Hadronic contribution to $\alpha(M_Z^2)$ and the anomalous magnetic moment of the muon$^a$

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In this contribution recent developments are discussed which lead to a significant reduction of the error for $\alpha(M_Z^2)$ and $(g-2)_{\mu}$.

The impressive amount of data mainly collected at LEP, SLC and TEVATRON has resulted in a very high precision sometimes even below the per mille level. The combination with accurate computations has made it possible to predict the mass of the top quark, $M_t$, before its actual experimental discovery. A similar scenario is nowadays applied to the Higgs boson mass. However, the dependence of the radiative corrections on $M_H$ is much weaker than on $M_t$ which has the consequence that the limits are much more loose. Nevertheless it is possible to derive an upper bound on $M_H$ of roughly 300 GeV at 95% confidence level.

A crucial role in the indirect determination of $M_H$ is taken over by the error on the electromagnetic coupling at the scale $M_Z$, $\alpha(M_Z^2)$. Adopting the value given in Ref. and expressing the error in terms of uncertainties in the weak mixing angle it actually dominates in the comparison with the other error sources (putting aside the error on $M_H$, of course). The authors of performed a very conservative analysis which exclusively relies on rather imprecise data below an center-of-mass energy, $\sqrt{s}$, of 40 GeV. Only for $\sqrt{s} \geq 40$ GeV perturbative QCD is used.

A similar situation can be found in the case of the anomalous magnetic moment of the muon. The error cited in reads $\pm 153 \times 10^{-11}$ which has to be compared with the aimed experimental error of $\pm 40 \times 10^{-11}$ in the E821 experiment at Brookhaven.

These numbers show that it is very important to improve both the error on $\alpha(M_Z^2)$ and on $(g-2)_{\mu}$. In the recent months several different options have been

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suggested which significantly reduced the uncertainties. In this contribution
they are briefly discussed and compared at the end.

The electromagnetic coupling at the scale $M_Z$ is given by

$$\alpha(s) = \frac{\alpha(0)}{1 - \Delta \alpha_{\text{lep}}(s) - \Delta \alpha_{\text{had}}^{(5)}(s) - \Delta \alpha_{\text{top}}(s)}.$$  

with $\alpha = \alpha(0) = 1/137.0359895$. The leptonic and the top quark contribution are both known up to the three-loop order where the errors are negligible:

$$\Delta \alpha_{\text{lep}}(M_Z^2) = 314.98 \times 10^{-4}, \quad \Delta \alpha_{\text{top}}(M_Z^2) = -0.70 \pm 0.05 \times 10^{-4}.$$  

The evaluation of the hadronic contribution requires the computation of the following dispersion integral.

$$\Delta \alpha_{\text{had}}^{(5)}(M_Z^2) = -\frac{\alpha M_Z^2}{3\pi} \Re \int_{4m_e^2}^{\infty} ds \frac{R(s)}{s(s - M_Z^2 - i\epsilon)},$$

with $R(s) = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$. It is not possible to use perturbation theory for $R(s)$ in the whole energy region. Thus one has to rely — to a more or less large extent — on experimental data.

Similarly the anomalous magnetic moment receives contributions from different areas:

$$\left( g - \frac{2}{2} \right)_{\mu} \equiv a_{\mu} = a_{\mu}^{\text{QED}} + a_{\mu}^{\text{had}} + a_{\mu}^{\text{ew}}.$$  

The QED contribution is known up to the four-loop order (plus the dominant terms from five loops) and $a_{\mu}^{\text{ew}}$ is computed up to the two-loop level:

$$a_{\mu}^{\text{QED}} = 116.584.705.6 \pm 2.9 \times 10^{-11}, \quad a_{\mu}^{\text{ew}} = 151 \pm 4 \times 10^{-11}.$$  

Both contributions exhibit a small error and are thus completely under control. The hadronic contribution again has to be obtained via a dispersion relation

$$a_{\mu}^{\text{had}} = \frac{\alpha^2(0)}{3\pi^2} \int_{4m_e^2}^{\infty} ds \frac{K(s)}{s} R(s),$$

where $K(s)$ denotes the QED kernel function. $K(s)$ is strongly peaked for small values of $\sqrt{s}$. Actually 98% of the contribution arises from energies $\sqrt{s} < 1.8$ GeV.

Let us now discuss the various improvements which intend to reduce the error on $\Delta \alpha_{\text{had}}^{(5)}$ and $a_{\mu}^{\text{had}}$. 

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In Ref. 7, data from ALEPH has been used in order to get more information about $R(s)$ for energies below roughly 1.8 GeV. The hypothesis of conserved vector current (CVC) in combination with isospin invariance relates, e.g., the vector part of the two-pion $\tau$ spectral function to the corresponding part of the isovector $e^+e^-$ cross section through the following relation

$$\sigma^{I=1} (e^+e^- \rightarrow \pi^+\pi^-) = \frac{4\pi\alpha_s^2(0)}{s} v_{J=1} (\tau \rightarrow \pi\pi\nu_{\tau}).$$

(7)

A similar equation holds for the four-pion final state. Their incorporation into the analysis has been performed in Ref. 7 leading to a slight reduction of the error on $\Delta\alpha^{(5)}_{\text{had}}$, however, to a significant reduction for $a^\mu_{\text{had}}$ (see table below). In the meantime new date from the Novosibirsk experiment CMD-2 became available. The main improvement results from the reduction of the systematic error from 2 to 1.5% in the energy region below 1.4 GeV. This brings the error for $a^\mu_{\text{had}}$ (obtained from $e^+e^-$ data only) down to the same order of magnitude as the one cited in Ref. 7. However, it remains to check if also the central values are in agreement.

Looking at the data in the energy region above 2 GeV one realizes that they are accompanied with large systematic uncertainties. Thus there is the temptation to replace the inaccurate data by predictions from perturbative QCD (pQCD) at least in those energy regions which are not affected by resonance phenomena. This is also supported from recent QCD analyses performed by ALEPH and OPAL using hadronic $\tau$ decays. Not only for the total rate but also for the spectral function towards the upper end the substitution of imprecise data by pQCD seems justified.

$R(s)$ can be calculated in the framework of pQCD up to order $\alpha_s^4$ if quark masses are neglected and up to $O(\alpha_s^2)$ with full quark mass dependence (see Ref. 11 and references therein). In Ref. 11 pQCD has been used down to an energy scale of $\sqrt{s} = 1.8$ GeV and it has been shown that the non-perturbative contributions are small. This leads to a further reduction of the error of about a factor two.

In Ref. 11 pQCD also has been applied down to small center-of-mass energies implementing the state-of-the-art corrections up to order $\alpha_s^3$. The full charm and bottom quark mass effects are taken into account at two-loop order. All formulae are available for arbitrary renormalization scale $\mu$ which allows to test the scale dependence of the final answer. This has been used to estimate the theoretical uncertainties from uncalculated higher orders. The details of the formalism can be found in Ref. 11.

Perturbative QCD is clearly inapplicable in the charm threshold region between 3.7 and 5 GeV where rapid variations of the cross section are ob-
erved. Data have been taken more than 15 years ago by the PLUTO, DASP, and MARK I collaborations. The systematic errors of 10 to 20% exceed the statistical ones significantly. They are reflected in a sizeable spread of the experimental results. In the experimental data are normalized to match the predictions of perturbative QCD both below 3.7 and above 5.0 GeV. Two models have been constructed which describe the differences of the normalization factors below and above the considered energy interval. Similar statements hold for the bottom threshold region in the range from 10.5 GeV to 11.2 GeV. There, however, the numerical contribution is much less significant.

A different approach for the evaluation of $\Delta \alpha^{(5)}_{\text{had}}$, based on QCD sum rules (SR), has been used in Global parton-hadron duality is used in order to reduce the influence of the data in the different intervals. This is achieved by choosing a proper polynomial, $Q_N(s)$, which is supposed to approximate the weight function $M_Z^2/s(M_Z^2 - s)$ as good as possible. Adding and subtracting $Q_N(s)$ in Eq. (3) and exploiting the analyticity of the subtracted term leads to

$$\int_{s_0}^{s_1} ds \frac{R(s)}{s(s - M_Z^2 - i\epsilon)} = \int_{s_0}^{s_1} ds \left\{ \frac{1}{s(s - M_Z^2 - i\epsilon)} - Q_N(s) \right\} + 6\pi i \oint_{|s|=s_1} ds \Pi^{QCD}_{QCD}(s)Q_N(s). \tag{8}$$

Thus the influence of the experimental data is significantly reduced in the first term of the r.h.s. and pQCD only has to be used for $|s| = s_1$ which is indicated by the superscript “QCD”. In the interval $[2m_\pi, 40 \text{ GeV}]$ has been subdivided into four parts leading to the values $\sqrt{s_1} = 3.1 \text{ GeV}, 9.46 \text{ GeV}, 30 \text{ GeV}$ and $40 \text{ GeV}$, respectively, for the upper integration bound. This subdivision is necessary as for large $N$ more and more stress is put on the unknown QCD-input in the second term of Eq. (8). The authors of applied similar methods to the energy intervals $[2m_\pi, 1.8 \text{ GeV}]$ and $[3.7 \text{ GeV}, 5.0 \text{ GeV}]$ in order to improve their previous analysis.

An approach complementary to the ones mentioned above has been chosen in. It is based on unsubtracted dispersion relations (UDR) which are used in order to evaluate the electromagnetic coupling in the $\overline{\text{MS}}$ scheme. For the energy region below 1.8 GeV the data analysis of is adopted. Then four-loop running is accompanied by three-loop matching in order to arrive at $\bar{\alpha}(M_Z^2)$, which subsequently has to be transformed to the on-shell quantity $\alpha(M_Z^2)$. Via this method no complications in connection with the $J/\Psi$ or $\Upsilon$ resonances occur. However, one encounters a much stronger dependence on the quark masses.
Table 1: Comparison of the recent improvements on the error of \( \Delta \alpha_{\text{had}}^{(5)}(M_Z^2) \) and \( \alpha_{\mu}^{\text{had}} \) with the values given in [2]. The column “comment” reminds on the different methods used in the analysis as described in the text. (\*\( \Delta \alpha_{\text{top}}(M_Z^2) \) subtracted; \*\( \text{value corresponding to } \alpha_{\mu}(M_Z^2) = 0.118 \) adopted.)

| \( \Delta \alpha_{\text{had}}^{(5)}(M_Z^2) \times 10^4 \) | \( \alpha_{\mu}^{\text{had}} \times 10^{-11} \) | Ref. | comment          |
|---------------------------------|----------------|-------|----------------|
| 280 ± 7                         | 7024 ± 153     |       | data           |
| 281.7 ± 6.2                     | 7011 ± 94      |       | \( \tau \) data|
| 278.4 ± 2.6\*                   | 6951 ± 75      |       | + pQCD         |
| 277.5 ± 1.7                     | —              |       | + “charm threshold” |
| 277.6 ± 4.1                     | —              |       | SR             |
| 277.3 ± 2.0\*\*                 | —              |       | \( \tau \) data + UDR |
| 277.0 ± 1.6\*\*                 | 6924 ± 62      |       | \( \tau \) data + pQCD + SR |

In Tab. 1 the recent evaluations of \( \Delta \alpha_{\text{had}}^{(5)}(M_Z^2) \) and \( \alpha_{\mu}^{\text{had}} \) are compared with the one of Ref. 2. A significant reduction of the error is observed. It is very impressive that the new analysis show very good agreement both in their central values and in their quoted errors. This development is very promising and the new values should be considered in the analyses of the precision data. Once more precise experimental input is available it can replace the theory-motivated parts in Refs. 7, 12, 5, 13, 14.

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References

1. G. D'Agostini and G. Degrassi, Report Nos.: DFPD-99/TH/02 and hep-ph/9902226.
2. S. Eidelman and F. Jegerlehner, Z. Phys. C 67, 585 (1995).
3. P. Gambino, Proceedings of the IVth Int. Symp. on Radiative Corrections (RADCOR 99), Barcelona, Spain, Sept. 8-12, 1998.
4. M. Steinhauser, Phys. Lett. B 429, 158 (1998).
5. J.H. Kühn and M. Steinhauser, Phys. Lett. B 437, 425 (1998).
6. A. Czarnecki and W.J. Marciano, Report Nos.: BNL-HET-98-43 and hep-ph/9810512, contribution to the 5th International Workshop on Tau Lepton Physics (TAU 98), Santander, Spain, Sept. 14-17, 1998.
7. R. Alemany, M. Davier, and A. Höcker, Eur. Phys. J. C 2, 123 (1998).
8. J. Thompson, private communication.
9. ALEPH Collaboration (R. Barate et al.), *Eur. Phys. J* C 4, 409 (1998).
10. OPAL Collaboration (K. Ackerstaff et al.), Report Nos.: CERN-EP-98-102 and hep-ex/9808019.
11. K.G. Chetyrkin, A.H. Hoang, J.H. Kühn, M. Steinhauser, and T. Teubner, *Eur. Phys. J.* C 2, 137 (1998).
12. M. Davier and A. Höcker, *Phys. Lett.* B 419, 419 (1998).
13. S. Groote, J.G. Körner, K. Schilcher, and N.F. Nasrallah, *Phys. Lett.* B 440, 375 (1998).
14. M. Davier and A. Höcker, *Phys. Lett.* B 435, 427 (1998).
15. J. Erler, *Phys. Rev.* D 59, 054008 (1999).