AN INVARIANT SET IN ENERGY SPACE
FOR SUPERCRITICAL NLS IN 1D

Scipio Cuccagna

Abstract. We consider radial solutions of a mass supercritical monic NLS and we
prove the existence of a set, which looks like a hypersurface, in the space of finite
ergy functions, invariant for the flow and formed by solutions which converge to
ground states.

§1 Introduction

We consider a monic supercritical NLS

\begin{equation}
    iu_t + u_{xx} + |u|^{p-1}u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \quad u(t,x) \equiv u(t,-x), \quad 5 < p < \infty.
\end{equation}

We ignore translation and consider only even solutions $u(t,x) \equiv u(t,-x)$ of (1.1): by

\begin{equation}
    H^1_r(\mathbb{R}, \mathbb{C})
\end{equation}

we will mean the space of finite energy even functions. (1.1) admits ground
states solutions $e^{i\omega t + i\gamma \phi_\omega(x)}$, with $\phi_\omega(x) = \omega^{\frac{1}{2(p-1)}(\frac{p+1}{2})}\text{sech}^{\frac{p}{p-2}}(\frac{p-1}{2}\sqrt{\omega x})$. Let

\begin{equation}
    G = \{e^{i\gamma \phi_\omega(x)} : \omega > 0; \gamma \in \mathbb{R}\} \subset H^1_r(\mathbb{R}, \mathbb{C}).
\end{equation}

For any initial datum $u(0,x) \in H^1_r(\mathbb{R}, \mathbb{C})$ close to $G$, for some time the corresponding
solution $u(t,x)$ remains close to $G$ and can be written in a canonical way as a varying
ground state plus a reminder term:

\begin{equation}
    u(t,x) = e^{i \int_0^t \omega(s)ds + i\gamma(t)\phi_{\omega(t)}(x) + r(t,x)}.
\end{equation}

The orbits in $G$ are unstable and $u(t,x)$ can blow up in finite time [BC], so (1.3) in
general does not persist for all $t$. We will prove:

Theorem 1.1. There exist a $X \subset H^1_r(\mathbb{R}, \mathbb{C})$ such that:

- $G \subset X$;
- $X$ is invariant by the flow;
- $X$ looks like a hypersurface, in the following sense: for any $g_0 \in G$ there exists
  a neighborhood $U$ of $g_0$ in $H^1_r(\mathbb{R}, \mathbb{C})$ such that there is $\tilde{X} \subseteq X \cap U$ with $\tilde{X}$ the
graph of a real valued function, non necessarily continuous, defined on a real closed hyperplane through \( g_0 \) in \( H^1_r(\mathbb{R}, \mathbb{C}) \):

- For any \( g_0 = e^{it_0} \phi_{\omega_0}(x) \in G \) there are \( C > 0 \) and \( \epsilon_0 > 0 \), which depend only on \( \omega_0 \), such that for any \( 0 < \epsilon < \epsilon_0 \) if we pick \( u_0 \in X \) with \( \| u_0 - g_0 \|_{H^1(\mathbb{R})} < \epsilon \) then the corresponding solution \( u(t, x) \) is globally defined and contained in \( X \), can be written in a canonical way in the form (1.3), and we have

\[
(1) \quad \| r(t) \|_{H^1(\mathbb{R}, \mathbb{C})} + \| r \|_{L^4_t(\mathbb{R}; L^\infty_x(\mathbb{R}, \mathbb{C}))} + \| (\omega_0, \gamma_0) - (\omega(t), \gamma(t)) \| < C \epsilon.
\]

The limit

\[
(2) \quad \lim_{t \to +\infty} (\omega(t), \gamma(t)) = (\omega_\infty, \gamma_\infty)
\]

exists and there exists \( r_\infty \in H^1_r(\mathbb{R}, \mathbb{C}) \) with \( \| r_\infty \|_{H^1(\mathbb{R}, \mathbb{C})} < C \epsilon \) such that

\[
(3) \quad \lim_{t \to +\infty} \| e^{i \int_0^t \omega(\tau)d\tau + i \gamma(t)} r(t) - e^{it \partial_x^2} r_\infty \|_{H^1(\mathbb{R}, \mathbb{C})} = 0.
\]

**Remark.** In the subspace of \( H^1_r \times H^1_r \) formed by pairs \((u, \overline{\pi})\), the hyperplane at \( g_0 = e^{it_0} \phi_{\omega_0}(x) \) is spanned by \( N_g(H_{\omega_0}) \oplus \mathbb{R} \sigma_1(\omega_0) \oplus L^2_c(H_{\omega_0}) \), with the various terms introduced in §2.

**Remark.** We emphasize that all the functions considered in this paper are even in \( x \).

Theorem 1.1 is related Tsai & Yau [TY], Schlag [S] and Krieger & Schlag [KS]. [KS] for (1.1) proves the existence of a Lipschitz hypersurface of initial data \( u_0 \) with \( \langle x \rangle u_0 \in H^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R}) < \infty \), such that the corresponding solutions \( u(t, x) \) converge to ground states. The stronger decay hypothesis on the initial data allows to control the rate of convergence of \( \omega(t) \) to its limit, and also the rate of convergence of the motion of the ground state to the inertial asymptotic motion. For data \( H^1(\mathbb{R}) \) or in the smaller space \( H^1_r(\mathbb{R}) \) the method in [KS] does not work. We consider only even initial data to eliminate spatial motion of the ground state. So the velocity is zero and we trivialize one of the difficulties. The problem with \( \omega(t) \) however remains. We obtain our result by means of Schauder fixed point theorem applied to an appropriate functional. Unfortunately, due to the fact that \( u_0 \in H^1(\mathbb{R}) \) and to the lack of sufficient control on \( \omega(t) \), we are not able to show that the functional is a contraction, which would yield \( X = \tilde{X} \) and some regularity for the hypersurface. It would be nice to prove that \( X \) is a continuous hypersurface, and then, given a small ball \( B \subset H^1_r(\mathbb{R}) \) of center \( g \in G \), to study the behavior of solutions which start in \( B \setminus X \). During the review process of this paper we learned of the work by Beceanu [B] which proves an analogous result to the present one for solutions \( u(t) \in H^1(\mathbb{R}^3) \cap L^2,1(\mathbb{R}^3) \), in the notation below, for the cubic NLS treated in
The result in [B] is stronger than ours in two respects: $X$ is indeed a Lipschitz hypersurface in $H^1(\mathbb{R}^3) \cap L^{2,1}(\mathbb{R}^3)$, and there is no requirement of spherical or other symmetries. The proof in [B] does not work in our 1 dimensional setting for solutions $u(t) \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$. We remark that the endpoint Strichartz estimate needed in [B] is a corollary of the transposition to linearizations of the NLS of the following material: Yajima’s $L^p$ theory of wave operators [Y1,Y2] transposed in [C4,C5]; Kato smoothness theory [K], applied in Proposition 4.1 [CPV]. Furthermore, in cases when they cannot be derived directly from bounds on wave operators, as for example Lemma 3.1 below, Strichartz estimates for the linearization $H_\omega$ in (2.2) can be proved with a standard $TT^*$ argument, using an appropriate bilinear form, see the proof of Lemma 3.1 in [C1,C3]. For other results related to the present paper see [Co, Ma] and references therein.

In the last section we list a series of errata in paper [C1]. In particular the present paper is based on [C3], which is a thorough revision of [C1].

We write $R_H(z) = (H - z)^{-1}$ and $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. We set $\|u\|_{H^{k,s}} := \|\langle x \rangle u\|_{H^k}$. We set $L^{2,s} = H^{0,s}$. We set $\langle f, g \rangle = \int f(x)g(x)dx$, with $f(x)$ and $g(x)$ column vectors and with $^tA$ the transpose. $W^{1,p}(\mathbb{R})$ is the set of tempered distributions $f(x)$ with derivative $f(x), f'(x) \in L^p(\mathbb{R})$. $W^{k,p}(\mathbb{R})$ is the space of tempered distributions $f(x)$ such that $(1 - \partial_x^2)^{k/2} f \in L^p(\mathbb{R})$. Recall that $W^{1,p}(\mathbb{R}) = W^{k,p}(\mathbb{R})$ exactly for $1 < p < \infty$. $\stackrel{\perp}{W}^{1,p}(\mathbb{R})$ is the set of tempered distributions $f(x)$ with derivative $f'(x) \in L^p(\mathbb{R})$.

§2 Linearization and Spectral Decomposition

We plug the ansatz (1.3) in (1.1) obtaining, for $n(r, \tau) = O(r^2), \quad (2.1)$

$$ir_t = -r_{xx} + \omega(t)r - \frac{p+1}{2} \phi_{\omega(t)}^{-1}r(t, y) - \frac{p-1}{2} \phi_{\omega(t)}^{p-1}r + \dot{\gamma}(t) (\phi_{\omega(t)} + r) - i\dot{\omega}(t) \partial_\omega \phi_{\omega(t)} + n(r, \tau)$$

Let $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The linearization is

$$H_\omega = \sigma_3 (-d^2/dx^2 + \omega) + \omega V(\sqrt{\omega}x)$$

(2.2)

$$V(x) = -(\sigma_3(p+1) - i\sigma_2(p-1))(p+1)2^{-2}\text{sech}^2\left(\frac{x}{2}\right).$$

By (2.1), for $^tR = (r, \bar{r}), ^t\Phi = (\phi_\omega, \phi_\omega)$ and $^tN(R) = (n(r, \tau), \bar{n}(r, \bar{r}))$

$$iR_t = H_{\omega(t)}R + \sigma_3 \dot{\gamma}R + \sigma_3 \dot{\gamma} \Phi - i\dot{\omega} \partial_\omega \Phi + N(R).$$

(2.3)

By implicit function theorem we impose $R(t) \in N_g(H_{\omega(t)}^*)$, with $N_g$ the generalized kernel. We state the following known result:
The continuous spectrum of $N$ is $\mathbb{R}\setminus(-\omega, \omega)$. 0 is an eigenvalue and there are two simple eigenvalues $\pm i\mu(\omega)$, with $\mu(\omega) > 0$.

$N_g(H_\omega)$ is spanned by $\{\sigma_3\Phi_\omega, \partial_\omega \Phi_\omega, \partial_x \Phi_\omega, \sigma_3 x \Phi_\omega\}$.

(3) $\pm \omega$ are not resonances and $\{0, i\mu(\omega), -i\mu(\omega)\}$ are the only eigenvalues.

For (1) and (2) see [W], for (3) see [KS]. Let $\xi(\omega, x)$ be an eigenvector of $i\mu(\omega)$. Notice that $\mu(\omega) = \omega\mu(1)$. Recalling that $\langle f, g \rangle = \int \Re f(x)g(x)dx$, we have:

**Lemma 2.2.** The eigenvector $\xi(\omega, x)$ can be chosen so that $\langle \xi(\omega), \sigma_3 \xi(\omega) \rangle = i\lambda_1$ with $\lambda_1 \in \mathbb{R}\setminus\{0\}$ a fixed number. The function $(\omega, x) \rightarrow \xi(\omega, x)$ is $C^2$ and $|\xi(\omega, x)| < c\sqrt{\omega e^{-a|\omega|}}$ for fixed $c > 0$ and $a > 0$. $\sigma_1 \xi(\omega, x) = \xi(\omega, x)$ generates $\ker(H_\omega + i\mu(\omega))$ with $\langle \sigma_1 \xi, \sigma_3 \sigma_1 \xi \rangle = -i\lambda_1$. We have $H_\omega$ invariant decompositions

(1) \[ L^2(\mathbb{R}, \mathbb{C}^2) = L_d^2(\omega) \oplus L^c_\omega(\omega) \quad \text{and} \quad L^2(\mathbb{R}, \mathbb{C}^2) = N_g(H_\omega) \oplus N_g^\perp(H_\omega^*) \]

with $L_d^2(\omega) = N_g(H_\omega) \oplus \left( \oplus_\pm \ker(H_\omega \mp i\mu(\omega)) \right)$ and $L^2_\omega(\omega) = \left[ \sigma_3 L^2_d(\omega) \right] \perp$.

**Proof.** The decomposition (1) is a consequence of Theorem 2.1. Let $\xi(x)$ be a generator of $\ker(H_1 - i\mu(1))$. Since both $\xi(x)$ and $\sigma_1 \xi(x) \in \ker(H_1 + i\mu(1))$, we can normalize $\xi(x)$ so that $\xi(x) = \sigma_1 \xi(x)$. Then $\hat{\xi}(x) = \langle v(x), \mathfrak{m}(x) \rangle$. Then $\langle \xi, \sigma_3 \xi \rangle = \int (\dot{v}^2 - \mathfrak{m}^2)dx = i\lambda_1$ with $\lambda_1 \in \mathbb{R}\setminus\{0\}$. Notice that $\lambda_1 \neq 0$ since otherwise $\langle \xi, \sigma_3 f \rangle = 0$ for any $f$ would follow from the fact that $\langle \xi, \sigma_3 f \rangle = 0$ for any $f \in N_g(H_1) \oplus L^2_\omega(1)$ and for $f = \sigma_1 \xi$. Finally set $\xi(\omega, x) = \sqrt{\omega} \xi(1, \sqrt{\omega} x)$. The rest is standard.

We denote by $P_d(\omega)$ (resp. $P_c(\omega)$) the projection on $L^2_d(\omega)$ (resp. $L^2_c(\omega)$) associated to the splitting in (1) Lemma 2.2. By $N_g(H^*_\omega) = \sigma_3 N_g(H_\omega)$, the condition $R(t, x) \in N_g^\perp(H^*_\omega(t))$ and (2.3) imply the modulation equations:

\[
\begin{align*}
&i\omega \ d(||\phi_\omega||^2_2) / d\omega = i\omega \langle R, \partial_\omega \Phi_\omega \rangle + \langle \sigma_3 \hat{\gamma} R + N(R), \Phi_\omega \rangle \\
&\hat{\gamma} \ d(||\phi_\omega||^2_2) / d\omega = i\omega \langle R, \sigma_3 \partial_\omega^2 \Phi_\omega \rangle - \langle \sigma_3 \hat{\gamma} R + N(R), \sigma_3 \partial_\omega \Phi_\omega \rangle.
\end{align*}
\]

By elementary computations, see [C2], there are real valued exponentially decreasing functions $\alpha(\omega, x)$ and $\beta(\omega, x)$ such that

\[
\mathcal{M}(\omega, R) \begin{bmatrix} i\omega \\ -\hat{\gamma} \end{bmatrix} = \begin{bmatrix} \langle n(r, \mathfrak{m}) - n(\mathfrak{m}, r), \phi_\omega \rangle \\ \langle n(r, \mathfrak{m}) + n(\mathfrak{m}, r), \partial_\omega \phi_\omega \rangle \end{bmatrix} \quad \text{with} \quad \mathcal{M}(\omega, R) = d(||\phi_\omega||^2_2) / d\omega + \begin{bmatrix} \langle r + \mathfrak{m}, \alpha(\omega) \rangle \\ \langle r - \mathfrak{m}, \beta(\omega) \rangle \end{bmatrix} \begin{bmatrix} \langle r - \mathfrak{m}, \phi_\omega \rangle \\ \langle r + \mathfrak{m}, \partial_\omega \phi_\omega \rangle \end{bmatrix}.
\]
Lemma 2.3. We can write \( R(t) = f(t) + \zeta(t) \) with \( f(t) \in L^2_c(\omega(t)) \) and \( \zeta(t, x) = z_+(t)\xi(\omega(t), x) + z_-\sigma_1 \xi(\omega(t), x) \). \( R = \sigma_1 R \) implies \( z_\pm(t) \in \mathbb{R} \) and \( f = \sigma_1 f \).

Proof. By \( R(t, x) \in \mathcal{N}_g(H^*_\omega(t)) \) and setting \( f = P_c(\omega(t))R \) we get \( R(t) = f(t) + \zeta(t) \) for an \( \zeta(t) = z_+(t)\xi(\omega(t)) + z_-\sigma_1 \xi(\omega(t)) \). \( z_\pm(t) \in \mathbb{R} \) and \( f = \sigma_1 f \) follow by \( \xi = \sigma_1 \tilde{x}, \sigma_1 \xi = \tilde{\xi}, \sigma_1 L^2_c(\omega(t)) = L^2_c(\omega(t)) \) and

\[
R(t) = f(t) + \zeta(t) = \sigma_1 R = \sigma_1 f = \sigma_1 f = \sigma_1 f.
\]

We have from (2.3) and Lemma 2.3

\[
\begin{align*}
if_t &= H_{\omega(t)}f + \sigma_3 \dot{\zeta}R + N(R) + \sigma_3 \Phi_{\omega(t)} - i\dot{\zeta} \Phi_{\omega(t)} \\
&+ i(z_+\mu(\omega(t)) - \dot{z}_+\xi(\omega(t)) - i(z_-\mu(\omega(t)) + \dot{z}_-\sigma_1 \xi(\omega(t))) \\
&- i\dot{\zeta}(z\dot{\xi}(\omega(t)) + z\sigma_1 \dot{\xi}(\omega(t))).
\end{align*}
\]

We apply \( \langle \cdot, \sigma_3 \xi \rangle \) and \( \langle \cdot, \sigma_3 \sigma_1 \xi \rangle \). Setting \( d_1 = -\lambda_1^{-1} \) with \( \lambda_1 \) the constant in Lemma 2.2, we get the discrete mode equations:

\[
\dot{z}_\pm(t) = \mu(\omega(t))z_\pm(t) = d_1 i\dot{\zeta}(f(t), \sigma_3 \sigma_1^{\frac{1}{2}} \partial_\omega \xi(\omega(t))) +
\]

\[
d_1 (\sigma_2 \dot{\zer}R + N(R) - i\dot{\zeta}(z_+(t) + z_-(t) \sigma_1 \partial_\omega \xi(\omega(t)), \sigma_3 \sigma_1^{\frac{1}{2}} \xi(\omega(t))).
\]

We fix an \( \omega_0 \). Setting \( \omega = \omega(t) \) and \( \ell(t) = \omega(t) - \omega_0 + \dot{\zeta}(t) \) we get

\[
[i\partial_t - (H_{\omega_0} + \ell(t)P_c(\omega_0)\sigma_3)] f = P_c(\omega)\sigma_3 \dot{\zeta}(z_+ + z_\sigma_1) \xi + N(R) +
\]

\[
(\omega_0 V(\sqrt{\omega_0} x) - \omega V(\sqrt{\omega} x)) f + i\partial_\omega P_c(\omega)f + \ell(t) (P_c(\omega) - P_c(\omega_0)) \sigma_3 f.
\]

To correct the fact that \( [P_c(\omega_0)\sigma_3, H_{\omega_0}] \neq 0 \), we split \( f \in L^2_c(\omega(t)) \) into

\[
f = f_d + f_c \in L^2(\omega_0) \oplus L^2_c(\omega_0).
\]
Then splitting $P_c(\omega_0) = P_+(\omega_0) + P_-(\omega_0)$, with the two terms the projections on the positive and the negative part of the continuous spectrum, see Lemma 5.12 [C1] or Appendix B [C3] or also [BP], we get

$$
\begin{align*}
& [i\partial_t - (H_{\omega_0} + \ell(t)(P_+(\omega_0) - P_-(\omega_0)))] f_c = P_c(\omega_0)\{P_c(\omega_0)\sigma_3 \gamma(z_+ + z_-\sigma_1)\xi
\end{align*}
$$
\tag{2.9}

+ N(R) + (\omega_0 V(\sqrt{\omega_0}x) - \omega V(\sqrt{\omega}x)) f + i\omega \partial_\omega P_c(\omega)f
\\
& + \ell(t) (P_c(\omega) - P_c(\omega_0))\sigma_3 f + \ell(t)(P_+(\omega_0) - P_-(\omega_0) - P_c(\omega_0)\sigma_3) f_c \}. 
\end{align*}

Now $[P_+(\omega_0) - P_-(\omega_0), H_{\omega_0}] = 0$. We will use the following elementary lemma.

**Lemma 2.4.** Fix $\alpha \in (0, 1)$. Then there exists a small $\delta(\alpha) > 0$ such that for any fixed $\omega_0 \in (\alpha, 1/\alpha)$ and for any $\omega$ with $|\omega - \omega_0| \leq \delta(\alpha)$ there exist constants $C_N(\alpha)$ such the following holds: for any $f_c \in L^2_c(\omega_0)$ there exists exactly one $f_d \in L^2_d(\omega_0)$ so that $f = f_c + f_d \in L^2_c(\omega)$ and for any $q \in [1, \infty]$ we have

$$
\|f_d\|_{L^q_c} \leq C_N(\alpha)|\omega - \omega_0| \|\langle x\rangle^{-N} f_c\|_{L^2_c}.
\tag{2.10}
$$

Furthermore, if $\overline{f_c} = \sigma_1 f_c$, then we have $\overline{f_d} = \sigma_1 f_d$.

**Proof.** For $f_c = P_c(\omega)f_c + P_c(\omega)f_c$ we seek $f_d \in L^2_d(\omega_0)$ with $P_d(\omega)f_d = -P_d(\omega)f_c$. We have $P_d(\omega)P_d(\omega_0) = P_d(\omega_0) + (P_d(\omega) - P_d(\omega_0))P_d(\omega_0)$. Since $P_d(\omega) - P_d(\omega_0) = O(\omega - \omega_0)$ in any norm, we see for the ranks, $\text{Rk}(P_d(\omega)P_d(\omega_0)) = \text{Rk}(P_d(\omega_0)) = \text{Rk}(P_d(\omega))$, so $P_d(\omega)P_d(\omega_0) : L^2_c(\omega_0) \to L^2_d(\omega)$ is an isomorphism and $f_d$ exists unique. Next,

$$
-P_d(\omega)f_c = (P_d(\omega_0) - P_d(\omega))f_c = f_d + (P_d(\omega) - P_d(\omega_0))f_d
\end{align*}
$$
implies $\|f_d\|_q(1 - C|\omega - \omega_0|) \leq \|(P_d(\omega_0) - P_d(\omega))f_c\|_q \lesssim |\omega - \omega_0| \|\langle x\rangle^{-N} f_c\|_{L^2_c}$.

Let $J$ be either $\sigma_1$ or the conjugation operator $Jh = \overline{h}$. Then, in either case $[P_d(\omega), J] = [P_c(\omega), J] = 0$ for any $\omega$. This implies $\overline{f_d} = \sigma_1 f_d$.

§3 Spacetime estimates for $H_\omega$

We will need the following estimates, proved in [C3].

**Lemma 3.1 (Strichartz estimate).** Let $W^{k,p}(\mathbb{R})$ be the space of tempered distributions $f(x)$ such that $(1 - \partial_x^2)^{k/2} f \in L^p(\mathbb{R})$. Then there exists a positive number $C = C(\omega)$ upper semicontinuous in $\omega$ such that for any $k \in [0, 2]$: 

(a) for any $f \in L^2_c(\omega)$, $\|e^{-itH_\omega} f\|_{L^1_tW^{k,\infty}_x \cap L^\infty_tH^k_x} \leq C\|f\|_{H^k}$

(b) for any $g(t, x) \in S(\mathbb{R}^2)$,

$$
\| \int_0^t e^{-i(t-s)H_\omega} P_c(\omega)g(s, \cdot)ds\|_{L^1_tW^{k,\infty}_x \cap L^\infty_tH^k_x} \leq C\|g\|_{L^{4/3}_tW^{k,1}_x + L^1_tH^k_x}.
$$
Lemma 3.2. For any $k$ and $\tau > 3/2 \exists C = C(\tau, k, \omega)$ upper semicontinuous in $\omega$ such that:
(a) for any $f \in S(\mathbb{R})$,
\[ \|e^{-it\omega} P_c(H_\omega)f\|_{L_t^2 H_x^{k,-\tau}} \leq C \|f\|_{H^k}. \]
(b) for any $g(t, x) \in S(\mathbb{R}^2)$
\[ \left\| \int_{\mathbb{R}} e^{it\omega} P_c(H_\omega)g(t, \cdot)dt \right\|_{H_x^k} \leq C \|g\|_{L_t^2 H_x^{k,\tau}}. \]

Lemma 3.3. For any $k$ and $\tau > 3/2 \exists C = C(\tau, k, \omega)$ as above such that $\forall g(t, x) \in S(\mathbb{R}^2)$
\[ \left\| \int_0^t e^{-i(t-s)\omega} P_c(H_\omega)g(s, \cdot)ds \right\|_{L_t^2 H_x^{k,-\tau}} \leq C \|g\|_{L_t^2 H_x^{k,\tau}}. \]

Lemma 3.4. $k$ and $\tau > 3/2 \exists C = C(\tau, k, \omega)$ as above such that $\forall g(t, x) \in S(\mathbb{R}^2)$
\[ \left\| \int_0^t e^{-i(t-s)\omega} P_c(H_\omega)g(s, \cdot)ds \right\|_{L_t^\infty L_x^2 \cap L_t^1(\mathbb{R}, W_x^{k,\infty})} \leq C \|g\|_{L_t^2 H_x^{k,\tau}}. \]

Lemma 3.5. In Lemmas 3.1 (b), 3.3 and 3.4 the estimates continue to hold if we replace in the integral $[0, t]$ with $[t, +\infty)$.

§4 Functional setting and integral formulation

From now on in the paper all the functions we consider are even in $x$. We want to build a set $X$ of special solutions of (1.1) which for all times are approximate ground states $u(t, x) = e^{itf_0^t \omega(s)ds+i\gamma(t)}(\phi_\omega(t)(x) + r(t, x))$ as in Ansatz (1.3). The reminder $^tR = (r, \tau)$ will be split as
\[ R = (z_+(t) + z_-(t)\sigma_1)\xi(\omega(t), x) + f_d(t, x) + f_c(t, x) \]
with $f_d + f_c$ the splitting in (2.8). In analogy to standard constructions of center and stable manifolds, we consider functional spaces where we will interpret $X$ as the set of fixed points of certain functionals.

For $p > 5$ the exponent in (1.1) and for $4/q = 1 - 1/p$ we set
\[ Z := L_t^1 L_x^\infty \cap L_t^4 W_x^{1,2p} \cap L_t^\infty H_x^1 \cap C_t^0 H_x^1 \cap H_x^{1,-2} L_t^2([0, \infty) \times \mathbb{R}, \mathbb{C}^2); \]
\[ \hat{X} := \{(z_+(t), z_-(t), \gamma(t), f_c(t, x)) : z_\pm(t) \in (L^1 \cap L^{\infty} \cap C^0)([0, \infty), \mathbb{R}) ; f_c(t, \cdot) \in L_c^2(\omega_0) \cap Z \text{ with } \int_{\mathbb{R}} f_c = \sigma_1 f_c; \gamma \in (L^1 \cap L^{\infty})([0, \infty), \mathbb{R}) \} \]
with, for $\hat{R} = (z_+, z_-, \gamma, f_c)$,

$$\|\hat{R}\|_\mathcal{X} = \|(z_+, z_-)\|_{(L^1 \cap L^\infty)[0, \infty)} + \|\gamma\|_{(W^1, \infty \cap W^{1,1})[0, \infty)} + \|f_c\|_Z.$$

Let $\mathcal{X} := ((W^{1,\infty} \cap \dot{W}^{1,1})[0, \infty)) \times \hat{\mathcal{X}}$ with elements $\mathcal{R} = (\omega, \hat{R})$. Fix $\alpha \in (0, 1)$ and $\omega_0 \in (\alpha, 1/\alpha)$. For $\epsilon \in (0, \epsilon_0]$ let

$$B_X(\omega_0, \gamma_0, \epsilon) = \{\mathcal{R} \in \mathcal{X} : \|\mathcal{R} - (\omega_0, 0, 0, \gamma_0, 0)\|_{\mathcal{X}} \leq \epsilon\}$$
$$B_{\hat{\mathcal{X}}}(\gamma_0, \epsilon) = \{\hat{R} \in \hat{\mathcal{X}} : \|\hat{R} - (0, 0, 0)\|_{\hat{\mathcal{X}}} \leq \epsilon\}.$$

For $\epsilon_0 < \delta(\alpha)$, with $\delta(\alpha)$ chosen to be the same of Lemma 2.4, in $B_X(\omega_0, \gamma_0, \epsilon)$ by definition we have $\|\omega(t) - \omega_0\|_\infty < \delta(\alpha)$. By Lemma 2.4 we define $f_d(t, x) \in L^2(\omega_0)$ with $|f_d| \ll |f_c|$, so that $f(t, x) = f_d(t, x) + f_c(t, x) \in L^2(\omega(t))$ and $\|f\|_Z \approx \|f_c\|_Z$. Then given $\mathcal{R} \in B_X(\omega_0, \gamma_0, \epsilon)$ we define $R(t, x, \mathcal{R})$ by formula (4.1). By construction, $R(t) \in N^*_g(H^{s*}(\omega))$ and $\|R\|_Z \leq C(\|(z_+, z_-)\|_{L^1 \cap L^\infty} + |f_c|_Z)$ for $C = C(\alpha)$. We fix $\omega(0) > 0$ (resp.$\gamma(0) \in \mathbb{R}$) close to $\omega_0$ (resp.$\gamma_0$) and for $R = R(\mathcal{R})$ we write

$$\text{(4.2)} \quad \omega(t) = \omega(0) + \tilde{\omega}(\mathcal{R}), \quad \tilde{\omega}(\mathcal{R}) := \int_0^t \tilde{\omega}(\mathcal{R})(s)ds,$$
$$\text{(4.3)} \quad \gamma(t) = \gamma(0) + \tilde{\gamma}(\mathcal{R}), \quad \tilde{\gamma}(\mathcal{R}) := \int_0^t \tilde{\gamma}(\mathcal{R})(s)ds,$$

where $\tilde{\omega}(\mathcal{R})$ and $\tilde{\gamma}(\mathcal{R})$ are as in (2.5). Schematically we have for $\mathcal{R} \in B_X(\omega_0, \gamma_0, \epsilon)$

$$\text{(4.4)} \quad \tilde{\omega}(\mathcal{R}) := \langle O(R^2(t)), \Phi_{\omega(t)} \rangle \quad \text{and} \quad \tilde{\gamma}(\mathcal{R}) := \langle O(R^2(t)), \partial_\omega \Phi_{\omega(t)} \rangle.$$ 

Let $\ell(t, \mathcal{R}) = \omega(t) - \omega_0 + \tilde{\gamma}(\mathcal{R})$. For a small $h_0 \in H^1(\mathbb{R}, C^2) \cap L^2(\omega_0)$, $h_0(-x) = h_0(x)$, with $h_0 = \sigma_1 h_0$, we write

$$\text{(4.5)} \quad P_{\pm}(\omega_0)f_c(t, x) = e^{-itH_{\omega_0}}e^{\mp i\int_0^t \ell(\tau, \mathcal{R})d\tau} P_{\pm}(\omega_0)h_0(x) - \tilde{f}_c(\mathcal{R}),$$
$$\tilde{f}_c(\mathcal{R}) := \int_0^\infty e^{-i(t-s)H_{\omega_0}}e^{\mp i\int_s^t \ell(\tau, \mathcal{R})d\tau} P_{\pm}(\omega_0)F(\mathcal{R})(s)ds,$$
$$F(\mathcal{R}) := P_c(\omega_0) \{ P_c(\omega)\sigma_3 \tilde{\gamma}(\mathcal{R})(z_+ + z_-\sigma_1)\xi + N(\mathcal{R}) + i\tilde{\omega}(\mathcal{R})\partial_\omega P_c(\omega)f \}
+ \ell(t, \mathcal{R})(P_c(\omega) - P_c(\omega_0))\sigma_3 f
+ \ell(t, \mathcal{R})(P_+(\omega_0) - P_-(\omega_0) - P_c(\omega_0)\sigma_3) f_c \}.$$
We write
\[ z_+(t) = \overline{z}_+(\mathcal{R}), \text{ where } \overline{z}_+(\mathcal{R}) := \]
(4.6) \[ d_1 \int_t^\infty ds e^{\int_s^t \mu(\omega(s')) ds'} \left\{ i\dot{\omega}(\mathcal{R}) \langle f(s), \sigma_3 \partial_\omega \xi(\omega(s)) \rangle + \langle \sigma_3 \dot{\gamma}(\mathcal{R}) R(s) \right\} \]
\[ + N(R(s)) - i\dot{\omega}(\mathcal{R}) \{ z_+(s) + z_-(s) \sigma_1 \partial_\omega \xi(\omega(s)), \sigma_3 \xi(\omega(s)) \} \].

For a \( z_-(0) \) small we write
\[ z_-(t) = e^{-\int_0^t \mu(\omega(s)) ds_0} z_-(0) + \overline{z}_-(\mathcal{R}), \text{ where } \overline{z}_-(\mathcal{R}) := \]
(4.7) \[ d_1 \int_0^t ds e^{-\int_0^s \mu(\omega(s')) ds'} \left\{ i\dot{\omega}(\mathcal{R}) \langle f(s), \sigma_1 \sigma_3 \partial_\omega \xi(\omega(s)) \rangle + \langle \sigma_3 \dot{\gamma}(\mathcal{R}) R(s) \right\} \]
\[ + N(R(s)) - i\dot{\omega}(\mathcal{R}) \{ z_+(s) + z_-(s) \sigma_1 \partial_\omega \xi(\omega(s)), \sigma_3 \xi(\omega(s)) \} \].

We interpret (4.2-7) as an equation in \( B_X(\omega_0, \gamma_0, \epsilon) \subset \mathcal{X} \).

**Proposition 4.1.** Fix \( \alpha \in (0, 1) \) and \( \omega_0 \in (\alpha, 1/\alpha) \), \( \gamma_0 \in \mathbb{R} \). Then \( \exists \epsilon_0 > 0, c(\alpha) \) and a \( C > 0 \) such that \( \forall (\omega(0), \gamma(0), z_-(0)) \), with \( z_-(0) \in \mathbb{R}, \gamma(0) \in \mathbb{R} \), with
\[ |\omega(0) - \omega_0| + |\gamma(0) - \gamma_0| + |z_-(0)| < \epsilon/5 \leq \epsilon_0/5 \]
and \( \forall h_0 \in H^1_r(\mathbb{R}, \mathbb{C}^2) \cap L^2_c(\omega_0) \) satisfying \( \overline{h_0} = \sigma_1 h_0 \) with \( \|h_0\|_{H^1_r(\mathbb{R})} < c(\alpha) \epsilon \), and if we define \( R(t, x) \) by (4.1) with \( f_\alpha(t, x) \) defined by Lemma 2.4, then, for \( 4/q = 1 - 1/p \) with \( p > 5 \) the exponent in (1.1), there exists a solution
(4.8) \[ (\omega(t), z_+(t), z_-(t), \gamma(t), f_c(t)) \in C^0([0, \infty), \mathbb{R}^4) \times \mathcal{Z} \]
of (4.2-7) such that \( \forall t \geq 0 \) we have \( f_c(t) = \sigma_1 \overline{f_c(t)} \) and
1. \[ |\omega(t) - \omega_0| \leq \epsilon, \ |\omega(t), z_-(t), \gamma(t) - (\omega(0), z_-(0), \gamma(0))| < C \epsilon^2, \]
2. \[ \|f_c\|_{\mathcal{Z}} \leq \epsilon, \]
3. \[ \|z_-(t)\|_{L^1(0, \infty)} \leq \epsilon, \ \|z_+(t)\|_{L^1(0, \infty)} < C \epsilon^2 \]
4. \[ \lim_{t \to \infty} (z_+(t), z_-(t)) = (0, 0). \]

There exist \( \gamma_\infty \in \mathbb{R}, \omega_\infty > 0 \) such that
(5) \[ \lim_{t \to \infty} (\omega(t), \gamma(t)) = (\omega_\infty, \gamma_\infty) \]
and for \( \ell(t) = \omega(t) - \omega_0 + \dot{\gamma}(t) \)
(6) \[ \lim_{t \to \infty} \|f(t) - e^{-it H_{\omega_0}} e^{i \int_0^t \ell(\tau) d\tau (P_-(\omega_0) - P_+(\omega_0))} h_0\|_{H^1(\mathbb{R}, \mathbb{C}^2)} = 0. \]
We have \( \sigma_1 R(t, x) = \overline{R(t, x)}, \) \( R(t, x) \) solves (2.3), the first entry \( r(t, x) \) of \( \overline{R} = (r, \overline{r}) \) solves (2.1) and \( u(t, x) \), defined in (1.3), solves (1.1).

There is an isomorphism between the space of the \( u(t, x) \) and the space of the \( U(t, x) = \overline{u(t, x)}, \overline{u}(t, x) \), so we think \( X \) in the latter space. The spirit of Proposition 4.1 is that we try to parametrize the set \( X \) by means of \( (\omega(0), \gamma(0), z_-(0), h_0) \). In fact we cannot exclude that for each choice of the parameter there are more than one solutions of the form (4.8). So we define the \( X \) in Theorem 1.1 as the union of the trajectories associated to all possible solutions of (4.2-7).

§5 Proof of Proposition 4.1

Set \( R = (\omega, \widehat{R}) \) with \( \widehat{R} = (z_+, z_-, \gamma, f_c) \). Set for \( \ell(t, R) = \omega(0) + \omega(t, x) + \gamma(t, x) \)

\[
L(R) := (\overline{z_+}(R), \overline{z_-}(R), \widehat{R}(R), \widehat{\gamma}(R));
\]

\[
G(\omega)(\widehat{R}) := (0, e^{-\int_0^t \mu(\omega(s))ds} \overline{z_-}(0), \overline{\gamma}(0), e^{-itH_\omega} p_+(\omega_0) e^{-i \int_0^t \ell(\tau, R)d\tau} + P_-(\omega_0) e^{i \int_0^t \ell(\tau, R)d\tau} h_0) + \bar{G}(\widehat{R});
\]

\[
(5.1) \quad \bar{G}(\omega)(\widehat{R}) := (\overline{z_+}(\omega, \widehat{R}), \overline{z_-}(\omega, \widehat{R}), \overline{\gamma}(\omega, \widehat{R}), \overline{f_c}(\omega, \widehat{R}));
\]

\[
F(\omega) := (\omega(0), \varphi - \int_0^t \mu(\omega(s))ds \overline{z_-}(0), \overline{\gamma}(0), e^{-itH_\omega} p_+(\omega_0) e^{-i \int_0^t \ell(\tau, R)d\tau} + P_-(\omega_0) e^{i \int_0^t \ell(\tau, R)d\tau} h_0) + \bar{F}(\widehat{R});
\]

\[
\bar{F}(\omega) := (\overline{\omega}(\omega), \bar{G}(\omega)(\widehat{R})).
\]

To prove Proposition 4.1 we look for fixed points of \( F(\omega) \). We are not able to show that \( F(\omega) \) is Lipschitz because of \( \omega(t) \) in the \( \ell(t, R) = \omega(t) - \omega_0 + \gamma(t, R) \) and the exponent \( \int_0^t \ell(\tau, R)d\tau \) in the definition (4.5) of \( \overline{f_c}(\omega) \). We split \( R = (\omega, \widehat{R}) \) and we solve the system by substitution, by first solving for \( \widehat{R} \) with \( \omega \) arbitrary but with \( \| \omega - \omega_0 \|_\infty \) small. Since \( F(\omega, \widehat{R}) \) is Lipschitz and a contraction in \( \widehat{R} \), with constant independent of \( \omega \), for each \( \omega \) we get a unique corresponding \( \widehat{R} = \widehat{R}(\omega) \) by the contraction principle. \( \widehat{R}(\omega) \) is continuous in \( \omega \). Substituting in the equation for \( \omega \), we obtain a fixed point problem in \( \omega \) which we solve by the Schauder fixed point theorem.

By Lemmas 3.1-2 we have:

**Lemma 5.1.** For \( \alpha \in (0, 1) \) there exists \( C(\alpha) > 0 \) such that \( \forall \omega_0 \in (\alpha, 1/\alpha) \) we have \( \| e^{-itH_\omega} p_c(\omega_0) h \|_Z < C(\alpha) \| h \|_{H^1} \); \( e^{-itH_\omega} p_c(\omega) \) is strongly continuous in \( H^1(\mathbb{R}, \mathbb{C}^2) \).

Next, we have:

**Lemma 5.2.** There exists a fixed \( C > 0 \) such that for all \( 0 < \epsilon < \epsilon_0 \), \( L(R) \) is \( C \) in \( B_X(\omega_0, \gamma_0, \epsilon) \) such that for \( L(R) = (\overline{z_+}(R), \overline{z_-}(R), \widehat{R}(R), \widehat{\gamma}(R)) \) and for any \( t_0 \geq 0 \)

\[
\| L(R) \|_{(L^1(\mathbb{R}) \cap L^\infty) \times (W^{1, \infty} \cap W^{1, 1})} \leq C \epsilon e^{\gamma \frac{\epsilon^2}{2} t_0} + \| (z_+, z_-) \|_{L^1(t_0, \infty)} + \| f_c \|_{L^2(t_0, \infty)}.
\]
Furthermore we have

\[ \|DL(R)\delta R\|_{((L^1 \cap L^\infty)^2 \times (W^{1,\infty} \cap W^{1,1})^2)(0,\infty)} \leq C\epsilon \|\delta R\|_{\mathcal{X}}. \]

Proof. Set \( \tilde{z}_+(t) = \tilde{z}_+(R)(t) = d_1 \int_t^{\infty} ds e^{f_s^t \mu(\omega(s'))} ds' Z_+(R)(s), \)

\[ Z_+(R) := \langle \sigma_3 \hat{\gamma} R + N(R) - i\hat{\omega}[z_+ + z_- \sigma_1] \partial_\omega \xi(\omega), \sigma_3 \xi(\omega) \rangle + i\hat{\omega} \langle f, \sigma_3 \partial_\omega \xi(\omega) \rangle. \]

In \( \mathcal{B}_X(\omega_0, \gamma_0, \epsilon) \) we have \( \mu(\omega(t)) > \alpha \mu(1) > 0 \). So for \( t \geq t_0 \)

\[ |\tilde{z}_+(t)| + \|\tilde{z}_+\|_{L^1(t_0, \infty)} \leq \int_t^{\infty} dse^{-\alpha \mu(1)|t-s|} |Z_+(s)| ds + \frac{1}{\alpha \mu(1)} \|Z_+\|_{L^1(t_0, \infty)}. \]

The above is \( \leq C_\alpha \|Z_+\|_{L^1(t_0, \infty)}. \) We have

\[ \|Z_+\|_{L^1(t_0, \infty)} \leq C\epsilon(\|(z_+, z_-)\|_{L^1(t_0, \infty)} + \|f_c\|_{L^2((t_0, \infty), L^{2,-2}_x)}). \]

So for \( t \geq t_0 \) we get

\[ |\tilde{z}_+(t)| + \|\tilde{z}_+\|_{L^1(t_0, \infty)} \leq C\epsilon(\|(z_+, z_-)\|_{L^1(t_0, \infty)} + \|f_c\|_{L^2((t_0, \infty), L^{2,-2}_x)}). \]

We have \( \tilde{z}_-(t) = \tilde{z}_-(R)(t) = d_1 \int_0^t ds e^{-f_s^t \mu(\omega(s'))} ds' Z_-(R)(s) \) with

\[ Z_-(R) = \langle \sigma_3 \hat{\gamma} R + N(R) - i\hat{\omega}[z_+ + z_- \sigma_1] \partial_\omega \xi(\omega), \sigma_1 \sigma_3 \xi(\omega) \rangle + i\hat{\omega} \langle f, \sigma_1 \sigma_3 \partial_\omega \xi(\omega) \rangle. \]

Then

\[ |\tilde{z}_-(t)| \leq \int_0^t dse^{-\alpha \mu(1)|t-s|} |Z_-(s)| ds, \]

\[ \|\tilde{z}_-\|_{L^1(t_0, \infty)} \leq \left\| \int_0^t dse^{-\alpha \mu(1)|t-s|} |Z_-(s)| ds \right\|_{L^1(t_0, \infty)}. \]

From the first we read for \( t \geq t_0 \)

\[ |\tilde{z}_-(t)| \leq C e^{-\alpha \mu(1)t/2} \|Z_-\|_{L^1(t_0, t_0/2)} + C \|Z_-\|_{L^1(t_0/2, \infty)}. \]

This yields for \( t \geq t_0 \)

\[ |\tilde{z}_-(t)| \leq C \epsilon^2 e^{-\alpha \mu(1)t/2} + C\epsilon(\|(z_+, z_-)\|_{L^1(t_0, \infty)} + \|f_c\|_{L^2((t_0, \infty), L^{2,-2}_x)}). \]

In a similar fashion we obtain

\[ \|\tilde{z}_-\|_{L^1(t_0, \infty)} \leq C \epsilon^2 e^{-\alpha \mu(1)t_0/2} + C\epsilon(\|(z_+, z_-)\|_{L^1(t_0, \infty)} + \|f_c\|_{L^2((t_0, \infty), L^{2,-2}_x)}). \]
Notice that (3-5) imply
\[(6) \quad \| (z_+, z_-) \|_{L^1([t_0, \infty))} \leq C e^{2\epsilon - \alpha \mu(1) t_0/2} + C \epsilon \| f_c \|_{L^2((t_0, \infty), L^2_{x_2})}.\]
By (4.4) we have
\[\| \hat{\omega}(R) \|_{L^1([t_0, \infty))} = \| \langle O(R^2(t)), \phi_\omega(t) \rangle \|_{L^1([t_0, \infty))} \leq C \| R \|_{L^2(t_0, \infty, L^2_{x_2})}^2.\]
Then
\[(7) \quad \| \hat{\omega}(R) \|_{L^1([t_0, \infty))} \leq C \epsilon (\epsilon e^{2\epsilon} + \| f_c \|_{L^2((t_0, \infty), L^2_{x_2})}).\]
Similarly
\[(8) \quad \| \hat{\gamma}(R) \|_{L^1([t_0, \infty))} \leq C \epsilon (\epsilon e^{2\epsilon} + \| f_c \|_{L^2((t_0, \infty), L^2_{x_2})}).\]
Then (3-5) and (7-8) yield (1).

We have \(\tilde{z}_+(\mathcal{R} + \delta \mathcal{R}) = \tilde{z}_+(\mathcal{R}) + \delta \tilde{z}_+(\mathcal{R} + \delta \mathcal{R}) = \tilde{z}_+(\mathcal{R}) + D\tilde{z}_+(\mathcal{R})\delta \mathcal{R} + O(\delta \mathcal{R}^2)\)
with \(D\tilde{z}_+(\mathcal{R})\delta \mathcal{R} = \int_s^{+\infty} ds' e^{\int_s^{t} \mu(\omega(s'))ds'} Z_+(\mathcal{R})(s) + DZ_+(\mathcal{R})\delta \mathcal{R}\).
So \(\|DZ_+(\mathcal{R})\delta \mathcal{R}\|_{L^1([0, \infty))} \leq \tilde{C}_0 \epsilon \| \delta \mathcal{R} \|_{\mathcal{X}}\) and \(\| (D\tilde{z}_+(\mathcal{R})\delta \mathcal{R})(t) + \| D\tilde{z}_+(\mathcal{R})\delta \mathcal{R}\|_{L^1([0, \infty))} \leq \tilde{C}_0 \epsilon \| \delta \mathcal{R} \|_{\mathcal{X}}\). For \(\| \delta \mathcal{R} \|_{\mathcal{X}} \leq \epsilon\), \(\| O(\delta \mathcal{R}^2)(t) + \| O(\delta \mathcal{R}^2)\|_{L^1([0, \infty))} \leq \epsilon \| \delta \mathcal{R} \|_{\mathcal{X}}\). Similar estimates hold for the \(\tilde{z}_-(\mathcal{R}), \tilde{\omega}(\mathcal{R})\) and \(\tilde{\gamma}(\mathcal{R})\). This yields (2).

Consider the ball \(B_{\mathcal{X}}(\omega_0, \epsilon)\) defined by \(\| \omega(t) - \omega_0 \|_{L^1([0, \infty))} < \epsilon\).

**Lemma 5.3.**

1. There is a fixed \(C > 0\) such that we have \(\| \tilde{\mathcal{f}}_c(\mathcal{R}) \|_{\mathcal{Z}} \leq C \epsilon^2\) for any \(\mathcal{R} \in B_{\mathcal{X}}(\omega_0, \gamma_0, \epsilon)\).
2. There is a fixed \(C > 0\) such that given any \(\omega \in \overline{B_{L^\infty}(\omega_0, \epsilon)}\) the map \(\tilde{\mathcal{R}} \in B_{\tilde{\mathcal{X}}}(\gamma_0, \epsilon) \rightarrow \tilde{\mathcal{f}}_c(\omega, \tilde{\mathcal{R}}) \in \mathcal{Z}\) is differentiable with \(\| D\tilde{\mathcal{f}}_c(\omega, \tilde{\mathcal{R}})\delta \tilde{\mathcal{R}}\|_{\mathcal{Z}} \leq C \epsilon \| \tilde{\mathcal{R}} \|_{\tilde{\mathcal{X}}}\).
3. Let \(\mathcal{R}_j = (\omega, \tilde{\mathcal{R}}_j)\) with \(\omega \in B_{L^\infty}(\omega_0, \epsilon)\) and \(\tilde{\mathcal{R}}_j \in B_{\tilde{\mathcal{X}}}(\gamma_0, \epsilon)\) for \(j = 1, 2\). Then
\[
\| e^{-iH_\omega t} P_\pm(\omega_0) \left( e^{\mp i \int_\tau^{t_0} (\tau, \mathcal{R}_1) d\tau} - e^{\mp i \int_\tau^{t_0} (\tau, \mathcal{R}_2) d\tau} \right) h_0 \|_{\mathcal{Z}} \leq C \epsilon \| \tilde{\mathcal{R}}_1 - \tilde{\mathcal{R}}_2 \|_{\tilde{\mathcal{X}}} \| h_0 \|_{H^2}.\]

**Proof.** (3) follows by
\[
\| e^{-iH_\omega t} P_\pm(\omega_0) e^{\mp i \int_\tau^{t_0} (\tau - \omega_0) d\tau} \left( e^{\mp i \int_\tau^{t_0} \hat{\gamma}(\mathcal{R}_1)(\tau) d\tau} - e^{\mp i \int_\tau^{t_0} \hat{\gamma}(\mathcal{R}_2)(\tau) d\tau} \right) h_0 \|_{\mathcal{Z}} \leq C \epsilon \| \tilde{\mathcal{R}}_1 - \tilde{\mathcal{R}}_2 \|_{\tilde{\mathcal{X}}} \| h_0 \|_{H^2}.\]
The first two claims of Lemma 5.3 are a consequence of Lemmas 5.4 and 5.5 below. We have a decomposition \(N(R) = O_{loc}(R^2) + N_2(f_c)\) with \(N_2(f_c) = O(f_c^p)\). We set \(F(R) = F_1(R) + F_2(R)\) with \(F_2(R) = N_2(f_c) = O(f_c^p)\).
Lemma 5.4. Let $\omega(t)$ be a function with values in $(\alpha, 1/\alpha)$. Then for a fixed $C = C(\alpha)$ we have $\|\tilde{f}_c(\mathcal{R})\|_Z \leq C \left( \|F_1(\mathcal{R})\|_{H^1_x L^2_t} + \|F_2(\mathcal{R})\|_{L^1_t H^1_x} \right)$.

Proof. By Lemmas 3.1, 3.4 and 3.5 for $t_0 \geq 0$

$$\|\tilde{f}_c(\mathcal{R})\|_{L^2_t((t_0,\infty),L^\infty_x) \cap L^1_t((t_0,\infty),W^{1,2} p) \cap L^\infty((t_0,\infty),H^2_x)} \leq C \left( \|F_1(\mathcal{R})\|_{L^2_t(t_0,\infty) H^1_x} + \|F_2(\mathcal{R})\|_{L^1_t((t_0,\infty),H^1_x)} \right).$$

Let $f_j(\mathcal{R})$ be defined by (4.5) with $F(\mathcal{R})$ replaced by $F_j(\mathcal{R})$. By Lemmas 3.3 and 3.5

$$\|\tilde{f}_1\|_{L^2_t(t_0,\infty)H^1_x} \leq C \|F_1\|_{L^2_t(t_0,\infty)H^1_x}$$

By Lemma 3.2, for a fixed $C$ and for $t_0 \geq 0$, $\|\tilde{f}_2\|_{L^2_t(t_0,\infty)H^1_x} \leq$

$$\leq \int_t^\infty ds \left| e^{-i(t-s)H_\omega} e^{\pm i t_0} f_0(t,\tau) d\tau \right| P_{\pm}(\omega_0) F_2(\mathcal{R})(s) \|H^1_x\|_{L^2(t_0,\infty)}$$

$$\leq \int_{t_0}^\infty ds \left| e^{-i(t-s)H_\omega} e^{\pm i t_0} f_0(t,\tau) d\tau \right| P_{\pm}(\omega_0) F_2(\mathcal{R})(s) \|H^1_x\|_{L^2(t_0,\infty)}$$

$$\leq \int_{t_0}^\infty ds \left| e^{\mp i t_0} F_2(\mathcal{R})(s) \|H^1_x\|_{L^2(t_0,\infty)}.$$
Schematically we have $F_1(\mathcal{R}) = O(\epsilon)\psi f + O_{loc}(\mathcal{R}^2)$ for an exponentially decreasing $\psi(x)$. Then by Lemma 5.2 we get (1). We have $F_2(\mathcal{R}) = O(f_c^p)$ and this yields

$$\|F_2\|_{L^q} \lesssim \left\| f_c^p \right\|_{L^q} \lesssim \left\| f_c \right\|_{W^{1,2p}} \left\| f_c \right\|_{L^p} \lesssim \left\| f_c \right\|_{L^q} \left\| f_c \right\|^{p-1}_{L^q}.$$ 

Since $q = \frac{4p}{p-1} < \frac{4p(p-1)}{3p+1} = q' - (p-1)$ by $p > 3$, then for some $0 < \vartheta < 1$ we get

$$\|F_2\|_{L^q} \lesssim \left\| f_c \right\|_{L^q} \left\| f_c \right\|^{1+\vartheta(p-1)}_{L^q} \left\| f_c \right\|^{(1-\vartheta)(p-1)}_{L^q}.$$ 

This yields (2). Proceeding similarly we get (3).

**Lemma 5.6.** Consider $G(\omega)$ defined by (5.1).

1. For every $\omega \in B_{L^\infty}(\omega_0, \epsilon) \ni \widehat{R}(\omega) = (z_+(\omega), z_-(\omega), \gamma(\omega), f_c, \omega) \in \hat{X}$, unique, such that $\widehat{R}(\omega, h_0) \in B_{\hat{X}}(\gamma_0, \epsilon/2)$ satisfies the fixed point problem $\widehat{R}(\omega) = G(\omega)\left(\widehat{R}(\omega)\right)$.

2. The map $\omega \in B_{L^\infty}(\omega_0, \epsilon) \rightarrow \widehat{R}(\omega) \in B_{\hat{X}}(\gamma_0, \epsilon)$ is continuous.

**Proof.** For $\epsilon \in (0, \epsilon_0)$ with $\epsilon_0 > 0$ small enough, $G(\omega)$ maps $B_{\hat{X}}(\gamma_0, \epsilon/2)$ into itself. By the estimates on the derivatives in Lemmas 5.2 and 5.5, $\|G(\omega)\hat{R}_1 - G(\omega)\hat{R}_2\|_{\hat{X}} \leq C\epsilon\|\hat{R}_1 - \hat{R}_2\|_{\hat{X}}$. There is a fixed point, which we denote by $\hat{R}(\omega)$, and which is unique. This yields (1). Let $C\epsilon < 1/2$. We have $\|\hat{R}(\omega_1) - \hat{R}(\omega_2)\|_{\hat{X}} \leq \|\hat{R}(\omega_1) - G(\omega_2)\|_{\hat{X}} + \|G(\omega_2)\hat{R}(\omega_1) - G(\omega_2)\hat{R}(\omega_2)\|_{\hat{X}} + C\epsilon\|\hat{R}(\omega_1) - \hat{R}(\omega_2)\|_{\hat{X}}$.

To complete Lemma 5.6 we need to show that $\omega \in B_{L^\infty}(\omega_0, \epsilon) \rightarrow G(\omega)\hat{R}_0 \in \hat{X}$ is continuous for fixed $\hat{R}_0$. In view of Lemma 5.2 it remains to show the following:

**Lemma 5.7.** The map $\mathcal{R} \in B_{X}(\omega_0, \gamma_0, \epsilon) \rightarrow \hat{f}_c(\mathcal{R}) \in Z$ is continuous.

**Proof.** We write $\mathcal{R} = (\hat{\omega}, \hat{\mathcal{R}})$ to distinguish between $\omega$ and $\hat{\mathcal{R}} = (z_+, z_-, \gamma, f_c)$. By Lemma 5.5, to complete the proof of the continuity of $\hat{f}(\mathcal{R})$ it is enough to show that for fixed $\mathcal{R}_0 = (\omega_0, \hat{\mathcal{R}}_0)$ and if we set $\mathcal{R}_1 = (\omega_0 + \delta \omega, \hat{\mathcal{R}}_0)$, for any $\epsilon > 0$ there is $0 < \delta > 0$ such that $|\hat{f}(\mathcal{R}_0) - \hat{f}(\mathcal{R}_1)| \leq \epsilon$ if $\|\delta \omega\|_{L^\infty} < \delta$. For $g(s) = e^{-i\int_0^s \tilde{\omega}(\tau) d\tau} P_{\pm}(\omega_0) F(\mathcal{R}_0)\left(\omega_0\right) s$ we need to show that for any $\epsilon > 0$ there exists $\delta > 0$ such that $\|\delta \omega\|_{L^\infty} < \delta$ implies

$$\left\| \int_t^\infty e^{-it-s} H_{\omega_0} \left( e^{it} f_c \tilde{\omega}(\tau) d\tau - 1 \right) g(s) ds \right\|_Z < \epsilon.$$ 

We fix a large number $M > 0$. Then, for $\delta > 0$ with $M\delta C\|F(\mathcal{R}_0)\|_Z \leq \epsilon/2$ and since $\|g\|_{L^1} \leq \|F(\mathcal{R}_0)\|_{L^1} \leq \|F(\mathcal{R}_0)\|_{L^1} + \|F(\mathcal{R}_0)\|_{L^2}$ for any interval $I$, we conclude

$$\left\| \int_t^{t+M} e^{-it-s} H_{\omega_0} \left( e^{it} f_c \tilde{\omega}(\tau) d\tau - 1 \right) g(s) ds \right\|_Z < \epsilon/2.$$
We have
\[
\left\| \int_{t}^{\infty} e^{-i(t-s)H_{\omega_{0}}} \left( e^{\pm i f_{\omega}^{r} \delta \omega(s) dt} - 1 \right) g(s) ds \right\|_{L^{\infty}} \leq C \| F(R_{0}) \|_{H_{x}^{1,2} L_{t}^{2}(M, \infty) + L_{t}^{1}((M, \infty), H_{x}^{1})} \to 0 \text{ for } M \not\to \infty.
\]

Having \( \hat{R}(\omega) \) for any \( \omega \in B_{L^{\infty}}(\omega_{0}, \varepsilon) \) we substitute \( \hat{R} = \hat{R}(\omega) \) in the system and we reduce to a fixed point problem in \( \omega \). We will denote by \( Z(t_{0}) \) the space defined like \( Z \) in §4 but with the time interval \((0, \infty)\) replaced by \((t_{0}, \infty)\). We get:

**Lemma 5.8.** There is a \( \omega(t) \in B_{L^{\infty}}(\omega_{0}, \varepsilon/2) \) such that, for \( R = (\omega, \hat{R}(\omega)) \) and \( R(t) = R(t, x, R) \),

\[
(1) \quad \omega(t) = \omega(0) + \tilde{\omega}(R)(t), \quad \tilde{\omega}(R)(t) = \int_{0}^{t} \langle O(R^{2}(s)), \Phi_{\omega(s)} \rangle ds.
\]

**Proof.** The map on the right side in (1) sends \( B_{L^{\infty}}(\omega_{0}, \varepsilon) \) into itself. Lemma 5.8 is a consequence of the Schauder fixed point theorem if we are able to show that the image of \( B_{L^{\infty}}(\omega_{0}, \varepsilon) \), which we denote by \( A \), has compact closure in \( B_{L^{\infty}}(\omega_{0}, \varepsilon) \). First of all, \( A \subset B_{L^{\infty}}(\omega_{0}, \varepsilon/3) \cap (W^{1, \infty} \cap \dot{H}^{1,1}) \). It will be enough to show that, for any \( \varepsilon > 0 \) there exists \( t_{0} = t_{0}(\varepsilon) \) such that for any \( \omega \in A \) we have \( \| \tilde{\omega} \|_{L_{t}^{1}(t_{0}, \infty)} < \varepsilon \). This reduces to showing that for any \( \varepsilon > 0 \) there is \( t_{0} > 0 \) such that for any \( \omega \in B_{L^{\infty}}(\omega_{0}, \varepsilon) \), given the corresponding \( \mathcal{R} = (\omega, \hat{R}(\omega)) \), we have \( \| \tilde{f}_{c}(\mathcal{R}) \|_{Z(t_{0})} < \varepsilon \). But by the proof of Lemma 5.4 and by (1-2) Lemma 5.5 we get

\[
\| \tilde{f}_{c}(\mathcal{R}) \|_{Z(t_{0})} \leq C \left( \| F_{1}(\mathcal{R}) \|_{H_{x}^{1,2} L_{t}^{2}(t_{0}, \infty)} + \| F_{2}(\mathcal{R}) \|_{L_{t}^{1}((t_{0}, \infty), H_{x}^{1})} \right) \leq C \varepsilon(\varepsilon e^{-2\alpha_{0}(1)}) = e^{-i\varepsilon t_{0}H_{\omega_{0}}^{t}h_{0}} \| \tilde{f}_{c}(\mathcal{R}) \|_{Z(t_{0})}
\]

which implies \( \| \tilde{f}_{c}(\mathcal{R}) \|_{Z(t_{0})} \leq C_{1} \varepsilon(\varepsilon e^{-2\alpha_{0}(1)}) + || e^{-i\varepsilon t_{0}H_{\omega_{0}}^{t}h_{0}} ||_{Z(t_{0})} \) and yields the desired result.

By Lemmas 5.6-8 we conclude that we have a solution \( \mathcal{R} = (\omega, \hat{R}) \in B_{X}(\omega_{0}, \gamma_{0}, \varepsilon) \) which yields the solution (4.8) of Proposition 4.1. Estimates (1-4) as well as the limits (5) follow from the definition of \( X \). Now we prove the remaining part of Proposition 4.1. We can define a smooth diffeomorphism from a neighborhood of \( (\omega_{0}, 0, 0, \gamma_{0}, 0) \in \mathbb{R}^{4} \times (H_{r}^{1}(\mathbb{R}, \mathbb{C}^{2}) \cap L^{2}(\omega_{0})) \) with values in a small neighborhood of \( e^{i\gamma_{0}^{0}} \phi_{\omega_{0}}(x) \in H_{r}^{1}(\mathbb{R}, \mathbb{C}) \) which associates to every \( \Pi = (\omega^{0}, z_{+}^{0}, z_{-}^{0}, \gamma^{0}, f_{c}^{0}(x)) \)

\[
u_{\Pi}(x) = e^{i\gamma^{0}} \left( \phi_{\omega^{0}}(x) + r_{\Pi}(x) \right)
\]
with $t(r_\Pi(x), \overline{\Pi}(x)) = R_\Pi(x)$ and, for $f_{d[\Pi]}(x)$ defined by Lemma 2.4, with

$$R_\Pi(x) = \left(z^{(0)}_+ + z^{(0)}_- \sigma_1\right) \xi(\omega^{(0)}, x) + f_{d[\Pi]}(x) + f_c^{(0)}(x).$$

Then given the solution in (4.8) and given $R(t)$ defined by (4.1), the corresponding point in $u(t) \in H^1_r(\mathbb{R}, \mathbb{C})$ is given by $t(u, \overline{\Pi}) = e^{it\sigma_3(\int_0^t \omega(s)ds + \gamma(t))}(\Phi_{\omega(t)} + R(t))$. In particular $u(t) \in C^0([0, \infty), H^1_r(\mathbb{R}, \mathbb{C}))$ and is the solution of (1.1) with $u(0) = u_\Pi$. By construction

$$\lim_{t \to \infty} \|R(t) - e^{-itH_{\omega_0}} e^{-i\int_0^t \ell(\tau)d\tau(P_+(\omega_0) - P_-(\omega_0))}h_0\|_{H^1(\mathbb{R}, \mathbb{C}^2)} = 0.$$

For $h_0 = W(\omega_0)\tilde{h}_0$ with $W(\omega_0) = \text{strong} - \lim_{t \to \infty} e^{itH_{\omega_0}} e^{-it\sigma_3(-\partial_x^2 + \omega_0)}$, see [C1],

$$\lim_{t \to \infty} \|f_c(t) - e^{-i\int_0^t \omega(\tau)d\tau + \gamma(t) - \gamma(0) - t\omega_0}\sigma_3 e^{it\sigma_3(-\partial_x^2 + \omega_0)}\tilde{h}_0\|_{H^1(\mathbb{R}, \mathbb{C}^2)} = 0.$$

So for $t(r_\infty, \overline{r}_\infty) = e^{i\gamma(0)\sigma_3}\tilde{h}_0$ and $t(r, \overline{r}) = R$ we conclude

$$\lim_{t \to \infty} \|e^{it\int_0^t \omega(\tau)d\tau + i\gamma(t)}r(t) - e^{it\partial_x^2}r_\infty\|_{H^1(\mathbb{R}, \mathbb{C})} = 0.$$

**Errata in paper [C1]**

Unfortunately paper [C1] has many mistakes. Fortunately all of them can be corrected. Among the various mistakes we list:

1. Various formulas between sections 5 and 8 are wrong, for example the formula for the Wronskian from §5 on.
2. In formula (8.2) in [C1] there is a missing term on the right hand side.
3. The really serious mistake is Lemma 5.4 [C1]: not only the proof is incorrect, but probably the statement is incorrect.

In [C3] we have revised [C1] simplifying considerably the argument. In particular the smoothing estimates in §3 [C1], which are analogues of estimates in [M], have been replaced by weaker estimates estimates in §3 [C3]. These new estimates are listed in §3 in the present paper and are simple to prove. The estimates in §3 [C3] are sufficient for the main result in [C1,C3]. In particular in [C3] most of the material in sections from 5 to 8 in [C1] is eliminated. In particular the statements in §3 [C3] are proved immediately in §3 [C3] with elementary arguments based on material already in the literature. [C3] relies more on [KS]. The statement that the linear part in [C1] is proven also when the matrix potential $V(x) = H_{\omega} - \sigma_3(-\partial_x^2 + \omega)$ is not necessarily even, does not stand any more, since [KS] assumes symmetry of $V(x)$ as an hypothesis. In fact the arguments from §5 to §8 in [C1] can be saved in a corrected form, and this is done in [CV]. However in the present paper we assume the results in [C3].

16
REFERENCES

[BC] H. Berestycki, T. Cazenave, *Instabilité des états stationnaires des les équations de Schrödinger et de Klein Gordon non linéaires*, C.R. Acad. Sci. Paris 293 (1981), 489–492.

[B] M. Beceanu, *A Centre-Stable Manifold for the Focussing Cubic NLS in $R^{1+3}$*, Comm. Math. Phys. 280 (2008), 145–205.

[BP] V. S. Buslaev, G. S. Perelman, *On the stability of solitary waves for nonlinear Schrödinger equations*, Nonlinear evolution equations (N. N. Uraltseva, eds.), Transl. Ser. 2, 164, Amer. Math. Soc., Providence, RI, 1995, pp. 75–98.

[Co] R. Cote, *Construction of solutions to the L2-critical KdV equation with a given asymptotic behaviour*, Duke Mathematical Journal 138 (2007), 487–532.

[C1] S. Cuccagna, *On asymptotic stability in energy space of ground states of NLS in 1D*, J. Differential Equations 245 (2008), 653–691.

[C2] ———, *On instability of excited states of the nonlinear Schrödinger equation*, http://arxiv.org/abs/0801.4237.

[C3] ———, *A revision of ”On asymptotic stability in energy space of ground states of NLS in 1D”, http://arxiv.org/.*

[C4] ———, *Stabilization of solutions to nonlinear Schrödinger equations*, errata: vol 58 (2005) p. 147, Comm. Pure App. Math. 54 (2001), 1110–1145.

[C5] ———, *Erratum: Stabilization of solutions to nonlinear Schrödinger equations*, Comm. Pure App. Math. 58 (2005), 147–147.

[CPV] S. Cuccagna, D. Pelinovsky, V. Vougalter, *Spectra of positive and negative energies in the linearization of the NLS problem*, Comm. Pure Appl. Math. 58 (2005), 1–29.

[CV] S. Cuccagna, N. Visciglia, *On asymptotic stability of ground states of NLS with a finite bands periodic potential in 1D*, in preparation.

[K] T. Kato, *Wave operators and similarity for some non-selfadjoint operators*, Math. Annalen 162 (1966), 258–269.

[KS] J. Krieger, W. Schlag, *Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension*, J. Amer. Math. Soc. 19 (2006), 815–920.

[Ma] I. Martel, *Asymptotic N-soliton-like solutions of subcritical and critical generalized KdV equations*, Amer. J. of Math. 127 (2005), 1103–1140.

[M] T. Mizumachi, *Asymptotic stability of small solitons to 1D NLS with potential*, http://arxiv.org/abs/math.AP/0605031.

[S] W. Schlag, *Stable manifolds for an orbitally unstable NLS*, [http://www.its.caltech.edu/~schlag/recent.html](http://www.its.caltech.edu/~schlag/recent.html) (2004).

[TY] T. P. Tsai, H. T. Yau, *Stable directions for excited states of nonlinear Schrödinger equations*, Comm. P.D.E. 27 (2002), 2363–2402.

[Y1] K. Yajima, *The $W^{k,p}$ continuity of wave operators for Schrödinger operators*, J. Math. Soc. Japan 47 (1995), 551–581.

[Y2] ———, *The $W^{k,p}$ continuity of wave operators for Schrödinger operators III, even dimensional case $m \geq 4$*, J. Math. Sci. Univ. Tokyo 2 (1995), 311–346.

[W] M. I. Weinstein, *Modulation stability of ground states of nonlinear Schrödinger equations*, Siam J. Math. Anal. 16 (1985), 472–491.

DISMI UNIVERSITY OF MODENA AND REGGIO EMILIA, VIA AMENDOLA 2, PADIGLIONE MORSELLI, REGGIO EMILIA 42100 ITALY

E-mail address: cuccagna.scipio@unimore.it

17