HOMOTOPY MOTIONS OF SURFACES IN 3-MANIFOLDS

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Abstract. We introduce the concept of a homotopy motion of a subset in a manifold, and give a systematic study of homotopy motions of surfaces in closed orientable 3-manifolds. This notion arises from various natural problems in 3-manifold theory such as domination of manifold pairs, homotopical behavior of simple loops on a Heegaard surface, and monodromies of virtual branched covering surface bundles associated to a Heegaard splitting.

Introduction

0.1. Homotopy motion groups and related groups

For a manifold $M$ and a compact subspace $\Sigma$, a motion of $\Sigma$ in $M$ is an ambient isotopy of $M$ of compact support that ends up with a homeomorphism preserving the subset $\Sigma$. The motion group $M(M, \Sigma)$ of $\Sigma$ in $M$ is the group made up of the equivalence classes of such motions where the product is defined by concatenation of ambient isotopies. The concept of a motion has its origin in the braid group, which can be regarded as the motion group of a finite set in the plane. In his 1962 PhD thesis [27] supervised by Fox, Dahm developed a general theory of motions and calculated the motion group of a trivial link in the Euclidean space. In [36], Goldsmith published an exposition of Dahm’s thesis, and in the succeeding paper [37], she obtained generators and relations of the motion groups of torus links in $S^3$. Since then (variations of) motion groups have been studied by many researchers in various settings. (See [14, 28, 33] and references therein.)

In the case where $M$ is a closed, orientable 3-manifold and $\Sigma$ is a Heegaard surface, Johnson-Rubinstein [54] and Johnson-McCullough [53] studied the (smooth) motion group $M(M, \Sigma)$ and its quotient group $G(M, \Sigma)$ defined by

$$G(M, \Sigma) = \{ [f] \in MCG(\Sigma) \mid \text{There exists a motion } \{f_t\}_{t \in I} \text{ with } j \circ f = f_1|_{\Sigma} \}$$

$$= \{ [f] \in MCG(\Sigma) \mid j \circ f : \Sigma \to M \text{ is ambient isotopic to } j. \},$$

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where $\text{MCG}(\Sigma)$ is the mapping class group of $\Sigma$ and $j : \Sigma \to M$ is the inclusion map. These groups are also intimately related to the pairwise mapping class group $\text{MCG}(M, \Sigma)$ first studied by Goeritz [35] in 1933, which has been attracting attention of various researchers. See Section 5 for a brief summary.

Motivated by Minsky’s question [38] on the homotopical behavior of simple loops on a Heegaard surface in the ambient 3-manifold (see Question 0.1 below) and the second author’s joint work with Donghi Lee [63] on the corresponding problem for 2-bridge spheres of 2-bridge links (see Subsection 0.2), we are naturally lead to a homotopy version of the motion group $\text{M}(M, \Sigma)$ and that of the group $\text{G}(M, \Sigma)$.

A homotopy motion of a closed surface $\Sigma$ in a compact 3-manifold $M$ is a homotopy $F = \{f_t\}_{t \in I} : \Sigma \times I \to M$, such that the initial end $f_0$ is the inclusion map $j : \Sigma \to M$ and the terminal end $f_1$ is an embedding with image $\Sigma$, where $f_t : \Sigma \to M$ ($t \in I = [0, 1]$) is the continuous map from $\Sigma$ to $M$ defined by $f_t(x) = F(x, t)$. The homotopy motion group $\Pi(M, \Sigma)$ is the group of equivalence classes of homotopy motions of $\Sigma$ in $M$, where the product is defined by concatenation of homotopies (see Section 3 for the precise definition). There is a natural homomorphism $\partial^+ : \Pi(M, \Sigma) \to \text{MCG}(\Sigma)$ which assigns (the equivalence class of) a homotopy motion with (the mapping class represented by) its terminal end. We denote the image of $\partial^+$ by $\Gamma(M, \Sigma)$. Then we have

$$\Gamma(M, \Sigma) = \{[f] \in \text{MCG}(\Sigma) \mid j \circ f : \Sigma \to M \text{ is homotopic to the inclusion map } j\}.$$ 

By denoting the kernel of $\partial^+$ by $\mathcal{K}(M, \Sigma)$, we have the following exact sequence.

(1)$$1 \longrightarrow \mathcal{K}(M, \Sigma) \longrightarrow \Pi(M, \Sigma) \longrightarrow \Gamma(M, \Sigma) \longrightarrow 1.$$ 

In the case where $M$ is a closed, orientable 3-manifold and $\Sigma$ is a Heegaard surface, the above exact sequence is a homotopy version of the following exact sequence studied by Johnson-McCullough [53].

(2)$$1 \longrightarrow \pi_1(\text{Diff}(M)) \longrightarrow \mathcal{M}(M, \Sigma) \longrightarrow \text{G}(M, \Sigma) \longrightarrow 1,$$ 

where $\text{Diff}(M)$ is the space of diffeomorphisms of $M$. (The smooth motion group $\mathcal{M}(M, \Sigma)$ corresponds to $\mathcal{H}_1(M, \Sigma)$ in [53], the fundamental group of the space $\mathcal{H}(M, \Sigma)$ of Heegaard surfaces equivalent to $(M, \Sigma)$.)

The purpose of this paper is to give a systematic study of the homotopy motion group $\Pi(M, \Sigma)$ and the related groups in the exact sequence (1) for a closed, orientable surface $\Sigma$ in a closed, orientable 3-manifold $M$.

0.2. Motivation

Before stating the main results, we explain our motivation. Let $\Sigma$ be a Heegaard surface of a closed, orientable 3-manifold $M$, and let $V_1$ and $V_2$ be the handlebodies obtained by cutting $M$ along $\Sigma$. Let $\Gamma(V_i)$ be the kernel of the homomorphism $\text{MCG}(V_i) \to \text{Out}(\pi_1(V_i))$ ($i = 1, 2$). Now, let $\mathcal{S}(\Sigma)$ be the set of the isotopy classes
of essential loops on \(\Sigma\). Let \(\Delta_i \subset S(\Sigma)\) be the set of (isotopy classes of) meridians, i.e., the essential loops on \(\Sigma\) that bound disks in \(V_i\). Set \(\Delta := \Delta_1 \cup \Delta_2\). Let \(Z \subset S(\Sigma)\) be the set of (isotopy classes of) essential loops on \(\Sigma\) that are null-homotopic in \(M\). In [38, Question 5.4], Minsky raised the following question.

**Question 0.1.** When is \(Z\) equal to the orbit \(\langle \Gamma(V_1), \Gamma(V_2) \rangle \Delta\)?

Note that the group \(\Gamma(V_i)\) is identified with the group \(\Gamma(V_1, \Sigma) = \partial_+ (\Pi(V_1, \Sigma)) < \text{MCG}(\Sigma)\) and so \(\langle \Gamma(V_1), \Gamma(V_2) \rangle\) is contained in the group \(\Gamma(M, \Sigma) = \partial_+ (\Pi(M, \Sigma))\). Moreover, since the action of \(\Gamma(M, \Sigma)\) on \(S(\Sigma)\) preserves the homotopy classes of loops in the ambient manifold \(M\), we have

\[
\langle \Gamma(V_1), \Gamma(V_2) \rangle \Delta \subset \Gamma(M, \Sigma) \Delta \subset Z.
\]

This suggests that it is natural to work with the group \(\Gamma(M, \Sigma)\) rather than the group \(\langle \Gamma(V_1), \Gamma(V_2) \rangle\) for Question 0.1 and we have the following refinement of the question.

**Question 0.2.** Let \(\Sigma\) be a Heegaard surface of a closed, orientable 3-manifold \(M\).

1. When is \(Z\) equal to the orbit \(\Gamma(M, \Sigma) \Delta\)?
2. Let \(\kappa : S(\Sigma)/\Gamma(M, \Sigma) \to S(\Sigma)/ \simeq_M\) be the projection, where \(\simeq_M\) is the equivalence relation on \(S(\Sigma)\) induced by homotopy in \(M\), namely two essential simple loops of \(\Sigma\) are equivalent with respect to \(\simeq_M\) if they are homotopic in \(M\). Then how far is the map \(\kappa\) from being injective? In particular, when is the restriction of \(\kappa\) to \((S(\Sigma) - Z)/\Gamma(M, \Sigma)\) injective?

The corresponding question for 2-bridge spheres for 2-bridge links were completely solved by Lee-Sakuma [63, 65], and applied the study of epimorphisms among 2-bridge knot groups [3, Theorem 8.1] and variations of McShane’s identity [64] (see [62] for summary). This paper, as well as Ohshika-Sakuma [78], is motivated by the natural question to what extent these results hold in general setting.

To explain the main question treated in this paper, we note the following facts that easily follow from [65] (cf. [60]). (Below, we use the same symbol \((M, \Sigma)\) for a 2-bridge decomposition by abusing notation.)

- If the Hempel distance of the 2-bridge sphere is \(\geq 3\), then
  \[
  \Gamma(M, \Sigma) = \langle \Gamma(V_1), \Gamma(V_2) \rangle,
  \]
  whereas if the Hempel distance of the 2-bridge sphere is \(2\), then
  \[
  \Gamma(M, \Sigma) \geq \langle \Gamma(V_1), \Gamma(V_2) \rangle.
  \]

In the latter case, the index \([\Gamma(M, \Sigma) : \langle \Gamma(V_1), \Gamma(V_2) \rangle]\) is \(2\), and the gap arises from the open book structure of the link complement whose binding is the axis of the 2-strand braid representing the 2-bridge torus link (see [63, p.5] and Section 5).
Moreover, in both cases, the image of \( \langle \Gamma(V_1), \Gamma(V_2) \rangle \) in the automorphism group of the curve complex of the 4-times punctured sphere is isomorphic to the free product of those of \( \Gamma(V_1) \) and \( \Gamma(V_2) \).

Thus the following question naturally arises.

**Question 0.3.** (1) When is the group \( \langle \Gamma(V_1), \Gamma(V_2) \rangle \) equal to \( \Gamma(M, \Sigma) \)?

(2) When is the group \( \langle \Gamma(V_1), \Gamma(V_2) \rangle \) equal to the free product \( \Gamma(V_1) \ast \Gamma(V_2) \)?

A partial answer to the second question was given by Bowditch-Olshiba-Sakuma in \[78, \text{Theorem B}\] (cf. Bestvina-Fujiwara \[9, \text{Section 3}\]), which says that if the Hempel distance is large enough, then the orientation-preserving subgroup \( \langle \Gamma^+(V_1), \Gamma^+(V_2) \rangle \) is equal to the free product \( \Gamma^+(V_1) \ast \Gamma^+(V_2) \).

**0.3. Main results**

A main purpose of this paper is to give the following partial answer to Question 0.3(1).

**Theorem 8.1.** Let \( M = V_1 \cup_\Sigma V_2 \) be a Heegaard splitting of a closed, orientable 3-manifold \( M \) induced from an open book decomposition. If \( M \) has an aspherical prime summand, then we have \( \langle \Gamma(V_1), \Gamma(V_2) \rangle \leq \Gamma(M, \Sigma) \).

To prove this theorem we construct a \( \mathbb{Z}_2 \)-valued invariant of \( \Gamma(M, \Sigma) \), i.e., a map \( \text{Deg} : \Gamma(M, \Sigma) \to \mathbb{Z}_2 \), such that its mod 2 reduction vanishes on the subgroup \( \langle \Gamma(V_1), \Gamma(V_2) \rangle \). This actually comes from a natural invariant \( \hat{\text{Deg}} : \Pi(M, \Sigma) \to \mathbb{Z}_2 \), where the well-definedness of \( \text{Deg} \) is equivalent to the vanishing of \( \hat{\text{Deg}} \) on the subgroup \( K(M, \Sigma) \).

An element \( \alpha \) of \( K(M, \Sigma) \) is represented by a homotopy motion \( F = \{f_t\}_{t \in I} : \Sigma \times I \to M \), such that both \( f_0 \) and \( f_1 \) are equal to the inclusion map \( j : \Sigma \to M \). Thus \( F \) determines a continuous map \( \hat{F} : \Sigma \times S^1 \to M \). Though the homotopy class of \( F \) is not always uniquely determined by \( \alpha \in K(M, \Sigma) \), its degree is uniquely determined by \( \alpha \), and so we have a homomorphism \( \text{deg} : K(M, \Sigma) \to \mathbb{Z} \) (see Definition 2.2). The map \( \text{Deg} : \Gamma(M, \Sigma) \to \mathbb{Z}_2 \) is well-defined if and only if the homomorphism \( \text{deg} : K(M, \Sigma) \to \mathbb{Z} \) vanishes (see the paragraph just before Proposition 8.11). The problem of whether this condition holds can be regarded as a refined version of the problem of dominations among 3-manifolds, which has been a subject of extensive literatures (see e.g. \[93, 61, 77\] and references therein).

**Definition 0.4.** We say that a closed, orientable surface \( \Sigma \) in a closed, orientable 3-manifold \( M \) (or a pair \( (M, \Sigma) \)) is dominated by \( \Sigma \times S^1 \) if there exists a map \( \phi : \Sigma \times S^1 \to M \) such that \( \phi|_{\Sigma \times \{0\}} \) is an embedding with image \( \Sigma \subset M \) and that the degree of \( \phi \) is non-zero.

Clearly, the homomorphism \( \text{deg} : K(M, \Sigma) \to \mathbb{Z} \) vanishes if and only if \( (M, \Sigma) \) is not dominated by \( \Sigma \times S^1 \).
We study the question of which Heegaard splitting \((M, \Sigma)\) is dominated by \(\Sigma \times S^1\), and give a complete answer for the case where \(M\) is irreducible (Theorem 7.1) and a partial answer for the generic case (Theorem 7.2). In particular, we show that if \(M\) has an aspherical prime summand then \((M, \Sigma)\) is not dominated by \(\Sigma \times S^1\) for any Heegaard surface \(\Sigma\) of \(M\). This guarantees the existence of the map \(\text{Deg} : \Gamma(M, \Sigma) \to \mathbb{Z}^2\) when \(M\) has an aspherical prime summand, and Theorem 8.1 is proved by using this fact.

We remark here that Theorems 7.1 and 7.2 are intimately related with the result of Kotschick-Neofytidis [61, Theorem 1], which says that a closed, orientable 3-manifold \(M\) is dominated by a product \(\Sigma \times S^1\) for some closed, orientable surface \(\Sigma\) if and only if \(M\) is finitely covered by either a product \(F \times S^1\), for some aspherical surface \(F\), or a connected sum \(#_g(S^2 \times S^1)\) for some non-negative integer \(g\). (In [61] and the present paper, we employ the usual convention that the empty connected sum \(#_0(S^2 \times S^1)\) represents \(S^3\).) Thus part of our non-existence result for domination follows from their result. However, the construction of dominating maps in Theorem 7.2 require more subtle arguments, for we impose that the product \(\Sigma \times S^1\) dominates not only the manifold \(M\) itself but also the pair \((M, \Sigma)\).

In this paper, we also study incompressible surfaces in Haken manifolds. In Theorem 3.3 and Corollary 3.6, we completely describe the structures of their homotopy motion groups and related groups. The proof of that theorem is inspired by the work of Jaco-Shalen [49] (see also [48, Chapter VII]), where they introduced the concept of a spatial deformation of a subset \(\Sigma\) in the boundary of a manifold. The concept of a homotopy motion is also regarded as a variation of that of a spatial deformation. As in [49] and [48, Chapter 5], the proof of Theorem 3.3 uses the covering spaces of compact 3-manifolds corresponding to the surface fundamental groups.

The opposite case where \(\Sigma\) is homotopically trivial, in the sense that the inclusion map \(j : \Sigma \to M\) is homotopic to the constant map, is studied as well (see Theorem 4.2). In that case, we prove that if \(M\) is aspherical then \(\Pi(M, \Sigma) \cong \pi_1(M) \times \text{MCG}(\Sigma)\): the factors \(\pi_1(M)\) and \(\text{MCG}(\Sigma)\) correspond to \(K(M, \Sigma)\) and \(\Gamma(M, \Sigma)\), respectively. Conversely, if \(\Gamma(M, \Sigma) = \text{MCG}(\Sigma)\) then \(\Sigma\) is homotopically trivial provided that \(M\) is irreducible (Corollary 4.4).

Our interest in the group \(\Gamma(M, \Sigma)\) has also its origin in the second author’s observation in [35, Addendum 1] (cf. [15, 72, 46]), called the virtual branched fibration theorem, which says that, for every Heegaard surface \(\Sigma\) of a closed, orientable 3-manifold \(M\), there exists a double branched covering of \(M\) which fibers over the circle, such that the inverse image of \(\Sigma\) is the union of two fiber surfaces. We show that this theorem is intimately related to the subgroup \(\langle \Gamma(V_1), \Gamma(V_2) \rangle\) of \(\Gamma(M, \Sigma)\). Let \(\mathcal{I}(V_i) (\subseteq \text{MCG}(\Sigma))\) be the set of torsion elements of \(\Gamma(V_i)\). By slightly refining the arguments of Zimmermann [94, Proof of Corollary 1.3], we can see that this is nothing but the set of vertical \(I\)-bundle involutions of \(V_i\) (Lemma 9.3). Here, a vertical \(I\)-bundle involution of a handlebody \(V\) is an involution \(h\) for which there
exists an $I$-bundle structure of $V$ such that $h$ preserves each fiber setwise and acts on it as a reflection. We then prove the following refinement of \cite[Addendum 1]{S5}.

**Theorem 9.1.** Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold $M$. Then there exists a double branched covering $p: \tilde{M} \to M$ that satisfies the following conditions.

(i) $\tilde{M}$ is a surface bundle over $S^1$ whose fiber is homeomorphic to $\Sigma$.

(ii) The preimage $p^{-1}(\Sigma)$ of the Heegaard surface $\Sigma$ is a union of two (disjoint) fiber surfaces.

Moreover, the set $D(M, \Sigma)$ of monodromies of such bundles is equal to the set $\{h_1 \circ h_2 \mid h_i \in I(V_i)\}$, up to conjugation and inversion.

0.4. Structure of the paper

This paper is organized as follows. In Section 1, we give formal definitions of the homotopy motion group $\Pi(M, \Sigma)$, its subgroup $K(M, \Sigma)$ and its quotient group $\Gamma(M, \Sigma)$. In Section 2, we present basic properties of these groups for surfaces in 3-manifolds. Section 3 is devoted to the case where $\Sigma$ is an incompressible surface in a Haken manifold $M$. Section 4 treats the opposite case where $\Sigma$ is homotopically trivial. The remaining sections are devoted to the case where $\Sigma$ is a Heegaard surface. In Section 5, we recall various natural subgroups of $\text{MCG}(\Sigma)$ associated with a Heegaard surface, and describe their relationships with the group $\Gamma(M, \Sigma)$. In Section 6, we consider the Heegaard splitting obtained from an open book decomposition, and introduce two homotopy motions, the half book rotation $\rho$ and the unilateral book rotation $\sigma$, which play key roles in the proofs of the main theorems given in the succeeding two sections. In Section 7, we study the problem of which Heegaard surface $(M, \Sigma)$ is dominated by $\Sigma \times S^1$. In Section 8, we discuss gaps between $\Gamma(M, \Sigma)$ and the subgroup $\langle \Gamma(V_1), \Gamma(V_2) \rangle$, and prove Theorem 8.1, which provides a partial answer to Question 0.3(1). In Section 9, we prove the branched fibration theorem (Theorem 9.1), which gives another motivation for defining and studying the group $\Gamma(M, \Sigma)$.

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greatly improved this paper. The authors would like to thank the anonymous referee for his or her valuable comments and suggestions that helped them to improve the exposition.

1. The homotopy motion groups

Let $X$ and $Y$ be topological spaces. We denote by $C(X,Y)$ the space of continuous maps from $X$ to $Y$, endowed with the compact-open topology. For a subspace $A$ of $X$, we denote by $J(A,X)$ the subspace of $C(A,X)$ consisting of embeddings of $A$ into $X$ with image $A = j(A)$, where $j : A \to X$ is the inclusion map. For subspaces $A_1, \ldots, A_n$ of $X$, let $\text{Homeo}(X,A_1,\ldots,A_n)$ denote the topological group of self-homeomorphisms of $X$ that preserves each $A_i$ $(1 \leq i \leq n)$. By $\text{MCG}(X,A_1,\ldots,A_n)$ we mean the mapping class group of $(X,A_1,\ldots,A_n)$, i.e., the group of connected components of $\text{Homeo}(X,A_1,\ldots,A_n)$. We usually do not distinguish notationally between $f \in \text{Homeo}(X,Y_1,\ldots,Y_n)$ and the element $[f] \in \text{MCG}(X,A_1,\ldots,A_n)$ represented by $f$. Note that we allow orientation-reversing homeomorphisms when $X$ is an orientable manifold, so our $\text{MCG}(X,A_1,\ldots,A_n)$ is what is often called the extended mapping class group. A “plus” symbol, as in $\text{MCG}^+(X,A_1,\ldots,A_n)$, indicates the subgroup, of index 1 or 2, consisting of the elements represented by orientation-preserving homeomorphisms of $X$.

Throughout the paper, we identify $S^1$ with $\mathbb{R}/\mathbb{Z}$. In our notation, we will not distinguish between an element of $S^1$ and its representative in $\mathbb{R}$.

Let $\Sigma$ be a subspace of a manifold $M$, and $j : \Sigma \to M$ the inclusion map. In this section, we first introduce formal definitions of the homotopy motion group $\Pi(M,\Sigma)$, its subgroup $\mathcal{K}(M,\Sigma)$, and the quotient group $\Gamma(M,\Sigma)$.

Let $C(\Sigma,M)$ be the space of continuous maps from $\Sigma$ to $M$, and $J(\Sigma,M)$ the subspace of $C(\Sigma,M)$ consisting of embeddings of $\Sigma$ into $M$ with image $j(\Sigma)$. We call a path

$$\alpha : (I,\{1\},\{0\}) \to (C(\Sigma,M),J(\Sigma,M),\{j\})$$

a homotopy motion of $\Sigma$ in $M$. We call the maps $\alpha(0)$ and $\alpha(1)$ from $\Sigma$ to $M$ the initial end and the terminal end, respectively, of the homotopy motion. Two homotopy motions $\alpha, \beta : (I,\{1\},\{0\}) \to (C(\Sigma,M),J(\Sigma,M),\{j\})$ are said to be equivalent if they are homotopic via a homotopy through maps of the same form. We remark that in that case, thinking of the codomains of the two maps $\alpha(1), \beta(1) \in J(\Sigma,M)$ as $\Sigma$, $\alpha(1)$ and $\beta(1)$ are isotopic as self-homeomorphisms of $\Sigma$. When there is no fear of confusion, we do not distinguish notationally between a homotopy motion $\alpha$ and the element $[\alpha]$ of $\Pi(M,\Sigma)$ represented by $\alpha$.

We define

$$\Pi(M,\Sigma) := \pi_1(C(\Sigma,M),J(\Sigma,M),j)$$
to be the set of equivalence classes of homotopy motions, as usual in the definition of relative homotopy groups $\pi_n(X, A, x_0)$ for $x_0 \in A \subset X$, where $X$ is a topological space. We equip $\Pi(M, \Sigma)$ with a group structure as in the following way.

Let $\alpha$ and $\beta$ be homotopy motions. Then the concatenation

$$\alpha \cdot \beta : (I, \{1\}, \{0\}) \to (C(\Sigma, M), J(\Sigma, M), \{j\})$$

of them is defined by

$$\alpha \cdot \beta(t) = \begin{cases} 
\alpha(2t) & (0 \leq t \leq 1/2) \\
\beta(2t - 1) \circ \alpha(1) & (1/2 \leq t \leq 1).
\end{cases}$$

We can easily check that the concatenation naturally induces a product of elements of $\pi_1(C(\Sigma, M), J(\Sigma, M))$. The identity motion $e : (I, \{1\}, \{0\}) \to (C(\Sigma, M), J(\Sigma, M), \{j\})$ defined by $e(t) = j$ ($t \in I$) represents the identity element of $\Pi(M, \Sigma)$. The inverse $\bar{\alpha}$ of a homotopy motion $\alpha$ is defined by

$$\bar{\alpha}(t) = \alpha(1 - t) \circ \alpha(1)^{-1},$$

where we regard $\alpha(1)$ as a self-homeomorphism of $\Sigma$, and $\alpha(1)^{-1}$ denotes its inverse. Then the inverse of $[\alpha]$ in the group $\pi_1(C(\Sigma, M), J(\Sigma, M))$ is given by $[\bar{\alpha}]$.

**Definition 1.1.** We call the group $\Pi(M, \Sigma)$ the homotopy motion group of $\Sigma$ in $M$.

**Remark 1.2.** When $\Sigma$ is a single point $x_0$, $\Pi(M, \Sigma)$ is nothing but the fundamental group $\pi_1(M, x_0)$ of $M$. Thus, the group $\Pi(M, \Sigma)$ is a sort of generalization of $\pi_1(M, x_0)$. See also Theorem 4.2 below.

**Notation 1.3.** For a homotopy motion $\alpha : (I, \{1\}, \{0\}) \to (C(\Sigma, M), J(\Sigma, M), \{j\})$ we employ the following notation.

1. We occasionally regard $\alpha$ as a continuous map $\Sigma \times I \to M$ defined by $\alpha(x, t) = \alpha(x)(t)$ (cf. [29, Theorem 6.5], [91, Introduction 1.9]).
2. When we regard $\alpha$ as a continuous family of maps, we occasionally write $\alpha = \{f_t\}_{t \in I}$ where $f_t = \alpha(t) : \Sigma \to M$.
3. When $\alpha$ is a closed path, i.e., $\alpha(1) = \alpha(0) = j$, $\alpha$ induces a continuous map $\Sigma \times S^1 \to M$, which we denote by $\hat{\alpha}$, that sends $(x, t) \in \Sigma \times S^1$ to $\alpha(t)(x) = \alpha(x, t)$. The homotopy class of this map relative to $\Sigma \times \{0\}$ is uniquely determined by the element $[\alpha] \in \pi_1(C(\Sigma, M), j)$.

Since the inclusion map $j$ is nothing but the identity if we think of the codomain of $j$ as $\Sigma$, $J(\Sigma, M)$ can be canonically identified with Homeo$(\Sigma)$. Thus, the terminal end $\alpha(1) = f_1$ of a homotopy motion $\alpha = \{f_t\}_{t \in I}$ can be regarded as an element of Homeo$(\Sigma)$. Therefore, we obtain a map

$$\partial_+ : \Pi(M, \Sigma) \to \text{MCG}(\Sigma)$$

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by taking the equivalence class of a homotopy motion $\alpha = \{f_t\}_{t \in I}$ to the mapping class of $\alpha(1) = f_1 \in \text{Homeo}(\Sigma)$. Clearly, this map is a homomorphism. (To be precise, this holds when we think of $\text{Homeo}(\Sigma)$ as acting on $\Sigma$ from the right: under the usual convention where $\text{Homeo}(\Sigma)$ acts on $\Sigma$ from the left, which we employ in this paper, the map $\partial_+$ is actually an anti-homomorphism.)

**Definition 1.4.** We denote the image of $\partial_+$ by $\Gamma(M, \Sigma)$. Namely, $\Gamma(M, \Sigma)$ is the subgroup of the mapping class group $\text{MCG}(\Sigma)$ defined by

$$\Gamma(M, \Sigma) = \{[f] \in \text{MCG}(\Sigma) \mid \text{There exists a homotopy motion } \{f_t\}_{t \in I} \text{ with } f = f_1.\}$$

The kernel of $\partial_+$ is denoted by $K(M, \Sigma)$: thus we have the exact sequence (1) in the introduction.

2. Basic properties of homotopy motion groups of surfaces in 3-manifolds.

In this section, we provide a few basic properties concerning the groups defined in the above for surfaces in 3-manifolds, by using elementary arguments in homotopy theory. Throughout the remainder of this paper, $\Sigma$ denotes a connected, closed, orientable surface embedded in a connected, orientable 3-manifold $M$, unless otherwise stated, and $j : \Sigma \to M$ denotes the inclusion. When we mention the degrees of maps, we assume that the surfaces and 3-manifolds are endowed with (suitable) orientations.

Note that we have the following long exact sequence.

$$\cdots \to \pi_1(J(\Sigma, M), j) \xrightarrow{\mathcal{J}} \pi_1(C(\Sigma, M), j) \to \pi_1(C(\Sigma, M), J(\Sigma, M), j) \to \pi_0(J(\Sigma, M)) \to \pi_0(C(\Sigma, M)).$$

Here, $\mathcal{J} : \pi_1(J(\Sigma, M), j) \to \pi_1(C(\Sigma, M), j)$ is the map induced from the inclusion $(J(\Sigma, M), j) \hookrightarrow (C(\Sigma, M), j)$. The boundary map $\pi_1(C(\Sigma, M), J(\Sigma, M), j) \to \pi_0(J(\Sigma, M))$ respects the group structures of $\Pi(M, \Sigma) = \pi_1(C(\Sigma, M), J(\Sigma, M), j)$ and $\text{MCG}(\Sigma) = \pi_0(J(\Sigma, M))$, and it is identical with the (anti-)homomorphism $\partial_+$. Thus we have the following description of the kernel $K(M, \Sigma)$.

**Lemma 2.1.** We have the isomorphism

$$K(M, \Sigma) \cong \pi_1(C(\Sigma, M), j)/\mathcal{J}(\pi_1(J(\Sigma, M), j)).$$

Moreover, if the genus of $\Sigma$ is at least 2, then we have

$$K(M, \Sigma) \cong \pi_1(C(\Sigma, M), j).$$

**Proof.** The first assertion is a direct consequence of the exact sequence. The second assertion follows from the fact that $J(\Sigma, M)$ can be canonically identified with $\text{Homeo}(\Sigma)$ as discussed before, and the result of Hamstrom [10] that $\pi_1(J(\Sigma, M), j)$ is the trivial group when the genus of $\Sigma$ is at least 2. \qed
As noted in Notation 1.3(3), a closed path \( \alpha : (I, \partial I) \to (C(\Sigma, M), \{j\}) \) determines a continuous map \( \hat{\alpha} : \Sigma \times S^1 \to M \) whose homotopy class is uniquely determined by the element \([\alpha] \in \pi_1(C(\Sigma, M), j)\). Thus we have a well-defined map \([\alpha] \mapsto \deg(\hat{\alpha})\) from \(\pi_1(C(\Sigma, M), j)\) to \(\mathbb{Z}\), which is obviously a homomorphism. If \([\alpha]\) belongs to the subgroup \(\mathcal{S}(\pi_1(J(\Sigma, M), j))\), then \(\hat{\alpha}\) is homotopic to a map with image(\(\hat{\alpha}\)) = \(\Sigma \times \{0\}\), and therefore \([\alpha]\) belongs to the kernel of the homomorphism. Hence it descends to a homomorphism \(K(M, \Sigma) \cong \pi_1(C(\Sigma, M), j)/\mathcal{S}(\pi_1(J(\Sigma, M), j)) \to \mathbb{Z}\).

**Definition 2.2.** We denote by \(\deg : K(M, \Sigma) \to \mathbb{Z}\) the homomorphism defined by
\[
\deg([\alpha]) = \deg(\hat{\alpha} : \Sigma \times S^1 \to M),
\]
where \(\alpha : (I, \partial I) \to (C(\Sigma, M), \{j\})\) is a closed path. We call \(\deg([\alpha])\) the **degree** of the element \([\alpha] \in K(M, \Sigma)\).

In order to prove further basic properties of the homotopy motion groups, we recall a few results from classical obstruction theory.

**Definition 2.3.** Let \(X\) and \(Y\) be arcwise-connected topological spaces, and let \(\theta_i : \pi_1(X, x_0) \to \pi_1(Y, y_i)\) \((i = 0, 1)\) be homomorphisms, where \(x_0 \in X\) and \(y_0, y_1 \in Y\). Then we say that \(\theta_0\) and \(\theta_1\) are **equivalent** if there is a path \(u : (I, 0, 1) \to (Y, y_0, y_1)\) such that \(\theta_1 = \tau_u \circ \theta_0\) where \(\tau_u : \pi_1(Y, y_0) \to \pi_1(Y, y_1)\) is the isomorphism induced by \(u\). In the case where \(y_0 = y_1\), \(\tau_u\) is the inner-automorphism induced by \([u] \in \pi_1(Y, y_0)\); so, we say that \(\theta_0\) and \(\theta_1\) are **conjugate** if they are equivalent.

When we are concerned with the equivalence class of a homomorphism \(\theta : \pi_1(X, x_0) \to \pi_1(Y, y_0)\), the choice of base point does not matter. So, we often drop the description of the base points and denote the homomorphism by \(\theta : \pi_1(X) \to \pi_1(Y)\).

**Proposition 2.4.** Let \(X\) be a connected \(n\)-dimensional CW-complex and \(Y\) an arcwise connected topological space.

1. **Suppose \(\pi_r(Y) = 0\) for every \(r\) with \(1 < r < n\). Then any homomorphism \(\theta : \pi_1(X) \to \pi_1(Y)\) is realized by a continuous map, namely there is a continuous map \(f : X \to Y\) such that \(f_*\) is equivalent to \(\theta\). To be precise, if \(\theta\) is a homomorphism from \(\pi_1(X, x_0)\) to \(\pi_1(Y, y_0)\) \((x_0 \in X, y_0 \in Y)\), then there is a continuous map \(f : (X, x_0) \to (Y, y_0)\) such that \(f_* = \theta\).

2. **Suppose \(\pi_r(Y) = 0\) for every \(r\) with \(1 < r \leq n\). Then two continuous maps \(f_0\) and \(f_1 : X \to Y\) are homotopic if and only if they induce equivalent homomorphisms between the fundamental groups.

**Proof.** (1) It is obvious that any homomorphism \(\theta\) is realized by a continuous map from the 2-skeleton of \(X\) to \(Y\). For \(r \) with \(1 < r < n\), the condition \(\pi_r(Y) = 0\) guarantees that any continuous map from the \(r\)-skeleton of \(X\) to \(Y\) extends over the \((r + 1)\)-skeleton (see [29] Theorem 7.1(1))). By applying this fact inductively, we obtain (1).
(2) This is obtained by a similar inductive argument by using [29] Theorem 7.12
(cf. [79] Theorem 25.3).

We also need the following relative version of Proposition 2.4(1).

**Proposition 2.5.** Let \((X, X_0)\) be a relative CW-complex of dimension \(n\) (i.e., \(X\) is a topological space obtained from \(X_0\) by attaching cells of dimension \(\leq n\)), and let \(x_0 \in X_0\). Let \(f : (X_0, x_0) \to (Y, y_0)\) be a continuous map to an arcwise connected topological space \(Y\) with base point \(y_0\). Suppose \(\pi_r(Y) = 0\) for every \(r\) with \(1 < r < n\). Then \(f\) extends to a continuous map from \(X\) to \(Y\) if and only if there is a homomorphism \(\theta : \pi_1(X, x_0) \to \pi_1(Y, y_0)\) such that \(f_* = \theta \circ i_* : \pi_1(X_0, x_0) \to \pi_1(Y, y_0)\) where \(i : X_0 \to X\) is the inclusion map.

**Proof.** This can be proved by an inductive argument using [79] Theorem 25.1. \(\square\)

The following result, which is a consequence of [80] Theorems IIA and IIb), refines Proposition 2.4(2) in the case where \(X\) and \(Y\) are closed, orientable \(n\)-manifolds.

**Proposition 2.6.** Let \(X\) and \(Y\) be connected, closed, oriented \(n\)-manifolds, and assume that \(\pi_r(Y) = 0\) for every \(r\) with \(1 < r < n\) and that \(\pi_1(Y)\) is finite. Then two continuous maps \(f_0\) and \(f_1 : X \to Y\) are homotopic if and only if the homomorphisms \((f_i)_* : \pi_1(X) \to \pi_1(Y)\) \((i = 0, 1)\) are equivalent and \(\deg(f_0) = \deg(f_1)\).

Moreover, for a given homomorphism \(\theta : \pi_1(X) \to \pi_1(Y)\), the set of \(\deg(f)\), where \(f : X \to Y\) runs over the continuous maps such that \(f_* : \pi_1(X) \to \pi_1(Y)\) is equivalent to \(\theta\), is of the form \(d + |\pi_1(Y)| \cdot \mathbb{Z}\) for some \(d \in \mathbb{Z}\).

Now we state two basic properties (Lemmas 2.7 and 2.9) of the homotopy motion groups and related groups, which are obtained by using the above results. The following lemma gives a characterization of the group \(\Gamma(M, \Sigma)\) in terms of the induced homomorphisms between the fundamental groups.

**Lemma 2.7.** Let \(\Sigma\) be a closed, orientable surface embedded in a 3-manifold \(M\), and let \(f\) be a self-homeomorphism of \(\Sigma\). Then the following hold.

1. If the mapping class \([f] \in \text{MCG}(\Sigma)\) belongs to the subgroup \(\Gamma(M, \Sigma)\), then the homomorphism \((j \circ f)_* : \pi_1(\Sigma) \to \pi_1(M)\) is equivalent to the homomorphism \(j_*\).

2. Suppose \(M\) is irreducible. Then the converse to the above also holds.

**Proof.** The first assertion is obvious from the definition of \(\Gamma(M, \Sigma)\). To prove the second assertion, assume that \(M\) is irreducible and let \([f] \in \text{MCG}(\Sigma)\) be a mapping class such that \((j \circ f)_*\) is equivalent to \(j_*\). Then by the sphere theorem we have \(\pi_2(M) = 0\). So, we can apply Proposition 2.4(2) to show that \(j \circ f\) is homotopic to \(j\). Hence \([f]\) belongs to \(\Gamma(M, \Sigma)\). \(\square\)

Now fix a base point \(x_0 \in \Sigma \subset M\), and consider a closed path \(\alpha : (I, \partial I) \to (C(\Sigma, M), \{j\})\). Let \(w\) be the element of \(\pi_1(\Sigma \times S^1, (x_0, 0)) = \pi_1(\Sigma, x_0) \times \pi_1(S^1, 0)\)
representing the canonical generator of $\pi_1(S^1, 0)$. Then $\hat{\alpha} \ast (w)$ belongs to the centralizer $Z(j_*(\pi_1(\Sigma, x_0)), \pi_1(M, x_0))$ of $j_*(\pi_1(\Sigma, x_0))$ in $\pi_1(M, x_0)$, and it is represented by the based loop $t \mapsto \alpha(t)(x_0)$. Since the homotopy class of $\hat{\alpha} : \Sigma \times S^1 \to M$ relative to $(x_0, 0)$ is uniquely determined by $[\alpha] \in \pi_1(\Sigma(M, x_0), j)$, we obtain the following homomorphism.

**Definition 2.8.** We denote by $\Phi$ the homomorphism

$$\Phi : \pi_1(C(\Sigma, M), j) \to Z(j_*(\pi_1(\Sigma, x_0)), \pi_1(M, x_0))$$

where $\alpha : (I, \partial I) \to (C(\Sigma, M), \{j\})$ and $u : (I, \partial I) \to (M, \{x_0\})$, $u(t) = \alpha(t)(x_0)$.

The next lemma plays important roles in the proofs of Theorems 3.3, 4.2 and 7.1.

**Lemma 2.9.** Let $\Sigma$ be a closed, orientable surface embedded in a 3-manifold $M$, and $x_0 \in \Sigma$. Then the following hold.

1. If $M$ is irreducible, then $\Phi$ is surjective.
2. If $M$ is aspherical, then $\Phi$ is injective.

**Proof.** (1) Assume that $M$ is irreducible, and let $[u]$ be an element of $Z(j_*(\pi_1(\Sigma, x_0)), \pi_1(M, x_0))$. Consider the pair $(X, X_0) = (\Sigma \times I, \Sigma \times \partial I \cup \{x_0\} \times I)$ and the map $F : X_0 \to M$ defined by $F(x, 0) = F(x, 1) = x \ (x \in \Sigma)$ and $F(x_0, t) = u(t) \ (t \in I)$. Put $\check{x}_0 = (x_0, 0) \in X$ and let $\theta : \pi_1(X, \check{x}_0) \to \pi_1(M, x_0)$ be the homomorphism induced by $j \circ p : (X, \check{x}_0) \to (M, x_0)$ where $p : X = \Sigma \times I \to \Sigma$ is the projection. Then, since $u$ represents an element of $Z(j_*(\pi_1(\Sigma, x_0)), \pi_1(M, x_0))$, we see that $F_* : \pi_1(X_0, \check{x}_0) \to \pi_1(M, x_0)$ is identical with $\theta \circ i_*$, where $i : X_0 \to X$ is the inclusion. See Figure 1. Since $\pi_2(M) = 0$ by the irreducibility of $M$, we see by Proposition 2.5 that the map $F$ extends over $X = \Sigma \times I$. The resulting map $F : \Sigma \times I \to M$ determines

![Figure 1. The maps $F$, $i$ and $p$.](image-url)
a closed path, \( \alpha \), in \( C(\Sigma, M) \) based on \( j \), and the image of \([\alpha] \in \pi_1(C(\Sigma, M), j)\) by \( \Phi \) is equal to \([u]\). Hence \( \Phi \) is surjective.

(2) Assume that \( M \) is aspherical, and let \([\alpha] \) be an element of \( \pi_1(C(\Sigma, M), j) \) which is contained in \( \ker \Phi \). Consider the maps \( \hat{\alpha} \) and \( \hat{e} : \Sigma \times S^1 \to M \) induced by \( \alpha \) and the identity motion \( e \). Since \([\alpha] \in \ker \Phi \), the homomorphisms \( \hat{\alpha}_* \) and \( \hat{e}_* : \pi_1(\Sigma \times S^1) \to \pi_1(M) \) are equivalent (in fact, identical). By Proposition 2.4(2), this implies that \( \hat{\alpha} \) and \( \hat{e} \) are homotopic, for \( M \) is aspherical. Hence \([\alpha] \) is conjugate to the identity element \([e]\) in \( \pi_1(C(\Sigma, M), j) \). Hence \([\alpha] = [e] \in \pi_1(C(\Sigma, M), j)\). □

3. The homotopy motion groups of incompressible surfaces in Haken manifolds

In this section, we consider the groups \( \Pi(M, \Sigma) \) and \( \Gamma(M, \Sigma) \) in the case where \( \Sigma \) is a closed, orientable, incompressible surface in a closed, orientable Haken manifold \( M \). Let us begin with two examples of non-trivial elements of \( \Pi(M, \Sigma) \). We will see soon in Theorem 3.3 that they are in fact the only elements necessary to generate \( \Pi(M, \Sigma) \).

**Example 3.1.** Let \( \varphi \) be an element of \( \text{MCG}(\Sigma) \). Consider the 3-manifold \( M := \Sigma \times \mathbb{R}/(x, t) \sim (\varphi(x), t+1) \), which is the \( \Sigma \)-bundle over \( \mathbb{S}^1 \) with monodromy \( \varphi \). We denote the image of \( \Sigma \times \{0\} \) in \( M \) by the same symbol \( \Sigma \) and call it a fiber surface. Then we have a natural homotopy motion \( \lambda = \{f_t\} \) of \( \Sigma \) in \( M \) defined by \( f_t(x) = [x, t] \), where \([x, t]\) is the element of \( M \) represented by \((x, t)\) (see Figure 2(i)). Its terminal end is equal to \( \varphi^{-1} \), because \( f_1(x) = [x, 1] = [\varphi^{-1}(x), 0] = \varphi^{-1}(x) \). Thus \( \varphi \) belongs to \( \Gamma(M, \Sigma) \).

![Figure 2](image)

**Figure 2.** (i) The homotopy motion \( \lambda \). (ii) The homotopy motion \( \mu \).

**Example 3.2.** Let \( h \) be an orientation-reversing free involution of a closed, orientable surface \( \Sigma \). Consider the 3-manifold \( N := \Sigma \times [0, 1]/(x, t) \sim (h(x), 1-t) \), which is the orientable twisted \( I \)-bundle over the closed, non-orientable surface \( \Sigma/h \). The boundary \( \partial N \) is identified with \( \Sigma \) by the homeomorphism \( \Sigma \to \partial N \) mapping \( x \)
to $[x, 0]$, where $[x, t]$ denotes the element of $N$ represented by $(x, t)$. Then we have a natural homotopy motion $\mu = \{f_t\}_{t \in I}$ of $\Sigma = \partial N$ in $N$, defined by $f_t(x) = [x, t]$. Its terminal end is equal to $h$, because $f_1(x) = [x, 1] = [h(x), 0] = h(x)$ for every $x \in \Sigma = \partial N$. Let $N'$ be any compact, orientable 3-manifold whose boundary is identified with $\Sigma$, i.e., a homeomorphism $\partial N' \cong \Sigma$ is fixed, and let $M = N \cup N'$ be the closed, orientable 3-manifold obtained by gluing $N$ and $N'$ along the common boundary $\Sigma$. Then the homotopy motion $\mu = \{f_t\}_{t \in I}$ of $\Sigma$ in $N$ defined as above can be regarded as that of $\Sigma$ in $M$, and thus $h$ is an element of $\Gamma(M, \Sigma)$ (see Figure 2(ii)). If $N'$ is also a twisted $I$-bundle associated with an orientation-reversing involution $h'$ of $\Sigma$, then we have another homotopy motion $\mu'$ of $\Sigma$ in $N'$ with terminal end $h' \in \Gamma(M, \Sigma)$.

The following theorem is proved by using the positive solution of Simon’s conjecture [89] concerning manifold compactifications of covering spaces, with finitely generated fundamental groups, of compact 3-manifolds, which in turn is proved by using the geometrization theorem established by Perelman [82, 83, 84] and the tameness theorem of hyperbolic manifolds established by Agol [12] and Calegari-Gabai [17] (see also Soma [90] and Bowditch [13]). A proof of Simon’s conjecture can be found in Canary’s expository article [18, Theorem 9.2], where he attributes it to Long and Reid.

**Theorem 3.3.** Let $M$ be a closed, orientable Haken manifold, and suppose that $\Sigma$ is a closed, orientable, incompressible surface in $M$. Then the following hold.

1. If $M$ is a $\Sigma$-bundle over $S^1$ with monodromy $\varphi$ and $\Sigma$ is a fiber surface, then $\Pi(M, \Sigma)$ is the infinite cyclic group generated by the homotopy motion $\lambda$ described in Example 3.1.

2. If $\Sigma$ separates $M$ into two submanifolds, $M_1$ and $M_2$, precisely one of which is a twisted $I$-bundle, then $\Pi(M, \Sigma)$ is the order-2 cyclic group generated by the homotopy motion $\mu$ described in Example 3.2.

3. If $\Sigma$ separates $M$ into two submanifolds, $M_1$ and $M_2$, both of which are twisted $I$-bundles, then $\Pi(M, \Sigma)$ is the infinite dihedral group generated by the homotopy motions $\mu$ and $\mu'$ described in Example 3.2.

4. Otherwise, $\Pi(M, \Sigma)$ is the trivial group.

To show the above theorem, we require the following two lemmas.

**Lemma 3.4.** Let $\Sigma$ be a closed, orientable surface of genus at least 1. Then

$$\Pi(\Sigma \times \mathbb{R}, \Sigma \times \{0\}) = 1.$$  

**Proof.** Consider the projection $q : \Sigma \times \mathbb{R} \to \Sigma \times \{0\}$. Then for any homotopy motion $\alpha = \{f_t\}_{t \in I}$ of $\Sigma \times \{0\}$ in $\Sigma \times \mathbb{R}$, the composition $\{q \circ f_t\}_{t \in I}$ is a homotopy of maps from $\Sigma \times \{0\}$ to itself with initial end $\text{id}_{\Sigma \times \{0\}}$ and terminal end $f_1$. It follows from Baer [5] (cf. [30, Theorem 1.12]) that $f_1$ is isotopic to $\text{id}_{\Sigma \times \{0\}}$. Hence $\Gamma(\Sigma \times \mathbb{R}, \Sigma \times \{0\})$ is trivial, and so $\Pi(\Sigma \times \mathbb{R}, \Sigma \times \{0\}) = K(\Sigma \times \mathbb{R}, \Sigma \times \{0\})$. 

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Suppose first that the genus of $\Sigma$ is at least 2. Then $K(\Sigma \times \mathbb{R}, \Sigma \times \{0\}) \cong \pi_1(C(\Sigma \times \{0\}, \Sigma \times \mathbb{R}), j)$ by Lemma 2.1 and this group is isomorphic to $Z(j_* (\pi_1(\Sigma \times \{0\})), \pi_1(\Sigma \times \mathbb{R}))) \cong Z(\pi_1(\Sigma))$ by Lemma 2.9. Since $Z(\pi_1(\Sigma)) = 1$, we have $K(\Sigma \times \mathbb{R}, \Sigma \times \{0\}) = 1$.

Suppose next that $\Sigma$ is the torus. Then, by Lemmas 2.1 and 2.10, $K(\Sigma \times \mathbb{R}, \Sigma \times \{0\})$ is isomorphic to the quotient of the centralizer $\{\Phi(J) \mid J \in Z(\pi_1(\Sigma \times \{0\}), \pi_1(\Sigma \times \mathbb{R})))\}$ by $\Phi(J(\Sigma \times \{0\}, \Sigma \times \mathbb{R})))$. Now, identify $\Sigma$ with $\mathbb{R}^2/\mathbb{Z}^2$, and denote by $[x, y]$ the point of $\Sigma$ represented by $(x, y, \cdot)$. For an element $(m, n) \in \mathbb{Z}^2 = \pi_1(\Sigma)$, let $\xi_{m,n} = \{g_t\}_{t \in I}$ be the ambient isotopy of $\Sigma$, defined by $g_t([x, y]) = [(x + mt, y + nt)]$, and regard it as an element of $\pi_1(J(\Sigma \times \{0\}, \Sigma \times \mathbb{R})))$. Then we can easily check that $\Phi(\xi_{m,n}) = (m, n)$. Thus $\Phi \circ \xi$ is surjective, and therefore we again have $K(\Sigma \times \mathbb{R}, \Sigma \times \{0\}) = 1$. Hence $\Pi(\Sigma \times \mathbb{R}, \Sigma \times \{0\}) = 1$ as desired. 

**Lemma 3.5.** Let $\Sigma$ be a closed, orientable surface of genus at least 1. For $t \in [0, 1]$, let $j_t : \Sigma \to \Sigma \times \mathbb{R}$ be the embedding defined by $j_t(x) = (x, t)$, and set $\kappa := \{j_t\}_{t \in I} : I \to C(\Sigma, \Sigma \times \mathbb{R})$. Then any continuous map $\alpha = \{f_t\}_{t \in I} : (I, \{1\}, \{0\}) \to (C(\Sigma, \Sigma \times \mathbb{R}), J(\Sigma \times \{1\}, \Sigma \times \mathbb{R}) \circ j_1, \{j_0\})$ is homotopic to $\kappa$ via a homotopy through maps of the same form.

**Proof.** Let $\alpha = \{f_t\}_{t \in I}$ be a continuous map from $(I, \{1\}, \{0\})$ to $(C(\Sigma, \Sigma \times \mathbb{R}), J(\Sigma \times \{1\}, \Sigma \times \mathbb{R}) \circ j_1, \{j_0\})$. Let $q : \Sigma \times \mathbb{R} \to \Sigma$ be the projection. Then, by Baer [5], $q \circ \alpha(1) = q \circ f_1$ is isotopic to the identity map as a self-homeomorphism of $\Sigma$. Thus deforming $\alpha$ by a homotopy through maps $(I, \{1\}, \{0\}) \to (C(\Sigma, \Sigma \times \mathbb{R}), J(\Sigma \times \{1\}, \Sigma \times \mathbb{R}) \circ j_1, \{j_0\})$ if necessary, we may assume that $\alpha(1) = f_1 = j_1$.

Consider the path $u : (I, \{1\}, \{0\}) \to (\Sigma \times \mathbb{R}, \{(x_0, 0), (x_0, 1)\}), \ t \mapsto f_t(x_0)$.

We see that the closed loop $q \circ u$ represents an element of the center $Z(\pi_1(\Sigma))$, by an argument similar to that in the paragraph preceding Definition 2.8. (Here, we use the map $\alpha : \Sigma \times S^1 \to \Sigma$ defined by $\alpha(x, t) = q(f_t(x))$.) If $\Sigma$ has genus at least 2, then $q \circ u$ represents the trivial element of $\pi_1(\Sigma)$. If $\Sigma$ is a torus, using the ambient isotopy $\xi_{m,n}$ for $(m, n) \in \mathbb{Z}^2$ defined in the proof of Lemma 3.4, we can deform $\alpha$ by a homotopy through maps $(I, \{1\}, \{0\}) \to (C(\Sigma, \Sigma \times \mathbb{R}), J(\Sigma \times \{1\}, \Sigma \times \mathbb{R}) \circ j_1, \{j_0\})$ so that $q \circ u$ represents the trivial element of $\pi_1(\Sigma)$.

Let $X = (\Sigma \times I) \times I$ and $X_0 = \partial X$, and consider the map $F : X_0 \to \Sigma \times \mathbb{R}$ defined by $F(x, t, 0) = \alpha(x, t), \ F(x, t, 1) = \kappa(x, t) = (x, t), \ F(x, 0, s) = (x, 0), \ F(x, 1, s) = (x, 1)$ $(x \in \Sigma, \ t, \ s \in I)$. Note that $F$ is a well-defined continuous map, because $\alpha(1) = j_1$. Put $\tilde{x}_0 = (x_0, 0, 0), \ \hat{x}_0 = (x_0, 0)$ and let $\theta : \pi_1(X, \hat{x}_0) \to \pi_1(\Sigma \times \mathbb{R}, \hat{x}_0)$ be the homomorphism induced by the map $X = (\Sigma \times I) \times I \to \Sigma \times \mathbb{R}$ obtained as the composition of the projection $(\Sigma \times I) \times I \to \Sigma \times I$ and the inclusion $\Sigma \times I \to \Sigma \times \mathbb{R}$. Then, since $q \circ u$ represents the trivial element of $\pi_1(\Sigma)$, we see that $F_* : \pi_1(X, \hat{x}_0) \to \pi_1(\Sigma \times \mathbb{R}, \hat{x}_0)$ is equal to $\theta \circ i_*$. Hence, by Proposition 2.3, the map
$F$ extends over $X = (\Sigma \times I) \times I$, and the resulting map $F : (\Sigma \times I) \times I \to \Sigma \times \mathbb{R}$ gives the desired homotopy between $\alpha$ and $\kappa$. \hfill \Box$

**Proof of Theorem [3.3]** Let $p : \tilde{M} \to M$ be the covering corresponding to $\pi_1(\Sigma) < \pi_1(M)$. Then, by the positive solution of Simon's conjecture (see [18] Theorem 9.2), $\tilde{M}$ admits a manifold compactification, that is, there exists a compact 3-manifold $M$ with boundary, such that $\tilde{M}$ is homeomorphic to $\tilde{M} - \tilde{C}$, where $\tilde{C}$ is a closed subset of $\partial \tilde{M}$. We actually have $\tilde{C} = \partial \tilde{M}$ and $M \cong \text{Int } M$, because $M$ is closed.

Brown’s theorem [16] Theorem 3.4] implies that $\tilde{M} \cong \Sigma \times [-\infty, \infty]$, where $[-\infty, \infty]$ is the closed interval that is obtained by compactifying $\mathbb{R}$. Thus $\tilde{M}$ is identified with $\Sigma \times \mathbb{R}$. We assume that the restriction of the covering projection $p$ to $\Sigma \times \{0\}$ is given by $p(x, 0) = x$. In other words, the inclusion map $j : \Sigma \to M$ has a lift $\tilde{j} : \Sigma \to \tilde{M} = \Sigma \times \mathbb{R}$ such that $\tilde{j}(x) = (x, 0)$.

Suppose that $\Pi(M, \Sigma)$ has a nontrivial element $\alpha = \{f_t\}_{t \in I}$ with $f_1 = f$. Let $\tilde{\alpha} = \{\tilde{f}_t\}_{t \in I}$ be the lift of $\alpha$ with $\tilde{f}_0 = \tilde{j}$, and let $\tilde{f}$ be the lift of $f$ defined by $\tilde{f} = \tilde{f}_1$. Since $f(\Sigma) = \Sigma$, the image $\tilde{f}(\Sigma)$ is a component of $p^{-1}(\Sigma)$ to which the restriction of $p$ is a homeomorphism onto $\Sigma$. In particular, we have either $\tilde{f}(\Sigma) = \Sigma \times \{0\}$ or $\tilde{f}(\Sigma) \cap (\Sigma \times \{0\}) = \emptyset$. If $\tilde{f}(\Sigma) = \Sigma \times \{0\}$, the map $\tilde{\alpha}$ is homotopic to the constant map from $I$ to $\tilde{j} \in C(\Sigma, \tilde{M})$ via a homotopy through maps $(I, \{1\}, \{0\}) \to (C(\Sigma, \tilde{M}), \tilde{\alpha}(\Sigma), \{\tilde{j}\})$ by Lemma 3.4. This homotopy projects to a homotopy from $\alpha$ to the trivial homotopy motion of $\Sigma \subset M$. This contradicts the assumption that $\alpha$ is a nontrivial element of $\Pi(M, \Sigma)$. Therefore we have $\tilde{f}(\Sigma) \cap (\Sigma \times \{0\}) = \emptyset$. Then, by [16], $\tilde{f}(\Sigma)$ is parallel to $\Sigma \times \{0\}$ in $\tilde{M} = \Sigma \times \mathbb{R}$. We may choose the product structure so that $\tilde{f}(\Sigma) = \Sigma \times \{1\}$. Since $p$ is a covering and since $\Sigma$ is incompressible, we see that $p^{-1}(\Sigma) \cap (\Sigma \times (0, 1))$ is a finite disjoint union of compact surfaces that are incompressible in $\tilde{M}$ and so in $\Sigma \times \mathbb{R}$. The result of [16] implies that all components of $p^{-1}(\Sigma) \cap (\Sigma \times (0, 1))$ are parallel to $\Sigma \times \{0\}$ in $\tilde{M}$. Hence there exists a component that is closest to $\Sigma \times \{0\}$. We choose $\alpha$ so that $\tilde{f}(\Sigma) = \Sigma \times \{1\}$ is the closest component, namely $p^{-1}(\Sigma) \cap (\Sigma \times (0, 1)) = \emptyset$.

Fix an orientation of the surface $\Sigma \subset M$, and orient the surfaces $\Sigma \times \{t\} \subset \tilde{M} = \Sigma \times \mathbb{R}$ ($t \in \mathbb{R}$) via the canonical identification with the oriented $\Sigma$. Consider the homeomorphism $\psi : \Sigma \times \{0\} \to \Sigma \times \{1\}$ defined by $\psi = (|p|_{\Sigma \times \{1\}})^{-1} \circ p|_{\Sigma \times \{0\}}$. It should be noted that $\psi$ is a “local covering transformation”, in the sense that $\psi$ extends to a homeomorphism between a neighborhoods of $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ that commutes with the covering projection $p$. Let $q : \Sigma \times \mathbb{R} \to \Sigma$ be the projection to the first factor, which is identified with the surface $\Sigma$ in $M$.

Case 1. Suppose that $\psi$ is orientation-preserving. Consider the 3-manifold $M' := (\Sigma \times [0, 1])/\sim (q(\psi(x)), 1)$. Then the restriction of the covering projection $p$ to $\Sigma \times [0, 1]$ descends to a continuous map $p' : M' \to M$, which is a local-homeomorphism at the image of $\Sigma \times (0, 1)$ in $M'$. The condition that $\psi$ is orientation-preserving implies that $p'$ is also a local homeomorphism at an open neighborhood
of the image of $\Sigma \times \{0\}$ (which is equal to that of $\Sigma \times \{1\}$) in $M'$. (Here, we use the fact that $\psi$ is a local covering transformation.) Thus $p' : M' \to M$ is a local homeomorphism. Since $M$ is a compact, connected manifold, it follows that $p'$ has the path-lifting property, and hence $p'$ is a covering (see e.g. Forster [31 Theorem 4.19]). Since $p^{-1}(\Sigma) \cap (\Sigma \times (0,1)) = \emptyset$, the preimage of a point in $\Sigma \subset M$ by $p'$ is a singleton. Hence $p'$ is a homeomorphism and so $M$ is identified with the $\Sigma$-bundle $(\Sigma \times \mathbb{R})/(x,t) \sim (\varphi(x), t + 1)$ over $S^1$, where the monodromy $\varphi$ is defined by $\varphi = q \circ \psi$.

Case 2. Suppose that $\psi$ is orientation-reversing. Consider the submanifold $M' := \Sigma \times [0,1]$ of $M$ and its image $M_1 := p(M')$ in $M$. Since $p^{-1}(\Sigma) \cap (\Sigma \times (0,1)) = \emptyset$, $p(\text{int } M')$ is disjoint from $\Sigma$. This together with the assumption that $\psi$ is orientation-reversing implies that $M_1$ is a submanifold of $M$ with boundary $\Sigma$. Moreover, as in Case 1, we can see that the restriction $p' : M' \to M_1$ of $p$ to $M'$ is a covering and that it has geometric degree 2. (Note that the preimage of a point $x \in \Sigma = \partial M_1$ by $p'$ consists of the two points $(x,0)$ and $(\psi(x),1)$ of $M' = \Sigma \times [0,1]$.) By [33 Theorem 10.3], this implies that $M_1$ is a twisted $I$-bundle, $\Sigma \times [0,1]/(x,t) \sim (h(x),t-1)$, associated with some orientation-reversing free involution $h$ of $\Sigma$.

The above arguments show that if $\Pi(M,\Sigma)$ is nontrivial, then either (i) $M$ is a $\Sigma$-bundle over $S^1$ or (ii) $\Sigma$ separates $M$ into two submanifolds, at least one of which is a twisted $I$-bundle. In particular, we obtain the assertion (4) of the theorem.

Suppose that the conclusion (i) holds, namely $M \cong (\Sigma \times \mathbb{R})/(x,t) \sim (\varphi(x), t + 1)$ for some $\varphi \in \text{MCG}_+ (\Sigma)$. Consider the map $\zeta : \Pi(M,\Sigma) \to \mathbb{Z}$ that sends the homotopy motion $\alpha = \{f_t\}_{t \in I}$ to $n \in \mathbb{Z}$ given by $f_t(\Sigma) = \Sigma \times \{n\}$, where $\tilde{\alpha} = \{\tilde{f}_t\}_{t \in I}$ is the lift of $\alpha$ with $\tilde{f}_0 = \tilde{j}$. Then $\zeta$ is injective, because if two homotopy motions $\alpha$ and $\alpha'$ are mapped to the same element $n \in \mathbb{Z}$, then the homotopy between $\tilde{\alpha}$ and $\tilde{\alpha}'$, given by Lemmas 3.4 and 3.5 according to whether $n = 0$ or not, projects to a homotopy which gives the equivalence of $\alpha$ and $\alpha'$ as elements of $\Pi(M,\Sigma)$. It is obvious that $\zeta$ is a group homomorphism and maps $\lambda$ to 1, where $\lambda$ is the homotopy motion described in Example 3.1. Hence $\Pi(M,\Sigma)$ is the infinite cyclic group generated by $\lambda$, proving the assertion (1).

Suppose that the conclusion (ii) holds, namely $M = M_1 \cup_\Sigma M_2$ and $M_1 = \Sigma \times [0,1]/(x,t) \sim (h(x), 1-t)$, where $h$ is an orientation-reversing involution of $\Sigma$. By the preceding argument, we may assume that $M = \Sigma \times \mathbb{R}$ and the restriction of the covering projection $p$ to $\Sigma \times [0,1]$ is the double covering of $M_1$ that maps $(x,t)$ to the point $[x,t] \in M_1$ it represents. Note that the homotopy motion $\mu = \{f_t\}_{t \in I}$ defined in Example 3.2 lifts to the map $\tilde{\mu} = \{\tilde{f}_t\}_{t \in I} : \Sigma \to \Sigma \times \mathbb{R}$ given by $\tilde{f}_t(x) = (x,t)$. Moreover, Lemma 3.5 implies if a homotopy motion $\alpha = \{f_t\}$ has a lift $\tilde{\alpha} = \{\tilde{f}_t\}_{t \in I} : \Sigma \to \Sigma \times \mathbb{R}$ such that $\tilde{f}_0(x) = (x,0)$ and $\text{image} (\tilde{f}_1) = \Sigma \times \{1\}$, then it is equivalent to $\mu$. We can easily see from the definition of the concatenation that $\mu \cdot \mu$ is equivalent to the identity motion, thus, the order of $\mu$ in $\Pi(M,\Sigma)$ is 2.
Suppose that there exists a nontrivial element $\beta = \{g_t\}_{t \in I} \in \Pi(\Sigma, M)$ that is not equivalent to $\alpha$. Then the previous arguments imply that $\tilde{g}_1(\Sigma)$ is equal to neither $\Sigma \times \{0\}$ nor $\Sigma \times \{1\}$, and so $\tilde{g}_1(\Sigma) \cap (\Sigma \times [0,1]) = \emptyset$. (Here, $\tilde{\beta} = \{\tilde{g}_t\}_{t \in I}$ is the lift of $\beta$ with $\tilde{g}_0 = \tilde{j}$.) By choosing $\beta = \{g_t\}_{t \in I}$ suitably, we may assume that $\tilde{g}_1(\Sigma)$ is the lift of $\Sigma$ closest to $\Sigma$ in $(\Sigma \times \{0\})$ in $\Sigma \times (\infty,0)$. Then the argument in Case 2 implies that $M_2$ is a twisted $I$-bundle and that the terminal end $g_1$ is (represented by) the involution $h'$ corresponding to the twisted $I$-bundle structure of $M_2$. This, in particular, proves the assertion (2). In order to prove the assertion (3), observe that $p : \tilde{M} \to M$ is a regular covering and that the covering transformation group is the infinite dihedral group generated by the two involutions $\gamma$ and $\gamma'$ of $M = \Sigma \times \mathbb{R}$ defined by

$$\gamma(x,t) = (h(x), 1-t), \quad \gamma'(x,t) = (h'(x), -1-t).$$

In this case, $p^{-1}(\Sigma) = \Sigma \times \mathbb{Z}$, and the argument for the case (i) implies that $\Pi(\Sigma, M)$ is the infinite dihedral group generated by the two elements $\mu$ and $\mu'$ of order 2. This completes the proof of the assertion (3).

**Corollary 3.6.** Let $M$ be a closed, orientable Haken manifold $M$, and suppose that $\Sigma$ is an orientable incompressible surface in $M$. Then the following hold.

1. If $M$ is a $\Sigma$-bundle over $S^1$ with monodromy $\varphi$ and $\Sigma$ is a fiber surface, then $\Gamma(M, \Sigma)$ is the cyclic group $\langle \varphi \rangle$, and $K(M, \Sigma)$ is the (possibly trivial) subgroup generated by $\lambda^n$ of the infinite cyclic group $\Pi(M, \Sigma) = \langle \lambda \rangle$, where $n$ is the order of $\varphi$. Moreover, the homomorphism $\deg : K(M, \Sigma) = \langle \lambda^n \rangle \to \mathbb{Z}$ is given by $\deg(\lambda^n) = n$ under a suitable orientation convention.

2. If $\Sigma$ separates $M$ into two submanifolds, $M_1$ and $M_2$, precisely one of which is a twisted $I$-bundle, then $\Gamma(M, \Sigma)$ is the order-2 cyclic group generated by the orientation-reversing involution of $\Sigma$ associated with the twisted $I$-bundle structure, and $K(M, \Sigma)$ is the trivial group.

3. If $\Sigma$ separates $M$ into two submanifolds, $M_1$ and $M_2$, both of which are twisted $I$-bundles, then $\Gamma(M, \Sigma)$ is the (finite or infinite, and possibly cyclic) dihedral group generated by the two orientation-reversing involutions $h_1$ and $h_2$ of $\Sigma$ associated with the twisted $I$-bundle structures, and $K(M, \Sigma)$ is the subgroup of the infinite dihedral group $\Pi(M, \Sigma) = \langle \mu, \mu' | \mu^2, \mu'^2 \rangle$ generated by $\langle \mu \mu' \rangle^n$, where $n$ is the order of $hh'$. Moreover, the homomorphism $\deg : K(M, \Sigma) = \langle \langle \mu \mu' \rangle^n \rangle \to \mathbb{Z}$ is given by $\deg(\langle \mu \mu' \rangle^n) = 2n$ under a suitable orientation convention.

4. Otherwise, both $\Gamma(M, \Sigma)$ and $K(M, \Sigma)$ are the trivial group.

**Proof.** The assertions except for those concerning the homomorphism $\deg : K(M, \Sigma) \to \mathbb{Z}$ follow immediately from Theorem 3.3 and the exact sequence (1). It is also easy to see that $\deg(\lambda^n) = n$. The identity $\deg(\langle \mu \mu' \rangle^n) = 2n$ can be verified by considering the double covering of $M$, which is the $\Sigma$-bundle over $S^1$ with monodromy $h_1h_2$. □
4. Homotopy motion groups of homotopically trivial surfaces

In this section, we study the case contrastive to that treated in the previous section. We say that a closed, orientable surface $\Sigma$ embedded in a closed, orientable 3-manifold $M$ is homotopically trivial if the inclusion map $j : \Sigma \to M$ is homotopic to a constant map.

**Lemma 4.1.** A closed, orientable surface $\Sigma$ embedded in a closed, orientable, irreducible 3-manifold $M$ is homotopically trivial if and only if $j_* : \pi_1(\Sigma) \to \pi_1(M)$ is the trivial homomorphism.

**Proof.** The “only if” part is obvious. The “if” part follows from Proposition 2.4(2). $\square$

We have the following theorem.

**Theorem 4.2.** Let $\Sigma$ be a closed, orientable surface embedded in a closed, orientable 3-manifold $M$. Then the following hold.

1. If $\Sigma$ is homotopically trivial and if $M$ is aspherical, then $\Pi(M, \Sigma) \cong \pi_1(M) \times \text{MCG}(\Sigma)$. To be more precise, $\Gamma(M, \Sigma) = \text{MCG}(\Sigma)$, and $K(M, \Sigma)$ is identified with the factor $\pi_1(M)$. Moreover, the homomorphism $\deg : K(M, \Sigma) \to \mathbb{Z}$ vanishes.

2. Conversely, if $\Gamma(M, \Sigma) = \text{MCG}(\Sigma)$ and if $M$ is irreducible, then $\Sigma$ is homotopically trivial.

**Proof.** (1) Suppose that $\Sigma$ is homotopically trivial and $M$ is aspherical. Pick a base point $x_0 \in \Sigma \subset M$, and define a homomorphism $\Psi : \Pi(M, \Sigma) \to \pi_1(M, x_0)$ as follows. For an element of $\Pi(M, \Sigma)$, choose a representative homotopy motion $\alpha$ such that $\alpha(1)(x_0) = x_0$. Then the element of $\pi_1(M, x_0)$ represented by the closed path

$$(I, \partial I) \to (M, x_0), \ t \mapsto \alpha(t)(x_0)$$

does not depend on the choice of a representative $\alpha$, by the following reason. Two such closed paths are related, up to homotopy relative to $\partial I$, by concatenation of a closed path on $\Sigma$ based at $x_0$. However, since $\Sigma$ is homotopically trivial in $M$, any closed path on $\Sigma$ is null-homotopic in $M$. Thus two such closed paths represent the same element of $\pi_1(M, x_0)$. We define $\Psi([\alpha]) \in \pi_1(M, x_0)$ to be that element.

Suppose first that the genus of $\Sigma$ is at least 2. By Lemma 2.1, $K(M, \Sigma)$ can be canonically identified with $\pi_1(C(\Sigma, M), j)$, and the restriction of $\Psi$ to $\pi_1(C(\Sigma, M), j)$ is nothing but the homomorphism $\Phi$ in Definition 2.8. Therefore, we have the following commutative diagram:

\[
\begin{array}{ccccccccc}
1 & \to & \pi_1(C(\Sigma, M), j) & \to & \Pi(M, \Sigma) & \to & \Gamma(M, \Sigma) & \to & 1 \\
\Phi & & \downarrow{\Psi \times \partial_j} & & \downarrow{\Phi} & & \downarrow{\text{id}} & & \downarrow{1} \\
1 & \to & \pi_1(M, x_0) & \to & \pi_1(M, x_0) \times \text{MCG}(\Sigma) & \to & \text{MCG}(\Sigma) & \to & 1,
\end{array}
\]

(2) Conversely, if $\Gamma(M, \Sigma) = \text{MCG}(\Sigma)$ and if $M$ is irreducible, then $\Sigma$ is homotopically trivial.
where the two rows are exact, and \( \iota : \Gamma(M, \Sigma) \to \text{MC}G(\Sigma) \) is the inclusion homomorphism. Lemmas \([2,7]\) and \([2,9]\) respectively, imply that \( \iota \) and \( \Phi \) are isomorphisms, so does \( \Psi \times \partial_+ \).

Suppose that the genus of \( \Sigma \) is less than 2. In that case, by replacing \( \pi_1(C(\Sigma, M), j) \) with \( \pi_1(C(\Sigma, M), j) / \mathcal{I}(\pi_1(J(\Sigma, M), j)) \), the same argument as above still works because the homomorphism \( \Phi \) vanishes on \( \mathcal{I}(\pi_1(J(\Sigma, M), j)) \) due to the assumption that \( \Sigma \) is homotopically trivial.

The vanishing of \( \deg : \mathcal{K}(M, \Sigma) \to \mathbb{Z} \) can be seen as follows. Suppose that \( \deg(\alpha) \neq 0 \) for some \( \alpha \in \mathcal{K}(M, \Sigma) \). Then the image of \( \alpha_* : \pi_1(\Sigma \times S^1) \to \pi_1(M) \) has finite index in \( \pi_1(M) \) (cf. \([43, \text{Lemma 15.12}]\)). Since \( M \) is an aspherical, closed, orientable 3-manifold, this implies that the homological dimension of \( \text{image}(\alpha_*) \) is 3. On the other hand, since \( \Sigma \) is homotopically trivial, \( \text{image}(\alpha_*) \) is cyclic. This is a contradiction, because the cohomological dimension of a cyclic group is 0, \( \infty \), or 1, according as it is trivial, nontrivial finite cyclic, or infinite cyclic. This completes the proof of (1).

(2) Note that the assertion is trivial when \( \Sigma = S^2 \). We assume that the genus of \( \Sigma \) is at least 1, and we show the assertion by induction on the genus \( g \) of \( \Sigma \). Suppose that \( g = 1 \) and \( \Gamma(M, \Sigma) = \text{MC}G(\Sigma) \). By Corollary \([3,6]\) \( \Sigma \) cannot be incompressible in \( M \). Thus, there exists an essential simple closed curve on \( \Sigma \) that is null-homotopic in \( M \). Since \( \text{MC}G(\Sigma) \) acts on the set \( \mathcal{S}(\Sigma) \) of (isotopy classes of) essential simple closed curves on \( \Sigma \) transitively, every simple closed curve on \( \Sigma \) is null-homotopic in \( M \). Thus, the homomorphism \( j_* : \pi_1(\Sigma) \to \pi_1(M) \) vanishes, which implies by Lemma \([4,1]\) that \( \Sigma \) is homotopically trivial in \( M \), as \( M \) is irreducible. For the inductive step, suppose that the assertion holds for any surface \( \Sigma \) with genus at most \( g \). Let \( \Sigma \) be a closed, orientable surface of genus \( g + 1 \) embedded in \( M \) so that \( \Gamma(M, \Sigma) = \text{MC}G(\Sigma) \). Again, by Corollary \([3,6]\) \( \Sigma \) is compressible. Let \( D \) be a compression disk for \( \Sigma \). If \( \partial D \) is non-separating in \( \Sigma \), the proof runs as in the case of \( g = 1 \), for \( \text{MC}G(\Sigma) \) acts on the set of (isotopy classes of) non-separating simple closed curves on \( \Sigma \) transitively, and \( \pi_1(\Sigma) \) is generated by elements represented by non-separating simple closed curves. Suppose that \( \partial D \) is separating. Let \( \Sigma_1 \) and \( \Sigma_2 \) be the connected surfaces obtained by compressing \( \Sigma \) along \( D \). Then we can see that \( \Gamma(M, \Sigma_i) = \text{MC}G(\Sigma_i) \) \((i = 1, 2)\) as follows. Let \( \Sigma'_i \) and \( \Sigma''_i \) \((i = 1, 2)\) be the closures of components of \( \Sigma - \partial D \). We regard \( \Sigma_i \) as \( \Sigma'_i \cup D \) \((i = 1, 2)\), so \( \Sigma_1 \cap \Sigma_2 = D \). Let \( f_1 \) be an arbitrary element of \( \text{MC}G(\Sigma_1) \). We show that \( f_1 \in \Gamma(M, \Sigma_1) \). We can assume that \( f_1(D) = D \). Then there exists an element \( f_2 \in \text{MC}G(\Sigma_2) \) such that \( f_2(D) = D \) and \( f_1|_D = f_2|_D \). Let \( f : \Sigma \cup D \to M \) be the map obtained by gluing \( f_1 \) and \( f_2 \), and let \( f \) be the restriction of \( f \) to \( \Sigma \). By the assumption, there exists a homotopy motion \( \alpha : \Sigma \times I \to M \) with terminal end \( f \). Then we can extend \( \alpha \) to a homotopy motion \( \tilde{\alpha} : (\Sigma \cup D) \times I \to M \) with terminal end \( \tilde{f} \) because \( M \) is irreducible and hence \( \pi_2(M) = 0 \). By restriction, \( \tilde{\alpha} \) determines a homotopy motion of \( \Sigma_1(\subset \Sigma \cup D) \) with terminal end \( f_1 \), which implies that \( \Gamma(M, \Sigma_1) = \text{MC}G(\Sigma_1) \). Clearly, the same
consequence holds for $\Sigma_2$. By the assumption of induction, both $\Sigma_1$ and $\Sigma_2$ are homotopically trivial in $M$. Hence the image of $\pi_1(\Sigma \cup D) \cong \pi_1(\Sigma_1) * \pi_1(\Sigma_2)$ in $\pi_1(M)$ is trivial. Thus $j_* : \pi_1(\Sigma) \to \pi_1(M)$ is the trivial homomorphism, and so $\Sigma$ is homotopically trivial in $M$, by Lemma 4.1.

Remark 4.3. The assumption that $M$ is aspherical in Theorem 4.2(1) is essential. In fact, in Theorem 7.1(2) we will see that when $\Sigma$ is a Heegaard surface of $S^3$, which is homotopically trivial, the kernel of the homomorphism $\Psi: \Pi(S^3, \Sigma) \to \pi_1(S^3) \times \text{MCG}(\Sigma) = \text{MCG}(\Sigma)$ defined in the above proof consists of infinitely many elements.

Corollary 4.4. Let $\Sigma$ be a closed, orientable surface embedded in a closed, orientable, irreducible 3-manifold $M$. Then $\Sigma$ is homotopically trivial and if and only if $\Gamma(M, \Sigma) = \text{MCG}(\Sigma)$.

Proof. The “if” part is nothing other than Theorem 4.2(2), and the “only if” part follows from Lemma 2.7. □

5. The group $\Gamma(M, \Sigma)$ for a Heegaard surface and its friends

From this section, we are going to study the homotopy motion group $\Pi(M, \Sigma)$ and related groups $\Gamma(M, \Sigma)$ and $\mathcal{K}(M, \Sigma)$ for a Heegaard surface $\Sigma$ of a closed, orientable 3-manifold $M$. Recall that a closed surface $\Sigma$ in a closed orientable 3-manifold $M$ is called a Heegaard surface if $\Sigma$ separates $M$ into two handlebodies $V_1$ and $V_2$. Such a decomposition $M = V_1 \cup_{\Sigma} V_2$ is then called a Heegaard splitting of $M$, and the genus of the splitting is defined to be the genus of $\Sigma$.

In this section, we mainly consider the group $\Gamma(M, \Sigma)$ rather than $\Pi(M, \Sigma)$. We recall various natural subgroups of $\text{MCG}(\Sigma)$ associated with a Heegaard surface, and describe their relationships with the group $\Gamma(M, \Sigma)$. We also describe the group $\Gamma(M, \Sigma)$ and give answers to Questions 0.2 and 0.3 for the very special cases where $\Sigma$ is either an arbitrary Heegaard surface of $M = S^3$ and where $\Sigma$ is a minimal genus Heegaard surface of $M = \#_g(S^2 \times S^1)$.

By definition, the group $\Gamma(M, \Sigma)$ is a subgroup of the extended mapping class group $\text{MCG}(\Sigma)$ of the Heegaard surface $\Sigma$. For a Heegaard splitting $M = V_1 \cup_{\Sigma} V_2$, many other (and similar) subgroups of $\text{MCG}(\Sigma)$ associated with the Heegaard splitting $M = V_1 \cup_{\Sigma} V_2$ has been studied as follows.

1. The handlebody group $\text{MCG}(V_i)$ of the handlebody $V_i$, which is identified with a subgroups of $\text{MCG}(\Sigma)$, by restricting a self-homeomorphism of $V_i$ to its boundary $\partial V_i = \Sigma$. This has been a target of various works (see a survey by Hensel [15] and references therein).

2. The intersection $\text{MCG}(V_1) \cap \text{MCG}(V_2)$, which is identified with $\text{MCG}(M, V_1, V_2)$. This group or its orientation-subgroup $\text{MCG}^+(M, V_1, V_2)$ is called the Goeritz group of the Heegaard splitting $M = V_1 \cup_{\Sigma} V_2$ and it has been extensively studied. In particular, the problem of when this group is finite, finitely
generated, or finitely presented attracts attention of various researchers (cf. Minsky \cite{Minsky:58} Question 5.1). The work on this problem goes back to Goeritz \cite{Goeritz:35}, which gave a finite generating set of the Goeritz group of the genus-2 Heegaard splitting of $S^3$. In these two decades, great progress was achieved by many authors \cite{Minsky:58, Minsky:61, Minsky:62, Minsky:63, Minsky:64, Minsky:65, Minsky:66, Minsky:67, Minsky:68, Minsky:69, Minsky:70, Minsky:71}, however, it still remains open whether the Goeritz group a Heegaard splitting of $S^3$ is finitely generated when the genus is at least 4.

(3) The group $\langle \text{MCG}(V_1), \text{MCG}(V_2) \rangle$ generated by $\text{MCG}(V_1)$ and $\text{MCG}(V_2)$. Minsky \cite{Minsky:58} Question 5.2] asked when this subgroup is the free product with amalgamated subgroup $\text{MCG}(V_1) \cap \text{MCG}(V_2)$. A partial answer to this question was given by Bestvina-Fujiwara \cite{Bestvina-Fujiwara:9}.

(4) The mapping class group $\text{MCG}(M, \Sigma)$ of the pair $(M, \Sigma)$. This contains $\text{MCG}(M, V_1, V_2)$ as a subgroup of index 1 or 2. The result of Scharlemann-Tomova \cite{Scharlemann-Tomova:74} says that the natural homomorphism from $\text{MCG}(M, \Sigma)$ to $\text{MCG}(M)$ is surjective if the Hempel distance $d(\Sigma)$ (see \cite{Hempel:69}) is greater than $2g(\Sigma)$. On the other hand, it is proved by Johnson \cite{Johnson:50}, improving the result of Namazi \cite{Namazi:69}, that the natural homomorphism $\text{MCG}(M, \Sigma) \to \text{MCG}(M)$ is injective if the Hempel distance $d(\Sigma)$ is greater than 3. Hence, the natural homomorphism gives an isomorphism $\text{MCG}(M, \Sigma) \cong \text{MCG}(M)$ if $g(\Sigma) \geq 2$ and $d(\Sigma) > 2g(\Sigma)$. Building on the work of McCullough-Miller-Zimmermann \cite{McCullough-Miller-Zimmermann:70} on finite group actions on handlebodies, finite group actions on the pair $(M, \Sigma)$ are extensively studied (see Zimmermann \cite{Zimmermann:95, Zimmermann:96} and references therein).

(5) The subgroup $G(M, \Sigma) := \text{ker}(\text{MCG}(M, \Sigma) \to \text{MCG}(M))$, which forms a subgroup of the group $\Gamma(M, \Sigma)$. We can write this group as

$$G(M, \Sigma) = \{ [f] \in \text{MCG}(\Sigma) \mid j \circ f \text{ is ambient isotopic to } j \},$$

where $j : \Sigma \to M$ is the inclusion map, and thus we can think of $\Gamma(M, \Sigma)$ as a “homotopy version” of $G(M, \Sigma)$. Johnson-Rubinstein \cite{Johnson-Rubinstein:51} gave systematic constructions of periodic, reducible, pseudo-Anosov elements in this group. Johnson-McCullough \cite{Johnson-McCullough:52} called this group the Goeritz group instead of the one we described in (2), and they used this group to study the homotopy type of the space of Heegaard surfaces. In particular, they prove that if $\Sigma$ is a Heegaard surface of a closed, orientable, aspherical 3-manifold $M$, then, except the case where $M$ is a non-Haken infranilmanifold, the exact sequence \cite{Johnson-McCullough:52} in the introduction is refined to the following exact sequence

$$1 \to Z(\pi_1(M)) \to \mathcal{M}(M, \Sigma) \to G(M, \Sigma) \to 1,$$

where $\mathcal{M}(M, \Sigma)$ is the smooth motion group of $\Sigma$ in $M$ \cite[Corollary 1]{Johnson-McCullough:52}.

(6) The group $\Gamma(V_i) := \text{ker}(\text{MCG}(V_i) \to \text{Out}(\pi_1(V_i)))$. As noted in the introduction, the group $\Gamma(V_i)$ is identified with the group $\Gamma(V_i, \Sigma) < \text{MCG}(\Sigma)$. It was shown by Luft \cite{Luft:67} that its index-2 subgroup $\Gamma^+(V_i) := \text{ker}(\text{MCG}^+(V_i) \to$
Out(π_1(V_i))) is the **twist group**, that is, the subgroup of MCG^+(V_i) generated by the Dehn twists about meridian disks. McCullough [69] proved that Γ(V_i) is not finitely generated by showing that it admits a surjection onto a free abelian group of infinite rank. A typical orientation-reversing element of Γ(V_i) (⊂ MCG(Σ)) is the restriction to Σ = ∂V_i of a vertical I-bundle involution of V_i.

(7) The group ⟨Γ(V_1), Γ(V_2)⟩ generated by Γ(V_1) and Γ(V_2), which is contained in Γ(M, Σ). It was proved by Bowditch-Ohshika-Sakuma in [78, Theorem B] (see also Bestvina-Fujiwara [9, Section 3]) that its orientation-preserving subgroup ⟨Γ^+(V_1), Γ^+(V_2)⟩ is the free product Γ^+(V_1) * Γ^+(V_2) if the Hempel distance d(Σ) is high enough. (The question of whether the same conclusion holds for ⟨Γ(V_1), Γ(V_2)⟩ is still an open question.)

In summary, the subgroups of MCG(Σ) introduced above are related as follows:

\[ G(M, Σ) < Γ(M, Σ) ∩ MCG(M, Σ) < Γ(M, Σ), \]
\[ ⟨Γ(V_1), Γ(V_2)⟩ < Γ(M, Σ) ∩ ⟨MCG(V_1), MCG(V_2)⟩ < Γ(M, Σ). \]

As noted in the introduction, our interest in Γ(M, Σ) was motivated by Minsky’s Question 0.1 and its refinement Question 0.2, and our main concern is Question 0.3(1) about the relationship between Γ(M, Σ) and its subgroup ⟨Γ(V_1), Γ(V_2)⟩. We end this section by giving an answer to Questions 0.2 and 0.3 in two very special cases.

**Example 5.1.** Let \(S^3 = V_1 ∪_Σ V_2\) be the genus-\(g\) Heegaard splitting of \(S^3\). Recall that the (orientation-preserving) mapping class group MCG^+(Σ) is generated by the Dehn twists about certain \(3g - 1\) simple closed curves on Σ by Lickorish [66], where \(g\) is the genus of Σ. Since we can find those simple closed curves in ∆, we have \(⟨Γ^+(V_1), Γ^+(V_2)⟩ = MCG^+(Σ)\). It is thus easy to see that

\[ ⟨Γ(V_1), Γ(V_2)⟩ = Γ(S^3, Σ) = MCG(Σ) \]

and

\[ ⟨Γ(V_1), Γ(V_2)⟩ Δ = Γ(S^3, Σ) Δ = S(Σ) = Z. \]

We note that the group Γ(M, Σ) detects the 3-sphere as in the following meaning.

**Proposition 5.2.** Let \(M = V_1 ∪_Σ V_2\) be a Heegaard splitting of a closed, orientable 3-manifold. Then we have Γ(M, Σ) = MCG(Σ) if and only if \(M = S^3\).

**Proof.** This is straightforward from Corollary [4.4] and the Poincaré conjecture proved by Perelman [82] [83] [84].

**Example 5.3.** Let \(M = #_g(S^2 × S^1)\), and \(M = V_1 ∪_Σ V_2\) the genus-\(g\) Heegaard splitting. In this case, \(M\) is the double of the handlebody \(V_1\), and thus \(Δ = Δ_i = Z\) (\(i = 1, 2\)). Further, we can check easily that

\[ Γ(V_i) = ⟨Γ(V_1), Γ(V_2)⟩ = Γ(M, Σ) \]
and
\[ \langle \Gamma(V_1), \Gamma(V_2) \rangle \Delta = \Gamma(M, \Sigma) \Delta = Z. \]

In the above easy examples, the group \( \langle \Gamma(V_1), \Gamma(V_2) \rangle \) coincides with the whole group \( \Gamma(M, \Sigma) \) in an obvious way. However, this is not the case in general, as indicated in the introduction and proved in Theorem 8.1.

6. OPEN BOOK ROTATIONS

In this section, we first recall the definition of an open book decomposition and the Heegaard splitting obtained from an open book decomposition. We then introduce two homotopy motions of the Heegaard surface, the “half book rotation” \( \rho \) and the “unilateral book rotation” \( \sigma \), which play key roles in the subsequent two sections.

Let \( M \) be a closed, orientable 3-manifold. Recall that an open book decomposition \( (L, \pi) \) of \( \partial \Sigma \) of \( \Sigma = \Sigma_0 \cup \Sigma_1 \) in an obvious way. However, this is not the case in general, as indicated in the introduction and proved in Theorem 8.1.

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by

$$\sigma(t)(x) = \begin{cases} r_t(x) & (x \in \Sigma_0) \\ x & (x \in \Sigma_{1/2}) \end{cases},$$

see Figure 4. We call $\rho$ and $\sigma$, respectively, the half book rotation and the unilateral book rotation associated with the open book decomposition $(L, \pi)$ (or $(\Sigma_0, \varphi)$).

The elements of the group $\Gamma(M, \Sigma)$ obtained as the terminal ends $\rho(1) = \partial_+ (\rho)$ and $\sigma(1) = \partial_+ (\sigma)$ play a key role in the proof of the main Theorem 8.1. We note that $\rho(1)$ is orientation-reversing whereas $\sigma(1)$ is orientation-preserving, and they are related as follows.

Lemma 6.1. $\sigma(1) = \rho(1) \circ h$, where $h$ is the restriction to $\Sigma$ of the vertical $I$-bundle involution on $V_1$ with respect to the natural $I$-bundle structure given by $(L, \pi)$.

Proof. Under the identification $(M, L) = (\Sigma_0 \times \mathbb{R}, \partial \Sigma_0 \times \mathbb{R})/\sim$, the following formulas hold for every $x \in \Sigma_0$.

$$h([x, 0]) = [x, 1/2], \quad h([x, 1/2]) = [x, 0],$$

$$\rho(1)([x, 0]) = [x, 1/2], \quad \rho(1)([x, 1/2]) = [x, 1] = [\varphi^{-1}(x), 0],$$

$$\sigma(1)([x, 0]) = [x, 1] = [\varphi^{-1}(x), 0], \quad \sigma(1)([x, 1/2]) = [x, 1/2].$$

Figure 3. The homotopy motion $\rho = \{f_t\}_{t \in I}$.

Figure 4. The homotopy motion $\sigma = \{g_t\}_{t \in I}$.
By using these formulas, we see that the following hold for every $x \in \Sigma_0$, which in turn imply the desired identity.

$$\rho(1) \circ h([x, 0]) = \rho(1)([x, 1/2]) = [\phi^{-1}(x), 0] = \sigma(1)([x, 0]),$$

$$\rho(1) \circ h([x, 1/2]) = \rho(1)([x, 0]) = [x, 1/2] = \sigma(1)([x, 1/2]).$$

□

For open book decompositions with trivial monodromies, we have the following lemma. Though it should be well-known, we provide a brief proof here, for this plays an important role in Section 7.

**Lemma 6.2.** Let $M$ be a closed, orientable 3-manifold that admits an open book decomposition $(\Sigma_0, \text{id}_{\Sigma_0})$ with trivial monodromy $\text{id}_{\Sigma_0}$, where $\Sigma_0$ is a compact, connected, orientable surface embedded in $M$. Let $\Sigma$ be the Heegaard surface of $M$ associated with the open book decomposition $(\Sigma_0, \text{id}_{\Sigma_0})$. Then the following hold.

1. $M \cong \#_g(S^2 \times S^1)$, where $g$ is the first Betti number of $\Sigma_0$, and $\Sigma$ is the unique minimal genus Heegaard surface of $M$.

2. The unilateral book rotation $\sigma$ associated with $(\Sigma_0, \text{id}_{\Sigma_0})$ determines a non-trivial element of $\mathcal{K}(M, \Sigma)$ of degree 1 under a suitable orientation convention.

**Proof.** Let $\{\delta_i\}_{1 \leq i \leq g}$ be a complete non-separating arc system of $\Sigma_0$, namely a family of disjoint non-separating arcs which cuts $\Sigma_0$ into a disk. Then the image of $\{\delta_i \times \mathbb{R}\}_{1 \leq i \leq g}$ in $M = (\Sigma_0 \times \mathbb{R}, \partial \Sigma_0 \times \mathbb{R})/\sim$ gives a family of disjoint non-separating spheres which cut $M$ into a 3-ball. Hence $M \cong \#_g(S^2 \times S^1)$ and $\Sigma$ is a genus $g$ Heegaard surface of $M$. Since $\#_g(S^2 \times S^1)$ admits a unique Heegaard splitting of genus $g$ by Waldhausen [92], Bonahon-Otal [12] and Haken [39], we obtain the assertion (1). Since the monodromy of the open book decomposition is the identity map, the terminal end $\sigma(1)$ of $\sigma$ is the identity map. Thus $\sigma$ determines an element of $\mathcal{K}(M, \Sigma)$. Obviously, $\deg(\sigma) = \deg(\hat{\sigma} : \Sigma \times S^1 \to M) = 1$, and so we obtain the assertion (2). □

7. **The group $\mathcal{K}(M, \Sigma)$ for Heegaard surfaces of closed orientable 3-manifolds**

Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold, and $j : \Sigma \to M$ the inclusion map. Recall the homomorphism $\deg : \mathcal{K}(M, \Sigma) \to \mathbb{Z}$ introduced in Definition 2.22 and the fact that this homomorphism does not vanish if and only if the pair $(M, \Sigma)$ is dominated by $\Sigma \times S^1$ (cf. Definition 0.4). In this section, we discuss the problem of which pair $(M, \Sigma)$ of a closed, orientable 3-manifold and its Heegaard surface $\Sigma$ is dominated by $\Sigma \times S^1$.

For irreducible 3-manifolds, we obtain the following complete information including the structure of the group $\mathcal{K}(M, \Sigma)$, where $\Phi : \mathcal{K}(M, \Sigma) \to \mathbb{Z}(\pi_1(M))$ is the the homomorphism introduced in Definition 2.28 (Note that, since $\Sigma$ is a Heegaard
surface of $M$, we have $j_*(\pi_1(\Sigma)) = \pi_1(M)$, thus, the codomain of $\Phi$ is the center $Z(\pi_1(M))$ of the fundamental group of $M$.)

**Theorem 7.1.** Let $M$ be a closed, orientable, irreducible 3-manifold and $\Sigma$ a Heegaard surface of $M$.

1. Suppose that $M$ is aspherical. Then $(M, \Sigma)$ is not dominated by $\Sigma \times S^1$. To be precise, $\Phi$ gives an isomorphism $K(M, \Sigma) \cong Z(\pi_1(M))$, and the homomorphism $\deg : K(M, \Sigma) \to \mathbb{Z}$ vanishes. Thus if $M$ is a Seifert fibered space with orientable base orbifold, then $K(M, \Sigma)$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$ according to whether $M$ is the 3-torus $T^3$ or not; otherwise, $K(M, \Sigma)$ is the trivial group.

2. Suppose that $M$ is non-aspherical, or equivalently, $M$ has the geometry of $S^3$. Then $(M, \Sigma)$ is dominated by $\Sigma \times S^1$. To be precise, the following holds.

   (i) If $g(\Sigma) \geq 2$, then the product homomorphism $\Phi \times \deg$ induces an isomorphism $K(M, \Sigma) \cong \pi_1(M) \times |\pi_1(M)| \cdot \mathbb{Z}$.

   (ii) If $g(\Sigma) \leq 1$, then the homomorphism $\deg$ induces an isomorphism $K(M, \Sigma) \cong \pi_1(M) \cdot \mathbb{Z}$.

The proof of Theorem 7.1 will be given in Subsections 7.1 and 7.2.

For 3-manifolds which are not necessarily irreducible, we obtain the following partial result, whose proof will be given in Subsection 7.3.

**Theorem 7.2.** Let $M$ be a closed, orientable 3-manifold and $\Sigma$ a Heegaard surface of $M$.

1. If $M$ contains an aspherical prime summand, then $(M, \Sigma)$ is not dominated by $\Sigma \times S^1$.

2. If $M = \#_g(S^2 \times S^1)$ for some $g \geq 1$, then $(M, \Sigma)$ is dominated by $\Sigma \times S^1$. To be precise, $\deg(K(M, \Sigma)) = \mathbb{Z}$.

3. If $M = \mathbb{RP}^3 \# \mathbb{RP}^3$, then $(M, \Sigma)$ is dominated by $\Sigma \times S^1$. To be precise, $\deg(K(M, \Sigma)) = 2\mathbb{Z}$.

By Kneser-Milnor prime decomposition theorem [58, 71] (cf. [43, 48]), every closed, orientable 3-manifold $M$ admits a unique prime decomposition, and by the geometrization theorem established by Perelman [82, 83, 84] (see [10, 19, 57, 74, 75] for exposition), each prime factor admits a unique decomposition by tori into geometric manifolds, i.e., those which have one of Thurston’s 8 geometries. This together with the sphere theorem implies that for a closed, orientable 3-manifold $M$, the following three conditions are equivalent: (i) $M$ is aspherical, (ii) $M$ is irreducible and $\pi_1(M)$ is not finite, (iii) the universal covering space of $M$ is homeomorphic to $\mathbb{R}^3$. If $M$ is non-aspherical then either $M$ admits the geometry of $S^3$ or $S^2 \times \mathbb{R}$, or $M$ is non-prime (cf. [10, Chapter 1]). Here $\mathbb{RP}^3 \# \mathbb{RP}^3$ is the unique geometric 3-manifold which is non-prime. Thus, Theorems 7.1 and 7.2 especially imply the following corollary.
Corollary 7.3. Let $M$ be a closed, orientable, 3-manifold which is either prime or geometric, and let $\Sigma$ a Heegaard surface of $M$. Then $(M, \Sigma)$ is dominated by $\Sigma \times S^1$ if and only if $M$ is non-asperpherical, namely $M$ admits the geometry of $S^3$ or $S^2 \times \mathbb{R}$.

We do not know, however, what happens when $M$ is non-prime and $M$ has no aspherical prime summand, in other words, each prime summand of $M$ is $S^2 \times S^1$, or has the geometry of $S^3$, except when $M = \#_g(S^2 \times S^1)$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$.

Question 7.4. Let $M = \#_{i=1}^n M_i$ ($n \geq 2$) be a closed, orientable non-prime 3-manifold such that each $M_i$ is either $S^2 \times S^1$ or admits the geometry of $S^3$. When is a Heegaard surface $\Sigma$ of $M$ dominated by $\Sigma \times S^1$?

The remainder of this section is devoted to the proof of Theorems 7.1 and 7.2.

7.1. Proof of Theorem 7.1(1)

Suppose that $M$ is aspherical. Since a closed, orientable, irreducible 3-manifold that admits Heegaard splitting of genus at most 1 is either $S^3$ or a lens space, which are not aspherical, we have $g(\Sigma) \geq 2$ and $\mathcal{K}(M, \Sigma) \cong \pi_1(C(\Sigma, M), j)$ by Lemma 2.1. We see by Lemma 2.9 that $\Phi : \mathcal{K}(M, \Sigma) \to Z(\pi_1(M))$ is an isomorphism. If $Z(\pi_1(M)) = 1$, there is nothing to prove. Suppose that $Z(\pi_1(M))$ is non-trivial. By the Seifert fibered space conjecture proved by Gabai [34] and Casson-Jungreis [20], $M$ is then a Seifert fibered space with orientable base orbifold. If $M$ is not the 3-torus $T^3$, then, $M$ admits a unique Seifert fibration with orientable base orbifold, and the center $Z(\pi_1(M)) \cong \mathbb{Z}$ is generated by an element represented by a regular fiber of the Seifert fibration of $M$, see Jaco [18, VI]. When $M = T^3$, we have $Z(\pi_1(M)) = \pi_1(M) \cong \mathbb{Z}^3$, and any primitive element of $Z(\pi_1(M))$ can be realized as a regular fiber of a Seifert fibration of $M$. In any case, let $z$ be a primitive element of $Z(\pi_1(M))$. Equip $M$ with a Seifert fibration where $z$ is represented by its regular fiber. Fix a faithful action of $S^1 = \mathbb{R}/\mathbb{Z}$ on $M$ that is compatible with the Seifert fibration. Let $\alpha_z$ be the homotopy motion of $\Sigma$ defined by $\alpha_z(t)(x) = t \cdot x$ for $t \in I$ and $x \in \Sigma$, where $t \cdot x$ is the image of $x$ by the action of $t \in S^1$. Then we see that $\alpha_z$ determines an element of $\mathcal{K}(M, \Sigma)$ and that $\Phi([\alpha_z]) = z$. Thus we have only to show that the degree of the map $\hat{\alpha}_z : \Sigma \times S^1 \to M$ is 0. To this end, let $Y_1$ be a spine of the handlebody $V_1$ in $M$ bounded by the Heegaard surface $\Sigma$, and let $\{r_t\}_{t \in I}$ be a strong deformation retraction of $V_1$ onto $Y_1$, namely $r_0 = \text{id}_{V_1}$, $r_t|_{\partial Y_1} = \text{id}_{Y_1}$ ($t \in I$), and $r_t(V_1) = Y_1$. Define a map $H = \{h_s\}_{s \in I} : (\Sigma \times S^1) \times I \to M$ by $H(x, t, s) = t \cdot r_s(x)$. Then $h_0 = \hat{\alpha}_z$ and $h_1(\Sigma \times S^1) = S^1 \cdot Y_1$. Since $Y_1$ is 1-dimensional, the image $h_1(\Sigma \times S^1)$ is a proper subset of $M$. Hence $\deg(\hat{\alpha}_z) = \deg(h_1) = 0$. This completes the proof of Theorem 7.1(1).
7.2. Proof of Theorem 7.1(2)

Suppose that $M$ is non-asperical. Since $M$ is irreducible by assumption of the theorem, the geometrization theorem implies that $M$ admits the geometry of $S^3$, namely $M \cong S^3/G$ for some finite subgroup $G$ of SO(4) acting freely on $S^3$.

Suppose first that $g(\Sigma) \geq 2$. Then $\mathcal{K}(M, \Sigma) \cong \pi_1(C(\Sigma, M), j)$ by Lemma 2.1. We show that the homomorphism $\Phi \times \deg : \pi_1(C(\Sigma, M), j) \to Z(G) \times \mathbb{Z}$ is injective and that its image is $Z(G) \times |G| \cdot \mathbb{Z}$.

To see the injectivity, pick an element $[\alpha] \in \ker(\Phi \times \deg)$, and consider the maps $\hat{\alpha}$ and $\hat{e} : \Sigma \times S^1 \to M$ induced by $\alpha$ and the identity motion $e$, respectively. Since $[\alpha] \in \ker \Phi$, the homomorphisms $\hat{\alpha}_* \times \pi_1(\Sigma \times S^1) \to \pi_1(M)$ are equivalent. Since $\deg(\hat{\alpha}) = \deg([\alpha]) = 0 = \deg([e]) = \deg(\hat{e})$, we see by Proposition 2.6 that $\hat{\alpha}$ and $\hat{e}$ are homotopic. Thus $[\alpha] + [e]$ are conjugate and so identical in $\pi_1(C(\Sigma, M), j)$. Hence $\Phi \times \deg$ is injective.

Next we show that the image of $\Phi \times \deg$ is equal to $Z(G) \times |G| \cdot \mathbb{Z}$. To this end, we need the lemma below. Though this should be well-known, we provide a brief proof here, for we could not find a reference.

**Lemma 7.5.** The center $Z(G)$ is the cyclic group generated by the homotopy class of a regular orbit of a circle action on $M = S^3/G$.

**Proof.** If $M$ is a lens space, then $M$ has a genus-1 Heegaard splitting $V_1 \cup V_2$, and there is a circle action on $M$ such that a regular orbit forms a core circle of $V_1$. Since $G = \pi_1(M)$ is the cyclic group generated by the homotopy class of the regular orbit, the lemma obviously holds. So we may assume $M$ is not a lens space. Then $M$ admits a circle action such that the orbit space $\mathcal{O}$ is the 2-dimensional spherical orbifold $S^2(p, q, r)$ where $(p, q, r)$ is $(2, 2, r)$ with $r \geq 2$ or $(2, 3, r)$ with $r \in \{3, 4, 5\}$ (see [SS]). Let $z$ be an element of $G$ represented by a regular orbit of the circle action. Then the subgroup $\langle z \rangle$ is contained in $Z(G)$ and the quotient $G/\langle z \rangle$ is identified with the orbifold fundamental group $\pi_1^{\text{orb}}(\mathcal{O})$. The center of this group is trivial unless $\mathcal{O} = S^2(2, 2, r)$ for some even integer $r \geq 2$. Thus the assertion holds except for this case.

Let $\psi : S^3 \to SO(3)$ and $\phi : S^3 \times S^3 \to SO(4)$ be the universal covering projection. Let $\mathbb{Z}_m$ and $\mathbb{D}_r$, respectively, be the order-$m$ cyclic subgroup and the order-$2r$ dihedral subgroup of $SO(3)$, which are unique up to conjugation. Set $\hat{\mathbb{Z}}_m = \psi^{-1}(\mathbb{Z}_m)$ and $\hat{\mathbb{D}}_r = \psi^{-1}(\mathbb{D}_r)$. Then, by [SS Theorem 4.11], we have $G = \phi(\hat{\mathbb{Z}}_m \times \hat{\mathbb{D}}_r)$ after conjugation, in the exceptional case where $\mathcal{O} = S^2(2, 2, r)$ with $r \geq 2$ even. Moreover, the subgroup $\langle z \rangle$ is identified with the subgroup $\phi(\hat{\mathbb{Z}}_m \times \{1\})$. It is easy to check that any element of $G$ that does not belong to $\phi(\hat{\mathbb{Z}}_m \times \{1\})$ is not central. □

By the above lemma, $Z(G)$ is generated by an element $z$ represented by a regular orbit of a circle action on $M$. Consider the homotopy motion $\alpha_z$ of $M$ generated by the circle action, as in Subsection 7.1 and consider the element $[\alpha_z]$ of
\[\pi_1(C(\Sigma, M), j)\) it represents. Then, by the argument in Subsection [7.1] we see \(\Phi([\alpha_x]) = z\) and \(\text{deg}([\alpha_x]) = 0\). Thus we have shown that the image of \(\Phi \times \text{deg}\) contains \(Z(G) \times \{0\}\).

Next, we show that the image of \(\Phi \times \text{deg}\) contains \(\{1\} \times |G| \cdot \mathbb{Z}\). Pick a point \(x\) in the interior of \(\Sigma \times I\), and modify the trivial motion \(e : \Sigma \times I \to M\) on a regular neighborhood \(N\) of \(x\) to the wedge sum \(N \vee S^3 = N \cup_{\{x\}} S^3\), such that (i) \(r|_{\partial N}\) is the identity map, (ii) \(r|_{N - N_0}\) is a homeomorphism onto \(N - \{x\}\), (iii) \(r|_{N_0}\) induces a homeomorphism from \(N_0/\partial N_0\) onto \(S^3\). Let \(p : S^3 \to M\) be the universal covering projection, such that, when regarded as a map from the subspace \(S^3\) of \(N \vee S^3\), it maps the point \(x = N \cap S^3 \subseteq S^3\) into the point \(e(x) \in M\). Then we obtain a continuous map \(e|_N \vee p : N \vee S^3 \to M\). See Figure 5. Let \(\beta : \Sigma \times I \to M\)

![Figure 5](image)

**Figure 5.** The pinching map \(r\) from \(N\) to \(N \vee S^3\), and the map \(e|_N \vee p\) from \(N \vee S^3\) to \(M\).

the continuous map obtained from \(e\) by redefining \(e\) on \(N\) to be the composition \((e|_N \vee p) \circ r\). Then \(\beta\) determines an element \(\pi_1(C(\Sigma, M), j)\) such that \(\Phi([\beta]) = 1\) and \(\text{deg}([\beta]) = |G|\). Thus the image of \(\Phi \times \text{deg}\) contains \(\{1\} \times |G| \cdot \mathbb{Z}\).

On the other hand, Proposition [24] implies, for each \(z \in Z(G)\), that the set \(\text{deg}([\alpha]) = \text{deg}(\hat{\alpha})\), where \([\alpha]\) runs over \(\Phi^{-1}(z)\), is of the form \(d + |G| \cdot \mathbb{Z}\) for some \(d \in \mathbb{Z}\), because the homomorphisms \(\hat{\alpha} : \pi_1(\Sigma \times S^1) \to \pi_1(M)\) induced by \([\alpha]\) \(\in \Phi^{-1}(z)\) are all equivalent. In a preceding paragraph, we have observed that \(\text{deg}([\alpha_x]) = 0\) for the element \([\alpha_x] \in \Phi^{-1}(z)\) (for the generator \(z\) of \(Z(G)\)). Thus we may assume \(d = 0\). Hence the image of \(\Phi \times \text{deg}\) is contained in \(Z(G) \times |G| \cdot \mathbb{Z}\).

Thus we have shown that the image of \(\Phi \times \text{deg}\) is contained in \(Z(G) \times |G| \cdot \mathbb{Z}\) and that it contains both \(Z(G) \times \{0\}\) and \(\{1\} \times |G| \cdot \mathbb{Z}\). Hence the image of \(\Phi \times \text{deg}\) is equal to \(Z(G) \times |G| \cdot \mathbb{Z}\). Since the injectivity of \(\Phi \times \text{deg}\) is already proved, \(\Phi \times \text{deg}\) induces an isomorphism from \(K(M, \Sigma) \cong \pi_1(C(\Sigma, M), j) / \mathcal{F}(\pi_1(J(\Sigma, M), j))\) onto \(Z(G) \times |G| \cdot \mathbb{Z}\), completing the proof of of Theorem [7.1](2) when \(g(\Sigma) \geq 2\).

Suppose that \(g(\Sigma) \leq 1\). Then \(K(M, \Sigma) \cong \pi_1(C(\Sigma, M), j) / \mathcal{F}(\pi_1(J(\Sigma, M), j))\) by Lemma [24]. Note that the preceding argument shows that \(\Phi \times \text{deg}\) induces
an isomorphism \( \pi_1(C(\Sigma, M), j) \cong Z(G) \times |G| \cdot \mathbb{Z} \), where \( \widetilde{\text{deg}} \) is the composition of \( \pi_1(C(\Sigma, M), j) \to \mathcal{K}(M, \Sigma) \) and \( \text{deg} : \mathcal{K}(M, \Sigma) \to \mathbb{Z} \). When \( g(\Sigma) = 1 \), by the argument in the proof of Lemma 3.4, we see that the image of the subgroup \( \mathcal{I}(\pi_1(J(\Sigma, M), j)) \) by the isomorphism is equal to \( Z(G) \). When \( g(\Sigma) = 0 \), \( \Sigma \) is the genus-0 Heegaard splitting of \( M = S^3 \), thus, \( Z(G) = G \). Hence, in any case, we see that \( \text{deg} \) induces an isomorphism \( \mathcal{K}(M, \Sigma) \to |G| \cdot \mathbb{Z} \) by Lemma 2.1, completing the proof of Theorem 7.1(2).

7.3. Proof of Theorem 7.2

Case 1. \( M \) contains a prime aspherical summand. Then \( M \) is a connected sum \( M_1 \# M_2 \) of an aspherical prime manifold \( M_1 \) and another 3-manifold \( M_2 \), which is possibly \( S^3 \). Suppose on the contrary that there is a Heegaard surface \( \Sigma \) of \( M \), such that \( (M, \Sigma) \) admits a \( \Sigma \)-domination \( \phi : (\Sigma \times S^1, \Sigma \times \{0\}) \to (M, \Sigma) \). By Haken’s theorem on Heegaard surfaces of composite manifolds (see [48, Theorem II.7]), \( (M, \Sigma) \) is a pairwise connected sum \( (M_1, \Sigma_1) \# (M_2, \Sigma_2) \) where \( \Sigma_i \) is a Heegaard surface of \( M_i \) \( (i = 1, 2) \). By pinching \( (M_2, \Sigma_2) \) into a point, we obtain from \( \phi \) a \( \Sigma_1 \)-domination of \( (M_1, \Sigma_1) \). This contradicts Theorem 7.1(1). Hence we obtain Theorem 7.2(1).

In order to treat the remaining cases, recall that a stabilization of a Heegaard splitting \( M = V_1 \cup_{\Sigma} V_2 \) (or a Heegaard surface \( \Sigma \subset M \)) is an operation to obtain a Heegaard splitting \( M = V_1' \cup_{\Sigma'} V_2' \) (or a Heegaard surface \( \Sigma' \subset M \)) of higher genus by adding \( V_1 \) a trivial 1-handle, that is, a 1-handle whose core is parallel to \( \Sigma \) in \( V_2 \), and removing that from \( V_2 \).

Lemma 7.6. Let \( M \) be a closed, orientable 3-manifold, \( \Sigma \) a Heegaard surface for \( M \), and \( \Sigma' \) a Heegaard surface obtained by a stabilization from \( \Sigma \). If there exists a degree-d \( \Sigma \)-domination of \( (M, \Sigma) \), then there exists a degree-d \( \Sigma' \)-domination of \( (M, \Sigma') \) as well.

Proof. Suppose that there exists a degree-d \( \Sigma \)-domination \( \phi : (\Sigma \times S^1, \Sigma \times \{0\}) \to (M, \Sigma) \). Without loss of generality we can assume that \( \phi(x, 0) = x \) for any \( x \in \Sigma \). Further, we can assume that the stabilization is performed in a 3-ball \( B \) in \( M \) that intersects \( \Sigma \) in a disk, thus, \( \Sigma - B = \Sigma' - B \). Then there exists a homotopy \( F = \{f_t\}_{t \in I} : \Sigma' \times I \to M \) such that

\begin{enumerate}
  \item \( f_0(x) = x \), for \( x \in \Sigma' \);
  \item \( f_t(x) = x \) for \( x \in \Sigma' - B \), \( t \in I \);
  \item \( f_t(x) \in B \) for \( x \in B \cap \Sigma \), \( t \in I \); and
  \item \( f_1(\Sigma') = \Sigma \).
\end{enumerate}
Using this homotopy, we can define a $\Sigma'$-domination $\phi' : (\Sigma' \times S^1, \Sigma' \times \{0\}) \to (M, \Sigma')$ by
\[
\phi'(x, \theta) = \begin{cases} 
  f_{3\theta}(x) & (0 \leq \theta \leq 1/3) \\
  \phi(f_1(x), 3\theta - 1) & (1/3 \leq \theta \leq 2/3) \\
  f_{3-3\theta}(x) & (2/3 \leq \theta \leq 1).
\end{cases}
\]
Since the homotopy $F$ moves $\Sigma'$ only inside the local 3-ball $B$, the degree of $\phi'$ is $d$.

\[\square\]

Case 2. $M = \#_g(S^2 \times S^1)$ for some $g \geq 1$. By Waldhausen [92], Bonahon-Otal [12] and Haken [39], any Heegaard splitting of $M$ is a stabilization of the unique genus-$g$ Heegaard splitting $M = V_1 \cup_\Sigma V_2$. Therefore, by Lemma 7.6 we may assume $\Sigma$ is the unique genus-$g$ Heegaard surface. Then by Lemma 5.2 there exists a degree-1 $\Sigma$-domination of $(M, \Sigma)$. Since $\deg : K(M, \Sigma) \to \mathbb{Z}$ is a homomorphism, we have $\deg(K(M, \Sigma)) = \mathbb{Z}$, completing the proof of Theorem 7.2(2).

**Remark 7.7.** It is proved in [61, Proposition 4] that there exists a double branched covering map from $\Sigma \times S^1$ to $\#_g(S^2 \times S^1)$ where $\Sigma$ is a closed, orientable surface of genus $g$. (See [77, Lemma 2.3] for a related interesting result.) That map actually gives a domination of the minimal genus Heegaard surface of $\#_g(S^2 \times S^1) \times \Sigma \times S^1$. However, this does not imply the full statement of Theorem 7.2(2), because the map has degree 2 and it gives domination of only the minimal genus Heegaard surface.

Case 3. $M = \mathbb{RP}^3 \# \mathbb{RP}^3$. By Montesinos-Saafont [73] and Haken [39], any Heegaard splitting of $M$ is a stabilization of the unique genus-2 Heegaard splitting $M = V_1 \cup_\Sigma V_2$. Therefore, by Lemma 7.6 we may assume $\Sigma$ is the unique genus-2 Heegaard surface.

We first show that there exists a degree-$d$ $\Sigma$-domination of $(M, \Sigma)$ for any even integer $d$. For this, it is enough to find a degree-2 $\Sigma$-domination of $(M, \Sigma)$.

Let $\tau$ be the antipodal map of $S^2$, and $\eta$ the involution of $S^1$ defined by $\eta(\theta) = -\theta$. Identify $M = \mathbb{RP}^3 \# \mathbb{RP}^3$ with $(S^2 \times S^1)/(\tau \times \eta)$, and let $p : S^2 \times S^1 \to M$ be the covering projection. Thus we can regard $M$ as an $S^2$-bundle over the orbifold $S^1/\eta$ with underlying space $[0, 1/2]$. Choose disjoint disks $D_-$ and $D_+$ in $S^2$ with $\tau(D_-) = D_+$. Let $R = I \times I$ be a rectangle in $\cl(S^2 - (D_+ \cup D_-))$ such that $R \cap D_- = \{0\} \times I$ and $R \cap D_+ = \{1\} \times I$. Then $\bar{V}_1 := ((D_- \cup D_+) \times S^1) \cup (R \times [1/6, 2/6]) \cup (\tau(R) \times [4/6, 5/6])$ is a $(\tau \times \eta)$-invariant handlebody of genus 3, and its exterior $\bar{V}_2 := \cl(M - \bar{V}_1)$ is also a $(\tau \times \eta)$-invariant handlebody of genus 3. Thus the pair $(\bar{V}_1, \bar{V}_2)$ determines a $(\tau \times \eta)$-invariant Heegaard splitting of $S^2 \times S^1$, and it projects to the genus-2 Heegaard splitting $(V_1, V_2)$ of $M$, where $V_i := p(\bar{V}_i) \ (i = 1, 2)$. See Figure 8. We are going to construct a domination of the Heegaard surface $\Sigma := V_1 \cap V_2$, by constructing an equivariant domination of the Heegaard surface $\bar{\Sigma} = p^{-1}(\Sigma) = \bar{V}_1 \cap \bar{V}_2$.

To this end, consider an annulus $A := \delta \times S^1 \subset \bar{\Sigma}$, where $\delta$ is an arc in $\partial D_-$ disjoint from the rectangle $R$. Then there is an open book decomposition $(L, \pi)$ of $S^2 \times S^1$ with $L = \partial A$, such that $A$ is the page $\pi^{-1}(0) \cup L$. Let $\{r_t\}_{t \in \mathbb{R}}$ be the book
rotation with respect to \((L, \pi)\). Now consider the conjugate of the above open book decomposition by the covering involution \(\tau \times \eta\), and let \(\{r_t'\}_{t \in \mathbb{R}}\) be the associated book rotation obtained from \(\{r_t\}_{t \in \mathbb{R}}\) through conjugation by \(\tau \times \eta\). Observe that the page \(A' := (\tau \times \eta)(A)\) is disjoint from \(A\). Since the monodromy of the open book decomposition \((L, \pi)\) is the identity, we can construct a \(\mathbb{Z}/2\mathbb{Z}\)-equivariant map \(\tilde{\phi} : \tilde{\Sigma} \times S^1 \to S^2 \times S^1\) by

\[
\tilde{\phi}(x, t) = \begin{cases} 
    r_t(x) & (x \in A) \\
    r'_t(x) & (x \in A') \\
    x & (x \in \tilde{\Sigma} \setminus (A \cup A')).
\end{cases}
\]

This map naturally induces a \(\Sigma\)-domination \(\phi : (\Sigma \times S^1, \Sigma \times \{0\}) \to (M, \Sigma)\) whose restriction to \(\Sigma \times \{0\}\) is a homeomorphism onto the Heegaard surface \(\Sigma\), that is actually the identity map under a natural identification of the two surfaces. Since the degree of each of \(\tilde{\phi}\), \(p\) and the map \(\tilde{\Sigma} \times S^1 \to \Sigma \times S^1\) is 2, the degree of \(\phi\) is 2.

To show the other direction, suppose that \(\phi\) is a degree-\(d\) map from \(\Sigma \times S^1\) to \(\mathbb{RP}^3 \# \mathbb{RP}^3\). Here we do not need to require that \(\phi(\Sigma \times \{0\})\) is a Heegaard surface of \(\mathbb{RP}^3 \# \mathbb{RP}^3\). Let \(p : \mathbb{RP}^3 \# \mathbb{RP}^3 \to \mathbb{RP}^3\) be a degree-1 map defined by pinching one summand \(\mathbb{RP}^3\) to a 3-ball in the other summand \(\mathbb{RP}^3\). Then the composition \(p \circ \phi\) is a degree-\(d\) map from \(\Sigma \times S^1\) to \(\mathbb{RP}^3\). From Hayat-Legrand-Wang-Zieschang [42, Theorem 2] it follows that \(d\) should be an even number. This completes the proof of Theorem 7.2(3).

8. Gap between \(\Gamma(M, \Sigma)\) and the subgroup \(\langle \Gamma(V_1), \Gamma(V_2) \rangle\)

In this section, we show the following theorem, which gives a partial answer to Question 0.3(2).
Theorem 8.1. Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold $M$ induced from an open book decomposition. If $M$ has an aspherical prime summand, then we have $\langle \Gamma(V_1), \Gamma(V_2) \rangle \preceq \Gamma(M, \Sigma)$.

In fact, we will see that neither $\rho(1)$ nor $\sigma(1)$, defined in Section 6, is contained in $\langle \Gamma(V_1), \Gamma(V_2) \rangle$ under the assumption of Theorem 8.1. To show this, we will define a $\mathbb{Z}^2$-valued invariant $\hat{\text{Deg}}(f)$ for elements $f$ of $\Gamma(M, \Sigma)$, and study its basic properties. We then show, by using Theorem 7.2(1) that it descends to an invariant for elements of $\Gamma(M, \Sigma)$ when $M$ satisfies the assumption of Theorem 8.1.

Remark 8.2. Let $\Sigma$ be a Heegaard surface of a closed, orientable 3-manifold $M$. The existence of a gap between $\langle \Gamma(V_1), \Gamma(V_2) \rangle$ and $\Gamma(M, \Sigma)$ given in the above theorem implies, in particular, that the Seifert-van Kampen-like theorem for the homotopy motion group $\Pi(M, \Sigma)$ is no longer valid as in the following meaning, though $\Pi(M, \Sigma)$ could be regarded as a generalization of the fundamental group (cf. Remark 1.2 and Theorem 4.2). Consider the homotopy motion groups $\Pi(V_1, \Sigma)$ and $\Pi(V_2, \Sigma)$. Recall that the group $\Gamma(V_i)$ $(i = 1, 2)$ is the image of the natural map $\partial_+ : \Pi(V_i, \Sigma) \to \text{MCG}(\Sigma)$. Since a homotopy motion of $\Sigma$ in $V_i$ is that of $\Sigma$ in $M$ as well, we have a canonical map $I_i : \Pi(V_i, \Sigma) \to \Pi(M, \Sigma)$. Since the manifold $M$ is obtained by gluing $V_1$ and $V_2$ along $\Sigma$, one might expect that

$$\langle I_1(\Pi(V_1, \Sigma)), I_2(\Pi(V_2, \Sigma)) \rangle = \Pi(M, \Sigma),$$

that is, $\Pi(M, \Sigma)$ is generated by elements of $I_1(\Pi(V_1, \Sigma))$ and $I_2(\Pi(V_2, \Sigma))$ as we see in the Seifert-van Kampen theorem. The incoincidence $\langle \Gamma(V_1), \Gamma(V_2) \rangle \preceq \Gamma(M, \Sigma)$, however, implies that this is not true because

$$\partial_+(\langle I_1(\Pi(V_1, \Sigma)), I_2(\Pi(V_2, \Sigma)) \rangle) = \langle \Gamma(V_1), \Gamma(V_2) \rangle$$

while

$$\partial_+(\Pi(M, \Sigma)) = \langle \Gamma(M, \Sigma) \rangle,$$

and they are different.

Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold $M$. We will adopt the following convention. Given an orientation of $M$, or equivalently, a fundamental class $[M] \in H_3(M)$, we always choose the fundamental classes $[V_i] \in H_3(V_i, \partial V_i)$ $(i = 1, 2)$ and $[\Sigma] \in H_2(\Sigma)$ so as to satisfy the following.

Convention 8.3. $[M] = [V_2] - [V_1]$ and $[\Sigma] = [\partial V_1] = [\partial V_2]$, where $[\partial V_i]$ is the one induced from $[V_i]$.

By $[I] \in H_1(I; \partial I)$ we always mean the fundamental class corresponding to the canonical orientation of $I$. 

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We define a map \( \widehat{\text{deg}} : \Pi(M, \Sigma) \to \mathbb{Z}^2 \) as follows. First, we fix an orientation of \( M \). Let \( \alpha : \Sigma \times I \to M \) be a homotopy motion. Consider the homomorphism

\[
\alpha_* : H_3(\Sigma \times I, \Sigma \times \partial I) \to H_3(M, \Sigma) \cong H_3(V_1, \partial V_1) \oplus H_3(V_2, \partial V_2),
\]

and let \((d_1, d_2)\) be the pair of integers such that \( \alpha_*([\Sigma \times I]) = d_1[V_1] + d_2[V_2] \), where \([\Sigma \times I]\) is the cross product of \([\Sigma]\) and \([I]\). This pair is uniquely determined by the equivalence class of \( \alpha \). We then define \( \widehat{\text{deg}}(\alpha) = (d_1, d_2) \). We note that this invariant does not depend on the orientation of \( M \) under the above convention. The following examples can be easily checked.

**Example 8.4.** Let \( f \) be a Dehn twist about a meridian of \( V_1 \). This is an element of \( \Gamma^+(V_1) \). In fact, we can construct a homotopy motion \( \alpha \) of \( \Sigma = \partial V_1 \) in \( V_1 \) with terminal end \( f \) as follows. As in Subsection 7.1, let \( Y \) be a spine of \( V_1 \) and let \( \{r_t\}_{t \in \mathbb{R}} \) be a strong deformation retraction of \( V_1 \) onto \( Y \). Define a homotopy motion \( \alpha \) of \( \Sigma \) in \( M = V_1 \cup V_2 \) by

\[
\alpha(t)(x) = \begin{cases} 
  r_{2t}(x) & (0 \leq t \leq 1/2) 
  \vspace{1em}
  r_{-2t}(f(x)) & (1/2 \leq t \leq 1).
\end{cases}
\]

Then we have \( \widehat{\text{deg}}(\alpha) = (0, 0) \).

**Example 8.5.** Let \( h \) be a vertical \( I \)-bundle involution on \( V_1 \), which is an element of \( \Gamma^-(V_1) \). Then the linear homotopy joining each \( x \in \Sigma \) with \( h(x) \) in the fiber \( I_x \) containing both \( x \) and \( h(x) \) determines a homotopy motion \( \alpha \) of \( \Sigma = \partial V_1 \) in \( V_1 \) with terminal end \( h|_{\Sigma} \). Regarding \( \alpha \) as a homotopy motion of \( \Sigma \) in \( M = V_1 \cup V_2 \), we have \( \widehat{\text{deg}}(\alpha) = (-2, 0) \) as shown below. Note that if \( x \in \Sigma \) projects to an interior point of the base surface of the \( I \)-bundle structure then the inverse image of the fiber \( I_x \) by \( \alpha \) is the disjoint union \((\{x\} \times I) \sqcup (\{h(x)\} \times I) \subset \Sigma \times I \). Moreover, for a small regular neighbourhood \( B \) of \( x \) in the interior of \( I_x \), the inverse image \( \alpha^{-1}(B) \) is the disjoint union of two 3-balls \( B_1 \) and \( B_2 \) such that the restriction of \( \alpha|_{B_i} : B_i \to B \) is a homeomorphism for \( i = 1, 2 \). By Convention 3.3, \( \alpha|_{B_i} \) is orientation-reversing with respect to the orientations induced from the fundamental classes \([\Sigma \times I]\) and \([V_1] \). Since the degree is equal to the sum of local degrees (see for example 11 Section 3.3, Exercise 8), we have \( \alpha_*([\Sigma \times I]) = -2[V_1] \) for the map \( \alpha : \Sigma \times I \to V_1 \). This implies \( \widehat{\text{deg}}(\alpha) = (-2, 0) \), because the image of the map \( \alpha : \Sigma \times I \to M \) does not contain \( V_2 \). Similarly, if \( h \) is a vertical \( I \)-bundle involution on \( V_2 \), then we have \( \widehat{\text{deg}}(\alpha) = (0, -2) \) for the corresponding homotopy motion \( \alpha \) of \( \Sigma \) in \( M \).

**Example 8.6.** Recall the homomorphism \( \deg : \mathcal{K}(M, \Sigma) \to \mathbb{Z} \) introduced in Definition 2.2. For each \( \alpha \in \mathcal{K}(M, \Sigma) \), we have

\[
\widehat{\text{deg}}(\alpha) = (-\deg(\alpha), \deg(\alpha)).
\]

**Example 8.7.** Suppose that \( M = V_1 \cup \Sigma V_2 \) is the Heegaard splitting induced from an open book decomposition, and let \( \rho \) and \( \sigma \), respectively, be the half rotation and
the unilateral rotation of $\Sigma$ associated with the open book decomposition. Then we check that

$$\hat{\text{Deg}}(\rho) = (-1, -1), \quad \hat{\text{Deg}}(\sigma) = (-1, 1)$$

by an argument similar to that in Example 8.5. However, since the orientation convention is slightly involved, we give a detailed explanation. As explained in Section 6, we have the natural identification $\hat{(M, L)} = (F \times \mathbb{R}, \partial F \times \mathbb{R}) / \sim$, where $L$ is the binding of the open book decomposition. We equip $\mathbb{R}$ with the standard orientation, and then we orient $F$ so that the orientation of $M$ is compatible with that of $F \times \mathbb{R}$, where we note that each chain in $C_*(F \times I)$ naturally projects to that in $C_*(M) = C_*(\hat{(F \times I) / \sim})$. By abuse of notation, we shall not distinguish notationally between an oriented manifold and an $n$-chain representing it. Under this identification, we can write the chains $V_1, V_2, \Sigma$ and $M$ according to Convention 8.3 as follows:

- $V_1 = -(F \times [0, 1/2]) / \sim$;
- $V_2 = (F \times [1/2, 1]) / \sim$; and
- $\Sigma = (F_0 - F_1/2) / \sim = \partial V_1 = \partial V_2$, where $F_t : = F \times \{t\}$.
- $M = (F \times I) / \sim = -V_1 + V_2$.

Now, consider the chain map $\rho_\# : C_3(\Sigma \times I) \rightarrow C_3(M)$ induced by the half-rotation $\rho$. By the definition of $\rho$, we have

$$\rho_\#(\Sigma \times I) = \rho_\#(((F_0 - F_1/2) / \sim) \times I) = \rho_\#((F_0 / \sim) \times I - (F_1/2 / \sim) \times I)$$

$$= ((F \times [0, 1/2]) / \sim) - ((F \times [1/2, 1]) / \sim) = -V_1 - V_2.$$

Hence $\hat{\text{Deg}}(\rho) = (-1, -1)$.

Similarly, for the unilateral rotation $\sigma$, we have

$$\sigma_\#(\Sigma \times I) = \sigma_\#((F_0 / \sim) \times I - (F_1/2 / \sim) \times I)$$

$$= ((F \times I) / \sim) - 0 = M = -V_1 + V_2.$$

Hence $\hat{\text{Deg}}(\sigma) = (-1, 1)$.

Examples 8.4, 8.5, and 8.7 allow us to predict that $\rho(1)$ and $\sigma(1)$ should give a gap between $\Gamma(M, \Sigma)$ and the subgroup $\langle \Gamma(V_1), \Gamma(V_2) \rangle$ for a Heegaard splitting $M = V_1 \cup \Sigma V_2$ induced from an open book decomposition. We are going to verify that when $M$ has an aspherical prime summand.

**Lemma 8.8.** The invariant $\hat{\text{Deg}} : \Pi(M, \Sigma) \rightarrow \mathbb{Z}^2$ has the following properties.

1. Let $\alpha$ be an element of $\Pi(M, \Sigma)$, and let $\hat{\text{Deg}}(\alpha) = (d_1, d_2)$. Then we have

$$d_1 + d_2 = -1 + \deg(\partial_+(\alpha)).$$

2. For any pair $\alpha, \beta$ of elements of $\Pi(M, \Sigma)$, we have

$$\hat{\text{Deg}}(\alpha \cdot \beta) = \hat{\text{Deg}}(\alpha) + \deg(\partial_+(\alpha)) \cdot \hat{\text{Deg}}(\beta).$$
In the above lemma, \( \deg(\partial_+(\alpha)) \in \{\pm 1\} \) is the degree of the terminal end \( \partial_+(\alpha) = \alpha(1) \in \Gamma(M, \Sigma) < \text{MCG}(\Sigma) \) of \( \alpha \), as a mapping class of the closed, orientable surface \( \Sigma \).

**Proof.** (1) Let \( \alpha \) be a homotopy motion of \( \Sigma \) in \( M \) with terminal end \( \alpha(1) = f \). Consider the following commutative diagram:

\[
\begin{array}{ccc}
H_3(\Sigma \times I, \Sigma \times \partial I) & \xrightarrow{\alpha_*} & H_3(M, \Sigma) \\
\downarrow & & \downarrow \\
H_2(\Sigma \times \partial I) & \xrightarrow{(\alpha|_{\Sigma \times \partial I})_*} & H_2(\Sigma),
\end{array}
\]

where the vertical arrows are the connecting homomorphisms. By the connecting homomorphism \( H_3(\Sigma \times I, \Sigma \times \partial I) \to H_2(\Sigma \times \partial I) \), the fundamental class \( [\Sigma \times I] \) is mapped to \(-[\Sigma \times \{0\}] + [\Sigma \times \{1\}]\). The homomorphism \((\alpha|_{\Sigma \times \partial I})_* : H_2(\Sigma \times \partial I) \to H_2(\Sigma)\) then takes this to

\[
j_*([-\Sigma]) + (j \circ f)_*([\Sigma]) = (-1 + \deg(f))[\Sigma],
\]

where \( j : \Sigma \to M \) is the inclusion map. On the other hand, the homomorphism \( \alpha_* : H_3(\Sigma \times I, \Sigma \times \partial I) \to H_3(M, \Sigma) \) takes \( [\Sigma \times I] \) to \( d_1[V_1] + d_2[V_2] \), and then, the connecting map \( H_3(M, \Sigma) \to H_2(\Sigma) \) takes this to \((d_1 + d_2)[\Sigma]\). This implies \( d_1 + d_2 = -1 + \deg(f)\).

(2) Let \( f = \partial_+(\alpha) = \alpha(1) \) be the terminal end of \( \alpha \). Then the concatenation \( \alpha \cdot \beta \) is given by

\[
\alpha \cdot \beta(x,t) = \begin{cases} 
\alpha(x,2t) & (0 \leq t \leq 1/2) \\
\beta(f(x),2t-1) & (1/2 \leq t \leq 1).
\end{cases}
\]

Let \( \overline{\deg}(\alpha) = (d_1,d_2) \) and \( \overline{\deg}(\beta) = (e_1,e_2) \). The assertion then follows from

\[
(\alpha \cdot \beta)_*([\Sigma \times I]) = \alpha_*([\Sigma \times I]) + (\beta_* \circ (f \times \text{id}_I)_*)([\Sigma \times I])
= (d_1[V_1] + d_2[V_2]) + \deg(f \times \text{id}_I)(e_1[V_1] + e_2[V_2])
= (d_1[V_1] + d_2[V_2]) + \deg(f) \cdot (e_1[V_1] + e_2[V_2]).
\]

The following corollary generalises Examples 8.4 and 8.5.

**Corollary 8.9.** Let \( \alpha \) be a homotopy motion of \( \Sigma \) in \( V_i \) \((i = 1 \text{ or } 2)\) with terminal end \( f \), and regard it as a homotopy motion of \( \Sigma \) in \( M \). Then \( \overline{\deg}(\alpha) = (-1 + \deg(f),0) \) or \((0,-1 + \deg(f))\) according to whether \( i = 1 \) or \( 2 \).

**Proof.** Put \((d_1,d_2) = \overline{\deg}(\alpha)\) and suppose that \( \alpha \) comes from a homotopy motion in \( V_1 \). Then the image of \( \alpha \) is equal to \( V_1 \), and so we have \( d_2 = 0 \). By Lemma 8.8(1), \( d_1 = -1 + \deg(f) \). Then \( d_2 = -1 + \deg(f) \), completing the proof for the case \( i = 1 \). The remaining case \( i = 2 \) is proved by the same argument. \( \square \)
The following corollary is a consequence of Lemma 8.8(2) and the definition of a semi-direct product.

**Corollary 8.10.** Let $C_2 = \{ \pm 1 \}$ be the order-2 cyclic group, and consider its action on $\mathbb{Z}^2$ defined by $(-1) \cdot (d_1, d_2) = (-d_1, -d_2)$. Let $\mathbb{Z}^2 \rtimes C_2$ be the semi-direct product determined by this action. Then the map $\Pi(M, \Sigma) \to \mathbb{Z}^2 \rtimes C_2$ defined by $\alpha \mapsto (\hat{\deg}(\alpha), \deg(\partial_+\alpha))$ is a group homomorphism.

By the above corollary we can define a map $\hat{\deg} : \Gamma(M, \Sigma) \to \mathbb{Z}^2$ so that the diagram

$$
1 \longrightarrow \mathcal{K}(M, \Sigma) \longrightarrow \Pi(M, \Sigma) \xrightarrow{\partial_+} \Gamma(M, \Sigma) \longrightarrow 1
$$

commutes if and only if $\hat{\deg}$ vanishes on $\mathcal{K}(M, \Sigma)$. By Example 8.6, the latter condition is satisfied if and only if the homomorphism $\deg : \mathcal{K}(M, \Sigma) \to \mathbb{Z}$ vanishes, namely $(M, \Sigma)$ is not dominated by $\Sigma \times S^1$. Hence, Theorems 7.1 and 7.2 imply the following proposition.

**Proposition 8.11.** Let $M = V_1 \cup_{\Sigma} V_2$ be a Heegaard splitting of a closed, orientable 3-manifold $M$. Then if $M$ has an aspherical prime summand, the map $\hat{\deg} : \Gamma(M, \Sigma) \to \mathbb{Z}^2$ is well-defined.

From the properties of $\hat{\deg}$ we have the following.

**Lemma 8.12.** Let $M = V_1 \cup_{\Sigma} V_2$ be a Heegaard splitting of a closed, orientable 3-manifold $M$, and assume that the map $\hat{\deg}$ vanishes on $\mathcal{K}(M, \Sigma)$ and so the map $\hat{\deg} : \Gamma(M, \Sigma) \to \mathbb{Z}$ is defined. Then the following hold.

1. Let $f$ be an element of $\Gamma(M, \Sigma)$, and suppose $\deg(f) = (d_1, d_2)$. Then we have $d_1 + d_2 = -1 + \deg(f)$.
2. For any $f, g \in \Gamma(M, \Sigma)$, we have $\deg(g \circ f) = \deg(f) + \deg(f) \cdot \deg(g)$.
3. For any $f \in \Gamma(V_1)$, we have $\deg(f) = (-1 + \deg(f), 0)$; and for any $f \in \Gamma(V_2)$, we have $\deg(f) = (0, -1 + \deg(f))$.
4. If $f \in \langle \Gamma(V_1), \Gamma(V_2) \rangle$, then $\deg(f) \equiv (0, 0) \pmod{2}$.

**Proof.** The assertions (1) and (2) follow from Lemma 8.8 and the assertion (3) follows from Corollary 8.9.

(4) If $f$ belongs to either $\Gamma(V_1)$ or $\Gamma(V_2)$, then the assertion holds by (3). Since

$$\deg(g \circ f) \equiv \deg(f) + \deg(g) \pmod{2}$$

by (2), we obtain the desired result for every $f \in \langle \Gamma(V_1), \Gamma(V_2) \rangle$.

**Remark 8.13.** Lemma 8.12(4) can refined as follows: If $f \in \langle \Gamma(V_1), \Gamma(V_2) \rangle$, then $\deg(f)$ is one of $(2k, -2k)$ and $(2k - 2, -2k)$ for some $k \in \mathbb{Z}$, according to whether $f$ is orientation-preserving or reversing.
Now we are ready to prove Theorem 8.1.

Proof of Theorem 8.1. Let \( M = V_1 \cup \Sigma V_2 \) be the Heegaard splitting of a closed, orientable 3-manifold \( M \) induced from an open book decomposition, and assume that \( M \) has an aspherical prime summand. Then by Proposition 8.11, the map \( \text{Deg} : \Gamma(M, \Sigma) \to \mathbb{Z}^2 \) is well-defined. Let \( \rho \) and \( \sigma \) be the half rotation and the unilateral rotation of \( \Sigma \), respectively. Then by Example 8.7, we have \( \text{Deg}(\rho(1)) = (-1, -1) \) and \( \text{Deg}(\sigma(1)) = (-1, 1) \). Therefore, \( \rho(1) \) and \( \sigma(1) \) do not belong to \( \langle \Gamma(V_1), \Gamma(V_2) \rangle \) by Lemma 8.12(4), as desired. \( \square \)

In the above proof, we have shown that neither \( \sigma(1) \) nor \( \rho(1) \) is contained in \( \langle \Gamma(V_1), \Gamma(V_2) \rangle \). Clearly, the same consequence holds for any odd power of \( \sigma(1) \) and \( \rho(1) \) by Lemma 8.12. We do not know, however, whether \( \sigma(1)^2 \) or \( \rho(1)^2 \) is contained in \( \langle \Gamma(V_1), \Gamma(V_2) \rangle \).

Question 8.14. Under the assumption of Theorem 8.1, is \( \sigma(1)^2 \) or \( \rho(1)^2 \) contained in \( \langle \Gamma(V_1), \Gamma(V_2) \rangle \)?

We see from a result in the companion paper [60] that, for the genus-1 Heegaard surface \( \Sigma \) of a lens space \( L(p, q) \), there is a gap between \( \Gamma(L(p, q), \Sigma) \) and \( \langle \Gamma(V_1), \Gamma(V_2) \rangle \) generically. This and Theorem 8.1 are the only examples of Heegaard splittings we know for which there are gaps between the two groups. By the way, the Hempel distance of a Heegaard splitting induced from an open book decomposition is at most 2. Thus, we pose the following question.

Question 8.15. Let \( \Sigma \) be a Heegaard surface of genus at least 2 of a closed, orientable 3-manifold \( M \). Is it true that \( \Gamma(M, \Sigma) = \langle \Gamma(V_1), \Gamma(V_2) \rangle \) if \( \Sigma \) has high Hempel distance?

9. The virtual branched fibration theorem and the group \( \langle \Gamma(V_1), \Gamma(V_2) \rangle \)

In this section, we give yet another motivation for studying the group \( \Gamma(M, \Sigma) \) and its subgroup \( \langle \Gamma(V_1), \Gamma(V_2) \rangle \) associated with a Heegaard splitting \( M = V_1 \cup \Sigma V_2 \). To describe this, let \( \mathcal{I}(V_i) \subset \text{MCG}(\Sigma) \) be the set of torsion elements of \( \Gamma(V_i) \). (In fact, this set will turn out to be equal to the set of vertical \( I \)-bundle involutions of \( V_i \) as shown in Lemma 9.3.) Then we have the following theorem, which refines the observation [55, Addendum 1] that every closed, orientable 3-manifold \( M \) admits a surface bundle as a double branched covering space.

Theorem 9.1. Let \( M = V_1 \cup \Sigma V_2 \) be a Heegaard splitting of a closed, orientable 3-manifold \( M \). Then there exists a double branched covering \( p : \tilde{M} \to M \) that satisfies the following conditions.

(i) \( \tilde{M} \) is a surface bundle over \( S^1 \) whose fiber is homeomorphic to \( \Sigma \).
(ii) The preimage \( p^{-1}(\Sigma) \) of the Heegaard surface \( \Sigma \) is a union of two (disjoint) fiber surfaces.
Moreover, the set $D(M, \Sigma)$ of monodromies of such bundles is equal to the set $\{h_1 \circ h_2 \mid h_i \in \mathcal{I}(V_i)\}$, up to conjugation and inversion.

**Example 9.2.** Let $M = \#_g(S^2 \times S^1)$, and $V_1 \cup_\Sigma V_2$ the genus-$g$ Heegaard splitting. Recall that $\Gamma(V_1) = \Gamma(V_2)$ (cf. Example 5.3). Pick an element $h_1 = h_2$ from $\mathcal{I}(V_1) = \mathcal{I}(V_2)$. Then $h_1 \circ h_2 = \text{id}_\Sigma$ and hence the above theorem implies that $\Sigma \times S^1$ is a double branched covering space of $M = \#_g(S^2 \times S^1)$, and so $\Sigma \times S^1$ dominates $\#_g(S^2 \times S^1)$. This gives the construction by Kotschick-Neofytidis [61, Proposition 4].

We first prove the lemma below, following and correcting the arguments of Zimmermann [94, Proof of Corollary 1.3].

**Lemma 9.3.** Let $V$ be a handlebody with $\partial V = \Sigma$. Then an element of $\Gamma(V) < \text{MCG}(\Sigma)$ is a nontrivial torsion element if and only if it is represented by (the restriction to $\Sigma = \partial V$ of) a vertical $I$-bundle involution of $V$.

**Proof.** Since the “if” part is clear, we prove the “only if” part. If the genus of $V$ is 0 or 1, then the assertion can be easily proved by using the facts that $\text{MCG}(B^3) = \mathbb{Z}_2$ and $\text{MCG}(S^1 \times D^2) \cong \text{MCG}^+(S^1 \times D^2) \rtimes \mathbb{Z}_2 \cong \mathbb{D}_\infty \rtimes \mathbb{Z}_2$, the semi-direct product of the infinite dihedral group $\mathbb{D}_\infty$ and the order-2 cyclic group $\mathbb{Z}_2$. Assume that the genus of $V$ is greater than 1. Let $h$ be a torsion element of $\Gamma(V) \subset \text{MCG}(\Sigma)$. Then, by the solution of the Nielsen realization problem (see Kerckhoff
there exists a conformal structure on $\Sigma = \partial V$ and a conformal (or anti-conformal) map $h'$ of the Riemann surface $\Sigma$ that is isotopic to $h$. By Bers [8, Theorem 3], the Riemann surface $\Sigma$ admits a Schottky uniformization, i.e., there is a Schottky group $G$ such that the Riemann surface $\Sigma$ is conformally equivalent to the Riemann surface $\partial \Omega(G)/G$, where $\Omega(G)$ is the domain of discontinuity of $G$, and such that the identification of $\Sigma$ with $\partial \Omega(G)/G$ extends to an identification of $V$ with $V(G) := (\mathbb{H}^3 \cup \Omega(G))/G$. By Marden's isomorphism theorem [68, Theorem 8.1], $h'$ extends to an isometry of $V(G)$, which we continue to denote by $h'$. Let $\hat{h}$ be the lift of $h'$ to $\mathbb{H}^3$. Then, by the assumption that $h \in \Gamma(V)$, the conjugation action of $\hat{h}$ on $G$ is an inner-automorphism of $G$, that is, there exists an element $k \in G$ such that $\hat{h} \circ g \circ \hat{h}^{-1} = k \circ g \circ k^{-1}$ for every $g \in G$. Thus $\hat{h} \circ k^{-1}$ belongs to the centralizer, $Z$, of $G$ in the isometry group $\text{Isom}(\mathbb{H}^3)$. Since $G$ is a free group of rank $\geq 2$, it follows that $Z$ is trivial except when $G$ preserves a hyperbolic plane. In the exceptional case, $Z$ is the order-2 cyclic group generated by the reflection in the hyperbolic plane, and so we may assume that $\hat{h}$ is the reflection in the hyperbolic plane preserved by $G$. This implies that the isometry $h'$ of $V(G)$ is a vertical I-bundle involution. This completes the proof of the lemma.

Remark 9.4. The assertion in the proof that there exists a Schottky group $G$ such that $h$ is realized by an isometry of $V(G)$ is proved by Zimmermann [94, Theorem 1.1] under a more general setting. In fact, [94, Theorem 1.1] says that any finite subgroup of $\text{MCG}(V)$ is realized as a subgroup of the isometry group of $V(G)$. His proof is based on Zieschang's partial solution of the Nielsen realization problem, which was available at that time, and some delicate consideration on the group structure, which guarantees that Zieschang's result is applicable to his setting. Since we only need to consider cyclic groups, we do not need the consideration of the group structure, or we may simply appeal to Kerckhoff's full solution of the Nielsen realization problem [55]. In our terminology, [94, Corollary 1.3] should be read as follows: the orientation-preserving subgroup of $\Gamma^+(V)$ of $\Gamma(V)$ is torsion-free. (A similar proof of this result was also given by Otal [81, Proposition 1.7], and an outline of a similar proof, suggested by Minsky, is included in [9, Introduction].) Thus Lemma 9.3 is a slight extension of [94, Corollary 1.3].

Lemma 9.5. Let $V$ be a handlebody with $\partial V = \Sigma$, and suppose that $h$ is an orientation-reversing involution of $\Sigma$ that extends to a vertical I-bundle involution of $V$. Then there exists a double branched covering projection $p: \Sigma \times [-1, 1] \to V$ satisfying the following conditions.

(i) $p(x, 1) = p(h(x), -1) = x \in \Sigma = \partial V$ for every $x \in \Sigma$.

(ii) The covering transformation is given by the involution $\hat{h} := h \times (-1)$ of $\Sigma \times [-1, 1]$ defined by $\hat{h}(x, t) = (h(x), -t)$. In particular, the branch set of $p$ is equal to the image of $\text{Fix}(h) \times \{0\} \subset \Sigma \times [-1, 1]$ in $V$. 

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Proof. Let $\hat{V}$ be the quotient of $\Sigma \times [-1, 1]$ by the orientation-preserving involution $\hat{h}$ defined by the formula in (ii), and let $\hat{p} : \Sigma \times [-1, 1] \to \hat{V}$ be the projection. Then $\hat{p}$ is a double branched covering projection with branched set the image of $\text{Fix}(h) \times \{0\} \subset \Sigma \times [-1, 1]$ in $\hat{V}$, and the restriction of $\hat{p}$ to $\Sigma \times \{1\}$ is a homeomorphism onto $\partial V$. We identify $\partial V$ with this homeomorphism, i.e., identify each $x \in \Sigma$ with $\hat{p}(x, 1) \in \partial V$. We show that the identification of $\Sigma = \partial V$ with $\partial V$ extends to a homeomorphism from $V$ to $\hat{V}$. To this end, recall the assumption that $h$ extends to a vertical $I$-bundle involution of $V$, that is, there exists an $I$-bundle structure of $V$ such that $h$ preserves each fiber setwise and acts on it as a reflection. Then the base space of the $I$-bundle structure is identified with the quotient surface $F := \Sigma/h$ and we can construct a complete meridian system of $V$ as follows. Pick a complete arc system $\{\delta_i\}_{i=1}^g$ of $F$. Then the preimages of these arcs by the $I$-bundle projection form a complete meridian disk system of $V$. Let $\{\alpha_i\}_{i=1}^g$ be the family of essential loops on $\Sigma = \partial V$ obtained as the boundaries of these meridian disks. Note that the involution $h$ preserves each $\alpha_i$ and that $\alpha_i/h = \delta_i \subset \Sigma/h = F$. This implies that the quotient $(\alpha_i \times [-1, 1])/h$ is a meridian disk of the handlebody $\hat{V} = \Sigma \times [-1, 1]/h$ bounded by the loop $\alpha_i \subset \Sigma = \partial \hat{V}$. Since the meridian loop $\alpha_i$ of $V$ remains to be a meridian loop of $\hat{V}$ under the identification of $\Sigma = \partial V$ with $\partial \hat{V}$, the identification homeomorphism extends to a homeomorphism from $V$ to $\hat{V}$. Thus the composition of the branched covering projection $\hat{p} : \Sigma \times [-1, 1] \to \hat{V}$ and the identification homeomorphism $\hat{V} \cong V$ determines the desired branched covering projection $p : \Sigma \times [-1, 1] \to V$. \hfill \qed

By using the result of Kim-Tollefson [56, Theorem A] on involutions of product spaces, we can obtain the following converse to Lemma 9.5.

**Lemma 9.6.** Let $V$ be a handlebody with $\partial V = \Sigma$, and $p : \Sigma \times [-1, 1] \to V$ a double branched covering projection such that the restriction $p|_{\Sigma \times \{1\}} : \Sigma \times \{1\} \to \partial V = \Sigma$ is the identity, i.e., $p(x, 1) = x$ for every $x \in \Sigma$. Then there exists an orientation-reversing involution $h$ of $\Sigma$ that extends to a vertical $I$-bundle involution of $V$ such that $p$ is equivalent to the covering projection constructed from $h$ as indicated in Lemma 9.5. To be precise, there exists a self-homeomorphism of $\Sigma \times [-1, 1]$ that fixes $\Sigma \times \{1\}$ such that the composition of this homeomorphism and $p$ is equal to the covering projection constructed in Lemma 9.5.

**Proof.** Let $g$ be the covering transformation of the double branched covering $p$. Since $g$ interchanges the two components of $\Sigma \times \partial I$, the result [56, Theorem A] implies that there exists an orientation-reversing involution $h$ of $\Sigma$ such that $g$ is equivalent to the involution $h \times (-1)$. To be more precise, we can see that $g$ is conjugate to $h \times (-1)$ by a self-homeomorphism of $\Sigma \times [-1, 1]$ that fixes $\Sigma \times \{1\}$. Thus we may assume that $g = h \times (-1)$. By the assumption, $\Sigma \times [-1, 1]/g$ is identified with the handlebody $V$ in such a way that the point $[x, 1]$ of $\Sigma \times [-1, 1]/g$ represented by $(x, 1)$ is identified
with the point \( x \in \Sigma = \partial V \) for every \( x \in \Sigma \). Now consider the involution \( h \times 1 \) of \( \Sigma \times [-1, 1] \). This map is commutative with the involution \( g = h \times (-1) \) and so it descends to an involution \( h \) of \( V = \Sigma \times [-1, 1]/g \). The restriction of \( h \) to \( \partial V = \Sigma \) is equal to \( h \). Moreover, \( h \) is a vertical \( I \)-bundle involution of \( V \), as shown below.

Note that \( V = \Sigma \times [-1, 1]/g = \Sigma \times [0, 1]/(x, 0) \sim (h(x), 0) \), and so there exists a deformation retraction of \( V \) onto the subspace \( F := \Sigma \times \{0\}/(x, 0) \sim (h(x), 0) \). Thus \( F \) is a compact surface with nonempty boundary, which is embedded in the interior of \( V \) and is a deformation retract of \( V \). Note that \( \text{Fix}(h) \) is equal the union of the image of \( \text{Fix}(h) \times [0, 1] \) and the image of \( \Sigma \times \{0\} \). The former is a disjoint union of annuli and the latter is equal to \( F \). Thus \( \text{Fix}(h) \) is a surface properly embedded in \( V \) and contains \( F \) as its deformation retract. This implies that \( h \) is an \( I \)-bundle involution, where \( \text{Fix}(h) \cong F \) is the base space of the \( I \)-bundle structure of \( V \). Thus we have proved that the involution \( h \) of \( \Sigma = \partial V \) extends to the vertical \( I \)-bundle involution \( \tilde{h} \) of \( V \). Since the covering involution \( g \) of the double branched covering projection \( p : \Sigma \times [-1, 1] \to V \) is given by \( g = h \times (-1) \), we can say that \( p \) is obtained from \( h \), satisfying the prescribed condition, as indicated in Lemma 9.5. □

**Proof of Theorem 9.1.** For \( i = 1, 2 \), pick an element \( h_i \) of \( \mathcal{I}(V_i) \subset \text{MCG}(\Sigma) \). By Lemma 9.3, \( h_i \) is represented by an orientation-reversing involution of \( \Sigma \) that extends to a vertical \( I \)-bundle involution of \( V_i \). We continue to denote the orientation-reversing involution of \( \Sigma \) by \( h_i \). Let \( p_i : \Sigma \times [-1, 1] \to V_i \) the double branched covering projection given by Lemma 9.5. Take two copies \([-1, 1]_i \) of \([-1, 1] \), and regard \( p_i \) as a map \( \Sigma \times [-1, 1]_i \to V_i \). Let \( M \) be the space obtained from the disjoint union \( \bigsqcup_{i=1}^2 \Sigma \times [-1, 1]_i \) through the identification

\[
(x, 1)_1 \sim (x, 1)_2, \quad (h_1(x), -1)_1 \sim (h_2(x), -1)_2 \quad (x \in \Sigma).
\]

Here \((x, t)_i \) denotes the point in \( \Sigma \times [-1, 1]_i \) corresponding to \((x, t) \in \Sigma \times [-1, 1] \). Then \( M \) is a \( \Sigma \)-bundle over \( S^1 \) with monodromy \( h_1^{-1} \circ h_2 = h_1 \circ h_2 \). Moreover we can glue the branched covering projections \( p_i : \Sigma \times [-1, 1]_i \to V_i \) \((i = 1, 2)\) together to obtain a continuous map \( p : M \to M = V_1 \cup_{\Sigma} V_2 \), because

\[
p_1((h_1(x), -1)_1) = p_1((x, 1)_1) = x = p_2((x, 1)_2) = p_2((h_2(x), -1)_2).
\]

Then \( p \) is a branched covering projection whose branch set is the union of those of \( p_1 \) and \( p_2 \). Hence the \( \Sigma \)-bundle over \( S^1 \) with monodromy \( h_1 \circ h_2 \) is a double branched covering space of \( M \). Moreover the preimage \( p^{-1}(\Sigma) \) of the Heegaard surface \( \Sigma = \partial V_1 = \partial V_2 \) is the image of \( \Sigma \times \partial [-1, 1]_1 \) (and so is that of \( \Sigma \times \partial [-1, 1]_2 \)) in \( M \). Thus, \( p^{-1}(\Sigma) \) is a union of two fiber surfaces. This completes the proof of the first assertion of Theorem 9.1, and the assertion \( \{h_1 \circ h_2 \mid h_i \in \mathcal{I}(V_i)\} \subset D(M, \Sigma) \).

We prove \( D(M, \Sigma) \subset \{h_1 \circ h_2 \mid h_i \in \mathcal{I}(V_i)\} \). To this end, let \( p : M \to M \) be a double branched covering satisfying the conditions (i) and (ii) of Theorem 9.1, and let \( \tau \) be the covering involution. By the condition (ii), \( p^{-1}(\Sigma) \) consists of two (distinct and so disjoint) fiber surfaces, \( \Sigma_0 \) and \( \Sigma_1 \), and \( \tau \) interchanges these two
components. Set $\tilde{V}_i = p^{-1}(V_i)$ ($i = 1, 2$). Then $\tilde{V}_1 \cap \tilde{V}_2 = \partial \tilde{V}_1 = \partial \tilde{V}_2 = \Sigma_0 \cup \Sigma_1$ and $\tilde{V}_1 \cong \tilde{V}_2 \cong \Sigma \times [-1, 1]$. We identify the fiber surface $\Sigma_0$ with the Heegaard surface $\Sigma$ via the restriction $p|_{\Sigma_0}$. Then there exists a homeomorphism $\psi_i : \tilde{V}_i \to \Sigma \times [-1, 1]$ such that $\psi_i(x) = (p(x), 1)$ for every $x \in \Sigma_0$. Let $\tau_i$ be the involution of $\Sigma \times [-1, 1]$ defined by $\psi_i \circ \tau_i \circ \psi_i^{-1}$. Then $p_i := p|_{\tilde{V}_i} \circ \psi_i^{-1} : \Sigma \times [-1, 1] \to V_i$ is a double branched covering whose restriction to $\Sigma \times \{1\}$ is the identity map onto $\Sigma = \partial V_i$. Hence, by Lemma 9.6 there exists an element $h_i \in \mathcal{I}(V_i)$ such that the covering $p_i := p|_{\tilde{V}_i} \circ \psi_i^{-1}$ of $\Sigma \times \{1\}$ is the identity map onto $\Sigma = \partial V_i$. This implies that the monodromy of the $\Sigma$-bundle $\tilde{M}$ is equal to $h_1 \circ h_2$.

The characterization of $D(M, \Sigma)$ in Theorem 9.1 reminds us of the result of A’Campo [1, Corollary 1] which says that the geometric monodromy of an isolated complex hypersurface singularity, which is defined by a real equation, is the composition of two orientation-reversing involutions of the fiber, one of which is the restriction of the complex conjugation. Brooks [15] and Montesinos [72] independently proved that $D(M, \Sigma)$ contains a pseudo-Anosov element whenever $g(\Sigma) \geq 2$. Hirose and Kin [16] studied the asymptotic behavior of the minimum of the dilatations of pseudo-Anosov elements contained in $D(S^3, \Sigma_g)$ as $g \to \infty$, where $\Sigma_g$ is the genus-$g$ Heegaard surface of $S^3$.

When $g(\Sigma) = 1$, we will see in the companion paper [60] that, for any element $\phi$ of $D(M, \Sigma)$, the minimum translation length $d(\phi)$ of the action of $\phi$ on the curve graph is comparable with $2d(\Sigma)$, where $d(\Sigma)$ is the Hempel distance of $\Sigma$. We expect that this toy example may be extended to a result for general Heegaard splittings.

**Question 9.7.** For a Heegaard splitting $M = V_1 \cup_\Sigma V_2$ of a closed, orientable 3-manifold $M$ and for an element $\phi \in D(M, \Sigma)$, is there an estimate of $d(\phi)$, the translation length or the asymptotic translation length of the action of $\phi$ on the curve graph of $\Sigma$, in terms of the Hempel distance $d(\Sigma)$?

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