Doubly Robust Estimation under Covariate-induced Dependent Left Truncation

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Abstract

In prevalent cohort studies with follow-up, the time-to-event outcome is subject to left truncation when only subjects with event time greater than enrollment time are included. In such studies, subjects with early event times tend not to be captured, leading to selection bias if simply ignoring left truncation. Conventional methods adjusting for left truncation tend to rely on the (quasi-)independence assumption that the truncation time and the event time are “independent” on the observed region. This assumption is subject to failure when there is dependence between the truncation time and the event time possibly induced by measured covariates. Inverse probability of truncation weighting leveraging covariate information can be used in this case, but it is sensitive to misspecification of the truncation model. In this work, we first apply the semiparametric theory to find the efficient influence curve of the expectation of an arbitrary transformed survival time in the presence of covariate-induced dependent left truncation. We then use it to further construct estimators that are shown to enjoy double-robustness properties: 1) model double-robustness, that is, they are consistent and asymptotically normal (CAN) when the estimators for the nuisance parameters are both asymptotically linear and one of the two
estimators is consistent, but not necessarily both; 2) rate double-robustness, that is, they are CAN when both of the nuisance parameters are consistently estimated and the error product rate under the two nuisance models is faster than root-$n$. Simulation studies demonstrate the finite sample performance of the estimators.

**Keywords:** Conditional quasi-independence; Efficient influence curve; Selection bias; Semiparametric theory.

1 Introduction

A time-to-event outcome in a prevalent cohort study is subject to left truncation because, starting from a well-defined time zero, such as the onset of a disease, only subjects with event times greater than the enrollment times are included in the data set. Subjects with early or short event times therefore tend not to be captured in the data; such a phenomenon is often referred to as left truncation. For example, in certain pregnancy studies women typically enroll after clinical recognition of their pregnancies, and women with early pregnancy losses tend not to be included in the data (Ying et al., 2020). As another example, an aging study using the data collected between 1965 and 2012 from the Honolulu Heart Program (HHP, 1965-1990) and the subsequent Honolulu Asia Aging Study (HAAS, 1991-2012) focused on subjects that were alive at the start of HAAS. When considering age as the time scale of interest, age at death was left truncated by age at the start of HAAS.

Left truncation can lead to selection bias and invalidate statistical analysis if not properly accounted for. Conventional methods for handling left truncation typically rely on the random truncation assumption that the truncation time and event time are independent (Lynden-Bell, 1971; Woodroofe, 1985; Wang et al., 1986; Wang, 1989, 1991; Gross, 1996; Gross and Lai, 1996; Shen, 2010). This can often be weakened to a quasi-independence assumption (Tsai, 1990). Such independence or quasi-independence assumptions may be violated in practice. For instance, in the retirement center example of Klein and Moeschberger (2003), the life lengths of the individuals in the retirement center are left truncated because individuals must survive long enough to enter the retirement center. However, individuals’ life lengths and entry times may be dependent because
individuals who entered the retirement home earlier may have received better medical attention and therefore lived longer (Chaieb et al., 2006).

In the presence of dependent left truncation, copula models (Chaieb et al., 2006; Emura et al., 2011; Emura and Wang, 2012) and structural transformation models (Efron and Petrosian, 1994; Chiou et al., 2019) have been proposed to handle the dependence between the left truncation time and the event time under different assumptions. Copula models rely on strong modeling assumptions for the dependency, whilst structural transformation approaches model a latent quasi-independent truncation time as a function of the observed dependent truncation time and the event time. Neither of the above methods make use of covariate information. There are also methods that incorporate left truncation time as a covariate in the event time model (Mackenzie, 2012; Cheng and Wang, 2015) which, as pointed out in Vakulenko-Lagun et al. (2021), is biologically unjustified if the left truncation time is study-specific and not related to the event process. A more detailed discussion of the above methods can be found in Vakulenko-Lagun et al. (2021). On the other hand, Vakulenko-Lagun et al. (2021) proposed two inverse probability weighted (IPW) estimators of survival probabilities that account for the dependence of the left truncation and the event time induced by covariates. The inverse probability weights of their methods are obtained from modeling the conditional distribution of left truncation time given the covariates.

IPW approaches (Rosenbaum and Rubin, 1983) are known to rely on a correct model specification for estimating the weights. They are also known to be inefficient. This motivated us to seek estimators that are more efficient and that provide extra protection against model misspecification. For the parameter of interest, we focus on the expectation of a transformed event time, which, for example, includes survival probabilities and restricted mean survival times. We will use semiparametric theory to find the efficient influence curve (EIC), which typically suggests estimators with good performance in both the missing data literature and the causal inference literature. Indeed as will be seen in our case, the EIC suggests a doubly robust estimator, as well as the IPW estimator and certain regression-based estimators as special cases. The doubly robust estimator is shown to enjoy model double-robustness (Smucler et al., 2019), that is, it is consistent and asymptotically normal (CAN) when one of the two nuisance parameters is consistently estimated at root-\(n\) rate, but not necessarily both; and rate double-robustness, that is, it is CAN and achieves the semipara-
metric efficiency bound when both nuisance parameters are consistently estimated and the product of the error rates under the two nuisance models is faster than root-$n$. Cross-fitting is applied when the nuisance parameters are estimated using machine learning or nonparametric methods.

We remark that there has been a growing literature on doubly robust estimators to handle missing or coarsened data including right censoring for time-to-event data, as well as non-randomized treatment assignment in causal inference (Robins et al., 1995; Bang and Robins, 2005; Tsiatis, 2006; Tchetgen Tchetgen et al., 2010; Robins and Rotnitzky, 2005; Robins, 1997, 1998, 1999; van der Laan and Robins, 2003, for examples). Our work contributes to that thread of literature by being the first, to the best of our knowledge, to explore doubly robust estimation in the presence of left truncation, a common type of selection bias for survival data.

The rest of the paper is organized as the following. In Section 2, we introduce notation and assumptions and show the parameter of interest can be identified from inverse probability weighting. In Section 3, we derive the EIC and construct an estimation function that is doubly robust in the sense that it has mean zero when one of the two nuisance parameters takes value at the truth. In Section 4, we show that the estimator from the doubly robust estimation function enjoys model double robustness and rate double robustness. Section 5 shows simulation results in finite sample. Section 6 contains further discussion.

2 Inverse Probability Weighting Identification

Let $Q, T, Z$ be random variables denoting the left truncation time, the event time of interest, and the baseline covariates, respectively. Suppose that $Q$ and $T$ are both absolutely continuous random variables. Let $F, G$ and $H$ be the conditional cumulative distribution function (CDF) of $T$ given $Z$, $Q$ given $Z$, and the CDF of $Z$ in full data, i.e., if there were no left truncation. Let $f, g$ and $h$ be the corresponding full data densities, or probability function in the case of discrete $Z$.

In the presence of left truncation, we observe $O := (Q, T, Z)$ only if $Q < T$. The observed data distribution is the conditional full data distribution of $O$ given $Q < T$. In this paper we will use $P^*$ to denote the probability operator of the full data, $E^*$ the expectation with respect to the full data distribution; $P$ the probability operator of the observed data, i.e. $P(\cdot) = P^*(\cdot|Q < T)$, and $E$
the expectation with respect to the observed data distribution. In addition, we will use \( P \) and \( p \) to denote the CDF and the density or probability function for the observed data, respectively; for example, \( P_{Q|Z}, p_{Q|Z}, P_{Q,Z} \) and \( p_{Q,Z} \) etc.

We focus on an estimand that is the mean of a transformed event time in the full data:

\[
\theta := \mathbb{E}^*\{\nu(T)\} = \int \nu(t)f(t|z)h(z) \, dt \, dz,
\]

where \( \nu \) is a given transformation such that \( \nu(T) < C \) almost surely for some constant \( C > 0 \).

**Remark 1.** Expression (1) include commonly considered parameters of interest for time-to-event data. For example, when \( \nu(t) = 1(t > t_0) \) for some fixed \( t_0 > 0 \), \( \theta = \mathbb{P}^*(T > t_0) \) is the survival probability; when \( \nu(t) = \min(t, t_0) \) for some \( t_0 > 0 \), \( \theta \) is the restricted mean survival time (RMST).

We make the following conditional quasi-independence assumption.

**Assumption 1 (Conditional quasi-independence).** The observed data density for \((Q, T, Z)\) satisfies

\[
p_{Q,T,Z}(q, t, z) = \begin{cases} 
f(t|z)g(q|z)h(z)/\beta, & \text{if } t > q, \\
0, & \text{otherwise},
\end{cases}
\]

where

\[
\beta = \mathbb{P}^*(Q < T) = \int \mathbb{1}(q < t)f(t|z)g(q|z)h(z) \, dt \, dq \, dz.
\]

**Remark 2.** Assumption 1 is a generalization of the quasi-independence assumption in Tsai (1990) to settings with covariates. It is weaker than the conditional independence assumption that \( Q \perp \perp T \mid Z \) for the full data, which was assumed in Shen (2003) and Vakulenko-Lagun et al. (2021). Unlike the conditional independence assumption, Assumption 1 only restricts the distribution of \((Q, T, Z)\) on the observed region \( \{(q, t, z) : q < t\} \). Notice that Assumption 1 does not imply that \( Q \) is independent of \( T \) given \( Z \) in observed data since \( p_{Q,T|Z} \neq p_Q|Z \cdot p_T|Z \), as shown in the Appendix.

In addition to the conditional quasi-independence assumption, we also need the following positivity assumption. This assumption ensures that there are enough data observed to identify the parameter of interest.

**Assumption 2 (Positivity).** \( \beta(Z) = \mathbb{P}^*(Q < T|Z) = \int \mathbb{1}(q < t)g(q|Z)f(t|Z) \, dq \, dt > 0 \) a.s.
Under Assumptions 1 and 2, θ can be identified from observed data. This can be shown by an IPW type of argument below.

Lemma 1. Under Assumption 1 and 2,

$$\theta = \frac{E\left\{ \nu(T) / G(T|Z) \right\}}{E\left\{ 1 / G(T|Z) \right\}}. \quad (4)$$

Moreover, G is identifiable from the observed data distribution. Therefore, θ is identifiable from the observed data distribution.

Based on Lemma 1, an estimator for θ from an observed random sample \((Q_i, T_i, Z_i)_{i=1}^n\) is the inverse probability of truncation weighted estimator

$$\hat{\theta}_{\text{IPW},Q} = \frac{\sum_{i=1}^n \nu(T_i) / G(T_i|Z_i)}{\sum_{i=1}^n 1 / G(T_i|Z_i)}, \quad (5)$$

where the \(1/G(T_i|Z_i)'s\) are the estimated inverse probability weights from the conditional distribution of \(Q\) given \(Z\) in the full data. This estimator coincides with one of the IPW estimators proposed in Vakulenko-Lagun et al. (2021) when estimating a full data survival probability.

## 3 Semiparametric Theory and Double Robustness

As mentioned in the introduction, the IPW estimator (5) above is known to rely on correct model specification and is known to be inefficient. Augmented IPW (AIPW) estimators are often developed in the literature to improve upon the IPW estimators over such weaknesses; that is, they are shown to provide extra robustness and to be more efficient.

In order to develop the improved estimators, we need to leverage the semiparametric theory and, as will become evident later, to better understand the truncation distribution.

### 3.1 On the distribution of \(Q\)

Due to the asymmetry between \(Q\) and \(T\), it is often useful to consider the “reverse time” for the distribution of \(Q\). In fact, without covariates the product-limit estimate of \(G\) can be derived by moving backwards in time (Wang, 1991; Gross, 1996).
Bickel et al. (1993) considered the “reverse time” counting processes, and here we extend them to the setting with covariates. For \( t \geq 0 \), let

\[
\bar{N}_Q(t) := 1(t \leq Q < T), \quad \bar{N}_T(t) := 1(t \leq T).
\]

with their natural history filtration \( \{\bar{F}_t\}_{t \geq 0} \) in the reversed time scale:

\[
\bar{F}_t := \sigma \{Z, 1(Q < T), 1(s \leq T), 1(s \leq Q < T) : s \geq t\}.
\]

Note that \( \bar{N}_Q \) is a decreasing process that starts at one and jumps to zero immediately after \( t = Q \) if \( Q < T \), and is always zero if \( Q \geq T \). Also \( \{\bar{F}_t\}_{t \geq 0} \) decreases as \( t \) increases.

Let \( \alpha \) be the reverse time hazard function of \( Q \) given \( Z \) for the full data; specifically,

\[
\alpha(q|z) := \lim_{h \to 0+} \frac{P^*(q - h < Q \leq q|Q \leq q, Z = z)}{h} = \lim_{h \to 0+} \frac{P^*(q - h < Q \leq q|Z = z)}{h \cdot P^*(Q \leq q|Z = z)} = \frac{g(q|z)}{G(q|z)}.
\]

It follows immediately that \( G(q|z) = \exp\{-\int_q^\infty \alpha(v|z)dv\} \) (see also the Appendix, in the proof of Lemma 1). This way either \( \alpha \) or \( G \) alone characterizes the full data distribution of \( Q \) given \( Z \).

Define the compensator \( \bar{A}_Q(t; G) \) for \( \bar{N}_Q \), where

\[
\bar{A}_Q(t; G) := \int_t^\infty 1(Q \leq s < T)\alpha(s|Z)ds = \int_t^\infty 1(Q \leq s < T)\frac{dG(s|Z)}{G(s|Z)}.
\]

We show in the Appendix that

\[
\bar{M}_Q(t; G) := \bar{N}_Q(t) - \bar{A}_Q(t; G)
\]

is a backwards martingale with respect to \( \{\bar{F}_t\}_{t \geq 0} \) if \( G \) is the true CDF of \( Q \) given \( Z \) in the full data. A stochastic process \( \{Y_t\}_{t \geq 0} \) is called a backwards martingale with respect to a set of decreasing \( \sigma \)-algebras \( \{\bar{F}_t\}_{t \geq 0} \) if \( \mathbb{E}(|Y_t|) < \infty \), \( Y_t \) is \( \bar{F}_t \)-measurable for all \( t \geq 0 \), and \( \mathbb{E}(Y_s|\bar{F}_t) = Y_t \) for all \( 0 \leq s \leq t \). If we reverse the time scale and define

\[
M_Q^\tau(t; G) := \bar{M}_Q(\tau - t; G), \quad \forall \ t \geq 0, \quad \tau > 0,
\]

\[
G^\tau_t := \bar{F}_{\tau-t}, \quad \forall \ t \geq 0,
\]

for some \( \tau > 0 \), then \( \{M_Q^\tau(t; G_0) : 0 \leq t \leq \tau\} \) is a martingale with respect to the filtration \( \{G^\tau_t\}_{0 \leq t \leq \tau} \).
3.2 Efficient influence curve

Semiparametric theory (Begun et al., 1983; Bickel et al., 1993; Tsiatis, 2006; Van der Vaart, 2000; Newey, 1994; Kosorok, 2008) is often leveraged to construct estimators with good properties, such as the AIPW estimators. In the following we also leverage the semiparametric theory to first compute the efficient influence curve (EIC) of \( \theta \), which in turn suggests a useful estimator that is shown to be doubly robust.

Consider the Hilbert space \( \mathcal{H} \) of all one-dimensional mean-zero measurable functions of \( O = (Q, T, Z) \) with finite second moments equipped with the covariance inner product, \( \langle h_1, h_2 \rangle = \mathbb{E}(h_1 h_2) \) for \( h_1, h_2 \in \mathcal{H} \). In the following, we first characterize the observed data tangent space in \( \mathcal{H} \). We then introduce and find an influence curve (IC); the projection of the IC onto the tangent space gives the EIC (Tsiatis, 2006; Bickel et al., 1993). Consider a regular (Bickel et al., 1993) parametric submodel \( \{P_\epsilon : \epsilon \in \mathbb{R}\} \) indexed by a real-valued parameter \( \epsilon \) for the distribution of \( (T, Q, Z) \), that satisfies Assumptions 1 and 2, and equals the true distribution of the observed data at \( \epsilon = 0 \). Its score function is

\[
S(O) = \left. \frac{d}{d\epsilon} \log p_\epsilon(O) \right|_{\epsilon=0};
\]

in the notation above as well as in the rest of the paper we suppress the dependence of \( S \) on the parametric submodel. The closure of the linear span of the score functions of all such regular parametric submodels is the observed data tangent space.

Denote

\[
L_2^0(P_{T,Z}) = \left\{ a \in L_2(P_{T,Z}) : \int a(t,z) \, dP_{T,Z}(t,z) = 0 \right\},
\]

\[
L_2^0(P_{Q,Z}) = \left\{ b \in L_2(P_{Q,Z}) : \int b(q,z) \, dP_{Q,Z}(q,z) = 0 \right\},
\]

\[
L_2^0(P_{Q,T,Z}) = \left\{ c \in L_2(P_{Q,T,Z}) : \int c(q,t,z) \, dP_{Q,T,Z}(q,t,z) = 0 \right\},
\]

where the above integrals are taken on the support of \( (T, Z) \), \( (Q, Z) \) and \( (Q, T, Z) \), respectively.

**Lemma 2.** The tangent space under the semiparametric model imposed by Assumptions 1 and 2 is

\[
\hat{P} = L_2^0(P_{T,Z}) + L_2^0(P_{Q,Z}).
\]
In addition, the orthogonal complement of the tangent space is

\[
(\dot{P})^\perp = \left\{ \xi \in L^0_2(P_{Q,T,Z}) : \begin{array}{l}
\mathbb{E}\{\xi(Q,T,Z)a(Q,Z)\} = 0, \quad \forall a \in L^0_2(P_{Q,Z}), \\
\mathbb{E}\{\xi(Q,T,Z)b(T,Z)\} = 0, \quad \forall b \in L^0_2(P_{T,Z}).
\end{array} \right\}.
\]

We now introduce the definition of IC. The parameter of interest, \(\theta = \theta(P)\), is pathwise differentiable with influence curve \(\varphi \in \mathcal{H}\) if, for any regular parametric submodel \(\{P_{\epsilon} : \epsilon \in \mathbb{R}\}\),

\[
\frac{d}{d\epsilon}\theta(P_{\epsilon}) \bigg|_{\epsilon=0} = \int \varphi(o)S(o)dP(o) = \mathbb{E}\{\varphi(O)S(O)\}.
\]

**Remark 3.** In the literature there are two practically exchangeable terms: influence function (IF) and influence curve (IC). Here we follow the convention as reviewed in Kennedy (2017), for example, to use IF for estimators in the next section, and use IC for parameters here. Note that if \(\varphi\) is the IF for an arbitrary regular asymptotically linear (RAL) estimator of \(\theta\), then the IF for any RAL estimator of \(\theta\) must lie in the space \(\{\varphi(O) + (\dot{P})^\perp\}\) (Tsiatis, 2006; Bickel et al., 1993).

For the rest of the paper, we make the following overlap assumption, which is stronger than Assumption 2.

**Assumption 3 (Overlap).** Suppose there exists \(0 < \tau_1 < \tau_2 < \infty\) such that \(T \geq \tau_1\) a.s., \(Q \leq \tau_2\) a.s.; also \(G(\tau_1|Z) \geq \delta_1\) a.s. and \(F(\tau_2|Z) \leq 1 - \delta_2\) a.s. for some constants \(\delta_1 > 0\) and \(\delta_2 > 0\).

Under Assumptions 1 and 3, we can show that \(\theta = \theta(P)\), as identified in Lemma 1 from observed data distribution, is pathwise differentiable with an influence curve as stated in Lemma 3. Recall that \(O = (Q, T, Z)\).

**Lemma 3.** Under the semiparametric model imposed by Assumptions 1 and 2 and at the law where Assumption 3 holds,

\[
\varphi(O; \theta, F, G, H) = \beta \cdot \left\{ \frac{\nu(T) - \theta}{G(T|Z)} - \int_0^\infty \frac{m(v, Z; F) - \theta F(v|Z)}{1 - F(v|Z)} \cdot \frac{dM_Q(v; G)}{G(v|Z)} \right\},
\]

is an influence curve for \(\theta\), where \(m(v, z; F) = \int_0^v \nu(t)dF(t|z)\) and \(\beta\) is defined in (3).

We can show that \(\varphi\) given in (9) lies in \(\dot{P}\), and is therefore the EIC.

**Corollary 1.** Under the semiparametric model imposed by Assumptions 1 and 2 and at the law where Assumption 3 holds, the influence curve \(\varphi\) given in (9) is the efficient influence curve (EIC) for \(\theta\), and the semiparametric efficiency bound for estimating \(\theta\) is \(\mathbb{E}(\varphi^2)\).
Remark 4. Chao (1987) derived the IF for the product-limit (PL) estimate of the survival function of $T$ under random left truncation, in the absence of covariates. Since the PL estimate is the nonparametric maximum likelihood estimator (NPMLE), which is known to be asymptotically efficient, the IF derived in Chao (1987) is the efficient IF (EIF). We show in the Appendix that when there are no covariates and $\theta = 1 - F(t_0)$, the (E)IC derived in this paper matches the (E)IF derived in Chao (1987).

3.3 Double robustness

The EIC in (9) requires the knowledge of $F$ and $G$, which are unknown in practice. However, in the following we derive a doubly robust score from the EIC which can be used to construct estimators with good properties.

Dropping the constant factor $\beta$ from the EIC (9), we have the estimating function:

$$U(\theta; F, G) = \frac{\nu(T) - \theta}{G(T|Z)} - \int_0^\infty \frac{m(v, Z; F) - \theta F(v|Z)}{1 - F(v|Z)} \cdot \frac{dM_Q(v; G)}{G(v|Z)}.$$  

(10)

It turns out that $U$ is a doubly robust estimating function in the sense described below. From here on we will allow $F$ to be different from the true full data CDF of $T$ given $Z$, and $G$ to be different from the true full data CDF of $Q$ given $Z$. Denote $\theta_0$ the true value of $\theta$, $F_0$ the true CDF of $T$ given $Z$ in the full data, and $G_0$ the true CDF of $Q$ given $Z$ in the full data.

Theorem 1 (Double robustness). Under Assumption 1, assuming $F$, $F_0$, $G$ and $G_0$ all satisfy Assumption 3, then

$$\mathbb{E}\{U(\theta_0; F, G)\} = 0$$

if either $F = F_0$ or $G = G_0$.

In the next section, we will construct estimators from $U$ above and show that they are consistent and asymptotically normal.

4 Estimation

Let $\{O_i\}_{i=1}^n$ be a random sample of size $n$ of the observed data, where $O_i = (Q_i, T_i, Z_i)$. According to Theorem 1, a natural estimating equation for $\theta$ would be $\sum_{i=1}^n U_i(\theta; F, G) = 0$ if $F$ and $G$ were
known, where $U_i$ is the $i$-th copy of $U$ with $O$ replaced by $O_i$. In practice, we would first estimate $F$ and $G$, and then use
\[
\sum_{i=1}^{n} U_i(\theta; \hat{F}, \hat{G}) = 0
\] (11)
to solve for $\theta$. Since $U(\theta; \hat{F}, \hat{G})$ is linear in $\theta$, we have an explicit solution to (11):
\[
\hat{\theta} = \left\{ \sum_{i=1}^{n} \frac{1}{G(T_i|Z_i)} - \int_{0}^{\infty} \frac{\hat{F}(v|Z_i)}{G(v|Z_i)(1 - F(v|Z_i))} d\hat{M}_{Q,i}(v; \hat{G}) \right\}^{-1} \cdot \left\{ \sum_{i=1}^{n} \frac{\nu(T_i)}{G(T_i|Z_i)} - \int_{0}^{\infty} \frac{m(v, Z_i; \hat{F})}{G(v|Z_i)(1 - F(v|Z_i))} d\hat{M}_{Q,i}(v; \hat{G}) \right\}.
\]

Before establishing the double robust properties we have immediately the following special cases in which we plug in trivial estimators for $F$ or $G$ in $\hat{\theta}$. The double robust properties in the next subsection will guarantee that they are CAN if conditions are met.

By plugging $\hat{F} \equiv 0$ into $\hat{\theta}$ above, we obtain the IPW estimator $\hat{\theta}_{IPW,Q}$ in (5), which only requires the estimation of $G$.

On the other hand, if we plug $\hat{G} \equiv 1$ into $\hat{\theta}$, we obtain
\[
\hat{\theta}_{Reg.T1} = \left( \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 - \hat{F}(Q_i|Z_i)} \right)^{-1} \left[ \frac{1}{n} \sum_{i=1}^{n} \nu(T_i)(1 - \hat{F}(Q_i|Z_i)) + m(Q_i, Z_i; \hat{F}) \right].
\] (12)

This can be seen as a regression based estimator because, conditioned on $Q$ and $Z$, we have
\[
\mathbb{E}^{*}\{\nu(T)|Q, Z\} = \mathbb{E}^{*}\{\nu(T)|T > Q, Q, Z\} \mathbb{P}^{*}(T > Q|Q, Z)
+ \mathbb{E}^{*}\{\nu(T)|T \leq Q, Q, Z\} \mathbb{P}^{*}(T \leq Q|Q, Z)
= \mathbb{E}^{*}\{\nu(T)|T > Q, Q, Z\} \{1 - F(Q|Z)\} + \int_{0}^{Q} \nu(t)dF(t|Z)
= \mathbb{E}\{\nu(T)|Q, Z\} \{1 - F(Q|Z)\} + m(Q, Z; \hat{F}).
\]

This way $\nu(T)\{1 - \hat{F}(Q|Z)\} + m(Q, Z; \hat{F})$ identifies the full data conditional expectation $\mathbb{E}^{*}\{\nu(T)|Q, Z\}$. Without left truncation we may take the simple average over $i = 1, ..., n$ to obtain an estimate of $\theta = \mathbb{E}^{*}\{\nu(T)\}$.

In the presence of left truncation, however, inverse probability weights $1/\{1 - \hat{F}(Q_i|Z_i)\}$ are needed and we therefore have (12).

Another regression based estimator is
\[
\hat{\theta}_{Reg.T2} = \left( \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 - \hat{F}(Q_i|Z_i)} \right)^{-1} \left[ \frac{1}{n} \sum_{i=1}^{n} \mu(Z_i; \hat{F}) \right].
\]
where \( \mu (Z_i; \hat{F}) = \int_{0}^{\infty} \nu (t) d \hat{F}(t|Z_i) \).

In the following we consider two scenarios for double robustness. In Section 4.1, we consider the scenario where both \( \hat{F} \) and \( \hat{G} \) are asymptotically linear. In Section 4.2, we construct estimators from a cross-fitting procedure, from which we can obtain \( \sqrt{n} \)-consistent estimators even if the two parameters \( F \) and \( G \) are estimated at a slower than \( \sqrt{n} \) rate.

4.1 Doubly robust estimator under asymptotic linearity

In this subsection, we consider the scenario where both \( \hat{F} \) and \( \hat{G} \) are asymptotically linear and the convergence rates are \( \sqrt{n} \). We first define some notation below:

\[
\| \hat{F} - F \|_{\sup, 2}^{2} := \mathbb{E} \left[ \sup_{t} \left| \hat{F}(t|Z) - F(t|Z) \right| \right],
\]

\[
\| \hat{G} - G \|_{\sup, 2}^{2} := \mathbb{E} \left[ \sup_{t} \left| \hat{G}(t|Z) - G(t|Z) \right| \right],
\]

where the expectations are with respect to the ‘hat’s and \( Z \). Furthermore, we define the cross integral products:

\[
\mathcal{D}_1(\hat{F}, \hat{G}; F, G) = \mathbb{E} \left[ \int_{0}^{\infty} \left\{ \hat{F}(t|Z) - F(t|Z) \right\} d \left\{ \hat{G}(t|Z) - G(t|Z) \right\} \right],
\]

\[
\mathcal{D}_2(\hat{F}, \hat{G}; F, G) = \mathbb{E} \left[ \int_{0}^{\infty} \left\{ \hat{G}(t|Z) - G(t|Z) \right\} d \left\{ \hat{F}(t|Z) - F(t|Z) \right\} \right].
\]

We denote \( \hat{\theta} = \hat{\theta}_{dr} \) when \( \hat{F} \) and \( \hat{G} \) are estimated from the entire observed sample. Under the assumptions below, we will show that \( \hat{\theta}_{dr} \) is doubly robust in the sense that it is consistent and asymptotically normal if both \( \hat{F} \) and \( \hat{G} \) satisfy the following and one of them is consistent.

**Assumption 4** (Uniform Convergence). We have

\[
\left\| \hat{F}(t|Z) - F^*(t|Z) \right\|_{\sup, 2} = o(1), \quad \left\| \hat{G}(t|Z) - G^*(t|Z) \right\|_{\sup, 2} = o(1),
\]

for some \( F^* \) and \( G^* \).

**Assumption 5** (Asymptotic Linearity). Suppose \( \hat{F} \) and \( \hat{G} \) are obtained from single index models.
such that

\[
\begin{align*}
\left\| \hat{F}(t|Z) - F^*(t|Z) - \left\{ \frac{1}{n} \sum_{i=1}^{n} \xi_{1,i}(t) \right\}^\top \zeta_1(Z) \right\|_{\sup,2} &= o(n^{-1/2}), \\
\left\| \hat{G}(t|Z) - G^*(t|Z) - \left\{ \frac{1}{n} \sum_{i=1}^{n} \xi_{2,i}(t) \right\}^\top \zeta_2(Z) \right\|_{\sup,2} &= o(n^{-1/2}),
\end{align*}
\]

where \( \xi_{1,i} \) and \( \xi_{2,i} \) \((i = 1, \ldots, n)\) are each i.i.d. random functions with mean zero and finite second moment, and \( \zeta_1 \) and \( \zeta_2 \) are transformations of \( Z \). In addition, we have either

\[ D_1(\hat{F}, \hat{G}; F^*, G^*) = o(n^{-1/2}), \] (15)

or

\[ D_2(\hat{F}, \hat{G}; F^*, G^*) = o(n^{-1/2}). \] (16)

Assumption 4 requires the uniform convergence of the estimators \( \hat{F} \) and \( \hat{G} \). Assumption 5 considers single index models which are often the cases for most parametric or semiparametric models used in practice. It requires both \( \hat{F} \) and \( \hat{G} \) to be asymptotically linear, and that one of the cross integral products converges to zero at a faster than \( \sqrt{n} \) rate. Assumptions on the cross integral products in (15) and (16) are required to handle the involvement of time \( t \) in \( F \) and \( G \). A similar regularity condition is also assumed in Hou et al. (2021, equation (12)).

The following Theorem formally states the double robustness of \( \hat{\theta}_{dr} \).

**Theorem 2.** Under Assumptions 1, 3 and 4, assume either \( F^* = F_0 \) or \( G^* = G_0 \). Then

\[ \hat{\theta}_{dr} \rightarrow_p \theta_0, \]

where \( \rightarrow_p \) indicates convergence in probability. In addition, if Assumptions 5 holds, then

\[ \sqrt{n}(\hat{\theta}_{dr} - \theta_0) \rightarrow_d N(0, \sigma^2), \]

where \( \rightarrow_d \) indicates convergence in distribution. Furthermore, when both \( F^* = F_0 \) and \( G^* = G_0 \), \( \sigma^2 \) can be consistently estimated by \( \hat{\beta}^2 \cdot \sum_{i=1}^{n} U_i^2(\hat{\theta}_{dr}, \hat{F}, \hat{G})/n \), where

\[ \hat{\beta} = \left\{ n^{-1} \sum_{i=1}^{n} 1/\hat{G}(T_i|Z_i) \right\}^{-1} \]

or

\[ \hat{\beta} = \left[ n^{-1} \sum_{i=1}^{n} 1/(1 - \hat{F}(Q_i|Z_i)) \right]^{-1}. \]
Remark 5. As a consequence of Theorem 2, the IPW estimator $\hat{\theta}_{IPW}$ is CAN when the conditions for $\hat{G}$ in Assumptions 4 and 5 are satisfied and $G^* = G_0$; and the regression based estimators $\hat{\theta}_{Reg.T1}$ and $\hat{\theta}_{Reg.T2}$ are CAN when the conditions for $\hat{F}$ in Assumptions 4 and 5 are satisfied and $F^* = F_0$. In addition, it can be shown that $\hat{\theta}_{Reg.T1}$ and $\hat{\theta}_{Reg.T2}$ are asymptotically equivalent.

4.2 Doubly robust estimator with cross-fitting

The results of the previous subsection mainly apply to parametric or semiparametric estimates of $F$ and $G$. When both models are wrong, we show in simulation studies of the next section that the bias of $\hat{\theta}_{dr}$ can be very large. As an alternative, we may consider using nonparametric or machine learning methods to estimate $F$ and $G$, which can be consistent under certain conditions. However, these estimators usually converge at a slower than $\sqrt{n}$ rate.

In order to incorporate estimators of $F$ and $G$ with convergence rates slower that $\sqrt{n}$, we utilize $K$-fold cross-fitting (Hasminskii and Ibragimov, 1978; Bickel, 1982; Robins et al., 2008; Chernozhukov et al., 2018). The details of the cross-fitted estimator $\hat{\theta}_{cf}$ is given in Algorithm 1. This estimator can be shown to have rate double robustness (Smucler et al., 2019; Hou et al., 2021) in the sense that will become clear below.

**Algorithm 1:** Estimation of $\theta$ via $K$-fold cross-fitting.

**Data:** split the data into $K$ folds of equal size with the index sets $I_1, \ldots, I_K$.

**for** each fold indexed by $k$ **do**

- estimate the nuisance parameters $F$ and $G$ using the out-of-$k$-fold data indexed by $I_{-k} = \{1, \ldots, n\} \setminus I_k$, and denote the estimates by $\hat{F}^{(-k)}$ and $\hat{G}^{(-k)}$.

**end**

**Result:** Obtain the estimator $\hat{\theta}_{cf}$ by solving

$$\sum_{k=1}^{K} \sum_{i \in I_k} U\{O_i; \theta, \hat{F}^{(-k)}, \hat{G}^{(-k)}\} = 0.$$
Let $O^\dagger = (Q^\dagger, T^\dagger, Z^\dagger)$ be an independent copy of the observed data. Define
\[
\|\hat{F} - F_0\|^2_{\dagger, \text{sup}, 2} := \mathbb{E}_{\dagger} \left( \sup_t \left| \hat{F}(t|Z^\dagger) - F_0(t|Z^\dagger) \right|^2 \right),
\]
\[
\|\hat{G} - G_0\|^2_{\dagger, \text{sup}, 2} := \mathbb{E}_{\dagger} \left( \sup_t \left| \hat{G}(t|Z^\dagger) - G_0(t|Z^\dagger) \right|^2 \right),
\]
where $\mathbb{E}_{\dagger}$ denotes the expectation taken with respect to $O^\dagger$ conditional on the observed data that are used to obtain $\hat{F}$ and $\hat{G}$. Furthermore, we define the following out-of-sample cross integral product:
\[
D_{\dagger,1}(\hat{F}, \hat{G}; F_0, G_0) = \mathbb{E}_{\dagger} \left[ \int_0^\infty \left\{ \hat{F}(t|Z^\dagger) - F_0(t|Z^\dagger) \right\} d \left\{ \hat{G}(t|Z^\dagger) - G_0(t|Z^\dagger) \right\} \right],
\]
\[
D_{\dagger,2}(\hat{F}, \hat{G}; F_0, G_0) = \mathbb{E}_{\dagger} \left[ \int_0^\infty \left\{ \hat{G}(t|Z^\dagger) - G_0(t|Z^\dagger) \right\} d \left\{ \hat{F}(t|Z^\dagger) - F_0(t|Z^\dagger) \right\} \right].
\]
We assume the following.

**Assumption 6** (Uniform Convergence). The uniform $L_2$ norms of $\hat{F} - F_0$ and $\hat{G} - G_0$ converges to zero,
\[
\|\hat{F} - F_0\|_{\dagger, \text{sup}, 2} = o(1), \quad \|\hat{G} - G_0\|_{\dagger, \text{sup}, 2} = o(1).
\]

**Assumption 7** (Rate condition). We have either
\[
D_{\dagger,1}(\hat{F}, \hat{G}; F_0, G_0) = o(n^{-1/2}),
\]

or
\[
D_{\dagger,2}(\hat{F}, \hat{G}; F_0, G_0) = o(n^{-1/2}).
\]

The rate double robustness is formally stated in the following theorem.

**Theorem 3.** Under Assumptions 1, 3, and 6, we have
\[
\hat{\theta}_{cf} \rightarrow_p \theta_0.
\]
In addition, if Assumptions 7 holds, then
\[
\sqrt{n}(\hat{\theta}_{cf} - \theta_0) \rightarrow_d N(0, \sigma^2).
\]
Furthermore, $\sigma^2$ can be consistently estimated by $\hat{\beta}_{cf}^2 \cdot \sum_{k=1}^K \sum_{i \in I_k} U_i^2 \{\hat{\theta}_{cf}, \hat{F}(-k), \hat{G}(-k)\}/n$, where
\[
\hat{\beta}_{cf} = \left\{ n^{-1} \sum_{k=1}^K \sum_{i \in I_k} 1/\hat{G}(-k)(T_i|Z_i) \right\}^{-1} \quad \text{or} \quad \hat{\beta}_{cf} = \left[ n^{-1} \sum_{k=1}^K \sum_{i \in I_k} 1/(1 - \hat{F}(-k)(Q_i|Z_i)) \right]^{-1}.
\]
The rate condition in Assumption 7 is a generalized version of the product rate condition assumed for rate doubly robust estimators in the literature (Smucler et al., 2019; Hou et al., 2021; Rava and Xu, 2021) to handle the involvement of time $t$ in $F$ and $G$. The condition only requires one of the out-of sample cross integral products to converge to zero at a rate faster than $n^{-1/2}$. Such a rate condition on the minimum of the two product rates is also assumed in Ghassami et al. (2022).

5 Simulation

In this section, we study the finite sample performance of the doubly robust estimator $\hat{\theta}_{dr}$. We also compare it with the IPW estimator $\hat{\theta}_{IPW,Q}$, the regression-based estimators $\hat{\theta}_{Reg,T1}$, and $\hat{\theta}_{Reg,T2}$, as well as the naive estimator and the full data estimator. The naive estimator is simply the average of $\nu(T_i)$’s from the observed data without accounting for left truncation, that is,

$$\hat{\theta}_{\text{naive}} = \frac{1}{n} \sum_{i=1}^{n} \nu(T_i),$$

The full data estimator $\hat{\theta}_{\text{full}}$ is the the average of $\nu(T_i)$’s from the full data. We consider sample size $n = 1000$ for the observed data sets, and 500 data sets are simulated, which give margin of error of about $+/−1.91\%$ for the coverage probability of nominal 95\% confidence intervals.

We generate data $(Q, T, Z)$, where $Z = (Z_1, Z_2)$ with $Z_1 \sim \text{Unif}(−1, 1)$ and $Z_2 \sim \text{Bernoulli}(0.5)−0.5$. We then generate $Q$ and $T$ for full data from seven different scenarios that are summarized in Table 1. The details for the seven scenarios are described in the Appendix. The subjects are included in the observed sample only if $Q < T$. In the seven scenarios, the left truncation rates are about 55\% for Scenarios 1, 2, 4 and 6, 66\% for Scenario 3, 75\% for Scenario 5, and 85\% for Scenario 7.

Our estimand is $\theta = \mathbb{P}^*(T > 7)$. It is difficult to calculate analytically the true value of $\theta$ in each scenario, so we approximate the truth by computing the average of $\nu(T_i)$’s from a simulated full data sample with sample size $10^7$. The approximated true value of $\theta$ is 0.2370 for Scenarios 1, 2 and 3; 0.2441 for Scenarios 4 and 6; and 0.0976 for Scenarios 5 and 7.

We use the Cox proportional hazards model as the working models for $T$ and $\tau − Q$, respectively.
Specifically, we fit

\[ \lambda_1(t|Z_1, Z_2) = \lambda_{01}(t)e^{\beta_{11}Z_1 + \beta_{21}Z_2} \]  

(17)

for \( T \) with the sample \((Q_i, T_i, Z_i)_{i=1}^n\), where \( T \) is left truncated by \( Q \); and fit

\[ \lambda_2(t|Z_1, Z_2) = \lambda_{02}(t)e^{\beta_{12}Z_1 + \beta_{22}Z_2} \]  

(18)

for \( \tau - Q \) with the sample \((\tau - T_i, \tau - Q_i, Z_i)_{i=1}^n\), where \( \tau - Q \) is left truncated by \( \tau - T \). For the seven scenarios we considered, the \( T \) model in (17) is correct for Scenarios 1, 2 and 3, moderately wrong for Scenarios 4 and 6, and severely wrong for Scenarios 5 and 7. The \( Q \) model in (18) is correct for Scenarios 1, 4 and 5, moderately wrong for Scenarios 2 and 6, and severely wrong for Scenarios 3 and 7.

We reporting the bias, the \( \% \text{bias} = (\text{bias/truth}) \times 100\% \), the empirical standard deviation (SD), the average of the (model-based) standard errors (SE), the average of the bootstrapped standard errors (boot SE), and the coverage probabilities (CP) of the nominal 95\% confidence intervals constructed using asymptotic normality and the standard errors. For \( \hat{\theta}_{dr} \), the model-based SE is obtained from \( \hat{\sigma} \) as defined in Theorem 2 with \( \hat{\beta} = \left\{ n^{-1} \sum_{i=1}^n 1/G(T_i|Z_i) \right\}^{-1} \). For the IPW estimator \( \hat{\theta}_{IPW,Q} \), the model-based SE is from the sandwich variance estimator assuming that the weights are known. For the naive estimator and the full data estimator, the SE’s are computed by the sample SD of the \( \nu(T_i) \)'s divided by the square root of sample size. Bootstraps are carried out using resampling with replacement, with 100 resamples in order to estimate the SE.

The simulation results are shown in Figure 1 as well as Table 2, with full details in the table. As seen from the results, the doubly robust estimator \( \hat{\theta}_{dr} \) has in general small biases in Scenarios 1 - 5, where at least one working model is correctly specified, and good coverage of the 95\% bootstrap CIs. When both working models are correctly specified (Scenario 1), the average SE is very close to the empirical SD and the corresponding coverage probability is close to 95\%. When either the \( Q \) model (Scenarios 2 and 3) or the \( T \) model is misspecified (Scenarios 4 and 5), the average SE is smaller than SD and the 95\% CIs constructed with SE are away from 95\%; more apparently so under severe model misspecification (Scenarios 3 and 5).

For \( \hat{\theta}_{IPW,Q} \), the bias is small in Scenarios 1, 4 and 5 where the \( Q \) model is correct, and the coverage of the 95\% bootstrap CIs is good; but otherwise the bias is large when the \( Q \) model is
misspecified (Scenarios 2, 3, 6 and 7) and the 95% bootstrap CIs under-cover, especially when the Q model is severely wrong (Scenarios 3 and 7). On the other hand, \( \hat{\theta}_{\text{Reg.T1}} \) and \( \hat{\theta}_{\text{Reg.T2}} \) have small biases when the T model is correct (Scenarios 1, 2 and 3), and good coverage of the 95% bootstrap CIs; and otherwise have large biases and under-covered 95% bootstrap CIs when the T model is misspecified (Scenarios 4, 5, 6 and 7), especially when the T model is severely misspecified (Scenarios 5 and 7).

The bias of the naive estimator is very large in all seven scenarios; it is substantially larger than the true values of \( \theta \), consistent with the direction of the left truncation bias as subjects with smaller event times tend not to be captured by the observed data, leading to over-estimation by the naive estimator.

6 Discussion

We derived the EIC and obtained a doubly robust estimating function for the mean of transformed survival time in the presence of covariate-induced dependent left truncation. Cross-fitting can be leveraged to incorporate nonparametric or high-dimensional models for estimating the involved nuisance parameters. Bickel et al. (1993) derived the space of all influence functions, the efficient influence function, and the semiparametric efficiency bound for the random left truncation model without covariates. To the best of our knowledge, semiparametric theory in the setting with covariate-induced non-random left truncation has not been studied yet.

As also pointed out in Bickel et al. (1993, page 252), truncation is quite different from the perhaps more familiar censoring. In particular, truncated data is not coarsened, i.e. partially observed. In this way, we cannot ‘augment’ an IPW estimator to obtained an improved and doubly robust estimator, for example. The second term in (10) can be written as

\[
\int_0^\infty \mathbb{E}^*\{\nu(t) - \theta | T < v, Z\} \frac{F(v|Z)}{1 - F(v|Z)} \cdot \frac{d\tilde{M}_Q(v; G)}{G(v|Z)},
\]

because \( \int_0^v \{\nu(t) - \theta\} dF(t|Z) = \mathbb{E}^*\{\nu(T) - \theta | T < v, Z\} \cdot F(v|Z) \). It can be shown that this does not lie in the tangent space of the model \( G \) for the conditional distribution of \( Q \) given \( Z \).

Time-to-event data are often subject to right censoring in practice. One straight forward extension to adjust for right censoring is to use inverse probability of censoring weighting (IPCW). This
approach, however, relies on a correct model for the censoring mechanism. It would be of interest to explore approaches that are robust against censoring model misspecification. In addition, although Assumption 1 is general, it may still be violated, in which case one may consider relaxing it by leveraging the proximal identification framework developed in Tchetgen Tchetgen et al. (2020).

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Table 1: Summary of the data-generating mechanisms; $\tau = 20$, $\tau_1 = 5$, $\tau_2 = 8$.

| Scenarios               | Data-generating mechanism                                                                                                                                 |
|-------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------|
| 1 Both models correct   | Cox: $T \sim 0.3Z_1 + 0.5Z_2$.                                                                                                                         |
|                         | Cox: $(\tau - Q) \sim 0.3Z_1 + 0.5Z_2$.                                                                                                             |
| 2 Q model moderately wrong | Cox: $T \sim 0.3Z_1 + 0.5Z_2$.                                                                                                                         |
|                         | Cox: $(\tau - Q) \sim 0.3Z_1 + 0.5Z_2 + 0.6(Z_1^2 - 1/3) + 0.5Z_1Z_2$.                                                                             |
| 3 Q model severely wrong | mixture: Cox: $(\tau - Q) \sim 0.3Z_1 + 0.5Z_2 + 0.6(Z_1^2 - 1/3) + 0.5Z_1Z_2$, if $0.3Z_1 + 0.5Z_2 < 0$.               |
|                         | AFT: $(\tau_2 - Q) \sim 0.3Z_1 + 0.5Z_2 + 0.6(Z_1^2 - 1/3) + 0.5Z_1Z_2$, if $0.3Z_1 + 0.5Z_2 \geq 0$.                                                        |
| 4 T model moderately wrong | Cox: $T \sim 0.3Z_1 + 0.5Z_2$.                                                                                                                         |
|                         | Cox: $(\tau - Q) \sim 0.3Z_1 + 0.5Z_2$.                                                                                                             |
| 5 T model severely wrong | mixture: AFT: $(T - \tau_1) \sim 0.3Z_1 + 0.5Z_2 + 0.6(Z_1^2 - 1/3) + 0.5Z_1Z_2$, if $0.3Z_1 + 0.5Z_2 < 0$.               |
|                         | Cox: $T \sim 0.3Z_1 + 0.5Z_2 + 0.6(Z_1^2 - 1/3) + 0.5Z_1Z_2$, if $0.3Z_1 + 0.5Z_2 \geq 0$.                                                                 |
|                         | Cox: $(\tau - Q) \sim 0.3Z_1 + 0.5Z_2$.                                                                                                             |
| 6 both models moderately wrong | Cox: $T \sim 0.3Z_1 + 0.5Z_2 + 0.6(Z_1^2 - 1/3) + 0.5Z_1Z_2$                                                                                      |
|                         | Cox: $Q \sim 0.3Z_1 + 0.5Z_2 + 0.6(Z_1^2 - 1/3) + 0.5Z_1Z_2$                                                                                   |
| 7 both models severely wrong | mixture: AFT: $(T - \tau_1) \sim 0.3Z_1 + 0.5Z_2 + 0.6(Z_1^2 - 1/3) + 0.5Z_1Z_2$, if $0.3Z_1 + 0.5Z_2 < 0$.               |
|                         | Cox: $T \sim 0.3Z_1 + 0.5Z_2 + 0.6(Z_1^2 - 1/3) + 0.5Z_1Z_2$, if $0.3Z_1 + 0.5Z_2 \geq 0$.                                                                 |
|                         | mixture: Cox: $(\tau - Q) \sim 0.3Z_1 + 0.5Z_2 + 0.6(Z_1^2 - 1/3) + 0.5Z_1Z_2$, if $0.3Z_1 + 0.5Z_2 < 0$.               |
|                         | AFT: $(\tau_2 - Q) \sim 0.3Z_1 + 0.5Z_2 + 0.6(Z_1^2 - 1/3) + 0.5Z_1Z_2$, if $0.3Z_1 + 0.5Z_2 \geq 0$.                                                                 |

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| Scenario | Model | Estimator | 0.002 | 0.011 | 0.029 | 0.734 | 0.000 | 0.000 | 0.000 | Coverage |
|----------|-------|-----------|-------|-------|-------|-------|-------|-------|-------|----------|
| 1: Both correct | Q | dr | 0.002 | 0.011 | 0.029 | 0.734 | 0.000 | 0.000 | 0.000 | 0.938 |
| 2: Q model moderately wrong | Q | dr | 0.002 | 0.011 | 0.029 | 0.734 | 0.000 | 0.000 | 0.000 | 0.734 |
| 3: Q model severely wrong | Q | dr | 0.002 | 0.011 | 0.029 | 0.734 | 0.000 | 0.000 | 0.000 | 0.734 |
| 4: T model moderately wrong | Q | dr | 0.002 | 0.011 | 0.029 | 0.734 | 0.000 | 0.000 | 0.000 | 0.734 |
| 5: T model severely wrong | Q | dr | 0.002 | 0.011 | 0.029 | 0.734 | 0.000 | 0.000 | 0.000 | 0.734 |
| 6: Both moderately wrong | Q | dr | 0.002 | 0.011 | 0.029 | 0.734 | 0.000 | 0.000 | 0.000 | 0.734 |
| 7: Both severely wrong | Q | dr | 0.002 | 0.011 | 0.029 | 0.734 | 0.000 | 0.000 | 0.000 | 0.734 |

Figure 1: Bias, empirical standard deviation, and coverage probability of 95% bootstrap CIs for the estimators under Scenarios 1 - 7.
Table 2: Simulation results for Scenarios 1-7; SD: standard deviation, SE: standard error, CP: coverage probability.

| Scenarios | Estimator     | bias   | %bias   | SD     | SE/boot SE | CP/boot CP |
|-----------|---------------|--------|---------|--------|------------|------------|
| 1         | dr            | -0.0015| -0.6    | 0.021  | 0.020/0.020| 0.946/0.938|
| Both models | IPW.Q        | -0.0004| -0.2    | 0.020  | 0.015/0.019| 0.864/0.946|
| correct   | Reg.T1        | -0.0004| -0.2    | 0.019  | -/0.019    | -/0.942    |
|           | Reg.T2        | -0.0005| -0.2    | 0.019  | -/0.019    | -/0.942    |
|           | naive         | 0.2414 | 101.9   | 0.016  | 0.016/0.016| 0.000/0.000|
|           | full          | -0.0005| -0.2    | 0.009  | 0.009/0.009| 0.954/0.946|
| 2         | dr            | -0.0017| -0.7    | 0.020  | 0.018/0.019| 0.930/0.948|
| Q model   | IPW.Q         | 0.0106 | 4.5     | 0.017  | 0.014/0.017| 0.802/0.894|
| moderately| Reg.T1        | -0.0007| -0.3    | 0.019  | -/0.019    | -/0.944    |
| wrong     | Reg.T2        | -0.0008| -0.3    | 0.019  | -/0.019    | -/0.946    |
|           | naive         | 0.2355 | 99.4    | 0.016  | 0.016/0.016| 0.000/0.000|
|           | full          | -0.0006| -0.2    | 0.009  | 0.009/0.009| 0.950/0.954|
| 3         | dr            | -0.0027| -1.1    | 0.023  | 0.018/0.024| 0.896/0.950|
| Q model   | IPW.Q         | 0.0528 | 22.3    | 0.018  | 0.015/0.018| 0.064/0.176|
| severely  | Reg.T1        | -0.0019| -0.8    | 0.023  | -/0.024    | -/0.948    |
| wrong     | Reg.T2        | -0.0020| -0.8    | 0.023  | -/0.024    | -/0.950    |
|           | naive         | 0.2877 | 121.4   | 0.016  | 0.016/0.016| 0.000/0.000|
|           | full          | -0.0001| -0.1    | 0.008  | 0.008/0.008| 0.956/0.946|
| 4         | dr            | -0.0015| -0.6    | 0.021  | 0.019/0.020| 0.930/0.948|
| T model   | IPW.Q         | -0.0007| -0.3    | 0.021  | 0.016/0.020| 0.872/0.944|
| moderately| Reg.T1        | 0.0113 | 4.6     | 0.018  | -/0.017    | -/0.894    |
| wrong     | Reg.T2        | 0.0110 | 4.5     | 0.018  | -/0.017    | -/0.894    |
|           | naive         | 0.2446 | 100.2   | 0.016  | 0.016/0.016| 0.000/0.000|
|           | full          | -0.0006| -0.2    | 0.009  | 0.009/0.009| 0.948/0.944|
| 5         | dr            | -0.0020| -2.0    | 0.011  | 0.015/0.012| 0.988/0.964|
| T model   | IPW.Q         | -0.0009| -0.9    | 0.009  | 0.007/0.010| 0.840/0.956|
| severely  | Reg.T1        | -0.0288| -29.5   | 0.008  | -/0.008    | -/0.070    |
| wrong     | Reg.T2        | -0.0293| -30.0   | 0.007  | -/0.008    | -/0.058    |
|           | naive         | 0.2329 | 238.7   | 0.014  | 0.015/0.015| 0.000/0.000|
|           | full          | -0.0001| -0.1    | 0.005  | 0.005/0.005| 0.956/0.956|
| 6         | dr            | 0.0230 | 9.4     | 0.019  | 0.017/0.018| 0.720/0.734|
| Both models | IPW.Q        | 0.0263 | 10.8    | 0.018  | 0.014/0.017| 0.546/0.662|
| moderately| Reg.T1        | 0.0262 | 10.7    | 0.018  | -/0.017    | -/0.672    |
| wrong     | Reg.T2        | 0.0260 | 10.6    | 0.018  | -/0.017    | -/0.678    |
|           | naive         | 0.2433 | 99.7    | 0.016  | 0.016/0.016| 0.000/0.000|
|           | full          | -0.0005| -0.2    | 0.009  | 0.009/0.009| 0.950/0.952|
| 7         | dr            | -0.0570| -58.4   | 0.006  | 0.006/0.007| 0.000/0.004|
| Both models | IPW.Q        | -0.0635| -65.0   | 0.006  | 0.003/0.006| 0.000/0.000|
| severely  | Reg.T1        | -0.0589| -60.4   | 0.006  | -/0.006    | -/0.000    |
| wrong     | Reg.T2        | -0.0590| -60.5   | 0.006  | -/0.006    | -/0.000    |
|           | naive         | 0.2217 | 227.2   | 0.015  | 0.015/0.015| 0.000/0.000|
|           | full          | 0.0000 | 0.0     | 0.004  | 0.004/0.004| 0.944/0.946|
A Useful Quantities and Preliminaries

In this section, we summarize the quantities used in the Appendix. Recall that

\[
\beta(z) := \int_0^\infty \mathbb{1}(q < t) f(t|z) g(q|z) \, dt \, dq,
\]

\[
m(v, Z; F) := \int_0^t \nu(t) dF(t|Z),
\]

\[
\mu(Z; F) := \int_0^\infty \nu(t) dF(t|Z).
\]

Denote

\[
\tilde{m}(v, Z; F) := \int_0^\infty \nu(t) dF(t|Z).
\]

Based on the joint density (2) for \((Q, T, Z)\) in observed data, we can get the following marginal densities and conditional densities for observed data distribution in terms of \(f, g\) and \(h\).

\[
p_Z(z) = \frac{\beta(z)}{\beta} h(z),
\]

\[
p_{T,Z}(t, z) = \frac{G(t|z)}{\beta} f(t|z) h(z),
\]

\[
p_{Q,Z}(q, z) = \frac{1 - F(q|z)}{\beta} g(q|z) h(z),
\]

\[
p_{T|Z}(t|z) = f(t|z) \cdot \frac{G(t|z)}{\beta(z)},
\]

\[
p_{Q|Z}(q|z) = \frac{1 - F(q|z)}{\beta(z)} g(q|z),
\]

\[
p_{Q,T,Z}(q, t, z) = \mathbb{1}(q < t) \frac{G(t|z)}{\beta(z)} f(t|z) g(q|z),
\]

\[
p_{Q,T}(q, t) = \frac{1 - F(q|z)}{G(t|z)} g(q|z),
\]

\[
p_{T|Q,Z}(t|q, z) = \frac{1 - F(q|z)}{1 - F(q|z)} f(t|z).
\]
Moreover, denote

\[ K(v|z) := P(Q \leq v < T|Z = z) = \frac{G(v|z)\{1 - F(v|z)\}}{\beta(z)}, \]

\[ L(v|z) := E\left[ \frac{\nu(T) - \theta}{G(T|z)} \right] \bigg| Z = z \]

\[ = \frac{1}{\beta(z)} \int_0^v \{\nu(t) - \theta\} f(t|z) dt = \frac{m(v, z; F) - \theta F(v|z)}{\beta(z)}, \]

\[ R(y, u|z) := \int_0^\infty L(v|z) L(v|z)^2 \cdot 1(y \leq v < u) \cdot p_{Q|Z}(v|z) dv \]

\[ = \int_0^\infty \frac{m(v, z; F) - \theta F(v|z)}{G(v|z)^2 \{1 - F(v|z)\}} \cdot 1(y \leq v < u) dG(v|z) \]

where \( m(v, z; F) = \int_0^v \nu(t) dF(t|z) \) as defined before. The quantities \( K, L \) and \( R \) appear during the computation to derive the influence curve. We have immediately

\[ \frac{L(Q|Z)}{K(Q|Z)} - R(Q, T|Z) \]

\[ = \frac{m(Q, F)}{G(Q|Z)} - \int_0^\infty \frac{m(v, z; F) - \theta F(v|z)}{G(v|z)^2 \{1 - F(v|z)\}} \cdot 1(Q \leq v < T) \cdot \frac{dG(v|z)}{G(v|z)} \]

\[ = - \int_0^\infty \frac{m(v, z; F) - \theta F(v|z)}{G(v|z)^2 \{1 - F(v|z)\}} dM_Q(v; G). \tag{23} \]

Throughout the paper, if we do not specify the limit for integrals, the integrals are taken on the support of the corresponding variables.

As previously defined, \( P_\epsilon \) denotes a regular parametric submodel for observed data and \( p_\epsilon(\cdot) \) the corresponding density. We have for example,

\[ S_{Q,Z}(q, z) = \partial \log p_{Q,Z}(q, z; \epsilon) / \partial \epsilon |_{\epsilon=0}, \]

\[ S_{Q|Z}(q|z) = \partial \log p_{Q|Z}(q|z; \epsilon) / \partial \epsilon |_{\epsilon=0}. \]

The following properties of the scores will be useful in the proofs. Because

\[ S(\cdot) = \frac{\partial}{\partial \epsilon} \log p_\epsilon(\cdot) \bigg|_{\epsilon=0} = \left\{ \frac{\partial p_\epsilon(\cdot)}{\partial \epsilon} / p_\epsilon(\cdot) \right\} \bigg|_{\epsilon=0} = \left\{ \frac{\partial}{\partial \epsilon} p_\epsilon(\cdot) \bigg|_{\epsilon=0} \right\} / p(\cdot), \]

we have

\[ \frac{\partial}{\partial \epsilon} p_\epsilon(\cdot) \bigg|_{\epsilon=0} = S(\cdot) p(\cdot). \tag{24} \]

In addition, \( \mathbb{E} [S_{Q,T,Z}(Q, T, Z)] = \mathbb{E} [S_{Q,T,Z}(Q, T, Z)|Z] = \mathbb{E} [S_{Q,Z}(Q, Z)] = 0, \) etc.
\textbf{B Identifiability}

\textbf{Proof of Lemma 1.} 1) we first show that (4) holds. Under Assumption 1 and 2,

\[
\mathbb{E} \left\{ \frac{\nu(T)}{G(T|Z)} \right\} = \int \frac{\nu(t)}{G(t|z)} \frac{f(t|z)g(q|z)h(z)\mathbb{1}(q < t)}{\beta} dq \, dt \, dz
\]

\[
= \frac{1}{\beta} \int \frac{\nu(t)}{G(t|z)} \left\{ \int g(q|z)\mathbb{1}(q < t) dq \right\} f(t|z)h(z) \, dt \, dz
\]

\[
= \frac{1}{\beta} \int \nu(t) \cdot f(t|z)h(z) \, dt \, dz
\]

\[
= \frac{\theta}{\beta}.
\]

Likewise, as a special case with \( \nu \equiv 1 \), we have

\[
\mathbb{E} \left\{ \frac{1}{G(T|Z)} \right\} = \frac{1}{\beta}.
\]

Therefore,

\[
\mathbb{E} \left\{ \frac{\nu(T)}{G(T|Z)} \right\} \bigg/ \mathbb{E} \left\{ \frac{1}{G(T|Z)} \right\} = \theta.
\]

2) We now show that \( G \) can be characterized by the reverse time hazard function \( \alpha \) defined in (6). By (6),

\[
\alpha(q|z) = \frac{g(q|z)}{G(q|z)} = \left\{ \frac{\partial}{\partial q} G(q|z) \right\} \bigg/ G(q|z).
\]

Besides, \( \lim_{q \to \infty} G(q|z) = 1 \). By solving the above differential equation, we get

\[
G(q|z) = \exp \left\{ - \int_q^{\infty} \alpha(v|z) dv \right\}.
\]

(25)

3) Next, we will show \( \alpha \) is identifiable from observed data distribution. Specifically, we will show that under Assumptions 1 and 2,

\[
\alpha(q|z) = \frac{p_{Q|Z}(q|z)}{\mathbb{P}(Q \leq q < T | Z = z)}.
\]

(26)

Recall from (20) and (21),

\[
p_{Q|Z}(q|z) = \frac{1 - F(q|z)}{\beta(z)} g(q|z),
\]

\[
p_{Q,T|Z}(q,t|z) = \frac{\mathbb{1}(q < t)}{\beta(z)} f(t|z)g(q|z).
\]
So

\[ P(Q \leq q < T|Z = z) = \int_0^\infty \int_0^\infty 1(v \leq q < u)p_{T,Q}(u,v|z) \, du \, dv \]

\[ = \int_0^\infty \int_0^\infty 1(v < q < u) \frac{1(v < u)}{\beta(z)} f(u|z)g(v|z) \, du \, dv \]

\[ = \{1 - F(q|z)\}G(q|z) \]

Therefore,

\[ \frac{p_{Q|Z}(q|z)}{P(Q \leq q < T|Z = z)} = \frac{g(q|z)}{G(q|z)}. \]
C  Martingale Theory

Lemma 4. Under Assumption 1 and 2, \( \{\tilde{M}_Q(t; G)\}_{t \geq 0} \) is a backwards martingale with respect to \( \{\tilde{F}_t\}_{t \geq 0} \) if \( G \) is the true full data CDF of \( Q \) given \( Z \).

Proof. In the following proof, we will first show that \( \tilde{M}_Q(t; G) \) is \( \tilde{F}_t \)-measurable and \( \mathbb{E} [ |\tilde{M}_Q(t; G)| ] < \infty \) for all \( t \geq 0 \). Then we will show \( \mathbb{E} [M_Q(s; G) | \tilde{F}_t] = M_Q(t; G) \) for all \( 0 < s \leq t \) if \( G \) is the true CDF of \( Q \) given \( Z \) in full data.

We first show that the at risk indicator \( 1 \{ Q \leq t < T \} \) is \( \tilde{F}_{t+} \)-measurable, where \( \tilde{F}_{t+} := \bigcup_{s \geq t} \tilde{F}_s \).

For any \( t \geq 0 \), since

\[
\{ T > t \} = \bigcup_{n=1}^{\infty} \left\{ T \geq t + \frac{1}{n} \right\} \in \tilde{F}_{t+},
\]

\[
\{ Q < T, Q \leq t \} = \bigcup_{n=1}^{\infty} \left\{ Q < T, Q < t + \frac{1}{n} \right\}
\]

\[
= \{ Q < T \} \cap \left[ \bigcup_{n=1}^{\infty} \left\{ t + \frac{1}{n} \leq Q < T \right\} \right] \in \tilde{F}_{t+},
\]

we have

\[
\{ Q \leq t < T \} = \{ T > t \} \cap \{ Q < T, Q \leq t \} \in \tilde{F}_{t+} \subseteq \tilde{F}_t,
\]

so \( \tilde{A}_Q(t; G) \) is \( \tilde{F}_{t+} \)-measurable, and \( \tilde{M}_Q(t; G) \) is \( \tilde{F}_t \)-measurable.

Now show that for any \( t \geq 0 \), \( \mathbb{E} \{ |\tilde{M}_Q(t; G)| \} < \infty \). It suffice to show that \( \mathbb{E} \{ |\tilde{M}_Q(t; G)| | Z \} \) is bounded. By (21),

\[
\mathbb{E} \{ |\tilde{M}_Q(t; G)| | Z \} \leq \mathbb{E} \{ N_Q(t) | Z \} + \mathbb{E} \{ \tilde{A}_Q(t; G_0) | Z \}
\]

\[
\leq 1 + \mathbb{E} \left\{ \int_0^T 1(Q \leq s < T) \alpha(s|Z) ds | Z \right\}
\]

\[
= 1 + \int_0^\infty \int_0^\infty \int_0^\infty 1(q \leq s < t) \frac{dG(s|Z)}{G(s|Z)} \cdot \frac{1}{\beta(Z)} dF(t|Z) dG(q|Z)
\]

\[
= 1 + \frac{1}{\beta(Z)} \int_0^\infty 1(s < t) dG(s|Z) dF(t|Z)
\]

\[
= 1 + \frac{1}{\beta(Z)} \cdot \beta(Z)
\]

\[
= 2.
\]
Therefore,

\[ \mathbb{E} \{ |\bar{M}_Q(t; G)| \} = \mathbb{E} \left[ \mathbb{E} \{ |\bar{M}_Q(t; G)| | Z \} \right] \leq 2 < \infty. \]

Suppose \( G \) is the true CDF of \( Q \) given \( Z \) in full data. We will show that \( \mathbb{E} \left\{ \bar{N}_Q(s) - \bar{N}_Q(t) | \bar{F}_t \right\} = \mathbb{E} \left\{ \bar{A}_Q(s; G) - \bar{A}_Q(t; G) | \bar{F}_t \right\} \), which implies \( \mathbb{E} \{ M_Q(s; G) | \bar{F}_t \} = \bar{M}_Q(t; G) \) for any \( 0 < s \leq t \). Note that

\[ \bar{F}_t = \sigma \{ Z, 1(Q < T), 1(s \leq Q < T), 1(s \leq T) : s \geq t \} \]

so it suffice to show that for any nonnegative measurable function \( \xi \) on \( \{0, 1\} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \) and any \( 0 < s \leq t \),

\[ \mathbb{E} \left[ \{ \bar{N}_Q(s) - \bar{N}_Q(t) \} \xi(1(Q < T), 1(Q < T)(Q \vee t), T \vee t, Z) \right] \]

\[ = \mathbb{E} \left[ \{ \bar{A}_Q(s; G) - \bar{A}_Q(t; G) \} \xi(1(Q < T), 1(Q < T)(Q \vee t), T \vee t, Z) \right]. \]

For any nonnegative measurable function \( \xi \) on \( \{0, 1\} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{Z} \), and any \( 0 < s \leq t \),

\[ \mathbb{E} \left[ \{ \bar{N}_Q(s) - \bar{N}_Q(t) \} \xi(1(Q < T), 1(Q < T)(Q \vee t), T \vee t, Z) | Z \right] \]

\[ = \mathbb{E} \{ 1(s \leq Q < t)1(Q < T)\xi(1(Q < T), Q \vee t, T \vee t, Z) | Z \} \]

\[ = \mathbb{E} \{ 1(s \leq Q < t)1(Q < T)\xi(1(Q < T), t, T \vee t, Z) | Z \} \]

\[ = \mathbb{E} \{ 1(s \leq Q < t)1(Q < T)\{ 1(T \geq t)\xi(1(Q < T), t, T, Z) + 1(T < t)\xi(1(Q < T), t, T, Z) \} | Z \} \]

\[ = \mathbb{E} \{ 1(s \leq Q < t)1(Q < T)1(T \geq t)\xi(1(Q < T), t, T, Z) | Z \} \]

\[ + \mathbb{E} \{ 1(s \leq Q < t)1(Q < T)1(T < t)\xi(1(Q < T), t, t, Z) | Z \} \]

\[ = \frac{1}{\beta(Z)} \int_0^\infty \int_0^\infty 1(s \leq v < t)1(u \geq t)\xi(1, t, u, Z)f(u|Z)g(v|Z)1(v < u) \, du \, dv \]

\[ + \frac{1}{\beta(Z)} \int_0^\infty \int_0^\infty 1(s \leq v < t)1(u \geq t)\xi(1, t, t, Z)f(u|Z)g(v|Z)1(v < u) \, du \, dv \]

\[ = \frac{1}{\beta(Z)} \{ G(t-|Z) - G(s-|Z) \} \int_0^\infty 1(u < t)\xi(1, t, u, Z)dF(u|Z) \]

\[ + \frac{\xi(1, t, t, Z)}{\beta(Z)} \int_0^\infty 1(s \leq u < t)dF(u|Z)dG(v|Z). \]
On the other hand,

\[
\mathbb{E} \left[ \{ \bar{A}_Q(s; G) - \bar{A}_Q(t; G) \} \cdot \xi(1(Q < T), 1(Q < T)(Q \lor t), T \lor t, Z) \mid Z \right ]
\]
\[
= \mathbb{E} \left\{ \int_s^t 1(Q \leq x < T) \frac{dG(x|Z)}{G(x|Z)} \cdot \xi(1(Q < T), 1(Q < T)(Q \lor t), T \lor t, Z) \mid Z \right\}
\]
\[
= \mathbb{E} \left\{ \int_0^\infty 1(s \leq x < t) 1(Q \leq x < T) \frac{dG(x|Z)}{G(x|Z)} \xi(1(Q < T), t, T \lor t, Z) \mid Z \right\}
\]
\[
= \mathbb{E} \left\{ \int_0^\infty 1(s \leq x < t) 1(Q \leq x < T) \frac{dG(x|Z)}{G(x|Z)} 1(T \geq t) \xi(1(Q < T), t, T, Z) \mid Z \right\}
\]
\[
+ \mathbb{E} \left\{ \int_0^\infty 1(s \leq x < t) 1(Q \leq x < T) \frac{dG(x|Z)}{G(x|Z)} 1(T < t) \xi(1(Q < T), t, Z) \mid Z \right\}
\]
\[
= \frac{1}{\beta(Z)} \int_0^\infty \int_0^\infty \int_0^\infty 1(s \leq x < t) 1(v \leq x < u) \frac{dG(x|Z)}{G(x|Z)} 1(u \geq t) \xi(1, t, u, Z) f(u|Z) g(v|Z) 1(v < u) \ du \ dv
\]
\[
+ \frac{1}{\beta(Z)} \int_0^\infty \int_0^\infty \int_0^\infty 1(s \leq x < t) 1(v \leq x < u) \frac{dG(x|Z)}{G(x|Z)} 1(u < t) \xi(1, t, t, Z) f(u|Z) g(v|Z) 1(v < u) \ du \ dv
\]
\[
= \frac{1}{\beta(Z)} \int_0^\infty \int_0^\infty 1(s \leq x < t) 1(u > x) 1(u \geq t) \xi(1, t, u, Z) \left\{ \int_0^\infty 1(v \leq x) g(v|Z) \ dv \right\} \frac{dG(x|Z)}{G(x|Z)} f(u|Z) \ du
\]
\[
+ \frac{\xi(1, s, s, Z)}{\beta(Z)} \int_0^\infty \int_0^\infty 1(s \leq x < t) 1(u > x) 1(u < t) \left\{ \int_0^\infty 1(v \leq x) g(v|Z) \ dv \right\} \frac{dG(x|Z)}{G(x|Z)} f(u|Z) \ du
\]
\[
= \frac{1}{\beta(Z)} \int_0^\infty \int_0^\infty 1(s \leq x < t) 1(u \geq t) \xi(1, t, u, Z) dG(x|Z) dF(u|Z)
\]
\[
+ \frac{\xi(1, t, t, Z)}{\beta} \int_0^\infty \int_0^\infty 1(s \leq u < t) dF(x|Z) dG(v|Z)
\]
\[
= \frac{1}{\beta(Z)} \{ G(t-|Z) - G(s-|Z) \} \int_0^\infty 1(u \geq t) \xi(1, t, u, Z) dF(u|Z)
\]
\[
+ \frac{\xi(1, t, t, Z)}{\beta} \int_0^\infty \int_0^\infty 1(s \leq u < t) dF(x|Z) dG(v|Z).
\]

Therefore,

\[
\mathbb{E} \left[ \{ \bar{N}_Q(s) - \bar{N}_Q(t) \} \cdot \xi(1(Q < T), 1(Q < T)(Q \lor t), T \lor t, Z) \mid Z \right ]
\]
\[
= \mathbb{E} \left[ \{ \bar{A}_Q(s) - \bar{A}_Q(t) \} \cdot \xi(1(Q < T), 1(Q < T)(Q \lor t), T \lor t, Z) \mid Z \right ]
\].
Let 

\[\bar{\tau} = \max(\tau, t)\]

and 

\[\bar{\tau}^2 = \max(\bar{\tau}, \tau^2)\]

for \(t \geq 0\). Therefore,

\[\mathbb{E}\{\bar{\tau} \in [0, t]\} = \frac{t}{\bar{\tau}}\]

and 

\[\mathbb{E}\{\bar{\tau}^2 \in [0, t]\} = \frac{t^2}{\bar{\tau}^2}\]

are finite. Therefore, \(\bar{\tau}\) and \(\bar{\tau}^2\) are finite for \(t \geq 0\).

Thus, \(\bar{\tau}\) and \(\bar{\tau}^2\) are finite for \(t \geq 0\). Therefore, \(\mathbb{E}\{\bar{\tau} \in [0, t]\} = \frac{t}{\bar{\tau}}\)

and 

\[\mathbb{E}\{\bar{\tau}^2 \in [0, t]\} = \frac{t^2}{\bar{\tau}^2}\]

are finite. Therefore, \(\bar{\tau}\) and \(\bar{\tau}^2\) are finite for \(t \geq 0\).

Therefore,

\[\mathbb{E}\{\bar{\tau} \in [0, t]\} = \frac{t}{\bar{\tau}}\]

and 

\[\mathbb{E}\{\bar{\tau}^2 \in [0, t]\} = \frac{t^2}{\bar{\tau}^2}\]

are finite. Therefore, \(\bar{\tau}\) and \(\bar{\tau}^2\) are finite for \(t \geq 0\).
D Derivation of the EIC

Proof of Lemma 2. (i) To prove $\dot{P} = L^0_2(P_{T,Z}) + L^0_2(P_{Q,Z})$, we will first show $\dot{P} \subseteq L^0_2(P_{T,Z}) + L^0_2(P_{Q,Z})$ and then show $\dot{P} \supseteq L^0_2(P_{T,Z}) + L^0_2(P_{Q,Z})$.

Recall that under Assumption 1, the joint density for $(Q,T,Z)$ in observed data is

$$p(q,t,z) = \frac{f(t|z)g(q|z)h(z) \cdot 1(q < t)}{\beta} = \frac{f(t|z)g(q|z)h(z) \cdot 1(q < t)}{\int 1(q < t)f(t|z)g(q|z)h(z) \, dt \, dq \, dz}.$$

For any regular parametric submodel $P_\epsilon$ satisfying assumption 1, the corresponding observed data density is

$$p_\epsilon(q,t,z) = \frac{f_\epsilon(t|z)g_\epsilon(q|z)h_\epsilon(z) \cdot 1(q < t)}{\int 1(q < t)f_\epsilon(t|z)g_\epsilon(q|z)h_\epsilon(z) \, dt \, dq \, dz}.$$

Denote

$$a(t,z) := \left. \frac{\partial}{\partial \epsilon} \log f_\epsilon(t|z) \right|_{\epsilon=0},$$

$$b(q,z) := \left. \frac{\partial}{\partial \epsilon} \log g_\epsilon(q|z) \right|_{\epsilon=0},$$

$$c(z) := \left. \frac{\partial}{\partial \epsilon} \log h_\epsilon(z) \right|_{\epsilon=0}.$$

and denote

$$L^0_2(P_Z) = \left\{ c \in L_2(P_Z) : \int_Z c(z) \, dP_Z(z) = 0 \right\}.$$
Then the score for observed data is

\[ \frac{\partial}{\partial \epsilon} \log p_\epsilon(q, t, z) \bigg|_{\epsilon=0} = \frac{\partial}{\partial \epsilon} \log f_\epsilon(t|z) \bigg|_{\epsilon=0} + \frac{\partial}{\partial \epsilon} \log g_\epsilon(q|z) \bigg|_{\epsilon=0} + \frac{\partial}{\partial \epsilon} \log h_\epsilon(z) \bigg|_{\epsilon=0} \]

\[ - \frac{\partial}{\partial \epsilon} \log \int 1(q < t) f_\epsilon(t|z) g_\epsilon(q|z) h_\epsilon(z) \, dq \, dt \, dz \bigg|_{\epsilon=0} \]

\[ = a(t, z) + b(q, z) + c(z) \]

\[ - \int 1(q < t)a(t, z)f(t|z)g(q|z)h(z) \, dt \, dq \, dz \]

\[ - \int 1(q < t)b(q, z)f(t|z)g(q|z)h(z) \, dq \, dt \, dz \]

\[ - \int 1(q < t)c(z)f(q|z)g(t|z)h(z) \, dq \, dt \, dz \]

\[ = a(t, z) + b(q, z) + c(z) - E[a(T, Z)] - E[b(Q, Z)] - E[c(Z)] \]

\[ = \{a(t, z) - E[a(T, Z)]\} + \{b(q, z) - E[b(Q, Z)]\} + \{c(z) - E[c(Z)]\} \]

\[ \in L_2^0(P_{T,Z}) + L_2^0(P_{Q,Z}) + L_2^0(P_Z). \]

Since \( L_2^0(P_Z) \subseteq L_2^0(P_{T,Z}) \), we have

\[ \dot{P} \subseteq L_2^0(P_{T,Z}) + L_2^0(P_{Q,Z}) + L_2^0(P_Z) \subseteq L_2^0(P_{T,Z}) + L_2^0(P_{Q,Z}). \]

On the other hand, we will shown that \( \dot{P} \subseteq L_2^0(P_{T,Z}) + L_2^0(P_{Q,Z}) \). Since bounded functions of \((T, Z)\) and \((Q, Z)\) are dense in \( L_2^0(P_{T,Z}) \) and \( L_2^0(P_{Q,Z}) \), respectively, it suffice to show that for any bounded functions \( a \in L_2^0(P_{T,Z}), b \in L_2^0(P_{Q,Z}) \), there exists a parametric submodel whose score is \( a(t, z) + b(q, z) \). For any bounded functions \( a \in L_2^0(P_{T,Z}), b \in L_2^0(P_{Q,Z}) \), let

\[ \tilde{a}(t, z) = a(t, z) - \int a(t, z)f(t|z)dt, \]

\[ \tilde{b}(q, z) = b(q, z) - \int b(q, z)g(q|z)dq, \]

\[ \tilde{c}(z) = \int a(t, z)f(t|z)dt + \int b(q, z)g(q|z)dq \]

\[ - \int \left\{ \int a(t, z)f(t|z)dt + \int b(q, z)g(q|z)dq \right\} h(z) \, dz. \]

Consider the parametric submodel

\[ p_\epsilon(q, t, z) = \frac{f_\epsilon(t|z)g_\epsilon(q|z)h_\epsilon(z) \cdot 1(q < t)}{\int 1(q < t)f_\epsilon(t|z)g_\epsilon(q|z)h_\epsilon(z) \, dt \, dq \, dz} \quad (27) \]
with
\[ f_\epsilon(t|z) = f(t|z)\{1 + \epsilon \tilde{a}(t, z)\}, \]
\[ g_\epsilon(q|z) = g(t|z)\{1 + \epsilon \tilde{b}(q, z)\}, \]
\[ h_\epsilon(z) = h(z)\{1 + \epsilon \tilde{c}(z)\}, \]
and \( \epsilon \) is the parameter chosen sufficiently small to guarantee that \( f_\epsilon, g_\epsilon \) and \( h_\epsilon \) are positive. Since
\[ \int f_\epsilon(t|z)dt = 1 + \epsilon \int \tilde{a}(t, z)f(t|z)dt = 1, \]
\[ \int g_\epsilon(q|z)dq = 1 + \epsilon \int \tilde{b}(q, z)g(q|z)dq = 1, \]
\[ \int h_\epsilon(z)dz = 1 + \epsilon \int \tilde{c}(z)h(z)dz = 1, \]
\( f_\epsilon, g_\epsilon, h_\epsilon \) are density functions and, at \( \epsilon = 0 \) they are (conditional) densities of of \( T \) given \( Z \), \( Q \) given \( Z \), and \( Z \), respectively. Therefore the class of functions given in (27) is a parametric submodel. Moreover, it can be verified that
\[ \frac{\partial}{\partial \epsilon} \log p_\epsilon(q, t, z) = a(t, z) + b(q, z). \]
Therefore,
\[ L_2^0(P_{T, Z}) + L_2^0(P_{Q, Z}) \subseteq \mathcal{P}. \]
Thus,
\[ \mathcal{P} = L_2^0(P_{T, Z}) + L_2^0(P_{Q, Z}). \]

(ii) We now characterize \( \mathcal{P}^\perp \). Since the Hilbert space is \( L_2^0(P_{Q, T, Z}) \), by the definition of orthogonal complement,
\[ \mathcal{P}^\perp = \left\{ \xi(q, t, z) \in L_2^0(P_{Q, T, Z}) : \mathbb{E} [\xi(Q, T, Z)\{a(Q, Z) + b(T, Z)\}] = 0, \quad \forall a \in L_2^0(P_{Q, Z}), \ b \in L_2^0(P_{T, Z}) \right\}. \]
We would like to find all functions \( \xi(Q, T, Z) \) such that
\[ \mathbb{E} [\xi(Q, T, Z)\{a(Q, Z) + b(T, Z)\}] = 0, \quad \forall a \in L_2^0(P_{T, Z}), \ b \in L_2^0(P_{Q, Z}). \] (28)
Since $0 \in L^0_2(P_{Q,Z}), 0 \in L^0_2(P_{T,Z})$, it can be verified that (28) holds if and only if the following two conditions hold:
\[
\begin{align*}
&\mathbb{E}\{\xi(Q,T,Z)a(Q,Z)\} = 0, \quad \forall a \in L^0_2(P_{Q,Z}), \\
&\mathbb{E}\{\xi(Q,T,Z)b(T,Z)\} = 0, \quad \forall b \in L^0_2(P_{T,Z}).
\end{align*}
\]
Therefore,
\[
(P^\perp) = \left\{ \xi \in L^0_2(P_{Q,T,Z}) : \mathbb{E}\{\xi(Q,T,Z)a(Q,Z)\} = 0, \quad \forall a \in L^0_2(P_{Q,Z}), \\
&\mathbb{E}\{\xi(Q,T,Z)b(T,Z)\} = 0, \quad \forall b \in L^0_2(P_{T,Z}). \right\}
\]

\[\Box\]

**Proof of Lemma 3.** To derive an influence curve, we consider a regular parametric submodel $P_\epsilon$ for the observed data $(Q,T,Z)$. The proof contains much algebra related to the computation under $P_\epsilon$.

Let $\mathbb{E}_\epsilon$ denote the expectation taken with respect to the distribution $P_\epsilon$. Then
\[
\theta(P_\epsilon) = \frac{\mathbb{E}_\epsilon\{\nu(T)/G(T|Z;\epsilon)\}}{\mathbb{E}_\epsilon\{1/G(T|Z;\epsilon)\}}.
\]
We want to find a function $\varphi(O) = \varphi(Q,T,Z)$ such that
\[
\left. \frac{\partial}{\partial \epsilon} \theta(P_\epsilon) \right|_{\epsilon=0} = \mathbb{E}\{\varphi(Q,T,Z)S_{Q,T,Z}(Q,T,Z)\},
\]
and
\[
\mathbb{E}\{\varphi(O)\} = 0
\]
because $\varphi \in \mathcal{H}$.

We will first compute $\frac{\partial \theta(P_\epsilon)}{\partial \epsilon}|_{\epsilon=0}$, and then try to write it in the form of the right hand side (RHS) of (29). The derivative
\[
\frac{\partial}{\partial \epsilon} \theta(P_\epsilon) = \frac{\partial}{\partial \epsilon} \mathbb{E}_\epsilon\left\{ \frac{\nu(T)}{G(T|Z;\epsilon)} \right\} - \mathbb{E}_\epsilon\left\{ \frac{\nu(T)}{G(T|Z;\epsilon)} \right\} \frac{\partial}{\partial \epsilon} \mathbb{E}_\epsilon\left\{ \frac{1}{G(T|Z;\epsilon)} \right\} \sqrt{\mathbb{E}_\epsilon\left\{ \frac{1}{G(T|Z;\epsilon)} \right\}^2}.
\]
When evaluated at $\epsilon = 0$,
\[
\left. \frac{\partial}{\partial \epsilon} \theta(P_\epsilon) \right|_{\epsilon=0} = \beta \frac{\partial}{\partial \epsilon} \mathbb{E}_\epsilon\left\{ \frac{\nu(T)}{G(T|Z;\epsilon)} \right\}|_{\epsilon=0} - \beta \theta \frac{\partial}{\partial \epsilon} \mathbb{E}_\epsilon\left\{ \frac{1}{G(T|Z;\epsilon)} \right\}|_{\epsilon=0}
\]
(30)
because in the proof of Lemma 1, we have shown that \( \mathbb{E} \{1/G(T|Z)\} = 1/\beta \) and \( \mathbb{E} \{\nu(T)/G(T|Z)\} = \theta/\beta \).

We will compute the two derivatives in (30) next.

\[
\frac{\partial}{\partial \epsilon} \mathbb{E}_\epsilon \left\{ \frac{\nu(T)}{G(T|Z)} \right\} = \frac{\partial}{\partial \epsilon} \int \frac{\nu(t)}{G(t|z;\epsilon)} p_{Q,T,Z}(q,t,z;\epsilon) \, dq \, dt \, dz
\]

\[
= \int \frac{\nu(t)}{G(t|z;\epsilon)} \left\{ \frac{\partial}{\partial \epsilon} p_{Q,T,Z}(q,t,z;\epsilon) \right\} \, dq \, dt \, dz
\]

\[
- \frac{\partial}{\partial \epsilon} G(t|z;\epsilon) \int \frac{\partial}{\partial \epsilon} G(t|z;\epsilon) \, p_{Q,T,Z}(q,t,z;\epsilon) \, dq \, dt \, dz.
\]

From (24) we have

\[
\left. \frac{\partial}{\partial \epsilon} p_{Q,T,Z}(q,t,z;\epsilon) \right|_{\epsilon=0} = S_{Q,T,Z}(q,t,z)p_{Q,T,Z}(q,t,z).
\]

So

\[
\frac{\partial}{\partial \epsilon} \mathbb{E}_\epsilon \left\{ \frac{\nu(T)}{G(T|Z)} \right\} \bigg|_{\epsilon=0} = \int \frac{\nu(t)}{G(t|z)} S_{Q,T,Z}(q,t,z) \, p_{Q,T,Z}(q,t,z) \, dq \, dt \, dz
\]

\[
- \int \frac{\nu(t)}{G(t|z)^2} \left\{ \left. \frac{\partial}{\partial \epsilon} G(t|z;\epsilon) \right|_{\epsilon=0} \right\} p_{Q,T,Z}(q,t,z) \, dq \, dt \, dz
\]

\[
= \mathbb{E} \left\{ \frac{\nu(T)}{G(T|Z)} S_{Q,T,Z}(Q,T,Z) \right\} - \mathbb{E} \left[ \frac{\nu(T)}{G(T|Z)^2} \cdot \left\{ \left. \frac{\partial}{\partial \epsilon} G(T|Z;\epsilon) \right|_{\epsilon=0} \right\} \right]\] (31)

As a special case with \( \nu \equiv 1 \), we have

\[
\frac{\partial}{\partial \epsilon} \mathbb{E}_\epsilon \left\{ \frac{1}{G(T|Z)} \right\} \bigg|_{\epsilon=0} = \mathbb{E} \left\{ \frac{1}{G(T|Z)} S_{Q,T,Z}(Q,T,Z) \right\} - \mathbb{E} \left[ \frac{1}{G(T|Z)^2} \cdot \left\{ \left. \frac{\partial}{\partial \epsilon} G(T|Z;\epsilon) \right|_{\epsilon=0} \right\} \right]\] (32)

Therefore, by (30), (31) and (32), we have

\[
\frac{\partial}{\partial \epsilon} \theta(P_\epsilon) \bigg|_{\epsilon=0} = \beta \cdot \mathbb{E} \left\{ \frac{\nu(T)-\theta}{G(T|Z)} S_{Q,T,Z}(Q,T,Z) \right\} - \beta \cdot \mathbb{E} \left[ \frac{\nu(T)-\theta}{G(T|Z)^2} \cdot \left\{ \left. \frac{\partial}{\partial \epsilon} G(T|Z;\epsilon) \right|_{\epsilon=0} \right\} \right].\] (33)

If we can write the second term of the RHS of (33) above into \( \beta \cdot \mathbb{E} \{ \varphi_2(Q,T,Z)S_{Q,T,Z}(Q,T,Z) \} \) for some function \( \varphi_2 \), then \( \varphi(Q,T,Z) = \beta \{ \nu(T) - \theta \}/G(T|Z) - \varphi_2(Q,T,Z) \) satisfies (29).

We now focus on finding \( \varphi_2 \). To compute \( \partial G(t|z;\epsilon)/\partial \epsilon|_{\epsilon=0} \) recall from (25), \( G(t|z;\epsilon) = \exp \{ -\int_t^\infty \alpha(v|z;\epsilon) \, dv \} \), so

\[
\frac{\partial}{\partial \epsilon} G(t|z;\epsilon) = -\exp \left\{ -\int_t^\infty \alpha(v|z;\epsilon) \, dv \right\} \frac{\partial}{\partial \epsilon} \int_t^\infty \alpha(v|z;\epsilon) \, dv
\]

\[
= -G(t|z) \int_t^\infty 1(v \geq t) \left\{ \frac{\partial}{\partial \epsilon} \alpha(v|z;\epsilon) \right\} \, dv.\] (34)
By (26), we have \( \alpha(v|z; \epsilon) = p_{Q|Z}(y|z; \epsilon)/\int \mathbb{1}(y < u) p_{T,Q|Z}(u, y|z; \epsilon) du dy \). In addition, by (24) we have

\[
\frac{\partial}{\partial \epsilon} \alpha(v|z; \epsilon) \bigg|_{\epsilon=0} = \frac{\partial}{\partial \epsilon} \left\{ \frac{p_{Q|Z}(v|z; \epsilon)}{\int \mathbb{1}(y < u) p_{T,Q|Z}(u, y|z; \epsilon) du dy} \right\} \bigg|_{\epsilon=0} \\
= \frac{1}{\int \mathbb{1}(y < u) p_{T,Q|Z}(u, y|z) du dy} \left[ p_{Q|Z}(v|z) \cdot \int \mathbb{1}(y < u) \left\{ \frac{\partial p_{T,Q|Z}(u, y|z; \epsilon)}{\partial \epsilon} \bigg|_{\epsilon=0} \right\} du dy \right] \\
= \frac{S_{Q|Z}(v|z) p_{Q|Z}(v|z)}{K(v|z)} - \frac{p_{Q|Z}(v|z) \cdot \int \mathbb{1}(y < u) S_{T,Q|Z}(u, y|z) p_{T,Q|Z}(u, y|z) du dy}{K(v|z)^2}.
\]

Therefore, from (34),

\[
\frac{\partial}{\partial \epsilon} G(t|z; \epsilon) \bigg|_{\epsilon=0} = -G(t|z) \int \mathbb{1}(v \geq t) \frac{S_{Q|Z}(v|z) p_{Q|Z}(v|z)}{K(v|z)} dv \\
+ G(t|z) \int \mathbb{1}(v \geq t) \frac{p_{Q|Z}(v|z) \cdot \int \mathbb{1}(y < u) S_{T,Q|Z}(u, y|z) p_{T,Q|Z}(u, y|z) du dy}{K(v|z)^2} dv,
\]

so we have

\[
\mathbb{E} \left[ \frac{\nu(T) - \theta}{G(T|Z)^2} \left\{ \frac{\partial}{\partial \epsilon} G(T|Z; \epsilon) \bigg|_{\epsilon=0} \right\} \right] = E_1 + E_2
\]
where $E_1, E_2$ are stated as follows. We have

\[
E_1 = - \int_Z \left( \int_0^\infty \frac{v(t) - \theta}{G(t|z)} \left\{ \int_0^\infty 1(v \geq t) \frac{S_{Q|Z}(v|z)p_{Q|Z}(v|z)}{K(v|z)} dv \right\} \cdot p_{T|Z}(t|z) dt \cdot p_Z(z) dz \right)
\]

\[
= - \int_Z \int_0^\infty \left\{ \int_0^\infty \frac{\nu(T) - \theta}{G(t|z)} \cdot p_{T|Z}(t|z) dt \right\} \frac{1}{K(v|z)} S_{Q|Z}(v|z)p_{Q|Z}(v|z) dv \cdot p_Z(z) dz
\]

\[
= - \int_Z \int_0^\infty \frac{L(v|z)}{K(v|z)} \cdot S_{Q|Z}(v|z)p_{Q|Z}(v|z) dv \cdot p_Z(z) dz
\]

\[
= -\mathbb{E} \left\{ \frac{L(Q|Z)}{K(Q|Z)} S_{Q|Z}(Q|Z) \right\}
\]

\[
= -\mathbb{E} \left\{ \frac{L(Q|Z)}{K(Q|Z)} S_{T,Q|Z}(T, Q|Z) \right\}
\]

\[
= -\mathbb{E} \left\{ \frac{L(Q|Z)}{K(Q|Z)} S_{Q,T,Z}(Q, T, Z) \right\} + \mathbb{E} \left\{ \frac{L(Q|Z)}{K(Q|Z)} S_{Z}(Z) \right\}
\]

\[
= -\mathbb{E} \left\{ \frac{L(Q|Z)}{K(Q|Z)} S_{Q,T,Z}(Q, T, Z) \right\} + \mathbb{E} \left[ \mathbb{E} \left\{ \frac{L(Q|Z)}{K(Q|Z)} \bigg| Z \right\} \right] \cdot S_{Z}(Z)
\]

\[
= -\mathbb{E} \left( \frac{L(Q|Z)}{K(Q|Z)} \right) - \mathbb{E} \left\{ \frac{L(Q|Z)}{K(Q|Z)} \bigg| Z \right\} \cdot S_{Q,T,Z}(Q, T, Z)
\]

(35)

where the (35) holds because $S_{T,Q|Z} = S_{T,Q,Z} + S_{Q|Z}$, and $\mathbb{E} \left\{ S_{T,Q,Z}(T|Q,Z)\big|Q,Z \right\} = 0$; (36) holds because $S_{T,Q|Z} = S_{T,Q,Z} - S_Z$; and (37) holds because because $S_{T,Q,Z} = S_{T,Q,Z} + S_Z$, and $\mathbb{E} \left\{ S_{T,Q|Z}(T,Q|Z)\big|Z \right\} = 0$. 

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\[
E_2 = \int_{\mathcal{Z}} \int_{0}^{\infty} \frac{\nu(t) - \theta}{G(t|z)} \left\{ \int_{0}^{\infty} \mathbb{1}(v \geq t) \frac{p_{Q|Z}(v|z)}{K(v|z)^2} \cdot \int_{0}^{\infty} \int_{0}^{\infty} \mathbb{1}(y \leq v < u) S_{T,Q|Z}(u,y|z) p_{T,Q|Z}(u,y|z) \, du \, dv \right\} \cdot p_{T|Z}(t|z) dt \cdot p_{Z}(z) dz
\]

\[= \int_{\mathcal{Z}} \int_{0}^{\infty} \int_{0}^{\infty} \left\{ \int_{0}^{\infty} \frac{\nu(T) - \theta}{G(t|z)} \mathbb{1}(1 \leq t \leq v) p_{T|Z}(t|z) dt \right\} \mathbb{1}(y \leq v < u) \frac{p_{Q|Z}(v|z)}{K(v|z)^2} p_{Q|Z}(v|z) dv \] \]

\[= \int_{\mathcal{Z}} \int_{0}^{\infty} \int_{0}^{\infty} \{ \int_{0}^{\infty} \frac{L(v|z)}{K(v|z)^2} \mathbb{1}(y \leq v < u) p_{Q|Z}(v|z) dv \} S_{T,Q|Z}(u,y|z) p_{T,Q|Z}(u,y|z) \, du \, p_{Z}(z) dz \]

\[= \int_{\mathcal{Z}} \int_{0}^{\infty} R(y,u|z) \cdot S_{T,Q|Z}(u,y|z) p_{T,Q|Z}(u,y|z) \, du \cdot p_{Z}(z) dz \]

\[= \mathbb{E} \{ R(Q,T|Z) S_{T,Q|Z}(T,Q|Z) \} \]

\[= \mathbb{E} \{ R(Q,T|Z) S_{Q,T,Z}(Q,T,Z) \} - \mathbb{E} \{ R(Q,T|Z) S_{Z}(Z) \} \]

\[= \mathbb{E} \{ |R(Q,T|Z) - \mathbb{E} \{ R(Q,T|Z)|Z \}| S_{Q,T,Z}(Q,T,Z) \}, \quad (38) \]

where (38) holds because because $S_{T,Q,Z} = S_{T,Q|Z} + S_{Z}$, and $\mathbb{E} \{ S_{T,Q|Z}(T,Q|Z)|Z \} = 0$.

Putting the above together, we have

\[\mathbb{E} \left[ \frac{\nu(T)}{G(T|Z)^2} \left\{ \frac{\partial}{\partial \epsilon} G(T|Z; \epsilon) \bigg|_{\epsilon=0} \right\} \right] = -\mathbb{E} \{ \varphi_2(Q,T,Z) S_{Q,T,Z}(Q,T,Z) \}, \]

where

\[\varphi_2(Q,T,Z) = \left[ \frac{L(Q|Z)}{K(Q|Z)} - \mathbb{E} \left\{ \frac{L(Q|Z)}{K(Q|Z)} \bigg| Z \right\} \right] - \left[ R(Q,T|Z) - \mathbb{E} \{ R(Q,T|Z)|Z \} \right] \]

\[= \left\{ \frac{L(Q|Z)}{K(Q|Z)} - R(Q,T|Z) \right\} - e(Z), \]

with

\[e(Z) = \mathbb{E} \left\{ \frac{L(Q|Z)}{K(Q|Z)} - R(Q,T|Z) \bigg| Z \right\}. \]

From (23) and by Lemma 5,

\[e(Z) = -\mathbb{E} \left\{ \int_{0}^{\infty} \frac{m(v,Z;F) - \theta F(v|Z)}{G(v|Z) \{1 - F(v|Z)\}} dM_{Q}(v;G) \bigg| Z \right\} = 0. \]

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Then Denote is an influence curve.

Thus, Proof of Corollary 1.

Recall that from the proof of Lemma 1, we have shown that \( E \) \( \phi \) \( \{ O \} = 0 \) by Lemma 5. Therefore,

\[
E \left[ \beta \left\{ \frac{\nu(T) - \theta}{G(T|Z)} + \varphi_2(Q, T, Z) \right\} S_{Q, T, Z}(Q, T, Z) \right] = 0.
\]

Therefore,

\[
\varphi_2(Q, T, Z) = - \int_0^\infty \frac{m(v, Z; F) - \theta F(v|Z)}{G(v|Z)\{1 - F(v|Z)\}} dB_Q(v; H).
\]

From the above,

\[
\frac{\partial}{\partial \epsilon}(P_{\epsilon}) \mid_{\epsilon=0} = E \left[ \beta \left\{ \frac{\nu(T) - \theta}{G(T|Z)} + \varphi_2(Q, T, Z) \right\} S_{Q, T, Z}(Q, T, Z) \right],
\]

Recall that from the proof of Lemma 1, we have shown that \( E \{ \nu(T) - \theta \}/G(T|Z) = 0 \). In addition, \( E \{ \varphi_2(Q, T, Z) \} = 0 \) by Lemma 5. Therefore,

\[
E \left[ \beta \left\{ \frac{\nu(T) - \theta}{G(T|Z)} + \varphi_2(Q, T, Z) \right\} \right] = 0.
\]

Thus,

\[
\varphi(O) = \varphi(Q, T, Z) = \beta \cdot \left\{ \frac{\nu(t) - \theta}{G(t|Z)} + \varphi_2(Q, T, Z) \right\}
\]

is an influence curve.

**Proof of Corollary 1.** We will show that the influence curve derived \( \varphi \) in Lemma 3 lies in the tangent space \( \hat{P} \). Therefore, \( \varphi \) is the efficient influence function.

Recall that the influence function is

\[
\varphi(O; \theta, F, G) = \beta \cdot \left[ \frac{\nu(T) - \theta}{G(T|Z)} - \int_0^\infty \frac{m(v, Z; F) - \theta F(v|Z)}{G(v|Z)\{1 - F(v|Z)\}} \{ dN_Q(v) - dA_Q(v; G) \} \right]
\]

\[
= \beta \cdot \left[ \frac{\nu(t) - \theta}{G(t|Z)} + \frac{m(Q, Z; F) - \theta F(Q|Z)}{G(Q|Z)\{1 - F(Q|Z)\}} \right.
\]

\[
\left. - \int_0^\infty \frac{m(v, Z; F) - \theta F(v|Z)1(Q \leq v < T)}{G(v|Z)^2\{1 - F(v|Z)\}} dB_Q(v; H) \right].
\]

Denote

\[
C(t|Z) := \int_0^t \frac{m(v, Z; F) - \theta F(v|Z)1(Q \leq v < T)}{G(v|Z)^2\{1 - F(v|Z)\}} dB_Q(v; H).
\]

Then

\[
\varphi(O; \theta, F, G, H) = \beta \cdot \left\{ \frac{\nu(T) - \theta}{G(T|Z)} - C(T|Z) \right\} + \beta \cdot \left\{ \frac{m(Q, Z; F) - \theta F(Q|Z)}{G(Q|Z)\{1 - F(Q|Z)\}} + C(Q|Z) \right\}
\]

\[
=: a(T, Z) + b(Q, Z),
\]

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where
\[
a(T, Z) = \beta \cdot \left\{ \frac{\nu(T) - \theta}{G(T|Z)} - C(T|Z) \right\},
\]
\[
b(Q, Z) = \beta \cdot \left\{ \frac{m(Q, Z; F) - \theta F(Q|Z)}{G(Q|Z)\{1 - F(Q|Z)\}} + C(Q|Z) \right\}.
\]

Since \( \mathbb{E}\{\varphi(O; \theta, F, G, H)\} = 0 \), we have \( \mathbb{E}\{a(T, Z)\} + \mathbb{E}\{b(Q, Z)\} = 0 \), so
\[
\varphi(O; \theta, F, G, H) = [a(T, Z) - \mathbb{E}\{a(T, Z)\}] + [b(Q, Z) - \mathbb{E}\{b(Q, Z)\}]
\in L_0^2(P_{T,Z}) + L_2^0(P_{Q,Z}) = \hat{P}.
\]

Therefore, by Theorem 4.3 on page 67 of Tsiatis (2006), the influence curve \( \varphi \) in Theorem 3 is the efficient influence curve, and thus, the semiparametric efficiency bound for estimating \( \theta \) is \( \mathbb{E}(\varphi^2) \). \hfill \box

\section{Double robustness of the estimating function}

\textbf{Proof of Theorem 1.} 1) We will first show that \( \mathbb{E}\{U(O; \theta_0, F, G_0)\} = 0 \). Recall that the estimating function is
\[
U(O; \theta, F, G) = \frac{\nu(T) - \theta}{G_0(T|Z)} - \int_0^\infty \frac{m(v, Z; F) - \theta F(v|Z)}{G_0(v|Z)\{1 - F(v|Z)\}} d\tilde{M}_Q(v; G_0).
\]
(39)

By Lemma 4, \( \{\tilde{M}_Q(t; G_0)\}_{t \geq 0} \) is a backwards martingale with respect to the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \). By Assumption 3, the integrand for the integral in (39) is bounded on \([0, \tau_2] \). Therefore, by Lemma 5,
\[
\mathbb{E}\left[ \int_0^\infty \frac{m(v, Z; F) - \theta F(v|Z)}{G_0(v|Z)\{1 - F(v|Z)\}} d\tilde{M}_Q(v; G_0) \right] = 0.
\]

In addition, we have shown that \( \mathbb{E}\{[\nu(T) - \theta_0]/G_0(T|Z)\} = 0 \) in the proof of Lemma 1. Thus,
\[
\mathbb{E}\{U(O; \theta_0, F, G_0)\} = 0.
\]

2) We will next show that \( \mathbb{E}\{U(O; \theta_0, F_0, G)\} = 0 \). We begin by rewriting the estimating function:
\[
U(O; \theta, F, G) = \frac{\nu(T) - \theta}{G(T|Z)} - \int_0^\infty \frac{m(v, Z; F) - \theta F(v|Z)}{G(v|Z)\{1 - F(v|Z)\}} \{d\tilde{N}_Q(v) - d\tilde{A}_Q(v; G)\}
= \frac{\nu(T) - \theta}{G(T|Z)} + \frac{m(Q, Z; F) - \theta F(Q|Z)}{G(Q|Z)\{1 - F(Q|Z)\}}
- \int_0^\infty \frac{m(v, Z; F) - \theta F(v|Z)}{G(v|Z)^2\{1 - F(v|Z)\}} \{1(Q \leq v < T)G(v|Z)\}.
\]
(40)
Replacing term \( \{ \nu(T) - \theta \} / G(T | Z) \) in (40) by the RHS of (41) and after some algebra, we have

\[
U(O; \theta, F, G) = A_1 - A_2,
\]

where

\[
A_1 = \frac{\nu(T) - \theta}{G(Q | Z)} + \frac{m(Q, Z; F) - \theta F(Q | Z)}{G(Q | Z) \{1 - F(Q | Z)\}},
\]

\[
A_2 = \int_0^\infty \left[ \nu(T) - \theta \right] + \frac{m(v, Z; F) - \theta F(v | Z)}{1 - F(v | Z)} \right] 1(T > v) \cdot 1(Q \leq v) \frac{dG(v | Z)}{G(v | Z)^2}.
\]

In the following, we will first compute \( \mathbb{E}\{U(O; \theta, F_0, G)|Q, Z\} \) and then show \( \mathbb{E}\{U(O; \theta, F_0, G)\} = \mathbb{E}\{\mathbb{E}\{U(O; \theta, F_0, G)|Q, Z\}\} = 0 \). The key quantities involved are \( \mathbb{E}\{\nu(T)|Q, Z\}, \mathbb{E}\{1(T > v)|Q, Z\} \) and \( \mathbb{E}\{\nu(T)1(T > v)|Q, Z\} \) for \( v \geq Q \). Denote

\[
\tilde{m}(v, Z; F) := \int_v^\infty \nu(t)dF(t|Z), \quad \mu(Z; F) := \int_0^\infty \nu(t)dF(t|Z).
\]

Then \( m(v, Z; F) + \tilde{m}(v, Z; F) = \mu(Z; F) \) for all \( v > 0 \).

Recall the observed data density of \( T \) given \( (Q, Z) \) from (22):

\[
p_{T|Q,Z}(t|q, z) = \frac{1(q < t)}{1 - F_0(q|z)} dF_0(t|z).
\]

We have

\[
\mathbb{E}\{\nu(T)|Q, Z\} = \int_0^\infty \nu(t) \cdot \frac{1(Q < t)}{1 - F_0(Q|Z)} dF_0(t|Z) = \frac{\tilde{m}(Q, Z; F_0)}{1 - F_0(Q|Z)}.
\]

Moreover, for \( v \geq Q \),

\[
\mathbb{E}\{1(T > v)|Q, Z\} = \int_0^\infty 1(t > v) \cdot \frac{1(Q < t)}{1 - F_0(Q|Z)} dF_0(t|Z)
\]
\[
= \frac{1}{1 - F_0(Q|Z)} \int_0^\infty 1(t > v) dF_0(t|Z)
\]
\[
= \frac{1}{1 - F_0(Q|Z)} \int_0^\infty 1(t > v) dF_0(t|Z)
\]

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and
\[
\mathbb{E} \{ \nu(T) \mathbb{1}(T > v) | Q, Z \} = \int_0^\infty \nu(t) \mathbb{1}(t > v) \cdot \frac{1}{1 - F_0(Q|Z)} dF_0(t|Z) \\
= \frac{1}{1 - F_0(Q|Z)} \int_0^\infty \nu(t) \mathbb{1}(t > v) dF_0(t|Z) \\
= \frac{\tilde{m}(v, Z, F_0)}{1 - F_0(Q|Z)},
\]

Therefore, when \( F = F_0 \),
\[
\mathbb{E}(A_1 | Q, Z) = \left\{ \frac{\tilde{m}(Q, Z; F_0)}{1 - F_0(Q|Z)} - \theta \right\} \cdot \frac{1}{G(Q|Z)} + \frac{m(Q, Z; F_0) - \theta F_0(Q|Z)}{G(Q|Z)(1 - F_0(Q|Z))} \\
= \frac{\mu(Z; F_0) - \theta}{G(Q|Z)(1 - F_0(Q|Z))},
\]

and
\[
\mathbb{E}(A_2 | Q, Z) = \int_0^\infty \left\{ \frac{\tilde{m}(v, Z, F_0)}{1 - F_0(v|Z)} - \theta \cdot \frac{1}{1 - F_0(v|Z)} \\
+ \frac{m(v, Z; F_0) - \theta F_0(v|Z)}{1 - F_0(v|Z)} \cdot \frac{1}{1 - F_0(v|Z)} \right\} \cdot \mathbb{1}(v \leq v) \frac{dG(v|Z)}{G(v|Z)^2} \\
= \int_0^\infty \frac{\mu(Z; F_0) - \theta}{1 - F_0(Q|Z)} \cdot \mathbb{1}(Q \leq v) \frac{dG(v|Z)}{G(v|Z)^2} \\
= \frac{\mu(Z; F_0) - \theta}{1 - F_0(Q|Z)} \cdot \int_Q^\infty \frac{dG(v|Z)}{G(v|Z)^2} \\
= \frac{\mu(Z; F_0) - \theta}{1 - F_0(Q|Z)} \cdot \left\{ \frac{1}{G(Q|Z)} - 1 \right\} \\
= \frac{\mu(Z; F_0) - \theta}{G(Q|Z)(1 - F_0(Q|Z))} - \frac{\mu(Z; F_0) - \theta}{1 - F_0(Q|Z)},
\]

where (42) holds because \( \lim_{v \to \infty} G(v|z) = 1 \) for all \( z \).

Combining the above we have
\[
\mathbb{E} \{ U(O; \theta, F_0, G) | Q, Z \} = \frac{\mu(Z; F_0) - \theta}{1 - F_0(Q|Z)}.
\]

Recall the observed data density of \((Q, Z)\) from (19):
\[
p_Q, Z(q, z) = \frac{1 - F_0(q|z)}{\beta} g_0(q|z) h_0(z).
\]

So we have
\[
\mathbb{E} \left\{ \frac{\mu(Z; F_0)}{1 - F_0(Q|Z)} \right\} = \int_0^\infty \frac{\mu(z; F_0)}{1 - F_0(q|z)} \cdot \frac{1 - F_0(q|z)}{\beta_0} dG_0(q|z) dH_0(z) = \frac{\theta_0}{\beta_0},
\]
\[
\mathbb{E} \left\{ \frac{1}{1 - F_0(Q|Z)} \right\} = \int_0^\infty \frac{1}{1 - F_0(q|z)} \cdot \frac{1 - F_0(q|z)}{\beta_0} dG_0(q|z) dH_0(z) = \frac{1}{\beta_0}.
\]
Therefore, when $\theta = \theta_0$,
\[
\mathbb{E} \{ U(O; \theta_0, F_0, G) \} = \frac{\theta_0}{\beta_0} - \theta_0 \cdot \frac{1}{\beta_0} = 0.
\]

\[
F \text{ Equivalence with IF in Chao (1987)}
\]

In this section, we compare the influence function in Chao (1987) with the EIC derived in this paper. Chao (1987) considered the setting without covariates, i.e., $Z = \emptyset$ in our notation; and they assumed the random left truncation assumption that $Q$ and $T$ are independent in full data. They derived the influence function for the product-limit estimator of the survival function. We will show in this section that in their setting, our EIC coincides with the influence function in their paper when estimating the probability of $T$ greater than $t_0$ in full data for some $t_0 > 0$. This is the parameter of interest $\theta$ defined in (1) with $\nu(t) = 1(t < t_0)$. We note that the random left truncation implies the quasi-independence assumption (Tsai, 1990), which is our Assumption 1 when $Z = \emptyset$. We follow the notation in this paper, and denote $F$ and $G$ be the CDF’s of $T$ and $Q$ respectively.

By equation (1.2) in Chao (1987), the influence function for the product-limit estimator of $\theta$ is
\[
\tilde{\varphi}(Q,T) = \{1 - F(t_0)\} \left[ \beta \int_0^{t_0} \frac{1(Q \geq s) - 1(T \geq s)}{G(s)\{1 - F(s)\}^2} dF(s) + \frac{\beta \cdot 1(T \leq t_0)}{G(T)\{1 - F(T)\}} \right].
\]

For observed data, we have $Q < T$, so
\[
\tilde{\varphi}(Q,T) = \beta \cdot (1 - \theta) \left[ - \int_0^{\infty} 1(s \leq t_0) \cdot \frac{1(Q < s \leq T)}{G(s)\{1 - F(s)\}^2} dF(s) + \frac{1(T \leq t_0)}{G(T)\{1 - F(T)\}} \right].
\]

On the other hand, by Lemma 3, the influence function derived in this paper under the setting without covariates is
\[
\varphi(Q, T) = \beta \left[ \frac{1(T \leq t_0) - \theta}{G(T)} + \frac{F(t_0 \land Q) - \theta F(Q)}{G(Q)(1 - F(Q))} - \int_0^\infty \frac{F(t_0 \land v) - \theta F(v)}{G(v)(1 - F(v))} \mathbb{1}(Q \leq v < T) \frac{dG(v)}{G(v)} \right]
\]

where

\[
\beta = \int_0^\infty \mathbb{1}(q < t) \frac{dF(t)}{dG(q)},
\]

\[
m(v; F) = \int_0^v \mathbb{1}(t \leq t_0) \frac{dF(t)}{dG(t)} = F(t_0 \land v),
\]

so we have

\[
\varphi(Q, T) = \beta \left[ \frac{1(T \leq t_0) - \theta}{G(T)} + \frac{F(t_0 \land Q) - \theta F(Q)}{G(Q)(1 - F(Q))} - \int_0^\infty \frac{F(t_0 \land v) - \theta F(v)}{G(v)(1 - F(v))} \mathbb{1}(Q \leq v < T) \frac{dG(v)}{G(v)} \right]
\]

\[
= \beta \left[ \frac{1(T \leq t_0) - \theta}{G(T)} + \frac{F(t_0 \land Q) - \theta F(Q)}{G(Q)(1 - F(Q))} - \int_Q^T \frac{F(t_0 \land v) - \theta F(v)}{G(v)(1 - F(v))^2} \frac{dG(v)}{G(v)} \right].
\]

Since \( Q < T \) for observed data, there are three possible situations: \( t_0 < Q < T, \ Q \leq t_0 \leq T, \) and \( Q < T < t_0. \) We will verify that \( \varphi = \tilde{\varphi} \) in all three situations. We note that \( F(t_0) = \theta. \)

When \( t_0 < Q < T, \)

\[
\tilde{\varphi}(Q, T) = 0,
\]

\[
\varphi(Q, T) = \beta \left[ -\frac{\theta}{G(T)} + \frac{F(t_0) - \theta F(Q)}{G(Q)(1 - F(Q))} - \int_Q^T \frac{F(t_0) - \theta F(v)}{G(v)(1 - F(v))^2} \frac{dG(v)}{G(v)} \right]
\]

\[
= \beta \left[ -\frac{\theta}{G(T)} + \frac{\theta(1 - F(Q))}{G(Q)(1 - F(Q))} - \int_Q^T \frac{\theta(1 - F(v))}{G(v)(1 - F(v))^2} \frac{dG(v)}{G(v)} \right]
\]

\[
= \beta \left[ -\frac{\theta}{G(T)} + \frac{\theta}{G(Q)} - \int_Q^T \frac{\theta}{G(v)^2} dG(v) \right] = 0.
\]

When \( Q \leq t_0 \leq T, \)

\[
\tilde{\varphi}(Q, T) = -\beta(1 - \theta) \int_Q^{t_0} \frac{dF(v)}{G(v)(1 - F(v))^2},
\]

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and

\[
\varphi(Q, T) = \beta \left[ -\frac{\theta}{G(T)} + \frac{F(Q) - \theta F(Q)}{G(Q)\{1 - F(Q)\}} - \int_Q^T \frac{F(t_0 \wedge v) - \theta F(v)}{G(v)^2\{1 - F(v)\}} dG(v) \right]
\]

\[
= \beta \left[ -\frac{\theta}{G(T)} + \frac{F(Q) - \theta F(Q)}{G(Q)\{1 - F(Q)\}} - \int_Q^T \frac{F(t_0) - \theta F(v)}{G(v)^2\{1 - F(v)\}} dG(v) \right]
\]

\[
= \beta \left[ -\frac{\theta}{G(T)} + \frac{F(Q) - \theta F(Q)}{G(Q)\{1 - F(Q)\}} - \int_Q^T \frac{\theta\{1 - F(v)\}}{G(v)^2\{1 - F(v)\}} dG(v) \right]
\]

\[
= \beta \left[ -\frac{\theta}{G(T)} + \frac{(1 - \theta)F(Q)}{G(Q)\{1 - F(Q)\}} - (1 - \theta)\int_Q^T \frac{F(v)}{G(v)^2\{1 - F(v)\}} dG(v) - \int_Q^T \frac{\theta}{G(v)^2} dG(v) \right]. \tag{43}
\]

By integration by parts, we have

\[
- \int_Q^T \frac{F(v)}{G(v)^2\{1 - F(v)\}} dG(v)
\]

\[
= \int_Q^T \frac{F(v)}{1 - F(v)} d\left\{ \frac{1}{G(v)} \right\}
\]

\[
= \left. \frac{F(v)}{G(v)\{1 - F(v)\}} \right|_{v = t_0}^Q - \int_Q^t \frac{1}{G(v)} \left[ \frac{dF(v)}{1 - F(v)} + \frac{F(v)dF(v)}{\{1 - F(v)\}^2} \right]
\]

\[
= \frac{F(t_0)}{G(t_0)\{1 - F(t_0)\}} - \frac{F(Q)}{G(Q)\{1 - F(Q)\}} - \int_Q^t \frac{dF(v)}{G(v)\{1 - F(v)\}^2}.
\]

Besides,

\[
- \int_t^T \frac{\theta}{G(v)^2} dG(v) = \frac{\theta}{G(T)} - \frac{\theta}{G(t_0)}. \tag{45}
\]
Plugging (44) and (45) into (43), we have

\[ \varphi(Q, T) = \beta \left[ -\frac{\theta}{G(T)} + \frac{(1 - \theta)F(Q)}{G(Q)\{1 - F(Q)\}} + \frac{\theta}{G(t_0)} - \frac{(1 - \theta)F(Q)}{G(Q)\{1 - F(Q)\}} - (1 - \theta) \int_Q^{t_0} \frac{dF(v)}{G(v)\{1 - F(v)\}^2} \right. \]

\[ + \frac{\theta}{G(T) - \frac{\theta}{G(t_0)}} \]

\[ = \beta \left[ -\frac{\theta}{G(T)} + \frac{(1 - \theta)F(Q)}{G(Q)\{1 - F(Q)\}} + \frac{\theta}{G(t_0)} - \frac{(1 - \theta)F(Q)}{G(Q)\{1 - F(Q)\}} - (1 - \theta) \int_Q^{t_0} \frac{dF(v)}{G(v)\{1 - F(v)\}^2} \right. \]

\[ \left. - \beta(1 - \theta) \int_Q^{t_0} \frac{dF(v)}{G(v)\{1 - F(v)\}^2} \right] \]

When \( Q < T < t_0 \), we have

\[ \tilde{\varphi}(Q, T) = \beta(1 - \theta) \left\{ - \int_Q^{T} \frac{dF(v)}{G(v)\{1 - F(v)\}^2} + \frac{1}{G(T)\{1 - F(T)\}} \right\} , \]

and

\[ \varphi(Q, T) = \beta \left[ \frac{1 - \theta}{G(T)} + \frac{F(Q) - \theta F(Q)}{G(Q)\{1 - F(Q)\}} - \int_Q^{T} \frac{F(v) - \theta F(v)}{G(v)^2\{1 - F(v)\}} dG(v) \right. \]

\[ = \beta \left[ \frac{1 - \theta}{G(T)} + \frac{(1 - \theta)F(Q)}{G(Q)\{1 - F(Q)\}} - (1 - \theta) \int_Q^{T} \frac{F(v)}{G(v)^2\{1 - F(v)\}} dG(v) \right] . \]

(46)

Again, by integration by parts, we have

\[ - \int_Q^{T} \frac{F(v)}{G(v)^2\{1 - F(v)\}} dG(v) = \frac{F(T)}{G(T)\{1 - F(T)\}} - \frac{F(Q)}{G(Q)\{1 - F(Q)\}} - \int_Q^{T} \frac{dF(v)}{G(v)\{1 - F(v)\}} \]

(47)

Plugging (47) into (46), we have

\[ \varphi(Q, T) = \beta \left[ \frac{1 - \theta}{G(T)} + \frac{(1 - \theta)F(Q)}{G(Q)\{1 - F(Q)\}} + \frac{(1 - \theta)F(T)}{G(T)\{1 - F(T)\}} - \frac{(1 - \theta)F(Q)}{G(Q)\{1 - F(Q)\}} - (1 - \theta) \int_Q^{T} \frac{dF(v)}{G(v)\{1 - F(v)\}^2} \right] \]

\[ = \beta(1 - \theta) \left[ \frac{1}{G(Q)\{1 - F(Q)\}} - \int_Q^{T} \frac{dF(v)}{G(v)\{1 - F(v)\}^2} \right] \]

Therefore, combining all three cases, we have \( \varphi = \tilde{\varphi} \).
G Details of the data generating mechanisms

- Scenario 1: the full data distribution of $T$ given $Z$ and $(\tau - Q)$ given $Z$ follow the following Cox proportional hazards models, respectively.

  \[
  \lambda_1(t|Z_1, Z_2) = \lambda_{01}(t)e^{0.3Z_1+0.5Z_2}, \quad (48)
  \]
  \[
  \lambda_2(t|Z_1, Z_2) = \lambda_{02}(t)e^{0.3Z_1+0.5Z_2}, \quad (49)
  \]

  where the baseline hazards

  \[
  \lambda_{01}(t) = \begin{cases} 
  0, & \text{if } 0 < t < \tau_1, \\
  2e^{-1}(t - \tau_1), & \text{if } t \geq \tau_1, 
  \end{cases} \quad (50)
  \]
  \[
  \lambda_{02}(t) = \begin{cases} 
  0, & \text{if } 0 \leq t < \tau - \tau_2, \\
  2e^{-1}\{t - (\tau - \tau_2)\}, & \text{if } t \geq \tau - \tau_2. 
  \end{cases} \quad (51)
  \]

  We note that in this case, $(T - \tau_1)$ follows a Weibull distribution with shape parameter 2 and scale parameter $e^{(1-0.3Z_1-0.5Z_2)/2}$, and $(\tau_2 - Q)$ follows a Weibull distribution with shape parameter 2 and scale parameter $e^{(1-0.3Z_1-0.5Z_2)/2}$.

- Scenario 2: $T$ given $Z$ follows the Cox proportional hazards model in (48) and (50), and $(\tau - Q)$ given $Z$ follows the following Cox proportional hazards model with quadratic and interaction terms:

  \[
  \lambda_2(t|Z_1, Z_2) = \lambda_{02}(t)e^{0.3Z_1+0.5Z_2+0.6(Z_1^2-1/3)+0.5Z_1Z_2}, \quad (52)
  \]

  where the baseline hazard $\lambda_{02}(t)$ is given in (51). Under this model, $(\tau_2 - Q)$ follows a Weibull distribution with shape parameter 2 and scale parameter $e^{(1-0.3Z_1-0.5Z_2-0.6(Z_1^2-1/3)-0.5Z_1Z_2)/2}$.

- Scenario 3: $T$ given $Z$ follows the Cox proportional hazards model in (48) and (50), and $(\tau - Q)$ follows a mixture model of Cox proportional hazards model with quadratic and interaction terms and AFT model with quadratic and interaction terms. Specifically, if $0.3Z_1+0.5Z_2 < 0$, $(\tau - Q)$ given $Z$ follows the Cox proportional hazards model with quadratic and interaction terms given in (52) and (51), otherwise, $(\tau_2 - Q)$ follows the following AFT model:

  \[
  \log(\tau_2 - Q) = -1 + 0.3Z_1 + 0.5Z_2 + 0.6(Z_1^2 - 1/3) + 0.5Z_1Z_2 + \epsilon_2, \\
  \epsilon_2 \sim \text{Weibull}(1.5, 1) - \Gamma(5/3). 
  \]
• Scenario 4: \((\tau - Q)\) given \(Z\) follows the Cox proportional hazards model in (49) and (51), and \(T\) given \(Z\) follows a Cox proportional hazards model with quadratic and interaction terms stated as follows.

\[
\lambda_1(t|Z_1, Z_2) = \lambda_{01}(t)e^{0.3Z_1 + 0.5Z_2 + 0.6(Z_1^2 - 1/3) + 0.5Z_1Z_2},
\]

where the baseline hazard \(\lambda_{01}(t)\) is given in (50). Under this model, \(T\) follows a Weibull distribution with shape parameter 2 and scale parameter \(e^{(1-0.3Z_1 - 0.5Z_2 - 0.6(Z_1^2 - 1/3) - 0.5Z_1Z_2)/2}.

- Scenario 5: \((\tau - Q)\) given \(Z\) follows the Cox proportional hazards model in (49) and (51), and \(T\) given \(Z\) follows a mixture model of Cox proportional hazards model with quadratic and interaction terms and AFT model with quadratic and interaction terms. Specifically, if \(0.3Z_1 + 0.5Z_2 \geq 0\), \(T\) given \(Z\) follows the Cox proportional hazards model with quadratic and interaction terms given in (53) and (50); otherwise, \((T - \tau_1)\) follows the following AFT model:

\[
\log(T - \tau_1) = -1 + 0.3Z_1 + 0.5Z_2 + 0.6(Z_1^2 - 1/3) + 0.5Z_1Z_2 + \epsilon_1, \quad \epsilon_1 \sim N(0, 1).
\]

- Scenario 6: \(T\) given \(Z\) follows the Cox proportional hazards model with quadratic and interaction terms in (53) and (50), and \((\tau - Q)\) given \(Z\) follows the Cox proportional hazards model with quadratic and interaction terms in (52) and (51).

- Scenario 7: \(T\) given \(Z\) follows the mixture model stated in Scenario 5, and \((\tau - Q)\) follows the mixture model stated in Scenario 3.