Sequence entropy for amenable group actions

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Abstract

We study the sequence entropy for amenable group actions and investigate systematically spectrum and several mixing concepts via sequence entropy both in measure-theoretic dynamical systems and topological dynamical systems. Moreover, we use sequence entropy pairs to characterize weakly mixing and null systems in the topological sense.

1. Introduction

By a $G$-topological dynamical system (or $G$-system for short) $(X,G)$ we mean a compact metric space $X$ with a (semi)-group $G$ acting continuously on $X$. If $G = \mathbb{Z}_+$ (resp. $\mathbb{Z}$) is the additive semigroup of non-negative integers (resp. the additive group of integers), then the classical $\mathbb{Z}_+$ (resp. $\mathbb{Z}$)-action can be induced by a continuous map (resp. a homeomorphism) $T: X \to X$, and we usually denote it by $(X, T)$. For a $G$-system $(X, G)$ if there exists a $G$-invariant Borel probability measure $\mu$ on $X$, it induces a measure-theoretic dynamical system $(X, B_X, \mu, G)$ (or $G$-MPS for short), where $B_X$ is the Borel $\sigma$-algebra on $X$. It is well known that, for a $G$-system, there are some groups such that the set of invariant probability measures may be empty, whereas, amenability of the group ensures the existence of invariant probability measures (see [1]).

For $\mathbb{Z}_+$-actions, discrete spectral and mixing properties for measure-theoretic dynamical systems have been investigated by many authors from kinds of viewpoints (see for example [2–5]). An important way to characterize them is by the sequence entropy concept. The related research can be traced back to Kushnirenko [6] who introduced the notion of sequence entropy along a given infinite sequence of $\mathbb{Z}_+$ for the measure-theoretical dynamical system and proved that an invertible measure-theoretical dynamical system has discrete spectrum if and only if the sequence entropy of the system with respect to any infinite sequence of $\mathbb{Z}_+$ is zero (some new proofs related to this result can be found in [7, 8]). Later, Hulse [9] studied the sequence entropy of measure-theoretical dynamical systems with quasi-discrete spectrum. In addition, Saleski [10] and Hulse [11] obtained some characterizations of weakly mixing and strongly mixing measure-theoretical dynamical systems by using the sequence entropy. Moreover, Hulse [12] gave the characterizations of the compact and weakly mixing extensions of measure-theoretical dynamical systems via conditional sequence entropy, and some results related to mild mixing and mildly mixing extension can be found in [13, 14] by Zhang. In [15], Coronel, Maass and Shao studied the relation between sequence entropy and the Kronecker and rigid algebras. As an application, they characterized compact, rigid and mixing extensions via conditional sequence entropy. More recently, directional sequence entropy was introduced to study the directional discrete spectrum by Liu and Xu [16] and directional weak mixing by Liu [17].

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From the viewpoint of topological dynamical systems, there are several ways to characterize topologically mixing properties (see for example [18–22]). In 1974, Goodman [23] introduced the notion of topological sequence entropy in a way analogous to that of Kushnirenko, and studied some properties of topologically null systems. More details of topological sequence entropy for $\mathbb{Z}$-actions can be found in [24]. In [25], Li first obtained a characterization of topological weak mixing by using topological sequence entropy. Moreover, Huang, Shao and Ye [8] further investigated systematically several topological mixing concepts via topological sequence entropy. Besides, Huang, Li, Shao and Ye [26] localized the notion of sequence entropy by defining sequence entropy pairs and proved that a topological dynamical system is topologically weakly mixing if and only if any pair not in the diagonal is a sequence entropy pair.

As the research progressed, many results in ergodic theory and topological dynamical systems were extended to larger groups, such as amenable groups. The reader may read a recent book [27] to learn more details about systems under amenable group actions. In 1987, Ornstein and Weiss [28] developed a method called quasi-tiling, which can serve as the substitute of the Rohlin lemma, and extended many known results of entropy theory from $\mathbb{Z}$-actions to countable infinite amenable group actions. Recently, discrete spectrum and mixing concepts for group actions were studied in different settings by many authors. For example, Huang, Ye and Zhang [29] studied systematically the local entropy theory for actions of a countable discrete amenable group, and used it to discuss transitivity, mild mixing, strong mixing, regionally proximal relation along sequence and weak mixing of all orders. In [30], Yu, Zhang and Zhang studied discrete spectrum for amenable group actions from the viewpoint of measure complexity. In [31], Yan, Liu and Zeng investigate systematically several mixing concepts for group actions via weak disjointness, return time sets and topological complexity functions. In [32], García–Ramos defined weaker forms of topological and measure-theoretical equicontinuity for topological dynamical systems, and studied their relationships with sequence entropy and systems with discrete spectrum. Kerr and Li [33, 34] established local combinatorial and linear-geometric characterizations of sequence entropy in both topological and measure-theoretical sense. The reader also can see a survey [35], section 8 for more details.

It is a natural question whether we can characterize discrete spectrum and mixing properties for amenable group actions using sequence entropy. Based on this motivation, in this paper, we shall introduce the notion of sequence entropy of the system with respect to a group action, and use it to investigate discrete spectral properties and kinds of mixing concepts in both measure-theoretical and topological sense. Let $G$ be a countable discrete amenable group. We prove that a $G$-MPS has discrete spectrum if and only if the sequence entropy of the system with respect to a fixed Følner sequence along any infinite subset of $G$ is zero. Moreover, we show that if a $G$-system $(X, G)$ is topologically strongly mixing then each non-trivial finite open cover of $X$ and any infinite subset $H$ of $G$, the topological sequence entropy with respect to a fixed Følner sequence along some infinite subset of $H$ is positive; topological mild mixing implies that for each non-trivial finite open cover of $X$ and any IP-set $H$ of $G$, the topological sequence entropy with respect to a fixed Følner sequence along some infinite subset of $H$ is positive. In addition, for an Abelian group action, we obtain that it is weakly mixing if and only if for each non-trivial finite open cover, the topological sequence entropy with respect to a fixed Følner sequence along some infinite subset is positive.

This paper is organized as follows. In section 2, we recall some basic notions and properties that we use in this paper. In section 3, we give characterizations of discrete spectrum and weak mixing measures via sequence entropy. In section 4, we investigate some notions of topological mixing via topological sequence entropy. In section 5, we introduce sequence entropy pairs for $G$-systems and use them to describe null systems and weakly mixing systems.

2. Preliminaries

Recall that an infinite countable discrete group $G$ is called amenable if there exists a sequence of finite subsets $F_n \subset G$ such that for every $g \in G$,

$$\lim_{n \to +\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0,$$

(2.1)

where $| \cdot |$ denotes the cardinality of a set and $\triangle$ stands for the symmetric difference of sets. A sequence satisfying condition (2.1) is called a Følner sequence (see [36]).

Throughout this paper, we let $X$ be a compact metric space with a metric $d$ and $G$ an infinite countable discrete group. By a $G$-system we mean a pair $(X, G)$, where $\Gamma: G \times X \to X, (g, x) \mapsto gx$ is a continuous mapping satisfying

1. $\Gamma(e_G, x) = x$ for each $x \in X$;
2. $\Gamma(g_1, \Gamma(g_2, x)) = \Gamma(g_1g_2, x)$ for each $g_1, g_2 \in G$ and $x \in X$. 


If X is a single point set then \((X, G)\) is said to be a trivial system.

If not explicitly stated, in this paper, we always suppose that G is an infinite countable discrete amenable group and \(F = \{ F_n \}_{n=1}^{\infty}\) is a Fø lner sequence of \(G\) with \(e_G \in F_1 \subseteq F_2 \subseteq \cdots \) and \(\bigcup_{n=1}^{\infty} F_n = G\).

For two dynamical systems \((X, G)\) and \((Y, G)\), their product system \((X \times Y, G)\) is defined by the diagonal action: \(g(x, y) = (gx, gy)\) for all \(x \in X, y \in Y\) and \(g \in G\). Higher order products are defined analogously.

Let \((X, G)\) be a G-system, \(B_X\) be the collection of Borel subsets of \(X\) and \(\mathcal{M}(X, G)\) be the set of all Borel probability measures on \(X\). We say that \(\mu \in \mathcal{M}(X, G)\) is G-invariant if \(\mu = \mu g^{-1}\) for each \(g \in G\); a G-invariant measure \(\nu \in \mathcal{M}(X)\) is called ergodic if \(\nu(\bigcup_{g \in G} A) = 0\) or 1 for any \(A \in B_X\). Denote by \(\mathcal{M}(X, G)\) and \(\mathcal{M}^e(X, G)\) the set of all G-invariant Borel probability measures and ergodic G-invariant Borel probability measures on \(X\), respectively. The amenability of \(G\) ensures that \(\mathcal{M}(X, G) = \emptyset\). Under the \(\ast\)-topology, \(\mathcal{M}(X, G)\) is a convex compact metric space, and \(\mu \in \mathcal{M}^e(X, G)\) if and only if it is an extreme point of \(\mathcal{M}(X, G)\) (see [37], Theorem 4.2).

Note that each probability measure \(\mu \in \mathcal{M}(X, G)\) induces a G-MPS \((X, B_X, \mu, G)\). Let \(H_A \subset L^2(X, B_X, \mu)\) be the closed subspace generated by all finite dimensional G-invariant subspaces of \(L^2(X, B_X, \mu)\). It is well known that for \(f \in L^2(X, B_X, \mu), f \in H_f\) if and only if \(f\) is an almost periodic function, that is, \(\{ U_g f : g \in G \}\) is precompact in \(L^2(X, B_X, \mu)\), where \(U_g f := f g\) for any \(g \in G\). By [38], Theorem 7.1, there exists a G-invariant sub-\(\sigma\)-algebra \(K_{\mu}\) of \(B_X\) such that \(H_f = L^2(X, K_{\mu}, \mu)\). We call \(K_{\mu}\) the Kothe algebra of \((X, B_X, \mu, G)\). We say that \(\mu\) has discrete spectrum if \(H_f = L^2(X, B_X, \mu)\) or equivalently \(K_{\mu} = B_X\).

We recall that a G-MPS \((X, B_X, \mu, G)\) is weakly mixing if the product system \((X \times X, B_X \times B_X, \mu \times \mu, G)\) is ergodic. For an amenable group action, weak mixing can be expressed in terms of asymptotic conditions involving averages along a Fø lner sequence (see [27], Theorem 4.21).

**Theorem 2.1.** Given a G-MPS \((X, B_X, \mu, G)\), it is weakly mixing if and only if

\[
\lim_{n \to \infty} \frac{1}{|F_n|} \sum_{g \in F_n} |\mu(A \cap g^{-1}B) - \mu(A) \mu(B)| = 0
\]

for all \(A, B \in B_X\).

Given a subset \(S\) of \(G\), the upper density and lower density of \(S\) with respect to a Fø lner sequence \(F = \{ F_n \}_{n=1}^{\infty}\) are defined by

\[
\bar{D}_F(S) = \limsup_{n \to \infty} \frac{|S \cap F_n|}{|F_n|}, \quad \underline{D}_F(S) = \liminf_{n \to \infty} \frac{|S \cap F_n|}{|F_n|}.
\]

We say that \(A\) has density \(D_F(S)\) with respect to \(F\) if \(D_F(S) = \bar{D}_F(S) = \underline{D}_F(S)\).

**Theorem 2.2.** A given G-MPS \((X, B_X, \mu, G)\) is weakly mixing if and only if for every pair of elements \(A, B \in B_X\), there is a subset \(S = \{ s_i \}_{i=1}^{\infty}\) of \(G\) with \(D_F(S) = 1\) such that

\[
\lim_{i \to \infty} \mu(A \cap s_i^{-1}B) = \mu(A) \mu(B).
\]

**Proof.** This directly follows the idea of the proof of [39], Theorem 1.20. \(\square\)

The following decomposition theorem is well-known (see [27], Theorem 2.24), which can be viewed as an amenable version of the Koopman-Von Neumann spectrum mixing theorem for \(\mathbb{Z}\)-actions (see [40, 41]).

**Theorem 2.3.** Let \((X, B_X, \mu, G)\) be a G-MPS. Then the Hilbert space \(H = L^2(X, B_X, \mu)\) can be decomposed as

\[
H = L^2(X, K_{\mu}, \mu) \oplus WM(X, G),
\]

where

\[
WM(X, G) = \left\{ f \in H: \lim_{n \to \infty} \frac{1}{|F_n|} \sum_{g \in F_n} |\langle U_g f, h \rangle| = 0 \text{ for all } h \in H \right\}.
\]

**Remark 2.4.** It follows directly from theorems 2.1 and 2.2 that \((X, B_X, \mu, G)\) is weakly mixing if and only if \(K_{\mu}\) is trivial, that is, \(K_{\mu} = \{ X, \emptyset \}\).

Let \((X, G)\) be a G-system and \(\mu \in \mathcal{M}(X, G)\). Given a finite measurable partition \(\alpha\) of \(X\) and a sub-\(\sigma\)-algebra \(A\) of \(B_X\), we define the conditional entropy of \(\alpha\) given \(A\) as

\[
\text{Ent}_{\alpha}\left(\frac{\mu}{A} \right) = -\sum_{s \in \mathcal{P}(\alpha)} \mu(s) \log \mu(s).
\]
\[ H_{\mu}(\alpha|A) = \sum_{A \in \alpha} \int_A -E(1_A|A) \log E(1_A|A) \, d\mu, \]

where \( E(1_A|A) \) denotes the conditional expectation of \( 1_A \) with respect to \( A \). It is well known that \( H_{\mu}(\alpha|A) \) increases with respect to \( \alpha \) and decreases with respect to \( A \). Set \( T = \{ \emptyset, X \} \) and define

\[ H_{\mu}(\alpha) := H_{\mu}(\alpha|T) = \sum_{A \in \alpha} -\mu(A) \log \mu(A). \]

It is easy to check that \( H_{\mu}(\alpha|\beta) = H_{\mu}(\alpha \lor \beta) - H_{\mu}(\beta) \) for any finite measurable partitions \( \alpha \) and \( \beta \). More generally, for a sub-\( \sigma \)-algebra \( A \subset B_X \), we have

\[ H_{\mu}(\alpha \lor \beta|A) = H_{\mu}(\beta|A) + H_{\mu}(\alpha|\beta \lor A). \]

A finite measurable partition \( \alpha \) is finer than another finite measurable partition \( \beta \) if for any \( A \in \alpha \), there exists \( B \in \beta \) such that \( A \subset B \), denoted by \( \alpha \triangleright= \beta \). It is well known that if \( \alpha \triangleright= \beta \) then \( H_{\mu}(\alpha) \geq H_{\mu}(\beta) \).

Given an infinite subset \( S \) of \( G \) and a Følner sequence \( F = \{ F_n \}_{n=1}^{\infty} \), for \( \mu \in \mathcal{M}(X, G) \) and a finite measurable partition \( \alpha \) of \( X \), we define the sequence entropy of \( \alpha \) with respect to \( \mu \) and \( F \) along \( S \) by

\[ h_{\mu}^{SF}(G, \alpha) = \lim_{n \to \infty} \frac{1}{|S \cap F_n|} H_{\mu}\left( \bigvee_{g \in S \cap F_n} g^{-1}\alpha \right). \]

The sequence entropy of \((X, B_X, \mu, G)\) with respect to \( F \) along \( S \) is defined by

\[ h_{\mu}^{SF}(G) = \sup_{\alpha} h_{\mu}^{SF}(G, \alpha), \]

where the supremum is taken over all finite measurable partitions of \( X \).

Following ideas in the proof of similar result for entropy, we can extend it to sequence entropy (see for example [39], theorem 4.22).

**Proposition 2.5.** Let \((X, B_X, \mu, G)\) be a G-MPS. Suppose that \( \{ \alpha_n \}_{n=1}^{\infty} \) is a sequence of finite measurable partitions of \( X \) with \( \alpha_n \triangleright \mathcal{B}_X \). Then for any infinite sequence \( S \) of \( G \), we have

\[ \lim_{n \to \infty} h_{\mu}^{SF}(G, \alpha_n) = h_{\mu}^{SF}(G). \]

Let \((X, G)\) be a G-system and \( S \) an infinite subset of \( G \). For a finite open cover \( \mathcal{U} \) of \( X \), let \( \mathcal{N}(\mathcal{U}) \) denote the minimal cardinality among all cardinalities of subcovers of \( \mathcal{U} \). Then we define the topological sequence entropy of \( \mathcal{U} \) with respect to \( F \) along \( S \) by

\[ h_{\mu}^{SF}(G, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{|S \cap F_n|} \log \mathcal{N}\left( \bigvee_{g \in S \cap F_n} g^{-1}\mathcal{U} \right). \]

The topological sequence entropy of \((X, G)\) with respect to \( F \) along \( S \) is defined by

\[ h_{\mu}^{SF}(G) = \sup_{\mathcal{U}} h_{\mu}^{SF}(G, \mathcal{U}), \]

where the supremum is taken over all finite open covers of \( X \).

**Remark 2.6.** In fact, the sequence entropy can be defined by just picking any Følner sequence of the group \( G \). The reason why we fix a Følner sequence to define the sequence is stated as follows:

1. For \( \mathbb{Z} \)-actions, the topological sequence entropy and metric entropy are defined by fixing the Følner sequence \( \{ [0, n] \}_{n=1}^{\infty} \) and so for \( G \)-actions, we also define them similarly.
2. The notions of mixing and spectrum can be defined by fixing a Følner sequence, and, in fact, the definitions are independent of the choice of Følner sequences. In this paper, we will establish the relation between these notions and sequence entropy. Therefore, to some extent, sequence entropy is independent of the choice of Følner sequences. Then, when we study some properties, we may just fix a single Følner sequence to study them instead of considering all Følner sequences of \( G \).
3. Discrete spectrum measures and sequence entropy

In this section, we give characterizations of discrete spectrum and weakly mixing measures via sequence entropy, which follow the ideas in [7] and [8].

Firstly, we discuss properties of the Kronecker algebra $\mathcal{K}_\mu$. Let us begin with the following lemmas.

**Lemma 3.1.** ([39], lemma 4.15) Let $(X, \mathcal{B}_X, \mu)$ be a Borel probability space and $r \geq 1$ be a fixed integer. For each $\epsilon > 0$, there exists $\delta > 0$ such that if $\alpha = \{A_1, A_2, \ldots, A_n\}$ and $\beta = \{B_1, B_2, \ldots, B_n\}$ are any two finite measurable partitions of $(X, \mathcal{B}_X, \mu)$ with $\sum_{i=1}^n \mu(A_i \triangle B_i) < \delta$, then $H_\mu(\alpha|\beta) + H_\mu(\beta|\alpha) < \epsilon$.

**Lemma 3.2.** Let $(X, \mathcal{B}_X, \mu, G)$ be a G-MPS. Then for any finite measurable partition $\alpha \subset \mathcal{K}_\mu$ and any infinite subset $S$ of $G$, we have $h^{SF}_\mu(G, \alpha) = 0$.

**Proof.** Let $\alpha = \{A_1, A_2, \ldots, A_n\}$ with $A_i \subset \mathcal{K}_\mu$ for all $i = 1, 2, \ldots, n$. Since $\bigvee_{i=1}^n \{A_i \setminus X\} \ni \alpha$, to show that $h^{SF}_\mu(G, \alpha) = 0$ for any infinite subset $S$ of $G$, it suffices to show that $h^{SF}_\mu(G, \{B, X\setminus B\}) = 0$ for any $B \subset \mathcal{K}_\mu$ and any infinite subset $S$ of $G$.

Let $B \subset \mathcal{K}_\mu$ and $\eta = \{B, X\setminus B\}$. Since $\{U_g \mid g \in G\}$ is precompact in $L^2(X, \mathcal{B}_X, \mu)$, for any $\delta > 0$, there exists $N \in \mathbb{N}$ such that for any $g \in S$,

$$\mu\left(g^{-1}B \triangle h_g^{-1}B\right) = \|U_g 1_B - U_{h_g} 1_B\| < \delta$$

for some $h_g \in F_N \cap S$. Thus, by lemma 3.1, one has that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $g \in S$,

$$H_\mu(g^{-1}\eta | h_g^{-1}\eta) + H_\mu(h_g^{-1}\eta | g^{-1}\eta) < \epsilon$$

for some $h_g \in F_N \cap S$. This implies that for any $g \in S$,

$$H_\mu(g^{-1}\eta | \bigvee_{h \in F_N \cap S} h^{-1}\eta) \leq H_\mu(g^{-1}\eta | h_g^{-1}\eta) < \epsilon.$$ 

Furthermore, we have

$$h^{SF}_\mu(G, \eta) = \lim_{n \to \infty} \frac{1}{|F_n \cap S|} H_\mu\left( \bigvee_{g \in F_n \cap S} g^{-1}\eta \right)$$

$$= \lim_{n \to \infty} \frac{1}{|F_n \cap S|} H_\mu\left( \bigvee_{g \in F_n \cap S} g^{-1}\eta \lor \bigvee_{g \in (F_n \setminus F_{n-1}) \cap S} g^{-1}\eta \right)$$

$$= \lim_{n \to \infty} \frac{1}{|F_n \cap S|} \left[ H_\mu\left( \bigvee_{g \in F_n \cap S} g^{-1}\eta \right) + H_\mu\left( \bigvee_{g \in (F_n \setminus F_{n-1}) \cap S} g^{-1}\eta \right) \right]$$

$$\leq \lim_{n \to \infty} \frac{|F_n|}{|F_n \cap S|} H_\mu(\eta) + \frac{1}{|F_n \cap S|} \sum_{h \in (F_n \setminus F_{n-1}) \cap S} H_\mu(h^{-1}\eta | \bigvee_{g \in F_n \cap S} g^{-1}\eta)$$

$$\leq \epsilon.$$ 

Since $\epsilon > 0$ is arbitrary, one has $h^{SF}_\mu(T, \eta) = 0$. This finishes the proof.

**Lemma 3.3.** Let $(X, \mathcal{B}_X, \mu, G)$ be a G-MPS. Then for any finite measurable partition $\alpha$ of $X$ and any infinite subset $S$ of $G$, we have $h^{SF}_\mu(G, \alpha) \leq H_\mu(\alpha|\mathcal{K}_\mu)$.

**Proof.** Since $(X, \mathcal{B}_X)$ is separable, there exist countably many finite measurable partitions $\{\eta_k : k \in \mathbb{N}\} \subset \mathcal{K}_\mu$ such that

$$\lim_{k \to \infty} H_\mu(\alpha|\eta_k) = H_\mu(\alpha|\mathcal{K}_\mu).$$
For a fixed $k \in \mathbb{N}$ and an infinite subset $S$ of $G$, by Lemma 3.2, one has
\begin{equation}
\lim_{n \to \infty} \frac{1}{|E_n \cap S|} I_{E_n} \left( \bigvee_{g \in E_n \cap S} g^{-1} \eta_k \right) = 0. \tag{3.1}
\end{equation}

Therefore, we have
\begin{align*}
h_{\mu}^{S,F}(G, \alpha) &= \limsup_{n \to \infty} \frac{1}{|E_n \cap S|} \left( \bigvee_{g \in E_n \cap S} g^{-1} \alpha \right) \\
&\leq \limsup_{n \to \infty} \frac{1}{|E_n \cap S|} \left( \bigvee_{g \in E_n \cap S} g^{-1} (\alpha \vee \eta_k) \right) \\
&= \limsup_{n \to \infty} \frac{1}{|E_n \cap S|} \left[ \left( \bigvee_{g \in E_n \cap S} g^{-1} \eta_k \right) + \left( \bigvee_{g \in E_n \cap S} g^{-1} |\alpha| \right) \right] \\
&\overset{(3.1)}{=} \limsup_{n \to \infty} \frac{1}{|E_n \cap S|} \left( \bigvee_{g \in E_n \cap S} g^{-1} |\alpha| \right) \\
&\leq \limsup_{n \to \infty} \frac{1}{|E_n \cap S|} \sum_{g \in E_n \cap S} h_\mu(g^{-1} |\alpha| g^{-1} \eta_k) \\
&= H_\mu(\alpha |\eta_k).
\end{align*}

By letting $k \to \infty$, one has $h_{\mu}^{S,F}(G, \alpha) \leq H_\mu(\alpha |K_\mu)$, which completes the proof. □

Furthermore, we have the following result.

**Theorem 3.4.** Let $(X, \mathcal{B}_X, \mu, G)$ be a $G$-MPS. Then for any finite measurable partition $\alpha$ of $X$, there is an infinite subset $S$ of $G$ such that $h_{\mu}^{S,F}(G, \alpha) = H_\mu(\alpha |K_\mu)$.

**Proof.** Note that for any $A \in \mathcal{B}_X$, $1_A - \mathbb{E}(1_A |K_\mu) \in WM(X, G)$. Thus, by Theorem 2.2, there exists a subset $S' = \{s_i\}_{i=1}^\infty$ of $G$ with $D_\mu(S') = 1$ such that
\begin{equation}
\lim_{i \to \infty} \langle U_{s_i} \cdot (1_A - \mathbb{E}(1_A |K_\mu)), 1_B \rangle = 0
\end{equation}
for all $B \in \mathcal{B}_X$.

Given a finite measurable partition $\beta$ of $X$, let $\alpha = \{A_1, A_2, \ldots, A_l\}$ and $\beta = \{B_1, B_2, \ldots, B_j\}$. Then, by the discussion above-mentioned, there exists a subset $S'' = \{s''_i\}_{i=1}^\infty$ of $G$ with $D_\mu(S'') = 1$ such that
\begin{equation}
\lim_{i \to \infty} \langle U_{s''_i} \cdot (1_{A_k} - \mathbb{E}(1_{A_k} |K_\mu)), 1_{B_j} \rangle = 0 \tag{3.2}
\end{equation}
for any $1 \leq k \leq l$ and $1 \leq j \leq t$. Hence
\begin{align*}
\liminf_{i \to \infty} H_\mu(s''_{i-1} \alpha |\beta) \\
&= \liminf_{i \to \infty} \sum_{k,j} \mu(s''_{i-1} A_k \cap B_j) \log \left( \frac{\mu(s''_{i-1} A_k \cap B_j)}{\mu(B_j)} \right) \\
&= \liminf_{i \to \infty} \sum_{k,j} \langle U_{s''_i} 1_{A_k}, 1_{B_j} \rangle \log \left( \frac{\langle U_{s''_i} 1_{A_k}, 1_{B_j} \rangle}{\mu(B_j)} \right) \\
&\overset{(3.2)}{=} \liminf_{i \to \infty} \sum_{k,j} \langle U_{s''_i} \mathbb{E}(1_{A_k} |K_\mu), 1_{B_j} \rangle \log \left( \frac{\langle U_{s''_i} \mathbb{E}(1_{A_k} |K_\mu), 1_{B_j} \rangle}{\mu(B_j)} \right).
\end{align*}
Let
\[ a_{ij}^i = -\langle U_{\alpha}^i \mathbb{E}(1_{A_i} | \mathcal{K}_\mu), 1_{B_j} \rangle \log \left( \frac{\langle U_{\alpha}^i \mathbb{E}(1_{A_i} | \mathcal{K}_\mu), 1_{B_j} \rangle}{\mu(B_j)} \right) \]
and
\[ \mu_{B_j} = \frac{\mu(\cdot \cap B_j)}{\mu(B_j)}. \]

By the concavity of \(-x \log x\), we conclude that
\[
\frac{a_{ij}^i}{\mu(B_j)} = \left( \int_{B_j} U_{\alpha}^i \mathbb{E}(1_{A_i} | \mathcal{K}_\mu) \frac{d\mu}{\mu(B_j)} \right) \log \left( \frac{\int_{B_j} U_{\alpha}^i \mathbb{E}(1_{A_i} | \mathcal{K}_\mu) \frac{d\mu}{\mu(B_j)}}{\mu(B_j)} \right) \\
\geq - \int_{B_j} U_{\alpha}^i \mathbb{E}(1_{A_i} | \mathcal{K}_\mu) \log (U_{\alpha}^i \mathbb{E}(1_{A_i} | \mathcal{K}_\mu)) \frac{d\mu}{\mu(B_j)} \\
= - \int_{B_j} U_{\alpha}^i \mathbb{E}(1_{A_i} | \mathcal{K}_\mu) \log (U_{\alpha}^i \mathbb{E}(1_{A_i} | \mathcal{K}_\mu)) \frac{d\mu}{\mu(B_j)}.
\]

Therefore, we have
\[
\sum_{k,j} a_{kj}^i \geq - \sum_{k,j} \int_{B_j} U_{\alpha}^i \mathbb{E}(1_{A_i} | \mathcal{K}_\mu) \log (U_{\alpha}^i \mathbb{E}(1_{A_i} | \mathcal{K}_\mu)) \frac{d\mu}{\mu(B_j)} \\
= - \sum_{k} \int_{X} U_{\alpha}^i \mathbb{E}(1_{A_i} | \mathcal{K}_\mu) \log (U_{\alpha}^i \mathbb{E}(1_{A_i} | \mathcal{K}_\mu)) \frac{d\mu}{\mu(B_j)} \\
= - \sum_{k} \int_{X} \mathbb{E}(1_{A_i} | \mathcal{K}_\mu) \log (\mathbb{E}(1_{A_i} | \mathcal{K}_\mu)) \frac{d\mu}{\mu(B_j)} \\
= H_\mu(\alpha | \mathcal{K}_\mu).
\]

This shows that
\[
\liminf_{i \to \infty} H_\mu(s_i^{\alpha - 1} | \beta) \geq H_\mu(\alpha | \mathcal{K}_\mu). \tag{3.3}
\]

By using (3.3) repeatedly, we can obtain inductively an infinite subset \(S = \{s_i\}_{i=1}^\infty\) of \(G\) such that for each \(i \in \mathbb{N}\), one has
\[ H_\mu \left( s_i^{\alpha - 1} \bigvee_{j=1}^{i-1} s_j^{\alpha} \right) \geq H_\mu(\alpha | \mathcal{K}_\mu) - \frac{1}{2^i}. \]

For each \(n \in \mathbb{N}\), there exists \(1 \leq i_1 < i_2 < \ldots < i_k \) such that
\[ E_n \cap S = \{s_{i_1}, s_{i_2}, \ldots, s_{i_k}\}. \]

Hence
\[
H_\mu \left( \bigvee_{g \in E_n \cap S} g^{-1} \alpha \right) = H_\mu \left( \bigvee_{j=1}^{k_x} s_{i_j}^{\alpha - 1} \right) \\
= H_\mu(s_{i_1}^{-1} \alpha) + H_\mu(s_{i_2}^{-1} \alpha | s_{i_1}^{-1} \alpha) + \ldots + H_\mu \left( s_{i_k}^{-1} \alpha \bigvee_{j=1}^{i_{k-1}} s_{i_j}^{-1} \alpha \right) \\
\geq \sum_{i=1}^{k_x} H_\mu(s_{i_j}^{-1} \alpha \bigvee_{j=1}^{i_{k-1}} s_{i_j}^{-1} \alpha) \\
\geq \sum_{i=1}^{k_x} \left( H_\mu(\alpha | \mathcal{K}_\mu) - \frac{1}{2^i} \right) \geq |E_n \cap S| \cdot H_\mu(\alpha | \mathcal{K}_\mu) - 1.
\]
Therefore, we have
\[
\tilde{h}^{S_F}_\mu(G, \alpha) = \limsup_{n \to \infty} \frac{1}{|E_n \cap S|} H_\mu \left( \bigvee_{g \in E_n \cap S} g^{-1} \alpha \right) \
\geq \limsup_{n \to \infty} \frac{|E_n \cap S|}{|E_n \cap S|} H_\mu(\alpha | \mathcal{K}_n) - 1 = H_\mu(\alpha | \mathcal{K}_n).
\]
This finishes the proof of theorem 3.4. \hfill \Box

We say that the G-MPS \((X, \mathcal{B}_X, \mu, G)\) is null with respect to a Følner sequence \(F\), if \(\tilde{h}^{S_F}_\mu(G) = 0\) for any infinite subset \(S\) of \(G\). By lemma 3.3 and theorem 3.4, we can obtain the following characterization of discrete spectrum via sequence entropy.

**Theorem 3.5.** Let \((X, \mathcal{B}_X, \mu, G)\) be a G-MPS. Then the following two conditions are equivalent:

1. \(\mu\) has discrete spectrum;
2. \((X, \mathcal{B}_X, \mu, G)\) is null with respect to any Følner sequence \(F\);
3. \((X, \mathcal{B}_X, \mu, G)\) is null with respect to some Følner sequence \(F\).

In particular, whether or not a G-MPS is null is independent of the choice of Følner sequences.

**Proof.** (1) \(\Rightarrow\) (2). Assume that \(\mu\) has discrete spectrum, that is, \(\mathcal{K}_\mu = \mathcal{B}_X\). Then for any Følner sequence \(F\), by lemma 3.3, \(\tilde{h}^{S_F}_\mu(G, \alpha) \leq H_\mu(\alpha | \mathcal{B}_X) = 0\) for any finite measurable partition \(\alpha\) and any infinite subset \(S\) of \(G\), which implies \((X, \mathcal{B}_X, \mu, G)\) is null with respect to \(F\).

(2) \(\Rightarrow\) (3). This is trivial.

(3) \(\Rightarrow\) (1). Assume that \((X, \mathcal{B}_X, \mu, G)\) is null with respect to some Følner sequence \(F\). Given \(B \in \mathcal{B}_X\), let \(\alpha = \{B, X \setminus B\}\). Then by theorem 3.4, there is an infinite subset \(S\) of \(G\) such that
\[
H_\mu(\alpha | \mathcal{K}_n) = \tilde{h}^{S_F}_\mu(G, \alpha) = 0,
\]
which implies \(B \in \mathcal{K}_\mu\). Therefore, \(\mathcal{K}_\mu = \mathcal{B}_X\), that is, \(\mu\) has discrete spectrum. \hfill \Box

**Remark 3.6.** Given a G-MPS \((X, \mathcal{B}_X, \mu, G)\), if its Koopman representation is compact (see definition 2.22 in [27]), then \(\mu\) has discrete spectrum and hence its sequence entropy is zero.

By a proof similar to that of theorem 3.5, we can obtain a characterization of weak mixing via sequence entropy.

**Theorem 3.7.** Let \((X, \mathcal{B}_X, \mu, G)\) be a G-MPS. Then the following three conditions are equivalent:

1. \((X, \mathcal{B}_X, \mu, G)\) is weakly mixing;
2. for any finite measurable partition \(\alpha\) of \(X\), there exists some infinite subset \(S\) of \(G\) such that \(h^{S_F}_\mu(G, \alpha) = H_\mu(\alpha)\);
3. for any non-trivial finite measurable partition \(\alpha\) of \(X\), there exists some infinite subset \(S\) of \(G\) such that \(h^{S_F}_\mu(G, \alpha) > 0\).

**Remark 3.8.** As an example of G-MPS with positive sequence entropy, we consider the Bernoulli actions. Let \(Y\) be a Polish space. We consider the product topological space \(Y^G\), which is also Polish. The product topology on \(Y^G\) is generated by the cylinder sets \(\prod_{s \in G} A_s\), where each \(A_s\) is open and \(A_s = Y\) for all \(s\) outside of a finite subset of \(G\). This generates the Borel \(\sigma\)-algebra on \(Y^G\). Let \(\nu\) be a Borel probability measure on \(Y\). One can show that there is a unique Borel probability measure \(\nu^G\) on \(Y^G\) by Kolmogorov’s extension theorem. Now we define the action \(G\) on \(Y^G\) by \((x_t)_s = x_{s+t}\) for all \(s, t \in G\) and \(x \in Y^G\). This action preserves the measure \(\nu^G\), and it is called a Bernoulli action. Then \((Y^G, \mathcal{B}_Y^G, \nu^G, G)\) is weak mixing (see for example [27], Page 38). Thus, by theorem 3.7, this system has positive sequence entropy.

### 4. Topological mixing and topological sequence entropy

In this section, we characterize topological weak mixing, and provide some necessary conditions of strong mixing and mild mixing for amenable group actions via topological sequence entropy.
4.1. Weak mixing
We say that a $G$-system $(X, G)$ is (topologically) transitive if for every two nonempty open subsets $U$ and $V$ of $X$,

$$N(U, V) = \{g \in G : U \cap g^{-1}V \neq \emptyset\}$$

is nonempty. A $G$-system $(X, G)$ is called (topologically) weakly mixing if the product system $(X \times X, G)$ is transitive, i.e., for any four nonempty open sets $U_1, U_2, V_1, V_2$ of $X$,

$$N(U_1 \times U_2, V_1 \times V_2) = N(U_1, V_1) \cap N(U_2, V_2) = \emptyset.$$

Or explicitly, there exists a $g \in G$ with $U_1 \cap g^{-1}V_1 = \emptyset$ and $U_2 \cap g^{-1}V_2 = \emptyset$. Clearly every weakly mixing system is transitive.

In order to characterize the weak mixing via sequence entropy, we need the following result.

**Theorem 4.1.** ([37], theorem 1.11 or [42]) For an Abelian group $G$ and a $G$-system $(X, G)$, the following conditions are equivalent:

1. $(X, G)$ is weakly mixing;
2. for any nonempty open sets $U, V$, $N(U, V)$ is nonempty and for every four nonempty open sets $U_1, U_2, V_1, V_2 \subset X$, there exist nonempty open sets $U, V$ with $N(U_1, V_1) \subset N(U, V) \cap N(U_2, V_2)$;
3. $N(U, U) \cap N(U, V) = \emptyset$ for every nonempty open sets $U, V$ of $X$;

Recall that an open cover $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ of $X$ is called non-trivial if $U_i$ is not dense in $X$ for every $1 \leq i \leq n$, and standard if $n = 2$. An open cover $\mathcal{U} = \{U_i\}_{i \in I}$ is called admissible if $U_i \setminus \left( \bigcup_{j \in I, i \neq j} U_j \right)$ has nonempty interior for each $i \in I$.

Now we provide a characterization of weakly mixing systems via sequence entropy.

**Theorem 4.2.** Let $G$ be an Abelian group. Then for every $G$-system $(X, G)$, the following statements are equivalent:

1. $(X, G)$ is weakly mixing;
2. for each admissible open cover $\mathcal{U}$, there exists an infinite $S \subset G$ such that $h_{\log}^{SF}(G, \mathcal{U}) = \log N(\mathcal{U})$;
3. for each non-trivial finite open cover $\mathcal{U}$, there exists an infinite $S \subset G$ such that $h_{\log}^{SF}(G, \mathcal{U}) > 0$;
4. for each standard open cover $\mathcal{U}$, there exists an infinite $S \subset G$ such that $h_{\log}^{SF}(G, \mathcal{U}) > 0$.

**Proof.** (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (1). By contradiction, we assume that $(X, G)$ is not weakly mixing. Then by theorem 4.1 there exist nonempty open sets $U_1$ and $U_2$ such that

$$N(U_1, U_2) \cap N(U_1, U_2) = \emptyset.$$ (4.1)

Clearly, $U_1 \cap U_2 = \emptyset$. Since $X$ is compact, for each $i = 1, 2$, we take a closed subset $V_i \subset U_i$ with nonempty interior. Then $V = \{X \setminus V_1, X \setminus V_2\}$ is a standard open cover of $X$. By (4.1), one has that for any $g \in G$ we have $U_1 \cap g^{-1}U_1 = \emptyset$ or $U_1 \cap g^{-1}U_1 = \emptyset$. Thus, for any $g \in G$, there exists $W_g = X \setminus V_1$ or $W_g = X \setminus V_2$ such that $V_i \subset g^{-1}W_g$.

For any infinite set $S \subset G$ and $n \in \mathbb{N}$, set $S \cap F_n = \{g_1, g_2, \ldots, g_n\}$. If a point $x$ does not belong to $\mathcal{V}_0 = \bigcap_{i=1}^n g_i^{-1}(X \setminus V_i)$, then there exists $i \in \{1, 2, \ldots, k_n\}$ such that $g_ix \not\in V_i$, which implies that for any $g \in G$, we have $g_ix \in g^{-1}W_g$. In particular, letting $g = g_i g_j^{-1} (j = 1, 2, \ldots, k_n)$ we have

$$g_i x \in g_i g_j^{-1} W_{g_i g_j^{-1}}$$

which implies that

$$x \in g_j^{-1} W_{g_i g_j^{-1}}.$$
Thus,

\[ x \in \mathcal{V}_i = \bigcap_{j=1}^{k_n} g_j^{-1}W_{g_j}^{-1}. \]

This set depends only on \( i \), so we have a subcover consisting of \( k_n + 1 \) sets \( \mathcal{V}_i, i = 0, 1, \ldots, k_n \).

Therefore, for all \( n \in \mathbb{N} \), we have

\[ \mathcal{N}\left( \bigvee_{g \in G \cap F_n} g^{-1}\mathcal{V}_i \right) \leq |S \cap F_n| + 1. \]

This implies that \( h_{\text{top}}^{SF}(G, \mathcal{V}) = 0. \)

(1) \( \Rightarrow \) (2). Assume that \((X, G)\) is weakly mixing and \( \mathcal{U} = \{ U_0, U_1, \ldots, U_l \}\) is an admissible open cover. Let

\[ W_i = \text{int}(U_i \setminus \bigcup_{j=1}^{l} U_j) \text{ for each } i = 1, 2, \ldots, l. \]

Then \( W_1, W_2, \ldots, W_l \) are pairwise disjoint nonempty open sets of \( X \).

Claim. There exists a sequence \( S = \{ g_n \}_{n=1}^{\infty} \) of distinct elements of \( G \) and an increasing sequence \( \{ m_n \}_{n=1}^{\infty} \) of positive integers such that, for each \( n \geq 1 \), \( g_n \in F_{m_n+1} \setminus F_{m_n} \) and, for any \( s \in \{ 1, 2, \ldots, l \}^n \),

\[ \bigcap_{i=1}^{n} g_i^{-1}W_i(\mathcal{U}) \neq \emptyset \text{ for all } s \in \{ 1, 2, \ldots, l \}^n. \]

Let \( m_{k+1} \) be large enough, so that \( g_1 \ldots g_k \in F_{m_{k+1}} \). By theorem 4.1 (2), we have

\[ \bigcap_{s \in \{ 1, 2, \ldots, l \}^n} \bigcap_{i=1}^{l} \mathcal{N}\left( \bigcap_{j=1}^{n} g_i^{-1}W_i(\mathcal{U}) \right) \]

is an infinite subset of \( G \), and hence there exists \( g_{k+1} \notin F_{m_{k+1}} \) with \( g_{k+1} \perp g_i \) for each \( i = 1, 2, \ldots, k \) such that

\[ g_{k+1} \in \bigcap_{s \in \{ 1, 2, \ldots, l \}^n} \bigcap_{i=1}^{l} \mathcal{N}\left( \bigcap_{j=1}^{n} g_i^{-1}W_i(\mathcal{U}) \right). \]

Through this iterative process, we obtain the sequence \( S = \{ g_n \}_{n=1}^{\infty} \) and an increasing sequence \( \{ m_n \}_{n=1}^{\infty} \) of positive integers satisfying the claim. \( \square \)

By the above claim, we have

\[ h_{\text{top}}^{SF}(G, \mathcal{U}) = \limsup_{n \to \infty} \frac{1}{|F_n \cap S|} \log \mathcal{N}\left( \bigvee_{g \in G \cap F_n} g^{-1}\mathcal{U} \right) \]

\[ \geq \limsup_{n \to \infty} \frac{1}{|F_{m_{n+1}} \cap S|} \log \mathcal{N}\left( \bigvee_{g \in G \cap F_{m_{n+1}}} g^{-1}\mathcal{U} \right) \]

\[ = \limsup_{n \to \infty} \frac{1}{|G_n|} \log \mathcal{N}\left( \bigvee_{g \in G_n} g^{-1}\mathcal{U} \right) \]

\[ = \limsup_{n \to \infty} \frac{1}{n} \log n = \log l = \log \mathcal{N}(\mathcal{U}). \]

Meanwhile, as \( h_{\text{top}}^{SF}(G, \mathcal{U}) \leq \log \mathcal{N}(\mathcal{U}) \), we have \( h_{\text{top}}^{SF}(G, \mathcal{U}) = \log \mathcal{N}(\mathcal{U}). \)

(1) \( \Rightarrow \) (3). Let \( \mathcal{V} = \{ V_0, V_2, \ldots, V_l \} \) be a non-trivial finite open cover. Take \( x_i \in \text{int}(X \setminus V_i) \) for every \( 1 \leq j \leq k \).

Set \( \{ y_1, y_2, \ldots, y_l \} = \{ x_1, x_2, \ldots, x_l \} \) with \( y_i = y_j \) for \( 1 \leq s < t \leq l \). Clearly, \( l \geq 2 \) and we can take pairwise disjoint closed neighborhood \( W_i \) of \( y_i \), \( i = 1, 2, \ldots, l \), such that the open cover \( \mathcal{U} = \{ X \setminus W_1, X \setminus W_2, \ldots, X \setminus W_l \} \) is coarser than \( \mathcal{V} \). Given any finite subset \( F \) of \( G \), let \( \mathcal{P} \) be the subcover of \( \bigvee_{g \in G} g^{-1}\mathcal{U} \) with the minimum cardinality. Thus, for any \( s \in \{ 1, 2, \ldots, n \}^{\mathcal{P}} \), if
then there exist \( V \in \mathcal{P} \) with the form

\[
V = \bigcap_{g \in F} X \setminus W_{(g)}
\]

for some \( s' \in \{1, 2, \ldots, n\}^{[F]} \) with \( s'(g) = s(g) \) for all \( g \in F \).

such that \( V \) covers \( \bigcap_{g \in F} g^{-1}W_{(g)} \). Note that such \( V \) cannot cover \( \bigcap_{g \in F} g^{-1}W_{(g)} = \emptyset \) with \( s(g) = s'(g) \) for some \( g \in F \). Thus, each such \( V \in \mathcal{P} \) covers at most \( (l - 1)^{|F|} \) sets for \( gF \) such that \( V \) covers \( \bigcap_{g \in F} g^{-1}W_{(g)} \). Note that such \( V \) cannot cover \( \bigcap_{g \in F} g^{-1}W_{(g)} = \emptyset \) for some \( g \). Thus, each such \( V \) covers at most \( (l - 1)^{|F|} \) sets for \( gF \) such that \( V \) covers \( \bigcap_{g \in F} g^{-1}W_{(g)} \).

By a proof of similar that of claim in \((1) \Rightarrow (2)\), there exists a sequence \( \{s_i\} = \infty_{i=1} \) of distinct elements of \( G \) and an increasing sequence \( \{m_n\} = \infty_{n=1} \) of positive integers such that for each \( n \geq 1 \), \( g_n \in F_{m_{n+1}} \setminus F_{m_n} \) and for any \( s \in \{1, 2, \ldots, l\}^l \),

\[
\bigcap_{i=1}^n g_i^{-1}W_{(i)} = \emptyset.
\]

Therefore, we have

\[
h_{\text{top}}^{S, F}(G, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{|F_n \cap S|} \log N\left( \bigvee_{g \in F \cap S} g^{-1}\mathcal{U} \right)
\]

\[
\geq \lim_{n \to \infty} \frac{1}{|F_{m_{n+1}} \cap S|} \log N\left( \bigvee_{g \in F_{m_{n+1}} \cap S} g^{-1}\mathcal{U} \right)
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \log N\left( \bigvee_{i=1}^n \bigwedge_{i=1}^n g_i^{-1}\mathcal{U} \right)
\]

\[
\geq \lim_{n \to \infty} \frac{1}{n} \log \left| \left\{ s \in \{1, 2, \ldots, l\}^l : \bigcap_{i=1}^n g_i^{-1}W_{(i)} = \emptyset \right\} \right|
\]

\[
= \log \frac{l}{l - 1} > 0.
\]

So \( h_{\text{top}}^{S, F}(G, \mathcal{U}) \geq h_{\text{top}}^{S, F}(G, \mathcal{U}) > 0. \)

### 4.2. Strong mixing

In this subsection, we discuss the relation between strong topological mixing and sequence entropy. Let us recall that a \( G \)-system \((X, G)\) is called strongly (topologically) mixing if for every two nonempty open subsets \( U \) and \( V \) of \( X, N(U, V) \) is cofinite, i.e., \( \{g \in G : U \cup g^{-1}V = \emptyset\} \) is finite. This definition is classical, see [43, 44]. The classical strongly mixing example is the topological Bernoulli system.

We have the following result.

**Theorem 4.3.** Let \((X, G)\) be a \( G \)-system. If \((X, G)\) is strongly mixing, then the following properties hold:

1. for each admissible open cover \( \mathcal{U} \) and any infinite subset \( H \) of \( G \), there exists an infinite subset \( S \subset H \) such that \( h_{\text{top}}^{S, F}(G, \mathcal{U}) = \log N(\mathcal{U}) \);
2. for each non-trivial finite open cover \( \mathcal{U} \) and any infinite subset \( H \) of \( G \), there exists an infinite subset \( S \subset H \) such that \( h_{\text{top}}^{S, F}(G, \mathcal{U}) > 0. \)
Proof. (1). Assume that \((X, G)\) is strongly mixing and let \(\mathcal{U} = \{U_1, U_2, \ldots, U_l\}\) be an admissible open cover. Let

\[
W_i = \operatorname{int} \left( U_i \setminus \bigcup_{j \neq i} U_j \right) \quad \text{for each} \quad i = 1, 2, \ldots, l.
\]

Then \(W_1, W_2, \ldots, W_l\) are pairwise disjoint nonempty open sets of \(X\).

Now we can prove the following claim.

Claim. There exists a sequence \(S = \{g_n\}_{n=1}^\infty\) of distinct elements of \(H\) and an increasing sequence \(\{m_n\}_{n=1}^\infty\) of positive integers such that for each \(n \geq 1\), \(g_j \in F_{m_{n+1}} \setminus F_{m_n}\) and, for any \(s \in \{1, 2, \ldots, l\}^n\),

\[
\bigcap_{i=1}^n g_i^{-1} W_{i(s)} = \emptyset.
\]

Proof. Proof of Claim.

We use induction on \(n\). It is obvious that the claim holds for \(n = 1\). Assume that the claim holds for \(n = k\), next we want to show that the claim also holds for \(n = k + 1\). By the assumption, there exist distinct elements \(g_1, g_2, \ldots, g_k\) of \(H\) and positive integers \(m_1 < m_2 < \ldots < m_k\) such that for each \(i \in \{1, 2, \ldots, k\}\), \(g_i \in F_{m_{i+1}} \setminus F_{m_i}\) and

\[
\bigcap_{i=1}^n g_i^{-1} W_{i(s)} = \emptyset \quad \text{for all} \quad s \in \{1, 2, \ldots, l\}^n.
\]

Let \(m_{k+1}\) be large enough, so that \(g_1, \ldots, g_k \in F_{m_{k+1}}\). Since \(N(U, V)\) is cofinite for any nonempty open sets \(U, V\) of \(X\), it follows that

\[
\bigcap_{s \in \{1, 2, \ldots, l\}^n} \bigcap_{j=1}^l N \left( \bigcap_{i=1}^n g_i^{-1} W_{i(s)} \right) W_j
\]

is also cofinite, and hence

\[
\bigcap_{s \in \{1, 2, \ldots, l\}^n} \bigcap_{j=1}^l N \left( \bigcap_{i=1}^n g_i^{-1} W_{i(s)} \right) \cap H
\]

is an infinite subset of \(G\). Thus, there exists \(g_{k+1} \in H \setminus F_{m_{k+1}}\) with \(g_{k+1} \not\equiv g_i\) for each \(i = 1, 2, \ldots, k\) such that

\[
g_{k+1} \in \bigcap_{s \in \{1, 2, \ldots, l\}^n} \bigcap_{j=1}^l N \left( \bigcap_{i=1}^n g_i^{-1} W_{i(s)} \right) W_j.
\]

Through this iterative process, we obtain the sequence \(S = \{g_n\}_{n=1}^\infty\) and an increasing sequence \(\{m_n\}_{n=1}^\infty\) of positive integers satisfying the claim. \(\square\)

By the above claim, we have

\[
h_{\text{top}}(G, \mathcal{U}) = \limsup_{n \to \infty} \frac{1}{|F_k \cap S|} \log N \left( \bigvee_{g \in F_k \cap S} g^{-1} \mathcal{U} \right) \geq \limsup_{n \to \infty} \frac{1}{|F_{m_{n+1}} \cap S|} \log N \left( \bigvee_{g \in F_{m_{n+1}} \cap S} g^{-1} \mathcal{U} \right) = \limsup_{n \to \infty} \frac{1}{|G_n|} \log N \left( \bigvee_{g \in G_n} g^{-1} \mathcal{U} \right) = \limsup_{n \to \infty} \frac{1}{n} \log l^n = \log l = \log N(\mathcal{U}).
\]

Since \(h_{\text{top}}(G, \mathcal{U}) \leq \log N(\mathcal{U})\), we have \(h_{\text{top}}(G, \mathcal{U}) = \log N(\mathcal{U})\).

Statement (2) is proved by the same argument as that of (1) \(\Rightarrow\) (3) in theorem 4.2. \(\square\)

Remark 4.4. The converse of theorem 4.3 is not true even if \(G = \mathbb{Z}\) (see for example [20, 45]).

4.3. Mild mixing

In this section, we investigate mild mixing via sequence entropy. Let us begin with some notations and definitions. Given a \(G\)-system \((X, G)\), a point \(x \in X\) is called

1. a transitive point of \((X, G)\) if \([gX : g \in G] = X\), and denote by \(\text{Trans}(X, G)\) the set of all transitive points;
2. a recurrent point of \((X,G)\) if for any neighborhood \(U\) of \(x\), \(\{g \in G : gx \in U\}\) is an infinite set, and denote by \(\text{Rec}(X, G)\) the set of all recurrent points.

A \(G\)-system \((X, G)\) is said to be \textit{transitive recurrent} if \(\text{Trans}(X, G) \cap \text{Rec}(X, G) = \emptyset\).

**Proposition 4.5.** A \(G\)-system \((X, G)\) is transitive recurrent if and only if it is infinite transitive, i.e., \(N(U, V)\) is infinite for every two nonempty open subsets \(U\) and \(V\) of \(X\).

**Proof.** Assume that \((X, G)\) is transitive recurrent and \(x \in \text{Trans}(X, G) \cap \text{Rec}(X, G)\). Then \(N(x, U)\) is infinite for every nonempty open subset \(U\) of \(X\), where \(N(x, U) = \{g \in G : gx \in U\}\). Thus, for every pair nonempty open subsets \(U\) and \(V\) of \(X\), \(N(U, V) = N(x, V)^{-1}\) is infinite.

Conversely, suppose that \((X, G)\) is infinite transitive, that is, \(N(U, V)\) is infinite for every two nonempty open subsets \(U\) and \(V\) of \(X\). Then \(N(x, U)\) is infinite for every \(x \in \text{Trans}(X, G)\) and every nonempty open subsets \(U\) of \(X\). Thus, \(\emptyset = \text{Trans}(X, G) \subseteq \text{Rec}(X, G)\), which implies that \((X, G)\) is transitive recurrent.

A \(G\)-system \((X, G)\) is called \textit{mildly mixing} if \((X \times Y, G)\) is transitive recurrent for any transitive recurrent system \((Y, G)\). From Proposition 4.5 and the definition, it is easy to check that

\[
\text{strong mixing} \Rightarrow \text{mild mixing} \Rightarrow \text{weak mixing.} \quad (4.2)
\]

**Remark 4.6.** (1) Recall that for a \(\mathbb{Z}_{\omega}\)-system \((X,T)\), the transitivity is defined by for any nonempty open subsets \(U\) and \(V\), \(
\{ n \in \mathbb{N} : U \cap T^{-n} V = \emptyset \}
\) is nonempty. It well known that if \((X, T)\) is transitive, then \(\text{Trans}(X, T)\) is a dense \(G_0\)-set of \(X\), and \(\text{Trans}(X, T) \subseteq \text{Rec}(X, T)\). Clearly, if \(\text{Trans}(X, T) \cap \text{Rec}(X, T) = \emptyset\) then \((X,T)\) is transitive. Thus, for \(\mathbb{Z}_{\omega}\)-actions, transitivity is equivalent to transitive recurrent.

(2) For a \(\mathbb{Z}_{\omega}\)-system \((X, T)\), it is called mild mixing if \((X \times Y, T \times S)\) is transitive for any transitive system \((Y, S)\) (see \([18]\) or \([20]\)). However, for group actions, the reviewer pointed out to us the following claim:

\textit{Claim.} The only topologically mildly mixing system (according to the definition of mild mixing for \(\mathbb{Z}_{\omega}\)-actions in \([18]\) or \([20]\)) is the trivial system.

**Proof.** Proof of the Claim.

Let \(G\) be any nontrivial countable group and let \(X\) be the one-point compactification of \(G\) viewed as a discrete set (if \(G\) is finite then we let \(X = G\)). If we agree that \(g \cdot \infty = \infty\) for any \(g \in G\), then \(G\) acts on \(X\) by homeomorphisms, via left multiplication. Clearly, \((X, G)\) is transitive, with any \(x = \infty\) (i.e., \(x \in G\)) being a transitive point.

Now, let \((Y, G)\) be any \(G\)-action, where \(Y\) has at least two different points. Suppose that \((Y, G)\) is topologically mildly mixing. Then the product \(X \times Y\) should be transitive under the product action of \(G\). Let \((x_0, y_0)\) be a transitive point in \(X \times Y\). Clearly, we have \(x_0 = \infty\), i.e., \(x_0 = g_0 \in G\) and there exists \(y_1 = y_2\) in \(Y\). By transitivity, exists a sequence \(\{g_n\}_{n \geq 1}\) in \(G\) such that \(g_n(x_0, y_0) = (g_n g_0, g_n y_0)\) converges to \((g_0, y_1)\). In particular, \(g_n g_0\) converges to \(g_0\). Since \(G\) is discrete, it must be that \(g_n = e_G\) (the unit of \(G\)) for large enough \(n\). But then \(g_n(x_0, y_0) = (x_0, y_0)\) which does not converge to \((x_0, y_1)\), and we have a contradiction.

Denote by \(\mathcal{P}_I(N)\) the set of all finite nonempty subsets of \(N\). Let \(G\) be a group and \(\{p_i\}_{i=1}^\infty\) be a sequence in \(G\). Define

\[
\text{FP}(\{p_i\}_{i=1}^\infty) = \left\{ \prod_{i \in F} p_i : F \in \mathcal{P}_I(N) \right\}, \quad \text{where} \quad \prod_{i \in F} p_i = p_{i_1} \cdots p_{i_k}
\]

for \(F = \{n_1, n_2, \ldots, n_k\} \in \mathcal{P}_I(N)\) with \(n_1 < n_2 < \cdots < n_k\). For each \(n \in \mathbb{N}\), the initial \(n\)-segment of \(\text{FP}(\{p_i\}_{i=1}^\infty)\) is defined as

\[
\text{FP}(\{p_i\}_{i=1}^n) = \left\{ \prod_{i \in F} p_i : F \in \mathcal{P}_I(\{1, 2, \ldots, n\}) \right\}.
\]

A subset \(S\) of \(G\) is called an \textit{infinite IP-set} if there exists a sequence \(\{p_i\}_{i=1}^\infty\) in \(G\) such that \(\text{FP}(\{p_i\}_{i=1}^\infty)\) is infinite and \(\text{FP}(\{p_i\}_{i=1}^\infty) \subseteq S\).

To study the mild mixing by sequence entropy, we need the following version of Furstenberg’s realization theorem of IP-sets, which was generalized from the additive semigroup of positive integers (see \([2]\), theorem 2.17) to discrete groups (see \([46]\), theorem 2.8).
Lemma 4.7. Let $G$ be an infinite discrete group and $A$ be an infinite IP-set of $G$. Then there exists a $G$-system $(X,G)$, $x \in \text{Rec}(X,G)$, and a neighborhood $U$ of $x$ such that
\[ \{ g \in G : gx \in U, \ g \neq e \} \subset A. \]

With the help of lemma 4.7, we have the following result to describe the set $N(U,V)$ for any nonempty open sets $U,V$ of $X$.

Lemma 4.8. Let $(X,G)$ be a $G$-system. If $(X,G)$ is mildly mixing, then $N(U,V) \cap AA^{-1}$ is infinite for every pair of nonempty open subsets $U,V$ of $X$ and every infinite IP-set $A$ of $G$.

Proof. Fix a pair of nonempty open subsets $U,V$ of $X$ and an infinite IP-set $A$ of $G$. By lemma 4.7, there exists a $G$-system $(Y,G)$, $y \in \text{Rec}(Y,G)$, and an open neighborhood $W$ of $y$ such that
\[ \{ g \in G : gy \in W, \ g \neq e \} \subset A. \]

Let $Z = \{ gy : g \in G \}$ be a $G$-invariant closed subset of $Y$. Then
\[ y \in \text{Rec}(Z,G) \cap \text{Trans}(Z,G) \]

(4.3)

and
\[ \{ g \in G : gy \in W \cap Z, \ g \neq e \} = \{ g \in G : gy \in W, \ g \neq e \} \subset A. \]

(4.4)

By (4.3), $(Z,G)$ is recurrent transitive, and hence $(X \times Z,G)$ is transitive recurrent, as $(X,G)$ is mildly mixing. Thus,
\[ N(U,V) \cap N(W,W) = N(U \times W, V \times W) \]

(4.5)

Now we prove that $N(W,W) \subset AA^{-1}$. Indeed, for any $g \in N(W,W)$, $W \cap g^{-1}W$ is a nonempty open subset. By (4.3), there exists $h \in G$ with $h = e_G$, $g^{-1}$ such that $hy \in W \cap g^{-1}W$, that is, $gy, ghy \in W$. Since $h, gh \in e_G$, follows from (4.4) that $h, gh \in A$, which implies that $g = (gh)h^{-1} \in AA^{-1}$. By (4.5), we have that $N(U,V) \cap AA^{-1}$ is infinite. This finishes the proof of the lemma.

With the help of lemma 4.8, we obtain the following description of topologically mildly mixing systems.

Theorem 4.9. Let $(X,G)$ be a $G$-system. If $(X,G)$ is mildly mixing, then the following properties hold:

1. for each admissible open cover $U$ and each infinite IP-set $H$, there exists an infinite subset $S \subset H$ such that $h^S_{\text{top}}(G,U) = \log N(U)$;

2. for each non-trivial finite open cover $U$ and each infinite IP-set $H$, there exists an infinite subset $S \subset H$ such that $h^S_{\text{top}}(G,U) > 0$.

Proof. Assume that $(X,G)$ is mildly mixing, let $U = \{ U_0, U_2, \ldots, U_l \}$ be an admissible open cover and let $H = \text{FP}(\{ p_i \})$ be an IP-set. Let
\[ W_i = \text{int} \left( U_i \setminus \bigcup_{j \neq i} U_j \right) \]

for each $i = 1, 2, \ldots, l$.

Then $W_0, W_2, \ldots, W_l$ are pairwise disjoint nonempty open sets of $X$.

Claim. Fix a finite subset $F$ of $G$. For any $n \in \mathbb{N}$, there exists a subset $C_n = \{ g_{n,n}, g_{2,n}, \ldots, g_{m,n} \} \subset H$ such that $C_n \cap F = \emptyset$ and
\[ \bigcap_{g \in C_n} g^{-1}W_{g(s)} = \emptyset \]

for any $s = \{ 1, 2, \ldots, l \}$.\]

Proof of Claim

We use induction on $n$. It is obvious that the claim holds for $n = 1$. Assume that the claim holds for $n = k$. Next we want to show that the claim also holds for $n = k + 1$. By the assumption, there exists a finite subset $C_k = \{ g_{k,k}, g_{2,k}, \ldots, g_{m,k} \} \subset H$ such that $C_k \cap F = \emptyset$ and
\[ \bigcap_{g \in C_k} g^{-1}W_{g(s)} = \emptyset \]

for any $s = \{ 1, 2, \ldots, l \}$.\]

Take $m_k \in \mathbb{N}$ large enough such that $C_k \subset \text{FP}(\{ p_i \})$. Let $H_{m_k} = \text{FP}(\{ p_i \})$. From (4.2), the system $(X,G)$ is weakly mixing, which together with theorem 4.1, implies that there exist two nonempty open sets $U,V$ with

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Thus, by lemma 4.8, we have

$$N(U, V) \subset \bigcap_{s \in \{1, 2, \ldots, l\}^{q_k}} \bigcap_{j=1}^{l} N_{s \in \mathbb{C}^{k}} \left( \bigcap_{g \in \mathbb{C}^{k}} g^{-1}W_{(s,g)} W_{j} \right).$$

is infinite, and then there exist $g_1, g_2 \in H_{m_k}$ such that

$$g_1 g_2^{-1} \in \bigcap_{s \in \{1, 2, \ldots, l\}^{q_k}} \bigcap_{j=1}^{l} N_{s \in \mathbb{C}^{k}} \left( \bigcap_{g \in \mathbb{C}^{k}} (g_1 g_2^{-1})^{-1}W_{(s,g)} \right), \quad g_1 g_2^{-1} \notin C_{k} \text{ and } g_1, g_1 g_2 \notin F.$$

Let $g_{k+1} = g_k g_2^{-1}, \quad i = 1, 2, \ldots, k$ and $g_{k+1, k+1} = g_{1}$. Then

$$C_{k+1} = \{g_{1, k+1}^{-1}, g_{2, k+1}, \ldots, g_{k+1, k+1}\} \subset H,$$

$$C_{k+1} \cap F = \emptyset \text{ and for any } s \in \{1, 2, \ldots, l\}^{q_k} \text{ and every } j = 1, 2, \ldots, l,$$

$$\bigcap_{g \in \mathbb{C}^{k}} g^{-1}W_{(s,g)} g_{2} g_{1}^{-1} W_{j} \neq \emptyset,$$

and so

$$\bigcap_{g \in \mathbb{C}^{k}} g^{-1}W_{(s,g)} = \emptyset \text{ for all } s \in \{1, 2, \ldots, l\}^{q_{k+1}}.$$

This shows the claim holds for $n = k + 1$ and so completes the proof of the claim. \hfill \Box

By claim, we can choose an increasing sequence $\{m_i\}_{i=1}^{\infty}$ of positive integers such that $C_{m_i} \subset F_{m_i+1}$, $C_{m_i} \cap F_{m_i} = \emptyset$ and $|C_{m_i}| = n_i = (i + 1)! - 1^!$. Let $S = \bigcup_{i=1}^{\infty} C_{m_i} \subset H$ and $S_i = \bigcup_{j=1}^{i} C_{m_j}$. Then we have

$$h_{\text{top}}^{S,F}(G, \mathcal{U}) = \lim_{n \to -\infty} \frac{1}{|F_n \cap S|} \log N\left( \bigvee_{g \in F_n \cap S} g^{-1}U \right) \geq \lim_{i \to -\infty} \frac{1}{|F_{m_i} \cap S|} \log N\left( \bigvee_{g \in F_{m_i} \cap S} g^{-1}U \right) = \lim_{i \to -\infty} \frac{1}{|S_i|} \log N\left( \bigvee_{g \in S_i} g^{-1}U \right) \geq \lim_{i \to -\infty} \frac{1}{|S_i|} \sum_{j=1}^{i} n_j \log l = \lim_{i \to -\infty} \frac{\log l_{n_i}}{\sum_{j=1}^{i} n_j} = \lim_{i \to -\infty} \frac{\log l_{n_i}}{\sum_{j=1}^{i} n_j} \log l = \log l = \log N(U).$$

Since $h_{\text{top}}^{S,F}(G, \mathcal{U}) \leq \log N(U)$, we have $h_{\text{top}}^{S,F}(G, \mathcal{U}) = \log N(U)$. This finishes the proof of (1).

Now we prove (2). Let $\mathcal{V} = \{V_1, V_2, \ldots, V_l\}$ be a non-trivial finite open cover. Take $x_j \in \operatorname{int}(X \setminus V_j)$ for every $1 \leq j \leq k$. Set $\{y_1, y_2, \ldots, y_{l}\} = \{x_1, x_2, \ldots, x_k\}$ with $y_t \neq y_t$ for $1 \leq s < t \leq l$. Clearly, $l \geq 2$ and we can take pairwise disjoint closed neighborhood $W_i$ of $y_i, i = 1, 2, \ldots, l$ such that the open cover $U = \{X \setminus W_1, X \setminus W_2, \ldots, X \setminus W_l\}$ is coarser than $\mathcal{V}$. By a proof similar to that of inequalities of the same type in theorem 4.2, we can obtain that

$$(l - 1)! l \log N\left( \bigvee_{g \in F} g^{-1}U \right) \geq \left| \left\{ s \in \{1, 2, \ldots, l\}, \bigvee_{g \in F} g^{-1}W_{(s,g)} = \emptyset \right\} \right|$$

for every finite subset $F$ of $G$. Moreover, by a proof similar to that of (1), we can find an increasing sequence $\{m_i\}_{i=1}^{\infty}$, an infinite subset $S = \bigcup_{i=1}^{\infty} C_{m_i} \subset H$ and $S_i = \bigcup_{j=1}^{i} C_{m_j}$ such that $C_{m_i} \subset F_{m_i+1}$, $C_{m_i} \cap F_{m_i} = \emptyset$, $|C_{m_i}| = (i + 1)! - 1^!$ and

$$|C_{m_i}| = (i + 1)! - 1!$$
\[ \bigcap_{g \in C_n} g^{-1} W_{(g)} = \emptyset \text{ for any } s \in \{1, 2, \ldots, l\}^C_n. \]

Thus,
\[
\begin{align*}
\htop^{SF}(G, \mathcal{U}) &= \limsup_{n \to \infty} \frac{1}{|g_n \cap S|} \log N \left( \bigvee_{g \in g_n \cap S} g^{-1} \mathcal{U} \right) \\
&= \limsup_{i \to \infty} \frac{1}{|S_i|} \log N \left( \bigvee_{g \in C_{n_i}} g^{-1} \mathcal{U} \right) \\
&\geq \limsup_{i \to \infty} \frac{1}{\sum_{j=1}^{n_{i_j}} n_{i_j}} \log \left( \sum_{g \in C_{n_i}} g^{-1} \mathcal{U} \right) \\
&= \limsup_{i \to \infty} \frac{1}{\sum_{j=1}^{n_{i_j}} n_{i_j}} \log \left( \sum_{g \in C_{n_i}} g^{-1} \mathcal{U} \right) \\
&= \lim_{i \to \infty} \frac{1}{\sum_{j=1}^{n_{i_j}} n_{i_j}} \log \left( \sum_{j=1}^{n_{i_j}} n_{i_j} \right) = \lim_{i \to \infty} \frac{\log \left( \sum_{j=1}^{n_{i_j}} n_{i_j} \right)}{\sum_{j=1}^{n_{i_j}} n_{i_j}} = \lim_{i \to \infty} \frac{l}{l} = \log \left( \frac{l}{l} \right) > 0.
\end{align*}
\]

Hence, \( \htop^{SF}(G, \mathcal{V}) \geq \htop^{SF}(G, \mathcal{U}) > 0. \)

5. Sequence entropy pairs and null systems

In this section, we use sequence entropy pairs to describe null systems and weakly mixing systems.

**Definition 5.1.** Let \((X, G, \mathcal{B})\) be a \(G\)-system.

1. We say that \((x_1, x_2) \in X \times X\) is a sequence entropy pair if \(x_1 \neq x_2\) and if whenever \(U_i\) are closed mutually disjoint neighborhoods of points \(x_i, i = 1, 2, \) there exists an infinite subset \(S \subset G\) such that \(\htop^{SF}(G, \{X \setminus U_1, X \setminus U_2\}) > 0.\)
2. We say that \((X, G)\) has a uniform positive sequence entropy (for short u.p.s.e.), if every point \((x_1, x_2) \in X \times X\), not in the diagonal \(\Delta = \{(x, x): x \in X\},\) is a sequence entropy pair.
3. We say that \((X, G)\) is a null system, if for any infinite subset \(S \subset G\) such that \(\htop^{SF}(G) = 0.\)

**Remark 5.2.** It is easy to see that \((X, G)\) has u.p.s.e. if and only if for any cover \(\mathcal{U} = \{U, V\}\) of \(X\) by two non-dense open sets, one has \(\htop^{SF}(G, \mathcal{U}) > 0\) for some infinite subset \(S \subset G.\)

Denote by \(\text{SE}(X, G)\) the set of all sequence entropy pairs. Following ideas in [47], we provide a sufficient condition for the existence of sequence entropy pairs.

**Lemma 5.3.** If there is a standard open cover \(\mathcal{U} = \{U, V\}\) of \(X\) with \(\htop^{SF}(G, \mathcal{U}) > 0\) for some infinite sequence \(S \subset G,\) then there are points \(x \in X \setminus U\) and \(x' \in X \setminus V\) such that \((x, x')\) is a sequence entropy pair.

**Proof.** Firstly, we show that one can find a strictly coarser cover \(\mathcal{U}_1 = \{U_i, V_i\}\) with \(\htop^{SF}(G, \mathcal{U}_1) > 0,\) having the property that \(\text{diam}(X \setminus U_i) < 1/2 \text{diam}(X \setminus U)\) and \(\text{diam}(X \setminus V_i) < 1/2 \text{diam}(X \setminus V),\) where \(\text{diam}(A) = \max \{d(x, y): x, y \in A\}\) for \(A \subset X.\) Note that \(X \setminus U\) cannot be a singleton, because in this case, \(\htop^{SF}(G, \mathcal{U}) = 0;\) if \(X \setminus U\) is a singleton, then \(\text{diam}(X \setminus U) < 1/2 \text{diam}(X \setminus V),\) which implies that \(\htop^{SF}(G, \mathcal{U}) = 0.\) Thus, there exist two distinct points \(y, y' \in X \setminus U.\) Fix \(\epsilon_1 \text{ such that } 0 < \epsilon_1 < \frac{1}{3} d(y, y'),\) and construct a finite cover of \(X \setminus U\) by open balls with radius \(\epsilon_1\) centered in \(X \setminus U;\) call is \(\mathcal{W} = \{U_1', \ldots, U_k'\},\) where \(k \geq 2.\) Let \(F_i' = \overline{U_i} \setminus U_i, i = 1, 2, \ldots, k.\) Consider the open cover \(\bigcup_{i=1}^k \{X \setminus F_i, V\}.\) Since \(\bigcap_{i=1}^k (X \setminus F_i) \subset U,\) it follows that \(\mathcal{U}\) is
coarser than \( \varnothing_{i=1}^k (X \setminus F_i, V) \). Thus,

\[
0 < h^S_{\text{top}}(G, \mathcal{U}) \leq h^S_{\text{top}}(G, \bigvee_{i=1}^k (X \setminus F_i, V)) \leq \sum_{k=1}^n h^S_{\text{top}}(G, (X \setminus F_i, V)),
\]

which implies that there exists \( j \in \{1, 2, \ldots, k\} \) such that \( h^S_{\text{top}}(G, (X \setminus F_j, V)) > 0 \). Denote \( U_1 = X \setminus F_j \). Then choosing a suitable \( \epsilon' \) and doing the same for \( V \), we obtain \( V_i \) such that \( U_i = \{U_i, V_i\} \) is a strictly coarser cover than \( \mathcal{U} \) with \( h^S_{\text{top}}(G, \mathcal{U}) > 0 \), having the property that \( \text{diam}(X \setminus U_i) < \frac{1}{2} \text{diam}(X \setminus \mathcal{U}) \)

Repeating this process infinitely many times, we obtain two decreasing sequences of closed subsets \( \{X \setminus U_i\}_{i=1}^\infty \) and \( \{X \setminus V_i\}_{i=1}^\infty \) with \( \text{diam}(X \setminus U_{i+1}) < \frac{1}{2} \text{diam}(X \setminus U_i) \) and \( \text{diam}(X \setminus V_{i+1}) < \frac{1}{2} \text{diam}(X \setminus V_i) \) for each \( i \in \mathbb{N} \) such that \( h^S_{\text{top}}(G, \mathcal{U}_i) > 0 \), where \( \mathcal{U}_i = \{U_i, V_i\} \) for each \( i \in \mathbb{N} \). Since \( X \) is compact, we deduce that

\[
\bigcap_{i=1}^\infty (X \setminus U_i) = \{x\} \quad \text{and} \quad \bigcap_{i=1}^\infty (X \setminus V_i) = \{x'\}.
\]

(5.1)

Now we prove that \( (x, x') \in \text{SE}(X, G) \). Let \( W, W' \) be two closed mutually disjoint neighborhoods of \( x, x' \), respectively. Choose \( \epsilon > 0 \) such that closed balls \( B(x, \epsilon) \subset W \) and \( B(x', \epsilon) \subset W' \). Then the open cover \( \{X \setminus B(x, \epsilon), X \setminus B(x', \epsilon)\} \) is coarser than \( \{X \setminus W, X \setminus W'\} \). By (5.1), there exists \( i \in \mathbb{N} \) such that \( X \setminus U_i \subset B(x, \epsilon) \) and \( X \setminus V_i \subset B(x', \epsilon) \), which implies that the open cover \( \mathcal{U}_i \) is coarser than \( \{X \setminus B(x, \epsilon), X \setminus B(x', \epsilon)\} \). Thus, \( \mathcal{U}_i \) is coarser than \( \{X \setminus W, X \setminus W'\} \), which implies that

\[
0 < h^S_{\text{top}}(G, \mathcal{U}_i) \leq h^S_{\text{top}}(G, (X \setminus W, X \setminus W')).
\]

Therefore, \( (x, x') \in \text{SE}(X, G) \), proving this lemma. \( \square \)

With the help of lemma 5.3, we provide a characterization of null systems via sequence entropy pairs.

**Theorem 5.4.** Let \((X, G)\) be a \( G \)-system. Then \((X, G)\) is null if and only if \( \text{SE}(X, G) = \emptyset \).

**Proof.** It is easy to see that if \((X, G)\) is null then \( \text{SE}(X, G) = \emptyset \). Conversely, if \((X, G)\) is not null then there exists a standard open cover \( \mathcal{U} \) such that \( h^S_{\text{top}}(G, \mathcal{U}) > 0 \), and hence by lemma 5.3, there exists \((x, x') \in \text{SE}(X, G)\). This finishes the proof of theorem 5.4. \( \square \)

Taking advantage of theorem 4.2, we immediately obtain the following description of weakly mixing systems via sequence entropy pairs.

**Theorem 5.5.** Let \( G \) be an Abelian group. Then the \( G \)-system \((X, G)\) is weakly mixing if and only if it is u.p.s.e.

### 6. Example

In this section, we provide an example to show that there exists a \( \mathbb{Z}^2 \)-system such that it has zero entropy, but there exists a sequence such that it has positive sequence entropy both in measure-theoretic and topological sense.

**Example 6.1.** Following ideas of Example 5.4 in [16], let \( Y = \{0, 1\} \) and \( \mu([0]) = \mu([1]) = 1/2 \). Let \( X = Y^\mathbb{Z} \) and \( \nu = \nu^\mathbb{Z} \). Define \( T_1 = I \) by \( X \to X \) by

\[
T_1(x_n)_{n \in \mathbb{Z}} = (x_n)_{n \in \mathbb{Z}} \text{ for any } (x_n)_{n \in \mathbb{Z}} \in X.
\]

Define \( T_2 : X \to X \) by

\[
T_2(x_n)_{n \in \mathbb{Z}} = (x_{n+1})_{n \in \mathbb{Z}} \text{ for any } (x_n)_{n \in \mathbb{Z}} \in X,
\]

that is, \( T_2 \) is the two-sided full shift. Define a \( \mathbb{Z}^2 \)-action \( T \) by

\[
T^{(m, n)} = T_1^m T_2^n \text{ for all } (m, n) \in \mathbb{Z}^2.
\]

Then \((X, T)\) is a \( \mathbb{Z}^2 \)-system, and \( \mu \) induces a \( \mathbb{Z}^2 \)-MPS. Let \( E_n = [0, n-1] \times [0, n-1] \) for each \( n \in \mathbb{N} \). Then \( F = \{E_n\}_{n=1}^\infty \) is a Følner sequence.
Now, we prove \((X,T)\) has zero topological entropy. Indeed, for any finite open cover \(\mathcal{U}\), we have that
\[
\mathcal{N}\left(\bigvee_{i,j=0}^{n-1} T^{-i}(j)\mathcal{U}\right) \leq n^{\mathcal{N}(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U})}
\]
for each \(n \in \mathbb{N}\).

Since \(T_1 = \text{Id}\) has zero topological entropy, it follows that
\[
h_{\text{top}}(T, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{|F_n|} \log \mathcal{N}\left(\bigvee_{i,j \in F_n} T^{-i}(j)\mathcal{U}\right) = 0.
\]
Since \(\mathcal{U}\) is arbitrary, we have that \(h_{\text{top}}(\mathbb{Z}^2) = \sup_{\mathcal{U}} h_{\text{top}}(T, \mathcal{U}) = 0\). However, we take \(S = \{(0, n)\}_{n=1}^\infty\). It is known that the topological entropy of the two-sided full shift is \(\log 2\). Thus,
\[
h_{\text{top}}(\mathbb{Z}^2) = \sup_{\mathcal{U}} \lim_{n \to \infty} \frac{1}{|F_n \cap S|} \log \mathcal{N}\left(\bigvee_{i,j \in F_n \cap S} T^{-i}(j)\mathcal{U}\right)
= \sup_{\mathcal{U}} \lim_{n \to \infty} \frac{1}{n} \log \mathcal{N}\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}\right) = \log 2.
\]

By an argument similar to that of the topological entropy, we also can prove that \(h_o(\mathbb{Z}^2) = 0\), but \(h_{\text{top}}(\mathbb{Z}^2) = \log 2\).

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Data availability statement

No new data were created or analysed in this study.

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