Cosmic Strings and Quintessence

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**Abstract**

We present a new Lorentz gauge invariant $U(1)$ topological field theory in Riemann-Cartan spacetime manifold $U_4$. By virtue of the decomposition theory of $U(1)$ gauge potential and the $\phi$-mapping topological current theory, it is proved that the $U(1)$ complex scalar field $\phi(x)$ can be looked upon as the order parameter field in our Universe, and the set of zero points of $\phi(x)$ create the cosmic strings as the spacetime defects in the early Universe. In the standard cosmology this complex scalar order parameter field possesses negative pressure, provides an accelerating expansion of Universe and be able to explain the inflation in the early Universe. Therefore this complex scalar field is not only the order parameter field created the cosmic strings, but also reasonably behaves as the quintessence, the dark energy.

**Key words:** Cosmic Strings, Quintessence, Gauge Field Theories, Topology, Defects, Cosmology

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1 Introduction

The topological spacetime defects such as cosmic strings may have been found at phase transitions in the early Universe, analogous to those examples found in some condensed matter systems—vortex line in liquid helium, flux tubes in type-II superconductors, or disclination lines in liquid crystals\cite{1,2}. Such phase transitions in the early Universe are usually viewed as being caused by the breaking of the symmetry of a Higgs field occurring in a spacetime whose geometry is Riemannian\cite{3}. However the Einstein-Cartan theory, which is formulated in a Riemann-Cartan spacetime manifold $U_4$, represents a viable gravity theory with torsion. So it is still an open debate if the spacetime manifold is Riemann-Cartan or not. Furthermore, the viewpoint that the early Universe is Riemann-Cartan spacetime in which cosmic strings are treated as

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line defects created directly from torsion[4], is very attractive, but it is not very perfect[1,2,5]. At the same time the order parameter of the Universe is not very clear yet. In this letter, by virtue of constructing a new Lorentz gauge invariant \( U(1) \) topological field from torsion, we present a theory of creating cosmic strings. Further this theory rigorously unites the geometry and topology of gauge theories of spacetime defects and is independent of concrete spacetime manifold. It’s also important to show that the order parameter field \( \phi = \phi^1 + i\phi^2 \) which creating cosmic strings, possesses negative pressure. Therefore the complex scalar order parameter field has the property of quintessence, the dark energy.

The theoretical framework in the present letter includes three basic aspects. In section 2, we present a new Lorentz gauge invariant \( U(1) \) topological field theory in \( U_4 \). This general theory is still valid in both special cases: the Riemann manifold and the Weitzenböck manifold. In section 3, by virtue of the decomposition theory of gauge potential and the \( \phi \)–mapping topological current theory[6,7], the cosmic strings as the spacetime defects are created naturally from the zero points set of the order parameter \( \phi \) and the length of these strings are topologically quantized by winding number of \( \phi \)–mapping with Planck length \( (L_p = \sqrt{\hbar G/c^3}) \). In section 4, The nonzero part of the complex scalar order parameter field \( \phi \) is shown as quintessence which possesses negative pressure and cause inflation of the early Universe.

2 A New Topological Invariant in Riemann-Cartan Manifold

It is well known that in a 4-dimensional Riemann-Cartan spacetime manifold \( U_4 \), the torsion 2-form is defined by

\[
T^a = De^a,
\]

where \( e^a \) is the vierbein 1-form with Lorentz group index \( a \) and

\[
e^a = e^a_\mu dx^\mu.
\]

The vierbein \( e^a_\mu \) is related with the metric tensor by

\[
g_{\mu\nu} = e^a_\mu e^a_\nu, \quad \mu, \nu, a = 0, 1, 2, 3.
\]

\( D \) is the covariant differential in Lorentz gauge theory

\[
D = D_\mu dx^\mu, \quad D_\mu = \partial_\mu - \omega_\mu,
\]

\[
\omega_\mu = \frac{1}{2} \omega_\mu^{ab} I_{ab},
\]
where $\omega_{\mu}^{ab}$ is the spin connection and $I_{ab}$ is the generator of lorentz group $SO(1, 3)$. The torsion 2-form can be expressed explicitly by

$$T^a = \frac{1}{2} T^a_{\mu\nu} dx^\mu \wedge dx^\nu,$$

in which the torsion tensor is

$$T^a_{\mu\nu} = D_\mu e^a_v - D_v e^a_\mu.$$

As in ref.[2], by analogy with the 't Hooft's viewpoint of $SU(2)$ gauge theory of monopole[8], using vierbein 1-form and torsion 2-form, we define a Lorentz gauge invariant antisymmetrical 2-form

$$K_a = T_a u^a - e^a \wedge Du^a,$$

where $u^a$ is a $SO(1, 3)$ unit vector field. It’s easy to see that $K$ can be expressed explicitly as

$$K = \frac{1}{2} K_{\mu\nu} dx^\mu \wedge dx^\nu,$$

and $K_{\mu\nu}$ is an antisymmetric tensor

$$K_{\mu\nu} = T_{\mu\nu} u^a - \left( e^a_\mu D_\nu u^a - e^a_\nu D_\mu u^a \right).$$

Substituting torsion tensor formula (7) into (10), we can find a simplified form for $K_{\mu\nu}$

$$K_{\mu\nu} = \partial_\mu \left( e^a_\nu u^a \right) - \partial_\nu \left( e^a_\mu u^a \right),$$

which shows that the dimensionless vector $e^a_\mu u^a$ in (11) can be looked upon as a $U(1)$ gauge potential. Since the usual $U(1)$ gauge potential $a_\mu$ in covariant derivative $D_\mu = \partial_\mu - ia_\mu$ must has dimension $[L]^{-1}$, we can define a $U(1)$ gauge potential

$$A_\mu = \frac{1}{L_P} e^a_\mu u^a,$$

where $L_P$ is the Plank length introduced to satisfy the dimension of $U(1)$ gauge potential. The antisymmetric tensor $K_{\mu\nu}$ exactly behaves as a $U(1)$ gauge field tensor

$$K_{\mu\nu} = L_P \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right).$$

And the $U(1)$ gauge transformation of $A_\mu$ is

$$A'_\mu = A_\mu + \partial_\mu \alpha,$$
where $\alpha(x)$ is an arbitrary scalar function.

Since for a given Riemann-Cartan manifold the vierbein $e^a_\mu$ is fixed, then the above $U(1)$ gauge transformation (14) can be only explained as the change of the unit vector $u^a \rightarrow u'^a$, therefore

$$e^a_\mu u'^a = e^a_\mu u^a + \partial_\mu \alpha,$$

which gives

$$u'^a = u^a + e^{\mu a} \partial_\mu \alpha,$$

where $e^{\mu a}$ is the inverse vierbein

$$e^a_\mu e^{\mu b} = \delta^{ab}.\quad (17)$$

It must be pointed out that the $SO(1,3)$ gauge invariant antisymmetry tensor $K_{\mu \nu}$ can also be defined as[1,2]

$$K_{\mu \nu} = T^a_{\mu \nu} u^a,\quad (18)$$

with a parallel $SO(1,3)$ unit vector field $u^a$ satisfied

$$Du^a = 0, \quad a = 0, 1, 2, 3.\quad (19)$$

Therefore we have presented a new Lorentz ($SO(1,3)$) gauge invariant $U(1)$ field theory in $U_4$. In this new $U(1)$ gauge theory the corresponding first Chern class in $U_4$ is a topological invariant with dimension $[L]$ and is expressed as

$$L = \frac{1}{2\pi} \int_{\Sigma} K,\quad (20)$$

where $\Sigma$ is an arbitrary 2-dimension surface in $U_4$. Substituting (9) and (10) into (20), it is easy to see that the complete expression for our new $U(1)$ topological invariant in $U_4$ is

$$L = \frac{1}{2\pi} \int_{\Sigma} \left\{ \frac{1}{2} \left[ T^a_{\mu \nu} u^a - \left( e^a_\mu D_\nu u^a - e^a_\nu D_\mu u^a \right) \right] \right\} dx^\mu \wedge dx^\nu.\quad (21)$$

In the last years, it is still an open debate whether the spacetime of the early Universe is Riemann-Cartan manifold or not. Here we should point out that in torsion free case

$$T^a_{\mu \nu} = 0, \quad D_\mu e^a_v = D_v e^a_\mu,\quad (22)$$

and from (21) we can find that the formula (13) and (20) still holds true. Therefore the theory mentioned above may be considered as a universal $U(1)$ topological invariant theory in Riemann-Cartan spacetime manifold $U_4$ and Riemann spacetime manifold $V_4$.  

4
3 Cosmic Strings

In order to introduce the decomposition theory of $U(1)$ gauge potential in $U_4$, let the complex scalar

$$ \phi = \phi^1 + i\phi^2 $$

be an ordered parameter field in our Universe and its covariant derivative corresponding gauge potentials $A_\mu$ (12)

$$ D_\mu \phi = \partial_\mu \phi - iA_\mu \phi. $$

By means of the decomposition theory of $U(1)$ gauge potential proposed by one of authors (Duan) [7], the gauge potential (12) can be decomposed [9] as following

$$ A_\mu = \epsilon_{AB} n^A \partial_\mu n^B $$

where $n^A$ ($A = 1, 2$) is 2-dimensional unit vector fields

$$ n^A = \frac{\phi^A}{\|\phi\|}, \quad n^A n^A = 1, \quad A = 1, 2. $$

Substituting (25) into (13), we have

$$ K_{\mu\nu} = L_p \epsilon_{AB} (\partial_\mu n^A \partial_\nu n^B - \partial_\nu n^A \partial_\mu n^B), $$

and the corresponding new topological invariant (20) is denoted by

$$ L = \frac{L_p}{2\pi} \int_\Sigma \epsilon_{AB} \partial_\mu n^A \partial_\nu n^B dx^\mu \wedge dx^\nu. $$

By making use of the $\phi$-mapping topological current theory [7] and the Green function equation in $\phi$-space

$$ \triangle_\phi \ln \|\phi\| = 2\pi \delta^2(\phi), \quad \triangle_\phi = \frac{\partial^2}{\partial \phi^A \partial \phi^A}, $$

we can find

$$ L = L_p \int_\Sigma \delta^2(\phi) D_{\mu\nu} \left( \frac{\phi}{x} \right) dx^\mu \wedge dx^\nu, $$

where $D_{\mu\nu} \left( \frac{\phi}{x} \right) = \epsilon_{ab} \partial_\mu \phi^A \partial_\nu \phi^B$ is the Jacobian tensor. The 2-dimension surface $\Sigma$ can be parameterized by

$$ x^\mu = x^\mu \left( u^1, u^2 \right), $$
and the integral $L$ (30) is expressed as

$$L = L_p \int_{\Sigma} \delta^2(\phi) D\left(\frac{\phi}{u}\right) \sqrt{g_u} d^2u.$$  (32)

It is obviously that the above integral $L$ (32) is non-zero only when

$$\phi^A(\vec{x}, t) = 0 \quad , \quad A = 1, 2.$$  (33)

Suppose that the order parameter field $\phi^A(x)$ ($A = 1, 2$) possesses $\ell$ zeroes, according to the implicit function theorem[10], when the zeroes are regular points of $\phi$-mapping at which the rank of Jacobian matrix $[\partial_u \phi^A]$ is 2, the $i$-th solution of $\phi^A(\vec{x}, t) = 0$ ($A = 1, 2$) can be expressed by

$$S_i : x^\mu = z^\mu_i (s, t) \quad , \quad i = 1, 2, \ldots, \ell.$$  (34)

We see that each solution of (34) is a world sheet of a string with string parameter $s$ and time $t$.

Using the expanding formula of $\delta^2(\phi)$ in the $\phi$-mapping topological current theory[11] we have

$$L = L_p \int_{\Sigma} \sum_{i=1}^{\ell} W_i \delta^2(u - z_i) d^2u,$$  (35)

where $z_i(i = 1, 2, \ldots, \ell)$ are the intersection points of strings $S_i$ with surface $\Sigma$ and $W_i$ is winding number of $\phi^A(x)$ ($A = 1, 2$) around string $S_i$ on $\Sigma$ at $z_i$. Then we have

$$L = \sum_{i=1}^{\ell} L_i \quad , \quad L_i = L_p W_i.$$  (36)

This is our $\phi$-mapping topological current theory of cosmic strings in $U_4$ and $V_4$, which shows that the cosmic strings are created from the zeros of complex scalar ordered parameter field $\phi^A(x)$ ($A = 1, 2$) and the length $L_i$ of each string is topologically quantized by winding numbers with Planck length $L_p$.

4 Complex Scalar Quintessence

Recent observation of type Ia supernova (SNe Ia) revealed that the expansion of the Universe is accelerating[12]. This means the pressure of Universe is negative. The result of BOOMERanG supports a flat Universe ($k = 0$) which leads to that the total density of the Universe is equal to the so-called critical density[13]. The properties of these observed have important implication for cosmology that the Universe consists of $1/200$th bright stars, $1/3rd$s dark
matter and $2/3$rds “dark energy” with a negative pressure as $p < -\rho c^2/3$ [14]. The latest observations of distant quasars distorted by massive invisible objects have provided fresh evidence that the Universe is mostly made up of mysterious “dark energy” [15]. In the light of these observations, some cosmologists called the exotic dark energy as quintessence, a slowly evolving scalar or complex scalar [16]. The main goal of this section is to show that the complex scalar field acting as the order parameter which creates the cosmic strings can also lead to both negative pressure and acceleration of Universe. Therefore the complex scalar order parameter field introduced in the last section can play an important role of quintessence, so-called dark energy.

In the standard cosmology our Universe is described by the Friedman-Robertson-Walker (FRW) metric:

$$ds^2 = c^2 dt^2 - a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2\right), \quad (37)$$

here $a(t)$ is the cosmic scale factor, $k$ is the signature of curvature. The cosmology dynamic equations are well-known as following

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} (\rho c^2 + 3p), \quad (38)$$

$$\ddot{a} + kc^2 = \frac{8\pi G}{3} \rho a^2. \quad (39)$$

The eq. (39) leads to

$$k = \frac{8\pi G}{3c^2} (\rho - \rho_c) a^2, \quad (40)$$

where $\rho_c$ is the critical density which is related to the hubble constant.

Suppose that the Lagrangian density for the complex scalar order parameter field is

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi^A \partial_\nu \phi^A - V(\phi), \quad (41)$$

where $\phi^A$ ($A = 1, 2$) are real and imaginary parts of the complex scalar order parameter field $\phi = \phi^1 + i\phi^2$ and $V(\phi)$ is a potential function of $\phi^*\phi = \phi^A\phi^A$. In the last section, using the $\phi$-mapping theory we have shown that the singularities of $\delta(\phi)$ can create the cosmic strings. As in the case of scalar field [17,18,19,20], the perfect fluid model in standard cosmology for Lagrangian (41) gives

$$\rho_\phi c^2 = \frac{1}{2} \phi^A \dot{\phi}^A + \frac{1}{2a^2} \nabla \phi^A \cdot \nabla \phi^A + V(\phi), \quad (42)$$

$$p_\phi = \frac{1}{2} \phi^A \dot{\phi}^A - \frac{1}{6a^2} \nabla \phi^A \cdot \nabla \phi^A - V(\phi). \quad (43)$$
As we know the scale factor $a(t)$ is very large in the present Universe, and the above two equations become

$$\rho_{\phi}c^2 = \frac{1}{2} \phi^A \dot{\phi}^A + V(\phi), \quad (44)$$

$$p_{\phi} = \frac{1}{2} \phi^A \dot{\phi}^A - V(\phi). \quad (45)$$

As usual $\phi^A$ ($A = 1, 2$) is considered as slowly evolving[21], i.e. approximately static

$$\dot{\phi}^A \approx 0, \quad (46)$$

then we have

$$\rho_{\phi}c^2 \approx + V(\phi), \quad p_{\phi} \approx -V(\phi), \quad (47)$$

and

$$p_{\phi} \approx -\rho_{\phi}c^2. \quad (48)$$

Thus, the complex scalar order parameter possesses a negative pressure $p_{\phi} < -1/3\rho_{\phi}c^2$, which leads to an accelerating expansion Universe by eq.(38).

The observations of the power spectrum of cosmic microwave background (CMB) indicate that the Universe is a spatially flat Universe ($k = 0$) and from (40) we have

$$\rho = \rho_c, \quad (49)$$

i.e. total density of the Universe is equal to the critical density $\rho_c$. As mentioned in the beginning of this section the recent observations implicate that the quintessence is enough to explain the unknown $2/3$ of critical density. Therefore the complex scalar order parameter field with both negative pressure and creating the cosmic strings is a reasonable candidate for the quintessence (dark energy).

It must be pointed out that from (48) and the equation of motion of the perfect fluid with FRW metric, we can find a inflation solution of the Universe

$$a(t) \propto e^{\chi t}, \quad (50)$$

where

$$\chi = \sqrt{\frac{8\pi G}{3}} \rho_c \sim 10^{34} \text{sec}^{-1}. \quad (51)$$

This shows that in a very small time interval

$$\Delta t = t - t_0 \approx 10^{-32}, \quad (52)$$
the Universe should be expanded $e^{100} \approx 10^{44}$ times, i.e.

$$a(t_i + \Delta t) = 10^{44}a(t_i).$$

Therefore our complex scalar quintessence can also give an explanation of the inflation in the early Universe.

5 Conclusion

We present a new Lorentz gauge invariant topological $U(1)$ field theory. By virtue of the decomposition theory of gauge potential and the $\phi$–mapping topological current theory, we prove that from the viewpoint of spacetime defects the set of zero points of $\phi^A (A = 1, 2)$ create cosmic strings. When $\phi^A \neq 0$ ($A = 1, 2$), the complex scalar order parameter field which has the property of negative pressure can reasonably play a role of quintessence i.e. so-called dark energy. We also want to point out that for the spacetime defects theory of cosmic strings it need naturally introduce the order parameter field that just behave as the quintessence of our Universe. We think that the further investigation of creating quarks and leptons from cosmic strings or quintessence is important.

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