Properties of anti-adjacency matrix of directed cyclic sun graph

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Abstract. In this paper we focus on the properties of anti-adjacency matrix of directed cyclic sun graph. Some of these properties are related to the characteristic polynomials and the eigenvalues of the anti-adjacency of its matrix. We will show the general form of characteristic polynomial of the anti-adjacency matrix of directed cyclic sun graph by figuring out the number of the directed induced-cyclic graphs and the directed induced-acyclic graphs. After we find out the general form of the characteristic polynomial, we can find the general form of the eigenvalues of its polynomial by using factorization and Horner methods.

1. Introduction

There are many ways to illustrate a directed graph. Other than making the depiction of the graph, the directed graph can be represented by an adjacency matrix or anti-adjacency matrix [1]. Since the anti-adjacency matrix is an $n \times n$ matrix then we can find the characteristic polynomial and the roots/eigenvalues of its matrix [2].

Some properties of anti-adjacency matrix are related to its characteristic polynomial and its eigenvalues. In Chemical graph theory, the eigenvalues of the anti-adjacency matrix of a graph can be used in accomplishing physical properties of chemical graphs [3]. However, the study about the properties of anti-adjacency matrix of directed cyclic sun graph, especially its eigenvalues or its characteristic polynomials has not been explored yet. Therefore, this adds the importances of the further study about the anti-adjacency matrix of directed cyclic sun graph, which is being discussed in this paper. Hence, we will figure out the general form of the graph's anti-adjacency matrix, its characteristic polynomial, and its eigenvalues. The matrix's coefficients of characteristic polynomial can be obtained by looking for how many directed cyclic-induced subgraphs and directed acyclic-induced subgraphs within the graph [4].

2. Preliminaries

The following section explains the definitions and theorems required to prove the main result.

Definition 2.1. A directed graph $G$ is a finite nonempty set of objects called vertices together with a (possibly empty) set of ordered pairs of distinct vertices of $G$ called arcs or directed edges. The set of vertices of $G$ is denoted by $V(G) = \{v_1, v_2, v_3, ..., v_{n-1}, v_n\}$ and the set of directed edge denoted by $E(G) = \{e_1, e_2, e_3, ..., e_m\}$ [5].
Definition 2.2. An adjacency matrix of directed graph $G$ with the set of vertices $V(G) = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ is an $n \times n$ matrix $A = [a_{ij}]$, where $a_{ij} = 1$, if there exists a directed edge from $v_i$ to $v_j$ and $a_{ij} = 0$ otherwise. \[1\]

Definition 2.3. An anti-adjacency matrix of directed graph $G$ with the set of vertices $V(G) = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ is an $n \times n$ matrix $B = J - A$, where $J$ is a $n \times n$ matrix which all the entries are 1 and $A$ is an adjacency matrix of its graph \[1\]

Definition 2.4. A cycle of order $n$ with $m$ pendant edges attached at each vertex i.e., $C_n \circ m K_1$, is called an $m$-crown with cycle of order $n$. A 1-crown or a sun graph is a cycle with exactly one pendant edge attached at each vertex of the cycle. We will use $C_n \circ K_1$ for this graph. \[6\]

Definition 2.5. Directed cyclic sun graph is a directed graph obtained by adding a directed edge and a vertex, which are called the outer-vertices, to every vertices in the cycle Graph $C_n$, which are called the inner-vertices. Directed cyclic sun graph is denoted by $C_n \circ K_1$. It has a set of vertices $V(C_n \circ K_1) = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n, v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$ and a set of edges $E(C_n \circ K_1) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_{n+1}, v_{n+1}v_{n+2}, v_{n+2}v_2, \ldots, v_{2n}v_n\}$, where $v_1, v_2, v_3, \ldots, v_{n-1}, v_n$ are the inner-vertices and $v_{n+1}, v_{n+2}, \ldots, v_{2n}$ are the outer-vertices of it. In this paper we focus on the directed cyclic sun graph which the directed edges of each outer-vertices $v_{n+i}$ always lead to the inner-vertices $v_i$ and the inner-vertices always form a clockwise-cycle graph for $i = 1, 2, \ldots, n$. The following is the illustration of it.

![Directed Cyclic Sun Graph](image)

The directed cyclic sun graph $C_n \circ K_1$ has an anti-adjacency $B(C_n \circ K_1) = [b_{ij}]$, where

\[
b_{ij} = \begin{cases} 0, & \{1 \leq i \leq n - 1, j = i + 1\}; \\ 1, & \{i = n, j = 1\}; \\ 1, & \{i = j + 1, 1 \leq j \leq n\} \\ 0, & otherwise \end{cases}
\]

Theorem 2.6. Let $G$ be a directed acyclic graph, with $V(G) = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ and $B$ is the anti-adjacency of $G$. Then $\det(B) = 1$, if $G$ has a Hamiltonian path and $\det(B) = 0$ otherwise. \[2\]

Theorem 2.7. Let $P(B(G)) = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_{n-1}\lambda + b_n$ be a characteristic polynomial of an anti-adjacency matrix $B(G)$ of a directed graph, let $\det(B(U)_{acyclic})^{(j_1)}$ be the determinant of an anti-adjacency matrix of a directed acyclic induced subgraph with $i$ vertices and $j_1 = 1, 2, \ldots, w_i$; where $w_i$ is the number of a directed acyclic-induced subgraphs $(U)_{acyclic}$ with $i$-vertices of directed cyclic graph $G$, and let $\det(B(U)_{cyclic})^{(j_2)}$ be the determinant of the anti-adjacency
Lemma 2.8. Let G be a directed graph with a set of vertices $V = \{v_1, v_2, ..., v_n\}$ and B be an anti-adjacency matrix of directed cyclic sun graph $G$. If there exists two vertices which their directed edges go to the same vertex then $\text{det}(B) = 0$.

Lemma 2.9. Let G be a directed graph with a set of vertices $V = \{v_1, v_2, ..., v_n\}$ and B be an anti-adjacency matrix of directed cyclic sun graph $G$. If there exists two vertices which their directed edges go to the same vertex then $\text{det}(B) = 0$.

3. Main Result
In the following section we will discuss about the theorems which are related to the main result. In Theorem 3.1, we will discuss about the coefficients $b_i$, $1 \leq i \leq 2n$, of the characteristic polynomial of the anti-adjacency matrix of directed cyclic sun graph $\overline{C_n \odot K_1}$. In Corollary 3.2, we will discuss about the general form of the characteristic polynomial of its anti-adjacency matrix. In Theorem 3.3, we will discuss about the real eigenvalues and complex eigenvalues of its anti-adjacency matrix.

**Theorem 3.1.** If $P(B(\overline{C_n \odot K_1})) = \lambda^n + b_1 \lambda^{n-1} + b_2 \lambda^{n-2} + \cdots + b_{2n-1} \lambda + b_2$ is the characteristic polynomial of the anti-adjacency of directed cyclic sun graph $\overline{C_n \odot K_1}$ then

a) $b_i = (-1)^i (2n)$, for $1 \leq i \leq n - 1$.

b) $b_i = (-1)^i (2n - 1)$, for $i = n$.

c) $b_i = 0$, for $n + 1 \leq i \leq 2n$.

Proof:
According to Theorem 2.7, $b_i = (-1)^i \left( \sum_{j=1}^{w_i} B(\overline{U})_{\text{acyclic}}^{(j)} \right) + \sum_{j=2}^{w_2} B(\overline{U})_{\text{acyclic}}^{(j)}$.

a) For $1 \leq i \leq n - 1$

• Acyclic Parts
The following is the illustration of its directed acyclic-induced subgraphs.

![Directed acyclic-induced subgraphs](image)

**Figure 2.** Directed acyclic-induced subgraphs for $i = k$, $1 \leq k \leq n - 1$.

Let B be an anti-adjacency matrix of the directed cyclic-induced subgraphs which induce the directed cyclic sun graph $\overline{C_n \odot K_1}$, for $i = k$, $1 \leq k \leq n - 1$. The set of directed cyclic-induced subgraphs which in the form of Hamiltonian paths is $\{v_1 - v_2 - \cdots - v_k, v_2 - v_3 - \cdots - v_{k+1}, ..., v_n - v_1 - v_2 - \cdots - v_{k-1}, v_{n+1} - v_1 - v_2 - \cdots - v_{k-1}, v_{n+2} - v_2 - v_3 - \cdots - v_k, ..., v_{n+n} - v_n - v_1 - v_2 - \cdots - v_{k-2}\}$. From the set of directed cyclic-induced subgraphs, see Fig. 2, we can see that there are $2n$-subgraphs of directed cyclic-induced subgraphs which is in the form of Hamiltonian paths.
Hence, according to Theorem 2.6 and Theorem 2.7; \( \det(B) = 1 \) and \( \sum_{f_1}^{w_1} \left| B\left( (U)_{acyclic} \right)_{i}^{(j_1)} \right| = (2n) \cdot 1 = 2n. \)

- **Cyclic Parts**
  Since making directed cyclic-induced subgraph in directed cyclic sun graph needs at least \( n \) -vertices, then in the directed cyclic sun graph \( \overrightarrow{C_n} \circ \overrightarrow{K_1} \) there does not exist a directed cyclic-induced subgraph which has \( i \) -vertices; for \( 1 \leq i \leq n - 1 \). Hence, \( \sum_{f_2}^{w_2} \left| B\left( (U)_{cyclic} \right)_{i}^{(j_2)} \right| = 0. \)
  So we can conclude that \( b_i = (-1)^i (2n + 0) = (-1)^i (2n) \), for \( 1 \leq i \leq n - 1 \).  

b) For \( i = n \)

- **Acyclic Parts**
The following is the illustration of its directed acyclic-induced subgraphs.

![Illustration of directed acyclic-induced subgraphs](image)

Let B be an anti-adjacency matrix of its directed acyclic-induced subgraphs which induce the directed cyclic sun graph \( \overrightarrow{C_n} \circ \overrightarrow{K_1} \), for \( i = n \). The set of directed acyclic-induced subgraphs which in the form of Hamiltonian paths is \( \{ v_{n+1} - v_1 - v_2 - \cdots - v_{n-1}, v_{n+2} - v_2 - v_3 - \cdots - v_{n+1}, \cdots, v_{2n} - v_n - v_1 - v_2 - \cdots - v_{n-2} \} \). From the set of directed acyclic-induced subgraphs, see Fig. 3, we can see that there are \( n \) -subgraphs of directed acyclic-induced subgraphs which is in the form of Hamiltonian paths. Hence, according to Theorem 2.6 and Theorem 2.7; \( \det(B) = 1 \) and \( \sum_{f_1}^{w_1} \left| B\left( (U)_{acyclic} \right)_{i}^{(j_1)} \right| = (n) \cdot 1 = n. \)

- **Cyclic Parts**
The number of directed cyclic-induced subgraphs which is in the form of directed cycle graph and has \( n \) -vertices is one subgraphs. Let B be an anti-adjacency matrix of its subgraph. Hence, according to Lemma 2.9 \( \det(B) = n - 1 \) and \( \sum_{f_2}^{w_2} \left| B\left( (U)_{cyclic} \right)_{i}^{(j_2)} \right| = 1 \cdot (n - 1) = n - 1. \)
  So we can conclude that \( b_i = (-1)^i (n + (n - 1)) = (-1)^i (2n - 1) \), for \( i = n \).  

c) For \( n + 1 \leq i \leq 2n \)

- **Acyclic Parts**
  Since making directed acyclic-induced subgraph in directed cyclic sun graph needs at most \( n \) -vertices then in the directed cyclic sun graph \( \overrightarrow{C_n} \circ \overrightarrow{K_1} \) there is not exist a directed acyclic-induced subgraph
which has \(i\)-vertices; for \(n + 1 \leq i \leq 2n\). Hence, \(\sum_{j=1}^{w_i} |B((U)_{\text{cyclic}})_i^{(j)}| = 0\)

- Cyclic Parts

The following is the illustration of its the directed cyclic-induced subgraphs.

**Figure 4.** Directed cyclic-induced subgraphs for \(n + 1 \leq i \leq 2n\).

Let \(B\) be an anti-adjacency matrix of its directed cyclic-induced subgraphs for \(n + 1 \leq i \leq 2n\). From Fig. 4 we can see that among the directed cyclic-induced subgraphs from directed cyclic sun graph \(C_n \circ K_1\), which has \((n + k)\)-vertices, there are two vertices which their directed edges go to the same vertex, those are \(v_{n+k+1} - v_k + 1\) and \(v_k - v_{k+1}\) for \(1 < k \leq n\), or \(v_{n+1} - v_1\) and \(v_n - v_1\) for \(k = 1\).

Then according to Lemma 2.8, \(\det(B) = 0\) and \(\sum_{j=1}^{w_i} |B((U)_{\text{cyclic}})_i^{(j)}| = 0\).

So we can conclude that \(b_i = (-1)^i(0 + 0) = 0\), for \(n + 1 \leq i \leq 2n\).

**Corollary 3.2.** Let \(B\left(C_n \circ K_1\right)\) be the anti-adjacency matrix of directed cyclic sun graph \(C_n \circ K_1\), then the characteristic polynomial of its matrix is

\[
P(B(C_n \circ K_1)) = \lambda^{2n} + \sum_{i=1}^{n-1} (-1)^i(2n)\lambda^{2n-i} + (-1)^n(2n - 1)\lambda^n, \quad \text{for } n \geq 3.
\]

**Proof:**

According to Theorem 3.1, the coefficients of the characteristic polynomial of the anti-adjacency matrix of directed cyclic sun graph \(C_n \circ K_1\) are \(b_i = (-1)^i(2n);\) for \(1 \leq i \leq n - 1, b_i = (-1)^i(2n - 1);\) for \(i = n,\) and \(b_i = 0,\) for \(n + 1 \leq i \leq 2n\). Hence, the characteristic polynomial is the equation (5).

**Theorem 3.3.** Let \(P(B(C_n \circ K_1)) = \lambda^{2n} + \sum_{i=1}^{n-1} (-1)^i(2n)\lambda^{2n-i} + (-1)^n(2n - 1)\lambda^n;\) for \(n \geq 3,\)

is the characteristic polynomial of the anti-adjacency matrix of directed cyclic sun graph. Then the eigenvalues of its characteristic polynomial are

a) \(\lambda_{1,2,...,2k} = 0, \quad \lambda_{2k+1} = (4k - 1), \quad \lambda_{2k+2} = 1, \quad \text{and } \lambda_{2k+3,2k+4,...,4k} = e^{i\left(\frac{m\pi}{k}\right)}; \quad \text{where } m = 1,2,3,...,2k - 2, \quad \text{for } n = 2k, k = 2,3,4,...
\]

b) \(\lambda_{1,2,...,2k+1} = 0, \quad \lambda_{2k+2} = (4k + 1), \quad \text{and } \lambda_{2k+3,2k+4,...,4k} = e^{i\left(\frac{\pi}{2k+1}\right)}; \quad \text{where } m = 1,2,3,...,2k \quad \text{for } n = 2k + 1, k = 1,2,3,...
\]

**Proof:**

a) For \(n = 2k, \quad k = 2,3,4,...,\) then the equation (1) is equal to the following:

\[
P(B(C_n \circ K_1)) = \lambda^{4k} + \sum_{i=1}^{2k-1} (-1)^i(4k)\lambda^{4k-i} + (-1)^{2k}(4k - 1)\lambda^{2k} = 0
\]
b) For \( n = 2k + 1; \ k = 1, 2, 3, \ldots \), the equation (1) is equal to the following:

\[
P \left( B \left( C_\mathbf{n} \circ \mathbf{K}_1 \right) \right) = \lambda^{2k+2} + \sum_{i=1}^{2k+1} (-1)^i (4k+2) \lambda^{2k+2-i} + (-1)^{2k+1} (4k+1) \lambda^{2k+1} = 0
\]

\[\Leftrightarrow P \left( B \left( C_\mathbf{n} \circ \mathbf{K}_1 \right) \right) = \lambda^{2k+1} \left[ \lambda^2 + \sum_{i=1}^{2k} (-1)^i (4k+2) \lambda^{2k-i} - (4k+1) \right] = 0 \quad \ldots (12)
\]

Similar to a), we use Horner method with the factor equals to \( \lambda = (4k+1) \) to simplify the equation (12). Then we will have the following:

\[
P \left( B \left( C_\mathbf{n} \circ \mathbf{K}_1 \right) \right) = \lambda^{2k+1}[\lambda - (4k + 1)] \left[ 1 + \sum_{i=1}^{2k} (-\lambda)^i \right] = 0 \quad \ldots (13)
\]

From the equation (13), we can see the polynomial \( 1 + \sum_{i=1}^{2k} (-\lambda)^i \) is a geometric series with the first term is \( a = 1 \), the ratio is \( r = -\lambda \), and the number of the terms is \( 2k+1 \). Hence, we will have

\[
1 + \sum_{i=1}^{2k} (-\lambda)^i = \frac{(-\lambda)^{2k+1} - 1}{-\lambda - 1}
\]

So we will have the following:

\[
P \left( B \left( C_\mathbf{n} \circ \mathbf{K}_1 \right) \right) = \lambda^{2k+1}[\lambda - (4k + 1)] \left[ \frac{(-\lambda)^{2k+1} - 1}{-\lambda - 1} \right] = 0 \quad \ldots (14)
\]

From the equation (14) we will have

\[
\lambda_{1,2,\ldots,2k+1} = 0 \quad \text{and} \quad \lambda_{2k+2} = 4k + 1
\]
\(-\lambda^{2k+1-1} = 0\) 
\[-\lambda^{-1} = 0\] 
\[\Rightarrow -\lambda^{2k+1} - 1 = 0\] 
\[\Leftrightarrow \lambda^{2k+1} = -1 \text{ and } \lambda \neq -1\]  
(15)

According to [7], the roots from the equation (15) are complex numbers and to solve the equation (15), let \(\lambda = r_0 e^{i\theta}\) be the solution of \(\lambda^{2k+1} = -1\). Hence, 
\((r_0 e^{i\theta})^{2k+1} = -1 \Leftrightarrow r_0^{2k+1} e^{i(2k+1)\theta} = -1 \Leftrightarrow r_0^{2k+1} e^{i(2k+1)\theta} = 1 \cdot e^{i(\pi+2m)\theta} \)  
(16)

From the equation (16) we will have \(r_0 = 1\) and \(\theta = \frac{\pi}{2k+1} + \frac{2m\pi}{2k+1}\)  
(17)

Hence, from the equation (15) and the equation (17) we will have 
\[\lambda_{2k+3,2k+4,...,4k+2} = e^{i\left(\frac{\pi}{2k+1} + \frac{2m\pi}{2k+1}\right)}\]  
for \(m = 1,2,3,...,2k\). \[\blacksquare\]

4. Conclusion

The general form of the anti-adjacency matrix, the coefficients of its characteristic polynomial, the general form of the characteristic polynomial of its matrix, and the eigenvalues of directed cyclic sun graph can be seen in the equation (1), (2), (3), (4), (5), (6), and the following table:

| The value of n \(n = 2k + 1; k = 1,2,3,...\) | Separation |
|---------------------------------------------|------------|
| \(\lambda_{1,2,...,2k+1} = 0\) and \(\lambda_{2k+2} = 4k + 1\) | Real Eigenvalues |
| \(\lambda_{2k+3,2k+4,...,4k+2} = e^{i\left(\frac{\pi}{2k+1} + \frac{2m\pi}{2k+1}\right)}\) | Complex Eigenvalues |
| where \(m = 1,2,3,...,2k\), | |

| The value of n \(n = 2k; k = 2,3,4,...\) | Separation |
|---------------------------------------------|------------|
| \(\lambda_{1,2,...,2k} = 0\), \(\lambda_{2k+1} = 4k - 1\), and \(\lambda_{2k+2} = 1\) | Real Eigenvalues |
| \(\lambda_{2k+3,2k+4,...,4k} = e^{i\left(\frac{\pi m}{\pi}\right)}\) | Complex Eigenvalues |
| where \(m = 1,2,3,...,2k - 2\) | |

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