Abstract:

We define a natural generalized symmetry of the Yang-Mills equations as an infinitesimal transformation of the Yang-Mills field, built in a local, gauge invariant, and Poincaré invariant fashion from the Yang-Mills field strength and its derivatives to any order, which maps solutions of the field equations to other solutions. On the jet bundle of Yang-Mills connections we introduce a spinorial coordinate system that is adapted to the solution subspace defined by the Yang-Mills equations. In terms of this coordinate system the complete classification of natural symmetries is carried out in a straightforward manner. We find that all natural symmetries of the Yang-Mills equations stem from the gauge transformations admitted by the equations.
1. Introduction.

Yang-Mills theory, by which we mean any non-Abelian gauge theory, has provided a fruitful area of study for both physicists and mathematicians. Physicists have used Yang-Mills theory to describe the strong and electroweak interactions [1]. Applications of the Yang-Mills equations in mathematics have been found in several areas; an important example is given by the recent discovery of an intimate relation between reductions of the Yang-Mills equations and a large class of integrable differential equations [2]. Whether one is interested in physical or mathematical applications of the Yang-Mills equations, there are certain basic structural properties of these equations that one would like to understand. One of the most fundamental properties to be examined is the class of generalized symmetries admitted by the equations [3], [4]. Roughly speaking, by generalized symmetries we mean infinitesimal transformations of the fields that map solutions to solutions. The transformations are to be constructed in a local fashion from the fields and their derivatives to any finite order [5]. Given a set of differential equations, the presence of symmetries is connected with the existence of conservation laws, the construction of solution generating techniques, and integrability properties of the equations [3], [4], [6], [7].

There are, of course, manifest symmetries that are built into the Yang-Mills equations, namely, the Poincaré and gauge symmetries. The Poincaré symmetry is responsible for ten first-order conservation laws, while the gauge symmetry leads to trivial conservation laws. In recent years it has been found that many non-linear differential equations admit “hidden symmetries”. For example, the Sine-Gordon equation in 1+1 dimensions is a non-linear wave equation with a built-in Poincaré symmetry group. Remarkably, this equation admits an infinite number of higher-order generalized symmetries and corresponding conservation laws [3], and this fact is intimately associated with the integrability of the Sine-Gordon equation. In light of such examples, and given the strong connection between the Yang-Mills equations and integrable systems, it is tempting to speculate that the Yang-Mills equations will admit higher-order symmetries and conservation laws.

In this paper we begin a classification of all generalized symmetries admitted by the Yang-Mills equations on a flat four-dimensional spacetime. Given the manifest gauge and Poincaré covariance of the Yang-Mills equations, it is reasonable to search for symmetries that are constructed in a gauge and Poincaré covariant manner from the Yang-Mills field strength and its gauge-covariant derivatives. We call such symmetries natural generalized symmetries. In order to classify natural symmetries we borrow techniques from a recent classification of all symmetries for the vacuum Einstein equations [8]. The principal tool used in [8] was an adapted set of spinor coordinates on the jet space of Einstein metrics. These coordinates derive, in part, from Penrose’s notion of an “exact set of fields” [9], [10]. As noted by Penrose, an exact set of fields exists for the Yang-Mills equations, and this leads, via a relatively quick and straightforward analysis which is very similar to that
of [8], to a complete classification of all natural symmetries of the Yang-Mills equations. Thus the power of combining spinor and jet space techniques has a more general scope than merely in gravitation theory.

In §2 we summarize the preliminary results needed for our analysis. The requirement that symmetries be built locally is handled by employing the jet bundle description of Yang-Mills theory, and it is on the jet bundle that the adapted spinor coordinates are defined. Various technical results needed for our analysis are also presented. In §3 we analyze the linearized Yang-Mills equations and classify the natural symmetries. We find that all natural symmetries of the Yang-Mills equations stem from the gauge transformations admitted by the equations. In §4 we comment on the generalizations needed to effect a complete classification of all symmetries of the Yang-Mills equations.

2. Preliminaries.

We choose spacetime to be the manifold \( M = R^4 \) equipped with a flat metric \( \eta_{ab} \) of signature \((-+++)\). The unique torsion-free derivative operator compatible with \( \eta_{ab} \) will be denoted by \( \partial_a \). To define the Yang-Mills field we consider a principal bundle \( \pi: P \to M \) over spacetime with the structure group given by any Lie group \( G \). Because every bundle over \( R^4 \) is trivial, we can globally represent a connection on \( \pi: P \to M \) by a 1-form \( A_a \) on \( M \) taking values in the Lie algebra \( g \) of \( G \). We call this 1-form the Yang-Mills field. The curvature of the connection is represented by a 2-form \( F_{ab} \) on \( M \) taking values in \( g \), which will be called the Yang-Mills field strength. The field strength is given in terms of the Yang-Mills field by

\[
F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b],
\]

(2.1)

where \([\cdot, \cdot]\) is the bracket of \( g \). If \( \tau_\alpha \) is a basis for \( g \), we write

\[
A_a = A_\alpha^a \tau_\alpha \quad \text{and} \quad F_{ab} = F_{\alpha}^{ab} \tau_\alpha.
\]

(2.2)

We then have

\[
F_{\alpha}^{ab} = \partial_a A_\alpha^b - \partial_b A_\alpha^a + \kappa_\gamma^\alpha \gamma^\delta \partial_\gamma A_\delta^a A_\gamma^b,
\]

(2.3)

where \( \kappa_\gamma^\alpha \gamma^\delta \) are the structure constants of the Lie algebra \( g \).

Given a representation \( \rho \) of the group \( G \) we have an associated vector bundle \( \pi: E \to M \). The Yang-Mills field defines a derivative operator \( \nabla_a \) on sections \( s: M \to E \) via

\[
\nabla_a s = \partial_a s + A_a \cdot s,
\]

(2.4)

where we use the raised dot (\( \cdot \)) to indicate the action of the Lie algebra on sections that is defined by \( \rho \). The Yang-Mills field strength measures the failure of this derivative operator to commute; we have the identity

\[
\nabla_\alpha \nabla_\beta s = \frac{1}{2} F_{\alpha\beta} \cdot s.
\]

(2.5)
The Yang-Mills field strength can be viewed as a (2-form-valued) section of the vector bundle defined by the adjoint representation of $G$. We thus have that
\[ \nabla_a F_{bc} = \partial_a F_{bc} + [A_a, F_{bc}], \quad (2.6) \]
and the field strength satisfies the Bianchi identities
\[ \nabla_{[a} F_{bc]} = 0. \quad (2.7) \]

The Yang-Mills field equations are given by
\[ \nabla^a F_{ab} = \partial^a F_{ab} + [A^a, F_{ab}] = 0. \quad (2.8) \]

In terms of the basis $\tau_\alpha$ we have
\[ \nabla^a F_\alpha^{ab} = \partial^a F_\alpha^{ab} + \kappa_\beta^\alpha A^\beta_a F_\gamma^{ab} = 0. \quad (2.9) \]

Let $\pi: \mathcal{Q} \to M$ be the bundle of $g$-valued 1-forms on $M$. A section $A: M \to \mathcal{Q}$ of this bundle is a Yang-Mills field $A_a(x)$. Let $J^k(\mathcal{Q})$ be the bundle of $k^{th}$-order jets of sections of $\mathcal{Q}$ [3], [11]. A point $\sigma$ in $J^k(\mathcal{Q})$ is defined by a spacetime point $x$, the Yang-Mills field at $x$ and all its derivatives to order $k$ at $x$. A section $A: M \to \mathcal{Q}$ lifts to give a section $j^k(A): M \to J^k(\mathcal{Q})$, which is called the $k$-jet of $A$. If we write
\[ A_{a,b_1\ldots b_k}(j^k(A)(x)) = \partial_{b_1} \partial_{b_2} \cdots \partial_{b_k} A_a(x), \quad (2.10) \]
then a point $\sigma \in J^k(\mathcal{Q})$ is given by
\[ \sigma = (x, A_a, A_{a,b_1}, \ldots, A_{a,b_1\ldots b_k}). \quad (2.11) \]

The total derivative $D_c f$ of a function
\[ f = f(x, A_a, A_{a,b_1}, \ldots, A_{a,b_1\ldots b_k}) \quad (2.12) \]
on $J^k(\mathcal{Q})$ is defined by
\[ D_c f = \frac{\partial f}{\partial x^c} + \frac{\partial f}{\partial A_a^\alpha} A_a^{\alpha,c} + \frac{\partial f}{\partial A_{a,b_1}^\alpha} A_{a,b_1}^{\alpha,c} + \cdots + \frac{\partial f}{\partial A_{a,b_1\ldots b_k}^\alpha} A_{a,b_1\ldots b_k}^{\alpha,c}. \quad (2.13) \]
The main property of the total derivative is that it represents on the jet bundle the effect of the derivative operator $\partial_a$ on fields. More precisely, if $f: J^{k-1}(\mathcal{Q}) \to R$ is a smooth function and $A: M \to \mathcal{Q}$ is a Yang-Mills field with $k$-jet $j^k(A): M \to J^k(\mathcal{Q})$, then we have the identity
\[ (D_a f) \circ j^k(A)(x) = \partial_a (f \circ j^{k-1}(A)(x)). \quad (2.14) \]
The field equations (2.8) define a submanifold
\[ \mathcal{R}^2 \hookrightarrow J^2(Q), \]
which we call the equation manifold. The \( k \)th (total) derivative of the field equations defines the prolonged equation manifold
\[ \mathcal{R}^{k+2} \hookrightarrow J^{k+2}(Q). \]
A generalized symmetry for the field equations (2.8) is an infinitesimal map, depending locally on the independent variables, the dependent variables, and the derivatives of the dependent variables to some finite order, which carries solutions to nearby solutions. Geometrically, a generalized symmetry of order \( k \) is a vector field on \( J^k(Q) \) which is tangent to \( \mathcal{R}^k \) and preserves the contact ideal associated to \( J^k(Q) \) [11]. A generalized symmetry of order \( k \) for the Yang-Mills equations can be represented as a map from \( J^k(Q) \) into the bundle of \( g \)-valued 1-forms on \( M \). We denote this map by \( C_a = C_a^\alpha \tau_\alpha \), and we write
\[ C_a = C_a(x, A_a, A_{a,b_1}, \ldots, A_{a,b_1\cdots b_k}). \] (2.15)
We say a generalized symmetry is trivial if it vanishes on the prolonged equation manifold. Two generalized symmetries are deemed equivalent if they differ by a trivial symmetry. Any generalized symmetry of the form (2.15) is equivalent to a generalized symmetry obtained by restricting (2.15) to \( \mathcal{R}^k \), that is, we can assume that \( C_a \) is a map from \( \mathcal{R}^k \) into the bundle of \( g \)-valued 1-forms on \( M \).

The following proposition is easily established from the theory of generalized symmetries [3].

**Proposition 2.1.** The functions
\[ C_a = C_a(x, A_a, A_{a,b_1}, \ldots, A_{a,b_1\cdots b_k}). \]
represent a \( k \)th-order generalized symmetry for the Yang-Mills field equations if and only if
\[ \nabla^b \nabla_b C_a - \nabla^b \nabla_a C_b + [C_b, F_{b,a}^b] = 0 \quad \text{on } \mathcal{R}^{k+2}, \] (2.16)
where
\[ \nabla_b C_a = D_b C_a + [A_b, C_a]. \] (2.17)
Note that the defining equations (2.16) for a generalized symmetry are the linearized field equations.

Familiar examples of symmetries of the Yang-Mills equations stem from the gauge and conformal invariance of these equations. If \( A^a_\alpha(x) \) is a solution to (2.8), and \( \phi: M \to M \) is a conformal isometry of the spacetime \( (M, \eta) \), then \( \phi^* A^a_\alpha(x) \) is also a solution to (2.8). Here we define \( \phi^* A^a_\alpha(x) \) to be the pull-back of \( A^a_\alpha \) in which \( A^a_\alpha(x) \) is viewed as a collection of 1-forms on \( M \). The infinitesimal form of this conformal symmetry leads to the following proposition.
Proposition 2.2. Let $\xi^a(x)$ be a conformal Killing vector field for the spacetime $(M, \eta)$, then

$$C_a = \xi^b(x)F_{ba}$$  \hspace{1cm} (2.18)

is a generalized symmetry of the Yang-Mills equations.

The infinitesimal transformation defined by (2.18) is the “gauge covariant Lie derivative” of $A_a$ along $\xi^b$.

Let $U: M \to P$ be a section of the principal bundle. If $A_a$ is a solution to the Yang-Mills equations (2.8) then

$$A_a^U = U^{-1}A_aU + U^{-1}\partial_aU$$  \hspace{1cm} (2.19)

is also a solution to (2.8). $A_a^U$ is called the gauge transformation of $A$. The infinitesimal form of the gauge transformations leads to the following proposition.

Proposition 2.3. Let $\Lambda(x) = \Lambda^\alpha(x)\tau_\alpha$ be a $g$-valued function on $M$, then

$$C_a = \nabla_a\Lambda = \partial_a\Lambda(x) + [A_a, \Lambda(x)]$$  \hspace{1cm} (2.20)

is a generalized symmetry of the Yang-Mills equations.

The gauge symmetry of Proposition 2.3 can be generalized to the case where $\Lambda$ is constructed locally from the Yang-Mills field and its derivatives to any order.

Proposition 2.4. Let

$$\Lambda = \Lambda(x, A_a, A_{a,b_1}, \ldots, A_{a,b_1\cdots b_{k-1}})$$

be a $g$-valued function on $J^{k-1}(Q)$, then

$$C_a = \nabla_a\Lambda = D_a\Lambda + [A_a, \Lambda]$$  \hspace{1cm} (2.21)

is a $k^{th}$-order generalized symmetry of the Yang-Mills equations.

We will call these symmetries generalized gauge symmetries.

In this paper we will classify natural generalized symmetries. These are generalized symmetries that have a simple behavior under Poincaré and gauge transformations of the Yang-Mills field. More precisely, the gauge transformations and isometries can be lifted (by prolongation [3]) to act on $J^k(Q)$, and, in terms of these lifted actions, we have the following definition of a natural generalized symmetry.
Definition 2.5. Let \( \phi : M \to M \) be an isometry of the spacetime \((M, \eta)\), and \( U : M \to P \) a section of the principal bundle. A natural generalized symmetry is a function
\[
C_a = C_a(x, A_a, A_{a,b_1}, \ldots, A_{a,b_1 \ldots b_k})
\]
satisfying (2.16), and such that for any \( \phi \)
\[
C_a(\phi^{-1}(x), \phi^* A_a, \phi^* A_{a,b_1}, \ldots, \phi^* A_{a,b_1 \ldots b_k}) = \phi^* C_a(x, A_a, A_{a,b_1}, \ldots, A_{a,b_1 \ldots b_k}),
\]
and for any \( U \)
\[
C_a(x, A_a^U, A_{a,b_1}^U, \ldots, A_{a,b_1 \ldots b_k}^U) = U^{-1} C_a(x, A_a, A_{a,b_1}, \ldots, A_{a,b_1 \ldots b_k}).
\]

We remark that, according to this definition, a generalized gauge symmetry can be a natural generalized symmetry, but the conformal symmetry of Proposition 2.2 is not a natural symmetry. We also note that we could have defined a natural symmetry using the full conformal group; by only using the Poincaré subgroup we put fewer restrictions on the allowed symmetries.

To elucidate the structure of a natural generalized symmetry we will construct a set of adapted coordinates for \( J^k(Q) \). To this end, let us define
\[
A_{b_0b_1 \ldots b_k}^\alpha = \partial_{(b_1} \cdots \partial_{b_k)} A_{b_0}^\alpha, \quad k = 0, 1, \ldots
\]
and
\[
Q_{b_0,b_1 \ldots b_k}^\alpha = \nabla_{(b_k} \nabla_{b_{k-1}} \cdots \nabla_{b_2} F_{b_1)}^\alpha, \quad k = 1, 2, \ldots.
\]
Both \( A_{b_0b_1 \ldots b_k}^\alpha \) and \( Q_{b_0,b_1 \ldots b_k}^\alpha \) depend on the Yang-Mills field and its first \( k \) derivatives; we denote these variables by \( A^k \) and \( Q^k \). Each of these variables is algebraically irreducible in the sense that
\[
A_{b_0b_1 \ldots b_k}^\alpha = A_{(b_0b_1 \ldots b_k)}^\alpha,
\]
\[
Q_{b_0,b_1 \ldots b_k}^\alpha = Q_{b_0,(b_1 \ldots b_k)}^\alpha \quad \text{and} \quad Q_{b_0,(b_1 \ldots b_k)}^\alpha = 0.
\]
We have the identity
\[
\partial_{b_1} \cdots \partial_{b_k} A_{b_0}^\alpha = A_{b_0b_1 \ldots b_k}^\alpha + \frac{k}{k+1} Q_{b_0,b_1 \ldots b_k}^\alpha + L_{b_0b_1 \ldots b_k}^\alpha,
\]
where \( L_{b_0b_1 \ldots b_k}^\alpha \) depends on the Yang-Mills field and its derivatives to order \( k - 1 \). From this identity it is straightforward to show that coordinates for \( J^k(Q) \) are given by
\[
(x, A_{b_0}^\alpha, A_{b_0b_1}^\alpha, \ldots, A_{b_0b_1 \ldots b_k}^\alpha, Q_{b_0,b_1 \ldots b_k}^\alpha).
\]
Here we have taken the convenient liberty of using the same symbols \( Q \) and \( A \) to denote the fields on spacetime and functions on jet space. Every generalized symmetry can be expressed as a function of the variables (2.29):
\[
C_a^\alpha = C_a^\alpha(x, A_{b_0}^\alpha, A_{b_0b_1}^\alpha, \ldots, A_{b_0b_1 \ldots b_k}^\alpha, Q_{b_0,b_1 \ldots b_k}^\alpha).
\]

We can now characterize natural generalized symmetries as follows.
Proposition 2.6. Let $C_a$ be a natural generalized symmetry of the Yang-Mills equations of order $k$. Then $C_a$ can be expressed as a function of the variables $Q_{b_0,b_1}^{\alpha}, Q_{b_0,b_1,b_2}^{\alpha}, \ldots, Q_{b_0,b_1\ldots b_k}^{\alpha}$.

Proof: We begin by analyzing the requirement (2.23). Let $C_a^\alpha$ be given as in (2.30). Let $U(t): R \times M \to P$ be a 1-parameter family of gauge transformations such that $U(0)$ is the identity transformation. The derivative

$$\Lambda = \frac{dU}{dt} \bigg|_{t=0}$$

is a $g$-valued function on $M$ defining an infinitesimal gauge transformation. Under an infinitesimal transformation $\Lambda$ associated to $U(t)$ we have that

$$\frac{d}{dt} A_{b_1\ldots b_k}^{\alpha} \bigg|_{t=0} = \Lambda_{b_1\ldots b_k}^{\alpha} + \{\ast\},$$

where $\Lambda_{b_1\ldots b_l}^{a}$, for $l = 0, 1, 2, \ldots, k$ (along with $x$) defines the $k$-jet of $\Lambda$, and \{\ast\} denotes terms involving $A^l$ and $\Lambda_{b_1\ldots b_l}^{a}$, for $l = 0, 1, 2, \ldots, k - 1$. We also have that

$$\frac{d}{dt} Q_{a,b_1\ldots b_k}^{\alpha} \bigg|_{t=0} = \kappa_{\gamma\beta}^{\alpha} \Lambda_{b_1\ldots b_k}^{\beta} Q_{\gamma}^{a}.$$}

We now demand that (2.23) holds for any $U(t)$ and differentiate this equation with respect to $t$ to find

$$\frac{\partial C_a^\alpha}{\partial A_{b_0\ldots b_k}^{\beta}} \Lambda_{b_0\ldots b_k}^{\beta} + R_{a}^{\alpha} = \Lambda_{b_0\ldots b_k}^{\gamma} \kappa_{\beta\gamma}^{\alpha} C_a^{\beta},$$

where $R_{a}^{\alpha}$ is independent of the variables $\Lambda_{b_0\ldots b_k}^{\beta}$. Equation (2.35) must hold for all values of $\Lambda_{b_0\ldots b_k}^{\beta}$ and this implies that

$$\frac{\partial C_a^\alpha}{\partial A_{b_0\ldots b_k}^{\beta}} = 0.$$}

A simple induction argument then establishes that

$$\frac{\partial C_a^\alpha}{\partial A_{b_0\ldots b_l}^{\beta}} = 0$$

for $l = 0, 1, \ldots, k$. Thus we have

$$C_a^\alpha = C_a^\alpha(x, Q_{b_0,b_1}^{\alpha}, Q_{b_0,b_1,b_2}^{\alpha}, \ldots, Q_{b_0,b_1\ldots b_k}^{\alpha}).$$

It remains to be shown that $C_a^\alpha$ is independent of $x$. Let $x^\mu$ be a global inertial coordinate chart on $M$, and let $\xi^\alpha$ be a translational Killing vector field. In the chart $x^\mu$ the components $\xi^\mu$ are any set of constants:

$$\frac{\partial \xi^\mu}{\partial x^\nu} = 0.$$
If we demand that (2.22) be satisfied for all translational isometries we have that
\[
C^\alpha_a (x^\mu - \xi^\mu, Q^\alpha_{b_0,b_1}, \ldots, Q^\alpha_{b_0,b_1b_2}) = C^\alpha_a (x^\mu, Q^\alpha_{b_0,b_1}, \ldots, Q^\alpha_{b_0,b_1b_2}) \tag{2.40}
\]
for any constants $\xi^\mu$, which implies that
\[
C^\alpha_a = C^\alpha_a (Q^\alpha_{b_0,b_1}, \ldots, Q^\alpha_{b_0,b_1b_2}). \tag{2.41}
\]

If $C_a = C_a (Q^\alpha_{b_0,b_1}, \ldots, Q^\alpha_{b_0,b_1b_2})$ is a natural generalized symmetry of the Yang-Mills equations, then it must satisfy the linearized equations (2.16) at each point of $\mathcal{R}^{k+2}$. To classify solutions to the linearized equations we will construct an explicit parametrization of the prolonged equation manifolds.

In the following proposition $[Q^\alpha_{b_0,b_1\ldots b_k}]_{\text{tracefree}}$ denotes the completely trace-free part of the tensor $Q^\alpha_{b_0,b_1\ldots b_k}$ with respect to the metric $\eta_{ab}$.

**Proposition 2.7.** The variables
\[
(x, A^\alpha_{b_0}, \ldots, A^\alpha_{b_1\ldots b_k}, [Q^\alpha_{b_0,b_1}]_{\text{tracefree}}, \ldots, [Q^\alpha_{b_0,b_1\ldots b_k}]_{\text{tracefree}}) \tag{2.42}
\]
form a global coordinate system for $\mathcal{R}^k$.

**Proof:** The prolonged equation manifold $\mathcal{R}^k$ can be defined by $k$-jets which satisfy
\[
\eta^{mn} \nabla_{(b_1} \ldots \nabla_{b_l)} \nabla_m F_{na} = 0 \tag{2.43}
\]
for $l = 0, 1, 2, \ldots, k - 2$. We express these equations in terms of the variables $Q^j$ via the identity
\[
\eta^{mn} \nabla_{(b_1} \ldots \nabla_{b_l)} \nabla_m F_{na} = \left(\frac{l+2}{l+3}\right) \eta^{mn} [Q_{a,mnb_1\ldots b_l} - Q_{n,amb_1\ldots b_l}] + L_{ab_1\ldots b_l}, \tag{2.44}
\]
where $L_{ab_1\ldots b_l} = L_{a(b_1\ldots b_l)}$ depends on $Q^j$ for $j = 1, 2, \ldots, l$. From (2.44) we have that
\[
\eta^{ab_1} L_{ab_1\ldots b_l} = 0 \quad \text{on } \mathcal{R}^l. \tag{2.45}
\]

Let $S^p$ denote the vector space of tensors with the algebraic symmetries (2.27) of $Q^p$. Denote by $S^p_0$ the subspace of totally trace-free tensors. Let $T^p$ be the vector space of tensors with the symmetries of $L_{ab_1\ldots b_p}$ and satisfying the trace condition (2.45). Define a linear map $\Psi^p : S^{p+2} \to T^p$ which takes $W_{a,mnb_1\ldots b_p} \in S^{p+2}$ into $V_{ab_1\ldots b_p} \in T^p$ by the rule
\[
V_{ab_1\ldots b_p} = \left(\frac{l+2}{l+3}\right) \eta^{mn} [W_{a,mnb_1\ldots b_p} - W_{n,amb_1\ldots b_p}]. \tag{2.46}
\]
It is straightforward to show that

$$\text{Ker } \Psi^p = S_0^{p+2},$$

and

$$\text{Im } \Psi^p = T^p.$$ (2.47)

By virtue of (2.47) and (2.48), each point in $\mathcal{R}^k$, $k = 2, 3, \ldots$, can be uniquely determined as follows. Let us begin with $\mathcal{R}^2$. Choose $x$, $A_a$, and $Q_{a,b_1}$, $A_{ab_1}$, and $A_{ab_1b_2}$ arbitrarily. In equation (2.44) with $l = 0$ we have $L_a = 0$, and so, from (2.47) and (2.48), we solve (2.43) by setting all traces of $Q^2$ to zero. $\mathcal{R}^2$ is thus parametrized by

$$(x, A_a, A_{ab_1}, Q_{a,b_1}, A_{ab_1b_2}, [Q_{a,b_1b_2}]_{\text{tracefree}}).$$ (2.49)

Now we consider $\mathcal{R}^3$. We choose the coordinates (2.49) and $A^3$ arbitrarily. In the identity (2.44) for $l = 1$ we have that $L_{ab_1}$ depends on $[Q^1]_{\text{tracefree}}$ only. By virtue of the surjectivity (2.48) of the map $\Psi^1$ we can solve (2.43). By virtue of (2.47) the solution will be uniquely parametrized by $[Q^3]_{\text{tracefree}}$, $A^3$, and the variables (2.49). By iterating this procedure, we can build every solution to (2.43), which is viewed as an equation on $J^k(Q)$, and the solutions will be uniquely parametrized by the variables (2.42).

In principle, the variables (2.42) can be used to analyze the linearized equation (2.16), but the resulting equations are still rather complicated. Considerable simplifications can be obtained by using a spinor representation of the variables $[Q^k]_{\text{tracefree}}$. Hence we now describe a spinorial coordinate system on $\mathcal{R}^k$. We remark that while all of the results presented to this point are essentially independent of the spacetime dimension, our use of spinors will limit the validity of subsequent results to a 4-dimensional spacetime.

We begin with a brief summary of notation; for more details on spinors, see [10]. The spacetime metric and associated derivative operator have the spinor representation

$$\eta_{ab} \longleftrightarrow \epsilon_{AB}\epsilon_{A'B'}$$

and

$$\partial_a \longleftrightarrow \partial_{AA'}.$$ (2.50)

The $\epsilon$ spinors are skew-symmetric and non-degenerate at each point of $M$. The Yang-Mills field and field strength have the spinor representation

$$A_b \longleftrightarrow A_{BB'}$$

and

$$F_{ab} \longleftrightarrow F_{AA'B'B'} = \Phi_{AB}\epsilon_{A'B'} + \overline{\Phi}_{A'B'}\epsilon_{AB}.$$ (2.50)
In (2.50) the $g$-valued spinor fields $\Phi$ and $\overline{\Phi}$ are symmetric,
\[
\Phi_{AB} = \Phi_{(AB)} \quad \text{and} \quad \overline{\Phi}_{A'B'} = \overline{\Phi}_{(A'B')},
\]
and correspond to the self-dual and anti-self-dual part of the field strength.

The Bianchi identities (2.7) take the spinor form
\[
\nabla_{A'} \Phi_{AB} = \nabla_{A} \overline{\Phi}_{A'B'},
\]
while the identities (2.5) become
\[
\nabla_{X'}(\nabla_{X} \Phi)_{AB} = -i \Phi_{AB} \cdot v
\]
and
\[
\nabla_{X}(\nabla_{X'} \Phi)_{AB} = -i \overline{\Phi}_{A'B'} \cdot v.
\]

Given the identities (2.52), the field equations (2.8) are equivalent to
\[
\nabla_{A'} \Phi_{AB} = 0 = \nabla_{A} \overline{\Phi}_{A'B'}.
\]

We now present a spinor representation of $[Q^k]_{\text{tracefree}}$.

**Proposition 2.8.** Let the $g$-valued tensor $Q^k$ be defined as in (2.25), and let $[Q^k]_{\text{tracefree}}$ have the spinor representation
\[
[Q_{b_0, b_1 \ldots b_k}]_{\text{tracefree}} \longleftrightarrow Q_{b_0, b_1 \ldots b_k}^{b'_0, b'_1 \ldots b'_k},
\]
then $[Q^k]_{\text{tracefree}}$ admits the unique spinor decomposition
\[
Q_{b_0, b_1 \ldots b_k}^{b'_0, b'_1 \ldots b'_k} = \epsilon_{B_0}^{b'_0}(B_1 \Phi_{B_0 B_1 \ldots B_k}^{b'_0 b'_1 \ldots b'_k}) + \epsilon_{B_0}(B_1 \overline{\Phi}_{B_2 \ldots B_k}^{b'_0 b'_1 \ldots b'_k}),
\]
where $\Phi_{B_1 \ldots B_{k+1}}^{b'_1 \ldots b'_{k+1}}$ and $\overline{\Phi}_{B_1 \ldots B_{k+1}}^{b'_1 \ldots b'_{k+1}}$ are the totally symmetric spinors
\[
\Phi_{B_1 \ldots B_{k+1}}^{b'_1 \ldots b'_{k+1}} = \nabla_{(B_1} \ldots \nabla_{B_{k+1}} \Phi_{B_k B_{k+1}}^{b'_1 \ldots b'_{k+1})},
\]
\[
\overline{\Phi}_{B_0 \ldots B_{k+1}}^{b'_1 \ldots b'_{k+1}} = \nabla_{(B_0} \ldots \nabla_{B_{k+1}} \overline{\Phi}_{B_k B_{k+1}}^{b'_1 \ldots b'_{k+1})}.
\]

**Proof:** From the first symmetry given in (2.27) and the trace-free requirement on the indices $b_1 \ldots b_k$, it is readily shown that
\[
Q_{B_0, B_1 \ldots B_k}^{b'_0, b'_1 \ldots b'_k} = Q_{B_0, B_1 \ldots B_k}^{B'_0, (B'_1 \ldots B'_k)} = Q_{B_0, (B_1 \ldots B_k)}^{B'_0, b'_1 \ldots b'_k},
\]
The requirement
\[
\eta^{b_0 b_1} Q_{b_0, b_1 \ldots b_k} = 0
\]
and the cyclic symmetry in (2.27) leads to the algebraic form (2.56). In (2.56) the spinors $\Phi^{B_0\cdots B_k}_{B_0\cdots B_k}$ and $\overline{\Phi}^{B_0\cdots B_k}_{B_0\cdots B_k}$ are uniquely defined by

\[ \Phi^{B_0\cdots B_k}_{B_0\cdots B_k} = \frac{k}{k+1} \epsilon^{B_0B_1}_{B_0B_1} Q^{B_0B_1}_{B_0B_1} \]  
(2.61)

and

\[ \overline{\Phi}^{B_0\cdots B_k}_{B_0\cdots B_k} = \frac{k}{k+1} \epsilon^{B_0B_1}_{B_0B_1} Q^{B_0B_1}_{B_0B_1} \]  
(2.62)

We use the decomposition (2.50) in the spinor representation of (2.25), then, using (2.56), (2.61), and (2.62), we can solve for $\Phi^{B_0\cdots B_k}_{B_0\cdots B_k}$ and $\overline{\Phi}^{B_0\cdots B_k}_{B_0\cdots B_k}$ to find (2.57) and (2.58).

From Propositions 2.7 and 2.8 we can now define a spinorial coordinate system on $\mathcal{R}^k$.

**Proposition 2.9.** The variables

\[ (x, A_{b_0}, A_{b_0b_1}, \ldots, A_{b_0\cdots b_k}, \Phi_{B_0B_1}, \overline{\Phi}^{B_0B_1}, \ldots, \Phi^{B_0\cdots B_k}_{B_0\cdots B_k}, \overline{\Phi}^{B_0\cdots B_k}_{B_0\cdots B_k}) \]  
(2.63)

define a global coordinate chart on $\mathcal{R}^k$.

We remark that to pass between the coordinates (2.42) and (2.63) we use any soldering form $\sigma^{AA'}_{a}$ such that

\[ \eta_{ab} = \sigma^{AA'}_{a} \sigma_{bAA'} \]  
(2.64)

The spinor variables $\Phi^{B_0\cdots B_k}_{B_0\cdots B_k}$ and $\overline{\Phi}^{B_0\cdots B_k}_{B_0\cdots B_k}$ will play a fundamental role in our symmetry analysis. Their role as coordinates for $\mathcal{R}^k$ stems from the fact that $\Phi^{AB}_{AB}$ and $\overline{\Phi}^{A'B'}_{A'B'}$ form what Penrose calls an “exact set of fields” for the Yang-Mills equations [10]. Henceforth we will call the fields (2.57) and (2.58) the *Penrose fields* and denote them by $\Phi^k$ and $\overline{\Phi}^k$.

By virtue of the identities (2.52), (2.53), (2.54), and equations (2.55), the Penrose fields satisfy the following *structure equation* on the prolonged equation manifolds. See [10] for details.

**Proposition 2.10.** The spinorial covariant derivative of $\Phi^{B_0\cdots B_k}_{B_0\cdots B_k}$, when evaluated on $\mathcal{R}^k$, is given by

\[ \nabla_A^{A'} \Phi^{B_0\cdots B_k}_{B_0\cdots B_k} = \Phi^{A'B_0\cdots B_k}_{A'B_0\cdots B_k} + \{\star\} \]  
(2.65)

where $\{\star\}$ denotes a spinor (and $g$) -valued function of the Penrose fields $\Phi^1$, $\overline{\Phi}^1$, $\ldots$, $\Phi^{k-1}$, $\overline{\Phi}^{k-1}$.

An analogous result holds for the complex conjugate Penrose fields $\overline{\Phi}^k$. Proposition 2.10 is central to our generalized symmetry analysis.

From Propositions 2.6–2.9 we now have the following restriction on the domain of natural generalized symmetries.
Proposition 2.11. The spinor components,

\[ C_a \leftrightarrow C_{AA'}, \]

of a natural generalized symmetry of order \( k \) are functions of the Penrose fields to order \( k \)

\[ C_{AA'} = C_{AA'}(\Phi_{B_0B_1}, \overline{\Phi}_{B_0'B_1'}, \ldots, \Phi_{B_0\ldots B_k}, \overline{\Phi}_{B_0'\ldots B_k'}). \]  

(2.66)

Let us note that the requirements (2.22) and (2.23) must still be satisfied by the generalized symmetry (2.66). In particular, the Lorentz invariance requirement implies that the spinor form of the generalized symmetry must be \( SL(2, \mathbb{C}) \) covariant. More precisely, if \( L_B^A \) is an element of \( SL(2, \mathbb{C}) \), then

\[ C_{BB'}(L \cdot \Phi, \overline{L} \cdot \overline{\Phi}) = L_B^A L_B'^{A'} C_{AA'}(\Phi, \overline{\Phi}), \]  

(2.67)

where \( L \cdot \Phi \) and \( \overline{L} \cdot \overline{\Phi} \) denote the action of \( SL(2, \mathbb{C}) \) on the Penrose fields, e.g.,

\[ [L \cdot \Phi]_{AB} = L_C^B L_D^C \Phi_{CD} \quad \text{and} \quad [\overline{L} \cdot \overline{\Phi}]_{A'B'} = \overline{L}_{A'}^{C'} \overline{L}_{B'}^{D'} \overline{\Phi}_{C'D'}. \]  

(2.68)

To take advantage of Proposition 2.11, we will use the following spinor form of the linearized equations (2.16).

Proposition 2.12. The spinor \((g)\) -valued functions on \( \mathcal{R}^k \)

\[ C_{AA'} = C_{AA'}(\Phi_{B_0B_1}, \overline{\Phi}_{B_0'B_1'}, \ldots, \Phi_{B_0\ldots B_k}, \overline{\Phi}_{B_0'\ldots B_k'}). \]

define a \( k^{th} \)-order generalized symmetry of the Yang-Mills equations if and only if

\[ \nabla_{BB'} \nabla_{BB'} C_{AA'} - \nabla_{BB'} \nabla_{AA'} C_{BB'} + [\Phi_B^B, C_{BA'}] + [\overline{\Phi}_{B'}^{B'}, C_{AB'}] = 0 \quad \text{on} \quad \mathcal{R}^{k+2}. \]  

(2.69)

Let us point out that in equation (2.69) the covariant derivatives are defined using total derivatives as in (2.17). In this regard it is worth noting that the gauge invariance requirement (2.23) implies that

\[ \nabla_{BB'} C_{AA'} = \frac{\partial C_{AA'}}{\partial \Phi_{C_0C_1}} \nabla_{BB'} \Phi_{C_0C_1} + \frac{\partial C_{AA'}}{\partial \overline{\Phi}_{C_0'C_1'}} \nabla_{BB'} \overline{\Phi}_{C_0'C_1'} \]

\[ + \cdots + \frac{\partial C_{AA'}}{\partial \Phi_{C_0C_1\ldots C_k}} \nabla_{BB'} \Phi_{C_0C_1\ldots C_k} + \frac{\partial C_{AA'}}{\partial \overline{\Phi}_{C_0'C_1'\ldots C_k'}} \nabla_{BB'} \overline{\Phi}_{C_0'C_1'\ldots C_k'}. \]  

(2.70)

Our analysis of the linearized equation (2.69) will involve its differentiation with respect to the Penrose fields. Thus we need an efficient way to deal with symmetric spinors of
arbitrary rank. This will be done by viewing spinors as multi-linear maps on complex two-dimensional vector spaces. If $T_{A_1...A_p}^{A'_1...A'_q}$ is a spinor of type $(p, q)$ we write

$$T(\alpha_1, \alpha_2, \ldots, \alpha_p, \alpha_1', \alpha_2', \ldots, \alpha_q') = T_{A_1...A_p}^{A'_1...A'_q} \cdot \alpha_1^{A_1} \cdot \alpha_2^{A_2} \cdots \alpha_p^{A_p} \cdot \alpha_1'^{A_1'} \cdot \alpha_2'^{A_2'} \cdots \alpha_q'^{A_q'}.$$  \hspace{1cm} (2.71)

If the spinor $S_{ABC}$ is symmetric in its first two indices, we write

$$S(\alpha, \beta, \gamma) = S_{ABC} \cdot \alpha^A \cdot \beta^B \cdot \gamma^C = S(\beta, \gamma),$$

where we have dropped the comma between symmetric arguments of $S$. Note that in this case $S$ is completely determined by the values of

$$S(\alpha^2, \beta) := S(\alpha, \beta),$$  \hspace{1cm} (2.73)

for all $\alpha$ and $\beta$. Here we have introduced an exponential notation for repeated symmetric arguments. More generally, if $V_{A_1...A_kB}$ is symmetric in its first $k$ indices, we will write

$$V(\alpha^k, \beta) = V_{A_1...A_kB} \cdot \alpha^{A_1} \cdot \beta^B.$$  \hspace{1cm} (2.74)

We extend our multi-linear map notation to $g$-valued spinors as follows. If $T = T^\alpha_{\tau\alpha}$ takes values in the Lie algebra $g$, we will write

$$T(v) = T^\alpha v_\alpha,$$  \hspace{1cm} (2.75)

where $v_\alpha$ are the components of an element $v$ of the dual vector space $g^*$ to $g$. If $S$ takes values in $g^*$ we will write

$$S(w) = S_\alpha w^\alpha,$$  \hspace{1cm} (2.76)

where $w^\alpha$ are the components of $w \in g$.

The anti-symmetric pairing of spinors defined by the $\epsilon$ spinors is denoted by

$$<\alpha, \beta> = \epsilon_{AB} \cdot \alpha^A \cdot \beta^B = \alpha_A \cdot \beta^A \quad \text{and} \quad <\bar{\alpha}, \bar{\beta}> = \bar{\epsilon}_{A'B'} \cdot \bar{\alpha}^{A'} \cdot \bar{\beta}^{B'} = \bar{\alpha}_{A'} \cdot \bar{\beta}^{A'}.$$ \hspace{1cm} (2.77)

Next we develop a notation for derivatives of functions on $J^k(Q)$ (or $R^k$) with respect to the Penrose fields. If

$$T_{c_1...c_p}^{c_1'...c_q'} = T_{c_1...c_p}^{c_1'...c_q'}(\Phi^1, \overline{\Phi}^1, \ldots, \Phi^k, \overline{\Phi}^k)$$

is a natural spinor of type $(p, q)$ and order $k$, then the partial derivative of $T_{c_1...c_p}^{c_1'...c_q'}$ with respect to $\Phi^l$ is a natural spinor of type $(p + l + 1, q + l - 1)$. We shall write

$$[\partial^l_{\Phi} \cdot T_{c_1...c_p}^{c_1'...c_q'}(\psi^1, \ldots, \psi^{l+1}, \overline{\psi}_1, \ldots, \overline{\psi}_{l-1}, v)] = \left. \frac{\partial T_{c_1...c_p}^{c_1'...c_q'}}{\partial \Phi^{A_1...A_{l+1}}} \right|_{\Phi = \Phi^l} \cdot v^{A_1} \cdot \psi^1 \cdot \ldots \cdot \psi^{l+1} \cdot \overline{\psi}_1 \cdot \ldots \cdot \overline{\psi}_{l-1}.$$  \hspace{1cm} (2.79)
Further, let \( \phi^1, \ldots, \phi^p \) and \( \bar{\phi}_1, \ldots, \bar{\phi}_q \) be arbitrary spinors of type \((1, 0)\) and \((0, 1)\) respectively; we shall write
\[
[\partial^l T](\psi_l^{l+1}, \bar{\psi}_l^{-1}, v; \phi^1, \ldots, \phi^p, \bar{\phi}_1, \ldots, \bar{\phi}_q) = [\partial^l T_{c_1 \cdots c_p}^{c'_1 \cdots c'_q}](\psi_l^{l+1}, \bar{\psi}_l^{-1}, v) \phi_{c_1}^1 \cdots \phi_{c_p}^p \bar{\phi}_{c'_1}^1 \cdots \bar{\phi}_{c'_q}^q.
\]
A semi-colon will always be used to separate arguments corresponding to derivatives with respect to the coordinates \( \Phi^k \). Partial derivatives, \( \partial^l \Phi \), with respect to \( \Phi_{A_1 \ldots A_l} \) will be similarly denoted.

We shall repeatedly need certain commutation relations between the partial derivative operators \( \partial \Phi \) and the covariant derivative operator \( \nabla_{c'} \).

**Lemma 2.13.** Let \( T^{c_1 \cdots c_p}_{c'_1 \cdots c'_q} = T^{c_1 \cdots c_p}_{c'_1 \cdots c'_q}(\Phi^1, \bar{\Phi}^1, \ldots, \Phi^m, \bar{\Phi}^m) \) be a natural spinor of order \( m \). Then, on \( \mathcal{R}^{m+1} \),
\[
[\partial^m \nabla_{c'} T^{c_1 \cdots c_p}_{c'_1 \cdots c'_q}](\psi^{m+2}, \bar{\psi}^m, v) = \psi^{c'} \bar{\psi}^c [\partial^m T^{c_1 \cdots c_p}_{c'_1 \cdots c'_q}](\psi^{m+1}, \bar{\psi}^m, v),
\]
and
\[
[\partial^m \nabla_{c'} T^{c_1 \cdots c_p}_{c'_1 \cdots c'_q}](\psi^m, \bar{\psi}^{m-1}, v) = [\nabla_{c'} \partial^m T^{c_1 \cdots c_p}_{c'_1 \cdots c'_q}](\psi^m, \bar{\psi}^{m-1}, v) + \psi^{c'} \bar{\psi}^c [\partial^m T^{c_1 \cdots c_p}_{c'_1 \cdots c'_q}](\psi^m, \bar{\psi}^{m-2}, v),
\]
and similarly,
\[
[\partial^m \nabla_{c'} T^{c_1 \cdots c_p}_{c'_1 \cdots c'_q}](\psi^m, \bar{\psi}^{m+2}, v) = \psi^{c'} \bar{\psi}^c [\partial^m T^{c_1 \cdots c_p}_{c'_1 \cdots c'_q}](\psi^{m-1}, \bar{\psi}^{m+1}, v),
\]
and
\[
[\partial^m \nabla_{c'} T^{c_1 \cdots c_p}_{c'_1 \cdots c'_q}](\psi^{m-1}, \bar{\psi}^{m+1}, v) = [\nabla_{c'} \partial^m T^{c_1 \cdots c_p}_{c'_1 \cdots c'_q}](\psi^{m-1}, \bar{\psi}^{m+1}, v) + \psi^{c'} \bar{\psi}^c [\partial^m T^{c_1 \cdots c_p}_{c'_1 \cdots c'_q}](\psi^{m-2}, \bar{\psi}^m, v).
\]

**Proof:** These formulas follow directly from equation (2.70) and the structure equations (2.65).

We conclude this section by presenting a couple of elementary results from spinor algebra which we shall use in our symmetry analysis. See [12] and/or [10] for proofs.

**Lemma 2.14.** Let \( P(\psi^k, \alpha) \) be a rank \((k + 1)\) spinor that is symmetric in its first \( k \) arguments. Then there are unique, totally symmetric spinors \( P^* \) and \( Q \), of rank \( k + 1 \) and \( k - 1 \) respectively, such that
\[
P(\psi^k, \alpha) = P^*(\psi^k \alpha) + <\psi, \alpha> Q(\psi^{k-1}).
\]
If $P$ is a natural spinor of the Penrose fields $\Phi^1, \overline{\Phi}^1, \ldots, \Phi^k, \overline{\Phi}^k$, then so are $P^*$ and $Q$.

**Lemma 2.15.** Let $P(\psi^k, \alpha)$ be a rank $(k + 1)$ spinor that is symmetric in its first $k$ arguments. If $P(\psi^k, \alpha)$ satisfies

$$P(\psi^k, \psi) = 0,$$

then there is a totally symmetric spinor $Q = Q(\psi^{k-1})$ such that

$$P(\psi^k, \alpha) = <\psi, \alpha > Q(\psi^{k-1}).$$

If $P$ is a natural spinor, then so is $Q$.

### 3. Symmetry Analysis.

We suppose that $C_{AA'} = C_{AA'}(\Phi^1, \overline{\Phi}^1, \ldots, \Phi^k, \overline{\Phi}^k)$ is a natural generalized symmetry of order $k$. Keeping with our multilinear map notation we write

$$C(\psi, \overline{\psi}, v) = C^{\alpha}_{AA'}\psi^A\overline{\psi}^{A'}v_{\alpha}.$$  \hspace{1cm} (3.2)

On $\mathcal{R}^{k+2}$ the linearized equation (2.69) is a gauge and $SL(2, \mathbb{C})$ invariant identity in the Penrose fields $\Phi^l$ and $\overline{\Phi}^l$ for $l = 1, \ldots, k + 2$. Our analysis consists of differentiating this identity with respect to the Penrose fields $\Phi^l$ and $\overline{\Phi}^l$ for $l = k, k + 1, k + 2$; we present the results in the following series of propositions. All equations in this section hold on $\mathcal{R}^{k+2}$, i.e., modulo the field equations.

**Proposition 3.1.** Let $C_{AA'} = C_{AA'}(\Phi^1, \overline{\Phi}^1, \ldots, \Phi^k, \overline{\Phi}^k)$ be a $k$th-order natural generalized symmetry of the Yang-Mills equations. Then there exist natural spinors $G(\psi^k, \overline{\psi}^{k-2}, v, w), H(\psi^{k-2}, \overline{\psi}^k, v, w), A(\psi^k, \overline{\psi}^k, v, w), B(\psi^{k+2}, \overline{\psi}^{k-2}, v, w), D(\psi^{k-2}, \overline{\psi}^{k+2}, v, w), E(\psi^k, \overline{\psi}^k, v, w)$ such that

$$[\partial^k_\phi C](\psi^{k+1}, \overline{\psi}^{k-1}, v; \alpha, \overline{\alpha}, w) = <\alpha, \psi > <\overline{\alpha}, \overline{\psi} > G(\psi^k, \overline{\psi}^{k-2}, v, w)$$

$$+ <\alpha, \psi > A(\psi^k, \overline{\psi}^{k-1}, \alpha, v, w)$$

$$+ <\overline{\alpha}, \overline{\psi} > B(\psi^{k+1}, \overline{\psi}^{-k-2}, v, w),$$

and

$$[\partial^k_\phi C](\psi^{k-1}, \overline{\psi}^{k+1}, v; \alpha, \overline{\alpha}, w) = <\alpha, \psi > <\overline{\alpha}, \overline{\psi} > H(\psi^{k-2}, \overline{\psi}^k, v, w)$$

$$+ <\alpha, \psi > D(\psi^{k-2}, \overline{\psi}^{k+1}, \alpha, v, w)$$

$$+ <\overline{\alpha}, \overline{\psi} > E(\psi^{k-1}, \overline{\psi}^k, v, w).$$
With the symmetries as indicated, the spinors $A, B, D, E, G,$ and $H$ are uniquely determined by $\partial^k_{\Phi} C$ and $\bar{\partial}^k_{\Phi} C$. When $k = 1$, equations (3.3) and (3.4) hold with $B = 0, D = 0, G = 0,$ and $H = 0$.

**Proof:** This proposition follows from an analysis of the dependence of (2.69) on $\Phi^{k+2}$ and $\bar{\Phi}^{k+2}$. To this end we use the commutation relations (2.81) to find

$$
[\partial^k_{\Phi} \nabla_{MM'} \nabla_{NN'} C^{\alpha}_{AA'}](\psi^{k+3}, \psi^{k+1}, v) = \psi_M \psi_N \bar{\psi}_{M'} \bar{\psi}_{N'} [\partial^k_{\Phi} C^{\alpha}_{AA'}](\psi^{k+1}, \bar{\psi}^{k-1}, v). \quad (3.5)
$$

We use this result to compute the derivative of (2.69) with respect to $\Phi^{k+2}$, and this implies that

$$
[\partial^k_{\Phi} C](\psi^{k+1}, \bar{\psi}^{k-1}, v; \psi, \bar{\psi}, w) = 0. \quad (3.6)
$$

Similarly, the derivative of (2.69) with respect to $\bar{\Phi}^{k+2}$ implies that

$$
[\partial^k_{\bar{\Phi}} C](\psi^{k-1}, \bar{\psi}^{k+1}, v; \psi, \bar{\psi}, w) = 0. \quad (3.7)
$$

We use Lemmas 2.14 and 2.15 to decompose $[\partial^k_{\Phi} C]$ and $[\partial^k_{\bar{\Phi}} C]$ into irreducible components. We then use (3.6) and (3.7) and arrive at (3.3) and (3.4). Uniqueness of the decompositions (3.3) and (3.4) is easily established.

**Proposition 3.2.** If $C_{AA'} = C_{AA'}(\Phi^1, \bar{\Phi}^1, \ldots, \Phi^k, \bar{\Phi}^k)$ is a natural generalized symmetry of the Yang-Mills equations, then $C_{AA'}$ is linear in the top-order Penrose fields $\Phi^k$ and $\bar{\Phi}^k$.

**Proof:** This result follows from the quadratic dependence of (2.69) on the Penrose fields $\Phi^{k+1}$ and $\bar{\Phi}^{k+1}$. From Lemma 2.13 we deduce that

$$
[\partial^k_{\Phi} \partial^{k+1}_{\Phi} \nabla_{MM'} \nabla_{NN'} C^{\alpha}_{AA'}](\chi^{k+2}, \bar{\chi}^{k}, u; \psi^{k+2}, \bar{\psi}^{k}, v) = (\psi_M \chi_N \bar{\psi}_{M'} \bar{\chi}_{N'} + \psi_N \chi_M \bar{\psi}_{N'} \bar{\chi}_{M'}) [\partial^k_{\Phi} \partial^k_{\Phi} C^{\alpha}_{AA'}](\chi^{k+1}, \bar{\chi}^{k-1}, u; \psi^{k+1}, \bar{\psi}^{k-1}, v). \quad (3.8)
$$

We now differentiate (2.69) twice with respect to $\Phi^{k+1}$ and use (3.8) to find

$$
2 \langle \psi, \chi \rangle \langle \bar{\psi}, \bar{\chi} \rangle [\partial^k_{\Phi} \partial^k_{\Phi} C](\chi^{k+1}, \bar{\chi}^{k-1}, u; \psi^{k+1}, \bar{\psi}^{k-1}, v; \alpha, \bar{\alpha}, w) \nonumber
- \langle \psi, \alpha \rangle \langle \bar{\psi}, \bar{\alpha} \rangle \langle \bar{\psi}, \bar{\chi}, \alpha \rangle [\partial^k_{\Phi} \partial^k_{\Phi} C](\chi^{k+1}, \bar{\chi}^{k-1}, u; \psi^{k+1}, \bar{\psi}^{k-1}, v; \chi, \bar{\chi}, w) \quad (3.9)
- \langle \chi, \alpha \rangle \langle \chi, \bar{\alpha} \rangle \langle \bar{\psi}, \bar{\psi}, \bar{\alpha} \rangle [\partial^k_{\Phi} \partial^k_{\Phi} C](\psi^{k+1}, \bar{\psi}^{k-1}, v; \chi^{k+1}, \bar{\chi}^{k-1}, u; \psi, \bar{\psi}, w) = 0.
$$

The last two terms of this equation vanish by virtue of equation (3.6) and we then have

$$
[\partial^k_{\Phi} \partial^k_{\Phi} C](\chi^{k+1}, \bar{\chi}^{k-1}, u; \psi^{k+1}, \bar{\psi}^{k-1}, v; \alpha, \bar{\alpha}, w) = 0. \quad (3.10)
$$

Similar computations, which involve applying $\partial^{k+1}_{\Phi} \partial^{k+1}_{\Phi}$ and $\partial^{k+1}_{\Phi} \partial^{k+1}_{\Phi}$ to the linearized equations, lead to

$$
[\partial^k_{\Phi} \partial^k_{\Phi} C](\chi^{k+1}, \bar{\chi}^{k-1}, u; \psi^{k-1}, \bar{\psi}^{k+1}, v; \alpha, \bar{\alpha}, w) = 0. \quad (3.11)
$$
and
\[ [\partial^k_\Phi \partial^{k+1}_\Phi C](\chi^{k-1}, \chi^{k+1}, u; \psi^{k-1}, \psi^{k+1}, v; \alpha, \bar{\alpha}, w) = 0. \] (3.12)

**Proposition 3.3.** The natural spinors $A, B, D, E$ in the decompositions (3.3) and (3.4) depend on the Penrose fields $\Phi^l$ and $\overline{\Phi}^l$ for $l \leq k - 2$.

**Proof:** Using Proposition 3.2 it is straightforward to show from the commutation relations (2.81) and (2.82) that
\[
[\partial^k_\Phi \partial^{k+1}_\Phi \nabla_{MM'} \nabla_{NN'} C_{AA'}^\alpha](\chi^{k+1}, \chi^{k-1}, u; \psi^{k+2}, \psi^k, v)
= \psi_M \psi_{N'} \psi_{M'} \psi_{N'} [\partial^k_\Phi \partial^{k+1}_\Phi C_{AA'}^\alpha](\psi^k, \psi^{-2}, v; \chi^{k+1}, \chi^{k-1}, u)
+ (\psi_M \psi_{N'} \psi_{M'} \psi_{N'}) [\partial^k_\Phi \partial^{k+1}_\Phi C_{AA'}^\alpha](\chi^k, \chi^{-2}, u; \psi^{k+1}, \psi^{-1}, v).
\]
(3.13)

We now differentiate the linearized equations (2.69) with respect to $\Phi^k$ and $\Phi^{k+1}$ and use (3.13) and (3.6) to find
\[
2 < \psi, \chi > < \bar{\psi}, \bar{\chi} > [\partial^k_\Phi \partial^{k+1}_\Phi C](\chi^k, \chi^{-2}, u; \psi^{k+1}, \psi^{k-1}, v; \alpha, \bar{\alpha}, w)
- < \psi, \alpha > < \bar{\psi}, \bar{\alpha} > [\partial^k_\Phi \partial^{k+1}_\Phi C](\psi^k, \psi^{-2}, v; \chi^{k+1}, \chi^{k-1}, u; \psi, \bar{\psi}, w)
- < \psi, \alpha > < \bar{\psi}, \bar{\alpha} > [\partial^k_\Phi \partial^{k+1}_\Phi C](\chi^k, \chi^{-2}, u; \psi^{k+1}, \psi^{k-1}, v; \chi, \bar{\chi}, w) = 0.
\]
(3.14)

We set $\alpha = \psi$ and find
\[
[\partial^k_\Phi \partial^{k+1}_\Phi C](\chi^k, \chi^{-2}, u; \psi^{k+1}, \psi^{k-1}, v; \psi, \bar{\psi}, w) = 0.
\]
(3.15)

In terms of the decomposition (3.3), this equation implies that
\[
[\partial^k_\Phi B](\chi^k, \chi^{-2}, u; \psi^{k+2}, \psi^{-2}, v, w) = 0,
\]
(3.16)
i.e., $B$ is independent of the Penrose fields $\Phi^{k-1}$. In a similar fashion, setting $\overline{\alpha} = \overline{\psi}$ leads to
\[
[\partial^k_\Phi A](\chi^k, \chi^{-2}, u; \psi^k, \psi^{-k}, v, w) = 0.
\]
(3.17)

Analogous computations, which involve applying the derivatives $\partial^k_\Phi \partial^{k+1}_\Phi$, $\partial^k_\Phi \partial^{k+1}_\Phi$, and $\partial^k_\Phi \partial^{k+1}_\Phi$ to (2.69), yield
\[
[\partial^k_\Phi D](\chi^k, \chi^{-2}, u; \psi^{k-2}, \psi^{k+2}, v, w) = 0,
[\partial^k_\Phi E](\chi^k, \chi^{-2}, u; \psi^k, \psi^{-k}, v, w) = 0,
\]
(3.18)
and

\[
[\partial_{\Phi}^{k-1} A](\chi^{k-2}, \chi^k, u; \psi^k, \bar{\psi}^k, v, w) = 0,
\]

\[
[\partial_{\Phi}^{k-1} B](\chi^{k-2}, \chi^k, u; \psi^{k+2}, \bar{\psi}^{k-2}, v, w) = 0,
\]

\[
[\partial_{\Phi}^{k-1} C](\chi^{k-2}, \chi^k, u; \psi^{k-2}, \bar{\psi}^{k+2}, v, w) = 0,
\]

\[
[\partial_{\Phi}^{k-1} D](\chi^{k-2}, \chi^k, u; \psi^k, \bar{\psi}^k, v, w) = 0.
\]

(3.19)

**Proposition 3.4.** Let \( C_{A'} \) be a natural generalized symmetry of order \( k > 1 \). Then there is a natural \( g \)-valued function of order \( k - 1 \),

\[
\Lambda = \Lambda(\Phi^1, \overline{\Phi}^1, \ldots, \Phi^{k-1}, \overline{\Phi}^{k-1}),
\]

such that, in the decompositions (3.3) and (3.4) for \( C_{A'} \), \( G \) and \( H \) are the gradients

\[
G(\psi^k, \bar{\psi}^{k-2}, v, w) = [\partial_{\Phi}^{k-1} \Lambda](\psi^k, \bar{\psi}^{k-2}, v; w)
\]

\[
H(\psi^{k-2}, \bar{\psi}^k, v, w) = [\partial_{\Phi}^{k-1} \Lambda](\psi^{k-2}, \bar{\psi}^k, v; w).
\]

(3.20)

**Proof:** We begin by deriving the integrability conditions for (3.20) from the linearized equations (2.69). We return to equation (3.14), which, on account of Propositions 3.1 and 3.3, reduces to

\[
[\partial_{\Phi}^{k-1} G](\chi^k, \bar{\chi}^{k-2}, u; \psi^k, \bar{\psi}^{k-2}, v, w) = [\partial_{\Phi}^{k-1} G](\psi^k, \bar{\psi}^{k-2}, v; \chi^k, \bar{\chi}^{k-2}, u, w).
\]

(3.21)

This is one of the integrability conditions needed to establish (3.20). The remaining integrability conditions,

\[
[\partial_{\Phi}^{k-1} H](\chi^{k-2}, \bar{\chi}^k, u; \psi^{k-2}, \bar{\psi}^k, v, w) = [\partial_{\Phi}^{k-1} H](\psi^{k-2}, \bar{\psi}^k, v; \chi^{k-2}, \bar{\chi}^k, u, w),
\]

(3.22)

and

\[
[\partial_{\Phi}^{k-1} G](\chi^{k-2}, \bar{\chi}^k, u; \psi^k, \bar{\psi}^{k-2}, v, w) = [\partial_{\Phi}^{k-1} H](\psi^k, \bar{\psi}^{k-2}, v; \chi^{k-2}, \bar{\chi}^k, u, w)
\]

(3.23)

are obtained in an analogous manner from the equations resulting from applying \( \partial_{\Phi}^k \partial_{\Phi}^{k+1} \), \( \partial_{\Phi}^k \partial_{\Phi}^{k+1} \), and \( \partial_{\Phi}^k \partial_{\Phi}^{k+1} \) to (2.69).

From these integrability conditions it is straightforward to verify that \( \Lambda \) can be expressed as a natural function of order \( k - 1 \) via

\[
\Lambda^\alpha = \int_0^1 dt \Phi_{B\beta_1 \cdots \beta_k}^{\beta A_1 \cdots A_k} G_{\alpha A_1 \cdots A_{k-2}}^{\beta A_1 \cdots A_k} (\Phi^1, \overline{\Phi}^1, \ldots, t\Phi^{k-1}, t\overline{\Phi}^{k-1})
\]

(3.24)

\[+ \int_0^1 dt \Phi_{A_1 \cdots A_k}^{\beta A_1 \cdots A_{k-2}} H_{\alpha A_1 \cdots A_k}^{\beta A_1 \cdots A_{k-2}} (\Phi^1, \overline{\Phi}^1, \ldots, t\Phi^{k-1}, t\overline{\Phi}^{k-1}).\]
Proposition 3.5. Let $C_{AA'}$ be a natural generalized symmetry of order $k$. Then there is a natural $g$-valued function $\Lambda = \Lambda(\Phi^1, \bar{\Phi}^1, \ldots, \Phi^{k-1}, \bar{\Phi}^{k-1})$ and a natural generalized symmetry $\hat{C}_{AA'}$ of order $k - 1$ such that

$$C_{AA'} = \hat{C}_{AA'} + \nabla_{AA'} \Lambda.$$  \hfill (3.25)

Proof: We choose $\Lambda$ as in Proposition 3.4 and define

$$\hat{C}_{AA'} = C_{AA'} - \nabla_{AA'} \Lambda.$$  \hfill (3.26)

By Proposition 2.4 and linearity of the equations (2.69), $\hat{C}_{AA'}$ is a generalized symmetry; by construction, $\partial^k_{\Phi} \hat{C}_{AA'}$ has the decomposition

$$[\partial^k_{\Phi} \hat{C}](\psi^{k+1}, \bar{\psi}^{k-1}, v; \alpha, \bar{\alpha}, w) = <\alpha, \psi > A(\psi^k, \bar{\psi}^{k-1} \alpha, v, w)$$

$$+ <\bar{\alpha}, \bar{\psi} > B(\psi^{k+1} \alpha, \bar{\psi}^{k-2}, v, w),$$  \hfill (3.27)

and $\partial^k_{\bar{\Phi}} \hat{C}_{AA'}$ has the decomposition

$$[\partial^k_{\bar{\Phi}} \hat{C}](\psi^{k-1}, \bar{\psi}^{k+1}, v; \alpha, \bar{\alpha}, w) = <\alpha, \psi > D(\psi^{k-2}, \bar{\psi}^{k+1} \alpha, v, w)$$

$$+ <\bar{\alpha}, \bar{\psi} > E(\psi^{k-1} \alpha, \bar{\psi}^k, v, w).$$  \hfill (3.28)

We now show that the linearized equations (2.69) force $A$, $B$, $D$, and $E$ to vanish, thus establishing (3.25). To this end, we consider the derivative of the linearized equations (2.69) with respect to $\Phi^{k+1}$. We use the commutation relation (2.82) and equation (3.6) to find that

$$2 \psi^A \bar{\psi}^{A'} \nabla_{AA'}[\partial^k_{\Phi} \hat{C}](\psi^{k+1}, \bar{\psi}^{k-1}, v; \alpha, \bar{\alpha}, w)$$

$$- <\alpha, \psi > <\bar{\alpha}, \bar{\psi} > w_{\alpha} \nabla_{AA'}[\partial^k_{\Phi} \hat{C}_{AA'}](\psi^k, \bar{\psi}^{k-1}, v)$$

$$- <\alpha, \psi > <\bar{\alpha}, \bar{\psi} > [\partial^k_{\Phi} \hat{C}](\psi^k, \bar{\psi}^{k-2}, v; \psi, \bar{\psi}, w) = 0.$$  \hfill (3.29)

In this equation we set $\alpha = \psi$ and substitute from (3.3) to obtain

$$\psi^A \bar{\psi}^{A'} [\nabla_{AA'} B](\psi^{k+2}, \bar{\psi}^{k-2}, v, w) = 0.$$  \hfill (3.30)

Similarly, setting $\bar{\alpha} = \bar{\psi}$ we obtain

$$\psi^A \bar{\psi}^{A'} [\nabla_{AA'} A](\psi^k, \bar{\psi}^k, v, w) = 0.$$  \hfill (3.31)

These equations imply that $A$ and $B$ are independent of the Penrose fields $\Phi^l$ and $\bar{\Phi}^l$, for $l = 1, \ldots, k - 2$. To see this, let us consider the spinor $A$. If we assume $A$ is a natural
spinor of order \( l \), then the derivative of (3.31) with respect to \( \Phi^{l+1} \) becomes, after using the commutation relation (2.81),

\[
< \chi, \psi > < \overline{\chi}, \overline{\psi} > [\partial^l \Phi](\chi^{l+1}, \overline{\chi}^{l-1}, u; \psi^k, \overline{\psi}^k, v, w) = 0, \quad (3.32)
\]

which implies

\[
\partial^l \Phi A = 0. \quad (3.33)
\]

A simple induction argument then shows that \( A \) is independent of all the Penrose fields \( \Phi^l \) for \( l = 1, \ldots, k - 2 \). An identical argument establishes that \( A \) is independent of \( \overline{\Phi}^l \) for \( l = 1, \ldots, k - 2 \). In a similar fashion we can show that \( B \) is independent of the Penrose fields \( \Phi^l \) and \( \overline{\Phi}^l \), for \( l = 1, \ldots, k - 2 \). We conclude that \( A \) and \( B \) are \( SL(2, \mathbb{C}) \) invariant spinors constructed solely from the \( \epsilon \) spinors. But there are no \( SL(2, \mathbb{C}) \) invariant spinors with the rank and symmetry of \( A \) or \( B \) built solely from the \( \epsilon \) spinors, so \( A \) and \( B \) must vanish.

If we differentiate the linearized equations for \( \hat{C} \) with respect to \( \overline{\Phi}^{k+1} \), a similar line of reasoning shows that \( D \) and \( E \) must also vanish.

We can now classify all natural generalized symmetries of the Yang-Mills equations.

**Theorem 3.6.** Let

\[
C_a = C_a(x, A_a, A_{a,b_1}, \ldots, A_{a,b_1 \ldots b_k})
\]

be a \( k^{th} \)-order natural generalized symmetry of the Yang-Mills equations. Then there is a natural \( g \)-valued function

\[
\Lambda = \Lambda(x, A_a, A_{a,b_1}, \ldots, A_{a,b_1 \ldots b_{k-1}}),
\]

such that, modulo the field equations,

\[
C_a = \nabla_a \Lambda.
\]

**Proof:** From Proposition 3.5 we have that every generalized symmetry of order \( k \) differs from a symmetry of order \( k - 1 \) by a generalized gauge symmetry. By induction, every generalized symmetry of order \( k \) differs from a gauge symmetry by a generalized symmetry of order 1, which we denote by \( C^{(1)}_{AA'} \). From Proposition 3.1 and 3.3, we can apply equation (3.29) to \( C^{(1)}_{AA'} \). From the discussion following (3.29) we conclude that \( C^{(1)}_{AA'} \) is in fact independent of the Penrose fields and is thus an \( SL(2, \mathbb{C}) \) invariant spinor of type \((1,1)\) constructed from the \( \epsilon \) spinors. But there are no such spinors, as can be seen, for example, by noting that such a spinor would define a Lorentz invariant vector field. And so it follows that \( C^{(1)}_{AA'} = 0 \).
4. Discussion.

We have shown that all natural generalized symmetries of the Yang-Mills equations are generalized gauge symmetries. These symmetries are physically trivial, and they give rise to trivial conservation laws. In order to extend our results to a complete classification of generalized symmetries of the Yang-Mills equations we will have to drop the requirements (2.22) and (2.23). Thus we must consider solutions of the linearized equations (2.16) which are (i) not gauge covariant, and (ii) not Poincaré covariant, i.e., $C_a$ is now allowed to be any function of the coordinates (2.29) or, better yet, the coordinates (2.63). In the gravitational case [8], the generalizations analogous to (i) and (ii) lead to no new types of symmetries. Preliminary computations imply that (i) is unlikely to lead to any new symmetries also in the Yang-Mills case for similar reasons to those found in [8]. On the other hand, the relaxation of Poincaré invariance may lead to new, non-trivial symmetries (beyond those of Proposition 2.2). Indeed, the putative generalized symmetries can be constructed using the conformal Killing vectors admitted by the underlying Minkowski spacetime, and this significantly changes the analysis beginning with Proposition 3.5. We will present the complete symmetry analysis elsewhere.

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