Moment maps, monodromy and mirror manifolds

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Abstract
Via considerations of symplectic reduction, monodromy, mirror symmetry and Chern-Simons functionals, a conjecture is proposed on the existence of special Lagrangians in the hamiltonian deformation class of a given Lagrangian submanifold of a Calabi-Yau manifold. It involves a stability condition for graded Lagrangians, and can be proved for the simple case of $T^2$.

1 Introduction

Just as explicit solutions of the Einstein and Hermitian-Yang-Mills equations exist only on spaces that are either low dimensional, non-compact and/or highly symmetric, so the equations for special Lagrangian (SLag) cycles, also important in physics, have the same properties. Physically there are also similarities in that we have two first order supersymmetric minimal energy equations (HYMs and SLag) implying the more standard second order equations (YMs and minimal volume equations).

There are powerful existence results of Calabi and Yau (and more recently Tian, Donaldson and others) for the Einstein equations, and of Donaldson-Uhlenbeck-Yau for the HYM equations, so long as we are on a Kähler (or projective) manifold; this often reduces an infinite dimensional problem in PDEs to a finite dimensional problem in linear algebra. Producing many Kähler-Einstein (e.g. Calabi-Yau) manifolds becomes trivial, and dealing with Hermitian-Yang-Mills connections requires only algebraic computations; in both cases the complicated role of the Kähler form and/or metric is almost removed. This can be thought of as possible because of the existence of some infinite dimensional geometry recasting the equations in terms of moment maps and symplectic reduction. A similar situation for SLags would therefore be highly desirable. In particular it might give a way of studying SLags using only Lagrangians and symplectic geometry, much as HYM connections are studied via stable bundles and algebraic geometry.

This paper explores the mirror symmetry of holomorphic bundles (on a Calabi-Yau 3-fold $M$, often referred to here as ‘the complex side’) and Lagrangians (on the
mirror Calabi-Yau 3-fold $W$, 'the symplectic side', known as the \( \text{Kähler side in the physics literature} \). Many people have worked and are still working on proving some kind of direct correspondence between such objects given an SYZ torus fibration \[\text{SYZ}\]; see for example \[\text{AP}, \text{BBHM}, \text{Ch}, \text{Fu1}, \text{Gr}, \text{LYZ}, \text{PZ}, \text{Ty}\], and see \[\text{MMM}\] for a review of this and many many more issues in mirror symmetry. Here, however, we work purely formally without reference to a particular pair of mirror manifolds, without worrying about what mirror symmetry might rigorously mean, and we will not try to give any explicit correspondence. Using mirror symmetry merely as motivation, we point out some similar structures on both sides of the mirror map. Under some conditions (in some ‘large complex structure’ or ‘semiclassical’ or somesuch limit) these structures might be genuinely dual; again it does not matter if they are not in general. For instance, physics \[\text{MMMS}, \text{DFR}\] predicts that one should consider not the HYM equations and slope but some perturbation of them away from the large complex structure limit; however these equations also come from a moment map and, conjecturally, a stability condition (for a discussion of such matters see \[\text{Le} \) or \[\text{T3}\]). So while the slope and phase of Lagrangians discussed below might not be exactly mirror to slope of bundles, it should be mirror to something with analogous properties and significance.

Loosely, we would like to think of submanifolds in a fixed homology class as mirror to connections on a fixed topological complex bundle (with Chern classes mirror to the homology class); then Lagrangians should correspond to holomorphic connections (i.e. integrable connections; those with no \((0,2)\)-curvature) and special Lagrangians to those with HYM curvature. These last two conditions should be stability conditions for the group actions of hamiltonian deformations and complex gauge transformations, respectively. The full picture is much more complicated, involving triangulated categories and so forth, as envisaged some six years ago in the seminal conjecture of Kontsevich \[\text{K}\]; we can ignore this in only using mirror symmetry as motivation. It could be noted, however, that the functionals defined below are additive under exact sequences of holomorphic vector bundles and sums of Lagrangians, so should extend to the derived category of coherent sheaves and the derived Fukaya category of Lagrangians respectively.

First note that while the connections side has a complex structure and a complex gauge group involved, the Lagrangian side needs complexifying. So motivated by Kontsevich \[\text{K}\] and by physics (e.g. \[\text{SYZ}\)] we add in connections on the submanifolds (which will later reduce to flat connections on Lagrangians). The dictionary we are aiming towards, much of which is already standard, is the following in the 3-dimensional case; all the terms used will be defined in due course.
| Complex side $M$ | Symplectic side $W$ |
|-----------------|----------------------|
| $\Omega = \Omega_M \in H^{3,0}$ | $\omega = \omega_W \in H^{1,1}$ |
| $H^{ev}$ | $H^3$ |
| Connections $A$ on a fixed $C^\infty$ complex bundle $E$; $v(E) = ch(E)\sqrt{Td X} \in H^{ev}$ | Submanifolds/cycles $L$ in a fixed class $[L] \in H^3$, with a connection on $\mathbb{C} \times L$ |
| $CS_{\mathbb{C}} \left( A = A_0 + a \right) = \frac{1}{4\pi^2} \int_M tr \left( \partial A_0 a \wedge a + \frac{3}{2} a \wedge \Omega \right)$ | $f_{\mathbb{C}} (A, L) = \int_{L_0}^L (F + \omega)^2$ |
| Critical points: $F_A^{0,2} = 0$ holomorphic bundles | Critical points: $\omega|_L = 0$, $F_A = 0$ Lagrangians + flat line bundles |
| Holomorphic Casson invariant | Counting SLags |
| Gauge group | $U(1)$ gauge group on $L$ |
| Complexified gauge group | Hamiltonian deformations |
| $\omega = \omega_M \in H^{1,1}$ | $\Omega = \Omega_W \in H^{3,0}$ |
| Moment map $F_A \wedge \omega^{n-1}$ | Moment map $\text{Im} \Omega|_L$ |
| Stability, slope $\mu = \frac{1}{rk E} \int tr F_A \wedge \omega^{n-1}$ | Stability, slope $\mu = \frac{1}{\text{vol}(L)} \int_L \text{Im} \Omega$ |

(In the fourth line, $v(E)$ is the Mukai vector of $E$; in the last line, $\text{vol}(L)$ is the cohomological volume measured with respect to $\text{Re} \Omega$. A SLag cycle (of phase $\phi$) in a Calabi-Yau is a Lagrangian with $\text{Im} \left( e^{-i\phi} \Omega \right)|_L \equiv 0$; then $\text{Re} \left( e^{-i\phi} \Omega \right)|_L$ is the Riemannian volume form on $L$ induced by the Ricci-flat metric on the Calabi-Yau. Obviously, rotating $\Omega$ by $e^{-i\phi}$ gives SLags in the more traditional sense of phase zero. The part of the theory to do with SLags will apply in all dimensions; it is only the functionals that are special to Calabi-Yau 3-folds.

We will partially justify the above table, though the symplectic structure and moment map give problems that will appear in due course. However we can derive enough to arrive at a conjecture about Lagrangians and SLags for which evidence
will be given by using monodromy and mirror symmetry from [ST] to interpret an example of Lawlor and Joyce [J].

Acknowledgements. The debt of any ex-student of Simon Donaldson writing a paper on moment maps should be clear. This work is also more immediately influenced by the papers [D], [J], [K]. In particular I was surprised to see the Lagrangian condition coming from a moment map in [D], [H], which does not fit into the scheme I always supposed was true. So the purpose of this paper, apart from trying to set a record for the number of m’s in a title, is to expand on that scheme and to try to get the special condition from a moment map instead. This paper was finished in the summer of 2000 and reported on in [T3]; since then exciting new ideas have appeared in physics [Do] and mathematics [KS] better explaining mirror symmetry. I would like to thank S.-T. Yau, C. H. Taubes and Harvard University for support, and Yi Hu, Albrecht Klemm, Ivan Smith and Xiao Wei Wang for useful conversations. Communications with Mike Douglas, Dominic Joyce, Paul Seidel, S.-T. Yau and Eric Zaslow have been extremely influential.

2 Chern-Simons-type functionals and critical points

Consider the space $\mathcal{A}$ of $(0,1)$-connections $A$ on a fixed complex bundle on a Calabi-Yau 3-fold $M$. This infinite dimensional space has a natural complex structure, with respect to which it admits a holomorphic functional, Witten’s holomorphic Chern-Simons functional [W1], [DT],

$$CS_{\mathcal{C}}(A = A_0 + a) = \frac{1}{4\pi^2} \int_M \text{tr} \left( \bar{\partial}_{A_0} a \wedge a + \frac{2}{3} a \wedge a \wedge a \right) \wedge \Omega,$$

where $\Omega$ is the holomorphic $(3,0)$-form. It is infinitesimally gauge-invariant (gauge transformations not homotopic to the identity can give periods to $CS_{\mathcal{C}}$) and its gradient is $F_A^{\partial A}$, with zeros the integrable connections. That is, after dividing by gauge equivalence (under which grad $CS_{\mathcal{C}}$ is invariant), the critical points of $CS_{\mathcal{C}}$ form the space of holomorphic bundles of the same topological type. As critical points of a functional, moduli of holomorphic bundles have virtual dimension zero, and one might try to make sense of counting them – a holomorphic Casson invariant [T1]. This is independent of deformations of the complex structure, but can have wall-crossing changes as the Kähler form varies. (This is because we count only stable bundles, and the notion of stability depends on a Kähler form.)

On the other hand, on a different Calabi-Yau 3-fold $W$ (for instance the mirror, in some situation where this makes sense), Lagrangians are the critical points of a
functional too, on the space of all 3-dimensional submanifolds (or cycles):

\[ f_{\mathbb{R}}(L) = \int_{L_0}^{L} \omega \wedge \omega, \]

where \( \omega \) is the symplectic form on \( W \). Here \( L_0 \) is a fixed cycle in the same homology class, and we integrate over a 4-chain with boundary \( L - L_0 \); the functional \( f_{\mathbb{R}} \) is invariant under the choice of different, homologous, 4-chains (picking non-homologous 4-chains can give periods to \( f_{\mathbb{R}} \)). It is invariant under deformations of \( L \) pulled back from Hamiltonian deformations of \( W \) (deformations generated by vector fields \( v \) on \( W \) whose contraction with \( \omega \) is exact \( v \cdot \omega = dh \) at each point in time) as \( \int_{L} \omega \wedge dh = 0 \), and its gradient is \( \omega|_{L} \). Thus its critical points are Lagrangian submanifolds. We would like to think of \( f_{\mathbb{R}} \) as mirror to \( CS_\mathbb{C} \), but to do so we must complexify it.

Thus we work on the space \( A \) of submanifolds \( L \) of \( W \) with \( U(1) \) connections \( A \) on the trivial bundle \( \mathbb{C} \times L \) on \( L \). Notice these submanifolds are not parameterised by a map of a real 3-manifold into \( W \); we are only interested in the image \( L \). From now on we shall restrict attention to smooth Lagrangian submanifolds. Formally, we consider the tangent space to \( A \) at a point \((A, L \subset W)\) to be

\[ \Omega^1(L; \mathbb{R}) \oplus \Omega^1(L; \mathbb{R}), \quad (2.1) \]

at least for those \( L \) with no \( J \)-invariant subspaces of its tangent spaces (\( J \) is the complex structure on \( W \), and this is reasonable since we are looking for Lagrangian submanifolds after all). The first factor is the obvious tangent space to the connections on \( L \); the second gives deformations of \( L \) via the vector fields produced by contracting with the Kähler form \( \omega \) on \( W \). That is, we use the metric on \( W \) to map \( \Omega^1(L) \) to \( \Omega^1(W)|_{L} \), then use the isomorphism provided by \( \omega \) to get a vector field along \( L \). Equivalently, using the metric on \( W \), we may think of one-forms on \( L \) as tangent vectors to \( L \), then apply the complex structure \( J \) on \( W \) to give \( W \)-vector fields on \( L \). We denote this map from one-forms to normal vector fields by

\[ \Omega^1(L) \to TW|_{L}, \quad \sigma \mapsto \sigma \cdot \omega^{-1}. \quad (2.2) \]

Connections on \( L \) are carried along by the vector field to connections on nearby cycles, and we are identifying the space of \( U(1) \) connections with \( i \Omega^1(L; \mathbb{R}) \).

There is a natural almost complex structure on \( A \), acting as

\[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]
with respect to the splitting $[2,1]$ of the tangent spaces. With respect to this we claim to have the following holomorphic functional

$$f_C(A, L) = \int_{L_0}^L (F + \omega)^2 = \int_{L_0}^L (F^2 + \omega^2) + 2 \int_{L_0}^L \omega \wedge F.$$ 

Here we have extended $A$ to a connection on the trivial bundle on the whole of $W$ (restricting to a fixed connection $A_0$ on $L_0$, and to $A$ on $L$) and taken its curvature form $F$. We have again picked a 4-cycle bounding $L - L_0$; because $F$ and $\omega$ are closed the resulting functional is independent of different homologous choices of the 4-cycle, and in general well defined up to the addition of some discrete periods. It is also (again) independent of hamiltonian isotopies of $L$. Notice that the $\int_{L_0}^L F^2$ term is just the real Chern-Simons functional $CS_R$ of the connection $A$ on $L$, whose critical points are well known to be flat connections. As pointed out to me by Eric Zaslow, the real and complex Chern-Simons functionals already appear in [V1] and [V4] as possible mirror partners (this is partially justified in [LYZ]), but without the terms in the symplectic form (and including instanton corrections from holomorphic discs which we are ignoring for our rough analogy). Asking for a real function to be equal to a complex one is possible when one restricts attention to a real slice such as the space of Lagrangian submanifolds in $A$; deforming within this space the imaginary part of $f_C$ remains constant and it reduces to $CS_R$. But allowing the imaginary counterparts to these real deformations the right functional to consider is $f_C$. Notice also that if $\omega/2\pi$ is integral, so that we can pick a connection $B$ with curvature $-i\omega$, then the action functional can be written in the more familiar looking Chern-Simons form

$$f_C(A, L) = \int_L (B + iA) \wedge d(B + iA) = \int_L C dC$$

for the ‘complexified connection’ $C = B + iA$ (a $\mathbb{C}^\times$-connection, instead of a $U(1)$-connection.) This makes more contact with the physics literature and allows one to extend the identification of $CS_R$ and $CS_C$ in [LYZ] to non Lagrangian sections, giving complex values. Tian has informed me that he and Chen have also considered the functional $f_C(A, L)$ [Ch].

Mirror symmetry should relate Lagrangians not just to bundles but the whole derived category. For Riemann surfaces $C \subset M$, for instance, there is a functional in [DT], [W2] rather like $f_R$ above:

$$\int_{C_0}^C \Omega$$

is formally holomorphic and has as critical points the holomorphic curves $C$. Similarly for four-manifolds $S \subset M$ with connections on them the following functional
formally similar to $f_C$)

$$
\int_{S_0} S \, \mathrm{tr} \, F \wedge \Omega
$$

has critical points the holomorphic surfaces with flat connection on them. Alternatively, as $CS_{\Sigma}$ is additive under extensions of bundles it does extend to the derived category. (Whether these two approaches are compatible; i.e. whether or not the functional associated to a curve or surface is the same as $CS_{\Sigma}$ applied to a locally free resolution of its structure sheaf, up to a constant, seems to not have been worked out.)

That $f_C$ is holomorphic follows from the computation that the derivative of $f_C$ down $a \in \Omega^1(L) \oplus 0$ (that only changes the connection $A \mapsto A + \delta a$) is $\int_L 2F \wedge ia + 2\omega \wedge ia$, while the derivative down $-Ja \in 0 \oplus \Omega^1(L)$, i.e. down the vector field $a \omega^{-1}$, is $\int_L 2a \wedge a + 2a \wedge F$. The second expression is $-i$ times the first, so the derivative is complex linear and $f$ is holomorphic. Equivalently we are saying that $df_C$ is the 2-form

$$
2i(F + \omega) \oplus 2(F + \omega),
$$

which pairs with the tangent space (2.1) by integration over $L$ to give a form of type $(1,0)$ on (2.1).

Thus critical points of the functional are Lagrangian cycles with flat line bundles on them: exactly the basic building blocks of the objects proposed in [K] to be mirror dual to the holomorphic bundles that are the critical points of $CS$. So this ties in three well known moduli problems of virtual dimension zero (i.e. with deformation theories whose Euler characteristic vanishes) – flat bundles on 3-manifolds, holomorphic bundles on Calabi-Yau 3-folds, and Lagrangians (up to hamiltonian deformation) in symplectic 6-manifolds.

So as mirror to [T1] one would like to count Lagrangians (up to hamiltonian deformations) plus flat line bundles on them, and this is what Joyce’s work [J] has begun to tackle (in the rigid case of $L$ being a homology sphere). Mirroring precisely the behaviour of the holomorphic Casson invariant this count appears to be independent of deformations of the Kähler form and to have wall-crossing changes as the complex structure varies.

### 3 Gauge equivalence and moment maps

In fact what Joyce is proposing to count is \emph{special} Lagrangian spheres with flat line bundles on them (hence the otherwise anomalous dependence on the complex structure), while [T1] counts \emph{stable} bundles (i.e. by Donaldson-Uhlenbeck-Yau, modulo the technicalities of polystable and non-locally-free sheaves, we count Hermitian-Yang-Mills connections; hence the dependence on the Kähler form). (Tyurin [Ty]...
was perhaps the first to suggest that the holomorphic Casson invariant should be related by mirror symmetry to the real Casson invariant (here the $U(1)$ Casson invariant) of SLag submanifolds.

The link should be, of course, that we want to consider holomorphic connections on one side, up to complex gauge equivalence, and Lagrangians on the other side, up to hamiltonian isotopy, and in both cases we try to do this by picking distinguished representatives of equivalence classes by the usual method of symplectic reduction. Bringing in a Kähler structure on the complex side, we get a moment map for the gauge group action, whose zeros give the HYM equations. Dually, we would like to bring in the holomorphic 3-form on the symplectic (Kähler) side, and get a complex group to act. So again complexify by adding flat line bundles: consider the critical points of the functional $f$ of the last section, i.e. the space

$$
\mathcal{Z} = \{(L, A) : L \subset W \text{ is Lagrangian, } A \text{ is a flat connection on } L\}
$$

(not up to gauge equivalence). In fact consider this space on a Calabi-Yau manifold $W$ of any dimension $n$. It has tangent space

$$
T_{(L,A)} \mathcal{Z} = Z^1(L) \oplus Z^1(L)
$$

($Z^1(L)$ denotes closed real one-forms on $L$), the first being tangent to the space of flat connections, the second giving normal vector fields (by contracting with $\omega^{-1}$) preserving the Lagrangian condition. We have an obvious almost complex structure

$$
J = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
$$

Then the real group $C^\infty(L; \mathbb{R})/\mathbb{R}$ acts as the Lie algebra to the group of gauge transformations on the flat line bundles (taking $d$ and adding to the connection) whose complexification $C^\infty(L; \mathbb{C})/\mathbb{C}$ acts complex linearly: the imaginary part $C^\infty(L; \mathbb{R})/\mathbb{R}$ acts by hamiltonian deformations through the normal vector field

$$
h \mapsto dh \omega^{-1}.
$$

Unfortunately, without using a metric this vector field is only defined up to the addition of tangent vector fields to $L$; the map \([2.2]\) is really a map to $(TW|_L)/TL$ which we have lifted to $TW|_L$ using the metric. (Equivalently we can extend $h$ to a first formal neighbourhood of $L$ in different ways to get a different vector field.) How we pick this alters how we carry the flat connection along with $L$, and how the almost complex structure \([3.1]\) acts. For instance suppose we are in the rather artificial case of $L$ being transverse to an SYZ $T^n$-fibration. Then we can carry $L$ and the flat connection up the fibres and identify the functions $C^\infty(L)$ from Lagrangian
to Lagrangian using the projection. Thus the group remains constant as $L$ moves (effectively what we are doing is extending functions from $L$ to a neighbourhood of $L$ in $W$ by pulling up along the SYZ fibres). This does not work when $L$ branches over the base of such a fibration. One can instead use the metric to define normal vector fields, but then identifying the Lie algebra $C^\infty(L)$ with a fixed $C^\infty(L_0)$ for all $L$ becomes difficult.

This problem is perhaps not so surprising – the moment the Lagrangian has branching over the base of an SYZ fibration simple explicit correspondences between Lagrangians and vector bundles (such as \cite{LYZ}) also break down due to our ignoring important holomorphic disc instanton corrections that appear in the physics. For instance recent work of Fukaya, Oh, Ohta and Ono \cite{FO2}, surveyed in \cite{Fu}, show these provide the obstructions mirror to those of deformations of holomorphic bundles \cite{FO} – one should not in general consider all (S)Lags (which are unobstructed) as mirror to holomorphic bundles, but only those whose Floer cohomology (whose definition involves holomorphic discs) is well defined.

However, what is clear is the totality of the group action, even if identifying individual elements causes problems, and this is all we really need. For instance in the $K3$ (or $T^4$) case one can get the same total group orbit, with a genuine fixed group acting, by hyperkähler rotating a construction due to Donaldson \cite{D}. The end result is that one considers parametrised Lagrangian embeddings $f$ from a Riemann surface $L$ into the $K3$ such that the pullback of Re $\Omega$ is a fixed symplectic form on $L$. Then the group of exact symplectomorphisms of $(L, f^* \text{Re } \Omega)$

provide a symmetry group of the space of maps $f$, which also carries a natural Kähler structure. Complexified orbits give hamiltonian deformations, and the moment map is $m(f) = f^* \text{Im } \Omega$. The connection with our construction is that after fixing a line bundle $\eta$ and connection with curvature $f^* \text{Re } \Omega$, an infinitesimal symplectomorphism $\phi$ induces a flat connection, via parallel transport and pull back, on the bundle $\eta \otimes \phi^* \eta^\ast$. Globally the action is different (this action has non-zero Lie bracket, for instance, and a fixed group) but the total group orbit and the moment map (see below) are the same.

In general it is clear that the problem of identifying the group for different embeddings of $L$ should be resolved by working with the space of maps from a fixed $L_0$ to $W$, and enlarging the group by including diffeomorphisms of $L_0$, giving a semi-direct product of Diff($L_0$) and $U(1)$ gauge transformations on $L_0$. Then the moment map for the diffeomorphism part of the total group would be the Lagrangian condition as in \cite{D}, and the problems we are encountering would come from the fact that the group is a semi-direct product and not a product, so that we cannot separate
the two out and divide by them separately, as in effect we have been trying to do. Unfortunately, I have not found the correct formulation of the problem, but it is not so important for follows.

So we shall not worry too much about whether the complex structure defined above is integrable, the group is fixed, or the symplectic structure below is closed. In 1 complex dimension it is trivial, in 2 we can use Donaldson’s picture, and in 3 dimensions we could either try to use an abstract SYZ fibration to deform and identify Lagrangians transverse to it, or take everything in this section as motivation for finding the stability condition for Lagrangians of the next section.

Fix a homology class of Lagrangians and multiply Ω by a unit norm complex number so that
\[ \int_L \Im \Omega = 0. \]
We induce a symplectic structure on \( Z \) from \( J \) and the following metric on the tangent space
\[ \langle a, b \rangle = \int_L a \wedge ((b \downarrow \omega^{-1}) \downarrow \Im \Omega), \]
for \( a, b \) closed 1-forms. A computation in local coordinates shows this is symmetric in \( a \) and \( b \); in fact it can be written as
\[ \int_L a \wedge (\bar{b} \downarrow \Re \Omega) = \int_L \cos \theta (a \wedge \ast b), \quad (3.2) \]
where \( \sim \) is the isomorphism \( T^* L \rightarrow TL \) set up by the induced metric on \( L, \Omega|_L = e^{i\theta} \vol_L \), and \( \vol_L \) the Riemannian volume form on \( L \) induced by the Ricci-flat metric. Thus for Lagrangians with \( \theta \in (-\pi/2, \pi/2) \), i.e. those for which \( \Re \Omega \) restricts to a nowhere vanishing volume form on \( L \) and so are not too far from being \( \text{SLag} (\theta \equiv 0) \), this gives a non-degenerate metric.

The symplectic form is invariant under the group action, and formally the moment map is indeed \( m(L, A) = \Im \Omega \) in the dual \( \Omega^n(L)_0 \) of the Lie algebra (i.e. \( n \)-forms on \( L \) with integral zero). This follows from the computation
\[ X \int_L h \Im \Omega = \int_L h d (X \downarrow \Im \Omega) = \int_L dh \wedge ((\sigma \downarrow \omega^{-1}) \downarrow \Im \Omega), \]
where \( X = \sigma \downarrow \omega^{-1} \) is a normal vector field to the Lagrangian \( L \) down which we compute the derivative of the Hamiltonian \( \int_L h \Im \Omega = \langle m(L, A), h \rangle \) for the infinitesimal action of \( h \). Here have extended \( h \) to a first-order neighbourhood of \( L \subset W \) so that it is constant in the direction of \( X = \sigma \downarrow \omega^{-1} \). Then the right hand side of the above equation is the pairing using the symplectic form of \( dh \) and \( \sigma \), as required.

Infinitesimally we can see the moment map interpretation very easily, and fitting naturally with the mirror bundle point of view. Deformations of holomorphic connections \( A \) modulo complex gauge equivalence are given by a ker \( \partial_A / \text{im} \partial_A \)
first cohomology group, related to deformations \( \ker \bar{\partial}_A \cap \ker \bar{\partial}_A^* \) of the HYM equations (modulo unitary gauge transformations) via Hodge theory, with the moment map equation providing the \( d^* = 0 \) slice to the imaginary part of the linearised group action. Similarly, deformations of Lagrangians are given by closed 1-forms \( \ker d : \Omega^1(L; \mathbb{R}) \to \Omega^2(L; \mathbb{R}) \), so that dividing by Hamiltonian deformations we get

\[
H^1(L) = \ker d / \text{im} d.
\]

If instead of dividing we impose the special condition, we get a \( \ker d^* \) slice

\[
H^1(L) = \ker d \cap \ker d^*.
\]

to the (imaginary) deformations (real deformations are given by changing the flat \( U(1) \) connection that can be incorporated into this).

**A symplectic example**

To motivate a guess at the correct definition of stability for Lagrangians, we expand on an example of Lawlor and Joyce ([J] Sections 6 and 7, building on work of [Ha], [L]; see also a similar example in [SV] that is studied in [TY]), explaining its relevance to mirror symmetry, and giving a simple example in algebraic geometry that mirrors it.

First define the pointwise phase \( \theta \) of a submanifold \( L \): we may write

\[
\Omega|_L = e^{i\theta} \text{vol}
\]

where vol is the Riemannian volume form on \( L \) induced by Yau’s Ricci-flat metric \([Y]\) on \( W \). Thus vol provides a (local) orientation for \( L \), and reversing its sign alters the phase \( \theta \) by \( \pi \). A SLag is a Lagrangian with constant phase \( \theta \).

At first sight \( \theta \) is multiply-valued; we always choose it to be a fixed single-valued function to \( \mathbb{R} \), lifting \( e^{i\theta} : L \to S^1 \) and thus providing the Lagrangian with a grading as introduced by Kontsevich \([K],[S2]\). Thus we only consider Lagrangians of vanishing Maslov class – for a Calabi-Yau this is the winding class \( \pi_1(L) \to \pi_1(S^1) \) of the phase map

\[
L \xrightarrow{e^{i\theta}} S^1,
\]

which of course vanishes for a SLag. (The definition of grading in \([K],[S2]\) is topological and uses the universal \( \mathbb{Z} \)-cover of the bundle of Lagrangian Grassmannians; here we first pass to the \( \mathbb{Z}/2 \) orientation cover of the Grassmannian, choosing an orientation of our Lagrangians, and then use a complex structure to pass to the universal \( \mathbb{Z} \)-cover of this. The two definitions are of course equivalent.)
Similarly we can define a kind of average phase $\phi = \phi(L)$ of a submanifold (or homology class) $L \subset W$ by
\[ \int_L \Omega = A e^{i\phi(L)}, \]
for some real number $A$; we then use $\text{Re}(e^{-i\phi(L)}\Omega|_L)$ to orient $L$. Reversing the sign of $A$ alters the phase by $\pi$ and reverses the orientation. Again for a graded Lagrangian $L = (L, \theta)$, and we will always implicitly assume a grading, $\phi(L)$ is canonically a real number (rather than $S^1$-valued). Shifting the grading $[2n] : \theta \mapsto \theta + 2n\pi$ gives a similar shift to the phase $\phi(L)$.

The terminology comes from the fact that if there is a submanifold in the same homology class as $L$ that is SLag with respect to some rotation of $\Omega$, then it is with respect to $e^{-i\phi(L)}\Omega$. Slope, which we define as
\[ \mu(L) := \tan(\phi(L)) = \frac{1}{\int_L \text{Re} \Omega} \int_L \text{Im} \Omega, \]
is defined independently of grading, is monotonic in $\phi$ in the range $(-\pi/2, \pi/2)$, and is invariant under change of orientation $\phi \mapsto \phi \pm \pi$. This agrees with the slope of a straight line SLag in the case of $T^2$, as featured in [PZ], and we think of it as mirror to the slope of a mirror sheaf, as is shown for tori in [PZ] (see [DFR] for corrections in higher dimensions away from the large complex structure limit).

Joyce describes examples of SLags which we interpret as follows. We have a family of Calabi-Yau 3-folds $W^t$ as $t$ ranges through (a small open subset of) the moduli space of complex structures on $W$ with fixed symplectic structure. That is, the holomorphic 3-form $\Omega^t$ varies with $t$, but the Kähler form $\omega$ is fixed. We also have a family of SLag homology 3-spheres $L^t_1$, $L^t_2 \subset W^t$ such that $L^t_1$ and $L^t_2$ intersect at a point. If we choose a rotation of $\Omega^t$ such that $L^t_2$ always has phase $\phi^t_2 \equiv 0$ (this is possible locally at least; in the family described later it will have to be modified slightly), then we are interested as $t$ varies only in the complex number
\[ \int_{L^t_1} \Omega^t = R^t e^{i\phi^t_1} \]
and its polar phase $\phi = \phi^t_1$; we plot this (i.e. the projection from the complex structure moduli space to $\mathbb{C}$ via this map) in Figure 1.

Then in Joyce’s example, for $\phi < 0$ (and $R^t > 0$) there is a SLag $L^t$ (of some phase $\phi$) in the homology class $[L^t] = [L^t_1] + [L^t_2]$, such that as $\phi \uparrow 0$, this degenerates to a singular union of SLags of the same phase $L^t = L^t_1 \cup L^t_2$ and then disappears for $\phi > 0$.

Most importantly, where $L^t$ exists as a smooth SLag ($\phi < 0$) we have the slope (and phase) inequality
\[ \mu^t_1 < \mu^t_2, \quad \text{i.e. } \phi^t_1 < \phi^t_2 \equiv 0; \] (3.3)
at $t = 0$, $L^t$ becomes the singular union of $L^t_1$ and $L^t_2$, with

$$\mu^t_1 = \mu^t_2 \ (\phi^t_1 = \phi^t_2);$$

then there is no SLag in $L$’s homology class for

$$\mu^t_1 > \mu^t_2 \ (\phi^t_1 > \phi^t_2),$$

though there is a Lagrangian, of course – the symplectic structure has not changed. Though we have been using slope $\mu$ in order to strengthen the analogy with the mirror (bundle) situation, from now on we shall use only the phase (lifted to $\mathbb{R}$ using the grading). While each is monotonic in the other for small phase (as $\tan \phi = \mu$), slope does not see orientation as phase does; reversing orientation adds $\pm \pi$ to the phase but leaves $\mu$ unchanged. This is related to the fact that we should really be working with complexes and so forth on the mirror side (the bundle analogy is too narrow) and changing orientation has no mirror analogue in terms of only stable bundles; it corresponds to shifting (complexes of) bundles by one place in the derived category. While slopes of bundles cannot go past infinity (without moving degree in the derived category at least), for Lagrangians they certainly can, and phase $\phi$ continues monotonically upwards as its slope $\tan \phi$ becomes singular and then negative.

Importantly, we can think of the various SLags as independent of time when thought of as Lagrangians in the fixed symplectic manifold $W^t$:

**Lemma 3.4** For $t > 0$ the SLags $L^t$ are all in the same hamiltonian deformation class. Similarly for $L^t_1$, $L^t_2$, and for $t < 0$.

**Proof** Now choosing the phase of $\Omega^t$ such that $\phi(L^t) \equiv 0$,

$$\int_L \frac{d}{dt} (\text{Im } \Omega^t) = \int_L \text{Im } \dot{\Omega}^t = 0. \quad (3.5)$$

To show this deformation preserves the hamiltonian class of $L$, we need to find a corresponding first order hamiltonian deformation $d\hbar \omega \omega^{-1}$ under which the change in $\text{Im } \Omega^t$,

$$L_{d\hbar \omega \omega^{-1}} \left( \text{Im } \Omega^t \right) |_L = d((d\hbar \omega^{-1}) \cdot \text{Im } \Omega^t) |_L,$$

is $-\text{Im } \dot{\Omega}^t |_L$. But as $\text{Re } \Omega^t |_L$ is the induced Riemannian volume form $\text{vol}^t$ on $L$, this means we want to solve

$$-\text{Im } \dot{\Omega}^t |_L = d(J(d\hbar \omega) \cdot \text{Re } \Omega^t |_L) = d(d\hbar \cdot \text{vol}^t) = d(*d\hbar) = \Delta(*h),$$

where $J$ is the complex structure and $\sim$ is the isomorphism $T^* L \rightarrow TL$ set up by the induced metric on $L$. So the equation has a solution by the Fredholm alternative
and (3.5).

Thus for $\phi > 0$ we consider the $L^t$’s as the same as Lagrangian submanifolds (up to hamiltonian deformation) in the fixed symplectic manifold $W^t$; it is only the SLag representative that changes as $\Omega^t$ varies. We think of this as mirror to a fixed holomorphic bundle in a fixed complex structure, with varying HYM connection as the mirror Kähler form changes.

**Lemma 3.6** In the analogous 2-dimensional situation of SLags in a $K3$ or abelian surface, the obstruction does not occur.

**Proof** Choose a real path of complex structures $W^t$, $t \in (-\epsilon, \epsilon)$ in complex structure moduli space such that there is a nodal SLag $L^0 = L_1^0 \cup L_2^0$ in $W^0$. Without loss of generality we can choose the phase of $\Omega^t$ so that both $\omega$ and $\text{Im} \Omega^t$ pair to zero on the homology class of $L^0$. Now hyperkähler rotate the complex structures so that instead the new $\text{Re} \Omega^t$ and $\text{Im} \Omega^t$ pair to zero on the homology class of $L^0$ for all $t$. $L^0$ is now a nodal holomorphic curve $C$ in the central $K3$. We can understand deformations of $C$ via deformations of the ideal sheaf $J_C$, with obstructions in

$$\text{Ext}^2(J_C, J_C) \to H^{0,2}(W) \cong \mathbb{C},$$

where the arrow is the trace map and is an isomorphism by Serre duality. Standard deformation theory shows the obstruction is purely cohomological – it is the derivative of the $H^{0,2}$-component of the class $[C] \in H^2(W) \cong H^{2,0}(W) \oplus H^{1,1}(W) \oplus H^{0,2}(W)$.

But we have fixed this to remain zero by the phase condition, so the curve deforms to all $t$ (really we should assume the family is analytic in $t$ here and extend to $t \in \mathbb{C}$, or just work with first order deformations). Hyperkähler unrotating gives back a family of SLags. □

There is a notion of connect summing Lagrangian submanifolds intersecting in a single point (probably due to Polterovich) – see for instance Appendix A of [S1] – which we claim gives the smoothings $L^t$ of the singular $L^0 = L_1 \cup L_2$. This follows by comparing the local models [J], [S1] for the Lagrangians; see [TY] where it is studied in more detail for a related purpose, and our conventions (slightly different from those of [S2]) are described. While topologically we are just connect summing $L_1$ and $L_2$ by removing a small 3-ball containing the intersection point from each and gluing the resulting boundary $S^3$'s (there are two ways, depending on orientation), symplectically the construction does not explicitly use orientations of
the submanifolds. (Effectively we are using their relative orientation – the canonical orientation of the sum of the tangent spaces of $L_1$, $L_2$ at the intersection point given by the symplectic structure.)

Giving $L_1$ and $L_2$ in that order produces a Lagrangian, well defined up to hamiltonian isotopy (this will be shown in Section 4 in more generality; see (4.1)),

$$L_1 \# L_2,$$

with the singular union $L_1 \cup L_2$ a limit point in the hamiltonian isotopy class, which is not itself hamiltonian isotopic to $L_1 \# L_2$ (we have seen a family of hamiltonian deformations which has limit $L_1 \cup L_2$, but the deformations are singular at this limit).

There is also an obvious notion of graded connect sum, which is in fact what we shall always mean by #. There is a unique grading on $L_1$ compatible with a fixed grading on $L_2$ such that we can give a (continuous) grading to the smoothing $L_1 \# L_2$. In the case of connect summing at multiple intersection points (Section 4) there is at most one such grading; in general the graded connect sum may not exist.

In $n$ dimensions, if $L_1$ and $L_2$ are graded such that $L_1 \# L_2$ exists, then on reversing the order of the $L_i$, the graded sum that exists is

$$L_2 \# (L_1[2-n]) \text{ in the homology class } [L_2] + (-1)^n[L_1]. \quad (3.7)$$

Here $L[m]$ means the graded Lagrangian $L$ with its grading changed by adding $m\pi$ to $\theta$, and the homology class of $L_1 \# L_2$ is $[L_1] + [L_2]$ using the orientations on the $L_i$s induced by the gradings.

This is closely related, as we shall see, to Joyce’s obstruction, and the lack of it in dimension 2 (Lemma 3.6). In 2 dimensions, $L_1 \# L_2$ and $L_2 \# L_1$ are in the same homology class, though by a result of Seidel [31] not in general in the same hamiltonian isotopy class,

$$L_1 \# L_2 \not\approx L_2 \# L_1,$$

importantly (we use $\approx$ to denote equivalence up to hamiltonian deformations). For $t > 0$ in the above family $L^t$ is in the constant hamiltonian deformation class of $L_1 \# L_2$, for $t < 0$ it is in the different class of $L_2 \# L_1$, and at $t = 0$ it is $L_1 \cup L_2$ – in neither class but in the closure of both. (For complex $t$ the symplectic structure is no longer constant like it is for $t \in \mathbb{R}$, as one can see by following through the hyperkähler rotation; thus we do not get a contradiction to the above statement by going round $t = 0$ in $\mathbb{C}$.) In 3 dimensions, however, the corresponding obvious choice for a SLag on the other side of the $\pi_1 = 0$ wall, $L_2 \# L_1[-1]$, is in the wrong homology class.

15
In the case that the $L_i$ are Lagrangian spheres we can see this by going round the wall
\[ \phi(L_1) = 0 \simeq \phi(L_2), \]
and using monodromy. In the 2-dimensional $K3$ or $T^4$ case this works as follows.

\[ \mathbb{C} = \left\{ \int_{L_1} \Omega = Re^{i\phi(L_1)} \right\} \]

Figure 1: $\left( \int_{L_1} \Omega \right)$-space, as $\Omega$ on $K3$ varies, with polar coordinates $(R, \phi(L_1))$

Consider a disc in complex structure moduli space over which the family of Kähler $K3$ surfaces (with constant Kähler form) degenerates at the origin to a $K3$ with an ordinary double point (ODP) with the Lagrangian $L_1 \cong S^2$ as vanishing cycle. A local model is the standard Kähler structure on $x^2 + y^2 + z^2 = u$, over the parameter $u$ in the unit disc in $\mathbb{C}$. Now base-changing by pulling back to the double cover in $u$, $u \mapsto u^2$, we get the 3-fold
\[ x^2 + y^2 + z^2 = u^2, \]
with a 3-fold ODP which has a small resolution at the origin putting in a holomorphic sphere resolving the central $K3$ fibre $u = 0$. Choosing a nowhere-zero holomorphic section $\Omega_u$ of the fibrewise $(2,0)$-forms (using the fact that the relative canonical bundle of either family is trivial), this restricts to zero on the exceptional $\mathbb{P}^1$ (which is homologous to the vanishing cycle $L_1$). Therefore the function
\[ \int_{L_1} \Omega_u \]
has a simple zero at $u = 0$, i.e. it vanishes to order 1 in $u$. (The same expression vanished only as $\sqrt{u}$ in the original family with the singular fibre, and as such its
sign was not well defined; this is because the class \([L_1]\) was defined globally only up to the monodromy \(T_{L_1}[L_1] = -[L_1]\), i.e. up to a sign. In our new family the monodromy action \(T^2_{L_1}\) is trivial on homology so it makes sense to talk about the homology class \([L_1]\) in any fibre, and \((3.9)\) is single valued.

Then our loop of complex structures is given by taking the loop \(u = e^{it}\) and setting \(\Omega' = \Omega_{ext}\). Pulling back the Kähler form from the original family, we get a locally trivial fibre bundle of symplectic manifolds over the circle whose monodromy is the Dehn twist \(T^2_{L_1}\) (since the monodromy round the un-base-changed loop is \(T_{L_1}\)). As the family no longer has a singular fibre this monodromy is trivial as a diffeomorphism, but it is a result of \([31], [32]\) that as a symplectic automorphism it is non-trivial. This is possible because although the family is a locally trivial bundle of symplectic manifolds over the punctured disc, over \(u = 0\) the symplectic form becomes degenerate since it was pulled back via the resolution map.

Measuring \([L_1]\) against \(\Omega_u\) as in \((3.9)\) we see a principle familiar in physics (in issues of ‘marginal stability’, and taught to me by Eric Zaslow) – we detect a monodromy, like the degree 1 map \(S^1 \rightarrow \mathbb{C} \times\) given by \(t \mapsto \int_{L_1} \Omega'\), or the phase \(\phi'_{1}\) hitting \(0 \simeq \phi'_{2}\).

(Here we can no longer choose the phase of \(\Omega\) such that \(\phi'_{2} = \phi(L^2_2) \equiv 0\) in the whole family, as the homology class of \(L_2\) is not preserved in the family:

\[
[T^2_{L_1}L_2] = [L_2] + 2[L_1].
\]

However, for a sufficiently small loop about the ODP, i.e. for \(|\int_{L_1} \Omega|\) sufficiently small, this will not affect us much and we can write \(\phi'_2 \simeq 0\): we are only interested in topological information like winding numbers and \(\phi'_{1}\) crossing the wall at \(\phi'_2 \simeq 0\), which are unaffected by small perturbations.)

So instead of going through the \(\phi(L'_1) = \phi(L^2_2) \simeq 0\) wall we can go round it. If the loop is sufficiently small we do not encounter any more walls where the homology class \([L_1]+[L_2]\) can be split into classes of the same phase to possibly make the SLag a singular union of distinct SLags of equal phase. For instance the wall at phase 0 does not extend past \(u = 0\) to phase \(\phi'_1 = \pi\) (even though there \(\mu'_1 = 0\) – the phase of \(L_1\) is not zero but \(\pi\), and is only zero for \(L_1\) with the opposite orientation, so it does not exist as a SLag (e.g. in the hyperkähler rotated situation, we are saying there is no complex curve in \(L_1\)’s homology class to possibly make \(L\) the nodal union of \(L_1\) and something else, there is only an anti-complex curve). So we really can go round the wall; it ends at \(u = 0\).

So this monodromy description shows that on the other \(t \uparrow 2\pi\) side of the wall the SLag deforming \(L_2 \cup L_1\) is in the hamiltonian deformation class

\[
T^2_{L_1}L = T^2_{L_1}(L_1\#L_2) = T^2_{L_1}(T^{-1}_{L_1}L_2) \approx T_{L_1}L_2 \approx L_2\#L_1,
\]

\((3.10)\)
as claimed (for the above equalities see [S1], [S2]).

Notice that the alternative connect sum description of the above Lagrangian

\[ L_2 \# L_1 = T_{L_1}^2 (L_1 \# L_2) \approx T_{L_1}^2 (L_1) \# T_{L_1}^2 (L_2) \approx L_1 [-2] \# T_{L_1}^2 (L_2), \quad (3.11) \]

does not violate the phase inequality to (3.3), as

\[ -2\pi + \epsilon \approx \phi(L_1 [-2]) < \phi(T_{L_1}^2 (L_2)) \simeq 0. \]

This is why it is important here to keep track of gradings – assigning the phase \( \epsilon \) to \( \phi(T_{L_1}^2 (L_1)) \) would give the opposite inequality, but one would not be able to form the above graded connect sum without also shifting the phase of \( T_{L_1}^2 (L_2) \) by \(-2\pi\).

\[
\mathbb{C} = \left\{ \int_{L_1} \Omega = Re^{i\phi(L_1)} \right\}
\]

Figure 2: \( \left( \int_{L_1} \Omega \right) \)-space, as \( \Omega \) on a 3-fold varies, with polar coordinates \((R, \phi(L_1))\)

The 3-fold case (which Dominic Joyce has also, independently, considered) is slightly different; we need only take a single Dehn twist \( T_{L_1} \) corresponding to the local family

\[ x^2 + y^2 + z^2 = u, \]

over \( u \in \mathbb{C} \) to get a winding number one loop in the phase of \( L_1 \). This is because

\[ T_{L_1} L_1 \approx L_1 [1 - n] \]

in dimension \( n \), so in 3 dimensions the homology class \([L_1]\) is preserved instead of being reversed. The corresponding picture is displayed in Figure 2.
Again there is a SLAG on the other side of the $\phi = 0$ wall, but it is in the wrong homology class $[L_2]$:  
$$T_{L_1}L \approx L_2. \quad (3.12)$$
Analogously to (3.11) this has a number of decompositions as connect sums induced by monodromy, 
$$T_{L_1}(L_1 \# L_2) \approx L_1[-2] \# (L_2 \# (L_1[-1])) \approx L_2 \approx (L_1 \# L_2) \# (L_1[1]),$$
none of which violate the phase inequality (3.3). The only other obvious choice for a SLAG on the other side of the $\phi = 0$ wall (given the K3 result) is $T_{L_1}^2(L_1 \# L_2) \approx L_2 \# (L_1[-1])$; this however is also in the wrong homology class, and in any case does violate (3.3) and so, by Joyce’s analysis, should not be represented by a SLAG. Thinking of $T_{L_1}^2$ as rotating through $-4\pi$ in Figure 2, it is at roughly $-3\pi$ that the phase inequality (3.3) gets violated, and the $-\pi$ rotation of $L_2$ splits as a SLAG into the union of the $-\pi$ rotations of $(L_1 \# L_2)$ and $L_1[1]$: these both have phase approximately zero.

A holomorphic bundle example

These phenomena are similar to wall-crossing in bundle theory on the complex side – in a real one-parameter family of Kähler forms, for fixed complex structure, stable holomorphic bundles for $t > 0$ can become semistable at $t = 0$ and unstable for $t < 0$.

An example that mirrors Joyce’s is the following. Suppose we have two stable bundles (or coherent sheaves) $E_1$ and $E_2$ with  
$$\text{Ext}^1(E_2, E_1) \cong \mathbb{C}.$$  
This is $H^1(E_1 \otimes E_2^\ast)$ in the case of bundles and is the mirror of the one dimensional Floer cohomology $HF^+(L_2, L_1) \cong \mathbb{C}$ that is defined by the single intersection point of $L_1$ and $L_2$ (see Section 3 for more details of this, and an explanation of why we are dealing with Ext$^1$ and HF$^1$ here). We then form $E$ from this extension class  
$$0 \to E_1 \to E \to E_2 \to 0. \quad (3.13)$$
Take a family of Kähler forms $\omega^t$ such that $\mu^t(E_2) - \mu^t(E_1)$ is the same sign as $t$ (here $\mu^t(F) = c_1(F) \cdot (\omega^t)^{n-1}/\text{rk } (F)$ is the slope of $F$ with respect to $\omega^t$). Supposing that the $E_i$ are stable for all $t \in (-\epsilon, \epsilon)$, we claim that $E$ is stable for sufficiently small $t > 0$, while it is destabilised by $E_1$ for $t \leq 0$. Without loss of generality take $\mu^t(E_2) = \mu$ fixed, and $\mu^t(E_1) = \mu - t$. As $E_2$ is stable, for $t$ sufficiently small there
are no subsheaves of $E_2$ of slope greater than $\mu - t$, so for any stable destabilising 
subsheaf $F$ of $E$, the composition 

$$F \hookrightarrow E \rightarrow E_2$$

cannot be an injection (unless it is an isomorphism, but (3.13) does not split. So 
$F \cap E_1 \neq 0$, and the quotient $Q = F/(F \cap E_1)$ has slope $\mu(Q) > \mu(F) > \mu - t$ by 
the stability of $F$ and instability of $E$. But $Q$ injects into $E_2$, which we know is 
impossible.

In the 2-dimensional case, by Serre duality $\operatorname{Ext}^1(E_1, E_2) \cong \operatorname{Ext}^1(E_2, E_1)^* \cong \mathbb{C}$ on $K3$ or $T^4$, so for $t < 0$ we can instead form an extension 

$$0 \rightarrow E_2 \rightarrow E' \rightarrow E_1 \rightarrow 0,$$ 

(3.14) 

to give a new bundle $E'$ which is also stable, and has the same Mukai vector 

$$v(E') = v(E_1) + v(E_2);$$

compare (3.7). At $t = 0$ we take the (polystable) bundle 

$$E_1 \oplus E_2.$$ 

This is because the semistable extension (3.13) no longer admits a Hermitian-Yang-
Mills metric, but $E_1 \oplus E_2$ does. Also, the algebraic geometry of the moduli problem 
shows that while a semistable bundle gets identified in the moduli space with the 
other (“S-equivalent”) sheaves in the closure of its gauge group orbit, there is a 
distinguished representative of its equivalence class – the polystable direct sum (of 
the Jordan-Hölder filtration, which here is $E_1 \oplus E_2$).

Thus, while the HYM connections vary, the bundle has only 3 different holomorphic 
structures – for $t > 0$, $t = 0$, and $t < 0$. Put another way (to spell out the 
analogy with the Lagrangians $L^t$, $L_1$, $L_2$ as $\omega_t$ varies with $t > 0$ we take different 
points in a fixed complexified gauge group orbit, and at $t = 0$ we take as limit point 
something in a different orbit that is nonetheless in the closure of the $t > 0$ (and 
t < 0) orbit. The stable deformations of the polystable $E_1 \oplus E_2$ (which we are thinking 
of as the mirror of the singular union $L_1 \cup L_2$, of course) are precisely (3.13) for 
t > 0 and (3.14) for $t < 0$.

In the 3-fold case, however, Serre duality gives $\operatorname{Ext}^2(E_1, E_2) \cong \operatorname{Ext}^1(E_2, E_1)^* \cong \mathbb{C}$ instead, and so no stable extension (3.14). In fact one would expect there to 
be no stable bundle with the right Chern classes; instead the one dimensional $\operatorname{Ext}^2$ 
gives us a complex $E'$ in the derived category $D^b(M)$ fitting into an exact sequence 
of complexes 

$$0 \rightarrow E_2 \rightarrow E' \rightarrow E_1[-1] \rightarrow 0,$$ 

20
where $E_1[-1]$ is $E_1$ shifted in degree by one place to the right as a complex. This has Mukai vector
\[ v(E') = v(E_2) - v(E_1), \]
compare (3.7). Thus, just as in the case of SLags, as we pass through $t = 0$ there is no natural stable object on the other side in the same homology class in 3 dimensions (though there is in 2 dimensions) and so an element of the appropriate moduli space disappears.

In fact, as in the Lagrangian example, the natural stable object on the other side of the wall is $E_2$ if we consider monodromy. The mirror of the symplectic Dehn twists of above are described in [ST] (in the case that the bundles $E_i$ are spherical in the sense of [ST]: $\text{Ext}^k(E_i, E_i) \cong H^k(S^n; \mathbb{C})$; this is the natural mirror analogue of the $L_i$s being spheres). These are the twists $T_{E_1}$ of $[ST]$ on the derived category of the Calabi-Yau that act on the extension bundle $E$ of (3.13) to give precisely the extension (3.14),
\[ T^k_{E_1} E = E' \]
(compare (3.10)), as a short calculation using [ST] shows. Similarly
\[ T_{E_1} E = E_2, \]
the analogue of (3.12). (In both of these calculations it is important to calculate this monodromy in the derived category; in the K3 case the action of $T^k_{E_1}$ is trivial on K-theory and cohomology, and we cannot distinguish between (3.13) and (3.14), but they are very different as holomorphic bundles and as elements of the derived category.)

The mirror wall crossing, with a SLag splitting into two and then disappearing, is interpreted in [DFR] (and in [SV] in a different case) as the state it represents decaying as we reach a point of ‘marginal stability’. Despite this dealing with only SLags (and so with only a priori stable Lagrangians in our mathematical sense of stability), this suggestive language does in fact have something to say about the stability, in our sense of group actions, of (non-special) Lagrangians, by considering the nodal limit $L_1 \cup L_2$ to be a semistable Lagrangian.

Thus the Lagrangian $L_1 \# L_2$ (which always exists as a Lagrangian as the complex structure varies with fixed Kähler form) becomes semistable at $t = 0$ and is represented by something in a different orbit of the hamiltonian deformation symmetry group (but in the closure of the original orbit), and is unstable for $t < 0$ so exists there only as a Lagrangian and not as a SLag. This, and the bundle analogue described above, leads us to think of the Lagrangian $L_1$ as destabilising $L = L_1 \# L_2$ when $\phi(L_1) \geq \phi(L_2)$. This motivates the now obvious definition of stability in Section 3: first we explain more about the connections to mirror symmetry, and generalisations to connect sums at more intersection points.
4 Relationship to Kontsevich’s mirror conjecture

The inspiration behind most of this paper is of course Kontsevich’s mirror conjecture \[ K \]. In particular, Kontsevich proposes that the graded vector spaces \( \text{Ext}^* \) and \( HF^* \) should be isomorphic for mirror choices of bundles \( E_i \) and graded Lagrangians \( L_i \) (or more exotic objects in their derived categories)

\[
HF^*(L_2, L_1) \cong \text{Ext}^*(E_2, E_1);
\]

this corresponds to the equality of (graded) morphisms on both sides. Here \( HF^* \) is Floer cohomology \[ Fl \] – a symplectic refinement of the intersection number of \( L_1 \) and \( L_2 \) – which can be \( \mathbb{Z} \)-graded for graded Lagrangians \[ S2 \], whenever it is defined \[ FO3, Fu1 \]. (More precisely it is the cohomology of a chain complex built out of the free vector space generated by the intersection points, with the differential defined by counting holomorphic discs with boundary in the Lagrangians running from one intersection point to another.) In mirror symmetry, and so in this paper, one should only really consider those Lagrangians whose Floer cohomology is well defined \[ Fu1 \].

Thus the point of intersection of the \( L_1 \) and \( L_2 \) of the last section define the Floer cohomology \( HF^*(L_2, L_1) \cong \mathbb{C} \), and the grading of \[ S2 \] is designed specifically so that \( L_1 \# L_2 \) can be \( \mathbb{Z} \)-graded for graded Lagrangians \[ S2 \], whenever it is defined \[ FO3, Fu1 \]. We then think of the connect sum \( L_1 \# L_2 \) as being mirror to the extension \[ 3.13 \] defined by \( \text{Ext}^1(E_2, E_1) \cong \mathbb{C} \). Fukaya, Seidel, and perhaps others have also proposed that Lagrangian connect sum should be mirror to extensions \[ Fu2, S3 \].

We also consider connect sums of Lagrangians intersecting at \( n \) points \( p_i \). Then the connect sum is not unique up to hamiltonian deformation: \( H^1 \) is added to the Lagrangian as loops between the intersection points, giving additional deformations of its hamiltonian isotopy class. The upshot is that there is a scaling of the neck of the connect sum at each intersection point; we denote any such resulting Lagrangian by \( L_1 \# L_2 \). Since we insist on all intersection points having Floer (Maslov) index one (so that the connect sum can be graded), the Floer differential vanishes in this case, and these scalings define a class in \( HF^1(L_2, L_1) \).

Deformations (up to those which are hamiltonian) as such a connect sum are given by the elements of

\[
H^1(L_1 \# L_2) \cong H_{n-1}(L_1 \# L_2)
\]

spanned by the \( S^{n-1} \) vanishing cycles \( S_i \) at the points of intersection \( p_i \in L_1 \cap L_2 \). Given a particular connect sum, the deformation represented by \( \sum_i a_i S_i \) simply scales the local gluing parameter in a Darboux chart around each \( p_i \) by a factor
(1 + a_i) (here a_i is considered to be infinitesimal). Since the sum of these spheres separates \(L_1 \# L_2\) into \(L_1 \setminus \{p_i\}\) and \(L_2 \setminus \{p_i\}\) and so is zero in homology

\[
\sum_i [S_i] = \pm \partial[L_1 \setminus \{p_i\}] = \mp \partial[L_2 \setminus \{p_i\}] = 0 \in H_{n-1}(L_1 \# L_2),
\]

the infinitesimal deformation represented by \(\sum_i S_i\) is zero (it is pure hamiltonian) and dividing out gives the projectivisation

\[
\mathbb{P}(\oplus_i \mathbb{R} \alpha_i).
\]

(Replace \(\mathbb{R}\) by \(\mathbb{C}\) when including flat bundles and their gluing parameters at the \(p_i\)s.) This explains the earlier claim that connect sums at one intersection point are uniquely defined up to hamiltonian deformations. More precisely, when holomorphic discs are taken into account and we consider only those Lagrangians whose Floer cohomology is defined [FO3], hamiltonian deformation classes of connect sums whose Floer cohomology can be defined should be parameterised by \(\mathbb{P}(HF^1(L_2, L_1))\). (On the mirror side isomorphism classes of extensions of \(E_2\) by \(E_1\) are parametrised by \(\mathbb{P}\) \(\mathbb{E}xt^1(E_2, E_1)\).)

We would then expect that the resulting connect sum has a canonical homomorphism from \(L_1\); that is there should be a canonical element

\[
\text{id}_{L_1} \in HF^0(L_1, L_1 \# L_2)
\]

for any graded Lagrangians \(L_i\) for which the graded connect sum exists. While a local model suggests this is true (see for instance [GY]), a complete proof is still not available. This homomorphism we think of as expressing \(L_1\) as a subobject of \(L_1 \# L_2\); i.e. as giving an injection. It should be emphasised that subobject does not make sense in a triangulated category such as the derived Fukaya category of Lagrangians; in the context of the derived category of sheaves, subobject only makes sense for an abelian category such as that of the sheaves themselves (i.e. complexes with cohomology in degree zero only). What we are proposing is that it also makes sense in the category of (complexes of sheaves mirror to) graded Lagrangians, and is vital to make definitions of stability (which involve such subobjects). While there are now more Homs to consider, in particular those of higher order (i.e. Homs to Lagrangians shifted in phase by some \(2\pi n\)), the targets of these Homs have higher phase and so do not disturb the definition of stability below – this is seemingly a huge piece of luck that means we can extend the stability condition for bundles to all Lagrangians. For similar reasons, the many connect sum decompositions of the \(L_i\)s given in the last section also do not destabilise them.

There are other operations, however, which can also be thought of as \(\mathbb{E}xt^1\)-type extensions. For instance, taking the product of a single Lagrangian curve \(L_1\)
in $T^2$ with a (graded) connect sum $L_2#L_3$ in another $T^2$, we get a Lagrangian $L_1 \times (L_2#L_3)$ in $T^4$ which is some kind of extension of the Lagrangians $L_1 \times L_2$ and $L_1 \times L_3$ in $T^4$. Supposing that the $L_1$s are mirror to some (complexes of) sheaves $E_i$, and that the connect sum $L_2#L_3$ is mirror to an extension represented by an element $e \in \text{Ext}^1(E_3, E_2)$. Then by the K"unneth formula for sheaf cohomology, we see that $L_1 \times (L_2#L_3)$ is indeed mirror to an extension

$$\text{id} \otimes e \in \text{Hom}(E_1, E_1) \otimes \text{Ext}^1(E_3, E_2) = \text{Ext}^1(E_1 \boxtimes E_3, E_1 \boxtimes E_2),$$

and so this sort of relative connect sum (which is not # on $T^4$: $L_1 \times L_2$ and $L_1 \times L_3$ do not intersect transversely) should also be considered.

So we consider Lagrangians $L_1, L_2$ intersecting cleanly (see e.g. [S1] Definition 2.1), that is $N = L_1 \cap L_2$ is a smooth submanifold, and $TN = TL|_N \cap TL_2|_N$. Basic results of Weinstein allow us to identify a neighbourhood of $N$ with a neighbourhood of the zero section $N$ in $T^*N \oplus E$, where the total space of $T^*N$ has its canonical symplectic structure, and

$$E \equiv (TL_1|_N)/TN \oplus (TL_2|_N)/TN$$

is the annihilator, under the symplectic form, of $TN \subset TX|_N$ (to which the symplectic form therefore descends, making $E$ a symplectic bundle).

Choosing a metric on $E$, compatible with its symplectic structure, such that its transverse subbundles $(TL_1|_N)/TN$, $(TL_2|_N)/TN$ are orthogonal, we can now perform the family connect sum of these, over the base $N$, since the local model in [S1] is $O(n)$ invariant. As before we insist that this can be compatibly graded again denote it by #; given a grading on $L_1$ there will be at most one grading on $L_2$ such that this graded relative connect sum exists.

It should be noted that although such a clean intersection could be hamiltonian isotoped to be transverse, the resulting intersection points would not necessarily all be of Floer/Maslov index one, and so the pointwise graded connect sum could not be formed at every point; we would end up with an immersed Lagrangian. Studying which immersed Lagrangians should be included in the Fukaya category, and which embedded Lagrangians they should be considered equivalent to, is an important part of mirror symmetry and will need to be better understood to refine our conjecture. For instance forming extensions of bundles which also have nonzero homorphisms between them would appear to be mirror to forming connect sums between graded Lagrangians at index one intersection points, leaving the index zero intersection points immersed. In general one would like to consider two objects of the Fukaya category to be equivalent if their Floer cohomologies with any other objects are the same. This would include hamiltonian deformation equivalence, but also more exotic equivalences for immersed Lagrangians (thanks to Paul Seidel for pointing this out
to me). A start in understanding the Floer cohomology of immersed Lagrangians is \[ Ak \]; in the present paper we are largely ignoring singularities.

5 Stability

Definition 5.1 Take graded Lagrangians \((L_1, \theta_1)\) and \((L_2, \theta_2)\), hamiltonian isotoped to intersect cleanly, and such that the graded (relative) Lagrangian connect sums \((L_1 \# L_2, \theta_1 \# \theta_2)\) exist as above. Then a Lagrangian \(L\) of Maslov class zero is said to be destabilised by the \(L_i\) if it is hamiltonian isotopic to such an \(L_1 \# L_2\), and the phases (real numbers, induced by the gradings) satisfy

\[ \phi(L_1) \geq \phi(L_2). \]

If \(L\) is not destabilised by any such \(L_i\) then it is called stable.

Remarks

- There is an obvious notion of a flux homomorphism for isotopies of smooth Lagrangians, taking a deformation to an element of \(H^1(L; \mathbb{R})\) (and linearising to give the usual deformation theory of Lagrangians). Namely, take a deformation \(\Phi_t(L)\) through a vector field \(X_t\), \(t \in [0, 1]\) to the one form

\[ \int_0^1 (X_t \downarrow \omega) dt \in H^1(L; \mathbb{R}). \]

Alternatively, the homomorphism takes a loop \(\gamma \subset L\), tracing out the 2-cycle \(f(\gamma \times [0, 1])\) in \(W\) under the isotopy, to the real number \(\int_{\gamma \times [0, 1]} \omega\). (See Chapter 10 of [MS] for the analogous map for symplectomorphisms.) If the isotopy \(\Phi_t\) is hamiltonian, the flux is zero; the converse is also easily proved using the methods of ([MS] Theorem 10.12): we may assume without loss of generality that the 1-form \(\int_0^1 X_t \downarrow \omega\) is identically zero in \(\Omega^1(L)\). [To see this, write the 1-form as \(d\phi\), and compose the deformation with the time one map of the hamiltonian flow with vector field \(d\phi \downarrow \omega^{-1}\); this does not alter the flux in \(H^1(L; \mathbb{R})\) or the property of being hamiltonian.] Then let \(\Sigma^s\) be the closed 1-form on \(L\) defined by

\[ \Sigma^s = \int_0^s X_t \downarrow \omega dt, \]

and let \(\Psi^s_t\) be the corresponding flow through time \(t\). Then the flow \(\phi_t = \Psi^t_0 \circ \Phi_t\) is the corresponding hamiltonian flow from \(\Phi_0(L)\) to \(\Phi_1(L)\); see [MS]. Thus it is not too hard to check if two Lagrangians are hamiltonian deformations of each other, at least through smooth Lagrangians, if we know they are...
deformations of each other as Lagrangians. This second condition, however, is harder to test, as the results of [S1] demonstrate.

- As mentioned in the last section, holomorphic discs are crucial in both mirror symmetry and Floer cohomology; thus one should perhaps restrict attention in the above definition to those Lagrangians whose Floer cohomology is defined [FO3],

- As pointed out to me by Conan Leung, this definition and the resulting conjecture below may only be reasonable close to the large complex structure limit point where the mirror symmetric arguments used to motivate the conjecture are most valid.

**Conjecture 5.2** A Lagrangian of Maslov class zero has a special Lagrangian in its hamiltonian deformation class if and only if it is stable, and this SLag representative is unique.

Again, we have been vague about singularities: which we allow, and what hamiltonian deformation equivalence would mean for them. We might also want to restrict attention to those Lagrangians whose Floer cohomology exists [FO3], and whose Oh spectral sequence \( H^*(L) \Rightarrow HF^*(L, L) \) degenerates; this will be discussed more in [TY]. We might also want to restrict to Lagrangians whose phase function varies only by a certain bounded amount; in the example worked out in [TY], this is required. In [TY], it is shown there that the gradient of the norm-squared \( |m|^2 \) of the moment map can be taken to be the mean curvature vector of the Lagrangian, so mean curvature flow (which is hamiltonian for Maslov class zero) should converge to this SLag representative if the Lagrangian is stable and the phase satisfies certain bounds.

### 6 The 2-torus

Everything works rather simply on \( T^2 \); Grayson [G], building on work of Gage, Hamilton and others (e.g. [GH]), has shown that mean curvature flow for curves (of Maslov class zero) converges to straight lines and so we get the mirror symmetric analogue of Atiyah’s classification [At] of sheaves on an elliptic curve – they are basically all sums of stable sheaves. The only exceptions are the non-trivial extensions of certain sheaves by themselves; these correspond to thickenings of the corresponding special Lagrangian (giving fat SLags, as they are known in Britain, or multiply-wrapped cycles in physics speak).
We give an example to demonstrate why one cannot form smooth unstable Lagrangians on $T^2$ in Figure 3. First, giving $L_1$ and $L_2$ the gradings such that their phases are 0 and $\pi/4$, we expect $L_1 \# L_2$ to be stable, and indeed we see it is hamiltonian deformation equivalent to the slope $1/2$ SLag mirror to the stable extension $E$ of $\mathcal{O}$ by $\mathcal{O}(p)$ (where $p$ is a basepoint of $T^2$ with corresponding line bundle mirror to the diagonal SLag drawn).

If one then tries to form an unstable SLag $L_2 \# L_1$, the graded connect sum does not exist – the phase would become discontinuous. To form $L_2 \# L_1$ we see from the diagram that we have to take the phase of $L_1$ to be $\pi$, thus reversing its orientation, and in fact forming $L_2 \# (L_1[1])$. Then the stability inequality (3.3) is not violated, and in fact this Lagrangian is stable and hamiltonian deformation equivalent to the SLag in $T^2$ represented by the vertical edge of the square (and so drawn with a little artistic license in Figure 3). Under the mirror map this corresponds to replacing the extension $\text{Ext}^1$ class by a Hom (as we have shifted complexes of sheaves by one place) and taking the cone of this in the derived category; this is the cokernel $\mathcal{O}_p$ of Figure 3.

As pointed out to me by Markarian and Polishchuk, one can play with lots of pictures of connect sums on tori to recover descriptions of certain moduli of sheaves, their special cycles (for instance where one connect-sum neck parameter goes to zero), and so forth, giving results similar to some of those in [FO].
This example can be extended to show that we cannot form the graded connect sum \( L_1 \# L_2 \) of any two Lagrangians (via a class in \( HF^*(L_1, L_2) \)) if \( \phi(L_1) > \phi(L_2) \). Namely, replace \( L_1 \) and \( L_2 \) by their Hamiltonian deformation equivalent SLag representatives, which are straight lines of constant phase \( \theta_i = \phi(L_i) \). As Figure 3 shows, \( L_1 \# L_2 \) can be compatibly graded about an intersection point if and only if we have the local inequalities

\[
\theta_2 > \theta_1 > \theta_2 - \pi.
\]

Thus we require \( \phi(L_2) > \phi(L_1) \). (We will explain this kind of phenomenon more generally in [TY] in terms of the grading on Floer cohomology.) Each intersection point is Floer coclosed since the Floer grading is the same as the relative orientation of the Lagrangians, mod 2, and this is the same at each intersection point of the straight lines. So each possible connect sum of the SLags defines a class in \( HF^* \), and any other connect sum, defined on Hamiltonian deformations of \( L_1 \) and \( L_2 \) by a class in \( HF^* \), will be Hamiltonian deformation equivalent to the appropriate connect sum of the SLags, and so satisfy the same phase inequality.

If two smooth Lagrangians have the same phase then their representative SLags will either be the same or disjoint parallel SLags. Either way there are no connect sums (though as mentioned above to account for the mirror symmetry of bundles one should also include non-trivial thickenings of SLags in the Fukaya category).

So unstable Lagrangians do not exist, and by the result of [3] mentioned earlier, the conjecture is true on \( T^2 \).

Thus complex dimension 1 is too simple – in trying to make the phase of one Lagrangian become larger than the phase of another, the two must cross, thus reversing their relative orientations and changing the order of the connect sum. Far more complicated phenomena arise in 2 and 3 dimensions, however.

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