Low-energy dynamics of a $\mathbb{C}P^1$ lump on the sphere

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Abstract

Low-energy dynamics in the unit-charge sector of the $\mathbb{C}P^1$ model on spherical space (space-time $S^2 \times \mathbb{R}$) is treated in the approximation of geodesic motion on the moduli space of static solutions, a six-dimensional manifold with non-trivial topology and metric. The structure of the induced metric is restricted by consideration of the isometry group inherited from global symmetries of the full field theory. Evaluation of the metric is then reduced to finding five functions of one coordinate, which may be done explicitly. Some totally geodesic submanifolds are found and the qualitative features of motion on these described.

1 Introduction

The $\mathbb{C}P^1$ model in flat space is a scalar field theory whose configuration space $Q$ consists of finite energy maps from Euclidean $\mathbb{R}^2$ to the complex projective space $\mathbb{C}P^1$, the energy functional being constructed naturally from the Riemannian structures of the base and target spaces (that is, the model is a pure sigma model in the broad sense). The requirement of finite energy imposes a boundary condition at spatial infinity, that the field approaches the same constant value, independent of direction in $\mathbb{R}^2$, so that the field may be regarded as a map from the one point compactification $\mathbb{R}^2 \cup \{\infty\} \cong S^2$ to $\mathbb{C}P^1$. Since $\mathbb{C}P^1 \cong S^2$ also, finite energy configurations are effectively maps $S^2 \to S^2$, the homotopy theory of which is well understood, and the configuration space is seen to consist of disconnected sectors $Q_n$ labelled by an integer $n$, the “topological charge” (degree),

$$Q = \bigcup_{n \in \mathbb{Z}} Q_n.$$  \hspace{1cm} (1)

Each configuration is trapped within its own sector because time evolution is continuous.

The Lorentz invariant, time-dependent model is not integrable but complete solution of the static problem has been achieved by means of a Bogomol’nyi argument and the general charge $n$ moduli space, the space of charge-$n$ static solutions $M_n \subset Q_n$, is known (that all static, finite energy solutions of the $\mathbb{C}P^1$ model saturate the Bogomol’nyi bound is a non-trivial result \cite{1}). Each static solution within the charge-$n$ sector has the same energy (minimum within that sector and proportional to $n$), and $M_n$ is parametrized by $4n + 2$ parameters (the moduli), so such a moduli space may be thought of as the $(4n + 2)$-dimensional level bottom of a potential valley defined on the infinite dimensional charge-$n$ sector, $Q_n$. Low energy dynamics may be approximated by motion restricted to this valley bottom, a manifold embedded in the full configuration space, and thus inheriting from it a non-trivial metric induced by the kinetic energy functional. The approximate dynamic problem is reduced to the geodesic problem with this metric, and has been investigated by several authors \cite{2,3}. In the unit-charge sector one here encounters a difficulty: certain components of the metric are singular and the approximation is ill defined. For example, unit-charge static solutions are localized lumps of energy with arbitrary spatial scale, so one of the six moduli of $M_1$ is a scale parameter. Motion which changes this parameter is impeded by infinite inertia in the geodesic approximation, a result in conflict with numerical evidence which suggests that lumps collapse under scaling perturbation \cite{4}.

This problem should not be present in the model defined on a compact two dimensional physical space. The obvious choice is the 2-sphere because the homotopic partition of the configuration space carries through...
unchanged. Also, $S^2$ with the standard metric is conformally equivalent to Euclidean $\mathbb{R}^2 \cup \{\infty\}$, and the static $\mathbb{C}P^1$ model energy functional is conformally invariant, so the whole flat space static analysis is still valid and all the moduli spaces are known. However, the kinetic energy functional does change and induces a new, well defined metric on the unit-charge moduli space. By means of the isometry group derived from the spatial and internal symmetries of the full field theory we can place restrictions on the possible structure of this metric, greatly simplifying its evaluation. The geodesic problem is still too complicated to be solved analytically in general, but by identifying totally geodesic submanifolds, it is possible to obtain the qualitative features of a number of interesting solutions. In particular, the possibilities for lumps travelling around the sphere are found to be surprisingly varied.

2 The $\mathbb{C}P^1$ model on $S^2$

The $\mathbb{C}P^1$ model on the 2-sphere is defined by the Lagrangian

$$L[W] = \int_{S^2} dS \frac{\partial_\mu W \partial_\nu \bar{W}}{(1 + |W|^2)^2} \eta^{\mu\nu}$$

where $W$ is a complex valued field, $dS$ is the invariant $S^2$ measure and $\eta^{\mu\nu}$ are the components of the inverse of the Lorentzian metric

$$\eta = dt^2 - d^2\Omega$$

on $\mathbb{R}(time) \times S^2$(space), $d^2\Omega$ being the natural metric on $S^2$. Although the language of the $\mathbb{C}P^1$ model is analytically convenient, the homotopic classification and physical meaning of the field configurations are more easily visualized if we exploit the well known equivalence to the $O(3)$ sigma model [5, 6]. In the latter, the scalar field is a three dimensional isovector $\phi$ constrained to have unit length with respect to the Euclidean $\mathbb{R}^3$ norm ($\phi \cdot \phi = 1$), that is, the target space is the 2-sphere of unit radius with its natural metric, which we will denote $S^2_{iso}$ for clarity. (The suffix refers to “isospace” in analogy with the internal space of nuclear physics models.) The $\mathbb{C}P^1$ field $W$ is then thought of as the stereographic image of $\phi$ in the equatorial plane, projected from the North pole, $(0,0,1)$. Explicitly,

$$\phi = \left( \frac{W + \bar{W}}{1 + |W|^2}, \frac{W - \bar{W}}{i(1 + |W|^2)} \frac{|W|^2 - 1}{1 + |W|^2} \right)$$

and

$$W = \phi_1 + i\phi_2 \frac{1}{1 - \phi_3}$$

Then

$$L[W] = L_{\sigma}[\phi] = \frac{1}{4} \int_{S^2} dS \partial_\mu \phi \cdot \partial_\nu \phi \eta^{\mu\nu}$$

the familiar $O(3)$ sigma model Lagrangian. A $W$ configuration, then, may be visualized as a distribution of unit length arrows over the surface of the physical 2-sphere $S^2_{sp}$. Each smooth map $S^2_{sp} \rightarrow S^2_{iso}$ falls into one of a discrete infinity of disjoint homotopy classes, each class associated with a unique integer which may be thought of as the topological degree of the map (see, for example [6]), so homotopic partition of the configuration space is built in to the model from the start.

We also choose stereographic coordinates $(x, y)$ on $S^2_{sp}$, in terms of which,

$$(\eta_{\mu\nu}) = \text{diag} \left( 1, \frac{-1}{(1 + r^2)^2}, \frac{-1}{(1 + r^2)^2} \right)$$

where $r = \sqrt{x^2 + y^2}$, $(x, y)$ takes all values in $\mathbb{R}^2$ and $x^0 = t$, $x^1 = x$, $x^2 = y$. The radius of $S^2_{sp}$ has been normalized to unity. The invariant measure is,

$$dS = dx \, dy \, \sqrt{|\det(\eta_{\mu\nu})|} = \frac{dx \, dy}{(1 + r^2)^2}$$

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and so,

$$L[W] = \int dx\,dy \frac{1}{(1 + |W|^2)^2} \left( \frac{|\dot{W}|^2}{(1 + r^2)^2} - \left| \frac{\partial W}{\partial x} \right|^2 - \left| \frac{\partial W}{\partial y} \right|^2 \right).$$

We identify kinetic energy,

$$T[W] = \int dx\,dy \frac{|\dot{W}|^2}{(1 + r^2)^2 (1 + |W|^2)^2}$$

and potential energy

$$V[W] = \int dx\,dy \frac{1}{(1 + |W|^2)^2} \left( \frac{|\partial W|^2}{\partial x} + \left| \frac{\partial W}{\partial y} \right|^2 \right).$$

Note that the potential energy is identical to that for flat space by virtue of the conformal invariance of the static model (stereographic projection is a conformal transformation). Thus the familiar Bogomol’nyi argument follows immediately and \((i, j, k\) run over 1, 2 and \(\times\) represents the \(\mathbb{R}^3\) vector product in \(\phi\) space):

$$\begin{align*}
0 & \leq \int dx\,dy \left( \partial_i \phi \pm \epsilon_{ij} \phi \times \partial_j \phi \right) \cdot \left( \partial_i \phi \pm \epsilon_{ik} \phi \times \partial_k \phi \right) \\
& = 2 \int dx\,dy \left[ \partial_i \phi \cdot \partial_j \phi \mp \epsilon_{ij} \left( \partial_i \phi \times \partial_j \phi \right) \cdot \phi \right], \\
\Rightarrow V[W] & = \frac{1}{4} \int dx\,dy \partial_i \phi \cdot \partial_i \phi \\
& \geq \frac{1}{2} \left| \int dx\,dy \frac{\partial \phi}{\partial x} \times \frac{\partial \phi}{\partial y} \right| \cdot \phi = 2\pi|n|,
\end{align*}$$

which, on substitution of \((14)\) becomes the Cauchy-Riemann condition for \(W\) to be an analytic function of \(z = x + iy\) (upper sign) or \(\bar{z} = x - iy\) (lower sign). The former (latter) case corresponds to static solutions of positive (negative) degree, and if \(W\) is single valued with finite degree \(n\), then it must be a rational map of degree \(n\) in \(z\) if \(n \geq 0\) or in \(\bar{z}\) if \(n < 0\). We shall deal with the unit charge moduli space, consisting of all rational maps of degree 1 in \(z\). Since the configuration space and moduli spaces of the flat space and spherical space models are diffeomorphic, we shall use the same notation \((Q,Q_n,M_n\text{ etc.})\) in both cases.

3 The unit-charge moduli space

The simplest static unit-charge solution is

$$W(z) = z$$

which we shall call the symmetric hedgehog because its \(\phi\) field points radially outwards at all points on \(S^2_z\). Its energy density is uniformly distributed, so it is not really a lump. Since the static model is conformally invariant, any configuration obtained from this by a Möbius transformation must be another point on the moduli space. In fact the orbit of \(W = z\) under the Möbius group is the space of degree 1 rational maps, each map being generated by one and only one group element. Thus we may identify the moduli space with the parameter space of the Möbius group.

There is a well known matrix representation of Möbius transformations \([8]\) which we denote thus:

$$W(z) = az + b \quad \frac{cz + d}{cz + d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ z = M \circ z$$

where \(M \in GL(2,\mathbb{C})\) so that \(\text{det} \ M \neq 0\). The last condition ensures the invertibility of the transformation and fixes the degree of \(W\) at 1. The Möbius group product becomes matrix multiplication,

$$M_2 \circ (M_1 \circ z) = (M_2 M_1) \circ z$$

(16)
where the left hand side means
\[ M_2 \odot (M_1 \odot z) = \frac{a_2(M_1 \odot z) + b_2}{c_2(M_1 \odot z) + d_2} \] (17)
in obvious notation. All matrices differing by a constant factor yield the same configuration, and \( \det M \neq 0 \) so when we divide by this scaling equivalence we can choose a unimodular matrix as the representative for each equivalence class. There are two such matrices possible for each distinct configuration because if \( M \) is unimodular, so is \(-M\). Thus \( SL(2, \mathbb{C}) \) is a double cover of the moduli space, which we recover by dividing out the equivalence \( M' \sim M \Leftrightarrow M' = -M \): the moduli space is \( SL(2, \mathbb{C})/\mathbb{Z}_2 \).

Coincidentally, \( SL(2, \mathbb{C}) \) is also a double cover of the proper orthochronous Lorentz group. The statement that any Lorentz transformation may be formed by a unique composition of a boost then a rotation (or vice versa) translates to the existence, for all \( M \in SL(2, \mathbb{C}) \), of \( U \in SU(2) \) and \( H \), a positive definite, unimodular, Hermitian \( 2 \times 2 \) matrix (call this set \( \mathcal{H} \)), satisfying
\[ M = UH \] (18)
both \( U \) and \( H \) being unique. It follows that the space \( SL(2, \mathbb{C}) \) is locally a product of \( S^3 \) (the group manifold of \( SU(2) \)) and \( \mathbb{R}^3 \) (the parameter space of \( \mathcal{H} \)), a result which generalizes globally, \( SL(2, \mathbb{C}) \cong S^3 \times \mathbb{R}^3 \).

We may choose local coordinates on \( SL(2, \mathbb{C}) \) by defining the standard Euler angles \( (\alpha, \beta, \gamma) \) on \( S^3 \),
\[ U = \begin{pmatrix} \cos \frac{\beta}{2} e^{i(\beta+\gamma)/2} & \sin \frac{\beta}{2} e^{-i(\beta-\gamma)/2} \\ -\sin \frac{\beta}{2} e^{-i(\beta+\gamma)/2} & \cos \frac{\beta}{2} e^{-i(\beta-\gamma)/2} \end{pmatrix} \] (19)
and expanding \( H \) in terms of Pauli matrices \( \tau \),
\[ H = \Lambda \mathbb{I} + \lambda \cdot \tau = \begin{pmatrix} \Lambda + \lambda_3 & \lambda_1 - i \lambda_2 \\ \lambda_1 + i \lambda_2 & \Lambda - \lambda_3 \end{pmatrix} \] (20)
\( \Lambda(\lambda) \) being chosen to ensure the unimodular and positive definite properties:
\[ \Lambda = \sqrt{1 + \lambda^2}. \] (21)
The 3-vector \( \lambda \) (modulus \( |\lambda| = \lambda \)) takes all values in \( \mathbb{R}^3 \), while \( \beta \in [0, 4\pi] \), \( \gamma \in [0, 2\pi] \) and \( \alpha \in [0, \pi] \). These ranges allow \( M \) to take all values in the double cover \( SL(2, \mathbb{C}) \). In analyzing the structure of the metric, it is convenient to work with \( SL(2, \mathbb{C}) \), checking that the metric is single valued under the identification of \( M \) with \(-M\). The true moduli space \( SL(2, \mathbb{C})/\mathbb{Z}_2 \) is charted by the same coordinates but with \( \beta \) lying in the reduced range \( [0, 2\pi] \), for \( U \) is then restricted to the “upper half” of \( S^3 \). The chart has a coordinate singularity at \( \alpha = 0 \) and at \( \alpha = \pi \). The explicit connexion between a point in \( M_1 \) and the corresponding static solution will be made in section 5, below.

4 The induced metric and its isometry group

Field dynamics of the \( \mathbb{C}P^1 \) model may be visualized as the dynamics of a point particle with “position” \( W : S^2_{\text{loop}} \rightarrow S^2_{\text{loop}} \) moving in an infinite-dimensional configuration space. A solution \( W(t, x, y) \) of the field equations is thought of as a trajectory in this space, motion on which is determined by metric \( T[W] \) and potential \( V[W] \). In the unit-charge sector, the Bogomol’nyi argument shows that there is a six-dimensional subspace on which the potential achieves its topological minimum value of \( 2\pi \), and that any perturbation departing from this subspace must involve increasing \( V \). If a configuration sitting at the bottom of this potential valley is given a small velocity tangential to it then we expect the ensuing time-evolved field to stay close to the valley bottom, for departure from it entails climbing up the valley walls. In the geodesic approximation we restrict motion to the valley bottom, assuming that orthogonal modes are insignificant.

Thus, at all times \( W(t, x, y) \) is a solution of the static model, but we allow the moduli \( \{ \lambda, \alpha, \beta, \gamma \} \), denoted collectively by \( \{ q^i : i = 1 \ldots 6 \} \), to vary with time in accordance with the inherited action principle. So,
\[ \dot{W} = \frac{\partial W}{\partial q^i} \dot{q}^i, \] (22)

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and the Lagrangian is
\[ L = T - V = \int \frac{dx
dy}{(1 + r^2)^2} \frac{\partial W}{\partial q^i} \frac{\partial \bar{W}}{\partial \bar{q}^j} \dot{q}^i \dot{q}^j - 2\pi. \] (23)

Defining the induced metric,
\[ g(q) = g_{ij}(q) dq^i dq^j = 2 \int \frac{dx
dy}{(1 + r^2)^2} \frac{\partial W}{\partial q^i} \frac{\partial \bar{W}}{\partial \bar{q}^j} dq^i dq^j \] (24)

and ignoring the irrelevant constant, the Lagrangian is recast as that of a free particle moving on a Riemannian manifold with metric \( g \):
\[ L = \frac{1}{2} g_{ij}(q) \dot{q}^i \dot{q}^j. \] (25)

The equations of motion are the geodesic equations. In principle all we need do is evaluate the integrals of (24), but these are 21 functions of 6 variables so as it stands this is intractable in practice. It is profitable to take a more circumspect approach, using symmetries of the model to place restrictions on the structure of \( g \).

Consider the rotation group \( SO(3) \) acting on \( S^2_{sp} \) and \( S^2_{iso} \). The former is the group of spatial rotations (under which \( W \), or equivalently \( \phi \), transforms as a scalar) while the latter is the group of global internal rotations (henceforth called “isorotations”) of the \( \phi \) field of which \( W \) is the stereographic image. Any such transformation \( T \) leaves invariant (a) the topological charge, so \( T \) is a bijection \( T : Q \rightarrow Q \), (b) the potential energy, so within \( Q \) static solutions are mapped to other static solutions, \( T : M \rightarrow M \), and (c) the kinetic energy, which induces the metric on \( M \). Hence \( T \) is an isometry of \( (M, g) \).

The action of the isorotation on the moduli space is simple:
\[ M = UH \mapsto LUH \mapsto UH \] (28)

produces an isorotation of the configuration \( W(z) = M \circ z \), while
\[ M = MR \mapsto W(z) \mapsto W(R \circ z) \] (27)

produces a spatial rotation, both isometries of the induced metric.

The action of the isorotation on the moduli space is simple:
\[ M = UH \mapsto LUH \mapsto UH \mapsto H. \]

The isometry takes the \( SU(2) \) left multiplication action on \( S^3/\mathbb{Z}_2 \) while leaving the \( \mathbb{R}^3 \) moduli \( \lambda \) unchanged. Using a technique standard in the analysis of isometries in general relativity \( \mathbb{E} \), we change from the coordinate basis on \( S^3/\mathbb{Z}_2 \), \( \{d\alpha, d\beta, d\gamma\} \), to a non-coordinate basis, in this case the left-invariant 1-forms of the Lie group \( SU(2) \). These may be found by expanding the left-invariant 1-form \( U^{-1} dU \) in terms of a convenient basis of the Lie algebra \( su(2) \), for example \( i\tau/2 \). Explicitly,
\[ U^{-1} dU = \sigma \cdot \left( \frac{i}{2} \tau \right) \] (29)

where
\[ \begin{align*}
\sigma_1 &= -\sin \gamma d\alpha + \cos \gamma \sin \alpha d\beta \\
\sigma_2 &= \cos \gamma d\alpha + \sin \gamma \sin \alpha d\beta \\
\sigma_3 &= \cos \alpha d\beta + d\gamma.
\end{align*} \] (30)

If we evaluate the metric at one particular point on \( S^3/\mathbb{Z}_2 \), for all possible \( \lambda \in \mathbb{R}^3 \), we can obtain the metric at all other points on \( S^3/\mathbb{Z}_2 \) because \( S^3/\mathbb{Z}_2 \) is the isorotation orbit of our base point, and isorotation is an
isometry, so the metric must remain constant (for each $\lambda$) over the entire orbit. “Constant” means unchanging when considered as a geometric object, not that the components with respect to the original coordinate basis are constant, because the basis vectors themselves transform non-trivially. The basis of $\overline{H}$ is invariant however, so the metric must be of the form

$$g = \mu_{ab}(\lambda)\sigma_a\sigma_b + \nu_{ab}(\lambda)\sigma_a d\lambda_b + \pi_{ab}(\lambda)d\lambda_a d\lambda_b,$$

where $a, b = 1, 2, 3$ and each of the component functions is independent of $(\alpha, \beta, \gamma)$.

Let us now consider the spatial rotations:

$$M = UH \Rightarrow Mr = URR^1HR$$

$$\Rightarrow U \Rightarrow UR$$

$$H \Rightarrow R^1HR \in \mathcal{H}.$$

The latter in terms of coordinates is

$$H = \sqrt{1 + \lambda^2 I + \lambda \cdot \tau} \Rightarrow \sqrt{1 + \lambda^2 I + R^1(\lambda \cdot \tau)R}.$$ (33)

The action of conjugation of the Hermitian, traceless matrix $\lambda \cdot \tau$ by a unitary matrix $R$ is well known — it is equivalent to a $SO(3)$ rotation of $\lambda$:

$$R^1(\lambda \cdot \tau)R = (\mathcal{R}\lambda) \cdot \tau$$ (34)

where $\mathcal{R} \in SO(3)$ with components $\mathcal{R}_{ab} = \frac{1}{2}tr(\tau_a R^1 \tau_b R)$. The action on the left-invariant 1-forms $\sigma$ is similar. Under $U \mapsto UR$,

$$U^{-1}dU \Rightarrow R^1(U^{-1}dU)R$$

$$\Rightarrow \frac{i}{2}\sigma \cdot \tau \Rightarrow \frac{i}{2}R^1(\sigma \cdot \tau)R = \frac{i}{2}(\mathcal{R}\sigma \cdot \tau)$$

$$\Rightarrow \sigma \mapsto \mathcal{R}\sigma$$

where $\mathcal{R}$ is the same $SO(3)$ matrix defined above. Thus both $\lambda$ and $\sigma$ transform as 3-vectors under spatial rotations and as scalars under isorotations.

The metric must be invariant under spatial rotations also, so the task is to construct from $\lambda$, $d\lambda$ and $\sigma$ the most general possible $(0, 2)$ tensor which is scalar under these rotations. This is

$$g = A d\lambda \cdot d\lambda + B(\lambda \cdot d\lambda)^2 + C\sigma \cdot \sigma + D(\lambda \cdot \sigma)^2$$

$$+ E\sigma \cdot d\lambda + F(\lambda \cdot d\lambda)(\lambda \cdot \sigma) + G(\lambda \times \sigma) \cdot d\lambda$$

$A$—$G$ being 7 unknown functions of $\lambda = |\lambda|$ only.

The metric may be restricted still further on consideration of a discrete isometry. The kinetic energy is invariant under the discrete “parity” transformations $P_z : z \mapsto \bar{z}$ and $P_W : W \mapsto \bar{W}$. However, neither is an isometry of the moduli space because each reverses the sign of the topological charge, mapping lumps to anti-lumps. The composite transformation $P_W \circ P_z$ is an isometry. Using the configuration of $[15]$,

$$W(z) = P_z \frac{az + b}{cz + d} P_W = \frac{\bar{a}z + \bar{b}}{\bar{c}z + \bar{d}} = \bar{M} \circ z.$$ (37)

In terms of the moduli, $M \mapsto \bar{M}$ is the transformation,

$$\sigma = (\sigma_1, \sigma_2, \sigma_3) \mapsto (-\sigma_1, \sigma_2, -\sigma_3)$$ (38)

$$\lambda = (\lambda_1, \lambda_2, \lambda_3) \mapsto (\lambda_1, -\lambda_2, \lambda_3).$$

This isometry removes two of the terms in $\overline{H}$ because under it,

$$\sigma \cdot d\lambda \mapsto -\sigma \cdot d\lambda,$$ (39)
and

\[
(\lambda \cdot d\lambda)(\lambda \cdot \sigma) \mapsto -(\lambda \cdot d\lambda)(\lambda \cdot \sigma)
\]  

(40)

so that \( E(\lambda) \equiv F(\lambda) \equiv 0 \).

The remaining five functions of \( \lambda \) are evaluated by choosing convenient orientations for \( \lambda \), positions on \( S^3/\mathbb{Z}_2 \) and tangent vectors (velocities), then calculating the kinetic energy and comparing with (36). Repeating this four times it is possible to extract the following (see figure 1):

\[
\begin{align*}
A &= 4\pi S_2(\chi) \\
B &= \frac{4\pi}{\lambda^2} \left[ \frac{4}{1+\lambda^2} S_1(\chi) - S_2(\chi) \right] \\
C &= \frac{\pi}{2} - 2\pi S_1(\chi) \\
D &= \frac{\pi}{\lambda^2} \left[ 6S_1(\chi) - \frac{1}{2} \right] \\
G &= A
\end{align*}
\]  

(41)

where,

\[
\begin{align*}
\chi(\lambda) &= \frac{\sqrt{1 + \lambda^2} + \lambda}{\sqrt{1 + \lambda^2} - \lambda} \quad (42) \\
S_1(\chi) &= \frac{\chi^2}{2(\chi^2 - 1)^3} \left[ (\chi^2 + 1) \log \chi^2 - 2\chi^2 + 2 \right] \\
S_2(\chi) &= \frac{\chi}{(\chi^2 - 1)^3} \left[ \chi^4 - 2\chi^2 \log \chi^2 - 1 \right].
\end{align*}
\]

Note that \( \chi \) is a strictly increasing function of \( \lambda \), and that \( \chi : [0, \infty) \to [1, \infty) \). There appear to be divergences of the functions \( A-D \) at \( \lambda = 0 \), but these are in fact removable singularities, so all the limits of vanishing \( \lambda \) exist. Although \( B \) and \( D \) are negative it is straightforward to show that this metric is positive definite, as of course it must be. The veracity of the statement \( G \equiv A \) is established by explicit calculation, there being no obvious symmetry argument in its favour. In summary then, the metric is

\[
g = A \, d\lambda \cdot d\lambda + B(\lambda \cdot d\lambda)^2 + C \, \sigma \cdot \sigma + D(\lambda \cdot \sigma)^2 + A(\lambda \times \sigma) \cdot d\lambda.
\]  

(43)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{The metric functions \( A, B, C \) and \( D \).}
\end{figure}
5 Some totally geodesic submanifolds

Before discussing geodesics of the metric (43) we must describe the connexion between a point on the moduli space \( (S^3/Z_2) \times \mathbb{R}^3 \) and its corresponding field configuration. Consider first the 3-dimensional submanifold defined by \( U = 1 \), parametrized by \( \lambda \in \mathbb{R}^3 \). Any point in this subspace may be written as \( \lambda = \mathcal{R} \lambda' \) where \( \lambda' = (0,0,\lambda) \), \( \lambda > 0 \) and \( \mathcal{R} \in SO(3) \). The lump represented by \( \lambda' \) is

\[
W''(z) = (\lambda' \cdot \tau) \circ z = \left( \sqrt{1 + \lambda'^2 + \lambda} \right) \left( \frac{1}{\sqrt{1 + \lambda'^2 - \lambda}} \right) z = \chi(\lambda) z. \tag{44}
\]

This is a distorted hedgehog with the arrows pulled towards the North pole. The larger \( \lambda \) is, so the larger \( \chi \) is and the greater is the distortion. Although it is usual to define the position of a lump as the position of maximum energy density, we shall refer to this as a lump of sharpness \( \lambda \) located at \( \hat{\lambda} = (0,0,1) \), the antipodal point to the energy density peak which occurs where the arrows are stretched apart. Obviously the motion of any point is trivially mirrored by its antipodal image, so this terminology makes sense. The lump represented by \( \lambda \) is

\[
W(z) = \left( \mathcal{R} \lambda' \cdot \tau \right) \circ z = \left[ R^3(\lambda' \cdot \tau) R \right] \circ z = \left[ R^3(\lambda' \cdot \tau) \right] \circ (R \circ z) = R^3 \circ \left[ W''(R \circ z) \right]. \tag{45}
\]

This configuration is formed by first performing a spatial rotation – taking the \( \phi \) arrow at the old point \( z \) and placing it at the new point \( R \circ z \) without changing its orientation – then performing the inverse isorotation. The result looks like the arrows have been fixed to \( S^2 \), which has then been rotated by \( \mathcal{R} \) which, as defined, has the action on \( S^2 \) equivalent to \( R^3 \), not \( R \), acting on \( CP^1 \) via \( \circ \). That is, if we define \( P \) to be stereographic projection, \( P : S^2 \rightarrow CP^1 \) so that \( P : \phi \mapsto W \), then \( P \circ \mathcal{R} \phi \mapsto W \circ R \circ z \). So the lump at the North pole is shifted to \( \hat{\lambda} = \mathcal{R} \hat{\lambda} \).

All other points on the moduli space are on the isorotation orbit of this submanifold, and isorotation, while changing the internal orientation of the lump, does not move the lump around on physical space. Thus we can always interpret \( \hat{\lambda} \) as the lump’s position, and \( \lambda \) as parametrizing its sharpness. The symmetric hedgehog has \( \lambda = 0 \), and large \( \lambda \) lumps have taller, narrower energy density peaks than small \( \lambda \) lumps.

One way of attacking the geodesic problem is to reduce its dimension by identifying totally geodesic submanifolds, that is, choosing initial value problems whose solution is simplified by some symmetry. The easiest method for identifying such submanifolds is to find fixed point sets of discrete groups of isometries. Any isometry maps geodesics to geodesics, so if there were a geodesic starting off in the fixed point set of the isometry and subsequently deviating from it, this would be mapped under the isometry to another geodesic, identical to the first throughout its length in the fixed point set, but deviating from the set in a different direction. This violates the uniqueness of solutions of ordinary differential equations, so no such geodesic may exist. If the initial data are a point on the fixed point set and a velocity tangential to it, then the geodesic must remain on the fixed point set for all subsequent time.

Examining (43) we see that \( \lambda \rightarrow -\lambda \) is an isometry. Its fixed point set is \( S^3/Z_2 \), the isorotation orbit of the symmetric hedgehog, on which the metric is

\[
g = \frac{\pi}{6} \sigma \cdot \sigma. \tag{46}\]

The kinetic energy is the rotational energy of a totally symmetric rigid body, moment of inertia \( \pi/6 \). The solutions are just isorotations of the symmetric hedgehog at constant frequency about some fixed axis. In this case isorotation is equivalent to spatial rotation because \( \lambda = 0 \Rightarrow H = I \).

A less trivial geodesic submanifold is the fixed point set of the parity transformation described above, \( M \rightarrow \tilde{M} \). This is a 3-dimensional manifold, the product of the plane \( \lambda_2 = 0 \) in \( \mathbb{R}^3 \) with the circle \( \{ \alpha \in [0,\pi], \beta = \gamma = 0 \} \cup \{ \alpha \in [0,\pi], \beta = \gamma = \pi \} \) in \( S^3/Z_2 \). The circle is more conveniently parametrized if we temporarily allow \( \alpha \) the domain \( [0,2\pi] \), for it is then \( \{ \alpha \in [0,2\pi], \beta = \gamma = 0 \} \). This space contains lumps of arbitrary sharpness located on a great circle through the poles of \( S^2 \), each lump having an internal phase, so certain of its geodesics may be candidates for “travelling lumps.” Introducing spherical polar coordinates for \( \lambda \),

\[
\lambda = \lambda(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \tag{47}
\]
the plane $\lambda_2 = 0$ is parametrized by $(\lambda, \theta)$ where $\theta \in [0, 2\pi]$, again gluing two semicircles together and extending the domain of $\theta$ to cover the whole circle in one go. The metric on this geodesic submanifold is

$$
g = (A + \lambda^2 B) d\lambda^2 + \lambda^2 A d\theta^2 + C d\alpha^2 - \lambda^2 A d\theta d\alpha. \quad (48)$$

So the kinetic energy is

$$
T = \frac{1}{2} \left[ (A + \lambda^2 B) \dot{\lambda}^2 + \frac{p_\theta^2 C}{\lambda^2 A(C - \lambda^2 A/4)} + \frac{p_\alpha (p_\alpha + p_\theta)}{C - \lambda^2 A/4} \right] \quad (49)
$$

where we have used the cyclicity of $\theta$ and $\alpha$ to eliminate $\dot{\theta}$ and $\dot{\alpha}$ in favour of their constant, canonically conjugate momenta,

$$
p_\theta = \lambda^2 A \left( \dot{\theta} - \frac{1}{2} \dot{\alpha} \right) \quad (50)
\quad p_\alpha = C \dot{\alpha} - \frac{1}{2} \lambda^2 A \dot{\theta}.
$$

Note that constant $p_\theta$ ($p_\alpha$) does not imply constant $\dot{\theta}$ ($\dot{\alpha}$), nor does $p_\theta = 0$ ($p_\alpha = 0$) imply $\dot{\theta} = 0$ ($\dot{\alpha} = 0$).

This system can be visualized as a point particle of position dependent mass $(A + \lambda^2 B)$ moving in a potential. It is the form of the potential which determines the broad qualitative features of its behaviour:

$$
V(\lambda) = p_\theta^2 V_\theta(\lambda) + p_\alpha (p_\alpha + p_\theta) V_\alpha(\lambda). \quad (51)
$$

As can be seen from figure 2 while $V_\theta(\lambda)$ is monotonically decreasing, $V_\alpha(\lambda)$ is monotonically increasing. This allows the possibility of potential minima where the forces $-p_\theta^2 V_\theta'(\lambda)$ outwards (in the sense of increasing $\lambda$) and $p_\alpha (p_\alpha + p_\theta) V_\alpha'(\lambda)$ inwards are in stable equilibrium. It certainly is not possible if $p_\alpha (p_\alpha + p_\theta) \leq 0$, for then $V(\lambda)$ as a whole is monotonically decreasing. This region in the $(p_\alpha, p_\theta)$ plane is shown shaded in figure 3. Whatever the initial conditions on $\lambda$, the lump always moves towards infinite $\lambda$ without passing through $\lambda = 0$ (which would correspond to the lump swapping hemispheres), reaching the singularity $\lambda = \infty$, an infinitely tall, sharp spike, in finite time. Thus $(M_1, g)$ is geodesically incomplete. This result follows from the rapid vanishing of the inertia to sharpening, $A + \lambda^2 B$, in the large $\lambda$ limit (see figure 4). For example, consider the simple case $p_\alpha = p_\theta = 0$ and let $\lambda(0)$ and $\dot{\lambda}(0)$ be strictly positive. It is easily seen that $t_\infty$, the time taken to reach the singular spike is proportional to the following integral:

$$
t_\infty \propto \int_{\lambda(0)}^{\infty} d\lambda \sqrt{A(\lambda) + \lambda^2 B(\lambda)}. \quad (52)
$$

The integrand is finite over the integration range (even if $\lambda(0) = 0$), so if $t_\infty$ diverges it can only be due to the large $\lambda$ behaviour. But the integrand vanishes like $(\log \lambda)/\lambda^2$ at large $\lambda$, fast enough to ensure convergence. The inclusion of repulsive potentials can only make matters worse, so this singular behaviour extends to the rest of the shaded area.
Figure 2: The potential functions $\mathcal{V}_\theta(\lambda)$ and $\mathcal{V}_\alpha(\lambda)$. 
In the unshaded region, one can define the positive constant \( \kappa = \frac{p_\theta^2}{(p_\alpha^2 + p_\alpha p_\theta)} \) such that

\[
V(\lambda) = p_\alpha(p_\alpha + p_\theta)(\kappa V_\theta(\lambda) + V_\alpha(\lambda)).
\] (53)

Then the forms of the functions \(-V_\theta'\) and \(V_\alpha'\) (see figure 5) suggest that for each \( \lambda \), there is one (and only one) value of \( \kappa \) (call it \( \bar{\kappa} \)) for which \( V \) has a minimum at \( \lambda \). The equilibrium condition is \( V'(\lambda) = 0 \), so

\[
\bar{\kappa}(\lambda) = -\frac{V_\alpha'(\lambda)}{V_\theta'(\lambda)}.
\] (54)

Inverting the definition of \( \kappa \) we find that there are two distinct values of \( p_\alpha/p_\theta \) for each \( \bar{\kappa} \). If \( p_\alpha/p_\theta \) takes one of these and \( \dot{\lambda}(0) = 0 \) then \( \lambda \) will not subsequently change and hence \( \dot{\alpha} \) and \( \dot{\theta} \) will also remain constant, allowing the lump to travel around a great circle on \( S^2_{\alpha\theta} \) with constant speed and shape while undergoing constant frequency isorotation. The two values are

\[
\nu_\pm(\lambda) \equiv -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{1}{\bar{\kappa}(\lambda)}}.
\] (55)

Substituting (50) we can find the corresponding pair of stable ratios \( \dot{\alpha}/\dot{\theta} \) as functions of \( \lambda \),

\[
\omega_\pm \equiv \frac{\lambda^2 A(\lambda) (2\nu_\pm(\lambda) + 1)}{2C(\lambda) + \lambda^2 A(\lambda)\nu_\pm(\lambda)}.
\] (56)
(see figure 6). Thus, for any lump sharpness $\lambda$ and travel speed $\dot{\theta}$ there are two possible isorotation frequencies $\dot{\alpha}$ which allow stable, uniform travel and these two stability “branches” never coincide. It is interesting to note that $\lim_{\lambda \to \infty} \omega_+(\lambda) = 1$ meaning that very tall, sharp lumps can travel uniformly with $\dot{\alpha} \approx \dot{\theta}$. Motion with constant $\lambda$ and $\dot{\alpha} = \dot{\theta}$ is simply constant speed spatial rotation carrying the lump around a great circle. So when the extent of the lump’s structure is negligible relative to the radius of curvature of $S^2_{\text{sp}}$, it can travel in analogous fashion to a flat-space $\mathbb{C}P^1$ lump.

Figure 5: The outward (shrinking) force, $-V'_\theta(\lambda)$ and the inward (spreading) force, $V'_\alpha(\lambda)$. The vertical scales of the two curves are different.

Figure 6: The stable frequency ratios, $\dot{\alpha}/\dot{\theta} = \omega_{\pm}(\lambda)$. The upper curve $\omega_-(\lambda)$ tends to infinity at large $\lambda$.

Since $\overline{\kappa}(\lambda)$ takes all positive values, whatever value $\kappa$ takes there is an equilibrium $\lambda$. If $\lambda(0)$ is near this value, then (assuming $|\dot{\lambda}(0)|$ is not too large) the shape of the lump will oscillate periodically about the preferred sharpness, and its speed of travel round the sphere will vary with the same period. If $|\dot{\lambda}(0)|$ is too large, or the lump is initially much too spread out for its $\kappa$, then it will escape to the singular spike in finite time.

Let us examine the concrete example $p_\theta = p_\alpha$. Figure 7 shows the potential $\mathcal{V}$ with its minimum and
the lump travel speed \( \dot{\theta} \) as functions of \( \lambda \). We imagine a particle of position dependent mass moving in this potential and for simplicity take \( \dot{\lambda}(0) = 0 \). Clearly, if we release the particle with \( \lambda(0) < \lambda_A \approx 0.626 \), it will move off to infinity, the last case mentioned above. But if \( \lambda(0) > \lambda_A \approx 0.626 \), oscillatory motion ensues. Even here there are two qualitatively different cases, because \( \dot{\theta}(\lambda) \) has an absolute minimum at \( \lambda_C \approx 2.096 \), a turning point which is only reached if \( \lambda(0) < \lambda_B \approx 0.789 \) (or \( \lambda(0) > \lambda_C \)). If \( \lambda_B < \lambda(0) < \lambda_C \) then the speed of travel oscillates in simple phase with the lump sharpness, going from fast, spread-out lump to slow, sharp lump and back again. But if \( \lambda_A < \lambda(0) < \lambda_B \), the speed undergoes an extra wobble during the middle of the sharpness cycle, speeding up then slowing down again as it passes through its maximum sharpness. This case corresponds to lumps whose shape oscillates more acutely.

![Figure 1: The p_\theta = p_\alpha case: potential V and speed \dot{\theta}.](image)

Other interesting geodesic submanifolds are generated by computing the fixed point sets \( \Sigma_{ab} \) of the isometries \( M \mapsto (i\tau_a)\dagger M(i\tau_b) \), simultaneous isorotation and spatial rotation by \( \pi \) about the \( a \) and \( b \) axes respectively:

\[
\begin{align*}
U & \mapsto (i\tau_a)\dagger U(i\tau_b), \\
H & \mapsto (i\tau_b)\dagger H(i\tau_b) \\
\Rightarrow \lambda_c & \mapsto \begin{cases} \\
l_c & c = b \\
-l_c & c \neq b 
\end{cases}.
\end{align*}
\]

Thus if \( q \in \Sigma_{ab} \) \( \lambda \) must point along the \( b \)-axis. On \( \Sigma_{bb} \), \( U = \exp(i\psi\tau_0/2) \) where \( \psi \in [0,2\pi] \), whereas if \( a = b \pm 1 \mod 3 \) then \( U = \exp(\pm i\pi\tau_c/4)\exp(i\psi\tau_0/2) \) where \( c = b \mp 1 \mod 3 \). It follows that \( \sigma \) also points along the \( b \)-axis, independent of \( a \). The \( a \neq b \) submanifolds are the images of \( \Sigma_{bb} \) under \( \pm \pi/2 \) isorotations about the three axes, so it suffices to solve the geodesic problem on \( \Sigma_{bb} \) - geodesics on \( \Sigma_{ab}, a \neq b \), are then obtained by acting with the appropriate isometry. The choice of \( b \) doesn’t matter, and we choose to study the cylinder \( \Sigma_{33} \cong S^1 \times \mathbb{R} \) consisting of lumps of every sharpness located at the North (South) pole if \( \lambda_3 > 0 \) (\( \lambda_3 < 0 \), arbitrarily rotated about the North-South axis. Note that \([U,H] = 0 \) on \( \Sigma_{33} \) so “isorotated” and “spatially rotated” mean the same thing in this case.

The kinetic energy on \( \Sigma_{33} \) is

\[
T = \frac{1}{2} \left[ (A + \lambda_3^2 B)(\dot{\lambda}_3^2 + \frac{p_\psi^2}{C + \lambda_3^2 D}) \right]
\]

where once again \( p_\psi \) is the momentum conjugate to \( \psi \),

\[
p_\psi = (C + \lambda_3^2 D)\dot{\psi},
\]
and is constant by virtue of the cyclicity of $\psi$. This looks like a particle in one dimension moving in a potential

$$p_\psi^2 V_\psi = \frac{p_\psi^2}{2(C + \lambda_3^2 D)} = \frac{1}{2} p_\psi \dot{\psi} \tag{60}$$

with postion dependent mass. From the potential (figure 8) we see that all motion is oscillatory and that $\lambda_3$ periodically changes sign. Thus a lump set spinning about its own axis will spread out, its rotation slowing, until it is uniformly spread over the sphere, whereupon it will shrink to its mirror image in the opposite hemisphere, regaining its original spin speed as it does so. The process then reverses and the lump “bounces” between antipodal points indefinitely.

$$\text{Figure 8: Potential, } V_\psi(\lambda_3).$$

Defining the new coordinate

$$s(\lambda_3) = \int_0^{\lambda_3} d\mu \sqrt{A(\mu) + \mu^2 B(\mu)}, \tag{61}$$

which takes values in a finite open interval $(-\rho, \rho)$ symmetric about $s = 0$, the metric on $\Sigma_{33}$ becomes

$$g = ds^2 + H(s) d\psi^2 \tag{62}$$

where $H(s(\lambda_3)) = C(\lambda_3) + \lambda_3^2 D(\lambda_3)$. Since $|dH/ds| < 1 \ \forall s$, the manifold may be embedded as a surface of revolution in $\mathbb{R}^3$ and geodesics on it can be visualized directly. Figure 9 is a sketch of the embedded surface, which is of finite length and sausage-shaped with its ends pinched to infinitely sharp spikes, the tips of which are the points $\lambda_3 = \pm \infty$ and so are missing. The coordinates $(s, \psi)$ are geodesic orthogonal coordinates: a curve of constant $\psi$ is a geodesic along the length of the cylinder, lying in a plane containing the cylinder’s axis, parametrized by arc length $s$, while a curve of constant $s$ is a circle of radius $H(s)$, lying in a plane orthogonal to the cylinder’s axis. Two such curves always intersect at right angles.

The spinning geodesics described above wind around the cylinder, never reaching the ends (this would violate conservation of “angular momentum”) but winding back and forth between two circles $s = \pm \tilde{\rho}$, $\tilde{\rho} < \rho$ which they touch tangentially. The angle $\varphi$ at which the geodesic intersects the circle $s = 0$ determines $\tilde{\rho}$. When $\varphi = 0$ the geodesic stays on the circle, $\tilde{\rho} = 0$ (a spinning symmetric hedgehog) and $\tilde{\rho}(\varphi)$ monotonically increases, tending to the supremum $\rho$ as $\varphi$ tends to $\pi/2$ ($\varphi = \pi/2$ is an irrotational geodesic between antipodal singular spikes). Note that the geodesic incompleteness already mentioned appears again, this time characterized by the finite length of the cylinder and the missing points $s = \pm \rho$ ($\lambda_3 = \pm \infty$).
6 Concluding remarks

The behaviour of isolated topological solitons in flat space is generally rather trivial, whereas, as we have seen, despite the homogeneity of $S^2$, the motion of a single lump on the sphere is surprisingly complicated. It does travel on great circles, but while doing so its shape may oscillate in phase with its speed, whose periodic variation is of one of two types depending on the violence of the shape oscillations, or it may collapse to an infinitely tall, thin spike in finite time. A lump sent spinning about its own axis spreads out then re-forms in the opposite hemisphere, endlessly commuting between antipodal points.

The infinities in the unit-charge metric in flat space can be attributed to the lumps' polynomial tail-off: the kinetic energy needed to rigidly spin or scale-deform a lump diverges because such motions involve changing the field at spatial infinity. The $CP^1$ model on any compact space should be free of this problem because the kinetic energy, being an integral over a space of finite volume, must be finite provided the kinetic energy density is non-singular. Conversely, one would expect the singularity to persist in the model defined on hyperbolic space.

The flat-space $CP^1$ model can be made more “physical” by adding a $(2 + 1)$-dimensional version of the Skyrme term to stabilize against lump collapse, and a potential to stabilize against spread. The Bogomol’nyi bound remains valid but unsaturable. The potential is somewhat arbitrary, but one interesting possibility gives a mass to small amplitude travelling waves of the $\phi$ field, termed pions in analogy with the Skyrme model, and gives the lump an exponential rather than polynomial tail. This allows the lump to rotate, a problem if one attempts a collective coordinate approximation to low-energy dynamics along the lines recently proposed in \[15\]. The idea is to restrict the field to the “Bogomol’nyi regime” moduli space (in this the space of static $CP^1$ solutions), introducing a potential and a perturbed metric (in \[15\] but not \[14\]) to account for the new interactions, which are assumed to be weak. There seems little hope of perturbing the singular flat-space metric such that rotations become possible, but the problem does not arise on the sphere.

The geodesic approximation could be used to investigate the interaction of two lumps moving on $S^2$. Right angle scattering in head on collisions emerges naturally from the geodesic approximation of many flat-space
models as a consequence of the classical indistinguishability of topological solitons. It would be interesting to see if there is some analogous behaviour on the sphere. However, evaluating the two-lump metric could be difficult since the action of the isometry group on the charge-2 moduli space is far less accessible than in the present case. Even in flat space, the scattering problem is sufficiently complicated to require considerable numerical effort.

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