Birational geometry of varieties fibred into complete intersections of codimension two

A. V. Pukhlikov

Abstract. In this paper we prove the birational superrigidity of Fano–Mori fibre spaces $\pi : V \to S$ all of whose fibres are complete intersections of type $d_1 \cdot d_2$ in the projective space $\mathbb{P}^{d_1 + d_2}$ satisfying certain conditions of general position, under the assumption that the fibration $V/S$ is sufficiently twisted over the base (in particular, under the assumption that the $K$-condition holds). The condition of general position for every fibre guarantees that the global log canonical threshold is equal to one. This condition also bounds the dimension of the base $S$ by a constant depending only on the dimension $M$ of the fibre (this constant grows like $M^2/2$ as $M \to \infty$). The fibres and the variety $V$ may have quadratic and bi-quadratic singularities whose rank is bounded below.

Keywords: Fano variety, Mori fibre space, birational map, birational rigidity, linear system, maximal singularity, quadratic singularity, bi-quadratic singularity

Introduction

0.1. Main results. We fix a pair of integers $(d_1, d_2)$ such that $d_2 \geq d_1 \geq 2$ and $d_2 \geq 27$. Put $M = d_1 + d_2 - 2$ and write $\mathbb{P}$ for the complex projective space $\mathbb{P}^{M+2}$ with homogeneous coordinates $(x_0 : \cdots : x_{M+2})$. The aim of this paper is to prove the birational rigidity of Fano–Mori fibre spaces $\pi : V \to S$ all of whose fibres are complete intersections of type $d_1 \cdot d_2$ in $\mathbb{P}$ satisfying some natural conditions of general position, under the assumption that the fibration $V/S$ is sufficiently twisted over the base. When the base $S$ is fixed, most families of fibre spaces satisfy the condition of twistedness. The condition of general position for every fibre bounds the dimension of the base $S$ by a constant depending only on $M$ (this constant grows like $M^2/2$ as the dimension $M$ of the fibre grows). The key property that must be possessed by every fibre of $V/S$ for our proof to work, is the boundedness, in a certain sense, of the singularities of every divisor. We now give precise statements.

Let $S$ be a non-singular projective rationally connected variety of positive dimension. By a Fano–Mori fibre space over $S$ we mean a surjective morphism $\pi : V \to S$ all of whose fibres are irreducible, reduced and of dimension $\dim V - \dim S \geq 3$.
where the variety $V$ is projective and factorial with at most terminal singularities and, moreover, we have an equality $\rho(V) = \rho(S) + 1$ of Picard numbers and the anti-canonical class $(-K_V)$ is ample on a general (and hence on every) fibre. (We assume throughout that the base of the fibre space is rationally connected and its total space is factorial since the total spaces of all fibre spaces considered in this paper are rationally connected and factorial.)

Let $\pi': V' \to S'$ be a \textit{rationally connected fibre space}, that is, a surjective morphism of non-singular projective varieties whose general fibre is an irreducible rationally connected variety and whose base $S'$ is rationally connected. We say that a birational map (when such a map exists, in particular, when $\dim V = \dim V'$)

$$\chi: V \dasharrow V'$$

is \textit{fibrewise} if there is a rational dominant map $\beta: S \dasharrow S'$ such that $\pi' \circ \chi = \beta \circ \pi$, that is, the following diagram of maps commutes:

$$
\begin{array}{ccc}
V & \xrightarrow{\chi} & V' \\
\pi \downarrow & & \downarrow \pi' \\
S & \xrightarrow{\beta} & S'.
\end{array}
$$

We emphasize that the property of being fibrewise does not mean that $\beta$ is birational: generally speaking, $\chi$ maps a fibre of $\pi$ onto a closed subset of a fibre of $\pi'$, so that the inverse map $\chi^{-1}$ may well not be fibrewise (more precisely, $\chi$ and $\chi^{-1}$ are both fibrewise if and only if $\beta$ is birational).

Now let $\pi': V' \to S'$ be a \textit{Mori fibre space}, that is, the singularities of $V'$ and $S'$ are terminal and $\mathbb{Q}$-factorial, the anticanonical class of $V'$ is relatively ample and we have an equality $\rho(V') = \rho(S') + 1$.

\textbf{Definition 0.1.} A Fano–Mori fibre space $V/S$ is \textit{birationally rigid} if, for every birational map $\chi: V \dasharrow V'$ onto the total space of a (any) Mori fibre space $V'/S'$, there is a birational map $\beta: S \dasharrow S'$ such that $\pi' \circ \chi = \beta \circ \pi$, that is, $\chi$ maps the fibre of general position of $\pi$ birationally onto the fibre of general position of $\pi'$. If, moreover, the restriction of $\chi$ to the general fibre of $\pi$ is always a biregular isomorphism, then the Fano–Mori fibre space $V/S$ is \textit{birationally superrigid}.

For $d \in \mathbb{Z}_+$ let $P_{d,M+3}$ be the linear space of homogeneous polynomials of degree $d$ in the variables $x_0, \ldots, x_{M+2}$ and let

$$\mathcal{F} = P_{d_1,M+3} \times P_{d_2,M+3}$$

be the corresponding space of pairs $(f_1, f_2)$. We say that the set $\{f_1 = f_2 = 0\} \subset \mathbb{P}$ of their common zeros is a complete intersection of codimension 2 with good singularities if the scheme-theoretic intersection of the hypersurfaces $\{f_1 = 0\}$ and $\{f_2 = 0\}$ is an irreducible reduced subvariety $F = F(f_1, f_2) \subset \mathbb{P}$ of codimension 2 and, moreover, for every point $o \in F$, one of the following three conditions holds:

1. The point $o \in F$ is non-singular;
2. $o$ is non-singular on one of the hypersurfaces $\{f_1 = 0\}, \{f_2 = 0\}$ and $F$ has a quadratic singularity of rank $\geq 5$ at $o$;
(3) \( o \) is a quadratic singularity on both \( \{ f_1 = 0 \} \) and \( \{ f_2 = 0 \} \) and if we decompose the inhomogeneous representatives of the polynomials \( f_1, f_2 \) with respect to an affine coordinate system \( z_1, \ldots, z_{M+2} \) centred at \( o \) into homogeneous components
\[
 f_1 = f_{1,2} + \cdots + f_{1,d_1}, \quad f_2 = f_{2,2} + \cdots + f_{2,d_2},
\]
where \( \deg f_{i,j} = j \), then the quadratic forms \( f_{1,2} \) and \( f_{2,2} \) satisfy a certain condition of general position. To state it more conveniently, we give a definition that will be useful in what follows.

**Definition 0.2.** The rank of a \( k \)-tuple \( q_1, \ldots, q_k \) of quadratic forms in some variables is the number
\[
 \text{rk}(q_1, \ldots, q_k) = \min \{ \text{rk}(\lambda_1 q_1 + \cdots + \lambda_k q_k) \mid (\lambda_1, \ldots, \lambda_k) \neq (0, \ldots, 0) \}
\]
(the minimum is taken over all \( (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k \setminus \{0\} \)).

The condition of general position for \( f_{1,2} \) and \( f_{2,2} \) now takes the form
\[
 \text{rk}(f_{1,2}, f_{2,2}) \geq 7.
\]

Points of type (3) are called bi-quadratic singularities of \( F \). Quadratic and bi-quadratic singularities will be considered in detail in §1.

If \( F \) is a complete intersection of codimension 2 with good singularities, then
\[
 \text{codim}(\text{Sing } F \subset F) \geq 4.
\]

Hence Grothendieck’s theorem [1] yields that \( F \) is a factorial complete intersection, \( \text{Pic } F = \mathbb{Z} H \), where \( H \) is the class of a hyperplane section, and \( K_F = -H \). In other words, \( F \) is a primitive Fano variety. It is easy to check that the singularities of \( F \) are terminal (see § 1.7). We write \( \mathcal{F}_{\text{bq}} \) for the open set of pairs \( (f_1, f_2) \in \mathcal{F} \) such that the common zero set \( \{ f_1 = f_2 = 0 \} \) is a complete intersection of codimension 2 with good singularities.

Now let \( S \) be a non-singular projective rationally connected variety of positive dimension and let \( \pi_X : X \to S \) be a locally trivial fibre space with fibre \( \mathbb{P} \). A subvariety \( V \subset X \) of codimension 2 is a fibre into complete intersections of type \( d_1 \cdot d_2 \) if the base \( S \) can be covered by Zariski-open subsets such that the fibration \( \pi_X \) is trivial over \( U \), \( \pi_X^{-1}(U) \cong U \times \mathbb{P} \), and for every \( U \) there is a map
\[
 \Phi_U : U \to \mathcal{F}_{\text{bq}}
\]
such that, for every point \( s \in U \), the subvariety of common zeros \( \{ \Phi_U(s) = 0 \} \subset \mathbb{P} \)
is \( V \cap \pi_X^{-1}(s) \) in the sense of the trivialization above. In other words, \( V \) is given over \( U \) by a pair of equations
\[
 f_1(x_*, s) = 0, \quad f_2(x_*, s) = 0
\]
whose coefficients are regular functions on \( U \) and, for every \( s \in U \),
\[
 f_1(x_*, s), f_2(x_*, s) \in \mathcal{F}_{\text{bq}}.
\]
The restriction of $\pi_X$ to $V$ is denoted by $\pi$. We shall prove in § 1.7 that the singularities of $V$ are terminal and $V$ is factorial. Moreover,

$$\text{Pic} \, V = \mathbb{Z}K_V \oplus \pi^* \text{Pic} \, S,$$

that is, $\pi: V \to S$ is a Fano–Mori fibre space over $S$.

To state our first main result, we recall another well-known definition.

**Definition 0.3.** A primitive Fano variety $F$ (that is, a factorial projective variety $F$ with terminal singularities whose Picard group is generated by the canonical class) is said to be **divisorially canonical** if, for every $n \geq 1$ and every effective divisor $D_F \in |-nK_F|$, the pair $(F, (1/n)D_F)$ is canonical, that is, the following inequality holds for every exceptional divisor $E$ over $F$:

$$\text{ord}_E D_F \leq n \cdot a(E, F).$$

We are now ready to state our first main result, a sufficient condition for the birational superrigidity of the Fano–Mori fibre space $V/S$ constructed above. We abuse notation somewhat and identify each pair $(f_1, f_2)$ with the corresponding subvariety $\{f_1 = f_2 = 0\}$. In particular, we write $F \in \mathcal{F}_{\text{bq}}$ for the fibre of $\pi$.

**Theorem 0.1.** Assume that the Fano–Mori fibre space $\pi: V \to S$ constructed above possesses the following properties.

(i) For every $s \in S$, the fibre $F_s = \pi^{-1}(s) \in \mathcal{F}_{\text{bq}}$ is a divisorially canonical variety.

(ii) For every effective divisor

$$D \in |-nK_V + \pi^*Y|,$$

$n \geq 1$, the class $Y$ is pseudoeffective on $S$.

(iii) For every mobile family $\mathcal{C}$ of irreducible rational curves on $S$ sweeping out an open dense subset of $S$ and for every curve $\overline{C} \in \mathcal{C}$, no positive multiple of the class of the algebraic cycle

$$-(K_V \cdot \pi^{-1}(\overline{C})) - F$$

(where $F$ is the class of a fibre of $\pi$) is effective. In other words, it is not rationally equivalent to an effective cycle of dimension $\dim F$ on $V$.

Then, for every rationally connected fibre space $V'/S'$, all birational maps $\chi: V \dashrightarrow V'$ are fibrewise and $V/S$ is birationally superrigid.

We emphasize that the two assertions in Theorem 0.1 are independent of each other. The fact that every birational map $\chi$ onto the total space of a rationally connected fibre space is fibrewise, neither implies birational superrigidity (or birational rigidity), nor is implied by it. Therefore Theorem 0.1 is an essential improvement of Theorem 1.1 in [2] (we will see in § 1 that the proof of Theorem 0.1 works without any changes for all the Fano–Mori spaces considered in [2] and accordingly strengthens the results of [2]). The hypotheses of Theorem 0.1 are somewhat different from those of Theorem 1.1 in [2]. The relations between them will be discussed in § 1. Here we only mention that assumption (ii) in Theorem 0.1 is the well-known
\textit{K-condition} for the fibration $V/S$. It says that the anticanonical class $(-K_V)$ is not contained in the interior of the pseudoeffective cone:

$$(-K_V) \notin \text{Int } A^1_+ V.$$

Theorem 0.1 motivates the following general conjecture.

**Conjecture 0.1.** Consider an arbitrary Fano-Mori fibre space satisfying the $K$-condition and assume that its fibre of general position is a birationally rigid (resp. birationally superrigid) Fano variety. Then this fibre space is birationally rigid (resp. birationally superrigis) and every birational map of its total space onto the total space of an arbitrary rationally connected fibre space (if such birational maps exist) is fibrewise. In particular, the total space of the original fibre space has no structure of a rationally connected fibre space over a base of dimension greater than the dimension of the original fibre space.

In addition to Theorem 0.1, this paper contains two other main results. They enable us to construct large classes of Fano–Morifibrepacessatisfyingthehypotheses of Theorem 0.1. Their precise statements will be given below.

**0.2. Conditions of general position.** Let $o \in \mathbb{P}$ be an arbitrary point and let $(z_1, \ldots, z_{M+2})$ be a system of affine coordinates with origin at $o$. Abusing notation, we write $f_1, f_2$ for the corresponding inhomogeneous polynomials in $z_*$. Assume that $f_1, f_2$ vanish at $o$ and write

$$f_i(z_*) = f_{i,1} + f_{i,2} + \cdots + f_{i,d_i},$$

where $i = 1, 2$ and the polynomials $f_{i,j}$ are homogeneous of degree $j$. For example, the system of equations $f_1 = f_2 = 0$ determines a non-singular complete intersection of codimension 2 in a neighbourhood of $o$ if and only if the linear forms $f_{1,1}, f_{2,1}$ are linearly independent. We order the pairs of indices $(i, j)$ lexicographically: $(i_1, j_1) < (i_2, j_2)$ if $j_1 < j_2$, or $j_1 = j_2$ and $i_1 < i_2$. Ordered in this way, the polynomials $f_{i,j} |_{\{f_{1,1}=f_{2,1}=0\}}$ with $j \geq 2$ constitute a sequence

$$f_{1,2} |_{\{f_{1,1}=f_{2,1}=0\}}, \ldots, f_{2,d_2} |_{\{f_{1,1}=f_{2,1}=0\}}.$$

We denote it by $S$. Removing the last $k \geq 1$ polynomials, we obtain a sequence $S[-k]$. Note that the linear subspace $\{f_{1,1} = f_{2,1} = 0\}$ has codimension 1 in the case of a quadratic singularity and is the whole space $\mathbb{C}^{M+2}_{z_1, \ldots, z_{M+2}}$ in the case of a bi-quadratic singularity.

We now state the conditions of general position (regularity conditions) needed in order to prove that assumption (i) of Theorem 0.1 holds for a sufficiently large class of complete intersections of codimension 2. Given a linear subspace $\mathcal{L} \subset \{f_{1,1} = f_{2,1} = 0\}$, we write $S[-k]|_\mathcal{L}$ for the restriction of $S[-k]$ to $\mathcal{L}$ (that is, the sequence of the restrictions to $\mathcal{L}$ of all the polynomials in $S[-k]$).

**The regularity condition at a non-singular point.** Assume that the linear forms $f_{1,1}$ and $f_{2,1}$ are linearly independent. The point $o$ is regular if the following condition (R1) holds.

(R1) For every linear subspace $\mathcal{L} \subset \{f_{1,1} = f_{2,1} = 0\}$
of codimension 2, the sequence $S[-5]|_{L}$ is regular, that is, the set of its common zeros in $\mathbb{P}(L) \cong \mathbb{P}^{M-3}$ is of dimension 2.

The regularity condition at a bi-quadratic point. Assume that the linear forms $f_{1,1}$ and $f_{2,1}$ are linearly dependent but not both identically equal to zero: $f_{1,1} = \alpha_1 \tau$ and $f_{2,1} = \alpha_2 \tau$, where $\tau(z_\ast)$ is a non-zero linear form and $(\alpha_1, \alpha_2) \neq (0, 0)$. In this case, we say that the point $o$ is regular if the following condition (R2) holds.

(R2) The quadratic form

$$(\alpha_2 f_{1,2} - \alpha_1 f_{2,2})|_{\{\tau = 0\}}$$

is of rank at least 13 and, for every linear subspace $L \subset \{\tau = 0\}$ of codimension 1 (with respect to the hyperplane $\{\tau = 0\}$), the sequence $S[-4]|_{L}$ is regular, that is, the set of its common zeros in $\mathbb{P}(L) \cong \mathbb{P}^{M-2}$ is of dimension 3.

The regularity conditions at a bi-quadratic point. Assume that the linear forms $f_{1,1}$ and $f_{2,1}$ are identically equal to zero. Let $\mathbb{P} \to \mathbb{P}$ be the blow-up of $o$ and $E_\mathbb{P}$ the exceptional divisor. The affine coordinates $(z_1, \ldots, z_{M+2})$ generate homogeneous coordinates

$$(z_1 : z_2 : \cdots : z_{M+2})$$
on $E_\mathbb{P} \cong \mathbb{P}^{M+1}$. The point $o$ is regular if the following conditions (R2.1), (R2.2) and (R2.3) hold. The second and third of these conditions depend on the value of the degree $d_1$. The first is the same for all values of $d_1$.

(R2.1) We have

$$\text{rk}(f_{1,2}, f_{2,2}) \geq 13$$

and, moreover, at least one of the quadratic forms $f_{1,1}$ and $f_{2,2}$ is of rank at least 18.

It will be proved in § 1.7 that the condition (R2.1) implies that the system of equations

$$f_{1,2} = f_{2,2} = 0$$
defines an irreducible reduced complete intersection $E \subset E_\mathbb{P}$ of codimension 2 satisfying the inequality

$$\text{codim}(\text{Sing } E \subset E) \geq 10.$$We now assume that $d_1 \geq 4$.

(R2.2) For every linear subspace $L \subset \mathbb{C}^{M+2}$ of codimension 2, the system of equations

$$f_{1,2}|_{L} = f_{2,2}|_{L} = f_{1}|_{L} = f_{2}|_{L} = 0$$
defines an irreducible reduced complete intersection of codimension 4 in $L \cong \mathbb{C}^{M}$.

(R2.3) For every linear subspace $L \subset \mathbb{C}^{M+2}$ with codim $L \in \{2, 3\}$, the sequence

$$S[-\text{codim } L - 1]|_{L}$$
is regular.

Assume that $d_1 = 3$. Then the second and third regularity conditions take the following form.

(R2.2) For every linear subspace $L \subset \mathbb{C}^{M+2}$ of codimension 2, the system of equations

$$f_{1,2}|_{L} = f_{1,3}|_{L} = f_{2,2}|_{L} = f_{2,3}|_{L} = f_{2}|_{L} = 0$$
defines an irreducible reduced complete intersection of codimension 5 in $L \cong \mathbb{C}^{M}$.
(R2.3) For every linear subspace $L \subset \mathbb{C}^{M+2}$ with codim $L \in \{2, 3\}$, the sequence

$$S[\text{codim } L]|_L$$

is regular.

Assume, finally, that $d_1 = 2$. Then the second and third conditions are as follows.

(R2.2) For every linear subspace $L \subset \mathbb{C}^{M+2}$ of codimension 2, the system of equations

$$f_{1,2}|_L = f_{2,2}|_L = f_{2,3}|_L = f_2|_L = 0$$

defines an irreducible reduced complete intersection of codimension 4 in $L \cong \mathbb{C}^M$, and the condition (R2.3) is the same as in the case $d_1 = 3$.

We say that the pair of polynomials $(f_1, f_2) \in \mathcal{F}$ is regular if every point of the set of their common zeros is regular in the sense of the corresponding condition (R1), (R2) or (R2'). The set of regular pairs is denoted by $\mathcal{F}_{\text{reg}}$. If $(f_1, f_2) \in \mathcal{F}_{\text{reg}}$, then the corresponding complete intersection $F(f_1, f_2)$ is also said to be regular. It is clear that a regular complete intersection is a complete intersection of codimension 2 with good singularities and, therefore, a primitive Fano variety.

The following assertion is our second main result.

**Theorem 0.2.** When $d_1 \neq 3$, the complement $\mathcal{F} \setminus \mathcal{F}_{\text{reg}}$ is of codimension at least

$$\frac{1}{2}(M^2 - 17M + 64)$$

in $\mathcal{F}$. When $d_1 = 3$, the codimension of the complement $\mathcal{F} \setminus \mathcal{F}_{\text{reg}}$ in $\mathcal{F}$ is at least

$$\frac{1}{2}(M^2 - 19M + 82).$$

It follows from Theorem 0.2 that if $d_1 \neq 3$ and

$$\dim S < \frac{1}{2}(M^2 - 17M + 64)$$

(with $d_1 = 3$, replace the right-hand side by $(M^2 - 19M + 82)/2$), then for any ambient projective bundle $\pi_X : X \to S$ and any general subvariety $V \subset X$ of codimension 2 which is given locally over $S$ by a pair of equations

$$f_1(x_*, s) = f_2(x_*, s) = 0$$

with $(f_1(s), f_2(s)) \in \mathcal{F}$, the variety $V$ is a Fano–Mori fibre space into complete intersections of type $d_1 \cdot d_2$ because we may assume that

$$(f_1(s), f_2(s)) \in \mathcal{F}_{\text{reg}} \subset \mathcal{F}_{\text{bq}}.$$ 

Theorem 0.1 applies to the Fano–Mori fibre space $\pi : V \to S$ in view of the following fact, which is our third main result.

**Theorem 0.3.** Every regular complete intersection $F \in \mathcal{F}_{\text{reg}}$ is divisorially canonical.
The following example shows that it is easy to check the conditions (ii) and (iii) of Theorem 0.1.

**Example 0.1.** Assume that $d_1 \neq 3$ and $m$ is an integer with

$$1 \leq m \leq \frac{1}{2} (M^2 - 17M + 62).$$

Consider the direct product $X = \mathbb{P}^m \times \mathbb{P}^{M+2}$ and a general subvariety $V$ of codimension 2 which is a complete intersection of two hypersurfaces of bidegrees $(l_1, d_1)$ and $(l_2, d_2)$ respectively. Denote the projection of $V$ to $\mathbb{P}^m$ by $\pi$. It is clear that $\pi: V \to \mathbb{P}^m$ is a Fano–Mori fibre space into complete intersections of type $d_1 \cdot d_2$. Let $H_S$ and $H_P$ be the classes of hyperplanes in $\mathbb{P}^m$ and $\mathbb{P}$ respectively. We use the same notation for their pullbacks to $X$ and for their subsequent restrictions to $V$. In view of this agreement, we have

$$-K_V = (m + 1 - l_1 - l_2)H_S + H_P.$$

It is easy to see that assumptions (ii) and (iii) of Theorem 0.1 hold provided that

$$((-K_V) \cdot \pi^{-1}(L) \cdot H_P^M) \leq 0,$$

where $L \subset \mathbb{P}^m$ is a line. Simple calculations show that this inequality is equivalent to the estimate

$$l_1 \left(1 - \frac{1}{d_1}\right) + l_2 \left(1 - \frac{1}{d_2}\right) \geq m + 1.$$

If the integer parameters $l_1, l_2 \in \mathbb{Z}_+$ satisfy this inequality, then the Fano–Mori fibre space $\pi: V \to \mathbb{P}^m$ is birationally superrigid and $V$ has no structures of a Mori fibre space other than $\pi$ (up to a fibrewise birational equivalence). Note that the condition (iii) already follows from the inequality

$$l_1 \left(1 - \frac{1}{d_1}\right) + l_2 \left(1 - \frac{1}{d_2}\right) > m,$$

and (ii) follows from the inequality $l_1 + l_2 \geq m + 1$ provided that

$$H_P \notin \text{Int} A_1^1 V.$$

On the other hand, if $l_1 + l_2 \leq m$, then the anticanonical class $-K_V$ is ample and the projection to $\mathbb{P}$ gives the structure of a Fano–Mori fibre space on $V$ which is ‘transversal’ to the original structure $\pi$. Hence the fibration $\pi: V \to \mathbb{P}^m$ is not birationally rigid when $l_1 + l_2 \leq m$, and our numerical conditions on $l_1, l_2$ that guarantee the birational superrigidity are close to optimal.

**Remark 0.1.** The assumption $d_2 \geq 27$, which gives a lower bound for the dimension of the complete intersections considered in this paper (since $M = d_1 + d_2 - 2$), is needed only because the estimate in Theorem 0.2 for smaller dimensions is not given by a uniform simple expression in the form of a quadratic polynomial in $M$. 
But the assumption on the rank of one of the quadratic forms $f_{1,2}, f_{2,2}$ in $(R^2,1)$ is essential. Its omission would force us to exclude all bi-quadratic points from consideration, which would in turn impose much stronger upper bounds on the admissible dimension of $S$. Therefore, in order to state the main results simply, we assume that $d_2 \geq 27$, so that we also have $M \geq 27$.

**0.3. Structure of the paper.** In §1 we prove Theorem 0.1. In fact, we establish a much more general result on the birational superrigidity of Fano–Mori fibre spaces satisfying certain additional assumptions. These assumptions hold automatically for the fibrations into Fano hypersurfaces of index 1 considered in [2], so that the proof of Theorem 0.1 significantly improves the main theorem of that paper by giving not only the property that all birational maps onto rationally connected fibre spaces are fibrewise, but also the birational superrigidity in its full strength, that is, the uniqueness of the structure of a Mori fibre space up to fibrewise birational modifications which are biregular on the generic fibre. We shall prove Theorem 0.1 in the form stated and then identify the conditions used in the proof. The general fact that follows from the proof of Theorem 0.1 can also be applied to Fano–Mori spaces whose fibres are multiple projective spaces of index 1, as recently studied in [3], [4]. These varieties can be realized as hypersurfaces in weighted projective spaces and, therefore, have hypersurface singularities, which may be assumed quadratic with rank bounded below.

In §2 we prove Theorem 0.2. Except for the condition $(R^2,2)$, a routine technique (used many times and described, for example, in [5]) enables us to estimate the codimension of the set of pairs $(f_1, f_2) \in \mathcal{F}$ not satisfying the regularity conditions at one or more points. However, the condition $(R^2,2)$ is a much harder problem. We consider the general question of estimating the codimension of the set of tuples of polynomials whose common zero set is reducible or non-reduced. Comparing the estimates of codimension under the violation of each of the regularity conditions, we obtain Theorem 0.2.

The remainder of the paper, §§3–5, is devoted to the proof of Theorem 0.3. This is the hardest part of the paper. We describe its main stages. In §3 we prove Theorem 0.3 under the assumption that certain claims, global and local, are true. Assuming that these claims hold, we argue by contradiction. Suppose that the pair $(F, (1/n)D_F)$ is not canonical, where $D_F \sim n(-K_F)$. To arrive at a contradiction, we use the technique of hypertangent divisors ([5], Ch. 3) based on the regularity conditions for the fibre $F$. Thus we reduce Theorem 0.3 to a number of facts in global and local geometry.

The global assertions, which belong to the geometry of complete intersections of two quadrics in a projective space, are proved in §4. They are needed for the local analysis of the fibres $F$ at bi-quadratic points. To prove them, we use an elementary, but geometrically non-trivial, technique which was developed in [6], §4, for a single quadric. In the case of two quadrics, it is substantially more difficult to study maximal linear subspaces on their intersection.

The local facts, which characterize the blow-up behavior of non-canonical singularities of the pair $(F, (1/n)D_F)$ at quadratic and bi-quadratic points, are proved in §5. Note that the local fact for quadratic singularities radically simplifies the proof of divisorial canonicity for Fano hypersurfaces of index 1 with quadratic
singularities in [2], Theorem 1.4, as well as for multiple projective spaces in [3]. A similar radical simplification holds for the main result of [7], which appears as Theorem 3.2 in [5], Ch. 7. See § 3.10 for a detailed discussion of how the local facts established in § 5 improve known results.

0.4. Historical remarks. A majority of the results on birational rigidity deal with the absolute case (Fano varieties regarded as Fano–Mori fibre spaces over a point) and Fano–Mori spaces over \( \mathbb{P}^1 \); see [5] and the bibliography therein. Except for Sarkisov’s theorem on conic bundles [8], [9], the only result of birational rigidity type for fibrations over a base of dimension greater than 1 known until recently was the theorem on Fano direct products [10] (reproduced in [5], Ch. 7). Note that Theorem 0.3 enables us to use Fano complete intersections of codimension 2 and index 1 as direct factors of Fano direct products that preserve the birational rigidity.

Of the greatest interest, however, are theorems on the birational rigidity of general Fano–Mori fibre spaces. They realize the idea that sufficient ‘twistedness’ over the base implies birational (super)rigidity and the uniqueness of the structure of a Fano–Mori fibre space. The first result of this type was obtained in [2] for fibrations into primitive Fano hypersurfaces and double spaces over a fixed base. One of the main difficulties in the study of fibrations over a base of dimension greater than 1 is that, on the one hand, the singularities of the fibres worsen over some points and subvarieties and, on the other, we need birational modifications of the base.

For one-dimensional fibres (conics), these difficulties were within reach 40 years ago. This is why the uniqueness of the structure of a conic bundle sufficiently twisted over the base (or having sufficiently large degeneracies) is an old theorem. A way to get around the troubles described above was found in [2]. It consists in considering fibrations into Fano varieties with restricted singularities stable under blow-ups and thus obtaining fibre spaces whose base can be blown up preserving the properties of a Fano–Mori fibre space. We also use this approach in the present paper, but now the singularities of complete intersections can be much worse. Therefore the technique developed in [2] must be considerably improved, and this is what we do in the present paper (§§ 2, 4, 5). As a reward, we simplify the proof of the main result in [2] and strengthen the results obtained there.

Theorem 0.3 can be stated in terms of global canonical thresholds: it says that \( ct(F) \geq 1 \) for all \( F \in F_{\text{reg}} \). But in the proof of Theorem 0.1 it suffices to know that each fibre \( F \) of the fibre space \( \pi: V \to S \) satisfies the following weaker conditions instead of (i): the equality \( \ellct(F) = 1 \) for the global log canonical threshold and the inequality \( mct(F) \geq 1 \) for the mobile canonical threshold (then the proof given in § 1 needs no changes). These weaker conditions follow from the divisorial canonicity. However, since the technique in the proof of Theorem 0.3 (in §§ 3–5) yields the divisorial canonicity, and replacing it by the two weaker conditions in no way simplifies the proof, we give condition (i) in its strongest form.

Computing and estimating the global canonical thresholds has recently become a popular topic; see, for example, [11], [12] and many other papers over the last two or three years. This is related to applications in complex differential geometry, including the existence of Kähler–Einstein metrics and the question of \( K \)-stability.
Another topic of current interest related to the theory of birational rigidity is the study of groups of birational self-maps of rationally connected varieties. After the remarkable result about the Jordan property of the groups of birational self-maps [13], [14], papers in this direction (which began with Serre’s paper [15]) appeared one after another. Note that Theorem 0.1 yields the following fact, which is so standard in the theory of birational rigidity that we did not state it explicitly: the group BirV of birational self-maps is trivial for a general fibre space V → S into complete intersections of codimension 2.

Theorem 0.1 on the birational superrigidity of fibrations into complete intersections of codimension 2 may be regarded as a fragment of the birational classification of rationally connected varieties. In this context, we mention a new approach to the proof of stable non-rationality in terms of the ‘decomposition of the diagonal’ developed by Voisin and used in many recent papers by various authors; see, for example, [16]–[21]. This method enables us to prove the stable non-rationality of a very general variety in a given family. Another approach, which is based on using the Grothendieck ring, was recently suggested in [22]; see also [23]. Note that the results obtained by the method of maximal singularities (and, first of all, the theorem on Fano direct products [10]) are ‘at the opposite pole’ from the papers listed above: we take divisorially canonical primitive Fano varieties (and not \( \mathbb{P}^N \)) for direct factors and obtain a birational classification stable under such direct products.

Finally, we mention another important point. There are two types of technique used in the theory of birational rigidity, linear and quadratic. The linear technique is based on studying the singularities of a general divisor that defines the birational map. This is the technique used in the present paper. It was first developed and used in [10]. The quadratic technique is based on studying the singularities of the self-intersection of a mobile linear system that determines the birational map (or, in other words, of the scheme-theoretic intersection of two general divisors in this linear system). The quadratic technique goes back to the paper [24] by Iskovskikh and Manin on the three-dimensional quartic. Almost all the results on birational rigidity in the absolute case and the case of Fano–Mori fibre spaces over \( \mathbb{P}^1 \) were obtained using this technique. We mention the recent papers [25]–[27] where it is used to prove birational rigidity. Both techniques were used in [28], [6]. It seems that the quadratic technique can relax the upper bounds for the dimension of the base S of the Fano–Mori fibre space. However, all attempts to apply it in the case when the base is of dimension at least 2 meet considerable difficulties. The strongest result obtained in this direction using the quadratic technique is the theorem in [29] on the birational geometry of fibrations into double spaces of index 1.

§1. Birationally rigid Fano–Mori fibre spaces

In this section we prove Theorem 0.1. In § 1.1 we explain the main idea of the proof and introduce its most important constructions, the mobile linear system \( \Sigma \) and the mobile family \( \mathcal{C} \) of irreducible rational curves on V. In § 1.2 we begin the study of the maximal singularities of \( \Sigma \). In § 1.3 we apply fibrewise modifications to \( V/S \) to make the centres of all maximal singularities cover divisors on the base. The first assertion of Theorem 0.1 (saying that any birational map is
Birational geometry of varieties

fibrewise in the rationally connected case) is proved in § 1.4, and the second (on birational rigidity) is proved in 1.5. Finally, in § 1.6 we consider the properties of $V/S$ used in the proof of Theorem 0.1 and state the most general theorem on the birational superrigidity of Fano–Mori fibre spaces, which is proved by repeating §§ 1.1–1.5 verbatim. Theorem 0.1 is a particular case of this theorem for fibrations into complete intersections of codimension 2.

1.1. Statement of the problem and plan of the proof. Fix a Fano–Mori fibre space $\pi: V \to S$ satisfying the hypotheses of Theorem 0.1. We consider the following two options simultaneously. The fibration $\pi': V' \to S'$ is

- either a rationally connected fibre space, that is, the base $S'$ and the fibre of general position are rationally connected (the rationally connected case),
- or a Mori fibre space (the case of a Mori fibre space).

Fix a fibration $\pi': V' \to S'$ belonging to either of the two cases, with $\dim V' = \dim V$. Assume that there is a birational map $\chi: V \to V'$ and fix it too. In the rationally connected case we have to show that $\chi$ is fibrewise, that is, there is a rational dominant map $\beta: S \dashrightarrow S'$ such that the diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\chi} & V' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
S & \xrightarrow{\beta} & S'
\end{array}
$$

commutes. In the case of a Mori fibre space we have to show in addition that $\beta$ is birational, so that $\chi$ induces a birational isomorphism between the fibres $F_s = \pi^{-1}(s)$ and $F'_{\beta(s)} = (\pi')^{-1}(\beta(s))$ for every point $s \in S$ of general position. This isomorphism is biregular in view of the birational superrigidity of the fibres $F_s$.

Remark 1.1. First of all, we note that the inverse birational map $\chi^{-1}: V \dashrightarrow V'$ can be fibrewise only when $\chi$ is fibrewise and the corresponding map $\beta: S \dashrightarrow S'$ of the base is birational. Indeed, assume that there is a rational dominant map $\beta': S' \dashrightarrow S$ such that $\beta' \circ \pi' = \pi \circ \chi^{-1}$, where $\beta'$ is not birational. Then $\dim S' > \dim S$ and, for a point $s \in S$ of general position, the map

$$
\pi' \circ \chi|_{F_s} : F_s \dashrightarrow (\beta')^{-1}(s)
$$

fibres the Fano variety $F_s$ over the positive-dimensional variety $(\beta')^{-1}(s)$ into rationally connected varieties birational to the fibres $F'_{t}$ of the projection $\pi'$ when $t \in (\beta')^{-1}(s)$. However, by hypothesis (i) of Theorem 0.1, every fibre $F_s$ is birationally superrigid and, therefore, has no structures of a rationally connected fibre space over a positive-dimensional base. This contradiction proves our claim.

In view of this remark, we assume that $\chi^{-1}: V \dashrightarrow V'$ is not fibrewise. However, $\chi$ can still be fibrewise, but this assumption must lead to a contradiction in the case of a Mori fibre space. We now describe the two main objects which play a key role in the proof of Theorem 0.1.

The first of these is the mobile linear system $\Sigma$ related to $\chi$. In the rationally connected case, this is the same system as in [2], §2. Consider a very ample system $\Sigma'$ on $S'$ and write $\Sigma'$ for its $\pi'$-pullback on $V'$. Then

$$
\Sigma = (\chi_*^{-1})\Sigma' \subset |-nK_V + \pi^*Y|
$$

(1)
is its strict transform on $V$, where $n \in \mathbb{Z}_+$. In the case of a Mori fibre space (recall that we are considering both cases simultaneously), let $\Sigma'$ be the very ample complete linear system
\[\left| -mK' + (\pi')^*Y' \right|\]
on $V'$, where $m' \geq 1$, $K'$ is the canonical class $K_{V'}$, and $Y'$ is a very ample divisor on $S'$. Define $\Sigma$ by the formula (1). This is a mobile linear system on $V$, but in the case of a Mori fibre space we have $n \geq 1$. To make the notation uniform, we put $m = 0$ in the rationally connected case. The following assertion is a key fact in the proof of Theorem 0.1.

**Theorem 1.1.** We have $n \leq m$.

Theorem 1.1 clearly yields Theorem 0.1 in the rationally connected case: we obtain $n = 0$, whence $\Sigma$ is pulled back from $S$ and $\chi$ is fibrewise. It is also not very hard to complete the proof of Theorem 0.1 in the case of a Mori fibre space provided that Theorem 1.1 holds. Therefore our main purpose is to prove the latter theorem. From now on (up to the end of the proof of Theorem 1.1), we assume that $n > m$. In particular, $\chi$ is not fibrewise in the rationally connected case.

The second main object in our proof is a family $C$ of curves on $V$. It is constructed in the same way as in [2], §2. We recall this construction. Let $\varphi : \widetilde{V} \to V$ be a resolution of singularities of $\chi$ and let $E_{\text{exc}}$ be the set of all prime $\varphi$-exceptional divisors $E$ on $\widetilde{V}$ whose image on $V'$ is divisorial and the prime divisor $[\chi \circ \varphi](E) \subset V'$ on $V'$ covers the base of $\pi'$, that is,
\[\pi'( [\chi \circ \varphi](E)) = S'.\]
We consider a family $C'$ of (irreducible rational) curves on $V'$ contracted by $\pi'$, sweeping out an open dense subset of $V'$, disjoint from the indeterminacy locus of the rational map
\[ [\chi \circ \varphi]^{-1} : V' \dashrightarrow \widetilde{V} \]
and intersecting every divisor $[\chi \circ \varphi](E), E \in E_{\text{exc}}$, transversally at points of general position; see [2], §2.1. The curves $C' \in C'$ lie in the fibres of $\pi'$. Let
\[ C = \varphi_* \circ [\chi \circ \varphi]^{-1}(C') \]
be the strict transform of $C'$ on $V$. This is a mobile family of (irreducible rational) curves on $V$ sweeping out an open dense subset of $V$. By Remark 1.1, the curves $C \in C$ do not lie in the fibres of $\pi$. Hence the image
\[ \pi_* C = \overline{C} \]
of this family on $S$ is a mobile family of curves on $S$. Given any $C \in C$, we write $\overline{C}$ for the image $\pi(C)$. The linear system $\Sigma \subset \left| -nK_V + \pi^* Y \right|$ and the family $C$ of curves are the main elements in our construction. Since $\Sigma$ is mobile, the divisor $Y$ is pseudoeffective by the hypothesis (ii) of Theorem 0.1. Hence, given any curve $C \in C$ and its image $\overline{C}$ on $S$, we have
\[ (C \cdot \pi^* Y) = (\overline{C} \cdot Y) \geq 0.\]
The main idea in the proof of Theorem 1.1 is to assume that $n > m$ and obtain a contradiction to hypothesis (iii) of Theorem 0.1. The class of the cycle

$$ (D \circ \pi^{-1}(C)) \sim -n(K_V \cdot \pi^{-1}(C)) + (C \cdot Y)F $$

is clearly effective for a general divisor $D \in \Sigma$. We will see that the scheme-theoretic intersection on the left-hand side contains sufficiently many fibres of $\pi$. Subtracting them, we obtain an effective cycle that contradicts hypothesis (iii).

1.2. Maximal singularities. Let $E_{exc}$ be the set of all prime $\varphi$-exceptional divisors whose image on $V'$ is divisorial and covers $S'$. We represent it as a disjoint union,

$$ E_{exc} = E_S \sqcup E_{div} \sqcup E, $$

where $E \in E_S$ (resp. $E_{div}$ and $E$) if and only if the centre $\varphi(E)$ of the divisor $E$ on $V$ covers $S$, that is,

$$ \pi[\varphi(E)] = S $$

(resp. covers a prime divisor and an irreducible closed subset of codimension at least 2 on $S$).

Putting $\tilde{K} = K_V$ and omitting $\varphi^*$ for simplicity of notation, we get

$$ \tilde{K} = K_V + \sum_{E \in E_{exc}} a_E E + (\ldots) $$

and for the strict transform $\tilde{\Sigma}$ of $\Sigma$ on $\tilde{V}$ we have

$$ \tilde{\Sigma} \subset \left\{-n\tilde{K} + \left(\pi^*Y - \sum_{E \in E_{exc}} \varepsilon(E)E\right) + (\ldots)\right\}, $$

where $\varepsilon(E) = b_E - na_E$, $b_E = \text{ord}_E \varphi^*\Sigma$, and $(\ldots)$ in both cases stands for a linear combination of prime $\varphi$-exceptional divisors not belonging to $E_{exc}$ (that is, either their image on $V'$ is not divisorial, or it is divisorial but does not cover $S'$). Such prime divisors are inessential for our construction. If $\varepsilon(E) > 0$, that is, the Noether–Fano inequality holds:

$$ b_E > na_E, $$

then the prime divisor $E$ is a maximal singularity in the strong sense of the linear system $\Sigma$ (the inessential prime divisors in $(\ldots)$ can also be maximal singularities satisfying the Noether–Fano inequality).

**Proposition 1.1.** We have $\varepsilon(E) > 0$ for at least one $E \in E_{exc}$, that is, maximal singularities in the strong sense exist.

**Proof.** This is Proposition 2.1 in [2]. □

We denote the set of all maximal singularities in the strong sense by $M$. The following assertion strengthens Proposition 2.2 in [2].

**Proposition 1.2.** We have an inclusion $M \subset E$, that is, the image of the centre of a maximal singularity $E \in M$ on $V$ under the projection $\pi$ is of codimension at least 2 on $S$. 
Proof. Assume the opposite: the centre of a maximal singularity $E$ covers the whole base $S$ or a prime divisor on $S$. Let $s \in \pi[\varphi(E)]$ be a point of general position. Since $\Sigma$ has no fixed components, the fibre $F_s = \pi^{-1}(s)$ is not contained in the base set of $\Sigma$. For a general divisor $D \in \Sigma$, the pair $(V,(1/n)D)$ is not canonical and, moreover, $E$ is a non-canonical singularity of this pair and $F_s$ is not contained in the support $|D|$ of the divisor $D$. We write $D_F$ for the restriction

$$D|_{F_s} = (D \circ F_s).$$

Then $D_F \sim -nK_{F_s}$. If $\text{codim}(\pi[\varphi(E)] \subset S) = 1$, then the pair $(F_s,(1/n)D_F)$ is not log canonical by inversion of adjunction. But if $\pi[\varphi(E)] = S$, then $(F_s,(1/n)D_F)$ is not canonical. In either case, this contradicts hypothesis (i) of Theorem 0.1. □

Remark 1.2. (i) In the case (considered in Proposition 2.2 of [2]) when the centre $\varphi(E)$ covers the base $S$, the divisor $D_F$ is a general divisor of a mobile linear system $\Sigma|_{F_s}$. To exclude this divisor, it suffices to assume that the mobile canonical threshold of the fibre is at least 1. In the case when $\varphi(E)$ covers a divisor on $S$, it suffices to assume that $\text{lct}(F_s) = 1$ (this condition is weaker than hypothesis (i) of Theorem 0.1). The divisorial canonicity is stronger than each of these two conditions.

(ii) The proof of Proposition 1.2 holds without change for any maximal singularity, that is, any prime $\varphi$-exceptional divisor $E \subset \widetilde{V}$ satisfying the Noether–Fano inequality

$$\text{ord}_E \varphi^* \Sigma > n \cdot a(E, V).$$

The property of having a divisorial image on $V'$ that covers $S'$ is not used in the proof. This remark will be needed below in the proof of the birational superrigidity of $V/S$.

Following § 2.2 in [2], we now consider a fibrewise modification of $V/S$. We need to ensure that the centre of every maximal singularity in the strong sense covers at least a divisor on the base of the fibre space.

1.3. Fibrewise birational modification. Let $\sigma_S: S^+ \to S$ be a composite of blow-ups with non-singular centres such that the centres of all the singularities $E \in \mathcal{E}$ on $V^+ = V \times_S S^+$ cover a prime divisor on $S^+$. This is the minimal requirement for the birational morphism $\sigma_S$. In what follows we will need the additional property that the base set of the strict transform of $\Sigma$ on $V^+$ contains no fibres of the modified Fano–Mori fibre space.

Let $\sigma: V^+ \to V$ and $\pi_+: V^+ \to S^+$ be the projections of the fibre product, so that the following diagram commutes:

$$
\begin{array}{ccc}
V^+ & \xrightarrow{\sigma} & V \\
\downarrow{\pi_+} & & \downarrow{\pi} \\
S^+ & \xrightarrow{\sigma_S} & S.
\end{array}
$$

The existence of such a modification is proved in [2], § 2.2. The morphism $\pi_+: V^+ \to S^+$ is a Fano–Mori fibre space: the variety $V^+$ is factorial and its singularities
are terminal. This follows easily from the construction of the fibre space $V/S$ in §0.1; see §1.7.

We now introduce some new notation. Let $T$ (resp. $\mathcal{T}$) be the set of prime $\sigma$-exceptional (resp. $\sigma_S$-exceptional) divisors on $V^+$ (resp. $S^+$). The projection $\pi_+$ gives a bijection $T \to \mathcal{T}$. We have an equality of discrepancies for every $T \in T$ and $\mathcal{T} = \pi_+(T) \in \mathcal{T}$:

$$a_T = a(T, V) = a(\mathcal{T}, S).$$

Moreover, we define the set $\mathcal{T}_{\text{div}}$ of prime divisors on $S$ that are the $\pi$-images of the subvarieties $\varphi(E) \subset V$ for $E \in \mathcal{E}_{\text{div}}$. Accordingly, let $\mathcal{T}_{\text{div}}$ be the set of all prime divisors on $V$ of the form $\pi^{-1}(T)$, where $T \in \mathcal{T}_{\text{div}}$.

Let $\Sigma^+$ be the strict transform of $\Sigma$ on $V^+$. We have

$$\Sigma^+ \subset \left| -nK_V + \pi^*Y - \sum_{T \in \mathcal{T}} b_T T \right|$$

for some $b_T \in \mathbb{Z}_+$ (we again omit the pullback symbol $\sigma^*$). As in [2], §2.3, we assume that the resolution $\varphi: \widetilde{V} \to V$ factors through $\sigma$, that is, it takes the form $\psi = \sigma \circ \varphi_+$, where $\varphi_+: \widetilde{V} \to V^+$ is a sequence of blow-ups with non-singular centres. By construction there is a map

$$\lambda: \mathcal{E} \to \mathcal{T},$$

where $\lambda(E) = T \in \mathcal{T}$ is the uniquely determined $\sigma$-exceptional divisor such that

$$\varphi_+(E) \subset T,$$

and we have $\pi_+ [\varphi_+(E)] = \mathcal{T} = \pi_+(T)$. In a similar way, we define a map

$$\lambda_{\text{div}}: \mathcal{E}_{\text{div}} \to \mathcal{T}_{\text{div}}.$$

There is an obvious equality for the discrepancies:

$$a_E = a_E^+ + a_T \cdot \text{ord}_E \varphi_+^* T,$$

where $T = \lambda(E)$ and $a_E^+ = a(E, V^+)$. It may happen that $a_E^+ = 0$. This occurs precisely when $E = T$ (and $a_E = a_T$). For exceptional divisors $E \in \mathcal{E}_S \sqcup \mathcal{E}_{\text{div}}$ we have the equality $a_E = a_E^+$. Furthermore, the mobile linear system $\mathcal{E}$ is a subsystem of the complete linear system

$$\left| \varphi_+^* \left( -nK_V + \pi^*Y - \sum_{T \in \mathcal{T}} b_T T \right) - \sum_{E \in \mathcal{E}_{\text{exc}}} b_E^+ E - (\ldots) \right|,$$

where $b_E^+ \in \mathbb{Z}_+$ and $(\ldots)$ has the same meaning as before. When $E \in \mathcal{E}$ we have the equality

$$b_E = b_E^+ + b_T \cdot \text{ord}_E \varphi_+^* T,$$

where $T = \lambda(E)$. Again, $b_E^+ = 0$ if and only if $E = T$.

**Proposition 1.3.** For every $E \in \mathcal{E}_{\text{exc}}$ we have

$$b_E^+ \leq n \cdot a_E^+.$$
Proof. We repeat the proof of Proposition 1.2 almost verbatim. Assume the opposite: \( b_E > n \cdot a_E \). Then \( b_E > 0 \) and, therefore, \( E \neq T \) for \( T = \lambda(E) \), that is, the centre of \( E \) has codimension at least 2 on \( V^+ \). Hence \( a_E > 0 \). Furthermore, the pair \((V^+, (1/n)D^+)\) is not canonical for \( D^+ \in \Sigma^+ \) by our assumption. Since the linear system \( \Sigma^+ \) is mobile, we can assume that, for a point \( s \in T \) of general position, the fibre \( F_s = \pi^{-1}(s) \) is not contained in the support of \( D^+ \). Hence \( a_E > 0 \). Furthermore, the pair \((V^+, (1/n)D^+)\) is not canonical for \( D^+ \in \Sigma^+ \) by our assumption.

**Corollary 1.1.** For every maximal singularity \( E \in M \) we have
\[
 b_T > n \cdot a_T,
\]
where \( T = \lambda(E) \).

**Proof.** This follows immediately from the inequality \( b_E > n \cdot a_E \), Proposition 1.3 and the explicit formulae (given above) for \( b_E \) and \( a_E \). □

We now return to the mobile family \( \mathcal{C} \) of irreducible rational curves on \( V \) constructed in § 1.1. It is the strict transform of the mobile family \( \mathcal{C}' \) on \( V' \). In the rationally connected case, the linear system \( \Sigma' \) is pulled back from \( S' \), so that for a divisor \( D' \in \Sigma' \) and a curve \( C' \in \mathcal{C}' \) we have
\[
 (C' \cdot D') = 0.
\]
In the case of a Mori fibre space we have
\[
 (C' \cdot D') = -m(C' \cdot K') \geq m,
\]
so that when \( l > m \) we get
\[
 (C' \cdot [D' + lK]) = (l - m)(C' \cdot K') \leq -(l - m).
\]

We write \( \tilde{C} \), \( C^+ \) and \( \overline{C}^+ \) for the strict transforms of \( C \) on \( \tilde{V} \), \( V^+ \) and the image of \( C^+ \) on \( S^+ \) respectively. The general curves are accordingly denoted by \( \tilde{C} \in \tilde{C}, C^+ \in C^+ \) and \( \overline{C}^+ \in \overline{C}^+ \) (where \( \overline{C}^+ = \pi_+(C^+) \)). The family \( \overline{C}^+ \) of curves on \( S^+ \) is mobile and sweeps out a dense open subset of the base. Hence for a general divisor \( D^+ \in \Sigma^+ \) we have a well-defined algebraic cycle of scheme-theoretic intersection
\[
 (D^+ \circ \pi^{-1}_+(\overline{C}^+) ),
\]
This is an effective cycle of dimension \( \dim F \) on \( V^+ \). We shall complete the proof of Theorem 1.1 by estimating the class of this cycle.

1.4. Restriction to the inverse image of a curve. Obviously,
\[
 (D^+ \circ \pi^{-1}_+(\overline{C}^+) ) \sim -n(\sigma^* K_V \cdot \pi^{-1}_+(\overline{C}^+) ) + bF,
\]
where
\[
 b = \left( \left[ \sigma^* Y - \sum_{T \in T} b_T T \right] \cdot \overline{C}^+ \right).
\]
The following assertion is of key importance.
Proposition 1.4. (i) In the rationally connected case we have
\[ b \leq -2n. \]

(ii) In the case of a Mori fibre space we have
\[ b \leq -n. \]

Proof. Since \((\pi_+)_* C^+ = \overline{C^+}\), we obtain
\[ b = \left(\pi^* Y - \sum_{T \in T} b_T T \right) \cdot C^+ \]
(omitting \(\sigma^*\), as usual). Here \((C^+ \cdot T) \neq 0\) only for the divisors \(T \in \lambda(\mathcal{E})\) since the curve \(C' \subseteq C'\) is disjoint from the indeterminacy locus of \((\chi \circ \varphi)^{-1}\) by construction. Since \(C\) and \(\tilde{C}\) sweep out dense open subsets of \(V\) and \(\tilde{V}\) respectively, we have
\[ (K_V \cdot C) < 0 \quad \text{and} \quad (\tilde{K} \cdot \tilde{C}) < 0 \]
for \(C = \sigma(C^+) \in \mathcal{C}\) and \(\tilde{C} \subseteq \tilde{C}\). On the other hand, for a general divisor \(\tilde{D} \in \tilde{\Sigma}\) we have
\[ (\tilde{D} \cdot \tilde{C'}) = (D' \cdot C') = 0 \]
in the rationally connected case and
\[ (\tilde{D} \cdot \tilde{C'}) = (D' \cdot C') = -m(K' \cdot C'). \]
in the case of a Mori fibre space. Since
\[ \tilde{D} \sim -n\tilde{K} + \left[ \pi^* Y - \sum_{E \in \mathcal{E}_{\text{exc}}} (b_E - na_E)E \right] + (\ldots), \]
where \((\ldots)\) is a linear combination of divisors disjoint from \(\tilde{C}\), we obtain the inequality
\[ \left(\pi^* Y - \sum_{E \in \mathcal{E}_{\text{exc}}} (b_E - na_E)E \right) \cdot \tilde{C} \leq -(n - m) \]
(recall that we put \(m = 0\) in the rationally connected case). However, for \(E \in \mathcal{E}_S \sqcup \mathcal{E}_{\text{div}}\) we have
\[ b_E - na_E = b_E^+ - na_E^+ \leq 0 \]
(see Proposition 1.2), and for \(E \in \mathcal{E}\) we have
\[ b_E - na_E = (b_E^+ - na_E^+) + \text{ord}_E \varphi^*_+ T \cdot (b_T - na_T), \]
where \(T = \lambda(E) \in \mathcal{T}\). In view of Proposition 1.3, it follows that
\[ \left(\pi^* Y - \sum_{T \in \lambda(\mathcal{E})} (b_T - na_T) \sum_{E \in \lambda^{-1}(T)} (\text{ord}_E \varphi^*_+ T)E \right) \cdot \tilde{C} \leq -(n - m). \]
However, for every \( T \in \lambda(\mathcal{E}) \), the divisor
\[
\varphi_+^* T - \sum_{E \in \lambda^{-1}(T)} (\text{ord}_E \varphi_+^* T) E
\]
is effective and has zero intersection with \( \tilde{C} \) since no irreducible component of it is divisorial on \( V' \). (Here the strict transform \( \tilde{T} \) of \( T \) on \( \tilde{V} \) can also occur in the sum if \( \tilde{T} \in \lambda^{-1}(T) \) and, in this case, we clearly have \( \text{ord}_{\tilde{T}} \varphi_+^* T = 1 \).) Hence the intersection number
\[
\left( \left[ \pi^* Y - \sum_{T \in \lambda(\mathcal{E})} (b_T - na_T) \varphi_+^* T \right] \cdot \tilde{C} \right) = \left( \left[ \pi^* Y - \sum_{T \in T} (b_T - na_T) T \right] \cdot C^+ \right)
\]
does not exceed \(-(n - m)\) (we recall that \( (T \cdot C^+) = 0 \) when \( T \in T \setminus \lambda(\mathcal{E}) \)). Therefore,
\[
b = \left( \left[ \pi^* Y - \sum_{T \in T} b_T T \right] \cdot C^+ \right) \leq -(n - m) - n \left( \sum_{T \in T} a_T T \right) \cdot C^+ \leq -2n + m.
\]
This completes the proof of Proposition 1.4. □

Thus,
\[
\sigma_*(D^+ \circ \pi_+^{-1}(\overline{C}^+)) \sim -n(K_V \cdot \pi^{-1}(\overline{C})) + bF,
\]
where \( b \leq -n - (n - m) \). Since the algebraic cycle on the left-hand side is effective, the class
\[
-(K_V \cdot \pi^{-1}(\overline{C})) - F
\]
turns out to be effective. This contradicts hypothesis (iii) of Theorem 0.1. We assumed that \( n > m \) and arrived at a contradiction. This completes the proof of Theorem 0.1.

In particular, in the rationally connected case (when \( m = 0 \)), we obtain the equality \( n = 0 \), that is, \( \chi \) is fibrewise. This proves the first (‘rationally connected’) assertion of Theorem 0.1. Let \( \beta: S \to S' \) be a rational dominant map such that \( \beta \circ \pi = \pi' \circ \chi \).

We now proceed to prove the second assertion of Theorem 0.1, that is, the birational superrigidity of the Fano–Mori fibre space \( V/S \).

We begin with the following observation.

**Remark 1.3.** It is easy to see from the proof of Proposition 1.2 or Proposition 1.3 that if a prime \( \varphi_+ \)-exceptional divisor \( E \subset \tilde{V} \) is a singularity of \( \Sigma^+ \), that is, we have an inequality
\[
b_+^E = \text{ord}_E \varphi_+^* \Sigma^+ > 0,
\]
and if the closed subset
\[
\pi_+^{-1}(\pi_+[\varphi_+(E)]) \subset V^+
\]
(consisting of all the fibres of \( \pi_+ \) that contain at least one point of the centre of \( E \) on \( V^+ \)) is not contained entirely in the base set \( \text{Bs} \Sigma^+ \), then
\[
b_E \leq na_E.
\]
Indeed, it follows from our assumption that, for a point
\[ s \in \pi_+[\varphi_+(E)] \subset S^+ \]
of general position, the fibre \( F_s \) is not contained in the support of a general divisor \( D^+ \in \Sigma^+ \). Hence the proof of Proposition 1.3 (or 1.2), which is based on inversion of adjunction, works without any changes.

1.5. Proof of the birational rigidity. We have to show that \( \beta \) is birational. As above, we assume that this is not the case. Since the general fibre of a Mori fibre space is irreducible, it follows that \( \dim S' < \dim S \) and the general fibre of \( \beta \) is an irreducible variety of positive dimension. We shall prove that this is impossible.

Using the notation in § 1.3, we assume in addition that the sequence \( \sigma_S : S^+ \to S \) of blow-ups resolves the singularities of \( \beta \), so that the composite
\[ \beta_+ = \beta \circ \sigma_S : S^+ \to S' \]
is a morphism (whose general fibre is irreducible). We denote the composite
\[ \chi \circ \sigma : V^+ \to V' \]
by \( \chi_+ \). Since we have completed the study of the rationally connected case, we are in the case of a Mori fibre space. Hence the linear system
\[ \Sigma' = \langle -mK' + (\pi')^*Y' \rangle \]
is a very ample complete linear system on \( V' \) and \( m \geq 1 \). Its strict transform on \( V \),
\[ \Sigma \subset \langle -nK_V + \pi^*Y \rangle, \]
satisfies the inequality \( n \leq m \) by Theorem 1.1. Consider the strict transform \( \Sigma^+ \) of \( \Sigma \) on \( V^+ \). In addition to our assumptions about the birational morphism \( \sigma_S \), we assume that \( \sigma_S \) flattens the singularities of \( \Sigma \) over \( S \) in the following sense. No fibre \( F_s, s \in S^+ \), of the projection \( \pi_+ \) is contained in the base set \( B_\Sigma \). Then we have
\[ \Sigma^+ \subset \langle -nK^+ + \pi_+^*Y^+ \rangle, \]
where \( K^+ = K_{V^+} \) and \( Y^+ \) is a divisor on \( S^+ \). We again consider the resolution \( \varphi_+: \tilde{V} \to V^+ \) of singularities of the map \( \chi_+ = \chi \circ \sigma : V^+ \to V' \), and let \( \varphi' = \chi_+ \circ \varphi_+: \tilde{V} \to V' \) be the corresponding birational morphism. We write \( \mathcal{E}' \) for the set of all prime \( \varphi' \)-exceptional divisors on \( \tilde{V} \) that cover \( S' \), and \( \mathcal{E}^+ \) for the set of all prime \( \varphi_+ \)-exceptional divisors on \( \tilde{V} \) (we no longer require the images of \( E \in \mathcal{E}^+ \) on \( V' \) to be divisorial). For the canonical class \( \tilde{K} = K_{\tilde{V}} \) we have
\[ \tilde{K} = K^+ + \sum_{E \in \mathcal{E}^+} a_E^+ E = K' + \sum_{E' \in \mathcal{E}'} a(E')E' + (\ldots), \tag{2} \]
where \( a_E^+ \) and \( a(E') \) are the discrepancies with respect to \( V^+ \) and \( V' \) respectively, and \( (\ldots) \) stands for an effective linear combination of prime \( \varphi' \)-exceptional divisors on \( \tilde{V} \) whose image on \( V' \) does not cover \( S' \).
Let $F' = F'_t = (\pi')^{-1}(t)$ be a fibre of general position of the projection $\pi'$. The divisors occurring in $(\ldots)$ are clearly disjoint from the strict transform $\widetilde{F'}$ on $\widetilde{V}$. We write $G$ for the subvariety

$$
\pi_+^{-1}(\beta_+^{-1}(t)) \subset V^+.
$$

Its strict transform $\widetilde{G}$ on $\widetilde{V}$ is clearly equal to $\widetilde{F'}$. For the strict transform $\widetilde{\Sigma}$ of $\Sigma^+$ on $\widetilde{V}$ we have

$$
\widetilde{\Sigma} \subset \left\{ -nK^+ + \pi^+_+ Y^+ - \sum_{E \in E^+} b_E^+ E \right\}
$$

(omitting the pullback symbols $\varphi^+_+$, as usual). Hence we obtain the following equality of divisorial classes:

$$
-mK' + (\pi')^* Y' = -nK^+ + \pi^+_+ Y^+ - \sum_{E \in E^+} b_E^+ E.
$$

Hence, using easy computations and (2), we arrive at the equality

$$(m-n)K^+ + (\pi^+_+ Y^+ - (\pi')^* Y') = \sum_{E \in E^+} (b_E^+ - ma_E^+) E + m \sum_{E' \in E'} a(E') E' + (\ldots), \quad (3)$$

where $(\ldots)$ has the same meaning as in (2). Since by construction no fibre of $\pi_+$ is contained in $\text{Bs} \Sigma^+$, we deduce from Remark 1.3 that

$$
b_E^+ \leq na_E^+ \leq ma_E^+.
$$

Let $F = F_s \subset G$ be a general fibre of $\pi_+$, and let $\widetilde{F}$ be its strict transform on $\widetilde{V}$. Since

$$
\pi^+_+ Y^+ |_F = 0 \quad \text{and} \quad (\pi')^* Y' |_{\tilde{G}} = (\ldots) |_{\tilde{G}} = 0,
$$

restricting (3) to $\widetilde{F}$ yields that

$$(m-n)K^+ |_{\tilde{F}} = \sum_{E \in E^+} (b_E^+ - ma_E^+) E |_{\tilde{F}} + m \sum_{E' \in E'} a(E') E' |_{\tilde{F}}.
$$

Applying the direct image operation $(\varphi_+)_*$ to both sides, we obtain an equality of divisorial classes on $F$ whose left-hand side is a negative divisor when $n < m$ and whose right-hand side is the effective divisor

$$
m \sum_{E' \in E'} a(E') \left( \varphi_+ \right)_* E' |_{\tilde{F}}.
$$

Therefore we conclude that $n = m$. Now, restricting (3) to $\widetilde{G}$, we have

$$
\pi^+_+ Y^+ |_{\tilde{G}} + \sum_{E \in E^+} (na_E^+ - b_E^+) E |_{\tilde{G}} = \sum_{E' \in E'} a(E') E' |_{\tilde{G}}. \quad (4)
$$

Applying $(\varphi_+)_*$ again and taking into account that all the discrepancies $a(E')$ are strictly positive, we see that the divisorial class

$$
(\pi^+_+ Y^+) | G = \pi^+_+ (Y^+ |_{\beta_+^{-1}(t)})
$$
is effective. Therefore both sides of (4) are effective divisors. However, all the
divisors $E'|\bar{G}$ are exceptional for the birational morphism $\varphi'|\bar{G}: \bar{G} \to F'$. Hence
effectivity any linear combination of them with positive coefficients is a fixed effective divisor.
We conclude that there are finitely many prime divisors $Y_i$, $i \in I$, on the variety
$\bar{G} = \beta_{+}^{-1}(t) = \pi_{+}(G) \subset S^+$ such that

$$Y^+|_{\bar{G}} \sim \sum_{i \in I} m_i Y_i$$

for some $m_i \geq 1$ and, moreover, the complete linear system $|\sum_{i \in I} m_i^* Y_i|$ consists of a single divisor (equal to $\sum_{i \in I} m_i^* Y_i$) for arbitrary numbers $m_i^* \in \mathbb{Z}_+$. Consider the $\mathbb{Q}$-vector space

$$\mathcal{A} = \text{Pic} \bar{G} \otimes \mathbb{Q}$$

and let $\mathcal{D} \subset \mathcal{A}$ be the subspace generated by the classes of the irreducible components of the divisors $E'|\bar{G}$ for all $E' \in \mathcal{E}'$. We clearly have

$$\mathcal{A} = \mathbb{Q}K' \oplus \mathcal{D}$$

since the Picard number of the fibre $F'$ is equal to 1. On the other hand, replace

$$\pi^+_Y|_{\bar{G}} \quad \text{by} \quad (\varphi^+_|\bar{G})^*(\pi^+_|G)^* \sum_{i \in I} m_i Y_i$$

in (4). Since the divisor on the right-hand side is fixed, we obtain physical equality (and not merely linear equivalence) of two effective divisors. It follows that the hyperplane $\mathcal{D} \subset \mathcal{A}$ is generated by the classes of irreducible components of the divisors $E|\bar{G}$, $E \in \mathcal{E}^+$, and the divisors

$$(\varphi^+_|\bar{G})^*(\pi^+_|G)^* Y_i, \quad i \in I.$$  

We now claim that the equality

$$\dim_{\mathbb{Q}} \mathcal{A}/\mathcal{D} = 1$$

is impossible. Indeed, the class of any ample divisor $\Delta$ on the variety $\bar{G}$ in the space $\text{Pic} \bar{G} \otimes \mathbb{Q}$ cannot belong to the subspace generated by $Y_i$, $i \in I$, because it is mobile. Therefore,

$$\Delta = (\pi^+_|G)^* \Delta \notin \mathcal{D}$$

(we again omit the symbol $(\varphi^+_|\bar{G})^*$ of pullback to $\bar{G}$). It follows that

$$\mathcal{A} = \mathbb{Q} \Delta \oplus \mathcal{D}.$$  

However, a curve $\Gamma \subset F$ of general position in a fibre $F$ of general position of the projection $\pi^+_|G$ is disjoint from the indeterminacy locus of the map

$$(\varphi^+_|\bar{G})^{-1}: G \dashrightarrow \bar{G}$$

and, therefore, its inverse image $\Gamma = \Gamma$ on $\bar{G}$ has zero intersection with any divisor whose class is contained in $\mathcal{D}$ and with any divisor pulled back from the base $\bar{G}$. 
Thus $\Gamma$ has zero intersection with any divisor on $\tilde{G}$, which is absurd. This contradiction shows that the rational dominant map $\beta: S \rightarrow S'$ is birational and, therefore, $\chi$ maps a fibre $F_s$ of general position birationally onto the fibre $F'_{\beta(s)}$ of general position. Since $F_s$ is birationally superrigid while $F'_{\beta(s)}$ is $\mathbb{Q}$-factorial with terminal singularities and $\rho(F'_{\beta(s)}) = 1$, we conclude that the birational map

$$\chi|_{F_s}: F_s \rightarrow F'_{\beta(s)}$$

is an isomorphism. Theorem 0.1 is proved.

1.6. A generalization of Theorem 0.1. In the proof of Theorem 0.1 (including the proof of Theorem 1.1) we never used the explicit construction of the fibre space $V/S$, but we used properties of it that follow from this construction. Theorem 0.1 holds for any Fano–Mori fibre space possessing these properties. Let $\pi: V \rightarrow S$ be an arbitrary Fano–Mori fibre space over a non-singular rationally connected base $S$, that is, $V$ is factorial with terminal singularities, we have

$$\text{Pic } V = \mathbb{Z}K_V \oplus \pi^*\text{Pic } S$$

and the anticanonical class $(-K_V)$ is relatively ample. (The varieties $V$ and $S$ are assumed to be projective.)

Definition 1.1. A Fano–Mori fibre space $\pi: V \rightarrow S$ all of whose fibres are irreducible and reduced is said to be stable under fibrewise birational modifications if, for every birational morphism $\sigma_S: S^+ \rightarrow S$, where $S^+$ is non-singular, the corresponding morphism

$$\pi^+: V^+ = V \times_S S^+ \rightarrow S^+$$

is a Fano–Mori fibre space, that is, $V^+$ is factorial and its singularities are terminal.

Then the proof of Theorem 0.1 in §§ 1.1–1.5 yields the following general fact.

Theorem 1.2. Assume that the Fano–Mori fibre space $\pi: V \rightarrow S$ all of whose fibres are irreducible and reduced is stable under fibrewise birational modifications. Also assume that the following conditions hold.

(i) For every point $s \in S$, the fibre $F_s = \pi^{-1}(s)$ has global log canonical threshold $\text{lct}(F_s) = 1$ and mobile canonical threshold $\text{mct}(F_s) \geq 1$.

(ii) The $K$-condition holds, that is, for every effective divisor

$$D \in |-nK_V + \pi^*Y|,$$

where $n \geq 1$, the divisorial class $Y$ is pseudoeffective on $S$.

(iii) For every mobile family $\mathcal{C}$ of irreducible rational curves on $S$ sweeping out a dense open subset of $S$ and for every curve $\mathcal{C} \in \mathcal{C}$, no positive multiple of the algebraic cycle

$$-(K_V \cdot \pi^{-1}(\mathcal{C})) - F$$

is effective, where $F$ is the class of a fibre of $\pi$.

Then, for any rationally connected fibre space $V'/S'$, every birational map $\chi: V \rightarrow V'$ (if such maps exist) is fibrewise, and the fibre space $V/S$ is birationally superrigid.
1.7. Quadratic and bi-quadratic singularities. We shall prove that the Fano–Mori fibre spaces $V/S$ constructed in §§ 0.1, 0.2 are stable under fibrewise birational modifications. The proof is not hard, but it is convenient to perform it in a more general context.

First of all, we recall that an algebraic variety $X$ is called a variety with at most quadratic singularities of rank at least $r$ if, in a neighbourhood of every point $o \in X$, this variety can be realized as a hypersurface in a non-singular variety $Y$ with a local equation at $o$ of the form

$$0 = \beta_1(u_*) + \beta_2(u_*) + \cdots,$$

where $(u_*)$ is a system of local parameters on $Y$ at $o$ and either the linear form $\beta_1 \not\equiv 0$ (that is, the point $o \in X$ is non-singular), or $\beta_1 \equiv 0$ and then the quadratic form $\beta_2$ is of rank at least $r$. It was proved in [2], § 3.1, that the property of having quadratic singularities of rank at least $r$ is stable under blow-ups in the following sense. Let $B \subset X$ be an irreducible subvariety. Then there is an open set $U \subset Y$ such that

1) $U \cap B \neq \emptyset$ and the variety $U \cap B$ is non-singular;

2) the strict transform of $X \cap U$ under the blow-up $\sigma_B : U_B \to U$ is a hypersurface $X_B \subset U_B$ which is again a variety with at most quadratic singularities of rank at least $r$.

It is easy to see that if $X$ is a variety with quadratic singularities of rank at least 5, then it is factorial and its singularities are terminal. The proof of the stability under blow-ups in [2] uses the following obvious fact. Let $X$ be a hypersurface in a non-singular variety $Y$, $Z \subset Y$ a non-singular hypersurface, and $o \in X \cap Z$ a point. Let $\beta(u_*) = 0$ be a local equation of $X$ at $o$ with respect to a system of parameters $(u_*)$ at that point. If the equation

$$\beta|_Z = 0$$

determines a hypersurface with a quadratic singularity of rank at least $r$ at $o$, then $X$ also has a quadratic singularity of rank at least $r$ at $o$. It follows from this observation that if $X \cap (Y \setminus Z)$ has at most quadratic singularities of rank at least $r$ and the same holds for the restriction $X|_Z$, then the hypersurface $X$ has at most quadratic singularities of rank at least $r$. Taking the exceptional divisor of the blow-up of $U \cap B$ for the hypersurface $Z$, we obtain the stability of such singularities under blow-ups.

We now consider bi-quadratic singularities.

Fix a pair $(r_1, r_2) \in \mathbb{Z}^2_+$ with $r_2 \geq r_1 + 2$.

We define the class of varieties with at most quadratic singularities of rank at least $r_1$ and bi-quadratic singularities of rank at least $r_2$. A variety $X$ belongs to this class if, in a neighbourhood of every point $o \in X$, it can be realized as a complete intersection of codimension 2 in a non-singular variety $Y$ with local equations at $o$ of the form

$$0 = \beta_{1,1}(u_*) + \beta_{1,2}(u_*) + \cdots,$$

$$0 = \beta_{2,1}(u_*) + \beta_{2,2}(u_*) + \cdots,$$

where $(u_*)$ is a system of local parameters on $Y$ at $o$ and precisely one of the following three conditions holds.
Theorem 1.3. Assume that $\beta_{rk}$ of local parameters in such a way that variety given by a pair of equations in a non-singular variety $B$ determines an irreducible reduced subvariety of codimension 2. It suffices to consider the case when $\beta_{rk}$ is non-singular at $o$. 

Proof. By the stability of quadratic singularities (proved in [2]), it suffices to consider the case when $B$ is entirely contained in the set of bi-quadratic singularities. Let $U$ be an open set such that $U \cap B$ is non-singular and the ranks $rk_{1,2}$ and $rk_{2,2}$ are constant along $U \cap B$. Regarding $U \cap B$ as a subvariety given by a pair of equations in a non-singular variety $Y$, we choose a system of local parameters in such a way that $B$ is given by the system of equations

$$u_1 = \cdots = u_k = 0.$$

A blow-up of $B \subset Y$ with exceptional divisor $E_B \subset \widetilde{Y}$ enables us to realize the blow-up $U_B$ as a subvariety of codimension 2 in $\widetilde{Y}$ given by a pair of equations $\widetilde{\beta}_1 = \widetilde{\beta}_2 = 0$. It suffices to verify that the singularities of $U_B$ on $U_B \cap E_B$ are either quadratic of rank at least $r_1$ or bi-quadratic or rank at least $r_2$. It is clear that the system of equations

$$\widetilde{\beta}_1|_{E_B} = \widetilde{\beta}_2|_{E_B} = 0$$

determines an irreducible reduced subvariety of codimension 2 in $E_B$ fibred over $B$, and the fibre over a point $b \in B$ is a complete intersection of two quadrics of rank at least $r_2$ in $\mathbb{P}^{k-1}$. We denote this fibre by $E_b$. If $p \in E_b$ is a non-singular point, then $U_B$ is also non-singular at $p$. But if $p \in E_b$ is a quadratic singularity, then its rank is at least $r_2 - 2 \geq r_1$ because the rank of a quadratic form decreases by at most 2 under restriction to a hyperplane. Then $U_B$ is either non-singular at $p$ or has a quadratic singularity of rank at least $r_1$. Finally, if $p \in E_b$ is a bi-quadratic singularity, then the rank of every quadratic form in the pencil

$$\lambda_1\widetilde{\beta}_1|_{p^p} + \lambda_2\widetilde{\beta}_2|_{p^p}$$

is at least $r_2$. Therefore, if $p \in U_B$ is a singular point, then it is either a quadratic singularity of rank at least $r_2 - 2 \geq r_1$ or a bi-quadratic singularity of rank at least $r_2$. $\square$
Note that singularities of type \((r_1, r_2)\) satisfy the same principle as quadratic singularities: if the restriction \(X|_Z\) to a non-singular divisor \(Z \subset Y\) has singularities of type \((r_1, r_2)\), then the same holds for the singularities of \(X\) at the points of \(X \cap Z\). We used this principle in the proof of the theorem.

If \(r_1 \geq 5\) and \(r_2 \geq 7\), then the theorem just proved yields that all singularities of type \((r_1, r_2)\) are factorial and terminal, as stated in § 0.1. In Theorem 0.1 we have \((r_1, r_2) = (5, 7)\) for the fibres of \(V/S\). This guarantees the factoriality and terminality of singularities of each fibre and, in view of the construction of \(V/S\) as a sub-fibration of a locally trivial bundle \(\pi: X \rightarrow S\) with fibre \(\mathbb{P}\), the factoriality and terminality of singularities of \(V\). If \(\sigma_S: S^+ \rightarrow S\) is a birational morphism, then \(V^+ = V \times_S S^+\) is a sub-fibration of the locally trivial \(\mathbb{P}\)-bundle

\[\pi_X^+: X^+ = X \times_S S^+ \rightarrow S^+\]

and, therefore, \(V^+\) is factorial with terminal singularities. This proves that the Fano–Mori space \(V/S\) constructed in § 0.1 is stable under fibrewise birational modifications. (The values \((r_1, r_2) = (9, 13)\) in § 0.2 are needed in the proof of divisorial canonicity in §§2, 3.)

We make some concluding remarks.

**Remark 1.4.** If the base \(S\) is one-dimensional, that is, \(S = \mathbb{P}^1\), then Theorem 0.1 can be improved. The birational map \(\chi: V \dashrightarrow V'\) onto the total space of the Mori fibre space \(V'/S'\) is a biregular isomorphism; see [6], Theorem 1, (iv). The proof is *verbatim* the same as in [6], § 1.5.

**Remark 1.5.** The hypotheses (ii) and (iii) of Theorem 0.1 (and Theorem 1.2) can be replaced by a single stronger condition (as was done in [2]): the class

\[-N(K_V \cdot \pi^{-1}(C)) - F\]

is ineffective for all \(N \geq 1\). This condition is easier to check; see Example 0.1. However, the \(K\)-condition seems to be closer to a criterion (see [30], [31] in the three-dimensional case for varieties with a pencil of del Pezzo surfaces of degree 1) and is one of the fundamental conditions for Fano–Mori fibre spaces.

**Remark 1.6.** In [2], § 2.3, there is a misprint in the displayed formula (2.2): the class \(\pi^*Y\) is missing. This does not affect the result of the computations as the missing class is accounted for and the final result at the end of § 2.3 is correct.

## §2. Regular complete intersections

In this section we prove Theorem 0.2 and obtain upper bounds for the multiplicities of the singular points of certain subvarieties of regular complete intersections \(F \in \mathcal{F}_{\text{reg}}\). In § 2.1 we reduce the proof of Theorem 0.2 to an estimate for the codimension of the complement of the set of tuples of homogeneous polynomials defining an irreducible reduced complete intersection of an appropriate dimension in the projective space (Theorem 2.1). In § 2.2 we prove this theorem. In § 2.3 we use the technique of hypertangent divisors based on the regularity conditions and obtain upper bounds for the multiplicity-to-degree ratio for prime divisors on the sections of \(F\) by linear subspaces in the projective space \(\mathbb{P}\).
2.1. Proof of Theorem 0.2. Let \( o \in \mathbb{P} \) be an arbitrary point and let \( \mathcal{F}(o) \subset \mathcal{F} \) be the closed set of pairs \((f_1, f_2)\) such that \( f_1(o) = 0, f_2(o) = 0.\) We fix a system of affine coordinates \( z_1, \ldots, z_{M+2} \) with origin at \( o.\) Given any pair \((f_1, f_2) \in \mathcal{F}(o)\) of homogeneous polynomials, we denote the corresponding inhomogeneous polynomials in the variables \( z_* \) again by \( f_1 \) and \( f_2.\) We write

\[
 f_i(z_*) = f_{i,1} + f_{i,2} + \cdots + f_{i,d_i},
\]

\( i = 1, 2, \) where \( f_{i,j}(z_*) \) is a homogeneous polynomial of degree \( j.\) Define the following subsets \( B(?) \subset \mathcal{F}(o) \) for

\[
 ? \in \{1, 2, 2^2.1, 2^2.2, 2^2.3\}.
\]

The subset \( B(1) \) consists of the pairs \((f_1, f_2)\) such that the linear forms \( f_{1,1} \) and \( f_{2,1} \) are linearly independent but the condition \((R1)\) does not hold. The subset \( B(2) \) consists of all \((f_1, f_2) \in \mathcal{F}(o)\) such that

\[
 \dim(\langle f_{1,1}, f_{2,1} \rangle) = 1
\]

but the condition \((R2)\) does not hold. Finally, the subsets \( B(2^2.\ast) \) consist of all \((f_1, f_2) \in \mathcal{F}(o)\) such that

\[
 f_{1,1} \equiv f_{2,1} \equiv 0
\]

but the corresponding condition \((R2^2.\ast)\) does not hold. Put \( \mathcal{B} = \cup B(?) \). Since the point \( o \in \mathbb{P} \) is arbitrary and \( \text{codim}(\mathcal{F}(o) \subset \mathcal{F}) = 2,\) we clearly have

\[
 \text{codim}(\mathcal{F} \setminus \mathcal{F}_{\text{reg}} \subset \mathcal{F}) \geq \text{codim}(\mathcal{B} \subset \mathcal{F}(o)) - M.
\]

Therefore, to prove Theorem 0.2, it suffices to obtain lower bounds for the codimensions

\[
 \text{codim}(B(?) \subset \mathcal{F}(o))
\]

and choose the worst of these bounds. We shall obtain them using the following well-known facts and methods:

- standard properties of binomial coefficients;
- the fact that the set of quadratic forms of rank at most \( r \) in \( N \) variables is of codimension \((N - r)(N - r + 1)/2\) in the space of all quadratic forms;
- the ‘projection method’ of estimating the codimension of the set of irregular sequences; see [5], Ch. 3, § 1.3.

We first consider the non-singular case.

Let \( \xi_1(z_*) \) and \( \xi_2(z_*) \) be linearly independent linear forms. We put

\[
 \mathcal{F}(o, \xi_1, \xi_2) = \{(f_1, f_2) \in \mathcal{F}(o) \mid f_{1,1} = \xi_1, f_{2,1} = \xi_2\}
\]

and \( B(1, \xi_1, \xi_2) = B(1) \cap \mathcal{F}(o, \xi_1, \xi_2).\) Clearly,

\[
 \text{codim}(B(1) \subset \mathcal{F}(o)) = \text{codim}(B(1, \xi_1, \xi_2) \subset \mathcal{F}(o, \xi_1, \xi_2)).
\]

Hence we can assume that the linear forms \( f_{1,1} \) and \( f_{2,1} \) are fixed. Let

\[
 \mathcal{L} \subset \{\xi_1 = \xi_2 = 0\}
\]
be a subspace of codimension 2 and let
\[ B(1, \xi_1, \xi_2, \mathcal{L}) \subset B(1, \xi_1, \xi_2) \]
be the set of all pairs \((f_1, f_2)\) such that the sequence \(S[-5]|_{\mathcal{L}}\) is irregular. The codimension
\[ \text{codim}(B(1, \xi_1, \xi_2) \subset \mathcal{F}(o, \xi_1, \xi_2)) \]
is clearly greater than or equal to
\[ \text{codim}(B(1, \xi_1, \xi_2, \mathcal{L}) \subset \mathcal{F}(o, \xi_1, \xi_2)) - 2(M - 2). \]
Hence it suffices to estimate the codimension of \(B(1, \xi_1, \xi_2, \mathcal{L})\). We shall use the projection method (taking into account that \(d_1 \leq d_2\)). This method was suggested in [32]. It estimates the codimension of the set of irregular sequences by fixing the first moment when the regularity is violated. Thus we obtain a bound for the codimension of the set of sequences for which the regularity is violated for the first time at the kth member of the sequence and then take the worst of all these bounds. See [27], §3, where this technique is described in detail. In our case, the codimension of \(B(1, \xi_1, \xi_2, \mathcal{L})\) is greater than or equal to the minimum of the following set of integers, which is split into two parts for convenience. We first assume that \(d_2 \geq d_1 + 5\). Then the first part consists of
\[ \binom{M - k}{k}, \quad k = 2, \ldots, d_1, \]
and the second consists of
\[ \binom{M - d_1}{d_1 + k}, \quad k = 1, \ldots, M - 2d_1 - 3. \]
Using standard properties of binomial coefficients, we can easily conclude that the minimum of this set is attained at one of the endpoints. Comparing the numbers
\[ \binom{M - 2}{2} \quad \text{and} \quad \binom{d_2 - 2}{3}, \]
we see by an elementary check that the first is smaller than the second. The same result holds in the five omitted cases when
\[ d_1 \leq d_2 \leq d_1 + 4. \]
For example, when \(d_2 = d_1\), we have \(k = 2, \ldots, d_1 - 3\) in the first part while the second part consists of a single number
\[ \binom{M - (d_1 - 3)}{3}. \]
When \(d_2 = d_1 + 1\), we have \(k = 2, \ldots, d_1 - 2\) in the first part while the second part is empty, and so on. The minimum of the sequence is always equal to
\[ \binom{M - 2}{2} = \frac{1}{2}(M - 2)(M - 3). \]
In view of the remarks above, we conclude that
\[
\operatorname{codim}(B(1) \subset F(o)) \geq \frac{1}{2}(M^2 - 9M + 14).
\]

We now consider the \textit{quadratic case}.

Arguing as above, we fix a pair of proportional linear forms \(\xi_1, \xi_2\), not both identically equal to zero, and define the subsets \(F(o, \xi_1, \xi_2)\) and \(B(2, \xi_1, \xi_2)\) by the same formulae as in the non-singular case above. We write
\[
\xi_1 = \alpha_1 \tau \quad \text{and} \quad \xi_2 = \alpha_2 \tau,
\]
where \(\tau(z_*)\) is a non-zero linear form and \((\alpha_1, \alpha_2) \neq (0, 0)\). Let \(\mathcal{L} \subset \{\tau = 0\}\) be a linear subspace of codimension 1 and let
\[
B(2, \xi_1, \xi_2, \mathcal{L}) \subset B(2, \xi_1, \xi_2)
\]
be the set of all pairs \((f_1, f_2)\) such that the sequence \(S[-4]|_{\mathcal{L}}\) is irregular. As in the non-singular case, the codimension of \(B(2, \xi_1, \xi_2)\) in \(F(o, \xi_1, \xi_2)\) is greater than or equal to
\[
\operatorname{codim}(B(2, \xi_1, \xi_2, \mathcal{L}) \subset F(o, \xi_1, \xi_2)) - M.
\]
Hence it suffices to estimate the last codimension. We omit the elementary computations based on the projection method. The codimension of \(B(2)\) in \(F(o)\) turns out to be greater than the codimension of \(B(2^2.3)\) that will be estimated in what follows.

We finally consider the \textit{bi-quadratic case}.

The identical vanishing of the linear forms \(f_{1,1}\) and \(f_{2,1}\) gives \(2(M + 2)\) independent conditions. We first estimate the codimension of \(B(2^2.1)\) because the condition \((R2^2.1)\) is independent of \(d_1\). If
\[
\operatorname{rk}(f_{1,2}, f_{2,2}) \leq 12,
\]
then either \(\operatorname{rk} f_{1,2} \leq 12\), or \(\operatorname{rk} f_{1,2} \geq 13\) and \(f_{2,2}\) lies on the following cone in \(P_{2, M+2}\): the vertex of this cone is the form \(f_{1,2}\) and the base is the closed set of all forms of rank at most 12. It follows that
\[
\operatorname{codim}(B(2^2.1) \subset F(o)) \geq \frac{1}{2}(M - 9)(M - 10) - 1 + 2(M + 2)
\]
because the conditions \(\operatorname{rk} f_{i,2} \leq 17, \ i = 1, 2\), together give \((M - 14)(M - 15)\) independent conditions for the pair \((f_{1,2}, f_{2,2})\) of quadratic forms, and this number of conditions is much greater.

We now consider the conditions \((R2^2.2)\) and \((R2^2.3)\). We first assume that \(d_1 \geq 4\). To estimate the codimension of \(B(2^2.3)\), let \(\mathcal{L} \subset \mathbb{C}^{M+2}\) be a linear subspace of codimension 2 or 3 and let \(B(2^2.3, \mathcal{L}) \subset F(o)\) be the set of all pairs \((f_1, f_2)\) such that \(f_{1,1} \equiv f_{2,1} \equiv 0\) and the sequence
\[
S[- \operatorname{codim} \mathcal{L} - 1]|_{\mathcal{L}}
\]
is irregular. The case $\text{codim} \mathcal{L} = 3$ gives the worst bound for the codimension, so we will take that case. Using the projection method as in the non-singular case, we obtain the inequality

$$\text{codim}(\mathcal{B}(2^2, 3, \mathcal{L}) \subset \mathcal{F}(o)) \geq \left(\frac{M - 1}{2}\right) + 2(M + 2).$$

Since $\mathcal{L}$ varies in the $(3(M - 1))$-dimensional Grassmannian, it follows that

$$\text{codim}(\mathcal{B}(2^2, 3) \subset \mathcal{F}(o)) \geq \frac{1}{2}(M^2 - 5M + 16).$$

We now consider the condition $(R2^2, 2)$. Here we need the following general fact. Let $\underline{m} = (m_1, \ldots, m_k)$ be a tuple of integers with

$$2 \leq m_1 \leq m_2 \leq \cdots \leq m_k$$

and let

$$\mathcal{P}(\underline{m}) = \prod_{i=1}^{k} \mathcal{P}_{m_i, N+1}$$

be the space of tuples $(g) = (g_1, \ldots, g_k)$ of homogeneous polynomials of degree $m_1, \ldots, m_k$ in $N + 1$ variables (we regard them as homogeneous polynomials on the projective space $\mathbb{P}^N$). We write

$$\mathcal{B}^*(\underline{m}) \subset \mathcal{P}(\underline{m})$$

for the set of all tuples $(g_1, \ldots, g_k)$ such that the scheme of their common zeros

$$V(g) = V(g_1, \ldots, g_k)$$

is not an irreducible reduced subvariety of codimension $k$ in $\mathbb{P}^N$.

**Theorem 2.1.** We have the inequality

$$\text{codim}(\mathcal{B}^*(\underline{m}) \subset \mathcal{P}(\underline{m})) \geq \frac{1}{2}(N - k - 1)(N - k - 4) + 2.$$

**Proof.** The proof is given in § 2.2. □

Using Theorem 2.1, we obtain the following inequality for the codimension of $\mathcal{B}(2^2, 2, \mathcal{L}) \subset \mathcal{F}(o)$, where $\mathcal{L}$ is a linear subspace of codimension 2 in $\mathbb{C}^{M+2}$ and $\mathcal{B}(2^2, 2, \mathcal{L})$ consists of all $(f_1, f_2)$ such that $f_{1,1} \equiv f_{2,1} \equiv 0$ and the condition $(R2^2, 2)$ does not hold:

$$\text{codim}(\mathcal{B}(2^2, 2, \mathcal{L}) \subset \mathcal{F}(o)) \geq \frac{1}{2}(M^2 - 15M + 58) + 2(M + 2).$$

Since the subspace $\mathcal{L}$ is arbitrary, we have

$$\text{codim}(\mathcal{B}(2^2, 2) \subset \mathcal{F}(o)) \geq \frac{1}{2}(M^2 - 15M + 66).$$
The worst of the resulting bounds for the codimension corresponds to the violation of \((R^2.2)\). Since the closed subset \(\mathcal{F}(o)\) has codimension 2 and the point \(o\) varies in the \((M + 2)\)-dimensional projective space, this completes the proof of Theorem 0.2 for \(d_1 \geq 4\).

We now consider the two remaining cases \(d_1 = 2, 3\). The codimension of \(\mathcal{B}(2^2.3)\) can be estimated by the projection method as above. If the codimension of \(\mathcal{L}\) is equal to 2, then the worst bound for the codimension

\[
\text{codim}(\mathcal{B}(2^2.3, \mathcal{L}) \subset \mathcal{F}(o))
\]

corresponds to the violation of regularity at the last member of the sequence \(S[\text{codim} \mathcal{L}]|_{\mathcal{L}}\) and is given by the number

\[
\left( \frac{d_2 - 2 + 2}{2} \right) + 2(M + 2).
\]

It follows that

\[
\text{codim}(\mathcal{B}(2^2.3) \subset \mathcal{F}(o)) \geq \left( \frac{d_2}{2} \right) - M + 7
\]

(where \(d_2 = M + 2 - d_1 \geq M - 1\)). This is better than what Theorem 0.2 claims.

We estimate the codimension of \(\mathcal{B}(2^2.2)\) using Theorem 2.1.

Suppose that \(d_1 = 3\). This case is special because \((R^2.2)\) requires that a complete intersection of codimension 5 is irreducible and reduced (in all other cases, this codimension is equal to 4). Using Theorem 2.1 for a fixed linear subspace \(\mathcal{L} \subset \mathbb{C}^{M+2}\), we obtain the inequality

\[
\text{codim}(\mathcal{B}(2^2.2, \mathcal{L}) \subset \mathcal{F}(o)) \geq \frac{1}{2}(M^2 - 17M + 74) + 2(M + 2).
\]

Since the subspace \(\mathcal{L}\) is arbitrary, it follows that

\[
\text{codim}(\mathcal{B}(2^2.2) \subset \mathcal{F}(o)) \geq \frac{1}{2}(M^2 - 17M + 82).
\]

Taking into account that the closed subset \(\mathcal{F}(o)\) is of codimension 2 and the point \(o\) varies in \(\mathbb{P}^{M+2}\), we complete the proof of Theorem 0.2 for \(d_1 = 3\).

When \(d_1 = 2\), the codimension of \(\mathcal{B}(2^2.2)\) can be estimated in exactly the same way as for \(d_1 \geq 4\) because the condition \((R^2.2)\) deals with complete intersections of codimension 4. This completes the proof of Theorem 0.2.

2.2. Irreducible reduced complete intersections. We now prove Theorem 2.1. Our arguments are similar to those in [27], §2, but are more general and more formal. The proof is by induction on \(k\). The case \(k = 1\) is almost obvious since the codimension of the set of reducible polynomials can easily be estimated. However, to perform the induction step, we consider tuples of polynomials that determine not only irreducible reduced subvarieties of the required codimension but also factorial subvarieties. This enables us to add one more polynomial. Our notation is close to that in [27], §2, whenever possible. Thus, let

\[
\mathcal{P}^{\geq j} = \prod_{i=j}^{k} \mathcal{P}_{m_i, N+1}
\]
be the space of truncated tuples \((g_j, \ldots, g_k)\). We denote these tuples by \(g_{[j,k]}\). Instead of the ascending induction on \(k\) we use the equivalent descending induction on \(j = k, \ldots, 1\). It is convenient to have the degree of the added polynomial greater than the degrees of the polynomials already in the tuple. We write \(\mathcal{P}^{\geq j}_{\text{mq}}\) for the Zariski-open subset of \(\mathcal{P}^{\geq j}\) consisting of all the tuples \(g_{[j,k]}\) whose scheme of common zeros

\[ V(g_{[j,k]}) \subset \mathbb{P}^N \]

is an irreducible reduced complete intersection of codimension \(k - j + 1\) with at most factorial multi-quadratic singularities (see [27], §2). We recall the meaning of the last condition. Let \(o \in V(g_{[j,k]})\) be an arbitrary point and let \(w_1, \ldots, w_N\) be a system of affine coordinates on \(\mathbb{P}^N\) with origin \(o\). We write

\[ g_a = g_{a,1} + g_{a,2} + \cdots + g_{a,m_a} \]

for the expansion of the inhomogeneous representative (again denoted by \(g_a\)) of the polynomial \(g_a\) into components homogeneous in \(w_\ast\). If

\[ \dim \langle g_{j,1}, \ldots, g_{k,1} \rangle = k - j + 1, \]

then the scheme \(V(g_{[j,k]})\) is a non-singular subvariety of codimension \(k - j + 1\) in a neighbourhood of \(o\). Assume that

\[ l = k - j + 1 - \dim \langle g_{j,1}, \ldots, g_{k,1} \rangle \geq 1. \]

Let \(\varphi_{\mathbb{P},o} : X \to \mathbb{P}^N\) be the blow-up of \(o\) with exceptional divisor \(E_X \cong \mathbb{P}^{N-1}\). We split the set of indices \(\{j, \ldots, k\}\) into a disjoint union \(I_1 \sqcup I_2\) in such a way that the linear forms \(g_{\alpha,1}, \alpha \in I_1\), are linearly independent and

\[ \langle g_{j,1}, \ldots, g_{k,1} \rangle = \langle g_{\alpha,1} \mid \alpha \in I_1 \rangle, \]

so that for \(\gamma \in I_2\) there are uniquely determined constants \(c_{\gamma, \alpha}\) such that

\[ g_{\gamma,1} = \sum_{\alpha \in I_1} c_{\gamma, \alpha} g_{\alpha,1}. \]

We put \(\Pi = \{g_{\alpha,1} = 0 \mid \alpha \in I_1\} \subset \mathbb{C}^N\) and construct the following quadratic forms for \(\gamma \in I_2\):

\[ g_{\gamma,2}^* = g_{\gamma,2} - \sum_{\alpha \in I_1} c_{\gamma, \alpha} g_{\alpha,2}. \]

If

\[ \text{rk}(g_{\gamma,2}^*|_{\Pi}, \gamma \in I_2) \geq 2l + 3, \quad (5) \]

then the scheme \(V(g_{[j,k]})\) is an irreducible complete intersection with a multi-quadratic singularity of type \(2^l\) in a neighbourhood of \(o\); see [27]; let \(V^+(g_{[j,k]})\) be the strict transform of \(V(g_{[j,k]})\) on \(X\) and let \(Q = V^+(g_{[j,k]}) \cap E_X\) be the exceptional divisor of the blow-up \(V^+(g_{[j,k]}) \to V(g_{[j,k]})\) of the point \(o\). Then \(Q \subset E_X\) is an irreducible reduced complete intersection of type \(2^l\) in the linear subspace

\[ \mathbb{P}(\Pi) = \{g_{\alpha,1} = 0 \mid \alpha \in I_1\} \subset E_X \]
of codimension \((k - j + 1 - l)\) and, by Lemma 2.1 in [27], §2, we have

\[
\text{codim}(\text{Sing} \ Q \subset Q) \geq 4.
\]

If the inequality (5) holds for every singular point \(o \in V(g_{[j,k]})\), then \(V(g_{[j,k]})\) is an irreducible reduced factorial variety of codimension \(k - j + 1\).

We shall prove Theorem 2.1 by estimating the codimension of the complement

\[
\text{codim}(\mathcal{P}^{\geq j} \setminus \mathcal{P}_{mq}^{\geq j} \subset \mathcal{P}^{\geq j})
\]

for \(j = k, \ldots, 1\). Thus the proof is by descending induction on \(j\).

**Base of the induction.** The closed set of reducible polynomials of degree \(m_k\) on \(\mathbb{P}^N\) is of codimension

\[
\binom{N + m_k - 1}{m_k} - N
\]

in \(\mathcal{P}_{m_k,N+1}\). Consider the condition on an irreducible polynomial \(g_k \in \mathcal{P}_{m_k,N+1}\) requiring that the hypersurface \(\{g_k = 0\}\) has at least one singular point which is not a quadratic singularity of rank at least 5 (that is, a factorial quadratic singularity). This requirement imposes at least

\[
\frac{1}{2}(N - 3)(N - 4) + 1
\]

independent conditions on the coefficients of \(g_k\). Since \(m_k \geq 2\), we have

\[
\text{codim}(\mathcal{P}^{\geq k} \setminus \mathcal{P}_{mq}^{\geq k} \subset \mathcal{P}^{\geq k}) \geq \frac{1}{2}(N - 3)(N - 4) + 1.
\]

**Induction step: irreducibility.** Suppose that

\[
g_{[j+1,k]} \in \mathcal{P}_{mq}^{\geq j+1}.
\]

Arguing as in [27], § 2.2, we see that the codimension in \(\mathcal{P}_{m_j,N+1}\) of the set of polynomials \(g_j \in \mathcal{P}_{m_j,N+1}\) such that \(V(g_{[j,k]})\) is reducible, non-reduced or of an ‘incorrect’ codimension \((k - j)\) in \(\mathbb{P}^N\), is greater than or equal to

\[
\binom{N + m_j - 1}{m_j} - N - (k - j).
\]

(This follows from the factoriality of \(V(g_{[j,k]})\) and Lefschetz’ theorem: the Picard group of this variety is generated by the hyperplane section, and every divisor is cut out by a hypersurface in \(\mathbb{P}^N\). It is here that we use the condition \(2 \leq m_1 \leq m_2 \leq \cdots \leq m_k\).) We now assume that the scheme \(V(g_{[j,k]})\) is an irreducible reduced complete intersection of codimension \((k - j + 1)\) in \(\mathbb{P}^N\).

**Induction step: factoriality.** We estimate the codimension of the set of tuples \(g_{[j,k]}\) such that the irreducible reduced variety \(V(g_{[j,k]})\) does not satisfy the condition of having factorial multi-quadratic singularities. It is easier to do this from the outset, that is, without assuming that \(V(g_{[j+1,k]})\) has factorial multi-quadratic singularities. Let \(o \in V(g_{[j,k]})\) be a point. We write \(Q(l)\) for the space of quadratic forms on the linear space \(\Pi\). Note that \(\dim \Pi = N + j - k + l - 1\). Since \(|I_2| = l\), instead of
the tuples \((g_{\gamma,2}^*|_{\Pi}, \gamma \in I_2)\) of quadratic forms varying independently of each other, we shall consider the tuples
\[ h_{[1,l]} = (h_1, \ldots, h_l) \in \mathcal{Q}(l)^\times l \]
in order to simplify the notation. We write \(\mathcal{R}_{\leq a}\) for the closed subset of quadratic forms of rank at most \(a\) in \(\mathcal{Q}(l)\),
\[ \text{codim}(\mathcal{R}_{\leq a} \subset \mathcal{Q}(l)) = \frac{1}{2}(N + j - k + l - a)(N + j - k + l - a - 1). \]
For every \(e \in \{1, \ldots, l\}\) let \(\mathcal{X}_{e,a} \subset \mathcal{Q}(l)^\times e\) be the closed set of all tuples \(h_{[1,e]} = (h_1, \ldots, h_e)\) such that \(\text{rk} \ h_{[1,e]} \leq a\).

**Lemma 2.1.** We have the inequality
\[ \text{codim}(\mathcal{X}_{e,a} \subset \mathcal{Q}(l)^\times e) \geq \text{codim}(\mathcal{R}_{\leq a} \subset \mathcal{Q}(l)) - e + 1. \]

**Proof.** This is identical to the proof of Lemma 2.2 in [27], §2. □

In particular, substituting \(e = l\) and \(a = 2l + 2\), we see that the codimension of the closed set \(\mathcal{X}_{l,2l+2}\) in \(\mathcal{Q}(l)^\times l\) (that is, precisely the set of tuples of quadratic forms \(g_{\gamma,2}^*|_{\Pi}, \gamma \in I_2\), that do not satisfy the inequality (5)) is at least
\[ \frac{1}{2}(N + j - k - l - 2)(N + j - k - l - 3) - l + 1. \] (7)

Returning to the complete intersection \(V(g_{\{j,k\}})\), which we assume to be irreducible and reduced, we estimate the codimension of the set of tuples
\[ g_{\{j,k\}} \notin \mathcal{P}_{mq}^{\geq j}. \]
Let \(o \in \mathbb{P}^N\) be a fixed point and let
\[ l \in \{1, \ldots, k - j + 1\} \]
be an integer. The set of tuples \(g_{\{j,k\}}\) such that \(o \in V(g_{\{j,k\}})\) and
\[ \text{dim} \langle g_{j,1}, \ldots, g_{k,1} \rangle = k - j + 1 - l, \]
is of codimension
\[ (k - j + 1) + l(N + j - k - 1 + l) \]
in \(\mathcal{P}_{mq}^{\geq j}\). Violation of the condition (5) adds the further codimension (7). Taking into account that \(o\) varies in \(\mathbb{P}^N\) but keeping \(l\) fixed, we obtain a quadratic function of \(l\). Its minimum is attained at \(l = 1\) and is equal to
\[ \frac{1}{2}(N + j - k - 2)(N + j - k - 5) + 2. \] (8)
This expression is a lower bound for the codimension of the complement \(\mathcal{P}_{mq}^{\geq j} \setminus \mathcal{P}_{mq}^{\geq j}\) in \(\mathcal{P}_{mq}^{\geq j}\) since it is certainly smaller than (6). Finally, regarding (8) as a function of \(j \in \{1, \ldots, k\}\), we see that the minimum is attained at \(j = 1\) and is equal to
\[ \frac{1}{2}(N - k - 1)(N - k - 4) + 2. \]
This completes the proof of Theorem 2.1.
Remark 2.1. The regularity condition (R2^2.2) is about a complete intersection in the affine space \( L \), but not in a projective space. Let us explain how Theorem 2.1 applies in the affine situation. Regard \( L \) as an affine chart \( \mathbb{C}^M \subset \mathbb{P}^M \). The equations \( f_{1,2}, f_{2,2} \) (which are homogeneous by construction) and \( f_1, f_2 \) (we return to the original homogeneous polynomials on \( \mathbb{P} \)) restricted to \( \mathbb{P}^M \) determine an irreducible reduced complete intersection of codimension 4 in \( \mathbb{P}^M \) if and only if the condition (R2^2.2) holds. If the same is true for the restrictions of these four polynomials to the ‘hyperplane at infinity’ \( \mathbb{P}^M \setminus \mathbb{C}^M \), then the condition (R2^2.2) holds. This hyperplane can be identified with the projectivization \( \mathbb{P}(L) \). Then we obtain a set of four homogeneous polynomials

\[ f_{1,2}|L, \ f_{2,2}|L, \ f_{1,d_1}|L, \ f_{2,d_2}|L \]

of degrees 2, 2, \( d_1 \), \( d_2 \) respectively. If these four polynomials determine an irreducible reduced complete intersection of codimension 4 in \( \mathbb{P}(L) \cong \mathbb{P}^{M-1} \), then the condition (R2^2.2) holds. Since the coefficients of these polynomials are four disjoint groups of the coefficients of the original polynomials \( f_1, f_2 \), Theorem 2.1 applies and yields a bound for the codimension of \( B(2^2.2) \).

2.3. Degrees and multiplicities. Using the regularity conditions, we obtain bounds for the multiplicities of the singular points of subvarieties \( Y \subset F \in \mathcal{F}_{\text{reg}} \) in the usual way (see [5], Ch. 3). From now on, we fix a regular complete intersection \( F \in \mathcal{F}_{\text{reg}} \). As usual, the ratio of the multiplicity \( \text{mult}_o Y \) to the degree \( \text{deg} Y \) (with respect to the embedding \( Y \subset F \subset \mathbb{P} \)) is denoted by

\[ \frac{\text{mult}_o}{\text{deg}} Y. \]

We first consider points that are non-singular on the complete intersection \( F \).

Proposition 2.1. Let \( \Delta \subset F \) be the section of \( F \) by an arbitrary linear subspace of codimension 2 in \( \mathbb{P} \), \( o \in \Delta \) a non-singular point, and \( Y \subset \Delta \) a prime divisor. Then

\[ \frac{\text{mult}_o}{\text{deg}} Y \leq \frac{2}{d_1 d_2}. \] (9)

Proof. Note that the inequality (9) is optimal: the section of \( \Delta \) by any hyperplane tangent to \( \Delta \) at \( o \) gives an equality. We assume the opposite:

\[ \frac{\text{mult}_o}{\text{deg}} Y > \frac{2}{d_1 d_2}, \]

aiming to arrive at a contradiction. Our argument uses the technique of hypertangent divisors and is completely analogous to that in [2], Ch. 3, §2. We discuss in detail only those of its fragments which need to be modified. By the conditions (R2) and (R2^2.1), we have \( \text{codim}(\text{Sing} F \subset F) \geq 10 \). Hence for the subvariety \( \Delta \subset F \) we have

\[ \text{codim}(\text{Sing} \Delta \subset \Delta) \geq 6. \]

Therefore, \( \Delta \subset \mathbb{P}^M \) is a factorial complete intersection of codimension 2 and \( \text{Pic} \Delta = \mathbb{Z}H_\Delta \), where \( H_\Delta \) is the class of a hyperplane section. We write \( |H_\Delta - 2o| \)
for the pencil of tangent hyperplanes at \( o \). Let \( D_{1,1} \in |H_\Delta - 2o| \) be an arbitrary divisor. We have \( \text{mult}_o D_{1,1} = 2 \) by the condition (R1). Hence \( D_{1,1} \neq Y \) and, therefore, the algebraic cycle \( (D_{1,1} \circ Y) \) of scheme-theoretic intersection of \( D_{1,1} \) and \( Y \) is well defined. This is an effective cycle of codimension 2 on \( \Delta \). Moreover,

\[
\text{mult}_o(D_{1,1} \circ Y) \geq 2 \text{mult}_o Y
\]

and \( \deg(D_{1,1} \circ Y) = \deg Y \). Hence this cycle contains an irreducible component \( Y_2 \) (where we put \( Y_1 = Y \)) such that

\[
\frac{\text{mult}_o Y_2}{\deg Y_2} > \frac{4}{d_1 d_2}.
\]

Let \( P \ni o \) be a general 7-dimensional subspace through \( o \) in \( \mathbb{P}^M \). Then \( \Delta_P = \Delta \cap P \) is a non-singular 5-dimensional complete intersection in \( P \cong \mathbb{P}^7 \). Hence the numerical Chow group of classes of cycles of codimension 2 on \( \Delta_P \) is

\[
A^2 \Delta_P = \mathbb{Z} H_P^2,
\]

where \( H_P \in \text{Pic} \Delta_P \) is the class of a hyperplane section. Let \( D_{2,1} \neq D_{1,1} \) be another element of the tangent pencil. Then we have

\[
(D_{1,1} \circ D_{2,1} \circ \Delta_P) \sim H_P^2.
\]

Hence \( (D_{1,1} \circ D_{2,1} \circ \Delta_P) = D_{1,1} \cap D_{2,1} \cap \Delta_P \) is an irreducible subvariety and, therefore, \( (D_{1,1} \circ D_{2,1}) = D_{1,1} \cap D_{2,1} \) is an irreducible subvariety of codimension 2 on \( \Delta \). By condition (R1), this subvariety satisfies the equation

\[
\text{mult}_o(D_{1,1} \circ D_{2,1}) = 4,
\]

whence \( Y_2 \neq (D_{1,1} \circ D_{2,1}) \). Since \( Y_2 \subset D_{1,1} \) by construction, we have \( Y_2 \nsubseteq D_{2,1} \). Hence the effective cycle \( (D_{2,1} \circ Y_2) \) of scheme-theoretic intersection of \( D_{2,1} \) and \( Y_2 \) is well defined. This cycle contains an irreducible component \( Y_3 \) such that

\[
\frac{\text{mult}_o Y_3}{\deg Y_3} > \frac{8}{d_1 d_2}.
\]

We now use the technique of hypertangent divisors exactly as in [5], Ch. 3. Define the hypertangent linear systems in terms of a system of affine coordinates \( z_1, \ldots, z_{M+2} \) by the formula

\[
\Lambda_j = \min\{j,d_1-1\} \sum_{a=1}^{j} f_{1,[1,a]} s_{1,j-a} + \sum_{a=1}^{j} f_{2,[1,a]} s_{2,j-a},
\]

where \( j = 2, \ldots, d_2 - 1 \) and \( f_{i,[1,a]} \) stands for the left segment of length \( a \) of the (inhomogeneous) polynomial \( f_i \),

\[
f_{1,[1,a]} = f_{i,1} + \cdots + f_{i,a},
\]

and the homogeneous polynomials \( s_{i,j-a}(z_a) \in \mathcal{P}_{j-a,M+2} \) are arbitrary and independent of each other. If \( j \leq d_1 - 1 \), then we choose two general divisors \( D_{1,j} \)
and $D_{2,j}$ in the linear system $\Lambda_j|\Delta$. But if $j \geq d_1$, then we choose one general divisor $D_{2,j}$ in the linear system $\Lambda_j|\Delta$. We order the resulting

$$d_1 + d_2 - 4 = M - 2$$

divisors on $\Delta$ lexicographically (as in § 0.2) and remove the very first and the five last divisors from this sequence. This yields a sequence of $M - 8$ hypertangent divisors. It is convenient to denote them by

$$Z_3, Z_4, \ldots, Z_{M-6}.$$ 

Now, starting with the subvariety $Y_3$ (already constructed), we construct by induction a sequence of irreducible subvarieties

$$Y_3, Y_4, \ldots, Y_{M-6}, Y_{M-5}$$
of codimension $\text{codim}(Y_k \subset \Delta) = k$ in the following way. If $Y_k$ has already been constructed, then the condition (R1) guarantees that $Y_k \not\subseteq |Z_k|$. Hence the algebraic cycle of scheme-theoretic intersection $(Z_k \circ Y_k)$ is well defined and has codimension $k + 1$. We take for $Y_{k+1}$ a component of this cycle with the maximal value of the ratio $\text{mult}_o/\text{deg}$. We do not give more details since this technique has been used many times and is well known. The subvariety $Y_{M-2}$ is three-dimensional and its ratio $\text{mult}_o/\text{deg}$ is strictly larger than

$$\frac{4}{3} \frac{(d_1 - 2)(d_2 - 3)}{d_1 d_2} \geq 1 \quad \text{when} \quad d_1 = d_2 \quad \text{or} \quad d_1 = d_2 - 1,$$
$$\frac{4}{3} \frac{(d_1 - 1)(d_2 - 4)}{d_1 d_2} \geq 1 \quad \text{when} \quad d_1 = d_2 - 2 \quad \text{or} \quad d_1 = d_2 - 3,$$
$$\frac{4}{3} \frac{d_2 - 5}{d_2} \geq 1 \quad \text{when} \quad d_1 \leq d_2 - 4.$$

In each case we arrive at a contradiction. □

We now consider quadratic points of the complete intersection $F$ and of its hyperplane sections.

**Proposition 2.2.** Let $o \in F$ be a quadratic singularity of $F$, $\Delta \subset F$ a hyperplane section of $F$ through $o$ such that the point $o \in \Delta$ is a quadratic singularity of $\Delta$, and $Y \subset \Delta$ a prime divisor. Then

$$\frac{\text{mult}_o}{\text{deg}} Y \leq \frac{4}{d_1 d_2}. \quad (10)$$

**Proof.** Since $o$ is a quadratic singularity of $F$ and of the hyperplane section $\Delta$, we have $\langle \Delta \rangle \neq \{\tau = 0\}$ in the notation of § 0.2, where $\langle \Delta \rangle \subset \mathbb{P}$ is the linear span, that is, the hyperplane that cuts out $\Delta$. Note that there is a unique hyperplane section of $\Delta$ (by the hyperplane $\tau|\langle \Delta \rangle = 0$) for which the inequality (10) becomes an equality. Hence the assertion of the proposition is optimal. We now assume the opposite: there is a prime divisor $Y \subset \Delta$ such that

$$\frac{\text{mult}_o}{\text{deg}} Y > \frac{4}{d_1 d_2}.$$
We claim that this assumption leads to a contradiction. Indeed, since the rank of a quadratic form decreases by at most 2 when the form is restricted to a hyperplane, the condition (R2) implies that the rank of the quadratic singularity $o$ of $\Delta$ is at least 7. Let $D_1$ be the hyperplane section of $\Delta$ described above, so that the inequality (10) becomes an equality for $D_1$. We have $D_1 \neq Y$. Hence the algebraic cycle $(D_1 \circ Y)$ of scheme-theoretic intersection is well defined. It contains an irreducible component $Y_2$ of codimension 2 (with respect to $\Delta$) such that

$$\frac{\text{mult}_o Y_2}{\text{deg} Y_2} > \frac{8}{d_1 d_2}.$$  

We now apply the technique of hypertangent divisors exactly as in the non-singular case using the regularity condition (R2). We obtain a subvariety of positive dimension in $\mathbb{P}$ whose multiplicity at $o$ is strictly larger than its degree. This contradiction completes the proof. □

We now consider bi-quadratic points of the complete intersection $F$. Fix an arbitrary bi-quadratic point $o \in F$.

**Proposition 2.3.** Let $\Delta \subset F$ be the section of $F$ by an arbitrary linear subspace of codimension 2 or 3 through $o$ and let $Y \subset \Delta$ be a prime divisor. If $\text{codim}(\Delta \subset F) = 2$, then

$$\frac{\text{mult}_o Y}{\text{deg} Y} \leq \frac{6}{d_1 d_2}.$$  

If $\text{codim}(\Delta \subset F) = 3$, then

$$\frac{\text{mult}_o Y}{\text{deg} Y} \leq \frac{8}{d_1 d_2}.$$  

**Proof.** The proof is carried out as in the non-singular and quadratic cases. We assume that there is a prime divisor $Y$ not satisfying the corresponding inequality. Then we use the technique of hypertangent divisors based on the regularity conditions (R2².2) and (R2².3) to construct (by means of successive intersections) a subvariety of positive dimension in $\mathbb{P}$ that gives a contradiction. Only the beginning of this procedure is non-standard. It is related to the condition (R2².2), and we consider these first steps in detail. Assume that $d_1 \geq 4$ and that $\Delta$ is the section of $F$ of a linear subspace $\langle \Delta \rangle \subset \mathbb{P}$ of codimension 2. The hypertangent system $\Lambda_2$ is the pencil of quadrics generated by the quadratic forms $f_{1,2}$ and $f_{2,2}$. The hypertangent divisor $D_{1,2} = \{f_{1,2}\Delta = 0\}$ is irreducible (it cannot be a sum of hyperplane sections by the condition (R2².1) or (R2².3)). Moreover, we have

$$\text{mult}_o D_{1,2} = 12 \quad \text{and} \quad \text{deg} D_{1,2} = 2d_1 d_2.$$  

Hence $D_{1,2} \neq Y$ and the scheme-theoretic intersection $(D_{1,2} \circ Y)$ is well defined. Choosing an irreducible component $Y_2$ (of codimension 2 on $\Delta$) with the maximal (in this cycle) ratio of multiplicity at $o$ to degree, we obtain

$$\frac{\text{mult}_o Y_2}{\text{deg} Y_2} > \frac{9}{d_1 d_2}.$$  

Let $D_{2,2} = \{f_{2,2}|\Delta = 0\}$ be the second hypertangent divisor. It follows from (R$^{2.2}$) that $(D_{1,2} \circ D_{2,2}) = D_{1,2} \cap D_{2,2}$ is an irreducible subvariety of codimension 2 on $\Delta$. By (R$^{2.1}$) (or (R$^{2.3}$)) we have

$$\frac{\text{mult}_o}{\text{deg}} (D_{1,2} \circ D_{2,2}) = \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{4}{d_1 d_2} = \frac{9}{d_1 d_2},$$

whence $Y_2 \neq (D_{1,2} \circ D_{2,2})$. However, $Y_2 \subset |D_{1,2}|$ by construction. Hence $Y_2 \not\subset |D_{2,2}|$ and, therefore, the effective algebraic cycle of scheme-theoretic intersection $(D_{2,2} \circ Y_2)$ is well defined and has codimension 3 in $\Delta$. Let $Y_3$ be a component of this cycle with maximal value of the ratio $\text{mult}_o / \text{deg}$. Then

$$\frac{\text{mult}_o}{\text{deg}} Y_3 > \frac{27}{2d_1 d_2}.$$

The remaining part of the procedure of applying the technique of hypertangent divisors is absolutely standard and we skip it. We use only (R$^{2.3}$) and choose one general divisor (not two) in the hypertangent system $\Lambda_3$. Then we proceed as in the non-singular and quadratic cases, thus completing the proof.

When $d_1 = 3$, the modified regularity condition (R$^{2.2}$) enables us to argue as above. We intersect $Y = Y_1$ successively with the hypertangent divisors

$$D_{1,2} = \{f_{1,2}|\Delta = 0\}, \quad D_{2,2} = \{f_{2,2}|\Delta = 0\}, \quad D_{2,3} = \{f_{2,[2,3]}|\Delta = 0\},$$

then skip $D_{2,4}$ and proceed in the standard way.

When $d_1 = 2$, we have $f_1 = f_{1,2}$. Hence we intersect $Y$ with the hypertangent divisors $D_{2,2}$ and $D_{2,3}$, skip $D_{2,4}$ and complete the construction in the standard way. This proves the proposition in the case when $\text{codim}(\Delta \subset F) = 2$.

When $\text{codim}(\Delta \subset F) = 3$, the assumption

$$\frac{\text{mult}_o}{\text{deg}} Y > \frac{8}{d_1 d_2}$$

is so strong that the standard technique of hypertangent divisors (choose one general divisor in the linear system $\Lambda_2|_\Delta$ and proceed as in the non-singular case) provides a contradiction. Note that the condition (R$^{2.3}$) consists of two parts corresponding to whether $\text{codim}(\mathcal{L} \subset \mathbb{C}^{M+2})$ is equal to 2 or 3. We use the part where

$$\text{codim}(\mathcal{L} \subset \mathbb{C}^{M+2}) = \text{codim}(\Delta \subset F).$$

\begin{flushright}
\begin{samepage}
\text{□}
\end{samepage}
\end{flushright}

\textbf{Remark 2.2.} The weaker inequality

$$\frac{\text{mult}_o}{\text{deg}} Y \leq \frac{56}{9d_1 d_2}$$

would suffice for our purposes in the case when $\text{codim}(\Delta \subset F) = 2$ (but the inequality

$$\frac{\text{mult}_o}{\text{deg}} Y \leq \frac{6}{d_1 d_2}$$

for the hyperplane section $\Delta \subset F$ is still necessary). However, this partial relaxation of the hypotheses of Proposition 2.3 neither simplifies the proof nor enlarges
the class of varieties covered by the main result (in the sense of improving the bound for the codimension of the set of varieties not satisfying the regularity conditions). It only makes the statement more complicated. Therefore we have stated Proposition 2.3 in the above form.

§ 3. Divisorially canonical complete intersections

In this section we prove Theorem 0.3, that is, the divisorial canonicity of complete intersections of codimension 2 satisfying the regularity conditions. Assuming the opposite, we fix a non-canonical singularity $E^*$ of the pair $(F, (1/n)D)$ and successively exclude all the options: when the centre $B$ of the singularity $E^*$ is not contained in the set $\text{Sing } F$ of singular points (§ 3.1); when it is contained in $\text{Sing } F$ but not in the set of bi-quadratic singularities of $F$ (§ 3.2); and finally when it is contained in the set of bi-quadratic points (§§ 3.3–3.9). The last case is the hardest and occupies most of the section. The proof uses some local facts (proved in § 5) and some facts from projective geometry (proved in § 4). In § 3.10 we briefly discuss how one of these local facts simplifies a number of previous papers.

3.1. Beginning of the proof. Non-singular points. We begin the proof of Theorem 0.3. Let $F \in \mathcal{F}_{\text{reg}}$ be a regular complete intersection and let $H \in \text{Pic } F$ be the class of the hyperplane section. Assume that there is an effective divisor $D \sim nH$ such that the pair $(F, (1/n)D)$ is not canonical, that is, the Noether–Fano inequality

$$\text{ord}_{E^*} D > n \cdot a(E^*)$$

holds for some divisor $E^*$ over $F$. The divisor $E^*$ (and other exceptional divisors that emerge in the proof and satisfy the Noether–Fano inequality or its log version $\text{ord}_{E^*} D > n \cdot (a(E^*) + 1)$) will also be called a non-canonical singularity of $(F, (1/n)D)$ or a maximal singularity of $D$ (respectively, a non log canonical singularity of $(F, (1/n)D)$ or a log maximal singularity of $D$ in the case of the log version of the Noether–Fano inequality). Since the Noether–Fano inequality is linear in $D$, we can assume that $D$ is a prime divisor. Let $B \subset F$ be the centre of the maximal singularity $E^*$ on $F$.

In order to prove Theorem 0.3, we shall show that the existence of such a divisor $D$ (possessing a maximal singularity $E^*$) leads to a contradiction. We write

$$\text{CS} \left( F, \frac{1}{n}D \right)$$

for the union of the centres of the maximal singularities of $D$. We assume $D$ to be fixed and assume in addition that $\bar{B}$ is an irreducible component of the closed set $\text{CS}(F, (1/n)D)$. In particular, in a neighbourhood of a generic point of $B$, the pair $(F, (1/n)D)$ is canonical outside $B$. We arrive at a contradiction (‘exclude the maximal singularity’) in different ways depending on whether a point $o \in \overline{B}$ in general position is a non-singular, quadratic or bi-quadratic point of the complete intersection $F \subset \mathbb{P}$. First of all, we shall prove the following fact.

**Proposition 3.1.** The pair $(F, (1/n)D)$ is canonical on the set $F \setminus \text{Sing } F$ of non-singular points, that is, $\text{CS}(F, (1/n)D) \subset \text{Sing } F$. 

Proof. Assume the opposite:  

\[ B \not\subset \text{Sing} F. \]

We begin by showing that \( B \) is of sufficiently high codimension.

**Lemma 3.1.** We have  

\[ \text{codim}(B \subset F) \geq 8. \]

**Proof of the lemma.** Assume the opposite:  

\[ \text{codim}(B \subset F) \leq 7. \]

Then  

\[ \text{codim}(B \subset \mathbb{P}) \leq 9. \]

Let \( P \subset \mathbb{P} \) be a general 11-dimensional linear subspace and let \( F_P = F \cap P \) be the corresponding section of \( F \). We recall that  

\[ \text{codim}(\text{Sing} F \subset F) \geq 10 \]

and, therefore,  

\[ \text{codim}(\text{Sing} F \subset \mathbb{P}) \geq 12. \]  

Hence \( F_P \) is a non-singular complete intersection of codimension 2 in \( P \cong \mathbb{P}^{11} \) and, moreover,  

\[ \dim B \cap P \geq 2. \]

By [33], Proposition 3.6, or [34], for every surface on \( F_P \), the multiplicity of the effective divisor  

\[ D_P = (D \circ F_P) = D|_{F \cap P} \sim nH_P \]

(where \( H_P \) stands for the class of a hyperplane section of \( F_P \subset \mathbb{P}^{11} \)) along this surface does not exceed \( n \). This contradicts the inequality  

\[ \text{mult}_{B \cap P} D_P = \text{mult}_B D > n, \]

which holds because \( B \not\subset \text{Sing} F \) is the centre of a maximal singularity of \( D \). \( \square \)

We return to the proof of Proposition 3.1. Let \( o \in B \) be a point in general position, so that \( o \not\in \text{Sing} F \). Consider a general 10-dimensional linear subspace \( P \subset \mathbb{P} \) through \( o \). Let \( F_P = F \cap P \) be the corresponding section of \( F \). Putting \( D_P = (D \circ F_P) \), we see that \( o \) is an isolated centre of a non-canonical singularity of the pair \((F_P, (1/n)D_P)\), that is, an irreducible component of \( \text{CS}(F_P, (1/n)D_P) \). It is well known (see, for example, [5], Ch. 7, Proposition 2.3) that this implies that either

\[ \nu = \text{mult}_o D > 2n \]

or the exceptional divisor \( E \cong \mathbb{P}^{M-1} \) of the blow-up \( \varphi: F^+ \to F \) of \( o \) contains the (uniquely determined) hyperplane \( \Theta \subset E \) such that

\[ \nu + \text{mult}_\Theta D^+ > 2n, \]

where \( D^+ \) is the strict transform of \( D \) on \( F^+ \). We consider only the second option because it formally includes the first. Write \( |H - \Theta| \) for the projectively two-dimensional linear system of hyperplane sections of \( F \) of which a general element is a divisor \( W \ni o \) which is non-singular at \( o \) and satisfies

\[ W^+ \cap E = \Theta \]
(in other words, $W^+$ cuts out $\Theta$ on $E$). For such a hyperplane section $W$, the restriction $D_W = (D \circ W)$ is an effective divisor on the factorial variety $W$ with
\[ \text{mult}_o D_W > 2n. \]
This inequality can be rewritten in the form
\[ \frac{\text{mult}_o}{\text{deg}} D_W > \frac{2}{d_1d_2}. \]
This contradicts Proposition 2.1. □

Note that we did not use the full strength of Proposition 2.1. We used it only for an effective divisor on a hyperplane section of $F$ while Proposition 2.1 holds for the divisors on the sections of $F$ by linear subspaces of codimension 2. The full strength of Proposition 2.1 will be used below to exclude the bi-quadratic case.

3.2. Quadratic singular points. We write $\text{Sing}^{(2 \cdot 2)} F$ for the closed set of bi-quadratic points of $F$. Hence $\text{Sing} F \setminus \text{Sing}^{(2 \cdot 2)} F$ is the set of quadratic singular points.

Proposition 3.2. The pair $(F, (1/n)D)$ is canonical on the Zariski-open set $F \setminus \text{Sing}^{(2 \cdot 2)} F$, that is,
\[ \text{CS} \left( F, \frac{1}{n}D \right) \subset \text{Sing}^{(2 \cdot 2)} F. \]

Proof. Assume the opposite:
\[ B \subset \text{Sing} F, \quad B \not\subset \text{Sing}^{(2 \cdot 2)} F. \]
Let $o \in B$ be a point in general position. It is a quadratic singularity of $F$ of rank at least 11. We write $P$ for a general 5-dimensional linear subspace through $o$ in $\mathbb{P}$ and put $F_P = F \cap P$. The point $o$ is a non-degenerate (in particular, isolated) quadratic singularity of the threefold $F_P$ (in fact, the only singular point of $F_P$). Since $B \subset \text{Sing} F$, we have $\text{codim}(B \subset F) \geq 10$. Hence the section $F_P$ can be constructed in two steps: first intersect $F$ with a general linear subspace $P' \ni o$ in $\mathbb{P}$ of dimension
\[ \text{codim}(B \subset F) + 2, \]
and then intersect the result with a general 5-dimensional subspace $P \subset P'$ containing $o$. This enables us to use inversion of adjunction and conclude that the pair $(F_P, (1/n)D_P)$, where $D_P = (D \circ F_P) \sim nH_P$ and $H_P$ is the class of a hyperplane seticon of $F_P$, is not log canonical but is canonical outside $o$. Let
\[ \varphi_P : P^+ \to P \]
be the blow-up of $o$ with exceptional divisor $E_P \cong \mathbb{P}^4$ and let $F_P^+ \subset P^+$ be the strict transform of $F_P$ ($F_P^+$ is clearly the blow-up of $F_P$ at $o$). Put
\[ E_P = F_P^+ \cap E_P. \]
This is a non-singular two-dimensional quadric \( \cong \mathbb{P}^1 \times \mathbb{P}^1 \) in some hyperplane \( \langle E_P \rangle \cong \mathbb{P}^3 \) of the projective space \( \mathbb{P}_P \). Obviously, 

\[
a(E_P, F_P) = 1.
\]

Therefore, writing \( D^+_P \sim nH_P - \nu E_P \) (where \( D^+_P \) is the strict transform of \( D_P \) on \( F^+_P \)), we obtain two options:

(Q1) \( \nu > 2n \), so that \( E_P \) is a non log canonical singularity of the pair \( (F_P, (1/n)D_P) \),

(Q2) \( n < \nu \leq 2n \) and then the closed set 

\[
\text{LCS}\left(\frac{1}{n}D_P, F^+_P\right),
\]

the union of the centres of the non log canonical singularities of the original pair \( (F_P, (1/n)D_P) \) on \( F^+_P \), is a connected closed subset of the non-singular quadric \( E_P \subset \langle E_P \rangle \). This subset can be 

(Q2.1) either a connected curve \( C_P \subset E_P \) (possibly reducible),

(Q2.2) or a point \( x_P \in E_P \).

Returning to the original variety \( F \), we consider the blow-ups

\[
\varphi_P : \mathbb{P}^+ \to \mathbb{P} \quad \text{and} \quad \varphi : F^+ \to F
\]

of \( o \) on \( \mathbb{P} \) and \( F \) respectively, where \( F^+ \subset \mathbb{P}^+ \) is the strict transform of \( F \). We denote the exceptional divisors of these blow-ups by \( E \) and \( E \). Hence \( \mathbb{E} \cong \mathbb{P}^{M+1} \) and \( E \subset \mathbb{E} \) is a quadratic hypersurface of rank at least \( 11 \) in the hyperplane \( \langle E \rangle \subset \mathbb{E} \). We write the strict transform in the form

\[
D^+ \sim nH - \nu E,
\]

where \( \text{mult}_o D = 2\nu \). Then it follows from Proposition 2.2 that the case (Q1) does not occur. It is also easy to exclude the case (Q2.2): since \( P \ni o \) is a general linear subspace, we see in case (Q2.2) that the quadric \( E \) contains a linear subspace \( \Lambda \) of codimension 2 (with respect to \( E \)) such that \( x_P = \Lambda \cap E_P \). However, there are no linear subspaces of codimension 2 on a quadric of rank at least 11 (it suﬃces to have \( \text{rk} E \geq 7 \)). Thus the case (Q2.1) occurs. Since

\[
\text{Pic} E = \mathbb{Z}H,
\]

where \( H \) is the class of a hyperplane section of \( E \), we see that every irreducible component of the connected curve \( C_P \subset E_P \) is of bi-degree \( (l, l) \) with respect to the representation \( E_P \cong \mathbb{P}^1 \times \mathbb{P}^1 \), where \( l \geq 1 \).

We now need the following local fact.

Let \( o \in \mathcal{X} \) be the germ of an isolated singularity with \( \dim \mathcal{X} \geq 3 \) and let \( \varphi_{\mathcal{X}} : \mathcal{X}^+ \to \mathcal{X} \) be the blow-up of \( o \). Assume that the following condition holds.

(G) The exceptional divisor \( Q = \varphi_{\mathcal{X}}^{-1}(o) \) is irreducible, reduced and non-singular in codimension 1, that is,

\[
\text{codim}(\text{Sing} Q \subset Q) \geq 2.
\]

Moreover, we have \( a(Q, \mathcal{X}) = 1 \).
Obviously, $\text{Sing} \, X^+ \cap Q \subset \text{Sing} \, Q$. We also assume that the pair $(X, (1/n)D)$ is not log canonical but is canonical outside $o$ and that some exceptional divisor $E \neq Q$ over $X$ is not a log canonical singularity of $(X, (1/n)D)$ whose centre on $X^+$ is a prime divisor $W \subset Q$. Moreover, we assume that

$$D^+ \sim -\nu_Q \, Q,$$

where $\nu_Q \leq 2n$ and $D^+$ stands for the strict transform of the effective divisor $D$ on $X^+$. Under these conditions and with this notation, we have the following assertion.

**Theorem 3.1.** (i) The prime divisor $W$ occurs in the scheme-theoretic intersection $(D^+ \circ Q)$ with multiplicity $\mu_W > n$.

(ii) The following inequality holds:

$$\nu_D + \text{mult}_W D^+ > 2n.$$

**Proof.** The proof is given in §5. □

It follows from part (i) of Theorem 3.1 that the curve $C_P$ is irreducible of bi-degree $(1,1)$. In other words, it is a plane section of the quadric $E_P$. Using this and part (ii) of Theorem 3.1, we see that the quadric $E$ contains a hyperplane section $\Lambda \subset E$ such that

$$\nu + \text{mult}_\Lambda D^+ > 2n.$$

Consider the pencil $|H - \Lambda|$ of hyperplane sections of $F$, general divisor $W \in |H - \Lambda|$ of which contains $o$, and its strict transform $W^+ \subset F^+$ cuts out $\Lambda$ on $E$:

$$W^+ \cap E = \Lambda.$$

The quadric $\Lambda$ is obviously the exceptional divisor of the blow-up

$$\varphi_W : W^+ \rightarrow W$$

of $o$ on $W$. We put $D_W = (D \circ W)$. For the strict transform of this divisor on $W^+$ we have

$$D_W^+ \sim nH_W - \nu_W \, \Lambda,$$

where $\nu_W = \nu + \text{mult}_\Lambda D^+ > 2n$ and $H_W$ is the class of a hyperplane section of $W$. Thus the effective divisor $D_W$ satisfies the inequality

$$\frac{\text{mult}_o D_W}{\deg D_W} > \frac{4}{d_1d_2}.$$

This contradicts Proposition 2.2. □

### 3.3. Bi-quadratic points: beginning of the proof.

The bi-quadratic case is most difficult to exclude. Let $B$ be an irreducible component of the closed set $\text{CS}(F, (1/n)D) \subset \text{Sing}^{(2,2)} F$ and let $o \in B$ be a point in general position. Since we have to consider sections of $F \subset \mathbb{P}$ by various linear subspaces, it is convenient to introduce the following notation. For every $k \geq 1$ let $P_k$ be a $k$-dimensional linear subspace through $o$ in $\mathbb{P}$. This subspace is assumed to be general in its family, which
will always be specified. It is either the family of all linear subspaces of dimension \( k \) through \( o \) in \( \mathbb{P} \), or a more special family, for example, the family of \( k \)-dimensional subspaces through \( o \) in some hyperplane. In either case, this generality will be sufficient for the intersection

\[
F_k = (F \circ P_k)_\mathbb{P} = F \cap P_k
\]

(where \(( \circ )_\mathbb{P}\) stands for the scheme-theoretic intersection in \( \mathbb{P} \)) to be an irreducible reduced complete intersection of type \( d_1 \cdot d_2 \) in \( \mathbb{P} \). The blow-up of \( o \) in \( \mathbb{P} \) is denoted by \( \mathbb{P}^+ \), and its exceptional divisor by \( \mathbb{E} \). We put \( E = (F^+ \circ \mathbb{E}) = F^+ \cap \mathbb{E} \), where \( F^+ \subset \mathbb{P}^+ \) is the strict transform of \( F \) on \( \mathbb{P}^+ \). \( E \subset \mathbb{E} \) is obviously a complete intersection of two quadrics in \( \mathbb{E} \cong \mathbb{P}^{M+1} \). The strict transform of a subspace \( P_k \) on \( \mathbb{P}^+ \) is denoted by \( P_k^+ \), so that \( P_k^+ \to P_k \) is the blow-up of the point \( o \in P_k \). The exceptional divisor of this blow-up is denoted by \( \mathbb{E}_k \), that is, \( \mathbb{E}_k = \mathbb{E} \cap P_k^+ \cong \mathbb{P}^{k-1} \). The strict transform of the subvariety \( F_k \) on \( P^+ \) is denoted by \( F^+_k \). In all the cases considered below, the generality of the subspace \( P_k \) is sufficient for the intersection

\[
E_k = (F_k^+ \circ \mathbb{E}_k) = (F_k^+ \circ \mathbb{E}) = F_k^+ \cap \mathbb{E}_k
\]

to be an irreducible reduced complete intersection of two quadrics in \( \mathbb{E}_k \). The pullback to \( F_k^+ \) of a divisorial class on \( F_k \) will be denoted by the same symbol, and the strict transform on \( F_k^+ \) of an effective divisor on \( F_k \) will be denoted by adding the superscript +. The restriction \( D|_{F_k} = (D \circ F_k) \) of a divisor \( D \) to \( F_k \) is denoted by \( D_k \). Hence the original pair \( (F, (1/n)D) \) generates the pair \( (F_k, (1/n)D_k) \) on the \((k-2)\)-dimensional variety \( F_k \). Since

\[
\text{codim}(\text{Sing } F \subset F) \geq 10
\]

and the condition \((R2.1)\) holds, we see that for a general subspace \( P_{11} \) we obtain a variety \( F_{11} \) with a unique singular point \( o \) and, moreover, the variety \( F_{11}^+ \) is non-singular and \( E_{11} \subset \mathbb{E}_{11} \cong \mathbb{P}^{10} \) is a non-singular intersection of two quadrics. Furthermore, the point \( o \) is a connected component of the set

\[
\text{LCS} \left( F_{11}, \frac{1}{n}D_{11} \right),
\]

that is, this pair has a non log canonical singularity with centre \( o \) and is canonical outside the point \( o \) in a neighbourhood of that point.

The same holds for the pair \( (F_6, (1/n)D_6) \), where \( P_6 \subset P_{11} \) is a general subspace (we recall that all the subspaces \( P_k \) contain \( o \) by default). We have \( \text{mult}_o D \leq 6n \) by Proposition 2.3. Hence

\[
D^+ \sim D - \nu E,
\]

where \( \nu \leq (3/2)n \) and, therefore, \( D^+_6 \sim D_6 - \nu E_6 \). Note that \( a(E_6, F_6) = 1 \), so that \( E_6 \) is not a non log canonical singularity of \( (F_6, (1/n)D_6) \). Let

\[
\text{LCS} \left( \left( F_6, \frac{1}{n}D_6 \right), E_6 \right)
\]
be the union of the centres of all the non log canonical singularities of \((F_6, (1/n)D_6)\) over the point \(o\) on the exceptional divisor \(E_6\). We mentioned above that

\[
\text{LCS}\left(\left(F_6, \frac{1}{n}D_6\right), E_6\right) \neq E_6.
\]

Hence \(\text{LCS}((F_6, (1/n)D_6), E_6)\) is a proper connected closed subset of the non-singular threefold \(E_6\). This subset can

- (B1) contain the surface \(S(P_6) \subset E_6\);
- (B2) be a connected curve \(C(P_6) \subset E_6\);
- (B3) be a point \(x(P_6) \in E_6\).

The case (B3) is clearly impossible. Indeed, if this case occurs, then the non-singular 8-dimensional intersection \(E_{11} \subset \mathbb{P}^{10}\) of two quadrics contains a linear subspace of dimension 5 while the numerical Chow group of cycles of codimension 3 is

\[
A^3 E_{11} \cong \mathbb{Z}H_{11}^3,
\]

where \(H_{11}\) is the class of a hyperplane section of \(E_{11} \subset \mathbb{P}^{10}\).

We claim that the case (B1) is also impossible. Indeed, all the hypotheses of Theorem 3.1 hold for the singularity \(o \in F_6\). Since the non-singular threefold \(E_6\) is a complete intersection of two quadrics in \(E_6 \cong \mathbb{P}^{5}\), we have

\[
A^1 E_6 = \text{Pic} E_6 = \mathbb{Z}H_6,
\]

where \(H_6\) is the class of a hyperplane section with respect to the embedding \(E_6 \subset E_6\). (In what follows, \(H_k\) always stands for the class of a hyperplane section of the variety \(E_k\) embedded in the projective space \(\mathbb{E}_k\).) We have

\[
(D_6^+ \circ E_6) \sim \nu H_6.
\]

However, by Theorem 3.1, the effective divisor \((D_6^+ \circ E_6)\) contains the prime divisor (a surface) \(S(P_6)\) with multiplicity strictly greater than \(n\). Since \(\nu \leq (3/2)n\), we conclude that \(S(P_6)\) is a hyperplane section of \(E_6\). Moreover,

\[
\nu + \text{mult}_{S(P_6)} D_6^+ > 2n
\]

by part (ii) of Theorem 3.1. Since \(P_{11}\) and \(P_6 \subset P_{11}\) are linear subspaces in general position through \(o\), we can conclude that there is a hyperplane section \(\Lambda\) of \(E\) such that

\[
\nu + \text{mult}_{\Lambda} D^+ > 2n.
\]

We now argue in almost the same way as we did when excluding the case (Q2.1) in the proof of Proposition 3.2. Consider a hyperplane section \(W\) of \(F\) that contains \(o\) and cuts out \(\Lambda\) on \(E\),

\[
W^+ \cap E = \Lambda.
\]

(There is a unique such section in the bi-quadratic case and a pencil of such sections in the quadratic case.) Putting \(D_W = (D \circ W)\), we obtain an effective divisor \(D_W\) on the hyperplane section \(W\) of \(F\) such that

\[
\frac{\text{mult}_o}{\deg D_W} > \frac{8}{d_1d_2}.
\]

This contradicts Proposition 2.3. The case (B1) is excluded.

Therefore we can assume that the case (B2) occurs.
3.4. Investigation of the case (B2). We recall that the pair \((F_6, (1/n)D_6)\) is canonical outside the point \(o\) (in a neighbourhood of that point) but has a non log canonical singularity whose centre on \(F_6^+\) is an irreducible curve \(C(F_6)\). We denote this curve by \(\Lambda_6\) for consistency of notation. We now need another local fact.

Let \(o \in X\) be the germ of an isolated singularity satisfying the condition \((G)\) in § 3.2. Assume that the pair \((X, (1/n)D)\) is not log canonical but is canonical outside \(o\) and \(E \neq Q\) is a non log canonical singularity of \((X, (1/n)D)\) whose centre on \(X^+\) is an irreducible subvariety \(W \subset Q\) not contained in \(\text{Sing } Q\), and

\[
\text{codim}(W \subset Q) \geq 2.
\]

Assume that \(D^+ \sim -\nu_D Q\), where \(\nu_D \leq 2n\). Under these conditions we have the following assertion (a proof is given in § 5).

**Theorem 3.2.** At least one of the following inequalities holds:

1. \(\text{mult}_W D^+ > n\);
2. \(\text{mult}_W (D^+ \circ Q) > 3n - \nu_D\).

Note that the scheme-theoretic intersection \((D^+ \circ Q)\) is the restriction of the effective divisor \(D^+\) to the effective divisor \(Q\). Hence,

\[
(D^+ \circ Q) \sim \nu_D H_Q,
\]

where \(H_Q = -(Q \circ Q)\) is the ‘hyperplane section’ of \(Q\).

Applying Theorem 3.2 to the pair \((F_6, (1/n)D_6)\), we obtain the following fact.

**Proposition 3.3.** We have

\[
\text{mult}_{\Lambda_6} D_6^+ > n.
\]

**Proof.** Assume the opposite:

\[
\text{mult}_{\Lambda_6} D_6^+ \leq n.
\]

By Theorem 3.2 we have

\[
\text{mult}_{\Lambda_6} (D_6^+ \circ E_6) > 3n - \nu.
\]

Recall that \(F_6\) is a section of \(F_{11}\) by a linear subspace \(P_6 \subset P_{11}\) in general position. Hence there is an irreducible subvariety \(\Lambda_{11} \subset E_{11}\) of codimension 2 such that \(\Lambda_6 = \Lambda_{11} \cap P_6^+ = \Lambda_{11} \cap F_6^+\), that is, \(\Lambda_6\) is the section of \(\Lambda_{11}\) by a linear 5-dimensional subspace in general position in \(E_{11} \cong \mathbb{P}^{10}\) and we have

\[
\text{mult}_{\Lambda_{11}} (D_{11}^+ \circ E_{11}) > 3n - \nu.
\]

However, \((D_{11}^+ \circ E_{11}) \sim \nu \mathbb{H}_{11}\) is an effective divisor on the non-singular complete intersection \(E_{11} \subset E_{11} \cong \mathbb{P}^{10}\) of two quadrics and this divisor is cut out on \(E_{11}\) by a hypersurface of degree \(\nu\), and \(\Lambda_{11} \subset E_{11}\) is an irreducible subvariety of dimension 6. Hence, by Proposition 3.6 in [33], we have

\[
\nu > 3n - \nu.
\]

Hence \(\nu > (3/2)n\), which is impossible. This contradiction completes the proof. ☐

Returning to the original variety \(F \subset \mathbb{P}\), we conclude that the exceptional divisor \(E \subset \mathbb{E}\) contains an irreducible subvariety \(\Lambda \subset E\) of codimension 2 such that

\[
\text{mult}_{\Lambda} D^+ > n.
\]
3.5. The secant variety. We need a simple but non-trivial fact from projective geometry, which will be proved in §4.

Let \( Q \subset \mathbb{P}^N \) be an irreducible complete intersection of two quadrics, where \( N \geq 16 \). More precisely, \( Q = Q_1 \cap Q_2 \), where each \( Q_i = \{ q_i = 0 \} \subset \mathbb{P}^N \) is a quadratic hypersurface. Assume that the following conditions of generality hold:

- (C1) \( \max\{ \text{rk } q_1, \text{rk } q_2 \} \geq 16 \);
- (C2) \( \text{rk}(q_1, q_2) \geq 10 \).

We saw in §1.7 that (C2) implies the inequality \( \text{codim}(\text{Sing } Q \subset Q) \geq 7 \).

Let \( X \subset Q \) be an irreducible subvariety of codimension 2. We introduce the following notation. Given a pair of points \( p \neq q \) in \( \mathbb{P}^N \), we write \([p, q]\) for the line connecting these points. To simplify some formulae, it is convenient to put \([p, p] = \emptyset\), so that we can use the notation \([p, q]\) without specifying that \( p \neq q \). Given a pair of distinct points \( p \neq q \) on \( Q \), we write \([p, q]_Q\) for the line \([p, q]\) provided that it is not contained entirely in \( Q \). Otherwise we put \([p, q] = \emptyset\). We also put \([p, p]_Q = \emptyset\) for convenience. Define the secant variety of a subvariety \( X \) on \( Q \) by the formula

\[ \text{Sec}(X \subset Q) = \bigcup_{p, q \in X} [p, q]_Q \]

(the bar means the closure).

**Theorem 3.3.** Precisely one of the following three options holds:

- (S1) \( \text{Sec}(X \subset Q) = Q \);
- (S2) \( \text{Sec}(X \subset Q) \) is a hyperplane section of \( Q \) on which \( X \) is cut out by a hyperplane of degree \( d_X \geq 2 \) in \( \mathbb{P}^N \);
- (S3) \( X = \text{Sec}(X \subset Q) \) is the section of \( Q \) by a linear subspace of codimension 2 in \( \mathbb{P}^N \).

In any case, \( \text{Sec}(X \subset Q) \) is the closure of the union of all lines \([p, q]_Q\) such that \([p, q]_Q \cap \text{Sing } Q = \emptyset\).

**Proof.** A proof of Theorem 3.3 is given in §4. \(\Box\)

Consider the secant variety \( \text{Sec}(\Lambda \subset E) \). The following assertion is almost obvious.

**Proposition 3.4.** The support \( |D^+| \) of the effective divisor \( D^+ \) contains the closed set \( \text{Sec}(\Lambda \subset E) \).

**Proof.** By the last assertion of Theorem 3.3 it suffices to show that, for any pair of distinct points \( p, q \in \Lambda \setminus \text{Sing } E \)

(in particular, \( p, q \notin \text{Sing } F^+ \)), we have the inclusion \([p, q]_E \subset D^+\).

Of course, we can assume that \([p, q]_E \neq \emptyset\). Assume the opposite:

\([p, q]_E \notin D^+\).
Restricting the effective Cartier divisor $D^+$ to the line $[p, q] = [p, q]_E$, we obtain

$$(p, q) \cdot D^+ = \nu \geq \text{mult}_p D^+ + \text{mult}_q D^+ > 2n.$$ 

This contradiction completes the proof. □

It follows from Proposition 3.4 that the option (S1) (in the sense of Theorem 3.3) is not realized for the secant variety $\text{Sec}(\Lambda \subset E)$, that is, $\text{Sec}(\Lambda \subset E) \neq E$. It is also easy to exclude the option (S3).

**Proposition 3.5.** The subvariety $\Lambda \subset E$ is not a section of $E$ by a linear subspace of codimension 2 in $\mathbb{E}$.

**Proof.** Assume the opposite:

$$\Lambda = \text{Sec}(\Lambda \subset E).$$

Write $|H - \Lambda|$ for the pencil of hyperplane sections of $F$ of which a general element $W$ contains $o$ and the strict transform $W^+$ of $W$ contains $\Lambda$. Thus the strict transform $|H - \Lambda|^+$ of the pencil $|H - \Lambda|$ on $F^+$ cuts out on $E$ a pencil of hyperplane sections of $E \subset \mathbb{E}$ containing $\Lambda$. Let $W \in |H - \Lambda|$ be a general element.

Let $D_W = (D \circ W) = D|_W$ be the restriction of the divisor $D$ to the hyperplane section $W$. Then we have

$$D_W^+ \sim nH_W - \nu E_W,$$

where $H_W$ is the class of a hyperplane section of $W \subset \mathbb{P}^{M+1}$ and $E_W = (E \circ W^+) = E \cap W^+$ is the exceptional divisor of the blow-up of $o$ on $W$ and, moreover,

$$\text{mult}_\Lambda D_W^+ = \text{mult}_\Lambda D^+ > n.$$

The divisor $D_W$ is effective. But it can be reducible and contain the subvariety

$$F_\Lambda = \text{Bs} |H - \Lambda|$$

as a component. This subvariety is the unique section of $F$ by a linear subspace through $o$ of codimension 2 in $\mathbb{P}$ such that

$$E \cap F_\Lambda^+ = \Lambda.$$

The subvariety $F_\Lambda$ is clearly a hyperplane section of $W$. We write

$$D_W = \Xi_W + aF_\Lambda,$$

where $a \in \mathbb{Z}_+$ and the effective divisor $\Xi_W$ on $W$ does not contain $F_\Lambda$ as a component. Since $F_\Lambda^+ \sim H_W - E_W$ and $\text{mult}_\Lambda F_\Lambda^+ = 1$, we obtain

$$\Xi_W^+ \sim (n - a)H_W - (\nu - a)E_W,$$

where $\text{mult}_\Lambda \Xi_W^+ > n - a$ and $\nu - a > n - a$. In particular, the divisor $\Xi_W^+$ is non-zero. Hence we have a well-defined effective divisor

$$D_\Lambda = (\Xi_W \circ F_\Lambda)$$
on the variety \( F_\Lambda \subset \mathbb{P}^M \), where \( D_\Lambda \sim (n - a)H_\Lambda \) (\( H_\Lambda \) is the class of a hyperplane section of \( F_\Lambda \)) and, moreover,

\[ D_\Lambda^+ \sim (n - a)H_\Lambda - \nu_\Lambda \Lambda, \]

where the coefficient \( \nu_\Lambda \) satisfies the inequality

\[ \nu_\Lambda \geq (\nu - a) + \text{mult}_\Lambda \Xi_W > 2(n - a). \]

Thus we have constructed an effective divisor \( D_\Lambda \) on the section \( F_\Lambda \) of \( F \) by a linear subspace of codimension 2 in \( \mathbb{P} \) such that

\[ \frac{\text{mult}_o D_\Lambda}{\deg D_\Lambda} > \frac{8}{d_1d_2}. \]

This contradicts Proposition 2.3. □

We conclude that only the second possibility (S2) (of the three options listed in Theorem 3.3) occurs. Thus, \( \text{Sec}(\Lambda \subset E) \) is a hyperplane section of \( E \subset E \) and \( \Lambda \) is cut out on it by a hypersurface of degree \( d_\Lambda \geq 2 \).

### 3.6. Restriction to a hyperplane section.

Let \( R \) be the uniquely determined hyperplane section of \( F \) such that \( o \in R \) and

\[ E_R = E \cap R^+ = (E \circ R^+) = \text{Sec}(\Lambda \subset E). \]

The support \( |D^+| \) of the effective divisor \( D^+ \) contains \( E_R \) by Proposition 3.4. We now estimate the multiplicity of \( D^+ \) at a point of general position on \( E_R \). To do this, we need another local fact.

Let \( o \in X \) be the germ of a three-dimensional isolated non-degenerate biquadratic singularity and let \( \varphi_X : X^+ \to X \) be the blow-up of \( o \). Then \( X^+ \) and the exceptional surface \( E = \varphi^{-1}(o) \) are non-singular and \( E \cong Q_1 \cap Q_2 \subset \mathbb{P}^4 \) is a Del Pezzo surface of degree 4. Let \( L \subset E \) be a line, \( p \neq q \) two distinct points of \( L \), \( D \) an effective divisor on \( X \), and \( D^+ \sim -\nu_D E \) its strict transform on \( X^+ \), where \( \nu_D \in \mathbb{Z}_+ \). Assume that

\[ \text{mult}_p D^+ = \text{mult}_q D^+ = \mu \geq 1. \]

**Theorem 3.4.** We have

\[ \text{mult}_L D^+ \geq \frac{1}{3}(2\mu - \nu_D). \]

**Proof.** The proof is given in §5. It is completely analogous to the proofs of Lemma 4.2 in [2] and Lemma 4.5 in [35], § 3.7. However, a simple reference to them is insufficient since the normal sheaf of \( L \) with respect to \( X^+ \) is not the same as in those lemmas and, therefore, the arguments in [35] and [2] require a small modification. □

Let \( p, q \in \Lambda \) be a general pair of distinct points such that the line \([p, q] \) is contained in \( E \) and is disjoint from \( \text{Sing} \ E \). Considering the section \( F_5 = F \cap P_5 \) of \( F \) by a general 5-dimensional subspace \( P_5 \ni o \) such that

\[ F_5^+ \supset [p, q], \]
and using Theorem 3.4, we see that the multiplicity of $D^+$ along $E_R$ is at least $(1/3)(2\mu - \nu)$, where $\mu = \text{mult}_A D^+ > n$. Therefore for the effective divisor
\[ D_R = D|_R = (D \circ R) \]
(the pair $(F, R)$ is canonical and $D$ is irreducible by assumption, so that $D \neq R$) we have
\[ D_R^+ \sim nH_R - \nu_R E_R, \]
where $H_R$ is the class of a hyperplane section of $R \subset \mathbb{P}^M$ and $\nu_R$ satisfies the inequality
\[ \nu_R \geq \nu + \frac{1}{3}(2\mu - \nu) = \frac{2}{3}(\nu + \mu) > \frac{4}{3}n. \]
Unfortunately, this inequality is insufficient to obtain a contradiction by means of the bounds in §2, as we did in previous cases. However, we have made a step forward since the pair $(R, (1/n)D_R)$ satisfies the inequality $\nu_R > (4/3)n$, which is considerably stronger than the inequality $\nu > n$ for the original pair $(F, (1/n)D)$. We still have $\nu_R \leq (3/2)n$ by Proposition 2.3. To exclude the maximal singularity whose centre is contained in the closed set $\text{Sing}^{(2,2)} F$ of bi-quadratic points, we shall study the singularities of the pair $(R, (1/n)D_R)$ at $o$ and show that the whole procedure of exclusion that was carried out above for the original pair $(F, (1/n)D)$ can be repeated (with simplifications if we use already-known facts) for the new pair $(R, (1/n)D_R)$. This will complete the proof of Theorem 0.3.

The hyperplane in $\mathbb{P}$ that cuts out $R$ on $F$ is the linear span of $R$. Therefore it is denoted by $\langle R \rangle$. The hyperplane in $E$ that cuts out $E_R$ on $E$, is denoted by $E_R$. Obviously,
\[ E_R = \langle E_R \rangle = \langle R \rangle^+ \cap E, \]
where $\langle R \rangle^+ \subset \mathbb{P}^+$ is the strict transform on the blow-up of $o$.

We recall a well-known fact. The dimension of the singular set of a complete intersection in a projective space increases by at most 1 under taking a hyperplane section; see [36], [37]. Therefore we have
\[ \text{codim}(\text{Sing} R \subset R) \geq 8. \]
Hence the section of $R \subset \langle R \rangle \cong \mathbb{P}^{M+1}$ by a general 9-dimensional linear subspace has no singularities.

The notation introduced at the beginning of §3.3 can be extended to sections of $R$. For a sufficiently general $k$-dimensional linear subspace $P_k \ni o$, the intersection $R \cap P_k$ is denoted by $R_k$. If $P_k \subset \langle R \rangle$ by construction, then $R_k = F_k$. In this case we clearly have
\[ E_k = E \cap P_k^+ = E_R \cap P_k^+. \]
The strict transform of $R_k$ on $P_k^+$ is denoted by $R_k^+$. Then we have
\[ E_k = F_k^+ \cap E_k = R_k^+ \cap E_k. \]
For the restriction of $D$ to $F_k$ we have
\[ D_k = D|_{F_k} = D_R|_{R_k}. \]
We will always specify the inclusion $P_k \subset \langle R \rangle$ because the last equalities are very useful. A general $k$-dimensional linear subspace (in $\mathbb{P}$ or $\langle R \rangle$) which is not required to contain $o$, is denoted by $\Pi_k$. 

The notation introduced at the beginning of §3.3 can be extended to sections of $R$. For a sufficiently general $k$-dimensional linear subspace $P_k \ni o$, the intersection $R \cap P_k$ is denoted by $R_k$. If $P_k \subset \langle R \rangle$ by construction, then $R_k = F_k$. In this case we clearly have
\[ E_k = E \cap P_k^+ = E_R \cap P_k^+. \]
The strict transform of $R_k$ on $P_k^+$ is denoted by $R_k^+$. Then we have
\[ E_k = F_k^+ \cap E_k = R_k^+ \cap E_k. \]
3.7. Non-singular points of $R$. We begin by establishing an analogue of Proposition 3.1 for the pair $(R, (1/n)D_R)$.

**Proposition 3.6.** The pair $(R, (1/n)D_R)$ is canonical outside the closed set $\text{Sing } R$.

**Proof.** We only sketch the main steps since the proof is similar to that of Proposition 3.1. Assume the opposite:

$$\text{CS}(R, \frac{1}{n}D_R) \not\subset \text{Sing } R.$$ 

Let $B^* \subset R$ be the centre of a maximal singularity $E^*_R$ of the pair $(R, (1/n)D_R)$ such that $B^* \not\subset \text{Sing } R$ and $B^*$ is an irreducible component of the closed set $\text{CS}(R, (1/n)D_R)$. Since the section $R \cap \Pi_9$ of $R$ by a general 9-dimensional subspace $\Pi_9 \subset \langle R \rangle$ is a non-singular complete intersection of codimension 2 in $\Pi_9 \cong \mathbb{P}^9$, we can argue as in the proof of Lemma 3.1 and conclude that

$$\text{codim}(B^* \subset R) \geq 6.$$ 

Let $p \in B^*$ be a point in general position. In particular, $p \not\in \text{Sing } R$ (the fixed bi-quadratic point is still denoted by $o$). Proceeding as in the proof of Proposition 3.1, we consider a general 8-dimensional linear subspace $\Pi_8 \subset \langle R \rangle$ through $p$. Then $p$ is an isolated centre of a non-canonical singularity of the pair

$$\left( R \cap \Pi_8, \frac{1}{n}D|_{R \cap \Pi_8} \right).$$

Hence, by Proposition 2.3 in Ch. 7 of [5], either $\text{mult}_p D_R > 2n$, or the exceptional divisor $E(p, R) \cong \mathbb{P}^{M-2}$ of the blow-up $R^{(p)} \rightarrow R$ of $p$ contains a uniquely determined hyperplane $\Theta(p) \subset E(p, R)$ such that

$$\text{mult}_p D_R + \text{mult}_{\Theta(p)} D^{(p)}_R > 2n,$$

where $D^{(p)}_R$ is the strict transform of $D_R$ on $R^{(p)}$. The second option includes the first. We write $|H_R - \Theta(p)|$ for the projectively two-dimensional linear system of hyperplane sections of $R$ whose general element $W$ contains $p$, is non-singular at $p$ and satisfies the equality

$$W^+ \cap E(p, R) = \Theta(p).$$

For a general divisor $W \in |H_R - \Theta(p)|$, the restriction $D_R|_W = (D_R \circ W)$ satisfies the inequality $\text{mult}_p (D_R \circ W) > 2n$. Hence,

$$\frac{\text{mult}_o}{\text{deg}} (D_R \circ W) > \frac{2}{d_1 d_2}.$$ 

However, $(D_R \circ W)$ is an effective divisor on the section $W$ of $F$ by a linear subspace of codimension 2 through $p$ and $W$ is non-singular at $p$. This contradicts Proposition 2.1. □

**Corollary 3.1.** For a general 9-dimensional subspace $\Pi_9 \subset \langle R \rangle$, the pair

$$\left( R \cap \Pi_9, \frac{1}{n}D|_{R \cap \Pi_9} \right)$$

is canonical.
3.8. Singularities of the pair \((R, (1/n)D_R)\) at the point \(o\). We return to the study of the bi-quadratic point \(o\). Recall that for a section \(F_{11}\) of \(F\) by a general 11-dimensional linear subspace \(P_{11} \ni o\), the point \(o\) is a connected component of the closed set \(\text{LCS}(F_{11}, (1/n)D_{11})\) and

\[ D^+_{11} \sim D_{11} - \nu E_{11} \sim nH_{11} - \nu E_{11}. \]

Since \(P_{11}\) is general, we have

\[ \text{mult}_{\{E_{11} \cap E_R\}} D^+_{11} = \text{mult}_{E_R} D^+, \]

where \(E_{11} \cap E_R\) is a hyperplane section of \(E_{11} \subset \mathbb{E}_{11} \cong \mathbb{P}^{10}\). Consider the linear subspace

\[ P_{10} = P_{11} \cap \langle R \rangle. \]

On the one hand, \(P_{10}\) is a general 10-dimensional linear subspace through \(o\) in \(\langle R \rangle\). On the other, \(P_{10}\) is a hyperplane in \(P_{11}\) and, therefore, \(F_{10} = F \cap P_{10}\) is a hyperplane section of \(F_{11}\). However, \(P_{10}\) is a specially selected (and, generally speaking, uniquely determined) hyperplane in \(P_{11}\). Hence it is not a hyperplane in general position through \(o\) in \(P_{11}\). Hence, by inversion of adjunction, we can only claim that

\[ o \in \text{LCS}\left(\left. F_{10}, \frac{1}{n} D_{10} \right\rangle, \right. \]

where \(D_{10} = (D \circ F_{10}) = (D_R \circ F_{10}) \sim nH_{10} - \nu_R E_{10}\). It is clear that \(E_{10} = E_{11} \cap E_R\). The variety \(F_{10}\) is a section of \(F_{11}\) by a specially selected hyperplane. Therefore \(\text{Sing} F_{10}\) can be larger than \(P_{10} \cap \text{Sing} F_{11}\) and, in particular, larger than \(\{o\}\). However, we have the inequality

\[ \text{codim}(\text{Sing} R \subset \langle R \rangle) \geq 10, \]

which implies that the singularities of \(F_{10} = R_{10}\) are zero-dimensional. In particular, the point \(o\) is an isolated singularity of \(F_{10} = R_{10}\).

**Proposition 3.7.** The closed set \(\text{CS}(F_{10}, (1/n)D_{10})\) is zero-dimensional. In particular, the pair \((F_{10}, (1/n)D_{10})\) is canonical outside \(o\) in a neighbourhood of \(o\), and \(o\) is the centre of a non log canonical singularity of the pair \((F_{10}, (1/n)D_{10})\).

**Proof.** Consider a hyperplane \(\Pi_9 \subset P_{10}\) in general position (in particular, not containing \(o\)). Since \(P_{10} \subset \langle R \rangle\) is a general 10-dimensional subspace through \(o\), the subspace \(\Pi_9 \subset P_{10}\) is a general 9-dimensional subspace, without any restrictions. Applying Corollary 3.1, we obtain the first assertion of the proposition. The other two follow directly from it. \(\square\)

We now have what we wanted: the properties of the pair \((R, (1/n)D_R)\) at the point \(o\) are completely analogous to those of the original pair \((F, (1/n)D)\) at \(o\).

For a general subspace \(P_6 \subset \langle R \rangle\), the pair \((R_6 = F_6, (1/n)D_6)\) is not log canonical at \(o\) but is canonical in a punctured neighbourhood of \(o\) and, moreover, we have \(a(E_6, R_6) = 1\) and \(\nu_R \leq (3/2)n\). Therefore we have three options (B1)\(_R\), (B2)\(_R\), (B3)\(_R\) for the closed set

\[ \text{LCS}\left(\left(\left. R_6, \frac{1}{n} D_6 \right\rangle, E_6 \right\rangle, \right. \]

...
They are identical to the options (B1), (B2), (B3) in § 3.3. The case $(B3)_R$ can be excluded in almost the same way as the case $(B3)$. Indeed, if it holds, then the non-singular 7-dimensional complete intersection $E_{10} \subset \mathbb{P}^9$ of two quadrics (for a general subspace $P_{10} \subset \langle R \rangle$) contains a linear subspace of codimension 3 with respect to $E_{10}$ while the numerical Chow group is

$$A^3E_{10} \cong \mathbb{Z} \mathbb{H}_{10}^3,$$

where $\mathbb{H}_{10}$ is the class of a hyperplane section of $E_{10} \subset \mathbb{P}^9$, a contradiction. The case $(B1)_R$ can also be excluded in almost the same way as the case $(B1)$ in § 3.3. Indeed, we deduce from the inequality $\nu_R \leq (3/2)n$ that the surface $S(P_6)$ is a hyperplane section of $E_6$. Hence, using part (ii) of Theorem 3.1 and taking into account that $P_6 \subset P_{10} \subset \langle R \rangle$ is a subspace in general position, we conclude that there is a hyperplane section $\Lambda(R) \subset E_R$ of $E_R$ such that

$$\nu_R + \text{mult}_{\Lambda(R)} D^+_R > 2n.$$

Consider the hyperplane section $W$ of $R$ containing $o$ and cutting out $\Lambda(R)$ on $E_R$: $W^+ \cap E_R = \Lambda(R)$. As a result, we obtain an effective divisor $(D \circ W)_R$ on $W$ such that

$$\frac{\text{mult}_o}{\deg (D \circ W)_R} > \frac{8}{d_1 d_2}.$$

This contradicts Proposition 2.3.

Thus the cases $(B1)_R$ and $(B3)_R$ are impossible and we can assume that the case $(B2)_R$ occurs.

3.9. Investigation of the case $(B2)_R$. Our arguments are similar to those in § 3.4. The pair $(R_6, (1/n)D_6)$ is canonical outside $o$ in a neighbourhood of $o$ but has a non log canonical singularity whose centre on $R_6^+$ is an irreducible curve $\Lambda_6(R)$.

**Proposition 3.8.** We have

$$\text{mult}_{\Lambda_6(R)} D^+_6 > n.$$

**Proof.** The proof is completely analogous to that of Proposition 3.3 (obviously replacing $\nu$ by $\nu_R$, $F_{11}$ by $F_{10} = R_{10}$, $P_{11}$ by $P_{10} \subset \langle R \rangle$, $A_{11} \subset E_{11}$ by $A_{10} \subset E_{10}$, $A_6$ by $A_6(R)$, and so on). We do not repeat it here. $\square$

Proceeding as in § 3.4, we return to the variety $R$ and conclude that the exceptional divisor $E_R \subset E_R$ contains an irreducible subvariety $\Lambda(R) \subset E_R$ of codimension 2 such that

$$\text{mult}_{\Lambda(R)} D^+_R > n.$$

The next step of our proof is parallel to the arguments in § 3.5. Since the rank of a quadratic form decreases by at most 2 when restricted to a hyperplane, Theorem 3.3 applies to the subvariety $\Lambda(R)$ of the complete intersection $E_R \subset E_R$ of two quadrics.

**Proposition 3.9.** The support $|D^+_R|$ of the effective divisor $D^+_R$ contains the closed set

$$\text{Sec}(\Lambda(R) \subset E_R).$$
\textbf{Proof.} The proof is identical to that of Proposition 3.4 and we omit it. □

Obviously, \( \text{Sec}(\Lambda(R) \subset E_R) \neq E_R \).

\textbf{Proposition 3.10.} The subvariety \( \Lambda(R) \subset E_R \) is not a section of \( E_R \) by a linear subspace of codimension 2 in \( \mathbb{E}_R \).

\textbf{Proof.} The proof is completely analogous to that of Proposition 3.5. However, the notation must be changed. Therefore we trace the argument briefly. Instead of the pencil \( |H - \Lambda| \) (in the proof of Proposition 3.5) we consider the pencil \( |H_R - \Lambda(R)| \), a general element \( W \) of which is a hyperplane section of \( R \) through \( o \) with \( W^+ \supset \Lambda(R) \). We again put \( D_W = (D_R \circ W) = (D \circ W) \)

and obtain \( D_W^+ \sim nH_W - \nu_R E_W \), where \( H_W \) is the class of a hyperplane section of \( W \) and

\( E_W = (E_R \circ W^+) = (E \circ W^+) = E \cap W^+ \),

where \( \mu_R = \text{mult} \Lambda(R) D_W^+ = \text{mult} \Lambda(R) D_R^+ > n \). Instead of \( F_\Lambda \), we consider the subvariety

\( R_{\Lambda(R)} = \text{Bs} |H_R - \Lambda(R)| \).

It is the section of \( R \) by a linear subspace of codimension 2 through \( o \) in \( \langle R \rangle \) and, moreover, \( R_{\Lambda(R)}^+ \cap E_R = \Lambda(R) \). The subvariety \( R_{\Lambda(R)} \) is clearly a hyperplane section of \( W \). Writing

\( D_W = \Xi_W + aR_{\Lambda(R)} \),

where \( a \in \mathbb{Z}_+ \) and \( R_{\Lambda(R)} \) is not a component of the effective divisor \( \Xi_W \), we obtain

\( \Xi_W^+ \sim (n - a)H_W - (\nu_R - a)E_W \),

where \( \text{mult} \Lambda(R) \Xi_W^+ > n - a \) and \( \nu_R - a > n - a \). Hence

\( D_{\Lambda(R)} = (\Xi_W \circ R_{\Lambda(R)}) \)

is a well-defined effective divisor on \( R_{\Lambda(R)} \) such that

\[ \frac{\text{mult}_o}{\deg} D_{\Lambda(R)} > \frac{8}{d_1 d_2} \].

This contradicts Proposition 2.3, which is now being used in its full strength. □

Thus, \( \text{Sec}(\Lambda(R) \subset E_R) \) is a hyperplane section of \( E_R \) and \( \Lambda(R) \) is cut out on it by a hypersurface of degree \( d_{\Lambda(R)} \geq 2 \). Let \( Z \) be the uniquely determined hyperplane section of \( R \) such that \( o \in Z \) and

\( E_Z = Z^+ \cap E_R = (Z^+ \circ E_R) = \text{Sec}(\Lambda(R) \subset E_R) \).

Applying Theorem 3.4 and arguing as in § 3.6, we see that the effective divisor

\( D_Z = D|Z = D_R|Z = (D_R \circ Z) \)
satisfies
\[ D_Z^+ \sim nH_Z - \nu_Z E_Z, \]
where \( H_Z \) is the class of a hyperplane section of \( Z \subset \mathbb{P}^M \) and \( \nu_Z \) satisfies the inequality
\[ \nu_Z \geq \nu_R + \frac{1}{3}(2\mu_R - \nu_R) = \frac{2}{3}(\nu_R + \mu_R) > \frac{14}{9}n > \frac{3}{2}n, \]
(we recall that \( \mu_R = \text{mult}_{\mathcal{M}(R)} D_R > n \)). This contradicts Proposition 2.3, which is again being used in its full strength, and completes the exclusion of the bi-quadratic case.

Theorem 0.3 is completely proved.

3.10. A remark on hypersurfaces of index 1. We conclude with a discussion of how Theorem 3.1 simplifies the proofs of a number of known results. The divisorial canonicity of Zariski-general non-singular hypersurfaces \( F \subset \mathbb{P}^{M+1} \) of degree \( M + 1 \) was proved in [10] for \( M \geq 5 \) (the symbol \( F \) is available after the proof of Theorem 0.3 has been completed). Moreover, it was shown, for example in [2], that the codimension of the set of hypersurfaces violating the conditions in general position (the regularity conditions) at one or more non-singular points, is bounded below by a quadratic function of \( M \) (this function grows like \( M^2/2 \)). However, every non-trivial family of hypersurfaces contains singular hypersurfaces. Therefore, a study of hypersurfaces \( F \subset \mathbb{P}^{M+1} \) of degree \( M + 1 \) with quadratic singularities was initiated in [7] by considering hypersurfaces with non-degenerate quadratic points for \( M \geq 8 \). The main result of [7] is the divisorial canonicity of \( F \) (that is, the canonicity of every pair \( (F, (1/n)D) \), where \( D \sim nH_F \) is an effective divisor and \( H_F \) is the class of a hyperplane section of \( F \)) provided that all the non-singular points of \( F \) satisfy the regularity conditions of [10] and every singular point \( o \in F \) is a non-degenerate quadratic singularity and satisfies the following conditions. Let \( (z_1, \ldots, z_{M+1}) \) be a system of affine coordinates with origin at \( o \) and let
\[ f = q_2 + \cdots + q_{M+1} = 0 \]
be the equation of the hypersurface \( F \) with respect to \( z_* \), decomposed into components homogeneous in \( z_* \). It is assumed that
- the sequence \( q_2, \ldots, q_{M+1} \) is regular;
- for every \( k \in \{2, 3, 4, 5\} \) and every linear subspace \( P \) of codimension 2 through \( o \), the set
\[ F \cap P \cap \{q_2 = 0\} \cap \cdots \cap \{q_k = 0\} \]
of codimension \( k + 1 \) with respect to \( F \) is irreducible and has multiplicity \((k+1)!\) at \( o \).

It is very difficult to estimate the codimension of the set of hypersurfaces violating these strong and complicated regularity conditions. Essentially, the divisorial canonicity was proved in [7] for hypersurfaces \( F \) with a unique singular point in general position. This enables us to use this result only for Fano fibre spaces \( V/\mathbb{P}^1 \) over a one-dimensional base. Note that the proof in [7] is technically very difficult. This applies both to §2 (‘The local analysis of a divisor at the quadratic point’) and §3 (‘Exclusion of a maximal singularity’).
In particular, the main theorem of [7] could not be applied directly to the study of the birational geometry of Fano hypersurfaces of index 2 in $\mathbb{P}^{M+2}$. This requires the divisorial canonicity of every hyperplane section, but they form an $(M + 2)$-dimensional family and the quadratic singularities of the section can degenerate considerably. Therefore, the arguments in [7] were modified and the conditions in general position were relaxed in [28], §§ 2.3–2.5, in such a way that they hold for every hyperplane section of a general hypersurface of index 2. However, the proof of the divisorial canonicity in [28] was based on the approach used in [7] and was still very complicated.

The next step in simplifying the proof of the divisorial canonicity was made in [2]. It was required from the equation $f = q_2 + \cdots + q_{M+1}$ that the rank of the quadratic form $q_2$ is at least 8, the divisor \{$q_3|\{q_2=0\} = 0$\} on the quadric $\{q_3 = 0\}$ is not the sum of three (not necessarily distinct) hyperplane sections in the same pencil and, finally, the sequence $q_2|\Pi, \ldots, q_{M-c}|\Pi$ is regular for every subspace $\Pi \subset \mathbb{C}^{M+1}_{z_1, \ldots, z_{M+1}}$ of codimension $c \in \{0, 1, 2\}$; see [2], § 3.3. However, a complicated technical argument was still necessary in order to exclude a maximal singularity whose centre is contained in the set of quadratic points; see [2], § 4.3. The same argument was used in [6], where every hyperplane section must be divisorially canonical in order to describe the birational geometry of Fano hypersurfaces of index 2 with quadratic singularities.

Theorem 3.1, which is stated in § 3.2 and proved in § 5, makes the exclusion of a maximal singularity in the quadratic case an easy exercise. It suffices to assume that the equation $q_2 + \cdots + q_{M+1} = 0$ of the hypersurface satisfies the following two conditions in general position: $\text{rk} q_2 \geq 7$ and the sequence $q_2|\Pi, \ldots, q_k|\Pi$ is regular when $k = \lceil(3M - 1)/4\rceil$ for every hyperplane $\Pi \subset \mathbb{C}^{M+1}_{z_1, \ldots, z_{M+1}}$. Repeating the proof of Proposition 3.2 (with simplifications), we exclude the maximal singularity whose centre is contained in the set of quadratic singular points of the hypersurface $F$.

§ 4. Projective geometry

In this section we prove Theorem 3.3. In § 4.1 we recall the classification of secant varieties in a projective space for closed subsets of codimension 2. In § 4.2 we study the intersection of an irreducible subvariety of a quadric and a general linear subspace of maximal dimension of that quadric. This is used in § 4.3 to classify the secant varieties on a quadric for irreducible subvarieties of codimension 2. We complete the proof of Theorem 3.3 in § 4.4.

4.1. The secant variety in the projective space. We use the notation introduced in § 3.5 for the statement of Theorem 3.3. Let $X \subset Q = Q_1 \cap Q_2 \subset \mathbb{P}^N$
be an irreducible subvariety of codimension 2 on a complete intersection of two quadrics satisfying the conditions (C1) and (C2). We prove Theorem 3.3 in three steps. First, we consider subvarieties of codimension 2 in the projective space (here the classification of secant varieties is trivial). Second, we prove some facts about the intersection of an irreducible subvariety of a quadric and a linear subspace of maximal dimension (this strengthens and sharpens the results of §4 in [6]). Third, we study the secant varieties on a complete intersection of two quadrics.

We begin with subvarieties of the projective space. Third, we study the secant varieties on a complete intersection of two quadrics. We prove Theorem 3.3 in three steps. First, we consider subvarieties of codimension 2 in the projective space (here the classification of secant varieties is trivial). Second, we prove some facts about the intersection of an irreducible subvariety of a quadric and a linear subspace of maximal dimension (this strengthens and sharpens the results of §4 in [6]). Third, we study the secant varieties on a complete intersection of two quadrics.

Let $Z \subset \mathbb{P}^m$ be an irreducible closed set of codimension 2. We put

\[ \text{Sec}(Z) = \bigcup_{(p, q) \in Z \times Z} [p, q] \]

(recall that $[p, p] = \emptyset$ by definition; see § 3.5). When considering the obvious case of an irreducible curve in $\mathbb{P}^3$, we easily see that there are three options:

(1.1) $Z = \text{Sec}(Z)$ is a linear subspace of codimension 2 in $\mathbb{P}^m$;

(1.2) $\text{Sec}(Z)$ is a hyperplane in $\mathbb{P}^m$ and $Z$ is a hypersurface of degree $d_Z \geq 2$ in this hyperplane;

(1.3) $\text{Sec}(Z) = \mathbb{P}^m$.

Moreover, if $\Xi \subset \mathbb{P}^m$ is a closed set of codimension at least 4 in the case (1.1), of codimension at least 3 in the case (1.2), and of codimension at least 2 in the case (1.3), then the line $[p, q]$ is disjoint from $\Xi$ for a general pair of points $(p, q) \in Z \times Z$ and, therefore, $\text{Sec}(Z)$ is the closure of the union of all the lines $[p, q]$ such that $(p, q) \in Z \times Z$ and $[p, q] \cap \Xi = \emptyset$.

We now assume that $Z \subset \mathbb{P}^m$ is a reducible subset:

\[ Z = \bigcup_{i \in I} Z_i, \]

where the $Z_i \subset \mathbb{P}^m$ are irreducible subvarieties of codimension 2, and $Z_i \neq Z_j$ for $i \neq j$ and $|I| \geq 2$. We easily see that there are the following options for the secant variety $\text{Sec}(Z)$ (which is swept out by the lines $[p, q]$, where $p, q$ can lie on the same or different components of $Z$):

(2.1) $\text{Sec}(Z)$ is a hyperplane in $\mathbb{P}^m$ containing all the subvarieties $Z_i$;

(2.2) $\text{Sec}(Z)$ is a union of several (at least 2) hyperplanes in $\mathbb{P}^m$;

(2.3) $\text{Sec}(Z) = \mathbb{P}^m$.

It is easy to make the option (2.2) more precise, but we will not need that here. Note that the simplest example of the case (2.2) is given by three lines in $\mathbb{P}^3$ that pass through a point but do not lie in a plane. Then the secant variety is the union of three planes. Note also that if $\Xi \subset \mathbb{P}^m$ is a closed set of codimension at least 3 in the cases (2.1) and (2.2) and of codimension at least 2 in the case (2.3), then $\text{Sec}(Z)$ is the closure of the union of all the lines $[p, q]$ such that $(p, q) \in Z \times Z$ and $[p, q] \cap \Xi = \emptyset$.

4.2. Intersections of subvarieties on a quadric. Let $G \subset \mathbb{P}^m$ be an irreducible quadric. Its vertex subspace $\text{Sing} G \subset \mathbb{P}^m$ is of dimension $m - \text{rk} G$, where $\text{rk} G = \text{rk} g$ is the rank of the quadratic form $g$ defining this quadric. If $\text{rk} G = m + 1$, then the dimension of $\text{Sing} G = \emptyset$ is $(-1)$. We write $G^{\text{sm}}$ for the set $G \setminus \text{Sing} G$ of non-singular points. Every point of the quadric lies on at least one linear subspace
$P \subset G$ of maximal dimension $m - \lceil (1/2) \text{rk } G \rceil$. Let $\mathcal{L}$ be the family of all linear subspaces $P \subset G$ of maximal dimension.

If $\text{rk } G \in 2\mathbb{Z}$, then $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ consists of two connected components each of which is a non-singular irreducible projective variety. If $\text{rk } G \notin 2\mathbb{Z}$, then $\mathcal{L}$ is a non-singular irreducible projective variety.

Let $Y \subset G$ be an irreducible subvariety not contained entirely in $\text{Sing } G$, and let $P \in \mathcal{L}$ be a linear subspace in general position (obviously, $P \supset \text{Sing } G$). The following assertion strengthens and sharpens Proposition 4.1 in [6].

**Proposition 4.1.** (i) The closed set

$$Y \cap (P \setminus \text{Sing } G)$$

is either empty or such that each of its irreducible components is of codimension $\text{codim}(Y \subset G)$ in $P$.

(ii) Assume that

$$\text{codim}(Y \subset G) \leq \left\lfloor \frac{1}{2} \text{rk } G \right\rfloor - 2.$$  

Then $Y \cap P$ is a non-empty closed set and each of its irreducible components is of codimension $\text{codim}(Y \subset G)$ in $P$. Moreover, the algebraic cycle $(Y \cap P)^{\text{sm}}$ of scheme-theoretic intersection of $Y$ and $P$ on the non-singular part $G^{\text{sm}}$ contains every irreducible component with multiplicity 1.

(iii) Assume that

$$\text{codim}(Y \subset G) \leq \left\lfloor \frac{1}{2} \text{rk } G \right\rfloor - 3.$$  

Then the closed set $Y \cap P$ is irreducible and the scheme-theoretic intersection $(Y \cap P)^{\text{sm}}$ is reduced. Moreover,

$$\text{Sing}(Y \cap P \cap G^{\text{sm}}) \subset \text{Sing } Y \cap P \cap G^{\text{sm}}$$

and, therefore, either $Y \cap P$ is non-singular outside $\text{Sing } G$ or the codimension of the singular set of $Y \cap P$ with respect to $P$ is

$$\text{codim}((\text{Sing } Y \cap G^{\text{sm}}) \subset G).$$

**Proof.** (i) Assume that

$$Y \cap (P \setminus \text{Sing } G) \neq \emptyset.$$

Consider the projection

$$\pi_P : \mathbb{P}^m \setminus P \to \mathbb{P}^{\frac{1}{2} \text{rk } G - 1 - 1}$$

from the subspace $P$. The closures of the fibres of $\pi_P$ are subspaces $\Lambda \supset P$ of dimension $\dim P + 1$. For such a subspace we have

$$G \cap \Lambda = P \cup G(P, \Lambda),$$

where $G(P, \Lambda) \in \mathcal{L}$, and if $P \in \mathcal{L}$ is a subspace in general position and $\Lambda = \pi_P^{-1}(s)$ for a point $s \in \mathbb{P}^{(1/2) \text{rk } G - 1}$ in general position, then $G(P, \Lambda)$ is also a subspace in general position. For every fibre $\Lambda$ of $\pi_P$, the intersection

$$P_\Lambda = P \cap G(P, \Lambda)$$
is a hyperplane in $P$ (and in $G(P, \Lambda)$) containing the vertex space $\text{Sing} G$, and it is easy to check that, varying the point $s \in \mathbb{P}^{r(1/2)} \cap G^{-1}$, we obtain the whole linear system of hyperplanes in $P$ containing $\text{Sing} G$ (and the same is true, of course, for $G(P, \Lambda)$). Restricting the morphism $\pi_P$ to the irreducible quasi-projective variety $Y \setminus (Y \cap P)$, we see that, for a point $s$ in general position, the closed affine set

$$Y \cap [G(P, \Lambda) \setminus P \Lambda]$$

is either empty or such that each of its irreducible components is of codimension $\text{codim}(Y \subset G)$ in $G(P, \Lambda)$. Since the subspaces $P$ and $G(P, \Lambda)$ are symmetric objects in our construction, we can conclude that, for a general point $s$, each irreducible component of the closed affine set

$$Y \cap [P \setminus P \Lambda]$$

is of codimension $\text{codim}(Y \subset G)$ in $P$ (because this set is non-empty in view of our assumption and the mobility of the hyperplane $P \Lambda \subset P$). But this means precisely that each irreducible component of the set

$$\overline{Y \cap (P \setminus \text{Sing} G)}$$

is of codimension $\text{codim}(Y \subset G)$. This proves part (i).

We now prove (ii). Since

$$\text{codim}(Y \subset \mathbb{P}^m) = \text{codim}(Y \subset G) + 1,$$

each component of the non-empty closed set $Y \cap P$ is of codimension at most $\text{codim}(Y \subset G) + 1$ in $P$. It follows from the hypothesis of part (ii) that

$$\dim P - \text{codim}(Y \subset G) - 1 \geq \dim \text{Sing} G + 1.$$

Hence every irreducible component of $Y \cap P$ neither coincides with $\text{Sing} G$ nor is contained in $\text{Sing} G$. Using part (i) and taking into account that (in the notation of the proof of part (i)) each component of a fibre in general position of the projection $\pi_P|_{Y \setminus (Y \cap P)}$ is reduced at the generic point, we obtain part (ii).

We now prove part (iii). It consists of three assertions: on irreducibility, on the property of being reduced and on singularities. We consider them in turn. To prove the irreducibility of $Y \cap P$, we proceed as in the proof of Proposition 4.1 in [6]. Assume the opposite: for a general subspace $P \in \mathcal{L}$ we have

$$Y \cap P = \bigcup_{i \in I} Y_i(P),$$

where $|I| \geq 2$ and the $Y_i(P)$ are the distinct irreducible components of $Y \cap P$. By part (i), each of them is of codimension $\text{codim}(Y \subset G)$ with respect to $P$. It follows from the hypothesis of part (iii) that

$$\dim Y_i(P) \geq \dim \text{Sing} G + 3.$$

Therefore (in the notation of the proof of part (i)), for a general fibre $\Lambda = \pi_P^{-1}(s)$, the closed set $Y_i(P) \cap P_{\Lambda}$ is irreducible (since the image of $Y_i(P)$ under the projection from the subspace $\text{Sing } G$ is of dimension at least 2) and we have the equality

$$\text{codim}((Y_i(P) \cap P_{\Lambda}) \subset P) = \text{codim}(Y \subset G) + 1.$$ 

Moreover, for a general fibre $\Lambda$ of $\pi_P$, the irreducible closed sets $Y_i(P) \cap P_{\Lambda}$, $i \in I$, are all distinct (because the hyperplane $P_{\Lambda} \subset P$ varies over the linear system of hyperplanes containing $\text{Sing } G$). Therefore, the irreducible components $Y_i(P)$ are identified by their hyperplane sections $Y_i(P) \cap P_{\Lambda}$.

Using the symmetry of $P$ and $G(P, \Lambda)$ in our construction, we see that there is a one-to-one correspondence between the irreducible components of $Y \cap P$ and those of $Y \cap G(P, \Lambda)$: two components correspond to each other if their intersections with $P_{\Lambda}$ are equal. Therefore, we can write

$$Y \cap G(P, \Lambda) = \bigcup_{i \in I} Y_i(G(P, \Lambda)).$$

But then the set

$$Y_i = \bigcup_{s \in U} Y_i(G(P, \pi_P^{-1}(s))),$$

where the union is taken over all the points of a non-empty Zariski-open subset $U \subset \mathbb{P}^{(1/2) \text{rk } G^\perp -1}$, is an irreducible component of $Y$ for every $i \in I$, and we have $Y_i \neq Y_j$ for $i \neq j$, $i, j \in I$. This contradicts the irreducibility of $Y$. Hence the first, set-theoretic, assertion of part (iii) is proved.

We proceed to prove that the scheme-theoretic intersection $(Y \cap P)^{\text{sm}}$ is reduced. This follows since fibres in general position of $\pi_P |_{Y \setminus P}$ are reduced. Indeed, as in the proof of part (i), we deduce that the scheme-theoretic intersection

$$(Y \cap [G(P, \Lambda) \setminus P_{\Lambda}])$$

is reduced. In view of the symmetry of $P$ and $G(P, \Lambda)$ in our construction, it follows that the scheme-theoretic intersection of $Y$ and $P \setminus P_{\Lambda}$ is reduced for a general hyperplane $P_{\Lambda} \subset P$ containing the subspace $\text{Sing } G$. This in its turn yields that $(Y \cap P)^{\text{sm}}$ is reduced.

We now prove the last assertion of part (iii), about the singularities of the quasi-projective variety $Y \cap P \cap G^{\text{sm}}$. By Bertini’s theorem, for a point $s \in \mathbb{P}^{(1/2) \text{rk } G^\perp -1}$ in general position, we have

$$\text{Sing}[Y \cap (G(P, \Lambda) \setminus P_{\Lambda})] = (\text{Sing } Y) \cap (G(P, \Lambda) \setminus P_{\Lambda}),$$

where $\Lambda = \pi_P^{-1}(s)$. Using the symmetry of our construction, we have

$$\text{Sing}[Y \cap (P \setminus P_{\Lambda})] = (\text{Sing } Y) \cap (P \setminus P_{\Lambda})$$

for a general hyperplane $P_{\Lambda} \supset \text{Sing } G$ in $P$. In view of part (i), the last assertion of part (iii) follows. □
4.3. Secant varieties on a quadric. Let $G \subset \mathbb{P}^m$ be a quadric of rank $\text{rk} \ G \geq 8$. Given any distinct points $p, q \in G$, we write $[p, q]_G$ for the line $[p, q]$ if it lies entirely in $G$, and put $[p, q]_G = \emptyset$ otherwise. As usual, $[p, p]_G = \emptyset$.

For any closed subset $Y \subset G$ we put

$$\text{Sec}(Y \subset G) = \bigcup_{(p, q) \in Y \times Y} [p, q]_G.$$

We shall take irreducible subvarieties of codimension 2 for $Y$. Let

$$\text{Sec}^*(Y \subset G)$$

be an irreducible component of $\text{Sec}(Y \subset G)$ with the following property. For a general linear subspace $P \in \mathcal{L}$ of maximal dimension, $\text{Sec}^*(Y \subset G)$ contains an irreducible component of the secant variety $\text{Sec}(Y \cap P)$ (in the sense of § 4.1).

**Proposition 4.2.** One of the following three options holds:

1. $Y = \text{Sec}(Y \cap G)$ is a section of $G$ by a linear subspace of codimension 2 in $\mathbb{P}^m$;
2. $\text{Sec}(Y \cap G)$ is a hyperplane section of $G$ (a quadric of rank at least 6) and $Y$ is cut out on this section by a hypersurface of degree $d_Y \geq 2$;
3. $\text{Sec}(Y \cap G) = G$.

**Proof.** Let $P \subset G$ be a general linear subspace of maximal dimension $m - \lceil(1/2) \text{rk} \ G \rceil$. By the assumption on the rank of the quadric $G$, part (ii) of Proposition 4.1 applies: every component of $Y \cap P$ is of codimension 2 in $P$, and the scheme-theoretic section of $Y$ and $P$ is reduced at a generic point of each of these components.

Assume that $Y \cap P$ is a linear subspace of codimension 2. Let $\Pi \subset P$ be a general two-dimensional plane. In particular, $\Pi \cap \text{Sing} \ G = \emptyset$. Then $Y$ and $\Pi$ intersect each other transversally at a single point, which is non-singular on $G$. Consider a general 7-dimensional linear subspace $R \subset \mathbb{P}^m$ containing the plane $\Pi$. It is obvious that $G_R = G \cap R$ is a non-singular 6-dimensional quadric. Hence the numerical Chow group of the classes of cycles of codimension 2 is

$$A^2G_R = \mathbb{Z}H_R^2$$

(where $H_R$ is the class of a hyperplane section of the quadric $G_R$), and $Y_R = Y \cap R$ is an irreducible subvariety of codimension 2 in $G_R$ and

$$Y_R \sim m_Y H_R^2$$

for some $m_Y \geq 1$. In fact,

$$1 = (Y_R \cdot \Pi)_{G_R} = m_Y.$$

Hence $\deg Y_R = \deg Y = 2$ and $Y$ is a quadratic hypersurface in its linear span $\langle Y \rangle \cong \mathbb{P}^{m-2}$. Therefore,

$$Y = G \cap \langle Y \rangle$$

is the section of $G$ by a linear subspace of codimension 2 in $\mathbb{P}^m$. 


We now assume that Sec\((Y \cap P)\) is a hyperplane in \(P\). If Sec\((Y \subset G) \neq G\), then Sec\(^*\)(\(Y \subset G\)) is a prime divisor on \(G\) which is cut out on the factorial quadric \(G\) by a hypersurface of degree \(s_Y \geq 1\) in \(\mathbb{P}^m\). The irreducible subvariety Sec\(^*\)(\(Y \subset G\)) of codimension 1 satisfies the hypotheses of part (iii) of Proposition 4.1 and, therefore,

\[
\text{Sec}^*(Y \subset G) \cap P
\]
is an irreducible hypersurface in \(P\) containing the hyperplane Sec\((Y \cap P)\). Therefore,

\[
\text{Sec}^*(Y \subset G) \cap P = \text{Sec}(Y \cap P)
\]

and \(s_Y = 1\). Hence Sec\((Y \subset G) = \text{Sec}^*(Y \subset G)\) is a hyperplane section of the quadric \(G\). Since \(\text{rk}G \geq 8\), this hyperplane section is factorial. Hence \(Y\) is cut out on it by a hypersurface of degree \(d_Y \geq 1\). However, the case \(d_Y = 1\) is impossible: in that case Sec\((Y \subset G) = Y\). Therefore, \(d_Y \geq 2\).

We now assume that Sec\((Y \cap P)\) is the union of at least two hyperplanes. Arguing as above, we see that if Sec\((Y \subset G) \neq G\), then the closed set

\[
\text{Sec}^*(Y \subset G) \cap P
\]
is one of these hyperplanes. Since \(Y\) cannot lie in two \textit{distinct} hyperplanes in \(\mathbb{P}^m\) (then we would have Sec\(^*\)(\(Y \subset G\)) = \(Y\) again, contrary to assumption), we conclude that

\[
\text{Sec}(Y \subset G) = G.
\]

Finally, if Sec\((Y \cap P) = P\), then we again have Sec\((Y \subset G) = G\). □

\textbf{Remark 4.1.} Since

\[
\text{codim}(\text{Sing} \ G \subset P) = \text{rk} \ G - r \frac{1}{2} \text{rk} \ G^\gamma,
\]
it follows from the assumption on the rank of \(G\) that

\[
\text{codim}(\text{Sing} \ G \subset P) \geq 4.
\]

Hence the subvariety Sec\((Y \subset G)\) is swept out by the secant lines \([p,q]_G\) for pairs \(p,q \in Y\) such that \([p,q]_G \cap \text{Sing} \ G = \emptyset\).

\textbf{4.4. Secant varieties on a complete intersection of two quadrics.} We now prove Theorem 3.3. We use the notation of §3.5. To be definite, we assume that \(\text{rk} \ q_1 \geq 16\), where \(Q_1 = \{q_1 = 0\}\). By part (iii) of Proposition 4.1, for a general subspace \(P \subset Q_1\) of maximal dimension, the intersection \(Q \cap P = Q_2 \cap P\) is an irreducible quadratic hypersurface and codim(Sing\((Q \cap P) \subset P\)) is greater than or equal to

\[
\min\{\text{codim}(\text{Sing} \ Q_1 \subset P), \text{codim}(\text{Sing} \ Q \subset Q_1)\} \geq 8.
\]

Hence \(\text{rk} \ q_2|_P \geq 8\). Let \(X \subset Q\) be an irreducible subvariety of codimension 2. Then codim\((X \subset Q_1) = 3\) and part (iii) of Proposition 4.1 applies. We see that \(X \cap P\) is an irreducible subvariety of codimension 3 in \(P\). It is contained in the quadric
Hence above that this section is a factorial complete intersection of codimension 2 in \( Q \cap P \). Applying Proposition 4.2, we see that one of the following three options holds:

1. \( X \cap P = \text{Sec}((X \cap P) \subset (Q \cap P)) \) is the section of the quadric \( Q \cap P \) by a linear subspace of codimension 2 in \( P \);
2. \( \text{Sec}((X \cap P) \subset (Q \cap P)) \) is a hyperplane section of \( Q \cap P \) and the irreducible subvariety \( X \cap P \) is cut out on this section by a hypersurface of degree \( d_X \geq 2 \) in \( P \);
3. \( \text{Sec}((X \cap P) \subset (Q \cap P)) = Q \cap P \).

Moreover, by Remark 4.1 and the arguments in § 4.1, the secant variety \( \text{Sec}((X \cap P) \subset (Q \cap P)) \) is swept out by the lines \([p,q]_{Q \cap P} \) disjoint from \( \text{Sing} Q_1 \cup \text{Sing} Q \).

In the case (3) we have \( \text{Sing}(X \subset Q) = Q \). Assume that the case (2) holds, where \( \text{Sec}(X \subset Q) \neq Q \). We again introduce the notation \( \text{Sec}^*(X \subset Q) \): this is an irreducible component of \( \text{Sec}(X \subset Q) \) whose intersection with a general subspace \( P \subset Q_1 \) of maximal dimension contains the set \( \text{Sec}((X \cap P) \subset (Q \cap P)) \). It is obvious that \( \text{Sec}^*(X \subset Q) \) is a prime divisor on \( Q \) and, by part (iii) of Proposition 4.1, the scheme-theoretic intersection of this divisor with \( P \) is irreducible, reduced and contains a hyperplane section of the quadric \( Q \cap P \). Hence it is equal to that hyperplane section. Since \( Q \subset \mathbb{P}^N \) is factorial, we have

\[
\text{Sec}^*(X \subset Q) \sim s_X H_Q,
\]

where \( s_X \geq 1 \) and \( H_Q \) is the class of a hyperplane section of \( Q \). Restricting to \( P \), we obtain \( s_X = 1 \). Hence,

\[
\text{Sec}^*(X \subset Q) = \text{Sec}(X \subset Q)
\]

is a hyperplane section of \( Q \), and \( X \) is a prime divisor on that section. However, the singular set of any hyperplane section of the complete intersection \( Q \) is of codimension at least 5 with respect to that section and, since the section itself is an irreducible complete intersection of two quadrics in \( \mathbb{P}^{N-1} \), it is a factorial variety. Therefore \( X \) is cut out on \( \text{Sec}(X \subset Q) \) by a hypersurface of degree \( d_X \geq 2 \) in \( \mathbb{P}^{N-1} \).

It remains to consider the case (1).

Let \( \Pi \subset [Q \cap P] \) be a two-dimensional plane in general position, and let \( R \) be the section of \( Q \subset \mathbb{P}^N \) by a general 7-dimensional linear subspace containing \( \Pi \). Then \( R \subset \mathbb{P}^7 \) is a non-singular complete intersection of two quadrics. Hence the numerical Chow group of classes of cycles of codimension 2 is

\[
A^2 R = \mathbb{Z} H_R^2
\]

and \( X_R = X \cap R = (X \circ R)_Q \sim d_{X,R} H_R^2 \) for \( d_{X,R} \geq 1 \). In particular, we have

\[
\deg X = \deg X_R = 4d_{X,R}.
\]

However, \( (X_R \cdot \Pi) = d_{X,R} = 1 \). Hence \( \deg X = 4 \).

Since the variety \( X \) is of dimension \( N-4 \), it is contained in a hyperplane in \( \mathbb{P}^N \). Therefore, \( X \) is a prime divisor on a hyperplane section of \( Q \). It was mentioned above that this section is a factorial complete intersection of codimension 2 in \( \mathbb{P}^{N-1} \). Hence \( X \) is the section of \( Q \) by a linear subspace of codimension 2.
It follows from the remarks above that $\operatorname{Sec}(X \subset Q)$ is swept out by the lines $[p, q]_Q$ which are disjoint from $\operatorname{Sing} Q$.
This completes the proof of Theorem 3.3.

§ 5. Local facts

In this section we prove the local assertions used in § 3 in order to prove the divisorial canonicity of complete intersections: Theorem 3.1 (in §§ 5.1–5.3), Theorem 3.2 (in § 5.4) and Theorem 3.4 (in § 5.5).

5.1. The oriented graph of the singularity $\mathcal{E}$. We begin the proof of Theorem 3.1. We use the notation of § 3.2: the map $\varphi_X : X^+ \to X$ is the blow-up of the isolated singularity $o \in X$, the exceptional divisor $Q = \varphi_X^{-1}(o)$ satisfies the condition $(G)$ and the pair $(X, (1/n)D)$ is not log canonical, but it is canonical outside the point $o$. This pair has a log maximal singularity $E$, which cannot coincide with the exceptional divisor $Q$ because $\nu_D \leq 2n$. By assumption, the centre $W$ of $E$ on $X^+$ is a prime divisor on $Q$. We have

$$\operatorname{ord}_E D > n \cdot (a_E + 1),$$

where $a_E = a(\mathcal{E}, X)$ is the discrepancy of $\mathcal{E}$ with respect to $X$. The first (and main) assertion of Theorem 3.1 says that the subvariety $W \subset Q$ occurs in the scheme-theoretic intersection $(D^+ \circ Q)$ with multiplicity $\mu_W > n$. Consider a resolution of the singularity $E$ in the sense of [5], Ch. 2, § 1. This is a sequence of blow-ups $\varphi_{i,i-1} : X_i \to X_{i-1}$, $i = 1, \ldots, N$, where $X_0 = X$, $X_1 = X^+$ and $\varphi_{1,0} = \varphi_X$, the birational morphism $\varphi_{i,i-1}$ blows up the centre $B_{i-1}$ of $\mathcal{E}$ on $X_{i-1}$ (hence $B_0 = o$ and $B_1 = W$), the exceptional divisor of the blow-up $\varphi_{i,i-1}$ is denoted by $E_{i}$ (no danger of confusion with the notation of § 1), so that $E_1 = Q$ and, finally, the exceptional divisor $E_N \subset X_N$ of the last blow-up realizes $\mathcal{E}$ (that is, the discrete valuations $\operatorname{ord}_E$ and $\operatorname{ord}_{E_N}$ coincide). By our assumptions, the varieties $B_1, \ldots, B_{N-1}$ are subvarieties of codimension 2 in $X_1, \ldots, X_{N-1}$ respectively and

$$B_i \not\subset \operatorname{Sing} X_i$$

for $i = 1, \ldots, N - 1$. Hence the exceptional divisor $E_i$ is a locally trivial $\mathbb{P}^1$-bundle over a non-empty Zariski-open subset of $B_{i-1}$ for $i = 2, \ldots, N$. Let $\Gamma = \Gamma_\mathcal{E}$ be the oriented graph of the resolution of the singularity $\mathcal{E}$ (see [5], Ch. 2, § 1). Its vertices are

$$1, \ldots, N,$$

and a pair $i, j$ of vertices is connected by a directed edge (an arrow) $i \to j$ if $i > j$ and

$$B_{i-1} \subset E_{j}^{i-1},$$

where a superscript $a$ stands for the strict transform on $X_a$ and, in particular, $E_{j}^{i-1}$ is the strict transform of the exceptional divisor $E_j \subset X_j$ on $X_{i-1}$. As usual, let $p_{ij}, i \neq j$, be the number of paths from the vertex $i$ to the vertex $j$ in the directed
graph $\Gamma$ (hence $p_{ij} = 0$ when $i < j$) and we put $p_{ii} = 1$ for all $i = 1, \ldots, N$. To simplify the notation, we write $p_i$ instead of $p_{Ni}$. Put

$$\mu_i = \text{mult}_{B_{i-1}} D^{i-1}$$

for $i = 2, \ldots, N$ and $\mu_1 = \nu_D$. Then our assumption that $E$ (or $E_N$) is a log maximal singularity of the pair $(X, (1/n)D)$, takes the explicit form of the log Noether–Fano inequality

$$\sum_{i=1}^N p_i \mu_i > n \cdot \left( \sum_{i=1}^N p_i + 1 \right).$$

Let $L_i \subset E_i$ be the fibre of the projection $E_i \to B_{i-1}$ over a general point of $B_{i-1}$. Intersecting the strict transform $D^N$ with the strict transform $L_i^N$, we see that the multiplicities $\mu_i$, $i \geq 2$, satisfy the inequalities

$$\mu_i \geq \sum_{j \to i} \mu_j. \quad (11)$$

We recall that $\mu_1 \leq 2n$ by assumption.

Since the variety $X_1$ is non-singular at a generic point of the subvariety $W = B_1$ of codimension 2, we obtain the inequality

$$\mu_W \geq \sum_{i=1} \mu_i.$$ 

Therefore part (i) of Theorem 3.1 is a corollary of the following fact.

**Proposition 5.1.** We have

$$\sum_{i=1} \mu_i > n.$$ 

We shall prove Proposition 5.1 in two steps. We first reduce it to an assertion of convex geometry, which in its turn will be reduced to a combinatorial assertion about the graph $\Gamma$.

5.2. Proof of Proposition 5.1: some convex geometry. Recall two well-known properties of the graph $\Gamma$. If $i \to j$ and $l$ is a vertex between $i$ and $j$, that is,

$$j < l < i,$$

then $i \to l$. (This follows since the image of $B_{i-1}$ on $X_{l-1}$ is $B_{l-1}$.) Since the centres of the blow-ups $B_1, \ldots, B_{N-1}$ are subvarieties of codimension 2, each of them can be contained in the strict transforms of at most two previous exceptional divisors. This yields the second property: for every vertex $i$, there are at most two outgoing arrows, one of which is $i \to (i-1)$ (when $i \geq 2$).

We now consider a real space $\mathbb{R}^N$ with coordinates $t_1, \ldots, t_N$. Define linear functions

$$\lambda_0^*(t_1, \ldots, t_N) = \sum_{i=1}^N p_i t_i.$$
and, for $i = 1, \ldots, N$,

$$
\lambda_i(t_1, \ldots, t_N) = t_i - \sum_{j \rightarrow i} t_j
$$

(in particular, $\lambda_N = t_N$). Put

$$
\lambda(t_1, \ldots, t_N) = \sum_{i \rightarrow 1} t_i = t_2 + \cdots + t_k,
$$

where $k \in \{2, \ldots, N\}$ is such that $\Gamma$ contains the arrows

$$
2 \rightarrow 1, \ldots, k \rightarrow 1
$$

and either $k = N$, or $(k + 1) \not\rightarrow 1$.

Consider the hyperplane

$$
\Pi^* = \left\{ \lambda^*_0(t_1, \ldots, t_N) = \sum_{i=1}^{N} p_i + 1 \right\}
$$

and define a convex compact set $\Delta^* \subset \Pi^*$ by the inequalities

$$
\lambda_i \geq 0 \quad \text{for} \quad i = 1, \ldots, N \quad \text{and} \quad t_1 \leq 2.
$$

By considering the point $(1/n)(\mu_1, \ldots, \mu_N) \in \mathbb{R}^N$, we see that Proposition 5.1 follows from the inequality

$$
\min_{\Delta^*} \lambda \geq 1.
$$

Furthermore, we put

$$
\lambda_0(t_2, \ldots, t_N) = \sum_{i=2}^{N} p_i t_i
$$

and rewrite the equation of the hyperplane $\Pi^*$ in the form

$$
\lambda_0(t_2, \ldots, t_N) = \sum_{i=2}^{N} p_i + 1 + p_1 (1 - t_1).
$$

We thus conclude that it suffices to consider the worst case $t_1 = 2$, that is, to prove the inequality

$$
\min_{\Delta^* \cap \{t_1=2\}} \lambda \geq 1.
$$

Therefore we consider the real space $\mathbb{R}^{N-1}$ with coordinates $t_2, \ldots, t_N$. Since

$$
p_1 = \sum_{i \rightarrow 1} p_i = p_2 + \cdots + p_k,
$$

we see that the set $\Delta^* \cap \{t_1 = 2\} \subset \mathbb{R}^{N-1}$ is the simplex $\Delta$ in the hyperplane

$$
\Pi = \left\{ \lambda_0(t_2, \ldots, t_N) = \sum_{i=k+1}^{N} p_i + 1 \right\}.$$
(if \( k = N \), then the right-hand side is equal to 1). This simplex is given by the system of linear inequalities

\[
\lambda_i(t_2, \ldots, t_N) \geq 0 \quad \text{for} \quad i = 2, \ldots, N.
\]  
(12)

Therefore, to prove Proposition 5.1, it suffices to show that

\[
\min_{\Delta} \lambda \geq 1.
\]  
(13)

We easily see that if \( k = N \), then

\[
p_2 = \cdots = p_k = 1
\]

and \( \lambda_0 = \lambda \), so that the regarded inequality (13) holds trivially. Therefore we assume that \( N \geq k + 1 \).

**Lemma 5.1.** We have

\[
\min_{\Delta \cap \{t_N = 0\}} \lambda \geq 1.
\]

**Proof.** Since (13) holds in the trivial case \( k = N \), we can use induction on \( N \). For the uppermost vertex \( N \), there are two options:

1. There is only one outgoing arrow \( N \to (N - 1) \);
2. There are two outgoing arrows: \( N \to (N - 1) \) and \( N \to l \) for some \( l \leq N - 2 \).

We consider the second option since the first is simpler.

Let \( \Gamma_1 \) and \( \Gamma_2 \) be the subgraphs of \( \Gamma \) with vertices \( 1, \ldots, N - 1 \) and \( 1, \ldots, l \) respectively. Since the first step of every path from \( N \) to \( i < N \) passes through either \( (N - 1) \) or \( l \) (for \( i \leq l \)), we have \( p_i = p_{N-1,i} + p_{l,i} \). For every point

\[
(b_2, \ldots, b_{N-1}, 0) \in \Delta \cap \{t_N = 0\},
\]

taking into account that \( p_N = p_{N,N} = 1 \), we have

\[
\lambda_0(b_2, \ldots, b_{N-1}, 0) = \sum_{i=2}^{N-1} p_{N-1,i} b_i + \sum_{i=2}^{l} p_{l,i} b_i = \sum_{i=k+1}^{N-1} p_{N-1,i} + \sum_{i=k+1}^{l} p_{l,i} + p_N + 1
\]

\[
= \left( \sum_{i=k+1}^{N-1} p_{N-1,i} + 1 \right) + \left( \sum_{i=k+1}^{l} p_{l,i} + 1 \right)
\]

(if \( l \leq k \), then the sum over \( k + 1 \leq i \leq l \) is equal to zero). Therefore at least one of the following inequalities holds:

\[
\sum_{i=2}^{N-1} p_{N-1,i} b_i \geq \sum_{i=k+1}^{N-1} p_{N-1,i} + 1,
\]

\[
\sum_{i=2}^{l} p_{l,i} b_i \geq \sum_{i=k+1}^{l} p_{l,i} + 1.
\]

Multiplying the vector \( (b_2, \ldots, b_{N-1}) \) by \( 1/(1 + \varepsilon) \) for some \( \varepsilon > 0 \), we can ensure that this inequality is an equality and apply the induction hypothesis. If option (1) holds, the proof is obvious. \( \square \)
The minimum of the function $\lambda$ on the simplex $\Delta$ is attained at one of the vertices which are obtained by replacing the inequality sign by the equality sign in all but one of the inequalities (12). By the lemma, it suffices to consider the vertex given by the equations $\lambda_i = 0$ for $i = 2, \ldots, N - 1$. Its coordinates $(a_2, \ldots, a_N)$ can easily be computed going ‘from top to bottom’. For example, $\lambda_{N-1}(t_{N-1}, t_N) = t_{N-1} - t_N$ and, therefore, $a_{N-1} = a_N$. More generally,

$$a_i = p_i a_N \quad \text{for} \quad i = 2, \ldots, N - 1,$$

where $a_N$ is computed from the relation

$$\left( \sum_{i=2}^{N} p_i^2 \right) a_N = \sum_{i=k+1}^{N} p_i + 1.$$

Therefore Proposition 5.1 is a corollary of the following combinatorial fact.

**Proposition 5.2.** We have

$$(p_2 + \cdots + p_k) \left( \sum_{i=k+1}^{N} p_i + 1 \right) \geq \sum_{i=2}^{N} p_i^2.$$  \hspace{1cm} (14)

5.3. **Proof of Proposition 5.2.** The proof is by induction on the number of vertices of $\Gamma$. We shall reduce the proposition to the corresponding assertion for the subgraph with vertices $a, \ldots, N$. For example, consider the case $k = 2$. Here (14) takes the form

$$p_2 \left( \sum_{i=3}^{N} p_i + 1 \right) \geq \sum_{i=2}^{N} p_i^2.$$

Assume that the vertices $3, \ldots, l$ are connected to the vertex 2 by arrows, but $(l + 1) \not\rightarrow 2$. Then

$$p_2 = p_3 + \cdots + p_l$$

and, by the induction hypothesis,

$$(p_3 + \cdots + p_l) \left( \sum_{i=l+1}^{N} p_i + 1 \right) \geq \sum_{i=3}^{N} p_i^2.$$  \hspace{1cm} (14)

Adding $p_2^2$ to both parts of this inequality completes the proof in the case $k = 2$.

Assume that $k \geq 3$. Furthermore, assume that

$$(k + 1) \not\rightarrow (k - 1).$$

Then there are no arrows from vertices $i \geq k + 1$ to vertices $j \leq k - 1$. Hence $p_2 = \cdots = p_k$. Assume that there are arrows

$$(k + 1) \rightarrow k, \quad \ldots, \quad l \rightarrow k,$$

but $(l + 1) \not\rightarrow k$. Hence $p_2 = \cdots = p_k = p_{k+1} + \cdots + p_l$. Here (14) takes the form

$$(k - 1)p_2 \left( p_2 + \sum_{i=l+1}^{N} p_i + 1 \right) \geq (k - 1)p_2^2 + \sum_{i=k+1}^{N} p_i^2.$$
This inequality obviously follows from the estimate
\[
(p_{k+1} + \cdots + p_l) \left( \sum_{i=l+1}^{N} p_i + 1 \right) \geq \sum_{i=k+1}^{N} p_i^2,
\]
which holds by the induction hypothesis.

Finally, we consider the case when
\[
(k + 1) \rightarrow (k - 1).
\]
Assume that \( \Gamma \) contains the arrows
\[
k \rightarrow (k - 1), \quad (k + 1) \rightarrow (k - 1), \quad \ldots, \quad l \rightarrow (k - 1),
\]
but \((l + 1) \not\rightarrow (k - 1)\). Then
\[
p_2 = \cdots = p_{k-1} = p_k + \cdots + p_l
\]
and the left-hand side of (14) can be transformed in the following way:
\[
((k - 3)p_2 + p_{k-1} + p_k) \left( \sum_{i=k+1}^{l} p_i + \sum_{i=l+1}^{N} p_i + 1 \right)
= ((k - 3)p_2 + p_k) \left( \sum_{i=k+1}^{N} p_i + 1 \right) + p_{k-1} \left( \sum_{i=k+1}^{l} p_i \right) + p_{k-1} \left( \sum_{i=l+1}^{N} p_i + 1 \right).
\]
Since we have
\[
p_{k-1} \left( \sum_{i=l+1}^{N} p_i + 1 \right) \geq \sum_{i=k}^{N} p_i^2
\]
by the induction hypothesis and
\[
\sum_{i=k+1}^{N} p_i + 1 \geq p_{k-1}
\]
by Lemma 2.7 in Ch. 2 of [5], the left-hand side of (14) is bounded below by the expression
\[
((k - 3)p_2 + p_k)p_{k-1} + (p_{k-1} - p_k)p_{k-1} + \sum_{i=k}^{N} p_i^2
= (p_2 + \cdots + p_{k-1})p_{k-1} + \sum_{i=k}^{N} p_i^2 = \sum_{i=2}^{N} p_i^2,
\]
as required.

This completes the proof of Proposition 5.2 and, therefore, of Proposition 5.1 and part (i) of Theorem 3.1.
Remark 5.1. Assume in addition that \( X^+ \) and \( Q \) are factorial and
\[
\text{Pic } Q = \mathbb{Z}H_Q,
\]
where \( H_Q \) is the intersection class \(-(Q \circ Q)\) regarded as a divisor on \( Q \). Then part (i) of Theorem 3.1 implies that \( W \sim H_Q \) is the ‘hyperplane section’ of the exceptional divisor \( Q \) (because \( \nu_D \leq 2n \)).

Proof of part (ii) of Theorem 3.1. The proof repeats verbatim that of Proposition 9 in [10], §3, or Proposition 2.4 in [5], Ch. 7. In the notation of § 5.1, part (ii) of Theorem 3.1 takes the form of the inequality
\[
\mu_1 + \mu_2 > 2n.
\]
We will not repeat these arguments here. Theorem 3.1 is completely proved. □

5.4. Proof of Theorem 3.2. Recall that, under the hypotheses of Theorem 3.2, the centre \( \mathcal{W} \subset \mathcal{Q} \) of the log maximal singularity \( \mathcal{E} \) on \( X^+ \) is an irreducible subvariety of codimension at least 2 (with respect to \( \mathcal{Q} \)) which is not contained in \( \text{Sing } \mathcal{Q} \), that is, \( \mathcal{Q} \) and \( X^+ \) are non-singular at a generic point of \( \mathcal{W} \). The other hypotheses are the same as in Theorem 3.1.

We again consider a resolution of the singularity \( \mathcal{E} \). It is now convenient to change the notation introduced in § 5.1. We put \( X_0 = X^+ \) and \( X_{i-1} = X_i \), that is, we shift the enumeration of the blow-ups \( \varphi_i, i_1 \) and the varieties \( X_i \) by one downwards. In particular, \( E_0 = Q \) and \( B_0 = \mathcal{W} \). Therefore, the resolution is the sequence of blow-ups
\[
X_{i-1} = X \leftarrow X_0 = X^+ \leftarrow X_1 \leftarrow \cdots \leftarrow X_N.
\]
Suppose that the blow-ups \( \varphi_{i,i-1}: X_i \rightarrow X_{i-1} \) correspond to the centres \( B_{i-1} \subset X_{i-1} \) of codimension at least 3 (with respect to \( X_{i-1} \)) for \( i = 0, 1, \ldots, L \) and we have \( \text{codim}(B_{i-1} \subset X_{i-1}) = 2 \) for \( i \geq L + 1 \). We again consider the graph \( \Gamma \) of the resolution of the singularity \( \mathcal{E} \). Its set of vertices is now
\[
0, \ 1, \ \ldots, \ N.
\]
We write \( p_{ij} \) and \( p_i = p_{Ni} \) for the numbers of paths in \( \Gamma \) introduced in § 5.1. The only difference is that \( i \) and \( j \) may be equal to 0. Define a number \( k \geq 1 \) by the condition
\[
B_{i-1} \subset E_0^{i-1}
\]
for \( i = 1, \ldots, k \) (that is, \( i \rightarrow 0 \)). Hence
\[
p_0 = p_1 + \cdots + p_k.
\]
Putting \( \mu_0 = \nu_D \) and \( \mu_i = \text{mult}_{B_{i-1}} D^{i-1} \), we obtain the following well-known explicit form of the log Noether–Fano inequality:
\[
p_0\mu_0 + \sum_{i=1}^{N} p_i \mu_i > \left( p_0 + \sum_{i=1}^{N} p_i \delta_i + 1 \right) n,
\]
where \( \delta_i = \text{codim}(B_{i-1} \subset X_{i-1}) - 1 \).
Lemma 5.2. If $\mu_1 \leq n$, then $k \geq L + 1$.

Proof. Assume the opposite: $k \leq L$. By the log Noether–Fano inequality,

$$
\sum_{i=1}^{k} p_i \mu_i + \sum_{i=k+1}^{L} p_i \mu_i + \sum_{i=L+1}^{N} p_i \mu_i
\geq \left( \sum_{i=1}^{k} p_i (\delta_i + 1 - \frac{\mu_0}{n}) + \sum_{i=k+1}^{L} p_i \delta_i + \sum_{i=L+1}^{N} p_i + 1 \right) n.
$$

Since $\mu_1 \leq n$, certainly $\mu_i \leq n$ for all $i \geq 1$. Hence the inequality above is impossible: when $i \in \{1, \ldots, k\}$ we have $\delta_i + 1 - \frac{\mu_0}{n} \geq 3 - \frac{\mu_0}{n} \geq 1$, and when $i \in \{k+1, \ldots, L\}$ we have $\delta_i \geq 2$. This contradiction proves the lemma. \(\square\)

Since the inequality $\mu_1 > n$ is precisely the inequality (1) of Theorem 3.2, we assume from now on that $\mu_1 \leq n$. Hence we have $k \geq L + 1$ by the lemma. The log Noether–Fano inequality can now be rewritten in the form

$$
\sum_{i=1}^{L} p_i \mu_i + \sum_{i=L+1}^{k} p_i \mu_i + \sum_{i=k+1}^{N} p_i \mu_i
\geq \left( \sum_{i=1}^{L} p_i (\delta_i + 1 - \frac{\mu_0}{n}) + \left( 2 - \frac{\mu_0}{n} \right) \sum_{i=L+1}^{k} p_i + \sum_{i=k+1}^{N} p_i + 1 \right) n. \quad (15)
$$

Note that $\delta_{E,i} = \delta_i - 1$ for $i = 1, \ldots, L$ is the discrepancy of the exceptional divisor $E_i \cap E_0 = E_i \cap Q^i$ of the blow-up

$$
\varphi_{E,i}^E: E_0^i \to E_0^{i-1}
$$

of the subvariety $B_{i-1} \subset E_0^{i-1}$ with respect to $E_0^{i-1}$:

$$
\delta_{E,i} = \text{codim}(B_{i-1} \subset E_0^{i-1}) - 1.
$$

Lemma 5.3. For every vertex

$$
a \in \{L + 1, \ldots, k\}
$$

there is only one outgoing arrow, $a \to (a-1)$. In particular, the subgraph of $\Gamma$ with vertices $L + 1, \ldots, k$ is a chain if $k \geq L + 2$.

Proof. By construction, we have $\text{codim}(B_{a-1} \subset X_{a-1}) = 2$ for these values of $a$. Since $B_{a-1} \subset E_{a-1}$ (for every $a$) and $B_{a-1} \subset E_0^{a-1}$ (because $a \leq k$), there is only one option:

$$
B_{a-1} = E_{a-1} \cap E_0^{a-1}.
$$

Hence $\varphi_{a,a-1}$ blows up a divisor on $E_0^{a-1}$ and, therefore, the $\varphi_{E,a-1}$ are identity maps. It follows that

$$
E_0^L \cong E_0^{L+1} \cong \cdots \cong E_0^k.
$$
All the previous blow-ups $\varphi_{i,i-1}^E$ with $i \leq L$ blow-up subvarieties of codimension at least 2 on $E_{0,i}$. Hence they are non-trivial and

$$E_L \cap E_0^L \neq (E_i \cap E_0^i)^L$$

for $i \leq L - 1$. Hence, for $a \in \{L + 1, \ldots, k\}$, we have

$$B_{a-1} = E_{a-1} \cap E_0^{a-1} \neq (E_i \cap E_0^i)^a$$

for $i \leq L - 1$. Since $B_{a-1} \subset E_0^{a-1}$, it follows that

$$B_{a-1} \not\subset E_i^{a-1},$$

that is, $a \not\rightarrow i$. \(\square\)

For any vertices $i < j < l$, $l \rightarrow i$ implies that $j \rightarrow i$. Hence the lemma yields that, besides the arrow $(k + 1) \rightarrow k$, there can be at most one arrow outgoing from $(k + 1)$, namely, $(k + 1) \rightarrow (k - 1)$. Thus we obtain the following assertion.

**Proposition 5.3.** (i) If $\Gamma$ contains an arrow $b \rightarrow a$, where $a \leq k$ and $b \geq k + 1$, then $a = k - 1$.

(ii) We have

$$p_L = \cdots = p_{k-1} \text{ and } p_{k-1} = p_k + \sum_{k \neq i \rightarrow (k-1)} p_i.$$

(iii) For $i = 1, \ldots, L$ we have

$$p_i = p_L \cdot p_{Li}.$$

**Proof.** Part (i) is obvious by Lemma 5.3. Part (ii) follows from part (i) since every path to a vertex $a \in \{L, \ldots, k - 1\}$ must go through $(k - 1)$. Finally, part (iii) follows since every path from $N$ to $i \in \{1, \ldots, L\}$ goes through $L$ and, therefore, consists of a path from $N$ to $L$ and a path from $L$ to $i$. \(\square\)

We now complete the proof of Theorem 3.2. By hypothesis, $\mu_i \leq n$ for $i \geq 1$. Therefore it follows from (15) that

$$p_L \left( \sum_{i=1}^{L} p_{Li} \mu_i + \sum_{i=L+1}^{k-1} \mu_i \right) + p_k \mu_k > p_L \sum_{i=1}^{L} p_{Li} \left( \delta_{E,i} + 2 - \frac{\mu_0}{n} \right) n.$$

Combining this with the obvious inequality $p_k \leq p_{k-1} = p_L$, we obtain the bound

$$\sum_{i=1}^{L} p_{Li} \mu_i + \sum_{i=L+1}^{k} \mu_i > \sum_{i=1}^{L} p_{Li} \left( \delta_{E,i} + 2 - \frac{\mu_0}{n} \right) n. \quad (16)$$

Putting

$$\varphi_{k,i} = \varphi_{i+1,i} \circ \cdots \circ \varphi_{k,k-1} : X_k \rightarrow X_i$$
for $i \leq k - 1$ and $\varphi_{k,k} = \text{id}_{\mathcal{X}_k}$, we consider the effective divisor

$$\mathcal{D}^k = \varphi_{k,0}^* \mathcal{D}^+ - \sum_{i=1}^{k} \mu_i \varphi_{k,i}^* E_i.$$ 

Since $\mathcal{D}^+$ does not contain $Q = E_0$ as a component, the restriction of $\mathcal{D}^k$ to $E_0^k$ is an effective divisor

$$(\mathcal{D}^k \circ E_0^k) = (\varphi_{k,0}^E)^* \mathcal{D}^+_E - \sum_{i=1}^{k} \mu_i (\varphi_{k,i}^E)^*(E_i \circ E_0^i),$$

where the meaning of the symbols $\varphi_{k,i}^E$ is obvious and $\mathcal{D}^+_E = (\mathcal{D}^+ \circ E_0^k)$ is the restriction of $\mathcal{D}^+$ to $Q$. In particular, the last exceptional divisor $(E_L \circ E_0^L)$ is subtracted from $(\varphi_{k,0}^E)^* \mathcal{D}^+_E$ with multiplicity

$$\sum_{i=1}^{L} p_L \mu_i + \sum_{i=L+1}^{k} \mu_i.$$ 

We now put

$$\nu_{E,i} = \text{mult}_{B_{i-1}} \mathcal{D}^{i-1}_E$$

for $i \in \{1, \ldots, L\}$ and deduce that $(\varphi_{L,0}^E)^* \mathcal{D}^+_E$ contains $(E_L \circ E_0^L)$ with multiplicity

$$\sum_{i=1}^{L} p_L \nu_{E,i} \geq \sum_{i=1}^{L} p_L \mu_i + \sum_{i=L+1}^{k} p_L \mu_i$$

(recall that $\varphi_{a,a-1}^E$ is the identity map for $a \in \{L + 1, \ldots, k\}$ and, therefore, $\varphi_{L,0}^E = \varphi_{k,0}^E$). It follows from (16) that

$$\sum_{i=1}^{L} p_L \nu_{E,i} > \sum_{i=1}^{L} p_L (\delta_{E,i} n + 2n - \mu_0).$$

Since $\delta_{E,i} \geq 1$ and the multiplicities $\nu_{E,i}$ are non-increasing, we conclude that

$$\nu_{E,1} = \text{mult}_W \mathcal{D}^+_E > 3n - \mu_0.$$ 

Since $\mu_0 = \nu_{\mathcal{D}}$, this completes the proof of Theorem 3.2.

5.5. **Proof of Theorem 3.4.** Here $o \in \mathcal{X}$ is the germ of a three-dimensional non-degenerate bi-quadratic singularity. We use the notation of § 3.6: $\mathcal{E} \subset \mathbb{P}^4$ is a non-singular Del Pezzo surface of degree 4, $L \subset \mathcal{E}$ is a line, and $p \neq q$ are distinct points of $L$. Since

$$(L^2)_\mathcal{E} = -1 \quad \text{and} \quad (\mathcal{E} \cdot L)_{\mathcal{X}^+} = -1,$$

we have $\mathcal{N}_{L/\mathcal{X}^+} \cong \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-1)$. Hence, for the blow-up $\varphi_L: \mathcal{X}_L \rightarrow \mathcal{X}^+$ of the line $L$, we see that the exceptional divisor

$$\mathcal{E}_L = \varphi_L^{-1}(L) \cong \mathbb{P}^1 \times \mathbb{P}^1$$
is isomorphic to the direct product of $L$ (the first factor) and the fibre $\mathbb{P}^1$. Let $s_L$ and $f_L$ be the classes of fibres of the projections of $\mathcal{E}_L$ onto the direct factors (where $f_L$ is the class of a fibre of $\pi_L: \mathcal{E}_L \to L$). Then

$$A^1\mathcal{E}_L = \text{Pic} \mathcal{E}_L = \mathbb{Z}s_L \oplus \mathbb{Z}f_L.$$ 

**Lemma 5.4.** The class of the scheme-theoretic intersection $(\mathcal{E}_L \circ \mathcal{E}_L)$ as a divisor on the surface $\mathcal{E}_L$ is $-s_L - f_L$.

**Proof.** Let $\mathcal{E} = \varphi_L^* \mathcal{E} - \mathcal{E}_L$ be the strict transform of $\mathcal{E}$ on $\mathcal{X}_L$, $\mathcal{E} \cong \mathcal{E}$. The surfaces $\mathcal{E}$ and $\mathcal{E}_L$ intersect each other transversally along a non-singular curve $C$ which is a section of the projection $\pi_L$. Obviously,

$$(\mathcal{E}_L \cdot C)_{\mathcal{X}_L} = (C^2)_{\mathcal{E}} = (L^2)_{\mathcal{E}} = -1.$$

It follows from the direct decomposition of the normal sheaf $\mathcal{N}_{\mathcal{L}/\mathcal{X}}$ (or from the numerical equalities $(C^2)_{\mathcal{E}_L} = (\mathcal{E} \cdot C)_{\mathcal{X}_L} = (\varphi_L^* \mathcal{E} \cdot C)_{\mathcal{X}_L} - (\mathcal{E}_L \cdot C)_{\mathcal{X}_L} = (\mathcal{E} \cdot L)_{\mathcal{X}} - (\mathcal{E}_L \cdot C)_{\mathcal{X}_L} = 0$) that $C \sim s_L$ on $\mathcal{E}_L$. We write

$$(\mathcal{E}_L \circ \mathcal{E}_L) \sim -s_L + xf_L$$

for some $x \in \mathbb{Z}$. By the equalities above, $x = (\mathcal{E}_L^2 \cdot \mathcal{E})_{\mathcal{X}_L}$. However,

$$(\mathcal{E}_L^2 \cdot \mathcal{E})_{\mathcal{X}_L} = (C^2)_{\mathcal{E}} = -1. \quad \square$$

We now consider the effective divisor $\mathcal{D}$ on $\mathcal{X}$ and its strict transforms $\mathcal{D}^+ \sim -\nu_D \mathcal{E}$ on $\mathcal{X}^+$ and $\mathcal{D}_L \sim -\nu_D \mathcal{E} - \nu_L \mathcal{E}_L$ on $\mathcal{X}_L$, where

$$\nu_L = \text{mult}_L \mathcal{D}^+.$$ 

Obviously,

$$\mathcal{D}_L|_{\mathcal{E}_L} \sim \nu_D f_L + \nu_L (s_L + f_L) = \nu_L s_L + (\nu_D + \nu_L) f_L.$$ 

On the other hand, the strict transform $\mathcal{D}_L$ has multiplicity $(\mu - \nu_L)$ along the fibres $\varphi_{L}^{-1}(p)$ and $\varphi_{L}^{-1}(q)$ of the projection $\pi_L$. Hence the $1$-cycle $\mathcal{D}_L|_{\mathcal{E}_L}$ contains these fibres with multiplicity at least $\mu - \nu_L$. Therefore we have

$$\nu_D + \nu_L \geq 2(\mu - \nu_L)$$

(since the pseudoeffective cone $A^1_+ \mathcal{E}_L$ of the surface $\mathcal{E}_L \cong \mathbb{P}^1 \times \mathbb{P}^1$ is obviously $\mathbb{Z}_+ s_L \oplus \mathbb{Z}_+ f_L$). It follows that

$$\nu_L \geq \frac{1}{3}(2\mu - \nu_D).$$

The proof of Theorem 3.4 is complete.
Acknowledgements. The author is grateful to the members of Divisions of Algebraic Geometry and Algebra at the Steklov Institute of Mathematics for their interest in this work, and also to colleagues in the Algebraic Geometry research group at the University of Liverpool for general support. The author is also grateful to the referees for their work and a number of useful suggestions.

Bibliography

[1] F. Call and G. Lyubeznik, “A simple proof of Grothendieck’s theorem on the parafactoriality of local rings”, *Commutative algebra: syzygies, multiplicities, and birational algebra* (South Hadley, MA 1992), Contemp. Math., vol. 159, Amer. Math. Soc., Providence, RI 1994, pp. 15–18.

[2] A.V. Pukhlikov, “Birationally rigid Fano fibre spaces. II”, *Izv. Ross. Akad. Nauk Ser Mat.* 79:4 (2015), 175–204; English transl., *Izv. Math.* 79:4 (2015), 809–837.

[3] A.V. Pukhlikov, “Canonical and log canonical thresholds of multiple projective spaces”, *Eur. J. Math.* 7:1 (2021), 135–162; arXiv:1906.11802.

[4] A.V. Pukhlikov, “Birationally rigid finite covers of the projective space”, *Algebra, number theory, and algebraic geometry*, Collection of papers. Dedicated to the memory of Academician Igor Rostislavovich Shafarevich, Trudy Mat. Inst. Steklova, vol. 307, MIAN, Moscow 2019, pp. 254–266; English transl., *Proc. Steklov Inst. Math.* 307 (2019), 232–244.

[5] A. Pukhlikov, *Birationally rigid varieties*, Math. Surveys Monogr., vol. 190, Amer. Math. Soc., Providence, RI 2013.

[6] A.V. Pukhlikov, “Birational geometry of singular Fano hypersurfaces of index two”, *Manuscripta Math.* 161:1-2 (2020), 161–203.

[7] A.V. Pukhlikov, “Birational geometry of singular Fano varieties”, *Multidimensional algebraic geometry*, Collection of papers. Dedicated to the memory of Corresponding Member of RAS Vasily Alekseevich Iskovskikh, Trudy Mat. Inst. Steklova, vol. 264, MAIK “Nauka/Interperiodika”, Moscow 2009, pp. 165–183; English transl., *Proc. Steklov Inst. Math.* 264 (2009), 159–177.

[8] V.G. Sarkisov, “Birational automorphisms of conic bundles”, *Izv. Akad. Nauk SSSR Ser. Mat.* 44:4 (1980), 918–945; English transl., *Math. USSR-Izv.* 17:1 (1981), 177–202.

[9] V.G. Sarkisov, “On conic bundle structures”, *Izv. Akad. Nauk SSSR Ser. Mat.* 46:2 (1982), 371–408; English transl., *Math. USSR-Izv.* 20:2 (1983), 355–390.

[10] A.V. Pukhlikov, “Birational geometry of Fano direct products”, *Izv. Ross. Akad. Nauk Ser. Mat.* 69:6 (2005), 153–186; English transl., *Izv. Math.* 69:6 (2005), 1225–1255.

[11] I.A. Cheltsov and K.A. Shramov, “Log canonical thresholds of smooth Fano threefolds”, *Uspekhi Mat. Nauk* 63:5(383) (2008), 73–180; English transl., *Russian Math. Surveys* 63:5 (2008), 859–958.

[12] I. Cheltsov, Jihun Park, and Joonyeong Won, “Log canonical thresholds of certain Fano hypersurfaces”, *Math. Z.* 276:1-2 (2014), 51–79.

[13] Yu. Prokhorov and C. Shramov, “Jordan property for groups of birational selfmaps”, *Compos. Math.* 150:12 (2014), 2054–2072.

[14] Yu. Prokhorov and C. Shramov, “Jordan property for Cremona groups”, *Amer. J. Math.* 138:2 (2016), 403–418.
[15] J.-P. Serre, “A Minkowski-style bound for the orders of the finite subgroups of the Cremona group of rank 2 over an arbitrary field”, Mosc. Math. J. 9:1 (2009), 183–198.

[16] J.-L. Colliot-Thélène and A. Pirutka, “Cyclic covers that are not stably rational”, Izv. Ross. Akad. Nauk Ser. Mat. 80:4 (2016), 35–48; English transl., Izv. Math. 80:4 (2016), 665–677.

[17] B. Hassett, A. Kresch, and Yu. Tschinkel, “Stable rationality and conic bundles”, Math. Ann. 365:3-4 (2016), 1201–1217.

[18] B. Totaro, “Hypersurfaces that are not stably rational”, J. Amer. Math. Soc. 29:3 (2016), 883–891.

[19] A. Auel, Ch. Böhning, and A. Pirutka, “Stable rationality of quadric and cubic surface bundle fourfolds”, Eur. J. Math. 4:3 (2018), 732–760.

[20] B. Hassett, A. Pirutka, and Yu. Tschinkel, “A very general quartic double fourfold is not stably rational”, Algebr. Geom. 6:1 (2019), 64–75.

[21] S. Schreieder, “Stably irrational hypersurfaces of small slopes”, J. Amer. Math. Soc. 32:4 (2019), 1171–1199.

[22] J. Nicaise and E. Shinder, “The motivic nearby fiber and degeneration of stable rationality”, Invent. Math. 217:2 (2019), 377–413.

[23] M. Kontsevich and Yu. Tschinkel, “Specialization of birational types”, Invent. Math. 217:2 (2019), 415–432.

[24] V. A. Iskovskih and Yu. I. Manin, “Three-dimensional quartics and counterexamples to the Lüroth problem”, Mat. Sb. 86(128):1(9) (1971), 140–166; English transl., Math. USSR-Sb. 15:1 (1971), 141–166.

[25] I. Krylov, “Birational geometry of del Pezzo fibrations with terminal quotient singularities”, J. Lond. Math. Soc. (2) 97:2 (2018), 222–246.

[26] H. Ahmadinezhad and I. Krylov, Birational rigidity of orbifold degree 2 del Pezzo fibrations, arXiv:1710.05328.

[27] D. Evans and A.V. Pukhlikov, “Birationally rigid complete intersections of high codimension”, Izv. Ross. Akad. Nauk Ser. Mat. 83:4 (2019), 100–128; English transl., Izv. Math. 83:4 (2019), 743–769.

[28] A.V. Pukhlikov, “Birational geometry of Fano hypersurfaces of index two”, Math. Ann. 366:1-2 (2016), 721–782.

[29] A.V. Pukhlikov, “Birational geometry of algebraic varieties fibred into Fano double spaces”, Izv. Ross. Akad. Nauk Ser. Mat. 81:3 (2017), 160–188; English transl., Izv. Math. 81:3 (2017), 618–644.

[30] M.M. Grinenko, “Birational properties of pencils of del Pezzo surfaces of degrees 1 and 2. II”, Mat. Sb. 194:5 (2003), 31–60; English transl., Sb. Math. 194:5 (2003), 669–695.

[31] M.M. Grinenko, “Fibrations into del Pezzo surfaces”, Uspekhi Mat. Nauk 61:2(368) (2006), 67–112; English transl., Russian Math. Surveys 61:2 (2006), 255–300.

[32] A.V. Pukhlikov, “Birational automorphisms of Fano hypersurfaces”, Invent. Math. 134:2 (1998), 401–426.

[33] A.V. Pukhlikov, “Birational geometry of algebraic varieties with a pencil of Fano complete intersections”, Manuscripta Math. 121:4 (2006), 491–526.

[34] F. Suzuki, “Birational rigidity of complete intersections”, Math. Z. 285:1-2 (2017), 479–492.

[35] A.V. Pukhlikov, “Birationally rigid Fano fibrations”, Izv. Ross. Akad. Nauk Ser. Mat. 64:3 (2000), 131–150; English transl., Izv. Math. 64:3 (2000), 563–581.
[36] V. A. Iskovskikh and A. V. Pukhlikov, “Birational automorphisms of multidimensional algebraic manifolds”, Algebraic geometry – 1, Itogi Nauki Tekhn. Ser. Sovrem. Probl. Mat. Temat. Obz., vol. 19, VINITI, Moscow 2001, pp. 5–139; English transl., J. Math. Sci. (N.Y.) 82:4 (1996), 3528–3613.

[37] A. V. Pukhlikov, “Fiber-wise birational correspondences”, Mat. Zametki 68:1 (2000), 120–130; English transl., Math. Notes 68:1 (2000), 103–112.

Aleksandr V. Pukhlikov
Department of Mathematical Sciences,
University of Liverpool
E-mail: pukh@liverpool.ac.uk

Received 27/JAN/21
Translated by THE AUTHOR

12/JUN/21