Nonminimal De Rham-Hodge Operators and Non-commutative Residue

Abstract

In this paper, we get a Kastler-Kalau-Walze type theorem associated to nonminimal de Rham-Hodge operators on compact manifolds with boundary. We give two kinds of operator-theoretic explanations of the gravitational action in the case of four dimensional compact manifolds with flat boundary.

Keywords: Nonminimal de Rham-Hodge operator; lower-dimensional volume; Noncommutative residue.

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1. Introduction

The noncommutative residue found in [1, 2] plays a prominent role in noncommutative geometry. For one dimensional manifolds, the noncommutative residue was discovered by Adler [3] in connection with geometric aspects of nonlinear partial differential equations. In [3], Connes used the noncommutative residue to derive a conformal four dimensional Polyakov action analogy. Moreover, in [4], Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action, which we call the Kastler-Kalau-Walze theorem. In [5], Kastler gave a brute-force proof of this theorem. Then Kalau and Walze also gave a proof of this theorem by using normal coordinates [7].

In [8], Fedosov etc. defined a noncommutative residue on Boutet de Monvel’s algebra and proved that it was a unique continuous trace. In [9], Schrohe gave the relation between the Dixmier trace and the noncommutative residue for manifolds with boundary. For an oriented spin manifold $M$ with boundary $\partial M$, by the composition formula in Boutet de Monvel’s algebra and the definition of $\overline{\text{Wres}}[(\pi + D^{-1})^2]$, should be the sum of two terms from interior and boundary of $M$, where $\pi + D^{-1}$ is an element in Boutet de Monvel’s algebra [10]. For lower-dimension spin manifolds with boundary and the associated Dirac operators, Wang computed the lower dimensional volume and got a Kastler-Kalau-Walze theorem in [11], [12], and [13]. In [14], Gilkey, Branson and Fulling obtained a formula about heat kernel expansion coefficients of nonminimal operators. In [15], we considered the non-commutative residue of nonminimal operators and got the Kastler-Kalau-Walze type theorems for nonminimal operators. The motivation of this paper is to generalize Theorem 2.2 in [15] to manifolds with boundary in the four dimensional case. For this purpose, we introduce the nonminimal de Rham-Hodge operators $\tilde{D} = ad + b\delta$ and nonminimal laplacian operators $\tilde{D}\tilde{D}^* = a^2d\delta + b^2\delta d$. Our main result is as follows:

**Theorem 3.5** The following equality holds:

$$\overline{\text{Wres}}[\pi^+\tilde{D}^{-1} \circ \pi^+(\tilde{D}^*)^{-1}] = 4\pi \int_M \sum_{k=0}^4 c_1(4, k, a, b)Rd\text{vol}(M) - \frac{23}{12}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\pi \int_{\partial M} K\Omega_3 dx', $$

(for related definition, see Section 3).

This paper is organized as follows: In Section 2, we define lower dimensional volumes of manifolds with boundary for nonminimal De-Rham Hodge operators. In Section 3, for four dimensional compact manifolds with boundary and the associated nonminimal De-Rham Hodge operators $\tilde{D} = ad + b\delta$ and $\tilde{D}^* = bd + a\delta$, we compute the lower dimensional volume $\text{Vol}^{(1,1)}_{1,1}$ and get a Kastler-Kalau-Walze type theorem in this case. In Section 4, for four dimensional compact manifolds with boundary and the associated nonminimal De-Rham Hodge operators $\tilde{D} = ad + b\delta$, we compute the lower dimensional volume. When $\partial M$ is flat, we give two kinds of operator theoretic explanations of the gravitational action on boundary.
2. Lower dimensional volumes of Riemannian manifolds with boundary

In this section we consider an $n$-dimensional oriented Riemannian manifold $(M, g^M)$ equipped with some spin structure. Let $M$ be an $n$-dimensional compact oriented manifold with boundary $\partial M$. We assume that the metric $g^M$ on $M$ has the following form near the boundary

$$g^M = \frac{1}{h(x_n)}g^{\partial M} + dx_n^2,$$

where $g^{\partial M}$ is the metric on $\partial M$. Let $U \subset M$ be a collar neighborhood of $\partial M$ which is diffeomorphic $\partial M \times [0,1)$. By the definition of $h(x_n) \in C^\infty([0,1))$ and $h(x_n) > 0$, there exists $\hat{h} \in C^\infty((-\varepsilon,1))$ such that $\hat{h}|_{(0,1)} = h$ and $\hat{h} > 0$ for some sufficiently small $\varepsilon > 0$. Then there exists a metric $\hat{g}$ on $\tilde{M} = M \cup_{\partial M} \partial M \times (-\varepsilon,0]$ which has the form on $U \cup_{\partial M} \partial M \times (-\varepsilon,0]$.

$$\hat{g} = \frac{1}{h(x_n)}g^{\partial M} + dx_n^2,$$

such that $\hat{g}|_M = g$. We fix a metric $\hat{g}$ on $\tilde{M}$ such that $\hat{g}|_M = g$.

Let $\nabla$ denote the Levi-civita connection about $g^M$. In the local coordinates $\{x_i; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{\hat{e}_1, \cdots, \hat{e}_n\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\nabla(\hat{e}_1, \cdots, \hat{e}_n) = (\hat{e}_1, \cdots, \hat{e}_n; \omega_{s,t}).$$

Let $c(e_i) = c(e_1) - \varnothing(e_i), \ \hat{c}(\hat{e}_i) = c(e_i) + \varnothing(e_i)$. Denote the exterior and interior multiplications by $c(e_j)$, $\varnothing(e_j)$ respectively, denote by $d + \delta : \wedge^*(T^*M) \to \wedge^*(T^*M)$ the signature operator. By [17], we have

$$d + \delta = \sum_{i=1}^n c(e_i) + \frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i) [\hat{c}(e_s)\hat{c}(e_t) - c(e_s)c(e_t)].$$

Then we define the nonminimal de Rham-Hodge operators as

$$\tilde{D} = ad + b\delta, \ \tilde{D}^* = bd + a\delta,$$

where $\tilde{D}^*$ is the adjoint operator of $\tilde{D}$ and $ab \neq 0$.

To define the lower dimensional volume, some basic facts and formulae about Boutet de Monvel’s calculus which can be found in Sec.2 in [10] and [16] are needed. Let

$$F : L^2(\mathbb{R}_x) \to L^2(\mathbb{R}_x); \ F(u)(v) = \int e^{-ivt}u(t)dt$$

denote the Fourier transformation and $\Phi(\mathbb{R}^r)$ (similarly define $\Phi(\mathbb{R}^{-r})$), where $\Phi(\mathbb{R})$ denotes the Schwartz space and

$$r^+ : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^r); \ f \to f|_{\mathbb{R}^r}; \ \mathbb{R}^r = \{x \geq 0; x \in \mathbb{R}\}.$$

We define $H^+ = \Phi(\mathbb{R}^r)$; $H_0^- = \Phi(\mathbb{R}^{-r})$ which are orthogonal to each other. We have the following property: $h \in H^+ (H_0^-)$ iff $h \in C^\infty(\mathbb{R})$, which has an analytic extension to the lower (upper) complex half-plane $\{\text{Im}\xi < 0\}$ $\{\text{Im}\xi > 0\}$) such that for all nonnegative integer $l$,

$$\frac{dh}{d\xi^l}(\xi) \sim \sum_{k=1}^{\infty} \frac{d^l}{d\xi^l}(C_k),$$

as $|\xi| \to +\infty, \text{Im}\xi \leq 0$ ($\text{Im}\xi \geq 0$).
Let $H'$ be the space of all polynomials and $H^- = H_0^- \oplus H'$; $H = H^+ \oplus H^-$. Denote by $\pi^+$ ($\pi^-$) respectively the projection on $H^+$ ($H^-$). For calculations, we take $H = \hat{H} = \{ \text{rational functions having no poles on the real axis} \}$ ($\hat{H}$ is a dense set in the topology of $H$). Then on $\hat{H}$,

$$\pi^+ h(\xi_0) = \frac{1}{2\pi i} \lim_{\nu \to 0} \int_{\Gamma^+} \frac{h(\xi)}{\xi_0 + i\nu - \xi} d\xi,$$

(2.1)

where $\Gamma^+$ is a Jordan close curve included $\text{Im} \xi > 0$ surrounding all the singularities of $h$ in the upper half-plane and $\xi_0 \in \mathbb{R}$. Similarly, define $\pi^-$ on $\hat{H}$,

$$\pi^- h = \frac{1}{2\pi} \int_{\Gamma^-} h(\xi) d\xi.$$

So, $\pi'(H^-) = 0$. For $h \in H \cap L^1(R)$, $\pi' h = \frac{1}{2\pi} \int_R h(v)dv$ and for $h \in H^+ \cap L^1(R)$, $\pi' h = 0$.

Denote by $\mathcal{B}$ Boutet de Monvel’s algebra, we recall the main theorem in [8].

**Theorem 2.1. (Fedosov-Golse-Leichtnam-Schrohe)** Let $X$ and $\partial X$ be connected, $\dim X = n \geq 3$, $A = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} \in \mathcal{B}$, and denote by $p$, $b$ and $s$ the local symbols of $P, G$ and $S$ respectively. Define:

$$\widetilde{\text{Wres}}(A) = \int_X \int_S \text{tr}_E \left[ p_{-n}(x, \xi) \right] \sigma(\xi) dx$$

$$+ 2\pi \int_{\partial X} \left\{ \text{tr}_E \left[ (tr \text{b}_{-n})(x', \xi') \right] + \text{tr}_F \left[ s_{1-n}(x', \xi') \right] \right\} \sigma(\xi') dx'.$$

(2.2)

Then a) $\widetilde{\text{Wres}}([A, B]) = 0$, for any $A, B \in \mathcal{B}$; b) It is a unique continuous trace on $\mathcal{B}/\mathcal{B}^{-\infty}$.

Let $p_1, p_2$ be nonnegative integers and $p_1 + p_2 \leq n$. Then by Sec 2.1 of [12], we have

**Definition 2.2.** Lower-dimensional volumes of Riemannian manifolds with boundary are defined by

$$\text{Vol}_n^{[p_1, p_2]} M := \widetilde{\text{Wres}}[\pi^+ \hat{D}^{-p_1} \circ \pi^+ (\hat{D}^*)^{-p_2}].$$

(2.3)

Denote by $\sigma_r(A)$ the $l$-order symbol of an operator $A$. For $n$ dimensional Riemannian manifolds with boundary, an application of (2.1.4) in [10] shows that

$$\widetilde{\text{Wres}}[\pi^+ \hat{D}^{-p_1} \circ \pi^+ (\hat{D}^*)^{-p_2}] = \int_M \int_{|\xi| = 1} \text{trace}_{\Lambda^r(T^*M)} [\sigma_{-n}(\hat{D}^{-p_1} (\hat{D}^*)^{-p_2})] \sigma(\xi) dx + \int_{\partial M} \Phi,$$

(2.4)

where

$$\Phi = \int_{|\xi'| = 1} \int_{|\xi| = 1} \sum_{j, k = 0}^{\infty} \frac{(-i)^{|\alpha| + j + k + 1}}{a!(j + k + 1)!} \text{trace}_{\Lambda^r(T^*M)} \left[ \partial_{x_n} \partial_{\xi_n} \sigma_r^{+}(\hat{D}^{-p_1})(x', 0, \xi', \xi_n) \right]$$

$$\times \partial_{x_n} \partial_{\xi_n} \sigma_r^{+}(\hat{D}^{-p_2})(x', 0, \xi', \xi_n) dx' \sigma(\xi') dx',$$

(2.5)

and the sum is taken over $r - k + |\alpha| + \ell - j - 1 = -n, r \leq -p_1, \ell \leq -p_2$.

3. A Kastler-Kalau-Walze type theorem of nonminimal de Rham-Hodge operators $\hat{D}$ and $\hat{D}^*$

In this section, we compute the lower dimension volume for four dimension compact connected manifolds with boundary associated to nonminimal de Rham-Hodge operators $\hat{D}$ and $\hat{D}^*$ and get a Kastler-Kalau-Walze type formula in this case. Let $M$ be an four dimensional compact oriented connected manifold with boundary $\partial M$, and the metric $g^M$ on $M$ as above. Note that

$$\hat{D} \hat{D}^* = a^2 d\delta + b^2 d\delta$$
is a nonminimal operator on $C^\infty(\Lambda^*(T^*M))$, then $[\sigma_{-4}((\tilde{D}\tilde{D}^*)^{-1})]_{_{\theta}}$ has the same expression with the case of without boundary in [13], so locally we can use Theorem 2.2 in [13] to compute the first term. Therefore

$$\int_M \int_{\xi=1} \text{tr} \Lambda_{\ast (T^*M)}[\sigma_{-4}((\tilde{D}\tilde{D}^*)^{-1})] \sigma(\xi) dx = 4\pi \int_M \sum_{k=0}^4 c_1(4, k, a, b) R_{\text{vol}}(M),$$

(3.1)

where $R$ is the scalar curvature and $c_1(4, k, a, b) = b^{-2}(\frac{1}{6}(\xi) - (\xi^2)_{a-1}) + (b^{-2} - a^{-2}) \sum_{j<k}(-1)^{j-k} \frac{1}{2}(\xi) - (\xi^2)_{j-1})$.

Hence we only need to compute $\int_{\partial M} \Phi$. Firstly, we compute the symbol $\sigma(\tilde{D}^{-1})$ and $\sigma((\tilde{D}^*)^{-1})$. Denote by $\tilde{c}(\xi) = a e(\xi) - b i(\xi), \tilde{\epsilon}(\xi) = be(\xi) - ai(\xi)$, then we have $\tilde{c}(e_j) = ae(e_j) - bi(e_j)$ and $\tilde{\epsilon}(e_j) = be(e_j) - ai(e_j)$. From the form of Signature operator

$$\tilde{D} = \sum_{i=1}^n \tilde{c}(e_i) \left[ e_i + \sum_{s,t} \omega_{s,t}(e_i) \left( \tilde{c}(e_s)\tilde{c}(e_t) - c(e_s)c(e_t) \right) \right],$$

(3.2)

we get

$$\sigma_1(\tilde{D}) = \sqrt{-1} \tilde{c}(\xi);$$

(3.3)

$$\sigma_0(\tilde{D}) = \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(e_i) \tilde{c}(e_i) \left[ \tilde{c}(e_s)\tilde{c}(e_t) - c(e_s)c(e_t) \right],$$

(3.4)

where $\xi = \sum_{i=1}^n \xi_i dx_i$ denotes the cotangent vector. Write

$$\tilde{D}_{\theta}^n = (-\sqrt{-1})^{n} \partial^n_{\theta}; \quad \sigma(\tilde{D}) = p_1 + p_0; \quad \sigma(\tilde{D}^{-1}) = \sum_{j=1}^\infty q_{-j}.$$

(3.5)

By the composition formula of psudodifferential operators, we have

$$1 = \sigma(\tilde{D} \circ \tilde{D}^{-1}) = \sum_{\alpha} \frac{1}{a!} \partial_{\alpha}^{\sigma(n)}[\sigma(\tilde{D})] D_{\alpha}^n[\sigma(\tilde{D}^{-1})]$$

$$= (p_1 + p_0)(q_{-1} + q_{-2} + q_{-3} + \cdots) + \sum_{j} (\partial_{\xi_j} p_1 + \partial_{\xi_j} p_0)(D_{x_j} q_{-1} + D_{x_j} q_{-2} + \cdots)$$

$$= p_1 q_{-1} + (p_1 q_{-2} + p_0 q_{-1} + \sum_{j} \partial_{\xi_j} p_1 D_{x_j} q_{-1}) + \cdots.$$

Thus, we obtain

$$q_{-1} = p_1^{-1}; \quad q_{-2} = -p_1^{-1} p_0 + \sum_{j} \partial_{\xi_j} p_1 D_{x_j} (p_1^{-1}).$$

(3.6)

By (3.2), (3.5) and direct computations, we have

**Lemma 3.1.** Let $\tilde{D}, \tilde{D}^*$ on $C^\infty(\Lambda^*(T^*M))$. Then

$$q_{-1}(\tilde{D}^{-1}) = \sqrt{-1} \frac{\tilde{c}(\xi)}{a\xi^2};$$

(3.7)

$$q_{-2}(\tilde{D}^{-1}) = \frac{\tilde{c}(\xi) p_0(\tilde{D}) \tilde{c}(\xi)}{a^2 b^2 |\xi|^4} + \frac{\tilde{c}(\xi)}{a^2 b^2 |\xi|^6} \sum_{j} \tilde{c}(dx_j) \left[ \partial_{x_j} [\tilde{c}(\xi)] |\xi|^2 - \tilde{c}(\xi) \partial_{x_j} |\xi|^2 \right];$$

(3.8)

$$q_{-1}(\tilde{D}^*)^{-1}) = \sqrt{-1} \frac{\tilde{c}(\xi)}{ab |\xi|^2};$$

(3.9)

$$q_{-2}(\tilde{D}^*)^{-1}) = \frac{\tilde{c}(\xi) p_0(\tilde{D}^*) \tilde{c}(\xi)}{a^2 b^2 |\xi|^4} + \frac{\tilde{c}(\xi)}{a^2 b^2 |\xi|^6} \sum_{j} \tilde{c}(dx_j) \left[ \partial_{x_j} [\tilde{c}(\xi)] |\xi|^2 - \tilde{c}(\xi) \partial_{x_j} |\xi|^2 \right],$$

(3.10)
where
\[
\tilde{p}_0 = \sigma_0(D)(x_0) = -\frac{1}{4} h'(0) \sum_{i=1}^{n-1} \tilde{c}(e_i) \tilde{c}(e_i) \tilde{c}(e_n)(x_0) + \frac{1}{4} h'(0) \sum_{i=1}^{n-1} \tilde{c}(e_i) c(e_i) c(e_n)(x_0); \tag{3.11}
\]
\[
\tilde{p}_0 = \sigma_0(D^*)(x_0) = -\frac{1}{4} h'(0) \sum_{i=1}^{n-1} \tilde{c}(e_i) \tilde{c}(e_i) \tilde{c}(e_n)(x_0) + \frac{1}{4} h'(0) \sum_{i=1}^{n-1} \tilde{c}(e_i) c(e_i) c(e_n)(x_0). \tag{3.12}
\]

Since $\Phi$ is a global form on $\partial M$, so for any fixed point $x_0 \in \partial M$, we can choose the normal coordinates $U$ of $x_0$ in $\partial M$ (not in $M$) and compute $\Phi(x_0)$ in the coordinates $U = U \times [0, 1)$ and the metric $\frac{1}{h(x_n)} g^{\text{dM}} + dx_n^2$. The dual metric of $g^{\text{dM}}$ on $\tilde{U}$ is $h(x_n)g^{\text{dM}} + dx_n^2$. Write $g^{\text{dM}}_{ij} = g^{\text{dM}}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}); \ g^{\text{dM}} = g^{\text{dM}}(dx_i, dx_j)$, then
\[
[g^{\text{dM}}_{ij}] = \begin{bmatrix}
\frac{1}{h(x_n)} g^{\text{dM}}_{ij} & 0 \\
0 & 0
\end{bmatrix}; \quad [g^{\text{dM}}_{ij}] = \begin{bmatrix}
h(x_n) g^{\text{dM}}_{ij} & 0 \\
0 & 1
\end{bmatrix},
\]
and
\[
\partial_{x_i} g^{\text{dM}}_{ij}(x_0) = 0, \quad 1 \leq i, j \leq n - 1; \quad g^{\text{dM}}_{ij}(x_0) = \delta_{ij}. \tag{3.13}
\]

By Lemma 2.2 in [12] and the normal coordinates $U$ of $x_0$ in $\partial M$ (not in $M$), we have

**Lemma 3.2.** With the metric $g^{\text{dM}}$ on $M$ near the boundary
\[
\partial_{x_i}(|\xi|^{2}_{g^{\text{dM}}})(x_0) = \begin{cases}
0, & \text{if } j < n; \\
h'(0)|\xi|^{2}_{g^{\text{dM}}}, & \text{if } j = n,
\end{cases} \tag{3.14}
\]
\[
\partial_{x_i} (\tilde{c}(\xi))(x_0) = \begin{cases}
0, & \text{if } j < n; \\
\partial x_i(\tilde{c}(\xi))(x_0), & \text{if } j = n.
\end{cases} \tag{3.15}
\]

where $\xi = \xi' + \xi_a dx_a$.

**Lemma 3.3.** If $i < n$, $\omega_{i,n}(\tilde{c}_i)(x_0) = \frac{1}{2} h'(0)$; and $\omega_{i,n}(\tilde{c}_i)(x_0) = -\frac{1}{2} h'(0)$, In other cases, $\omega_{i,n}(\tilde{c}_i)(x_0) = 0$.

**Lemma 3.4.** By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then we have the equalities:
\[
\text{tr}[\tilde{c}(\xi') \tilde{p}_0 \tilde{c}(\xi') \epsilon(dx_n)](x_0)|_{\xi'|=1} = 6a^2 b h'(0); \quad \text{tr}[\tilde{c}^2(\xi') \tilde{p}_0 \epsilon(dx_n)](x_0)|_{\xi'|=1} = -6a^2 b h'(0);
\]
\[
\text{tr}[\tilde{c}(\xi') \tilde{p}_0 \epsilon(dx_n)](x_0)|_{\xi'|=1} = -6 b^2 a h'(0); \quad \text{tr}[\tilde{c}(\xi') \tilde{p}_0 \epsilon(dx_n)](x_0)|_{\xi'|=1} = 6 a^2 b h'(0);
\]
\[
\text{tr}[\tilde{c}(\xi') \tilde{p}_0 \epsilon(dx_n)](x_0)|_{\xi'|=1} = -2 a^2 b h'(0); \quad \text{tr}[\tilde{c}(\xi') \tilde{p}_0 \epsilon(dx_n)](x_0)|_{\xi'|=1} = 10 a^2 b h'(0);
\]
\[
\text{tr}[\tilde{c}(\xi') \tilde{p}_0 \epsilon(dx_n)](x_0)|_{\xi'|=1} = -10 b^2 a h'(0); \quad \text{tr}[\tilde{c}(\xi') \tilde{p}_0 \epsilon(dx_n)](x_0)|_{\xi'|=1} = 2 a^2 b h'(0). \tag{3.16}
\]

**others vanishes.**

**Proof.** By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then
\[
\text{tr}[\tilde{c}(\xi') \tilde{p}_0 \epsilon(dx_n)](x_0)|_{\xi'|=1} = \text{tr}[(a \epsilon(\xi') - b \epsilon(\xi')) \tilde{p}_0 \epsilon(dx_n)](x_0)|_{\xi'|=1} = ab \text{tr}[(\epsilon(\xi') \tilde{c}(\xi') \tilde{p}_0 \epsilon(dx_n)](x_0)|_{\xi'|=1} + \text{tr}[(\tilde{c}(\xi') \epsilon(\xi') \tilde{p}_0 \epsilon(dx_n)](x_0)|_{\xi'|=1}
\]
\[
= ab \text{tr}[\tilde{p}_0 \epsilon(dx_n)](x_0)|_{\xi'|=1}, \tag{3.17}
\]

where
\[
\text{tr}[\tilde{p}_0 \epsilon(dx_n)] = \text{tr}[(-\frac{1}{4} h'(0) \sum_{i=1}^{n-1} \tilde{c}(e_i) \tilde{c}(e_i) \tilde{c}(e_n) + \frac{1}{4} h'(0) \sum_{i=1}^{n-1} \tilde{c}(e_i) c(e_i) c(e_n)) \epsilon(dx_n)]
\]
\[
= -\frac{1}{4} h'(0) \sum_{i=1}^{n-1} \text{tr}[\epsilon(dx_n)](e_i) \tilde{c}(e_i) \tilde{c}(e_n) + \frac{1}{4} h'(0) \sum_{i=1}^{n-1} \text{tr}[\epsilon(dx_n)](e_i) c(e_i) c(e_n)]. \tag{3.18}
\]
By the relation of the Clifford action and \( \epsilon(e_i)\epsilon(e_j) + i\epsilon(e_j)\epsilon(e_i) = \delta_{ij} \), we obtain
\[
\sum_{i=1}^{n-1} \text{tr}[\epsilon(dx_n)\tilde{c}(e_i)\tilde{c}(e_i)\tilde{c}(e_n)] = a \sum_{i=1}^{n-1} \text{tr}[\epsilon(e_i)\epsilon(e_i)\epsilon(e_n)] - b \sum_{i=1}^{n-1} \text{tr}[\epsilon(e_i)\epsilon(e_i)\epsilon(e_n)]
\]
\[
= \frac{a}{2} \sum_{i=1}^{n-1} \text{tr}[\epsilon(e_i)\epsilon(e_i)] - b \sum_{i=1}^{n-1} \text{tr}[\epsilon(e_i)\epsilon(e_i)] = 12(a - b),
\]
(3.19)
and
\[
\sum_{i=1}^{n-1} \text{tr}[\epsilon(dx_n)\tilde{c}(e_i)c(e_i)c(e_n)] = 12(a + b).
\]
(3.20)
Combining (3.17)-(3.20), we have
\[
\text{tr}[\tilde{c}(\xi')\tilde{p}_0\tilde{c}(\xi')\epsilon(dx_n)](x_0)|_{\xi'|=1} = 6ab^2h'(0).
\]
(3.21)
Others are similarly.

Let us now turn to compute \( \Phi \) (see formula (2.5) for definition of \( \Phi \)). Since the sum is taken over 
\(-r - \ell + k + j + |\alpha| = 3, r, \ell \leq -1\), then we have the following five cases:

**Case a (I):** \( r = -1, \ell = -1, k = j = 0, |\alpha| = 1 \)

From (2.5) we have
\[
\text{Case a (I)} = -\int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{tr} \left[ \partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_+ (\tilde{D}^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'.
\]
(3.22)
Then an application of Lemma 3.2 shows that,
\[
\partial_{\xi'} \sigma_+ (\tilde{D}^{-1})(x_0) = \partial_{\xi'} \left( \frac{\sqrt{-1}\tilde{c}(\xi)}{ab|\xi|^2} \right) (x_0) = \frac{\sqrt{-1}\partial_{\xi_n}[\tilde{c}(\xi)](x_0)}{ab|\xi|^2} = \frac{\sqrt{-1}\tilde{c}(\xi)\partial_{\xi_n}(|\xi|^2)(x_0)}{ab|\xi|^4} = 0,
\]
(3.23)
so Case a (I) vanishes.

**Case a (II):** \( r = -1, \ell = -1, k = |\alpha| = 0, j = 1 \)

From (2.5) we have
\[
\text{Case a (II)} = -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_+ (\tilde{D}^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'.
\]
(3.24)
From Lemma 3.1 and Lemma 3.2, we have
\[
\partial_{\xi_n} \sigma_+ (\tilde{D}^{-1})(x_0)|_{\xi'|=1} = \frac{\sqrt{-1}\partial_{\xi_n}[\tilde{c}(\xi)](x_0)}{ab|\xi|^2} = \frac{\sqrt{-1}\tilde{c}(\xi)h'(0)}{ab|\xi|^4},
\]
(3.25)
By the Cauchy integral formula, we obtain
\[
\pi_{\xi_n}^+ \left[ \frac{1}{(1 + \xi_n^2)^2} \right] (x_0)|_{\xi'|=1} = \frac{1}{2\pi i} \lim_{u \to 0} \int_{\Gamma^+} \frac{1}{(\eta_n + i(\xi_n^2 + u) - \eta_n)^2} d\eta_n = -\frac{i\xi_n + 2}{4(\xi_n - i)^2},
\]
(3.26)
and
\[
\pi_{\xi_n}^+ \left[ \frac{\sqrt{-1}\partial_{\xi_n}[\tilde{c}(\xi')]}{|\xi'|^2} \right] (x_0)|_{\xi'|=1} = \frac{\partial_{\xi_n}[\tilde{c}(\xi')](x_0)}{2(\xi_n - i)}.
\]
(3.27)
From Lemma 3.1 and Lemma 3.2, we get
\[
\frac{\partial_x \sigma^+}{\partial x} = \frac{i\epsilon'(x)\epsilon(x)}{2ab(\xi_n - i)} + \frac{i\epsilon'(x)}{4(\xi_n - i)^2} + \frac{i\epsilon'(x)}{4(\xi_n - i)^2}.
\]

From Lemma 3.2, we have
\[
\text{Case a (III)}
\]
\[
\int_{\xi_n}^{a} \frac{i\epsilon'(x)}{4(\xi_n - i)^2} - \frac{b\epsilon'(x)\epsilon(x)}{4(\xi_n - i)^2} + \frac{a\epsilon'(x)}{4(\xi_n - i)^2} = \frac{b\epsilon'(x)\epsilon(x)}{4(\xi_n - i)^2}.
\]

By the relation of the Clifford action and trace, we have the equalities:
\[
\text{trace}[\cos \pi_n \epsilon - (D^{-1})(x)] = 8; \text{trace}[\cos \epsilon d(x_n)\epsilon(dx_n)] = 8; \text{trace}[\cos \epsilon (\epsilon')\epsilon(x)\epsilon(dx_n)] = 8h'(x);
\]
\[
\text{trace}[\cos \epsilon \epsilon(dx_n)\epsilon(dx_n)] = 0.
\]

Combining (3.28), (3.29) and (3.30), we have
\[
\text{trace}[\partial_x \sigma^+ \sigma_1 ((D^*)^{-1})(x)]_{\xi_n = 1} = \frac{h'(x)}{a} \left( \frac{8(-i\xi_n - i\xi_3)}{(\xi_n - i)^2(1 + \xi_3)^3} \right) + \frac{h'(x)}{b} \left( \frac{8(-1 - 2i\xi_n + 3\xi_3 + 2i\xi_3)}{(\xi_n - i)^2(1 + \xi_3)^3} \right).
\]

Hence
\[
\text{Case a (II)}
\]
\[
\int_{\xi_n}^{a} \frac{i\epsilon'(x)}{4(\xi_n - i)^3} - \frac{b\epsilon'(x)\epsilon(x)}{4(\xi_n - i)^3} + \frac{a\epsilon'(x)}{4(\xi_n - i)^3} = \frac{b\epsilon'(x)\epsilon(x)}{4(\xi_n - i)^3}.
\]

where \(\Omega_3\) is the canonical volume of \(S^3\).

Case a (III): \(r = -1, \ell = -1, j = |a| = 0, k = 1\)

From (2.5) we have
\[
\text{Case a (III)} = \frac{-1}{2} \int_{|\xi_n| = 1}^{+\infty} \text{trace} \left[ \partial_x \sigma^+ \sigma_1 ((D^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'.
\]

From Lemma 1 and Lemma 2, we get
\[
\partial_x \sigma^+ \sigma_1 ((D^*)^{-1})(x_0)_{|\xi_n| = 1} = \frac{\epsilon'(x) + i\epsilon(dx_n)}{2ab(\xi_n - i)^2} = \frac{a\epsilon'(x) - b\epsilon'(x) + i(\epsilon(dx_n) + \epsilon(dx_n))}{2ab(\xi_n - i)^2}.
\]
and
\[
\partial_{\xi_n} \partial_{\xi_n} \sigma_1((\hat{D}^*)^{-1})(x_0)|_{\xi'=1} = \frac{-\sqrt{-1}h'(0)}{ab} \left[ \frac{\tilde{c}(dx_n)}{\xi_0} - 4\xi_0 \frac{\tilde{c}(\xi') + \xi_0 \tilde{c}(dx_n)}{|\xi|^6} \right] - 2\frac{\xi_0 \sqrt{-1} \partial_{\xi_n} \tilde{c}(\xi')(x_0)}{ab|\xi|^4} \\
= \frac{2i\xi_0 \partial_{\xi_n}[\tilde{c}(\xi')(x_0)]}{b|\xi|^4} + \frac{i h'(0)}{ab} \left[ \frac{2i \xi_0 \tilde{c}(\xi')}{|\xi|^6} + \frac{4a \xi_0 t(\xi')}{|\xi|^6} \right] \frac{b(|\xi|^2 - 4\xi_n^2)\tilde{c}(dx_n) + a(|\xi|^2 - 4\xi_n^2)\tilde{c}(dx_n)}{|\xi|^6}.
\]

Combining (3.34) and (3.35), we obtain
\[
\text{trace} \left[ \partial_{\xi_n} \pi^+_{\xi_n} \sigma_1((\hat{D}^*)^{-1}) \times \partial_{\xi_n} \partial_{\xi_n} \sigma_1((\hat{D}^*)^{-1})(x_0) \right]|_{\xi'=1} = \frac{h'(0)}{a^2} \frac{4(1 + 4i\xi_n - 3\xi_n^2)}{(\xi_n - i)^2(1 + \xi_n^2)^3} - \frac{h'(0)}{b^2} \frac{4(-1 - 2i\xi_n + 3\xi_n^2 + 2\xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^3)^3}.
\]

Then
\[
\text{Case a (III)}
\]
\[
= \frac{-1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{h'(0)}{2a^2} \frac{4(1 + 4i\xi_n - 3\xi_n^2)}{(\xi_n - i)^2(1 + \xi_n^2)^3} d\xi_n \sigma(\xi') dx'
\]
\[
= \frac{1}{2} \frac{h'(0)}{a^2} \frac{2\pi i}{4!} \left( -12i \right) dx' + \frac{1}{2} \frac{h'(0)}{b^2} \frac{2\pi i}{4!} \frac{2\pi i}{4!} (2i) dx'
\]
\[
= \left( \frac{1}{2a^2} + \frac{1}{2b^2} \right) \pi h'(0) \Omega_3 dx'.
\]

\[
\text{Case b: } r = -2, \ell = -1, k = j = |\alpha| = 0
\]
From (2.5) we have
\[
\text{Case b: } -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi^+_{\xi_n} \sigma_2((\hat{D}^*)^{-1}) \times \partial_{\xi_n} \sigma_1((\hat{D}^*)^{-1})(x_0) \right] d\xi_n \sigma(\xi') dx'.
\]
By Lemma 3.1 and Lemma 3.2, we obtain
\[
\partial_{\xi_n} \sigma_2((\hat{D}^*)^{-1})(x_0) = \frac{\sqrt{-1}}{ab} \left( \frac{2\xi_n^2 \tilde{c}(dx_n) + 2c_0 \tilde{c}(\xi')}{|\xi|^4} \right)
\]
\[
= \frac{\sqrt{-1}}{ab} \left[ \frac{-2b_0 t(\xi')}{(1 + \xi_n^2)^2} + \frac{2a \xi_0 t(\xi')}{(1 + \xi_n^2)^2} \right] \frac{b(1 - \xi_n^2)\tilde{c}(dx_n) + a(1 - \xi_n^2)\tilde{c}(dx_n)}{|\xi|^2(1 + \xi_n^2)^2}
\]
and
\[
\sigma_2((\hat{D}^*)^{-1})(x_0) = \frac{c(\xi')\tilde{p}_0(x_0) + \tilde{c}(\xi')}{a^2 b^2 |\xi|^4} \tilde{c}(dx_n) \left[ \partial_{\xi_n} [\tilde{c}(\xi')](x_0) \right] \frac{\tilde{c}(dx_n)}{|\xi|^2} - \frac{c\xi_0}{a^2 b^2 (1 + \xi_n^2) |\xi|^2} \\
\]
\[
\frac{\xi_0}{a^2 b^2 (1 + \xi_n^2)} \right] \frac{\tilde{c}(dx_n)}{|\xi|^2} - \frac{c\xi_0}{a^2 b^2 (1 + \xi_n^2) |\xi|^2} \\
\]
\[
\text{Then}
\]
\[
\pi_{\xi_n}^+ \sigma_2((\hat{D}^*)^{-1})(x_0) \big|_{\xi'=1} = \pi_{\xi_n}^+ \left[ \frac{c(\xi')\tilde{p}_0(x_0) + \tilde{c}(\xi')}{a^2 b^2 (1 + \xi_n^2) |\xi|^2} \right] - \frac{c(\xi') \tilde{c}(dx_n) - \tilde{c}(\xi')}{a^2 b^2 (1 + \xi_n^2) |\xi|^2} \\
\]
\[
=: B_1 - B_2,
\]
\[
(3.41)
\]
From (3.39) and (3.42), we have

\[
B_1 = \frac{-1}{4a^2b^2(\xi_n - i)^2} \left[ (2 + i\xi_n)\tilde{c}(\xi')\tilde{p}_0\tilde{c}(\xi') + i\xi_n\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(dx_n) \right] + \frac{1}{4a^2(\xi_n - i)^2} \left[ (2 + i\xi_n)\tilde{c}(\xi')\tilde{p}_0\tilde{c}(\xi') + i\xi_n\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(dx_n) \right] + \frac{1}{4a^2(\xi_n - i)^2} \left[ (2 + i\xi_n)\tilde{c}(\xi')\tilde{p}_0\tilde{c}(\xi') + i\xi_n\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(dx_n) \right]
\]

Combining (3.43) and (3.45), we have

\[
\frac{1}{4b(\xi_n - i)^2} \left[ (2 + i\xi_n)\epsilon(dx_n)\partial_{\xi_n}[i(\xi')] f(\xi'')(x_0) + \frac{b(2 + i\xi_n)i(\xi')\epsilon(dx_n)\partial_{\xi_n}[i(\xi')] f(\xi'')(x_0) - i\partial_{\xi_n}[i(\xi')] f(\xi'')(x_0) \right] = \frac{1}{4a(\xi_n - i)^2} (3.42)
\]

From (3.39) and (3.42), we have

\[
\text{trace} \left[ B_1 \times \dot{\partial}_{\xi_n} \sigma_{-1} (D^*)^{-1} \right] (x_0)_{\xi' = 1} = \frac{h'(0)}{a^2} \times \frac{-i(3 + 5i\xi_n - 3\xi_n^2 + 3\xi_n^4)}{(\xi_n - i)^2(1 + \xi_n^2)} + \frac{h'(0)}{b^2} \times \frac{i(-1 - 23i\xi_n + \xi_n^2 + 2i\xi_n^4)}{(\xi_n - i)^2(1 + \xi_n^2)}.
\]

On the other hand,

\[
B_2 = \frac{h'(0)\pi^+_{\xi_n}}{a^2b^2(\xi_n - i)^3 (1 + \xi_n^2)} \left[ -\xi_n^2 c(dx_n) - 2\xi_n c(\xi') + c(dx_n) \right] = \frac{h'(0)}{2ab} \left[ \frac{\tilde{c}(dx_n)}{4i(\xi_n - i)} + \frac{\tilde{c}(dx_n) - i\tilde{c}(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3}[i\tilde{c}(\xi') - \tilde{c}(dx_n)] \right] = \frac{3 + i\xi_n}{8b(\xi_n - i)^3} \frac{h'(0)i(\xi')}{3 + i\xi_n} + \frac{3 - 3\xi_n + 4i}{8a(\xi_n - i)^3} \frac{h'(0)i(\xi')}{3 - 3\xi_n + 4i}
\]

\[
\text{trace} \left[ B_2 \times \dot{\partial}_{\xi_n} \sigma_{-1} (D^*)^{-1} \right] (x_0)_{\xi' = 1} = \frac{h'(0)}{a^2} \times \frac{-i(4i - 9\xi_n - 7i\xi_n^2 + 3\xi_n^4 + i\xi_n^4)}{(\xi_n - i)^3(1 + \xi_n^2)} + \frac{h'(0)}{b^2} \times \frac{i(4i - 9\xi_n - 7i\xi_n^2 + 3\xi_n^4 + i\xi_n^4)}{(\xi_n - i)^3(1 + \xi_n^2)}.
\]

Combining (3.43) and (3.45), we have

\[
\text{Case b)} = -i \int_{[\xi' = 1]} \int_{-\infty}^{+\infty} \text{trace} \left[ (B_1 - B_2) \times \dot{\partial}_{\xi_n} \sigma_{-1} (D^*)^{-1} \right] (x_0) dx_0 \sigma(\xi') dx' = \frac{1}{8a_i^2} + \frac{11}{8b^2} (\pi h'(0)\Omega_3) dx'.
\]

\[
\text{Case c)}: r = -1, k = j = \alpha = 0 \quad \text{From (2.5) we have)
\]

\[
\text{Case c = } -i \int_{[\xi' = 1]} \int_{-\infty}^{+\infty} \text{trace} \left[ \pi^+_{\xi_n} (D^*)^{-1} \times \dot{\partial}_{\xi_n} \sigma_{-2} (D^*)^{-1} \right] (x_0) dx_0 \sigma(\xi') dx'.
\]

By Lemma 3.1, Lemma 3.2 and Lemma 3.4, we obtain

\[
\pi^+_{\xi_n} (D^*)^{-1} (x_0)_{\xi' = 1} = \frac{\epsilon(\xi') + i\epsilon(dx_n)}{2b(\xi_n - i)} = \frac{\epsilon(\xi')}{2b(\xi_n - i)} - \frac{i(\xi')}{2a(\xi_n - i)} + \frac{i\epsilon(dx_n)}{2b(\xi_n - i)} - \frac{i\epsilon(dx_n)}{2a(\xi_n - i)}.
\]
Combining (3.48) and (3.49), we have

\[ \begin{align*}
\frac{1}{a^2b^2(1 + \xi_n^2)^3} \left[ (2\xi_n - 2\xi_n^3)\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(dx_n) + (1 - 3\xi_n^2)\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(\xi') \right. \\
\left. + (1 - 3\xi_n^2)\tilde{c}(\xi')\tilde{p}_0\tilde{c}(dx_n) - 4\xi_n\tilde{c}(\xi')\tilde{p}_0\tilde{c}(\xi') + \left( 3\xi_n^2 - 1 \right) ab\partial_{x_n}\tilde{c}(\xi') - 4\xi_n\tilde{c}(\xi')\tilde{c}(dx_n)\partial_{x_n}\tilde{c}(\xi') \right] \\
+ 2ab\tilde{h}'(0)\tilde{c}(\xi') + 2ab\tilde{h}'(0)\xi_n\tilde{c}(dx_n) \right] + 6\xi_n\tilde{h}'(0)\tilde{c}(\xi') \quad a^2b^2(1 + \xi_n)^3 \\
\end{align*} \]

Then similarly to computations of the case b), we have

\[ \begin{align*}
\text{Wres}[\pi^+\tilde{D}^{-1} \circ \pi^+(\tilde{D}^*)^{-1}](x_0) \big|_{|\xi'|=1} = \\
\begin{split}
\frac{h'(0)}{a^2} \left[ -2(-7 - 21i\xi_n + 26\xi_n^2 + 6i\xi_n^3 + 9\xi_n^4 + 3\xi_n^5) \right] \\
+ \frac{h'(0)}{b^2} \left[ -2(-1 - 25i\xi_n + 26\xi_n^2 + 21\xi_n^3 + 3\xi_n^4 + 3\xi_n^5) \right] \\
\end{split}
\end{align*} \]

(3.50)

Then similarly to computations of the case b), we have

\[ \text{case c) } = \frac{5}{2a^2}\pi h'(0)\Omega_3dx'. \]

(3.51)

Since \( \Phi \) is the sum of the case a, b and c,

\[ \Phi = \frac{23}{8} \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \pi h'(0)\Omega_3dx'. \]

(3.52)

Then we have

**Theorem 3.5.** Let \( M \) be a four dimensional compact connected manifold with the boundary \( \partial M \) and the metric \( g^M \) as above, and \( \tilde{D}, \tilde{D}^* \) are the nonminimal de Rham-Hodge operators on \( C^\infty(\Lambda^*(T^*M)) \), then

\[ \text{Wres}[\pi^+\tilde{D}^{-1} \circ \pi^+(\tilde{D}^*)^{-1}] = 4\pi \int_M \sum_{k=0}^{4} c_1(4, k, a, b) Rdvol(M) - \frac{23}{12} \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \pi \int_{\partial M} K\Omega_3dx', \]

(3.53)

where \( R \) is the scalar curvature and \( c_1(4, k, a, b) = b^{-2}\left( \frac{1}{6}k^4 - \frac{1}{2}k^2 \right) + (b^{-2} - a^{-2}) \sum_{j<k}(-1)^{j-k} \frac{4}{j^4} \left( \frac{1}{k^4} - \frac{1}{j^4} \right) \).

Let us now consider the Einstein-Hilbert action for four dimensional manifolds with boundary. Recall the Einstein-Hilbert action for manifolds with boundary [12],

\[ \begin{align*}
I_{Gr} &= \frac{1}{16\pi} \int_M Rdvol_M + 2 \int_{\partial M} Kdvol_{\partial M} := I_{Gr,1} + I_{Gr,2},
\end{align*} \]

(3.54)
where

\[ K = \sum_{1 \leq i, j \leq n-1} K_{i,j} \delta_{\partial M}^j \cdot K_{i,j} = -\Gamma^n_{i,j}, \]  

(3.55)

and $K_{i,j}$ is the second fundamental form, or extrinsic curvature. Take the metric in Section 2, $K_{i,j}(x_0) = -\Gamma^n_{i,j}(x_0) = -\frac{1}{2} h'(0)$, when $i = j < n$, otherwise is zero.

\[
\begin{align*}
\widetilde{\text{Wres}}[\pi^+(\tilde{D})^{-1} & \circ \pi^+(\tilde{D}^*)^{-1}] = \widetilde{\text{Wres}}_{\pi^+ \tilde{D}^{-1} \circ \pi^+(\tilde{D}^*)^{-1}] + \widetilde{\text{Wres}}_{\pi^+ \tilde{D}^{-1} \circ \pi^+(\tilde{D}^*)^{-1}]}, \\
\text{where } & \\
\text{and } & \\
\end{align*}
\]

(3.56)

where

\[
\begin{align*}
\widetilde{\text{Wres}}_{\pi^+ (\tilde{D})^{-1} \circ \pi^+(\tilde{D}^*)^{-1}] = & \int_M \int_{|\xi| = 1} \text{trace}_{\Lambda^*(T^*M)}[\sigma_{-4}(\tilde{D}^*\tilde{D}^{-1})]\sigma(\xi) dx \\
\text{and } & \\
\end{align*}
\]

(3.57)

and

\[
\begin{align*}
\widetilde{\text{Wres}}_{\pi^+ (\tilde{D})^{-1} \circ \pi^+(\tilde{D}^*)^{-1}] = & \int_{\partial M} \int_{|\xi| = 1} \sum_{j,k=0}^{+\infty} \sum_{\alpha = 0}^{k+1} \frac{(-i)^{\alpha+j+k+1}}{\alpha!(j+k+1)!} \times \text{trace}_{\Lambda^*(T^*M)}[\partial^j_{\xi^j} \partial^k_{\xi^k} \sigma_{-4}']((\tilde{D}^*\tilde{D}^{-1})(x', 0, 0', 0, 0, 0) \partial\xi_{\alpha} \sigma(\xi') dx' \\
\end{align*}
\]

(3.58)

denote the interior term and boundary term of $\widetilde{\text{Wres}}[\pi^+(\tilde{D})^{-1} \circ \pi^+(\tilde{D}^*)^{-1}]$. Combining (3.42), (3.43) and (3.45), we obtain

**Theorem 3.6.** Let $M$ be a four dimensional compact manifold with the boundary $\partial M$ associated to non-minimal de Rham-Hodge operators $\tilde{D}$ and $\tilde{D}^*$. Assume $\partial M$ is flat, then

\[
\begin{align*}
I\text{Gr},a = & \frac{1}{64\pi \xi_i(4, k, a, b)} \widetilde{\text{Wres}}_{\pi^+ (\tilde{D})^{-1} \circ \pi^+(\tilde{D}^*)^{-1}]}; \\
I\text{Gr},b = & \frac{-24}{23(\frac{1}{\alpha^2} + \frac{1}{\beta^2})\pi \Omega_3} \widetilde{\text{Wres}}_{\pi^+ (\tilde{D})^{-1} \circ \pi^+(\tilde{D}^*)^{-1}]}. \\
\end{align*}
\]

(3.59)

4. The Kastler-Kalau-Walze type theorem of the nonminimal de Rham-Hodge operators $\tilde{D}$

In this section, we compute the lower dimension volume for four dimension compact connected manifolds with boundary associated to nonminimal de Rham-Hodge operators $\tilde{D}$ and get a Kastler-Kalau-Walze type theorem in this case. Let $M$ be an four dimensional compact oriented connected manifold with boundary $\partial M$, and the metric $g^M$ on $M$ as above. Note that $[\sigma_{-4}(\tilde{D})^{-2}]|_{M}$ has the same expression with the case of without boundary in [12], so locally we can use Theorem 3.1 in [12] to compute the first term. Therefore

\[
\int_M \int_{|\xi| = 1} \text{trace}_{\Lambda^*(T^*M)}[\sigma_{-4}(\tilde{D}^{-2})]\sigma(\xi) dx = \frac{8\Omega_4}{3ab} \int_M R \text{vol} M, 
\]

(4.1)

where $R$ is the scalar curvature.

Let us now turn to compute $\Phi$ (see formula (2.5) for definition of $\Phi$). Since the sum is taken over $-r - \ell + k + j + |\alpha| = 3$, $r, \ell \leq -1$, then we have the following five cases:

**Case a (I):** $r = -1$, $\ell = -1$, $k = j = 0$, $|\alpha| = 1$

From (2.5) we have

\[
\text{Case a (I)} = - \int_{|\xi| = 1} \int_{-\infty}^{+\infty} \sum_{|\alpha| = 1} \text{trace}[\partial^2_{\xi^i} \sigma_{-1}(\tilde{D}^{-1}) \times \partial^2_{\xi^j} \sigma_{-1}(\tilde{D}^{-1})](x_0) d\xi_0 \sigma(\xi') dx'.
\]

(4.2)
By the Cauchy integral formula we obtain
\[ \partial_{z_1} \sigma_1 (\tilde{D}^{-1}) (x_0) = \partial_{z_1} \left( \frac{\sqrt{-1} i \partial_{x_1} \tilde{e} (\xi)}{ab | \xi |^2} \right) (x_0) = \frac{\sqrt{-1} i \partial_{x_1} \tilde{e} (\xi) (x_0)}{ab | \xi |^2} - \frac{\sqrt{-1} i \partial_{x_1} \tilde{e} (| \xi |^2) (x_0)}{ab | \xi |^4} = 0, \tag{4.3} \]
so Case a (I) vanishes.

**Case a (II):** \( r = -1, \ell = -1, k = |\alpha| = 0, j = 1 \)

From (2.5) we have
\[ \text{Case a (II)} = -\frac{1}{2} \int_{| \xi | = 1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} [\partial_{x_n} \pi_{\xi_n}^+ \sigma_1 (\tilde{D}^{-1}) \times \partial_{\xi_n} \sigma_1 (\tilde{D}^{-1})] (x_0) d\xi_n \sigma (\xi') d\xi'. \tag{4.4} \]

By Lemma 3.1, Lemma 3.2, we have
\[ \partial_{x_n} \sigma_1 (\tilde{D}^{-1}) (x_0) |_{| \xi | = 1} = \frac{\sqrt{-1} i \partial_{x_n} e (\xi) (x_0)}{ab | \xi |^2} - \frac{\sqrt{-1} i e (| \xi |^2) (x_0)}{ab | \xi |^4}, \tag{4.5} \]

By the Cauchy integral formula we obtain
\[ \pi_{\xi_n}^+ \left[ \frac{1}{(1 + | \xi |^2)^2} \right] (x_0) |_{| \xi | = 1} = \frac{1}{2 \pi i} \lim_{\eta \to 0} \int_{\Gamma^+} \frac{1}{( \eta + i \xi) (\eta - i \xi)^2} d\eta_n = -\frac{i \xi_n + 2}{4 (\xi_n - i)^2}, \tag{4.6} \]
and
\[ \pi_{\xi_n}^+ \left[ \frac{\sqrt{-1} i \partial_{x_n} e (\xi')}{| \xi |^2} \right] (x_0) |_{| \xi | = 1} = \frac{\partial_{x_n} e (\xi') (x_0)}{2 (\xi_n - i)}. \tag{4.7} \]

Then
\[ \partial_{x_n} \pi_{\xi_n}^+ \sigma_1 (\tilde{D}^{-1}) (x_0) |_{| \xi | = 1} = \frac{\partial_{x_n} e (\xi') (x_0)}{2ab (\xi_n - i)} + \frac{b i h' (0)}{ab} \frac{| \xi |^2}{(4 \xi_n - i)^2}, \]
\[ = -\frac{\partial_{x_n} [i \xi_n (\xi')] (x_0)}{2ab (\xi_n - i)} + \frac{b i h' (0)}{ab} \frac{| \xi |^2}{(4 \xi_n - i)^2} + \frac{a i e (dx_n)}{4 (\xi_n - i)^2} - \frac{b i e (dx_n)}{4 (\xi_n - i)^2} \tag{4.8} \]

By Lemma 3.1 and Lemma 3.2, we have
\[ \partial_{x_n} \pi_{\xi_n}^+ \sigma_1 (\tilde{D}^{-1}) (x_0) |_{| \xi | = 1} = \frac{\partial_{x_n} e (dx_n)}{ab} \frac{| \xi |^4}{(4 \xi_n - i)^2} + \frac{8 \xi_n^2 e (\xi)}{4 (\xi_n - i)^2}, \]
\[ = \frac{i (6 \xi_n^2 - 4) e (\xi')}{b (1 + | \xi |^2)^3} - \frac{i (6 \xi_n^2 - 4) e (\xi')}{a (1 + | \xi |^2)^3} + \frac{i (2 \xi_n^2 - 6 \xi_n e (dx_n)}{b (1 + | \xi |^2)^3} - \frac{i (2 \xi_n^2 - 6 \xi_n e (dx_n)}{a (1 + | \xi |^2)^3}. \tag{4.9} \]

Combining (4.5) and (4.9), we have
\[ \text{trace} [\partial_{x_n} \pi_{\xi_n}^+ \sigma_1 (\tilde{D}^{-1}) \times \partial_{\xi_n} \sigma_1 (\tilde{D}^{-1})] (x_0) |_{| \xi | = 1} = \frac{h' (0)}{ab} \frac{8 (-1 - 3 i \xi_n + 3 \xi_n^2 + i \xi_n^3)}{(\xi_n - i)^2 (1 + | \xi |^2)^3}, \tag{4.10} \]

Hence
\[ \text{Case a (II)} = -\frac{1}{2} \int_{| \xi | = 1}^{+\infty} \int_{-\infty}^{+\infty} \frac{h' (0)}{ab} \frac{8 (-1 - 3 i \xi_n + 3 \xi_n^2 + i \xi_n^3)}{(\xi_n - i)^2 (1 + | \xi |^2)^3} d\xi_n \sigma (\xi') d\xi', \]
\[ = -\frac{1}{2} \frac{h' (0)}{ab} \Omega_3 \int_{1+} 8 (-1 - 3 i \xi_n + 3 \xi_n^2 + i \xi_n^3) d\xi_n d\xi', \]
\[ = -\frac{1}{2} \frac{h' (0)}{ab} \Omega_3 \frac{2 \pi i}{4} \left[ \frac{8 (-1 - 3 i \xi_n + 3 \xi_n^2 + i \xi_n^3)}{(\xi_n - i)^4} \right] |_{\xi_n = \imath} d\xi', \]
\[ = -\frac{1}{2} \frac{h' (0)}{ab} \Omega_3 \frac{2 \pi i}{4} (-36 i) d\xi', \]
\[ = -\frac{3}{2ab} \pi h' (0) \Omega_3 d\xi'. \tag{4.11} \]
Case a (III): \( r = -1, \ell = -1, j = |a| = 0, k = 1 \)

From (2.5) we have

\[
\text{Case a (III)} = -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-1}(\tilde{D}^{-1}) \right](x_0) d\xi_n \sigma(\xi') dx'.
\] (4.12)

From Lemma 3.1 and Lemma 3.2 we obtain

\[
\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1})(x_0)|_{|\xi'|=1} = \frac{\tilde{c}(\xi') + i \tilde{c}(dx_n)}{2ab(\xi_n - i)^2}
\] (4.13)

and

\[
\partial_{\xi_n} \partial_{x_n} \sigma_{-1}(\tilde{D}^{-1})(x_0)|_{|\xi'|=1} = \frac{-\sqrt{-1}h'(0)}{ab} \left[ \frac{\tilde{c}(dx_n)}{|\xi'|^4} - 4\xi_n \tilde{c}(\xi') + \xi_n \tilde{c}(dx_n) \right] - \frac{2\xi_n \sqrt{-1} \partial_{\xi_n} \tilde{c}(\xi')(x_0)}{ab|\xi'|^4}
\] (4.14)

From (4.13) and (4.14), we have

\[
\text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-1}(\tilde{D}^{-1}) \right](x_0)|_{|\xi'|=1} = \frac{h'(0)}{ab} \frac{8i(-i + 3\xi_n + 3i\xi_n^2 - \xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3}.
\] (4.15)

Then

\[
\text{Case a (III)} = -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{h'(0)}{ab} \frac{8i(-i + 3\xi_n + 3i\xi_n^2 - \xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3} d\xi_n \sigma(\xi') dx'.
\]

\[
= -\frac{1}{2} \frac{h'(0)}{ab} \Omega_3 \int_{\Gamma^+} 8i(-i + 3\xi_n + 3i\xi_n^2 - \xi_n^3) \frac{d\xi_n}{(\xi_n - i)^2(1 + \xi_n^2)^3}
\]

\[
= -\frac{1}{2} \frac{h'(0)}{ab} \Omega_3 \frac{2\pi i}{4!} (36i) dx'
\]

\[
= \frac{3}{2ab} \pi h'(0) \Omega_3 dx'.
\] (4.16)

Case b: \( r = -1, \ell = -2, k = j = |a| = 0 \)

From (2.5) and the Leibniz rule, we obtain

\[
\text{Case b} = -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1}) \times \partial_{\xi_n} \sigma_{-1}(\tilde{D}^{-1}) \right](x_0) d\xi_n \sigma(\xi') dx'
\]

\[
= i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1}) \times \sigma_{-2}(\tilde{D}^{-1}) \right](x_0) d\xi_n \sigma(\xi') dx'.
\] (4.17)
From Lemma 3.1, Lemma 3.2 and Lemma 3.4 we obtain
\[
\sigma^{-2}(\bar{D}^{-1})(x_0) = \frac{\tilde{c}(\xi)\tilde{p}_0(\xi)\tilde{c}(\xi)}{a^2b^2|\xi|^4} + \frac{\tilde{c}(\xi)}{a^2b^2|\xi|^4}\tilde{c}(dx_n)\left[\partial_{\nu_n}[\tilde{c}(\xi')](x_0)|\xi|^2 - \tilde{c}(\xi)h'(0)|\xi|^2\right] \\
= \frac{1}{a^2b^2(1 + \xi_n^2)^2}\left[\tilde{c}(\xi')\tilde{p}_0(\xi') + \xi_n\tilde{c}(dx_n)\tilde{p}_0(\xi') + \xi_n\tilde{c}(\xi')\tilde{p}_0(\xi) + \xi_n\tilde{c}(\xi')\tilde{p}_0(\xi_n)\right] \\
+ \frac{2\xi_nh'(0)e(\xi')}{b(1 + \xi_n^2)^3} - \frac{2\xi_nh'(0)e(\xi')}{b(1 + \xi_n^2)^3} + \frac{(\xi_n^2 - 1)h'(0)e(dx_n)}{b(1 + \xi_n^2)^3} \\
+ \frac{(1 - \xi_n^2)h'(0)e(dx_n)}{a(1 + \xi_n^2)^3} + \frac{\xi_n\partial_{\nu_n}e(\xi')}{a(1 + \xi_n^2)^3} - \frac{\xi_n\partial_{\nu_n}e(\xi')}{a(1 + \xi_n^2)^3} \\
= \frac{\xi_n\partial_{\nu_n}e(\xi')(x_n)}{a^2(1 + \xi_n^2)^2}.
\]

From (4.13) and (4.18), we obtain
\[
\text{trace}[\partial_{\nu_n}\pi^+_{\xi_n}\sigma_{-1}(\bar{D}^{-1}) \times \sigma_{-2}(\bar{D}^{-1})](x_0)|_{|\xi'|=1} = \frac{h'(0)2(8i + 12\xi_n + 3i\xi_n^2 + 4\xi_n^3 + 3\xi_n^4)}{(\xi_n - i)^2(1 + \xi_n^2)^3}
\]

Then similarly to computations of the case a), we have
\[
\text{case b)} = -\frac{6}{ab}h'(0)\Omega_3 dx'.
\]

**Case c:** \(r = -2, \ell = -1, k = j = |\alpha| = 0\)

From (2.5) we have
\[
\text{Case c) } = -i \int^{+\infty}_{-\infty} \text{trace}[\pi^+_{\xi_n}\sigma_{-2}(\bar{D}^{-1}) + \sigma_{-2}(\bar{D}^{-1})](x_0)d\xi_n\sigma(\xi')dx'.
\]

By the Leibniz rule, trace property and "+ +" and "- -" vanishing after the integration over \(\xi_n\), then
\[
\int^{+\infty}_{-\infty} \text{trace}[\pi^+_{\xi_n}\sigma_{-2}(\bar{D}^{-1}) + \sigma_{-2}(\bar{D}^{-1})]d\xi_n
\]

The nal result is
\[
\text{case c) } = \text{case b)} - i \int^{+\infty}_{-\infty} \text{trace}[\partial_{\nu_n}\sigma_{-1}(\bar{D}^{-1}) \times \sigma_{-2}(\bar{D}^{-1})]d\xi_n\sigma(\xi')dx'.
\]

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By Lemma 3.1, Lemma 3.2, we obtain

\[
\begin{align*}
\partial_{\xi_n} \sigma^{-1}(\tilde{D}^{-1})[(x_0)_{|\xi'|=1}] &= \frac{\sqrt{-1}}{ab} \left( -2\xi_n^2 \bar{c}(dx_n) + 2\xi_n \bar{c}(\xi') + \bar{c}(dx_n) \right) \\
&= -\frac{2i\xi_n \epsilon(\xi')}{b(1 + \xi_n^2)} + \frac{2i\xi_n \epsilon(\xi')}{2(1 + \xi_n^2)} + \frac{i(1 - \xi_n^2) \epsilon(dx_n)}{b(1 + \xi_n^2)} + \frac{i(\xi_n^2 - 1) \epsilon(dx_n)}{a(1 + \xi_n^2)}.
\end{align*}
\]

(4.22)

From (4.18) and (4.22), we have

\[
- i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \sigma^{-1}(\tilde{D}^{-1}) \times \sigma^{-2}(\tilde{D}^{-1})] d\xi_n \sigma(\xi') dx' = \frac{12}{ab} \pi h'(0) \Omega_3 dx'.
\]

(4.23)

By (4.19), (4.21) and (4.23), we have

\[
\text{case } c = \frac{6}{ab} \pi h'(0) \Omega_3 dx'.
\]

(4.24)

Since \(\Phi\) is the sum of the case a, b and c, so is zero. Then we have

**Theorem 4.1.** Let \(M\) be a four dimensional compact connected manifold with the boundary \(\partial M\) and the metric \(g^M\) as above, and \(D\) are the nonminimal de Rham-Hodge operators on \(C^\infty(\Lambda^* (T^* M))\), then

\[
\overline{\text{Wres}}[(\pi^+ \tilde{D}^{-1})^2] = \frac{8\Omega_4}{3ab} \int_M R \text{dvol}_M,
\]

(4.25)

where \(R\) is the scalar curvature and \(\Omega_4\) is the canonical volume of \(S^3\).

5. Lower dimensional volumes for three dimensional spin manifolds with boundary

For an odd dimensional manifolds with boundary, as in Section 5-7 in [10], we have the formula

\[
\overline{\text{Wres}}[\pi^+ \tilde{D}^{-1} \circ \pi^+ (\tilde{D}^*)^{-1}] = \int_{\partial M} \Phi.
\]

(5.1)

When \(n = 3\), then in (2.5), \(r - k - |\alpha| + l - j - 1 = -3\), \(r, l \leq -1\), so we get \(r = l = -1\), \(k = |\alpha| = j = 0\),

\[
\Phi = \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}_{S(TM)}[\sigma^{-1}_+(\tilde{D}^{-1})](x', 0, \xi', \xi_n) \times \partial_{\xi_n} \sigma^{-1}_+((\tilde{D}^*)^{-1})(x', 0, \xi', \xi_n)] d\xi_3 \sigma(\xi') dx'.
\]

(5.2)

By Lemma 3.1 and Lemma 3.2, we have

\[
\sigma^{-1}_+(\tilde{D}^{-1})|_{|\xi'|=1}(x_0)|_{|\xi'|=1} = \frac{\bar{c}(\xi') + i\bar{c}(dx_n)}{2ab(\xi_n - i)}
\]

\[
= \frac{ae(\xi') - be(\xi') + i(ae(dx_n) - be(dx_n))}{2ab(\xi_n - i)},
\]

(5.3)

and

\[
\partial_{\xi_n} \sigma^{-1}_-((\tilde{D}^*)^{-1})(x_0)|_{|\xi'|=1} = \frac{\sqrt{-1} e(dx_n)}{ab(1 + \xi_n^2)} - \frac{2\sqrt{-1} \xi_n \bar{c}(\xi)}{ab(1 + \xi_n^2)^2}
\]

\[
= \frac{\sqrt{-1}(be(dx_n) - ae(dx_n))}{ab(1 + \xi_n^2)} - \frac{2\sqrt{-1} \xi_n (be(\xi) - ae(\xi))}{ab(1 + \xi_n^2)^2}.
\]

(5.4)
For $n = 3$, we take the coordinates as in Section 2. Locally $S(TM)|_{\tilde{U}} \cong \tilde{U} \times \Lambda_C^{even}(2)$. Let $(f_1, f_2)$ be an orthonormal basis of $\Lambda_C^{even}(2)$ and we will compute the trace under this basis. By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then we have the equalities:

$$\text{tr}[e(\xi')\text{tr}(\xi')] = 4; \quad \text{tr}[e(dx_n)\xi(dx_n)] = 4.$$  

Form (5.3) (5.4) and (5.5), we get

$$\text{tr}[e(\xi')\text{tr}(\xi')] = 4; \quad \text{tr}[e(dx_n)\xi(dx_n)] = 4.$$  

By (5.2) and (5.6) and the Cauchy integral formula, we get

$$\Phi = \left(-\frac{1}{2} + \frac{\xi}{a^2} + \frac{1}{b^2}\right)\pi \Omega_2 \text{vol}_{\partial M},$$  

where $\text{vol}_{\partial M}$ denotes the canonical volume form of $\partial M$. Then we obtain

**Theorem 5.1.** Let $M$ be a three dimensional compact spin manifold with the boundary $\partial M$ and the metric $g^M$ as in Section 2, and $\tilde{D}$ be the nonminimal de Rham-Hodge operators on $\tilde{M}$, then

$$\overline{\text{Wres}}[\pi^+\tilde{D}^{-1} \circ \pi^+(\tilde{D}^*)^{-1}] = \left(-\frac{1}{2} + \frac{\xi}{a^2} + \frac{1}{b^2}\right)\pi \Omega_2 \text{vol}_{\partial M},$$  

where $\text{vol}_{\partial M}$ denotes the canonical volume of $\partial M$.

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