The Off-Shell (3/2, 2) Supermultiplets Revisited

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ABSTRACT

Using superspace projection operators we provide a classification of (3/2, 2) off-shell supermultiplets which are realized in terms of a real axial vector superfield, with or without compensating superfields. Any linearized supergravity action is shown to be a superposition of those corresponding to (i) old minimal supergravity, (ii) new minimal supergravity and (iii) the novel (3/2, 2) off-shell supermultiplet with 12 + 12 degrees of freedom obtained in [hep-th/0201096].

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1 Introduction and Outlook

Different off-shell realizations of four-dimensional $\mathcal{N} = 1$ supergravity and their couplings to supersymmetric matter were thoroughly studied in the late 1970s – early 1980s, and by now these issues have become a subject of two comprehensive textbooks [1, 2] (see also [3] for an introduction to old minimal supergravity). Therefore one could hardly expect to say anything new about off-shell $\mathcal{N} = 1$ supergravity. A surprise came a year ago. While investigating superfield models for the massive superspin-$\frac{3}{2}$ multiplet, Buchbinder et al. [4] found a new version of linearized supergravity when setting the mass parameter to zero. To the best of our knowledge, this model had been overlooked in all previous investigations. Its existence indicates that along with the old minimal, new minimal and non-minimal supergravity formulations, there may exist a new realization, with possibly interesting properties. Its existence also naturally calls upon working out a classification of all free $(3/2, 2)$ supermultiplets. The present note is aimed at deriving such a classification. An unexpected outcome of our analysis is that the novel formulation of [4] taken together with the old and new minimal formulations are the main building blocks for generating all possible linearized supergravity actions.

Our approach in this paper is very similar, in spirit, to the work [6] which provided the classification of minimal free $(1, 3/2)$ supermultiplets (massless gravitino multiplet) in terms of an unconstrained spinor superfield $\Psi_\alpha$ and its conjugate $\bar{\Psi}^{\dot{\alpha}}$. This was accomplished via the use of $\mathcal{N} = 1$ superfield projector operators $\mathcal{P}_{\alpha \dot{\alpha}}$ as formulated in [8]. In the case of massless $(3/2, 2)$ supermultiplet, we are going to consider a general linearized local action for a real axial vector superfield $H_{\alpha \dot{\alpha}}$ and then rewrite it in terms of relevant superprojectors. Gauge invariant models emerge if one requires that some superprojectors are not present in the action. As is shown below, there exist gauge invariant actions with two, three and four superprojectors present. Not all of such actions, however, describe a pure linearized supergravity multiplet. Some of them may represent a particular coupling of linearized supergravity to supersymmetric matter, and these should be discarded. The most tedious part of the classification problem is to select those models which indeed describe a single massless $(3/2, 2)$ supermultiplet.

One of the main motivations for the present work was the desire to achieve a better understanding of supersymmetric higher spin multiplets, both in the massless and massive cases. Superstring theory predicts the existence of an infinite tower of higher

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5See also [5] for the study of off-shell realizations of the massive gravitino multiplet.
spins supermultiplets, and therefore it seems important to have their manifestly supersymmetric field theoretic description(s). On the other hand, massless higher spin supermultiplets are of some importance in the framework of the AdS/CFT correspondence (see, e.g. [9, 10, 11, 12, 13, 14] and references therein). As concerns the massive case, off-shell higher spin supermultiplets have never been constructed. In the massless case, for each superspin \( s > 3/2 \) there exist two dually equivalent off-shell realizations in 4D \( \mathcal{N} = 1 \) flat superspace [15] (see [2] for a review) and anti-de Sitter superspace [16]. We also know the structure of 4D \( \mathcal{N} = 2 \) off-shell higher spin supermultiplets [17] as well as a generating superfield action for arbitrary superspin massless multiplets in 4D \( \mathcal{N} = 1 \) anti-de Sitter superspace [18]. One thing still missing is a classification of massless higher spin supermultiplets (say, those involving, for a half-integer superspin \( s = p + 1/2 \), a real tensor superfield \( H(\alpha_1 \ldots \alpha_p)(\dot{\alpha}_1 \ldots \dot{\alpha}_p) \) along with some compensators).

In the case of half-integer superspin, for \( s = 3/2 \) one of the families constructed in [15] reduces to linearized old minimal supergravity, while the other – to linearized \( n = -1 \) non-minimal supergravity. Is there a series of massless higher spin supermultiplets\(^6\) that terminates at linearized new minimal supergravity (or the novel formulation given in [1]) for \( s = 3/2 \)? Presently, we do not know an answer to this question. Since superstring compactifications are often believed to favor the new minimal formulation of supergravity, the question does not seem to be of purely academic interest. We plan to address this and related questions in a future publication. There are some grounds to believe that the analysis undertaken in the present note, can be generalized to provide a classification of higher spin massless supermultiplets.

This note is organized as follows. Section 2 is devoted to describing the super-projector setup which is crucial for our subsequent analysis. In section 3 we derive all minimal gauge invariant actions which involve two superprojectors. In section 4 the models with three projectors are analyzed. We demonstrate that all such models describe linearized supergravity coupled to a scalar multiplet. Finally, in section 4 we consider gauge invariant actions with four projectors and show that they describe linearized non-minimal supergravity parameterized by a complex parameter \( n \). The case of real \( n \) is studied in detail.

\(^6\)Recently, Engquist et al. [19] have formulated, following generalizations of the approach pioneered in [20], nonlinear equations for interacting massless higher spin \( \mathcal{N} = 1 \) superfields in four space-time dimensions. No analysis was given, however, as to the relationship between the spectrum of their model at the linearized level and the free massless supermultiplets in 4D \( \mathcal{N} = 1 \) anti-de Sitter superspace introduced in [16, 17, 18].
2 Setup

We start with the most general linearized action for $H_\underline{a}$ required to be local, CPT even and of fourth order in spinor derivatives:

$$
S = \int d^8 z \{ \alpha_1 H^a D^\beta \overleftrightarrow{D}_\beta H_\underline{a} + \alpha_2 H^a \square H_\underline{a} + \alpha_3 H^a \partial_\underline{a} \partial_\underline{b} H_\underline{a} + \alpha_4 H^a [D_\alpha, \overleftrightarrow{D}_\beta] [D_\beta, \overleftrightarrow{D}_\beta] H_\underline{a} \},
$$

(1)

with $\alpha_1, \ldots, \alpha_4$ constant dimensionless parameters. The gravitational superfield can then be represented as a superposition of SUSY irreducible components,

$$
H_\underline{a} = (\Pi_0^L + \Pi_1^{1/2} + \Pi_1^T + \Pi_2^{1/2} + \Pi_3^{3/2}) H_\underline{a},
$$

(2)

by making use of the relevant superprojectors[3][4][5]

$$
\begin{align*}
\Pi_0^L H_\underline{a} &= -\frac{1}{\sqrt{2}} \square^{-2} \partial_\underline{a} \{ D^2, \overleftrightarrow{D}^2 \} \partial_\underline{b} H_\underline{b}, \\
\Pi_1^{1/2} H_\underline{a} &= \frac{1}{\sqrt{2}} \square^{-2} \partial_\underline{a} \overleftrightarrow{D}^2 D_\beta \partial_\underline{b} H_\underline{b}, \\
\Pi_1^{1/2} H_\underline{a} &= \frac{1}{38} \square^{-2} \partial_\underline{a} \alpha [D_\beta \overleftrightarrow{D}^2 D_\beta \partial_\underline{b} H_\underline{b} + D_\alpha \overleftrightarrow{D}^2 D_\beta \partial_\underline{b} \beta H_\underline{b}] \\
\Pi_1^T H_\underline{a} &= \frac{1}{38} \square^{-2} \partial_\underline{a} \beta \{ D^2, \overleftrightarrow{D}^2 \} \partial_\underline{b} \beta H_\underline{b}, \\
\Pi_3^{3/2} H_\underline{a} &= -\frac{1}{38} \square^{-2} \partial_\underline{a} \alpha \beta \{ D^2, \overleftrightarrow{D}^2 \} \partial_\underline{b} \gamma \partial_\underline{b} \beta H_\underline{b}. 
\end{align*}
$$

(3)

Here the superscripts $L$ and $T$ denote longitudinal and transverse projectors, while the subscripts $0, 1/2, 1, 3/2$ stand for superspin. One can readily express the action in terms of the superprojectors. It is a D-algebra exercise to show

$$
\begin{align*}
D^\gamma \overleftrightarrow{D}^2 D_\gamma H_\underline{a} &= -8 \square (\Pi_1^{1/2} + \Pi_1^T + \Pi_3^{3/2}) H_\underline{a}, \\
\partial_\underline{a} \partial_\underline{b} H_\underline{a} &= -2 \square (\Pi_0^L + \Pi_1^{1/2}) H_\underline{a}, \\
[D_\alpha, \overleftrightarrow{D}_\beta] [D_\alpha, \overleftrightarrow{D}_\beta] H_\underline{a} &= + \square (8 \Pi_0^L - 24 \Pi_1^T) H_\underline{a},
\end{align*}
$$

(4)

and therefore the action [1] takes the form

$$
S = \int d^8 z H_\underline{a} \square \{ (\alpha_2 - 2 \alpha_3 + 8 \alpha_4) \Pi_0^L + (-8 \alpha_1 + \alpha_2 - 2 \alpha_3) \Pi_1^{1/2} + (\alpha_2 \Pi_1^T + (-8 \alpha_1 + \alpha_2 - 24 \alpha_4) \Pi_1^{1/2} + (-8 \alpha_1 + \alpha_2) \Pi_3^{3/2}) \} H_\underline{a}.
$$

(5)

It is the projector $\Pi_3^{3/2}$ which singles out the superspin-3/2 part of $H_\underline{a}$, and the corresponding projection $\Pi_3^{3/2} H_\underline{a}$ is invariant under the gauge transformation

$$
\delta H_\underline{a} = \overleftrightarrow{D}_\alpha L_\alpha - D_\alpha \overleftrightarrow{L}_\alpha,
$$

(6)

We use a slightly simplified notation for the superprojectors as compared with [3]. The precise dictionary is the following: $\Pi_0^L = \Pi_{0,0}^L$, $\Pi_1^{1/2} = \Pi_{1,0}^{1/2}$, $\Pi_1^T = \Pi_{1,1/2}^T$, $\Pi_3^{3/2} = \Pi_{3,1/2}^T$, $\Pi_3^{3/2} = \Pi_{3,1/2}^T$.
which is typical of linearized conformal supergravity. This also means that the coefficient multiplying this projector in the action must never vanish.

However, since $\Pi_{T_{3/2}}$ is non-local, any local action involving this projector must contain at least one other projector. The other projectors do not respect the gauge freedom (6) with unconstrained $L_\alpha$, as is seen from explicit variations

$$\delta \int d^8z H^a \Pi^a T_{1/2} H_a = -\frac{i}{4} \int d^8z \partial_\alpha H_a \left[ D^\alpha D^2 L_\alpha - \overline{D}_\alpha D^2 \overline{L}_\alpha \right];$$

(7)

$$\delta \int d^8z H^a \Pi^a T_{1/2} H_a = -\frac{1}{4} \int d^8z \left[ D_\alpha, D_\beta \right] H_a \left[ D^\alpha D^2 L_\alpha + \overline{D}_\alpha D^2 \overline{L}_\alpha \right];$$

(8)

$$\delta \int d^8z H^a \Pi^a T_{1/2} H_a = -\frac{i}{4} \int d^8z \left[ \partial_\alpha \beta H^a \overline{D}_\alpha D_\beta L_\alpha + \delta_\alpha \beta H^a D^2 D_\beta \overline{L}_\beta \right].$$

(9)

Therefore, given a gauge invariant action of the form (5), with $\alpha_2 - 8\alpha_1 \neq 0$, the gauge parameter $L_\alpha$ in (6) should obey some constraints. The gauge freedom thus becomes model dependent. This may be avoided at the cost of introducing special lower spin compensator(s) $\Upsilon$ with a nontrivial transformation law under (6) such that imposing the admissible gauge condition $\Upsilon = 0$ is equivalent to restoring the original constraints on the gauge parameter. This is of course the standard compensator philosophy in supersymmetric theories [1].

In what follows, we will uncover all gauge invariant models of the form (5) which describe the free $(3/2, 2)$ multiplet. We will set $\alpha_1 = -\frac{1}{16}$ to ensure canonical normalization.

### 3 Minimal Models with Two Projectors

There exist three minimal theories which involve two superprojectors and realize the free $(3/2, 2)$ multiplet with 12 + 12 off-shell degrees of freedom.

Consider first an action of the form (5) which contains the projectors $\Pi^c_{3/2}$ and $\Pi^c_0$ only. All the coefficients turn out to be uniquely fixed: $\alpha_2 = 0$, $\alpha_3 = 1/4$ and $\alpha_4 = 1/48$. As follows from (7), the action possesses the gauge invariance (6) with the gauge parameter constrained by

$$\overline{D}_\alpha D^\alpha L_\alpha = 0. \tag{11}$$

The constraint can be avoided by introducing a chiral scalar compensator $\sigma$, $\overline{D}_\alpha \sigma = 0$, with the following gauge transformation

$$\delta \sigma = -\frac{1}{4\alpha_2} \overline{D}^2 D^\alpha L_\alpha. \tag{12}$$
With the compensator present, the action turns into the linearized action of old minimal supergravity \([1, 2]\):

\[
S_{\text{Old-Min}} = \int d^8 z \left\{ H^a_\alpha \Box (-\frac{1}{2} \Pi^L_0 + \frac{1}{2} \Pi^T_{3/2}) H^a_\alpha - i(\sigma - \bar{\sigma}) \partial_\alpha H^a_\alpha - 3\sigma \bar{\sigma} \right\} .
\]  

(13)

Consider next an action of the form (5) which contains the projectors \(\Pi^T_{3/2}\) and \(\Pi^L_{1/2}\) only. It is singled out by the conditions \(\alpha_2 = 0, \alpha_3 = 1/4\) and \(\alpha_4 = 1/16\). As follows from (9), the action possesses the gauge invariance (6) provided the gauge parameter is constrained by

\[
D^\alpha \overline{D}^2 L_\alpha + \overline{D}_\alpha D^2 \overline{L}^\alpha = 0 .
\]  

(14)

The constraint can be eliminated at the cost of introducing a linear real compensator \(U = \overline{U}, \overline{D}^2 U = 0\), with the gauge transformation

\[
\delta U = \frac{i}{12} (D^\alpha \overline{D}^2 L_\alpha + \overline{D}_\alpha D^2 \overline{L}^\alpha) .
\]  

(15)

As a result, we end up with the linearized action of new minimal supergravity (see, e.g. [2])

\[
S_{\text{New-Min}} = \int d^8 z \left\{ H^a_\alpha \Box (-\Pi^T_{1/2} + \frac{1}{2} \Pi^T_{3/2}) H^a_\alpha + \frac{1}{2} U [D_\alpha, \overline{D}_\alpha] H^a_\alpha + \frac{3}{2} U^2 \right\} .
\]  

(16)

We now turn to the case when only the projectors \(\Pi^T_{3/2}\) and \(\Pi^L_{1/2}\) appear in the action (5). This corresponds to the choice \(\alpha_2 = 0, \alpha_3 = 1/12\) and \(\alpha_4 = 1/48\). In accordance with (8), the action possesses the gauge invariance (6) provided the gauge parameter is constrained by

\[
D^\alpha \overline{D}^2 L_\alpha - \overline{D}_\alpha D^2 \overline{L}^\alpha = 0 .
\]  

(17)

The constraint can be eliminated at the cost of introducing a linear real compensator \(U = \overline{U}, \overline{D}^2 U = 0\), with the gauge transformation

\[
\delta U = \frac{i}{12} (D^\alpha \overline{D}^2 L_\alpha - \overline{D}_\alpha D^2 \overline{L}^\alpha) .
\]  

(18)

As a result, we arrive at the following model

\[
S = \int d^8 z \left\{ H^a_\alpha \Box (\frac{1}{4} \Pi^L_{1/2} + \frac{1}{2} \Pi^T_{3/2}) H^a_\alpha + U \partial_\alpha H^a_\alpha + \frac{3}{2} U^2 \right\} ,
\]  

(19)

which was derived in [1] and which partially prompted the present investigation.

The above three models are known to be classically equivalent. The auxiliary action connecting the old minimal and new minimal models is given in [2], while the auxiliary action relating the old minimal model to (19) is given in [3].
It remains to consider the case when the action \ref{eq:action} involves the projectors \( \Pi_{3/2}^T \) and \( \Pi_{1}^T \) only. It corresponds to the choice \( \alpha_2 = 1, \alpha_3 = 3/4 \) and \( \alpha_4 = 1/16 \). As is seen from \ref{eq:constraint} the gauge freedom is now constrained by
\[
\overline{D}^2 D_{(\alpha} L_{\beta)} = 0 .
\] (20)

Such gauge symmetry turns out to be too restrictive to correspond to a pure \((3/2, 2)\) multiplet. The simplest way to understand this is to look at the component structure of all the models so far discussed.

With no compensators present, we have to deal with a restricted set of gauge transformations \ref{eq:gauge_transformations} such that \( L_\alpha \) obeys some constraint which can be symbolically represented as follows (the precise position of spinor derivatives as well as the index structure are not essential for our consideration at this point)
\[
\alpha \overline{D}^2 D L + \beta D^2 \overline{D} L = 0 ,
\] (21)

with \( \alpha \) and \( \beta \) constant parameters. This constraint does not impose any restrictions on the component fields of \( L \) of order \( \theta^n \), where \( n = 0, 1, 2 \). Therefore, we can gauge away all the component fields of \( H \) of order \( \theta^n \), with \( n = 0, 1 \), resulting with the Wess-Zumino gauge
\[
H_{\alpha\dot{\alpha}}(\theta, \bar{\theta}) = \theta^2 S_{\alpha\dot{\alpha}} + \bar{\theta}^2 \bar{S}_{\dot{\alpha}\dot{\alpha}} + \theta^3 \bar{\theta}^3 E_{\alpha\dot{\alpha},\dot{\beta}\dot{\beta}} + \theta^2 \bar{\theta}^2 A_{\alpha\dot{\alpha}} + \text{fermions} .
\] (22)

Were \( L_\alpha \) unconstrained, one could have gauged away \( S_{\alpha\dot{\alpha}} \) completely. When \( L_\alpha \) is constrained as above, some part of \( S_{\alpha\dot{\alpha}} \) can still be gauged away while the rest survives as an auxiliary field.

Let us concentrate on the component field \( E_{\alpha\dot{\alpha},\dot{\beta}\dot{\beta}} \) which is to be identified with a spin-2 field. Its Lorentz irreducible components are
\[
\begin{align*}
& h_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} , \quad h & \quad & \text{massless spin-2 field} ; \\
& \Omega_{(\alpha\beta)} , \quad \overline{\Omega}_{(\dot{\alpha}\dot{\beta})} & \quad & \text{gauge degrees of freedom} .
\end{align*}
\] (23)

The gauge superfield parameter \( L_\alpha \) should not be over-constrained in the sense that it should contain enough component parameters to be able to gauge \( \Omega_{(\alpha\beta)} \) away. The relevant part of \( L_\alpha \) occurs at the level \( \theta^2 \bar{\theta} \) which is
\[
L_\alpha(\theta, \bar{\theta}) \propto \bar{\theta}^2 \theta_\alpha (f + i g) + \bar{\theta}^2 \theta^3 \Upsilon_{(\alpha\beta)} .
\] (24)

It is the parameter \( \Upsilon_{(\alpha\beta)} \) which allows us to gauge \( \Omega_{(\alpha\beta)} \) away. In the case of constraints \ref{eq:constraint1}, \ref{eq:constraint2} and \ref{eq:constraint3}, the gauge parameter \( \Upsilon_{(\alpha\beta)} \) remains completely arbitrary.
On the other hand, the constraint (20) implies $\Upsilon_{(\alpha\beta)} = 0$, and hence there is no gauge freedom at all to eliminate $\Omega_{(\alpha\beta)}$. As a result, the fourth model considered above does not describe the free $(3/2, 2)$ multiplet. The same is actually true for any action involving the projector $\Pi_1$. This is why we will set $\alpha_2 = 0$ in what follows.

To get an idea of the main difference between the models (16) and (19), which otherwise look very similar, it is worth pursuing the component analysis a bit further. With the compensators present, we can choose a stronger Wess-Zumino gauge than the one given in (22):

$$H_{\alpha\dot{\alpha}}(\theta, \bar{\theta}) = \theta^\beta \bar{\theta}^{\dot{\beta}} \left( h_{(\alpha\beta)(\dot{\alpha}\dot{\beta})} + \varepsilon_{\alpha\beta} \varepsilon_{\dot{\alpha}\dot{\beta}} h \right) + \theta^2 \bar{\theta}^2 A_{\alpha\dot{\alpha}} + \text{fermions} .$$

(25)

The remaining independent gauge parameters are

$$L_{\alpha}(\theta, \bar{\theta}) = i \bar{\theta}^{\dot{\alpha}} \zeta_{\alpha\dot{\alpha}} + \bar{\theta}^2 \theta_{\alpha} \left( f + i \bar{g} \right) + \text{fermions} .$$

(26)

Here the parameter $\zeta_{\alpha\dot{\alpha}}$ generates linearized spin-2 gauge transformations. The parameters $f$ and $g$ turn out to play rather different roles in the models (16) and (19).

In the new minimal model, the parameter $f$ can be used to gauge away the leading ($\theta$-independent) component of the compensator $U$. After that, $U$ possesses a single bosonic component $G_{\alpha\dot{\alpha}}$:

$$U(\theta, \bar{\theta}) = \theta^\alpha \bar{\theta}^{\dot{\alpha}} G_{\alpha\dot{\alpha}} + \text{fermions} ,$$

(27)

which is the field strength of a gauge two-form, $\partial^a G_a = 0$. The parameter $g$ generates local $\gamma_5$-transformations for which $A_m$ is a gauge field, $\delta A_m \propto \partial_m g$.

(28)

The fields $G_m$ and $A_m$ appear in the action as follows

$$S_{\text{New-Min}} \propto \int d^4x \left( A_m G^m + G_m G^m \right) ,$$

(29)

with the local $\gamma_5$-invariance being manifest. It is only with the choice $\alpha_4 = 1/16$, which is characteristic of the new minimal supergravity, that no quadratic term $A^2$ is present in the action.

In the model of [4], the parameter $g$ can be used to gauge away the leading ($\theta$-independent) component of the compensator $U$,

$$U(\theta, \bar{\theta}) = \theta^\alpha \bar{\theta}^{\dot{\alpha}} G_{\alpha\dot{\alpha}} + \text{fermions} ,$$

(30)
with $G_{\alpha\dot{\alpha}}$ the field strength of a gauge two-form, $\partial^a G_a = 0$. As to the parameter $f$, now it can be used to gauge away the trace component $h$ of the gravitational field, see (29),

$$\delta h = f .$$

(31)

Setting $h = 0$, the only remaining gauge symmetry is linearized general coordinate transformations generated by $\zeta_m$. The corresponding transformation law of $G_m$ is

$$\delta G_m \propto \partial^a (\partial_m \zeta_a - \partial_a \zeta_m) .$$

(32)

In this formulation, the gravitational field is described by a symmetric traceless tensor $h_{mn}$, $h_{mn} = h_{nm}$, $h_{mm} = 0$

(33)

together with an antisymmetric tensor $B_{mn}$, $B_{mn} = -B_{nm}$

(34)

which generates the field strength $C^m = \frac{1}{2} \varepsilon^{mnr} \partial_n B_{rs}$. The two fields, $h_{mn}$ and $B_{mn}$, can be combined into a general traceless tensor $h_{mn} + B_{mn}$. Essentially, what we end up with is a dual formulation for the linearized massless spin-2 action, in which the trace component $h$ of the gravitational field is traded for a gauge two-form.

4 No Irreducible Models with Three Projectors

We now turn to the study of actions with the projector $\Pi^T_{3/2}$ and two others. Since we have set $\alpha_2 = 0$, there are three possible cases, using different combinations of the three projectors $\Pi^L_0$, $\Pi^T_{1/2}$ and $\Pi^T_{3/2}$. Technically, such models appear to look like a sum of two of the minimal theories discussed in the previous section. Thus, we will simply present the gauge invariant actions with compensators included.

With $\alpha_3 \neq \frac{1}{12}, \frac{1}{4}$ and $\alpha_4 = \frac{1}{48}$ we arrive at the theory containing $\sigma$ and $U$:

$$S_{\sigma, U} = \int d^8z \left\{ H^a_{\alpha\dot{\alpha}} \left[ -2(\alpha_3 - \frac{1}{12}) \Pi^L_0 - 2(\alpha_3 - \frac{1}{4}) \Pi^L_{1/2} + \frac{1}{2} \Pi^T_{3/2} \right] + \partial^a \left[ H^a_{\alpha\dot{\alpha}} \right] + 6[i(\alpha_3 - \frac{1}{12})(\sigma - \bar{\sigma}) + (\alpha_3 - \frac{1}{4})U] \partial_a H^a_{\alpha\dot{\alpha}} \right. \right.$$  
\[ \left. - 18(\alpha_3 - \frac{1}{12}) \sigma \bar{\sigma} - 9(\alpha_3 - \frac{1}{4})U^2 \right\} . \]

(35)

If we set $\alpha_3 = \frac{1}{4}$ and $\alpha_4 \neq \frac{1}{36}, \frac{1}{16}$ we are lead to the action:

$$S_{\sigma, U} = \int d^8z \left\{ H^a_{\alpha\dot{\alpha}} \left[ +8(\alpha_4 - \frac{1}{16}) \Pi^L_0 - 24(\alpha_4 - \frac{1}{36}) \Pi^T_{1/2} + \frac{1}{2} \Pi^T_{3/2} \right] H^a_{\alpha\dot{\alpha}} \right.$$  
\[ \left. - 12[(\alpha_4 - \frac{1}{16})(\sigma + \bar{\sigma}) - (\alpha_4 - \frac{1}{36})U][D_\alpha, \bar{D}_\dot{\alpha}] H^a_{\alpha\dot{\alpha}} \right. \right.$$  
\[ \left. + 72(\alpha_4 - \frac{1}{16}) \sigma \bar{\sigma} + 36(\alpha_4 - \frac{1}{36})U^2 \right\} . \]

(36)
Finally, by setting $\alpha_3 \neq \frac{1}{4}, \frac{1}{12}$ and $\alpha_4 = \frac{1}{4}\alpha_3$ we can construct an action containing only the linear compensators:

$$S_{U,U} = \int d^8z \left\{ \frac{H^2}{\Box} \left[ -2(\alpha_3 - \frac{1}{4})\Pi^L_{1/2} - 6(\alpha_3 - \frac{1}{12})\Pi^T_{1/2} + \frac{1}{2}\Pi^T_{3/2} \right] H^2 
+ 3(\alpha_3 - \frac{1}{12})U[D_a, \overline{D}_a]H^2 - 6(\alpha_3 - \frac{1}{4})U\partial_\alpha H^2 
+ 9(\alpha_3 - \frac{1}{12})U^2 - 9(\alpha_3 - \frac{1}{4})U^2 \right\}. \quad (37)$$

These actions describe $16 + 16$ off-shell degrees of freedom.

The above models prove to be equivalent as they are related to each other by superfield duality transformations. The following two dual actions connect (35) to (37) and (36) to (37), respectively:

$$S^{(1)}_{\text{Aux}} = \int d^8z \left\{ \frac{H^2}{\Box} \left[ -2(\alpha_3 - \frac{1}{4})\Pi^L_{1/2} - 6(\alpha_3 - \frac{1}{12})\Pi^T_{1/2} + \frac{1}{2}\Pi^T_{3/2} \right] H^2 
- 6(\alpha_3 - \frac{1}{4})U\partial_\alpha H^2 - 9(\alpha_3 - \frac{1}{4})U^2 
+ W[3(\alpha_3 - \frac{1}{12})][D_a, \overline{D}_a]H^2 + 9(\alpha_3 - \frac{1}{12})W - 18(\alpha_3 - \frac{1}{12})(\sigma + \overline{\sigma}) \right\}; \quad (38)$$

$$S^{(2)}_{\text{Aux}} = \int d^8z \left\{ \frac{H^2}{\Box} \left[ -2(\alpha_3 - \frac{1}{4})\Pi^L_{1/2} - 6(\alpha_3 - \frac{1}{12})\Pi^T_{1/2} + \frac{1}{2}\Pi^T_{3/2} \right] H^2 
+ 3(\alpha_3 - \frac{1}{12})U[D_a, \overline{D}_a]H^2 + 9(\alpha_3 - \frac{1}{12})U^2 
+ X[-6(\alpha_3 - \frac{1}{4})\partial_\alpha H^2 - 9(\alpha_3 - \frac{1}{4})X - 6i(\alpha_4 - \frac{1}{12})(\sigma - \overline{\sigma})] \right\}. \quad (39)$$

If we vary these actions with respect to the real unconstrained superfields $W$ and $X$, we are led to (35) and (36). If instead we vary the actions with respect to $\sigma$, then $W$ and $X$ become real linear superfields, and both (38) and (39) turn into (37).

Since the models (35), (36) and (37) are dually equivalent, it is sufficient to analyze the multiplet structure, say, of the model (35). The important point here is that the leading ($\theta$-independent) components of $\sigma$, $\overline{\sigma}$ and $U$ are (pseudo) scalars of mass dimension +1. If these fields cannot be algebraically gauged away, the theory contains propagating scalars and, therefore, does not describe a pure $(3/2, 2)$ multiplet. In the Wess-Zumino gauge (26), there are only two local parameters, $f$ and $g$ in (26), which can be used for gauging away the (pseudo) scalars under consideration. These gauge parameters are not enough to do the job.

There might be a way out provided all the dependence of (35) on the superfields $\sigma$, $\overline{\sigma}$ and $U$ could be expressed via a single unconstrained real superfield of the form $V = i(\alpha_3 - \frac{1}{12})(\sigma - \overline{\sigma}) + (\alpha_3 - \frac{1}{4})U$. If this were possible, there would be enough gauge freedom to remove the dangerous leading component of $V$. A direct analysis shows this is not the case. Actually, in the action (35) one can trade the compensators $\sigma$, $\overline{\sigma}$ and $U$ for, say, a complex superfield $M = i(\alpha_3 - \frac{1}{12})\sigma + \frac{1}{2}(\alpha_3 - \frac{1}{4})U$ and its conjugate.
\[ \overline{M} \text{ under the constraints} \]
\[ D^2 M = 0 , \ D^2 D\alpha (M - \overline{M}) = 0 . \] (40)

Now, one is in a position to gauge away the leading components of \( M \) and \( \overline{M} \). Unfortunately, \( M \) still contains a transverse vector field, at the \( \theta \overline{\theta} \) level, which is the field strength of a two form gauge potential. The gauge two-form enters the action as a propagating field.

Our conclusion is that the models (35), (36) and (37) do not describe an irreducible \((3/2, 2)\) multiplet. In fact, at the full non-linear level, we have the following equivalences:

- (35) = old minimal SUGRA coupled to a tensor multiplet;
- (36) = new minimal SUGRA coupled to a chiral multiplet;
- (37) = new minimal SUGRA coupled to a tensor multiplet.

This is in agreement with the the old result [21] that 16 + 16 supergravity [22, 23] is reducible, and represents only a particular coupling of supergravity to a tensor multiplet.

5 Non-minimal Models with Four Projectors

It remains to analyze gauge invariant actions involving four projectors. With the standard choice \( \alpha_1 = -\frac{1}{16} \), \( \alpha_2 = 0 \) and with the compensators included, the corresponding action reads

\[
S = \int d^8 z \left\{ H_{\alpha} \overline{\Pi} \left[ (-2\alpha_3 + 8\alpha_4)\Pi_{0}^L + \left( \frac{1}{2} - 2\alpha_3 \right) \Pi_{1/2}^L + \left( \frac{1}{2} - 24\alpha_4 \right) \Pi_{1/2}^T + \frac{1}{2} \Pi_{3/2}^T \right] H_{\bar{\alpha}} \right\} + 3 \left[ i(-2\alpha_3 + 8\alpha_4)(\sigma - \bar{\sigma}) + (\frac{1}{2} - 2\alpha_3) U \right] \partial_{\alpha} H^2 - \frac{1}{2} (\frac{1}{2} - 24\alpha_4) U \left[ D_{\alpha}, \overline{D}_{\bar{\alpha}} \right] H^2 \\
+ 9(-2\alpha_3 + 8\alpha_4)\sigma\bar{\sigma} - \frac{3}{2} (\frac{1}{2} - 24\alpha_4) U^2 + \frac{9}{2} (\frac{1}{2} - 2\alpha_3) U^2 \right\} , \] (41)

and presents itself a sum of the three minimal theories derived in section 3. At first sight, such models look even more hopeless, in the sense of describing a pure \((3/2, 2)\) multiplet, than the ones considered in the previous section. Fortunately, the situation is not really that bad and there is a way out. The point is that the compensators \( \sigma \), \( U \) and \( \bar{U} \) can now be arranged into a complex linear superfield \( \Sigma = a \sigma + b U + i c U \), for some real coefficients \( a, b \) and \( c \), under the constraint

\[ \overline{D}^2 \Sigma = 0 . \] (42)
In the Wess-Zumino gauge (25), there is still enough gauge freedom to remove the
dangerous θ-independent components of Σ and \( \Sigma \) (as well as a spinor component
of \( \Sigma \) at the next-to-leading order); the remaining component fields of Σ and \( \Sigma \) enter
the action simply as auxiliaries. By expressing (41) via Σ and \( \Sigma \) (along with implementing
a field redefinition of the form \( \Sigma \to \Sigma + \kappa D_\alpha D_\alpha H^a \), for some parameter \( \kappa \)), one should
end up with the linearized action of non-minimal supergravity parameterized by a
complex parameter \( n \) [24]. Since the case of real-\( n \) non-minimal supergravity is more
familiar [1, 2], we are going to describe in more detail the procedure of how it occurs
in the present framework. The latter case turns out to correspond to a specific choice
in conjunction with a special relationship between \( \alpha_3 \) and \( \alpha_4 \).

To express the action (41) via Σ and \( \Sigma \), we should first match the
\( H^a \) couplings. These terms necessarily take the form:
\[
i(\Sigma - \overline{\Sigma}) \partial_a H^a = \left\{ ia (\sigma - \overline{\sigma}) - 6b U \right\} \partial_a H^a
\]
\[
\int d^8 z (\Sigma + \overline{\Sigma})(D_\alpha, \overline{D_\alpha}) H^a = -2 \int d^8 z \left\{ ia (\sigma - \overline{\sigma}) \partial_a H^a - b U[D_\alpha, \overline{D_\alpha}] H^a \right\} .
\]
Furthermore, since there is no coupling between \( U \) and \( \mathcal{U} \) the kinetic terms for Σ
must take the following form:
\[
\int d^8 z \Sigma \overline{\Sigma} = \int d^8 z \left\{ a^2 \sigma \overline{\sigma} + b^2 (\mathcal{U}^2 + 9 U^2) \right\} ,
\]
\[
\int d^8 z \left\{ \Sigma^2 + \overline{\Sigma}^2 \right\} = 2 b^2 \int d^8 z \left\{ \mathcal{U}^2 - 9 U^2 \right\} .
\]
Choosing the Σ-dependent part of the Lagrangian to be of the form
\[
i(\Sigma - \overline{\Sigma}) \partial_a H^a + \alpha (\Sigma + \overline{\Sigma})[D_\alpha, \overline{D_\alpha}] H^a + \beta \Sigma \overline{\Sigma} + \gamma (\Sigma^2 + \overline{\Sigma}^2) ,
\]
we can readily get all the coefficients right. For the parameters \( a \) and \( b \) in (43) we obtain
\[
a = -\frac{1}{4} \frac{(-2 \alpha_3 + 8 \alpha_4)(\frac{1}{2} - 2 \alpha_3)}{(-2 \alpha_3 + 24 \alpha_4)} , \quad b = -\frac{1}{2} (\frac{1}{2} - 2 \alpha_3) .
\]
Next, for the parameters \( \alpha \) and \( \gamma \) in (46) we get
\[
\alpha = \frac{1}{2} \left( \frac{1}{2} - 24 \alpha_4 \right) , \quad \gamma = \frac{(2 - 2 \alpha_3 - 72 \alpha_4)}{(\frac{1}{2} - 2 \alpha_3)} ,
\]
while for \( \beta \) there occur two different expressions,
\[
\beta = \frac{(-2 \alpha_3 + 24 \alpha_4)^2}{(-2 \alpha_3 + 8 \alpha_4)(\frac{1}{2} - 2 \alpha_3)^2} = \frac{(72 \alpha_4 - 2 \alpha_3 - 1)}{(\frac{1}{2} - 2 \alpha_3)^2} ,
\]
and these imply

\[ \alpha_4 = \frac{2\alpha_3}{64\alpha_3 + 8} . \]  \hspace{1cm} (50)

Setting \( \alpha_3 = -\frac{n+1}{8n} \) gives \( \alpha_4 = \frac{n+1}{32} \). This is the exact solution for linearized non-minimal supergravity given in \[2\].

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