Classical versus quantum completeness

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The notion of quantum-mechanical completeness is adapted to situations where the only adequate description is in terms of quantum field theory in curved space-times. It is then shown that Schwarzschild black holes, albeit being geodesically incomplete, are quantum complete.

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I. INTRODUCTION

Completeness is a very important concept in classical and quantum physics. The classical motion on a half-line is called complete at the endpoint if there are no initial conditions such that the trajectory runs off to the endpoint in a finite time. If the potential satisfies certain regularity conditions, then the classical motion is complete at the endpoint if and only if the potential grows unbounded from above near the endpoint \( t_0 \). In general relativity, a space-time is called geodesically complete, if any maximal geodesic is defined on the entire real line. If the space-time is time-like or null geodesic incomplete, it is said to be singular \( [2] \). The physical relevance of this geometrical notion is provided upon identifying geodesics with trajectories of free test particles. In quantum mechanics on a half-line, a time-independent potential is called quantum-mechanically complete \( [1] \), if the associated Hamiltonian is essentially self-adjoint on the space of \( C^\infty \)-functions of compact support on the half-line with the origin excluded.

Horowitz and Marolf \( [3] \) showed that there are geodesically incomplete static space-times, with time-like curvature singularities, which are quantum-mechanically complete. Their work stimulated a lot of research concerning geodesically incomplete but quantum-mechanically complete spacetimes, e.g. \( [4] [5] \). As a working analogue, they suggested the non-relativistic hydrogen atom. The classical motion of the electron in the Coulomb potential is incomplete at the origin, because the potential is bounded from above near the origin and thus the origin can be reached by the electron in a finite time. The Coulomb potential is, however, quantum-mechanically complete when probed by the non-relativistic bound-state electron. In other words, the classical singularity of the Coulomb potential is not reflected in any observable related to the bound-state electron.

Quantum field theory in a static, globally hyperbolic space-time allows to define a consistent quantum theory for a single relativistic particle, where the energy of each one-particle state is equal to that of the corresponding classical field \( [2] \). Horowitz and Marolf showed that this is still the case for certain static space-times with time-like singularities \( [2] \). Their result is based on a work by Wald \( [6] [7] \), who proved that the problem of defining the evolution of a Klein-Gordon scalar field in an arbitrary static space-time (with arbitrary singularities consistent with statics) can be reformulated as the problem of constructing self-adjoint extensions of the spatial part of the wave operator.

For a general time-dependent space-time, there is no consistent quantum theory of a single free particle, and the only adequate description is in terms of quantum field theory. In the standard approach this requires to study the evolution of classical test fields in a singular space-time. In static space-times, the evolution of quantum fields is unitary and represents an endomorphism of the physical Hilbert space. In particular, unitarity preserves state normalisation. If dynamical space-times are treated as external backgrounds, the quantum theory does not require a unitary evolution \( [10] \). Therefore, the notion of quantum-mechanical completeness needs to be adapted to include this case.

In discussing geodesic completeness, it usually suffices to consider geodesics defined on \( (0, t_0) \), right endpoints can be treated similarly. A convenient topological criterion for the inextendibility of a geodesic \( \gamma(t), t \in (0, t_0) \) is the following: There is a parameter sequence \( \{t_n\} \to 0 \) such that \( \{\gamma(t_n)\} \) does not converge. As is well known, geodesic parametrisations have geometric significance. If a curve has a reparametrisation as a geodesic, it is called a pregeodesic. In particular, any space-like or time-like curve is pregeodesic if and only if its reparametrisation by its arc-length yields a geodesic. A space-like or time-like pregeodesic \( \alpha(t), t \in (0, t_0) \) is complete (to the left) if and only if it has infinite length \( [11] \).

We call a globally hyperbolic space-time quantum complete (to the left) with respect to a free field theory, if the Schrödinger wave functional of the free test fields can be normalised at the initial time \( t_0 \), and if the normalisation is bounded from above by its initial value for any \( t \in (0, t_0) \). Note that neglecting backreaction is no severe restriction here, since if backreaction becomes important, the question whether a previously given space-time is quantum complete becomes obsolete.

For a Schwarzschild black hole, a Cauchy hypersurface is given by \( \{t_0\} \times \mathbb{R} \times S^2 \), where \( t_0 \in (0, 2M) \) and \( M \) denotes the black hole mass. It follows that the black hole interior is globally hyperbolic and foliated by smooth
space-like Cauchy hypersurfaces \([\Sigma_t]\).

The purpose of this article is to show that the interior of a Schwarzschild black hole is quantum complete, albeit it is geodesically incomplete.

II. SET-UP

We briefly review the functional Schrödinger formulation for quantum field theory in generic space-times, when backreaction can be neglected (for a detailed discussion in Minkowski space-time, see [13]). This formulation will prove to be efficient for investigating qualitative features such as the stability of ground states and the quantum (in)completeness of generic space-times.

Due to a theorem by Geroch [14], a globally hyperbolic space-time is diffeomorphic to \(\mathbb{R} \times \Sigma\), and foliates into hypersurfaces \(\Sigma_t, t \in \mathbb{R}\). In the \((1 + 3)\)-split formulation, the classical theory for a free scalar field is given by the Hamiltonian

\[
H = \int_{\Sigma_t} d\mu(x) \left( N_\perp H^\perp + N_\parallel a H^a \right).
\]

(1)

Here, \(d\mu(x) \equiv d^3x / \sqrt{det(q)}\), with \(q\) denoting the spatial part of the metric, \(H^a = \pi \partial_a \Phi / \sqrt{det(q)}\),

\[
H^\perp = \frac{1}{2} \left[ \frac{1}{det(q)} \pi^2 + q^{ab} \partial_a \phi \partial_b \phi + (m^2 + \zeta R) \phi^2 \right],
\]

(2)

and all tensors are pulled-back to the hypersurface \(\Sigma_t\). Adapting the space-time coordinates to the foliation, \(N_\parallel = 0\) and \(N_\perp = \sqrt{-g_{tt}}\).

Each hypersurface \(\Sigma_t\) is equipped with a Fock space. In the Schrödinger representation, the basis of this Fock space is constructed from the time-independent operator \(\Phi(x)\). Its spectrum contains the classical fields \(\phi(x)\) as eigenvalues [12]. The \(\phi\)-representation of an arbitrary state \(|\Psi\rangle\) in the Fock space is a (nonlinear) wave functional \(|\Psi[\phi](t)\rangle\).

\[\Phi[\phi](t)\]

satisfies a functional generalisation of the Schrödinger equation,

\[
i \partial_t |\Psi[\phi](t)\rangle = H[\Phi[\phi](t)] |\Psi[\phi](t)\rangle,
\]

(3)

\[
H[\Phi[\phi](t)] = \int_{\Sigma_t} d\mu(x) \mathcal{H}(\Phi(\phi); t, x),
\]

(4)

where \(H[\Phi[\phi](t)]\) denotes an operator valued functional constructed from the Hamilton density

\[
\mathcal{H} = \frac{1}{2} \left[ \frac{1}{det(q)} \pi^2 + q^{ab} \partial_a \phi \partial_b \phi + (m^2 + \zeta R) \phi^2 \right].
\]

(5)

Note that the any explicit dependence on \((t, x)\) is due to the space-time geometry, which can be thought of as an external source non-minimally coupled to the quantum field.

Wave functionals are normalised in the usual sense,

\[
\|\Psi\|^2(t) = \int D\phi \, \Psi^*[\phi](t) \Psi[\phi](t),
\]

(6)

where \(D\phi\) denotes the measure over all field configurations in \(\Sigma_t\). Stability of the state populated with \(\phi(x)\) requires that the norm of the wave functional is time-independent. This corresponds to a unitary evolution.

On a dynamical space-time, considered as an external background, however, the evolution is not required to be unitary, i.e. \(H[\Phi](t)\) needs not be a self-adjoint operator on the space of wave functionals. Intuitively, probability can be lost to the background (like for dissipative systems when the interaction causing the friction is not fully resolved in the participating degrees of freedom). Consistency of the dynamics is more subtle in this case. Let \(\|\Psi[\phi]\|^2(t_0)\) denote the probability density (with respect to the space of field configurations) at the initial hypersurface, and consider the interval \([0, t_0]\) with zero marking the left endpoint. We call the evolution consistent, even if it violates unitarity, provided that \(\|\Psi[\phi]\|^2(t) \leq \|\Psi[\phi]\|^2(t_0)\) \(\forall t \in (0, t_0)\). Intuitively, probability must not be gained from a background which is not resolved in dynamical degrees of freedom. If the above consistency relation is violated, then backreaction effects are relevant, and the original space-time geometry is obsolete.

For the time-dependent ground state, a generalised Gaussian ansatz is motivated following the example of the harmonic oscillator in quantum mechanics:

\[
\Psi^{(0)}[\phi](t) = N^{(0)}(t) \mathcal{G}^{(0)}[\phi](t),
\]

(7)

\[
\mathcal{G}^{(0)}[\phi](t) = \exp \left[ -\frac{1}{2} \int_{\Sigma_t} d\mu(x) d\mu(y) \phi(x) K(x, y, t) \phi(y) \right].
\]

The functional Schrödinger equation gives for the \(\phi\)-independent factor \(N^{(0)}(t)\) an evolution equation that can be directly integrated,

\[
N^{(0)}(t) = N_0 \exp \left[ -\frac{1}{2} \int_{t_0}^t dt' \int_{\Sigma_{t'}} \sqrt{-g_{tt}} d\mu(z) K(z, z', t') \right],
\]

while the evolution for the kernel \(K(x, y, t)\) is described by a \(\phi\)-dependent nonlinear integro-differential equation. Acting with two functional derivatives \(\delta^2 / \delta \phi(x) \delta \phi(y)\) on the kernel equation, we find

\[
\frac{i \partial_t}{\sqrt{det(q)}(x)} \left[ \sqrt{det(q)(x)} \sqrt{det(q)(y)} K(x, y, t) \right] = \int_{\Sigma_{t'}} \sqrt{-g_{tt}(z)} d\mu(z) K(x, z, t) K(z, y, t) + \sqrt{-g_{tt}(x)} (\Delta - m^2 - \zeta R) \delta^{(3)}(x, y).
\]

(8)

Our convention for the Dirac distribution is as follows:

\[
\sqrt{det(q)(x)} \delta^{(3)}(x, y) \equiv \delta^{(3)}(x - y).
\]

III. CALCULATION

In this section, we specialise to the interior of Schwarzschild black holes. In the usual Schwarzschild coordinate neighbourhood, the Schwarzschild function is
given by \( h(\tau) = (2 - \tau)/\tau \), where \( \tau \equiv 2t/r_g \) is dimensionless, and \( r_g \equiv 2M \) denotes the Schwarzschild radius (\( G_N \equiv 1 \)). The warped product line element for the Schwarzschild black hole becomes

\[
g = -h^{-1}(\tau) dt^2 + h(\tau) dr^2 + (\tau r_g)^2 ds^2/4 \, ,
\]

where by this normalisation, in each rest-space \( t = \text{constant} \), the surface \( r = \text{constant} \) has the induced line element \((\tau r_g)^2 ds^2/4\), and is thus the two-sphere of radius \( \tau r_g/2 \) with Gaussian curvature \( 4/(\tau r_g)^2 \) and area \( \pi (\tau r_g)^2 \).

Since the Schwarzschild space-time is spherically symmetric, the kernel \( K \) introduced in (7) is a function \( K(x - y, \tau) \). Our convention for Fourier transforms is

\[
K(z, \tau) = \int \frac{d^3k}{(2\pi)^3} \exp(ik\cdot z) \hat{K}(k, \tau) \, ,
\]

where \( \hat{\phi}(k, \tau) \equiv q^a k_a \hat{\phi}^* \). The Fourier amplitudes \( \hat{K} \equiv \hat{K}/\text{det}(q) \) satisfy a Riccati equation,

\[
\frac{i}{\sqrt{\text{det}(q)}} \partial_\tau \hat{K}(k, \tau) = \sqrt{\text{det}(q)} (\hat{K}^2(k, \tau) - \Omega^2(k, \tau)) \, .
\]

(11)

The inhomogeneous contribution \( \Omega^2(k, \tau) \equiv g^aM_k k_b + m^2 \) is just the dispersion relation of the free fields.

The kernel can alternatively be described as follows. Suppose \( \phi(x', t') \) is a solution of the equation of motion for the free fields. It is related to a solution at a later time \( t > t' \) by Huygens’ principle [13, 16],

\[
\phi(x, t) = \int_{t_0}^t dt' h^{-1/2} \int_{\Sigma_{t'}} \sqrt{\text{det}(q)} iK(x - x', t') \phi(x', t') \, .
\]

(12)

Indeed, a kernel fulfilling Huygens’ principle for the time-dependent fields \( \phi \) is a solution of the kernel equation [7]. Moreover,

\[
(\Box - \Omega^2(k, \tau)) \hat{\phi}(k, \tau) = 0 \, .
\]

(13)

Of course, from the solutions of (13) the kernel can be calculated directly,

\[
\hat{K}(k, \tau) = \frac{-i}{\sqrt{\text{det}(q)}} \partial_\tau \ln \hat{\phi}(k, \tau) \, ,
\]

(14)

but it should be clear that this is a less efficient approach than solving the kernel equation. With the kernel representation (14), however, it is straightforward to show that the time dependence of \( \|\Psi^{(0)}\| \) is not fictitious, even without solving (13). Using (14) in (8), we find

\[
\|\Psi^{(0)}(\tau)\|^2 = |N_0|^2 \exp \left(-\frac{v(\Sigma)}{2} \int \frac{d^3k}{(2\pi)^3} \ln \left| \frac{\hat{\phi}(k, \tau)}{\phi(k, \tau)} \right|^2 \right) \, ,
\]

(15)

where \( v(\Sigma) \) denotes the time-independent coordinate volume of the hypersurfaces. Furthermore,

\[
\|G(0)\| (\tau) = \left( \text{Det} \left( \frac{\text{det}(q)}{\sqrt{\text{det}(q)}} W(\hat{\phi}, \hat{\phi}^*) \right) \right)^{-1/2} \, ,
\]

(16)

with \( W(\hat{\phi}, \hat{\phi}^*) \equiv \hat{\phi} \partial_\tau \hat{\phi}^* \) denoting the Wronskian of the solution and its complex conjugate, and \( \text{Det} \) is the functional determinant. From this result, we can draw two important immediate conclusions. First, for Friedman space-times, Abel’s differential equation identity [17] gives that \( \sqrt{\text{det}(q)} W(\hat{\phi}, \hat{\phi}^*) \) is time-independent. As a consequence, \( \|\Psi^{(0)}\| \) is time-independent (the time-dependent contributions to [18] and [19] cancel), and the ground state is stable in Friedman space-times. By our definition, Friedman space-times are quantum complete, albeit they are geodesically incomplete. Second, for a Schwarzschild black hole, the situation is different, because \( g^aM_k k_b + m^2 \equiv 0 \) is time-dependent in this case. Hence, the ground state cannot be stable, but the Schwarzschild black hole can still be quantum complete (with respect to free fields).

In order to show that Schwarzschild black holes are indeed quantum complete, we transform the Riccati equation (11) for the Fourier amplitudes \( \hat{K} \) to a homogeneous, second-order ordinary differential equation in normal form,

\[
\partial^2_\tau f(k, \tau) + \omega^2(k, \tau) f(k, \tau) = 0 \, ,
\]

(17)

\[
\omega^2(k, \tau) \equiv \frac{r_g^2}{16\ln(\tau)} (1 - 2g_{tt}(\tau) + g_{tt}^2(\tau)) - g_{tt}(\tau)M^2 \Omega^2(k, \tau) \, .
\]

(18)

The Fourier amplitudes \( \hat{K} \) are related to \( f \) as follows:

\[
\hat{K}(k, \tau) = -i\frac{1}{\omega^2} \partial_\tau \ln(\sigma(\tau) f(k, \tau)) \, ,
\]

(19)

with \( \sigma(\tau) \equiv -iM \sqrt{\text{det}(q)}/\text{det}(q) \).

The dispersion relation for \( f \) is singular at the horizon, \( \tau = 2 \), and at the classical black hole singularity, \( \tau = 0 \). For our purposes, it suffices to expand \( f \) near \( \tau = 0 \). Let us first give a quick argument and justify it posteriori. The leading singularity in the dispersion relation around \( \tau = 0 \) is given by \( \omega_0 = 1/(2\tau) \), with corrections \( O(1/\sqrt{\tau}) \). Near \( \tau = 0 \), the dynamics is governed by the background, i.e. the dominant contribution in the dispersion relation is momentum-independent. In this regime, \( f(\tau) \rightarrow C^2 \sqrt{\tau} (C + \ln \tau) \), which translates to

\[
\text{Im} \left( \hat{K}(\tau) \right) \rightarrow \frac{-1}{M^2 \sin(\theta)} \frac{1}{\tau \ln|\tau|} \, ,
\]

\[
\text{Re} \left( \hat{K}(\tau) \right) \rightarrow \left| \text{Im}(C) \right| \frac{\ln|\hat{K}(\tau)|}{\ln|\tau|} \, ,
\]

(20)

near the black hole singularity. Here, \( C, C' \in \mathbb{C} \) are constants of integration. Note that \( \text{Re}(\hat{K}(\tau)) \ll \text{Im}(\hat{K}) \) near the singularity. The real part is taken into account since
the dominant contribution gives a phase factor for $G^{(0)}$. Using (20) in (8), the normalisation $N^{(0)}$ goes to zero like

$$N^{(0)}(\tau) \to |\ln\tau|^{-\frac{1}{2} v(\Sigma) \Lambda}.$$  \hspace{1cm} (21)

The evaluation requires a volume as well as a ultraviolet cut-off. We simply introduced a coordinate volume and an ultraviolet regularisation. We find an irrelevant phase factor. Of course, this evaluation requires a volume as well as an ultraviolet cut-off, $v(\Sigma)$ and $\Lambda$, respectively, since the regularisation details have no impact on the limit $N^{(0)}(\tau) \to 0$ as $\tau \to 0$. For $G^{(0)}$, we find

$$G^{(0)}(\tau) = \exp \left( -\frac{1}{2} \text{Re} \left( \hat{K} \right) (\tau) \int_{\Sigma} d\mu(x) \phi^2(x) \right)$$  \hspace{1cm} (22)

times an irrelevance phase factor.

It is more rigorous to take all contributions in the dispersion relation into account that are singular at $\tau = 0$,

$$\omega_s^2(k, \tau) = \omega_0^2(\tau) + (k^2_\tau + \frac{1}{2}) \omega_0(\tau) + \mathcal{O}(\tau^0),$$  \hspace{1cm} (23)

where $k^2_\tau \equiv (\tau r_g)^2 \sigma^2(k, k)/4$. Introducing the variable $z \equiv \sqrt{1 + 2k^2_\tau} \sqrt{\tau}$, we find

$$f(k, \tau) \to -\frac{\pi}{2} (J_0(z) - 2i K_0(iz))$$  \hspace{1cm} (24)

near the black hole singularity, with $J_0$ denoting the Bessel function of the first kind, and $K_0$ denoting the modified Bessel function of the second kind. This combination shows the same behaviour near $\tau = 0$ as $f$ subject to the dispersion relation $\omega_k$. The momenta $k_\tau$ in angular directions appear only in an overall factor $\geq 1$ and do not modify the dominant behaviour near $\tau = 0$.

Therefore, it is safe to conclude that

$$\left\| \Psi^{(0)}(\tau) \right\|^2(\tau) \to |\ln(\tau)|^{-\frac{1}{2} v(\Sigma) \Lambda} \left( \tau^{3/4} |\ln(\tau)| \right)^{N(\Lambda)} \to 0$$  \hspace{1cm} (25)

as the black hole singularity is approached. Here, $N(\Lambda)$ denotes the number of momentum modes with $|k| \in [0, \Lambda^{1/3}]$. The limit (25) is our main result. In fact, already $\Psi^{(0)}(0) \rightarrow 0$ as $\tau \rightarrow 0$, i.e. the wave functional has vanishing support towards the singularity.

Let us stress again that we were interested in examining the quantum completeness of Schwarzschild black holes with respect to free quantum fields. The answer to this question is insensitive to the details of volume and short-distance regularisation, both of which are required, in principal.

**IV. CONCLUSION & DISCUSSION**

In this article we adapted the notion of quantum-mechanical completeness to situations where the only adequate description is in terms of a quantum theory of fields in generic space-times. We showed that according to the advanced consistency criterion, a Schwarzschild black hole is quantum complete with respect to free scalar fields (in the ground state). Moreover, the wave functional has vanishing support towards the black hole singularity.

There are two types of Non-Gaussianities that can be introduced to describe processes associated with deviations from free fields in the ground state. First of all, excitations of the ground state can be considered. It should be clear that the term excitation is strictly appropriate for static backgrounds. In general, excitations will depend on eigenfunctions $\epsilon$ of $(\Delta - m^2)$ in the background geometry. Excited states are of the form $\Psi^{(n)}(\phi)(\tau) = N_m(\phi, \epsilon(\tau))\Psi^{(0)}(\phi(\tau))$. So excitations are reflected in a (functional) renormalisation of $N_0(\tau)$. The difference between the ground state and the excited states is the following: $\Psi^{(0)}(\phi(\tau))$ populates the ground state with field configurations that need not satisfy any on-shell criteria. What matters is the spatial support of the scalar fields and the correlation between two fields as communicated by the kernel function. This is why the completeness concept used here poses a rather strong consistency requirement on the kernel function. In contrast, excited states are sensitive, in addition, to the moderated overlap between an arbitrary field configuration and fields obeying on-shell conditions. Moderation indicates that the overlap is evaluated using $\phi(x)K(x, y, \tau)\phi(y)$. Intuitively, excitations show an increasing sensitivity on the on-shell conditions.

Secondly, interactions of the Klein-Gordon field with itself and with other fields can be introduced in the Hamiltonian. In this case, we choose the initial data such that the interactions can be treated in the usual perturbative framework. If the Schwarzschild black hole fails to be quantum complete with respect to interacting fields, then the participating fields necessarily entered a strong coupling regime, because the space-time is quantum complete with respect to free fields.

Perhaps not surprisingly, Schwarzschild black holes are enjoying a clash of completeness concepts. The obvious question is how to qualify the importance of quantum completeness relative to classical completeness. We think that this question is related to the measurement process. Let $\gamma(t)$, $t \in [0, t_0]$ be a geodesic, and $\{t_n\} \rightarrow 0$ denote a parameter sequence such that $\{\gamma(t_n)\}$ does not converge. The inextendibility of the geodesic can be observed by measuring any classical observable $\mathcal{O}$ along $\gamma$: $\{\mathcal{O}(\gamma(t_n))\} \subset \mathcal{R}$ does not converge. Hence, geodesic incompleteness is observable, provided the measurement process associated with $\mathcal{O}$ is known. Certainly the measurement process will involve quantum theory at a more or less obvious but essential level. We can ask whether the geodesic incompleteness has an impact on the quantum theory underlying the measurement process. For instance, if black holes are quantum incomplete with respect to the degrees of freedom employed in the measurement device, then $\mathcal{O}$ cannot be measured, and the geodesic incompleteness is not observable. If this holds for any observable, then the geodesic incompleteness is unobservable in principal. This may sound impractical
as a criterion. Measurement processes, however, rely on a few principles and are realised via universal principles such as minimal coupling. This makes it relatively easy to pass from unobservable to unobservable in principal.

We found that Schwarzschild black holes are quantum complete, and, moreover, the ground state does not support field configurations near the singularity. The logical conflict with the measurement process as described above has a well-known resolution: Near the black hole singularity, observables necessarily are part and parcel of the quantum theory. So consistency of the quantum theory is not only essential for the measurement device, but already for the very construction of observables.

In our opinion, and in conclusion, the concept of quantum completeness as suggested in this work has physical relevance, and presents a physical characterisation of space-time singularities and their impact.

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