Statistics of quantum transport in weakly non-ideal chaotic cavities

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We consider statistics of electronic transport in chaotic cavities where time-reversal symmetry is broken and one of the leads is weakly non-ideal, i.e. it contains tunnel barriers characterized by tunneling probabilities \( \Gamma_i \). Using symmetric function expansions and a generalized Selberg integral, we develop a systematic perturbation theory in \( 1 - \Gamma_i \), valid for any number of channels, and obtain explicit formulas up to second order for the average and variance of the conductance, and for the average shot-noise. Higher moments of the conductance are considered to leading order.

I. INTRODUCTION

At low temperatures and applied voltage, provided that the average electron dwell time is well in excess of the Ehrenfest time, statistical properties of electronic transport in mesoscopic cavities exhibiting chaotic classical dynamics are universal. Random matrix theory (RMT) has proven very successful in describing this universality. In this approach, the scattering \( S \)-matrix of the cavity is modeled by a random unitary matrix \( U \) (see Refs. 4,6 for the most recent analytical results on distribution of \( S \)).

We consider a chaotic cavity attached to two leads, having \( N_1 \) and \( N_2 \) (with \( N_1 \leq N_2 \)) open channels in each lead, and denote by \( N = N_1 + N_2 \) the total number of channels. The \( S \) matrix can be written in the usual block form \( S = \begin{pmatrix} r & t^\dagger \\ t & r^\dagger \end{pmatrix} \), in terms of reflection and transmission matrices. Landauer-Büttiker theory expresses most physical observables in terms of the eigenvalues \( \{ T_1, \ldots, T_N \} \) of the hermitian matrix \( tt^\dagger \). A figure of merit is the conductance \( G(T) = G_0 \sum_{i=1}^{N_1} T_i \), where \( G_0 = 2e^2/h \) is the conductance quantum. The assumption that \( S \) is a random matrix implies that the \( T_j \)'s are correlated random variables characterized by a certain joint probability density (jpd), and as a consequence every observable becomes a random variable whose statistics is of paramount interest.

When the leads attached to the cavity are ideal, \( S \) is uniformly distributed in one of Dyson’s circular ensembles of random matrices, which are labeled by a parameter \( \beta \): it is unitary and symmetric for \( \beta = 1 \) (corresponding to systems that are invariant under time-reversal), just unitary for \( \beta = 2 \) (broken time-reversal invariance) and unitary self-dual for \( \beta = 4 \) (anti-unitary time-reversal invariance). In this ideal case the jpd of reflection eigenvalues \( R_i = 1 - T_i \) is given by the Jacobi ensemble of RMT, namely

\[
P^{(0)}_\beta(R) \propto |\Delta(R)|^\beta \prod_{i=1}^{N_1} (1 - R_i)^{\frac{1}{2}(N_2 - N_1 + 1) - 1},
\]

where \( \Delta(R) = \prod_{j<k} (R_k - R_j) \) is the Vandermonde determinant. The average and variance of the conductance were studied, using perturbation theory in \( 1/N \), long ago. In particular, as \( N \to \infty \) this variance becomes a constant depending only on the symmetry class, a phenomenon that has been dubbed universal conductance fluctuations. More recently, a fruitful approach based on the theory of the Selberg integral was developed and afterwards extended to compute transport statistics non-perturbatively. The full distribution of \( G \), known to be strongly non-Gaussian for a small number of channels, was studied. The statistics of other observables was studied.

A more generic situation occurs when the leads are not ideal but contain tunnel barriers. A simple barrier, which does not mix transversal modes, is represented by a set of tunneling probabilities \( \{ \Gamma_i \} \), one for each open channel. In this case the distribution of \( S \) is given by the so-called Poisson kernel,

\[
P_\beta(S) = \left[ \det(1 - SS^\dagger) \det(1 - S^\dagger S) \right]^{\beta/2 - 1 - \beta N/2},
\]

which depends only on \( \beta \) and the average scattering matrix \( \bar{S} \) (whose singular values are determined by the tunneling probabilities of each lead). In the limit \( \Gamma_i \to 1 \) we have \( S = 0 \) and one recovers the ideal case. Even though controllable barriers in the leads are by now an established experimental protocol, few explicit theoretical predictions are available due to the complicated nature of the Poisson kernel. For instance, the average and variance of conductance are only known perturbatively in the limit \( \Gamma_i N \to 1 \) for all. Semiclassical studies of transport in the large \( N \) limit and non-ideal setting have also recently appeared.

A more systematic RMT theoretical investigation was initiated when Vidal and Kanzieper obtained the jpd of reflection eigenvalues for \( \beta = 2 \) and only one non-ideal lead. In this work we characterize those \( N_1 \) non-ideal channels by a diagonal matrix \( \gamma = \text{diag}(\gamma_1, \ldots, \gamma_{N_1}) \), with \( \gamma_i = 1 - \Gamma_i \) (these are not the same \( \gamma_i \) which appear in Ref. 39, the definitions differ by a square root). The other lead is kept ideal. Our goal is to use symmetric function expansions and a generalized Selberg integral to develop a systematic perturbation theory in \( \gamma \) of this
If we define \( F_\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(i + 1)\Gamma(j + 1)} \) be the hypergeometric function. Let \( F \) be the \( N \times N \) matrix whose elements are \( F_{ij} = 2F_1(N_2 + 1, N_2 + 1; \gamma_j R_j) \).

When the non-ideal lead supports \( N_1 \) channels, the jpd of reflection eigenvalues is given by

\[
P^{2(\gamma)}(R) = N! \sum_{\lambda} \Delta(\lambda R) \frac{\Delta(R)}{\Delta(\gamma)} \prod_{i=1}^{N_1} (1 - R_i)^{N_2 - N_1},
\]

where \( \Delta(\lambda) = \prod_{i=1}^{\lambda} (i - 1) \) is the rising factorial and \( \Delta(R) = \prod_{i=1}^{N} (1 - R_i) \) is the falling factorial.

The expression is hardly operational. We therefore start by writing it in a perturbative way, i.e. as an infinite series in \( \gamma \).

Let a non-increasing sequence of positive integers \( \lambda_1, \lambda_2, \ldots \) be called a partition of \( n \) if \( \sum \lambda_i = n \) and let this be denoted by \( \lambda \vdash n \). The number of parts in \( \lambda \) is \( \ell(\lambda) \) and we assume \( \lambda_0 = 0 \) if \( m > \ell(\lambda) \). Partitions can be used to label a very important set of symmetric polynomials known as Schur polynomials, which are denoted by \( s_\lambda \). Assuming \( N_1 \) variables, they are defined by

\[
s_\lambda(x) = \frac{1}{\Delta(x)} \det \left( x^{\lambda_i - i + N_1} \right).
\]

For example, the first few such polynomials are given by

\[
s_0(x) = 1, \quad s_1(x) = \sum_{i=1}^{N_1} x_i, \quad s_2(x) = \sum_{i<j}^{N_1} x_i x_j, \quad s_3(x) = \sum_{i=1}^{N_1} x_i^3.
\]

If we define

\[
\alpha_\lambda = \prod_{i=1}^{N_1} \left( \frac{N + \lambda_i - i}{N_2} \right)^2,
\]

the following expansion can be established:

\[
\det(F) = \Delta(\gamma) \Delta(R) \sum_{\lambda} \alpha_\lambda s_\lambda(\gamma) s_\lambda(R),
\]

where the infinite sum is over all possible partitions. This follows from the nice structure of \( F_{ij} \), which depends on the indices \( ij \) only through the combination \( \gamma_i R_j \). An account of this and similar identities can be found for example in the book by Hua.

In order to use (11) to express the jpd of reflection eigenvalues, it is useful to factor out the \( \alpha_0 \) term and notice that

\[
\frac{\alpha_\lambda}{\alpha_0} = \frac{[N]^2}{[N_1]^2},
\]

where

\[
[N]_\lambda = \prod_{i=1}^{\ell(\lambda)} \frac{(N + \lambda_i - i)!}{(N - i)!}
\]

is a generalization of the rising factorial. The normalization constant then simplifies as

\[
Z' = Z \alpha_0 = \prod_{i=1}^{N_1} \frac{(N - i)!}{(N_1 - i)!(N_2 - i)!}.
\]

This is precisely the normalization constant missing from (11). Finally, combining (11), (5), (11) and (12) we get the final result,

\[
P^{2(\gamma)}(R) = Z' \sum_{\lambda} \frac{[N]^2}{[N_1]^2} s_\lambda(\gamma) s_\lambda(R).
\]

### III. Computing Observables

Since any observable is a symmetric function of the reflection eigenvalues, it must be expressible as a linear combination of Schur polynomials; hence it suffices to obtain the average value of \( s_\mu(R) \) for an arbitrary partition \( \mu \). In this way we are led to consider the multiple integral

\[
\int_{0}^{1} \Delta^2(R) s_\lambda(R) s_\mu(R) \prod_{i=1}^{N_1} (1 - R_i)^{N_2 - N_1} dR,
\]

(where \( dR = \prod_{i=1}^{N_1} dR_i \)) which is a generalization of Selberg’s integral. However, this is difficult to evaluate directly. One way to proceed is to express the product of two Schur polynomials again as a linear combination of Schur polynomials,

\[
s_\lambda(R) s_\mu(R) = \sum_{\nu} C^\nu_{\lambda\mu} s_\nu(R),
\]

where the constants \( C^\nu_{\lambda\mu} \) are known as Littlewood-Richardson coefficients. There is no explicit formula for
appears as the limit to products. The result is all its arguments are equal to unity.

By means of the Littlewood-Richardson coefficients, we only need to consider the simpler integral

$$I_\nu = \int_0^1 \Delta^2(R) s_\nu(R) \prod_{i=1}^{N_1} (1 - R_i)^{N_2 - N_1} dR,$$

which is known to be given by

$$I_\nu = s_\nu(1^{N_1}) \prod_{i=1}^{N_1} \frac{(N_1 + \nu_i - i)!(N_2 - i)!}{(N_1 + \nu_i - i)!},$$

where $s_\nu(1^{N_1})$ is the value of a Schur polynomial when all its arguments are equal to unity.

V. STATISTICS OF CONDUCTANCE UP TO SECOND ORDER

Using the approach presented here, the average value of any observable can in principle be found to any order in $\gamma$. Consider for instance the average conductance. In the ideal case, it is given by $\langle G \rangle_0 = N_1 N_2 / N$. Up to second order in $\gamma$, the calculations we just outlined provide

$$\langle G \rangle_\gamma \approx \frac{N_1 N_2}{N} - \frac{N_2^2}{N^2 - 1} + \frac{N_2^2}{(N^2 - 1)(N^2 - 4)}(2\text{Tr}^2 \gamma - N\text{Tr}^2 \gamma^2).$$

IV. THE LEADING ORDER

The jpd (15) equals the jpd of the ideal case (11) times a correction which can be systematically expanded in powers of $\gamma$. In this way any observable in the finite-$\gamma$ regime can be expressed in terms of observables computed in the ideal regime. For example, to leading order we have

$$\frac{P^{(\gamma)}_2(R)}{P^{(0)}_2(R)} \approx \left[ 1 + \frac{N}{N_1}(\frac{N}{N_1}s_1(R) - N_1) \text{Tr}\gamma \right].$$

As a first application, let $\langle G^n \rangle_\gamma$ be the average value of the $n$th moment of the conductance in the non-ideal case. Using (25) and the fact that $s_1(R) = N_1 - G$, it is easy to see that the difference between the weakly non-ideal case and the ideal case is given to leading order by

$$\langle G^n \rangle_\gamma - \langle G^n \rangle_0 \approx \frac{N}{N_1} \text{Tr}\gamma \left[ \frac{N_2}{N_1} \langle G^n \rangle_0 - \frac{N}{N_1} \langle G^{n+1} \rangle_0 \right].$$

A similar estimate holds for other transport statistics.
identically and we have

For instance, when $N_1 = 1$ and $N_2 = 5$. Solid and dashed lines are, respectively, exact results and our approximations. For the average, the difference is minimal. For the variance, the approximation predicts a non-physical negative result at high $\gamma$, but is excellent up to moderate values of $\gamma$.

In the non-ideal case, up to second order in $\gamma$, it becomes

$$\var_G(\gamma) \approx \frac{N_1^2 N_2^2}{N^2(N^2 - 1)} + \frac{2N_1^2(N_1 - N_2)^22\gamma_2 + N_1^2(N_1 + N_2)^22\gamma_2}{N(N^2 - 1)(N^2 - 4)}$$

$$+ \frac{N_2^2[A_1 N(\gamma_2)^2 + B_1(N^2 - 1)\gamma_2^2]}{N(N^2 - 1)^2(N^2 - 4)(N^2 - 9)}, \quad (31)$$

where

$$A_1 = (N_1 - 2N_2)(3N_1 - 8N_2)N^2 + 20N_1N_2 - 37N_2^2 - 3,$$

and

$$B_1 = (N_1 - 2N_2)(N_2N^2 + N_2 - 5N_1). \quad (32)$$

Again, the result is exact for any $N_1, N_2$. Notice that each order in $\gamma$ attains a finite value in the large $N$ limit. For instance, when $N_1 = N_2$, the first order vanishes identically and we have

$$\lim_{N_1 = N_2 \to \infty} \var_G(\gamma) \approx \frac{1}{16} + \frac{5(\gamma_2)^2 - 4\gamma_2^2}{64}. \quad (34)$$

which perfectly matches the first terms of the expansion in $\gamma$ of formula 6.24 in Ref.[34], where in our notation $g_1 = N_1 - \gamma_2, g_2 = N_1 - 2\gamma_2 + \gamma_2, g_3 = N_1 - 3\gamma_2 + 3\gamma_2^2 - \gamma_2^3$ and $g_p' = N_2$ for all $p$.

As a further check, we consider the case $N_1 = 1$, for which the full density of conductance is known in terms of a single scalar opacity parameter $\gamma$,

$$f_{\gamma}(G) = N_2 G^{N_2 - 1} \phi_{\gamma}(G), \quad (35)$$

where

$$\phi_{\gamma}(G) = (1 - \gamma)^{N_2 + 1} 2F_1(N_2 + 1, N_2 + 1; 1; \gamma(1 - G)). \quad (36)$$

The average conductance (and similarly for the variance) is given by the integral $\langle G \gamma \rangle = \int_0^1 dG G_\gamma(G)$. Expanding $\phi_\gamma(G)$ up to second order in $\gamma$ and computing the integral order by order we obtain

$$\langle G \gamma \rangle \approx \frac{N_2}{1 + N_2} - \frac{N_2}{2 + N_2} \gamma - \frac{N_2}{2 + N_2} (3 + N_2) \gamma^2 \quad (37)$$

in full agreement with (25) with $N_1 = 1$.

In Figure 1 we plot the average and variance of conductance as functions of $\gamma$ when $N_1 = 1$ and $N_2 = 5$, comparing the exact integration of formula (35) and our approximate expansions (25) and (31). The approximation is excellent for the average, while for the variance the quality deteriorates for $\gamma$ close to 1.

**VI. AVERAGE SHOT-NOISE UP TO SECOND ORDER**

Another important quantity that can be measured in the transport context is shot-noise. This is related to fluctuations of the electric current as a time series. Since it is evaluated at zero temperature, it is of quantum nature, arising from the granularity of electric charge. In terms of reflection eigenvalues, shot noise is given by

$$p(R) = \sum_{i=1}^{N_1} R_i(1 - R_i) = s_1(R) - s_2(R) + s_11(R). \quad (38)$$
Its average value is known in the ideal case. Using the present approach, we include \(\gamma\)-effects up to second order. The result is

\[
\langle p \rangle_\gamma \approx \frac{N^2 N_1^2}{N(N^2 - 1)} + \frac{N^2(N_1 - N_2)^2}{(N^2 - 1)(N^2 - 4)} + \frac{N^2[A_2(\text{Tr} \gamma)^2 - B_2 \text{Tr} \gamma^2]}{(N^2 - 1)(N^2 - 4)(N^2 - 9)},
\]

(39)

where

\[
A_2 = N(5N_2^2 - 4N_1 N_2 + N_1^2) + N_1 - 5N_2,
\]

(40)

and

\[
B_2 = N^2N_1^2 + 4N_1^2 - 14N_2 N_1 + 2N_1^2 + 3.
\]

(41)

The limit of large numbers of channels, with \(N_1 = N_2\), can easily be obtained as

\[
\lim_{N_1 = N_2 \to \infty} \frac{\langle p \rangle_\gamma}{N_1} \approx \frac{1}{8} + \frac{(\text{tr} \gamma)^2 - \text{tr} \gamma^2}{16}.
\]

(42)

We compare the approximation (39) against the exact result for \(N_1 = 1\) and \(N_2 = 5\) in Figure 2 (the exact result is obtained by numerical integration of \(G(1 - G)\) times the density (35)). The approximation is not able to account for the fact that the noise vanishes at \(\gamma = 1\) (since all particles are surely reflected), but it can be very good for moderate \(\gamma\).

VII. CONCLUSION

In summary, combining the theory of symmetric functions and generalized Selberg integrals we presented a systematic perturbation theory in the opacity matrix \(\gamma\) for the jpd of reflection eigenvalues in chaotic cavities with \(\beta = 2\) and supporting one ideal and one non-ideal leads. This jpd is found to be given by the standard Jacobi ensemble (1), valid for the ideal case, times a correction that can be systematically expanded in \(\gamma\) (see (15)). Using this result, we computed the average and variance of conductance, as well as average shot-noise, up to the second order in \(\gamma\) and moments of conductance to leading order.

Our results are valid for arbitrary \(N_1, N_2\), in contrast with previously available results which are exact in \(\gamma\) but perturbative in \(N_1, N_2\) and often limited to the leading order term as \(N \to \infty\). Comparison with numerics for \(N_1 = 1\) showed that our perturbative expressions are generally rather accurate for moderate \(\gamma\), and have the advantage of a complete analytical tractability.

Naturally, it would be interesting to extend this calculation to higher orders in \(\gamma\). However, the expressions become quite cumbersome. This may be related to the asymmetric role of the parameters \(N_1\) and \(N_2\). Therefore, it would be even more desirable to be able to consider both leads as non-ideal. Extensions to other symmetry classes is another challenging open problem.

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