The Logarithmic Sobolev Inequality on the Heisenberg group for the Infinite Dimensional Gibbs measure.

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Abstract

We are interested in the $q$ Logarithmic Sobolev inequality for infinite dimensional measures on the Heisenberg group. We assume that the one site boundary free measure satisfies either a $q$ Log-Sobolev inequality or a U-Bound inequality, and we determine conditions so that the infinite dimensional Gibbs measure satisfies a $q$ Log-Sobolev inequality.

Keywords Logarithmic Sobolev inequality·Gibbs measure·Heisenberg group·Infinite dimensions

Mathematics Subject Classification (2000) 60E15·26D10·39B62·22E30·82B20

1 Introduction

We focus on the $q$ Logarithmic Sobolev Inequality (LSq) for the infinite dimensional Gibbs measure related to systems of spins with values on the Heisenberg group. More specifically, we extend the already know results for real valued spins with interactions $V$ that satisfy $\|\nabla_i \nabla_j V(x_i, x_j)\|_\infty \leq \infty$ to the case of the Heisenberg group. We investigate two cases, that of a boundary free one site measure that satisfies an (LSq) inequality and that of a one site measure with interactions that satisfies a non uniform U-Bound inequality. In both cases we determine conditions so that the infinite dimensional Gibbs measure satisfies an (LSq) inequality.

In this section we present the main definitions as well as some of the most relevant past results concerning the Log-Sobolev inequality on the Heisenberg group and the infinite dimensional setting.

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1.1 (LSq) on the Heisenberg Group.

The Heisenberg group is one of the simplest sub-Riemannian settings in which we can define non-elliptic Hörmander type generators. We can then consider coercive inequalities associated to such generators.

Most of the attention has been focused on the case of elliptic generators, for which there are some very powerful methods for proving such inequalities. For $L$ being the generator of a Markov semigroup $P_t$ and

$$\Gamma(f, f) = \frac{1}{2} (L(f^2) - 2fLf)$$
$$\Gamma_2(f, f) = \frac{1}{2} [L\Gamma(f, f) - 2\Gamma(f, Lf)].$$

we can apply the $CD(\rho, \infty)$ or $\Gamma_2$ criterion ([B]), that is

$$\Gamma_2(f, f) \geq \rho \Gamma(f, f)$$

for some constant $\rho \in \mathbb{R}$. It is a well know result that the $CD(\rho, \infty)$ condition implies the Log-Sobolev inequality (see [B], [B-E]). When $L$ is elliptic the condition holds in many situations. In the case when $M$ is a complete connected Riemannian manifold, and $\nabla$ and $\Delta$ are the standard Riemannian gradient and Laplace-Beltrami operators respectively, taking $L = \Delta$, the condition reads

$$|\nabla \nabla f|^2 + \text{Ric}(\nabla f, \nabla f) \geq \rho |\nabla f|^2$$

(see [A-B-C], [B]). This holds for some $\rho \in \mathbb{R}$ when $M$ is compact, or for $\rho = 0$ when $M = \mathbb{R}^n$ with the usual metric, since Ricci $= 0$ in the last case. However, we can still consider non-elliptic Hörmander generators. For such generators these methods do not work, since the $CD(\rho, \infty)$ condition does not hold (see [B-B-B-C]). Indeed, the Ricci tensor of our generators can be thought of as being $-\infty$ almost everywhere. For the case of the Heisenberg group in particular, for $L = \Delta$, following [B-B-B-C] we can calculate the $\Gamma$ and $\Gamma_2$ operators for this generator explicitly

$$\Gamma_2(f, f) = (X^2 f)^2 + (Y^2 f)^2 + \frac{1}{2} (XY + YX) f)^2 + \frac{1}{2} (Z f)^2$$
$$+ 2 ((XZ f)Y f - (YZ f)X f)$$

while

$$\Gamma(f, f) = (X f)^2 + (Y f)^2$$

Because of the presence of $XZ(f)$ and $YZ(f)$ in the expression of $\Gamma_2$ one can see that there cannot exist a constant $\rho \in \mathbb{R}$ such that $\Gamma_2 \geq \rho \Gamma$ in terms of quadratic forms.
**The Heisenberg group.** The Heisenberg group, \( \mathbb{H} \), can be described as \( \mathbb{R}^3 \) with the following group operation:

\[
x \cdot \tilde{x} = (x_1, x_2, x_3) \cdot (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (x_1 + \tilde{x}_1, x_2 + \tilde{x}_2, x_3 + \tilde{x}_3 + \frac{1}{2}(x_1 \tilde{x}_2 - x_2 \tilde{x}_1))
\]

\( \mathbb{H} \) is a Lie group, and its Lie algebra \( \mathfrak{h} \) can be identified with the space of left invariant vector fields on \( \mathbb{H} \) in the standard way. By direct computation we see that this space is spanned by

\[
X_1 = \partial_{x_1} - \frac{1}{2} x_2 \partial_{x_3} \\
X_2 = \partial_{x_2} + \frac{1}{2} x_1 \partial_{x_3} \\
X_3 = \partial_{x_3} = [X_1, X_2]
\]

From this it is clear that \( X_1, X_2 \) satisfy the Hörmander condition (i.e. \( X_1, X_2 \) and their commutator \([X_1, X_2]\) span the tangent space at every point of \( \mathbb{H} \)). It is also easy to check that the left invariant Haar measure (which is also the right invariant measure since the group is nilpotent) is the Lebesgue measure \( dx \) on \( \mathbb{R}^3 \).

On \( C_0^\infty(\mathbb{H}) \), define the sub-gradient to be the operator given by

\[
\nabla := (X_1, X_2)
\]

and the sub-Laplacian to be the second order operator given by

\[
\Delta := X_1^2 + X_2^2
\]

\( \nabla \) can be treated as a closed operator from \( L^2(\mathbb{H}, dx) \) to \( L^2(\mathbb{H}; \mathbb{R}^2, dx) \). Similarly, since \( \Delta \) is densely defined and symmetric in \( L^2(\mathbb{H}, dx) \), we may treat \( \Delta \) as a closed self-adjoint operator on \( L^2(\mathbb{H}, dx) \) by taking the Friedrich extension. We introduce the Logarithmic Sobolev Inequality on \( \mathbb{H} \) in the following way.

**The q Log-Sobolev Inequality (LSq) on \( \mathbb{H} \).** Let \( q \in (1, 2] \), and let \( \mu \) be a probability measure on \( \mathbb{H} \). \( \mu \) is said to satisfy a \emph{q Logarithmic Sobolev inequality (LSq)} on \( \mathbb{H} \) if there exists a constant \( c > 0 \) such that for all smooth functions \( f : \mathbb{H} \to \mathbb{R} \)

\[
\mu \left( |f|^q \log \frac{|f|^q}{\mu |f|^q} \right) \leq c \mu (|\nabla f|^q)
\]

where \( \nabla \) is the sub-gradient on \( \mathbb{H} \).
Remark 1.1. Since we have the sub-gradient on the right hand side, \((1.1)\) is a Logarithmic Sobolev inequality corresponding to a Hörmander type generator. Indeed, if \(\mu(dx) = e^{-U}dx\) then it is clear that \(L = \Delta - \nabla U.\nabla\) is a Dirichlet operator satisfying
\[
\mu(f \mathcal{L} f) = -\mu(|\nabla f|^2)
\]
where \(\Delta\) is the sub-Laplacian, and \(\nabla\) the sub-gradient.

In [H-Z] the authors were able to show that a related class of measures on \(\mathbb{H}\) satisfy \((LSq)\) inequalities (see Theorem 1.5 below). To describe these we first need to introduce the natural distance function on \(\mathbb{H}\), which is the so-called Carnot-Carathéodory distance. This distance is more natural than the usual Euclidean one, since it takes into account the extra structure that the Heisenberg group possesses. We define the Carnot-Carathéodory distance between two points in \(\mathbb{H}\) by considering only admissible curves between them in the following sense. A Lipschitz curve \(\gamma : [0, 1] \to \mathbb{H}\) is said to be admissible if \(\gamma'(s) = a_1(s)X_1(\gamma(s)) + a_2(s)X_2(\gamma(s))\) almost everywhere with measurable coefficients \(a_1, a_2\) i.e. if \(\gamma'(s) \in sp\{X_1(\gamma(s)), X_2(\gamma(s))\}\) a.e. Then the length of \(\gamma\) is given by
\[
l(\gamma) = \int_0^1 (a_1^2(s) + a_2^2(s))^{1/2} ds
\]
and we define the Carnot-Carathéodory distance between two points \(x, y \in \mathbb{H}\) to be
\[
d(x, y) := \inf\{l(\gamma) : \gamma\text{ is an admissible path joining } x \text{ to } y\}.
\]
Write \(d(x) = d(x, e)\), where \(e\) is the identity.

Remark 1.2. This distance function is well defined as a result of Chow’s theorem, which states that every two points in \(\mathbb{H}\) can be joined by an admissible curve (see for example [B-L-U],[Grom]).

Geodesics are smooth, and are helices in \(\mathbb{R}^3\). We also have that \(x = (x_1, x_2, x_3) \mapsto d(x)\) is smooth for \((x_1, x_2) \neq 0\), but not at points \((0, 0, x_3)\), so that the unit ball has singularities on the \(x_3\)-axis. In our analysis, we will frequently use the following two results. The first is the well-known fact that the Carnot-Carathéodory distance satisfies the eikonal equation (see for example [Mo]):

Proposition 1.3. Let \(\nabla\) be the sub-gradient on \(\mathbb{H}\). Then \(|\nabla d(x)| = 1\) for all \(x = (x_1, x_2, x_3) \in \mathbb{H}\) such that \((x_1, x_2) \neq 0\).

We must be careful in dealing with the notion of \(\Delta d\), since it will have singularities on the \(x_3\)-axis. However, the following proposition from [I-P] provides some control of these singularities.
**Proposition 1.4.** Let $\Delta$ be the sub-Laplacian on $\mathbb{H}$. There exists a constant $K_0$ such that $\Delta d \leq \frac{K_0}{d}$ in the sense of distributions.

The following result concerning the $q$ Log-Sobolev inequality can be found in [H-Z].

**Theorem 1.5.** ([H-Z]) Let $\mu_p$ be the probability measure on $\mathbb{H}$ given by

$$
\mu_p(dx) = \frac{\int_{\mathbb{H}} e^{-\beta d_p(x)} dx}{\int_{\mathbb{H}} e^{-\beta d_p(x)} dx} d_x
$$

where $p \geq 2$, $\beta > 0$, $d_x$ is the Lebesgue measure on $\mathbb{R}^3$ and $d(x)$ is the Carnot-Carathéodory distance. Then $\mu_p$ satisfies an $(LS_q)$ inequality, where $\frac{1}{p} + \frac{1}{q} = 1$.

In order to prove the Log-Sobolev inequality, the following inequality, denoted as U-bound, was first shown.

$$
\mu_p \left| f \right|^q d^p(x) \leq C \mu_p \left| \nabla f \right|^q + D \mu_p \left| f \right|^q
$$

for some constants $C, D > 0$. More generally, for arbitrary measures

$$
\mu(dx) = \frac{\int_{\mathbb{H}} e^{-U(x)} dx}{\int_{\mathbb{H}} e^{-U(x)} dx} dx
$$

the authors in [H-Z] associated the $q$ Log-Sobolev Inequality with the following U-bound inequality:

$$
\mu \left| f \right|^q (\left| \nabla U \right|^q + U) \leq C \mu \left| \nabla f \right|^q + D \mu \left| f \right|^q
$$

for constants $C, D > 0$. Concerning the weaker Spectral Gap inequality the associated inequality is

$$
\mu \left| f \right|^q \eta \leq C' \mu \left| \nabla f \right|^q + D' \mu \left| f \right|^q
$$

for some non negative non decreasing function $\eta$ and constants $C', D' > 0$.

### 1.2 Infinite dimensional setting.

In this section we present the infinite dimensional setting as well as past results for the infinite dimensional Gibbs measure.

**Infinite dimensional analysis.** In the more standard Euclidean model the problem has been extensively discussed. Regarding the Log-Sobolev Inequality for the local specification $\{E^x, \omega\}_{A \in Z^d, \omega \in \Omega}$ on a $d$-dimensional Lattice, criterions and
examples of measures $E^{\Lambda,\omega}$ that satisfy the Log-Sobolev -with a constant uniformly on the set $\Lambda$ and the boundary conditions $\omega$— are investigated in [Z2], [B-E], [B-L], [Y] and [B-H]. For $\|\nabla_i \nabla_j V(x_i, x_j)\|_\infty < \infty$ the Log-Sobolev is proved when the phase $\phi$ is strictly convex and convex at infinity. Furthermore, in [G-R] the Spectral Gap Inequality is proved to be true for phases beyond the convexity at infinity.

For the measure $E^{\{i\},\omega}$ on the real line, necessary and sufficient conditions are presented in [B-G], [B-Z] and [R-Z], so that the Log-Sobolev Inequality is satisfied uniformly on the boundary conditions $\omega$.

The problem of the Log-Sobolev inequality for the Infinite dimensional Gibbs measure on the Lattice is examined in [G-Z], [Z1] and [Z2]. The first two study the LS for measures on a d-dimensional Lattice for bounded spin systems, while the third one looks at continuous spins systems on the one dimensional Lattice.

In [M] and [O-R], criterions are presented in order to pass from the Log-Sobolev Inequality for the single-site measure $E^{\{i\},\omega}$ to the (LS2) for the Gibbs measure $\nu_N$ on a finite N-dimensional product space. Furthermore, using these criterions one can conclude the Log-Sobolev Inequality for the family $\{\nu_N, N \in \mathbb{N}\}$ with a constant uniformly on $N$.

In [L-Z] a similar situation is studied, in that the authors consider a system of Hörmander generators in infinite dimensions and prove logarithmic Sobolev inequalities as well as some ergodicity results. The main difference between the present set up and their situation is that we consider a non-compact underlying space, namely the Heisenberg group, in which the techniques of [L-Z] cannot be applied. Concerning the same problem for the LSq ($q \in (1, 2]$) inequality in the case of Heisenberg groups with quadratic interactions in [I-P] a similar criterion is presented for the Gibbs measure based on the methods developed in [Z1] and [Z2].

Our general setting is as follows:

The Lattice. When we refer to the Lattice we mean the 1-dimensional Lattice $\mathbb{Z}$.

The Configuration space. We consider continuous unbounded random variables in $\mathbb{H}$, representing spins. Our configuration space is $\Omega = \mathbb{H}^\mathbb{Z}$. For any $\omega \in \Omega$ and $\Lambda \subset \mathbb{Z}$ we denote

$$\omega = (\omega_i)_{i \in \mathbb{Z}}, \omega_{\Lambda} = (\omega_i)_{i \in \Lambda}, \omega_{\Lambda^c} = (\omega_i)_{i \in \Lambda^c}, \text{ and } \omega = \omega_{\Lambda} \circ \omega_{\Lambda^c}$$

where $\omega_i \in \mathbb{H}$. When $\Lambda = \{i\}$ we will write $\omega_i = \omega_{\{i\}}$. Furthermore, we will write $i \sim j$ when the nodes $i$ and $j$ are nearest neighbours, that means, they are connected with a vertex, while we will denote the set of the neighbours of $k$ as $\{\sim k\} = \{r : r \sim k\}$.

The functions of the configuration. We consider integrable functions $f$ that depend on a finite set of variables $\{x_i\}, i \in \Sigma_f$ for a finite subset $\Sigma_f \subset \subset \mathbb{Z}$. The symbol $\subset \subset$ is used to denote a finite subset.
The Measure on $\mathbb{Z}$. For any subset $\Lambda \subset \subset \mathbb{Z}$ we define the probability measure $E^{\Lambda, \omega}(dx_{\Lambda}) = \frac{e^{-H^{\Lambda, \omega}}}{Z^{\Lambda, \omega}} dx_{\Lambda}$ where

- $x_{\Lambda} = (x_i)_{i \in \Lambda}$ and $dx_{\Lambda} = \prod_{i \in \Lambda} dx_i$
- $Z^{\Lambda, \omega} = \int e^{-H^{\Lambda, \omega}} dx_{\Lambda}$
- $H^{\Lambda, \omega} = \sum_{i \in \Lambda} \phi(x_i) + \sum_{i \in \Lambda, j \sim i} J_{ij} V(x_i, z_j)$
- $\phi \geq 0$ and $V \geq 0$

and

- $z_j = x_{\Lambda} \circ \omega_{\Lambda^c} = \begin{cases} x_j, & i \in \Lambda \\ \omega_j, & i \notin \Lambda \end{cases}$

We call $\phi$ the phase and $V$ the potential of the interaction. For convenience we will frequently omit the boundary symbol from the measure and will write $E^{\Lambda} \equiv E^{\Lambda, \omega}$. Furthermore we will assume that

$(H^*)$ There exist constants $B_*(L), B^*(L) \in (0, \infty)$ such that

$$\int_{B_L \oplus B_L} e^{-H^{\{i\}}_L} dX_{\{i\}} \geq \frac{1}{B^*(L)} \text{ for } \omega_j \in B_L, j \in \{i - 2, i, i + 2\}$$

and

$$e^{-H^{\{i\}}_L} \geq \frac{1}{B_*(L)}$$

for $\omega_j \in B_L, j \in \{i - 2, i, i + 2\}$ and $x_j \in B_L, j \in \{i - 1, i + 1\}$

where for any $R \geq 0$, $B_R = \{x \in \mathbb{Z} : d(x) \leq R\}$.

Remark 1.6. The hypothesis $(H^*)$ is a technical condition which essentially does not allow singularities on $\phi$ and $V$.

The Infinite Volume Gibbs Measure. The Gibbs measure $\nu$ for the local specification $\{E^{\Lambda, \omega}\}_{\Lambda \subset \subset \mathbb{Z}, \omega \in \Omega}$ is defined as the probability measure which solves the Dobrushin-Lanford-Ruelle (DLR) equation

$$\nu E^{\Lambda, \star} = \nu$$

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for finite sets Λ ⊂ Z (see [Pr]). For conditions on the existence and uniqueness of the Gibbs measure see e.g. [B-HK] and [D]. It should be noted that \( \{ E^\Lambda,\omega \} \) always satisfies the DLR equation, in the sense that

\[
E^\Lambda,\omega E^M,\omega^* = E^\Lambda,\omega
\]

for every \( M \subset \Lambda \). [Pr].

The gradient \( \nabla \) for continuous spins systems. For any subset \( \Lambda \subset Z \) we define the gradient

\[
|\nabla f|^q = \sum_{i \in \Lambda} |\nabla_i f|^q
\]

the sub-gradient \( \nabla_i \) corresponds to the i’th variable

\[
\nabla_i := (X_i^1, X_i^2)
\]

When \( \Lambda = Z \) we will simply write \( \nabla = \nabla_Z \). We denote

\[
E^\Lambda,\omega f = \int f dE^\Lambda,\omega(x_\Lambda)
\]

Under this specific higher dimensional setting the \( q \) Logarithmic Sobolev and \( q \) Spectral Gap inequalities are defined for measures of the local specification \( \{ E^\Lambda,\omega \} \)

**The \( q \) Log-Sobolev Inequality (LSq) on \( \mathbb{H}^Z \).** We say that the measure \( E^\Lambda,\omega \) satisfies the \( q \) Log-Sobolev Inequality for \( q \in (1, 2] \), if there exists a constant \( C_{LS} \) such that for any function \( f \), the following holds

\[
E^\Lambda,\omega |f|^q \log \frac{|f|^q}{E^\Lambda,\omega |f|^q} \leq C_{LS} E^\Lambda,\omega |\nabla f|^q
\]

with the constant \( C_{LS} \in (0, \infty) \) uniformly on the set \( \Lambda \) and the boundary conditions \( \omega \).

**The \( q \) Spectral Gap Inequality on \( \mathbb{H}^Z \).** We say that the measure \( E^\Lambda,\omega \) satisfies the \( q \) Spectral Gap Inequality for \( q \in (1, 2] \), if there exists a constant \( C_{SG} \) such that for any function \( f \), the following holds

\[
E^\Lambda,\omega |f|^q \leq C_{SG} E^\Lambda,\omega |\nabla f|^q
\]

with the constant \( C_{SG} \in (0, \infty) \) uniformly on the set \( \Lambda \) and the boundary conditions \( \omega \).

**Remark 1.7.** We will frequently use the following two well known properties about the Log-Sobolev and the Spectral Gap Inequality. If the probability measure \( \mu \) satisfies the Log-Sobolev Inequality with constant \( c \) then it also satisfies the Spectral
Gap Inequality with a constant $\hat{c} = \frac{4c}{\log 2}$. More detailed, in the case where $q = 2$ the optimal constant is less or equal to $\frac{c}{2} < \hat{c}$, while in the case $1 < q < 2$ it is less or equal to $\frac{4c}{\log 2}$. The constant $\hat{c}$ does not depend on the value of the parameter $q \in (1, 2]$.

Furthermore, if for a family $I$ of sets $\Lambda_i \subset \mathbb{Z}$, $\text{dist}(\Lambda_i, \Lambda_j) > 1$, $i \neq j$ the measures $\mathbb{E}^{\Lambda, \omega}, i \in I$ satisfy the Log-Sobolev Inequality with constants $c_i, i \in I$, then the probability measure $\mathbb{E}^{\cup_{i \in I} \Lambda_i, \omega}$ also satisfies the (LS) Inequality with constant $c = \max_{i \in I} c_i$. The last result is also true for the Spectral Gap Inequality. The proofs of these two properties can be found in [Gros] and [G-Z] for $q = 2$ and in [B-Z] for $1 < q < 2$.

Concerning the $q$ Log-Sobolev inequality for spins on the Heisenberg group, in [I-P] the inequality was proven for a specific class of Hörmander type generators on the Heisenberg group. Under the three hypothesis bellow, the main result of [I-P] for the infinite volume Gibbs measure $\nu$ follows. For the local specification

$$
\mathbb{E}^{\Lambda, \omega}(dx) = \frac{e^{-H^{\Lambda, \omega}(x)}}{\int e^{-H^{\Lambda, \omega}(x)}dx}dx
$$

(1.2)

where $dx$ is the Lebesgue product measure we consider the following hypothesis:

(H0): The one dimensional measures $\mathbb{E}^{\Lambda, \omega}$ satisfies the Log-Sobolev-q Inequality with a constant $c$ uniformly with respect to the boundary conditions $\omega$.

(H1'): The interactions are such that: $\|V''\|_{\infty} < \infty$.

(H3): The coefficients $J_{i,j}$ are such that $|J_{i,j}| \in [0, J]$ for some $J < 1$ sufficiently small.

**Theorem 1.8. ([I-P])** Let $\nu$ be a Gibbs measure corresponding to the local specification defined by (1.2). Let $q$ be dual to $p$ i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Then if (H0), (H1') and (H3) are satisfied the $\nu$ is unique and satisfies the $(LS_q)$ inequality i.e. there exists a constant $C$ such that

$$
\nu(|f|^q \log |f|^q) \leq \nu|f|^q \log \nu|f|^q + C\nu \left( \sum_i |\nabla_i f|^q \right)
$$

for all $f \in C^\infty$ for which the right-hand side is well defined and (H0) is true.

We briefly mention some consequences of this result.
Corollary 1.9. Let \( \nu \) be as in Theorem 1.8. Then \( \nu \) satisfies the \( q \)-spectral gap inequality. Indeed

\[
\nu |f - \nu f|^q \leq \frac{4C}{\log 2} \nu \left( \sum_i |\nabla_i f|^q \right)
\]

where \( C \) is as in Theorem 1.8.

The proofs of the next two can be found in [B-Z].

Corollary 1.10. Let \( \nu \) be as in Theorem 1.8 and suppose \( f : \Omega \to \mathbb{R} \) is such that \( \|\|\nabla f\|\|_\infty < 1 \). Then

\[
\nu \left( e^{\lambda f} \right) \leq \exp \left\{ \lambda \nu(f) + \frac{C}{q'(q-1)} \lambda^q \right\}
\]

for all \( \lambda > 0 \) where \( C \) is as in Theorem 1.8. Moreover, by applying Chebyshev’s inequality, and optimising over \( \lambda \), we arrive at the following ‘decay of tails’ estimate

\[
\nu \left\{ \left| f - \int f \, dv \right| \geq h \right\} \leq 2 \exp \left\{ -\frac{(q-1)p}{Cp-1} h^p \right\}
\]

for all \( h > 0 \), where \( \frac{1}{p} + \frac{1}{q} = 1 \).

Corollary 1.11. Suppose that our configuration space is actually finite dimensional, so that we replace \( \mathbb{Z}^d \) by some finite graph \( G \), and \( \Omega = (\mathbb{H})^G \). Then Theorem 1.8 still holds, and implies that if \( L \) is a Dirichlet operator satisfying

\[
\nu(fL^2) = -\nu(\|\nabla f\|^2)
\]

then the associated semigroup \( P_t = e^{tL} \) is ultracontractive.

Furthermore as shown on the next theorem, hypothesis (H0) was proven to hold for a specific class of local specification as in (1.2) with

\[
H^{\Lambda,\omega}(x_{\Lambda}) = \alpha \sum_{i \in \Lambda} d^p(x_i) + \epsilon \sum_{\{i,j\} \cap \Lambda \neq \emptyset : j \sim i} (d(x_i) + \rho d(\omega_j))^2
\]

(1.3)

for \( \alpha > 0, \epsilon, \rho \in \mathbb{R} \), and \( p \geq 2 \), where as above \( x_i = \omega_i \) for \( i \notin \Lambda \).

Theorem 1.12. ([I-P]) Let \( \frac{1}{q} + \frac{1}{p} = 1 \), and \( \epsilon \rho > 0 \) with \( \epsilon > \frac{\rho}{2N} \) if \( p = 2 \) for the local specification defined by (1.2) and (1.3). Then there exists a constant \( c \), independent of the boundary conditions \( \omega \in \Omega \) such that

\[
\mathbb{E}^{\omega}(|f|^q \log |f|^q) \leq c\mathbb{E}^{\omega}(\|\nabla f\|^q)
\]

for all smooth \( f : \Omega \to \mathbb{R} \).
Remark 1.13. In the case when $p = 2$, we must have that $\varepsilon > -\frac{\alpha}{2N}$ to ensure that $\int e^{-H_{\Lambda}} dx_{\Lambda} < \infty$.

The proof of Theorem 1.12 was based on proving the following U-bound

$$E_{i,\omega}^{i,\omega} \left( |f|^q \left( d^{p-1} + \sum_{j:j \sim i} d(\omega_j) \right) \right) \leq A_1 E_{i,\omega}^{i,\omega} |\nabla_i f|^q + B_1 E_{i,\omega}^{i,\omega} |f|^q$$

for all smooth $f : \Omega \to \mathbb{R}$, and some constants $A_1, B_1 \in (0, \infty)$ independent of $\omega$ and $i$. One of the purposes of the current paper is to present criterions which will allow to obtain the (LS$q$) inequality for infinite dimensional Gibbs measure in the case where the measures $E_{i,\omega}^{i,\omega}$ satisfy a U-bound inequality as above with a constant $B_1$ which is not independent of the boundary conditions $\omega$. This will be the subject of Theorem 2.3.

All the pre mentioned developments refer to measures with interactions $V$ that satisfy $\|\nabla_i \nabla_j V(x_i, x_j)\|_\infty < \infty$. The question that arises is whether similar assertions can be verified for the infinite dimensional Gibbs measure in the case where $\|\nabla_i \nabla_j V(x_i, x_j)\|_\infty = \infty$. In [Pa], under (H0), such a result was presented under the two additional hypothesis:

(H1): The restriction $\nu_{\Lambda(k)}$ of the Gibbs measure $\nu$ to the $\sigma-$algebra $\Sigma_{\Lambda(k)}$,

$$\Lambda(k) = \{k - 2, k - 1, k, k + 1, k + 2\}$$

satisfies the Log-Sobolev-q Inequality with a constant $C \in (0, \infty)$.

(H2): For some $\epsilon > 0$ and $\hat{K} > 0$

$$\nu_{\Lambda(i)} e^{2\varepsilon + 2\varepsilon V(x_r, x_s)} \leq \hat{K} \quad \text{and} \quad \nu_{\Lambda(i)} e^{2\varepsilon + 2\varepsilon |\nabla_r V(x_r, x_s)|^q} \leq \hat{K}$$

for $r, s \in \{i - 2, i - 1, i, i + 1, i + 2\}$

The main theorem follows.

Theorem 1.14. ([Pa]) If hypothesis (H0), (H1), (H2) and (H3) are satisfied, then the infinite dimensional Gibbs measure $\nu$ for the local specification $\{E_{\Lambda,\omega}\}_{\Lambda \subset \subset \mathbb{Z}, \omega \in \Omega}$ satisfies the q Log-Sobolev inequality

$$\nu |f|^q \log \frac{|f|^q}{\nu |f|^q} \leq \mathcal{C}_\nu \nu |\nabla f|^q$$

for some positive constant $\mathcal{C}$.
As a consequence of the last Theorem, the analogues of Corollaries 1.9-1.11 follow. As an example of a measure that satisfies (H0) with non quadratic interaction on the Heisenberg group one can think of a measure similar with that on (1.3) but with interactions of higher growth, i.e.

\[ H^\Lambda,\omega(x_\Lambda) = \alpha \sum_{i \in \Lambda} d^p(x_i) + \varepsilon \sum_{\{i,j\} \cap \Lambda \neq \emptyset, j \sim i} (d(x_i) + \rho d(\omega_j))^s \]

for \( \alpha > 0, \varepsilon, \rho \in \mathbb{R} \), and \( p > s > 2 \), where as above \( x_i = \omega_i \) for \( i \notin \Lambda \). The proof of this follows exactly as Theorem 1.12 with the use of uniform U-Bounds.

2 Main result.

We focus on the Logarithmic Sobolev Inequality (LS\( g \)) for measures related to systems with values on the Heisenberg group on the one dimensional Lattice with nearest neighbour interactions. The aim is to investigate the conditions under which the inequality can be extended from the one dimensional measure to the Infinite volume Gibbs measure.

In this paper we apply the same ideas as in [H-Z] and [I-P] to investigate cases of measures were the U-bound inequalities do not hold uniformly on the boundary conditions but still the infinite volume Gibbs measure ultimately satisfies the Log-Sobolev inequality. We will be concerned with two cases.

Case 1: A Perturbation property. The first case is actually a perturbation result on the measures obtained in [H-Z]. We recall that according to [H-Z], the measure on \( \mathbb{H} \) given by

\[ \mu_p(dx) = \frac{e^{-\beta d^p(x)}}{\int_{\mathbb{H}} e^{-\beta d^p(x)}dx}dx \]

where \( p \geq 2, \beta > 0, \) satisfies an (LS\( g \)) inequality, where \( \frac{1}{p} + \frac{1}{q} = 1 \). We try to address the following question. If we perturb this measure with interactions to obtain the following local specifications

\[ E^{\{i\},\omega}(dx_i) = \frac{e^{-\beta d^p(x)} - \sum_{j \sim i} \delta_j V(x_i, x_j)}{Z^{\{i\},\omega}} \]  

under which conditions does the infinite volume Gibbs measure \( \nu \) for the local specification \( \{E^{\Lambda,\omega}\}_{\Lambda \subset \subset \mathbb{Z}, \omega \in \Omega} \) satisfies the Log-Sobolev inequality?

In both [I-P] and [Pa], the main assumption was (H0), which is that the one dimensional measures \( E^{\{i\},\omega} \) satisfies the q Log-Sobolev Inequality with a constant \( c \) uniformly with respect to the boundary conditions \( \omega \). In this paper we want to relax the main hypothesis (H0) for \( E^{\{i\},\omega} \) to the same assumption for the boundary
free one dimensional measure. In other words we want to address the following problem.

Consider the local specification

$$E_{\Lambda,\omega}(dx_{\Lambda}) = \frac{e^{-\sum_{i \in \Lambda} \phi(x_i) - \sum_{i \in \Lambda} \sum_{j \sim i} J_{ij} V(x_i, \omega_j)} dX_{\Lambda}}{Z_{\Lambda,\omega}}$$

with \( \|\partial_x \partial_y V(x, y)\|_{\infty} \leq \infty \) \hspace{1cm} (2.1)

and assume that

(H0'): The one site measures \( \mu(dx_i) = \frac{e^{-\phi(x_i)} dx_i}{\int e^{-\phi(x_i)} dx_i} \) satisfies the q Log-Sobolev Inequality with a constant \( c \).

Under which conditions does the infinite volume Gibbs measure \( \nu \) corresponding to the local specification \{ \( E_{\Lambda,\omega}(dx_{\Lambda}) \) \} for \( \Lambda \subset \subset \mathbb{Z} \) and \( \omega \in \Omega \) satisfy the Log-Sobolev inequality? We present a strategy to solve this problem. As we will see, hypothesis (H0'), together with (H1), (H2), (H3), mentioned before, as well as

(H4): \( \nu_{\Lambda(i)} e^{2^{10} d(x_i)} < \hat{K} \), where \( d \) denotes the distance of the space.

imply the q Log-Sobolev inequality for the infinite dimensional Gibbs measure. We will focus on measures on the one dimensional lattice, but our result can also be easily extended on trees.

**Case 2: Non uniform U-Bound.** As explained in the introduction, the U-bound inequalities introduced in [H-Z] are an essential tool in proving the Spectral Gap and the Logarithmic Sobolev inequality, under the framework of the Heisenberg group. In the case of the specific example examined in [I-P], for the proof of both the Spectral Gap and the Log-Sobolev inequality the basic step was again the U-bound inequalities. In order to obtain the two coercive inequalities uniformly on the boundary conditions, the two U-bounds had to be proven to hold also independently of the boundary conditions of the measure \( \mathbb{E}^{t,\omega} \).

Here we investigate cases were weaker U-bound inequalities hold for \( \mathbb{E}^{t,\omega} \). In particular we concentrate on these cases were one of the constants depends on the boundary conditions \( \omega \). For the local specification

$$E_{\Lambda,\omega}(dx_{\Lambda}) = \frac{e^{-H_{\Lambda,\omega}} dX_{\Lambda}}{Z_{\Lambda,\omega}}$$

for \( \Lambda \subset \subset \mathbb{Z} \) and \( \omega \in \Omega \), with

$$H_{\Lambda,\omega} = \sum_{i \in \Lambda} \phi(x_i) + \sum_{i \in \Lambda} \sum_{j \sim i} J_{ij} V(x_i, \omega_j) \text{ with } \|\partial_x \partial_y V(x, y)\|_{\infty} \leq \infty$$
we consider the following hypothesis:

\( (H_0') \): Non uniform U-bound.

\[
\mathbb{E}^{(\sim i)} \omega | f|^q (|\nabla^{(\sim i)} H^{(\sim i)} \omega| + H^{(\sim i)} \omega) \leq \hat{C} \mathbb{E}^{(\sim i)} \omega \ |\nabla^{(\sim i)} f|^q + \hat{D}_{(\sim i)} (\omega) \mathbb{E}^{(\sim i)} \omega | f|^q
\]

for functions \( f \in C^\infty \) for which the right-hand side is well defined, with \( \nu e^{\hat{D}_{(\sim i)} (\omega)} \leq \hat{K} \) where \( \hat{D}_{(\sim i)} (\omega) \) is a function of \( \omega_{i-2}, \omega_i, \omega_{i+2} \). What we will show is that even when the Log-Sobolev inequality does not hold for the one site measure \( \mathbb{E}^{i} \omega \) with a constant uniformly on the boundary, we can still obtain the inequality for the infinite dimensional Gibbs measure. On section \( \S \) we present examples of measures that satisfy this weaker non-uniform U-bound inequality.

Before we present the two main results a useful remark concerning the conditions will follow.

**Remark 2.1.** From Hypothesis \( (H2) \) and Hölder inequality it follows that

\[
\nu e^{(|F(r)| + \mathbb{E}^{(r)} \omega |F(r)|)^q} \leq \hat{K}, \quad \text{for } r = i - 2, i - 1, i, i + 1, i + 2
\]

where the functions \( F(r) \) are defined by

\[
F(r) = \begin{cases} 
\nabla_r V(x_{i-1}, x_i) + \nabla_r V(x_{i+1}, x_i) & \text{for } r = i - 1, i, i + 1 \\
\nabla_r V(x_s, x_r) I_{\{s = r, s \in \{i-3, i+3\}\}} & \text{for } r = i - 2, i + 2 
\end{cases}
\]

and the sets \( S(r) \) by

\[
S(r) = \begin{cases} 
\{\sim i\} & \text{for } r = i - 1, i, i + 1 \\
\{i + 3, i + 4, \ldots\} & \text{for } r = i + 2 \text{ and } s = i + 3 \\
\{\ldots, i - 4, i - 3\} & \text{for } r = i - 2 \text{ and } s = i - 3 
\end{cases}
\]

These bounds will be frequently used throughout this and the next chapter.

The main two theorems follow. The first refers to the \( (H0') \) hypothesis for the one site measure.

**Theorem 2.2.** If the set of conditions \( A = \{(H0'), (H1) - (H4)\} \) for the local specification \( \{E^{\Lambda \omega}\}_{\Lambda \subset \subset Z, \omega \in \Omega} \) is satisfied, then the infinite dimensional Gibbs measure \( \nu \) for the local specification \( \{E^{\Lambda \omega}\}_{\Lambda \subset \subset Z, \omega \in \Omega} \) satisfies the Log-Sobolev q inequality

\[
\nu |f|^q \log |f|^q \leq \nu |f|^q \log \nu |f|^q + C \nu |\nabla f|^q
\]

for some positive constant \( C \in (0, \infty) \) independent of the function \( f \), for all functions \( f \in C^\infty \) for which the right-hand side is well defined and \( (H0') \) is true.
The next theorem assumes the (H0″) hypothesis.

**Theorem 2.3.** If the set of conditions $\mathcal{B} = \{(H0″), (H1) - (H4)\}$ is satisfied, then the infinite dimensional Gibbs measure $\nu$ satisfies the Log-Sobolev $q$ inequality

$$
\nu |f|^q \log |f|^q \leq \nu |f|^q \log \nu |f|^q + \mathcal{C} \nu |\nabla f|^q
$$

for some positive constant $\mathcal{C} \in (0, \infty)$ independent of the function $f$, for all functions $f \in C^\infty$ for which the right-hand side is well defined and (H0″) is true.

For computational reasons we set $\hat{K} := e^K$ and

$$
\eta(i, \omega) = d(x_{i-1}) + d(x_{i+1}) + \sum_{j \sim \{i-1, i+1\}} d(\omega_j)
$$

Aside from hypothesis (H0″), (H0″) and (H4) the rest of the assumptions are the same as in [Pa]. Concerning hypothesis (H4), in most cases where the interaction $V$ is a polynomial of high growth, it should be weaker than hypothesis (H2).

In order to prove Theorem 2.2 and Theorem 2.3 we will use the methods developed by Zegarlinski in [Z1] and [Z2]. The main idea is based on approximating the infinite dimensional Gibbs measure $\nu$ for the local specification $\{E_{\Lambda, \omega}\}_{\Lambda \subseteq \mathbb{Z}, \omega \in \Omega}$ by a sequence which involves components in the local specification that satisfy the Log-Sobolev inequality. This method was used in [Pa] and [I-P] where the one dimensional measures $E_{\{i\}, \omega}$ satisfied the Log-Sobolev inequality uniformly on the boundary conditions $\omega$. In the two cases examined here where either the one dimensional boundary-free measure $\mu(dx_i) = \frac{e^{-\Phi(x_i)}dx_i}{\int e^{-\Phi(x_i)}dx_i}$ satisfies an (LSq) or $E_{\{i\}, \omega}$ a non-uniform U-bound, we will replace under our assumptions $A$ or $B$ the property (H0) of the Log-Sobolev inequality for the measure $E_{\{i\}, \omega}$ by a similar but weaker inequality that maintains most of the properties of the Log-Sobolev inequality. This Log-Sobolev type inequality will be

$$
\nu E_{\{\sim i\}, \omega}( |f|^q \log \frac{|f|^q}{E_{\{\sim i\}, \omega} |f|^q} ) \leq R \sum_{r=i-2}^{i+2} \nu |\nabla_r f|^q + R \sum_{r=3}^{\infty} J_0^{r-2} \nu |\nabla_{i+r} f|^q \quad (2.3)
$$

We will prove a similar inequality to replace Spectral Gap inequality. This will be

$$
\nu E_{\{\sim i\}, \omega} |f - E_{\{\sim i\}, \omega} f|^q \leq M \sum_{r=i-2}^{i+2} \nu |\nabla_r f|^q + M \sum_{r=3}^{\infty} J_0^{r-2} \nu |\nabla_{i+r} f|^q \quad (2.4)
$$

where $J_0 < 1$ is a constant depending on $J$. The q Log-Sobolev type inequality (2.3) will be shown in Proposition 4.1 and Proposition 5.3 under the hypothesis (H0″) and (H0″) respectively. The q Spectral Gap type inequality (2.4) will be
proven in Proposition 3.3 for both the cases of hypothesis (H0') and (H0''). In addition, an analogue of the product property for the Log-Sobolev inequality is proven in Proposition 2.4 for the inequality (2.3). The proof of Theorem 2.2 and Theorem 2.3 follows.

**Proof of Theorem 2.2 and Theorem 2.3.** We want to extend the Log-Sobolev Inequality from the one site measure to the infinite dimensional Gibbs measure for the local specification \( \{ E^{\Lambda, \omega} \} _{\Lambda \subset \subset \mathbb{Z}, \omega \in \Omega} \) on the entire one dimensional lattice. Define the following sets
\[
\Gamma_0 = \text{even integers}, \quad \Gamma_1 = \mathbb{Z} \setminus \Gamma_0
\]
One can notice that \( \{ \text{dist}(i, j) > 1, \forall i, j \in \Gamma_k, k = 0, 1 \} \), \( \Gamma_0 \cap \Gamma_1 = \emptyset \) and \( \mathbb{Z} = \Gamma_0 \cup \Gamma_1 \). For convenience we will write \( E^{\Gamma_0} = E^{\Gamma_1} \) for \( i = 0, 1 \). Denote \( P = E^{\Gamma_0} E^{\Gamma_1} \) \( (2.5) \)

In order to prove the Log-Sobolev Inequality for the measure \( \nu \), we will express the entropy with respect to the measure \( \nu \) as the sum of the entropies of the measures \( E^{\Gamma_0} \) and \( E^{\Gamma_1} \). Assume \( f \geq 0 \). We can write
\[
\nu( f^q \log \frac{f^q}{E^{\Gamma_0} f^q} ) = \nu E^{\Gamma_0} ( f^q \log \frac{f^q}{E^{\Gamma_0} f^q} ) + \nu E^{\Gamma_1} ( E^{\Gamma_0} f^q \log \frac{E^{\Gamma_0} f^q}{E^{\Gamma_1} E^{\Gamma_0} f^q} ) + \\
\nu( E^{\Gamma_1} E^{\Gamma_0} f^q \log E^{\Gamma_1} E^{\Gamma_0} f^q ) - \nu( f^q \log \nu f^q ) \quad (2.6)
\]
The following proposition gives a Log-Sobolev type inequality for the product measures \( E^{\Gamma_k, \omega}, k = 0, 1 \).

**Proposition 2.4.** If conditions A or B are satisfied then for \( J \) sufficiently small the following Log-Sobolev type inequality holds
\[
\nu ( f^q | f^q \log \frac{| f^q |}{E^{\Gamma_k} f^q} ) \leq \tilde{C} \nu | \nabla_{\Gamma_k} f^q | + \tilde{C} \nu | \nabla_{\Gamma_1} f^q | \quad \text{for } k = 0, 1, \text{ and some positive constant } \tilde{C}.
\]

The proof of Proposition 2.4 will be the subject of Section 6. If we use the Proposition 2.4 for \( E^{\Gamma_i}, i = 0, 1 \), we get
\[
(2.6) \leq \tilde{C} \nu | \nabla_{\Gamma_0} f^q | + \tilde{C} \nu | \nabla_{\Gamma_1} f^q | + \tilde{C} \nu | \nabla_{\Gamma_1} ( E^{\Gamma_0} f^q ) |^q
\]
\[
+ \nu ( E^{\Gamma_1} E^{\Gamma_0} f^q \log E^{\Gamma_1} E^{\Gamma_0} f^q ) - \nu( f^q \log \nu f^q ) \quad (2.7)
\]
For the fourth term on the right hand side of (2.7) we can write
\[
\nu( \mathcal{P} f^q \log \mathcal{P} f^q ) = \nu E^{\Gamma_0} ( \mathcal{P} f^q \log \frac{\mathcal{P} f^q}{E^{\Gamma_0} \mathcal{P} f^q} ) + \nu E^{\Gamma_1} ( E^{\Gamma_0} \mathcal{P} f^q \log \frac{E^{\Gamma_0} \mathcal{P} f^q}{E^{\Gamma_1} E^{\Gamma_0} \mathcal{P} f^q} ) + \\
+ \nu ( E^{\Gamma_1} E^{\Gamma_0} \mathcal{P} f^q \log E^{\Gamma_1} E^{\Gamma_0} \mathcal{P} f^q )
\]
If we use again Proposition 2.4 for the measures $\mathbb{E}^{\Gamma_i}, i = 0, 1$ we get
\[
\nu(\mathcal{P}^{f_q} \log \mathcal{P}^{f_q}) \leq \tilde{C} \nu \left| \nabla_{\Gamma_0} (\mathcal{P} f_q)^\frac{1}{q} \right|^q + \tilde{C} \nu \left| \nabla_{\Gamma_1} (\mathcal{P}^{\Gamma_0} \mathcal{P}^{f_q})^{\frac{1}{q}} \right|^q + \nu(\mathcal{P}^{f_q} \log \mathcal{P}^{f_q})
\] (2.8)

If we work similarly for the last term $\nu(\mathcal{P}^{2} f_q \log \mathcal{P}^{2} f_q)$ of (2.8) and inductively for any term $\nu(\mathcal{P}^k f_q \log \mathcal{P}^k f_q)$, then after $n$ steps (2.7) and (2.8) will give
\[
\nu(f_q \log \frac{f_q}{\nu f_q}) \leq \nu(\mathcal{P}^n f_q \log \mathcal{P}^n f_q) - \nu(f_q \log \nu f_q) + \tilde{C} \nu \left| \nabla_{\Gamma_1} f \right|^q + \tilde{C} \nu \left| \nabla_{\Gamma_0} f \right|^q
\] (2.9)

In order to calculate the third and fourth term on the right-hand side of (2.9) we will use the following proposition

**Proposition 2.5.** Suppose that hypothesis A or B are satisfied. Then the following bound holds
\[
\nu \left| \nabla_{\Gamma_i} (\mathbb{E}^{\Gamma_j} f^q)^{\frac{1}{q}} \right|^q \leq C_1 \nu \left| \nabla_{\Gamma_i} f \right|^q + C_2 \nu \left| \nabla_{\Gamma_j} f \right|^q
\] (2.10)
for $\{i, j\} = \{0, 1\}$ and constants $C_1 \in (0, \infty)$ and $0 < C_2 < 1$.

The proof of Proposition 2.5 will be the subject of Section 6. If we apply inductively the bound (2.10) $k$ times to the third and the fourth term of (2.9) we obtain
\[
\nu \left| \nabla_{\Gamma_0} (\mathcal{P}^k f_q)^{\frac{1}{q}} \right|^q \leq C_2^{2k-1} C_1 \nu \left| \nabla_{\Gamma_1} f \right|^q + C_2^{2k} \nu \left| \nabla_{\Gamma_0} f \right|^q
\] (2.11)
and
\[
\nu \left| \nabla_{\Gamma_1} (\mathbb{E}^{\Gamma_0} \mathcal{P}^k f_q)^{\frac{1}{q}} \right|^q \leq C_2^{2k} C_1 \nu \left| \nabla_{\Gamma_1} f \right|^q + C_2^{2k+1} \nu \left| \nabla_{\Gamma_0} f \right|^q
\] (2.12)

If we plug (2.11) and (2.12) in (2.9), we get
\[
\nu(f_q \log \frac{f_q}{\nu f_q}) \leq \nu(\mathcal{P}^n f_q \log \mathcal{P}^n f_q) - \nu(f_q \log \nu f_q) + \tilde{C} \nu \left| \nabla_{\Gamma_1} f \right|^q
\]
\[
+ \tilde{C} (\sum_{k=0}^{n-1} C_2^{2k-1}) C_1 \nu \left| \nabla_{\Gamma_1} f \right|^q + \tilde{C} (\sum_{k=0}^{n-1} C_2^{2k}) \nu \left| \nabla_{\Gamma_0} f \right|^q
\]
\[
+ \tilde{C} (\sum_{k=0}^{n-1} C_2^{2k}) C_1 \nu \left| \nabla_{\Gamma_1} f \right|^q + \tilde{C} (\sum_{k=0}^{n-1} C_2^{2k+1}) \nu \left| \nabla_{\Gamma_0} f \right|^q
\] (2.13)

If we take the limit of $n$ to infinity in (2.13) the first two term on the right hand side cancel with each other, as explained in the proposition bellow.
Proposition 2.6. Under hypothesis $A$ or $B$, $\mathcal{P}^n f$ converges $\nu$-almost everywhere to $\nu f$, where $\mathcal{P}$ as in (2.5).

The proof of this proposition will be presented in Section 7. So, taking the limit of $n$ to infinity in (2.13) leads to

$$
\nu(f^q \log \frac{f^q}{\nu f^q}) \leq \left( \tilde{C} + cA \left( \frac{C_1}{C_2} + C_2 + C_1 \right) \right) \nu |\nabla_{1} f|^q + \tilde{C} A \nu |\nabla_{0} f|^q
$$

where $A = \lim_{n \to \infty} \sum_{k=0}^{n-1} C_2^k < \infty$ for $C_2 < 1$, and the theorem follows for a constant

$$
\mathcal{C} = \max \left\{ \left( \tilde{C} + \tilde{C} A \left( \frac{C_1}{C_2} + C_2 + C_1 \right) \right) , cA \right\}
$$

\[\square\]

3 q Poincaré type Inequality.

In this section we present the proof of the q Spectral Gap type inequality (2.4). In the case of quadratic interactions $V(x, y) = (x - y)^2$ one can calculate

$$
\mathbb{E}^{i, \omega} \left( f^2 (\nabla_j V(x_i - x_j) - \mathbb{E}^{i, \omega} \nabla_j V(x_i - x_j))^2 \right)
$$

(see [B-H] and [H]) with the use of the Deuschel-Stroock relative entropy inequality (see [D-S]) and the Herbst argument (see [L] and [H]). Herbst’s argument states that if a probability measure $\mu$ satisfies the (LS2) inequality and a function $F$ is Lipschitz continues with $\|F\|_{\text{Lips}} \leq 1$ and such that $\mu(F) = 0$, then for some small $\epsilon$ we have

$$
\mu e^{\epsilon F^2} < \infty
$$

For $\mu = \mathbb{E}^{i, \omega}$ and $F = \frac{\nabla_j V(x_i - x_j) - \mathbb{E}^{i, \omega} \nabla_j V(x_i - x_j)}{2}$ we then obtain

$$
\mathbb{E}^{i, \omega} e^{\frac{\epsilon}{4} (\nabla_j V(x_i - x_j) - \mathbb{E}^{i, \omega} \nabla_j V(x_i - x_j))^2} < \infty
$$

uniformly on the boundary conditions $\omega$, because of hypothesis (H0). In the more general case however examined in this work, where interactions may be non quadratic and the (H0) property does not hold, the Herbst argument cannot be applied. In this and next sections, following [Pa], we show how one can bound exponential quantities like the last one with the use of the projection of the infinite dimensional Gibbs measure and hypothesis (H1) and (H2).

For every probability measure $\mu$, we define the correlation function

$$
\mu(f; g) \equiv \mu(fg) - \mu(f) \mu(g)
$$
For the function $h_k := f - \mathbb{E}^{(\sim i)} f$ we define

$$Q(u, k) \equiv \nu_{\Lambda(u)} \left( |\nabla_{\Lambda(u)} (\mathbb{E}^{M(u)} |h_k|^q)|^{\frac{1}{q}} \right)$$

where the set $\Lambda(k) = \{k - 2, k - 1, k, k + 1, k + 2\}$ and $M(k) = \mathbb{Z} \setminus \Lambda(k)$. This quantity will be frequently used in the remaining section to bound the variance and the entropy. The following proposition presents a useful bound for $Q(k, k)$ under the hypothesis (H1), (H2) and (H3). The proof of this proposition can be found in [Pa].

**Proposition 3.1.** Suppose that hypothesis (H1), (H2) and (H3) are satisfied. Then

$$Q(k, k) \leq J^q S \nu |f - \mathbb{E}^{k-1} \mathbb{E}^{k+1} f|^q + S \sum_{r=k-2}^{k+2} \nu |\nabla_r f|^q + S \sum_{r=3}^{\infty} J_0^{-2} \nu |\nabla_{k+r} f|^q$$

for some positive constant $S$ and $J_0 = J^{\frac{q}{q-1}}$.

The next lemma shows the Poincaré inequality for the two site measure $\mathbb{E}^{(\sim i)}$, $i \in \mathbb{Z}$ on the ball.

**Lemma 3.2.** For any $L > 0$ the following Poincaré inequality holds

$$\mathbb{E}^{(\sim i)} \otimes \hat{\mathbb{E}}^{(\sim i)} |f - \tilde{f}|^q \mathcal{I}_{\{\eta(i, \omega) + \tilde{\eta}(i, \omega) \leq L\}} \leq D_L \mathbb{E}^{(\sim i)} |\nabla_{(\sim i)} f|^q$$

where $\eta(i, \omega) = d(x_{i-1}) + d(x_{i+1}) + \sum_{j \sim i, i + 1} d(\omega_j)$ and $\mathcal{I}_A$ is the indicator function of set $A$.

**Proof.**

$$I_1 := \mathbb{E}^{(\sim i)} \otimes \hat{\mathbb{E}}^{(\sim i)} |f - \tilde{f}|^q \mathcal{I}_{\{\eta(i, \omega) + \tilde{\eta}(i, \omega) \leq L\}}$$

$$= \int \int |f - \tilde{f}|^q \mathcal{I}_{\{\eta(i, \omega) + \tilde{\eta}(i, \omega) \leq L\}} \rho_i \hat{\rho}_i dX^{(\sim i)} d\hat{X}^{(\sim i)}$$

$$\leq \int_{\{\eta(i, \omega) \leq L\}} \int_{\{\tilde{\eta}(i, \omega) \leq L\}} |f - \tilde{f}|^q \rho_i \hat{\rho}_i \mathcal{I}_{\{\eta(i, \omega) \leq L\}} \mathcal{I}_{\{\tilde{\eta}(i, \omega) \leq L\}} dX^{(\sim i)} d\hat{X}^{(\sim i)}$$

where $\rho_i = \frac{e^{-H^{(\sim i), \omega}}}{\int e^{-H^{(\sim i), \omega}} dX_{(\sim i)}}$. Since on $\{\eta(i, \omega) \leq L\}$ we have $d(x_j) \leq L$, $j = i - 1, i + 1$ and $\sum_{j \sim (i-1, i+1)} d(\omega_j) \leq L$, according to hypothesis (H*) we can bound $\int e^{-H^{(\sim i), \omega}} dX_{(\sim i)}$ from below independently on the boundary conditions $\omega$. This leads to

$$I_1 \leq \frac{D^*(L)}{\int e^{-H^{(\sim i), \omega}} dX_{(\sim i)}} \times$$

$$\times \int_{\{\eta(i, \omega) \leq L\}} \int_{\{\tilde{\eta}(i, \omega) \leq L\}} |f - \tilde{f}|^q \mathcal{I}_{\{\eta(i, \omega) \leq L\}} \mathcal{I}_{\{\tilde{\eta}(i, \omega) \leq L\}} dX^{(\sim i)} d\hat{X}^{(\sim i)} \quad (3.1)$$
for some positive constant $D^*(L)$. If we set $B_R = \{x \in \mathbb{H} : d(x) \leq R\}$ then (3.1) gives

$$I_1 \leq \frac{D^*(L)}{e^Ht_{(\omega)}} \int_{B_L} \int_{B_L} |f\mathcal{I}_{\{\eta(i,\omega) \leq L\}} - \hat{f}\mathcal{I}_{\{\hat{\eta}(i,\omega) \leq L\}}|^q dX_{\{\sim i\}} d\hat{X}_{\{\sim i\}}$$

$$\leq \frac{D^*(L)A_L}{e^Ht_{(\omega)}} \int_{B_L} \nabla_{\{\sim i\}} f|^q \mathcal{I}_{\{\eta(i,\omega) \leq L\}} dX_{\{\sim i\}}$$

(3.2)

where above we used the Poincaré Inequality in the Carnot-Carathéodory ball on the Heisenberg group with respect to the Haar measure (see [V-SC-C]) with constant $A_L$ depending only on the radius. From (3.2) and hypothesis (H'), we obtain

$$I_1 \leq D_*(L)D^*(L)A_L \int_{\{\eta(i,\omega) \leq L\}} \nabla_{\{\sim i\}} f|^q d\mathbb{E}^{\{\sim i\}}$$

$$\leq D_*(L)D^*(L)A_L \mathbb{E}^{\{\sim i\}} |\nabla_{\{\sim i\}} f|^q$$

And the lemma follows for appropriate constant $D_L$. \hfill \Box

The following proposition gives a Spectral Gap type inequality for the measure $\mathbb{E}^{(i,\omega)}$.

**Proposition 3.3.** If conditions (H1), (H2) and (H3) are satisfied, then the following Spectral Gap type inequality

$$\nu \mathbb{E}^{\{\sim i\}} |f - \mathbb{E}^{\{\sim i\}} f|^q \leq M \sum_{r=1-2}^{i+2} \nu |\nabla_r f|^q + M \sum_{r=3}^{\infty} J_r^{-2} \nu |\nabla_{k^2 r} f|^q$$

holds for a positive constant $M$.

**Proof.** If $\hat{\mathbb{E}}^{\{\sim i\}}$ is an isomorphic copy of $\mathbb{E}^{\{\sim i\}}$ we can then write

$$\nu |f - \mathbb{E}^{\{\sim i\}} f|^q = \nu \mathbb{E}^{\{\sim i\}} |f - \mathbb{E}^{\{\sim i\}} f|^q \leq \nu \mathbb{E}^{\{\sim i\}} \otimes \hat{\mathbb{E}}^{\{\sim i\}} |f - \hat{f}|^q$$

$$= \nu \mathbb{E}^{\{\sim i\}} \otimes \hat{\mathbb{E}}^{\{\sim i\}} |f - \hat{f}|^q \mathcal{I}_{\{\eta(i,\omega) + \hat{\eta}(i,\omega) \leq L\}}$$

$$+ \nu \mathbb{E}^{\{\sim i\}} \otimes \hat{\mathbb{E}}^{\{\sim i\}} |f - \hat{f}|^q \mathcal{I}_{\{\eta(i,\omega) + \hat{\eta}(i,\omega) > L\}}$$

(3.3)

where we have denoted \(\eta(i, \omega) = d(x_{i-1}) + d(x_{i+1}) + \sum_{j \sim \{i-1,i+1\}} d(\omega_j)\)

and $\mathcal{I}_A$ the indicator function of set $A$. For the first term on the right hand side of (3.3) we can use Lemma 3.2 to obtain

$$\mathbb{E}^{\{\sim i\}} \otimes \hat{\mathbb{E}}^{\{\sim i\}} |f - \hat{f}|^q \mathcal{I}_{\{\eta(i,\omega) + \hat{\eta}(i,\omega) \leq L\}} \leq D_L \mathbb{E}^{\{\sim i\}} \otimes \hat{\mathbb{E}}^{\{\sim i\}} |\nabla_{\{\sim i\}} f|^q$$

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If we apply the Gibbs measure at the last inequality we obtain
\[ \nu E^{(\sim i)} \otimes \tilde{E}^{(\sim i)} | f - \tilde{f} | q \mathbb{I}_{\{\eta(i,\omega) + \tilde{\eta}(i,\omega) \leq L\}} \leq DL\nu \left| \nabla E^{(\sim i)} f \right| ^q \]  
(3.4)

For the second term in (3.3) we can write
\[
\nu E^{(\sim i)} \otimes \tilde{E}^{(\sim i)} | f - \tilde{f} | q \mathbb{I}_{\{\eta(i,\omega) + \tilde{\eta}(i,\omega) > L\}} \leq \nu E^{(\sim i)} \otimes \tilde{E}^{(\sim i)} | f - \tilde{f} | q \frac{\eta(i,\omega) + \tilde{\eta}(i,\omega)}{L}
\]
\[
\leq \frac{2q2}{L} \nu E^{(\sim i)} \left( | f - E^{(\sim i)} f | q \eta(i,\omega) \right) + \frac{2q2}{L} \nu E^{(\sim i)} | f - E^{(\sim i)} f | q \mathbb{E}^{(\sim i)} \eta(i,\omega)
\]
\[
= \frac{2q2}{L} \nu \left( | f - E^{(\sim i)} f | q \left( \eta(i,\omega) + E^{(\sim i)} \eta(i,\omega) \right) \right)
\]
(3.5)

where above we used that \( \eta(i,\omega) + E^{(\sim i)} \eta(i,\omega) \) is localised in \( \Lambda(i) \) and that \( M(i) = Z \setminus \Lambda(i) \). On the right hand side of (3.5) we can use the following Deuschel-Stroock entropic inequality (see [D-S])
\[
\forall t > 0, \mu(uv) \leq \frac{1}{t} \log \left( \mu(e^{tu}) \right) + \frac{1}{t} \mu(v \log v)
\]
(3.6)

for any measure \( \mu \) and \( v \geq 0 \) such that \( \mu(v) = 1 \). Then from (3.5) and (3.6) we will obtain
\[
\nu E^{(\sim i)} \otimes \tilde{E}^{(\sim i)} | f - \tilde{f} | q \mathbb{I}_{\{\eta(i,\omega) + \tilde{\eta}(i,\omega) > L\}}
\]
\[
\leq \frac{2q2}{\epsilon L} \nu_{\Lambda(i)} | f - E^{(\sim i)} f | q \log \frac{| f - E^{(\sim i)} f | q}{\nu_{\Lambda(i)} | f - E^{(\sim i)} f | q}
\]
\[
+ \frac{2q2}{\epsilon L} \left( \log \nu_{\Lambda(i)} e^{(\eta(i,\omega) + E^{(\sim i)} \eta(i,\omega))} \right) \nu_{\Lambda(i)} E^{M(i)} | f - E^{(\sim i)} f | q
\]
(3.7)

The first term on the right hand side of (3.7) can be bounded by the Log-Sobolev inequality for \( \nu_{\Lambda(i)} \) from hypothesis (H1)
\[
\nu_{\Lambda(i)} | f - E^{(\sim i)} f | q \log \frac{| f - E^{(\sim i)} f | q}{\nu_{\Lambda(i)} | f - E^{k+1} f | q}
\]
\[
\leq C \nu_{\Lambda(i)} \left| \nabla_{\Lambda(i)} (E^{M(i)}) | f - E^{(\sim i)} f | \right| ^q = CQ(i,i)
\]
(3.8)

If we combine (3.7) and (3.8) together with hypothesis (H4) we get
\[
\nu E^{(\sim i)} \otimes \tilde{E}^{(\sim i)} | f - \tilde{f} | q \mathbb{I}_{\{\eta(i,\omega) + \tilde{\eta}(i,\omega) > L\}}
\]
\[
\leq \frac{2q2}{\epsilon L} Q(i,i) + \frac{2q2K}{\epsilon L} \nu_{\Lambda(i)} E^{M(i)} | f - E^{(\sim i)} f | q
\]
(3.9)
If we put together relationships (3.3), (3.4) and (3.9) we obtain

\[ \nu |f - \mathbb{E}[(\sim i)f]|^q \leq D_L B |\nabla_{(\sim i)} f|^q + \frac{2q^2 C}{\epsilon L} Q(i, i) + \frac{2q^2 K}{\epsilon L} \nu |f - \mathbb{E}[(\sim i)f]|^q \]  
(3.10)

where the constant \( K \) is as in (H4). If we use the bound for \( Q(i, i) \) from Proposition 3.1, then (3.10) gives

\[ \nu |f - \mathbb{E}[(\sim i)f]|^q \leq D_L B |\nabla_{(\sim i)} f|^q + \frac{2q^2 CS}{\epsilon L} \nu |\nabla_r f|^q + 2q^2 S \frac{K}{\epsilon L} \sum_{r=3}^{\infty} J_r^{-2} \nu |\nabla_{i\pm r} f|^q \]  
(3.11)

For \( L \) sufficiently large such that \( 1 - \frac{2q^2 K + J^q S^2 q^2 C}{\epsilon L} > \frac{1}{2} \) we obtain

\[ \left( 1 - \frac{2q^2 K + J^q S^2 q^2 C}{\epsilon L} \right) \nu |f - \mathbb{E} f|^q \leq D_L B |\nabla_{(\sim i)} f|^q + \frac{2q^2 CS}{\epsilon L} \sum_{r=1}^{i-2} \nu |\nabla_r f|^q \\
+ \frac{2q^2 S}{\epsilon L} \sum_{r=3}^{\infty} J_r^{-2} \nu |\nabla_{i\pm r} f|^q \]

Which implies

\[ \nu |f - \mathbb{E}[(\sim i)f]|^q \leq M \sum_{r=1}^{i-2} \nu |\nabla_r f|^q + M \sum_{r=3}^{\infty} J_r^{-2} \nu |\nabla_{i\pm r} f|^q \]

for some constant \( M > 0 \).

If we combine together Proposition 3.1 and Proposition 3.3 the following explicit bound for \( Q(k, k) \) directly follows.

**Corollary 3.4.** Suppose that hypothesis (H1), (H2) and (H3) are satisfied. Then

\[ Q(k, k) \leq D \sum_{r=k-2}^{\infty} \nu |\nabla_r f|^q + 2 \sum_{r=3}^{\infty} J_r^{-2} \nu |\nabla_{k\pm r} f|^q \]

where the constant \( D = J^q S M + S \) and \( J_0 = \frac{J^{q+\frac{1}{2}}}{4} \).
4 (LSq) type inequality under (H0').

Bellow an analogue result for the Log-Sobolev type inequality for $E_{\sim i, \omega}$ is presented assuming hypothesis A.

**Proposition 4.1.** If conditions A are satisfied then for $J$ sufficiently small, the following Log-Sobolev type inequality holds

$$\nu(|f|^q \log \frac{|f|^q}{E_{\sim i, \omega}|f|^q}) \leq R_1 \sum_{r=1-2}^{i+2} \nu |\nabla_r f|^q + R_1 \sum_{r=3}^{\infty} J_{r-2}^2 \nu |\nabla_{i\pm r} f|^q$$

for some positive constant $R_1$.

**Proof.** Assume $f \geq 0$. We will use the Log-Sobolev Inequality for the $\mu$ measure to derive conditions for the Log-Sobolev inequality for the measure $E_{\sim i, \omega}$. From hypothesis (H0') and Remark 1.7 the product measure $\mu(dx_{i+1}) \otimes \mu(dx_{i-1})$ satisfies the LSq with constant $c$.

$$\mu(dx_{i+1}) \otimes \mu(dx_{i-1})(g^q \log \frac{g^q}{\mu(dx_{i+1}) \otimes \mu(dx_{i-1})g^q}) \leq c\mu(dx_{i+1}) \otimes \mu(dx_{i-1})|\nabla g|^q$$

(4.1)

for $g \geq 0$. Define the function

$$h^i = -\sum_{j={i-1, i+1}, i\sim j} J_{j,i}V(x_j, \omega_t)$$

The function $h^i$ is localized in $\Lambda(i)$. We also denote

$$\Phi^i = \phi(x_{i-1}) + \phi(x_{i+1})$$

Then inequality (4.1) for $g = e^{\Phi^i}, f \geq 0$ gives

$$\int e^{-\Phi^i}(e^{h^i}f^q \log \frac{e^{h^i}f^q}{\int e^{-\Phi^i}(e^{h^i}f^q)dx_{i-1}dx_{i+1}\int e^{-\Phi^i}dx_{i-1}dx_{i+1}})dx_{i-1}dx_{i+1}$$

$$\leq c \sum_{j=i-1, i+1} \int e^{-\Phi^i} \left|\nabla_j (e^{h^i}f)\right|^q dx_{i-1}dx_{i+1}$$

(4.2)
Denote by \(I_r\) and \(I_t\) the right and left hand side of (4.2) respectively. If we use the Leibnitz rule for the gradient on the right hand side of (4.2) we have

\[
I_r \leq 2^{q-1} c \int e^{-\Phi^i} \left| \frac{h^i}{E} \nabla_j f \right|^q dx_{i-1} dx_{i+1} + 2^{q-1} c \sum_{j=i-1,i+1} \int e^{-\Phi^i} \left| f \nabla_j e \right|^q dx_{i-1} dx_{i+1} = \\
\left( \int e^{-\Phi^i + h^i} f^q dx_{i-1} dx_{i+1} \right) c 2^{q-1} \left( E^{(\sim i)} \omega \left| \nabla_j f \right|^q + \frac{1}{q^q} E^{(\sim i)} \omega f^q \sum_{j=i-1,i+1} \left| \nabla_j h^i \right|^q \right) \tag{4.3}
\]

On the left hand side of (4.2) we form the entropy for the measure \(\mathbb{E}^{(\sim i)} \omega\) measure with phase \(\Phi^i - h^i\).

\[
I_t = \int e^{-\Phi^i + h^i} f^q \log \frac{f^q}{\int e^{-\Phi^i + h^i} f^q dx_{i-1} dx_{i+1}} \frac{f^q}{\int e^{-\Phi^i + h^i} dx_{i-1} dx_{i+1}} dx_{i-1} dx_{i+1} + \int e^{-\Phi^i + h^i} f^q \log \left( \int e^{-\Phi^i + h^i} dx_{i-1} dx_{i+1} \right) e^{h^i} \\
\left( \int e^{-\Phi^i + h^i} f^q dx_{i-1} dx_{i+1} \right) \left( E^{(\sim i)} \omega \left( f^q \log \frac{f^q}{E^{(\sim i)} \omega f^q} \right) + E^{(\sim i)} \omega (f^q h^i) \right) \\
+ \int e^{-\Phi^i + h^i} f^q \log \left( \frac{e^{-\Phi^i + h^i} dx_{i-1} dx_{i+1}}{e^{-\Phi^i + h^i} dx_{i-1} dx_{i+1}} \right) e^{h^i} dx_{i-1} dx_{i+1} \tag{4.4}
\]

Since \(h^i\) is negative, because of hypothesis (H3), the last equality leads to

\[
I_t \geq \left( \int e^{-\Phi^i + h^i} f^q dx_{i-1} dx_{i+1} \right) \left( E^{(\sim i)} \omega \left( f^q \log \frac{f^q}{E^{(\sim i)} \omega f^q} \right) + E^{(\sim i)} \omega (f^q h^i) \right) \tag{4.5}
\]

Combining (4.2) together with (4.3) and (4.5) we obtain

\[
E^{(\sim i)} \omega (f^q \log \frac{f^q}{E^{(\sim i)} \omega f^q}) \leq 2^{q-1} c \sum_{j=i-1,i+1} E^{(\sim i)} \omega \left| \nabla_j f \right|^q + E^{(\sim i)} \omega \left( f^q \left( \frac{c 2^{q-1} \sum_{j=i-1,i+1} \left| \nabla_j h^i \right|^q}{q^q} - h^i \right) \right) 
\]

If we apply the Gibbs measure in the last relationship we have

\[
\nu (f^q \log \frac{f^q}{E^{(\sim i)} \omega f^q}) \leq 2^{q-1} c \sum_{j=i-1,i+1} \nu \left| \nabla_j f \right|^q + \nu \left( f^q \left( \frac{c 2^{q-1} \sum_{j=i-1,i+1} \left| \nabla_j h^i \right|^q}{q^q} - h^i \right) \right) \tag{4.6}
\]
From [B-Z] and [R], for $1 < q < 2$ and $q = 2$ respectively, the following estimate of the entropy holds

\[
\mathbb{E}^{\{i\}, \omega}(|f|^q \log \frac{|f|^q}{\mathbb{E}^{\{i\}, \omega} |f|^q}) \leq A \mathbb{E}^{\{i\}, \omega} |f - \mathbb{E}^{\{i\}, \omega} f|^q
\]

\[
+ \mathbb{E}^{\{i\}, \omega} |f - \mathbb{E}^{\{i\}, \omega} f|^q \log \frac{|f - \mathbb{E}^{\{i\}, \omega} f|^q}{\mathbb{E}^{\{i\}, \omega} |f - \mathbb{E}^{\{i\}, \omega} f|^q}
\]

(4.7)

for some positive constant $A$. If we apply the Gibbs measure at the last inequality we get

\[
\nu(|f|^q \log \frac{|f|^q}{\mathbb{E}^{\{i\}, \omega} |f|^q}) \leq A \nu |f - \mathbb{E}^{\{i\}, \omega} f|^q
\]

\[
+ \nu(|f - \mathbb{E}^{\{i\}, \omega} f|^q \log \frac{|f - \mathbb{E}^{\{i\}, \omega} f|^q}{\mathbb{E}^{\{i\}, \omega} |f - \mathbb{E}^{\{i\}, \omega} f|^q})
\]

(4.8)

We can now use (4.6) to bound the second term on the right hand side of (4.8). Then we will obtain

\[
\nu(f^q \log \frac{f^q}{\mathbb{E}^{\{i\}, \omega} f^q}) \leq A \nu |f - \mathbb{E}^{\{i\}, \omega} f|^q + 2^{q-1} c \sum_{j=i-1,i+1} \nu |\nabla_j f|^q
\]

\[
+ \nu \left( |f - \mathbb{E}^{\{i\}, \omega} f|^q \left( \frac{c 2^{q-1} \sum_{j=i-1,i+1} |\nabla_j h_i|^q}{q^q} - h_i \right) \right)
\]

\[
= A \nu |f - \mathbb{E}^{\{i\}, \omega} f|^q + 2^{q-1} c \sum_{j=i-1,i+1} \nu |\nabla_j f|^q
\]

\[
+ \nu_{\Lambda(i)} \left( (\mathbb{E}^M(i) |f - \mathbb{E}^{\{i\}, \omega} f|^q) \left( \frac{c 2^{q-1} \sum_{j=i-1,i+1} |\nabla_j h_i|^q}{q^q} - h_i \right) \right)
\]

(4.9)

where the last equality holds due to the fact that $h_i$ is localised in $\Lambda(i)$. We can bound the last term on the right hand side of (4.9) with the use of the entropic inequality (3.6) and the Log-Sobolev inequality for $\nu_{\Lambda(i)}$ from (H1), in the same
way we worked in Proposition 3.3. Then we will get

$$
\nu(f^q \log \frac{f^q}{\mathbb{E}^{(\sim i), \omega} f^q}) \leq A \nu |f - \mathbb{E}^{(\sim i), \omega} f|^q + 2c \sum_{j=i-1, i+1} \nu |\nabla_j f|^q \\
+ \frac{C}{\epsilon} \nu \lambda(i) \left| \nabla \lambda(i) \left( \frac{c^q - 1 \sum_{j=i-1, i+1} |\nabla_j f|^q}{\epsilon^q} \right) \right|^q \\
+ \frac{1}{\epsilon} \left( \log \nu e^{\frac{c^q - 1 \sum_{j=i-1, i+1} |\nabla_j f|^q}{\epsilon^q}} \right) \nu |f - \mathbb{E}^{(\sim i), \omega} f|^q \\
\leq (A + K) \nu |f - \mathbb{E}^{(\sim i), \omega} f|^q + \frac{C}{\epsilon} Q(i, i) + 2c \sum_{j=i-1, i+1} \nu |\nabla_j f|^q
$$

where at the last inequality we used hypothesis (H2) to bound

$$
\nu e^{\frac{c^q - 1 \sum_{j=i-1, i+1} |\nabla_j f|^q}{\epsilon^q}}. \quad \text{We can now use Corollary 3.3 to bound } Q(i, i) \text{ in (4.10) as well as Proposition 3.3 to bound } \nu |f - \mathbb{E}^{(\sim i), \omega} f|^q. \text{ We will then obtain}
$$

$$
\nu(f^q \log \frac{f^q}{\mathbb{E}^{(\sim i), \omega} f^q}) \leq (\frac{DC}{\epsilon} + AM + \frac{MK}{\epsilon}) \sum_{r=3}^{\infty} J_{r-2}^r \nu |\nabla_{i \pm r} f|^q \\
+ (\frac{DC}{\epsilon} + AM + \frac{MK}{\epsilon}) \sum_{r=i-2}^{i+2} \nu |\nabla_r f|^q + 2c \sum_{j=i-1, i+1} \nu |\nabla_j f|^q
$$

The lemma follows for appropriate choice of the constant $R_1$. \qed

5 (LSq) type inequality under (H0$''$).

The proofs in this section follow closely mainly the methods used in [H-Z], but also in [I-P]. We start with a proposition that shows how the non uniform U-bound and the Log-Sobolev inequality are related.

**Proposition 5.1.** Suppose that the measure

$$
d\mathbb{E}^{(\sim i), \omega} = \frac{e^{-H^{(\sim i), \omega}} dX^{(\sim i)}}{\int e^{-H^{(\sim i), \omega}} dX^{(\sim i)}}
$$

satisfies the following non uniform U-bound

$$
\mathbb{E}^{(\sim i), \omega} |f|^q \left( |\nabla^{(\sim i)} H^{(\sim i), \omega}|^q + H^{(\sim i), \omega} \right) \leq \tilde{C} \mathbb{E}^{(\sim i), \omega} |\nabla_i f|^q + \tilde{D}_{(\sim i)}(\omega) \mathbb{E}^{(\sim i), \omega} |f|^q
$$

(5.1)
for some positive constant $\hat{C}$ and a function $\hat{D}_{\{\sim i\}}(\omega)$ of $\omega$, both independent of $f$. Then the following defective Log-Sobolev inequality holds

$$\mathbb{E}^{\sim i}\omega |f|^q \log \frac{|f|^q}{\mathbb{E}^{\sim i}\omega |f|^q} \leq C \mathbb{E}^{\sim i}\omega |\nabla_{\{\sim i\}} f|^q + C\mathbb{E}^{\sim i}\omega |f|^q + D_i(\omega)\mathbb{E}^{\sim i}\omega |f|^q$$

where $C$ is a constant and $D_i(\omega) = \max\{\frac{2^{q-1}(q+\epsilon)\alpha}{\epsilon q^q}, 1\} \hat{D}_{\{\sim i\}}(\omega)$.

**Proof.** Without loss of generality we can assume that $f \geq 0$. We set $\rho_i = \frac{e^{-H(\sim i)} - \rho}{f e^{-H(\sim i)} dX_{\sim i}}$ and $g = f^{\frac{1}{q}}$. We also assume that

$$\int g^q dX_{\sim i} = \mathbb{E}^{\sim i}\omega f^q = 1$$

Then we can write

$$\int (g^q \log g^q) dX_{\sim i} = \frac{q}{\epsilon} \int g^q(log g^q) dX_{\sim i} \leq \frac{q + \epsilon}{\epsilon} \frac{q}{q} \log \left( \int g^{q+\epsilon} dX_{\sim i} \right)^{\frac{q}{q+\epsilon}}$$

where above we used the Jensen’s inequality. In order to bound the last expression we can use the Classical Sobolev (C-S) inequality for the Lebesgue measure $dX_{\sim i}$ (see [V-SC-C]),

$$\left( \int |f|^{q+\epsilon} dX_{\sim i} \right)^{\frac{q}{q+\epsilon}} \leq \alpha \int |\nabla f|^q dX_{\sim i} + \beta \int |f|^q dX_{\sim i}$$

(C-S)

for positive constants $\alpha, \beta$. We will then obtain

$$\int (g^q \log g^q) dX_{\sim i} \leq \frac{q + \epsilon}{\epsilon} \log \left( \alpha \int |\nabla_{\sim i} g|^q dX_{\sim i} + \beta \int |g|^q dX_{\sim i} \right)$$

$$\leq \frac{(q + \epsilon)\alpha}{\epsilon} \int |\nabla_{\sim i} g|^q dX_{\sim i} + \frac{(q + \epsilon)\beta}{\epsilon} \int |g|^q dX_{\sim i}$$

(5.2)

where in the last inequality we used that $\log x \leq x$ for $x > 0$. For the first term on the right hand side of (5.2) we have

$$\int |\nabla_{\sim i} g|^q dX_{\sim i} = \int |\nabla_{\sim i}(f \rho_{\frac{1}{q}})|^q dX_{\sim i}$$

$$\leq 2^{q-1}\mathbb{E}^{\sim i}\omega |\nabla_{\sim i} f|^q + 2^{q-1} \int |f \nabla_{\sim i}(\rho_{\frac{1}{q}})|^q dX_{\sim i}$$

(5.3)

We have

$$\int |f \nabla_{\sim i}(\rho_{\frac{1}{q}})|^q dX_{\sim i} = \int |\rho_{\frac{1}{q}} \rho_{\frac{1}{q}}^{-\frac{1}{q}} f \nabla_{\sim i}(\rho_{\frac{1}{q}})|^q dX_{\sim i} = \mathbb{E}^{\sim i}\omega f^q |\rho_{\frac{1}{q}} \nabla_{\sim i}(\rho_{\frac{1}{q}})|^q$$

$$= \frac{1}{q^q} \mathbb{E}^{\sim i}\omega f^q |\nabla_{\sim i} H^{\sim i}\omega|^q$$

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If we replace the uniform U-bound (5.1), the inequality (5.7) gives

\[ \int |\nabla_{\{\sim\}} g|^q dX_{\{\sim\}} \leq 2^{q-1} E^{\{\sim\}, \omega} |\nabla_{\{\sim\}} f|^q + \frac{2^{q-1}}{q^q} E^{\{\sim\}, \omega} f^q |\nabla_{\{\sim\}} H^{\{\sim\}, \omega}|^q \]  

(5.4)

If we combine inequalities (5.2) and (5.4), we get

\[ \int (g^q \log g^q) dX_{\{\sim\}} \leq \frac{2^{q-1}(q + \epsilon)\alpha}{\epsilon} E^{\{\sim\}, \omega} |\nabla_{\{\sim\}} f|^q + \frac{(q + \epsilon)\beta}{\epsilon} E^{\{\sim\}, \omega} f^q + \frac{2^{q-1}(q + \epsilon)\alpha}{\epsilon q^q} E^{\{\sim\}, \omega} f^q |\nabla_{\{\sim\}} H^{\{\sim\}, \omega}|^q \]  

(5.5)

For the left hand side of (5.5), since \( H^{\{\sim\}, \omega} \geq 0 \), we have

\[ \int (g^q \log g^q) dX_{\{\sim\}} = \int \left( \frac{e^{-H^{\{\sim\}, \omega}}}{\int e^{-H^{\{\sim\}, \omega}} dX_{\{\sim\}}} f^q \log \frac{e^{-H^{\{\sim\}, \omega}}}{\int e^{-H^{\{\sim\}, \omega}} dX_{\{\sim\}}} \right) dX_{\{\sim\}} \]

\[ = E^{\{\sim\}, \omega}(f^q \log f^q) + E^{\{\sim\}, \omega} \left( f^q \log \int e^{-H^{\{\sim\}, \omega}} dX_{\{\sim\}} \right) \]

\[ = E^{\{\sim\}, \omega}(f^q \log f^q) - E^{\{\sim\}, \omega}(f^q H^{\{\sim\}, \omega}) \]

\[ \geq E^{\{\sim\}, \omega}(f^q \log f^q) - E^{\{\sim\}, \omega}(f^q H^{\{\sim\}, \omega}) \]  

(5.6)

If we combine (5.5) and (5.6), we obtain

\[ E^{\{\sim\}, \omega}(f^q \log f^q) \leq \hat{\alpha} E^{\{\sim\}, \omega} |\nabla_{\{\sim\}} f|^q + \hat{\gamma} E^{\{\sim\}, \omega} f^q \]

\[ + \hat{\beta} E^{\{\sim\}, \omega} f^q \left( |\nabla_{\{\sim\}} H^{\{\sim\}, \omega}|^q + H^{\{\sim\}, \omega} \right) \]  

(5.7)

where \( \hat{\alpha} = 2^{q-1}(q + \epsilon)\alpha \), \( \hat{\beta} = \max\left\{ \frac{2^{q-1}(q + \epsilon)\alpha}{\epsilon q^q}, 1 \right\} \) and \( \hat{\gamma} = \frac{(q + \epsilon)\beta}{\epsilon} \). If we use the non uniform U-bound (5.1), the inequality (5.7) gives

\[ E^{\{\sim\}, \omega}(f^q \log f^q) \leq (\hat{\alpha} + \hat{\beta} \hat{C}) E^{\{\sim\}, \omega} |\nabla_{\{\sim\}} f|^q + \hat{\beta} \hat{D}_{\{\sim\}}(\omega) E^{\{\sim\}, \omega} f^q + \hat{\gamma} E^{\{\sim\}, \omega} f^q \]

If we replace \( f \) with \( \frac{f}{E^{\{\sim\}, \omega} f} \) which has mean equal to one we obtain the result. \( \square \)

**Corollary 5.2.** If condition \((H0')\) is satisfied then the following inequality is true.

\[ E^{\{\sim\}, \omega} \left( |f|^q \log \frac{|f|^q}{E^{\{\sim\}, \omega} |f|^q} \right) \leq C E^{\{\sim\}, \omega} |\nabla_{\{\sim\}} f|^q + (A + C) E^{\{\sim\}, \omega} |f - E^{\{\sim\}, \omega} f|^q \]

\[ + D_i(\omega) E^{\{\sim\}, \omega} |f - E^{\{\sim\}, \omega} f|^q \]  

(5.8)
Proof. We recall (4.7).

\[ \mathbb{E}^{\{\sim i\}, \omega} \left( |f|^q \log \frac{|f|^q}{\mathbb{E}^{\{\sim i\}, \omega} |f|^q} \right) \leq A \mathbb{E}^{\{\sim i\}, \omega} |f - \mathbb{E}^{\{\sim i\}, \omega} f|^q \]

\[ + \mathbb{E}^{\{\sim i\}, \omega} |f - \mathbb{E}^{\{\sim i\}, \omega} f|^q \log \frac{|f - \mathbb{E}^{\{\sim i\}, \omega} f|^q}{\mathbb{E}^{\{\sim i\}, \omega} |f - \mathbb{E}^{\{\sim i\}, \omega} f|^q} \]

If we use Proposition 5.1 to bound the second term on the right hand side, we get

\[ \mathbb{E}^{\{\sim i\}, \omega} \left( |f|^q \log \frac{|f|^q}{\mathbb{E}^{\{\sim i\}, \omega} |f|^q} \right) \leq C \mathbb{E}^{\{\sim i\}, \omega} |\nabla{\sim i} f|^q + (A + C) \mathbb{E}^{\{\sim i\}, \omega} |f - \mathbb{E}^{\{\sim i\}, \omega} f|^q \]

\[ + D_i(\omega) \mathbb{E}^{\{\sim i\}, \omega} |f - \mathbb{E}^{\{\sim i\}, \omega} f|^q \]

\[ \Box \]

Bellow we prove the Log-Sobolev type inequality (2.3).

**Proposition 5.3.** If conditions \( B \) are satisfied then the following Log-Sobolev type inequality holds

\[ \frac{1}{R_2} \nu(|f|^q \log \frac{|f|^q}{\mathbb{E}^{\{\sim i\}, \omega} |f|^q}) \leq \sum_{i=2}^{i+2} \nu |\nabla f|^q + \sum_{r=3}^{\infty} J_0^{-2} \nu |\nabla \pm f|^q \]

for some positive constant \( R_2 \) independent of \( f \) and \( i \).

Proof. If we apply the Gibbs measure at the Log-Sobolev type inequality (5.8), from Corollary 5.2 we have

\[ \nu(|f|^q \log \frac{|f|^q}{\mathbb{E}^{\{\sim i\}, \omega} |f|^q}) \leq C \nu |\nabla{\sim i} f|^q + (A + C) \nu |f - \mathbb{E}^{\{\sim i\}, \omega} f|^q \]

\[ + \nu D_i(\omega) |f - \mathbb{E}^{\{\sim i\}, \omega} f|^q \]

\[ = C \nu |\nabla{\sim i} f|^q + (A + C) \nu |f - \mathbb{E}^{\{\sim i\}, \omega} f|^q \]

\[ + \nu \Lambda(i) D_i(\omega) \mathbb{E}^{\Lambda(i)} |f - \mathbb{E}^{\{\sim i\}, \omega} f|^q \tag{5.9} \]

where we used that \( D_i(\omega) \) is localised in \( \Lambda(i) \) and that \( M(i) = Z \setminus \Lambda(i) \). If we use again the entropic inequality (3.6) as we did in Proposition 5.3 to bound the last

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term on the right hand side of (5.9), we obtain
\[
\nu(\|f\|^q \log \frac{\|f\|^q}{\mathbb{E}(\sim i, \omega) \|f\|^q}) \leq C\nu \left| \nabla_{\sim i} f \right|^q + (A + C)\nu \left| f - \mathbb{E}(\sim i, \omega) f \right|^q
\]
+ \frac{C}{\epsilon} \nu_{\Lambda(i)} \mathbb{E}(\sim i, \omega) \left| f - \mathbb{E}(\sim i, \omega) f \right|^q \log \frac{\mathbb{E}(i) \left| f - \mathbb{E}(\sim i, \omega) f \right|^q}{\nu_{\Lambda(i)} \mathbb{E}(i) \left| f - \mathbb{E}(\sim i, \omega) f \right|^q}
+ \frac{1}{\epsilon} \left( \log \nu_{\Lambda(i)} \epsilon(D(\omega)) \right) \nu_{\Lambda(i)} \mathbb{E}(\sim i, \omega) \left| f - \mathbb{E}(\sim i, \omega) f \right|^q
\leq C\nu \left| \nabla_{\sim i} f \right|^q + (A + C + \frac{K}{\epsilon})\nu \left| f - \mathbb{E}(\sim i, \omega) f \right|^q
+ \frac{C}{\epsilon} Q(i, i) \quad (5.10)
\]

Where to obtain the last inequality we used the Log-Sobolev inequality for the measure \(\nu_{\Lambda(i)}\) from hypothesis (H1). If we use Proposition 3.3 and Corollary 3.4 to bound the third and fourth term on the right hand side of (5.10), we finally get
\[
\nu(\|f\|^q \log \frac{\|f\|^q}{\mathbb{E}(\sim i, \omega) \|f\|^q}) \leq C\nu \left| \nabla_{\sim i} f \right|^q + (A + C + \frac{K}{\epsilon})\nu \left| f - \mathbb{E}(\sim i, \omega) f \right|^q
+ \frac{DC}{\epsilon} \sum_{r=1}^{i+2} \nu \left| \nabla_r f \right|^q + \frac{DC}{\epsilon} \sum_{r=3}^{\infty} J_0^{-2} \nu \left| \nabla_{r \pm i} f \right|^q
\]
which gives the result for appropriate constant \(R_2\) and \(0 < J_0 < 1\).

6 Proof of Proposition 2.4 and Proposition 2.5.

We first present some useful lemmata. The first lemma provides a technical result for the correlation. The proof of the lemma can be found in [Pa].

**Lemma 6.1.** For any function \(v_k\) localised in \(\Lambda(k)\), the following inequality holds
\[
\mathbb{E}(\sim k) \left( \|f\|^q ; v_k \right) \leq c_0 \left( \mathbb{E}(\sim k) \left| f \right|^q \right)^{\frac{1}{2}} \left( \mathbb{E}(\sim k) \left( \|f - \mathbb{E}(\sim k) f\|^q \left( \|v_k\|^q + \mathbb{E}(\sim k) \|v_k\|^q \right) \right)^{\frac{1}{2}}
\]
for some constant \(c_0\) uniformly on the boundary conditions.

The next lemma presents an estimate involving \(Q(k, k)\).
Lemma 6.2. Suppose that hypothesis (H1) is satisfied. Then
\[
\nu \left( \mathbb{E}^{(-k)} |f|^q \right)^{\frac{2}{p}} \left| \mathbb{E}^{(-k)} (|f|^q; v_k) \right|^q \leq \frac{c_0^2 C}{\epsilon} Q(k, k) + \frac{c_0^2}{\epsilon} \left( \log \nu e^{(v_k^q + \mathbb{E}^{(-k)} |v_k|^q)} \right) \nu |f - \mathbb{E}^{(-k)} f|^q
\]
for any function \( v_k \) localised in \( \Lambda(k) \).

Proof. We can start with the bound from Lemma 6.1
\[
\nu \left( \mathbb{E}^{(-k)} |f|^q \right)^{\frac{2}{p}} \left| \mathbb{E}^{(-k)} (|f|^q; v_k) \right|^q \leq c_0^2 \nu \mathbb{E}^{(-k)} (|f - \mathbb{E}^{(-k)} f|^q (|v_k|^q + \mathbb{E}^{(-k)} |v_k|^q))
\]
(6.1)
because \(|v_k|^q + \mathbb{E}^{(-k)} |v_k|^q\) is localised in \( \Lambda(k) \). If we use the relative entropic inequality (3.6) as we did in Proposition 3.3, together with hypothesis (H1) we can bound (6.1) by
\[
(6.1) \leq \frac{c_0^2 C}{\epsilon} Q(k, k) + \frac{c_0^2}{\epsilon} \left( \log \nu e^{(v_k^q + \mathbb{E}^{(-k)} |v_k|^q)} \right) \nu_{\Lambda(k)} \mathbb{E}^{(-k)} |f - \mathbb{E}^{(-k)} f|^q
\]
(6.1)

Before we prove the sweeping out relations of Lemma 6.5 and Lemma 6.6 we present two lemmata whose proof can be found in [Pa].

Lemma 6.3. The following inequality is satisfied
\[
\nu \left| \nabla_j (\mathbb{E}^{(-i)} |f|^q)^{\frac{1}{q}} \right|^q \leq c_1 \nu |\nabla_j f|^q
\]
\[
+ \frac{Jq^q}{q^q} \nu \left( \mathbb{E}^{(-i)} |f|^q \right)^{\frac{2}{p}} \left| \mathbb{E}^{(-i)} (|f|^q; \sum_{t \in \{i-2,i,i+2\}, i \sim j} \nabla_j V(x_t, \omega_j)) \right|^q
\]
for \( j = i - 2, i, i + 2 \).

Lemma 6.4. Under hypothesis (H1), for any functions \( u \) localised in \( \Lambda(k) \) the following inequality is satisfied
\[
\nu |\mathbb{E}^{k-1} \mathbb{E}^{k+1} (f; u)|^q \leq \frac{C}{\epsilon} Q(k, k) + \frac{1}{\epsilon} \left( \log \nu_{\Lambda(k)} e^{(u - \mathbb{E}^{k-1} \mathbb{E}^{k+1} u)^q} \right) \nu |f - \mathbb{E}^{k-1} \mathbb{E}^{k+1} f|^q
\]
for \( \epsilon > 0 \).
Lemma 6.5. Suppose that hypothesis A or B are satisfied. Then
\[
\nu \left| \nabla_{\Gamma_i}(E^{\Gamma_j} f) \right|^q \leq D_1 \nu \left| \nabla_{\Gamma_j} f \right|^q + D_2 \nu \left| \nabla_{\Gamma_j} f \right|^q
\]
for \( \{i, j\} = \{0, 1\} \) and constants \( D_1 > 0 \) and \( 0 < D_2 < 1 \).

Proof. Without loss of generality assume \( i = 0 \) and \( j = 1 \). We have
\[
\nu \left| \nabla_{\Gamma_i}(E^{\Gamma_0} f) \right|^q = \sum_{\Gamma_i} \nu \left| \nabla_{\Gamma_i}(E^{\Gamma_0} f) \right|^q \leq \sum_{\Gamma_i} \nu \left| \nabla_{\Gamma_i}(E^{\Gamma_1} f) \right|^q
\]
(6.2)
If we denote \( \rho_i = \frac{e^{-H(x_{i-1})}e^{-H(x_{i+1})}}{\int e^{-H(x_{i-1})}dx_i \int e^{-H(x_{i+1})}dx_i} \) the density of the measure \( E^{\Gamma_1} \) we can then write
\[
\nu \left| \nabla_{\Gamma_i}(E^{\Gamma_1} f) \right|^q = \nu \left( \int \int \rho_i f dx_{i-1} dx_{i+1} \right)^q \leq
\]
\[
2^{q-1} \nu \left( \int \int (\nabla_{\Gamma_i}) \rho_i dx_{i-1} dx_{i+1} \right)^q + 2^{q-1} \nu \left( \int \int f \nabla_{\Gamma_i} \rho_i dx_{i-1} dx_{i+1} \right)^q \leq
\]
\[
c_1 \nu \left| \nabla^{\Gamma_1} f \right|^q + c_1 J^q \nu \left| \nabla^{\Gamma_1} f - E^{\Gamma_1} f \right|^q
\]
(6.3)
where in (6.3) we used hypothesis (H3) to bound the coefficients \( J_{i,j} \) and we have set \( c_1 = 2^{q} \). If we apply the Hölder Inequality to the first term of (6.3) and Lemma 6.4 to the second term, we obtain
\[
\nu \left| \nabla_{\Gamma_i}(E^{\Gamma_1} f) \right|^q \leq c_1 \nu \left| \nabla_{\Gamma_i} f \right|^q + \frac{c_1 J^q}{\epsilon} Q(i, i) + \frac{c_1 K J^q}{\epsilon} \nu \left| f - E^{\Gamma_1} f \right|^q
\]
(6.4)
where the constant \( K \) as in hypothesis (H2).

If we use Corollary 3.3 to bound \( Q(i, i) \) and Proposition 3.3 to bound the last term on the right hand side of (6.3) we obtain
\[
\nu \left| \nabla_{\Gamma_i}(E^{\Gamma_1} f) \right|^q \leq c_1 \nu \left| \nabla_{\Gamma_i} f \right|^q + \frac{c_1 J^q DC}{\epsilon} \sum_{r=i-2}^{i+2} \nu \left| \nabla_{\Gamma_r} f \right|^q + \frac{c_1 J^q DC}{\epsilon} \sum_{r=i}^{\infty} J_0 r^{-2} \nu \left| \nabla_{\Gamma_i} f \right|^q
\]
\[
+ \frac{c_1 KM J^q}{\epsilon} \sum_{r=i-2}^{i+2} \nu \left| \nabla_{\Gamma_r} f \right|^q + \frac{c_1 KM J^q}{\epsilon} \sum_{r=i}^{\infty} J_0 r^{-2} \nu \left| \nabla_{\Gamma_i} f \right|^q
\]
(6.5)
for the constants \( D \) as in Corollary 3.3 and \( M \) as in Proposition 3.3. From (6.2) and (6.5) we have
\[
\nu \left| \nabla_{\Gamma_i}(E^{\Gamma_0} f) \right|^q \leq c_1 \nu \left| \nabla_{\Gamma_i} f \right|^q + \frac{(MK + D) c_1 J^q}{\epsilon} \sum_{i \in \Gamma_1} \sum_{r=i-2}^{i+2} \nu \left| \nabla_{\Gamma_r} f \right|^q
\]
\[
+ \frac{(MK + D) c_1 J^q}{\epsilon} \sum_{i \in \Gamma_1} \sum_{r=i}^{\infty} J_0 r^{-2} \nu \left| \nabla_{\Gamma_i} f \right|^q
\]

The last one implies
\[
\nu |\nabla_{\Gamma_1}(\mathbb{E}^{\Gamma_0} f)|^q \leq c_1 \nu |\nabla_{\Gamma_1} f|^q + \frac{(MK + D) c_1 J^q}{\epsilon} (3 + \sum_{n=0}^{\infty} J_0^{2n}) \nu |\nabla_{\Gamma_1} f|^q
\]
\[
+ \frac{(MK + D) c_1 J^q}{\epsilon} (2 + \sum_{n=1}^{\infty} J_0^{2n-1}) \nu |\nabla_{\Gamma_0} f|^q
\]
If we choose \( J \) in (H3) sufficiently small such that \( J_0 < 1 \) we finally obtain
\[
\nu |\nabla_{\Gamma_1}(\mathbb{E}^{\Gamma_0} f)|^q \leq J^q \frac{(MK + D) c_1 J^q}{\epsilon} (2 + \frac{J_0}{1 - J_0^2}) \nu |\nabla_{\Gamma_0} f|^q
\]
\[
+ \left( c_1 + \frac{(MK + D) c_1 J^q}{\epsilon} \right) (3 + \frac{1}{1 - J_0^2}) \nu |\nabla_{\Gamma_1} f|^q
\]
and the lemma follows for
\[
D_1 = c_1 + \frac{(MK + D) c_1 J^q}{\epsilon} \left( 3 + \frac{1}{1 - J_0^2} \right)
\]
and \( J \) sufficiently small such that
\[
D_2 = J^q \frac{(MK + D) c_1 J^q}{\epsilon} (2 + \frac{J_0}{1 - J_0^2}) < 1
\]
\( \square \)

We continue with another sweeping out property which will play the basis of the proof of Proposition 2.5.

**Lemma 6.6.** Suppose that hypothesis \( A \) or \( B \) are satisfied. Then
\[
\nu |\nabla_{\Gamma_1}(\mathbb{E}^{\Gamma_0} f)|^q \leq H \nu |\nabla_{\Gamma_1} f|^q + H J^q \sum_{r=2}^{\infty} J_0^{-r-2} \nu |\nabla_{\Gamma_1} f|^q
\]
for \( j = i - 2, i, i + 2 \) and some positive constant \( H \).

**Proof.** Assume \( f \geq 0 \). For \( j = i - 2, i, i + 2 \), from Lemma 6.3 we have
\[
\nu |\nabla_{\Gamma_1}(\mathbb{E}^{\Gamma_0} f)|^q \leq c_1 \nu |\nabla_{\Gamma_1} f|^q +
\]
\[
\frac{J^q c_1}{q^q} \nu (\mathbb{E}^{\Gamma_1} f)^{\frac{q}{\pi}} |\mathbb{E}^{\Gamma_1} (f_{t_{\sim i}} \sum_{t \in \{i-2,i,i+2\}: t \sim j} \nabla_j V(x_t, \omega_j))|^q
\]
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If we bound the last term from Lemma 6.6 we obtain

\[ \nu \left| \nabla_j \left( \mathbb{E}^{\{\sim i \}} f^q \right) \right|^q \leq c_1 \nu \left| \nabla_j f \right|^q + \frac{J^q c_0^q C_1}{eq^q} Q(i, i) \]

\[ + \frac{J^q c_0^q}{eq^q} \left( \log \nu \Lambda(i) e^{c \left( |W_i|^q + \mathbb{E}^{\{\sim i \}} |W_i|^q \right)} \right) \nu |f - \mathbb{E}^{\{\sim i \}} f|^q \]  \hspace{1cm} (6.6)

where above we denoted \( W_i = \sum_{t \in \{i-2, i, i+2\}; t \sim j} \nabla_j V(x_t, \omega_j) \). We can make use of hypothesis (H2) to bound

\[ \log \nu \Lambda(i) e^{c \left( |W_i|^q + \mathbb{E}^{\{\sim i \}} |W_i|^q \right)} < K \]

as well as use Proposition 3.3 and Corollary 3.4 to bound \( \nu |f - \mathbb{E}^{\{\sim i \}} f|^q \) and \( Q(i, i) \). Then (6.6) becomes

\[ \nu \left| \nabla_j \left( \mathbb{E}^{\{\sim i \}} f^q \right) \right|^q \leq c_1 \nu \left| \nabla_j f \right|^q + \frac{J^q c_0^q C(M + D)c_1}{eq^q} \sum_{r=1-2}^{i+2} \nu \left| \nabla_r f \right|^q \]

\[ + \frac{J^q c_0^q C(M + D)c_1}{eq^q} \sum_{r=3}^{\infty} J_r^{-2} \nu \left| \nabla_{i \pm r} f \right|^q \]  \hspace{1cm} (6.7)

The proof of Lemma 6.6 is complete for appropriate choice of constant \( H \). \( \square \)

We will finish this section with the Proposition 2.5.

**Proof of Proposition 2.5.** Assume \( f \geq 0 \). We can write

\[ \nu \left| \nabla \Gamma_1 \left( \mathbb{E}^{\Gamma_0} f^q \right) \right|^q = \sum_{i \in \Gamma_1} \nu \left| \nabla_i \left( \mathbb{E}^{\Gamma_0} f^q \right) \right|^q \leq \sum_{i \in \Gamma_1} \nu \left| \nabla_i \left( \mathbb{E}^{\{\sim i \}} f^q \right) \right|^q \]  \hspace{1cm} (6.8)

If we substitute in (6.8) the bound from Lemma 6.6 we obtain

\[ \nu \left| \nabla \Gamma_1 \left( \mathbb{E}^{\Gamma_0} f^q \right) \right|^q \leq H \sum_{i \in \Gamma_1} \nu \left| \nabla_i f \right|^q + J^q H \sum_{i \in \Gamma_1} \sum_{r=2}^{\infty} J_r^{-2} \nu \left| \nabla_{i \pm r} f \right|^q \]

\[ = H \nu \left| \nabla \Gamma_1 f \right|^q + J^q H \left( \sum_{r=0}^{\infty} J_0^R \right) \nu \left| \nabla \Gamma_0 f \right|^q + \nu \left| \nabla \Gamma_1 f \right|^q \]

\[ = (H + \frac{J^q}{1 - J_0}) \nu \left| \nabla \Gamma_1 f \right|^q + \frac{J^q H}{1 - J_0} \nu \left| \nabla \Gamma_0 f \right|^q \]

For \( J \) in (H3) sufficiently small such that \( \frac{J^q H}{1 - J_0} < 1 \), the proposition follows for constants

\[ C_1 = H + \frac{J^q H}{1 - J_0} \quad \text{and} \quad C_2 = \frac{J^q H}{1 - J_0} < 1 \]

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We can now prove Proposition 2.4.

Proof of Proposition 2.4. We will prove Proposition 2.4 for \( k = 1 \), that is

\[
\nu \mathbb{E}^{\Gamma_1}( f^q \log \frac{f^q}{\mathbb{E}^{\Gamma_1} f^q} ) \leq \hat{C} \nu |\nabla \Gamma_0 f|^q + \hat{C} \nu |\nabla \Gamma_1 f|^q
\]

for \( f \geq 0 \). Define the sets

\[
a_0 \equiv \{ \sim 0 \} \quad \text{and} \quad a_k \equiv \begin{cases} \{ \sim (2k + 2) \} & \text{for } k \in \mathbb{N} \text{ odd} \\ \{ \sim (2k) \} & \text{for } k \in \mathbb{N} \text{ even} \end{cases}
\]

and consider the following representation of the odd integers \( \Gamma_1 \)

\[
\Gamma_1 = \bigcup_{k=0}^{\infty} a_k = \{ \sim 0 \} \cup \{ \sim 4 \} \cup \{ \sim (-4) \} \cup \{ \sim 8 \} \cup \{ \sim (-8) \} \cup \ldots
\]

where we have denoted \( \{ \sim k \} = \{ j \in \mathbb{Z} : j \sim k \} = \{ k - 1, k + 1 \} \). Then we can write

\[
\nu \mathbb{E}^{\sim 0} ( f^q \log \frac{f^q}{\mathbb{E}^{\sim 0} f^q} ) = \nu \mathbb{E}^{\sim 0} ( f^q \log \frac{f^q}{\mathbb{E}^{\sim 0} f^q} ) + \sum_{k=1}^{\infty} \nu \mathbb{E}^{a_k} ( \mathbb{E}^{a_{k-1}} \ldots \mathbb{E}^{a_0} f^q \log \frac{\mathbb{E}^{a_{k-1}} \ldots \mathbb{E}^{a_0} f^q}{\mathbb{E}^{a_k} \ldots \mathbb{E}^{a_0} f^q} ) \quad (6.9)
\]

If we use Proposition 4.1 and Proposition 5.3 in the case of hypothesis A and B respectively to bound the first term in (6.9) we have

\[
\nu \mathbb{E}^{\sim 0} ( f^q \log \frac{f^q}{\mathbb{E}^{\sim 0} f^q} ) \leq R \sum_{r=-2}^{2} \nu |\nabla \Gamma_r f|^q + R \sum_{r=3}^{\infty} J_0^{r-2} \nu |\nabla \pm \Gamma f|^q \quad (6.10)
\]

where \( R = \max\{ R_1, R_2 \} \) for the constants \( R_1 \) and \( R_2 \) as in Proposition 4.1 and Proposition 5.3. For the terms in the sum in the last term of (6.9), for \( k \) odd we have

\[
\nu \mathbb{E}^{a_k} ( \mathbb{E}^{a_{k-1}} \ldots \mathbb{E}^{a_0} f^q \log \frac{\mathbb{E}^{a_{k-1}} \ldots \mathbb{E}^{a_0} f^q}{\mathbb{E}^{a_k} \ldots \mathbb{E}^{a_0} f^q} ) \leq R \sum_{r=2k}^{2k+4} \nu |\nabla \Gamma_r (\mathbb{E}^{a_{k-1}} \ldots \mathbb{E}^{a_0} f^q)^{\frac{1}{q}}|^q \\
+ R \sum_{r=3}^{\infty} J_0^{r-2} \nu |\nabla_{2k+2 \pm r} (\mathbb{E}^{a_{k-1}} \ldots \mathbb{E}^{a_0} f)^{\frac{1}{q}}|^q \quad (6.11)
\]
while for $k$ even we have

$$\nu \mathbb{E}^{ak} (\mathbb{E}^{a_{k-1}} \cdots \mathbb{E}^{a_0} f^q \log \frac{\mathbb{E}^{a_{k-1}} \cdots \mathbb{E}^{a_0} f^q}{\mathbb{E}^{a_k} \cdots \mathbb{E}^{a_0} f^q}) \leq R \sum_{r=-2k-2}^{-2k+2} \nu \left| \nabla_r (\mathbb{E}^{a_{k-1}} \cdots \mathbb{E}^{a_0} f^q) \frac{1}{q} \right|^q$$

$$+ R \sum_{r=3}^{\infty} J_{0}^{r-2} \nu \left| \nabla_{-2k+r} (\mathbb{E}^{a_{k-1}} \cdots \mathbb{E}^{a_0} f^q) \frac{1}{q} \right|^q$$

\[(6.12)\]

For the quantities involved in \((6.11)\) and \((6.12)\), if we define $m_k = \min \{i : i \in \bigcup_{j=0}^{k-1} a_j \}$ and $M_k = \max \{i : i \in \bigcup_{j=0}^{k-1} a_j \}$, we then have

$$\nu \left| \nabla_s (\mathbb{E}^{q_s} f^q) \right|^q \leq \begin{cases} \nu \left| \nabla_s (\mathbb{E}^{q_s} f^q) \right|^q & \text{if } s \text{ is even, } s \in (m_k, M_k) \\ \nu \left| \nabla_s (\mathbb{E}^{q_m} f^q) \right|^q & \text{if } s = m_k - 1 \\ \nu \left| \nabla_s (\mathbb{E}^{q_M} f^q) \right|^q & \text{if } s = M_k + 1 \\ 0 & \text{if } s \text{ is odd, } s \in [m_k, M_k] \\ \nu \left| \nabla_s f \right|^q & \text{if } s \notin [m_k, M_k] \end{cases}$$

\[(6.13)\]

From relationships \((6.9)\) - \((6.13)\) we derive that the right hand side of \((6.9)\) is reduced to an infinite sum of the following terms

$$J_0^s \nu \left| \nabla_s (\mathbb{E}^{t_s} f^q) \right|^q \text{ and } J_0^s \nu \left| \nabla_t (\mathbb{E}^{t_s} f^q) \right|^q \text{ for } t = \{s - 2, s + 2\}$$

for every $s \in \Gamma_0$ and $r \in \mathbb{N}$. For every $s$ and $t$ the above terms are repeated at most two times for every different $r$. So, $\nu \left| \nabla_s (\mathbb{E}^{t_s} f^q) \right|^q$ and $\nu \left| \nabla_t (\mathbb{E}^{t_s} f^q) \right|^q$ occur in the sum at most $2 \sum_{n=0}^{+\infty} J_0^n$ times each. Thus, we finally obtain

$$\nu \mathbb{E}^{1/2} (f^q \log \frac{f^q}{\mathbb{E}^{a_1} f^q}) \leq R \sum_{r=-2}^{2} \nu \left| \nabla_r f \right|^q + R \sum_{r=3}^{\infty} J_{0}^{r-2} \nu \left| \nabla_{\pm r} f \right|^q$$

$$+ 2 \sum_{n=0}^{+\infty} J_0^n R \sum_{s \in \mathbb{Z}} \nu \left| \nabla_{2s} (\mathbb{E}^{t_{s+2}} f^q) \right|^q$$

$$+ 2 \sum_{n=0}^{+\infty} J_0^n R \sum_{s \in \mathbb{N}} \nu \left| \nabla_{2s+2} (\mathbb{E}^{t_{s+2}} f^q) \right|^q + \nu \left| \nabla_{2s-2} (\mathbb{E}^{t_{s+2}} f^q) \right|^q)$$

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If we choose $J$ in (H3) sufficient small such that $J_0 = J^\frac{1}{2} < 1$, the last leads to

$$\nu\mathbb{E}^{\Gamma_1} (f^q \log \frac{f^q}{\mathbb{E}^{\Gamma_1} f^q}) \leq R \sum_{r=-2}^{2} \nu |\nabla_r f|^q + R \sum_{r=3}^{\infty} J_0^{-2} \nu |\nabla_{\pm r} f|^q$$

$$+ \frac{2R}{1-J_0} \sum_{s \in \mathbb{Z}} \nu \left| \nabla_{2s}(\mathbb{E}^{(\sim 2s)} f^q)^{\frac{1}{q}} \right|^q$$

$$+ \frac{2R}{1-J_0} \sum_{s \in \mathbb{N}} \left( \nu \left| \nabla_{2s+2}(\mathbb{E}^{(\sim 2s)} f^q)^{\frac{1}{q}} \right|^q + \nu \left| \nabla_{2s-2}(\mathbb{E}^{(\sim 2s)} f^q)^{\frac{1}{q}} \right|^q \right)$$

(6.14)

In order to bound the terms involved in the summations in (6.14) we will apply Lemma 6.6 to bound the right hand side of (6.14), from which we obtain

$$\nu\mathbb{E}^{\Gamma_1} (f^q \log \frac{f^q}{\mathbb{E}^{\Gamma_1} f^q}) \leq R \sum_{r=-2}^{2} \nu |\nabla_r f|^q + R \sum_{r=3}^{\infty} J_0^{-2} \nu |\nabla_{\pm r} f|^q$$

$$+ \frac{4HR}{1-J_0} \sum_{s \in \mathbb{Z}} \left( \nu |\nabla_{2s} f|^q + \sum_{r=2}^{\infty} J_0^{-2} \nu |\nabla_{2s \pm r} f|^q \right)$$

$$\leq R \sum_{r=-2}^{2} \nu |\nabla_r f|^q + \left( \frac{4HR}{1-J_0} (1 + \sum_{r=0}^{\infty} J_0^{2r}) + R \right) \sum_{s \in \mathbb{Z}} \nu |\nabla_{2s} f|^q$$

$$+ \left( \frac{4HR}{1-J_0} (1 + \sum_{r=0}^{\infty} J_0^{r}) + R \right) \sum_{s \in \mathbb{Z}} \nu |\nabla_{2s+1} f|^q$$

where the two sums are finite since $J_0 < 1$. The proposition follows for

$$\hat{C} = \frac{4HR}{1-J_0} (1 + \sum_{r=0}^{\infty} J_0^{r}) + 2R$$

\[\square\]

7 Proof of Proposition 2.6.

In Proposition 4.1 and Proposition 5.3 we showed a Log-Sobolev type inequality for the one site measure $\mathbb{E}^{(\sim i)}$, under hypothesis A and B respectively, while in Proposition 3.3 a Spectral Gap type inequality was shown for both cases. In the following lemma the Spectral Gap type inequality (2.4) will also be extended to the product measure $\mathbb{E}^{\Gamma_i}$, $i = 0, 1$. What we will show is that (2.3) for $\mathbb{E}^{\Gamma_i}$, $i = 0, 1$ actually implies (2.4) for $\mathbb{E}^{\Gamma_i}$, $i = 0, 1$, a basic result for the usual Log-Sobolev and Spectral Gap inequalities.

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Lemma 7.1. If conditions A or B are satisfied then the following Spectral Gap type inequality holds

\[ \nu \left| f - \mathbb{E}^{\Gamma, \omega} f \right|^q \leq \tilde{R} \nu \left| \nabla f \right|^q \]

for \( i = 0, 1 \) and some positive constant \( \tilde{R} \), where \( \nabla := \nabla_Z \).

Proof. To show the lemma we will follow the steps of the proof of the usual LSq implying the SGq inequality (see [B-Z]). We will show the inequality for \( i = 0 \).

From Proposition 2.4 we have

\[ \nu \left( \left| f \right|^q \log \left| f \right|^q \right) \leq \tilde{C} \nu \left| \nabla f \right|^q \quad (7.1) \]

Assume without loss of generality that the function \( f \) has median zero and denote \( f^+ = \max(f, 0) \) and \( f^- = \min(f, 0) \). Then, according to Lemma 2.2 from [B-Z] and the proof of Theorem 2.1 from the same paper, we obtain

\[ \mathbb{E}^{\Gamma_0, \omega} ((f^+)^q \log \left( \frac{(f^+)^q}{\mathbb{E}^{\Gamma_0, \omega} (f^+)^q} \right)) \geq \log 2 \mathbb{E}^{\Gamma_0, \omega} ((f^+)^q I_{\left\{ f > 0 \right\}}) \]

as well as

\[ \mathbb{E}^{\Gamma_0, \omega} ((f^-)^q \log \left( \frac{(f^-)^q}{\mathbb{E}^{\Gamma_0, \omega} (f^-)^q} \right)) \geq \log 2 \mathbb{E}^{\Gamma_0, \omega} ((f^-)^q I_{\left\{ f < 0 \right\}}) \]

If we apply the Gibbs measure \( \nu \) to the last two inequalities we get

\[ \nu ((f^+)^q \log \left( \frac{(f^+)^q}{\mathbb{E}^{\Gamma_0, \omega} (f^+)^q} \right)) \geq \log 2 \nu ((f^+)^q I_{\left\{ f > 0 \right\}}) \quad (7.2) \]

and

\[ \nu ((f^-)^q \log \left( \frac{(f^-)^q}{\mathbb{E}^{\Gamma_0, \omega} (f^-)^q} \right)) \geq \log 2 \nu ((f^-)^q I_{\left\{ f < 0 \right\}}) \quad (7.3) \]

If we use (7.1) to bound from above the right hand sides of (7.2) and (7.3) we obtain

\[ \tilde{C} \nu \left( \left| \nabla f^+ \right|^q I_{\left\{ f > 0 \right\}} \right) \geq \log 2 \nu ((f^+)^q I_{\left\{ f > 0 \right\}}) \]

and

\[ \tilde{C} \nu \left( \left| \nabla f^- \right|^q I_{\left\{ f < 0 \right\}} \right) \geq \log 2 \nu ((f^-)^q I_{\left\{ f < 0 \right\}}) \]

If we add the last two and use the estimates \( |\nabla f^+|^q \leq |\nabla f|^q \) and \( |\nabla f^-|^q \leq |\nabla f|^q \) for the gradient, we get

\[ \nu (|f|^q) \leq \frac{\tilde{C}}{\log 2} \nu |\nabla f|^q \]

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The last relationship for \( f - \mathbb{E}^{\Gamma_0, \omega} f \) in place of \( f \) gives

\[
\nu | f - \mathbb{E}^{\Gamma_0, \omega} f |^q \leq \frac{\tilde{C}}{\log 2} \nu | \nabla_{\Gamma_1} (f - \mathbb{E}^{\Gamma_0, \omega} f) |^q + \frac{\tilde{C}}{\log 2} \nu | \nabla_{\Gamma_0} (f - \mathbb{E}^{\Gamma_0, \omega} f) |^q 
\]

\[
\leq 2^q \frac{\tilde{C}}{\log 2} | \nabla_{\Gamma_1} f |^q + 2^q \frac{\tilde{C}}{\log 2} \nu | \nabla_{\Gamma_1} (\mathbb{E}^{\Gamma_0, \omega} f) |^q + \frac{\tilde{C}}{\log 2} | \nabla_{\Gamma_0} f |^q 
\]

(7.4)

If we use Lemma 6.5 to bound the second term in the right hand side of (7.4) we get

\[
\nu | f - \mathbb{E}^{\Gamma_0, \omega} f |^q \leq (2^q \frac{\tilde{C}}{\log 2} D_1 + 2^q \tilde{C}) \nu | \nabla_{\Gamma_1} f |^q + (2^q \frac{\tilde{C}}{\log 2} D_2 + \tilde{C}) \nu | \nabla_{\Gamma_0} f |^q 
\]

and the lemma follows for \( \tilde{R} = \max \{ 2^q \frac{\tilde{C}}{\log 2} D_1 + 2^q \tilde{C}, 2^q \frac{\tilde{C}}{\log 2} D_2 + \tilde{C} \} \).

Now we can prove Proposition 2.6.

**Proof of Proposition 2.6.** In order to show Proposition 2.6 we can follow [G-Z] as in [I-P] and [Pa] for the case of quadratic and non quadratic interactions respectively. In both these two cases the local specification satisfied (H0), which implied that \( \mathbb{E}^{\Gamma_0, \omega} \) satisfied the Log-Sobolev \( q \) inequality and thus the \( q \) Spectral Gap inequality. In the case of Proposition 2.6 we have assumed the weaker assumptions (H0') and (H0''), in which case we can use the weaker result of Lemma 7.1

\[
\nu | f - \mathbb{E}^{\Gamma_0, \omega} f |^q \leq \tilde{R} \nu | \nabla_{\Gamma_1} f |^q
\]

for \( \{ i, j \} = \{ 0, 1 \} \). For \( i \neq j \) we then have that

\[
\nu | \mathbb{E}^{\Gamma_i} f - \mathbb{E}^{\Gamma_j} f |^q = \nu | \mathbb{E}^{\Gamma_i} ( \mathbb{E}^{\Gamma_j} f - \mathbb{E}^{\Gamma_i} f ) |^q
\]

\[
\leq \tilde{R} \nu | \nabla_{\Gamma_i} ( \mathbb{E}^{\Gamma_j} f ) |^q
\]

(7.5)

the last inequality from Lemma 7.1 for the measures \( \mathbb{E}^{\Gamma_0} \) and \( \mathbb{E}^{\Gamma_1} \). If we use Lemma 6.5 to bound the last term in the right hand side of (7.5) we get

\[
\nu | \mathbb{E}^{\Gamma_i} f - \mathbb{E}^{\Gamma_j} f |^q \leq \tilde{R} D_1 \nu | \nabla_{\Gamma_0} f |^q + \tilde{R} D_2 \nu | \nabla_{\Gamma_1} f |^q
\]

From the last inequality we obtain that for any \( n \in \mathbb{N} \),

\[
\nu | \mathbb{P}^n f - \mathbb{E}^{\Gamma_0} \mathbb{P}^n f |^q \leq \tilde{R} D_1 \nu | \nabla_{\Gamma_0} ( \mathbb{E}^{\Gamma_0} \mathbb{P}^{n-1} f ) |^q + \tilde{R} D_2 \nu | \nabla_{\Gamma_1} ( \mathbb{E}^{\Gamma_0} \mathbb{P}^{n-1} f ) |^q
\]

\[
= \tilde{R} D_2 \nu | \nabla_{\Gamma_1} ( \mathbb{E}^{\Gamma_0} \mathbb{P}^{n-1} f ) |^q
\]

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If we use Lemma 6.5 to bound the last expression we have the following

\[ \nu|\mathcal{P}^n f - \mathbb{E}^{\Gamma_0} \mathcal{P}^n f|^q \leq \hat{R}D_2^n (D_1 \nu |\nabla_{\Gamma_1} f|^q + D_2 \nu |\nabla_{\Gamma_0} f|^q) \]  

(7.6)

Similarly we obtain

\[ \nu|\mathbb{E}^{\Gamma_0} \mathcal{P}^n f - \mathcal{P}^{n+1} f|^q \leq \hat{R}D_2^n (D_1 \nu |\nabla_{\Gamma_1} f|^q + D_2 \nu |\nabla_{\Gamma_0} f|^q) \]  

(7.7)

Consider the sequence \( \{Q^n\}_{n \in \mathbb{N}} \) defined as

\[ Q^n f = \begin{cases} \mathcal{P}^{\frac{n}{2}} f & \text{if } n \text{ even} \\ \mathbb{E}^{\Gamma_0} \mathcal{P}^{\frac{n-1}{2}} f & \text{if } n \text{ odd} \end{cases} \]

for every \( n \in \mathbb{N} \). Hence, if we define the sets

\[ A_n = \{ |Q^n f - Q^{n+1} f| \geq \left( \frac{1}{2} \right)^n \} \]

we obtain

\[ \nu(A_n) = \nu \left( \{ |Q^n f - Q^{n+1} f| \geq \left( \frac{1}{2} \right)^n \} \right) \leq 2^n \nu |Q^n f - Q^{n+1} f|^q \]

by Chebyshev inequality. If we use (7.6) and (7.7) to bound the last we have

\[ \nu(A_n) \leq (2^n D_2^\frac{1}{2})^n \hat{R} (D_1 \nu |\nabla_{\Gamma_1} f|^q + D_2 \nu |\nabla_{\Gamma_0} f|^q) \]

We can choose \( J \) sufficiently small such that \( 2^n D_2^\frac{1}{2} < \frac{1}{2} \) in which case we get that

\[ \sum_{n=0}^{\infty} \nu(A_n) \leq \left( \sum_{n=0}^{\infty} \frac{1}{2} \right)^n \hat{R} (D_1 \nu |\nabla_{\Gamma_1} f|^q + D_2 \nu |\nabla_{\Gamma_0} f|^q) < \infty \]

From the Borel-Cantelli lemma, only finite number of the sets \( A_n \) can occur, which implies that the sequence

\[ \{Q^n f\}_{n \in \mathbb{N}} \]

is a Cauchy sequence and that it converges \( \nu \)-almost surely. Say

\[ Q^n f \to \theta(f) \quad \nu \text{ a.e.} \]

We will first show that \( \theta(f) \) is a constant, i.e. it does not depend on variables on \( \Gamma_0 \) or \( \Gamma_1 \). To show that, first notice that \( Q^n f \) is a function on \( \Gamma_1 \) and \( \Gamma_0 \) when \( n \) is even and odd respectively, which implies that the limits

\[ \theta_o(f) = \lim_{n \text{ odd}, n \to \infty} Q^n f \text{ and } \theta_e(f) = \lim_{n \text{ even}, n \to \infty} Q^n f \]
do not depend on variables on $\Gamma_0$ and $\Gamma_1$ respectively. Since both the subsequences $\{Q^nf\}_{n\text{ even}}$ and $\{Q^nf\}_{n\text{ odd}}$ converge to $\theta(f)$ $\nu$–a.e. we have that

$$\theta_{\nu}(f) = \theta(f) = \theta_{\nu}(f)$$

which implies that $\theta(f)$ is a constant. From that we obtain that

$$\nu(\theta(f)) = \theta(f) \quad (7.8)$$

Since the sequence $\{Q^nf\}_{n\in\mathbb{N}}$ converges $\nu$–almost, the same holds for the sequence $\{Q^nf - \nu Q^nf\}_{n\in\mathbb{N}}$. We have

$$\lim_{n\to\infty} (Q^nf - \nu Q^nf) = \theta(f) - \nu(\theta(f)) = \theta(f) - \theta(f) = 0$$

where above we used (7.8). On the other side, we also have

$$\lim_{n\to\infty} (Q^nf - \nu Q^nf) = \lim_{n\to\infty} (Q^nf - \nu f) = \theta(f) - \nu(f) \quad (7.9)$$

From (7.8) and (7.9) we get that

$$\theta(f) = \nu(f)$$

We finally get

$$\lim_{n\to\infty} P^nf = \lim_{n\text{ even}, n\to\infty} Q^nf = \nu f, \ \nu \ a.e.$$ 

\[\square\]

8 Paradigms of U-bounds.

In the case of hypothesis B we focus on measures on the Heisenberg group that satisfy the non uniform U-bound ($H^{0''}$). In this section we want to find examples of measures

$$E^{i, \omega}(dx_i) = \frac{e^{-H^{i, \omega}} dx_i}{\int e^{-H^{i, \omega}} dx_i} \quad (8.1)$$

where

$$H^{i, \omega} = \phi(x_i) + \sum_{j\sim i} J_{ij} V(x_i, \omega_j) \quad (8.2)$$

that satisfy the non uniform U-bound ($H^{0''}$):

$$E^{\{\sim i\}, \omega} |f|^q (|\nabla_{\{\sim i\}} H^{\{\sim i\}, \omega}|^q + H^{\{\sim i\}, \omega}) \leq \hat{C} E^{\{\sim i\}, \omega} |\nabla_{\{\sim i\}} f|^q + \hat{D}_{\{\sim i\}}(\omega) E^{\{\sim i\}, \omega} |f|^q$$
with $\nu e^{\hat{D}_i(\omega)} \leq e^{K}$. Because the condition involves a product measure $E^{(\sim_1)\omega} = E^{(\sim_1-1)\omega} \otimes E^{(\sim_1+1)\omega}$, it is sufficient to have the following inequality for the one site measure
\[E^{(\sim_1)\omega} |f|^q (|\nabla_i H^{(\sim_1)\omega}|^q + H^{(\sim_1)\omega}) \leq \hat{C} E^{(\sim_1)\omega} |\nabla_i f|^q + \hat{D}_i(\omega) E^{(\sim_1)\omega} |f|^q\]
where $D_i(\omega)$ a function of $\omega_{-1}, \omega_{+1}$. Then (H0") will follow for $\hat{D}_i(\omega) = \hat{D}_{i-1}(\omega) + \hat{D}_{i+1}(\omega)$.

**Lemma 8.1.** For $E^{(\sim_1)\omega}$ as in (8.1) and (8.2) the following inequality holds
\[E^{(\sim_1)\omega} (|f|^q \nabla_i d(x_i) \cdot \nabla_i H^{(\sim_1)\omega}) \leq A_1 E^{(\sim_1)\omega} |\nabla_i f|^q + A_1 E^{(\sim_1)\omega} |f|^q\]
for all differentiable $f : \Omega \to \mathbb{R}$, and some constant $A_1 \in (0, \infty)$ independent of $\omega$.

**Proof.** Without loss of generality assume $f \geq 0$. By the Liebniz rule, we have
\[(\nabla_i f) e^{-H^{(\sim_1)\omega}} = \nabla_i (f e^{-H^{(\sim_1)\omega}}) + f \nabla_i H^{(\sim_1)\omega} e^{-H^{(\sim_1)\omega}}.\tag{8.3}\]

We can multiply (8.3) on both sides by $d(x_i) \nabla_i d(x_i)$. If we integrate now we see that
\[
\int_{\mathbb{H}} f d\nabla_i d(x_i) \nabla_i H^{(\sim_1)\omega} e^{-H^{(\sim_1)\omega}} dx_i \leq \int_{\mathbb{H}} d|\nabla_i d||\nabla_i f| e^{-H^{(\sim_1)\omega}} dx_i - \int_{\mathbb{H}} d\nabla_i d. \nabla_i \left( f e^{-H^{(\sim_1)\omega}} \right) dx_i
\]
\[
= \int_{\mathbb{H}} d|\nabla_i f| e^{-H^{(\sim_1)\omega}} dx_i + \int_{\mathbb{H}} f \nabla_i \cdot (d\nabla_i d) e^{-H^{(\sim_1)\omega}} dx_i,
\]
where we have used integration by parts once more. From Proposition 1.4 we obtain
\[\nabla_i \cdot (d\nabla_i d) = |\nabla_i d|^2 + d\Delta_i d \leq 1 + K_0\]
in terms of distributions (see [I-P]). Therefore we have
\[
\int_{\mathbb{H}} f d\nabla_i d(x_i) \nabla_i H^{(\sim_1)\omega} e^{-H^{(\sim_1)\omega}} dx_i \leq \int_{\mathbb{H}} d|\nabla_i f| e^{-H^{(\sim_1)\omega}} dx_i + (1 + K_0) \int_{\mathbb{H}} f e^{-H^{(\sim_1)\omega}} dx_i.
\]
Replacing $f$ by $f^q$ in this inequality, and using Young’s inequality again, we arrive at
\[
\int_{\mathbb{H}} f^q d\nabla_i d(x_i) \nabla_i H^{(\sim_1)\omega} e^{-H^{(\sim_1)\omega}} dx_i \leq \frac{1}{\tau} \int_{\mathbb{H}} |\nabla_i f|^q e^{-H^{(\sim_1)\omega}} dx_i + \frac{q}{p} \int_{\mathbb{H}} f^p d e^{-H^{(\sim_1)\omega}} dx_i + (1 + K_0) \int_{\mathbb{H}} f^q e^{-H^{(\sim_1)\omega}} dx_i,\tag{8.4}\]
for all $\tau > 0$. We finally obtain that there exist constant $A \in (0, \infty)$ independent of $\omega$ such that
\[
\int_{\mathbb{H}} f^q d\nabla_i d(x_i) \cdot \nabla_i H^{(\sim_1)\omega} e^{-H^{(\sim_1)\omega}} dx_i \leq A_1 \int |\nabla_i f|^q e^{-H^{(\sim_1)\omega}} dx_i + A_1 \int f^q e^{-H^{(\sim_1)\omega}} dx_i
\]
\[\square\]
Corollary 8.2. If hypothesis (H1)-(H4) are satisfied and there exist constants $a \geq 0$ and $b(\omega) \geq 0$ such that

$$|\nabla_i H^i|^q + H^i \leq a d\nabla_i d(x_i) \cdot \nabla_i H^i + b(\omega)$$  \hspace{1cm} (8.5)

holds, with $\nu e^b(\omega) \leq e^{K_1}$ then the infinite dimensional Gibbs measure $\nu$ is unique and satisfies the Logarithmic Sobolev $q$ inequality.

$$\nu |f|^q \log \frac{|f|^q}{\nu |f|^q} \leq C$$

for some positive constant $C$.

Proof. We begin by applying Theorem [23]. To prove the Corollary, it is sufficient to show that condition (H0$''$) is satisfied. Inequality (8.5) implies

$$E i,\omega |f|^q \left(|\nabla_i H^i|^q + H^{(\sim)} \right) \leq aE i,\omega |f|^q \left(d\nabla_i d(x_i) \cdot \nabla_i H^i + b(\omega)E i,\omega |f|^qight)$$

We can bound the right hand side from Lemma 8.1 to get

$$E i,\omega |f|^q \left(|\nabla_i H^i|^q + H^{(\sim)} \right) \leq aA_1 E i,\omega |\nabla_i f|^q + (aA_1 + b(\omega))E i,\omega |f|^q$$

with $\nu e^{(aA_1 + b(\omega))} \leq e^{e^{A_1}e^{K_1}} < \infty$, which is the (H0$''$) condition for $K = e^{e^{A_1}e^{K_1}}$.

We will now present examples of interactions for which the local specification $E i,\omega$ satisfies condition (H0$''$), or equivalently inequality (8.5).

Example 1. Consider phase $\phi(x) = d(x)^s$ for $0 \leq s < 2$ and interaction $V(x, y) = (d(x) - d(y))^2$.

We can rewrite the Hamiltonian as

$$H^i = \tilde{H}^i + J \sum_{j:j \sim i} d^2(\omega_j)$$

where

$$\tilde{H}^i(x_i) = d^s(x_i) + J d^2(x_i) - 2J d(x_i) \sum_{j:j \sim i} d(\omega_j)$$

We have

$$|\nabla_i H^i|^2 \leq (s - 1)^2 d^{s-1}(x_i) d_{j:j \sim i} d(x_i) + J 2 \sum_{j:j \sim i} (d(x_i) - d(\omega_j)) |\nabla_i d(x_i)|^2$$

$$\leq 2(s - 1)^2 d^{s-2}(x_i) |\nabla_i d(x_i)|^2 + 16J^2 \sum_{j:j \sim i} (d(x_i) - d(\omega_j))^2 |\nabla_i d(x_i)|^2$$

$$\leq 2(s - 1)^2 d^s(x_i) + 16J^2 \sum_{j:j \sim i} (d(x_i) - d(\omega_j))^2 + c''$$

$$\leq \max\{2(s - 1)^2, 16J\} \cdot H^i + c''$$  \hspace{1cm} (8.6)
for some positive constant \(c''\), where above we used that \(|\nabla_i d(x_i)| = 1\), as well as that \(s < 2\). We have

\[
d\nabla_i d(x_i) \nabla_i H^{i,\omega} = d\nabla_i d(x_i) \nabla_i (\tilde{H}^{i,\omega} + J \sum_{j:j\sim i} d^2(\omega_j)) = d\nabla_i d(x_i) \nabla_i \tilde{H}^{i,\omega}
\]

\[
= d\nabla_i d(x_i) sd^{s-1}(x_i) \nabla_i d(x_i) + d\nabla_i d(x_i) J \sum_{j:j\sim i} d(\omega_j) \nabla_i d(x_i)
\]

\[
= sd^s(x_i) + 2Jd^2(x_i) - 2Jd(x_i) \sum_{j:j\sim i} d(\omega_j)
\]

where above we used again that \(|\nabla_i d(x_i)| = 1\). Because \(s \geq 1\) we finally obtain

\[
d\nabla_i d(x_i) \nabla_i H^{i,\omega} \geq \tilde{H}^{i,\omega} = H^{i,\omega} - J \sum_{j:j\sim i} d^2(\omega_j)
\]  

(8.7)

From inequality (8.6) we get

\[
|\nabla_i H^{i,\omega}|^2 + H^{i,\omega} \leq (c' + 1) \cdot H^{i,\omega} + c''
\]

where \(c' = \max\{2(s-1)^2, 16J\}\). If we now use (8.7) the last gives

\[
|\nabla_i H^{i,\omega}|^2 + H^{i,\omega} \leq (c' + 1) \cdot (d\nabla_i d(x_i) \nabla_i H^{i,\omega} + J \sum_{j:j\sim i} d^2(\omega_j)) + c''
\]

which is (8.5) for \(a = c' + 1\) and \(b(\omega) = (c' + 1) J \sum_{j:j\sim i} d^2(\omega_j) + c''\).

Example 2. Consider phase \(\phi(x) = x^s\) for \(0 \leq s < p\) and interaction \(V(x, y) = (d(x) + d(y))^p\), where \(\frac{1}{p} + \frac{1}{q} = 1\).

We can rewrite the Hamiltonian as

\[
H^{i,\omega} = \tilde{H}^{i,\omega} + J \sum_{j:j\sim i} d^p(\omega_j)
\]

where

\[
\tilde{H}^{i,\omega}(x_i) = d^s(x_i) + J \sum_{j:j\sim i} \sum_{k=0}^{p-1} \binom{p}{k} d^{p-k}(x_i) d^k(\omega_j)
\]
We have
\[ |\nabla_i H^{i,\omega}|^q = \left| (s - 1)d^{s-1}(x_i)\nabla_i d(x_i) + Jp \sum_{j:j\sim i} (d(x_i) + d(\omega_j))^p \nabla_i d(x_i) \right|^q \]
\[ \leq 2q^{-1}(s - 1)^q d^{s-q}(x_i) |\nabla_i d(x_i)|^q + 2^{2q-2} J^q p^q \sum_{j:j\sim i} (d(x_i) + d(\omega_j))^{q-q} |\nabla_i d(x_i)|^q \]
\[ \leq 2q^{-1}(s - 1)^q d^s(x_i) + 2^{2q-2} J^q p^q \sum_{j:j\sim i} (d(x_i) + d(\omega_j))^p + c'' \]
\[ \leq \max\{2q^{-1}(s - 1)^q, 2^{2q-2} J^q p^q\} \cdot H^{i,\omega} + c'' \quad (8.8) \]
for some positive constant $c''$, where above we used that $|\nabla_i d(x_i)| = 1$, as well as that $\frac{1}{p} + \frac{1}{q} = 1$ and $s < p$. We have
\[ d\nabla_i d(x_i).\nabla_i H^{i,\omega} = d\nabla_i d(x_i).\nabla_i (\tilde{H}^{i,\omega} + J \sum_{j:j\sim i} d'(\omega_j)) = d\nabla_i d(x_i).\nabla_i \tilde{H}^{i,\omega} \]
\[ = d\nabla_i d(x_i). (sd^{s-1}(x_i)\nabla_i d(x_i) \]
\[ + \sum_{j:j\sim i} \sum_{k=0}^{r-1} (r-k) \binom{r}{k} d^{-k-1}(x_i)\nabla_i d(x_i)d^k(\omega_j) \}
\[ = sd^s(x_i) + \sum_{j:j\sim i} \sum_{k=0}^{r-1} (r-k) \binom{r}{k} d^{-k}(x_i)d^k(\omega_j) \]
where above we used again that $|\nabla_i d(x_i)| = 1$. Because $s, p - k \geq 1$ we finally obtain
\[ d\nabla_i d(x_i).\nabla_i H^{i,\omega} \geq \tilde{H}^{i,\omega} = H^{i,\omega} - J \sum_{j:j\sim i} d^p(\omega_j) \quad (8.9) \]
From inequality (8.8) we get
\[ |\nabla_i H^{i,\omega}|^q + H^{i,\omega} \leq (c' + 1) \cdot H^{i,\omega} + c'' \]
where $c' = \max\{2^{q-1}(s - 1)^q, 2^{2q-2} J^q p^q\}$. If we now use (8.9) the last gives
\[ |\nabla_i H^{i,\omega}|^q + H^{i,\omega} \leq (c' + 1) \cdot (d\nabla_i d(x_i).\nabla_i H^{i,\omega} + J \sum_{j:j\sim i} d^p(\omega_j)) + c'' \]
which is (8.3) for $a = c' + 1$ and $b(\omega) = (c' + 1) J \sum_{j:j\sim i} d^p(\omega_j) + c''$.

9 Conclusion.

In the present work, we focus on the $q$ Logarithmic Sobolev Inequality (LSq) for the infinite dimensional Gibbs measure related to systems of spins with values on the Heisenberg group.
We considered two cases, that of a one site boundary-free measure that satisfies a $q$ Log-Sobolev inequality and that of a one site measure with boundary conditions that satisfies a non uniform $U$-bound. In both cases we determined conditions for the infinite volume Gibbs measure to satisfy the Log-Sobolev Inequality.

In this way, the work of [H-Z] was extended to the infinite dimensional setting. In particular we have relaxed the conditions obtained in [Pa] about a similar problem where one site measures that satisfied an $\mathcal{H}(0)$ condition where considered.

Furthermore, the criterion presented in Theorem 2.2 and Theorem 2.3 can in particular be applied in the case of local specifications $\{\mathcal{E}_\Lambda, \omega\}_{\Lambda \subset \subset \mathbb{Z}, \omega \in \Omega}$ with no quadratic interactions for which

$$\|\nabla_i \nabla_j V(x_i, x_j)\|_\infty = \infty$$

Thus, we have shown that our results can go beyond the usual uniform boundness of the second derivative of the interactions considered in [Z1] and [O-R] for real valued variables as well as in [I-P] for spins on the Heisenberg group.

Concerning the additional conditions (H1) and (H2) placed here to handle the interactions, they refer to finite dimensional measures with no boundary conditions which are easier to handle than the $\{\mathcal{E}_\Lambda, \omega\}_{\Lambda \subset \subset \mathbb{Z}, \omega \in \Omega}$ measures or the infinite dimensional Gibbs measure $\nu$.

In fact, the following results concerning the conditions can be proven. This is a work in progress that will consist the material of a forthcoming paper.

**Proposition 9.1.** The hypothesis $(\mathcal{H}0'(\prime)/(\mathcal{H}0'(\prime)), (H2), (H3) and (H4)$ imply hypothesis $(H1)$.

Consequently, the main result of Theorem 2.2 and Theorem 2.3 are then reduced to the following

**Theorem 9.2.** If hypothesis $(\mathcal{H}0'(\prime)/(\mathcal{H}0'(\prime)), (H2), (H3)$ and $(H4) are satisfied, then the infinite dimensional Gibbs measure $\nu$ for the local specification $\{\mathcal{E}_\Lambda, \omega\}_{\Lambda \subset \subset \mathbb{Z}, \omega \in \Omega}$ satisfies the $q$ Log-Sobolev inequality

$$\nu |f|^q \log \frac{|f|^q}{\nu |f|^q} \leq \mathcal{C} \nu |\nabla f|^q$$

for some positive constant $\mathcal{C}$ independent of $f$.

The main idea of the proof of the Proposition 9.1 follows in main lines the method followed in the current paper. Although some of the details are more involved because of the lack of hypothesis $(H1)$, the fact that in Proposition 9.1 the Gibbs measure is localised and thus the approximation procedure starts from a finite set compensates for the loss of the $LS_q$ for $\nu_{\Lambda(i)}$. 46
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