Simple implementation of high fidelity controlled-iSWAP gates and quantum circuit exponentiation of non-Hermitian gates

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The $i$SWAP gate is an entangling swapping gate where the qubits obtain a phase of $i$ if the state of the qubits is swapped. Here we present a simple implementation of the controlled-$i$SWAP gate. The gate can be implemented with several controls and works autonomously. The gate time is independent of the number of controls, and we find high fidelities for any number of controls. We discuss an implementation of the gates using superconducting circuits, however, the ideas presented in this paper are not limited to such implementations. An exponentiation of quantum gates is desired in some quantum information schemes and we therefore also present a quantum circuit for probabilistic exponentiating the $i$SWAP gate and other non-Hermitian gates.

I. INTRODUCTION

In order to perform non-trivial quantum computations it is often necessary to change the operation applied to one set of qubits depending upon the values of some other set of qubits. Some well known controlled gates are the controlled-NOT (CNOT), controlled-SWAP (Fredkin), and controlled-controlled-NOT (Toffoli) \cite{1}. While these gates are used in many quantum information schemes, such as quantum computing \cite{2–4}, error-correction \cite{5–8}, cryptography \cite{9–11}, fault tolerant quantum computing \cite{12, 13}, and measurement \cite{14, 15}, they are not necessarily the most experimentally feasible ones \cite{16}.

Equivalent (in the sense that they both constitute a universal set of gates together with the set of one-qubit operations) to the CNOT gate is the $i$SWAP gate which we denote $\hat{S} = |00\rangle\langle 00| + i(|10\rangle\langle 01| + |01\rangle\langle 10|) + |11\rangle\langle 11|$. The $i$SWAP gate is a perfect entangling version of the SWAP gate, which is why it is equivalent to the CNOT gate. However, the $i$SWAP gate has the advantage over the CNOT gate that it occurs naturally in systems with $XY$-interaction or Heisenberg models, such as solid state systems \cite{17, 18} and superconducting circuits \cite{19}. Other implementations of the $i$SWAP gate include linear optics \cite{20, 21} and nuclear spin using qubits \cite{22}.

Despite several attempts of implementing the $i$SWAP gate \cite{23–25}, the Fredkin gate \cite{26–34}, and other controlled-swapping gates \cite{35–37}, no one have embarked in the implementation of a controlled-$i$SWAP gate, to the best of our knowledge.

Here we present a simple implementation of a multi-qubit controlled-$i$SWAP gate which we call $C^n i$SWAP, where the $n$ indicates the number of control qubits. For a single control qubit this is essentially an $i$Fredkin gate, i.e., a Fredkin gate with a phase of $i$ on the swapping part. The implementation is based using the control qubits to tune the target qubits in and out of resonance by following the approach presented in Refs. \cite{38, 39}, and can be realized using different schemes for quantum information processing. We include circuit design for an implementation of the $C^n i$SWAP gate in superconducting circuits as well as for the $C^2 i$SWAP gate in the appendix. The gate is autonomous in the sense that it does not require any outside driving, and the gate time is thus independent of the number of control qubits. When neglecting the decoherence of the qubits we find a fidelity above 0.998 for one control qubit. When including decoherence of the qubits the fidelity stays above 0.99 for up to four control qubits.

Being able to exponentiate quantum gates can be useful in different quantum information schemes such as in continuous variable (CV) systems \cite{40}, where exponentiated gates, such as $\exp(i\theta \hat{X})$, can be used to operate on the systems \cite{41, 42}. Another scheme which might benefit from being able to exponentiate non-Hermitian quantum gates is quantum random walks \cite{43}, where non-unitary operations is needed for, e.g., graph coloring \cite{44, 45}. We therefore present a quantum circuit for probabilistic exponentiating of non-Hermitian operators, based on the method by \cite{46} which works for exponentiating Hermitian operators. Our method is exact for cyclic operator, i.e., operators fulfilling $\hat{T}^n = 1$, while it is approximate for all other non-Hermitian operators.

The paper is organized as follows: In Section II we present a simple Hamiltonian and show how it yields an $C^n i$SWAP gate. We discuss the effectiveness of the gate exploring the single qubit controlled-$i$SWAP gate as an example in Section II A. We further, in Section III, present an implementation using superconducting circuits of the $i$SWAP gate and discuss how to expand it to more controls. In Section IV we show how to expand the implementation of the controlled-$i$SWAP gate into controlling swapping of an array of qubits. In Section V we present a quantum circuit for probabilistic exponentiating cyclic quantum gates, and discuss its range of validity. In Section VI we provide a summary and outlook for future work.

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II. IMPLEMENTATION OF THE CONTROLLED-iSWAP GATE

Consider $n + 2$ qubits each with frequency $\Omega_i$. The first $n$ qubits are connected to the second to last qubit by an Ising couplings $J^z_i$ and to the last qubit by an Ising coupling $-J^z_i$, where $i$ refers to which of the first $n$ qubits. The last two qubits are further connected to each other by a transversal coupling $J^z$. We denote the last two target qubits $T1$ and $T2$. The Hamiltonian for the system is

$$\hat{H} = \hat{H}_0 - \Delta (\sigma^z_{T1} - \sigma^z_{T2}) + \sum_{i=1}^{n} J^z_i (\sigma^z_{T1} - \sigma^z_{T2}) \sigma^z_i$$

$$+ \frac{1}{2} J^x (\sigma^x_{T1} \sigma^x_{T2} + \sigma^y_{T1} \sigma^y_{T2}),$$

where $\sigma^{x,y,z}$ denotes the Pauli matrices, and the non-interacting part of the Hamiltonian is given as

$$\hat{H}_0 = - \sum_{i=1}^{n} \Omega_i \sigma^z_i - \tilde{\Omega} (\sigma^z_{T1} + \sigma^z_{T2}),$$

where $\tilde{\Omega} = (\Omega_{T1} + \Omega_{T2})/2$ is the average of the target qubits, and $\Delta = (\Omega_{T1} - \Omega_{T2})/2$. Changing to the interaction picture using the transformation $\hat{U}_\text{int}(t) = \exp(i \hat{H}_0 t)$, the Hamiltonian takes the form

$$\hat{H}_I = - \Delta (\sigma^z_{T1} - \sigma^z_{T2}) + \sum_{i=1}^{n} J^z_i (\sigma^z_{T1} - \sigma^z_{T2}) \sigma^z_i$$

$$+ J^x (\sigma^x_{T1} \sigma^x_{T2} + \sigma^y_{T1} \sigma^y_{T2}).$$

In order to realize the behavior of the controlled-iSWAP gate we must require the detuning to be

$$\Delta = - \sum_{i=1}^{n} J^z_i,$$

and $J^z_i \gg J^x$ for all $i$. Thus the energy shift due to the first $n$ qubits must be large enough to bring the last two qubit in and out of resonance, making the first $n$ qubits the control qubits and the last two the swapping qubits.

Changing into the frame rotating with the diagonal part of the Hamiltonian we obtain

$$\hat{H}_{\text{rot}} = J^x \left[ \sigma^+_{T1} \sigma^-_{T2} e^{i \sum_{i=1}^{n} J^z_i (1 + \sigma^z_i) t} + \text{H.c.} \right].$$

With the condition that $J^x \gg J^z$ both terms of $\hat{H}_{\text{rot}}$ will rotate rapidly, and can thus be neglected using the rotating wave approximation, unless all of the control qubits are in the state $|1\rangle$. The means that the Hamiltonian effectively becomes

$$\hat{H}_{\text{rot}} = J^x |\tilde{1}\rangle_{C} \otimes \left[ \sigma^+_{T1} \sigma^-_{T2} + \sigma^+_{T2} \sigma^-_{T1} \right],$$

denotes the state where all control qubits are in the state $|1\rangle$.

We can calculate the time evolution operator by taking the matrix exponential, $\hat{U}(t) = \exp(i \hat{H}_{\text{rot}} t)$, which yields

$$\hat{U}(t) = \hat{I}_C \otimes \hat{I}_T + |\tilde{1}\rangle_{C} \langle \tilde{1}|_{C}$$

$$\otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(Jt) & -i \sin(Jt) & 0 \\ 0 & -i \sin(Jt) & \cos(Jt) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\hat{I}_C$ denotes the reduced identity of the control qubits where the states $|\tilde{1}\rangle_{C}$ have been removed. The identity of the target qubits is denoted $\hat{I}_T$.

From Eq. (7) we see that for times $t = (2m + 1) \pi / 2 J^x$, $m \in \mathbb{Z}$ the time evolution operator takes the form of a controlled-iSWAP gate.

$$\hat{U}(t = T) = \hat{I}_C \otimes \hat{I}_T + |\tilde{1}\rangle_{C} \langle \tilde{1}|_{C} \otimes \hat{S}_T,$$

where $\hat{S}_T$ is the two-qubit iSWAP gate on the target qubits, which swaps the target qubit with a phase of $\pm i$. The phase on the target qubit depends on the sign of $\mp J^x$. For completeness we note that for times $T' = (2m + 1) \pi / 4 J^x$ we obtain the controlled-$\sqrt{\text{iSWAP}}$ gate [47].

A. Example: The single controlled-iSWAP gate

In order to illuminate the performance of the system worked as a $c^n$-iSWAP gate we explore the example of the single controlled-iSWAP gate. We chose this example since not only is it the simplest non-trivial example, it is also closely related to the Fredkin gate. A schematic presentation of the model yielding the controlled-iSWAP gate can be seen in Fig. 3(a), which corresponds to Eq. (1) with $n = 1$.

We characterize the performance of the gate by calculating the average process fidelity, which is defined as [1, 48–50]:

$$F = \int \psi^* \hat{U}^\dagger \mathcal{E}(\psi) \hat{U} \psi,$$

where integration is performed over the subspace of all possible initial states and $\mathcal{E}$ is the quantum map realized by our system. We simulate the system using the Lindblad Master equation and the interaction Hamiltonian of Eq. (3) using the QuTiP Python toolbox [51]. The result is then transformed into the frame rotating with the diagonal of the Hamiltonian, and then the average fidelity is calculated.

For all simulations we have $J^z / 2 \pi = 50 \text{ MHz}$, while we change the transversal coupling, $J^x / 2 \pi$, from 5 to 25 MHz. The average fidelity of the simulation can be seen in Fig. 1 together with the gate time. The figure shows both the average fidelity without any decoherence and
with a decoherence time of $T_1 = T_2 = 30\,\mu s$ [52]. Without any decoherence we find that the average fidelity increases asymptotically towards 1 as the driving decreases, with the only expense being an increase in gate time. Since decoherence increases over time, a longer gate time means lower fidelity, which is exactly what we observe when including decoherence in the simulations. In this case we find that the fidelity peaks at $\sim 0.995$ around $J^z/J^x \sim 4$, which yields a gate time of $T \sim 25\,\mu s$. However, we note that the fidelity are dependent on the parameters $J^x$ and $J^z$ thus changing these will change the fidelity. We also see that for just $J^z = 2J^x$ we obtain an average fidelity above 0.99 for a gate time $T \sim 15\,\mu s$. The oscillation of the average fidelity is due to a small mismatch in the phase of the evolved state compared to the desired matrix in Eq. (7), which disappears when $J^z/J \in \mathbb{Z}$.

We simulate the $C^n\text{iswap}$ gate for different $n$ in the optimal ratio between couplings, $J^z/J^x \sim 4$. The result of this simulation is seen in Fig. 2. We observe that the fidelity stays above 0.998 for up to $n = 4$ control qubits when decoherence is not included. The reason for this is that for larger $n$ the gate resembles the identity more. This is due to the fact that the identity operation is applied to the control qubits, meaning that for a large number of control qubits, the gate will perform the identity on the control qubits and the swapping operation will only be performed on the target qubits. When decoherence is included the average fidelity decreases for larger $n$ as it should, however, we still find a fidelity above 0.99 for up to 4 controls.

### III. EXPERIMENTAL IMPLEMENTATIONS

A possible implementation of the controlled-$\text{iswap}$ gate using superconducting circuits can be seen in Fig. 3(b). The circuit consists of three transmon qubits [53, 54], where two of them are connected by a capacitance and the third qubit is connected to the other two by Josephson junctions, with as small a parasitic capacitance as possible. By using the diabatic gate approach in Ref. [55] one can minimize the leakage of the capacitive coupling between the two target qubits. Such a circuit has the Hamiltonian

$$
\hat{H} = \frac{1}{2} \hat{p}^T K^{-1} \hat{p} - \sum_{i=1}^{n} E_i \cos \hat{\phi}_i - \sum_{i=\{T1,T2\}} E_i \cos \hat{\phi}_i - \sum_{i=1}^{n} E_z [\cos(\hat{\phi}_{T1} - \hat{\phi}_i) + \cos(\hat{\phi}_{T2} - \hat{\phi}_i)],
$$

where $\phi_i$ are the node fluxes and $\hat{p}^T = (p_{T1}, p_{T2}, p_1)$ are the conjugate momenta. The Hamiltonian is the general Hamiltonian for a $C^n\text{iswap}$ gate, and in the case of Fig. 3(b) one needs $n = 1$. As the capacitive couplings yields transversal $XX$-couplings when truncating to a Ising-type model, we are not interested in the capacitive couplings between the control qubit and the target qubits, and thus we require $C_z \ll C_i, C_{T1}$ which will leave the
capacitance matrix being approximately diagonal, with
the exception of the desired capacitance between
the target qubits. This leaves only longitudinal ZZ-couplings
between the control and target qubits. This limit where
the longitudinal coupling dominates over the transversal
couplings is within experimental reach [56]. An other
way of reaching high-contrast ZZ-couplings could be to
use a combination of transmon and flux qubit, and then
engineering opposite sign anharmonicities as in Ref. [57].
When truncating the Hamiltonian in Eq. (10) to a two
level system one reaches the Hamiltonian in Eq. (3). A
detailed calculation going from the circuit design to the
technical frequency of the target qubits, and the subscript
indicates the
i
th control qubit. If we require that the Ising
couplings have the strengths
J
i
, then at times
T
i
= \Delta
i
, and that
J
z
i
= -\Delta
i
, and require that
J
z
i
 \gg J
x
i
 for all
i
, then at times
T = (2m + 1)\pi/(2J
x
),
m \in \mathbb{Z}
the time evolution operator for the
n = 3 case takes the form
\begin{align}
\hat{U}(T) = \hat{I}_C \otimes \hat{I}_T &+ |100\rangle \langle 100|_C \otimes \hat{S}_{12} \\
&+ |010\rangle \langle 010|_C \otimes \hat{S}_{23} + |001\rangle \langle 001|_C \otimes \hat{S}_{13},
\end{align}
where \hat{I}_C denotes the reduced identity of the control qubits
where the states \langle 100| = \langle 000|, \langle 010| = \langle 001|, and \langle 001| = \langle 000|.

From the time evolution operator in Eq. (12) we see
that we have complete control over which qubits we wish
to swap, depending on the three ancilla qubits, i.e., if we
wish to swap qubit \( i \) and \( j \) we must have qubit \( C_{ij} \) in the
\langle 1 \rangle state and the remaining control qubits in the \langle 0 \rangle state,
and so fourth. This means that we need an ancilla qubit
for each possible swap. If we require all-to-all swapping
with \( n \) qubits, we would need \( N = n(n - 1)/2 \) ancilla
qubits in order to control all couplings.

In order to bring the number of ancilla qubits down we only
couple one ancilla qubit to each qubit in the array.
The Hamiltonian then takes the form
\[
\mathcal{H} = - \sum_{i=1}^{n} [(\Omega + \Delta_i)\sigma_3^i \sigma_3^i + \Omega \sigma_3^i \sigma_3^i] + \sum_{i=1}^{n} J_i^x \sigma_i^z \sigma_{i+1}^z + \frac{1}{4} \sum_{j,j'=1}^{n} J_j^x \sigma_j^z \sigma_{j'}^z. \tag{13}
\]

A schematic representation of the model for \( n = 3 \) can be seen in Fig. 4(c). If we require that the Ising couplings have the strengths \( J_i^z = -\Delta_i \), and require that \( J_i^z \gg J^x \) for all \( i \), then at times \( T = (m + 1)\pi/(2J^x), m \in \mathbb{Z} \) the time evolution operator for the \( n = 3 \) case takes the form
\[
\hat{U}(T) = \hat{I}_C \otimes \hat{I}_T + |110\rangle\langle 110|_C \otimes \hat{S}_{12}
+ |011\rangle\langle 011|_C \otimes \hat{S}_{23} + |101\rangle\langle 101|_C \otimes \hat{S}_{13} \tag{14}
+ |111\rangle\langle 111|_C \otimes \hat{S}_{123}.
\]

Again we obtain full control over swapping of the target qubits, however, this time we need control qubits \( Ci \) and \( Cj \) to be in the \(|1\rangle\) state and remaining control qubits to be in the state \(|0\rangle\), in which case with the \( \pm \text{iSWAP} \) -operators \( \hat{S}_{ij} \) swaps the state of the two qubits \( i \) and \( j \). We note that we also obtain a three-way swapping operator when all control qubits are in the \(|1\rangle\) state. In its matrix representation the three-way swap-operator is an \( 8 \times 8 \) matrix and takes the form
\[
\hat{S}_{123} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \hat{S}_1 & 0 & 0 \\
0 & 0 & \hat{S}_2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \tag{15}
\]
where the two operators \( \hat{S}_1 \) and \( \hat{S}_2 \) are \( 3 \times 3 \) matrices and operate on the three dimensional subspaces of one and two excitation number, of the target subspace, respectively. In their matrix representation these take the same form
\[
\hat{S}_{1,2} = \frac{1}{3} e^{iJ^x t/2} \begin{pmatrix}
3 \cos(3J^x t/2) & -i\sin(3J^x t/2) & 2i\sin(3J^x t/2) & 2i\sin(3J^x t/2) \\
2i\sin(3J^x t/2) & 3 \cos(3J^x t/2) & 2i\sin(3J^x t/2) & 2i\sin(3J^x t/2) \\
2i\sin(3J^x t/2) & 2i\sin(3J^x t/2) & 3 \cos(3J^x t/2) & -i\sin(3J^x t/2) \\
i\sin(3J^x t/2) & i\sin(3J^x t/2) & i\sin(3J^x t/2) & i\sin(3J^x t/2)
\end{pmatrix}. \tag{16}
\]

\[
\hat{S}_{1,2} = \frac{1}{3} e^{i\pi/6} \begin{pmatrix}
-1 & 2 & 2 & 2 \\
2 & -1 & 2 & 2 \\
2 & 2 & -1 & 2 \\
2 & 2 & 2 & -1
\end{pmatrix}. \tag{17}
\]

This operator can be used to create state belonging to the same non-biseparable classes of three-qubit states as the \( W \) state [58].

In Fig. 5(a) we show the model for a four qubit swapping array with all to all couplings corresponding to Hamiltonian in Eq. (13) with \( n = 4 \). In Fig. 5(b) we present the corresponding gate of the model coming from making the time evolution operator from the Hamiltonian. As above we obtain fully controllable two-qubit swapping between all of the four qubits. We further obtain four three-qubit entangling gates, similar to the one in Eq. (16) and one single four-qubit entangling gate.

In order to test the viability of our analysis we simulate the Hamiltonian in Eq. (13) using the Python toolbox QuTiP using the same approach as in Section II A. Using parameters \( J_i^z/(2\pi) \in \{-20, 20, 60\} \) MHz and \( J^x = \min_i |J_i^z|/5 \) we find a fidelity of 0.993 at time \( T = \pi/(2J^x) = 62.5 \) ns without including decoherence, and a fidelity of 0.98 when including a decoherence time of \( T_1 = T_2 = 30 \) μs.

V. PROBABILISTIC EXPONENTIATING OF CYCLIC NON-HERMITIAN QUANTUM GATES

In this section we present an exact probabilistic method for exponentiating cyclic non-Hermitian gates using an explicit quantum circuit. While our method is exact for cyclic operators it is approximate for non-cyclic operators. The controlled-\( i\text{SWAP} \) gate presented in this paper is in fact a cyclic non-Hermitian gate. Note that exponentiating non-Hermitian gates leads to non-unitary gates.

Unitary Hermitian gates can be exponentiated using the method developed by Marvian and Lloyd [46]. Albeit they only present their method for the controlled-\( \text{SWAP} \) gate, it works for all unitary Hermitian gates. Here we extend their method in order to exponentiate non-Hermitian gates. Our method is exact for a gate, \( \hat{T} \), for which \( \hat{T}^n = 1 \) for \( n \in \mathbb{Z} \) and approximately correct if this is not the case. We call gates where \( \hat{T}^n = 1 \) for cyclic gates with cyclic order \( n \). For \( n > 2 \) all cyclic gates become non-Hermitian, due to the fact that all eigenvalues of Hermitian matrices must be real and a diagonal matrix \( D \) fulfilling the Spectral theorem such that \( \hat{T} = \hat{U}D\hat{U}^{-1} \), where \( \hat{U} \) is a unitary, must then fulfill \( \hat{D}^n = 1 \).

Our result become interesting as soon as you want to exponentiate some sort of phase gate, with a phase other than \( -1 \), in which case the gate becomes non-Hermitian. This means that the result of such exponentiating will be non-unitary for \( n > 2 \). In Table I we mention a few often used non-Hermitian gates and their cyclic order.
We note that in order to use our method we must be able to perform a controlled version of the gate we wish to exponentiate, i.e., if we wish to exponentiate an $i$SWAP we would need a controlled-$i$SWAP, as discussed above.

Suppose we have a controlled cyclic gate $\hat{T}$ working on an arbitrary number of qubits. In order to create a circuit for exponentiating such an operator we must first Taylor expand the exponential

$$e^{i\theta \hat{T}} = \sum_{j=0}^{\infty} \frac{1}{(nj)!}(i\theta)^{nj}1 + \sum_{j=0}^{\infty} \frac{1}{(nj+1)!}(i\theta)^{nj+1}\hat{T} + \cdots$$

$$+ \sum_{j=0}^{\infty} \frac{1}{(n+1)(j+1)!}((i\theta)^{(n+1)}j^{-1})\hat{T}^{j-1}$$

$$= \sum_{k=0}^{n-1} \sum_{j=0}^{\infty} \frac{1}{(nj+k)!}(i\theta)^{nj+k}\hat{T}^k.$$  

In total this yields $n$ Taylor terms. This means that our quantum circuit would need $n - 1$ ancilla qubits to perform the controls. We then apply the controlled gate $n - 1$ times, each time controlled by a different ancilla qubit. The quantum circuit can be seen in Fig. 6.

We must now prepare the ancilla qubits in the state

$$|\tilde{\varphi}\rangle = N \sum_{k=0}^{n-1} \sum_{j=0}^{\infty} \frac{1}{(nj+k)!}(i\theta)^{nj+k}|\bar{k}\rangle,$$

where $N$ is a normalization which depends on $\theta$, and the state $|\bar{k}\rangle$ indicates a state with $k$ excitations, i.e., we have $|0\rangle = |00\cdots00\rangle$, and $|1\rangle = |01\cdots00\rangle$, $|0\rangle = |00\cdots01\rangle$, etc.

Let $|\gamma\rangle$ be the initial state of the target qubits. If we act with the $n - 1$ controlled-$\hat{T}$ gates on the initial state $|\tilde{\varphi}\rangle|\gamma\rangle$, as in Fig. 6 we arrive at the state

$$|\tilde{\varphi}\rangle|\gamma\rangle \rightarrow N \sum_{k=0}^{n-1} \sum_{j=0}^{\infty} \frac{1}{(nj+k)!}(i\theta)^{nj+k}\hat{T}^k|\bar{k}\rangle|\gamma\rangle.$$  

If we measure the $n - 1$ ancillae in the $\{|\pm\rangle\} = \{|0\rangle + |1\rangle\}/\sqrt{2}$ basis, there is a probability of around $1/2^n$ that we measure $|+\rangle$ in all of the ancillae, if we require $\theta$
to be small. This means that the total state becomes
\[ | + \cdots + \rangle N \sum_{k=0}^{n-1} \sum_{j=0}^{\infty} \frac{1}{(n_j + k)!} (i\theta)^{n_j + k} \hat{T}^k | \gamma \rangle \]
\[ = | + \cdots + \rangle N e^{i\theta \hat{T}} | \gamma \rangle, \]
which is the desired result. If this state is not measured the experiment must be repeated until the desired result is obtained.

We note that if the gate is not cyclic our method works approximately as long as \( \Theta \) is small, in which case the first terms of the Taylor expansion will dominate. This means that we can choose the number of terms we want in our Taylor expansion as the number of ancillae we include in our quantum circuit.

### A. Example

For an example of the Hermitian \( n = 2 \) case see Ref. [46]. Here we consider the case \( n = 4 \). This could, for example, be a controlled-\texttt{swap}. The exponential in this case becomes

\[
e^{i\theta \hat{T}} = \frac{1}{2} (\cos \theta + \cosh \theta) \hat{1} + \frac{i}{2} (\sin \theta + \sinh \theta) \hat{T}
+ \frac{1}{2} (\cos \theta - \cosh \theta) \hat{T}^2 + \frac{i}{2} (\sin \theta - \sinh \theta) \hat{T}^3.
\]

Remember that the operator in the exponent is \textit{not} Hermitian, and thus we are \textit{not} dealing with a unitary. This means that if \( \Theta \) becomes large, then the hyperbolic functions will blow up. Therefore we keep \( \Theta \) small. Notwithstanding, we prepare three ancillae in the state

\[
| \tilde{\varphi} \rangle = \frac{N}{2} \left[ (\cos \Theta + \cosh \Theta) |0\rangle + i (\sin \Theta + \sinh \Theta) |1\rangle 
+ (\cos \Theta - \cosh \Theta) |2\rangle + i (\sin \Theta - \sinh \Theta) |3\rangle \right] \]
\[ = A |000\rangle + B |001\rangle + C |011\rangle + D |111\rangle. \tag{20} \]

All normalization is included in \( N \). Note that we could have chosen other states such as \( |001\rangle \) and \( |101\rangle \) in the second and third term of \( | \tilde{\varphi} \rangle \) as well, as this choice can be made without loss of generality.

Now preforming the three controlled \texttt{T}-gates on the qubits we arrive at the state

\[
| \tilde{\varphi} \rangle | \gamma \rangle \rightarrow \frac{N}{2} \left[ (\cos \Theta + \cosh \Theta) |0\rangle + i (\sin \Theta + \sinh \Theta) |1\rangle \hat{T}
+ (\cos \Theta - \cosh \Theta) |2\rangle \hat{T}^2 + i (\sin \Theta - \sinh \Theta) |3\rangle \hat{T}^3 \right] | \gamma \rangle.
\]

By measuring in the \( \{ |\pm\rangle \} \)-basis there is a probability that we will measure the state \( | + + + \rangle \) which means that we have achieved matrix exponentiation by arriving at the state \( | + + + \rangle N e^{i\theta \hat{T}} | \gamma \rangle \).

### B. Measuring probability

In order to investigate the probability of measuring the correct state, we consider the state \( | \tilde{\varphi} \rangle \) in Eq. (20). In the \( \{ |\pm\rangle \} \)-basis it takes the form

\[
| \tilde{\varphi} \rangle = [A + B + C + D] | + + + \rangle
+ [A - B - C - D] | + + - \rangle
+ [A + B + C + D] | + - + \rangle
+ [A - B + C - D] | + - - \rangle
+ [A - B + C + D] | - + + \rangle
+ [A + B + C - D] | - + - \rangle
+ [A + B - C + D] | - - + \rangle
+ [A - B + C - D] | - - - \rangle, \tag{21}\]

We wish to measure a state with a coefficient \( A + B + C + D \), and thus we want to measure the state \( | ++ + \rangle \). Note that if we chose our \( | \tilde{k} \rangle \) states as superpositions, such as \( |1\rangle = a |001\rangle + b |010\rangle + c |100\rangle \), then there is no state in the \( \{ |\pm\rangle \} \)-basis with a coefficient \( A + B + C + D \), since the normalization then require the \( B \) and \( C \) coefficients to be normalized by the superposition coefficients \( a \), \( b \), and \( c \), which means that we get an imbalance between the \( B \) and \( C \) coefficients and the \( A \) and \( D \) coefficients.

We plot the probabilities of measuring the eight states as a function of \( \Theta \) to see how they behave. The result is seen in Fig. 7. Unfortunately we observe that the probability of measuring the state \( | ++ + \rangle \) decreases exponentially with \( \Theta \). This supports our previous understanding that we should indeed keep \( \Theta \) small.

### VI. CONCLUSION

We have proposed a simple implementation of a controlled-\texttt{swap} -gate, and shown that these exhibit a high fidelity. We have discussed an implementation of our
gates using superconducting circuits, however, the implementation is not limited to this scheme. While the difficulty of implementing our gates does increase with the number of controls, we do believe that our gates will be superior in certain types of quantum computations, especially compared to equivalent circuits built from one- and two-qubit gates, which often become quite deep. Our controlled-iSWAP can easily be extended to swapping between more qubits, such that it is possible to control swapping between three, four, and so on qubits. We also propose a quantum circuit for probabilistic exponentiating of non-Hermitian quantum gates, which is exact for cyclic gates and approximately exact given small parameters for all other non-Hermitian gates. These results could enhance the performance of near-term quantum computing experiments on algorithms that require multi-qubit swapping gates and exponentiating of gates.

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[1] M. A. Nielsen, Physics Letters A 303, 249 (2002).
[2] L. M. K. Vandersypen, M. Steffen, G. Breyta, C. S. Yannoni, M. H. Sherwood, and I. L. Chuang, Nature 414, 883 (2001).
[3] E. Martín-López, A. Laing, T. Lawson, R. Alvarez, X.-Q. Zhou, and J. L. O’Brien, Nature Photonics 6, 773 EP (2012).
[4] B. P. Lanyon, T. J. Weinhold, N. K. Langford, M. Barbieri, D. F. V. James, A. Gilchrist, and A. G. White, Phys. Rev. Lett. 99, 250505 (2007).
[5] I. L. Chuang and Y. Yamamoto, Phys. Rev. Lett. 76, 4281 (1996).
[6] A. Barenco, A. Berthiaume, D. Deutsch, A. Ekert, R. Jozsa, and C. Macchiavello, SIAM Journal on Computing 26, 1541 (1997).
[7] D. G. Cory, M. D. Price, W. Maas, E. Knill, R. Laflamme, W. H. Zurek, T. F. Havel, and S. S. Somaroo, Phys. Rev. Lett. 81, 2152 (1998).
[8] P. Schindler, J. T. Barreiro, T. Monz, V. Nebendahl, D. Nigg, M. Chwalla, M. Hennrich, and R. Blatt, Science 332, 1059 (2011).
[9] H. Buhrman, R. Cleve, J. Watrous, and R. de Wolf, Phys. Rev. Lett. 87, 167902 (2001).
[10] R. T. Horn, S. A. Babichev, K.-P. Marzlin, A. I. Lvovsky, and B. C. Sanders, Phys. Rev. Lett. 95, 150502 (2005).
[11] D. Gottesman and I. Chuang, “Quantum digital signatures,” (2001), arXiv:quant-ph/0105032.
[12] E. Dennis, Phys. Rev. A 63, 052314 (2001).
[13] A. Paetznick and B. W. Reichardt, Phys. Rev. Lett. 111, 090505 (2013).
[14] A. K. Ekert, C. M. Alves, D. K. L. Oi, M. Horodecki, P. Horodecki, and L. C. Kwek, Phys. Rev. Lett. 88, 217901 (2002).
[15] J. Fiurášek, M. Dušek, and R. Filip, Phys. Rev. Lett. 89, 190401 (2002).
[16] N. Schuch and J. Siewert, Phys. Rev. A 67, 032301 (2003).
[17] T. Tanamoto, K. Maruyama, Y.-x. Liu, X. Hu, and F. Nori, Phys. Rev. A 78, 062313 (2008).
[18] T. Tanamoto, Y.-x. Liu, X. Hu, and F. Nori, Phys. Rev. Lett. 102, 100501 (2009).
[19] A. M. Zagorodnik, S. Ashhab, J. R. Johansson, and F. Nori, Phys. Rev. Lett. 97, 077001 (2006).
[20] H.-F. Wang, X.-Q. Shao, Y.-F. Zhao, S. Zhang, and K.-H. Yeon, J. Opt. Soc. Am. B 27, 27 (2010).
[21] M. Bartkowiak and A. Miranowicz, J. Opt. Soc. Am. B 27, 2369 (2010).
[22] C. Godfrin, R. Ballou, E. Bonet, M. Ruben, S. Klyatskaya, W. Wernsdorfer, and F. Balestro, npj Quantum Information 4, 53 (2018).
[23] D. C. McKay, S. Filipp, A. Mezzacapo, E. Magesan, J. M. Chow, and J. M. Gambetta, Phys. Rev. Applied 6, 064007 (2016).
[24] A. Dewes, F. R. Ong, V. Schmitt, R. Lauro, N. Boulant, P. Bertet, D. Vion, and D. Esteve, Phys. Rev. Lett. 108, 057002 (2012).
[25] Y. Salathé, M. Mondal, M. Oppliger, J. Heinsoo, P. Kupiers, A. Potočnik, A. Mezzacapo, U. Las Heras, L. Lamata, E. Solano, S. Filipp, and A. Wallraff, Phys. Rev. X 5, 012107 (2015).
[26] G. L. Milburn, Phys. Rev. Lett. 62, 2124 (1989).
[27] H. F. Chau and F. Wilczek, Phys. Rev. Lett. 75, 748 (1995).
[28] J. Fiurášek, Phys. Rev. A 73, 062313 (2006).
[29] J. Fiurášek, Phys. Rev. A 78, 032317 (2008).
[30] Y.-X. Gong, G.-C. Guo, and T. C. Ralph, Phys. Rev. A 78, 012305 (2008).
[31] R. B. Patel, J. Ho, F. Ferreyrol, T. C. Ralph, and G. J. Pryde, Science Advances 2, 10.1126/sciadv.1501531 (2016).
[32] T. Ono, R. Okamoto, M. Tanida, H. F. Hofmann, and S. Takeuchi, Scientific Reports 7, 45353 EP (2017), article.
[33] J. A. Smolin and D. P. DiVincenzo, Phys. Rev. A 53, 2855 (1996).
[34] T. Bækkegaard, L. B. Kristensen, N. J. S. Loft, C. K. Andersen, D. Petrosyan, and N. T. Zinner, Scientific Reports 9, 13389 (2019).
[35] S. Poletto, J. M. Gambetta, S. T. Merkel, J. A. Smolin, J. M. Chow, A. D. Córdcoles, G. A. Keefe, M. B. Rothwell, J. R. Rozen, D. W. Abraham, C. Rigetti, and M. Steffen, Phys. Rev. Lett. 109, 240505 (2012).
[36] S. E. Rasmussen, K. S. Christensen, and N. T. Zinner, Phys. Rev. B 99, 134508 (2019).
[37] N. J. S. Loft, M. Kjaergaard, C. K. Andersen, T. W. Larsen, S. Gustavsson, W. D. Oliver, and N. T. Zinner, “Quantum interference device for controlled two-qubit operations,” (2019), arXiv:1809.09049.
where the first and third summation is understood as the summation over $T$. 

M. A. Nielsen and I. L. Chuang, A. M. Childs, R. Cleve, E. Deotto, E. Farhi, S. Gutmann, H.-K. Lau and M. B. Plenio, Phys. Rev. Lett. 117, 100501 (2016).

H.-K. Lau and M. B. Plenio, Phys. Rev. Lett. 118, 080501 (2017).

J. Kempe, Contemporary Physics 44, 307 (2003).

D. Dervovic, M. Herbster, P. Mountney, S. Severini, N. Usher, and L. Wossnig, “Quantum linear systems algorithms: a primer,” (2018), arXiv:1802.08227.

A. M. Childs, R. Cleve, E. Deotto, E. Farhi, S. Gutmann, H.-K. Lau, R. Pooser, G. Siopsis, and C. Weedbrook, Phys. Rev. Lett. 118, 080501 (2017).

I. Marvian and S. Lloyd, “Universal quantum emulator,” (2016), arXiv:1606.02734.

F. Krantz, M. Kjaergaard, F. Yan, T. P. Orlando, S. Gustavsson, and W. D. Oliver, Applied Physics Reviews 6, 021318 (2019).

M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information: 10th Anniversary Edition (Cambridge University Press, 2010).

M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A 60, 1888 (1999).

B. Schumacher, Phys. Rev. A 54, 2614 (1996).

J. Johansson, P. Nation, and F. Nori, Computer Physics Communications 183, 1760 (2012).

G. Wendin, Reports on Progress in Physics 80, 106001 (2017).

J. Koch, T. M. Yu, J. Gambetta, A. A. Houck, D. I. Schuster, J. Majer, A. Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, Phys. Rev. A 76, 042319 (2007).

J. A. Schreier, A. A. Houck, J. Koch, D. I. Schuster, B. R. Johnson, J. M. Chow, J. M. Gambetta, J. Majer, L. Frunzio, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, Phys. Rev. B 77, 180502 (2008).

R. Barends, C. M. Quintana, A. G. Petukhov, Y. Chen, D. Kafri, K. Kechedzhi, R. Collins, O. Naaman, S. Boixo, F. Arute, K. Arya, D. Buell, B. Burkett, Z. Chen, B. Chiaro, A. Dunsworth, B. Foxen, A. Fowler, C. Girdev, M. Giustina, R. Graff, T. Huang, E. Jeffrey, J. Kelly, P. V. Klimov, F. Kostritsa, D. Landhuis, E. Lucero, M. McEwen, A. Megrant, X. Mi, J. Mutus, M. Neeley, C. Neill, E. Ostby, P. Roushan, D. Sank, K. J. Satzinger, A. Vainsencher, T. White, J. Yao, P. Yeh, A. Zalcman, H. Neven, V. N. Smelyanskiy, and J. M. Martinis, Phys. Rev. Lett. 123, 210501 (2019).

M. Kounalakis, C. Dickel, A. Bruno, N. K. Langford, and G. A. Steele, npj Quantum Information 4, 38 (2018).

P. Zhao, P. Xu, D. Lan, X. Tan, H. Yu, and Y. Yu, “High contrast ZZ interaction using multi-type superconducting qubits,” (2020), arXiv:2002.07560.

W. Dirr, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000).

M. H. Devoret, in Fluctuations Quantiques/Quantum Fluctuations: Les Houches Session LXIII, edited by S. Reynaud, E. Giacobino, and J. Zinn-Justin (Elsevier, 1997) p. 351.

U. Vool and M. H. Devoret, International Journal of Circuit Theory and Applications 45, 897 (2017).

Appendix A: Analysis of the superconducting circuit

Following the procedure of Refs. [59, 60] we obtain the following Lagrangian from the circuit diagram in Fig. 3(b)

$$L = 2 \sum_{i=T1,T2,1}^{n} C_i \dot{\phi}_i^2 + 2C_\pi (\dot{\phi}_{T1} - \dot{\phi}_{T2})^2 + 2 \sum_{i=1}^{n} C_{z,i} \left[ (\dot{\phi}_i - \dot{\phi}_{T1})^2 + (\dot{\phi}_i - \dot{\phi}_{T2})^2 \right]$$

$$+ \sum_{i=T1,T2,1}^{n} E_i \cos \phi_i + \sum_{i=1}^{n} E_{z,i} \left[ \cos(\phi_{T1} - \phi_i) + \cos(\phi_{T2} - \phi_i) \right],$$

where the first and third summation is understood as the summation over $T1, T2, 1, 2 \ldots n$. The first line of terms comes from the capacitors and are interpreted as kinetic terms, while the remaining terms come from the Josephson junctions and are interpreted as potential terms. The $n$ indicates the number of blue islands on the circuit diagram, i.e., for the $c_{iS}W$ in Fig. 3(b) $n = 1$. The capacitance matrix in this case is

$$K = \begin{bmatrix} C_1 + 2C_{z,1} & -C_{z,1} & -C_{z,1} & -C_{z,1} \\ -C_{z,1} & C_{T1} + C_{z,1} + C_2 & -C_{z,1} & -C_{z,1} \\ -C_{z,1} & -C_{z,1} & C_{T2} + C_{z,1} + C_2 & -C_{z,1} \\ \end{bmatrix}.$$ 

For the $c^2iS_{W}$ (see Fig. 8(b) for circuit diagram of this gate) we need $n = 2$ and the capacitance matrix takes the form

$$K = \begin{bmatrix} C_1 + 2C_{z,1} & 0 & -C_{z,1} & -C_{z,1} \\ 0 & C_1 + 2C_{z,2} & -C_{z,1} & -C_{z,1} \\ -C_{z,1} & -C_{z,2} & C_{T1} + C_{z,1} + C_{z,2} + C_x & C_{z,1} + C_{z,2} + C_x \\ -C_{z,1} & -C_{z,2} & -C_{z,2} & C_{T2} + C_{z,1} + C_{z,2} + C_x \end{bmatrix}. $$
We now do the canonical quantization \( \zeta \) with impedance of the harmonic oscillator Hamiltonian \( \hat{H} \) are close to resonant, we can treat their detuning as part of the perturbation. The total Hamiltonian is then the sum

\[
H = \sum_{i=1}^{n} E_i \left[ \frac{1}{2} \varphi_i^2 - \frac{1}{24} \varphi_i^4 \right] + \sum_{i=1}^{n} E_{z,i} \left[ \frac{1}{2} (\varphi_i - \varphi_{T1})^2 - \frac{1}{24} (\varphi_i - \varphi_{T1})^4 \right] + \frac{1}{2} (\varphi_i - \varphi_{T2})^2 - \frac{1}{24} (\varphi_i - \varphi_{T2})^4 \right].
\]

By collecting terms we can write the full Hamiltonian as

\[
H = \sum_{i=T_1,T_2,1}^{n} \left[ \frac{1}{2} E_i \hat{\varphi}_i^2 + \frac{1}{2} E'_i \hat{\varphi}_i^2 - \frac{1}{24} E_i \varphi_i^4 \right] + \frac{1}{2} \sum_{i \neq j = T_1,T_2,1}^{n} (K^{-1})_{(i,j)} p_i p_j
\]

\[
+ \sum_{i=1}^{n} \sum_{j=1}^{2} E_{z,i} \left[ -\frac{1}{4} \varphi_i \varphi_{Tj} - \varphi_i \varphi_{Tj} + \frac{1}{6} (\varphi_i \varphi_{Tj} + \varphi_i \varphi_{Tj}^3) \right],
\]

where the effective energy of the capacitances is \( E_i^C = (K^{-1})_{(i,i)}/8 \). The second summation is understood as the sum over \( i \) and \( j = T_1, T_2, 1, \ldots, n \), where \( i \) and \( j \) is never equal. Note that there is a capacitive coupling between all of the qubits regardless of whether there actually is a capacitor between them. The effective Josephson energies are

\[
E_i^J = E_i + 2E_{z,i},
\]

\[
E_i^{J_{T}} = E_{Ti} + \sum_{i=1}^{n} E_{z,i}.
\]

We now do the canonical quantization \( \varphi_i \rightarrow \hat{\varphi}_i \) and \( p_i \rightarrow \hat{p}_i \), requiring that \( [\hat{p}_i, \hat{\varphi}_j] = \delta_{ij} \). This allows us to change into step operators

\[
\hat{\varphi}_i = \sqrt{\frac{\zeta_i}{2}} \left( \hat{b}_i^+ + \hat{b}_i \right), \quad \hat{p}_i = \frac{i}{\sqrt{2\zeta_i}} \left( \hat{b}_i - \hat{b}_i^+ \right),
\]

with impedance \( \zeta_i = \sqrt{(K^{-1})_{(i,i)}}/E_i^J \), the Hamiltonian takes the form

\[
\hat{H} = \sum_{i=T_1,T_2,1}^{n} \left[ \sqrt{8E_i^C} E_i^J \hat{b}_i \hat{\varphi}_i - \frac{1}{12} E_i^C \hat{\varphi}_i^4 \right] - \frac{1}{2} \sum_{i \neq j = T_1,T_2,1}^{n} \frac{(K^{-1})_{(i,j)}}{\sqrt{\zeta_i \zeta_j}} \left( \hat{b}_i - \hat{b}_j \right) \left( \hat{b}_i^+ - \hat{b}_j^+ \right)
\]

\[
+ \sum_{i=1}^{n} \sum_{j=1}^{2} E_{z,i} \left[ -\frac{1}{24} \zeta_i \zeta_{Tj} \left( \hat{b}_i^+ - \hat{b}_j \right) \left( \hat{b}_j^+ + \hat{b}_j \right)^2 - \frac{1}{2} \sqrt{\zeta_i \zeta_{Tj}} \left( \hat{b}_i - \hat{b}_j \right) \left( \hat{b}_j^+ + \hat{b}_j \right) \left( \hat{b}_j^+ + \hat{b}_j \right) + \frac{1}{24} \left( \zeta_i \sqrt{\zeta_i \zeta_{Tj}} \left( \hat{b}_i^+ - \hat{b}_j^+ \right)^3 \hat{b}_j^+ + \hat{b}_j \right) \right],
\]

If we operate the circuit in the weak coupling regime \( E_{z,i} \ll E_i \) and \( C_{z,i} \ll C_j \) for all \( i \) and \( j \) we can view the system as \( n + 2 \) harmonic oscillators perturbed by their interactions. If we also assume that the modes of oscillator \( T1 \) and \( T2 \) are close to resonant, we can treat their detuning as part of the perturbation. The total Hamiltonian is then the sum of the harmonic oscillator Hamiltonian \( \hat{H}_0 \) and a perturbation \( \hat{V} \). If we introduce the number operator \( \hat{n} = \hat{b}^\dagger \hat{b} \) and swap operator \( \hat{X}_{ij} = \hat{b}_j \hat{b}_i^+ + \hat{b}_j^+ \hat{b}_i \), we can write the two parts of the Hamiltonian as

\[
\hat{H}_0 = \sum_{i=1}^{n} \Omega_i \hat{n}_i + \Omega(\hat{n}_{T1} + \hat{n}_{T2}), \quad \hat{V} = \delta(\hat{n}_{T1} - \hat{n}_{T2}) - \frac{1}{2} \sum_{i=T_1,T_2,1}^{n} E_i^C \hat{n}_i (\hat{n}_i - 1) + \sum_{i=1}^{n} \sum_{j=1}^{2} g_{Tj}^z \hat{n}_i \hat{n}_{Tj} + \sum_{i \neq j = T_1,T_2,1}^{n} \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{2} \left( \zeta_i \left( \hat{X}_{ij} \hat{n}_i + \hat{n}_i \hat{X}_{ij} \right) + \zeta_{Tj} \left( \hat{X}_{Tj} \hat{n}_{Tj} + \hat{n}_{Tj} \hat{X}_{Tj} \right) \right),
\]
where the qubit frequencies are then given as

\[ \Omega_i = \sqrt{8E_i^C}E_i - \frac{1}{12}E_{z,i}\zeta_i(\zeta_{T1} + \zeta_{T2}) \quad \text{for } i = 1, 2, \ldots n, \]  

\[ \Omega_{Tj} = \sqrt{8E_{Tj}^C} - \frac{1}{12}E_{z,i}\zeta_{Tj} \sum_{i=1}^{n} \zeta_i \quad \text{for } j = 1, 2, \]

\[ \bar{\Omega} = \frac{1}{2} (\Omega_{T1} + \Omega_{T2}), \]

\[ \delta = \frac{1}{2}(\Omega_{T1} - \Omega_{T2}), \]

and the coupling strengths are given as

\[ g_{iTj} = -\frac{1}{4}E_{z,i}\zeta_{iTj}, \quad \text{for } j = 1, 2 \text{ and } i = 1, 2, \ldots, n, \]  

\[ g_{ij}^x = -\frac{1}{16}E_{z,i}\zeta_{ij}, \quad \text{for } j = 1, 2 \text{ and } i = 1, 2, \ldots, n, \]  

\[ g_{ij}^y = \frac{(K^{-1})(i,j)}{2\sqrt{\zeta_{ij}}}, \quad \text{for } j = 1, 2,  \ldots, n, \]  

\[ g_{iTj}^x = -\frac{1}{2}(K^{-1})(i,Tj)\sqrt{\zeta_{iTj}} - \frac{1}{16}E_{z,i}(\zeta_i + \zeta_{Tj})\sqrt{\zeta_{iTj}}, \quad \text{for } j = 1, 2 \text{ and } i = 1, 2, \ldots, n. \]  

If we only consider the two lowest lying states of each oscillator, the uncoupled Hamiltonian has a degenerate spectrum with \(2^{n+2}\) states. If we require the detunings \(\Delta_{ij} = \Omega_i - \Omega_j\) between each of the control qubits to be much larger than the transversal couplings in Eq. (A8c), we can ignore first order excitations swaps between the control qubits. If we further require that the control qubits are detuned from the target qubits in such a way that \(\Delta_{iTj} = \Omega_i - \Omega_{Tj}\) is much larger than the transversal coupling in Eq. (A8e) we can also neglect first order excitation swaps between the target qubits and the control qubits. This leaves only one transversal coupling in Eq. (A8d).

If the anharmonicity is sufficiently larger than the transversal coupling between the target qubits we can justify projecting the final effective Hamiltonian into the two lowest states of each qubit. This projection is done using degenerate second order perturbation theory. In this case each degenerate subspace is well described by an effective interaction

\[ \hat{P}\hat{V}\hat{P} = \hat{P}\hat{V}\hat{P}\frac{1}{E_D - QH_0Q}\hat{Q}\hat{V}\hat{P}, \]

where \(\hat{P}\) projects onto the degenerate subspace consisting of the \(2^{n+2}\) lowest lying states and \(\hat{Q} = 1 - \hat{P}\) projects onto the orthogonal complement. Doing so yields an effective interaction between the qubits given by

\[ \hat{V}_{\text{eff}} = -\frac{1}{2}\Delta_{T1}\sigma_{T1}^z - \frac{1}{2}\Delta_{T2}\sigma_{T2}^z + \sum_{i=1}^{n}(J_{iT1}\sigma_{T1}^z + J_{iT2}\sigma_{T2}^z)\sigma_i^z + \sum_{i=1}^{n}(J_i^x + J_i^y)\sigma_i^z(\sigma_{T1}^z\sigma_{T2}^z + \sigma_{T1}^z\sigma_{T2}^z). \]

Note that the coupling strength of the swapping term (the last term) depends on the state of the control qubit via \(J_i^z\). This could be useful if one can engineer the system such that the coupling strength is largest when the control qubit is in the \(|1\rangle\) state. This means that the coupling strength is largest when we do want the gate to perform a swap, and it is smallest when we do not want it to swap. Thus the coupling supports the gate operation.

The qubit frequencies can then be calculated and the second order matrix elements are

\[ \Delta_{T1} = -\delta + \sum_{i=1}^{n}\frac{g_{iT1}^2}{2} - \sum_{i=1}^{n}\frac{g_{iT1}^2 - g_{iT1}^2(\zeta_i + 2\zeta_{T1})}{\Delta_{iT}}, \]

\[ \Delta_{T2} = \delta + \sum_{i=1}^{n}\frac{g_{iT2}^2}{2} - \sum_{i=1}^{n}\frac{g_{iT2}^2 - g_{iT2}^2(\zeta_i + 2\zeta_{T2})}{\Delta_{iT}}, \]

\[ \Delta_i = \frac{g_{iTj}^2}{2} - \sum_{j=1}^{2}\frac{g_{iTj}^2 - g_{iTj}^2(\zeta_i + 2\zeta_{Tj})}{\Delta_{iT}}. \]
where $\Delta_{iT} = \Omega_i - \Omega$ is the detuning of the target qubits with respect to the $i$th control qubit. The longitudinal coupling between the target qubits and the control qubits are

$$J^z_{iTj} = \frac{g_{iTj}^z}{4} + \frac{g_{iTj}^z(\zeta_i - 2\zeta_Tj)}{2\Delta_{iTj}}. \tag{A12}$$

As described in the main text, the purpose of this longitudinal coupling is to tune the target qubits in and out of resonance, depending on the state of the control qubits. We thus require this coupling to be significantly larger than the coupling between the target qubits. The coupling strength between the two target qubits are

$$J^z_i = g_i^z - \frac{(g_{iT1}^z + g_{iT1}^z(\zeta_i - 3\zeta_T1))(g_{iT2}^z + g_{iT2}^z(\zeta_i - 3\zeta_T2))}{\Delta_{iT}}. \tag{A13a}$$

$$J^{xz}_i = \frac{2g_{iT1}^zg_{iT2}^z\zeta_{T2} + 2g_{iT2}^zg_{iT1}^z\zeta_{T1} + g_{iT1}^zg_{iT2}^z[8\zeta_{T1}\zeta_{T2} - 2\zeta_i(\zeta_{T1} + \zeta_{T2})]}{\Delta_{iT}}. \tag{A13b}$$