Reduction of the wave packet of a relativistic charged particle by emission of a photon

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Abstract

The problem of reduction of the wave packet of a relativistic charged particle by emission of a photon is studied with help of the path integral approach. A general expression for arbitrary order correlation function of the electromagnetic field is obtained. As a specific example an ultrarelativistic electron circulating in a storage ring is considered. It is shown that the longitudinal width of the electron wave packet defined by characteristic difference of time between registrations of two photons emitted by the electron is of the order of the wave length of the photons.

1 Introduction

When detecting a photon emitted by the charged particle we measure the coordinate of the particle. Following the general principles of quantum measurement theory it results to the reduction of the wave packet of the particle. However in our case the reduction of the wave packet occurs in the process of emission of the photon and not in the process of measurement, because the photodetector may be placed sufficiently far from the particle path. The goal of the paper is to study which way the reduction of the wave packet of the particle occurs and to calculate the scale of the localization of the wave packet.

The technique used in the paper is the functional integral formulation of relativistic quantum mechanics. Within this approach the interaction of the particle with quantum electromagnetic field can be described with the help of the ”influence functional” technique of Feynman and Vernon [1, 2].

2 Correlation function of electromagnetic field

We consider a relativistic charged particle moving in external classical potentials and interacting with quantum electromagnetic field. The state of the system is described by a wave function which is the function of the coordinate of the particle \( q \) and the functional of variables of quantum electromagnetic field

\[
\Psi(q, t, \{A\}) \equiv \langle q, A|\Psi \rangle .
\]  

(1)

The wave function is the projection of the state-vector of the system \(|\Psi\rangle\) onto the state of particle with given coordinate \( \langle q \rangle \) and the state of electromagnetic field – eigenstate of the field operator \( \hat{A}(x) \) with the eigenvalue \( A(x) \):

\[
\langle A|\hat{A}(x) = \langle A|A(x) .
\]

(2)

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The field operator $\hat{A}(x)$ has the following standard expansion over the operators of creation $a_{k\lambda}^+$ and annihilation $a_{k\lambda}$ of the photons with given momentum $k$ and polarization $\lambda$

$$\hat{A}(x) = \hat{A}^-(x) + \hat{A}^+(x) = \sum_{k\lambda} (a_{k\lambda} e^{ikx} + a_{k\lambda}^+ e^{-ikx}) \frac{\varepsilon_{k\lambda}}{\sqrt{2\omega_k V}} , \quad (3)$$

where $\hat{A}^+(x)$ and $\hat{A}^-(x)$ are the operators of creation and annihilation of the photon at the point $x$, $\varepsilon_{k\lambda}$ is the polarization vector, $\omega_k = |k|$ – the energy of the photon and $V$ – the normalization volume. The sum over momenta means the integration

$$\sum_k = \int \frac{d^3k V}{(2\pi)^3} , \quad (4)$$

The speed of light $c$ and Plank constant $\hbar$ in the paper considered to be equal unity $\hbar = c = 1$.

The time evolution of the wave function of the system is controlled by the equation

$$i\partial_t \Psi(q, t, \{A\}) = \left\{ \sqrt{-i\nabla - e(A^{(ex)}(q, t) + A(q))}^2 + M^2 + eU^{(ex)}(q, t) \right\} \Psi(q, t, \{A\}) + \langle q, A | H_{em} | \Psi \rangle , \quad (5)$$

where $e$ and $M$ are the electrical charge and mass of the particle, $A^{(ex)}(q, t)$ and $U^{(ex)}(q, t)$ are the external vector and scalar potentials, and

$$H_{em} = \sum_{k\lambda} \omega_k a_{k\lambda}^+ a_{k\lambda} \quad (6)$$

is the hamiltonian of quantum electromagnetic field. The external vector potential is given in Coulomb gauge

$$\nabla A^{(ex)} = 0 . \quad (7)$$

Let the initial wave function of the system be factorized, i.e. the particle be at pure state described by the wave function $\psi_0(q)$ and electromagnetic field be at a state $|\Phi_0\rangle$. The solution of the evolution equation (5) can be expressed in terms of the functional integral over paths of the particle:

$$\Psi(q_f, t, \{A_f\}) = \int dq_0 |\psi_0(q_i)\rangle \int_{q_i}^{q_f} Dq e^{iS_0[q]} \langle A_f | T \exp\left\{ -i \int_0^t d\tau [H_{em} - e\dot{q}(\tau) \hat{A}(q)] \right\} |\Phi_0\rangle . \quad (8)$$

Here $T$ is the symbol of chronological product. The action of relativistic particle $S_0[q]$ is given by the expression

$$S_0[q] = \int_0^t d\tau (-M \sqrt{1 - \dot{q}^2} + e\dot{q} A^{(ex)}(q, \tau) - eU^{(ex)}(q, \tau)) . \quad (9)$$

The functional integrals in (8) are evaluated over paths $q(\tau)$ with endpoints $q(0) = q_i$ and $q(t) = q_f$.

We will consider only the case when characteristic energy of photons emitted by the particle is much less than the energy of the particle

$$\omega_0 \gamma^3 \ll M \gamma , \quad (10)$$
where $\omega_0$ is characteristic frequency of motion of the particle in external fields and 
$\gamma = (1 - \dot{q}^2)^{-\frac{1}{2}}$ is the relativistic factor. Otherwise one has to consider the quantum field theory with arbitrary number of particle-antiparticle pairs.

The other restriction is that we are not allowed to consider too short intervals of time. The integral over the paths $q(\tau)$ in (8) is defined as a product of the integrals over the coordinates $q(\tau_n)$ at discrete moments of time $\tau_n = dtn, n = 1, 2, 3...$ The functional integral (8) is the solutions of the equation (5) only in the limit

$$M \frac{dt}{\gamma} \gg 1.$$ (11)

Furthermore, if $dtM/\gamma \sim 1$ the functional integral is not well-defined quantity. If the condition (11) is satisfied one has to evaluate the path integral (8) by the steepest descent method in semiclassical approximation by expanding the action (9) up to the quadratic term over the small deviation of the path $q(\tau)$ from the classical, saddle path.

On the other hand the time interval $dt$ is to be much less than the characteristic time $\sim (\omega_0 \gamma)^{-1}$ of emission of the photon. The possibility to choose the time interval $dt$ which satisfies both conditions $\gamma/M \ll dt \ll (\omega_0 \gamma)^{-1}$ is provided by the inequality (10).

Let the initial state of quantum electromagnetic field be the ground state $|0\rangle$. At each given path $q(\tau)$ the state

$$|\alpha(t, [q])\rangle \equiv T \exp\{-i \int_0^t d\tau [H_A - e\dot{q}(\tau)\dot{A}(q)]\}|0\rangle$$ (12)

which arises in (8) is a coherent state. The state is eigenstate of the annihilation operator $\hat{A}^-(x)$

$$\hat{A}^-(x)|\alpha(t, [q])\rangle = A^{(cl)}(x, t, [q])|\alpha(t, [q])\rangle$$ (13)

with eigenvalue

$$A^{(cl)}(x, t, [q]) = i \int \frac{d^3k}{(2\pi)^3} \int_0^t d\tau \frac{e^{-i\omega_k\tau}}{2\omega_k} [\hat{q}_\tau - n(\hat{q}_\tau n)] e^{i k(x - q_\tau \tau) - i\omega_k(t - \tau)},$$ (14)

where we denote $n \equiv \frac{k}{|k|}$ and for short $q_\tau \equiv q(\tau)$. As one can see, the field $A^{(cl)}(x, t, [q])$ is the positive frequency part of the classical vector potential induced by the motion of a point charge along the path $q(\tau)$ [4].

We are interested in the two-photon correlation of radiation produced by the ultrarelativistic particle. Let a photodetector be placed at the point $x_d$ on the tangent to the classical path of the particle. According to well-known results of quantum optics [3], the probability $W(\Delta t)$ to detect two photons emitted by the particle with time interval between the registrations $\Delta t$ is proportional to the second order correlation function of electromagnetic field

$$W(\Delta t) \propto \int dt_0 \langle \hat{A}^+(x_d, t_0)\hat{A}^+(x_d, t_0 + \Delta t)\hat{A}^-(x_d, t_0 + \Delta t)\hat{A}^-(x_d, t_0)\rangle.$$ (15)

Here we suppose that the time resolution of the photodetector is much less than the characteristic time $\Delta t$ of changing of the correlation function in r.h.s. (15). The averaging $\langle . . \rangle$ in (13) is taken over the initial state of the system. The Heisenberg’s creation $\hat{A}^+(x, t)$
and annihilation \( \hat{A}^- (x, t) \) operators are expressed through the Schrödinger’s ones by standard way

\[
\hat{A}^\pm (x, t) = \{ \hat{T} e^{i \int_0^t \mathrm{d} r H_{\text{tot}}(r)} \} \hat{A}^\pm (x) \{ T e^{-i \int_0^t \mathrm{d} r H_{\text{tot}}(r)} \},
\]

(16)

where \( \hat{T} \) is the anti-chronological product symbol. The total Hamiltonian of the system \( H_{\text{tot}} \) depends on time only if the external potentials are time-dependent.

Since the state \(|\alpha(\tau, [q])\rangle \) (12) is the eigenstate of the operator \( \hat{A}^- (x) \) one can easily obtain the expression for the second order correlation function of electromagnetic field

\[
\langle \hat{A}^+ (x_d, t_0) \hat{A}^+ (x_d, t_0 + \Delta t) \hat{A}^- (x_d, t_0 + \Delta t) \hat{A}^- (x_d, t_0) \rangle =
\]

\[
\int \mathrm{d} q \int \mathrm{d} q' \psi_0^* (q_0) \psi_0 (q) \int_{q_0}^{q'} \mathrm{d} q_f D q_f e^{-i q_f |q - [q_0]|} F[q, q'] \times
\]

\[
A^{(cl)} (x_d, t_0, [q']) A^{(cl)} (x_d, t_0 + \Delta t, [q']) A^{(cl)} (x_d, t_0 + \Delta t, [q]) A^{(cl)} (x_d, t_0, [q]),
\]

(17)

where the influence functional is given by the following expression

\[
F[q, q'] \equiv \langle \alpha(t, [q']) | \alpha(t, [q]) \rangle =
\]

\[
\exp \left( \int \frac{d^3 k}{(2\pi)^3} \frac{2\omega_k}{2\omega_k} \int_0^T \mathrm{d} \tau \int_0^\tau \mathrm{d} \tau' d s \{ \hat{q}_s \hat{q}_s' - (n\hat{q}_s)(n\hat{q}_s') \} e^{-ik(q_s - q_s') + i\omega_k (\tau - s)} \right) \times
\]

\[
\exp \left( \int \frac{d^3 k}{(2\pi)^3} \frac{2\omega_k}{2\omega_k} \int_0^T \mathrm{d} \tau \int_0^\tau \mathrm{d} \tau' d s \{ \hat{q}_s \hat{q}_s' - (n\hat{q}_s)(n\hat{q}_s') \} e^{ik(q_s - q_s') - i\omega_k (\tau - s)} +
\]

\[
\{ \hat{q}_s \hat{q}_s' - (n\hat{q}_s)(n\hat{q}_s') \} e^{-ik(q_s - q_s') + i\omega_k (\tau - s)} \right) \}
\]

(18)

The upper limit of time \( t \) in the path integrals (17) is an arbitrary time more than the largest argument of correlation function \( t_0 + \Delta t \).

The expression (17) can be evidently generalized to obtain the correlation function of arbitrary order.

3 Reduction of the wave packet of ultrarelativistic electron in storage ring by synchrotron radiation

Let us consider an ultrarelativistic electron circulating in a storage ring with frequency of revolution \( \omega_0 \) along an circular orbit of radius \( R \). The equilibrium circular path \( q_e (\tau) \) considered to be stable.

Suppose that a point-like classical electron emits two photons while passing through the sector of the orbit where radiation is formed (i.e. when the angle between the velocity vector of the particle and the vector "particle - photodetector" is little more than \( 1/\gamma \)). These two photons will be registered by the detector with the typical time interval between photocounts being of the order \( \Delta t \sim \lambda/c \), where \( \lambda \sim R/\gamma^3 \) is the characteristic wave length of the synchrotron radiation. For a bunch of the point-like electrons with the bunch length \( \Delta l \gg \lambda \) the time interval between photocounts is defined by the width of the bunch \( \Delta t \sim \Delta l/c \). Thus, the characteristic scale \( \Delta t \) of the two-photon correlation function \( W(\Delta t) \) (15) gives us information about the longitudinal width of the wave packet of single quantum electron.

Our basic statement is that the longitudinal "size" of the electron defined by characteristic difference of time between registrations of two photons emitted by the particle is of the
order of the wave length $\lambda$ of emitted photons. It can be treated in the following way: the wave packet of the electron is reduced by the emission of first photon up to the wave length of the photon.

Recently the two-photon function $W(\Delta t)$ of synchrotron radiation was measured experimentally [5]. It was obtain that the characteristic width $\Delta t$ of the function coincides with the resolution time of the measurement system which is much more than $\lambda/c$ but much less than the "natural bunch length".

Now let us prove our statement. Consider the product of classical fields which appears in the indegrand (17)

$$A^{(cl)}(x_d, t_0 + \Delta t, [q]) A^{(cl)}(x_d, t_0, [q])$$

at the time moment $t_0$ when the quantity $A^{(cl)}(x_d, t_0, [q])$ approaches its maximum. The basic point of our analysis is that both classical fields in the product (19) depend on the same path $q(\tau)$. If the path $q(\tau)$ is not too different from the equilibrium circular path $q_e(\tau)$ the expression (19) would have a narrow peak at $\Delta t \sim \lambda/c$ and vanish as $1/\Delta t^4$ at $\lambda/c \ll \Delta t \ll \omega^{-1}$. The same one can say about analogous to (19) product of classical fields in (17) which depends on the path $q'(\tau)$.

Hence, to prove that the two-photon correlation function $W(\Delta t)$ (15) has a peak at $\Delta t \sim \lambda/c$ we need to show that the dominating paths in the functional integrals in (17) are such, that the product of classical fields (19) differs only slightly from that with $q = q_e$.

To be specific, we have to show that characteristic longitudinal $\delta v_\parallel$ and transverse $\delta v_\perp$ deviation of the velocity $\dot{q}(\dot{q})$ from the equilibrium velocity $\dot{q}_e$ satisfy the conditions

$$\langle \delta v_\parallel \rangle^2 \ll \frac{1}{\gamma^4} \quad ; \quad \langle \delta v_\perp \rangle^2 \ll \frac{1}{\gamma^2} .$$

The conditions (20) mean that fluctuations of the deviation angle of velocity vector from the equilibrium velocity is much less than $1/\gamma$ and fluctuations of the relativistic factor (and consequently the energy) are small $\delta \gamma / \gamma \ll 1$.

The functional integrals analogous to that in (17) but without the product of classical fields in the integrand were considered by author in the work [6]. In semiclassical approximation, when the difference of the paths $\delta q \equiv q - q'$ in the functional integrals (17) considered to be much less than the radius of curvature of classical path (i.e. the radius of accelerator $R$), the presence of the influence functional (18) in integrand (17) leads to (infinite) renormalization of the particle mass and appearance of the well-known [4] radiation friction forces and so-called fluctuating forces in equation on the classical, saddle path $\bar{q}_{cl}$, where $\bar{q} \equiv (q + q')/2$. The fluctuating forces, which accounts for the quantum discrete character of radiation, have been originally introduced in the works [4, 5] on statistical grounds.

The fluctuating forces build up the synchratron oscillations of the electron in accelerator. On the other hand, the forces of radiation friction leads to damping of the oscillations. The uncertainty of the energy $\delta \varepsilon \sim \gamma^2 \sqrt{\omega_0 M}$ [4, 5] associated with these oscillations is much less than the energy of the electron $\varepsilon = M \gamma$ provided by the condition (10). Hence, in the functional integrals (17) the characteristic deviation of the velocity of the classical path $\bar{q}_{cl}$ from the equilibrium velocity satisfies the requirement (20).

Now, let us evaluate the quantum fluctuations of the velocity. We calculate the functional integral over the paths $Dq(Dq')$ in (17) by steepest descent method near the classical,
saddle path \( \bar{q}_{d} \). Each integration over the small deviation \( (q(\tau_{n}) - \bar{q}_{d}(\tau_{n})) \) at time \( \tau_{n} \) is performed with the integrand which contains the multiplier

\[
\exp\{-idtM[(\delta v_{\parallel})^{2}\gamma^{3} + (\delta v_{\perp})^{2}\gamma]/2\},
\]

(21)

arising from expansion of the kinetic part of the action \( (\ref{eq:kinetic}) \) over the deviation \( \delta v \equiv \dot{q} - \dot{\bar{q}}_{d} \) up to the quadratic term. Because of the inequality \( (\ref{eq:inequality}) \), the corresponding projections \( \delta v_{\parallel} \) and \( \delta v_{\perp} \) of typical quantum fluctuation of the velocity satisfy the requirements \( (\ref{eq:requirements}) \).

4 Conclusion

On the basis of the expression \( (\ref{eq:correlation}) \) for the correlation function of the electromagnetic field one can obtain the two-photon \( W(\Delta t) \) (or many-photon) correlation function for arbitrary system of external fields. For example, one can consider the scattering of the sufficiently extended wave packet of the particle on some potential. From the experience of previous section it can be concluded that the "size" of the particle defined by characteristic difference of time between registrations of two photons emitted by the particle during the scattering process is of the order of wave length of emitted photons. One can say that the wave packet of the particle is localized by the emission of the photon up to the wave length of the photon.

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