ON THE CONTINUOUS COHOMOLOGY OF
DIFFEOMORPHISM GROUPS

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ABSTRACT. Suppose that $M$ is a connected orientable $n$-dimensional manifol-
don and $m > 2n$. If $H^i(M, \mathbb{R}) = 0$ for $i > 0$, it is proved that for each
$m$ there is a monomorphism $H^m(W_n, O(n)) \to H^m_{\text{cont}}(\text{Diff} M, \mathbb{R})$. If $M$
is closed and oriented, it is proved that for each $m$ there is a monomorphism $H^m(W_n, O(n)) \to H^m_{\text{cont}}(\text{Diff}_+ M, \mathbb{R})$, where $\text{Diff}_+ M$ is a group of preserving
orientation diffeomorphisms of $M$.

1. INTRODUCTION

Let $M$ be a connected orientable $n$-dimensional manifold. Denote by $D$ the
group $\text{Diff} M$ of diffeomorphisms of $M$ and by $D_+$ the group of diffeo-
morphisms of $M$ preserving the orientation of $M$ whenever $M$ is oriented. Later we consider $D$ or $D_+$ as an infinite-dimensional Lie group by [12] and use the calculus of differential
forms and vector fields on infinite-dimensional manifolds developed in this book.

In the present paper we study the continuous cohomology $H^*_{\text{cont}}(D, \mathbb{R})$ and
$H^*_{\text{cont}}(D_+, \mathbb{R})$ of the groups $D$ and $D_+$ with values in the trivial $D$ and $D_+$-module
$\mathbb{R}$. The main known result about the cohomology $H^*_{\text{cont}}(G_+, \mathbb{R})$ is Bott’s theorem
([4]) about $(n+1)$-cocycles on the group $D_+$ for the closed oriented $M$. These
cocycles are obtained from some $(2n+1)$-cocycles of the complex $C^*(W_n, O(n))$
of relative with respect to the group $O(n)$ cochains of the Lie algebra of formal
vector fields $W_n$ and are expressed via integrals using a Riemannian metric on $M$.
Moreover, for $M = S^1$ it is known ([5]) that the cohomology $H^*_{\text{cont}}(D_+, \mathbb{R})$ is a ring
with two generators $\alpha, \beta \in H^2(D_+, \mathbb{R})$ which satisfy the only relation $\beta^2 = 0$.

Next, for brevity, put $H^p(D) = H^p_{\text{cont}}((D, \mathbb{R}), H^p(D_+) = H^p_{\text{cont}}(D_+, \mathbb{R})$, and
$H^p(M) = H^p(M, \mathbb{R})$. Let $H^p(M) = 0$ for $i > 0$. We prove that in this case, for
each $m > 2n$, there is a monomorphism $H^m(W_n, O(n)) \to H^m(D)$. Let $M$
be closed and oriented. We prove that in this case, for each $m > 2n$, there is a
monomorphism $H^m(W_n, O(n)) \to H^{m-n}(D_+)$. In particular, in the last case
Bott’s cocycles define a part of the monomorphism above for $m = 2n + 1$.

The main idea of the proof is the following. Denote by $\Omega^*(M)$ the de Rham
complex of the manifol $M$ and by Vect $M$ the Lie algebra of vector fields on $M$.
First we interpret $H^*(D)$ in terms of some double complex. Then we compare the
cohomology of this double complex with the diagonal cohomology of the double
complex $C^*_\Delta(\text{Vect} M; \Omega^*(M))$ of the Lie algebra Vect $M$ with values in the Vect $M$-
module $\Omega^*(M)$ and use the known facts on this cohomology.

Section 2 contains the main algebraic constructions which will be used after-
wards. Namely, one introduces a double complex $C^*_\Delta(\text{cont} G; K)$ for a topological

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cohomology $H^*(W_n, O(n))$, the space $S(M)$ of frames of infinite order of $M$ and the canonical Gelfand-Kazhdan form with values in $W_n$ on $S(M)$.

In section 3 one proves the theorem showing how to construct the cocycles of the groups $D$ and $D_+$ from cocycles of the complex $H^*(W_n, O(n))$ (Theorem 3.3, corollaries 3.4 and 3.5). Moreover, one recalls the definitions of the diagonal cohomology for the Lie algebra $M$ with values in $\mathbb{R}$ and $\Omega^*(M)$ and the relationship between the cohomology of the double complex $C^*_\Delta(V M; \Omega^*(M))$ and the diagonal cohomology $H^*_\Delta(V M)$.

Section 4 contains the definition of the diagonal complex $\Omega^*_{\Delta}(D \times M)$, the filtration of $\Omega^*_\Delta(D \times M)$ and the corresponding spectral sequence, and the proof of the cohomology isomorphism $H^*(\Omega^*_\Delta(D \times M)^p) = H^*_\Delta(V M; \Omega^*(M))$ (Lemma 4.2). Finally, one obtains the main result on the cohomology $H^*(D)$, when $H^*(M) = 0$ for $i > 0$ (Corollary 4.10), and on the cohomology $H^*(D_+)$, when $M$ is a closed oriented manifold (Corollary 4.11).

Throughout the paper $M$ is a connected $n$-dimensional oriented manifold with countable base of $C^\infty$-class, smooth map means $C^\infty$-map and, for a finite or infinite dimensional manifold $X$, $\Omega^*(X) = (\Omega^p(X))$ means the de Rham complex of $X$.

2. Preliminaries

2.1. A double complex. Let $G$ be a topological group and let $K$ be a left topological $G$-module. Recall (see, for example, [8]) that the standard complex $C^*_c(G, K) = \{ C^p_c(G, K), d \}$ of continuous nonhomogeneous cochains with the differential $d = \{ d^p \}$ is defined as follows: for $p > 0$, $C^p_c(G, K)$ is the space of continuous maps from $G^p$ to $K$, $C^0(G, K) = K$, and, for $c \in C^p_c(G, K)$, we have

\[(d^pc)(g_1, \ldots, g_{p+1}) = g_1c(g_2, \ldots, g_{p+1}) + \sum_{i=1}^p (-1)^{i-1}c(g_1, \ldots, g_ig_{i+1}, \ldots, g_{p+1}) + (-1)^p c(g_1, \ldots, g_p),\]

where $g_1, \ldots, g_{p+1} \in G$. The $p$th cohomology group of this complex is denoted by $H^p_c(G, K)$ and is called the $p$th continuous cohomology group of $G$ with values in $K$.

Let $G$ be a topological group, $K = \{ K^q, d^q \}$ a cochain complex such that each $K^q$ is a left topological $G$-module, and $d^q : K^q \to K^{q+1}$ a $G$-equivariant continuous homomorphism of modules. Later we consider the following construction of a double complex $C^*_c(G, K)$ and some of its applications ([14],[15]).

Let $d^{p,q} : C^p_c(G, K^q) \to C^{p+1}_c(G, K^{q+1})$ be the differential of the standard complex $C^p_c(G, K)$. By the standard way we will consider $C^*_c(G, K)$ as a double complex putting $\delta_1 = \{ \delta_1^{p,q} \}$, where $\delta_1^{p,q} = d^{p,q}$, and defining the second differential $\delta_2 = \{ \delta_2^{p,q} \}$ in the following way: for $c \in C^p_c(G, K^q)$ we put

\[\delta_2^{p,q}c(\cdot) = (-1)^pd^{p,q}c(\cdot).\]

Then $C^*_c(G, K) = \{ C^p_c(G, K^q) \}$ is a cochain complex with respect to the total differential $\delta = \delta_1 + \delta_2$ and the total grading $C^m(G, K) = \oplus_{p+q=m} C^p_c(G, K^q)$. We denote this complex by $C^*_c(G; K)$ and denote by $H^p_c(G; K)$ the $p$th cohomology group of this complex.

Let $K^G$ be the subcomplex of $G$-invariant cochains of $K$. Evidently $K^G \subset C^0(G, K)$ is a subcomplex of the complex $C^*_c(G; K)$ and we have the corresponding cohomology homomorphism $H^*(K^G) \to H^*(G; K)$. This cohomology homomorphism will play an important role in the following constructions.

Next we mainly consider the case when $K = \{ K^q \}$ is a differential graded algebra (briefly DG-algebra) and the differential $d = \{ d^q \}$ is an antiderivation of this algebra.
of degree 1. Then $C^*_{\text{cont}}(G;K)$ is a DG-algebra also and the total differential is an antiderivation of this algebra of degree 1.

Let $f : K' \to K''$ be a $G$-equivariant homomorphism of topological $G$-complexes. It is easy to check that $f$ induces a homomorphism of the corresponding double complexes and, therefore, a homomorphism of complexes $C^*(G;K') \to C^*(G;K'')$.

2.2. The cohomology $H^*(W_n)$. Let $\mathfrak{g}$ be a topological Lie algebra and let $E$ be a left topological $\mathfrak{g}$-module. Recall (see, for example, [8]) that the complex $C^*_{\text{cont}}(\mathfrak{g},E) = \{C^q_{\text{cont}}(\mathfrak{g},E),d^q\}$ of standard continuous cochains of $\mathfrak{g}$ with values in $E$ is defined as follows: $C^q_{\text{cont}}(\mathfrak{g},E)$ is the space of continuous skew-symmetric $q$-forms induced by the product in $\mathfrak{g}$, and the differential $d^q : C^q_{\text{cont}}(\mathfrak{g},E) \to C^{q+1}_{\text{cont}}(\mathfrak{g},E)$ is defined by the following formula:

$$(d^q c)(\xi_1, \ldots, \xi_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i-1} \xi_i c(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{q+1}) + \sum_{i < j} (-1)^{i+j} c([\xi_i, \xi_j], \xi_1, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_{q+1}),$$

where $c \in C^q_{\text{cont}}(\mathfrak{g},E)$, $\xi_1, \ldots, \xi_{q+1} \in \mathfrak{g}$, and, as usual, \(\hat{\xi}\) means that the term $\xi$ is omitted. We denote the cohomology of this complex by $H^*_{\text{cont}}(\mathfrak{g},E) = \{H^p_{\text{cont}}(\mathfrak{g},E)\}$.

If $E$ is an algebra and the action of $\mathfrak{g}$ on $E$ is compatible with this algebra structure, the complex $C^*_{\text{cont}}(\mathfrak{g},E)$ is a graded algebra with respect to the exterior product of forms induced by the product in $E$ and the differential $d = \{d^q\}$ is an antiderivation of the graded algebra $C^*(\mathfrak{g},E)$ of degree 1.

Let $W_n$ be the algebra of formal vector fields in $n$ variables, i.e. the topological vector space of smooth vector fields on $\mathbb{R}^n$ with the bracket induced by the Lie bracket of vector fields on $\mathbb{R}^n$. Consider $\mathbb{R}$ as a trivial $W_n$-module. For brevity, put $C^*(W_n) = C^*_{\text{cont}}(W_n,\mathbb{R})$ and $H^*(W_n) = H^*_{\text{cont}}(W_n,\mathbb{R})$.

Recall some facts about the cohomologies $H^*(W_n)$ and $H^*(W_n,\text{GL}_n(\mathbb{R}))$ ([1], [5], [7]). Consider the complex $C^*_{\text{cont}}(W_n) = \{C^q(W_n),d^q\}$ of standard continuous cochains of $W_n$ with values in the trivial $W_n$-module $\mathbb{R}$. By definition, $C^*(W_n)$ is a DG-algebra and the differential $d$ is an antiderivation of degree 1. For $\xi^i \in \mathbb{R}[[\mathbb{R}^n]]$ and $\xi = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i} \in W_n$, put

$$c^i_{j_1 \ldots j_r}(\xi) = \frac{\partial^r \xi^i}{\partial x^{j_1} \ldots \partial x^{j_r}}(0),$$

where $x^i (i = 1, \ldots, n)$ are the standard coordinates in $\mathbb{R}^n$. By definition, we have $c^i_{j_1 \ldots j_r} \in C^1(W_n)$. Moreover, $c^i_{j_1 \ldots j_r}$ for $r = 0, 1, \ldots$ and $i, j_1 \ldots j_r = 1, \ldots, n$ are generators of the DG-algebra $C^*(W_n)$. Since $d = \{d^q\}$ is an antiderivation of degree 1 of $C^*(W_n)$, it is uniquely determined by the following conditions:

$$dc^i_{j_1 \ldots j_r} = \sum_{a \leq k \leq r} \sum_{s_1 < \ldots < s_k} \sum_{l=1}^n c^i_{l j_1 \ldots j_{s_1} \ldots j_{s_k} \ldots j_r} \wedge c^l_{j_{s_1} \ldots j_{s_k}}.$$  \hspace{1cm} (2.1)

The group $\text{GL}_n(\mathbb{R})$ acts naturally on $C^*(W_n)$ and its Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ as the Lie algebra of vector fields with linear components is a subalgebra of $W_n$. Then we have the complex $C^*(W_n,\text{GL}_n(\mathbb{R}))$ of relative cochains of the Lie algebra $W_n$ with respect to $\text{GL}_n(\mathbb{R})$ and the cohomology $H^*(W_n,\text{GL}_n(\mathbb{R}))$ of this complex. Similarly, we have the complex $C^*(W_n,\text{O}(n))$ of relative cochains of the Lie algebra $W_n$ with respect to the orthogonal group $\text{O}(n) \subset \text{GL}_n(\mathbb{R})$ and the cohomology $H^*(W_n,\text{O}(n))$. 

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Put
\[ \gamma = (c^i_j), \quad \Psi^j_i = \sum_{k=1}^{n} c^i_k \wedge c^k, \quad \text{and} \quad \Psi = (\Psi^j_i). \]

It is known that
\[ \Psi_p = \text{tr}(\Psi_{\cdots \wedge \Psi}) \quad (p = 1, \ldots, n). \]

are cocycles of \( C^*(W_n, \text{GL}_n(\mathbb{R})) \) and the cohomology classes of these cocycles generate \( H^*(W_n, \text{GL}_n(\mathbb{R})) \). The cohomology class of \( \Psi_k \) is called \( k \)-th formal Pontrjagin class.

Put
\[ \gamma_p = \text{tr}(\gamma_{\cdots \wedge \gamma}) \quad (p = 1, \ldots, n). \]

By definition, \( \gamma_p \in C^{2p-1}(W_n) \). Consider the inclusion \( \mathfrak{gl}_n(\mathbb{R}) \subset W_n \). It is known that \( \gamma_p \) as a cochain of the complex \( C^*(\mathfrak{gl}_n(\mathbb{R}), \mathbb{R}) \) is a cocycle and the ring \( H^*(\mathfrak{gl}_n(\mathbb{R}), \mathbb{R}) \) is the exterior algebra of its subspace spanned by the cohomology classes \( \gamma_p \) for \( p = 1, \ldots, n \). Moreover, there is a \((2p-1)\)-cochain \( \Gamma_p \) of the complex \( C^*(W_n) \) such that the restriction of \( \Gamma_p \) to \( \mathfrak{gl}_n(\mathbb{R}) \) equals \( \gamma_p \) and \( d\Gamma_p = \Psi_p \). Consider the DG-subalgebra of DG-algebra \( C^*(W_n) \) generated by \( \Gamma_p \) and \( \Psi_p \) for \( p = 1, \ldots, n \).

Then the inclusion of this subalgebra into \( C^*(W_n) \) induces an isomorphism of the cohomologies. Moreover, the cohomology classes of the cocycles
\[ \Gamma_{p_1} \wedge \cdots \wedge \Gamma_{p_l} \wedge \Psi_{r_1} \wedge \cdots \wedge \Psi_{r_m} \tag{2.2} \]
for \( 1 \leq p_1 < \cdots < p_l \leq n \), \( 1 \leq r_1 \leq \cdots \leq r_m \leq n \), \( p_1 \leq r_1 \), \( r_1 + \cdots + r_m \leq n \), and \( p_1 + r_1 + \cdots + r_m > n \) give a basis of \( H^m(W_n) \) (the so-called Vey basis) as a vector space. This implies that \( H^m(W_n) = 0 \) whenever \( m < 2n + 1 \) or \( m > n(n + 2) \).

2.3. The space of frames of infinite order and the Gelfand-Kazhdan form.
Let \( M \) be a connected orientable \( n \)-dimensional smooth manifold. Denote by \( S(M) \) the space of frames of infinite order of \( M \), i.e. \( \infty \)-jets at 0 of germs at 0 of diffeomorphisms from \( \mathbb{R}^n \) into \( M \). It is known that \( S(M) \) is a manifold with model space \( \mathbb{R}^\infty \) ([1]). Recall that we denote by \( D \) the group of diffeomorphisms \( \text{Diff} M \) of \( M \). We put, for \( g_1, g_2 \in D \), \( g_1 g_2 = g_2 \circ g_1 \). Then the standard action of \( D \) on \( M \) is a right action. Evidently this action of \( D \) is naturally extended to an action of \( D \) on \( S(M) \).

Define the canonical Gelfand-Kazhdan 1-form \( \omega \) with values in \( W_n \) on \( S(M) \) ([6] and [1]). Let \( \tau \) be a tangent vector at \( s \in S(M) \) and let \( s(t) \) be a path on \( S(M) \) such that \( \tau = \frac{d}{dt}(0) \). One can represent \( s(t) \) by a smooth family \( k_t \) of germs at 0 of diffeomorphisms \( \mathbb{R}^n \to M \), i.e. \( s(t) = j_0^\infty k_t \). Then put
\[ \omega(\tau) = -j_0^\infty \frac{d}{dt}(k_0^{-1} \circ k_t)(0). \]

**Theorem 2.4.** ([1], [6], [10]) The form \( \omega \) satisfies the following conditions:
(1) \( \omega \) induces a topological isomorphism between the tangent space \( T_s \) of \( S(M) \) at \( s \in S(M) \) and \( W_n \);
(2) \( d\omega = -\frac{1}{2}[\omega, \omega] \) (the Maurer-Cartan condition);
(3) The form \( \omega \) is \( D \)-invariant.

This theorem and the theorem on the covering isotopy of imbeddings of disks ([2], [18]) implies the following

**Corollary 2.5.** The group \( D \) acts transitively on \( S(M) \).
Let $c \in C^q(W_n)$. For each $s \in S(M)$ and $X_1, \ldots, X_q \in T_s$, put
\[ \omega_c(X_1, \ldots, X_q) = c(\omega(X_1), \ldots, \omega(X_q)). \]

By theorem 2.4, $c \mapsto \omega_c$ is an injective homomorphism of the complexes $\alpha : C(W_n) \to \Omega^*(S(M))$. By theorem 2.4 and corollary 2.5, there is a one-to-one correspondence between the space of $D$-invariant forms on $S(M)$ and the space of continuous skew-symmetric forms on the tangent space $T_s$ at $s \in S(M)$. Then
\[ \alpha(C(W_n)) = \Omega^*(S(M))^D, \]

where $\Omega^*(S(M))^D$ is the subcomplex of $D$-invariant forms from $\Omega^*(S(M))$. Moreover, we have $\alpha(C(W_n, O(n))) = \Omega^*(S(M)/O(n))^D$.

It is easy to check that $\theta_j = -\alpha(c_j^s))$ is a connection form on a principal $\text{GL}_n(\mathbb{R})$-bundle $S(M) \to S(M)/\text{GL}_n(\mathbb{R})$.

Consider a Riemannian metric on $M$ and the corresponding Levi-Civita connection. Denote by $O(M)$ the principal $O(n)$-bundle of orthogonal tangent frames on $M$. For each frame $r \in O(M)$ at $x \in M$, denote by $\sigma(r)$ the infinity-Jet at $x$ of the inverse of the geodesic chart with center at $x$ uniquely determined by $r$. Then we have a smooth map $\sigma : O(M) \to S(M)$.

Denote by $\eta = (\theta^i)$ the canonical 1-form $\theta = (\theta^i)$ with values in $\mathbb{R}^n$ on $O(M)$ and by $\eta = (\eta^i_j)$ the form of the Levi-Civita connection. It is easy to check that, for $\omega^i_j = \alpha(c^s_j)$ and $\omega^i_j = \alpha(c^s_j)$, we have
\[ \sigma^*\omega^i = -\theta^i \quad \text{and} \quad \sigma^*\omega^i_j = -\theta^i_j. \quad (2.3) \]

Let $c^s_i_{j_1 \ldots j_r} \in C^1(W_n)$. For a vector field $X$ on $M$, denote by $\tilde{X}$ the corresponding extension of $X$ to $S(X)$. Put $\omega^i_{j_1 \ldots j_r}(\tilde{X}(s)) = -\frac{\partial^r X^i}{\partial x^{j_1} \ldots \partial x^{j_r}}(0)$, \quad (2.4)

where $x \in M$ and the right hand side is calculated in the coordinates $x^i$ presenting $s \in S(M)$.

2.6. The cohomology $H^*(W_n, O(n))$. Consider the basic cocycles $\gamma_p$ and $\Gamma_p$ so that the restriction of $\gamma$ is a cocycle of $C^*(W_n, O(n))$ and $\Gamma_p \in C^*(W_n, O(n)) ([5], [7])$. In this case we replace such $\gamma_p$ and $\Gamma_p$ by $\lambda_p$ and $\Lambda_p$, respectively.

Consider the $DG$-subalgebra of $DG$-algebra $C^*(W_n, O(n))$ generated by $\Lambda_p$ for odd $p \leq n$ and $\Psi_p$ for $p = 1, \ldots, n$. Then the inclusion of this subalgebra into $C^*(W_n, O(n))$ induces an isomorphism of the cohomologies. Besides, the cohomology classes of cocycles
\[ c_{p_1 \ldots p_l r_1 \ldots r_k} = \Lambda_{p_1} \wedge \ldots \wedge \Lambda_{p_l} \wedge \Psi_{r_1} \wedge \ldots \wedge \Psi_{r_k} \quad (2.5) \]

for odd $p_1$ such that $1 \leq p_1 < \cdots < p_l \leq n$, $1 \leq r_1 \leq \cdots \leq r_k \leq n$, $p_1 \geq r_1, r_1 + \cdots + r_k \leq n$, and $p_1 + r_1 + \cdots + r_k > n$ give a basis of the cohomology $H^*(W_n, O(n))$ as a vector space in dimensions $> 2n$. In particular, for each dimension $m > 2n$ we have $H^m(W_n, O(n)) \subset H^m(W_n)$. Moreover, for $m > n$ the cohomology group $H^m(W_n, O(n))$ may be nontrivial only if $2n + 1 \leq m \leq \frac{n(n+3)}{2}$, when $n$ is even, and $2n + 1 \leq m \leq \frac{n(n+3)}{2}$, when $n$ is odd.

Next we find the explicit expressions for $\Lambda_p$ for odd $p$.

Put $\lambda^i_j = \frac{1}{2}(c^i_j + c^i_j)$, $\lambda = (\lambda^i_j)$, and
\[ \lambda_p = tr(\lambda \wedge \cdots \wedge \lambda) \quad (p = 1, \ldots, n). \]

It is easy to check that the restriction of $\lambda_p$ to $\text{gl}_n(\mathbb{R})$ belongs to $C^*(\text{gl}_n(\mathbb{R}), O(n))$ and $\lambda_p = 0$ for even $p$. 


Let $P$ be a homogeneous $\text{GL}_n(\mathbb{R})$-invariant polynomial of degree $p$ on the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$. We denote by $\tilde{P}$ the symmetric $p$-linear form on $\mathfrak{gl}_n(\mathbb{R})$ such that $\tilde{P}(X,\ldots,X) = P(X)$. Then for $X,X_1,\ldots,X_p \in \mathfrak{gl}_n(\mathbb{R})$ we have
\[
\sum_{i=1}^{p} \tilde{P}(X_1,\ldots,[X,X_i],\ldots,X_p) = 0. \tag{2.6}
\]
If $\omega$ is a linear form with values in $\mathfrak{gl}_n(\mathbb{R})$ and, for $i = 1,\ldots,p$, $\omega_i$ is a $k_i$-linear form with values in $\mathfrak{gl}_n(\mathbb{R})$, (2.6) implies the equality
\[
\sum_{i=1}^{p} (-1)^{k_1+\cdots+k_{i-1}+1} P(\omega_1,\ldots,[\omega,\omega_i],\ldots,\omega_p) = 0. \tag{2.7}
\]
Put $\alpha_j = \frac{1}{2} (c_j^i - c_j^j)$ and $\alpha = (\alpha_j)$. By definition, $\lambda_j, \alpha_j \in C^1(W_n)$. By (2.1), we have
\[
d\lambda_j = \sum_{k=1}^{n} \left( \lambda_k^i \wedge \psi_j^k + \psi_k^i \wedge \lambda_j^k \right) + \frac{1}{2} (\psi_j + \psi_j^i).
\]
This formula could be rewritten as follows:
\[
d\lambda = [\alpha, \lambda] + \frac{1}{2} (\Psi + \Psi^i). \tag{2.8}
\]
Similarly, we have
\[
d\Psi = [\lambda, \Omega] + [\alpha, \Psi], \tag{2.9}
\]
\[
d\Psi^i = -[\lambda, \Psi^i] + [\alpha, \Psi^i]. \tag{2.10}
\]
Let $P$ be a $\mathfrak{gl}_n(\mathbb{R})$-invariant $(a+b+c)$-linear form on $\mathfrak{gl}_n(\mathbb{R})$. Then
\[
P(\lambda,\ldots,\lambda,\Psi,\ldots,\Psi,\Psi^i,\ldots,\Psi^i)
\]
is a cochain of $\text{C}^{a+2b+2c}(W_n, \text{O}(n))$. Since $[\lambda,\lambda,\lambda] = 0$, by (2.7), (2.8), (2.9), and (2.10), we have
\[
dP(\lambda,\ldots,\lambda,\Psi,\ldots,\Psi,\Psi^i,\ldots,\Psi^i)
\]
\[
= \sum_{i=1}^{a} (-1)^{i-1} P(\lambda,\ldots,\lambda,\Psi+\Psi^i,\ldots,\lambda,\Psi,\ldots,\Psi,\Psi^i,\ldots,\Psi^i)
\]
\[
+ (-1)^{a} \sum_{j=1}^{b} P(\lambda,\ldots,\lambda,\Psi,\ldots,[\lambda,\Psi],\ldots,\Psi,\Psi^i,\ldots,\Psi^i)
\]
\[
- (-1)^{a} \sum_{k=1}^{c} P(\lambda,\ldots,\lambda,\Psi,\ldots,\Psi,\Psi^i,\ldots,[\lambda,\Psi^i],\ldots,\Psi^i). \tag{2.11}
\]
Let $Q$ be a homogeneous invariant polynomial of degree $p$ on $\mathfrak{gl}_n(\mathbb{R})$. For brevity, put $Q(X,X') = Q(X,X',\ldots,X')$ and $Q(X,X',X'') = Q(X,X',X'',\ldots,X'')$. Let $Q_p(X) = \text{tr} X^p$. Then we have
\[
\bar{Q}_p(X_1,\ldots,X_p) = \frac{1}{p!} \sum_{\sigma \in S_p} \text{tr} (X_{\sigma(1)} \ldots X_{\sigma(p)}) ,
\]
where $S_p$ is the symmetric group.
For odd $p$, put $\Psi(t) = \frac{1}{2} t \Psi + \frac{1}{2} (t-1) \Psi^i + (t-t^2)[\lambda,\lambda]$. It is evident that
\[
\int_0^1 \frac{d}{dt} Q_p(\Psi(t)) dt = Q_p\left(\frac{1}{2} \Psi \right) + Q_p\left(\frac{1}{2} \Psi^i \right) = \frac{1}{2^{p-1}} \Psi_p.
\]
On the other hand, by (2.8), (2.9), (2.10), (2.7), and (2.11) we have
\[
\int_0^1 \frac{d}{dt} Q_p(\Psi(t)) \, dt = p \int_0^1 \left( Q_p(2^{-1}(\Psi + \Psi'), \Psi(t)) + (1 - 2t)Q_p([\lambda, \chi], \Psi(t)) \right) \, dt
\]
\[
= p \int_0^1 \left( Q_p(2^{-1}(\Psi + \Psi'), \Psi(t)) - 2^{-1}(p-1)(2t-1)Q_p(\lambda, \chi', \Psi(t)) - \frac{(p-1)(t-1)}{2}(2t-1)Q_p(\lambda, \chi', \Psi(t)) \right) \, dt
\]
\[
= p \frac{d}{dt} \int_0^1 Q_p(\lambda, \Psi(t)) \, dt.
\]

It is easy to see that the restriction of \(\int_0^1 Q_p(\lambda, \Psi(t)) \, dt\) to \(C^*(\mathfrak{gl}_n(\mathbb{R}), \mathbb{R})\) equals \(\langle \eta_{p+1}, \ldots, \eta_{2p+1} \rangle \lambda_p\). Therefore, one can put
\[
\Lambda_p = \frac{p! 2^{p-1}}{(p+1) \ldots (2p+1)} \int_0^1 Q_p(\lambda, \Psi(t)) \, dt. \tag{2.12}
\]

### 3. Cocycles on Diffeomorphism Groups

#### 3.1. The definition of cocycles

Later we need the following theorem.

**Theorem 3.2.** Let \(M\) be a connected orientable manifold. Then, in the category of topological vector spaces, for each \(p > 0\) we have the following decomposition
\[
\Omega^p(M) = d\Omega^{p-1}(M) \oplus H^p(M) \oplus \Omega^p(M)/Z^p(M),
\]
where \(Z^p(M)\) is the space of closed \(p\)-forms. If \(H^p(M) = 0\), \(d\Omega^{p-1}(M) = Z^p(M)\) and \(\Omega^p(M)/Z^p(M)\) are Fréchet spaces.

**Proof.** For a compact \(M\) the statement follows from the Hodge decomposition for the identity operator 1 on \(\Omega^p(M)\): \(1 = d \circ \delta \circ G + H^p(M) \oplus \delta \circ d \circ G\) (see, for example, [20]).

For a noncompact \(M\), the statement follows from Palamodov’s theorem ([17], Proposition 5.4).

The action of \(D\) on \(M\) induces the natural right actions of \(D\) on \(S(M)\) and \(S(M)/O(n)\) and the induced left actions on the de Rham complexes \(\Omega^*(S(M))\) and \(\Omega^*(S(M)/O(n))\). Consider the \(D\)-module \(\Omega^*(S(M)/O(n))\) and the corresponding complex \(C^*_\text{cont}(D; \Omega^*(S(M)/O(n)))\). If the manifold \(M\) is oriented, denote by \(D_+\) the subgroup of \(D\) of diffeomorphisms preserving the orientation of \(M\).

Further we consider the complex \(\Omega^*(S(M)/O(n))^D\) as a subcomplex of the complex \(C^*_\text{cont}(D; \Omega^*(S(M)/O(n)))\).

The proof of the next theorem shows how to construct cocycles on the group \(D\) from the cocycles of the complex \(\Omega^*(S(M)/O(n))^D\).

Consider a structure of \(D\)-module on a cohomology group \(H^i(M)\) induced by the action of the group \(D\) on \(\Omega^*(M)\).

**Theorem 3.3.** Let \(\nu\) be an \(m\)-cocycle of the complex \(\Omega^*(S(M)/O(n))^D\) and \(0 \leq p \leq m - 1\). Assume that \(H^{m-p}(M) = \cdots = H^{m-1}(M) = 0\), \(H^{m-p-1}(M) \neq 0\). Then the following statements are true:

1. There are cochains \(\nu_{i,m-i-1} \in C^\text{cont}_{i-1}(D; \Omega^m(S(M)/O(n)))\) \((i = 0, \ldots, p)\) such that \(\nu + \delta(\nu_{0,m-1} + \cdots + \nu_{p,m-p-1}) = \delta \nu_{p,m-p-1}\);
2. \(c(\nu) = \delta \nu_{p+1,m-p-1} \in C^\text{cont}_{p+1}(D; \Omega^{m-p-1}(S(M)/O(n)))\) defines a \((p+1)\)-cocycle of the complex \(C^\text{cont}_{p+1}(D, \Omega^{m-p-1}(S(M)/O(n)))\). The cohomology class \(h(\nu)\) of the cocycle \(c(\nu)\) depends only on the cohomology class of the cocycle \(\nu\) in the complex \(\Omega^*(S(M)/O(n))^D\);
3. The map \(\nu \mapsto h(\nu)\) induces a linear map
\[
H^m(\Omega^*(S(M)/O(n))^D) \to H^p_{\text{cont}}(D; H^{m-p-1}(M)).
\]
Proof. Denote by $R(M)$ the space of $\infty$-jets of of germs of Riemannian metrics at points of $M$. Let $g_x$ be the $\infty$-jet at $x \in M$ of a germ at $x$ of Riemannian metric $g$. For an orthogonal frame $r$ at $x$, denote by $s(r)$ the $\infty$-jet at the center $x$ of the inverse of the geodesic chart defined by $r$ and the Riemannian metric $g$. It is evident that $s(r)$ is uniquely determined by the $\infty$-jet $g_x$. Then the map $r \mapsto s(r)$ induces a diffeomorphism $R(M) \to S(M)/O(n)$. Next we identify $R(M)$ and $S(M)/O(n)$ by this diffeomorphism.

Consider some Riemannian metric $g_0$ on $M$ and the smooth map $\sigma : O(M) \to S(M)$ defined by $g_0$ in 2.3. Let $\bar{\sigma} : M \to S(M)/O(n)$ be the smooth map induced by $\sigma$. Consider a smooth map $F : [0, 1] \times R(M) \to R(M)$ defined by

$$(t, g_x) \mapsto (tg_0 + (1-t)g_x) = t g_{0,x} + (1-t)g_x,$$

where $g_{0,x}$ and $g_x$ are the $\infty$-jets at $x$ of germs at $x$ of Riemannian metrics $g_0$ and $g$.

It is clear that $F$ is a smooth homotopy between the identity map of $R(M)$ and the composition of the projection $p : R(M) \to M$ and the section $\bar{\sigma}$. Then we have the standard homotopy operator $\Phi^p : \Omega^p(S(M)/O(n)) \to \Omega^{p-1}(S(M)/O(n))$ given by

$$\omega \mapsto \int_0^1 i(\frac{d}{dt}) F^* \omega dt,$$

where $i(X)$ denote the inner product by a vector field $X$. In particular, we have $H^*(S(M)/O(n)) = H^*(M)$.

First we indicate some canonical construction of the sequence of cochains

$$\nu_{i,m-i-1} \in C^i_{cont}(D; \Omega^{m-i-1}(S(M)/O(n)))$$

satisfying the conditions of the theorem.

By assumption, we have $\Phi^1(M) = 0$ and $p \geq 0$. Suppose that $m > n$. Then, for the $(m-1)$-form $\nu_{0,m-1} = -H^m(\nu)$, we have $\nu = -\delta_2 \nu_{0,m-1}$. If $m \leq n$, we use theorem 3.2 to get the $(m-1)$-form $\nu_{0,m-1}$ satisfying the same equality $\nu = -\delta_2 \nu_{0,m-1}$. In the both cases one can assume that $\nu_{0,m-1} \in C^0_{cont}(D; \Omega^{m-1}(M))$. Then we have $\nu = \delta_1 \nu_{0,m-1} = \delta_1 \nu_{0,m-1}$ and $\delta_2 \nu_{0,m-1} = -\delta_1 \delta_2 \nu_{0,m-1} = \delta_1 \nu = 0$.

Let $p \geq 1$ and, therefore, $H^{m-1}(M) = 0$. Suppose that $m-1 > n$. Then, for the cochain $\nu_{1,m-2} = \Phi^{m-1}(\nu_{1,m-1})$, we have $\delta_1 \nu_{0,m-1} = -\delta_2 \nu_{1,m-2}$. If $m-1 \leq n$, we use theorem 3.2 to get a cochain $\nu_{1,m-2} \in C^1_{cont}(D; \Omega^{m-2}(M))$ satisfying the same equality $\delta_1 \nu_{0,m-1} = -\delta_2 \nu_{1,m-2}$. Thus, we have $\nu = \delta_1 \nu_{0,m-1} + \nu_{1,m-2} = \delta_1 \nu_{1,m-2}$ and

$$\delta_2 \delta_1 \nu_{1,m-2} = -\delta_1 \delta_2 \nu_{1,m-2} = \delta_1^2 \nu_{0,m-1} = 0.$$

Using the conditions

$$H^{m-2}(S(M)/O(n)) = \cdots = H^{m-p}(S(M)/O(n)) = 0$$

and proceeding in the same way we get for $i = 1, \ldots, p$ the cochains $\nu_{i,m-i-1} \in C^i_{cont}(D; \Omega^{m-i-1}(S(M)/O(n)))$ such that

$$\delta_1 \nu_{i-1,m-i} + \delta_2 \nu_{i,m-i-1} = 0 \quad (3.1)$$

and so

$$\nu = \delta_1 \nu_{0,m-1} + \cdots + \nu_{p,m-p-1} = \delta_1 \nu_{p,m-p-1} \in C^{p+1}(D, \Omega^{n-p-1}(S(M)/O(n))).$$

Moreover, we have

$$\delta_2 \delta_1 \nu_{p,m-p-1} = -\delta_1 \delta_2 \nu_{p,m-p-1} = \delta_1^2 \nu_{p-1,m-p} = 0.$$

Consider $H^{m-p-1}(S(M)/O(n)) = H^{m-p-1}(M)$ as a $D$-module with respect to the natural action of $D$. Then the cochain $\delta_1 \nu_{p,m-p-1}$ defines a $(p+1)$-cocycle $c(\nu)$ on $D$ with values in $H^{m-p-1}(M)$. The cohomology class of $c(\nu)$ is denoted by $h(\nu)$. We claim that the cohomology class $h(\nu)$ depends only on the cohomology class of $\nu$ in the complex $\Omega^*(S(M)/O(n))^D$. 


If we replace the form $\nu$ by a form $\nu + d\nu_1$, where $\nu_1 \in \Omega^{m-1}(S(M)/O(n)) \cap \Omega^*(S(M)/O(n))^D$, one can replace the sequence $\nu_{0,m-1}, \ldots, \nu_{p,m-p-1}$ by the sequence $\nu_{0,m-1} - \nu_1, \nu_{m-2}, \ldots, \nu_{p,m-p-1}$ and obtain the same cochain $\nu_{p,m-p-1}$ at the end.

Consider another sequence $\tilde{\nu}_{0,m-1}, \ldots, \tilde{\nu}_{p,m-p-1}$ ($i = 0, \ldots, p$) such that

$$\nu = -\delta_2 \nu_{0,m-1} \quad \text{and} \quad \delta_1 \tilde{\nu}_{i-1,m-i} + \delta_2 \tilde{\nu}_{i,m-i-1} = 0$$

for $i = 1, \ldots, p$. The same arguments as above show that

$$\tilde{\nu}_{0,m-1} = \nu_{0,m-1} + \delta_2 \sigma_{0,m-2},$$

where $\sigma_{0,m-2} \in C^0(D; \Omega^{m-2}(M))$. If $p = 1$, we have $\delta_1 \tilde{\nu}_{0,m-1} = \delta_1 \nu_{0,m-1} - \delta_2 \delta_1 \sigma_{0,m-2}$ and we are done. If $p > 1$ we have

$$\delta_1 \tilde{\nu}_{0,m-1} = \delta_1 \nu_{0,m-1} - \delta_2 \delta_1 \sigma_{0,m-2} = -\delta_2 (\nu_{1,m-2} + \delta_1 \sigma_{0,m-2}) = -\delta_2 \tilde{\nu}_{1,m-2}.$$

For $i = 1, \ldots, p - 1$ proceeding in the same way we get the cochains $\sigma_{i,m-i-2} \in C^i_{\text{cont}}(D; \Omega^{m-i-2}(S(M)/O(n)))$ such that

$$\tilde{\nu}_{i,m-i-1} = \nu_{i,m-i-1} + \delta_1 \sigma_{i-1,m-i-1} + \delta_2 \sigma_{i,m-i-2}.$$

In particular, we have

$$\nu_{p,m-p-1} = \nu_{p,m-p-1} + \delta_1 \sigma_{p-1,m-p-1} + \delta_2 \sigma_{p,m-p-2}$$

and

$$\delta_1 \nu_{p,m-p-1} = \delta_1 \nu_{p,m-p-1} - \delta_2 \sigma_{p,m-p-2}.$$

Thus, the cocycles $\delta_1 \tilde{\nu}_{p,m-p-1}$ and $\delta_1 \nu_{p,m-p-1}$ define the same cohomology class of $H^{p+1}_c(D, H^{m-p-1}(M))$.

The last statement of the theorem follows from the construction. \qed

Theorem 3.3 implies the following corollaries.

**Corollary 3.4.** Let $H^i(M) = 0$ for $i > 0$. Then, for each nontrivial $m$-cocycle $\nu$ of the complex $C^*(W_n, O(n)) = \Omega^*(S(M)/O(n))^D$, the cocycle $c(\nu)$ is an $m$-cocycle of the complex $C^*(D)$. The map $\nu \mapsto h(\nu)$ induces a linear map

$$H^m(\Omega^*(S(M)/O(n))^D) \to H^m(D).$$

**Corollary 3.5.** Let $M$ be a closed oriented $n$-dimensional manifold and let $D_+$ be the group of diffeomorphisms of $M$ preserving the orientation of $M$. For $m > n$ and each $m$-cocycle $\nu$ of the complex $\Omega^*(S(M)/O(n))^D$, $\int_M c(\nu)$ is a $(m-n)$-cocycle of the complex $C^*(D_+)$ presenting the cohomology class $h(\nu)$. The map $\nu \mapsto h(\nu)$ induces a linear map $H^m(\Omega^*(S(M)/O(n))^D) \to H^{m-n}(D_+)$.

**Proof.** Since $H^i(M) = 0$ for $i > n$ and $H^n(M) = \mathbb{R}$ one could apply theorem 3.3. Evidently, all statements of theorem 3.3 are true if we replace the group $D$ by the group $D_+$. But then $\int_M c(\nu)$ presents the cohomology class $h(\nu)$. \qed

The main aim of the paper is to prove that, for $m > 2n$, the linear maps

$$H^m(W_n, O(n)) = H^m(\Omega^*(S(M)/O(n))^D) \to H^m(D)$$

and

$$H^m(W_n, O(n)) = H^m(\Omega^*(S(M)/O(n))^D) \to H^{m-n}(D_+)$$

induced by the maps $\nu \mapsto h(\nu)$ and $\nu \mapsto \int_M c(\nu)$ from corollaries 3.4 and 3.5 are monomorphisms.
3.6. The diagonal cohomology of the Lie algebra \( \text{Vect} M \). Denote by \( \text{Vect} M \) the Lie algebra of smooth vector fields on \( M \). We consider \( \text{Vect} M \) as a topological vector space with respect to the \( C^\infty \)-topology. Consider the complex \( C^\ast(\text{Vect} M) \) of standard continuous cochains of \( \text{Vect} M \) with values in the trivial \( \text{Vect} M \)-module \( \mathbb{R} \) and its subcomplex formed by chains \( c \in C^\ast(\text{Vect} M) \) such that for \( X_1, \ldots, X_q \in \text{Vect} M \) we have \( c(X_1, \ldots, X_q) = 0 \) whenever \( \cap_{i=1}^q \text{supp} X_i = \emptyset \). This subcomplex is denoted by \( C^\ast_\Delta(\text{Vect} M) \) and is called the diagonal subcomplex of \( C^\ast(\text{Vect} M) \) ([5]). The cohomology of \( C^\ast_\Delta(\text{Vect} M) \) is denoted by \( H^\ast_\Delta(\text{Vect} M) \) and is called the diagonal cohomology of \( \text{Vect} M \) with values in \( \mathbb{R} \).

Consider the left action of the Lie algebra \( \text{Vect} M \) on the de Rham complex \( \Omega^\ast(M) \) by the Lie derivatives: \( \omega \mapsto L_X \omega \), where \( \omega \in \Omega^\ast(M) \), \( X \in \text{Vect} M \), and \( L_X \) is the Lie derivative with respect to the vector field \( X \). Let \( C^\ast(\text{Vect} M, \Omega^\ast(M)) \) be the complex of standard cochains of the topological Lie algebra \( \text{Vect} M \) with values in the \( \text{Vect} M \)-module \( \Omega^\ast(M) \).

A cochain \( c \in C^p(\text{Vect} M, \Omega^p(M)) \) is called diagonal if, for \( X_1, \ldots, X_p \in \text{Vect} M \), the value \( c(X_1, \ldots, X_p) \) at \( x \in M \) depends only on the germs of \( X_1, \ldots, X_p \) at \( x \). By [19], this condition is equivalent to the following one: for \( X_1, \ldots, X_p \in \text{Vect} M \) and \( x \in M \), the value \( c(X_1, \ldots, X_p) \) at \( x \in M \) depends only on \( \infty \)-jets of \( X_1, \ldots, X_p \) at \( x \). Denote by \( C^\ast_\Delta(\text{Vect} M, \Omega^\ast(M)) \) the set of diagonal cochains of the complex \( C^\ast(\text{Vect} M, \Omega^\ast(M)) \). It is easy to see that \( C^\ast_\Delta(\text{Vect} M, \Omega^\ast(M)) \) is a subcomplex of the complex \( C^\ast(\text{Vect} M, \Omega^\ast(M)) \) The subcomplex \( C^\ast_\Delta(\text{Vect} M, \Omega^\ast(M)) \) is called the complex of diagonal cochains of the topological Lie algebra \( \text{Vect} M \) with values in the \( \text{Vect} M \)-module \( \Omega^\ast(M) \). The cohomology of \( C^\ast_\Delta(\text{Vect} M, \Omega^\ast(M)) \) is denoted by \( H^\ast_\Delta(\text{Vect} M, \Omega^\ast(M)) \) and is called the diagonal cohomology of \( \text{Vect} M \) with values in \( \Omega^\ast(M) \).

Denote by \( \text{Vect}_c M \) the Lie algebra of vector fields on \( M \) with compact supports. By definition, each diagonal cochain cochain \( c \in C^p(\text{Vect} M, \Omega^p(M)) \) is uniquely determined by its values on the subalgebra \( \text{Vect}_c M \) of the Lie algebra \( \text{Vect} M \). Define the complex \( C^\ast_\Delta(\text{Vect}_c M, \Omega^\ast(M)) \) of diagonal cochains of the topological Lie algebra \( \text{Vect}_c M \) with values in the \( \text{Vect}_c M \)-module \( \Omega^\ast(M) \) similarly. Remark that the map \( C^\ast_\Delta(\text{Vect} M, \Omega^\ast(M)) \to C^\ast_\Delta(\text{Vect}_c M, \Omega^\ast(M)) \) induced by the inclusion \( \text{Vect}_c M \subset \text{Vect} M \) is an isomorphism of complexes.

Since the Lie derivative and the exterior derivative on \( \Omega^\ast(M) \) commute, one can endow \( C^\ast_\Delta(\text{Vect} M, \Omega^\ast(M)) \) with the second differential induced by the exterior derivative on \( \Omega^\ast(M) \). We denote by \( C^\ast_\Delta(\text{Vect} M; \Omega^\ast(M)) \) the corresponding double complex with respect to the total differential and denote by \( H^\ast_\Delta(\text{Vect} M; \Omega^\ast(M)) \) the cohomology of this complex. By the remark above, we see that the inclusion \( \text{Vect}_c M \subset \text{Vect} M \) induces an isomorphism of double complexes

\[ C^\ast_\Delta(\text{Vect}_c M; \Omega^\ast(M)) \to C^\ast_\Delta(\text{Vect} M; \Omega^\ast(M)). \]

It is clear that \( C^\ast_\Delta(\text{Vect} M; \Omega^\ast(M)) \) is a DG-algebra and its differential is an antiderivation of degree 1.

Let \( M \) be a closed oriented \( n \)-dimensional manifold. Consider a map

\[ \psi : C^\ast_\Delta(\text{Vect} M; \Omega^\ast(M)) \to C^\ast(\text{Vect} M) \]

defined as follows: for \( 0 \leq q < n \) put \( \psi(c) = 0 \), for \( q = n \) and \( c \in C^n_\Delta(\text{Vect} M; \Omega^n(M)) \), put \( \psi(c) = \int_M c \).

**Theorem 3.7.** ([9], [13]). The map \( \psi \) is a homomorphism of complexes which induces an isomorphism \( H^p_\Delta(\text{Vect} M; \Omega^\ast(M)) = H^p(\text{Vect} M) \) for any \( p \geq 0 \).
4. The main double complex

4.1. The complex $\Omega^*(D \times M)$ and its diagonal subcomplex. By [12], $\text{Vect}_c M$ is the tangent space $T_e(D)$ of $D$ at the neutral element $e \in D$. Then, for $X \in \text{Vect}_c M$ and $g \in D$, the corresponding left invariant vector field $X^l$ on $D$ is the map $g \mapsto (L_g)_*X$ and the corresponding right invariant vector field $X^r$ on $D$ is the map $g \mapsto (R_g)_*X$, where $L_g$ is a left translation and $R_g$ is a right translation on $D$. Note that, by our assumption, our multiplication in $D$ differs from the multiplication in [12] by the order of factors. By [12], for $X, Y \in \text{Vect}_c M$ we have $[X^r, Y^r] = -[X, Y]^r$.

Consider the infinite-dimensional manifold $D \times M$. Then we may consider the Lie algebra $V(D \times M) = \text{Vect}_c^r M \oplus \text{Vect} M$ as a subalgebra of the Lie algebra of vector fields on $D \times M$.

Put $\text{Vect}_c M = \{-X^r + X; X \in \text{Vect}_c M\}$ and $\tilde{X} = -X^r + X$. Then $\tilde{\text{Vect}_c M}$ is a subalgebra of the Lie algebra of vector fields on $D \times M$, which is isomorphic to the Lie algebra $\text{Vect}_c M$. Evidently, we have $V(D \times M) = \text{Vect}_c M \oplus \text{Vect} M$.

Consider the de Rham complex $\Omega^r(D \times M)$. It is clear that each $m$-form $\omega \in \Omega^r(D \times M)$ is uniquely determined by the corresponding continuous $m$-linear skew-symmetric form on $V(D \times M)$ with values in the ring of smooth functions on $D \times M$.

Introduce a bigrading of $\Omega^r(D \times M) = \oplus_{p,q} \Omega^{pq}(D \times M)$ induced by the decomposition $V(D \times M) = \text{Vect}_c M \oplus \text{Vect} M$. Then $\Omega^{pq}(D \times M)$ consist of forms $\omega \in \Omega^r(D \times M)$ which are uniquely determined by their values $\omega(\tilde{X}_1, \ldots, \tilde{X}_p, Y_1, \ldots, Y_q)$, where $X_1, \ldots, X_p \in \text{Vect}_c M$ and $Y_1, \ldots, Y_q \in \text{Vect} M$. The exterior derivative $d = \{d^{pq}\}$ on $\Omega^r(D \times M)$ is uniquely determined by the following standard formulas:

\[ (d^{pq})\omega(\tilde{X}_1, \ldots, \tilde{X}_{p+1}; Y_1, \ldots, Y_q) = \sum_{i=1}^{p+1} (-1)^{i-1} \left( \tilde{X}_i \omega(\tilde{X}_1, \ldots, \tilde{X}_{i-1}, \tilde{X}_{i+1}, \ldots, \tilde{X}_{p+1}; Y_1, \ldots, Y_q) \right) 
\]

\[ - \sum_{j=1}^{q} \omega(\tilde{X}_1, \ldots, \tilde{X}_{j-1}, \tilde{X}_{j+1}, \ldots, \tilde{X}_{p+1}; Y_1, \ldots, [X_i, Y_j], \ldots, Y_q) \]

\[ + \sum_{i<j} (-1)^{i+j} \omega([\tilde{X}_i, \tilde{X}_j], \ldots, \tilde{X}_{i+1}, \ldots, \tilde{X}_{j-1}, \tilde{X}_{j+1}, \ldots, \tilde{X}_{p+1}; Y_1, \ldots, Y_q), \quad (4.1) \]

\[ (d^{pq})\omega(\tilde{X}_1, \ldots, \tilde{X}_p; Y_1, \ldots, Y_{q+1}) = (-1)^p \left( \sum_{i=1}^{q} (-1)^{i-1} Y_i \omega(\tilde{X}_1, \ldots, \tilde{X}_p; Y_1, \ldots, \tilde{Y}_i, \ldots, Y_{q+1}) \right) 
\]

\[ + \sum_{i<j} (-1)^{i+j} \omega(\tilde{X}_1, \ldots, \tilde{X}_p; [Y_i, Y_j], Y_1, \ldots, \tilde{Y}_i, \ldots, \tilde{Y}_j, \ldots, Y_{q+1}), \quad (4.2) \]

where $\omega \in \Omega^{pq}(G \times M)$, $X_1, \ldots, X_p, Y_{p+1} \in \text{Vect}_c M$ and $Y_1, \ldots, Y_{q+1} \in \text{Vect} M$.

A form $\omega \in \Omega^{pq}(D \times M)$ is diagonal if the function $\omega(\tilde{X}_1, \ldots, \tilde{X}_p; Y_1, \ldots, Y_q)$ at $(g, x) \in D \times M$ vanishes whenever the germ of at least one of the vector fields $X_1, \ldots, X_p$ at $x$ equals 0. Denote by $\Omega_{\Delta}^{pq}(D \times M)$ the set of diagonal $(p, q)$-forms and put $\Omega_{\Delta}^r(D \times M) = \bigoplus_{p+q=n} \Omega_{\Delta}^{pq}(D \times M)$. It is easy to check that $\Omega_{\Delta}^r(D \times M) = \{\Omega_{\Delta}^{pq}(D \times M)\}$ is a subcomplex of $\Omega^r(D \times M)$.

Consider an action of the group $D$ on $D \times M$ induced by its action on $D$ by right translations and the trivial action on $M$. This action induces a left action of $D$ on $\Omega^r(D \times M)$. By definition, for $\omega \in \Omega^{pq}(D \times M)$ and $g, h \in D$, we have

\[ (h \omega)(\tilde{X}_1, \ldots, \tilde{X}_p; Y_1, \ldots, Y_q)(g, x) = \omega(\tilde{X}_1, \ldots, \tilde{X}_p; Y_1, \ldots, Y_q)(h \circ g, x). \]
It is easy to see that this action of $D$ preserves the bigrading of $\Omega^*_\Delta(D \times M)$ and the subcomplex $\Omega^*_\Delta(D \times M)$ is $D$-stable. Denote by $H^*_\Delta(D \times M)$ the cohomology of the complex $\Omega^*_\Delta(D \times M)$ and by $\Omega^*_\Delta(D \times M)^D$ the subcomplex of $D$-invariant forms of the complex $\Omega^*_\Delta(D \times M)$.

**Lemma 4.2.** There is a natural isomorphism of complexes

$$\Omega^*_\Delta(D \times M)^D = C^*(\text{Vect} M; \Omega^\q(M)).$$

**Proof.** Note that, for $\omega \in \Omega^p_\Delta(D \times M)^D$ and $X, X_1, \ldots, X_p, Y_1, \ldots, Y_q$, we have

$$\hat{X}\omega(\hat{X}_1, \ldots, \hat{X}_p; Y_1, \ldots, Y_q) = X\omega(\hat{X}_1, \ldots, \hat{X}_p; Y_1, \ldots, Y_q).$$

Consider an action of the Lie algebra $\text{Vect} M$ on $\Omega^\q(M)$ as follows: for $\hat{X} \in \text{Vect} M$, define the operator $L_X$ of the form $\omega \in \Omega^\q(M)$ along a vector field $X \in \text{Vect} M$ as

$$(L_X \omega)(Y_1, \ldots, Y_q) = X\omega(Y_1, \ldots, Y_q) - \sum_{i=1}^q \omega(Y_1, \ldots, [X, Y_i], \ldots, Y_q), \quad (4.3)$$

where $Y_1, \ldots, Y_q \in \text{Vect} M$.

Evidently, the form $\omega \in \Omega^p_\Delta(D \times M)^D$ can be considered as a cochain of the complex $C^p_\Delta(\text{Vect} M, \Omega^\q(M))$. Then, for $\omega \in C^p_\Delta(D \times M)^D$, using (4.1), (4.2), and (4.3), we get

$$(d^p\omega)(\hat{X}_1, \ldots, \hat{X}_{p+1}; \ldots) = \sum_{i=1}^{p+1} (-1)^{i-1} L_{\hat{X}_i} \omega(\hat{X}_1, \ldots, \hat{X}_{i-1}, \hat{X}_{i+1}; \ldots)$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([\hat{X}_i, \hat{X}_j], \ldots, \hat{X}_i, \hat{X}_{i+1}, \ldots, \hat{X}_{p+1}; \ldots), \quad (4.4)$$

and

$$(d^p\omega)(\hat{X}_1, \ldots, \hat{X}_p; Y_1, \ldots, Y_{q+1}) = (-1)^p \left( \sum_{i=1}^{q+1} Y_i \omega(\hat{X}_1, \ldots, \hat{X}_p; Y_1, \ldots, \hat{Y}_i, \ldots, Y_{q+1}) \right.$$  

$$+ \sum_{i<j} (-1)^{i+j} \omega(\hat{X}_1, \ldots, \hat{X}_p; [Y_i, Y_j], Y_1, \ldots, \hat{Y}_i, \ldots, \hat{Y}_j, \ldots, Y_{q+1}) \right). \quad (4.5)$$

Since $X \mapsto \hat{X}$ is an isomorphism $\text{Vect} M = \text{Vect} M$, comparing the differential of the complex $C^*_\Delta(\text{Vect} M; \Omega^\q(M))$ with the differential of the complex $\Omega^*_\Delta(D \times M)^D$ defined by formulas (4.4) and (4.5), we see that these complexes are naturally isomorphic. \hfill $\square$

**Lemma 4.3.** For any $m, p, q$ we have

1. $H^0_{\text{cont}}(D, \Omega^p_\Delta(D \times M)) = 0$ for $m > 0$;
2. $H^m_{\text{cont}}(D, \Omega^p_\Delta(D \times M)) = C^m_{\text{cont}}(\text{Vect} M; \Omega^q(M)).$

**Proof.** Define the standard operator $B = \{B^pq\}_{pq}$, where

$$B^pq : C^m_{\text{cont}}(D, \Omega^p_\Delta(D \times M)) \rightarrow C^{m-1}_{\text{cont}}(D, \Omega^p_\Delta(D \times M)),$$

as follows: for $m > 0$ and $c \in C^m_{\text{cont}}(D, \Omega^p_\Delta(D \times M))$, put

$$(B^pqc)(g_1, \ldots, g_{m-1})(\cdot)(g, x) = c(g, g_1, \ldots, g_{m-1})(\cdot)(e, x),$$

where $g, g_1, \ldots, g_{m-1} \in D$ and $x \in M$. For $p = 0$, put $B^0q = 0.$
It is easy to check that for $m > 0$ we have
\[
d^{m-1} \circ B^p_m + B^p_{m+1} \circ d^m = \text{id}.
\]
For $c \in C^0_{\text{cont}}(D; \Omega^p(D \times M))$, we have
\[
(B^p_m \circ d^p)(c)(g, x) = c(g, x) - c(e, x).
\]
Thus, $B = (B^p_m)$ is the homotopy operator of the identity map of the complex $C^*_{\text{cont}}(D; \Omega^p(D \times M))$ and the map which is trivial on the cochains of positive dimension and is equal $c(g, x) \mapsto c(e, x)$ on $C^0_{\text{cont}}(D; \Omega^p(D \times M))$. Later we identify $c(e, x)$ with the cochain $\tilde{c} \in C^0_{\text{cont}}(D; (\Omega^p(D \times M))^D)$ given by $\tilde{c}(g, x) = c(g, x)$.

Since $B(C^*_{\text{cont}}(D; \Omega^p(D \times M)) \subset C^*_{\text{cont}}(D; \Omega^p(D \times M))$, the corresponding statements are true for $C^*(D; \Omega^*_\Delta(D \times M))$. Note that one can consider the map $c(g, x) \mapsto c(e, x)$ on $C^0_{\text{cont}}(D; (\Omega^p(D \times M))^D)$ as the map
\[
C^0_{\text{cont}}(D; \Omega^p(D \times M))^D \to \Omega^p(D \times M)^D = C^0_{\Delta \text{f}}(\text{Vect} M; \Omega^p(M)).
\]
These statements imply the claims of the lemma.

By lemma (4.2), we have the following composition of homomorphisms of complexes
\[
C^*_\Delta(\text{Vect} M; \Omega^*(M)) = \Omega^*_\Delta(D \times M)^D \subset C^0_{\text{cont}}(D; \Omega^*_\Delta(D \times M)) \subset C^*_{\text{cont}}(D; \Omega^*_\Delta(D \times M)).
\]

**Corollary 4.4.** The inclusion $C^*_\Delta(\text{Vect} M; \Omega^*(M)) \subset C^*_{\text{cont}}(D; \Omega^*_\Delta(D \times M))$ induces an isomorphism $H^*_\Delta(\text{Vect} M; \Omega^*(M)) = H^*(D; \Omega^*_\Delta(D \times M))$.

**Proof.** Consider the second filtration of the double complex $C^*(D; \Omega^*_\Delta(D \times M))$ and the corresponding spectral sequence $E_{2,r}$. By lemma 4.3 we have $E^p_{2,1} = 0$ for $p > 0$ and $E^p_{2,1} = C^0_{\Delta f}(\text{Vect} M; \Omega^*(M))$. It is evident that the differential $d_1 : E^0_{2,1} \to E^0_{2,1}$ coincides with the differential of the complex $C^*_\Delta(\text{Vect} M; \Omega^*(M))$. Then $E^0_{2,1} = E^0_{2,1} = 0$ for $p > 0$ and $E^0_{2,1} = E^0_{2,1} = H^0_{\Delta f}(\text{Vect} M; \Omega^*(M))$.

This implies the statement of the corollary. \hfill \qed

### 4.5. A filtration of $C^*_\Delta(\text{Vect} M; \Omega^*(M))$ and the corresponding spectral sequence

Denote by $\Omega^*_n$ the DG-algebra of formal differential forms in $n$ variables, i.e. the DG-algebra of $\infty$-jets at $0 \in \mathbb{R}^n$ of differential forms on $\mathbb{R}^n$. It is clear that $\Omega^*_n$ is a $W_n$-module with respect to the action of $W_n$ by the formal Lie derivatives $L_\xi$, where $\xi \in W_n$. Consider the complex $C^*(W_n, \Omega^*_n)$ of standard cochains of $W_n$ with values in $\Omega^*_n$ and endow it with the second differential induced by the formal exterior derivative on $\Omega^*_n$. We denote by $C^*(W_n; \Omega^*_n)$ the corresponding DG-algebra with respect to the total differential $D$ and the total grading. For $f^i \in \mathbb{R}[[\mathbb{R}^n]]$ and $\xi = \sum_{i=1}^{n} f^i \frac{\partial \xi^i}{\partial x^i} \in W_n$, put
\[
f^i_{j_1 \ldots j_r}(\xi) = \frac{\partial^r \xi^i}{\partial x^{j_1} \ldots \partial x^{j_r}},
\]
where $x^i$ for $i = 1, \ldots, n$ are the standard coordinates in $\mathbb{R}^n$. It is clear that $f^i_{j_1 \ldots j_r} \in C^1(W_n; \Omega^0_n)$. Moreover, $f^i_{j_1 \ldots j_r}$ and $dx^i$ for $r = 0, 1, \ldots$ and $i, j_1, \ldots j_r = 1, \ldots, n$ are generators of the DG-algebra $C^*(W_n; \Omega^*_n)$. Since $D$ is an antiderivation of degree 1 of DG-algebra $C^*(W_n; \Omega^*_n)$, it is uniquely determined by the following conditions:
\[
D f^i_{j_1 \ldots j_r} = \sum_{1 \leq k_1 \leq r} \sum_{s_1 < \ldots < s_k} \sum_{l=1}^{n} f^i_l j_{l j_1 \ldots j_{s_1} \ldots j_{s_k} \ldots j_r} \wedge f^i_{j_{s_1} \ldots j_{s_k}} - \sum_{l=1}^{n} f^i_{j_{s_1} \ldots j_{s_k}} \wedge dx^l
\]
and $D(dx^i) = f^i_j \wedge dx^j$. 


Consider the DG-algebra $C^*(W_n) \otimes \Lambda^*((\mathbb{R}^n)')$, where $(\mathbb{R}^n)'$ is a dual vector space for $\mathbb{R}^n$, with the differential which equals the differential of the complex $C^*(W_n)$ on the first factor and trivial on the second factor.

Consider the generators $c^{i_1}_{j_1}, \ldots, c^{i_r}_{j_r}$ of the DG-algebra $C^*(W_n)$ as cochains of the complex $C^*(W_n) \otimes \Lambda^*((\mathbb{R}^n)')$. Then $c^{i_1}_{j_1}, \ldots, c^{i_r}_{j_r}$ and $dx^i$ are the generators of the DG-algebra $C^*(W_n) \otimes \Lambda^*((\mathbb{R}^n)')$.

Consider a morphism $\mu : C^*(W_n; \Omega^*_q) \to C^*(W_n) \otimes \Lambda^*((\mathbb{R}^n)')$ of graded algebras defined by the following conditions:

$$\mu(f) = df + c^i, \quad \mu(f^{i_1}_{j_1}, \ldots, f^{i_r}_{j_r}) = c^{i_1}_{j_1}, \ldots, c^{i_r}_{j_r}, \quad \text{and} \quad \mu(dx^i) = -c^i,$$

where $r = 1, 2, \ldots$ and $i, j_1, \ldots, j_r = 1, \ldots, n$.

**Lemma 4.6.** The morphism $\mu$ is an isomorphism of DG-algebras.

*Proof.* It is clear that $\mu$ is an isomorphism of graded algebras. The formulas above for the differential $D$ of the complex $C^*(W_n; \Omega^*_q)$ and (2.1) imply that $\mu$ is an isomorphism of DG-algebras. □

Consider again the DG-algebra $C^*_\Delta(Vect M; \Omega^*(M))$. First assume that $M = U$ is a connected open subset of $\mathbb{R}^n$. For $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i} \in Vect U$, where $X^i$ is a smooth function on $U$, put

$$\varphi^i_{j_1, \ldots, j_r} = \frac{\partial^r X^i}{\partial x^{j_1} \cdots \partial x^{j_r}}.$$

By definition, we have $\varphi^i_{j_1, \ldots, j_r} \in C^0_\Delta(Vect U; \Omega^q(U))$. By definition, the DG-algebra $C^*_\Delta(Vect U; \Omega^*(U))$ is generated by the ring of smooth functions on $U$, the chains $\varphi^i_{j_1, \ldots, j_r}$ for $i, j_1, \ldots, j_r = 1, \ldots, n$ and $r \geq 0$, and the forms $dx^i + \varphi^i$ for $i = 1, \ldots, n$. Then the differential $d$ of the complex $C^*_\Delta(Vect U; \Omega^*(U))$ is uniquely defined by the following conditions:

1. For a smooth function $f$ on $U$ we have $df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} (dx^i + \varphi^i)$;
2. For $r \geq 0$ we have

$$\begin{align*}
\langle dx^i + \varphi^i \rangle = & \sum_{0 \leq k \leq r} \sum_{s_1, \ldots, s_k} \sum_{l=1}^n \varphi^i_{j_1, \ldots, j_r} \wedge \varphi^i_{j_{s_1}, \ldots, j_{s_k}} \wedge \varphi^i_{j_{s_1}, \ldots, j_{s_k}} \\
& - \sum_{l=1}^n \varphi^i_{j_1, \ldots, j_r} \wedge (dx^i + \varphi^i);
\end{align*}
$$

3. $d(dx^i + \varphi^i) = 0$.

It is clear that each cochain $c \in C^*_\Delta(Vect U; \Omega^q(U))$ is uniquely represented as a sum of the cochains of the following types:

$$f \varphi^i_{j_1, \ldots, j_{r_1}} \wedge \cdots \wedge \varphi^i_{j_{r_p-1}, \ldots, j_{r_p}} \wedge dx^{k_1} \wedge \cdots \wedge dx^{k_q},$$

(4.6)

where $f \in \Omega^p(U)$ and $0 \leq r_1 \leq \cdots \leq r_p$.

If the germ at $x \in U$ of the cochain (4.6) is nonzero the number $r_p$ is called the order of the cochain (4.6) at $x$. If $c \in C^*_\Delta(Vect M; \Omega^*(M))$ is a sum of cochains of types (4.6), the maximal order of the summands at $x$ is called the order of $c$ at $x$ and is denoted by $\text{ord}_x(c)$. By definition, we have $\text{ord}_x(c_1 + c_2) \leq \max\{\text{ord}_x(c_1), \text{ord}_x(c_2)\}$. It is easy to see that $\text{ord}_x(c)$ is independent of the choice of coordinates in $U$.

Let $M$ be an arbitrary manifold and let $c \in C^*_\Delta(M; \Omega^q(M))$ and $c \neq 0$. Then $\text{ord}(c)$ is the maximum of orders of $c$ for any $x \in M$. If $c = 0$, we put $\text{ord}(c) = \infty$.

Define a filtration $F_s$ of the complex $C^*_\Delta(M; \Omega^*(M))$ as follows: let $c \in C^*_\Delta(M; \Omega^*(M))$, then $c \in F_s$ if $\text{ord}(c) \leq p + q - s$. It is clear that $F_s$ is a
subalgebra of the differential algebra $C^\ast_{\Delta}(\text{Vect} M; \Omega^\ast(M))$, $F_{s_1} \wedge F_{s_2} \subset F_{s_1+s_2}$, and $F_s \subset F_{s-1}$.

Consider the spectral sequence $E_r$ induced by this filtration. First consider the term $E_0$ for $M = U \subset \mathbb{R}^n$. By definition, we can identify $E_0^{pq}$ with the vector space generated by cochains (4.6) of order $s$ and the differential $d_0 : E_0 \to E_0$ is uniquely determined by the following conditions:

1. For a smooth function $f$ on $M$, we have $d_0 f = 0$.
2. For $r \geq 0$ we have

$$d_0 \phi^i_{j_1 \ldots j_r} = \sum_{0 \leq k \leq r} \sum_{s_1 < \ldots < s_k} \sum_{l=1}^n \phi^i_{j_1 \ldots j_{s_1} \ldots j_{s_k} \ldots j_r} \wedge \phi^j_{j_1 \ldots j_{s_1} \ldots j_{s_k} \ldots j_r}$$

$$- \sum_{l=1}^n \phi^i_{j_1 \ldots j_r} \wedge (dx^l + \phi^l);$$

3. $d_0(dx^i + \phi^i) = 0$.

Then, comparing the differentials of the generators $c^i, c^i_{j_1 \ldots j_r}$, and $dx^i$ of the complex $C^\ast(W_n) \otimes \Lambda^\ast((\mathbb{R}^n)')$ with the differentials of the generators $\phi^i_{j_1 \ldots j_r}$ and $dx^i + \phi^i$ of the complex $(E_0, d_0)$, we see that the complex $(E_0, d_0)$ is isomorphic to the tensor product of the complex complex $C^\ast(W_n; \Omega^\ast)$ and the complex $\Omega^0(U)$ with the trivial differential, i.e.

$$E_1 = \Omega^0(U) \otimes H^\ast(W_n) \otimes \Lambda^\ast((\mathbb{R}^n)'),$$

where $dx^i + \phi^i$ are generators of the exterior algebra $\Lambda^\ast((\mathbb{R}^n)')$.

Let $M$ be an arbitrary manifold. Using the partition of unity on $M$ one can prove that $E_1 = \Omega^\ast(M) \otimes H^\ast(W_n)$. It is clear that the differential $d_1 : E_1 \to E_1$ is trivial on $H^\ast(W_n)$ and is equal to the exterior derivative on $\Omega^\ast(M)$. Then we have $E_2 = H^\ast(M) \otimes H^\ast(W_n)$.

**Remark 4.7.** The differentials $d_r$ for $r \geq 2$ of this spectral sequence are calculated in [13] and [14]. Since the odd Pontrjagin classes of $M$ are trivial, from this calculation it follows that for $m > 2n$ all elements of $H^m(W_n, O(n)) \subset H^m(W_n) \subset E_2$ live in $E_\infty$, i.e. $H^m(W_n, O(n)) \subset E_\infty$.

4.8. The proof of main results. Consider a Riemannian metric on $M$ and the corresponding bundle $O(M)$ of orthogonal frames of $M$. Let $\sigma : O(M) \to S(M)$ be a map defined in 2.3.

Consider the map $f_\sigma : D \times O(M) \to S(M)$ given by $(g, r) \mapsto g(\sigma(r))$, where $r \in O(M)$ and $g \in D$. Consider the actions of $D$ on $D \times O(M)$ induced by the action of $D$ on $D$ by right translations and the trivial action of $D$ on $M$. Then the map $f_\sigma$ is $D$-equivariant. Moreover, the map $\sigma$ is $O(n)$-equivariant with respect to the natural actions of the group $O(n)$ on $O(M)$ and $S(M)$. Then the map $\sigma$ induces a smooth map $\tilde{\sigma} : M = O(M)/O(n) \to S(M)/O(n)$. Denote by $f_\sigma$ the map $D \times M \to S(M)/O(n)$ induced by the map $f_\sigma$. Consider the map $f_\sigma^* : \Omega^\ast(S(M)/O(n)) \to \Omega^\ast(D \times M)$ induced by $f_\sigma$. Since the actions of $D$ and $O(n)$ on $D \times O(M)$ commute, the map $f_\sigma^*$ is $D$-equivariant homomorphism of complexes. It is easy to see that $f_\sigma^*(\Omega^\ast(S(M)/O(n))) \subset \Omega^\ast_{\Delta}(D \times M)$. Then we get a homomorphism of double complexes

$$C^\ast_{\text{cont}}(D; \Omega^\ast(S(M)/O(n))) \to C^\ast_{\text{cont}}(D; \Omega^\ast_{\Delta}(D \times M))$$

induced by $f_\sigma^*$ and, by corollary 4.4, the corresponding cohomology homomorphism

$$H^\ast_{\text{cont}}(D; \Omega^\ast(S(M)/O(n))) \to H^\ast_{\Delta}(\text{Vect} M; \Omega^\ast(M)). \quad (4.7)$$
Moreover, we have the homomorphism

\[ \Omega^*(S(M)/O(n))^D \rightarrow C^*_\text{cont}(D; \Omega^*(S(M)/O(n))) \]

and, hence, the homomorphism

\[ F_\sigma : C^*(W_n, O(n)) = \Omega^*(S(M)/O(n))^D \rightarrow C^*_\Delta(\text{Vect} M; \Omega^*(M)). \]

Now we prove the main theorem.

**Theorem 4.9.** For \( m > 2n \), the map \( H^m(W_n, O(n)) \rightarrow H^*(\text{Vect} M; \Omega^*(M)) \) induced by \( F_\sigma \circ \alpha \) is a monomorphism.

**Proof.** Consider the homomorphism of complexes

\[ f_\sigma^* : \Omega^*(S(M)/O(n))^D \rightarrow \Omega^*_\Delta(D \times M)^D. \]

It will be convenient to us to treat \( \Omega^*_\Delta(D \times M)^D \) as a subcomplex of the de Rham complex \( \Omega^*(D \times O(M)) \). Moreover, we will consider the forms of \( \Omega^*_\Delta(D \times O(M)) \) as skew-symmetric multilinear forms on the Lie algebra of vector fields on \( D \times O(M) \).

Let \( X \) be the horizontal lift of a vector field \( X \in \text{Vect} M \) with respect to the Levi-Civita connection. Put \( \text{Vect} M \spadesuit \{ X; X \in \text{Vect} M \} \). Since we are interested only in forms from \( \Omega^*_\Delta(D \times M)^D \), it suffices to us to consider these forms only as multilinear functions on the vector space \( \text{Vect}_c M \oplus \text{Vect} M \).

Let \((g, r) \in D \times O(M)\). Consider the linear map

\[ (f_\sigma)_* : T_{(g, r)}(D \times O(M)) \rightarrow T_{\sigma(g, r)}S(M). \]

For \( X \in \text{Vect} M \), denote by \( \tilde{X} \) a vector field on \( S(M) \) induced by \( X \). Then we have

\[ (f_\sigma)_*(X^i)(g, r) = g_*\tilde{X}(\sigma(r)), \quad (4.8) \]

where \( X \in \text{Vect}_c M, g \in D, \) and \( r \in O(M) \). For the Gelfand-Kazhdan form \( \omega \), by refG-K we get

\[ f_\sigma^*\omega(X^i)(g, r) = (g^*\omega)(\tilde{X}(\sigma(r)) = \omega(\tilde{X})(\sigma(r)). \quad (4.9) \]

Recall that \( \omega^i_{j\ldots j_r} = \alpha(c^i_{j\ldots j_r}) \). Denote by \( \nabla_\tau \) the operator of covariant derivative with respect to the Levi-Civita connection. It is known that, for \( X \in \text{Vect} M, r \in O(M) \), and the geodesic coordinates \( x^i \) corresponding to \( r \), we have \( \frac{\partial X^i}{\partial x^\tau}(0) = (\nabla_\tau X^i)(0) \). Then, by (2.4), on \( \sigma(M) \) in any coordinates on \( M \) we have

\[ \omega^i_j(\tilde{X}) = -\nabla_jX^i = -\left( \frac{\partial X^i}{\partial x^\tau} + \sum_{k=1}^n \Gamma^i_{jk}X^k \right), \quad (4.10) \]

where \( \Gamma^i_{jk} \) are the Christoffel symbols for the Levi-Civita connection.

Define two 1-forms \( \nabla^i \) and \( \nabla_j^\tau \) on \( D \times O(M) \) by the equations \( \nabla^i(X^r)(g, r) = X^i(r), \nabla^i(Y)(g, r) = 0, \nabla_j^\tau(X^r)(g, r) = \nabla_jX^i(r), \) and \( \nabla_j^\tau(Y)(g, r) = 0 \) where \( X \in \text{Vect}_c M, Y \in \text{Vect} M, X^i(r) \) and \( \nabla_jX^i(r) \) are the components of \( X \) and its covariant derivative with respect to the frame \( r \in O(M) \). Then, by the definition of covariant derivative, (2.3), (2.4) and (4.9), we get

\[ (f_\sigma^*\omega^i)(X^r + Y)(g, r) = -\nabla^i(X^r)(g, \sigma(r)) - \theta^i(\tilde{Y})(\sigma(r)), \quad (4.11) \]

\[ (f_\sigma^*\omega^i_j)(X^r + Y)(g, r) = -\nabla^i_j(X^r)(\sigma(r)), \quad (4.12) \]

where \( X \in \text{Vect}_c M, Y \in \text{Vect} M, \) and we identify the form \( \theta^i \) on \( O(M) \) with the corresponding form on \( D \times O(M) \). Since \( \theta^i(\tilde{Y})(\sigma(r)) = Y^i(\sigma(r)) \), we get

\[ (f_\sigma^*\omega^i)(\tilde{X} + \tilde{Y}) = -(\theta^i + \nabla^i)(\tilde{X} + \tilde{Y}), \quad (4.13) \]

\[ (f_\sigma^*\omega^i_j)(\tilde{X} + \tilde{Y}) = -\nabla^i_j(\tilde{X} + \tilde{Y}), \quad (4.14) \]
where $\hat{X} = -X^r + \hat{X}$.

Consider the cochains $\psi^i_j \in C^2(W, GL_n(\mathbb{R}))$ and $\Psi = (\psi^i_j)$ introduced in 2.3 and put $\hat{\Psi} = \alpha(\Psi)$. Define a 1-form $\nabla_{jk}^i$ on $D \times \mathcal{O}(M)$ by the equations $\nabla_{jk}^i(X^r)(g, r) = \nabla_j \nabla_k X^r(\sigma(r))$ and $\nabla_{jk}^i(Y)(g, r) = 0$, where $X \in \text{Vect}(M)$ and $Y \in \text{Vect} M$. By the definition of covariant derivative and the properties of the curvature tensor of a Riemannian manifold, one could check that we have
\begin{equation}
\begin{split}
f_\sigma^* \hat{\Psi}(\tilde{X}_1 + \tilde{Y}_1, \tilde{X}_2 + \tilde{Y}_2) = -\left( \sum_k \nabla_{jk}^i \wedge (\theta^k + \nabla^k) - \sum_{k,l} R^i_{jkl} \nabla^k \wedge \nabla^l \right) (\tilde{X}_1 + \tilde{Y}_1, \tilde{X}_2 + \tilde{Y}_2),
\end{split}
\end{equation}
where $R^i_{jkl}$ are the components of the curvature tensor of the Levi-Civita connection.

Put $C_{p_1 \ldots p_l, r_1 \ldots r_k} = \alpha(C_{p_1 \ldots p_l, r_1 \ldots r_k})$, where $c_{p_1 \ldots p_l, r_1 \ldots r_k}$ is the basic cocycle of the complex $C^*(W, O(n))$ defined by (2.5). Consider the cocycles $f_\sigma^i(C_{p_1 \ldots p_l, r_1 \ldots r_k})$ of the complex $\Omega_\infty^m(D)$. Since $\sigma^* \theta^i = dx^i$, by (4.10), (4.11), (4.12), and (4.15), the leading term (with respect to the order of cochains) of $f^i_\sigma(C_{p_1 \ldots p_l, r_1 \ldots r_k})$ equals the cocycle $C_{p_1 \ldots p_l, r_1 \ldots r_k} \in C^*(W, O(n)) \subset E_0$, where $E_0 = \Omega^0(M) \otimes \Lambda^*(\mathbb{R}^n) \otimes H^*(W, O(n))$ is the zero term of the spectral sequence studied in 4.5. It is easy to see that $f_\sigma^i(C_{p_1 \ldots p_l, r_1 \ldots r_k})$ is a basic cocycle of the second term $E_2$ of the spectral sequence. By the remark 4.7, for $m > 2n$ we have $H^m(W, O(n)) \subset E_\infty$. Hence, for $m > 2n$, the map $F_\sigma \circ \alpha$ induces a monomorphism of $H^m(W, O(n))$ into $H^*(\text{Vect} M; \Omega^*(M))$. \hfill \Box

Theorems 4.9 and 3.7, corollaries 3.4 and 3.5 imply immediately the following corollaries.

**Corollary 4.10.** Assume that $H^p(M) = 0$ for $p > 0$. For each $m > 2n$ the map $F_\sigma \circ \alpha$ induces a monomorphism of $H^m(W, O(n))$ into $H^m(D)$. In particular, $c(C_{p_1 \ldots p_l, r_1 \ldots r_k})$ is a nontrivial $m$-cocycle of the complex $C^*(D)$ for $m = 2(p_1 + \cdots + p_l + r_1 + \cdots + r_k) - l$.

**Corollary 4.11.** Assume that $M$ is a closed oriented manifold. For each $m > 2n$ the map $F_\sigma \circ \alpha$ induces a monomorphism of $H^m(W, O(n))$ into $H^m(D_+)$. In particular, $c(C_{p_1 \ldots p_l, r_1 \ldots r_k})$ is a nontrivial $(m - n)$-cocycle of the complex $C^*(D_+)$ for $m = 2(p_1 + \cdots + p_l + r_1 + \cdots + r_k) - l$.

There is a problem to find explicit expressions for the cocycles $c(C_{p_1 \ldots p_l, r_1 \ldots r_k})$ in the cases of corollaries 4.10 and 4.11. In principal, it is possible under the conditions of corollary 4.10 whenever the manifold $M$ is contractible and the homotopy for this contraction is given and under the conditions of corollary 4.11. For this one need to use formulas (2.12) and the procedure for the constructing of the group $D$ in theorem 3.3. It is clear that thus we will get an expression for each cocycle $c(C_{p_1 \ldots p_l, r_1 \ldots r_k})$ via integrals and a Riemannian metric on $M$.

**Example 4.12.** Let $M = \mathbb{R}$ and let $x$ be the standard coordinate on $\mathbb{R}$. Then $s \in S(\mathbb{R})$ is an $\infty$-jet $j_0^\infty k$, where $k(t)$ is a regular at 0 map $\mathbb{R} \to \mathbb{R}$. We take for the coordinates on $S(M)$ the derivatives $x_i = k(t)(0)$ for $i \geq 0$. We put $y = x_0$, $y^1 = \log |x^1|$, and $y^2 = \frac{x^2}{x^1}$. Consider the cocycle of the Godbillon-Vey class $c_{1,1}$. It is easy to check that we have
\begin{equation}
C_{1,1} = \alpha(c_{1,1}) = dy \wedge dy^1 \wedge dy^2.
\end{equation}
Applying the procedure of theorem 3.3 one could get
\begin{equation}
c(C_{1,1}) = \int_x^f \log |h'(g(t))|d\log |g'(t)|,
\end{equation}
where \( f, g, h \in \text{Diff} \mathbb{R} \). It is easy to see that the cohomology class of the cocycle \( c(C_1) \) given by (4.16) in the complex \( C^*_{\text{cont}}(\text{Diff} \mathbb{R}, \mathbb{R}) \) is independent of the choice of \( x \in \mathbb{R} \).

**Example 4.13.** Let \( M \) be a closed oriented Riemannian manifold, \( \theta \) the form of the Levi-Civita connection, and \( v \) the volume form on \( M \). For any \( g \in D_+ \), put \( \xi(g) = g^*\theta - \theta \) and define the function \( \mu \) by the condition \( g^*v = \mu v \). Consider the cocycles \( c_{s_1+\cdots+s_k} \), where \( s_1 + \cdots + s_k = n \). It is clear that the cohomology classes of the cocycles \( c_{s_1+\cdots+s_k} \) for all \( s_1, \ldots, s_k \) define a part of the cohomology \( H^{2n+1}(W_n, O(n)) \). The Bott cocycle \( c(C_{s_1+\cdots+s_k}) \) is defined by the following formula
\[
c(C_{s_1+\cdots+s_k}) = \int_M \text{Alt}_n \left( \log(\mu) \text{tr}(\xi_1 \cdots \xi_{s_1}) \wedge \cdots \wedge \text{tr}(\xi_{s_1+\cdots+s_{k-1}+1} \cdots \xi_n) \right),
\]
where \( g_1 \ldots g_n \in D_+ \), \( g_i = g_1 \circ \cdots \circ g_1 \), \( \xi_i = \xi(g_i) \), and \( \text{Alt}_n \) is the alternation operator in \( 1, \ldots, n \). By ([14]), the cocycle \( c(C_{s_1+\cdots+s_k}) \), given by this formula, is obtained from the cocycle \( c_{s_1+\cdots+s_k} \) by the procedure of theorem 3.3. Hence, the cohomology classes of Bott’s cocycles \( c(C_{s_1+\cdots+s_k}) \) are linearly independent. In particular, all of them are nontrivial.

Unfortunately, for the more complicated cocycles \( C_{p_1, \ldots, p_l, r_1, \ldots, r_k} \), as fine formulas as for Bott’s cocycles are not known.

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