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Article:
Applebaum, D. and van Neerven, J. (2015) Second quantisation for skew convolution products of infinitely divisible measures. Infinite Dimensional Analysis, Quantum Probability and Related Topics (idaqp), 18 (1). ISSN 0219-0257

https://doi.org/10.1142/S0219025715500034

Electronic version of an article published as Infinite Dimensional Analysis, Quantum Probability and Related Topics, 18, 1, 2015 10.1142/S0219025715500034 © copyright World Scientific Publishing Company http://www.worldscientific.com/worldscinet/idaqp

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SECOND QUANTISATION FOR SKEW CONVOLUTION
PRODUCTS OF INFINITELY DIVISIBLE MEASURES

DAVID APPLEBAUM AND JAN VAN NEERVEN

Abstract.

1. Introduction

Let $E_i$, $i = 1, 2$ be Banach spaces equipped with Radon probability measures $\mu_1$ and $\mu_2$, respectively. A Borel measurable mapping $T : E_1 \to E_2$ is called a skew map for the pair $(\mu_1, \mu_2)$ if there exists a Radon probability measure $\rho$ on $E_2$ so that $\mu_2$ is the convolution of $\rho$ with the image of $\mu_1$ under the action of $T$. In this case we obtain a linear contraction $P_T : L^p(E_2, \mu_2) \to L^p(E_1, \mu_1)$ given by

$$P_T f(x) = \int_{E_2} f(T(x) + y) \rho(dy).$$

Such constructions arise naturally in the study of Mehler semigroups, linear stochastic partial differential equations driven by additive Lévy noise and operator self-decomposable measures (see [2]). In this context, the problem of “second quantisation” is to find a functorial manner of expressing $P_T$ in terms of $T$. The reason for this name is that the first work on this subject [3], within the context of Gaussian measures, exploited constructions that were similar to those that are encountered in the construction of the free quantum field from one-particle space (see e.g. [7]) wherein the $n$th chaos spanned by multiple Wiener-Itô integrals corresponds to the $n$-particle space within the Fock space decomposition. In our previous paper [2] we implemented this programme and constructed $P_T$ as the second quantisation of $T$ in the two cases where for $i = 1, 2$, $\mu_i$ are Gaussian (generalising [3] and [6]), and are infinitely divisible measures of pure jump type (generalising [8]). In this article, we complete the programme by dealing with the case where the $\mu_i$’s are general infinitely divisible measures, and so are convolutions of the cases previously considered.

2. Background

Let $\mu$ be an infinitely divisible Radon probability measure defined on a (separable) Banach space $E$. It is well-known that the generic such measure may be written as the convolution $\mu = \mu_G * \mu_F$ where $\mu_G$ is a Gaussian measure (see e.g. [5, 4]). In fact, it follows from the Lévy-Itô decomposition of [9] that $\mu$ may always be realised as the law of an $E$-valued random variable $X$ defined on some probability space $(\Omega, \mathcal{F}, P)$ for which $X = X_1 + X_2$, where the summands $X_1$ and $X_2$ are independent. Here $X_1$ is Gaussian and has law $\mu_G$, while $X_2$ is controlled by a Poisson random measure on $\mathbb{E}$ whose intensity measure is a Lévy measure $\nu$, and $X_2$ has law $\mu_F$. From [2], we know that we can effectively realise the second
quantisation of twist maps of $\mu_G$ in the symmetric Fock space $\Gamma(H)$ of the reproducing kernel space $H$ of $\mu_G$ which is naturally isomorphic to $L^2(E, \mu_G)$. To second quantise twist maps of $\mu_P$, we use $L^2(E, \mu_P) \simeq \Gamma(L^2(E, \nu))$. To unify these two approaches we make use of the following:

$$L^2(E, \mu) = L^2(E, \mu_G \ast \mu_P) \hookrightarrow L^2(E, \mu_G) \otimes L^2(E, \mu_P) \simeq \Gamma(H) \otimes \Gamma(L^2(E, \nu)) \simeq \Gamma(H \oplus L^2(E, \nu)).$$

We give a more detailed account of these embeddings and isomorphisms in the sequel.

3. Main result

Suppose $\mu$ is an infinitely divisible measure, say $\mu = \gamma \ast \Pi$ with $\gamma$ centred Gaussian and $\Pi$ as in [2] 1. For a function $f \in L^2(\mu)$ let $F_f(x, y) := f(x + y)$.

Using the fact that $L^2(\gamma) \hat{\otimes} L^2(\Pi) = L^2(\gamma \times \Pi)$ isometrically (with $\hat{\otimes}$ indicating the Hilbert space tensor product) it is immediate to verify that

$$\|f\|_{L^2(\mu)}^2 = \int_E \int_E |f(x + y)|^2 \, d\gamma(x) \, d\Pi(y) = \|F_f\|_{L^2(\gamma) \hat{\otimes} L^2(\Pi)}^2.$$

As a result the mapping $f \mapsto F_f$ is an isometry from $L^2(\mu)$ into $L^2(\gamma) \hat{\otimes} L^2(\Pi)$. This brings us to the setting with independence structure as discussed in [1]. Following that reference, on the algebraic tensor product $L^2(\gamma) \otimes L^2(\Pi)$ we define

$$D := D_\gamma \otimes I + I \otimes D_\Pi,$$

where we denote the ‘Gaussian’ and the ‘pure jump’ derivatives with subscripts $\gamma$ and $\Pi$, respectively.

Consider the Hilbert spaces

$$\mathcal{H}_n := \bigoplus_{j, k \geq 0} H^{\otimes j} \hat{\otimes} L^2(\nu)^{\otimes k}. $$

Then,

$$L^2(\gamma \times \Pi) = L^2(\gamma) \hat{\otimes} L^2(\Pi) = \bigoplus_{j, k \geq 0} H^{\otimes j} \hat{\otimes} L^2(\nu)^{\otimes k} = \bigoplus_{n=0}^{\infty} \bigoplus_{j+k=n} H^{\otimes j} \hat{\otimes} L^2(\nu)^{\otimes k} \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

may be viewed as the associated Wiener-Itô decomposition. We define the $n$-fold stochastic integral on $I_n : \mathcal{H}_n \to L^2(\Omega)$ by

$$I_n(f \otimes g) := I_{j, \gamma} f \otimes I_{k, \Pi} g$$

for $f \in H^{\otimes j}$ and $g \in L^2(\nu)^{\otimes k}$ with $j + k = n$, where we denote the ‘Gaussian’ and the ‘pure jump’ integrals with subscripts $\gamma$ and $\Pi$, respectively.

---

1 Be more precise
Proposition 3.1. For all $F \in L^2(\gamma \times \Pi)$,
\[ F = \sum_{m=0}^{\infty} \frac{1}{m!} I_m(\mathbb{E}(D^m F)). \]

Proof. Let $F = f \otimes g$ with $f \in H^{(\otimes j)}$ and $g \in L^2(\nu)^{\otimes k}$. By Leibniz’s rule,
\[
\sum_{m=0}^{\infty} \frac{1}{m!} I_m(\mathbb{E} D^m F) = \sum_{m=0}^{\infty} \frac{1}{m!} I_m(\mathbb{E} \sum_{\ell=0}^{m} \binom{m}{\ell} D^\ell f \otimes D^{m-\ell} g)
= \sum_{m=0}^{\infty} \sum_{\ell=0}^{m} \frac{1}{\ell!(m-\ell)!} I_m(\mathbb{E}(D^\ell f \otimes D^{m-\ell} g))
= \sum_{m=0}^{\infty} \sum_{\ell=0}^{m} \frac{1}{\ell!(m-\ell)!} I_{\ell,\gamma}(\mathbb{E} D^\ell f) \otimes I_{m-\ell,\Pi}(D^{m-\ell} g)
= \sum_{j=0}^{\infty} \frac{1}{j!} I_{j,\gamma}(\mathbb{E} D^j f) \otimes \sum_{k=0}^{\infty} \frac{1}{k!} I_{k,\Pi}(\mathbb{E} D^k g),
\]
using the Last-Penrose type decompositions for $\gamma$ and $\Pi$ in the second last identity.

Suppose now that two measures $\mu_1$ and $\mu_2$ are given as above, on Banach spaces $E_1$ and $E_2$, respectively, say $\mu_i = \gamma_i \ast \Pi_i$ for $i = 1, 2$. Let $T : E_1 \to E_2$ be a linear skew mapping with respect to both $(\gamma_1, \gamma_2)$ and $(\Pi_1, \Pi_2)$ with skew factors $\rho_\gamma$ and $\rho_\Pi$. Recall that this means that $T\gamma_1 \ast \rho_\gamma = \gamma_2$ and $T\Pi_1 \ast \rho_\Pi = \Pi_2$.

Set $\rho := \rho_\gamma \ast \rho_\Pi$.

Lemma 3.2. Under these assumptions, $T$ is skew with respect to $(\mu_1, \mu_2)$ with skew factor $\rho$.

Proof. Since for any two measures on $E_1$ one has $T(\nu_1 \ast \nu_2) = (T\nu_1) \ast (T\nu_2)$, this follows from
\[
T\mu_1 \ast (\rho_\gamma \ast \rho_\Pi) = (T\gamma_1 \ast T\Pi_1) \ast (\rho_\gamma \ast \rho_\Pi) = (T\gamma_1 \ast \rho_\gamma) \ast (T\Pi_1 \ast \rho_\Pi) = \gamma_2 \ast \Pi_2 = \mu_2.
\]

It follows from the lemma that we may define $P_T : L^2(E_2, \mu_2) \to L^2(E_1, \mu_1)$ by
\[
P_T f(x) := \int_{E_2} f(Tx + y) \, d\rho(y), \quad x \in E_1,
\]
where $\rho$ is the skew factor on $E_2$, i.e., $T\mu_1 \ast \rho = \mu_2$. Similarly we can define an operator $P_T \otimes P_T : L^2(\gamma_2) \otimes L^2(\Pi_2) \to L^2(\gamma_1) \otimes L^2(\Pi_1)$ in the obvious way (with an apology for the abuse of notation) and we then have:

Lemma 3.3. Under the above assumptions, $F_{P_T f} = (P_T \otimes P_T) F_f$. 

Proof. For \((\gamma \times \Pi)\)-almost all \(x,y \in E_2\) we have
\[
(P_T \otimes P_T)(\phi \otimes \psi)(x,y) = (P_T\phi \otimes P_T\psi)(x,y)
\]
\[
= \int_{E_2} \phi(Tx + z) d\rho_\gamma(z) \int_{E_2} \psi(Ty + z) d\rho_\Pi(z)
\]
\[
= \int_{E_2} \int_{E_2} (\phi \otimes \psi)(Tx + z_1, Ty + z_2) d\rho_\gamma(z_1) d\rho_\Pi(z_2).
\]
Now suppose that \(G_n = F_f\) in \(L^2(\gamma \times \Pi)\), where each \(f_n\) belongs to the algebraic tensor product \(L^2(\gamma) \otimes L^2(\Pi)\). By the above identity and linearity it follows, after passing to a subsequence if necessary, that for \((\gamma \times \Pi)\)-almost all \(x,y \in E_2\) we have
\[
(P_T \otimes P_T)G_n(x,y) = \lim_{n \to \infty} (P_T \otimes P_T)G_n(x,y)
\]
\[
= \lim_{n \to \infty} \int_{E_2} \int_{E_2} G_n(Tx + z_1, Ty + z_2) d\rho_\gamma(z_1) d\rho_\Pi(z_2)
\]
\[
= \int_{E_2} \int_{E_2} F_f(Tx + z_1, Ty + z_2) d\rho_\gamma(z_1) d\rho_\Pi(z_2)
\]
\[
= \int_{E_2} \int_{E_2} f(Tx + Ty + z_1 + z_2) d\rho_\gamma(z_1) d\rho_\Pi(z_2)
\]
\[
= \int_{E_2} f(Tx + Ty + z) d\rho_\gamma(z)
\]
\[
= P_Tf(x + y)
\]
\[
= F_{P_T}(x, y).
\]

For \(h \in H\) and \(y_1, \ldots, y_n \in E\) and \(h \in H\) we define
\[
D_{h,y_1,\ldots,y_n} := D_h \otimes I + I \otimes D_{y_1,\ldots,y_n}.
\]

Lemma 3.4. For all \(f \in L^2(E_2, \mu_2), h \in H, \) and \(y_1, \ldots, y_n \in E_1\),
\[
E_{\gamma_1 \times \Pi_1} D_{h,y_1,\ldots,y_n}^{\mu_2} F_{P_T} f = E_{\gamma_1 \times \Pi_1} D_{y_1,\ldots,y_n}^{\mu_2} T_h T_{y_1,\ldots,y_n} F_f.
\]

Proof. We approximate \(F_f\) by finite sums of elementary tensors as in the proof of the previous lemma. For such functions \(G_n\) the identity follows from the results in [2] for the Gaussian and Poissonian case.

Take care of details, closedness argument needed?

Our \(D\)'s are unbounded. 

Can we define a derivative \(D\) in \(L^2(\mu)\) satisfying the requirement
\[
E_\mu D^n f = E_{\gamma_1 \times \Pi_1} D^n F_f ?
\]

(On the right, this is the \(D\) defined previously on \(L^2(\gamma) \otimes L^2(\Pi)\),
extended (by closability? check) to a closed operator on \(L^2(\gamma \times \Pi)\)).
That would clean up the lemma as well as the commuting diagram.
For Hilbert spaces \( H \) and \( H \) we note that
\[
\Gamma(H, \oplus H) = \bigoplus_{n=0}^{\infty} \left( \bigoplus_{j+k=n} H^\otimes j \otimes H^\otimes k \right).
\]

**Theorem 3.5.** Putting everything together, under the above assumptions the following diagram commutes:

\[
\begin{array}{ccc}
L^2(E_2, \mu_2) & \xrightarrow{Pr} & L^2(E_1, \mu_1) \\
\downarrow f \mapsto Ff & & \downarrow f \mapsto Ff \\
\bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} E_{\gamma_2 \times \Pi_2} \tilde{D}^n & \xrightarrow{Pr \otimes Pr} & \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} E_{\gamma_1 \times \Pi_1} \tilde{D}^n \\
\Gamma(L^2(E_2, \nu_2) \oplus H_2) & \xrightarrow{\bigoplus_{n=0}^{\infty} (T^*)^n} & \Gamma(L^2(E_1, \nu_1) \oplus H_1)
\end{array}
\]

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