On the Large Mass Limit of the Continuum Theories in Kaplan’s Formulation

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Abstract
Being inspired by Kaplan’s proposal for simulating chiral fermions on a lattice, we examine the continuum analog of his domain-wall construction for two-dimensional chiral Schwinger models. Adopting slightly unusual dimensional regularization, we explicitly evaluate the one-loop effective action in the limit that the domain-wall mass goes to infinity. For anomaly-free cases, the effective action turns out to be gauge invariant in two-dimensional sense.

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1. Introduction

The \((2n+1)\)-dimensional Dirac fermion with the domain-wall mass has interesting properties as discussed by Callan and Harvey [1]. Here the domain-wall mass means the mass depending on the extra coordinate \(\sigma\): it behaves asymptotically like \(M(\sigma) \to \pm M\) as \(\sigma \to \pm \infty\) and vanishes at \(\sigma = 0\). A \(2n\)-dimensional chiral fermion can arise from the Dirac fermion and is bound to the domain wall \textit{i.e.} the \(2n\)-dimensional hyperplane specified by \(\sigma = 0\). It was argued that, when the \((2n+1)\)-dimensional fermion is coupled to external gauge fields, the gauge anomaly produced by this chiral mode is compensated with the current flow of the massive modes and the gauge invariance is kept intact as a whole.

The use of such chiral mode on the domain wall was recently proposed by Kaplan [2] to simulate chiral fermions on a lattice, which has been a long-standing problem because of the species doubling [3][4][5]. In fact, putting naïvely the Dirac fermion on a lattice, we find not only the chiral mode but also their doublers on the domain wall. In this case, however, we can introduce a Wilson term for the \((2n+1)\)-dimensional Dirac fermion in a gauge invariant way. Then it is possible to show that such doublers can be removed at least for free theories. More recently, the lattice analog of the Callan-Harvey analysis [6][7] shows that the same mechanism of the anomaly cancellation also works in the lattice setup even though the detail of the cancellation depends on the ratio between the Wilson coupling and the domain wall height \(M\).

Although it is a clever idea, there are several questions for the proposal. First of all, it is not clear how to reduce the freedom of the \((2n+1)\)-dimensional gauge bosons to \(2n\)-dimensional ones. The second one is how to separate the chiral zero-mode of the opposite chirality on the anti domain wall. Such an extra mode exists because of the periodicity condition on the compactified extra dimension which is a necessary condition practically to perform numerical simulation. The third question is, besides the anomaly cancellation, how the massive modes including doublers affect the low energy theory, \textit{i.e.} what about the decoupling of such modes.

The first and second questions are rather serious. Since the gauge field can propagate in the extra direction and feel both of chiralities, we are afraid that this theory may become vector-like. Especially, this is actually the case, in particular, if we make the gauge field independent of the extra coordinate \(\sigma\). Thus, keeping the dependence of the gauge field on
σ, we should carefully introduce the gauge field. In order to do so, we can conceive of at least two possibilities: one is that we prevent the gauge field from propagating in the extra direction. The other is that we make the gauge field couple only to the chiral zero-mode on the wall but not to the one on the anti-wall. This also needs an additional scalar field for the gauge invariance.

The first possibility can be realized, if the gauge coupling $\beta_\sigma$ in the extra direction goes to zero. Then all the four-dimensional gauge bosons at every $\sigma$ become independent each other. The gauge field coupled to the chiral zero-mode on the wall is different from that coupled to the chiral zero-mode on the anti-wall. But for small $\beta_\sigma$, in the mean-field approximation there emerges the layered phase and the fermion is entirely confined to the four-dimensional layers. Then the fermion propagator is found to be vector-like.

The latter possibility was discussed in [10], and was investigated in detail by the authors of [11]. They concluded that there does not exist the phase where the mirror mode on the waveguide could decouple. This result is also disappointing.

As for the third question, an example of the effect of doublers can be seen in [6][7] that the number of chiral fermions on the domain wall depends on the ratio between the Wilson coupling and the domain-wall height $M$. Aoki and Hirose [12] have evaluated the one-loop effective action for the domain-wall fermions in the would-be two-dimensional chiral Schwinger models and found a mass-like term for the gauge boson. As far as the gauge field depends on the extra coordinate $\sigma$, such a term exists. This term might indicate that, in the resultant low energy theory which we expect to be our target theory, we cannot obtain the two-dimensional gauge invariance even in an anomaly-free chiral gauge theory. Certainly such invariance in the two-dimensional sense is not guaranteed from the outset. They obtained this result keeping the finite lattice spacing for the extra direction finite, since it was still unclear how the low energy limit can be obtained. So there may exist the possibility that this mass-like term turns to vanish in such a limit.

In this paper we do not try to overcome the first two problems. Rather we address the third question, especially the problem of the mass-like term. In fact, this is also not clear even in the continuum theory. So our question here is whether such a mass-like term found by Aoki and Hirose, arises or not, even in the low energy limit of the continuum theory. If

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1 An alternative approach have been proposed by Narayanan and Neuberger, where the gauge field is $2n$-dimensional one and the extra dimension are not compactified. They have shown that their overlap formula reproduces the correct chiral anomaly and the effect of instanton.
this term does not appear in this case, the lattice regularization accounts for such breaking of the two-dimensional gauge invariance. Therefore we re-examine the system considered by Callan and Harvey i.e. a continuum analog of Kaplan’s formulation; in fact, using the explicit fermion propagator coupled to the domain-wall mass, we will evaluate the one-loop effective action for the domain-wall fermions.

To perform the calculation in the continuum theory, we adopt the dimensional regularization slightly different from the ordinary ones in the treatment of the Dirac gamma matrices as follows: to the two-dimensional space parallel to the domain wall, we apply the usual dimensional regularization, and for the remaining direction parametrized by $\sigma$, the Dirac gamma matrix is held to be $\gamma^2 = i\gamma^5 = i\gamma^0\gamma^1$. Once we perform loop-integrations, we can find results finite. We can remove the regularization i.e. $D \rightarrow 2$. Then we define the low energy limit as letting the domain-wall height $M$ go to infinity, the large mass limit. In this way, we show that the effective action in the anomaly-free case turns out to be gauge invariant as a two-dimensional theory; there is no mass-like term for the gauge boson in the large mass limit.

This paper is organized as follows: In section 2, we present all solutions to the Dirac equation coupled to the domain-wall mass including both the chiral modes and the massive modes. Then, adopting the canonical quantization, we derive the domain-wall fermion propagator. In section 3, we check that the above-mentioned dimensional regularization respects the three-dimensional gauge invariance. Chandrasekharan [13] recently calculated a similar propagator in Euclidean space and showed the anomaly cancellation explicitly. We can see from the Lagrangian extended to extra dimensions that this regularization keeps the gauge invariance. But the above-mentioned propagator fails to be the kernel of the domain-wall Dirac equation under this regularization. So this seems not so obvious. Our main result is contained in section 4, where we present the one-loop effective action for the domain-wall fermion in the large mass limit explicitly. Our conclusion and discussion are given in section 5. In Appendix A, we present the properties of the functions appearing in the domain-wall fermion propagator. For the convenience in the perturbation calculation, we define a set of functions and give their large mass limit in Appendix B. In Appendix C, the calculation in section 4 is explained in some detail.
2. Propagator for Domain-Wall Fermions

Following Kaplan [2], we consider a Dirac fermion with a domain-wall mass

\[ M(\sigma) = M\epsilon(\sigma), \quad (2.1) \]

where

\[ \epsilon(\sigma) = \begin{cases} 
1 & (\sigma > 0) \\
-1 & (\sigma < 0) 
\end{cases}, \quad (2.2) \]

in the three-dimensional Minkowsky space parametrized by \((x^0, x^1, x^2) = (x^0, x^1, \sigma)\), with metric \(\eta_{IJ} = \text{diag.}(1, -1, -1)\) \((I, J = 0, 1, 2)\). The Dirac equation is given by

\[ 0 = \left[ i\gamma^I \partial_I - M(\sigma) \right] \psi(x, \sigma) \]
\[ = \left[ i\gamma^\mu \partial_\mu - \left( \gamma^5 \frac{\partial}{\partial \sigma} + M(\sigma) \right) \right] \psi(x, \sigma). \quad (2.3) \]

\((I = 0, 1, 2; \mu = 0, 1)\)

Note that if \(\psi(x, \sigma)\) is a solution of (2.3), then \(\gamma^5 \psi(x, -\sigma)\) is, too. Therefore we can decompose the solutions into odd/even eigenstates under the operation

\[ \psi(x, \sigma) \rightarrow \psi^I(x, \sigma) = \gamma^5 \psi(x, -\sigma), \quad (2.4) \]

as

\[ \psi^L(x, \sigma) = \frac{1}{2} (\psi(x, \sigma) - \gamma^5 \psi(x, -\sigma)) \]
\[ = -\gamma^5 \psi^L(x, -\sigma), \quad (2.5) \]

which we call a left-handed eigenstate, and

\[ \psi^R(x, \sigma) = \frac{1}{2} (\psi(x, \sigma) + \gamma^5 \psi(x, -\sigma)) \]
\[ = +\gamma^5 \psi^R(x, -\sigma). \quad (2.6) \]

which we call a right-handed eigenstate. They have the same name of chiralities on the domain wall \(i.e. \) at \(\sigma = 0\),

\[ \gamma^5 \psi^L(x, \sigma = 0) = \pm \psi^R(x, \sigma = 0). \quad (2.7) \]

Now we present all the normalizable solutions of (2.3). We discard solutions that exponentially grow up as \(\sigma \rightarrow \pm \infty\). The chiral zero modes which we expect to represent a
two-dimensional chiral fermion, and which we hereafter call right-handed Weyl type wave functions, are given as

\[ U^R_W(p; \sigma) = \sqrt{2M e^{-M|\sigma|}} \begin{pmatrix} -i\sqrt{p_0} \\ 0 \end{pmatrix} \theta(-p) \]  

which correspond to the positive-energy solutions, and

\[ V^R_W(p; \sigma) = \sqrt{2M e^{-M|\sigma|}} \begin{pmatrix} +i\sqrt{p_0} \\ 0 \end{pmatrix} \theta(-p) \]  

which correspond to the negative-energy solutions, where \( p_0 = |p| \), and \( p \) denotes the momentum \( p_1 \) conjugate to the coordinate \( x^1 \). By the exponential factor, these chiral modes are seen to be bound to the domain wall. Note that these solutions are only right-handed eigenstates and have the ordinary right-handed chirality at any value of \( \sigma \). Such chirality for the chiral zero mode is determined by the sign of the domain-wall mass.

On the other hand, there exist both right- and left-handed eigenstates in the massive modes which we call Dirac type wave functions. Such right-handed eigenstates are given as

\[ U^R_D(p, \omega; \sigma) = \frac{1}{\sqrt{p_+}} \begin{pmatrix} \omega \cos \omega \sigma - M(\sigma) \sin \omega \sigma \\ -ip_+ \sin \omega \sigma \end{pmatrix} \]  

with positive energy, and

\[ V^R_D(p, \omega; \sigma) = \frac{1}{\sqrt{p_+}} \begin{pmatrix} \omega \cos \omega \sigma - M(\sigma) \sin \omega \sigma \\ +ip_+ \sin \omega \sigma \end{pmatrix} \]  

with negative energy. While the left-handed eigenstates are given as

\[ U^L_D(p, \omega; \sigma) = \frac{1}{\sqrt{p_-}} \begin{pmatrix} -ip_- \sin \omega \sigma \\ \omega \cos \omega \sigma + M(\sigma) \sin \omega \sigma \end{pmatrix} \]  

with positive energy, and

\[ V^L_D(p, \omega; \sigma) = \frac{1}{\sqrt{p_-}} \begin{pmatrix} +ip_- \sin \omega \sigma \\ \omega \cos \omega \sigma + M(\sigma) \sin \omega \sigma \end{pmatrix} \]  

with negative energy. Here

\[ p_0 = E = \sqrt{p^2 + \omega^2 + M^2}, \quad (p = p_1) \]  

and \( p_\pm = p_0 \pm p_1 = E \pm p \). \( \omega \) denotes the momentum \( p_2 \) conjugate to the extra coordinate \( \sigma \). Note that all these solutions are odd functions with respect to \( \omega \), so that the solutions with \( \pm \omega \) are not independent of each other.
These wave functions are normalized such that
\[
\int d\sigma U_D^{A\dagger}(p,\omega';\sigma)U_D^B(p,\omega;\sigma) = \int d\sigma V_D^{A\dagger}(p,\omega';\sigma)V_D^B(p,\omega;\sigma)
= 2E \times \frac{1}{2}(2\pi)\delta^{AB} (\delta(\omega - \omega') - \delta(\omega + \omega')) , \quad (A, B = L, R)
\] (2.15)

\[
\int d\sigma U_W^{R\dagger}(p;\sigma)U_W^R(p;\sigma) = \int d\sigma V_W^{R\dagger}(p;\sigma)V_W^R(p;\sigma)
= 2|p| , \quad (2.16)
\]
and otherwise vanish. And they satisfy the following completeness relation:
\[
\frac{1}{2p_0} \left[ U_W^R(p,\sigma')U_W^{R\dagger}(p,\sigma) + V_W^R(p,\sigma')V_W^{R\dagger}(p,\sigma) \right] = M e^{-M(|\sigma'|+|\sigma|)} P_R . \quad (2.17)
\]

\[
\int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)^2} \frac{1}{2p_0} \sum_{A=R,L} \left[ U_D^A(p,\omega;\sigma')U_D^{A\dagger}(p,\omega;\sigma) + V_D^A(-p,\omega;\sigma')V_D^{A\dagger}(-p,\omega;\sigma) \right]
= \delta(\sigma' - \sigma) - M e^{-M(|\sigma'|+|\sigma|)} P_R , \quad (2.18)
\]

with \( P_{(R)} = \frac{1 \pm \gamma_5}{2} \).

Now we are ready to derive the domain-wall fermion propagator following the canonical quantization. The field operator \( \psi(x,\sigma) \) is expanded as follows:
\[
\psi(x,\sigma) = \psi_W^R(x,\sigma) + \psi_D^R(x,\sigma) + \psi_D^L(x,\sigma) , \quad (2.19)
\]
where
\[
\psi_W^R(x,\sigma) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}\sqrt{2|p|}} \left[ b(p)e^{-ip_\mu x^\mu} U_W^R(p,\sigma) + d(p)e^{ip_\mu x^\mu} V_W^R(p,\sigma) \right] , \quad (2.20)
\]
which consists of the chiral zero modes,
\[
\psi_D^R(x,\sigma) = \int_{-\infty}^{\infty} \frac{dpd\omega}{(2\pi)^2} \left[ B_R(p,\omega)e^{-ip_\mu x^\mu} U_D^R(p,\omega;\sigma) + D_R(p,\omega)e^{ip_\mu x^\mu} V_D^R(p,\omega;\sigma) \right] , \quad (2.21)
\]
which consists of the massive right-handed modes, and

$$\psi_D^L(x, \sigma) = \int_{-\infty}^{\infty} \frac{dp d\omega}{(2\pi)\sqrt{2E}} \left[ B_L(p, \omega) e^{-ip \cdot x'} U_D^L(p, \omega; \sigma) + D_L^\dagger(p, \omega) e^{ip \cdot x''} V_D^R(p, \omega; \sigma) \right],$$

(2.22)

which consists of the massive left-handed modes. Since the wave functions $U_D$ and $V_D$ are odd under $\omega \leftrightarrow -\omega$, the annihilation (creation) operators $B^{(\dagger)}$ and $D^{(\dagger)}$ defined here satisfy the following relations:

$$B^{(\dagger)}_L(p, \omega) = -B^{(\dagger)}_L(p, -\omega),$$

(2.23)

$$D^{(\dagger)}_L(p, \omega) = -D^{(\dagger)}_L(p, -\omega).$$

(2.24)

Because of these dependence relations, the $\omega$-integrations in (2.21) and (2.22) seem to count double the independent modes. But the double counting is properly avoided by the factor $\frac{1}{2}$ put in the normalization condition (2.15). From the equal-time commutation relation

$$\{ \psi(t, x^1, \sigma), \psi(t, x'^1, \sigma') \} = \delta(x^1 - x'^1)\delta(\sigma - \sigma'),$$

(2.25)

we can derive the domain-wall fermion propagator [13], in a similar fashion to the ordinary fermion case:

$$\langle 0 | T \psi(x, \sigma) \bar{\psi}(x', \sigma') | 0 \rangle = S_D^W(x, \sigma; x', \sigma') = S_D^F(x, \sigma; x', \sigma') + S_D^L(x, \sigma; x', \sigma'),$$

(2.26)

where

$$S_D^W(x, \sigma; x', \sigma') = \langle 0 | T \psi_W^R(x, \sigma) \bar{\psi}_W^R(x', \sigma') | 0 \rangle$$

$$= \int \frac{d^2p}{4\pi} \frac{1}{-p^2 - i\epsilon} e^{-ip \cdot (x-x')} \mathcal{P}_R \hat{\phi},$$

(2.27)

which represents the two-dimensional chiral fermion propagator localized on the domain wall and comes from only the chiral zero modes. On the other hand,

$$S_D^F(x, \sigma; x', \sigma') = \sum_{A,B=R,L} \langle 0 | T \psi_D^A(x, \sigma) \bar{\psi}_D^B(x', \sigma') | 0 \rangle$$

$$= \int \frac{d^2p d\omega}{(2\pi)^3} \frac{1}{M^2 + \omega^2 - p^2 - i\epsilon} e^{-ip \cdot (x-x')}$$

$$\left[ \mathcal{P}_R (\xi(\omega; \sigma', \sigma) + \hat{\phi}_+(\omega; \sigma, \sigma')) + \mathcal{P}_L (\xi(\omega; \sigma', \sigma) + \hat{\phi}_-(\omega; \sigma, \sigma')) \right].$$

(2.28)
which comes from the massive modes, where

\[
\xi(\omega; \sigma, \sigma') \equiv f_+(\omega, \sigma) \sin \omega \sigma' - f_-(\omega, \sigma') \sin \omega \sigma, \quad (2.29)
\]
\[
\varphi_\pm(\omega; \sigma, \sigma') \equiv \frac{1}{\omega^2 + M^2} f_\pm(\omega, \sigma) f_\pm(\omega, \sigma') + \sin \omega \sigma \sin \omega \sigma', \quad (2.30)
\]
\[
f_\pm(\omega, \sigma) \equiv \omega \cos \omega \sigma \pm M(\sigma) \sin \omega \sigma. \quad (2.31)
\]

The properties of these functions are shown in Appendix A. Note that this propagator expression is valid in any \((2n + 1)\) dimensions, if \(p_\mu\) is understood to be \(2n\)-dimensional one.

3. Anomaly Cancellation

We couple an external Abelian gauge field to this system and investigate the gauge anomaly. Of course, it is obvious that there is no gauge anomaly in odd dimensions, because in such dimensions, all fermions are Dirac fermions so that gauge anomalies are automatically cancelled. Here we are not interested in whether this anomaly is cancelled or not, but in how this cancellation occurs. Since we have the right-handed chiral modes on the domain wall, it seems at first sight that this mode produces the gauge anomaly. Callan and Harvey[1] resolved this seemingly paradox. The gauge anomaly produced by this chiral mode is compensated with the current flow of the massive modes and the gauge invariance is maintained. Their argument is qualitatively clear and elegant, without using the explicit propagator which we derived as above. But for our purpose, we need to investigate the behavior of the massive modes explicitly, and make it clear whether this theory is chiral or not in the large mass limit \(M \to \infty\).

In calculating this gauge anomaly, we adopt the dimensional regularization in such a manner as explained in the introduction; namely, we use the usual dimensional regularization only for the first two dimensions and keep the \(\sigma\) dimension intact.

In this regularization scheme, our propagator is not the kernel of the free Dirac equation \([i \gamma^\alpha \partial_\alpha - (\gamma^5 \partial_{\sigma} + M(\sigma))]\), where \(\alpha = (\mu, j)\) and \('j'\) denotes the extra components due to the dimensional regularization. The reason is that it includes the “chiral” projection operators \(\mathcal{P}_L^{(\bar{L})}\) with \(\gamma^5\) fixed to be \(\gamma^0\gamma^1\) and so a similar situation occurs to that of usual chiral fermions; not all the components of dimensionally extended \(\gamma^\alpha\) anti-commute with
the directions parallel to the domain wall,

\[ S_F(x, \sigma; x', \sigma') = \int \frac{d^Dp}{i(2\pi)^D} e^{-ip(x-x')} S_F(p; \sigma, \sigma'). \]  

(3.1)

Then the deviation from the Dirac equation can be verified as that

\[
\left[ \not{\gamma} + i\gamma^2 \frac{\partial}{\partial \sigma} - M(\sigma) \right] S_F(p; \sigma, \sigma') = -\delta(\sigma - \sigma') + \gamma^5 \not{\gamma} \Delta_F(p; \sigma, \sigma'),
\]

(3.2)

and

\[
S_F(p; \sigma', \sigma) \left[ \not{\gamma} - i\gamma^2 \frac{\partial}{\partial \sigma} - M(\sigma) \right] = -\delta(\sigma - \sigma') + 2\gamma \Delta_F(p; \sigma', \sigma),
\]

(3.3)

where \( \not{\gamma} = \gamma^\alpha p_\alpha = \not{p} + \not{\mu} \) with \( \not{p} = \gamma^\mu p_\mu \) and \( \not{\mu} = \gamma^j p_j \), and

\[ \Delta_F(p; \sigma, \sigma') = \Delta^W_F(p; \sigma, \sigma') + \Delta^D_F(p; \sigma, \sigma'), \]

(3.4)

\[
\begin{aligned}
\Delta^W_F(p; \sigma, \sigma') &= Me^{-M(|\sigma|+|\sigma'|)} \frac{1}{-p^2-i\epsilon} \not{\gamma}, \\
\Delta^D_F(p; \sigma, \sigma') &= \int \frac{d\omega}{2\pi} \frac{1}{M^2 + \omega^2 - p^2 - i\epsilon} \left[ \langle \xi(\omega; \sigma', \sigma) - \xi(\omega; \sigma, \sigma') \rangle \right. \\
&\hspace{2cm} - \langle \not{\gamma}(\varphi_-(\omega; \sigma, \sigma') - \varphi_+(\omega; \sigma, \sigma')) \rangle,
\end{aligned}
\]

(3.5)

and

\[ \tilde{\Delta}_F(p; \sigma', \sigma) = \tilde{\Delta}^W_F(p; \sigma', \sigma) + \tilde{\Delta}^D_F(p; \sigma', \sigma), \]

(3.6)

\[
\begin{aligned}
\tilde{\Delta}^W_F(p; \sigma', \sigma) &= -Me^{-M(|\sigma|+|\sigma'|)} \frac{1}{-p^2-i\epsilon} M(\sigma) \mathcal{P}_R, \\
\tilde{\Delta}^D_F(p; \sigma', \sigma) &= \int \frac{d\omega}{2\pi} \frac{1}{M^2 + \omega^2 - p^2 - i\epsilon} \left[ \mathcal{P}_R \langle \xi(\omega; \sigma, \sigma') - M(\sigma) \varphi_-(\omega; \sigma, \sigma') \rangle \\
&\hspace{2cm} + \mathcal{P}_L \langle \xi(\omega; \sigma', \sigma) - M(\sigma) \varphi_+(\omega; \sigma, \sigma') \rangle \right].
\end{aligned}
\]

(3.7)

This situation is different from that for the ordinary three-dimensional Dirac fermions, which does not have such deviation. This fact makes the calculation for the anomaly cancellation nontrivial.

We proceed to the calculation for the gauge anomaly. The gauge current is defined, as usual;

\[ J^I(x, \sigma) = \bar{\psi}(x, \sigma) \gamma^I \psi(x, \sigma). \]

(3.8)
Then the total divergence of this current turns out to be

\[
\partial_I \langle J^I(x, \sigma) \rangle = ie \int d\sigma' \int \frac{d^2 p}{(2\pi)^2} e^{-ipx} A_I(p, \sigma') \int \frac{d^D k}{i(2\pi)^D} \times \left[ \text{tr} \left\{ \left( \frac{\not{p} + \not{k}}{} + i\gamma^2 \frac{\partial}{\partial \sigma} - M(\sigma) \right) S_F(p + k; \sigma, \sigma') \gamma^I S_F(k; \sigma', \sigma) \right\} 
- \text{tr} \left[ S_F(p + k; \sigma, \sigma') \gamma^I S_F(k; \sigma', \sigma) \right] \right],
\]

where \( A_I(p, \sigma) \) \((I = 0, 1, 2)\) is the external Abelian gauge field with the coordinates \( x^\mu \) Fourier-transformed to the momenta \( p^\mu \) and \( e \) is the coupling constant.

In fact, this anomaly \( \partial_I \langle J^I(x, \sigma) \rangle \) vanishes, since the massive modes give the same contribution with the opposite sign as the chiral zero mode which has the exponential damping factor. In order to see this explicitly, firstly, we evaluate the first term in the square bracket of (3.9). Then, we distinguish the contribution to this anomaly into four types, as follows:

\[
\int \frac{d^D k}{(2\pi)^D i} \text{tr} \left[ \gamma^I \frac{\not{k} \Delta F}{\not{k} \Delta F} (p + k; \sigma, \sigma') \gamma^I S_F(k; \sigma', \sigma) \right]
- \text{tr} \left[ S_F(p + k; \sigma, \sigma') \gamma^I \frac{\not{k} \Delta F}{\not{k} \Delta F} (k; \sigma', \sigma) \right],
\]

They are evaluated to be

\[
\int \frac{d^D k}{(2\pi)^D i} \text{tr} \left[ \gamma^I \frac{\not{k} \Delta W}{\not{k} \Delta W} (p + k; \sigma, \sigma') \gamma^I S_F^W(k; \sigma', \sigma) \right]
\]
\begin{equation}
\int \frac{d^D k}{(2\pi)^D} \text{tr} \left[ \gamma^5 k \Delta^W_F(p + k; \sigma, \sigma') \gamma^I S^D_F(k; \sigma', \sigma) \right]
= - \frac{1}{4\pi} M^2 e^{-2M(|\sigma| + |\sigma'|)} [\epsilon^{2\mu I} + g^{\mu I}] p_{\mu}.
\end{equation}

\begin{equation}
\int \frac{d^D k}{(2\pi)^D} \text{tr} \left[ \gamma^5 k \Delta^D_F(p + k; \sigma, \sigma') \gamma^I S^W_F(k; \sigma', \sigma) \right]
= - \frac{1}{4\pi} M e^{-M(|\sigma| + |\sigma'|)}
\times [\epsilon^{2\mu I} (2\delta(\sigma - \sigma') - M e^{-M(|\sigma| + |\sigma'|)}) - g^{\mu I} M e^{-M(|\sigma| + |\sigma'|)}] p_{\mu}.
\end{equation}

\begin{equation}
\int \frac{d^D k}{(2\pi)^D} \text{tr} \left[ \gamma^5 k \Delta^D_F(p + k; \sigma, \sigma') \gamma^I S^W_F(k; \sigma', \sigma) \right]
= \frac{1}{4\pi} M^2 e^{-2M(|\sigma| + |\sigma'|)} [\epsilon^{2\mu I} + g^{\mu I}] p_{\mu}.
\end{equation}

\begin{equation}
\int \frac{d^D k}{(2\pi)^D} \text{tr} \left[ \gamma^5 k \Delta^D_F(p + k; \sigma, \sigma') \gamma^I S^D_F(k; \sigma', \sigma) \right]
= \frac{1}{4\pi} M e^{-M(|\sigma| + |\sigma'|)}
\times [\epsilon^{2\mu I} (2\delta(\sigma - \sigma') - M e^{-M(|\sigma| + |\sigma'|)}) - g^{\mu I} M e^{-M(|\sigma| + |\sigma'|)}] p_{\mu}.
\end{equation}

From these equations, we see that (3.11) and (3.13) cancel each other, and so do (3.12) and (3.14). As is different from the ordinary Dirac fermion, the contribution (3.14) has the exponential damping factor and is of chiral type, despite that it comes from massive Dirac-type modes alone. We see that similar things also happen for the other remaining terms in the square bracket of (3.9). Thus, the gauge anomaly \( \partial_I \langle J^I(x, \sigma) \rangle \) turns out to be zero [13].

\begin{equation}
\partial_I \langle J^I(x, \sigma) \rangle = 0.
\end{equation}

4. One-Loop Effective Action

Usually, if we adopt the ordinary dimensional regularization for chiral fermions, we can not get a gauge invariant answer even in anomaly-free chiral gauge theories. The reason is well known to be that such a regularization does not respect the gauge invariance. On the other hand in the case under consideration, the contribution from the chiral modes are summed up with that from the massive modes to yield a three-dimensional gauge invariant result, as we have seen in the previous section. What happens there is the following; the functions \( \varphi_{\pm} \left( \omega; \sigma, \sigma' \right) \) included in the Dirac type propagator represent “the densities of the massive modes” for the right- and left-handed eigenstate, respectively. Then the difference between \( \varphi_{\pm} \left( \omega; \sigma, \sigma' \right) \)’s accounts for the mismatching of the number of modes between right-
and left-handed ones. Thus we may expect that the contribution from the massive modes can be essentially decomposed into two parts: one from a massive Dirac fermion and the other from the just-mentioned difference; that is, in the large mass limit \( M \to \infty \), only the former decouples, while the difference survives the limit and is summed up with the chiral mode contribution to yield a gauge invariant answer even in our dimensional regularization.

We show in this section that this is the case in the present model, evaluating the one-loop effective action in the gauge \( A_2 = 0 \) as follows:

\[
e^{i\Gamma[A]} \equiv \text{Det} \left[ i\not\!D - M(\sigma) + eA(x, \sigma) \right].
\]

\[
\Gamma[A] = -\frac{e^2}{2} \int \frac{d^2p}{(2\pi)^2} d\sigma d\sigma' A_\mu(-p, \sigma) \Pi^{\mu\nu}(p; \sigma, \sigma') A_\nu(p, \sigma') + \ldots
\]  

with

\[
\Pi^{\mu\nu}(p; \sigma, \sigma') \equiv \int \frac{d^Dk}{i(2\pi)^D} \text{tr} \left[ \gamma^\mu S_F(p + k; \sigma, \sigma') \gamma^\nu S_F(k; \sigma', \sigma) \right],
\]

where the dots denotes the higher order terms in \( A_\mu \). The vacuum polarization \( \Pi^{\mu\nu}(p; \sigma, \sigma') \) has three distinct contributions depending on whether the two internal fermion lines are Weyl-type \( S_F^W \) or Dirac-type \( S_F^D \):

\[
\Pi_W^{\mu\nu}(p; \sigma, \sigma') = \int \frac{d^Dk}{(2\pi)^D} \text{tr} \left[ \gamma^\mu S_F^W(p + k; \sigma, \sigma') \gamma^\nu S_F^W(k; \sigma', \sigma) \right],
\]

\[
\Pi_M^{\mu\nu}(p; \sigma, \sigma') = \int \frac{d^Dk}{(2\pi)^D} \text{tr} \left[ \gamma^\mu S_F^W(p + k; \sigma, \sigma') \gamma^\nu S_F^D(k; \sigma', \sigma) \right],
\]

\[
\Pi_D^{\mu\nu}(p; \sigma, \sigma') = \int \frac{d^Dk}{(2\pi)^D} \text{tr} \left[ \gamma^\mu S_F^D(p + k; \sigma, \sigma') \gamma^\nu S_F^D(k; \sigma', \sigma) \right].
\]

We calculate these contributions in our regularization scheme. Since these quantities turn out to be finite, we can remove the regularization, i.e., \( D \to 2 \). After that, we take the magnitude \( M \) of the domain-wall mass to infinity.

As we mentioned above, the ordinary dimensional regularization breaks the gauge invariance in the chiral gauge theories, so we cannot get even the parity-even contribution of e.g., the vacuum polarization, in a gauge invariant way. This is the case also here, but only in the pure Weyl-contribution \( \Pi_W^{\mu\nu}(p; \sigma, \sigma') \), which is expected to represent the vacuum polarization of the two-dimensional chiral gauge theory. However we will see later that the large mass limit of the total vacuum polarization \( \Pi^{\mu\nu}(p; \sigma, \sigma') \) becomes gauge invariant in the two-dimensional sense, if we consider anomaly-free chiral gauge theories.

In order to see this, we would like to show how these contributions are summed up to yield a gauge invariant result. So we evaluate each of the contributions separately now.
4.1. Pure Weyl-type contribution $\Pi^\mu\nu_W(p;\sigma,\sigma')$

Removing the dimensional regularization, the pure Weyl-type contribution $\Pi^\mu\nu_W(p;\sigma,\sigma')$ is seen to be

$$\Pi^\mu\nu_W(p;\sigma,\sigma') = \frac{1}{4\pi} M^2 e^{-2(|\sigma|+|\sigma'|)} \frac{1}{p^2} T^\mu\nu\rho\sigma(p)p\rho p\sigma$$

$$\to_{M\to\infty} \left( \frac{1}{4\pi} \right) \frac{1}{p^2} \left[ 2(p^\mu p^\nu - g^\mu\nu p^2) - (p^\mu \epsilon^\nu p + p^\nu \epsilon^\mu p) p^\rho \right] \delta(\sigma)\delta(\sigma')$$

$$+ \frac{1}{4\pi} (g^\mu\nu + \epsilon^\mu\nu) \delta(\sigma)\delta(\sigma'),$$

where

$$T^\mu\nu\rho\sigma_{(L\,R)} = \text{tr}[\mathcal{P}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma)],$$

and in the large mass limit $M\to\infty$,

$$\lim_{M\to\infty} Me^{-2M|\sigma|} = \delta(\sigma).$$

From (4.3), we can see that even the parity-even part of this quantity is not gauge invariant.

4.2. Mixed-type contribution $\Pi^\mu\nu_M(p;\sigma,\sigma') + \Pi^\mu\nu_M(-p;\sigma',\sigma)$

We have only to evaluate the part $\Pi^\mu\nu_M(p;\sigma,\sigma')$ of the mixed-type contribution. After the loop-integration over the momentum $k$, we can remove the regularization, as mentioned above. Since this quantity has no infra-red singularity, we can Taylor-expand it with respect to the momentum $p^\mu$. We can see from the dimensional analysis that all the terms other than the leading order one in this expansion, turn to be zero in the large mass limit $M\to\infty$.

$$\Pi^\mu\nu_M(p;\sigma,\sigma') = -(\frac{1}{4\pi}) Me^{-2M|\sigma|}(g^\mu\nu - \epsilon^\mu\nu)\delta(\sigma - \sigma')$$

$$- (\frac{1}{4\pi}) Me^{-2M(|\sigma|+|\sigma'|)} T^\mu\nu\rho\sigma_{(L\,R)} p\rho p\sigma$$

$$\times \int_1^0 d\alpha(1-\alpha) \int \frac{d\omega}{2\pi} \frac{\varphi_-(\omega;\sigma,\sigma')}{\omega^2 + M^2 - (1-\alpha)p^2}$$

$$+ O(\frac{p^2}{M}),$$

with $\alpha$ being Feynman parameter. Since the second term in (4.8) is also seen to vanish in the limit $M\to\infty$, 

$$Me^{-2M(|\sigma|+|\sigma'|)} \int_0^1 d\alpha(1-\alpha) \int \frac{d\omega}{2\pi} \frac{\varphi_-(\omega;\sigma,\sigma')}{\omega^2 + M^2} \to_{M\to\infty} 0,$$
we obtain
\[ \Pi^\mu_\nu_M(p; \sigma, \sigma') \rightarrow_\infty \frac{1}{\mu^2} \frac{1}{(g^{\mu\nu} - \epsilon^{\mu\nu}) \delta(\sigma) \delta(\sigma')} \] (4.10)

Therefore the mixed-type contribution turns out to be
\[ \Pi^\mu_\nu_M(p; \sigma, \sigma') + \Pi^\mu_\nu_M(-p; \sigma', \sigma) \rightarrow_\infty \frac{1}{\mu^2} \frac{1}{(g^{\mu\nu} + \epsilon^{\mu\nu}) \delta(\sigma) \delta(\sigma')} \] (4.11)

4.3. Pure Dirac-type contribution \( \Pi^\mu_\nu_D(p; \sigma, \sigma') \)

The calculation for this pure Dirac-type contribution \( \Pi^\mu_\nu_D(p; \sigma, \sigma') \) is rather involved. Here we only present the result in the large mass limit. The derivation are given in detail in Appendix C.

\[ \Pi^\mu_\nu_D(p; \sigma, \sigma') \]
\[ \frac{\Gamma(2 - \frac{D}{2})}{(4\pi)^{\frac{D}{2}}} \frac{1}{M^2 + \alpha \omega^2 + (1 - \alpha) \omega'^2 - \alpha(1 - \alpha)p^2} \int_0^\infty d\alpha \int \frac{d\omega d\omega'}{(2\pi)^2} \]
\[ \times \left\{ \frac{1}{[M^2 + \alpha \omega^2 + (1 - \alpha) \omega'^2 - \alpha(1 - \alpha)p^2]^{1 - \frac{D}{2}}} \right. \]
\[ \times \left\{ \text{tr}(P_L \gamma^\mu \gamma^\nu)(\varphi_+ (\omega; \sigma, \sigma') - \varphi_-(\omega; \sigma, \sigma')) \varphi_+ (\omega'; \sigma', \sigma) \right. \]
\[ + \text{tr}(P_R \gamma^\mu \gamma^\nu)(\varphi_-(\omega; \sigma, \sigma') - \varphi_+(\omega; \sigma, \sigma')) \varphi_-(\omega'; \sigma', \sigma) \right\} \] (4.12)
\[ - \frac{1}{[M^2 + \alpha \omega^2 + (1 - \alpha) \omega'^2 - \alpha(1 - \alpha)p^2]^{2 - \frac{D}{2}}} \]
\[ \times \frac{\partial}{\partial \sigma} \left\{ M(\sigma) [\text{tr}(P_L \gamma^\mu \gamma^\nu) \varphi_+ (\omega; \sigma, \sigma') \varphi_+ (\omega'; \sigma', \sigma) \right. \]
\[ - \text{tr}(P_R \gamma^\mu \gamma^\nu) \varphi_-(\omega; \sigma, \sigma') \varphi_-(\omega'; \sigma', \sigma)] \right\}, \]

where we neglect the irrelevant terms in the large mass limit. Using the formulae of the \( \omega \)-integration for \( \varphi_\pm \) in Appendix A, the first term are seen to be finite at \( D = 2 \) and can be easily evaluated. As for the second term, the result are derived in Appendix C. As we can see there, this term is also finite. In the large mass limit, we obtain
\[ \Pi^\mu_\nu_D(p; \sigma, \sigma') \rightarrow_\infty \frac{1}{\mu^2} \left[ (g^{\mu\nu} - \epsilon^{\mu\nu}) \delta(\sigma) \delta(\sigma') + \epsilon^{\mu\nu} \frac{\partial}{\partial \sigma} (\epsilon(\sigma) \delta(\sigma - \sigma')) \right] \] (4.13)
4.4. The total vacuum polarization and the one-loop effective action

Summing up three contributions evaluated in the previous subsections, we find the total vacuum polarization $\Pi^{\mu\nu}(p; \sigma, \sigma')$ in the large mass limit to be

$$\Pi^{\mu\nu}(p; \sigma, \sigma') \sim \Pi_\text{W}^{\mu\nu}(p; \sigma, \sigma') + \Pi_\text{M}^{\mu\nu}(p; \sigma, \sigma') + \Pi_\text{D}^{\mu\nu}(p; \sigma, \sigma') \mathrel{\xrightarrow{M \to \infty}} \frac{1}{2\pi} \frac{1}{p^2} (p^\mu p^\nu - g^{\mu\nu} p^2) \delta(\sigma) \delta(\sigma') - \frac{1}{4\pi} \frac{1}{p^2} (p^\mu \epsilon^{\mu\rho} + p^\nu \epsilon^{\nu\rho}) p_\rho \delta(\sigma) \delta(\sigma') + \frac{1}{4\pi} \epsilon^{\mu\nu} \frac{\partial}{\partial \sigma} (\epsilon(\sigma) \delta(\sigma - \sigma')).$$

Finally, when the magnitude $M$ of the domain-wall mass goes to infinity, the one-loop effective action $\Gamma[A]$ has a limit

$$\Gamma[A] \mathrel{\xrightarrow{M \to \infty}} - \frac{\epsilon^2}{4\pi} \int \frac{d^2 p}{(2\pi)^2} \left[ \frac{1}{p^2} A_\mu(-p, \sigma = 0) (p^\mu p^\nu - g^{\mu\nu} p^2) A_\nu(p, \sigma = 0) + \frac{1}{p^2} p^\rho A_\rho(-p, \sigma = 0) \epsilon^{\mu\nu} p_\mu A_\nu(p, \sigma = 0) + \int d\sigma \frac{1}{2} \epsilon^{\mu\nu} A_\mu(-p, \sigma) \epsilon(\sigma) \frac{\partial}{\partial \sigma} A_\nu(p, \sigma) \right] + \cdots. \tag{4.15}$$

We see that the first term, the parity-even part, is gauge invariant in two-dimensional sense, which is expected as the contribution in two-dimensional chiral gauge theories. However note that this part could not be made gauge invariant, if the contribution came from only the chiral zero mode. On the other hand, we can regard the second term as the chiral anomaly from a two-dimensional chiral fermion, while the third term represents the current flow or the Chern-Simons term, which may be compared with the Goldstone-Wilczek current discussed by Callan and Harvey [1].

If we consider anomaly-free chiral Schwinger models instead, the second and third terms in (4.15) are absent, and so a gauge invariant one-loop effective action results. It may be interesting to compare this result with the lattice version [12] by Aoki and Hirose, though in the latter the contribution from the doublers might be important.
5. Conclusions and Discussions

We analyzed the would-be chiral Schwinger models in the continuum version of the Kaplan’s formulation \[2\]. At first we derived the domain-wall fermion propagator in the explicit form. It consists of two parts: the propagator of the chiral mode bound to the domain wall and the propagator of the massive modes. Using this propagator, we performed the perturbative expansion for the one-loop effective action of the domain-wall fermion. In this calculation, we adopt the dimensional regularization explained in the text, which respects the gauge invariance. In fact we have concretely shown how the divergence of the gauge current vanishes in this regularization. After we have verified that the super-renormalizability of this theory allows us to remove the regularization, we made the domain-wall height \( M \) go to infinity in the effective action. Then we have shown that this action turns out to be gauge invariant in two-dimensional sense in the large mass limit, if we consider anomaly-free cases. This is an interesting result, because the three-dimensional gauge invariance reduce to the two-dimensional one in the low energy theory without adding any noninvariant counterterm; namely, the whole domain wall system serves as a gauge-invariant regularization for the two-dimensional chiral Schwinger models. On the other hand for anomalous case, we got both the anomaly term which is expected from the chiral zero mode and the Chern-Simons term from the massive modes as discussed by Callan and Harvey \[1\], besides the above gauge invariant parity-even term.

As we mentioned in the introduction, Aoki and Hirose \[12\] have calculated the one-loop effective action in the lattice counterpart of the models discussed here. Since the extra space remains discretized in their calculation, we cannot directly compare their results with ours. But our results indicate the possibility that there exists the lattice counterpart of the large mass limit we discussed here, and that the gauge boson mass-like term vanishes in such a limit. Otherwise, the nondecoupling effect of the doublers should account for the mass-like term. So it is an interesting attempt to identify such a limit in their effective action.

If we can succeed in it, the problem of the mass-like term will be harmless in thinking of the gauge field dependent on the extra coordinate. Then the remaining problems are how to reduce the freedom of the \((2n + 1)\)-dimensional gauge bosons to \(2n\)-dimensional ones and how to separate the chiral zero-mode of the opposite chirality on the anti domain wall, as mentioned in the introduction. We surely need further consideration on these issues.
So far, we have discussed as target theories the two-dimensional models which are super-renormalizable. If we consider the four-dimensional models, we encounter the UV divergence. Then we cannot avoid to modify the definition of the large mass limit. Therefore our results can not be extended to that case straightforwardly. We would like to discuss this problem elsewhere.

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Appendix A. Properties of the functions appearing in the propagator

The functions

\[ f_\pm(\omega, \sigma) \equiv \omega \cos \omega \sigma \pm M(\sigma) \sin \omega \sigma, \]  
\[ \varphi_\pm(\omega; \sigma, \sigma') \equiv \frac{1}{\omega^2 + M^2} f_\pm(\omega, \sigma) f_\pm(\omega, \sigma') + \sin \omega \sigma \sin \omega \sigma', \]  
\[ \xi(\omega; \sigma, \sigma') \equiv f_+(\omega, \sigma) \sin \omega \sigma' - f_-(\omega, \sigma') \sin \omega \sigma \]
\[ = -\omega \sin \omega (\sigma - \sigma') + (M(\sigma) + M(\sigma')) \sin \omega \sigma \sin \omega \sigma'. \]  

appearing in the propagator, satisfy the following differential equations:

\[ \left[ \frac{\partial}{\partial \sigma'} + M(\sigma') \right] \xi(\omega; \sigma, \sigma') = (\omega^2 + M^2) \varphi_+(\omega; \sigma, \sigma'). \]  
\[ \left[ \frac{\partial}{\partial \sigma} - M(\sigma) \right] \xi(\omega; \sigma, \sigma') = -(\omega^2 + M^2) \varphi_-(\omega; \sigma, \sigma'). \]  
\[ \left[ \frac{\partial}{\partial \sigma} + M(\sigma) \right] \varphi_-(\omega; \sigma, \sigma') = \xi(\omega; \sigma, \sigma'). \]  
\[ \left[ \frac{\partial}{\partial \sigma} - M(\sigma) \right] \varphi_+(\omega; \sigma, \sigma') = -\xi(\omega; \sigma', \sigma). \]  

Some useful formulae, in particular, for proving the completeness of the wave functions are:

\[ e^{\pm i\omega|\sigma|} = \cos \omega \sigma \pm i \epsilon(\sigma) \sin \omega \sigma. \]  
\[ \int \frac{d\omega}{2\pi} \varphi_+ (\omega; \sigma, \sigma') = \delta(\sigma - \sigma') \]  
\[ \int \frac{d\omega}{2\pi} \varphi_- (\omega; \sigma, \sigma') = \delta(\sigma - \sigma') - Me^{-M(|\sigma|+|\sigma'|)} \]  

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Appendix B. Definition of the functions appearing in the perturbation calculation and their limiting form in $M \to \infty$

When we calculate the one-loop effective action after integrating over the loop momentum parallel to the domain wall, we encounter the following function

$$I_n(\sigma; a) \equiv M \int_0^\infty \tau^{n-1} \exp \left( -\tau - \frac{M^2}{4\tau} \sigma^2 a \right) d\tau, \quad \text{(for } a > 0) \quad (B.1)$$

which may be compared with the modified Bessel function $K_\nu(x)$.

In the limit $M \to \infty$,

$$I_n(\sigma; a) \to \left( \frac{4\pi}{a} \right)^{\frac{1}{2}} \Gamma(n + \frac{1}{2}) \delta(\sigma). \quad (B.2)$$

Another function we meet is:

$$J_n(\sigma, \sigma'; a) \equiv M^2 \int_0^\infty d\tau \tau^{n-1} \exp \left( -\tau - \frac{M^2}{4\tau} \left( \frac{\sigma^2}{a} + \frac{\sigma'^2}{1-a} \right) \right). \quad \text{(for } a > 0) \quad (B.3)$$

This has a limit,

$$J_n(\sigma, \sigma'; a) \to M \to \infty 4\pi \Gamma(n + 1)(a(1-a))^{\frac{1}{2}} \delta(\sigma) \delta(\sigma'). \quad (B.4)$$

In addition, the following relations hold:

$$\frac{\partial}{\partial \alpha} f^\frac{1}{2}(\alpha) I_{n-1}(\sigma; f(\alpha)) = \frac{2}{M^2} \left( \frac{\partial}{\partial \alpha} f^{-\frac{1}{2}}(\alpha) \right) \frac{\partial^2}{\partial \sigma^2} I_n(\sigma; f(\alpha)), \quad (B.5)$$

$$\frac{\partial}{\partial \alpha} \left( \frac{1}{f(\alpha)} \right)^\frac{1}{2} J_{n-1}(\sigma, \sigma'; f(\alpha)) = \frac{1}{M^2} \left( \frac{1}{f(\alpha)} \right)^\frac{1}{2} \left( \frac{\partial}{\partial \alpha} f(\alpha) \right) \left( \frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \sigma'^2} \right) J_n(\sigma, \sigma'; f(\alpha)), \quad (B.6)$$

where $f(\alpha)$ is an arbitrary function of $\alpha$.

Appendix C. Calculation for the pure Dirac-type contribution $\Pi_{D}^{\mu\nu}$

In this appendix, we outline the derivation of (4.13). The pure Dirac-type contribution is defined by

$$\Pi_{D}^{\mu\nu}(p; \sigma, \sigma') = \int \frac{d^Dk}{(2\pi)^D} \text{tr} \left[ \gamma^\mu S_F(p+k; \sigma, \sigma') \gamma^\nu S_F(k; \sigma', \sigma) \right]. \quad (C.1)$$
Performing the loop-integration, we obtain

\[
\Pi^{\mu \nu}_D(p; \sigma, \sigma') = \frac{\Gamma(2 - \frac{D}{2})}{(4\pi)^{\frac{D}{2}}} \int_0^1 d\alpha \int \frac{d\omega d\omega'}{(2\pi)^2} \times [\Pi^{\mu \nu}_a(p; \omega, \omega'; \alpha) + \Pi^{\mu \nu}_b(p; \omega, \omega'; \alpha) + \Pi^{\mu \nu}_c(p; \omega, \omega'; \alpha)]
\]

with the Feynman parameter \( \alpha \), where

\[
\Pi^{\mu \nu}_a(p; \omega, \omega'; \alpha) = -\frac{1}{[M^2 + \alpha \omega^2 + (1 - \alpha)\omega'^2 - \alpha(1 - \alpha)p^2]^{1 - \frac{D}{2}}} \times \left\{ \text{tr}[P_L \gamma^{\mu} \gamma^{\nu}] \varphi_- (\omega; \sigma, \sigma') \varphi_+ (\omega'; \sigma', \sigma) \right. \\
+ \text{tr}[P_R \gamma^{\mu} \gamma^{\nu}] \varphi_+ (\omega; \sigma, \sigma') \varphi_- (\omega'; \sigma', \sigma) \right\} 
\]

\[
\Pi^{\mu \nu}_b(p; \omega, \omega'; \alpha) = -\frac{\alpha(1 - \alpha)}{[M^2 + \alpha \omega^2 + (1 - \alpha)\omega'^2 - \alpha(1 - \alpha)p^2]^{2 - \frac{D}{2}}} \\
\times \left\{ \text{tr}[P_L \gamma^{\mu} \gamma^{\nu}] \varphi_- (\omega; \sigma, \sigma') \varphi_- (\omega'; \sigma', \sigma) \\+ \text{tr}[P_R \gamma^{\mu} \gamma^{\nu}] \varphi_+ (\omega; \sigma, \sigma') \varphi_+ (\omega'; \sigma', \sigma) \right\} 
\]

\[
\Pi^{\mu \nu}_c(p; \omega, \omega'; \alpha) = -\frac{1}{[M^2 + \alpha \omega^2 + (1 - \alpha)\omega'^2 - \alpha(1 - \alpha)p^2]^{2 - \frac{D}{2}}} \\
\times \left\{ \text{tr}[P_L \gamma^{\mu} \gamma^{\nu}] \xi (\omega; \sigma', \sigma) \xi (\omega'; \sigma', \sigma) \\+ \text{tr}[P_R \gamma^{\mu} \gamma^{\nu}] \xi (\omega; \sigma, \sigma') \xi (\omega'; \sigma, \sigma') \right\}. 
\]

From the differential equations in Appendix A, we can verify the following equations:

\[
\begin{align*}
\left\{ \xi (\omega; \sigma', \sigma) \xi (\omega'; \sigma', \sigma) \right\} &= \frac{1}{2} \frac{\partial^2}{\partial \sigma^2} (\varphi_\pm (\omega; \sigma, \sigma') \varphi_\pm (\omega'; \sigma', \sigma)) \\
+ \frac{\partial}{\partial \sigma} (M(\sigma) \varphi_\pm (\omega; \sigma, \sigma') \varphi_\pm (\omega'; \sigma', \sigma)) \\
+ \frac{1}{2} (\omega^2 + \omega'^2 + 2M^2) (\varphi_\pm (\omega; \sigma, \sigma') \varphi_\pm (\omega'; \sigma', \sigma)).
\end{align*}
\]

We substitute these equations into \( \Pi^{\mu \nu}_c(p; \omega, \omega'; \alpha) \). Then note that the third term in (C.6) can be replaced by \((\omega^2 + M^2)\varphi_\pm \varphi_\pm\) since the difference \(\propto (\omega^2 - \omega'^2)\) vanishes in
the integral \((C.2)\). Further we can rewrite it into

\[
\frac{1}{[M^2 + \alpha \omega^2 + (1 - \alpha)\omega'^2 - \alpha(1 - \alpha)p^2]^{1-D/2}} \times \left\{ \begin{array}{l}
\text{tr}[\mathcal{P}_L \gamma^{\mu} \gamma^{\nu}] (\varphi_+(\omega; \sigma, \sigma') - \varphi_-(\omega; \sigma, \sigma')) \varphi_+(\omega'; \sigma', \sigma) \\
+ \text{tr}[\mathcal{P}_R \gamma^{\mu} \gamma^{\nu}] (\varphi_-(\omega; \sigma, \sigma') - \varphi_+(\omega; \sigma, \sigma')) \varphi_-(\omega'; \sigma', \sigma)
\end{array} \right. \tag{C.8}
\]

This is the first term in \((4.12)\). Using the formulae for the \(\omega\)-integration of \(\varphi_\pm\) in Appendix A, we can see that this part in \((C.2)\) is finite as \(D \to 2\) and gives

\[
\frac{\Gamma(2 - D/2)}{(4\pi)^{D/2}} \int_0^1 d\alpha \int \frac{d\omega d\omega'}{(2\pi)^2} \frac{1}{[M^2 + \alpha \omega^2 + (1 - \alpha)\omega'^2 - \alpha(1 - \alpha)p^2]^{1-D/2}} \times \left\{ \begin{array}{l}
\text{tr}[\mathcal{P}_L \gamma^{\mu} \gamma^{\nu}] (\varphi_+(\omega; \sigma, \sigma') - \varphi_-(\omega; \sigma, \sigma')) \varphi_+(\omega'; \sigma', \sigma) \\
+ \text{tr}[\mathcal{P}_R \gamma^{\mu} \gamma^{\nu}] (\varphi_-(\omega; \sigma, \sigma') - \varphi_+(\omega; \sigma, \sigma')) \varphi_-(\omega'; \sigma', \sigma)
\end{array} \right. \tag{C.9}
\]

The remaining terms in \(\Pi_\mu^{\mu}\), on the other hand, are summed with \(\Pi_\sigma^{\mu}\) to yield \(\Pi_\omega^{\mu}\) and \(\Pi_1^{\mu} + \Pi_2^{\mu}\), where

\[
\Pi_\omega^{\mu}(p; \omega, \omega'; \alpha) = -\frac{1}{[M^2 + \alpha \omega^2 + (1 - \alpha)\omega'^2 - \alpha(1 - \alpha)p^2]^{2-D/2}} \times \frac{\partial}{\partial \sigma} \left\{ \begin{array}{l}
M(\sigma) \left\{ \text{tr}[\mathcal{P}_L \gamma^{\mu} \gamma^{\nu}] \varphi_+(\omega; \sigma, \sigma') \varphi_+(\omega'; \sigma', \sigma) \\
- \text{tr}[\mathcal{P}_R \gamma^{\mu} \gamma^{\nu}] \varphi_-(\omega; \sigma, \sigma') \varphi_-(\omega'; \sigma', \sigma) \right\} \right. \tag{C.10}
\]

\(^2\) This can be seen by noting the even property of the multiplied factor in \((C.2)\) under \(\omega \leftrightarrow \omega'\) and \(\alpha \leftrightarrow 1 - \alpha\).
Using the function (C.13), we can write

\[ \Pi_1^{\mu\nu}(p;\omega,\omega';\alpha) = \frac{1}{(1 - \frac{D}{2})^\alpha} \frac{\partial}{\partial \alpha} \frac{1}{[M^2 + \alpha \omega^2 + (1 - \alpha)\omega'^2 - \alpha(1 - \alpha)p^2]^{1 - \frac{\alpha}{2}}} \times \left\{ \text{tr}[P_L \gamma^\mu \gamma^\nu] \varphi_+(\omega;\sigma,\sigma') \varphi_+(\omega';\sigma',\sigma) + \text{tr}[P_R \gamma^\mu \gamma^\nu] \varphi_-(\omega;\sigma,\sigma') \varphi_-(\omega';\sigma',\sigma) \right\}, \] (C.11)

\[ \Pi_2^{\mu\nu}(p;\omega,\omega';\alpha) = \frac{1}{[M^2 + \alpha \omega^2 + (1 - \alpha)\omega'^2 - \alpha(1 - \alpha)p^2]^{2 - \frac{\alpha}{2}}} \times \left\{ \text{tr}[P_L \gamma^\mu \gamma^\nu] \left( \frac{\partial^2}{2 \partial \sigma^2} + \alpha^2 p^2 \right) - \alpha(1 - \alpha) \text{tr}[P_R \gamma^\mu \gamma^\nu p] \right\} \times \varphi_+(\omega;\sigma,\sigma') \varphi_+(\omega';\sigma',\sigma) \right\} + \left\{ \text{tr}[P_R \gamma^\mu \gamma^\nu] \left( \frac{\partial^2}{2 \partial \sigma^2} + \alpha^2 p^2 \right) - \alpha(1 - \alpha) \text{tr}[P_L \gamma^\mu \gamma^\nu p] \right\} \times \varphi_-(\omega;\sigma,\sigma') \varphi_-(\omega';\sigma',\sigma) \right\}. \] (C.12)

This \( \Pi_{\text{CS}}^{\mu\nu} \) is the second term in (4.12).

In order to perform \( \omega \)-integration in (C.2) for these integrand, we have only to evaluate the following type of integral:

\[ \tilde{\Phi}_\pm^{(\kappa)}(p^2;\sigma,\sigma';\alpha) = \int \frac{d\omega d\omega'}{(2\pi)^2} \frac{\varphi_\pm(\omega;\sigma,\sigma') \varphi_\pm(\omega';\sigma,\sigma')}{[M^2 + \alpha \omega^2 + (1 - \alpha)\omega'^2 - \alpha(1 - \alpha)p^2]^{\kappa - \frac{D}{2}}}, \] (C.13)

where \( \kappa = 1, 2 \). This is Taylor-expanded with respect to \( p^2 \) as

\[ \sum_{n=0}^{\infty} \frac{\Gamma(n + \kappa - \frac{D}{2})}{\Gamma(\kappa - \frac{D}{2}) n!} (\alpha(1 - \alpha)p^2)^n \frac{M}{(M^2)^n - \frac{D}{2} + \kappa} \Phi_\pm^{(n+\kappa)}(\sigma,\sigma';\alpha), \] (C.14)

where

\[ \Phi_\pm^{(n)}(\sigma,\sigma';\alpha) = \frac{(M^2)^{n - \frac{D}{2}}}{M} \int \frac{d\omega d\omega'}{(2\pi)^2} \frac{\varphi_\pm(\omega;\sigma,\sigma') \varphi_\pm(\omega';\sigma,\sigma')}{[M^2 + \alpha \omega^2 + (1 - \alpha)\omega'^2]^{n - \frac{D}{2}}}. \] (C.15)

Using the function (C.13), we can write

\[
\int \frac{d\omega d\omega'}{(2\pi)^2} \Pi_{\text{CS}}^{\mu\nu}(p;\omega,\omega';\alpha) = -\frac{\partial}{\partial \sigma} \left\{ M(\sigma) \left[ \text{tr}[P_L \gamma^\mu \gamma^\nu] \tilde{\Phi}_+^{(2)}(p^2;\sigma,\sigma';\alpha) + \text{tr}[P_R \gamma^\mu \gamma^\nu] \tilde{\Phi}_-^{(2)}(p^2;\sigma,\sigma';\alpha) \right] \right\}, \] (C.16)
\[
\int \frac{d\omega d\omega'}{(2\pi)^2} \Pi_{\mu\nu}^1(p; \omega, \omega'; \alpha) \left\{ \text{tr}[\mathcal{P}_L \gamma^\mu \gamma^\nu] \tilde{\Phi}_{+}^{(1)}(p^2; \sigma, \sigma'; \alpha) \right. \\
+ \left. \text{tr}[\mathcal{P}_R \gamma^\mu \gamma^\nu] \tilde{\Phi}_{-}^{(1)}(p^2; \sigma, \sigma'; \alpha) \right\}, \\
\int \frac{d\omega d\omega'}{(2\pi)^2} \Pi_{\mu\nu}^2(p; \omega, \omega'; \alpha) \left\{ \text{tr}[\mathcal{P}_L \gamma^\mu \gamma^\nu] \left( \frac{1}{2} \frac{\partial^2}{\partial \sigma^2} + \alpha^2 p^2 \right) - \alpha(1 - \alpha) \text{tr}[\mathcal{P}_R \gamma^\mu \gamma^\nu \hat{p}] \right\} \\
\times \tilde{\Phi}_{+}^{(2)}(p^2; \sigma, \sigma'; \alpha) \\
+ \left\{ \text{tr}[\mathcal{P}_R \gamma^\mu \gamma^\nu] \left( \frac{1}{2} \frac{\partial^2}{\partial \sigma^2} + \alpha^2 p^2 \right) - \alpha(1 - \alpha) \text{tr}[\mathcal{P}_L \gamma^\mu \gamma^\nu \hat{p}] \right\} \\
\times \tilde{\Phi}_{-}^{(2)}(p^2; \sigma, \sigma'; \alpha) \right\}. 
\] 

We introduce another expression for \( \varphi_\pm(\omega; \sigma, \sigma') \):

\[
\varphi_\pm(\omega; \sigma, \sigma') = \cos \omega(\sigma - \sigma') - \frac{M^2}{\omega^2 + M^2} h_\pm(\omega; \sigma, \sigma'), 
\]

where

\[
h_\pm(\omega; \sigma, \sigma') = \begin{align*}
P_+(\sigma, \sigma') &\left[ 1 \pm \epsilon(\sigma') \frac{1}{M} \frac{\partial}{\partial \sigma} \right] \cos \omega(\sigma + \sigma') \\
+ P_-(\sigma, \sigma') &\left[ 1 \mp \epsilon(\sigma') \frac{1}{M} \frac{\partial}{\partial \sigma} \right] \cos \omega(\sigma - \sigma')
\end{align*} 
\] 

with

\[
P_\pm(\sigma, \sigma') = \frac{1}{2}(1 \pm \epsilon(\sigma) \epsilon(\sigma')). 
\]

These \( P_\pm(\sigma, \sigma') \) have the property of the projection operators:

\[
P_\pm(\sigma, \sigma') P_\pm(\sigma, \sigma') = P_\pm(\sigma, \sigma'), \quad \text{and} \quad P_\pm(\sigma, \sigma') P_\mp(\sigma, \sigma') = 0. 
\] 

Substituting these expressions into \( \Phi_{\pm}^{(n)} \), we have

\[
\Phi_{\pm}^{(n)}(\sigma, \sigma'; \alpha) = \Omega_{\pm}^{(n)}(\sigma, \sigma'; \alpha) - C_{\pm}^{(n)}(\sigma, \sigma'; \alpha) \\
- C_{\pm}^{(n)}(\sigma, \sigma'; 1 - \alpha) + H_{\pm}^{(n)}(\sigma, \sigma'; \alpha), 
\]
where

\[
\Omega^{(n)}_\pm (\sigma, \sigma'; \alpha) = \frac{(M^2)^{n-\frac{D}{2}}}{M} \int \frac{d\omega d\omega'}{(2\pi)^2 \omega'^2 + M^2} \frac{M^2}{\omega'^2 + M^2} \cos \omega (\sigma - \sigma') \sin \omega' (\sigma - \sigma'), \quad (C.24)
\]

\[
C^{(n)}_\pm (\sigma, \sigma'; \alpha) = \frac{(M^2)^{n-\frac{D}{2}}}{M} \int \frac{d\omega d\omega'}{(2\pi)^2 \omega'^2 + M^2} \cos \omega (\sigma - \sigma') \sin \omega' (\sigma - \sigma') \frac{h_\pm (\omega'; \sigma, \sigma')}{{M^2 + \alpha \omega^2 + (1 - \alpha)\omega'^2}} \quad (C.25)
\]

\[
H^{(n)}_\pm (\sigma, \sigma'; \alpha) = \frac{(M^2)^{n-\frac{D}{2}}}{M} \int \frac{d\omega d\omega'}{(2\pi)^2 \omega'^2 + M^2} \frac{M^4}{\omega'^2 + M^2} \cos \omega (\sigma - \sigma') \sin \omega' (\sigma - \sigma') \frac{h_\pm (\omega; \sigma, \sigma') h_\pm (\omega'; \sigma, \sigma')}{[M^2 + \alpha \omega^2 + (1 - \alpha)\omega'^2]} \quad (C.26)
\]

In order to evaluate these functions, we first use the Feynman-parameter technique to unify

\[
\frac{1}{\omega^2 + M^2} \left[ \omega^2 + (1 - \alpha)\omega'^2 + M^2 \right]^{n-\frac{D}{2}} \quad (C.27)
\]

with

\[
\frac{1}{\omega^2 + M^2} \quad \text{or} \quad \frac{1}{\omega'^2 + M^2}. \quad (C.28)
\]

Next we exponentiate these factors as usual via the formula

\[
\frac{1}{Q^n} = \frac{1}{\Gamma(n)} \int_0^\infty d\tau \tau^{n-1} \exp[-\tau Q], \quad (C.29)
\]

and perform \(\omega\)-(\(\omega'\))-integrations using the formula

\[
\int d\omega \frac{e^{-\tau \omega^2} \cos \omega \sigma}{2\pi} = \left( \frac{1}{4\pi\tau} \right)^{\frac{3}{4}} e^{-\frac{\sigma^2}{4\tau}}. \quad (C.30)
\]

Then, rescaling \(\tau\)'s to \((\tau/M^2)\)'s, we obtain the following results:

\[
\Omega^{(n)}_\pm (\sigma, \sigma'; \alpha) = \frac{1}{\Gamma(n-\frac{D}{2})} \frac{(M^2)^{n-\frac{D}{2}}}{M} \int_0^\infty d\tau \tau^{(n-\frac{D}{2})-1} e^{-\tau M^2} \quad (C.31)
\]

\[
C^{(n)}_\pm (\sigma, \sigma'; \alpha) = \frac{1}{\Gamma(n-\frac{D}{2})} \frac{(M^2)^{n-\frac{D}{2}}}{M} \int_0^1 d\beta \beta^{n-\frac{D}{2}-1} \int_0^\infty d\tau \tau^{(n-\frac{D}{2})} e^{-\tau M^2}
\]

\[
= \frac{1}{4\pi} \frac{1}{\Gamma(n-\frac{D}{2})} \frac{1}{\alpha(1-\alpha)} \left( \frac{1}{\alpha(1-\alpha)} \right)^{\frac{1}{2}} I_{(n-\frac{D}{2}-1)}(\sigma - \sigma'; 1/\alpha(1-\alpha)). \quad (C.31)
\]
\[
\begin{align*}
\Gamma_\mu^a_{\nu} &= \frac{1}{4\pi} \frac{1}{\Gamma(n-D)} \int_0^1 \frac{d\beta n^{D-1}}{2\pi} \left\{ \frac{1}{M} P_+(\sigma, \sigma') \left[ 1 \pm \epsilon(\sigma') \frac{1}{2M} \left( \frac{\partial}{\partial \sigma} + \frac{\partial}{\partial \sigma'} \right) J_{(n-\frac{D}{2})}(\sigma - \sigma'; \sigma + \sigma'; \alpha \beta) \right] \\
&\quad + P_-(\sigma, \sigma') \left[ 1 \mp \epsilon(\sigma') \frac{\alpha \beta}{2M} \frac{\partial}{\partial \sigma'} I_{(n-\frac{D}{2})}(\sigma - \sigma'; 1) \right] \right\}, \\
H_\pm^{(n)}(\sigma, \sigma'; \alpha) &= \left( \frac{M^2}{\Gamma(n-D)} \right)^{n-D-3} \int_0^1 \frac{d\beta_1 d\beta_2 n^{D-1}}{2\pi} \int_0^\infty \frac{d\tau}{\tau^{n-D-1}} \right\} \\
&\quad \times \frac{d\omega'}{2\pi} \int_{\Gamma(n-D)} \frac{d\omega}{2\pi} e^{-\tau(1-\alpha \beta)\omega^2} h_\pm(\omega; \sigma, \sigma') \\
&\quad \times \left\{ (1 \pm \epsilon(\sigma') \frac{\alpha \beta}{M} \frac{\partial}{\partial \sigma'} I_{(n-\frac{D}{2})}(\sigma + \sigma'; 1) \right\} \\
&\quad + P_-(\sigma, \sigma') \left[ 1 \mp \epsilon(\sigma') \frac{\alpha \beta}{2M} \frac{\partial}{\partial \sigma'} I_{(n-\frac{D}{2})}(\sigma - \sigma'; 1) \right] \right\}. \\
\end{align*}
\]
\[- \frac{M}{4\pi} \int_0^1 d\alpha \frac{\partial}{\partial \sigma} \left\{ \epsilon(\sigma) \left[ \text{tr} [P_L \gamma^\mu \gamma^\nu] \Phi_+^{(2)}(\sigma, \sigma'; \alpha) - \text{tr} [P_R \gamma^\mu \gamma^\nu] \Phi_-^{(2)}(\sigma, \sigma'; \alpha) \right] \right\} \rightarrow \frac{1}{4\pi} \frac{\partial}{\partial \sigma} \left\{ \epsilon^\mu \epsilon(\sigma) \delta(\sigma - \sigma') \right\}. \quad (C.35)\]

This is the second term in (4.13).

Finally, we examine the term with \( \Phi_\pm^{(1)} \):

\[ \frac{M}{1 - \frac{D}{2}} \alpha \frac{\partial}{\partial \alpha} \left\{ \text{tr} [P_L \gamma^\mu \gamma^\nu] \Phi_\pm^{(1)}(\sigma, \sigma'; \alpha) + \text{tr} [P_R \gamma^\mu \gamma^\nu] \Phi_\pm^{(1)}(\sigma, \sigma'; \alpha) \right\} \quad (C.36)\]

in (C.17). Note that we can set \( D \) equal to 2 in (C.36) thanks to the factor \( \frac{1}{\Gamma(1 - \frac{D}{2})} \) contained in \( \Phi_\pm^{(1)} \). If we perform the \( \alpha \)-differentiation in (C.36) using the relations (B.5) and (B.6) in Appendix B, we find that all the terms in \( \Phi_\pm^{(1)} \), except one type of terms, yield the factor \( \frac{1}{M^2} \) and give no contributions in the large mass limit. The exceptional terms are

\[ \pm \frac{\alpha}{4\pi} \frac{\partial}{\partial \bar{\alpha}} C_\pm^{(1)}(\sigma, \sigma'; \alpha) \quad (C.37)\]

in \( \frac{M}{1 - \frac{D}{2}} \alpha \frac{\partial}{\partial \alpha} C_\pm^{(1)}(\sigma, \sigma'; \alpha) \) and

\[ \pm \frac{\alpha}{4\pi} \frac{\partial}{\partial \bar{\alpha}} C_\pm^{(1)}(\sigma, \sigma'; \alpha) \quad (C.38)\]

in \( \frac{M}{1 - \frac{D}{2}} \alpha \frac{\partial}{\partial \alpha} C_\pm^{(1)}(\sigma, \sigma'; 1 - \alpha) \). In the large mass limit, however, these terms cancel each other in (C.36) and do not contribute, either. The term (C.36) thus vanishes in the large mass limit.

In summary, only the terms from (C.9) and (C.35) can contribute to the pure Dirac-type contribution \( \Pi^{\mu\nu}_D(p; \sigma, \sigma') \) in the large mass limit. Thus,

\[ \Pi^{\mu\nu}_D(p; \sigma, \sigma') \rightarrow \frac{1}{4\pi} \left( g^{\mu\nu} - \epsilon^{\mu\nu} \right) \delta(\sigma) \delta(\sigma') + \epsilon^{\mu\nu} \frac{\partial}{\partial \sigma} \left( \epsilon(\sigma) \delta(\sigma - \sigma') \right) \quad (C.39)\]

This is (1.13).
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