"Hodge strings" and elements of K.Saito’s theory of the Primitive form

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Abstract

The "Hodge strings" construction of solutions to associativity equations is proposed. From the topological string theory point of view this construction formalizes the "integration over the position of the marked point" procedure for computation of amplitudes. From the mathematical point of view the "Hodge strings" construction is just a composition of elements of harmonic theory (known among physicists as a $t$-part of $t - t^*$ equations) and the K.Saito construction of flat coordinates (starting from flat connection with a spectral parameter).

We also show how elements of K.Saito theory of primitive form appear naturally in the "Landau-Ginzburg" version of harmonic theory if we consider the holomorphic pieces of germs of harmonic forms at the singularity.

1. Introduction and summary

The first aim of this article is to explain how and why some version of the Hodge (harmonic) theory leads to the specific map from tensor powers of the vector space to the cohomologies of the Deligne-Mumford compactification of the moduli space of rational curves with $n$ marked points. In physics such a map is called "generalized amplitudes in topo-

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logical strings”; in mathematics, a particular case of this map is called ”Gromov-Witten” invariants.

The second aim is to explain how elements of the K.Saito theory of Primitive form (that, among other things, provides solutions to associativity (WDVV) equation [BV]) naturally arise from the ”Landau-Ginzburg” version of Hodge (harmonic) theory.

In section 2, we remind that ”generalized amplitudes” in genus zero (in a ”topological string without gravitational descendents”) are in a one-to-one correspondence with the solutions to the WDVV equation (4), thus our construction (called ”Hodge strings”) should end with the solution to this equation.

Sections 3 and 4 contain motivations coming from the topological string theory for the construction (they partly answer the question ”why?”) and could be omitted by a reader who is a mathematician. In section 3, we review the concept and structures of ”conformal topological strings”, and in section 4 we describe the ”integration over the position of the marked point” procedure of computation of the ”amplitudes” in genus zero. Along these lines we explain the origin of the $QG_-$-system, that contains $\mathbb{Z}_2$ graded vector space $H$, odd operators $Q$ and $G_-$, even commuting $Q$-closed operators $\Phi_i$ (all operators are acting on $H$) and the bilinear pairing $<>$ on $H$. The ”integration over the marked point” procedure shows what kind of structure should we expect to see.

Section 5 is axiomatic: here we introduce the notion and general properties of an abstract $QG_-$ system. Then we show that starting with the $QG_-$-system $(H, Q, G_-, \Phi_i, <>)$ having Hodge property, Pairing of the cohomology property and the Primitive element property one can canonically construct a solution to the WDVV equation. The construction is done in two steps. First, by comparing two flat connections (the ”Hodge” connection and the ”Gauss-Manin” connection) on the bundle of $Q(t) + zG_-$ cohomologies, we show the existence of a flat connection with the spectral parameter ($\nabla H - z^{-1}C$), which is known in physics as the $t$-part of $t - t^*$ equations [CV]. Then, using the Primitive element property, we construct a solution to the WDVV equations, like K.Saito did in the theory of Primitive form. This section answers the question ”how” and formally is independent of the previous sections. Nevertheless, we try to comment ”why” the construction goes this way by referring to section 4.

In section 6, we review the so called ”Landau-Ginzburg” realization of the ”Hodge strings” input (in physics such a system is known as $N = 2$ supersymmetric Landau-Ginzburg quantum mechanics).

Then, in section 7, we start out by briefly reviewing (in subsection
7.1) elements of K. Saito’s theory of primitive form in the form of "good section" and in terms of $QG_{-}$ systems. (In the Appendix we relate it to the original formulation). To reach K. Saito’s theory of Primitive form from the "Landau-Ginzburg" system (for quasihomogeneous case) we first pass from the smooth quickly vanishing forms of the "Landau-Ginzburg" system to the non-holomorphic germs of forms at a singularity, and then take holomorphic pieces of germs. We find that holomorphic pieces of germs coming from the development of harmonic forms of the "Landau-Ginzburg" theory satisfy two of K. Saito’s conditions for a "good section" and, with a quasihomogeneous "antiholomorphic superpotential $\bar{U}$", satisfy the third condition (this third condition is not necessary for construction of the solution to the WDVV equations).

We explain the ambiguity of the solutions for K. Saito’s conditions for a "good section" as coming from the "antiholomorphic superpotential $\bar{U}$" that disappears in the "taking holomorphic pieces" procedure (this phenomena in topological strings is called the "holomorphic anomaly").

We expect that methods developed here could be useful in the understanding of non-quasihomogeneous systems.

Conventions. The sum over repeating indexes (the physicist’s convention) is adopted in the text.

2. "Compact" topological strings and the associativity equation

"Topological string theory" [Wi, DW, VV, DVV, Wi2, KM, BCOV] studies genus $q$ "generalized amplitudes" $GA_{q}$, taking values in cohomologies of the Deligne-Mumford compactification $\bar{M}_{q,n}$ of the moduli space of complex structures of genus $q$ Riemann surfaces with $n$ marked points. Pairing between $GA_{q}$ and the cycle $C \in \bar{M}_{q,n}$ is given by the functional integral [Wi, DW, KM]

$$ (GA_{q}, C)(V_{1}, \ldots, V_{n}) = \int_{C \in \bar{M}_{q,n}} \int D\phi V_{1}(\phi(z_{1})) \ldots V_{n}(\phi(z_{n})) \exp(S_{TS}(\phi)), $$

(1)

fields $V_{i}(\phi(z))$ are called "vertex operators" and ordinary "amplitudes" $A_{q}(V_{1}, \ldots, V_{n})$ correspond to $C = M_{q,n}$.

Deligne-Mumford compactification $\bar{M}_{0,n}$ is a union of $M_{0,n}$ (a set of $n$ noncoincident points on $CP_{1}$ moduli $SL(2, C)$ action) and the compactification divisor $Comp$. The divisor $Comp$ is a union of components $C(S)$, where $S$ partitions $n$ marked points into two groups consisting of $n_{1}(S)$ and $n_{2}(S)$ points, $n_{i} > 1$. A surface corresponding to a general point in $C(S)$ is a union of two spheres having one common point with $n_{1}(S)$
marked points on the first sphere and \( n_2(S) \) on the second. The set of general points in \( C(S) \) form the space \( M_{0,n_1+1} \otimes M_{0,n_2+1} \).

In this paper, we will consider a class of "compact" topological string theories that have no gravitational descendents \([W]\) among its "vertex operators" and have a nondegenerate pairing on the space of "vertex operators". This class of theories includes, for example, topological sigma models of type A on compact Kahler manifolds and twisted unitary superconformal theories. It is believed that these theories play the same role among all theories as smooth compact manifolds among all manifolds. It is expected that, in "compact" topological theories, the functional integral for surfaces corresponding to points in \( C(S) \) factorizes:

\[
(GA_0, C(S))(V_{i_1}, \ldots, V_{i_n}) = \eta^{jk} A_0(V_{i_1}, \ldots, V_{i_{n_1}}, V_j) A_0(V_{i_{n_1+1}}, \ldots, V_{i_{n_2+n_1}}, V_k)
\] (2)

where \( \eta \) is a matrix of symmetric bilinear nondegenerate products on vertex operators.

Keel \([K]\) found that the homologue ring \( H_\ast \) of \( \bar{M}_{0,k} \) is generated by cycles \( C(S) \). He described relations between these cycles in homologies leading to constraints on \( GA_0 \) because of (2).

An elegant way of formulating these constraints uses the generating function for "amplitudes". Introducing formal parameters \( T_i \), we define the germ \( F(T) \):

\[
F(T) = \sum_{k=3}^{\infty} \frac{1}{k!} A_0(T_{i_1} V_{i_1}, \ldots, T_{i_k} V_{i_k})
\] (3)

Then, Keel’s relations lead to:

\[
\frac{\partial^3 F(T)}{\partial T_i \partial T_j \partial T_k} \eta^{kl} \frac{\partial^3 F(T)}{\partial T_l \partial T_p \partial T_q} = \frac{\partial^3 F(T)}{\partial T_i \partial T_p \partial T_k} \eta^{kl} \frac{\partial^3 F(T)}{\partial T_l \partial T_j \partial T_q}
\] (4)

Using the factorization property and Keel’s description of homologies of moduli space, we can reconstruct \( GA_0 \) from \( A_0 \) \([KM]\), see also \([DV]\).

### 3. Amplitudes in conformal topological strings theory

The "Hodge string" construction generalizes the "integration over the position of the marked point" procedure \([V], [DV], [LO], [LP], [BCOV], [L2]\) of computation of amplitudes in "conformal topological theory coupled to topological gravity" also known as "conformal topological string theory".

The general covariant action \( S_m \) of topological field theory is a sum of a "topological" (metric independent) \( Q \)-closed term \( S_{top} \) and a \( Q \)-exact
term for a fermionic scalar symmetry $Q$:

$$S_m = S_{\text{top}}(\phi) + Q(R(\phi), g),$$

where $g$ denotes the metric on the Riemann surface. The energy-momentum tensor $T$ is $Q$-exact:

$$T = Q\left(\frac{\delta R}{\delta g}\right) = Q(G)$$

We call topological field theory conformal, if $R$ is conformally invariant, i.e. $G$ is traceless.

We introduce fermionic two-tensor fields $\psi$, such that functions of $g, \psi$ are forms on the space of metrics. An external differential on these forms could be written as follows: $Q_g = \psi \frac{\delta}{\delta g}$.

The action for a topological theory coupled to topological gravity is

$$S_{TS} = S_m + \psi G = S_{\text{top}} + (Q + Q_g)(R).$$

The functional integral $Z(g, \psi)$ over the set of fields $\phi$ with the action $S_{TS}$ is a closed form on the space of metrics. Since $G$ is traceless, $Z$ is a horizontal [DVV, Di] form with respect to the action of conformal transformations of metrics and diffeomorphisms of the Riemann surface; thus, it defines a closed form on the moduli space of conformal (=complex) structures on the genus $q$ Riemann surface.

To construct generalized amplitudes we insert fields (zero-observables =”vertex operators”) $V_i$ at marked points on Riemann surface. They should satisfy

$$Q(V_i) = 0, G_{0,-}(V_i) = 0.$$  \hspace{1cm} (6)

Here $G_{0,-}$ is the superpartner of the component of the energy-momentum tensor $T_{0,-}$ that corresponds to the rotation with the constant phase $z \to e^{i\theta} z$ of the local coordinate at the marked point. The first condition in (6) is needed to construct a closed form on the space of metrics, while the second provides horizontality of the corresponding form with respect to diffeomorphisms that leave marked points fixed but rotate local coordinates [All, DN, DVV, Eg, Di].

4. Integration over positions of marked points

The "integration over marked points" procedure reduces all genus zero amplitudes to the three point amplitude:

$$F_{ijk} = A_0(V_i, V_j, V_k),$$

\[\text{†Differential form on the principal bundle is called horizontal if its contraction with the vertical (tangent to fiber) vector is zero. Closed horizontal forms on the total space correspond to closed forms on the base of the bundle.}\]
which can be computed from topological matter theory.

In conformal topological theory, we associate a two-observable $V_i^{(2)} = G_{L,-1} G_{R,-1} V_i$ to a zero observable $V_i$. Thus, we deform topological theory to a family of theories parametrized by $t$, with the action $S_m(t) = S_m + t_i V_i^{(2)}$; thus, zero-observables $V$ form a tangent bundle to this space of theories [DVV].

If, in the functional integral that computes the $n$-point amplitude, we first pick up one of the marked points (we will call it a “moving point”), integrate over the position of the moving point, and only then take the functional integral, the $n$-point amplitude becomes the derivative in $t$ of the $n - 1$ point amplitude.

In the process of integration, we should take special care about the region where the moving point tends to hit a fixed point because the geometry there is not a naive one. The contribution from this region(contact terms [VV, Lo1, Lo2, LP, D1, BCOV]) leads to a specific contact term connection on the bundle of zero-observables over the space of theories and thus on the tangent space to the space of theories.

Repeating this procedure again and again, we can recover amplitudes from $F_{i,j,k}(t)$. The amplitudes should be symmetric and independent of the order of integration over positions of marked points.

In other terms, generating parameters $T$ from (3) should become the so-called special coordinates on the space of theories, the derivatives with respect to the special coordinates should become covariantly constant sections of the contact term connection, and symmetric tensor $F_{i,j,k}$ (in the special coordinate frame) should be a third derivative of $F(T)$. Moreover, $F(T)$ has to solve the WDVV equations (4).

All this implies that the contact term connection is quite a special one!

To gain better understanding of this connection, we will study the space of states in 2d theory associated with the circle (considered as a component of the boundary of the Riemann surface). Moreover, we will restrict ourselves to the subspace $H$ of these states that are invariant under constant rotation of the circle.

Fermionic symmetry $Q$ of the theory and $G_{0,-}$ reduce to odd anticommuting operators $Q$ and $G_-$ on $H$.

Zero-observables $V_i$ (being inserted at the middle of the punctured disc) generate states $h_i$ that are $Q$ and $G_-$ closed:

$$Qh_i = G_- h_i = 0,$$

the zero observable 1 generates the distinguished state $h_0$. The operation of sewing two discs together corresponds to the bilinear pairing $<, >$. Integrals of zero observables along the boundary give operators $\Phi_i = \int_{S_1} V_i d\sigma$. 

One can show from the functional integral that the objects defined above have the following properties:

\[ Q^2 = G_+^2 = QG_+ + G_+Q = 0, \quad [Q, \Phi_i] = 0, \quad [\Phi_i, \Phi_j] = 0, \quad (8) \]

\[ Q^T = EQ, \quad G^T = -EG, \quad \Phi^T = \Phi \quad (9) \]

Here transposition "T" is taken with respect to the pairing \(<,>\), and operator \(E\) commutes with \(\Phi\) and anticommutes with \(Q\) and \(G_+\).

In the deformed theory, \(Q(t) = Q + [G_-, t_i \Phi_i]\) in the first order in \(t\). To ensure it globally we will take for simplicity\[‡\]

\[ [[G_-, \Phi_i], \Phi_j] = 0. \quad (10) \]

The contribution from the region near the place where the "moving" \(i\)-th point hits the marked \(j\)-th one gives the "cancelled propagator argument" (CPA) connection on states \(h_j\) over the space of theories \([V, D, I, LO]\):

\[ \delta^{(CPA)} h_j = G_- \int_0^\infty d\tau G_{0,+} \exp(-\tau T_{0,+}) \Phi_i h_j, \quad (11) \]

thus \(\delta^{(CPA)} h\) is \(G_-\)-exact. Here \(T_{0,+}\) is the Hamiltonian acting on the space \(H\), and \(G_{0,+}\) is its superpartner: \(T_{0,+} = Q(G_{0,+})\).

Covariantly constant sections\[§\] of the CPA connection will be denoted as \(h_i(t)\). This connection induces the connection on the space of zero-observables: covariantly constant sections of contact term connection \(V_i(t) = u_i^j(t)V_j\) are such that, being inserted in the middle of the disc in the \(t\)-deformed theory, they produce covariantly constant sections \(h_i(t)\):

\[ h_i(t) = \lim_{r \to 0} u_i^j(t)r T_{0,+} \Phi_j h_0(t). \quad (12) \]

Let us denote as \(C_i(t)\) the linear operator representing the action of \(\Phi_i\) in \(Q(t)\)-cohomologies. Then, the relation (12) reads:

\[ [h_i(t)]_{Q(t)} = u_i^j(t)C_j(t)[h_0(t)]_{Q(t)} \quad (13) \]

here and below \([h]_{Q}\) stands for a class of a \(Q\)-closed element \(h\) in \(Q\)-cohomologies.

\[†\]In general case one has to go in for Kodaira-Spencer type arguments, see [BCOV].

\[§\]Flatness of CPA connection is necessary for the consistency of the procedure.
From the functional integral we get:

\[ F_{ijk}(t) = \langle h_i(t), \Phi_j h_k(t) \rangle u^l_j(t) \]  \hspace{1cm} (14)

While the string origin of the described procedure is quite natural, its consistency is far from being obvious.

5. The "Hodge string" \(QG_-\)-system

5.1 General facts about \(QG_-\) systems

Definition. The \(QG_-\) system \((Q, G_-, \Phi, H)\) is a collection of \(Z_2\)-graded vector space \(H\), odd operators \(Q\) and \(G_-\), and a set of even operators \(\Phi_i\), \(i = 1, \ldots, \mu\), acting on this space, that have the properties (8,10).

Given a \(QG_-\) system, one can construct a family \(Q(t)\) of nilpotent odd operators in \(H\):

\[ Q(t) = Q + t_i [G_-, \Phi_i] \]  \hspace{1cm} (15)

over a deformation space with coordinates \(t_i\).

Definition. Cohomologies of \(QG_-\)-systems.

Let \(H_{Q(t)}\) be the space of \(Q(t)\) cohomologies in \(H\). Let

\[ Q(t, z) = Q(t) + zG_- \]  \hspace{1cm} (16)

- Let \(H_{Q(t)}\) be the space of cohomologies of \(Q(t)\) in \(H\).
- Let \(\hat{H}_{Q(t,z)}\) be the space of cohomologies of \(Q(t, z)\) in \(H \otimes C[[z]]\).
- Let \(H_{Q(t,z)}\) be the space of cohomologies of \(Q(t, z)\) in the space \(H \otimes C << z >>\), where \(C << z >>\) is the space of Laurent expansions in \(z\).
- Let \(H^l_{Q(t,z)} \subset \hat{H}_{Q(t,z)}\) be the space of "little" cohomologies, defined as those classes in \(\hat{H}_{Q(t,z)}\) that have representatives in \(H\):

\[ H^l_{Q(t,z)} = \{ [\omega]_{Q(t,z)} \in \hat{H}_{Q(t,z)} | \omega \in H, Q\omega = G_- \omega = 0 \} \]  \hspace{1cm} (17)

Remark. The space \(\hat{H}_{Q(t,z)}\) has a natural decreasing filtration by powers of \(z\):

\[ \hat{H}_{Q(t,z)} = \hat{H}^{(0)} \supset \hat{H}^{(1)} \supset \ldots \]  \hspace{1cm} (18)

a class is in \(\hat{H}^{(k)}\) if it contains element \(z^k\omega\). The inclusion \(H^l_{Q(t,z)} \subset \hat{H}_{Q(t,z)}\) induces the decreasing filtration on "little" cohomologies.
Remark from string theory. In string theory of the general type the space of "little" cohomologies corresponds to the space of states created by "vertex operators". One can show \[\text{[VV, Lo1, Eg, Lo2]}\] that states from \(H_{Q(t,z)}^{k(k)}\) are created by "vertex operators" that are \(k\)-th gravitational descendents.

Definition. Let \(C_i(t) : H_Q(t) \rightarrow H_Q(t)\) be a linear operator representing the action of \(\Phi_i\) in \(Q(t)\) cohomologies:

\[C_i(t)[\omega]_{Q(t)} = [\Phi_i \omega]_{Q(t)}\]  \hspace{1cm} (19)

Remark. Operators \(C_i(t)\) should be considered as components of the one-form on the deformation space with values in \(\text{End}H_Q(t)\). From the definition it follows that these operators commute with each other:

\[[C_i(t), C_j(t)] = 0\]  \hspace{1cm} (20)

Definition. A morphism of \(QG_\beta\)-systems

\[(Q^1, G^1_\beta, \Phi^1_\beta, H^1) \rightarrow (Q^2, G^2_\beta, \Phi^2_\beta, H^2)\]  \hspace{1cm} (21)

is a morphism \(H^1 \rightarrow H^2\) commuting with the action of operators \(Q^\beta, G^\beta_\beta, \Phi^\beta_\beta\) in \(H^\beta, \beta = 1, 2\).

It is clear that the morphism of \(QG_\beta\) systems induces the morphism of cohomologies of \(QG_\beta\) systems.

Definition. A morphism of \(QG_\beta\)-systems will be called a quasiisomorphism of \(QG_\beta\) systems if it induces an isomorphism in all cohomologies of \(QG_\beta\) systems.

Definition. By the "Gauss-Manin" connection in a \(QG_\beta\) system, we call a canonical flat connection \(\nabla^GM\) in \(H_{Q(t,z)}\) over \(C[[t]]\), whose horizontal sections \([\omega^GM(t,z)]_{Q(t,z)}\) satisfy the following:

\([\omega^GM(t,z)]_{Q(t,z)} = [\exp(-t_i \Phi_i/z)\omega^GM(0,z)]_{Q(t,z)}\]  \hspace{1cm} (22)

i.e. their representatives solve the following differential equation:

\[\frac{\partial}{\partial t_i} \omega^GM(t) + z^{-1}\Phi_i \omega^GM(t) \in \text{Im}(Q(t,z))\]  \hspace{1cm} (23)

Remark. We call this canonical connection "Gauss-Manin" following K.Saito (see the Appendix).

Remark. It is clear that morphisms of \(QG_\beta\) systems induce morphisms of the "Gauss-Manin" connections, and quasiisomorphisms induce the isomorphism of these connections.
Below we will list some additional properties that the \( QG_- \) system could have and that would be important for the "Hodge string" system.

**The Hodge property.**

\[
\text{Im}Q \cap \text{Ker}G_- = \text{Im}G_- \cap \text{Ker}Q = \text{Im}(QG_-) \tag{24}
\]

**Statement 5.1** It follows from the Hodge property that \( \text{dim}H_Q = \text{dim}H_{G_-} \) and there exists a set \( \{ h_a \} \) of \( Q \) and \( G_- \) closed elements of \( H \) (these elements are unique up to \( \text{Im}QG_- \)), such that classes \( [h_a]_Q \) and \( [h_a]_{G_-} \) form bases in \( Q \) and \( G_- \) cohomologies.

**Remark.** In harmonic theory such elements are just harmonic forms; that is why below we will call these elements "harmonic".

**Statement 5.2** If the \( QG_- \) system has a Hodge property, then

\[
H_{Q(0,z)} \cong \hat{H}_{Q(0,z)} \otimes C[z^{-1}] \cong H_{Q(0)} \otimes C \ll z >> \tag{25}
\]

and

\[
H^I_{Q(0,z)} \cong H_{Q(0)}. \tag{26}
\]

**Proof.** Classes in the first line are identified by considering \( hP(z) \) and \( hP(z, z^{-1}) \) for "harmonic" \( h \) as representatives of classes in \( \hat{H}_{Q(0,z)} \) and \( H_{Q(0,z)} \) respectively. (Here, \( P \) are polynomials.) The statement on the second line becomes clear as a generalization of the following reasoning: \( [\omega_1] = z[\omega_2] \), if \( \omega_1 = Q\omega' \) and \( \omega_2 = -G_-\omega' \). But from the Hodge property it follows that \( \omega_i \in \text{Im}QG_- \), and thus \( \omega_i \in \text{Im}(Q + zG_-) \). \( \square \)

**Remark.** The difference between \( H^I_{Q(0,z)} \) and \( H_{Q(0)} \) is one of the criteria showing the failure of the Hodge property. From the string theory point of view this difference means that corresponding string theory has gravitational descendents among its vertex operators, and is "noncompact" in the sense of section 2.

The \( QG_- \) system with pairing is a \( QG_- \) system with the bilinear pairing \( <,> \) satisfying property (9).

**Remark.** From (9) it follows that the pairing \( <,> \) descends to both \( Q(t) \) and \( G_- \) cohomologies, i.e. for \( Q(t) \)-closed \( \omega_2 \in H \)

\[
< Q(t)\omega_1, \omega_2 >= < \omega_2, Q(t)\omega_1 >= 0, \tag{27}
\]

and for \( G_- \) closed \( \omega_2 \)

\[
< G_-\omega_1, \omega_2 >= < \omega_2, G_-\omega_1 >= 0, \tag{28}
\]

**Pairing of the cohomology property.**
The pairing \( <,> \) is non-degenerate when restricted to \( Q \)-cohomologies.
**Primitive element property.** \( \text{Dim} H_{Q(0)} = \mu \) and there is a class \([h_0]_Q\) in \(Q\)-cohomologies (that we will call a primitive class) such that the set \(\{C_i([h_0]), i = 1, \ldots, \mu\}\) forms a basis in \(H_{Q(0)}\).

**Remark.** The primitive element is not unique. One can easily see that if the \(QG_-\) system has a primitive class, almost all classes are primitive\(^*\).

**5.2 Solution to the WDVV equations from the ”Hodge string” \(QG_-\)-system.**

**Definition.** The ”Hodge string” system is a \(QG_-\)-system with pairing that has the Hodge property, pairing of cohomologies property and the primitive element property.

**Theorem.** There is a canonical construction of a solution to the WDVV equation from the ”Hodge string” system and the choice of a Primitive element.

**Summary of the construction.** The construction is made in two steps. In the first step, we start from the ”Hodge string” system and construct a flat connection with the spectral parameter. Physicists know this connection as the \(t\)-part of the \(t - t^*\) equations [CV]. The Primitive element property is not used in the first step.

The flat connection with the spectral parameter appears from the comparison of two flat connections. The first connection would be the ”Gauss-Manin” connection. The second connection comes as a formalization of the CPA-connection on the space of states (see section 4) - variation of its covariantly constant section will be \(G_-\) exact. Since such a connection exists canonically due to the ”Hodge” property, it will be called below the ”Hodge” connection. This connection could be extended to connection in \(Q(t, z)\) cohomologies, and we will call this extension Hodge connection too. The difference between these two connections turns out to be one-form \(C_i(t)\) (introduced in subsection (5.1)) divided by \(z\) (that becomes a spectral parameter).

In the second step, with the help of the Primitive element property, we induce a flat connection on the tangent bundle to the deformation space from the ”Hodge” connection constructed in step one (i.e., we induce a connection on the space of zero-observables from the connection on the space of states, as in (13)). Then, we integrate covariantly constant vector fields of this connection to special coordinates \(T\) on the deformation space and, finally, construct \(F(T)\). This step was first done by K.Saito (for a \(QG_-\) system coming from the family of hypersurfaces near the singularity).

\(^*\)The work [Kr] implicitly assumes that it is possible to construct solutions to WDVV equation starting from any primitive element.
Construction.

**Statement. Step 1.** The “Hodge String” system leads canonically to the matrix-valued 1-form on the deformation space \( C_{i,ab}(t) \), (first constructed by K. Saito \[Sa\] in a slightly different context) such that the following differential operators commute

\[
\frac{\partial}{\partial t_i} \delta_{ab} - z^{-1} C_{i,ab}(t),
\]

and \( C_{i,ab} = C_{i,ba} \).

**Proof of Step 1.** The proof of the Step 1 is divided in two parts, 1A and 1B.

**1A. Hodge connection in** \( H_{Q(t)} \): The idea of construction of the “Hodge” connection is as follows. The Hodge property canonically identifies \( Q(0) \)-cohomologies and \( G_- \)-cohomologies. While the operator \( Q(t) \) changes with \( t \), the operator \( G_- \) remains fixed. Since variation of \( Q(t) \) is \( G_- \)-exact, it is possible to identify canonically \( Q(t) \) and \( G_- \)-cohomologies over \( C[[t]] \), and construct such a flat connection in \( Q(t) \) cohomologies that the image in \( G_- \) cohomologies of its covariantly constant sections (that are taken to be \( G_- \) closed) is constant. We will call such a connection the “Hodge” connection.

Specifically, let us define \( \Phi = \frac{t}{\tau} \Phi_i \), and consider the following equation:

\[
(Q(0) + \tau [G_-, \Phi]) h_a(t) = 0, \quad h_a(t) = h_a + \sum_{k=1}^{\infty} \tau^k \omega_k \quad (30)
\]

with \( h_a \) being the \( Q(0) \) and \( G_- \) closed element of \( H \). This leads to

\[
Q \omega_1 = -G_- \Phi h_a, \quad Q \omega_{k+1} = -G_- \Phi \omega_k \quad (31)
\]

and we should like to solve these equations for \( G_- \)-exact \( \omega_k \).

Due to the Hodge property, it is possible to solve recursively the equations above for any \( Q(0) \) and \( G_- \) closed element \( h_a \). The solution \( h_a(t) \) is a germ in \( t \) defined up to \( Im(Q(t)G_-) \) (considered as the germ in \( t \)).

The germ \( h_a(t) \) satisfies the following differential equation:

\[
\frac{\partial}{\partial t_i} h_a(t) - G_- \frac{1}{Q(t)} (\Phi_i h_a(t) - C^b_{i,a} h_b(t)) \in Im(Q(t)G_-); \quad (32)
\]

here, \( C^b_{i,a} \) are matrix elements of the operator \( C_i(t) \) written in the basis \( \{ [h_a(t)]_{Q(t)} \} \) in \( H_{Q(t)} \), \( \frac{1}{Q(t)} \) has to be considered as a germ in \( t \) and operation \( \frac{1}{Q} \) is defined only on the \( Q \)-closed elements of \( H \) and stands for taking some preimage of \( Q \). This ends the construction of “Hodge” connection.
**Statement.** The "Hodge" connection preserves the bilinear pairing $<,>$ on $Q(t)$ cohomologies.

**Proof.** The bilinear pairing descends not only to $Q(t)$ but also to $G_-$ cohomologies. However, the class of $G_-$ cohomologies of $h_a(t)$ does not depend on $t$. $\square$

1B. Comparison of "Gauss-Manin" and "Hodge" connections

In order to compare two connections, we will lift the "Hodge" connection to the connection $\nabla^H$ in the bundle of $H_{Q(t,z)}$ cohomologies. The covariantly constant sections of the lifted connection are taken to be equal to $[h_a(t)P_a(z)]_{Q(t,z)}$, where $P_a(z)$ is a $t$ independent element of $C << z >>$.

Due to the identity

$$G_\frac{1}{Q(t)}\omega = -z^{-1}\omega + (Q(t) + zG_-)z^{-1}\frac{1}{Q(t)}\omega$$

we find that the relation between "Gauss-Manin" and "Hodge" connections in $H_{Q(t,z)}$ takes the following form:

$$\nabla_i^{GM} = \nabla_i^H - z^{-1}C_i$$

where $C_i$ is the matrix representing the action of $\Phi_i$ in $Q(t)$ cohomologies.

Now let us rewrite the above relation in the basis of covariantly constant sections of the "Hodge" connection:

$$\nabla_i^{GM} = \delta^b_a \frac{\partial}{\partial t_i} - z^{-1}C(t)^b_{i,a}$$

Since the bilinear pairing is preserved by the "Hodge" connection, it is represented by a $t$ - independent bilinear form in the basis we are working with. Moreover, the bilinear form is non-degenerate (due to a corresponding property) and from the property (9) we conclude that $C_i^T = C_i$. Making a change in the basis that puts the pairing into the form $\delta_{ab}$, we prove the assertion made in the step 1.

**Step 2.** From the assertion in Step 1 we conclude that

$$\frac{\partial}{\partial t_i} C_{j,ab}(t) = \frac{\partial}{\partial t_j} C_{i,ab}(t)$$

so there exists a symmetric matrix $\tau_{ab}(t)$, such that

$$C_{i,ab}(t) = \frac{\partial}{\partial t_i} \tau_{ab}(t).$$

**Definition.** Fix the Primitive element $[h_0]_{Q(0,0)}$. Let $h_0$ be the harmonic representative of $[h_0]_{Q(0,0)}$. Let $h_0(t)$ be the result of the transport of this
element by the "Hodge" connection, so that \( h_0(t) = h_{0,b} h_b(t) \). Here \( h_b(t) \) stands for the basis in covariantly constant sections of "Hodge" connection. Note, that coefficients \( h_{0,b} \) are \( t \)-independent. Let us define the auxiliary special coordinates \( \theta_a \) on the deformation space as:

\[
\theta_a(t) = \tau_{ab}(t) h_{0,b}.
\]

**Statement.** There exists a function \( F(\theta) \) of the auxiliary special coordinates defined by

\[
\frac{\partial^2 F(\theta)}{\partial \theta_a \partial \theta_b} = \tau_{ab}(t(\theta))
\]

such that it satisfies the WDVV equations with \( \eta^{ab} = \delta^{ab} \).

**Proof.** Explicit check.

**Definition.** We define the special coordinates \( \parallel T_i \) as linear combinations of \( \theta_a \) by:

\[
\theta_a = T_i \tau_{i,ab}(0) h_{0,b}.
\]

**The result of the "Hodge string" construction**

The function \( F(\theta_a(T_i)) \) is the desired function that solves the associativity equations with \( \eta^{ij} \), such that its inverse is given by:

\[
(\eta^{-1})_{ij} = \langle h_0, C_i(0) C_j(0) h_0 \rangle = h_{0,a}(C_i(0) C_j(0))_{ab} h_{0,b}
\]

Below, we will present some explicit formulas. Let us define the coefficients \( C_{ij...j_n,ab} \) as

\[
C_{i,ab}(t) = \sum C_{ij...j_n,ab} \frac{t_{j_1} \cdots t_{j_n}}{n!},
\]

Then, we have the following formulas for the "amplitudes"

\[
A_0(V_i, V_j, V_k) = \langle h_0, C_i C_j C_k h_0 \rangle,
\]

\[
A_0(V_i, V_j, V_k, V_l) = \langle h_0, C_i [C_j, C_{kl}] h_0 \rangle
\]

\[
A_0(V_i, V_j, V_k, V_l, V_m) = \langle h_0, C_i [C_{jkl}, C_m] h_0 \rangle + \langle h_0, [C_{im}, C_j] C_{kl} h_0 \rangle + \langle h_0, [C_{im}, C_l] C_{jk} h_0 \rangle + \langle h_0, [C_{im}, C_k] C_{lj} h_0 \rangle
\]

6. The "Landau-Ginzburg" realization of Hodge data

---

\( ^{\parallel} \) the coordinates \( T_i \) integrate vector fields \( u_i \) introduced in (12,13)
Here, we present the realization of the ”Hodge string” system coming from the $N = 2$ supersymmetric quantum mechanics on $C^d$ with superpotential, see [CGP, CG, CV] and references therein.

Let us denote as $X^A$ the holomorphic coordinates on $C^d$ and let us take two polynomials: a holomorphic polynomial $W(X)$ (in physics it is called superpotential) and an antiholomorphic one $\bar{U}(\bar{X})$.

Let the space $H^F$ be the space of smooth forms $\omega$ on $C^d$:

$$\omega = \omega(X, \bar{X})_{A_1...A_p, \bar{A}_1...\bar{A}_q} dX^{A_1} \ldots dX^{\bar{A}_q}$$

such that any finite number of derivatives of their coefficients $\omega(X, \bar{X})_{A_1...\bar{A}_q}$ vanish when $|X| \to \infty$ faster than any negative power of $|X|$ (F stands for ”fast vanishing”). We take the odd operators $Q$ and $G_-$ to be:

$$Q = \bar{\partial} + \partial W; G = \partial + \bar{\partial} \bar{U},$$

here and below, $\partial = dX^A \frac{\partial}{\partial X^A}$ and $\bar{\partial}$ is obtained from $\partial$ by complex conjugation, $\partial W$ stands for external multiplication by the $(1, 0)$ form $dX^A \frac{\partial W}{\partial X^A}$.

Let $\Phi_i = \Phi_i(X)$ be the set of polynomials that form a basis in the ring $J(W) = C[X]/I(W)$, where the ideal $I(W)$ is generated by partial derivatives of $W$.

We will define pairing $<>$ between two forms from $H^F$ as:

$$<\omega_1, \omega_2> = \int_{C^d} \omega_1 C \omega_2,$$

where the Weil operator

$$C = (\sqrt{-1})^{(\hat{p}-\hat{q})},$$

operators $\hat{p} = p, \hat{q} = q$ when acting on the $(p, q)$ forms. This pairing has property (9) with $E = \sqrt{-1}(-1)^{(\hat{p}+\hat{q})}$.

We will study the properties of the $QG_-$ system defined above with the help of some version of Harmonic theory.

Specifically, let us introduce two auxiliary operators:

$$Q' (\bar{U}) = * (\partial - \bar{\partial} \bar{U}) *$$

and

$$G'_-(W) = * (\bar{\partial} - \partial W) *;$$

here $*$ is a Hodge operation associated with the standard flat Kahler metric on $C^d$.

** here and below $\bar{X}$ denotes the complex conjugate of $X$
It is easy to check that operators $Q, Q', G_-, G'_-$ have the same commutation relations as $\bar{\partial}, \bar{\partial}^+, \partial, \partial^+$ on a compact Kahler manifold with the standard Laplacian $\Delta$ being replaced by operator $\Delta(W, \bar{U})$, namely:

\[
\{Q(W), Q'(\bar{U})\} = \{G'_-(W), G_-(\bar{U})\} = \Delta(W, \bar{U})
\]

\[
\Delta(W, \bar{U}) = \Delta - \delta_{BB} \left( \frac{\partial \bar{U}}{\partial X^B} \frac{\partial W}{\partial X^B} + \frac{\partial^2 W}{\partial X^A \partial X^B} dX^A \frac{\partial}{\partial X^B} + \frac{\partial^2 \bar{U}}{\partial X^A \partial X^B} dX^A \frac{\partial}{\partial X^B} \right).
\]

Here, $\iota_v$ denotes the operator of contraction of the form with the vector field $v$.

We will begin with the case $\bar{U} = (W)^*$.

**Statement 6.1** The harmonic forms (forms from the $Ker \Delta(W, (W)^*)$) form bases in spaces of $Q(W), Q'(W^*), G_-(W^*), G'_-(W)$ cohomologies, and a pair $Q$ and $G_-$ has the Hodge property.

**Idea of the proof.** Operator $\Delta(W, (W)^*)$ is Hermitean and has a discrete spectrum; thus, it is a complete analogue of the ordinary Laplasian on Kahler manifolds, so the same proofs are valid.

Now let us examine the structure of cohomologies of $Q(W)$.

**Statement 6.2** The operator $Q(W)$ has cohomologies only in the middle dimension, and the dimension of the space of these cohomologies is equal to the dimension of the ring $J(W)$.

**Sketch of the proof.** 1) $Q$ has no cohomologies below the middle dimension. Proof. Cohomologies of the operator of multiplication by $\partial W$ are non zero only in $(d, k)$ components of the space of forms. Thus, 1) follows from the spectral sequence argument.

2) $Q$ has no cohomologies above the middle dimension. Proof. The Hodge $\ast$ operation identifies harmonic forms above the middle dimension with those below the middle dimension (up to the change of sign of $W$). Thus, 2) follows from 1) and the Statement 6.1.

3) $\text{Dim} H_Q \geq \text{Dim} J$. Proof. Given a polynomial $\Phi_i$ representing an element of $J$, one easily constructs a form $\omega_i \in H^F$

\[
\omega_i = \exp\{-\{Q(W), R\}\} \Phi_i dX^1 \ldots dX^d
\]

where operator $R = \ast \bar{\partial} \bar{W} \ast$.

4) There are no other cohomologies. Proof. Consider the deformation of $W$ into a polynomial having only simple critical points widely separated from each other (their number would be equal to $\text{dim} J$). Then, from quasiclassical arguments one can show that the number of harmonic forms is not greater than the number of critical points. The number of harmonic forms is invariant under the deformation due to 1), 2) since there is an index $(Tr(-1)^{p+q})$ that is invariant under the deformation.
From the description of cohomologies it is obvious that the "Landau-Ginzburg" $QG_-$ system has the Primitive element property.

**Statement 6.3** Pairing $<>$ is nondegenerate.

**Proof.** Consider the value of pairing on the forms $\omega_i$ introduced in the proof of Statement 6.2. After some calculation we get:

$$<\omega_i,\omega_j> = \sum_{\alpha} \text{Res}_\alpha \Phi_i \Phi_j dX^1 \ldots dX^d \prod_{A=1}^{d} \frac{\partial W}{\partial X_A}. \tag{53}$$

here sum is taken over all critical points of $W$, and $\text{Res}_\alpha$ is the residue at the critical point $\alpha$. $\square$

Thus we found that for $U = W^*$ "Landau-Ginzburg" system is a "Hodge string" system.

To see the other options, we will introduce the following definition:

**Definition.** The polynomial $\bar{U}$ is called good for the polynomial $W$, if the Hodge property is satisfied.

**Statement 6.4** If the operator $\Delta(W, \bar{U})$ acting on $H^F$ has a discrete spectrum, and $\dim \ker \Delta(W, \bar{U}) = \dim(H_{Q(W)})$, then polynomial $\bar{U}$ is good for polynomial $W$.

**Proof.** For operators with a discrete spectrum the eigenspaces with non zero eigenvalues are finite dimensional and contain no cohomologies (that is why if $\Delta(W, \bar{U})$ has a discrete spectrum, $\dim \ker \Delta(W, \bar{U})$ can not be smaller than $\dim(H_{Q(W)})$ or than $\dim(H_{G(\bar{U})})$ . These eigenspaces are preserved by $Q$ and $G_-$ and the pair $Q$ and $G_-$ restricted to these spaces has the Hodge property. Thus, the pair $Q$ and $G_-$ may not have the Hodge property only when being restricted to an eigenspace with a zero eigenvalue; however, this space is just annihilated by $Q$ and $G_-$. $\square$

**Corollary 1.** Polynomial $M^2(W)^*$ is good for a polynomial $W$ for any real number $M$.

**Proof.**

$$\Delta(W, M^2(W)^*) = M^{-\hat{q}} \Delta(MW, M(W)^*) M^{(\hat{q})}; \tag{54}$$

here, $\hat{q}$ stands for the operator that acts on $(p, q)$ forms as multiplication by $q$.

**Corollary 2.** Consider the space $A$ of polynomials $\bar{U}$ such that

$$\text{Re} \left( \frac{\partial W}{\partial X_A} \frac{\partial \bar{U}}{\partial X_A} \right) \to +\infty \tag{55}$$

as $|X| \to +\infty$. Then, there is an open set in this space consisting of polynomials $\bar{U}$ that are good for $W$.

**Sketch of the proof.** For polynomials from $A$ the operator $\Delta(W, \bar{U})$ has a discrete spectrum ( forms with the bounded real parts of eigenvalues...
are confined in $C^d$ by the growing potential at infinity; their derivatives are also confined due to the Laplasian). The dimension of the space of harmonic forms for such $\Delta(W, \bar{U})$ can increase only when a non-zero eigenvalue comes down to zero; this could happen only outside an open set containing $\bar{U} = W^*$. Corollary 2 shows that there are many polynomials $\bar{U}$ that are good for $W$, and thus good enough to produce the "Hodge string" system.

7. From LG harmonic theory to "good section" of K.Saito

7.1 K.Saito $QG_-$ system and conditions for a "good section".

We will start with a short sketch of K.Saito theory of primitive form in the form of a "good section" ([Sa], part 4) in terms of $QG_-$-systems.

Let $W(X, t) = W(X) + t_i \Phi_i(X)$ be a versal deformation of the isolated singularity $W(X)$ at $X = 0$. Let $H^S$ be the space of germs of holomorphic $(k, 0)$ forms at a singularity.

K.Saito’s theory represents operators of the $QG_-$ system as follows (K.Saito’s representation is denoted by the superscript $S$):

$$Q^S(t) = \partial W(X, t), \quad G^S_- = \partial, \quad Q^S(t, z) = z\partial + \partial W(X, t).$$

(56)

Here and below the superscript indicating the type of $QG_-$-system will be used only once, i.e. we will write $H^S_{Q(t,z)}$ instead of $H^S_{Q^S(t,z)}$.

One can easily show that $Q^S(t)$ has cohomologies only in holomorphic top forms, i.e. in the $(d, 0)$ component, and $H^S_{Q(t)}$ is non-canonically isomorphic to $J(W)$:

$$H^S_{Q(t)} = H^S / \{\omega' dW(t)\}$$

(57)

where $\omega'$ is a holomorphic $(d-1, 0)$-form.

The K.Saito’s $QG_-$ system definitely has no Hodge property, since the operator $G^S_-$ has no cohomologies. That is why relations between different cohomology groups introduced in subsection 5.1 drastically differ from those (Statement 5.2) that follow from the Hodge property.

Really, it is easy to show that $H^S_{Q(t,z)}$ has a decreasing filtration (a class of $H^S_{Q(t,z)}$ is in $H(t)^{(k)}$ if being considered as a class in $H^S_{Q(t,z)}$ it has a representative $z^k \omega$):

$$H^S_{Q(t,z)} = H(t)^{(0)} \supset H(t)^{(1)} \supset \ldots$$

(58)
and

\[ 0 \to H(t)^{(k+1)} \to H(t)^{(k)} \xrightarrow{\pi_k} H_{Q(t)}^S \to 0 \]  \quad (59)

**Example.** For \( d = 1 \)

\[
H(t)^{(k)} = \{ \partial W(t) \int \ldots \partial W(t) \int P(X)dX \}/\{\partial(W(t)^m), m \in \mathbb{N} \}. \quad (60)
\]

Thus, we see that

\[
H_{Q(t,z)}^{S,l} \cong H_{Q(t)}^S \otimes C[[z]] \cong \hat{H}_{Q(t,z)}^S,
\]

which is much larger than simply \( H_{Q(t)}^S \), and the isomorphisms in the relation above are not canonical.

**Definition.** We define a "section" as a map \( V(t) : H_{Q(t)}^S \to H_{Q(t,z)}^{S,l} \) that inverts the projection \( \pi_1 \).

Having a "section" \( V(t) \), we can invert projections \( \pi_k \) by maps \( z^k V(t) \), thus identifying \( H_{Q(t,z)}^{S,l} \sim C[[z]] \otimes \text{Im}V(t) \), and \( H_{Q(t,z)}^S \sim C \ll z \gg \otimes \text{Im}V(t) \).

Finally, the higher residue pairings \( K_S^{(k)} \) are defined as a set of \( C \)-bilinear pairings on \( H_{Q(t,z)}^{S,l} \).

It is instructive to consider the formal generating function

\[
K_S = \sum_{k \geq 0} z^k K_S^{(k)}
\]

as a pairing

\[
K_S : \hat{H}_{Q(t,z)}^S \otimes \hat{H}_{Q(t,z)}^S \to C[[z]].
\]

For original definition see [5a2]; here, we will just mention some general properties of this pairing and present formulas for \( K^{(0)} \) and \( K^{(1)} \).

\[
K_S(z^k[\omega_1], z^l[\omega_2]) = (-1)^k z^{k+l} K_S([\omega_1], [\omega_2])
\]  \quad (64)

and

\[
K_S^{(0)}([\omega_1], [\omega_2]) = \text{Res}_s P_1 P_2 dX^1 \ldots dX^d \prod_{A=1}^d \frac{\partial W}{\partial X^A}
\]

\[
K_S^{(1)}([\omega_1], [\omega_2]) = \text{Res}_s \sum_{A=1}^d 1/2(P_2 \partial_{X^A} P_1 - P_1 \partial_{X^A} P_2) dX^1 \ldots dX^d \prod_{B=1}^d \frac{\partial W}{\partial X^B}
\]  \quad (65)\quad (66)

where \([\omega_\alpha] \in \hat{H}_{Q(t,z)}^S\) for \( \alpha = 1, 2 \); \( P_\alpha dX_1 \ldots dX_d \) is a representative of the class \([\omega_\alpha] \).

**Definition.** K.Saito defines the notion of a "good section" as a "section" \( V(t) \) satisfying the following conditions\[^{††} \]:

\[^{††}\] The action of \( W \) on classes in (iii) is well defined since \( W\text{Im}QG_- \in \text{Im}QG_- \), i.e. \( W\partial\omega\partial W = \partial(W\omega)\partial W \)
Using the notion of a "good section" K.Saito defines an improved connection $\nabla^S$ on $\hat{H}^S_{Q(t,z)}$ as follows: when acting on $H^{(k)}(t)$ for $k > 0$, 
\[
\nabla^S = \nabla^{GM}.
\]
For $[\omega(t)]_{Q(t,z)} \in ImV(t)$
\[
\nabla^S_i[\omega(t)]_{Q(t,z)} = \nabla^{GM}_i[\omega(t)]_{Q(t,z)} - z^{-1}V([\Phi_i;\omega(t)]_{Q(t)}).
\]
(67)
and he proves (among other things) that if conditions (i,ii) are satisfied his connection $\nabla^S$ is integrable, and being restricted to $ImV(t)$ preserves the pairing $K^{(0)}_S$.

**Remark from string theory.** In "noncompact" string theories $\nabla^S_i$ coincides with the connection on the space of all states (including states, corresponding to descendants, [Lo2]).

**7.2 The strategy for reducing "Landau-Ginzburg" to K.Saito theory.** We can observe the striking similarity between K.Saito’s connection $\nabla^S$ acting on $ImV$ and the "Hodge" connection for the "Landau-Ginzburg" realization of "Hodge strings" system. Using this analogy, one can guess that a "good section" should somehow correspond to harmonic forms.

Naively, K.Saito’s theory is as far from theory with the Hodge property as one can even imagine: the operator $G^S_-$ has no cohomologies at all. The second problem is that K.Saito’s theory is local while the global issues seem to be crucial in the harmonic theory. Below we will overcome these difficulties as follows.

Here we will discuss only the quasihomogeneous case ( $W$ has only one critical point) but we expect that our methods could be applied also for the general case. We will consider (in subsection 7.3) the "Landau-Ginzburg" operators $Q$ and $G_-$ as operators acting on the space of non-holomorphic germs of forms at the critical point of $W$. Below, we will call it a "local LG-system" (omitting the letters $QG_-$ for brevity). Thus, we have a natural morphism of $QG_-$ systems, which induces an isomorphism of "Gauss-Manin" connections.

However, the local LG-system does not have the Hodge property and does not have a proper pairing.

To study the properties of the local LG-system, we will introduce the morphism $Hol$ that maps a non-holomorphic germ to the holomorphic piece of that germ. The morphism $Hol$ maps the local LG-system to
the K.Saito’s $QG_-$ system and is a quasiisomorphism of $QG_-$ systems. Thus, absence of the Hodge property in a local LG-system is illustrated by the obvious absence of the Hodge property in the K.Saito’s system.

Nevertheless, it is possible to construct a ”quasihodge” connection, (that is a ”pushforward” of the ”Hodge” connection in the global LG-system), whose covariantly constant sections are germs of covariantly constant sections of the ”Hodge” connection in the global LG-system.

The operation $Hol$ maps the ”quasihodge” covariantly constant section into the image of some ”section” of K.Saito cohomologies. We claim (in subsection 7.3) that a ”section” obtained in this way is a ”good section” in the K.Saito sense and show that it satisfies the condition (ii) for a ”good section” (see subsection 7.1).

To establish further relations between local and global LG-systems, we need a pairing on $\hat{H}$ cohomologies in the local system. The pairing (46) is definitely not defined on all germs, so we have to replace it by another pairing, which should coincide with (46) on germs of global harmonic forms. In doing this, we will discover higher residue pairings and their vanishing on germs of harmonic forms. The pairing we defined turns out to be invariant under the $Hol$ operation and gives the K.Saito higher residue pairings on holomorphic germs of forms. The vanishing of higher residue pairings on the holomorphic pieces of germs of harmonic forms mean that they satisfy the condition (i) imposed by K.Saito on a ”good section”.

Thus, we conclude that (up to condition (iii)) the image of a ”good sections” in the K.Saito’s system is spanned by classes of holomorphic pieces of germs of harmonic forms of the global LG system.

An important issue here is that K.Saito’s theory depends on $\bar{U}$ - but in an obscure way, since after passing to holomorphic pieces of germs, the ”antiholomorphic superpotential” $\bar{U}$ naively disappears from the problem. Still it ”shows up” in the choice of a ”good section”. We will discuss this and condition (iii) in the subsection 7.5, were we find that condition (iii) is satisfied if $\bar{U}$ is quasihomogeneous.

7.3 Maps $I$ and $Hol$, and the condition (ii) for a ”good section.” Suppose $W(X, 0)$ has an isolated critical point at zero. Let $H^q$ be a space of non-holomorphic germs of forms at zero. There is a natural map $I : H^F \rightarrow H^q$ given by expansion of a form at zero. The operators $Q(W)$, $G_-(\bar{U})$ and $\Phi_i$ in the global LG system considered as operators on germs lead to operators $Q^g(W)$, $G^g(\bar{U})$ and $\Phi_i^g$ in the local LG system. Thus, $I$ is a morphism of $QG_-$ systems.

**Statement.** The morphism $I$ induces an isomorphism in $H_{Q(z,t)}$ cohomologies and induces an isomorphism of ”Gauss-Manin” connections in
these cohomologies.

What about the Hodge property? If $\bar{U}$ also has zero as a critical point (and $\dim J(W) = \dim J(\bar{U})$), then one can even find simultaneously $Q^g(W)$ and $G^g(\bar{U})$ closed elements in $H^g$, but this is not enough - the local LG system does not have the Hodge property!

To show this we will introduce the operation $\text{Hol}$ that takes a holomorphic piece of a germ.

**Definition.** We define the linear map $\text{Hol}$ from the space $H^g$ to the space $H^S$ of germs of the K.Saito $QG_-$ system as follows: $\text{Hol}$ sends $(p, q)$-forms to zero if $q \neq 0$, and

$$\text{Hol}(\Omega(X, \bar{X})_{A_1...A_p} dX^{A_1} \ldots dX^{A_p}) = \Omega(X, 0)_{A_1...A_p} dX^{A_1} \ldots dX^{A_p}. \quad (68)$$

It is clear that

$$\text{Hol} \circ Q^g(t, z) = Q^S(t, z) \circ \text{Hol}. \quad (69)$$

Note that $\text{Hol}$ wipes out $\bar{U}$.

**Statement.** The map $\text{Hol}$ is a quasiisomorphism between the local LG-system and the K.Saito $QG_-$ system.

This follows from the following Lemma.

**Lemma.** For any germ of holomorphic form $\omega \in H^S$ there is a germ $\omega' \in H^g$, such that $\omega + \omega'$ considered as an element of $H^g$ is both $Q^g$ and $G^g_-$ closed, i.e. represents an element of $H^g_{Q(t, z)}$.

**Idea of the proof.** Below we will show it for the case $d = 1$; this gives the "idea" why it happens.

Let us introduce the parameter $\tau$ in front of $\bar{U}$, and solve equations $Q^g(\omega + \omega') = G^g_-(\omega + \omega') = 0$, expanding $\omega'$ in $\tau$. Specifically, let

$$\omega' = \sum_k \tau^k (P_k(X, \bar{X}) dX + R_k(X, \bar{X}) d\bar{X}), \quad (70)$$

then we have to solve the following system:

$$\frac{\partial P_k}{\partial X} - R_k \frac{\partial W}{\partial X} = 0; \quad (71)$$

$$\frac{\partial \bar{U}}{\partial X} P_k - \frac{\partial R_{k+1}}{\partial X} = 0. \quad (72)$$

If $P_k, R_k$ are known, we can get $R_{k+1}$ from the second equation, and then $P_{k+1}$ from the first. Note that degrees of polynomials $P$ and $R$ are constantly increasing in these iterations; that is why (in germ topology) the seria in $\tau$ is convergent.$\square$
From the previous statement it follows that the local LG-system does not have the Hodge property (since the K.Saito system does not have it). Nevertheless, if \( W \) has only one critical point, one can find a substitute to the "Hodge" connection in \( H^g_{Q(t, z)} \); let us call it the "quasihodge" connection.

**Definition.** Let \( \omega^H_a(t) \) be harmonic elements in \( H^F \) that are covariantly constant with respect to "Hodge" connection. Let us define "quasiharmonic" elements in \( H^g \) as germs of expansion at zero of harmonic elements in \( H^F \): \( \omega^QH_a(t) = I\omega^H_a(t) \). "Quasiharmonic" elements determine the "quasihodge" connection \( \nabla^QH \) in \( H^g_{Q(t, z)} \) through its covariantly constant sections \( [P_a(z, z^{-1})\omega^QH_a(t)]_{Qg(t, z)} \), where \( P \) are polynomials.

"Quasiharmonic" elements satisfy equation (that is obtained by expansion at zero of the corresponding equation in global LG system, see subsection 5.2):

\[
\frac{\partial}{\partial t}\omega^QH_a(t) + z^{-1}(\Phi^H_i\omega^QH_a(t) - C_{i,a}^b(t)\omega^QH_b(t)) \in \text{Im}Qg(t, z) \quad (73)
\]

Relation between "quasihodge" and "Gauss-Manin" connections is exactly the same as in global LG-system:

\[
\nabla^QH_i = \nabla^QH_i - z^{-1}C_i. \quad (74)
\]

Let us define the "quasiharmonic" elements \( \omega^QHS_a(t) \in H^S \) as images of \( Hol \) acting on "quasiharmonic" elements in \( H^g \):

\[
\omega^QHS_a(t) = Hol\omega^QH_a(t) = Hol \circ I\omega^H_a(t) \quad (75)
\]

As in local LG-system "quasiharmonic" elements determine the "quasihodge" connection \( \nabla^QHS \) in \( H^S_{Q(t, z)} \) that is related to the "Gauss-Manin" connection \( \nabla^GMS \) like in local LG-system (just apply \( Hol \) to \( (73) \)):

\[
\nabla^GMS_i = \nabla^QHS_i - z^{-1}C_i. \quad (76)
\]

Classes of "quasiharmonic" elements in \( H^S_{Q(t, z)} \) span the vector space \( H^QHS(t) \subset H^S_{Q(t, z)} \) that projects onto the space \( H^S_{Q(t)} \), i.e. the space \( H^QHS(t) \) could be considered as the image of the "section" \( V^QHS(t) \):

\[
H^QHS(t) = \text{Im}V^QHS(t) \quad (77)
\]

From \( (76) \) we conclude that the "section" \( V^QHS(t) \) satisfies the K.Saito’s condition (ii) for a "good section" (over the space \( C[[t]] \)).

Now we need a pairing.
7.4 Higher pairings. In this subsection we continue to assume that $W(X,0)$ has only one critical point at $X = 0$. We will try to define a pairing on $H^{g,l}_{Q(t,z)}$ that coincides with the pairing (46) on germs of forms from $H^F$.

**Definition.** Consider the space $H^P$ of forms on $C^d$ whose coefficients grow not faster than a polynomial as $|X| \to +\infty$. Let us take an operator $R = \ast \bar{\partial} W \ast$, where $\ast$ is a Hodge operation. Take a positive real number $\epsilon$. Then, we define the bilinear pairing $\langle , \rangle$ on $H^P$ with values in $C[[z]]$ as:

$$\langle \omega_1, \omega_2 \rangle_P(\epsilon, z) = \int_{C^d} \omega_1 C \exp(-\epsilon^{-1}\{Q(t) + zG_-, R\}) \omega_2$$

(78)

Here $C$ is a Weil operator (see (46)).

**Statement 7.4.1** The asymptotic expansion at $\epsilon = 0$ of the pairing $\langle \epsilon, \epsilon \rangle_P(\epsilon, z)$ leads to a pairing $\langle \epsilon, \epsilon \rangle_g(\epsilon, z)$ defined on germs at $X = 0$.

**Proof.** Fix the power $n$ of $z$, and then use the saddle point estimation of the pairing on forms whose coefficients are monomials. The $\epsilon$ expansion of the pairing has the form: $(\epsilon)^{(m-k)/L} S(\epsilon^1/L)$, where $k$ and $L$ are some integers depending on the polynomial $W$ and the integer $n$, the positive integer $m$ depends on the monomial, and $S(u)$ is some Taylor seria. As the power of the monomial grows, $m$ tends to infinity. $\square$

**Definition-Statement 7.4.2** The value of the pairing $\langle \omega_1, \omega_2 \rangle_g(\epsilon, z)$ on $Q(t)$ and $G_-$ closed germs is independent of $\epsilon$ and defines the pairing $\langle \omega_1, \omega_2 \rangle_{g, \bar{U}}(z)$ between $Q(t) - zG_-$ and $Q(t) + zG_-$ cohomologies in the space of germs.

**Proof.** Take the derivative in $\epsilon$. This brings down the $Q(t) + zG_-$ exact term. After commutation with $C$, the operator $Q(t) + zG_-$ turns into $Q(t) - zG_-$. Thus, the derivative in $\epsilon$ is equal to zero. Similarly one can prove that the change of $\omega_2$ by $Im(Q(t) + zG_-)$ or of $\omega_1$ by $Im(Q(t) - zG_-)$ does not change the value of the pairing. $\square$

**Statement 7.4.3** Main property of the pairing $\langle \omega_1, \omega_2 \rangle_{g, \bar{U}}(z)$: the value of this pairing on germs of harmonic forms from $H^F$ is independent on $z$ and coincides with the value of the pairing (46) on corresponding harmonic forms:

$$\langle I\omega_a^H, I\omega_b^H \rangle_{g, \bar{U}}(z) = \langle \omega_a^H, \omega_b^H \rangle$$

(79)

**Proof.** Consider the value of the pairing $\langle \epsilon, \epsilon \rangle_P(\epsilon, z)$ on harmonic forms from $H^F$ as a smooth function of $\epsilon$, taking values in $C[[z]]$. As in the proof of Statement 7.4.2 we conclude that such a function is independent of $\epsilon$. In order to evaluate this function we can go to the limit $\epsilon \to +\infty$. Such a limit exists and is equal to the pairing $\langle \epsilon, \epsilon \rangle$ on the harmonic forms from $H^F$; so, the $z$ dependence disappears. $\square$
Definition. Let us define the pairing $<,>_{\text{Hol}} (z)$ on holomorphic germs of $(d,0)$-forms as a pairing $<,>_{g,0} (z)$. Expanding in $z$, we get a set of higher pairings $<,>_{\text{Hol}} (z) = \sum_k <,>_{\text{Hol}}^{(k)} z^k$.

Statement 7.4.4

$$< \omega_1, \omega_2 >_{g,U} (z) = < \text{Hol} (\omega_1), \text{Hol} (\omega_2) >_{\text{Hol}} (z).$$

(80)

Proof. Consider the antiholomorphic dilatation $D_\lambda: X \to X$, $\bar{X} \to \lambda \bar{X}$. The dilatation leaves $Q(t)$ invariant but transforms $G_-(\bar{U})$ into $G_-(D_\lambda \bar{U})$. It also transforms $\{Q(t) + zG_-(\bar{U}), R\}$ to $\{Q(t) + zG_-(D_\lambda \bar{U}), D_\lambda R\}$. The change in $R$ does not change the value of the pairing on $Q(t)$ and $G_-$ closed germs of forms (if this value is defined) by the argument of "bringing down" a $Q(t,z)$ exact term from the exponent; thus,

$$< \omega_1, \omega_2 >_{g,U} (z) = < D_\lambda \omega_1, D_\lambda \omega_2 >_{g,D_\lambda U} (z).$$

(81)

Now, taking $\lambda$ to zero, we have proved the statement. $\Box$

Conjecture. The K.Saito pairing $K^S (z)$ coincides with $<,>_{\text{Hol}} (z)$.

Arguments in favor of the conjecture:
1) Both pairings are in some sense natural in $Q(t,z)$ cohomologies.
2) By explicit computations one can show that the first two terms in the expansion in $z$ coincide for these two pairings.
3) The $(-1)^k$ factor in K.Saito’s pairing could be explained by observing that, after commutation with the Weil operator $\mathcal{C}$, $Q(t) + zG_-$ passes to $Q(t) - zG_-$. $\Box$

Putting everything together, we see that higher pairings $<,>_{\text{Hol}}^{(k)}$, $k > 0$ vanish on holomorphic pieces of germs of covariantly constant sections of the "Hodge" connection:

$$< \omega^{QHS}, \omega^{QHS} >_{\text{Hol}}^{(k)} = < I(\omega^H), I(\omega^H) >_{g,\bar{U}}^{(k)} = 0, k > 0.$$  

(82)

Thus the "section" determined by $V^{QHS}$ (i.e. whose image is spanned by classes of holomorphic pieces of germs of harmonic forms) satisfies the condition (i) of K.Saito for a "good section" (if we assume that conjecture above is correct).

7.5 "Holomorphic anomaly" and K.Saito’s condition (iii) Here we continue to assume that $W(X,0)$ has only one critical point at $X = 0$.

From results of subsections (7.3) and (7.4) we see that taking any $\bar{U}$ that is good for $W$ we can construct $V^{QHS}(t, \bar{U})$ that determines a "section", that satisfies conditions (i) and (ii). Now we will study how $V^{QHS}(t, \bar{U})$ depends on $\bar{U}$. 

Consider the family inside the space of polynomials $\bar{U}$ that are good for $W(X,t)$:

$$ U(X,t') = U(X) + t'\Psi(X) $$

(83)

where $\Psi(X)$ is a polynomial, depending on $X$, $T'$ is a base of the family and parameter $t'$ is a coordinate on the base.

Harmonic elements of the pair $(Q(W(t)), G_-(\bar{U}(t')))$ form a bundle over $T'$. There is a "Hodge" connection in this bundle such that the classes in $Q(t)$-cohomologies of its covariantly constant sections do not depend on $t'$. We denote these covariantly constant sections as $\omega^H_a(t',t)$, where index $a$ labels some basis in $Q(t)$-cohomologies.

Operator $Q(t) + zG_-(t')$ we will denote as $Q(t,t',z)$.

Let us define classes $[\omega^{QH'S}_a(t',t)]_{Q^S(t,z)}$ as:

$$ [\omega^{QH'S}_a(t',t)]_{Q^S(t,z)} = [Hol \circ I\omega^H_a(t',t)]_{Q^S(t,z)} $$

(84)

Note, that $H^{QH'S}(t,\bar{U}(t')) = \text{Span}\{[\omega^{QH'S}_a(t',t)]_{Q^S(t,z)}\}$.

Since polynomial $\Psi$ (considered as an operator acting in the space of forms by multiplication) commutes with $G_-(t')$, this polynomial leads to a linear operator acting in the space of $G_-(t')$ cohomologies. This action is represented by a linear operator $\tilde{C}(\Psi,t',t)$ whose matrix elements in the basis $[\omega^H_a(t',t)]_{G_-(t')}$ we will denote as $\tilde{C}(\Psi,t',t)^b_a$.

$$ \Psi\omega^H_a(t',t') = C(\Psi,t',t)^b_a\omega^H_b(t',t') + I\text{m}G_-(t') $$

(85)

**Statement 7.5.1** In $Q^S(t,z)$ cohomologies

$$ \frac{\partial}{\partial t'}[\omega^{QH'S}_a(t',t)]_{Q^S(t,z)} = z(\tilde{C}(\Psi,t',t)^b_a - \Psi(0)\delta^b_a)[\omega^{QH'S}_b(t',t)]_{Q^S(t,z)} $$

(86)

**Proof.** By reasoning like in the proof of step 1 in subsection 5.1 we get

$$ \frac{\partial}{\partial t'}\omega^H_a(t',t) + z(\Psi(\bar{X})\omega^H_a(t',t) - \tilde{C}(\Psi)^b_a(t',t)\omega^H_b(t',t)) \in \text{Im}(Q(t,t',z)) $$

(87)

After passing to germs and taking the holomorphic piece we are left only with $\Psi(0)$. \square

**Remark.** A version of statement 7.5.1 is known in the physics literature as $t - t^*$ equations \cite{CV}.

Up to now we ignored condition (iii) of K.Saito. Now we are in position to study restrictions it imposes on $\bar{U}$.

**Statement 7.5.2** Let $u = \bar{U}(0,t')$. Then,

$$ [W(t)\omega^{QH'S}_a(t,t')]_{Q^S(t,z)} \in H^{QH'S}(t,t') + zH^{QH'S}(t,t') + z^2(\bar{C}(\bar{U}(t')) - u)H^{QH'S}(t,t') $$

(88)
Proof. Let us fix $t$ and $t'$. Consider auxiliary family $T''$ of pairs of polynomials $\bar{U}(t', t'') = (1 + t'')\bar{U}$, $W(t, t'') = (1 + t'')^{-1}W(t)$; $t''$ is a coordinate in $T''$. Now take two connections ($\nabla^A$ and $\nabla^B$) in the bundle of harmonic forms over $T''$. First we describe connection $\nabla^A$.

Consider the manifold of "good pairs" of polynomials $W$ and $\bar{U}$, and a bundle of harmonic forms over this manifold. There is a "sum" Hodge connection in this bundle, that is defined as follows. The restriction of the "sum" Hodge connection to submanifolds of varying $W$ for fixed $\bar{U}$ is a Hodge connection (see section 5.2), and its restriction to submanifolds of varying $\bar{U}$ for fixed $W$ is a Hodge connection that was already defined in this subsection. The family $T''$ is a submanifold in the space of "good pairs" and connection $\nabla^A$ is a restriction of the "sum" Hodge connection to $T''$.

Connection $\nabla^B$ follows from (54) and is induced by multiplication with $(1 + t'')\hat{q}$, where $\hat{q}$ is an operator acting on a $(p, q)$-form as multiplication by $q$. From comparison of the two connections at $t'' = 0$ we get:

$$\nabla^A = \nabla^B \left( \text{Span} \{ \omega^H_a(t, t') \} \right) \subset \left( \text{Span} \{ \omega^H_a(t, t') \} \right)$$

Evaluation of this relation up to the image of $Q(t, t', z)$ gives:

$$z(\bar{U}(X)\omega^H_a(t, t') - \bar{C}(\bar{U})_b\omega^H_b(t, t')) - (z^{-1})(W(X, t)\omega^H_a(t, t') - C(W(t))_b\omega^H_b(t, t')) - -\hat{q}\omega^H_a(t, t') \in \text{Span} \{ \omega^H_c \} + \text{Im}Q(t, t', z)$$

Now application of $\text{Hol} \circ I$ proves the statement.\(\square\)

Corollary. K.Saito condition (iii) (for a case with one critical point) is satisfied if $\bar{U}$ is quasihomogeneous.

Proof. If $\bar{U}$ is quasihomogeneous, $\bar{U}$ is $G_-$ exact and acts by zero in $G_-$ cohomologies, i.e. $\bar{C}(\bar{U})_a = 0$. Then the condition (iii) of K.Saito follows from statement 7.5.2.\(\square\)

Note, that condition (iii) of K.Saito leaves open a possibility to get different "good sections" from different quasihomogeneous $\bar{U}$ (phenomena of "holomorphic anomaly").

Example of holomorphic anomaly. Consider $W(X, Y) = X^4 + Y^4$, and

$$\bar{U}(X, Y, t') = \bar{X}^4 + \bar{Y}^4 + t'\bar{X}^2\bar{Y}^2.$$  \hspace{1cm} (91)

Here "good section" $V^{QHS}(t')$ does depend on $t'$. In particular,

$$V([X^2Y^2dXdY]_{Q^S}; t') = [X^2Y^2dXdY + zc(t')dXdY]_{Q^S(z)},$$  \hspace{1cm} (92)

with some function $c(t')$ that is not zero (but $c(0) = 0$). Really, $\bar{X}^2\bar{Y}^2$ is non-zero in $G_-$ cohomologies, and statement of holomorphic anomaly
follows from the Statement 7.5.1. One can also directly check that the family of sections (92) do satisfy (i) and (iii) requirements of K.Saito.

8. Conclusion

In this paper we have shown that genus zero ”amplitudes” in ”compact” topological string theories are completely determined by the corresponding ”Hodge string” $QG_-$ system.

It seems that $QG_-$ systems deserve the study on their own. They form a category similar to the category of manifolds. ”Hodge string” $QG_-$ systems (like global ”Landau-Ginzburg” system) look like compact manifolds, i.e. there are theorems of existence of different structures but it is hard to compute them. The $QG_-$ systems of the K.Saito type are like affine manifolds, and the choice of ”good section” resembles compactification.

Appendix: From the family of hypersurfaces to $z\partial + \partial W(t)$ cohomologies.

In this appendix we will show (following K.Saito) how to get from the family of hypersurfaces to the bundle of $z\partial + \partial W(t)$ cohomologies, and how the Gauss-Manin connection in the cohomologies of the fiber in K.Saito interpretation leads to canonical ”Gauss-Manin” connection (subsection (5.1)).

In this Appendix $\partial$ stands for partial derivatives in $X$ only.

Consider the family of affine hypersurfaces in $C^d$ over $S \otimes T$ defined by equation:

$$W(X,t) - s = W(X) + t_i \Phi_i(X) - s = 0.$$ (93)

Here, parameters of equation $s, t_i$ are considered as algebraic coordinates on $S$, and $T$ respectively. This surface is degenerate on the discriminant $Dis$, defined as the submanifold in $S \otimes T$, such that

$$(W(t) - s) \in C[X]/I(W),$$ (94)

where $I(W)$ is the ideal generated by partial derivatives of $W$ with respect to $X$. For $(s,t) \in S \otimes T - Dis$ the surface is nondegenerate, and the holomorphic (d-1) forms on it are given as residues of the meromorphic forms

$$\frac{\Omega(t,s)}{W(t) - s}.$$ (95)
where Ω is a holomorphic $d$-form on $C^d$. If

$$\Omega = \partial W(t) \partial \omega,$$  

then, its periods vanish (just integrate $\partial \omega$ by parts); so, it is exact. Thus, $(d-1,0)$-cohomologies $H(s, t)$ of the surface over $S \otimes T - Dis$ are equal to:

$$H^{(0)}(s, t) = \{\Omega(s, t)\}/\{(W(t) - s)\Omega(s) + \partial \omega(s)\partial W\}.$$  

(97)

If $(s, t) \in S \otimes T - Dis$, there is another way to represent $(d-1,0)$-forms on the hypersurface as a residue:

$$\frac{\omega(s, t)\partial W(t)}{W(t) - s}.$$  

(98)

Thus, we can define

$$H^{(1)}(s, t) = \left\{\omega(s, t)\partial W(t)\right\}/\{((W(t) - s)\Omega(s, t) + \partial \omega'(s, t)\partial W(t)) \cap \omega(s, t)\partial W(t)\},$$  

and there is a map

$$b(s, t) : H^{(1)}(s, t) \to H^{(0)}(s, t), [\omega(s, t)\partial W(t)]_{(1)} \mapsto [\omega(s, t)\partial W(t)]_{(0)}$$  

(100)

that is an isomorphism for $(s, t) \in S \otimes T - Dis$. Here, $[\ ]_{(i)}$ mean equivalence classes in $H^{(i)}$.

There is a Gauss-Manin connection, defined as a unique flat connection in the bundle of $(d-1,0)$ cohomologies of the family of the non singular hypersurfaces, such that the periods of its covariantly constant sections are constant (as functions of parameters).

This connection acting on cohomologies (written in the form $H^{(0)}(s, t)$) has the following form:

$$\nabla^GM_s[\Omega(s, t)]_{(0)} = [\partial \omega(s, t)]_{(0)}$$  

(101)

$$\nabla^GM_i[\Omega(s, t)]_{(0)} = [\Phi, \partial \omega(s, t)]_{(0)}$$  

(102)

for $[\omega(s, t)\partial W(t)]_{(1)} = b^{(-1)}(s, t)([\Omega(s, t)]_{(0)}).$

The definitions of $H^{(i)}$ could be easily extended to $(s, t) \in Dis$, while (as one can expect) for such values of $(s, t)$, the morphism $b(s, t)$ is just an inclusion, and (as one could expect from the very beginning) the Gauss-Manin connection is not defined. It becomes a Gauss-Manin operator

\[\nabla^GM \text{ for } d = 1 \partial \omega'(s, t) \text{ has to be replaced by a } X\text{-independent function of } s \text{ and } t \text{ because } \partial \text{ has cohomology in the space of holomorphic 0-forms.}\]
(that we will still denote by the same letter $\nabla^{GM}$ ) acting from $H^{(1)}(s,t)$ to $H^{(0)}(s,t)$ considered as modules over $\mathcal{O}(S \otimes T)$ . K.Saito introduced the decreasing filtration on the module $H^{(0)}(s,t)$: class $[\Omega]_{(0)} \in H^{(k)}$ iff $(\nabla^{GM})^k[\Omega]_{(0)}$ is well defined.

In order to have Gauss-Manin connection defined everywhere K.Saito extends the space $H^{(0)}$ to the space

$$H(\delta^{-1}) = C << \delta^{-1} >> \otimes_{C[\delta^{-1}]} H(s,t)$$

and $\delta^{-1}$ acts on $H(s,t)$ as an inverse of the Gauss-Manin connection $\nabla^{GM}$.

Now the Gauss-Manin connection acts on the space $H(\delta^{-1})$ as follows:

$$\nabla^{GM}_{\delta^{-k}}[\omega]_{(0)} = \delta^{-(k-1)}[\omega]_{(0)}, \quad \nabla^{GM}_{\delta^{-k}}[\omega]_{(0)} = \delta^{-(k-1)}[\Phi_{\delta}][\omega]_{(0)}$$

In order to make contact with the subsection 7.1 we first observe that over $C[s]$ the space $H^{(0)}$ could be identified with

$$\{\Omega(t)\}/\{\partial W(t) \partial \omega(t)\}$$

and $s$ acting as multiplication with $W$.

Recalling that the space of holomorphic top forms was named in subsection 7.1 as $H^S$, and that operators $\partial$ and $\partial W(t)$ were named as $G^S$ and $Q^S(t)$ respectively, we identify

$$H^{(0)} \cong H^S/ImQ^S(t)G^S$$

i.e. $H^{(0)} \cong H^{(0)}$. One can see that K.Saito’s filtration (described in this Appendix) corresponds to filtration by powers of $z$ that is induced on $H^{(t)}(s,t)$ from its inclusion in $\hat{H}_{Q(t,z)}$ (see section 5.1).

Thus, identifying $\delta^{-1}$ with a formal parameter $-z$ from subsection 5.1, one easily finds that K.Saito’s $H(\delta^{-1})$ corresponds to $H^S_{Q(t,z)}$ and Gauss-Manin connection acting $H(\delta^{-1})$ corresponds to canonical ”Gauss-Manin” connection (see subsection (5.1)) .

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