RIGIDITY OF CONES WITH BOUNDED RICCI CURVATURE

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Abstract. We show that the only metric measure space with the structure of an $N$-cone and with two-sided synthetic Ricci bounds is the Euclidean space $\mathbb{R}^{N+1}$ for $N$ integer. This is based on a novel notion of Ricci curvature upper bounds for metric measure spaces given in terms of the short time asymptotic of the heat kernel in the $L^2$-transport distance.

Moreover, we establish a beautiful rigidity results of independent interest which characterize the $N$-dimensional standard sphere $\mathbb{S}^N$ as the unique minimizer of

$$\int_X \int_X \cos d(x, y) \, m(dy) \, m(dx)$$

among all metric measure spaces with dimension bounded above by $N$ and Ricci curvature bounded below by $N - 1$.

1. Introduction

The theory of synthetic curvature-dimension bounds for non-smooth space has been very active and successful in the last decades. It was initiated in the works of Bakry–Émery [5] from the point of view of abstract Markov semigroups and Lott–Villani [17] and Sturm [20] from the point of view of optimal transport and metric measure space. Generalized lower bounds on the Ricci curvature and upper bounds on the dimension lead to a large number of geometric and functional inequalities and powerful control on the underlying diffusion process. By now, many precise analytic and geometric results for metric measure spaces under curvature-dimension bounds have been established such as Li–Yau type estimates for heat semigroup [10] and splitting and rigidity results [11, 15] and a clear picture of the fine structure of such spaces is emerging [18]. Recently, significant progress has been made in developing more detailed synthetic control on the Ricci curvature in a non-smooth context. Gigli [12] and Han [14] provide a definition of the full Ricci tensor on metric measure spaces, building upon a similar contruction in the context of $\Gamma$-calculus by Sturm [21]. Naber [19] characterized two-sided bounds on the Ricci curvature in terms of functional inequalities in the path space, see also recent work of Cheng–Thalmaier [8] and of Wu [24].

A drawback of the previous approaches to detailed controls on Ricci is that they do not see curvature concentrated in singular sets such as the tip of a cone. One goal of the present article is to analyze a different concept of synthetic upper Ricci bounds introduced recently by the first author [22] and to impose a remarkable rigidity: the only metric measure spaces with cone structure and with Ricci curvature bounded above and below are Euclidean spaces $\mathbb{R}^N$. We will work in the setting of RCD$^*$($K', N'$) metric measure spaces, see Section 2 for definitions and references. In this setting an equivalent definition of lower Ricci bound $K$ is the contraction estimate

$$W_2(\hat{P}_t \mu, \hat{P}_t \nu) \leq e^{-Kt} W_2(\mu, \nu),$$

for the dual heat flow $\hat{P}_t$ in $L^2$-Wasserstein distance. The central object in [22] to define upper Ricci bounds is a reverse estimate asymptotically for short times. More precisely, consider for a

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Let \((Y, d_Y, m_Y)\) be a mm-space satisfying the curvature-dimension condition \(RCD^*(K', N')\) for some \(K' \in \mathbb{R}\) and \(N' \in [0, \infty)\) and assume that it is the \(N\)-cone over a mm space \((X, d_X, m_X)\) for some \(N \geq 1\). Then:

(i) either \(\vartheta^+(o, y) = +\infty\) for any \(y \in Y\) and \(o\) the tip of the cone,
(ii) or \(N\) is an integer and \((Y, d_Y, m_Y)\) is isomorphic to Euclidean space \(\mathbb{R}^{N+1}\) with the Euclidean distance and a multiple of the Lebesgue measure.

In particular, (up to isomorphism) the only \(N\)-cone with bounded Ricci curvature among all mm-spaces is the \(N\)-dimensional Euclidean space for \(N\) integer.

An important ingredient to establish the rigidity of cones will be a novel class of rigidity results characterizing the standard sphere \(S^N\) which will be applied to characterize the base of the cone. They are of independent interest and form the second goal of this article.

Let \(f : [0, \pi] \to \mathbb{R}\) be continuous and strictly increasing and put for a mm-space \((X, d, m)\) with \(m(X) < \infty\) and \(\text{diam}(X) \leq \pi\):

\[
\begin{align*}
M_f(X) &:= \frac{1}{m(X)^2} \int_X \int_X f(d(x, y)) \, dm(x) \, dm(y) , \\
M_{f,N}^* &:= \left[ \int_0^\pi f(r) \sin(r)^{N-1} dr \right] / \left[ \int_0^\pi \sin(r)^{N-1} dr \right].
\end{align*}
\]

**Theorem 1.2.** Let \((X, d, m)\) be an \(RCD^*(N - 1, N)\) space with \(N \geq 1\), \(\text{diam}(X) \leq \pi\). Then we have \(M_f(X) \leq M_{f,N}^*\). Moreover, the following are equivalent:

(i) \(M_f(X) = M_{f,N}^*\),
(ii) \(N\) is an integer and \(X\) is isomorphic to the sphere \(S^N\) with the round metric and a multiple of the volume measure.

In particular, we see that for \(N \in \mathbb{N}\) the standard sphere \(S^N\) is the unique maximizer of the expected distance between points and of the variance among \(RCD^*(N - 1, N)\) spaces, choosing \(f(r) = r\) or \(f(r) = r^2\) respectively. We also establish a corresponding almost rigidity theorem, see Theorem 3.1. It is easy to see that the extremum of \(M_f\) among \(RCD^*(N - 1, N)\) spaces is attained also for non-integer \(N\). It would be an interesting question to characterize the extremizers in this case.

The proof of Theorem 1.2 will rely on the maximal diameter theorem obtained by Ketterer [15] which in turn stems from Gigli’s non-smooth splitting theorem [11]. In fact, we will see that (i) will imply that \(m\text{-a.e.} \text{ point in } X\) will have a partner at the maximal distance \(\pi\). Also, the other known rigidity results for \(RCD^*(K, N)\) spaces with \(K > 0\), namely Ketterer’s non-smooth Obata theorem [16] for spaces with extremal spectral gap and the rigidity of spaces saturating the Levy–Gromov isoperimetric inequality [6], are based on the maximal diameter theorem.

An analogous statement (with \(M_f(X) \geq M_{f,N}^*\) in the place of \(M_f(X) \leq M_{f,N}^*\)) holds for strictly decreasing \(f\). Of particular interest is the case \(f = \cos\) which leads to \(M_{\cos,N}^* = 0\).

**Corollary 1.3.** Let \((X, d, m)\) be an \(RCD^*(N - 1, N)\) space with \(N \geq 1\), \(\text{diam}(X) \leq \pi\). Then the following are equivalent:

(i) \(\int_X \int_X \cos d(x, y) \, m(dx) \, m(dy) \leq 0\)
(ii) $N$ is an integer and $X$ is isomorphic to the sphere $S^N$ with the round metric and a multiple of the volume measure.

Note the condition $\text{diam}(X) \leq \pi$ is only requested in the case $N = 1$. In the case $N > 1$, it already follows from the RCD*(N − 1, N)-condition.

In order to obtain Theorem 1.1 from this Corollary, note that the distance on the cone $Y$ is built from the distance on $X$ via the law of cosines. We will show that as soon as

$$a := \int_X \cos (d(x, y)) m(dy) > 0$$

for some point $x \in X$ we have for $p = (r, x)$ in the cone $Y$ that

$$W_2(\hat{P}_t \delta_o, \hat{P}_t \delta_p)^2 \leq d(o, p)^2 - ca\sqrt{t} + O(t),$$

for some constant $c > 0$, which implies that $\vartheta(o, p) = +\infty$.

**Organization.** The article is organized as follows. In Section 2 we recall definitions and results concerning synthetic curvature-dimension bounds for metric measure spaces, as well as the notion of upper bounds on the Ricci curvature considered here. The proof of Theorem 1.1 will be given in Section 3 together with corresponding almost rigidity statements. In Section 4 we give the proof of Theorem 1.1.

## 2. Preliminaries

### 2.1. Synthetic Ricci bounds for metric measure spaces.

We briefly recall the main definitions and results concerning synthetic curvature-dimension bounds for metric measure spaces that will be used in the sequel.

A metric measure space (mm-space for short) is a triple $(X, d, m)$ where $(X, d)$ is a complete and separable metric space and $m$ is a locally finite Borel measure on $X$. In addition, we will always assume the integrability condition $\int_X \exp(-cd(x_0, x)^2)dm(x) < \infty$ for some $c > 0$ and $x_0 \in X$. We denote by $\mathcal{P}_2(X)$ the space of Borel probability measures on $X$ with finite second moment and by $W_2$ the $L^2$-Kantorovich-Wasserstein distance.

The Boltzmann entropy of $\mu \in \mathcal{P}(X)$ is defined by $\text{Ent}(\mu) = \int \rho \log \rho dm$ provided $\mu = \rho m$ is absolutely continuous w.r.t. $m$ and $\int \rho (\log \rho)_+ dm < \infty$; otherwise $\text{Ent}(\mu) = +\infty$. The Cheeger energy of $f \in L^2(X, m)$ is defined by

$$\text{Ch}(f) = \liminf_{g \rightharpoonup f \text{ in } L^2(X, m)} \frac{1}{2} \int |\nabla g|^2 dm,$$

where $|\nabla g|$ denotes the local Lipschitz constant. A mm-space is called *infinitesimally Hilbertian* if Ch is quadratic. In this case, Ch gives rise to a strongly local Dirichlet form. The associated generator $\Delta$ is called the Laplacian and the associated Markov semigroup $(P_t)_{t \geq 0}$ on $L^2(X, m)$ is called the heat flow on $(X, d, m)$, see [2] for more details.

For $\kappa \in \mathbb{R}$ and $\theta \geq 0$ define the functions

$$s_\kappa(\theta) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} \theta), & \kappa > 0, \\ \theta, & \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} \theta), & \kappa < 0, \end{cases}$$

and $c_\kappa(\theta) = \frac{d}{d\theta} s_\kappa(\theta)$. Moreover, for $t \in [0, 1]$ define the distortion coefficients

$$\sigma^{(t)}_\kappa(\theta) = \begin{cases} \frac{s_\kappa(t \theta)}{s_\kappa(\theta)}, & \kappa \theta^2 \neq 0 \text{ and } \kappa \theta^2 < \pi^2, \\ t, & \kappa \theta^2 = 0, \\ +\infty, & \kappa \theta^2 \geq \pi^2. \end{cases}$$
Definition 2.1. i) A metric measure space satisfies the condition $CD^*(K, N)$ with $K \in \mathbb{R}$ and $N \in [1, \infty)$ if for each pair $\mu_0 = \rho m$ and $\mu_1 = \rho_1 m \in \mathcal{P}_2(X)$ there exists an optimal coupling $q$ of $\mu_0, \mu_1$ and a geodesic $\mu_t = \rho_t m$ connecting them such that
\[
\int \rho_t \frac{1}{\sqrt{m}} \rho dm \geq \left[\sigma_{K/N'}((d(x_0, x_1))\rho_0 + \sigma_{K/N'}(d(x_0, x_1))\rho_1 \frac{1}{\sqrt{m}}\right] dq(x_0, x_1)
\]
holds for all $t \in [0, 1]$ and all $N' \geq N$, see [4].

ii) A mm-space satisfies the condition $RCD^*(K, N)$ for $K \in \mathbb{R}$ and $N \in [1, \infty)$ if it is infinitesimally Hilbertian and satisfies $CD^*(K, N)$.

It has been shown in [9] that the $RCD^*(K, N)$ condition can be formulated equivalently in terms of Evolution Variational Inequalities. In particular, for each $\mu_0 \in \mathcal{P}_2(X)$ there exists a (unique) EVI gradient flow emanating in $\mu_0$, denoted by $\dot{P}_t \mu_0$ and called the heat flow acting on measures. For $\mu_0 = \rho m$ with $f \in L^2(X, m)$ it coincides with the heat flow [2], i.e. $\dot{P}_t (f m) = (P_t f) m$. It has been shown ([3, Thm. 6.1], [1, Thm. 7.1]) that the CD condition entails several regularization properties for $P_t$. For instance, $P_t f(x) = \int f d\dot{P}_t \delta_x$ holds for $m$-a.e. for every $f \in L^2(X, m)$. This representative of $P_t f$ has the strong Feller property, that is $x \mapsto \int f d\dot{P}_t \delta_x$ is bounded and continuous for any bounded $f \in L^2(X, m)$. In particular, we have the following estimate for the quadratic variation.

Lemma 2.2. Let $X$ be an $RCD^*(0, N)$ space. Then we have for $\mu \in \mathcal{P}_2(X)$ and all $t > 0$:
\[
W_2(\dot{P}_t \mu, \mu)^2 \leq 2N t .
\]

Proof. Choosing $K = 0, \nu = \mu$ and $s = 0$ (or more precisely, considering the limit $s \searrow 0$) in [9, Thm. 4.1] yields the claim. 

The CD$^*(K, N)$ condition is a priori slightly weaker than the original condition $CD(K, N)$ given in [20], where the coefficients $\sigma_{K/N}(\theta)$ are replaced by $\tau_{K/N}(\theta) = t^{1/N} \sigma_{K/(N-1)}(\theta)^{1-1/N}$.

Recently, however, Cavaletti and Milman [6] succeeded to show that the condition CD$^*(K, N)$ is in fact equivalent to CD$(K, N)$ provided $(X, d, m)$ is non-branching – which in particular will be the case if it is infinitesimally Hilbertian. Thus in turn RCD$^*(K, N)$ will imply the sharp Bonnet-Myers diameter and Bishop-Gromov volume comparison estimates, see also [7, 20] for an alternative argument. Given $x_0 \in \text{supp}[m]$ and $r > 0$ we denote by $v(r) := m(\overline{B}_r(x_0))$ the volume of the closed ball of radius $r$ around $x_0$ and by
\[
s(r) := \limsup_{\delta \to 0} \frac{1}{\delta} m(\overline{B}_{r+\delta}(x_0) \setminus B_r(x_0))
\]
the volume of the corresponding sphere.

Proposition 2.3. Assume that $(X, d, m)$ is non-branching and satisfies CD$^*(K, N)$. Then each bounded closed subset of $\text{supp}[m]$ is compact and has finite volume. For each $x_0 \in \text{supp}[m]$ and $0 < r \leq R \leq \pi \sqrt{N}/(K \wedge 0)$ we have
\[
\frac{s(r)}{s(R)} \geq \left(\frac{\mathcal{H}_{K/(N-1)}(r)}{\mathcal{H}_{K/(N-1)}(R)}\right)^{N-1} \quad \text{and} \quad \frac{v(r)}{v(R)} \geq \frac{\int_r^R \mathcal{H}_{K/(N-1)}(t)^{N-1} dt}{\int_0^R \mathcal{H}_{K/(N-1)}(t)^{N-1} dt} .
\]

Moreover, if $K > 0$ then $\text{supp}[m]$ is compact and its diameter is bounded by $\pi \sqrt{N}/K$.

2.2. Upper Ricci bounds. Here, we briefly introduce the synthetic notion of upper Ricci curvature bounds considered in this paper. For more details we refer to [22]. Let us mention that there also other approaches in terms the behaviour of the entropy along Wasserstein geodesics and their relations are discussed.
Let \((X, d, m)\) be an \(\text{RCD}^\ast(K', N')\) mm-space and let \(\hat{P}_t\) denote the dual heat flow acting on measures. For points \(x, y \in X\) we set
\[
\vartheta^+(x, y) := \liminf_{t \to 0} \frac{1}{t} \log \left( \frac{W_2(\hat{P}_t \delta_x, \hat{P}_t \delta_y)}{d(x, y)} \right), \quad \vartheta^*(x) := \limsup_{y, z \to x} \vartheta^+(y, z).
\]

It is shown in [22, Thm. 2.10] that a lower bound \(\vartheta^+(x, y) \geq K\) is equivalent to the \(\text{RCD}^\ast(K, \infty)\) condition and in particular to the Wasserstein contraction estimate \(W_2(\hat{P}_t \mu, \hat{P}_t \nu) \leq e^{-Kt} W_2(\mu, \nu)\) for all \(\mu, \nu \in \mathcal{P}_2(X)\) and all \(t > 0\).

If \((X, d, m) = (M, d, e^{-V} \text{vol})\) is a smooth weighted Riemannian manifold we have the following precise estimate on \(\vartheta^+\) in terms of the Bakry–Émery Ricci curvature \(\text{Ric}_f = \text{Ric} + \text{Hess} f\).

**Theorem 2.4** ([22, Thm. 3.1]). For all pairs of non-conjugate points \(x, y \in M\)
\[
\text{Ric}_f(\gamma) \leq \vartheta^+(x, y) \leq \text{Ric}_f(\gamma) + \sigma(\gamma) \tan^2 \left( \sqrt{\sigma(\gamma) d(x, y) / 2} \right),
\]
where \(\gamma = (\gamma^a)_{a \in [0, 1]}\) is the (unique) constant speed geodesic connecting \(x\) and \(y\),
\[
\text{Ric}_f(\gamma) = \frac{1}{d(x, y)^2} \int_0^1 \text{Ric}_f(\gamma^a, \dot{\gamma}^a) \, da,
\]
and \(\sigma(\gamma)\) denotes the maximal modulus of the Riemannian curvature along the geodesic \(\gamma\).

In particular one sees that an upper bound \(\text{Ric}_f \leq K\) for some \(K \in R\) is equivalent to the estimate \(\vartheta^*(x) \leq K\) for all \(x \in M\). This motivates the following definition.

**Definition 2.5.** We say that a number \(K \in \mathbb{R}\) is a synthetic upper Ricci bound for the mm-space \((X, d, m)\) if \(\forall x \in X\)
\[
\vartheta^*(x) \leq K.
\]

### 2.3. Cones and suspensions

We recall the construction of cones for metric measure spaces.

**Definition 2.6.** For a metric measure space \((X, d_X, m_X)\) and \(K \geq 0, N \geq 1\) the \((K, N)\)-cone \(\text{Con}^N_K(X) = (C, d_C, m_C)\) over \((X, d, m)\) is the defined by
\[
C = \begin{cases} \left[0, \frac{\pi}{\sqrt{K}}\right] \times X/\{(0, 0) \times X\}, & K > 0, \\ \left[0, \infty\right) \times X/\{0\} \times X, & K = 0, \end{cases}
\]
with \(m_C(dr, dx) = \mathcal{S}_K(r)^N dr \, m_X(dx)\) and \(d_C\) given for \((r, x), (s, y) \in C\) by
\[
d_C((r, x), (s, y)) = \begin{cases} c_K^{-1} \left[ c_K(r) c_K(s) + K \mathcal{S}_K(r) \mathcal{S}_K(s) \cos \big( d_X(x, y) \wedge \pi \big) \right], & K > 0, \\ \sqrt{r^2 + s^2 - 2rs \cos \big( d_X(x, y) \wedge \pi \big)}, & K = 0. \end{cases}
\]

We refer to the \((1, N)\)-cone as the spherical suspension of \(X\).

Curvature-dimension bounds for cones are intimately related to curvature-dimension bounds for the base space. We recall the following result by Ketterer [15].

**Theorem 2.7.** Let \((X, d_X, m_X)\) be a metric measure space and let \(K \geq 0\) and \(N \geq 1\). Then the \((K, N)\)-cone \(\text{Con}^N_K(X)\) satisfies \(\text{RCD}^\ast(KN, N + 1)\) if and only if \(X\) satisfies \(\text{RCD}^\ast(N - 1, N)\) and \(\text{diam}(X) \leq \pi\).

In fact, any curvature-dimension bound on the cone is sufficient to infer bounds on the base space as we will show here. More precisely, the following generalization holds.

**Theorem 2.8.** Let \((X, d_X, m_X)\) be a metric measure space and let \(N \geq 1\). Then the following statements are equivalent:
(i) The \((K, N)\)-cone \(\text{Con}^N_K(X)\) satisfies \(\text{RCD}^\ast(K', N')\) for some \(K' \in \mathbb{R}\) and \(N' \geq N + 1\).
(ii) \(X\) satisfies \(\text{RCD}^\ast(N - 1, N)\) and \(\text{diam}(X) \leq \pi\).

In this case \(\text{Con}^N_K(X)\) satisfies \(\text{RCD}^\ast(KN, N + 1)\).
A close inspection of the proof in [15] reveals that, at least in the case of the Euclidean cone $\mathcal{K} = 0$, the arguments there already yield that $\text{RCD}^*(0, N')$ on the cone implies $\text{RCD}^*(N - 1, N)$ on the base space although this is not explicitly stated. Since the argument is quite technical and involved we sketch the main steps for the reader’s convenience and highlight the modifications. See the proof of [15, Thm. 1.2] for more details. To obtain the statement in the case $K > 0$ and under the relaxed curvature bound $K'$ we provide additional arguments.

**Proof of Thm. 2.8.** We only need to treat the implication (i)$\Rightarrow$(ii). We proceed in three steps

**Step 1:** Let us first consider the case $K = 0$ and assume that $\text{Con}_0^N(X)$ satisfies $\text{RCD}^*(0, N')$.

a) Following the argument of Bacher and Sturm in [4] one finds that the $\text{CD}^*(0, N')$ condition for $C = \text{Con}_0^N(X)$ implies that $\text{diam}X \leq \pi$ and hence $C$ coincides with the warped product $[0, \infty) \times_{id} X$. Corollary 5.15 in [15] yields that $X$ is infinitesimally Hilbertian. Prop. 5.11, Cor. 5.12 in [15] show that the Cheeger energy of $C$ coincides with the skew product of the Dirichlet forms on $[0, \infty)$ and $X$ and that the intrinsic distance of the latter coincides with $d_C$.

Moreover, with $I = [0, \infty)$ one has $C^0_c(I) \otimes D(\Gamma^X_2) \subset D(\Gamma^C_2)$ and $1 \otimes D^{b,2}(L^X) \subset D^{b,2}(L^C)$. Finally, [15, Thm. 4.26] yields that the Bakry–Émery condition $\text{BE}(0, N')$ holds for the Dirichlet form on $C$.

b) Following the proof of [15, Thm. 3.23], using the explicit expression of the $\Gamma_2$-operator on $C$ (see (27) in [15]):

$$
\Gamma^C_2(u \otimes v) = 
\left((u')^2 + \frac{N}{r^2}(u')^2\right)v^2
+ \frac{1}{r^2}u^2\Gamma^X_2(v) - \frac{N - 1}{r^2}u^2\Gamma^X(v)
+ \frac{2}{r^2}u'\text{L}^X(v)v
+ \left(\frac{r^2}{\sqrt{2}}(u')^2 - \frac{4}{r^2}u'u + \frac{2}{r^2}u^2\right)\Gamma^X(v),
$$

choosing in particular $u(r) = r$ locally, and using the Bochner inequality with parameters $(0, N')$ in $C$ one arrives at the following integrated estimate for $v \in D(\Gamma^X_2)$ and test function $\phi \in D^{b,2}(L^X)$:

$$
\int L^X\phi\Gamma^X(v)dm_X - \int \Gamma^X(v, L^Xv)\phi dm_X 
\geq (N - 1) \int \Gamma^X(v)\phi dm_X + \frac{1}{N} \int \left(L^X v + N v\right)^2 \phi dm_X
- \int \phi\left(v^2 N + 2vL^X v\right) dm_X
= (N - 1) \left(\Gamma^X(v)\phi dm_X + \frac{1}{N} \int \left(L^X v + N v\right)^2 \phi dm_X
- \int \phi\left(v^2 N + 2vL^X v\right) dm_X - \frac{N - 1}{N'} \int \left(L^X v + N v\right)^2 \phi dm_X
= (N - 1) \left(\Gamma^X(v)\phi dm_X + \frac{1}{N} \int \left(L^X v + N v\right)^2 \phi dm_X
- \frac{N - 1}{N'} \int \left(L^X v + N v\right)^2 \phi dm_X\right). (2.1)
$$

c) It remains to get rid of the last term in (2.1) in order to conclude that $X$ satisfies $\text{RCD}^*(N - 1, N)$. For a given point $x_0$ one could simply replace $v$ by $v - 1/NL^Xv(x_0)$ in order to make the last term vanish at $x_0$ leaving all other terms invariant. However, since the Bochner inequality is an integrated estimate, more care is needed.
One deduces from (2.1) the gradient estimate
\[
|\nabla P_t^X v|^2 + \frac{c(t)}{N} \left( (L^X P_t^X v)^2 - \frac{N - N'}{N} P_t^X (L^X v + N v)^2 \right) \leq P_t^X |\nabla v|^2 .
\]
From here one can follow the argument in [15] further to conclude the usual gradient estimate without the extra term \( - \frac{N - N'}{N} P_t^X (L^X v + N v)^2 \) which in turn implies the RCD\(^*(N - 1, N)\) condition.

**Step 2:** Let us still consider the case \( K = 0 \) but assume that \( \text{Con}^N_0(X) \) satisfies RCD\(^*(K', N')\) for some \( K' \in \mathbb{R} \). For \( \lambda > 0 \) consider the homothety \( \Phi_\lambda \) of \( \text{Con}^N_0(X) \) given by \( \Phi_\lambda(s, y) = (\lambda s, y) \) and note that it maps geodesics to geodesics. Consequently, also the induced map from \( \mathcal{P}(\text{Con}^N_0(X)) \) to itself acting by push-forward maps \( W_2 \)-geodesics to \( W_2 \)-geodesics. Let \( (\mu_t)_{t \in [0, 1]} \) be a \( W_2 \)-geodesic and let \( \mu_0 = (\Phi_\lambda)_{#} \mu_t \). By the RCD\(^*(K', N')\) condition the entropy is \((K', N')\)-convex along the geodesic \( \mu_\lambda^t\). One finds that \( \text{Ent}(\mu_\lambda^t) = \text{Ent}(\mu_t) - (N + 1) \log \lambda \) and that \( W_2(\mu_0^\lambda, \mu_1^\lambda) = \lambda W_2(\mu_0, \mu_1) \). This implies \((K'\lambda^2, N')\)-convexity of the entropy along the original geodesic \( (\mu_t) \). Since, \( (\mu_t) \) was arbitrary, letting \( \lambda \to 0 \) yields that \( \text{Con}^N_0(X) \) satisfies RCD\(^*(0, N')\) and we conclude from the first step.

**Step 3:** Let us finally consider the case \( K > 0 \) and assume that \( \text{Con}^N_K(X) \) satisfies RCD\(^*(K', N')\). The result will follow from a simple blow-up argument. Note that the pointed rescaled spaces \( (\text{Con}^N_{K/\mu^2}(X), o)) \) converge in pointed measured Gromov-Hausdorff sense to the pointed Euclidean cone \( (\text{Con}^N_0(X), o)) \) and that the they satisfy RCD\(^*(K', N')\). By the stability of the conditions CD\(^*(K, N)\) and RCD\(^*(K, \infty)\) under pointed measured Gromov-Hausdorff convergence (see [23, Thm. 29.25] and [13, Thm. 7.2, Prop. 3.33]) we obtain that \( \text{Con}^N_0(X) \) satisfies RCD\(^*(0, N')\). From the first part of the proof we infer that \( X \) satisfies RCD\(^*(N - 1, N)\).

### 3. Rigidity of the Standard Sphere

Here, we give the proof of the rigidity theorem for the standard sphere, Theorem 1.2. Then, we formulate an almost rigidity statement.

**Proof of Theorem 1.2.** Without restriction we can assume that \( m(X) = 1 \). We consider the case that \( f : [0, \pi] \to \mathbb{R} \) is continuous and strictly increasing. The case of decreasing \( f \) then follows by considering \( -f \). Possibly adding a constant to \( f \) we can assume without restriction that \( f \geq 0 \).

Recall that the Bishop–Gromov volume comparison Proposition 2.3 asserts that for any \( x \in X \):
\[
\frac{m_X(\{\bar{B}_r(x)\})}{m_X(\{\bar{B}_R(x)\})} \geq \frac{\int_0^t \sin(t)^{N-1} \, dt}{\int_0^R \sin(t)^{N-1} \, dt} =: \frac{V_{\pi}^*}{V_{R}^*} .
\]

Fix \( x \in X \) and put \( g(y) = f(d_X(x, y)) \). Using that \( m_X(X) = 1 \) and \( \text{diam}(X) \leq \pi \) we can estimate
\[
\int_X g(y) \, dm_X(y) = \int_0^\infty m_X(\{g \geq s\}) \, ds = \int_0^{\text{diam}(X)} m_X(\bar{B}_{f^{-1}(s)}(x)) \, ds
\]
\[
= \int_0^{\text{diam}(X)} 1 - m_X(\bar{B}_{f^{-1}(s)}(x)) \, ds \leq \int_0^\pi 1 - \frac{V_{f^{-1}(s)}^*}{V_\pi^*} \, ds
\]
\[
= \left[ \int_0^\pi f(r) \sin(r)^{N-1} \, dr \right] / \left[ \int_0^\pi \sin(r)^{N-1} \, dr \right] = M_{f,N}^* .
\]
Integrating over \( x \) then yields the first statement.

Let us now prove the rigidity statement. From the above argument we obtain also that the equality \( M_f(X) = M_{f,N}^* \) implies that for \( m_X \) a.e. point \( x \) there must exist a point \( x' \) with \( d_X(x, x') = \pi \). This implies that \( N \) is an integer and that \( X \) is isomorphic to \( \mathbb{S}^N \) by iteratively
applying the maximal diameter theorem [15, Thm. 1.4]. Indeed, recall that the existence of points $x_1, x'_1$ with $d_X(x_1, x'_1) = \pi$ implies that

(a) if $N \in [1, 2)$ then either $X$ is isomorphic to the interval $[0, \pi]$ or $N = 1$ and $X$ is isomorphic to the circle $S^1$ with normalized Hausdorff measure.

(b) if $N \geq 2$, then $X$ is isomorphic to a spherical suspension $\text{Con}^{N-1}_r(Y)$ for some $\text{RCD}^*(N-2, N-1)$ space $(Y, d_Y, m_Y)$ with $\text{diam} Y \leq \pi$ and $m(Y) = 1$.

In case (a), we must have $N = 1$ and $X$ isomorphic to $S^1$ since otherwise there would be points that do not have a partner at distance $\pi$. In case (b) we pick $x_2 \in X$ of the form $x_2 = (\pi/2, y_2)$ and $x'_2$ such that $d_X(x_2, x'_2) = \pi$. Then we have $x'_2 = (\pi/2, y'_2)$ and $d_Y(y_2, y'_2) = \pi$. We then repeat the previous argument inductively. After $\lceil N \rceil$ steps we arrive at case (a). Thus, we conclude that $N$ is an integer and that $X$ is the $N-1$ fold spherical suspension over $S^1$, i.e. isomorphic to $S^N$.

We have the following almost rigidity statement.

**Theorem 3.1.** For all $\epsilon > 0$ and $N \geq 1$ there exists $\delta > 0$ depending only on $\epsilon$ and $N$ such that the following holds: If $X$ is an $\text{RCD}^*(N-1-\delta, N+\delta)$ space with $m(X) = 1$ and $M_f(X) \leq \delta$, then $N$ is an integer and $d_{mGH}(X, S^N) \leq \epsilon$, where $S^N$ is the standard $N$-sphere with normalized volume.

**Proof.** Assume on the contrary that there is $\delta_0 > 0$ and a sequence $X_n$ of normalized $\text{RCD}^*(N-1-1/n, N+1/n)$ spaces with $M_f(X_n) \leq 1/n$ and $d_{mGH}(X_n, S^N) \geq \epsilon_0$ for all $n$. By compactness of the class of $\text{RCD}^*(K, N)$ spaces, there exist a normalized $\text{RCD}^*(N-1, N)$ space $X$ such that $X_n$ converges to $X$ in $mGH$-sense along a subsequence. Obviously, we still have $d_{mGH}(X, S^N) > \epsilon_0$.

On the other hand, since $M_f$ is readily checked to be continuous w.r.t. measured Gromov–Hausdorff convergence, $M_f(X) = \lim_n M_f(X_n) = 0$. But then, by the rigidity result Theorem 1.2, we have that $N$ is an integer and that $X$ is isomorphic to $S^N$, a contradiction.

Let us give an alternative proof of Theorem 1.2 in the special case $f = \cos$ that will yield the rigidity of cones with bounded Ricci curvature. In this case $M_{\text{cos},N}^* = 0$. The proof is based on a slightly different induction argument, noting the the condition $M_{\text{cos}}(X) = 0$ directly implies $M_{\text{cos}}(Y) = 0$ if $X$ is a suspension over $Y$.

**Proof of Theorem 1.2 for $f = \cos$.** First note that by Bishop–Gromov volume comparison we have that for any $x_0 \in X$:

$$\int_X \cos \left( d_X(x_0, y) \right) m_X(dy) \geq 0.$$  

Indeed, denote by $s(r)$ the volume of the sphere of radius $r$ around $x_0$ in $X$. Since $X$ satisfies $\text{RCD}^*(N-1, N)$ the Bishop–Gromov volume comparison Proposition 2.3 asserts that for all $0 < r \leq R \leq \pi$:

$$\frac{s(r)}{s(R)} \geq \left( \frac{\sin(r)}{\sin(R)} \right)^{N-1}. \tag{3.2}$$

Thus we obtain

$$\int_X \cos \left( d_X(x_0, y) \right) m_X(dy) = \int_0^\pi \cos(r) s(r) dr \tag{3.3}$$

$$= \int_0^{\pi/2} \cos(r) s(r) dr + \int_{\pi/2}^\pi \cos(r) s(r) dr$$

$$= \int_0^{\pi/2} \cos(r) [s(r) - s(\pi - r)] dr \geq 0.$$  

Here we have used that $\cos(r) = -\cos(\pi - r)$ and that $s(r) \geq s(\pi - r)$ for $r \leq \pi/2$ by (3.2).
The previous argument also shows that in order for $M_{\cos}(X) = 0$ to hold, for a.e. $x \in X$ there must exist a point $x' \in X$ at maximal distance, i.e. with $d_X(x, x') = \pi$. The maximal diameter theorem [15, Thm. 1.4] again yields that one of the two cases (a), (b) above must hold and that in case (a) we must have that $N = 1$ and $X$ is isomorphic to $S^1$.

In the case (b), we have from the definition of distance and measure in the spherical suspension:

\[
0 = \int_X \int_X \cos(d(x, y)) m_X(dx)m_X(dy) = \int_0^{\pi} \int_Y \int_0^{\pi} \int_Y \left[ \cos(r) \cos(s) + \sin(r) \sin(s) \cos(d_Y(\theta, \phi)) \right] \\
\quad \times \sin(s)^{N-1} \sin(r)^{N-1} ds \, dr \, m_Y(d\theta) \, m_y(d\phi)
\]

\[
= A^2 \int_Y \int_Y \cos(d_Y(\theta, \phi)) m_Y(d\theta) \, m_y(d\phi),
\]

with

\[
A = \int_0^{\pi} \sin(s)^N ds > 0.
\]

This implies that also $M_{\cos}(Y) = 0$ holds and we repeat the previous argument inductively.

After $\lfloor N \rfloor$ steps we arrive at case (a) and conclude that $N$ is an integer and that $X$ is the $N - 1$ fold spherical suspension over $S^1$, i.e. isomorphic to $S^N$.

\[\square\]

4. RIGIDITY OF CONES WITH BOUNDED RICCI CURVATURE

Here, we give the proof of the rigidity result for cones with bounded Ricci curvature, Theorem 1.1.

A crucial ingredient in the proof will be the relation between the vanishing of the integral

\[
\int_X \int_X \cos(d(x, y)) m_X(dx)m_X(dy) = 0 \quad (4.1)
\]

and the asymptotic behaviour as $t \to 0$ of $W_2(\hat{P}_t \delta_o, \hat{P}_t \delta_q)$ for the vertex $o$ of the cone and any other point $q$. We will first prove the following pointwise equivalence which is somewhat stronger than what is needed in the proof of Theorem 1.1.

**Proposition 4.1.** Let $(X, d_X, m_X)$ be an $RCD^*(N - 1, N)$ space with $N \geq 1$ and $\text{diam}(X) \leq \pi$. Then for any $p_0 = (r_0, x_0) \in \text{Con}_0^N(X)$ and $o$ the vertex one of the following statements holds:

(i) $\int_X \cos(d_X(x_0, y)) m_X(dy) = 0$ and $\vartheta^+(o, p_0) = 0$.

(ii) $\int_X \cos(d_X(x_0, y)) m_X(dy) > 0$ and $\vartheta^+(o, p_0) = +\infty$.

**Proof. Step 1:** Let us fix $p_0 = (r_0, x_0) \in \text{Con}_0^N(X)$. Recall from (3.3) that

\[
\int_X \cos(d_X(x_0, y)) m_X(dy) \geq 0.
\]

**Step 2:** Let us assume first that $a := \int_X \cos(d_X(x_0, y)) m_X(dy) > 0$. We claim that as $t \to 0$ we have

\[
W_2(\hat{P}_t \delta_o, \hat{P}_t \delta_{p_0})^2 \leq d_c(o, p_0)^2 - O(\sqrt{t}), \quad (4.2)
\]

which immediately implies that $-\partial_t^- |_{t=0} \log W_2(\hat{P}_t \delta_o, \hat{P}_t \delta_{p_0}) = +\infty$. To this end, denote by $\nu_p^t = \hat{P}_t \delta_p$ the heat kernel measure at time $t$ centered at $p = (r, x) \in \text{Con}_0^N(X)$. Denote by $\nu_p^t$ its marginal in the radial component. Further we consider the desintegration $\nu_{p,s}^t \in P(X)$ of $\nu_p^t$ after $\nu_p^t$, i.e.

\[
\nu_{p,s}^t(ds, dy) = \nu_{t}^t(ds) \nu_{p,r}^t(dy).
\]
Lemma 4.2 gives that for $p = o$ we have that $\nu^t_{o,x} = m_X$ is the uniform distribution on $X$. Let now, $\pi = \nu^t_{p_0} \otimes \nu^t_0$ be the product coupling. We obtain

\[ W_2(\hat{P}_t \delta_o, \hat{P}_t \delta_{p_0})^2 \leq \int d_C^2 d\pi \]

\[ = \int \left[ r^2 + s^2 - 2rs \cos (d_X(x, y)) \right] \nu^t_{p_0}(dr, dx) \nu^t_0(ds, dy) \]

\[ = \int r^2 \nu^t_{r_0}(dr) + \int s^2 \nu^t_0(ds) - 2 \int rs \cos (d_X(x, y)) \nu^t_{p_0}(dr, dx) d\nu^t_0(ds)m_X(dy) \]

\[ = \int r^2 \nu^t_{r_0}(dr) + \int s^2 \nu^t_0(ds) - 2 \int \int f d\nu^t_{p_0} \left( \int s \nu^t_0(ds) \right), \]

where we have set $f(r, x) = r \int_X \cos (d_X(x, y)) m_X(dy)$. Obviously, $f$ is a continuous function on $\text{Con}_0^N(X)$ with at most linear growth. Since $W_2(\nu^t_{p_0}, \delta_{p_0}) \to 0$ as $t \to 0$ by Lemma 2.2 we have that

\[ \int f d\nu^t_{p_0} = f(p_0) + o(1) = r_0 \int_X \cos (d_X(x, y)) m_X(dy) + o(1) = r_0 a + o(1). \]

Thus, using the moment estimates from Lemma 4.2 we obtain

\[ W_2(\hat{P}_t \delta_o, \hat{P}_t \delta_{p_0})^2 \leq r_0^2 + 2Ct - 2c\sqrt{t}(r_0a + o(1)). \]

for suitable constants $C, c > 0$. This proves (4.2).

**Step 3:** Let us now assume that $\int_X \cos (d_X(x, y)) m_X(dy) = 0$. We claim that

\[ W_2(\hat{P}_t \delta_o, \hat{P}_t \delta_{p_0}) \geq d_C(o, p_0) + O(t), \quad (4.3) \]

which immediately implies that $-\partial_t^- \big|_{t=0} \log W_2(\hat{P}_t \delta_o, \hat{P}_t \delta_{p_0}) \leq 0$. To this end, consider the function $\phi : \text{Con}_0^N(X) \to \mathbb{R}$ given by $\phi(s, y) = s \cos (d_X(x_0, y))$. By Lemma 4.3, $\phi$ is 1-Lipschitz w.r.t. the cone distance. Hence, by Kantorovich–Rubinstein duality, we obtain

\[ W_2(\hat{P}_t \delta_o, \hat{P}_t \delta_{p_0}) \geq W_1(\hat{P}_t \delta_o, \hat{P}_t \delta_{p_0}) \geq \int_{\text{Con}_0^N(X)} \phi d(\nu^t_{p_0} - \nu^t_0) = \int \phi \, d\nu^t_{p_0} =: g(t). \]

Using the definition of the cone distance we write

\[ 2r_0 g(t) = -\int d_C(p_0, \cdot)^2 d\nu^t_{p_0} + r_0^2 + \int s^2 \nu^t_{p_0}(ds, dy) \]

\[ = -\int d_C(p_0, \cdot)^2 d\nu^t_{p_0} + 2r_0^2 + Nt. \]

By Lemma 2.2 we have as $t \to 0$ that

\[ \int d_C(p_0, \cdot)^2 d\nu^t_{p_0} = O(t). \]

Thus, $g(t) = r_0 + O(t)$ which yields (4.3).

**Step 4:** Finally, recall that the RCD$^\ast(0, N + 1)$ property of $\text{Con}_0^N(X)$ implies the contraction estimate

\[ W_2(\hat{P}_t \delta_p, \hat{P}_t \delta_q) \leq d_C(p, q) \quad \forall p, q \in C, \]

which implies that $-\partial_t^- \big|_{t=0} \log W_2(\hat{P}_t \delta_o, \hat{P}_t \delta_{p_0}) \geq 0$. \qed
Proof of Thm. 1.1. (i) ⇒ (ii): By Theorem 2.8, X satisfies RCD*(N − 1, N). Moreover, the assumption that Ric Y < K implies that there exists q such that
\[-\partial_t^\|_{t=0} \log W_2(\hat{P}_t\delta_o, \hat{P}_t\delta_p) < +\infty .\]
Thus, Proposition 4.1 yields that
\[\int_X \cos (d_X(x, y)) m_X(dy) = 0 \quad \forall x \in X ,\]
and in particular (4.1) holds. Theorem 1.2 with f = cos yields that N is an integer and X is isomorphic to S^N with the round metric and a multiple of the volume measure. Hence X is isomorphic to \( \mathbb{R}^{N+1} \) with Euclidean distance and a multiple of the Lebesgue measure.

(ii) ⇒ (i): If Y is isomorphic to \( \mathbb{R}^{N+1} \), it satisfies RCD*(0, N + 1) and it is isomorphic to the N-cone Con_0^N(S^N). Moreover, we have that
\[ W_2(\hat{P}_t\delta_p, \hat{P}_t\delta_q) = d_Y(p, q) \]
for all p, q and hence (i) follows. □

Lemma 4.2. Let \( \nu^t_p = \hat{P}_t\delta_p \) for p = \((r, x)\) and denote by \( \bar{\nu}^t_r \) its marginal in the radial component. Further, let \( \nu^t_{p,s} \in P(X) \) be the desintegration of \( \nu^t_p \) after \( \bar{\nu}^t_r \), i.e.
\[ \nu^t_{p,s}(ds, dy) = \bar{\nu}^t_r(ds)\nu^t_{p,r}(dy) . \]
Then there are constants \( c, C > 0 \) such that
\[ \int s^2d\bar{\nu}^t_r(s) \leq r^2 + Ct , \]
\[ \int sd\bar{\nu}^t_o(s) = c\sqrt{t} . \]
More precisely, the constants are given by \( C = \lceil N \rceil \) and \( c = \int sd\bar{\nu}^1_o(s) \). Furthermore, for \( p = o \) we have that \( \nu^t_{o,r} = m_X \) is the uniform distribution on X.

Proof. First note that \( \bar{\nu}^t_r \) coincides with the heat flow in the RCD*(0, N + 1) space \( B = ([0, \infty), |\cdot|, r^N dr) \). To see this, recall that \( \nu^t_r \) satisfies the Evolution Variational Inequality EVI on Con_N(X). One can check from this that the projection \( \bar{\nu}^t_r \) satisfies EVI_{0,\infty} on B and conclude by recalling that the heat flow is the unique solution to EVI.

Now, the Cheeger energy on B is given by the closure of the quadratic form \( \mathcal{E}^B \) on \( C_c^\infty(0, \infty) \) given by
\[ \mathcal{E}^B(u) = \int_0^\infty \|u^t_r\|^2(r)r^N dr , \]
see e.g. [15, Sec. 2.3]. It follows that \( \bar{\nu}^t_r \) coincides with the law of the N-dimensional Bessel process started from r. To obtain the first estimate, one uses that the second moment of the N-dimensional Bessel process is controlled from above by the one of the M-dimensional Bessel process if \( N \leq M \) and that for \( N \in \mathbb{N} \) the N-dim. Bessel process is obtained as the absolute value of a N-dimensional Brownian motion.

To obtain the second statement, one employs the scaling property of the Bessel process \( (X_t) \) starting form 0. Namely, the law of \( X_t \) coincides with the image of the law of \( X_1 \) under the homothety \( r \mapsto \sqrt{tr} \).

Finally, the last statement is obtained by noting that the measures \( \mu^t \) on Con_N(X) given by
\[ \mu^t(ds, dy) := \bar{\nu}^t_o(ds)m_X(dy) \]
satisfy EVI_{0,\infty} which follows from the corresponding property of \( \bar{\nu}^t_o \).

□
Lemma 4.3. Let $(X,d_X,m_X)$ be a metric measure space with $\text{diam}(X) \leq \pi$ and $x \in X$. Then the function $\phi: \text{Con}_0^N(X) \to \mathbb{R}$ given by 
\[ (s,y) \mapsto s \cos \left( d_X(x,y) \right) \]
is 1-Lipschitz w.r.t. the cone distance.

Proof. Let $(s, y), (s', y') \in C = \text{Con}_0^N(X)$ and set $\alpha = d_X(x, y)$, $\alpha' = d_X(x, y')$ and $\beta = d_X(y, y')$. Note that $\alpha, \alpha', \beta \leq \pi$ and $\beta \geq |\alpha - \alpha'|$. Let $p, p' \in \mathbb{R}^2$ be points at angle $\alpha$ and $\alpha'$ with the first coordinate axis respectively and $||p|| = s$, $||p'|| = s'$. Now we have that 
\[ d_C((s, y), (s', y')) = s^2 + (s')^2 - 2ss' \cos \beta \geq s^2 + (s')^2 - 2ss' \cos |\alpha - \alpha'| \]
\[ = ||p - p'||^2. \]

On the other hand, we find that 
\[ ||\phi(s, y) - \phi(s', y')|| = |s \cos \alpha - s' \cos \alpha'| = ||q - q'|| \leq ||p - p'||, \]
where $q$ and $q'$ are the projections of $p$ and $p'$ respectively onto the first coordinate axis. \qed

Example 4.4. Consider the special case $X = \mathbb{S}^2(1/\sqrt{3}) \times \mathbb{S}^2(1/\sqrt{3})$ equipped with the Cartesian product of the standard Riemannian distances on the spheres $\mathbb{S}^2(1/\sqrt{3})$ with radius $1/\sqrt{3}$ and the normalized product measure, which is an RCD$^*(3,4)$ space. Hence, the 4-cone over $\mathbb{S}^2(1/\sqrt{3}) \times \mathbb{S}^2(1/\sqrt{3})$ is an RCD$^*(0,5)$ space with Ricci curvature $+\infty$ at the tip.

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