Adaptive stochastic synchronization of delayed reaction–diffusion neural networks

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Abstract
In this paper, we deal with the adaptive stochastic synchronization for a class of delayed reaction–diffusion neural networks. By combining Lyapunov–Krasovskii functional, drive-response concept, the adaptive feedback control scheme, and linear matrix inequality method, we derive some sufficient conditions in terms of linear matrix inequalities ensuring the stochastic synchronization of the addressed neural networks. The output coupling with delay feedback and the update laws of parameters for adaptive feedback control are proposed, which will be of significance in the real application. The novel Lyapunov–Krasovskii functional to be constructed is more general. The derived results depend on the measure of the space, diffusion effects, and the upper bound of derivative of time-delay. Finally, an illustrated example is presented to show the effectiveness and feasibility of the proposed scheme.

Keywords
Adaptive stochastic synchronization, delayed, reaction–diffusion neural networks

Introduction
During the past two decades, chaos synchronization has been widely studied since it was introduced by Pecora and Carroll¹ in 1990. Research on the synchronization and control of coupled chaotic systems has received considerable attention due to its potential applications in many different areas including secure communication, chemical and biological systems, chaos generators design, biological systems, information science, image processing, human heartbeat regulation, and so on.²–⁸ As we all know, chaotic systems exhibit sensitive dependence on initial conditions. In view of this, chaotic systems are difficult to be synchronized or controlled. Thus, to make two or more chaotic systems achieve synchronization has been an interesting and challenging issue. To date, a wide variety of approaches for the synchronization or control of chaotic systems have been investigated mainly including adaptive control,²⁴ PI-type learning algorithm,⁷ feedback control,⁵ and so on. In Huyhn et al.,⁷ learning algorithm was applied to online tune the parameters and a Lyapunov function was used to guarantee the considered system’s stability. The proposed control system was employed to synchronize two chaotic systems and to control an inverted pendulum.

Recently, there has been increasing interest in the potential applications of the dynamics of neural networks (NNs) in signal and image processing. Among the most popular models in the literature of NNs is continuous time model described by a system for ordinary differential equations.⁹,¹⁰ NNs have been verified to exhibit chaotic behaviors if the NNs’ parameters and time delays are appropriately chosen. As this chaotic behavior may affect the synchronization, there has been a growing research interest in the study of the synchronization of chaotic NNs. It is inspiring that many important works on the synchronization of chaotic NNs have been reported in the literature.⁵,¹¹,¹² Strictly speaking, diffusion effects cannot be avoided in the NNs when electrons are moving in asymmetric electromagnetic fields. Therefore, it should be considered that the activations vary in space as well as in time. In the literature,¹³–²¹ the stability of NNs with reaction diffusion terms has been considered. The synchronization control problems of NNs with reaction–diffusion terms have been widely studied.²²–²⁶ In Li et al.,²² an
adaptive synchronization controller was obtained to achieve exponential synchronization for a class of reaction–diffusion neural networks (RDNNs) with time-varying and distributed delays using the Lyapunov functional method and Young’s inequality. In Wang and Ding,25 the synchronization schemes for delayed non-autonomous reaction–diffusion fuzzy cellular NNs were considered. Based on the simple adaptive controller, some sufficient conditions to guarantee the synchronization were obtained.

However, stochastic phenomenon usually appears in the electrical circuit design of NNs. NNs could be stabilized or destabilized by certain stochastic inputs.20,21 Thus, it is important to consider stochastic effects to the chaos synchronization control of NNs with delays. Based on the above analysis, for synchronization analysis of RDNNs, the diffusion effects cannot be neglected. So, the synchronization criteria including diffusion parameters are more reasonable. However, to the best of our knowledge, there are few, or even no, results dependent on diffusion terms concerning adaptive synchronization of the delayed RDNNs, which is very important in both theories and applications and also is a very challenging problem.

In this paper, we will investigate the adaptive stochastic synchronization of RDNNs with mixed time delays. By constructing suitable Lyapunov–Krasovskii functional, combing the adaptive control and output coupling with delay feedback, drive-response concept, some analysis techniques, and linear matrix inequality (LMI) approach, the sufficient conditions in terms of analysis techniques, and linear matrix inequality coupling with delay feedback, drive-response concept, stochastic synchronization of RDNNs with mixed time delays is described by the following partial differential equations

\[
d\Omega(t, x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( D_i \frac{\partial \Omega(t, x)}{\partial x_i} \right) dt \\
+ \left[ -A\Omega(t, x) + Bf(u(t, x)) \right] \\
+ C f(u(t - d(t), x)) + \tilde{E} \int_{-\infty}^{t} K(t-s)f(u(s, x))ds + J \]

\[
dt, \quad t \geq 0, x \in \Omega
\]

\[
u(t, x) = 0, \quad (t, x) \in (-\infty, +\infty) \times \partial \Omega
\]

\[
u(t, x) = \varphi(s, x), \quad (s, x) \in (-\infty, 0] \times \Omega
\]

where \( x = (x_1, x_2, \ldots, x_n)^T \in \Omega \), \( u(t, x) = (u_1(t, x), \ldots, u_n(t, x))^T \) denotes the state vector associated with the \( n \) neurons at time \( t \) and in space \( x \); \( A = \text{diag}\{a_1, \ldots, a_n\} \) is a diagonal matrix with \( a_i > 0 \); \( B = (b_{ij})_{n \times n} \); \( C = (c_{ij})_{n \times n} \), and \( \tilde{E} = (\tilde{e}_i)_{n \times n} \) are the connection weight matrix, the time-varying delay connection weight matrix, and the distributed delay connection weight matrix, respectively; \( f(u(t, x)) = (f_1(u_1(t, x)), \ldots, f_n(u_n(t, x)))^T \) is the neuron activation function, and \( J = (J_1, J_2, \ldots, J_n)^T \) is a constant external input vector. \( d(t) \) denotes the time-varying delay, and \( d(t) \) is assumed to satisfy \( 0 \leq d(t) \leq d \) and \( 0 \leq d(t) \leq \mu < 1 \), where \( d \) and \( \mu \) are constants; \( K(t-s) = \text{diag} \left[ k_1(t-s), \ldots, k_n(t-s) \right] \), and the delay kernel \( k_i(\cdot) \) is a real value nonnegative continuous function defined on \([0, +\infty)\), where \( \int_{0}^{\infty} k_i(\theta)d\theta = 1 \); \( D_\gamma = \text{diag}(D_{\gamma 1}, D_{\gamma 2}, \ldots, D_{\gamma n}) \) with \( D_\gamma = \sum_{i=1}^{n} D_{\gamma i} \geq 0 \) stands for transmission diffusion operator along the \( \gamma \)th neuron. \( \varphi(s, x) = (\varphi_1(s, x), \ldots, \varphi_n(s, x))^T \in C[(-\infty, 0] \times \Omega; R^n] \), \( i, j = 1, 2, \ldots, n \) and \( l = 1, 2, \ldots, m \).

In order to observe the synchronization behavior of drive system (1), the response system with stochastic perturbation is designed as
\[
du(t, x) = \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left( D_i \frac{\partial \hat{u}(t, x)}{\partial x_i} \right) dt + \left[ -A\hat{u}(t, x) + Bf(\hat{u}(t, x)) \right. \\
+ \left. Cj(\hat{u}(t - d(t), x)) + E \int_{-\infty}^{t} K(t - s) f(\hat{u}(x(s)) ds + J + u'(t, x) \right] dt \\
+ \sigma \left( t, e(t, x), e(t - d(t), x), \int_{-\infty}^{t} K(t - s) \hat{f}(e(x(s)) ds \right) dw(t), t \geq 0, x \in \Omega \\
\hat{u}(t, x) = 0, (t, x) \in (-\infty, +\infty) \times \partial \Omega \\
\hat{u}(x, s) = \hat{\phi}(x, s), (x, s) \in (-\infty, 0] \times \Omega 
\]

where \( \hat{\phi}(x, s) = (\hat{\phi}_1(x, s), \ldots, \hat{\phi}_n(x, s))^T \in C^2_{\Omega}([-\infty, 0] \times \Omega; \mathbb{R}^n) \), the synchronization error is \( e(t, x) = \hat{u}(t, x) - u(t, x) \), and \( u'(t, x) \) is the control vector. The noise perturbation \( \sigma(\cdot) \) is the noise intensity matrix; \( w(t) = (w_1(t), \ldots, w_n(t))^T \) is an \( n \)-dimensional standard Brownian motion defined on a complete probability space \( (\Omega, F, \{F_t\}_{t \geq 0}, P) \) with a filtration \( \{F_t\}_{t \geq 0} \) generated by \( \{w(s) : 0 \leq s \leq t\} \), where we associate \( \hat{\Omega} \) with the canonical space generated by all \( \{w(t)\} \) and denote by \( F \) the associated \( \sigma \)-algebra generated by \( \{w(t)\} \) with the probability measure \( P \).

Define \( f(e(\cdot)) = f(e(\cdot) + u(\cdot)) - f(u(\cdot)) \), and the control input in the response system (2) is designed as follows

\[
u'(t, x) = -\Xi(t, x)f(e(t, x)) - \hat{\Xi}(t, x) f(e(t - d(t), x))
\]

where \( \Xi(t, x) = \text{diag}(\Xi_1(t, x), \ldots, \Xi_n(t, x)) \) and \( \hat{\Xi}(t, x) = \text{diag}(\hat{\Xi}_1(t, x), \ldots, \hat{\Xi}_n(t, x)) \) are the unknown gain matrices to be scheduled. Substituting the control law (3) into system (2), the error dynamics between systems (1) and (2) can be expressed by

\[
de(t, x) = \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left( D_i \frac{\partial \hat{f}(e(t, x))}{\partial x_i} \right) dt + \left[ -Ae(t, x) + (B - \Xi(t)) \right) \\
+ \left. (C - \hat{\Xi}(t)) f(e(t - d(t), x)) + E \int_{-\infty}^{t} K(t - s) f(e(x(s)) ds \right] dt \\
+ \sigma \left( t, e(t, x), e(t - d(t), x), \int_{-\infty}^{t} K(t - s) \hat{f}(e(x(s)) ds \right) dw(t), t \geq 0, x \in \Omega \\
e(t, x) = 0, (t, x) \in (-\infty, +\infty) \times \partial \Omega \\
e(x, s) = \hat{\phi}(x, s) - \hat{\phi}(x, s), (x, s) \in (-\infty, 0] \times \Omega 
\]

Remark 1. In many applications, the authors in the literature are interested in the state-feedback controller or time-delay feedback controller designed as \( u'(t, x) = \Xi e(t, x) \) and \( u'(t, x) = \hat{\Xi} e(t, x) + \hat{\Xi} e(t - d(t), x) \), respectively. However, in many real networks, only output signals \( \hat{f}(e(\cdot)) \) can be measured. Li et al.27 investigated the synchronization analysis of a class of delayed chaotic ordinary differential NNs with stochastic perturbations and designed the control input

\[
u'(t) = K_1(\hat{e}(t)) + K_2(\hat{e}(t - d(t))), \quad \text{where} \ K_1 \text{ and } K_2 \text{ are the gain matrices to be scheduled. Motivated by the above works, we develop the idea to a class of coupled delayed stochastic RDNNs. As far as we know, this extension has not been investigated in the literature works at the present stage. Therefore, we investigate the adaptive controller (3) with update law in the response system, which will be of significance in the real application. In this paper, we refer to this as output coupling with delay feedback and propose a novel synchronization scheme for drive-response systems via output coupling with delay feedback.

Remark 2. Comparing with the model recently discussed in Wang and Cao,24 our model is more general than those given in Wang et al.26 Actually, if we take \( \hat{E} = 0 \) and omit stochastic effects, then our model is same as in Wang et al.26

In this paper, the following assumptions are made.

Assumption 1. Activation functions \( f(\cdot) \) are bounded and \( f(0) = 0 \); there exists a positive diagonal matrix \( L = \text{diag}(L_1, \ldots, L_n) \) such that

\[
0 \leq f(\xi_1) - f(\xi_2) \leq L_j
\]

for all \( \xi_1, \xi_2 \in \mathbb{R}, \xi_1 \neq \xi_2, j = 1, 2, \ldots, n. \)

Assumption 2. The matrix \( \sigma(\cdot) \) is local Lipschitz continuous and satisfies the linear growth condition as well. Furthermore, there exist positive definite matrices \( \Sigma_1, \Sigma_2, \Sigma_3 \) such that

\[
\text{tr} \left[ (\sigma(t, \xi_1, \xi_2), \xi_3))T(\sigma(t, \xi_1, \xi_2, \xi_3)) \right] \\
\leq \xi_1^T \Sigma_1 \xi_1 + \xi_2^T \Sigma_2 \xi_2 + \xi_3^T \Sigma_3 \xi_3
\]

where \( \xi_1, \xi_2, \xi_3 \in \mathbb{R}^n \).

For the sake of simplicity, we denote \( u(t, x), e(t, x), \sigma(t, e(t, x), e(t - d(t), x), \int_{-\infty}^{t} K(t - s) f(e(x(s)) ds \) by \( u(t), e(t), \sigma(\cdot), \) respectively.

Let \( C_\beta^2(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+ \) define the family of all nonnegative functions \( V(t, e(t)) \) on \( \mathbb{R}^+ \times \mathbb{R}^n \) which are continuously twice differentiable in \( e \) and continuously differentiable in \( t \). If \( V \in C_\beta^2(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+ \) then along the trajectory of the system (4), we denote an operator \( LV \) from \( \mathbb{R}^+ \times \mathbb{R}^n \) to \( \mathbb{R} \) by
Lemma 1. The drive system (1) and the response system (2) are said to be asymptotically synchronized if there exists a constant $\delta > 0$ satisfying that $\lim_{t \to \infty} \|u(t,x) - \hat{u}(t,x)\|_2^2 = 0$ when $t > 0$ and $\|\varphi(0) - \varphi(0)\|_2^2 < \delta$.

Lemma 2. For any real matrices $X$ and $Y$ and a positive definite matrix $Q$ with compatible dimensions, the following matrix inequality holds:

$$X^T Y + Y^T X \leq X^T Q X + Y^T Q^{-1} Y$$

Main results

Theorem 1. Under Assumptions 1 and 2, the two coupled RDNNs (1) and (2) can be synchronized, if the feedback strength $\mathcal{E}(t,x)$ and $\mathcal{E}_f(t,x)$ are updated by the following law

$$\frac{d \mathcal{E}(t,x)}{dt} = \gamma_i \mathcal{E}(e_i(t,x))$$
and $M_i$; positive definite symmetric matrices $P_2, P_3, G_1,$ and $G_2$; and scalar $\rho > 0$, such that the following LMI holds

$$P_1 \leq \rho I$$

where

$$P_{11} = -2P_1D^* - 2P_1A + \rho \sum_{i=1}^n \frac{j}{1 - \mu} P_2,$$

$$P_{22} = \rho \sum_{i=2}^n - P_2,$$

$$P_{44} = G_1 - P_3,$$

$$P_{33} = Q + \frac{1}{1 - \mu} P_3 - 2M_1,$$

$$P_{55} = G_2 + \rho \sum_{i=3}^n - \tilde{Q},$$

$$P_{14} = -2P_1e^*,$$

$$D^* = \text{diag}\left\{ \sum_{i=1}^n \frac{j}{1 - \mu} D_i, \ldots, \sum_{i=1}^n \frac{j}{1 - \mu} D_i \right\},$$

$$\varepsilon = \text{diag}\{ \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \},$$

$$e^* = \text{diag}\{ e_1^*, e_2^*, \ldots, e_n^* \}$$

$\varepsilon_i$ and $e_i^*$ are constants, $i = 1, 2, \ldots, n$.

Proof. Consider Lyapunov–Krasovskii functional as

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t)$$

where

$$V_1(t) = \int_{\Omega} e^T(t) P_1 e(t) dx$$

$$V_2(t) = \int_{\Omega} \left[ \sum_{i=1}^n \frac{P_{11}}{L_i \gamma_i} (\mathcal{E}(t,x) - \varepsilon_i)^2 + \sum_{i=1}^n \frac{P_{44}}{L_i \gamma_i} (\mathcal{E}_f(t,x) - e_i^*)^2 \right] dx$$

$$V_3(t) = \frac{1}{1 - \mu} \int_{t-d(t)}^t e^T(s) P_2 e(s) ds dx$$

$$V_4(t) = \frac{1}{1 - \mu} \int_{t-d(t)}^t \mathcal{E}(e(s))^T P_3 \mathcal{E}(e(s)) ds dx$$

$$V_5(t) = \int_{t_0}^t \int_{t_0}^\infty K_j(\theta) \mathcal{E}_f^T(e(s)) ds d\theta dx$$

in which $P_1 = \text{diag}\{ p_{11}, \ldots, p_{1n} \}$ and $Q = \text{diag}\{ q_1, \ldots, q_n \}$ are positive diagonal matrices, $P_2$ and $P_3$
are positive definite symmetric matrices, \( L = \text{diag}(L_1, \ldots, L_n) \) is given in Assumption 1, \( c_i \) and \( c_i^* \) are constants to be determined, \( \gamma_i \) and \( \gamma_i^* \) are arbitrary positive constants. Here, the terms \( V_1(t) \) and \( V_2(t) \) extend the Lyapunov–Krasovskii functional construction of Huynh et al.\(^7\) to stochastic RDNNs. The terms \( V_3(t) \) and \( V_4(t) \) extend the constructions of the literature.\(^1\)\(^1\)\(^1\)\(^1\)\(^2\)\(^2\)\(^7\) The term \( V_5(t) \) is added to \( V(t) \) to treat the distributed delay.

One can calculate \( LV(t, e(t)) \) along trajectories of system (4), then we obtain

\[
LV(t, e(t)) = \int_0^\infty \left\{ 2e(t)^T P_1 \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( D_i \frac{\partial e(t)}{\partial x_i} \right) - A e(t) + (B - \Xi(t)) \tilde{f}(e(t)) \right\} ds \\
+ (C - \Xi(t)) \tilde{f}(e(t - d(t))) + \tilde{E} \left( \int_0^t K(t - s) \tilde{f}(e(s)) ds \right) \\
+ \sum_{i=1}^n \frac{2p_{ii}}{L_i} (\Xi_i(t) - c_i) \tilde{f}_i(e_i(t)) + \sum_{i=1}^n \frac{2p_{ii}}{L_i} (\Xi_i(t) - c_i^*) \tilde{f}_i(e_i(t)) \tilde{f}_i(e_i(t - d(t))) \\
+ \text{trace} \left[ \sigma^T V_\sigma \sigma \right] \right\} dx + \int_0^\infty \left\{ \frac{1}{1 - \mu} e(t)^T P_2 e(t) - \frac{1 - d(t)}{1 - \mu} e(t - d(t))^T P_2 e(t - d(t)) \right\} dx \\
+ \int_0^\infty \left\{ \frac{1}{1 - \mu} \tilde{f}(e(t))^T P_2 \tilde{f}(e(t)) - \frac{1 - d(t)}{1 - \mu} \tilde{f}(e(t - d(t)))^T P_2 \tilde{f}(e(t - d(t))) \right\} dx \\
+ \int_0^\infty \sum_{i=1}^n \int_0^\infty K_i(\theta) \tilde{f}_i(e_i(t, \theta)) d\theta dx - \int_0^\infty \sum_{i=1}^n \int_0^\infty K_i(\theta) \tilde{f}_i(e_i(t - \theta)) d\theta dx 
\]

Using Assumption 1, for some constants \( c_i \) and \( c_i^* \), it yields

\[
\sum_{i=1}^n \frac{2p_{ii}}{L_i} (\Xi_i(t) - c_i) \tilde{f}_i(e_i(t)) = 2 \tilde{f}(e(t))^T L^{-1} P_1 \\
(\Xi_i(t) - c_i) \tilde{f}_i(e_i(t)) \\
\leq 2e(t)^T P_1 (\Xi_i(t) - c_i) \tilde{f}_i(e_i(t)) \\
\sum_{i=1}^n \frac{2p_{ii}}{L_i} (\Xi_i(t) - c_i^*) \tilde{f}_i(e_i(t, \theta)) \tilde{f}_i(e_i(t - d(t))) \\
= 2 \tilde{f}(e(t))^T L^{-1} P_1 (\Xi(t) - c^*) \tilde{f}(e(t - d(t))) \\
\leq e(t)^T 2P_1 (\Xi(t) - c^*) \tilde{f}(e(t - d(t))) 
\]

By \( d(t) \leq \mu < 1 \), one can get that

\[
- \frac{1 - d(t)}{1 - \mu} \leq -1 
\]

From Lemma 2, we have

\[
2e(t)^T P_1 C \tilde{f}(e(t - d(t))) \leq e(t)^T P_1 C \tilde{f}_i(e(t)) \\
+ \tilde{f}(e(t - d(t)))^T G_i \tilde{f}(e(t - d(t))) \\
2e(t)^T P_1 E \int_0^{-\infty} K(t - s) \tilde{f}(e(s)) ds \leq \\
e(t)^T P_1 E G_2^{-1} E^T P_1 e(t) + \left( \int_0^{-\infty} K(t - s) \tilde{f}(e(s)) ds \right)^T \\
G_2 \left( \int_0^{-\infty} K(t - s) \tilde{f}(e(s)) ds \right) 
\]
\[
\begin{align*}
\int_{\Omega} \sum_{j=1}^{n} q_j \, K_j(\theta) \, (c_j(t)) \, d\theta dx \\
- \int_{\Omega} \sum_{j=1}^{n} q_j \, K_j(\theta) \, f_j(c_j(t-\theta)) \, d\theta dx \\
= \int_{\Omega} \tilde{f}(e(t))^T \tilde{Q} \tilde{f}(e(t)) \, dx \\
- \int_{\Omega} \sum_{j=1}^{n} q_j \, \left[ K_j(\theta) \tilde{f}(c_j(t-\theta)) \right] \, d\theta dx \\
\leq \int_{\Omega} \tilde{f}(e(t))^T \tilde{Q} \tilde{f}(e(t)) \, dx \\
- \int_{\Omega} \left( \int K(t-\theta) \tilde{f}(c(t)) \, d\theta \right)^T \tilde{Q} \left( \int K(t-\theta) \tilde{f}(c(t)) \, d\theta \right) \, dx \\
(16)
\end{align*}
\]

From Green's formula, the Dirichlet boundary conditions, and Lemma 1, we get
\[
\begin{align*}
\sum_{j=1}^{m} e_j(t) \, \frac{\partial}{\partial x_j} \left( D_{\theta} \tilde{f}(e(t)) \right) \, dx &= - \sum_{j=1}^{m} D_{\theta} \left( \frac{\partial c_j(t)}{\partial x_j} \right) \, dx \\
\sum_{j=1}^{m} \sum_{i=1}^{n} D_{\theta} \left( \frac{\partial c_j(t)}{\partial x_j} \right) \, dx &\leq - \sum_{j=1}^{m} D_{\theta} \left( \frac{\partial e_j(t)}{\partial x_j} \right) \, dx \\
(17)
\end{align*}
\]

From Assumption 1, we have
\[
2\tilde{f}(e(t))^T M_1 L e(t) - 2\tilde{f}(e(t))^T M_1 \tilde{f}(e(t)) \geq 0 \\
(18)
\]
where \(M_1\) is positive definite diagonal matrix.

Substituting equations (9) and (11)–(18) into equation (10), and then by Assumption 2, we can be derived
\[
\begin{align*}
LV(t,e(t)) &\leq \left[ e(t)^T (-2P_2 D^T - 2P_2 A + P_1 C G_1^T C^T P_1 + P_1 E G_2^T E^T P_1 + \rho \sum_{i=1}^{n} \right. \\
&+ \left. \frac{1}{1-\mu_2} p \right] e(t) + e(t-d(t))^T \left( \rho \sum_{i=1}^{n} - P_2 \right) e(t-d(t)) \\
&+ \frac{1}{1-\mu_2} \tilde{f}(e(t))^T \tilde{Q} \tilde{f}(e(t)) + e(t-d(t))^T (G_3 - P_3) e(t-d(t)) \\
&+ e(t-d(t))^T (2P_2 B - 2P_2 e + 2M_1) e(t-d(t)) \\
&+ e(t-d(t))^T (2P_2 B - 2P_2 e + 2M_1) e(t-d(t)) \\
&\left. \left( K(t-s) \tilde{f}(c(t)) \right) ds \right) + \left[ G_5 + \rho \sum_{j=1}^{N} \right. \\
&\left. \left( \int K(t-s) \tilde{f}(c(t)) \, ds \right) \right] dx \\
- \eta \Sigma \eta \, dx < 0 \\
(19)
\end{align*}
\]

where
\[
\eta = \left( e(t)^T e(t-d(t))^T \tilde{f}(e(t))^T \tilde{f}(e(t-d(t)) \right)^T \\
\left( \int K(t-s) \tilde{f}(c(t)) \, ds \right)^T \\
\end{align*}
\]

By Schur complement and our assumption, \(\Sigma < 0\) if and only if \(\Xi_1 < 0\). According to the conditions of Theorem 1, if \(\eta(t,x) \neq 0\), we can obtain
\[
LV(t,e(t)) \leq \lambda_{\min}(P_1) EV(t,e(t)) \\
\leq \lambda_{\min}(P_1) \left( \int K(t-s) \tilde{f}(c(t)) \, ds \right)^T \\
\end{align*}
\]

So, there exists \(\delta > 0\) such that \(\|\phi(0) - \phi(0)\|^2 < \delta\) implies \(\|\varphi(t,x) - \varphi(t,x)\|^2 < \delta, \forall t \geq 0\). Moreover, according to the LaSalle invariant principle of stochastic differential equation proposed in Mao,\(^{2-23}\) we can conclude \(\Sigma(t,x) \rightarrow e_i \) and \(\Sigma(t,x) \rightarrow e_i^\ast \) as \(t \rightarrow \infty\). We can see that the drive-response systems (1) and (2) are asymptotically synchronized. This completes the proof.

Remark 3. It is well-known that the choice of an appropriate Lyapunov–Krasovskii functional is crucial for deriving stability and bounded real criteria and, as a result, for obtaining a solution to various control problems. Special forms of Lyapunov–Krasovskii functionals lead to simpler delay-independent and delay-dependent finite dimensional conditions in terms of LMIs. In proof of Theorem 1, the new Lyapunov–Krasovskii functional to be constructed is more general. The adaptive synchronization criteria in the literature\(^{22-24}\) are independent on the measure of the space and diffusion effects. However, in this paper, the obtained results are dependent on the measure of the space and diffusion effects. It is noted that Theorem 1
only depends on the upper bound of derivative of time-delay. Therefore, it is shown that the newly obtained results are less conservative and more applicable than the existing results.\textsuperscript{22–24}

**Remark 4.** The direct Lyapunov method is a powerful tool for studying systems stability. It remains helpful for the determination of synchronization conditions when some parameters are uncertain. In Theorem 1, we propose the adaptive synchronization for a class of stochastic RDNNs with time-varying delays and unbounded distributed delays. To the best of authors’ knowledge, up to now, this model has not been investigated for the stochastically adaptive asymptotical synchronization problem. Thus, our results are general and significant in practice.

When the smooth operator $D_{q} = 0$, systems (1) and (2) degenerate into the following models (equations (20) and (21)), respectively


d\dot{u}(t) = \left[ -Au(t) + Bf(u(t)) + Cf(u(t - d(t))) \\
+ \int_{-\infty}^{t} K(t-s)f(u(s))ds + J_1 + u(t) \right] dt, \quad t \geq 0

\quad u(s) = \phi(s), \quad s \in (-\infty, 0]

\tag{20}

and

\begin{align*}
\dot{u}'(t) &= \left[ -A\dot{u}(t) + B(f'(u(t)) + C\dot{f}(u(t - d(t))) \\
+ \int_{-\infty}^{t} K(t-s)f'(u(s))ds + J_2 \\
+ \sigma'(t)dv(t) \right] dt, \quad t \geq 0

\dot{u}(s) = \phi(s), \quad s \in (-\infty, 0]
\tag{21}
\end{align*}

We have the following corollary.

**Corollary 1.** Under Assumptions 1 and 2, the two coupled NNs (equations (20) and (21)) can be synchronized in the mean square sense if the feedback strengths $\Xi_1(t)$ and $\Xi_2(t)$ are updated by the following law

\begin{align}
\Xi_1(t) &= \gamma J_1^2(e(t)) \\
\Xi_2(t) &= \gamma J_2^2(e(t)) (e(t-d(t)))
\end{align}

(22)

and there exist positive definite diagonal matrices $P_1$, $Q_1$, and $M_1$; positive definite diagonal matrices $P_2$, $P_3$, $G_1$, and $G_2$; and scalar $\rho > 0$, such that the following LMIs hold

\begin{align}
P_1 \leq \rho I
\end{align}

(23)

\begin{align}
\Xi_1 = \begin{bmatrix}
P_{11} & 0 & P_{13} & 0 & P_1C & P_1E \\
+ & + & + & 0 & 0 & 0 & 0 \\
+ & + & + & 0 & 0 & 0 & 0 \\
+ & + & + & + & 0 & 0 & 0 \\
+ & + & + & + & + & -G_1 & 0 \\
+ & + & + & + & + & + & -G_2 \\
\end{bmatrix} < 0
\end{align}

where $\prod_{i=1}^M = -2P_1A + \rho \sum_{i=1}^M + 1/(1-\mu)P_2$ and other notations are similarly given in Theorem 1.

**Remark 5.** When the smooth operator $D_{q} = 0$, model (1) becomes the delayed NNs analyzed in the literature.\textsuperscript{11,12} Li and Cao\textsuperscript{11} and Sun et al.\textsuperscript{12} have presented some exponential synchronization schemes for systems (20) and (21) via the adaptive feedback controller or time-delay feedback controller. In this letter, we give a novel adaptive asymptotical synchronization scheme in the mean square sense for systems (20) and (21) using Lyapunov–Krasovskii functional, drive-response concept and LMI approach as well as the adaptive feedback control technique. Therefore, our results and those established in Li and Cao\textsuperscript{11} and Sun et al.\textsuperscript{12} complement each other.

**A numerical example**

In this section, we give an example with numerical simulations to illustrate the effectiveness of the theoretical results obtained above.

**Example 1.** Consider the drive-delayed NNs (equation (1)) with reaction–diffusion terms, the response stochastic delayed NNs (equation (2)) with reaction–diffusion terms, and the error system (4) with the network parameters given as follows

\begin{align*}
A &= \begin{bmatrix} 1.5 & 0 \\
0 & 1.3 \end{bmatrix},\quad B = \begin{bmatrix} -2.1 & 1.3 \\
0.8 & -1.5 \end{bmatrix},

C &= \begin{bmatrix} 0.3 & 0.4 \\
0.2 & 0.1 \end{bmatrix},\quad E = \begin{bmatrix} 1.1 & 0 \\
0 & 1.1 \end{bmatrix},

\Sigma_1 = \Sigma_2 = \Sigma_3 = I
\end{align*}

\begin{align*}
f(u(t)) &= \tanh(u(t)), \quad d(t) = 0.5 + 0.5 \sin 2t,

K_j(s) &= e^{-s}, \quad D_1 = I, \quad e = \bar{e} = 0.3 I
\end{align*}

\begin{align*}
\sigma(t, e(t), e(t-d(t)), t) &= \int_{-\infty}^{t} K(t-s)f(e(s, x)) ds \\
&= 0.1 e(t, x) + 0.2 e(t-d(t), x)
+ 0.3 \int_{-\infty}^{t} K(t-s)f(e(s, x)) ds, J = (1, 2)^T
\end{align*}
By simple computation, we have $D^* = L = I$, $\mu = 0.5$. By applying the MATLAB LMI control Toolbox to solve the LMIs (8) and (9) in Theorem 1, we obtain a set of feasible solutions as

\[
P_1 = \begin{bmatrix}
0.1582 & 0 \\
0 & 0.1582
\end{bmatrix},
\]

\[
P_2 = \begin{bmatrix}
0.1390 & -0.0348 \\
-0.0348 & 0.1390
\end{bmatrix},
\]

\[
P_3 = \begin{bmatrix}
0.1666 & -0.0618 \\
-0.0618 & 0.0902
\end{bmatrix},
\]

\[
M_1 = \begin{bmatrix}
0.5416 & 0 \\
0 & 0.5416
\end{bmatrix},
\]

\[
G_1 = \begin{bmatrix}
0.8857 & -0.2151 \\
-0.2151 & 0.4984
\end{bmatrix},
\]

\[
G_2 = \begin{bmatrix}
0.2241 & 0.0323 \\
0.0323 & 0.2319
\end{bmatrix},
\]

\[
\bar{Q} = \begin{bmatrix}
0.4280 & 0 \\
0 & 0.4280
\end{bmatrix},
\]

$\rho = 1.2831$

Let the initial conditions be $u_1(t,x) = u_2(t,x) = 2\sin \pi x$, $\dot{u}_1(t,x) = \dot{u}_2(t,x) = 5\sin \pi x$, $\mathcal{E}_1(t,x) = \mathcal{E}_2(t,x) = \sin \pi x$, and $\mathcal{E}_1(t,x)^T = \mathcal{E}_2(t,x)^T = -0.2\sin \pi x$, and the boundary conditions are set as Dirichlet boundary conditions. Let $e_i(t,x) = \dot{u}_i(t,x) - u_i(t,x), \quad i = 1, 2$.

After 10 times operations, the average simulation results can be shown in Figures 1–14, where Figures 1 and 2 show the state surfaces of $u_1(t,x)$ and $u_2(t,x)$ in system (1), respectively. Figures 3 and 4 exhibit the synchronization errors $e_1(t,x)$ and $e_2(t,x)$, respectively. Figure 5 shows synchronization error $e_1(t,x)$ of addressed system with $x = 0.5$. Figures 7 and 8 depict the control surface of system (2), respectively. Figures 9 and 10 exhibit dynamic curve of the parameter estimation $\mathcal{E}_1(t,x)$ and $\mathcal{E}_2(t,x)$, respectively. Figures 11 and 12 show the parameters update laws of $\mathcal{E}_1(t,x)$ with $x = 0$ and $\mathcal{E}_2(t,x)$ with $x = -0.6$, respectively. Figures 13 and 14 show dynamic curve of the parameter estimation $\mathcal{E}_1(t,x)^T$ and $\mathcal{E}_2(t,x)^T$, respectively. According to Theorem 1, the drive system (1) and the response system (2) are asymptotically synchronized as shown in Figures 3–6. The numerical simulations clearly verify the effectiveness of the developed adaptive feedback control approach to the asymptotical synchronization of chaotic delayed NNS with reaction-diffusion terms and Dirichlet boundary conditions.

**Conclusion**

In this paper, we have investigated the adaptive synchronization problem for a class of stochastic RDNNs with time-varying delays and unbounded distributed delays. We have proposed a novel adaptive control scheme for asymptotic synchronization by utilizing Lyapunov–Krasovskii functional theory with stochastic analysis and LMI approaches as well as the adaptive feedback control technique. The issue considered in this paper is more general in many aspects, for it...
incorporates as special cases various problems which have been studied extensively in the literature. Some remarks and one numerical example have been applied to demonstrate the effectiveness of the obtained results.

Figure 4. Synchronization error $e_2(t,x)$ in Example 1.

Figure 5. Synchronization error $e_1(t,x)$ when $x = -0.6$ in Example 1.

Figure 6. Synchronization error $e_2(t,x)$ when $x = 0.5$ in Example 1.

Figure 7. Control surface of $u_1(t,x)$ in system (2) in Example 1.

Figure 8. Control surface of $u_2(t,x)$ in system (2) in Example 1.

Figure 9. Dynamic curve of the parameter estimation $\Xi_1(t,x)$.
In fact, it is worth mentioning that there are still some important problems to solve for delayed RDNNs. There are two common phenomena in some evolving networks: delay effects and stochastic effects. Stochastic phenomenon usually appears in the electrical circuit design of NNs. In addition, NNs could be stabilized or destabilized by certain stochastic inputs. Since delays and stochastic inputs can heavily affect the dynamical behaviors of NNs, it is necessary to investigate both delay and stochastic effects on the synchronization of NNs. Furthermore, efforts have been made to study the complex dynamics of partial differential system in recent years. In Eisenberg et al.,\textsuperscript{30} stochastic trajectories are described that underly classical diffusion between known concentrations. In Morita,\textsuperscript{31} the stochastic Verhulst equation with a white noise term is treated, in order to calculate the average value of the population as a function of time, which is expressed analytically in a
nite series with an arbitrary initial value of the population. Unfortunately, to the best of our knowledge, there are few results concerning average value analysis of stochastic trajectories for delayed RDNNs have not yet been proposed. These are interesting problems and will become our future investigative direction.

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