Poisson algebras and Yang-Baxter equations

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Dedicated to Kazem Mahdavi, my kind mentor and friend.

Abstract. We connect generalizations of Poisson algebras with the classical and associative Yang-Baxter equations. In particular, we prove that solutions of the classical Yang-Baxter equation on a vector space $V$ are equivalent to “twisted” Poisson algebra structures on the tensor algebra $TV$. Here, “twisted” refers to working in the category of graded vector spaces equipped with $S_n$ actions in degree $n$. We show that the associative Yang-Baxter equation is similarly related to the double Poisson algebras of Van den Bergh. We generalize to $L_\infty$-algebras and define “infinity” versions of Yang-Baxter equations and double Poisson algebras. The proofs are based on the observation that $Lie$ is essentially unique among quadratic operads having a certain distributivity property over the commutative operad; we also give an $L_\infty$ generalization. In the appendix, we prove a generalized version of Schur-Weyl duality, which is related to the use of nonstandard $S_n$-module structures on $V^n$.

1. Twisted Poisson algebras and the CYBE

Throughout, we will work over a characteristic-zero field $k$. The tensor algebra $TV = T_kV$ satisfies the following twisted-commutativity property: each graded component $V^n$ is equipped with an $S_n$-module structure by permutation of components, and given homogeneous elements $v, w \in TV$ of degrees $|v|, |w|$, we have

\[(1.1) \quad w \otimes v = (21)^{|v|,|w|}(v \otimes w),\]

where $(21)^{|v|,|w|} \in S_{|v|+|w|}$ is the permutation of the two blocks $\{1, \ldots, |v|\}, \{|v|+1, \ldots, |v|+|w|\}$. We thus say that $TV$ is a twisted commutative algebra.\(^2\)

Similarly, we may define twisted Lie algebras. Again let $A = \bigoplus_{m \geq 0} A_m$ together with an $S_m$ action on $A_m$ for all $m$. A twisted Lie algebra is $A$ together with a graded bracket $\{ , \} : A \otimes A \to A$ satisfying

\[(1.2) \quad \{w, v\} = (21)^{|v|,|w|}\{v, w\},\]

\[(1.3) \quad \{u, \{v, w\}\} + (231)^{|v|,|w|,|u|}\{v, \{w, u\}\} + (312)^{|w|,|u|,|v|}\{w, \{u, v\}\} = 0,\]

where $\sigma = (i_1 i_2 \ldots i_n) \in S_n$ denotes the element $\sigma(j) = i_j$, and given $\tau \in S_3$, $\tau^{a,b,c} \in S_{a+b+c}$ denotes the permutation acting by permuting the blocks $\{1, \ldots, a\}, \{a+1, \ldots, a+b\}, \{a+b+1, \ldots, a+b+c\}$. (We will not use cycle notation in this paper.)

The motivating observation of this paper is as follows: If $A = TV$ is endowed with a twisted Lie algebra structure satisfying the Leibniz rule,

\[(1.4) \quad \{u \otimes v, w\} = u \otimes \{v, w\} + (213)^{|v|,|u|,|w|}(v \otimes \{u, w\}),\]

then the Jacobi identity restricted to degree one, $V \otimes V \otimes V \to T^3V$, says that the bracket yields a skew solution of the well-known classical Yang-Baxter equation (CYBE)\(^3\) interpreted as a map $r : V \otimes V \to$

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\(^1\)The notion of twisted algebras is an old notion from topology dating to at least the 1950’s; see, e.g., [Bar78, Joy86].

\(^2\)The CYBE is a central equation in physics and the study of quantum groups.
\( V \otimes V \), we have

\begin{align}
(1.5) \quad r &= -r^{21}, \\
(1.6) \quad [r^{12}, r^{13}] - [r^{23}, r^{12}] + [r^{13}, r^{23}] &= 0,
\end{align}

where here \( r^{ij} \) denotes \( r \) acting in the \( i \)-th and \( j \)-th components (e.g., \( r^{23} = \text{Id}_V \otimes r \)). The starting point for this paper is then

**Theorem 1.7.** Let \( V \) be any vector space. Skew solutions \( r \in \text{End}(V \otimes V) \) of the CYBE are equivalent to twisted Poisson algebra structures on \( TV \), equipped with its usual twisted commutative multiplication \( \otimes \).

Here, a **twisted Poisson** structure on \( TV \) is the same as a twisted Lie algebra structure satisfying (1.4).

The proof is based on the twisted generalization of the following well-known fact: a Poisson algebra structure on \( Sym^V \) is the same as a Lie algebra structure on \( V \) (Proposition 1.10). Precisely, recall that an \( S \)-module is a graded vector space \( V = \bigoplus_{m \geq 0} V_m \) together with \( S_m \)-actions on each \( V_m \). \( S \)-modules form a symmetric monoidal category, and the notion of \( Sym^V \) (the free commutative monoid in the category of \( S \)-modules) makes sense, and yields a twisted commutative algebra. In the case that \( V \) is concentrated in degree zero, the twisted commutative algebra \( Sym^V \), viewed as an ordinary vector space with the induced multiplication map, is the usual symmetric algebra \( Sym^V_0 \). In the case \( V \) is concentrated in degree one, \( Sym^V \), viewed as an ordinary vector space with an associative multiplication, is the usual tensor algebra \( TV \).

Then, as explained in (1.1) below, a standard proof that Poisson structures on \( Sym^V \) are the same as Lie algebra structures on \( V \) carries over to the twisted setting, and yields Theorem 1.7.

**Remark 1.8.** P. Etingof pointed out to the author a connection with the Lie algebra \( \mathfrak{t} \mathfrak{r} \) from BEER05 (which is generated by \( r_{ij} \) subject to the universal relations satisfied by \( r^{ij} \) for any skew solution \( r \) of the CYBE). More precisely, the universal enveloping algebra of \( \mathfrak{t} \mathfrak{r} \) contains the space of all possible operations \( V \otimes^m \rightarrow V \otimes^m \) obtainable from the twisted Lie structure on \( TV \) (in terms of an indeterminate \( r \)).

**1.1. Proof of Theorem 1.7** We recall first the definition of the symmetric monoidal structure on the category of \( S \)-modules: Given \( S \)-modules \( V = \bigoplus_{n \geq 0} V_n \) and \( W = \bigoplus_{n \geq 0} W_n \),

\[
(V \otimes W)_p := \bigoplus_{m+n=p} \text{Ind}_{S_m \times S_n}^{S_p}(V_m \otimes W_n).
\]

As before, let \( Sym^V \) denote the free commutative monoid generated by \( V \), in this category. Theorem 1.7 will follow from the following more general result:

**Proposition 1.10.** A multiplication \( \{ , \} : V \otimes V \rightarrow Sym^V \) satisfying (1.2), (1.3) extends uniquely to a twisted Poisson structure on \( Sym^V \).

In the proof below, it is helpful to have in mind the usual case when \( V \) is concentrated in degree zero; here the proof is one of the most obvious ones of the well-known fact that a Lie structure on \( V \) extends uniquely to a Poisson structure on \( Sym^V \).

We will need to introduce the following notation for technical convenience (and it is not needed in the case \( V \) is concentrated in degree zero):

**Notation 1.11.** Given any product of operations applied to symbols \( x_1, \ldots, x_m \), which represent elements of degrees \( |x_1|, \ldots, |x_m| \), let \( \sigma'_{x_1, \ldots, x_m} \in S_{|x_1|+\ldots+|x_m|} \) be the permutation which corresponds to rearranging the symbols in the order \( x_1, \ldots, x_m \) via a permutation of blocks of sizes \( |x_1|, \ldots, |x_m| \). For example, in the case \( |x_1| = 2, |x_2| = 3 \), we have \( \sigma'(x_3, x_2) = (34512)[x_3, x_2] \). Also, let us allow \( \sigma' \) to be extended linearly to linear combinations of such expressions.

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\(^3\)More conceptually, the forgetful functor from \( S \)-modules to vector spaces is a monoidal, although not symmetric monoidal, functor, which is why twisted (commutative or associative) algebras may also be viewed as ordinary associative algebras. These observations have been carried much further in, e.g., [Sto93, PR04].
The author first came upon the aforementioned observations after reading Van den Bergh’s paper \cite{VdB04} on double Poisson algebras. These algebras formalize Poisson geometry for noncommutative algebras such as path algebras of quivers (see Example 2.10 below). They are defined by the following axioms, which are quite similar to those for twisted Lie algebras:

**Definition 2.1.** VdB04 A double Poisson algebra is an associative algebra $A$ with a $k$-linear map $\{ \}\ : A \otimes A \to A \otimes A$ satisfying:

\begin{align}
\{a, b\} & = -(21)\{b, a\}, \\
\sum_{i=0}^{2} (231)^i \circ \{ -, -, - \} \circ (231)^{-i} & = 0, \\
\{a, bc\} & = (b \otimes 1)\{a, c\} + \{a, b\}(1 \otimes c).
\end{align}

Dropping the Poisson condition, we define:

**Definition 2.5.** Let $V$ be any $k$-vector space. A double Lie bracket is a $k$-linear map $\{ \}\ : V \otimes V \to V \otimes V$ satisfying (2.2) and (2.3).

We prove that double Lie algebras are the same as solutions of the associative Yang-Baxter equation (AYBE), which was introduced in \cite{Agu00, Agu01} and independently in \cite{Pol02}:

$$r^{12}r^{13} - r^{31}r^{12} + r^{13}r^{23} = 0.$$ 

Note that, when $r$ is skew ($r = -r^{21}$), then the AYBE implies the CYBE (this is an special case of \cite{Agu01} \textit{Theorem 3.5}). Namely, let $CYBE(r)$ denote the LHS of (1.6) and let $AYBE(r)$ denote the LHS of (2.6). Then, if $r$ is skew, we have

$$CYBE(r) = AYBE(r) - (132) \circ AYBE(r) \circ (132).$$

**Theorem 2.8.**

(i) Let $V$ be a vector space. Double Lie algebra structures on $V$ are equivalent to skew solutions $r \in \text{End}(V \otimes V)$ of the AYBE. Hence, any double Lie algebra $V$ yields a twisted Poisson algebra structure on $T_kV$ using its tensor product multiplication;

(ii) Let $A$ be an associative algebra. Suppose that $r \in \text{End}(A \otimes A)$ satisfies (2.4). If, furthermore, $r$ satisfies the CYBE, then, letting $AYBE(r)$ denote the LHS of (2.6), one has

$$(a \otimes 1 \otimes 1)AYBE(r) = (1 \otimes 1 \otimes a)AYBE(r), \quad \forall a \in A.$$

(iii) In particular, if $A$ is a prime and noncommutative associative algebra, then (2.9) implies that $AYBE(r) = 0$, so twisted Poisson structures on $T_kA$ satisfying (2.4) (where multiplication is taken in $A$) are equivalent to double Poisson structures on $A$. 

In part (iii), “prime” means that, for all nonzero \( a, b \in A \), there exists \( c \in A \) such that \( abc \neq 0 \). This is a standard generalization of integral domains to noncommutative rings.

**Example 2.10.** Consider any quiver (= directed graph) \( Q \). We recall that the path algebra, \( kQ \), of \( Q \), is the algebra which, as a \( k \)-vector space, is the set of \( k \)-linear combinations of paths in the graph, and whose multiplication is given by concatenation of paths. To be explicit, we may say that, for paths \( p \) and \( q \), \( pq \) is the concatenation if the terminal vertex of \( p \) equals the initial vertex of \( q \), and otherwise \( pq = 0 \). This multiplication is extended \( k \)-linearly to all of \( kQ \). If \( Q \) is strongly connected, which means that for any two vertices \( i \) and \( j \), there is a path from \( i \) to \( j \), then \( kQ \) is prime. If \( Q \) additionally has either at least two vertices or at least two edges, then \( kQ \) is noncommutative. In this case, the theorem shows that double Poisson structures on \( kQ \) are equivalent to twisted Poisson structures on \( T_kkQ \) satisfying (2.4).

We will say that a quiver \( Q \) is (extended) Dynkin if the underlying undirected graph (forgetting orientations, but remembering multiplicities) is (extended) Dynkin of type \( A, D, \) or \( E \).

**Example 2.11.** For any quiver \( Q \), another important algebra is called the preprojective algebra of \( Q \), whose definition we recall as follows. Let \( \overline{Q} \supset Q \) be the double quiver, which is the quiver with the same set of vertices as \( Q \), but with twice as many edges: for each edge \( e \in Q \), we include not merely \( e \), but also an edge \( e^* \in \overline{Q} \) which has the same endpoints as \( e \) but points in the opposite direction. Then, \( \Pi_Q \) is defined by \( \Pi_Q := kQ/(\sum_{e \in Q} ee^* - e^*e) \).

Then, by Proposition 9.2.23, for any non-Dynkin quiver \( Q \), \( \Pi_Q \) is prime, and provided \( Q \neq \tilde{A}_0 \), it is clear that \( \Pi_Q \) is noncommutative. Thus, the theorem applies also to this case.

**Example 2.12.** The deformed preprojective algebra \( \Pi_Q^\delta := kQ/(\lambda - \sum_{e \in Q} ee^* - e^*e) \) is filtered by degree, and its associated graded is \( \Pi_Q \). Hence, it is also prime and noncommutative when \( \Pi_Q \) is, and thus the theorem applies.

**Proof of Theorem 2.8** (i) Using the obvious correspondence between elements \( r \in \text{End}(V \otimes V) \) and double brackets \( V \otimes V \rightarrow V \otimes V \), the skew-symmetry condition (2.2) becomes the condition that \( r \) is skew. Then, (2.13) becomes

\[
r^{12}r^{23} + r^{23}r^{31} + r^{31}r^{12} = 0.
\]

If we permute the first and third components, multiply by \(-1\), and apply skew-symmetry, we get (2.16). This proves the first statement. The second statement then follows from the aforementioned fact that the AYBE implies the CYBE for skew elements \( r \).

(ii) Let us write

\[
AYBE(r) = r^{13}r^{12} - r^{12}r^{23} + r^{23}r^{13},
\]

so that \( CYBE(r) = AYBE(r) - AYBE'(r) \). Then, using the derivation property for \( r \), one may verify that

\[
CYBE(r)(a \otimes (b_1b_2) \otimes c) = (b_1 \otimes 1 \otimes 1)AYBE(r)(a \otimes b_2 \otimes c) - (1 \otimes 1 \otimes b_1)AYBE'(r)(a \otimes b_2 \otimes c) + CYBE(r)(a \otimes b_1 \otimes c) \cdot (1 \otimes b_2 \otimes 1),
\]

so that, if \( CYBE(r) = 0 \), then

\[
(a \otimes 1 \otimes 1)AYBE(r) = (1 \otimes 1 \otimes a)AYBE'(r),
\]

However, since \( CYBE(r) = 0 \), one also has \( AYBE(r) = AYBE'(r) \), so

\[
(\otimes 1 \otimes 1)AYBE(r) = (1 \otimes 1 \otimes a)AYBE(r),
\]

(iii) Assume that (2.17) holds. Then,

\[
(\otimes 1 \otimes 1)AYBE(r) = (1 \otimes 1 \otimes a)AYBE(r) = (1 \otimes 1 \otimes ba)AYBE'(r) = (ba \otimes 1 \otimes 1)AYBE(r).
\]

We deduce that \((ab - ba) \otimes 1 \otimes 1)AYBE(r) = 0\). So, if \([A, A]x = 0 \) implies \( x = 0 \) for all \( x \in A \), then \( AYBE(r) = 0 \). This follows because, for arbitrary \( y_1, y_2, y_3 \in A \), if we write \( AYBE(r)(y_1 \otimes y_2 \otimes y_3) = \sum v_i \otimes v'_i \), where the \( v_i \in A \otimes A \) are all linearly independent, and \( v_i \in A \), then \((ab - ba) \otimes 1 \otimes 1)AYBE(r) = 0\) implies that \((ab - ba)v_i = 0 \) for all \( i \) and all \( a, b \). Hence, if \([A, A]x = 0 \) implies \( x = 0 \) for all \( x \), then \( v_i = 0 \) for all \( i \), and hence \( AYBE(r)(y_1 \otimes y_2 \otimes y_3) = 0 \). Since \( y_1, y_2, y_3 \in A \) were arbitrary, \( AYBE(r) = 0 \) as well.
On the other hand, to say that $[A, A]a = 0$ implies $a = 0$ is the same as saying that the left ideal generated by $[A, A]$ annihilates only zero. But, the left ideal generated by $[A, A]$ is a two-sided ideal: $w[x, y]z = zw[x, y] + [w[x, y], z]$. So, for $[A, A]a = 0$ to imply $a = 0$, it is enough that $([A, A])a = 0$ implies $a = 0$. If $A$ is prime and noncommutative, then $[A, A]Aa = 0$ implies $a = 0$, hence $([A, A])a = 0$ implies $a = 0$, and so $AYBE(r) = 0$, as desired.

**Remark 2.19.** One may be curious what happens if $A$ is commutative (such as $\Pi_{Ad}^0$). In this case, any $r \in \text{End}(A \otimes A)$ satisfying (2.20) satisfies

$$\text{(2.20)} \ (a \otimes 1 - 1 \otimes a)(b \otimes c) = (c \otimes 1 - 1 \otimes c)r(b \otimes a) = (b \otimes 1 - 1 \otimes b)r(c \otimes a) = (a \otimes 1 - 1 \otimes a)r(c \otimes b),$$

and moreover, for any $a_1, a_2 \in A$,

$$\text{(2.21)} \ (a_1 \otimes 1 - 1 \otimes a_1)(a_2 \otimes 1 - 1 \otimes a_2)r(b \otimes c) = (b \otimes 1 - 1 \otimes b)(c \otimes 1 - 1 \otimes c)r(a_1 \otimes a_2).$$

So this puts special restrictions on $r$. For instance, if $A$ is a polynomial algebra over a field, then so is $A \otimes A$, and by unique factorization, $\text{(2.21)}$ implies that $r(b \otimes c) = (b \otimes 1 - 1 \otimes b)(c \otimes 1 - 1 \otimes c)f = r(c \otimes b)$ for all $b, c \in A$ and some fixed $f$ in the quotient field of $A \otimes A$. (Furthermore, one must have $f \in A \otimes A$ unless $A$ is a polynomial algebra in only one variable $x$, in which case $f \in (A \otimes A) \cdot (x \otimes 1 - 1 \otimes x)^{-2}$.) Also, for $r$ to be skew, $f$ must be skew, and in this case, the image of $r$ lies in $A \wedge A$, so one deduces that $r_{12}^{13} = -(213)_{o}(r_{12}^{13})_{o} = r_{23}^{12}(213)$. In the one-variable case, one deduces that $CYBE(r)(x \otimes x \otimes x) = 0$, and the latter is true (using the Poisson condition) iff $AYBE(r) = 0$ and (hence $CYBE(r) = 0$). One may then deduce inductively that these are all zero iff $r(x \otimes x) = \lambda(x \otimes 1 - 1 \otimes x)$ for some $\lambda \in k$. It would be interesting to see if there are other solutions in more variables (and whether $AYBE$ and $CYBE$ have the same solutions). Note that $\Pi_{Ad}^0$ is the two-variable case.

**Remark 2.22.** Many solutions of the AYBE with (graded and spectral) parameters $u, v$, related to solutions of the CYBE and QYBE, have been classified in [Pol02, Sch03, Pol06]. For example, any “associative Belavin-Drinfeld structure” gives rise to a (trigonometric) such solution which additionally satisfies the QYBE and CYBE with spectral parameters. One may interpret a solution of the AYBE and skew-symmetry (unitarity) with parameters as a sort of graded version of double Lie algebra; to make it Poisson, one would need to find a compatible multiplication (if it exists).

## 3. Operadic generalization

We ask the question: Is Theorem 1.7 a special property of Lie algebras, or can it be generalized to other operads (suitably replacing the CYBE with other equations)? To make sense of this question, for any twisted-commutative operad $A$ equipped with an additional binary operation $\star : A \otimes A \to A$, we generalize the Leibniz rule (3.1) to

$$\text{(3.1)} \ (uv) \star w = u(v \star w) + (213)^{[v], [u], [w]}v(u \star w).$$

Let $\mathcal{O}$ be any operad generated by a single element $m \in O(2)$. If $A$ as above is an $O$-algebra (with $a \star b := m(a \otimes b)$) satisfying (3.1), we call it a twisted distributive $O$-algebra.

Let $\mathcal{F}$ be the operad freely generated by an element also denoted by $m \in F(2)$. By a quadratic operad $\mathcal{O} = \mathcal{F}/(R)$, we mean one such that $R = R_2 \oplus R_3$, with $R_2 \subset F(2)$ and $R_3 \subset F(3)$. In particular, the only possible relations from $R_2$ are symmetry conditions on $m$, namely, that $(21)m = \pm m$ as elements of $\mathcal{O}$. Also, $R_3$ consists of relations which are quadratic in $m$. We then prove the following result (in [3.1]):

**Definition 3.2.** Let $\text{Lie-adm}$ denote the Lie-admissible operad, which is the operad whose ordinary algebras are vector spaces together with a binary operation whose skew-symmetrization is a Lie bracket.

**Theorem 3.3.** The only quadratic operads $\mathcal{O} = \mathcal{F}/(R)$ for which distributive $O$-algebra structures on $\text{Sym} V$ are equivalent to $O$-algebra structures on $V$, for every vector space $V$, are the following five:

- $\mathcal{O} = \text{Lie}$, the Lie operad
- $\mathcal{O} = \text{Lie-adm}$, the Lie-admissible operad
- $\mathcal{O} = \mathcal{F}/(m + (21)m)$, the operad whose algebras are vector spaces with a skew-symmetric binary operation
- $\mathcal{O} = \mathcal{F}/(m - (21)m)$, the operad whose algebras are vector spaces with a symmetric binary operation
\[ \mathcal{O} = F, \] where the free operad generated by \( m \), the operad whose algebras are magmas.

These are also exactly the operads for which twisted distributive \( \mathcal{O} \)-algebra structures on \( \text{Sym} V \) are equivalent to operations \( V \otimes V \rightarrow \text{Sym} V \) satisfying the relations \( R \), for every \( \mathbb{S} \)-module \( V \).

After proving this theorem, we will generalize to quadratic operads which are not generated by only a single operation, and find that \( L_x \)-algebras (heuristically, Lie algebras up to homotopy) are the prototypical example of algebras with operations satisfying the desired distributivity property (Theorem 3.15 and Remark 3.17).

### 3.1. Proof of Theorem 3.3

As in Proposition 1.10, it is not difficult to show that the listed operads have the desired property. We show the converse. Let \( F \) be as in Theorem 3.1 and let \( \mathcal{O} = F/(R) \) with \( R = R_2 \oplus R_3 \), where \( R_2 \subset F(2) \) and \( R_3 \subset F(3) \). We show that, if \( \mathcal{O} \)-algebra structures on every vector space are equivalent to distributive \( \mathcal{O} \)-algebra structures on \( \text{Sym} V \), then \( \mathcal{O} \) is one of the listed operads.

Assume first that \( R_2 = 0 \), i.e., there is no (skew)-symmetry axiom for \( \mathcal{O} \)-algebras. Take an arbitrary element of \( R_3 \): this is equivalent to a quadratic axiom for \( \mathcal{O} \)-algebras \((V, \ast)\). Let us write it as

\[ \sum_{\sigma \in S_3} \lambda_{\sigma,1} b_{\sigma(1)} \ast (b_{\sigma(2)} \ast b_{\sigma(3)}) + \lambda_{\sigma,2} (b_{\sigma(1)} \ast b_{\sigma(2)}) \ast b_{\sigma(3)} = 0, \]

for some constants \( \lambda_{\sigma,i} \).

If \( \mathcal{O} \)-algebra structures \((V, \ast)\) on \( V \) are equivalent to distributive \( \mathcal{O} \)-algebra structures on \( \text{Sym} V \) for all \( V \), then if we expand the above axiom for \((b_1, b_2, b_3) = (b_1', b_2', b_3')\) using the Leibniz rule 3.31, the terms of the form \((x \ast y)(z \ast w)\) must cancel identically, for \( \{x, y, z, w\} = \{b_1', b_2', b_2, b_3\} \). That is, the following expression must be identically zero:

\[ \sum_{\sigma(1)=1} (\lambda_{\sigma,1} (b_2 \ast b_1')(b_3' \ast b_3) + (b_1 \ast b_3) (b_2'' \ast b_2)) \]

\[ + \sum_{\sigma(2)=1} ((\lambda_{\sigma,1} + \lambda_{\sigma,2}) (b_{\sigma(2)} \ast b_1') (b_1'' \ast b_{\sigma(3)}) + (b_{\sigma(2)} \ast b_1') (b_2' \ast b_{\sigma(3)}) + (b_{\sigma(2)} \ast b'_2) (b_2'' \ast b_{\sigma(3)}) + \sum_{\sigma(3)=1} \lambda_{\sigma,1} (b_2 \ast b_1')(b_3' \ast b_3) + (b_3 \ast b_1') (b_2'' \ast b_2)). \]

This can only happen if the following equations are satisfied:

\[ \lambda_{id,2} = -\lambda_{(32),2}, \lambda_{(21),2} = -\lambda_{(31),1}, \lambda_{(31),2} = -\lambda_{(32),1}, \lambda_{(23),1} = -\lambda_{(32),1}. \]

Similarly, if we plug in instead \((b_1, b_2, b_3) = (b_1, b_2 b_3', b_3)\) or \((b_1, b_2, b_3) = (b_1, b_2, b_3 b_3')\), we obtain additionally the following conditions:

\[ \lambda_{(21),2} = -\lambda_{(31),2}, \lambda_{id,2} = -\lambda_{(32),1}, \lambda_{(32),2} = -\lambda_{(31),1}, \lambda_{(23),1} = -\lambda_{(31),1}. \]

(These can also be obtained from (3.6) by applying the action of \( S_3 \).)

We deduce that the only possible element of \( R_3 \) is a multiple of the associated Lie relation, which proves the theorem in the case that \( R_3 = 0 \).

If \( R_2 \neq 0 \), then the above computation simplifies. Suppose that \( R \) is spanned by \( m - \varepsilon(21)m \) for \( \varepsilon \in \{-1, 1\} \). Take an arbitrary element of \( R_3 \) and write the corresponding axiom for \( \mathcal{O} \)-algebras as

\[ \lambda_1 b_1 \ast (b_2 \ast b_3) + \lambda_2 b_2 \ast (b_3 \ast b_1) + \lambda_3 b_3 \ast (b_1 \ast b_2) = 0. \]

Next, we plug in \( b_1 = b_1' b_1'' \) and gather all terms on the LHS of the form \((x \ast y)(z \ast w)\) where \( \{x, y, z, w\} = \{b_1', b_2', b_2, b_3\} \):

\[ \lambda_2 ((b_2 \ast b_1') (b_3 \ast b_1'') + (b_2 \ast b_1'') (b_3 \ast b_1')) + \lambda_3 ((b_3 \ast b_1') (b_1'' \ast b_2) + (b_3 \ast b_1'') (b_1') (b_1' \ast b_2)). \]

The above must be zero, in order for \( \mathcal{O} \) to have the desired property. Using the symmetry condition \( x \ast y = \varepsilon y \ast x \), we may rewrite (3.10) as

\[ (\lambda_2 + \varepsilon \lambda_3) ((b_2 \ast b_1') (b_3 \ast b_1'') + (b_3 \ast b_1') (b_2' \ast b_3'')). \]

For this to be zero in general, we require that \( \lambda_2 = -\varepsilon \lambda_3 \). Similarly, setting \((b_1, b_2, b_3) = (b_1, b_2 b_3', b_3)\), we conclude that \( \lambda_1 = -\varepsilon \lambda_3 \), and finally we conclude that \( \lambda_1 = -\varepsilon \lambda_2 \). Hence, if \( \varepsilon = 1 \), then \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \), and if \( \varepsilon = -1 \), then \( \lambda_1 = \lambda_2 = \lambda_3 \) can be arbitrary. This proves the desired result.
For the final statement, it suffices to generalize Proposition 1.10 to the listed operads. This is straightforward and omitted. □

3.2. $L_\infty$ generalization and arbitrary quadratic operads. In this subsection we will drop the assumption on $O$ that it be generated by a single binary operation. Suppose instead that $O$ is generated by any operations, of any arity, satisfying quadratic and linear relations.

The simplest example of this is the case where $O$ includes a differential $d \in O(1)$, satisfying $d^2 = 0$. We see that, already, it is not true that $d^2|V = 0$ implies $d^2|\text{Sym} V = 0$. The solution to this problem is well-known: make $V$ a graded vector space, take $d$ to be an operator of degree $-1$, and make $\text{Sym} V$ the supersymmetric algebra generated by $V$, i.e., $vw = (-1)^{|v||w|}vw$.

It then turns out that Theorems 1.7, 3.3 generalize, roughly, by replacing Lie algebras with $L_\infty$ algebras (heuristically, these are Lie algebra up to homotopy).

We note that the “super” grading above is independent of the twisted grading, so that when one has both, $V$ is bigraded. To simplify things, we restrict to ordinary (not twisted) algebras with a single grading. All of the results generalize to the twisted setting, by working with $S$-modules and adding permutations $\sigma'$ (using Notation 1.4) to the beginning of formulas.

Recall that an $L_\infty$ algebra is a $\mathbb{Z}$-graded vector space $A$ equipped with a differential $d = \{ \}$, satisfying axioms (3.12), there is a unique extension using the Leibniz rule

\[
\sum_{i+j=m+1, i,j \geq 1} \sigma \in S_m \left( -1 \right)^+ \text{sign}_{\text{odd}}(a, \sigma) \left\{ a_{\sigma(1)}, \ldots, a_{\sigma(i)} \right\}_i, a_{\sigma(i+1)}, \ldots, a_{\sigma(i+j-1)} \right\}_j = 0,
\]

for all $m \geq 1$, where $\text{sign}_{\text{odd}}(a, \sigma)$ is the sign of the permutation $\sigma_{|a_1|+1, |a_2|+1, \ldots |a_m|+1} \in S_{|a_1|+1, |a_2|+1, \ldots |a_m|+1}$ obtained from $\sigma$ by acting on blocks of the sizes $|a_1| + 1, \ldots, |a_m| + 1$.

In particular, this includes the axiom that $d$ is a differential, and that the Jacobi identity for $\{ \}$ is satisfied up to chain homotopy.

We will use the notation

\[
b := (a_1, a_2, \ldots, a_k, a'_k, a''_k, a_{k+1}, \ldots, a_m)\]

We define the Leibniz rule for an $L_\infty$ algebra endowed with an additional supercommutative multiplication as

\[
\{ a_1, a_2, \ldots, a_k, a'_k, a''_k, a_{k+1}, \ldots, a_m \}_m
\]

\[
= (-1)^{\left| a'_k \right| (|a_1| + |a_2| + \cdots + |a_{k-1}|)} a'_k \{ a_1, a_2, \ldots, a_{k-1}, a''_k, a_{k+1}, \ldots, a_m \}_m
\]

\[
+ (-1)^{\left| a''_k \right| (|a_1| + |a_2| + \cdots + |a_{k-1}| + |a'_k|)} a''_k \{ a_1, a_2, \ldots, a_{k-1}, a'_k, a_{k+1}, \ldots, a_m \}_m.
\]

**Theorem 3.15.** An $L_\infty$ structure on SuperSym $V$ satisfying the Leibniz rule (3.14) is equivalent to operations $\{ \}$ on $V \otimes V$ satisfying the $L_\infty$ axioms (3.12).

**Proof.** This is similar to the proof of Proposition 1.10. We need to show that, given an operation $V \otimes V \to \text{SuperSym} V$ satisfying the $L_\infty$ axioms (3.12), there is a unique extension using the Leibniz rule (3.13) to a multiplication $\text{SuperSym} V \otimes \text{SuperSym} V \to \text{SuperSym} V$, and this also satisfies the $L_\infty$ axioms. The fact that there is a unique extension is easy. To show it satisfies the $L_\infty$ axioms, it is enough inductively to verify that, for any $m$-tuple of the form $(a_1, a_2, \ldots, a_m) = (a'_1, a'_2, a_3, \ldots, a_m) \in (\text{SuperSym} V)^m$, then the $L_\infty$-axioms for $(a'_1, a'_2, a_3, \ldots, a_m)$ and $(a''_1, a'_2, a_3, \ldots, a_m)$ imply the $L_\infty$-axioms for $(a_1, a_2, \ldots, a_m)$. This follows by expanding the expression, for all $m \geq 1$:

\[
\sum_{i,j \geq 1, i+j=m+1, \sigma \in S_m} (-1)^{\left| a'_k \right| (|a_1| + |a_2| + \cdots + |a_{i-1}|)} a'_i \{ a_1, a_2, \ldots, a_{i-1}, a''_i, a_{i+1}, \ldots, a_m \}_m
\]

\[
\sum_{i,j \geq 1, i+j=m+1, \sigma \in S_m} (-1)^{\left| a''_k \right| (|a_1| + |a_2| + \cdots + |a_{i-1}| + |a'_i|)} a''_i \{ a_1, a_2, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots, a_m \}_m.
\]

using (3.11), and verifying that the terms of the form $\pm \{ \}$, $\{ \}$ cancel. In more detail, it is equivalent to sum above not over all permutations $\sigma \in S_m$, but only over the $i, j-1$-shuffles: that is, $\sigma$ such that $\sigma(\ell) < \sigma(\ell+1)$ for all $\ell \neq 1$ (these are the permutations that leave the order of $1, 2, \ldots, i$ and $i+1, i+2, \ldots, i+j-1 = m$ unchanged). Then, there is a canonical bijection between $i, j-1$ shuffles $\sigma$ such that $\sigma(1) = 1$ and $j, i-1$ shuffles $\sigma'$ such that $\sigma'(1) = 1$, and the terms of the form $\pm \{ \}$, $\{ \}$ that appear in the expansion of the
The expansion of the summand of (3.16) corresponding to \((i, j, \sigma)\) cancel with the terms of the form \(\pm\{\ldots\}\{\ldots\}_i\) that appear in the expansion of the summand of (3.16) corresponding to \((j, i, \sigma')\). We omit further details. 

\[\sum_{\sigma \in S_{|o_i|+|o_j|}-1} \text{sign}_{\text{odd}}(a, \sigma) \left( (-1)^{|\sigma|}\{\{a_{\sigma(1)}, \ldots, a_{\sigma(|o_i|)}\}_i, a_{\sigma(|o_i|+1)}, \ldots, a_{\sigma(|o_i|+|o_j|)-1}\}_j \right) + (-1)^{|\sigma|}\{\{a_{\sigma(1)}, \ldots, a_{\sigma(|o_j|)}\}_j, a_{\sigma(|o_j|+1)}, \ldots, a_{\sigma(|o_j|+|o_j|)-1}\}_i = 0,\]

where \{\}, {\} denote applying the operations \(o_i, o_j\) to the given arguments. The proof is similar to the proofs of Theorems 3.3 and 3.14. If \(O\) is not generated by totally skew-symmetric operations, then the only allowable quadratic relations are those specifying that the skew-symmetrization of the generating operations satisfy certain relations as above. We omit the details.

4. Yang-Baxter-infinity equations and double Poisson-infinity algebras

In view of the fact that \(L_\infty\) algebras also have the distributivity property of Theorem 3.3, we explain here the double Poisson analogue of twisted distributive \(L_\infty\) structures on \(TV\), which we call “double Poisson-infinity algebras” (Definition 4.1). Here, “infinity” refers to relaxing the Jacobi identity up to higher homotopies: double Poisson-infinity algebras, as we define them, still include an honest associative algebra and the bracket is still skew-symmetric.

Further, using twisted and double Poisson-infinity algebras, we define infinity versions of the classical and associative Yang-Baxter equations (Definition 4.6), by analogy with Theorem 1.7. The CYBE\(_\infty\) yields equations for sequences of elements \(r_n \in g^\otimes n\) where \(g\) is any graded Lie algebra, and the AYBE\(_\infty\) yields equations for \(r_n \in A^\otimes n\), where \(A\) is any associative algebra. We do not know if there exists a corresponding notion of quantum Yang-Baxter equation-infinity.

Specifically, we will (abusively) call a (twisted) commutative and \(L_\infty\) algebra satisfying (3.14) a (twisted) “Poisson-infinity” algebra. As an application of our comparison of twisted and double Poisson algebras, it makes sense to define double Poisson-infinity algebras. To do this, we need only define a “double” version of the Jacobi-infinity identity (3.12). As in the usual setting, we do this by replacing sums over all permutations by sums over only cyclic permutations:

**Definition 4.1.** A double Poisson-infinity algebra is a \(Z\)-graded associative algebra \(A\) together with brackets \{\} : \(A^\otimes n \to A^\otimes n\) of degree 2 - \(n\), for all \(n \geq 1\), satisfying the identities (for all \(n \geq 1\)):

\[\text{Skew-symmetry: } \sigma\{a_{\sigma(1)}, \ldots, a_{\sigma(n)}\}_{\alpha} = \text{sign}(a, \sigma)\{a_1, a_2, \ldots, a_n\}, \forall \sigma \in S_n,\]

\[\text{Jacobi}_\infty : \sum_{i+j=n+1} \sum_{\sigma \in S/(i+j-1)} (-1)^{i\text{sign}_{\text{odd}}(a, \sigma)}\{\{a_{\sigma(1)}, \ldots, a_{\sigma(i)}\}_i, a_{\sigma(i+1)}, \ldots, a_{\sigma(i+j-1)}\}_{j} = 0,\]

\[\text{Double Leibniz: } \{a_1, a_2, \ldots, a_{n-1}, a'_na''_n\} = (-1)^{|a'_n|\{([a_1]+\ldots+[a_{n-1}])a''_n\{a_1, \ldots, a_{n-1}, a''_n\} + (-1)^{|a'_n|\{a_1, a'_n\}a''_n.\]

In a future paper, we hope to explain a double version of Kontsevich’s formality theorem [Kon03], where the above will replace \(L_\infty\) for the Poisson side (the differential operator side will use [CS06]).
Finally, we obtain infinity versions of the Yang-Baxter equations by writing down the Poisson-infinity conditions in terms of elements $r_n \in \text{End}(V^\otimes n)$ (which we may generalize to $g^\otimes n, A^\otimes n$). In order to make the sum over as few terms as possible, and to specialize to the ordinary Yang-Baxter equations, we use the

**Notation 4.5.** Let $Sh_{i,j} \subset S_{i+j}$ denote the set of $i,j$-shuffles: this means permutations $(k_1k_2\cdots k_{i+j})$ such that $k_1 < k_2 < \cdots < k_i$ and $k_{i+1} < k_{i+2} < \cdots < k_{i+j}$.

**Definition 4.6.** (i) Let $g$ be a graded Lie algebra. The classical Yang-Baxter-infinity equations for elements $\{r_n \in g^\otimes n\}_{n \geq 1}$ of degrees $2 - n$ are, for all $n \geq 1$,

$$\sum_{i+j=n+1} (-1)^i \sum_{\sigma \in Sh_{i,i,j-1}} [r_i^{\sigma(1)}, r_j^{\sigma(2)}, \ldots, r_n^{\sigma(i)\sigma(i+\cdots\sigma(i+j-1)}] = 0.$$  

(ii) Let $A$ be a graded associative algebra. The associative Yang-Baxter-infinity equations for elements $\{r_n \in A^\otimes n\}_{n \geq 1}$ of degrees $2 - n$ are, for all $n \geq 1$,

$$\sum_{i+j=n+1} (-1)^i \sum_{\sigma \in Z/n} [r_i^{\sigma(1)}, r_j^{\sigma(2)}, \ldots, r_n^{\sigma(i)\sigma(i+\cdots\sigma(i+j-1)}] = 0.$$  

We remark that the graded condition is not essential for the AYBE$_\infty$ to make sense, although it is needed for CYBE$_\infty$ since we need supercommutators.

It would be interesting to see if there is any reasonable $\infty$-analogue of the quantum Yang-Baxter equation, obtained by somehow “quantizing” the above equation.

### 5. Non-Poisson twisted algebra structures on $TV$

Let $V$ be a vector space. One interpretation of Theorem 4.3 is that the Leibniz rule (4.1) is not a good condition to impose for many types of algebra structures on $TV$. For example, the result holds for neither associative algebras nor commutative algebras. In this section, in the form of remarks, we briefly explain how twisted associative algebra structures on $TV$ (without the Leibniz condition) are related to the quantum rather than the classical Yang-Baxter equation.

Let $V$ be a representation $\rho$ of a Hopf algebra $H$, which endows $V^\otimes m$ (using $\rho^\otimes m$) and hence $TV$ with a canonical structure of $H$-representation. We look for twisted associative algebra structures on $TV$ given by a single element $J \in H \otimes H$, by the rule

$$a \cdot b = \rho^{[a]} \otimes \rho^{[b]}(J)(a \otimes b).$$

We obtain the following:

**Proposition 5.2.** The formula (5.1) yields a twisted associative algebra structure iff $J$ is a twist:

$$\Delta \otimes \rho(1)(J \otimes 1) = (1 \otimes \Delta)(J)(1 \otimes J).$$

This has a generalization to algebras over any operad $O$, where $J$ is replaced by an element satisfying an $O$-version of (5.3).

We may generalize the above to the case where there is no universal element $J$. First, note that the induced multiplication operation on $TV$ is described by the restrictions to $W_1 \otimes W_2 \rightarrow TV$, where $W_1, W_2$ are irreducible representations of $H$ occurring in $TV$. This is still true without an element $J$. These maps need only satisfy associativity for triples $W_1, W_2, W_3$, and preserve a restricted form of the permutation action.

Furthermore, we don’t need to be given an $H$, since we can always take $H = U(\mathfrak{gl}(V))$. Let $\text{Rep}_{alg}(\text{End}(V))$ denote the category of algebraic representations of $\text{End}(V)$ viewed as a $k$-algebraic monoid, which is equivalent to the category of representations of $H$ occurring in $TV$. (This includes all representations up to twisting by the trace representation of $\mathfrak{gl}(V)$ (and up to isomorphism).) By Schur-Weyl duality, the $S_n$ action on $V^\otimes n$ spans all of $\text{End}_H(V^\otimes n)$. In the language of category theory, we obtain the

**Proposition 5.4.** Twisted associative algebra structures on $TV$ are the same as monoidal structures on the fiber functor $\text{Rep}_{alg}(\text{End}(V)) \rightarrow \text{Vect}$.  

9
The above also naturally generalizes to the case of algebra structures on \( TV \) that use a modified permutation action, given by an element \( R \in \text{End}(V \otimes V) \) (so, \( R^{21} = \text{Id} \) and \( R \) satisfies the quantum Yang-Baxter equation, \( R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \)). In this case, one has a natural Hopf algebra \( H_R \), defined in \([RTF90]\) (see also the appendix), which makes \( V \) a canonical comodule. Using a generalized version of Schur-Weyl duality (Theorem \( A.7 \)), we obtain the

**Proposition 5.5.** Twisted associative algebra structures on \((TV, R)\) are the same as monoidal structures on the fiber functor \( \text{Comod}(H_R) \to \text{Vect} \).

6. Acknowledgements

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Appendix A. Generalized Schur-Weyl duality

A.1. The \([RTF90]\) construction. We recall the definition of the coquasitriangular Hopf algebra \( H_R \). Beginning with any solution \( R \) of the QYBE, \( R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \), Faddeev, Reshetikhin, Takhtajan, and Sklyanin \([RTF90]\) constructed the following bialgebra, which is like a \( R \)-twisted version of \( \mathcal{O}(\text{End}(V)) \), the commutative bialgebra of functions on the multiplicative monoid \( \text{End}(V) \). It has a coquasitriangular structure from which one recovers \( R \).

**Definition A.1.** \([RTF90]\) Define \( H_R \) to be the quotient of the free algebra \( F := k \langle L_{ij} \rangle_{i,j \in \{1, \ldots, n\}} \) by the following relations. Set \( L = \sum_{i,j \in \{1, \ldots, n\}} e_{ij} \otimes L_{ij} \in \text{End}(V) \otimes F \). Then \( H_R := F/I_R \), where \( I_R \) is the ideal generated by the relations

\[
R^{12}L^{13}L^{23} = L^{23}L^{13}R^{12} \in \text{End}(V) \otimes \text{End}(V) \otimes F.
\]

We also let \( L \) denote its own image under the quotient \( F \to H_R \). Then, the coproduct \( \Delta \) and counit \( \epsilon \) are defined by

\[
\Delta(L) = L^{12}L^{13}, \quad \epsilon(L) = 1.
\]

**Theorem A.4.** \([RTF90]\) (cf. \([Kas95]\)) The preceding definition makes sense and defines a bialgebra (with a unique coquasitriangular structure inducing \( R \)).

By “coquasitriangular structure inducing \( R \),” we mean a map \( H_R \otimes H_R \to k \), which satisfies the dual of the quasitriangularity axioms (replacing multiplication by convolution), and whose action on the standard comodule is \( R \). (See \([Kas95]\) for details.)

We will need the standard comodule:

**Definition A.5.** The “standard comodule” \( V \) of \( H_R \) is given by the element \( L \in \text{End}(V) \otimes H_R \cong \text{Hom}(V, V \otimes H_R) \). That is, the map \( \Delta : V \to V \otimes H_R \) is given by \( \Delta(e_j) = \sum_{i=1}^{\dim V} e_i \otimes L_{ij} \). Let \( V^{\otimes m} \) denote the comodules obtained by the tensor power of this one.

A.2. Generalized Schur-Weyl duality. In this subsection we prove a generalization of Schur-Weyl duality to the bialgebras \( H_R \). This says that the irreducible subrepresentations of \( V^{\otimes m} \) are given by Young diagrams using the \( R \)-symmetric action. This result should not be too surprising, given that the relations \((A.2)\) are defined in terms of \( R \), and it is possible that the result even motivated the definition of \( H_R \). However, since the author could not find it in the literature, the result is given here. This result shows that \( \text{Comod}(H_R) \) is the same as the category \( \mathcal{SV}(V) \) studied in \([GM00]\) (under certain conditions on \( R \)).

**Notation A.6.** For any permutation \( \sigma \in S_n \), let \( \tau_\sigma : V^{\otimes n} \to V^{\otimes n} \) denote the permutation of components (i.e., the standard permutation action).
Theorem A.7. Assume char(k) = 0 and R is a unitary solution of the QYBE. Let SRm ⊂ End(V⊗m) be the image of the R-symmetric action of Sm (generated by elements τ_{b,b+1}R^{b,b+1}), and let HRm ⊂ End(V⊗m) be the span of linear maps of the form φ ⊗ Δ, where Δ : V⊗m → V⊗m ⊗ H is the comodule action, and φ ∈ End(HR,k) is any linear map. Then, one has

(A.8) \[ \text{End}_{SRm}(V⊗m) = HRm, \]
(A.9) \[ \text{End}_{HRm}(V⊗m) = SRm, \]
(A.10) \[ V⊗m \cong \bigoplus_{\lambda} \rho_{\lambda,S_m} \otimes \rho_{\lambda,H_R}, \]
where the sum is over Young diagrams λ parametrizing irreducible representations of Sm, and ρ_{\lambda,S_m} is the corresponding irreducible representation of Sm. The space ρ_{\lambda,H_R} is the H_R-submodule of V⊗m equal to c_λ(R)ν⊗m, where c_λ (cf. [FH91]) is the Young symmetrizer in k[S_m] corresponding to λ (so that k[S_m]a_λ = ρ_{\lambda,S_m}), and c_λ(R) is the corresponding element of End(V⊗m) given by the “R-permutation action” *_R of Sm:

(A.11) \[ (b,b+1)_R v := \tau_{(b,b+1)}R^{b,b+1} \cdot v. \]
Furthermore, ρ_{\lambda,H_R} is an irreducible H_R-comodule, and (A.10) is the multiplicity-free decomposition of V⊗m into irreducible SRm ⊗ H_R-modules.

Proof. We claim that (A.8) is true. First, note that H_R is graded (setting T_{i,j} to have degree 1), and that Δ(V⊗m) ⊂ V⊗m ⊗ H_R[m]. Next, consider H_R[1] = F[1] to be End(V)*, by the pairing (e_{i,j}, T_{k,ℓ}) = δ_{i,k}δ_{j,ℓ}. This means that H_R[m]* ⊂ F[m]* = End(V⊗m), which is the subspace respecting the relation (A.9).

Then, the LHS of (A.2), R^{12}L^{13}L^{23}, is identified, as a subspace of End(V) ⊗ End(V) ⊗ F[2] ∼ End(V) ⊗ End(V) ⊗ (End(V)*)* ⊗2, with R^{12}(Id_{End(V)}Id_{End(V)})2

(A.12) \[ \phi^{34}(R^{12}Id_{End(V)}Id_{End(V)})2 \rangle = R\phi \in \text{End}(V \otimes V). \]
Similarly, φ^{34} applied to the RHS of (A.2) (considered as an element of End(V)*⊗2 ⊗ (End(V)*)* ⊗2) is identified with φ^{21}R. So, the condition for φ to be an element of H_R[2]* is

(A.13) \[ \phi(\tau_{(12)} \circ R) = (\tau_{(12)} \circ R)\phi, \]
which says that φ commutes with the R-permutation action. Since H_R is presented by the quadratic relation (A.2), H_R[m]* consists of φ that satisfy (A.13) when applying (\tau_{(12)} \circ R) to any components i, i + 1 for 1 ≤ i ≤ m − 1. This proves (A.8).

Since Sm is completely reducible over k (k has characteristic zero), the above means that V⊗m decomposes, as a Sm-representation, into a sum of the form

(A.14) \[ V⊗m \cong \bigoplus_{\lambda} \rho_{\lambda,S_m} \otimes V_{m,\lambda,R}, \]
where ρ_{\lambda,S_m} ⊗ V_{m,\lambda,R} corresponds to the ρ_{\lambda,S_m}-isotypical part of V⊗m, with respect to the R-permutation action of Sm, and V_{m,\lambda,R} has trivial Sm-action. Then, it immediately follows that H_Rm = \bigoplus Id ⊗ End(V_{m,\lambda,R}) with respect to this decomposition, and since the ρ_{\lambda,S_m} are distinct irreducible representations, (A.9) follows, and hence also (A.10). The remaining statements are immediate, and the theorem is proved.

Remark A.15. The first part of the above theorem, (A.8), is still true if the characteristic of k is not zero, or if the condition on unitarity is dropped and Sm is replaced by B_m, both using the same proof as above. However, the next two parts may not generalize (since B_m is not finite, its representations are not completely reducible in general, so the double commutant arguments fail). Nonetheless, the inclusion ≥ in (A.10) is still true and well-known in all cases (i.e., (A.11) defines endomorphisms of comodules).

Remark A.16. As a special case of the above, we immediately get the usual Schur-Weyl duality for R = 1 (and it is essentially the same as Weyl’s original proof), except that the usual statement also says that H_Rm is generated by the diagonal action of GL(V). To get this last fact, as in Weyl’s proof, one may use the fact that the symmetric elements of W⊗m are generated by the diagonal elements w⊗m, for any vector space W over an infinite field.
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