Exponential Fermi Acceleration in a Switching Billiard

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Received: 21 October 2021 / Accepted: 15 August 2022
Published online: 4 September 2022 – © The Author(s), under exclusive licence to Springer-Verlag GmbH
Germany, part of Springer Nature 2022

Abstract: In this paper we show the existence of an infinite measure set of exponentially
escaping orbits for a resonant Fermi accelerator, which is realised as a square billiard
with a periodically oscillating platform. We use normal forms to describe the energy
change in a period and employ techniques from the theory of hyperbolic systems with
singularities to show the exponential drift given by these normal forms on a divided
time-energy phase.

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The authors gratefully acknowledge the enlightening discussion with Dmitry Dolgopyat, which has greatly
improved the structure of the proof in the paper. The authors also sincerely thank the anonymous referees for
the valuable comments and suggestions which help to improve the quality of the paper and make it more
accessible.
1. Introduction

The Fermi acceleration model has been extensively investigated numerically and theoretically by many physicists and mathematicians in the past decades (c.f. [6, 8, 17] for a survey on the subject). In 1949 Fermi [7] proposed a model where charged particles travel in a moving magnetic field, as an attempt to explain the existence of high-energy particles in the outer universe. Later in 1961 Ulam [23] extracted the effective dynamics in this situation, i.e. a ball bounces between a fixed wall and a periodically oscillating wall, and he conjectured the existence of escaping orbits (those whose energy grows to infinity in time) based on his numerical experiment with piecewise linear models. Since then, numerous efforts have been made toward locating escaping orbits as well as bounded (those whose energy remains bounded) and oscillatory ones (those whose energy has finite lim inf but infinite lim sup) in the Fermi acceleration models and their variations.

For sufficiently smooth wall motions, KAM theory has implied that the prevalence of invariant curves forces the energy of every orbit to be bounded [15, 20, 21]. Zharnitsky [26] generalized the situation to KAM quasi-periodic motions with Diophantine frequencies. Kunze and Ortega [14] showed that the set of escaping orbits has zero measure for wall motions with rationally independent frequencies.

The Fermi acceleration exhibits richer dynamical phenomena if we allow singularities in the wall motions. Zharnitsky [25] found linearly escaping orbits in Ulam’s piecewise linear models. De Simoi and Dolgopyat [4] showed in a piecewise smooth model with one singularity that the system changes from elliptic behavior to hyperbolic behavior as a critical system parameter varies: in the elliptic regime the set of escaping orbits has infinite measure while in the hyperbolic regime it has zero measure but full Hausdorff dimension.

When the fixed wall is replaced by background potential, escaping orbits are shown to have an infinite measure for periodic analytic wall motions with gravity [19]. However, if we allow one singularity in the gravity model and impose hyperbolicity assumptions, the escaping orbits constitute a null set and they co-exist with bounded orbits at arbitrarily high energy levels [28]. Non-constant background potential models have also been examined and we refer to [3, 5, 18] for examples.

The billiard model is a natural realization of the Fermi acceleration model in two dimensions. For instance, unbounded orbits have been located in a billiard model with smoothly breathing boundaries [11, 13, 16]. Exponentially escaping orbits have been constructed in a non-autonomous billiard model [10] and are conjectured to be generic in an oscillating mushroom model [9]. However in general it remains challenging to construct a Fermi accelerator with a full measure or even positive measure set of exponentially accelerating orbits.

In this paper, we study a unit square billiard with a vertically oscillatory slit of infinite mass on the left. A ball bounces elastically on the boundaries of the square and the slit. The slit has length $\lambda$ and its motion (height) is described by $f(t) \in C^3$ which is periodic in two $f(t+2) = f(t)$ (c.f. Fig. 1). The horizontal speed of the ball is 1 so the horizontal motion of the ball is in 1:1 resonance with the vertical movement of the slit.

We acquire this model from [22] where Shah, Turaev and Rom-Kedar proposed an ideally probabilistic approximation that the probability of jumping up or down is
proportional to the lengths of openings which well matched the numerically observed exponential growth rate in the non-resonant case and the escape rate in the resonant case was numerically observed to be significantly higher than that in the non-resonant case.

This paper addresses an open question from [27] where the non-switching case with 1:1 resonance was discussed. In [27] when the relative positions of two oscillating slits are different at two critical jumps in a period, a trapping region, either the upper or the lower chamber of the table, is therefore created so that the ball can never escape to the other chamber once it enters the trapping chamber and in this situation almost every high energy orbit eventually starts to gain energy exponentially fast. However, in the current setting when there is only one slit, trapping regions do not exist and hence the old mechanism of exponential acceleration no longer works.

Let $x_0$ denote the starting horizontal position of the ball. We assume without loss of generality that the ball starts from the left part of the table, i.e. $0 < x_0 < \lambda$. We name as the interacting region the slit on the left and the floor on the right (c.f. the red region in Fig. 1). We record the time $t$ and the vertical velocity $v$ of the ball immediately after the collision at the interacting region. Let $F$ denote the collision map which sends a collision $(t, v)$ to the next $(\tilde{t}, \tilde{v})$. We exclude from our discussion the collisions at the edge of the slit, which constitute a null set from the $(t, v)$ phase cylinder. It takes time $\Delta t = 2$ for the ball to finish a complete revolution. We denote by $\mathcal{F}$ the dynamics of a complete revolution and $P$ a half revolution, whose definitions are made precise in Sect. 3. We are interested in the measure of the set of the initial conditions $(t_0, v_0)$ producing escaping orbits in this billiard system and we show that for a large collection of slit motions $f$ there exists an infinite measure set of initial conditions producing exponentially escaping orbits.

The paper is structured as follows. First we state the main result in Sect. 2. Then in Sect. 3 we derive the normal forms for the half-revolution map $P$ and discuss the structure of the phase cylinders on which our dynamics sits. The proof of the main result is dissected into 4 sections: In Sect. 4 we show that under our assumptions the dynamical system is uniformly hyperbolic with an invariant unstable cone; in Sect. 5 we give an estimate on the control of the short unstable curves in the form of the growth lemmas as an outcome of the strong hyperbolic behavior of our system; then in Sect. 6 we show the system possesses on average an exponential drift in energy in finite time; and finally in Sect. 7 we prove the eventual exponential acceleration as time goes to infinity.
2. Main Result

We recall that $x_0$ is the starting horizontal position of the ball and $\lambda$ is the length of the slit. We may assume without loss of generality that we start from the left chamber, i.e. $0 < x_0 < \lambda$. Since the horizontal motion of the ball is in 1:1 resonance with the vertical oscillation of the slit, in a period $\Delta t = 2$ the ball switches from the left chamber to the right at the moment $t_1^* = \lambda - x_0$ and later from right to left at $t_2^* = 2 - \lambda - x_0$. We note that the two jumping moments $t_1^*, t_2^*$ are constant due to the 1:1 resonance once the parameters $x_0, \lambda$ are fixed. We denote $f_i := f(t_i^*)$ and $\dot{f}_i := \dot{f}(t_i^*)$ ($i = 1, 2$) etc.

**Theorem 1** (Main Theorem).

Assume that $f(t) \in C^3$ is periodic in two $f(t + 2) = f(t)$, and that $f_1 \neq f_2$. Let

$$\mathcal{E} = (1 - f_2) \log \frac{1 - f_2}{1 - f_1} + f_2 \log \frac{f_2}{f_1}.$$ 

Assume also that for all $t$, $c \leq f(t) \leq 1 - c$ for some constant $c \in (0, 1/2)$. Then there exists $\dot{f}_* > 0$ large so that if $|\dot{f}_1|, |\dot{f}_2| > \dot{f}_*$ then one can choose an energy threshold $V_* = V_\alpha(f_1, f_2, c, \mathcal{E})$ in such a way that there exist positive constants $\alpha, \beta, T_\alpha$ and $t > 1$, depending on the constants $\mathcal{E}, c$, so that for every $V_1, V_2$ and $T \geq T_\alpha$ with $V_\alpha T < V_1 \ll V_2, V_2 - V_1 \geq 1$, we have

$$\frac{|\{v_0 \in [V_1, V_2] : v_t \geq e^{\alpha t} v_0, \forall t \geq T\}|}{|V_2 - V_1|} \geq 1 - e^{-\beta T}$$

where $|\cdot|$ is the Lebesgue measure of the initial conditions $(t_0, v_0)$ in the $(t, v)$-phase cylinder and $v_t$ is the velocity of the ball at time $t$.

It follows from the Gibbs’ inequality that the quantity $\mathcal{E} > 0$ if and only if $f_1 \neq f_2$. The quantity $\mathcal{E}$ is the expected exponential energy growth rate of the randomized system considered by Shah et al. [22] and they numerically detected exponentially escaping orbits when $\mathcal{E} > 0$. The requirement $V_1 \ll V_2$ in Theorem 1 is understood as follows: the window of initial conditions $\{(t_0, v_0) : v_0 \in [V_1, V_2]\}$ needs to cover at least one fundamental domain of the normal forms (c.f. the green/red/yellow boxes in Fig. 3) in Sect. 3.2. It is clear from Theorem 1 that the higher the initial energy, the larger the density of the exponentially escaping orbits.

Without loss of generality we may assume, throughout the discussion that follows, that $f_2 > f_1$, so that the ball accelerates when it enters the lower chamber and decelerates when it enters the upper chamber (c.f. Sect. 3). The case when $f_2 < f_1$ can be proven analogously.

The main tool we use to prove the exponential acceleration is the adiabatic normal forms derived in Sect. 3, which describes how the energy of the ball changes after a period when it enters either the upper chamber or the lower chamber. The time-energy phase cylinder for these normal forms is divided into the accelerating lower route and the decelerating upper route (c.f. Fig. 3) under the assumption $f_1 \neq f_2$. Our strategy of the proof of the Main Theorem 1 proceeds as follows: we first show that under the assumptions in Theorem 1 our system enjoys strong uniform hyperbolicity so that ideally if we start with an expanding curve it would get stretched considerably and we can estimate the accelerating and decelerating portions in its image and repeat the process. However, since our phase cylinder is divided, a long expanding curve will get cut into many pieces some of which might remain short for a long time so that the whole picture after a long time will be highly fragmented and we cannot carry out a uniform estimate.
We fix the problems by proving some growth lemmas which control the size of short curves and revive the analysis. The other technical difficulty is that the normal forms only hold above a certain energy threshold $V^*$ and the analysis collapses when the energy of the ball falls below it. We take care of this issue by first proving the result for a modified system whose dynamics are always well-defined and then we argue that a considerable portion of exponentially accelerating orbits of the modified system coincide with those of the original system.

More precisely, the proof of the Main Theorem 1 is structured as follows. We first derive the adiabatic normal forms in Sect. 3 which captures how the energy of the ball changes after one period when it enters either the upper chamber or lower chamber. Then in Sect. 4 we show that our system is uniformly hyperbolic under the assumptions in Theorem 1 in the sense that the adiabatic normal forms share an invariant unstable cone. Next in Sect. 5 we estimate the size of the short (“bad”) unstable curves by proving the growth lemmas as a consequence of the strong uniform hyperbolicity of the system. Then we prove for the modified system in Sect. 6 that the assumption $f_2 \neq f_1$ yields exponential acceleration in finite time for sufficiently long expanding curves (called “long curves”). Finally, in Sect. 7 we prove exponential acceleration for an infinite time by using a moment estimate argument combined with the quantitative growth lemma and the large deviation estimate proven in Sect. 5.

3. Preliminaries

In this section we discuss the structure of our dynamics: the map of a complete revolution $(\Delta t = 2)$ and the divided phase cylinder it sits on. Starting with sufficiently high initial energy, the ball experiences a large number of collisions during one revolution, so we aim to derive an effective tool to study the behavior of the system in one period, i.e. the first return map $F$ to the right floor.

We denote the two singularity curves in the $(t, v)$-phase cylinder as $S_i = \{ t = t^*_i \}$ ($i = 1, 2$): these are the collisions occurring at the edge of the slit and hence the ball motion is not well-defined.

For $i = 1, 2$, let $R_i$ denote the singular strip bounded by $S_i$ and its image $f(S_i)$ respectively. We note that $R_1$ collects the collisions at the right floor immediately after the ball exits the left chamber and similarly $R_2$ the first collisions at the slit after the ball enters the left chamber. We further subdivide $R_2$ into $R^+_2$ and $R^-_2$ for the collisions in the left upper and the left lower chambers. We denote their preimages as $\tilde{R}_i = f^{-1}(R_i)$. We also denote as $f_i = f(t^*_i)$, $\dot{f}_i = \dot{f}(t^*_i)$, etc. We aim to derive the first return map to the right floor, i.e. $F : R_1 \varsubsetneq$.

The most important result in this section is the following Proposition 3.1 which describes how the energy of the ball changes under $F$. During a period $\Delta t = 2$, the ball either chooses the upper left chamber (called the Upper Route) or the lower left chamber (called the Lower Route). Roughly speaking, if the ball follows the Upper Route, then its velocity gets rescaled by a factor close to $1 - f_2/f_1 < 1$ after one revolution; if it follows the Lower Route, then its velocity gets rescaled by a factor close to $f_2/f_1 > 1$ after one revolution. More precisely, we introduce $l(t)$ as the distance from the interacting region to the ceiling:

$$l(t) = \begin{cases} 1 - f(t) & 0 \leq t \leq t^*_1, t^*_2 \leq t \leq 2 \\ 1 & \text{otherwise} \end{cases}.$$
and the following \((\sigma, \mathcal{H})\)-coordinate on \(R_1\)

\[
\sigma = v \int_{t_1^*}^t \frac{ds}{l(s)^2}, \quad \mathcal{H} = v/2.
\]

For a point \((\sigma_0, \mathcal{H}_0) \in R_1\), we denote by \((\sigma_n, \mathcal{H}_n) = F_n(\sigma_0, \mathcal{H}_0)\).

**Proposition 3.1.** Under the assumptions of Theorem 1, there exist constants \(V_* = V_*(\dot{f}_1, \dot{f}_2, c, E)\) and \(\bar{I} = \bar{I}(\dot{f}_1, \dot{f}_2, c, E) < 1\) such that for any \((\sigma_0, \mathcal{H}_0) \in R_1\) with \(\mathcal{H}_0 > V_*\) following the Upper Route we have

\[
\frac{1}{1 - \dot{f}_1} \leq \frac{\mathcal{H}_1}{\mathcal{H}_0} \leq \frac{1 - \dot{f}_2}{1 - \dot{f}_1},
\]

and for any \((\sigma_0, \mathcal{H}_0) \in R_1\) with \(\mathcal{H}_0 > V_*\) following the Lower Route we have

\[
\frac{\dot{f}_2}{\dot{f}_1} \leq \frac{\mathcal{H}_1}{\mathcal{H}_0} \leq \frac{1 - \dot{f}_2}{1 - \dot{f}_1}.
\]

The proof of Proposition 3.1 follows straight from the adiabatic normal forms derived in Sect. 3.2, which are based on the adiabatic coordinates presented in Sect. 3.1. We first describe in Sect. 3.1 how the energy of the ball remains almost constant when it stays in one chamber and we then describe in Sect. 3.2 how it changes when the ball jumps from left to right and vice versa. Finally in Sect. 3.3 we briefly describe the divided phase cylinders that the adiabatic normal forms sit on and give a proof of Proposition 3.1. The adiabatic coordinates and the normal forms in this Section follow from those derived in [27] if we think of the right slit in [27] to be always on the floor, thus we omit the derivation and refer the readers to [27] for details.

### 3.1. The adiabatic coordinates.

In this section, we show that the energy of the ball remains almost constant when it stays in either the left or right chamber. We tailor the adiabatic coordinates from [27] for our case. We have three sets of adiabatic coordinates: one in the upper left chamber, one in the lower left chamber and one in the right chamber. We introduce the notation \(\psi = O_s(v^{-n})\) when \(v^n \psi\) is bounded and so are its derivatives up to \(s\)-order.

First, we assume that the ball hits the slit from above in the upper left chamber and it does not switch chamber in the next collision. We need the normalizing constant \(L_* = \int_0^2 \frac{ds}{l(s)^2}\). We have the following adiabatic coordinate \((\theta, I)\) in the upper left chamber.

**Lemma 3.1.** For \((t, v) \in \{0 \leq t < t_1^*, t_2^* < t \leq 2, \} \setminus R_1 \cup R_3 \cup \tilde{R}_1 \cup \tilde{R}_2\) and \(v \gg 1\), there exists an adiabatic coordinate \((\theta, I) = \Psi_U(t, v) \in \mathbb{R}/2\mathbb{Z} \times \mathbb{R}_+\) such that

\[
\theta_{n+1} = \theta_n + \frac{2}{I_n} + O_3 \left(\frac{1}{I_n^3}\right), \quad I_{n+1} = I_n + O_3 \left(\frac{1}{I_n^3}\right).
\]

In fact, \(\theta = \theta(t) = \frac{2}{L_*} \int_0^t \frac{ds}{l(s)^2} \mod 2, I = I(t, v) = \frac{L_*}{2} \left(lv + \dot{l}t + \frac{l^2}{3v}\right)\).
Similarly, if the ball hits the slit from below in the lower left chamber and it does not
switch chamber at the next collision. We introduce \( m(t) \) as the “distance” from the slit
to the floor of the table

\[
m(t) = \begin{cases} 
-f(t) & 0 \leq t \leq t_1^*, t_2^* \leq t \leq 2 \\
-1 & \text{otherwise}
\end{cases}
\]

We need the normalising constant \( \mathcal{M}_* = \int_0^2 m(s)^{-2} \, ds \). We have the following adiabatic
coordinate \((\xi, J)\) in the lower left chamber.

**Lemma 3.2.** For \((t, v) \in \{0 \leq t < t_1^*, t_2^* < t \leq 2\} \setminus \{R_1 \cup R_2 \cup \tilde{R}_1 \cup \tilde{R}_2\} \) and \( v \ll -1 \),
there exists an adiabatic coordinate \((\xi, J) = \Psi_L (t, v) \in \mathbb{R}/2\mathbb{Z} \times \mathbb{R}^+ \) such that

\[
\xi_{n+1} = \xi_n + 2 \frac{J_n}{J_n} + O_3 \left( \frac{1}{J_n^3} \right), \quad J_{n+1} = J_n + O_3 \left( \frac{1}{J_n^3} \right).
\]

In fact, \( \xi = \xi(t) = \frac{2}{M_*} \int_0^t \frac{ds}{m(s)^2} \mod 2, J = J(t, v) = \frac{M_*}{2} \left( mv + \dot{m} + \frac{m^2 \ddot{m}}{3v} \right) \).

Finally, let us present the adiabatic coordinate \((\theta, H)\) in the right chamber.

**Lemma 3.3.** For \((t, v) \in \{t_1^* < t < t_2^*\} \setminus \{R_1 \cup R_2 \cup \tilde{R}_1 \cup \tilde{R}_2\} \) and \( v \gg 1 \),
there exists an adiabatic coordinate \((\theta, H) = \Psi_F (t, v) \in \mathbb{R}/2\mathbb{Z} \times \mathbb{R}^+ \) such that

\[
\theta_{n+1} = \theta_n + \frac{2}{L_n}, \quad H_{n+1} = H_n.
\]

In fact, \( \theta = \theta(t) = \frac{2}{L_*} \int_0^t \frac{ds}{l(s)^2} \mod 2, H = H(t, v) = L_* v / 2. \)

### 3.2. The adiabatic normal forms.

The adiabatic coordinates presented in the previous section show that the energy of the ball remains almost constant if the ball stays in
one chamber and in this section we examine the behavior of the ball when it switches
chambers. We tailor the adiabatic normal forms from [27] for our case and all the
adiabatic normal forms are valid for sufficiently large initial energies \( v_0 > V_* \).

During a complete revolution (\( \Delta t = 2 \)), the ball switches chambers twice, thus we
decompose the Poincaré map \( \mathcal{F} \) on \( R_1 \) (i.e. the first return map to the right floor) into
two maps \( P \) between the two singular strips \( R_1, R_2 \) in the right/left chambers. Our goal
is to obtain an adiabatic normal form for each \( P \) as follows

\[
P = G + H + R
\]

where \( G \) is the linear part, \( H \) is the first-order correction and \( R \) is the error term of second order. We recall that the ball may follow either the Upper or the Lower Route, thus we will end up with multiple adiabatic normal forms which share the same format
described above and differ only by constants depending on \( f \) up to the second derivative.
The linear parts \( G \) in these normal forms will play the most important role in the future
sections.
3.2.1. The upper route In the Upper Route, $\mathcal{F} = \mathcal{F}^U$ decomposes into the following two maps $P^U_{12} \circ P^U_{21}$ where $P^U_{12} : R_1 \to R_2^+$ sends the ball from the right floor to the upper left chamber and $P^U_{21} : R_2^+ \to R_1$ sends it back.

We recall the $(\sigma, \mathcal{H})$-coordinate on $R_1$ (defined above Proposition 3.1) and we introduce a new pair of variables $(\tau, \mathcal{I})$ on $R_2^+$

$$\tau = I(\theta - \theta^*_2), \quad \mathcal{I} = \frac{I}{\mathcal{L}_*}$$

where $\theta^*_2 = \theta(t^*_2) = \frac{2}{\mathcal{L}_*} \int_0^{t^*_2} \frac{ds}{l(s)^2}$.

We need the following constants $(i = 1, 2)$:

$$\Delta_i = \frac{1}{2} \frac{l_1^+ l_1^-}{l_1^+} \left( l_1^- \mathcal{L}_* - l_1^+ \mathcal{L}_* \right),$$

$$\Delta_i' = \frac{1}{8} \left( l_1^+ \right)^2 \left( l_1^- \mathcal{L}_* - l_1^+ \mathcal{L}_* \right),$$

$$\Delta_i'' = \frac{1}{24} l_1^- l_1^+ \left( l_1^- \mathcal{L}_* - l_1^+ \mathcal{L}_* \right),$$

where $l_1^+ = l(t^*_1)$, etc.

Now we present the adiabatic normal forms for the Upper Route $P^U_{12}$ and $P^U_{21}$.

**Proposition 3.2** (Upper Route). Suppose that $(\sigma, \mathcal{H}) \in R_1$ with $\mathcal{H} > V_*$ and that $f_2 \lesssim (\mathcal{L}_* \mathcal{H}(\theta^*_2 - \theta^*_1) - \sigma)_2 \lesssim 2 - f_2$,

where $\lesssim$ means the inequality holds up to an error of order $O(\frac{1}{\mathcal{H}})$, and $(\bullet)_2 = \bullet \mod 2$.

Then the Poincaré map $P^U_{12} : R_1 \to R_2^+$ is given by

$$(\tau, \mathcal{I}) = G_{12}^U(\sigma, \mathcal{H}) + H_{12}^U(\sigma, \mathcal{H}) + O_3(\mathcal{H}^{-2})$$

where

$$G_{12}^U(\sigma, \mathcal{H}) = \left( -\frac{1}{l_2} \left( \mathcal{L}_* \mathcal{H}(\theta^*_2 - \theta^*_1) - \sigma \right)_2 + \frac{1 + l_2}{l_2}, l_2 \mathcal{H} + \Delta_2(\tau - 1) \right)$$

and

$$H_{12}^U(\sigma, \mathcal{H}) = \left( 0, \Delta'_2(\tau - 1)^2/\mathcal{H} + \Delta''_2/\mathcal{H} \right).$$

Similarly, suppose that $(\tau, \mathcal{I}) \in R_2^+$ with $\mathcal{I} > V_*$. Then the Poincaré map $P^U_{21} : R_2^+ \to R_1$ is given by

$$(\sigma, \mathcal{H}) = G_{21}^U(\tau, \mathcal{I}) + H_{21}^U(\tau, \mathcal{I}) + O_3(\mathcal{I}^{-2})$$

where

$$G_{21}^U(\tau, \mathcal{I}) = \left( -l_1 \left( \mathcal{L}_* \mathcal{I}(2 + \theta^*_1 - \theta^*_2) - \tau \right)_2 + 1 + l_1, \frac{1}{l_1} \mathcal{I} + \Delta_1(\sigma - 1) \right)$$

and

$$H_{21}^U(\tau, \mathcal{I}) = \left( 0, \Delta'_1(\sigma - 1)^2/\mathcal{I} + \Delta''_1/\mathcal{I} \right).$$
3.2.2. The lower route  

In the Lower Route, for technical reasons we need to differentiate whether the ball travels the long trajectory (hitting the ceiling first) or the short one (hitting the floor first) when it switches chambers (c.f. Fig. 2), and consequently \( \mathcal{F} = \mathcal{F}^L \) decomposes into four possible combinations \( P_{21}^{Ll} \circ P_{12}^{Ll}, P_{21}^{Ll} \circ P_{12}^{Ls}, P_{21}^{Ls} \circ P_{12}^{Ll} \) and \( P_{21}^{Ls} \circ P_{12}^{Ls} \) where \( P_{12}^{Ll/Ls} : R_1 \rightarrow R_2^- \) sends the ball from the right floor to the lower left chamber via the Long/Short Entry while \( P_{21}^{Ll/Ls} : R_2^- \rightarrow R_1 \) sends it back via the Long/Short Exit.

We introduce a new pair of variables \((\rho, J)\) on the lower singular strip \( R_2^- \), which is the counterpart of \((\tau, I)\) on \( R_2^+ \), as follows

\[
\rho = J(\zeta - \zeta^*_2), \quad J = J/M^*_2
\]

where \( \zeta^*_i = \frac{2}{M^*_2} \int_0^{t^*_i} ds \frac{m(s)}{m(s)^2} \).

First we describe the Long/Short Entry from the right floor to the left lower chamber.

We need the following constants for the Long Entry:

\[
\begin{align*}
\kappa_l &= \frac{1}{2} m_+ \dot{m}_+ \\
\kappa'_l &= \frac{1}{24} m_+^2 \ddot{m}_+ \\
\kappa''_l &= \frac{1}{8} m_+^2 \dddot{m}_+
\end{align*}
\]

where all functions above take values at the moment \( t = t^*_2 \).

**Proposition 3.3** (Long Entry). Assume that \((\sigma, \mathcal{H}) \in R_1^+ \) with \( \mathcal{H} > V_2 \) and that

\[
\{ \mathcal{L}_s \mathcal{H}(\theta_2^* - \theta_1^*) - \sigma \} \geq 2 - f_2
\]

Then the Poincaré map \( P_{12}^{Ll} : R_1 \rightarrow R_2^- \) is given by

\[
(\rho, J) = G_{12}^{Ll}(\sigma, \mathcal{H}) + H_{12}^{Ll}(\sigma, \mathcal{H}) + O_3(\mathcal{H}^{-2})
\]

where

\[
G_{12}^{Ll}(\sigma, \mathcal{H}) = \left( -\frac{1}{f_2} \{ \mathcal{L}_s \mathcal{H}(\theta_2^* - \theta_1^*) - \sigma \} + \frac{f_2 + 2}{f_2}, \frac{f_2 + 2}{f_2}, f_2 \mathcal{H} + \kappa_l (\rho - 1) \right)
\]

and

\[
H_{12}^{Ll}(\sigma, \mathcal{H}) = \left( 0, \kappa'_l \frac{\rho - 1}{\mathcal{H}} - \kappa''_l (\rho - 1)^3 \right).
\]
Meanwhile we need the following constants to describe the Short Entry:

\[
\kappa'_s = \frac{1}{8} m_+^2 \dot{m}_+ \\
\kappa''_s = -\frac{1}{24} m_+ m_+ \ddot{m}_+ 
\]

where all functions above take values at the moment \( t = t^*_2 \).

**Proposition 3.4** (Short Entry). Assume that \((\sigma, \mathcal{H}) \in R_1\) with \( \mathcal{H} > V_* \) and that

\[
f_2 \gtrsim \{ \mathcal{L}_* \mathcal{H}(\theta^*_2 - \theta^*_1) - \sigma \}_2.
\]

Then the Poincaré map \( P_{Ls}^{12} : R_1 \rightarrow R^-_2 \) is given by

\[
(\rho, \mathcal{J}) = G_{Ls}^{12}(\sigma, \mathcal{H}) + H_{Ls}^{12}(\sigma, \mathcal{H}) + O(\mathcal{H}^{-2})
\]

where

\[
G_{Ls}^{12}(\sigma, \mathcal{H}) = \left( -\frac{1}{f_2} \{ \mathcal{L}_* \mathcal{H}(\theta^*_2 - \theta^*_1) - \sigma \}_2 + 1, f_2 \mathcal{H} + \kappa_l(\rho - 1) \right)
\]

and

\[
H_{ULII}^{12}(\sigma, \mathcal{H}) = \left( 0, -\kappa'_s (\rho - 1)^2 \mathcal{H}^{-1} - \frac{\kappa''_s}{\mathcal{H}} \right)
\]

Next we describe the Long/Short Exit from the lower left chamber to the right floor. We need the following constants for the Long Exit:

\[
\chi_l = -\frac{1}{2} \dot{m}_- / m_-
\]

\[
\chi'_l = \frac{1}{8} \ddot{m}_-(1 - \frac{1}{3} m_-^2)
\]

\[
\chi''_l = -\frac{1}{4} \dddot{m}_-
\]

where all functions above take values at the moment \( t = t^*_1 \).

**Proposition 3.5** (Long Exit). Assume that \((\rho, \mathcal{J}) \in R^-_2\) with \( \mathcal{J} > V_* \) and that

\[
\{ \mathcal{M}_* \mathcal{J}(2 + \xi^*_1 - \xi^*_2) - \rho \}_2 \gtrsim 1.
\]

Then the Poincaré map \( P_{Ll}^{21} : R^-_2 \rightarrow R_1 \) is given by

\[
(\sigma, \mathcal{H}) = G_{Ll}^{21}(\rho, \mathcal{J}) + H_{Ll}^{21}(\rho, \mathcal{J}) + O(\mathcal{H}^{-2})
\]

where

\[
G_{Ll}^{21}(\rho, \mathcal{J}) = \left( -f_1 \{ \mathcal{M}_* \mathcal{J}(2 + \xi^*_1 - \xi^*_2) - \rho \}_2 + 2 + f_1, \frac{1}{f_1} \mathcal{J} + \chi_l(\sigma - 2) \right)
\]

and

\[
H_{ULII}^{21}(\rho, \mathcal{J}) = \left( 0, \frac{\chi'_l}{\mathcal{J}} + \chi''_l \frac{\sigma - 1}{\mathcal{J}} - \frac{\chi''_l}{2} \frac{(\sigma - 1)^2}{\mathcal{J}} \right).
\]
Finally we need the following constants to describe the Short Exit:

\[ \chi_s' = \frac{1}{4} \ddot{m}_- \]
\[ \chi_s'' = \frac{1}{24} m_-(m_2 \dddot{m}_- - 3 \dddot{m}_-) \]

where all functions above take values at the moment \( t = t^*_1 \).

**Proposition 3.6** (Short Exit). Assume that \((\rho, J) \in R^-_2\) with \( J > V_*\) and that

\[ \{M_* J(2 + \xi^*_1 - \xi^*_2) - \rho\}_2 \lesssim 1. \]

Then the Poincaré map \( P^{Ls}_{21} : R^-_2 \to R_1 \) is given by

\[
(\sigma, H) = G^{Ls}_{21}(\rho, J) + H^{Ls}_{21}(\rho, J) + O_3(J^{-2})
\]

where

\[
G^{Ls}_{21}(\rho, J) = \left(-f_1 \{M_* J(2 + \xi^*_1 - \xi^*_2) - \rho\}_2 + f_1, \frac{1}{f_1} J + \chi_s \sigma\right)
\]

and

\[
H^{Ls}_{21}(\rho, J) = \left(0, \frac{\chi_s'}{2} \sigma - \frac{1}{J} + \frac{\chi_s''}{2} \frac{(\sigma - 1)^2}{J} - \frac{\chi_s'''}{J}\right).
\]

We note that the derivatives of the linear parts \( G \) of \( P^{Ll}_{12} \) and \( P^{Ls}_{12} \) are identical, and so are those of \( P^{Ll}_{21} \) and \( P^{Ls}_{21} \).

3.3. The divided phase cylinder. In this section we describe the general behavior of our system using the normal forms derived in the previous section and give a proof of Proposition 3.1. In Section 3.3.1 we describe the fundamental domains (“boxes”) on the phase cylinders. Then in Section 3.3.2 we provide a proof of Proposition 3.1 using the adiabatic normal forms in Sect. 3.2.

3.3.1. The fundamental domains Let the ball enter the left chamber in the next collision. Depending on the ball’s choice of entering the upper or lower left chamber, the \((\sigma, H)\)-phase cylinder on the right floor \( R_1 \) is divided into three parts: the points in

\[ \mathcal{L}^L_{en} := \{L_* H(\theta^*_2 - \theta^*_1) - \sigma \}_2 \gtrsim 2 - f_2 \]

enter the lower left chamber following the Long Route, the points in

\[ \mathcal{L}^S_{en} := \{L_* H(\theta^*_2 - \theta^*_1) - \sigma \}_2 \lesssim f_2 \]

enter the lower left chamber following the Short Route, and the points in

\[ \mathcal{U}_{en} := \{f_2 \lesssim L_* H(\theta^*_2 - \theta^*_1) - \sigma \}_2 \lesssim 2 - f_2 \]

enter the upper left chamber (c.f. Fig. 3 on the top).

Next when the ball is already in the upper left chamber \( R^+_2 \) in \((\tau, I)\)-cylinder, it then has no choice but to return to the floor on the right (c.f. Fig. 3 on the bottom left). However,
if we suppose the ball is in the lower left chamber \( R_2^- \) in the \((\rho, J)\)-cylinder, then it may return to the right floor following either the Long Exit for points in \( \mathcal{L}_{ex}^l := \{ M_*(J(2 + \xi_1^* - \xi_2^*) - \rho \} \geq 1 \) or the Short Exit for points in \( \mathcal{L}_{ex}^s := \{ M_*(J(2 + \xi_1^* - \xi_2^*) - \rho \} \leq 1 \) (c.f. Fig. 3 on the bottom right).

Note that each of \( \mathcal{L}_{en/ax}, \mathcal{L}_{en/sx} \) and \( \mathcal{U}_{en/ax} \) consists of almost identical connected components (“boxes”) indexed by integers \( m \in \mathbb{N} \): for instance, a connected component \( \mathcal{U}_{en,m} \) of \( \mathcal{U}_{en} \) is defined by (see Fig. 3)

\[ f_2 + m \lesssim f_2 \mathcal{L}_{ex}(\theta_2^* - \theta_1^*) - \sigma \lesssim 2 - f_2 + m. \]

**Definition 1 (Fundamental domain).** We call a connected component (“box”) \( \mathcal{U}_{en,m} \) of \( \mathcal{U}_{en} \) a fundamental domain for the map \( P_{12}^U \). Similarly, we can define the fundamental domains for all the other half-revolution maps.

When jumping from one fundamental domain in a phase cylinder to another one in another phase cylinder, an ascending (viewed from left to right) strip is mapped into a narrow but long descending (viewed from left to right) strip. More precisely, the top boundary \( \{ \mathcal{L}_{ex}(\theta_2^* - \theta_1^*) - \sigma \} \approx 2 - f_2 \) and the bottom boundary \( \{ \mathcal{L}_{ex}(\theta_2^* - \theta_1^*) - \sigma \} \approx f_2 \) of \( \mathcal{U}_{en} \) are mapped by \( P_{12}^U \) into the left boundary \( \{ \tau = 0 \} \) and the right boundary \( \{ \tau = 2 \} \) respectively, etc. (c.f. Fig. 3).

### 3.3.2. Energy evolution after one period

Under the assumption \( f_1 < f_2 \), it is easy to observe that the ball accelerates when it follows the Lower Route and decelerates when it follows the Upper Route.

We collectively denote the decelerating chambers (upper chambers) as

\[ S_{-1,V_\ast} = \bigcup_{m: \mathcal{U}_{en,m} \subset [0,2) \times (V_\ast, \infty)} \mathcal{U}_{en,m} \]

and the accelerating chambers (lower chambers) as

\[ S_{+1,V_\ast} = [0, 2] \times (V_\ast, \infty) \setminus S_{+1,V_\ast}. \]

For ease of presentation, we may drop the term \( V_\ast \) in the notations \( S_{-1,V_\ast} \) and \( S_{+1,V_\ast} \).

We collect all the half-revolutions and denote as

\[ \mathcal{P} = \{ P_{12}^U, P_{21}^U, P_{12}^{Ll}, P_{21}^{Ll}, P_{12}^{Ls}, P_{21}^{Ls}, P_{21}^{Ll} \}. \]

From the discussion in Sect. 3.2 half-revolution \( P \in \mathcal{P} \) from one strip to another is decomposed into a linear part \( G \), a first-order correction \( H \) and a second-order error term:

\[ P = G + H + \mathcal{O}_3. \]

Finally we are ready to prove Proposition 3.1.
Proof of Proposition 3.1. We observe that in the definition of the Poincare maps $P$ the parameters $\Delta_1, \Delta_2, \kappa_1$, etc. do not depend on the energy of the ball, and hence it follows that for some number $D_0 = D_0(f_1, f_2) > 0$ we have for all $(\sigma_0, \mathcal{H}_0) \in S_{+1}$

$$\left(\frac{f_2}{f_1}\right) \mathcal{H}_0 - D_0 \leq \mathcal{H}_2 \leq \left(\frac{f_2}{f_1}\right) \mathcal{H}_0 + D_0,$$

and respectively for the upper route $(\sigma_0, \mathcal{H}_0) \in S_{-1}$

$$\left(\frac{1 - f_2}{1 - f_1}\right) \mathcal{H}_0 - D_0 \leq \mathcal{H}_2 \leq \left(\frac{1 - f_2}{1 - f_1}\right) \mathcal{H}_0 + D_0,$$

Note that since $E > 0$ then there exists a number $a = a(E) > 1$, so that $f_2/f_1 > a$. In a similar way we have $\frac{1 - f_2}{1 - f_1} < a_1$, for some $a_1 = a(E, c) < 1$. Thus, if we take a number

---

**Fig. 3.** The divided phase cylinder
t close to 1 and \( V_\ast = V_\ast (\hat{f}_1, \hat{f}_2) \) large enough, we will arrive at (1) and (2). Note that \( t \) will be dependant on \( \mathcal{E} \) and \( c \) and not on \( f_1, f_2 \).

We note that the normal forms and all the estimates obtained in the previous sections are valid only for large values of \( V_\ast \). Therefore, when the energy drops below \( V_\ast \) our analysis may collapse. To avoid technical problems connected with this, we modify our dynamics near the threshold \( V_\ast \) in a way that whenever the ball’s energy falls near \( V_\ast \), it will be pushed up in the next period by following the accelerating Lower Route.

More precisely, we choose a number \( V_0 \gg V_\ast \) and for all the decelerating fundamental domains \( U_{en,m} \), that lie entirely below \( V_0 \) on the \( (\sigma, \mathcal{H}) \)-phase cylinder, we replace \( P_{12}^U \) by \( P_{12}^{L_1} \) and elsewhere untouched, and hence we obtain a modified system as follows:

\[
P_0(x, y) = \begin{cases} 
P_{12}^{L_1}(x, y), & (x, y) \in U_{en,m}, \text{ for } m \text{ with} \\
P(x, y), & \text{otherwise.}
\end{cases}
\]

(3)

From now on we will only consider the modified dynamical system \( \mathcal{F}_0 \). We will first prove the exponential acceleration result for the modified system \( \mathcal{F}_0 \) and then in the last Section we will show that the velocity of a considerable portion of these exponentially accelerating orbits never drops below \( V_\ast \), and hence they are the also the true orbits of the original unmodified system \( \mathcal{F} \).

4. Uniform Hyperbolicity

In this section we show that under the assumptions in Theorem 1 our system is uniformly hyperbolic in the sense that all the six half-revolution maps \( P \) share a common unstable invariant cone, provided that \( \hat{f}_1, \hat{f}_2 \) are sufficiently large.

**Proposition 4.1 (Invariant Cone).** Assume that \( |\hat{f}_1|, |\hat{f}_2| > \hat{f}_\ast \). There exists \( V_\ast \gg 1 \) such that if the initial energy \( v_0 > V_\ast \), then \( P_{12}^U, P_{21}^U, P_{12}^{L_1}, P_{21}^{L_1} \) share a common invariant unstable cone \( \mathcal{C}_u^* \). Moreover, the expansion rates \( \lambda_{12}^U, \lambda_{12}^{L_1} \) of the first half-revolution maps \( P_{12}^U, P_{12}^{L_1} \) in the cone are of order \( \mathcal{O}(|\hat{f}_2|) \), and the expansion rates \( \lambda_{21}^U, \lambda_{21}^{L_1} \) of the second half-revolution maps \( P_{21}^U, P_{21}^{L_1} \) are of order \( \mathcal{O}(|\hat{f}_1|) \).

The proof of Proposition 4.1 proceeds as follows. We show that the linear parts \( G \) of the six half-revolution maps \( P \) share a common invariant cone and then we transfer it to \( P \)’s themselves through the robustness of uniform hyperbolicity under \( C^1 \)-perturbations.

We recall that the Long Enter and the Short Enter have the same derivatives for their linear parts, and so do the Long Exit and the Short Exit, thus we use the same notations for their derivatives respectively: \( DG_{12}^U := DG_{12}^{L_1} = DG_{21}^{L_1} \) and \( DG_{21}^U := DG_{21}^{L_1} = DG_{21}^{L_2} \).

Moreover, these derivatives take the following forms:

\[
DG_{12}^U = \left( \frac{1}{l_2} - \frac{L_s(\theta_1^* - \theta_2^*)}{l_2^2}, \frac{\Delta_1}{l_2} - \frac{\Delta_2 L_s(\theta_1^* - \theta_2^*)}{l_2^2} \right),
\]

\[
DG_{21}^U = \left( \frac{l_1}{l_1} - \frac{\Delta_1 L_s(2 + \theta_1^* - \theta_2^*)}{l_1}, \frac{1}{l_1} - \frac{\Delta_1 L_s(2 + \theta_1^* - \theta_2^*)}{l_1} \right)
\]
\[ DG_{12}^L = \begin{pmatrix} \frac{1}{f_2} & -\mathcal{L}_s(\theta^*_2 - \theta^*_1) \\ \frac{\kappa L}{f_2} & f_2 - \frac{\kappa f_2}{f_2} \end{pmatrix}, \]

and

\[ DG_{21}^U = \begin{pmatrix} f_1 & -\mathcal{M}_s(2 + \xi^*_1 - \xi^*_2)f_1 \\ \chi_l f_1 & \frac{1}{f_1} - \chi_l \mathcal{M}_s(2 + \xi^*_1 - \xi^*_2)f_1 \end{pmatrix}. \]

First we observe that \( \det DG_{12}^U = \det DG_{21}^U = \det DG_{12}^L = \det DG_{21}^L = 1. \)

Now we show that our linear maps \( G_{12}^U, G_{21}^U, G_{12}^L, G_{21}^L \) share a common unstable invariant cone provided that \( \dot{f}_1, \dot{f}_2 \) are sufficiently large. We prove the claim in two steps: first we show in Lemma 4.1 that the unstable eigenvector of a hyperbolic matrix is almost vertical and that the stable eigenvector remains a positive angular distance away from the vertical direction, provided that the bottom entries are significantly larger than the top entries; then we show in Proposition 4.2 that the unstable cones around the almost vertical unstable eigenvectors of our linear maps \( G_{12}^U, G_{21}^U, G_{12}^L, G_{21}^L \) have a nontrivial intersection, which is the common invariant unstable invariant cone we aim for.

**Lemma 4.1.** Let \( A_n \in SL(2, \mathbb{R}) \)

\[ A_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \]

with nonzero entries. Assume that \( a_n, b_n \) are uniformly bounded, \( b_n/a_n \to M \) for some constant \( M \neq 0 \) and \( c_n, d_n \to \infty \) as \( n \to \infty \). Then the unstable eigenvector \( e_u \to (0, 1) \) and the stable eigenvector \( e_s \to (-M, 1)/\sqrt{M^2 + 1} \) as \( n \to \infty \).

**Proof.** The unstable eigenvalue of the matrix \( A_n \) is

\[ \lambda_n = \frac{(a_n + d_n) + \sqrt{(a_n + d_n)^2 - 4}}{2}, \]

and the eigenvectors (not necessarily unit vectors) are

\[ v_u^n = (b_n, \lambda_n - a_n), \quad v_s^n = (b_n, 1/\lambda_n - a_n). \]

We note, as \( n \to \infty \) that

\[ \frac{b_n}{\lambda_n - a_n} = \frac{2b_n}{d_n - a_n + \sqrt{(a_n + d_n)^2 - 4}} \to 0 \]

and that

\[ \frac{b_n}{1/\lambda_n - a_n} \to -M, \]

which concludes our proof.

**Proposition 4.2.** There exists \( \dot{f}_* \gg 1 \) such that if \( |\dot{f}_1|, |\dot{f}_2| > \dot{f}_* \), then \( G_{12}^U, G_{21}^U, G_{12}^L, G_{21}^L \) share a common invariant unstable cone \( C_u^* = C_u(\dot{f}_*) \). Moreover, the expansion rates \( \lambda^U_{12}, \lambda^L_{12}^{1/s} \) of the first half-revolution maps \( G_{12}^U, G_{21}^U, G_{12}^L, G_{21}^L \) are of order \( \mathcal{O}(1) \), and the expansion rates \( \lambda^U_{21}, \lambda^L_{21}^{1/s} \) of the second half-revolution maps \( G_{12}^U, G_{21}^U, G_{21}^L \) are of order \( \mathcal{O}(1) \).
Proof. First we note that the top entries of the matrices $DG_{12}^U$, $DG_{21}^U$, $DG_{12}^L$, $DG_{21}^L$ are of order 1 and that the bottom entries contain $\Delta_2, \Delta_1, k_\ell$ and $\chi_\ell$ respectively. We recall that

$$\Delta_1 = -\frac{1}{2} l_1^l, \quad \Delta_2 = \frac{1}{2} l_2^l, \quad k_\ell = \frac{1}{2} f_2 f_\ell, \quad \chi_\ell = -\frac{1}{2} f_\ell,$$

which can be made arbitrarily large by choosing $|f_1|, |f_2| \gg 1$.

Thus by Lemma 4.1 we know that if $|f_1|, |f_2| \to \infty$ the unstable eigenvectors of the four matrices $DG_{12}^U$, $DG_{21}^U$, $DG_{12}^L$, $DG_{21}^L$ tend to the vertical direction while the stable eigenvectors tend to $e^s_1 = (\mathcal{L}_s(\theta_2^* - \theta_1^*), 1)$, $e^s_2 = (\mathcal{L}_s(2 + \theta_1^* - \theta_2^*), 1)$, $e^s_3 = (\mathcal{L}_s(\theta_2^* - \theta_1^*), 1)$ and $e^s_4 = (\mathcal{M}_s(2 + \xi_1^* - \xi_2^*), 1)$ respectively.

We recall from Sect. 3.1 that the constants $\mathcal{L}_s, \mathcal{M}_s, \theta_1^*, \theta_2^*$ are defined by the integrals of $1/f^2$ and $(1 - f)^2$ over $(0, 2)$ or $(t_1^*, t_2^*)$ etc., thus they are uniformly bounded in magnitude by some constant $M = M(c)$, where $c$ is the upper bound of the distance that the slit $f(t)$ remains from the ceiling/floor as in Theorem 1. Therefore the four vectors $e^1, e^2, e^s_3, e^s_4$ all remain a uniform angular distance $\Theta = \Theta(c)$ from the vertical direction. On the other hand, there exists $\bar{f}_s \gg 1$ such that if $|f_1|, |f_2| > \bar{f}_s$, the corresponding stable eigenvectors of $DG_{12}^U$, $DG_{21}^U$, $DG_{12}^L$, $DG_{21}^L$ all lie within an angular distance $\Theta_1 = \Theta_1(c)$ to the horizontal direction, whereas the corresponding unstable eigenvectors all lie within an angular distance $\Theta_2 = \Theta_2(\bar{f}_s)$ to the vertical direction, and

$$\Theta_1 + 3\Theta_2 < \frac{\pi}{2}, \quad \frac{1}{2} \left( \frac{\pi}{2} - \Theta_1 + \Theta_2 \right) > 4\Theta_2, \quad (4)$$

so that for each map $G$ we can choose the opening gauge $k = k(2\Theta_2)$ which gives a cone $C_{u,k}(G)$ of angular width $2\Theta_2$ around the unstable eigenvector of $G$. We note that (4) guarantees that $k > 1$ so the cones are indeed expanding, and (4) can be always achieved since $\Theta_1 < \pi/2$ and $\Theta_2 \to 0$ as $f_s \to \infty$. We also observe that such choice of $k$ guarantees that all the four cones $C_{u,k}(G)$’s of the four maps $G$’s have a nontrivial intersection $C_u$ (c.f. Fig. 4 on the left: the stable eigenvectors lie in the green region, the unstable eigenvectors lie in the blue region, the dotted lines cut out the region where $k > 1$).

We claim that this nontrivial intersection $C_u$ is invariant under the four maps $G_{12}^U$, $G_{21}^U$, $G_{12}^L$, $G_{21}^L$. Indeed, for any $\varrho$ in the phase cylinder and any vector $d\varrho \in C_u(\varrho)$, any map $G$ among the four maps $G_{12}^U$, $G_{21}^U$, $G_{12}^L$, $G_{21}^L$ will map $d\varrho$ closer to the corresponding unstable eigenvector $e_u$, thus still remains in $C_u$ (c.f. Fig. 4 on the right).

It follows from a straightforward computation that for each map $G$ its expansion rate $\lambda_G$ in the cone $C_u^*$ has a lower bound by $\frac{k_u^u - 1}{k_u + 1}$, where $\lambda_u$ is the unstable eigenvector of $DG$. It follows from a quick observation that $\lambda_u \sim \mathcal{O}(\bar{f}_{1/2})$, thus the expansion rates $\lambda_G$’s are of the same order.

Finally, we recall from Sect. 3.2 that the half-revolution maps $P$ are $C^2$ perturbations of their linear parts $G$ if the initial energy $\mathcal{H}_0$ are sufficiently large and the perturbation is of order $\mathcal{O}(\mathcal{H}_0^{-1})$. The classical literature on the robustness of uniform hyperbolicity under $C^1$ perturbations (for example c.f. Appendix A in [24]) guarantees that the map $P$’s also have the common invariant unstable cone with the desired properties in Proposition 4.2. With an abuse of notation, we also denote the common invariant unstable cone as $C_u^*$.\[\square\]
5. Growth Lemmas and Deviation Estimates

We observe from Sect. 4 that our system enjoys strong uniform expansion for \( \dot{f}_1, \dot{f}_2 \gg 1 \). If we start with an “expanding” curve on the phase cylinder, then on the next iterate it will get considerably stretched so that a portion of nearly \( 1 - f_2 \) of the points on the image will enter the decelerating upper chamber and the other portion of nearly \( f_2 \) will enter the accelerating lower chamber (c.f. the top phase cylinder in Fig. 3). However, this estimate cannot be carried out forever as the phase cylinders are divided by infinitely many singularity lines which cut long curves short during iterates and some “bad” short pieces may remain short for a long time. Nonetheless, we may gain some hope from the strong uniform hyperbolicity of the system by showing that these “bad” pieces cannot show up too often and we can revive the analysis with “long enough” pieces (c.f. Definition 3 for long curves).

First of all, let us be precise on the definition of the “expanding curves”.

**Definition 2 (Unstable curves).** We say \( \gamma \) is an unstable curve if it is \( C^2 \), it lies above the threshold \( V^* \) and the slope of any point in \( \gamma \) lies in the unstable cone \( C^* \), i.e. \( \forall \Theta = (x, y) \in \gamma, J_{\gamma, \Theta} \in C^* \) where \( J_{\gamma} \) is the directional derivative.

The goal of this section is to provide detailed controls on the size of the (“bad”) short unstable curves (c.f. the preparatory lemmas in Sect. 5.1) and the time it takes for an unstable curve to land on long curves, i.e. Lemmas 5.5 and 5.6, which will later be used towards the proof of the Main Theorem.

This section is organized as follows. In Sect. 5.1 we prove some preparatory growth lemmas on the size of short unstable curves. In Sect. 5.2 we introduce the notion of a “long curve” (c.f. Definition 3) and provide explicit estimates on the time it takes for a short unstable curve to grow into a long curve. Then in Sect. 5.3 we prove a quantitative growth lemma and a large deviation estimate which have a certain delay: namely, for some large \( N_0 \) we first iterate the unstable curve \( N_0 \) many times and then we start measuring the size and the growth of unstable curves. The magnitude of the delay time \( N_0 \) and its necessity will be made clear in Sect. 6.

We employ a similar strategy to control the growth of unstable curves which has been used to handle hyperbolic systems with singularities such as the Sinai billiard, where the growth lemmas on unstable curves play a vital role, and we refer the readers to [1,2] for a detailed discussion on this subject.

5.1. Preparatory lemmas. In this subsection we present three preparatory growth lemmas which will be used later towards the proof of the key estimates Lemma 5.5 and Lemma 5.6 in the Sect. 5.3.
First we begin with some quick observation of the behavior of our maps on unstable curves. We recall from Sect. 3.1 that our half-revolution maps $P$’s are $C^2$ perturbations of their linear parts $G$’s and hence they are not far away from being linear themselves.

**Proposition 5.1 (Distortion Control).** Suppose $\gamma$ is an unstable curve and that $I, J \subseteq \gamma$ are two connected components of $F_n^0 \gamma$. Then there exists $K > 1$, so that

$$\frac{1}{K} |I| \leq |F_n^0 I| \leq K |I|$$

and $\lim_{j_n \to \infty} K = 1$, where $j_n$ is the minimal energy threshold and $|\cdot|$ is the curve length.

Next we discuss the maximum number an unstable curve of proper size can be cut into along iterates. We denote the complexity by $\kappa_n(\delta)$, i.e. the maximal number of pieces an unstable curve $\gamma$ can be cut into under the map $P_n^0$. It is easy to see from Fig. 3 that for any $\delta_0 < \min\{c/\lambda_{12}^U, c/\lambda_{12}^L\}$, where $c$ is the uniform bound in distance that the slit remains from the ceiling/floor as in Theorem 1, there exists a finite constant $M$ such that $\kappa_1(\delta) < M$ for any $\delta < \delta_0$ (if our maps $P$ are perfectly linear, then $M = 4$ for such choice of $\delta_0$ as illustrated in Fig. 5).

Now we present three preparatory lemmas. Let $\gamma$ be an unstable curve and $\varrho \in \gamma$ is a point on the unstable curve. We denote by $r_n(\varrho)$ the distance from $F_0^0(\varrho)$ to the boundary of the connected component $\gamma_n \subseteq F_n^0(\gamma)$ containing $\varrho_n = F_n^0(\varrho)$. We also denote the minimal expansion of the complete revolution as $\lambda_{F_0} := \min\{\lambda_{12}^U, \lambda_{12}^L\}$.

**Lemma 5.1 (First Growth Lemma).** There exist $\vartheta_1 := \frac{\kappa_1 K}{\lambda_{F_0}} < 1$ and $C_2 := 2K/\delta_0(1 - \vartheta_1) > 0$ such that for any $\epsilon > 0$

$$m_\gamma(r_n(\varrho) < \epsilon) \leq \vartheta_1^n m_\gamma(\varrho(0) < \epsilon) + C_2 \epsilon |\gamma|$$

where $m_\gamma$ is the Lebesgue measure restricted to $\gamma$.

**Proof** We prove a slightly stronger result. We need to control the number of pieces while iterating an unstable curve, thus we cut a long curve into several pieces of length equal to or shorter than $\delta_0$, which comes from the complexity. We update to $r_n^*$ the distance from $\varrho_n$ to the real or artificial boundary (the latter introduced by the chopping procedure).

We claim that

$$m_\gamma(r_{n+1}^* < \epsilon) \leq \vartheta_1^n m_\gamma(r_n^* < \epsilon) + \frac{2K \epsilon}{\delta_0} |\gamma|$$
Lemma 5.2 (Second Growth Lemma). For any unstable curve $\gamma$ and any $\epsilon > 0$, there exists $C_3$ such that for $n > \log |\gamma| / \log \vartheta_1$

$$m_\gamma (r_n < \epsilon) \leq C_3 \epsilon |\gamma|.$$ 

Proof It is easy to see that $\vartheta_1^n < |\gamma|$ for $n > \log |\gamma| / \log \vartheta_1$.

Also, $m_\gamma (r_n < \epsilon) \leq \min\{2\epsilon, |\gamma|\}$. Thus the Second Growth Lemma follows from the First Growth Lemma 5.1 with $C_3 := 1 + C_2$. \hfill \Box

As a direct consequence, after waiting long enough, we have the following second growth lemma.

Lemma 5.2 (Second Growth Lemma). For any unstable curve $\gamma$ and any $\epsilon > 0$, there exists $C_3$ such that for $n > \log |\gamma| / \log \vartheta_1$

$$m_\gamma (r_n < \epsilon) \leq C_3 \epsilon |\gamma|.$$ 

Proof It is easy to see that $\vartheta_1^n < |\gamma|$ for $n > \log |\gamma| / \log \vartheta_1$.

Also, $m_\gamma (r_n < \epsilon) \leq \min\{2\epsilon, |\gamma|\}$. Thus the Second Growth Lemma follows from the First Growth Lemma 5.1 with $C_3 := 1 + C_2$. \hfill \Box

Finally we prove that a short piece from an unstable curve cannot remain too short for too long in time.

Lemma 5.3 (Third Growth Lemma). There exist $\epsilon_0$, $b_1$, $\vartheta_2 < 1$ such that for any $k \in \mathbb{N}$

$$\min\{\vartheta_1^{k-1}, \epsilon_0\} \leq b_1 \vartheta_2^k$$

and that $\vartheta_3 := \vartheta_2 (1 + C_3 K b_1) < 1$. Then for any $n_2 > n_1 > \log |\gamma| / \log \vartheta_1$

$$m_\gamma \left( \max_{n_1 \leq n \leq n_2} r_n (\varrho) < \epsilon_0 \right) \leq \vartheta_3^{n_2-n_1} |\gamma|.$$ 

Proof We define a descending sequence of “unlucky” sub-curves $\tilde{\gamma}_1$ on $\gamma$ as follows.

First, we define

$$\tilde{\gamma}_1 = \{ \varrho \in \gamma : r_n (\varrho) < \epsilon_0 \}. $$

$F_0^{n_1} \gamma_1 = \bigcup j \gamma_{1,j}$ consists of finitely many pieces and each piece has length $|\gamma_{1,j}| < 2\epsilon_0$.

Then for any $\varrho \in \tilde{\gamma}_{1,j} := F_0^{n_1} (\gamma_{1,j}) \subseteq \gamma$, we define $k_1 (\varrho) := \lfloor \log |\gamma_{1,j}| / \log \vartheta_1 \rfloor + 1$. If $n_1 + k_1 \geq n_2$ on some piece $\tilde{\gamma}_{1,j}$, then we update the definition to $k_1 := n_2 - n_1$ on
that piece. We also introduce the time counters $t_1 := n_1$, $t_2(\varrho) := t_1 + k_1(\varrho)$ for any $\varrho \in \tilde{\gamma}_{1,j}$.

$k_1$ is a piecewise constant function on $\tilde{\gamma}_1$. Fix $k_1 \in \mathbb{N}$ and consider all those pieces $\tilde{\gamma}_{1,j}$ with $k_1(\varrho) = k_1$, then by the Second Growth Lemma 5.2

$$m_\gamma(k_1(\varrho) = k_1) \leq C_3 \varepsilon_0 |\gamma|.$$ 

Meanwhile by the definition of $k_1$ we have that $|\gamma_{1,j}| < \vartheta_1^{k_1-1}$, so by the Second Growth Lemma 5.2

$$m_\gamma(k_1(\varrho) = k_1) \leq C_3 \vartheta_1^{k_1-1} |\gamma|.$$ 

We choose $b_1, \vartheta_2 \ll 1$ (to be specified later) so small that for all $k_1 \in \mathbb{N}$

$$\min\{\vartheta_1^{k_1-1}, \varepsilon_0\} \leq b_1 \vartheta_2.$$ 

Then

$$m_\gamma(k_1(\varrho) = k_1) \leq C_3 b_1 \vartheta_2^{k_1} |\gamma|.$$ 

We continue the process of going forward by $[\log |\gamma_{1,j}|/ \log \vartheta_1] + 1$ steps each time and inductively we define a descending sequence of sub-curves $\tilde{\gamma}_{i,j} \subseteq \gamma$ and the functions $k_i, t_i$ on each piece $\tilde{\gamma}_{i,j}$. Also on each piece $\gamma_{i,j} \subseteq \mathcal{F}_j(\gamma)$ we have

$$m_{\gamma_{i,j}}(k_{i+1}(\varrho) = k_{i+1}) \leq C_3 b_1 \vartheta_2^{k_{i+1}} |\gamma_{i,j}|$$

Pulling back this estimate to $\gamma$ and considering distortion, on each piece $\tilde{\gamma}_{i,j}$

$$m_{\tilde{\gamma}_{i,j}}(k_{i+1}(\varrho) = k_{i+1}) \leq C_3 K b_1 \vartheta_2^{k_{i+1}} |\tilde{\gamma}_{i,j}|$$

where $K$ comes from the distortion control in Proposition 5.1.

Next we define

$$\tilde{\gamma}_{k_1,\ldots,k_i} := \{\varrho \in \gamma : k_1(\varrho) = k_1, \ldots, k_i(\varrho) = k_i\}.$$ 

Then by the above estimate

$$m_\gamma(\tilde{\gamma}_{k_1,\ldots,k_i} \cap \{k_{i+1}(\varrho) = k_{i+1}\}) \leq C_3 K b_1 \vartheta_2^{k_{i+1}}.$$ 

Finally we fix a sequence of natural numbers $k_1, \ldots, k_m$ with $k_1 + \cdots + k_m = n_2 - n_1$. Then

$$m_\gamma(\tilde{\gamma}_{k_1,\ldots,k_i}) = |\gamma| \frac{m_\gamma(k_1(\varrho) = k_1)}{|\gamma|} \frac{m_\gamma(\tilde{\gamma}_{k_1} \cap \{k_2(\varrho) = k_2\})}{|\tilde{\gamma}_{k_1}|} \cdots \frac{m_\gamma(\tilde{\gamma}_{k_1,\ldots,k_{m-1}} \cap \{k_m(\varrho) = k_m\})}{|\tilde{\gamma}_{k_1,\ldots,k_{m-1}}|}$$

$$\leq |\gamma| C_3 K b_1 \vartheta_2^{k_1} \cdot C_3 K b_1 \vartheta_2^{k_2} \cdots C_3 K b_1 \vartheta_2^{k_m}$$

$$\leq (C_3 K b_1)^m \vartheta_2^{n_2 - n_1} |\gamma|$$

Now summing over all such possible sequences of natural numbers, we obtain

$$m_\gamma \left( \max_{n_1 \leq n \leq n_2} r_n(\varrho) < \varepsilon_0 \right) \leq \sum_{m=1}^{n_2 - n_1} \left( \begin{array}{c} n_2 - n_1 - 1 \\ m - 1 \end{array} \right) (C_3 K b_1)^m \vartheta_2^{n_2 - n_1} |\gamma|$$
\[ \leq (1 + C_3 K b_1)^{n_2 - n_1 - 1} C_3 K b_1 \vartheta_2^{n_2 - n_1} |\gamma| \]
\[ \leq (\vartheta_2 (1 + C_3 K b_1))^{n_2 - n_1} |\gamma|. \]

By definition \( \vartheta_3 := \vartheta_2 (1 + C_3 K b_1) \). Now it remains to show that we can choose \( \epsilon_0, b_1, \vartheta_2 \) such that \( \vartheta_3 < 1 \) and that will finish the proof.

Indeed, we fix some large \( k_* \in \mathbb{N} \). We take \( \vartheta_2 = \vartheta_1^{1/2} \) and \( b_1 = \vartheta_1^{k/2 - 1} \). Then for \( k \geq k_* \) we have \( \vartheta_1^{k_1 - 1} \leq b_1 \vartheta_2^{k_1} \) for such choice of \( b_1, \vartheta_2 \). Next we take \( \epsilon_0 \leq \vartheta_1^{k_1 - 1} \). Then for all \( k_1 \in \mathbb{N} \)
\[ \min\{\vartheta_1^{k_1 - 1}, \epsilon_0\} \leq b_1 \vartheta_2^{k_1}. \]

For \( k_* \) sufficiently large, \( \vartheta_2 (1 + C_3 K b_1) < 1 \). \( \square \)

**Remark 5.1** In fact, we may take \( k_* = 2 \) so that \( \vartheta_3 = \vartheta_1^{1/2}, b_1 = 1 \) and \( \epsilon_0 \leq \vartheta_1 \), and Lemma 5.3 holds for such choice parameters as long as we the minimal expansion \( \lambda_{\min} \gg 1 \) is sufficiently large, which can be achieved by choosing \( f_* \) large.

### 5.2. Quantitative estimate on long curves

In the previous section, we provide estimates on the size of “bad” short unstable curves, and in this section, we introduce the notion of long curves on which we can iterate our dynamics effectively and show exponential acceleration in Sects. 6 and 7. We also provide an estimate on the time we need to wait for an unstable curve to grow into a long curve (c.f. Lemma 5.4).

We recall from the First Growth Lemma 5.1 that \( \vartheta_1 = \frac{k_1 K}{\lambda_{F_0}} \).

**Definition 3** (Long curve). An unstable curve \( \gamma \) is called long if it has size \( \frac{\vartheta_1}{2} \leq |\gamma| \leq \vartheta_1 \).

Suppose \( \gamma \) is a long curve. For \( \varrho \in \gamma \) we denote by \( \tilde{N}(\varrho, \gamma) \) the first time \( t = t(\varrho) > 0 \) when \( F_t \varrho \) enters a long curve.

**Lemma 5.4** (Quantitative Growth lemma). Let \( \gamma \) be a long curve. Then there exists \( b_2 > 0 \) and \( \vartheta_4 < 1 \) so that
\[ m_\gamma\left( \varrho : \tilde{N}(\varrho, \gamma) = N \right) \leq b_2 \vartheta_4^N |\gamma|. \]

**Proof** We denote by \( n_\gamma = \left\lfloor \frac{\log |\gamma|}{\log \vartheta_1} \right\rfloor + 1 \). It follows from the Third Growth Lemma 5.3, by taking \( \epsilon_0 = |\gamma|/2 \), that for \( N > n_\gamma \)
\[ m_\gamma\left( \max_{n_\gamma \leq n \leq N} r_n(\varrho) < |\gamma|/2 \right) \leq \vartheta_4^{N-n_\gamma} |\gamma|. \]

For \( N \leq n_\gamma \), it follows from the First Growth Lemma 5.1, by taking \( \epsilon = |\gamma|/2 \), that
\[ m_\gamma(r_N(\varrho) < |\gamma|/2) \leq \vartheta_1^N m_\gamma(r_0(\varrho) < |\gamma|/2) + C_2 |\gamma|/2 |\gamma| \]
\[ \leq \vartheta_1^N |\gamma| + C_2 |\gamma|/2 |\gamma| \]
\[ \leq \left( \vartheta_1^N + \frac{C_2 |\gamma|}{2} \right) |\gamma| \]
Now we choose $\vartheta_4 \in (\vartheta_3, 1)$ such that $\vartheta_3^{N-n_\gamma} \leq \vartheta_4^N$ for all $N > n_\gamma$. Then we choose

$$b_2 := \max_{2 \leq N \leq n_\gamma} \left\{ \vartheta_4^N + C_2 |\gamma|/2 \right\}.$$ 

Therefore we conclude that for all $N > 1$ we have

$$m_\gamma \left( \varrho : \hat{N}(\varrho, \gamma) = N \right) \leq b_2 \vartheta_4^N |\gamma|.$$ 

5.3. Growth lemmas with delay. We fix some $N_0 \geq 1$ (the size of $N_0$ and its necessity will be made clear in Sects. 6 and 7). We now let the dynamics run for $N_0$ many times and then provide estimates on the growth of short/long curves.

Let $\gamma$ be a long curve. We define a map $\hat{N}(\varrho) : \gamma \to N$ as follows: for each $\varrho \in \gamma$, let $\hat{N}(\varrho) = N_0 + \tilde{N} \left( \mathcal{F}_{N_0}^0 \varrho, \gamma' \right)$, where $\gamma'$ is the unstable curve that contains $\mathcal{F}_{N_0}^0 \varrho$.

We have the following estimate on the time it takes for $\gamma$ to grow long again after a delay of $N_0$ times.

**Lemma 5.5** (Delayed Quantitative Growth Lemma). For all $\ell > N_0$ and every long curve $\gamma$ we have that

$$m_\gamma (\varrho : \hat{N}(\varrho) = \ell) \leq b_2 \vartheta_4^{\ell-N_0} |\gamma|,$$

Moreover, the constants $b_2, \vartheta_4$ only depend on $\dot{f}_\ast$ and $c$ in Theorem 1.

**Proof** Note that the set $\{ \varrho : \hat{N}(\varrho) = \ell \}$ consists of two types of points. First, the ones that never visit a long curve throughout their journey up to time $\ell$. The measure of this set can be estimated by the Third Growth Lemma 5.3 as $b_2 \vartheta_4^\ell$.

Secondly, for $\varrho \in \gamma$ let $\{ \gamma_{\varrho} \}$ be the collection of all long curves in the image $\mathcal{F}_{N_0}^0 \gamma$ and suppose $\mathcal{F}_{N_0}^0 \varrho \in \gamma_1$ for some long curve $\gamma_1 \subseteq \mathcal{F}_{N_0}^0 \gamma$, then inductively we set $\hat{N}_k(\varrho) = \hat{N}_{k-1}(\mathcal{F}_{N_0}^{N_{k-1}} \varrho, \gamma_1)$ ($k \geq 2$).

We now iterate the long curves obtained in the previous Lemma 5.5. Namely, we define $\hat{N}_k$ inductively as follows: we set $\hat{N}_1 = \hat{N}$ (c.f. the definition of $\hat{N}$ before Lemma 5.4 in the previous Sect. 5.2); now we let $\{ \gamma_{\varrho} \}$ be the collection of all long curves in the image $\mathcal{F}_{N_0}^0 \gamma$ and suppose $\mathcal{F}_{N_0}^0 \varrho \in \gamma_1$ for some long curve $\gamma_1 \subseteq \mathcal{F}_{N_0}^0 \gamma$, then inductively we set $\hat{N}_k(\varrho) = \hat{N}_{k-1}(\mathcal{F}_{N_0}^{N_{k-1}} \varrho, \gamma_1)$ ($k \geq 2$).
Lemma 5.6 (Delayed Large Deviation Estimate). There exists $a = a(N_0) > 0$ and $\vartheta_5 < 1$ such that

$$m_\gamma (q : \hat{N}_n(q) > an) \leq \vartheta_5^n |\gamma|.$$  

Proof First note that for any $c > 0$ we can write

$$E_\gamma [e^{c\hat{N}_n}] \leq \sum_{k=N_0}^{\infty} 3\vartheta_4^{k-N_0} e^{ck} < 3\vartheta_4^{-N_0} \sum_{k=N_0}^{\infty} (\vartheta_4 e^c)^k.$$  

If $c$ is small enough then $\vartheta_4 e^c < 1$. Hence

$$E[e^{c\hat{N}_n}] \leq Cb\vartheta_4^{-N_0}(\vartheta_4 e^c)^{N_0} = \rho < \infty.$$  

Let $A_k$ be the $\sigma$-algebra generated by the partition of the long curve at step $k$ by the intervals of constancy of $\hat{N}_k$. Then

$$E[e^{c\hat{N}_n}] = E[E[e^{c\hat{N}_n} | A_{n-1}]] = E[e^{c\hat{N}_{n-1}} E[e^{c(\hat{N}_n - \hat{N}_{n-1})} | A_{n-1}]] \leq E[e^{c\hat{N}_{n-1}}] \rho \leq \rho^n.$$  

Thus for any $A > 0$

$$e^A m_\gamma (q : e^{c\hat{N}_n}(q) > e^A) \leq \rho^n |\gamma|.$$  

Choose $A = nq$. Then

$$m_\gamma (q : e^{c\hat{N}_n}(q) > e^{nq}) = m_\gamma (q : c\hat{N}_n(q) > nq) \leq \left( \frac{\rho}{e^q} \right)^n.$$  

Taking $q$ so large that $e^q > \rho$ and setting $\vartheta_4 = \rho/e^q$ and $a = q/c$ we get that

$$m_\gamma (q : \hat{N}_n(q) > na) \leq \vartheta_4^n |\gamma|.$$  

$\square$

6. Energy Growth in Finite Time

In the previous Sect. 5, we have provided controlled estimates on the growth of long unstable curves after a long delay $N_0 \gg 1$ (c.f. the Delayed Quantitative Growth Lemma 5.5 and the Delayed Large Deviation Estimate 5.6). In this section we show that our system starts to show an exponential drift on average towards acceleration after the long wait $N_0$.

Proposition 6.1 Assume that $f_1 < f_2$. Then for all sufficiently large $N_0$, one can choose $\hat{f}_* = \hat{f}_*(E, c)$ and $V_* = V_*(f_1, \hat{f}_2, c, E)$ so large that for the modified dynamical system $\mathcal{F}_0$, we have that for every long curve $\gamma$ on the $R_1$ cylinder

$$E_\gamma \left[ \frac{\ln \mathcal{H}_{N_0}(q) - \ln \mathcal{H}_0}{N_0} \right] \geq \frac{E}{3},$$

where

$$E = (1 - f_2) \log \frac{1 - f_2}{1 - f_1} + f_2 \log \frac{f_2}{f_1}.$$  

The rest of the section is devoted to the proof of Proposition 6.1 and is structured as follows. In Sect. 6.1 we present an estimate on the size of the orbits on any prescribed itinerary. Then in Sect. 6.2 we present the proof of Proposition 6.1 with the help of Lemma 6.1 in Sect. 6.1.
6.1 An estimate on prescribed itineraries. The purpose of this section is to prove Lemma 6.1 below, where for every long curve $\gamma$, we estimate the measure of all initial conditions $\varrho \in \gamma$ for which the energy change follows a prescribed itinerary.

First, let us introduce the notion of complete curves.

**Definition 4 (Complete curves).** An unstable curve $\gamma$ is called complete if it runs across an entire fundamental domain (“box”) vertically (c.f. Fig. 6).

We recall that $S_{+1}$ and $S_{-1}$ are the collections of accelerating and decelerating fundamental domains on the phase cylinder $R_1$ respectively. We denote $q_1 = f_2$, $q_{-1} = 1 - f_2$.

For an unstable curve $\gamma \in R_1$ and a length-$n$ itinerary $t^n = (t_0, t_1, \ldots, t_{n-1})$ with $t_k = \pm 1$ ($0 \leq k \leq n - 1$) (we may drop the superscript $n$ whenever it is clear from the context), we collect all the points $\varrho$ on $\gamma$ whose orbits up to time $n - 1$ follow the itinerary $t^n$:

$$A_{\gamma, t^n} = \{ \varrho = (\sigma_0, \mathcal{H}_0) \in \gamma : \mathcal{H}_k \in S_{t_k}, \ 0 \leq k \leq n - 1 \}. \quad (5)$$

We note that for a complete curve $\gamma$, $t_0 = t_0(\gamma)$ will be determined by $\gamma$. We have the following estimate on the proportion of the points on a complete curve $\gamma$ following an itinerary $t^n = (t_0(\gamma), t_1, \ldots, t_{n-1})$.

**Lemma 6.1** For every $n \geq 1$, $\varepsilon > 0$, $\hat{f}_s$ can be taken so large that if $|\hat{f}_1|, |\hat{f}_2| > \hat{f}_s$, then for every complete curve $\gamma$ and any itinerary $t^n = (t_0(\gamma), t_1, \ldots, t_{n-1})$ we have that

$$m_\gamma(A_{\gamma, t^n}) \leq (q_{t_1} \cdots q_{t_{n-1}} + \varepsilon)|\gamma|.$$

**Remark 6.1** In this Lemma, $\hat{f}_s = \hat{f}_s(n)$ needs to be taken larger as $n$ increases. However, in the proof of Proposition 6.1 we will apply Lemma 6.1 with a fixed $n = N_0$ and $\hat{f}_s = \hat{f}_s(N_0)$ will be fixed then.

**Proof** We start by proving the case $t = (t_0(\gamma), t_1)$.

We may assume without loss of generality that the case $\gamma \subset S_{-1}$ (so $t_0(\gamma) = -1$), as the other case $\gamma \subset S_{+1}$ can be proven analogously.

Note that the curve $P^U_{12}\gamma$ will first be mapped into $R_2$-cylinder (c.f. Fig. 3). It will be of size close to $|\hat{f}_2||\gamma|$. $P^U_{12}\gamma$ meets multiple singularity lines in $R_2$ and we artificially cut
it into complete and non-complete curves. Note that there are at most two non-complete curves at the ends of $P^U_{12} \gamma$. Clearly, we can choose $\hat{f}_* \gamma$ so large that the preimage of the non-complete tails of $P^U_{12} \gamma$ takes a proportion less than $\varepsilon/3$ on $\gamma$.

Now for each complete curve $\gamma'$ on $P^U_{12} \gamma$ in $R_2$-cylinder, $P^U_{21} \gamma'$ will be an unstable curve of size $|\hat{f}_1|/|\gamma'|$ in $R_1$-cylinder. We note that $|\gamma'| \sim O(1)$, and hence again by neglecting non-complete tails and taking $\hat{f}_*$ sufficiently large, the preimages on $\gamma'$ of $P^U_{21} \gamma' \cap S_1$ and $P^U_{21} \gamma' \cap S_{-1}$ for each and hence all complete $\gamma''$s on $P^U_{12} \gamma$ are respectively of proportion $q_1$ and $q_{-1}$ with an error in total less than $2 \varepsilon/3$.

This completes the proof for $t = (t_0(\gamma), t_1)$.

Next we prove the statement for itineraries of longer length.

We wish to show that for every $\varepsilon > 0$ and every $m \in [1, n]$, one can take $\hat{f}_* \gamma$ so large that

$$m_{\gamma}(A_{\gamma,tm})/|\gamma| \leq \left( \prod_{1 \leq s \leq m} q_{ts} \right) + \varepsilon, \tag{6}$$

and there exist a collection $\Gamma_m$ of complete curves in $F^m_0 \gamma$ so that

$$m_{\gamma}(A_{\gamma,tm} \setminus \bigcup_{\gamma_m \in \Gamma_m} F^{-m}_{\gamma_m}) \leq \varepsilon |\gamma|. \tag{7}$$

We proceed by induction.

Assume we have already established (6), (7) for all itineraries of length $m < k$.

For $m = k$ and $t = (t_1, \ldots, t_k)$, we consider the collection $\Gamma_{k-1}$ of complete curves in the image at step $k-1$. For a complete curve $\gamma_{k-1} \in \Gamma_{k-1}$, we consider the set $A_{\gamma_{k-1},t_k}$ of the points on $\gamma_{k-1}$ which will visit $S_k$ in the next iterate.

By (6) the proportion of the set $A_{\gamma_{k-1},t_k}$ inside $\gamma_{k-1}$ will be not greater than $q_{tk} + \varepsilon$. Hence

$$m_{\gamma}(A_{\gamma,t_k}) \leq m_{\gamma}(A_{\gamma,t_{k-1}} \setminus \bigcup_{\gamma_{k-1} \in \Gamma_{k-1}} F^{-k}_{0} \gamma_{k-1}) + \left( \prod_{s=1}^{k-1} q_{ts} + \varepsilon \right) (q_{tk} + \varepsilon) K |\gamma|,$$

where the factor $K$ is to account for the distortion.

The first term is small due to (7) while the second term can be made arbitrarily close to $\left( \prod_{s=1}^{k} q_{ts} \right) |\gamma|$. As for the collection $\Gamma_k$, we note that

$$m_{\gamma}(A_{\gamma,t_k} \setminus \bigcup_{\gamma_k \in \Gamma_k} F^{-k}_{0} \gamma_k) \leq \varepsilon |\gamma| + \left( \prod_{s=1}^{k-1} q_{tk} + \varepsilon \right) K \varepsilon |\gamma|,$$

where $\varepsilon$ is the contribution of points outside of the set $A_{\gamma,t_{k-1}} \setminus \bigcup_{\gamma_{k-1} \in \Gamma_{k-1}} \gamma_{k-1}$, while the second term is the contribution of the set outside of $A_{\gamma_{k-1},(t_k)} \setminus \Gamma_{\gamma_{k-1},k}$.

It is now clear that if we had started the procedure with $\varepsilon$ small enough then the estimate for $m = k$ would follow.

Repeating the procedure until $m = n$ and taking $\hat{f}_*$ large enough, we will arrive at the result. 

\[\square\]
6.2. Exponential escape in finite time. In the previous section we are able to estimate the probability of the energy of the ball following a given itinerary \( t \) by taking \( \dot{f}_s \) large. Note that we also know from Proposition 3.1 how the energy changes when the ball follows the trajectory given by \( t \). We now combine these two pieces of information to estimate the probability that the ball will exhibit exponential acceleration after \( N_0 \) iterations. The idea is to approximate the dynamics by a Markov chain.

Proof of Proposition 6.1 We first prove the statement for complete curves and then for long curves (see Definition 3).

Now suppose that \( \gamma \) is a complete curve in \( R_1 \).

We consider an itinerary \( t = (t_0(\gamma), t_1, \ldots, t_{N_0-1}) \) of length \( N_0 \). We define a function \( T(t) = \sum_{k=1}^{N_0-1} t_k \) which counts the number of entries taking value 1 in the itinerary \( t \).

For the (modified) deterministic system \( F_0 \), we obtain from Proposition 3.1, that for any \((\sigma_0, H_0) \in \gamma\) following the itinerary \( t \), we have that

\[
H_{N_0}(t) \geq \left( t \left( \frac{f_2}{f_1} \right) \right)^{N_0-T(t)} \left( t \left( \frac{1-f_2}{1-f_1} \right) \right) H_0.
\]

By Lemma 6.1, we also have a bound on the measure of the set \( A_{\gamma,t} \) of points in \( \gamma \) following the itinerary \( t \) up to time \( N_0 - 1 \)

\[
m_{\gamma}(A_{\gamma,t}) \leq \left( \prod_{k=1}^{N_0-1} q_{t_k} + \varepsilon \right) |\gamma|,
\]

where \( \varepsilon > 0 \) can be made arbitrarily small by taking \( \dot{f}_s \) large.

On the other hand, we can approximate the deterministic system with a random walk in the following way.

We take \( D_0 = \min\{H_0 : H_0 \in \gamma\} \), and then we define

\[
D_{n+1} = D_n X_n,
\]

where \( \{X_n\} \) is an i.i.d. sequence of discrete random variables with

\[
P\left(X_n = t \frac{f_2}{f_1}\right) = f_2, \quad P\left(X_n = t \frac{1-f_2}{1-f_1}\right) = 1 - f_2.
\]

As we already remarked in Sect. 2, by the Gibbs’ inequality, \( \mathcal{E} = 0 \) if and only if \( f_1 = f_2 \), and then \( \mathcal{E} > 0 \) under the assumption \( f_1 \neq f_2 \). Hence if \( t \) is sufficiently close to 1 (which can always be achieved by taking \( V_s \) sufficiently large), then

\[
E[\ln X_n] = (1 - f_2) \log \left( t \frac{1-f_2}{1-f_1} \right) + f_2 \log \left( t \frac{f_2}{f_1} \right) = \mathcal{E} + \log t > 0.
\]

Then by the Hoeffding’s inequality [12], for every \( r \in (0, \mathcal{E} + \log t) \) we have that

\[
P\left( \sum_{k=0}^{n-1} \ln X_k < (\mathcal{E} + \log t - r)n \right) \leq e^{-2nr^2},
\]

or equivalently

\[
P\left( D_n < e^{(\mathcal{E}+\log t-r)n} D_0 \right) \leq e^{-2nr^2}. \tag{8}
\]
If we consider the probability $p_{N_0}(t)$ that the random walk $D_n$ follows the itinerary $t$, then clearly

$$p_{N_0}(t) = \prod_{k=1}^{N_0-1} q_{t_k}$$

and the random walk at step $N_0$ takes value

$$D_{N_0}(t) = \left( t \left( \frac{f_2}{f_1} \right) \right)^{T(t)} \left( t \left( \frac{1 - f_2}{1 - f_1} \right) \right)^{N_0-T(t)} D_0.$$

Since $D_0 \leq H_0$ for every $\varrho = (\sigma_0, H_0) \in \gamma$, $D_{N_0}(t) \leq H_{N_0}(t)$. Now let $r \in (0, E + \log ı)$ be small and $N_0$ be the collection of all itineraries $t$ of length $N_0$ for which there exists $\varrho \in A_{\gamma,t}$ so that $H_{N_0}(t) < e^{N_0(E + \log ı) - r} H_0$. Then

$$P_{\gamma}(H_{N_0} < e^{E + \log ı} - r) \leq \sum_{t \in \Omega_{N_0}} |A_{\gamma,t}| \leq \sum_{t \in \Omega_{N_0}} \left( \prod_{k=1}^{N_0-1} q_{t_k} \right) + \epsilon \leq P(D_{N_0} < e^{E + \log ı} - r) H_0 + \epsilon,$$

where the estimate in the last line was due to the fact that $D_{N_0}(t) \leq H_{N_0}(t)$ for any itinerary $t$.

We note that the vertical height of $\gamma$ is less than 1 (c.f. Fig. 3 for the heights of fundamental domains), then $H_0 \leq 2D_0$ for every $\varrho = (\sigma_0, H_0) \in \gamma$. We take $r' = r/\sqrt{2}$ and $N_0 > \frac{\log 2}{r-r'}$, and then by (8) we have

$$P(D_{N_0} < e^{E + \log ı - r}) \leq P(D_{N_0} < e^{E + \log ı - r} 2D_0) \leq P(D_{N_0} < e^{E + \log ı - r'} D_0) \leq e^{-2N_0 r'^2} = e^{-N_0 r^2}.$$

Now we take $\epsilon$ so small (by taking $f_*$ large) that

$$P_{\gamma}(H_{N_0} < e^{E + \log ı - r}) \leq C e^{-N_0 r^2}.$$

For $r$ sufficiently small and $r$ sufficiently close to 1 (by taking $V_*$ large), we have that

$$P_{\gamma}(H_{N_0} < e^{(2E/3)N_0 H_0}) \leq C e^{-N_0 r^2}.$$  \hfill (9)

We denote the set of points on $\gamma$ which do not accelerate exponentially at time $N_0$ as

$$B_{N_0} = \{ \varrho \in \gamma : \frac{\ln H_{N_0} - \ln H_0}{N_0} < \frac{2E}{3} \}$$

Then $P_{\gamma}(B_{N_0}) < C e^{-N_0 r^2}$.

Also, we recall from Proposition 3.1 that the energy of the ball drops at most by a factor of $ı \frac{1 - f_2}{1 - f_1} < 1$ each time.
Therefore
\[
E_\gamma \left[ \ln \mathcal{H}_{N_0} - \ln \mathcal{H}_0 \over N_0 \right] \geq P_\gamma(B_{N_0}) \log \left( {1 - f_2 \over 1 - f_1} \right) + \left( 1 - P_\gamma(B_{N_0}) \right) {2E \over 3} \geq E \over 2,
\]
for sufficiently large \( N_0 \).

We now extend the above estimate to the case of long curves.

We recall from Sect. 5.2 that an unstable curve \( \gamma \) is long if it has size \( |\gamma| \in [\vartheta_1/2, \vartheta_1] \), and \( \vartheta_1 = \frac{\kappa_1 K}{k \gamma_0} \sim O(1/|\hat{f}_1 \hat{f}_2|) \). Roughly speaking, a long curve \( \gamma \) would grow into size \( |\mathcal{F}_0 \gamma| \sim O(1) \) after one complete revolution and then into size \( |\mathcal{F}_0^2 \gamma| \sim O(|\hat{f}_1 \hat{f}_2|) \) after another complete revolution, and the estimates (9), (10) would hold on the complete curves in the image at the second complete revolution, which would take up most of \( \mathcal{F}_0^2 \gamma \), hence the Proposition 6.1 holds for such long curve \( \gamma \).

More precisely, let \( \gamma \) be a long curve. Then \( |\mathcal{F}_0 \gamma| = L \) for some constant \( L \sim O(1) \) and \( \mathcal{F}_0 \gamma \) consists of at most \( \kappa_1 = \kappa_1(\delta_0) \) pieces (c.f. the discussion on complexity \( \kappa_1 \) in Sect. 5.1), some of which might be long while others short. We call a piece \( \gamma_1 \subseteq \mathcal{F}_0 \gamma \) bad if it has length \( |\gamma_1| < \frac{1}{|f_2|} \), otherwise we say \( \gamma_1 \) is good. A good piece might run across several fundamental domains in \( R_1 \) and we artificially cut it by singularity lines if necessary, which creates (multiple) good piece(s) of size at most \( O(1) \) and at most two bad tails at the two ends. Together in \( \mathcal{F}_0 \gamma \), we end up with (multiple) good piece(s) and the bad ones, the latter of which take up at most a portion of

\[
P_\gamma \left( \gamma_1 \subseteq \mathcal{F}_0 \gamma \text{ with } |\gamma_1| < \frac{1}{|f_2|} \right) < \frac{3K\kappa_1}{L|\hat{f}_2|}, \tag{11}
\]

where \( K \) comes from the distortion control.

Next we consider a good piece \( \gamma_1 \subseteq \mathcal{F}_0 \gamma \) (after artificial cutting) in \( R_1 \)-cylinder. By construction, \( P_{12} \gamma_1 \) is one piece in \( R_2 \)-cylinder and \( C \leq |P_{12} \gamma_1| \leq C|\hat{f}_2| \) for some constant \( C \). We note that \( P_{12} \gamma_1 \) might run across several fundamental domains in \( R_2 \) and we artificially cut it in by singularity lines if necessary. After the artificial cutting, we say a piece \( \gamma_{1.5} \subseteq P_{12} \gamma_1 \) is unlucky if it has size \( |\gamma_{1.5}| < \frac{1}{\sqrt{|f_1|}} \), otherwise we say \( \gamma_{1.5} \) is lucky. We observe that the height of a fundamental box lies in \([c, 1]\) (c.f. Fig. 3) and hence on each good \( \gamma_1 \) the unlucky pieces take up a portion on \( \gamma_1 \) of

\[
P_{\gamma_1} \left( \gamma_{1.5} \subseteq P_{12} \gamma_1 \text{ with } |\gamma_{1.5}| < \frac{1}{\sqrt{|f_1|}} \right) < \frac{2K}{c \sqrt{|f_1|}}. \tag{12}
\]

Finally, for a lucky \( \gamma_{1.5} \subseteq P_{12} \gamma_1 \) in \( R_2 \), by construction, \( P_{21} \gamma_{1.5} \) is a long piece in \( R_1 \) and \( C' \sqrt{|\hat{f}_1|} \leq |P_{21} \gamma_{1.5}| \leq C' |\hat{f}_1| \) for some constant \( C' \), which, after the artificial cutting by singularity lines in \( R_1 \), consist of many complete curves and at most two non-complete curves at the ends that takes up a proportion on \( \gamma_{1.5} \) of

\[
P_{\gamma_{1.5}}(\gamma_2 \subseteq P_{21} \gamma_{1.5} \text{ non-complete}) \leq \frac{2K}{C' \sqrt{|\hat{f}_1|}}. \tag{13}
\]

Now the estimates (9), (10) hold on the complete curves \( \gamma_2 \subseteq \mathcal{F}_0^2 \gamma \) (with \( \tilde{N}_0 = N_0 + 2 \)), which, by pulling back the estimates (11), (12), (13) onto \( \gamma \), take up a portion on \( \gamma \) of

\[
P_\gamma(\gamma_2 \subseteq \mathcal{F}_0^2 \gamma \text{ complete}) \geq 1 - \frac{C''}{\sqrt{|\hat{f}_*|}}.
\]
Therefore by taking \( \dot{f}_s \) sufficiently large, we prove the Proposition 6.1 for long curves. \( \square \)

7. Proof of Main Theorem 1

In the previous section we showed that for any sufficiently large \( N_0 \) and for an appropriate choice of \( \dot{f}_s \) and \( V_s \), the expected exponential energy growth can be achieved for a substantial portion on every long curve \( \gamma \) above \( V_s \). After \( N_0 \) many iterations \( \mathcal{F}_0^{N_0} \gamma \) will consist of multiple unstable curves, some of which will be long and others short. If we had only long curves, then after another \( N_0 \) iterations each of these long curves would produce some further exponential acceleration. Continuing like this we would get acceleration at each step for a large set of initial conditions. Understandably, this cannot continue forever and there will eventually emerge some short curves on which the analysis may fail. In order to overcome this problem, we need to iterate these curves until they start to produce long curves. The tool for keeping track of the waiting times in this process is the delayed growth lemmas proven in Sect. 5.3. One can recognize the procedure described above as the first step \( \mathcal{N}_1(\varrho) \) defined in Sect. 5.3. After this, we will repeat the procedure and subsequently define \( \mathcal{N}_2(\varrho), \mathcal{N}_3(\varrho) \) etc.

Observe, however, that if the energy of the ball spends too much time in a short curve, the energy of the ball may start to change uncontrollably and the energy may eventually drop below the threshold \( V_s \) where the adiabatic normal forms no longer hold for the original system. However, we will show that this will not happen too often and the exponential energy growth obtained in finite intervals of length \( N_0 \) will eventually persist in infinite time for a set of initial conditions of considerable measure.

In this section, we present the proof of the Main Theorem 1, which proceeds as follows: in Sect. 7.1 we prove the Main Theorem 1 given a moment estimate (i.e. Lemma 7.1), and in Sect. 7.2 we prove the moment estimate.

7.1. Proof of main Theorem 1. First, let us state the moment estimate.

Suppose \( \gamma \) is a long curve and for every \( \varrho \in \gamma \), let \( \Delta(\varrho) = \ln \mathcal{H}_{\mathcal{N}(\varrho)} - \ln \mathcal{H}_0 \). Then we have the following moment estimate.

**Lemma 7.1** The parameters \( N_0, \dot{f}_s \) and \( V_s \) in Proposition 6.1 can be chosen in such a way that there exist constants \( \kappa > 0, \vartheta < 1 \) (depending on \( E \) and \( c \)) so that for the modified map \( \mathcal{F}_0 \) and for every long curve \( \gamma \) in \( R_1 \) we have that

\[
E_{\gamma}[e^{-\kappa \Delta}] \leq \vartheta.
\]

Now we prove the Main Theorem 1 given the moment estimate above, whose proof can be found in the next section.

**Proof of Main Theorem 1** Recall the definitions of the quantities \( \mathcal{N} \) and \( \mathcal{N}_n \) from Sect. 5.3.

We first choose \( \dot{f}_s \) so large that we have invariant unstable cones as in Proposition 4.1 and the constants in the growth Lemma 5.5 are uniform for all \( |\tilde{f}_1|, |\tilde{f}_2| > \dot{f}_s \) for the modified dynamical system \( \mathcal{F}_0 \). We choose \( \dot{f}_s \) and \( V_s = V_s(\ell_1, \ell_2, c, \mathcal{E}) \) in such a way that the conditions of Lemma 7.1 are satisfied. Now, for \( n \geq 1 \), let \( \Delta_n = \ln \mathcal{H}_{\mathcal{N}_n(\varrho)} - \ln \mathcal{H}_{\mathcal{N}_{n-1}(\varrho)} \). Set \( t_n = \Delta_n + \cdots + \Delta_1 = \ln \mathcal{H}_{\mathcal{N}_n(\varrho)} - \ln \mathcal{H}_0 \). Let \( \mathcal{F}_k \) be the
\( \sigma \) algebra generated by the partition of \( \gamma \) by the constancy intervals of \( \hat{N}_k \) up to time \( k \). Then, by Lemma 7.1

\[
E[ e^{-\kappa(t_n-t_1)} ] = E[ E[ e^{\kappa(t_{n-1}-t_1)} | F_{n-1} ] ] = E[ e^{\kappa \Delta_k | F_{n-1} } ] \leq \vartheta E[ e^{-\kappa(t_n-t_1)} ].
\]

Iterating

\[
E[ e^{-\kappa(t_n-t_1)} ] \leq \vartheta^n.
\]

Then for any \( A > 0 \)

\[
e^A m[ e^{-\kappa t_n} \geq e^A ] \leq \vartheta^n |\gamma|.
\]

Take \( A = -\kappa \alpha n \). Then

\[
m( e^{-\kappa t_n} \geq e^{-\kappa \alpha n} ) \leq e^{\kappa \alpha n \vartheta n} |\gamma|.
\]

Hence

\[
m(t_n < \alpha n) \leq e^{\kappa \alpha n \vartheta n} |\gamma| = e^{n(\kappa \alpha - |\ln \vartheta|)} |\gamma|.
\]

Obviously, if we take \( \alpha \) so small that \( a' = - (\kappa \alpha - |\ln \vartheta|) > 0 \), then

\[
m( \Delta_1 + \cdots + \Delta_n \leq \alpha n ) \leq e^{-a'n} |\gamma|;
\]

or equivalently

\[
m( \varrho : \mathcal{H}_{\hat{N}_n(\varrho)} \leq \alpha n \mathcal{H}_0 ) \leq e^{-a'n} |\gamma|.
\]

Hence, there exists \( c_0, N_0 > 0 \) such that for all \( N \geq N_0 \)

\[
m( \varrho = (\sigma_0, \mathcal{H}_0) : \mathcal{H}_{\hat{N}_n(\varrho)} > e^{an} \mathcal{H}_0 \text{ for all } n \geq N ) \geq (1 - e^{-c_0 N}) |\gamma|.
\]

Thus we have shown that there is energy acceleration at times \( \{ \hat{N}_n(\varrho) \} \) on a set of large measure.

We now show that the acceleration will persist between the times \( \hat{N}_n(\varrho) \) and \( \hat{N}_{n+1}(\varrho) \) as well.

If the energy between these times drops very low, then the interval \( [\hat{N}_n(\varrho), \hat{N}_{n+1}(\varrho)] \) will have to be very large. But this cannot happen too often due to the Delayed Quantitative Growth Lemma 5.5. Once the energy acceleration is achieved for all times \( \hat{N} \), it will then remain to use the delayed deviation estimate in Lemma 5.6 to transition from times \( \hat{N}_n(\varrho) \) to \([an]\).

By Lemma 5.6 we have that

\[
m( \varrho : \hat{N}_n(\varrho) > an ) \leq \vartheta^2 |\gamma|.
\]

Hence, as above there exists \( c_1, N_1 > 0 \) so that for all \( N \geq N_1 \)

\[
m( \varrho : \hat{N}_n(\mathcal{H}_0) > an : \text{ for all } n \geq N ) \geq (1 - e^{-c_1 N}) |\gamma|.
\]
Combining the estimates above, we can find constants $c_2, N_2$, so that for all $N \geq N_2$ there exists a subset $A_N \subset \gamma$ so that $\mathbb{P}_{\gamma}(A_N) \geq (1 - e^{-c_2N})$ and that for all $\varrho \in A_N$ and all $n \geq N$

$$\hat{N}_n(\varrho) \leq an, \quad \text{and } \mathcal{H}_{\hat{N}_n(\varrho)} > e^{an} \mathcal{H}_0.$$  

Let $\alpha_1 < \alpha$ and assume that for some $\varrho \in A_N$ and $n \geq N$ there is an integer $m(\varrho) \in [1, \hat{N}_{n+1}(\varrho) - \hat{N}_n(\varrho)]$ so that

$$\mathcal{H}_{\hat{N}_n(\varrho) + m(\varrho)} < e^{n\alpha_1} \mathcal{H}_0.$$  

Then by Proposition 3.1, for $d = \iota \left( \frac{1 - f_2}{1 - j_1} \right)$ we have that

$$d^m e^{n\alpha} \mathcal{H}_0 \leq d^m \mathcal{H}_{\hat{N}_n} \leq \mathcal{H}_{\hat{N}_n + m} \leq e^{n\alpha_1} \mathcal{H}_0.$$  

Then $d^m \leq e^n(\alpha_1 - \alpha)$, and hence

$$m \geq \frac{n(\alpha_1 - \alpha)}{\ln 1/d} =: \xi n.$$  

Let $\gamma_n(\varrho)$ be the long curve that contains $\varrho$ at time $\hat{N}_n(\varrho)$. Then by Lemma 5.5 we have that

$$m_{\gamma_n}(\varrho : m(\varrho) \geq \xi n) \leq m_{\gamma_n}(\varrho : \hat{N}_{n+1}(\varrho) \geq \xi n) \leq b\vartheta_4^{\xi n} - N_0.$$  

Thus we have that for some $c_3 = c_3(\vartheta_4, N_0)$

$$m_{\gamma_n}(\varrho : \exists m \text{ so that } \mathcal{H}_{\hat{N}_n(\varrho) + m} < e^{n\alpha_1} \mathcal{H}_0) \leq C\vartheta_4^{\xi n} (\vartheta_4^{\xi n})^n \leq e^{-c_3n}.$$  

Thus, there exists $c_4, N_4 > 0$ so that for all large $N \geq N_4$

$$m_{\gamma}(\varrho : \text{there exists } n \geq N \text{ and } m \geq 1, \text{ so that } \mathcal{H}_{\hat{N}_n(\varrho) + m} < e^{n\alpha_1} \mathcal{H}_0) \leq e^{-c_4N}.$$  

Adding this to the sets above, we see that there exist constants $c_5, N_5 > 0$ such that for all $N \geq N_5$ one can find a subset $B_N \subset A_N$ so that $|B_N| \geq 1 - e^{-c_5N}$ and that for all $\varrho \in B_N$ we have for all $n \geq N$ that

$$\hat{N}_n(\varrho) < an, \quad \mathcal{H}_{\hat{N}_n(\varrho)} > e^{n\alpha} \mathcal{H}_0,$$  

and for all $n \geq N$ and $\ell \in (\hat{N}_n, \hat{N}_{n+1})$ we have

$$\mathcal{H}_{\ell} \geq e^{\alpha_1 n} \mathcal{H}_0.$$  

Now consider the times $[an]$ for $n \geq N$. There exists $k$ so that $\hat{N}_k \leq [an] \leq \hat{N}_{k+1}$. Then for every $\varrho \in B_N$ and $n \geq N$

$$\mathcal{H}_{[an]}(\varrho) \geq e^{k\alpha_1} \mathcal{H}_0(\varrho).$$  

But we have that $[an] > \hat{N}_n(\varrho)$, hence $k \geq n$. Thus

$$\mathcal{H}_{[an]} \geq e^{k\alpha_1} \mathcal{H}_0 \geq e^{[an] \frac{\alpha_1}{N}} \mathcal{H}_0.$$
Between the times \([an] \leq s \leq [a(n + 1)]\), \(\mathcal{H}_k\) can decrease by at most \(d^a\). Hence for all \([an] \leq s \leq [a(n + 1)]\) and \(\varrho \in B_N\) we have that

\[ \mathcal{H}_s \geq d^a e^{[an]_a \mathcal{H}_0} \geq e^{s_2/a} \mathcal{H}_0 = e^{a_0} \mathcal{H}_0. \]

To get rid of \(a\) in front of \(N\) we will take \(N\) larger. Thus we have shown that there exists \(c_6, N_6 > 0\) so that for all \(N \geq N_6\) there exists a set \(B_N \subset \gamma\), with \(m_\gamma(B_N) > 1 - e^{-c_6 N}\) and for all \(\varrho \in B_N\) we have for any \(n \geq N\) that

\[ \mathcal{H}_n \geq \mathcal{H}_0 e^{(a_0/a)n}. \]

To summarize, by setting \(\beta = c_6, \alpha = a_0/a\) and \(N_* = N\), we get

\[ m_\gamma(\varrho = (\sigma_0, \mathcal{H}_0)) : \mathcal{H}_n \geq e^{an} \mathcal{H}_0, \text{ for all } n \geq N \geq (1 - e^{-\eta N})|\gamma|. \]

It now remains to transition from the modified system \(\mathcal{F}_0\) to the original system \(\mathcal{F}\). Recall that by construction the two systems coincide above \(V_0 \gg V_*\). By the estimate above we have for every long curve \(\gamma\) on the vertical line \([\sigma = \sigma_0] (\mathcal{H} \geq V_*, \sigma_0 \in (0, 2))\) that there is a set \(A_N \subset \gamma\), with \(m_\gamma(A_N) > (1 - e^{-\beta N})|\gamma|\), so that for all \(n \geq N\) and \(\varrho = (x_0, \mathcal{H}_0) \in A_N\) we have that

\[ \mathcal{H}_n(\varrho) \geq e^{an} \mathcal{H}_0. \]

Note that for all \(n \geq N\) we have that \(\mathcal{H}_n(\varrho) \geq e^{aN} \mathcal{H}_0 > \mathcal{H}_0\). Hence, \(\mathcal{H}_n > \mathcal{H}_0\) for all \(n \geq N\).

If we now take \(\mathcal{H}_0 \geq V_0(1/d)^N\), then for all \(n \leq N\) we will have that \(\mathcal{H}_n \geq V_0\). Thus, for all \(\varrho \in A_N\) and \(n \geq 1\) we will have that \(\mathcal{H}_n(\varrho) > V_0\). This means that the energy of the ball will never go below \(V_0\), so the dynamics will coincide with that of the original system \(\mathcal{F}\). If we now take \(\varepsilon\) so that \(V_0(1/d)^N = V_\varepsilon t_N\) and consider a foliation of \(R_1 \cap [V_1, V_2]\) by long curves, the result will follow.

The dependence of \(\alpha\) on \(\mathcal{E}\), \(c\) follows from Lemma 7.1. Note that the parameters \(\beta, \alpha\) and \(N_*\) depend on \(\kappa\) and \(\vartheta\), which in turn depend on \(\mathcal{E}\) and \(c\), if \(f_*\) and \(V_*\) are sufficiently large. \(\square\)

7.2. The moment estimate. Finally in this section we prove the moment estimate stated in the previous section.

Proof of Lemma 7.1 Let \(\kappa = \eta/N_0\) where \(\eta\) is such that for \(|s| \leq 2\eta\) we have

\[ e^{-s} \leq 1 - s + s^2. \]

Then

\[
\begin{align*}
\mathcal{E}_\gamma[e^{-\kappa \Delta}] &= \mathcal{E}_\gamma[e^{-\kappa \Delta}1_{\hat{N} \leq 2N_0}] + \mathcal{E}_\gamma[e^{-\kappa \Delta}1_{\hat{N} > 2N_0}] \\
&\leq 1 - \mathcal{E}_\gamma[\kappa \Delta 1_{\hat{N} \leq 2N_0}] + \mathcal{E}_\gamma[(\kappa \Delta)^2 1_{\hat{N} \leq 2N_0}] + \mathcal{E}_\gamma[e^{-\kappa \Delta}1_{\hat{N} > 2N_0}].
\end{align*}
\]

By Proposition 3.1 we have that \(|\kappa \Delta| = \kappa |\ln \mathcal{H}_{\hat{N}(x)} - \ln \mathcal{H}_0| \leq \kappa c \hat{N}\) for some finite \(c > 0\). Then

\[
\mathcal{E}_\gamma[e^{-\kappa \Delta}] \leq 1 - \mathcal{E}_\gamma[\kappa \Delta] + 4\eta^2 + \mathcal{E}_\gamma[(e^{-\kappa \Delta} + \kappa \Delta)1_{\hat{N} > 2N_0}].
\]
Next

\[ E_{\gamma}[\Delta] = E_{\gamma} \left[ \ln \mathcal{H}_{N_0} - \ln \mathcal{H}_0 \right] + E_{\gamma} \left[ \ln \mathcal{H}_{N} - \ln \mathcal{H}_{N_0} \right] \]

For the first term, by Proposition 6.1 we have that

\[ E_{\gamma} \left[ \ln \mathcal{H}_{N_0} - \ln \mathcal{H}_0 \right] \geq \frac{\mathcal{E}}{3} N_0. \]

For the second term by Lemma we have that 5.5

\[ E_{\gamma} \left[ \ln \mathcal{H}_{N} - \ln \mathcal{H}_{N_0} \right] \leq C b \sum_{m=N_0}^{\infty} (m - N_0) \vartheta_4^{m-N_0} < \infty. \]

Thus

\[ E_{\gamma}[\Delta] = \frac{\mathcal{E} N_0}{2} + O(1) = \left( \frac{\mathcal{E}}{3} + o(1) \right) N_0. \]

On the other hand, we note that

\[ E_{\gamma}[(e^{-\kappa \Delta} + \kappa \Delta)1_{\mathcal{N} \geq 2N_0}] \leq 2E[e^{\kappa |\Delta|}1_{\mathcal{N} \geq 2N_0}] \]

\[ \leq C_0 b \sum_{m \geq 2N_0} e^{c\kappa (m-N_0)} \vartheta_4^{(m-N_0)} = C_0 b \sum_{m \geq N_0} (e^{c\kappa} \vartheta_4)^m \]

(14)

We now take \( N_0 \) so large that \( e^{c\kappa} \) is so small that \( e^{c\kappa} \vartheta_4 < 1 \).

Note that \( N_0 \) can be increased without affecting the values of \( c, b, \vartheta_4 \) (however it will affect the size of a long curve). We first take \( \bar{f}_s \) so large that we have the unstable cone invariance and that the constants in Lemma 5.5 are uniform for all \( |\bar{f}_1|, |\bar{f}_2| > \bar{f}_s \). Thus the constants \( c, b, \vartheta_4 \) will be fixed for all large \( \bar{f}_s \). Therefore to achieve Proposition 6.1, it remains to further increase the values \( \bar{f}_s \) and \( V_s \).

We obtain from above that

\[ E_{\gamma}[(e^{-\kappa \Delta} + \kappa \Delta)1_{\mathcal{N} \geq 2N_0}] \leq C^* (e^{c\kappa} \vartheta_4)^{N_0}. \]

To sum it up, we have

\[ E_{\gamma}[e^{-\kappa \Delta}] \leq 1 - \left( \frac{\mathcal{E}}{2} + o(1) \right) \eta + 4\eta^2 + C^* e^{c\eta} \vartheta_4^{N_0}. \]

We observe that the parameters \( \kappa, \eta \) are independent and they can be made arbitrarily small at the expense of taking \( N_0 \) large. We first take \( \eta \) so small that \(- \left( \frac{\mathcal{E}}{2} + o(1) \right) \eta + 4\eta^2 \) is negative. Then we keep \( \eta \) fixed and increasing \( N_0 \), so that \( e^{c\eta} \vartheta_4^{N_0} \) becomes small compared with the remaining terms. As a result, \( E_{\gamma}[e^{-\kappa \Delta}] \) becomes less than a number of size \( 1 - \eta \frac{\mathcal{E}}{2} \). Note that as long as \( \bar{f}_s \) and \( V_s \) are large, \( \alpha \) depends only on \( \mathcal{E}, c \), as in Proposition 6.1 the constant \( \bar{f}_s \) depends only on \( \mathcal{E}, c \).
Declarations

Conflict of interest We have no conflict of interest to disclose.

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Communicated by C. Liverani