Random polymers and delocalization transitions
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In these proceedings, we first summarize some general properties of phase transitions in the presence of quenched disorder, with emphasis on the following points: the need to distinguish typical and averaged correlations, the possible existence of two correlation length exponents \( \nu \), the general bound \( \nu_{FS} \geq \frac{2}{d} \), the lack of self-averaging of thermodynamic observables at criticality, the scaling properties of the distribution of pseudo-critical temperatures \( T_c(i, L) \) over the ensemble of samples of size \( L \). We then review our recent works on the critical properties of various delocalization transitions involving random polymers, namely (i) the bidimensional wetting (ii) the Poland-Scheraga model of DNA denaturation (iii) the depinning transition of the selective interface model (iv) the freezing transition of the directed polymer in a random medium.

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I. INTRODUCTION

In these proceedings, we review our recent works on the critical properties of various delocalization transitions involving random polymers, namely (i) the bidimensional wetting (ii) the Poland-Scheraga model of DNA denaturation (iii) the depinning transition of the selective interface model (iv) the freezing transition of the directed polymer in a random medium.

The paper is organized as follows. In Section II we summarize some general properties of random critical points. The other Sections are devoted to the various models : the wetting and Poland-Scheraga models in Section III the selective interface model in Section IV, and finally the directed polymer model in Section V.

The remainder of this Introduction contains a brief presentation of the various models we will discuss.

A. Wetting and Poland-Scheraga models

Wetting transitions are in some sense the simplest phase transitions, since they involve linear systems \[42\]. Let us consider a one-dimensional random walk (RW) of \( 2L \) steps, starting at \( z(0) = 0 \), with increments \( z(\alpha+1) - z(\alpha) = \pm 1 \). The random walk is constrained to remain in the upper half plane \( z \geq 0 \), but gains an adsorption energy \( \epsilon_\alpha \) if \( z(\alpha) = 0 \).

More precisely, the model is defined by the partition function

\[
Z_{\text{wetting}}(2L) = \sum_{\text{RW}} \exp \left( \sum_{1 \leq \alpha \leq N} \beta \epsilon_\alpha \delta_{z_\alpha, 0} \right)
\]

with inverse temperature \( \beta = 1/T \).

In the pure case \( \epsilon_\alpha = \epsilon_0 \), there exists a continuous phase transition between a localized phase at low temperature, characterized by an extensive number of contacts at \( z = 0 \), and a delocalized phase at high temperature. The critical temperature is simply the point where \( e^{\beta \epsilon_0} = 2 \), i.e. where the energy gain \( \epsilon_0 \) of a contact exactly compensates for its entropy loss \( T \ln 2 \). At \( T_c \), the wall \( z = 0 \) is exactly reflexive, being attractive (resp. repulsive) for \( T < T_c \) (resp. \( T > T_c \)).

The Poland-Scheraga (PS) model of DNA denaturation \[79\] is closely related to the wetting model. It describes the configuration of the two complementary chains as a sequence of bound segments and open loops. Each loop of length \( l \) has a polymeric entropic weight \( N(l) \sim \mu^l / l^c \), whereas each contact at position \( \alpha \) has a Boltzmann weight \( e^{-\beta \epsilon_\alpha} \). We assume that the two chains are bound at \( \alpha = 1 \) and \( \alpha = L \). The partial partition function \( Z_{PS}(\alpha) \) with bound ends then satisfies the simple recursion relation

\[
Z_{PS}(\alpha) = e^{-\beta \epsilon_\alpha} \sum_{\alpha'=1}^{\alpha-1} N(\alpha - \alpha') Z_{PS}(\alpha')
\]

with
The wetting model (11) corresponds to a Poland-Scheraga model with loop exponent \( q = 3/2 \) (this exponent comes from the first return distribution of a one-dimensional random walk). For DNA denaturation, the appropriate value of the loop exponent \( c \) has been the source of some debate. Gaussian loops in \( d = 3 \) dimensions are characterized by \( c = d/2 = 3/2 \). The role of self avoidance within a loop was taken into account by Fisher (41), and yields the bigger value \( c = dv_{SAW} \approx 1.76 \), where \( v_{SAW} \) is the SAW radius of gyration exponent in \( d = 3 \). More recently, Kafri et al. (59, 60) pointed out that the inclusion of the self avoidance of the loop with the rest of the chain further increased \( c \) to a value \( c > 2 \). In the pure case \( \epsilon_\alpha = \epsilon_0 \), the transition between the low temperature bound phase and the high temperature unbound phase is discontinuous for \( c > 2 \), in marked contrast to the wetting case. The discontinuous character of the transition was in fact previously found in Monte Carlo simulations of self avoiding walks (22). The value \( c \approx 2.11 \) was subsequently measured (3, 5, 21).

For the wetting and the Poland-Scheraga models, the question is how disorder in the contact energies \( \epsilon_\alpha \) modifies the critical properties of the pure phase transition.

### B. Selective interface model

Heteropolymers containing both hydrophobic and hydrophilic components are of particular interest in biology. In a polar solvent, these heteropolymers prefer conformations where the hydrophilic components are in contact with the polar solvent, whereas hydrophobic components avoid contacts with the solvent. The behavior of such heteropolymers in the presence of an interface separating two selective solvents, one favorable to the hydrophobic components and the other to the hydrophilic components, is less obvious, and has been much studied recently. In the initial work of ref. (49), the following model was proposed for a polymer carrying random charges \( \{q_\alpha\} \), at the selective interface between two solvents, such that monomers with positive charges prefer to be in the upper fluid (sgn \( z \) > 0). The partition function

\[
Z_{SI}(L) = \sum_{RW} \exp \left( \beta \sum_\alpha q_\alpha \text{sgn} z_\alpha \right)
\]

is over all random walks \( \{z_\alpha\} \) of \( L \) steps, with increments \( z_{\alpha+1} - z_\alpha = \pm 1 \) and (bound-bound) boundary conditions \( z_1 = 0 = z_L \). Furthermore, it is convenient to choose \( q_{2\alpha+1} = 0 \) for all \( \alpha \), so that there is no frustration at zero temperature (each monomer \( z_{2\alpha} \) being in its preferred solvent). Even charges \( q_{2\alpha} \) are random and drawn from the Gaussian distribution with mean value \( q_0 \)

\[
P(q_{2\alpha}) = \frac{1}{\sqrt{2\pi \Delta^2}} e^{-\frac{(q_{2\alpha} - q_0)^2}{2\Delta^2}}
\]

The initial work (49) found that for \( q_0 = 0 \) the chain is localized around the interface at any finite temperature, whereas for \( q_0 > 0 \) there is a phase transition at a critical temperature \( T_c \approx O(\Delta^2/q_0) \) separating a localized phase at low temperature from a delocalized phase into the most favorable solvent at high temperature. Here, in contrast with the wetting and Poland-Scheraga models, there is strictly speaking no corresponding ‘pure’ phase transition, since an homogeneous chain \( q_\alpha = q_0 \) will be always delocalized in its preferred solvent. To obtain a phase transition with a non-disordered chain, one has to consider a periodic structure of charges of both signs, the simplest case being an alternate sequence \( q_A > 0, -q_B < 0 \) (see Ref. (17, 71) and references therein). Another important difference with the wetting and Poland-Scheraga models is that here the disorder is felt by all monomers, whereas in the wetting and Poland-Scheraga models, the loops \( \{z_\alpha\} > 0 \) do not feel the disorder.

### C. Directed polymers in random media: transition towards a disorder-dominated phase

The model of directed polymer in a \( 1 + d \) random medium is defined by the following partition function

\[
Z_L(\beta) = \sum_{RW} \exp \left( \beta \sum_{1 \leq \alpha \leq L} \epsilon(\alpha, \bar{r}(\alpha)) \right)
\]

over \( d \)-dimensional random walks \( \bar{r}(\alpha) \), where the independent random energies \( \epsilon(\alpha, \bar{r}) \) define the random medium. This model has attracted a lot of attention because it is directly related to non-equilibrium properties of growth models (54). Within the field of disordered systems, it is also very interesting on its own because it represents a ‘baby-spin-glass’ model (51, 52, 54, 58). At low temperature, there exists a disorder dominated phase, where the
order parameter is an ‘overlap’ \[27, 33, 68\]. In finite dimensions, a scaling droplet theory was proposed \[44\], in direct correspondence with the droplet theory of spin-glasses \[13\], whereas in the mean-field version of the model on the Cayley, a freezing transition very similar to the one occurring in the Random Energy Model was found \[33\]. The phase diagram as a function of space dimension \(d\) is the following \[54\]. In dimension \(d \leq 2\), there is no free phase, i.e. any initial disorder drives the polymer into the strong disorder phase, whereas for \(d > 2\), there exists a phase transition between the low temperature disorder dominated phase and a free phase at high temperature \[28, 57\], where the free energy has its annealed value. This phase transition has been studied exactly on a Cayley tree \[33\]. In finite dimensions, bounds on the critical temperature \(T_c\) have been derived \[28, 34, 57\]: \(T_0(d) \leq T_c \leq T_2(d)\). The upper bound \(T_2(d)\) corresponds to the temperature above which the ratio \(Z_L^c/(Z_L)^2\) remains finite as \(L \to \infty\). The lower bound \(T_0\) corresponds to the temperature below which the annealed entropy becomes negative.

For the directed polymer model, one is thus interested both in the properties of the low temperature disorder dominated phase in any dimension \(d = 1, 2, ...\) and in the critical properties of the transition that exists for \(d \geq 3\).

II. GENERAL PROPERTIES OF RANDOM CRITICAL POINTS (A PHYSICIST’S POINT OF VIEW)

In this Section, we summarize the general properties of random critical points that will be useful to analyse the various random polymer models considered in the other Sections.

A. Harris criterion to determine disorder relevance near second order pure critical points

The stability of pure critical points with respect to weak bond disorder is governed by the Harris criterion \[55\]: near a second order phase transition in dimension \(d\), the bond disorder is irrelevant if the correlation length exponent \(\nu_p = \nu_{\text{pure}} > 2/d\), or equivalently, using the hyperscaling relation \(f \sim 1/\xi^d\), if the specific exponent \(\alpha = 2 - d\nu_p\) is negative \(\alpha < 0\). On the contrary if \(\nu_p < 2/d\) or \(\alpha > 0\), disorder is relevant and drives the system towards a random fixed point characterized by new critical exponents.

A simple argument to understand Harris criterion is the following. The pure system at a temperature \(T \neq T_c\) is characterized by a correlation length \(\xi(T) \sim t^{-\nu_p}\), where \(t = |T_c^{\text{pure}} - T|\) represents the distance to criticality, and \(\nu_p\) the correlation length exponent. The pure system can be divided into nearly independent subsamples of volume \(V \sim \xi^d(T) \sim t^{-d\nu_p}\). In the presence of an additional weak bond disorder, the averaged bond value \((1/V) \sum_{i \in V} J_i\) seen in a volume \(V\) will present fluctuations of order \(1/\sqrt{V}\). So the fluctuations of critical temperatures among the volumes of size \(V\) will be of order \(\Delta T_c(V) \sim 1/\sqrt{V} \sim \xi^{-d/2}(T)\). Disorder will be irrelevant if these fluctuations \(\Delta T_c(V) \sim t^{d\nu_p/2}\) becomes negligible with respect to \(t = |T_c^{\text{pure}} - T|\) in the limit \(t \to 0\) where the critical point is approached.

B. Correlation functions in disordered systems

In disordered systems, it is well known that the partition function \(Z\) has a very broad distribution, which becomes peaked, in the thermodynamic limit, around the typical (typ) value \(Z_{\text{typ}} \sim e^{\overline{\ln Z}}\), whereas the averaged (av) partition function \(Z_{\overline{\text{av}}} = \overline{Z}\) is usually atypical and dominated by rare samples.

Correlation functions are, from this point of view, very similar to partition functions. It is especially clear in one dimensional spin systems \[24, 32, 64\], where correlation functions can be expressed as product of random numbers. More generally in any disordered system, the averaged correlation is expected to differ from the typical correlation. However in contrast with partition functions where the averaged value \(Z_{\overline{\text{av}}}\) has usually no physical meaning, both the typical and averaged correlations are actually important, depending on the physical quantities one wants to study \[24, 32, 40\]. The non self-averaging properties of correlations have also been studied in higher dimensional systems, such as the two-dimensional (2D) McCoy-Wu model \[46, 60\] (see below), in 2D random \(q\)-state Potts model \[62\] and in the 3D random field Ising model \[77\].

As a consequence, the exponential decay at large distance of the typical and averaged correlations lead to define two distinct correlation lengths. It turns out that close to a phase transition, these two correlation lengths may have different critical behaviors. The best understood example of the existence of two different correlation length exponents is the random transverse field Ising chain (this quantum 1D model is equivalent to the 2D classical Ising model with columnar disorder introduced by McCoy and Wu \[68\]), which has been studied in great details by D. Fisher via a strong disorder renormalization approach \[46\]: the exponent \(\nu_{\text{typ}} = 1\) governs the decay of the typical correlation at
large distance $r$

$$\ln C(r) \sim -\frac{r}{\xi_{typ}}$$

(6)

whereas $\nu_{av} = 2$ governs the decay of the averaged correlation

$$\ln(C(r)) \sim -\frac{r}{\xi_{av}}$$

(7)

Exactly at criticality, the typical and averaged correlations are also very different, since the typical correlation decays as $C_{typ}(r) \sim e^{-w \sqrt{r}}$, where $w$ is a random variable of order 1, whereas the averaged correlation is dominated by rare events and decays algebraically $C(r) \sim 1/r^{(3-\sqrt{5})/2}$.

C. General bound $\nu_{FS} \geq 2/d$ for random systems, and the possible existence of two distinct exponents $\nu$

There exists a general bound for the finite-size correlation length exponent $\nu_{FS} \geq 2/d$ in disordered systems [24], which essentially means that a random critical point should itself be stable with respect to the addition of disorder, as in the Harris criterion argument given above. However, this general bound has to be understood with the subtleties explained in [24]. In so-called ‘conventional’ random critical points, there is a single correlation length exponent $\nu = \nu_{FS}$ and this single exponent is expected to satisfy the bound. However, there are also ‘unconventional’ random critical points, where there are two different correlation length exponents. In this case, the typical correlation exponent $\nu_{typ}$ can be less than $2/d$, whereas the bound holds for the finite-size exponent $\nu_{FS} \geq 2/d$. For instance in the random transverse field Ising chain where there are two diverging correlation lengths (Eqs 6 and 7), the typical correlation exponent $\nu_{typ} = 1$ is less than $2/d = 2$, whereas the finite-size exponent $\nu_{FS} = \nu_{av} = 2$ that has to satisfy the bond $\nu_{FS} \geq 2/d = 2$ actually saturates it. Another important example discussed in [24, 46] is the case of a first order transition that remains first order in the presence of quenched disorder: this first order transition in dimension $d$ is associated to the typical exponent $\nu_{typ} = 1/d$, which is less than $2/d$, whereas the finite-size exponent satisfies the bound $\nu_{FS} = 2/d$. The interpretation given in Sec. VII A of Ref. [46] is the following: the exponent $\nu_{typ} = 1/d$ is expected to describe the rounding of the transition in a typical sample, whereas $\nu_{FS} = 2/d$ describes the rounding of the transition of the distribution of samples. Other critical points with two different correlation length exponents are discussed in [15, 46, 76, 78, 82].

D. Lack of self-averaging at random critical points

In disordered systems, the densities of extensive thermodynamic observables are self-averaging off-criticality, because the finiteness of the correlation length $\xi$ allows to divide a large sample into independent large sub-samples. At criticality however, this ‘subdivision’ argument breaks down because of the divergence of $\xi$ at $T_c$, and a lack of self-averaging has been found at criticality whenever disorder is relevant [1, 87, 88]. More precisely, for a given observable $X$, it is convenient to define its normalized width as

$$R_X(T, L) \equiv \frac{X^2(T; L) - (X(T; L))^2}{(X(T; L))^2}$$

(8)

To be more specific, in ferromagnets, the observable $X$ can be the magnetization $M$, the susceptibility $\chi$, the singular parts of the energy or of the specific heat [88]. In terms of the correlation length $\xi(T)$, the following behaviour of $R_X(T, L)$ is expected [1, 88]:

(i) off criticality, when $L \gg \xi(T)$, the system can be divided into nearly independent sub-samples and this leads to ‘Strong Self-Averaging’

$$R_X(T, L) \sim \frac{1}{L^d} \text{ off criticality for } L \gg \xi(T)$$

(9)

(ii) in the critical region, when $L \ll \xi(T)$, the system cannot be divided anymore into nearly independent sub-samples. In particular at $T_c(\infty)$ where $\xi = \infty$, one can have either ‘Weak Self-Averaging’

$$R_X(T_c(\infty), L) \sim \frac{1}{L^d - \nu_P} \text{ for irrelevant disorder } (\nu_P > 2/d)$$

(10)
or ‘No Self-Averaging’

\[ R_X(T_c(\infty), L) \sim Cst \] for random critical points

To understand the origin of this lack of self-averaging, it is useful to introduce the notion of sample-dependent pseudo-critical temperatures, as we now explain.

**E. Distribution of pseudo-critical temperatures**

Important progresses have been made recently in the understanding of finite size properties of random critical points \[ \text{[1, 2, 8, 23, 78, 87, 88]} \]. To each disordered sample \( (i) \) of size \( L \), one should first associate a pseudo-critical temperature \( T_c(i, L) \) \[ \text{[8, 78, 87, 88]} \]. Various definitions can be used, but one expects that the scaling properties do not depend on the details of the definition. For instance, in spin systems, one may define the pseudo-critical temperature \( T_c(i, L) \) as the temperature \( T \) where the susceptibility of the sample \( (i) \) is maximum. For the case of ‘conventional’ random fixed points that are characterized by a single correlation length exponent \( \nu_R \), the disorder averaged pseudo-critical critical temperature \( T_c^{av}(L) \equiv \bar{T}_c(i, L) \) satisfies

\[ T_c^{av}(L) - T_c(\infty) \sim L^{-1/\nu_R} \] (12)

which generalizes the analogous relation for pure systems

\[ T_c^{pure}(L) - T_c(\infty) \sim L^{-1/\nu_P} \] (13)

The physical meaning of these equations is simply that a sample of length \( L \) can be considered at criticality when the correlation length \( \xi \sim (T_c(T) - T_c)^{-\nu_R} \) reaches the size \( L \) of the system.

The nature of the disordered critical point then depends on the width \( \Delta T_c(L) \) of the distribution of the pseudo-critical temperatures \( T_c(i, L) \)

\[ \Delta T_c(L) \equiv \sqrt{\text{Var}[T_c(i, L)]} = \sqrt{T_c^2(i, L) - (T_c(i, L))^2} \] (14)

When the disorder is irrelevant, the fluctuations of these pseudo-critical temperatures obey the scaling of a central limit theorem as in the Harris argument:

\[ \Delta T_c(L) \sim L^{-d/2} \] for irrelevant disorder (15)

This behaviour was first believed to hold in general \[ \text{[78, 87]} \], but was later shown to be wrong in the case of random fixed points. In this case, it was argued \[ \text{[1, 88]} \] that eq. (15) should be replaced by

\[ \Delta T_c(L) \sim L^{-1/\nu_R} \] for random critical points (16)

i.e. the scaling is the same as the \( L \)-dependent shift of the averaged pseudo-critical temperature (Eq. 12). The fact that these two temperature scales remain the same is then an essential property of random fixed points that leads to the lack of self-averaging at criticality.

Up to now, to our knowledge, the distribution of \( T_c(i, L) \) or of another sample-dependent critical parameter has been studied for various disordered spin models \[ \text{[1, 8, 78, 88]} \], for elastic lines in random media \[ \text{[15]} \], for Poland-Scheraga models \[ \text{[71]} \], for the selective interface model \[ \text{[72]} \], and for the directed polymer in a random medium of dimension \( 1 + 3 \) \[ \text{[75]} \].

**F. Finite-size scaling in disordered systems**

In pure systems, the finite-size scaling theory relates the critical exponents of the phase transition in the thermodynamic limit to finite-size effects that can be measured in numerical simulations \[ \text{[20]} \]. In short, this theory says that the only important variable is the ratio between the size \( L \) of the finite system and the correlation length that diverges at the critical point \( \xi(T) \sim |T - T_c|^{-\nu} \). So the data \( X_L(T) \) for various sizes \( L \) should be analysed in terms of the appropriate rescaled variable \( \tau = (T - T_c)L^{1/\nu} \) to obtain a master curve of the form \( L^yX_L(T) = \phi(\tau) \), where \( y \) is the exponent describing the decay the observable \( X \) exactly at criticality \( X_T(T_c) \sim 1/L^y \). Note that using the
In random systems, one has instead data \( X^i_L(T) \) measured at temperature \( T \) for various disordered samples \((i)\) of size \( L \), and the question is : what is the best way to analyse these data? The usual procedure consists in averaging over the samples \((i)\) at fixed \((T,L)\) to apply the pure procedure to these disorder averaged quantities : one tries to find a master curve \( L^\nu X^i_L(T) = \phi(\tau) \) in terms of the variable \( \tau = (T - T_c(L))/L^{1/\nu} \). However, this procedure leads to extremely large sample-to-sample fluctuations in the critical region, as a consequence of the width of the distribution of pseudo-critical temperatures : at a given temperature \( T \), the samples having their pseudo-critical temperature \( T_c(i,L) > T \) are effectively in the low temperature phase, whereas the samples having \( T_c(i,L) < T \) are effectively in the high temperature phase. This mixing of samples in the critical regions makes it very difficult to obtain clean results on critical exponents.

To avoid these difficulties, the following alternative procedure has been proposed [8, 78, 88] : the data for a given sample \((i)\) should be analysed in terms of the rescaling variable \( \tau_{\text{new}} = (T - T_c(i,L))/L^{1/\nu} \) with respect to its own pseudo-critical temperature \( T_c(i,L) \). Since \( T_c(i,L) = T_c^{av}(L) + u_1 \Delta T_c(L) \), where \( u_1 \) is a random variable of order one, and where the mean \( T_c^{av}(L) \) and the variance \( \Delta T_c(L) \) follow the respective behaviours (12) and (10), it is clear that the two procedures are not equivalent, since \( \tau_{\text{new}} = \tau + a + u_1 \) is not a simple translation of the constant \( a \) as in the pure case, because the random variable \( u_1 \) remains present for arbitrary \( L \). This new way of analysing the data allows to reduce very significantly the sample-to-sample fluctuations, as shown for spin models [8, 78, 88], for the non-equilibrium depinning transition of elastic lines in random media [40], and for disordered polymer models [71].

### III. DISORDERED POLAND-SCHERAGA MODEL WITH VARIOUS LOOP EXPONENT \( c \)

#### A. Pure critical properties and disorder relevance : role of exponent \( c \)

In the pure case \( \epsilon_a = \epsilon_0 \), the model is of course exactly solvable, and the critical properties are determined by the value of the loop exponent \( c \) : for \( c > 2 \), the transition is first order with exponent \( \nu_P = 1 \), whereas for \( 1 < c < 2 \) the transition is second order with exponent \( \nu_P = 1/(c-1) \). A simple argument to understand these properties is that the loop distribution at \( T_c \) has a power-law decay involving the exponent \( c \) that enters the definition of the model

\[
P_{T_c}^{\text{pure}}(l) \sim \frac{1}{l^c}
\]

(17)

For \( c > 2 \), the averaged length \( <l> = \int dl \, l P_{T_c}^{\text{pure}}(l) \) is finite, so that the number \( n(T_c) \) of contacts with the substrate is extensive (\( n(T_c) \sim L \)) ; the transition is therefore first order. For \( 1 < c < 2 \), the averaged length \( <l> \) diverges, and the Lévy sum of \( n \) independent variables \( l_i \) drawn from the distribution \( \nu \) scales as \( l_1 + \ldots + l_n \sim n^{1/(c-1)} \). As a consequence at criticality, the number of contacts \( n_L^{\text{pure}}(T_c) \) in a sample of length \( L \) scales as

\[
n_L^{\text{pure}}(T_c) \sim L^{c-1}
\]

(18)

and the transition is second order.

The Harris criterion concerning the stability of pure second order transitions with respect relies on the sign of the specific heat exponent

\[
\alpha_P = 2 - \nu_P = \frac{2c-3}{c-1}
\]

(19)

An equivalent way to decide whether disorder is relevant consists in a simple power-counting analysis of the disorder perturbation exactly at \( T_c \) : the pure finite-size contact density \( \delta \) has the form \( n_L^{\text{pure}}(T_c)/l \sim L^{c-2} \) of Eq. (18) yields that the perturbation due to the presence of a small disorder in the contact energies \( \epsilon_i = \epsilon_0 + \delta \epsilon_i \) scales as

\[
\sum_{i=1}^{L} \delta \epsilon_i < \delta \epsilon_i >_{\text{pure}} \sim L^{1/2} \times L^{c-2} = L^{c-\frac{3}{2}}
\]

(20)

Disorder is thus irrelevant for \( 1 < c < \frac{3}{2} \) and relevant for \( \frac{3}{2} < c < 2 \). Poland-Scheraga models are thus particularly interesting to study disorder effects on pure phase transitions, since the parameter \( c \) allows to study, within a single model, the various cases of second order transition with respectively marginal/relevant disorder according to the Harris criterion, or first-order transition. From this point of view, it is reminiscent of the 2D Potts model, where the
pure critical properties vary with the parameter $q$: the transition is second order for $q < 4$, the Ising case $q = 2$ corresponding to the marginal case of the Harris criterion, whereas the transition becomes first order for $q > 4$.

The marginal case $c = \frac{3}{2}$ has been studied for a long time [31, 37, 47, 50, 61, 84] and is of special interest since it corresponds to two-dimensional wetting as explained in the Introduction. On the analytical side, efforts have focused on the small disorder limit: Ref [47] finds a marginally irrelevant disorder where the quenched critical properties are the same as in the pure case, up to subleading logarithmic corrections. Other studies have concluded that the disorder is marginally relevant [9, 35, 61]. On the numerical side, the same debate on the disorder relevance took place. The numerical studies of Ref. [47] and Ref. [30] have concluded that the critical behaviour was indistinguishable from the pure transition. On the other hand, the numerical study of [35] pointed towards a negative specific heat exponent ($\alpha < 0$), and finally Ref. [84] has been interpreted as an essential singularity in the specific heat, that formally corresponds to an exponent $\alpha = -\infty$.

As explained in the Introduction, the case $c > 2$ where the pure transition is first order is of interest for DNA denaturation. The effect of disorder on this transition has been recently debated [25, 51, 52, 71].

### B. Definition of a sample-dependent pseudo-critical temperature

In the magnetic systems studied in [37, 88], the pseudo-critical temperature $T_c(i, L)$ of the sample $i$ was identified to the maximum of the susceptibility. In the PS model, one can not follow the same path and we have tried two different definitions in [71]. Here we present the simplest one based on the free-energy.

In the pure PS model with bound-bound boundary conditions, the behavior of the partition function as a function of temperature reads

$$Z_L^{\text{pure}}(T < T_c) \xrightarrow{L \gg 1/(T - T_c)^\nu_P} 2^L/(T - T_c)^{\nu_P L}$$

$$Z_L^{\text{pure}}(T_c) \approx \frac{2^L}{L^{2-c}}$$

$$Z_L^{\text{pure}}(T > T_c) \xrightarrow{L \gg 1/(T - T_c)^\nu_P} \frac{2^L}{(T - T_c)^2 L^c}$$

with $\nu_P = 1/(c - 1)$. A finite-size pseudo-critical temperature $T_c^{\text{pure}}(L)$ can then be defined as the temperature where the free-energy reaches the extensive delocalized value $F_{\text{deloc}} = -TL \ln 2$, i.e. $T_c^{\text{pure}(f)}(L)$ is the solution of the equation

$$F_L^{\text{pure}(f)}(L, T) + TL \ln 2 = 0$$

This definition introduces a logarithmic factor

$$T_c^{\text{pure}(f)}(L) - T_c(\infty) \sim \left(\frac{\ln L}{L}\right)^{1/\nu_P}$$

with respect to the purely algebraic factor usually expected (Eq. [13]). This logarithmic factor comes the finite-size free-energy value exactly at criticality $F_L^{\text{pure}}(L, T_c) = -T_c L \ln 2 + (2 - c) T_c \ln L$ (Eq. [21]). In the disordered case, we may similarly define a sample-dependent pseudo-critical temperature $T_c^{(f)}(i, L)$ as the solution of the equation

$$F_L^{(f)}(L, T) + TL \ln 2 = 0$$

Logarithmic corrections are to be expected to appear in the shift (Eq. [12]). This definition of the pseudo-critical temperature thus uses the fact that the free-energy density of the delocalized phase is exactly known.

### C. Distribution of pseudo-critical temperatures in Poland-Scheraga models

In [71], the distribution of pseudo-critical temperatures disordered Poland-Scheraga models with different loop exponents $c$, corresponding to either (i) a pure second order transition with marginal disorder $c = 3/2$ (wetting case); (ii) a pure second order transition with relevant disorder $c = 1.75$ (iii) a pure first order transition $c > 2$ (DNA denaturation)
In there three cases \( c = 1.5, 1.75 \) and \( 2.15 \), the distributions of pseudo-critical temperatures were found to follow the scaling form

\[
P_L(T_c(i, L)) \simeq \frac{1}{\Delta T_c(L)} g\left( x = \frac{T_c(i, L) - T_c^{av}(L)}{\Delta T_c(L)} \right)
\]

where the scaling distribution \( g(x) \) is simply Gaussian

\[
g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}
\]

Note however that this Gaussian distribution is not generic but seems specific to these Poland-Scheraga models, since in the selective interface \( [72] \), the corresponding scaling distribution was found to be very asymmetric. The rescaling \( [25] \) means that the important scalings of the pseudo-critical temperatures distribution are the behaviours of its average \( T_c^{av}(L) \) and width \( \Delta T_c(L) \) as \( L \) varies.

For the margival case \( c = 3/2 \) corresponding to two-dimensional wetting, both the width \( \Delta T_c(L) \) and the shift \( [T_c(\infty) - T_c^{av}(L)] \) are found to decay as \( L^{-\nu_R} \), where the exponent is very close to the pure exponent \( (\nu_R \approx 2 = \nu_{pure}) \) but disorder is nevertheless relevant since it leads to non self-averaging of the contact density at criticality \([71]\). For relevant disorder \( c = 1.75 \), the width \( \Delta T_c(L) \) and the shift \( [T_c(\infty) - T_c^{av}(L)] \) decay with the same new exponent \( L^{-1/\nu_R} \) (where \( \nu_R \approx 2.7 > 2 = \nu_{pure} \) ) and there is again no self-averaging at criticality. Finally for the value \( c = 2.15 \), of interest in the context of DNA denaturation, the width \( \Delta T_c(L) \sim L^{-1/2} \) dominates over the shift \( [T_c(\infty) - T_c^{av}(L)] \sim L^{-1} \), i.e. there are two correlation length exponents \( \nu = 2 \) and \( \tilde{\nu} = 1 \). This is reminiscent of what happens at strong disorder fixed points \([40, 50]\), where the typical and averaged correlation exponents are different as explained in Eqs \( [6, 7] \).

### D. Discussion on the nature of the transition for \( c = 2.15 \) in the presence of disorder

The distribution of pseudo-critical temperatures for \( c = 2.15 \) shows that the transition is an unconventional random critical point with two different correlation length exponents \( \nu = 2 \) and \( \tilde{\nu} = 1 \). This is in contrast with usual random critical points, arising from second order transitions with relevant disorder, where the same exponent is expected to govern the width and the shift (Eq. \([16]\), but this is reminiscent of what happens at strong disorder fixed points \([40, 50]\). The question is now which correlation exponent appears in a given observable. In the random transverse field Ising chain where many exacts results are known for exponents and scaling distribution functions \([40]\), it is well understood how the two exponents \( \nu = 2 \) and \( \tilde{\nu} = 1 \) govern respectively the averaged/typical correlations. Here in the disordered PS model, the analog of the correlation function is the loop distribution. To simplify the discussion, let us more specifically consider the probability of an end-to-end loop of length \( L \) in sample \( i \) of length \( L \), which is directly related to the partition function \( Z_L^{(i)}(T) \) of sample \( i \)

\[
P_L^{(i)}(L, T) = \frac{2L}{L^c} \frac{1}{Z_L^{(i)}(T)}
\]

Introducing for each sample \( i \) the difference between the free-energy density \( F_L^{(i)}(T)/L = -T \ln Z_L^{(i)}(L, T)/L \) and the delocalized value \( f_{deloc} = -T \ln 2 \)

\[
f^{(i)}(L, T) \equiv \frac{-T \ln Z_L^{(i)}(L, T)}{L} + T \ln 2
\]

one gets

\[
\ln P_L^{(i)}(L, T) = -c \ln L + L \beta f^{(i)}(L, T)
\]

The self-averaging property of the free energy means that \( f_L^{(i)}(T) \) converge for large \( L \) to a non-random value \( f(T) \) for any sample \( i \) with probability one

\[
f_L^{(i)}(T) \xrightarrow{L \to \infty} f(T)
\]
where $f(T)$ is the free-energy difference between the localized phase and the delocalized phase: $f(T < T_c) < 0$ and $f(T > T_c) = 0$. This translates immediately into the corresponding statement \( \xi(T) \equiv -\frac{1}{\beta f(T)} \)

\[
\lim_{L \to \infty} \frac{\ln P_L^{(i)}(L, T)}{L} \to \beta f(T)
\]

for any sample \((i)\) with probability one. Since the typical correlation length $\xi_{typ}(T)$ is defined as the decay rate of the logarithm of the correlation, we obtain here that it is simply given by the inverse of the free-energy $f(T)$,

\[
\frac{1}{\xi_{typ}(T)} = \lim_{L \to \infty} \left( \frac{\ln P_L^{(i)}(L, T)}{L} \right) = -\beta f(T)
\]

The dominance of the variance $\Delta T_c(L) \sim L^{-1/2}$ over the shift $[T_c^{av}(L) - T_c(\infty)] \sim L^{-1}$ indicates that asymptotically for large $L$, half of the samples \((i, L)\) are still localized at $T_c(\infty)$, whereas the other half is already delocalized. This suggests that the contact density is finite at criticality, as we have numerically found in [51, 71]. Similarly, the Monte-Carlo study of 3D Self-Avoiding Walks with random pairing energies [22] point towards a finite energy density at $T_c$.

The free-energy of the disordered Poland-Scheraga model is thus expected to vanish linearly

\[
f(T) \to (T_c - T)
\]

The typical correlation length then involves the exponent $\nu_{typ} = 1$

\[
\xi(T) \to (T_c - T)^{-\nu_{typ}} \quad \text{with} \quad \nu_{typ} = 1
\]

Let us now consider the decay of the averaged end-to-end loop distribution that defines an a priori different correlation length $\xi_{av}(T)$

\[
\lim_{L \to \infty} \left( \frac{\ln P_L^{(i)}(L, T)}{L} \right) = \frac{1}{\xi_{av}(T)}
\]

This correlation length $\xi_{av}(T)$ determines the divergence of high moments of the averaged loop distribution. At a given temperature $T < T_c$, these moments will actually be dominated by the rare samples of length $L$ which are already delocalized at $T$, i.e. the samples having $T_c(i, L) < T$. Since our numerical results indicate that the distribution of the pseudo-critical temperature $T_c(i, L)$ is a Gaussian with mean and width given respectively by $[T_c^{av}(L) - T_c(\infty)] \sim L^{-1}$ and $\Delta T_c(L) \sim L^{-1/2}$, we obtain that the fraction of delocalized samples presents the following exponential decay in $L$

\[
\text{Prob}[T_c(i, L) < T] \sim e^{-(T_c^{\infty} - T)^2 L}
\]

This measure of the rare delocalized samples will govern the decay of the averaged loop distribution, and the correlation length defined in [33] thus involves the exponent $\nu_{av} = 2$

\[
\xi_{av}(T) \to (T_c - T)^{-\nu_{av}} \quad \text{with} \quad \nu_{av} = 2
\]

in contrast with the typical correlation length $\xi_{typ}$.

To better understand the emergence of two different correlation lengths, we have numerically measured the distribution over the samples \((i)\) of the free-energy $f^{(i)}(L, T)$ defined in Eq. [28]. We obtain that for $T < T_c$

\[
f_L^{(i)}(T) = f(T) + \frac{a_T}{L} + \frac{\sigma_T u_i}{\sqrt{L}}
\]

where $a_T$ is temperature dependent and $u_i$ is a Gaussian random variable of zero mean and of variance 1

\[
G(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}
\]

The averaged end-to-end loop distribution then reads

\[
P_L^{(i)}(L, T) = \frac{1}{L^c} e^{L \beta f^{(i)}(L, T)} = \frac{1}{L^c} e^{L \beta f(T)} \int_{-\infty}^{+\infty} du \ G(u) e^{\sqrt{T} \sigma_T u} = \frac{1}{L^c} e^{L \beta f(T) + L \beta^2 \sigma^2 / 2}
\]

\[
\text{Prob}[T_c(i, L) < T] \sim e^{-(T_c^{\infty} - T)^2 L}
\]
The difference between the correlation length $\xi_{av}(T)$ \[^{35}\] and the typical correlation length $\xi_{typ}(T)$ \[^{62}\] is due to the variance $\sigma_T^2$:

\[
\frac{1}{\xi_{av}(T)} = \frac{1}{\xi_{typ}(T)} - \frac{\beta^2 \sigma_T^2}{2} \quad (41)
\]

In particular, to obtain the scaling $\frac{1}{\xi_{av}(T)} \sim (T_c - T)^2$ different from $\frac{1}{\xi_{typ}(T)} \sim (T_c - T)$, the variance term in $\sigma_T^2$ has to cancel exactly the leading order in $(T_c - T)$ on the left hand-side.

So the picture that emerges of the present analysis is very reminiscent of what happens at strong disorder fixed points \[^{10, 50}\] : the exponents $\nu_{typ}$ = 1 and $\nu_{av}$ = 2 govern respectively the decay of typical/averaged loop distribution. Our conclusion is thus that the exponent $\nu_{typ}$ = 1 governs the free-energy \[^{33}\] that corresponds to a Lyapunov exponent, i.e. it describes the critical behavior of any typical sample, whereas the exponent $\nu_{av}$ = $2 = 2/d$ is the finite-size scaling exponent of Chayes et al \[^{24}\] and is related to the variance of the distribution of pseudo-critical temperatures. The numerical results concerning the contact density (Figure 8 of \[^{51}\]) may be now interpreted as follows : for each sample, the critical region has a width of order $1/L$, whereas the contact density averaged over the samples decay on a much wider scale $1/\sqrt{L}$ that represents the sample-to-sample fluctuations of the pseudo-critical temperatures $T_c(i, L)$.

**IV. SELECTIVE INTERFACE MODEL**

As explained in the introduction, the selective interface model \[^{31}\] is expected to undergo a phase transition at a critical temperature $T_c \simeq O(\Delta^2/q_0)$ between a localized phase and a delocalized phase in the upper fluid. A real space renormalization group study, based on rare events, was proposed in \[^{69}\]. Mathematicians have also been interested in this model. The localization at all temperatures for the symmetric case was proven in ref. \[^{4, 81}\]. In the asymmetric case, the existence of a transition line in temperature vs asymmetry plane was proven in ref. \[^{12, 10}\]. More recent work can be found in \[^{14, 17, 52, 57}\]. In the following, we describe the results on the distribution of pseudo-critical temperatures $T_c(i, L)$.

**A. Definition of a sample dependent $T_c(i, L)$**

As for the wetting and Poland-Scheraga models, the free energy of the delocalized phase is known. If one forgets the boundary conditions at $(1, \alpha)$, the partition function characterizing the delocalized phase in the (+) solvent ($q_0 > 0$) would simply be for each sequence of charges

\[
Z^{\text{deloc}}(\alpha) = 2^{\alpha-1} e^{\beta V(\alpha)} \quad \text{with} \quad V(\alpha) \equiv \sum_{\alpha'} q_i \quad (42)
\]

For each sample $(i)$ of length $L$, we may thus define a pseudo-critical temperature $T_c(i, L)$ as the temperature where the free energy $F^{(i)}(L, T) \equiv -T \ln Z^{(i)}(L, T)$ reaches the delocalized value $F^{(i)}_{\text{deloc}}(L, T) = -T(L - 1) \ln 2 - V(L)$ (Eq. \[^{62}\]), i.e. $T_c(i, L)$ is the solution of the equation

\[
F^{(i)}(L, T) + T(L - 1) \ln 2 + V(L) = 0 \quad (43)
\]

As explained in the previous section on Poland-Scheraga models, this definition of pseudo-critical critical temperatures, together with bound-bound boundary conditions, introduces logarithmic correction in the convergence towards $T_c(\infty)$. Eq. \[^{12}\] is accordingly replaced by

\[
T_c^{av}(L) - T_c(\infty) \sim \left( \frac{\ln L}{L} \right)^{1/\nu_R} \quad (44)
\]

**B. Scaling form of the probability distribution**

Our data for the distribution of pseudo-critical temperatures \[^{72}\] follow the scaling form

\[
P_L(T_c(i, L)) \simeq \frac{1}{\Delta T_c(L)} g \left( x = \frac{T_c(i, L) - T_c^{av}(L)}{\Delta T_c(L)} \right) \quad (45)
\]

\[
\frac{1}{\xi_{av}(T)} = \frac{1}{\xi_{typ}(T)} - \frac{\beta^2 \sigma_T^2}{2} \quad (41)
\]
where the scaling distribution $g(x)$ (normalized with $<x> = 0$ and $<x^2> = 1$) is now very asymmetric, in marked contrast with the Gaussian form obtained for the wetting and Poland-Scheraga models.

C. Scaling properties of the shift and of the width

We now discuss the numerical results obtained for the width $\Delta T_c(L)$ and for the average $T_{cav}(L)$ of the distribution, as $L$ varies. The width $\Delta T_c(L)$ follows the power law

$$\Delta T_c(L) \sim \left(\frac{1}{L}\right)^{0.26} \quad (46)$$

and the average $T_{cav}(L)$ can be fitted with the generalized form of eq. (44)

$$T_c(\infty) - T_{cav}(L) \sim \left(\frac{\ln(L)}{L}\right)^{0.26} \quad (47)$$

The value $T_c(\infty) \approx 0.838 \Delta^2 q_0$ is in agreement with the numerical estimate of ref [19]. The result for the exponent in eqs (46) and (47) indicates that the transition can be described as a random critical point with a single correlation exponent

$$\frac{1}{\nu_R} \approx 0.26 \quad (48)$$

A similar value has been observed in numerical simulations by the authors of ref. [19] (private communication).

Our data rule out the possibility of an infinite order transition based on rare negatively charged sequences that would lead to a smaller value of the critical temperature $T_{cav}(\infty) = (2/3) \Delta^2 q_0$. The present results suggest that the excursions in the unfavorable fluid that are important for the transition, are of finite length.

V. FREEZING TRANSITION OF THE DIRECTED POLYMER IN A RANDOM MEDIUM

We first summarize some properties of the low temperature phase, before turning our attention towards the critical properties.

A. Statistics of excitations above the ground state

The droplet theory for directed polymers [44], is similar to the droplet theory of spin-glasses [43]. It is a scaling theory that can be summarized as follows. At very low temperature $T \to 0$, all observables are governed by the statistics of low energy excitations above the ground state. An excitation of large length $l$ costs a random energy

$$\Delta E(l) \sim l^\theta u \quad (49)$$

where $u$ is a positive random variable distributed with some law $Q_0(u)$ having some finite density at the origin $Q_0(u = 0) > 0$. The exponent $\theta$ is the exponent governing the fluctuation of the energy of the ground state is exactly known in one-dimension $\theta(d = 1) = 1/3$ [39, 58, 63, 80] and for the mean-field version on the Cayley tree $\theta(d = \infty) = 0$ [33]. In finite dimensions $d = 2, 3, 4, 5, ...$, the exponent $\theta(d)$ has been numerically measured, and we only quote here the results of the most precise study we are aware of [67] for dimensions $d = 2, 3 : \theta(d = 2) = 0.244$ and $\theta(d = 3) = 0.186$.

From (49), the probability distribution of large excitations $l \gg 1$ reads within the droplet theory

$$d\rho(E = 0, l) \sim \frac{dl}{l} e^{-\beta \Delta E(l)} \sim \frac{dl}{l} e^{-\beta \theta^\theta u} \quad (50)$$

where the factor $dl/l$ comes from the notion of independent excitations [43]. In particular, its average over the disorder follows the power-law

$$d\rho(E = 0, l) \sim \int_0^{+\infty} du Q_0(u) \frac{dl}{l} e^{-\beta \theta^\theta u} = TQ(0) \frac{dl}{l^{1+\theta}} \quad (51)$$
This prediction describes very well the numerical data in the regime \(1 \ll l \ll L\) in dimensions \(d = 1, 2, 3\).

Since correlation functions at large distance are directly related to the probability of large excitations, we already see that the low temperature phase is very non-trivial from the point of view of correlations lengths: the typical exponential decay \(\xi_{yp}(T)\) indicates a finite typical correlation length \(\xi_{av}(T)\) whereas the averaged power-law behavior \(\xi_{av}(T)\) means that the averaged correlation length \(\xi_{av}(T)\) is actually infinite in the whole low temperature phase

\[
\xi_{av}(0 < T \leq T_c) = \infty
\]  

In addition to the general discussion of Section II B, this shows once again why it is crucial to distinguish between typical and averaged correlation functions in disordered systems. Note that within the droplet theory of spin-glasses \([43]\), the correlation length \(\xi_{av}(T)\) is also infinite in the whole low temperature phase for the same reasons.

### B. Low temperature phase governed by a zero-temperature fixed point

According to the droplet theory, the whole low temperature phase \(0 < T < T_c\) is governed by a zero-temperature fixed point. However, many subtleties arise because the temperature is actually ‘dangerously irrelevant’. The main conclusions of the droplet analysis \([44]\) can be summarized as follows. The scaling \((49)\) governs the free energy cost of an excitation of length \(l\), provided one introduces a correlation length \(\xi(T)\) to rescale the length

\[
\Delta F(l) = \left(\frac{l}{\xi(T)}\right)^\theta u
\]  

Here as before, \(u\) denotes a positive random variable distributed with some law \(Q(u)\) having some finite density at the origin \(Q(u = 0) > 0\). Moreover, this droplet free energy is a near cancellation of energy and entropy contributions that scale as \([43, 44]\)

\[
\Delta E(l) \sim l^{1/2} w
\]  

where \(w\) is a random variable of order \(O(1)\) and of zero mean. The argument is that the energy and entropy are dominated by small scale contributions of random sign \([13, 44]\), whereas the free energy is optimized on the coarse-grained scale \(\xi(T)\). These predictions for the energy and entropy have been numerically checked in \([44, 85]\).

### C. Logarithmic fluctuations of the free energy at criticality

Let us now consider what happens for \(T = T_c\). Forrest and Tang \([48]\) have conjectured from their numerical results on a growth model in the KPZ universality class and from the exact solution of another model \([13]\) that the fluctuations of the height of the interface were logarithmic at criticality. For the directed polymer model, this translates into a logarithmic behavior of the free energy fluctuations at \(T_c\)

\[
\Delta F(L, T_c) \sim (\ln L)^\sigma v
\]  

where \(v\) is a positive random variable of order one distributed with some law \(R(v)\), and where the exponent was measured to be in \(d = 3\) \([48, 62]\)

\[
\sigma = \frac{1}{2}
\]  

Further theoretical arguments in favour of this logarithmic behavior can be found in \([36, 83]\). The argument of \([36]\) is that the power-law behavior \(F(L, T_c) \sim L^{\theta_c}\) is impossible at criticality so that \(\theta_c = 0\). From the scaling relation \(\theta_c = 2\zeta_c - 1\) between exponents \([38]\), the roughness exponent \(\zeta\) is expected to be exactly \(\zeta_c = 1/2\) \([31]\), and a renormalization argument then leads to logarithmic fluctuations of the free energy \([83]\).

### D. Location of the critical temperature

#### 1. Exact bounds on \(T_c\) derived by Derrida and coworkers

Let us first recall the physical meaning of the exact bounds for the critical temperature derived by Derrida and coworkers \([28, 34, 37]\)

\[
T_0(d) \leq T_c \leq T_2(d)
\]
The upper bound \( T_2(d) \) corresponds to the temperature above which the ratio
\[
\mathcal{R}_L(T) = \frac{Z_L^2}{(Z_L)^2}
\]
remains finite as \( L \to \infty \). The lower bound \( T_0 \) corresponds to the temperature below which the annealed entropy becomes negative.

In dimensions \( d = 1, 2 \), the upper bound is at infinity \( T_2 = \infty \), whereas for \( d \geq 3 \), the upper bound \( T_2 \) is finite. The interpretation is as follows \[37\]. The ratio \( (58) \) can be decomposed according to the probability \( P_L(m) \) that two independent usual random walks in dimension \( d \) meet \( m \) times before time \( L \)
\[
\mathcal{R}_L(T) = \sum_{m=1}^{L} P(m) B^m
\]
where the factor
\[
B(T) = \frac{e^{2\beta \epsilon}}{(e^{\beta \epsilon})^2}
\]
can be explicitly computed for any distribution of the site disorder variable \( \epsilon \). In dimensions \( d = 1, 2 \), two random walks meet an infinite number of times as \( L \to \infty \), whereas for \( d \geq 3 \), they meet a finite number \( m \) of times as \( L \to \infty \).

The distribution of \( m \) decays exponentially
\[
P(m) \sim (1 - A)A^m
\]
where \( (1 - A) \) is the finite probability of never meeting again. \( T_2 \) is defined as the temperature where
\[
AB(T_2) = 1
\]
For \( T > T_2 \), \( B(T) < B(T_2) = 1/A \), and the ratio \( \mathcal{R}_L(T) \) has a finite limit
\[
\mathcal{R}_\infty(T > T_2) = \frac{1 - A}{1 - AB(T)}
\]
For \( T < T_2 \), \( \mathcal{R}_L(T) \) is a geometric series of parameter \( AB(T) > 1 \), and it thus diverges exponentially in \( L \)
\[
\mathcal{R}_L(T < T_2) \sim (1 - A) \sum_{m=1}^{L} (AB(T))^m \sim (AB(T))^L
\]
Exactly at \( T_2 \), the ratio diverges but not exponentially
\[
\mathcal{R}_L(T_2) = (1 - A) \sum_{m=1}^{L} 1 \sim L
\]

2. Interpretation in terms of the probability distribution of free energies

Let us now interpret the above results of the ratio \( \mathcal{R}_L(T) \) in terms of the probability distribution \( P_L(F) \) of the free energy \( F = -kT \ln Z_L \) over the samples of length \( L \). By definition \[38\], one has
\[
\mathcal{R}_L(T) = \frac{\int dF P_L(F)e^{-2\beta F_L}}{(\int dF P_L(F)e^{-\beta F_L})^2}
\]
For \( T > T_2(d) \), the ratio \( \mathcal{R}_\infty(T) \) is finite : this means that the fluctuations of the free energy over the samples
\[
[\Delta F_L]_{\text{samples}}^2 = \int dF P_L(F)F^2 - \left( \int dFP_L(F)F \right)^2
\]
remain of order \( O(1) \) in the limit \( L \to \infty \).
On the other hand, for the directed polymer in the low temperature phase $T < T_c$, the fluctuations of free energies over the samples is expected to have the same scaling as the fluctuations of free energies within the same sample when the end-point varies: the fluctuations of free energy over the samples are thus governed by the droplet exponent $\theta$

$$\left[\Delta F_L \right]_{\text{samples}} (T < T_c) \sim \left[\Delta F_L \right]_{\text{droplet}} (T < T_c) \sim L^\theta$$

(68)

Let us now recall Zhang’s argument [54] that allows to determine the exponent $\eta$ of the tail of the free energy distribution

$$P_L(F \to -\infty) \sim e^{-\left(\frac{\sigma}{\theta}\right)^\eta}$$

(69)

Moments of the partition function can be then evaluated by the saddle-point method, with a saddle value $F^*$ lying in the negative tail [60]

$$Z_L = \int dF P_L(F)e^{-\beta n_0 F_L} \sim \int dF e^{-\left(\frac{F}{\theta}\right)^\eta} e^{-\beta n_0 F_L} \sim e^{\beta \eta (n_0) L^\eta}$$

(70)

Since these moments of the partition function have to diverge exponentially in $L$, the exponent $\eta$ of the tail [60] reads in terms of the droplet exponent

$$\eta = \frac{1}{1 - \theta}$$

(71)

3. Debate on the location of $T_c$ for the directed polymer in finite dimensions

At $T_c$, the fluctuations of the free energy are expected to be logarithmic, as discussed around Eq [55]

$$\Delta F \sim (\ln L)^\sigma \quad \text{with} \quad \sigma = \frac{1}{2}$$

(72)

From these logarithmic fluctuations, it seems rather difficult to obtain an exponential divergence in $L$ of the ratio $R_L(T_c)$ [60] if the free-energy distribution $P_{T_c}(F)$ decays more rapidly than exponentially as $F \to -\infty$. On the contrary, if $T_c = T_2$, it is very natural to obtain the divergence found for the ratio at $T_2$ [63]

$$R_L(T_2) \sim L \sim e^{\ln L}$$

(73)

Moreover, to obtain the linear divergence [65], the saddle-point method described above for the low temperature phase [60] gives that the tail of the free energy distribution should be at criticality

$$P_{T_c}(F \to -\infty) \sim e^{-\left(\frac{\sigma}{\theta}\right)^\eta} \quad \text{with} \quad \eta_c = \frac{1}{1 - \sigma}$$

(74)

The value $\sigma = 1/2$ corresponds to the tail exponent $\eta_c = 2$.

We have thus proposed in [74] that the critical temperature $T_c$ in finite dimension $d$ satisfying $\theta(d) > 0$ coincides with the temperature $T_2(d)$. Explicit expressions for $T_2(d)$ in terms of usual integrals appearing in the theory of random walks can be found in [28, 34] for site and bond disorder respectively.

However, other arguments are in favor of the strict inequality $T_c < T_2$ in finite dimensions. In particular a new upper bound $T^*$ based on the entropy of the random walk was recently proposed in [11]. Moreover, the strict inequality $T_c < T_2$ is satisfied for the directed polymer on the Cayley tree that plays the role of a mean-field version of the model ($T_c$ coincides with the lower bound $T_0$ [57] below which the annealed entropy becomes negative [28]). It is useful to discuss the behavior of the free-energy distribution on the Cayley tree to compare with the finite dimensional case. Let us first consider the low-temperature phase $T < T_c$. On the Cayley tree, the low-temperature phase is characterized by $\theta = 0$ and $\Delta F = O(1)$ whereas Zhang’s argument above is consistent only if $\eta = 1/(1 - \theta) > 1$ to ensure the convergence in the presence of the exponential term $e^{-\beta n_0 F}$ (Eq [74]). When $\theta = 0$, the tail of $P_L(F \to -\infty)$ is also an exponential $e^{\alpha F / (\Delta F)}$ as in the Random Energy Model [33] and one has to take into account the minimal free energy that can be obtained for a finite size $L$. From a physical point of view, the reason could be that the configurations of two polymers in the same disordered sample are very different. In finite dimensions, contacts and loops alternate extensively, whereas on the tree, the loops simply do not exist: the two polymers may only coincide over some distance and then never meet again. Since the exponential tail found for the Cayley tree actually corresponds to
the universal Gumbel tail for the minimum of independent variables, this shows that the non-exponential tail found in finite dimensions for $T < T_c$ reflects the importance of correlations between the free energies of paths due to the presence of loops. Let us now discuss what happens in the high temperature phase $T > T_c$. On the Cayley tree, the tail of the free-energy distribution $P_L(F \to -\infty)$ is known to be also an exponential $e^{\alpha F/(\Delta F)}$ \[3\], and this is why the ratio \[3\] can diverge exponentially in the region $T_c < T < T_2$ even if $\Delta F = O(1)$. In finite dimension $d$, the free-energy fluctuations are expected to be of order $\Delta F = O(1)$ for $T > T_c$. The debate between the two possibilities $T_c = T_2$ or $T_c < T_2$ in finite dimensions thus depends on the tail the free-energy distribution $P_L(F \to -\infty)$ for $T > T_c$. If the tail is exponential as on the Cayley tree, then $T_c < T_2$, whereas if $P_L(F \to -\infty)$ decays more rapidly than exponentially, then $T_c = T_2$. For instance, in the mean-field Sherrington-Kirkpatrick model of spin-glasses, one has $T_c = T_2$ and the distribution of the free-energy fluctuations for $T > T_c$ is known to be Gaussian \[8\]. We are presently studying the directed polymer in $d = 3$ numerically \[72\]. Various indicators based on geometrical or thermodynamical behaviors points towards a critical temperature $T_c$ in the vicinity of $T_2$, and thus the possibility $T_c = T_2$ cannot be presently excluded within the numerical precision. Concerning the free-energy distribution, we obtain numerically that it is Gaussian in the high temperature phase for $T \gg T_2$, but we are not aware of any argument discussing the form of this distribution in the literature. In conclusion, we have the feeling that the most clear way to solve the debate on the value of $T_c$ in finite dimension would be to obtain results on the probability distribution of the free-energy in the whole high-temperature phase $T > T_c$.

4. Why $T_c$ is different from $T_2$ in other finite dimensional disordered systems

The fact that the fluctuations of free energies over the samples have the same scaling as the droplet excitations within one given sample \[65\] is very specific to the directed polymer model. In other disordered models, such as spin-glasses for instance, the fluctuations of free energies over the samples scale instead as \[18, 86\]. This scaling simply reflects the Central-Limit fluctuations of the distribution of the free-energy in the whole high-temperature phase $T > T_c$.

\[\frac{\Delta F_L}{\text{samples}} \sim L^{d/2} \text{ } (75)\]

at any temperature. This scaling simply reflects the Central-Limit fluctuations of the $L^d$ disorder variables defining the sample. The directed polymer escapes from these normal fluctuations because it is a one-dimensional path living in a $1 + d$ disordered sample: each configuration of the polymer only sees $L$ random variables among the $L^{1 + d}$ disorder variables that define the sample, and the polymer can 'choose' the random variables it sees. So for other disordered systems having fluctuations over the samples governed by \[65\], the ratio $R_L(T)$ will diverge exponentially as any temperature. The temperature $T_2$ is thus infinite

$T_2 = \infty \text{ } (76)$

and has nothing to do with any critical temperature. However, the droplet exponent $\theta$ is expected to govern the correction to the extensive part of the mean value \[18\]

$\bar{F}_L \sim L^d f_0 + L^\theta f_1 \text{ } (77)$

It can for instance be measured in the free energy difference upon a change of boundary conditions that forces the introduction of some domain wall in the sample \[43\].

E. Description of the transition in terms of the loop distribution between two polymers in the same sample

For $T < T_c$, the number of contacts of two independent polymers $x(i)$ and $y(i)$ in the same disordered sample

$n_L(T) = \sum_{i=1}^{L} < \delta_{x(i),y(i)} > \text{ } (78)$

is extensive, and the density of contacts, also called the overlap, is precisely the order parameter of the low temperature phase \[27, 33, 65\].

$q(T) = \lim_{L \to \infty} \left( \frac{n_i(T)}{L} \right) \text{ } (79)$
Note that on the Cayley tree where $\theta = 0$, the distribution of this overlap is made of two delta peaks at $q = 0$ and $q = 1$, whereas in finite dimensions with $\theta > 0$, the distribution of this overlap is expected to be a single delta function at $q(T)$.

One may thus analyse the configuration of two polymers in the same sample in terms of contacts separated by loops. For $T < T_c$, the distribution of large loops follows a scaling form based on the free energy scaling of a droplet of length $l$:

$$dlP_{\text{large}}(l, T) = N(T)\frac{dl}{l} e^{-\beta \Delta F(l)} = N(T)\frac{dl}{l} e^{-\beta \left(\frac{1}{\xi(T)}\right)^\sigma}$$  \hspace{1cm} (80)

At $T_c$, the logarithmic scaling suggests that the loop distribution still exists and follows the form

$$dlP_{T_c}(l) = \frac{dl}{l} e^{-\beta \Delta F(l)} = \frac{dl}{l} e^{-\beta_c \left(\ln l\right)^\sigma_v}$$  \hspace{1cm} (81)

The normalization factor $N(T)$ of large loops of the low temperature phase (Eq. 80) can be found by a matching procedure on scale $l \sim \xi(T)$ with the critical distribution (81) as explained in details in [74]. For the value $\sigma = 1/2$ measured in $d = 3$ [48, 62], the resulting critical behavior for the free-energy and the overlap are respectively

$$f(T) - f(T_c) \sim \frac{1}{\xi(T)} \sim e^{-\left(\frac{1}{\xi(T)}\right)^2} + ...$$  \hspace{1cm} (82)

$$q(T) \sim e^{-\left(\frac{1}{\xi(T)}\right)^2} + 2 \ln \frac{1}{\xi(T)} + ...$$  \hspace{1cm} (83)

where $K$ is some constant.

In conclusion, the logarithmic behavior with $\sigma = 1/2$ is responsible for the unusual critical properties, whereas the usual power-laws in pure phase transitions correspond to the value $\sigma = 1$. Since the droplet scaling theory of the low temperature phase was initially developed for spin-glasses [43], this raises the question of the existence of an exponent $0 < \sigma < 1$ at the spin-glass transition. Some consequences of this possibility are discussed in [74].

VI. CONCLUSION

In these proceedings, we have first summarized some general properties of phase transitions in the presence of quenched disorder. We have then reviewed our recent works on the critical properties of various delocalization transitions involving random polymers. In the wetting and Poland-Scheraga models, the delocalization transition already exists in the pure case, and pure critical properties depends on the value of the loop exponent. The Harris criterion is a convenient measure of the relevance of the disorder, and we have presented results for second order with either marginal/relevant disorder, as well as for first order with relevant disorder. In the selective interface model, we have explained why the scenario based on rare large loops in the minority solvent seems now numerically ruled out. Further work is thus needed to elucidate the precise mechanism of the transition. Finally, for the directed polymer model, which is of special interest in connection with spin glasses in finite dimensions, we have shown that the droplet picture leads to a very unusual critical behavior. We have also discussed whether the critical temperature could coincide with the upper bound $T_2$ derived by Derrida and coworkers, depending on the negative tail of the free-energy distribution in the high temperature phase.

VII. ACKNOWLEDGEMENTS

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