The ★-value Equation and Wigner Distributions in Noncommutative Heisenberg algebras∗

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Abstract

We consider the quantum mechanical equivalence of the Seiberg-Witten map in the context of the Weyl-Wigner-Groenewold-Moyal phase-space formalism in order to construct a quantum mechanics over noncommutative Heisenberg algebras. The formalism is then applied to the exactly soluble Landau and harmonic oscillator problems in the 2-dimensional noncommutative phase-space plane, in order to derive their correct energy spectra and corresponding Wigner distributions. We compare our results with others that have previously appeared in the literature.

1 Introduction

There is a fairly deep understanding in theoretical physics on the microscopic structure of matter, but very little is known concerning the microscopic structure of the space-time. We know, for instance, that to distances of the order of $10^{-17} m.$ the space-time is a continuum but we do not know what happens to distances arbitrarily smaller than that. So, one of the most important open problems in theoretical physics is to understand the microscopic structure of the space-time, i.e. how to build a quantum theory of gravity.

By means of a simple heuristic argument, based on Heisenberg’s Uncertainty Principle, the Einstein Equivalence Principle and the Schwarzschild metric, it is easy to show that the Planck length seems to be a lower limit to the possible precision of measurement of position, and that shorter distances do not appear to have any operational meaning. It would then appear reasonable the need to extend the phase-space noncommutativity of quantum mechanics to a noncommutativity of space-time in order to quantize gravity. Furthermore, under these premises the very concept of manifold as an underlying mathematical structure in the construction of unified physical theories, applicable to distances of the order of the Planck length, becomes questionable and some people have been convinced that a new paradigm of geometric space is needed.

∗Dedicated to Mike Ryan on his sixtieth birthday, who as a scientist always understood that it is nice to be good, but that it is better to be nice.
that would allow us to incorporate into our theoretical formalisms completely different small-scale structures from those to which we are usually accustomed. Among physicists some options for this paradigm are embodied in topological quantum field theory, dynamical triangulations, string theory (and efforts in this context to develop a nonperturbative formulation that could allow us to reach Planck scale physics) and loop quantum gravity. See e.g. [1] for a collection of these different directions of research.

Among mathematicians mainly one such outstanding paradigm is the noncommutative geometry invented by Connes, which considers a new calculus, the so called spectral calculus, based on operators in Hilbert space and the use of the tools of spectral analysis [2]. This geometry has among its features that it includes ordinary Riemannian space; discrete spaces are treated on the same footing as the continuum, thus allowing for a mixture of the two; it allows for the possibility of noncommuting coordinates; and even though quite different from the geometry arising in string theory, it is not incompatible with it.

Although none of the above mentioned apparently conceptually different approaches and their variants are anywhere near a final theory of grand unification, and probably no single one of this directions will succeed in producing it, there appears to be emerging a common denominator of noncommutativity in some of their ingredients which points to the fact that when considering the problem of coordinates below the Planck length, there is no good reason to presume that the texture of space-time will still have a 4-dimensional continuum. Further evidence along this line of thought has been provided by recent developments in string theory where noncommutative geometry appears in the low energy effective theory of brane configurations and in the matrix model of M-theory. It has also been shown recently that in noncommutative field theories the Seiberg-Witten map can be interpreted as a field dependent gravitational background [3]. In fact, it is not difficult to show that a similar interpretation can be carried out even at the level of quantum mechanics on noncommutative phase-space.

These recent results, as well as others (c.f. examples of noncommutative geometry in field theory listed in [4]), have generated a considerable interest to understand the role played by noncommutative geometry in different theoretical sectors of physics.

In quantum field theory noncommutativity can be formulated mathematically in two different ways:

1) By means of the $\star$-product on the space of $c$-functions

$$f \star g = \exp \left( \frac{i}{2} \theta_{ij} \partial_x \partial_y \right) f(x)g(y) |_{x=y}, \quad (1)$$

or

2) By defining the field theory on an operator space that is intrinsically noncommutative. Although formally well defined, the operator approach is hard to implement in explicit calculations. Hence the analysis of the noncommutative
effects is usually performed by expanding the $\star$-product perturbatively.

Moreover, since single particle quantum mechanics can be seen, in the free field or weak coupling limit, as a mini-superspace sector of quantum field theory where most degrees of freedom have been frozen (i.e., as a one-particle sector of field theory), the above mentioned results from field theory as well as others suggest that a more detailed study of exactly solvable models in noncommutative quantum mechanics will be helpful both for the understanding of the effects of noncommutativity in field theory, as well as of its possible phenomenological consequences in space.

¿From the intrinsically noncommutative operator point of view, the development of a formulation for noncommutative quantum mechanics requires first a specification of a representation for the phase-space algebra, second a specification of the Hamiltonian which governs the time evolution of the system and last a specification of the Hilbert space on which these operators and the other observables of the theory act. Regarding the choice of a representation for the intrinsic Heisenberg noncommutative phase-space algebra, several works that have appeared lately in the literature have suggested using a quantum mechanical equivalent to the Seiberg-Witten map 5, whereby the noncommutative Heisenberg algebra is mapped into a commutative one 6, 7, 8, 9. Since in all generality this map admits many possible realizations, one could have in principle also many possible resulting self-consistent quantum mechanics of which the proper one could only be discerned by experiment.

As for the choice of the Hilbert space, however, a reasonable assumption is that it can be taken to be the same as that for the corresponding commutative system, for any of the realizations of the noncommutative Heisenberg algebra in terms of the position and momentum operators for the commutative one 10.

The purpose of this paper is to show that a noncommutative quantum mechanics based on the Weyl-Wigner-Groenewold-Moyal formalism, extended to noncommutative phase-space by means of the quantum mechanical equivalent of the Seiberg-Witten map, can provide an interesting frame for further investigating the above mentioned approaches. In particular, we analyze the so called Weyl-Moyal correspondence procedure as symbolized by 11, when applying it to two exactly solvable models: the Landau problem and the harmonic oscillator in both noncommutative configuration and phase-space. We argue that this procedure leads to the correct quantum mechanics for the case of Heisenberg algebras where noncommutativity is restricted to configuration space and then only when the c-Weyl equivalent to the quantum observables is the same as the ordinary function that would be obtained by replacing the operators of the commutative Heisenberg algebra by their corresponding canonical dynamical variables. In addition, we also show through these examples what we consider is the correct procedure for applying the $\star$-value equation (see equation 18 below) to the case of non-commutative spaces and for the derivation of the Wigner distribution function in this case.
In order to make our presentation as self-contained as possible, we begin our discussion in Sec. 2 with a brief review of the Weyl-Wigner-Groenewold-Moyal formalism for ordinary quantum mechanics. We then turn to show how this formalism can be extended to noncommutative Heisenberg algebras by resorting to what could be considered a quantum mechanical equivalent of the Seiberg-Witten map, which we discuss there. In Secs. 3 and 4 we apply the formalism to calculate the energy spectrum and Wigner functions for the Landau and harmonic oscillator problems in noncommutative phase-space as a basis for a comparison with the results derived by an application of the Weyl-Moyal correspondence and for the analysis of the particular circumstances when both procedures are equivalent. We conclude the paper in Sec. 5 with some general remarks on this issues and with suggestions for further work.

2 Weyl functions and Wigner distributions in commutative and noncommutative phase spaces

Let

\[
\begin{align*}
[Q_i, Q_j] &= 0, \\
[\Pi_i, \Pi_j] &= 0, \\
[Q_i, \Pi_j] &= i\hbar \delta_{ij},
\end{align*}
\]  

(2)

be the commutative Heisenberg algebra of ordinary quantum mechanics. Making use of the Baker-Campbell-Hausdorff (BCH) theorem one can readily show that the set of operators \((2\pi\hbar)^{-\frac{d}{2}} \exp\left[\frac{i}{\hbar}(x \cdot \Pi + y \cdot Q)\right]\) satisfy the orthonormality condition

\[
(2\pi\hbar)^{-d}\text{Tr}\{\exp\left[\frac{i}{\hbar}(x \cdot x') \cdot \Pi + (y - y') \cdot Q)\right]\} = \delta(x - x')\delta(y - y'),
\]  

(3)

where \(x, y\) are \(c\)-vectors and \(d\) is the dimension of the configuration space. Thus they form a complete set and any quantum operator \(A(\Pi, Q, t)\) can be written as

\[
A(\Pi, Q, t) = \int \int d\mathbf{x} \, d\mathbf{y} \alpha(\mathbf{x}, \mathbf{y}, t) \exp\left[\frac{i}{\hbar}(\mathbf{x} \cdot \Pi + \mathbf{y} \cdot Q)\right],
\]  

(4)

where, by (3), the \(c\)-function \(\alpha(\mathbf{x}, \mathbf{y}, t)\) is determined by

\[
\alpha(\mathbf{x}, \mathbf{y}, t) = (2\pi\hbar)^{-d}\text{Tr}\{A(\Pi, Q, t) \exp\left[-\frac{i}{\hbar}(\mathbf{x} \cdot \Pi + \mathbf{y} \cdot Q)\right]\}. 
\]  

(5)

Define now the Weyl function corresponding to the quantum operator \(A(\Pi, Q, t)\) by

\[
A^W(p, q, t) = \int \int d\mathbf{x} \, d\mathbf{y} \alpha(\mathbf{x}, \mathbf{y}, t) \exp\left[\frac{i}{\hbar}(\mathbf{x} \cdot p + \mathbf{y} \cdot q)\right],
\]  

(6)
and consider the expectation value of the product of two quantum operators relative to the pure state $\ket{\Psi}$. The Weyl-Wigner-Groenewold-Moyal phase space formalism then shows that

$$
\langle \Psi | A_1(\Pi, Q, t) A_2(\Pi, Q, t) | \Psi \rangle = \int \int dp \, dq \rho_W(p, q, t) A_1^W(p, q, t) \ast A_2^W(p, q, t),
$$

(7)

where

$$
\rho_W(p, q, t) = (2\pi \hbar)^{-d} \int d\mathbf{z} \exp\left[ i \frac{\hbar}{2} \mathbf{z} \cdot \mathbf{p} \right] \langle \mathbf{q} - \frac{\mathbf{z}}{2} | \rho | \mathbf{q} + \frac{\mathbf{z}}{2} \rangle,
$$

(8)

is the Wigner quasi-probability distribution function, $\rho = |\Psi \rangle \langle \Psi|$ is the von Neumann density matrix for a pure quantum state, and

$$
\ast = \exp\left[ i \frac{\hbar}{2} \Lambda \right] := \exp\left[ i \frac{\hbar}{2} (\nabla_q \cdot \nabla_p - \nabla_p \cdot \nabla_q) \right],
$$

(9)

is the Moyal bidifferential $\ast$-operator.

To complete this brief summary of the Weyl-Wigner-Groenewold-Moyal formalism note that if $A(\Pi, Q, t)$ is a Heisenberg operator then

$$
A^W(p, q, t) = \exp\left\{ - \frac{2t}{\hbar} H^W \sin\left( \frac{\hbar}{2} \Lambda \right) \right\} A^W(p, q, 0),
$$

(10)

so setting $A^W(p, q, 0)$ equal to $p$ and $q$ we get

$$
\dot{p} = \dot{p}^W(0) = H^W \Lambda p^W(0) = -\nabla_q H^W,
$$

(11)

$$
\dot{q} = \dot{q}^W(0) = H^W \Lambda q^W(0) = \nabla_p H^W,
$$

(12)

respectively. Thus the c-numbers $p$ and $q$ satisfy Hamilton’s equations of motion, and may be interpreted as classical dynamical variables.

Note also, as it may be readily seen from (5), that the Wigner distribution function is everywhere real and its projection on configuration and momentum space gives the correct quantum mechanical configuration and momentum probabilities, respectively. Hence its designation as a quasi-probability density function.

Parallel to the classical phase-space integral equation (7), for the case when $|\Psi\rangle$ is a pure energy state there is a stronger equation, known as the $\ast$-value equation which can be derived directly from the energy eigenvalue equation

$$
H(\Pi, Q)|\Psi\rangle = E|\Psi\rangle.
$$

(13)

Indeed, using the fact that the c-Weyl function of a product of two operators is equal to the Moyal product of their corresponding c-Weyl functions (cf. (7)), we have that

$$
(H(\Pi, Q)\rho)^W = H^W \ast \rho^W
$$

(14)

where $\rho^W$ on the right side of (14) stands for the c-Weyl equivalent to the density matrix $\rho = |\Psi\rangle \langle \Psi|$. Now, by (5) and (9), we can write

$$
H(\Pi, Q)\rho = \int \alpha(x, y)e^{\frac{i}{\hbar}(x \cdot \Pi + y \cdot Q)} dx \, dy
$$

(15)
from where we derive
\[
\alpha = (2\pi \hbar)^{-d} \langle \Psi | e^{-\frac{i}{\hbar} (x\Pi + yQ)} H | \Psi \rangle
\]
\[
= (2\pi \hbar)^{-d} E \int dq' \psi^\dagger(q') e^{-\frac{i}{\hbar} y(q' - \frac{x}{2})} \psi(q' - x),
\]
(16)
and
\[
(H\rho)^W = E(2\pi \hbar)^{-d} \int \int \int dx dy dq' e^{\frac{i}{\hbar} (x p + y q)} \times \psi^\dagger(q') e^{-\frac{i}{\hbar} y(q' - \frac{x}{2})} \psi(q' - x).
\]
(17)
Integrating over \(y\) and \(q'\) and comparing with (8) we see that \(\rho_W\) is precisely the \(c\)-Weyl function corresponding to \(\rho\), so it immediately follows that
\[
H_W(p, q) \ast \rho_W(p, q) = E \rho_W(p, q),
\]
(18)
We emphasize here that \(H_W(p, q)\) is in general not equal to the \(c\)-function obtained by replacing the momentum and position operators in the original quantum Hamiltonian by their corresponding classical dynamical variables. This will be only true for Hamiltonians of the form \(\Pi^2/2m + V(Q)\), and will be an important proviso in our subsequent discussions.

Note also that by making use of the integral representation
\[
A_1^W(p, q) \ast A_2^W(p, q) = (2\pi \hbar)^{-2d} \int \ldots \int dp' dp'' dq' dq'' A_1^W(p', q') A_2^W(p'', q'') \exp[-\frac{2i}{\hbar} (p \cdot (q' - q'') + p' \cdot (q'' - q) + p'' \cdot (q - q'))],
\]
(19)
it immediately follows that
\[
\int \int dp dq H_W(p, q) \ast \rho_W(p, q) = \int \int dp dq H_W(p, q) \rho_W(p, q) = E,
\]
(20)
which is consistent with (17).

Let us now turn to the noncommutative Heisenberg algebra
\[
[R_i, R_j] = i\hbar \theta_{ij},
\]
(21)
\[
[P_i, P_j] = i\hbar \bar{\theta}_{ij},
\]
(22)
\[
[R_i, P_j] = i\hbar \delta_{ij},
\]
(23)
where \(\theta_{ij}\) and \(\bar{\theta}_{ij}\) are evidently antisymmetric matrices reflecting the noncommutativity of phase space. In order to study the quantum mechanics associated with operators which are arbitrary functions of \(\mathbf{R}\) and \(\mathbf{P}\), and in particular their eigenvalues and eigenstates in the context of the Weyl-Wigner-Groenewold-Moyal phase space formalism, we need first to apply the quantum mechanical
equivalent of the Seiberg-Witten map to express the algebra of operators \((21, 22, 23)\) in terms of their “commutative” counterparts \((2)\). To this end, and making use of the results in \([7]\) (cf. also \([8], [11], [9]\)), we write a linear representation of the algebra \((21, 22, 23)\) as

\[
Q_i = a_{ij}R_j + b_{ij}\Pi_j,
\]

\[
P_i = c_{ij}R_j + d_{ij}\Pi_j,
\]

(24)

Substituting this expressions into \((21, 22, 23)\) and using \((2)\) one obtains the matrix equations

\[
AB^T - BA^T = \Theta
\]

\[
CD^T - DC^T = \bar{\Theta}
\]

\[
AD^T - BC^T = 1,
\]

(25)

where the notation is self-evident. The solution of the above conditions determine the structure of the mapping \((24)\). For our present purposes we shall not be concerned with the problem of finding and classifying general solutions to this problem. It will suffice to consider one of the possible solutions which can be readily found by choosing \(A = \lambda 1, D = \mu 1\), and also assuming that \(B\) and \(C\) are antisymmetric matrices. It is then easy to show that

\[
B = -\frac{1}{2\lambda}\Theta,
\]

(26)

and

\[
C = \frac{1}{2\mu}\bar{\Theta},
\]

(27)

subject to the constraint

\[
\bar{\Theta}\Theta = \Theta\bar{\Theta} = 4\lambda\mu(\lambda\mu - 1)1.
\]

(28)

Thus we write

\[
R_i = \lambda Q_i - \frac{1}{2\lambda}\theta_{ij}\Pi_j,
\]

(29)

\[
P_i = \mu\Pi_i + \frac{1}{2\mu}\bar{\theta}_{ij}Q_j,
\]

(30)

where \(\lambda\) and \(\mu\) are constants. Note that if we require \(R\) and \(P\) to be Hermitian, then \(\lambda, \mu, \theta_{ij}\) and \(\bar{\theta}_{ij}\) have to be real.

Let us now investigate the implications of this specific noncommutative phase-space quantization scheme by considering two exactly soluble problems.
3 The Landau problem in noncommutative phase-space

Neglecting spin, consider the 2-dimensional noncommutative phase-space quantum Hamiltonian for an electron moving in a magnetic field \( B \) in the direction normal to the quantum plane \((R_1, R_2)\):

\[
H(P, R) = \frac{1}{2m} (P + \frac{e}{c} A)^2. \tag{31}
\]

In the symmetric gauge

\[
A = (-\frac{B}{2} R_2, \frac{B}{2} R_1), \tag{32}
\]

equation (31) reads, after substituting (29), (30),

\[
H(P, R) = \hat{H}(\Pi, Q) = \frac{1}{2m} \left[ (\mu + \kappa \lambda) p_1 + \frac{eB \lambda}{2c} q_2 \right]^2 + \frac{1}{2m} \left[ (\mu + \kappa \lambda) p_2 - \left( \frac{\bar{\theta}}{2 \mu} - \frac{eB \lambda}{2c} \right) q_1 \right]^2, \tag{33}
\]

where we have also used \( \theta_{ij} = \epsilon_{ij} \theta \) and \( \bar{\theta}_{ij} = \epsilon_{ij} \bar{\theta} \).

Note now that by virtue (7) the Weyl function associated with the Hamiltonian (33) is

\[
H_W(p, q) = \frac{1}{2m} \left[ (\mu + \kappa \lambda) p_1 + \frac{eB \theta}{4c} \right] (\Pi_1 + \left( \frac{\bar{\theta}}{2 \mu} - \frac{eB \lambda}{2c} \right) Q_2)^2 + \frac{1}{2m} \left[ (\mu + \kappa \lambda) p_2 - \left( \frac{\bar{\theta}}{2 \mu} - \frac{eB \lambda}{2c} \right) Q_1 \right]^2, \tag{34}
\]

where

\[
\kappa := - \frac{eB \theta}{4c}. \tag{35}
\]

We can now use this expression together with (18) to solve the \( \star \)-value equation for the Wigner distribution function. We thus have the second order differential equation

\[
H_W \star \rho_W = \left\{ \frac{1}{2m} \left[ (\mu + \kappa \lambda) p_1 + \left( \frac{\bar{\theta}}{2 \mu} - \frac{eB \lambda}{2c} \right) \right] (q_2 + \frac{i \hbar}{2} \partial p_2)^2 + \frac{1}{2m} \left[ (\mu + \kappa \lambda) p_2 - \left( \frac{\bar{\theta}}{2 \mu} - \frac{eB \lambda}{2c} \right) q_1 \right]^2 \right\} \rho_W. \tag{36}
\]

Separating the real and imaginary parts in the above expression, results in

\[
\left\{ \frac{\hbar^2}{8m} \left[ (\mu + \frac{\kappa}{\lambda})^2 \nabla_q \cdot \nabla_q + \left( \frac{\bar{\theta}}{2 \mu} - \frac{eB \lambda}{2c} \right)^2 \nabla_p \cdot \nabla_p + \frac{2(\mu + \frac{\kappa}{\lambda})}{2 \mu} \left( \frac{eB \lambda}{2c} \right) (\partial p_1, \partial q_2 - \partial q_2, \partial q_1) \right] \right\} \rho_W = E \rho_W, \tag{37}
\]

\[8\]
\[
-\frac{i\hbar}{2m}[(\mu + \frac{\kappa}{\lambda})^2 \mathbf{p} \cdot \nabla \eta + (\mu + \frac{\kappa}{\lambda})(\frac{\bar{\theta}}{2\mu} - \frac{eB\lambda}{2c})(q_2 \partial_{q_1} - q_1 \partial_{q_2} + p_2 \partial_{p_1} - p_1 \partial_{p_2})]
\]
\[
-((\frac{\bar{\theta}}{2\mu} - \frac{eB\lambda}{2c})^2 (\mathbf{q} \cdot \nabla \mathbf{p}))|\rho_W = 0.
\]

(38)

Now, since the time evolution of the Wigner function is given by
\[
\frac{\partial \rho_W}{\partial t} = 2\hbar H^W \sin \left( \frac{\hbar \Lambda}{2} \right) \rho_W,
\]
and, since for a stationary system the density matrix \( \rho = |\Psi\rangle \langle \Psi | \) commutes with the Hamiltonian \( \hat{H}(\Pi, Q) \), it clearly follows that the right side of (39) has to be zero. Furthermore, since the Weyl function \( H^W \) for the Landau Hamiltonian is at most quadratic in the classical dynamical variables (cf. (34)) only the first term in the series expansion of the operator \( \sin \left( \frac{\hbar \Lambda}{2} \right) \) contributes to (39). Hence
\[
H^W \left( \frac{\hbar \Lambda}{2} \right) \rho_W = 0.
\]

(40)

But this is precisely equation (38). Noting, in addition, that (40) would be identically satisfied if we require that \( \rho_W \) be a function of \( H^W \), we shall now make this ansatz and use (37) to evaluate \( \rho_W \). By a rather direct, albeit tedious calculation, we arrive at
\[
-\frac{\hbar^2}{m^2}[(\mu + \frac{\kappa}{\lambda})^2 \left( \frac{\bar{\theta}}{2\mu} - \frac{eB\lambda}{2c} \right)^2 \left( \xi \frac{\partial^2}{\partial \xi^2} + \frac{\partial}{\partial \xi} \right) + \xi \rho_W = E \rho_W,
\]

(41)

where we have set \( \xi := H^W \).

Moreover, letting
\[
\tau := \frac{\hbar}{2m}(\mu + \frac{\kappa}{\lambda}) \left( \frac{\bar{\theta}}{2\mu} - \frac{eB\lambda}{2c} \right),
\]
and introducing the new variable \( \eta := \frac{\xi}{\tau} \) we get, from (41):
\[
\eta \frac{\partial^2 \rho_W}{\partial \eta^2} + \frac{\partial \rho_W}{\partial \eta} - (\frac{\eta}{4} - \frac{E}{4\tau}) \rho_W = 0.
\]

(43)

Making the additional change of dependent variable
\[
\rho_W = e^{-\frac{\eta}{2}} \omega,
\]

(44)
equation (43) takes the form of Laguerre’s differential equation
\[
\left[ \eta \frac{\partial^2}{\partial \eta^2} + (1 - \eta) \frac{\partial}{\partial \eta} + \frac{E}{4\tau} - \frac{1}{2} \right] \omega = 0,
\]

(45)
which, for integral values of \( \frac{E}{\hbar} - \frac{1}{2} = n \), has a solution in terms of Laguerre polynomials

\[
\omega = L_n(\eta) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \eta^k / k!.
\]

The energy spectrum for the Landau problem is then given by

\[
E = \frac{2\hbar}{m} (\mu + \frac{\kappa}{4})(\frac{\ddot{\theta}}{2\mu} - \frac{eB\lambda}{2c})(n + \frac{1}{2}),
\]

and the corresponding Wigner distribution function by

\[
\rho_W = \exp \left( -\frac{H_W^2}{2\tau} \right) L_n \left( \frac{H_W}{\tau} \right),
\]

with \( \tau(\theta, \ddot{\theta}, \lambda, \mu) \) given by (42).

Let us now compare the above results with others appearing in the literature for the Landau and similar problems obtained by applying a certain deformation quantization prescription to the point product of a classical Hamiltonian and the Wigner function. To be more specific, in the Landau problem for example (see e.g. [12]), the classical Hamiltonian is taken to be the one determined by (31) and (32) with the operators \( R \) and \( P \) replaced by the classical phase-space variables and, in order to take care of the noncommutativity of the phase-space, the \( \star^\prime \)-value equation (18) is replaced by the prescription

\[
H(p, q) \star^\prime \rho_W = E\rho_W,
\]

where

\[
\star^\prime \equiv \star_h \star_\theta \star_\bar{\theta},
\]

\[
\star_\theta = \exp \left[ \frac{i\hbar}{2} \sum_{i,j} \theta_{ij} (\overleftarrow{\partial}_{q_i} \cdot \overrightarrow{\partial}_{q_j} - \overrightarrow{\partial}_{q_i} \cdot \overleftarrow{\partial}_{q_j}) \right],
\]

\[
\star_\bar{\theta} = \exp \left[ \frac{i\hbar}{2} \sum_{i,j} \bar{\theta}_{ij} (\overleftarrow{\partial}_{p_i} \cdot \overrightarrow{\partial}_{p_j} - \overrightarrow{\partial}_{p_i} \cdot \overleftarrow{\partial}_{p_j}) \right],
\]

and \( \star_h \) is the Moyal \( \star \)-operator defined in (49). Note that this approach hinges on the criterion that the noncommutative algebra (21, 22, 23) can be derived via the composition \( \star_\theta \star_\bar{\theta} \) in (50).

For the particular case when \( \ddot{\theta} = 0 \) \([P_i, P_j] = 0\), the energy eigenstates and Wigner function for the 2-dimensional Landau problem obtained with the prescription \( \star^\prime \) and those obtained with the formalism described before (Eqs. 41 and 42) turn out to be the same. The reason becomes obvious when we note that when acting with the operator \( \star_\theta \) from the right on the classical Hamiltonian yields the operator

\[
\hat{H}_{nc} = \frac{1}{2m} \left[ (1 + \kappa)\hat{p}_1 - \frac{eB}{2c}\hat{q}_2 \right]^2 + \left[ (1 + \kappa)\hat{p}_2 + \frac{eB}{2c}\hat{q}_1 \right]^2,
\]

(53)
where $\hat{p}_1$, $\hat{p}_2$ and $\hat{q}_1$, $\hat{q}_2$ are momenta and position operators, respectively, in the coordinate representation. Defining an effective $c$-Hamiltonian by replacing these operators by their corresponding $c$-dynamical variables, results in the effective $c$-number Hamiltonian

$$H_{\text{eff}} = \frac{(1 + \kappa)^2}{2m} \hat{p}^2 + \frac{m\omega^2}{8} \hat{q}^2 + \frac{(1 + \kappa)\omega}{2}(q_1\hat{p}_2 - q_2\hat{p}_1),$$

(54)

with $\omega = \frac{eB}{mc}$. But (54) is for this particular case the same as the Weyl function that we would get from the Weyl-Wigner-Groenewold-Moyal formalism. Indeed, by virtue of the condition (28), the constants $\lambda$ and $\mu$, appearing as a result of the transformations (29) and (30), are related by

$$\mu = \frac{1 \pm \sqrt{1 - \theta \bar{\theta}}}{2\lambda}.$$  

(55)

So that when $\bar{\theta} = 0$, $\mu$ and $\lambda$ both need necessarily be equal to 1, and the Weyl function $H^W$ derived in (54) turns out to be the same (after setting $\mu = 1$, $\lambda = 1$, and $\bar{\theta} = 0$) to the effective Hamiltonian (54). This is of course not true for the more general cases where the Weyl equivalent to a quantum operator is different from the classical operator.

Furthermore, when $\theta, \bar{\theta} \neq 0$ it also follows clearly from (55) that $\lambda$ and $\mu$ can not be chosen simultaneously to be equal to 1. Hence the results obtained for the energy eigenvalues and the Wigner function will be quantitatively quite different for the two approaches (compare with results in [12]), and in fact the correct procedure is the one which uses the mappings (29), (30) and the $\star$-value equation (18) leading to equations (47) and (48).

4 The harmonic oscillator in noncommutative phase-space

Another quantum mechanical problem on the noncommutative plane that has been extensively considered in the literature is that of a particle in an external central potential described by the Hamiltonian

$$H(P, R) = \frac{P^2}{2m} + V(|R|^2),$$

(56)

where $P$ and $R$ satisfy the algebra (21, 22, 23). Note in particular that for a free particle the mapping (30) leads back to the Landau problem considered in the previous section when we identify $\bar{\theta}$ with the external constant magnetic field.

From the extended noncommutative phase-space point of view of the Weyl-Wigner-Groenewold-Moyal formalism, general solutions to (56) for the energy
spectrum and Wigner functions can become quite complicated depending on the form of the potential. One reason for this is that even when \( V(|R|^2) = V((\lambda Q_i - \frac{1}{2\lambda} \theta_{ij} \Pi_j)^2) \) is of a polynomial form in the argument, it clearly follows that

\[
(|R|^{2m+2n})^W = (|R|^{2m})^W \ast (|R|^{2n})^W \neq (|R|^{2m})^W \ast (|R|^{2n})^W
\]

except for the case when \( m = n \). Hence the Weyl \( c \)-functions corresponding to the potential part of the Hamiltonian are not, in general, just the classical functions resulting from replacing the operators \( Q \) and \( \Pi \) by their corresponding classical canonical variables. This will only be so for polynomial functions of the form \( V(|R|^2) = \sum_n a_n |R|^{2n} \). It is not our objective here however to pursue the discussion for the general case, as it will suffice for our purposes to concentrate on the problem of the harmonic oscillator in the noncommutative phase-space plane. We shall therefore consider the quantum Hamiltonian

\[
H(P, R) = \frac{P^2}{2m} + \frac{m \omega^2}{2} |R|^2 = \frac{1}{2m} (\mu \Pi_1 + \frac{1}{2\mu} \bar{\theta} Q_2)^2 + \frac{1}{2m} (\mu \Pi_2 - \frac{1}{2\mu} \bar{\theta} Q_1)^2
+ \frac{m \omega^2}{2} (\lambda Q_1 - \frac{1}{2\lambda} \theta \Pi_2)^2 + \frac{m \omega^2}{2} (\lambda Q_2 + \frac{1}{2\lambda} \theta \Pi_1)^2.
\]

Rearranging terms, (57) reads

\[
H(P, R) = \alpha^2 Q^2 + \beta^2 \Pi^2 + \left( \frac{\bar{\theta}}{2m} + \frac{m \omega^2 \theta}{2} \right) (\Pi_1 Q_2 - \Pi_2 Q_1),
\]

where

\[
\alpha^2 = \left( \frac{\lambda^2 m \omega^2}{2} + \frac{\bar{\theta}^2}{8m \mu^2} \right),
\]

\[
\beta^2 = \left( \frac{\mu^2}{2m} + \frac{m \omega^2 \theta^2}{8 \lambda^2} \right).
\]

Introducing now the creation and annihilation operators

\[
\hat{a}_i^\dagger = \frac{\alpha}{\sqrt{2 \hbar \alpha \beta}} Q_i - i \frac{\beta}{\sqrt{2 \hbar \alpha \beta}} \Pi_i,
\]

\[
\hat{a}_i = \frac{\alpha}{\sqrt{2 \hbar \alpha \beta}} Q_i + i \frac{\beta}{\sqrt{2 \hbar \alpha \beta}} \Pi_i,
\]

we can write (58) as

\[
H(P, R) = 2\hbar \alpha \beta (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1)
- i\hbar (\frac{\bar{\theta}}{2m} + \frac{m \omega^2 \theta}{2}) (\hat{a}_1 \hat{a}_2^\dagger - \hat{a}_2 \hat{a}_1^\dagger).
\]

Note that in the above

\[
[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij},
\]
\[
[\hat{a}_i, \hat{a}_j] = [\hat{a}^\dagger_i, \hat{a}^\dagger_j] = 0. \tag{65}
\]

Note also that the angular momentum term
\[
L = (\hat{\alpha}_1 \hat{\beta} - \hat{\alpha}_2 \hat{\beta}) \tag{66}
\]

in (63) commutes with the number operator \( N = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \) and so it is a constant of the motion, and they both together form a complete set of commuting observables. Indeed, introducing the new annihilation and creation operators
\[
\hat{A}_\pm = \frac{1}{\sqrt{2}} (\hat{a}_1 \mp i \hat{a}_2), \tag{67}
\]
\[
\hat{A}_\pm^\dagger = \frac{1}{\sqrt{2}} (\hat{a}_1^\dagger \pm i \hat{a}_2^\dagger), \tag{68}
\]

which satisfy the commutation relations
\[
[\hat{A}_\pm, \hat{A}_\pm] = [\hat{A}_\pm^\dagger, \hat{A}_\pm^\dagger] = 0, \tag{69}
\]
\[
[\hat{A}_\pm, \hat{A}_\pm^\dagger] = [\hat{A}_\pm^\dagger, \hat{A}_\pm^\dagger] = 0, \tag{70}
\]
\[
[\hat{A}_\pm^\dagger, \hat{A}_\pm] = 1, \ [\hat{A}_\mp, \hat{A}_\mp^\dagger] = 1, \tag{71}
\]
\[
[\hat{A}_\mp, \hat{A}_\pm^\dagger] = 0, \ [\hat{A}_\pm, \hat{A}_\mp^\dagger] = 0. \tag{72}
\]

we have that the number operators
\[
N_+ = \hat{A}_+^\dagger \hat{A}_+, \tag{73}
\]
\[
N_- = \hat{A}_-^\dagger \hat{A}_-, \tag{74}
\]

form a complete set of commuting observables, whose spectra is the sequence of non-negative integers
\[
n_+ = 0, 1, \ldots \quad n_- = 0, 1, \ldots,
\]

respectively. Their common eigenstates are
\[
|n_+ n_-\rangle = (n_+!n_-!)^{-\frac{1}{2}} (\hat{A}_+^\dagger)^{n_+} (\hat{A}_-^\dagger)^{n_-} |00\rangle, \tag{75}
\]
such that
\[
N_\pm |n_+ n_-\rangle = n_\pm |n_+ n_-\rangle. \tag{76}
\]

We can therefore write
\[
H(\mathbf{P}, \mathbf{R}) = 2\hbar \alpha \beta (N_+ + N_- + 1) - \hbar (\frac{\bar{\theta}}{2m} + \frac{m\omega^2}{2}) (N_+ - N_-), \tag{77}
\]
and

\[ H(P, R)|_{n+ n-} = \langle n+ n- | H(\text{P}, \text{R}) | n+ n- \rangle = \langle n+ n- | \left[ 2\hbar \alpha \beta (n_+ + n_- + 1) - \hbar (\frac{\hat{\theta}}{2m} + \frac{m\omega^2}{2}) (n_+ - n_-) \right] | n+ n- \rangle. \quad (78) \]

Let us now denote by \( \bar{A}_\pm \) and \( A_\pm \) the classical Weyl-equivalents to the operators \( \hat{A}^\dagger_\pm, \hat{A}_\pm \), respectively. In this holomorphic coordinates the Moyal \( \star \)-operator is given by

\[ \star = \exp \left[ \frac{1}{2} \left( \frac{\hat{\theta}}{2m} \frac{\partial}{\partial A_\pm} - \frac{\hat{\theta}}{2m} \frac{\partial}{\partial \bar{A}_\pm} - \frac{\hat{\theta}}{2m} \frac{\partial}{\partial A_-} + \frac{\hat{\theta}}{2m} \frac{\partial}{\partial \bar{A}_-} \right) \right]. \quad (79) \]

We thus have that

\[ \bar{A}_\pm \star A_\pm = \bar{A}_\pm A_\pm - \frac{1}{2} \gamma \]

and the Weyl c-function corresponding to (77) is

\[ H^W(\xi_1, \xi_2) = 2\hbar \alpha \beta (\xi_1 + \xi_2) - \hbar (\frac{\hat{\theta}}{2m} + \frac{m\omega^2}{2}) (\xi_1 - \xi_2), \quad (81) \]

where \( \xi_1 := \bar{A}_+ A_+ \) and \( \xi_2 := \bar{A}_- A_- \).

Setting

\[ \gamma := \left( \frac{\hat{\theta}}{2m} + \frac{m\omega^2}{2} \right), \]

and rearranging terms, we can write

\[ H^W(\xi_1, \xi_2) = (2\hbar \alpha \beta - \hbar \gamma)\xi_1 + (2\hbar \alpha \beta + \hbar \gamma)\xi_2. \quad (83) \]

It is easy to see that for this particular form of the Weyl-Hamiltonian function the \( \star \)-value equation (79) yields

\[ H^W(\xi_1, \xi_2) \star \rho_W = [(2\hbar \alpha \beta - \hbar \gamma)\xi_1 + (2\hbar \alpha \beta + \hbar \gamma)\xi_2 + \frac{1}{2} (2\hbar \alpha \beta - \hbar \gamma)(\bar{A}_+ \frac{\partial}{\partial A_+} - A_+ \frac{\partial}{\partial \bar{A}_+}) + \frac{1}{2} (2\hbar \alpha \beta + \hbar \gamma)(\bar{A}_- \frac{\partial}{\partial A_-} - A_- \frac{\partial}{\partial \bar{A}_-}) - \frac{1}{4} (2\hbar \alpha \beta - \hbar \gamma) \frac{\partial^2}{\partial A_+ \partial A_+} - \frac{1}{4} (2\hbar \alpha \beta + \hbar \gamma) \frac{\partial^2}{\partial A_- \partial A_-} - E] \rho_W = 0. \quad (84) \]

The above equation can now be readily solved for the energy spectrum and Wigner function by separation of variables and by following a procedure similar to that used in Sec.3. We thus get the set of ordinary differential equations

\[ [\xi_1 - \frac{1}{4} (\frac{\partial}{\partial \xi_1} + \xi_1 \frac{\partial^2}{\partial \xi_1^2}) - \varepsilon_1] U(\xi_1) = 0, \quad (85) \]
\[ [\xi_2 - \frac{1}{4}(\frac{\partial}{\partial \xi_2} + \xi_2 \frac{\partial^2}{\partial \xi_2^2}) - \varepsilon_2]V(\xi_2) = 0, \quad (86) \]

where \( \rho_W = U(\xi_1)V(\xi_2) \), and \((2\hbar\alpha\beta - \hbar\gamma)\varepsilon_1 + (2\hbar\alpha\beta + \hbar\gamma)\varepsilon_2 = E \).

The explicit solutions to (85) and (86) in terms of Laguerre polynomials are

\[ U(\xi_1) = e^{-2\xi_1} L_{n_1}(4\xi_1), \quad (87) \]
\[ V(\xi_2) = e^{-2\xi_2} L_{n_2}(4\xi_2), \quad (88) \]

where \( n_1, n_2 \) are non-negative integers, and

\[ \varepsilon_1 = (n_1 + \frac{1}{2}), \quad \varepsilon_2 = (n_2 + \frac{1}{2}). \quad (89) \]

Hence

\[ E = (2\hbar\alpha\beta)(n_1 + n_2 + 1) + \hbar\gamma(n_1 - n_2), \quad (90) \]

and, in terms of canonical phase-space dynamical variables,

\[ \rho_W = \exp[\frac{\alpha}{\beta}(q^2 + \frac{\beta}{\alpha}p^2)]L_{n_1}(\frac{2}{\hbar}(\frac{\alpha}{\beta}q^2 + \frac{\beta}{\alpha}p^2 + 2(q_1p_2 - q_2p_1))) \times L_{n_2}(\frac{2}{\hbar}(\frac{\alpha}{\beta}q^2 + \frac{\beta}{\alpha}p^2 - 2(q_1p_2 - q_2p_1))). \quad (91) \]

Substituting (59), (60) and (82) into (90) we arrive at the final following expression for the energy spectrum of the harmonic oscillator problem in noncommutative phase-space

\[ E = \frac{\hbar}{2} \sqrt{4\omega^2 + (m\omega^2\theta - \frac{\theta}{m})^2} (n_1 + n_2 + 1) + (\frac{\theta}{m} + m\omega^2\theta)(n_1 - n_2). \quad (92) \]

This expression is in agreement with that reported in the literature by other authors (see e.g. [6], [14]) who derived it essentially by splitting the algebra (21), (22), (23) into two independent subalgebras and solving the quantum energy eigenvalue equation after performing a Bogolyubov transformation. There are however a few remarks that should be made here. First, in the above mentioned papers the authors consider three possible cases which, in our notation, correspond to \( \kappa = 1 - \theta \bar{\theta} = 0 \) (the so called “critical point” case), and \( \kappa > 0, \kappa < 0 \).

The solution (92) corresponds to the case \( \kappa > 0 \). For the “critical point” case (\( \kappa = 1 - \theta \bar{\theta} = 0 \)) we obtain, as so do the authors in the above mentioned references, that \( \alpha\beta = \frac{1}{4}(m\omega^2\theta + \frac{\theta}{m}) \) so that the energy spectrum reduces to that of a single harmonic oscillator:

\[ E = \hbar(m\omega^2\theta + \frac{1}{m\theta})(n_1 + \frac{1}{2}). \quad (93) \]
Furthermore, the phase-space volume elements in the coordinates $\mathbf{R}, \mathbf{P}$ in the noncommutative Heinsenberg algebra are related to the commutative ones $\mathbf{Q}, \mathbf{Π}$, by
\[
dR_1 \, dR_2 \, dP_1 \, dP_2 = \left| J \left( \frac{\mathbf{R}, \mathbf{P}}{\mathbf{Q}, \mathbf{Π}} \right) \right| \, dQ_1 \, dQ_2 \, dΠ_1 \, dΠ_2, \tag{94}
\]
and since by \((29), (30)\), the Jacobian turns out to be zero for this case it follows that the density of states for a fixed energy becomes degenerate. Hence the designation of “critical point” for this particular situation.

Note on the other hand that the additional restriction \((55)\) implied by the mappings \((29), (30)\) in our formalism, precludes the case $\kappa<0$, since the parameters $\lambda$ and $\mu$ are required by hermicity to be real. Furthermore, while these parameters are irrelevant to the energy spectrum problem, as they do not appear in the final expression, this is clearly not so for the Wigner function \((91)\) and the energy eigenstates for the problem. We thus have a complete 1-free-parameter set of solutions which lead to the same energy spectrum for the harmonic oscillator problem but, by virtue of the expectation value equation \((c.f. \ (7))\)
\[
\langle \Omega(\mathbf{P}, \mathbf{R}) \rangle = \int \int dp dq \, \rho W \, \Omega W(p, q), \tag{95}
\]
the spectrum of other observables of the theory may be dependent on this parameter. This, as well as its possible physical implications, remain to be investigated.

In \[15\] the harmonic oscillator problem is considered from the point of view of quantum deformation via the prescription \[14\]. Here again, as in the case of the Landau problem in noncommutative phase-space, discussed at the end of the previous section, we are met both with the same conceptual and computational differences in the derivation of the energy spectrum and the Wigner function. First, of all the algebra $[R_1, R_2] = iℏ\theta$, $[P_1, P_2] = -iℏ\theta$, $[Q_i, P_j] = iℏδ_{ij}$, initially considered in that work is incompatible with the mappings \((29), (30)\) and the condition \((55)\). Second the calculation of the Wigner function obtained for this case as well as for the more general noncommutative phase-space algebra by means of the Weyl-Moyal correspondence \[29\] leads to results quite different to \([71]\), for the same reasons as those discussed at the end of Sec.3. It is \([91]\) that gives the correct quantum mechanics for the problem, which again exemplifies our contention that it is the extended Weyl-Wigner-Groenewold-Moyal formalism the correct procedure to follow when considering these type of problems.

5 Discussion and conclusions

We have constructed a quantum mechanics over the noncommutative phase-space $\{\mathbf{R}, \mathbf{P}\}$, whose algebra is given by the commutators in \((21), (22), (23)\), by extending the Weyl-Wigner-Groenewold-Moyal formalism with the mappings
which can be viewed as the quantum mechanical equivalent of the Seiberg-Witten map in field theory. In this way operators defined over the quantum variables \( \{ R, P \} \) are first re-expressed in terms of the ordinary quantum mechanics position and momentum operators \( \{ Q, \Pi \} \) and then their corresponding \( c \)-\( \text{Weyl} \) equivalents are constructed by following the usual procedures of the Weyl-Wigner-Groenewold-Moyal formalism.

In particular, given a quantum Hamiltonian \( H(P, R, t) \) which determines the time evolution of the system, the above procedure can be used to obtain its \( c \)-\( \text{Weyl} \) equivalent which in turn can be used in the \( \star \)-value equation (18) to derive the Wigner distribution function for the problem under consideration. We stress here the fact, as was elaborated in the text, that the \( c \)-\( \text{Weyl} \) equivalent of the original Hamiltonian quantum operator is not in general equal to the \( c \)-function resulting from replacing the operators \( Q, \Pi \) in the former by their corresponding classical canonical dynamical variables.

We have applied the above considerations to two exactly soluble problems and have specifically shown that the use of the Weyl-Moyal equivalence, as given in (49), leads to different results for the energy spectrum and the Wigner function for these problems, thus verifying our contention that it is either the intrinsically noncommutative operator space approach or the extended Weyl-Wigner-Groenewold-Moyal formalism the appropriate ones for constructing the quantum mechanics over the noncommutative phase-space. Furthermore, as noted in the introduction, the former is hard to implement in explicit calculations for non-exactly soluble problems, the study of the noncommutative effects by means of perturbations can be best carried out via series expansions of the Weyl and Wigner functions in the extended Weyl-Wigner-Groenewold-Moyal formalism.

The essential difference between the approach advocated here of extending to noncommutative phase-space the Weyl-Wigner-Groenewold-Moyal formalism, and the prescription for deformation quantization contained in equations (49) and (50), is that the former is unequivocal in the sense that to a given quantum operator with arguments in the algebra (21, 22, 23) there corresponds a unique Weyl function determined by (5) and (6). For a given quantum Hamiltonian, it is this Weyl function and the \( \star \)-value equation (18) that we claim give the correct Wigner function and energy eigenvalues for the problem under consideration.

We have shown that when \( \bar{\Theta} = 0 \) in the Heisenberg algebra (21, 22, 23), there is at least one solution \( (\lambda = \mu = 1) \) of equations (24) for which the Wey-Moyal correspondence (49), (50), gives quantizations equivalent to the extended Weyl-Wigner-Groenewold-Moyal formalism, for the problems considered. We have also shown, however, that this is a consequence of the particular situation stemming from the fact that the \( c \)-\( \text{Weyl} \) functions related to the specific quantum Hamiltonians are indeed those resulting from replacing the \( \Pi \) and \( Q \) operators by their corresponding classical dynamical variables. In more general cases the two quantization schemes would not be equivalent, even for the \( \bar{\Theta} = 0 \) noncommutative Heisenberg algebra. For the \( \Theta \neq 0 \) case there seems not to be much
sense in using (49) to derive the Wigner function, since by this procedure the first two $\star$-products in the composition (50) lead to no effective classical Hamiltonian in terms of canonical dynamical variables that would give sense to the third Moyal product in the composition and hence to a phase-space quantum mechanics.

Another issue that was mentioned cursively in the text and needs further investigation is the analysis and classification of the more general solutions to the set of conditions (25), and their possible physical implications.

We conclude by remarking that deformation quantization would be of course the natural procedure to follow when given a classical Hamiltonian over classical phase-space one would try to infer the corresponding noncommutative quantum one by some $\star$-operator. In the context of deformation quantization one starts from a pair of $c$-functions of the classical dynamical variables and quantum deforms its point product by means of a $\star$-multidifferential operator. There are many possible choices for these operators that satisfy the usual properties of associativity, classical and semi-classical limits. The universal one being the Kontsevich product. Here one would also have to deal with the associated operator ordering problems, in addition to the different possible choices of the $\star$-product.

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