NONPARAMETRIC HYPERSURFACES MOVING BY POWERS OF GAUSS CURVATURE

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ABSTRACT. We study asymptotic behavior of nonparametric hypersurfaces moving by $\alpha$ powers of Gauss curvature with $\alpha > 1/n$. Our work generalizes the results of V. Oliker [Oli91] for $\alpha = 1$.

1. Introduction

Let $\Omega$ be a bounded strictly convex domain in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $\partial \Omega$. We consider a solution of the following initial boundary problem

$$
\begin{align*}
\frac{\partial u}{\partial t} &= \left[ \det(u_{ij}) \right]^\alpha \left(1 + |\nabla u|^2 \right)^{\alpha\beta} \quad \text{in } \Omega \times (0, \infty), \\
\frac{\partial u}{\partial t} &= 0 \quad \text{in } \partial \Omega \times (0, \infty), \\
u(x, t) &= 0 \quad \text{is strictly convex for each } t \geq 0,
\end{align*}
$$

where $\alpha > 1/n$ and $\beta \geq 0$ are constants and

$$
u_t := \frac{\partial u}{\partial t}, \quad \nu_{ij} := \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad \nabla u := \left( \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right).
$$

Equation (1.1) describes the graphs $(x, u(x, t)), (x, t) \in \Omega \times [0, \infty)$ evolving in $\mathbb{R}^{n+1}$ with relative boundaries $(x, u(x, t))|_{\partial \Omega}$ remain fixed. When $\beta = \frac{n+2-\frac{1}{2}}{2}$, the normal speed of the point $(x, u(x, t))$ is equal to $\alpha$ powers of the Gauss curvature of the graph. Such parabolic Monge-Ampère equations have been studied by many authors in recent years. See, for instance, [HL06] [DS12]. On the other hand, in the parametric setting, flow by Gauss curvature or its powers have received considerable interests, see [Tso85] [Cho85] [Cho91] [And99] [And00] [GN] [AGN] and the references therein.

V. Oliker considered (1.1) with $\alpha = 1$ in [Oli91]. He analyzed the asymptotic behavior of smooth convex solutions of (1.1). It turned out that solutions with different $\beta$ all have the same asymptotic behavior. Moreover, if $\Omega$ is centrally symmetric or rotationally symmetric, then the solution $u(x, t)$ asymptotically becomes centrally symmetric or rotational symmetric, regardless of its initial shape.

The goal of this paper is to generalize V. Oliker’s results in [Oli91] to any power $\alpha > 1/n$. We investigate the asymptotic behavior of a smooth convex solution of (1.1) and show

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that, by comparing with self-similar solutions of \([1]\) with \(\beta = 0\), the solution \(u(x,t)\) asymptotically converges to the solution of the following nonlinear elliptic problem:

\[
[\det(\psi_{ij})]^\alpha = \frac{1}{1 - n\alpha} \psi \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial\Omega,
\]

\(\psi\) is strictly convex and \(\psi < 0\) in \(\Omega\).

Furthermore, our estimate implies geometric properties of the flow by \(\alpha\) powers of the Gauss curvature. For instance, the asymptotic behavior of \(u(x,t)\) reflects the symmetries of \(\Omega\). More precisely, if \(\Omega\) is centrally or rotationally symmetric, then the solution \(u(x,t)\) asymptotically becomes centrally or rotational symmetric, regardless of its initial shape, and we also give sharp estimates on the rate of this process.

Throughout the paper, we denote by \(M\) the Monge-Ampère operator \(M(u) := \det(u_{ij})\) and \(M^\alpha(u) := [\det(u_{ij})]^\alpha\).

2. Main Results

Consider the following initial boundary problem:

\[
\begin{align*}
\partial_t u = M^\alpha(u) & \quad \text{in } \Omega \times (0, \infty), \\
u(x,t) = 0 & \quad \text{in } \partial\Omega \times (0, \infty), \\
u(x,t) & \quad \text{is strictly convex for each } t \geq 0.
\end{align*}
\]

(2.1)

We seek for self-similar solutions of (2.1) of the form

\[
u(x,t) = \varphi(t)\psi(x),
\]

(2.2)

where \(\varphi(t) \in C^\infty([0, \infty))\) and \(\psi(x) = C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})\). By convexity of \(u(x,0) = \varphi(0)\psi(x)\), we have either \(\varphi(0) < 0\) and \(\psi(x) > 0\) in \(\Omega\) and concave or \(\varphi(0) > 0\) and \(\psi(x) < 0\) in \(\Omega\) and convex. Since both cases are equivalent for our purpose, we always deal with the latter one. Substituting (2.2) into (2.1) yields

\[
\frac{\varphi(t)}{\varphi^{n\alpha}} = \frac{M^\alpha(\psi)}{\psi} = \lambda = \text{constant}.
\]

Noting that \(\psi(x) < 0\) and convex in \(\Omega\), we get \(\lambda \leq 0\) and

\[
\varphi(t) = (\varphi(0)^{1-n\alpha} - (n\alpha - 1)\lambda t)^{\frac{1}{1-n\alpha}},
\]

(2.3)

\[
M(\psi) = (\lambda\psi)^{\frac{1}{\alpha}} \quad \text{in } \Omega \quad \text{and} \quad \psi = 0 \quad \text{on } \partial\Omega.
\]

(2.4)

An easy argument shows that \(\lambda = 0\) implies \(u(x,t) \equiv 0\). Thus we only consider the case \(\lambda < 0\). By scaling, it suffices to consider one negative value of \(\lambda\) and thus we fix \(\lambda = \frac{1}{1-n\alpha} < 0\) for convenience. The following result establishes the existence of self-similar solutions to (2.1).

**Theorem 2.1.** Let \(\Omega\) be a bounded strictly convex domain with smooth boundary \(\partial\Omega\). Then problem (2.1) admits a self-similar solution in \(\overline{\Omega} \times (0, \infty)\) given by

\[
u(x,t) = (1 + t)^{\frac{1}{1-n\alpha}} \psi(x),
\]

(2.5)
where $\psi$ is the unique solution in $C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$ of the equation

\begin{equation}
M(\psi) = \left( \frac{-\psi}{1 - n\alpha} \right)^\frac{1}{\alpha} \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega,
\end{equation}

$\psi$ is strictly convex and $\psi < 0$ in $\Omega$,

and $\sup_{\Omega} |\psi(x)|$ admits an estimate depending only on $n$, $\alpha$ and the domain $\Omega$. Furthermore, if $\tilde{u}(x,t) = \varphi(t)\tilde{\psi}(x)$ is an arbitrary self-similar solution of (2.1), then there exists a unique $c > 0$ such that $\tilde{\psi}(x) = c\psi(x)$ and

\begin{equation}
\tilde{u}(x,t) = u(x,t) \left\{ \frac{1 + t}{|\varphi'(0)|^{1-n\alpha} + t} \right\} \frac{1}{n\alpha - 1}.
\end{equation}

The main theorem concerning the asymptotic behavior of the solution is the following:

**Theorem 2.2.** Let $u(x,t) \in C^2(\overline{\Omega} \times (0, \infty))$ be a solution of the problem

\begin{equation}
\begin{aligned}
& u_t = \frac{M^\alpha(u)}{1 + |\nabla u|^2} \text{ in } \Omega \times (0, \infty), \\
& u(x,t) = 0 \text{ in } \partial\Omega \times (0, \infty), \\
& u(x,t) \text{ is strictly convex for each } t \geq 0,
\end{aligned}
\end{equation}

where $\alpha > 1/n$ and $\beta \geq 0$ are constants. If $\beta = 0$, then there exists positive constant $C_1$ depending only on dimension $n$, $\alpha$, $\Omega$ and $u(x,0)$, such that for all $t \geq 0$,

\begin{equation}
\sup_{\Omega} \left| (1 + t)\frac{1}{n\alpha - 1} u(x,t) - \psi(x) \right| \leq \frac{C_1}{1 + t}.
\end{equation}

If $\beta > 0$, then

\begin{equation}
\psi \leq (1 + t)\frac{1}{n\alpha - 1} u(x,t) - \psi(x) \leq \frac{-C_3\psi}{1 + t},
\end{equation}

where $C_2$ and $C_3$ are positive constants depending only on dimension $n$, $\alpha$, $\Omega$, $u(x,0)$ and $G = \inf_{\Omega} \left( 1 + |\nabla u(x,0)|^2 \right)^{-\alpha\beta}$. Moreover,

\begin{equation}
\lim_{t \to \infty} (1 + t)\frac{1}{n\alpha - 1} u(x,t) = \psi(x) \text{ uniformly on } \overline{\Omega}.
\end{equation}

We have gradient estimates for solutions of (2.8).

**Corollary 2.3.** Suppose the same conditions as in Theorem 2.2 holds. Then for all $t \geq 0$,

\begin{equation}
\sup_{\Omega} |\nabla u(x,t)| \leq G\frac{1}{1 + t} \sup_{\partial\Omega} \psi(x)(C_4 + t)^{-\alpha\beta}
\end{equation}

where $\psi_\nu$ is the derivative in the direction of the outward unit normal to $\partial\Omega$, and $C_4$ depends only on $u(x,0)$.

An interesting geometric consequence of Theorem 2.2 is the following:

**Theorem 2.4.** If $\Omega$ is a ball in $\mathbb{R}^n$ and $u(x,t) \in C^2(\overline{\Omega} \times (0, \infty))$ is a solution of (2.8). Then

\begin{equation}
(1 + t)\frac{1}{n\alpha - 1} u(x,t) \to \psi(|x|) \text{ uniformly on } \overline{\Omega} \text{ as } t \to \infty.
\end{equation}
This theorem implies that, \( u(x, t) \) asymptotically becomes radially symmetric regardless of the initial shape. More generally, if \( \Omega \) is centrally symmetric, then
\[
(1 + t)^\frac{1}{n-1} u(x, t) \to \psi(x) \quad \text{uniformly on } \overline{\Omega} \quad \text{as } t \to \infty,
\]
where \( \psi(x) = \psi(-x) \). The proof of Theorem 2.4 is the same as in [Oli91 Section 6] and we omit it here.

3. Proof of Theorem 2.4

Proof. It was shown in [Tso90 Corollary 4.2, in which (2) should read as (1.2)] that for any \( \alpha > 1/n \), problem (2.6) admits a unique strictly convex solution \( \psi \) in \( C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega}) \).

Direct calculation shows \( u(x, t) = (1+t)\psi(x) \) solves (2.4) with initial data \( u_0(x) = \psi(x) \).

Next we prove \( \sup_{\Omega} |\psi(x)| \) depends only on \( n, \alpha \) and \( \Omega \). Since \( \psi \) is strictly convex and vanishes on \( \partial \Omega \), there exists a point \( x_0 \in \Omega \) such that \( \sup_{\Omega} |\psi| = |\psi(x_0)| \). Consider a cone \( K \) generated by the linear segments joining the vertex \((x_0, \psi(x_0))\) with points on \( \partial \Omega \). Denote \( \theta(x), x \in \Omega \), the function whose graph is \( K \).

On the other hand, the Aleksandrov-Bakelman-Pucci maximum principle (see, for instance, [Gut01 Theorem 1.4.5]) says \( M\theta(\Omega) \geq \omega_n|\psi(x_0)|^{n/(\text{diam} \Omega)^{-n}} \), where \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \). Thus
\[
(3.2) \quad \sup_{\Omega} |\psi(x)| = |\psi(x_0)| \leq \left( \frac{|\psi(x_0)|^{n/(\text{diam} \Omega)^{-n}}}{\omega_n} \right)^{\frac{n}{n-1}}.
\]

Finally, the proof of (2.7) parallels that in [Oli91 Section 4.3]. \( \square \)

Remark 3.1. One can prove Theorem 2.4 without using the existence results from [Tso90]. V. Oliker [Oli91] proved that (2.6) has a unique solution in \( C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega}) \) when \( \alpha = 1 \).

A careful examination of his proof shows it works indeed for all \( \alpha > 1/n \).

Remark 3.2. When \( \alpha = 1/n \), it was shown by P. L. Lions [Lio83] that
\[
(3.3) \quad M(\psi) = \mu(-\psi)^n \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial \Omega
\]

admits a unique solution pair \((\mu, \psi)\) in the sense that if \((\nu, \phi)\), where \( \nu \) is positive and \( \phi \) is convex, solves (1.3), then we must have \( \mu = \nu \) and \( \phi \) is a constant multiple of \( \psi \). The number \( \mu \) is called the first (in fact the only) eigenvalue of the Monge-Ampère operator \( M \), and the corresponding (normalized) eigenfunction is in \( C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega}) \). The asymptotic behavior for \( \alpha = 1/n \) remains interesting and open.

Remark 3.3. When \( 0 < \alpha < 1/n \), K. Tsu [Tso90] proved that (2.6) admits a convex solution in \( C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega}) \). The uniqueness, however, is not known. In this case, the reader will see easily from the comparison with self-similar supersolutions in Section 4 that smooth convex solutions of (2.3) must vanish at finite time.
4. Proof of Theorem 2.2

In this section, we determine the asymptotic behavior of $u$ by comparing with self-similar solutions of $(2.1)$. A direct generalization of the proof given by V. Oliker in [Oli91] works for $\alpha \geq 2/n$. New estimates are introduced in the following lemma to take care of the case $1/n < \alpha < 2/n$.

**Lemma 4.1.** Let $F : (0, S) \times [0, \infty) \to (0, \infty), S < \infty$ be defined by

$$F(s, t) = \left( \frac{1 + t}{s + t} \right)^{\frac{n + \alpha}{\alpha}} \equiv \left( 1 + \frac{1 - s}{s + t} \right)^{\frac{n}{n - \alpha}}.$$ 

Then we have for all $t \geq 0$,

$$F(s, t) \leq 1 + \frac{1}{n\alpha - 1} \frac{1 - s}{s(1 + t)}, \quad \text{if } s \leq 1, \alpha \geq 2/n;$$

$$F(s, t) \leq 1 + \frac{1}{n\alpha - 1} \left( \frac{1}{s} \right)^{\frac{n}{n - \alpha}} \frac{1 - s}{1 + t}, \quad \text{if } s \leq 1, \alpha \leq 2/n;$$

$$F(s, t) \geq 1 - \frac{s - 1}{1 + t}, \quad \text{if } s \geq 1, \alpha \geq 2/n;$$

$$F(s, t) \geq 1 - \frac{1}{n\alpha - 1} \frac{s - 1}{1 + t} \quad \text{if } s \geq 1, \alpha \leq 2/n.$$

**Proof.** This lemma follows from elementary calculus. When $\alpha \geq 2/n$, $\gamma := \frac{1}{n\alpha - 1} \leq 1$. Then (4.2) follows from $(1 + x)^{\gamma} \leq 1 + \gamma x$ for all $x \geq 0$ and (4.3) follows from $x^{\gamma} \geq x$ for all $0 \leq x \leq 1$. When $\alpha \leq 2/n$, $\gamma := \frac{1}{n\alpha - 1} \geq 1$. Now (4.3) is a consequence of $(1 + x)^{\gamma} \leq 1 + (1 + \alpha)^{\gamma - 1}x$ for all $0 \leq x \leq \alpha$ and (4.5) is a consequence of $(1 + x)^{\gamma} \geq 1 + \gamma x$ for all $1 < x \leq 0$.

**Proof of Theorem 2.2.** First of all, a uniform estimate of $|\nabla u(x, t)|$ is obtained similarly as in [Oli91]. For any $t \geq 0$,

$$\sup_{\Omega} |\nabla u(x, t)| \leq \sup_{\partial \Omega} |\nabla u(x, t)| = \sup_{\partial \Omega} |u_{\nu}(x, t)| \leq \frac{1}{\alpha} |u_{\nu}(x, 0)|.$$ 

Self-similar subsolution and supersolution are then constructed as follows: Let

$$G = \inf_{\Omega} (1 + |\nabla u(x, 0)|^{2})^{-\alpha \beta}.$$ 

Clearly we have $0 < G \leq 1$. It follows from (4.6) that

$$GM^\alpha(u) \leq (1 + |\nabla u(x, t)|^{2})^{-\alpha \beta} M^\alpha(u) = u_t \text{ in } \Omega \times (0, \infty).$$

Put $\overline{u}(x, t) = GM^\alpha(u)$ and $\overline{\varphi}(t) = \overline{\varphi}(t) \psi(x)$, where $\psi$ is the solution of (2.1) and

$$\varphi(t) = (\varphi(0)^{1 - \alpha} + t)^{\frac{1}{1 - \alpha}},$$

$$\overline{\varphi}(t) = (\overline{\varphi}(0)^{1 - \alpha} + t)^{\frac{1}{1 - \alpha}}.$$ 

Then $\overline{u}$ and $\overline{\varphi}$ satisfy $\overline{u} = GM^\alpha(\overline{u})$ and $\overline{\varphi} = M^\alpha(\overline{\varphi})$ in $\Omega \times (0, \infty)$, respectively. Finally we define $\tilde{u}(x, t) = \overline{u}(x, t) - u(x, t)$ and it satisfies

$$\tilde{u}_t = GM^\alpha(u) - (1 + |\nabla u(x, t)|^{2})^{-\alpha \beta} M^\alpha(u) \leq GM^\alpha(u) - GM^\alpha(u) \text{ in } \Omega \times (0, \infty).$$
Observe that the operator \( L(\tilde{u}) = M^{\alpha}(u) - M^{\alpha}(u) \) is elliptic since
\[
L(\tilde{u}) = \sum_{ij} \left( \int_0^1 \alpha \det(u_{\tau ij})^{\alpha - 1} \cof(u_{\tau ij}) d\tau \right) \tilde{u}_{ij},
\]
where \( u_r(x, t) = \tau u(x, t) + (1 - \tau) u(x, t) \) is strictly convex and the cofactor matrix \( \cof(u_{\tau ij}) \) is positive definite on any compact subset of \( \Omega \times (0, T) \) for any \( T < \infty \). Next we choose \( \varphi(0) \) and \( \tilde{\varphi}(0) \) so that \( \varphi(0) \psi(x) \leq u(x, 0) \leq \tilde{\varphi}(0) \psi(x) \) on \( \Omega \). Then
\[
(4.9) \quad \tilde{u}(x, 0) \leq 0 \text{ in } \tilde{\Omega} \text{ and } \tilde{u}(x, t) = 0 \text{ in } \partial \Omega \times [0, \infty),
\]
and we can then apply the classical maximum principle to conclude that \( \tilde{u}(x, t) = u(x, t) - u(x, t) \leq 0 \) on \( \tilde{\Omega} \times [0, \infty) \). Consequently,
\[
(4.10) \quad \{ (1 + t)^{-\frac{1}{\alpha}} (G(\tilde{\varphi}(0)^{1 - \alpha} + t))^{\frac{1}{1 - \alpha}} - 1 \} \psi(x) \leq (1 + t)^{-\frac{1}{\alpha}} u(x, t) - \psi(x).
\]
Similarly, one derives that \( u(x, t) \leq \tilde{u}(x, t) \), namely,
\[
(4.11) \quad (1 + t)^{-\frac{1}{\alpha}} u(x, t) - \psi(x) \leq \{ (1 + t)^{-\frac{1}{\alpha}} (\tilde{\varphi}(0)^{1 - \alpha} + t) \} \psi(x)
\]
Without loss of generality we may assume \( \tilde{\varphi}(0) \geq 1 \) and \( \tilde{\varphi}(0) \leq 1 \). Thus by Lemma 4.1,
\[
F(\tilde{\varphi}(0)^{1 - \alpha}, t) \leq 1 + C_2/(1 + t)
\]
\[
F(\tilde{\varphi}(0)^{1 - \alpha}, t) \geq 1 - C_3/(1 + t),
\]
where \( C_2, C_3 \) depend on \( n, \alpha \) and \( u_0(x) \). Combining now (4.10) and (4.11), we arrive at that for all \( t \geq 0 \) and \( x \in \tilde{\Omega} \),
\[
(4.12) \quad \left[ \frac{C_2}{1 + t} + G^{\frac{1}{1 - \alpha}} - 1 \right] \psi \leq (1 + t)^{-\frac{1}{\alpha}} u(x, t) - \psi \leq \frac{-C_3 \psi}{1 + t},
\]
If \( \beta = 0 \), then \( G = 1 \) and (4.12) implies (2.3) with \( C_1 = \max\{C_2, C_3\} \sup_{\tilde{\Omega}} |\psi| \). If \( \beta > 0 \), one needs to estimate \( |\nabla u(x, t)| \) more carefully as V. Oliker did [Oli91] Pages 255-256]. Take an increasing sequence \( t_m \to \infty \) and let \( G_m = \inf_{\Omega} (1 + |\nabla u(x, t_m)|^2)^{-\alpha \beta m} \). The same argument as in deriving (4.12) yields for all \( t \geq t_m \) and \( x \in \tilde{\Omega} \),
\[
(4.13) \quad \left[ \frac{c_m}{1 + t} + G_m^{1 - \frac{1}{\alpha}} - 1 \right] \psi \leq (1 + t)^{-\frac{1}{\alpha}} u(x, t) - \psi \leq \frac{-C_3 \psi}{1 + t},
\]
where \( c_m = (1 - \tilde{\varphi}(t_m)^{1 - \alpha}) \tilde{\phi}(t_m)^{\alpha}(m \alpha - 1) < \infty \) uniformly in \( m \) due to (4.10) . The same argument as in [Oli91] allows one to let \( t_m \to \infty \) and deduce (2.11), hence completing the proof of Theorem 2.2. \( \square \)

**Remark 4.1.** Similarly to [APS81] one sees the sharpness of the estimate (4.13) by considering the function \( u(x, t) = (s + t)^{-\frac{1}{\alpha}} \psi(x) \) for any \( s > 0 \).

**Remark 4.2.** Corollary 2.3 with \( C_4 = \tilde{\varphi}(0)^{1 - \alpha} \) follows from \( \tilde{\varphi}(x, t) \leq u(x, t) \), namely,
\[
G^{-1 - \frac{1}{\alpha}}(\tilde{\varphi}(0)^{1 - \alpha} + t)^{-\frac{1}{\alpha}} \psi(x) \leq u(x, t).
\]
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