A NOTE ON THE LIMIT OF ORLICZ NORMS

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A well-known result in measure theory is that if \((X, \mathcal{M}, \mu)\) is a measure space, and if for some \(r < \infty\),
\[ \|f\|_r = \left( \int_X |f(x)|^r \, d\mu \right)^{\frac{1}{r}} < \infty, \]
then
\[ \lim_{p \to \infty} \|f\|_p = \|f\|_\infty. \]
(See Rudin [5, p. 73].) The purpose of this note is to generalize this result to the scale of Orlicz spaces.

To state our main result, we begin with some definitions and basic facts about Young functions and Orlicz spaces. For complete information, see [3, 4]; for a briefer summary, see [1, Section 5.1]. Let \(A : [0, \infty) \to [0, \infty)\) be a Young function: that is, \(A\) is continuous, convex and increasing, \(A(0) = 0\), and \(A(t)/t \to \infty\) as \(t \to \infty\). Given a measurable, real-valued function on \(f\), define the Orlicz norm (more properly, the Luxemburg norm) by
\[ \|f\|_A = \inf \left\{ \lambda > 0 : \int_X A\left( \frac{|f(x)|}{\lambda} \right) \, d\mu \leq 1 \right\}. \]
When \(A(t) = t^p\), \(1 \leq p < \infty\), then \(\|f\|_A = \|f\|_p\). Given two Young functions \(A\) and \(B\), we say \(A \lesssim B\) if there exists a constant \(c\) such that for all \(t > 0\), \(A(t) \leq B(ct)\). If \(A \lesssim B\) and \(B \lesssim A\), we write \(A \approx B\). If \(A \lesssim B\), then there exists a constant \(C > 0\) depending only on \(A\) and \(B\), such that \(\|f\|_A \leq C\|f\|_B\).

We are interested in the Orlicz norms defined with respect to the Young functions
\[ B_{pq}(t) = t^p \log(e_0 + t)^q, \]
where \(1 \leq p < \infty\), \(q > 0\), and \(e_0 = e - 1\). Our result is the following.

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Theorem 1. Given a measure space \((X, \mathcal{M}, \mu)\) and measurable function \(f\), if for some \(p \geq 1\) and \(q_0 > 0\), \(\|f\|_{B_{pq_0}} < \infty\), then

\[
\lim_{q \to \infty} \|f\|_{B_{pq}} = \|f\|_{\infty}.
\]

Before giving the proof, we make three remarks. First, we emphasize that the significant feature of Theorem 1 is that the power \(p\) remains fixed, and it is only the logarithm term that grows.

Second, we first considered this result during an (unsuccessful) attempt to generalize Moser iteration to the scale of Orlicz spaces (see [1], where we instead implemented De Giorgi iteration in this setting). For Moser iteration we needed to control the norm as \(q \to \infty\). We were surprised that this result was not known, but a search of the literature did not find it.

Third, in applications the Young functions \(B_{pq}\) are often defined with \(e\) in place of \(e_0\) (see, for instance, [1]), and if we define

\[
\tilde{B}_{pq}(t) = t^p \log(e_0 + t)^q,
\]

then \(B \approx \tilde{B}\). However, in order to get an equality in the conclusion of Theorem 1 we are required to assume that \(B_{pq}(1) = 1\).

Proof. We prove (1) in two steps. We will first show that

\[
\liminf_{q \to \infty} \|f\|_{B_{pq}} \geq \|f\|_{\infty}.
\]

If \(\|f\|_{\infty} = 0\), there is nothing to prove. Now suppose that \(0 < \|f\|_{\infty} < \infty\). Since for some \(q_0 > 0\), \(\|f\|_{B_{pq_0}} < \infty\), the level sets of \(f\), \(S(\lambda) = \{x \in X : |f(x)| > \lambda\}, \lambda > 0\), have finite measure: this follows from the corresponding result for \(L^p\) spaces, since

\[
\|f\|_p \lesssim \|f\|_{B_{pq_0}}.
\]

Fix \(\epsilon > 0\) and let \(M = S(\|f\|_{\infty} - \epsilon)\). Then \(0 < \mu(M) < \infty\), and so

\[
\|f\|_{B_{pq}} \geq \|f\chi_M\|_{B_{pq}} \geq (\|f\|_{\infty} - \epsilon)\|\chi_M\|_{B_{pq}}.
\]

Since \(\epsilon > 0\) is arbitrary, to prove inequality (2) it will suffice to show

\[
\liminf_{q \to \infty} \|\chi_M\|_{B_{pq}} \geq 1.
\]

To estimate \(\|\chi_M\|_{B_{pq}}\), fix \(q > q_0\). By the definition of the Orlicz norm,

\[
\lambda = \|\chi_M\|_{B_{pq}} = B^{-1}_{pq} \mu(M)^{-1^{-1}}.
\]

Hence,

\[
\mu(M)^{-1} = B_{pq}(\lambda^{-1}) = \lambda^{-p} \log(e_0 + \lambda^{-1})^q.
\]

If we always have that \(\lambda \geq 1\), then (3) clearly holds. If for some \(q\), \(\lambda < 1\), fix \(\delta > 0\) such that \(\log(e_0 + \lambda^{-1}) = 1 + \delta\); by the binomial theorem,

\[
\lambda^{-p} \log(e_0 + \lambda^{-1})^q = (e^{1+\delta} - e_0)^p(1 + \delta)^q \geq 1 + q\delta > q\delta.
\]
Thus, $\delta < \frac{\mu(M)^{-1}}{q}$. If we solve for $\lambda$ in the definition of $\delta$, we get

$$\|X_M\|_{B_{pq}} = \lambda > \left[ \exp \left( 1 + \frac{\mu(M)^{-1}}{q} \right) - e_0 \right]^{-1}.$$ 

Therefore, if we take the limit infimum as $q \to \infty$, we get (3).

Finally, suppose $f$ is unbounded. For any $N > 1$ define $f_N = \min(|f|, N)$. Then the above argument shows that

$$\liminf_{q \to \infty} \|f\|_{B_{pq}} \geq \liminf_{q \to \infty} \|f_N\|_{B_{pq}} \geq \|f_N\|_\infty = N,$$

and if we let $N \to \infty$, (2) follows.

We will now show that

$$\limsup_{q \to \infty} \|f\|_{B_{pq}} \leq \|f\|_\infty. \tag{4}$$

We may assume that $0 < \|f\|_\infty < \infty$; otherwise (4) is immediate. Since $\|f\|_{B_{p0}} < \infty$, by the definition of the norm we have that there exists $\lambda > 0$ such that

$$\left( \frac{|f(x)|}{\lambda} \right)^p \log \left( e_0 + \frac{|f(x)|}{\lambda} \right)^{q_0} \in L^1(X).$$

Moreover, this is true for any $\lambda$, $0 < \lambda < \infty$; this follows at once from the fact that for any $c > 0$, the function

$$\frac{\log(e_0 + t)}{\log(e_0 + ct)}$$

is bounded and bounded away from 0 for all $t > 0$.

In particular, if we fix $\epsilon > 0$ and let $\lambda = (1 + \epsilon)\|f\|_\infty$, then for almost every $x \in X$, and for all $q > q_0$, since $e_0 + (1 + \epsilon)^{-1} < e$,

$$B_{pq} \left( \frac{|f(x)|}{\lambda} \right) \leq B_{p0} \left( \frac{|f(x)|}{\lambda} \right).$$

Furthermore,

$$B_{pq} \left( \frac{|f(x)|}{\lambda} \right) \leq (1 + \epsilon)^{-p} \log(e_0 + (1 + \epsilon)^{-1})^q.$$

Again since $e_0 + (1 + \epsilon)^{-1} < e$, the right-hand side tends to 0 as $q \to \infty$. Hence, by the dominated convergence theorem, for all $q > q_0$ sufficiently large,

$$\int_X B_{pq} \left( \frac{|f(x)|}{\lambda} \right) d\mu \leq 1.$$

Therefore, by the definition of the Orlicz norm, for all such $q$,

$$\|f\|_{B_{pq}} \leq (1 + \epsilon)\|f\|_\infty,$$

and inequality (4) follows at once. This completes the proof. □
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