WEAK COMPACTNESS AND ESSENTIAL NORMS OF INTEGRATION OPERATORS

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Abstract. Let $g$ be an analytic function on the unit disc and consider the integration operator of the form $T_g f(z) = \int_0^z f(\zeta)g'(\zeta) d\zeta$. We show that on the spaces $H^1$ and $BMOA$ the operator $T_g$ is weakly compact if and only if it is compact. In the case of $BMOA$ this answers a question of Siskakis and Zhao. More generally, we estimate the essential and weak essential norms of $T_g$ on $H^p$ and $BMOA$.

1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$ and $g : \mathbb{D} \to \mathbb{C}$ an analytic function. We consider the generalized Volterra integration operator $T_g$ defined by

$$T_g f(z) = \int_0^z f(\zeta)g'(\zeta) d\zeta, \quad z \in \mathbb{D},$$

for functions $f$ analytic in $\mathbb{D}$. As special cases this includes the classical Volterra operator for $g(z) = z$ and the Cesàro operator for $g(z) = -\log(1-z)$.

In the general form such operators were first introduced by Pommerenke [15] to study exponentials of $BMOA$ functions. He observed, in particular, that $T_g$ is bounded on the Hardy space $H^2$ if and only if $g$ belongs to $BMOA$, the space of analytic functions with bounded mean oscillation on the unit circle. A detailed study of the operators $T_g$ was later initiated by Aleman and Siskakis [4], who showed that Pommerenke’s boundedness characterization is valid on each $H^p$ for $1 \leq p < \infty$ and that $T_g$ is compact on $H^p$ if and only if $g \in VMOA$.

Subsequently a number of authors have extended this line of research to a variety of other spaces and contexts; we refer the reader to the surveys [2, 17] for more information and further references. In particular, in [18] Siskakis and Zhao considered $T_g$ as an operator acting on $BMOA$ and characterized its boundedness and compactness in terms of logarithmically weighted $BMOA$ and $VMOA$ conditions placed on $g$ (see below for precise statements).

The main purpose of this paper is to address the weak compactness of $T_g$ on $H^1$ and $BMOA$. We will namely show that on each of these spaces $T_g$ is weakly compact precisely when it is compact. In the setting of $BMOA$ this result provides a negative answer to a question posed by Siskakis and Zhao in [18, Sec. 3]. More generally, we will derive estimates for the essential and weak essential norms of $T_g$ on $H^p$ and $BMOA$, extending an earlier result of Rättyä [16] for the $H^2$ case. These results are contained in Theorems 1 and 2 below.

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Recall here that a linear operator $T$ on a Banach space is weakly compact if it maps the unit ball of the space into a set whose closure is compact in the weak topology. The essential and weak essential norms of $T$, denoted by $\|T\|_{e}$ and $\|T\|_{w}$, are the distances of $T$ (in the operator norm) from the closed ideals of compact and weakly compact operators, respectively.

As usual, we define $BMOA$ as the space of analytic functions $g : \mathbb{D} \to \mathbb{C}$ such that

$$(1.1) \quad \|g\|_{*} = \sup_{a \in \mathbb{D}} \|g \circ \sigma_{a} - g(a)\|_{H^{2}} < \infty,$$

where $\| \cdot \|_{H^{2}}$ is the standard norm of $H^{2}$ and $\sigma_{a}(z) = (a - z)/(1 - \bar{a}z)$ is the conformal automorphism of $\mathbb{D}$ that interchanges 0 and $a$. Equivalently, $g \in H^{2}$ and the boundary values of $g$ have bounded mean oscillation on the unit circle. Introducing the norm $|g(0)| + \|g\|_{*}$ makes $BMOA$ a Banach space. Its closed subspace $VMOA$ consists of those $g$ for which $\|g \circ \sigma_{a} - g(a)\|_{H^{2}} \to 0$ as $|a| \to 1$. For detailed accounts on $BMOA$ and $VMOA$ we refer the reader to \[5\] [8] [9].

Throughout the paper we use the notation $A \lesssim B$ to indicate that $A \leq cB$ for some positive constant $c$ whose value may change from one occurrence into another and which may depend on $p$. If $A \lesssim B$ and $B \lesssim A$, we say that the quantities $A$ and $B$ are equivalent and write $A \simeq B$.

**Theorem 1.** Let $g \in BMOA$. Then, for $1 \leq p < \infty$,

$$\|T_{g} : H^{p} \to H^{p}\|_{c} \simeq \dist(g, VMOA),$$

and

$$\|T_{g} : H^{1} \to H^{1}\|_{w} \simeq \dist(g, VMOA).$$

In particular, $T_{g}$ is weakly compact on $H^{1}$ if and only if it is compact, or equivalently, $g \in VMOA$.

The logarithmic $BMOA$ space, denoted here by $LMOA$, consists of those analytic functions $g : \mathbb{D} \to \mathbb{C}$ for which

$$(1.2) \quad \|g\|_{*, \log} = \sup_{a \in \mathbb{D}} \lambda(a)\|g \circ \sigma_{a} - g(a)\|_{H^{2}} < \infty,$$

where $\lambda(a) = \log(2/(1 - |a|))$. It is a Banach space under the norm $|g(0)| + \|g\|_{*, \log}$. The logarithmic $VMOA$ space, denoted by $LMOA_{0}$, is defined by the corresponding “little-oh” condition. Note that $LMOA \subset VMOA$.

Siskakis and Zhao \[18\] proved that $T_{g}$ is bounded (resp. compact) on $BMOA$, or equivalently on $VMOA$, if and only if $g \in LMOA$ (resp. $g \in LMOA_{0}$). Alternative proofs can be found in \[14\] [20]. We extend these results as follows.

**Theorem 2.** Let $g \in LMOA$. Then $T_{g} : BMOA \to BMOA$ satisfies

$$\|T_{g}\|_{e} \simeq \|T_{g}\|_{w} \simeq \dist(g, LMOA_{0}).$$

In particular, $T_{g}$ is weakly compact on $BMOA$ if and only if it is compact, or equivalently, $g \in LMOA_{0}$. The same estimates hold for the restriction $T_{g} : VMOA \to VMOA$.

Observe that the distances in Theorems \[1\] and \[2\] can be calculated in terms of the respective seminorms (1.1) and (1.2) because the constant functions belong to $VMOA$ and $LMOA_{0}$. There are various function-theoretic formulas for estimating such distances and we collect a pair of these in Section \[2\]. The proof of Theorem \[1\] is then given in Section \[3\] and the proof of Theorem \[2\] in Section \[4\].
Lemma 4. For The lower estimate for Proof. in the scale 0 because the arguments in \([6, 19]\) do not seem to be directly adapatable to exponents furnished by Lemma 4 below. For completeness we briefly sketch its proof, especially BMOA general expression defining (or equivalent to) the BMOA operators that are not weakly compact and weakly compact operators that are not compact.

\[ \|g\|_* \simeq \sup_{a \in B} \|g \circ \sigma_a - g(a)\|_{H^p}, \]

where the proportionality constants depend on \(p\). Likewise, \(g \in VMOA\) if and only if \(\|g \circ \sigma_a - g(a)\|_{H^p} \to 0\) as \(|a| \to 1\). (See e.g. \([5, \text{Corollary 3}]\).)

Lemma 4. For \(g \in BMOA\) and \(0 < p < \infty\),

\[ \text{dist}(g, VMOA) \simeq \limsup_{|a| \to 1} \|g \circ \sigma_a - g(a)\|_{H^p}. \]

Proof. The lower estimate for \(\text{dist}(g, VMOA)\) is an easy consequence of (2.1) and the corresponding characterization of \(VMOA\) functions.

To prove the upper estimate, one approximates \(g\) by the \(VMOA\) functions \(g_r(z) = g(rz)\) for \(0 < r < 1\). Fix \(0 < \eta < 1\). It is easy to check that \(g_r \circ \sigma_a - g_r(a)\) converges to \(g \circ \sigma_a - g(a)\) in \(H^p\) uniformly for \(|a| \leq \eta\) as \(r \to 1\). Hence, by (2.1),

\[ \text{dist}(g, VMOA) \leq \limsup_{r \to 1} \|g - g_r\|, \]

\[ \simeq \limsup_{r \to 1} \sup_{|a| > \eta} \|g \circ \sigma_a - g(a) - [g_r \circ \sigma_a - g_r(a)]\|_{H^p} \]

\[ \leq \sup_{|a| > \eta} \|g \circ \sigma_a - g(a)\|_{H^p} + \sup_{r > \eta} \|g_r \circ \sigma_a - g_r(a)\|_{H^p}. \]

We may write

\[ g_r \circ \sigma_a - g_r(a) = [g \circ \sigma_r - g(ra)] \circ \psi_{r,a}, \]

where \(\psi_{r,a} = \sigma_r \circ \sigma_a\) is an analytic self-map of \(D\) that fixes the origin. Therefore the Littlewood subordination theorem (see e.g. \([24, \text{Thm 1.7}]\)) yields \(\|g_r \circ \sigma_a - g_r(a)\|_{H^p} \leq \|g \circ \sigma_r - g(ra)\|_{H^p}\). The upper estimate follows as \(\eta \to 1\). □

In the case of logarithmic \(BMOA\) we have the following analogue.
Lemma 5. For \( g \in LMOA \),
\[
\text{dist}(g, LMOA_0) \asymp \limsup_{|a| \to 1} \lambda(a) \| g \circ \sigma_a - g(a) \|_{H^2}.
\]

Proof. The lower estimate is obtained by an application of the triangle inequality and the definition of \( LMOA_0 \) as in the proof of Lemma 4.

The proof of the upper estimate follows the previous idea as well, but requires a refined version of the subordination argument. Indeed, we again approximate \( g \) by the functions \( g_r(z) = g(rz) \) (belonging to \( LMOA_0 \) by [15, Lemma 3.5]) to get the estimate
\[
\text{dist}(g, LMOA_0) \leq \sup_{|a| > \eta} \lambda(a) \| g \circ \sigma_a - g(a) \|_{H^2}
\]
\[
+ \sup_{r > \eta} \lambda(a) \| g_r \circ \sigma_a - g_r(a) \|_{H^2}.
\]

(2.3)

Applying the weighted subordination principle of [12, Prop. 2.3] to (2.2), we obtain
\[
\| g_r \circ \sigma_a - g_r(a) \|_{H^2} \lesssim \| \psi_{r,a} \|_{H^2} \| g \circ \sigma_a - g(a) \|_{H^2}.
\]

A calculation shows that \( \| \psi_{r,a} \|_{H^2}^2 = r^2(1 - |r|^2)/(1 - r^4|a|^2) \), so, for \( r \) and \( |a| \) sufficiently close to 1, we have
\[
\lambda(a) \| g_r \circ \sigma_a - g_r(a) \|_{H^2} \lesssim \frac{\lambda(a) \sqrt{1 - |a|^2}}{\sqrt{1 - |r|^2}} \| g \circ \sigma_a - g(a) \|_{H^2}
\]
\[
\leq \lambda(r) \| g \circ \sigma_a - g(a) \|_{H^2}
\]
because \( \lambda(s) \sqrt{1 - s^2} \) is decreasing on, say, \( (\frac{1}{r}, 1) \). Combining this with (2.2) and letting \( \eta \to 1 \) yields the required estimate. \( \square \)

3. Proof of Theorem 1

We will make use of the standard test functions in \( H^p \), \( 1 \leq p < \infty \), defined by
\[
f_a(z) = \left[ \frac{1 - |a|^2}{(1 - az)^2} \right]^{1/p}
\]
for each \( a \in \mathbb{D} \). Note that \( \| f_a \|_{H^p} = 1 \). For \( p = 2 \) this is just the normalized reproducing kernel of \( H^2 \). According to a theorem of Aleman and Cima [3, Thm 3], there exists a constant \( c_{p,q} > 0 \) such that
\[
\| T_g f_a \|_{H^p} \geq c_{p,q} \| g \circ \sigma_a - g(a) \|_{H^p}
\]
whenever \( 0 < q < p/2 \) (for example, \( q = p/4 \)).

In order to deal with the weak essential norm of \( T_g \) on \( H^1 \) a localization argument is needed. Let \( m \) be the normalized Lebesgue measure on the unit circle \( \mathbb{T} = \partial \mathbb{D} \). We will utilize the classical Dunford–Pettis criterion (see e.g. [11, Thm 5.2.9]) which says that a set \( F \subset \ell^1(m) \) is relatively compact in the weak topology of \( \ell^1(m) \) if and only if it is uniformly integrable, i.e.
\[
\sup_{f \in F} \int_E |f| \, dm \to 0 \quad \text{as} \quad m(E) \to 0.
\]
The application of this criterion in our setting is based on the following lemma.
Lemma 6. For non-zero \(a \in \mathbb{D}\), let \(I(a) = \{e^{i\theta} : |\theta - \arg a| < (1 - |a|)^{1/6}\}\) and \(f_a(z) = (1 - |a|^2)/(1 - \bar{a}z)^2\). Then

\[
\lim_{|a| \to 1} \int_{I(a)} |T_g f_a| \, dm = 0.
\]

Proof. We may assume \(0 < a < 1\) (by rotation-invariance) and \(g(0) = 0\). It is easy to check that \(|1 - \arg a| \geq c|\theta|\) for all \(0 \leq r < 1\) and \(|\theta| \leq \pi\), where \(c > 0\) is an absolute constant. Thus, for \(0 \leq r < 1\) and \((1 - |a|)^{1/6} \leq |\theta| \leq \pi\), we have the uniform estimates

\[
|f_a(re^{i\theta})| \lesssim \frac{1 - a}{|1 - a e^{i\theta}|^2} \lesssim \frac{1 - a}{|\theta|^2} \lesssim (1 - a)^{2/3},
\]

\[
|f_a'(re^{i\theta})| \lesssim \frac{1 - a}{|1 - a e^{i\theta}|^3} \lesssim \frac{1 - a}{|\theta|^3} \lesssim (1 - a)^{1/2}.
\]

The functions \(g\) and \(T_g f_a\) have radial limits at almost every point of \(\mathbb{T}\). Therefore, for a.e. \(\zeta \in \mathbb{T} \setminus I(a)\), we may use integration by parts and the above estimates to get

\[
|T_g f_a(\zeta)| = \left| \int_0^1 f_a(r \zeta)g'(r \zeta) \zeta \, dr \right| \leq |f_a(\zeta)g(\zeta)| + \int_0^1 |f_a'(r \zeta)|g(r \zeta) \, dr 
\lesssim (1 - a)^{2/3} |g(\zeta)| + (1 - a)^{1/2} \int_0^1 |g(r \zeta)| \, dr.
\]

Since \(\text{BMOA}\) is contained in the Bloch space and consequently \(g\) has at most logarithmic growth, the last integral here is bounded by a constant multiple of \(\|g\|_s\). Hence

\[
\int_{\mathbb{T} \setminus I(a)} |T_g f_a| \, dm \lesssim (1 - a)^{2/3} \|g\|_{H^1} + (1 - a)^{1/2} \|g\|_s.
\]

This yields the required result. \(\square\)

Proof of Theorem 4. For every \(1 \leq p < \infty\), the upper estimate for \(\|T_g\|_e\) follows easily from the linearity of \(T_g\) with respect to \(g\). Indeed, for each \(h \in \text{VMOA}\), the operator \(T_h\) is compact and hence \(\|T_g\|_e \leq \|T_g - T_h\| = \|T_g - h\| \lesssim \|g - h\|_e\) by [4] Thm 1.

To establish the lower estimates, we first consider the case \(1 < p < \infty\). Define functions \(f_a\) by (3.1). Since \(f_a \to 0\) weakly in \(H^p\) as \(|a| \to 1\), for every compact operator \(K\) on \(H^p\) we have \(\|K f_a\|_{H^p} \to 0\). Consequently

\[
\|T_g\|_e \geq \limsup_{|a| \to 1} \|T_g f_a\|_{H^p} \gtrsim \limsup_{|a| \to 1} \|g \circ \sigma_a - g(a)\|_{H^p} \equiv \|g\|_{H^p+},
\]

where the last estimate follows from (3.2). But the right-hand side here is equivalent to \(\text{dist}(g, \text{VMOA})\) by Lemma 4.

We finally consider the case \(p = 1\) and derive the lower bound for \(\|T_g\|_w\) on \(H^1\). Let \(S\) be an arbitrary weakly compact operator on \(H^1\) and, as before, consider the test functions \(f_a(z) = (1 - |a|^2)/(1 - \bar{a}z)^2\) for \(a \in \mathbb{D}\). Since \(\|f_a\|_{H^1} = 1\), we have the estimate

\[
\|T_g\|_w \geq \|(T_g - S)f_a\|_{H^1} \geq \int_{I(a)} |T_g f_a| \, dm - \int_{I(a)} |S f_a| \, dm,
\]

where \(I(a)\) is the arc of Lemma 3. Since the set \(\{S f_a : a \in \mathbb{D}\}\) is relatively weakly compact in \(H^1\) and hence uniformly integrable in \(L^1(m)\), the last integral on the right-hand side tends to zero as \(|a| \to 1\). Hence Lemma 6 yields \(\|T_g\|_w \geq \limsup_{|a| \to 1} \|T_g f_a\|_{H^1}\). The argument is then finished as above. \(\square\)
4. Proof of Theorem 2

We start by recalling that \( BMOA \) functions admit a characterization in terms of Carleson measures (see e.g. [8, VI.3]). For any arc \( I \subset \mathbb{T} \), write \( |I| = m(I) \) and let \( S(I) = \{ z \in \mathbb{D} : 1 - |z| < |I|, \ z/|z| \in I \} \) denote the Carleson window determined by \( I \). Given an analytic function \( g : \mathbb{D} \to \mathbb{C} \), define

\[
\mu(g, I) = \int_{S(I)} |g'(z)|^2 (1 - |z|^2) \, dA(z),
\]

where \( A \) denotes the normalized Lebesgue area measure on \( \mathbb{D} \). Then

\[
\|g\|_* \sim \sup_{I \subset \mathbb{T}} \left( \frac{\mu(g, I)}{|I|} \right)^{1/2}
\]

with the understanding that \( g \in BMOA \) if and only if the right-hand side is finite. Also,

\[
g \in VMOA \iff \lim_{|I| \to 0} \frac{\mu(g, I)}{|I|} = 0.
\]

Furthermore, for logarithmic \( BMOA \) we have the following equivalence that is contained in the proof of [18, Lemma 3.4]:

\[
\limsup_{|a| 
rightarrow 1} (\lambda(a)\|g \circ \sigma_a - g(a)\|_{H^2}) \sim \limsup_{|I| \to 0} \left( \log \frac{2}{|I|} \right) \left( \frac{\mu(g, I)}{|I|} \right)^{1/2}.
\]

In view of Lemma 3 this gives another estimate for the distance of a function \( g \in LMOA \) from \( LMOA_0 \).

A key tool in the proof of Theorem 2 is an idea of Leibov [13] on how to construct isomorphic copies of the sequence space \( c_0 \) inside \( VMOA \). As usual, here \( c_0 \) denotes the Banach space of all complex sequences converging to zero equipped with the supremum norm. The following reformulation of Leibov’s result is taken from [12].

**Lemma 7** [11, Prop. 6]. Let \( (f_n) \) be a sequence in \( VMOA \) such that \( \|f_n\|_* \sim 1 \) and \( \|f_n\|_{H^2} \to 0 \) as \( n \to \infty \). Then there is a subsequence \( (f_{n_j}) \) which is equivalent to the natural basis of \( c_0 \); that is, the map \( \iota : (\lambda_j) \to \sum_j \lambda_j f_{n_j} \) is an isomorphism from \( c_0 \) into \( VMOA \).

The utility of this lemma lies in the fact that \( c_0 \) has the Dunford–Pettis property: any weakly compact linear operator from \( c_0 \) into any Banach space maps weak-null sequences into norm-null sequences (see e.g. [11, Sec. 5.4]).

After these preparations we are ready to carry out the proof of Theorem 2.

**Proof of Theorem 2**. We first consider the case of \( T_g \) acting on \( BMOA \). Recall that \( \|T_g\|_w \leq \|T_g\|_e \).

To derive the upper estimate for \( \|T_g\|_* \), we argue as in the proof of Theorem 1 for each \( h \in LMOA_0 \), the operator \( T_h \) is compact on \( BMOA \) and hence \( \|T_g - T_h\| \leq \|T_g - T_h\| \approx \|g - h\|_* \) by Theorem 3.1 and Lemma 3.4 of [18].

To prove the lower estimate for \( \|T_g\|_* \), we first choose a sequence \( (I_n)_{n=1}^\infty \) of subarcs of \( \mathbb{T} \) such that \( |I_n| \to 0 \) and

\[
\lim_{n \to \infty} \left( \log \frac{2}{|I_n|} \right)^2 \frac{\mu(g, I_n)}{|I_n|} = \limsup_{|I| \to 0} \left( \log \frac{2}{|I|} \right)^2 \frac{\mu(g, I)}{|I|} \equiv \alpha.
\]
In view of Lemma 3 and equivalence 13 it is enough to show that \( \|T_g\|_w \gtrsim \sqrt{\alpha} \). Note that \( \alpha \) is finite because \( g \in LMOA \).

For \( n \geq 1 \), let \( u_n = (1 - |I_n|)\xi_n \) where \( \xi_n \) is the midpoint of \( I_n \). By passing to a subsequence if necessary, we may assume that \( (u_n) \) is a Cauchy sequence. Define \( f_n(z) = \log(1 - \frac{u_n}{n}z) \) for \( z \in \mathbb{D} \). A calculation shows that
\[
|f_n(z)| \geq c \log \frac{2}{|I_n|}, \quad z \in S(I_n),
\]
for all \( n \geq 1 \) and a uniform constant \( c > 0 \). Hence
\[
\mu(T_g f_n, I_{n+1}) \leq \frac{c_0}{42}
\]
for all \( n \geq 1 \). Let \( h_n = f_{n+1} - f_n \). Then, by combining 4.4 and 4.5 and applying the triangle inequality, we get, for \( n \geq N \),
\[
\frac{c_0}{42} \leq \mu(T_g (f_{n+1} - f_n), I_{n+1}) \leq C\|T_g h_n\|_w^2,
\]
where \( C > 0 \) is a constant that stems from 4.2. Thus \( \|h_n\|_w^2 \geq c_0/16C\|T_g\| > 0 \).

On the other hand,
\[
\|h_n\|_w^2 = \sum_{k=1}^{\infty} \left| \frac{u^k - u^{k+1}}{k^2} \right|^2 \to 0
\]
as \( n \to \infty \). Therefore, by Lemma 7 there is a subsequence \( (h_{n_j}) \) such that the map \( c: (\lambda_j) \to \sum_{j=1}^{\infty} \lambda_j h_{n_j} \) is an isomorphism from \( c_0 \) into \( BMO \).

Let now \( S \) be any weakly compact operator on \( BMO \). Then \( S \alpha \) is weakly compact from \( c_0 \) to \( BMO \) and since the standard unit vector basis \((e_j)\) of \( c_0 \) converges to zero weakly in \( c_0 \), the Dunford–Pettis property of \( c_0 \) implies \( \|Sh_{nj}\|_w = \|(S \circ e_j)_j\|_w \to 0 \) as \( j \to \infty \). Since \( (h_n) \) is bounded in \( BMO \), we have, by 4.6,
\[
\|T_g - S\| \gtrsim \|T_g h_{nj} - Sh_{nj}\|_w \geq \frac{1}{4C} \sqrt{\alpha}/\sqrt{\alpha} - \|Sh_{nj}\|_w.
\]
This yields that \( \|T_g - S\| \gtrsim \sqrt{\alpha} \) as \( j \to \infty \). Hence \( \|T_g\|_w \gtrsim \sqrt{\alpha} \) and the proof of the lower estimate is complete.

Finally consider \( T_g \) as an operator on \( VMOA \). The upper estimate for \( \|T_g\|_c \) is obtained exactly as in the \( BMOA \) case because the compact approximants \( T_h, h \in LMOA \), take \( VMOA \) into itself. Moreover, since the test functions \( f_n \) and \( h_n \) above belong to \( VMOA \), our argument for the lower estimate of \( \|T_g\|_w \) works in the \( VMOA \) case as well. This finishes the proof of Theorem 2.

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