Optimality conditions and constraint qualifications for cardinality constrained optimization problems

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Abstract. The cardinality constrained optimization problem (CCOP) is an optimization problem where the maximum number of nonzero components of any feasible point is bounded. In this paper, we consider CCOP as a mathematical program with disjunctive subspaces constraints (MPDSC). Since a subspace is a special case of a convex polyhedral set, MPDSC is a special case of the mathematical program with disjunctive constraints (MPDC). Using the special structure of subspaces, we are able to obtain more precise formulas for the tangent and (directional) normal cones for the disjunctive set of subspaces. We then obtain first and second order optimality conditions by using the corresponding results from MPDC. Thanks to the special structure of the subspace, we are able to obtain some results for MPDSC that do not hold in general for MPDC. In particular, we show that the relaxed constant positive linear dependence (RCPLD) is a sufficient condition for the metric subregularity/error bound property for MPDSC which is not true for MPDC in general. Finally, we show that under all constraint qualifications presented in this paper, certain exact penalization holds for CCOP.

Key Words. cardinality constrained optimization problems, disjunctive subspaces constraints, necessary optimality conditions, constraint qualifications, metric subregularity, error bounds property, RCPLD.

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1 Introduction

In this paper, we consider the cardinality constrained optimization problem (CCOP) in the following form:

\[
\begin{align*}
\min_x & \quad f(x) \\
\text{s.t.} & \quad g(x) \leq 0, \ h(x) = 0, \ \|x\|_0 \leq s,
\end{align*}
\]

where \( f : \mathbb{R}^n \to \mathbb{R} \), \( g : \mathbb{R}^n \to \mathbb{R}^m \) and \( h : \mathbb{R}^n \to \mathbb{R}^p \) and \( \|x\|_0 \) is the number of nonzero elements in the vector \( x \) (also called \( l_0 \)-norm). We assume \( s < n \), otherwise the cardinality constraint would be superfluous. Unless otherwise mentioned, we assume that all functions are smooth. Two simple examples about the cardinality constraint are given in Figure[1]

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CCOPs have many applications such as image processing, portfolio optimization, machine learning, and other related problems; see e.g. [38] and the references within for a survey of recent developments.

CCOPs are intrinsically nonconvex since even when all functions involved are convex, the feasible region is still nonconvex. Moreover problem (1) cannot be treated as a nonlinear program since $\|x\|_0$ is a discontinuous function. For the case where $X := \{x \in \mathbb{R}^n \mid g(x) \leq 0, h(x) = 0\} = \mathbb{R}^n$, Beck and Eldar [6] introduced three concepts of stationarity conditions for CCOP, and proposed some efficient numerical algorithms aimed at finding these stationary points. The results are extended to include a so-called symmetric set constraint in [7, 28]. Pan et al. [35] studied CCOP where $X = \mathbb{R}^n$ by reformulating the problem as $\min_x f(x)$ s.t. $x \in S$, where $S := \{x \in \mathbb{R}^n \mid \|x\|_0 \leq s\}$. They then give the expressions for the Bouligand/Clarke tangent cone and the regular/Clarke normal cone to $S$ and present corresponding first and second order optimality conditions. In [5], the formula for the limiting normal cone to $S$ is given. Recently Pan et al. [36], formulated CCOP as the following mathematical program:

$$\min_x f(x)$$

s.t. $g(x) \leq 0, h(x) = 0, x \in S := \bigcup_{I \in \mathcal{I}_s} \mathbb{R}_I,$

(2)

where $\mathcal{I}_s := \{I \subseteq \{1, 2, \ldots, n\} \mid |I| = s\}$ represents all subsets of $\{1, 2, \ldots, n\}$ with cardinality equal to $s$ and $\mathbb{R}_I := \text{span}\{e_i \mid i \in I\}$. By using some tools from variational analysis, they obtained some first and second order optimality conditions and studied constraint qualifications for the necessary optimality conditions. Note that each $\mathbb{R}_I$ is a subspace and hence problem (2) is a special case of the mathematical program with disjunctive constraints (MPDC). Recently Burdakov et al. [16] introduced a relaxed complementarity-type reformulation of CCOP and proposed some numerical algorithms. Based on this continuous reformulation, some constraint qualifications and stationarity conditions for CCOP were introduced in Červinka et al. [17]. Some second-order necessary and sufficient optimality conditions for CCOP based on such reformulation are also introduced in Bucher and Kanzow [15]. The relaxed complementarity-type reformulation is more computationally friendly than its original formulation. But due to the extra variable introduced in the reformulation, the relaxed complementarity-type reformulation are not equivalent to the original problem in the sense of local optimality [16].

In this paper, we study optimality conditions and constraint qualifications for problem (1) by its equiv-
alent reformulation \cite{2}. Recently there are many developments in the optimality conditions and constraint qualifications for MPDC \cite{18,21,39}, see \cite{27} and the references within for a survey.

The main contributions of this paper are summarized as follows:

(i) We study the mathematical programs with disjunctive subspaces constraints (MPDSC) which include CCOP as well as the mathematical programs with switching constraints (MPSC) \cite{31} as special cases. We explore the structure of the disjunctive set of subspaces to obtain formulas for various tangent and normal cones which do not hold if one of the subspaces is only a convex polyhedral set. Unlike in smooth systems with equality and inequality constraints, RCPLD may not imply the error bound property for the disjunctive system. Recently Xu and Ye \cite{39, Section 5.3} proposed RCPLD and the piecewise RCPLD for MPDCs, and showed that the piecewise RCPLD is a sufficient condition for the error bound property. Inspired by the fact that RCPLD coincides with the piecewise RCPLD for MPSC \cite{39, Section 5.3}, we show that this result actually holds for any system that can be formulated as a constraint system of MPDSC. Based on such result, we prove that for MPDSC, RCPLD is a sufficient condition for error bounds.

(ii) Applying the results from MPDSC, we obtain first and second order optimality conditions for CCOP and some new sufficient conditions for the error bound for CCOP. The first order necessary and the second order necessary and sufficient optimality conditions we present are sharper and hold under weaker constraint qualifications than the corresponding results in Pan et al. \cite{36}. Moreover under all constraint qualifications presented in this paper, the exact penalty holds for CCOP. That is, if \( x^* \) is a local minimizer of CCOP and one of the constraint qualifications discussed in this paper holds, then there exists a constant \( \mu \geq 0 \) such that \( x^* \) is also a local minimizer of the exact penalty problem:

\[
\min_x f(x) + \mu \left( \sum_{i=1}^m \max\{g_i(x), 0\} + \sum_{i=1}^p |h_i(x)| \right)
\]

\[
\text{s.t. } \|x\|_0 \leq s.
\]

The remainder of this paper is organized as follows. In Section 2, we derive formulas for various tangent and normal cones that will be used in this paper. In Section 3, we study optimality conditions and constraint qualifications for MPDSC. We also show that RCPLD is a sufficient condition for error bounds for MPDSC, an important result which does not hold for MPDC in general. In Section 4, we first reformulate CCOP as MPDSC, then apply the results in Section 3 to CCOP.

2 Notation and preliminary results

The notations we adopt are standard. Given a point \( x \in \mathbb{R}^n \), \( B_\varepsilon(x) \) stands for the open ball of radius \( \varepsilon \) centered at \( x \), while the symbol \( B \) simply stands for the open unit ball centered at the origin. We denote by \( \nabla f(x) \) the gradient of a continuously differentiable function \( f : \mathbb{R}^n \to \mathbb{R} \) at \( x \) and by \( d_\Omega(x) \) the distance between the point \( x \) and the set \( \Omega \). Unless otherwise stated, \( \| \cdot \| \) denotes an arbitrary norm in \( \mathbb{R}^n \) and the
notation \( \langle \cdot, \cdot \rangle \) denotes the inner product. For a given nonempty set \( A \subseteq \mathbb{R}^n \), we use notations \( \text{cl}A \), \( \text{cone}A \), and \( \text{span}A \) to represent the closure of \( A \), the conic hull of \( A \), and the span of \( A \), that is, the smallest subspace of \( \mathbb{R}^n \) comprising \( A \). For any given set \( B \) with finite elements, we denote the number of elements in \( B \) by \( |B| \). Given finite index sets \( I, J \), a pair \( (\{v_i\}_{i \in I}, \{u_i\}_{i \in J}) \) of family of vectors \( \{v_i\}_{i \in I}, \{u_i\}_{i \in J} \) is said to be positive linearly dependent if there exist scalars \( \{\alpha_i\}_{i \in I} \) and \( \{\beta_i\}_{i \in J} \) with \( \alpha_i \geq 0 \) for any \( i \in I \), not all equal to zero such that \( \sum_{i \in I} \alpha_i v_i + \sum_{i \in J} \beta_i u_i = 0 \).

The main purpose of this section is to derive some formulas for various tangent cones and normal cones to the disjunctive set of subspaces \( S = \bigcup_{r=1}^{R} S_r \) where each \( S_r \) is a subspace of \( \mathbb{R}^n \). These formulas will be needed in the next sections of the paper.

### 2.1 Formulas for tangent and normal cones

Given a closed set \( \Omega \subseteq \mathbb{R}^n \) and \( x \in \Omega \), we denote the Bouligand tangent cone, the Fréchet/regular normal cone and the Mordukhovich/limiting normal cone to \( \Omega \) at \( x \) by \( T_{\Omega}(x) \), \( \hat{N}_{\Omega}(x) \) and \( N_{\Omega}(x) \), respectively. When \( \Omega \) is convex, we denote by \( N_{\Omega}(x) \) the normal cone in the sense of convex analysis. We refer the reader to the standard reference of variational analysis in \cite{14,18,33,34,37} for their precise definitions.

**Proposition 2.1.** Let \( x^* \in S \). Then

\[
T_S(x^*) = \bigcup_{r \in I(x^*)} T_{S_r}(x^*) = \bigcup_{r \in I(x^*)} S_r, \tag{4}
\]

\[
\hat{N}_S(x^*) = \bigcap_{r \in I(x^*)} N_{S_r}(x^*) = \bigcap_{r \in I(x^*)} S_r^\perp, \tag{5}
\]

\[
N_S(x^*) = \bigcup_{r \in I(x^*)} N_{S_r}(x^*) = \bigcup_{r \in I(x^*)} S_r^\perp, \tag{6}
\]

respectively, where \( I(x^*) := \{r \in \{1, \ldots, R\} \mid x^* \in S_r\} \) is the active index set for \( S \) at \( x^* \).

**Proof.** The first equation in (4) is from \cite{1} Table 4.1 and the second equation in (4) is obvious since each \( S_r \) is a subspace and hence \( T_{S_r}(x^*) = S_r \) provided that \( x^* \in S_r \). The proof of (5) follows from the one for the tangent cone by polarization.

Now it remains to prove that (6) holds. The second equation in (6) is obvious from the fact that each \( S_r \) is a subspace for \( r = 1, \ldots, R \). We only need to prove the first equation. The inclusion \( N_S(x^*) \subseteq \bigcup_{r \in I(x^*)} N_{S_r}(x^*) \) follows from \cite{30} Lemma 2.2. Now we prove the converse inclusion. First from (13) and (14) in Adam et al. \cite{3}, there exists \( \delta > 0 \) such that

\[
N_S(x^*) = \bigcup_{x \in B_{\delta}(x^*)} \hat{N}_S(x). \tag{7}
\]

For each \( r \in I(x^*) \), since there are only finitely many subspaces we can pick a unit vector \( u_r \) such that \( u_r \in S_r \) but \( u_r \notin S_j \) for other \( j \neq r \). Consider the point \( x = x^* + tu_r \) such that \( t < \delta \). Then \( x \in \mathbb{B}_{\delta}(x^*) \) and
\( x \in S \), but \( x \notin S_j \) for other \( j \neq r \). In this way, for every \( i \in I(x^*) \) we have

\[
N_{S_i}(x^*) = S^\perp_r = N_{S_r}(x) = \hat{N}_S(x) \subseteq \bigcup_{x \in \mathbb{B}_S(x^*)} \hat{N}_S(x),
\]

where the third equality follows from the first equation in (5) taking into account that \( I(x) = \{r\} \). Combing (7) and (8), we obtain the inclusion \( \bigcup_{r \in I(x^*)} N_{S_r}(x^*) \subseteq N_S(x^*) \) as desired, which completes the proof.

In the following proposition, we will compute the regular normal cone to tangent cone \( T_S(x^*) \) at a tangent direction \( d \).

**Proposition 2.2.** (The regular normal cone to the tangent cone) Let \( x^* \in S \) and \( d \in T_S(x^*) \). Then the regular normal cone to \( T_S(x^*) \) at \( d \) is calculated as

\[
\hat{N}_{T_S(x^*)}(d) = \bigcap_{r \in I(x^*) \cap I(d)} S^\perp_r.
\]

**Proof.** We denote \( S(x^*) := \bigcup_{r \in I(x^*)} S_r \). Combing (4) with (5), we have

\[
\hat{N}_{T_S(x^*)}(d) = \hat{N}_S(x^*) = \bigcap_{r \in I(x^*) \cap I(d)} N_{S_r}(d) = \bigcap_{r \in I(x^*) \cap I(d)} S^\perp_r,
\]

which completes the proof.

Recently a directional version of the limiting normal cone has been introduced by Ginchev and Mordukhovich [23].

**Definition 2.1.** (Directional normal cones) [23, Definition 2.3] Given a closed set \( \Omega \subseteq \mathbb{R}^n \), \( x^* \in \Omega \) and \( d \in \mathbb{R}^n \). The limiting normal cone to \( \Omega \) at \( x^* \) in direction \( d \) is defined by

\[
N_{\Omega}(x^*; d) := \left\{ v \in \mathbb{R}^n \mid \exists t_k \downarrow 0, d_k \rightarrow d, v_k \rightarrow v, \text{ s.t. } v_k \in \hat{N}_{\Omega}(x^* + t_k d_k) \right\}.
\]

From the definition we have \( N_{\Omega}(x^*; 0) = N_{\Omega}(x^*) \), \( N_{\Omega}(x^*; d) \subseteq N_{\Omega}(x^*) \) and \( N_{\Omega}(x^*; d) = \emptyset \) if \( d \notin T_{\Omega}(x^*) \).

If \( C \) is a disjunctive set which is the union of finitely many convex polyhedral sets, by [21, Lemma 2.1] we have the inclusion

\[
N_C(x^*; d) \subseteq \{ v \in N_C(x^*) \mid v^T d = 0 \}.
\]

In the following proposition, we will show that the above inclusion holds as an equality if \( C = S \), the union of finitely many subspaces.

**Proposition 2.3.** (Directional normal cone) Let \( x^* \in S \) and \( d \in T_S(x^*) \). Then the directional normal cone to \( S \) at \( x^* \) in direction \( d \) is calculated as

\[
N_S(x^*; d) = \{ v \in N_S(x^*) \mid v^T d = 0 \} = \bigcup_{r \in I(x^*) \cap I(d)} S^\perp_r.
\]
We define the sequences

\[ \{v \in N_S(x^*) \mid v^T d = 0\} = \bigcup_{r \in I(x^*)} \{v \in N_{S_r}(x^*) \mid v^T d = 0\}. \] (12)

By \cite{21} Lemma 2.1, since \( S_r \) is convex,

\[ \{v \in N_{S_r}(x^*) \mid v^T d = 0\} = N_{T_{S_r}(x^*)}(d) \quad \forall r \in I(x^*). \] (13)

Further, since \( T_{S_r}(x^*) = S_r \) for each \( r \in I(x^*) \), we have

\[ \bigcup_{r \in I(x^*)} N_{T_{S_r}(x^*)}(d) = \bigcup_{r \in I(x^*)} N_{S_r}(d) = \bigcup_{r \in I(x^*) \cap I(d)} S_r^\bot. \] (14)

Combing (10), (12), (13) and (14), we establish the inclusion

\[ N_S(x^*; d) \subseteq \{v \in N_S(x^*) \mid v^T d = 0\} = \bigcup_{r \in I(x^*) \cap I(d)} S_r^\bot. \]

Reversely, for any \( v \in \bigcup_{r \in I(x^*) \cap I(d)} S_r^\bot \), there exists \( r' \in I(x^*) \cap I(d) \) such that \( v \in S_{r'}^\bot \). Since there are only finitely many subspaces, we can find a unit vector \( u' \) such that \( u' \in S_{r'} \) but \( u' \notin S_j \) for other \( j \neq r' \). We define the sequences \( \{d^k\} \) as \( d^k := d + \frac{1}{k} u' \). Then \( d^k \to d \) as \( k \to \infty \). Since \( r' \in I(x^*) \cap I(d) \) we have \( x^* \in S_{r'} \) and \( d \in S_{r'} \), which implies that \( d^k \in S_{r'} \) but \( d^k \notin S_j \) for other \( j \neq r' \). For any sequence \( \{t_k\} \) such that \( t_k \to 0^+ \) as \( k \to \infty \), it follows that \( x^* + t_k d^k \in S_{r'} \) but \( d^k \notin S_j \) for other \( j \neq r' \). Therefore, we have

\[ v \in S_{r'}^\bot = N_{S_{r'}}(x^* + t_k d^k) = N_S(x^* + t_k d^k), \]

where the last equation is from \cite{5}. From the definition of the directional normal cone in Definition 2.1, we have \( v \in N_S(x^*; d) \), which implies \( \bigcup_{r \in I(x^*) \cap I(d)} S_r^\bot \subseteq N_S(x^*; d) \) as desired. \( \square \)

### 2.2 The generator sets for disjunctive set of subspaces

Let \( A \) be a set with finitely many linearly independent vectors and \( D \) be a subspace. We say \( A \) is the generator of \( D \) if \( D = \mathcal{G}(A) := \text{span}(A) \).

Now we describe the regular normal cone to the disjunctive set of subspaces by its generator set. Without loss of generality, we assume that each \( S_r \) is represented by

\[ S_r := \{x \in \mathbb{R}^n \mid \langle a_j^r, x \rangle = 0, j \in \mathcal{E}_r\}, \] (15)

where \( \mathcal{E}_r \) is a finite index set and vectors \( \{a_j^r\}_{j \in \mathcal{E}_r} \) are linearly independent. For \( x^* \in S_r \), let

\[ A_{S_r} := \{a_j^r \mid j \in \mathcal{E}_r\} \]
denote the generator of \( S_r \). Then \( N_{S_r}(x^*) = G(A_{S_r}) \).

By (5), the regular normal cone to set \( S \) at \( x^* \in S \) is the intersection of finitely many subspaces, hence it is still a subspace. So we may assume that \( \hat{N}_S(x^*) \) is generated by a set of linearly independent vectors denoted by \( \hat{A}_S(x^*) \). That is, we have

\[
\hat{N}_S(x^*) = G(\hat{A}_S(x^*)). \tag{16}
\]

We call \( \hat{A}_S(x^*) \) satisfying (16) the generator set of the regular normal cone to \( S \) at \( x^* \).

By (7) and (16) we have that for some \( \delta > 0 \),

\[
N_S(x^*) = \bigcup_{x \in B_\delta(x^*)} \hat{N}_S(x) = \bigcup_{x \in B_\delta(x^*)} G(\hat{A}_S(x)). \tag{17}
\]

We also define the set

\[
A_S(x^*) := \bigcup_{x \in B_\delta(x^*)} \hat{A}_S(x), \tag{18}
\]

where \( \delta > 0 \) is the constant satisfying condition (17).

By (6), we know that

\[
N_S(x^*) = \bigcup_{r \in I(x^*)} N_{S_r}(x^*). \tag{19}
\]

Therefore an interesting question is, what is the relationship between generator set \( A_S(x^*) \) and the union of all \( A_{S_r} \) where \( r \in I(x^*) \). While it is shown that the inclusion \( \subseteq \) holds if each \( S_r \subseteq \mathbb{R}^d \) with \( d = 1, 2 \) is convex polyhedral by Lemma 2.1 in [39], in the following lemma, we show that the equality holds if each \( S_r \) is a subspace.

**Lemma 2.1.** Let \( S = \bigcup_{r=1}^R S_r \subseteq \mathbb{R}^q \) where \( S_r \) is a subspace. Then for any \( x^* \in S \), we have

\[
A_S(x^*) = \bigcup_{r \in I(x^*)} A_{S_r}. \tag{20}
\]

**Proof.** By the definition, \( A_S(x^*) = \bigcup_{x \in B_\delta(x^*)} \hat{A}_S(x) \). By (5), we have

\[
\hat{N}_S(x) = \bigcap_{r \in I(x)} N_{S_r}(x) \quad \forall x \in S, \tag{21}
\]

which implies the inclusion

\[
\hat{A}_S(x) \subseteq \bigcup_{r \in I(x)} A_{S_r} \quad \forall x \in S. \tag{22}
\]

For any \( v \in A_S(x^*) = \bigcup_{x \in B_\delta(x^*)} \hat{A}_S(x) \), there exists \( x' \in B_\delta(x^*) \) such that

\[
v \in \hat{A}_S(x') \subseteq \bigcup_{r \in I(x')} A_{S_r} \subseteq \bigcup_{r \in I(x^*)} A_{S_r}. \tag{23}
\]

Indeed, the first inclusion in (23) is from (22) directly. It is not difficult to see that we can take \( \delta > 0 \)
sufficiently small to guarantee \( I(x') \subseteq I(x^*) \). Hence we justify the second inclusion in (23). Therefore,

\[
A_S(x^*) = \bigcup_{x \in \mathcal{B}_S(x^*)} \hat{A}_S(x) \subseteq \bigcup_{r \in I(x^*)} A_{S_r}.
\] (24)

Now we prove the reverse inclusion. Take any \( v \in \bigcup_{r \in I(x^*)} A_{S_r} \). Then there exists \( r \in I(x^*) \) such that \( v \in A_{S_r} \). Since the set \( S \) is the union of finitely many subspaces, we can pick a unit vector \( u_r \) such that \( u_r \in S_r \) but \( u_r \notin S_j \) for other \( j \neq r \). In this way, for any \( 0 < t < \delta \) we know that the point \( x_r := x^* + tu_r \in \mathcal{B}_S(x^*) \) satisfies \( x_r \in S_r \) but \( x_r \notin S_j \) for other \( j \neq r \). Therefore, it follows from (5) that \( \hat{N}_S(x_r) = N_{S_r}(x_r) \), which implies that \( \hat{A}_S(x_r) = A_{S_r} \) and hence we have

\[
v \in A_{S_r} = \hat{A}_S(x_r) \subseteq \bigcup_{x \in \mathcal{B}_S(x^*)} \hat{A}_S(x).
\]

Then, we have

\[
\bigcup_{r \in I(x^*)} A_{S_r} \subseteq \bigcup_{x \in \mathcal{B}_S(x^*)} \hat{A}_S(x) = A_S(x^*),
\] (25)

and hence the proof is complete.

3 Optimality conditions and constraint qualifications for MPDSC

In this section we study optimality conditions for MPDSC of the form:

\[
\min_x f(x) \quad \text{s.t.} \quad g(x) \leq 0, h(x) = 0, \Phi_i(x) \in S, i = 1, \ldots, l,
\] (26)

where \( f : \mathbb{R}^n \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}^m, h : \mathbb{R}^n \to \mathbb{R}^p, \Phi_i : \mathbb{R}^n \to \mathbb{R}^q \), and constraint set \( S := \bigcup_{r=1}^R S_r \subseteq \mathbb{R}^q \) with \( S_r \) being a subspace, \( r = 1, \ldots, R \). Unless otherwise mentioned, we assume that all functions are smooth. We denote the feasible region of problem (26) by \( \mathcal{F} \) and the active set for inequality constraints at \( x^* \) by \( I_g(x^*) := \{ i \in \{1, \ldots, m\} \mid g_i(x^*) = 0 \} \). Problem (26) is more general than what we will need to study CCOP in the form (2). We could also study CCOP in the more general form as in (26). But for simplicity in comparison with other results for CCOP, we do not state CCOP in the more general form.

3.1 Optimality conditions for MPDSC

According to the popular terminology in the mathematical program with equilibrium constraints (MPEC), one normally associate the strong (S-) stationarity with the regular normal cone and the Mordukhovich (M-) stationarity with the limiting normal cone (see e.g. [19, Definition 1]). Hence if \( x^* \) is a feasible point of
MPDSC we say that \( x^* \) is S-/M-stationary if there exists \( \lambda = (\lambda^g, \lambda^h, \lambda^\Phi) \) with \( \lambda^g_i \geq 0, \forall i \in I_g(x^*) \) satisfying

\[
\nabla f(x^*) + \sum_{i \in I_g(x^*)} \lambda^g_i \nabla g_i(x^*) + \sum_{i = 1}^p \lambda^h_i \nabla h_i(x^*) + \sum_{i = 1}^l \nabla \Phi_i(x^*)^T \lambda^\Phi_i = 0, \tag{27}
\]

such that

\[
\lambda^\Phi_i \in \dot{N}_S(\Phi_i(x^*)) \equiv \bigcap_{r \in I(\Phi_i(x^*))} S_r^\perp, \forall i = 1, \ldots, l; \tag{28}
\]

\[
\lambda^\Phi_i \in N_S(\Phi_i(x^*)) \equiv \bigcup_{r \in I(\Phi_i(x^*))} S_r^\perp, \forall i = 1, \ldots, l, \tag{29}
\]

respectively.

Recently the directional versions of S- and M-stationarity for MPDC have been introduced and studied by Gfrerer [21]. We denote the linearization cone of MPDSC (26) at \( x^* \) as

\[
L^{lin}_{x^*} = \begin{cases} 
    d \in \mathbb{R}^n \mid & \nabla g_i(x^*)d \leq 0, \quad i \in I_g(x^*) \\
    & \nabla h_i(x^*)d = 0, \quad i \in \{1, \ldots, p\} \\
    & \nabla \Phi_i(x^*)d \in \bigcup_{r \in I(\Phi_i(x^*))} S_r, \quad i \in \{1, \ldots, l\}
\end{cases},
\]

and the critical cone at \( x^* \) as

\[
C(x^*) := \{ d \in L^{lin}_{x^*} \mid \nabla f(x^*)d \leq 0 \}.
\]

We also denote the active set for inequality constraints at \( x^* \) in direction \( d \) by

\[
I_g^*(d) := \{ i \in I_g(x^*) \mid \nabla g_i(x^*)d = 0 \}. \tag{30}
\]

According to [27, Propositions 3.8 and 3.6], we may define the directional S-/M-stationarity as follows.

**Definition 3.1.** Let \( x^* \) be a feasible point of MPDSC (26) and \( d \in C(x^*) \). We say \( x^* \) is S-/M-stationary \((d)\) if there exists \( \lambda = (\lambda^g, \lambda^h, \lambda^\Phi) \) with \( \lambda^g_i \geq 0, \forall i \in I_g^*(d) \) satisfying

\[
\nabla f(x^*) + \sum_{i \in I_g^*(d)} \lambda^g_i \nabla g_i(x^*) + \sum_{i = 1}^p \lambda^h_i \nabla h_i(x^*) + \sum_{i = 1}^l \nabla \Phi_i(x^*)^T \lambda^\Phi_i = 0, \tag{31}
\]

such that

\[
\lambda^\Phi_i \in \dot{N}_{I_g^*(\Phi_i(x^*))}(\nabla \Phi_i(x^*))d \equiv \bigcap_{r \in I(\Phi_i(x^*)) \cap I(\nabla \Phi_i(x^*))d} S_r^\perp, \forall i = 1, \ldots, l;
\]

\[
\lambda^\Phi_i \in N_{I_g^*(\Phi_i(x^*))}(\nabla \Phi_i(x^*))d \equiv \bigcup_{r \in I(\Phi_i(x^*)) \cap I(\nabla \Phi_i(x^*))d} S_r^\perp, \forall i = 1, \ldots, l,
\]

respectively. We call the above multiplier \( \lambda \) the S-/M- multiplier in direction \( d \) respectively.

In fact, if we take direction \( d = 0 \) in Definition 3.1 then since \( I_g^*(0) = I_g(x^*),I(0) = \{1, \ldots, R\} \), we will
recover S-/M-stationarity for MPDSC [26].

We now discuss under what conditions a local minimizer are S-stationary \((d)\) or M-stationary \((d)\), respectively. When \(d \neq 0\), it is easy to see that the M-stationarity \((d)\) are stronger than the standard M-stationarity while S-stationarity are stronger than the standard S-stationarity \((d)\).

First we recall the following well-known condition.

**Remark 3.1.** Since the metric subregularity is an abstract condition, in practice, one needs to use some verifiable sufficient conditions to ensure its validity. There are some verifiable sufficient conditions in the literature such as the (directional) quasi-/pseudo normality [1] Definition 4.1] and the first/second-order sufficient condition for metric subregularity [20 Theorem 4.3].

We now give the first/second-order sufficient condition for metric subregularity introduced in [20 Theorem 4.3] and the directional MPDC-LICQ [21 Definition 3.6] to our problem setting.

**Definition 3.3.** Let \(x^* \in F\).

(a) We say that the MPDSC first-order sufficient condition for metric subregularity (MPDSC-FOSCMS) holds at \(x^*\) in direction \(d \in L_{\text{lin}}^1 (x^*)\) if there is no nonzero vector \((\lambda^g, \lambda^h, \lambda^p)\) \(\in \mathbb{R}^m \times \mathbb{R}^n \times \prod_{i=1}^l \mathbb{R}^q\) with \(\lambda^g \geq 0\), \(\forall i \in T_g^1 (d)\) and \(\lambda^p \in \bigcup_{r \in I(\Phi_i(x^*))(\nabla \Phi_i(x^*))d} S^+_r\) satisfying

\[
\sum_{i \in T_g^1 (d)} \lambda^g_i \nabla g_i (x^*) + \sum_{i=1}^p \lambda^h_i \nabla h_i (x^*) + \sum_{i=1}^l \nabla \Phi_i (x^*)^T \lambda^p_i = 0. \tag{32}
\]

(b) Suppose that all functions are twice continuously differentiable. We say that the MPDSC second-order sufficient condition for metric subregularity (MPDSC-SOSCMS) holds at \(x^*\) in direction \(d \in L_{\text{lin}}^1 (x^*)\) if there is no nonzero vector \((\lambda^g, \lambda^h, \lambda^p)\) \(\in \mathbb{R}^m \times \mathbb{R}^n \times \prod_{i=1}^l \mathbb{R}^q\) with \(\lambda^g \geq 0\), \(\forall i \in T_g^1 (d)\) and \(\lambda^p \in \bigcup_{r \in I(\Phi_i(x^*))(\nabla \Phi_i(x^*))d} S^+_r\) satisfying \(32\) and the following second-order condition

\[
d^T \nabla^2 \mathcal{L}^0 (x^*, \lambda^g, \lambda^h, \lambda^p) d \geq 0,
\]

where

\[
\mathcal{L}^0 (x, \lambda^g, \lambda^h, \lambda^p) := \langle \lambda^g, g(x) \rangle + \langle \lambda^h, h(x) \rangle + \langle \lambda^p, \Phi(x) \rangle.
\]
We say that MPDSC linear independence constraint qualification (MPDSC-LICQ) holds at \( x^* \) in direction \( d \in L^\perp_{\text{lin}}(x^*) \) if there is no nonzero vector \((\lambda^g, \lambda^h, \lambda^\Phi)\) with \( \lambda^\Phi \in \sum_{r \in I(\Phi_i(x^*)) \cap I(\nabla \Phi_i(x^*) d)} S_r^\perp \) such that (32) holds.

When \( d = 0 \), MPDSC-FOSCMS (d) reduces to MPDSC-NNAMCQ, MPDSC-LICQ (d) reduces to MPDSC-LICQ defined in Definition 3.4 respectively. It is obvious that MPDSC-FOSCMS (d) and MPDSC-LICQ (d) with \( d \neq 0 \) are weaker than NNAMCQ and MPDSC-LICQ, respectively.

We denote the Lagrangian function of problem (26) by
\[
\mathcal{L}(x, \lambda^g, \lambda^h, \lambda^\Phi) := f(x) + \langle \lambda^g, g(x) \rangle + \langle \lambda^h, h(x) \rangle + \langle \lambda^\Phi, \Phi(x) \rangle.
\]

Similarly as in [27, Section 4], the following theorem can be obtained directly from the optimality conditions for the disjunctive program originally obtained by Gfrerer in [21, Theorems 3.3 and 3.17, Corollary 3.20] and summarized in [12, Theorem 6.1] with the calculus rules of Cartesian product rules of tangent and directional normal cones in [41, Proposition 3.3].

**Theorem 3.1.** Let \( x^* \) be a local optimal solution of MPDSC (26) and \( d \in C(x^*) \).

(i) If MSCQ in direction \( d \) holds at \( x^* \), then \( x^* \) must be M-stationary (d). Moreover if all functions are twice continuously differentiable, then there exists an M-stationary (d) multiplier \((\lambda^g, \lambda^h, \lambda^\Phi)\) such that the second-order necessary optimality condition holds:
\[
d^T \nabla^2 x \mathcal{L}(x^*, \lambda^g, \lambda^h, \lambda^\Phi) d \geq 0.
\] (33)

(ii) If MPDSC-LICQ (d) is fulfilled at \( x^* \), then \( x^* \) must be S-stationary (d). Moreover if all functions are twice continuously differentiable, then there exists an S-stationary (d) multiplier \((\lambda^g, \lambda^h, \lambda^\Phi)\) such that the second-order necessary optimality condition \((33)\) holds.

Conversely, let \( x^* \) be a feasible solution of MPDSC (26). Suppose that all functions are twice continuously differentiable and for every nonzero critical direction \( 0 \neq d \in C(x^*) \) there exists an S-stationary (d) multiplier \((\lambda^g, \lambda^h, \lambda^\Phi)\) such that the second-order condition strictly holds:
\[
d^T \nabla^2 x \mathcal{L}(x^*, \lambda^g, \lambda^h, \lambda^\Phi) d > 0.
\]

Then \( x^* \) is a strict local minimizer of MPDSC (26).

### 3.2 Constraint qualifications for MPDSC from MPDC

Now using the formulas for tangent and normal cones in Proposition 2.1, we obtain some constraint qualifications for MPDSC from the corresponding ones for MPDC (see e.g., [39, Definitions 3.1 and 4.2]).

**Definition 3.4.** Let \( x^* \in \mathcal{F} \) be a feasible point for MPDSC (26).
1. We say that \( x^* \) satisfies MPDSC linear independence constraint qualification (MPDSC-LICQ) if there is no nonzero vector \( (\lambda^g, \lambda^h, \lambda^\Phi) \in \mathbb{R}^m \times \mathbb{R}^n \times \prod_{i=1}^{l} \mathbb{R}^q \) with \( \lambda^\Phi_i \in \sum_{r \in \Phi_i(x^*)} S^\perp_r \) satisfying

\[
0 = \sum_{i \in I_g(x^*)} \lambda^g_i \nabla g_i(x^*) + \sum_{i=1}^{p} \lambda^h_i \nabla h_i(x^*) + \sum_{i=1}^{l} \nabla \Phi_i(x^*)^T \lambda^\Phi_i.
\] (34)

2. We say that \( x^* \) satisfies MPDSC no nonzero abnormal multiplier constraint qualification (MPDSC-NNAMCQ) if there is no nonzero vector \( (\lambda^g, \lambda^h, \lambda^\Phi) \in \mathbb{R}^m \times \mathbb{R}^n \times \prod_{i=1}^{l} \mathbb{R}^q \) with \( \lambda^g_i \geq 0, i \in I_g(x^*), \lambda^\Phi_i \in \bigcup_{r \in \Phi_i(x^*)} S^\perp_r \) satisfying (34).

3. We say that \( x^* \) satisfies MPDSC relaxed constant positive linear dependence constraint qualification (MPDSC-RCPLD) if the following conditions hold.

   (i) The vectors \( \{\nabla h_i(x)\}_{i=1}^{p} \) have the same rank for all \( x \in \mathbb{B}_\varepsilon(x^*) \) for some \( \varepsilon > 0 \);

   (ii) Let \( J \subseteq \{1, \ldots, p\} \) be such that the set of vectors \( \{\nabla h_i(x^*)\}_{i \in J} \) is a basis for \( \text{span}\{\nabla h_i(x^*)\}_{i=1}^{p} \).

   If there exist index sets \( I \subseteq I_g(x^*) \), a nonzero vector \( (\lambda^g, \lambda^h, \lambda^\Phi) \in \mathbb{R}^m \times \mathbb{R}^n \times \prod_{i=1}^{l} \mathbb{R}^q \) with \( \lambda^\Phi_i \in \bigcup_{r \in \Phi_i(x^*)} S^\perp_r, i \in \{1, \ldots, l\} \) satisfying

\[
0 = \sum_{i \in I} \lambda^g_i \nabla g_i(x^*) + \sum_{i \in J} \lambda^h_i \nabla h_i(x^*) + \sum_{i=1}^{l} \nabla \Phi_i(x^*)^T \lambda^\Phi_i;
\] (35)

then the set of vectors

\[
\{\nabla g_i(x^k)\}_{i \in I} \cup \{\nabla h_i(x^k)\}_{i \in J} \cup \bigcup_{\beta_i \in A_i, i \in \{1, \ldots, l\}} \{\nabla \Phi_i(x^k)^T \beta_i\}
\] (36)

is linearly dependent for \( k \) sufficiently large, for all sequences \( \{x^k\} \) satisfying \( x^k \to x^*, x^k \neq x^* \) as \( k \to \infty \) and any set of linearly independent vectors \( A_i \) with the following conditions:

\[
\text{if } \lambda^\Phi_i \neq 0, \text{ then } A_i \subseteq \mathcal{A}_{\Phi_i}(x^*), \lambda^\Phi_i \in \mathcal{G}(A_i) \subseteq \bigcup_{r \in \Phi_i(x^*)} S^\perp_r; \\
\text{if } \lambda^\Phi_i = 0, \text{ then } A_i = \emptyset.
\] (37)

4. We say that \( x^* \) satisfies MPDSC constant positive linear dependence constraint qualification (MPDSC-CPLD) if all condition (ii) in MPDSC-RCPLD hold with the index set \( J \) is taken as a arbitrary subset of \( \{1, \ldots, p\} \).

5. We say that \( x^* \) satisfies MPDSC constant rank constraint qualification (MPDSC-CRCQ) if for every index sets \( I \subseteq I_g(x^*), J \subseteq \{1, \ldots, p\}, L \subseteq \{1, \ldots, l\} \) and \( \lambda^\Phi_i \in \bigcup_{r \in \Phi_i(x^*)} S^\perp_r, i \in L \), the set of vectors

\[
\{\nabla g_i(x^*)\}_{i \in I} \cup \{\nabla h_i(x^*)\}_{i \in J} \cup \bigcup_{\beta_i \in A_i, i \in L} \{\nabla \Phi_i(x^*)^T \beta_i\}
\]
and the set of vectors
\[
\{\nabla g_i(x^k)\}_{i \in I} \cup \{\nabla h_i(x^k)\}_{i \in J} \cup \bigcup_{\beta_i \in A_i, i \in L} \{\nabla \Phi_i(x^k)^T \beta_i\}
\]

have the same rank for \(k\) sufficiently large, for all sequences \(\{x^k\}\) satisfying \(x^k \to x^*, x^k \neq x^*\) as \(k \to \infty\) and any set of linearly independent vectors \(A_i\) satisfying condition (37).

6. We say that \(x^*\) satisfies MPDSC relaxed constant rank constraint qualification (MPDSC-RCRCQ) if the index set \(J\) is taken as \(\{1, \ldots, p\}\) in MPDSC-CRCQ.

7. We say that \(x^*\) satisfies MPDSC-ERCPLD if the following conditions hold.

(i) The vectors \(\{\nabla h_i(x)\}_{i = 1}^p\) have the same rank for all \(x \in B_\varepsilon(x^*)\) for some \(\varepsilon > 0\);
(ii) Let \(J \subseteq \{1, \ldots, p\}\) be such that the set of vectors \(\{\nabla h_i(x^*)\}_{i \in J}\) is a basis for \(\text{span}\{\nabla h_i(x^*)\}_{i = 1}^p\).
If there exist index sets \(I \subseteq I_g(x^*)\) and \(L \subseteq \{1, \ldots, l\}\), \(\lambda_i^\Phi \in \bigcup_{r \in I_g(x^*)} S_r^\perp\), \(i \in L\) such that the set of vectors
\[
\{\nabla g_i(x^*)\}_{i \in I} \cup \{\nabla h_i(x^*)\}_{i \in J} \cup \bigcup_{\beta_i \in A_i, i \in L} \{\nabla \Phi_i(x^*)^T \beta_i\}
\]
is positive linearly dependent, then the set of vectors
\[
\{\nabla g_i(x^k)\}_{i \in I} \cup \{\nabla h_i(x^k)\}_{i \in J} \cup \bigcup_{\beta_i \in A_i, i \in L} \{\nabla \Phi_i(x^k)^T \beta_i\}
\]
is linearly dependent for \(k\) sufficiently large, for all sequences \(\{x^k\}\) satisfying \(x^k \to x^*, x^k \neq x^*\) as \(k \to \infty\) and any set of linearly independent vectors \(A_i\) satisfying condition (37).

3.3 RCPLD as a sufficient condition for MSCQ

Unlike the differentiable nonlinear program, RCPLD for MPDC is not a sufficient condition for MSCQ in general. In order to obtain a RCPLD type sufficient condition for MSCQ, Xu and Ye introduced MPDC-piecewise RCPLD and show that it is a sufficient condition for error bounds [39, Theorem 4.2]. It turns out that for MPDSC many constraint qualifications such as CRCQ, RCRCQ, CPLD, ERCPLD, and RCPLD coincide with their piecewise versions, respectively. Here we only focus on the discussions on RCPLD and the piecewise RCPLD and omit discussions on other constraint qualifications as well as their piecewise versions since the derivation would be similar. Consequently, since RCPLD coincides with piecewise RCPLD, RCPLD implies error bounds for MPDSC.

First we present a simple example to illustrate the equivalence of RCPLD and the piecewise RCPLD for MPDSC.

Example 3.1. Consider the following cardinality constrained system in \(\mathbb{R}^3\):
\[
g(x) \leq 0, \ h(x) = 0, \ ||x||_0 \leq 2. \quad (38)
\]
From Figure 1 we know that problem (38) is equivalent to

\[ g(x) \leq 0, \ h(x) = 0, \ \Phi(x) := x \in S, \]

with \( S = S_1 \cup S_2 \cup S_3 \) such that \( S_1 = \{0\} \times \mathbb{R} \times \mathbb{R}, \ S_2 = \mathbb{R} \times \{0\} \times \mathbb{R} \) and \( S_3 = \mathbb{R} \times \mathbb{R} \times \{0\} \). Assume that point \( x^* = (0,0,1) \) is feasible for system (38). According to Definition 3.4(7), we say that RCPLD holds at \( x^* \) if

1. The vectors \( \{\nabla h_i(x)\}_i \) have the same rank for all \( x \in \overline{B}_\varepsilon(x^*) \) for some \( \varepsilon > 0 \);

2. Let \( J \subseteq \{1, \ldots, p\} \) be such that the set of vectors \( \{\nabla h_i(x^*)\}_i \) is a basis for \( \text{span}\{\nabla h_i(x^*)\}_i \). If there exist an index set \( I \subseteq \mathcal{I}_g(x^*) \), a multiplier \( \lambda = (\lambda^g, \lambda^h, \lambda^\Phi) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^3 \) with \( \lambda_i^g \geq 0, i \in I \) and \( \lambda^\Phi \in \mathbb{R}^n \) such that

\[
0 = \sum_{i \in I} \lambda_i^g \nabla g_i(x^*) + \sum_{i \in J} \lambda_i^h \nabla h_i(x^*) + \lambda^\Phi,
\]

then the set of vectors

\[
\{\nabla g_i(x^*)\}_i \cup \{\nabla h_i(x^*)\}_i \cup A
\]

is linearly dependent for \( k \) sufficiently large, for all sequences \( \{x^k\} \) satisfying \( x^k \rightarrow x^* \), \( x^k \neq x^* \) as \( k \rightarrow \infty \) and any set of linearly independent vectors \( A \subseteq A_g(x^*) = \{e_1, e_2\} \) such that

\[
\begin{cases}
\text{if } \lambda^g \neq 0 & \text{then } \lambda^g \in \mathcal{G}(A) \subseteq S_1^+ \cup S_2^+, \\
\text{if } \lambda^g = 0 & \text{then } A = \emptyset.
\end{cases}
\]

If \( \lambda^g \neq 0 \), then since \( \lambda^g \in S_1^+ \cup S_2^+ \), either \( \lambda^g \in \Phi(e_1) \subseteq S_1^+ = \mathbb{R} \times \{0\} \times \{0\} \) or \( \lambda^g \in \Phi(e_2) \subseteq S_2^+ = \{0\} \times \mathbb{R} \times \{0\} \). Hence the set \( A \) above must be taken as \( e_1 \) if \( \lambda^g \in S_1^+ \) and \( e_2 \) if \( \lambda^g \in S_2^+ \).

We now consider the piecewise RCPLD. There are three partitions of index set \( \{1\} \) into sets \( P = \{P_1, P_2, P_3\} \): (i) \( P_1 = \{1\}, P_2 = \emptyset, P_3 = \emptyset \); (ii) \( P_1 = \emptyset, P_2 = \{1\}, P_3 = \emptyset \); (iii) \( P_1 = \emptyset, P_2 = \emptyset, P_3 = \{1\} \). Since \( x^* \notin S_3 \), we only have two possible subsystems. Therefore, the piecewise RCPLD holds for system (38) at \( x^* \) if RCPLD holds for each of the following two subsystems at \( x^* \):

\[
\begin{cases}
\text{(P1)} & g(x) \leq 0, \\
\text{and} & h(x) = 0, \\
\text{x} & x \in S_1.
\end{cases}
\]

But RCPLD for subsystem \( P_1 \) holding at \( x^* \) if condition (i) above holds and the following conditions hold. Let \( J \subseteq \{1, \ldots, p\} \) be such that the set of vectors \( \{\nabla h_i(x^*)\}_i \) is a basis for \( \text{span}\{\nabla h_i(x^*)\}_i \). If there exist an index set \( I \subseteq \mathcal{I}_g(x^*) \), a multiplier \( \lambda = (\lambda^g, \lambda^h, \lambda^\Phi) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^3 \) with \( \lambda_i^g \geq 0, i \in I \) and \( \lambda^\Phi \in S_1^+ \) such that \( 39 \) holds, then the set of vectors \( \{\nabla g_i(x^k)\}_i \cup \{\nabla h_i(x^k)\}_i \cup A \) is linearly dependent for \( k \) sufficiently large, for all sequences \( \{x^k\} \) satisfying \( x^k \rightarrow x^* \), \( x^k \neq x^* \) as \( k \rightarrow \infty \) and any set of linearly independent
strings $A \subseteq A_{S_1} = \{e_1\}$ such that

$$\begin{align*}
\text{if } \lambda^g \neq 0 & \quad \text{then } \lambda^g \in \mathcal{G}(A) \subseteq S_1^+, \\
\text{if } \lambda^g = 0 & \quad \text{then } A = \emptyset.
\end{align*}$$

RCPLD for subsystem $P_2$ holding at $x^*$ if the RCPLD for subsystem $P_1$ with $S_1^+$ replaced by $S_2^+$ and $e_1$ replaced by $e_2$. Hence obviously RCPLD coincides with the piecewise RCPLD.

Now we consider the general case and let $x^*$ be feasible for MPDSC \[26\]. Let sets $P_1, \ldots, P_R$ (sometimes some of them may be empty) be a partition of $\{1, \ldots, l\}$. We denote such partition by $P := \{P_1, \ldots, P_R\}$ and consider the subsystem for partition $P$:

$$
\begin{align*}
g(x) \leq 0, \quad h(x) &= 0, \\
\Phi_i(x) &\in S_1, \quad i \in P_1, \\
& \vdots \\
\Phi_i(x) &\in S_R, \quad i \in P_R.
\end{align*}
$$

We denote the feasible region of subsystem \[40\] by $F_P$. Applying the definition for MPDC \[39\] Definition 4.1] to MPDSC we have the following definition.

**Definition 3.5.** (MPDSC-PRCPLD) We say that the piecewise RCPLD holds for MPDSC \[26\] at $x^* \in F$, if MPDSC-RCPLD holds for subsystem \[40\] for any partition $P = \{P_1, \ldots, P_R\}$ such that $x^* \in F_P$. That is, the following conditions hold for any partition $P = \{P_1, \ldots, P_R\}$ such that $x^* \in F_P$.

(i) The vectors $\{\nabla h_i(x)\}_{i=1}^P$ have the same rank for all $x \in \mathbb{B}_\varepsilon(x^*)$ for some $\varepsilon > 0$;

(ii) Let $J \subseteq \{1, \ldots, p\}$ be such that the set of vectors $\{\nabla h_i(x^*)\}_{i \in J}$ is a basis for $\text{span}\{\nabla h_i(x^*)\}_{i=1}^P$. If there exist index sets $I \subseteq I_g(x^*)$, a nonzero vector $(\lambda^g, \lambda^h, \lambda^\Phi)T \in \mathbb{R}^m \times \mathbb{R}^n \times \prod_{i=1}^m \mathbb{R}^q$ with $\lambda_i^g \geq 0$, $i \in I$ and $\lambda_i^\Phi \in S_1^+$ for $i \in P_r$, $r = 1, \ldots, R$ such that

$$0 = \sum_{i \in I} \lambda_i^g \nabla g_i(x^*) + \sum_{i \in J} \lambda_i^h \nabla h_i(x^*) + \sum_{i \in P_1} \nabla \Phi_i(x^*)^T \lambda_i^\Phi + \cdots + \sum_{i \in P_R} \nabla \Phi_i(x^*)^T \lambda_i^\Phi, \quad (41)$$

then the set of vectors

$$\{\nabla g_i(x^k)\}_{i \in I} \cup \{\nabla h_i(x^k)\}_{i \in J} \bigcup_{\beta_i^r \in A_{S_r}^r, i \in P_r, r=1, \ldots, R} \{\nabla \Phi_i(x^k)^T \beta_i^r\}$$

is linearly dependent for $k$ sufficiently large, for all sequences $\{x^k\}$ satisfying $x^k \to x^*$, $x^k \neq x^*$ as $k \to \infty$ and any set of linearly independent vectors $A_i^r$ with $r \in I(\Phi_i(x^*))$ satisfying the following condition

$$\begin{align*}
\text{if } \lambda^g \neq 0, & \quad \text{then } \lambda_i^g \in \mathcal{G}(A_i^r) \subseteq S_r^+, \quad A_i^r \subseteq A_{S_r}^r, \\
\text{if } \lambda^g = 0, & \quad \text{then } A_i^r = \emptyset.
\end{align*}$$

\(15\)
Now we show that MPDSC-RCPLD coincides with MPDSC-piecewise RCPLD. As we can see as follows, the following two equalities from Proposition 2.1 and Lemma 2.1 play key roles in the proof of Theorem 3.2:

\[ N_S(\Phi_i(x^*)) = \bigcup_{r \in I(\Phi_i(x^*))} N_{S_r}(\Phi_i(x^*)), \tag{44} \]
\[ A_S(\Phi_i(x^*)) = \bigcup_{r \in I(\Phi_i(x^*))} A_{S_r}. \tag{45} \]

In fact, we can give a more accurate description of generator sets \( A_i \), \( i = 1, \ldots, l \) when we consider MPDSC instead of general MPDC. The following lemma is useful in what follows.

**Lemma 3.1.** For \( 0 \neq \lambda \in N_S(y) = \bigcup_{r \in I(y)} S_r^\perp \), suppose that \( \lambda \) is generated by a set of linearly independent vectors \( A \) which is a subset of the generator set of the limiting normal cone of \( S \) at \( y \), i.e.,

\[ \lambda \in G(A) \subseteq N_S(y) = \bigcup_{r \in I(y)} S_r^\perp, \quad A \subseteq A_S(y) = \bigcup_{r \in I(y)} A_{S_r}. \]

Then if \( \lambda \in N_{S_r}(y) = S_r^\perp \), \( \lambda \) is also generated by a subset of linearly independent vectors \( A \) which is a subset of the generator of \( S_r \), i.e.,

\[ \lambda \in G(A) \subseteq G(A_{S_r}) = S_r^\perp, \quad A \subseteq A_{S_r}. \tag{46} \]

**Proof.** Suppose on the contrary that (46) does not hold. Without loss of generality, we assume there are \( r_1, r_2 \in I(y) \) such that

\[ A \subseteq A_{S_{r_1}} \cup A_{S_{r_2}} \text{ but } A \nsubseteq A_{S_{r_1}} \text{ and } A \nsubseteq A_{S_{r_2}}. \]

Since both \( S_{r_1} \) and \( S_{r_2} \) are subspaces, it follows that

\[ G(A) \subseteq \text{span}\{S_{r_1}^\perp \cup S_{r_2}^\perp\} \text{ but } G(A) \nsubseteq S_{r_1}^\perp \text{ and } G(A) \nsubseteq S_{r_2}^\perp, \]

which contradicts the fact that

\[ G(A) \subseteq N_S(y) = \bigcup_{r \in I(y)} S_r^\perp. \]

In this way, we complete the proof. \( \square \)

**Theorem 3.2.** For (MPDSC), RCPLD coincides with the piecewise RCPLD.

*Proof.** Without loss of generality assume that the number of subspaces are more than 2. Condition (i) is identical for both RCPLD and the piecewise RCPLD. In condition (ii), (35) is also identical to (41). Hence (35) holds for \( \lambda = (\lambda^g, \lambda^h, \lambda^\Phi) \) with \( \lambda^\Phi \in \bigcup_{r \in I(\Phi_i(x^*))} S_r^\perp \) if and only if (41) for the same \( \lambda \) when \( \lambda^\Phi \in S_r^\perp, r \in I(\Phi_i(x^*)) \). The rest of the proof follows from applying Lemma 3.1. \( \square \)

The discussions on other constraint qualifications are similar as that of RCPLD, hence we have the following corollary.
Corollary 3.1. For mathematical programs with disjunctive subspaces constraints (MPDSC), their constraint qualifications such as CRCQ, RCRCQ, CPLD, ERCPLD, and RCPLD coincide with their piecewise versions, respectively.

Based on Theorem 3.2 now we show that for MPDSC the constraint qualification RCPLD implies error bounds for MPDSC.

Theorem 3.3. Suppose that MPDSC-RCPLD holds at $x^*$ which is feasible for problem (26). Then, the error bound property holds at $x^*$.

Proof. The proof is rather straightforward by combining Theorem 3.2 and [39, Theorem 4.2]. The constraint qualification MPDSC-RCPLD implies the error bound property since MPDSC-RCPLD coincides with MPDSC-piecewise RCPLD.

We conclude Section 3 with Figure 2 which summarizes the relations among various constraint qualifications for MPDSC.
4 Optimality conditions and sufficient condition for error bounds for CCOP

In this section we apply main results obtained for MPDSC to CCOP. Recall that \( \mathcal{I}_s := \{ \mathcal{I} \subseteq \{1, 2, \ldots, n\} | |\mathcal{I}| = s \} \) and \( \mathbb{R}_\mathcal{I} := \text{span}\{e_i | i \in \mathcal{I}\} \). In this section \( S := \bigcup_{\mathcal{I} \in \mathcal{I}_s} \mathbb{R}_\mathcal{I} \) and use the notation \( I_\pm(x) := \{ i \in \{1, \ldots, n\} | x_i \neq 0 \} \) and \( I_0(x) := \{ i \in \{1, \ldots, n\} | x_i = 0 \} \) for a given vector \( x \in \mathbb{R}^n \).

The following proposition is immediate from Propositions 2.1-2.3 taking into account that \( r \in I(x^*) \) here means that \( x^* \in \mathbb{R}_\mathcal{I} \) and \( I_\pm(x^*) \subseteq I \in \mathcal{I}_s \).

**Proposition 4.1.** Let \( x^* \) and \( d \in \mathbb{R}^n \). Then, we have the expressions of various tangent and normal cones in Table 1.

| Cones          | \(|I_\pm(x^*)| < s \) (or \(|I_\pm(x^*) \cup I_\pm(d)| < s \)) | \(|I_\pm(x^*)| = s \) (or \(|I_\pm(x^*) \cup I_\pm(d)| = s \)) |
|----------------|-------------------------------------------------|-------------------------------------------------|
| \( T_S(x^*) \) | \( \bigcup_{I \in \mathcal{I}_s} \mathbb{R}_I \) | \( \mathbb{R}_{I_\pm(x^*)} \) |
| \( \bar{N}_S(x^*) \) | \( \{0\} \) | \( \mathbb{R}_{I_\pm(x^*)}^\perp \) |
| \( N_S(x^*) \) | \( \bigcup_{I \in \mathcal{I}_s} \mathbb{R}_I^\perp \) | \( \mathbb{R}_{I_\pm(x^*)}^\perp \) |
| \( N_S(x^*;d) \) | \( \bigcup_{I \in \mathcal{I}_s} \mathbb{R}_I^\perp \) | \( \mathbb{R}_{I_\pm(x^*) \cup I_\pm(d)}^\perp \) |
| \( \hat{N}_{I_0}(x^*) \) | \( \{0\} \) | \( \mathbb{R}_{I_\pm(x^*) \cup I_\pm(d)}^\perp \) |

**Table 1:** Various cones to set \( S \) in CCOP

Now let us write down the constraint qualifications for CCOP from the ones for MPDSC in Definition 3.4. In order to do that we need to specify the condition \( (37) \) to CCOP. For problem CCOP written in the form \( (2), l = 1, \Phi(x) = x, S_r = \mathbb{R}_\mathcal{I} \) and \( \lambda_{N^\Phi} = \eta \). Moreover, \( r \in I(\Phi_i(x^*)) \) here means that \( x^* \in \mathbb{R}_\mathcal{I} \) and \( I_\pm(x^*) \subseteq I \in \mathcal{I}_s \). Hence the condition \( (37) \) is

\[
\text{if } \eta \neq 0, \quad \text{then } A \subseteq A_S(x^*), \quad \eta \in \mathcal{G}(A) \subseteq \bigcup_{I \in \mathcal{I}_s} \mathbb{R}_I^\perp; \\
\text{if } \eta = 0, \quad \text{then } A = \emptyset.
\]

Since for \( I_\pm(x^*) \subseteq I \in \mathcal{I}_s \), we have \( N_{\mathbb{R}_\mathcal{I}}(x^*) = \mathbb{R}_\mathcal{I}^\perp \),

\[ A_{\mathbb{R}_\mathcal{I}} = \{ e_i | i \in K \text{ such that } K \subseteq I_0(x^*) \text{ and } |K| \leq n - s \}. \]

Hence by \( (20) \),

\[ A_S(x^*) = \bigcup_{I_\pm(x^*) \subseteq I \in \mathcal{I}_s} A_{\mathbb{R}_\mathcal{I}} \]
and so in the case where $0 \neq \eta \in \mathcal{G}(A)$, by Lemma 3.1 we can take the generator set $A$ above as

$$A = \{e_i | i \in K \text{ such that } I_{\pm}(\eta) \subseteq K \subseteq I_0(x^*) \text{ and } |K| \leq n-s\}.$$ 

Therefore, we obtain the following definition. Note that the constraint qualification CCOP-LICQ is the same as the one proposed in [30, Section 5.3].

**Definition 4.1.** (Constraint qualifications for CCOP) Let $x^*$ be feasible for CCOP. We say that $x^*$ satisfies

1. **CCOP-LICQ** if the family of vectors $\{\nabla g_i(x^*)\}_{i \in \mathcal{I}_g(x^*)} \cup \{\nabla h_i(x^*)\}_{i=1}^P \cup \{e_i\}_{i \in I_0(x^*)}$ is linearly independent;

2. **CCOP-NNAMCQ** if there exists no nonzero vector $(\lambda^g, \lambda^h, \eta) \in \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^n$ with $\lambda^g_i \geq 0, \forall i \in \mathcal{I}_g(x^*)$ and

   $$\eta_i = 0, \forall i \in I_{\pm}(x^*) \text{ and } \|\eta\|_0 \leq n-s.$$ (47)

   satisfying

   $$\sum_{i \in \mathcal{I}_g(x^*)} \lambda^g_i \nabla g_i(x^*) + \sum_{i=1}^P \lambda^h_i \nabla h_i(x^*) + \eta = 0;$$ (48)

3. **CCOP-RCPLD** if the following conditions hold.

   (i) The vectors $\{\nabla h_i(x)\}_{i=1}^P$ have the same rank for all $x \in \mathbb{B}_\varepsilon(x^*)$ for some $\varepsilon > 0$;

   (ii) Let $J \subseteq \{1, \ldots, p\}$ be such that the set of vectors $\{\nabla h_i(x^*)\}_{i \in J}$ is a basis for $\text{span}\{\nabla h_i(x^*)\}_{i=1}^P$. If there exist an index set $I \subseteq \mathcal{I}_g(x^*)$, a nonzero vector $(\lambda^g, \lambda^h, \eta) \in \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^n$ with $\lambda^g_i \geq 0, \forall i \in \mathcal{I}_g(x^*)$ and $\eta$ such that (47) and

   $$\sum_{i \in I} \lambda^g_i \nabla g_i(x^*) + \sum_{i \in J} \lambda^h_i \nabla h_i(x^*) + \eta = 0,$$ (49)

   hold then the set of vectors

   $$\{\nabla g_i(x^k)\}_{i \in I} \cup \{\nabla h_i(x^k)\}_{i \in J} \cup \{e_i\}_{i \in K}$$

   is linearly dependent for all sequences $\{x^k\}$ satisfying $x^k \to x^*$, $x^k \neq x^*$ as $k \to \infty$ and all set $K$ satisfying

   if $\eta \neq 0$, then $I_{\pm}(\eta) \subseteq K \subseteq I_0(x^*)$, $|K| \leq n-s$,

   if $\eta = 0$, then $K = \emptyset.$ (50)

4. **CCOP-CPLD** if all condition (ii) in CCOP-RCPLD hold with the index set $J$ is taken as a arbitrary subset of $\{1, \ldots, p\}$.

5. **CCOP-CRCQ** if for every index sets $I \subseteq \mathcal{I}_g(x^*)$, $J \subseteq \{1, \ldots, p\}$ and any $\eta$ satisfying (47) such that the family of vectors

   $$\{\nabla g(x^*)\}_{i \in I} \cup \{\nabla h(x^*)\}_{i \in J} \cup \{e_i\}_{i \in K}$$

   19
and the set of vectors
\[ \{\nabla g(x^k)\}_{i \in I} \cup \{\nabla h(x^k)\}_{i \in J} \cup \{e_i\}_{i \in K} \]

have the same rank for all sequences \{x^k\} satisfying \(x^k \to x^*, x^k \neq x^*\) as \(k \to \infty\) and any set \(K\) satisfying (50).

6. CCOP-RCRCQ if the index set \(J\) is taken as \(\{1, \ldots, p\}\) in CCOP-CRCQ.

7. CCOP-ERCPLD if the following conditions hold.

(i) The vectors \(\{\nabla h_i(x)\}_{i=1}^p\) have the same rank for all \(x \in \mathbb{R}_\varepsilon(x^*)\) for some \(\varepsilon > 0\);

(ii) Let \(J \subseteq \{1, \ldots, p\}\) be such that the set of vectors \(\{\nabla h_i(x^*)\}_{i \in J}\) is a basis for \(\text{span}\{\nabla h_i(x^*)\}_{i=1}^p\).

If there exists an index set \(I \subseteq I_g(x^*), \eta\) satisfying (47) such that the set of vectors
\[ \{\nabla g_i(x^*)\}_{i \in I} \cup \{\nabla h_i(x^*)\}_{i \in J} \cup \{e_i\}_{i \in K} \]
is positive linearly dependent, then the set of vectors
\[ \{\nabla g_i(x^k)\}_{i \in I} \cup \{\nabla h_i(x^k)\}_{i \in J} \cup \{e_i\}_{i \in K} \]
is linearly dependent for all sequences \(\{x^k\}\) satisfying \(x^k \to x^*, x^k \neq x^*\) as \(k \to \infty\), and any set \(K\) satisfying (50).

In Figure 3 we summarize the relations among constraint qualifications for CCOP we discussed above.

We now study optimality conditions for CCOP. The critical cone of CCOP at \(x^*\) is
\[ \mathcal{C}(x^*) = \left\{ d \in \mathbb{R}^n \left| \begin{array}{ll} \nabla g_i(x^*)d \leq 0, & i \in I_g(x^*) \\
abla h_i(x^*)d = 0, & i \in \{1, \ldots, p\} \\
abla \Phi_i(x^*)d \in \bigcup_{I \in I_g} \mathbb{R}_\gamma, & i \in \{1, \ldots, l\} \\
\nabla f(x^*)d \leq 0 & \end{array} \right. \right\}. \]

Applying Definition 3.1 to CCOP (2), with the help of Proposition 4.1, we obtain the following directional S-/M-stationary conditions for CCOP. Recall that the active set \(I_g^*(d)\) is defined in (30).

**Definition 4.2.** Let \(x^* \in \mathbb{R}^n\) be a feasible point of CCOP and \(d \in \mathcal{C}(x^*)\).

(i) We say \(x^*\) is S-stationary in direction \(d\) if there exists \((\lambda^g, \lambda^h, \eta) \in \mathbb{R}_\gamma^m \times \mathbb{R}_\gamma^p \times \mathbb{R}^n\) with \(\lambda_i^g \geq 0, \forall i \in I_g^*(d)\) satisfying
\[ \nabla f(x^*) + \sum_{i \in I_g^*(d)} \lambda_i^g \nabla g_i(x^*) + \sum_{i=1}^p \lambda_i^h \nabla h_i(x^*) + \eta = 0, \quad (51) \]
such that

(a) if \(|I_+(x^*) \cup I_-(d)| = s\), then \(\eta_i = 0, \forall i \in I_+(x^*) \cup I_-(d)\).
Figure 3: Relations among new constraint qualifications for CCOP

(b) if $|I_\pm(x^* \cup I_\pm(d)| < s$, then $\eta_i = 0$, $\forall i \in \{1, \ldots, n\}$.

(ii) We say $x^*$ is M-stationary (d) if there exists $(\lambda^g, \lambda^h, \eta) \in \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^n$ with $\lambda^g_i \geq 0, \forall i \in I^*_g(d)$ satisfying (51) such that

$$
\eta_i = 0, \forall i \in I_\pm(x^*) \cup I_\pm(d) \text{ and } \|\eta\|_0 \leq n - s.
$$

Moreover we call $\lambda = (\lambda^g, \lambda^h, \eta)$ satisfying (i) and (ii) S-/M-stationary (d) multiplier respectively.

Taking direction $d = 0$ in Definition 4.2 we have $I(d) = \emptyset$. Hence we recover the S-/M-stationary condition for CCOP which was first proposed by Pan et al. in [36, Definition 3.1] under the name B-/M-KKT condition. From Definition 4.2 it is clear that the directional M-stationary condition is in general sharper than the standard M-stationary condition while the S-stationary condition is in general stronger than the directional S-stationary condition.

We now specify the three conditions for MPDSC in Definition 3.3 to CCOP.

**Definition 4.3.** Let $x^*$ be a feasible solution to CCOP.

(a) We say that CCOP-FOSCMS holds at $x^*$ in direction $d \in C(x^*)$ if there exists no nonzero vector $(\lambda^g, \lambda^h, \eta) \in \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^n$ with $\lambda^g_i \geq 0, \forall i \in I^*_g(d)$ and $\eta_i = 0, \forall i \in I_\pm(x^*) \cup I_\pm(d)$ and $\|\eta\|_0 \leq n - s$
satisfying
\[
\sum_{i \in I^*_g(d)} \lambda^g_i \nabla g_i(x^*) + \sum_{i=1}^P \lambda^h_i \nabla h_i(x^*) + \eta = 0. \tag{52}
\]

(b) Suppose that all functions are twice continuously differentiable. We say that the CCOP-SOSCMS holds at \(x^*\) in direction \(d \in C(x^*)\) if there is no nonzero vector \((\lambda^g, \lambda^h, \eta) \in \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^n\) with \(\lambda^g_i \geq 0, \forall i \in I_g(d)\) and \(\eta_i = 0, \forall i \in I_\pm(x^*) \cup I_\pm(d)\) and \(\|\eta\|_0 \leq n - s\) satisfying (52) and the following second-order condition
\[
d^T \nabla^2_x L_0(x^*, \lambda^g, \lambda^h) d \geq 0,
\]
where \(L_0(x, \lambda^g, \lambda^h) := \langle \lambda^g, g(x) \rangle + \langle \lambda^h, h(x) \rangle\).

(c) We say that CCOP-LICQ holds at \(x^*\) in direction \(d \in C(x^*)\) if the family of vectors
\[
\{\nabla g_i(x^*)\}_{i \in I_g(x^*)} \cup \{\nabla h_i(x^*)\}_{i=1}^P \cup \{e_i\}_{i \in I_0(x^*) \cap I_0(d)}
\]
is linearly independent;

When \(d = 0\), since \(I_0^*(0) = I_0(x^*)\) CCOP-FOSCMS reduces to CCOP-NNAMCQ, CCOP-LICQ in direction \(d\) reduces to CCOP-LICQ defined in Definition 4.1 repsectively. It is obvious that in general we have
\[
\text{CCOP-NNAMCQ} \implies \text{CCOP-FOSCMS}(d) \implies \text{CCOP-SOSCMS}(d) \quad \tag{53}
\]
\[
\text{CCOP-LICQ} \implies \text{CCOP-LICQ}(d). \quad \tag{54}
\]

We denote the Lagrangian function of CCOP by
\[
\mathcal{L}(x, \lambda^g, \lambda^h) := f(x) + \langle \lambda^g, g(x) \rangle + \langle \lambda^h, h(x) \rangle.
\]

The following optimality conditions are the CCOP version of Theorem 3.1.

**Theorem 4.1.** Let \(x^*\) be a local optimal solution of CCOP. Then the following first and second order necessary optimality conditions hold:

(i) Suppose that MSCQ holds at \(x^*\) in direction \(d \in C(x^*)\). Then \(x^*\) is M-stationary in direction \(d\) and there exists an M-stationary \((d)\) multiplier \(\lambda = (\lambda^g, \lambda^h, \eta)\) such that the second-order condition holds:
\[
d^T \nabla^2_x L_0(x^*, \lambda^g, \lambda^h) d \geq 0 \tag{55}
\]

(ii) For \(d \in C(x^*)\), assume that CCOP-LICQ \((d)\) is fulfilled at \(x^*\). Then, there exists an S-stationary \((d)\) multiplier \(\lambda = (\lambda^g, \lambda^h, \eta)\) such that the second-order condition \((55)\) holds.

Conversely, let \(x^*\) be a feasible solution of CCOP. Suppose that for every nonzero critical direction \(0 \neq d \in C(x^*)\) there exists an S-stationary \((d)\) multiplier \(\lambda = (\lambda^g, \lambda^h, \eta)\) such that the second-order condition strictly
holds:
\[ d^T \nabla_x^2 \mathcal{L}(x^*, \lambda^g, \lambda^h) d > 0. \]

Then \( x^* \) is a strict local minimizer of \( \text{CCOP} \).

Since direction M-stationary condition \((d)\) implies M-stationary condition while S-stationary condition implies S-stationary condition \((d)\) respectively, we have the following corollary.

**Corollary 4.1.** Let \( x^* \) be a local optimal solution of \( \text{CCOP} \). Then the following first and second order necessary optimality conditions hold: Suppose that either \( \text{CCOP-SOSCMS} \) holds at \( x^* \) in direction \( d \in \mathcal{C}(x^*) \) or one of the constraint qualification presented in Figure 3 holds. Then \( x^* \) is M-stationary and there exists an M-stationary multiplier \( \lambda = (\lambda^g, \lambda^h, \eta) \) such that the second-order condition (55) holds.

Conversely, let \( x^* \) be a feasible solution of \( \text{CCOP} \). Suppose that for every nonzero critical direction \( 0 \neq d \in \mathcal{C}(x^*) \) there exists an S-stationary \((d)\) multiplier \( \lambda = (\lambda^g, \lambda^h, \eta) \) associated with \( x^* \) such that the second-order condition (55) strictly holds. Then \( x^* \) is a strict local minimizer of \( \text{CCOP} \).

Corollary 4.1 has improved the first order necessary optimality conditions in Pan et al. in [36, Theorems 3.2](ii) since \( \text{CCOP-SOSCMS} \) is weaker than R-MFCQ. It also improved the second order sufficient in Pan et al. in [36, Theorems 4.2] since S-stationarity implies S-stationarity \((d)\).

We now conclude this section with the following application of our error bound results.

**Theorem 4.2.** Let \( x^* \in \mathcal{F}_{\text{CCOP}} \) where \( \mathcal{F}_{\text{CCOP}} \) denotes the feasible region of \( \text{CCOP} \). If one of the constraint qualification presented in Figure 3 holds, then there exist \( \alpha \geq 0 \) and \( \rho > 0 \) such that
\[
d_{\mathcal{F}_{\text{CCOP}}}(x) \leq \alpha \left( \sum_{i=1}^m \max\{g_i(x), 0\} + \sum_{i=1}^p |h_i(x)| \right) \quad \forall x \in \mathbb{B}_\rho(x^*) \cap S.
\]

Moreover if \( x^* \) is a local optimal solution of \( \text{CCOP} \) then it is also a local optimal solution of the exact penalty problem (3) for any \( \mu \geq L_f \alpha \) where \( L_f \) is the Lipschitz constant of \( f \) at \( x^* \).

**Proof.** The weakest constraint qualification in Figure 3 is \( \text{CCOP-RCPLD} \). The error bound property holds by Theorem 3.3. With the error bound property and the continuous differentiability of \( f \), the exact penalty result follows by using the Clarke’s exact penalty principle; see [18 Proposition 2.4.3] or [40 Theorem 4.2].

**References**

[1] J-P. Aubin and H. Frankowska. *Set-valued Analysis*. Boston, Birkhäuser, 2009.

[2] W. Achtziger and C. Kanzow. *Mathematical programs with vanishing constraints: optimality conditions and constraint qualifications*. Math. Program., 114, (2008), 69-99.

[3] L. Adam, M. Červinka, and M. Pištěk. *Normally admissible stratifications and calculation of normal cones to a finite union of polyhedral sets*. Set-Valued Var. Anal., 24, (2016), 207-229.
[4] K. Bai, J. J. Ye, and J. Zhang. *Directional quasi-/pseudo-normality as sufficient conditions for metric subregularity*. SIAM J. Optim., 29, (2019), 2625-2649.

[5] H. H. Bauschke, D. R. Luke, H. M. Phan, and X. Wang. *Restricted normal cones and sparsity optimization with affine constraints*. Found. Comput. Math., 14, (2014), 63-83.

[6] A. Beck and Y. C. Eldar. *Sparsity constrained nonlinear optimization: Optimality conditions and algorithms*. SIAM J. Optim., 23, (2013), 1480-1509.

[7] A. Beck and N. Hallak. *On the minimization over sparse symmetric sets projections, optimality conditions, and algorithms*. Math. Oper. Res., 41, (2016), 196-223.

[8] M. Benko, M. Červinka, and T. Hoheisel. *Sufficient conditions for metric subregularity of constraint systems with applications to disjunctive and ortho-disjunctive programs*. Set-Valued Var. Anal., 30, (2022), 143-177.

[9] M. Benko and H. Gfrerer. *On estimating the regular normal cone to constraint systems and stationarity conditions*. Optim., 66, (2017), 61-92.

[10] M. Benko and H. Gfrerer. *New verifiable stationarity concepts for a class of mathematical programs with disjunctive constraints*. Optim., 67, (2018), 1-23.

[11] M. Benko, H. Gfrerer, and J. Outrata. *Calculus for directional limiting normal cones and subdifferentials*. Set-Valued Var. Anal., 27, (2019), 713-745.

[12] M. Benko, H. Gfrerer, J. J. Ye, J. Zhang, and J. Zhou. *Second-order optimality conditions for general nonconvex optimization problems and variational analysis of disjunctive systems*. arXiv:2203.10015

[13] D. Bertsimas and R. Shioda. *Algorithm for cardinality-constrained quadratic optimization*. Comput. Optim. Appl., 43, (2009), 1-22.

[14] J. F. Bonnans and A. Shapiro. *Perturbation Analysis of Optimization Problems*. Springer, New York, 2000.

[15] M. Bucher and A. Schwartz. *Second-order optimality conditions and improved convergence results for regularization methods for cardinality-constrained optimization problems*. J. Optim. Theory Appl., 178, (2018), 383-410.

[16] O. P. Burdakov, C. Kanzow, and A. Schwartz. *Mathematical programs with cardinality constraints: reformulation by complementarity-type conditions and a regularization method*. SIAM J. Optim., 26, (2016), 397-425.

[17] M. Červinka, C. Kanzow, and A. Schwartz. *Constraint qualifications and optimality conditions for optimization problems with cardinality constraints*. Math. Program., 160, (2016), 353-377.
[18] F. H. Clarke. *Optimization and Nonsmooth Analysis*. Classics Appl. Math. 5, SIAM, Philadelphia, PA, 1990.

[19] M. L. Flegel, C. Kanzow, and J. Outrata. *Optimality conditions for disjunctive programs with application to mathematical programs with equilibrium constraints*. Set-Valued Var. Anal., 15, (2007), 139-162.

[20] H. Gfrerer. *On directional metric subregularity and second-order optimality conditions for a class of nonsmooth mathematical programs*. SIAM J. Optim. 23, (2013), 632-665.

[21] H. Gfrerer. *Optimality conditions for disjunctive programs based on generalized differentiation with application to mathematical programs with equilibrium constraints*. SIAM J. Optim., 24, (2014), 898-931.

[22] H. Gfrerer. *Linearized M-stationarity conditions for general optimization problems*. Set-Valued Var. Anal., 27, (2019), 819-840.

[23] I. Ginchev and B. S. Mordukhovich. *On directionally dependent subdifferentials*. C.R. Bulg. Acad. Sci., 64, (2011), 497–508.

[24] R. Henrion and J. Outrata. *On calculating the normal cone to a finite union of convex polyhedra*. Optim., 57, (2008), 57-78.

[25] C. Kanzow, A. B. Raharja, and A. Schwartz. *Sequential optimality conditions for cardinality-constrained optimization problems with applications*. Comput. Optim. Appl., 80, (2021), 185-211.

[26] E.H.M. Krulikovski, A.A. Ribeiro and M. Sachine. *On the weak stationarity conditions for mathematical programs with cardinality constraints: a unified approach*. Appl. Math. Optim., 84, (2021), 3451-3473.

[27] Y.-C. Liang and J. J. Ye. *Optimality conditions and exact penalty for mathematical programs with switching constraints*. J. Optim. Theory Appl., 190, (2021), 1-31.

[28] Z. Lu. *Optimization over sparse symmetric sets via a nonmonotone projected gradient method*, ArXiv: 1509.08581, (2015).

[29] Z. Q. Luo, J. S. Pang, and D. Ralph. *Mathematical Programs with Equilibrium Constraints*. Cambridge University Press, Cambridge, 1996.

[30] P. Mehlitz. *On the linear independence constraint qualification in disjunctive programming*. Optim., 69, (2020), 2241-2277.

[31] P. Mehlitz. *Stationarity conditions and constraint qualifications for mathematical programs with switching constraints*. Math. Program., 181, (2020), 149-186.

[32] P. Mehlitz. *Asymptotic stationarity and regularity for nonsmooth optimization problems*. J. Nonsmooth Anal. Optim., 1, (2020), 6575.

[33] B. S. Mordukhovich. *Variational Analysis and Generalized Differentiation, Vol. 1: Basic Theory, Vol. 2: Applications*. Springer, Berlin, 2006.
[34] B. S. Mordukhovich. Variational Analysis and Applications. Monographs in Mathematics, Springer, Cham, Switzerland, 2018.

[35] L. L. Pan, N. H. Xiu, and S. L. Zhou. On solutions of sparsity constrained optimization. J. Oper. Res. Soc. China, 3, (2015), 421-439.

[36] L. L. Pan, N. H. Xiu, and J. Fan. Optimality conditions for sparse nonlinear programming. Sci. China Math., 60, (2017), 759-776.

[37] R. T. Rockafellar and R. J. Wets. Variational Analysis. Springer, Berlin, 1998.

[38] A. M. Tillmann, D. Bienstock, A. Lodi, and A. Schwartz. Cardinality minimization, constraints, and regularization: a survey. Preprint arXiv:2106.09606.

[39] M. Xu and J. J. Ye. Relaxed constant positive linear dependence constraint qualification for disjunctive programs. Preprint arXiv:2204.09869.

[40] J.J. Ye. Exact penalty principle. Nonlinear. Anal., 75, (2012), 1642-1654.

[41] J. J. Ye and J. Zhou. Verifiable sufficient conditions for the error bound property of second-order cone complementarity problems. Math. Program., 171, (2018), 361-395.