The Effect of Brick Walls on the Black Hole Radiation

Sang Pyo Kim\textsuperscript{a} *, Sung Ku Kim\textsuperscript{b} †, Kwang-Sup Soh\textsuperscript{c} ‡, and Jae Hyung Yee\textsuperscript{d} §

\textsuperscript{a} Department of Physics, Kunsan National University, Kunsan 573-701, Korea
\textsuperscript{b} Department of Physics, Ewha Womans University, Seoul 120-750, Korea
\textsuperscript{c} Department of Physics Education, Seoul National University, Seoul 151-742, Korea
\textsuperscript{d} Department of Physics and Institute for Mathematical Sciences, Yonsei University, Seoul 120-749, Korea

Abstract

In order to understand the physical effect of the brick wall boundary condition, we compute the distribution of the zero-point energy of the massless scalar fields minimally coupled to the Schwarzschild and Reissner-Nordström black hole backgrounds. We find that the black hole radiation spectrum depends on the positions of the brick wall and the observer, and reveals the interference effect due to the reflected field by the brick wall.
Since the Bekenstein’s introduction of the idea of black hole entropy [1] and the subsequent discovery of the black hole radiation phenomena by Hawking [2], many attempts have been made to understand the origin of the black hole entropy. One of such proposals was ’t Hooft’s brick wall near the horizon, across which no quantum fields can propagate [3]. Recently the brick wall method has been successfully utilized in understanding the divergence structure of the entropy contribution of quantum fields propagating outside the black hole horizon [4], and the fact that such divergences can be absorbed as the renormalization of the gravitational and cosmological constants [3,4].

Although the introduction of brick wall enabled us to understand many aspects of black hole entropy, it raises a puzzling question as to whether it really gives the correct physics since the existence of brick wall prohibits the propagation of matter across the horizon whereas it is known that black holes can absorb matter. To understand this question, in this letter, we compute the radiation spectra of black holes with brick wall boundary conditions imposed.

There exist many different methods to evaluate the radiation spectrum of black hole background [7]. For the purpose of understanding the effect of brick walls on black hole radiation, it is convenient to use the interpretation that the gravitational field of black hole deforms the energy spectrum of the zero-point field and makes it appear as a thermal radiation spectrum [8,9].

We consider a massless scalar field $\phi(x)$ minimally coupled to a black hole background. Then the vacuum expectation value of energy density observed by an observer at $x^\mu = x^\mu(\tau)$ is given by

$$\mathcal{E} = \frac{1}{2}(\xi^\mu \xi_\mu)^{1/2} \langle \phi \frac{d^2 \phi}{d\tau^2} - \frac{d^2 \phi}{d\tau^2} \phi + 2 \frac{d\phi}{d\tau} \frac{d\phi}{d\tau} \rangle, \tag{1}$$

where $\tau$ is the proper time of the observer and $\xi^\mu$ is the time-like Killing vector. Using the definition of the Wightman functions $D^\pm(x^\mu, x'^\mu)$, evaluated at two points $x^\mu = x^\mu(\tau + \sigma/2)$ and $x'^\mu = x^\mu(\tau - \sigma/2)$ along the world-line of the observer, the vacuum energy density can be written as [8]

$$\mathcal{E} = \frac{1}{\pi}(\xi^\mu \xi_\mu)^{1/2} \int_0^\infty d\omega \omega^2 [\tilde{D}^+(\omega, \tau) + \tilde{D}^-(\omega, \tau)], \tag{2}$$
where the Fourier transforms of the Wightman functions are defined by

\[
\tilde{D}^\pm(\omega, \tau) = \int_{-\infty}^{\infty} d\sigma e^{i\omega\sigma} D^\pm(\tau + \sigma/2, \tau - \sigma/2).
\]

(3)

The distortion of the zero-point energy (i.e., the Hawking radiation of black hole) can then be evaluated by Eq. (2) from the appropriate Wightman functions for the scalar field in the black hole background spacetime.

We first consider the Schwarzschild black hole

\[
ds^2 = -\frac{2M}{r} e^{-r/2M} d\tau d\nu
\]

(4)

where \( M \) is the black hole mass and the Kruskal coordinates are defined as

\[
\bar{\tau} = -4M e^{-u/4M}, \quad u = t - r^*;
\]

\[
\bar{\nu} = 4M e^{v/4M}, \quad v = t + r^*,
\]

(5)

where \( r^* \) is defined by \( dr^* = dr/(1 - 2M/r) \). Here we only consider the radial motion and the system to be 2-dimensional. The metric (4) admits a time-like Killing vector \( \xi^\mu \) with magnitude \((\xi^\mu \xi_\mu)^{1/2} = (1 - 2M/r)^{1/2}\), and the world-line of a detector at \( r = r_0 \) can be represented by

\[
\bar{\tau} = -\frac{1}{b} e^{-a\tau + br_0^*}
\]

\[
\bar{\nu} = \frac{1}{b} e^{a\tau + br_0^*}
\]

(6)

where \( \tau \) is the proper time of the detector, \( r_0^* \) is the position of the detector in tortoise coordinates and

\[
a = \left(1 - \frac{2M}{r_0}\right)^{-1/2}
\]

\[
b = \frac{1}{4M}.
\]

(7)

The Wightman functions in the open \((\bar{\tau}, \bar{\nu})\) space are given by [7,8]

\[
D^\pm(\bar{\tau}, \bar{\nu}; \bar{u}', \bar{v}') = -\frac{1}{8\pi} \ln[(\bar{u} - \bar{u}') \mp i\epsilon](\bar{v} - \bar{v}' \mp i\epsilon)].
\]

(8)

3
't Hooft's brick wall method consists of introducing a boundary condition

\[ \phi(x) = 0, \quad \text{for} \quad r \leq 2M + h \] (9)

where \( h \ll 2M \). Thus with the brick wall boundary condition, the Wightman functions must satisfy the conditions

\[ D^\pm(x, x')|_{r=2M+h} = 0 = D^\pm(x, x')|_{r'=2M+h}. \] (10)

By using the image method we find the correct Wightman functions \( D^\pm_h(x, x') \) satisfying the brick wall boundary condition (10),

\[ D^\pm_h(\tau, \sigma/2, \tau - \sigma/2) = -\frac{1}{8\pi} \ln \left[ \left( \frac{\tau - \bar{u} \mp \iota \epsilon}{\| \tau - \bar{v} \mp \iota \epsilon \|} \right) \left( \frac{\tau + \bar{u} \mp \iota \epsilon}{\| \tau + \bar{v} \mp \iota \epsilon \|} \right) \right] 
+ \frac{1}{8\pi} \ln \left[ \left( \frac{\pi + \frac{1}{b^2v'} e^{2br*} - \iota \epsilon}{\| \pi + \frac{1}{b^2v'} e^{2br*} \mp \iota \epsilon \|} \right) \left( \frac{\pi + \frac{1}{b^2u'} e^{2br*} \mp \iota \epsilon}{\| \pi + \frac{1}{b^2u'} e^{2br*} \mp \iota \epsilon \|} \right) \right] 
- \frac{1}{8\pi} \ln \left[ e^{4br*} \left( -\frac{1}{b^2u'} + \frac{1}{b^2v'} \mp \iota \epsilon \right) \left( -\frac{1}{b^2u'} + \frac{1}{b^2v'} \mp \iota \epsilon \right) \right], \] (11)

where \( r^*_h = r^*|_{r=2M+h} \). Using Eq. (8) we find

\[ D^\pm_h(\tau + \sigma/2, \tau - \sigma/2) = -\frac{1}{2\pi} \ln \left[ \frac{2}{b} e^{br^*_h} \sinh \left( \frac{a\sigma}{2} \mp \iota \epsilon \right) \right] 
+ \frac{1}{4\pi} \ln \left[ \frac{4}{b^2} e^{2br^*_h} \sinh \left( \frac{a\sigma}{2} \mp b(r_0^* - r^*_h) \mp \iota \epsilon \right) \right] 
\times \sinh \left( \frac{a\sigma}{2} - b(r_0^* - r^*_h) \mp \iota \epsilon \right). \] (12)

The Fourier transforms of the Wightman functions (12) are

\[ \tilde{D}^\pm(\omega, \tau) = \frac{1}{\omega e^{2\pi\omega/a} - 1} \left[ 1 - \cos \left( \frac{\omega(r_0^* - r^*_h)}{2Ma} \right) \right], \] 
\[ \tilde{D}^-(\omega, \tau) = \frac{1}{\omega e^{2\pi\omega/a} - 1} \left[ 1 - \cos \left( \frac{\omega(r_0^* - r^*_h)}{2Ma} \right) \right], \] (13)

for positive \( \omega \). Using Eq. (4) we thus find the energy density of the zero-point field,

\[ \mathcal{E} = \frac{2}{\pi} \left( 1 - \frac{2M}{r_0} \right)^{1/2} \int_0^\infty d\omega \omega \left[ \frac{1}{2} + \frac{1}{e^{2\pi\omega/a} - 1} \right] \left[ 1 - \cos \left( \frac{\omega(r_0^* - r^*_h)}{2Ma} \right) \right]. \] (14)
As in the case of the Schwarzschild background without a brick wall \[8\], we find the original energy of the zero-point energy plus an additional Planckian distribution term at the temperature

\[
kT = \frac{a}{2\pi} = \left(1 - \frac{2M}{r_0}\right)^{-1/2} \frac{1}{8\pi M}.
\]

(15)

The result, however, has an extra factor, the last factor of Eq. (14), due to the existence of the brick wall boundary condition which prevents matter fields from crossing the wall. This factor depends on the positions of the brick wall and the detector, and vanishes when \(r_0 = r_h\), i.e., when the detector is right at the brick wall.

We now consider the Reissner-Nordström black hole

\[
ds^2 = -\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)e^{-2br^*}d\bar{m}/d\bar{m},
\]

(16)

where \(r^\pm = M \pm \sqrt{M^2 - Q^2}\), \(M\) and \(Q\) are the mass and the charge of the black hole, respectively, and the Kruskal coordinates are defined by

\[
\bar{u} = -\frac{1}{b} e^{-bu} + \frac{1}{b}, \quad u = t - r^*
\]

\[
\bar{v} = \frac{1}{b} e^{bv} - \frac{1}{b}, \quad v = t + r^*
\]

(17)

with \(r^*\) defined by \(dr^*/dr = (1 - 2M/r + Q^2/r^2)\). We also consider only the radial motion and the system to be 2-dimensional. The metric (16) admits a time-like Killing vector \(\xi^\mu\) with magnitude \((\xi^\mu \xi_\mu)^{1/2} = (1 - 2M/r + Q^2/r^2)^{1/2}\), and the world-line of a detector at \(r = r_0\) is represented by

\[
\bar{u} = -\frac{1}{b} e^{-a\tau + b\tau_0^*} + \frac{1}{b}
\]

\[
\bar{v} = \frac{1}{b} e^{a\tau + b\tau_0^*} - \frac{1}{b}
\]

(18)

where \(\tau_0^*\) is the position of the detector in tortoise coordinates and

\[
b = \frac{r_+ - r_-}{2r_+^2},
\]

\[
a = b\left(1 - \frac{2M}{r_0} + \frac{Q^2}{r_0^2}\right)^{-1/2}.
\]

(19)
The Wightman functions $\Delta_h^\pm(x,x')$ satisfying the brick wall boundary conditions
\[
\Delta_h^\pm(x,x')|_{r=r_++h} = 0 = \Delta_h^\pm(x,x')|_{r'=r_++h}
\]
are given by
\[
\Delta_h^\pm(u,u';v,v') = \frac{1}{8\pi} \ln \left[ (u-v\mp i\epsilon)(u-v'+i\epsilon) \right] \\
+ \frac{1}{8\pi} \ln \left[ -\frac{1}{b^2(u+\frac{1}{b})} e^{2br_*^h} - u + \frac{1}{b} \mp i\epsilon \right] - \frac{1}{b^2(u-\frac{1}{b})} e^{2br_*^h} - v - \frac{1}{b} \mp i\epsilon \right] \\
+ \frac{1}{8\pi} \ln \left[ \frac{1}{b} + \frac{1}{b^2(v+\frac{1}{b})} e^{2br_*^h} \mp i\epsilon \right] \left[ u + \frac{1}{b} + \frac{1}{b^2(u-\frac{1}{b})} e^{2br_*^h} \mp i\epsilon \right] \\
- \frac{1}{8\pi} \ln \left[ e^{4br_*^h} \left( -\frac{1}{b^2(u+\frac{1}{b})} + \frac{1}{b^2(v+\frac{1}{b})} \mp i\epsilon \right) \right] \times \left( -\frac{1}{b^2(u-\frac{1}{b})} + \frac{1}{b^2(v'-\frac{1}{b})} \mp i\epsilon \right) \\
\times \left( -\frac{1}{b^2(v'-\frac{1}{b})} + \frac{1}{b^2(u'-\frac{1}{b})} \mp i\epsilon \right) \\
\times \left( -\frac{1}{b^2(u'-\frac{1}{b})} + \frac{1}{b^2(v'-\frac{1}{b})} \mp i\epsilon \right)
\]
where $r_*^h = r^*|_{r=r_++h}$. This result is exactly the same as the Schwarzschild case except for
the fact that $\overline{u}$ and $\overline{v}$ of Eq. (14) are replaced by $\overline{u}-1/b$ and $\overline{v}+1/b$, respectively, and the
constants $a$ and $b$ are defined now by Eq. (19). We thus obtain the energy density in the
Reissner-Nordström black hole background,
\[
\mathcal{E} = \frac{2}{\pi} \left( 1 - \frac{2M}{r_0} + \frac{Q^2}{r_0^2} \right)^{1/2} \int_0^\infty d\omega \omega \left( \frac{1}{2} + \frac{1}{e^{2\pi\omega/a} - 1} \right) \left[ 1 - \cos \left( \frac{2\omega(r_0^* - r_*^h)}{a} \right) \right].
\]
This implies that the black hole radiates at the temperature
\[
kT = \frac{a}{2\pi} = \frac{r_* - r_+}{4\pi r_*^2} \left( 1 - \frac{2M}{r_0} + \frac{Q^2}{r_0^2} \right)^{-1/2}
\]
which is consistent with that of other methods [10], but the radiation spectrum is modified
due to the existence of the brick wall boundary condition as in the case of the Schwarzschild black hole.

We have shown that, for both the Schwarzschild and Reissner-Nordström black holes,
the radiation spectra are drastically modified by the brick wall boundary conditions. The
brick wall provides a mirror-like boundary condition, and there appear reflected fields. The
cosine terms in the last factors (14) and (22) represent the interference effects due to the
reflected field by the brick wall. Therefore the introduction of the brick wall deforms the
local distribution of the energy spectra, even though the overall thermodynamic entropy may look similar to the one without a brick wall. This implies therefore that, although the brick wall method enabled us to clarify many aspects of black hole entropy, the presence of brick wall near horizon may considerably modify other aspects of black hole physics.

ACKNOWLEDGMENTS

This work was supported in part by the Korea Science and Engineering Foundation under Grant No. 951-0207-56-2, 95-0701-04-01-3, and in part by the Basic Science Research Institute Program, Ministry of Education under Project No. BSRI-96-2418, BSRI-96-2425, BSRI-96-2427, and by the Center for Theoretical Physics, Seoul National University. SKK was also supported in part by Non-Directed Research Fund of the Korea Research Foundation.
REFERENCES

[1] J. D. Bekenstein, Phys. Rev. D 9, 3292 (1974).

[2] S. W. Hawking, Commun. Math. Phys. 43, 199 (1975).

[3] 't Hooft, Nucl Phys. B256, 727 (1985).

[4] S. P. de Alwis and N. Ohta, Phys. Rev. D 52, 3529 (1995).

[5] J-G. Demers, R. Lafrance, and R. C. Myers, Phys. Rev. D 52, 2245 (1995).

[6] S. P. Kim, S. K. Kim, K.-S. Soh, and J. H. Yee, ”Remarks on Renormalization of Black Hole Entropy”, Seoul National University Preprint, SNUTP-96032 (1996)
    S. P. Kim, S. K. Kim, K.-S. Soh, and J. H. Yee, ”Renormalized Thermodynamic Entropy of Black Holes”, Seoul National University Preprint, SNUTP-96033 (1996).

[7] N. D. Birrel and P. C. W. Davies, Quantum Fields in Curved Spaces (Cambridge University, Cambridge, 1982) and references therein.

[8] S. Hacyan, A. Sarmiento, G. Cocho, and F. Soto, Phys. Rev. D 32, 914 (1985).

[9] S. K. Kim, K. S. Soh, and J. H. Yee, Phys. Rev. D 35, 557 (1987)
    J. S. Kim, K. S. Soh, S. K. Kim, and J. H. Yee, Phys. Rev. D 36, 3700 (1987).

[10] A. Gosh and P. Mitra, ”Temperatures of Extremal Black Holes”, gr-qc/9507032 and references therein.