Peculiar properties of the cluster-cluster interaction
induced by the Pauli exclusion principle

Gennady Filippo\textsuperscript{\textcopyright} and Yuliya Lashko\textsuperscript{\textcopyright}

Bogolyubov Institute for Theoretical Physics
14-b Metrolohichna Str., Kiev-143, Ukraine
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Role of the Pauli principle in the formation of both the discrete spectrum and multi-channel states of the binary nuclear systems composed of clusters is studied in the Algebraic Version of the resonating-group method. Solutions of the Hill-Wheeler equations in the discrete representation of a complete basis of the Pauli-allowed states are discussed for $^4\text{He}+\text{n}$, $^3\text{H}+^3\text{H}$, and $^3\text{He}+^4\text{He}$ binary systems. An exact treatment of the antisymmetrization effects are shown to result in either an effective repulsion of the clusters, or their effective attraction. It also yields a change in the intensity of the centrifugal potential. Both factors significantly affect the scattering phase behavior. Special attention is paid to the multi-channel cluster structure $^6\text{He}+^3\text{He}$ as well as to the difficulties arising in the case when the two clustering configurations, $^5\text{He}+^3\text{He}$ and $^4\text{He}+^3\text{He}$, are taken into account simultaneously. In the latter case the Pauli principle, even in the absence of a potential energy of the cluster-cluster interaction, leads to the inelastic processes and secures an existence of both the bound state and resonance in the $^{12}\text{Be}$ compound nucleus.

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I. INTRODUCTION

Studying an interaction of the light nuclei, it is necessary to take into account the Pauli exclusion principle which significantly influences a behavior of the nuclei at small distances between them. Thus the so-called "forbidden" states, which can not be realized in reality, appear due to the Pauli principle. Furthermore, the eigenvalues of the Pauli-allowed states, which are not equal to unity, change a contribution of the latter states to the wave function of the cluster system. At last, matrix elements of the Hamiltonian and other operators between allowed states contain their eigenvalues; and in this respect they differ from the matrix elements of the same operators, but calculated leaving out of the account the Pauli principle.

There are several different approaches to taking into account the Pauli exclusion principle in the problem of collision between compound nuclear systems. The first approach (see, for instance, \textsuperscript{1,2,3,4}) takes into consideration the fact that the Pauli principle does not allow identical nucleons to be at the same point. And that is why a simulation of action of the antisymmetrizer is realized by introducing an additional repulsive potential between clusters. The choice of such potential is not uniquely determined. As usual, in the approach stated above the main requirement to be satisfied by a model potential is that it should make a wave function of the nucleon system vanish, provided the coordinates of the identical nucleons coinciding. In the papers \textsuperscript{1,2,3,4}, where the problems of $\alpha$-$\alpha$ scattering and scattering of $\alpha$-particle by the $^{12}\text{C}$, $^{16}\text{O}$, $^{20}\text{Ne}$ nuclei have been considered, the Pauli principle has been simulated with a phenomenological repulsive potential of infinite intensity. The parameters of this potential have been chosen to reproduce the data on the continuous spectrum of the corresponding compound systems – the energies and widths of their resonance states. The authors of paper \textsuperscript{3} simulated the Pauli principle for the systems composed of several clusters within the method of hyperspherical functions. As in \textsuperscript{1,2,3,4}, they used a phenomenological repulsive potential. But in this approximation there is no guarantee that the forbidden states are excluded completely, and a contribution of the allowed states meets all the requirements of the Pauli principle.

To take into account action of the Pauli principle and to eliminate the forbidden states, Saito suggested the orthogonality condition model \textsuperscript{5}. In this model the allowed states are found from the requirement of their orthogonality to the forbidden states. In articles \textsuperscript{6,7,8} phenomenological potentials of a special kind were used. Such potentials contain the operators of projection onto the forbidden states and suppress these states by passage to the limit of high intensity. To construct these potential, an explicit form of the wave functions forbidden by the Pauli principle should be used. The latter functions can be easily found, provided that the interaction of the simplest nuclei being studied. As regards the allowed states, they are not considered for constructing the pseudopotentials; and, therefore, an influence of the Pauli principle on these states is not taken into account, although it may be essential.

Enumerated approaches turned to be fruitful and provided an important information about discrete and continuum states of the light nuclei composed of clusters.
However, a fundamental problem of constructing a complete basis of the states satisfying the Pauli exclusion principle in the generator-parameter space and of analyzing the influence of these states on the dynamics of cluster systems (especially, multichannel ones) has not been resolved yet.

It is worth mentioning that in all the cases of approximate treatment of antisymmetrization effects the Pauli principle was simulated with a repulsive potential, and its role has been restricted to the elimination of the forbidden states from the wave function. Meanwhile, an antisymmetrization operator affects also Pauli-allowed states generating the matrix of an effective potential which has a diagonal form in the representation of the latter basis. By definition, an allowed state is an orthonormalized eigenfunction of a norm kernel, provided that its eigenvalue is non-zero. The eigenvalues of the norm kernel are always positive. In case the eigenfunctions of the norm kernel are normalized to the number of states, their eigenvalues show how this normalization changes effectively under the influence of the antisymmetrization operator. Change of the normalization of the allowed states produces a direct effect on the cluster dynamics.

The requirements of the Pauli principle can be accurately fulfilled within the resonating-group method (RGM). But traditional form of the norm kernel and Hamiltonian kernel commonly used for deducing the RGM dynamical equations complicates analysis of the effects induced by the influence of the Pauli principle, and thus many peculiarities of the cluster behavior in collision are left beyond the scope of the study.

Detailed analysis of the exchange effects related to the antisymmetrization can be performed with the use of the algorithm outlined in \[10\,11\]. Following the procedure described there, we first define a complete discrete basis of the harmonic-oscillator states allowed by the Pauli principle (Section II). Then we derive the Hill-Wheeler equations in the representation of the discrete basis (Section III). General properties of the solutions obeying the latter equations are discussed in Section IV.

In a consistent microscopic approach the forbidden states are excluded, because their eigenvalues equal zero. And that is why they do not enter an expression for the norm kernel, expansion of which supplies a complete information on the spectrum of the allowed states. Therefore, an effective potential related to the antisymmetrization affects only the allowed states; and, as will be shown later, it may not be a repulsive potential. A repulsion arises in the states whose eigenvalues are less than unity, whereas an attraction appears in the states with the eigenvalues exceeding unity.

An algorithm of accounting the Pauli principle in calculation of the matrix elements of the Hamiltonian and derivation of the effective potential induced by the antisymmetrization operator is suggested for a binary cluster system on the ground of the generator-coordinate method and the Hill-Wheeler equation. Properties of such a potential are analyzed and the conditions fulfilling of which, by virtue of exact treatment of the antisymmetrization effects, leads either to an effective repulsion of the approaching clusters, or to their effective attraction are established (Section V).

By giving some examples of the binary cluster systems with one open channel, a physical interpretation of the phenomena directly related to an antisymmetrization of the wave function is suggested (Section VI).

From the very beginning we use the allowed states only and do not resort to the explicit form of the forbidden states neither to introduce a pseudopotential nor to appeal to the orthogonality condition model. Assuming a certain cluster structure of the system studied or considering several cluster configurations simultaneously, a complete basis of the Pauli-allowed states (classified with the use of SU(3) symmetry indices and, in case of the SU(3) degeneracy, additional quantum numbers) is considered along with a complete set of their eigenvalues. The latter ones indicate an existence of the leading SU(3) irreps that dominate in the continuum states with the low above-threshold energy, and the amplitudes of which are maximal for the discrete states of the cluster system. Basis states of the leading representations have the largest eigenvalues.

Antisymmetrization effects in a binary cluster system with several open channels are studied on the example of the continuum states of $^{12}\text{Be}$ that are able to decay through $^6\text{He}+^6\text{He}$ and $^8\text{He}+^4\text{He}$ channels (Section VII).

II. COMPLETE BASIS OF THE ALLOWED STATES

As starting theses we shall use those which are initial for the derivation of the Hill–Wheeler integral equations. At first, a generating function of the system under consideration should be determined. A form of this function depends on the degrees of freedom, dynamics of which are of interest for us. These degrees of freedom correspond to a complete basis of the allowed states. Resort to the Hill–Wheeler integral equations implies the transformation from the coordinate (or momentum) representation to the representation of generator parameters.

Let $\Phi(R, r)$ be the generator function of the Hill–Wheeler method, antisymmetric with respect to a permutation of the nucleon coordinates. Here $r$ is the set of nucleon vectors, $R$ is the set of generator parameters describing dynamics of the degrees of freedom which are of interest. We construct this function as the Slater determinant composed of nucleon orbitals to ensure its proper permutation symmetry. Such function can be a linear superposition of the Pauli-allowed states only. But we should show how to define a set of quantum numbers $\{n\}$ for the basis of orthonormal states $\{\phi_n(r)\}$ generated by the function $\Phi(R, r)$. Also it is necessary to find an explicit form of the basis functions. The simplest way to do this is to introduce instead of the allowed basis function $\{\phi_n(r)\}$ defined in the coordinate space its map...
\{ \psi_n(\mathbf{R}) \} in the generator parameter representation. The simplification is attained due to the fact that the number of generator parameters \( \mathbf{R} \) is significantly less than the number of single-particle variables \( \mathbf{r} \) of the functions \( \{ \phi_n(\mathbf{r}) \} \).

In order to construct functions \( \psi_n(\mathbf{R}) \), let us introduce an expression

\[
I(\mathbf{S}, \mathbf{R}) = \int \Phi(\mathbf{S}, \mathbf{r})\Phi(\mathbf{R}, \mathbf{r})d\tau, \tag{1}
\]

which is usually called the norm kernel or, in other words, the overlap integral. Integration in (1) is over all single-particle vectors. The norm kernel is symmetric with respect to permutations of the generator parameters \( \mathbf{R} \) and \( \mathbf{S} \). That is why it can be treated as a kernel of the integral equation

\[
\Lambda \psi(\mathbf{R}) = \int I(\mathbf{S}, \mathbf{R})\psi(\mathbf{S}^*)d\mu_\mathbf{S}. \tag{2}
\]

Symmetry of the kernel ensures an existence of its nontrivial eigenfunctions \( \psi_n(\mathbf{R}) \) and eigenvalues \( \Lambda_n \). Here \( n \) is a set of quantum numbers of the basis functions. All that remains is to define the integration domain of the generator parameters \( \mathbf{R} \) and \( \mathbf{S} \) as well as the measure \( d\mu_\mathbf{R} \).

Both problems can be solved, provided the Slater determinant \( \Phi(\mathbf{R}, \mathbf{r}) \) is composed of the Bloch-Brink orbitals which are known to be the generating functions for the single-particle harmonic-oscillator basis. The determinant \( \Phi(\mathbf{R}, \mathbf{r}) \) needs a special discussion. For the first time it was used as a trial function (but in a slightly different form) in the \( \alpha \)-cluster Brink model, that is, in the variational calculation of the spectrum of \(^8\text{Be}\) excited states. Our determinant differs from the Brink determinant in two respects. Firstly, it is generating function for the many-particle harmonic-oscillator basis of the Pauli allowed states. Secondly, its vector generator parameters take complex values, and we treat them as independent variables of the allowed states defined in the Fock-Bargmann representation.

If we restrict ourselves with one complex vector

\[
\mathbf{R} = \frac{\xi + i\eta}{\sqrt{2}},
\]

where \( \xi \) and \( \eta \) are real vectors, then an expression for the Bargmann measure takes the form

\[
d\mu_\mathbf{R} = \exp\{-\mathbf{(R \cdot R^*)}\} \frac{d\xi d\eta}{(2\pi)^{3/2}}.
\]

The eigenvalues and eigenfunctions of the kernel (1) are uniquely defined. Moreover, kernel (2) is a sum of orthogonal degenerate kernels. Each of them corresponds to definite values of the number of oscillator quanta \( \nu \) and SU(3) symmetry indices \( (\lambda, \mu) \) that makes these kernels be orthogonal. Therefore, solving of the equation (2) is reduced to a standard algebraic procedure for the integral equation with degenerate kernel. In the most general case and only because the generator functions are constructed as the Slater determinants composed of the Bloch-Brink orbitals, the eigenfunctions of the norm kernel are labeled by the total number of the oscillator quanta \( \nu \), SU(3) symmetry indices \( (\lambda, \mu) \), an additional quantum number \( \alpha(\lambda, \mu) \) when several different \( (\lambda, \mu) \) multiplets exist, the angular momentum \( L \), its projection \( M \), and, if necessary, one more additional quantum number \( \alpha_L \). The latter is needed to label the states with the same \( L \) in a given \( (\lambda, \mu) \) multiplet. Then the Hilbert–Schmidt expansion of the kernel of the integral equation (2) is

\[
I(\mathbf{S}, \mathbf{R}) = \sum_n \Lambda_n \psi_n(\mathbf{S})\psi_n(\mathbf{R}), \tag{3}
\]

where each of the eigenvalues \( \Lambda_n \) of the norm kernel corresponds to the eigenfunction \( \psi_n(\mathbf{R}) \). Naturally, the eigenfunctions of the kernel (2) are orthogonal with Bargmann measure and normalized to the dimensionality of the irreducible representation \( (\lambda, \mu) \) (irrep) [12]

\[
\int d\mu_\mathbf{R} \psi_{(\lambda, \mu)}(\mathbf{R})\psi_{(\lambda, \mu)}(\mathbf{R}^*) = \frac{(\nu + 1)(\mu + 1)(\lambda + \mu + 2)}{2}.
\]

The second-order Casimir operator of the SU(3) group commutes with the operator of permutation of the nucleon position vectors. Hence, SU(3) symmetry indices naturally appear as the quantum numbers of the eigenfunctions \( \psi_n(\mathbf{R}) \). Only such quantum numbers ensure a diagonal form of the Hilbert–Schmidt expansion. Any other classification of the basis states disrupts the diagonal form of the expansion [3]. For instance, keeping \( \nu \) as a quantum number, instead of the SU(3) symmetry indices \( (\lambda, \mu) \) quantum numbers of the angular momentum-coupled ("physical") basis can be introduced. The states of the latter basis (referred to as "\( l \)-basis" in what follows) are labeled by the number of quanta \( \nu \), the angular momenta of each of the clusters \( l_1 \) and \( l_2 \), and the angular momentum of their relative motion \( l \) (see, for example, [16, 11]). Then unitary transform should be applied to the functions \( \psi_n(\mathbf{R}) \). But it results in an off-diagonal form of the expansion [3], because all the eigenvalues \( \Lambda_n \) are not equal to unity. Only in the asymptotic limit of the large number of oscillator quanta \( \nu \) the eigenvalues appear to be close to unity; and the unitary transformation from the SU(3) basis to the \( l \)-basis leaves the expansion (3) intact, thus allowing to solve continuum state problems in either basis [11].

In the absence of an SU(3) degeneracy the eigenfunctions of the kernel (1) can be straightforwardly constructed with the use of algebraic methods. In case of the SU(3) degeneracy the eigenfunctions, along with their additional quantum numbers \( \alpha(\lambda, \mu) \), are found by solving an integral equation with the degenerate kernel (see, for example, [11]), which is reduced to a set of homogeneous algebraic equations with the rank equal to the degree of the SU(3) degeneracy.

The dependence of the eigenvalues \( \Lambda_n \) on the quantum numbers of the allowed states is known. They take
only positive values, and they are the same for all the states with definite SU(3) symmetry indices \((\lambda, \mu)\). In the presence of the SU(3) degeneracy the eigenvalues for different SU(3) multiplets with the same symmetry indices \((\lambda, \mu)\) do differ. At a given \(\nu\) we shall have several identical SU(3) irreps, i.e. SU(3) degeneration. Then an additional index is necessary to distinguish branches of the states with the same SU(3) symmetry. At that, the eigenvalues belonging to the same branch should change smoothly with \(\nu\).

The eigenvalues appear to be nonzero starting with some minimal number of quanta \(\nu_{\text{min}}\). As long as only the binary cluster configuration with \(\nu = \nu_{\text{min}}\) is considered, there exists only one allowed SU(3) multiplet \((\lambda_0, \mu_0)\) which corresponds to the Elliott’s scheme for the translation-invariant oscillator shell model spectrum generated by the leading representation \([12]\). But, in addition to the orthonormalized basis of this scheme, we found also the eigenvalue \(\Lambda_{(\lambda_0, \mu_0)}\). In case the basis of the leading representation only is employed, this eigenvalue has not practical influence on the results of calculations. It comes into play when we invoke all the other multiplets with \(\nu > \nu_{\text{min}}\) along with their eigenvalues instead of restricting ourselves with the multiplet \((\lambda_0, \mu_0)\).

Irreps with the different number of quanta can be distributed among several branches, with all the states of the same branch having the same symmetry index \(\mu\) and differing only in value of the first index \(\lambda\). That is, the irreps \((\lambda_0 + \nu - \nu_{\text{min}}, \mu_0)\) are assumed to belong to the first branch; while \((\lambda_1 + \nu - \nu_{\text{min}} - 2, \mu_1)\), to the second one, etc. Hierarchy among these irreps is established by the magnitude of the eigenvalues \(\Lambda_{\nu, (\lambda, \mu)}\). The irreps with the maximal values of \(\Lambda_{\nu, (\lambda, \mu)}\) are leading ones. In particular, the irreps \((\lambda_0 + \nu - \nu_{\text{min}}, \mu_0)\) belong to the leading irreps.

For many of the cluster systems (for instance, \(^8\)He+\(^4\)He and \(^6\)He+\(^3\)He), although not for all, the least symmetric SU(3) irreps correspond to the branch \((\lambda_0 + \nu - \nu_{\text{min}}, \mu_0)\). As for the most symmetric ones, they appear at \(\nu = \nu_1\).

In a three-cluster (or multi-cluster) system the number of the Pauli-allowed SU(3) irreps infinitely increases with \(\nu\).

### III. Deduction of the Hill-Wheeler Equations

After the generating invariants \(\Phi(S, r)\) and \(\Phi(R, r)\) having constructed, we can express the Hamiltonian kernel of the cluster system under consideration in terms of the basis functions as follows

\[
H(S, R) = \int \Phi(S, r) \hat{H} \Phi(R, r) dr = \sum_n \sum_{\tilde{n}} \psi_n(S) < n | \hat{H} | \tilde{n} > \psi_{\tilde{n}}(R). \tag{4}
\]

Let us express the wave function \(\Phi(r)\) of the generator-coordinate method to be defined as the Hill-Wheeler integral (see, for example, \([14]\))

\[
\Phi(r) = \int C(R^*) \Phi(R, r) d\mu_R,
\]

containing a new unknown function \(C(R^*)\). The equation for the latter follows from the variational principle for the functional

\[
\int \left[ C(S^*) [H(S, R) - EI(S, R)] C(R^*) d\mu_S d\mu_R = 0, \tag{5}\right.
\]

where the energy \(E\) makes sense of the Lagrange multiplier.

But here we shall deviate from a traditional deduction procedure for the Hill-Wheeler equation. Instead, in order to reduce the functional to an algebraic expression containing the expansion coefficients \(C_n^* (C_{\tilde{n}})\) of an unknown functions \(C(R^*) [C(S^*)]\) in the basis of the Pauli-allowed states, we shall make use of the expansions \([16]\) and \([17]\) for the norm kernel and the Hamiltonian kernel, correspondingly, as well as the orthonormality condition for the Pauli-allowed eigenfunctions.

Let

\[
C(R^*) = \sum_n C_n^* \psi_n(R^*), \quad C(S^*) = \sum_{\tilde{n}} C_{\tilde{n}} \psi_{\tilde{n}}(S^*).\]

Then

\[
\sum_n \sum_{\tilde{n}} C_n^* \left( < n | \hat{H} | \tilde{n} > - E \Lambda_n \delta_{n, \tilde{n}} \right) C_{\tilde{n}} = 0. \tag{6}\]

Now we have two possibilities. Variation of the functional \([19]\) brings us to one of them that implies solving a set of the algebraic equations

\[
\sum_{\tilde{n}} < n | \hat{H} | \tilde{n} > C_{\tilde{n}} - E \Lambda_n C_n = 0. \tag{7}\]

Certainly, \(n\) takes all the values permissible for the Pauli-allowed basis functions. Another possibility consists in a diagonalization of the two quadratic forms containing in the left-hand side of the equation \([20]\).
IV. SOLUTION OF THE HILL-WHEELER EQUATION

First, let us discuss general properties of solutions of the set of equations (7). For a binary cluster system the components of the eigenvectors of the discrete states with the energy \( E_n = -\kappa^2/2 < 0 \) decrease exponentially with the number of radial quanta \( \nu = 2k \) obeying the law
\[
C_n^\kappa = A_n^\kappa \sqrt{2 \exp \left( -\sqrt{2 |E_n|} \sqrt{4k + 2l + 3} \right)} / \sqrt{\kappa} \sqrt{4k + 2l + 3}.
\]

Here \( \{ \kappa \} \) are the quantum numbers of the \( l \)-basis, \( l \) is the angular momentum of cluster relative motion, \( r_0 \) is the oscillator length, \( A_n^\kappa \) is the asymptotic normalization coefficient [15].

Asymptotic behavior (at large values of the number of quanta \( \nu \)) of the eigenvectors for the continuum states \( \{ C_n(E) \} \) with the energy \( E > 0 \) is expressed in terms of the Hankel functions of the first and second kind, and the scattering \( S \)-matrix elements; or the Bessel and the Neumann functions, and the \( K \)-matrix elements [17,11].

The eigenvectors of the different states are orthonormalized, but the orthonormality conditions contain the eigenvalues \( \Lambda_n \) as the weight coefficients,
\[
\sum_n \Lambda_n C_n^{\kappa} C_n^{\kappa'} = \delta_{\kappa,\kappa'}, \quad (8a)
\]
\[
\sum_n \Lambda_n C_n^{\kappa}(E) C_n^{\kappa'}(E') = \delta(E - E'), \quad (8b)
\]
\[
\sum_n \Lambda_n C_n^{\kappa}(E) C_n^{\kappa}(E) = 0. \quad (8c)
\]

A non-standard form of the orthonormality conditions [5] comes out from the fact that the coefficients \( C_n^\kappa \) and \( C_n(E) \) are the eigenvectors of (7), the so-called generalized eigenvalue problem [10].

The eigenvectors having been determined, it is easy to proceed either to the eigenfunctions \( \Phi_n(r), \Phi_E(r) \) in the coordinate representation or to the functions \( \Psi_n(R), \Psi_E(R) \) in the Fock-Bargmann space. Then for the discrete states we have
\[
\Psi_n(R) = \sum_n \sqrt{\Lambda_n} C_n^\kappa \psi_n(R), \quad (9)
\]
while for the continuum states we obtain
\[
\Psi_E(R) = \sum_n \sqrt{\Lambda_n} C_n^E \psi_n(R). \quad (10)
\]

Now let us address another version of the procedure of constructing the solutions of the Hill-Wheeler equation, namely, to the diagonalization of the two quadratic forms in (6). In order to reduce them to a diagonal form simultaneously, it is worthwhile redefining the coefficients \( C_n^\kappa \) \( (C_n) \) setting
\[
C_n^\kappa = \sqrt{\Lambda_n} C_n^\kappa, \quad C_n = \sqrt{\Lambda_n} C_n.
\]

Then, instead of (9), the following equation for the coefficients is obtained
\[
\sum_n \sum_{\tilde{n}} \left\{ C_n^\kappa \frac{\langle n|\hat{H}|\tilde{n}\rangle}{\sqrt{\Lambda_n\Lambda_{\tilde{n}}}} C_{\tilde{n}} - E \delta_{n,\tilde{n}} C_n^\kappa C_{n} \right\} = 0. \quad (11)
\]

After that it remains to make the renormalized matrix of the Hamiltonian
\[
\frac{\langle n|\hat{H}|\tilde{n}\rangle}{\sqrt{\Lambda_n\Lambda_{\tilde{n}}}}
\]
be diagonalized by means of the unitary transformation. Then an expression for the density matrix in the Fock-Bargmann representation can be written as follows
\[
\rho(S,R) = \sum_\kappa \left\{ \sum_n \sqrt{\Lambda_n} C_n^{\kappa} \psi_n(S) \sum_{\tilde{n}} \sqrt{\Lambda_{\tilde{n}}} C_{\tilde{n}}^{\kappa} \psi_{\tilde{n}}(R) \right\} + \int dE \left\{ \sum_n \sqrt{\Lambda_n} C_n^{\kappa}(E) \psi_n(S) \sum_{\tilde{n}} \sqrt{\Lambda_{\tilde{n}}} C_{\tilde{n}}(E) \psi_{\tilde{n}}(R) \right\}. \quad (13)
\]

The summation in Eq. (13) is over the discrete states, while the integration is over the continuum states. This density matrix gives us the information about the behavior of the density distribution function in the phase space. Note, that the number of independent variables in the expression (13) is significantly less than in the distribution function \( \rho\{ \{ r \} \} \) defined in the coordinate space. This is the main advantage of the generator-coordinate method, in general; and of the Fock-Bargmann representation, in particular [24].

Having integrated Eq. (13) over the phase space, we can reduce the diagonalized density matrix of the system of interacting clusters to the following sum
\[
\int \rho(R,R') d\mu_R = \sum_\kappa \sum_n |C_n^\kappa|^2 \Lambda_n,
\]
provided that all the states of the system belong to the discrete part of the spectrum.

Here
\[
(C_n^\kappa)^2 \Lambda_n = \int |\psi_n(R)^* \Psi_n(R) d\mu_R|^2
\]
is the realization probability of the state $\psi_n(R)$ in the wave function of the cluster system $\Psi_n(R)$.

So, employing basis of the Pauli-allowed states, we deal with a discrete representation when for each state of the system, belonging either to the discrete or continuum spectrum, its eigenvector should be constructed. The latter is a set of the expansion coefficients $\{\sqrt{\lambda_n}C_n\}$ of the wave function of this state in the basis of the Pauli-allowed states. Absolute value of a coefficient squared gives us a realization probability for the corresponding Pauli-allowed basis state; and a convergence holds both for the bound states and continuum.

V. HOW DOES AN ANTISYMMETRIZATION OPERATOR ACT

The RGM wave function belonging to the continuous spectrum of the system composed of several clusters has a remarkably simple form at large inter-cluster distances. In this region the potential energy of cluster-cluster interaction can be neglected and there is no need of taking into account an antisymmetrization operator. Then the wave function is given by the product of the intrinsic cluster wave functions and the wave function of free motion of their center of masses.

With the inter-cluster distance decreasing, the wave function of the system changes and is no longer reducible to the simple product. Nucleons of different clusters are not isolated from each other anymore and the region of their localization increases that results in changing the internal cluster energies and the energy of cluster relative motion, even if the potential energy of cluster-cluster interaction is not taken into consideration.

Among the factors coming into play at this stage, the most important are those which are directly related to the influence of the Pauli exclusion principle. As a result, a general picture of the phenomenon becomes intricate; and to interpret it completely, each factor determining the final result should be carefully analyzed. Further we shall concentrate our attention on the phenomena directly conditioned by an antisymmetrization of the RGM wave function and which reveal characteristic features of the latter function at small inter-cluster distances. With that end in view, some considerable simplifications will be introduced for the description of the potential energy of cluster-cluster interaction. Summarizing, if we take into consideration only the Pauli-allowed states, we should answer, at least, the two questions: (1) At which inter-cluster distances an interaction generated by the Pauli exclusion principle appears? (2) What are the main features of such interaction?

To understand the results of action of the antisymmetrization operator, first, let us remind the above discussed set of the algebraic equations (7) where only the operator of the kinetic energy of the relative motion of clusters (in the c.o.m. frame) is retained:

$$\sum_n <n|\hat{T}|n> C_n - E\Lambda_n C_n = 0. \quad (14)$$

Due to a particular simplicity of the kinetic energy operator its generating matrix element can be written in the form

$$T(R, S) = \hat{T}_R J(R, S) = \hat{T}_R \sum_n \Lambda_n \psi_n(R)\psi_n(S),$$

where

$$\hat{T}_R = -\frac{\hbar^2}{4mr_0^2} \left( R^2 - 2(R \cdot \nabla R) - 3 + \nabla^2_R \right) \quad (15)$$

is the Fock-Bargmann map of the kinetic energy operator (see, for example, [10]). $m$ stands for the nucleon mass from now on. Eq. (15) is immediately followed by the conclusion that the kinetic energy matrix in the harmonic-oscillator representation is tridiagonal, whence

$$T(R, S) = \sum_{\nu} \{\Lambda_{\nu-2}T_{\nu-2,\nu}\psi_{\nu-2}(R) + \Lambda_{\nu}T_{\nu,\nu}\psi_{\nu}(R) + \Lambda_{\nu}T_{\nu,\nu+2}\psi_{\nu+2}(R)\} \psi_{\nu}(S).$$

Here $T_{\nu,\nu}$ are the matrix elements of the kinetic energy operator $\hat{T}_R$ between the functions $\psi_{\nu}(R)$ normalized to unity with the Bargmann measure,

$$T_{\nu,\nu} = \int dR\psi_{\nu}^*(R)\hat{T}_R\psi_{\nu}(R).$$

For simplicity, the single-channel case is considered here. For such a case the basis functions differ only in the number of oscillator quanta $\nu$.

The nonlocal kinetic energy operator is obtained. This is an important stage in a procedure of deduction of a set of equations for the coefficients $C_n$. A typical equation of the set (14) looks like

$$\Lambda_{\nu-2}T_{\nu,\nu-2}C_{\nu-2} + \Lambda_{\nu}(T_{\nu,\nu} - E)C_{\nu} + \Lambda_{\nu}T_{\nu,\nu+2}C_{\nu+2} = 0. \quad (16)$$

A standard system of the discrete representation which leaves out of account the requirements of the Pauli principle,

$$T_{\nu,\nu}C_{\nu} + (T_{\nu,\nu} - E)C_{\nu} + T_{\nu,\nu+2}C_{\nu+2} = 0, \quad (17)$$

differs from the set of equations (16) in two aspects. Firstly, only the matrix elements between the allowed states enter the latter. Therefore, there is no need to introduce any additional potential in the initial Hamiltonian neither to remove the forbidden states, nor to make the wave function vanish as the identical nucleons approach each other. Such a term would give zero contribution in the set of equations (16), because the equations of our discrete representation are constructed with these requirements being properly accounted for.
Secondly, elimination of the forbidden states still does not resolve on the problem in toto and the matrix of the set of Schrödinger differential equation of free motion with definite angular momentum \( l \) is an evidence for such a statement. It contains the eigenvalues \( \Lambda_\nu \) of the Pauli-allowed states, and for this reason it is not identical with the matrix of the kinetic-energy operator of the cluster free motion.

Equations of the set (16) are known to take asymptotic form at large values of \( \nu \) and to be reduced to the Schrödinger differential equation of free motion with definite angular momentum \( l \) (see [17] for details). Equations of the set (16) are of the same form at large \( \nu \) when all the eigenvalues \( \Lambda_\nu \) equal unity. But at such values of \( \nu \), that the eigenvalues differ from unity and start to decrease or to increase (remaining positive, of course), asymptotic form of the equations becomes complicated; and they are transformed into equations of motion in the field of the potential generated by the antisymmetrizer. Our purpose is to find these equations in order to reveal the main features of the cluster-cluster interaction in their collision.

Equations of the set (16) for collision of the clusters in the state with angular momentum \( l \) can be written in the form of the finite-difference equations

\[
-\frac{1}{2} \left\{ \left( 1 + \frac{\Lambda_{\nu-2}}{\Lambda_\nu} \right) \left( \nu + \frac{3}{2} \frac{(2l+1)^2}{8\nu} \right) + 1 - \frac{\Lambda_{\nu-2}}{\Lambda_\nu} \right\} \\
\times \frac{1}{4} \left( C_{\nu+2} - 2C_\nu + C_{\nu-2} \right) \\
-\frac{1}{2} \left\{ 1 + \frac{\Lambda_{\nu-2}}{\Lambda_\nu} \right\} \left( \nu + \frac{3}{2} \frac{(2l+1)^2}{8\nu} \right) \\
\times \frac{1}{4} \left( C_{\nu+2} - C_{\nu-2} \right) \\
+ \left\{ \left( 1 + \frac{\Lambda_{\nu-2}}{\Lambda_\nu} \right) \frac{(2l+1)^2}{32\nu} \right. \\
+ \frac{1}{4} \left( 1 - \frac{\Lambda_{\nu-2}}{\Lambda_\nu} \right) \left( \nu + \frac{1}{2} \right) \right\} \left( \nu \frac{ \Delta \nu }{ \nu + \frac{1}{2} } \right) \right\}
\]

\( \nu = \frac{mv^2}{\hbar^2} EC_\nu. \)

The latter is transformed into the Bessel differential equation in the limit \( \nu \gg 1 \) when the eigenvalues \( \Lambda_\nu \) can be set to unity:

\[
\left( \frac{d^2}{dy^2} + \frac{1}{y} \frac{d}{dy} - \frac{(2l+1)^2}{4} \frac{1}{y^2} + \frac{m \nu^2}{\hbar^2} \frac{2E}{y} \right) C(y) = 0; (18)
\]

\( y = \sqrt{2\nu + 2l + 3}. \)

The diagonal matrix \( \langle U_{\nu,\nu}^{\text{Pauli}} \rangle \),

\[
U_{\nu,\nu}^{\text{Pauli}} = \frac{1}{2} \left( 1 + \frac{\Lambda_{\nu-2}}{\Lambda_\nu} \right) \frac{(2l+1)^2}{16\nu} + \frac{1}{4} \left( 1 - \frac{\Lambda_{\nu-2}}{\Lambda_\nu} \right) \left( \nu + \frac{1}{2} \right),
\]

can be considered as the matrix of the operator of the effective cluster-cluster interaction generated by the Pauli principle. Physical meaning of the first term in (19) is quite simple. It is the centrifugal potential that is renormalized with the factor

\[
\frac{1}{2} \left( 1 + \frac{\Lambda_{\nu-2}}{\Lambda_\nu} \right).
\]

The latter, along with the eigenvalues, tends to unity as \( \nu \to \infty \). If the eigenvalues approach unity from below as \( \nu \) increases, the renormalization factor does not exceed unity; therefore, partial suppression of the centrifugal potential is observed. In other case, if the eigenvalues approach unity from above, some strengthening of the centrifugal potential occurs.

The second term,

\[
\frac{1}{4} \left( 1 - \frac{\Lambda_{\nu-2}}{\Lambda_\nu} \right) \left( \nu + \frac{1}{2} \right),
\]

represents a finite-range potential generated by the Pauli principle (referred to as "effective potential related to the antisymmetrization" in what follows). Its intensity decreases in magnitude as the difference \( \Lambda_\nu - \Lambda_{\nu-2} \) vanishes. If the latter remains negative as \( \nu \) grows (the eigenvalues monotonically approach unity from above), then the effective potential turns to be attractive. If the difference of the eigenvalues remains positive, provided that \( \nu \) increases (the eigenvalues monotonically approach unity from below), then the effective potential is repulsive. Obviously, the range of such interaction depends on the width of the interval where the eigenvalues deviate from unity.

Indeed, as was shown in Section IV of the wave function of the binary cluster system can be presented in the form of the expansion (9) or (10). If \( \Lambda_\nu < 1 \), the eigenvalues suppress the terms with small values of \( \nu \) in the expansions (9) that can be naturally interpreted as the action of effective repulsion forces at small inter-cluster distances. This leads to a decreasing the ground state energy by absolute value and to a corresponding change in scattering phases due to an appearance of the additional effective repulsion. But if \( \Lambda_\nu > 1 \), then the terms with small \( \nu \), on the contrary, turn to be more preferred that can be considered as an effective attraction.

So, as follows from all stated above, the antisymmetrization effects result not only in the elimination of the forbidden states, but also in changing the kinetic energy of cluster relative motion as clusters approach each other. Let us begin with the remark on the cluster behavior in the state with definite angular momentum \( l \) in the simple case, that any cluster-cluster interaction is absent and influence of the Pauli exclusion principle on the cluster motion is not considered. Then a centrifugal potential is the only factor that changes the velocity of the cluster relative motion in their collision. It decreases the kinetic energy of the relative motion of the clusters until they stop at the turning point \( r_\nu = r_0 \sqrt{2\nu + 2l + 3} \) and then begin to fly away.

Antisymmetrization renormalizes intensity of the centrifugal potential and this factor should be the first that affects cluster velocity as clusters approach each other.
With the inter-cluster distance decreasing, the effective potential related to the antisymmetrization also comes into play.

If the eigenvalue $\Lambda_\nu$ is monotone increasing function of $\nu$, then some suppression of the centrifugal potential occurs. As a result, velocity of the cluster relative motion decreases a lesser degree than in the field of the centrifugal potential of the free motion. But then clusters come within the range of the effective potential related to the antisymmetrization. In the case considered this potential corresponds to the repulsion: and therefore, it decreases the kinetic energy of cluster relative motion. If the eigenvalue $\Lambda_\nu$ is monotone decreasing function of $\nu$, then some strengthening of the centrifugal potential on the distances, that the Pauli principle comes into play, slows down cluster motion a greater degree than in the case of free motion. But after that the relative velocity of the clusters increases due to the action of the effective attractive potential.

It should be noted also that in the low-energy region the elastic scattering phase is formed by the effective antisymmetrization potential. As for changing a centrifugal barrier caused by the Pauli principle, it affects, mainly, an asymptotic behavior of the phase shift.

To understand what is the range of influence of the antisymmetrizer on the structure of the wave functions, it is appropriate to consider two limit values of the number of quanta, $\nu_{\text{min}}$ and $\nu_{\text{max}}$. As long as $\nu < \nu_{\text{min}}$, there are no the Pauli-allowed states belonging to the branch under consideration; and all the eigenvalues equal zero. If the number of quanta satisfy the inequality $\nu_{\text{min}} \leq \nu \leq \nu_{\text{max}}$, then the eigenvalues become positive, but deviate from unity. Finally, provided that the inequality $\nu > \nu_{\text{max}}$ holds, the eigenvalues can be considered approximately equal unity. The limit number of quanta $\nu_{\text{max}}$ is defined in rather a relative way that demonstrates a diffuseness of the antisymmetrization operator range.

The intensity of the effective cluster-cluster interaction induced by the Pauli principle is determined by the natural combination of the parameters (with the dimension of energy) entering the problem considered,

$$E_x = \frac{\hbar^2}{mr_0^2(2\nu_{\text{max}} + 2l + 3)}.$$

Here $r_0 \sqrt{(2\nu_{\text{max}} + 2l + 3)}$ defines the inter-cluster distance where the antisymmetrization effects come into play; and the quantity $E_x$ determines the energy range where for the branches with $\Lambda_\nu > 1$ one can expect occurrence of the resonance phenomena caused by the influence of the antisymmetrization potential or, at least, a maximum of the scattering phase. Obviously, the larger is the value $\nu_{\text{max}}$, the larger is the range of the antisymmetrization potential, and the lower are the energies which correspond to possible resonant states.

It is appropriate to compare the range of the antisymmetrization operator with that of the cluster-cluster interaction generated by the nucleon-nucleon potential. At large values of the number of quanta $\nu$ the matrix of the potential energy operator $U(r)$ of cluster-cluster interaction is known to be equivalent to the diagonal matrix $|\delta_{\nu,\nu} U(r_0 \sqrt{2\nu + 2l + 3})|$ (see [18]) that significantly simplifies the above-mentioned comparison. For a central nucleon-nucleon potential having a Gaussian form we will have

$$U(r_0 \sqrt{2\nu + 2l + 3}) = U_0 \exp \left\{ \frac{2r_0^2(2\nu + 2l + 3)}{b_0^2} \right\}$$

(20)

where $b_0$ is the radius of the Gaussian potential, and $U_0$ is its intensity.

At large values of the number of oscillator quanta $\nu$ the effective cluster-cluster interaction induced by the antisymmetrization vanishes as the eigenvalues $\Lambda_\nu$ tend to unity. Thus, the range of such interaction can be estimated with the help of the relation

$$\Lambda_\nu - 1 \sim \beta(\nu) \alpha^{-\nu} = \beta(\nu) \exp(-\nu \ln \alpha).$$

(21)

Parameter $\alpha > 1$ is completely determined by the type of clustering of the nuclear system considered; and that is why it is the same for all the branches of the SU(3) irreps. As for the parameter $\beta(\nu)$, its dependence on the number of quanta obeys a power law and differs for different SU(3) irreps even within one cluster configuration.

It follows from (20) and (21) that the effective interaction induced by the antisymmetrization decreases exponentially with the number of quanta, and so is the cluster-cluster interaction generated by the nucleon-nucleon forces. Comparing the decrements of the expressions (20) and (21), it is possible to establish which of the potentials has larger range. It appears to be that

$$\frac{4r_0^2}{b_0^2} \gg \ln \alpha$$

even for the simplest binary cluster systems composed of $s$-clusters, such as $^4\text{He} + n$, $^4\text{He} + ^4\text{He}$, etc. This means that as the clusters approach each other, they first experience an influence of the effective interaction caused by the Pauli exclusion principle. And the potential of the cluster-cluster interaction generated by the nucleon-nucleon forces comes into play only at the smallest inter-cluster distances.

Further, on the examples of different binary systems composed of $s$-clusters and $p$-clusters we shall demonstrate how the eigenvalues (and, therefore, the parameters of the effective cluster-cluster interaction induced by the antisymmetrizer) depends on the number of nucleons of compound system and on the type of clustering being considered.

**VI. EXAMPLES OF THE EFFECTIVE POTENTIALS: SINGLE-CHANNEL CLUSTER SYSTEMS**

Our immediate purpose is to demonstrate the influence of the effective potential related to the antisymmetrization on the behavior of scattering phases and
wave functions in the discrete representation (expansion coefficients of the wave functions in the complete basis of the Pauli-allowed states) addressing the simplest examples of the cluster systems with one open channel. We will also draw attention to the comparison of our results and those which can be obtained by simulation of the Pauli principle with some phenomenological potentials of the optical model.

Formally, the equations of the optical model with forbidden states for the interaction of compound systems are the most similar to our approach. In this model an antisymmetrization is performed by the aid of the procedure of elimination of the forbidden states, for what a pseudopotential is introduced in the equations. In our procedure of elimination of the forbidden states, for what antisymmetrization is performed by the aid of the proportion of the Pauli-allowed states or, in other words, the allowed states belong to the branch of SU(3) irreps with \((\lambda, \mu) = (\nu, 0)\) and their role depends on changing the eigenvalues of the Pauli-allowed states with the number of quanta.

The cluster systems considered in this section have a remarkable property: any given number of quanta \(\nu\) corresponds with only one SU(3) multiplet \((\nu, 0)\). That rather simplifies an explicit form of the Pauli-allowed orthonormal basis functions in the Fock-Bargmann space (they are constructed from even powers of one complex vector) and provides only two sets of the eigenvalues: one for the even states and the other, for the odd ones. Besides, the matrix elements of the kinetic energy operator take a particularly simple form in this case. Really, all the allowed states belong to the branch of SU(3) irreps with \((\lambda, \mu) = (\nu, 0)\); and so the kinetic energy matrix, in addition to the diagonal elements, contains only those which couple SU(3) irreps \((\nu, 0)\) and \((\nu + 2, 0)\).

Now our purpose is to make the situation under consideration more realistic and, at the same time, reveal all that directly relates to the antisymmetrization. Towards this end, we shall discuss the results obtained in the approximation of zero-range nuclear forces, which takes into account a potential energy operator only in the most compact configuration for each cluster system; or without this operator at all when trying to emphasize the role of the antisymmetrization.

Behavior of the wave functions in discrete representation at the relatively small number of quanta (where the influence of the antisymmetrization is the most noticeable) is of special interest. As to the phases of the elastic scattering, we concentrate our attention on the region of energies up to 50 MeV (in the c.o.m. frame). In the absence of the operator of potential energy of the nucleon-nucleon interaction the two different modes in the behavior of the scattering phases are observed in this region. Later we shall see that at low energies the eigenvalues approaching unity from above are correspondent with a positive scattering phase; while those approaching unity from below, with a negative one.

A. System n+\(^4\)He

The scattering of a neutron and \(\alpha\)-particle is one of the simplest examples considered in the frame of the Hill-Wheeler method. We aim at giving a demonstration that even for this system the scattering phase and the wave function can not be reproduced in full by any optical model potential which simulates an action of the Pauli principle.

The norm kernel for \(\alpha+n\) system generating a complete basis of the Pauli-allowed states or, in other words, the
kernel of the integral equation looks like

\[ I_{n+\alpha}(R, S) = \exp(R \cdot S) - \exp \left[ -\frac{1}{4} (R \cdot S) \right]. \] (22)

The norm kernel does not contain the Pauli-forbidden states. They are already excluded. Therefore, it can be compared with the result of action of the model repulsive potential which eliminates the forbidden states. But we should also take account of the eigenvalues of the allowed states. They are not equal to unity and enter both the norm kernel and the matrix elements of the Hamiltonian between the allowed states. Therefore, provided that the Pauli principle is simulated by some phenomenological potential, at this stage of solving the problem there should be introduced some additional interaction to specify dynamics of the system after elimination of the forbidden states. Later we will attend to the question what should be the main features of such an additional phenomenological potential.

Let us start with calculating the eigenvalues. Eq. (22) is immediately followed by the Hilbert–Schmidt expansion

\[ I_{n+\alpha}(R, S) = \sum_{\nu=1}^{\infty} \left\{ 1 - \left( -\frac{1}{4} \right)^\nu \right\} \frac{1}{\nu!} (R \cdot S)^\nu. \] (23)

The expression

\[ I^{(\nu,0)}(R, S) = \frac{1}{\nu!} (RS)^\nu \]

is the norm kernel for the irrep \((\nu,0)\), which is normalized to the number of states. It can also be considered as a kernel of the integral equation

\[ \psi_{\nu,L,M}(R) = \int I^{(\nu,0)}(R, S) \psi_{\nu,L,M}(S^* )d\mu S. \]

At that, all the eigenvalues of the kernel \(I^{(\nu,0)}\) are equal to unity for any given quantum numbers \(\nu, L, M\). That is why the Hilbert–Schmidt expansion of the norm kernel (23) takes the form

\[ I_{n+\alpha}(R, S) = \sum_{\nu=1}^{\infty} \left\{ 1 - \left( -\frac{1}{4} \right)^\nu \right\} \times \sum_{L,M} \psi_{\nu,L,M}(R) \psi_{\nu,L,M}(S). \]

Therefore, the eigenvalues of the norm kernel for \(\alpha+n\) system are equal to

\[ \Lambda_\nu = 1 - \left( -\frac{1}{4} \right)^\nu. \]

Obviously, in the limit \(\nu \to \infty\) the eigenvalues tend to unity from below, if the number of quanta is even \((\nu = 2k)\); and from above, if the number of quanta is odd \((\nu = 2k + 1)\)

\[ \Lambda_{2k} = 1 - \left( \frac{1}{16} \right)^k, \quad \Lambda_{2k+1} = 1 + \frac{1}{4} \left( \frac{1}{16} \right)^k. \]

The minimal number of quanta \(\nu_{\text{min}}\) which corresponds to the lowest Pauli-allowed basis state is equal to 1. It means that the branch which belongs to the SU(3) irrep \((2k + 1, 0)\) appears first and its eigenvalues take the highest possible values.

To demonstrate a validity of the stated above conclusions about the characteristic features of the effective interaction induced by the Pauli principle on the concrete examples, it is appropriate to consider the phases of the elastic scattering of a neutron by \(\alpha\)-particle [30] in the states with \(L^x = 0^+\) [Fig. 1 (a)] and \(L^x = 1^-\) [Fig. 1 (b)]. Zero angular momentum corresponds to the eigenvalues \(\Lambda_{2k} < 1\), while the eigenvalues \(\Lambda_{2k+1}\) which exceed unity correspond to the momentum \(L = 1\). The behavior of the scattering phases \(\delta_{L=0}\) and \(\delta_{L=1}\) for the case when only an antisymmetrization is taken into account and in the approximation of zero-radius for nuclear force is presented in Fig. 1. In this approximation the potential energy of the interaction of the neutron and \(\alpha\)-particle is simulated with the single parameter, the diagonal matrix element of the potential energy operator \(\hat{U}\) in the state with the minimal number of even (for \(L = 0\)) or odd (for \(L = 1\)) quanta, i.e.

\[
\langle (\nu, 0) | U | (\nu', 0) \rangle = \begin{cases} U_0 = -5.52 \text{ MeV} & \text{if } \nu = \nu' = \nu_{\text{min}} = 2, L = 0 \\ U_0 = -11.05 \text{ MeV} & \text{if } \nu = \nu' = \nu_{\text{min}} = 1, L = 1 \\ 0 & \text{otherwise} \end{cases}
\]

The parameter \(U_0\) was fitted to reproduce a position of the maximum of the total cross-section of the elas-
tic scattering $E_r = 0.92 \pm 0.04$ MeV \cite{19}. The half-width $\Gamma = 1.3$ MeV also agrees well with the experimental value of $\Gamma = 1.2$ MeV, although the maximum value of the total cross section twice as large as its experimental value of 7.6 barns \cite{14}. As regards the parameter $U_0$, it was chosen to provide a reasonable description of the experimentally observed phase of the $\alpha+n$ elastic scattering with $L = 0$ at low energies \cite{20}.

As long as the energy $E$ of the relative motion of clusters is small, the phase of the elastic scattering with angular momentum $L$ obeys the law

$$\delta_L \sim \pi n - a_L (2E)^{L+1/2},$$

i.e. as for a standard short-range potential. Here $n$ is the number of bound states. If $L = 0$, then the factor $a_0$ is called the scattering length. There are no bound states in the system $\alpha+n$, but there exists one Pauli-forbidden state at $k = 0$ with angular momentum $L = 0$, which is known to have the same influence on the behavior of the scattering phase $\delta_0$ as a bound state \cite{21}. Therefore, at zero energy the scattering phase $\delta_0$ is naturally counted off from $\pi$. For the odd number of quanta there are no forbidden states and we set the scattering phase $\delta_1$ be equal to zero at zero energy.

The positive sign of the scattering length conforms to the known general consideration that Pauli principle can be simulated by a repulsive potential. Of course, the attraction $U_0$ appeared not to be strong; otherwise it would change the signs of the scattering phase and scattering length.

As it can be seen from Fig. 1(b), $a_{L=1} < 0$. Therefore, an effective interaction caused by the Pauli principle for the states with angular momentum $L = 1$ is attractive. Note that in the low-energy range up to 20 MeV the scattering phase is positive both for the case with the potential $U_0$ and with the antisymmetrization effects being taken into account only. We comparing the scattering phases with angular momentum $L = 1$ without a potential and with the potential $U_0$, the intensity of the attraction induced by the antisymmetrization effects only is not high enough to assure an existence of the experimentally observed $1^-$-resonance in the continuum of the $^3$He nucleus. But a contribution from the antisymmetrization is not negligible, because the range of the potential induced by the Pauli principle exceeds that of the nuclear forces.

Stress that the energy dependence of the scattering phase $\delta_1$ obtained with the account of antisymmetrization effects can not be reproduced by simulation of action of the Pauli principle with a soft or hard core. Such an approximation would be bad, because it could not explain a positive sign of the phase at low energy.

Now let us resort to the expansion coefficients of the wave functions of the continuous spectrum at different energies to understand the behavior of these functions in different states. This opens a prospect for a more detailed investigation of the structure of the states for which only the energy dependence of the scattering phases has been available so far.

First, let us consider the wave functions of the channel $L^z = 0^+$. Wave functions at the energy $E = 12.61$ MeV (at this energy the scattering phase calculated in the approximation of zero-range for nuclear force equals to $\pi/2$) are shown in Fig. 2(a) for the variants with and without the potential $U_0$. This value of the scattering phase is notable for the absence of the Bessel function in a corresponding expression for the wave function at large value of the number of quanta $k$; only the Neumann function $N_{1/2}(\sqrt{2Er_0}\sqrt{4k+3})$ remains there. The wave functions for both versions almost do not differ except a small shift of the wave to the origin in the case with the potential. This shift is induced by action of the weak attractive potential $U_0$.

The wave function of the $L^z = 1^-$ state at the energy corresponding to the maximum value of the scattering phase is also of interest. As is well known, taking into account a spin-orbit interaction leads to the splitting of the state $L^z = 1^-$; and, as a result, the two scattering phases $\delta_{3/2}$ and $\delta_{1/2}$ appear, each of them showing a resonance behavior. In our case it is possible to conclude about only one resonance located between the mentioned above. The phase $\delta_1$ calculated in the version with the potential $U_0$ reaches its maximum value at $E = 2.31$ MeV. Then the wave function \cite[see Fig. 2(b)]{3} concentrates in the vicinity of $r_{k_{\text{min}}}$ rather than at other $r_k$. Its amplitude almost four times as large as the value of the wave functions in the next extremums. In the potential-free version the phase, expectedly, reaches maximum at higher energy ($E = 7.41$ MeV). Again the wave function concentrates near $r_{k_{\text{min}}}$, although not as much as in the previous case.

B. System $^3\text{H}^+ + ^3\text{H}$

Now let us consider a collision of the two nuclei $^3\text{H}$ with opposite spins. In this case the norm kernel contains a complete basis set of the Pauli-allowed basis functions and their eigenvalues which correspond to the two different values of the total spin: $S = 0$ and $S = 1$,

$$I_{^3\text{H}^+ + ^3\text{H}}(R, S) = I_{^3\text{H}^+ + ^3\text{H}}^{S=0}(R, S)D_{00}^0(\sigma) + I_{^3\text{H}^+ + ^3\text{H}}^{S=1}(R, S)D_{00}^1(\sigma). \quad (24)$$

Here $D_{MM'}^S(\sigma)$ are the Wigner $D$-functions, $\sigma$ is a set of spin variables.

As the total isospin of the system is equal to unity, the even number of quanta, obviously, corresponds to the singlet states,

$$I_{^3\text{H}^+ + ^3\text{H}}^{S=0}(R, S) \propto \cosh(R \cdot S) - \cosh\left[\left(\frac{R \cdot S}{3}\right)\right]$$

$$= \sum_{k=1}^{\infty} \left[1 - \left(\frac{1}{3}\right)^{2k}\right] \frac{1}{(2k)!} (RS)^{2k};$$
In contrast to the system $n+\alpha$, the wave function of two identical bosons (which result by the antisymmetrization effects on the number of nucleons in the interacting clusters can be drawn.

As has already been mentioned in the previous section, at large number of quanta the behavior of an effective potential related to the antisymmetrization is determined by the expression (21). In a general case, at small inter-cluster distances an effective potential has rather a cumbersome form and depends on several exponentially decreasing terms,

$$A_\nu^{(\lambda,\mu)} - 1 = \sum_j \gamma_j^{(\lambda,\mu)}(\nu) \exp\left\{-\nu \ln \alpha_j^{(\lambda,\mu)}\right\},$$

However, in the problem of scattering of two $s$-clusters at a given $\nu$ the SU(3) irreps $(\nu, 0)$ appears to be the only possible representation; and in the sum (27) only one term remains. In Table II the parameters of an effective interaction induced by the Pauli principle are presented for the binary nuclear systems composed of the $s$-clusters.

By analyzing the data listed in Table II some general conclusions about dependence of the effective interaction caused by the antisymmetrization effects on the number of nucleons in the interacting clusters can be drawn.

As has already been mentioned in the previous section, one of the quantities defining the range of the antisymmetrization operator is the minimal number of quanta

$$I_{n+\alpha}(R, S) = \sinh(R \cdot S) - 3 \sinh \left(\frac{R \cdot S}{3}\right)$$

where the odd number of quanta, to the triplet ones,

$$I_{n+\alpha}^{S=1}(R, S) = \sinh(R \cdot S) - 3 \sinh \left(\frac{R \cdot S}{3}\right)$$

At that, the eigenvalues are given by

$$\Lambda_{\nu=2k} = 1 - \left(\frac{1}{3}\right)^\nu, \quad \Lambda_{\nu=2k+1} = 1 - \left(\frac{1}{3}\right)^{(\nu-1)}.$$  

In contrast to the system $n+\alpha$, the eigenvalues with the even and odd number of quanta are identical and tend to unity from below. As to the minimal number of quanta allowed by the Pauli principle, it increases by one comparing to the previous case.

Again, as for the system $n+\alpha$ in the states with $\Lambda_{2k} < 1$, the scattering phase behavior determined only by the influence of the antisymmetrized kinetic energy operator corresponds to the repulsion of the clusters at small distances between them. Because of the existence of the sole forbidden state in both the channels, the scattering phases at zero energy are counted off from $\pi$ (see Fig. 3).

The singlet scattering phase falls faster than the triplet one, because the former has lower value of the angular momentum. For the singlet state and $L = 0$ a version with zero-range attractive potential has also been considered, assuming that the interaction can be reproduced by just one diagonal matrix element

$$< (2, 0) | \hat{U} | (2, 0) >= -34.76 \text{ MeV}.$$  

Such model potential gives a correct value for the observed threshold energy of the disintegration of the $^6\text{He}$ nuclear system into the channel $^3\text{H}+^3\text{H}$ (12.3 MeV [12]) and close to the experimental value the root-mean-square radius of $^6\text{He}$ nucleus, equal to 2.24 fm. The singlet phase of the elastic scattering for the potential which assures an existence of one bound state is counted off from $2\pi$ (Fig. 3). This scattering phase is larger than the scattering phases obtained in the potential-free version in the energy range being considered.

C. System $\alpha+\alpha$

Finally, let us address a well-known example of the scattering of two $\alpha$-particles; or, in other words, let us consider $^8\text{Be}$ nuclear system in the $\alpha$-cluster model:

$$I_{\alpha+\alpha}(R, S) = 8 \sinh^4 \left(\frac{R \cdot S}{4}\right).$$

As seen from (26), the norm kernel $I_{\alpha+\alpha}$ contains only the basis functions with the even number of quanta, because the wave function of two identical bosons (which

| $\nu$ | $\Lambda_{\nu=1}$ | $\nu=2k$ | $\nu=2k+1$ |
|-----|----------|----------|----------|
| $n+\alpha$ | 1 | $1/4$ | 16 | -1 | 16 | $1/4$ |
| $^3\text{H}+^3\text{H}$ | 2 | $-1/9$ | 9 | -1 | 9 | -1 |
| $\alpha+\alpha$ | 4 | $-1/4$ | 4 | -4 | 0 | 0 |

are the $\alpha$-particles) should be symmetric with respect to interchange of the clusters as a whole,

$$I_{\alpha+\alpha}(R, S) = \sum_{k=2}^\infty \frac{1}{(2k)!} \left(1 - \frac{4}{2^{2k}}\right) (R \cdot S)^{2k}.$$  

An expression for the eigenvalues follows from (26).

$$\Lambda_{2k} = 1 - 4 \left(\frac{1}{4}\right)^k, \quad \Lambda_{2k+1} = 0.$$  

$$\Lambda_{2k} < 1$$ for any given number of quanta, and the minimal allowed number of quanta takes maximum value $\nu_{\text{min}} = 4$ among all the cases considered in this section. As a result, the Pauli principle leads to the repulsion of clusters; and the behavior of the scattering phase is similar to the above discussed one for the states with $L = 0$.  

D. Comparison of the eigenvalues for different nuclear systems

As has already been discussed in the previous section, at large number of quanta the behavior of an effective potential related to the antisymmetrization is determined by the expression (21). In a general case, at small inter-cluster distances an effective potential has rather a cumbersome form and depends on several exponentially decreasing terms,

$$A_\nu^{(\lambda,\mu)} - 1 = \sum_j \beta_j^{(\lambda,\mu)}(\nu) \exp\left\{-\nu \ln \alpha_j^{(\lambda,\mu)}\right\},$$

However, in the problem of scattering of two $s$-clusters at a given $\nu$ the SU(3) irreps $(\nu, 0)$ appears to be the only possible representation; and in the sum (27) only one term remains. In Table II the parameters of an effective interaction induced by the Pauli principle are presented for the binary nuclear systems composed of the $s$-clusters.

By analyzing the data listed in Table II some general conclusions about dependence of the effective interaction caused by the antisymmetrization effects on the number of nucleons in the interacting clusters can be drawn.

As has already been mentioned in the previous section, one of the quantities defining the range of the antisymmetrization operator is the minimal number of quanta

$$< (2, 0) | \hat{U} | (2, 0) >= -34.76 \text{ MeV}.$$  

Such model potential gives a correct value for the observed threshold energy of the disintegration of the $^6\text{He}$ nuclear system into the channel $^3\text{H}+^3\text{H}$ (12.3 MeV [12]) and close to the experimental value the root-mean-square radius of $^6\text{He}$ nucleus, equal to 2.24 fm. The singlet phase of the elastic scattering for the potential which assures an existence of one bound state is counted off from $2\pi$ (Fig. 3). This scattering phase is larger than the scattering phases obtained in the potential-free version in the energy range being considered.

C. System $\alpha+\alpha$

Finally, let us address a well-known example of the scattering of two $\alpha$-particles; or, in other words, let us consider $^8\text{Be}$ nuclear system in the $\alpha$-cluster model:

$$I_{\alpha+\alpha}(R, S) = 8 \sinh^4 \left(\frac{R \cdot S}{4}\right).$$

As seen from (26), the norm kernel $I_{\alpha+\alpha}$ contains only the basis functions with the even number of quanta, because the wave function of two identical bosons (which

| $\nu$ | $\Lambda_{\nu=1}$ | $\nu=2k$ | $\nu=2k+1$ |
|-----|----------|----------|----------|
| $n+\alpha$ | 1 | $1/4$ | 16 | -1 | 16 | $1/4$ |
| $^3\text{H}+^3\text{H}$ | 2 | $-1/9$ | 9 | -1 | 9 | -1 |
| $\alpha+\alpha$ | 4 | $-1/4$ | 4 | -4 | 0 | 0 |
which corresponds to the first non-vanishing eigenvalue. It is easy to understand that \( \nu \) increases with the number of nucleons of the system under study, because the number of the occupied states grows. Actually, in the system \( n+\alpha \) the first allowed state appears already at \( \nu = 1 \), while in the case of the two-\( \alpha \)-particles it appears only at \( \nu = 4 \).

With the number of nucleons increasing, the parameter \( \alpha \) defining the order of decrease for an effective potential also essentially decreases. At the same time, the parameter \( \beta \) rather increases that is an evidence of increasing the range of antisymmetrization forces. A negative sign of \( \beta \) indicates that an effective interaction is repulsive one, except for the states with the odd number of quanta in the system \( ^4\text{He}+n \) where it is attractive. Note, that in the latter case exactly those basis states that correspond to the eigenvalues \( \Lambda_\nu > 1 \) dominate in the wave function of \( ^4\text{He}+n \).

Finally, one more important parameter determining the intensity of an effective potential related to the antisymmetrization is the value of this potential at the minimal number of quanta. As seen from Table II the intensity of an effective interaction is maximum in the states with \( \nu = \nu_{\text{min}} \) and increases with the number of nucleons for the systems considered. Thus, a repulsive interaction in the states with the even number of quanta for the system \( ^4\text{He}+n \) intensifies for the \( ^3\text{H}+^3\text{H} \) nuclear system and becomes even stronger in the system \( \alpha+\alpha \).

VII. MULTI-CHANNEL BINARY CLUSTER SYSTEMS

A. System \( ^6\text{He}+^6\text{He} \)

Cluster configuration \( ^6\text{He}+^6\text{He} \) is a relatively simple multi-channel system, by giving an example of which it is possible to understand a role of the Pauli principle in the formation of coupling between different channels.

Each of the clusters of the system \( ^6\text{He}+^6\text{He} \) has open \( p \)-shell; and, therefore, studying a collision of these clusters it is natural to take account of their excitations accompanied by the transition from the ground \( 0^+ \) state to the \( 2^+ \) excited state. This is a distinctive feature of the \( ^6\text{He}+^6\text{He} \) nuclear system in comparison with the above discussed. It consists, in particular, in the fact that now at a given even value of the total number of quanta \( \nu = 2k > 8 \) a basis of the Pauli-allowed states with the total orbital angular momentum \( L = 0 \) belongs to the five \( \text{SU}(3) \) irreps with even symmetry indices \((\lambda, \mu)\): \((2k-2,0),(2k,2),(2k-4,4),(2k,2),(2k+4,0)\). Notice that the multiplets \((2k,2)\) and \((2k,2)\) have the same \( \text{SU}(3) \) symmetry indices, but different eigenvalues \( \Lambda_{(2k,2)} \) and \( \Lambda_{(2k,2)} \).

As a result, the norm kernel \( I_{^6\text{He}+^6\text{He}} \) for the states with \( L = 0 \) takes the form

\[
I_{^6\text{He}+^6\text{He}} = \sum_{k=2}^{\infty} \Lambda_{(2k-2,0)} \tilde{\psi}_{(2k-2,0)} \tilde{\psi}_{(2k-2,0)} + \sum_{k=3}^{\infty} \Lambda_{(2k,2)} \tilde{\psi}_{(2k,2)} \tilde{\psi}_{(2k,2)} + \sum_{k=3}^{\infty} \Lambda_{(2k-4,4)} \tilde{\psi}_{(2k-4,4)} \tilde{\psi}_{(2k-4,4)} + \sum_{k=4}^{\infty} \Lambda_{(2k,2)} \tilde{\psi}_{(2k,2)} \tilde{\psi}_{(2k,2)} + \sum_{k=5}^{\infty} \Lambda_{(2k+4,0)} \tilde{\psi}_{(2k+4,0)} \tilde{\psi}_{(2k+4,0)}.
\]

These eigenfunctions, along with their eigenvalues, were defined in [11]. Hence, here we will restrict ourselves with only those results from [11] which directly relate the problem discussed.

Such a plenty of the irreducible representations puts a question of their classification with the aim of determining the most important irreps, at least, for the low above-threshold energies. First, let us present in Fig. 4 the dependence of the eigenvalues belonging to the five different branches (according to their definition introduced in Section III) on \( k \). As seen from this figure, all the eigenvalues except \( \Lambda_{(2,0)} \) are less than unity; and, therefore, generate an effective repulsive potential. The largest eigenvalues belong to the branch \((2k-2,0)\); while the smallest ones, to the branch \((2k+4,0)\). In the states of the latter branch a cluster repulsion caused by the action of the Pauli principle is maximal, as well as the range of the antisymmetrization effects. Only if \( k \geq 14 \), eigenvalues are close to unity. Besides, this branch starts with \( k = 5 \), i.e. later than the others. The repulsion in the states of the branch \((2k,2)\), for which the minimal number of \( k \) equals 4, is somewhat less intensive; and its eigenvalues can be set to unity, if \( k \geq 13 \). The repulsion for the branches \((2k-4,4)\) and \((2k,2)\), which appear at \( k = 3 \), is even less pronounced. The eigenvalues of these branches are rather close to unity, if \( k \geq 10 \). The fact, that in the absence of degeneracy the higher \( \text{U}(3) \)-symmetry (the larger the eigenvalues of the second-order Casimir operator of \( \text{U}(3) \)-group), the smaller the eigenvalues, attracts attention.

Of course, the most remarkable feature of the basis of the Pauli-allowed states for the \( ^6\text{He}+^6\text{He} \) system is that this basis corresponds to the five different channels. Above some threshold energy \((E = 3.6 \text{ MeV})\) all these channels are open. But there is such energy range in the continuous part of the spectrum where two or only one channel is open. The influence of the Pauli principle on the system considered manifests itself in making all the five channels coupled at small inter-cluster distances. The radius of this domain is determined by the requirement that on its border the five different eigenvalues of the allowed states are almost equal to unity. Below we
shall specify at what real values of $k$ (and, therefore, $r_k$) it occurs. As soon as all the eigenvalues approach unity, a unitary transformation from the SU(3) basis to the $l$-basis, which allows us to make the coupling of different channels of the latter basis via the kinetic energy operator vanish, becomes possible [11].

Provided that the eigenvalues of different SU(3) branches are not identical, coupling of the channels via matrix elements of the kinetic energy operator directly results in an appearance of the off-diagonal elements of the scattering S-matrix; and, hence, in an occurrence of the inelastic processes in the collision of the two $^6$He nuclei. Certainly, a potential energy of cluster interaction also can influence the inelastic scattering cross-sections.

$$
\langle (2k - 2, 0) | U | (2k' - 2, 0) \rangle = \begin{cases} 
U_0 = -44.2 \text{ MeV} & \text{if } k = k' = 2 \\
U_1 = -28.7 \text{ MeV} & \text{if } k = k' = 3 \\
0 & \text{otherwise}
\end{cases}
$$

These values were fitted to the experimental values of the r.m.s. radius of $^{12}$Be ($2.59 \pm 0.06 \text{ fm}$ [22]) in its ground state, and the $^6$He+$^6$He decay threshold energy (10.11 MeV [23]). The oscillator length was fixed to 1.37 fm.

Comparing the coefficients of different branches, we come to a conclusion that the same inequality holds both for them and for their eigenvalues,

$$L(2k-2,0) \geq L(2k,2), L(2k-4,4) \geq L(2k,2) \geq L(2k+4,0),$$

with the only small exception: the coefficients of the branch $(2k - 4, 4, 4)$ are somewhat larger than the coefficients of the branch $(2k, 2, 2)$, if $r_k > 6.5 \text{ fm}$.

Let us consider the wave function for one more state (Fig. 6) obtained by diagonalization of the Hamiltonian on the SU(3) basis states, with the maximal value of $k$ being equal to 64. This state was chosen in such a way to make its energy $E = 0.885 \text{ MeV}$ be above the threshold of the $^{12}$Be decay into two $^6$He nuclei in their ground state ($E = 0$); but less than the threshold energy ($E = 1.8 \text{ MeV}$) of the decay of $^{12}$Be into the channel, one of the clusters being in $^2\Sigma^+$ excited state [31]. As long as $r_k < 6 \text{ fm}$, the behavior of the coefficients is determined by the intensity of repulsion in the corresponding SU(3) branches, like in the g.s. wave function. However, then a rearrangement of their values occurs. The coefficients of the irreps $(2k, 2)$ appear to be leading ones, and they are followed by the irreps $(2k - 4, 4)$ and $(2k - 2, 0)$. Enumerated coefficients correlate in such a way that by projecting on the states of $l$-basis we arrive at the basis functions of the open channel. At large $r_k$ the basis functions of the irreps $(2k, 2)$ contain only those functions of the $l$-basis which correspond to the closed channels. That is why the expansion coefficients belonging to the branch $(2k, 2)$ exponentially tend to zero with $r_k$ increasing.

However, as before, we shall restrict our analysis to only that contribution to the nuclear reaction cross-section which is made by the effects of exchange of the nucleons belonging to the different clusters. A simple potential energy operator used here does not couple different channels.

An information about the intensity of repulsion in the states belonging to the different branches is provided by the expansion coefficients (Fig. 6) of the ground state wave function of $^{12}$Be+$^6$He calculated in the zero-range approximation for nuclear force [11]. We assume that the interaction can be reproduced by just two diagonal matrix elements in the SU(3) representation $(2k - 2, 0)$, i.e.

Finally, turn to the state with the energy $E = 3.3 \text{ MeV}$ (Fig. 7). This state is above the threshold for the decay of $^{12}$Be into the two $^6$He nuclei; provided that one of them being in the ground state, while the other being excited. In this energy region all the states with zero angular momentum are two-fold degenerate, as the two channels are open. For the expansion coefficients the same behavior is observed as in the previous case, granting the fact that in this case the coefficients are nonzero for only those SU(3) irreps which contain the basis functions of the two open channels.

Multi-channel situation make analysis of the scattering eigenphases difficult. The question is from what value the eigenphases should be counted off at zero threshold energy of the corresponding channel, to make the eigenphases vanish at high energy $E$. At first, notice that after a transformation to the $l$-basis the total number of oscillator quanta $2k$ used for the SU(3) symmetry indices of different branches is appropriate to represent as a sum $2k = 2k' + l$, where $l$ is even angular momentum of cluster relative motion, $k'$ is the number of radial quanta. Let us begin with the fact that the first (regarding the number of quanta) Pauli-allowed state possesses SU(3) symmetry $(2, 0)$. After a transformation to the $l$-basis it appears to contain the functions

$$\Phi_2^{(0,0,0)}, \Phi_2^{(2,2,0)}, \frac{1}{\sqrt{2}} \left\{ \Phi_2^{(2,2,0)} + \Phi_2^{(0,2,2)} \right\}, \text{ and } \Phi_2^{(2,2,2)}.$$

If the angular momentum $l$ of the relative motion equals zero, the Pauli-forbidden states are

$$\Phi_0^{(0,0,0)}, \Phi_1^{(0,0,0)}; \text{ and } \Phi_0^{(2,2,0)}, \Phi_1^{(2,2,0)}.$$

Hence, the eigenphases of these channels are appropriate to be counted off from $2\pi$ at zero energy in the corre-
sponding channel. Even though $l = 2$, then in the channel where only one of the two $^6\text{He}$ nuclei is excited the two states are forbidden

$$\frac{1}{\sqrt{2}} \left\{ \Phi_1^{(2,0,2)} + \Phi_2^{(0,2,2)} \right\} \text{ and } \frac{1}{\sqrt{2}} \left\{ \Phi_1^{(2,0,2)} - \Phi_2^{(0,2,2)} \right\}. $$

Therefore, in this channel the eigenphase will also be counted off from $2\pi$. One more forbidden state, $\Phi_1^{(2,2,2)}$, specifies counting off from $\pi$ for the eigenphase in the channel with both $^6\text{He}$ being excited. At last, the channel $l = 4$ also has only one forbidden state $\Phi_2^{(2,4,4)}$, as this function appears only at $k = 3$ along with the functions of the irreps $(6,2)_1$ and $(2,4)$. Hence, the eigenphase $\delta_{l=4}$ will be counted off from $\pi$. To summarize, note that there are eight different forbidden states of the $l$-basis. Besides, because of existence of the sole bound state in the zero-approximation for nuclear force, all the eigenphases should be moved up by $\pi$ more.

Important information about the multi-channel continuous spectrum of the $^6\text{He}+^6\text{He}$ system is provided by its five (according to the maximal number of the open channels) eigenphases, presented in Fig. 8. Just above the corresponding threshold $E_{\text{thr}}$, where one or another channel opens, the eigenphases $\delta_l$ obey the law

$$\delta_l(E) = \delta_l(E_{\text{thr}}) + \text{const}E^{l+1/2}. $$

Here $\delta_l(E_{\text{thr}})$ is the value divisible by $\pi$. In Fig. 8 the three quasi-intersections of the phase curves are seen. Eigenphases can not intersect, because the fact of intersection would contradict the unicity theorem for a solution of a wave equation. After the intersection point each of the eigenphases moves in the direction along which the other eigenphase moved before the intersection point. Fall of the eigenphases with the energy indicates a repulsion due to the antisymmetrization effects, and is not compatible with the assumption that any resonance exists in the system in question.

B. $^4\text{He}+^8\text{He}$ and $^6\text{He}+^6\text{He}$ clustering: coupled-channel approach

Along with the clustering of $^{12}\text{Be}$ into the channels with both $^6\text{He}$ nuclei being in the ground or excited state, let us consider also the $^4\text{He}+^8\text{He}$ cluster structure. The latter allows an excitation of the $^8\text{He}$ nucleus to its $2^+$ state. Taking into account the $^4\text{He}+^8\text{He}$ clustering results in the important corrections to the results of calculations with due regard of the $^6\text{He}+^6\text{He}$ clustering only.

First of all, the number of the allowed states increases and an additional SU(3) degeneration appears. As a result, the number of the channels with the total angular momentum $L = 0$, total spin $S = 0$, and isospin $T = 2$ grows to seven. The $^4\text{He}+^8\text{He}$ clustering provides for the two additional branches of the basis states having SU(3) symmetry $(2k - 2,0)$ and $(2k,2)$. Basis states of different cluster configurations with the same SU(3) symmetry indices are not orthogonal, while the two states $(2,0)$ are identical. Hence, we considering the two clustering modes simultaneously; the new basis states (the eigenfunctions of the norm kernel for the two coupled cluster configurations) representing a superposition of the basis functions of the $^4\text{He}+^8\text{He}$ and $^6\text{He}+^6\text{He}$ channels are obtained.

Let us start with the Hilbert-Schmidt expansion of the new norm kernel $I_{(8,4)+(6,6)}$ for the states with $L = 0$. Seven different sums are written down in a descending order of the eigenvalues $\Lambda_{(\lambda,\mu)}$,

$$I_{(8,4)+(6,6)} = \sum_{k=2}^{\infty} \Lambda_{(2k-2,0),\lambda} \tilde{\psi}^{(2k-2,0),\lambda}_1 \tilde{\psi}^{(2k-2,0),\lambda}_1 $$

$$+ \sum_{k=3}^{\infty} \Lambda_{(2k,2),\lambda} \tilde{\psi}^{(2k,2),\lambda}_1 \tilde{\psi}^{(2k,2),\lambda}_1 $$

$$+ \sum_{k=3}^{\infty} \Lambda_{(2k-4,4),\lambda} \tilde{\psi}^{(2k-4,4),\lambda}_1 \tilde{\psi}^{(2k-4,4),\lambda}_1 $$

$$+ \sum_{k=3}^{\infty} \Lambda_{(2k-2,0),\lambda} \tilde{\psi}^{(2k-2,0),\lambda}_2 \tilde{\psi}^{(2k-2,0),\lambda}_2 $$

$$+ \sum_{k=4}^{\infty} \Lambda_{(2k,2),\lambda} \tilde{\psi}^{(2k,2),\lambda}_2 \tilde{\psi}^{(2k,2),\lambda}_2 $$

$$+ \sum_{k=4}^{\infty} \Lambda_{(2k,4,4),\lambda} \tilde{\psi}^{(2k,4,4),\lambda}_3 \tilde{\psi}^{(2k,4,4),\lambda}_3 $$

$$+ \sum_{k=5}^{\infty} \Lambda_{(2k,4,4),\lambda} \tilde{\psi}^{(2k,4,4),\lambda}_4 \tilde{\psi}^{(2k,4,4),\lambda}_4. $$

Now the states $(2k - 2,0)$ become two-fold degenerate, while the degree of degeneracy for the states $(2k,2)$ raises to three. The dependence on the number of quanta of the eigenvalues belonging to the different branches is shown in Fig. 8. Its remarkable feature is that now the eigenvalues of the two branches, $\Lambda_{(2k-2,0),\lambda}$ and $\Lambda_{(2k,2),\lambda}$, exceed unity. An effective potential related to antisymmetrization in the states of the aforementioned branches corresponds to the attraction. This directly affects the structure of the g.s. wave function of the $^{12}\text{Be}$ nucleus,

because now the expansion coefficients belonging to the SU(3) irreps $(2k - 2,0)$ dominate in the expression for the wave function of the $^{12}\text{Be}$ ground state. The basis functions of these irreps on equal footing contain both the states of the $^6\text{He}+^6\text{He}$ and $^4\text{He}+^8\text{He}$ clustering.

Now in order to reproduce the threshold energy (8.95 MeV) of the $^{12}\text{Be}$ break-up into $^4\text{He}$ and $^8\text{He}$ in their ground states and the root-mean-square radius of the $^{12}\text{Be}$ nucleus, $r_{\text{rms}} = 2.66$ fm, the parameter $U_0$ of the model zero-range potential should be reduced in absolute magnitude to the value of 42.2 MeV; while the parameter $U_1$ should even be set to zero. Such a change of the potential parameters seems to be natural, because the basis of the $^6\text{He}+^6\text{He}$ configuration is supplemented with the basis states of the $^4\text{He}+^8\text{He}$ configuration, with a contribution of the latter ones to the energy of the ground state being considerable.
The coefficients $\sqrt{\lambda(\lambda, \mu)} C_{E_k}^{(\lambda, \mu)}(r_k)$ are presented in Fig. \textbf{10}. With the small exception, the smaller are the eigenvalues of the basis functions, the smaller are the corresponding coefficients.

The chosen version of the potential provides for an existence of the second bound state at the energy $E_1 = -0.386$ MeV. The coefficients $\sqrt{\lambda(\lambda, \mu)} C_{E_k}^{(\lambda, \mu)}(r_k)$ are presented in Fig. \textbf{11}. The coefficients of the irreps $(2k, 2)_1$ appear to be dominating for this state, while the coefficients of the $(2k-2, 0)_1$ irreps take the second place where they compete with the coefficients of the irreps $(2k-4, 4)$. The contribution of the other coefficients again correlates with the magnitude of their eigenvalues.

Moreover, now even without any cluster interaction potential generated by the nucleon-nucleon forces the bound state of the $^{12}$Be nuclear system, with the energy $E_0 = -0.75$ MeV and the root-mean-square radius $r_{\text{rms}} = 4$ fm, appears. In Fig. \textbf{12} the expansion coefficients of the wave function for this state are shown. Those of the coefficients that correspond to the irreps $(2k-2, 0)_1$ and $(2k, 2)_1$ possess the largest values among all the coefficients at small $r_k$. The bound state appeared due to the attraction for which the two latter branches of the SU(3) irreps are responsible. For simplicity, the five threshold energies are assumed to be equal. But even then the question of the correct value assignment for each of the seven eigenphases at zero energy remains to be clarified. An answer to this question is reduced to the calculation of the total number of the forbidden states of the $l$-basis. The $^8$He+$^4$He configuration supplies the three forbidden states. One of them corresponds to the channel with $^8$He and $^4$He clusters being in their ground states, zero angular momentum of their relative motion and zero number of quanta, i.e. $k = 0$. The other forbidden state corresponds to the same channel, but with $k = 1$. The last forbidden state belongs to the channel with $k = 1$ and $^8$He nucleus excited.

Granting all the above-mentioned concerning the forbidden states of the $^6$He+$^8$He clustering, we arrive at the conclusion about an existence of the eleven forbidden states. As regards the magnitude of the eigenphases at zero energy, with due regard of the antisymmetrization effects only the eigenphases of the three channels with $l = 0$ begin with $3\pi$ like the eigenphase for one of the three channels with $l = 2$. The eigenphases for the other channels with $l = 2$ and the eigenphase of the channel with $l = 4$ begin with $2\pi$. The energy dependence for all the eigenphases is presented in Fig. \textbf{13}.

Let us remind that for the $^6$He+$^8$He clustering all the eigenphases decrease monotonically with the energy. For the case of simultaneous consideration of the $^6$He+$^6$He and $^4$He+$^4$He configurations, but without a nucleon-nucleon potential [Fig. \textbf{13} (a)], two of the seven eigenphases with the angular momentum $l = 2$ of the relative cluster motion first ascend, reach maximum within the energy range up to 1 MeV, and only after that begin to decrease monotonically. We have already observed such a behavior for $\alpha+n$ scattering phase with the angular momentum $L = 1$. To understand, whether it is possible to conclude about resonances owing to these maxima, let us consider the wave function in the continuum for the energy of 0.22 MeV obtained by diagonalization of the Hamiltonian (see Fig. \textbf{13}). The chosen energy is close to that one at which one of the eigenphases has a pronounced peak. In this wave function the states corresponding to the aforementioned channels with $l = 2$ prevail that is compatible with the assumption about an existence of the resonance.

Inclusion of the zero-range potential pulls down the resonance under the break-up threshold of $^{12}$Be into $^9$He+$^4$He [Fig. \textbf{13} (b)]. As a result, along with the ground state of the $^{12}$Be nucleus ($E_0 = -8.95$ MeV), the excited state appears at the energy $E_1 = -0.386$ MeV. Further behavior of the eigenphases with the energy increasing is similar to that which has already been discussed for the $^6$He+$^8$He clustering.

VIII. CONCLUSIONS

The Pauli principle influence on the structure of the continuum states of the compound-systems populated at the intermediate stage of collisions between light nuclei was studied within the algebraic version of resonating group method. The exchange effects generated by the antisymmetrization operator were analyzed on the ground of the discrete representation of the complete basis of the Pauli-allowed many-particle harmonic-oscillator states classified with the use of SU(3) symmetry indices. The eigenvalue problem for the norm kernel (the overlap integral of the antisymmetric generating functions of the Hill-Wheeler method) was reduced to a solution of the degenerate integral equations in the Fock-Bargmann space.

The influence of the Pauli exclusion principle on the collision of clusters was shown to be reducible to three factors which affect the dynamics of the cluster-cluster interaction. Firstly, elimination of the forbidden states drastically increases the maximum amplitude of the scattering phase that may be simulated by a repulsive potential at small inter-cluster distances. The larger is the number of the forbidden states, the larger should be the intensity and the radius of such a model potential. Secondly, out of the core (in the region where the eigenvalues of the Pauli-allowed states differ from unity) some additional effective potential (repulsive or attractive) appears. The latter can significantly affect the scattering phase behaviour. Finally, decreasing or increasing the centrifugal potential occurs in the same region. It also influences the phase shift, especially, at high energy. The inter-cluster distances, within which the Pauli principle is important, depend on the cluster structure and can be several times more than the radius of the cluster-cluster interaction induced by the nucleon-nucleon potential.

If there are several open channels, exchange of the nucleons which belong to the colliding clusters forms the
cross-sections of the inelastic scattering. The eigenphases of the multi-channel systems that define nature of the inelastic collisions have been calculated; and the fact of quasi-intersection of the eigenphases was established.

A considerable intensification of the antisymmetrization effects is observed in the case when different cluster configurations (such as, for example, $^6\text{He}+^6\text{He}$ and $^4\text{He}+^8\text{He}$ which are actual for the $^{12}\text{Be}$ compound nucleus) are taken into account simultaneously. This phenomenon relates to an appearance of the new branches of excitation with the especially large, greater than unity, eigenvalues of the allowed states. As a result, an effective attraction induced by the antisymmetrization effects appears to be strong enough to ensure an existence both the bound state and the resonance even without a participation of the nucleon-nucleon interaction between nucleons of different clusters.

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FIG. 1: Phases of the $\alpha+n$ scattering for the states with (a) $L = 0$ and (b) $L = 1$. Solid curve: phases obtained in the approximation of zero-radius for nuclear force. Dashed curve: phases obtained by granting the Pauli principle only.

FIG. 2: Continuum states of $^5\text{He}$: coefficients of the w.f. expansion in the SU(3) basis (a) $L = 0$, $E = 12.61$ MeV and (b) $L = 1$, line A: $E = 7.41$ MeV, line B: $E = 2.31$ MeV. Solid curve: w.f. obtained in the approximation of zero-radius for nuclear force. Dashed curve: w.f. obtained by granting the Pauli principle only.
FIG. 3: Phases of the $^3\text{H}+^3\text{H}$ scattering. Solid curve: the phase obtained in the approximation of zero-radius for nuclear force. Dotted and dashed curves: the phases obtained by granting the Pauli principle only.

FIG. 4: Eigenvalues $\Lambda_{(\lambda,\mu)}$ of the norm kernel for the system $^6\text{He}+^4\text{He}$ versus the number of quanta $k$. SU(3) symmetry indices $(\lambda,\mu)$ are shown near the curves.
FIG. 5: Ground state of $^{12}\text{Be}=^{6}\text{He}+^{6}\text{He}$: coefficients $\sqrt{\Lambda_{(\lambda,\mu)}}C_{E_0}^{(\lambda,\mu)}(r_k)$ of the w.f. expansion in the SU(3) basis, half-logarithmic scale. SU(3) symmetry indices $(\lambda, \mu)$ are shown near the curves.

FIG. 6: Coefficients $\sqrt{\Lambda_{(\lambda,\mu)}}C_{E}^{(\lambda,\mu)}(r_k)$ of the expansion of the continuum states of $^{12}\text{Be}=^{6}\text{He}+^{6}\text{He}$ in the SU(3) basis at $E = 0.885$ MeV. SU(3) symmetry indices $(\lambda, \mu)$ are shown near the curves.
FIG. 7: Coefficients $\sqrt{\Lambda_{(\lambda,\mu)}} C_{k}^{(\lambda,\mu)}(r_{k})$ of the expansion of the continuum states of $^{12}$Be=$^{4}$He+$^{6}$He in the SU(3) basis at $E = 3.3$ MeV. SU(3) symmetry indices $(\lambda, \mu)$ are shown near the curves.

FIG. 8: Eigenphases $\delta_{l}(E)$ of the $^{6}$He+$^{6}$He system obtained in the approximation of zero-range for nuclear force. Values of the angular momentum $l$ of the cluster relative motion are shown near the curves.
FIG. 9: Eigenvalues $\Lambda_{(\lambda, \mu)}$ of the norm kernel for the $^{12}$Be system versus the number of quanta $k$. SU(3) symmetry indices $(\lambda, \mu)$ are shown near the curves.

FIG. 10: Ground state of the $^{12}$Be system: coefficients $\sqrt{\Lambda_{(\lambda, \mu)}} C_{k_{0}}^{(\lambda, \mu)}(r_{k})$ of the w.f. expansion in the SU(3) basis, half-logarithmic scale. SU(3) symmetry indices $(\lambda, \mu)$ are shown near the curves.
FIG. 11: Coefficients $\sqrt{\Lambda_{(\lambda,\mu)}} C^{(\lambda,\mu)}_{E_1}(r_k)$ of the expansion of the second bound state of $^{12}$Be system in the SU(3) basis at $E_1 = -0.386$ MeV. SU(3) symmetry indices $(\lambda, \mu)$ are shown near the curves.

FIG. 12: Ground state of the $^{12}$Be system: coefficients $\sqrt{\Lambda_{(\lambda,\mu)}} C^{(\lambda,\mu)}_{E_0}(r_k)$ of the w.f. expansion in the SU(3) basis. SU(3) symmetry indices $(\lambda, \mu)$ are shown near the curves.
FIG. 13: Eigenphases $\delta_l(E)$ of the $^{12}$Be system. (a) The eigenphases obtained by granting the antisymmetrization effects only. (b) The eigenphases obtained in the zero-range approximation for nuclear force. Values of the angular momentum $l$ of the cluster relative motion are shown near the curves.
FIG. 14: Coefficients $C_{E_1}^{(l_1,l_2,l)}(r_k)$ of the expansion of the continuum states of the $^{12}$Be system in the $l$-basis at $E = 0.22$ MeV. $^6$He+$^6$He clustering: line A – $l_1 = l_2 = l = 0$; line B – $l_1 = l_2 = 2, l = 0$; line C – $l_1 = 2(0), l_2 = 0(2), l = 0$; line D – $l_1 = l_2 = l = 2$; line E – $l_1 = l_2 = 2, l = 4$; $^8$He+$^8$He clustering: line G – $l_1 = l_2 = l = 0$; line F – $l_1 = l = 2, l_2 = 0$. 