Fluid Dynamical Profiles and Constants of Motion from D-Branes

R. Jackiw

Center for Theoretical Physics
Massachusetts Institute of Technology
Cambridge, MA 02139-4307

Abstract

Various fluid mechanical systems, governed by nonlinear differential equations, enjoy a hidden, higher-dimensional dynamical Poincaré symmetry, which arises owing to their descent from a Nambu-Goto action. Also, for the same reason, there are equivalence transformations between different models. These interconnections are discussed in this lecture, and are summarized in Fig. 3 below.

Having attended a few years ago the dedication of the Bogolyubov Institute for Theoretical Physics, I am happy to be here again to commemorate the 90th anniversary of this eminent Kiev mathematician/physicist. These days, fueled by the string program, there is a vigorous interchange between mathematics and physics, which Bogolyubov would have been happy to witness. His activities encompassed field theory, nonlinear systems, and kinetic theory. Just these subjects have become linked in my work, which I shall describe in this lecture, dedicated to his memory.

I shall speak about several nonlinear equations of mathematical physics in arbitrary spatial dimensions, which possess remarkable hidden symmetries and unexpected constants of motion that allow construction of solutions, even complete integration. Moreover, the equations enjoy relationships with each other that provide mappings of one onto another. In the language currently

*This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under contract #DE-FC02-94ER40818. MIT-CTP #2926
in use, there are “dualities” that relate completely different models. Finally, I shall describe how all these properties derive from an “Ur-formulation” in terms of extended objects – d-branes – in a higher dimension. The nonlinear equations that I shall discuss are pre-existing and well known – they are not constructed to illustrate the theory. The observations I make about them are mostly new, though some results in low spatial dimensionality were known previously. The work I describe was performed in collaboration with Bazeia [1] and Polychronakos [2]; it is based on initial observations in this area by Bordermann and Hoppe [3], as well as by Jevicki [4].

1 Nonrelativistic Model

The first equation that I shall consider describes nonrelativistic fluid motion in $d$-spatial dimensions, $(d,1)$ space-time. The matter density of the fluid is $\rho(t, r)$; its local velocity is $v(t, r)$ and the current $j(t, r) = v(t, r)\rho(t, r)$ is linked to the density by a continuity equation

$$\dot{\rho} + \nabla \cdot (v\rho) = 0$$  \hspace{1cm} (1) 

(Over-dot denotes differentiation with respect to time.) The velocity satisfies the Euler equation, which relates the material time derivative of $v$ to a force (per unit volume), that is, to the gradient of the pressure $P$ divided by $\rho$

$$\dot{v} + v \cdot \nabla v = f = -\frac{1}{\rho} \nabla P$$  \hspace{1cm} (2)

For the nonrelativistic application, I shall be interested in a very special force-law $f$: its properties are, first, it arises from a pressure $P$ that is a function only of $\rho$ (this corresponds to isentropic flow); second, $P$ is of the polytropic form ($P \propto$ power of $\rho$), and finally, third, the specific power law is the inverse power; that is,

$$P(\rho) = -\frac{2\lambda}{m \rho}$$  \hspace{1cm} (3)

This is called the “Chaplygin gas” and corresponds to a sound speed $\sqrt{2\lambda/m / \rho}$ (hence we take $\lambda \geq 0$) and enthalpy $\lambda/m\rho^2$ ($m$ is the mass) [5].

The Euler equation for the Chaplygin gas reads

$$\dot{v} + v \cdot \nabla v = \frac{-2\lambda}{m \rho^3} \nabla \rho$$  \hspace{1cm} (4)
It is consistent to look for solutions without vorticity \((\nabla \times \mathbf{v} = 0)\), so we write
\[
\mathbf{v} = \nabla \theta / m
\]
and replace (4) by Bernoulli’s equation for the velocity potential \(\theta\), with a source term given by the enthalpy.
\[
\dot{\theta} + \frac{(\nabla \theta)^2}{2m} = \frac{\lambda}{\rho^2}
\]
The gradient of (3) reproduces (4).

In summary, we are studying the irrotational and isentropic motion of the Chaplygin gas.

Eqs. (1) and (6) possess an action formulation with a first-order (in time) Lagrangian.
\[
L_{NR} = \int d\mathbf{r} \left[ \dot{\theta} \dot{\rho} - \left( \rho \frac{(\nabla \theta)^2}{2m} + \frac{\lambda}{\rho} \right) \right]
\]
Evidently the Hamiltonian is
\[
H_{NR} = \int d\mathbf{r} \mathcal{H}_{NR} = \int d\mathbf{r} \left( \rho \frac{(\nabla \theta)^2}{2m} + \frac{\lambda}{\rho} \right)
\]
and the canonical 1-form
\[
\int d\mathbf{r} \theta \, d\rho
\]
leads to the Poisson bracket
\[
\{ \theta(t, \mathbf{r}), \rho(t, \mathbf{r}') \} = \delta(\mathbf{r} - \mathbf{r}')
\]
It is straightforwardly verified that Eqs. (1) and (5) are a Hamiltonian system with the above bracket.

This model being nonrelativistic possesses the appropriate nonrelativistic symmetry, namely, Galileo invariance, and as a consequence of Noether’s theorem, there are constants of motion, which generate via bracketing infinitesimal Galileo transformations. For future reference, I record these. Time translation, space translation, and rotation act on the coordinates \((t, \mathbf{r})\) in the obvious fashion and the transformed fields \((\rho, \theta)\) are evaluated on the
transformed coordinates. The corresponding constants of motion are energy $E$, momentum $\mathbf{P}$, and angular momentum $L^{ij}$, given by the formulas

\begin{align*}
\text{energy: } E &= H = \int \! dr \, \mathcal{H} \\
\text{(energy density: } \mathcal{H} &= \rho (\nabla \theta)^2 / 2m + \lambda / \rho) & \quad (11) \\
\text{momentum: } \mathbf{P} &= \int \! dr \, \mathbf{P} \\
\text{(momentum density: } \mathbf{P} &= \rho \nabla \theta) & \quad (12) \\
\text{angular momentum: } L^{ij} &= \int \! dr (r^i \mathbf{P}^j - r^j \mathbf{P}^i) & \quad (13)
\end{align*}

Additionally there are the Galilean boosts, which boost the spatial coordinate by a velocity $\mathbf{u}$

\begin{equation}
\mathbf{r} \to \mathbf{R} \equiv \mathbf{r} - t \mathbf{u} \quad (14)
\end{equation}

While the density field transforms simply

\begin{equation}
\rho(t, \mathbf{r}) \to \rho_u(t, \mathbf{r}) = \rho(t, \mathbf{R}) \quad (15)
\end{equation}

the velocity potential undergoes an affine transformation,

\begin{equation}
\theta(t, \mathbf{r}) \to \theta_u(t, \mathbf{r}) \equiv \theta(t, \mathbf{R}) + m (\mathbf{u} \cdot \mathbf{r} - u^2 t / 2) \quad (16)
\end{equation}

which has the consequence that the velocity acquires (as expected) a boost

\begin{equation}
\mathbf{v}(t, \mathbf{r}) \to \mathbf{v}_u(t, \mathbf{r}) = \mathbf{v}(t, \mathbf{R}) + \mathbf{u} \quad (17)
\end{equation}

The associated constant of motion is the boost generator

\begin{equation}
\text{boost generator: } \mathbf{B} = t \mathbf{P} - m \int \! dr \, \mathbf{r} \rho \quad (18)
\end{equation}

Also matter is conserved, as a consequence of invariance against a shift of $\theta$ by constant

\begin{align*}
\theta &\to \theta + \text{constant} \quad (19) \\
\rho &\to \rho \quad (20)
\end{align*}
with associated constant of motion

\[ N = \int \! dr \, \rho \]  

(21)

One can verify that the action \( I_{NR} = \int \! dt \, L_{NR} \) is invariant against all these transformations and consequently a transformation of a solution to the equations of motions (1) and (6) is again a solution. The generators, which can be obtained from the Lagrangian (7) by Noether’s theorem, are all time-independent, as can be verified by differentiating them with respect to time, and evaluating \((\dot{\rho}, \dot{\theta})\) from the equations of motion (1) and (6).

Note there is a total of \( \frac{1}{2}(d+1)(d+2) + 1 \) generators, the correct number for the (centrally extended) Galileo group, where the extension 1-cocycle and 2-cocycle are responsible for the inhomogeneous term in (16) and lead to the generator \( N \) of (21).

The remarkable fact about the Chaplygin gas is that (in any number of dimensions) it possesses further symmetries. First of all one can rescale time

\[ t \to T = e^{\omega t} \]  

(22)

The fields undergo an additional and opposite rescaling

\[ \theta(t, r) \to \theta_\omega(t, r) = e^{\omega t}(T, r) \]  

(23)

\[ \rho(t, r) \to \rho_\omega(t, r) = e^{-\omega t}(T, r) \]  

(24)

The time-independent generator reads

\[ D = tH - \int \! dr \rho \theta \]  

(25)

Furthermore, a peculiar field-dependent diffeomorphism, which mixes independent variables \((t, r)\) and dependent fields \((\rho, \theta)\) also leaves the action invariant. The transformation is parameterized by a d-component vector \( \omega \). On coordinates this acts as

\[ t \to T = t + \omega \cdot r + \frac{1}{2} \omega^2 \theta(T, R)/m \]  

(26)

\[ r \to R = r + \omega \theta(T, R)/m \]  

(27)

Fields transform according to

\[ \theta(t, r) \to \theta_\omega(t, r) = \theta(T, R) \]  

(28)

\[ \rho(t, r) \to \rho_\omega(t, r) = \rho(T, R) \frac{1}{|J|} \]  

(29)
Here $|J|$ is the Jacobian of the transformation

$$J = \det \begin{bmatrix} \frac{\partial T}{\partial t} & \frac{\partial \mathbf{R}}{\partial t} \\ \frac{\partial T}{\partial \mathbf{r}} & \frac{\partial \mathbf{R}}{\partial \mathbf{r}} \end{bmatrix}$$

(30)

The vectorial, $d$-component generator is

$$\mathbf{G} = \int \! d\mathbf{r} \{ \mathbf{r} \mathcal{H} - \theta \mathbf{P} / m \}$$

(31)

Just as with the conventional transformations/symmetries, the above transformations leave the action invariant, and thus take solutions into new solutions; the additional generators $(D, \mathbf{G})$, $d + 1$ in number, are gotten by Noether’s theorem and are time-independent by virtue of the equations of motion.

Using the canonical commutator (10) and the explicit formulas for the generators, one may compute their Lie algebra. As is expected, the $\frac{1}{2}(d + 1)(d + 2) + 1$ Galileo generators (11), (13), (18), and (21), close on the (extended) Galileo algebra, in $(d, 1)$ space-time. Supplemenitng these with the $(d + 1)$ additional generators (23) and (31) one arrives at a total of $\frac{1}{2}(d + 2)(d + 3)$ generators, and their algebra closes on the Poincaré group in one dimension higher, namely, $(d + 1, 1)$ space-time. Moreover, one establishes that the quantities $(t, \theta, \mathbf{r})$ transform as light-cone components of a $(d + 2)$ Lorentz vector, with $t$ acting as the $+$ component, and $\theta$ as the $-$ component [6].

Thus we conclude that the nonrelativistic, Galileo invariant Chaplygin gas in $(d, 1)$ space-time possesses a hidden dynamical Poincaré symmetry appropriate to $(d + 1, 1)$ space-time, which is realized nonlinearly with field-dependent diffeomorphisms.

Using symmetry one can generate new solutions from old ones. Of course when the transformations are of the familiar Galileo form, the “new” solutions bear an obvious relation to the old ones; they are time or space translated, space rotated or boosted, or $\theta$-shifted. However, when the transformations belong to the hidden symmetry, the new solutions take a new and unexpected form.

For example, when $d > 1$, a simple solution to (4) and (9) is

$$(d > 1): \theta = \frac{-m r^2}{2(d - 1)t}, \quad \rho = \frac{\sqrt{2\lambda}}{m d (d - 1)} \frac{|t|}{r}$$

(32)
This corresponds to a velocity and current

\[
(d > 1): \mathbf{v} = -\frac{1}{(d-1)} \frac{\mathbf{r}}{t}, \quad \mathbf{j} = -\epsilon(t) \sqrt{\frac{2\lambda}{md^2}} \hat{r} \tag{33}
\]

[\epsilon(t) is the step function \( t/|t| \).] The (\( \theta, \rho \)) profiles in (32) are invariant against the time-rescaling transformation (22)–(24), but the field-dependent diffeomorphism (24)–(29) alters the solution drastically. The analytic expression for the transformed profile is uninformative, a plot conveys the situation more clearly. In Figs. 1 and 2, the 2-dimensional solutions are plotted: Fig. 1 presents \( \rho \) of (32); in Fig. 2 the transformed \( \rho \) is exhibited [7].

In one spatial dimension, the equations are completely integrable [8]. A hint for this is seen in the special solution, where the current exhibits a soliton profile.

\[
(d = 1): \quad \theta = -\frac{m}{2k^2 t} \cosh^2 kx
\]
Figure 2: The transformed density $\rho(t, r)$.

$$\rho = \sqrt{\frac{2\lambda}{m}} \frac{k|t|}{\cosh^2 kx}$$

$$j = -\epsilon(t) \sqrt{\frac{2\lambda}{m}} \tanh kx \quad (34)$$

To conclude this Section, let me remark that the new symmetries, which we have uncovered, act equally well on the noninteracting ($\lambda = 0$) model, for which in fact a complete solution can be given in terms of initial data for $\rho$ and $v$.

$$\rho(t, r)|_{t=0} = \rho_0(r) \quad v(t, r)|_{t=0} = v_0(r) \quad (35)$$

Define the “retarded” position of $q(t, r)$ by

$$q + tv_0(q) = r \quad (36)$$

and the solution to (1), (2) without interaction reads

$$v(t, r) = v_0(q) \quad (37)$$
\[ \rho(t, r) = \rho_0(q) \left| \det \frac{\partial q^i}{\partial r^j} \right| \tag{38} \]

When \( \lambda \neq 0 \), one can eliminate \( \rho \) in favor of \( \theta \) and present a Lagrangian equivalent to (7) as

\[ L_\lambda = -2\lambda \int dr \sqrt{\dot{\theta} + (\nabla \theta)^2/2m} \tag{39} \]

Note that the equations of motion for \( \theta \) no longer involve \( \lambda \), which serves merely to normalize the Lagrangian. In spite of its peculiar appearance (39) defines a Galileo invariant theory, which also possesses the additional symmetries (23) and (28).

### 2 Relativistic Model

It is possible to give a relativistic generalization of the Chaplygin gas. The appropriate Lagrangian in \( d \)-spatial dimensions is

\[ L_R = \int dr \left[ \theta \dot{\rho} - (\sqrt{\rho^2c^2 + a^2 \sqrt{m^2c^2 + (\nabla \theta)^2}}) \right] \]

\[ = \int dr \left[ \theta \dot{\rho} - \mathcal{H}_R \right] \tag{40} \]

Here \( a \) is a measure of the interaction. When \( a = 0 \), the above is a relativistic generalization (7) with \( \lambda = 0 \). Retaining \( a \), and letting \( c \to \infty \) one finds that

\[ L_R = -Nmc^2 + L_{NR}|_{\lambda=ma^2/2} \tag{41} \]

Although not manifestly so, \( L_R \) is Lorentz and Poincaré invariant in \((d, 1)\) space-time (see below), and also matter conservation is respected. Thus there are \( \frac{1}{2}(d+1)(d+2) + 1 \) generators, where the first number counts the Poincaré generators and “+1” refers to \( N \).

When \( a \) vanishes, the model is free and elementary; a complete solution exists. Indeed the equations of motion take the form (I) and (2) (with \( f \) set to zero) but \( v \) is not \( \nabla \theta/m \), rather it is \( c \nabla \theta/\sqrt{m^2c^2 + (\nabla \theta)^2} \), so that

\( \nabla \theta/m = v/\sqrt{1 - v^2/c^2} \). Consequently the solutions take the same form as in (35)–(38).

Just as in the nonrelativistic case, when \( a \neq 0 \), \( \rho \) can be eliminated, leaving the Lagrangian of a “Born-Infeld”-type model.

\[ L_a = -a \int dr \sqrt{m^2c^2 - (\partial_\mu \theta)^2} \tag{42} \]
And again the coupling strength disappears from the equation of motion for $\theta$, serving merely to normalize the Lagrangian. Manifest Poincaré symmetry is now evident, and its generators can be constructed in the usual fashion from the energy momentum tensor for (12).

In view of its analogies to the nonrelativistic model, there is no surprise that the relativistic model also admits additional hidden symmetries which leave the action invariant and take solutions into new solutions. Once can reparameterize time through a field-dependent transformation, which depends on the scalar $\omega$

$$t \rightarrow T$$

$$T = \frac{t}{\cosh mc^2 \omega} + \frac{\theta(T, \mathbf{r})}{mc^2} \tanh mc^2 \omega$$

$$\theta(t, \mathbf{r}) \rightarrow \theta_{\omega}(t, \mathbf{r})$$

$$\theta_{\omega}(t, \mathbf{r}) = \frac{\theta(T, \mathbf{r})}{\cosh mc^2 \omega} - mc^2 \tanh mc^2 \omega$$

The associated conserved generator is

$$D = \int d\mathbf{r} \left( m^2 c^4 t \rho + \theta \mathcal{H}_R \right)$$

Also there is a spatial reparameterization, governed by the d-component vector $\omega$ ($\omega \equiv |\omega|$).

$$\mathbf{r} \rightarrow \mathbf{R}$$

$$\mathbf{R} = \mathbf{r} - \omega \theta(t, \mathbf{R}) \frac{\tan mc\omega}{mc\omega}$$

$$\quad + \omega \cdot \mathbf{r} \frac{1 - \cos mc\omega}{\omega^2 \cos mc\omega}$$

$$\theta(t, \mathbf{r}) \rightarrow \theta_{\omega}(t, \mathbf{r})$$

$$\theta_{\omega}(t, \mathbf{r}) = \frac{\theta(t, \mathbf{R})}{\cos mc\omega} - mc\omega \cdot \mathbf{r} \frac{\tan mc\omega}{\omega}$$

with conserved vectorial generator

$$G = \int d\mathbf{r} \left( m^2 c^2 \rho + \theta \mathcal{P} \right)$$

Only the $\theta$ transformation law is exhibited; the one for $\rho$ can be deduced from the equation of motion for $\theta$, which follows from (10).
The additional symmetries give us \( d + 1 \) further generators, which supplement the previously described \( \frac{1}{2}(d + 1)(d + 2) + 1 \) generators, for a total of \( \frac{1}{2}(d + 2)(d + 3) \) — just the right number for the Poincaré group in \((d + 1, 1)\) space-time. And indeed upon computing the canonical Lie algebra brackets of all the generators one finds that the totality of generators closes on the \((d + 1, 1)\) Poincaré group. The computation is based on the same bracket \( [10] \) as in the nonrelativistic case [because the canonical 1-form of \( [40] \) coincides with that of \( [7] \), and is given in \( [9] \)]. Moreover the set of quantities \( (t, \theta, r) \) transforms as a \((d + 2)\)-Lorentz vector in Cartesian components \( [6] \).

As in the nonrelativistic case, one may use the additional hidden symmetry transformations to map solutions into new solutions with different properties. Additionally, one may use the relativistic–nonrelativistic connection to obtain solutions of the Chaplygin gas problem by taking the \( c \to \infty \) limit of the Born-Infeld solutions \( [9] \). The Born-Infeld model in one spatial dimension is completely integrable \( [10] \).

### 3 Common Ancestry

The “hidden” symmetries and the associated haphazard transformation laws may be given a coherent setting by considering the Nambu-Goto action for a d-brane in \((d + 1)\) spatial dimensions, moving on \((d + 1, 1)\) dimensional space-time. [A d-brane is a d-dimensional extended object: 1-brane is a string, 2-brane is a membrane, and so on. A d-brane in \((d + 1)\) space divides that space in two.]

The Nambu-Goto action reads

\[
I_{\text{NG}} = \int d\phi^0 d\phi^1 \cdots d\phi^d \sqrt{G} \tag{49}
\]

\[
G = (-1)^d \det \frac{\partial X^\mu}{\partial \phi^\alpha} \frac{\partial X^\mu}{\partial \phi^\beta} \tag{50}
\]

Here \( X^\mu \) is a \((d + 1, 1)\) target space-time (d-brane) variable, with \( \mu \) extending over the range \( \mu = 0, 1, \ldots, d, d + 1 \). The \( \phi^\alpha \) are variables describing the extended object with \( \alpha \) ranging \( \alpha = 0, 1, \ldots, d \); \( \phi^\alpha, \alpha = 1, \ldots, d \), parameterizes the d-dimensional d-brane, while the extended object evolves in \( \phi^0 \).

The Nambu-Goto action is parameterization invariant, and we shall show that two different parameterizations (“light-come” and “Cartesian”) result in the Chaplygin gas and Born-Infeld actions. The parameterizations are fixed
as follows. For both parameterizations we choose \((X^1, \ldots, X^d)\) to coincide with \((\phi^1, \ldots, \phi^d)\) and rename them \(\mathbf{r}\) (a \(d\)-dimensional vector).

For the light cone parameterization we define \(X^\pm\) as \(\frac{1}{\sqrt{2}}(X^0 \pm X^{d+1})\). \(X^+\) is renamed \(t\) and identified with \(\sqrt{2\lambda m}\phi^0\). This completes the fixing of the parameterization and the remaining variable is \(X^-\), which is a function of \(\phi^0\) and \(\phi\), or after redefinitions, of \(t\) and \(\mathbf{r}\). \(X^-\) is renamed \(\theta\) and identified with \(\sqrt{2}\lambda m \phi^0\). This completes the fixing of the parameterization and the remaining variable is \(X^-\), which is a function of \(\phi^0\) and \(\phi\), or after redefinitions, of \(t\) and \(\mathbf{r}\). \(X^-\) is renamed \(\theta\) and identified with \(\sqrt{2}\lambda m \phi^0\). This completes the fixing of the parameterization and the remaining variable is \(X^-\), which is a function of \(\phi^0\) and \(\phi\), or after redefinitions, of \(t\) and \(\mathbf{r}\).

For the second, Cartesian parameterization \(X^0\) is renamed \(ct\) and identified with \(amc\phi^0\). The remaining target space variable \(X^{d+1}\), a function of \(\phi^0\) and \(\phi\), equivalently of \(t\) and \(\mathbf{r}\), is renamed \(\theta(t, \mathbf{r})/mc\). Then the Nambu-Goto action \((49)\) reduces to the Born-Infeld action \(\int dt L, (42)\). The relation to the Nambu-Goto action explains the origin of the hidden \((d+1, 1)\) Poincaré group in our two nonlinear models on \((d, 1)\) space-time: Poincaré invariance is what remains of the reparameterization invariance of the Nambu-Goto action after choosing either the light-cone or Cartesian parameterizations. Also the nonlinear, field dependent form of the transformation laws \((22)–(24), (26)–(29), (13), (14), (16), (17)\) is understood: it arises from the identification of some of the dependent variables \(X^\mu\) with the independent variables \(\phi^\alpha\).

The complete integrability of the \(d = 1\) Chaplygin gas and Born-Infeld model is a consequence of the fact that both descend from a string in 2-space. But the Nambu-Goto theory for that system is completely integrable \([1]\).

We observe that in addition to the nonrelativistic descent from the Born-Infeld theory to the Chaplygin gas, there exists a mapping of one system on another, and between solutions of one system and the other, because both have the same d-brane ancestor. The mapping is achieved by passing from the light-cone parameterization to the Cartesian, or vice-versa \([9]\). Specifically this is accomplished as follows:

Chaplygin gas \(\rightarrow\) Born-Infeld: Given \(\theta_{NR}(t, \mathbf{r})\), a nonrelativistic solution, determine \(T(t, \mathbf{r})\) from the equation

\[
T + \frac{1}{mc^2}\theta_{NR}(T, \mathbf{r}) = \sqrt{2}t
\]  

(51)

Then the relativistic solution is

\[
\theta_R(t, \mathbf{r}) = \frac{1}{\sqrt{2}}mc^2T - \frac{1}{\sqrt{2}}\theta_{NR}(T, \mathbf{r})
\]  

(52)
Born-Infeld → Chaplygin gas: Given $\theta_R(t, r)$, a relativistic solution, find $T(t, r)$ from

$$T + \frac{1}{mc^2} \theta_R(T, r) = \sqrt{2} t$$

(53)

Then the nonrelativistic solution is

$$\theta_{NR}(t, r) = \frac{1}{\sqrt{2}} mc^2 T - \frac{1}{\sqrt{2}} \theta_R(T, r)$$

(54)

All the relationships are summarized in Fig. 3.

**Figure 3: Dualities and other relations between nonlinear equations.**

One cannot establish the connection of our two nonlinear equations to the Nambu-Goto action in absence of the interaction, neither in the nonrelativistic ($\lambda = 0$) nor relativistic ($a = 0$) cases: one cannot eliminate $\rho$ in favor of $\theta$, because in the absence of an interaction $\rho$ no longer appears in the equation of motion for $\theta$. Equivalently one sees that the $\theta$-Lagrangians $L_\lambda$ (39) and $L_a$ (42) vanish with $\lambda$ and $a$, respectively. The Nambu-Goto action is normalized by the d-brane tension, which has been scaled to unity in (49). Thus the nonrelativistic and relativistic free models ($\lambda = 0 = a$) in their $\rho-\theta$ forms, (7) and (40) respectively, may be viewed as a parameterized description of "tension-less" d-branes.

Finally we remark that the emergence of the $(d + 1, 1)$ Poincaré group from the $(d, 1)$ Galileo group can also be understood in Kaluza-Klein-like construction [11].
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