An anisotropic hybrid non-perturbative formulation for 4D $\mathcal{N} = 2$ supersymmetric Yang-Mills theories.

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Abstract

We provide a simple non-perturbative formulation for non-commutative four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theories. The formulation is constructed by a combination of deconstruction (orbifold projection), momentum cut-off and matrix model techniques. We also propose a moduli fixing term that preserves lattice supersymmetry on the deconstruction formulation. Although the analogous formulation for four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theories is proposed also in [1], our action is simpler and better suited for computer simulations. Moreover, not only for the non-commutative theories, our formulation has a potential to be a non-perturbative tool also for the commutative four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theories.
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1 Introduction

It is very important to find promising numerical frameworks for non-perturbative studies of supersymmetric gauge theories. Although lattice formulations are natural candidates, it is not so straightforward to apply them for the purpose. This is because there tends to be many parameters which must be fine-tuned due to the SUSY breaking by the lattice cut-off.

So far, $\mathcal{N} = 1$ pure supersymmetric Yang-Mills theories without scalar fields in three [2] and four [3] dimensions have been shown to be free from fine-tunings. For one-dimensional supersymmetric matrix quantum mechanics, a "non-lattice" technique is applicable [4]. By using this method, supersymmetric matrix quantum mechanics has been extensively investigated. In
particular, quantitative agreement with the gauge/gravity duality conjecture has been obtained \cite{5,7,1} and qualitative consistency with the lattice calculations are also obtained in \cite{12,13}.

However, in supersymmetric field theories with scalar fields, there tends to be a plethora of the relevant operators violating supersymmetry whose coefficients must be fine-tuned. For example, scalar mass terms are difficult to exclude without fermionic symmetries. To overcome such difficulties, several lattice formulations which preserve a partial SUSY on the lattice, are proposed in \cite{14-24}. Although these models have succeeded to be free from fine-tunings in two or three-dimensional cases thanks to the super-renormalizability, extended supersymmetric four-dimensional theories with a finite rank of gauge group were out of reach\footnote{There are also other numerical studies \cite{10,11} in the context of gauge/gravity duality.}

The method we will use to avoid fine-tunings in four dimensions is to introduce anisotropic regularization. This approach was used by Hanada et al. \cite{1,36,37}, who first obtained non-perturbative four-dimensional formulations free from fine-tunings. They considered the lattice regularization of the mass-deformed two-dimensional supersymmetric Yang-Mill theory, which provides an additional two dimensions as a Fuzzy 2-sphere \cite{38,39}. They performed lattice regularization along the original two dimensions by utilizing the balanced topological field theory formalism in \cite{18,42}. On the lattice, a partial SUSY is preserved. The emergent Fuzzy 2-sphere directions are regularized by the non-commutative parameter $\Theta$ (and thus UV cut-off $\Lambda$) and the Fuzzy sphere radius $R_f$. To take the target four-dimensional continuum limit with no fine-tunings, they take the following steps:

1. Taking the continuum limit along the original two-dimensional lattice directions.
2. After that, taking the decompactified limit of the Fuzzy sphere, $R_f \to \infty (\Lambda \to \infty)$.
3. Finally taking commutative limit $\Theta \to 0$.

During the first step, the theory is regarded as a two-dimensional theory with super-renormalizability. Hence the target intermediate continuum theory can be reached without fine-tunings, where two of the four dimensions of the intermediate theory are kept as a Fuzzy 2-sphere with finite $\Theta, \Lambda, R_f$. And during the subsequent steps, we can use symmetries restored by the first step to circumvent dangerous corrections. By this series approach, we can finally get the target continuum limit without fine-tunings.

From this study, we can see following advantages of anisotropic formulations with taking the limits in a stepwise manner\footnote{The first attempt to construct a lattice formulation for the $\mathcal{N} = 2$ theory is discussed in \cite{25}. An $\mathcal{N} = (2, 2)$ $SU(N)$ formulation of the orbifold lattice gauge theory is discussed in \cite{26}. There is also another approach without employing the exact SUSY on the lattice \cite{27}. And there are several numerical studies on the two-dimensional theories \cite{28,31}.}

- In the first step, the theory can be regarded as a low-dimensional theory. Thanks to the super-renormalizability, it is easy to get a desirable intermediate theory from which we can approach the final target theory.

\footnote{For the large $N$, in the planar limit, four-dimensional theories can be obtained \cite{32,33} by using the large $N$ reduction \cite{34}.}

\footnote{In \cite{14}, the number of fine-tunings in an $\mathcal{N} = 4$ four-dimensional lattice model has been estimated. And the number of fine-tunings has turned out to be 1 by one-loop perturbative calculations in \cite{35}.}

\footnote{The construction of the four-dimensional non-commutative space from the zero dimensions has been discussed in \cite{10,41}.}

\footnote{Actually, the potency of the anisotropic continuum limit has been already mentioned in \cite{33} to reduce the number of fine-tunings.}
In later steps, symmetries recovered in earlier steps help to protect from dangerous corrections.

In this paper, we will construct a non-perturbative formulation for the non-commutative four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mill theories by further developing such an anisotropic treatment. Although an analogous formulation is constructed based on the BTFT formalism in [1] in a beautiful way, the formulation is rather complicated and not so easy to put on a computer. To construct a more convenient action, we employ a combination of orbifolding [15–17], momentum cut-off [4–9,44] and the Fuzzy sphere techniques. We start from the mass-deformed one-dimensional supersymmetric matrix model with 8 supercharges [45], and we perform the orbifold projection on it with keeping the one dimension continuum. After that the continuum dimension is regularized by the momentum cut-off, then we obtain the action for the mass-deformed two-dimensional supersymmetric Yang-Mills theory on $R^1_\Lambda \times R^1_\text{orb}$, where $R^1_\Lambda$ is the one dimensions regularized by the momentum cut-off and $R^1_\text{orb}$ is the regularized one dimensions which is generated by the orbifolding. For later use, we will call the two-dimensional theory regularized by the hybrid of the orbifolding and the momentum cut-off as the "hybrid regularization theory". The construction of our formulation is completed by uplifting the hybrid regularization theory to the four dimensions by taking the Fuzzy sphere solution, which is employed in the same way as [1]. Basically our formulation is obtained by replacing the lattice regularization in [1] with the hybrid regularization of orbifolding and momentum cut-off. Thus our construction uses $R^1_\Lambda \times R^1_\text{orb} \times \text{Fuzzy } S^2$, as compared to [1]'s $R^2_{\text{lat}} \times \text{Fuzzy } S^2$. Clearly our theory is more anisotropic than [1]. Remarkable properties and advantages of our approach are:

- There has not been an $\mathcal{N} = 2$ four-dimensional supersymmetric lattice gauge theory based on the orbifold projection because there are too few scalar fields. In our formulation, the large number of scalar fields is not required, since the orbifold projection is performed along only one direction. Thus we are able to get $\mathcal{N} = 2$ four-dimensional theories for the first time.

- Since the orbifold projection is performed along only one direction, $8 \times \left(\frac{1}{2}\right)^1 = 4$ supercharges can be preserved. The theory has more supersymmetry than [1].

- We can introduce a moduli stabilizing mass term that preserves 2 of 4 preserved SUSY on the lattice. This is the first moduli fixing term without breaking lattice SUSY.

- Because our formulation is constructed by the orbifold projection from the one-dimensional matrix model, it can be easily embedded into the one-dimensional matrix model with 8 supercharges. A matrix model with 8 supercharges is already numerically studied in [44]. So it would be easy to apply their numerical technique to this formulation.

The outline of this paper and the basic steps to reach the target theory are described in the next two subsections.

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7 A lattice formulation of four-dimensional $N=2$ supersymmetric gauge theory is constructed by Sugino [18]. Essentially the same formulation is obtained by Damgaard and Matsuura [19] by combining orbifolding and the truncation technique developed in [17].
1.1 Basic steps to reach the target theory and usage of our formulation

This section aims to clarify the order of taking limits to reach the target theory without fine-tunings. We start with a mass-deformed $U(klN)$ one-dimensional matrix model, with mass-deformed parameter $M$ and 8 supercharges, which we refer to as the mother theory. The four dimensions in the formulation are regularized different manners, namely:

- $\mathbb{R}^1_\Lambda$: Regularized by the momentum cut-off $\Lambda$.
- $\mathbb{R}^1_{\text{orb}}$: Regularized by orbifold projection with lattice spacing $a$, and its number of site is characterized by $N$. The moduli stabilizing mass terms are also introduced with parameter $\nu_1$.
- Fuzzy $S^2$: Regularized by UV cut-off $\hat{\Lambda} \sim \frac{Ml}{3}$, radius $R_f \sim 3/M$, and the non-commutative parameter $\Theta \sim \frac{1}{(M^2l)}$.

Among these, $\mathbb{R}^1_\Lambda \times \mathbb{R}^1_{\text{orb}}$ corresponds to $\mathbb{R}^2_{\text{lat}}$ of $\mathbb{R}^2_{\text{lat}} \times$ Fuzzy $S^2$ in [1]. In [1], to avoid the fine-tunings, the continuum limit of the lattice gauge theory on $\mathbb{R}^2_{\text{lat}}$ is taken first before managing Fuzzy $S^2$. This indicates that we need to manage $\mathbb{R}^1_\Lambda \times \mathbb{R}^1_{\text{orb}}$ before Fuzzy $S^2$. In our more anisotropic formulation, we have to choose between $\mathbb{R}^1_\Lambda$ or $\mathbb{R}^1_{\text{orb}}$ as the first direction to be managed. When choosing the order in which to relax regulators, we should start with a crude regularization which breaks much symmetry. This is because such a regulator is easiest to manage in an early low-dimensional stage, when the theory will be super-renormalizable. In our set-up, the momentum cut-off $\Lambda$ breaks both supersymmetry and gauge symmetry while the orbifolding can preserve a partial SUSY and gauge symmetry. Thus we deal with the momentum cut-off first. Then we will take the following steps to get the target non-commutative $\mathcal{N} = 2$ four-dimensional supersymmetric Yang-Mills continuum limit without any fine-tunings:

1. Taking the $\Lambda \to \infty$ keeping other parameters fixed.
2. $a \to 0$, $N \to 0$ with keeping $aN$, $\nu_1$ and regularization parameters of Fuzzy sphere fixed.
3. $aN \to \infty$, $\nu_1 \to 0$ with $k,l,(m = kl), M, \Theta, R_f, \hat{\Lambda}$ fixed.
4. $l \to \infty$ ($R_f, \hat{\Lambda} \to \infty$, $M \to 0$) with $\Theta, k$ fixed.

At the 1st step, since the system can be regarded as a one-dimensional system without UV divergences, both gauge symmetry and a part of supersymmetry are automatically recovered only by taking the $\Lambda \to \infty$. Without any fine-tunings, the orbifold lattice gauge theory on $\mathbb{R}^1_\Lambda \times \mathbb{R}^1_{\text{orb}} \times$ Fuzzy $S^2$ is obtained as an intermediate theory from our non-perturbative formulation on $\mathbb{R}^1_\Lambda \times \mathbb{R}^1_{\text{orb}} \times$ Fuzzy $S^2$. The orbifold lattice theory has 4 SUSY in the UV region and the 2 of 4 supercharges are exactly preserved on the lattice. We will explain the 1st step in Sec. 3.1. As the 2nd step, we will take the continuum limit $a \to 0$ with keeping the moduli stabilizing parameter $\nu_1$ and the volume $aN$ fixed. After that we take the 3rd step $\nu_1 \to 0, aN \to \infty$. By the 2nd and 3rd steps, without any fine-tunings, we will obtain the non-commutative supersymmetric Yang-Mills theory with 8 SUSY on $\mathbb{R}^2 \times$ Fuzzy $S^2$. Full 8 supersymmetry and $SO(2)$ rotational symmetry on $\mathbb{R}^2$ are recovered in these steps. We will explain the 2nd and 3rd steps in Sec. 3.2. By the 4th step, from the theory on $\mathbb{R}^2 \times$ Fuzzy $S^2$, we will obtain the final target theory which is the non-commutative $\mathcal{N} = 2$ four-dimensional supersymmetric Yang-Mills theory on $\mathbb{R}^2 \times \mathbb{R}^2_\Theta$. Here the $\mathbb{R}^2_\Theta$ is the two-dimensional non-commutative Moyal plane with the non-commutative
parameter $\Theta$. The theory on $\mathbb{R}^2 \times \text{Fuzzy } S^2$ is connected smoothly to the one on $\mathbb{R}^2 \times \mathbb{R}^2_\Theta$. We will explain the 4th step in Sec. 3.3. In the above way, we will get the target theory by these 4 steps without any fine-tunings. We also illustrate these steps by Table 1.

| 4d space | Theories | Symmetries |
|----------|----------|------------|
| $\mathbb{R}^2_\lambda \times \mathbb{R}^4_{\text{orb}} \times S^2_f$ | Our non-perturbative formulation (Hybrid regularization theory with Fuzzy $S^2$ solution) | $Q$ | $\text{Gauge } \text{SO}(2)$ |
| $r \times r \times S^2_f$ | $\mathcal{N} = 2 \text{ 4d Non-commutative } U(k) \text{ SYM on } \mathbb{R}^2 \times S^2_f$ | 2 (UV 4) | $\circ \times \times$ |
| $c \times r \times S^2_f$ | $\mathcal{N} = 2 \text{ 4d Non-commutative } U(k) \text{ SYM on } \mathbb{R}^2 \times S^2_f$ | $\circ \times \circ$ |
| $\mathbb{R}^2 \times S^2_f$ | $\mathcal{N} = 2 \text{ 4d Non-commutative } U(k) \text{ SYM on } \mathbb{R}^2 \times S^2_f$ | $\circ \circ \circ$ |

$S^2_f$: Fuzzy $S^2$
$c$: The direction is regularized
$r$: The direction is continuum
$Q$: The number of supercharges
$SO(2)$: Rotational symmetry on $\mathbb{R}^2$
$\circ$: The symmetry is restored
$\times$: The symmetry is still broken

Table 1: The flowchart of steps to take the target theory limit.

Our formulation is a powerful non-perturbative tool for the $\mathcal{N} = 2$ non-commutative four-dimensional supersymmetric Yang-Mills theories. Non-commutative gauge theory is an important subject of research in order to clarify non-perturbative aspects of gauge theories. For example, the singularity in the instanton moduli space is resolved by the non-commutativity. So numerical studies of non-commutative four-dimensional supersymmetric Yang-Mills theories by using our formulation will give a strong instrument to reveal the non-perturbative structure of supersymmetric gauge theories.

Not only for the non-commutative gauge theories, our formulation has a potential to be a non-perturbative tool also for the commutative four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theories. Although the non-commutative four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory is expected not to be continuously connected to the commutative $\mathcal{N} = 2$ theory \[48\], there is a discussion that the non-commutative $\mathcal{N} = 2$ theory may flow to the ordinary commutative theory in the infrared $[49]$. Since still there is a possibility that $\Theta \to 0$ smoothly connects to commutative theory in the infrared region, the formulation has a potential to be a non-perturbative formulation also for the commutative gauge theories.

1.2 Outlines of this paper

The outline of this paper is as follows: In section 2 we will explain how to construct our non-perturbative formulation which is the hybrid regularization theory with the Fuzzy 2-sphere
solution. In section 3 we will explain how to take the target $N = 2$ non-commutative supersymmetric Yang-Mills limit on $\mathbb{R}^2 \times \mathbb{R}^2_\Theta$. The absence of fine-tunings in the steps is explained in this section. Section 4 is the summary.

2 How to construct the non-perturbative formulation

Here we will explain how to construct our formulation. We start from the mass-deformed one-dimensional matrix model with 8 supersymmetry. First we obtain the orbifold lattice theory on $\mathbb{R}^1(\text{continuum}) \times \mathbb{R}^1_{\text{orb}}$ by performing the orbifold projection on the matrix model. In the subsection 2.1 we will explain the orbifold lattice gauge theory. Second we perform the momentum cut-off regularization on the orbifold action, then we get the hybrid regularization theory on $\mathbb{R}^1_\Lambda \times \mathbb{R}^1_{\text{orb}}$. We explain the momentum cut-off regularization in Sec. 2.2. And to complete the construction of our formulation, we expand the hybrid regularization theory around the Fuzzy 2-sphere solution. We explain it in Sec 2.3.

2.1 Two-dimensional orbifold lattice gauge theory with keeping one dimension continuum

2.1.1 Mother theory

We start from the following Euclidean one-dimensional $U(mN)$ matrix model with 8 supersymmetry, which we call as the "mother theory",

$$S^{\text{mat}} = \frac{2}{g^2} \int dx_1 \text{Tr} \left[ \frac{1}{2} D_1 X^I D_1 X^I - \frac{1}{4} [X^I, X^J]^2 + \frac{i}{2} \overline{\Psi} \gamma^K [X_K, \Psi] + \frac{1}{2} \overline{\Psi} D_1 \Psi - \frac{i}{6} \overline{\Psi} \gamma^{23} \Psi + \frac{iM}{3} X^3 D_1 X^2 + \frac{1}{2} \left( \frac{M}{3} \right)^2 (X^a)^2 + \frac{iM}{3} \epsilon_{abc} X^a X^b X^c \right]$$

(2.1)

where $I, J, K = 2, \ldots, 6$ and $a = 4, 5, 6$, labeling the bosonic fields $X^I, X^a$. The integer $m$ is defined as $m = kl$. $M$ is the mass deformation parameter. Here we compactify the $x_1$ direction with the length $R_1$, $x_1 \sim x_1 + R_1$. And we impose the periodic boundary condition on all fields. The fermion $\Psi$ is an 8-component spinor and each component of the $\Psi$ is written by the $\Psi^{(0)}$ as

$$\Psi^{(0)} = \left( \psi_{+1}, \psi_{+2}, \chi_{+}, \frac{1}{2} \eta_{+}, \psi_{-1}, \psi_{-2}, \chi_{-}, \frac{1}{2} \eta_{-} \right)^T,$$

(2.2)

with

$$\Psi = U_8 \Psi^{(0)},$$

(2.3)
and $U_8$ is

$$U_8 = \frac{1}{2} \begin{pmatrix}
0 & 0 & -1 & i & 1 & i & 0 & 0 \\
1 & -i & 0 & 0 & 0 & 0 & -1 & i \\
0 & 0 & i & 1 & i & -1 & 0 & 0 \\
-i & -1 & 0 & 0 & 0 & -i & -1 & 1 \\
i & 0 & 0 & 0 & 0 & 1 & i & 0 \\
0 & 1 & i & 1 & -i & 0 & 0 \\
-i & 1 & 0 & 0 & 0 & i & -1 & 0 \\
0 & 0 & -i & 1 & i & 1 & 0 & 0 
\end{pmatrix}. \quad (2.4)$$

Here the all fields are expanded by a basis of the representation $T^{\tilde{a}}$ ($\tilde{a} = 1, \cdots, \dim(u(mN))$) as $\Psi = \Psi^{\tilde{a}}T^{\tilde{a}}, \cdots$, and they are in the adjoint representation of the $U(mN)$ gauge group. Hence, they are $mN \times mN$ matrix valued quantities. Here $g$ is the one-dimensional gauge coupling with mass dimension $3/2$. The gamma matrices $\gamma_K$ are written in the appendix \ref{app:gamma}. The covariant derivative $D_1$ is defined as $D_1 = \partial x_1 + i[v^1, \cdots]$ where the $v^1$ is a gauge field along the $x_1$ direction.

The action (2.1) is derived from the 8 supersymmetry analog of the plane wave matrix model \cite{50}. It is also obtained by the dimensional reduction from the mass-deformed Euclidean two-dimensional supersymmetric Yang-Mills theory with 8 supercharges, which is described in eq. (A.28) of \cite{1}.

The mother theory (2.1) preserves the following supersymmetry with 8 supercharges,

$$\delta v^1 = \epsilon^T \Psi, \quad \delta X^I = \epsilon^T \gamma_I \Psi, \quad \delta \Psi = \left( -(D_1 X^I) \gamma_I + \frac{i}{2} [X^I, X^J] \gamma_{IJ} - \frac{M}{3} X^a \gamma_a \gamma_{456} \right) \epsilon. \quad (2.5)$$

The SUSY parameter $\epsilon$ is independent of the coordinate $x_1$ while the SUSY parameter in the plane wave matrix model \cite{50} depends on the coordinate $x_1$.

### 2.1.2 Balanced topological field theory form of the mother theory

Among the 8 supercharges of the mother theory (2.5), we pick up two supercharges $Q_+$ and $Q_-$. Each $Q_+$ and $Q_-$ is associated with the parameter $\epsilon_+$ and $\epsilon_-$ respectively. The $\epsilon_\pm$ are

$$\epsilon_\pm = \epsilon_\pm^T U_8^{-1}, \quad (\epsilon_\pm : \text{Grassmann numbers}). \quad (2.6)$$

with

$$\epsilon_+ = \begin{pmatrix}
\epsilon_+ \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 
\end{pmatrix}, \quad \epsilon_- = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\epsilon_- \\
0 \\
0 
\end{pmatrix}. \quad (\epsilon_\pm : \text{Grassmann numbers}). \quad (2.7)$$

\*PP wave matrix strings are discussed in \cite{51, 52}.
For later use, we define following complexified fields as
\[ Y = iX_2 + X_3, \quad Y^\dagger = -iX_2 + X_3, \]
\[ C = 2X^4, \quad \phi_\pm = X^5 \pm iX^6, \]
\[ \xi_\pm = i\psi_\pm + \chi_\pm, \quad \xi_\pm^\dagger = -i\psi_\pm + \chi_\pm. \]
\[ (2.8) \]
\[ (2.9) \]
\[ (2.10) \]

Off-shell \( Q_\pm \) transformations are
\[
Q_{\pm} v_1 = \psi_{\pm1}, \quad Q_{\pm} \psi_{\pm1} = \pm iD_1 \phi_{\pm}, \quad Q_{\pm} \psi_{\pm1} = \frac{i}{2} D_1 C \mp H_1, \\
Q_{\pm} H_1 = [\phi_{\pm}, \psi_{\mp1}] \pm \frac{1}{2} [C, \psi_{\pm1}] \mp \frac{i}{2} D_1 \eta_{\pm} + \frac{M}{3} \psi_{\pm1}, \\
Q_{\pm} X_2 = \psi_{\pm2}, \quad Q_{\pm} \psi_{\pm2} = \pm [\phi_{\pm}, X^2], \quad Q_{\mp} \psi_{\pm2} = \frac{1}{2} [C, X^2] \mp H_2, \\
Q_{\pm} H_2 = [\phi_{\pm}, \psi_{\pm2}] \pm \frac{1}{2} [X^2, \eta_{\pm}] \mp \frac{1}{2} [C, \psi_{\pm2}] + \frac{M}{3} \psi_{\pm2}, \\
Q_{\pm} X^3 = \chi_{\pm}, \quad Q_{\pm} \chi_{\pm} = \pm [\phi_{\pm}, X^3], \quad Q_{\mp} \chi_{\pm} = \frac{1}{2} [C, X^3] \mp H, \\
Q_{\pm} H = [\phi_{\pm}, \chi_{\mp}] \pm \frac{1}{2} [X^3, \eta_{\pm}] \mp \frac{1}{2} [C, \chi_{\pm}] + \frac{M}{3} \chi_{\pm}, \quad (2.11) \\
Q_{\pm} Y = \xi_{\pm}, \quad Q_{\pm} \xi_{\pm} = \pm [\phi_{\pm}, Y], \quad Q_{\mp} \xi_{\pm} = \frac{1}{2} [C, Y] \mp H_y, \\
Q_{\pm} H_y = [\phi_{\pm}, \xi_{\mp}] \pm \frac{1}{2} [Y, \eta_{\pm}] \mp \frac{1}{2} [C, \xi_{\pm}] + \frac{M}{3} \xi_{\pm}, \\
Q_{\pm} Y^\dagger = \xi_\pm^\dagger, \quad Q_{\pm} \xi_\pm^\dagger = \pm [\phi_{\pm}, Y^\dagger], \quad Q_{\mp} \xi_\pm^\dagger = \frac{1}{2} [C, Y^\dagger] \mp H_\dagger_y, \\
Q_{\pm} H_\dagger_y = [\phi_{\pm}, \xi_\pm^\dagger] \pm \frac{1}{2} [Y^\dagger, \eta_{\pm}] \mp \frac{1}{2} [C, \xi_\pm^\dagger] + \frac{M}{3} \xi_\pm^\dagger, \\
Q_{\pm} C = \eta_{\pm}, \quad Q_{\pm} \eta_{\pm} = \pm [\phi_{\pm}, C] + \frac{2M}{3} \phi_{\pm}, \quad Q_{\mp} \eta_{\pm} = \mp [\phi_{\pm}, \phi_{\mp}] \pm \frac{M}{3} C, \\
Q_{\pm} \phi_{\pm} = 0, \quad Q_{\mp} \phi_{\pm} = \mp \eta_{\pm}. \quad (2.12)
\]

Here
\[ H_y = i\tilde{H}_2 + H, \quad H_\dagger_y = -i\tilde{H}_2 + H, \quad (2.13) \]
and \( \tilde{H}_1 \) are auxiliary fields.

The mother theory action (2.1) is also invariant under the \( SU(2)_R \) transformations whose
Here the gauge invariant quantity \( F \)

We see that \( \phi \) is a singlet. \( \psi \) and \( \eta \) transform as doublets and \( \xi \) and \( \xi \) transform as a triplet under the \( SU(2) \) algebra.

\[
\begin{align*}
J_{++} &= \int dx_1 \left[ \psi_+^a(x_1) \frac{\delta}{\delta \psi_-^a(x_1)} + \xi_+^a(x_1) \frac{\delta}{\delta \xi_-^a(x_1)} + \bar{\xi}_+^a(x_1) \frac{\delta}{\delta \bar{\xi}_-^a(x_1)} - \eta_+^a(x_1) \frac{\delta}{\delta \eta_-^a(x_1)} \right] \\
&\quad + 2\phi_+^a(x_1) \frac{\delta}{\delta C^a(x_1)} - C^a(x_1) \frac{\delta}{\delta \phi_-^a(x_1)}
\end{align*}
\]

\[
J_{--} = \int dx_1 \left[ \psi_-^a(x_1) \frac{\delta}{\delta \psi_+^a(x_1)} + \xi_-^a(x_1) \frac{\delta}{\delta \xi_+^a(x_1)} + \bar{\xi}_-^a(x_1) \frac{\delta}{\delta \bar{\xi}_-^a(x_1)} - \eta_-^a(x_1) \frac{\delta}{\delta \eta_+^a(x_1)} \right] \\
&\quad - 2\phi_-^a(x_1) \frac{\delta}{\delta C^a(x_1)} + C^a(x_1) \frac{\delta}{\delta \phi_+^a(x_1)}
\]

\[
J_0 = \int dx_1 \left[ \psi_+^a(x_1) \frac{\delta}{\delta \psi_-^a(x_1)} - \psi_-^a(x_1) \frac{\delta}{\delta \psi_+^a(x_1)} + \xi_+^a(x_1) \frac{\delta}{\delta \xi_-^a(x_1)} + \bar{\xi}_+^a(x_1) \frac{\delta}{\delta \bar{\xi}_-^a(x_1)} - \eta_+^a(x_1) \frac{\delta}{\delta \eta_-^a(x_1)} \right] \\
&\quad + \xi_-^a(x_1) \frac{\delta}{\delta \xi_+^a(x_1)} + \bar{\xi}_-^a(x_1) \frac{\delta}{\delta \bar{\xi}_+^a(x_1)} - \xi_-^a(x_1) \frac{\delta}{\delta \xi_+^a(x_1)} + \bar{\xi}_-^a(x_1) \frac{\delta}{\delta \bar{\xi}_+^a(x_1)} \\
&\quad + \eta_-^a(x_1) \frac{\delta}{\delta \eta_+^a(x_1)} - \eta_-^a(x_1) \frac{\delta}{\delta \eta_+^a(x_1)} + 2\phi_+^a(x_1) \frac{\delta}{\delta \phi_-^a(x_1)} - 2\phi_-^a(x_1) \frac{\delta}{\delta \phi_+^a(x_1)} \right].
\]

The \( Q_\pm \) satisfy the following nilpotency relations,

\[
Q_+^2 = (\text{infinitesimal gauge transformation with parameter } \phi_+) + \frac{M}{3} J_{++},
\]

\[
Q_-^2 = (\text{infinitesimal gauge transformation with parameter } -\phi_-) - \frac{M}{3} J_{--},
\]

\[
\{Q_+, Q_-\} = (\text{infinitesimal gauge transformation with parameter } C) - \frac{M}{3} J_0,
\]

which satisfy the \( SU(2) \) algebra,

\[
[J_0, J_{\pm \pm}] = \pm 2 J_{\pm \pm}, \quad [J_{++}, J_{--}] = J_0.
\]

We see that \( (\psi_+, \psi_-), (\xi_+, \xi_-), (\xi_+^\dagger, \xi_-^\dagger), (\eta_+, -\eta_-) \) and \( (Q_+, Q_-) \) transform as doublets and \( (\phi_+, C, -\phi_-) \) as a triplet under the \( SU(2)_R \) transformation.

Using \( Q_\pm \), we can write down the mother theory action as \(^9\),

\[
S_{\text{mat}} = \left( Q_+ Q_- - \frac{M}{3} \right) F_{\text{mat}},
\]

where \( F_{\text{mat}} \) is

\[
F_{\text{mat}} = \frac{1}{g^2} \int dx_1 \text{Tr} \left[ Y D_1 Y^\dagger - \psi_+^\dagger \psi_- - \frac{1}{2} \xi_+^\dagger \xi_- - \frac{1}{2} \xi_+ \xi_-^\dagger - \frac{1}{4} \eta_+ \eta_- \right].
\]

Here the gauge invariant quantity \( F_{\text{mat}} \) is an \( SU(2)_R \) singlet,

\[
J_{\pm \pm} F_{\text{mat}} = J_0 F_{\text{mat}} = 0.
\]

\(^9\)This kind of deformation is discussed for various supersymmetric Yang-Mills models in [13].
and $SU(2)_R$ transformations on a doublet $(Q_+, Q_-)$ are

$$J_{\pm \pm} Q_\mp = Q_\mp, \quad J_0 Q_\pm = \pm Q_\pm.$$  \hfill (2.20)

Then we can see the $Q_\pm$ invariance of the action as

$$Q_+ S^\text{mat} = Q_+^2 Q_- F^\text{mat} - \frac{M}{3} Q_+ F^\text{mat}$$

$$= \frac{M}{3} J_{++} Q_- F^\text{mat} - \frac{M}{3} Q_+ F^\text{mat} = 0,$$

$$Q_- S^\text{mat} = \left( \{ Q_+, Q_- \} Q_- - Q_+ Q_-^2 \right) F^\text{mat} - \frac{M}{3} Q_- F^\text{mat}$$

$$= -\frac{M}{3} J_{0} Q_- F^\text{mat} + \frac{M}{3} Q_+ J_{--} F^\text{mat} - \frac{M}{3} Q_- F^\text{mat} = 0.$$  \hfill (2.21)

We should note that if a general gauge invariant quantity $\tilde{F}$ is an $SU(2)_R$ singlet, the following quantity

$$\left( Q_+ Q_- - \frac{M}{3} \right) \tilde{F},$$  \hfill (2.22)

is always $Q_\pm$ invariant.

The action (2.1) is obtained by integrating out the auxiliary fields $\tilde{H}_1, H_y, H_y^\dagger$, and it is written by complexified fields as

$$S^\text{mat} = \int dx \left( L_B^\text{mat} + L_F^\text{mat} \right),$$  \hfill (2.23)

$$L_B^\text{mat} = \frac{1}{g^2} \text{Tr} \left[ D_1 Y D_Y Y^\dagger + \frac{1}{4} (D_1 C)^2 + D_1 \phi_+ D_1 \phi_- + \frac{1}{4} [\phi_+, \phi_-]^2 + \frac{1}{4} [C, \phi_+] [\phi_-, C] + \frac{1}{2} [\phi_+, Y] [Y^\dagger, \phi_-] + \frac{1}{2} [\phi_-, Y] [Y^\dagger, \phi_+] + \frac{1}{4} [C, Y] [Y^\dagger, C] + \frac{1}{4} [Y, Y^\dagger]^2 \right]$$

$$- \frac{M}{3} Y D_1 Y^\dagger + \frac{M}{2} C [\phi_+, \phi_-] + \left( \frac{M}{3} \right)^2 \left( \frac{1}{4} C^2 + \phi_+ \phi_- \right),$$  \hfill (2.24)

$$L_F^\text{mat} = \frac{1}{g^2} \text{Tr} \left[ \xi_+^\dagger D_1 \xi_+ - \xi_- D_1 \xi_-^\dagger + i \eta_+ D_1 \psi_- + i \eta_- D_1 \psi_+ + i \xi_- [Y^\dagger, \psi_+] - i \xi_+ [Y, \psi_+] - i \xi_+ [Y^\dagger, \psi_-] + i \xi_- [Y, \psi_-] + \frac{1}{2} \eta_- [Y^\dagger, \xi_+] - \frac{1}{2} \eta_+ [Y, \xi_-] - \frac{1}{2} \eta_+ [Y^\dagger, \xi_-] + \frac{1}{2} \eta_- [Y, \xi_+] + \psi_- [C, \psi_+] - \psi_1 [\phi_+, \psi_-] + \psi_1 [\phi_-, \psi_+] + \frac{1}{4} \eta_+ [C, \eta_-] - \frac{1}{4} \eta_- [\phi_+, \eta_-] + \frac{1}{4} \eta_+ [\phi_-, \eta_+] + \frac{1}{2 \xi_- [\phi_+, \xi_-] - \frac{1}{2} \xi_+ [\phi_+, \xi_+] + \frac{1}{2} \xi_- [\phi_-, \xi_-] + \frac{1}{2} \xi_+ [\phi_-, \xi_+] + \frac{1}{2} \xi_- [C, \xi_+] + \frac{1}{2} \xi_+ [C, \xi_-] + \frac{1}{2} \xi_- [\phi_+, \xi_+] + \frac{1}{2} \xi_+ [\phi_-, \xi_-] + \frac{1}{2} \xi_- [\phi_+, \xi_+] + \frac{1}{2} \xi_+ [\phi_-, \xi_-] + \frac{2 M}{3} \psi_1 \psi_- + \frac{M}{3} \xi_+ \xi_+ + \frac{M}{3} \xi_- \xi_- - \frac{M}{6} \eta_+ \eta_- \right].$$  \hfill (2.25)
The mother theory action and $Q_{\pm}$ are invariant under the $U(1)_r$ transformation which acts on generic fields $O_{\text{mat}}$ as

$$O_{\text{mat}} \rightarrow e^{ir\theta}O_{\text{mat}}$$

(2.26)

where $\theta$ is a transformation parameter and $r$ represents the $r$-charge of $O_{\text{mat}}$. Each field has each $r$-charge described in the Table 2. We can confirm the compatibility with $Q_{\pm}$ by directly checking each transformation law in (2.11), (2.12). For example, let us see $Q_{\pm}$ transformations acting on the $r = 1$ fields $Y, \xi_{\pm}, H_y$,

$$Q_+ Y = \xi_+, \quad Q_+ \xi_\pm = \pm [\phi_\pm, Y], \quad Q_- \xi_\pm = \frac{1}{2} [C, Y] + H_y,$$

$$Q_\pm H_y = [\phi_\pm, \xi_\pm] \pm \frac{1}{2} [Y, \eta_{\pm}] \mp \frac{1}{2} [C, \xi_\pm] + \frac{M}{3} \xi_\pm.$$  

(2.27)

We can see that also the right hand sides still keep the same $r$-charge $r = 1$ as the ones of $Y, \xi_{\pm}, H_y$. Thus $Q_{\pm}$ commute with the $U(1)_r$ transformation. The $U(1)_r$ invariance of the mother theory action follows from the neutrality of the $F_{\text{mat}}$, thanks to the $U(1)_r$ invariance of the $Q_{\pm}$. By utilizing the $U(1)_r$ symmetry (2.26), we can perform the orbifolding [15–17] on the mother theory. The orbifolding will be explained in following sections.

### 2.1.3 Orbifold projection

We define the orbifold projection operator $\hat{\Gamma} \in \mathbb{Z}_N$ acting on a mother theory field $O_{\text{mat}}^{(r)}$ with $r$-charge $r = \bar{r}$ as

$$\hat{\Gamma}(O_{\text{mat}}^{(r)}) = e^{i\frac{2\pi}{N}} C(O_{\text{mat}}^{(r)}) C^{-1},$$

(2.28)

where $C \in \mathbb{Z}_N \subset U(mN)$ is a tensor product of the $N \times N$ matrix and $m \times m$ matrix as

$$C = \Omega \otimes 1_m.$$  

(2.29)

Here $\Omega$ is the $N \times N$ diagonal matrix

$$\Omega = \text{diag}(e^{i\frac{2\pi}{N}}, e^{i\frac{4\pi}{N}}, \ldots, e^{i\frac{2N\pi}{N}}).$$

(2.30)

Now we decompose $mN \times mN$ matrix elements into $N \times N$ blocks of the $m \times m$ submatrices. Indices of an $mN \times mN$ matrix valued field $(O_{\text{mat}}^{(r)})$ are represented as $(O_{\text{mat}}^{(r)})_{\bar{a}n_2, \bar{b}n_2'}$, where $\bar{a}, \bar{b}$ represent the $m \times m$ parts and the $n_2, n_2'$ represent the $N \times N$ parts.

The orbifold projection is removing matrix components except the ones satisfying

$$\left(\hat{\Gamma}(O_{\text{mat}}^{(r)})\right)_{\bar{a}n_2, \bar{b}n_2'} = (O_{\text{mat}}^{(r)})_{\bar{a}n_2, \bar{b}n_2'}.$$  

(2.31)

We perform this projection on the all fields of the mother theory. By replacing every mother theory field by its projected field, we obtain the orbifold lattice gauge theory action. Under
the projection, an \(mN \times mN\) matrix is reduced to \(N\) sets of \(m \times m\) matrices. For instance, among \(mN \times mN\) matrix components \((\mathcal{O}^{(r)}_{\text{mat}})_{\hat{a}n_2,\hat{b}n'_2}\) only the ones with \(n'_2 = n_2 + \bar{r}\) can remain as non-zero, and they are described by the \(N\) non-zero \(m \times m\) blocks as \(\mathcal{O}^{(r)}_{n_2}\) whose matrix components are \((\mathcal{O}^{(r)}_{n_2})_{\hat{a}\hat{b}} = (\mathcal{O}^{(r)}_{\text{mat}})_{\hat{a}n_2,\hat{b}n_2+\bar{r}}\). After the projection, indices \(n_2\) are regarded as the label of \(N\) sites in the \(N\) periodic lattice. And \(\mathcal{O}^{(r)}_{n_2}\) is interpreted as the \(m \times m\) matrix valued link field pointing from the site \(n_2\) to \(n_2 + \bar{r}\). So, for the \(r = 0\) fields, \(N\) non-zero \(m \times m\) matrices \(C_{n_2}, \psi_{\pm n_2}, \eta_{\pm n_2}, \phi_{\pm n_2}, \bar{H}_{1;n_2}, v_{n_2}^\dagger\) become the site fields on the \(n_2\). And for the \(r = 1\) fields, \(Y_{n_2}, \xi_{\pm n_2}, H_{y,n_2}\) are link fields pointing from \(n_2\) to \(n_2 + 1\). (Here their Hermitian conjugates \(Y_{n_2}^\dagger, \xi_{\pm n_2}^\dagger, H_{y,n_2}^\dagger\) are regarded as the link fields from \(n_2 + 1\) to \(n_2\).) The gauge symmetry of the mother theory \(U(mN)\) is broken to the \(U(m)^N\) by the orbifold projection.

After the projection, among the 8 supercharges of the mother theory (2.5), only 4 supercharges with \(r = 0\) can be preserved on the lattice. The \(Q_{\pm}\) are 2 of the 4 preserved supercharges on the lattice. The 4 supercharges are associated with the fermions on the sites. Among the 8 SUSY parameters \(\epsilon^T = \epsilon^T U_8^{-1}\) in (2.5), following 4 components become the SUSY parameter on the lattice,

\[
\epsilon_{\text{orb}}^T = (\epsilon_+, 0, 0, \epsilon\alpha, \epsilon_-, 0, 0, \epsilon\beta)
\]

(2.32)

Please note our orbifold lattice gauge theory can preserve not only the \(Q_{\pm}\) but also other 2 supercharges associated with \(\epsilon\alpha, \epsilon\beta\). In the lattice formulation in [11], only the \(Q_{\pm}\) can be preserved on the lattice.

The orbifolded action can also be written as

\[
S^{\text{orb}} = \left(Q_+ Q_- - \frac{M}{3}\right) \mathcal{F}^{\text{orb}},
\]

(2.33)

where \(\mathcal{F}^{\text{orb}}\) is

\[
\mathcal{F}^{\text{orb}} = \frac{1}{g^2} \int dx_1 \sum_{n_2} \text{tr} \left[ Y_{n_2} \mathcal{D}_1 Y_{n_2}^\dagger - \psi_{+1;n_2} \psi_{-1;n_2} - \frac{1}{2} \xi_{+;n_2} \xi_{-;n_2} - \frac{1}{2} \xi_{+;n_2} \xi_{+;n_2} - \frac{1}{2} \eta_{+;n_2} \eta_{-;n_2} \right],
\]

(2.34)

in the same way as (2.17). Here the "tr" denotes trace over the \(m \times m\) matrix. The covariant derivative \(\mathcal{D}_1\) is defined for link fields as

\[
\mathcal{D}_1 Y_{n_2} \equiv \partial_1 Y_{n_2}^\dagger + i v_{n_2}^1 Y_{n_2} - i Y_{n_2} v_{n_2+1}^1, \quad \mathcal{D}_1 Y_{n_2}^\dagger \equiv \partial_1 Y_{n_2}^\dagger + i v_{n_2+1} Y_{n_2}^\dagger - i Y_{n_2}^\dagger v_{n_2}.
\]

(2.35)

\footnote{We can also choose the definition \((\mathcal{O}^{(r)}_{n_2})_{\hat{a}\hat{b}} = (\mathcal{O}^{(r)}_{\text{mat}})_{\hat{a}n_2-\bar{r},\hat{b}n'_2}\). For the \(r = -1\) fields \(Y^\dagger, \xi_{\pm}^\dagger, H^\dagger\), we employ this choice.}
After the orbifold projection, the $Q_{\pm}$ transformations are written as

\[
Q_{\pm}^{1}_{n_2} = \psi_{\pm 1; n_2}, \quad Q_{\pm}^{2}_{n_2} = \pm i D_{1} \phi_{\pm 1; n_2}, \quad Q_{\mp}^{1}_{n_2} = \frac{i}{2} D_{1} C_{n_2} = \bar{H}_{1; n_2},
\]

\[
Q_{\pm} H_{1; n_2} = [\phi_{\pm; n_2}, \psi_{\mp; n_2}] + \frac{i}{2} [C_{n_2}, \psi_{\pm; n_2}] + \frac{i}{2} D_{1} \eta_{\pm; n_2} + \frac{M}{3} \psi_{\pm; n_2},
\]

\[
Q_{\pm} Y_{n_2} = \xi_{\pm; n_2}, \quad Q_{\pm} \xi_{\pm; n_2} = \pm (\phi_{\pm; n_2} Y_{n_2} - Y_{n_2} \phi_{\pm; n_2} + 1),
\]

\[
Q_{\mp} \xi_{\pm; n_2} = \frac{1}{2} (C_{n_2} Y_{n_2} - Y_{n_2} C_{n_2} + 1) = H_{2; n_2},
\]

\[
Q_{\pm} H_{y; n_2} = (\phi_{\pm; n_2} \xi_{\pm; n_2} - \xi_{\mp; n_2} \phi_{\pm; n_2} + 1) + \frac{1}{2} (C_{n_2} \xi_{\pm; n_2} - \xi_{\mp; n_2} C_{n_2} + 1)
\]

\[
\pm \frac{1}{2} (Y_{n_2} \eta_{\pm; n_2} - \eta_{\pm; n_2} Y_{n_2}) + \frac{M}{3} \xi_{\pm; n_2},
\]

\[
Q_{\pm} Y_{n_2} = \xi_{\mp; n_2}^\dagger, \quad Q_{\pm} \xi_{\mp; n_2} = \pm (\phi_{\pm; n_2} + 1) Y_{n_2}^\dagger - Y_{n_2} \phi_{\pm; n_2},
\]

\[
Q_{\mp} \xi_{\pm; n_2} = \frac{1}{2} (C_{n_2} Y_{n_2}^\dagger - Y_{n_2}^\dagger C_{n_2}) = H_{y; n_2}^\dagger,
\]

\[
Q_{\pm} H_{Y; n_2} = (\phi_{\pm; n_2} + 1) \xi_{\mp; n_2}^\dagger - \xi_{\mp; n_2} \phi_{\pm; n_2} + 1) + \frac{1}{2} (C_{n_2} + 1) \xi_{\mp; n_2}^\dagger - \xi_{\mp; n_2} C_{n_2} + 1)
\]

\[
\pm \frac{1}{2} (Y_{n_2}^\dagger \eta_{\pm; n_2} - \eta_{\pm; n_2} Y_{n_2}^\dagger) + \frac{M}{3} \xi_{\pm; n_2},
\]

\[
Q_{\mp} C_{n_2} = \eta_{\pm; n_2}, \quad Q_{\pm} \phi_{\pm; n_2} = 0, \quad Q_{\mp} \phi_{\pm; n_2} = \mp \eta_{\pm; n_2},
\]

\[
Q_{\mp} \eta_{\pm; n_2} = \pm [\phi_{\pm; n_2}, C_{n_2}] + \frac{2 M}{3} \phi_{\pm; n_2}, \quad Q_{\mp} \eta_{\pm; n_2} = \mp [\phi_{\pm; n_2}, \phi_{\mp; n_2}] + \frac{M}{3} C_{n_2}. \tag{2.36}
\]

### 2.1.4 Tree level continuum limit

For the discussion of the moduli fixing terms in the next sub-subsection, we will consider the tree level continuum limit. To see the tree level continuum limit, we perform the deconstruction, which is expanding the bosonic link fields around $\frac{1}{a}$,

\[
Y_{n_2} = \frac{1}{a} + s_{3;n_2} + i v_{2;n_2}, \quad Y^\dagger_{n_2} = \frac{1}{a} + s_{3;n_2} - i v_{2;n_2}, \tag{2.37}
\]

where the $a$ is regarded as lattice spacing. We also identify the site fields as

\[
\phi_{+; n_2} = s_{5;n_2}^5 + i s_{6;n_2}^6, \quad \phi_{-; n_2} = s_{5;n_2}^5 - i s_{6;n_2}^6, \quad C_{n_2} = s_{4;n_2}^4, \tag{2.38}
\]

where the $s_{i}^i$ $(i = 3, 4, 5, 6)$ are the scalar fields in the two-dimensional super Yang-Mills theory with 8 supercharges. By substituting the above into the orbifold lattice action \([2.33]\) and performing the Taylor expansion with respect to the lattice spacing $a$ around $a = 0$, we can see the target continuum limit at the tree level. During the procedure, we interpret $g^2 a = g_{2d}^2$ as the two-dimensional gauge coupling and it is fixed under the limit $a \to 0$. The target continuum limit ($a \to 0$) is

\[
S_{2d,0} = \frac{2}{g_{2d}^2} \int d^2 x \text{tr} \left[ \frac{1}{2} F_{12}^2 + \frac{1}{2} D_{\mu} s^i D_{\mu} s^i - \frac{1}{4} [s^i, s^j]^2 + \frac{i}{2} \Psi^T \gamma^i [s_1, \Psi] + \frac{1}{2} \Psi^T (D_{1} + \gamma_2 D_{2}) \Psi 
\]

\[
- \frac{i}{6} M \Psi^T \gamma^{23} \Psi + \frac{i M}{3} s^a F_{12} + \frac{1}{2} \left( \frac{M}{3} \right)^2 (s^a)^2 + i \frac{M}{3} \epsilon_{abc} s^a s^b s^c \right], \tag{2.39}
\]
where the subscripts $\mu$ run 1,2 and $i,j = 3,4,5,6$. The field $v_2$ is regarded as the gauge field along the $x_2$ direction and the $F_{12}$ is the gauge field strength, $F_{12} = \partial_1 v^2 - \partial_2 v^1 + i[v^1,v^2]$. Fermionic fields on the lattice are recombined to be the two-dimensional 8 component spinor $\Psi$ in the same way as (2.22)-(2.24), and the $\xi_{\pm}$ are reinterpreted as $\xi_{\pm} = i\psi_{\pm 2} + \chi_{\pm}$, $\xi^T_{\pm} = -i\psi_{\pm 2} + \chi_{\pm}$. The continuum action (2.39) is the mass-deformed two-dimensional supersymmetric Yang-Mills theory, which is same as (3.7) or (A.28) in the [1].

2.1.5 Stabilization of the vacuum without breaking lattice SUSY

It is necessary to justify the expansion around the point

$$Y_{n_2} = Y_{n_2}^\dagger = 1/a$$  \hspace{1cm} (2.40)

to make a lattice interpretation of our action (2.33) with the target continuum limit (2.39). In order to justify, quantum fluctuation around the classical vacua must be small enough compared to the classical value $1/a$. Actually, as pointed out also in [1], there is a flat direction along $s_3$ which allows the large fluctuation spoiling the lattice interpretation. In a conventional way to lift the degeneracy of the vacua, soft SUSY breaking mass terms might be introduced. Although such mass terms do not alter UV divergences, they break the supersymmetry on the lattice.

In this paper, as an alternative way, we would like to propose new moduli fixing terms which will not break the supersymmetry on the lattice,

$$S_{\text{mass}}^\text{orb} = \frac{a^2 \nu_1}{g^2} \left( Q_+ Q_- - \frac{1}{3} M \right) \int dx \sum_{n_2} \text{tr} \left( Y_{n_2} Y_{n_2}^\dagger - \frac{1}{a^2} \right)^2,$$  \hspace{1cm} (2.41)

where the $\nu_1$ is a mass parameter with mass dimension 1. These are very analogous to the mass terms of the $B(x)$ in the [1]. We should note that $Y_{n_2}, Y_{n_2}^\dagger$ are singlet under the $SU(2)_R$ symmetry generated by (2.14), and the lattice spacing $a$ is a non-dynamical quantity vanishing under the $Q_{\pm}$ transformation. Of course $a$ itself is invariant under the $SU(2)_R$. Since $\text{tr} \left( Y_{n_2} Y_{n_2}^\dagger - \frac{1}{a^2} \right)^2$ is a gauge singlet as well as an $SU(2)_R$ singlet, we can see that the mass terms (2.41) are invariant under the $Q_{\pm}$ from (2.19)-(2.21). Namely $Q_{\pm}$ are still preserved on the lattice in presence of the new moduli fixing terms (2.41).

The fixing terms (2.41) include the auxiliary fields. If we integrate out the auxiliary fields after summing up the (2.33) and (2.41), the terms depending on the $\nu_1$ become

$$S_{\text{mass}}^\text{orb} = \frac{1}{a^2} \int dx \sum_{n_2} \text{tr} \left[ 2a^2 \nu_1 \left( -\xi_{-;n_2}^\dagger \xi_{+;n_2} + \xi_{+;n_2}^\dagger \xi_{-;n_2} \right) Y_{n_2} 
- 2a^2 \nu_1 \left( \xi_{-;n_2} Y_{n_2}^\dagger + Y_{n_2}^\dagger \xi_{-;n_2} \right) \left( \xi_{+;n_2} Y_{n_2}^\dagger + Y_{n_2}^\dagger \xi_{+;n_2} \right) 
+ 2a^2 \nu_1 \left( D_1 Y_{n_2} Y_{n_2}^\dagger - Y_{n_2} Y_{n_2} D_1 Y_{n_2}^\dagger \right) 
- \frac{Ma^2 \nu_1}{3} \sum_{n_2} \left( y_{n_2}^2 - 4a^4 \nu_1^2 Y_{n_2} Y_{n_2}^\dagger Y_{n_2} Y_{n_2}^\dagger \right) \right],$$  \hspace{1cm} (2.42)

where $Y_{n_2} = Y_{n_2} Y_{n_2}^\dagger - \frac{1}{a^2}$. At the continuum limit, these mass terms become

$$S_{\text{mass}}^\text{orb} \xrightarrow{a \to 0} \frac{1}{g_{2d}^2} \int d^2 x \text{tr} \left( -16 \nu_1^2 - \frac{4M \nu_1}{3} \right) s^2 + 8i \nu_1 s_3 F_{12} - 8i \nu_1 \chi_{-\chi_+}$$  \hspace{1cm} (2.43)
these are the same as the eq. (3.12) in the [1]. Here each $\chi_+$ and $\chi_-$ is a spinor component in the $\Psi$ written in (2.2)-(2.4).

In our case, we also have to take care of the IR divergence $\sim g_2^2 \log(a\tilde{\nu})$ where the $\tilde{\nu}$ is the IR cut-off. To keep the divergence much smaller than the classical values, $a^2g_2^2 \log(a\tilde{\nu}) \ll 1$ must be satisfied. In order to take the continuum limit with keeping $a^2g_2^2 \log(a\tilde{\nu}) \ll 1$, we need to separate the procedure of taking the limits into the following two steps:

- In the orbifold lattice theory, first we should take $a \to 0$ and $N \to \infty$, with keeping $\nu_1$ (or $\tilde{\nu}_1, \tilde{\nu}_2$) and $aN$ fixed.

- After that, $\nu_1 \to 0$ as well as $aN \to \infty$ (or $\tilde{\nu}_1, \tilde{\nu}_2 \to 0$) to recover the full 8 supercharges.

In this way, the step 2 and the step 3, which are a part of the steps of taking the limits explained in Sec. [1.1] get separated from each other.

There is also another constraint on the parameter $\nu_1$. In order for $s_3$ to have a positive mass squared, $\nu_1$ must satisfy

$$-\frac{M}{12} < \nu_1 < 0.$$  

(2.44)

So in presence of the SUSY preserving moduli fixing terms (2.41), we also have to be careful about parameter region of the $\nu_1$ [11].

For a practical usage, if one prefers a simpler treatment which does not include fermionic terms, also adding conventional soft breaking mass terms

$$\tilde{S}_{\text{mass}} = \frac{1}{g^2} \int dx_1 \sum_{n_2} \left( a^2\tilde{\nu}_1^2 \left( Y_{n_2} Y_{n_2}^\dagger - \frac{1}{a^2} \right)^2 + a^2\tilde{\nu}_2^2 \left| \text{tr}(Y_{n_2} Y_{n_2}^\dagger) - \frac{1}{a^2} \right|^2 \right),$$  

(2.45)

could work. Because these moduli fixing terms are just soft SUSY breaking terms, they would not alter the UV divergences. But conclusions of the numerical calculation would be more or less obscured since the soft breaking terms break the lattice SUSY. The SUSY preserving fixing terms (2.41) would help to get more concrete conclusions.

2.1.6 Orbifold lattice action with the moduli fixing terms

Finally, by adding the moduli fixing terms (2.42), we complete the construction of the orbifold lattice action as

$$S_{\text{orb}} = S_{B\text{orb}} + S_{F\text{orb}},$$  

(2.46)

[11] This mass term will provide not only the mass term for the $SU(m)$ part, but also the $U(1)$ part decoupling from the $SU(m)$ part.
where the $S_{B}^{\text{orb}}$ is the bosonic part described as

$$S_{B}^{\text{orb}} = \frac{1}{g^2} \int dx \sum_{n_2} \text{tr} \left[ (\partial_{x_1} Y_{n_2} + i v_{n_2} Y_{n_2} - i Y_{n_2} v_{n_2}) (\partial_{x_1} Y_{n_2}^+ + i v_{n_2+1} Y_{n_2}^+ - i Y_{n_2} v_{n_2}^+) \right]$$

$$+ \frac{1}{4} (D_1 C_{n_2})^2 + D_1 \phi_{+;n_2} D_1 \phi_{-;n_2} + \frac{1}{4} [\phi_{+;n_2}, \phi_{-;n_2}]^2 + \frac{1}{4} [C_{n_2}, \phi_{+;n_2}][\phi_{-;n_2}, C_{n_2}]$$

$$+ \frac{1}{2} (\phi_{+;n_2} Y_{n_2} - Y_{n_2} \phi_{+;n_2+1})(Y_{n_2}^+ \phi_{-;n_2} - \phi_{-;n_2+1} Y_{n_2}^+)$$

$$+ \frac{1}{2} (\phi_{-;n_2} Y_{n_2} - Y_{n_2} \phi_{-;n_2+1})(Y_{n_2}^+ \phi_{+;n_2} - \phi_{+;n_2+1} Y_{n_2}^+)$$

$$+ \frac{1}{4} (C_{n_2} Y_{n_2} - Y_{n_2} C_{n_2+1})(Y_{n_2} C_{n_2} - C_{n_2+1} Y_{n_2}^+)$$

$$- \frac{1}{4} (Y_{n_2} Y_{n_2}^+ - Y_{n_2-1} Y_{n_2-1})(Y_{n_2-1} Y_{n_2-1} - Y_{n_2} Y_{n_2}^+)$$

$$- \frac{M}{3} \left( Y_{n_2} \partial_{x_1} Y_{n_2}^+ + i Y_{n_2} v_{n_2+1} Y_{n_2}^+ - i Y_{n_2} Y_{n_2} v_{n_2}^+ \right)$$

$$- \frac{M}{2} C_{n_2} [\phi_{+;n_2}, \phi_{-;n_2}] + \left( \frac{M}{3} \right)^2 \left( \frac{1}{4} C_{n_2}^2 + \phi_{+;n_2} \phi_{-;n_2} \right)$$

$$+ 2 a^2 \nu_1 \left( D_1 Y_{n_2} Y_{n_2}^+ Y_{n_2} - Y_{n_2} Y_{n_2} D_1 Y_{n_2}^+ \right)$$

$$- \frac{M a^2 \nu_1}{3} Y_{n_2}^2 - 4 a^4 \nu_2 Y_{n_2} Y_{n_2}^+ Y_{n_2} Y_{n_2}^+ \right]. \quad (2.47)$$

The fermionic part $S_{F}^{\text{orb}}$ is

$$S_{F}^{\text{orb}} = \int dx \left( L_{F1}^{\text{orb}} + L_{F2}^{\text{orb}} + L_{F3}^{\text{orb}} + L_{mF}^{\text{orb}} \right), \quad (2.48)$$

where

$$L_{F1}^{\text{orb}} = \frac{1}{g^2} \sum_{n_2} \text{tr} \left[ + i \eta_{+;n_2} D_1 \psi_{-;n_2} + i \eta_{-;n_2} D_1 \psi_{+;n_2} + \xi_{+;n_2} \partial_{x_1} \xi_{+;n_2} - \xi_{-;n_2} \partial_{x_1} \xi_{+;n_2} \right.$$  

$$+ i \xi_{+;n_2} (v_{1;n_2} \xi_{+;n_2} - \xi_{+;n_2} v_{1;n_2+1}) - i \xi_{-;n_2} (v_{1;n_2+1} \xi_{+;n_2} - \xi_{+;n_2} v_{1;n_2})$$

$$+ i \xi_{-;n_2} (Y_{n_2} \psi_{-;n_2} - \psi_{-;n_2} Y_{n_2}^+) - i \xi_{+;n_2} (Y_{n_2} \psi_{+;n_2+1} - \psi_{+;n_2+1} Y_{n_2}^+)$$

$$- i \xi_{+;n_2} (Y_{n_2} \psi_{-;n_2} - \psi_{-;n_2+1} Y_{n_2}^+) + i \xi_{+;n_2} (Y_{n_2} \psi_{-;n_2+1} - \psi_{-;n_2} Y_{n_2}^+) \right]. \quad (2.49)$$

$$L_{F2}^{\text{orb}} = \frac{1}{g^2} \sum_{n_2} \text{tr} \left[ - \frac{1}{2} (\eta_{-;n_2} Y_{n_2} - Y_{n_2} \eta_{-;n_2+1}) \xi_{+;n_2}^+ - \frac{1}{2} (\eta_{-;n_2+1} Y_{n_2}^+ - Y_{n_2}^+ \eta_{-;n_2}) \xi_{+;n_2} \right.$$  

$$- \frac{1}{2} (\eta_{+;n_2} Y_{n_2} - Y_{n_2} \eta_{+;n_2+1}) \xi_{-;n_2}^+ - \frac{1}{2} (\eta_{+;n_2+1} Y_{n_2}^+ - Y_{n_2}^+ \eta_{+;n_2}) \xi_{-;n_2}$$

$$+ \psi_{-;n_2} [C_{n_2}, \psi_{+;n_2}] - \psi_{-;n_2} \phi_{+;n_2}, \psi_{-;n_2}] + \psi_{+;n_2} \phi_{+;n_2}, \psi_{-;n_2}]$$

$$- \frac{1}{4} \eta_{+;n_2} [C_{n_2}, \eta_{-;n_2}] - \frac{1}{4} \eta_{-;n_2} [\phi_{+;n_2}, \eta_{-;n_2}] + \frac{1}{4} \eta_{+;n_2} [\phi_{-;n_2}, \eta_{+;n_2}], \quad (2.50)$$

16
Since the orbifold projection is just picking up a part of the $mN$ condition on all fields. The lattice theory (2.46) also. Here implemented in the [4–9, 44].

Continuum limit without any fine-tunings. Here we apply the momentum cut-off regularization even if we regularize a one-dimensional system in a naive way, it is easy to recover the target a later section Sec. 3.1, UV divergences are usually absent in one-dimensional systems. Then

\[ \frac{1}{g^2} \sum_{n_2} \text{tr} \left[ -\frac{1}{2} \xi^\dagger_{-n_2} (\phi_{+;n_2} \xi_{-;n_2} - \xi_{-;n_2} \phi_{+;n_2+1}) - \frac{1}{2} \xi_{-n_2} (\phi_{+;n_2+1} \xi^\dagger_{-;n_2} - \xi^\dagger_{-;n_2} \phi_{+;n_2}) \right. \\
+ \frac{1}{2} \xi^\dagger_{+;n_2} (\phi_{-;n_2} \xi_{+;n_2} - \xi_{+;n_2} \phi_{-;n_2+1}) + \frac{1}{2} \xi_{+;n_2} (\phi_{-;n_2+1} \xi^\dagger_{+;n_2} - \xi^\dagger_{+;n_2} \phi_{-;n_2}) \\
+ \frac{1}{2} \xi^\dagger_{-;n_2} (C_{n_2} \xi_{+;n_2} - \xi_{+;n_2} C_{n_2+1}) + \frac{1}{2} \xi_{-;n_2} (C_{n_2+1} \xi^\dagger_{+;n_2} - \xi^\dagger_{+;n_2} C_{n_2}) \right], \\
\]  
(2.51)

\[ \frac{1}{g^2} \sum_{n_2} \text{tr} \left[ \frac{2M}{3} \psi_{+;1n_2} \psi_{-;1n_2} + \frac{M}{3} \xi_{+;n_2} \xi^\dagger_{-;n_2} + \frac{M}{3} \xi^\dagger_{+;n_2} \xi_{-;n_2} - \frac{M}{6} \eta_{+;n_2} \eta_{-;n_2} \\
+ 2a^2 \nu_1 \left( -\xi_{-;n_2} \xi^\dagger_{+;n_2} + \xi_{+;n_2} \xi^\dagger_{-;n_2} \right) Y_{n_2} \\
- 2a^2 \nu_1 \left( \xi_{-;n_2} Y^\dagger_{n_2} + Y_{n_2} \xi^\dagger_{-;n_2} \right) \left( \xi^\dagger_{+;n_2} Y_{n_2} + Y_{n_2} \xi^\dagger_{+;n_2} \right) \right]. \\
\]  
(2.52)

2.2 Momentum cut-off regularization on the orbifold lattice gauge theory (the hybrid regularization theory)

To perform numerical studies, we need to regularize the continuum $x_1$ direction of the orbifold lattice theory [2,36] also. Here $x_1$ is periodic $x_1 \sim x_1 + R_1$, and we impose the periodic boundary condition on all fields.

At finite $mN$, the orbifold lattice gauge theory can be regarded as a one-dimensional system. Since the orbifold projection is just picking up a part of the $mN \times mN$ matrices, the action can be embedded into a one-dimensional $mN \times mN$ matrix model system. As we explain in a later section Sec. 3.1, UV divergences are usually absent in one-dimensional systems. Then even if we regularize a one-dimensional system in a naive way, it is easy to recover the target continuum limit without any fine-tunings. Here we apply the momentum cut-off regularization implemented in the [4,9,11,12].

To implement the momentum cut-off regularization, first we need to gauge-fix the $U(m)^N$ gauge symmetry. Details of the gauge fixing are described in the appendix [3]. Here we will impose the static diagonal gauge,

\[ v_{n_2}^1(x_1) = \frac{1}{R_1} \text{diag}(\alpha_{1n_2}, \ldots, \alpha_{an_2}, \ldots, \alpha_{mn_2}). \]  
(2.53)

Here the $\alpha_{an_2}$ are dimensionless constants with respect to $x_1$. By this gauge fixing, we need to include the following Fadeev-Popov determinant term,

\[ S_{FP} = -\sum_{\tilde{a} < b} \sum_{n_2} 2 \log \left| \sin \frac{\alpha_{\tilde{a}n_2} - \alpha_{bn_2}}{2} \right|. \]  
(2.54)

Here the ghost fields are sitting on sites. We can set the domain of $\alpha_{n_2}^\tilde{a}$ as $\max(\alpha_{n_2}^\tilde{a}) - \min(\alpha_{n_2}^\tilde{a}) \leq 2\pi$ by fixing the residual large gauge transformations with non-zero winding numbers [3]. Here

\[ ^{12} \text{There is also another choice to employ the lattice regularization like in [54].} \]

\[ ^{13} \text{It is necessary to fix the large gauge transformations to justify the momentum cut-off. If they are not fixed, there is a risk to allow the momentum to go beyond the cut-off $A$ since the transformations have an effect to shift the momentum.} \]
the integration measure is taken to be uniform. After performing the gauge fixing, we make a Fourier expansion with the UV cut-off $\Lambda$ as

$$\Phi_{n_2}(x_1) = \sum_{p_1 = -\Lambda}^{\Lambda} \Phi_{n_2,p_1} e^{ip_1 \omega x_1},$$

(2.55)

where the $\Phi$ denotes general field variables in the orbifold lattice gauge theory and $\omega = \frac{2\pi}{R_1}$ and $p_1$ is integer. By substituting the Fourier expansion into the action, we will obtain the (1+1)-dimensional hybrid regularization theory action as

$$S^\Lambda = S^\Lambda_B + S^\Lambda_F.$$

(2.56)

Here the bosonic part $S^\Lambda_B$ is

$$S^\Lambda_B = S^\Lambda_{Bp} + S^\Lambda_{B0} + S^\Lambda_{Bm} + S^\Lambda_{B\nu_1p} + S^\Lambda_{B\nu_10},$$

(2.57)

where

$$S^\Lambda_{Bp} = \frac{1}{g^2} \sum_{n_2} \sum_{a,b} \sum_{p_1 = -\Lambda}^{\Lambda} \left[ \left( ip_1 \omega + i \frac{\alpha_{n_2}^a - \alpha_{n_2+1}^b}{R_1} \right) \left( -ip_1 \omega + i \frac{\alpha_{n_2+1}^b - \alpha_{n_2}^a}{R_1} \right) \hat{Y} \hat{\phi}^\dagger \hat{Y} \hat{\phi}^\dagger \right]_{n_2,p_1 - p_1},$$

$$+ \frac{1}{4} \left( ip_1 \omega + i \frac{\alpha_{n_2}^a - \alpha_{n_2}^b}{R_1} \right) \left( -ip_1 \omega + i \frac{\alpha_{n_2}^b - \alpha_{n_2}^a}{R_1} \right) \hat{C} \hat{\phi}^\dagger \hat{C} \hat{\phi}^\dagger,$$

$$+ \left( -\frac{M}{3} + 4\nu_1 \right) \left( -ip_1 \omega + i \frac{\alpha_{n_2+1}^b - \alpha_{n_2}^a}{R_1} \right) \hat{Y} \hat{\phi}^\dagger \hat{Y} \hat{\phi}^\dagger \right],$$

(2.58)

$$S^\Lambda_{B0} = \frac{1}{g^2} \sum_{n_2} \text{tr} \left[ \frac{1}{4} \left( \hat{\phi}_{+;n_2} \hat{\phi}_{-;n_2} \right)^2 \right]_0 + \frac{1}{4} \left( \hat{C}_{n_2} \hat{\phi}_{+;n_2} \hat{\phi}_{-;n_2} \hat{C}_{n_2} \right)_0$$

$$+ \frac{1}{2} \left( \hat{\phi}_{-;n_2} \hat{Y}_{n_2} - \hat{Y}_{n_2} \hat{\phi}_{-;n_2} \right) \left( \hat{Y}^\dagger_{n_2} \hat{\phi}_{-;n_2} \hat{Y}^\dagger_{n_2} \right)_0,$$

$$+ \frac{1}{2} \left( \hat{\phi}_{+;n_2} \hat{Y}_{n_2} - \hat{Y}_{n_2} \hat{\phi}_{+;n_2} \right) \left( \hat{Y}^\dagger_{n_2} \hat{\phi}_{+;n_2} \hat{Y}^\dagger_{n_2} \right)_0$$

$$+ \frac{1}{4} \left( \hat{C}_{n_2} \hat{Y}_{n_2} - \hat{Y}_{n_2} \hat{C}_{n_2} \right) \left( \hat{Y}^\dagger_{n_2} \hat{C}_{n_2} \hat{Y}^\dagger_{n_2} \right)_0,$$

$$- \frac{1}{4} \left( \hat{Y}_{n_2} \hat{Y}^\dagger_{n_2} \hat{Y}_{n_2} \right) \left( \hat{Y}_{n_2} \hat{Y}^\dagger_{n_2} \hat{Y}_{n_2} \hat{Y}^\dagger_{n_2} \right)_0 \right],$$

(2.59)

$$S^\Lambda_{Bm} = \frac{1}{g^2} \sum_{n_2} \text{tr} \left[ -\frac{M}{2} \hat{C}_{n_2} \hat{\phi}_{+;n_2} \hat{\phi}_{-;n_2} \right]_0 + \left( \frac{M}{3} \right)^2 \left( \hat{C}_{n_2}^2 \right)_0 + \left( \hat{\phi}_{+;n_2} \hat{\phi}_{-;n_2} \right)_0 \right],$$

(2.60)

14 In the $U(m)^N$ theory, there are decoupling $U(1)$ zero mode of site fields. It is required to remove them for the simulation, in particular we need to remove the fermionic decoupling modes to avoid the zero fermion determinant.
\[
S_{B_{n1}}^\Lambda = \frac{2a^2\nu_1}{g^2} \sum_{n_2} \sum_{a,b} \sum_{p_1=-\Lambda}^{\Lambda} \left[ \frac{2g^2}{\nu_1} \left( \hat{\gamma}_{n_2,p_1} \left( \hat{\gamma}_{n_2} \hat{\gamma}_{n_2} \right)_{-p_1} \right) \frac{\alpha_{n_2} - \alpha_{n_2+1}}{R_1} \right] \left( ip_1\omega + \frac{\alpha_{n_2} - \alpha_{n_2+1}}{R_1} \right) \hat{\gamma}_{n_2,p_1} \left( \hat{\gamma}_{n_2} \hat{\gamma}_{n_2} \right)_{-p_1} \\
- \left( ip_1\omega + \frac{\alpha_{n_2} - \alpha_{n_2+1}}{R_1} \right) \hat{\gamma}_{n_2,p_1} \left( \hat{\gamma}_{n_2} \hat{\gamma}_{n_2} \right)_{-p_1} \right],
\]
\[
S_{B_{n0}}^\Lambda = \frac{1}{g^2} \sum_{n_2} \left[ - \frac{Ma^2\nu_1}{3} \left( \hat{\gamma}_{n_2} \right)_0 - 4a^4\nu_1^2 \left( \hat{\gamma}_{n_2} \hat{\gamma}_{n_2} \right)_0 \right].
\]

Here of course the momentum along the \(x_1\) direction is conserved at each vertex, and each subscript \(0\) and \(-p_1\) means the sum of the all assignments of the momentum such that the total momentum is 0 or \(-p_1\) respectively. The fermionic part of the action \(S_{F}^\Lambda\) is
\[
S_{F}^\Lambda = S_{F_{1p}}^\Lambda + S_{F_{10}}^\Lambda + S_{F_{20}}^\Lambda + S_{mF}^\Lambda + S_{\nu_1}^\Lambda,
\]
where
\[
S_{F_{1p}}^\Lambda = \frac{1}{2g^2} \sum_{n_2} \sum_{a,b} \sum_{p_1=-\Lambda}^{\Lambda} \left[ \frac{2g^2}{\nu_1} \left( \hat{\gamma}_{n_2,p_1} \left( \hat{\gamma}_{n_2} \hat{\gamma}_{n_2} \right)_{-p_1} \right) \frac{\alpha_{n_2} - \alpha_{n_2+1}}{R_1} \right] \left( ip_1\omega + \frac{\alpha_{n_2} - \alpha_{n_2+1}}{R_1} \right) \hat{\gamma}_{n_2,p_1} \left( \hat{\gamma}_{n_2} \hat{\gamma}_{n_2} \right)_{-p_1} \\
+ \left( ip_1\omega + \frac{\alpha_{n_2} - \alpha_{n_2+1}}{R_1} \right) \hat{\gamma}_{n_2,p_1} \left( \hat{\gamma}_{n_2} \hat{\gamma}_{n_2} \right)_{-p_1} \right],
\]
\[
S_{F_{10}}^\Lambda = \frac{1}{g^2} \sum_{n_2} \left[ \frac{2g^2}{\nu_1} \left( \hat{\gamma}_{n_2} \hat{\gamma}_{n_2} \right)_0 - \frac{2g^2}{\nu_1} \left( \hat{\gamma}_{n_2} \hat{\gamma}_{n_2} \right)_0 \right] \frac{\alpha_{n_2} - \alpha_{n_2+1}}{R_1} \left( ip_1\omega + \frac{\alpha_{n_2} - \alpha_{n_2+1}}{R_1} \right) \hat{\gamma}_{n_2,p_1} \left( \hat{\gamma}_{n_2} \hat{\gamma}_{n_2} \right)_{-p_1} \\
- \left( ip_1\omega + \frac{\alpha_{n_2} - \alpha_{n_2+1}}{R_1} \right) \hat{\gamma}_{n_2,p_1} \left( \hat{\gamma}_{n_2} \hat{\gamma}_{n_2} \right)_{-p_1} \\
- \left( ip_1\omega + \frac{\alpha_{n_2} - \alpha_{n_2+1}}{R_1} \right) \hat{\gamma}_{n_2,p_1} \left( \hat{\gamma}_{n_2} \hat{\gamma}_{n_2} \right)_{-p_1} \right],
\]
\[
S_{F_{20}}^\Lambda = \frac{1}{g^2} \sum_{n_2} \left[ \frac{2g^2}{\nu_1} \left( \hat{\gamma}_{n_2} \hat{\gamma}_{n_2} \right)_0 - \frac{2g^2}{\nu_1} \left( \hat{\gamma}_{n_2} \hat{\gamma}_{n_2} \right)_0 \right] \frac{\alpha_{n_2} - \alpha_{n_2+1}}{R_1} \left( ip_1\omega + \frac{\alpha_{n_2} - \alpha_{n_2+1}}{R_1} \right) \hat{\gamma}_{n_2,p_1} \left( \hat{\gamma}_{n_2} \hat{\gamma}_{n_2} \right)_{-p_1} \\
- \left( ip_1\omega + \frac{\alpha_{n_2} - \alpha_{n_2+1}}{R_1} \right) \hat{\gamma}_{n_2,p_1} \left( \hat{\gamma}_{n_2} \hat{\gamma}_{n_2} \right)_{-p_1} \\
- \left( ip_1\omega + \frac{\alpha_{n_2} - \alpha_{n_2+1}}{R_1} \right) \hat{\gamma}_{n_2,p_1} \left( \hat{\gamma}_{n_2} \hat{\gamma}_{n_2} \right)_{-p_1} \right] \\
- \left( ip_1\omega + \frac{\alpha_{n_2} - \alpha_{n_2+1}}{R_1} \right) \hat{\gamma}_{n_2,p_1} \left( \hat{\gamma}_{n_2} \hat{\gamma}_{n_2} \right)_{-p_1} \right],
\]
(2.61)
The above solution uplifts the theory from two dimensions to four dimensions.

Then we complete the construction of the hybrid regularization theory. For later discussion, we should keep in mind that there are no derivative couplings other than the 4-point bosonic vertices with one derivative in $S_{B\nu_1l}$.

### 2.3 Completing the non-perturbative formulation by uplifting the two-dimensional theory to the four-dimensional theory

Let us complete the construction of the non-perturbative formulation for non-commutative $\mathcal{N} = 2$ four-dimensional $U(k)$ supersymmetric Yang-Mills theories. In the hybrid regularization theory \eqref{eq:hybrid}, we expand fields around the following minimum of $k$-coincident Fuzzy 2-sphere,

$$C_{n_{2,c}} = \frac{2M}{3} L_3, \quad \phi_{\pm; n_{2,c}} = \frac{M}{3} (L_1 \pm iL_2),$$

where $m \times m$ matrices $L_i (i = 1, 2, 3)$ are decomposed to tensor products of $l \times l$ and $k \times k$ as

$$L_i = L_i^{(l)} \otimes 1_k, \quad m = lk.$$

Here the $L_i^{(l)}$ are $SU(2)$ generators in the $l(= 2j + 1)$-dimensional irreducible representation,

$$[L_i^{(l)}, L_j^{(l)}] = i\varepsilon_{ijk} L_k^{(l)}.$$  \hspace{1cm} (2.72)

The above solution uplifts the theory from two dimensions to four dimensions\textsuperscript{15}. The emergent two dimensions by the \eqref{eq:massive} are regularized by the non-commutative parameter $\Theta$, UV cut-off $\hat{\Lambda}$ and radius of the sphere $R_f$. Here the non-commutative parameter is $\Theta = \frac{1}{M^2 l}$ and the UV cut-off is $\hat{\Lambda} \sim 2j/R_f \sim \frac{M}{2} \cdot 2j$, and $R_f = \frac{3M}{1}$. The four-dimensional space is consisting of $\mathbb{R}^4_\Lambda$ which

\textsuperscript{15}Good references about Fuzzy 2-sphere described here are \cite{32,39,55}.
is regularized by the momentum cut-off, $\mathbb{R}^1_{\text{orb}}$, which is regularized by the orbifold projection, and the Fuzzy $S^2$. The final form of our non-perturbative formulation for the four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theories is obtained by the expansion of fields around the solution (2.70) in the hybrid regularization theory.

Let us see how the field variables in the hybrid regularization theory are expanded around the solution (2.70). A two-dimensional momentum $\hat{p}$ modes on $\mathbb{R}^1_{\Lambda} \times \mathbb{R}^1_{\text{orb}}$ of field variables in the $U(m)$ hybrid regularization theory, say $B$, are expanded further by the spherical harmonics,

$$\tilde{B}(\hat{p}) = \sum_{J=0}^{2j} \sum_{J_3=-J}^{J} \tilde{h}^{(jj)}_{JJ_3} \otimes b_{JJ_3}(\hat{p}),$$

(2.73)

where $\tilde{h}^{(jj)}_{JJ_3}$ is an $l \times l$ matrix corresponding to the Fuzzy spherical harmonic, and $b_{JJ_3}(\hat{p})$ is a $k \times k$ matrix becoming a field variable on the target four-dimensional theory. We truncate the sum of the spherical harmonic over the spin at the level spin $j$, the $j$ gives the UV cut-off for the Fuzzy 2-sphere directions as $\hat{\Lambda} = 2j/R_f$.

By this uplifting, we have completed the non-perturbative formulation for the non-commutative $\mathcal{N} = 2$ four-dimensional supersymmetric Yang-Mills theories on $\mathbb{R}^2 \times \mathbb{R}^2_\Theta$. In the formulation, the four dimensions are regularized as $\mathbb{R}^1_{\Lambda} \times \mathbb{R}^1_{\text{orb}} \times \text{Fuzzy } S^2$.

### 3 How to take the target theory limit

In this section, we will explain how we take the target non-commutative $\mathcal{N} = 2$ supersymmetric Yang-Mills theory limit from our non-perturbative formulation constructed in the previous section. We will explain how the following steps lead to the target theory:

1. $\Lambda \rightarrow \infty$.
2. $a \rightarrow 0$ with $aN$ and $\nu_1$ fixed.
3. $\nu_1 \rightarrow 0$, $aN \rightarrow \infty$.
4. $l, m \rightarrow \infty$, $(M \rightarrow 0, \hat{\Lambda}, R_f \rightarrow \infty)$ with $k, \Theta$ fixed.

#### 3.1 1st step: From the non-perturbative formulation on $\mathbb{R}^1_{\Lambda} \times \mathbb{R}^1_{\text{orb}} \times \text{Fuzzy } S^2$ to the orbifold lattice theory on $\mathbb{R}^1_{\Lambda} \times \mathbb{R}^1_{\text{orb}} \times \text{Fuzzy } S^2$

Here we will explain how the orbifold lattice gauge theory (2.46) is recovered from the starting non-perturbative formulation only by taking the $\Lambda \rightarrow \infty$ without any parameter fine-tunings. Although a one-dimensional system is expected not to have UV divergences in general, there is also a case that dangerous UV divergences requiring fine-tunings show up even in one-dimensions as discussed in [54,56]. So to rigorously confirm the absence of dangerous quantum corrections, we will carefully discuss the UV divergences here.

Here we will check the all diagrams with UV divergences, since all quantum corrections come from diagrams with UV divergences. In the non-perturbative formulation action, which is (2.56) expanded around the Fuzzy 2-sphere solution (2.70), only the bosonic 4-point vertices in $S_{Br_1p}$
can be derivative couplings. Then the degree of UV divergence \( D \) of each diagram is estimated as follows,

\[
D = L_{oop} - 2I_B - I_F + V_{d4}, \quad L_{oop} = 1 + I_B + I_F - V_{d4} - V_{bff} - V_{bbff} - \sum_{n \geq 3} V_n, \quad (3.1)
\]

namely

\[
D = 1 - I_B - V_{bff} - V_{bbff} - \sum_{n \geq 3} V_n. \quad (3.2)
\]

And

\[
E_B + 2I_B = \sum_{n \geq 3} (nV_n) + 4V_{d4} + V_{bff} + 2V_{bbff}, \quad E_F + 2I_F = 2V_{bff} + 2V_{bbff}. \quad (3.3)
\]

Here \( L_{oop} \) is the number of loops and each \( I_B \) and \( I_F \) is the number of internal lines of bosons and fermions respectively. The \( V_n(n \geq 3) \) are the number of bosonic \( n \)-point vertices without derivative and \( V_{d4} \) is the number of bosonic 4-point vertices with one derivative, each \( V_{bff} \) and \( V_{bbff} \) is the number of boson-fermion-fermion vertices and boson-boson-fermion-fermion interaction terms respectively. And each \( E_B \) and \( E_F \) is the number of external bosons and fermions respectively. Only following three diagrams can have UV divergences with \( D \geq 0 \),

\[
I_F = E_B = V_{bff} = 1, \quad E_F = I_B = V_n = V_{d4} = V_{bbff} = 0, \quad (3.4)
\]

\[
E_B = 2, \quad I_F = V_{bbff} = 1, \quad E_F = I_B = V_n = V_{d4} = V_{bff} = 0, \quad (3.5)
\]

and

\[
E_B = 2, \quad V_{d4} = I_B = 1, \quad V_{bff} = V_{bbff} = E_F = V_n = I_F = 0. \quad (3.6)
\]

(3.4) is the bosonic tadpole diagram with fermionic 1-loop, and (3.5) is bosonic 2-point function with fermionic 1-loop whose vertex is a boson-boson-fermion-fermion vertex. And (3.6) is the bosonic 2-point function with bosonic 4-point derivative coupling and bosonic 1-loop. The UV divergent parts of the (3.4) and (3.5) are

\[
\sim \int_{-\Lambda}^{\Lambda} dk_1 \frac{1}{k_1} \quad (3.7)
\]

where \( k_1 \) is one-dimensional momentum and the \( \frac{1}{k_1} \) comes from the fermion propagator. And the UV divergent part of (3.6) is

\[
\sim \int_{-\Lambda}^{\Lambda} dk_1 \frac{k_1}{k_1^2} \quad (3.8)
\]

where the factor in the denominator \( k_1^2 \) comes from the bosonic propagator and the factor \( k_1 \) in the numerator comes from the derivative coupling. So we can see that all the UV divergent parts of these diagrams are the momentum integration of the odd function of the momentum. Then if we set the integration domain as symmetric, \([-\Lambda, \Lambda]\), the UV divergent parts become zero. They will not give even finite corrections at \( \Lambda \to \infty \). So we can see that there are no quantum corrections blocking from reaching the orbifold lattice gauge theory (2.46) at \( \Lambda \to \infty \).

In the appendix C we also discuss the \( \Lambda \to \infty \) limit in the case of employing the soft SUSY breaking mass terms (2.45).

Then we have shown that the orbifold lattice theory on \( \mathbb{R}^1 \times \mathbb{R}^1_{orb} \times \text{Fuzzy } S^2 \) is obtained from the starting non-perturbative formulation on \( \mathbb{R}^1_{\Lambda} \times \mathbb{R}^1_{orb} \times \text{Fuzzy } S^2 \) by the 1st step without any fine-tunings.
3.2 2nd and 3rd steps: From the orbifold lattice theory to the non-commutative supersymmetric Yang-Mills on $\mathbb{R}^2 \times $ Fuzzy $S^2$.

Next let us discuss the 2nd and 3rd steps starting from the orbifold lattice theory. The tree level target continuum theory of the orbifold theory (2.16) is

$$S_{2d, \nu_1} = \frac{2}{g_{2d}^2} \int d^2 x \text{tr} \left[ \frac{1}{2} F_{12}^2 + \frac{1}{2} D_{\mu} s^i D_{\mu} s^i - \frac{1}{4} [s^i, s^j]^2 + i \frac{1}{2} \Psi^T \gamma^i [s_i, \Psi] + \frac{1}{2} \Psi^T (D_1 + \gamma_2 D_2) \Psi 
- i \frac{M}{6} \Psi^T \gamma^{23} \Psi + \left( \frac{M}{3} + 4 \nu_1 \right) i s^3 F_{12} + \frac{1}{2} \left( \frac{M}{3} \right)^2 (s^a)^2 + i \frac{M}{3} \epsilon_{abc} s^a s^b s^c 
+ \left( -8 \nu_1 - \frac{2M \nu_1}{3} \right) s_3^2 - 4 \nu_1 \chi - \chi^+ \right].$$

(3.9)

(Here we described $SU(m)$ part only.) Let us confirm possible perturbative quantum corrections, which could prevent from reaching the target continuum theory, are absent. Since the moduli fixing terms (2.41) do not break $Q_\pm$ and the $SU(2)_R$ symmetry, the argument goes completely parallel to the one in [1][33]. We will consider local operators with positive mass dimension $p$ near the continuum limit,

$$\mathcal{O}_p = \tilde{M}^q \varphi^a \partial^b \psi^{2\gamma}, \quad p = q + \alpha + \beta + 3\gamma,$$

(3.10)

where $\varphi$, $\psi$ and $\partial$ denote bosonic fields, fermions fields, and derivatives, respectively. $\tilde{M}$ represents mass parameter $M$ or $\nu_1$. And $q, \alpha, \beta, \gamma = 0, 1, 2, \cdots$.

From dimensional analysis, radiative corrections to the $\mathcal{O}_p$ have the form

$$\left( \frac{1}{g_{2d}^2} c_0 a^{p-4} + c_1 a^{p-2} + g_{2d}^2 a^p + \ldots \right) \int d^2 x \mathcal{O}_p$$

(3.11)

where the first, second and third terms and "..." in the parenthesis are contributions from tree level, 1-loop, 2-loop and higher-loops respectively. The $c_0, c_1, c_2$ are the dimensionless numerical constants. Since relevant or marginal operators appear with the negative power of lattice spacing, we only have to focus on terms up to 1-loop order. In order for 1-loop coefficients $c_1 a^{p-2}$ to be relevant or marginal, $p$ must be $p = 1, 2$. Only the operators $\text{tr}(\varphi), \tilde{M} \varphi, \varphi^2$ can be mass dimensions $p = 1, 2$ among the dynamical operators. But these candidates are not allowed to show up due to the preserved supersymmetry $Q_\pm$ and $SU(2)_R$ on the lattice. So, in the 2nd step $a \to 0$, the quantum continuum theory (3.9) is obtained without any fine-tuning.

Because there are no dynamical relevant or marginal operators which respect neither $8$ supersymmetry nor $SO(2)$ rotational symmetry, the both symmetries are recovered automatically only by taking 3rd step $\nu_1 \to 0$ in the quantum continuum theory (3.9). By taking the 3rd step $\nu_1 \to 0$, we obtain the non-commutative supersymmetric Yang-Mills theory on $\mathbb{R}^2 \times $ Fuzzy $S^2$,

$$S_{2d,0} = \frac{2}{g_{2d}^2} \int d^2 x \text{tr} \left[ \frac{1}{2} F_{12}^2 + \frac{1}{2} D_{\mu} s^i D_{\mu} s^i - \frac{1}{4} [s^i, s^j]^2 + i \frac{1}{2} \Psi^T \gamma^i [s_i, \Psi] + \frac{1}{2} \Psi^T (D_1 + \gamma_2 D_2) \Psi 
- i \frac{M}{6} \Psi^T \gamma^{23} \Psi + \frac{i M}{3} s^3 F_{12} + \frac{1}{2} \left( \frac{M}{3} \right)^2 (s^a)^2 + i \frac{M}{3} \epsilon_{abc} s^a s^b s^c \right],$$

(3.12)

with full 8 SUSY and the $SO(2)$ rotational symmetry on $\mathbb{R}^2$. 

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Although the orbifold lattice theory on $\mathbb{R}^1 \times \mathbb{R}^1_{orb} \times \text{Fuzzy } S^2$ is an anisotropic one which does not possess even finite point subgroups of $SO(2)$ rotational symmetry of (3.12), possible $SO(2)$ breaking corrections are all irrelevant thanks to the super-renormalizability and other preserved symmetries on the lattice [43].

Even if we introduce the soft SUSY breaking moduli fixing terms (2.45) instead of the SUSY preserving terms (2.41), discussion will not be changed. Because they are just soft breaking terms which will not alter UV divergences.

Then we have shown that the 2nd and 3rd steps also do not require any fine-tunings.

3.3 Final step: From the theory on Fuzzy 2-sphere to the one on the Moyal plane

After the first three steps, we will undertake to manage the Fuzzy sphere directions. Here we will take the $\hat{\Lambda} \to \infty$ by taking $l \to \infty$ with fixed $\Theta$ and $k$. In the limit, the Fuzzy sphere is decompactified to be flat Moyal plane $\mathbb{R}_\Theta^2$ because

$$R_F \sim 1/M \sim l^{1/2} \to \infty, \quad \hat{\Lambda} \sim l^{1/2} \to \infty. \quad (3.13)$$

By taking the limit, the theory becomes the $\mathcal{N} = 2$ $U(k)$ four-dimensional gauge theory on $\mathbb{R}^2 \times \mathbb{R}_\Theta^2$. The four-dimensional gauge coupling is given by the non-commutative parameter as

$$g_{4d}^2 = 2\pi \Theta g_{2d}^2. \quad (3.14)$$

After taking the limit (3.13), the Fuzzy spherical harmonic expansion (2.73) can be transcribed to the one by the plane waves on $\mathbb{R}_\Theta^2$ as

$$\tilde{B}(\mathbf{p}) = \int \frac{d^2 \tilde{p}}{(2\pi)^2} e^{i\tilde{p} \cdot \hat{x}} \otimes \tilde{b}(\mathbf{p}, \tilde{p}). \quad (3.15)$$

The set $(\mathbf{p}, \tilde{p})$ represents the four-momenta on $\mathbb{R}^2 \times \mathbb{R}_\Theta^2$ where $\mathbf{p}$ is the two-momenta on $\mathbb{R}^2$ coming from $\hat{\mathbf{p}}$ on $\mathbb{R}_\Lambda^1 \times \mathbb{R}_{orb}^1$, and $\tilde{p}$ is the two-momenta on $\mathbb{R}_\Theta^2$. $\tilde{h}^{(ij)}_{J\bar{J}}$ in (2.73) are transcribed into the plane waves $e^{i\tilde{p} \cdot \hat{x}}$ on $\mathbb{R}_\Theta^2$, and the modes $\tilde{b}(\mathbf{p}, \tilde{p})$ on $\mathbb{R}^2 \times \mathbb{R}_\Theta^2$ come from the $k \times k$ matrix $b_{J\bar{J}}(\mathbf{p})$. The inner product $\hat{\text{Tr}}$ between the plane waves on $\mathbb{R}_\Theta^2$ is defined as

$$\hat{\text{Tr}} \left( e^{i\tilde{p} \cdot \hat{x}} \otimes e^{i\tilde{q} \cdot \hat{x}} \right) = 2\pi \Theta \delta^2(\tilde{p} + \tilde{q}), \quad (3.16)$$

where the inner product depends on the $\Theta$.

Then finally we have shown that the target non-commutative four-dimensional $\mathcal{N} = 2$ supersymmetric theory on $\mathbb{R}^2 \times \mathbb{R}_\Theta^2$ can be reached with no fine-tunings by taking the following steps:

1. $\Lambda \to \infty$.
2. $a \to 0$ with $aN$ and $\nu_1$ fixed.
3. $\nu_1 \to 0, aN \to \infty$.
4. $l, m \to \infty$, $(M \to 0, \hat{\Lambda}, R_f \to \infty)$ with $k, \Theta$ fixed.

If $\Theta \to 0$ can be continuously connected to the commutative theory, our formulation can be a non-perturbative formulation also for the commutative $\mathcal{N} = 2$ supersymmetric Yang-Mills theories. But we should note that more investigation is needed to clarify whether $\Theta \to 0$ can be continuously connected to the commutative theories or not.

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4 Summary

In this paper we proposed a non-perturbative formulation for the non-commutative \( \mathcal{N} = 2 \) four-dimensional supersymmetric Yang-Mills theories. We made the formulation from the one-dimensional mass-deformed matrix model with 8 supercharges by performing the regularization which is a combination of the orbifold projection, the momentum cut-off and the generation of the Fuzzy 2-sphere. Similar to the formulation in [1], our formulation enables numerical studies of non-commutative four-dimensional \( \mathcal{N} = 2 \) supersymmetric Yang-Mills theories on \( \mathbb{R}^2 \times \mathbb{R}^2_\Theta \). The absence of any parameter fine-tuning was confirmed at all order of perturbation. We should note that non-commutative gauge theory plays an important role to clarify non-perturbative aspects of the gauge theories. Therefore our formulation will play a very important role to uncover non-perturbative structures of the supersymmetric gauge theories through numerical studies of \( \mathcal{N} = 2 \) non-commutative supersymmetric Yang-Mills theories.

Our formulation has several advantages. First this formulation is simpler than the similar model [1], and easier to put on a computer. Hence it is easy to check the absence of fine-tunings at non-perturbative level. Moreover, after we take the \( \Lambda \to \infty \) limit, our formulation can possesses more supersymmetry than the formulation in [1] in the UV region, so it would be easier to control UV divergences. In contrast to conventional moduli fixing terms in the orbifold lattice gauge theories, we proposed a new moduli fixing terms (2.41) with keeping a partial SUSY on the lattice. The new fixing terms could replace SUSY breaking mass terms, and the new terms would make conclusions of simulations more concrete. These advantages are owed to the highly anisotropic nature of the formulation. This demonstrates that anisotropic treatments can be very efficient to deal with four-dimensional theories.

The \( \Theta \to 0 \) limit of the four-dimensional \( \mathcal{N} = 2 \) supersymmetric Yang-Mills theory is expected not to be continuously connected to the commutative \( \mathcal{N} = 2 \) theory due to the UV/IR mixing [48]. This is in contrast to the \( \mathcal{N} = 4 \) four-dimensional supersymmetric Yang-Mills theories. There is a discussion, however, that the non-commutative four-dimensional \( \mathcal{N} = 2 \) theory may flow to the ordinary commutative theory in the infrared [49]. So there is still a chance for our formulation to be a numerical tool also for the commutative four-dimensional \( \mathcal{N} = 2 \) theories. To investigate the possibility, it would be also interesting to do numerical studies of behavior of infrared quantities on the \( \mathbb{R}^2 \times \mathbb{R}^2_\Theta \). These studies might be able to provide some insight to reach the commutative \( \mathcal{N} = 2 \) supersymmetric Yang-Mills theories.

The two-dimensional mass-deformed theory with 8 supercharges itself is also interesting on its own. In the previous simulations, and in particular simulations on two-dimensional deconstruction formulations, the conclusion has been more or less obscured because SUSY is broken by the moduli fixing terms. But by the new moduli fixing terms (2.41), it would be possible to get more concrete result since the new terms keep the lattice supersymmetry.

It is also interesting to study about the relationship between the mechanism for uplifting flat-directions and the \( \Omega \)-deformation [57,58]. The \( \Omega \)-deformation has been used to regularize the instanton moduli space of four-dimensional \( \mathcal{N} = 2 \) supersymmetric Yang-Mills theory, and the deformation uplifts the flat moduli space with keeping the SUSY. In fact, in [59], it has been shown that a deconstruction lattice formulation for the two-dimensional \( \mathcal{N} = (4, 4) \) supersymmetric gauge theory can be embedded into the topological matrix model with \( \Omega \)-deformation in the [60], and the flat directions of the deconstruction theory are uplifted by the \( \Omega \)-background with keeping the SUSY. Studies on the relationship of lattice formulations to such a mathematical method would provide a clue as to how to construct more sophisticated formulations for
numerical studies of supersymmetric gauge theories.

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A Gamma matrices

The gamma matrices $\gamma_i$ ($i = 2, \cdots, 6$) are $8 \times 8$ matrices satisfying $\{\gamma_i, \gamma_j\} = -2 \delta_{ij}$ and $\gamma_2 \cdots \gamma_6 = -i 1_8$. Their explicit form we use is

$$
\gamma_2 = -i \begin{pmatrix}
\sigma_3 & \sigma_3 & \sigma_3 & \sigma_3 \\
\sigma_3 & -\sigma_2 & \sigma_3 & \sigma_3 \\
\sigma_3 & \sigma_3 & \sigma_2 & -\sigma_2 \\
\sigma_3 & \sigma_3 & \sigma_3 & \sigma_2
\end{pmatrix}, \quad
\gamma_3 = \begin{pmatrix}
\sigma_2 & \sigma_2 & \sigma_2 & \sigma_2 \\
\sigma_2 & -\sigma_2 & \sigma_2 & \sigma_2 \\
\sigma_2 & \sigma_2 & -\sigma_2 & \sigma_2 \\
\sigma_2 & \sigma_2 & \sigma_2 & \sigma_2
\end{pmatrix},
$$

$$
\gamma_4 = -i \begin{pmatrix}
\sigma_1 & \sigma_1 & \sigma_1 & \sigma_1 \\
\sigma_1 & -\sigma_2 & \sigma_1 & \sigma_1 \\
\sigma_1 & \sigma_1 & \sigma_2 & -\sigma_2 \\
\sigma_1 & \sigma_1 & \sigma_1 & \sigma_2
\end{pmatrix}, \quad
\gamma_5 = \begin{pmatrix}
\sigma_1 & \sigma_1 & \sigma_1 & \sigma_1 \\
\sigma_1 & -\sigma_2 & \sigma_1 & \sigma_1 \\
\sigma_1 & \sigma_1 & \sigma_2 & -\sigma_2 \\
\sigma_1 & \sigma_1 & \sigma_1 & \sigma_2
\end{pmatrix}, \quad
\gamma_6 = \begin{pmatrix}
\sigma_1 & \sigma_1 & \sigma_1 & \sigma_1 \\
\sigma_1 & -\sigma_2 & \sigma_1 & \sigma_1 \\
\sigma_1 & \sigma_1 & \sigma_2 & -\sigma_2 \\
\sigma_1 & \sigma_1 & \sigma_1 & \sigma_2
\end{pmatrix},
$$

(A.1)

where $\sigma_{1,2,3}$ are Pauli matrices.

B Gauge fixing

The orbifold lattice gauge theory is an $mN \times mN$ matrix model with $U(m)^N$ gauge symmetry. The $N$ gauge fields $v_{n_2}(x_1)$ as well as $U(m)^N$ gauge transformation matrix $U_{n_2}$ are embedded in an $mN \times mN$ matrix block diagonally as

$$
v^1(x_1) = \begin{pmatrix}
v_1^1(x_1) \\
v_2^1(x_1) \\
\vdots \\
v_N^1(x_1)
\end{pmatrix}, \quad U(x_1) = \begin{pmatrix}
U_1(x_1) & & \\
& U_2(x_1) & \\
& & \ddots \\
& & & U_N(x_1)
\end{pmatrix}.
$$

(B.1)

All site fields are embedded in the above manner. Here we will consider following loop operators $W_{n_2}^c(x_1)$ on a site $n_2$,

$$
W_{n_2}^c(x_1) \equiv \exp \left( i \int_{x_1}^{x_1+R_1} dx'_1 v_{n_2}^1(x'_1) \right),
$$

(B.2)

which transform under the $U(m)^N$ in the adjoint representation,

$$
U_{n_2}(x_1) W_{n_2}^c(x_1) U_{n_2}(x_1)^\dagger.
$$

(B.3)
The loop operators are also embedded into an \( mN \times mN \) matrix block diagonally in the same way as (B.1). Here the \( x_1 \) direction is taken to be periodic, \( x_1 \sim x_1 + R_1 \). Next we define the \( N \) sets of \( m \times m \) constant diagonal matrices \( \hat{\alpha}_{n_2} \) residing on the sites,

\[
\hat{\alpha}_{n_2} = \text{diag}(e^{i\alpha_{a_1 n_2}}, \ldots, e^{i\alpha_{a_{mN} n_2}}, \ldots, e^{i\alpha_{mN n_2}}). \tag{B.4}
\]

These \( \hat{\alpha}_{n_2} \) are constant with respect to the \( x_1 \), while they are not constant with respect to \( n_2 \). And they should satisfy \( \sum_{n_2} \sum_{\hat{a}} \alpha_{n_2,\hat{a}} = 0 \) to make them consistent with the gauge fields which are traceless in the sense of \( mN \times mN \) matrix \( \text{Tr}(v^1) = \sum_{n_2} \text{tr}(v^1_{n_2}) = 0 \). By using \( W^c \) and \( \hat{\alpha} \), we will add the gauge fixing terms in the BRS exact form

\[
\frac{1}{g^2} \mathcal{Q} \int dx_1 \sum_{n_2} \text{tr} \left( \hat{c}_{n_2}(x_1) \left( W^c_{n_2}(x_1) - \hat{\alpha}_{n_2} \right) \right)
= \frac{1}{g^2} \int dx_1 \sum_{n_2} \text{tr} \left( \mathcal{B}_{n_2}(x_1) \left( W^c_{n_2}(x_1) - \hat{\alpha}_{n_2} \right) \right) + \frac{i}{g^2} \int dx_1 \sum_{n_2} \text{tr} \left( \hat{c}_{n_2}(x_1) [\hat{\alpha}_{n_2}, c_{n_2}(x_1)] \right), \tag{B.5}
\]

where \( \mathcal{Q} \) is the BRS charge different from the supercharges. Here all ghost \( c_{n_2} \), anti-ghost \( \tilde{c}_{n_2} \) and Nakanishi-Lautrup fields \( \mathcal{B}_{n_2} \) are sitting on the sites. If we explicitly write down the FP term with the \( m \times m \) indices, it becomes

\[
\frac{i}{g^2} \int dx_1 \sum_{n_2} \sum_{\hat{a} \neq \hat{b}} \left( \tilde{c}_{n_2}^\hat{a}(x_1) (e^{i\alpha_{a_{n_2}} - e^{i\alpha_{b_{n_2}}}}) c_{n_2}^\hat{b}(x_1) \right)
= -\frac{2}{g^2} \int dx_1 \sum_{n_2} \sum_{\hat{a} \neq \hat{b}} \left[ \exp \left( \frac{i\alpha_{\hat{a} n_2} + i\alpha_{\hat{b} n_2}}{2} \right) \sin \left( \frac{\alpha_{\hat{a} n_2} - \alpha_{\hat{b} n_2}}{2} \right) c_{n_2}^\hat{a}(x_1) c_{n_2}^\hat{b}(x_1) \right]. \tag{B.6}
\]

Here due to the property \( \sum_{n_2} \sum_{\hat{a}} \alpha_{n_2,\hat{a}} = 0 \), it becomes

\[
\exp \left( \sum_{n_2} \sum_{\hat{a} \neq \hat{b}} \left( \frac{i\alpha_{\hat{a} n_2} + i\alpha_{\hat{b} n_2}}{2} \right) \right) = 1. \tag{B.7}
\]

So each momentum mode of the ghost provides the FP determinant term (we factor out the parts independent of \( \alpha_{a_{n_2}} \))

\[
\exp (-S_{FP}) = \exp \left( 2 \sum_{n_2} \sum_{\hat{a} < \hat{b}} \log \left| \sin \left( \frac{\alpha_{\hat{a} n_2} - \alpha_{\hat{b} n_2}}{2} \right) \right| \right). \tag{B.8}
\]

Actually the above gauge fixing (B.5) is not enough to justify the momentum cut-off regularization. Following \( U(1)^{mN} \) transformations of the Cartan elements of the \( v^1_{n_2} \) remain as residual gauge symmetry

\[
v^1_{n_2,\hat{a}} \rightarrow v^1_{n_2,\hat{a}} + \partial_1 \phi_{n_2\hat{a}}, \tag{B.9}
\]

with

\[
\phi_{n_2\hat{a}}(x_1 + R_1) - \phi_{n_2\hat{a}}(x_1) = 2\pi n_{\hat{a}}, \tag{B.10}
\]

where \( n_{\hat{a}} \in \mathbb{Z} \). These are the large gauge transformations generating non-zero winding numbers. These gauge transformations have an effect to shift \( v^1_{n_2\hat{a}} \rightarrow v^1_{n_2\hat{a}} + 2\pi n_{\hat{a}}, \alpha_{n_2}^\hat{a} \rightarrow \)
$\alpha_{n_2} + 2\pi n_{n_2}$. Please note that gauge fixing function is unchanged under the shift because of $e^{2\pi n_{n_2}} = 1$. Through the covariant derivative $D_1$ (or $D_1$) this shift is transcribed into the shift of the momentum along the $x_1$. For instance, if we consider the covariant derivative of $Y$,

$$D_1 Y_{n_2}^{ab} \sim \left( ip_1 \omega + i \frac{\alpha_{n_2}^{a} - \alpha_{n_2 + 1}^{b}}{R_1} \right) Y_{n_2, p_1}^{ab}, \quad (B.11)$$

after a large gauge transformation, it is transformed as

$$\left( ip_1 \omega + i \frac{\alpha_{n_2}^{a} - \alpha_{n_2 + 1}^{b}}{R_1} \right) \rightarrow \left( ip_1 \omega + i \frac{R_1}{2\pi} \omega + i \frac{\alpha_{n_2}^{a} - \alpha_{n_2 + 1}^{b}}{R_1} \right)$$

$$= \left( ip_1 + n_{n_2} - n_{n_2 + 1} \right) \omega + i \frac{\alpha_{n_2}^{a} - \alpha_{n_2 + 1}^{b}}{R_1}, \quad (B.12)$$

then eventually the momentum is shifted as $p_1 \rightarrow p_1 + n_{n_2} - n_{n_2 + 1}$. This transformation may spoil the momentum cut-off regularization, because it allows the momentum to go beyond the momentum cut-off $\Lambda$. Therefore to justify the momentum cut-off regularization, we need to fix these large gauge transformations. To fix the large gauge transformation, we restrict the domain of the $\alpha_{n_2}^{a}$ as

$$\max(\alpha_{n_2}^{a}) - \min(\alpha_{n_2}^{b}) \leq 2\pi. \quad (B.13)$$

### C $\Lambda \rightarrow \infty$ limit in the case of employing the soft SUSY breaking mass terms (2.45)

Let us consider $\Lambda \rightarrow 0$ limit in the case of employing the soft SUSY breaking mass term (2.45) instead of SUSY preserving terms (2.41). In this case, there are no derivative couplings because there are not $S_{B_{\nu 1p}}$. The absence of the derivative couplings are owed to the gauge fixing also. In this case, the degree of UV divergence of each diagram is estimated as

$$D = L_{oop} - 2I_B - I_F, \quad L_{oop} = 1 + I_B + I_F - V_{bff} - \sum n V_n, \quad (C.1)$$

$$E_B + 2I_B = \sum n (nV_n) + V_{bff}, \quad E_F + 2I_F = 2V_{bff}. \quad (C.2)$$

Only the following one diagram can be $D \geq 0$ with UV divergences,

$$I_F = E_B = V_{bff} = 1, \quad E_F = I_B = V_n = 0. \quad (C.3)$$

The diagram (C.3) is the same as the diagram (3.4), and its UV divergence is guaranteed to vanish. So also in this case, there are no quantum corrections blocking from recovering the orbifold lattice gauge theory at $\Lambda \rightarrow \infty$.

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