{3}$ as we did in the proof of Lemma 5.8 and if such entry is circled we remove this circle from $T_2$ and put on the same box at $T_1$. For example, for $T = \begin{array}{ccccccc} \boxed{1} & 2 & 2 & 3 & 0 & \boxed{1} & \boxed{3} \\ 2 & 2 & 3 & 0 & \boxed{1} & \boxed{3} & \end{array}$ we have $T_1 = \begin{array}{ccccccc} \boxed{1} & 2 & 2 & 3 & 0 & \boxed{1} & \boxed{3} \\ 2 & 2 & 3 & 0 & \boxed{1} & \boxed{3} & \end{array}$ and $T_2 = \begin{array}{ccccccc} \boxed{1} & 2 & 2 & 3 & 0 & \boxed{1} & \boxed{3} \\ 2 & 2 & 3 & 0 & \boxed{1} & \boxed{3} & \end{array}$. Taking leading terms we also have a decomposition $0 \equiv m_{T_2} \circ m_{T_1} = \pm m_T + ($lower terms$)$. 

This leads us the following parallel results.

**Corollary 6.13.**

(1) $S^c_{\lambda,\mu}$ is spanned by $m_\lambda$ over $\Gamma_\lambda$, so that it is isomorphic to a quotient superalgebra of $\Gamma_\lambda$.

(2) $S^c_{\lambda,\mu}$ = 0 unless $\lambda \geq \mu$.

**Theorem 6.14.** $H_n^c$ and $S^c_{\mu,\nu}$ are standardly filtered algebras over $\{\Gamma_\lambda\}$ on compositions $\lambda$.

Note that we have a natural map $M^c_{\lambda,\mu} \rightarrow \text{Hom}^c_{H_n^c}(M^c_{\lambda,\mu}, M^c_{\lambda,\mu})$ where the right-hand side is the set of homomorphisms in the quotient category naïvely defined by using the whole category $H_n^c$-$\text{Mod}$, but in general this map is not surjective nor injective when the assumption in Proposition 6.10 is not satisfied. Although we can also define a standard filter using the right-hand side, this filter is ill-behaved with extension of scalars. In contrast, we have $S^c_{\lambda,\mu} = S^c_{\lambda,\mu}(\mathbb{Z}[a,q]) \otimes_{\mathbb{Z}[a,q]} k$ as desired since $M^c_{\lambda,\mu}$ has a free basis. So $S^c_{\lambda,\mu}$ is certainly the right definition.

In these modules we have the local transformation lemma by a similar proof as before.

**Lemma 6.15.** Suppose we have an equation $\sum T c_T m_T \equiv 0$ in $S^c_{\lambda,\mu}$ for some $c_T \in k$. For each $T \in \text{Tab}_{\lambda,\mu}$ let $T^*$ be the tableau obtained by adding a new common row at the top (resp. the bottom) of $T$. Then we have $\sum_T c_T m_T^* \equiv 0$.

### 6.6. Shifted semistandard circled tableaux

In this subsection, we consider the case $k = \mathbb{Q}(a,q)$. Recall that a composition $\lambda$ is called a strict partition if it is strictly decreasing: $\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0 = \lambda_{r+1} = \lambda_{r+2} = \cdots$.

**Definition 6.16.** A row-semistandard circled tableau $T \in \text{STab}_{\lambda,\mu}$ is called shifted semistandard if its shape $\lambda$ is a strict partition and it does not contain any of the patterns

\[
\begin{array}{cccc}
\boxed{1} & \boxed{2} & \boxed{3} \\
\boxed{1} & \boxed{2} & \boxed{3} \\
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
\boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} \\
\boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} \\
\end{array}
\]

for $i < j$ (in particular, its underlying tableau $T^\times$ is semistandard). In other words, its entries are also weakly increasing along with each diagonal line so that a non-circled number does not continue. We denote by $\text{STab}_{\lambda,\mu}$ the set of shifted semistandard circled tableaux of shape $\lambda$ of weight $\mu$. 66
A similar notion of generalized shifted tableau is introduced by Sagan [43]. The only difference is that he uses the order

\[ 1 < 1 < 2 < 1 < 3 < \cdots \]

instead of ours. The set of shifted semistandard circled tableaux is clearly in bijection with that of his generalized shifted tableaux by the following circle moving:

\[
\begin{align*}
    \begin{array}{c}
        1 \circ \\
        2 \circ \\
    \end{array} & \mapsto \begin{array}{c}
        1 \circ \\
        2 \circ \\
    \end{array}, \\
    \begin{array}{c}
        1 \circ \\
        3 \circ \\
    \end{array} & \mapsto \begin{array}{c}
        1 \circ \\
        3 \circ \\
    \end{array}.
\end{align*}
\]

Lemma 6.17. Let \( m, k \in \mathbb{N} \) such that \( m \geq k \). Let \( \lambda := (m, k) \) and \( \mu := (k, m) \). Then in \( S^c_{\lambda, \mu} \) we have

\[
\begin{align*}
    \begin{array}{cc}
        1 & 2 \\
        2 & 2 \\
    \end{array} & \equiv \begin{array}{cc}
        1 & 2 \\
        2 & 2 \\
    \end{array}, \\
    \begin{array}{cc}
        1 & 2 \\
        2 & 2 \\
    \end{array} & \equiv \begin{array}{cc}
        1 & 2 \\
        2 & 2 \\
    \end{array}.
\end{align*}
\]

Proof. By Lemma 6.14 and the assumption \( q \in k^\times \), we have

\[
\begin{align*}
    \gamma_{\mu, 2} \cdot \begin{array}{cc}
        1 & 2 \\
        2 & 2 \\
    \end{array} & = (-1)^k q^{-k} \gamma_{\mu, 2} \cdot \begin{array}{cc}
        1 & 1 \\
        1 & 1 \\
    \end{array} \\
    & = (-1)^k q^{-k} \begin{array}{cc}
        2 & 2 \\
        1 & 1 \\
    \end{array} \\
    & = (-1)^k q^{-k} \begin{array}{cc}
        2 & 2 \\
        1 & 1 \\
    \end{array} \cdot \gamma_{\lambda, 1} \\
    & = \begin{array}{cc}
        1 & 1 \\
        2 & 2 \\
    \end{array} \cdot \gamma_{\lambda, 1}.
\end{align*}
\]

If \( m = k \), we have

\[
\gamma_{\lambda, 2} \cdot \begin{array}{cc}
    1 & 1 \\
    2 & 2 \\
\end{array} = \begin{array}{cc}
    1 & 1 \\
    2 & 2 \\
\end{array}, \quad \begin{array}{cc}
    1 & 2 \\
    2 & 2 \\
\end{array} \gamma_{\lambda, 1} = \begin{array}{cc}
    1 & 2 \\
    2 & 2 \\
\end{array}.
\]

Otherwise both-hand sides can be computed by Lemma 6.7 as

\[
\begin{align*}
    \gamma_{\mu, 2} \cdot \begin{array}{cc}
        1 & 2 \\
        2 & 2 \\
    \end{array} & = \begin{array}{cc}
        1 & 2 \\
        2 & 2 \\
    \end{array} + q^k \begin{array}{cc}
        1 & 2 \\
        2 & 2 \\
    \end{array}, \\
    \gamma_{\lambda, 1} & = \begin{array}{cc}
        1 & 2 \\
        2 & 2 \\
    \end{array}.
\end{align*}
\]

so these equations also imply the statement.

Lemma 6.18. \( S^c_{\lambda, \mu} = 0 \) unless \( \lambda \) is a strict partition.

Proof. If \( \lambda \) is not a partition it holds by the same reason as the non-super case. Otherwise if \( \lambda \) is not a strict partition, it contains \( \lambda_i = \lambda_{i+1} > 0 \). So it suffices to prove for \( \lambda = (k, k) \). By the lemma above we have

\[
(\gamma_{\lambda, 1} - \gamma_{\lambda, 2}) \cdot \begin{array}{cc}
    1 & 1 \\
    2 & 2 \\
\end{array} = \begin{array}{cc}
    1 & 1 \\
    2 & 2 \\
\end{array} - \begin{array}{cc}
    1 & 1 \\
    2 & 2 \\
\end{array} \equiv 0.
\]

On the other hand, \( (\gamma_{\lambda, 1} - \gamma_{\lambda, 2})^2 = 2q[k] \in k^\times \). Hence we have \( m_{\lambda} \equiv 0 \) in \( S^c_{\lambda, \mu} \).

Lemma 6.19. \( S^c_{\lambda, \mu} \) is spanned by \( \{ m_T \mid T \in \text{STab}^c_{\lambda, \mu} \} \).
Proof. If \( \lambda \) is not a strict partition the statement is clear by the previous lemma, so we may assume so. First we prove the statement for two special cases.

Case 1: \( \lambda = (m, k) \) and \( \mu = (k, k, m - k) \). Then every good but non-shifted-semistandard circled tableaux have an underlying tableau

\[
\begin{array}{c}
1 & 1 & 3 \quad 3 \\
2 & 2 & 2 \\
\end{array}
\]

and can be made from this tableau by multiplying \( \gamma_{\mu, \ell} \). By Lemma 6.17

\[
\begin{array}{c}
1 & 1 & 3 \quad 3 \\
2 & 2 & 2 \\
\end{array}
\] \( \equiv \) \[
\begin{array}{c}
1 & 1 & 3 \quad 3 \\
2 & 2 & 2 \\
\end{array}
\] + (lower terms).

Hence the statement holds since \((\gamma_{\mu, 1} - \gamma_{\mu, 2})^2 = 2a[k][l]\) is invertible again and every good tableau which has smaller length is shifted semistandard.

Case 2: \( \lambda = (m, k) \) and \( \mu = (k, l, m - l - l', l') \) where \( l < k \) and \( l' < m - k \), so

\[
\begin{array}{c}
1 & 1 & 3 \quad 4 \\
2 & 2 & 3 \quad 3 \\
\end{array}
\] \quad and \quad \[
\begin{array}{c}
1 & 1 & 3 \quad 4 \\
2 & 2 & 3 \quad 3 \\
\end{array}
\]

are all the good but non-shifted-semistandard circled tableaux. Similar to above,

\[
\begin{array}{c}
1 & 1 & 3 \quad 4 \\
2 & 2 & 3 \quad 3 \\
\end{array}
\] \( \equiv \) \[
\begin{array}{c}
1 & 1 & 3 \quad 4 \\
2 & 2 & 3 \quad 3 \\
\end{array}
\] + \( \gamma \) \[
\begin{array}{c}
1 & 1 & 3 \quad 4 \\
2 & 2 & 3 \quad 3 \\
\end{array}
\] + \cdots.

Then by multiplying \( \gamma_{\mu, 1} \) and \( \gamma_{\mu, 2} \) from left respectively, we obtain

\[
a[k][l] \begin{array}{c}
1 & 1 & 3 \quad 4 \\
2 & 2 & 3 \quad 3 \\
\end{array} \equiv a[l] \begin{array}{c}
1 & 1 & 3 \quad 4 \\
2 & 2 & 3 \quad 3 \\
\end{array} + \gamma \begin{array}{c}
1 & 1 & 3 \quad 4 \\
2 & 2 & 3 \quad 3 \\
\end{array} + \cdots
\]

and

\[
-\begin{array}{c}
1 & 1 & 3 \quad 4 \\
2 & 3 & 3 \quad 3 \\
\end{array} \equiv a[l'] \begin{array}{c}
1 & 1 & 3 \quad 4 \\
2 & 2 & 3 \quad 3 \\
\end{array} + \gamma \begin{array}{c}
1 & 1 & 3 \quad 4 \\
2 & 2 & 3 \quad 3 \\
\end{array} + \cdots
\]

so that

\[
a[k][l] + a[l'] \begin{array}{c}
1 & 1 & 3 \quad 4 \\
2 & 2 & 3 \quad 3 \\
\end{array} \equiv \gamma \begin{array}{c}
1 & 1 & 3 \quad 4 \\
2 & 2 & 3 \quad 3 \\
\end{array} - \gamma \begin{array}{c}
1 & 1 & 3 \quad 4 \\
2 & 2 & 3 \quad 3 \\
\end{array} + \cdots
\]

with \( a[k][l] + a[l'] \in k^x \). Multiplying \( \gamma_{\lambda, 1} \) from right we can similarly decompose the second tableau above.

Now we proceed to a general case. Let \( T \in \text{Tab}^{\mu}_{\lambda} \) which is not shifted semistandard. Then there is a prohibited pattern at boxes \((k, l + 1)\) and \((k + 1, l)\). Choose \((k, l)\) so that \( T \) has no such patterns at the bottom right boxes of \((k, l)\) except for it. We prove that we can replace the element \( m_T \) by a linear combination of \( m_R \) where either \( R \) has no such patterns in this region or \( R \) satisfies \( \ell(R^1) < \ell(T^1) \). Then by induction it can be written as a linear combination of shifted standard ones.

First consider the case \( T(k + 1, l) = i, \emptyset \) and \( T(k, l + 1) = j, \emptyset \) with \( i < j \). Similar to the proof of Lemma 5.16 we define

\[
v := (a_1, \ldots, a_{k-1}, l, l, a_k - l, a_{k+1} - l, a_{k+2}, a_{k+3}, \ldots)
\]

68
and \( T_1 \in \text{Tab}_{2^\kappa}, \ T_2 \in \text{Tab}_{c^\kappa} \) by
\[
T_1(i,j) = \begin{cases} 
   i & \text{if } i < k \text{ or } (i = k, \ j \leq l) \text{ or } (i = k + 1, \ j \leq l), \\
   i + 2 & \text{otherwise},
\end{cases}
\]
and
\[
T_2(i,j) = \begin{cases} 
   T(i,j) & \text{if } i \leq k + 1, \\
   T(i - 2, j + l) & \text{if } i = k + 2 \text{ or } i = k + 3, \\
   T(i - 2, j) & \text{if } i > k + 3.
\end{cases}
\]

Then the leading term of \( m_{T_1} \circ m_{T_2} \) is \( m_T \). Applying the decomposition in Case 1 above to \( m_{T_1} \), we can replace \( m_{T_1} \circ m_{T_2} \) by a linear combination of tableaux with smaller lengths so the induction goes forward.

Next consider the other case \( T(k + 1, l) = i \) and \( T(k, l + 1) = i \) or \( \emptyset \). Let \((k + 1, l_1 + 1)\) and \((k, l_2)\) be the ends of the bars \( T \) which start from \((k + 1, l)\) and \((k, l + 1)\) respectively. In this case we define
\[
\nu := (\lambda_1, \ldots, \lambda_{k-1}, l, l_1, l_2 - l_1, \lambda_k - l_2, \lambda_{k+1} - l, \lambda_{k+2}, \lambda_{k+3}, \ldots)
\]
and \( T_1 \in \text{Tab}_{2^l}, \ T_2 \in \text{Tab}_{c^l} \) by
\[
T_1(i,j) = \begin{cases} 
   i & \text{if } i < k \text{ or } (i = k, \ j \leq l) \text{ or } (i = k + 1, \ j \leq l), \\
   k + 2 & \text{if } (i = k, l < j \leq l_2) \text{ or } (i = k + 1, l_1 < j \leq l), \\
   i + 3 & \text{otherwise},
\end{cases}
\]
and
\[
T_2(i,j) = \begin{cases} 
   T(i,j) & \text{if } i \leq k + 1, \\
   T(k + 1, j + l_1) & \text{if } i = k + 2, \ j \leq l_1 \\
   T(k, j + l_1) & \text{if } i = k + 2, \ j > l_1 \\
   T(k, j + l_2) & \text{if } i = k + 3, \\
   T(k + 1, j + l) & \text{if } i = k + 4, \\
   T(i - 3, j) & \text{if } i > k + 4.
\end{cases}
\]

In addition, if \( T(k, l_2) = \emptyset \) is circled, we remove the corresponding circle from \( T_2 \) and move it to \( T_1(k, l_2) = i \). Then the top term of \( m_{T_2} \circ m_{T_1} \) is again \( \pm m_T \). For example, when
\[
T = \begin{bmatrix}
   1 & 1 & 1 & 2 & 3 & 4 \\
   1 & 2 & 3 & 3 & 3 \\
  \end{bmatrix}
\]
and \((k, l) = (1, 3)\), by picking up entries at
\[
\begin{bmatrix}
   * & * & * & 1 \\
   * & * & * & * \\
  \end{bmatrix}
\]
then moving a circle from \( T_2 \) to \( T_1 \) we obtain
\[
T_1 = \begin{bmatrix}
   1 & 1 & 1 & 3 & 3 & 3 & 4 \\
   2 & 3 & 3 & 3 & 3 & 3 \\
   6 & 6 \\
  \end{bmatrix}
\]
and \( T_2 = \begin{bmatrix}
   1 & 1 & 1 \\
   2 & 2 & 2 \\
   3 & 3 & 3 \\
  \end{bmatrix}.
\]

69
Now according to Case 2 above, up to lower terms, we can replace $T_1$ by a linear combination of $m_S$ such that $S^\times = T_1^\times$, $S(k+1,l) = \otimes$ and $S$ has circles only at the boxes $(k,1),\ldots,(k,\lambda_k)$ and $(k+1,1),\ldots,(k+1,l)$. Then the top term of $m_T \circ m_S$ is $\pm m_R$, where $R$ also satisfies that $R^\times = T^\times$, $R(k+1,l) = \emptyset$ and the positions of circles of $R$ and $T$ only differ at these boxes. Hence $R$ also does not have bad patterns at the bottom right region of $(k,l)$. This completes the induction. \qed

Let $\text{STab}^{\prime}_{C_{\lambda}}$ be the subset of $\text{STab}^{\prime}_{C_{\lambda}}$ consisting of tableaux whose entries in the rightmost of each rows are not circled. Then clearly $\# \text{STab}^{\prime}_{C_{\lambda}} = 2^{l(\lambda)} \cdot \# \text{STab}^{\prime}_{C_{\lambda}}$ where $l(\lambda)$ is the number of non-zero components of $\lambda$.

**Corollary 6.20.** $S^{\prime}_{C_{\lambda}}$ is spanned by $\{m_T \mid T \in \text{STab}^{\prime}_{C_{\lambda}}\}$ over $\Gamma_\lambda$.

By a similar proof we can prove that $S^{\prime}_{C_{\lambda}} : M^{(\lambda)}_{\mu,\lambda}$ is also spanned by $\{m_T \mid T \in \text{STab}^{\prime}_{C_{\lambda}}\}$.

**Theorem 6.21.** When $k = Q(a,q)$, $M^{\prime}_{C_{\lambda}}$ has a basis

$$\bigg\{ m_S \circ m_T \mid S \in \text{STab}^{\prime}_{C_{\lambda}}, T \in \text{STab}^{\prime}_{C_{\lambda}} \bigg\}. $$

**Proof.** The proof of that this set spans $M^{\prime}_{C_{\lambda}}$ is same as that of Theorem [5.17]. For that of linear independence we use the one-to-one correspondence $\text{Tab}^{\prime}_{C_{\lambda}} \leftrightarrow \bigg\{ \text{STab}^{\prime}_{C_{\lambda}} \times \text{STab}^{\prime}_{C_{\lambda}} \bigg\}$ induced by Sagan’s shifted Knuth correspondence [43, Theorem 8.1]. \qed

**Corollary 6.22.**

1. $S^{\prime}_{C_{\lambda}}$ has a basis $\{m_T \mid T \in \text{STab}^{\prime}_{C_{\lambda}}\}$. In particular,

$$S^{\prime}_{C_{\lambda}} \simeq \begin{cases} \Gamma_\lambda & \text{if } \lambda \text{ is a strict partition}, \\ 0 & \text{otherwise}. \end{cases}$$

2. The product

$$\circ_{\lambda}: S_{C_{\lambda}} \otimes S_{C_{\lambda}} \to M^{(\lambda)}_{\mu,\lambda}$$

is injective.

3. $H^c_k$ and $A_{r,n}$ are standardly based algebras. \qed

**Remark 6.23.** The basis theorem above for $H^c_k$ (i.e. $\lambda = \mu = (1^n)$) also holds in the following more weaker conditions: $k$ is an arbitrary commutative ring and $2aq \in k^\times$, and the $q^2$-integers $\ll k \rr$ are also invertible for $1 \leq k \leq n/2$. Note that we need not to use Case 2 in the proof of Lemma [6.19]. This implies that $H^c_k$ is also standardly based over $[\Gamma_\lambda]$ in these conditions.

70
6.7. Modules in non-integral rank

Now let \( k \) be an arbitrary commutative ring again. Hereafter in this section, we assume that \( q \in k^\times \). In an unpublished work [37] of the author, it is proved that there exists a supercategory \( H_n^c\text{-Mod} \) which can be regarded as “the module category of \( H_n^c \) for \( t \not\in \mathbb{N} \)”. It covers all ordinary module categories \( H_n^c\text{-Mod} \) for all \( n \in \mathbb{N} \), in the sense that there is a full and surjective functor \( H_n^c\text{-Mod} \to H_n^c\text{-Mod} \). This is an analogue of the Deligne’s category [7] for the symmetric group \( S_\lambda \). Here we list results for this category we need for our purpose.

A fakecomposition of \( n \in \mathbb{N} \) is a pair \( \lambda = (\lambda_1,\lambda') \) of a number \( \lambda_1 \in \mathbb{Z} \) and a composition \( \lambda' \) such that \( n = \lambda_1 + |\lambda'| \). Here \( \lambda_1 \) can be a negative number. For each \( \lambda \), we let \( \lambda_i := \lambda_{i-1}' \) for each \( i \geq 2 \) and write \( \lambda = (\lambda_1,\lambda_2,\lambda_3,\ldots) \). For a natural number \( d \geq |\lambda'| \), let \( \lambda_{d+i} := (d-|\lambda'|,\lambda_2,\lambda_3,\ldots) \) be an ordinary composition of \( d \). For two fakecompositions \( \lambda,\mu \) of \( n \), we define the set of row-standard circled tableaux of shape \( \lambda \) of weight \( \mu \) by

\[
\text{Tab}^c_{\lambda,\mu} := \lim_{d \to} \text{Tab}^c_{\lambda d,\mu}.
\]

Here the map \( \text{Tab}^c_{\lambda d,\mu} \to \text{Tab}^c_{\lambda d+1,\mu} \) is inserting \( \lambda \) on the first row of the tableau from left. The set \( \text{Tab}^c_{\lambda d,\mu} \) is stable for a large number \( d \gg 0 \), so the direct limit converges to a finite set which contains \( \text{Tab}^c_{\lambda,\mu} \). Intuitively we think of Young diagrams whose first rows are very long:

![Diagram](image)

For example, the set \( \text{Tab}^c_{(3,3),(1,3,2)} \) has tableaux such as

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 2 & 3 & 3 & 1 & & 1
\end{array}
\quad\text{and}\quad
\begin{array}{cccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 3 & 3 & & & &
\end{array}
\]

which are not in \( \text{Tab}^c_{(3,1),(1,2,1)} \). The statements below are proved by the author.

**Theorem 6.24.** Let \( \lambda, \mu, \) and \( \nu \) denote fakecompositions of \( n \in \mathbb{N} \).

1. For each pair of \( \lambda, \mu \), there exists a supermodule \( M^c_{\lambda,\mu} \) which has a basis \( \{m_S \mid S \in \text{Tab}^c_{\lambda,\mu}\} \).
2. For each triple \( \lambda, \mu, \nu \), there exists a product map

\[
o^c_{\lambda,\mu,\nu}: \text{Tab}^c_{\lambda,\mu} \otimes \text{Tab}^c_{\lambda,\mu} \to \text{Tab}^c_{\lambda,\mu}
\]

where the coefficient of \( m^\lambda \otimes m^\mu \) on each \( m^\nu \) is given by a polynomial in the variable \( [n] \) over \( \mathbb{Q}[a, q] \) contained in \( \mathbb{Q}[a, q^{-1}] \) for all \( n \in \mathbb{N} \).
3. If \( \lambda \) and \( \mu \) are compositions (i.e. \( \lambda_1,\mu_1 \geq 0 \)) there exists a surjective map

\[
P: \text{Tab}^c_{\lambda,\mu} \to \text{Tab}^c_{\lambda,\mu}
\]

\[
m_S \mapsto \begin{cases} 
m_S & \text{if } S \in \text{Tab}^c_{\lambda,\mu} \\
0 & \text{otherwise}
\end{cases}
\]

which respects the product above.

Note that the number \( \#_{ij}(S^c) \) is also well-defined for \( (i, j) \neq (1, 1) \). We also define \( \#_{11}(S^c) \) by

\[
\#_{11}(S^c) := n - \sum_{(i,j) \neq (1,1)} \#_{ij}(S^c).
\]
More precisely, the map $P$ sends $m_S \in M_{c,\lambda}^\varepsilon$ to the corresponding $m_S \in M_{v,\mu}^\varepsilon$ if $#(S^\varepsilon) > 0$ or $#(S^\varepsilon) = 0$ and $S$ has no $\mathfrak{1}$ in its first row, and otherwise zero. For example, when $\lambda = \mu = (1, 1)$, $P$ is given by

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
1 & 1 & 1 & \mapsto & 1 & 1 & 1 & \mapsto & 1 & 1 \\
2 & 2 & 1 & \mapsto & 1 & 2 & 0 & \mapsto & 2 & 2 \\
1 & 1 & 0 & \mapsto & 2 & 0 & 1 & \mapsto & 0 & 2 \\
1 & 0 & 1 & \mapsto & 0 & 0 & 1 & \mapsto & 0 & 0 \\
\end{array}
\]

The superalgebra $\Gamma_1$ which has generators $\gamma_{1,1}, \gamma_{1,2}, \gamma_{1,3}, \ldots$ is defined similarly as $\Gamma_1$, but the relation $\gamma_{1,1} = 0$ is omitted even if $\lambda_1 = 1$. Hence as an abstract superalgebra, we have

\[\Gamma_1 \cong \mathbb{C}_1(a[a_1]) \otimes \Gamma_{\lambda'}.
\]

Then there is an inclusion of superalgebra $\Gamma_1 \hookrightarrow M_{c,\lambda}^\varepsilon$, so that the product can be defined as

\[\circ_{\mu} : M_{c,\lambda}^\varepsilon \otimes M_{v,\mu}^\varepsilon \to M_{c,\lambda'}^\varepsilon.
\]

We define the dominance order on the set of fakecompositions so that $\lambda \leq \mu$ if and only if $\lambda_d \leq \mu_d$ for all $d > 0$, each $\lambda_d < 0$, then the reversed dominance order is still well-founded. According to this dominance order we similarly define the quotient supermodules $M_{c,\lambda}^\varepsilon \to M_{v,\mu}^\varepsilon$ and $S_{c,\lambda}^\varepsilon \cong M_{v,\mu}^\varepsilon$. Then for compositions the surjective map $P : S_{c,\lambda}^\varepsilon \to S_{v,\mu}^\varepsilon$ is well-defined. Since the same proof works, it satisfies the following standardly filtered property again.

**Theorem 6.25.**

1. $S_{c,\lambda}^\varepsilon$ is spanned by $m_\lambda$ over $\Gamma_{\lambda}$.  
2. $S_{c,\lambda}^\varepsilon = 0$ unless $\lambda \geq \mu$.  

We say that a fakecomposition $\lambda$ is a fakepartition if $\lambda'$ is a partition, and say it is strict if so is $\lambda'$. The sets $STab_{\lambda}^c$ and $STab_{\lambda'}^c$ are defined by direct limits similarly as above. Then the standardly based structure of the category of parabolic fakemodules is obtained by the same proofs as for the ordinary case.

**Theorem 6.26.** Assume $k = \mathbb{Q}(a, q)$. Then

1. $M_{c,\lambda}^\varepsilon$ has a basis

\[\{ m_S \circ_{\mu} m_T | S \in STab_{\lambda}^c, T \in STab_{\lambda'}^c \} \]

2. $S_{c,\lambda}^\varepsilon$ has a basis $\{ m_\lambda | T \in STab_{\lambda}^c \}$ so

\[S_{c,\lambda}^\varepsilon = \begin{cases} \Gamma_1 & \text{if } \lambda \text{ is a strict fakepartition}, \\ 0 & \text{otherwise}. \end{cases}
\]

3. The product

\[\circ_{\mu} : S_{c,\lambda}^\varepsilon \otimes S_{v,\mu}^\varepsilon \to M_{v,\mu}^\varepsilon
\]

is injective.
6.8. Identification of the quotient superalgebras

In order to classify the simple modules, we first determine the quotient superalgebra $M^e_{\lambda,2} \twoheadrightarrow S^e_{\lambda,1}$. In this computation the superalgebras $M^e_{\lambda,2} \twoheadrightarrow S^e_{\lambda,1}$ which cover them are used.

**Lemma 6.27.** Let $\lambda = (m, k)$ with $m > k$. Then in $S^e_{\lambda,1}$ we have

$$\begin{array}{|c|c|c|}
\hline
1 & 1 & 0 \\
\hline
1 & 1 & 1 \hline
\end{array} \equiv (-1)^k q^{(1)} (\lambda_{1,1} - q^{m-k} \lambda_{1,2}) m_{1,1}.$$ 

**Proof.** By (5.14) we have

$$\begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
1 & 1 & 2 \hline
\end{array} \equiv (-1)^k q^{(1)} m_{1,2}.$$ 

Hence the equation is implied by

$$\gamma_{1,2} \cdot \begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
1 & 1 & 2 \\
\hline
\end{array} = \begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
1 & 1 & 2 \hline
\end{array}$$

and

$$\begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
1 & 1 & 2 \\
\hline
\end{array} \cdot \gamma_{1,1} = \begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
1 & 1 & 2 \hline
\end{array} + q^{m-k} \begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
1 & 1 & 2 \hline
\end{array}. \quad \boxempty$$

**Corollary 6.28.** For a fakepartition $\lambda = (m, k)$, in $S^e_{\lambda,1}$ we have

$$\begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
1 & 1 & 2 \hline
\end{array} \equiv (-1)^k q^{(1)} (\lambda_{1,1} - q^{m-k} \lambda_{1,2}) m_{1,1}.$$ 

**Lemma 6.29.** Let $\lambda$ be a partition. Then

1. if $\lambda_1 > \lambda_2$ then we have $S^e_{\lambda,1} \cong S^e_{\lambda,2}$
2. if $\lambda_1 = \lambda_2$ then $\text{Ker}(S^e_{\lambda,1} \twoheadrightarrow S^e_{\lambda,2})$ is generated by $(\lambda_{1,1} - \lambda_{1,2}) m_{1,1}$ as a 2-sided ideal.

**Proof.** For the case (2), as in the proof of Lemma 6.18 we have $(\lambda_{1,1} - \lambda_{1,2}) m_{1,1} \equiv 0$ in $S^e_{\lambda,1}$ so the kernel contains $(\lambda_{1,1} - \lambda_{1,2}) m_{1,1}$. We prove the converse inclusions.

$\text{Ker}(M^e_{\lambda,1} \twoheadrightarrow M^e_{\lambda,2})$ is spanned by $m_T$ for $T \in \text{Tab}^e_{\lambda,1}$ which satisfies either of the condition that $\#_{11}(T^c) < 0$ or that $\#_{11}(T^c) = 0$ with $\bigcirc$ in its first row. If $\lambda_1 > \lambda_2$, we have $\lambda_1 - \#_{11}(T^c) > \lambda_2$ for such $T$ so that $m_T \equiv 0$ in $S^e_{\lambda,1}$ as we did in the proof of Lemma 5.20. This implies that $\text{Ker}(M^e_{\lambda,1} \twoheadrightarrow M^e_{\lambda,2})$ is already zero in $S^e_{\lambda,1}$; in other words, $S^e_{\lambda,1} \cong S^e_{\lambda,2}$. In the other case $\lambda_1 = \lambda_2$, we also have $m_T \equiv 0$ if $\#_{11}(T^c) < 0$. Otherwise by applying local transformation on the second row or below $m_{\lambda_2}$ can be transformed into a linear combination of tableaux $S$ such that $T(1, j) = S(1, j)$ for all $j$ and $#_{21}(S^c) = \lambda_2$, that is, which is in the form

$$\begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
1 & 1 & 1 \hline
\end{array} \text{ or } \begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
1 & 1 & 1 \hline
\end{array}$$

By the corollary above, for the special case we have

$$\begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
1 & 1 & 1 \hline
\end{array} \equiv (-1)^k q^{(1)} (\lambda_{1,1} - \lambda_{1,2}) m_{1,1}.$$
in $S^c\gamma_{i,j}$, so that 

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
3 & 3 & 3 & 3 & 3 & 3 \\
1 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
3 & 3 & 3 & 3 & 3 & 3 \\
\end{array}
\equiv (-1)^k q^{(\gamma)_k}(\gamma_{i,j} - \gamma_{i+1,j})m_{\gamma_{i,j}}.
\]

Every such $S$ above can be made by multiplying elements to these tableaux from left. Hence the image of $\text{Ker}(M^c_{i,j} \rightarrow M^c_{i,j})$ in $S^c\gamma_{i,j}$ is generated by $(\gamma_{i,j} - \gamma_{i+1,j})m_{\gamma_{i,j}}$ as a 2-sided ideal.

**Lemma 6.30.** For a fakepartition $\lambda = (\lambda_1, \lambda')$, we have $S^c\gamma_{i,j} \simeq C_1(a[\lambda_1]) \otimes S^c\gamma_{i,j}$.

**Proof.** Since these modules are preserved by extension of scalars, it suffices to prove for the universal ring $k = \mathbb{Z}[a, q^\pm]$. Let

\[ V := \sum_{\nu > \lambda, \nu_1 = 1} M^c\gamma_{i,j} \quad \text{and} \quad W := \sum_{\nu > \lambda, \nu_1 = 1} M^c\gamma_{i,j} \]

so that $M^c\gamma_{i,j} = V + W$. On the other hand, let

\[ T := \{T \in \text{Tab}^c_{i,j} \mid T^c(1,j) = 1 \text{ for all } j \}
\]

and

\[ X := \mathbb{k}\{m_{\gamma} \mid T \in \text{Tab}^c_{i,j} \setminus T\}, \quad Y := \mathbb{k}\{m_{\gamma} \mid T \in T\}
\]

so that $M^c_{i,j} = X \oplus Y$. Since $\Gamma_i(m_{\gamma}, w \subset Y$ we have $M^c_{i,j} = V + Y$. Hence

\[ S^c_{i,j} = M^c_{i,j}/M^c\gamma_{i,j} = (V + Y)/(V + W) \simeq Y/((V \cap Y) + W).
\]

For a $\mathbb{Z}[a, q^\pm]$-module $M$, let $\tilde{M} := M \otimes_{\mathbb{Z}[a, q^\pm]} \mathbb{Q}(a, q)$ be its localization. By the cellular basis theorem, we have $\dim \tilde{V} + \dim \tilde{W} = \dim M^c_{i,j}$ and

\[ \dim \tilde{W} = \sum_{\nu > \lambda, \nu_1 = 1} \#\text{STab}^c_{i,j} \cdot \#\text{STab}^a_{i,j}.
\]

On the other hand, since we can view $\lambda_1$ as a sufficiently large number, we have a natural one-to-one correspondence

\[ \{\text{strict fakepartition } \nu \mid \nu > \lambda, \nu_1 = \lambda_1\} \xymatrix{\overset{1:1}{\longleftrightarrow}} \{\text{strict partition } \nu' \mid \nu' > \lambda'\}
\]

and for such $\nu$,

\[ \text{STab}^c_{i,j} \xymatrix{\overset{1:1}{\longleftrightarrow}} \{1, \emptyset\} \times \text{STab}^a_{i,j} \quad \text{and} \quad \text{STab}^a_{i,j} \xymatrix{\overset{1:1}{\longleftrightarrow}} \text{STab}^c_{i,j}.
\]

Then by using the shifted Knuth correspondence for $\lambda'$, we obtain $\dim \tilde{Y} = \dim \tilde{W}$. Since localization of modules is exact, from the exact sequence

\[ 0 \rightarrow V \cap Y \rightarrow V \oplus Y \rightarrow M^c_{i,j} \rightarrow 0
\]

we obtain

\[ 0 \rightarrow \tilde{V} \cap \tilde{Y} \rightarrow \tilde{V} \oplus \tilde{Y} \rightarrow \tilde{M}^c_{i,j} \rightarrow 0.
\]

By comparison of dimensions we have $\tilde{V} \cap \tilde{Y} = 0$. Since $V \cap Y \subset Y$ is a torsion-free module over the integral domain $\mathbb{Z}[a, q^\pm]$ it implies $\tilde{V} \cap \tilde{Y} = 0$. Hence we have

\[ S^c_{i,j} \simeq \tilde{Y}/\tilde{W} \simeq C_1(a[\lambda_1]) \otimes S^c_{i,j}.
\]

\[ \square \]

74
These two lemmas bring us the following identification of the superalgebras $S_{\lambda,t}^c$ and $S_{\lambda,t}^c$.

**Theorem 6.31.** Recall the assumption $q \in \mathbb{k}^\times$.

1. For a partition $\lambda$, the 2-sided ideal $\text{Ker}(\Gamma_1 \rightarrow S_{\lambda,1}^c)$ is generated by $\gamma_i - \gamma_j$ for all $i, j$ such that $\lambda_i = \lambda_j$.
2. For a fakepartition $\lambda$, the 2-sided ideal $\text{Ker}(\Gamma_1 \rightarrow S_{\lambda,1}^c)$ is generated by $\gamma_i - \gamma_j$ for all $i, j$ such that $\lambda_i = \lambda_j$ and $i, j \geq 2$.

**Proof.** We use a mutual induction for (1) and (2) on the number of components of $\lambda$. First let $\lambda$ be a fakepartition and suppose that (1) holds for $\lambda'$. Then

$$\text{Ker}(\Gamma_1 \rightarrow S_{\lambda,1}^c) \cong C_1(a\mathbb{A}_t) \otimes \text{Ker}(\Gamma_1' \rightarrow S_{\lambda',1}^c)$$

has a generating set above. Next let $\lambda$ be a partition of $n > 0$ and suppose (2) holds for $\lambda$. We have a commutative square

$$\begin{array}{ccc}
\Gamma_1 & \longrightarrow & S_{\lambda,1}^c \\
\downarrow & & \downarrow \\
\Gamma_1 & \longrightarrow & S_{\lambda,1}^c
\end{array}$$

where $\Gamma_1 \cong \Gamma_1$ since $\lambda_1 > 0$. Hence as a generating set of the kernel of $\Gamma_1 \rightarrow S_{\lambda,1}^c$ we can take the union of that of $\Gamma_1 \cong \Gamma_1 \rightarrow S_{\lambda,1}^c$ and that of $S_{\lambda,1}^c \rightarrow S_{\lambda,1}^c$. \qed

Consequently we obtain the following classification of simple modules of $\mathcal{R}_{r,n}^c$. We remark that $S_{\lambda,\mu}^c$ is not free over $\mathbb{k}$ in general even if in this case $q \in \mathbb{k}^\times$.

**Theorem 6.32.** Suppose $q \in \mathbb{k}^\times$. For a partition $\lambda$, let $\Theta_\lambda$ be the 2-sided ideal generated by $\gamma_{\lambda,i} - \gamma_{\lambda,j}$ above. Then there is a one-to-one correspondence

$$\text{Irr}(\mathcal{R}_{r,n}^c) \leftrightarrow_{\cong} \prod_{v=(v_1,\ldots,v_\mu)} \text{Irr}(\Gamma_1/\Theta_\lambda).$$

Note that for a partition $\lambda = (k, k)$, we have

$$\Gamma_1/\Theta_\lambda \cong \Gamma_k \otimes (k/2a[k]/k)$$

since $a[k] = \gamma_{1,1}^2 \equiv \gamma_{1,1} \gamma_{2,2} \equiv -\gamma_{1,2} \gamma_{2,1} \equiv -\gamma_{1,1}^2 = -a[k]$. In addition, clearly $2a[k]/k + 2a[l]/k = 2a \gcd(k,l)/k$. Thus for a general partition $\lambda$, let $\mu$ be the strict partition obtained by removing duplicate components of $\lambda$ and let $k_1, \ldots, k_\mu$ be such components, then

$$S_{\lambda,\mu}^c \cong \Gamma_\mu/\Theta_\lambda \cong \Gamma_\mu \otimes (k/2a[k]/k).$$

In particular, $S_{\lambda,\mu}^c = 0$ if and only if $2a[k]/\gcd(k_1,\ldots,k_\mu) \in \mathbb{k}^\times$.

Remember that when $\mathbb{k}$ is a field the Clifford superalgebra $\Gamma_\lambda$ has a unique simple module up to parity change. For $e \geq 2$, we say that a partition $\lambda$ is $e$-strict if $\lambda_i = \lambda_j, \ i \neq j$ implies $e|\lambda_i$. For convention the word $\infty$-strict stands for strict. For a superalgebra $A$, let $\text{Irr}(A)/\Pi$ be a quotient set of $\text{Irr}(A)$ on which $V \in \text{Irr}(A)$ is identified with its parity change $\Pi V$. 

75
Corollary 6.33. Suppose that \( k \) is a field and \( 2aq \in k^\times \). Let \( e_2 \) be the \( q^2 \)-characteristic of \( k \). Then there is a one-to-one correspondence

\[
\text{Irr}(\mathcal{F}_{e_2})/\Pi \overset{1:1}{\longleftrightarrow} \{v = (v_1, \ldots, v_r)\}: e_2\text{-strict partition of } n.
\]

The case \( 2a = 0 \) is easier.

Corollary 6.34. Suppose that \( k \) is a field and \( q \in k^\times, 2a = 0 \). Then there is a one-to-one correspondence

\[
\text{Irr}(\mathcal{F}_{e_2})/\Pi \overset{1:1}{\longleftrightarrow} \{v = (v_1, \ldots, v_r)\}: \text{partition of } n.
\]

6.9. Identification of the ideals

We keep assuming that \( q \in k^\times \). Finally, we reach to the classification of simple modules of the Hecke–Clifford superalgebra \( H_n \). Now let \( J_j \subset \Gamma_j \) be the pullback of the 2-sided ideal \( m_j \cdot S_j' \subset S_j' \) via the surjective map \( \Gamma_j \twoheadrightarrow \Gamma_j/\Theta_j \cong S_j' \). We determine this ideal as follows.

For \( n \in \mathbb{N} \), let \( K_n \subset k \) be the ideal generated by the elements

\[
[(\frac{dq_1}{2})^s][n]| 0 \leq s \leq \lfloor n/2 \rfloor.
\]

Then by Lemma 6.7 we have \( m_n C_n m_n = K_n m_n \oplus K_{n-1} \gamma_n \). Since it is a 2-sided ideal of \( \gamma_n m_n \), the following statement holds.

Lemma 6.35. There are inclusions \( a \| n \| K_{n-1} \subset K_n \subset K_{n-1} \).

Lemma 6.36. Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) be a partition. For each \( i \), let \( \Delta_{\lambda,i} \subset \Gamma_\lambda \) be the supermodule

\[
\Delta_{\lambda,i} := K_{\lambda-\lambda_i,1} \oplus K_{\lambda-\lambda_i,\lambda_1} \gamma_{\lambda,i+1} \oplus K_{\lambda-\lambda_i-1}(\gamma_{\lambda_1} - q^{l_{\lambda-\lambda_i,1}} \gamma_{\lambda_{i+1}}) \\
\quad \oplus K_{\lambda-\lambda_i-1}(\gamma_{\lambda_{i+1}} - q^{l_{\lambda-\lambda_i,1}} \gamma_{\lambda_{i+1}}) \gamma_{\lambda_{i+1}}
\]

and let \( \Delta := \Delta_{\lambda,1} \cdots \Delta_{\lambda,r} \Delta_{\lambda_i} \). Then \( \Delta \subset \Gamma_\lambda \) is a 2-sided ideal.

Proof. For simplicity we write \( \Delta = \Delta_1, \Delta_i = \Delta_{\lambda,i}, \Delta_{\lambda,i} = \gamma_{\lambda,i} \). First we prove

\[
\Delta \gamma_i \subset \Delta_i + \gamma_{i+1} \Delta_i, \quad \Delta_i \gamma_i \subset \Delta_i, \quad \Delta_i \gamma_j = \gamma_j \gamma_i \quad \text{for } j \neq i, i+1.
\]

The second and the third inclusions are clear so we prove the first one. Since \( K_n \subset K_{n-1} \), the inclusion \( K_{\lambda-\lambda_i,1} \gamma_i \subset \Delta_i \) is also obvious. We also have

\[
(\gamma_i - q^{l_{\lambda-\lambda_i,1}} \gamma_{i+1}) \gamma_i - q^{l_{\lambda-\lambda_i,1}} \gamma_{i+1}(\gamma_i - q^{l_{\lambda-\lambda_i,1}} \gamma_{i+1}) = \gamma_i^2 - q^{2(l_{\lambda-\lambda_i,1})} \gamma_{i+1}^2
\]

so that \( K_{\lambda-\lambda_i-1}(\gamma_i - q^{l_{\lambda-\lambda_i,1}} \gamma_{i+1}) \gamma_i \subset \Delta_i + \gamma_{i+1} \Delta_i \) by \( a \| n \| K_{n-1} \subset K_n \). Putting them together we obtain \( \Delta_i \gamma_i \subset \Delta_i + \gamma_{i+1} \Delta_i \) as desired. Then

\[
\Delta \gamma_i = \Delta_i \cdots \Delta_{i-1} \gamma_i \cdots \Delta_i \subset \Delta_i \cdots \Delta_{i-1} \Delta_i = \Delta
\]

for \( i \geq 2 \), and

\[
\Delta \gamma_i \subset \Delta + \Delta \gamma_2 \Delta_i \subset \Delta + \Delta \gamma_1 \Delta_2 \Delta_i \subset \cdots \subset \Delta
\]

so \( \Delta \) is a right ideal. By the equation above we also have inclusions

\[
\gamma_i \Delta_i \subset \Delta_i, \quad \gamma_i \Delta_i \subset \Delta_i + \Delta \gamma_i
\]

which imply that \( \Delta \) is also a left ideal in a similar manner.
Lemma 6.37. For \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) above, we have \( \Delta'_i + Q \subseteq J'_i \subseteq \Delta_i + Q \).

Proof. Parallel to the proof of Lemma 5.20. So first we prove \( J'_i \subseteq \Delta_i + Q \). Take an arbitrary \( T \in \text{Tab}'_i \). Let \( \mu := (\lambda_1, 1^{n-i}) \) and define \( S \in \text{Tab}'_{\lambda_1} \) which has underlying tableau \( S^* = T^0_{\lambda_1} \) and for each its bar \( \boxed{1} \) it has a circle if and only if there are odd number of circles in the corresponding boxes in \( T \). Let \( k := \#_1(S^*) \) and let \( p := 0 \) if \( S \) does not have \( \circ \) in its first row, and otherwise \( p := 1 \). Then by Lemma 6.7 we have

\[
m'_\mu \cdot m_T \in K_{k-p}m_S.
\]

If \( k < \lambda_1 - \lambda_2 \), we have \( m_S \equiv 0 \). If \( k = \lambda_1 - \lambda_2 \) and \( p = 1 \), \( m_S \) can be transformed into a linear combination of tableaux generated by

\[
\begin{array}{c}
1 1 1 1 1 0 2 2 2 2 \\
1 1 1 1 0 3 3 3 3 \\
1 1 1 0 4 4 4 4 \\
\end{array}
\equiv (-1)^{i}q^{-i}m_{(\gamma_1,1)}q - q^{i-1}q \gamma_{(2,2,2)}
\]

or

\[
\begin{array}{c}
1 1 1 1 1 0 2 2 2 2 \\
1 1 1 0 3 3 3 3 \\
1 1 0 4 4 4 4 \\
\end{array}
\equiv (-1)^{i}q^{-i}m_{(\gamma_1,1)}q\gamma_{(2,2,2,2)}
\]

as we did before. Hence

\[
K_{k-p}m_S \subseteq (\Pi_{\mu'}^{\lambda_1} \ast S_{\gamma'}^i) \cdot \Delta_{1,1}.
\]

In the other cases we have \( K_{k-p} \subseteq K_{k+i} \) so the inclusion above also holds. By induction we may assume that \( m_{\mu'} \cdot S_{\gamma'}^i \subseteq m_{\mu} \Delta_{r+1} \). This implies \( m_{\mu} \cdot m_T \in m_{\mu} \Delta_{i,1} \) in \( S_{\gamma'}^i \).

We can prove the other inclusion by using circled tableaux whose underlying tableau is \( R \) in the proof of Lemma 5.20. By putting circles on suitable boxes of \( R \) we can make arbitrary elements of

\[
(\Delta_{1,r} \cdots \Delta_{i,1})(\Delta_{1,i} \cdots \Delta_{1,2}) \cdots (\Delta_{i,1,\lambda_i-1}) \Delta_{i,r} \setminus \Delta'_{i,r}.
\]

For example, for \( \lambda = (6,4,1) \)

\[
m_{(\gamma_1,1,\gamma_2)} = [2][3]! (1)(2) = \gamma_{(1,2,1)} = \gamma_{(1,2,3,3)}
\]

\[
= \cdots = q^6m_{3} \cdot \gamma_{3} \cdot [3]! \cdot (1 \cdot q^3 \gamma_{2,3}) \cdot [2] \gamma_{2,3} \cdot q^3 \gamma_{3} \cdot 1
\]

where \( \gamma_1 = q^3 \gamma_2 \in \Delta_{1,3} \) and \( \gamma_3 \in \Delta_{1,3} \). Hence we conclude that

\[
m_{\mu} \cdot S_{\lambda_{1,3}} \supseteq m_{\mu} \Delta'_{i,3}.
\]

We state again the main theorem of this paper.

Theorem 6.38. When \( q \in \mathbb{K} \), there is a one-to-one correspondence

\[
\text{Irr}(H_{n}^0) \longleftrightarrow \text{Irr}_{\Theta_{\lambda}}^{\Delta_{i} + \Theta}((\Gamma_{i})).
\]
Now assume that $k$ is a field. By specializing this theorem we obtain several classifications. First consider the case $q \neq -1$. In this case simply $K_n = [n]! \cdot k$. Let $e$ (resp. $e_2$) be a $q$-characteristic (resp. $q^2$-) of $k$. Then we have

$$e_2 = \begin{cases} e & \text{if } e \text{ is odd,} \\ e/2 & \text{if } e \text{ is even.} \end{cases}$$

Let $\lambda$ be a partition. If $\lambda_i > \lambda_{i+1} + e$, we have $\Delta_i = 0$ as before. On the other hand if $\lambda_i < \lambda_{i+1} + e$ we have $\Delta_i = \Gamma_i$. So suppose $\lambda_i = \lambda_{i+1} + e$ so that $K_{\lambda_i - \lambda_{i+1}} = 0$ but $K_{\lambda_i - \lambda_{i-1} - 1} = k$. If $e_2|\lambda_i$ then $\gamma_{e_i}$ and $\gamma_{e_i+1}$ are central nilpotent so that they are contained in the Jacobson radical of $\Gamma_j$. Otherwise

$$(\gamma_{e_i,1} - q^{1-i}\gamma_{e_i,2})^2 = a\llbracket A_1 \rrbracket + aq^{2(1-i)\gamma_{e_i,2}}\llbracket A_2 \rrbracket = 2a\llbracket A_1 \rrbracket$$

is invertible if and only if $2a \neq 0$. When $2a = 0$, $K_{\lambda_i - \lambda_{i-1}}(\gamma_{e_i,1} - q^{1-i}\gamma_{e_i,2})$ generates a nilpotent ideal so is in the Jacobson radical also in this case.

Summarizing the above, we obtain the following results. We say that an $e_2$-strict partition $\lambda$ is $e$-restricted if

$$\begin{cases} \lambda_i - \lambda_{i+1} < e & \text{if } e_2|\lambda_i, \\ \lambda_i - \lambda_{i+1} \leq e & \text{otherwise.} \end{cases}$$

Corollary 6.39. Suppose $k$ is a field and $2aq[2] \neq 0$. Then there is a one-to-one correspondence

$$\text{Irr}(H^n_k)/\Pi \overset{1:1}{\longleftrightarrow} \{ e \text{-restricted } e_2 \text{-strict partition of } n \}.$$  

The result is now coincides with the crystal $B(A_0)$ of type $A_{n-1}^{(2)}$ for odd $e$ or of type $D_{e/2}^{(2)}$ for even $e$ whose descriptions are obtained by Kang [21] and Hu [17] respectively.

Corollary 6.40. Suppose $k$ is a field and $q[2] \neq 0$, $2a = 0$. Then there is a one-to-one correspondence

$$\text{Irr}(H^n_k)/\Pi \overset{1:1}{\longleftrightarrow} \{ e \text{-restricted partition of } n \}.$$  

Next consider the case $q = -1$, so that $[k] = -(k \mod 2)$ and $[k] = k$. First assume that $2a \neq 0$. Let $p$ be the (ordinary) characteristic of $k$. Then we have $K_n = k$ if $n < 2p$ and otherwise $K_n = 0$. Hence by a similar arguments as above we obtain the following.

Corollary 6.41. Suppose $k$ is a field of characteristic $p \neq 2$ and $q = -1$, $a \neq 0$. Then there is a one-to-one correspondence

$$\text{Irr}(H^n_k)/\Pi \overset{1:1}{\longleftrightarrow} \{ 2p \text{-restricted partitions of } n \}.$$  

Hence it corresponds to the crystal $B(A_0)$ of type $D^{(2)}_p$ again. S. Tsuchioka informed the author that this result can be also obtained from the supercategorification by the \textit{cyclotomic quiver Hecke superalgebra} given in [22]. As described in [24, Theorem 3.13 and Lemma 4.8] it is considered equivalent to $H^n_k$ after localization in some sense. See also [24] §4.6] (beware that our $q$ is their $q^2$).

Now finally let $q = -1$ and $2a = 0$, so that $K_0 = K_1 = k$ and $K_n = 0$ for $n \geq 2$. Similar to the case above for $2a = 0$, we obtain the following.

Corollary 6.42. Suppose $k$ is a field and $q = -1$, $2a = 0$. Then there is a one-to-one correspondence

$$\text{Irr}(H^n_k)/\Pi \overset{1:1}{\longleftrightarrow} \{ 2 \text{-restricted partition of } n \}.$$  

In fact, the two results for $2a = 0$ are already obtained in Remark [6.6].
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