Partition Functions and Stability Criteria of Topological Insulators

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Abstract

The non-chiral edge excitations of quantum spin Hall systems and topological insulators are described by means of their partition function. The stability of topological phases protected by time-reversal symmetry is rediscussed in this context and put in relation with the existence of discrete anomalies and the lack of modular invariance of the partition function. The $\mathbb{Z}_2$ characterization of stable topological insulators is extended to systems with interacting and non-Abelian edge excitations.
1 Introduction

The study of topological phases of matter has considerably grown in recent years and new systems have been investigated both theoretically and experimentally [1]. It is now apparent that some remarkable topological features of quantum Hall states can occur in a wider set of systems and thus be more universal and robust. In particular, non-chiral topological states, such as those of the quantum spin Hall effect, of topological insulators and of topological superconductors, do not require strong magnetic fields and exist in three space dimensions [2][3].

A characteristic feature of topological states is the existence of massless edge excitations that are well accounted for by low-energy effective field theory descriptions [4]. The response of topological states to external disturbances does not occur in the gapped bulk, but manifests itself through the edge dynamics. While the edge excitations of chiral states, such as quantum Hall states and Chern insulators, are absolutely stable, those of non-chiral topological states can interact and become gapful, leading to the decay into topologically trivial phases. In some cases, edge interactions are forbidden by the presence of (discrete) symmetries: we then speak of symmetry protected topological phases of matter [5].

Topological insulators protected by time-reversal (TR) symmetry have been first analyzed in free fermion systems using band theory [6][7]. These systems were found to be characterized by a topological bulk quantity equal to the $\mathbb{Z}_2$ index $(-1)^N$, where $N$ is the number of fermion edge modes of each chirality. The odd (even) $N$ cases were shown to be stable (unstable) upon using a remarkable flux argument by Fu, Kane and Mele [6]. The stability analysis was then reformulated in terms of the dynamics of edge excitations in Refs. [8][9][10], by using conformal field theory (CFT) methods [11]. It was shown that the $\mathbb{Z}_2$ classification extends to interacting systems that are described by Abelian edge excitations within the so-called $K$-matrix formalism [4].

The analysis of Levin and Stern led to the following simple and general result for the $\mathbb{Z}_2$ index [8]:

\[ (-1)^{2\Delta S}, \quad 2\Delta S = \frac{\sigma_{sH}}{e^*}, \quad (1.1) \]

where $\sigma_{sH}$ is the spin Hall conductance and $e^*$ is the minimal fractional charge for one of the spin (chiral) components, in units of $e/2\pi$ and $e$, respectively. Their ratio measures the spin $\Delta S$ of an excitation created at the edge, as explained later. An odd (even) ratio corresponds to stable (unstable) systems, as this quantity reduces to the number of fermion modes in the non-interacting case.

In this paper, we rederive and further extend the $\mathbb{Z}_2$ stability criterion (1.1) by studying the partition function of edge excitations. In earlier work, we obtained the general form of the partition function for both Abelian and non-Abelian quantum Hall states [12][13][14]. Upon generalizing this analysis to the quantum spin Hall effect and topological insulators,
we can apply the flux argument for stability to any type of interacting topological insulator and prove the general validity of (1.1).

Our analysis clarifies that the stability is associated to the presence of an anomaly in the $Z_2$ symmetry of fermion number parity at the edge, also equal to the edge spin parity $(-1)^{2S}$. Actually, the $U(1)_S$ spin symmetry of the quantum spin Hall effect is explicitly broken to $(-1)^{2S}$ by spin-orbit interaction and other TR invariant relativistic corrections that are present in generic topological insulators [1]. The $Z_2$ anomaly is thus the remnant of the $U(1)_S$ anomaly of the spin Hall effect, which is analogous to the $U(1)_Q$ charge anomaly in the Hall effect [4].

The partition function is obtained in the double periodic geometry of the torus made by one circular edge and compact Euclidean time for temperature [11][12]. This function contains anyonic sectors for each chirality, that involve sums over charged and neutral edge modes. This structure allows one to disentangle the Abelian charged mode that determines the $Z_2$ anomaly, from the neutral modes, either Abelian or non-Abelian, that are transparent. We can thus analyze the stability (anomaly) for general interacting system.

The response of the topological state, i.e. of its edge dynamics, to external perturbations can be easily described by using the partition function. One can see how its different sectors transforms among themselves by inserting fluxes, i.e. under the effect of an electromagnetic background. Furthermore, the behaviour under modular transformations, the discrete coordinate changes respecting the periodicities of the torus, yield the response to (some type of) gravitational backgrounds: they amount, e.g. to adding momentum to the system or to swapping space and time (respectively, the $T$ and $S$ modular transformations).

We find that the stability (instability) of non-chiral edge states is associated to the impossibility (possibility) of having a modular invariant partition function that is consistent with time reversal symmetry. The modular non-invariance of stable topological states is actually a kind of gravitational anomaly accompanying the spin parity anomaly. In some systems, we find that flux insertions and modular transformations act in similar way on the four “spin sectors” always present in fermionic systems, corresponding to periodic and antiperiodic boundary conditions in space and time [11][15]. In other systems, the two backgrounds have different effects owing to the different couplings of charged and neutral modes.

Modular non-invariance and stability of topological states have already been discussed in a paper by Ryu and Zhang [16] that actually motivated our work. They considered the topological superconductors, whose neutral Majorana edge modes cannot be analyzed by flux arguments, and argued that modular non-invariance of the partition function could be used as a criterion of stability. Then they showed that the system of $N_f$ Majorana fermions is unstable for $N_f = 0$ modulo 8, finding agreement with the study of possible interactions [17][18]. Our analysis reproduces this result, but cannot presently provide a
general approach to interacting topological superconductors. This open issue is discussed in the conclusions.

The paper is organized as follows. In Section two, we briefly describe the setting of the problem and recall the flux insertion argument by Fu, Kane and Mele. In Section three, we introduce the partition function of topological insulators with one edge mode per spin, described by the Luttinger liquid CFT with central charge $c = 1$. We discuss the effect of flux insertions and modular transformations, and rederive the $Z_2$ anomaly and the stability argument in this context. In Section four, we extend the analysis to general non-chiral edge CFTs and discuss some interesting examples. In Section five, we present our conclusions. The Appendix contains the details on modular transformations of Abelian and non-Abelian theories.

2 Flux argument and stability analysis

2.1 Laughlin flux insertion

We start by recalling the Laughlin argument for the quantization of the Hall current in the annulus geometry (see Fig. 1(a)) [19]. Upon the adiabatic insertion of one quantum of flux $\Phi_0$, the bulk Hamiltonian returns to itself while states in the spectrum drift one into another, leading to the so-called spectral flow. This amounts to the transport of a charge equal to the Hall conductivity between the two edges.

$$
\Phi \rightarrow \Phi + \Phi_0, \quad H [\Phi + \Phi_0] = H [\Phi],
$$

$$
Q \rightarrow Q + \Delta Q = \nu.
$$

From the point of view of the conformal field theory describing one edge of the annulus, the spectral flow corresponds to the non-conservation of charge, namely a $U(1)_Q$ chiral anomaly in two dimensions [4][20]. The associated index theorem for the integrated anomaly reads:

$$
\Delta Q = \int_0^\infty dt \int_0^{2\pi R} dx \partial_t J^0_R = \frac{e\nu}{2\pi} \int F = \nu \epsilon n. \quad n \in \mathbb{Z}.
$$

The last term in this expression is a topological quantity, the first Chern class of the two-dimensional electromagnetic field, $F = \frac{1}{2} F_{ij} dx^i \wedge dx^j$, whose integer values count the number of inserted flux quanta. We also recall that the chiral anomaly is an universal effect that is exact for any strength of the interaction [11].

We can interpreted the chiral anomaly of the edge as the response of the topological bulk to an electromagnetic background, as well explained in the associated description in terms of the effective hydrodynamic Chern-Simons theory [4]. Other chiral topological phases are also associated to anomalies, leading to the breaking of continuous symmetries and to anomalous currents. These correspond to non-vanishing values of transport coefficients,
such as the Hall conductivity $\sigma_H$ and/or thermal transport coefficient $\kappa_H$. The general analysis of topological phases characterized by electromagnetic, gravitational and mixed chiral anomalies in two and higher dimensions has been carried out in Ref. [21]; these results have identified some $\mathbb{Z}$ classes of topological states in $d$ dimensions that actually fit within the tenth-fold classification of non-interacting topological systems [22], thus extending it to interacting cases.

The next step is to discuss the Laughlin argument for the non-chiral topological state of the quantum spin Hall effect (see Fig. 1(b)). Consider the system made of two copies of the $\nu = 1$ Hall effect having opposite spin and chiralities. The time reversal (TR) transformation $T$ acts on up and down spin electrons, resp. $\psi_\uparrow$ and $\psi_\downarrow$, as follows,

$$
T : \psi_k^\uparrow \rightarrow \psi_{-k}^\downarrow, \quad \psi_k^\downarrow \rightarrow -\psi_{-k}^\uparrow,
$$

(2.3)

thus leaving the system invariant.

The addition of one flux causes the drift of up and down electrons in opposite directions with respect to the Fermi surface at each edge (Fig. 2). From the point of view of the CFT at one edge, say the outer one, the effect is to create a neutral excitation with spin one [23]:

$$
\Delta Q = \Delta Q_\uparrow + \Delta Q_\downarrow = 0, \quad \Delta S = \frac{1}{2} - \left( -\frac{1}{2} \right) = 1,
$$

(2.4)

where $\Delta S = \Delta Q_\uparrow = \nu^\uparrow$, i.e. the spin Hall current is equal the Hall current of one chiral component (in appropriate units).

Arguing as in the chiral case, we can characterize this topological state by the spin chiral anomaly $U(1)_S$ corresponding to an integer spin Hall conductivity. However, the quantum spin Hall effect is a rather academic model of topological insulator: in general, spin-orbit coupling and other relativistic effects cannot be neglected, that only conserve the total angular momentum; thus, the $U(1)_S$ symmetry is explicitly broken, the spin current is not defined and the spin Hall conductivity vanishes.
Figure 2: Flux insertion in the QSHE: up and down spins are displaced w.r.t. the Fermi surfaces at $L$ and $R$ edges of the annulus (dashed-dotted lines).

In the following, we want to discuss general topological insulators that only possess time reversal symmetry and cannot be characterized by anomalies of continuum symmetries; these represent interacting topological phases of a different kind. In our analysis we shall mostly keep the spin-quantum Hall description of topological insulators, that can be investigated by CFT methods, corresponding to the the ideal limit of spin-orbit coupling switched off. Nevertheless, we shall discuss properties that do not rely on the spin being conserved.

2.2 Fu-Kane-Mele flux argument

A topological insulator with a single free fermion edge mode per spin (chirality) is stable because the mass term coupling the two chiralities is forbidden by TR symmetry, as follows:

$$\mathcal{T} : H_{\text{int}} = m \int \psi_\uparrow \psi_\downarrow + h.c. \rightarrow -H_{\text{int}}.$$  \hspace{1cm} (2.5)

Another TR invariant mass term can be written that couples two fermions modes per spin, but a single fermion always remains massless in a system with an odd number of modes [1]. Of course, if TR symmetry is broken all edge excitations become gapful (and the insulator trivial).

The analysis of gapful instabilities due to more general, non-quadratic interactions compatible with TR can be done in some cases, but we consider here another criterion for
stability that is associated with a symmetry and a discrete $\mathbb{Z}_2$ anomaly. This is the Fu-Kane-Mele flux insertion argument called the “spin pump” (a cyclic adiabatic process) [6]. We mostly follow the presentation of Ref.[8].

The insertion of magnetic flux breaks TR symmetry, owing to:

$$T H[\Phi] T^{-1} = H[-\Phi]. \tag{2.6}$$

This relation together with the periodicity $H[\Phi] = H[\Phi + \Phi_0]$, implies that the bulk Hamiltonian is TR invariant for a discrete set of flux values:

$$\Phi = 0, \frac{\Phi_0}{2}, \frac{3\Phi_0}{2}, \ldots \tag{2.7}$$

The Fu-Kane-Mele analysis of band insulators let them to define an index called “TR invariant polarization”, $(−1)^{P_θ} = ±1$, that enjoys the following properties:

i) It is a bulk topological quantity, conserved by TR symmetry.

ii) Its value is equal to the spin parity (fermion number) at the edge,

$$(-1)^{P_θ} = (-1)^{N_↑ + N_↓} = (-1)^{2S}. \tag{2.8}$$

iii) In a stable topological insulator, it changes value between TR invariant points (2.7) separated by half flux $\Delta \Phi = \Phi_0/2$.

The energy change of the edge ground state as the flux is varied from zero to $\Phi_0/2$ is shown in Fig. 4. At $\Phi_0/2$, the evolved ground state $|Ω⟩$ necessarily meets with one excited state $|ex⟩$ owing to Kramers theorem. Going back to $\Phi = 0$, the excited state must have an energy $O(1/R)$ from the work done by adding a flux quantum in a system of size $R$. It then follows that the existence of a Kramers (spin one-half) pair at the edge for $\Phi = \Phi_0/2$ implies the presence of a gapless excitation at $\Phi = 0$ that is protected by TR symmetry. In the case of two fermion modes, the corresponding spin one excitation created at the boundary would not be protected by Kramers theorem against energy splitting from its TR companions. The argument then extend to odd and even numbers of fermion modes.

This completes the argument for stability of the topological phase with an odd number of fermion modes, leading to the $\mathbb{Z}_2$ classification of topological insulators in the free fermion case. Let us add some remarks:

*The local version of the Kramers theorem at each edge of the system has been discussed in [8].
\[ \Phi = 0 : (-1)^{2S} = 1 \rightarrow \Phi = \frac{\Phi_0}{2} : (-1)^{2S} = -1. \] (2.9)

ii) The spin parity is conserved by TR symmetry, being just another way to state the Kramers theorem. This \( \mathbb{Z}_2 \) invariance is the remnant of the continuous \( U(1)_S \) symmetry of the quantum spin Hall effect that gets broken by relativistic effects.

iii) At the two TR symmetric points, \( \Phi = 0, \frac{\Phi_0}{2} \), the spin parity takes different values without having included TR breaking terms in the Hamiltonian. Therefore, this quantity is anomalous.

In conclusions, TR invariant topological insulators are associated with the \( \mathbb{Z}_2 \) spin parity symmetry \( (-1)^{2S} \) that is anomalous. The full spin symmetry \( U(1)_S \) would also be anomalous but it is explicitly broken in general topological insulators. We note that the Fu-Kane-Mele argument is a generalization of the Laughlin argument that makes it apparent the \( \mathbb{Z}_2 \) spin parity anomaly\(^\dagger\). The goal of this paper is to generalize the stability analysis

\(^\dagger\)Discrete anomalies in topological insulators have also been discussed in [24]; for a general introduction,
to interacting systems through the study of partition functions.

3 Partition functions of topological insulators

3.1 Chiral edge system

We first recall from Ref. [12] the construction of the partition function of the quantum Hall effect for the Laughlin states, \( \nu = 1/p, \) \( p \) odd; next, we generalize it to topological insulators and discuss the \( \mathbb{Z}_2 \) anomaly and the stability criterion in this setting.

We consider the grand-canonical partition function of states at the outer edge of the annulus with radius \( R \) (see Fig. 5). This circle and the time period \( \beta \) realize the geometry of the torus. The energy and momentum of chiral excitations are expressed by the eigenvalues of the Virasoro generator \( L_0, E = P = L_0/R \) (we set the Fermi velocity \( v = 1 \) for simplicity). The trace over the states with Gibbs weight involving chemical and electric potential decomposes into orthogonal sectors \( \mathcal{H}^{(\lambda)} \), corresponding to a given value of fractional charge plus any number of electrons, \( Q = \lambda/p + n, n \in \mathbb{Z} \); there are \( p \) sectors for \( \lambda = 0, 1, \ldots, p-1 \).

The partition function for one sector takes the form:

\[
K_\lambda(\tau, \zeta; p) = \text{Tr}_{\mathcal{H}^{(\lambda)}} \left[ \exp \left( i2\pi \tau L_0 + i2\pi \zeta Q \right) \right] = \frac{F(\tau, \zeta)}{\eta(\tau)} \sum_{n \in \mathbb{Z}} \exp \left( i2\pi \left( \tau \frac{(np + \lambda)^2}{2p} + \zeta \frac{np + \lambda}{p} \right) \right); \quad (3.1)
\]

see [25].
Figure 5: Torus geometry with periods $(2\pi R, \beta)$.

This function is parameterized by the two complex numbers,

$$\tau = \frac{i\beta}{2\pi R} + t, \quad \zeta = \frac{\beta}{2\pi} (iV_o + \mu),$$

(3.2)

that are the modular parameter $\tau$ and the “coordinate” $\zeta$. $\text{Im}\tau > 0$ is the ratio of the two periods and $\text{Re}\tau$ is the torsion parameter conjugate to momentum $P$; $\zeta$ contains the electric $V_o$ and chemical $\mu$ potentials.

The expression of $K_\lambda$ (3.1) involves a sum of characters of the representations of the $U(1)$ current algebra of the $c = 1$ CFT with charges $Q = \lambda/p + n$ and conformal weight $h = (\lambda + pn)^2/2p$, as is apparent in the exponent of (3.1) [11]. It can be obtained by canonical quantization [20] as well as by using CFT representation theory supplemented by some physical conditions on the charge and statistics of electron excitations [12]. The formula involves the ratio of a theta function with characteristics $\Theta^{[\lambda/p]}(\zeta|p\tau)$ and the Dedekind function $\eta(\tau)$ describing particle-hole excitations. The non-holomorphic prefactor $F = \exp \left[ -\pi (\text{Im}z)^2/p \text{Im}\tau \right]$ is explained in Ref.[12]. Altogether, the partition function of one chiral edge is given by the multiplet of functions $K_\lambda$, for $\lambda = 1, \ldots, p$, having periodicity $K_{\lambda+p} = K_\lambda$, that correspond to the $p$ anyon sectors.

The torus geometry is left invariant by the modular transformations, the discrete coordinate changes consistent with the double periodicity. These act on the modular parameter $\tau$ and the coordinate $\zeta$ as follows [12]:

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \zeta \rightarrow \frac{\zeta}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1,$$

(3.3)

and span the group $\Gamma = SL(2,\mathbb{Z})/\mathbb{Z}_2$. The modular group is infinite dimensional and is generated by two transformations, $T : \tau \rightarrow \tau + 1, \ \zeta \rightarrow \zeta$ and $S : \tau \rightarrow -1/\tau, \ \zeta \rightarrow -\zeta/\tau$. 


obeying the relations $S^2 = (ST)^3 = C$, where $C$ is the charge conjugation matrix, $C^2 = 1$ [11]. In addition, there are the two periodicities of the coordinate $\zeta$ at $\tau$ fixed: $U: \zeta \to \zeta + 1$ and $V: \zeta \to \zeta + \tau$.

The modular transformations belong to the group of two-dimensional diffeomorphisms of the torus, being the global transformations not connected to the identity: they are the “large” gauge transformations of the conformal theory placed in a gravitational background. In similar way, the flux insertions are large gauge transformations of the electromagnetic background. The lack of invariance of the partition function signals the presence of gravitational and gauge anomalies, respectively. These do not led to inconsistencies if the backgrounds are classical, i.e. not quantized, as in our case, but nevertheless characterize the low-energy physics of the system [22].

The multiplet of $K_\lambda$ transforms linearly under the modular group and each generator has physical significance, as we now recall [12]. The $S$ transformation reads:

$$S: \quad K_\lambda \left(\frac{-1}{\tau}, \frac{-\zeta}{\tau}\right) = e^{i\varphi} \sum_{\mu=1}^{p} S_{\lambda \mu} K_\mu(\tau, \zeta), \quad S_{\lambda \mu} = \frac{1}{\sqrt{p}} \exp \left( i\frac{2\pi \lambda \mu}{p} \right),$$

(3.4)

where $S_{\lambda \mu}$ is the modular $S$-matrix and $\varphi$ is an overall phase. As is well know in the CFT literature, the $S$ unitary transformation expresses a completeness condition on the spectrum of the theory; it also determines the fusion rules of excitations via the Verlinde formula [11].

The invariance under $T^2$ transformation up to a global phase,

$$T^2: \quad K_\lambda(\tau + 2, \zeta) = \exp \left( i4\pi h_\lambda \right) K_\lambda(\tau, \zeta), \quad h_\lambda = \frac{\lambda^2}{2p},$$

(3.5)

implies that the electron excitations being summed up in each anyon sector have odd integer statistics (half integer conformal dimension). On the other hand, the $U$ invariance,

$$U: \quad K_\lambda(\tau, \zeta + 1) = \exp \left( i2\pi \lambda/p \right) K_\lambda(\tau, \zeta),$$

(3.6)

implies electrons with integer charge. The $T^2$ and $U$ transformations are readily checked from the expression (3.1).

Finally, the $V$ transformation,

$$V: \quad K_\lambda(\tau, \zeta + \tau) = e^{i\phi} K_{\lambda+1}(\tau, \zeta), \quad \Delta \Phi = \Phi_0,$$

(3.7)

(with $\phi$ another global phase) is very important because it realizes the change of electric potential following the addition of one flux quantum $\Phi_0$. The shift of index $\lambda \to \lambda + 1$ corresponds to the spectral flow $Q \to Q + \nu$, $\nu = 1/p$, discussed in the previous section.

Altogether the single edge is described by a multiplet of partition functions $K_\lambda$, that is not modular invariant, meaning that the chiral anomaly, i.e. the spectral flow, implies a discrete gravitational anomaly. More precisely, the $K_\lambda$ transform in a linear unitary vector
representation of \( \Gamma_\theta = \text{span}(T^2, S) \), a subgroup of \( \Gamma \) including only \( T^2 \) transformations and thus allowing fermionic statistics [12]. The dimension \( p \) of the representation is equal to the value of the Wen topological order [4].

### 3.2 Non-chiral edge

The partition function for the quantum spin Hall system made by a pair of Laughlin states is obtained by combining chiral and antichiral sectors for up and down spins, respectively, thus obtaining the expressions \( K^\mu_\lambda \tilde{K}^\mu_{-\lambda} \). In the case of the Hall effect of charge and spin, the presence of extended states in the bulk of the annulus allows for the matching of fractional charges between opposite edges. In the case of topological insulators, the bulk is insulating, thus the fractional charges should be matched locally at each edge (the \( U \) condition). We thus obtain the following expression for the partition function of a single edge:

\[
Z^{NS}(\tau, \zeta) = \sum_{\lambda=1}^{p} K^\mu_\lambda \tilde{K}^\mu_{-\lambda}.
\]  

This is invariant under \( S, T^2, U, V \). It turns out that this quantity is formally equal to the quantum Hall effect partition function for the system of two edges, i.e. for the whole annulus [12]. However, the physical interpretation in the case of topological insulators is rather different, since it only describes a single edge of the annulus and the different charge sectors do not correspond to bulk anyonic excitations, but just describe degenerate ground states at one edge.

The expression \( Z^{NS} \) is not completely modular invariant, because the transformations in the quotient \( \Gamma / \Gamma^0 \sim S_3 \) deform it into other quantities according to a general pattern. Indeed, the partition functions of fermionic systems always involve four terms corresponding to the four spins structures needed for defining spinors on the torus [15]. These amount to choosing periodic (\( P \)) and antiperiodic (\( A \)) boundary conditions for fermion fields in each direction (in general there are \( 2^{2g} \) terms on a genus \( g \) surface). These terms are known as the Neveu-Schwarz (\( NS \)) and Ramond (\( R \)) sectors and their tildes, as follows:

\[
NS, \tilde{NS}, R, \tilde{R}, \text{ respectively : } (AA), (AP), (PA), (PP).
\]  

The expression (3.8) is identified with the Neveu-Schwarz sector since the natural fermionic boundary conditions are antiperiodic:

\[
Z^{NS} = \text{Tr}_A \left[ \exp \left( i2\pi \tau L_0 + i2\pi \zeta Q + \text{h.c.} \right) \right].
\]  

The other expressions are defined as:

\[
Z^R = \text{Tr}_P \left[ \exp \left( i2\pi \tau L_0 + i2\pi \zeta Q + \text{h.c.} \right) \right],
\]

\[
Z^{\tilde{NS}} = \text{Tr}_A \left[ (-1)^{N^+_1+N^+_2} \exp \left( i2\pi \tau L_0 + i2\pi \zeta Q + \text{h.c.} \right) \right],
\]

\[
Z^{\tilde{R}} = \text{Tr}_P \left[ (-1)^{N^+_1+N^+_2} \exp \left( i2\pi \tau L_0 + i2\pi \zeta Q + \text{h.c.} \right) \right],
\]  

(3.11)
where periodic conditions in time introduce the sign $(-1)^F = (-1)^{N_1+N_k}$.

The modular transformations among the four terms are depicted in Fig. 6(b) and are explicitly checked in the Appendix. They form a triplet, $Z^{NS}, Z^{\tilde{NS}}, Z^R$, and a singlet, $Z^{\tilde{R}}$. Each one of the four spin sectors is made of $p$ “anyonic” sectors:

$$Z^s = \sum_{\lambda=1}^{p} K^s_{\lambda} K^{-s}_{-\lambda}, \quad s = NS, \tilde{NS}, R, \tilde{R}. \quad (3.12)$$

Furthermore, these $Z^\sigma$ are invariant under $T^2$, i.e. possess fermions excitations, under $U$ for charge matching and under $V$ for the spectral flow among the $p$ anyon sectors.

Regarding the $\tilde{NS}$ sector, the partition function and its anyon sectors are defined as follows,

$$Z^{\tilde{NS}}(\tau, \zeta) = Z^{NS}(\tau + 1, \zeta),$$
$$K^{\tilde{NS}}_\lambda(\tau, \zeta) = e^{i\theta_\lambda} K^{NS}_\lambda(\tau + 1, \zeta). \quad (3.13)$$

In this expression, the phase $\theta_\lambda = 2\pi \left( \frac{\lambda^2}{2} - \frac{\lambda}{2p} + \frac{1}{24} \right)$ is included for convenience; we also write $K^{NS}_\lambda \equiv K^S_\lambda$ and omit spin arrows for simplicity. Note that the term brought by the $T$ transformation in the summation over fermions inside $K^{NS}_\lambda$, i.e. $e^{i2\pi L_0}$, is proportional to $(-1)^F$ owing to the half integer conformal dimension of fermions.

The Ramond sector is similarly obtained by acting with $ST$ on $Z^{NS}$ and $Z^{\tilde{R}}$ is defined by inserting the $(-1)^F$ sign into the Ramond expression. Explicit examples will be given later and are collected in the Appendix. Altogether, the pattern in Fig. 6(b) shows the response of the topological insulator edge to modular transformations, i.e discrete diffeomorphisms not connected to the identity. The four spin sectors, each one made of $p$ anyon sectors, realize a kind of decoupling of anyonic and fermionic properties, as it will be more clear in the following.

### 3.3 Stability analysis

We now consider the response to adding magnetic fluxes and recover the Fu-Kane-Mele stability analysis in the context of partition functions. As already said, each spin sector is invariant under $V$, the addition of one flux quantum. The addition of half flux $V^{1/2}$ transforms the sectors as shown in Fig. 6(a). In particular, the Neveu-Schwarz sector is mapped into the Ramond sector,

$$V^{1/2} : Z^{NS}(\tau, \zeta) \to Z^{NS} \left( \tau, \zeta + \frac{\tau}{2} \right) = Z^{R}(\zeta, \tau). \quad (3.14)$$

The addition of any half integer number of fluxes $V^{1/2+n}$ yields the same result, up to a reshuffling of anyon sectors within $Z^R$ (cf. Eq.(3.12)). In order to disentangle the anyonic degeneracy from the electron degeneracy relevant for the Fu-Kane-Mele stability analysis [6],
we follow the extension of the argument due to Levin-Stern [8]. We observe that the addition of $p$ fluxes creates an electron excitation within the same anyon sector, as it corresponds to a symmetry of each $K_\lambda$,

$$V^p : K_\lambda \rightarrow K_{\lambda+p} = K_\lambda, \quad \Delta Q^\dagger = \frac{p}{p} = 1. \quad (3.15)$$

Therefore, the addition of $p/2$ fluxes will create a spin one-half excitation in the topological insulator edge, $\Delta S = \Delta Q^\dagger = 1/2$, while staying in the same anyonic sector. It is therefore convenient to define the Ramond sectors by the action of $V^{\frac{p}{2}}$:

$$V^{\frac{p}{2}} : K_\lambda (\tau, \zeta) \rightarrow K_{\lambda} \left(\tau, \zeta + \frac{p\tau}{2}\right) \sim K_{\lambda + \frac{p}{2}} (\tau, \zeta) = K_R^R (\tau, \zeta), \quad (3.16)$$

where ($\sim$) stands for equality up to a global phase.

We can now extend the stability analysis discussed in the free fermion case (Section two). Upon applying $p/2$ fluxes, the Neveu-Schwarz ground state $|\Omega\rangle_{NS}$, the lowest state in $K_0 \overline{K}_0$, evolves in the Ramond ground state $|\Omega\rangle_R$ present in $K_0^R \overline{K}_0^R = K_{p/2} \overline{K}_{p/2}$. This becomes
a spin-half partner of a degenerate Kramers pair: following the argument of Section two, this proves the stability of the topological phase in the system made by a pair of Laughlin states, for any \( p \) value.

The existence of the Kramers pair and the behaviour of the spectrum shown in Fig. 4 can be easily checked by inspecting the low lying states contained in \( K_0 \bar{K}_0 \) and \( K_0^R \bar{K}_0^R \); one expands to lowest order in \( q^\lambda \), with \( q = \exp(i2\pi \tau) \), and checks the terms to \( O(w^0w^0) \), \( w = \exp(i2\pi \zeta) \), i.e. not involving additional particles.

We can also recompute the spin parity of the Neveu-Schwarz and Ramond ground states:

\[
(-1)^{2S} |\Omega\rangle_{NS} = |\Omega\rangle_{NS} \rightarrow (-1)^{2S} |\Omega\rangle_R = - |\Omega\rangle_R , \quad \Delta S = \Delta Q^+ = \frac{1}{2} . \tag{3.17}
\]

The validity of the Levin-Stern stability index discussed in the Introduction (Eq.(1.1)) is thus verified for this system:

\[
2\Delta S = 2\Delta Q^+ = \frac{\sigma_{sH}}{e^*} = 1 , \quad (-1)^{2\Delta S} = -1 , \tag{3.18}
\]

where \( \sigma_{sH} = \nu^\dagger = 1/p \) and \( 1/e^* = p \) is the number of charge sectors, i.e. the periodicity of \( K_\lambda \). The (would-be) spin transport \( \Delta S \) involved in this index, equal to the Hall current of one chiral component, is that relative to the addition of half of the fluxes needed for creating an electron excitation within a given anyon sector.

In conclusion, we have found the important fact that the spin parity of the Ramond ground state is different from that of the Neveu-Schwarz ground state. This is a manifestation of the discrete anomaly \( \mathbb{Z}_2 \): different sectors of the path integral (Eq.(3.10) and (3.11)) have associated different quantum numbers [25]; in the same way, the chiral sectors \( K_\lambda \) have associated different charges due to the \( U(1)_Q \) anomaly (Eq.(3.7)).

### 3.4 Stability and modular non-invariance

In a fermionic non-chiral system composed of the four spin sectors (3.10),(3.11) it is always possible to find a modular invariant partition function by summing over all sectors,

\[
Z_{\text{Ising}} = Z^{NS} + \bar{Z}^{\bar{NS}} + Z^R + \bar{Z}^{\bar{R}} . \tag{3.19}
\]

This is the so-called Ising projection because it occurs in CFTs applied to statistical models like the Ising model, its supersymmetric generalizations etc. [11]. This quantity is \( S,T,U,V \frac{1}{2} \) invariant.

Nonetheless, we would like to argue that the theory defined by \( Z_{\text{Ising}} \) may not be consistent with the TR symmetry of topological insulators, implying spin parity conservation. In presence of the \( \mathbb{Z}_2 \) anomaly, the partition function (3.19) sums spin sectors with different values of the ground state spin parity, thus violating TR symmetry. Therefore, this symmetry is explicitly broken or not defined in that theory.
If we insist on preserving TR symmetry, we should not sum over spin sectors and leave them as the components of a four-dimensional vector,

\[ Z_{\text{TR}} = \left( Z^{NS}, Z^{\tilde{NS}}, Z^R, Z^{\tilde{R}} \right), \]

that carries a non-trivial representation of \( \Gamma/\Gamma_\theta \sim S_3 \). Therefore, the \( \mathbb{Z}_2 \) anomaly is associated to a \( S_3 \) gravitational anomaly. In the theory described by this set of partition functions, \( Z^{NS} \) represent the TR invariant edge system, while the other functions, \( Z^{\tilde{NS}}, Z^R, Z^{\tilde{R}} \), are excited states of the system in presence of electromagnetic or gravitational backgrounds.

We thus obtain the following result, later shown to be valid in general:

\[ TR \text{ symmetry } + \text{ anomaly } \leftrightarrow \text{ no modular invariance } \leftrightarrow \text{ topological insulator}, \]

\[ TR \text{ symmetry } + \text{ modular invariance } \leftrightarrow \text{ no anomaly } \leftrightarrow \text{ trivial insulator}. \] (3.21)

Let us add some remarks:

i) As in the case of the quantum Hall effect, the anomaly can be cancelled globally on the whole system by combining the partition functions of the two edges of the annulus, leading to a global modular invariant.

ii) In the quantum Hall state, the chiral partition functions like \( K_\lambda \) cannot be combined into a modular invariant for a single edge (unless a special case with \( c = 24 \)). In the topological insulator, they can or cannot be combined depending on the fate of the \( \mathbb{Z}_2 \) symmetry; this is a manifestation of the symmetry protection of the topological phase, because this can became trivial if the symmetry is not enforced. A related statement is that the stability of these systems is not solely determined by CFT properties but by the way 2 + 1-dimensional bulk symmetries are attached to CFT edge states.

iii) In the \( c = 1 \) Laughlin spin state considered in this section, the two partition functions \( Z_{\text{TR}} \) and \( Z_{\text{Ising}} \) can be obtained by canonical quantization of the following Lagrangians. \( Z_{NS} \) is obtained by independent quantization of two copies of the chiral boson theory [20],

\[ S_1 = \frac{1}{4\pi} \int \partial_x \xi (\partial_t \xi - v \partial_x \xi) + \frac{1}{4\pi} \int \partial_x \chi (\partial_t \chi + v \partial_x \chi), \] (3.22)

with rational compactified radius \( r^2 = p/q, p, q \in \mathbb{Z} \), and antiperiodic boundary conditions. The spectrum of Virasoro states is given by \( h = n^2/2pq \) in one chiral theory and \( \tilde{h} = m^2/2pq \) in the other one, with \( n, m \in \mathbb{Z} \).

The partition function \( Z_{\text{Ising}} \) is obtained by the quantization of the non-chiral compactified boson [11],

\[ S_2 = \frac{1}{4\pi} \int (\partial_t \phi)^2 - v^2 (\partial_x \phi)^2. \] (3.23)

One obtains the standard spectrum \( (h, \tilde{h}) = \left( \frac{n^2}{2r^2} + mr \right)^2 /2, \left( \frac{n^2}{2r^2} - mr \right)^2 /2 \right), \) with \( r^2 = p/2q \), that spans a two-dimensional even self-dual Lorentzian lattice, characteristic of compactified bosonic theories [26] [27]. The partition function resulting by summing over the
lattice is modular invariant and can be rewritten in the form (3.19):

\[
Z_{\text{Ising}} = \frac{1}{|\eta(\tau)|^2} \sum_{n,m \in \mathbb{Z}^2} q^{\frac{1}{2}(\frac{n}{\tau} + mr)^2} q^{\frac{1}{2}(\frac{m}{\tau} + nr)^2}.
\]  

(3.24)

Of course, the choice of other boundary conditions in the theory \( S_1 \) leads to the other three sectors, \( Z^{\tilde{NS}}, Z^R, Z^{\tilde{R}} \). The Ising projection then corresponds to imposing the condition \((-1)^{N^\uparrow + N^\downarrow} = 1\) that eliminates fermionic excitations from both the \((P)\) and \((A)\) spectrum. Namely, \( Z_{\text{Ising}} \) does not describe fermionic excitations and for this reason it contains less states than \( Z^{NS} \). On the other hand, the \( Z_{\text{Ising}} \) possesses additional states of the Ramond spectrum. In conclusion, the stable topological system described by \( Z^{NS} \) contains single fermion excitations and can be derived from \( S_1 \); the unstable, non TR symmetric phase has only bosonic states and is naturally described by \( S_2 \).

iv) Another indication that TR symmetry is not present in \( Z_{\text{Ising}} \) is given by the violation of the spin-statistics relation in the Ramond sector. Since \( Z^R \) and \( Z^{\tilde{R}} \) are \( T \) invariant, they only contain states with integer conformal dimension, i.e. bosonic statistics. On the other hand, their spin is half integer, upon summing \( S = 1/2 \) for Ramond ground state to those of excitations present in \( Z^R + Z^{\tilde{R}} \), i.e. \( \Delta S = (N^\uparrow + N^\downarrow)/2 \in \mathbb{Z} \). Therefore, the spectrum does not respect spin-statistics. On the other hand, the partition function \( Z'_{\text{Ising}} = Z^{NS} + Z^{\tilde{NS}} + Z^R - Z^{\tilde{R}} \) would obey the charge-statistics relation and also be modular invariant, but it is not invariant under \( V^{1/2} \).

4 General stability analysis and examples

4.1 General partition functions

The conformal theories of general quantum Hall edge states possess not only charged excitations but also neutral modes that can be Abelian or non-Abelian. These theories have the affine symmetry \( U(1) \times G/H \), where \( U(1) \) is the charge symmetry and \( G \) is another (non-Abelian) symmetry characterizing the neutral part (possibly a coset \( G/H \)). The electron field in this theory is represented by the product of a chiral vertex operator for the charge part and a chiral neutral field \( \psi_e \) of \( G/H \):

\[
\Psi_e = e^{i\alpha \varphi} \psi_e.
\]  

(4.1)

The field \( \psi_e \) should also have Abelian fusion rules with all fields in the theory: this property is needed for the electrons to have integer statistics with all excitations and for the ground state wavefunction to be unique, being written as a CFT correlator\(^\dagger\).

\(\dagger\)There are exceptions to the unique identification between wavefunction and correlators, but they will not be considered here [28].
The field $\psi_e$, called a simple current in the CFT literature [11], can be used to build a modular invariant that couples neutral and charged parts non-trivially and fulfills the physical conditions on charge and statistics of the edge spectrum. The general expression of the partition function for the Hall edge states obtained in this way is determined uniquely by two inputs: the choice of neutral $G/H$ theory and of the Abelian field $\psi_e$ that represents the electron neutral part. These simple-current modular invariant partition functions were shown to reproduce earlier results obtained by physical insight in many models and to build new ones [14].

The construction starts from the one-edge partition sum for an anyon sector, generalizing the $K_\lambda$ of the $c=1$ theory introduced in Section 3.1 (Eq.(3.1)): this involves again a basic anyon plus any number of electrons added to it, with charge $Q = \lambda/p + n$, $n \in \mathbb{Z}$. It is characterized by $\lambda$, and the neutral quantum numbers $(m, \alpha)$. In terms of states and fields, the electron excitations are obtained by fusing the basic anyon field with many electron fields, always getting a unique output owing to the Abelian fusion. This conserves the charge $\lambda$ and another neutral additive number $m$ of the simple current (assume $m$ modulo $k$ for simplicity, i.e. a $\mathbb{Z}_k$ neutral charge). Such partition function takes the form [14]:

$$
\Theta^\alpha_\lambda(\tau, \zeta) = \sum_{a=1}^{k} K_{\lambda + ap}(\tau, k\zeta; kp) \chi^\alpha_{\lambda + ap \mod k}(\tau, 0).
$$

(4.2)

The $K_{\lambda}(\tau, k\zeta; kp)$ are the characters for the charge part, while the $\chi^\alpha_m(\tau, 0)$ are sums of $G/H$ characters for the neutral part, that are labelled by the Abelian number $m$ and other, possibly non-Abelian, quantum numbers collectively denoted by $\alpha$. The explicit form of the neutral characters $\chi^\alpha_m$ is not needed, only their symmetries and modular transformations are relevant.

Equation (4.2) can be explained as follows. The basic anyon has quantum numbers $(\lambda, m, \alpha)$, with $m$ modulo $k$ and $\lambda$ modulo $kp$ owing to the periodicity:

$$
K_{\lambda}(\tau, k\zeta; kp) = K_{\lambda + kp}(\tau, k\zeta; kp).
$$

(4.3)

After adding one electron, the quantum numbers changes into $(\lambda + p, m + p, \alpha)$; then, after adding $k$ electrons these numbers return to those of the basic anyon. This explains the $k$ terms in the sum (4.2). The difference with respect to the $c=1$ case (3.1) is that $n$-electron states couple to different neutral parts for $n$ modulo $k$; actually, each $K_{\lambda}(\tau, k\zeta; kp)$ in (4.2) only sums electrons with $Q = \lambda/p + kn$, owing to its different charge normalization. Another way to state this fact is that physical $\lambda$ and neutral $m$ charges are related by a $\mathbb{Z}_k$ parity rule [29].

We can use pairs of these edge theories to model general interacting topological insulators. The functions $\Theta^\alpha_\lambda(\tau, \zeta)$ enjoy similar properties under modular transformations as the $K_{\lambda}$ of Section 3.1 and the partition function $Z_{NS}$, can be written accordingly, that couple the
up/down spin modes at one edge, though the $U$ charge condition (3.6):

$$Z^{NS} = \sum_{\lambda,\alpha} \Theta_\lambda^\alpha \overline{\Theta}_{-\lambda}^\alpha. \quad (4.4)$$

In this sum, the range for the $(\lambda, \alpha)$ values is given by the Wen topological order. In earlier works [12][13][29][14], we showed that known expressions for the partition functions of multicomponent Abelian theories in the $K$-matrix formalism can be recast in the form of (4.2), (4.4); non-Abelian states are also written in this form, as e.g. the Read-Rezayi parafermion states with $G/H = SU(2)/U(1)$.

Analogous expressions are obtained for the other spin sectors $\tilde{N}S, R, \tilde{R}$, by acting with the $T$ and $S$ transformations as explained before: these partition functions take the same form (4.2) with signs for the $(-)^F$ weight and slightly different $Z_k$ pairing in the Ramond sectors. Examples will be given later in this Section and in the Appendix.

### 4.2 Stability argument

The charge part $K_\lambda$ of the sectors $\Theta^\alpha_\lambda$ in (4.2) is parameterized by two independent numbers $(k,p)$, whose meaning can be understood from the expression (3.1):

i) The values of the fractional charge are $Q = k\lambda/kp = \lambda/p$, $\lambda = 0, \ldots, p-1$, and the minimal charge is equal to $e^* = 1/p$.

ii) the Hall current (spin current) is obtained by applying the $V$ transformation on (4.2), that acts on the charge part $K_\lambda$, causing the shift of quantum numbers:

$$V : \zeta \rightarrow \zeta + \tau, \quad \lambda \rightarrow \lambda + k, \quad \Delta Q^\uparrow = \nu^\uparrow = \frac{k}{p}, \quad (4.5)$$

while the neutral characters in (4.2) are not affected. We see that $\nu^\uparrow$ is parameterized by the ratio of the numbers $(k,p)$, but these might have common factors that are relevant in the stability argument.

As in Section 3, we should find the number of fluxes that creates an electron excitation in the same anyon sector (fractional charge) and same neutral sector (neutral quantum numbers), such that all fusion rules stay unchanged [8]: owing to the periodicity of $K_\lambda$ in (4.3), this number is given by $p$. Then, the Fu-Kane-Mele flux argument considers the change in spin parity due to adding $p/2$ fluxes. This is given by,

$$V^{\frac{p}{2}} : \Delta S = \Delta Q^\uparrow = \frac{p}{2} \nu^\uparrow = \frac{k}{2}. \quad (4.6)$$

Therefore, the Levin-Stern index (3.18) for the spin parity anomaly is:

$$2\Delta S = \frac{\nu^\uparrow}{e^*} = k. \quad (-1)^{2\Delta S} = (-1)^k.$$

$$2\Delta S = \frac{\nu^\uparrow}{e^*} = k. \quad (-1)^{2\Delta S} = (-1)^k. \quad (4.7)$$
The stability analysis then continues by observing that for odd values of \( k \), the action of \( V^k \) creates a Kramers pair at the edge that is protected by TR symmetry; then, this cannot be gapped and the topological insulator is stable.

The action on the anyon sectors (4.2) is,

\[
V^k : \Theta^\alpha_{\lambda}(\tau, \zeta) \rightarrow \sum_{a=1}^{k} K_{\lambda+ap+kp/2}(\tau, k\zeta; kp) \chi_\alpha^a \mod k(\tau, 0) \sim \Theta^\alpha_{\lambda'}(\tau, \zeta),
\]

where the values of \((\lambda', \alpha')\) depends on the specific theory considered through the symmetries of its characters. Looking at the expressions (4.2),(4.8), it is clear that the neutral characters \( \chi_m^\alpha \) do not enter in the stability argument, i.e. in the determination of the index (4.7); only the charge parts are relevant. Thus, the result (4.7) holds for both Abelian and non-Abelian edge theories of topological insulators. In particular, the relevant parameters \((k, p)\) are independent of the value of Wen topological order. In conclusion, we have extended the Levin-Stern stability criterion to any interacting topological insulator with time reversal symmetry.

In the case of multicomponent Abelian theories, the works [8][9][10] found the explicit edge interactions that gap the system in the unstable cases, thus double checking the result of the flux argument. Such analysis of gapping interactions is not yet available for general non-Abelian theories; however, we can provide the following argument. Some well-known non-Abelian states have been described as projections of so-called “parent” Abelian states [29]: for example, the (331) state of distinguishable electrons and its \( k \)-component generalizations are parents of the Pfaffian and \( \mathbb{Z}_k \) parafermion Hall states, respectively. The non-Abelian theories are obtained by projecting the Abelian theories to states that have identical electrons. Since this projection does not affect the TR invariance of states, it commutes with the analysis of TR-invariant interactions in the Abelian theory and extends it to these non-Abelian theories. In particular, we shall see later that the topological insulator made by Pfaffian states is unstable. Clearly, the value of the Levin-Stern index is equal in the unprojected (Abelian) and projected (non-Abelian) theories.

4.2.1 Stability and modular non-invariance

The stability of general topological insulators, corresponding to the \( \mathbb{Z}_2 \) spin parity anomaly, is again accompanied by modular non-invariance of the partition function. However, electromagnetic and gravitational responses are not always equivalent as in the \( c = 1 \) case (neutral modes are clearly sensible to coordinate changes but not to flux additions).

We should distinguish the following cases, according to the parities of \((k, p)\):

i) For \( p \) odd, the action of \( V^k \) is not a symmetry of each spin sector and maps them one into another. The transformations between Neveu-Schwarz and Ramond sectors and among
their tildes are the same as those of the c = 1 theory (see Fig. 6(a) and Eq.(3.16)). The anyon sector \( \Theta_0^0 \) containing the NS ground state is naturally mapped into \( \Theta_0^R \) including the Ramond ground state. The modular invariant and non-invariant partition functions are as in the \( c = 1 \) case:

\[
Z_{\text{Ising}} = Z^{NS} + Z^{\tilde{NS}} + Z^R + Z^{\tilde{R}}, \quad k \text{ even, unstable,}
\]

\[
Z_{\text{TR}} = \left( Z^{NS}, Z^{\tilde{NS}}, Z^R, Z^{\tilde{R}} \right), \quad k \text{ odd, stable.}
\]

(4.9)

ii) For \( p \) even, the action of \( V_{\frac{k}{2}}^2 \) maps each spin sector into itself and thus differs from the modular transformations (see Fig. 7). For \( k \) odd, the \( Z_2 \) anomaly manifests itself within each spin sector, as a difference in spin parity between the ground state and another “anyon” ground state (actually degenerate). The TR symmetry of the theory then requires to splitting each spin sector in two subsectors, \( Z^\sigma \to (Z_1^\sigma, Z_2^\sigma) \), \( \sigma = NS, \tilde{NS}, R, \tilde{R} \), that are related by \( V_{\frac{k}{2}}^2 \): \( Z_2^\sigma = V_{\frac{k}{2}}^2 (Z_1^\sigma) \) and collect anyon sectors of same spin parity. These subsectors carry a eight-dimensional representation of the modular group, namely the associated gravitational anomaly is slightly stronger. Finally, for \( k \) and \( p \) both even, there is no anomaly and the \( Z_{\text{Ising}} \) partition function is consistent with TR symmetry. Summarizing, in all cases modular non-invariance is associated to stability and the \( Z_2 \) anomaly.

4.3 Examples

4.3.1 Jain-like topological insulators

The Jain states are the prominent multicomponent Abelian states in the fractional quantum Hall effect. For \( k \) components, their \( c = k \) CFT involves a lattice of conformal dimensions (specified by the so-called \( K \) matrix) in which the Abelian symmetry \( \hat{U}(1) \) is enhanced to \( \hat{U}(1) \times SU(k) \) \([4]\). The fusion rules are Abelian but there are manifestations of the \( SU(k) \) symmetry and its center \( \mathbb{Z}_k \).

Consider the topological insulator made by a pair of spin up and spin down Jain states; the filling fraction, minimal charge and stability index are:

\[
\nu^\uparrow = \frac{k}{2nk + 1}, \quad e^* = \frac{1}{2nk + 1}, \quad 2\Delta S = \frac{\nu^\uparrow}{e^*} = k, \quad (-1)^{2\Delta S} = (-1)^k,
\]

(4.10)

showing that the system is stable (unstable) for \( k \) odd (even).

The Jain partition function in the NS sector has been studied extensively \([12]\). The previous formulas identify the parameters entering the stability analysis are \( k \) and \( p = 2nk + 1 \); they have no common factor, \((k,p) = 1\), and \( p \) is always odd. Moreover, the topological order is equal to \( p \). Thus, this case is very similar to that of Laughlin states, and the pattern of flux insertions and modular transformations among the four spin sectors is the same as that already discussed in Section 3 (Fig. 6). On the other hand, there are
Figure 7: Actions of (a) $p/2$ flux insertions and (b) modular transformations on the four spin sectors $Z^{NS}, Z^{\tilde{NS}}, Z^R, Z^{\tilde{R}}$ (even $p$ case).

$(k - 1)$ neutral modes and the partition functions have the simple-current form discussed in the previous Section.

The anyon sectors in the $NS$ sector take the form Eq.(4.2) [12]:

$$
\Theta^{NS}_\lambda(\tau, \zeta) = \sum_{a=0}^{k-1} K_{\lambda+ap}(\tau, k\zeta; kp) \chi_{\lambda+ap \mod k}(\tau, 0), \quad \lambda = 1, \ldots, kp. \quad (4.11)
$$

In this expression, $\chi_\beta$ are the characters of the $\widehat{SU}(k)_1$ affine algebra with Abelian label $\beta$ modulo $k$ [11]. It is apparent that charged and neutral sectors are paired by the $\mathbb{Z}_k$ section rule $\lambda = \beta \mod k$, whose origin can be understood as follows [12][13]. The Abelian $k$-dimensional lattice of conformal dimensions specified by the $K$ Gram matrix is not orthogonal, i.e. the charged and neutral excitations are not independently generated. The decomposition underlying the expression (4.11) is obtained by embedding the Abelian lattice into a thinner one that is orthogonal in the charge and neutral directions. In this last lattice, the “physical” points are those obeying the $\mathbb{Z}_k$ selection rule.

The anyon sectors are periodic by $\Theta^{NS}_{\lambda+p} = \Theta^{NS}_\lambda$, due to $p = 1$ modulo $k$; thus there are
Let us present more explicit formulas in the $k = 2$ case (unstable); the partition functions of the four spin sectors are:

$$
\begin{align*}
\Theta^{NS}_{2a}(\tau, \zeta) &= K_{2a} \chi_0 + K_{2a+p} \chi_1, \quad a = 1, \ldots, p, \\
\tilde{\Theta}^{NS}_{2a}(\tau, \zeta) &= K_{2a} \chi_0 - K_{2a+p} \chi_1, \\
\Theta^R_{2a}(\tau, \zeta) &= K_{2a} \chi_1 + K_{2a+p} \chi_0, \\
\tilde{\Theta}^R_{2a}(\tau, \zeta) &= -K_{2a} \chi_1 + K_{2a+p} \chi_0.
\end{align*}
$$

(4.12)

where $K_{\lambda}(\tau, 2\zeta; 2p) = K_{\lambda+2p}(\tau, 2\zeta; 2p)$ and $\chi_\sigma = \chi_\sigma+2$. Note that first (second) $K_{\lambda}$ in each expression includes an odd (even) number of fermion excitations; then, the relative sign appearing in the $\tilde{\Theta}^{NS}$ and $\tilde{\Theta}^R$ expressions accounts for the $(-1)^F$ factor in the trace (3.11).

Note also that the $\mathbb{Z}_2$ parity rule between charge and neutral quantum numbers that is different in the $NS$ and $R$ sectors. The transformation properties of these sectors are given in Appendix A. The partition functions of the four sectors are:

$$
Z^{(\sigma)} = \sum_{a=1}^{p} \Theta^{\sigma}_{2a} \tilde{\Theta}^{-\sigma} = \Theta^{NS}, \tilde{\Theta}^{NS}, R, \tilde{R}.
$$

(4.13)

The partition function for this unstable system is given by $Z_{\text{Ising}}$.

### 4.3.2 Multicomponent Abelian topological insulators

We briefly discuss the general Abelian edge theories that have been studied extensively in [8][9][10] and show how they fit in the present analysis. Abelian conformal theories with central charge $c = n$ are characterized by a $2n$-dimensional lattice of conformal weights and charges, that specifies the statistical phases $\theta/\pi$, the electric charge $Q$ and spin $S$ of excitations through the following formulae,

$$
\frac{\theta}{\pi} = \sum_{i,j=1}^{2n} n_i K^{-1}_{ij} n_j, \quad Q = \sum_{i,j=1}^{2n} t_i K^{-1}_{ij} n_j, \quad S = \sum_{i,j=1}^{2n} s_i K^{-1}_{ij} n_j, \quad n_i \in \mathbb{Z}^{2n}.
$$

(4.14)

These involve the Gram matrix $K^{-1}$ of the lattice, the charge and spin vectors, $t$, $s$, and the integer vector $n_i$ specifying the excitations. In particular, excitations with integer statistics, i.e. electrons and their compounds, are described by the dual lattice of $K$. The Gram matrix is expressed by the basis vectors $v_i$ of the lattice as $K^{-1}_{ij} = v_i \cdot \eta \cdot v_j$, where $\eta_{ij} = \delta_{ij}\sigma_i$, $\sigma_i = \pm 1$ is the signature of the Lorentzian metric, expressing the chirality of excitations [30][4]. Basis vectors $(v_i)_\alpha$ and rescaled eigenvectors $\sqrt{\lambda_\alpha}(u_\alpha)$, form equal matrices, up to transposition.

In time reversal invariant systems, there is an equal number of positive and negative chiralities. Next, we can choose a basis in which the first (second) $n$ components describe
the spin-up (down) modes. Then the action of the TR transformation $T$ and the other quantities take a two-dimensional block form:

$$
T = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad
t = \begin{pmatrix}
t \\
t
\end{pmatrix}, \quad
s = \begin{pmatrix}
s \\
-s
\end{pmatrix}, \quad
n = \begin{pmatrix}
n^t \\
n^t
\end{pmatrix}.
$$

(4.15)

The TR symmetry of the spectrum implies the condition $K = -TKT$, with solution:

$$
K = \begin{pmatrix}
K & W \\
W^T & -K
\end{pmatrix}, \quad
t T = K, \quad
t = -W.
$$

(4.16)

It follows that $K$ has eigenvectors in pairs $(\lambda, -\lambda)$, that are integer valued [8][9][10].

The case of Jain states considered before corresponds to $W = 0$, and, moreover, to a positive definite metric $K$, because excitations of same spin have same chirality:

$$
K = \begin{pmatrix}
K & 0 \\
0 & -K
\end{pmatrix}, \quad
n_i^t K_{ij} n_j^t \geq 0.
$$

(4.17)

The partition functions of topological insulators made of these Abelian theories can be obtained within the $K$-matrix formalism, and the expressions for anyon and spin sectors are given in Appendix A. The decomposition of anyon sectors in simple-current form (4.2), schematically $\Theta^t = \sum K^t \chi^t$, is obtained by separating the lattice into charged and neutral sublattices, whose points are coupled by a parity rule as explained before.

Next we discuss the systems with neutral modes counter-propagating with respect to the charged mode of the same spin type. These are described by the block-diagonal form (4.17), where $K$ has mixed signature, i.e. is not positive definite: an example is given by the Jain states with $\nu = k/(2nk - 1)$. As discussed in [12], the simple current expression (4.2) of partition functions is still valid with the replacement $\chi_a \rightarrow \bar{\chi}_a$ for the opposite neutral chirality, schematically $\Theta^t = \sum K^t \bar{\chi}^t$.

Finally, the case $W \neq 0$ describes interacting excitations with up and down spins. Indeed, let us choose the standard basis specified by $t = (1, 1)$ and $s = (1, -1)$; the statistical phases are computed from $\theta/\pi = \ell_i K_{ij} \ell_j$, where the $i$-th electron is represented by $\ell_j = e(i) = \delta_{ij}$. Then, the correlation of up and down electrons, $\langle \Psi^t_j(x) \Psi^d_k(y) \rangle$, has a phase proportional to $W_{jk} \neq 0$, for $1 \leq j \leq n$ and $n < k \leq 2n$. This is due to the coupling of the charge mode, e.g. spin up, with neutral modes of opposite spin. In this case, the simple current decomposition of anyon sectors (4.2) applies in the form $\Theta^t = \sum K^t \chi^t$. In conclusion, the simple current form of the partition function is valid in all Abelian cases; the stability argument discussed earlier applies equally, because it does not involve the properties of neutral modes.

Let us finish the discussion of Abelian theories by considering the example of type (4.17) with two-component $K$ matrix,

$$
K = \begin{pmatrix}
3 & 1 \\
1 & 5
\end{pmatrix}.
$$

(4.18)
and charge vector \( t = (1, 1) \). Using previous formulas, we obtain the charge spectrum, minimal charge and filling fraction of the corresponding Hall state:

\[
Q = \frac{2n_1 + n_2}{7}, \quad n_1, n_2 \in \mathbb{Z}, \quad e^* = \frac{1}{7}, \quad \nu^\uparrow = \frac{3}{7},
\]

while the topological order is \( \det K = 14 \).

Next we consider the topological insulator made by a pair of these chiral states. The parameters entering the stability analysis are \((k, p) = (3, 7)\), thus the topological state is stable,

\[
2\Delta S = \frac{\nu^\uparrow}{e^*} = 3, \quad (-1)^{2\Delta S} = -1.
\]

This example shows that the stability is independent of the number of fermion modes and of the value of the topological order.

### 4.3.3 (331) and Pfaffian topological states

The Pfaffian state is the simplest example of non-Abelian quantum Hall states [31]. Following the previous discussion of stability, it can be analyzed together with its parent Abelian state, the so-called (331) which has the same charge spectrum and filling fraction (see Ref. [29] for a complete analysis of this relation). From the matrix \( K = ((3, 1), (1, 3)) \), we find:

\[
\nu^\uparrow = \frac{1}{2}, \quad e^* = \frac{1}{4}, \quad 2\Delta S = 2, \quad (-1)^{2\Delta S} = 1.
\]

Thus the topological insulators made by pairs of these Hall states are unstable. The parameters entering the stability analysis are \((k, p) = (2, 4)\).

Let us first discuss the Abelian state. Its topological order is \( \det K = 8 \), thus there are 8 anyon sectors and a \( \mathbb{Z}_2 \) parity rule. In the \( NS \) sector, they read:

\[
\Theta_\lambda = K_\lambda \chi_\lambda + K_{\lambda+4} \chi_{\lambda+2}, \quad \lambda = 1, \ldots, 8,
\]

where \( K_\lambda = K_\lambda(\tau, 2\zeta; 8) \) and \( \chi_\lambda = K_\lambda(\zeta, 0; 4) \). The Neveu-Schwarz partition function reads [13]:

\[
Z_{NS}^{(331)} = \sum_{a=0}^3 |K_a \chi_a + K_{a+4} \chi_{a+2}|^2 + |K_a \chi_{a+2} + K_{a+4} \chi_a|^2.
\]

Note that the four charge sectors appears twice in the spectrum coupled to different neutral parts. The expressions for the other spin sectors are given in Appendix.

In the Pfaffian state, the neutral part is provided by the characters of the Ising model, i.e. the \( \mathbb{Z}_2 \) parafermions, \( \chi_a^\ell \), labelled by \( a \) modulo 4 and \( \ell = 0, 1, 2 \), non-vanishing for \( a = \ell \) modulo 2, and obeying \( \chi_{a+2}^\ell = \chi_2^{-\ell} \chi_a^\ell \) [14]. There are only three independent values, \( \chi_0^0 = \chi_2^2 = \phi, \chi_1^1 = \chi_3 = \sigma \) and \( \chi_2^0 = \phi = \psi \), denoted as the corresponding conformal fields. The anyon sectors are (\( NS \) sector):

\[
\tilde{\Theta}_a^\ell = K_a \chi_a^\ell + K_{a+4} \chi_{a+2}^\ell, \quad a = 0, 1, 2, 3, \quad \ell = 0, 1, 2, \quad a = \ell \mod 2.
\]
It is apparent that the charge parts in the (331) and Pfaffian states are equal, while the neutral parts of the latter obey some symmetries that reduce the topological order from 8 to 6; one indeed checks that $\Theta_1^1 = \Theta_5^1$ and $\Theta_3^1 = \Theta_7^1$ [13]. The $NS$ partition function reads [14]:

$$Z_{NS}^{Pf} = \sum_{a=0,2} |K_a \chi_a^0 + K_{a+4} \chi_{a+2}|^2 + |K_a \chi_a^0 + K_{a+4} \chi_a^0|^2 + (K_1 + K_{-3}) \chi_1^1|^2 + (K_3 + K_{-1}) \chi_1^1|^2,$$

where charge sectors appear again twice, but are coupled differently to neutral states. In particular, the two (331) sectors with neutral charge $a = 1$ (resp. $a = -1$) are projected to a single one. This example clarifies that the Fu-Kane-Mele stability argument is the same in both theories, since it only deals with the charge parts. The expressions of the other spin sectors for the Pfaffian topological state are also given in the Appendix.

These unstable theories are characterized by $(k,p)$ both even, thus flux insertions and modular transformations act differently on the four spin sectors (Fig. 7). As discussed before, $V_1^2 = V^2$ is an isometry of each spin sector. Since there is no associated $\mathbb{Z}_2$ anomaly, the modular invariant $Z_{Ising}$ partition function is consistent with TR symmetry.

### 4.3.4 Read-Rezayi parafermionic states

The Read-Rezayi states [32] are generalization of the Pfaffian state involving neutral modes of the $\mathbb{Z}_k$ parafermions that can be described by the coset $\frac{SU(2)_k}{U(1)}$ [14]. The quantities entering in the stability index (1.1) are

$$\nu^\uparrow = \frac{k}{kM+2}, \quad e^* = \frac{1}{kM+2}, \quad 2\Delta S = k, \quad (\Delta S)^2 = (-1)^k,$$

where $k = 3, 4 \ldots$ and $M = 1, 3, \ldots$. In this case, $(k,p) = (k,kM+2)$, thus the topological insulators made by a pair of these states is stable (unstable) for $k$ odd (even). Note that $k$ and $p$ have the same parity:

i) For $k$ and $p$ odd, these numbers have no common factor, and the flux insertions and modular transformations follow the pattern of the stable, odd $k$ Jain states and of the $c = 1$ theory (Fig. 6). The modular non-invariant partition function takes the form $Z_{TR}$ in (4.9).

ii) For $k$ and $p$ even, they have a common factor of 2 and the flux insertions and modular transformations are the same as those of the $k = 2$ case discussed in the previous paragraph (Fig. 7). The modular invariant partition function is $Z_{Ising}$.

All partition functions and modular transformations are described in Appendix. Let us briefly discuss the form of the Neveu-Schwarz anyon sectors, taken from Ref.[14]. These read:

$$\Theta^\ell_a = \sum_{b=1}^k K_{a+bp}(\tau, k\zeta; kp) \chi^\ell_{a+2b}(\tau, 0, 2k), \quad a = \ell \mod 2, \quad p = 2 + kM.$$

25
The charge characters $K_\lambda$ with periodicity $kp$, are coupled to the $\mathbb{Z}_k$ parafermion characters $\chi_{m}^{\ell}$, that are specified by the $SU(2)_k$ quantum number $\ell = 0, 1, \ldots, k$, and the Abelian number $m$ modulo $2k$. The $\mathbb{Z}_k$ parity rule between the two Abelian numbers is $\lambda = m$ modulo $k$ (note that $p = 2$ modulo $k$). The parafermion characters obey the periodicities $\chi_{m}^{\ell} = \chi_{m+2k}^{\ell} = \chi_{m+k}^{k-\ell}$ and vanish for $m + \ell = 1$ modulo 2. Taking into account these properties, one finds the periodicity $\Theta_{a+p}^{\ell} = \Theta_{a}^{k-\ell}$, implying $p(k+1)/2$ independent anyon sectors, the topological order.

5 Conclusions

In this paper we have obtained the general form of the partition function of edge excitations in non-chiral topological states protected by time-reversal symmetry, such as the quantum spin Hall effect and topological insulators. The study of partition functions has allowed us to discuss the flux argument for the stability of the topological phase in great generality and to extend it to interacting systems possessing non-Abelian edge excitations.

We have emphasized the anomaly in the $\mathbb{Z}_2$ edge spin parity that is associated to stable topological phases. We considered the modular transformations of the partition function that map the four spin sectors, Neveu-Schwarz, Ramond and their tildes, among themselves. We have found that the $\mathbb{Z}_2$ anomaly is accompanied by modular non-invariance, that is a kind of discrete gravitational anomaly. We also discussed the cases in which the electromagnetic and gravitational responses are equivalent and when they are different.

Among the possible directions of future progress, we mention the extension of our stability analysis to topological superconductors [1]. Let us briefly discuss this point. From the point of view of edge dynamics, topological superconductors amount to systems of (interacting) $N_f$ neutral chiral Majorana modes with spin up and $N_f$ ones with opposite spin and chirality, thus making $N_f$ Ising CFTs [11]. The topological phases are protected by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry of independent edge spin parity for each chirality, $(-1)^{2S^z} = (-1)^{N_f}$ and $(-1)^{2S^i} = (-1)^{N_f}$. This follows from TR symmetry and the absence of spin-flip terms in the bulk Hamiltonian [1]. Since this symmetry does not allow any mass term, the non-interacting topological insulators are classified by the $\mathbb{Z}$ index equal to $N_f$.

On the other hand, for $N_f = 8$ a non-trivial quartic interaction was found that gaps the system without breaking the symmetry explicitly or spontaneously [17][18]. This interaction is possible because the Ising CFTs have spin $\sigma$ and disorder $\mu$ fields of dimensions $h = 1/16$, such that the product of eight of them has dimension $h = 1/2$ and can bosonized and refermionized to obtain a quartic fermionic term respecting the symmetry. Therefore, the classification of interacting topological superconductors is associated to a $\mathbb{Z}_8$ index (at least).

For even $N_f$, the free Majorana fermions have $c = N_f/2$ and yield the same fermion systems considered in this paper. However, the neutral modes cannot be probed by flux
insertions for studying stability. In an interesting paper [16], Ryu and Zhang suggested
to consider the modular non-invariance as a test of stability of topological supercon-
ductors. In practice, they associated the instability to the modular invariance of the parti-
tion function for the singlet sector of the theory with respect to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry,
$(-1)^N_\up = (-1)^N_\down = 1$, i.e. involving only bosonic chiral states. Indeed, this projected parti-
tion function is modular invariant for $N_f = 8$, as it corresponds to the so-called Gliozzi-
Scherk-Olive projection $Z_{\text{GSO}}$ in superstring theory [15]. This results provided a symmetry
argument for the instability of $N_f = 8$ topological superconductors [17][18] (also extended
to other systems in [33]).

In our analysis, we considered the partition function of the full theory $Z_{\text{Ising}}$ that is always
modular invariant, but argued that it might not be consistent with TR invariance, owing to
anomalies in some of its sectors. The two criteria of stability do not seem to be equivalent
in general, but at least they agree on the instability of the $N_f = 8$ case. The argument
goes as follows. In Section 3, we have seen that in free fermionic systems, the half flux
insertion maps Neveu-Schwarz into Ramond sectors, i.e. we have the equivalence $V^{1/2} \sim ST$,
between electromagnetic and gravitational responses. Moreover, $V^{1/2}$ does the correct flux
insertion for measuring the anomaly. We then consider the chiral spin of the excitation in
the $R$ sector obtained from the $NS$ sector by the $ST$ transformation: for $N_f = 2N$ fermion
modes, this is $\Delta S^\up = \Delta S^\down = N/4$. The first $\mathbb{Z}_2 \times \mathbb{Z}_2$ anomaly free case is for $N_f = 8$, with
chiral indices:

$$
(-1)^{N^\up} = (-1)^{N^\down} = 1, \quad N_f = 8.
$$

Therefore, the modular invariant partition function $Z_{\text{Ising}}$ is free of $\mathbb{Z}_2 \times \mathbb{Z}_2$ anomalies for
$N_f = 8$, thus confirming the instability of this system in our analysis.

More general results cannot be provided at present, owing to two main difficulties: the
correct measure of spin anomaly in other systems and the form of the partition function to
be associated to the edge of interacting topological superconductors. Regarding the second
point, the absence of the $U$ condition, matching the charges of the two chiralities, allows
many possible expressions for the edge partition function, of the form $Z = \sum N_\mu \mathcal{K}_\mu \mathcal{K}_\mu^*$,
with $\mathcal{N}$ any symmetric positive integer matrix commuting with the $S$ and $T$ transformations
[11].

Another possible development concerns the comparison with the stability criterion pro-
posed in [34], which is based on properties of fusion rules, again related to modular trans-
formations through the Verlinde formula. Finally, there is the study of stability of three
dimensional systems: in particular, recent works [35] suggest the existence of topological
states at the $2d$ surface of a $3d$ topological insulators, whose stability could be different
from that of strictly $2d$ systems considered here.

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A  Modular transformation and spin structures

In the following we give the expressions of partition functions for the four spin sectors $NS$, $\tilde{NS}$, $R$, $\tilde{R}$, that describe the edge excitations of topological insulator models examined in the main text. We describe their behavior under flux insertion and modular transformations.

A.1 Laughlin states

The Laughlin states discussed in Section 3 are described by the $c = 1$ CFT [11]. The anyon sectors for the chiral modes of the $NS$ and $\tilde{NS}$ spin sectors are, for $\lambda = 1, \cdots, p$, $p$ odd, [12]:

$$K^{NS}_\lambda(\tau, \zeta; p) = \frac{F(\tau, \zeta)}{\eta(\tau)} \sum_{n \in \mathbb{Z}} \exp \left( i 2\pi \left( \frac{\tau (np + \lambda)^2}{2p} + \zeta \frac{np + \lambda}{p} \right) \right),$$

$$K^{\tilde{NS}}_\lambda(\tau, \zeta; p) = \frac{F(\tau, \zeta)}{\eta(\tau)} \sum_{n \in \mathbb{Z}} (-1)^{pn} \exp \left( i 2\pi \left( \frac{\tau (np + \lambda)^2}{2p} + \zeta \frac{np + \lambda}{p} + \frac{\lambda}{2} \right) \right),$$

with $F = \exp \left[ -\pi (\text{Im} \zeta)^2 / \text{Im} \tau \right]$ is a non-holomorphic prefactor and $\eta(\tau)$ the Dedekind function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \quad q = \exp(i2\pi \tau).$$

The anyon sectors of the $R$ and $\tilde{R}$ spin sectors are defined by:

$$K^R_\lambda = K^{NS}_{\lambda + \frac{1}{2}},$$

$$K^{\tilde{R}}_\lambda = K^{\tilde{NS}}_{\lambda + \frac{1}{2}}.$$ (A.3)

The edge partition functions for each spin sector (Eq.(3.10) and (3.11)) are obtained by matching the fractional charge of the chiral and antichiral anyon sectors locally at the edge (see $U$ condition (3.6)),

$$Z^{(\sigma)} = \sum_{\lambda=1}^{p} K^{(\sigma)}_\lambda K^{(\sigma)}_{-\lambda}, \quad \sigma = NS, \tilde{NS}, R, \tilde{R}.$$ (A.4)

The transformation of the anyon sectors ((A.1) and (A.3)) under the modular group, generated by $S$ and $T$, and for the insertion of one and $p/2$ fluxes through the annulus, the $V$ and $V^p/2$ transformations, respectively, are obtained by extending the calculations of Ref. [12] [14]. Altogether they are:
\( S \)

\[
K^N_{\lambda}(\frac{1}{\tau}, \frac{\zeta}{\tau}) = e^{i\varphi} \sum_{\lambda'} S_{\lambda\lambda'} K^N_{\lambda'}(\tau, \zeta),
\]

\( K^S_{\lambda}(\frac{1}{\tau}, \frac{\zeta}{\tau}) = e^{i\varphi} \sum_{\lambda'} S_{\lambda\lambda'} K^S_{\lambda'}(\tau, \zeta),
\]

\( K^R_{\lambda}(\frac{1}{\tau}, \frac{\zeta}{\tau}) = e^{i\varphi} \sum_{\lambda'} S_{\lambda\lambda'} K^S_{\lambda'}(\tau, \zeta),
\]

\( K^R_{\lambda}(\frac{1}{\tau}, \frac{\zeta}{\tau}) = \exp\left(2\pi i \frac{p}{4}\right) e^{i\varphi} \sum_{\lambda'} S_{\lambda\lambda'} K^R_{\lambda'}(\tau, \zeta),
\]

with

\[
S_{\lambda\lambda'} = \frac{1}{\sqrt{p}} \left(2\pi i \frac{\lambda\lambda'}{p}\right), \quad e^{i\varphi} = \exp\left(\frac{i\pi}{p} \text{Re}\left(\frac{2}{\tau}\right)\right).
\]

\( T \)

\[
K^N_{\lambda}(\tau + 1, \zeta) = \exp\left(-2\pi i \frac{\lambda}{2}\right) T_{a} K^N_{\lambda}(\tau, \zeta),
\]

\[
K^S_{\lambda}(\tau + 1, \zeta) = \exp\left(2\pi i \frac{\lambda}{2}\right) T_{a} K^S_{\lambda}(\tau, \zeta),
\]

\[
K^R_{\lambda}(\tau + 1, \zeta) = T_{a} T_{b} K^R_{\lambda}(\tau, \zeta),
\]

\[
K^S_{\lambda}(\tau + 1, \zeta) = T_{a} T_{b} K^S_{\lambda}(\tau, \zeta),
\]

with

\[
T_{a} = \exp\left(2\pi i \left(\frac{\lambda^2}{2p} - \frac{1}{24}\right)\right), \quad T_{b} = \exp\left(2\pi i \left(\frac{p}{8} + \frac{\lambda}{2}\right)\right).
\]

\( V \)

\[
K^N_{\lambda}(\tau, \zeta + \tau) = V_{\Phi_{0}} K^N_{\lambda+1}(\tau, \zeta),
\]

\[
K^S_{\lambda}(\tau, \zeta + \tau) = -V_{\Phi_{0}} K^S_{\lambda+1}(\tau, \zeta),
\]

\[
K^R_{\lambda}(\tau, \zeta + \tau) = V_{\Phi_{0}} K^R_{\lambda+1}(\tau, \zeta),
\]

\[
K^S_{\lambda}(\tau, \zeta + \tau) = -V_{\Phi_{0}} K^S_{\lambda+1}(\tau, \zeta),
\]

with

\[
V_{\Phi_{0}}(\tau, \zeta) = \exp\left(-2\pi i \frac{1}{p} \left(\text{Re}\frac{\tau}{2} + \text{Re}\zeta\right)\right)
\]

\( V^p \)

\[
K^N_{\lambda}(\tau, \zeta + \frac{p\tau}{2}) = V^p_{\Phi_{0}} K^R_{\lambda},
\]

\[
K^S_{\lambda}(\tau, \zeta + \frac{p\tau}{2}) = V^p_{\Phi_{0}} \exp\left(-2\pi i \frac{p}{4}\right) K^R_{\lambda},
\]

\[
K^R_{\lambda}(\tau, \zeta + \frac{p\tau}{2}) = V^p_{\Phi_{0}} K^N_{\lambda},
\]

\[
K^S_{\lambda}(\tau, \zeta + \frac{p\tau}{2}) = V^p_{\Phi_{0}} \exp\left(-2\pi i \frac{p}{4}\right) K^N_{\lambda},
\]
with
\[ V_2 \Phi_0 = \exp \left( -2\pi i \left( \frac{p}{8} \Re \tau + \frac{1}{2} \Re \zeta \right) \right) . \] (A.12)

Upon using these formulas, we obtain that the transformations of partition functions (A.4) illustrated in Fig.(6). As discussed in Section 3.4, the modular invariant partition function \( Z_{\text{Ising}} \) (3.19) is not consistent with TR symmetry owing the presence of the \( Z_2 \) anomaly, then the Laughlin-type topological insulators are stable.

### A.2 Jain states

The topological insulators with two-component Jain edge systems discussed in Section 4.3.1, are described by the \( c = 2 \) CFT with symmetry \( \widehat{U}(1) \times \widehat{SU}(2)_1 \); the two parameters that enter the stability argument (see Section 4.2) are \((k,p) = (2, 4n + 1)\). The characters of the \( \widehat{U}(1) \) charge part are expressed by the functions \( K_{2a+\alpha p}(\tau, 2\zeta; 2p) \) of the previous section (Eq.(3.1)), where \( \alpha = 0, 1 \) and \( a = 1, \cdots, p \). The \( \widehat{SU}(2)_1 \) neutral characters are:

\[ \chi_{\alpha=0} = \frac{1}{\eta(q)} \sum_{n \in \mathbb{Z}} q^{n^2}, \quad \chi_{\alpha=1} = \frac{1}{\eta(q)} \sum_{n \in \mathbb{Z}} q^{(2n+1)^2/4}. \] (A.13)

These have the periodicity \( \chi_{\alpha+2}(\tau, 0) = \chi_{\alpha}(\tau, 0) \) and obey the modular transformations (with \( k = 2 \)) [12]

\[ T : \quad \chi_{\alpha}(\tau + 1, 0) = \exp \left( 2\pi i \left( \frac{\alpha(k - \alpha)}{2k} - \frac{(k - 1)}{24} \right) \right) \chi_{\alpha}(\tau, 0), \] (A.14)

\[ S : \quad \chi_{\alpha}(-\frac{1}{\tau}, 0) = \frac{1}{\sqrt{k}} \sum_{\alpha'} \exp \left( -2\pi i \frac{\alpha \alpha'}{k} \right) \chi_{\alpha'}(\tau, 0). \]

Then, the chiral anyon sectors for the four spin sectors are:

\[ \Theta^{\Sigma}_{2a}(\tau, \zeta) = K_{2a} \chi_0 + K_{2a+p} \chi_1, \] (A.15)

\[ \Theta^{\tilde{\Sigma}}_{2a}(\tau, \zeta) = K_{2a} \chi_0 - K_{2a+p} \chi_1, \]

\[ \Theta^R_{2a}(\tau, \zeta) = K_{2a} \chi_1 + K_{2a+p} \chi_0, \]

\[ \Theta^{\tilde{R}}_{2a}(\tau, \zeta) = -K_{2a} \chi_1 + K_{2a+p} \chi_0. \]

For each spin sectors the edge partition functions are obtained by coupling the chiral and antichiral modes and summing over \( a = 1, \cdots, p \), as follows (see Eq.(4.13)):

\[ Z^{(\sigma)} = \sum_{a=1}^{p} \Theta^{(\sigma)}_{2a} \Theta^{(\sigma)}_{-2a}, \quad \sigma = NS, \widetilde{NS}, R, \widetilde{R}. \] (A.16)

Under the modular group and the insertion of fluxes, the anyon sectors (A.15) possess the following transformation properties:
\[ \begin{align*}
\Theta_{2a}(\frac{-1}{\tau}, \frac{-\zeta}{\tau}) &= e^{i\varphi} \sum_{a'=1}^{p} S_{a,a'} \Theta_{2a'}^{NS}(\tau, \zeta), \\
\Theta_{2a}^{NS}(\frac{-1}{\tau}, \frac{-\zeta}{\tau}) &= e^{i\varphi} \sum_{a'=1}^{p} S_{a,a'} \Theta_{2a'}^{R}(\tau, \zeta), \\
\Theta_{2a}^{R}(\frac{-1}{\tau}, \frac{-\zeta}{\tau}) &= e^{i\varphi} \sum_{a'=1}^{p} S_{a,a'} \Theta_{2a'}^{\tilde{N}S}(\tau, \zeta), \\
\Theta_{2a}^{\tilde{R}}(\frac{-1}{\tau}, \frac{-\zeta}{\tau}) &= -e^{i\varphi} \sum_{a'=1}^{p} S_{a,a'} \Theta_{2a'}^{\tilde{R}}(\tau, \zeta),
\end{align*} \]

with
\[ S_{a,a'} = \frac{1}{\sqrt{p}} \exp \left( 2\pi i \frac{2aa'}{p} \right), \quad e^{i\varphi} = \exp \left( i \frac{\pi}{2p} \text{Re} \left( \frac{(2\zeta)^2}{\tau} \right) \right). \]  

\[ \begin{align*}
\Theta_{2a}^{NS}(\tau + 1, \zeta) &= T_a \Theta_{2a}^{\tilde{N}S}(\tau, \zeta), \\
\Theta_{2a}^{\tilde{N}S}(\tau + 1, \zeta) &= T_a \Theta_{2a}^{R}(\tau, \zeta), \\
\Theta_{2a}^{R}(\tau + 1, \zeta) &= \exp \left( i \frac{\pi}{2} \right) T_a \Theta_{2a}^{\tilde{R}}(\tau, \zeta), \\
\Theta_{2a}^{\tilde{R}}(\tau + 1, \zeta) &= \exp \left( i \frac{\pi}{2} \right) T_a \Theta_{2a}^{\tilde{R}}(\tau, \zeta),
\end{align*} \]

with
\[ T_a = \exp \left( 2\pi i \left( a \frac{a^2}{p} - \frac{1}{12} \right) \right). \]

\[ \begin{align*}
\Theta_{2a}(\tau, \zeta + \tau) &= V_{\Phi_0} \Theta_{2a+2}^{NS}(\tau, \zeta), \\
\Theta_{2a}^{NS}(\tau, \zeta + \tau) &= V_{\Phi_0} \Theta_{2a+2}^{\tilde{N}S}(\tau, \zeta), \\
\Theta_{2a}^{R}(\tau, \zeta + \tau) &= V_{\Phi_0} \Theta_{2a+2}^{R}(\tau, \zeta), \\
\Theta_{2a}^{\tilde{R}}(\tau, \zeta + \tau) &= V_{\Phi_0} \Theta_{2a+2}^{\tilde{R}}(\tau, \zeta),
\end{align*} \]

with
\[ V_{\Phi_0} = \exp \left( -2\pi i \frac{2}{p} \left( \text{Re} \frac{\tau}{2} + \text{Re} \zeta \right) \right). \]

\[ \begin{align*}
\Theta_{2a}^{NS}(\tau, \zeta + \frac{p\tau}{2}) &= V_{\frac{p}{2} \Phi_0} \Theta_{2a}^{R}(\tau, \zeta), \\
\Theta_{2a}^{\tilde{N}S}(\tau, \zeta + \frac{p\tau}{2}) &= V_{\frac{p}{2} \Phi_0} \Theta_{2a}^{\tilde{R}}(\tau, \zeta), \\
\Theta_{2a}^{R}(\tau, \zeta + \frac{p\tau}{2}) &= V_{\frac{p}{2} \Phi_0} \Theta_{2a}^{\tilde{N}S}(\tau, \zeta), \\
\Theta_{2a}^{\tilde{R}}(\tau, \zeta + \frac{p\tau}{2}) &= V_{\frac{p}{2} \Phi_0} \Theta_{2a}^{R}(\tau, \zeta),
\end{align*} \]
with

\[ V_2^{\Phi_0} = \exp\left( -2\pi i \left( \frac{p}{2} \text{Re} \tau + \text{Re} \zeta \right) \right). \]  

(A.24)

Using these transformations, we obtain that the partition functions for the Jain states (A.16) transform as in the \( c = 1 \) case under the \( V_2^\Phi \) and modular transformations, because the number of charge sectors \( p \) is odd for the Jain theory (see Fig. (6) and Section 4.2.1). Owing to the absence of \( Z_2 \) anomaly, the modular invariant partition function \( Z_{\text{Ising}} \) is consistent with TR symmetry, then the topological phase is instable.

In the following, we also give the partition functions for the general multicomponent Jain theory with \( c = k \) and symmetry \( \hat{U}(1) \times \hat{SU}(k) \). The stability parameters are \( (k, p) = (k, 2nk + 1), \ n \in \mathbb{N} \). The form of the transformations depends on the parity of the \( k \) parameter:

i) For even \( k \), we remember from Ref. [12] that the charge characters are given by \( K_{ka+\alpha p}(\tau, k\zeta; kp) \), with \( \alpha = 1, \cdots, k \) and \( a = 1, \cdots, p \). The neutral characters \( \chi(\tau, 0) \) belong to the \( \hat{SU}(k)_1 \) affine algebra and have the periodicity and modular transformations (A.14). Starting from the anyon partition function of the Neveu-Schwarz sector (4.11), we can act with the \( T \) and \( ST \) modular transformations to obtain the other spin sectors. Altogether they are:

\[
\Theta_{ka}^{NS}(\tau, \zeta; k) = \sum_{\alpha=1}^{k} K_{ka+\alpha p}(\tau, k\zeta; kp) \chi_\alpha(\tau, 0),
\]

(A.25)

\[
\Theta_{ka}^{\tilde{NS}}(\tau, \zeta; k) = \sum_{\alpha=1}^{k} (-1)^\alpha K_{ka+\alpha p}(\tau, k\zeta; kp) \chi_\alpha(\tau, 0),
\]

\[
\Theta_{ka}^{R}(\tau, \zeta; k) = \sum_{\alpha=1}^{k} K_{ka+\alpha p}(\tau, k\zeta; kp) \chi_{\alpha+\frac{k}{2}}(\tau, 0),
\]

\[
\Theta_{ka}^{\tilde{R}}(\tau, \zeta; k) = \sum_{\alpha=1}^{k} (-1)^{\alpha+\frac{k}{2}} K_{ka+\alpha p}(\tau, k\zeta; kp) \chi_{\alpha+\frac{k}{2}}(\tau, 0).
\]

Note that for \( k = 2 \) we obtain the expressions given before (Eq.(A.15)).

ii) For odd \( k \), the charge characters are different for each spin sector (as in the \( c = 1 \) case). For the \( NS \) and \( \tilde{NS} \) spin sectors they read:

\[
K_{ka+\alpha p}^{NS}(\tau, k\zeta; kp) = K_{ka+\alpha p}(\tau, k\zeta; kp),
\]

(A.26)

\[
K_{ka+\alpha p}^{\tilde{NS}}(\tau, k\zeta; kp) = F(\tau, \zeta) \sum_{n \in \mathbb{Z}} (-1)^{pkn} \exp \left[ 2\pi i \left( \frac{\tau}{2kp} \left( pkn + ka + \alpha p \right)^2 + \frac{\zeta}{p} \left( pkn + 2a + \alpha p \right) + \frac{ka}{2} + \frac{\alpha}{2} \right) \right].
\]
The corresponding expressions for the $R$ and $\tilde{R}$ spin sectors are defined by
\begin{align}
K_{ka+\alpha p}^R &= K_{ka+\alpha p+\frac{kp}{2}}^{NS}, \quad (A.27) \\
K_{ka+\alpha p}^{\tilde{R}} &= K_{ka+\alpha p+\frac{kp}{2}}^{\tilde{NS}}.
\end{align}

Upon combining them with the neutral characters $\chi_\alpha(\tau,0)$ of the $SU(k)_1$ affine algebra, the anyon sectors take the general form for the simple current modular invariants, as explained in Section 4:
\begin{equation}
\Theta_\alpha^{(\sigma)}(\tau,\zeta;k) = \sum_{\alpha=1}^{k} K_{ka+\alpha p}^{(\sigma)}(\tau,k\zeta;k)\chi_\alpha(\tau,0), \quad \sigma = NS, \tilde{NS}, R, \tilde{R}. \quad (A.28)
\end{equation}

The transformations under the modular group and the insertion fluxes are those shown in Fig.6, since $p$ is always odd in the Jain theory.

### A.3 General $U(1)^n$ Abelian states

Let us discuss the general Abelian states whose spectrum of charge and statistics is described by the $K$-matrix formalism (see Section 4.3.2). In the case that the $n$ chiral (spin-up) and $n$ antichiral (spin-down) edge modes are independent, the $K$ matrix takes the block diagonal form (4.17) and the $n \times n$ block $K$ is definite positive, integer valued and with $K_{ii}$ odd, $i = 1, \ldots, n$. In the basis in which the charge vector is $t = (t,t)$, where $t = (1, \ldots, 1)$ $n$-dimensional, the characters of the chiral modes have been written in Ref.[12]. The chiral anyon sectors modes of the $NS$ and $\tilde{NS}$ spin sectors are (with $\lambda \in \mathbb{Z}^n/K\mathbb{Z}^n$):
\begin{align}
K_\lambda^{NS}(\tau,\zeta) &= \frac{F(\tau,\zeta)}{\eta(q)^n} \sum_{\ell \in \mathbb{Z}^n} \exp \left( 2\pi i \left[ \frac{\tau}{2} (K\ell + \lambda)^T K^{-1} (K\ell + \lambda) + \zeta T (\ell + K^{-1}\lambda) \right] \right), \quad (A.29) \\
K_\lambda^{\tilde{NS}}(\tau,\zeta) &= \frac{F(\tau,\zeta)}{\eta(q)^n} \times \\
&\sum_{\ell \in \mathbb{Z}^n} (-1)^{\ell T b} \exp \left( 2\pi i \left[ \frac{\tau}{2} (K\ell + \lambda)^T K^{-1} (K\ell + \lambda) + \zeta T (\ell + K^{-1}\lambda) + \frac{1}{2} \lambda^T K^{-1} b \right] \right),
\end{align}

with the prefactor $F(\tau,\zeta) = \exp \left( -\pi t^T K^{-1} t (\text{Im}\zeta)^2 / \text{Im}\tau \right)$; those of the $R$ and $\tilde{R}$ sectors are related to these by
\begin{align}
K_\lambda^R &= K_\lambda^{NS}^{\lambda+b}, \quad (A.30) \\
K_\lambda^{\tilde{R}} &= K_\lambda^{\tilde{NS}}^{\lambda+b}.
\end{align}

In these equations, $b$ is a $n$-dimensional vector whose components are the diagonal elements of the $K$ matrix:
\begin{equation}
b = \begin{pmatrix} K_{11} \\ \vdots \\ K_{nn} \end{pmatrix}. \quad (A.31)
\end{equation}
The edge partition functions for each spin sectors are

\[ Z^\sigma(\tau, \zeta) = \sum_{\lambda \in \mathbb{Z}^n/K\mathbb{Z}^n} K^{\sigma}_\lambda(\tau, \zeta) \overline{K^{\sigma}_\lambda(\tau, \zeta)}, \quad \sigma = NS, \overline{NS}, R, \overline{R}. \tag{A.32} \]

As in previous cases, we give the transformation rules of the anyon sectors (A.29) and (A.30):

- **S**

\[
K^{NS}_\lambda(-\frac{1}{\tau}, -\frac{\zeta}{\tau}) = e^{i\varphi} \sum_{\lambda' \in \mathbb{Z}^n/K\mathbb{Z}^n} S_{\lambda, \lambda'} K^{NS}_{\lambda'}(\tau, \zeta), \tag{A.33}
\]

\[
K^{\overline{NS}}_\lambda(-\frac{1}{\tau}, -\frac{\zeta}{\tau}) = \exp\left(2\pi i \frac{\lambda^T K^{-1} b}{2}\right) e^{i\varphi} \sum_{\lambda' \in \mathbb{Z}^n/K\mathbb{Z}^n} S_{\lambda, \lambda'} K^{\overline{NS}}_{\lambda'}(\tau, \zeta),
\]

\[
K^R_\lambda(-\frac{1}{\tau}, -\frac{\zeta}{\tau}) = e^{i\varphi} \sum_{\lambda' \in \mathbb{Z}^n/K\mathbb{Z}^n} S_{\lambda, \lambda'} K^\overline{NS}_{\lambda'}(\tau, \zeta),
\]

\[
K^{\overline{R}}_\lambda(-\frac{1}{\tau}, -\frac{\zeta}{\tau}) = \exp\left(2\pi i \left(\lambda^T K^{-1} b + \frac{1}{4} b^T K^{-1} b\right)\right) e^{i\varphi} \sum_{\lambda' \in \mathbb{Z}^n/K\mathbb{Z}^n} S_{\lambda, \lambda'} K^{\overline{R}}_{\lambda'}(\tau, \zeta),
\]

with

\[
S_{\lambda, \lambda'}(\tau, \zeta) = \exp\left(\frac{2\pi i \lambda^T K^{-1} \lambda'}{\sqrt{\det K}}\right), \quad e^{i\varphi} = \exp\left(i\pi t^T K^{-1} t \Re\left(\frac{\zeta^2}{\tau}\right)\right). \tag{A.34}
\]

- **T**

\[
K^{NS}_\lambda(\tau + 1, \zeta) = \exp\left(-2\pi i \frac{1}{2} \lambda^T K^{-1} b\right) T_a K^{NS}_\lambda(\tau, \zeta), \tag{A.35}
\]

\[
K^{\overline{NS}}_\lambda(\tau + 1, \zeta) = \exp\left(2\pi i \frac{1}{2} \lambda^T K^{-1} b\right) T_a K^{\overline{NS}}_\lambda(\tau, \zeta),
\]

\[
K^R_\lambda(\tau + 1, \zeta) = T_b K^R_\lambda(\tau, \zeta),
\]

\[
K^{\overline{R}}_\lambda(\tau + 1, \zeta) = T_b K^{\overline{R}}_\lambda(\tau, \zeta),
\]

with

\[
T_a = \exp\left(2\pi i \left(\frac{1}{2} \lambda^T K^{-1} \lambda - \frac{n}{24}\right)\right), \quad T_b = \exp\left(2\pi i \left(\frac{1}{2} \lambda^T K^{-1} b + \frac{1}{8} b^T K^{-1} b\right)\right). \tag{A.36}
\]

In order to obtain these transformations, we used the symmetry of the $K$ matrix and the odd parity of its diagonal elements, leading to:

\[
\exp\left[2\pi i \frac{1}{2} \ell^T K \ell\right] = \exp\left[2\pi i \frac{1}{2} K_{ii} \ell^2_i\right] = (-1)^{\ell^T b}. \tag{A.37}
\]
\[ K_{\lambda}^{\text{NS}}(\tau, \zeta + \tau) = V_{\Phi_0} K_{\lambda + t}^{\text{NS}}(\tau, \zeta), \] (A.38)

\[ K_{\lambda}^{\text{NS}}(\tau, \zeta + \tau) = \exp \left( -2\pi i \frac{1}{2} t^T K^{-1} b \right) V_{\Phi_0} K_{\lambda + t}^{\text{NS}}(\tau, \zeta), \]

\[ K_{\lambda}^{R}(\tau, \zeta + \tau) = V_{\Phi_0} K_{\lambda + t}^{R}(\tau, \zeta), \]

\[ K_{\lambda}^{\tilde{R}}(\tau, \zeta + \tau) = \exp \left( -2\pi i \frac{1}{2} t^T K^{-1} b \right) V_{\Phi_0} K_{\lambda + t}^{\tilde{R}}(\tau, \zeta), \]

with

\[ V_{\Phi_0}(\tau, \zeta) = \exp \left( -2\pi i t^T K^{-1} t \left( Re\frac{\tau}{2} + Re\zeta \right) \right). \] (A.39)

In this theory, the partition functions (A.32) transform under the modular group as depicted in Fig.(6)(b). We have to more careful about the \( V^{p}_{2} \) transformation: according to the discussion in Section (4.2.1), there are two different results depending on the parity of \( p \). This number can be obtained from the minimal value \( e^{*} = 1/p \) in the charge spectrum. In general, we can write:

\[ K_{\lambda}^{\text{NS}}(\tau, \zeta + \frac{p}{2} \tau) = V_{\frac{p}{2}} K_{\lambda + \frac{p}{2} t}^{\text{NS}}(\tau, \zeta), \] (A.40)

\[ K_{\lambda}^{\text{NS}}(\tau, \zeta + \frac{p}{2} \tau) = V_{\frac{p}{2}} \exp \left( -2\pi i \frac{p}{4} t^T K^{-1} b \right) K_{\lambda + \frac{p}{2} t}^{\text{NS}}(\tau, \zeta), \]

\[ K_{\lambda}^{R}(\tau, \zeta + \frac{p}{2} \tau) = V_{\frac{p}{2}} K_{\lambda + \frac{p}{2} t}^{R}(\tau, \zeta), \]

\[ K_{\lambda}^{\tilde{R}}(\tau, \zeta + \frac{p}{2} \tau) = V_{\frac{p}{2}} \exp \left( -2\pi i \frac{p}{4} t^T K^{-1} b \right) K_{\lambda + \frac{p}{2} t}^{\tilde{R}}(\tau, \zeta), \]

with

\[ V_{\frac{p}{2}} = \exp \left( -2\pi i t^T K^{-1} t \left( \frac{p^2}{8} Re\tau + \frac{p}{2} Re\zeta \right) \right). \] (A.41)

We distinguish the two cases:

i) If \( p \) is odd, the \( V^{p}_{2} \) transformation maps \( \text{NS} \) sectors into \( \text{R} \) sectors, up to a reshuffling of the anyon label, \( \lambda \rightarrow \lambda' \), as follows:

\[ V^{p}_{2} : K_{\lambda}^{\text{NS}} \rightarrow K_{\lambda'}^{\text{R}}, \] (A.42)

and similarly for the other sectors. Then the partition function of the Neveu-Schwarz sector is mapped in that of the Ramond sector and so on, as depicted in Fig.(6)(a).

ii) If \( p \) is even, every spin sector partition function returns to itself, up to a reshuffling of anyon sectors in Eq.(A.32),

\[ V^{p}_{2} : Z^{\sigma}(\tau, \zeta) \rightarrow Z^{\sigma}(\tau, \zeta + \frac{p}{2} \tau) = Z^{\sigma}(\tau, \zeta), \quad \sigma = \text{NS}, \tilde{\text{NS}}, \text{R}, \tilde{\text{R}}, \] (A.43)

as is pictorially represented in Fig.(7)(a).
A.4 (331) and Pfaffian states

For the (331) state, we recall from the main text that \((k, p) = (2, 4)\), the charge character is \(K_\lambda(\tau, 2\zeta; 8)\), with \(\lambda = 1, \cdots, 8\), and the Abelian neutral character is \(\chi_\lambda(\tau, 4) = K_\lambda(\tau, 0; 4)\) [13]. The anyon sectors for the chiral (spin-up) modes are:

\[
\begin{align*}
\Theta^{NS}_\lambda(\tau, \zeta; 2) &= K_\lambda \chi_\lambda + K_{\lambda+4} \chi_{\lambda+2} \\
\Theta^{\bar{NS}}_\lambda(\tau, \zeta; 2) &= K_\lambda \chi_\lambda - K_{\lambda+4} \chi_{\lambda+2} \\
\Theta^R_\lambda(\tau, \zeta; 2) &= K_\lambda \chi_{\lambda+1} + K_{\lambda+4} \chi_{\lambda+3} \\
\Theta^{\bar{R}}_\lambda(\tau, \zeta; 2) &= K_\lambda \chi_{\lambda+1} - K_{\lambda+4} \chi_{\lambda+3}.
\end{align*}
\]

Then, the edge partition functions for the Neveu-Schwarz and Ramond spin sectors of the (331) topological states can be written as:

\[
Z^{NS/\bar{NS}}_{(331)} = \sum_{\lambda=0}^{3} \left\{ |K_\lambda \chi_\lambda \pm K_{\lambda+4} \chi_{\lambda+2}|^2 + |K_\lambda \chi_{\lambda+2} \pm K_{\lambda+4} \chi_\lambda|^2 \right\},
\]

\[
Z^{R/\bar{R}}_{(331)} = \sum_{\lambda=0}^{3} \left\{ |K_\lambda \chi_{\lambda+1} \pm K_{\lambda+4} \chi_{\lambda+3}|^2 + |K_\lambda \chi_{\lambda+3} \pm K_{\lambda+4} \chi_{\lambda+1}|^2 \right\}.
\]

The (331) anyon sectors (A.44) transform under the modular group and the insertion of fluxes according to the Abelian formulas Eq.(A.33-A.40) with \(n = 2\), and \(p = 4\) even.

The edge partition function for the Pfaffian states in the Neveu-Schwarz spin sector is given in (4.25) [14]. We obtain the partition functions of the other spin sectors acting with \(T\) and \(ST\) on \(Z^{NS}_{Pf}\) (4.25) as described in Section (3.2). They read:

\[
Z^{NS/\bar{NS}}_{Pf} = |K_0 I \pm K_4 \psi|^2 + |K_0 \psi \pm K_4 I|^2 + |(K_1 \pm K_{-3})\sigma|^2,
\]

\[
|K_2 I \pm K_{-2} \psi|^2 + |K_2 \psi \pm K_{-2} I|^2 + |(K_3 + K_{-1})\sigma|^2,
\]

\[
Z^{R/\bar{R}}_{Pf} = |K_3 I \pm K_{-1} \psi|^2 + |K_3 \psi \pm K_{-1} I|^2 + |(K_0 \pm K_4)\sigma|^2,
\]

\[
|K_{-3} I \pm K_1 \psi|^2 + |K_{-3} \psi \pm K_1 I|^2 + |(K_2 + K_{-2})\sigma|^2.
\]

We have written the neutral characters in (A.46) with the same symbol of the Ising fields

\[
\chi^0_0 = \chi^1_2 = I, \quad \chi^1_1 = \chi^1_3 = \sigma, \quad \chi^0_2 = \chi^2_0 = \psi,
\]

that model the neutral excitations of this system [11] [31].

The spin sectors partition functions of (331) and Pfaffian topological states, Eqs. (A.45) and (A.46), respectively, transform in the same manner under \(V_2^T\) and the modular group because the two theories have the same \((k, p)\) parameters entering the stability analysis, as discussed in Section (4.3.3). Since \(p\) is even, these transformations are shown in Fig.(7). The absence of the \(Z_2\) anomaly in both cases allows for the \(Z_{Ising}\) modular invariant (3.19) to be TR symmetric, such that the two topological phases are unstable.
A.5 Read-Rezayi states

The Read-Rezayi models [32] are based on the neutral $\mathbb{Z}_k$ parafermion conformal theories with central charge $c = 2(k-1)/(k+2)$, described by the coset construction $SU(2)_k/U(1)_{2k}$ [14]. For these theories the values of the stability parameters are $p = kM + 2$ with $M = 1, 3, 5, \cdots$ and $k = 2, 3, \cdots$. The structure of partition functions depends on the parity of $k$.

A.5.1 Partition functions for RR states with even $k$

The charge characters are given by the functions (3.1) $K_{\lambda}(\tau, k\zeta, kp)$ with periodicities $K_{\lambda+kp} = K_{\lambda}$. The $\mathbb{Z}_k$ parafermionic characters that describe the neutral part are denoted by $\chi_m(\tau; 2k)$, and have the following periodicities and modular transformations [14],

$$\chi_{m}^{\ell} = \chi_{m+2k}^{\ell} = \chi_{m+k}^{k-\ell}, \quad m = \ell \mod 2, \quad \ell = 0, 1, \cdots, k.$$  \hspace{1cm} (A.48)

$$S: \quad \chi_m^{\ell}(-1, 0; 2k) = \frac{1}{\sqrt{2k}} \sum_{\ell' = 0}^{k-1} \sum_{m' = 1}^{2k} \exp\left(-2\pi i \frac{mm'}{2k}\right) s_{\ell,\ell'} \chi_{m'}^{\ell}(\tau; 2k), \quad \ell = 0, 1, \cdots, k,$$

$$T: \quad \chi_{m}^{\ell}(1, 0; 2k) = \exp\left(-\frac{2\pi i}{k} (\ell^2 + 1) \right) \chi_{m}^{\ell}(\tau; 2k).$$

Starting from the Neveu-Schwarz anyon sectors given in Ref.[14] and acting with $T$ and $ST$, we find those of the other spin sectors. Altogether they read:

$$\Theta_{\ell}^{NS}(\tau, \zeta; k) = \sum_{b=1}^{k} K_{a+b}(\tau, k\zeta; kp) \chi_{a+2b}^{\ell}(\tau; 2k), \quad a = 0, 1, \cdots, p-1, \quad \ell = 0, 1, \cdots, k.$$  \hspace{1cm} (A.50)

$$\Theta_{\ell}^{\tilde{NS}}(\tau, \zeta; k) = \sum_{b=1}^{k} (-1)^b K_{a+b}(\tau, k\zeta; kp) \chi_{a+2b}^{\ell}(\tau; 2k),$$

$$\Theta_{\ell}^{R}(\tau, \zeta; k) = \sum_{b=1}^{k} K_{a+b}(\tau, k\zeta; kp) \chi_{a+2b+k/2}^{\ell}(\tau; 2k),$$

$$\Theta_{\ell}^{\tilde{R}}(\tau, \zeta; k) = \sum_{b=1}^{k} (-1)^b K_{a+b}(\tau, k\zeta; kp) \chi_{a+2b+k/2}^{\ell}(\tau; 2k),$$

where $a = 0, 1, \cdots, p-1$, and $\ell = 0, 1, \cdots, k$. The partition functions in the corresponding spin sectors are:

$$Z_{RR}^{(\sigma)} = \sum_{\ell=0}^{k} \sum_{a=1}^{p} \Theta_{a}^{(\sigma)}(\tau) \Theta_{-a}^{(\sigma)}(\tau), \quad \sigma = NS, \tilde{NS}, R, \tilde{R}. \quad \hspace{1cm} (A.51)$$
We note that the partition functions of the Pfaffian state (Eq. A.46) are obtained by choosing \( M = 1 \) and \( k = 2 \) in the previous formulas. The transformations of the anyon sectors (A.50) and the partition functions (A.51) under the insertion of fluxes and the modular group are the same of the Pfaffian state, represented in Fig.(7). In this case the \( Z_2 \) anomaly is absent, the modular invariant partition function \( Z_{\text{Ising}} \) (3.19) is consistent with TR symmetry and the topological phases are unstable.

A.5.2 Partition functions for RR states with odd \( k \)

It is necessary to introduce different charge characters for each spin sector. For the \( NS \) and \( \tilde{NS} \) sectors they are:

\[
K^\lambda_{NS}(\tau, k\zeta; kp) = K(\tau, k\zeta; kp), \\
K^\lambda_{\tilde{NS}}(\tau, k\zeta; kp) = \frac{F(\tau, \zeta)}{\eta(\tau)} \sum_{n \in \mathbb{Z}} (-1)^{nkp} \exp \left[ 2\pi i \left( \frac{\tau}{2kp} (nkp + \lambda)^2 + \frac{\zeta}{p} (nkp + \lambda) + \frac{\lambda}{2} \right) \right].
\]

Those of the \( R \) and \( \tilde{R} \) spin sectors are defined by:

\[
K^R_\lambda = K^{NS}_{\lambda + \frac{kp}{2}}, \\
K^{\tilde{R}}_\lambda = K^{\tilde{NS}}_{\lambda + \frac{kp}{2}}.
\]

These charge characters have the same periodicities (4.3), i.e. \( K_\lambda = K_{\lambda + kp} \). Making use of \( \mathbb{Z}_k \) parafermionic characters \( \chi^{s}_m(\tau; 2k) \), the anyon sectors for four spin sector take the usual form of simple current invariants:

\[
\Theta^{(\sigma)}_a(\tau, \zeta; k) = \sum_{b=1}^{k} K^{(\sigma)}_{a+bp}(\tau, k\zeta; kp) \chi^{s}_{a+2b}(\tau; 2k), \quad \sigma = NS, \tilde{NS}, R, \tilde{R}, \quad (A.54)
\]

The edge partition functions of the four spin sector have the same expression of those in the even \( k \) case (A.51). Because the values of \((k, p)\) are odd, as discussed in Section (4.2.1), it is easy to show that under the \( V^k \) transformation and the modular group the anyon sectors and the partition functions transform as represented in Fig.(6). Owing to the presence of \( Z_2 \) anomaly, the modular invariant partition function \( Z_{\text{Ising}} \) is inconsistent with TR symmetry, and these non-Abelian topological phases are stable.

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