Classification of three-state Hamiltonians solvable by the coordinate Bethe ansatz

N Crampé\textsuperscript{1,2}, L Frappat\textsuperscript{3} and E Ragoucy\textsuperscript{3}

\textsuperscript{1} CNRS, Laboratoire Charles Coulomb L2C UMR 5221, Place Eugène Bataillon—CC069, F-34095 Montpellier Cedex 5, France
\textsuperscript{2} Université Montpellier II, Laboratoire Charles Coulomb, Place Eugène Bataillon—CC069, F-34095 Montpellier Cedex 5, France
\textsuperscript{3} Laboratoire de Physique Théorique LAPTH, CNRS and Université de Savoie, BP 110, F-74941 Annecy-le-Vieux Cedex, France

E-mail: nicolas.cramp@univ-montp2.fr, luc.frappat@lapth.cnrs.fr and eric.ragoucy@lapth.cnrs.fr

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Abstract

We classify ‘all’ Hamiltonians with rank 1 symmetry and nearest-neighbour interactions, acting on a periodic three-state spin chain, and solvable through (generalization of) the coordinate Bethe ansatz (CBA). In this way we obtain four multi-parametric extensions of the known 19-vertex Hamiltonians (such as Zamolodchikov–Fateev, Izergin–Korepin and Bariev Hamiltonians). Apart from the 19-vertex Hamiltonians, there exist 17-vertex and 14-vertex Hamiltonians that cannot be viewed as subcases of the 19-vertex ones. In the case of 17-vertex Hamiltonians, we get a generalization of the genus 5 special branch found by Martins, plus three new ones. We also get two 14-vertex Hamiltonians. We solve all these Hamiltonians using CBA, and provide their spectrum, eigenfunctions and Bethe equations. Special attention is given to provide the specifications of our multi-parametric Hamiltonians that give back known Hamiltonians.

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1. Introduction

In his seminal paper [1], Bethe provided a method, now known as the coordinate Bethe ansatz (CBA), to compute the eigenvalues and the eigenfunctions for the Heisenberg (or XXX 1/2-spin) models [2]. The same idea has also been used intensively in the context of the Bose gas with δ-interaction when the particles have no spin and with the periodic boundary condition [3]. When they carry a spin [4, 5], the famous Yang–Baxter equation shows up through the same techniques, and actually appeared for the first time in this context. When open boundaries are imposed, the procedure needs to be modified but still applies [6, 7], even when the boundaries...
are not diagonal [8]. The Hubbard model [9] is another example where the CBA has been used successfully [10]. We also remark that, depending on the context, different generalizations of the CBA have been considered, see e.g. [11–14].

However, this method was considered to be deprecated in the eighties in favour of the quantum inverse scattering method (QISM) [15, 16]. This latter method is more algebraic and provides a full set of commuting Hamiltonians. It lies on the explicit numerical solution of the Yang–Baxter equation and on representations of the underlying algebra. Nevertheless, to the best of our knowledge, one does not know, in general, if the set of models solved by the QISM and those solved by the CBA are equivalent.

In this paper, we classify the most general Hamiltonians with nearest-neighbour interaction and acting on a three-state spin chain with rank 1 symmetry (i.e. the 19-vertex models) that can be solved by the CBA. We also compare our results with the classification of the solutions of the Yang–Baxter equation for the 19-vertex model [17–19]. To solve this type of model by the CBA, the historical method must be generalized following the lines of [11, 20] where particular 19-vertex models (the Izergin–Korepin and the Zamolodchikov–Fateev (ZF) models) were investigated, or of [13] where higher spin chains have been solved. This type of computation has been initiated in [21] but the huge algebraic equations the authors obtained did not allow them to provide a classification. Here, with the use of formal mathematical software, we succeed in obtaining a complete classification. We recover as subcases all the models discovered by solving the Yang–Baxter equation and, as an important by-product, we get the eigenvalues and eigenfunctions, which were not previously known, for the models obtained in [19] (see section 4.1). We also obtain four 17-vertex models, one of them being a generalization of the special branch found in [19], and two new 14-vertex models. These 17-vertex and 14-vertex Hamiltonians are not subcases of 19-vertex Hamiltonians.

The paper is organized as follows. In section 2, we present the general Hamiltonian we want to solve and the symmetries one may consider. We give the outlines of the CBA in section 3: we derive the Bethe ansatz equations (BAE) and determine sets of constraint equations to be satisfied by the parameters entering the Hamiltonian. Our results are collected in the proposition 3.1. In section 4, we provide the complete classification of the Hamiltonians we can solve by CBA and give in each case the eigenvalues and the eigenfunctions. Finally, in section 5, we present simplified versions of the Hamiltonians, including explicit $9 \times 9$ matricial expressions with physically relevant parameters, and, where possible, connect them with known models.

In association with this classification, we constructed an interactive web page [22] that can test any 19-vertex Hamiltonian to determine if it is solvable by the CBA. If so, it also provides the connection with the models we present in this paper, as well as the physical data of the model.

2. General settings

2.1. Hamiltonian

We consider a $U(1)$-invariant Hamiltonian $H$ acting on a spin chain of length $L$, where each site carries a $\mathbb{C}^3$ vector space (i.e. we deal with three-state models). We assume nearest-neighbour interactions, that is

$$H = \sum_{j=1}^{L} H_{j,j+1}, \quad (2.1)$$

2 See remark 4.2 for the name ‘19-vertex’.
Under these requirements, the most general two-site Hamiltonian takes the form

\[ H_{12} = \sum_{i,j} h_{i,j} E_{i,j} \otimes E_{i,j} \]

\[ = pE_{00} \otimes E_{10} + qE_{10} \otimes E_{01} + t_1 E_{21} \otimes E_{01} + s_1 E_{12} \otimes E_{10} + t_2 E_{01} \otimes E_{21} + s_2 E_{10} \otimes E_{12} \]

\[ + t_3 E_{12} \otimes E_{21} + s_3 E_{21} \otimes E_{12} + t_4 E_{02} \otimes E_{20} + s_4 E_{20} \otimes E_{02} + \sum_{i,j} v_i E_{ii} \otimes E_{jj}, \]  

where \( E_{ij} \) denote the elementary \( 3 \times 3 \) matrices with entry 1 in position \((i, j)\) and zero elsewhere. In matricial form, it reads

\[ H_{12} = \begin{pmatrix}
  v_{00} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & v_{01} & 0 & p & 0 & 0 & 0 & 0 \\
  0 & 0 & v_{02} & 0 & t_2 & 0 & t_p & 0 \\
  0 & q & 0 & v_{10} & 0 & 0 & 0 & 0 \\
  0 & 0 & s_2 & 0 & v_{11} & 0 & s_1 & 0 \\
  0 & 0 & 0 & 0 & v_{12} & 0 & t_5 & 0 \\
  0 & 0 & s_p & 0 & t_1 & 0 & v_{20} & 0 \\
  0 & 0 & 0 & 0 & s_3 & 0 & v_{21} & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & v_{22} & 0
\end{pmatrix}. \]  

We aim at finding the most general Hamiltonian of the form (2.4) that is solvable by a generalized CBA. This will lead to an exhaustive classification of the possible sets of constraints on the parameters entering \( H \); see section 4.

Before performing this calculation, we use the symmetries of the problem to keep only physically relevant parameters.

### 2.2. Symmetries and transformations

The Hamiltonian (2.4) exhibits some symmetries that allow us to simplify it.

- **Telescopic terms.** For any matrix \( A \), let us consider the following transformation of the local Hamiltonian:

\[ H'_{j,j+1} = H_{j,j+1} + A_j - A_{j+1}. \]  

Then the periodicity condition implies that

\[ H' = \sum_{j=1}^{L} H'_{j,j+1} = \sum_{j=1}^{L} H_{j,j+1} = H. \]  

Demanding the \( U(1) \) invariance to be preserved forces the matrix \( A \) to be diagonal: \( A = \text{diag}(a_1, a_2, a_3) \).

\footnote{Strictly speaking, the spin is \( L - S' \).}
The transformation (2.5) for the diagonal matrix, which involves only two independent parameters, say \(a_1 - a_2\) and \(a_1 - a_3\), leads to the following invariant combinations of the parameters:

\[
V = v_{01} + v_{10} - 2v_{00}, \quad X_{11} = v_{11} - v_{00} - V, \quad Y = v_{02} + v_{20} - 2v_{00} - 2V, \\
X_{12} = v_{12} + v_{20} - v_{10} - v_{00} - 2V, \quad X_{21} = v_{21} + v_{02} - v_{01} - v_{00} - 2V, \\
X_{22} = v_{22} - v_{00} - 2V.
\] (2.7)

Note that this choice is not unique: in fact, the combinations above appear naturally when dealing with the CBA; see section 3.

- As already mentioned, we are considering \(U(1)\)-invariant Hamiltonians. This implies in particular that the entries of \(H_{ij}\) satisfy \(h_{i_1 j_2}^{(2)} = 0 \) if \(i_1 + i_2 \neq j_1 + j_2\), as can be checked from equation (2.4). In order to get a symmetry of rank 1 only, one has to impose \((t_1, t_2, s_1, s_2) \neq (0, 0, 0, 0)\), a condition that we assume to be satisfied in the whole paper. Indeed, the rank of the symmetry algebra determines the form of the CBA one should use. Hence, it is necessary to fix this rank. The only diagonal generators that commute with the Hamiltonian are then the identity matrix \(I\) and the \(S^z\) component of the total spin given in section 2.1. This property can be used to set the zero of the energy, for example. A particularly interesting choice is to consider \(H_{12} = \frac{1}{2} V(s_1^z + s_2^z)\).

Of course, a further diagonal element can be removed from the Hamiltonian using the identity. In the following we choose \(v_{00} = 0\).

One can then consider the following transformations.

- **Parity transformation (P).** \(h_{i_1 j_2}^{(2)} \rightarrow h_{i_1 j_2}^{(2)}\) (that is \(H_{12} \rightarrow H_{21}\)), which corresponds to the following correspondence between the parameters \((X_{11}, Y, X_{22})\) are invariant:

\[
p \leftrightarrow q, \quad t_1 \leftrightarrow t_2, \quad s_1 \leftrightarrow s_2, \quad t_3 \leftrightarrow s_3, \quad t_p \leftrightarrow s_p, \quad X_{12} \leftrightarrow X_{21}.
\] (2.8)

The Hamiltonians \(H_{12}\) and \(H_{21}\) lead to systems where the chain is oriented from right to left instead of left to right. Therefore, the set of solutions that lead to a solvable Hamiltonian has to be invariant under the correspondence (2.8).

- **Time reversal (T).** \(h_{i_1 j_2}^{(2)} \rightarrow h_{i_1 j_2}^{(2)}\) (that is \(H_{12} \rightarrow H_{12}'\)), which corresponds to the following correspondence between the parameters (all diagonal terms are invariant):

\[
p \leftrightarrow q, \quad t_1 \leftrightarrow t_1, \quad t_2 \leftrightarrow s_2, \quad t_3 \leftrightarrow s_3, \quad t_p \leftrightarrow s_p.
\] (2.9)

- **Charge conjugation (C).** \(h_{i_1 j_2}^{(2)} \rightarrow h_{i_1 j_2}^{(2)}\) (i.e. indices 0 and 2 are exchanged and index 1 is invariant), which corresponds to the following correspondence between the parameters:

\[
p \leftrightarrow s_3, \quad q \leftrightarrow t_3, \quad t_1 \leftrightarrow t_2, \quad s_1 \leftrightarrow s_2, \quad t_p \leftrightarrow s_p,
\]

\[
V \leftrightarrow -V - Y - 2X_{22} + X_{12} + X_{21}, \quad X_{11} \leftrightarrow X_{11} + Y + X_{22} - X_{12} - X_{21},
\]

\[
Y + X_{22} \leftrightarrow 5(Y + X_{22}) - 4(X_{12} + X_{21}), \quad Y - X_{22} \leftrightarrow Y - X_{22},
\]

\[
X_{12} + X_{21} \leftrightarrow 6(Y + X_{22}) - 5(X_{12} + X_{21}), \quad X_{12} - X_{21} \leftrightarrow X_{21} - X_{12}.
\] (2.10)

The action of the charge conjugation is equivalent to choosing as pseudo-vacuum \(|\Omega\rangle = \bigotimes_{i=1}^{N} |2\rangle\) instead of \(|\Omega\rangle = \bigotimes_{i=1}^{N} |0\rangle\), exchanging the roles of the vectors \(|0\rangle\) and \(|2\rangle\). The solution to the problem obtained thanks to the CBA can be reproduced \(mutatis\ mutandis\), but by taking into account the correspondence (2.10). We will use this property in section 4.3.

The action of these three transformations on solvable Hamiltonians is displayed in table A1; see the appendix. In the following we will work modulo these transformations.
3. The coordinate Bethe ansatz

We construct Hamiltonian eigenvectors using a generalization of the original CBA, following the techniques developed in [11, 13].

3.1. Results

Since the $S_z$ component of the total spin commutes with the Hamiltonian, one can decompose the space of states $\mathcal{H}$ into subspaces with a fixed $S_z$-eigenvalue

$$\mathcal{H} = \bigoplus_{M=0}^{2L} \mathcal{V}_M, \quad S_z \varphi_M = M \varphi_M, \quad \forall \varphi_M \in \mathcal{V}_M,$$

and look for eigenvectors of $H$ in a given subspace $\mathcal{V}_M$.

For $M = 0$, we have a one-dimensional subspace corresponding to a particular eigenvector of the Hamiltonian, called the pseudo-vacuum, defined here as $|\Omega\rangle = \bigotimes_{i=1}^{L} |0\rangle$. It is easy to see that, since we made the choice $v_{00} = 0$, $|\Omega\rangle$ is an eigenvector of $H$ with eigenvalue zero.

Then, in $\mathcal{V}_M$, one considers states with $M$ pseudo-excitations obtained by acting with the raising operator on the pseudo-vacuum. More precisely, an elementary state with $M$ pseudo-excitations is given by

$$|x_1, \ldots, x_M\rangle = |0\rangle \otimes \cdots \otimes |0\rangle \otimes |m_1\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle \otimes |m_2\rangle \otimes \cdots \otimes |0\rangle \otimes \cdots$$

$$\otimes |0\rangle \otimes \cdots \otimes |0\rangle \otimes |m_3\rangle \otimes \cdots \otimes |0\rangle \otimes \cdots,$$

where $1 \leq x_1 \leq x_2 \leq \cdots \leq x_M \leq L$.

The $x_i$ are the locations of the pseudo-excitations along the chain, and $m_k \in \{1, 2\}$ such that $\sum m_k = M$. For $j = 1 + m_1 + \cdots + m_{k-1}$, one has $m_k = 2$ if $x_{j+1} = x_j$, and $m_k = 1$ otherwise.

These states form the basis of the subspace $\mathcal{V}_M$ of states with a given number $M$ of pseudo-excitations.

An eigenstate $\Psi_M$ for the Hamiltonian in a given sector with $M$ pseudo-excitations is obtained as a linear combination of the elementary states (3.1) with coefficients $a(x_1, \ldots, x_M)$, which are complex-valued functions to be determined:

$$\Psi_M = \sum_{1 \leq x_1 \leq \ldots \leq x_M \leq L} a(x_1, \ldots, x_M)|x_1, \ldots, x_M\rangle.$$

We assume a plane wave decomposition for the functions $a(x_1, \ldots, x_M)$:

$$a(x_1, \ldots, x_M) = \sum_{\sigma \in \mathfrak{S}_M} A_\sigma^{(j_1, \ldots, j_P)}(k_1, \ldots, k_M) \exp \left( \sum_{n=1}^{M} ik_\sigma(n)x_n \right)$$

$$= \sum_{\sigma \in \mathfrak{S}_M} A_\sigma^{(j_1, \ldots, j_P)}(k) \omega_k^{\sigma(x)},$$

where $\mathfrak{S}_M$ is the permutation group of $M$ elements and $A_\sigma^{(j_1, \ldots, j_P)}(k_1, \ldots, k_M)$ are functions on the symmetric group algebra depending on some parameters $k$ to be determined later (these are solutions of the so-called BAE). The indices $(j_1, \ldots, j_P)$ correspond to double excitations, i.e. indices such that $x_{j_{k+1}} = x_j$ for $k = 1, \ldots, P$. When there are no double excitations, the indices $(j_1, \ldots, j_P)$ are of course omitted.
Proposition 3.1. The Hamiltonian $H$ given in (2.4) is solvable by the CBA provided its parameters obey the constraints given in (3.18), (3.21) and (3.24). The complete set of solutions to these equations is given in section 4, modulo the symmetries mentioned in section 2.2 and displayed in table A1.

For CBA-solvable Hamiltonians $H$, the state (3.2) is an eigenvector of $H$, with energy

$$E_M = MV + \sum_{n=1}^{M} (q e^{i k_n} + p e^{-i k_n})$$

provided the coefficients $A_n$ and $A^{(j_1, \ldots, j_p)}_n$, $P = 1, \ldots, \frac{M}{2}$, are related by

$$\frac{A_{\sigma}^{(j_1, \ldots, j_p)}(\vec{k})}{A_{\sigma}(\vec{k})} = S(k_{\sigma(j)}, k_{\sigma(j+1)}) = \frac{-\Lambda(k_{\sigma(j)}, k_{\sigma(j+1)})}{\Lambda(k_{\sigma(j+1)}, k_{\sigma(j)})},$$

$$A_{\sigma}^{(j_1, \ldots, j_p)}(\vec{k}) = \prod_{n=1}^{P} N(k_{\sigma(j_n)}, k_{\sigma(j_n+1)}),$$

where $T_j \in \Theta_M$ denotes the transposition $(j, j + 1)$ and

$$\Lambda(k_j, k_n) = e^{i k_j (s_1 + s_2 e^{i j \tau + i k_n}) (t_2 + t_1 e^{i j \tau + i k_n})} - (Y e^{i k_j + i k_n} - q e^{i k_j + i k_n} (e^{i j \tau} + e^{i k_j}) - p (e^{i j \tau} + e^{i k_n}) + s_p e^{2i j \tau + 2i k_n} + t_p) \times (X_{11} e^{i k_j} - q e^{i k_j + i k_n} - p).$$

The momenta $k_j$ must also obey the Bethe equations

$$e^{i k_j} = \prod_{n \neq j} S(k_n, k_j), \quad j = 1, \ldots, M.$$  

We remark that when $p = q = 0$, the energy depends only on the number of pseudo-excitations. In this case, one needs to consider another vacuum to get a complete spectrum; see section 4.3. When $p$ and $q$ are both non-vanishing, the energy can be rewritten as

$$E_M = MV + \sqrt{pq} \sum_{n=1}^{M} \left( z_n \sqrt{\theta} + \frac{1}{z_n \sqrt{\theta}} \right) \quad \text{where } \theta = q/p \quad \text{and } \quad z_n = e^{i k_n}.$$

In this case, after eliminating the constant term $MV$ thanks to the $S_z$ operator and rescaling of the Hamiltonian by $\sqrt{pq}$, the energy clearly depends only on the parameter $\theta$ (and those of the $S$ matrix through the Bethe equations).

3.2. Proofs

Since $H$ is a sum of two-site operators $H_{j, j+1}$, one has to single out only the following configurations:

1. **Configurations leading to the determination of eigenvalues and eigenvectors:**
   - the $x_j$ are far from each other (‘generic case’),
   - $x_{j+1} = x_j + 1$ for some $j$ and the other $x_n$ are far from each other,
   - $x_{j+1} = x_j$ for some $j$ and the other $x_n$ are far from each other,
   - $x_{j+1} = x_j$, $x_n + 1 < x_{j+2}$ for $k = 1, \ldots, P$, and the other $x_n$ are far from each other.
2. Configurations leading to constraints on the parameters of the models:
- \( x_{j+1} = x_j \) and \( x_{j+2} = x_j + 1 \) for a given \( j \), the other \( x_n \) are far from each other,
- \( x_{j+1} = x_j \) and \( x_{j-1} = x_j - 1 \) for a given \( j \), the other \( x_n \) are far from each other,
- \( x_{j} = x_j \) and \( x_{j+1} = x_{j+2} = x_j + 1 \), the other \( x_n \) are far from each other.

3. Configurations leading to Bethe equations and/or already known equations:
- \( x_1 = 1 \) and the other \( x_n \) are far from each other,
- \( x_M = L \) and the other \( x_n \) are far from each other,
- \( x_1 = 1, x_M = L \) and the other \( x_n \) are far from each other,
- \( x_1 = x_2 = 1 \) (or equivalently \( x_{M-1} = x_M = L \) and the other \( x_n \) are far from each other,
- \( x_1 = x_2 = 1 \) and \( x_M = L \) (or equivalently \( x_1 = 1 \) and \( x_{M-1} = x_M = L \)), and the other \( x_n \) are far from each other,
- \( x_1 = x_2 = 1 \) and \( x_{M-1} = x_M = L \), and the other \( x_n \) are far from each other.

3.2.1. Configurations leading to energy and eigenstates.
- **Configuration where the \( x_j \) are generic**, i.e. are far from each other and from the edges: \( 1 < x_1 < \cdots < x_n < x_{n+1} < \cdots < x_M < L \). Projecting the Schrödinger equation on it, we get
  \[
  \sum_{\sigma \in \mathcal{O}_M} A_\sigma(\vec{k}) e^{i\vec{k} \cdot \vec{x}} \left( MV + \sum_{n=1}^{M} (q e^{ik_n} + p e^{-ik_{n-1}}) \right) = E_M \sum_{\sigma \in \mathcal{O}_M} A_\sigma(\vec{k}) e^{i\vec{k} \cdot \vec{x}}
  \]
  (3.11)
  which leads to the value (3.4) of the energy of the state \( \Psi_M \).
- **Configuration where \( x_{j+1} = x_j + 1 \) for a given \( j \) (not on the edges)**, the other \( x_n \) being far from each other and from the edges. Then one gets
  \[
  \sum_{\sigma \in \mathcal{O}_M} e^{i\vec{k} \cdot \vec{x}} (A_\sigma(\vec{k})) (X_{11} - q e^{ik_{j+1}} - p e^{-ik_{j+1}}) + A^j_\sigma(\vec{k}) (s_2 e^{ik_{j+1}} + s_1 e^{-ik_{j+1}}) = 0.
  \]
  (3.12)
  Note that, since \( x_{j+1} = x_j + 1 \), one has here
  \[
  e^{i\vec{k} \cdot \vec{x}} = e^{i\vec{k} \cdot (j+1)} \exp \left( i (k_{\sigma(j)} + k_{\sigma(j+1)}) x_j + \sum_{\sigma \not \in \{j,j+1\}} i k_{\sigma(n)} x_n \right),
  \]
  which implies a symmetrization in the exchange \( j \leftrightarrow j+1 \) before projecting onto the independent states (3.1). Hence one gets, where \( T_j \in \mathcal{O}_M \) denotes the transposition \( (j, j+1) \),
  \[
  A_\sigma(\vec{k}) e^{i\vec{k} \cdot (j+1)} (X_{11} - q e^{ik_{j+1}} - p e^{-ik_{j+1}}) + A_{\sigma T_j}(\vec{k}) e^{i\vec{k} \cdot j} (X_{11} - q e^{ik_{j+1}} - p e^{-ik_{j+1}}) \\
  + (A^j_\sigma(\vec{k}) + A^j_{\sigma T_j}(\vec{k}))(s_2 e^{ik_{j+1}} + s_1 e^{-ik_{j+1}}) = 0.
  \]
  (3.13)
- **Configuration where \( x_{j+1} = x_j \) for a given \( j \) (not on the edges)**, the other \( x_n \) being far from each other and from the edges. Then one has
  \[
  \sum_{\sigma \in \mathcal{O}_M} e^{i\vec{k} \cdot \vec{x}} (A^j_\sigma(\vec{k}) (Y - q e^{ik_{j+1}} - q e^{ik_{j+1}} - p e^{-ik_{j+1}} - p e^{-ik_{j+1}}) + s_2 e^{ik_{j+1}} + s_1 e^{-ik_{j+1}}) \]
  \[
  + t_p e^{-ik_{j+1} - ik_{j+1}} + A_\sigma(\vec{k})(t_1 e^{ik_{j+1}} + t_2 e^{-ik_{j+1}}) = 0.
  \]
  (3.14)
3.2.2 Configurations leading to constraints on parameters.

After symmetrization in \((j, j+1)\) as above, one obtains

\[
(A^{(j)}_\sigma (\vec{k}^+ + A^{(j)}_\sigma (\vec{k}^-))(Y - q e^{ikx_{1j}} - q e^{i\bar{k}x_{1j}} - p e^{-i\bar{k}x_{1j}} - p e^{-ikx_{1j}} + s_p e^{ikx_{1j}} + t_p e^{-i\bar{k}x_{1j}} + A_\sigma (\vec{k}^+ t_1 e^{ikx_{1j}} + t_2 e^{-i\bar{k}x_{1j}}) + A_\sigma (\vec{k}^-)(t_1 e^{ikx_{1j}} + t_2 e^{-i\bar{k}x_{1j}}) = 0.
\] (3.15)

Without any loss of generality, we choose\(^6\) to impose \(A^{(j)}_\sigma (\vec{k}^-) = A^{(j)}_\sigma (\vec{k}^+)\). Then, using equations (3.13) and (3.15), one gets the relations (3.5) and (3.6).

- **Configuration where** \(x_{\hat{k},+1} = x_{\hat{k},\hat{k}}\) for \(k = 1, \ldots, P\) and the other \(x_n\) being far from each other and from the edges. One gets

\[
\sum_{\vec{k} \in \mathcal{H}} e^{i\vec{k} \cdot \vec{r}} \left\{ A^{(j_1, \ldots, j_p)}_\sigma (\vec{k}) \left( PY + \sum_{k=1}^{P} t_p e^{-ikx_{1j} - ikx_{1j+1}} - p e^{-i\bar{k}x_{1j}} - p e^{-ikx_{1j+1}} \right) + s_p e^{ikx_{1j}} + t_p e^{-i\bar{k}x_{1j}} + A^{(j_1, \ldots, j_p)}_\sigma \left( t_1 e^{ikx_{1j}} + t_2 e^{-i\bar{k}x_{1j}} \right) \right\} = 0
\] (3.16)

where \(A^{(j_1, \ldots, j_p)}_\sigma (\vec{k})\) means that \(x_{\hat{k},+1} = x_{\hat{k},\hat{k}}\) for \(n = 1, \ldots, P\) and \(n \neq k\).

Moreover, after projection onto the states (3.1), one needs to symmetrize (independently) on each pair \((j_n, j_n + 1)\). One is led to a recursion relation linking \(A^{(j_1, \ldots, j_p)}_\sigma (\vec{k})\) and \(A^{(j_1, \ldots, j_p-1)}_\sigma (\vec{k})\) that can be solved, and one gets (3.7).

3.2.2. Configurations leading to constraints on parameters.

- **Configuration where** \(x_{j+1} = x_{j}\) and \(x_{j+2} = x_{j+1}\) for a given \(j\) (not on the edges), the other \(x_n\) being far from each other and from the edges. One obtains

\[
\sum_{\vec{k} \in \mathcal{H}} e^{i\vec{k} \cdot \vec{r}} \left( A^{(j)}_\sigma (\vec{k})(X_{21} - q e^{ikx_{1j}} - q e^{i\bar{k}x_{1j}} - p e^{-i\bar{k}x_{1j}} - p e^{-ikx_{1j}} + s_p e^{ikx_{1j}} + t_p e^{-i\bar{k}x_{1j}} + A^{(j+1)}_\sigma (\vec{k})x_3 e^{ikx_{1j+1}} + A^{(j+1)}_\sigma (\vec{k})x_2 e^{-i\bar{k}x_{1j}} = 0.
\] (3.17)

Here one has \(e^{i\vec{k} \cdot \vec{r}} = e^{ikx_{1j}} \exp \left( \sum_{n,j,j+1,j+2} \chi_{\sigma(n)} x_{n} + \sum_{n,j,j+1,j+2} \chi_{\sigma(n)} x_{n} \right)\) given the configuration. Therefore, projecting onto the states (3.1), it is now necessary to symmetrize on \((j, j+1, j+2)\). Taking into account the relations (3.5) and (3.6) that allow one to express all \(A\) functions in terms of \(A_\sigma (\vec{k})\) only, one gets now

\[
\sum_{\vec{k} \in \mathcal{H}} \mathcal{E}_{21}(k_\sigma(j), k_\sigma(j+1), k_\sigma(j+2)) = 0,
\] (3.18)

where

\[
\mathcal{E}_{21}(k_\sigma(j), k_\sigma(j+1), k_\sigma(j+2)) = A_\sigma (\vec{k}) e^{ikx_{1j}} \left( N(k_\sigma(j), k_\sigma(j+1))(X_{21} - q e^{ikx_{1j}} - q e^{i\bar{k}x_{1j}} - p e^{-i\bar{k}x_{1j}} - p e^{-ikx_{1j}} + s_p e^{ikx_{1j}} + t_p e^{-i\bar{k}x_{1j}} + A^{(j+1)}_\sigma (\vec{k})x_3 e^{ikx_{1j+1}} + A^{(j+1)}_\sigma (\vec{k})x_2 e^{-i\bar{k}x_{1j}} \right) = 0.
\] (3.19)

Then projecting the above constraint onto the monomials in the variables \(e^{ikx_{1j}}, \ell = j, j+1, j+2\), one gets a first set of 32 constraint equations. For the sake of simplicity, we avoid writing these equations here.

\(^6\) Another possible choice \([21]\) is to impose \(A^{(j)}_\sigma (\vec{k}) = S(k_\sigma(j), k_\sigma(j+1) A^{(j)}_\sigma (\vec{k})\). One goes from one choice to another through the renormalization \(A^{(j)}_\sigma (\vec{k}) \rightarrow (k_\sigma(j) - k_\sigma(j+1)\Lambda(k_\sigma(j), k_\sigma(j+1) A^{(j)}_\sigma (\vec{k})\).
The projection of the constraint equation onto the monomials in the variables $e^ {i k_{s}(i)}$, $\ell = j - 1$, $j$, $j + 1$ leads to a second set of 32 constraint equations.

**Configuration where $x_{j-1} = x_{j}$ and $x_{j+1} = x_{j} + 1$, the other $x_{\sigma}$ being far from each other and from the edges.**

One gets

$$
\sum_{\sigma \in \Phi_4} C^{j-1,j,j+1}_\sigma \left( \sum_{m \neq j-1,j,j+1} t_x e^{- i k_{s}(i)-i k_{s}(j)} - q e^{i k_{s}(i)} - q e^{i k_{s}(j)} - q e^{i k_{s}(j+1)} - q e^{i k_{s}(j+2)} \\
+ s_p e^{i k_{s}(j+1)+i k_{s}(j+2)} - p e^{- i k_{s}(i)} - p e^{- i k_{s}(j)} - p e^{- i k_{s}(j+1)} - p e^{- i k_{s}(j+2)} \\
+ A^{j+1}_\sigma (\bar{k}_1 t_x e^{- i k_{s}(j+1)} + A^{j+1}_\sigma (\bar{k}_1 t_x e^{i k_{s}(j+1)}) = 0.
\right)
$$

(3.23)

Since now $e^{k_{s}(i)} = e^{i k_{s}(i)+i k_{s}(j)} e^{i k_{s}(j)+i k_{s}(j+1)} e^{i k_{s}(j+1)+i k_{s}(j+2)} e^{i k_{s}(j+2)+i k_{s}(j+3)}$, one symmetrizes on $(j-1,j,j+1,j+2)$, and gets

$$
\sum_{\sigma \in \Phi_4} E_{22}(k_{s}(j-1), k_{s}(j), k_{s}(j+1), k_{s}(j+2)) = 0,
$$

(3.24)

where

$$
E_{22} = A_\sigma (\bar{k}_1 t_x e^{i k_{s}(j+1)+i k_{s}(j+2)} (N(k_{s}(j-1), k_{s}(j)) N(k_{s}(j+1), k_{s}(j+2)) \\
\times (X_{22} + t_x e^{- i k_{s}(j)-i k_{s}(j+1)} - q e^{i k_{s}(j)} - q e^{i k_{s}(j+1)} - q e^{i k_{s}(j+2)} - q e^{i k_{s}(j+3)} \\
+ s_p e^{i k_{s}(j+1)+i k_{s}(j+2)} - p e^{- i k_{s}(i)} - p e^{- i k_{s}(j)} - p e^{- i k_{s}(j+1)} - p e^{- i k_{s}(j+2)} \\
+ A^{j+1}_\sigma (\bar{k}_1 t_x e^{- i k_{s}(j+1)} + A^{j+1}_\sigma (\bar{k}_1 t_x e^{i k_{s}(j+1)}) e^{i k_{s}(j+2)}).
$$

(3.25)

The projection of the constraint equation onto the monomials in the variables $e^{i k_{s}(i)}$, $\ell = j - 1$, $j$, $j + 1$, $j + 2$ finally leads to a third set of constraint equations.

The solutions to the sets of equations (3.18), (3.21) and (3.24) give the necessary conditions to be satisfied among the parameters defining the two-site Hamiltonian (2.4) to ensure the solvability of the chain. This leads to a classification of three-state integrable models as shown in the next section.
3.2.3 Configurations leading to the Bethe equations. We now concentrate on configurations with at least one excitation lying on the chain edges 1 and/or \( k \). Using the periodicity condition of the chain, this will allow us to derive the equations that determine the admissible values of the parameters \( k \) entering into the definition of the plane wave (3.3), i.e. the BAE.

**Configuration where \( x_1 = 1 \) and the other \( x_n \) are far from each other and from the edges:**

\[
1 = x_1 \ll \cdots \ll x_n \ll x_{n+1} \ll \cdots \ll x_M < L. \quad \text{Then one gets, provided that} \quad p \neq 0,
\]

\[
\sum_{\sigma \in \mathcal{S}_M} A_\sigma (\vec{k}) \left( \exp \left( \sum_{n=2}^{M} i k_{\sigma(n)} x_n \right) - \exp \left( i k_{\sigma(1)} L + \sum_{n=1}^{M} i k_{\sigma(n-1)} x_n \right) \right) = 0. \tag{3.26}
\]

Performing the transformation \( \sigma \rightarrow \sigma T_1 \ldots T_{M-1} \) in the second term, one gets

\[
A_\sigma (\vec{k}) = e^{-ik_{\sigma(M)} L} A_\sigma (\vec{k}). \tag{3.27}
\]

Now, performing the transformation \( \sigma \rightarrow \sigma T_{M-1} \ldots T_1 \) in the second term, one gets

\[
A_\sigma (\vec{k}) = e^{ik_{\sigma(1)} } A_\sigma (\vec{k}),
\]

which also leads to equation (3.9).

Note that since we excluded the values \( p = q = 0 \), the BAE (3.9) holds in any case.

**Configuration where \( x_1 = 1 \), \( x_M = L \) and the other \( x_n \) are far from each other:**

\[
1 = x_1 \ll \cdots \ll x_n \ll x_{n+1} \ll \cdots \ll x_M = L. \quad \text{One obtains}
\]

\[
\sum_{\sigma \in \mathcal{S}_M} A_\sigma (\vec{k}) \left( \exp \left( i k_{\sigma(1)} + i k_{\sigma(M)} L + \sum_{n=2}^{M-1} i k_{\sigma(n)} x_n \right) (X_1 - q e^{ik_{\sigma(M)} } - p e^{-ik_{\sigma(1)} }) \right)

+ A_\sigma^{(M-1)} (\vec{k}) \left( \exp \left( i k_{\sigma(M-1)} L + i k_{\sigma(M)} L + \sum_{n=2}^{M-1} i k_{\sigma(n-1)} x_n \right) \right)

+ A_\sigma^{(1)} (\vec{k}) \left( \exp \left( i k_{\sigma(1)} + i k_{\sigma(2)} + \sum_{n=2}^{M-1} i k_{\sigma(n+1)} x_n \right) \right) = 0. \tag{3.28}
\]

One then performs the transformations \( \sigma \rightarrow \sigma T_1 \ldots T_{M-1} \) (second term) and \( \sigma \rightarrow \sigma T_{M-1} \ldots T_1 \) (third term) and uses the relations (3.5) and (3.6). After the necessary symmetrization on the pair \( (1, M) \) and projection onto the states (3.1), one is left with an equation expressed in terms of \( A_{\sigma} (\vec{k}) \) only, the \( S \) matrix and the decay coefficient \( N \). Plugging the BAE (3.9) into the obtained equation, it appears that no further condition is required.

**Other ‘edge configurations’.** They correspond to the following cases:

(i) \( x_1 = x_2 = 1 \) (or equivalently \( x_{M-1} = x_M = L \)),

(ii) \( x_1 = x_2 = 1 \) and \( x_M = L \) (or equivalently \( x_1 = 1 \) and \( x_{M-1} = x_M = L \)),

(iii) \( x_1 = x_2 = 1 \) and \( x_{M-1} = x_M = L \), and the other \( x_n \) are far from each other.
The approach is similar to the previous case. After obtaining the Schrödinger equation for the particular configuration under consideration using the periodicity conditions, one performs the suitable transformations on the permutations and writes all functions $A(k)$ in terms of the running $A_r(k)$ only, products of $S$-matrices and decay coefficients $N$. If necessary, one symmetrizes on the indices which are left after the projection on the elementary states (3.1). In each case, plugging the BAE (3.9) into the equation that is finally obtained leads to some constraint equation belonging to one of the sets (3.18), (3.21) or (3.24). No further condition is eventually needed.

4. Solutions of the constraint equations

In this section, we present all the non-trivial solutions to equations (3.18), (3.21) and (3.24), described in section 3. It provides a classification of three-state models solvable by the CBA. We used formal calculation software to solve completely these equations, and found four 19-vertex, four 17-vertex and two 14-vertex models, up to the transformations under parity, time reversal and charge conjugation; see section 2.2. If one includes the images of the irreducible solutions under these transformations, one gets 22 solutions; see table A1.

Remark 4.1. Of course, when directly solving the equations (3.18), (3.21) and (3.24), one finds many more solutions, but most of them are subcases of these ten ‘irreducible’ solutions. We developed software that, starting from any given Hamiltonian of the form (2.4), can analyse whether the Hamiltonian is solvable by the CBA, and, if so, to which one of the ten irreducible solutions it corresponds. This program is freely accessible on our web page [22]. Let us stress that the correspondence may be sensitive to the choice of free parameters that is used. This is taken into account by the software. However, in this paper we made (arbitrarily) one specific choice. The other ones are found through the image under parity, charge conjugation and/or time reversal transformations of the choice we present here. We illustrate this in a particular case; see section 4.1.4.

In the following, we classify the models that have $(t_1, t_2, s_1, s_2) \neq (0, 0, 0, 0)$ and $(p, q, t_3, s_3) \neq (0, 0, 0, 0)$. Because we work modulo $P, C$ and $T$ transformation, it is enough to present the solutions with $(p, q) \neq (0, 0)$ and $(t_1, t_2) \neq (0, 0)$.

(i) Since we are considering $U(1)$ invariant models, to get a symmetry of rank 1 only (not rank 2), we are led to $(t_1, t_2, s_1, s_2) \neq (0, 0, 0, 0)$. Now suppose we get a solution with $(t_1, t_2) = (0, 0)$. Then, this solution has $(s_1, s_2) \neq (0, 0)$. But the image of this solution under time reversal is also a solution and has $(t_1, t_2) \neq (0, 0)$ and $(s_1, s_2) = (0, 0)$: since we are working modulo this transformation, we can choose to present solutions with $(t_1, t_2) \neq (0, 0)$.

(ii) To be able to construct the CBA on the vacuum $|\Omega\rangle$ or $|\bar{\Omega}\rangle$, one needs to have $(p, q, t_3, s_3) \neq (0, 0, 0, 0)$. Now, since charge conjugation (2.10) exchanges $(p, q)$ and $(t_3, s_3)$, we can suppose that $(p, q) \neq (0, 0)$.

These requirements exclude all the cases obtained in [17] apart from models 7 and 10: the remaining ones are models with rank 2 symmetry, or diagonal Hamiltonians, or not solvable through the CBA. They also exclude the model based on Temperley–Lieb algebra [23], for which another type of CBA is needed [12].

We introduce the following reduced parameters:

\[ \tau_p = t_p/p, \quad \tau_2 = t_2/p, \quad \tau_3 = t_3/p, \quad \theta = q/p, \quad \Upsilon = Y/p, \quad \sigma = s_1 t_2/p^2, \quad \mu = t_1/t_2. \]

(4.1)
These reduced parameters are the only ones that are part of the physical data of the models: scattering matrix $S$, decay coefficient $N$, energy $E$ and BAES. Hence the other ones can be eliminated from the model through gauge transformations and/or telescoping terms, as is done in section 5. We chose to present here our ‘raw’ Hamiltonians to be easily compared with any given Hamiltonian.

These ‘raw’ Hamiltonians are defined whatever the values of the free parameters, provided they lead to well-defined expressions for the other parameters. The reduced parameters are valid for generic values of the free parameters and can be ill-defined for some specific values; see remark 5.1 in section 5.

We define $J$ as one solution of the equation $J^2 + J + 1 = 0$.

4.1. Nineteen vertices

In this subsection, we focus on the solutions for which all off-diagonal parameters entering in the Hamiltonian are non-zero.

**Remark 4.2.** We will call the corresponding Hamiltonian a ‘19-vertex’ one, since we get 19 non-vanishing entries for $H_{12}$ when adding the nine diagonal parameters to the ten off-diagonal ones. Note however that one can always cancel some of the diagonal entries using the symmetries as discussed in section 2.2.

We would also like to stress that the name ‘19-vertex’ is not related to the terminology used for $R$ matrix formalism.

4.1.1. The generalized Zamolodchikov–Fateev model (gZF). The parameters which are left free are $p$, $t_p$, $t_2$, $s_1$. The remaining parameters entering the off-diagonal part of the Hamiltonian are given by

$$q = s_3 = \frac{p^3}{t_p}, \quad t_1 = \frac{p^2t_2}{t_p}, \quad t_3 = p, \quad s_2 = \frac{p^2s_1}{t_p}, \quad s_p = \frac{p^4}{t_p},$$

while on the diagonal we get:

$$X_{11} = 0, \quad Y = \frac{2p^2}{t_p}, \quad X_{12} = X_{21} = \frac{3p^2 - s_1t_2}{t_p}, \quad X_{22} = \frac{4p^2 - 2s_1t_2}{t_p}.$$ (4.3)

The $S$ matrix depends only on the reduced parameters $\tau_p$ and $\sigma$:

$$S(z_1, z_2) = -\frac{z_1z_2 - \tau_p(z_1 + z_2) - \sigma z_2}{z_1z_2 - \tau_p(z_1 + z_2) - \sigma z_1 + \tau_p^2}$$

and the decay coefficient $N$ reads

$$N(z_1, z_2) = \frac{\tau_2\tau_p(z_1 - z_2)}{2(z_1z_2 - \tau_p(z_1 + z_2) - \sigma z_1 + \tau_p^2)}.$$ (4.5)

**Remark 4.3.** The $PT$-invariant models of [18], branch 1A, are obtained as particular cases of this one. More precisely, setting

$$p = \frac{2k^2}{k^2 - 1}, \quad t_p = -\frac{2\epsilon_1k^2}{k^2 - 1}, \quad t_2 = s_1 = \pm e^{-\frac{\pi i}{4}(1-\epsilon_1)} \frac{2k}{k^2 - 1},$$ (4.6)

one recovers the branch 1A Hamiltonians $H_{1A}^{\epsilon_1}(k, \epsilon_1)$ of [18] which is associated with the ZF model [24].
4.1.2. The generalized Izergin–Korepin model (gIK). The parameters which are left free are $p, t_{p}, t_{2}$. The remaining parameters entering the off-diagonal part of the Hamiltonian are given by

$$s_{p} = v^{4} \frac{p^{4}}{t_{p}^{2}}, \quad s_{3} = q = v^{4} \frac{p^{3}}{t_{p}^{2}}, \quad t_{1} = p, \quad t_{1} = u_{\pm}^{-1} \frac{p^{2} t_{2}}{t_{p}^{2}}$$  \hspace{1cm} (4.7)

$$s_{1} = v(v - 1) \frac{p^{2}}{t_{2}}, \quad s_{2} = u_{\pm}^{-1} v(v - 1) \frac{p^{2}}{t_{2} t_{p}^{2}}$$  \hspace{1cm} (4.8)

while on the diagonal we get:

$$X_{11} = v(v + 1) \frac{p^{2}}{t_{p}} \quad \text{and} \quad Y = \frac{v^{2} + 1}{t_{p}} \quad \text{for} \quad X_{22} = 2(v + 1) \frac{p^{2}}{t_{p}}$$  \hspace{1cm} (4.9)

$$X_{12} = (v^{2} + 1 - u_{\pm}^{-1}) \frac{p^{2}}{t_{p}}, \quad X_{21} = (v^{2} + 1 - u_{\pm}^{-1}) \frac{p^{2}}{t_{p}},$$  \hspace{1cm} (4.10)

where $v$ is a free parameter and $u_{\pm}$ are the two solutions of the equation

$$v^{4} Z^{2} + (1 + 2v - v^{2}) Z + 1 = 0.$$  \hspace{1cm} (4.11)

The $S$ matrix depends only on the reduced parameter $\tau_{p}$ and $v$:

$$S(z_{1}, z_{2}) = \frac{(v^{2} z_{1} z_{2} - \tau_{p}(z_{1} + v z_{2}) + \tau_{p}^{2})}{(v^{2} z_{1} z_{2} - \tau_{p}(z_{1} + v z_{1}) + \tau_{p}^{2})} \frac{(v^{2} z_{1} z_{2} - \tau_{p}(1 + v) z_{2} + \tau_{p}^{2})}{(v^{2} z_{1} z_{2} - \tau_{p}(1 + v) z_{1} + \tau_{p}^{2})}$$  \hspace{1cm} (4.12)

and the decay coefficient $N$ reads

$$N(z_{1}, z_{2}) = \frac{\tau_{2} \tau_{p}(z_{1} - z_{2})(u_{\pm}^{-1} z_{1} z_{2} + \tau_{p}^{2})}{2(v^{2} z_{1} z_{2} - \tau_{p}(z_{1} + v z_{1}) + \tau_{p}^{2})} \frac{(v^{2} z_{1} z_{2} - \tau_{p}(1 + v) z_{1} + \tau_{p}^{2})}{(v^{2} z_{1} z_{2} - \tau_{p}(1 + v) z_{2} + \tau_{p}^{2})}.$$  \hspace{1cm} (4.13)

Remark 4.4. The $\mathbb{P}\mathbb{T}$-invariant models of [18], branch 2A, which are also linked to the Izergin–Korepin model [25], are obtained as particular cases of this one. More precisely, setting

$$p = \frac{2k^{2}}{k^{4} - 1}, \quad t_{p} = \frac{2k^{4}}{(k^{6} + \epsilon_{1})(k^{2} - \epsilon_{1})}, \quad t_{2} = \pm \text{e}^{-\frac{\tau}{1 - \epsilon_{1}}} \frac{2k}{k^{6} + \epsilon_{1}},$$  \hspace{1cm} (4.14)

one recovers the branch 2A Hamiltonians $H_{2A}^{\pm}(k, \epsilon_{1}, \epsilon_{2})$ of [18]. Note that each branch 2A Hamiltonian is related to the two models corresponding to the choices $u_{\pm}$ with $d = \epsilon_{1} k^{-2 \epsilon_{2}}$ or $d = \epsilon_{2} k^{2 \epsilon_{2}}$ through the following parametrization

$$v = \frac{d}{d^{2} - d + 1}, \quad u_{-} = -(d^{2} - d + 1)^{2}, \quad u_{+} = -\frac{(d^{2} - d + 1)^{2}}{d^{2}}.$$  \hspace{1cm} (4.15)

4.1.3. Generalization of the Bariev model (gB). This model appears to be a multi-parametric interpolation of three known models, one of them being the Bariev model; see remarks 4.5 and 4.6.

The parameters which are left free are $p, q, t_{1}, t_{2}, t_{p}$. The remaining parameters entering the off-diagonal part of the Hamiltonian are given by

$$s_{1} = j^{2} \frac{t_{1}^{2} - p q t_{2}^{2}}{t_{1} t_{2}^{2}}, \quad s_{2} = j^{2} \frac{t_{1}^{2} - p q t_{2}^{2}}{t_{1}^{2}}, \quad s_{3} = -j^{2} \frac{p t_{1}^{2}}{t_{2}^{2}},$$

$$t_{3} = -j^{2} \frac{q t_{2}^{2}}{t_{1}}, \quad s_{p} = j^{2} \frac{t_{1}^{2} t_{p}}{t_{2}^{2}},$$  \hspace{1cm} (4.16)
while on the diagonal we get:
\[
Y = \frac{p^2t_1^2t_2 + Jpq_1t_1^2 + J^2q_2^2t_2 - J^2t_1^2t_2^2}{t_1^2t_2^2}, \quad X_{22} = \frac{p^2t_1^2t_2 + Jpq_1t_1^2 + J^2q_2^2t_2}{t_1^2t_2^2},
\]
\[
X_{11} = \frac{Jq_1t_1t_2}{t_2}, \quad X_{12} = \frac{p^2t_1^2t_2 + Jpq_1t_1^2 + J^2q_2^2t_2 + t_1^2t_2^2}{t_1^2t_2^2}, \quad X_{21} = \frac{p^2t_1^2t_2 + Jpq_1t_1^2 + J^2q_2^2t_2 + Jq_1t_1t_2^2}{t_1^2t_2^2}.
\] (4.17)

The S matrix depends only on the reduced parameters \(\tau_p\), \(\theta\) and \(\mu\):
\[
S(z_1, z_2) = \frac{\Lambda(z_1, z_2)}{\Lambda(z_2, z_1)}
\]
where
\[
\Lambda(z_1, z_2) = J\mu^4\tau_p^2z_1^2z_2^2 - \mu^2\tau_p\theta z_1z_2(z_1 + z_2) - J^2\mu^3\tau_p z_1^2z_2 + (\mu - \theta)(\mu - J^2\theta)z_1z_2
\]
and the decay coefficient \(N\) reads
\[
N(z_1, z_2) = \frac{\tau_2\tau_p\mu^2(z_1 - z_2)(1 + \mu z_1z_2)}{2\Lambda(z_2, z_1)}.
\] (4.20)

When \(q = J\frac{\tau_2^2}{p_2^2}\) (i.e. \(\theta = J\mu^2\tau_2\)), the S matrix and the decay coefficient \(N\) simplify and one gets
\[
S(z_1, z_2) = \frac{J\mu^2\tau_p^2z_1^2z_2 - J^2\mu\tau_p z_2 + 1}{J\mu^2\tau_p^2z_1z_2 - J^2\mu\tau_p z_1 + 1}
\]
and
\[
N(z_1, z_2) = \frac{\tau_2\tau_p(z_1 - z_2)(1 + \mu z_1z_2)}{2(z_1 - \tau_2)(z_2 - \tau_p)(J\mu^2\tau_p^2z_1z_2 - J^2\mu\tau_p z_1 + 1)}.
\] (4.21) (4.22)

**Remark 4.5.** The Bariev model [21, 26] is obtained as a particular case of one. More precisely, setting \(p = q = 1, J = j, t_1 = -\frac{J^2}{\sqrt{t_2^2 - 1}}, t_2 = j\sqrt{t_2^2 - 1}\), where \(j = \exp(\frac{2\pi}{3})\), one gets for the other parameters \(t_1 = s_3 = 1, s_p = t_p, X_{11} = -t_p, Y = t_p + \frac{1}{t_p}, X_{12} = -\mu_p + \frac{1}{t_p}, X_{21} = -J^2t_p + \frac{1}{t_p}, X_{22} = \frac{1}{t_p}\) which are the values of [26]. In that case, the S matrix reads
\[
S(z_1, z_2) = \frac{-\tau_2^2\tau_p^2z_1^2 - \tau_p^2z_1z_2 - \tau_2^2\tau_p z_2 + \tau_p^2}{\tau_2^2\tau_p^2z_1^2 + \tau_p^2z_1z_2 - \tau_2^2\tau_p z_2 + \tau_p^2},
\]
and the decay coefficient \(N\) takes the form
\[
N(z_1, z_2) = \frac{\tau_2\tau_p(z_1 - z_2)(1 - \mu z_1z_2)}{2(\tau_2^2\tau_p^2z_1^2 - \tau_p^2z_1z_2 - \tau_2^2\tau_p z_2 + \tau_p^2)}.\] (4.23) (4.24)

**Remark 4.6.** The PT-invariant models of [18], branch 2B, are obtained as particular cases of this one. More precisely, setting \(p = q = -2i, \quad t_1 = \pm \epsilon_1 \epsilon_2 e^{-2i\epsilon_3 \pi/3}, \quad t_2 = \pm \epsilon_1 \epsilon_2 e^{2i\epsilon_3 \pi/3}, \quad t_p = -i \epsilon_1 \sqrt{3}\) (4.25) one recovers the branch 2B Hamiltonians \(H_{2B}^{\pm}(\epsilon_1, \epsilon_2)\) of [18] (with \(\epsilon_1, \epsilon_2 \in \{-1, +1\}\)).

The main branch genus 5 model of [19] is also a particular case of this one. More precisely, setting \(p = -\epsilon_2, \quad q = -\epsilon_1, \quad t_1 = \frac{\sqrt{3} \pm i}{2} - \sqrt{\epsilon_1 \epsilon_2} - 1, \quad t_2 = \frac{\sqrt{3} \pm i}{2} \sqrt{\epsilon_1 \epsilon_2} - 1\) (4.26) one recovers the main branch genus 5 Hamiltonians \(H_{MB5}^{\pm}(\epsilon_1, \epsilon_2)\) of [19] (here we remind the reader that \(\epsilon_1\) and \(\epsilon_2\) are free parameters).
4.1.4. Generalization of the Hamiltonian built on $U_q(sl(2))$ special representation at roots of unity (SpR). The parameters which are left free are $p, q, t_p, t_2, t_3$. The remaining parameters entering the off-diagonal part of the Hamiltonian are given by

$$
t_1 = \frac{qt_2}{p}, \quad s_1 = \frac{pt_3}{t_2}, \quad s_2 = \frac{qt_3}{t_2}, \quad s_3 = \frac{qt_3}{p}, \quad s_p = \frac{q(t_1^2 - t_3 p + p^2)}{pt_p},$$

while on the diagonal we get:

$$X_{11} = 0, \quad Y = X_{12} = X_{21} = X_{22} = \frac{t_1^2 - t_3 p + p^2}{t_p} + \frac{qt_p}{p}.$$  \hspace{1cm} (4.28)

**Remark 4.7.** Note that equations (4.27) and (4.28) look as if some of the free parameters, say $t_p$, cannot be set to zero. This is due to the choice of parametrization made. However, one can choose alternative presentations. For instance, we could use as free parameters $p, q, s_p, t_1, s_3$. In that case, the remaining parameters take the form

$$t_2 = \frac{pt_1}{q}, \quad s_1 = \frac{ps_3}{t_1}, \quad s_2 = \frac{qs_3}{t_1}, \quad t_1 = \frac{ps_3}{q}, \quad t_p = \frac{p(s_1^2 - s_3 q + q^2)}{qs_p},$$

and

$$X_{11} = 0, \quad Y = X_{12} = X_{21} = X_{22} = \frac{s_1^2 - s_3 q + q^2}{s_p} + \frac{ps_p}{q}.$$  \hspace{1cm} (4.30)

Clearly, (4.29) shows that we can now set $t_p = 0$ without any problems. This new choice of parametrization is in fact the image under parity of the previous choice.

The $S$ matrix depends only on the reduced parameters $t_p$ and $t_3$:

$$S(z_1, z_2) = -\frac{t_2}{(t_2^2 - t_3 + 1)z_1 z_2 - t_p(z_1 + z_2 - t_3 z_1) + t_3^2},$$

and the decay coefficient $N$ reads

$$N(z_1, z_2) = \frac{t_3 t_p(z_1 - z_2)}{2[(t_2^2 - t_3 + 1)z_1 z_2 - t_p(z_1 + z_2 - t_3 z_1) + t_3^2]},$$

**Remark 4.8.** The $PT$-invariant models of [18], branch 1B, which are linked to the models associated to special representation of $U_q(sl(2))$ at roots of unity [27–29], are obtained as particular cases of this one. More precisely, setting

$$p = q = -2i, \quad t_2 = \pm 2, \quad t_3 = 2i, \quad t_p = -2i\epsilon_1 \sqrt{3},$$

one recovers the branch 1B Hamiltonians $H_{1B}^{\epsilon_1}(\epsilon_1)$ of [18].

4.2. Generalization of the special branch genus 5 model (SB3)

The parameters which are left free are $p, q, t_2, Y$. The remaining parameters entering the off-diagonal part of the Hamiltonian are given by

$$t_p = s_p = 0, \quad t_1 = \frac{qt_2}{p}, \quad s_1 = -J^2 p^2, \quad s_2 = -J^2 p, \quad s_3 = -J q,$$

while on the diagonal we get:

$$X_{11} = 0, \quad X_{12} = X_{21} = X_{22} = Y.$$  \hspace{1cm} (4.35)

The $S$ matrix depends only on the reduced parameters $\theta$ and $\Upsilon$:

$$S(z_1, z_2) = \frac{\theta \bar{z}_1 z_2 (z_1 - J^2 z_2) - \Upsilon \bar{z}_1^2 z_2 + z_1 - J_{2z_2}}{\theta \bar{z}_1 z_2 (z_1 - J^2 z_1) - \Upsilon \bar{z}_1 z_2 + z_2 - J_{z_1}}.$$  \hspace{1cm} (4.36)
and the decay coefficient $N$ reads

$$N(z_1, z_2) = -\frac{t_2(z_1 - z_2)(\theta z_1 z_2 + 1)}{2(\theta z_1 z_2 - J^2 z_1) - Y z_1 z_2 + z_2 - J z_1).}$$

**Remark 4.9.** The special branch genus 5 model of [19] is also a particular case of this one. More precisely, setting

$$p = t_2 = e^{\pi 2i/3}, \quad q = -1, \quad Y = 4\Lambda$$

one recovers the special branch genus 5 Hamiltonians $H^{\pm}_{5BS}(\Lambda)$ of [19].

### 4.3. Other models (17- and 14-vertex models)

The terminology ‘17-vertex’ and ‘14-vertex’ follows that used for 19-vertex; see the explanation in remark 4.2.

As explained above, equations (3.18), (3.21) and (3.24) provide a set of constraints (denoted $C_{(\Omega)}$ hereafter) on the parameters entering the Hamiltonian. Solving these equations, we get a set of solutions, each of them determining a Hamiltonian solvable through the CBA.

Then, the BAE (3.9) allows us to compute the eigenvalues (3.4) and eigenvectors (3.2) of the model using the $S$ matrix and the decay coefficient $N$. The construction is based on the choice of a particular eigenvector of $H$: the pseudo-vacuum.

Since we are performing a classification, one gets the same set of solutions whatever the choice of the pseudo-vacuum. In the case of the three-state Hamiltonian we are studying, there are two pseudo-vacua $|\Omega\rangle = \bigotimes_{i=1}^{4}x_i |0\rangle$ and $|\tilde{\Omega}\rangle = \bigotimes_{i=1}^{4}x_i |2\rangle$. Deploying the CBA machinery for each pseudo-vacuum leads to two distinct sets of constraint equations, $C_{(\Omega)}$ and $C_{(\tilde{\Omega})}$, the latter being obtained by applying the charge conjugation\(^7\) (2.10) to the former. It follows in light of the foregoing that each solution of $C_{(\tilde{\Omega})}$ should satisfy the equations coming from $C_{(\Omega)}$.

As can be checked in table A1, it is indeed the case for the previous models.

However, there are cases where a solution to $C_{(\Omega)}$ does not identically solve $C_{(\tilde{\Omega})}$, but rather leads to additional constraints on the parameters. At this stage, the additional constraints could be interpreted as a failure in the CBA method: eigenvectors built on $C_{(\tilde{\Omega})}$ are not eigenvectors of the Hamiltonian based on a solution of $C_{(\Omega)}$. In fact, it just indicates that the eigenvectors obtained by the CBA on $C_{(\tilde{\Omega})}$ do not provide a complete basis of eigenvectors. One needs to consider a second pseudo-vacuum $|\tilde{\Omega}\rangle$ to get a (tentatively) complete basis. Thus, it is the full set of constraints $C_{(\Omega)}$ and $C_{(\tilde{\Omega})}$ that need to be considered. That is what we did for the class of models presented in this section. In practice, it is simpler, but equivalent, to apply the transformation (2.10) to a given solution to the initial constraints $C_{(\tilde{\Omega})}$, to impose the transformed solution to also be a solution of $C_{(\Omega)}$ (hence leading to more constraints on the parameters) and then to pull back the charge conjugation transformation (2.10) on the result to get the final answer.

#### 4.3.1. Model 17V$_1$

Solving the constraints $C_{(\Omega)}$, the parameters which are left free are $p, q, t_p, t_1, t_2, s_1, X_{22}$ and the constraints on the parameters are given by

$$s_1 = s_2 = 0, \quad s_3 = \frac{p q}{t_p}, \quad t_1 = \frac{q t_2}{p}$$

and

$$X_{11} = 0, \quad Y = \frac{p^2}{t_p} + \frac{q t_p}{p}, \quad X_{12} = \frac{p^2}{t_p} + \frac{q t_p}{p} + \frac{p t_3}{t_p}, \quad X_{21} = \frac{p^2}{t_p} + \frac{q t_p}{p} + \frac{t_p s_3}{p}.$$

Solving now the constraints $C_{(\tilde{\Omega})}$, one gets two inequivalent possibilities.

\(^7\) In fact, (2.10) is just the gauge transformation relating $|\Omega\rangle$ to $|\tilde{\Omega}\rangle$.  

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\cite{19}
The diagonal terms now reading
\[ X_{12} = Y + \epsilon \frac{p^2}{t_p}, \quad X_{21} = Y + \epsilon \frac{q t_p}{p}, \]
\[ \text{(4.42)} \]

**Model 17V_{1b}**. The additional conditions are
\[ q = \frac{I p^3}{t_p^2}, \quad t_3 = t p, \quad s_3 = \frac{p^3}{t_p^2}, \quad X_{22} = (1 + I) \frac{p^2}{t_p}, \]
\[ \text{(4.43)} \]
where \( I \) is one solution of \( I^2 + 1 = 0 \). It leads to a redefinition of the parameters:
\[ s_p = \frac{I p^4}{t_p}, \quad t_1 = \frac{I p^2 t_2}{t_p}, \quad Y = (1 + I) \frac{p^2}{t_p}, \quad X_{12} = (2I + 1) \frac{p^2}{t_p}, \quad X_{21} = (I + 2) \frac{p^2}{t_p}. \]
\[ \text{(4.44)} \]

For both models 17V_{1a} and 17V_{1b} the S matrix is trivial, \( S(z_1, z_2) = -1 \), and the decay coefficient \( N \) reads
\[ N(z_1, z_2) = \frac{\tau_2 \tau_p (z_1 - z_2)}{2(z_1 - \tau_p)(z_2 - \tau_p)}. \]
\[ \text{(4.45)} \]

**4.3.2. Model 17V_2**. Solving the constraints \( C_{\{z\}} \), the parameters which are left free are \( p, q, t_p, t_2, t_3, s_3 \) and the constraints on the parameters are given by
\[ s_1 = s_2 = 0, \quad s_p = \frac{pq}{t_p}, \quad t_1 = -\frac{p^2 t_2}{t_p} \]
\[ \text{(4.46)} \]
and
\[ X_{11} = Y = \frac{p^2}{t_p} + \frac{q t_p}{p}, \quad X_{12} = 2Y - \frac{q t_p^3}{p^2}, \quad X_{21} = 2Y - \frac{p^3 t_3}{q t_p}, \quad X_{22} = 2Y. \]
\[ \text{(4.47)} \]
Solving now the constraints \( C_{\{z\}} \), the additional conditions are
\[ s_3 = q, \quad t_3 = p, \]
\[ \text{(4.48)} \]
the diagonal terms now reading
\[ X_{12} = \frac{2 p^2}{t_p} + \frac{q t_p}{p}, \quad X_{21} = \frac{p^2}{t_p} + \frac{2 q t_p}{p}. \]
\[ \text{(4.49)} \]
The S matrix depends only on the reduced parameters \( \tau_p \) and \( \theta \):
\[ S(z_1, z_2) = \frac{\frac{\theta \tau_p z_1 z_2}{\theta \tau_p z_1 z_2 - (\theta \tau_p^2 + 1) z_1 + \tau_p}}{\theta \tau_p z_1 z_2 - (\theta \tau_p^2 + 1) z_1 + \tau_p} \]
\[ \text{(4.50)} \]
and the decay coefficient \( N \) reads
\[ N(z_1, z_2) = -\frac{\tau_2 (z_1 - z_2)(z_1 z_2 - \tau_p^2)}{2(\theta \tau_p z_1 z_2 - (\theta \tau_p^2 + 1) z_1 + \tau_p)(z_1 - \tau_p)(z_2 - \tau_p)}. \]
\[ \text{(4.51)} \]
4.3.3. Model 14V₁. Solving the constraints \( C|Ω_1⟩ \), the parameters which are left free are \( p, t_p, t_2, t_3, X_{21}, X_{22} \) and the constraints on the parameters are given by

\[
q = s_1 = s_2 = s_3 = s_p = 0, \quad t_1 = -\frac{p^2 t_2}{t_p^2}, \quad X_{11} = Y = \frac{p^2}{t_p}, \quad X_{12} = \frac{2p^2}{t_p}. \tag{4.52}
\]

Solving now the constraints \( C|\tilde{Ω}_1⟩ \), the additional conditions are

\[
t_3 = \epsilon p, \quad X_{21} = X_{22} = \frac{p^2}{t_p}, \quad \text{with } \epsilon = \pm 1. \tag{4.53}
\]

The \( S \) matrix and the decay coefficient \( N \) read:

\[
S(z_1, z_2) = -\frac{z_2 - \tau_p}{z_1 - \tau_p}, \quad N(z_1, z_2) = \frac{\tau_2(z_1 - z_2)(z_1 z_2 - \tau_p^2)}{2(\tau_1 - \tau_p)(z_2 - \tau_p)}. \tag{4.54}
\]

4.3.4. Model 14V₂. Solving the constraints \( C|Ω_2⟩ \), the parameters which are left free are \( p, t_p, t_1, t_2 \) and the constraints on the parameters are given by

\[
q = s_1 = s_2 = s_3 = s_p = 0, \quad t_3 = -\frac{p^2 t_1}{t_p^2} \tag{4.55}
\]

and

\[
X_{11} = 0, \quad X_{12} = Y = \frac{p^2}{t_p}, \quad X_{21} = X_{22} = \frac{p^2 t_2 - t_p^2 t_1}{t_p t_2}. \tag{4.56}
\]

Solving now the constraints \( C|\tilde{Ω}_2⟩ \) leads to one additional condition

\[
t_1 = \frac{p^2 t_2}{t_p^2} \tag{4.57}
\]

which gives \( t_3 = -p \) and \( X_{21} = X_{22} = 0 \).

The \( S \) matrix is trivial, \( S(z_1, z_2) = -1 \), and the decay coefficient \( N \) reads:

\[
N(z_1, z_2) = \frac{\tau_2(z_1 - z_2)(z_1 z_2 + \tau_p^2)}{2\tau_1(z_1 - \tau_p)(z_2 - \tau_p)}. \tag{4.58}
\]

5. Reduced Hamiltonians

In this section, we use telescoping terms and gauge transformations (see section 2.2) to get a simple expression \( \tilde{H} \) for the Hamiltonians described in section 4. For all Hamiltonians, in a first step, we change the normalization and perform a gauge transformation:

\[
H_{red} = N_0 \ G \otimes \ G \left( H - \frac{V}{2} (s_1 + s_2) \right) G^{-1} \otimes G^{-1}, \tag{5.1}
\]

where \( N_0 \) is a constant, and \( G \) a \( 3 \times 3 \) diagonal matrix. Their exact form depends on the model we consider. In this way we get Hamiltonians that depend only on the physical parameters. The transformation is valid only for generic values of the free parameters; see remark 5.1 below.
Remark 5.1. Note that this transformation is not valid for $s_1 = 0$, although the ‘raw’ Hamiltonian is well-defined in this case, see section 4.1.1. This is due to the fact that the physical parameter $\sigma$ vanishes for this particular value of $s_1$. One can however work on the raw Hamiltonian to then obtain a reduced Hamiltonian containing only the physical parameter $\tau_p$.

To compare with existing models, we furthermore modify it to

$$\tilde{H} = \frac{-2k^2}{k^4-1}H_{\text{red}} - \frac{k^4+1}{k^4-1}(s_1^z + s_2^z) \quad \text{with} \quad \sigma = \left(\frac{k^2 + 1}{k}\right)^2$$

and we get

$$\tilde{H} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{k^4+1}{k^4-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2k^2 + 2e^2}{1-k^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2k^2}{1-k^2} & 0 & \frac{k^4+1}{k^4-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{2k^2}{1-k^2} & 0 & 0 & 0 \\
0 & \frac{2k}{1-k^2} & 0 & \frac{2k^2 + 2e^2}{1-k^2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2k^2}{1-k^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{k^4+1}{k^4-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \quad (5.4)$$

When $\tau_p = -1$, we recover the ZF model [24] (or spin-1 XXZ spin chain). When $\tau_p = -\epsilon_1$, $k = \exp\left(\frac{\epsilon_1}{\epsilon_2} (1 - \epsilon_1)\right)$ and $\epsilon_1 = \pm 1$ is left free, we get the models ‘branch 1A’ described in [18]. The models 7 and 10 of [17] are also obtained in the same way.

For completeness, let us add that for $\tau_p = -1$, the Hamiltonian (5.4) is related to the one based on $U_q(B_1^{(1)})$ given in [30] by $H^{U_q(B_1^{(1)})} / k^2 = \tilde{H}(k) + (1 \otimes e_{22} - e_{22} \otimes 1) + 2(1 \otimes e_{33} - e_{33} \otimes 1)$ (the $R$ matrix of $U_q(B_1^{(1)})$ we consider is normalized such that $R_{11}^1 = 1$).

5.1.2. The generalized Izergin–Korepin model. From the Hamiltonian $H$ given in section 4.1.2, the transformation (5.1) with

$$N_0 = \frac{1}{p^2} \quad \text{and} \quad G = \text{diag}\left(1, \sqrt{\frac{\epsilon_2}{p}} \left(\frac{u_-}{v} - 1\right)^{1/4}, 1\right)$$

leads to a Hamiltonian $H_{\text{red}}$ depending on $\tau_p$ and $v$ only. To compare with the existing model, we first change the variable $v = \frac{1}{k^2 - k + 1}$, $u_- = -(k^2 - k + 1)^2$ and define

$$\tilde{H} = \frac{1}{(k^2 - 1)(k^2 - k + 1)} \left(- (k^2 - k + 1)^2 H_{\text{red}} + \frac{1}{2} (k^2 + 1)(k^2 - k + 1)(s_1^z + s_2^z)\right)$$
constant gauge transformation and constant telescopic terms.

\[ H = \frac{1}{2} (k^2 - 1)(k^2 - k + 1)(I \otimes e_{22} - e_{22} \otimes I) \]
\[ + \frac{1}{2} (k - 1)^2(k + 1)(I \otimes e_{33} - e_{33} \otimes I) \]

(5.6)

that is, with \( \tau_p' = \tau_p/v \),

\[
\tilde{H} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\kappa}{k^2 - 1} & 0 & -\kappa' \sqrt{\tau} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\tau}{k^2 + (k-1)} & 0 & -\frac{\tau}{k^2 + (k-1)} & 0 & 0 & 0 & 0 \\
0 & -\frac{k}{(k^2 - 1)\tau_p} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\kappa}{k^2 - 1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\tau}{k^2 + (k-1)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\kappa'}{k^2 + (k-1)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\tau_p}{(k^2 - 1)\tau_p} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(5.7)

For generic values of \( k \) and \( \tau_p \), the Hamiltonian \( \tilde{H} \) is conjugated to the one based on \( U_q(A^{(2)}_2) \), and given in [30] (the \( R \) matrix of \( U_q(A^{(2)}_2) \) we consider is normalized such that \( R^{(1)}_{11} = 1 \):

\[ \tilde{H}(k) = \tilde{F} H^{A^{(2)}_2}(k) \tilde{F}^{-1} \quad \text{with} \quad \tilde{F} = \text{diag}(u_1, \sqrt{u_1 u_3}, u_3) \otimes \text{diag}(u_1, \sqrt{u_1 u_3}, u_3) \frac{\tau_p}{v} \]

(5.8)

Note that the Hamiltonian \( H^{A^{(2)}_2}(k) \) is related to the Izergin–Korepin model [25] through a constant gauge transformation and constant telescopic terms.

For completeness, let us add that the Hamiltonian \( H^{A^{(2)}_2} \) is related to the Branch 2A of [18] through the following transformation:

\[ -2 H^{A^{(2)}_2}(\epsilon_1 k^2) + (I \otimes e_{22} - e_{22} \otimes I) + 2(I \otimes e_{33} - e_{33} \otimes I) = F H^{2A}(k) F^{-1} \]

(5.9)

with

\[ F = \text{diag}\left(1, \frac{1}{\sqrt{\epsilon_1}}, \sqrt{-\epsilon_2 \epsilon_3}, \sqrt{\epsilon_1}, \frac{1}{\sqrt{-\epsilon_2 \epsilon_3}}, \sqrt{-\epsilon_2 \epsilon_3}, \frac{1}{\sqrt{\epsilon_1}}, 1\right) \]

(5.10)

Note that in the correspondence (5.9), \( H^{2A} \) corresponds to the branch 2A Hamiltonian \( H_{12} \) of [18] for \( \epsilon_2 = 1 \), while it corresponds to \( H_{21} \) when \( \epsilon_2 = -1 \).

5.1.3. The generalized Bariev model. From the Hamiltonian \( H \) given in section 4.1.3, we perform the transformation (5.1) with

\[ N_0 = \frac{1}{p} \sqrt{\frac{\tau}{1}} \quad \text{and} \quad G = \text{diag}\left(1, \sqrt{\frac{\tau}{p}}, 1\right) \]

We get a Hamiltonian \( H_{\text{red}} \) depending on \( \tau_p, \theta \) and \( \mu \) only.

8 We remind the reader of the correspondence with the notation of [18]: \( u = \exp(\lambda), k = \exp(p/2 + \frac{\mu}{2}(1 - \epsilon_1)). \)
This Hamiltonian can be related to that of the main branch of \cite{19}, $H^{MB_5}$. One defines, with $\delta = \mu^2 + J\theta \mu + J^2 \theta^2$

$$\tilde{H} = -\frac{J}{\tau p \sqrt{\mu}} H_{\text{red}} + \delta (s^2_1 + s^2_2 - 1) + \frac{1}{2} (J - J^{-1}) (I \otimes e_{33} - e_{33} \otimes I)$$

(5.12)

that is,

$$\tilde{H} = \begin{bmatrix}
-\delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{J}{\tau \mu} & 0 & 0 & 0 & 0 \\
0 & 0 & -\delta - J^2 & 0 & -\frac{J}{\tau \mu} & 0 & -J \mu^{-1} & 0 \\
0 & 0 & -\frac{J^2}{\tau \mu} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$  

(5.13)

Then one gets

$$\tilde{H} = F H^{MB_5} F^{-1}$$

(5.14)

with $F = U \otimes U'$ where $U$ and $U'$ are expressed in terms of the free parameters $u_1, u_3$:

$$U = \text{diag} \left( u_1, Z \left( \frac{\theta - J \tau^2_2 \mu^2}{\mu} \right)^{1/4} \sqrt{u_1 u_3}, u_3 \right)$$

(5.15)

and

$$U' = \text{diag} \left( -i J \sqrt{\mu} u_1, \frac{1}{Z} \left( \frac{\theta - J \tau^2_2 \mu^2}{\mu} \right)^{1/4} \sqrt{u_1 u_3}, -u_3 \frac{1}{i J \sqrt{\mu}} \right).$$

(5.16)

The parameters $(\epsilon_1, \epsilon_2)$ entering into the definition of $H^{MB_5}$ are given by one of these relations:

$$\epsilon_1 = \frac{i \theta}{\tau \mu^{3/2}}, \quad \epsilon_2 = -\frac{i J^2}{\tau \mu^{1/2}} \quad \text{with the choice} \quad Z = 1$$

$$\epsilon_1 = -\frac{i J^2}{\tau \mu^{1/2}}, \quad \epsilon_2 = \frac{i \theta}{\tau \mu^{3/2}} \quad \text{with the choice} \quad Z = (\frac{J \theta}{\mu})^{1/2}$$

(5.17)

and $J$ is related to the parameter $\gamma_0 = \epsilon \frac{\tau^2}{\mu} (\epsilon = \pm 1)$ of $H^{MB_5}$ by $J = -e^{-2\gamma_0}$.

5.1.4. Generalization of the Hamiltonian associated with the special representation of $U_q(sl(2))$ at roots of unity. From the Hamiltonian $H$ given in section 4.1.4, the transformation (5.1) with

$${\cal N}_0 = \frac{t_p}{p^2} \quad \text{and} \quad G = \text{diag} \left( 1, \sqrt{\frac{t_p}{p}}, 1 \right)$$

(5.18)
produces a Hamiltonian $H_{\text{red}}$ depending on $\tau_p$, $\theta$ and $\tau_3$ only, that is, with $\delta = \tau_3^2 - \tau_3 + 1 + \tau_p^2 \theta$, 

$$H_{\text{red}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \delta & 0 & \tau_p & 0 & \tau_p^2 \theta & 0 & 0 \\ 0 & \tau_p \theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau_p \tau_3 \theta & 0 & 0 & \tau_p \tau_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \delta & 0 & \tau_p \tau_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_p \theta & 0 & \frac{1}{2} \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_p \tau_3 \theta & 0 & \frac{1}{2} \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta \end{bmatrix} . \quad (5.19)$$

The branch $1B$ Hamiltonians of [18] can be related to the Hamiltonian (5.18) through the following transformation:

$$\frac{2(k^2 - e^{2\gamma_0})}{(k^2 - 1)(1 - e^{2\gamma_0})} H_{\text{red}} = \frac{k^2 + 1}{k^2 - 1} \left( s_1^2 + s_2^2 \right) = F H^{1B}(k, \epsilon_1, \epsilon_2) F^{-1} \quad (5.20)$$

with $\gamma_0 = \epsilon_1 t_1$, $(\epsilon_1, \epsilon_2) \in \{-1, +1\}$, and $F = U \otimes U'$ where $U$ and $U'$ are expressed in terms of the free parameters $u_1, u_3$:

$$U = \text{diag} \left( u_1, \sqrt{\frac{u_1 u_3}{\Delta}}, u_3 \right) \quad \text{and} \quad U' = \text{diag} \left( u_1 \sqrt{\theta}, \sqrt{\frac{u_1 u_3}{\Delta}}, \frac{u_3}{\sqrt{\theta}} \right) , \quad (5.21)$$

$$\Delta = \frac{e^{\gamma_0}}{e^{\gamma_0} - \frac{1}{2}}, \quad \text{and the parameters } \theta, \tau_p \text{ and } \tau_3 \text{ are linked to } k \text{ by the relations}$$

$$\tau_3 = \frac{e^{\gamma_0} (k^2 - 1)}{k^2 - e^{2\gamma_0}} \quad \text{and} \quad \tau_p \sqrt{\theta} = \frac{k(1 - e^{2\gamma_0})}{k^2 - e^{2\gamma_0}} . \quad (5.22)$$

Note that the transformation (5.20) only holds when the parameters $\theta$, $\tau_p$ and $\tau_3$ are related to $k$ by (5.22). Hence this model appears as a generalization of the branch $1B$ models of [18].

### 5.2. Generalization of the special branch genus 5 model

From the Hamiltonian $H$ given in section 4.2, we perform the transformation (5.1) with

$$\mathcal{N}_0 = \frac{1}{\gamma} \quad \text{and} \quad G = \text{diag} \left( 1, \sqrt{\frac{t_2}{p}}, 1 \right) , \quad (5.23)$$

which leads to a Hamiltonian $H_{\text{red}}$ depending on $\theta$ and $\Upsilon$ only.

The genus 5 special branch Hamiltonian of [19] can be related to the Hamiltonian (5.23). Indeed, one defines

$$\tilde{H} = \frac{\Upsilon}{4 \sqrt{-\theta}} (4H_{\text{red}} - (s_1^2 + s_2^2) + 1) \quad (5.24)$$
with \( \gamma_0 = \pm \frac{i\pi}{3} \), \( J = e^{-2\gamma_0} \), that is
\[
\tilde{H} = \begin{bmatrix}
\frac{\gamma}{2\sqrt{-\theta}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2\sqrt{-\theta}} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\gamma}{2\sqrt{-\theta}} & 0 & \frac{1}{2\sqrt{-\theta}} & 0 & 0 & 0 \\
0 & -\frac{\gamma}{\sqrt{-\theta}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{-\theta} & 0 & -\frac{\gamma}{\sqrt{-\theta}} & 0 & \frac{J}{\sqrt{-\theta}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\gamma}{\sqrt{-\theta}} \\
0 & 0 & 0 & 0 & 0 & -\frac{\gamma}{\sqrt{-\theta}} & 0 & \frac{\gamma}{\sqrt{-\theta}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{-\theta} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\gamma}{\sqrt{-\theta}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]
(5.25)

Then one gets
\[
\epsilon\tilde{H} = FH_{SR5}(\Lambda, e^{i\omega})F^{-1}
\]
(5.26)
with \( F = U \otimes U' \) where \( U \) and \( U' \) are expressed in terms of the free parameters \( u_1, u_3 \):
\[
U = \text{diag}(u_1, \sqrt{u_1u_3}, u_3) \quad \text{and} \quad U' = \text{diag} \left( \epsilon J^2 \sqrt{-\theta} u_1, \sqrt{u_1u_3}, \frac{\epsilon u_3}{J^2 \sqrt{-\theta}} \right),
\]
(5.27)
the parameter \( \Lambda \) being linked to \( \Upsilon \) and \( \theta \) by \( \Lambda = \frac{\epsilon \Upsilon}{2\sqrt{-\theta}} \) (with \( \epsilon = \pm 1 \)).

### 5.3. Other models

#### 5.3.1. Model 17V_{1a}

From the Hamiltonian \( H \) given in section 4.3.1 (case 1a), we perform the transformation (5.1)
\[
\mathcal{N}_0 = \frac{t_p}{p^2} \quad \text{and} \quad G = \text{diag} \left( 1, \frac{t_p}{\sqrt{p}}, 1 \right)
\]
(5.28)
which leads to a Hamiltonian \( H_{\text{red}} \) depending on \( \theta \) and \( \tau_p \) only, that is,
\[
H_{\text{red}} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \tau_p & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1+\theta^2}{2} & 0 & \tau_p & 0 & \tau_p^2 & 0 \\
0 & \theta \tau_p & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1+\theta^2}{2} + \epsilon & \epsilon \tau_p \\
0 & 0 & \theta & 0 & \theta \tau_p & 0 & \frac{1+\theta^2}{2} & 0 \\
0 & 0 & 0 & 0 & \epsilon \theta \tau_p & 0 & \frac{1+\theta^2}{2} + \epsilon \theta \tau_p^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & (1+\epsilon)(1+\theta \tau_p^2) \\
\end{bmatrix}.
\]
(5.29)
5.3.2. Model 17V₁b. From the Hamiltonian $H$ given in section 4.3.1 (case 1b), we perform the transformation (5.1) with

\[
N_0 = \frac{\tau_p}{p^2} \quad \text{and} \quad G = \text{diag} \left( 1, \sqrt{\frac{\tau_p}{p}}, 1 \right)
\] (5.30)

which leads to a Hamiltonian $H_{\text{red}}$ depending on $\tau_p$ only, that is

\[
H_{\text{red}} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \tau_p & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2}(1 + I) & 0 & \tau_p & 0 & \tau_p^2 & 0 & 0 \\
0 & \tau_p^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1 + 3I) & 0 & I\tau_p & 0 \\
0 & I\tau_p^{-2} & 0 & I\tau_p^{-1} & 0 & \frac{1}{2}(1 + I) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tau_p^{-1} & 0 & \frac{1}{2}(3 + I) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \tau_p & 0 & 0 & 1 + I \\
\end{bmatrix}
\] (5.31)

5.3.3. Model 17V₂. From the Hamiltonian $H$ given in section 4.3.2, we perform the transformation (5.1) with

\[
N_0 = \frac{\tau_p}{p^2} \quad \text{and} \quad G = \text{diag} \left( 1, \sqrt{\frac{\tau_p}{p}}, 1 \right)
\] (5.32)

which leads to a Hamiltonian $H_{\text{red}}$ depending on $\theta$ and $\tau_p$ only, that is

\[
H_{\text{red}} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \tau_p & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1 + \theta \tau_p^2}{2} & 0 & \tau_p & 0 & \tau_p^2 & 0 & 0 \\
0 & \tau_p \theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 + \theta \tau_p^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{3 + \theta \tau_p^2}{2} & 0 & \tau_p & 0 & 0 \\
0 & \theta & 0 & -\tau_p^{-1} & 0 & \frac{1 + \theta \tau_p^2}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tau_p \theta & 0 & \frac{1 + 3 \theta \tau_p^2}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2(1 + \theta \tau_p^2) & 0 & 0 \\
\end{bmatrix}
\] (5.33)

5.3.4. Model 14V₁. From the Hamiltonian $H$ given in section 4.3.3, we perform the transformation (5.1) with

\[
N_0 = \frac{\tau_p}{p^2} \quad \text{and} \quad G = \text{diag} \left( 1, \sqrt{\frac{\tau_p}{p}}, 1 \right)
\] (5.34)
which leads to a Hamiltonian $H_{\text{red}}$ depending on $\tau_p$, $\xi = X_{22}/p$ and $\epsilon = \pm 1$ only, that is

$$H_{\text{red}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau_p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \tau_p & 0 & \tau_p^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{2} \tau_p & 0 & \epsilon \tau_p & 0 \\ 0 & 0 & 0 & -\tau_p^{-1} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau_p \xi - \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau_p \xi \end{bmatrix}. \quad (5.35)$$

5.3.5. Model $14V_2$. From the Hamiltonian $H$ given in section 4.3.4, we perform the transformation (5.1) with

$$N_0 = \frac{\tau_p}{p^2} \quad \text{and} \quad G = \text{diag} \left( 1, \sqrt{\frac{t_2}{p}}, 1 \right) \quad \text{(5.36)}$$

which leads to a Hamiltonian $H_{\text{red}}$ depending on $\tau_p$ only, that is

$$H_{\text{red}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau_p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \tau_p & 0 & \tau_p^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{2} \tau_p & 0 & -\tau_p & 0 \\ 0 & 0 & 0 & \tau_p^{-1} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (5.37)$$

6. Conclusion

In this paper we have provided a classification of ‘all’ the Hamiltonians with rank 1 symmetry and nearest-neighbour interactions, acting on a periodic three-state spin chain, and solvable through (a generalization of) the coordinate Bethe ansatz (CBA).

Of course, the search for an $R$ matrix formulation of the new models presented here should be done, but many directions of generalizations can also be planned. First of all, the case with rank 2 symmetry algebra can also be easily done using the same method. Next, the integrable Hamiltonians that are not solvable through the CBA, such as those obtained from Temperley–Lieb algebras, should be classified as well. Finally, a similar classification for models solvable through the algebraic Bethe ansatz would help to give a better understanding of the connection between these two approaches.

There is also a natural question that arises from this classification: what possible extensions of this work can be envisioned for $n$-state Hamiltonians? A priori, the method becomes rather intricate when increasing the number of states on each site, so there is little hope that this can be done in the same way. However, increasing the rank of the symmetry algebra at the
same time could provide some simplification. This question concerns the relevance to recent developments in ultracold gases in optical lattices, such as the achievement of cooling five Ytterbium isotopes to quantum degeneracy \cite{31}, which exhibit an enlarged $SU(6)$ symmetry.

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Appendix. $P, C, T$ transformations

| Table A1. Actions of $P, C, T$ on the Hamiltonians. |
|----------|----------|----------|----------|----------|
| Model    | # vertices | $P$ action $C$ action | $T$ action $P, C, T$ |
| gZF      | 19         | gZF       | gZF       | gZF       | $P, C, T$ |
| gIK      | 19         | gIK       | gIK       | gIK       | $P, C, T$ |
| gB       | 19         | gB        | gB        | gB        | $P, C, T$ |
| SpR      | 19         | SpR       | SpR       | SpR       | $P, C, T$ |
| SB5      | 17         | SB5       | C(SB5) $P, C, T$ |
| 17V_{a}  | 17         | 17V_{a}   | 17V_{a}   | $P, C$ |
| 17V_{b}  | 17         | 17V_{b}   | C(17V_{b}) $T(17V_{b})$ |
| 17V_{c}  | 17         | 17V_{c}   | 17V_{c}   | $P, C$ |
| 14V_{1}  | 14         | $P(14V_{1})$ $C(14V_{1}) = P(14V_{1})$ $T(14V_{1})$ |
| 14V_{2}  | 14         | $P(14V_{2})$ $C(14V_{2}) = P(14V_{2})$ $T(14V_{2})$ |

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