Consistent sets of lines with no colorful incidence

Boris Bukh
Carnegie Mellon University, Department of Mathematical Sciences
Pittsburgh, PA 15213, USA.

Xavier Goaoc
Université Paris-Est, LIGM
UMR 8049, CNRS, ENPC, ESIEE, UPEM, F-77454, Marne-la-Vallée, France.

Alfredo Hubard
Université Paris-Est, LIGM
UMR 8049, CNRS, ENPC, ESIEE, UPEM, F-77454, Marne-la-Vallée, France.

Matthew Trager
Inria and École Normale Supérieure, CNRS, PSL Research University.

Abstract
We consider incidences among colored sets of lines in $\mathbb{R}^d$ and examine whether the existence of certain concurrences between lines of $k$ colors force the existence of at least one concurrence between lines of $k + 1$ colors. This question is relevant for problems in 3D reconstruction in computer vision.

2012 ACM Subject Classification Theory of computation → Randomness, geometry and discrete structures, Computing methodologies → Artificial intelligence → Computer vision → Computer vision tasks → Scene understanding

Keywords and phrases Incidence geometry, image consistency, probabilistic construction, algebraic construction, projective configuration

Acknowledgements We thank Éric Colin de Verdière and Vojta Kalusza for discussions at an early stage of this work.

1 Introduction
A central problem in computer vision is the reconstruction of a three-dimensional scene from multiple photographs. Trager et al. [16, Definition 1] defined a set of images as consistent if they represent the same scene from different points of view. They constructed examples (like that of Figure 1) of a set of images which is pairwise consistent while being altogether inconsistent. They also showed [16] Proposition 4] that under a certain convexity hypothesis, images that are consistent three at a time are globally consistent. In this paper we drop the convexity condition and consider these affairs from the point of view of incidence geometry.

Problem statement. An incidence is a set of lines that meet at a single point. Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \ldots \cup \mathcal{L}_m$ be a set of lines of $m$ colors in $\mathbb{R}^d$ (where each $\mathcal{L}_i$ is a color class).
Given $S \subset \{1, 2 \ldots m\}$, an $S$-incidence in $\mathcal{L}$ is an incidence between lines of every color in $S$. This paper focuses on the following notions:

► **Definition 1.** For $1 \leq k \leq m$, a $k$-incidence in $\mathcal{L}$ is a $S$-incidence where $|S| = k$. A colorful incidence in $\mathcal{L}$ is an incidence that contains lines of every color.

► **Definition 2.** The set $\mathcal{L}$ is $k$-consistent if for every $k$-tuple of colors $S \subset \{1, 2 \ldots m\}$, every line in $\bigcup_{i \in S} \mathcal{L}_i$ belongs to an $S$-incidence. The set $\mathcal{L}$ is consistent if every line belongs to (at least) one colorful incidence.

Instead of wondering if $k$-consistency implies consistency, we aim for a more modest goal:

► **Problem 3.** Under which conditions does the $k$-consistency assumption imply the existence of a $(k + 1)$-incidence?

The main results of this paper are two constructions of (infinite families of) finite sets of lines which are $k$-consistent and have no colorful incidence. Thus, consistency does not propagate.

► **Remark.** Unless indicated otherwise, the set $\mathcal{L}$ is assumed to be finite. We also assume throughout that the lines in $\mathcal{L}$ are pairwise distinct. This has no consequence on Problem 3: repeating a line in a color class is useless, and if two lines of distinct colors coincide, then the $k$-consistency assumption trivially implies that this line has a $(k + 1)$-incidence.

**Relation to photograph consistency.** Let us explain how our initial image consistency question relates to Problem 3. Firstly, we ignore color or intensity information, and treat the scene as a set of opaque objects and the images as their projections onto certain planes. In this setting, images are consistent if and only if there exists a subset $R \subset \mathbb{R}^3$ that projects into each of them. Assuming that light travels along straight lines, the set of 3D points that are mapped to a given image point is a ray, or more conveniently a line, in $\mathbb{R}^3$. Starting with $m$ photographs, if we let $\mathcal{L}_i$ denote the lines that are pre-images of the projection on the $i$th photograph, then the photographs are consistent if and only if $\bigcup_{i=1}^m \mathcal{L}_i$ is consistent: $R$ is the set of points of colorful incidences.
Setting. In the basic set-up for computer vision, all lines used to project the scene onto a given image plane pass through a “pinhole”. We therefore define a color class $L_i$ as concurrent if it consists of concurrent lines. We consider, however, the problem more generally since it is possible to build other imaging systems. For example, there are cameras that use the lines secant to two fixed skew lines; other cameras use the lines secant to an algebraic curve $\gamma$ and to a line intersecting $\gamma$ in $\deg \gamma - 1$ points. For a discussion of the geometry of families of lines arising in the modeling of imaging systems, see [2] [17] and the references therein.

We focus in this paper on the consistency of finite sets of lines. This restriction is technically convenient and remains relevant to the initial motivation on continuous sets of lines. On the one hand, our constructions for the finite problem turn out to readily extend to infinite families of lines (see Section 4). On the other hand, the finite problem is already relevant to 3D reconstruction, when one has to recover the camera parameters (settings or position) used in the photographs. Indeed, this recovery is typically done by identifying pixels in different images that are likely to be the projection of the same 3D element, and using the incidence structure of their inverse images to infer the position of the camera; this process is called structure from motion [13]. The number of lines required to determine the cameras is typically 5 to 7 per image. Although pixels are usually matched across pairs of images, there are good reasons for wanting to match them across more images, firstly for robustness to noise, but also because this avoids ambiguities in the reconstruction in the case of degenerate camera configurations (for example, pairwise matches are never sufficient to reconstruct a scene from images when all the camera pinholes are exactly aligned [11, Chapter 15.4.2]). Understanding the consistency propagation may simplify the certification of such matchings.

1.1 Results

We focus on Problem 3 for $k \geq 3$ because examples of tricolor sets of lines that are 2-consistent but without a colorful incidence are relatively easy to build:

Example 4. Let $(\vec{x}_0, \vec{x}_1, \vec{x}_2)$ be a basis of $\mathbb{R}^3$. Let $p_0, p_2, \ldots, p_{3n}$ be a set of points where $p_0$ is arbitrary, $p_{i+1} \in p_i + \mathbb{R} \vec{x}_i (\text{mod } 3)$ and $p_{3n} = p_0$. For each $i \in \{0, 1, 2\}$ define $L_i$ to be the set of lines in coordinate direction $i$ that are incident to points $p_j$ with $j \equiv i$ (mod 3) and $j \equiv i - 1$ (mod 3). If desired, we may apply a projective transformation that turns parallelism into concurrence.

Constructions from higher-dimensional grids. We present two constructions of arbitrary large sets of lines in $\mathbb{R}^d$ of $k + 1$ colors that are $k$-consistent and have no colorful incidence, for every $k \geq 3$ and $k + 1 \geq d \geq 2$. Both constructions are based on selecting subsets of lines from a regular grid in $\mathbb{R}^{k+1}$. In one case, the selection is probabilistic (Theorems 5), while in the other case it uses linear algebra over finite vector spaces (Theorem 6). In both constructions, every color class is concurrent. The probabilistic argument is asymptotic and proves the existence of configurations where every line is involved in many $k$-incidences for every choice of $k - 1$ other colors. The algebraic construction is explicit and is minimal in the sense that removing any line breaks the $k$-consistency.

Restrictions on higher-dimensional grids. We then test the sharpness and potential of constructions from higher-dimensional grids. On the one hand, we examine the number of lines of such constructions. The algebraic selection method picks at least $2^{k^2 - k - 1}$ lines of each color (we leave aside the probabilistic selection method as its analysis is asymptotic).
This construction has the property that the lines meeting at a \( k \)-incidence are not “flat”, in the sense that they are not contained in a \( k - 1 \)-dimensional subspace. We show, using the polynomial method [10], that for any construction with this property, the number of lines must be at least exponential in \( k \) (Proposition 7). On the other hand, we examine the possibility of designing similar constructions for models of cameras in which the lines are not all concurrent. We observe that when every color class is secant to two fixed lines, lines from two color classes cannot form a complete bipartite intersection graph (Proposition 8).

**Small configurations.** We also investigate small-size configurations of lines in \( \mathbb{R}^3 \) with 4 colors that are 3-consistent but have no colorful incidence. The smallest example provided by our constructions has 32 lines per color, which says little for applications like structure from motion, where each color class has very few lines. Figure 2 shows two non-planar examples with 12 lines each. We prove that they are the only non-planar constructions with these parameters (Theorem 11). We also show that any configuration with these parameters and concurrent color classes must have at least 24 lines or be planar (Theorem 10).

**Figure 2** Two non-planar examples of 12 lines in 4 colors that are 3-consistent and have no 4-incidence. (Left) A variation around Desargues’ configuration. (Right) A subset of the \((12, 16_3)\) configuration of Reye; note that triples of parallel lines intersect at infinity.

### 1.2 Related work

The study of consistent families of colored lines relates most prominently to classical questions in computer vision and in discrete geometry.

**In computer vision.** The simplest and most extensively studied setting for consistency deals with families where each color class has a single line. The study of \( n \)-tuples of lines that are incident at a point (or “point correspondences”), is central in multi-view geometry [11], that is the foundation of 3D-reconstructions algorithms. In this setting, consistency propagates trivially: \( n \) lines are concurrent if and only if any three of them are (even better: \( n \) lines not all coplanar are concurrent if and only if every pair of them is). Concurrency constraints are traditionally expressed algebraically as polynomials in image coordinates (see, e.g. [5]).

A more systematic study of consistency for silhouettes (i.e., for infinite families of lines) was proposed to design reconstruction methods based on shapes more complex than points or lines [3, 12]. Pairwise consistency for silhouettes can be encoded in a “generalized epipolar constraint”, which can be viewed as an extension of the epipolar constraint for points, and
expresses 2-consistency in terms of certain simple tangency conditions \[1, 16\]. There is no known similar characterization for \(k\)-consistency with \(k > 2\). Consistency propagation is only known for convex silhouettes: 3-consistency implies consistency \[16\].

In the dual, consistency expresses conditions for a family of planar sets to be sections of the same 3D object \[16\], a question classical in geometric tomography or stereology. We are not aware of any relevant result on consistency in these directions.

**Discrete geometry.** As evidenced by Figure 2, our analysis of small configurations relates to the classical configurations of Reye and Desargues in projective geometry. Our problem and results for larger configurations relate to various lines of research in incidence geometry. Inspired by the Sylvester–Gallai theorem, Erdős \[4\] asked for the largest number of collinear \(k\)-tuples in a planar point set with no collinear \(k + 1\)-tuple. The best construction for \(k = 3\) come from irreducible cubic curves \[1\]. For higher \(k\) the best construction was given by Solymosi and Stojakovíc \[14\] and are projections of higher-dimensional subsets of the regular grid (selected, unlike ours, by taking concentric spheres). In the plane, our problem is dual to a colorful variant of Erdős’s question. An intermediate between Erdős’s problem and the one treated here would ask for the existence of a set of lines \(L\) in which each line is involved in many (colorless) \(k\)-incidences but there are no (colorless) \(k + 1\)-incidences. Since the Solymosi-Stojakovíc construction provides \(n^2 - \sqrt{n}\) aligned \(k\) tuples of points, it is not hard to see, using a greedy deletion argument, that this alternative problem is essentially equivalent to Erdős’s original one.

In higher dimensions, the question of finding sets of lines with many \(k\)-rich points (in the terminology of \[9\]) is interesting even without the condition of having no \((k + 1)\)-rich point. Much of the recent research around this question has followed the solution to the joint problem \[10\] and has been driven by algebraic considerations (see \[9\] and the references therein). Here, we also ask for many \(k\)-rich points, but our questions are driven by combinatorial considerations. Our assumptions trade the usual density requirements (we assume linearly many, rather than polynomially many, \(k\)-rich points) for structural hypotheses in the form of conditions on the colors. On the other hand, we can use some of the algebraic methods; the proof of Proposition \[12\] is, for instance, modeled on the upper bound on the number of joints of Guth and Katz \[10\].

### 2 Probabilistic construction

In this section we prove:

**Theorem 5.** For any \(k \geq 3, k + 1 \geq d \geq 2\), and arbitrarily large \(N \in \mathbb{N}\), there exists a finite set of lines in \(\mathbb{R}^d\) of \(k + 1\) colors that is \(k\)-consistent, has no \((k + 1)\)-incidence, and in which each color class consists of between \(N\) and \(3N\) lines, all concurrent.

We describe our construction in \(\mathbb{R}^{k+1}\) with color classes consisting of parallel lines. We then apply an adequate projective transform (to turn parallelism into concurrence) and a generic projection to a \(d\)-dimensional space; both transformations preserve incidences and therefore the properties of the construction.

---

\[4\] This case is closely connected with the famous orchard problem recently solved in its asymptotic version \[7\]
Consistent sets of lines with no colorful incidence

Construction. Consider the finite subset \([n]^{k+1} = \{1, 2, \ldots, n\}^{k+1} \subset \mathbb{R}^{k+1}\) of the integer grid. We make our construction in two stages:

- Consider the set \(L^k_i\) of \(n^k\) lines that are parallel to the \(i\)th coordinate axis and contain at least one point of our grid. We pick a random subset \(L'_i\), where each line from \(L^k_i\) is chosen to be in \(L'_i\) independently with probability \(p \overset{\text{def}}{=} 2n^{-\frac{1}{k+1}}\) (the value of \(p\) is chosen with foresight).

- We then delete from \(L'_i\) all lines that are concurrent with \(k\) other lines from \(\cup_{j \neq i} L'_j\) and denote the resulting set \(L_i\).

We let \(L\) denote the colored set of lines \(L = L_1 \cup L_2 \cup \ldots \cup L_{k+1}\). The second stage of the construction ensures that \(L\) has no \((k + 1)\)-incidence. To prove Theorem \(^5\), it thus suffices to show that with positive probability, \(L\) is \(k\)-consistent and each \(L_i\) has the announced size.

Let us clarify that all lines considered in the proof are in \(\cup_{i=1}^{k+1} L^k_i\) unless stated otherwise.

Consistency. Let us argue that \(L\) is \(k\)-consistent with high probability. For a set \(I \subset [k+1]\), let

\[
S_I \overset{\text{def}}{=} \{Q \in [n]^{k+1} : \forall i \in I \text{ there is a line of } L_i \text{ containing } Q\},
\]

\[
S'_I \overset{\text{def}}{=} \{Q \in [n]^{k+1} : \forall i \in I \text{ there is a line of } L'_i \text{ containing } Q\}.
\]

We say that \(\ell \in L^k_i\) is \(j\)-bad (for \(j \neq i\)) if \(\ell\) contains no point of \(S_{[k+1]\setminus\{i,j\}}\). Note that \(L_i\) is not \(k\)-consistent precisely when some \(\ell \in L^k_i\) is \(j\)-bad and \(\ell\) ends up in \(L_i\).

Let \(\ell \in L^k_i\) and let \(L \subset L^k_i\) be any set containing \(\ell\). Let \(j \neq i\). We shall estimate \(P[(\ell \in L_i) \wedge (\ell \text{ is } j\text{-bad}) \mid L'_i = L]\). For ease of notation, we may assume that \(i = k + 1\), \(j = k\) and \(\ell\) is the line \(\{(1,1,\ldots,1,x) : x \in \mathbb{R}\}\). Call a point \(Q \in [n]^{k+1}\) regular if \(Q \notin \ell\).

The randomness in the construction comes from \((k + 1)\)\(n^k\) random choices, one for each line in \(\cup_{i=1}^{k+1} L^k_i\). We refer to these random choices as ‘coin flips’ since we can think of each as being a result of a toss of a (biased) coin.

Let \(\ell_{r,x}\) denote the line \(\{(1,1,\ldots,1,y,1,\ldots,1,1,x) : y \in \mathbb{R}\}\), where \(y\) is at position \(r\). If a line \(\ell' \notin L^k_i\) intersects \(\ell_{r,x}\) in point \((1,1,\ldots,1,y,1,\ldots,1,1,x)\), then all points of \(\ell'\) have \(y\) in the \(r\)th position. Note that a point \((1,1,\ldots,1,y,1,\ldots,1,1,x)\) is regular if \(y \neq 1\). A crucial observation is that if a line \(\ell' \notin L^k_{k+1}\) intersects \(\ell_{r,x}\) in a regular point and a line \(\ell'' \notin L^k_{k+1}\) intersects \(\ell'_{r',x'}\) in a regular point and \((r,x) \neq (r',x')\), then \(\ell'\) is different from \(\ell''\). This implies that sets of coin flips on which the events of the form

\[
\text{“there is a regular } Q \in \ell_{r,x} \text{ Q } S'_{[k]\setminus\{r\}}\]
\]

are disjoint for distinct \((r,x)\), apart from the flips associated to the lines in \(L^k_{k+1}\).

\(^5\) Deleting one line per concurrence of size \(k + 1\) would suffice, but deleting all lines as we do simplifies the analysis and suffices for our purpose.
For a point \( Q \in [n]^{k+1} \), let \( \lambda(Q) \) be the line in \( \mathcal{L}_{k+1}^{\#} \) containing \( Q \). Hence,

\[
P \left[ (\ell \in \mathcal{L}_{k+1}' \land (\ell \text{ is } k\text{-bad})) \mid \mathcal{L}_{k+1}' = L \right]
\]

\[
= P \left[ (\ell \in \mathcal{L}_{k+1}' \land \bigwedge_{x \in [n]} (1, 1, \ldots, 1, x) \notin S_{[k-1]} \mid \mathcal{L}_{k+1}' = L \right]
\]

\[
= P \left[ (\ell \in \mathcal{L}_{k+1}' \land \bigwedge_{x \in [n]} (\exists r \in [k-1] \ell_{r,x} \notin \mathcal{L}_{r}) \mid \mathcal{L}_{k+1}' = L \right]
\]

\[
= P \left[ (\ell \in \mathcal{L}_{k+1}' \land \bigwedge_{x \in [n]} (\exists r \in [k-1] \ell_{r,x} \notin \mathcal{L}_{r} \lor (\exists Q \in \ell_{r,x} \cap S'_{[k+1]})) \mid \mathcal{L}_{k+1}' = L \right]
\]

In this last formula, the point \( Q \) can be assumed to be regular because \( \ell \in L \), by assumption. Now we may drop \( \ell \in \mathcal{L}_{k+1}' \) to obtain that the above is

\[
\leq P \left[ \bigwedge_{x \in [n]} (\exists r \in [k-1] \ell_{r,x} \notin \mathcal{L}_{r} \lor (\exists \text{ reg. } Q \in \ell_{r,x} \cap S'_{[k+1]})) \mid \mathcal{L}_{k+1}' = L \right]
\]

Observe that if \( \ell_{r,x} \in \mathcal{L}_{r} \) then \( Q \in \ell_{r,x} \cap S'_{[k+1]} \) holds if and only if \( Q \in \ell_{r,x} \cap S'_{[k]\{r\}} \) and \( \lambda(Q) \in L \). By the observation above, the set of coin flips on which these latter events depend for different \( x \) are disjoint, so this probability is

\[
= \prod_{x \in [n]} P \left[ \exists r \in [k-1] \ell_{r,x} \notin \mathcal{L}_{r} \lor (\exists \text{ reg. } Q \in \ell_{r,x} \cap S'_{[k+1]} \land \lambda(Q) \in L) \mid \mathcal{L}_{k+1}' = L \right]
\]

\[
= \prod_{x \in [n]} \left( 1 - P \left[ \forall r \in [k-1] \ell_{r,x} \in \mathcal{L}_{r} \land (\forall \text{ reg. } Q \in \ell_{r,x} \setminus S'_{[k]\{r\}} \lor \lambda(Q) \notin L) \mid \mathcal{L}_{k+1}' = L \right] \right)
\]

\[
= \prod_{x \in [n]} \left( 1 - \prod_{r \in [k-1]} \left( p \cdot \prod_{\text{reg. } Q \in \ell_{r,x} \land \lambda(Q) \in L} P \left[ Q \notin S'_{[k]\{r\}} \right] \right) \right)
\]

Call \( L \subseteq \mathcal{L}_{k+1}^{\#} \) \textit{unbiased} if for every pair \((r, x) \in [k-1] \times [n]\) the number of points \( Q \in \ell_{r,x} \) such that \( \lambda(Q) \in L \) is at most \( 2pm \). For unbiased \( L \), we obtain that the above is

\[
\leq \left( 1 - (p \cdot (1 - p^{k-1} 2m)^{k-1}) \right)^n \leq \left( 1 - \left( \frac{2}{3} \right)^{k-1} \right)^n \leq e^{-n \left( \frac{2}{3} \right)^{k-1}} = e^{-n \frac{1}{3^{k-1}}}
\]

If we pick \( L \) uniformly at random, then, for every \((r, x) \in [k-1] \times [n]\), the number of points \( Q \in \ell_{r,x} \) such that \( \lambda(Q) \in L \) is a binomial random variable. With help from Chernoff’s bound, we then obtain

\[
P \left[ (\ell \in \mathcal{L}_{k+1}' \land (\ell \text{ is } k\text{-bad})) \right]
\]

\[
\leq P \left[ \mathcal{L}_{k+1} \text{ is biased} \right] + \sum_{\text{unbiased } L} P \left[ \mathcal{L}_{k+1}' = L \right] P \left[ (\ell \in \mathcal{L}_{k+1}' \land (\ell \text{ is } k\text{-bad})) \mid \mathcal{L}_{k+1}' = L \right]
\]

\[
\leq \sum_{(r, x) \in [k-1] \times [n]} e^{-n (p/n)^2/2n} + e^{-n \frac{1}{3^{k-1}}} = e^{-cn \frac{1}{3^{k-1}}}.
\]
By taking the union bound over all \(i, j\) and \(\ell\) we obtain that
\[
P[\mathcal{L} \text{ is not } k\text{-consistent}] \leq P\left[\exists i, j \exists \ell \in \mathcal{L}_\ell^\# \left((\ell \in \mathcal{L}_\ell') \land (\ell \text{ is } j\text{-bad})\right)\right]
\leq (k + 1)^2 n^k e^{-cn^{\frac{1}{2k^2}}} \leq e^{-c'n^{\frac{1}{2k^2}}}.
\]

**Size.** We now analyze the probability that \(\mathcal{L}_1\) is large (the bound will hold for each \(\mathcal{L}_i\)). Let us write \(\mathcal{L}' = \cup_{i=1}^{k+1} \mathcal{L}_i'\) and label \(\ell_1, \ell_2, \ldots, \ell_n\) the lines parallel to the 1st coordinate axis that intersect our grid. Put \(X_i = \mathbb{1}_{\ell_i \in \mathcal{X}}\) and let \(X = |\mathcal{L}_1| = X_1 + X_2 + \ldots + X_n\). We have
\[
E[X_i] = P[X_i = 1] = P[\ell_i \in \mathcal{L}'] P[\ell_i \in \mathcal{L} \mid \ell_i \in \mathcal{L}'] = p(1 - p^k)^n,
\]
so
\[
E[X] = n^k p(1 - p^k)^n = \left(1 - n^{-\frac{2k}{k-1}}\right)^n n^{k-\frac{k}{k-1}} \geq \left(1 - \frac{1}{n}\right)^n n^{k-\frac{k}{k-1}} \geq \frac{1}{4} n^{k-\frac{k}{k-1}}.
\]
Thus \(E[X] = N \in \left[\frac{1}{4} n^{k-\frac{2k}{k-1}}, n^{k-\frac{2k}{k-1}}\right]\). We next use a concentration inequality to pass from \(E[X]\) to an estimate on the probability that \(X\) is large. The second step introduces some dependency between some of the variables \(X_i\), so we use Chebyshev's inequality:
\[
P[|X - E[X]| > \lambda E[X]] \leq \frac{\text{Var}[X]}{\lambda^2 E[X]^2}.
\]
Taking \(\lambda = 1/2\), we get
\[
P\left[|X - E[X]| > \frac{1}{2} E[X]\right] \leq \frac{\text{Var}[X]}{N^2} \leq 64 \text{Var}[X] n^{-\left(2k-\frac{k}{k-1}\right)}.
\]
Recall that
\[
\text{Var}[X] = \sum_{i=1}^{n^k} \text{Var}[X_i] + \sum_{1 \leq i < j \leq n^k} \text{Cov}[X_i, X_j].
\]
Since \(X_i\) takes values in \(\{0, 1\}\), the first sum in the right-hand term is bounded by \(n^k\). Moreover, there are \(O(n^{k+1})\) pairs of variables \(X_i\) and \(X_j\) with non-zero covariance, since this requires the two lines \(\ell_i\) and \(\ell_j\) to belong to a common axis-aligned 2-plane. Again, each non-zero covariance is at most 1. Altogether, \(\text{Var}[X] = O(n^{k+1})\), so
\[
P\left[|X - E[X]| > \frac{1}{2} E[X]\right] = O\left(n^{\frac{2k+3}{k-1} - k}\right).
\]
For \(k \geq 3\), the probability that \(X\) is in \(\left[\frac{1}{8} n^{k-\frac{2k}{k-1}}, \frac{3}{2} n^{k-\frac{2k}{k-1}}\right]\) goes to 1 as \(n\) goes to infinity.

## 3 Algebraic construction

In this section we prove:

**Theorem 6.** For any \(k \geq 3, k+1 \geq d \geq 2,\) and arbitrarily large \(N\), there exists a finite set of lines in \(\mathbb{R}^d\) with \(k+1\) colors that is \(k\)-consistent, has no \((k+1)\)-incidence, and in which each color class consists of \(N\) lines, all concurrent.

As in Section 2 we describe our construction in \(\mathbb{R}^{k+1}\) with parallel families of lines, and obtain the desired configuration by an adequate projective transformation and a projection. We again consider the finite portion of the integer grid \([n]^{k+1} \subset \mathbb{R}^{k+1}\) and the axis-aligned lines that intersects it. Unlike in Section 2 we give an explicit way to select some of these lines to achieve the desired configuration.
**Construction.** We work with axis-aligned lines that intersect in points of our grid. Hence, identifying each line with the subset of the grid that it contains does not affect incidences. We fix a prime number $p$ and parameterize $[n]$ by the vector space $V = (\mathbb{Z}/p\mathbb{Z})^{k-1}$; this restricts the choice of $n$ to certain prime powers, but still allows to make it arbitrarily large. We use this parametrization to describe the lines in our configuration as solutions of well-chosen linear equations.

Let $v_1, v_2, \ldots, v_k \in V$ such that $v_1 + v_2 + \ldots + v_k = 0$ and any proper subset of them are linearly independent. Let $\cdot$ denote the inner product of the vector space $V$. For $i = 1 \ldots k$, our set $L_i$ consist of all the lines parallel to the $i$th coordinates and passing through a point with parameters $(X_1, \ldots, X_{k+1}) \in V^{k+1}$ such that

$$v_{i-1} \cdot X_1 + v_{i-1} \cdot X_2 + \ldots + v_{i-1} \cdot X_{i-1} + v_i \cdot X_{i+1} + \ldots + v_i \cdot X_{k+1} = 0.$$  

(Keep in mind that each $X_i$ is a vector in $(\mathbb{Z}/p\mathbb{Z})^{k-1}$.) We define $L_{k+1}$ similarly but replace Equation (1) by

$$v_k \cdot X_1 + v_k \cdot X_2 + \ldots + v_k \cdot X_k = 1.$$  

(2)

**No $(k+1)$-incidence.** Any $(k+1)$-incidence is a point of the grid whose parameters $(X_1, \ldots, X_{k+1})$ satisfy the system:

$$
\begin{align*}
&v_1 \cdot X_2 + v_1 \cdot X_3 + \ldots + v_1 \cdot X_k + v_1 \cdot X_{k+1} = 0 \\
v_1 \cdot X_1 + v_2 \cdot X_2 + \ldots + v_2 \cdot X_k + v_2 \cdot X_{k+1} = 0 \\
v_2 \cdot X_1 + v_2 \cdot X_2 + \ldots + v_3 \cdot X_k + v_3 \cdot X_{k+1} = 0 \\
v_3 \cdot X_1 + v_3 \cdot X_2 + \ldots + v_4 \cdot X_k + v_4 \cdot X_{k+1} = 0 \\
\vdots \\
v_{k-1} \cdot X_1 + v_{k-1} \cdot X_2 + v_{k-1} \cdot X_3 + \ldots + v_k \cdot X_{k+1} = 0 \\
v_k \cdot X_1 + v_k \cdot X_2 + v_k \cdot X_3 + \ldots + v_k \cdot X_k = 1 
\end{align*}
$$

Summing all these conditions yields

$$
\left(\sum_{i=1}^{k} v_i\right) \cdot \left(\sum_{i=1}^{k+1} X_i\right) = 1,
$$

which contradicts $v_1 + v_2 + \ldots + v_k = 0$. So there is no $(k+1)$-incidence.
\section{More on grid-like examples}

Both Theorems \ref{thm:grid} and \ref{thm:grid2} construct examples as projections of subsets of a regular grid in higher dimension. We discuss here the properties of such constructions.

\textbf{Number of lines.} Consider a colored set of lines $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \ldots \cup \mathcal{L}_{k+1}$ in $\mathbb{R}^d$. We say that a $k$-incidence of $\mathcal{L}$ is flat if the lines meeting there are contained in an affine subspace of dimension at most $\min(d, t) - 1$. In any grid-like construction such as those in Theorems \ref{thm:grid} and \ref{thm:grid2}, every $k$-incidence is non-flat.

\begin{proposition}
Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \ldots \cup \mathcal{L}_{m}$ be a $k$-consistent colored set of lines in $\mathbb{R}^k$ with no $(k+1)$-incidence. If no $k$-incidence of $\mathcal{L}$ is flat, then
\[
\sum_{i=1}^{m} |\mathcal{L}_i| \geq \frac{\binom{n-1}{k} + k - 1}{\binom{n}{k} - 1}.
\]
\end{proposition}

\textbf{Proof.} The proof essentially follows the argument of Guth and Katz \cite{GuthKatz} for bounding the number of joint among $n$ lines; the main difference is that the consistency assumption makes their initial pruning step unnecessary. We spell it out for completeness.

Let $P$ denote a set of concurrency centers witnessing all the $k$-incidences required by the $k$-consistency. We choose $P$ of minimum size, so that
\[
|P| \leq \left( \sum_{i=1}^{m} |\mathcal{L}_i| \right) \binom{n-1}{k-1}.
\]  (3)

Let $f(x_1, \ldots, x_k)$ be a nontrivial (not necessarily homogeneous) polynomial that vanishes at every point of $P$ and has minimal total degree. We claim that
\[
\binom{n-1}{k-1} \leq \deg f \quad \text{and} \quad \binom{\deg f + k - 1}{k} \leq |P|.
\]  (4)

Note that inequalities \ref{eq:inequality1} and \ref{eq:inequality2} together imply the statement.

It remains to prove inequalities \ref{eq:inequality1}. The second inequality follows from the minimality of $\deg f$; if it were false, we would be able to find a non-zero polynomial of degree $\deg f + k - 1$ vanishing on $P$ by solving for its coefficients. For the first inequality, we argue \textit{ab absurdum}. Assume that it fails. Then every line in $\cup_{i} \mathcal{L}_i$ intersects $\{ f = 0 \}$ in strictly more points than the degree of $f$. This implies that every line in $\cup_{i} \mathcal{L}_i$ is contained in $\{ f = 0 \}$. Since no
incidence is flat, this in turn implies that \( \nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_2}) \) vanishes at every point of \( P \). So, every polynomial of the form \( \frac{\partial f}{\partial x_i} \) vanishes on all of \( P \); all these polynomials have total degree strictly smaller than \( f \). Since \( f \) is non-constant, then at least one of these polynomials is nontrivial. This contradicts the minimality of the total degree of \( f \).

For \( m = k + 1 \), the bound of Proposition \[\text{is} \left(\frac{2k}{k}\right) \], so the number of lines required grows exponentially with \( k \).

**Non-concurrent colors.** Theorems \[6\] and \[7\] both use a grid in \( \mathbb{R}^{k+1} \) to start with \( k + 1 \) color classes, each of size \( n^k \), where every line is involved in \( n \) colorful incidences. Recall that in this setup, every color class is concurrent (it consists of parallel lines). This is in fact important, perhaps essential. To see this, note that any two of our starting color classes contain arbitrarily large subsets whose intersection graph is dense. This is impossible, generically, if we try to work with families of lines that are secant to two skew lines in \( \mathbb{R}^3 \).

**Proposition 8.** For \( i = 1, 2 \), let \( \Gamma_i \) denote the set of lines secant to two fixed lines \( s_i \) and \( s_i' \) in \( \mathbb{R}^3 \). Let \( A \) and \( B \) be two sets of \( n \) lines from \( \Gamma_1 \) and \( \Gamma_2 \), respectively. If the lines \( s_1, s_1', s_2 \) and \( s_2' \) are in generic position then the intersection graph of \( A \) and \( B \) has \( O(n^{4/3}) \) edges.

**Proof.** First, note that the intersection graph of \( A \) and \( B \) is semi-algebraic: parametrizing \( \Gamma_i \) by \( s_i \times s_i' \cong \mathbb{R}^2 \) makes the incidence an algebraic relation, as can be deduced from the bilinearity of incidence in Plücker coordinates.

Next, remark that if this graph contains a complete bipartite subgraph \( K_{3,3} \), then the lines \( \{s_1, s_1', s_2, s_2'\} \) are in a special position. Indeed, in the generic case, these two triples of lines come from the two families of rulings of a quadric surface \[15, 10\]; the lines \( s_1, s_1', s_2, s_2' \) are also rulings of that quadric, so both \( s_1 \) and \( s_1' \) intersect both \( s_2 \) and \( s_2' \). In the non-generic cases, the six lines must be coplanar with \( s_1 \) and \( s_2 \).

Now, we apply the semi-algebraic version of the Kővári–Sós–Turán theorem \[6\], and obtain that the number of edges of our graph is \( O(n^{4/3}) \).

We see the previous result as indication that a straight forward adaptation of our probabilistic construction to the case of two-slit is impossible. We would like to improve the bound in Proposition \[8\] from \( O(n^{4/3}) \) to \( O(n) \).

**Remark.** Note that the genericity assumption in Proposition \[8\] is on the sets \( \Gamma_i \), not on their subsets. The analogue for concurrent sets of lines would be to require that the centers of concurrence are in generic position; this clearly does not prevent finding arbitrarily large subsets with dense intersection graphs.

**Extension to continuous sets of lines.** The constructions of Theorems \[5\] and \[6\] can be turned into continuous families of lines as follows.

First, we follow either construction up to the point where we have a family \( \mathcal{L} \) of lines of \( k + 1 \) colors in \( \mathbb{R}^{k+1} \) that is \( k \)-consistent, without colorful incidence, and where each color class is parallel. Consider a parameter \( \epsilon > 0 \), to be fixed later. For every \( i \), we build a set \( \mathcal{L}_i(\epsilon) \) by considering every line \( \ell \in \mathcal{L}_i \) in turn, and adding to \( \mathcal{L}_i \) every line \( \ell' \) parallel to \( \ell \) such that the distance between \( \ell \) and \( \ell' \) is most \( \epsilon \). Note that for \( \epsilon < 1/2 \) the family \( \mathcal{L}(\epsilon) \) is \( k \)-consistent and without colorful incidence.

---

\[6\] This choice is motivated by the design of two-slit cameras \[17, 2\].
Now, consider a generic projection $f: \mathbb{R}^{k+1} \rightarrow \mathbb{R}^d$ for the desired $d$. For any $\epsilon > 0$ the family $\mathcal{L}(e)$ is $k$-consistent. We observe that for $\epsilon > 0$ small enough, it also remains without colorful incidence. Let $\tau$ denote the minimum distance, in the projection, between a $k$-incidence and a line (of any color) not involved in that incidence. Every $k$-incidence in $\mathcal{L}$ gives rise, in $\mathcal{L}(e)$, to $k$ tubes that intersect in a bounded convex set $B$ of size $O(\epsilon)$. Choosing $\epsilon > 0$ such that the diameter of $f(B)$ is less than $\tau/2$ ensures that the corresponding family $f(\mathcal{L}(e))$ has no colorful incidence.

For a given family of colored lines $\mathcal{L}$ define the set $P_S$ to be the set of points incident to at least one line of each of the color classes in $S$; see Figure 1. Notice that in our examples, for each set $S$ of $k$ colors the set $P_S$ is highly disconnected. As mentioned in the introduction, Trager et al. \cite{13} showed that if a family of sets of lines is 3-consistent and for each $S$ of size 3, the set $P_S$ is convex, then the whole family is consistent. An interesting open question is whether an analogue theorem holds if instead of convexity, we assume that for every set $S$ of size $k$, the set $P_S$ is sufficiently connected.

5 Constructions with few lines

The configurations constructed in Sections 2 and 3 have at least 32 lines per color. This is considerably larger than the sets of lines involved in some of the questions around consistency that arise in computer vision. In the example of structure-from-motion mentioned in introduction, when the camera is central, every color class has only 5 to 7 lines. It turns out that for sufficiently small configurations, $k$-consistency does imply some colorful incidences:

\begin{itemize}
  \item \textbf{Lemma 9.} Any 3-consistent colored set of lines $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4$ in $\mathbb{R}^d$ with $|\mathcal{L}_1| = |\mathcal{L}_2| = |\mathcal{L}_3| = |\mathcal{L}_4| = 2$ contains a colorful incidence.
\end{itemize}

\textbf{Proof.} Let us prove the case where $d = 2$; the general case follows by projecting onto a generic 2-plane. Let $P'$ denote the dual of $\mathcal{L}_i$, and let $P = P_1 \cup P_2 \cup P_3 \cup P_4$. Assume, by contradiction, that $\mathcal{L}$ contains no colorful incidence, i.e. that no line intersects every $P_i$. Let $P' = P_2 \cup P_3 \cup P_4$ and let us apply a projective transform to map the points of $P_i$ to the horizontal and vertical directions, respectively. We call a line that contains a point of each of $P_2$, $P_3$ and $P_4$ a rainbow line.

Since $\mathcal{L}$ is 3-consistent, for any point $x \in P$ and any choice of 2 other colors, there is a line through $x$ that contains a point of each of these colors. Since $\mathcal{L}$ has no colorful incidence, there must exist three horizontal lines and three vertical lines that intersect $P'$, and each must contain exactly two points of $P'$ of distinct colors. Moreover, no rainbow line can be horizontal or vertical. But this implies that out of the 9 intersections between horizontal and vertical lines, only 5 (the corners and the center) can be on a rainbow line. This contradicts $|P'| = 6$.

We prove here a slightly stronger lower bound:

\begin{itemize}
  \item \textbf{Theorem 10.} Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4$ be a 3-consistent colored set of lines in $\mathbb{R}^3$ with no colorful incidence and concurrent colors. If $|\mathcal{L}| < 24$, then $\mathcal{L}$ is contained in a 2-plane.
\end{itemize}

We do not know whether the constant 24 is best possible.

\textbf{Classification.} We also provide a characterization of 3-consistent, 4-colored sets of lines in $\mathbb{R}^3$ with no colorful incidence and 3 lines per color. Forgetting for a moment about colors, any such configuration must consist of 12 lines and 12 points, every point on 3 lines and every line through 3 points; in the classical tabulation of projective configurations, they are
called \((12_3)\) configurations. It turns out that there are 229 possible incidence structures meeting this description, and that every single one of them is realizable in \(\mathbb{R}^3\) \(^8\). To analyze what happens when we add back the colors and the consistency assumption, we consider two special \((12_3)\) configurations:

- A **Reye-type configuration** \(^7\) is a configuration obtained by selecting 12 out of the 16 lines supporting the 12 edges and four long diagonals of a cube, in a way that produces a \((12_3)\) configuration.
- A **Desargues-type** configuration is defined from six planes \(\Pi_1, \Pi_2, \ldots, \Pi_6\) in \(\mathbb{R}^3\) where (i) each of \(\{\Pi_1, \Pi_2, \ldots, \Pi_5\}\) and \(\{\Pi_1, \Pi_2, \Pi_3, \Pi_6\}\) is in general position, and (ii) \(\Pi_4, \Pi_5\) and \(\Pi_6\) intersect in a line. The configuration consists of all lines that are contained in exactly two planes.

Here is our classification:

**Theorem 11.** Let \(\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4\) be a 3-consistent colored set of lines in \(\mathbb{R}^3\) with no colorful incidence. If every color class has size 3, and \(\mathcal{L}\) is not contained in a 2-plane, then it is a Desargues-type or a Reye-type configuration colored as in Figure \(\[8\].

### 5.1 Classification: Proof of Theorem 11

Let \(\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4\) be a 3-consistent colored set of lines in \(\mathbb{R}^3\) with no colorful incidence and that is not coplanar. We start with a few simple observations:

1. Every line \(\ell\) in a color class \(X\) intersects the lines from any other color class \(Y\) in at least two distinct points. Indeed, \(\ell\) needs to intersect the lines of \(Y\) to satisfy the 3-consistency condition, and if it does so in a single point, then that point is a colorful incidence.
2. If all lines of a color class are coplanar, then the configuration is coplanar. This follows readily from the previous observation.
3. Given two lines \(\ell, \ell'\) and a point \(x\) in \(\mathbb{R}^3\), if \(\ell\) and \(\ell'\) are skew and \(x \not\in \ell \cup \ell'\), then there is a unique line through \(x\) that intersects \(\ell\) and \(\ell'\).

#### 5.1.1 Geometric analysis

We first restrict the geometry of a color class of size three. We denote the lines of \(\mathcal{L}_1\) by \(a_1, a_2, a_3\) (\(b_i\) for \(\mathcal{L}_2\), \(c_i\) for \(\mathcal{L}_3\) and \(d_i\) for \(\mathcal{L}_4\)). We write \(\langle a, b \rangle\) for the plane spanned by lines \(a\) and \(b\) and \(\langle a, b, c \rangle\) for a common incidence of \(a\), \(b\) and \(c\), and \(p_{abc}\) for the point of concurrency of \(a\), \(b\) and \(c\).

**Lemma 12.** If \(\mathcal{L}_1 = \{a_1, a_2, a_3\}\) is a color class of size three, then

1. the lines in \(\mathcal{L}_1\) are pairwise skew, or
2. the lines in \(\mathcal{L}_1\) are concurrent and not coplanar, or
3. \(a_1\) intersects both \(a_2\) and \(a_3\) and every line in \(\mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4\) is contained in \(\langle a_1, a_2 \rangle\) or \(\langle a_1, a_3 \rangle\).

Moreover, every line in \(\mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4\) intersects the lines of \(\mathcal{L}_1\) in exactly two points.

\(^7\) The \((12_4 16_3)\) configuration of Reye consists of 12 points and 16 lines in \(\mathbb{R}^3\) such that every point is on 4 lines and every line contains 3 points; its realizations are projectively equivalent to the 16 lines supporting the 12 edges and four long diagonals of a cube, together with that cube’s vertices and center and the 3 points at infinity in the directions of its edges.
Consistent sets of lines with no colorful incidence

Proof. Assume that we are not in case (1) and that, wlog, the lines $a_1$ and $a_2$ are coplanar. We first claim that there is a concurrence $bcd$ of lines from $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ such that $p_{bcd} \notin <a_1, a_2>$. Indeed, as the configuration is not planar, we can assume that $b$ is not contained in $<a_1, a_2>$. The line $b$ intersects two lines of $\mathcal{L}_1$, it must intersect $a_1$ or $a_2$; this point cannot be $p_{bcd}$ as it would make a colorful incidence, so $b \notin <a_1, a_2>$ forces $p_{bcd} \notin <a_1, a_2>$. 

Now, $a_3$ must be coplanar with $a_1$ or $a_2$. Indeed, each of $b$, $c$ and $d$ has at most one point of intersection with $a_1 \cup a_2$, and therefore intersects $a_3$. This implies that $b$, $c$ and $d$ are coplanar, namely in the plane spanned by $p_{bcd}$ and $a_3$. This implies that the intersections of $b$, $c$ and $d$ with $<a_1, a_2>$ are three distinct, aligned points. This forces the intersection between the plane $p_{bcd} \cup a_3$ and $<a_1, a_2>$ to be one of the lines $a_1$ or $a_2$.

Let us say, wlog, that $a_3$ is coplanar with $a_1$. If $a_3$ is also coplanar with $a_2$ then all three lines are either coplanar or concurrent. The former would imply that the entire configuration is coplanar, so it must be the latter; this corresponds to case (2).

The remaining case is when $a_3$ is coplanar with $a_1$ but not with $a_2$. Consider a line $x$ in $\mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4$ and let $bcd$ be a concurrence among these three colors that involves $x$. As observed above, either $p_{bcd} \in <a_1, a_2>$ and $x$ is contained in $<a_1, a_2>$, or $p_{bcd} \notin <a_1, a_2>$ and $x$ is contained in $<a_1, a_3>$. We are thus in case (3).

Finally, let us consider a line $x \in \mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4$ and count its intersections with the lines of $\mathcal{L}_1$. We already observed that this number is at least two. In cases (2) and (3) it is straightforward that it is also at most two. So assume that we are in case (1), where the lines of $\mathcal{L}_1$ are pairwise skew. Assume that there exists a line $b \in \mathcal{L}_2$ that intersects the lines from $\mathcal{L}_1$ in three points. Let $bcd$ be an incidence with lines of $\mathcal{L}_3$ and $\mathcal{L}_4$. The point $p_{bcd}$ cannot belong to a line in $\mathcal{L}_1$, so there is a unique line from $p_{bcd}$ that intersects any two given lines in $\mathcal{L}_1$. Since $c \neq b$, $c$ intersects at most one line of $\mathcal{L}_1$. This contradicts one of our initial observations.

Remark that Lemma 12 only assumed that one color class has size 3. When all color classes have size three, we can eliminate case (3):

Lemma 13. If every color class has size 3, then every color class consists of lines that are concurrent and not coplanar, or pairwise skew.

Proof. The task is to show that case (3) in Lemma 12 cannot occur. We argue by contradiction, and assume that $a_1$ intersects both $a_2$ and $a_3$ and that every line in $\mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4$ is contained in $\Pi_2 = <a_1, a_2>$ or in $\Pi_3 = <a_1, a_3>$.

Wlog we can assume that $\Pi_2$ contains exactly one line of $\mathcal{L}_2$ and one line of $\mathcal{L}_3$. Indeed, each line $a_i$ must intersect lines from each other color class in at least two points. Thus, each of $\Pi_1$ and $\Pi_2$ must contain at least a line from each color class. Let $b_1$ and $c_1$ denote the lines of $\mathcal{L}_2$ and $\mathcal{L}_3$ contained in $\Pi_2$.

The line $a_2$ must intersect lines from each of $\mathcal{L}_2$ and $\mathcal{L}_3$ in two distinct points. The only point in which $a_2$ can meet lines from $\Pi_3$ is $a_1 \cap a_2$. Thus, two lines $b_2 \in \mathcal{L}_2$ and $c_2 \in \mathcal{L}_3$, both contained in $\Pi_3$, pass through $a_1 \cap a_2$, and neither $b_1$ nor $c_1$ goes through that point.

The line $b_1$ must intersect lines from $\mathcal{L}_3$ in two distinct points, so one of them has to be $a_1 \cap b_1$ and the other $b_1 \cap c_1$. In particular, $b_1$ and $c_1$ do not meet on $a_1$.

Now, the remaining lines $b_3 \in \mathcal{L}_2$ and $c_3 \in \mathcal{L}_3$ meet, respectively, $c_1$ and $b_1$ on $a_1$. Again, this follows from the fact that $b_1$ (resp. $c_1$) must intersect lines from $\mathcal{L}_4$ (resp. $\mathcal{L}_2$) in two distinct points.

We now have our contradiction: on the line $a_1$, every intersection with a line of $\mathcal{L}_2$ is also on a line of $\mathcal{L}_4$. The concurrence of $a_1$ with lines of $\mathcal{L}_2$ and $\mathcal{L}_4$ is therefore colorful.
From now on, let us assume that $|L_1| = |L_2| = |L_3| = |L_4| = 3$. Lemmas 12 and 13 force the incidence structure to be quite regular:

**Lemma 14.** Let $L_i$ and $L_j$ be two color classes. Every line of $L_i$ intersects exactly two lines of $L_j$. For any two lines of $L_j$, there is exactly one line of $L_i$ that intersects them both.

**Proof.** Let $\ell$ be a line of $L_i$. We know that $\ell$ intersects the lines of $L_j$ in exactly two points. Whether $L_j$ is pairwise skew or concurrent and non-coplanar, exactly one line of $L_j$ must pass through each of these points.

Now, consider the graph on $L_i \cup L_j$ where there is an edge between any two lines of different colors that intersect. This graph is bipartite, has three vertices per class, and every vertex has degree two (by the first statement). It must be a 6-cycle, and the second statement follows.

**Corollary 15.** At most one class consists of concurrent lines.

**Proof.** Assume that $L_1$ and $L_2$ both consist of concurrent lines. For any $a_i, a_j \in L_1$, there is a line of $L_2$ that is coplanar with $a_i$ and $a_j$ (since it intersects them in two points). The center of concurrence of $L_2$ therefore lies on the intersection of the three planes spanned by pairs of lines of $L_1$. That intersection is the center of concurrence of $L_1$. This implies that any line of $L_2$ can only intersect the lines of $L_1$ in a single point, a contradiction.

### 5.1.2 Characterization of the 3-incidences

We now focus on the combinatorial incidence structure defined by $L$. We represent a combinatorial configuration using a set of cubic monomials in $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3$. Here each variable represents a line in $L$, and a monomial $xyz$ means that the lines $x, y, z$ are concurrent. For convenience, we also write $A$ for $L_1$, $B$ for $L_2$, etc.

Let $C$ be an incidence structure for $L$ (that is, a set of monomials). If we fix a color, say $A$, we can construct a graph $C_A$ with colored nodes and labeled edges, defined as follows. The nodes of $C_A$ are incidence points of $C$ that involve lines in $A$. Finally, each edge is uniquely associated with a line from $B, C, D$: from Lemma 13 we see that each line from $B, C, D$ has exactly two incidences with lines in $A$, so it can be used to define an edge joining two vertices of $C_A$. From properties of $C$ (and using again Lemma 14) we deduce the following:

- Every node of $C_A$ has degree two.
- Any edge of $C_A$ joins vertices on different lines of $A$, and has neighbors of different color.
- For $s \neq 1$ and any $i \neq j$, there there exists a unique edge of class $L_s$ joining vertices on $a_i$ and $a_j$.

From these observations, we see that edges in $C_A$ form (one or several) cycles, and that any three consecutive edges have different colors. In particular, the colors alternate cyclically, so every cycle has length a multiple of three. An elementary case-analysis (e.g. distinguishing whether $C_A$ contains two consecutive edges that join $a_1$ and $a_2$) reveals that, up to relabeling, $C_A$ must be one of two graphs:
We claim that the only possible incidence structures are isomorphic to:

\[
\begin{align*}
(I) & : a_1b_2c_3 \ a_1b_3d_2 \ a_1c_2d_3 \ b_1c_3d_2 \ a_1b_2c_3 \ a_1b_3d_2 \ a_1c_2d_3 \ b_1c_1d_1 \\
(II) & : a_2b_3c_1 \ a_2b_1d_3 \ a_2c_3d_1 \ b_2c_1d_3 \ a_3b_1c_2 \ a_3b_2d_1 \ a_3c_1d_2 \ b_3c_3d_3.
\end{align*}
\]

Each graph specifies all incidences that involve \( A \), so we have to complete the list by those between \( B, C \) and \( D \), and look for isomorphisms. Remark that these incidences must occur outside \( A \), to avoid creating a colorful incidence. Note the following dichotomy:

- If the lines of \( A \) are concurrent and not coplanar, then two lines in \( B \cup C \cup D \) that intersect different pairs of lines of \( A \) cannot intersect outside of \( A \).
- If the lines of \( A \) are pairwise skew, then two lines in \( B \cup C \cup D \) that intersect the same pair of lines of \( A \) cannot intersect outside of \( A \).

Consider the graph on the left. The incidences involving \( A \) readily match \( (I) \) and \( (II) \). If the lines of \( A \) are concurrent and not coplanar, then we must have \( b_1c_1d_1, b_2c_2d_2, b_3c_3d_3 \), and therefore \( (II) \). If the lines of \( A \) are pairwise skew, then the only possibility avoiding a colorful incidence is to have \( b_1c_2d_2, b_2c_1d_3 \) and \( b_3c_2d_1 \), and therefore \( (I) \). Consider now the graph on the right. If the lines of \( A \) are concurrent and not coplanar, then the 3-consistency requires that a line from \( D \) passes through \( b_3 \cap c_2 \), which makes that incidence colorful since this point is already on \( a_1 \). Hence, the lines of \( A \) can only be pairwise skew, and the only choice avoiding a colorful incidence is \( b_1c_2d_3, b_2c_3d_1, b_3c_1d_2 \); Applying the cyclic permutation \( a \rightarrow d \rightarrow c \rightarrow b \rightarrow a \) gives \( (II) \).

**Remark.** The set of incidences \( (I) \) can be obtained from the monomials with positive sign in

\[
\det \begin{bmatrix}
  a_1 & b_1 & c_1 & d_1 \\
  a_2 & b_2 & c_2 & d_2 \\
  a_3 & b_3 & c_3 & d_3 \\
  1 & 1 & 1 & 1
\end{bmatrix}.
\]

### 5.1.3 Geometric realization

**Lemma 16.** The incidence structures \( (I) \) and \( (II) \) can be realized geometrically and their only non-planar geometric realizations are, respectively, a Reye-type configuration and a Desargues-type configuration.

**Proof.** The fact that the two incidence structures are realizable can be seen directly from Figure 2. The second part of statement are essentially two “geometric theorems”: it means the 12 incidences of triples of lines from \( (I) \) and \( (II) \) automatically guarantee that certain other collinearities or incidences always hold.

Let us consider first the incidence structure \( (II) \). Remark that the lines \( b_1 \) and \( c_1 \) intersect outside of a line of \( A \) (since their intersection is on \( d_1 \)) and intersect the same pair of lines of \( A \), namely \( \{a_2, a_3\} \). This implies that in any non-planar realization, the lines of \( A \) must be concurrent and not coplanar. Moreover, in each of the triples \( \{b_2, c_3, d_1\}, \{b_3, c_1, d_2\} \) and \( \{b_1, c_2, d_3\} \), the lines intersect pairwise (we see that from their incidences) but are not concurrent (because any pairwise intersection occurs on a line of \( A \)); each of these triples is therefore coplanar. All three planes must contain the points of incidence of \( b_1c_1d_1, b_2c_2d_2 \) and \( b_3c_3d_3 \). As \( A \) is already concurrent, all other color classes consist of pairwise skew lines, and these points are distinct. It follows that the three planes intersect in a line, and we have a Desargues-type configuration.
Now let us consider (I). We first observe that eight well-chosen incidence points are sufficient to determine all lines. For example, we may take the eight incidences to be

\begin{align*}
  a_2 b_3 c_1 &= a_1 c_2 d_3 \\
  a_3 b_1 c_2 &= a_2 c_3 d_1 \\
  a_1 b_3 d_2 &= b_1 c_3 d_3 \\
  a_3 b_2 d_1 &= b_2 c_1 d_3
\end{align*}

(5)

Fixing these points arbitrarily in $\mathbb{R}^3$ yields a configuration of lines that satisfies 8 out of the 12 required incidences. We now assume that three of the remaining four incidence points (e.g., $a_1 b_2 c_3$, $a_2 b_1 d_3$, $a_3 c_1 d_2$) are at infinity, and correspond to orthogonal directions. We can assume this because any incidence properties of a geometric realization is preserved by projective transformations. Enforcing the corresponding incidences (or collinearity of incidence points) forces the eight points in (5) to be the vertices of a parallelepiped. This fixes a unique projective configuration, and there is no more freedom to impose the convergence of the remaining triple of lines ($b_3 c_2 d_1$). On the other hand, this last incidence is always automatically satisfied: this geometric result is equivalent to the fact the diagonals of a parallelepiped meet at a single point. All of these alignments guarantee that the 12 incidence points are vertices in a Reye configuration.

$\blacktriangleright$ Remark. The problem of determining whether or not a combinatorial incidence structure admits a “coordinatization” can be addressed algorithmically, by enforcing appropriate algebraic constraints on coordinates using a computer algebra system (see, e.g., [15]). We discovered the Reye realization of the incidence structure (I) using this method. This approach also yields a computational proof that there is in fact a unique projective realization.

5.2 Proof of Theorem [10]

We now prove Theorem [10] which states that any 3-consistent colored set of lines $L = L_1 \cup L_2 \cup L_3 \cup L_4$ in $\mathbb{R}^3$ with no colorful incidence and concurrent colors has at least 24 lines or is contained in a 2-plane. Let us start with a decomposition lemma:

$\blacktriangleright$ Lemma 17. Let $L = L_1 \cup L_2 \cup L_3 \cup L_4$ be a 3-consistent colored set of lines in $\mathbb{R}^3$ with no colorful incidence and concurrent color classes. If two color classes, each concurrent, are contained in two planes, then $L$ is contained in these planes and the subset of $L$ contained in each plane is 3-consistent.

Proof. Consider two color classes $L_1$ and $L_2$ contained in two planes $\Pi_1$ and $\Pi_2$. If all lines of $L_1$ are coplanar, then any line of $L \setminus L_1$ must intersect that plane into at least two distinct points, and the configuration is fully planar. Thus, each of $\Pi_1$ and $\Pi_2$ contains some lines of $L_1$ and $L_2$.

Every line $\ell \in L \setminus (L_1 \cup L_2)$ must be contained in $\Pi_1$ or $\Pi_2$. To see this, assume otherwise. Then $\ell$ intersects the two planes in at most two points. These are the only points in which $\ell$ can intersect the lines of $L_1 \cup L_2$. It must then be that both of these points are on a line of $L_1$ and on a line of $L_2$. Thus, any concurrence of $\ell$ and a line of $L_1$ lies on a line of $L_2$; we cannot have 3-consistency without a colorful incidence.

Now, observe that $\Pi_1 \cap \Pi_2 = c_1 c_2$. This implies that no 3-incidence can occur on that line, as this would force a line from $L_1$ to contain $c_2$ (or vice versa) and $c_2$ would be a 4-incidence. Thus, each of the subsets of $L$ contained in $\Pi_1$ and $\Pi_2$ is already 3-consistent.

We also use the following strengthening of Lemma [9].
Lemma 18. Let \( \mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{L}_4 \) be a 3-consistent colored set of lines in \( \mathbb{R}^d \) with \( |\mathcal{L}_1| = 2 \). If every color class is concurrent and \( \mathcal{L} \) contains no colorful incidence, then \( |\mathcal{L}_2| = |\mathcal{L}_3| = |\mathcal{L}_4| \) is even and at least 4.

Before we prove Lemma 18, let us see how it implies Theorem 10.

Proof of Theorem 10. First, observe that the 3-consistency and absence of colorful incidence implies the following fact: every line intersects every other color in at least two distinct points. Since \( |\mathcal{L}| < 24 \) there exists a color class of size at most 5, say \( \mathcal{L}_1 \).

Let \( c_2 \) denote the center of concurrency of \( \mathcal{L}_2 \). We examine the planes spanned by the lines of \( \mathcal{L}_1 \) with \( c_2 \). Any such plane must contain two lines of \( \mathcal{L}_1 \). Indeed, any line \( \ell_1 \in \mathcal{L}_1 \) intersects at least one line in \( \mathcal{L}_2 \), which in turn intersects at least another line \( \ell'_1 \) in \( \mathcal{L}_1 \). The planes spanned by the lines of \( \mathcal{L}_1 \) with \( c_2 \) thus coincide by pairs, so there are at most 2 such planes. Moreover, every line of \( \mathcal{L}_2 \) intersects some line of \( \mathcal{L}_1 \), and is therefore contained in one of these planes. By Lemma 17, \( \mathcal{L} \) decomposes into two disjoint subsets, each of which is also 3-consistent, without 4-incidence, and with concurrent color classes. Now, observe that each subset must have at least 12 lines: this is immediate for a subset with all color classes of size 3 or more, and follows from Lemma 18 for subsets with a color class of size 2.

5.3 Proof of Lemma 18

To analyze the situations where a color class has size two, it is convenient to switch to a dual setting. Given a colored point set \( P = P_1 \cup P_2 \cup P_3 \cup P_4 \), let a colorful alignment in \( P \) is a line containing points of \( P \) of all colors. We say that \( P \) is 3-consistent if for any point \( x \in P \) and any choice of 2 other colors, there is a line through \( x \) that contains a point of each of these colors. Now, a 4-colored set of lines \( \mathcal{L} \) in \( \mathbb{R}^d \) is 3-consistent if and only if projecting \( \mathcal{L} \) to a generic 2-plane and taking the dual of these projected lines yields a 3-consistent colored point set. Similarly, colorful incidences are mapped, by generic projection and duality, to colorful alignment.

Proof of Lemma 18. We prove the statement in its dual formulation. So, let \( P = P_1 \cup P_2 \cup P_3 \cup P_4 \) be a colored planar point set that is 3-consistent, has no colorful incidence and where each color class \( P_i \) is contained in a line, denoted \( \ell_i \). The 3-consistency and lack of colorful alignment in \( P \) implies that the \( P_i \) are pairwise disjoint and that for any \( i \neq j \) we have \( \ell_i \cap P_j = \emptyset \).

Let us write \( P_1 = \{ v, h \} \) and put \( P' = P_2 \cup P_3 \cup P_4 \). We define a graph \( G \) with vertex set \( P' \) and an edge in \( G \) between \( p \) and \( q \) if they have different colors and are aligned with a point of \( P_1 \). We orient the graph \( G \) by orienting the edges from \( P_2 \) to \( P_3 \), from \( P_3 \) to \( P_4 \), and from \( P_4 \) to \( P_2 \). Every vertex in \( G \) has exactly two edges, one incoming and one outgoing; this follows from the 3-consistency and the fact that \( \ell_i \cap P_j = \emptyset \) for \( i \neq j \). Thus, \( G \) is a disjoint union of cycles.

To every edge \( (p, q) \) in \( G \) we associate two parameters: the pair \( (i, j) \) of color classes of \( p \) and \( q \), respectively, and the vertex of \( P_1 \) that is collinear with \( p \) and \( q \). For \( (i, j) \in \{(2,3), (3,4), (4,2)\} \) and \( c \in P_1 \) we let \( p^c_{i,j} \) denote the projection from \( \ell_i \) to \( \ell_j \) with center \( c \). Let \( T \) denote the set of vertices in \( P_3 \) that are aligned with their successors and the point \( h \) from \( P_1 \). Along a cycle, \( (i, j) \) is 3-periodic, by the choice of orientation of the edges, and \( x \) is 2-periodic, by 3-consistency and lack of 4-alignment. Thus, every cycle in \( G \) has length some multiple of 6 and has a vertex in \( T \). Moreover, this vertex must be a fixed point of some
power of the function $f$ defined by:

$$f : \begin{cases} \ell_3 \to \ell_3 \\
(x, y) \mapsto p_{2,3}^v \circ p_{3,4}^v \circ p_{2,3}^h \circ p_{4,2}^v \circ p_{3,4}^h(x, y) \end{cases}$$

To compute $f$ it is convenient to apply a projective transform that maps $v$ and $h$ to infinity, to respectively the vertical and horizontal directions. This preserves incidences in $P$, being understood that a line contains $v$ (resp. $h$) if and only if it is vertical (resp. horizontal). In particular, none of $\ell_2, \ell_3$ or $\ell_4$ is horizontal or vertical, so we can parametrize $\ell_i$ by $(x, \alpha_i x + \beta_i)$. We then have:

$$p_{i,j}^v(x, y) = (x, \alpha_i x + \beta_j) \quad \text{and} \quad p_{i,j}^h(x, y) = \left(\frac{y - \beta_j}{\alpha_j}, y\right).$$

Let us put the origin of our frame at $\ell_2 \cap \ell_3$ so that $\beta_2 = \beta_3 = 0$. An elementary computation then yield that the point $f(x, \alpha_3 x + \beta_3)$ has $x$-coordinate $x + c$ where

$$c = \frac{\alpha_3 - \alpha_2}{\alpha_3 \alpha_2} \beta_4.$$

If $c \neq 0$ then no power of $f$ has a fixed point, so we must have $c = 0$. Since $\alpha_2 \neq \alpha_3$, we must have $\beta_4 = 0$ and $\ell_4$ goes through $\ell_2 \cap \ell_3$.

Altogether, we proved that the lines supporting $P_2$, $P_3$ and $P_4$ are concurrent, and that $P'$ decomposes into some number $r$ of 6-cycles $\gamma_1, \ldots, \gamma_r$ of the form

$$a \in \ell_3 \to b \in p_{3,4}^h(a) \to c \in p_{4,2}^v(b) \to d \in p_{2,3}^v(c) \to e \in p_{3,4}^h(d) \to f \in p_{4,2}^v(e) \to a \in p_{2,3}^v(f).$$

In particular, $|P_2| = |P_3| = |P_4| = 2r$. Lemma 9 already established that we must have $r > 1$.

Interestingly, for every $r > 1$, there exists a 3-consistent configuration with $|P_1| = 2$ and $|P_2| = |P_3| = |P_4| = 2r$ that has no colorful alignment.

References

1. Kalle Åström, Roberto Cipolla, and Peter J Giblin. Generalised epipolar constraints. In European Conference on Computer Vision, pages 95–108. Springer, 1996.
2. Guillaume Batog, Xavier Goaoc, and Jean Ponce. Admissible linear map models of linear cameras. In Computer Vision and Pattern Recognition (CVPR), 2010 IEEE Conference on, pages 1578–1585. IEEE, 2010.
3. Edmond Boyer. On using silhouettes for camera calibration. Computer Vision–ACCV 2006, pages 1–10, 2006.
4. Paul Erdős and George Purdy. Some extremal problems in geometry. Discrete Math, pages 305–315, 1974.
5. Olivier Faugeras and Bernard Mourrain. On the geometry and algebra of the point and line correspondences between n images. In Computer Vision, 1995. Proceedings., Fifth International Conference on, pages 951–956. IEEE, 1995.
6. Jacob Fox, János Pach, Adam Sheffer, Andrew Suk, and Joshua Zahl. A semi-algebraic version of Zarankiewicz’s problem. Journal of the European Mathematical Society, 19:1785–1810, 2017.
7. Ben Green and Terence Tao. On sets defining few ordinary lines. Discrete & Computational Geometry, 50(2):409–468, 2013.
Consistent sets of lines with no colorful incidence

8 Harald Gropp. Configurations and their realization. *Discrete Mathematics*, 174(1-3):137–151, 1997.
9 Larry Guth. Ruled surface theory and incidence geometry. In *A Journey Through Discrete Mathematics*, pages 449–466. Springer, 2017.
10 Larry Guth and Nets Hawk Katz. Algebraic methods in discrete analogs of the Kakeya problem. *Advances in Mathematics*, 225(5):2828–2839, 2010.
11 Richard Hartley and Andrew Zisserman. *Multiple view geometry in computer vision*. Cambridge university press, 2003.
12 Carlos Hernández, Francis Schmitt, and Roberto Cipolla. Silhouette coherence for camera calibration under circular motion. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 29(2), 2007.
13 Onur Özyeşil, Vladislav Voroninski, Ronen Basri, and Amit Singer. A survey of structure from motion. *Acta Numerica*, 26:305–364, 2017.
14 József Solymosi and Miloš Stojaković. Many collinear k-tuples with no k + 1 collinear points. *Discrete & Computational Geometry*, 50(3):811–820, 2013.
15 Bernd Sturmfels. Computational algebraic geometry of projective configurations. *Journal of symbolic computation*, 11(5-6):595–618, 1991.
16 Matthew Trager, Martial Hebert, and Jean Ponce. Consistency of silhouettes and their duals. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, pages 3346–3354, 2016.
17 Matthew Trager, Bernd Sturmfels, John Canny, Martial Hebert, and Jean Ponce. General models for rational cameras and the case of two-slit projections. In *CVPR 2017 - IEEE Conference on Computer Vision and Pattern Recognition*, Honolulu, United States, July 2017. URL: [https://hal.archives-ouvertes.fr/hal-01506996](https://hal.archives-ouvertes.fr/hal-01506996).
18 Oswald Veblen and John Wesley Young. *Projective geometry*, volume 1. Ginn, 1918.