A tighter Z-eigenvalue localization set for tensors and its applications

Jianxing Zhao∗

College of Data Science and Information Engineering, Guizhou Minzu University, Guiyang 550025, P.R. China

Abstract. A new Z-eigenvalue localization set for tensors is given and proved to be tighter than those presented by Wang et al. (Discrete and Continuous Dynamical Systems Series B 22(1): 187-198, 2017) and Zhao (J. Inequal. Appl., to appear, 2017). As an application, a sharper upper bound for the Z-spectral radius of weakly symmetric nonnegative tensors is obtained. Finally, numerical examples are given to verify the theoretical results.

Keywords: Z-eigenvalue; localization set; nonnegative tensors; spectral radius; weakly symmetric

AMS Subject Classification: 15A18; 15A42; 15A69.

1 Introduction

For a positive integer \( n \), \( n \geq 2 \), \( N \) denotes the set \( \{1, 2, \cdots, n\} \). \( \mathbb{C} (\mathbb{R}) \) denotes the set of all complex (real) numbers. We call \( \mathcal{A} = (a_{i_1i_2\cdots i_m}) \) a real tensor of order \( m \) dimension \( n \), denoted by \( \mathbb{R}^{[m,n]} \), if

\[
a_{i_1i_2\cdots i_m} \in \mathbb{R},
\]

where \( i_j \in N \) for \( j = 1, 2, \cdots, m \). \( \mathcal{A} \) is called nonnegative if \( a_{i_1i_2\cdots i_m} \geq 0 \). \( \mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]} \) is called symmetric [1] if

\[
a_{i_1\cdots i_m} = a_{\pi(i_1\cdots i_m)}, \quad \forall \pi \in \Pi_m,
\]

where \( \Pi_m \) is the permutation group of \( m \) indices. \( \mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]} \) is called weakly symmetric [2] if the associated homogeneous polynomial

\[
\mathcal{A}x^m = \sum_{i_1,i_2,\cdots,i_m \in N} a_{i_1i_2\cdots i_m}x_{i_1}x_{i_2}\cdots x_{i_m}
\]

satisfies \( \nabla \mathcal{A}x^m = mA^{m-1} \). It is shown in [2] that a symmetric tensor is necessarily weakly symmetric, but the converse is not true in general.

Given a tensor \( \mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]} \), if there are \( \lambda \in \mathbb{C} \) and \( x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{C}^n \setminus \{0\} \) such that

\[
\mathcal{A}x^m - 1 = \lambda x \text{ and } x^T x = 1,
\]

∗Corresponding author. E-mail: zjx810204@163.com; zhaojianxing@gzmu.edu.cn (Jianxing Zhao)
then $\lambda$ is called an $E$-eigenvalue of $A$ and $x$ an $E$-eigenvector of $A$ associated with $\lambda$, where $Ax^{m-1}$ is an $n$ dimension vector whose $i$th component is $$(Ax^{m-1})_i = \sum_{i_2,\ldots,i_m \in \mathbb{N}} a_{i_2 \ldots i_m} x_{i_2} \cdots x_{i_m}.$$ If $\lambda$ and $x$ are all real, then $\lambda$ is called a $Z$-eigenvalue of $A$ and $x$ a $Z$-eigenvector of $A$ associated with $\lambda$; for details, see [1,3]. Here, we define the $Z$-spectrum of $A$, denoted $\sigma(A)$, to be the set of all $Z$-eigenvalues of $A$. Assume $\sigma(A) \neq \emptyset$, then the $Z$-spectral radius [2] of $A$, denoted $\rho(A)$, is defined as $$\rho(A) := \sup \{|\lambda| : \lambda \in \sigma(A)\}.$$ Recently, many people have focused on locating all $Z$-eigenvalues of tensors and bounding the $Z$-spectral radius of nonnegative tensors in [2,4–11]. In 2017, Wang et al. [4] established the following Geršgorin-type $Z$-eigenvalue inclusion theorem for tensors.

**Theorem 1.** [4, Theorem 3.1] Let $A = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]}$. Then $$\sigma(A) \subseteq \mathcal{K}(A) = \bigcup_{i \in \mathbb{N}} \mathcal{K}_i(A),$$ where $$\mathcal{K}_i(A) = \{z \in \mathbb{C} : |z| \leq R_i(A)\}, \quad R_i(A) = \sum_{i_2,\ldots,i_m \in \mathbb{N}} |a_{i_2 \ldots i_m}|.$$ To get a tighter $Z$-eigenvalue inclusion set than $\mathcal{K}(A)$, Wang et al. [4] gave the following Brauer-type $Z$-eigenvalue localization set for tensors.

**Theorem 2.** [4, Theorem 3.2] Let $A = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]}$. Then $$\sigma(A) \subseteq \mathcal{L}(A) = \bigcup_{i \in \mathbb{N}} \bigcap_{j \in \mathbb{N}, j \neq i} \mathcal{L}_{i,j}(A),$$ where $$\mathcal{L}_{i,j}(A) = \{z \in \mathbb{C} : (|z| - (R_i(A) - |a_{i_1 \ldots j}|)|z| \leq |a_{i_1 \ldots j}|R_j(A)\}.$$ Very recently, Zhao [5] presented another Brauer-type $Z$-eigenvalue localization set for tensors and proved that this set is tighter than those in Theorem 1 and Theorem 2.

**Theorem 3.** [5, Theorem 3] Let $A = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]}$. Then $$\sigma(A) \subseteq \Psi(A) = \bigcup_{i \in \mathbb{N}} \bigcap_{j \in \mathbb{N}, j \neq i} \Psi_{i,j}(A),$$ where $$\Psi_{i,j}(A) = \{z \in \mathbb{C} : (|z| - r^\Delta_i (A))|z| \leq r^\Delta_j (A)R_j(A)\},$$ $$r^\Delta_i (A) = \sum_{j \in \{i_2,\ldots,i_m\}} |a_{i_2 \ldots i_m}|, \quad r^\Delta_j (A) = \sum_{j \notin \{i_2,\ldots,i_m\}} |a_{i_2 \ldots i_m}|.$$
As we know, one can use eigenvalue inclusion sets to obtain the upper bound of the spectral radius of nonnegative tensors; for details, see [1, 12, 15]. Therefore, the main aim of this paper is to give a new \( Z \)-eigenvalue inclusion set for tensors and prove that the new set is tighter than those in Theorems 1-3. And as an application, a new upper bound for the \( Z \)-spectral radius of weakly symmetric nonnegative tensors is obtained and proved to be sharper than some existing upper bounds.

2 Main results

In this section, we give a new Brauer-type \( Z \)-eigenvalue localization set for tensors, and establish the comparison between the new set with those in Theorems 1-3.

**Theorem 4.** Let \( A = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]} \). Then

\[
\sigma(A) \subseteq \Omega(A) = \left( \bigcup_{i \in N} \bigcap_{i,j \in N, j \neq i} \Omega_{i,j}(A) \right) \cup \left( \bigcup_{i \in N} \bigcap_{i,j \in N, j \neq i} \tilde{\Omega}_{i,j}(A) \right) \cap \mathcal{K}_i(A),
\]

where

\[
\tilde{\Omega}_{i,j}(A) = \left\{ z \in \mathbb{C} : |z| < \frac{1}{r_i^\Delta}(A), |z| < \frac{1}{r_j^\Delta}(A) \right\}
\]

and

\[
\tilde{\Omega}_{i,j}(A) = \left\{ z \in \mathbb{C} : (|z| - \frac{1}{r_i^\Delta}(A))(|z| - \frac{1}{r_j^\Delta}(A)) \leq r_i^\Delta r_j^\Delta(A) \right\}.
\]

**Proof.** Let \( \lambda \) be a \( Z \)-eigenvalue of \( A \) with corresponding \( Z \)-eigenvector \( x = (x_1, \cdots, x_n)^T \in \mathbb{C}^n \setminus \{0\} \), i.e.,

\[
Ax^{m-1} = \lambda x, \quad |x|_2 = 1.
\]

Let \( |x_t| = \max_{i \in N} |x_i| \). Obviously, \( 0 < |x_t|^{m-1} \leq |x_i| \leq 1 \). For \( \forall j \in N, j \neq t \), from [1], we have

\[
\lambda x_t = \sum_{j \in \{i_2, \cdots, i_m\}} a_{t \bar{i}_2 \cdots \bar{i}_m} x_{i_2} \cdots x_{i_m} + \sum_{j \notin \{i_2, \cdots, i_m\}} a_{t \bar{i}_2 \cdots \bar{i}_m} x_{i_2} \cdots x_{i_m}.
\]

Taking modulus in the above equation and using the triangle inequality gives

\[
|\lambda| |x_t| \leq \sum_{j \in \{i_2, \cdots, i_m\}} |a_{t \bar{i}_2 \cdots \bar{i}_m}| |x_{i_2}| \cdots |x_{i_m}| + \sum_{j \notin \{i_2, \cdots, i_m\}} |a_{t \bar{i}_2 \cdots \bar{i}_m}| |x_{i_2}| \cdots |x_{i_m}|
\]

\[
\leq \sum_{j \in \{i_2, \cdots, i_m\}} |a_{t \bar{i}_2 \cdots \bar{i}_m}| |x_j| + \sum_{j \notin \{i_2, \cdots, i_m\}} |a_{t \bar{i}_2 \cdots \bar{i}_m}| |x_t|
\]

\[
= r_j^\Delta(A) |x_j| + r_t^\Delta(A) |x_t|,
\]

i.e.,

\[
(|\lambda| - r_j^\Delta(A)) |x_t| \leq r_t^\Delta(A) |x_t|.
\]

\[ \tag{2} \]
If \(|x_j| = 0\), then \(|\lambda| - r_i^{\xi'}(A) \leq 0\) as \(|x_l| > 0\). When \(|z| - r_j^{\Delta j}(A) \geq 0\), we have

\[
(|\lambda| - r_i^{\xi'}(A)) (|\lambda| - r_j^{\Delta j}(A)) \leq r_i^{\xi'}(A) r_j^{\xi}(A),
\]

which implies \(\lambda \in \bigcap_{j \in N, j \neq t} \hat{\Omega}_{t,j}(A) \subseteq \Omega(A)\) from the arbitrariness of \(j\). When \(|z| - r_j^{\Delta j}(A) < 0\), from the arbitrariness of \(j\), we have \(\lambda \in \bigcap_{j \in N, j \neq t} \hat{\Omega}_{t,j}(A) \subseteq \Omega(A)\).

Otherwise, \(|x_j| > 0\). From (1), we can get

\[
|\lambda| x_j \leq \sum_{j \in \{i_1, \ldots, i_m\}} a_{ji_2 \cdots i_m} x_{i_2} \cdots x_{i_m} + \sum_{j \notin \{i_2, \ldots, i_m\}} a_{ji_2 \cdots i_m} x_{i_2} \cdots x_{i_m} = r_j^{\Delta j}(A) x_j + r_j^{\xi}(A) x_l,
\]

i.e.,

\[
(|\lambda| - r_j^{\Delta j}(A)) |x_j| \leq r_j^{\xi}(A) |x_l|. \tag{3}
\]

By (2), it is not difficult to see \(\lambda \in K_t(A)\). When \(|\lambda| - r_i^{\xi'}(A) \geq 0\) or \(|\lambda| - r_j^{\Delta j}(A) \geq 0\) holds, multiplying (2) with (3) and noting that \(|x_l||x_j| > 0\), we have

\[
(|\lambda| - r_i^{\xi'}(A)) (|\lambda| - r_j^{\Delta j}(A)) \leq r_i^{\xi'}(A) r_j^{\xi}(A),
\]

which implies \(\lambda \in \bigcap_{j \in N, j \neq t} (\hat{\Omega}_{t,j}(A) \cap K_i(A)) \subseteq \Omega(A)\) from the arbitrariness of \(j\). And when \(|\lambda| - r_i^{\xi'}(A) < 0\) and \(|\lambda| - r_j^{\Delta j}(A) < 0\) hold, from the arbitrariness of \(j\), we have \(\lambda \in \bigcap_{j \in N, j \neq t} \hat{\Omega}_{i,j}(A) \subseteq \Omega(A)\). Hence, the conclusion \(\sigma(A) \subseteq \Omega(A)\) follows immediately from what we have proved.

Next, a comparison theorem is given for Theorems 1, 4.

**Theorem 5.** Let \(A = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}\). Then

\[
\Omega(A) \subseteq \Psi(A) \subseteq L(A) \subseteq K(A).
\]

**Proof.** From Theorem 5 in [5], we have \(\Psi(A) \subseteq L(A) \subseteq K(A)\). Hence, here only \(\Omega(A) \subseteq \Psi(A)\) is proved. Let \(z \in \Omega(A)\). Then \(z \in \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \hat{\Omega}_{i,j}(A)\) or \(z \in \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \left(\hat{\Omega}_{i,j}(A) \cap K_i(A)\right)\).

We next divide the proof into two cases.

Case I: If \(z \in \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \hat{\Omega}_{i,j}(A)\), then there is one index \(i \in N\) such that \(|z| < r_i^{\xi'}(A)\) and \(|z| < r_j^{\Delta j}(A), \forall j \in N, j \neq i\). Then, it is easy to see that

\[
(|z| - r_i^{\xi'}(A)) |z| \leq r_i^{\xi'}(A) R_j(A), \forall j \in N, j \neq i,
\]
which implies that \( z \in \bigcap_{j \in N, j \neq i} \Psi_{i,j}(A) \subseteq \Psi(A) \). This implies \( \Omega(A) \subseteq \Psi(A) \).

Case II: If \( z \in \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \left( \tilde{\Omega}_{i,j}(A) \cap K_i(A) \right) \), then there is one index \( i \in N \), for any \( j \in N, j \neq i \), such that

\[
|z| \leq R_i(A),
\]  

and

\[
(|z| - r_i^{\Delta_j}(A))(|z| - r_j^{\Delta_j}(A)) \leq r_i^{\Delta_j}(A)r_j^{\Delta_j}(A). \tag{5}
\]

(i) If \( r_i^{\Delta_j}(A)r_j^{\Delta_j}(A) = 0 \), then \(|z| \leq r_i^{\Delta_j}(A)\) or \(|z| \leq r_j^{\Delta_j}(A)\). When \(|z| \leq r_i^{\Delta_j}(A)\), we have

\[
(|z| - r_i^{\Delta_j}(A))|z| \leq 0 \leq r_i^{\Delta_j}(A)R_j(A),
\]

which implies that \( z \in \bigcap_{j \in N, j \neq i} \Psi_{i,j}(A) \subseteq \Psi(A) \) from the arbitrariness of \( j \). When \(|z| \leq r_j^{\Delta_j}(A)\), we have

\[
|z| \leq R_j(A). \tag{6}
\]

From (4), we can get

\[
|z| - r_i^{\Delta_j}(A) \leq r_i^{\Delta_j}(A). \tag{7}
\]

Multiplying (6) and (7), we have

\[
(|z| - r_i^{\Delta_j}(A))|z| \leq r_i^{\Delta_j}(A)R_j(A),
\]

which also implies that \( z \in \bigcap_{j \in N, j \neq i} \Psi_{i,j}(A) \subseteq \Psi(A) \), consequently, \( \Omega(A) \subseteq \Psi(A) \).

(ii) If \( r_i^{\Delta_j}(A)r_j^{\Delta_j}(A) > 0 \), then dividing both sides by \( r_i^{\Delta_j}(A)r_j^{\Delta_j}(A) \) in (5), we have

\[
\frac{|z| - r_i^{\Delta_j}(A)}{r_i^{\Delta_j}(A)} \frac{|z| - r_j^{\Delta_j}(A)}{r_j^{\Delta_j}(A)} \leq 1. \tag{9}
\]

From (4), we can get (7) and furthermore \( \frac{|z| - r_i^{\Delta_j}(A)}{r_i^{\Delta_j}(A)} \leq 1 \). When \( \frac{|z| - r_i^{\Delta_j}(A)}{r_i^{\Delta_j}(A)} \leq 1 \), then (6) holds.

Multiplying (6) and (7), we can get (8), which implies that \( z \in \bigcap_{j \in N, j \neq i} \Psi_{i,j}(A) \subseteq \Psi(A) \), consequently, \( \Omega(A) \subseteq \Psi(A) \).

And when \( \frac{|z| - r_j^{\Delta_j}(A)}{r_j^{\Delta_j}(A)} > 1 \), we can obtain \(|z| > R_j(A)\). Let \( a = |z|, b = r_j^{\Delta_j}(A) - |a_{jj\ldots}|, c = |a_{jj\ldots}| \) and \( d = r_j^{\Delta_j}(A) \). By Lemma 2.3 in [12], we have

\[
\frac{|z|}{R_j(A)} = \frac{a}{b + c + d} \leq \frac{a - (b + c)}{d} = \frac{|z| - r_j^{\Delta_j}(A)}{r_j^{\Delta_j}(A)}. \tag{10}
\]
If $|z| > r_i^{\Delta_j}(A)$, by (9) and (10), we have
\[
\frac{|z| - r_i^{\Delta_j}(A)}{r_i^{\Delta_j}(A)} \leq \frac{|z| - r_i^{\Delta_j}(A) |z| - r_j^{\Delta_j}(A)}{r_j^{\Delta_j}(A)} \leq 1,
\]
equivalently,
\[
(|z| - r_i^{\Delta_j}(A)) |z| \leq r_j^{\Delta_j}(A) r_j(A),
\]
which implies that $z \in \bigcap_{j \in N, j \neq i} \Psi_{i,j}(A) \subseteq \Psi(A)$ from the arbitrariness of $j$. If $|z| \leq r_i^{\Delta_j}(A)$, we have
\[
(|z| - r_i^{\Delta_j}(A)) |z| \leq 0 \leq r_i^{\Delta_j}(A) r_j(A).
\]
This also leads to $z \in \bigcap_{j \in N, j \neq i} \Psi_{i,j}(A) \subseteq \Psi(A)$, consequently, $\Omega(A) \subseteq \Psi(A)$. The conclusion follows from Case I and Case II.

**Remark 1.** Theorem 5 shows that the set $\Omega(A)$ in Theorem 4 is tighter than $K(A)$ in Theorem 1, $L(A)$ in Theorem 2 and $\Psi(A)$ in Theorem 3, that is, $\Omega(A)$ can capture all $Z$-eigenvalues of $A$ more precisely than $K(A)$, $L(A)$ and $\Psi(A)$.

Now, an example is given to verify the fact in Remark 2.

**Example 1.** Let $A = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$ be a symmetric tensor defined by
\[
a_{1111} = 1, \quad a_{1112} = 1, \quad a_{1122} = 0.25, \quad a_{2222} = 5, \quad \text{and } a_{ijkl} = 0 \text{ elsewhere}.
\]
By computation, we get that all the $Z$-eigenvalues of $A$ are $-0.2044, -0.2044, 5.0000$ and $5.0000$. By Theorem 4, we have
\[
K(A) = \{ z \in \mathbb{C} : |z| \leq 6.7500 \}.
\]
By Theorem 3, we have
\[
L(A) = \{ z \in \mathbb{C} : |z| \leq 6.4827 \}.
\]
By Theorem 2, we have
\[
\Psi(A) = \{ z \in \mathbb{C} : |z| \leq 6.3161 \}.
\]
By Theorem 4, we have
\[
\Omega(A) = \{ z \in \mathbb{C} : |z| \leq 5.0000 \}.
\]
The $Z$-eigenvalue inclusion sets $K(A)$, $L(A)$, $\Psi(A)$, $\Omega(A)$ and the exact $Z$-eigenvalues are drawn in Figure 1, where $K(A)$, $L(A)$, $\Psi(A)$ and $\Omega(A)$ are represented by black dashed boundary, green solid boundary, blue point line boundary and red solid boundary, respectively. The exact eigenvalues are plotted by black “+”. It is easy to see $\sigma(A) \subseteq \Omega(A) \subset \Psi(A) \subset L(A) \subset K(A)$, that is, $\Omega(A)$ can capture all $Z$-eigenvalues of $A$ more precisely than $\Psi(A)$, $L(A)$ and $K(A)$. 

6
3 A sharper upper bound for the $Z$-spectral radius of weakly symmetric nonnegative tensors

As the $Z$-spectral radius of weakly symmetric nonnegative tensors plays a fundamental role in the symmetric best rank-one approximation [10,16], recently, many people focus on bounding the $Z$-spectral radius of weakly symmetric nonnegative tensors. As an application of the set in Theorem 4, we in this section give a sharper upper bound for the $Z$-spectral radius of weakly symmetric nonnegative tensors.

**Theorem 6.** Let $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly symmetric nonnegative tensor. Then

$$\varrho(\mathcal{A}) \leq \Omega_{\max}(\mathcal{A}) = \max \{ \hat{\Omega}_{\max}(\mathcal{A}), \bar{\Omega}_{\max}(\mathcal{A}) \},$$

where

$$\hat{\Omega}_{\max}(\mathcal{A}) = \max_{i \in \mathbb{N}} \min_{j \in \mathbb{N}, j \neq i} \min \{ r_{i}^\Delta(A), r_{j}^\Delta(A) \},$$

$$\bar{\Omega}_{\max}(\mathcal{A}) = \max_{i \in \mathbb{N}} \min_{j \in \mathbb{N}, j \neq i} \min \{ R_i(A), \bar{\Omega}_{i,j}(A) \},$$

and

$$\bar{\Omega}_{i,j}(\mathcal{A}) = \frac{1}{2} \left\{ r_{i}^\Delta(A) + r_{j}^\Delta(A) + \sqrt{(r_{i}^\Delta(A) - r_{j}^\Delta(A))^2 + 4r_{i}^\Delta(A)r_{j}^\Delta(A)} \right\}.$$
Then we have $\Omega_{i,j}(A)$ or $\Omega_{i,j}(A) \subset \Omega_{i,j}(A) \cap K_i(A)$.

If $\varrho(A) \in \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \Omega_{i,j}(A)$, then there is one index $i \in N$ such that

$$\varrho(A) < r_i^{\overline{X}_j}(A) \text{ and } \varrho(A) < r_j^{\Delta_j}(A), \forall j \in N, j \neq i.$$  

Then we have $\varrho(A) \leq \min_{j \in N, j \neq i} \min \{ r_i^{\overline{X}_j}(A), r_j^{\Delta_j}(A) \}$. Furthermore, we have

$$\varrho(A) \leq \max_{i \in N} \min_{j \in N, j \neq i} \min \{ r_i^{\overline{X}_j}(A), r_j^{\Delta_j}(A) \}.$$  

If $\varrho(A) \in \bigcup_{i \in N} \bigcap_{j \in N, j \neq i} \left( \Omega_{i,j}(A) \cap K_i(A) \right)$, then there is one index $i \in N$, for any $j \in N, j \neq i$, such that

$$\varrho(A) \leq R_i(A)$$  

and

$$\left( \varrho(A) - r_i^{\overline{X}_j}(A) \right) \left( \varrho(A) - r_j^{\Delta_j}(A) \right) \leq r_i^{\Delta_j}(A) r_j^{\overline{X}_j}(A).$$

Solving $\varrho(A)$ in above inequality gives

$$\varrho(A) \leq \frac{1}{2} \left\{ r_i^{\overline{X}_j}(A) + r_j^{\Delta_j}(A) + \sqrt{\left( r_i^{\overline{X}_j}(A) - r_j^{\Delta_j}(A) \right)^2 + 4r_i^{\Delta_j}(A) r_j^{\overline{X}_j}(A)} \right\} = \Omega_{i,j}(A).$$

Combining (11) and (12), and by the arbitrariness of $j$, we have

$$\varrho(A) \leq \min_{j \in N, j \neq i} \min \{ R_i(A), \Omega_{i,j}(A) \} \leq \max_{i \in N} \min_{j \in N, j \neq i} \min \{ R_i(A), \Omega_{i,j}(A) \}.$$  

The conclusion follows from what we have proved. \hfill \Box

By Corollary 4.1 of [3], Theorem 6 of [5] and Theorem 5 of [6] and Corollary 4.5 of [7], the following comparison theorem can be derived easily.

**Theorem 7.** Let $A = (a_{i_1 \ldots i_n}) \in \mathbb{R}^{[m,n]}$ be a weakly symmetric nonnegative tensor. Then the upper bound in Theorem 6 is smaller than those in Theorem 5 of [5], Theorem 4.5 of [7] and Corollary 4.5 of [6], that is,

$$\Omega_{\max}(A) \leq \max_{i \in N} \min_{j \in N, j \neq i} \frac{1}{2} \left\{ r_i^{\overline{X}_j}(A) + \sqrt{\left( r_i^{\overline{X}_j}(A) \right)^2 + 4r_i^{\Delta_j}(A) R_j(A)} \right\}$$

$$\leq \max_{i \in N} \min_{j \in N, j \neq i} \frac{1}{2} \left\{ R_i(A) - a_{i_1 \ldots j} + \sqrt{(R_i(A) - a_{i_1 \ldots j})^2 + 4a_{i_1 \ldots j} R_j(A)} \right\}$$

$$\leq \max_{i \in N} R_i(A).$$
Finally, we show that the upper bound in Theorem 6 is smaller than those in [4–10] by the following example.

Example 2. Let \( A = (a_{ijk}) \in \mathbb{R}^{[3,3]} \) be a weakly symmetric nonnegative tensor with entries defined as follows:

\[
A(:, :, 1) = \begin{pmatrix} 3 & 3 & 0 \\ 3 & 2 & 2.5 \\ 0.5 & 2.5 & 0 \end{pmatrix}, \quad A(:, :, 2) = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 0 & 3 \\ 2.5 & 3 & 1 \end{pmatrix}, \quad A(:, :, 3) = \begin{pmatrix} 1 & 3 & 0 \\ 2.5 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

By Corollary 4.5 of [6] and Theorem 3.3 of [7], we both have \( \rho(A) \leq 19 \).

By Theorem 3.5 of [5], we have \( \rho(A) \leq 18.6788 \).

By Theorem 4.6 of [4], we have \( \rho(A) \leq 18.6603 \).

By Theorem 4.5 of [4] and Theorem 6 of [9], we both have \( \rho(A) \leq 18.5656 \).

By Theorem 4.7 of [4], we have \( \rho(A) \leq 18.3417 \).

By Theorem 2.9 of [10], we have \( \rho(A) \leq 17.2063 \).

By Theorem 5 of [5], we obtain \( \rho(A) \leq 15.2580, \)

By Theorem 6, we obtain \( \rho(A) \leq 14.9410. \)

This example shows that the bound in Theorem 6 is the smallest.

Remark 2. From Example 1, it is not difficult to see that the upper bound in Theorem 6 could reach the true value of \( \rho(A) \) in some cases.

4 Conclusion

In this paper, we present a new Z-eigenvalue localization set \( \Omega(A) \) and prove that this set is tighter than those in [4,5]. As an application, we obtain a new upper bound \( \Omega_{\text{max}}(A) \) for the Z-spectral radius of weakly symmetric nonnegative tensors, and show that this bound is sharper than those in [4–10] in some cases by a numerical example.

Acknowledgments

This work is supported by National Natural Science Foundations of China (Grant No.11501141), Foundation of Guizhou Science and Technology Department (Grant No.[2015]2073) and Natural Science Programs of Education Department of Guizhou Province (Grant No.[2016]066).
References

[1] L. Qi, Eigenvalues of a real supersymmetric tensor, *J. Symb. Comput.*, 40 (2005), pp. 1302-1324.

[2] K.C. Chang, K.J. Pearson, T. Zhang, Some variational principles for $Z$-eigenvalues of nonnegative tensors, *Linear Algebra Appl.*, 438 (2013), pp. 4166-4182.

[3] L.-H. Lim, Singular values and eigenvalues of tensors: a variational approach, in *Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP 05)*, 13-15 Dec. 2005, pp. 129-132. DOI: 10.1109/CAMAP.2005.1574201.

[4] G. Wang, G. Zhou, L. Caccetta, $Z$-eigenvalue inclusion theorems for tensors, *Discrete and Continuous Dynamical Systems Series B*, 22 (1) (2017), pp. 187-198.

[5] J. Zhao, A new $Z$-eigenvalue localization set for tensors, *J. Inequal. Appl.*, to appear, April 4 2017, 11 pages. DOI: 10.1186/s13660-017-1363-6.

[6] Y. Song, L. Qi, Spectral properties of positively homogeneous operators induced by higher order tensors, *SIAM J. Matrix Anal. Appl.*, 34 (2013), pp. 1581-1595.

[7] W. Li, D. Liu, S.-W. Vong, $Z$-eigenpair bounds for an irreducible nonnegative tensor, *Linear Algebra Appl.*, 483 (2015), pp. 182-199.

[8] J. He, Bounds for the largest eigenvalue of nonnegative tensors, *J. Comput. Anal. Appl.*, 20 (7) (2016), pp. 1290-1301.

[9] J. He, Y.-M. Liu, H. Ke, J.-K. Tian, X. Li, Bounds for the $Z$-spectral radius of nonnegative tensors, *Springerplus*, 5 (2016), Article no. 1727, 8 pages. DOI: 10.1186/s40064-016-3338-3.

[10] Q. Liu, Y. Li, Bounds for the $Z$-eigenpair of general nonnegative tensors, *Open Math*, 14 (1) (2016), pp. 181-194.

[11] J. He, T.-Z. Huang, Upper bound for the largest $Z$-eigenvalue of positive tensors, *Appl. Math. Lett.*, 38 (2014), pp. 110-114.

[12] C. Li, Y. Li, An eigenvalue localization set for tensor with applications to determine the positive (semi-)definiteness of tensors, *Linear Multilinear Algebra*, 64 (4) (2016), pp. 587-601.

[13] C. Li, Y. Li, X. Kong, New eigenvalue inclusion sets for tensors, *Numer. Linear Algebra Appl.*, 21 (2014), pp. 39-50.

[14] C. Li, J. Zhou, Y. Li, A new Brauer-type eigenvalue localization set for tensors, *Linear Multilinear Algebra*, 64 (4) (2016), pp. 727-736.

[15] C. Li, Z. Chen, Y. Li, A new eigenvalue inclusion set for tensors and its applications, *Linear Algebra Appl.*, 481 (2015), pp. 36-53.

[16] E. Kofidis, P. A. Regalia, On the best rank-1 approximation of higher-order supersymmetric tensors, *SIAM J. Matrix Anal. Appl.*, 23 (2002), pp. 863-884.

[17] X.-M. Gu, T.-Z. Huang, X.-L. Zhao, H.-B. Li, L. Li, Strang-type preconditioners for solving fractional diffusion equations by boundary value methods, *J. Comput. Appl. Math.*, 277 (2015), pp. 73-86.