Highest Posterior Distribution (HPD) Control Chart for Individual Observation

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Abstract. Highest Posterior Distribution (HPD) is one of the methods to determine the interval estimator of data having both symmetric and asymmetric distributions. On the symmetric distribution, the Lower Control Limit (LCL) and the Upper Control Limit (UCL) of Individual control chart will automatically have the equilibrium density value. However, the LCL and the UCL of Individual control chart for asymmetric distribution will never have the equilibrium density value. This research is aimed to propose HPD control chart based on the concept of equilibrium density. The simulation studies provide the control limits of Individual and HPD control charts for both symmetric and asymmetric distributions of data. The Average Run Length (ARL) of both Individual and HPD control charts are calculated by generating the data which follow normal distribution and some asymmetric distributions such as weibull, gamma, beta, and lognormal. The in-control ARL (ARL\(_0\)) of Individual control chart for various asymmetric distributions violate the ARL\(_0\) target. Even for asymmetric distribution, the ARL\(_0\) of HPD control chart is convergent to certain value of ARL\(_0\) target. Hence, HPD control chart is appropriate to monitor both symmetric and asymmetric distribution of data.

Keywords: Asymmetric, Average run length, highest posterior distribution, individual control chart

1. Introduction

Point estimator and interval estimator are two measurements which is commonly used in inferential statistics. Interval estimator is widely used due to its advantage to encompass the margin error of point estimator. In Statistical Process Control (SPC), the concept of interval estimator is adopted as the control limits of a control chart to determine whether the observations are within the process control or not. Some control charts had been developed under normal distribution assumption such as Individual (I) chart, \(\bar{X}\) chart [1], Exponentially Weighted Moving Average (EWMA) chart [2] and Cumulative Sum (CUSUM) chart [3]. In addition, some control charts such as p chart, np chart [4], and Laney p’ chart [5,6] are developed to monitor attribute process.

One of the interesting things that need to be concerned in the development of control chart under normal distribution assumption is subgroup size. A quality characteristic of a product tends to indicate the out-of control condition if it is monitored using Individual control chart \((n=1)\). However, this quality characteristic tends to be in-control when it is monitored using \(\bar{X}\) control chart with subgroup observation \((n>1)\). Even for the larger subgroup size, the smaller chance of quality characteristic to indicate the out-of control condition. This indicates that the subgroup size will affect the monitoring
result. Montgomery [7] pointed out that rational subgroup is a subgroup size which minimizes the variability within subgroup and maximizes the variability between subgroup.

Many researchers have developed control charts under the assumption of normal distribution. Even for non-normal distribution, Box-Cox transformation [8] is oftentimes selected to normalize the data. Thus, transformed data can be easily analyzed using conventional control chart based on normal distribution assumption. However, not all kinds of data can be transformed into normal distribution using Box-Cox transformation. For example, the data that follow uniform distribution cannot be transformed into normal distribution using Box-Cox transformation [9–11].

Several studies have been developed to implement the control chart for the data that follow non-normal distribution. Höppner and Wolff [12] developed a Fuzzy-Shewhart control chart to accommodate the data which contain a vagueness and have unclear distribution. Willemain and Runger [13] proposed a control chart based on an empirical reference distribution. This research shows that central limit theorem cannot be used for a data which have skew distribution. Furthermore, [14] developed a control chart based on non-normal distribution, especially for Burr distribution [15] which has skew pattern.

The normality of a process is affected by either the changes in both mean and variability or the change in one parameter separately [16]. Borror et al. [17] found that the ARL₀ of EWMA control chart is dramatically affected by non-normal data. This research is then developed by [18] for the data with individual observation. In addition, the difficulty in monitoring the data that follow non-normal distribution causes the development of non-parametric control chart. A study based on nonparametric control chart had been developed by [13]. However, this research proved that nonparametric control chart was less powerful when it was used to monitor the process following normal distribution.

Iriawan [19] developed the concept of equilibrium density or known by Highest Posterior Distribution (HPD) to estimate the interval estimator of both symmetric and asymmetric distributions. On the data which have a symmetric pattern, the lower and the upper limits of interval estimator will automatically have the equilibrium density value. However, the lower and the upper limits of the interval estimator of the asymmetric data will never have the equilibrium density value. Fortunately, the lower and the upper limits of the interval estimator of the asymmetric data which is estimated using HPD method will meet the equilibrium density value. Skewness due to extreme data is one of the causes of asymmetric pattern. On the symmetric distribution of data, the central tendency measures such as average, median, and mode are located at the same point. However, the average measure will be shifted away from the mode measure if the observations have a lot of extreme data. This indicates that the average measure is very vulnerable to the extreme data. Therefore, the mode measure is selected as the central tendency in the concept of equilibrium density.

This research is aimed to propose HPD control chart for individual observation based on the concept of equilibrium density. Furthermore, the performance of proposed HPD control chart is evaluated using ARL. The performance of HPD control chart is compared with Individual (I) control chart. Using HPD control chart is expected to obtain two advantages at once. First, the result is more naturally fit with the data pattern and free from normal distribution assumption. Second, HPD control chart is efficient because it can be applied to individual observation. Finally, the proposed HPD control chart is applied to simulation data from various distributions.

2. Highest Posterior Distribution (HPD) Control Chart

Highest Posterior Distribution (HPD) is one of the methods to estimate the confidence interval of asymmetric distribution [20,21]. Joseph et al. [22] calculated the sample size of binomial proportion using HPD method. Chen and Shao (1999) estimated the HPD interval using Monte Carlo procedure. In addition, [19] gets the equilibrium density value for the lower and the upper limits of an interval estimator for both symmetric and asymmetric distribution of data.

Suppose that there is a probability distribution function \( f(x) \) of a random variable \( X \). The principal concept to build a confidence interval using the concept of equilibrium density is basically solving two simultaneous equations. First, there is the value of \( b \) and \( a \) respectively as the lower limit and the upper limits of the confidence interval in the domain of \( -\infty < x < \infty \) so that \( f(b) = f(a) \).
Second, the value of $b$ and $a$ respectively as the lower limit and the upper limit of the confidence interval have to guarantee a certain area of probability distribution function such that \[ \int_{b}^{a} f(x)\,dx = 1 - \alpha, \] for a particular significance level $\alpha$.

The basic idea to build a confidence interval using the concept of equilibrium density is solving two simultaneous equations that mathematically can be written according to the following equation:

\[
\begin{align*}
\int_{a}^{b} f(x)\,dx &= 1 - \alpha, \\
\int_{a}^{b} f(x)\,dx &= 1 - \alpha.
\end{align*}
\]

By determine the significance level $\alpha$, the solution of equation (1) and equation (2) can be obtained simultaneously as the lower limit and the upper limit of a confidence interval respectively as $x = b$ and $x = a$.

Figure 1 displays the illustration of equilibrium density value from normal standard distribution. Defined that the significance level $\alpha$ equal to 0.05 so that the 0.95 of confidence interval is illustrated by white region. The lower and the upper limits of this confidence interval have the equilibrium density value which connected by horizontal line. Furthermore, the left grey area of the lower limit is equal to the right grey area of the upper limit of symmetric normal distribution.

Figure 1.b presents the illustration of equilibrium density value from skew normal, Fernandez distribution [24], for skew parameter $\gamma$ equal to 1.5. The white region illustrates the 0.95 of confidence interval. Since the lower and the upper limits are obtained by solving equation (1) and equation (2) simultaneously, the left blue area of the lower limit is not equal to the right blue area of the upper limit of asymmetric Fernandez distribution. However, the lower and the upper limits of the confidence interval guarantee the equilibrium density value which is connected by horizontal line. Based on the basic idea of equilibrium density, HPD control chart is developed by improving the control limits of Individual chart.

Algorithm 1. Estimate the Control Limits of HPD Chart

**Step 1.** Determine the distribution function and the probability distribution function of a data.

**Step 2.** Calculate the mode statistic and define as a Center Line (CL).

**Step 3.** Solve equation (1) and equation (2) simultaneously then rename solution $b$ and solution $a$ as the LCL and the UCL of HPD control chart respectively.

3. Average Run Length

Highest Posterior Distribution (HPD) is one of the methods to estimate the confidence interval of asymmetric distribution [20,21]. Joseph et al. [22] calculated the sample size of binomial proportion using HPD method. Chen and Shao [23] estimated the HPD interval using Monte Carlo procedure.

Average Run Length (ARL) is one of the famous measures used to evaluate the performance of a control
chart. ARL is defined as the average number of samples until finding the first out-of-control signal [7].
The in-control ARL or ARL\(_0\) is calculated while the observation is actually in-control. On the contrary,
the out-of-control ARL or ARL\(_1\) is calculated while the observation is actually shifted to an out-of-control
condition. Hence, for equal ARL\(_0\) value, the smaller value of ARL\(_1\) indicates the better performance of
a control chart. If the process is in control, the ARL could be calculated as follows:
\[
\text{ARL}_0 = \frac{1}{\alpha}. \tag{3}
\]
While the process is out-of control, the ARL could be calculated as:
\[
\text{ARL}_1 = \frac{1}{1-\beta}. \tag{4}
\]

In this paper, the ARLs of both Individual and HPD control charts are calculated using Algorithm 2 as
follows:

Algorithm 2. The ARLs of Individual and HPD control charts

**Step 1.** Specify the significance level \(\alpha\) and number of sample \(m\).

**Step 2.** Determine the LCL and UCL of the control chart.

**Step 3.** Specify the parameter \(\mu\) and parameter \(\sigma\).

**Step 4.** For \(N\) replications, do these steps:
   a. Generate \(m\) samples from certain distribution with parameter \(\mu\) and parameter \(\sigma\).
   b. Calculate the Run Length (RL), number of samples until finding the first observation
      which larger than the UCL or smaller than the LCL.

**Step 5.** Calculate the ARL\(_0\), average of RL over \(N\) replications.

**Step 6.** Specify the shift of mean process \(\Delta\).

**Step 7.** For parameter \(\mu_{k+1} = \mu_k + \Delta, k = 0, 1, 2, \ldots\), repeat step 4.

**Step 8.** Calculate the ARL\(_1\), average of RL over \(N\) replications.

**Step 9.** Save the calculated ARL\(_0\) and ARL\(_1\).

4. HPD Control Limit

The simulation studies are conducted in order to obtain the control limits of HPD control chart using
several distributions of data such as normal, lognormal, weibull, gamma, and beta. The probability
distribution function of those distributions can be written as follows:

- **Normal**\((\mu, \sigma^2)\): \(f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, -\infty < x < \infty. \tag{5}\)
- **Lognormal**\((\mu, \sigma^2)\): \(f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\ln x - \frac{\mu}{\sigma}\right)^2\right\}, 0 < x < \infty. \tag{6}\)
- **Weibull**\((\mu, \sigma)\): \(f(x) = \frac{\sigma}{\mu} x^{\sigma-1} \exp\left\{-\frac{x}{\mu}\right\}^{\sigma}, 0 \leq x < \infty. \tag{7}\)
- **Gamma**\((\mu, \sigma)\): \(f(x) = \frac{1}{\Gamma(\mu\sigma)} x^{\mu-1} \exp\left\{-\frac{x}{\sigma}\right\}, 0 < x < \infty. \tag{8}\)
- **Beta**\((\mu, \sigma)\): \(f(x) = \frac{\Gamma(\mu + \sigma)}{\Gamma(\mu)\Gamma(\sigma)} x^{\mu-1}(1-x)^{\sigma-1}, 0 < x < 1. \tag{9}\)

The simulation studies use significance level \(\alpha\) equal to 0.00273, except for lognormal distribution uses
\(\alpha\) equal to 0.05. The control limits of HPD chart are calculated using Algorithm 1 while the control
limits of Individual chart are calculated using Individual control chart formula [7] for 1,000 samples
of simulation data. Table 1 shows the summary of the control limits for both Individual and HPD control
charts. For symmetric normal distribution of data, the control limits of Individual control chart are not
significantly different from the control limits of HPD control chart. However, the LCL of HPD control chart for weibull, gamma, beta, and lognormal distributions are positive and close to zero value since there are no negative observation from these distributions. Moreover, the control limits of Individual control chart for weibull, gamma, beta, and lognormal distributions are developed based on normal assumption. As a result, the LCL of Individual control chart for those distributions are negative in spite of there are no negative observation value from those distributions.

**Table 1** Control Limits Comparison of Individual and HPD Control Chart

| Distribution of Data | Individual Control Chart | HPD Control Chart |
|----------------------|---------------------------|------------------|
|                      | LCL | CL  | UCL | LCL | CL  | UCL |
| Normal (0,1)         | -2.863 | 0.126 | 3.115 | -3.000 | 0.086 | 3.000 |
| Normal (3,1)         | -0.023 | 3.026 | 6.075 | 0.000 | 3.086 | 6.000 |
| Normal (6,1)         | 2.741  | 6.040 | 9.339 | 3.000 | 6.030 | 9.000 |
| Lognormal (0,1)      | -0.723 | 1.531 | 1.545 | 0.026 | 1.005 | 5.187 |
| Lognormal (1,1)      | -3.265 | 5.050 | 13.365 | 0.071 | 3.043 | 14.100 |
| Lognormal (2,1)      | -4.434 | 13.184 | 30.802 | 0.193 | 8.970 | 38.327 |
| Weibull (2,1)        | -2.960 | 1.880 | 6.730 | 0.000 | 1.334 | 11.829 |
| Weibull (2,2)        | -1.182 | 1.667 | 4.517 | 0.013 | 1.496 | 4.870 |
| Weibull (2,4)        | 0.234  | 1.843 | 3.542 | 0.369 | 1.786 | 3.190 |
| Gamma (2,3)          | -4.550 | 8.490 | 21.530 | 0.496 | 7.891 | 23.702 |
| Gamma (2,4)          | -4.790 | 5.950 | 16.690 | 0.130 | 5.074 | 20.019 |
| Gamma (2,5)          | -1.220 | 4.930 | 11.080 | 0.425 | 4.716 | 13.369 |
| Beta (2,4)           | -0.280 | 0.297 | 0.875 | 0.000 | 0.289 | 0.842 |
| Beta (1,4)           | -0.294 | 0.168 | 0.630 | 0.000 | 0.124 | 0.772 |
| Beta (1,5)           | -0.222 | 0.172 | 0.567 | 0.000 | 0.137 | 0.694 |

**Table 2** ARLs of Individual and HPD Control Chart for Normal Distribution

| Mean Shift | Normal (0,1) | Normal (3,1) | Normal (6,1) |
|------------|--------------|--------------|--------------|
|             | Individual HPD | Individual HPD | Individual HPD |
| 0.0         | 335.68       | 335.41       | 331.65       | 333.66       | 352.54       | 358.04       |
| 0.2         | 302.57       | 302.57       | 281.14       | 281.99       | 296.77       | 300.29       |
| 0.4         | 208.83       | 208.83       | 206.41       | 207.37       | 200.48       | 202.36       |
| 0.6         | 119.26       | 119.19       | 118.47       | 118.55       | 114.71       | 115.90       |
| 0.8         | 72.26        | 71.96        | 69.72        | 70.11        | 75.52        | 76.02        |
| 1.0         | 44.83        | 44.80        | 42.26        | 42.31        | 43.08        | 43.43        |
| 1.2         | 28.33        | 28.33        | 28.25        | 28.26        | 28.76        | 29.03        |
| 1.4         | 18.35        | 18.35        | 18.62        | 18.68        | 18.91        | 19.05        |
| 1.6         | 12.92        | 12.92        | 11.69        | 11.69        | 12.34        | 12.37        |
| 1.8         | 8.75         | 8.74         | 8.09         | 8.11         | 8.41         | 8.43         |
| 2.0         | 6.17         | 6.17         | 5.92         | 5.93         | 6.43         | 6.43         |
| 2.2         | 4.49         | 4.49         | 4.88         | 4.89         | 4.83         | 4.86         |
| 2.4         | 3.60         | 3.60         | 3.45         | 3.45         | 3.61         | 3.61         |
| 2.6         | 2.93         | 2.92         | 2.94         | 2.94         | 2.88         | 2.88         |
| 2.8         | 2.33         | 2.33         | 2.45         | 2.45         | 2.39         | 2.39         |
| 3.0         | 1.96         | 1.96         | 2.01         | 2.01         | 1.98         | 1.98         |

5. Performance Evaluation of HPD Control Chart

In this section, the performance of proposed HPD control chart is compared with the performance of Individual (I) control chart using ARL criteria. The ARL is obtained using Algorithm 2 from various distributions such as normal, weibull, gamma, beta, and lognormal. For each distribution, \( m = 1000 \) samples are generated with \( N = 1000 \) replications. The control limits are first specified as in Table 1 before calculating the ARLs of both HPD and Individual control charts. Table 2 and Table 3 present the ARLs of both Individual and HPD control charts for normal and weibull distribution respectively. For symmetric normal distribution, the performance of HPD control chart is equal to that of Individual
control chart because the ARLs of those control charts are not significantly different. In addition, the ARL₀ of HPD control chart for normal and weibull distribution tends to converge at about 370, which is theoretically true for α=0.0027. However, the ARL₀ of Individual control chart for weibull distribution is not stable at about 370. This points out that applying conventional Individual control chart for monitoring weibull distribution is not appropriate.

Table 3 shows that the ARL₀ of Individual control chart for weibull (2,1) and weibull (2,2) distributions are smaller than the ARL₀ target. This indicates that applying Individual control chart for monitoring both weibull (2,1) and weibull (2,2) distributions can increase the false alarm rate. Moreover, the increase of parameter σ in weibull distribution yields the higher ARL₀ value of Individual control chart. Even for weibull (2,4) distribution, the ARL₀ of Individual control chart is two times larger than that of HPD control chart. Consequently, the ARL₁ of HPD control chart for weibull (2,4) distribution is smaller than that of Individual control chart. In fact, the ARL₁ of HPD and Individual control charts are not comparable since the ARL₀ of both control charts are significantly different.

Table 3 ARLs of Individual and HPD Control Chart for Weibull Distribution

| Mean Shift | Weibull (2,1) | Weibull (2,2) | Weibull (2,4) |
|------------|---------------|---------------|---------------|
|            | Individual    | HPD           | Individual    | HPD           | Individual    | HPD           |
| 0.0        | 38.93         | 355.62        | 173.69        | 354.71        | 695.73        | 335.65        |
| 0.2        | 28.01         | 197.46        | 70.62         | 61.33         | 43.02         | 22.10         |
| 0.4        | 15.96         | 91.01         | 21.10         | 32.96         | 15.41         | 9.58          |
| 0.8        | 13.36         | 67.22         | 13.87         | 20.73         | 7.72          | 5.42          |
| 1.0        | 11.64         | 50.28         | 10.43         | 14.73         | 4.68          | 3.60          |
| 1.2        | 10.06         | 39.44         | 7.43          | 10.40         | 3.36          | 2.72          |
| 1.4        | 8.75          | 32.27         | 5.78          | 7.22          | 2.58          | 2.17          |
| 1.6        | 7.61          | 27.15         | 4.73          | 6.19          | 2.09          | 1.82          |
| 1.8        | 7.10          | 23.28         | 4.15          | 5.05          | 1.86          | 1.67          |
| 2.0        | 6.35          | 18.80         | 3.55          | 4.36          | 1.62          | 1.48          |
| 2.2        | 5.87          | 16.39         | 3.17          | 3.81          | 1.48          | 1.38          |
| 2.4        | 5.35          | 15.86         | 2.95          | 3.55          | 1.43          | 1.35          |
| 2.6        | 5.03          | 13.15         | 2.54          | 2.91          | 1.33          | 1.25          |
| 2.8        | 4.69          | 11.31         | 2.44          | 2.83          | 1.25          | 1.21          |
| 3.0        | 4.38          | 10.64         | 2.32          | 2.68          | 1.23          | 1.20          |

Table 4 ARLs of Individual and HPD Control Chart for Gamma Distribution

| Mean Shift | Gamma (2,3) | Gamma (2,4) | Gamma (1,5) |
|------------|-------------|-------------|-------------|
|            | Individual  | HPD         | Individual  | HPD         | Individual  | HPD         |
| 0.0        | 68.20       | 344.60      | 85.29       | 359.15      | 94.33       | 343.51      |
| 0.2        | 58.71       | 262.19      | 67.90       | 281.59      | 78.21       | 260.14      |
| 0.4        | 47.44       | 206.24      | 54.03       | 229.95      | 66.02       | 223.57      |
| 0.6        | 34.32       | 154.31      | 44.89       | 186.04      | 57.04       | 180.95      |
| 0.8        | 28.05       | 124.79      | 34.23       | 151.76      | 43.09       | 145.52      |
| 1.0        | 23.81       | 95.80       | 30.67       | 122.09      | 38.20       | 123.41      |
| 1.2        | 19.87       | 75.47       | 25.89       | 97.89       | 29.16       | 95.29       |
| 1.4        | 17.01       | 62.06       | 20.95       | 78.47       | 24.59       | 76.77       |
| 1.6        | 13.83       | 51.27       | 17.58       | 62.27       | 22.37       | 66.81       |
| 1.8        | 11.27       | 43.44       | 15.96       | 56.61       | 20.11       | 56.83       |
| 2.0        | 10.27       | 33.84       | 13.97       | 44.03       | 17.61       | 50.12       |
| 2.2        | 9.05        | 30.05       | 11.96       | 37.73       | 14.13       | 39.78       |
| 2.4        | 7.62        | 23.03       | 10.25       | 33.06       | 12.81       | 35.13       |
| 2.6        | 6.59        | 20.53       | 8.78        | 27.75       | 11.50       | 29.85       |
| 2.8        | 6.07        | 17.42       | 8.36        | 24.08       | 10.09       | 26.37       |
| 3.0        | 5.40        | 15.10       | 6.78        | 19.74       | 9.02        | 22.73       |
Table 5 ARLs of Individual and HPD Control Chart for Beta Distribution

| Mean Shift | Individual HPD | Individual HPD | Individual HPD | Individual HPD |
|------------|----------------|----------------|----------------|----------------|
| 0.0        | 560.87         | 339.54         | 84.85          | 347.02         | 70.76          | 347.67         |
| 0.2        | 502.35         | 292.61         | 60.85          | 240.89         | 48.16          | 241.50         |
| 0.4        | 442.76         | 244.39         | 48.79          | 180.41         | 35.08          | 183.87         |
| 0.6        | 390.86         | 211.26         | 34.93          | 133.99         | 25.91          | 141.13         |
| 0.8        | 326.56         | 168.72         | 28.43          | 88.57          | 17.04          | 81.04          |
| 1.0        | 292.02         | 149.39         | 22.68          | 53.60          | 12.27          | 51.39          |
| 1.2        | 244.27         | 119.83         | 16.59          | 31.55          | 13.92          | 28.03          |
| 1.4        | 214.67         | 101.54         | 12.36          | 23.19          | 10.29          | 20.52          |
| 1.6        | 184.43         | 87.66          | 9.23           | 17.64          | 7.68           | 15.96          |
| 1.8        | 159.35         | 78.80          | 6.54           | 12.36          | 5.19           | 10.22          |
| 2.0        | 142.85         | 67.18          | 5.28           | 8.88           | 4.34           | 7.68           |
| 2.2        | 121.26         | 50.76          | 4.61           | 6.76           | 3.70           | 5.19           |
| 2.4        | 107.04         | 35.07          | 3.96           | 5.76           | 3.03           | 4.34           |
| 2.6        | 100.20         | 28.05          | 3.24           | 4.34           | 2.30           | 3.70           |
| 2.8        | 83.92          | 21.70          | 2.42           | 3.70           | 1.76           | 3.03           |
| 3.0        | 77.66          | 15.12          | 1.71           | 2.42           | 1.10           | 2.30           |

The ARLs comparison of both Individual and HPD control charts for gamma distribution can be shown at Table 4. The ARL₀ of HPD control chart for gamma distribution tends to convergent at about 370 whereas the ARL₀ of Individual control chart for beta distribution is more than 500. That is why the ARL₁ of HPD and Individual control charts are not comparable. Furthermore, monitoring the data which follow beta (1,4) and beta (1,5) distribution using Individual control chart produces the higher false alarm rate because of its lower ARL₀. The ARLs of both Individual and HPD control charts for

Table 6: ARLs of Individual and HPD Control Chart for Lognormal Distribution

| Mean Shift | Lognormal (0,1) | Lognormal (1,1) | Lognormal (2,1) |
|------------|----------------|----------------|----------------|
| 0.0        | 13.07          | 20.03          | 13.02          |
| 0.2        | 9.42           | 13.42          | 9.23           |
| 0.4        | 6.72           | 9.33           | 6.60           |
| 0.6        | 4.61           | 6.54           | 4.86           |
| 0.8        | 3.66           | 4.90           | 3.70           |
| 1.0        | 3.12           | 3.96           | 3.03           |
| 1.2        | 2.42           | 2.93           | 2.30           |
| 1.4        | 2.02           | 2.44           | 2.05           |
| 1.6        | 1.71           | 2.01           | 1.76           |
| 1.8        | 1.52           | 1.76           | 1.56           |
| 2.0        | 1.42           | 1.58           | 1.42           |
| 2.2        | 1.28           | 1.40           | 1.28           |
| 2.4        | 1.20           | 1.30           | 1.20           |
| 2.6        | 1.14           | 1.21           | 1.12           |
| 2.8        | 1.09           | 1.15           | 1.10           |
| 3.0        | 1.06           | 1.09           | 1.07           |

The ARLs comparison of both Individual and HPD control charts for gamma distribution can be shown at Table 4. The ARL₀ of HPD control chart for gamma distribution tends to convergent at about 370 whereas the ARL₀ of Individual control chart for beta distribution is more than 500. That is why the ARL₁ of HPD and Individual control charts are not comparable. Furthermore, monitoring the data which follow beta (1,4) and beta (1,5) distribution using Individual control chart produces the higher false alarm rate because of its lower ARL₀. The ARLs of both Individual and HPD control charts for
lognormal distribution are presented at Table 6. The simulation study for lognormal distribution uses significance level $\alpha=0.05$ in order to investigate the performance of HPD control chart for different significance level. The ARL of HPD control chart for lognormal distribution is convergent to 20, which is theoretically true for $\alpha=0.05$. However, the ARL of Individual control chart for lognormal distribution is stable at about. This indicates that Individual control chart will detect an out-of control signal quickly than its nature.

6. Simulation Studies
Simulation studies are carried out in order to compare the performance of Individual and HPD control charts for monitoring simulation data. The simulation data are generated from symmetric normal distribution and some asymmetric distributions such as weibull, gamma, and beta. The simulation studies scenario is presented at Table 7. For each dataset, 700 samples of in-control process and 300 samples of shifted process are generated. Each dataset is then monitored using both Individual and HPD control charts. The control limits of those charts are estimated based on in-control process so that the shifted process indicates a shift in the parameter $\mu$.

| No | Data       | In-control Process | Shifted Process |
|----|------------|--------------------|-----------------|
| 1  | Dataset 1  | Normal (0,1)       | Normal (3,1)    |
| 2  | Dataset 2  | Weibull (2,1)      | Weibull (5,1)   |
| 3  | Dataset 3  | Gamma (2,4)        | Gamma (4,4)     |
| 4  | Dataset 4  | Beta (1,4)         | Beta (4,4)      |

Figure 2 Monitoring Result of Dataset 1 using (a) Individual Chart and (b) HPD Chart

Dataset 1 which is generated from normal distribution is monitored using Individual and HPD control charts as displayed at Figure 2. It can be known that the performance of both control charts is not significantly different. In the first 700 samples, there is no out-of control signal issued by those control charts because the actual process is in-control. Almost all of the last 300 samples of both Individual and HPD control charts are greater than the CL indicating the shift in the parameter $\mu$.

Figure 3 Monitoring Result of Dataset 2 using (a) Individual Chart and (b) HPD Chart
Figure 3 exhibits the monitoring result of dataset 2 using Individual and HPD control charts. There are some out-of control signals detected in the first 700 samples of Individual control chart in spite of the actual process is in-control (see Figure 3.a). As displayed at Figure 3.b, there are only three out-of control signals detected by HPD control chart for actual in-control process. This explains the ARL₀ of Individual control chart for weibull (2,1) distribution in Table 3 which smaller than 50 and the ARL₀ of HPD control chart which convergent to about 370. Hence, monitoring the data which follow weibull distribution using Individual control chart is not appropriate because it yields high false alarm rate.

The Individual and HPD control charts for dataset 3 are displayed at Figure 4. The HPD control chart in Figure 4.b detects two out-of control signals from 700 samples of actual in-control process which follows gamma (2,4) distribution. On the contrary, the individual control chart for dataset 3 yields many false alarms (see the first 700 samples of Figure 4.a). These conditions illustrate the ARL₀ of both control charts for gamma (2,4) distribution as in Table 4. For actual in-control process, the Individual control chart detects the first out-of control signal at about 68-th sample whereas HPD control chart detects the first out-of control signal at about 344-th sample.

The monitoring result of dataset 4 is presented at Figure 5. The performance of both Individual and HPD control charts for dataset 4 is not significantly different with that for dataset 3 and dataset 2. For monitoring the data which follow asymmetric pattern such as weibull, gamma, and beta distributions, Individual control chart yields more false alarms than HPD control chart. Thus, the proposed HPD control chart is more effective to monitor both symmetric and asymmetric distributions of data.

7. Conclusions
The simulation studies had been carried out to yield the control limits of Individual and HPD charts for normal, weibull, gamma, beta, and lognormal distributions. For symmetric normal distribution, the control limits of HPD chart are not significantly different with the control limits of Individual chart. The ARL₀ of HPD control chart tends to convergent to a certain value of ARL₀ target while the ARL₀ of Individual control chart is very varying. The proposed HPD control chart is appropriate to monitor the data having both symmetric and asymmetric distributions. On the contrary, monitoring the data which follow asymmetric distribution using Individual control chart produces high false alarm rate.
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