The Minimal Perimeter of a Log-Concave Function

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Abstract: Inspired by the equivalence between isoperimetric inequality and Sobolev inequality, we provide a new connection between geometry and analysis. We define the minimal perimeter of a log-concave function and establish a characteristic theorem of this extremal problem for log-concave functions analogous to convex bodies.

Keywords: isoperimetric problem; minimal perimeter; log-concave functions; isotropic measure

1. Introduction

The isoperimetric inequality is an important inequality in geometry which originated from the well-known isoperimetric problem. The isoperimetric inequality has a profound influence on each branches of mathematics. The breakthrough works of Federer and Fleming [1] and Mazya [2] discovered independently the connection between the isoperimetric problem and the Sobolev embedding problem. They established the sharp Sobolev inequality by using the isoperimetric inequality. This exciting connection has motivated a number of studies in recent years about interactions of geometric and analytic inequalities. In this paper, we further study the connection between geometry and analysis.

Let us recall some facts about convex bodies. Let $K$ be a convex body (i.e., compact, convex subset with non-empty interior) in the $n$-dimensional Euclidean space $\mathbb{R}^n$, the family $\{TK : T \in SL(n)\}$ of its positions are studied by many mathematicians. Introducing the right position of the unit ball $K_X$ of a finite dimensional normed space $X$ is one of the main problems in the asymptotic theory. There exist many celebrated positions for different purposes, for example isotropic position, $M$-position, John’s position, the $\ell$-position and so on, see [3,4].

Our purpose is to study the isotropic position of log-concave functions. Hence, we first recall some geometric backgrounds and these are our motivations. Let $K$ be a convex body in $\mathbb{R}^n$ with centroid at the origin and volume equal to one. A convex body $K$ is in isotropic position if

\[ \int_K (x, \theta)^2 dx = L^2_K, \quad \forall \theta \in S^{n-1}, \]

where $(\cdot, \cdot)$ is the usual inner product in $\mathbb{R}^n$ and $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$. It’s worth noting that every convex body $K$ with volume one has an isotropic position, and this position is uniqueness (up to an orthogonal transformation), see, e.g., [3]. Isotropic positions have been used to study the classical convexity problems, for example, the minimal surface area of a convex body and its extension [5,6], the minimal mean width of a convex body and its extension [7,8]. Other contributions include e.g., [9–11] among others.

We recall two specific examples on isotropic positions. Let $K$ be a convex body and denote by $S(K)$ its surface area. If $S(K) \leq S(TK)$ for every $T \in SL(n)$, then $K$ has minimal surface area (see, e.g., [5]).
Petty [5] obtained the following characterization of the minimal surface area position: a convex body \( K \) has minimal surface area if and only if its surface area measure \( \sigma_K \) is isotropic, i.e.,

\[
\int_{S^{n-1}} \langle u, \theta \rangle^2 d\sigma_K(u) = \frac{S(K)}{n}, \quad \forall \theta \in S^{n-1}.
\]

As a second example, the minimal mean width will be recalled which was defined by Giannopoulos and Milman [7]. Let \( K \) be a convex body in \( \mathbb{R}^n \), the mean width \( w(K) \) of \( K \) is define as

\[
w(K) = 2 \int_{S^{n-1}} h_K(u) d\sigma(u),
\]

where \( h_K(u) := \sup_{y \in K} \langle u, y \rangle \) is the support function of \( K \) and \( \sigma \) is the rotationally invariant probability measure on \( S^{n-1} \). For every \( T \in \text{SL}(n) \), if \( w(K) \leq w(TK) \) then \( K \) has minimal mean width (see, e.g., [7]). Giannopoulos and Milman [7] showed that if the support function of \( K \) is twice continuously differentiable, then \( K \) has minimal mean width if and only if the measure \( dv_K = h_K d\sigma \) is isotropic, i.e.,

\[
\int_{S^{n-1}} h_K(u)(u, \theta)^2 d\sigma(u) = \frac{w(K)}{2n}, \quad \forall \theta \in S^{n-1}.
\]

Within the last few years, many geometric results have been generalized to their corresponding functional versions, including but not limited to the functional version Blaschke-Santaló inequality and its reverse [12–16], the functional affine surface areas [17–19], Minkowski problem for functions [20–22], and analytic inequalities with geometric background [23–28].

In this paper, we consider the log-concave functions in \( \mathbb{R}^n \). A function \( f : \mathbb{R}^n \to \mathbb{R} \) is log-concave if for any \( x, y \in \mathbb{R}^n \) and \( \lambda \in [0, 1] \), it holds

\[
f(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda}.
\]

A typical example of log-concave functions is the characteristic function of convex bodies, \( 1_K \) (which is defined as \( 1_K(x) = 1 \) when \( x \in K \) and \( 1_K(x) = 0 \) when \( x \notin K \)). Let \( J(f) \) denote the total mass functional of \( f : \mathbb{R}^n \to \mathbb{R} \), namely

\[
J(f) = \int_{\mathbb{R}^n} f(x) dx.
\]

For any \( t > 0 \) and log-concave functions \( f, g : \mathbb{R}^n \to \mathbb{R} \), Colesanti and Fragalà [21] defined the first variation of \( J \) at \( f \) along \( g \) as

\[
\delta J(f, g) = \lim_{t \to 0^+} \frac{J(f \oplus t \cdot g) - J(f)}{t}, \quad (2)
\]

where \( t \cdot g(x) = g^t(x/t) \) for \( t > 0 \) and \( x \in \mathbb{R}^n \), and \( f \oplus g \) the Asplund sum of functions \( f \) and \( g \), i.e.,

\[
[f \oplus g](x) = \sup_{x=x_1+x_2} f(x_1)g(x_2), \quad x \in \mathbb{R}^n.
\]

It was proved that if \( f \) and \( g \) are restricted to a subclass of log-concave functions, then the first variation \( \delta J(f, g) \) precisely turns out to be \( L_p \) mixed volume of convex bodies (see Proposition 3.13 in [21]). In particular, the perimeter of \( f \) is defined as (see [21])

\[
P(f) = \delta J(f, \gamma_n),
\]

where \( \gamma_n(x) = e^{-\|x\|^2/2} \) is the Gaussian function and \( \|x\| \) is the Euclidean norm of \( x \in \mathbb{R}^n \).
Motivated by the work of Giannopoulos and Milman [7], we consider the extremal problems of log-concave functions instead of convex bodies, and our purpose is to discuss the possibility of an isometric approach to these questions. We introduce the notion of minimal perimeters of log-concave functions. Assume that $f$ is a log-concave function, we call $f$ has minimal perimeter if $P(f) \leq P(f \circ T)$ for every $T \in \text{SL}(n)$. Furthermore, we derive the following characteristic theorem of the minimal perimeter.

**Theorem 1.** If $f : \mathbb{R}^n \to [0, \infty)$ is a log-concave function, then $f$ has minimal perimeter if and only if

$$\frac{\text{tr}(T)}{n} P(f) = \frac{1}{2} \int_{\mathbb{R}^n} \langle x, Tx \rangle d\mu_f(x)$$

for every $T \in \text{GL}(n)$. Here $\text{tr}(T)$ denotes the trace of $T$, and $\mu_f = (\nabla u)(f \mathcal{H}^n)$ is a Borel measure on $\mathbb{R}^n$ (where $\mathcal{H}^n$ is the $n$-dimensional Hausdorff measure and $u = -\log f$).

Theorem 1 implies that the log-concave function $f$ has minimal perimeter if and only if $\mu_f(\cdot)$ is isotropic, and provides a further example of the connections between the theory of convex bodies and that of functions.

We remark that our works belong to the asymptotic theory of log-concave functions which parallel to that of convex bodies. From a geometric and analytic view of point to study convex bodies is the asymptotic theory of convex bodies which emphasize the dependence of various parameters on the dimension. Isotropic positions for convex bodies play important roles in the asymptotic theory of convex bodies. We are not aware of the related results for log-concave functions. Hence, our work in this paper presents a new connection between convex bodies and log-concave functions and it also leads to a new topic in the study of geometry of log-concave functions. We hope that our work provides some useful tools or ideas in the development of geometry of log-concave functions.

2. Preliminaries

In this section, we provide some preliminaries and notations required for functions. More details can be found in [3,4].

A function $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex if

$$u((1 - \lambda)x + \lambda y) \leq (1 - \lambda)u(x) + \lambda u(y)$$

for any $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$. Let

$$\text{dom}(u) = \{x \in \mathbb{R}^n : u(x) \in \mathbb{R}\}.$$

Since the convexity of $u$, $\text{dom}(u)$ is a convex set. If $\text{dom}(u) \neq \emptyset$, then $u$ is said proper. The function $u$ is called of class $C^2$ if it is twice differentiable on $\text{int(dom}(u))$, with a positive definite Hessian matrix. The Fenchel conjugate of $u$ is the convex function defined by

$$u^*(y) = \sup_{x \in \mathbb{R}^n} \{\langle x, y \rangle - u(x)\}, \quad \forall y \in \mathbb{R}^n.$$

Clearly, $u(x) + u^*(y) \geq \langle x, y \rangle$ for all $x, y \in \mathbb{R}^n$. The equality holds if and only if $x \in \text{dom}(u)$ and $y$ is in the subdifferential of $u$ at $x$. Hence, one can checked that

$$u^*(\nabla u(x)) + u(x) = \langle x, \nabla u(x) \rangle.$$
From the definition of log-concave functions (1), we known that a log-concave function \( f : \mathbb{R}^n \to \mathbb{R} \) has the form \( f(x) = e^{-u(x)} \) where \( u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is convex. Writing

\[
\mathcal{L} = \left\{ u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \mid u \text{ is proper and convex, } \lim_{\|x\| \to +\infty} u(x) = +\infty \right\},
\]

\[
\mathcal{A} = \left\{ f : \mathbb{R}^n \to \mathbb{R} \mid f = e^{-u}, u \in \mathcal{L} \right\}.
\]

For \( u,v \in \mathcal{L} \), the inf-convolution of \( u,v \) is defined by

\[
(u\boxplus v)(x) = \inf_{y \in \mathbb{R}^n} [u(y) + v(y - x)], \quad \forall x \in \mathbb{R}^n,
\]

and the right scalar multiplication \( ut \) is defined by

\[
(ut)(x) := \begin{cases} 
  tu \left( \frac{x}{t} \right), & \text{if } t > 0 \\
  l_{\{0\}}, & \text{if } t = 0.
\end{cases}
\]

Note that these operations are convexity preserving, and \( l_{\{0\}} \) acts as the identity element in (5). It is proved that \( (us\boxplus vt)(x) \in \mathcal{L} \) for \( u,v \in \mathcal{L} \) and \( s,t \geq 0 \) (see [21]). Let \( f = e^{-u}, g = e^{-v} \in \mathcal{A} \) and \( t > 0 \). Form (5), the Asplund sum (defined in (3)) can be rewritten as

\[
f \boxplus g = e^{-[u\boxplus v]},
\]

and \( t \cdot g = e^{-vt} \). Let \( f = e^{-u} \in \mathcal{A} \). The support function, \( h_f \), of \( f \) is defined as (see, e.g., [28])

\[
h_f(x) = u^*(x).
\]

We recall that a probability measure \( \mu \) is called isotropic if it satisfies \( \int_{\mathbb{R}^n} x \, d\mu(x) = 0 \) and

\[
\int_{\mathbb{R}^n} \langle x, \theta \rangle^2 d\mu(x) = \frac{1}{n}, \quad \forall \theta \in S^{n-1}.
\]

For a measure \( \mu \) with \( \int_{\mathbb{R}^n} x \, d\mu(x) = 0 \), the following claims are equivalent (see, e.g., [3]):

(a) \( \mu \) is isotropic;

(b) For any \( T \in \text{GL}(n) \), one has

\[
\int_{\mathbb{R}^n} \langle x, Tx \rangle \, d\mu(x) = \frac{\text{tr}(T)}{n};
\]

(c)

\[
\int_{\mathbb{R}^n} x_i x_j \, d\mu(x) = \frac{1}{n} \delta_{ij} \quad \text{for all } \ i, j = 1, \cdots, n.
\]

3. Minimal Perimeter of Log-Concave Functions

In this section, we consider the properties of the minimal perimeter of log-concave functions. Let \( f = e^{-u} \in \mathcal{A} \), the perimeter \( P(f) \) has an integral expression (see [21,29]):

\[
P(f) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{\|\nabla f\|^2}{f} \, dx.
\]
We define that $f$ has minimal perimeter if $P(f) \leq P(f \circ T)$ for every $T \in \text{SL}(n)$. This is, if $f$ has minimal perimeter, then

$$\int_{\mathbb{R}^n} \frac{\|\nabla f\|^2}{f} \, dx \leq \int_{\mathbb{R}^n} \frac{\|\nabla (f \circ T)\|^2}{f \circ T} \, dx,$$

for any $T \in \text{SL}(n)$.

The Borel measure $\mu_f$ on $\mathbb{R}^n$ of a log-concave function $f = e^{-u}$ is defined by (see [21])

$$\mu_f = (\nabla u)_!(f \mathcal{H}^n).$$

Here $\mathcal{H}^n$ denotes the $n$-dimensional Hausdorff measure. For any continuous function $g : \mathbb{R}^n \to \mathbb{R}$, one has

$$\int_{\mathbb{R}^n} g(y) d\mu_f(y) = \int_{\mathbb{R}^n} g(\nabla u(x)) f(x) dx. \quad (10)$$

The Borel measure $\mu_f$ plays the same role for $f$ as the surface area measure for the convex body.

**Proposition 1.** If $f \in A$, then

$$P(f) = \int_{\mathbb{R}^n} h_{\gamma_n}(x) d\mu_f(x).$$

**Proof.** From Equations (9), (10) and (7), we have

$$P(f) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{\|\nabla f\|^2}{f} \, dx = \int_{\mathbb{R}^n} f(x) \|\nabla u(x)\|^2 \, dx = \int_{\mathbb{R}^n} f(x) h_{\gamma_n}(\nabla u(x)) \, dx = \int_{\mathbb{R}^n} h_{\gamma_n}(x) d\mu_f(x).$$

We recall that the gauge function of a convex body $K$ is defined by

$$\|x\|_K = \min\{\alpha \geq 0 : x \in \alpha K\}. \quad (11)$$

It is clear that

$$\|x\|_K = 1 \text{ whenever } x \in \partial K, \quad (12)$$

where $\partial K$ is the boundary of $K$.

We note that the minimal perimeter of a log-concave function $f$ is equivalent to considering the minimization problem:

$$\min_{T \in \text{SL}(n)} P(f \circ T). \quad (13)$$
For $T \in SL(n)$, we write $\gamma_T$ for $\gamma_n \circ T$. From (9) and the fact that $\nabla_x (f \circ T) = T^t \nabla_T f$ for $T \in SL(n)$ and $x \in \mathbb{R}^n$, we have

$$P(f \circ T) = \delta (f \circ T, \gamma) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{\|\nabla_x (f \circ T)\|^2}{f(Tx)} dx = \frac{1}{2} \int_{\mathbb{R}^n} \frac{\|T^t \nabla_T f(Tx)\|^2}{f(Tx)} dx = \frac{1}{2} \int_{\mathbb{R}^n} \frac{\|T^t \nabla_T f(x)\|^2}{f(x)} dx = \delta (f, \gamma_T).$$

Therefore, we can reformulate problem (13) as follows:

$$\min \{ \delta (f, \gamma_T) : T \in SL(n) \}. \quad (14)$$

**Proposition 2.** There exists a unique (up to an orthogonal matrix) $T_0 \in SL(n)$ such that it solves the minimization problem (14).

**Proof.** We can limit our attention to $T \in SL(n)$ when $T$ is a positive definite symmetric matrix, since any $T \in SL(n)$ can be represented in the form $T = PQ$ where $P \in SL(n)$ is a positive definite symmetric matrix and $Q$ is an orthogonal matrix. In this case, we can write the function

$$\gamma_T(x) = \gamma_n(Tx) = e^{-\|Tx\|^2} \quad \text{as} \quad \gamma_T(x) = e^{-\|x\|^2/2}, \quad \text{where} \quad E = T^t B \text{ is an origin-centered ellipsoid and } E^o \text{ is the polar body of } E \text{ defined as } E^o = \{ x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in E \}. \text{ There exists a } z_E \in S^{n-1} \text{ such that the diameter of } E \text{ satisfies } \text{diam} (E) \leq \|x\|_{E^o}. \text{ Let } \{ T_k \} \in SL(n) \text{ be a minimizing sequence for the problem (14), namely,}$$

$$\lim_{k \to \infty} \delta (f, \gamma_{T_k}) = \min \{ \delta (f, \gamma_T) : T \in SL(n), T \text{ is a positive definite symmetric matrix} \}. \quad (15)$$

From (15) and the fact that $\min \{ \delta (f, \gamma_T) : T \in SL(n) \} \leq \delta (f, \gamma_n)$, we have

$$\frac{\text{diam} (E_k^o)^2}{8} \min_{z \in S^{n-1}} \int_{\mathbb{R}^n} |\langle z, x \rangle|^2 d\mu_f(x) \leq \int_{\mathbb{R}^n} \frac{\|x\|_{E_k}^2}{2} d\mu_f(x) = \delta (f, \gamma_{T_k}) \leq \delta (f, \gamma_n).$$

Since $\|x\|_{E_k} < \text{diam} (E_k^o)|x|$, therefore the upper bound of the convex function $\left(\frac{\|T_k x\|^2}{2}\right)^\ast$ is depended only on $f$. According to Theorem 10.9 in [30], there exist a function $\gamma_{T_0}$ such that the Legendre conjugate of a minimizing sequence of functions for problem (14) converge to $\gamma_{T_0}$. Due to Theorem 11.34 in [31], we known that a minimizing sequence of functions for problem (14) converge to $\gamma_{T_0}^\ast$. According to the dominated convergence theorem, there exists a solution to problem (14).

Next, we prove the uniqueness of $T_0$. Assume there are two different solutions $T_1, T_2 \in SL(n)$ to the considered problem which satisfy $T_1 \neq aT_2$ for all $a > 0$. If there exists a $a_0 > 0$ such that $T_1 = a_0 T_2$, then

$$\delta (f, \gamma_{T_1}) = \delta (f, \gamma_{a_0 T_2}) = \delta (f, \gamma_{T_2}) \implies a_0 = 1.$$

This contradicts to $T_1 \neq T_2$. The Minkowski inequality for symmetric positive definite matrices shows that

$$\det \left( \frac{T_1 + T_2}{2} \right)^\frac{1}{2} > \frac{1}{2} \det (T_1)^\frac{1}{2} + \frac{1}{2} \det (T_2)^\frac{1}{2} = 1.$$
Let
\[ T_3 = \det \left( \frac{T_1^{-1} + T_2^{-1}}{2} \right)^{\frac{1}{2}}. \]

Then \( T_3 \in \text{SL}(n) \) and \( \gamma T_3 < \gamma \left( \frac{T_1^{-1} + T_2^{-1}}{2} \right)^{-1} \), i.e., \( h_{\gamma T_3} < h_\gamma \left( \frac{T_1^{-1} + T_2^{-1}}{2} \right)^{-1} \). This deduces that
\[
\delta J(f, \gamma T_3) = \int_{\mathbb{R}^n} h_{\gamma T_3}(x) d\mu_f(x) < \int_{\mathbb{R}^n} h_\gamma \left( \frac{T_1^{-1} + T_2^{-1}}{2} \right)^{-1} d\mu_f(x) = \int_{\mathbb{R}^n} \frac{\| \frac{T_1^{-1} + T_2^{-1}}{2} x \|^2}{2} d\mu_f(x).
\]

By the convexity of the square of the Euclidean normal, we have
\[
\delta J(f, \gamma T_3) < \int_{\mathbb{R}^n} \frac{1}{2} \| T_1^{-1} x \|^2 + \frac{1}{2} \| T_2^{-1} x \|^2 d\mu_f(x) = \frac{1}{2} \delta J(f, \gamma T_1) + \frac{1}{2} \delta J(f, \gamma T_2) = \delta J(f, \gamma T_1) = \delta J(f, \gamma T_2).
\]

However, from the assumption on \( T_1 \) and \( T_2 \), we have
\[
\delta J(f, \gamma T_3) \geq \delta J(f, \gamma T_1) = \delta J(f, \gamma T_2),
\]
which is a contradiction. \( \square \)

Proposition 2 implies that the minimal perimeter of log-concave functions is well-defined. Namely,

**Corollary 1.** For a log-concave function \( f : \mathbb{R}^n \to \mathbb{R} \), there exists a unique (up to an orthogonal matrix) \( T_0 \in \text{SL}(n) \) such that \( f \circ T_0 \) has minimal perimeter.

Next we are in the position to consider the proof of Theorem 1.

**Proof of Theorem 1.** Let \( T \in \text{GL}(n) \), and \( \epsilon > 0 \) be a suitably small real number. Then
\[ T_\epsilon = [\det(I + \epsilon T)]^{-\frac{1}{2}} (I + \epsilon T)^{\frac{1}{2}} \in \text{SL}(n), \]
and this implies that \( P(f) \leq P(f \circ T_\epsilon) \), i.e.,
\[
\int_{\mathbb{R}^n} \frac{\| \nabla f \|^2}{f} dx \leq \int_{\mathbb{R}^n} \frac{\| \nabla (f \circ T_\epsilon) \|^2}{f \circ T_\epsilon} dx.
\]
By the fact that $\nabla_x(u \circ T) = T^t \nabla_{Tx}u$, then
\[
\int_{\mathbb{R}^n} \frac{\|\nabla f\|^2}{f} \, dx \leq \int_{\mathbb{R}^n} \frac{\|\nabla (f \circ T_x)\|^2}{f \circ T_x} \, dx
= \int_{\mathbb{R}^n} \frac{\|T^t \nabla_{Tx}(f(T_x))\|^2}{f(T_x)} \, dx
= \int_{\mathbb{R}^n} \frac{\|T^t \nabla f\|^2}{f} \, dx,
\]
i.e.,
\[
[\det(I + \epsilon T)]^\frac{2}{n} \int_{\mathbb{R}^n} \frac{\|\nabla f\|^2}{f} \, dx \leq \int_{\mathbb{R}^n} \frac{\|(I + \epsilon T) \nabla f\|^2}{f} \, dx.
\]
Because
\[
\|(I + \epsilon T) \nabla f\|^2 = \|\nabla f\|^2 + 2\epsilon \langle \nabla f, T \nabla f \rangle + o(\epsilon^2)
\]
and
\[
[\det(I + \epsilon T)]^\frac{2}{n} = 1 + 2\epsilon \frac{\text{tr}(T)}{n} + o(\epsilon^2),
\]
when letting $\epsilon \to 0^+$, we obtain
\[
\frac{\text{tr}(T)}{n} \int_{\mathbb{R}^n} \frac{\|\nabla f\|^2}{f} \, dx \leq \int_{\mathbb{R}^n} \frac{\langle \nabla f, T \nabla f \rangle}{f} \, dx. \tag{16}
\]
Replacing $T$ by $-T$ in (16), we conclude that there must be equality in (4) for every $T \in \text{GL}(n)$.

On the other hand, assume that (4) is satisfied and let $T \in \text{SL}(n)$. Since $\frac{\text{tr}(T)}{n} \geq 1$ for symmetric positive-definite metric, (9) and $\nabla_x(f \circ T) = T^t \nabla_{Tx}f$, we have
\[
P(f \circ T) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{\|\nabla (f \circ T)\|^2}{f \circ T} \, dx
= \frac{1}{2} \int_{\mathbb{R}^n} \frac{\|T^t \nabla_{Tx}f\|^2}{f \circ T} \, dx
= \frac{1}{2} \int_{\mathbb{R}^n} \frac{\langle T \nabla f, T \nabla f \rangle}{f} \, dx
= \frac{1}{2} \int_{\mathbb{R}^n} \frac{\langle \nabla f, T^t T \nabla f \rangle}{f} \, dx
\geq \frac{1}{2} \int_{\mathbb{R}^n} \frac{\|\nabla f\|^2}{f} \, dx
= P(f).
\tag{17}
\]
This shows that $f$ has minimal perimeter. Moreover, the equality in (17) holds only if $T$ is the identity matrix. This proves that the uniqueness of the minimal perimeter position (up to $U \in \text{O}(n)$). □

**Corollary 2.** From Theorem 1 and the definition of isotropic measure, the log-concave function $f \in \mathcal{A}$ has minimal perimeter if and only if $\frac{1}{\text{tr}(T)} \mu_f$ is an isotropic measure.

Next, we prove that Theorem 1 recovers the $L_2$ surface area measure of $K$, $dS_2(K, \cdot) = h_K(\cdot)^{-1} \sigma_K(\cdot)$, is an isotropic measure on $S^{n-1}$. 
Corollary 3. Let $K$ be a convex body in $\mathbb{R}^n$ containing the origin in its interior. If $f(x) = e^{-\|x\|_k}$ for $x \in \mathbb{R}^n$, then

$$P(f) = \frac{1}{2} \Gamma(n) S_2(K),$$

and Theorem 1 includes the fact that $L_2$ surface area measure of $K$, $S_2(K, \cdot)$, is an isotropic measure.

Proof. For a convex body $K$ in $\mathbb{R}$, let $\nabla K$ denote the normalized cone volume measure of $K$, which is given by

$$d\nabla K(z) = \frac{\langle z, \nu(z) \rangle}{nV(K)} d\mathcal{H}^{n-1}(z) \quad \text{for} \quad z \in \partial K.$$

Here $\nu(z)$ is the outer unit normal of $K$ at the boundary point $z$, and $\mathcal{H}^{n-1}$ is the $(n-1)$ dimensional Hausdorff measure. For any $x \in \mathbb{R}^n$, we write $x = rz$, with $z \in \partial K$ and $dx = nV(K) r^{n-1} dr d\nabla K(z)$. Since the map $x \mapsto \nabla \|x\|_K$ is 0-homogeneous, and (12), we have

$$P(f) = \frac{1}{2} \int_{\mathbb{R}^n} \|\nabla \|x\|_K\|^2 e^{-\|x\|_K} dx = \frac{1}{2} \frac{nV(K)}{\|\nu\|_K} \int_0^\infty r^{n-1} \int_{\partial K} \|\nabla \|z\|_K\|^2 e^{-r} d\nabla K(z) dr = \frac{1}{2} \Gamma(n) nV(K) \int_{\partial K} \|\nabla \|z\|_K\|^2 d\nabla K(z),$$

where $\Gamma(\cdot)$ is the Gamma function. We need the fact that $\nabla \|z\|_K = \frac{\nu(z)}{\|z\|_K}$ when $z \in \partial K$ (see, e.g., [4]). Therefore,

$$P(f) = \frac{1}{2} \Gamma(n) nV(K) \int_{\partial K} \|\nabla \|z\|_K\|^2 d\nabla K(z) = \frac{1}{2} \Gamma(n) \int_{\partial K} h_K(\nu(z))^{-1} d\mathcal{H}^{n-1}(z) = \frac{1}{2} \Gamma(n) S_2(K).$$

From the fact that the map $x \mapsto \nabla \|x\|_K$ is 0-homogeneous, (12) and (10), we have

$$\frac{1}{2} \int_{\mathbb{R}^n} (\nabla \|x\|_K, T \nabla \|x\|_K) e^{-\|x\|_K} dx = \frac{1}{2} \int_{\mathbb{R}^n} (\nabla \|z\|_K, T \nabla \|z\|_K) e^{-r} d\nabla K(z) dr = \frac{1}{2} \Gamma(n) \int_{\partial K} \langle \nu(z), T \nu(z) \rangle h_K(\nu(z))^{-1} d\mathcal{H}^{n-1}(z) = \frac{1}{2} \Gamma(n) \int_{S^{n-1}} \langle u, Tu \rangle dS_2(K, u).$$

Hence, (4) implies that

$$\frac{\text{tr}(T)}{n} S_2(K) = \int_{S^{n-1}} \langle u, Tu \rangle dS_2(K, u)$$

for every $T \in \text{GL}(n)$. This means that the $L_2$ surface area measure of $K$, $S_2(K, \cdot)$, is an isotropic measure on $S^{n-1}$. \qed

4. Conclusions

Many outstanding works showed that the log-concave function is closely linked to the convex body. This paper presents a new connection between the theory of convex bodies and that of
log-concave functions. We study the minimal perimeter of a log-concave function which can be viewed as a functional version of the minimal $L_2$ surface area measure of a convex body. A characteristic theorem (Theorem 1) shows that a log-concave function $f$ has minimal perimeter if and only if the Borel measure $\frac{1}{\mu(f)} \mu(\cdot)$ is isotropic. The work done in this paper is mainly to propose a special position for log-concave functions and provides a new idea for the study of optimal problems for log-concave functions.

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