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EXPONENTIAL MIXING OF THE 3D STOCHASTIC NAVIER-STOKES EQUATIONS DRIVEN BY MILDLY DEGENERATE NOISES

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Abstract. We prove the strong Feller property and exponential mixing for 3D stochastic Navier-Stokes equation driven by mildly degenerate noises (i.e. all but finitely many Fourier modes are forced) via Kolmogorov equation approach.

1. Introduction

The ergodicity of SPDEs driven by degenerate noises have been intensively studied in recent years (see for instance [7], [13], [6], [14], [21]). For the 2D stochastic Navier-Stokes equations (SNS), there are several results on ergodicity, among which the most remarkable one is by Hairer and Mattingly ([13]). They proved that the 2D stochastic dynamics has a unique invariant measure as long as the noise forces at least two linearly independent Fourier modes. As for the 3D SNS, most of ergodicity results are about the dynamics driven by non-degenerate noises (see [3], [11], [19], [21], [18]). In the respect of the degenerate noise case, as noises are essentially elliptic setting of which all but finite Fourier modes are driven, [23] obtained the ergodicity by combining Markov selection and Malliavin calculus. As the noises are truly hypoelliptic ([13]), ergodicity is still open.

In this paper, we shall still study the 3D SNS driven by essentially elliptic noises as above, but our approach is essentially different from that in [23]. Rather than Markov selection and cutoff technique, we prove the strong Feller property by studying some Kolmogorov equations with a large negative potential, which was developed in [3]. Comparing with the method in [3] and [5], we cannot apply the Bismut-Elworthy-Li formula ([8]) due to the degeneracy of the noises. To fix this problem, we follow the ideas in [7] and split the dynamics into high and low frequency parts, applying the formula to the dynamics at high modes and Malliavin calculus to those at low ones. Due to the degeneracy of the noises again, when applying Duhamel formula as in [3] and [5], we shall encounter an obstruction of not integrability (see (5.1)). Two techniques are developed in Proposition 5.1 and 5.2 to conquer this problem, and the underlying idea is to trade off the spatial regularity for the time integrability. Using the coupling method of [17], in which

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the noises have to be non-degenerate, we prove the exponential mixing and find that the construction of the coupling can be simplified. Finally, we remark that the large coefficient $K$ in front of the potential (see (2.11)), besides suppressing the nonlinearity $B(u, u)$ as in [3] and [5], also conquers the crossing derivative flows (see (3.9) and (3.10)).

Let us discuss the further application of the Kolmogorov equation method in [3], [5] and this paper. For another essentially elliptic setting where sufficiently large (but still finite) modes are forced ([13], section 4.5), due to the large negative potential, it is easy to show the asymptotic strong Feller ([13]) for the semigroup $S_t^n$ (see (2.13)). There is a hope to transfer this asymptotic strong Feller to the semigroup $P_t^n$ (see (2.14)) using the technique in Proposition 5.2. If $P_t^n$ satisfies asymptotic strong Feller, then we can also prove the ergodicity. This is the further aim of our future research in 3D SNS.

The paper is organized as follows. Section 2 gives a detailed description of the problem, the assumptions on the noise and the main results (Theorems 2.4 and 2.5). Section 3 proves the crucial estimate in Theorem 3.1, while the fourth section 4 applies Malliavin calculus to prove the important Lemma 3.5. Section 5 gives a sketch proof for the main theorems, and the last section contains the estimate of Malliavin matrices and the proof of some technical lemmas.

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2. Preliminary and main results

2.1. Notations and assumptions. Let $T^3 = [0,2\pi]^3$ be the three-dimensional torus, let

$$H = \{ x \in L^2(T^3, \mathbb{R}^3) : \int_{T^3} x(\xi) d\xi = 0, \ \text{div} x(\xi) = 0 \},$$

and let

$$P : L^2(T^3, \mathbb{R}^3) \to H$$

be the orthogonal projection operator. We shall study the equation

\begin{equation}
\begin{cases}
  dX + [\nu AX + B(X, X)]dt = QdW_t, \\
  X(0) = x,
\end{cases}
\end{equation}

where

- $A = -P\Delta$, $D(A) = H^2(T^3, \mathbb{R}^3) \cap H$.
- The nonlinear term $B$ is defined by
  $$B(u, v) = (u \cdot \nabla)v, \quad B(u) = B(u, u) \quad \forall \ u, \ v \in H^1(T^3, \mathbb{R}^3) \cap H.$$
- $W_t$ is the cylindrical Brownian motion on $H$ and $Q$ is the covariance matrix to be defined later.
- We shall assume the value $\nu = 1$ later on, as its exact value will play no essential role.
Define $Z^3_+ = \{ k \in \mathbb{Z}^3; k_1 > 0 \} \cup \{ k \in \mathbb{Z}^3; k_1 = 0, k_2 > 0 \} \cup \{ k \in \mathbb{Z}^3; k_1 = 0, k_2 = 0, k_3 > 0 \}$, $Z^3_- = -Z^3_+$ and $Z^3 = Z^3_+ \cup Z^3_-$, for any $n > 0$, denote

$$Z_i(n) = [-n, n]^3 \setminus (0, 0, 0)$, \hspace{1cm} Z_k(n) = Z^3_+ \setminus Z_i(n).$$

Let $k^\perp = \{ \eta \in \mathbb{R}^3; k \cdot \eta = 0 \}$, define the projection $\mathcal{P}_k : \mathbb{R}^3 \rightarrow k^\perp$ by

$$(2.2) \hspace{1cm} \mathcal{P}_k \eta = \eta - \frac{k \cdot \eta}{|k|^2} k \hspace{1cm} \eta \in \mathbb{R}^3.$$  

Let $e_k(\xi) = \cos k \xi$ if $k \in \mathbb{Z}^3_+$, $e_k(\xi) = \sin k \xi$ if $k \in \mathbb{Z}^3_-$ and let $\{e_{k,1}, e_{k,2}\}$ be an orthonormal basis of $k^\perp$, denote

$$e_k^0(\xi) = e_k(\xi)e_{k,1}, \hspace{1cm} e_k^1(\xi) = e_k(\xi)e_{k,2} \hspace{1cm} \forall \ k \in \mathbb{Z}^3_+;$$

$\{e_k^i; k \in \mathbb{Z}^3_+, i = 1, 2\}$ is a Fourier basis of $H$ (up to the constant $\sqrt{2}/(2\pi)^{3/2}$). With this Fourier basis, we can write the cylindrical Brownian motion $W$ on $H$ by

$$W_t = \sum_{k \in \mathbb{Z}^3_+} w_k(t)e_k = \sum_{k \in \mathbb{Z}^3_+} \sum_{i=1}^{2} w^i_k(t)e^i_k$$

where each $w_k(t) = (w^1_k(t), w^2_k(t))^T$ is a 2-d standard Brownian motion. Moreover,

$$B(u, v) = \sum_{k \in \mathbb{Z}^3_+} B_k(u, v)e_k$$

where $B_k(u, v)$ is the Fourier coefficient of $B(u, v)$ at the mode $k$. Define

$$\hat{B}(u, v) = B(u, v) + B(v, u), \hspace{1cm} \hat{B}_k(u, v) = B_k(u, v) + B_k(v, u).$$

We shall calculate $\hat{B}_k(a_j\eta_j, a_l\eta_l)$ with $a_j \in j^\perp, a_l \in l^\perp$ in Appendix 6.1.

Furthermore, given any $n > 0$, let $\pi_n : H \rightarrow H$ be the projection from $H$ to the subspace $\pi_n H := \{ x \in H; x = \sum_{k \in Z_i(n)} x_k e_k \}$.

**Assumption 2.1 (Assumptions for $Q$).** We assume that $Q : H \rightarrow H$ is a linear bounded operator such that

(A1) (Diagonality) There are a sequence of linear maps $\{q_k\}_{k \in \mathbb{Z}^3_+}$ with $q_k : k^\perp \rightarrow k^\perp$ such that

$$Q(\eta e_k) = (q_k \eta) e_k \hspace{1cm} \eta \in k^\perp.$$  

(A2) (Finitely Degeneracy) There exists some nonempty sublattice $Z_i(n_0)$ of $\mathbb{Z}^3_+$ such that

$$q_k = 0 \hspace{1cm} k \in Z_i(n_0).$$  

(A3) $(I_d - \pi_{n_0})A^\tau Q$ is bounded invertible on $(I_d - \pi_{n_0})H$ with $1 < r < 3/2$ and moreover $\text{Tr}[A^1+\sigma QQ^*] < \infty$ for some $\sigma > 0$.

**Remark 2.2.** Under the Fourier basis of $H$, $Q$ has the following representation

$$(2.3) \hspace{1cm} Q = \sum_{k \in Z_i(n_0)} \sum_{l=1}^{2} q^i_k e^i_k \otimes e^j_l$$

where $x \otimes y : H \rightarrow H$ is defined by $(x \otimes y)z = \langle y, z \rangle x$ and $(q^i_k)$ is a matrix representation of $q_k$ under some orthonormal basis $(e_{k,1}, e_{k,2})$ of $k^\perp$. By (A3),
\( \text{rank}(q_k) = 2 \) for all \( k \in \mathbb{Z}_i(n_0) \). Take \( Q = (Id - \pi_{n_0})A^{-r} \) with some \( 5/4 < r < 3/2 \), it clearly satisfies (A1)-(A3).

With the above notations and assumptions, equation (2.1) can be represented under the Fourier basis by

\[
\begin{align*}
\begin{cases}
  dX_k + [|k|^2 X_k + B_k(X)]dt = q_k dw_k(t), & k \in \mathbb{Z}_i(n_0) \\
  dX_k + [|k|^2 X_k + B_k(X)]dt = 0, & k \in \mathbb{Z}_i(n_0) \\
  X_k(0) = x_k, & k \in \mathbb{Z}^2_+
\end{cases}
\end{align*}
\]

where \( x_k, X_k, B_k(X) \in k^1 \).

We further need the following notations:

- \( B_0(B) \) denotes the Borel measurable bounded function space on the given Banach space \( B \). \( || \cdot ||_B \) denotes the norm of a given Banach space \( B \).
- \( || \cdot || \) and \( \langle \cdot, \cdot \rangle \) denote the norm and the inner product of \( H \) respectively.
- Given any \( \phi \in C(D(A), \mathbb{R}) \), we denote

\[
D_h \phi(x) := \lim_{\varepsilon \to 0} \frac{\phi(x + \varepsilon h) - \phi(x)}{\varepsilon},
\]

provided the above limit exists, it is natural to define \( D\phi(x) : D(A) \to \mathbb{R} \) by \( D\phi(x)h = D_h \phi(x) \) for all \( h \in D(A) \). Clearly, \( D\phi(x) \in D(A^{-1}) \). We call \( D\phi \) the first order derivative of \( \phi \), similarly, one can define the second order derivative \( D^2 \phi \) and so on. Denote \( C^k_0(D(A), \mathbb{R}) \) the set of functions from \( D(A) \) to \( \mathbb{R} \) with bounded 0-th, \( \ldots \), \( k \)-th order derivatives.

- Let \( B \) be some Banach space and \( k \in \mathbb{Z}_+ \), define \( C_k(D(A), B) \) as the function space from \( D(A) \) to \( B \) with the norm

\[
||\phi||_{k} := \sup_{x \in D(A)} \frac{|\phi(x)|_B}{1 + |Ax|^k} \quad \phi \in C_k(D(A), B).
\]

- For any \( \gamma > 0 \) and \( 0 \leq \beta \leq 1 \), define the Hölder’s norm \( || \cdot ||_{2, \beta} \) by

\[
||\phi||_{2, \beta} = \sup_{x,y \in D(A)} \frac{|\phi(x) - \phi(y)|}{|A^\gamma(x - y)|^\beta(1 + |Ax|^2 + |Ay|^2)^\gamma}
\]

and the function space \( C^\beta_{2, \gamma}(D(A), \mathbb{R}) \) by

\[
(2.6) \quad C^\beta_{2, \gamma}(D(A), \mathbb{R}) = \{ \phi \in C_2(D(A), \mathbb{R}); ||\phi||_{C^\beta_{2, \gamma}} = ||\phi||_2 + ||\phi||_{2, \beta} < \infty \}.
\]

2.2. Main results. The following definition of Markov family follows that in [5].

**Definition 2.3.** Let \( (\Omega, \mathcal{F}, P) \) be a family of probability spaces and let \( (X(\cdot, x))_{x \in D(A)} \) be a family of stochastic processes on \( (\Omega, \mathcal{F}, P) \). Let \( (\mathcal{F}_t^Y)_{t \geq 0} \) be the filtration generated by \( X(\cdot, x) \) and let \( \mathcal{P}_x \) be the law of \( X(\cdot, x) \) under \( P_x \). The family of \( (\Omega, \mathcal{F}, P_x, X(\cdot, x))_{x \in D(A)} \) is a Markov family if the following condition hold:

(1) For any \( x \in D(A), t \geq 0 \), we have

\[
P_x(X(t, x) \in D(A)) = 1.
\]
The map \( x \rightarrow \mathcal{P}_x \) is measurable. For any \( x \in D(A) \), \( t_0, \cdots, t_n \geq 0 \), \( A_0, \cdots, A_n \subset D(A) \) Borel measurable, we have
\[
\mathbb{P}_x(X(t + \cdot) \in A|\mathcal{F}^t_x) = \mathcal{P}_{X(t,x)}(A)
\]
where \( A = \{(y(t_0), \cdots, y(t_n)); y(t_0) \in A_0, \cdots, y(t_n) \in A_n\} \).
The Markov transition semigroup \((P_t)_{t \geq 0}\) associated to the family is then defined by
\[
P_t \phi(x) = \mathbb{E}_x[\phi(X(t,x))], \quad x \in D(A) \quad t \geq 0.
\]
for all \( \phi \in \mathcal{B}(D(A), \mathbb{R}) \).

The main theorems of this paper are as the following, and will be proven in Section 5.

**Theorem 2.4.** There exists a Markov family of martingale solution \((\Omega_x, \mathcal{F}_x, \mathbb{P}_x, X(\cdot,x))_{x \in D(A)}\) of the equation \[(2.1)\]
Furthermore, the transition semigroup \((P_t)_{t \geq 0}\) is stochastically continuous.

**Theorem 2.5.** The transition semigroup \((P_t)_{t \geq 0}\) in the previous theorem is strong Feller and irreducible. Moreover, it admits a unique invariant measure \(\nu\) supported on \(D(A)\) such that, for any probability measure \(\mu\) supported on \(D(A)\), we have
\[
||P_t^\mu - \nu||_{\text{var}} \leq Ce^{-ct} \left( 1 + \int_H |x|^2 \mu(dx) \right)
\]
where \(||\cdot||_{\text{var}}\) is the total variation of signed measures, and \(C, c > 0\) are the constants depending on \(Q\).

### 2.3. Kolmogorov equations for Galerkin approximation.

Let us consider the Galerkin approximations of the equation \[(2.4)\]
\[
\begin{align*}
\begin{cases}
    dX_m = -[AX_m + B_m(X_m)]dt + Q_m dW_t \\
    X_m(0) = x_m
\end{cases}
\end{align*}
\]
where \(x_m \in \pi_m D(A)\), \(B_m(x) = \pi_m B(\pi_m x)\) and \(Q_m = \pi_m Q\). The Kolmogorov equation for \[(2.8)\]
is
\[
\begin{align*}
\begin{cases}
    \partial_t u_m = \frac{1}{2} Tr[Q_m Q_m^* D^2 u_m] - \langle Ax + B_m(x), D u_m \rangle \\
    u_m(0) = \phi
\end{cases}
\end{align*}
\]
where \(\phi\) is some suitable test function and
\[
\mathcal{L}_m := \frac{1}{2} Tr[Q_m Q_m^* D^2] - \langle Ax + B_m(x), D \rangle
\]
is the Kolmogorov operator associated to \[(2.8)\]. It is well known that \[(2.9)\] is uniquely solved by
\[
u_m(t,x) = \mathbb{E}[\phi(X_m(t,x))], \quad x \in \pi_m D(A).
\]
Now we introduce an auxiliary Kolmogorov equation with a negative potential \( -K|Ax|^2 \) as
\[
\begin{align*}
\begin{cases}
    \partial_t v_m = \frac{1}{2} Tr[Q_m Q_m^* D^2 v_m] - \langle Ax + B_m(x), D v_m \rangle - K|Ax|^2 v_m, \\
v_m(0) = \phi
\end{cases}
\end{align*}
\]
which is solved by the following Feynman-Kac formula
\begin{equation}
    v_m(t, x) = \mathbb{E} \left[ \phi(X_m(t, x)) \exp \left\{ -K \int_0^t |AX_m(s, x)|^2 ds \right\} \right].
\end{equation}

Denote
\begin{align}
    S_t^m \phi(x) &= v_m(t, x), \\
    P_t^m \phi(x) &= u_m(t, x),
\end{align}
for any \( \phi \in \mathcal{B}(\pi_m D(A)) \), it is clear that \( S_t^m \) and \( P_t^m \) are both contraction semigroups on \( \mathcal{B}(\pi_m D(A)) \). By Duhamel’s formula, we have
\begin{equation}
    u_m(t) = S_t^m \phi + K \int_0^t S_{t-s}^m ||Ax||^2 u_m(s) ds.
\end{equation}

For further use, denote
\begin{equation}
    \mathcal{E}_{m,K}(t) = \mathbb{E} \left[ \exp \left\{ -K \int_0^t |AX_m(s)|^2 ds \right\} \right],
\end{equation}
which plays a very important role in section 3. The \( K > 1 \) in (2.16) is a large but fixed number, which conquers the crossing derivative flows (see (3.9) and (3.10)). We will often use the trivial fact
\begin{equation}
    \mathcal{E}_{m,K_1+K_2}(t) = \mathcal{E}_{m,K_1}(t)\mathcal{E}_{m,K_2}(t) \quad \text{and} \quad \mathcal{E}_{m,K}(t) \leq \frac{1}{K} (1 - \mathcal{E}_{m,K}(t)).
\end{equation}

3. Gradient estimate for the semigroups \( S_t^m \)

In this section, the main result is as follows, and it is similar to Lemma 3.4 in [5] (or Lemma 4.8 in [3]).

**Theorem 3.1.** Given any \( T > 0 \) and \( k \in \mathbb{Z}_+ \), there exists some \( p > 1 \) such that for any \( \max\{\frac{1}{2}, r - \frac{1}{2}\} < \gamma \leq 1 \) with \( \gamma \neq 3/4 \) and \( r \) defined in Assumption 2.1, we have
\begin{equation}
    || A^{-\gamma} DS_t^m \phi || \leq C t^{-\alpha} ||\phi||_k \quad 0 < t \leq T
\end{equation}
for all \( \phi \in \mathcal{C}_0^1(D(A), \mathbb{R}) \), where \( C = C(k, \gamma, r, T, K) > 0 \) and \( \alpha = p + \frac{1}{2} + r - \gamma \).

**Remark 3.2.** The condition \( '\gamma \neq 3/4' \) is due to the estimate (6.11) about the nonlinearity \( B(u, v) \).

[3] and [5] proved the estimate (3.1) by applying the identity
\begin{equation}
    D_h S_t^m \phi(x) = \frac{1}{t} \mathbb{E} \left[ \mathcal{E}_{m,K}(t) \phi(X^m(t, x)) \right] \int_0^t \langle Q^{-1} D_h X^m(s, x), dW_s \rangle
\end{equation}
\begin{equation}
    + 2K \mathbb{E} \left[ \mathcal{E}_{m,K}(t) \phi(X^m(t, x)) \int_0^t (1 - \frac{s}{t}) \langle AX^m(s, x), AD_h X^m(s, x) \rangle ds \right],
\end{equation}
and bounding the two terms on the r.h.s. of (3.2). Since the \( Q \) in Assumption 2.1 is degenerate, the formula (3.2) is not available in our case. Alternatively, we apply the idea in [7] to fix this problem, i.e. splitting \( X_m(t) \) into the low and high frequency parts, and applying Malliavin calculus and Bismut-Elworthy-Li formula on the them respectively.
Let $n \in \mathbb{N}$ be a fixed number throughout this paper which satisfies $n > n_0$ and will be determined later ($n_0$ is the constant in Assumption 2.1). We split the Hilbert space $H$ into the low and high frequency parts by
\begin{equation}
\pi^i H = \pi_n H, \quad \pi^s H = (\text{Id} - \pi_n) H.
\end{equation}
(We remark that the technique of splitting frequency space into two pieces is similar to the well known Littlewood-Paley projection in Fourier analysis.) Then, the Galerkin approximation (2.8) with $m > n$ can be divided into two parts as follows:
\begin{align}
dX^i_m + [AX^i_m + B^i_m(X_m)] dt = Q^i_m dW^i_t \\
dX^s_m + [AX^s_m + B^s_m(X_m)] dt = Q^s_m dW^s_t
\end{align}
where $X^i_m = \pi^i X_m, X^s_m = \pi^s X_m$ and the other terms are defined in the same way. In particular,
\begin{align}
Q^i_m & = \sum_{k \in Z(n) \setminus Z(n_0)} \sum_{i,j=1}^2 q_{ij}^k e_k \otimes e^j_k, \\
Q^s_m & = \sum_{k \in Z(m) \setminus Z(n)} \sum_{i,j=1}^2 q_{ij}^k e_k \otimes e^j_k,
\end{align}
with $x \otimes y : H \rightarrow H$ defined by $(x \otimes y)z = (y, z)x$.

With such separation for the dynamics, it is natural to split the Frechet derivatives on $H$ into the low and high frequency parts. More precisely, for any stochastic process $\Phi(t, x)$ on $H$ with $\Phi(0, x) = x$, the Frechet derivative $D_h \Phi(t, x)$ is defined by
\[ D_h \Phi(t, x) := \lim_{\epsilon \rightarrow 0} \frac{\Phi(t, x + \epsilon h) - \Phi(t, x)}{\epsilon} \quad h \in H, \]
provided the limit exists. The map $D(\Phi(t, x)) : H \rightarrow H$ is naturally defined by $D(\Phi(t, x))h = D_h \Phi(t, x)$ for all $h \in H$. Similarly, one can easily define $D^i \Phi(t, x), D^s \Phi(t, x), D^i \Phi^i(t, x), D^s \Phi^i(t, x)$ and so on, for instance, $D^i \Phi(t, x) : \pi^i H \rightarrow \pi^i H$ is defined by
\[ D^i \Phi(t, x)h = D_h \Phi(t, x) \quad \forall h \in \pi^i H. \]
with $D_h \Phi(t, x) = \lim_{\epsilon \rightarrow 0} [\Phi(t, x + \epsilon h) - \Phi(t, x)]/\epsilon$.

Recall that for any $\phi \in C^1_0(D(A), \mathbb{R})$ one can define $D\phi$ by (2.5), in a similar way as above, $D^i \phi(x)$ and $D^s \phi(x)$ can be defined (e.g. $D^i \phi(x)h = \lim_{\epsilon \rightarrow 0} [\phi(x + \epsilon h) - \phi(x)] / \epsilon$).

**Lemma 3.3.** Denote $Z(t) = \int_0^t e^{-(t-s)} Q dW_s$, for any $T > 0$ and $\varepsilon < \sigma/2$ with the $\sigma$ as in Assumption 2.1, one has
\begin{equation}
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |A^{1+\varepsilon} Z(t)|^{2k} \right] \leq C(\alpha) T^{2k(\sigma - 2\varepsilon - 2\alpha)}
\end{equation}
where $0 < \alpha < \sigma/2 - \varepsilon$ and $k \in \mathbb{Z}_+$. Moreover, as $K > 0$ is sufficiently large, for any $T > 0$ and any $k \geq 2$, we have
\begin{equation}
\mathbb{E} \left[ \sup_{0 \leq t \leq T} E_{m,K}(t) |AX_m(t)|^k \right] \leq C(k, T)(1 + |Ax|^k).
\end{equation}
Proof: The proof of (3.6) is standard (see Proposition 3.1 of [5]). Writing \( X_m(t) = Y_m(t) + Z_m(t) \), and differentiating \(|AY_m(t)|^2\) (or seeing (3.1) in Lemma 3.1 of [3]), we have
\[
\mathcal{E}_{m,K}(t)|AY_m(t)|^2 \leq |Ax|^2 + \sup_{0 \leq t \leq T} |AZ_m(t)|^2.
\]
as \( K > 0 \) is sufficiently large. Hence,
\[
\mathcal{E}_{m,K}(t)|AX_m(t)|^2 \leq \mathcal{E}_{m,K}(t)|AY_m(t)|^2 + \mathcal{E}_{m,K}(t)|AZ_m(t)|^2 \leq |Ax|^2 + 2 \sup_{0 \leq t \leq T} |AZ(t)|^2.
\]
Hence, by (3.6) and the above inequality, we immediately have (3.7).

The main ingredients of the proof of Theorem 3.1 are the following two lemmas, and they will be proven in Appendix 6.3 and Section 4.2 respectively.

Lemma 3.4. Let \( x \in D(A) \) and let \( X_m(t) \) be the solution to (2.8). Then, for any \( \max\{\frac{1}{2}, r - \frac{1}{2}\} < \gamma \leq 1 \) with \( \gamma \neq \frac{3}{4} \), \( h \in \pi_m H \) and \( v \in L^2_{loc}(0, \infty; H) \), as \( K \) is sufficiently large, we have almost surely
\[
\begin{align*}
|A^\gamma D_h X_m(t)|^2 \mathcal{E}_{m,K}(t) + & \int_0^t |A^{1+2\gamma} D_h X_m(s)|^2 \mathcal{E}_{m,K}(s) ds \leq |A^\gamma h|^2 \\
|A^\gamma D_h X_m^t(t)|^2 \mathcal{E}_{m,K}(t) \leq & \frac{C}{K}|A^\gamma h|^2 \\
|A^\gamma D_h X_m^t(t)|^2 \mathcal{E}_{m,K}(t) \leq & \frac{C}{K}|A^\gamma h|^2 \\
\int_0^t |A^\gamma D_h X_m(s)|^2 \mathcal{E}_{m,K}(s) ds \leq & C t^{1-2(\gamma - \frac{1}{2})} |A^\gamma h|^2 \\
\mathbb{E}[\mathcal{E}_{m,K}(t) \int_0^t (v(s), dW(s))] \leq & \mathbb{E}\left[\int_0^t \mathcal{E}_{m,K}(s) |v(s)|^2 ds\right]
\end{align*}
\]
where all the \( C = C(\gamma) > 0 \) above are independent of \( m \) and \( K \).

Lemma 3.5. Given any \( \phi \in C^1_0(D(A)) \) and \( h \in \pi'H \), there exists some \( p > 1 \) (possibly very large) such that for any \( k \in \mathbb{Z}_+ \), we have some constant \( C = C(p,k) > 0 \) such that
\[
\mathbb{E}[D^t \phi(X_m(t))D_h X_m^t(t,x) \mathcal{E}_{m,K}(t)] \leq C t^{-p} e^{C t} ||\phi||_k (1 + |Ax|^k)|h|
\]

Proof of Theorem 3.1. For the notational simplicity, we shall drop the index in the quantities if no confusion arises. For \( S_t \phi(X(s)) \), applying Itô formula to \( X(s) \) and the equation (2.11) to \( S_{-s} \), (differentiating on \( s \)), we have
\[
d[S_{-s} \phi(X(s)) \mathcal{E}_K(s)] = L_m S_{-s} \phi(X(s)) \mathcal{E}_K(s) ds + DS_{-s} \phi(X(s)) \mathcal{E}_K(s) QdW_s
\]
\[
- L_m S_{-s} \phi(X(s)) \mathcal{E}_K(s) ds + K |AX(s)|^2 S_{-s} \phi(X(s)) \mathcal{E}_K(s) ds
\]
\[
- S_{-s} \phi(X(s)) K |AX(s)|^2 \mathcal{E}_K(s) ds
\]
\[
= DS_{-s} \phi(X(s)) \mathcal{E}_K(s) QdW_s
\]
where \( L_m \) is the Kolmogorov operator defined in (2.9), thus
\[
\phi(X(t)) \mathcal{E}_K(t) = S_t \phi(x) + \int_0^t DS_{t-s} \phi(X(s)) \mathcal{E}_K(s) QdW_s
\]
Given any \( h \in \pi_m H \), by (A3) of Assumption 2.1 and (3.14), we have \( y^t \) := \((Q^t)^{-1} D_{h^t} X^t(t)\) so that

\[
E[\phi(X(t))E_K(t) \int_0^{t/2} \langle y_s^t, dW_s^t \rangle] = \mathbb{E} \int_0^t DS_{t-s} \phi(X(s))E_K(s)Q^s dW_s^t \int_0^{t/2} \langle (Q^t)^{-1} D_{h^t} X^t(s), dW_s^t \rangle
\]

\[(3.15)\]

\[
= \int_0^{t/2} \mathbb{E}[D^t S_{t-s} \phi(X(s))D_{h^t} X^t(s)E_K(s)] ds,
\]

hence,

\[
\int_0^{t/2} \mathbb{E}[D_{h^t} S_{t-s} \phi(X(s))E_K(s)] ds = \mathbb{E} \int_0^{t/2} \langle (Q^t)^{-1} D_{h^t} X^t(s), dW_s^t \rangle
\]

\[(3.16)\]

\[
\quad + \int_0^{t/2} \mathbb{E}[D^t S_{t-s} \phi(X(s))D_{h^t} X^t(s)E_K(s)] ds.
\]

By the fact \( S_t \phi(x) = \mathbb{E}[S_{t-s} \phi(X(s))E_K(s)] \), (3.15) and (3.16), we have

\[
D_{h^t} S_{t} \phi(x) = \frac{2}{t} \int_0^{t/2} D_{h^t} \mathbb{E}[S_{t-s} \phi(X(s))E_K(s)] ds
\]

\[
= \frac{2}{t} \mathbb{E} \int_0^{t/2} \langle (Q^t)^{-1} D_{h^t} X^t(s), dW_s^t \rangle
\]

\[
+ \frac{2}{t} \int_0^{t/2} \mathbb{E}[D^t S_{t-s} \phi(X(s))D_{h^t} X^t(s)E_K(s)] ds
\]

\[
- \frac{4K}{t} \int_0^{t/2} \mathbb{E}[S_{t-s} \phi(X(s))E_K(s)] \int_s^t \langle AX(r), AD_{h^t} X(r) \rangle dr ds
\]

\[
= \frac{2}{t} I_1 + \frac{2}{t} I_2 - \frac{4K}{t} I_3.
\]

We now fix \( T, \gamma, k, r \) and let \( C \) be constants depending on \( T, \gamma, k \) and \( r \) whose values can vary from line to line, then \( I_1, I_2 \) and \( I_3 \) above can be estimated as follows:

\[
|I_1| \leq || \phi ||_k \mathbb{E} \left( \mathcal{E}_{K/2}(t)(1 + |AX(t)|^k) \mathcal{E}_{K/2}(t) \int_0^{t/2} \langle (Q^t)^{-1} D_{h^t} X^t(s), dW_s^t \rangle \right)
\]

\[
\leq || \phi ||_k \mathbb{E} \left( \sup_{0 \leq s \leq T} (1 + |AX(s)|^k) \mathcal{E}_K(s) \right)^{1/2} \mathbb{E} \left( \mathcal{E}_{K/2}(t) \int_0^{t/2} \langle (Q^t)^{-1} D_{h^t} X^t(s), dW_s^t \rangle \right)^{1/2}
\]

\[
\leq C || \phi ||_k (1 + |Ax|^k) \left( \mathbb{E} \left( \int_0^{t/2} |A^t D_{h^t} X^t(s)|^2 \mathcal{E}_K(s) ds \right) \right)^{1/2}
\]

\[
\leq C t^{1/2 - (r-\gamma)} || \phi ||_k (1 + |Ax|^k) |A^t h|
\]
where the last two inequalities are by (3.7), (3.12) and (3.11) in order. By (3.9) and (3.7),

$$|I_2| \leq \frac{C}{K} \int_0^{t/2} ||A^{-\gamma} D^i S_{t-s} \phi||_k \mathbb{E} \left[(1 + |AX(s)|^k)E_{K/2}(s)\right] ds |A^\gamma h|$$

$$\leq \frac{C}{K} \int_0^{t/2} ||A^{-\gamma} DS_{t-s} \phi||_k ds (1 + |Ax|^k) |A^\gamma h|.$$  

By Markov property of $X(t)$ and (3.7), we have

$$|I_3| = \int_0^{t/2} \mathbb{E} \left\{ \mathbb{E}[\phi(X(t))e^{-K \int_0^t |AX(r)|^2 dr} |\mathcal{F}_s] E_K(s) \int_0^s \langle AX(r), AD_h X(r) \rangle dr \right\} ds$$

$$\leq C ||\phi||_k \int_0^t \mathbb{E}[(1 + |AX(s)|^k)E_{K/2}(s) \int_0^s E_{K/2}(r)|AX(r)| \cdot |AD_h X(r)| dr] ds,$$

moreover, and by Hölder inequality, Poincaré inequality $|A^{\gamma+\frac{1}{2}} x| \geq |Ax|$ (2.17) and (3.8),

$$\int_0^s E_{K/2}(r)|AX(r)| \cdot |AD_h X(r)| dr$$

$$\leq \left( \int_0^s E_{K/2}(r)|AX(r)|^2 dr \right)^{\frac{1}{2}} \left( \int_0^s |AD_h X(r)|^2 E_{K/2}(r) dr \right)^{\frac{1}{2}}$$

$$\leq \left[ \int_0^s E_{K/2}(r)|AX(r)|^2 dr \right]^{\frac{1}{2}} \leq |A^\gamma h|;$$

hence, by (3.7) and the above,

$$|I_3| \leq C t ||\phi||_k (1 + |Ax|^k) |A^\gamma h|.$$  

Collecting the estimates for $I_1$, $I_2$ and $I_3$, we have

(3.17)

$$|D_h S_t \phi(x)| \leq C \left\{ \left( t^{-\frac{1}{2}-(r-\gamma)} + K \right) ||\phi||_k + \frac{1}{Kt} \int_0^{t/2} ||A^{-\gamma} DS_{t-s} \phi||_k ds \right\} \left( 1 + |Ax|^k \right) |A^\gamma h|$$

For the low frequency part, according to Lemma 3.5, we have

(3.18)

$$|D_{h^i} S_{t/2} \phi(x)| = |D_{h^i} S_{t/2} (S_{t/2} \phi)(x)|$$

$$\leq |\mathbb{E}[D^i S_{t/2} \phi(X(t/2))D_h X^i(t/2)E_K(t/2)]|$$

$$+ |\mathbb{E}[D^i S_{t/2} \phi(X(t/2))D_h X^i(t/2)E_K(t/2)]|$$

$$+ \mathbb{E}[|S_{t/2} \phi(X(t/2))|E_K(t/2)K \int_0^{t/2} |AX(s)||AD_h X(s)| ds]$$

$$\leq C \left\{ \frac{1}{K} ||A^{-\gamma} DS_{t/2} \phi||_k + t^{-p} e^{Ct} ||\phi||_k + K ||\phi||_k \right\} (1 + |Ax|^k) |A^\gamma h|$$

where the last inequality is due to (3.10), (3.7) and (3.13), and to the following estimate (which is obtained by the same argument as in estimating $I_3$):

$$\mathbb{E}[S_{t/2} \phi(X(t/2))E_K(t/2) \int_0^{t/2} |AX(s)||AD_h X(s)| ds] \leq C ||\phi||_k (1 + |Ax|^k) |A^\gamma h|$$
Denote \( \alpha = p + \frac{1}{2} + r - \gamma \) and
\[
\phi_T = \sup_{0 \leq t \leq T} t^\alpha |A^{-\gamma} DS_t \phi|_k,
\]
by (3.8) and the similar argument as estimating \( I_3 \), we have
\[
|D_hS_t(\phi)| = |D_hE[\phi(X(t))\mathcal{E}_K(t)]|
\leq \mathbb{E} \left[ |A^{-\gamma} D\phi(X(t))\mathcal{E}_{\mathcal{F}_K}(t)|A^\gamma D_hX(t)\mathcal{E}_{\mathcal{F}_K}(t) \right]
+ 2K\mathbb{E} \left[ |\phi(X(t))\mathcal{E}_{\mathcal{F}_K}(t)\mathcal{E}_{\mathcal{F}_K}(t) \right] \int_0^t |AX(s)||AD_hX(s)|ds
\leq C(T, K, \gamma, k)(|A^{-\gamma} D\phi|_k + ||\phi||_k)(1 + |Ax|^k)|A^\gamma h|,
\]
which implies \( |A^{-\gamma} DS_t \phi|_k \leq C(T, K, \gamma, k)(|A^{-\gamma} D\phi|_k + ||\phi||_k) \), thus \( \phi_T < \infty \).

Combine (3.17) and (3.18), we have for every \( t \in [0, T] \)
\[
t^\alpha |A^{-\gamma} DS_t \phi|_k
\leq Ct^\alpha ||\phi||_k + \frac{C}{K} t^{\alpha-1} \int_0^{t/2} (t-s) -\alpha (t-s)^\alpha |A^{-\gamma} DS_{t-s} \phi|_k ds
+ CKt^\alpha ||\phi||_k + \frac{C}{K} t^{\alpha-1} \int_0^{t/2} (t-s) -\alpha ds
+ CKt^\alpha ||\phi||_k + \phi_T C \frac{C}{K} + CKt^\alpha e^{CT} ||\phi||_k
\leq \phi_T C + KCe^{CT} ||\phi||_k,
\]
this easily implies
\[
\phi_T \leq \phi_T C + KCe^{CT} ||\phi||_k.
\]
As \( K > 0 \) is sufficiently large, we have for all \( t \in [0, T] \)
\[
t^\alpha |A^{-\gamma} DS_t \phi|_k \leq \frac{K}{1 - C/K} Ce^{CT} ||\phi||_k,
\]
from which we conclude the proof. \( \square \)

4. Malliavin Calculus

4.1. Some preliminary for Malliavin calculus. Given a \( v \in L^2_{\text{loc}}(\mathbb{R}^+, \pi_mH) \), the Malliavin derivative of \( X_m(t) \) in direction \( v \), denoted as \( \mathcal{D}_v X_m(t) \), is defined by
\[
\mathcal{D}_v X_m(t) = \lim_{\epsilon \to 0} \frac{X_m(t, W + \epsilon V) - X_m(t, W)}{\epsilon}
\]
where \( V(t) = \int_0^t v(s)ds \), provided the above limit exists. \( v \) can be random and is adapted with respect to the filtration generated by \( W \).

Recall \( \pi \nu H = \pi_mH \) and \( Z_{\nu} = [-n, n]^3 \setminus (0, 0, 0) \) with \( n_0 < n < m \) to be determined in Proposition 4.6. The Malliavin derivatives on the low and high frequency parts of \( X_m(t) \), denoted by \( \mathcal{D}_v X_m^l(t) \) and \( \mathcal{D}_v X_m^h(t) \), can be defined in a
similar way as above. Moreover, $D_vX^i_m(t)$ and $D_vX^b_m(t)$ satisfy the following two SPDEs respectively:

\begin{equation}
(4.1) \quad \partial_t D_vX^i_m + AD_vX^i_m + \hat{B}^i_m(D_vX^i_m, X_m) + \hat{B}^i_m(D_vX^b_m, X_m) = Q^i_m v^i
\end{equation}

with $D_vX^i_m(0) = 0$, and

\begin{equation}
(4.2) \quad \partial_t D_vX^b_m + AD_vX^b_m + \hat{B}^b_m(D_vX^i_m, X_m) + \hat{B}^b_m(D_vX^b_m, X_m) = Q^b_m v^i
\end{equation}

with $D_vX^b_m(0) = 0$, where $\hat{B}(u, v) = B(u, v) + B(v, u)$. Moreover, we define a flow between $s$ and $t$ by $J^m_s, t$ ($s \leq t$), where $J^m_s, t \in \mathcal{L}(\pi^H, \pi^H)$ satisfies the following equation: \( \forall h \in \pi^H \)

\begin{equation}
(4.3) \quad \partial_t J^m_s, t h + AJ^m_s, t h + \hat{B}^i_m(J^m_s, t h, X_m(t)) = 0
\end{equation}

with $J^m_s, s = Id \in \mathcal{L}(\pi^H, \pi^H)$. It is easy to see that the inverse $(J^m_s, t)^{-1}$ exists and satisfies

\begin{equation}
(4.4) \quad \partial_t (J^m_s, t)^{-1} h - (J^m_s, t)^{-1}[Ah + \hat{B}^i_m(h, X_m(t))] = 0.
\end{equation}

Simply writing $J^m_t = J^m_0, t$, clearly, $J^m_s, t = J^m_t(J^m_s)^{-1}$.

We shall follow the ideas in section 6.1 of [7] to develop a Malliavin calculus for $X_m$, one of the key points for this approach is to find an adapted process $v \in L^{2}_{loc}(\mathbb{R}^2; \pi_m H)$ such that

\begin{equation}
(4.5) \quad Q^i_m v^i(t) = \hat{B}^i_m(D_vX^i_m(t), X_m(t)),
\end{equation}

which, combining with (4.2), implies $D_vX^b_m(t) = 0$ for all $t > 0$. More precisely,

**Proposition 4.1.** There exists some $v \in L^{2}_{loc}(\mathbb{R}^2; \pi_m H)$ satisfying (4.5), and

\begin{equation}
D_vX^i_m(t) = J^m_t \int_0^t (J^m_s)^{-1} Q^i_m v^i(s)ds, \quad D_vX^b_m(t) = 0.
\end{equation}

**Proof.** When $D_vX^b_m(t) = 0$ for all $t \geq 0$, the equation (4.1) is simplified to be

\begin{equation}
\partial_t D_vX^i_m + [AD_vX^i_m + \hat{B}^i_m(D_vX^i_m, X_m)] = Q^i_m v^i
\end{equation}

with $D_vX^i_m(0) = 0$, which is solved by

\begin{equation}
(4.6) \quad D_vX^i_m(t) = \int_0^t J^m_s, t Q^i_m v^i(s)ds = J^m_t \int_0^t (J^m_s)^{-1} Q^i_m v^i(s)ds.
\end{equation}

Due to (A3) of Assumption 2.1, there exists some $v \in L^{2}_{loc}(\mathbb{R}^2; \pi_m H)$ so that $v^i$ satisfies (4.5), therefore, (4.2) is a homogeneous linear equation and has a unique solution $D_vX^b_m(t) = 0$, \( \forall t > 0 \). \hfill \Box

With the previous lemma, we see that the Malliavin derivative is essentially restricted in low frequency part. Take

\[ N := 2[(2n + 1)^3 - 1] \]

vectors $v_1, \ldots, v_N \in L^{2}_{loc}(\mathbb{R}^2; \pi_m H)$ with each satisfying Proposition 4.1 ($N$ is the dimension of $\pi^H$). Denote

\begin{equation}
(4.7) \quad v = [v_1, \ldots, v_N],
\end{equation}

we have

\begin{equation}
(4.8) \quad D_vX^i_m = 0, \quad D_vX^b_m(t) = J^m_t \int_0^t (J^m_s)^{-1} Q^i_m v^i(s)ds,
\end{equation}

where $Q_m^t$ is defined in (3.5). In particular, $D_nX_m^t(t)$ is an $N \times N$ matrix. Choose

$$v^t(s) = [(J_s^m)^{-1}Q_m^t]^*$$

and denote

$$\mathcal{M}_m^t = \int_0^t [(J_s^m)^{-1}Q_m^t][(J_s^m)^{-1}Q_m^t]^*ds,$$

$\mathcal{M}_m^t$ is called Malliavin matrix, and is clearly a symmetric operator in $\mathcal{L}(\pi^t H, \pi^t H)$. For any $\eta \in \pi^t H$, we have by Parseval’s identity

$$\langle \mathcal{M}_t\eta, \eta \rangle = \int_0^t \langle [(J_s^m)^{-1}Q_m^t]^*\eta, [(J_s^m)^{-1}Q_m^t]^*\eta \rangle ds$$

$$= \sum_{k \in Z(n)} \sum_{i=1}^2 \int_0^t |\langle (J_s^m)^{-1}Q_m^t e_k, \eta \rangle|^2 ds$$

$$= \sum_{k \in Z(n)} \sum_{i=1}^2 \int_0^t |\langle (J_s^m)^{-1}q_k e_k, \eta \rangle|^2 ds$$

where $q_k$ is the $i$-th column vector of the $2 \times 2$ matrix $q$ (recall (2.3)).

The following lemma is crucial for proving Lemma 3.5 and will be proven in Appendix 6.3.

**Lemma 4.2.** 1. For any $h \in \pi^t H$, we have

$$|J_s^m h|^2 \mathcal{E}_{m,K}(t) \leq |h|^2,$$

$$|D_h X^t(t)|^2 \mathcal{E}_{m,K}(t) \leq |h|^2,$$

$$(J_s^m)^{-1}h \mathcal{E}_{m,K}(t) \leq C e^{Ct} |h|^2$$

$$|\mathcal{E}_{m,K}(t)(J_s^m)^{-1} - Id|_{\mathcal{L}(H)} \leq t^{1/2} C e^{Ct}$$

$$\mathbb{E} \left( \int_0^t |\langle (J_s^m)^{-1}Q_m^t \rangle h|^2 \mathcal{E}_{m,K}(s) ds \right) \leq t e^{Ct} tr[Q_m^t(Q_m^t)^*]|h|.$$n

where the above $C = C(n) > 0$ can vary from line to line and the $n$ is the size of $\pi^t H$ defined in (3.3).

2. Suppose that $v_1, v_2$ satisfy Proposition 4.1 and $h \in \pi^t H$, we have

$$|AD_v X_m^t(t)|^2 \mathcal{E}_{m,K}(t) \leq C \int_0^t e^{1/2(t-s)} |v_1(s)|^2 \mathcal{E}_{m,K}(s) ds$$

$$|D_{v_1} D_h X_m^t(t)|^2 \mathcal{E}_{m,K}(t) \leq C e^{Ct} |h|^2 \left( \int_0^t |v_1(s)|^2 \mathcal{E}_{m,K}(s) ds \right)$$

$$|D_{v_1} v_2 X_m^t(t)|^2 \mathcal{E}_{m,K}(t) \leq C e^{Ct} \left( \int_0^t |v_1(s)|^2 \mathcal{E}_{m,K}(s) ds \right) \left( \int_0^t |v_2(s)|^2 \mathcal{E}_{m,K}(s) ds \right)$$

where the above $C = C(n) > 0$ can vary from line to line and the $n$ is the size of $\pi^t H$ defined in (3.3).
4.2. Hörmander’s systems and proof of Lemma 3.5. We consider the SPDE about $X_m^\ell$ in Stratanovich form as

$$dX_m^\ell + [A_m^\ell X_m^\ell + B_m^\ell(X)]dt = \sum_{k \in Z_i(n)} \sum_{i=1}^2 q_k^i \circ dw_k^i(t)e_k$$

where $A^\ell$ is the Stokes operator restricted on $\pi^\ell H$ and $q_k^i$ is the $i$-th column vector in the $2 \times 2$ matrix $q_k$ (under the orthonormal basis $(e_k^1, e_k^2)$ of $k^\perp$). Given any two Banach spaces $B_1$ and $B_2$, denote $P(B_1, B_2)$ the collections of functions from $B_1$ to $B_2$ with polynomial growth. We introduce the Lie bracket on $\pi^\ell H$ as follows: \( \forall K_1 \in P(\pi_m H, \pi^\ell H), \ K_2 \in P(\pi_m H, \pi^\ell H), \) define \([K_1, K_2]\) by

$$[K_1, K_2](x) = DK_1(x)K_2(x) - DK_2(x)K_1(x) \ \forall \ x \in \pi_m H.$$  

The brackets \([K_1, K_2]\) will appear when differentiating $J^{-1}_i K_1(X(t))$ in the proof of Lemma 4.7.

**Definition 4.3.** The Hörmander’s system $K$ for equation (4.19) is defined as follows: given any $y \in \pi_m H$, define

- $K_0(y) = \{q_k^i e_k; k \in Z_i(n) \setminus Z_i(n_0), i = 1, 2\}$
- $K_1(y) = \{[A_m^\ell y + B_m^\ell(y, y), q_k^i e_k]; k \in Z_i(n) \setminus Z_i(n_0), i = 1, 2\}$
- $K_2(y) = \{[q_k^i e_k, K(y)]; K \in K_1(y), k \in Z_i(n) \setminus Z_i(n_0), i = 1, 2\}$

and $K(y) = \text{span}(K_0(y) \cup K_1(y) \cup K_2(y))$, where each $q_k^i$ is the column vector defined in (2.3).

**Definition 4.4.** The system $K$ satisfies the restricted Hörmander condition if there exist some $\delta > 0$ such that for all $y \in \pi_m H$

$$\sup_{K \in K} |\langle K(y), \ell \rangle| \geq \delta|\ell|, \quad \ell \in \pi^\ell H.$$  

The following lemma gives some inscription for the elements in $K_2$ (see (4.21)) and plays the key role for the proof of Proposition 4.6.

**Lemma 4.5.** For each $k \in Z_i(n_0)$, define mixing set $Y_k$ by

$$Y_k = \left\{ \tilde{B}_{m,k}(q_j^l e_j, q_k^l e_k) : j, l \in Z_i(n_0); \ell_j \in l_1^\perp, \ell_k \in l_2^\perp \right\},$$

where $\tilde{B}_{m,k}(x, y)$ is the Fourier coefficient of $\tilde{B}_m(x, y)$ at the mode $k$. For all $k \in Z_i(n_0)$, span$\{Y_k\} = k^\perp$.

**Proposition 4.6.** $K$ in Definition 4.3 satisfies the restricted Hörmander condition.

**Proof.** It suffices to show that for each $k \in Z_i(n_0)$, the Lie brackets in Definition 4.3 can produce at least two linearly independent vectors of $Y_k$ in Lemma 4.5. (We note that [19] proved a similar proposition).

As $k \in Z_i(n_0) \cap Z_i^+$, by Lemma 4.5, $Y_k$ has at least two linearly independent vectors $h_k^1 \parallel h_k^2$. Without lose of generality, assume $h_k^1 = \tilde{B}_k(q_j^1 e_j, q_k^1 e_k)$ and $h_k^2 = \tilde{B}_k(q_j^2 e_j, q_k^2 e_k)$ with $j_k - l_k \in Z_i(n_0)$ and $j_k + l_k = k$. We can easily have

$$[q_j^l e_j, [A^{\ell} y + B^{\ell}(y, y), q_k^l e_k]] = -\tilde{B}(q_j^l e_l, q_k^l e_j),$$

(4.21)
and by (6.1)-(6.3),
\[ [q_j^1 e_j, [A' y + B' (y, y), q_k^1 e_k]] = -\frac{1}{2} \tilde{B}_{j-l}(q_j^1 e_j, q_k^1 e_k) - \frac{1}{2} \tilde{B}_k(q_j^1 e_j, q_k^1 e_k). \]

Clearly, \( j-l \in \mathbb{Z}_n(n_0) \), by (A2) of Assumption 2.1, \( \tilde{B}_{j-l}(q_j^1 e_j, q_k^1 e_k) \) and \( \tilde{B}_k(q_j^1 e_j, q_k^1 e_k) \) must both be equal to a linear combination of \( q_j^1 e_j \) and \( q_k^1 e_k \) (i = 1, 2). Combining this observation with the fact \( \tilde{B}_k(q_j^1 e_j, q_k^1 e_k) \) and \( \tilde{B}_k(q_j^1 e_j, q_k^1 e_k) \) one immediately has that \( [q_j^1 e_j, [A' y + B' (y, y), q_k^1 e_k]] \) and \( q_j^1 e_j \) span \( k \).

Similarly, we have the same conclusion for \( j_k, l_k, j_k + l_k, j_k - l_k \in \mathbb{Z}_n(n) \) for all \( k \in \mathbb{Z}_n(n_0) \). \( \square \)

With Proposition 4.6, we can show the following key lemma (see the proof in Section 6.2).

**Lemma 4.7.** Suppose that \( X_m(t,x) \) is the solution to equation (2.8) with initial data \( x \in \pi m \). Then \( M^m_t \) is invertible almost surely. Denote \( \lambda_{min}(t) \) the minimal eigenvalue of \( M^m_t \), then there exists some constant \( q > 0 \) (possibly very large), for all \( p > 0 \), we have a constant \( C = C(p) \) such that
\[
 \mathbb{P} \left\{ \frac{1}{\lambda_{min}(t)} \geq \frac{1}{\varepsilon^q} \right\} \leq \frac{C \varepsilon^p}{t^p}.
\]

**Proof of Lemma 3.5.** We shall simply write \( X(t) = X_m(t), J_t = J^m_t, M_t = M^m_t, Q^t = Q^m_t \) and \( E_K(t) = E_{m,K}(t) \) for the notational simplicity. Under an orthonormal basis of \( \pi^t H \), the operators \( J_t, M_t, D_v X^t(t) \) with \( v \) defined in (4.6), and \( D^t X^t(t) \) can all be represented by \( N \times N \) matrices, where \( N \) is the dimension of \( \pi^t H \). Noticing \( D_v X^t(t) = J_t M_t \) (see (4.8)), the following \( \phi_{v,t} \) is well defined:
\[
 \phi_{v,t}(X(t)) = \phi(X(t)) \sum_{j=1}^N ([D_v X^t(t)]^{-1})_{ij} [D^t X^t(t)]_{jl} E_K(t) \quad i, l = 1, \ldots, N,
\]
where \( v \) is defined in (4.7) with \( v^t(t) = (J_t^{-1} Q^t) \). For any \( h \in \pi^t H \), by our special choice of \( v \), we have
\[
 D_{ch} \phi_{v,t}(X(t)) = D^t \phi(X(t)) [D_v X^t(t)]_{bh} \sum_{j=1}^N ([D_v X^t(t)]^{-1})_{ij} [D^t X^t(t)]_{jl} E_K(t)
\]
\[
 + \phi(X(t)) \sum_{j=1}^N D_{ch} \left\{ ([D_v X^t(t)]^{-1})_{ij} [D^t X^t(t)]_{jl} \right\} E_K(t)
\]
\[
 - 2K \phi_{v,t}(X(t)) \int_0^t \langle AX(s), AD_{ch} X(s) \rangle ds
\]

Note that \( \pi^t H \) is isomorphic to \( \mathbb{R}^N \) under the orthonormal basis. Take the standard orthonormal basis \( \{ h_i; i = 1, \ldots, N \} \) of \( \mathbb{R}^N \), which is a representation of the
orthonormal basis of $\pi' H$. Set $h = h_i$ in (4.23) and sum over $i$, we obtain

\begin{equation}
\mathbb{E}(D^i\phi(X(t))D_{h_i}X^i(t)E_K(t)) = \mathbb{E}\left(\sum_{i=1}^{N} D_{v_i,\phi_{il}}(X(t))\right) - \mathbb{E}\left(\sum_{i,j=1}^{N} \phi(X(t)) D_{v_i} \left\{ [D_{v_i} X^i(t)]_{ij} [D^i X^i(t)]_{jl} \right\} E_K(t)\right) + 2K\mathbb{E}\left(\sum_{i=1}^{N} \phi_{il}(X(t)) \int_{0}^{t} \langle AX(s), AD_{v_i} X(s) \rangle ds \right)
\end{equation}

Let us first bound the first term on the r.h.s. of (4.24) as follows: By Bismut formula (simply write $v_i = vh_i$), (3.7) and the identity $D_{v} X^i(t) = J_t M_{t}$, one has

\begin{equation}
\mathbb{E}\left(\sum_{i=1}^{N} D_{v_i,\phi_{il}}(X(t))\right) = \mathbb{E}\left(\sum_{i,j=1}^{N} \phi(X(t)) [M_t^{-1} J_t^{-1}]_{ij} [D^i X^i(t)]_{jl} E_K(t) \int_{0}^{t} \langle v'_i, dW_s \rangle \right) \\
\leq C||\phi||_k (1 + |Ax|^k) \sum_{i,j=1}^{N} \mathbb{E}\left(\frac{\mathcal{E}_{K/2}(t)}{\lambda_{\min}} |J_t^{-1} h_j||D_{h_i} X^i(t)|| \int_{0}^{t} \langle v'_i, dW_s \rangle \right),
\end{equation}

moreover, by Hölder’s inequality, Burkholder-Davis-Gundy’s inequality, (4.22), (4.13), (4.12) and (4.15) in order,

\begin{equation}
\mathbb{E}\left(\frac{\mathcal{E}_{K/2}(t)}{\lambda_{\min}} |J_t^{-1} h_j||D_{h_i} X^i(t)|| \int_{0}^{t} \langle v'_i, dW_s \rangle \right) \\
\leq \mathbb{E}\left(\frac{1}{\lambda_{\min}^{6q}} \right)^{\frac{p}{2}} \left[ \mathbb{E}( |J_t^{-1} h_j|^6 \mathcal{E}_{K}(t)) \right]^{\frac{p}{2}} \left[ \mathbb{E}( |D_{h_i} X^i(t)|^6 \mathcal{E}_{K}(t)) \right]^{\frac{p}{6}} \left[ \mathbb{E}( \int_{0}^{t} \mathcal{E}_{K}(s)(J_s^{-1} Q_s)^* h_i^2 ds \right]^{\frac{1}{2}} \\
\leq \frac{C e^{Ct}}{t^p}
\end{equation}

where $p > 6q + 1$ and $C = C(p)$. Combining (4.25) and (4.26), we have

\begin{equation}
\mathbb{E}\left(\sum_{i=1}^{N} D_{v_i,\phi_{il}}(X(t))\right) \leq \frac{C e^{Ct}}{t^p} ||\phi||_k (1 + |Ax|^k)
\end{equation}

where $C = C(p, k) > 0$. By the similar method but more complicate calculations (using Lemma 4.7 and the estimates in Lemma 4.2), we have the same bounds for the other two terms on the r.h.s. of (4.24). Hence,

\begin{equation}
\mathbb{E}\left[D^i\phi(X(t))D_{h_i}X^i(t)E_K(t)\right] \leq t^{-p} C e^{Ct} ||\phi||_k (1 + |Ax|^k)
\end{equation}

for all $t > 0$. Since the above argument is in the frame of $\pi' D(A)$ with the orthonormal base $\{h_l; 1 \leq l \leq N\}$, we have

\begin{equation}
\mathbb{E}\left[D^i\phi(X(t))D_{h_i}X^i(t)E_K(t)\right] \leq t^{-p} C e^{Ct} ||\phi||_k (1 + |Ax|^k) h_i \quad h \in \pi' H.
\end{equation}
5. Proof of the main theorems

5.1. Gradient estimates of $u_m(t)$. To prove the strong Feller of the semigroup $P_t^m$ (recall $P_t^m \phi = u_m(t)$) and the later limiting semigroup $P_t$, a typical method is to show that $P_t^m$ has a gradient estimate similar to (3.1). In [5], one has the same estimate as (3.1) but with $\alpha = \frac{1}{2} + r - \gamma$ therein, thanks to the property $0 < \frac{1}{2} + r - \gamma < 1$, one can easily show

$$||A^{-\gamma}Du_m(t)||_2 \leq C(t^{-\frac{1}{2} - r + \gamma} + 1)||\phi||_0,$$

this is exactly the second inequality in Proposition 3.5 of [5].

In our case, by the same method as in [5] (i.e. applying (3.1) to bound the r.h.s. of (2.15)), we formally have

$$||A^{-\gamma}Du_m(t)||_2 \leq Ct^{-\alpha}||\phi||_0 + KC \int_0^t (t-s)^{-\alpha} ds ||\phi||_0,

$$

however, the integral on the r.h.s. of (5.1) blows up due to $\alpha > 1$ in (3.1).

We have two ways to overcome the problem of not integrability in (5.1). One is by an interpolation argument (see Proposition 5.1), the other is by some more delicate analysis (see Proposition 5.2). The underlying ideas of the two methods are the same, i.e. trading off the regularity of the space for the integrability of the time.

**Proposition 5.1.** Given $T > 0$, for any $0 < t \leq T$, $\max\{\frac{1}{2}, r - \frac{1}{2}\} < \gamma \leq 1$ and $0 < \beta < 1$, if $\phi \in C^1_t(D(A), \mathbb{R})$, then $S_t^m \phi$ and $u_t^m$ are both functions in $C^{\beta/\alpha}_{2,\gamma}(D(A), \mathbb{R})$, which is the Hölder space defined by (2.6). Moreover,

$$||S_t^m \phi||_{C^{\beta/\alpha}_{2,\gamma}} \leq Ct^{-\beta}||\phi||_2,

(5.3)

$$

where $\alpha = p + \frac{1}{2} + r - \gamma$ is defined in (3.1) and $C = C(T, \alpha, \beta, \gamma) > 0$.

**Proof.** On the one hand, for any $x \in D(A)$, by (2.13) and (3.7), one clearly has

$$|S_t^m \phi(x)| \leq ||\phi||_2 E[(1 + |AX^m(t)|^2)E_{m,K}(t)] \leq C(1 + |Ax|^2)||\phi||_2

where $C > 0$ is independent of $m, t$ and $x$. Hence, $S_t^m : C_2(D(A), \mathbb{R}) \rightarrow C_2(D(A), \mathbb{R})$ has the estimate

$$||S_t^m \phi||_2 \leq C||\phi||_2.

On the other hand, by (3.1), one has $S_t^m : C_2(D(A), \mathbb{R}) \rightarrow C^{\beta/\alpha}_{2,\gamma}(D(A), \mathbb{R})$ with

$$||S_t^m \phi||_{C^{\beta/\alpha}_{2,\gamma}} \leq Ct^{-\alpha}||\phi||_2.

By a simple calculation with the the above two estimates, we have

$$||S_t^m \phi||_{C^{\beta/\alpha}_{2,\gamma}} \leq C||S_t^m \phi||_{C^{\beta/\alpha}_{2,\gamma}} ||S_t^m \phi||_2^{1-\beta/\alpha} \leq Ct^{-\beta}||\phi||_2,

for any $0 \leq \beta \leq \alpha$. Take any $0 < \beta < 1$, applying the above estimate on the Duhamel formula (2.15) and the clear fact $||u_m(t)||_0 \leq ||\phi||_0$, we immediately have (5.3).
Proposition 5.2. Given any $T > 0$, there exists some $C = C(T, \alpha, \gamma) > 0$ such that
\begin{equation}
||A^{-\gamma}Du_m(t)||_{2\alpha} \leq Ct^{-\alpha}||\phi||_0
\end{equation}

where $\max\{r - \frac{1}{2}, \frac{1}{3}\} < \gamma \leq 1$ with $\gamma \neq \frac{3}{4}$.

Proof. The idea of the proof is to split the integral $\int_0^1 |D_h S_{t-s}^m(\|Ax\|^2 u_m(s))|ds$ into two pieces, $\int_0^{\beta t} \cdots$ and $\int_{\beta t}^t \cdots$ with some special $\beta \in (0, 1)$, applying (3.1) to the first piece and the probability presentation of $S_{t-s}^m$ to the other. Roughly speaking, $\int_0^{\beta t} \cdots$ takes away the singularity of $(t-s)^{-\alpha}$ at $s = t$, while $\int_{\beta t}^t \cdots$ conquers the extra polynomial growth of $|Ax|^2$ in $S_{t-s}^m||Ax|^2 u_m(s)$. However, we have to pay a price of an extra polynomial growth of $|Ax|^2$ for $Du_m(t)$.

For the notational simplicity, we shall drop the index $m$ of the quantities if no confusions arise. Denote
\begin{equation}
\beta = 1 - \frac{1}{K^2(1 + |Ax|^2)}.
\end{equation}

by (3.1) with $k = 2$, we have
\begin{align*}
|A^{-\gamma}Du(t, x)| &\leq Ct^{-\alpha}||\phi||_2(1 + |Ax|^2) + KC \int_0^{\beta t} (t-s)^{-\alpha}ds ||\phi||_0(1 + |Ax|^2) \\
&\quad + K \int_{\beta t}^t |A^{-\gamma}DS_{t-s}(|Ax|^2 u(s))|ds \\
&\leq Ct^{-\alpha}||\phi||_0(1 + |Ax|^2) + K^{2\alpha+1}Ct^{-\alpha+1}(1 + |Ax|^{2\alpha})||\phi||_0 \\
&\quad + K \int_{\beta t}^t |A^{-\gamma}DS_{t-s}(|Ax|^2 u(s))|ds,
\end{align*}

thus
\begin{align}
t^{\alpha} |A^{-\gamma}D_h u(t, x)| &\leq C||\phi||_0(1 + |Ax|^2) + K^{2\alpha+1}Ct||\phi||_0(1 + |Ax|^{2\alpha}) \\
&\quad + K t^{\alpha} \int_{\beta t}^t |A^{-\gamma}DS_{t-s}(|Ax|^2 u(s))|ds.
\end{align}

Define
\begin{equation}
u_{\phi, T} = \sup_{0 \leq s \leq T} s^\alpha ||A^{-\gamma}Du(s)||_{2\alpha},
\end{equation}

let us estimate the integral on the r.h.s. of (5.5) in the following way: it is easy to see that
\begin{align}
\int_{\beta t}^t |D_h S_{t-s}(|Ax|^2 u(s))|ds \\
= \int_{\beta t}^t |\mathbb{E}(D_h |AX(t-s)|^2u(s, X(t-s))\mathcal{E}_K(t-s)||ds \\
+ \int_{\beta t}^t |\mathbb{E}(|AX(t-s)|^2u(s, X(t-s))D_h \mathcal{E}_K(t-s)||ds \\
+ \int_{\beta t}^t |\mathbb{E}(|AX(t-s)|^2D_h u(s, X(t-s))\mathcal{E}_K(t-s)||ds.
\end{align}
By the same argument as estimating \( I_3 \) in the proof of Theorem 3.1 and the easy fact \( ||u(t)||_0 \leq ||\phi||_0 \) for all \( t \geq 0 \), the first two integrals on the r.h.s. of (5.6) can both be bounded by
\[
C(1 + |Ax|^2)||\phi||_0|A^n h|.
\]
The last integral can be estimated as follows: By (3.7), (3.8) and the definition of \( u_{T,\phi} \), one has
\[
\int_{\beta t}^t |E(||AX(t-s)|^2D_hu(s, X(t-s))E_K(t-s)||) ds
\]
\[
\leq \int_{\beta t}^t E[(1 + |AX(t-s)|^{2+2\alpha})E_K(t-s)||A^{-\gamma}Du(s)||_{2\alpha}E_K(t-s)|A^n D_hX(t-s)||] ds
\]
\[
\leq C(1 + |Ax|^{2+2\alpha})|A^n h| \int_{\beta t}^t ||A^{-\gamma}Du(s)||_{2\alpha} ds
\]
\[
\leq C(1 + |Ax|^{2+2\alpha})|A^n h| \left( \int_{\beta t}^t \alpha^{-\alpha} ds \right) u_{T,\phi}
\]
\[
\leq \frac{Ct^{-\alpha+1}}{K^2} u_{T,\phi}(1 + |Ax|^{2\alpha})|A^n h|.
\]
Collecting the above three estimates, we have
\[
\int_{\beta t}^t |A^{-\gamma}DS_{t-s}(|Ax|^2 u(s))|| ds \leq C(1 + |Ax|^2)||\phi||_0 + \frac{Ct^{-\alpha+1}}{K^2} u_{T,\phi}(1 + |Ax|^{2\alpha}).
\]
Plugging this estimate into (5.5) and dividing the both sides of the inequality by \((1 + |Ax|^{2+2\alpha})\), one has
\[
u_{T,\phi} \leq C||\phi||_0 + CK^{2\alpha+1}T||\phi||_0 + CKT^{\alpha}||\phi||_0 + \frac{CT}{K} u_{T,\phi}.
\]
As \( K > 0 \) is sufficiently large, \( u_{T,\phi} \leq \frac{C(1 + K^{2\alpha+1}T + KT^{\alpha})}{1 - CT/K} ||\phi||_0 \),
from this inequality, we immediately have (5.4).
\[\square\]

5.2. Proof of Theorem 2.4. One can pass to the Galerkin approximation limit of \( u_m(t) \) by the same procedures as in [5]. For the completeness, we sketch out the main steps as following.

The following proposition is nearly the same as Proposition 3.6 in [5], only with a small modification in which (5.3) plays an essential role.

**Proposition 5.3.** Let \( \phi \in C^1_b(D(A), \mathbb{R}) \) and \( T > 0 \). For any \( 0 < \beta < 1/2, t_1 \geq t_2 > 0, m \in \mathbb{N} \) and \( x \in D(A) \), we have some \( C(T, \beta) > 0 \) such that
\[
|u_m(t_1, x) - u_m(t_2, x)| \leq C||\phi||_{C^1_{2,1,b}}(1 + |Ax|^6)||A^t_x - A^t_{x}||_{2\alpha} + |A(e^{-At_2} - e^{-At_1})x|.
\]

Define \( K_R = \{ x \in D(A); |Ax| \leq R \} \), which is compact in \( D(A^n) \) for any \( \gamma < 1 \), we have the following lemma (which is Lemma 4.1 in [5]) by applying Proposition 5.3.

**Lemma 5.4.** Assume \( \phi \in C^1_b(D(A), \mathbb{R}) \), then there exists a subsequence \((u_{m_k})_{k \in \mathbb{N}}\) of \((u_m)\) and a function \( u \) on \([0, T] \times D(A)\), such that
Lemma 5.5. For any \( \delta \in (1/2, 1 + \sigma] \), there exists some constant \( C(\delta) > 0 \) such that for any \( x \in H \), \( m \in \mathbb{N} \), and \( t \in [0, T] \), we have

\[
(1) \quad \mathbb{E}[|X_m(t, x)|^2] + \int_0^t |A^{1/2} X_m(s, x)|^2 ds \leq |x|^2 + tr(QQ^*) t.
\]

\[
(2) \quad \mathbb{E} \int_0^T \frac{|A^{1/2} X_m(s, x)|^2}{(1 + |A X_m(s, x)|^2)^{\gamma_\delta}} ds \leq C(\delta), \text{ with } \gamma_\delta = \frac{2}{20 - 1} \text{ if } \delta \leq 1 \text{ and } \gamma_\delta = \frac{2 \delta + 1}{20 - 1} \text{ if } \delta > 1.
\]

By (1) of Lemma 5.5, we can prove that the laws \( \mathcal{L}(X_m(\cdot, x)) \) is tight in \( L^2([0, T], D(A^{s/2})) \) for \( s < 1 \) and in \( C([0, T], D(A^{-\alpha})) \) for \( \alpha > 0 \). By Skohorod’s embedding Theorem, one can construct a probability space \((\Omega_x, \mathcal{F}_x, \mathbb{P}_x)\) with a random variable \( X(\cdot, x) \) valued in \( L^2([0, T], D(A^{s/2})) \cap C([0, T], D(A^{-\alpha})) \) such that for any \( x \in D(A) \) there exists some subsequence \( \{X_{m_k}\} \) satisfying

\[
X_{m_k}(\cdot, x) \rightarrow X(\cdot, x) \quad \text{d}\mathbb{P}_x \text{ a.s.}
\]

in \( L^2([0, T], D(A^{s/2})) \cap C([0, T], D(A^{-\alpha})) \). Moreover, by (3) of Lemma 5.5, for \( x \in D(A) \) we have (see (7.7) in [3])

\[
X_{m_k}(t, x) \rightarrow X(t, x) \quad \text{in } D(A) \quad \text{d}t \times \text{d}\mathbb{P}_x \text{ a.s. } [0, T] \times \Omega_x.
\]

Note that the subsequence \( \{u_{m_k}\} \) in Lemma 5.4 depends on \( \phi \), by the separable property of \( C(D(A), \mathbb{R}) \), we can find a subsequence \( \{m_k\} \) of \( \{m\} \), independent of \( \phi \), such that \( \{u_{m_k}\} \) converges. That is, we have the following lemma, which is Lemma 7.5 of [3].

Lemma 5.6. There exists a subsequence \( \{m_k\} \) of \( \{m\} \) so that for any \( \phi \in C_b^1(D(A), \mathbb{R}) \), one has a function \( u^\phi \in C_b([0, T] \times D(A)) \) satisfying

\[
\lim_{k \to \infty} u_{m_k}^\phi(t, x) = u^\phi(t, x) \quad \text{for all } (t, x) \in (0, T] \times D(A)
\]

and

\[
u_{m_k}^\phi(t, x) \rightarrow u^\phi(t, x) \quad \text{uniformly in } [\delta, T] \times K_R \text{ for any } \delta > 0, R > 0.
\]

where \( u_{m_k}^\phi(t, x) = \mathbb{E}[\phi(X_{m_k}(t, x))] \).

Take the subsequence \( \{m_k\} \) in Lemma 5.6 and define

\[
P^t \phi(x) = u^\phi(t, x).
\]

for all \( (t, x) \in [0, T] \times D(A) \), where \( u^\phi \) is defined by (5.10). By Riesz Representation Theorem for functionals ([12], page 223) and the easy fact \( P^t_1 = 1 \), (5.11) determines a unique probability measure \( P^t \delta_x \) supported on \( D(A) \). By (5.8), for any
In pp. 938 of [φ ∈ C^1(∥A∥, [x, y])],

\[ P_t \phi(x) = E_x[\phi(X(t, x))] \]

for all φ ∈ C^1 (D(A), R) ∩ C^0 (∥A∥, [x, y]), since the measure P_t \cdot \delta_x is supported on D(A), \mathbb{P}_x (X(t, x) \in D(A)) = 1, which is (1) of Definition 2.3. By a classic approximation (B_0(D(A), R) can be approximated by C(D(A), R)), we have

\[ P_t \phi(x) = E_x[\phi(X(t, x))] \]

is well defined for all φ ∈ B_0(D(A), R).

With the above observation, we can easily prove Theorem 2.4 as follows:

**Proof of Theorem 2.4.** Since X_{m_k} (·, x) → X (·, x) a.s. \mathbb{P}_x in C([0, T], D(A^-α)) and the map x → \mathcal{P}_{\mathcal{M}_k} is measurable (\mathcal{P}_{\mathcal{M}_k} is the law of X_{m_k} (·, x)), the map x → \mathcal{P}_x is also measurable. The following lemma is exactly Lemma 4.5 in [3] and expressed as

**Lemma 5.7.** Let X (·, x) be the limit process of a subsequence \{X_{m_k}\}_k. Then, for any M, N ∈ \mathbb{N}, t_1, \ldots , t_n ≥ 0 and \{f_k\}_{k=0}^M with each f_k ∈ C_c^\infty (\pi_N H, R), we have

\[ E_x [f_0 (X(0, x)) f_1 (X(t_1, x)) \cdots f_M (X(t_1 + \cdots + t_M, x))] = f_0 (x) P_t [f_1 P_t (f_2 P_t (f_3 \cdots ))] (x) \]

where each f_k (x) = f_k (\pi_N x) and P_t is defined by (5.12).

One can easily extend (5.13) from C_c^\infty (\pi_N H, R) to B_0 (D(A), R), which easily implies the Markov property of the family (Ω_x, \mathcal{F}_x, \mathbb{P}_x, X (·, x))_{x \in D(A)}.

5.3. **Proof of Theorem 2.5.** To prove the ergodicity, we first prove that (2.1) has at least one invariant measure, and then show the uniqueness by Doob’s Theorem. With the ergodic measure, we follow the coupling method in [17] to prove the exponential mixing property (2.7).

**Lemma 5.8.** Each approximate stochastic dynamics X_{m_k} (t) has a unique invariant measure \nu_{m_k}.

**Proof.** By Proposition 5.1 or Proposition 5.2, we can easily obtain that P_t \cdot \nu_{m_k} is strong Feller. The existence of the invariant measures for X_{m_k} (t) is standard (see [4]), and it is easy to prove that 0 is the support of each invariant measure (see Lemma 3.1, [6]). Therefore, by Corollary 3.17 of [13], we conclude the proof.

The following lemma is the same as Lemma 7.6 in [3] (or Lemma 5.1 in [5]).

**Lemma 5.9.** There exists some constant C > 0 so that

\[ \int_{H} ||Ax||^2 + ||A^{1/2}x||^2 + ||A^{1/2}x||^2 + ||A^{1/2}x||^{1+σ}/(1+8σ) || ν_{m_k} (dx) < C \]

where σ > 0 is the same as in Assumption 2.1.

With the above lemma, it is easy to see that \{ν_{m_k}\} is tight on D(A), and therefore there exists a limit measure \nu which satisfies ν(D(A)) = 1. Taking any φ ∈ C^0 (D(A), R), we can check via the Galerkin approximation (or see the detail in pp. 938 of [3]) that

\[ \int_{H} P_t \phi(x) ν(dx) = \int_{H} \phi(x) ν(dx) \]

for any t > 0. Hence ν is an invariant measure of P_t.
Proposition 5.10. The system $X(t)$ is irreducible on $D(A)$. More precisely, for any $x, y \in D(A)$, we have

$$P_t[1_{B_\delta(y)}](x) > 0.$$ \hfill (5.16)

for arbitrary $\delta > 0$, where $B_\delta(y) = \{z \in D(A); |Az - Ay| \leq \delta\}$.

Proof. We first prove that the following control problem is solvable: Given any $T > 0, x, y \in D(A)$ and $\varepsilon > 0$, there exist $\rho_0 = \rho_0(|Ax|, |Ay|, T)$, $u$ and $w$ such that

- $w \in L^2([0, T]; H)$ and $u \in C([0, T]; D(A))$,
- $u(0) = x$ and $|Au(T) - Ay| \leq \varepsilon$,
- $\sup_{t \in [0, T]} |Au(t)| \leq \rho_0$,

and $u$, $w$ solve the following problem,

$$\partial_t u + Au + B(u, u) = Qw,$$

where $Q$ is defined in Assumption 2.1.

This control problem is exactly Lemma 5.2 of [23] with $\alpha = 1/4$ therein, but we give the sketch of the proof for the completeness. Firstly, it is easy to find some $z \in D(A^{5/2})$ with $|Ay - Az| \leq \varepsilon/2$, therefore it suffices to prove there exists some control $w$ so that

$$|Au(T) - Az| \leq \varepsilon/2.$$ \hfill (5.18)

Secondly, decompose $u = u^k + u'$ where $u^k = (I - \pi_{n_0})u$ and $u' = \pi_{n_0}u$ and $n_0$ is the number in Assumption 2.1, then equation (5.17) can be written as

$$\partial_t u' + Au' + B'(u, u) = 0,$$ \hfill (5.19)

$$\partial_t u^k + Au^k + B^k(u, u) = Q^k w.$$ \hfill (5.20)

We prove (5.18) in the following four steps:

1. **Regularization of the initial data**: Let $w \equiv 0$ on $[0, T_1]$, by some classical arguments about the regularity of Navier-Stokes equation, one has $u(T_1) \in D(A^{5/2})$, where $T_1 > 0$ depends on $|Ax|$.

2. **High modes lead to zero**: Choose a smooth function $\psi$ on $[T_1, T_2]$ such that $0 \leq \psi \leq 1$, $\psi(T_1) = 1$ and $\psi(T_2) = 0$, and set $u^k(t) = \psi(t)u^k(T_1)$ for $t \in [T_1, T_2]$. Plugging this $u^k$ into (5.20), we obtain

$$w(t) = \psi(t)(Q^k)^{-1}u^k(T_1) + \psi(t)(Q^k)^{-1}Au^k(T_1) + (Q^k)^{-1}B^k(u(t), u(t)).$$

3. **Low modes close to $z'$**: Let $u^l(t)$ be the linear interpolation between $u'(T_2)$ and $z'$ for $t \in [T_2, T_3]$. Write $u(t) = \sum u_k(t)e_k$, then (5.19) in Fourier coordinates is given by

$$\dot{u}_k + |k|^2u_k + B_k(u, u) = 0, \quad k \in Z_L(N_0),$$

where $B_k(u, u) = B_k(u', u') + B_k(u^k, u^k) + B_k(u^l, u'^l) + B_k(u^l, u^k)$. We can choose a suitable simple $u^k$ to $B_k(u', u') = B_k(u^k, u^k) = 0$ and make the above equation explicitly solvable.

4. **High modes close to $z'$**: In the interval $[T_3, T]$ we choose $u^k$ as the linear interpolation between $u^k(T_3)$ and $z'$. By continuity, as $T - T_3$ is sufficiently small (thanks to that $T_3 \in (T_2, T)$ can be arbitrary), $u'(T)$ is still close to $z'$.
From the above four steps, we can see that \( \sup_{T_1 \leq t \leq T} |A^{3/2}u(t)| < \infty \). Moreover, since \( w \equiv 0 \) on \([0, T]\) and \( \sup_{0 \leq t \leq T} |Au(t)|^2 < \infty \), by differentiating \( |Au(t)|^2 \) and applying (6.12), we have the energy inequality
\[
|Au(T_1)|^2 + \int_0^{T_1} |A^{3/2}u(s)|^2 ds \leq C \int_0^{T_1} |Au(s)|^4 ds + |Ax|^2 < \infty
\]
Hence \( u \in L^2([0, T], D(A^{3/2})) \). With this observation and the controllability, we can apply Lemma 7.7 in [3] to obtain the conclusion (Note that our control \( w \) is different from the \( \tilde{w} \) in [3], this is the key point that we can apply the argument there with \( Q \) not invertible.)

Alternatively, with the solvability of the above control problem, we can apply the argument in the proof of Proposition 5.1 in [23] to show irreducibility. \( \square \)

From Proposition 5.1 or Proposition 5.2, \( P_t \) is strong Feller. By the irreducibility, there exists a unique invariant measure \( \nu \) for \( X(t) \) by Doob’s Theorem.

Finally, let us prove the exponential mixing property (2.7). To show this, it suffices to prove that
\[
||(P^m_t)^* \mu - \nu_m||_{var} \leq Ce^{-ct} \left( 1 + \int_H |x|^2 \mu(dx) \right)
\]
where \( c, C > 0 \) are independent of \( m \), and \( \nu_m \) is the unique measure of the approximate dynamics (see Lemma 5.8). We follow exactly the coupling method in [17] to prove (5.22), let us sketch out the key point as follows.

For two independent cylindrical Wiener processes \( W \) and \( \tilde{W} \), denote \( X_m \) and \( \tilde{X}_m \) the solutions of the equation (2.8) driven by \( W \) and \( \tilde{W} \) respectively. For any fixed \( 0 < T \leq 1 \), given any two \( x_1, x_2 \in D(A) \), we construct the coupling of the probabilities \( (P^m_T)^* \delta_{x_1} \) and \( (P^m_T)^* \delta_{x_2} \) as follows
\[
(V_1, V_2) = \begin{cases} 
\left( X_m(T, x_0), X_m(T, x_0) \right) & \text{if } x_1 = x_2 = x_0, \\
\left( Z_1(x_1, x_2), Z_2(x_1, x_2) \right) & \text{if } x_1, x_2 \in B_{D(A)}(0, \delta) \text{ with } x_1 \neq x_2, \\
\left( X_m(T, x_1), \tilde{X}_m(T, x_2) \right) & \text{otherwise,}
\end{cases}
\]
where \( (Z_1(x_1, x_2), Z_2(x_1, x_2)) \) is the maximal coupling of \((P^m_T)^* \delta_{x_1}\) and \((P^m_T)^* \delta_{x_2}\) (see Lemma 1.14 in [17]) and \( B_{D(A)}(0, \delta) = \{ x \in D(A); |Ax| \leq \delta \} \). It is clear that \( (V_1, V_2) \) is a coupling of \((P^m_T)^* \delta_{x_1}\) and \((P^m_T)^* \delta_{x_2}\). We construct \( (X^1, X^2) \) on \( TN \) by induction: set \( X^i(0) = x^i (i = 1, 2) \) and define
\[
X^i((n + 1)T) = V_i(X^1(nT), X^2(nT)) \quad i = 1, 2.
\]
The key point for using this coupling to show the exponential mixing is the following lemma, which plays the same role as Lemma 2.1 in [17], but we prove it by a little simpler way.

**Lemma 5.11.** There exist some \( 0 < T, \delta < 1 \) such that for any \( m \in \mathbb{N} \), one has a maximal coupling \( (Z_1(x_1, x_2), Z_2(x_1, x_2)) \) of \((P^m_T)^* \delta_{x_1}\) and \((P^m_T)^* \delta_{x_2}\) which satisfies
\[
\mathbb{P}(Z_1(x_1, x_2) = Z_2(x_1, x_2)) \geq 3/4
\]
if \( |Ax_1| \vee |Ax_2| \leq \delta \) with \( \delta > 0 \) sufficiently small.
Proof. Since \((Z_1(x_1, x_2), Z_2(x_1, x_2))\) is maximal coupling of \((P_T^m)^*\delta_{x_1}\) and \((P_T^m)^*\delta_{x_2}\) (see Lemma 1.14 in [17]), one has
\[
\|\{(P_T^m)^*\delta_{x_1} - (P_T^m)^*\delta_{x_2}\}\|_{\text{var}} = \mathbb{P}\{Z_1(x_1, x_2) \neq Z_2(x_1, x_2)\}.
\]
It is well known that
\[
\|\{(P_T^m)^*\delta_{x_1} - (P_T^m)^*\delta_{x_2}\}\|_{\text{var}} = \sup_{\|g\|_\infty = 1} |\mathbb{E}[g(X_m(T, x_1))] - \mathbb{E}[g(X_m(T, x_2))]| = \sup_{\|g\|_\infty = 1} |P_T^m g(x_1) - P_T^m g(x_2)|,
\]
where \(|\cdot|\|_\infty \) is the supremum norm. By Proposition 5.2, (noticing \(|g|_0 = |g|\|_\infty = 1\) with \(|\cdot|\|_0 \) defined in section 2), one has
\[
|P_T^m g(x_1) - P_T^m g(x_2)| \leq \int_0^1 A^{-1} D P_T^m g(\lambda x_1 + (1 - \lambda)x_2)||Ax_1 - Ax_2||d\lambda \\
\leq C T^{-\alpha}(1 + |Ax_1| + |Ax_2|)^{2+2\alpha}|A(x_1 - x_2)| \leq 1/4
\]
if choosing \(\delta = T^\beta\) with \(\beta > 0\) sufficiently large. Hence \(\mathbb{P}(Z_1 = Z_2) = 1 - \mathbb{P}(Z_1 \neq Z_2) \geq \frac{3}{4}.\)

With this lemma, one can prove the exponential mixing (2.7) by exactly the same procedure as in [17].

6. Appendix

6.1. Some calculus for \(\tilde{B}_k\) and Proof of Lemma 4.5.

Some calculus for \(\tilde{B}_k\). By \(B(u, v) = \mathbb{P}[(u \cdot \nabla)v]\), we have
\[
B(a_j \cos \xi, a_j \sin \xi) = \frac{1}{2}(l \cdot a_j)\mathbb{P}[a_j \cos(j + l)\xi] + \frac{1}{2}(l \cdot a_j)\mathbb{P}[a_j \cos(j - l)\xi],
\]
\[
B(a_j \sin \xi, a_j \cos \xi) = \frac{1}{2}(l \cdot a_j)\mathbb{P}[a_j \cos(j + l)\xi] - \frac{1}{2}(l \cdot a_j)\mathbb{P}[a_j \cos(j - l)\xi],
\]
where \(\mathbb{P}\) is the projection from \(L^2(\mathbb{T}^3, \mathbb{R}^3)\) to \(H\). If \(j, -l \in \mathbb{Z}_+^3\) with \(j + l \in \mathbb{Z}_+^3\), \(\forall a_j \in j^+, a_l \in l^+\), we have from the above two expressions
\[
(6.1) \quad \tilde{B}_{j-l}(a_j e_j, a_l e_l) = \frac{1}{2}[(l \cdot a_j)\mathbb{P}_{j-l} a_l - (j \cdot a_l)\mathbb{P}_{j-l} a_j],
\]
\[
(6.2) \quad \tilde{B}_{j+l}(a_j e_j, a_l e_l) = \frac{1}{2}[(l \cdot a_j)\mathbb{P}_{j+l} a_l + (j \cdot a_l)\mathbb{P}_{j+l} a_j],
\]
\[
(6.3) \quad \tilde{B}_k(a_j e_j, a_l e_l) = 0 \quad \text{if} \quad k \neq j + l, j - l.
\]
where the projection \(\mathbb{P}_k : \mathbb{R}^3 \to k^\perp\) is defined by (2.2). For the case of \(j, l \in \mathbb{Z}_+^3\) with \(j - l \in \mathbb{Z}_+^3\), we can calculate \(\tilde{B}_{j-l}(a_j e_j, a_l e_l), \tilde{B}_{j-l}(a_j e_j, a_l e_l)\) and so on by the same method. \(\square\)

Proof of Lemma 4.5. As \(k \in \mathbb{Z}_+(n_0) \cap \mathbb{Z}_+^3\), for any \(j, l \in \mathbb{Z}_+^3\) such that
\[
(6.4) \quad j \in \mathbb{Z}_+(n_0) \cap \mathbb{Z}_+^3, \quad l \in \mathbb{Z}_+(n_0) \cap \mathbb{Z}_+^3, \quad j \neq l, \quad |j| \neq |l|, \quad j + l = k;
\]
taking an orthogonal basis \(\{k, h_1, h_2\}\) of \(\mathbb{R}_3\) where \(\{h_1, h_2\}\) is an orthogonal basis of \(k^\perp\) with \(h_1\) defined by
\[
h_1 = l \quad \text{if} \quad k \cdot l = 0, \quad h_1 = j - \frac{j \cdot k}{k \cdot l} \quad \text{otherwise}.
\]
Let \( p_j \in j^+ \), \( p_l \in l^\perp \) be represented by \( p_j = ak + b_1h_1 + b_2h_2 \) and \( p_l = \alpha k + \beta_1h_1 + \beta_2h_2 \). Clearly, \( j, l \perp h_2 \), by some basic calculation, we have
\[
(j \cdot p_l)P_k p_j + (l \cdot p_j)P_k p_l = \begin{cases}
- \left[ \frac{(k)^2 - q_j^2}{(k)^2} \right] h_1 + (ab_2 + \beta_2\alpha)h_2 & \text{if } h_1 = j - \frac{k}{b_2}l \\
- \left[ \frac{k}{b_2} \right] h_1 + (ab_2 + \beta_2\alpha)h_2 & \text{if } h_1 = l
\end{cases}
\]
Since \( b_2, \beta_2, a, \alpha \in \mathbb{R} \) can be arbitrarily chosen, one clearly has
\[
\{j \cdot p_l\}P_k p_j + (l \cdot p_j)P_k p_l : \ p_j \in j^+, \ p_l \in l^\perp \} = k^+.
\]
By (A3) of Assumption 2.1, we have \( \text{rank}(q_j), \text{rank}(q_l) = 2 \), therefore, by (6.2) and (6.5),
\[
\{ \tilde{B}_k (q_j \ell_j e_j, q_l \ell_l e_l) : \ell_j \in j^+, \ell_l \in l^\perp \} = k^+.
\]
Hence, \( \text{span}\{Y_k\} = k^+ \). For \( k \in Z_i(n_0) \cap Z_i^2 \), we have the same conclusion by the same argument as above. \( \square \)

### 6.2. Proof of Lemma 4.7

The key points for the proof are Proposition 4.6 and the following Norris’ Lemma, which is exactly Lemma 4.1 in [16].

**Lemma 6.1. (Norris’ Lemma)** Let \( a, y \in \mathbb{R} \). Let \( \beta, \gamma_t = (\gamma^1_t, \ldots, \gamma^m_t) \) and \( u_t = (u^1_t, \ldots, u^m_t) \) be adaptive processes. Let
\[
a_t = a + \int_0^t \beta_sds + \int_0^t \gamma^i_s dw_s, \quad Y_t = y + \int_0^t a_s ds + \int_0^t u^i_s dw_s,
\]
where \( (u^1, \ldots, u^m) \) are i.i.d. standard Brownian motions. Suppose that \( T < t_0 \) is a bounded stopping time such that for some constant \( C < \infty \):
\[
|\beta_t|, |\gamma_t|, |a_t|, |u_t| \leq C \quad \text{for all } t \leq T.
\]
Then for any \( r > 8 \) and \( \nu > \frac{r}{r-8} \)
\[
P\left( \int_0^T \int_0^t Y^2_s dt < \epsilon, \int_0^T (|a_t|^2 + |u_t|^2) dt \geq \epsilon \right) < C(t_0, q, \nu)e^{-\frac{\nu}{C}}.
\]

**Proof of Lemma 4.7**. We shall drop the index \( m \) of the quantities if no confusions arise. The idea of the proof is from Theorem 4.2 of [16], it suffices to show the inequality in the lemma, which is equivalent to
\[
P\left( \inf_{\eta \in S} \langle M_t \eta, \eta \rangle \leq \epsilon^q \right) \leq \frac{C(p)\epsilon^p}{t^p} \quad (\forall \ p > 0)
\]
where \( S' = \{ \eta \in \pi' H ; |\eta| = 1 \} \). From (4.10), (6.6) is equivalent to
\[
P\left( \inf_{\eta \in S} \sum_{k \in Z_i(n) \setminus Z_i(n_0)} \sum_{j=1}^2 \int_0^t (\langle J_s^{-1}(q_k^j e_k), \eta \rangle)^2 ds \leq \epsilon^q \right) \leq \frac{C(p)\epsilon^p}{t^p},
\]
(recall \( q_k^j \) is the \( j \)-th column vector of the matrix \( q_k \), see Assumption 2.1), which is implied by
\[
D_0 \sup_{j} \sup_{\eta \in D_j} P\left( \int_0^t \sum_{k \in Z_i(n) \setminus Z_i(n_0)} \sum_{j=1}^2 (\langle J_s^{-1}(q_k^j e_k), \eta \rangle)^2 ds \leq \epsilon^q \right) \leq \frac{C(p)\epsilon^p}{t^p}
\]
where \( \{ \mathcal{D}_j \}_j \) is a finite \( \theta \)-radius disk cover of \( \mathcal{S}' \) (due to the compactness of \( \mathcal{S}' \)) and \( \mathcal{D}_0 = \# \{ \mathcal{D}_j \} \). Define a stopping time \( \tau \) by
\[
\tau = \inf \{ s > 0 \mid |E_K(s) J^{-1}_s - \text{Id}|_{\mathcal{L}(H)} > c \}.
\]
where \( c > 0 \) is a sufficiently small but fixed number. It is easy to see that (6.7) holds as long as for any \( \eta \in \mathcal{S}' \), we have some neighborhood \( \mathcal{N}(\eta) \) of \( \eta \) and some \( k \in \mathbb{Z} \), \( i \in \{ 1, 2 \} \) so that
\[
(6.9) \quad \sup_{\eta' \in \mathcal{N}(\eta)} P \left( \int_0^{t \wedge \tau} |(J^{-1}_s(q^i_k \epsilon_k), \eta')|^2 ds \leq \varepsilon^q \right) \leq \frac{C(p)\varepsilon^p}{tp} \quad (\forall \ p > 0).
\]
The above argument is according to [16] (see Claim 1 of the proof of Theorem 4.2), one may see the greater details there.

Let us prove (6.9). According to the restricted Hörmander condition and Definition 4.3, for any \( \eta \in \mathcal{S}' \), there exists a \( K \in \mathbf{K} \) satisfying for all \( y \in \pi_m H \)
\[
|\langle K(y), \eta \rangle|^2 \geq \delta|\eta|^2
\]
where \( \delta > 0 \) is a constant independent of \( y \). Without loss of generality, assume that \( K \in \mathbf{K}_2 \), so there exists some \( q^i_k \epsilon_k \) and \( q^i_l \epsilon_l \) such that
\[
K_0 := q^i_k \epsilon_k, \ K_1 := |A_m y + B'_m(y,y), q^i_k \epsilon_k|, \ K = K_2 := [ q^i_l \epsilon_l, K_1 ].
\]
Take
\[
Y(t) = (J^{-1}_t K_1(X(t)), \eta), \ u(t) = 0, \ a(t) = (J^{-1}_t K_2(X(t)), \eta),
\]
applying Norris lemma with \( t_0 = 1 \) therein, we have
\[
P \left( \int_0^{t \wedge \tau} |(J^{-1}_s K_1(X(s)), \eta)|^2 ds \leq \varepsilon^r, \int_0^{t \wedge \tau} |(J^{-1}_s K_2(X(s)), \eta)|^2 ds \geq \varepsilon \right) \leq C(p, \nu)e^{-\frac{c\varepsilon}{t}}.
\]
On the other hand, by (4.14), (6.8) and Chebyshev’s inequality, it is easy to have
\[
P \left( \int_0^{t \wedge \tau} |(J^{-1}_s K_2(X(s)), \eta)|^2 ds \leq \varepsilon \right)
= P \left( \int_0^{t \wedge \tau} \frac{1}{|E_K(s)|^2} |(E_K(s) J^{-1}_s K_2(X(s)), \eta)|^2 ds \leq \varepsilon \right)
\leq P \left( \int_0^{t \wedge \tau} |(E_K(s) J^{-1}_s K_2(X(s)), \eta)|^2 ds \leq \varepsilon \right)
\leq P \left( \tau \leq \frac{c\varepsilon}{t} \right) \leq \frac{C(p)\varepsilon^p}{tp}.
\]
Hence,
\[
P \left( \int_0^{t \wedge \tau} |(J^{-1}_s K_1(X(s)), \eta)|^2 ds \leq \varepsilon^r \right) \leq \frac{C(p)\varepsilon^p}{tp}.
\]
By a similar but simpler arguments, (recalling \( K_0 = q^i_k \epsilon_k \)), we have
\[
(6.10) \quad P \left( \int_0^{t \wedge \tau} |(J^{-1}_s(q^i_k \epsilon_k), \eta)|^2 ds \leq \varepsilon^q \right) \leq \frac{C(p)\varepsilon^p}{tp}
\]
for all \( p > 0 \).

Hence, for any \( \eta \in \mathcal{S}' \), we have some \( q^i_k \epsilon_k \) satisfying (6.10). Take the neighborhood \( \mathcal{N}(\eta) \) small enough and \( q = r^2 \), by the continuity, we have (6.9) immediately. \( \square \)
6.3. **Proof of some technical lemmas.** In this subsection, we need a key estimate as follows (see Lemma D.2 in [11]): For any $\gamma > 1/4$ with $\gamma \neq 3/4$, we have

\begin{equation}
|A^{\gamma-1/2}B(u, u)| \leq C(\gamma)|A^\gamma u|^2 \quad \text{for any } u \in D(A^\gamma).
\end{equation}

By (6.11) and Young’s inequality, we have

\begin{equation}
|\langle A^\gamma u, A^\gamma B(u, v) \rangle| \leq |A^{\gamma+1/2}u||A^{\gamma-1/2}B(u, v)| \leq |A^{\gamma+1/2}u|^2 + C(\gamma)|A^\gamma u|^2|A^\gamma v|^2
\end{equation}

**Proof of Lemma 3.4.** We shall drop the index $m$ of quantities if no confusions arise. By Itô formula, we have

\begin{equation}
d \left[ |A^\gamma D_h X(t)|^2 \mathcal{E}_K(t) \right] + 2|A^{1/2+\gamma}D_h X(t)|^2 \mathcal{E}_K(t) + 2\langle A^\gamma D_h X(t), A^\gamma \dot{B} [D_h X(t), X(t)]\rangle \mathcal{E}_K(t) dt + K|A^\gamma D_h X(t)|^2 |AX(t)|^2 \mathcal{E}_K(t) dt = 0
\end{equation}

where $\dot{B} (D_h X(t), X(t)) = B(D_h X(t), X(t)) + B(X(t), D_h X(t))$. Thus, one has by (6.12)

\begin{equation}
|A^\gamma D_h X(t)|^2 \mathcal{E}_K(t) + \int_0^t |A^{\gamma+1/2} D_h X(s)|^2 \mathcal{E}_K(s) ds 
\end{equation}

\begin{equation}
\leq |A^\gamma h|^2 + \int_0^t C |A^\gamma D_h X(s)|^2 |A^\gamma X(s)|^2 \mathcal{E}_K(s) ds - K \int_0^t |A^\gamma D_h X(s)|^2 |AX(s)|^2 \mathcal{E}_K(s) ds
\end{equation}

By Poincaré inequality, we have $|Ax| \geq |A^\gamma x|$, and therefore as $K \geq C$,

\begin{equation}
|A^\gamma D_h X(t)|^2 \mathcal{E}_K(t) + \int_0^t |A^{\gamma+1/2} D_h X(s)|^2 \mathcal{E}_K(s) ds \leq |A^\gamma h|^2.
\end{equation}

As to (3.9) and (3.10), we only prove (3.9), similarly for the other. By an estimate similar to (6.14), (3.8) and (2.17) (noticing $D_h X(t)(0) = 0$), we have

\begin{equation}
|A^\gamma D_{h^2} X(t)|^2 \mathcal{E}_K(t) + \int_0^t |A^{\gamma+1/2} D_{h^2} X(s)|^2 \mathcal{E}_K(s) ds \leq \int_0^t \left[ C |A^{\gamma+1/2} X(s)|^2 |A^\gamma X(s)|^2 - 2K |A^{\gamma+1/2} X(s)|^2 |AX(s)|^2 \right] \mathcal{E}_K(s) ds
\end{equation}

\begin{equation}
\leq C \int_0^t \left[ |A^\gamma D_{h^2} X(s)|^2 \mathcal{E}_K(s) \right] \left[ |AX(s)|^2 \mathcal{E}_E(s) \right] ds
\end{equation}

\begin{equation}
\leq C |A^\gamma h|^2 \int_0^t |AX(s)|^2 \mathcal{E}_E(s) ds \leq \frac{2C}{K} |A^\gamma h|^2.
\end{equation}

As to (3.11), by the classical interpolation inequality

\begin{equation}
|A^\gamma D_h X(s)|^2 \leq |A^\gamma D_h X(s)|^{2(1-2(r-\gamma))} |A^{1/2+\gamma} D_h X(s)|^{4(r-\gamma)},
\end{equation}

Hölder’s inequality and (3.8), we have

\begin{equation}
\int_0^t |A^\gamma D_h X(s)|^2 \mathcal{E}_K(s) ds \leq \left[ \int_0^t |A^{\gamma+1/2} D_h X(s)|^2 \mathcal{E}_K(s) ds \right]^{(r-\gamma)} \left[ \int_0^t |A^\gamma D_h X(s)|^2 \mathcal{E}_K(s) ds \right]^{1-2(r-\gamma)} \leq C t^{1-2(r-\gamma)} |A^\gamma h|^2.
\end{equation}
(3.12) immediately follows from applying Itô formula to \(|\mathcal{E}_K(t) \int_0^t (v, dW_s)|^2\). □

**Proof of Lemma 4.2.** By (4.3) and the evolution equation governing \(D_t X_t\), using the same method as proving (3.8), we immediately have (4.11) and (4.12). Recall that \(\mathcal{J}_t\) and \(\mathcal{J}_t^{-1}\) are both the dynamics in \(\pi H\), thus the operator \(\mathcal{J}_t\) is bounded invertible. Let \(C \) be some constant only depends on \(n \) (see (3.3)), whose values can vary from line to line. By the fact \(|A|_{\mathcal{L}(\pi H)} \leq C \) and (4.4), for any \(h \in \pi H\), we have by differentiating \(|J_t^{-1}h|^2\mathcal{E}_K(t)\)

(6.16)

\[
|J_t^{-1}h|^2\mathcal{E}_K(t) + K \int_0^t |J_s^{-1}h|^2|AX(s)|^2\mathcal{E}_K(s)ds \\
\leq |h|^2 + 2 \int_0^t |J_s^{-1}h||J_s^{-1}Ah|\mathcal{E}_K(s)ds \\
+ C \int_0^t |J_s^{-1}h||J_s^{-1}A^{-\frac{1}{2}}|_{\mathcal{L}(\pi H)} \cdot |A^{1/2}B'(h, X(s))|\mathcal{E}_K(s)ds \\
\leq |h|^2 + C \int_0^t |J_s^{-1}|_{\mathcal{L}(H)}^2|h|^2\mathcal{E}_K(s)ds + C \int_0^t |J_s^{-1}|_{\mathcal{L}(H)}^2|AX(s)||h|^2\mathcal{E}_K(s)ds,
\]

where the last inequality is by (6.11). Hence,

\[
|J_t^{-1}|_{\mathcal{L}(H)}^2\mathcal{E}_K(t) + K \int_0^t |J_s^{-1}|_{\mathcal{L}(H)}^2|AX(s)|^2\mathcal{E}_K(s)ds \leq 1 + C \int_0^t |J_s^{-1}|_{\mathcal{L}(H)}^2(1+|AX(s)|^2)\mathcal{E}_K(s)ds,
\]

as \(K\) is sufficiently large, we have \(|J_t^{-1}|_{\mathcal{L}(H)}^2\mathcal{E}_K(t) \leq 1 + C \int_0^t |J_s^{-1}|_{\mathcal{L}(H)}^2\mathcal{E}_K(s)ds\), which immediately implies (4.13).

To prove (4.14), by (4.4), we have

(6.17)

\[
|\mathcal{E}_K(t)J_t^{-1}h - h| \leq \int_0^t |J_s^{-1}Ah|\mathcal{E}_K(s)ds + \int_0^t |J_s^{-1}A^{-1/2}|_{\mathcal{L}(\pi H)}A^{1/2}B'(h, X(s))|\mathcal{E}_K(s)ds \\
\leq C \int_0^t |J_s^{-1}|_{\mathcal{L}(H)}|h|\mathcal{E}_K(s)ds + C \int_0^t |J_s^{-1}|_{\mathcal{L}(H)}|h||AX(s)||\mathcal{E}_K(s)ds,
\]

thus, by (4.13) and (2.17),

\[
|\mathcal{E}_K(t)J_t^{-1} - Id|_{\mathcal{L}(H)} \leq C \int_0^t \mathcal{E}_K(s)|J_s^{-1}|_{\mathcal{L}(H)}ds + \int_0^t \mathcal{E}_K(s)J_s^{-1}|s||AX(s)|ds \\
\leq Ct^{\frac{1}{2}} \left[ \int_0^t \mathcal{E}_K(s)|J_s^{-1}|_{\mathcal{L}(H)}^2ds \right]^{\frac{1}{2}} + t^{\frac{1}{2}} \left[ \int_0^t \mathcal{E}_K(s)|J_s^{-1}|_{\mathcal{L}(H)}^2\mathcal{E}_K(s)|AX(s)|^2ds \right]^{\frac{1}{2}} \\
\leq t^{1/2}C e^{\frac{1}{2}t}.
\]
where the last inequality is due to (4.13). As for (4.15), by Parseval’s identity and (4.13),

\[
(6.18) \\
\mathbb{E} \left[ \int_0^t \mathcal{E}_K^2(s) |(J^{-1}_s Q')^* h|^2 ds \right] = \sum_{k \in \mathcal{Z}(n)} \sum_{i=1}^2 \mathbb{E} \left[ \int_0^t \mathcal{E}_{2K}(s) |(J^{-1}_s (q_k e_k), h)|^2 ds \right] \\
\leq \sum_{k \in \mathcal{Z}(n)} \sum_{i=1}^2 \mathbb{E} \left[ \int_0^t \mathcal{E}_{2K}(s) |J^{-1}_s (q_k e_k)|^2 ds \right] |h|^2 \leq t C e^{Ct} \sum_{k \in \mathcal{Z}(n) \setminus \mathcal{Z}(n_0)} \sum_{i=1}^2 |q_k e_k|^2 |h|^2.
\]

By an estimate similar to (6.14), we have

\[
|AD_v X'(t)|^2 \mathcal{E}_K(t) + \int_0^t |A^{3/2} \mathcal{D}_v X'(t)|^2 \mathcal{E}_K(s) ds \\
\leq \frac{1}{2} \int_0^t |AD_v X'(s)|^2 \mathcal{E}_K(s) ds + \frac{1}{2} \int_0^t |A Q^\prime v'(s)|^2 \mathcal{E}_K(s) ds \\
\leq \frac{1}{2} \int_0^t |AD_v X'(s)|^2 \mathcal{E}_K(s) ds + C \int_0^t |v'(s)|^2 \mathcal{E}_K(s) ds
\]

which implies (4.16) by Gronwall’s inequality.

As to (4.17), write down the differential equation for $D_v \mathcal{D}_h X'(t)$, and apply Itô formula, we have

\[
|D_v D_h X'(t)|^2 \mathcal{E}_K(t) + 2 \int_0^t |A^{1/2} \mathcal{D}_v D_h X'(t)|^2 \mathcal{E}_K(s) ds \\
\leq \int_0^t |D_v D_h X'(s)| \left( |\hat{B}'(D_v D_h X'(s), X(s))| + |\hat{B}'(D_h X'(t), D_v X(s))| \right) \mathcal{E}_K(s) ds \\
- K \int_0^t |D_v D_h X'(s)|^2 \mathcal{E}_K(s) ds.
\]

By (6.11) and Young’s inequality,

\[
|D_v D_h X'(t)|^2 \mathcal{E}_K(t) + \int_0^t |A^{1/2} \mathcal{D}_v D_h X'(t)|^2 \mathcal{E}_K(s) ds \\
\leq \int_0^t |D_v D_h X'(s)|^2 \left( |A^{3/2} X(s)| + 1 \right) \mathcal{E}_K(s) ds + \int_0^t |A^{3/2} D_v X'(s)|^2 |A^{3/2} D_h X'(s)|^2 \mathcal{E}_K(s) ds \\
- K \int_0^t |D_v D_h X'(s)|^2 |AX(s)|^2 \mathcal{E}_K(s) ds,
\]

as $K$ is sufficiently large, by Poincare inequality, $|A|_{L^2(H)} \leq C$, (4.16) and (3.8), we have

\[
|D_v D_h X'(t)|^2 \mathcal{E}_K(t) + \int_0^t |A^{1/2} \mathcal{D}_v D_h X'(t)|^2 \mathcal{E}_K(s) ds \\
\leq \int_0^t |D_v D_h X'(s)|^2 \mathcal{E}_K(s) ds + C \int_0^t |D_v X'(s)|^2 |D_h X'(s)|^2 \mathcal{E}_{K/2}(s) ds \\
\leq \int_0^t |D_v D_h X'(s)|^2 \mathcal{E}_K(s) ds + C |h|^2 \int_0^t e^{Ct-s} |v'(s)|^2 \mathcal{E}_{K/2}(s) ds.
\]
By Gronwall’s inequality, we obtain (4.17) immediately. Similarly, (4.18) can be obtained by
\[
|D_{v_1v_2} X'(t)|^2 \mathcal{E}_K(t) \leq C \int_0^t \left[ |A D_{v_1} X'(s)|^2 \mathcal{E}_{K/2}(s) \right] ds
\]
\[
\leq t C e^{C t} \left[ \int_0^t |v'_1(s)|^2 \mathcal{E}_{K/2}(s) ds \right] \left[ \int_0^t |v'_2(s)|^2 \mathcal{E}_{K/2}(s) ds \right].
\]
where the last inequality is due to (4.16).

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