A REGULARITY THEORY FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH A SUPER-LINEAR DIFFUSION COEFFICIENT AND A SPATIALLY HOMOGENEOUS COLORED NOISE

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Abstract. Existence, uniqueness, and regularity of a strong solution are obtained for stochastic PDEs with a colored noise and its super-linear diffusion coefficient:

$$du = \left( a^{ij}u_{ix_j} + b^i u_{x_i} + cu \right) dt + \xi |u|^{1+\lambda} dF, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d,$$

where $\lambda \geq 0$ and the coefficients depend on $(\omega, t, x)$. The strategy of handling nonlinearity of the diffusion coefficient is to find a sharp estimation for a general Lipschitz case, and apply it to the super-linear case. Moreover, investigation for the estimate provides a range of $\lambda$, a sufficient condition for the unique solvability, where the range depends on the spatial covariance of $F$ and the spatial dimension $d$.

1. Introduction

In the research on nonlinear stochastic partial differential equations (SPDEs), problems with nonlinear diffusion coefficients and their solution behaviors have been studied [2, 9, 22, 23, 24, 25, 26, 28, 29] and applied to various other fields, such as chemistry [1], neurophysiology [12, 20, 27], and population dynamics [5]. In this article, we investigate the existence, uniqueness, and regularity of a solution to

$$du = \left( a^{ij}u_{ix_j} + b^i u_{x_i} + cu \right) dt + \xi |u|^{1+\lambda} dF, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d; \quad u(0, \cdot) = u_0,$$ (1.1)

where $\lambda \geq 0$. The coefficients $a^{ij}, b^i,$ and $c$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable and twice continuously differentiable in $x$. The coefficient $\xi$ is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable and bounded. The initial data $u_0(x)$ is random and nonnegative. The noise $F(t, x)$, referred to as the spatially homogeneous colored noise [6], is a centered Gaussian noise that is white in time and homogeneously colored in space, with covariance $f$ given by

$$E[F(t, x)F(s, y)] = \delta_0(t-s)f(x-y),$$

where $\delta_0$ is the centered Dirac delta distribution. The distribution $f$ is nonnegative and nonnegative definite; see Definition 2.1.

Next, we provide some motivation and history for this problem. An SPDE with a super-linear diffusion coefficient has been studied [11, 15, 21]:

$$du = (au_{xx} + bu_x + cu) dt + \xi |u|^{1+\lambda} dW, \quad t > 0; \quad u(0, \cdot) = u_0,$$ (1.2)

where $\lambda$ is in $[0, 1/2)$, $u_0 \in C^\infty$ is nonnegative, and $W$ is a Gaussian noise that is white both in time and space. In [21], C. Mueller proves the long-time existence of solutions for the particular case where $a = 1, b = c = 0$, and $\xi = 1$. In [15] Section 8.4, N. Krylov

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proves the existence and regularity of solutions to equation (1.2) with random space and time-dependent coefficients. In [11], the unique solvability in weighted Sobolev spaces is obtained and boundary behavior of the solution on a finite interval is studied.

Yet, it turns out that if the spatial dimension $d \geq 2$, an SPDE (1.2) with the space-time white noise cannot have a real-valued solution; see e.g. [27]. To overcome this weakness, for higher-dimensional cases, the spatially homogeneous colored noise has been studied [3, 4, 6, 7], as an alternation of the space-time white noise. In [6], Dawson introduces the notion of the spatially homogeneous colored noise $F$ and proves the existence and uniqueness of a solution to

$$
\frac{du}{dt} = \Delta u + \sigma(u) dF; \quad u(0, \cdot) = u_0 \tag{1.3}
$$

with nonnegative constant initial data $u_0 = c \geq 0$ and a linear diffusion coefficient $\sigma(u) = u$. In [3, 4], $F(dt, dx)$ is interpreted as a martingale measure by employing Walsh’s approach [27]. Here, the existence and uniqueness of a solution are obtained with $u_0 = c$ and $\sigma(u)$ satisfying the Lipschitz condition. In [7], inspired by [15], the $L_p$-solvability is proved under the following conditions on $\sigma(u)$: for any $x \in \mathbb{R}^d$ and $u \in \mathbb{R}$, $\sigma(\omega, t, x, u)$ is predictable and

$$
|\sigma(\omega, t, x, u) - \sigma(\omega, t, x, v)| \leq N|u - v|,
$$

where $N$ is independent of $\omega, t, x, u,$ and $v$.

It should be remarked that, to the best of our knowledge, the studies on SPDEs with the colored noise $F$ have proceeded to equations with the Lipschitz diffusion coefficient only. A key objective of this paper is to extend the research scope and study SPDE (1.1) with a super-linear diffusion coefficient by employing the approach of [15].

There are three notable achievements of this study. Firstly, the conditions for the initial data and the coefficients assumed in [3, 4, 6] are generalized: the initial data $u_0$ is random, and the coefficients $a^{ij}, b^i, c$ and $\xi$ depend on $\omega, t, x,$ and $u$.

Secondly, we prove the existence and uniqueness of a strong solution and provide maximal Hölder regularity of the solution. The following basic example clarifies the maximal regularity of the strong solution to SPDE (1.1) in a multi-dimensional space: assume that $\lambda \in [0, 1/d)$ and the spatial covariance function $f$ is nonnegative and nonnegative definite. If $u_0$ is nonnegative and in $L_p(\Omega; H_p^{1/2-\lambda}(\mathbb{R}^d)) \cap L_1(\Omega; L_1(\mathbb{R}^d))$ for all $p > 2$, then almost surely for any $T < \infty$ and small $\varepsilon > 0$,

$$
u \in C^{\frac{2}{1+\varepsilon} - \varepsilon - \gamma \varepsilon}_{t,x}([0, T] \times \mathbb{R}^d), \tag{1.4}
$$

where $\gamma = \frac{1}{2} \wedge (1 - \lambda d)$. It is worthwhile to note that if we consider a general covariance, the number of regularity changes; see Corollary 3.16.

Lastly, a range of $\lambda$ and conditions on $f$ are obtained as a sufficient condition for unique solvability of (1.1) in $L_p$-spaces. To see this, we investigate relations between the $L_p$-norm of diffusion coefficients and the $L_p$-norm of a solution, which leads to a sharp estimate on the diffusion coefficients. Moreover, it is found out that summability of the coefficient $\xi$ has a critical effect on the range of $\lambda$; see Remark 5.7.

This paper is organized as follows. Section 2 introduces preliminary definitions and properties. In Section 2, major results are presented for a general Lipschitz diffusion coefficient and for the super-linear coefficient. In Sections 4 and 5, proofs for the Lipschitz case and the super-linear case are elaborated, respectively.

We finish this section with the notations used in the article. $\mathbb{N}, \mathbb{Z},$ and $\mathbb{C}$ denote the set of natural numbers, integers, and complex numbers, respectively. $\mathbb{R}$ is the set of all real numbers and $\mathbb{R}^d$ stands for the $d$-dimensional Euclidean space of points $x = (x^1, \ldots, x^d)$.
for \( x^i \in \mathbb{R}, i = 1, 2, \ldots, d \). We use ‘:=’ to denote a definition. Let \( M \) and \( C_c^\infty = C_c^\infty(\mathbb{R}^d) \) denote the set of nonnegative Borel measures on \( \mathbb{R}^d \), and the set of real-valued infinitely differentiable functions with compact support on \( \mathbb{R}^d \), respectively. \( S = S(\mathbb{R}^d) \) stands for the space of Schwartz functions on \( \mathbb{R}^d \). \( S'_R \) denotes a set of real-valued Schwartz functions on \( \mathbb{R}^d \). Let \( S' = S'(\mathbb{R}^d) \) be the space of tempered distributions on \( \mathbb{R}^d \). For \( f, g \in S \), let us denote by

\[
\mathcal{F}(f)(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) dx, \quad \mathcal{F}^{-1}(g)(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi \cdot x} g(\xi) d\xi,
\]

the Fourier transform of \( f \) in \( \mathbb{R}^d \) and inverse Fourier transform of \( g \), respectively. For \( p \in [1, \infty) \), a normed space \( F \), and a measure space \( (X, \mathcal{M}, \mu) \), a function space \( L_p(X, \mathcal{M}, \mu; F) \) denotes the space of all \( F \)-valued \( \mathcal{M}^\mu \)-measurable functions \( u \) such that

\[
\|u\|_{L_p(X, \mathcal{M}, \mu; F)} := \left( \int_X \|u(x)\|^p_F \mu(dx) \right)^{1/p} < \infty,
\]

where \( \mathcal{M}^\mu \) is the completion of \( \mathcal{M} \) with respect to the measure \( \mu \). For \( \alpha \in (0, 1] \) and \( T > 0 \), let \( C^\alpha([0, T]; F) \) denote the set of \( F \)-valued continuous functions \( u \) such that

\[
|u|_{C^\alpha([0, T]; F)} := \sup_{t \in [0, T]} |u(t)|_F + \sup_{s, t \in [0, T], s \neq t} \frac{|u(t) - u(s)|_F}{|t - s|^{\alpha}} < \infty.
\]

For \( a, b \in \mathbb{R}, a \land b := \min\{a, b\}, a \lor b := \max\{a, b\} \). For \( a \in \mathbb{C} \), \( \overline{a} \) denotes the complex conjugate of \( a \). For a function \( \phi \) on \( \mathbb{R}^d \), we define \( \overline{\phi}(x) := \phi(-x) \). A generic constant is denoted by \( N \). Writing \( N = N(a, b, \cdots) \) implies that the constant \( N \) depends only on \( a, b, \cdots \). Finally, for functions depending on \( \omega, t, \) and \( x \), the argument \( \omega \in \Omega \) is omitted.

2. Preliminaries

In this section, basic definitions and related notions are introduced on distributions, spatially homogeneous colored noises, and stochastic Banach spaces.

2.1. Definitions and properties related to distributions.

**Definition 2.1.**

(i) A \( \mathbb{C} \)-valued continuous function \( f \) is **nonnegative definite** if

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x - y) \psi(x) \overline{\psi}(y) dx dy = (f, \psi \ast \overline{\psi}) \geq 0 \quad \forall \psi \in S.
\]

(ii) A distribution \( f \in S' \) is **nonnegative** if

\[
(f, \psi) \geq 0 \quad \text{for all nonnegative } \psi \in S_R.
\]

(iii) A distribution \( f \in S' \) is **nonnegative definite** if

\[
(f, \psi \ast \overline{\psi}) \geq 0 \quad \forall \psi \in S.
\]

(iv) A nonnegative Borel measure \( \mu \in M \) is a **tempered measure** if there exists \( k \in [0, \infty) \) such that

\[
\int_{\mathbb{R}^d} \frac{1}{(1 + |x|^2)^k/2} \mu(dx) < \infty.
\]

Let \( C_{nd}, S'_{nd}, S'_{nd}, \) and \( M_T \) denote the sets containing the objects defined in (i), (ii), (iii) and (iv), respectively.

Below we collect some well-known results on the objects defined in Definition 2.1. See \[8\].
Theorem 2.2. (i) Let $M_F$ be the set of finite Borel measures and $L : M_F \to C_{nd}$ be a mapping such that

$$(L\mu)(x) := \int_{\mathbb{R}^d} e^{i\xi \cdot x} \mu(d\xi).$$

Then the mapping $L$ is bijective.

(ii) There exists an one-to-one correspondence between $S'_{nd}$ and $M_T$. Precisely, for any $f \in S'_{nd}$, there exists a unique $\mu \in M_T$ such that

$$(f, \psi) = \int_{\mathbb{R}^d} \overline{\psi}(\xi) \nu(d\xi) \quad \forall \psi \in S.$$

Conversely, if $\mu \in M_T$ is given, there exists a unique $f \in S'_{nd}$ satisfying (2.1). For a given $f \in S'_{nd}$, the corresponding $\mu = \mu_f$ is referred to as the corresponding measure of $f$.

(iii) There exists an one-to-one correspondence between $S'_{nd}$ and $M_T$. Precisely, for any $f \in S'_{nd}$, there exists a unique $\nu \in M_T$ such that

$$(f, \psi) = \int_{\mathbb{R}^d} \overline{\psi}(\xi) \nu(d\xi) \quad \forall \psi \in S.$$

Conversely, if $\nu \in M_T$ is given, there exists a unique $f \in S'_{nd}$ satisfying (2.2). For a given $f \in S'_{nd}$, the corresponding $\nu = \nu_{F(f)}$ is referred to as the corresponding measure of $F(f)$.

(iv) For $f \in S'_{nd}$, there exist $h \in C_{nd}$ and $l \in [0, \infty)$ such that

$$f = (1 - \Delta)^{l/2} h \quad \text{and} \quad h(x) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} \frac{1}{(1 + |\xi|^2)^l/2} \nu(d\xi),$$

where $\nu$ is referred to as the corresponding measure of $F(f)$.

Proof. See [8].

Remark 2.3. (i) For any given $\mu \in M_T$, $S \subseteq L_p(\mu)$ for all $p \in [1, \infty]$.

(ii) If $f \in S'_{nd}$, then $f$ is a real-valued, nonnegative, symmetric, continuous, and bounded function. Thus, (2.1) can be written as

$$(f, \phi) = \int_{\mathbb{R}^d} \phi(x) \mu(dx) = \int_{\mathbb{R}^d} \phi(x) f(x) dx \quad \forall \phi \in S.$$

For more details, see [8].

2.2. Spatially homogeneous colored noise. Here and thereafter, let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions, $\mathcal{P}$ be the predictable $\sigma$-field related to $\mathcal{F}_t$, and $f$ be a distribution such that $f \in S'_{nd}$.

Definition 2.4. The random noise $F(t, A)$ is called a spatially homogeneous colored noise with covariance $f$ if $F(t, A) := F(1_{[0,t]} \cdot A)$ is a centered Gaussian process and its covariance is given by

$$E[F(t, A)F(s, B)] = t \wedge s(f, 1_A * 1_B),$$

where $A$ and $B$ are bounded Borel sets and $t \geq 0$. 

For more details, see [8].
Remark 2.5. The framework of Walsh [27] has been used to define a stochastic integral with respect to $F(dt, dx)$; see [3, 4]. On the other hand, the integral can also be written as an infinite summation of Itô stochastic integral; for predictable process $X(t, \cdot)$ such that

$$X(t, x) = \zeta(x)1_{[\tau_1, \tau_2]}(t)$$

where $\tau_1$, $\tau_2$ are bounded stopping times, $\{\tau_1, \tau_2\} := \{(\omega, t) : \tau_1(\omega) < t \leq \tau_2(\omega)\}$, and $\zeta \in C_c^\infty$, we have

$$\int_0^t \int_{\mathbb{R}^d} X(s, x)F(ds, dx) = \sum_{k=1}^\infty \int_0^t \int_{\mathbb{R}^d} (f \ast e_k)(x)X(s, x)dwdw^k_s,$$

where $w_s^k, k \in \mathbb{N}$, is a one-dimensional independent Wiener process and $e_k(x), k \in \mathbb{N}$ is an infinitely differentiable function induced by $f$. The construction and properties of $\{e_k, k \in \mathbb{N}\}$ are described in Remarks 2.6 and 2.7.

Remark 2.6. (i) Let $\nu$ be the corresponding measure of $\mathcal{F}(f)$ and define

$$\langle \phi, \psi \rangle_{\mathcal{H}} := (f, \phi \ast \bar{\psi}) = (\phi, f \ast \bar{\psi}),$$

for $\phi, \psi \in \mathcal{S}$. Then, if $\ell \ll \nu$, the operator $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is an inner product on $\mathcal{S}$. To see this, it suffices to show that $\langle \psi, \psi \rangle_{\mathcal{H}} = 0$ yields $\psi = 0$ ($\ell$-a.e.) since the operator $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is sesquilinear on $\mathcal{S}$. By Theorem 2.2 (iii), we have

$$\langle \psi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^d} |\mathcal{F}(\psi)(\xi)|^2 \nu(d\xi),$$

and thus $\langle \psi, \psi \rangle_{\mathcal{H}} = 0$ implies $\mathcal{F}(\psi) = 0$ ($\nu$-a.e.). Since $\ell \ll \nu$, $\mathcal{F}(\psi) = 0$ ($\ell$-a.e.) and thus $\psi = 0$ ($\ell$-a.e.).

(ii) Define $\mathcal{H}$ as the closure of $\mathcal{S}$ under the norm $\| \cdot \|^2_{\mathcal{H}} := \langle \cdot, \cdot \rangle_{\mathcal{H}}$. Then there exists a complete orthonormal basis $\{e_k : k \in \mathbb{N}\} \subseteq \mathcal{S}_R$ of $\mathcal{H}$, since $\mathcal{S}$ is dense in $L_2(\nu)$ and $L_2(\nu)$ is isomorphic to $\mathcal{H}$.

(iii) If $\ell \ll \nu$, the existence of a complete orthonormal basis of $\mathcal{H}$ can be shown similarly by identification with an equivalence relation; see [4, 8].

Remark 2.7. (i) Since $f \in S'_n \cap S'_{nd}$ and $e_k \in \mathcal{S}_R$, the real-valued function $v_k := f \ast e_k$ is infinitely differentiable.

(ii) The function $v_k$ is bounded. Indeed, by Theorem 2.2 (iv), we have

$$|v_k(x)| = |f \ast e_k(x)| = |(f, e_k(x - \cdot))| = |(1 - \Delta)^{1/2}h, e_k(x - \cdot)|$$

$$= |(h, (1 - \Delta)^{1/2}e_k(x - \cdot))| \leq \sup_{x \in \mathbb{R}^d} |h(x)||1 - \Delta|^{1/2}e_k|_{L_1}.$$

2.3. Stochastic Banach spaces. This section provides definitions and properties of Sobolev spaces (Bessel potential spaces) and stochastic Banach spaces; see [10, 15, 17].

Definition 2.8. For $p > 1$ and $\gamma \in \mathbb{R}$,

(i) let $H^\gamma_{L^p} = H^\gamma_{L^p}(\mathbb{R}^d)$ denote the class of tempered distributions $u$ on $\mathbb{R}^d$ such that

$$\|u\|_{H^\gamma_{L^p}} := \|(1 - \Delta)^{\gamma/2}u\|_{L_p} = \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{\gamma/2}\mathcal{F}(u)(\xi)]\|_{L_p} < \infty,$$

(ii) let $H^\gamma_{L^2}(l_2) = H^\gamma_{L^2}(\mathbb{R}^d; l_2)$ denote the class of $l_2$-valued tempered distributions $g = (g^1, g^2, \cdots)$ on $\mathbb{R}^d$ such that

$$\|g\|_{H^\gamma_{L^2}(l_2)} := \|(1 - \Delta)^{\gamma/2}g\|_{L^2} = \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{\gamma/2}\mathcal{F}(g)(\xi)]\|_{L^2} < \infty.$$
Remark 2.9. It is well known (e.g., [17]) that for $\gamma \in (0, \infty)$,
\[
(1 - \Delta)^{-\gamma/2}u(x) = \int_{\mathbb{R}^d} R_\gamma(x - y)u(y)dy,
\]
where
\[
R_\gamma(x) = c(\gamma, d)|x|^{\gamma-d} \int_0^\infty t^{-d+\gamma-2} e^{-|x|^2t^{-\frac{1}{2}}} dt.
\]
Observe that (e.g., [10, Proposition 1.2.5.])
\[
R_\gamma(x) \leq N_1 e^{-\frac{|x|}{\sqrt{2}}} 1_{|x| \geq 2} + N_2 A_\gamma(x) 1_{|x| < 2},
\]
where
\[
A_\gamma(x) := (O(|x|^{\gamma-d+2}) + |x|^{\gamma-d} + 1)1_{0 < \gamma < d}
\]
\[
+ (\log(2/|x|) + 1 + O(|x|^2))1_{\gamma=d} + (1 + O(|x|^\gamma-d))1_{\gamma>d},
\]
and $N_i \ (i = 1, 2)$ depends only on $\gamma$ and $d$.

Below some useful facts on $H_p^\gamma$ are described.

Lemma 2.10. Let $\gamma \in \mathbb{R}$ and $p > 1$.

(i) The spaces $C_\gamma$ and $S$ are dense in $H_p^\gamma$.

(ii) Let $\gamma - d/p = n + \nu$ for some $n = 0, 1, \cdots$ and $\nu \in (0, 1]$. Then for any $i \in \{0, 1, \cdots, n\}$, we have
\[
|D^i u|_C + |D^\nu u|_{C^0} \leq N\|u\|_{H_p^\gamma}.
\]

(iii) Let
\[
\kappa \in [0, 1], \quad p_i \in (1, \infty), \quad \gamma_i \in \mathbb{R}, \quad i = 0, 1,
\]
\[
\gamma = \kappa \gamma_1 + (1 - \kappa)\gamma_0, \quad 1/p = \kappa/p_1 + (1 - \kappa)/p_0.
\]
Then we have
\[
\|u\|_{H_p^\gamma} \leq \|u\|_{H_p^{\gamma_1}}^{1-\kappa} \|u\|_{H_p^{\gamma_0}}^{\kappa}.
\]

Proof. For (i) and (ii), see [17]. The multiplicative inequality (iii) follows from [13].

Recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions and $\mathcal{P}$ is the predictable $\sigma$-field related to $\mathcal{F}_t$.

Definition 2.11. For $\tau \leq T$, let us denote $\{(\omega, t) : 0 < t \leq \tau(\omega)\}$.

(i) For $\tau \leq T$, Stochastic Banach spaces are defined by
\[
\mathbb{H}_p^\gamma(\tau) := L_p((0, \tau], \mathcal{P}, d\mathbb{P} \times dt; H_p^\gamma),
\]
\[
\mathbb{H}_p^\gamma(\tau, l_2) := L_p((0, \tau], \mathcal{P}, d\mathbb{P} \times dt; H_p^\gamma(l_2)),
\]
\[
U_p^\gamma := L_p(\Omega, \mathcal{F}_0, d\mathbb{P}; H_p^{\gamma-2/p}).
\]
For convenience, we write $\mathbb{L}_p(\tau) := \mathbb{H}_p^0(\tau)$ and $\mathbb{L}_p(\tau, l_2) := \mathbb{H}_p^0(\tau, l_2)$.

(ii) The norm of each stochastic Banach space is defined in the usual way, e.g.,
\[
\|u\|_{\mathbb{H}_p^\gamma(\tau)}^p := E \int_0^\tau \|u(t)\|_{H_p^\gamma}^p dt.
\]

Definition 2.12. Let $\tau$ be a bounded stopping time and $u \in \mathbb{H}_p^\gamma(\tau)$.
Thus, we prove that the equation
\[ u \in \mathcal{H}_p^\gamma(\tau) \] if \( u_0 \in U_p^\gamma \) and there exists \((f,g) \in H_p^{\gamma-2}(\tau) \times H_p^{\gamma-1}(\tau, l_2)\) such that
\[ du = fdt + \sum_{k=1}^{\infty} g^k dw^k_t, \quad t \in (0, \tau]; \quad u(0, \cdot) = u_0 \]
in the sense of distributions, i.e., for any \( \phi \in C_c^\infty \), the equality
\[ (u(t, \cdot), \phi) = (u_0, \phi) + \int_0^t (f(s, \cdot), \phi) ds + \sum_{k=1}^{\infty} \int_0^t (g^k(s, \cdot), \phi) dw^k_s \quad (2.9) \]
holds for all \( t \in [0, \tau] \) almost surely. In this case, we write
\[ \mathbb{D}u := f, \quad Su := g. \]

(ii) The norm of \( \mathcal{H}_p^\gamma(\tau) \) is defined by
\[ \|u\|_{\mathcal{H}_p^\gamma(\tau)} := \|u\|_{\mathcal{H}_p^\gamma(\tau)} + \|\mathbb{D}u\|_{H_p^{\gamma-2}(\tau)} + \|Su\|_{H_p^{\gamma-1}(\tau, l_2)} + \|u(0, \cdot)\|_{U_p^\gamma}. \]

**Definition 2.13.** Let \( \tau \) be a stopping time. We write \( u \in \mathcal{H}_{p,loc}^\gamma(\tau) \) if there exists a sequence of bounded stopping times \( \tau_n \uparrow \tau \) such that \( u \in \mathcal{H}_p^\gamma(\tau_n) \) for each \( n \). We say \( u = v \) in \( \mathcal{H}_{p,loc}^\gamma(\tau) \) if there are bounded stopping times \( \tau_n \uparrow \tau \) such that \( u = v \) in \( \mathcal{H}_p^\gamma(\tau_n) \) for each \( n \). For convenience, \( \tau \) is omitted when \( \tau = \infty \).

**Remark 2.14.** Definition 2.12 can be defined with \( \psi \in \mathcal{S} \), instead of \( \phi \in C_c^\infty \); see, e.g., \cite[Remark 3.4]{15}.

**Remark 2.15.** Main equation (1.1) can be understood in the sense of distribution. In other words, for any \( \phi \in C_c^\infty \), the equality
\[ (u(t, \cdot), \phi) = (u_0, \phi) + \int_0^t \left( a^{ij}(s, \cdot)u_{x^i x^j}(s, \cdot) + b^i(s, \cdot)u_{x^i}(s, \cdot) + c(s, \cdot)u(s, \cdot), \phi \right) ds \]
\[ + \int_0^t \int_{\mathbb{R}^d} \xi(s, x)|u(s, x)|^{1+\lambda} \phi(x) F(ds, dx) \]
holds for all \( t \in [0, \tau] \) almost surely. Due to Remark 2.5 and \cite[Theorem 3.10]{15}, if \( u \in \mathcal{H}_{p,loc}^\gamma \) for some \( \gamma > 0 \) and \( p \geq 2 \), then we have
\[ \int_0^t \int_{\mathbb{R}^d} \xi(s, x)|u(s, x)|^{1+\lambda} \phi(x) F(ds, dx) = \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^d} (f * e_k)(x) \xi(s, x)|u(s, \cdot)|^{1+\lambda} \phi(x) dw^k_t. \]
Thus, we prove that the equation
\[ du = (a^{ij}u_{x^i x^j} + b^i u_{x^i} + cu)dt + \sum_{k=1}^{\infty} \xi |u|^{1+\lambda} (f * e_k) dw^k_t, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d; \quad u(0, \cdot) = u_0, \]
has a unique solution \( u \in \mathcal{H}_{p,loc}^\gamma \).

Below embedding theorems for stochastic Banach spaces are introduced.

**Theorem 2.16.** Let \( \tau \leq T \) be a bounded stopping time and \( \gamma \in \mathbb{R} \).

(i) For any \( p \geq 2 \), \( \mathcal{H}_p^\gamma(\tau) \) is a Banach space with the norm \( \| \cdot \|_{\mathcal{H}_p^\gamma(\tau)} \).
(ii) If $p > 2$ and $\frac{1}{p} < \alpha < \beta < \frac{1}{2}$, then for any $u \in \mathcal{H}_p^\gamma(\tau)$, we have $u \in C^{\alpha - \frac{1}{p} ([0, \tau]; H_p^{\gamma - 2\beta})}$ (a.s.) and
\[
\mathbb{E}|u|^p_{C^{\alpha - \frac{1}{p} ([0, \tau]; H_p^{\gamma - 2\beta})}} \leq N(d, p, \alpha, \beta, \tau)\|u\|^p_{\mathcal{H}_p^\gamma(\tau)}.
\] (2.10)

(iii) If $p = 2$, then inequality (2.10) holds for $\alpha = \beta = 1/p = 1/2$, i.e., $u \in C([0, \tau]; H_2^{\gamma - 1})$ (a.s.) and
\[
\mathbb{E}\sup_{t \leq \tau} \|u\|_{H_2^{\gamma - 1}}^2 \leq N(d, T)\|u\|_{H_2^\gamma(\tau)}^2.
\] (2.11)

(iv) If $p \geq 2$ and $\tau \equiv T$, then for any $t \leq T$,
\[
\|u\|_{H_2^{\gamma - 1}(t)}^p \leq N(d, p, T) \int_0^t \|u\|_{H_2^\gamma(s)}^p ds.
\] (2.12)

Proof. Theorem 2.16 (i)-(iii) are proved in [15]; for (i), see Section 3; for (ii) and (iii), see Section 7. In the case of (iv), use (2.8) and inequalities (2.10) and (2.11). □

Corollary 2.17. Let $\tau \leq T$ be a bounded stopping time and $p > 2$, $\gamma \in (0, \frac{1}{2})$, $\alpha, \beta \in (\frac{1}{p}, \frac{1}{2})$ such that
\[
\frac{1}{p} < \alpha < \beta < \frac{\gamma}{2} - \frac{d}{2p}.
\] (2.13)

Then for any $\delta \in [0, \gamma - 2\beta - \frac{d}{2p})$, we have
\[
\mathbb{E}|u|^p_{C^{\alpha - \frac{1}{p} ([0, \tau]; C^{\gamma - 2\beta - \frac{d}{2p} - \delta}(\mathbb{R}^d))}} \leq N(d, p, \alpha, \gamma, \tau)\|u\|^p_{\mathcal{H}_p^\gamma(\tau)}.
\] (2.14)

Proof. Note that Lemma 2.10 (i) implies
\[
|u|_{C^{\gamma - 2\beta - \frac{d}{2p} - \delta}(\mathbb{R}^d)} \leq N\|u\|_{H_p^{\gamma - 2\beta - \delta}} \leq N\|u\|_{H_p^{\gamma - 2\beta}}.
\] (2.15)

Applying inequality (2.15) to the definition of the norm $| \cdot |_{C^{\alpha - \frac{1}{p} ([0, \tau]; C^{\gamma - 2\beta - \frac{d}{2p} - \delta}(\mathbb{R}^d))}}$ and employing inequality (2.10) prove (2.14). The corollary is proved. □

3. Main results

In this section, we describe some major results on a solution to equation
\[
du = (a^{ij}u_{x^i x^j} + b^i u_{x^i} + cu) dt + \sum_{k=1}^{\infty} \xi|u|^{1+\lambda}(f * e_k) dw_t^k, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d; \quad u(0, \cdot) = u_0(\cdot),
\] (3.1)

where $\lambda \geq 0$ and $u_0 \geq 0$. The existence, uniqueness, and regularity of a solution in $\mathcal{H}_p^\gamma$ are proved and H"{o}lder regularity of the solution is achieved. In the case of $\lambda = 0$, we consider more general equation than (3.1); see equation (3.2). Investigation on equation (3.2) is employed to prove the super-linear case; see Section 3.2. Specifically, it turns out that $u \geq 0$, when $\lambda > 0$, by maximum principle. Then by taking integration with respect to $x$ in (3.1) (at least formally), one can check that $\|u(t, \cdot)\|_{L_1}$ is a continuous local martingale. Since $\|u(t, \cdot)\|_{L_1}$ is nonnegative, its paths are bounded almost surely. If $\xi := \xi|u|^{1+\lambda}/u(0/0 := 0)$ agrees with the summability condition of Theorem 3.5, the unique solvability of (3.1) can be obtained.

It should be remarked that more regularity of the solution can be obtained when the colored noise $F$ is simple in some sense; for example, the covariance $f$ is smooth or the
spatial dimension $d$ is small. Furthermore, we discuss the range of $\lambda$, a sufficient condition for the unique solvability, in relation to $f$ and $d$.

### 3.1. The Lipschitz case

In this section, consider an SPDE with Lipschitz diffusion coefficients; for a bounded stopping time $\tau \leq T$,

$$
\begin{align*}
\text{d}u &= (a^{ij}u_{x^i} + b^i u_x + cu)\text{d}t + \sum_{k=1}^{\infty} \xi_h(u) (f \ast e_k) \text{d}w^k_t, \quad (t, x) \in (0, \tau) \times \mathbb{R}^d; \quad u(0, \cdot) = u_0(\cdot),
\end{align*}
$$

where $a^{ij}, b^i, c$ and $\xi$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable functions and $h(u)$ satisfies Lipschitz condition in $u$; for more details, see Assumptions 3.1-3.3. An estimate, existence, uniqueness, and regularity of a solution to (3.2) in $\mathcal{H}_p^\omega$ are provided, and Hölder regularity of the solution is obtained; see Theorem 3.5. In particular, these results cover the case $\lambda = 0$ in (3.1).

**Assumption 3.1.**

(i) The coefficients $a^{ij}, b^i, c$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable.

(ii) There exist constants $\kappa_0, K > 0$ such that

$$
\kappa_0 |\eta|^2 \leq a^{ij}(t, x)\eta^i \eta^j \leq K |\eta|^2 \quad \forall \omega, t, x, \eta
$$

and

$$
|a^{ij}(t, \cdot)|_{C^2(\mathbb{R}^d)} + |b^i(t, \cdot)|_{C^1(\mathbb{R}^d)} + |c(t, \cdot)|_{C^2(\mathbb{R}^d)} < K \quad \forall \omega, t.
$$

**Assumption 3.2** ($\tau, s$). The coefficient $\xi$ is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable, and there exists a constant $K > 0$ such that

$$
||\xi(t, \cdot)||_{L_s} \leq K \quad \forall \omega \in \Omega, \ t \in [0, \tau] \cap [0, \infty).
$$

**Assumption 3.3** ($\tau, p$). The function $h = h(\omega, t, x, u)$ is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R})$-measurable, $h(0) = h(\omega, t, x, 0) \in L_p(\tau)$, and there exists a constant $K > 0$ such that for any $\omega \in \Omega, \ t \in [0, \tau], \ u, v \in \mathbb{R}$ and $x \in \mathbb{R}^d$,

$$
|h(t, x, u) - h(t, x, v)| \leq K|u - v|.
$$

**Remark 3.4.** Obviously, $h(u) = u$ or $h(u) = |u|$ satisfy Assumption 3.3.

Now, we describe the results on the Lipschitz case (3.2).

**Theorem 3.5.** Let $\tau \leq T$ be a bounded stopping time and $f \in \mathcal{S}_n \cap \mathcal{S}_{nd}$. Assume that $d, \gamma, s, p, \sigma$ and $f$ satisfy one of the following conditions:

(i) $d = 1$, $\gamma \in [0, 1/2)$, $s \in (\frac{1}{1-2\gamma}, \infty)$, $p \geq \frac{2s}{s-1} \left(\frac{2s}{\infty} := 2\right)$ and $\sigma = 2s$;

(ii) $d \geq 2$, $\gamma \in [0, 1)$, $s \in (\frac{d-\gamma}{d-1}, \infty)$, $p \geq \frac{2s}{s-1} \left(\frac{2s}{\infty} := 2\right)$, $\sigma = 2s$ and

$$
\int_{|x| < 1} |x|^{\frac{(1-\gamma-d)}{2}} \mu(dx) < \infty \quad \left(1-\gamma-d\right)_{\infty} := 1-\gamma-d,
$$

where $\mu$ is the corresponding measure of $f$;

(iii) $d \geq 1$, $\gamma \in [0, 1)$, $s \in (\frac{1}{d-\gamma}, \infty)$, $p \geq \frac{2s}{s-1} \left(\frac{2s}{\infty} := 2\right)$, $\sigma = s$ and $f \in C_{nd}$.

Suppose that Assumptions 3.2, 3.3 ($\tau, \sigma$) and 3.5 ($\tau, p$) hold. Then, for $u_0 \in U_0^\gamma$, equation (3.2) with initial data $u(0, \cdot) = u_0$ has a unique solution $u$ in $\mathcal{H}_p^\omega(\tau)$. The solution satisfies

$$
||u||_{\mathcal{H}_p^\omega(\tau)} \leq N(||h(0)||_{L_p(\tau)} + ||u_0||_{U_0^\gamma}),
$$

where $N = N(T, d, p, \gamma, \kappa_0, K)$. Furthermore, for any $\alpha, \beta$ and $\delta$ satisfying

$$
\frac{1}{p} < \alpha < \beta < \gamma - \frac{d}{2p}, \quad 0 \leq \delta < \gamma - 2\beta - \frac{d}{p},
$$
we have
\[ |u|_{C^{\alpha - \frac{1}{p}}([0,T];C^{\gamma - 2\beta - \frac{d}{p} - \delta}({\mathbb R}^d))} < \infty \]
for all \( T < \infty \) (a.s.).

**Proof.** See Section 4. \( \square \)

**Remark 3.6.** Note that Theorem 3.5 holds for \( s = \infty \) and \( h(u) = |u| \), which is equivalent to equation (3.1) on \( t \leq \tau \) with \( \lambda = 0 \).

### 3.2. The super-linear case.

This section provides unique solvability of (3.1) and regularities of a solution to equation (3.1). In addition, maximal Holder regularity is obtained.

Before we state the major results, assumptions on coefficients are introduced. Note that the coefficients can be dependent on \( \omega, t, \) and \( x \).

**Assumption 3.7.** The coefficient \( \xi \) is \( \mathcal{P} \times \mathcal{B}(\mathbb{R}^d) \)-measurable and there exists \( K > 0 \) such that
\[ \|\xi(t, \cdot)\|_{L^\infty} < K \quad \forall \omega, t. \]

**Definition 3.8.** For \( d \in \mathbb{N} \), let
\[
\gamma_0 = \gamma_0(d, \lambda) := \frac{1}{2} - 2d \left( \lambda - \frac{1}{4d} \right) 1_{\{\lambda > \frac{1}{4d}\}}, \\
\gamma_1 = \gamma_1(d, \lambda) := \frac{1}{2} - d \left( \lambda - \frac{1}{2d} \right) 1_{\{\lambda > \frac{1}{2d}\}}.
\]

Now, the unique solvability of equation (3.1) in \( H^{\gamma}_{p,\text{loc}} \) is presented. Besides, \( L_p \) and Hölder regularities of the solution are obtained.

**Theorem 3.9.** Suppose \( f \in \mathcal{S}' \cap \mathcal{S}'_{\text{nd}} \) and Assumptions 3.1 and 3.7 hold. The constants \( \gamma_0 \) and \( \gamma_1 \) are defined in Definition 3.8. Assume that \( d, \lambda, \gamma, p \) and \( f \) satisfy one of the following conditions:

(i) \( d = 1, 0 \leq \lambda < \frac{1}{2}, 0 < \gamma < \frac{1}{2} - \lambda, \) and \( p > \frac{3}{2} \);
(ii) \( d \geq 2, 0 \leq \lambda < \frac{1}{2d}, 0 < \gamma < \gamma_0, \) and \( p > \frac{d+2}{\gamma} \), and
\[
\int_{|x|<1} |x|^{\frac{1-\gamma-d}{1-2\lambda}} \mu(dx) < \infty, \quad (3.8)
\]
where \( \mu \) is the corresponding measure of \( f \);
(iii) \( d \geq 1, 0 \leq \lambda < \frac{1}{d}, 0 < \gamma < \gamma_1, \) and \( p > \frac{d+2}{\gamma} \), and \( f \in C_{\text{nd}} \).

Then, for nonnegative \( u_0 \) in \( U^\gamma_p \cap L_1(\Omega; L_1(\mathbb{R}^d)) \), equation (3.1) with initial data \( u(0, \cdot) = u_0 \) has a unique solution \( u \) in \( H^{\gamma}_{p,\text{loc}} \). In addition, for any \( \alpha, \beta \) and \( \delta \) satisfying
\[
\frac{1}{p} < \alpha < \beta < \frac{\gamma}{2} - \frac{d}{2p}, \quad 0 < \delta < \gamma - 2\beta - \frac{d}{p}, \quad (3.9)
\]
we have
\[ |u|_{C^{\alpha - \frac{1}{p}}([0,T];C^{\gamma - 2\beta - \frac{d}{p} - \delta}({\mathbb R}^d))} < \infty \]
for all \( T < \infty \) (a.s.).
Proof. In the case of $\lambda = 0$, we use Theorem 3.5. Let $\tau$ be a bounded stopping time. Note that the conditions in Theorem 3.9 with $\lambda = 0$ imply those in Theorem 3.5 for $s = \infty$, and thus there exists $u = u_\tau(t, x) \in H^\gamma_p(\tau)$. Similarly, for a stopping time $\tilde{\tau}$ such that $\tau \leq \tilde{\tau}$, we can find $\tilde{u} \in H^\gamma_p(\tilde{\tau})$. From the uniqueness of solutions in $H^\gamma_p(\tau)$, we have $\tilde{u} = u$ in $H^\gamma_p(\tau)$. Thus, by the definition of $H^\gamma_p$, the Lipschitz case is proved. It should be remarked that the proof of the Lipschitz case does not require the condition nonnegativity of the initial data $u_0$. For the case of $\lambda > 0$, see Section 5. □

Remark 3.10. The range of $\lambda$ in each condition of Theorem 3.9 is a sufficient condition for the unique solvability. More details on the range of $\lambda$ are discussed in Section 5; see Remark 5.7.

Remark 3.11. Due to the assumptions on $\gamma$ and $p$, there always exist $\alpha$ and $\beta$ satisfying inequality (3.9). Besides, the assumptions make the number $\gamma - 2\beta - \frac{d}{p} - \delta$ in (3.10) positive.

Remark 3.12. By (3.10), Hölder regularity of the solution is obtained and it varies according to the choice of $\alpha, \beta, \gamma$ and $p$. For example, with sufficiently small $\varepsilon > 0$, if $\alpha$ and $\beta$ close enough to $1/p$ and $\delta = 0$, then

$$\sup_{t \leq T} \|u(t, \cdot)\|_{C^{\gamma - \frac{d+2}{p} - \varepsilon}} < \infty,$$

for all $T < \infty$ almost surely. On the other hand, if $\alpha$ and $\beta$ close enough to $\frac{1}{2} - \frac{d}{2p}$ and $\delta = 0$, then

$$\sup_{x \in \mathbb{R}^d} \|u(\cdot, x)\|_{C^{\frac{\gamma - d}{2} - \varepsilon}} < \infty,$$

for all $T < \infty$ almost surely. Note that the solution $u$ depends not only on $(\omega, t, x)$ but also on $(\gamma, p)$.

Example 3.13 (Riesz kernel). Let $f_\alpha(x) = |x|^{-\alpha}$ for some $\alpha \in (0, d)$. Since the covariance $f_\alpha \in S'_n \cap S'_{nd}$, Theorem 3.9 can be applied.

(i) If $d = 1$ and $\lambda \in [0, 1/2)$, Theorem 3.9 holds for any $\alpha \in (0, 1)$.

(ii) If $d \geq 2$ and $\lambda \in [0, \frac{1}{2d})$, Theorem 3.9 holds for any

$$\alpha \in \left(0, \frac{1 - \gamma - 2\lambda d}{1 - 2\lambda}\right). \quad (3.11)$$

To see (3.11), note that the corresponding measure of $f$ is $\mu(dx) = |x|^{-\alpha} dx$ (e.g. 8). To satisfy inequality (3.8), we need $\frac{1 - \gamma - d}{1 - 2\lambda} - \alpha + d > 0$, which implies inequality (3.11).

Example 3.14 (Gaussian kernel). Let $f(x) = e^{-c|x|^2}$. Since $f \in S'_n \cap C_{nd}$, Theorem 3.9 (iii) holds for any Gaussian kernels.

To obtain maximal Hölder regularity of the solution by extending Theorem 3.9, the following theorem is required.

Theorem 3.15. Let $\gamma_0$, $\gamma_1$, $d$, $\lambda$, $\gamma$, $p$, $f$, and $u_0$ satisfy the conditions in Theorem 3.9 and $u \in H^\gamma_{p,loc}$ be the solution. Then if $q > p$ and $u_0 \in U^\gamma_q \cap L_1(\Omega; L_1(\mathbb{R}^d))$, the solution $u$ also belongs to $H^\gamma_{q,loc}$.

Proof. See Section 5. □
Consequently, combining Theorems 3.9 and 3.15 leads to maximal Hölder regularity of the solution to (3.1):

**Corollary 3.16.** Let $\gamma_0$ and $\gamma_1$ be defined in Definition 3.8. Suppose that $d$, $\lambda$, $\gamma_*$, and $f$ satisfy one of the following conditions;

(i) $d = 1$, $0 \leq \lambda < \frac{1}{2}$, and $\gamma_* = \frac{1}{2} - \lambda$;

(ii) $d \geq 2$, $0 \leq \lambda < \frac{1}{2d}$, $0 < \gamma_* < \gamma_0$, and

$$\int_{|x| < 1} |x|^{\frac{1-\gamma_*-d}{1-2d}} \mu(dx) < \infty,$$

where $\mu$ is the corresponding measure of $f$;

(iii) $d \geq 1$, $0 \leq \lambda < \frac{1}{d}$, $\gamma_* = \gamma_1$, and $f \in S'_\alpha \cap C^{\alpha d}$.

Suppose nonnegative $u_0$ is in $U^r_p \cap L_1(\Omega; L_1(\mathbb{R}^d))$ for any $p > \frac{d+2}{\gamma}$. Then for any $\gamma \in (0, \gamma^*)$ and $p > \frac{d+2}{\gamma}$, equation (3.11) with $u(0, \cdot) = u_0$ has a unique solution $u_p$ in $\mathcal{H}^p_{\gamma, p, loc}$. In addition, for fixed $\gamma \in (0, \gamma^*)$, all the solutions $u_p$ coincide with $u$ for all $p > \frac{d+2}{\gamma}$. Furthermore, for any $T > 0$ and $0 < \varepsilon < \frac{\gamma}{2}$ (a.s.),

$$u \in C^{\gamma - \varepsilon, \gamma - \varepsilon}_{T, x}([0, T] \times \mathbb{R}^d). \quad (3.12)$$

**Proof.** To see the first assertion, we use Theorem 3.9. Let $\gamma \in (0, \gamma^*)$ and $p > \frac{d+2}{\gamma}$. Since $U^r_p \subseteq U^r_p$, we have $u_0 \in U^r_p \cap L_1(\Omega; L_1(\mathbb{R}^d))$. Besides, in the case of (i), observe that $|x|^{\frac{1-\gamma_*-d}{1-2d}} |x| < 1 \in L(\mu)$ for any $\gamma \in (0, \gamma^*)$. Thus, Theorem 3.9 implies the first assertion.

The second assertion follows from Theorem 3.15. For the last assertion, take $\gamma$ close enough to $\gamma^*$ and $p > \frac{d+2}{\gamma}$. By (3.11), for any $\alpha$, $\beta$ such that $\frac{1}{p} < \alpha < \beta < \frac{\gamma}{2} - \frac{d}{2p}$, we have

$$|u|_{C^{\alpha, \beta}([0, T]; C^{\gamma-2\beta, \gamma-\beta}(\mathbb{R}^d))} < \infty \quad \forall T < \infty \text{ (a.s.)}.$$ 

Fix $T > 0$ and $\varepsilon \in (0, \frac{\gamma}{2})$. By taking $\alpha, \beta$ close to $1/p$ and sufficiently large $p$, Hölder regularity with respect to spatial variable is obtained. On the other hand, by choosing $\alpha, \beta$ close to $\frac{1}{p} - \frac{d}{2p}$ and sufficiently large $p$, we get Hölder regularity in time variable. Therefore, we have (3.12). The corollary is proved. \qed

4. **Proof for case with the general Lipschitz diffusion coefficient**

In this section, we provide a proof of Theorem 3.5 which presents existence, uniqueness, and $L_p$-estimate of a solution to Lipschitz case (3.2). For readers’ convenience, we recall equation (3.2):

$$du = (a^{ij}u_{x_i}x_j + b^i u_{x_i} + cu) dt + \sum_{k=1}^{\infty} \xi h(u)(f * e_k) dw^k_t, \quad 0 < t \leq \tau; \quad u(0, \cdot) = u_0, \quad (4.1)$$

where $h(u) = h(\omega, t, x, u)$ satisfies Lipschitz condition in $u$; see Assumption 3.3(\tau, p).

To prove Theorem 3.5, we employ Theorem 4.2 that provides a unique solvability and estimate of a solution to the linear equation

$$du = (a^{ij}u_{x_i}x_j + b^i u_{x_i} + cu) dt + \sum_{k=1}^{\infty} g^k(u) dw^k_t, \quad 0 < t \leq \tau; \quad u(0, \cdot) = u_0, \quad (4.2)$$
where $\tau \leq T$ is a bounded stopping time. Here, the diffusion coefficient $g^k(u) = g^k(\omega, t, x, u)$ satisfies Assumption (4.1) (τ). To show that the diffusion coefficients for the Lipschitz case (4.1) satisfy Assumption (4.1) (τ), we employ Lemma 4.4, which presents an estimate for the coefficients. Detailed proof for Theorem 4.5 is described at the end of this section.

Assumption 4.1 (τ). (i) For any $u \in H^p_\gamma$, the function $g(t, x, u)$ is a $H^{p−1}(l_2)$-valued predictable function, and $g(0) \in H^{p−1}(\tau, l_2)$, where $\tau$ is a stopping time. (ii) For any $\varepsilon > 0$, there exists a constant $N_{\varepsilon}$ such that\
\[
\|g(t, u) - g(t, v)\|_{H^{p−1}(l_2)} \leq \varepsilon \|u - v\|_{H^p_\gamma} + N_{\varepsilon} \|u - v\|_{H^{p−1}_\gamma},
\]
for any $u, v \in H^p_\gamma$, $\omega \in \Omega$ and $t \in [0, \tau]$.

Theorem 4.2. Let $\tau \leq T$ be a bounded stopping time, $|\gamma| \leq 1$, and $p \geq 2$. Suppose that Assumptions 3.1, and 4.1(τ) hold. Then for any $u_0 \in U^\gamma_0$, equation (4.2) with $u(0, \cdot) = u_0$ has a unique solution $u$ in $H^\gamma_0(\tau)$. The solution satisfies\
\[
\|u\|_{H^\gamma_0(\tau)} \leq N(\|g(0)\|_{H^{p−1}_\gamma(\tau)} + \|u_0\|_{U^\gamma_0}),
\]
where $N = N(d, p, T, \kappa_0, K, \gamma)$.

Proof. Note that by [15] Theorem 5.1 the claim holds if $\tau$ is a constant. Otherwise, consider $\tilde{g}(t, u) := 1_{t \leq \tau} g(t, u)$.

Since $\tilde{g}$ satisfies (4.3) for all $t \leq T$, applying [15] Theorem 5.1 leads to $u \in H^{\gamma+1}_0(T)$, which satisfies equation (4.2) and estimate (4.4) with $\tilde{g}$, in place of $g$. Then, one can check that $u \in H^{\gamma+1}_0(\tau)$ and it satisfies equation (4.2). The existence is proved.

To prove the uniqueness, assume that $w \in H^\gamma_0(\tau)$ is a solution to the Cauchy problem (4.2) with initial data $u_0$. From [15] Theorem 5.1, there exists $\tilde{u} \in H^\gamma_0(T)$, the unique solution of equation\
\[
d\tilde{u} = (a^{ij} \tilde{u}_{x^i x^j} + b^i \tilde{u}_x + c \tilde{u}) dt + \sum_{k=1}^{\infty} \tilde{g}^k(w) dw^k_t, \quad t > 0; \quad \tilde{u}(0, \cdot) = u_0.
\]

Then $v := w - \tilde{u}$ satisfies\
\[
dv = (a^{ij} v_{x^i x^j} + b^i v_x + cv) dt, \quad 0 < t \leq \tau; \quad v(0, \cdot) = 0.
\]

Therefore, $\tilde{u} = w$ by the uniqueness of a solution to deterministic parabolic equations (e.g. [18] Theorem 5.1). Consequently, $\tilde{g}(w)$ in (4.5) can be replaced with $\tilde{g}(\tilde{u})$. From the uniqueness result in $H^\gamma_0(T)$, $u = \tilde{u}$ in $H^\gamma_0(\tau)$. The theorem is proved.

Lemma 4.3 describes behavior of $R^\gamma_{1−\gamma} \ast R^\gamma_1$.

Lemma 4.3. Let $g$ and $h$ be nonnegative, radial and decreasing functions. Assume that $R(x) \in L_1(\mathbb{R}^d)$ satisfies\
\[
R(x) = O(g(x)) \quad \text{as} \quad |x| \to \infty,
\]
and\
\[
|R(x)| \leq Nh(x) \quad (a.e.),
\]
where $N$ is independent of $x$. Then\
\[
(R \ast R)(x) = O(g(x/3)) \quad \text{as} \quad |x| \to \infty,
\]
and
\[ |(R * R)(x)| \leq N'h(x/2) \quad (a.e.), \tag{4.8} \]
where \( N' \) is independent of \( x \).

Furthermore, if \( h(x) = |x|^{-\alpha} \) for some \( 0 < \alpha < d/2 \), then \( R * R \in L_\infty(\mathbb{R}^d) \).

**Proof.** Since \( R(x) = O(g(x)) \) as \( |x| \to \infty \), there exist \( c, M > 0 \) such that \( |R(x)| \leq Mg(x) \) for all \( |x| > c \). Then for \( |x| > 3c \), we have
\[
|R * R(x)| \leq \int_{|y| > |x|/2} |R(x - y)R(y)|dy + \int_{|y| \leq |x|/2} |R(x - y)R(y)|dy
\leq M \left( \int_{|y| > |x|/2} |R(x - y)|g(y)dy + \int_{|y| \leq |x|/2} |R(x - y)|g(x - y)dy \right)
\leq Mg(x/3) \left( \int_{|y| > |x|/2} |R(x - y)|dy + \int_{|y| > |x|/2} |R(y)|dy \right)
\leq 2M\|R\|_{L_1} g(x/3),
\]
which implies (4.7).

To prove (4.8), observe that
\[
|R * R(x)| \leq \int_{|y| < |x|/2} |R(x - y)R(y)|dy + \int_{|y| \geq |x|/2} |R(x - y)R(y)|dy
\leq N \left( \int_{|y| > |x|/2} |R(y)|h(x - y)dy + \int_{|y| \geq |x|/2} |R(x - y)|h(y)dy \right)
\leq Nh(x/2) \left( \int_{|y| > |x|/2} |R(y)|dy + \int_{|y| \geq |x|/2} |R(x - y)|dy \right)
\leq 2N\|R\|_{L_1} h(x/2),
\]
which implies (4.8).

For the last assertion, note that \( R \in L_2(\mathbb{R}^d) \) if \( 0 < \alpha < d/2 \), and apply Hölder’s inequality. The lemma is proved. \( \square \)

Employing Lemma 4.3 yields Lemma 4.4 that describes relations between diffusion coefficients and solutions to equation (4.1). Recall that for \( f \in S'_n \), there exists the corresponding measure \( \mu \in M_T \) such that
\[
(f, \phi) = \int_{\mathbb{R}^d} \phi(x)\mu(dx) \quad \text{for any} \quad \phi \in L_1(\mu),
\]
and
\[
\int_{\mathbb{R}^d} \frac{1}{(1 + |x|^2)^k/2} \mu(dx) < \infty
\]
for some \( k \in [0, \infty) \); see Theorem 2.2.
Lemma 4.4. (a) Let \( f \in S'_n \cap S'_{nd} \). Assume that \( d, \gamma, \) and \( s \) satisfy one of the following conditions:

(i) \( d = 1, \gamma \in [0, 1/2), \) and \( s \in \left(\frac{1}{1-2\gamma}, \infty\right] \);

(ii) \( d \geq 2, \gamma \in [0, 1), \) and \( s \in \left(\frac{d}{1-\gamma}, \infty\right] \), and

\[
\int_{|x|<1} |x|^{\frac{1-\gamma-d}{s-1}} \mu(dx) < \infty \quad \left( (1-\gamma-d)\frac{\infty}{\infty} := 1-\gamma-d \right), \tag{4.9}
\]

where \( \mu \) is the corresponding measure of \( f \).

Take \( r, p < \infty \) such that

\[
\frac{1}{s} + \frac{1}{r} = 1 \quad \left( 1_\infty := 0 \right) \quad \text{and} \quad p \geq 2r \geq 2.
\]

If \( \xi \in L_{2s}(\mathbb{R}^d) \) and \( u \in L_p(\mathbb{R}^d) \), then

\[
\|\xi uv\|_{H^s_p(t_2)} \leq A^{1/2s}I^{1/2r}\|\xi\|_{L_{2s}}\|u\|_{L_p},
\]

where \( v = (v_1, v_2, \ldots), v_i(x) = (f \ast e_i)(x), \)

\[
A = \int_{\mathbb{R}^d} \frac{1}{1 + |x|^2} \mu(dx), \quad \text{and} \quad I = \int_{\mathbb{R}^d} (R_{1-\gamma}^r \ast R_{1-\gamma}^r)(x)(1 + |x|^2)^{(r-1)/2} \mu(dx).
\]

(b) Let \( f \in S'_n \cap C_{nd} \). Take \( s \in (1, \infty) \) and \( \gamma \in [0, 1) \) such that

\[
\frac{d}{1-\gamma} < s.
\]

If \( \xi \in L_s(\mathbb{R}^d) \) and \( u \in L_p(\mathbb{R}^d) \), then

\[
\|\xi uv\|_{H^s_p(t_2)} \leq N\|\xi\|_{L_s}\|R_{\gamma}\|_{L_r}\|u\|_{L_p},
\]

where \( N = \sup_{x \in \mathbb{R}^d} |f(x)|. \)

Proof. Consider \( s < \infty \), since the case \( s = \infty \) can be handled similarly. By Fubini’s theorem and Parseval’s identity,

\[
\|(1 - \Delta)^{-\frac{1-\gamma}{2}}(\xi uv)(x)\|^2_{L_2} = \sum_{k=1}^{\infty} \|(1 - \Delta)^{-\frac{1-\gamma}{2}}(\xi uv_k)(x)\|^2
\]

\[
= \sum_{k=1}^{\infty} |R_{1-\gamma} \ast (\xi u(f \ast e_k))(x)|^2
\]

\[
= \sum_{k=1}^{\infty} |(R_{1-\gamma}(x-\cdot)\xi u, f \ast e_k)|^2 \quad \tag{4.10}
\]

\[
= \sum_{k=1}^{\infty} |(f, (R_{1-\gamma}(x-\cdot)\xi u) \ast \tilde{e}_k)|^2
\]

\[
= \sum_{k=1}^{\infty} \langle R_{1-\gamma}(x-\cdot)\xi u, e_k \rangle_H^2 = \|R_{1-\gamma}(x-\cdot)\xi u\|_H^2.
\]

Observe that the last term of (4.10) equals to

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} R_{1-\gamma}(x-(y-z))R_{1-\gamma}(x-z)u(y-z)\check{u}(z)\xi(y-z)\check{\xi}(z)dzd\mu(dy). \tag{4.11}
\]
Proof of (a). From (4.10) and (4.11), applying Hölder’s inequality and change of variables lead to
\[
| (1 - \Delta)^{-\frac{1}{2}} (\xi uv)(x) |^2_{l_2} 
\leq (J_r(x))^{1/r} \times \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |\xi(y - z)\xi(-z)|^s (1 + |y|^2)^{-k/2} dz \mu(dy) \right)^{1/s},
\]
which implies that
\[
\| \xi \|_{L_2}^2 A^{1/s}(J_r(x))^{1/r},
\]
where
\[
J_r(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} | R_{1-\gamma}(y - z) R_{1-\gamma}(z) |^r |u(y - z + x)u(x - z)|^r (1 + |y|^2)^{(r-1)/2} dz \mu(dy).
\]
Hence,
\[
\| \xi uv \|^p_{H^\gamma_{p-1}(l_2)} = \int_{\mathbb{R}^d} | (1 - \Delta)^{-\frac{1}{2}} (\xi uv)(x) |^p_{l_2} dx \leq \| \xi \|_{L_2}^p A^{p/2s} \int_{\mathbb{R}^d} (J_r(x))^{p/2r} dx.
\]
Since \( p \geq 2r \), by Minkowski’s inequality,
\[
\int_{\mathbb{R}^d} (J_r(x))^{p/2r} dx \leq \| u \|_{L_p}^p I^{p/2r},
\]
where
\[
I = \int_{\mathbb{R}^d} (R_{1-\gamma}^r \ast R_{1-\gamma}^r)(x) (1 + |x|^2)^{(r-1)/2} \mu(dx).
\]
It remains to show that \( I < \infty \). Since \( R_{1-\gamma}^r(x) \) exponentially decays as \( |x| \to \infty \), \( R_{1-\gamma}^r \ast R_{1-\gamma}^r \) decreases exponentially by Lemma 4.3. Therefore, it is enough to show that
\[
R_{1-\gamma}^r \ast R_{1-\gamma}^r(x)
\]
is \( \mu \)-integrable near zero. In the case of (i), observe that \( R_{1-\gamma} \in L_{2r}(\mathbb{R}) \). Indeed, \( R_{1-\gamma} \) has exponential decay as \( |x| \to \infty \), and
\[
\int_{|x|<1} | R_{1-\gamma}(x) |^{2r} dx \leq N(r, \gamma) \int_{|x|<1} |x|^{-2r\gamma} dx < \infty
\]
by the choice of constants. Therefore, by Hölder’s inequality, we have
\[
\sup_{x \in \mathbb{R}} | R_{1-\gamma}^r \ast R_{1-\gamma}^r(x) | \leq \| R_{1-\gamma}^r \|_{L_{2r}(\mathbb{R})}^2,
\]
which implies that \( R_{1-\gamma}^r \ast R_{1-\gamma}^r \) is \( \mu \)-integrable near zero.

In the case of (ii), by Lemma 4.3, \( R_{1-\gamma} \ast R_{1-\gamma}^r(x) \) decays exponentially as \( |x| \to \infty \). In addition, \( R_{1-\gamma}^r \ast R_{1-\gamma}^r(x) \) is bounded by \( |x|^r(1-\gamma-d) = |x|^{\frac{1}{r(1-\gamma-d)}} \) near zero. By (4.9), we have
\[
\int_{|x|<1} (R_{1-\gamma}^r \ast R_{1-\gamma}^r)(x)(1 + |x|^2)^{(r-1)/2} \mu(dx)
\leq N(d, k, r, \gamma) \int_{|x|<1} |x|^{\frac{1}{r(1-\gamma-d)}} \mu(dx) < \infty,
\]
which implies that \( R_{1-\gamma}^r \ast R_{1-\gamma}^r \) is \( \mu \)-integrable near zero.
Proof of (b). By Remark 2.3 $f$ is a bounded function. By boundedness of $f$, applying H"older’s inequality and Young’s inequality leads to
\begin{equation}
(1 - \Delta)^{-\frac{(1-\gamma)}{2s}} (\xi u v)(x)_{L_2}^2 \leq N \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |\xi(y-z)\xi(-z)|^s dz dy \right)^{1/s} \times (J'_r(x))^{1/r} \tag{4.13}
\end{equation}
where
\begin{equation}
J'_r(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |R_{1-\gamma}(y-z)R_{1-\gamma}(z)|^r |u(y-z+x)u(x-z)|^r dz dy
\end{equation}
and $N = \sup_{x \in \mathbb{R}^d} |f(x)|$. Therefore, by (4.13)
\begin{equation}
\|\xi u v\|^p_{H^\gamma_p L^{r-1} (t_2)} = \int_{\mathbb{R}^d} (1 - \Delta)^{-\frac{(1-\gamma)}{2s}} (\xi u v)(x)_{L_2}^p dx \leq N^{p/2} \|\xi\|^p_{L_\infty} \int_{\mathbb{R}^d} (J'_r(x))^{p/2r} dx.
\end{equation}
Since $p \geq 2r$, by Minkowski’s inequality and Young’s inequality, we have
\begin{equation}
\int_{\mathbb{R}^d} (J'_r(x))^{p/2r} dx \leq \|u\|^p_{L_p} \|R_{1-\gamma}\|^p_{L_r}.
\end{equation}
The lemma is proved. \qed

By employing Theorem 4.2 and Lemma 4.4, we can prove Theorem 3.5.

**Proof of Theorem 3.5.** To prove the first assertion, it suffices to show that the assumptions in Theorem 4.2 hold. In other words, we only need to prove that for any $\varepsilon > 0$, there exists a constant $N_\varepsilon > 0$ such that
\begin{equation}
\| (h(u) - h(w)) \xi v \|_{H^{\gamma - 1}_p (t_2)} \leq \varepsilon \| u - w \|_{H^\gamma_p} + N_\varepsilon \| u - w \|_{H^\gamma_p}
\end{equation}
for all $\omega \in \Omega$, $t \leq \tau$ and $u, w \in H^\gamma_p$.

By Lemma 4.4, we have
\begin{equation}
\| (h(u) - h(w)) \xi v \|_{H^{\gamma - 1}_p (t_2)} \leq N \| u - w \|_{L_p} \quad \text{and} \quad \| \xi h(0) v \|_{H^{\gamma - 1}_p (\tau)} \leq N \| h(0) \|_{L_p (\tau)},
\end{equation}
where $N$ is independent of $u, w$ and $h(0)$. Then applying Theorem 2.10 (iii) and Young’s inequality leads to (4.14). The theorem is proved. \qed

5. **Proof for Case with the Super-linear Diffusion Coefficient**

This section provides a proof of Theorem 3.9 which describes unique solvability and $L_p$-regularity of equation (5.1). For readers’ convenience, we recall equation (5.1):
\begin{equation}
du = (a^2 u + b^2 u + cu) dt + \sum_{k=1}^{\infty} \xi |u|^{1+\lambda} f u_k dt, \quad (t, x) \in [0, \infty) \times \mathbb{R}^d; \quad u(0, \cdot) = u_0(\cdot),
\end{equation}
where $\lambda \geq 0$. Theorem 5.1, Lemma 5.3, and Lemma 5.6 are used to prove Theorem 3.9. Based on investigation described in Section 4, Lemma 5.3 suggests a local solution to equation (5.1). Then a global solution is constructed from the local solution with the help of Lemma 5.6, which proves the existence and regularity of a solution for (5.1). In the construction of the global solution, Theorem 5.1 is employed to show that $\| u(t, \cdot) \|_{L_1(\mathbb{R}^d)}$ is a continuous local martingale. Detailed proof for Theorem 3.9 including the uniqueness of a
solution is described at the end of this section.

**Theorem 5.1** (Maximum principle). Let \( \tau \leq T \) be a bounded stopping time. Assume that all the conditions of Theorem 3.5 are satisfied with \( s = \infty \) and \( h(u) = u \). If \( u \in H^p_{\tau}(\tau) \) is the solution described in Theorem 3.5, then \( u(t, \cdot) \) is nonnegative for all \( t \leq \tau \) almost surely.

**Proof.** Although similar to the proof of \([11, \text{Theorem 4.2}]\), the proof of Theorem 5.1 is given in Appendix A for the sake of completeness. \( \square \)

Lemma 5.3 is used to construct a sequence of functions that approximate a solution to equation (5.1).

**Assumption 5.2** (\( \tau \)). The coefficient \( \xi \) is \( \mathcal{P} \times \mathcal{B}(\mathbb{R}^d) \)-measurable. Also, there exists a constant \( K > 0 \) such that

\[
\|\xi(t, \cdot)\|_{L^\infty} \leq K \quad \forall \omega \in \Omega, \forall t \leq \tau. \tag{5.2}
\]

**Lemma 5.3.** Let \( \tau \leq T \) be a bounded stopping time, \( f \in S'_n \cap S'_{nd} \) and \( m \in \mathbb{N} \). Suppose Assumptions 3.1 and 5.2(\( \tau \)) hold. Assume that \( d, \lambda, \gamma, p \) and \( f \) satisfy one of the conditions (i)-(iii) in Theorem 3.5. Assume \( u_0 \) is in \( U^\tau_p \) and \( u_0 \) is nonnegative. Consider

\[
du = (a^{ij}u_{x^i,x^j} + b^i u_{x^i} + cu)dt + \sum_{k=1}^{\infty} \xi_k(( -m ) \vee u \wedge m)^{1+\lambda}v_k dw^k_t, \quad 0 < t \leq \tau; \quad u(0, \cdot) = u_0. \tag{5.3}
\]

Then, equation (5.3) with initial data \( u(0, \cdot) = u_0 \) has a unique solution \( u_m \in H^p_{\tau}(\tau) \) and \( u_m \) is nonnegative. Moreover, if Assumption 5.2(\( \tau \)) is replaced with Assumption 5.2(\( \tau, \infty \)), then

\[
u_m \in H^p_{\tau}(T')
\]

for any \( T' < \infty \).

**Proof.** By the mean-value theorem,

\[
|(-m) \vee u \wedge m|^{1+\lambda} - |(-m) \vee w \wedge m|^{1+\lambda} \leq (1 + \lambda)2m^\lambda|u - w|.
\]

Thus, \( \xi \) and \( h(u) = (-m) \vee u \wedge m \) satisfy Assumptions 3.2(\( \tau, \infty \)) and 3.3(\( \tau, p \)), respectively. In the virtue of Theorem 3.5, equation (5.3) has a unique solution \( u_m \in H^p_{\tau}(\tau) \).

To see the nonnegativity of \( u_m \), define

\[
\xi_m(t, x) := \xi(t, x)(( -m ) \vee u_m(t, x) \wedge m)^{1+\lambda} u_m(t, x) \quad (0 := 0). \tag{5.5}
\]

Note that \( \xi_m \) is bounded, i.e., \( \sup_{\omega \in \Omega} \sup_{t \leq \tau} \|\xi_m(t, \cdot)\|_{L^\infty} < \infty \). Observe that \( u_m \) satisfies

\[
dw = Lw \, dt + \sum_{k=1}^{\infty} \xi_m w v_k dw^k_t, \quad 0 < t \leq \tau; \quad w(0, \cdot) = u_0, \tag{5.6}
\]

where \( Lw = a^{ij}w_{x^i,x^j} + b^i w_{x^i} + cw \). Therefore, by Theorem 5.1 we conclude that \( u_m \) is nonnegative.

To prove the last assertion, notice that there exists a unique solution \( \bar{u}_m \in H^p_{\tau}(T') \) for \( T' \geq \tau \), by the first assertion. From the uniqueness of a solution in \( H^p_{\tau}(\tau) \), we have \( \bar{u}_m = u_m \in H^p_{\tau}(\tau) \). The lemma is proved. \( \square \)
Remark 5.4. Suppose Assumption 5.2(∞) holds. Let \( u_m \) be the solution of equation (5.3). Then, for fixed \( \phi \in C_c^\infty \),

\[
M_t := \sum_{k=1}^{\infty} \int_0^t (\xi |u_m \land m|^{1+\lambda} v_k, \phi) du_t^k
\]
is a square integrable martingale. Indeed,

\[
\sum_{k=1}^{\infty} \mathbb{E} \int_0^t (\xi |u_m \land m|^{1+\lambda} v_k, \phi)^2 dt \leq N \sum_{k=1}^{\infty} \int_0^t (v_k, \phi)^2 dt
\]

\[
= tN \sum_{k=1}^{\infty} (f * e_k, \phi)^2 = tN \| \phi \| H_t^2 < \infty,
\]

where \( N = N(K', m, \lambda) \).

Note that the solution \( u_m \) from Lemma 5.3 satisfies equation (5.1) in \( H_{\gamma p}(\tau_m) \), where

\[
\tau_m := \inf \{ t \geq 0 : \sup_x |u_m(t, x)| \geq m \}.
\]

In the proof of Theorem 3.9, Lemma 5.6 is used to show the divergence of stopping times. Lemma 5.5 is applied to prove Lemma 5.6.

Lemma 5.5. For \( k = 1, 2, \cdots \), denote

\[
\psi_k(x) := \frac{1}{\cosh(|x|/k)}.
\]

Suppose Assumption 3.1 holds and let \( K \) is the constant described in Assumption 3.1. Then for all \( \omega, x, t, \)

\[
(a^{ij} \psi_k)_{x^i x^j} - (b^i \psi_k)_{x^i} + (c - 4K) \psi_k
\]

\[
= a^{ij} \psi_{k x^i x^j} + (2a^{ij} - b^i) \psi_{k x^i} + (a^{ij} - b^i + c - 4K) \psi_k \leq 0. \tag{5.7}
\]

Proof. Observe that

\[
\psi_{k x^i}(x) = -\psi_k(x) \frac{x^i \tanh(|x|/k)}{k|x|}
\]

and

\[
\psi_{k x^i x^j}(x) = \psi_k(x) \frac{x^i x^j}{k^2 |x|^2} \tanh^2(|x|/k) - \psi_k(x) \frac{x^i x^j}{k^2 |x|^2 \cosh^2(|x|/k)}
\]

\[
+ \psi_k(x) \left( \frac{x^i x^j}{|x|^3} - \frac{\delta_{ij}}{|x|^2} \right) \frac{\tanh(|x|/k)}{k},
\]
where $\delta_{ij}$ is the Kronecker delta. Condition (3.3) and the fact that $\tanh(|x|) \leq |x| \wedge 1$ lead to

$$a_{ij}^{i}w_{kx^i x^j}(x) = \psi_k(x)\frac{a_{ij}^{i}x^i x^j}{k^2|x|^2} \tanh^2(|x|/k) - \psi_k(x)\frac{a_{ij}^{i}x^i x^j}{k^2|x|^2 \cosh^2(|x|/k)}$$

$$+ \psi_k(x)\left(\frac{a_{ij}^{i}x^i x^j}{|x|^3} - \frac{a_{ii}^i}{|x|}\right) \tanh(|x|/k)$$

$$\leq \psi_k(x)\frac{K}{k^2} \tanh(|x|/k) - \psi_k(x)\frac{\kappa_0}{k^2 \cosh^2(|x|/k)} + \psi_k(x)\frac{(K - d\kappa_0) \tanh(|x|/k)}{k|x|}$$

$$\leq \frac{K\psi_k(x)}{k^2} \tanh(|x|/k) + \frac{K\psi_k(x)}{k^2}$$

$$\leq \frac{2K\psi_k(x)}{k^2}.$$

and

$$(2a_{ij}^{i} - b^j)w_{kx^i}(x) = -(2a_{ij}^{i} - b^j)\psi_k(x)\frac{x^i \tanh(|x|/k)}{k|x|}$$

$$\leq \frac{(2|a_{ij}^{i}| + |b^j|)}{k} \psi_k(x).$$

Thus by assumption (3.4),

$$a_{ij}^{i}w_{kx^i x^j}(x) + (2a_{ij}^{i} - b^j)w_{kx^i}(x) + (a_{ij}^{i}x^i x^j - b_{x^i}^j + c - 4K)\psi_k(x)$$

$$\leq \psi_k(x)\left[\frac{2K}{k^2} + \frac{2|a_{ij}^{i}| + |b^j|}{k} + |a_{ij}^{i}x^i x^j| + |b_{x^i}^j| + |c| - 4K\right] \leq 0.$$

The lemma is proved. 

\[\square\]

**Lemma 5.6.** Suppose the conditions of Theorem 3.9 are satisfied. Let $u_m$ be the solution described in Lemma 5.5. Then for any $T < \infty$,

$$\lim_{R \to \infty} \sup_{m} \mathbb{P}\left(\sup_{t \leq T \in \mathbb{R}} |u_m(t, x)| \geq R \right) = 0. \quad (5.8)$$

**Proof.** Let $K$ be the constant described in assumption (3.3). By Itô’s formula, $w := u_m e^{-4Kt}$ satisfies

$$dw = (Lw - 4Kw)dt + \sum_{k=1}^{\infty} \xi e^{-4Kt}|u_m \wedge m|^{1+\lambda} v_k dw_k^k, \quad 0 < t \leq \tau; \quad w(0, \cdot) = u_0,$$

where $Lw = a_{ij}^{i}w + b^jw + cw$. Let $\psi_k$ be from Lemma 5.5. By multiplying $\psi_k$ (see Remark 2.14), using integration by parts, taking expectation, and applying inequality (5.7), we have the following: for any stopping time $\tau \leq T$,

$$\mathbb{E}\left[(u_m(\tau)e^{-4K\tau}, \psi_k)\right] \leq \mathbb{E}\left[(u_m(\tau)e^{-4K\tau}, \psi_k)\right]$$

$$= \mathbb{E}\left[(u_0, \psi_k)\right] + \mathbb{E}\left[\int_{0}^{\tau} (u_m(t), (a_{ij}^{i}\psi_k)_{x^i x^j} - (b^j\psi_k)_{x^i} + (c - 4K)\psi_k)e^{-4Kt}dt\right]$$

$$\leq \mathbb{E}\left[(u_0, \psi_k)\right] \leq \mathbb{E}\left[||u_0||_{L_1}\right] = ||u_0||_{L_1(\Omega; L_1)}. $$
Then for any $\gamma \in (0, 1)$ (e.g. [14, Theorem III.6.8]),
\[
E \sup_{t \leq T} \left( \int_{\mathbb{R}^d} u_m(t, x) \psi_k(x) dx \right)^\gamma \leq e^{4\gamma KT} \frac{2 - \gamma}{1 - \gamma} \| u_0 \|_{L^1(\Omega : L_1)}^\gamma.
\]
By the monotone convergence theorem,
\[
E \sup_{t \leq T} \| u_m(t, \cdot) \|_{L_1}^{1/2} \leq 3e^{2KT} \| u_0 \|_{L^1(\Omega : L_1)}^{1/2} =: N_1.
\]
(5.9)
Thus, Chebyshev’s inequality and inequality (5.9) yield
\[
P \left( \sup_{t \leq T} \| u_m(t, \cdot) \|_{L_1} \geq S \right) \leq \frac{N_1}{\sqrt{S}},
\]
where $N_1$ is independent of $m$ and $S$. Fix $m$, $S > 0$ and define
\[
\tau_m(S) := \inf \{ t \geq 0 : \| u_m(t, \cdot) \|_{L_1} \geq S \}.
\]
(5.10)
Note that $\tau_m(S)$ is a stopping time by inequality (5.9). Besides, for any $T < \infty$, $u_m$ satisfies
\[
\begin{align*}
dw &= Lw \, dt + \sum_{k=1}^{\infty} \xi_m wv_k \, dw_k^k, \quad 0 < t \leq \tau_m(S) \wedge T; \quad w(0, \cdot) = u_0, \quad (5.11)
\end{align*}
\]
where $\xi_m$ is defined by (5.5). Due to the definition of $\xi_m$, we can apply Theorem 3.5 with equation (5.11). Indeed, if either condition (i) or (ii) in Theorem 3.5 holds,
\[
\begin{align*}
p > \frac{d + 2}{\gamma} > \frac{2}{1 - 2\lambda} = \frac{2s}{s - 1},
\end{align*}
\]
for $s := \frac{1}{2\lambda}$. Thus for $t \leq \tau_m(S)$,
\[
\| \xi_m(t, \cdot) \|_{L^{2s}}^{2s} \leq \int_{\mathbb{R}^d} |\xi(t, x)|^{2s} |u_m(t, x)|^{2\lambda s} dx \leq K^{2s} S.
\]
Hence, in each case, one can apply Theorem 3.5 with $h(u) = u$ since $\xi_m$ satisfy Assumption 3.2 ($\tau, 2s$).

On the other hand, if condition (iii) holds,
\[
\begin{align*}
p > \frac{d + 2}{\gamma} > \frac{2}{1 - \lambda} = \frac{2s}{s - 1},
\end{align*}
\]
where $s := \frac{1}{\lambda}$. Then for $t \leq \tau_m(S)$,
\[
\| \xi_m(t, \cdot) \|_{L^s}^{s} \leq \int_{\mathbb{R}^d} |\xi(t, x)|^{s} |u_m(t, x)|^{\lambda s} dx \leq K^{s} S.
\]
Thus, applying Theorem 3.5 with $h(u) = u$ and $\xi_m$ gives
\[
\| u_m \|_{\mathcal{H}^p_{d}(\tau_m(S) \wedge T)}^{p} \leq N_S \| u_0 \|_{U^p}^{p},
\]
(5.12)
where $N_S$ is independent of $m$. The estimate (5.12) and Corollary 2.17 lead to
\[
E \sup_{t \leq \tau_m(S) \wedge T} \sup_{x \in \mathbb{R}^d} | u_m(t, x) |^{p} \leq N_S \| u_0 \|_{U^p}^{p}.
\]
Therefore, by Chebyshev’s inequality,
\[ \mathbb{P}\left( \sup_{t \leq T} \sup_{x \in \mathbb{R}^d} |u_m(t, x)| \geq R \right) \]
\[ \leq \mathbb{P}\left( \sup_{t \leq \tau_m(S) \wedge T} \sup_{x \in \mathbb{R}^d} |u_m(t, x)| \geq R \right) + \mathbb{P}(\tau_m(S) \leq T) \]
\[ \leq \frac{N_S}{R} \|u_0\|_{L^p_{1-\gamma}}^p + \mathbb{P}\left( \sup_{t \leq T} \|u_m(t, \cdot)\|_{L^1} \geq S \right) \]
\[ \leq \frac{N_S}{R} \|u_0\|_{L^p_{1-\gamma}}^p + \frac{N_1}{\sqrt{S}}. \]

Observe that \( N_1 \) is independent of \( m, R \) ans \( S \). Taking supremum with respect to \( m \) and letting \( R \to \infty \) and \( S \to \infty \) in order yield the final result \([5.8]\). The lemma is proved. \( \square \)

**Remark 5.7.** In Theorem \([5.9]\), each range of \( \lambda \) is a sufficient condition for the unique solvability. To obtain motivation for the condition, assume that the covariance \( f \) is in \( S'_n \cap S'_{nd} \) and \( u \) is a solution to \([5.1]\) with smooth initial data \( u_0 \). Then for \( t \leq \tau_m(s) \) (see \([5.10]\)), \( u \) satisfies (at least formally)

\[ dv = (a^{ij}v_{x^i x^j} + b^i v_{x^i} + cv)dt + \xi(f \ast e_k) \frac{|u|^{1+\lambda}}{u} |dv|dw_t \]

\( (t, x) \in (0, \infty) \times \mathbb{R}^d, \quad \) (5.13)

with \( u(0, \cdot) = u_0 \) \( (0 := 0) \). To handle equation \((5.13)\), we consider Theorem \([5.5]\). Since \( \sup_{t \leq T} \|u(t, \cdot)\|_{L^1(\mathbb{R}^d)} \leq S \) and the coefficient \( \xi(f \ast e_k) \frac{|u|^{1+\lambda}}{u} \) have to satisfy Assumption \([3.2]\) \((\tau_m(S), 2s)\), we have

\[ \lambda = \frac{1}{2s}. \]  \( (5.14) \)

Thus, we consider such \( \lambda \) satisfying \([5.14]\). In the case of \( f \in S'_{nd} \cap C_{nd} \), equation \((5.14)\) will be replaced with \( \lambda = \frac{1}{2s} \).

By employing Lemmas \([5.3]\) and \([5.6]\) we prove Theorem \([3.9]\).

**Proof of Theorem \([3.9]\).**

**Step 1. Uniqueness.** Suppose \( u, v \in \mathcal{H}_{p,loc}^\gamma \) are solutions of equation \((5.1)\). By the definition of \( \mathcal{H}_{p,loc}^\gamma \), there are bounded stopping times \( \tau_m, m = 1, 2, \cdots \) such that

\[ \tau_m \uparrow \infty \quad \text{and} \quad u, v \in \mathcal{H}_{p,loc}^\gamma(\tau_m). \]

For fixed \( m \), denote \( \bar{\tau} = \tau_m \). By Corollary \([2.17]\), there exists small \( \varepsilon > 0 \) such that

\[ \mathbb{E}|u|^p_{C^\varepsilon([0, \bar{\tau}] \times \mathbb{R}^d)} + \mathbb{E}|v|^p_{C^\varepsilon([0, \bar{\tau}] \times \mathbb{R}^d)} < \infty. \]  \( (5.15) \)

For \( n > 0 \), define

\[ \tau'_n := \inf\{t \leq \bar{\tau} : \sup_{x \in \mathbb{R}^d} |u(t, x)| > n\}, \]

\[ \tau''_n := \inf\{t \leq \bar{\tau} : \sup_{x \in \mathbb{R}^d} |v(t, x)| > n\}, \]

and \( \bar{\tau}_n := \tau'_n \wedge \tau''_n \). Due to \((5.15)\), \( \tau'_n \) and \( \tau''_n \) are stopping times, and thus \( \bar{\tau}_n \) is a stopping time. Observe that \( \bar{\tau}_n \uparrow \bar{\tau} \) (a.s.). Now, define \( \xi \) by

\[ \xi := \frac{\xi(t, x)|u|^{1+\lambda}}{u} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0. \]
Notice that
\[ \sup_{\omega \in \Omega} \sup_{t \leq \tilde{\tau}_n} \| \xi \|_\infty < \infty. \]

Besides, \( u \) is a solution to equation
\[ dw = Lw \, dt + \sum_{k=1}^{\infty} \xi w_k dW^k, \quad 0 < t \leq \tilde{\tau}_n; \quad w(0, \cdot) = u_0. \]

By Theorem 5.1, \( u \) is nonnegative for all \( t \leq \tilde{\tau}_n \) (a.s.). Similarly, \( v \) is also nonnegative for all \( t \leq \tilde{\tau}_n \) (a.s.). Therefore,
\[ |(-n) \vee u \land n| = u \quad \text{and} \quad |(-n) \vee v \land n| = v, \]
for all \( t \leq \tilde{\tau}_n \) (a.s.). By the uniqueness result in Lemma 5.3, we conclude that \( u = v \) in \( H^\gamma_p(\tilde{\tau}_n) \) for each \( n \). The monotone convergence theorem yields \( u = v \) in \( H^\gamma_p(\tilde{\tau}) \).

**Step 2. Existence.** By Lemma 5.3 for any \( T < \infty \), there exists nonnegative \( u \in H^\gamma_p(T) \) such that
\[ du = Lu \, dt + \sum_{k=1}^{\infty} \xi |u \land m|^{1+\lambda} v_k \, dw^k, \quad 0 < t \leq \tau_m; \quad u(0, \cdot) = u_0. \tag{5.16} \]

For nonnegative integers \( m, R \) such that \( m \geq R \), define
\[ \tau^R_m := \inf \{ t > 0 : \sup_{x \in \mathbb{R}^d} |u_m(t, x)| \geq R \}. \tag{5.17} \]

Since \( \sup_{x \in \mathbb{R}^d} |u_m(t, x)| \leq R \) for \( t \leq \tau^R_m \), we have \( u_m \land m = u_m \land m \land R = u_m \land R \) for \( t \leq \tau^R_m \).

Thus, both \( u_m \) and \( u_R \) satisfy
\[ du = Lu \, dt + \sum_{k=1}^{\infty} \xi |u \land R|^{1+\lambda} v_k \, dw^k, \quad 0 < t \leq \tau^R_m \land T; \quad u(0, \cdot) = u_0. \]

On the other hand, since \( R \leq m \), \( u_R \land R = u_R \land R \land m = u_R \land m \) for \( t \leq \tau^R_m \). Then \( u_m \) and \( u_R \) satisfy
\[ du = Lu \, dt + \sum_{k=1}^{\infty} \xi |u \land m|^{1+\lambda} v_k \, dw^k, \quad 0 < t \leq \tau^R_m \land T; \quad u(0, \cdot) = u_0. \]

By the uniqueness result in Lemma 5.3, \( u_m = u_R \) in \( H^\gamma_p((\tau^R_m \lor \tau^R_R) \land T) \) for any positive integer \( T \). Therefore, we can conclude that \( \tau^R_R = \tau^R_m \leq \tau^m_m \) (a.s.). Indeed, for \( t < \tau^R_m \),
\[ \sup_{s \leq t} \sup_{x \in \mathbb{R}^d} |u_R(s, x)| = \sup_{s \leq t} \sup_{x \in \mathbb{R}^d} |u_m(s, x)| \leq R, \]
which implies \( \tau^m_m \leq \tau^R_R \). Similarly, we can obtain \( \tau^R_R \geq \tau^R_R \). Now, observe that by Lemma 5.6
\[ \limsup_{m \to \infty} \mathbb{P}(\tau^m_m \leq T) = \limsup_{m \to \infty} \mathbb{P} \left( \sup_{t \leq T, x \in \mathbb{R}^d} |u_m(t, x)| \geq m \right) \]
\[ \leq \limsup_{m \to \infty} \mathbb{P} \left( \sup_{t \leq T, x \in \mathbb{R}^d} |u_n(t, x)| \geq m \right) \to 0, \]
which implies \( \tau^m_m \to \infty \) in probability. Since \( \tau^m_m \) is increasing, we conclude that \( \tau^m_m \uparrow \infty \) (a.s.).
Lastly, define \( \tau_m := \tau_m^n \land m \) and 
\[
    u(t, x) := u_m(t, x) \quad \text{for } t \in [0, \tau_m).
\]
Observe that \(|u(t)| \leq m\) for \(t \leq \tau_m\) and \(u\) satisfies (5.1) for all \(t < \infty\). Since \(u = u_m\) for \(t \leq \tau_m\) and \(u_{m} \in \mathcal{H}^{\gamma}_p(\tau_m)\), it follows that \(u \in \mathcal{H}^{\gamma}_p(\tau_m)\) for any \(m\), and thus \(u \in \mathcal{H}^{\gamma}_{p,loc}\). The theorem is proved.

\[\square\]

**Proof of Theorem 3.15.**
Without loss of generality, we may assume \(q > p\). From the proof of Theorem 3.9, observe that there exists a sequence of bounded stopping times \(\tau_n \uparrow \infty\) (e.g. \(\tau_n^x\) in (5.17)) such that \(u \in \mathcal{H}^{\gamma}_p(\tau_n), u \in \mathcal{H}^{\gamma}_{p,q}(\tau_n)\) are the solutions of equation
\[
    dw = Lw dt + \sum_{k=1}^{\infty} \xi|w \wedge n|^{1+\lambda}v_k dw_t^k, \quad 0 < t \leq \tau_n; \quad w(0, \cdot) = u_0,
\]
where \(Lw = a^{ij}w_{x^i x^j} + b^i w_{x^i} + cw\). Note that
\[
    \sup_{t \leq \tau_n} \sup_{x \in \mathbb{R}^d} |u(t, x)| \leq n \quad (a.s.), \quad (5.18)
\]
which implies
\[
    \int_0^{\tau_n} \int_{\mathbb{R}^d} |u(t, x)|^q dx dt = \int_0^{\tau_n} \int_{\mathbb{R}^d} |u(t, x)|^p |u(t, x)|^{q-p} dx dt \\
    \leq \left( \sup_{t \leq \tau_n} \sup_{x \in \mathbb{R}^d} |u(t, x)| \right)^{q-p} \int_0^{\tau_n} \|u(t, \cdot)\|^p_{L^p} dt \\
    \leq n^{q-p} \int_0^{\tau_n} \|u(t, \cdot)\|^p_{\mathcal{H}^p} dt < \infty \quad (a.s.).
\]
Therefore, we can define a bounded stopping time
\[
    \tau_{n,k} := \tau_n \land \inf \left\{ t > 0 : \int_0^t \|u(t, \cdot)\|^q_{L^q} dt > k \right\}
\]
such that \(\tau_{n,k} \uparrow \tau_n\) as \(k \to \infty\) and \(u, \bar{u} \in \mathbb{L}^q(\tau_{n,k})\) for each \(k \in \mathbb{N}\). By Lemma 4.4 and (5.18),
\[
    \|\xi|u|^{1+\lambda}v\|_{\mathbb{L}^{1+\epsilon}(\tau_{n,k},I_d)} \leq N\|u\|_{\mathbb{L}^{q(1+\lambda)}(\tau_{n,k})} \leq Nn^{\lambda}\|u\|_{\mathbb{L}^{q}(\tau_{n,k})}
\]
for all \(\epsilon \in [0, 1)\). By Definition 2.12 we have \(u \in \mathcal{H}^0_q(\tau_{n,k})\). By Theorem 3.5, equation
\[
    dw = Lw dt + \sum_{k=1}^{\infty} \xi|u|^{1+\lambda}v_k dw_t^k, \quad 0 < t \leq \tau_{n,k}; \quad w(0, \cdot) = u_0.
\]
has a unique solution \(v \in \mathcal{H}^{\epsilon}_q(\tau_{n,k})\). Since \(u - v \in \mathcal{H}^0_q(\tau_{n,k})\) satisfies the equation
\[
    dw = Lw dt, \quad 0 < t \leq \tau_{n,k}; \quad w(0, \cdot) = 0,
\]
by Theorem 4.2 \(u = v\) almost all in \((0, \tau] \times \mathbb{R}^d\). Therefore, \(u \in \mathcal{H}^{\epsilon}_q(\tau_{n,k})\) for all \(\epsilon \in (0, 1)\). In particular, \(u \in \mathcal{H}^{\gamma}_q(\tau_{n,k})\). By the uniqueness result of Theorem 3.5, \(u = \bar{u}\) in \(\mathcal{H}^{\gamma}_q(\tau_{n,k})\).

The theorem is proved. \[\square\]
APPENDIX A. Proof of maximum principle (Theorem 5.1)

Without loss of generality, we may assume \( \tau = T \) (for example, consider \( \xi_1 \leq \tau \) in place of \( \xi \)). Observe that there exists a sequence of nonnegative functions \( u_0^m \in U_1^{1,\tau} \) such that \( u_0^m(\omega, \cdot) \in C^\infty_c \) for each \( \omega \in \Omega \) and \( u_0^m \to u_0 \) in \( U_1^{1,\tau} \). For \( m = 1, 2, \ldots, \), set

\[
\mathbf{v} = (v_1, v_2, \ldots), \quad \mathbf{v}_m = (v_1, v_2, \ldots, v_m, 0, 0, \ldots),
\]

\[
g(u) = \xi u \mathbf{v} = (\xi uv_1, \xi uv_2, \ldots),
\]

\[
g_m(u) = \xi u \mathbf{v}_m = (\xi uv_1, \xi uv_2, \ldots, \xi uv_m, 0, 0, \ldots).
\]

Obviously,

\[
\|g_m(u) - g_m(v)\|_{H_p^{-1}(t_2)} \leq \|g(u) - g(v)\|_{H_p^{-1}(t_2)}. \tag{A.1}
\]

From the proof of Theorem 3.5 one can check that \( g_m \) satisfies Assumption 4.1(4)(T). Therefore, there exists a unique solution \( u_m \in \mathcal{H}_p(T) \) to equation

\[
dw = Lw \, dt + \sum_{k=1}^{\infty} g^k_m(w) \, dw^k_t, \quad 0 < t \leq T; \quad w(0, \cdot) = u_0^m,
\]

where \( Lw = a^{ij}w_{x_i x_j} + b^i w_{x_i} + cw \).

Now, we prove that \( u_m \to u \) in \( \mathcal{H}_p(T) \) and \( u_m \) is nonnegative.

First, we show that \( u_m \to u \) in \( \mathcal{H}_p(T) \). Note that \( w_m := u - u_m \) satisfies

\[
dw = Lw \, dt + \sum_{k=1}^{\infty} (g^k(u) - g^k_m(u_m)) \, dw^k_t, \quad 0 < t \leq T; \quad w(0, \cdot) = u_0 - u_0^m,
\]

and

\[
g(u) - g_m(u_m) = g_m(u) - g_m(u_m) + \xi u(v - v_m).
\]

By Theorem 4.2 with \( \tau = t \), (4.14) and (A.1), for any \( t \leq T \),

\[
\|u - u_m\|_{H_p^0(t)}^p \leq N \|u_0 - u_0^m\|_{U_p^1}^p + N \|g_m(u) - g_m(u_m)\|_{H_p^{-1}(t_2)}^p + N \|\xi u(v - v_m)\|_{H_p^{-1}(t_2)}^p
\]

\[
\leq N \|u_0 - u_0^m\|_{U_p^1}^p + \frac{1}{2} \|u - u_m\|_{H_p^0(t)}^p + N \|u - u_m\|_{H_p^{-1}(t)}^p + N \|\xi u(v - v_m)\|_{H_p^{-1}(t_2)}^p.
\]

Therefore,

\[
\|u - u_m\|_{H_p^0(t)}^p \leq N \|u_0 - u_0^m\|_{U_p^1}^p + N \|u - u_m\|_{H_p^{-1}(t)}^p + N \|\xi u(v - v_m)\|_{H_p^{-1}(t_2)}^p,
\]

for any \( t \leq T \). By Gronwall’s inequality and Theorem 2.16 (1X),

\[
\|u - u_m\|_{\mathcal{H}_p(T)}^p \leq N \|u_0 - u_0^m\|_{U_p^1}^p + N \|\xi u(v - v_m)\|_{H_p^{-1}(T_2)}^p, \tag{A.2}
\]

where \( N \) is independent of \( m \) and \( u \). Since \( \|u_0 - u_0^m\|_{U_p^1} \to 0 \) as \( m \to \infty \), it is enough to show that \( \|\xi u(v - v_m)\|_{H_p^{-1}(T_2)}^p \) goes to zero as \( m \to \infty \). Note that

\[
(1 - \Delta)^{-1/2}(\xi u(v - v_m))(t, x)^2_{L_2} = \sum_{k=m+1}^{\infty} \langle R_{1-\gamma}(x - \cdot)(\xi u(t), e_k)^2_{H_p} \rangle_{L_2}.
\]

By the proof of Lemma 4.4 we have

\[
\|\xi(t)(u(t)(v - v_m))\|_{H_p^{-1}(t_2)} \leq N \|u(t)\|_{H_p^0},
\]

where \( N \) is independent of \( m \).
Therefore, by the dominated convergence theorem,
\[
\|\xi(t)u(t)(v - v_m)\|_{H^{-1}(l_2)} \to 0
\]
as \(m \to \infty\). Again, by the dominated convergence theorem,
\[
\|\xi u(v - v_m)\|_{H^{-1}(T,l_2)} \to 0 \quad \text{as} \quad m \to \infty.
\]

Next, we prove that \(u_m\) is nonnegative. Observe that \(u_m \in H^1_p(T) \subseteq L^p(T)\). Since \(\xi\) and \(v_k\) are bounded, we conclude that \(g_m(u_m) := \xi u_m v_k \in L^p(T,l_2)\).

Since \(u_0^m \in U^1_p\), by Theorem 4.2 there exists a unique solution \(\bar{u}_m \in H^1_p(T)\) to
\[
dw = Lwdt + \sum_{k=1}^{m} g^k_m(u_m)dw^k_t, \quad 0 < t \leq T; \quad w(0, \cdot) = u_0^m.
\]

Observe that \(u_m\) and \(\bar{u}_m\) are in \(H^1_p(T)\) and solutions of equation (A.4). By the uniqueness in \(H^1_p(T)\), we have \(u_m = \bar{u}_m \in H^1_p(T)\). Thus, \(u_m\) is in \(H^1_p(T)\) and \(u_m\) satisfies
\[
dw = Lwdt + \sum_{k \leq m} \xi w_k dw^k_t, \quad t \leq T; \quad w(0, \cdot) = u_0^m.
\]

Since \(\xi v_k\) is bounded for each \(k = 1, 2, \ldots\), by maximum principle (e.g. [16, Theorem 1.1]), we conclude that \(u_m(t)\) is nonnegative for all \(t \leq T\) almost surely. The theorem is proved. \(\square\)

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