Idealization of Ganster–Reilly decomposition theorems

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Abstract

In 1990, Ganster and Reilly [6] proved that a function \( f: (X, \tau) \to (Y, \sigma) \) is continuous if and only if it is precontinuous and \( LC \)-continuous. In this paper we extend their decomposition of continuity in terms of ideals. We show that a function \( f: (X, \tau, I) \to (Y, \sigma) \) is continuous if and only if it is \( I \)-continuous and \( I-LC \)-continuous. We also provide a decomposition of \( I \)-continuity.

1 Introduction to topological ideals

In [3, 7, 8], Ganster and Reilly gave several new decompositions of continuity.

Let \( A \) be a subset of a topological space \( (X, \tau) \). Following Kronheimer [12], we call the interior of the closure of \( A \), denoted by \( A^+ \), the consolidation of \( A \). Sets included in their consolidation are called preopen or locally dense [3]. If \( A \) is the intersection of an open and a closed (resp. regular closed) set, then \( A \) is called locally closed (resp. \( A \)-set [18]). A function \( f: (X, \tau) \to (Y, \sigma) \) is called precontinuous (resp. \( LC \)-continuous [5], \( A \)-continuous [18]) if the preimage of every open set is preopen (resp. locally closed, \( A \)-set). The following theorem is due to Ganster and Reilly [3, Theorem 4 (iv) and (v)].
Theorem 1.1 [4] For a function $f: (X, \tau) \to (Y, \sigma)$ the following conditions are equivalent:

1. $f$ is continuous.
2. $f$ is precontinuous and $\mathcal{A}$-continuous.
3. $f$ is precontinuous and $\text{LC}$-continuous.

The aim of this paper is to present an idealized version of the Ganster–Reilly decomposition theorem.

A nonempty collection $\mathcal{I}$ of subsets on a topological space $(X, \tau)$ is called an ideal on $X$ if it satisfies the following two conditions:

1. If $A \in \mathcal{I}$ and $B \subseteq A$ (heredity).
2. If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$ (finite additivity).

A $\sigma$-ideal on a topological space $(X, \tau)$ is an ideal which satisfies:

3. If $\{A_i: i = 1, 2, 3, \ldots\} \subseteq \mathcal{I}$, then $\bigcup \{A_i: i = 1, 2, 3, \ldots\} \in \mathcal{I}$ (countable additivity).

If $X \not\in \mathcal{I}$, then $\mathcal{I}$ is called a proper ideal. The collection of the complements of all elements of a proper ideal is a filter, hence proper ideals are sometimes called dual filters.

The following collections form important ideals on a topological space $(X, \tau)$: the ideal of all finite sets $\mathcal{F}$, the $\sigma$-ideal of all countable sets $\mathcal{C}$, the ideal of all closed and discrete sets $\mathcal{CD}$, the ideal of all nowhere dense sets $\mathcal{N}$, the $\sigma$-ideal of all meager sets $\mathcal{M}$, the ideal of all scattered sets $\mathcal{S}$ (here $X$ must be $T_0$) and the $\sigma$-ideal of all Lebesgue null sets $\mathcal{L}$ (here $X$ is the real line).

An ideal topological space is a topological space $(X, \tau)$ with an ideal $\mathcal{I}$ on $X$ and is denoted by $(X, \tau, \mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}) = \{x \in X: \text{for every } U \in \tau(x), U \cap A \not\in \mathcal{I}\}$ is called the local function of $A$ with respect to $\mathcal{I}$ and $\tau$ [10, 13]. We simply write $A^*$ instead of $A^*(\mathcal{I})$ in case there is no chance for confusion. Note that often $X^*$ is a proper subset of $X$. The hypothesis $X = X^*$ was used by Hayashi in [4], while the hypothesis $\tau \cap \mathcal{I} = \emptyset$ was used by Samuels in [7]. In fact, those two conditions are equivalent [10, Theorem 6.1] and we call the ideal topological spaces which satisfy this hypothesis Hayashi-Samuels spaces. Note that $(X, \tau, \{\emptyset\})$ and $(X, \tau, \mathcal{N})$ are always Hayashi-Samuels spaces; also $(\mathbb{R}, \tau, \mathcal{F})$ is a Hayashi-Samuels space, where $\tau$ denotes the usual topology on the real line $\mathbb{R}$. 

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For every ideal topological space \((X, \tau, I)\), there exists a topology \(\tau^*(I)\), finer than \(\tau\), generated by the base \(\beta(I, \tau) = \{U \setminus I : U \in \tau \text{ and } I \in I\}\). In general, \(\beta(I, \tau)\) is not always a topology \([1]\). When there is no chance for confusion, \(\tau^*(I)\) is denoted by \(\tau^*\). Observe additionally that \(\text{Cl}^*(A) = A \cup A^*\) defines a Kuratowski closure operator for (the same topology) \(\tau^*(I)\).

Recall that \(A \subseteq (X, \tau, I)\) is called \(*\)-dense-in-itself \([9]\) (resp. \(*\)-closed \([10]\), \(*\)-perfect \([9]\)) if \(A \subseteq A^*\) (resp. \(A^* \subseteq A, A = A^*\)).

It is interesting to note that \(A^*(I)\) is a generalization of closure points, \(\omega\)-accumulation points and condensation points. Recall that the set of all \(\omega\)-accumulation points of subset \(A\) of a topological space \((X, \tau)\) is \(A^\omega = \{x \in X : U \cap A \text{ is infinite for every } U \in \mathcal{N}(x)\}\). The set of all condensation points of \(A\) is \(A^k = \{x \in X : U \cap A \text{ is uncountable for every } U \in \mathcal{N}(x)\}\). It is easily seen that \(\text{Cl}(A) = A^*(\emptyset)\), \(A^\omega = A^*(\mathcal{F})\) and \(A^k = A^*(\mathcal{C})\). Note here that in \(T_1\)-spaces the concepts of \(\omega\)-accumulation points and limit points coincide.

In 1990, D. Janković and T.R. Hamlett introduced the notion of \(I\)-open sets in ideal topological spaces. Given an ideal topological space \((X, \tau, I)\) and \(A \subseteq X\), \(A\) is said to be \(I\)-open \([1]\) if \(A \subseteq \text{Int}(A^*)\). We denote by \(IO(X, \tau, I) = \{A \subseteq X : A \subseteq \text{Int}(A^*)\}\) or simply write \(IO(X, \tau)\) or \(IO(X)\) when there is no chance for confusion with the ideal. A subset \(F \subseteq (X, \tau, I)\) is called \(I\)-closed if its complement is \(I\)-open. Note that \(X\) need not be an \(I\)-open subset. Thus, not only are \(I\)-open and \(\tau^*\)-open sets are different concepts, but the former do not give a topology. In the extreme case when \(I\) is the maximal ideal of all subsets of \(X\), only the void subset is \(I\)-open.

A function \(f : (X, \tau, J_1) \rightarrow (Y, \sigma, J_2)\) is said to be \(I\)-continuous (resp. \(I\)-open, \(I\)-closed) if for every \(V \in \sigma\) (resp. \(U \in \tau\), \(U\) closed in \(X\)), \(f^{-1}(V) \in IO(X, \tau)\) (resp. \(f(U) \in IO(X, \tau)\), \(f(U)\) is \(I\)-closed). The definitions are due to Monsef et al. \([1]\).

In \([15]\), a topology \(\tau^\alpha\) has been introduced by defining its open sets to be the \(\alpha\)-sets, that is the sets \(A \subseteq X\) with \(A \subseteq \text{Int}(\text{Cl}(	ext{Int}(A)))\). Observe that \(\tau^\alpha = \tau^*(\mathcal{N})\).

2 Pre-\(I\)-open sets
Definition 1 A subset of an ideal topological space \((X, \tau, \mathcal{I})\) is called pre-\(\mathcal{I}\)-open if \(A \subseteq \text{Int}(\text{Cl}^*(A))\).

We denote by \(\text{PIO}(X, \tau, \mathcal{I})\) the family of all pre-\(\mathcal{I}\)-open subsets of \((X, \tau, \mathcal{I})\) or simply write \(\text{PIO}(X, \tau)\) or \(\text{PIO}(X)\) when there is no chance for confusion with the ideal. We call a subset \(A \subseteq (X, \tau, \mathcal{I})\) pre-\(\mathcal{I}\)-closed if its complement is pre-\(\mathcal{I}\)-open.

Although \(\mathcal{I}\)-openness and openness are independent concepts [1, Examples 2.1 and 2.2], pre-\(\mathcal{I}\)-openness is related to both of them as the following two results show.

Proposition 2.1 Every \(\mathcal{I}\)-open set is pre-\(\mathcal{I}\)-open.

Proof. Let \((X, \tau, \mathcal{I})\) be an ideal topological space and let \(A \subseteq X\) be \(\mathcal{I}\)-open. Then \(A \subseteq \text{Int}(A^*) \subseteq \text{Int}(A^* \cup A) = \text{Int}(\text{Cl}^*(A)).\) \(\square\)

Proposition 2.2 Every open set is pre-\(\mathcal{I}\)-open.

Proof. Let \(A \subseteq (X, \tau, \mathcal{I})\) be open. Then \(A \subseteq \text{Int}A \subseteq \text{Int}(A^* \cup A) = \text{Int}(\text{Cl}^*(A)).\) \(\square\)

The converse in the proposition above is not necessarily true as shown by the following two examples.

Example 2.3 A pre-\(\mathcal{I}\)-open set, even an open set, need not be \(\mathcal{I}\)-open. Let \(X = \{a, b, c, d\},\) \(\tau = \{\emptyset, \{a, c\}, \{d\}, \{a, c, d\}, X\}\) and \(\mathcal{I} = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}\). Set \(A = \{a, c, d\}\). Then \(A \in \tau\) and hence \(A \in \text{PIO}(X)\) but \(A \notin \text{IO}(X)\) [1, Example 2.2].

Example 2.4 Let \((X, \tau)\) be the real line with the usual topology and let \(\mathcal{F}\) be as mentioned before the ideal of all finite subsets of \(X\). Let \(\mathbb{Q}\) be the set of all rationals. Since \(\mathbb{Q}^*(\mathcal{F}) = X\), then \(\mathbb{Q}\) is pre-\(\mathcal{I}\)-open (even \(\mathcal{I}\)-open). But clearly \(\mathbb{Q} \notin \tau\).

Our next two results together with Proposition 2.1 and Proposition 2.2 shows that the class of pre-\(\mathcal{I}\)-open sets is properly placed between the classes of \(\mathcal{I}\)-open and preopen sets as well as between the classes of open and preopen sets.
Proposition 2.5 Every pre-$\mathcal{I}$-open set is preopen.

Proof. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let $A \in PIO(X)$. Then $A \subseteq \text{Int}(\text{Cl}^*(A)) = \text{Int}(A^* \cup A) \subseteq \text{Int}(\text{Cl}(A) \cup A) = \text{Int}(\text{Cl})(A)$. \qed

Example 2.6 A preopen set need not be pre-$\mathcal{I}$-open. Every singleton (for example) in an indiscrete topological space with cardinality at least two is preopen but if we set $\mathcal{I}$ to be the maximal ideal, i.e., $\mathcal{I} = \mathcal{P}(X)$, then it is easy to see that none of the singletons is pre-$\mathcal{I}$-open.

Proposition 2.7 For an ideal topological space $(X, \tau, \mathcal{I})$ and $A \subseteq X$ we have:

(i) If $\mathcal{I} = \emptyset$, then $A$ is pre-$\mathcal{I}$-open if and only if $A$ is preopen.

(ii) If $\mathcal{I} = \mathcal{P}(X)$, then $A$ is pre-$\mathcal{I}$-open if and only if $A \in \tau$.

(iii) If $\mathcal{I} = \mathcal{N}$, then $A$ is pre-$\mathcal{I}$-open if and only if $A$ is preopen.

Proof. (i) Necessity is shown in Proposition 2.3. For sufficiency note that in the case of the minimal ideal $A^* = \text{Cl}(A)$.

(ii) Necessity: If $A \in PIO(X)$, then $A \subseteq \text{Int}(\text{Cl}^*(A)) = \text{Int}(A^* \cup A) = \text{Int}(A \cup \emptyset) = \text{Int}$. Sufficiency is given in Proposition 2.2.

(iii) By Proposition 2.3 we need to show only sufficiency. Note that the local function of $A$ with respect to $\mathcal{N}$ and $\tau$ can be given explicitly \[19\]. We have:

$$A^*(\mathcal{N}) = \text{Cl}(\text{Int}(\text{Cl}(A))).$$

Thus $A$ is pre-$\mathcal{I}$-open if and only if $A \subseteq \text{Int}(A \cup \text{Cl}(\text{Int}(\text{Cl}(A))))$. Assume that $A$ is preopen. Since always $\text{Int}(\text{Cl}(A)) \subseteq A \cup \text{Cl}(\text{Int}(\text{Cl}(A)))$, then $A \subseteq \text{Int}(A \cup \text{Cl}(\text{Int}(\text{Cl}(A)))) = \text{Int}(A \cup A^*(\mathcal{N})) = \text{Int}(\text{Cl}^*(A))$ or equivalently $A$ is pre-$\mathcal{I}$-open. \quad \Box

The intersection of even two pre-$\mathcal{I}$-open sets need not be pre-$\mathcal{I}$-open as shown in the following example.

Example 2.8 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Set $A = \{a, c\}$ and $B = \{b, c\}$. Since $A^* = B^* = X$, then both $A$ and $B$ are pre-$\mathcal{I}$-open. But on the other hand $A \cap B = \{c\} \notin PIO(X)$.
Lemma 2.9 \[\text{[10, Theorem 2.3 (g)]}\] Let \((X, \tau, \mathcal{I})\) be an ideal topological space and let \(A \subseteq X\). Then \(U \in \tau \Rightarrow U \cap A^* = U \cap (U \cap A)^* \subseteq (U \cap A)^*\). \(\Box\)

Proposition 2.10 Let \((X, \tau, \mathcal{I})\) be an ideal topological space with \(\Delta\) an arbitrary index set. Then:

(i) If \(\{A_\alpha : \alpha \in \Delta\} \subseteq \text{PIO}(X)\), then \(\bigcup\{A_\alpha : \alpha \in \Delta\} \in \text{PIO}(X)\).

(ii) If \(A \in \text{PIO}(X)\) and \(U \in \tau\), then \(A \cap U \in \text{PIO}(X)\).

(iii) If \(A \in \text{PIO}(X)\) and \(B \in \text{SO}(X)\), then \(A \cap B \in \text{SO}(A)\).

(iv) If \(A \in \text{PIO}(X)\) and \(B \in \text{SO}(X)\), then \(A \cap B \in \text{PO}(B)\).

Proof. (i) Since \(\{A_\alpha : \alpha \in \Delta\} \subseteq \text{PIO}(X)\), then \(A_\alpha \subseteq \text{Int}(\text{Cl}^*(A_\alpha))\) for every \(\alpha \in \Delta\). Thus \(\bigcup_{\alpha \in \Delta} A_\alpha \subseteq \bigcup_{\alpha \in \Delta} \text{Int}(\text{Cl}^*(A_\alpha)) \subseteq \text{Int}(\bigcup_{\alpha \in \Delta} \text{Cl}^*(A_\alpha)) = \text{Int}(\bigcup_{\alpha \in \Delta} A_\alpha^*) \cup \bigcup_{\alpha \in \Delta} A_\alpha \subseteq \text{Int}(\bigcup_{\alpha \in \Delta} A_\alpha) \cap \text{Int}(U) = \text{Int}((A \cap U) \cap U) = \text{Int}((A \cap U) \cap U) \subseteq \text{Int}((A \cap U)^* \cup (A \cap U)) = \text{Int}(\text{Cl}^*(A \cap U))\).

(ii) By assumption \(A \subseteq \text{Int}(\text{Cl}^*(A))\) and \(U \subseteq \text{Int}(U)\). Thus applying Lemma 2.9, \(A \cap U \subseteq \text{Int}(\text{Cl}^*(A)) \cap \text{Int}(U) \subseteq \text{Int}(\text{Cl}^*(A) \cap U) = \text{Int}((A \cap U) \cap U) \subseteq \text{Int}((A \cap U)^* \cup (A \cap U)) = \text{Int}(\text{Cl}^*(A \cap U))\).

(iii) Since the intersection of a preopen set and an \(\alpha\)-set is always a preopen set, then the claim is clear due to Proposition 2.5.

(iv) and (v) It is proved in \[\text{[10]}\] that the intersection of a preopen and a semi-open set is a preopen subset of the semi-open set and a semi-open subset of the preopen set. Thus the claim follows from Proposition 2.5. \(\Box\)

Corollary 2.11 (i) The intersection of an arbitrary family of pre-\(\mathcal{I}\)-closed sets is a pre-\(\mathcal{I}\)-closed set.

(ii) The union of a pre-\(\mathcal{I}\)-closed set and a closed set is pre-\(\mathcal{I}\)-closed. \(\Box\)

Recall that \((X, \tau)\) is called submaximal if every dense subset of \(X\) is open.

Lemma 2.12 \[\text{[14, Lemma 5]}\] If \((X, \tau)\) is submaximal, then \(\text{PO}(X, \tau) = \tau\). \(\Box\)
Corollary 2.13 If \((X, \tau)\) is submaximal, then for any ideal \(\mathcal{I}\) on \(X\), \(\tau = PIO(X)\). \(\square\)

Remark 2.14 By Proposition 2.10, the intersection of a pre-\(\mathcal{I}\)-open set and an open set is pre-\(\mathcal{I}\)-open. However, the intersection of a pre-\(\mathcal{I}\)-open set and an \(\mathcal{I}\)-open set is not necessarily pre-\(\mathcal{I}\)-open, since in Example 2.8 \(\{c\} = A \cap B\) is not pre-\(\mathcal{I}\)-open, although \(A\) is pre-\(\mathcal{I}\)-open (even \(\mathcal{I}\)-open) and \(B\) is \(\mathcal{I}\)-open.

Remark 2.15 (i) In an ideal topological space \((X, \tau, \mathcal{I})\), the subset \(X\) need not always be \(\mathcal{I}\)-open. However, \(X\) is always pre-\(\mathcal{I}\)-open.

(ii) If \(A \subseteq (X, \tau, \mathcal{I})\) is \(\ast\)-perfect, then \(A \in \tau\) if and only if \(A \in IO(X)\) if and only if \(A \in P IO(X)\).

Problem. The class of ideal topological spaces \((X, \tau, \mathcal{I})\) with \(PIO(X, \tau, \mathcal{I}) \subseteq \tau^*(\mathcal{I})\) is probably of some interest. Call these spaces \(\mathcal{I}\)-strongly irresolvable. It is not difficult to observe that in the trivial case \(\mathcal{I} = \{\emptyset\}\), we have the class of strongly irresolvable spaces which were introduced in 1991 by Foran and Liebnitz [4]. Note also that in the case of the maximal ideal \(\mathcal{P}(X)\), every ideal topological space is \(\mathcal{P}(X)\)-strongly irresolvable. It is the author’s belief that further study of \(\mathcal{I}\)-strongly irresolvable spaces is worthwhile.

3 A decomposition of \(\mathcal{I}\)-continuity

Definition 2 A function \(f: (X, \tau, \mathcal{I}) \to (Y, \sigma)\) is called pre-\(\mathcal{I}\)-continuous if for every \(V \in \sigma\), \(f^{-1}(V) \in P IO(X, \tau)\).

In the notion of Proposition 2.2 we have the following result:

Proposition 3.1 Every continuous function \(f: (X, \tau, \mathcal{I}) \to (Y, \sigma)\) is pre-\(\mathcal{I}\)-continuous. \(\square\)

The converse is not true in general as shown in the following example.
Example 3.2 Consider first the classical Dirichlet function \( f : \mathbb{R} \to \mathbb{R} \):

\[
f(x) = \begin{cases} 
1, & x \in \mathbb{Q} \\
0, & \text{otherwise.}
\end{cases}
\]

Let \( \mathcal{F} \) be the ideal of all finite subsets of \( \mathbb{R} \). The Dirichlet function \( f : (\mathbb{R}, \tau, \mathcal{F}) \to (\mathbb{R}, \tau) \) is pre-\( \mathcal{I} \)-continuous, since every point of \( \mathbb{R} \) belongs to the local function of the rationals with respect to \( \mathcal{F} \) and \( \tau \) as well as to the local function of the irrationals. Hence \( f \) is even \( \mathcal{I} \)-continuous. But on the other hand the Dirichlet function is not continuous at any point of its domain.

Due to Proposition 2.1 we have the next result:

**Proposition 3.3** Every \( \mathcal{I} \)-continuous function \( f : (X, \tau, \mathcal{I}) \to (Y, \sigma) \) is pre-\( \mathcal{I} \)-continuous. \( \square \)

The reverse is again not true as the following example shows.

**Example 3.4** Let \( (X, \tau, \mathcal{I}) \) be the space from Example 2.3 and let \( \sigma = \{\emptyset, \{a, c, d\}, X\} \). Then the identity function \( f : (X, \tau, \mathcal{I}) \to (X, \sigma) \) is pre-\( \mathcal{I} \)-continuous but not \( \mathcal{I} \)-continuous.

From Proposition 2.3 we have:

**Proposition 3.5** Every pre-\( \mathcal{I} \)-continuous function \( f : (X, \tau, \mathcal{I}) \to (Y, \sigma) \) is precontinuous. \( \square \)

**Example 3.6** A precontinuous function need not be pre-\( \mathcal{I} \)-continuous. Let \( (X, \tau) \) be the real line with the indiscrete topology and \( (Y, \sigma) \) the real line with the usual topology. The identity function \( f : (X, \tau, \mathcal{P}(X)) \to (Y, \sigma) \) is precontinuous but not pre-\( \mathcal{I} \)-continuous.

**Proposition 3.7** For a function \( f : (X, \tau, \mathcal{I}) \to (Y, \sigma) \) the following conditions are equivalent:

1. \( f \) is pre-\( \mathcal{I} \)-continuous.
2. For each \( x \in X \) and each \( V \in \sigma \) containing \( f(x) \), there exists \( W \in \operatorname{PIO}(X) \) containing \( x \) such that \( f(W) \subseteq V \).
3. For each \( x \in X \) and each \( V \in \sigma \) containing \( f(x) \), \( \operatorname{Cl}^*(f^{-1}(V)) \) is a neighborhood of \( x \).
4. The inverse image of each closed set in \( (Y, \sigma) \) is pre-\( \mathcal{I} \)-closed.
Proof. (1) ⇒ (2) Let \( x \in X \) and let \( V \in \sigma \) such that \( f(x) \in V \). Set \( W = f^{-1}(V) \). By (1), \( W \) is pre-\( \mathcal{I} \)-open and clearly \( x \in W \) and \( f(W) \subseteq V \).

(2) ⇒ (3) Since \( V \in \sigma \) and \( f(x) \in V \), then by (2) there exists \( W \in PIO(X) \) containing \( x \) such that \( f(W) \subseteq V \). Thus, \( x \in W \subseteq \text{Int}(\text{Cl}^*(W)) \subseteq \text{Int}(\text{Cl}^*(f^{-1}(V))) \subseteq \text{Cl}^*(f^{-1}(V)) \).

Hence, \( \text{Cl}^*(f^{-1}(V)) \) is a neighborhood of \( x \).

(3) ⇒ (1) and (1) ⇔ (4) are obvious. □

The composition of two pre-\( \mathcal{I} \)-continuous functions need not be always pre-\( \mathcal{I} \)-continuous as the following example shows.

Example 3.8 Let \( \mathbb{R} \) be again the real line and \( \tau \) the usual topology. Note that the identity function \( g: (\mathbb{R}, \tau, P(X)) \to (\mathbb{R}, \tau, F) \) is pre-\( \mathcal{I} \)-continuous and also the Dirichlet function \( f: (\mathbb{R}, \tau, F) \to (\mathbb{R}, \tau) \) is pre-\( \mathcal{I} \)-continuous (Example 3.2). But their composition \( (f \circ g): (\mathbb{R}, \tau, P(X)) \to (\mathbb{R}, \sigma) \) is not pre-\( \mathcal{I} \)-continuous, since (for example) \( f^{-1}\{(0, 2)\} = \mathbb{Q} \not\in PIO(\mathbb{R}, \tau, P(X)) \).

However the following result holds.

Proposition 3.9 Let \( f: (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J}) \) and \( g: (Y, \sigma, \mathcal{J}) \to (Z, \upsilon) \) be two functions, where \( \mathcal{I} \) and \( \mathcal{J} \) are ideals on \( X \) and \( Y \) respectively. Then:

(i) \( g \circ f \) is pre-\( \mathcal{I} \)-continuous, if \( f \) is pre-\( \mathcal{I} \)-continuous and \( g \) is continuous.

(ii) \( g \circ f \) is precontinuous, if \( g \) is continuous and \( f \) is pre-\( \mathcal{I} \)-continuous.

Proof. Obvious. □

Hayashi [9] defined a set \( A \) to be \( \ast \)-dense-in-itself if \( A \subseteq A^*(\mathcal{I}) \). We say that a function \( f: (X, \tau, \mathcal{I}) \to (Y, \sigma) \) is \( \ast \)-\( \mathcal{I} \)-continuous if the preimage of every open set in \( (Y, \sigma) \) is \( \ast \)-dense-in-itself in \( (X, \tau, \mathcal{I}) \). In what follows, we try to decompose \( \mathcal{I} \)-continuity but before that we will give a decomposition of \( \mathcal{I} \)-openness. Our next two examples (the ones after Proposition 3.10 and Proposition 3.11) will show that pre-\( \mathcal{I} \)-continuity and \( \ast \)-\( \mathcal{I} \)-continuity are independent concepts.
Proposition 3.10 For a subset $A \subseteq (X, \tau, I)$ the following conditions are equivalent:

1. $A$ is $I$-open.
2. $A$ is pre-$I$-open and $\star$-dense-in-itself.

Proof. (1) By Proposition 2.1, every $I$-open set is pre-$I$-open. On the other hand $A \subseteq \text{Int}(A^*) \subset A^*$, which shows that $A$ is $\star$-dense-in-itself.

(2) $\Rightarrow$ (1) By assumption $A \subseteq \text{Int}(\text{Cl}^*(A)) = \text{Int}(A^* \cup A) = \text{Int}(A^*)$ or equivalently $A$ is $I$-open. ✷

Thus we have the following decomposition of $I$-continuity:

Theorem 3.11 For a function $f: (X, \tau, I) \to (Y, \sigma)$ the following conditions are equivalent:

1. $f$ is $I$-continuous.
2. $f$ is pre-$I$-continuous and $\star$-$I$-continuous. ✷

Example 3.12 The identity function $f: (\mathbb{R}, \tau, \mathcal{P}(X)) \to (\mathbb{R}, \tau)$, where $\tau$ stands for the usual topology on the real line is pre-$I$-continuous as mentioned in Example 3.8 but not $\star$-$I$-continuous, since the local function of every subset of $\mathbb{R}$ with respect to $\mathcal{P}(X)$ and $\tau$ coincides with the void set.

Example 3.13 Note that in the case of the minimal ideal every function is $\star$-$I$-continuous, since the local function of every set coincides with its closure. But since not every function is precontinuous, then $\star$-$I$-continuity does not always imply pre-$I$-continuity.

Remark 3.14 Of course a very appropriate example would be the construction of a space with a fixed ideal on it and finding topologies on the space such that certain functions would show the independence of pre-$I$-continuity and $\star$-$I$-continuity as well as the fact that they are both weaker than $I$-continuity. Such an example is the following: Let $X = \{a, b, c\}$, $\mathcal{I} = \emptyset, \{c\}$, $\mathcal{T} = \emptyset, \{b\}, X$, $\mathcal{S} = \emptyset, \{c\}, X$, $\nu = \emptyset, \{a\}, X$. The identity function $f: (X, \tau, I) \to (X, \nu, I)$ is $\star$-$I$-continuous but neither $I$-continuous nor pre-$I$-continuous. On the other hand the identity function $g: (X, \sigma, I) \to (X, \sigma, I)$ is pre-$I$-continuous but neither $I$-continuous nor $\star$-$I$-continuous.
In the case when $N$ is the ideal of all nowhere dense subsets precontinuity coincides with pre-$\mathcal{I}$-continuity, while $\beta$-continuity is equivalent to $\star\mathcal{I}$-continuity due to Proposition 2.7. Recall that a function $f: (X, \tau) \to (Y, \sigma)$ is called $\beta$-continuous (or sometimes semi-precontinuous) if the preimage of every open set in $(Y, \sigma)$ is $\beta$-open in $(X, \tau)$, where a set $A$ is called $\beta$-open if $A \subseteq \text{Cl}(\text{Int}(\text{Cl}(A)))$. It is clear, since every preopen set is $\beta$-open but not vice versa, that the family of all pre-$\mathcal{I}$-open subsets of an ideal topological space $(X, \tau, \mathcal{I})$ is a proper subset of the family of all $\beta$-open sets.

Consider next the ideal of all meager subsets. Recall that a set is meager if it is a countable union of nowhere dense sets. Meager sets are called often sets of first category. If a set is not meager it is said to be of second category. The points of second category of $A$ are the points of $A^*(\mathcal{M})$. In 1922 Blumberg called a point $x$ of a space $(X, \tau)$ inexhaustibly approached by $A \subseteq X$ if $x \in A^*(\mathcal{M})$. If we call the set $A$ inexhaustibly approached when every point of $A$ is inexhaustibly approached by $A$, then clearly a function is $\star\mathcal{M}$-continuous if and only if the inverse image of every open set is inexhaustibly approached.

4 Idealized Ganster–Reilly decomposition theorem

A subset $A$ of an ideal topological space $(X, \tau, \mathcal{I})$ is called $\mathcal{I}$-locally closed if $A = U \cap V$, where $U \in \tau$ is $\star$-perfect. Note that in the case of the minimal ideal, $\mathcal{I}$-locally closed is equivalent to locally closed, while $N$-locally closed is equivalent to the Tong’s notion of an $\mathcal{A}$-set from [18].

Proposition 4.1 For a subset $A \subseteq (X, \tau, \mathcal{I})$ of a Hayashi-Samuels space the following conditions are equivalent:

1. $A$ is open.
2. $A$ is pre-$\mathcal{I}$-open and $\mathcal{I}$-LC-continuous.

Proof. (1) $\Rightarrow$ (2) The first part is Proposition 22. For the second part, note that $A = A \cap X$, where $A \in \tau$ and $X$ is $\star$-perfect.

(2) $\Rightarrow$ (1) By assumption $A \subseteq \text{Int}(\text{Cl}^*(A)) = \text{Int}(\text{Cl}^*(U \cap V))$, where $U \in \tau$ and $V$ is $\star$-perfect. Hence, $A = U \cap A \subseteq U \cap (\text{Int}(\text{Cl}^*(U)) \cap \text{Int}(\text{Cl}^*(V))) = U \cap \text{Int}(V \cup V^*) = \text{Int}(U) \cap \text{Int}(V) = \text{Int}(U \cap V) = \text{Int}(A)$. This is shows that $A \in \tau$. $\square$
**Definition 3** A function \( f : (X, \tau, \mathcal{I}) \to (Y, \sigma) \) is called \( \mathcal{I}\text{-LC-continuous} \) if for every \( V \in \sigma \), \( f^{-1}(V) \) is \( \mathcal{I}\text{-LC-closed} \).

**Proposition 4.2** Let \((X, \tau, \mathcal{I})\) be a Hayashi-Samuels space. Then, every continuous function \( f : (X, \tau, \mathcal{I}) \to (Y, \sigma) \) is \( \mathcal{I}\text{-LC-continuous} \). \( \square \)

The converse is not true in general, since in the case of the minial ideal \((X, \tau, \mathcal{I})\) is a Hayashi-Samuels space but (usual) \( LC\text{-continuous} \) functions need not be \( LC\text{-continuous} \) [4].

Now, in the notion of Proposition 4.1, we have the following idealized decomposition of continuity:

**Theorem 4.3** Let \((X, \tau, \mathcal{I})\) be a Hayashi-Samuels space. For a function \( f : (X, \tau, \mathcal{I}) \to (Y, \sigma) \) the following conditions are equivalent:

1. \( f \) is continuous.
2. \( f \) is pre-\( \mathcal{I} \)-continuous and \( \mathcal{I}\text{-LC-continuous} \). \( \square \)

**Remark 4.4** From the particular cases \( \mathcal{I} = \{\emptyset\} \) and \( \mathcal{I} = \mathcal{N} \) in Theorem 4.3 we derive the well-known Ganster–Reilly decomposition Theorem [1].

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