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University Mathematics Students’ Learning Difficulties

Johan Lithner*

Abstract
The processes of learning mathematics are immensely complex and we largely lack insights into these processes. This is especially problematic when it comes to tertiary mathematics education, which has been much less researched than primary and secondary mathematics education. It is thus far from possible to clarify all relevant issues related to university mathematics learning difficulties. This paper will discuss the notion of learning difficulties and some related insights.

Keywords: university mathematics, learning difficulties

Background
The number of students commencing university mathematical studies in Sweden has risen dramatically in the last 20 years, as well as in large parts of the world. In a longer perspective, one could say that undergraduate mathematics was an elite education 30-40 years ago, but today it is a mass education. Of each Swedish age group, 98 percent study mathematics in upper secondary school for one to three years. About 15 percent of each age group study undergraduate mathematics, mainly as a service subject within programmes in technology, natural science, and computer science. A serious problem for those of us arranging undergraduate courses in mathematics, and probably for many engaged in teaching mathematics at any level and in any place in the world, is that we are unable to sufficiently help many students reach a desired level of mathematical competence. In Sweden some 20-30 percent of undergraduate mathematics students fail the initial courses, although they fulfil the entry qualifications. Even among the majority who pass, there are clear signs of difficulties and deficient mathematical competence and it seems clear that this is an international problem. At the same time, there is a large and increasing demand from society for individuals with different kinds of mathematically intense academic education.

The notion of learning difficulties
What is meant by learning difficulties in mathematics? We can mean different things and the problem is not just that we might misunderstand each other, but also that different meanings may lead to different conclusions and actions.

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The most severe difficulties, such as those of students with mental or physical disabilities or of students in very problematic social situations, require special attention and are not the focus of this paper. One may note that there is, for example, the doubtful notion of dyscalculia (Sjöberg, 2006) that seems to float between disabilities and difficulties that are not handicap-related.

Even though students often see “too difficult tasks” as the most painful aspect of mathematics learning (Lithner, 2002a,b), the struggles that follow naturally with mathematics are not considered as learning difficulties here.

A common notion is to see an inability to achieve basic educational goals as learning difficulties. This seems reasonable, but there could be several problems with this. For example, are the goals proper? Can we talk about learning difficulties if the goals are impossible to reach for large groups of students?

One may also question if we should really call it learning difficulties if it is the teaching or some other aspect of the learning environment that is not functioning well. “Teaching difficulties” is sometimes a better notion.

Regardless of the perspective that is chosen, it is clear that “learning difficulties” is a relative notion. And even the best have difficulties: “Do not worry about your difficulties in mathematics; I can assure you that mine are still greater” (Albert Einstein)

This paper will not provide a comprehensive overview of the notion of learning difficulties, but focuses on the ways we (teachers and students) handle the subject mathematics itself in teaching and learning contexts. In particular, the tendency to take the reduction of mathematical complexity too far into inefficient rote learning will be discussed.

Detecting learning difficulties

Trying to detect learning difficulties can also be done with different perspectives and there are several general questions to consider, for example:

- Are the measures proper, e.g. examinations and other tests? Ordinary assessment tools are often so limited in scope that only a few, lower order, competencies are normally assessed (Niss, 2007). It is possible that someone who has learned well will fail the test and, conversely, that someone who has not learned well will pass. For example, in tests after written university examinations asking students to explain what they have answered the result is often disappointing.

- What are the common informal indications? As teachers we often have other sources of information, e.g. from discussions with our students. But this could be more problematic at the tertiary level where we often have much
less contact with students than primary and secondary teachers have. And perhaps our intuition is not the sharpest tool:

– Do we need special diagnostic tools than the ones that seem more common in primary school?

– Can students recognise and understand their own difficulties? Of course, most students with learning difficulties surely realise this in some way. But I have met students who sincerely claim that they understand everything the teachers say and everything that is written in the textbook, and that the only thing they do not understand is why their exam score is zero. What lies behind this? A lack of self-insight? Completely different conceptions of understanding? An inability, e.g. through stress, to perform in high-stake examinations? I have also met students with top scores in exams who claim they do not understand the course. I have asked students about their learning difficulties in more or less systematic ways but often obtain very limited and non-informative answers. It may be that my ways of asking are not sharp enough, but it is also probably difficult enough for a student to understand why you succeed and actually much more difficult to understand why you do not succeed.

– What can we learn from research? Concerning primary and secondary education there is a large and growing body of research related to learning difficulties in mathematics, although in general terms much is still unknown (Niss, 2007). Regarding tertiary mathematics there is much less research. There are many papers describing different tests (resembling fairly traditional examinations) given to students, indicating lacking abilities, knowledge or competence in some aspect. But many of these papers are descriptive reports that are not formalised as research (Artigue et al., 2007). There are very few studies that explicitly address tertiary mathematics learning difficulties, but several do this implicitly. This in particular applies to theories of concept formation (Sfard, 1991; Asiala et al., 1996; Tall, 2004) that in different ways treat difficulties of coming to understand mathematical concepts. Some brief summaries and a few specific examples of research insights will be given below.

**Causes and characteristics of learning difficulties**

The characteristics of learning difficulties can be related to educational goals that some students fail to reach to various degrees. These goals can, in a similar way as in for example the NCTM Standards (NCTM, 2000) for K-12 education, be seen as comprising both content and processes to master.

**Content understanding difficulties**

Many learning and achievement difficulties are directly related to inherent mathematical difficulties within specified concepts. For example, concerning calculus the diffi-
cultures found in the research literature seem related to all of its fundamental notions: e.g. variable, function, limit, derivative, integral and differential equation. There are other literature examples related to linear algebra, abstract algebra and real analysis. It is clear that students’ concept images often differ substantially from the concept definitions, not only in the sense that they are incomplete but also in characteristics. These difficulties are at least partially explained by theories indicating the complexity of mental concept formations. It is also clear that a conceptual understanding is not sufficient for mathematical proficiency.

One of the more specific results of research in mathematics education is, according to Niss (1999), the key role of domain specificity:

“For a student engaged in learning mathematics, the specific nature, content and range of a mathematical concept that he or she is acquiring or building up are, to a large part, determined by the set of specific domains in which that concept has been concretely exemplified and embedded [...] For example, even if students who are learning calculus or analysis are presented with full theoretical definitions [...], and even if it is explicitly stated in the textbook and by the teacher that the aim is to develop these concepts in a general form [...], students actual notions and concept images will be shaped, and limited, by the examples, problems, and tasks on which they are actually set to work.”

One aspect of this is treated by Tall and Vinner (1981) when introducing the notion of “concept image” and describing how these may differ substantially from the corresponding concept definitions both in scope and character. For example, many students believe that all functions $y = f(x)$ are continuous (Harel and Dubinsky, 1992). Artigue (1996) summarises research on students’ difficulties with the conceptual field of analysis. One of the main problems is that the basic objects of the field (real numbers and functions) are not stabilised for students when they enter the field, although they have studied these basic objects earlier. Another problem is the students’ difficulties in fully understanding the central concept of limit, where primitive “pre-understandings” may have been sufficient in earlier social or scholar contexts but may in fact hinder the necessary development towards deeper insights. Extensive discussions about educational research on (among other things) students’ difficulties with university mathematics can be found in (Tall, 1996) and (Holton, 2001).

**Difficulties with mathematical processes**

There are findings related to difficulties with what in the NCTM Principles and Standards (NCTM, 2000) are denoted processes, e.g. non-routine problem-solving, proof and proving, reasoning, representing and modelling. Two of the more central, and recurrent, findings in research on problem-solving are: (i) students’ focus on the rote learning of routine procedures, which is often not complemented by the development of other task-solving approaches; and (ii) students’ extensive difficulties in solving non-routine problems (Schoenfeld, 1985; Lester, 1994; Selden et al., 1994). This unbalance seems to align poorly with most mathematics curricula goals. Although
this has been well known for quite a while, this unbalance seems persistent at all educational levels (Hiebert, 2003).

There are many studies on different aspects of learning, understanding and implementing proof (Hanna and Jahnke, 1996; Yackel and Hanna, 2003). Students have difficulties in differing proofs from other less rigorous types of argumentation (Chazan, 1993; Hoyles, 1997), understanding proof statements (Selden and Selden, 1995), making the transition from informal to formal reasoning (Tall, 1999) and constructing proofs. Even among university students, empirical sources of conviction (e.g. evidence from one or a few examples) dominate over more stable formal deductive reasoning (Balacheff, 1988; Harel and Sowder, 2007). There are also difficulties related to less formal but still central forms of reasoning, which will be exemplified below.

**Reasons behind learning difficulties**

**Reduction of complexity**

A large part of the research results dealing with the reasons behind the difficulties discussed above can be characterised as an unwarranted and far too extensive reduction of the complexity of mathematical concepts, processes and other ideas. This seems to be done in different situations by teachers, textbook writers and/or students in order to cope with curricula goals that are (too?) hard to reach.

Students are inclined to answer questions with a suspension of sense-making, and often use short-cut strategies (Schoenfeld, 1991). There is pressure from students to reduce ambiguity and risk, and to improve classroom order, by reducing the academic demands in tasks (Doyle, 1988). In a historical perspective, McGinty et al. (1986) analysed grade 5 arithmetic textbooks from 1924, 1944 and 1984 and found that the number of word problems had decreased, the number of drill problems had increased, and that word problems had also become shorter and less rich. A brief comparison between some older calculus textbooks, for example (Courant and John, 1965), (de La Vallée Poussin, 1954) and some newer textbooks (Edwards and Penney, 2002), (Adams, 2006) indicates that the proportion of exercises that have more or less complete solution methods provided (e.g. worked examples that are very similar to the exercises) has risen considerably.

Vinner (1997) suggests a theoretical framework where two of the main notions are “pseudo-conceptual” and “pseudo-analytical”. They are defined as thought processes that are not conceptual and analytical, respectively, but might give the impression of being so and could even produce correct solutions. Students’ difficulties may often be better understood if they are interpreted within this “non-cognitive” framework than if they are seen as misconceptions within the domain of meaningful contexts: What may be a true learning and problem-solving situation for the teacher may not be so for the student. Because of the didactic contract (Brousseau, 1997) students may, consciously or not, try to please the education system with behaviour that, perhaps only superficially, is considered acceptable by the system. Leron and Hazzan (1997)
emphasise additional non-cognitive means of trying to cope: attempts to guess and to find familiar surface clues for action, and the need to meet the expectations of the teacher or researcher.

**A procedural focus**

The most frequent type of reduction of complexity is to focus the teaching and learning on algorithmic procedures that can be carried out in order to solve advanced tasks without the need for conceptual understanding or constructive reasoning. The reasons behind students’ focus on learning and applying routine procedures are discussed by Tall (1996) in an article on functions and calculus under the heading “Procedural consequences of conceptual difficulties”:

“When faced with conceptual difficulties, the student must learn to cope. In previous elementary mathematics, this coping involves learning computational and manipulative skills to pass exams. If the fundamental concepts of calculus (such as the limit concept underpinning differentiation and integration) prove difficult to master, one solution is to focus on the symbolic routines of differentiation and integration. At least this resonates with earlier experiences in arithmetic and algebra in which a sequence of manipulations are performed to get an answer. The problem is that such routines become just that – routine – so that students begin to find it difficult to answer questions that are conceptually challenging. The teacher compensates by setting questions on examinations that students can answer and the vicious circle of procedural teaching and learning is set in motion.”

**An example of rote learning: trying to recall a rule**

A first-semester university mathematics student asks:

“Is $a^5 \cdot a^3 = 2a^8$ correct? I think that you should add or multiply 5 and 3. And there are two $a$:s, or...? Anyway, I have forgotten how to do it.”

My response is: “No, just add the exponents, $a^5 \cdot a^3 = a^{5+3} = a^8$.”

“Thanks, then I understand”, the student says and I turn to help another student.

There are several central issues related to this simple example of a general and far-reaching phenomenon, for example what does the student mean by ‘understand’? What can she learn from my poor response above memorising a meaningless rule? And why does the student ask me this question in the first place, instead of reasoning in the following way? If $m$ is a natural number, then $a^m$ is just an economic way of writing repeated multiplication $a\cdot a\cdot a\cdots a$ with $m$ factors. So $a^5 \cdot a^3$ just means $a\cdot a\cdot a\cdot a\cdot a\cdot a\cdot a\cdot a\cdot a$, which can be rewritten as $a^8$.

There seems to be two main types of answers. One is obvious, that the student has not learnt what $a^m$ means, but this answer is not very informative and leads to a deeper question: The student has seen this definition several times in secondary school and also at university. Why hasn’t she learnt it? One explanation is that the tasks she actually works with do not concern the fundamental meaning of definition but mainly the handful of algorithmic rules for powers that are consequences of the
definition, like the one for multiplication that she tries to recall (Lithner, 2004). The other main type of explanation is related to beliefs: that the student knows what am means, but has formed the belief that mathematics is all about applying rules provided by someone else and that it is not within her mathematical world view that she could try to reason herself (Schoenfeld, 1985).

**Creative and imitative reasoning**

The above example is too simple to capture the complexity we often see in students’ reasoning, but sufficient to illustrate some aspects of the reduction of complexity. There are two different ways to reason in learning situations that lead to fundamentally different possibilities to learn.

The student in the previous example attempts what can be denoted imitative reasoning, or more specifically in this case, algorithmic reasoning (Lithner, 2008). An algorithm is a finite sequence of instructions that allows one to find a definite result for a given class of problems. It can be determined in advance, the n:th transition does not depend on any circumstance unforeseen in the n-1:st transition. It has high reliability and speed since it is designed to avoid meaning (Brousseau, 1997).

What we really want, especially when algorithmic reasoning does not work, can be characterised as Creative Mathematically Founded reasoning (CMR): A new (to the reasoner) sequence of reasoning is (re-) created and the conclusions are based on arguments anchored in relevant mathematical properties of the components one is reasoning about (Lithner, 2008). The creativity shall be seen in an elementary and not in a genius sense, as ”an orientation or disposition toward mathematical activity that can be fostered broadly in the general school population” (Silver, 1997, p. 75). It could be seen as similar to common definitions of non-routine problem-solving reasoning, but one difference is that the latter includes a challenge aspect while CMR can be easy (as in the second solution to the a^5 · a^3 task above). A main point is that in CMR the construction and anchored argumentation is impossible to do without considering the meaning of the components involved in the reasoning.

In our studies of mainly secondary and tertiary mathematics education algorithmic reasoning is completely dominating in most parts of the learning environment. Comparing with international research, which often treats similar phenomena but with other terminology like rote learning, procedural learning, memorising, etc., it seems clear that this is one of the main aspects of learning difficulties in mathematics at all educational levels (Hiebert 2003). Fundamental competencies such as conceptual understanding and problem-solving ability are not developed by imitative reasoning, since it avoids both creativity and meaning. One main reason for its dominance seems to be that it is the easiest way to overcome learning difficulties in a very short perspective. Algorithmic reasoning is surely an essential part of mathematical competence, but if it dominates completely then learning difficulties are only made worse in the long-term perspective.
Teaching, textbooks and tests

If students’ reasoning is dominated by imitative approaches that are far from the educational goals, one must ask why? Do we not provide our students with opportunities to develop fundamental competencies? To answer this question it is not sufficient to consider what our goals are and not even how we as teachers present mathematics. A key issue is what kind of activities we invite our students to engage in. Swedish mathematics students normally spend 50-100% of their study time with textbook tasks. The following example from an American calculus textbook common in Sweden is more extensively analysed in (Lithner, 2004).

Exercise formulation:

“In Exercises 3-40, evaluate the limit or explain why it does not exist.

14. \[ \lim_{t \to 2} \frac{t^2 + 3t - 10}{t^2 - 4} \]

There are, of course, very many ways to solve it, but it can be done imitatively by mimicking the following textbook example.

“EXAMPLE Evaluate: (a) \[ \lim_{x \to -2} \frac{x^2 + x - 2}{x^2 + 5x + 6} \] (b) […] (c) […]

SOLUTION Each of these limits involves a fraction whose numerator and denominator are both 0 at the point where the limit is taken.

(a) \[ \lim_{x \to -2} \frac{x^2 + x - 2}{x^2 + 5x + 6} \]

 Undefined at \( x = -2 \). Factor numerator and denominator.

\[ = \lim_{x \to -2} \frac{(x+2)(x-1)}{(x+2)(x+3)} \text{ Cancel common factors.} \]

\[ = \lim_{x \to -2} \frac{x-1}{x+3} \text{ Evaluate this limit by substituting } x = -2. \]

\[ = \frac{(-2-1)}{(-2+3)} = -3. \]

This example is in effect an algorithm that can be imitated. If this template is followed, the target knowledge (limits) is missed. The only mathematical activities needed to engage in concern grade 10 elementary algebra: factorising a simple polynomial, evaluating a simple function, and fraction arithmetic. Such templates ‘help’ students that are unable to reach the real educational goals so that they and we may believe that they are dealing with advanced mathematics. As an extreme example, I have easily taught 7-year-old kids to differentiate simple polynomials. They had, of course, no understanding of derivatives whatsoever, but would have received half a point or so an a university examination.

In some common American calculus textbooks 70% of the exercises may be done with such imitative reasoning, 20% by essentially copying a solution but with a minor
modification by local CMR, and 10% require global CMR (Lithner, 2004). The latter are generally the most difficult exercises at the end of each exercise section, which few students ever attempt. Similar distributions are found in studies of Swedish grade 5-12 textbooks. Imitative reasoning is also the dominating type of student reasoning found in the individual or small-group learning situations (Lithner, 2000, 2003, 2008). A similar 70-20-10 distribution is found in Swedish university mathematics examination, where it is possible to pass and even receive high grades by imitative reasoning alone (Bergqvist 2007). In interviews concerning this distribution, teachers claim that this is necessary in order to cope with extensive learning difficulties and heterogeneous groups.

Thus, the idea is to help students but it seems that we take the reduction of complexity too far. This also affects students’ beliefs about what mathematics is: The search for algorithms becomes mathematics instead of being a part of it. An important observation is that this 70-20-10 distribution is the same for both secondary and tertiary mathematics education (Palm et al., 2011).

The transition from secondary to tertiary mathematics

An issue of intense debate concerns how well-prepared students entering tertiary mathematics are and how the students and the tertiary mathematics teachers handle the transition. The difficulties are related to individual, social, and institutional phenomena (Gueudet, 2008). As noted above, more students are entering and there are indications in the literature of decreasing knowledge levels, more negative attitudes towards learning mathematics, and political and budgetary pressure to help more students enter tertiary mathematics, leading to reduced entrance qualifications and course requirements (Selden, 2005). In order to cope with this situation, the way that we treat mathematics can be altered, e.g. by the reduction of complexity discussed above. Some claim that the situation is further made worse by reform pedagogy, while the reform proponents claim that secondary students are now better in other areas than those traditionally assessed, e.g. in non-routine problem-solving. Judging from the research literature, it seems that reforms, e.g. in the sense of NCTM Standards, can lead to substantial improvements. Some claim reforms are extensive in secondary school, but this may be varying. In Sweden, we have had a reform in the official syllabus documents but, as it seems, not so much in practice.

However, there are specific difficulties related to the transition from upper secondary school to university that are often emphasised (Forgasz and Leder, 2000; Tall, 1992). A general qualitative step in this transition is with respect to an increased level of abstraction, a difficult transition from intuitively-based concepts to formal definitions. This level is in a sense increasing continuously through the whole educational system, but is seen by many as a crucial difference between upper secondary school and university (see, for example, several of the articles in Tall (1991)). In addition, at the university level there are bigger requirements on the students’ responsibility
and autonomy, which many claim are one of the main reasons behind the learning and achievement problems (Guzman et al., 1998). Proof and strict rigour are at least meant to be more extensive, and students have extensive difficulties with this (see the discussion above). Concerning routine tasks the mere number of notions and methods make routinisation more difficult (Artigue et al., 2007).

What to do about tertiary learning difficulties?
This is the big question, but unfortunately it seems very difficult to provide well-founded suggestions for what the measures should be. I will just mention a few things to consider.

- Tertiary education needs to be more flexible and adjust to what can be seen as new societal assignments, e.g. mass education.
- More research is needed to clarify the extent, characteristics and causes of learning difficulties. In the debate there are many more or less speculative responses, which indicate that research is still far from clarifying this and convince actors on the educational arena.
- Better connections between research and development. As has been pointed out by, for example, Burkhardt and Schoenfeld (2003) and Plomp (2009), educational research and educational development are too separated.
- We can learn from primary and secondary reforms. I see no reason why the path set out by NCTM Principles and Standards (NCTM, 2000) and other similar frameworks should not be useful. Although there are differences, there seems to be much in common between primary, secondary and tertiary mathematics education.

One main hurdle for the development of mathematics education is that we often tend to see teaching (e.g. of mathematics) as a simple task, and thus look for simple solutions. But, in spite of decades of research and development, no one has found this simple solution to the extensive mathematics learning difficulties. One of my favourite quotes (by searching the interne, it seems that various versions of it are attributed to many different persons) that often comes to my mind when considering the educational debate is that “To every difficult problem there is a simple solution that is wrong”. If many of the seriose student performance difficulties and learning environment inadequacies discussed above have been known for at least a couple of decades, and if teachers and researchers have shown fruitful ways of improvement, why are the changes not more profound? One reason is the immense complexity of mathematics learning (Niss, 1999) and, according to Artigue (1998), that research seldom shows extensive improvements via simple changes: “On the contrary, most research based designs require more engagement, expertise from teachers, and significant changes in practices” (Dubinsky et al., 1997).
The easiest way to design mathematics teaching is to focus on algorithmic procedures. They are powerful both in the sense that they reduce the complexity of the task and that they can be learnt without the need to struggle with conceptual understanding, since they are designed to avoid meaning (Brousseau, 1997). And, indeed, procedures have central roles in learning and using mathematics. For example, in two-digit addition by hand, in many applications using a calculator or computer (e.g. to draw a graph), in finding a mean value or in calculating the length of a triangle’s side. Suitable procedures are very powerful in mathematic, and one of their main roles is to make mathematics more efficient (both regarding precision and speed). However, while a main advantage of a procedure is that, when using mathematics for applications it can be carried out without references to meaning, this advantage in an application situation is at the same time a big disadvantage in a learning situation since procedures can be learnt by rote. This is highly ‘efficient’ in the short-term perspective in the sense that it is easy to learn how to solve a task that may involve advanced concepts (e.g. to differentiate a polynomial) but may be devastating in the long run since understanding may be absent. There are clear indications that many students spend their time with mathematics mainly trying to learn procedures by rote imitation (Hiebert, 2003; Lithner, 2008; Tall, 1996; Bergqvist et al., 2010a,b).

We do not know exactly, e.g. in a psychological perspective, how students’ focus on algorithmic reasoning is created. But the main components of the traditional mathematics learning environment, teaching and textbook, at least reinforces but probably creates this behaviour among students. Assuming that students may learn what they have the opportunity to learn (Hiebert, 2003) and combining this assumption with the research showing that students are mainly provided with opportunities to learn mathematics by superficial imitation, one main conclusion is that it may be a misleading perspective to see the problems as learning difficulties. Instead, if what the students do is actually to adjust well to the teaching (including textbooks) then we should change the perspective towards analysing the problem in terms of teaching difficulties. For example, temporarily incomplete or faulty conceptions in the form of obstacles are in The Theory of Didactical Situations (Brousseau, 1997) not generally seen as failures but are often inevitable and constitutive of knowledge. An obstacle produces correct responses within a particular, frequently experienced context but not outside it and may withstand both occasional contradictions and the establishment of a better piece of knowledge. Clarifying obstacles helps the student see the necessity for learning, not by explaining what the obstacle is but to help her discover it. Good problems will permit her to overcome the obstacles. The teacher may (e.g. to reduce complexity) try to overcome the obstacle and force learning by devolving less of the problem to the student. Brousseau exemplifies this by the the Topaze effect, when the teacher lets the teaching act collapse by taking responsibility for the student’s work and letting the target knowledge disappear.
Telling the student that an automatic method exists relieves her of the responsibility for their intellectual work, thus blocking the devolution of a problem. If this is the normal didactic situation the student meets, then the didactical contract is formed accordingly, which may not be the teacher’s intention. The teacher expects the student to learn problem-solving reasoning, while the student expects that an algorithm should be provided that relieves her of the responsibility of engaging in the adidactical situation. This avoids dealing with the obstacle that can therefore become insurmountable.

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EDITORIAL

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