Hypergeometric solutions to the symmetric $q$-Painlevé equations

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Abstract

We consider the symmetric $q$-Painlevé equations derived from the birational representation of affine Weyl groups by applying the projective reduction and construct the hypergeometric solutions. Moreover, we discuss continuous limits of the symmetric $q$-Painlevé equations to Painlevé equations together with their hypergeometric solutions.

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1 Introduction

The discrete Painlevé equations, together with the Painlevé equations, are now widely recognized as one of the most important families of the integrable systems (see, for example, [7]). Originally, the discrete Painlevé equations were discovered in the form of single second-order difference equations and identified as the discrete analogues of the Painlevé equations [1–3,33,38]. Then they were generalized to simultaneous first-order equations by a singularity confinement criterion [8,38]. A typical example is the following equation known as a discrete Painlevé II equation [33,38]:

$$x_{n+1} + x_{n-1} = \frac{(an + b)x_n + c}{1 - x_n^2},$$

(1.1)

where $x_n$ is the dependent variable, $n$ is the independent variable, and $a$, $b$, $c \in \mathbb{C}$ are parameters. We note that (1.1) was found as the equation satisfied by the constant terms of the orthogonal polynomials on the unit circle [33]. By applying the singularity confinement criterion, (1.1) is generalized to

$$x_{n+1} + x_{n-1} = \frac{(an + b)x_n + c + (-1)^n d}{1 - x_n^2},$$

(1.2)

where $d$ is a parameter, with its integrability preserved. Introducing the dependent variables $X_n$ and $Y_n$ by

$$X_n = x_{2n}, \quad Y_n = x_{2n-1},$$

(1.3)
then (1.2) can be rewritten as

\[
Y_{n+1} + Y_n = \frac{(2an + b)X_n + c + d}{1 - X_n^2}, \quad X_{n+1} + X_n = \frac{(a(2n + 1) + b)Y_{n+1} + c - d}{1 - Y_{n+1}^2}.
\]  

Equation (1.4) is known as a discrete Painlevé III equation since it admits a continuous limit to the Painlevé III equation [4, 30, 40]. Conversely, (1.1) can be recovered from (1.4) by putting \(d = 0\) and (1.3). This procedure is referred to as “symmetrization” of (1.4), which comes from the terminology of the Quispel–Roberts–Thompson (QRT) mapping [35, 36]. After this terminology, (1.4) is sometimes called the “asymmetric” discrete Painlevé II equation, and (1.1) is called the “symmetric” discrete Painlevé III equation [21].

It appears as though the symmetrization is a simple specialization on the level of the equation, but the following problems were known: (i) According to Sakai’s theory [42], the Painlevé and discrete Painlevé equations are classified by the underlying space of initial conditions. Moreover, the discrete Painlevé equations arise as the birational mappings corresponding to the translations of the affine Weyl groups associated with the space of initial conditions. The asymmetric discrete Painlevé equations are characterized in this manner, however, it was not known how to characterize the symmetric equations as the action of affine Weyl groups. (ii) The Painlevé and discrete Painlevé equations admit the particular solutions expressible in terms of the hypergeometric type functions (hypergeometric solutions) when some of the parameters take special values (see, for example, [14, 15] and references therein). However, the hypergeometric solutions to the symmetric discrete Painlevé equation cannot be obtained by the naïve specialization of those to the corresponding asymmetric equation. For example, (1.1) has the hypergeometric solution expressible in terms of the parabolic cylinder function (Weber function) [13, 18]. On the other hand, (1.4) admits the hypergeometric solution in terms of the confluent hypergeometric function [16, 22, 32]. The crucial point is that although the former function is expressed as a specialization of the latter, this specialization is not consistent with the symmetrization.

In [16], the mechanism of the symmetrization was investigated in detail by taking an example of \(q\)-Painlevé equation with the affine Weyl group symmetry of type \((A_2 + A_1)^{(1)}\). Then it was shown that in general, various discrete dynamical systems of Painlevé type can be obtained from elements of infinite order that are not necessarily translations in the affine Weyl group by taking the projection on appropriate subspaces of the parameter spaces. Such a procedure is called a projective reduction, and the symmetrization can be understood as a kind of projective reduction. Moreover, the above nontrivial inconsistency among the hypergeometric solutions are explained by the factorization of the linear difference operators associated with the three-term relation of the hypergeometric functions.

In spite of understanding the mechanism of the symmetrization or the projective reduction, it is still nontrivial what function will appear in the hypergeometric solutions to the symmetric discrete Painlevé equations even if the hypergeometric solutions to the corresponding asymmetric equations are known. Since the continuous limit of the symmetric discrete Painlevé equation is different from the corresponding asymmetric one, it is also important to consider the continuous limit of the symmetric discrete Painlevé equations together with their hypergeometric solutions.

The purpose of this paper is to construct the simplest hypergeometric solutions to each of the symmetric \(q\)-Painlevé equations. In [14, 15] the simplest hypergeometric solutions to all possible \(q\)-Painlevé equations in Sakai’s list have been constructed. Similarly, in this paper, we present the list of the simplest hypergeometric solutions to the symmetric \(q\)-Painlevé equations reduced from
the asymmetric ones with the affine Weyl group symmetry of type $E_8^{(1)}$, $E_7^{(1)}$, $E_6^{(1)}$, $D_5^{(1)}$, $A_4^{(1)}$ and $(A_2 + A_1)^{(1)}$. We also aim to consider the continuous limits of those symmetric $q$-Painlevé equations to the Painlevé equations, together with their hypergeometric solutions.

This paper is organized as follows: in Section 2, we derive the symmetric $q$-Painlevé equations by applying the projective reduction to $q$-$P((A_2 + A_1)^{(1)})$, $q$-$P(A_4^{(1)})$, $q$-$P(D_5^{(1)})$, $q$-$P(E_6^{(1)})$, $q$-$P(E_7^{(1)})$ and $q$-$P(E_8^{(1)})$, where the $q$-Painlevé equation with the affine Weyl group of type $X$ is denoted by $q$-$P(X)$. In Section 3, we construct the hypergeometric solutions to the symmetric $q$-Painlevé equations derived in Section 2. In Section 4, we discuss the continuous limits of the symmetric $q$-Painlevé equations and their hypergeometric solutions. In Section 5, we prove that hypergeometric function appearing in the hypergeometric solution to the symmetric $q$-$P(A_4^{(1)})$ actually reduces to the Weber function by applying the saddle point method to its integral representation. Some concluding remarks are given in Section 6.

2 Symmetric $q$-Painlevé equations

In this section, we apply the projective reduction to the $q$-Painlevé equations $q$-$P((A_2 + A_1)^{(1)})$, $q$-$P(A_4^{(1)})$, $q$-$P(D_5^{(1)})$, $q$-$P(E_6^{(1)})$, $q$-$P(E_7^{(1)})$ and $q$-$P(E_8^{(1)})$, respectively, to obtain their symmetric forms. In the following, we use the notations

$$
\tilde{t} = qt, \quad t = q^{-1}t, \quad f = f(t), \quad \tilde{f} = f(\tilde{t}), \quad f' = f(t).
$$

Type $(A_2 + A_1)^{(1)}$

$q$-$P((A_2 + A_1)^{(1)})$ is given by [14, 15, 21, 39]

$$
\tilde{g}g = qc^2 \frac{1 + tf}{(t + f)f}, \quad \tilde{f}f = qc^2 \frac{1 + a_2g}{(a_2t + g)g}.
$$

(2.2)

The continuous limit yields the Painlevé III equation. To apply the projective reduction, we put

$$
f(t) = X(t), \quad g(t) = X(p^{-1}t), \quad a_2 = p,
$$

(2.3)

then (2.2) is reduced to the symmetric $q$-$P((A_2 + A_1)^{(1)})$ [16, 25, 27, 37]

$$
\tilde{X}X = p^2 c^2 \frac{1 + tX}{(t + X)X}.
$$

(2.4)

The continuous limit yields the Painlevé II equation.

The system described in (2.2) is a discrete dynamical system arising from the birational action of an element $T_1$ of the affine Weyl group of type $(A_2 + A_1)^{(1)}$, which is a translation on the corresponding root lattice and parameter space. We introduce another element of the affine Weyl group $R_1$ satisfying $R_1^2 = T_1$. We note that $R_1$ is not a translation on the root lattice or on the full parameter space. However, by taking the projection on an appropriate subspace of the parameter space, it becomes a translation on the subspace, which gives (2.4). The specialization from (2.2) to (2.4) corresponds to this projection, which is an example of a projective reduction. We refer to [16] for details.
**Type $A_4^{(1)}$**

$q$-P($A_4^{(1)}$) [43]:

$$ (g_1 f - 1)(g f - 1) = r^2 \frac{(f + a_1)(f + a_1^{-1})}{f + a_2 t}, \quad (g f - 1)(g f_1 - 1) = q^{-1} r^2 \frac{(g + a_1)(g + a_1^{-1})}{g + a_3 t}. \quad (2.5) $$

Symmetric $q$-P($A_4^{(1)}$) [37,43]:

$$ (X X - 1)(X \tilde{X} - 1) = r^2 \frac{(X + a_1)(X + a_1^{-1})}{X + a_2 t}, \quad (2.6) $$

where

$$ f(t) = X(t), \quad g(t) = X(p^{-1} t), \quad a_3 = p^{-1} a_2. \quad (2.7) $$

The continuous limits of (2.5) and (2.6) yield the Painlevé V and IV equations, respectively.

**Type $D_5^{(1)}$**

$q$-P($D_5^{(1)}$) [11,14,39,41]:

$$ g g = \frac{(f - q^{1/2} a_1 t)(f - q^{1/2} a_1^{-1} t)}{(f - a_2)(f - a_2^{-1})}, \quad f f = \frac{(g - a_3 t)(g - a_3^{-1} t)}{(g - a_4)(g - a_4^{-1})}. \quad (2.8) $$

Symmetric $q$-P($D_5^{(1)}$) [38]:

$$ X \tilde{X} = \frac{(X - a_1 \tilde{t})(X - a_1^{-1} \tilde{t})}{(X - a_2)(X - a_2^{-1})}. \quad (2.9) $$

where

$$ f(t) = X(t), \quad g(t) = X(p^{-1} t), \quad a_3 = a_1, \quad a_4 = a_2. \quad (2.10) $$

The continuous limits of (2.8) and (2.9) yield the Painlevé VI and III equations, respectively.

**Type $E_6^{(1)}$**

$q$-P($E_6^{(1)}$) [14,15,24,39]:

$$ \begin{align*}
\left\{ \begin{array}{l}
(g f - 1)(g f_2 - 1) = r^2 \frac{(f - b_1)(f - b_2)(f - b_3)(f - b_4)}{(f - b_5 t)(f - b_5^{-1} t)}, \\
(g f - 1)(g f_3 - 1) = q^{-1} r^2 \frac{(g - b_1^{-1})(g - b_2^{-1})(g - b_3^{-1})(g - b_4^{-1})}{g - b_6 u(g - b_6^{-1} u)},
\end{array} \right. \quad (2.11)
\end{align*} $$

where

$$ b_1 b_2 b_3 b_4 = 1, \quad u = p^{-1} t. \quad (2.12) $$

Symmetric $q$-P($E_6^{(1)}$) [38]:

$$ (X \tilde{X} - 1)(X \tilde{X} - 1) = r^2 \frac{(X - b_1 t)(X - b_1^{-1} t)(X - b_3)(X - b_3^{-1})}{(X - b_5 t)(X - b_5^{-1} t)}. \quad (2.13) $$
where
\[ f(t) = X(t), \quad g(t) = X(p^{-1}t), \quad b_1b_2 = 1, \quad b_3b_4 = 1, \quad b_5b_6 = 1. \]  

Direct continuous limit of (2.11) to the Painlevé equations does not exist, since it has more parameters than the Painlevé VI equation. On the other hand, (2.13) admits a continuous limit to the Painlevé V equation.

**Type \( E_7^{(1)} \)**

\( q\)-P(\( E_7^{(1)} \)) [14, 15, 39]:

\[
\begin{align*}
\frac{(g f - \bar{u})(g f - t^2)}{(g f - \bar{u})(g f - 1)} &= \frac{(f - b_1 t)(f - b_2 t)(f - b_3 t)(f - b_4 t)}{(f - b_3)(f - b_6)(f - b_7)(f - b_8)}, \\
\frac{(g f - t^2)(g f - t)}{(g f - \bar{u})(g f - 1)} &= \frac{(g - b_1^{-1} t)(g - b_2^{-1} t)(g - b_3^{-1} t)(g - b_4^{-1} t)}{(g - b_5^{-1})(g - b_6^{-1})(g - b_7^{-1})(g - b_8^{-1})},
\end{align*}
\]

where
\[ b_1b_2b_3b_4 = q, \quad b_5b_6b_7b_8 = 1. \]  

Symmetric \( q\)-P(\( E_7^{(1)} \)) [6]:

\[
\frac{(X - \bar{u}^2)(X - t^2)}{(X - \bar{u})(X - 1)} = \frac{(X - b_1 t)(X - b_2 t)(X - b_3 t)(X - b_4 t)}{(X - b_5)(X - b_6)(X - b_7)(X - b_8)}.
\]

where
\[
\begin{align*}
f(t) = X(t), \quad g(t) = X(p^{-1}t), \quad b_1b_2 = p, \quad b_3b_4 = p, \quad b_5b_6 = 1, \quad b_7b_8 = 1.
\end{align*}
\]

Direct continuous limit of (2.15) to the Painlevé equations does not exist, while that of (2.17) yields the Painlevé VI equation.

**Type \( E_8^{(1)} \)**

\( q\)-P(\( E_8^{(1)} \)) [14, 15, 24, 29, 39]:

\[
\begin{align*}
\frac{(u - f)(u - t^2 - 1)}{(u^{-1} - \bar{u}^{-1} t^{-1} g - f - (1 - u^{-2} t^{-2})(1 - u^{-2} t^{-2})} &= \frac{P(f, t, m_1, \cdots, m_7)}{\bar{P}(f, t^{-1}, m_7, \cdots, m_1)}, \\
\frac{(u - g)(u - t^2 - 1)}{(u^{-1} - \bar{u}^{-1} t^{-1} g - f - (1 - u^{-2} t^{-2})(1 - u^{-2} t^{-2})} &= \frac{P(g, u, m_7, \cdots, m_1)}{\bar{P}(g, u^{-1}, m_1, \cdots, m_7)}.
\end{align*}
\]

where
\[
P(f, t, m_1, \cdots, m_7) = f^4 - m_1 t f^3 + (m_2 t^2 - 3 - t) f^2 + (m_7 t^3 - m_3 t^3 + 2 m_1 t) f + t^8 - m_6 t^6 + m_4 t^4 - m_2 t^2 + 1.
\]
Here \( m_j \) \((j = 1, \cdots, 8)\) are the elementary symmetric functions of \( j \)-th degree in \( b_k \) \((k = 1, \cdots, 8)\), \( m_8 = 1 \), and \( u = p^{-1}t \). Symmetric \( q\)-P(E\( _8^{(1)} \)):

\[
\frac{(itX - X)(ttX - X) - (t^2 t^2 - 1)(t^2 t^2 - 1)}{(t^{-1}t^{-1}X - X)(t^{-1}t^{-1}X - X) - (t^{-2}t^{-2} - 1)(t^{-2}t^{-2} - 1)} = \frac{P(X, t, m_1, m_2, m_3, m_4, m_5, m_6, m_7)}{P(X, t^{-1}, m_1, m_2, m_3, m_4, m_5, m_6, m_7)},
\]

where

\[
f(t) = X(t), \quad g(t) = X(p^{-1}t), \quad m_i = m_{8-i} \quad (i = 1, 2, 3).
\]

Direct continuous limits of (2.19) and (2.21) to the Painlevé equations are not known.

### 3 Hypergeometric solutions to the symmetric \( q \)-Painlevé equations

In this section, we construct the hypergeometric solutions to the symmetric \( q \)-Painlevé equations. We use the following conventions of \( q \)-analysis [5]. The basic hypergeometric series \( \psi_r \) is defined by

\[
\psi_r \left( \frac{a_1, \cdots, a_s}{b_1, \cdots, b_r} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_s ; q)_n}{(b_1, \cdots, b_r ; q)_n(q ; q)_n} \frac{(-1)^n q^{n(n-1)/2}}{(1 + r - s)} z^n,
\]

where

\[
(a_1, \cdots, a_s ; q)_n = \prod_{i=1}^{s} (a_i ; q)_n, \quad (a ; q)_k = \prod_{i=1}^{k} (1 - a q^{i-1}),
\]

are the \( q \)-shifted factorials. The following special case of the basic hypergeometric series is known as the very-well-poised basic hypergeometric series

\[
r_{+1} W_r (a_1, a_4, a_5, \cdots, a_{r+1} ; q, z) = r_{+1} \psi_r \left( a_1, q a_1^{1/2}, -q a_1^{1/2}, a_4, \cdots, a_{r+1} \right).
\]

We also use the Jacobi theta function

\[
\Theta(a ; q) = (a ; q)_\infty (qa^{-1} ; q)_\infty,
\]

which satisfies the \( q \)-difference equation

\[
\Theta(qa ; q) = -q^{-1} \Theta(a ; q).
\]

#### 3.1 Symmetric \( q \)-P(\((A_2 + A_1)^{(1)}\))

**Proposition 3.1** Symmetric \( q \)-P(\((A_2 + A_1)^{(1)}\)) (2.4) admits the hypergeometric solution

\[
\bar{X} = p^{1/2} \frac{\bar{G}}{G},
\]

\[
G = A e^{(\pi i/2) \log t / \log p} \psi_1 \left( 0 ; p, ip^{3/2} t \right) + B e^{(-\pi i/2) \log t / \log p} \psi_1 \left( 0 ; -p, -ip^{3/2} t \right).
\]
with
\[ c = 1. \] (3.8)

Here, A and B are quasi-constants satisfying \( A(t) = A(pt) \) and \( B(t) = B(pt) \), respectively.

**Proof.** Substituting
\[
\tilde{X} = \frac{P(t)X + Q(t)}{R(t)X + S(t)}, \quad X = \frac{-S(p^{-1}t)X + Q(p^{-1}t)}{R(p^{-1}t)X - P(p^{-1}t)},
\] (3.9)
in (2.4), we find that (2.4) admits a specialization to the discrete Riccati equation
\[
\tilde{X} = -p \frac{1 + tX}{X},
\] (3.10)
when \( c = 1 \). Then putting as (3.6)), (3.10) is linearized to the three-term relation for \( G \)
\[
\tilde{G} + p^{3/2}t \tilde{G} + G = 0.
\] (3.11)
Moreover, substituting the power series expression
\[
G = \sum_{n=0}^{\infty} C_n t^{n+i}, \quad (\rho, \ C_n \in \mathbb{C}),
\] (3.12)
into (3.11), we obtain (3.7). \qed

In the following, we present the hypergeometric solutions to other cases. Since the results are verified by direct calculations, we omit the proof showing only the discrete Riccati equations and the linearized three-term relations. We also assume that \( A \) and \( B \) are quasi-constants.

### 3.2 Symmetric \( q - P(A_{4}^{(1)}) \)

**Proposition 3.2** Symmetric \( q - P(A_{4}^{(1)}) \) (2.6) admits the hypergeometric solution
\[
\tilde{X} = \frac{F}{\tilde{F}},
\] (3.13)
\[
F = (pa_2^{-1}t; p)_{\infty} \binom{a_2^2, 0}{-p; pa_2^{-1}t} + B (-1)^{\log t/\log p} \binom{-a_2^2, 0}{-p; pa_2^{-1}t},
\] (3.14)
with
\[ a_1 = p^{-1}a_2^2. \] (3.15)

The discrete Riccati equation and the linearized three-term relation are given by
\[
\tilde{X} = \frac{1 - pa_2^{-1}t}{X + a_2t},
\] (3.16)
and
\[
(pa_2^{-1}t - 1)\tilde{F} + a_2tF + F = 0,
\] (3.17)
respectively.
3.3 Symmetric q-P\((D_5^{(1)})\)

For convenience, we set
\[
X = -iW, \quad a_1 = ip^{\nu_1/2}, \quad a_2 = ip^{\nu_2/2}.
\]
(3.18)

Symmetric q-P\((D_5^{(1)})\) (2.9) can be rewritten as
\[
\tilde{W}W = \frac{(W + p^{\nu_1/2} \tilde{t})(W - p^{-\nu_1/2} \tilde{t})}{(W + p^{\nu_2/2})(W - p^{-\nu_2/2})}.
\]
(3.19)

**Proposition 3.3** ([17, 19]) Symmetric q-P\((D_5^{(1)})\) (3.19) admits the hypergeometric solution
\[
W = \frac{G}{G - p^{\nu_2/2}},
\]
(3.20)
\[
G = A J_{\nu_2}^{(1)}(2ip^{3/4}t^{1/2}; p) + B J_{\nu_2}^{(1)}(2ip^{3/4}t^{1/2}; p),
\]
(3.21)

with
\[
\nu_1 = \nu_2 + 1.
\]
(3.22)

Here, \(J_{\nu}^{(1)}(x; q)\) is Jackson’s q-Bessel function [5]
\[
J_{\nu}^{(1)}(x; q) = \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} \left(\frac{x}{2}\right)^{\nu} \left(0, 0; q^{\nu+1}; q^2, -\frac{x^2}{4}\right).
\]
(3.23)

The discrete Riccati equation and the linearized three-term relation are given by
\[
\tilde{W} = p^{-\nu_2/2} \frac{W + p^{(\nu_2+1)/2} \tilde{t}}{W + p^{\nu_2/2}},
\]
(3.24)
\[
\tilde{G} - (p^{\nu_2/2} + p^{-\nu_2/2})\tilde{G} + (1 - p^{1/2}\tilde{t})G = 0,
\]
(3.25)

respectively.

3.4 Symmetric q-P\((E_6^{(1)})\)

**Proposition 3.4** Symmetric q-P\((E_6^{(1)})\) (2.13) admits the hypergeometric solution
\[
X = (b_1^{-1}b_5t - 1)\frac{G}{G} + b_5t,
\]
(3.26)
\[
G = A_{\varphi_1} \left(\frac{b_1^{-1}b_5^2, -b_1^{-1}}{-p, b_1b_5^{-1}\tilde{t}}; p, b_1b_5\right) + B_{\varphi_1} \left(p\Theta(t; p)\Theta(-t; p)\right)_{\varphi_1} \left(-b_1^{-1}b_5^2, b_1^{-1} \right)\left(-p, b_1b_5^{-1}\tilde{t}\right),
\]
(3.27)

with
\[
b_5^2 = pb_1b_3.
\]
(3.28)

The discrete Riccati equation and the linearized three-term relation are given by
\[
\tilde{X} = \frac{b_1tX + b_1b_5 - (pb_1^2 + b_5^2)t}{b_1b_5(X - b_5t)},
\]
(3.29)
\[
(b_5t - b_1)b_5\tilde{G} + b_1(b_5^2 - 1)tG - b_1(b_1t - b_5)G = 0,
\]
(3.30)

respectively.
3.5 Symmetric $q$-$P(E_7^{(1)})$

**Proposition 3.5** Symmetric $q$-$P(E_7^{(1)})$ (2.17) admits the hypergeometric solution

\[
\frac{X - b_1^{-1} \tilde{t}}{X - b_5} = \frac{(b_3 b_5 \tilde{t} - 1)}{(b_5^2 - 1)(\tilde{t} - b_3 b_5)} \left( \frac{\tilde{G}}{G} + \tilde{t} - b_1^{-1} b_5 \right), \tag{3.31}
\]

\[G = AG_1 + BG_2,
\]

\[G_1 = \frac{(b_1 b_5 \tilde{t}, -b_3 \tilde{t}; p)_\infty (b_1 b_3 b_5) \log \tilde{t} / \log p}{(b_1^{-1} b_5^{-1} \tilde{t}, -b_3^{-1} \tilde{t}; p)_\infty}
\times \frac{W_7 (-b_3 b_5; p^{1/2} b_3, -p^{1/2} b_3, -b_1 b_3 b_5, b_3 b_5 \tilde{t}, b_3 b_5 \tilde{t}; p, pb_1^{-1} b_3^{-1} b_5^{-1})}{W_7 (b_1^{-1} b_3^{-1} b_5^{-1} \tilde{t}^2; b_1^{-1} b_2^{-1} \tilde{t}^2, p^{1/2} \tilde{t}, pb_1^{-1} b_3^{-1} b_5^{-1}; p, -b_5),}
\]

\[G_2 = \frac{(b_1 b_5 \tilde{t} - 1)(b_3 b_5 \tilde{t} - 1)}{(b_1 b_3 b_5 - 1)(1 + b_5)} \left( \frac{\tilde{G} - G}{G} + (\tilde{t} - 1)G + \frac{b_1 b_3 b_5 (\tilde{t} - b_3 b_5)(\tilde{G} - G)}{b_1 b_3 b_5 - 1}(1 + b_5) \right) = 0, \tag{3.35}
\]

The discrete Riccati equation and the linearized three-term relation are given by

\[
XX - \tilde{t}^2 = \frac{(X - b_1^{-1} \tilde{t})(X - b_3^{-1} \tilde{t})}{(X - b_3)(X - b_1^{-1} b_3^{-1} b_5^{-1})}, \tag{3.34}
\]

respectively. To obtain the solution of (3.35), the following proposition is useful:

**Proposition 3.6** ([10]) The three-term relation for $G = AG_1 + BG_2$, where

\[G_1 = \frac{(az^{-1}) \log u / \log p (abu, acu, adu, bcdz^{-1} u)_\infty}{(bcu, bdu, cdu,azu)_\infty}
\times \frac{W_7 (p^{-1} bcdu^{-1}, bz^{-1}, cz^{-1}, dz^{-1}, p^{-1} abcdiu, u^{-1}; p, pa^{-1} z)}{W_7 (bcdzu^2; bcu, bdu, cdu, pu, pa^{-1} z; p, az),} \tag{3.36}
\]

\[G_2 = \frac{(az) \log u / \log p (abcdiu^2, pbzu, pczu, pdzu, bcdzu)_\infty}{(bcu, bdu, cdu, pu, pbcdzu^2)_\infty}
\times \frac{W_7 (bcdzu^2; bcu, bdu, cdu, pu, pa^{-1} z; p, az)}{W_7 (bcdu^2; bcu, bdu, cdu, pu, pa^{-1} z; p, az),} \tag{3.37}
\]

is given by

\[
\frac{(1 - abu)(1 - acu)(1 - adu)(1 - p^{-1} abcdiu)}{a(1 - p^{-1} abcdiu^2)(1 - abcdiu^2)} (G(pu) - G(u)) + (a + a^{-1} - z - z^{-1})G(u)
\]

\[
- \frac{a(1 - p^{-1} bcdu)(1 - p^{-1} bcdiu)(1 - u)}{(1 - p^{-2} abcdiu^2)(1 - p^{-1} abcdiu^2)} (G(u) - G(p^{-1} u)) = 0. \tag{3.38}
\]

Here $A$ and $B$ are quasi-constants with respect to $u$.  

9
Substituting
\[ a = ib_1^{1/2}b_3^{1/2}t, \quad b = ip^{1/2}b_1^{-1/2}b_3^{1/2}, \quad c = -ip^{1/2}b_1^{-1/2}b_3^{1/2}, \]
\[ d = -ib_1^{1/2}b_3^{1/2}t, \quad z = ib_1^{-1/2}b_3^{-1/2}, \quad u = b_3^{-1}b_5^{-1}t, \]

in Proposition 3.6, we obtain the solution of (3.35).

For later convenience, we rewrite the solution in the following manner:

\[ F = \frac{(b_1b_5^{-1}t, -b_3^{-1}t; p)_\infty}{(b_1b_5t, -b_3t; p)_\infty(b_1b_3b_7)^{\log t/\log p} G}. \]

Then Proposition 3.5 is rephrased as follows:

**Proposition 3.7** Symmetric \( q\)-P(E\(_7^{(1)} \)) (2.17) admits the hypergeometric solution

\[ \frac{X - b_1^{-1}t}{X - b_5} = \frac{b_2b_5t - 1}{b_1(b_5^2 - 1)(t - b_3b_5)} \left\{ \frac{\tilde{t} - b_1b_5(\tilde{t} + b_3)}{b_3\tilde{t} + 1} F + b_1\tilde{t} - b_5 \right\}, \]

where

\[ F = AF_1 + BF_2, \]

\[ F_1 = 8W_7 \left(-b_3^{-1}b_5; p^{1/2}b_5, -p^{1/2}b_5, -b_1b_3b_5, b_3b_5t, b_3b_5\tilde{t}; p, pb_1^{-1}b_3^{-1}b_5^{-1} \right), \]

\[ F_2 = \frac{(b_1b_5t, -b_3\tilde{t}, p^{1/2}t, -p^{1/2}t, b_3^{-1}\tilde{t}, b_3^{-1}t, b_3b_5^{-1}t, b_3^{-1}t; p)_\infty}{(b_1b_3b_7)^{\log t/\log p}} \times 8W_7 \left(b_1^{-1}b_3^{-1}b_5^{-1}t; b_1^{-1}b_3^{-1}t, p^{-1/2}t, -p^{-1/2}t, b_3^{-1}t, pb_1^{-1}b_3^{-1}b_5^{-1}; p, -b_5 \right). \]

when

\[ b_1b_3b_5b_7 = 1. \]

We note that the linear three-term relation for \( F \) is given by

\[ \frac{(b_1b_5t - \tilde{t})(b_3b_5\tilde{t} - 1)}{(b_1b_3b_5 - 1)(1 + b_5)}\left\{ \frac{(b_1b_5 - \tilde{t})(b_3 + \tilde{t})}{(1 - b_1b_5\tilde{t})(1 + b_3\tilde{t})} - \tilde{F} + F \right\} + (\tilde{t}^2 - 1)F \]

\[ + \frac{(\tilde{t} - b_1b_5)(\tilde{t} - b_3b_5)}{(b_1b_3b_5 - 1)(1 + b_5)}\left\{ F - \frac{(1 - b_1b_5\tilde{t})(1 + b_3\tilde{t})}{(b_1b_5 - \tilde{t})(b_3 + \tilde{t})} \right\} = 0. \]

### 3.6 Symmetric \( q\)-P(E\(_8^{(1)} \))

**Proposition 3.8** Symmetric \( q\)-P(E\(_8^{(1)} \)) (2.21) admits the hypergeometric solution

\[ \frac{X - \beta_3}{X - \beta_1} = \frac{(\alpha_3 - \alpha_5)(\beta_1 - \beta_3)}{(\alpha_1 - \alpha_5)(\beta_3 - \beta_5)} \left[ (\alpha_1 - \alpha_3) \frac{\tilde{G}}{G} + \alpha_3 - \tilde{\beta}_1 \right]. \]
\[ G = AG_1 + BG_2, \]
\[ G_1 = \frac{(b_1b_3\tilde{t}^2, b_1b_5\tilde{t}^2, -\tilde{t}^2, b_1^{-1}b_3^{-1}b_5^{-1}\tilde{t}^2; p^2)_\infty}{(-\tilde{t}^2, b_1^{-1}b_3^{-1}b_5^{-1}\tilde{t}^2, b_1^{-1}b_3^{-1}b_5^{-1}\tilde{t}^2, b_1b_3b_5\tilde{t}^2; p^2)_\infty} \cdot (-b_1)^{\log t / \log p} \]
\[ \times_8 W_7(p^{-2}b_1^{-1}b_3^{-2}b_5^{-2}; b_1^{-1}, b_3^{-1}, -b_1^{-1}b_3^{-1}b_5^{-1}, b_3^{-1}b_5^{-1}\tilde{t}^2, b_3^{-1}b_5^{-1}\tilde{t}^2; p^2, -p^2b_1^{-1}), \quad (3.46) \]
\[ G_2 = \frac{(\tilde{t}^2, -b_3\tilde{t}^2, -b_5\tilde{t}^2, b_1^{-1}\tilde{t}^2, b_1^{-1}b_3^{-1}b_5^{-1}\tilde{t}^2; p^2)_\infty}{(-\tilde{t}^2, b_1^{-1}b_3^{-1}b_5^{-1}\tilde{t}^2, b_1^{-1}b_3^{-1}b_5^{-1}\tilde{t}^2, b_1b_3b_5\tilde{t}^2; p^2)_\infty} \cdot \frac{b_1^{\log t / \log p}}{b_1^{\log t / \log p}} \]
\[ \times_8 W_7(-b_1^{-1}\tilde{t}^2, -\tilde{t}^2, b_1^{-1}b_3^{-1}\tilde{t}^2, b_1^{-1}b_3^{-1}\tilde{t}^2, b_1b_3b_5\tilde{t}^2, -p^2b_1^{-1}; p^2, b_1), \]

with
\[ b_1b_3b_5 = p^{-2}. \quad (3.47) \]

Here, \( \alpha_i \) and \( \beta_i \) are given by
\[ \alpha_i = b_i \tilde{t} + \frac{1}{b_i \tilde{t}}; \quad \beta_i = \frac{t}{b_i} + \frac{b_i}{t}. \quad (3.48) \]

The discrete Riccati equation and the linearized three-term relation are given by
\[ \mathbf{X} = \begin{pmatrix} 1 & \alpha_1 & \alpha \beta_1 \\ 1 & \alpha_3 & \alpha \beta_3 \\ 1 & \alpha_5 & \alpha \beta_5 \end{pmatrix}, \quad \mathbf{X}^\ast = \begin{pmatrix} 1 & \beta_1 & \alpha \beta_1 \\ 1 & \beta_3 & \alpha \beta_3 \\ 1 & \beta_5 & \alpha \beta_5 \end{pmatrix}. \quad (3.49) \]

\[ \frac{(b_1b_3\tilde{t}^2 - 1)(b_1b_5\tilde{t}^2 - 1)(\tilde{t}^2 - b_3b_5)}{(\tilde{t}^2 - 1)(\tilde{t}^2\tilde{t}^2 - 1)} (G - G) + b_3b_5(b_1^2 - 1)G + \frac{(\tilde{t}^2 - b_1b_3)(\tilde{t}^2 - b_1b_5)(b_3b_5\tilde{t}^2 - 1)}{(\tilde{t}^2 - 1)(\tilde{t}^2\tilde{t}^2 - 1)} (G - G) = 0, \quad (3.50) \]

respectively. We note that (3.46) is obtained by substituting
\[ a = -ib_1, \quad b = ib_5^{-1}, \quad c = ib_3^{-1}, \quad d = -ib_1^{-1}b_3^{-1}b_5^{-1}, \quad z = i, \quad u = b_3b_5\tilde{t}^2, \quad (3.51) \]
in Proposition 3.6.

## 4 Continuous limits

In this section, we discuss the continuous limits of the symmetric \( q \)-Painlevé equations and their hypergeometric solutions. We introduce the hypergeometric series \( sF_r \) defined by
\[ sF_r \left( \frac{a_1, \ldots, a_r}{b_1, \ldots, b_r}; z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_r)_n n!} z^n, \quad (4.1) \]

where \((a)_n = a(a + 1) \cdots (a + n - 1)\).
4.1 Symmetric $q$-$P((A_2 + A_1)^{(1)})$

It is known that symmetric $q$-$P((A_2 + A_1)^{(1)})$ (2.4) yields the Painlevé II equation

$$z'' = 2z^3 + sz + \alpha,$$

(4.2)

where $z' = dz/ds$, by putting

$$t = -2e^{-se^{2/4 - \epsilon e^{3/4}}}, \quad c = e^{(1+2\epsilon)e^{3/4}}, \quad X = e^{-se^{2/4+\epsilon e^{3/4}}(1 + \epsilon z)}, \quad p = e^{-\epsilon^{3/4}},$$

(4.3)

and taking the limit $\epsilon \to +0$ [37]. The limit of the hypergeometric solution is given as follows:

**Proposition 4.1** *Under the parametrization (4.3), the hypergeometric solution in Proposition 3.1 is reduced to the following solution to (4.2):*

$$z = \frac{G'}{G},$$

(4.4)

$$G = A \text{Ai}(2^{-1/3}se^{-\pi i/3}) + B \text{Ai}(2^{-1/3}se^{\pi i/3}),$$

(4.5)

with

$$\alpha = -\frac{1}{2}.$$  

(4.6)

*Here A and B are constants, and Ai(x) is the Airy function*

$$\text{Ai}(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{iu^{3/2} + iux} du.$$  

(4.7)

We note that (3.10) and (3.11) are reduced to the following equations

$$z' = -z^2 - \frac{s}{2}$$

(4.8)

$$G'' + \frac{s}{2}G = 0,$$

(4.9)

respectively. We also note that (4.5) is obtained from (3.7) by using the following proposition:

**Proposition 4.2** ([9]) *With the substitutions*

$$q = e^{-\delta^{3/2}}, \quad T = -2ie^{-(u/2)\delta^2},$$

(4.10)

*as $\delta \to +0$, it follows that*

$$\left._1\Phi_1\begin{pmatrix} 0 \\ -q \end{pmatrix}; q, qT \right) = 2\pi^{1/2}\delta^{-1/2}e^{\log 2(\pi i/\delta^3)-(\pi i/2\delta)u-\pi i/12} \left[ \text{Ai}(ue^{-\pi i/3}) + O(\delta^2) \right],$$

(4.11)

$$\left._1\Phi_1\begin{pmatrix} 0 \\ -q \end{pmatrix}; -q, qT \right) = 2\pi^{1/2}\delta^{-1/2}e^{-\log 2(\pi i/\delta^3)+(\pi i/2\delta)u+\pi i/12} \left[ \text{Ai}(ue^{\pi i/3}) + O(\delta^2) \right],$$

(4.12)

*for u in any compact domain of $\mathbb{C}$.*
4.2 Symmetric \( q \)-P\( (A_4^{(1)}) \)

Putting

\[
t = 2i e^{i \varepsilon - 3 \alpha^2/2}, \quad a_1 = -e^{-2^{-1/2} i \beta^{1/2} \varepsilon^2}, \quad a_2 = i e^{-\alpha \varepsilon^2/2}, \quad X = 1 - i \varepsilon, \quad p = e^{-\varepsilon^2},
\]

and taking the limit \( \varepsilon \to 0 \), symmetric \( q \)-P\( (A_4^{(1)}) \) \( (2.6) \) yields the Painlevé IV equation [37]

\[
z'' = \left( \frac{z'}{2z} \right)^2 + \frac{3}{2} z^3 + 4s z^2 + 2(s^2 - \alpha)z + \frac{\beta}{z}
\]

(4.14)

**Proposition 4.3** Under the parametrization (4.13), the hypergeometric solution in Proposition 3.2 is reduced to

\[
z = \frac{F'}{F},
\]

\[
F = A e^{-s^2} D_{-\alpha}(2^{1/2} s) + B D_{\alpha-1}(2^{1/2} is), \quad s \notin \mathbb{R}, \sqrt{-1} \mathbb{R},
\]

(4.15)

with

\[
\alpha = 1 + 2^{-1/2} i \beta^{1/2}.
\]

(4.17)

Here \( A \) and \( B \) are constants, and \( D_{\alpha}(x) \) is the Weber function

\[
D_{\alpha}(x) = \int_{\gamma} e^{-i x/2 - xu} u^{-\alpha-1} du,
\]

(4.18)

where the path of integration (denote it by \( \gamma \)) runs from \(-\infty \) to \( +\infty \) so that \( u = 0 \) lies to the right of the path.

**Remark 4.4** The hypergeometric solution to the Painlevé IV equation given in Proposition 4.3 is consistent with the solution appeared in [28, 31].

Note that (3.16) and (3.17) are reduced to the following equations

\[
z' = -z^2 - 2sz + 2(\alpha - 1),
\]

(4.19)

\[
F'' + 2sF' + 2(1 - \alpha)F = 0.
\]

(4.20)

respectively. We also note that (4.16) is obtained from (3.14) by using the following lemmas:

**Lemma 4.5** With the substitutions (4.13), it follows that

\[
\begin{align*}
2\mathcal{F}_1 \left( \begin{array}{c} a_2^2, 0 \\ -p \\ pa_2^{-1} t \end{array} \right) - (a_2^{-2}; p)_{\infty} e^{\alpha i \log(pa_2^{-1})/\log p} \\
- \left( a_2^{-2}; p \right)_{\infty} (-1)^{\log t/\log p} e^{2\pi i /2} \mathcal{F}_1 \left( \begin{array}{c} -a_2^{-2}, 0 \\ -p \\ pa_2^{-1} t \end{array} \right)
\end{align*}
\]

(4.21)

\[
= \frac{(-1)^{3\alpha/2} 2^{-\alpha/2} \alpha^{\alpha_2 \log(2\pi^2/4)} e^{-\pi i /\varepsilon}}{\pi^{1/2} e^{1-\alpha}} \int_L e^{-u^2/2 -2^{1/2} su} u^{\alpha-1} [1 + O(\varepsilon)] du,
\]

as

\[
\varepsilon \to +0 \quad \text{(when Im}(s) < 0),
\]

(4.22)

\[
\varepsilon \to -0 \quad \text{(when Im}(s) > 0).
\]

(4.23)

The path \( L \) runs from \(-\infty \) to \( +\infty \) when \( \text{Im}(s) < 0 \) and \( +\infty \) to \(-\infty \) when \( \text{Im}(s) > 0 \) so that \( u = 0 \) lies to the right of the path.
Lemma 4.6 With the substitutions (4.13), it follows that
\[
2\varphi_1\left(-a_2^2, 0; p, pa_2^{-1}t\right)
\]
\[
= \frac{(-a_2^2; p)_\infty e^{(2 \log(-pa_2) \log a_2 - \log(pa_2^{-1}) \log(-a_2^{-2})) / \log p}}{a_2^4(a_2^2; p)_\infty}
\]
\[
= (-1)^{-1} - \pi^{-1/2}2^{1/2+\epsilon/2}e^{-\epsilon^2 + 2\pi i \log 2/\epsilon^2}
\int_{L} e^{-u^2/2-2^{1/2}iu} \frac{1}{u} [1 + O(\epsilon)] du,
\]
as
\[
\epsilon \to +0 \quad (\text{when Re}(s) < 0), \tag{4.24}
\]
\[
\epsilon \to -0 \quad (\text{when Re}(s) > 0). \tag{4.25}
\]

The path \(L\) runs from \(-\infty\) to \(+\infty\) when Re\((s) > 0\) and \(+\infty\) to \(-\infty\) when Re\((s) < 0\) so that \(u = 0\) lies to the right of the path.

The proofs of Lemma 4.5 and Lemma 4.6 will be given in the next section. For these proofs, we use the saddle point method \([9, 34, 45]\). Moreover, for the asymptotic expansions of the \(q\)-shifted factorials as \(q \to 1^-\) \((q = 1 - \epsilon, \epsilon > 0, \epsilon \to 0)\), we use the following proposition:

Proposition 4.7 (\([23, 34]\)) As \(q \to 1^-\), the \(q\)-shifted factorials have an asymptotic expansions,
\[
\log(t; q)_\infty = \frac{L_i(t)}{\log q} + \frac{\log(1 - t)}{2} + O(\log q), \tag{4.26}
\]
\[
\log(q; q)_\infty = \frac{\pi^2}{6 \log q} + \frac{1}{2} \log \left(-\frac{2\pi}{\log q}\right) + O(\log q), \tag{4.27}
\]
\[
\log(-q; q)_\infty = -\frac{\pi^2}{12 \log q} - \frac{\log 2}{2} + O(\log q). \tag{4.28}
\]

This is uniform for \(t \in \mathbb{C}\) such that \(|\arg(1 - t)| < \pi\). Here \(L_i(t)\) is the Euler dilogarithm defined by
\[
L_i(t) = -\int_{0}^{t} \frac{\log(1 - u)}{u} du \tag{4.29}
\]
\[
= \sum_{k=1}^{\infty} \frac{t^k}{k^2} \quad (|t| < 1). \tag{4.30}
\]

4.3 Symmetric \(q\)-P\((D_5^{(1)})\)

Putting
\[
t^{1/2} = \frac{i(1 - p)}{2} s, \quad W = \frac{\epsilon s}{2} z, \quad p = 1 + \epsilon, \tag{4.31}
\]
and taking the limit \(\epsilon \to -0\), symmetric \(q\)-P\((D_5^{(1)})\) (3.19) yields the Painlevé III equation \([38]\)
\[
z'' = \frac{(z')^2}{z} + \frac{z'}{s} + \frac{2(v_1 - v_2 z^2)}{s} - z^3 + \frac{1}{z} = 0. \tag{4.32}
\]
**Proposition 4.8 ([17, 19])** Under the parametrization (4.31), the hypergeometric solution in Proposition 3.3 is reduced to

\[
z = \frac{G'}{G} - \frac{v_2}{s},
\]

\[G = A J_{v_2}(s) + B J_{-v_2}(s),\]

with

\[v_1 = v_2 + 1.\]

Here, \(A\) and \(B\) are constants and \(J_v(x)\) is the Bessel function

\[J_v(x) = \frac{x^v}{2^{v+1} \Gamma(v+1)} {}_0F_1\left( -\frac{x^2}{4} \right).\]

We note that (3.24) and (3.25) are reduced to the following equations

\[z' + z^2 + \frac{1 + 2v_2}{s} z + 1 = 0,\]

\[G'' + \frac{G'}{s} + \left( 1 - \frac{v_2^2}{s^2} \right) G = 0,\]

respectively.

### 4.4 Symmetric \(q\)-P\(E_{6}(1)\)

Putting

\[t = \frac{(y^2 \epsilon^2 - 16)^{1/2}}{4 \epsilon} (1 + \epsilon)^{-\log s/\epsilon}, \quad b_1 = -1 + 2^{1/2} \beta \epsilon, \quad b_3 = 1 + 2^{1/2} \alpha \epsilon,\]

\[b_5 = \frac{(y^2 \epsilon^2 - 16)^{1/2}}{y \epsilon - 4}, \quad X = \frac{z + 1}{z - 1}, \quad p = 1 + \epsilon,\]

and taking the limit \(\epsilon \to -0\), symmetric \(q\)-P\(E_{6}(1)\) (2.13) yields the Painlevé V equation [37]

\[z'' = \left( \frac{1}{2z} + \frac{1}{z - 1} \right) (z')^2 - \frac{z'}{s} + \frac{(z-1)^2}{s^2} \left( \alpha^2 z - \beta^2 \right) + \frac{\gamma z}{s} - \frac{2z(z+1)}{z-1}.\]

**Proposition 4.9** Under the parametrization (4.39), the hypergeometric solution in Proposition 3.4 is reduced to the following solution to (4.40):

\[z = \frac{sG' - 2^{1/2} \beta G}{sG' - (2^{1/2} \beta + 2s)G}.\]

\[G = A \; {}_2F_0\left( 2^{1/2} \beta + \frac{\gamma}{2}, 2^{1/2} \beta, -\frac{1}{2s} \right) + \hat{B} e^{2s} s^{-1/2} \beta^{3/2} - 1 \; {}_2F_0\left( 1 - 2^{1/2} \beta - \frac{\gamma}{2}, 1 - 2^{1/2} \beta, -\frac{1}{2s} \right),\]

with

\[\gamma = 2(1 + 2^{1/2} \alpha - 2^{1/2} \beta).\]

Here, \(A\) and \(\hat{B}\) are constants.
The hypergeometric solution in Proposition 3.4 is reduced to the following solution to (4.40)

\[ s z' + \left( 2^{1/2} \beta + \frac{\gamma}{2} - 1 \right) z' - \left( 2^{3/2} \beta + \frac{\gamma}{2} + 2s - 1 \right) z + 2^{1/2} \beta = 0, \]  

\[ G'' + \frac{2 - \gamma - 2^{5/2} \beta - 4s}{2s} G' + \frac{\beta \gamma + 2^{3/2} \beta^2}{2^{1/2} s^2} G = 0, \]  

respectively. We note that (4.42) is obtained from (3.27) by the following procedure. By using Heine’s transformation [20]

\[ 2 \varphi_1 \left( \frac{a, b}{c}; q, z \right) = \frac{(abc^{-1}, z; q)_\infty}{(z; q)_\infty} 2 \varphi_1 \left( \frac{a^{-1} c, b^{-1} c}{c}; q, abc^{-1} z \right), \]

and setting

\[ \hat{B} = \frac{\Theta(t; p) \Theta(-b_1 b_5^{-1} e_t; p)(b_1^{-1} b_5 t, pb_1 b_5^{-1} t^{-1}; p)_\infty}{\Theta(-t; p) \Theta(b_1^{-1} b_5 e t, p)(b_1 b_5^{-1} t, b_1^{-1} b_5 t^{-1}; p)_\infty} B, \]

(3.27) is rewritten as

\[ G = A 2 \varphi_1 \left( \frac{b_1^{-1} b_5^{-2}, -b_1^{-1}}{-p}; p, b_1 b_5^{-1} t \right) + \hat{B} \frac{\Theta(b_1^{-1} b_5 e t, p)(b_1^{-1} b_5 t^{-1}; p)_\infty}{\Theta(-b_1 b_5^{-1} e_t; p)(pb_1 b_5^{-1} t^{-1}; p)_\infty} 2 \varphi_1 \left( \frac{pb_1 b_5^{-2}, -pb_1}{-p}; p, b_1^{-1} b_5 t \right). \]

Then (4.48) yields the hypergeometric series in (4.42) by taking a term-by-term limit. We use Proposition 4.7 for the limit of the coefficient.

**Remark 4.10** Setting

\[ s = -\frac{t}{2}, \quad z(s) = 1 - \frac{1}{f(t)}, \quad G(s) = e^{2s} s^{a_2} \phi(t), \quad \alpha = -2^{-1/2} a_1, \quad \beta = 2^{-1/2} a_2, \]

we can rewrite the Riccati equation, the dependent variable transformation and the linear differential equation as

\[ tf' - tf(1 - f) + (a_1 + a_2)f - a_1 = 0, \]

\[ f = \frac{\phi'}{\phi}, \]

\[ t \phi'' + (a_1 + a_2 - t) \phi' - a_1 \phi = 0, \]

respectively. These coincide with the result in [22, 32]. We note here that in [22, 32] the solution to (4.52) is given by the hypergeometric series around zero while in this section that is given by the series at infinity.
4.5 Symmetric $q$-$P(E_7^{(1)})$

Putting
\[ b_1 = e^{(\rho+1/2)\epsilon}, \quad b_3 = -e^{(\sigma+1/2)\epsilon}, \quad b_5 = e^{\sigma\epsilon}, \quad b_7 = -e^{-\gamma\epsilon}, \quad p = e^{\epsilon}, \] (4.53)
and taking the limit $\epsilon \to -0$, symmetric $q$-$P(E_7^{(1)})$ (2.17) yields Painlevé VI equation
\[
X'' = \frac{1}{2} \left( \frac{1}{X + 1} + \frac{1}{X - 1} + \frac{1}{X + t} + \frac{1}{X - t} \right) (X')^2 \\
- \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{t + 1} + \frac{1}{X - t} - \frac{1}{X + t} \right) X' \\
+ \frac{(X^2 - t^2)(X^2 - 1)}{t^2(t^2 - 1)} \left( \frac{4\alpha^2 - 1}{X + t} - \frac{4\beta^2 - 1}{X - t} - \frac{\gamma^2}{(X + 1)^2} + \frac{\delta^2}{(X - 1)^2} \right),
\] (4.54)
where $X' = dX/dt$ [6].

**Proposition 4.11** Under the parametrization (4.53), the hypergeometric solution in Proposition 3.5 is reduced to the following solution to (4.54):

\[
X(t) = \frac{(t - 1)F' - (1 + \alpha + \beta + t(\alpha + \beta + 2\delta + 1))F}{(t - 1)F' - (1 + \alpha + \beta + 2\delta + t(\alpha + \beta + 1))F},
\] (4.55)

\[
F = A \, _2F_1 \left( \frac{\alpha + 1, \alpha + \beta + \delta + 1}{\alpha + \delta + 1}; \left( \frac{1 + t}{1 - t} \right) \right) \\
+ \hat{B} \frac{(1 - t)^{2(1+2\alpha+\beta+\delta)}}{(1 + t)^{2(\alpha+\beta)^2+\alpha+\beta}} \, _2F_1 \left( \frac{-\alpha - \beta - \delta, -\alpha}{-\alpha - \beta}; \left( \frac{4t}{(1 - t)^2} \right) \right),
\] (4.56)

with
\[ \alpha + \beta + \delta + \gamma + 1 = 0. \] (4.57)

Here, $A$ and $\hat{B}$ are constants.

We note that (3.34) and (4.44) are reduced to the following equations
\[
X' = \frac{1 + \alpha + \beta + 2\delta - t(\alpha - \beta)}{t(t^2 - 1)} X^2 + \frac{\alpha + \beta + 1}{t} X + \frac{\alpha - \beta - t(\alpha + \beta + 2\delta + 1)}{t^2 - 1},
\] (4.58)

\[
F'' = \frac{2 + \alpha + \beta + 2(2 + 3\alpha + \beta + 2\delta)t + (\alpha + \beta)t^2}{t(t^2 - 1)} F' + \frac{4(\alpha + 1)(\alpha + \beta + 2\delta + 1)}{t(t - 1)^2} F = 0,
\] (4.59)
respectively. We also note that (4.56) is obtained by the following procedure. By using the transformation (see, (III.23) in [5])

\[
\text{8W}_7 \left( \alpha; b, c, d, e, f; q, \frac{q^2 a^2}{bcdef} \right) = \left( \begin{array}{c} q^a, qa, q^2 a^2, \frac{q^2 a^2}{bcdef} \end{array} \right)_{\infty} = \text{8W}_7 \left( \frac{q a^2}{b c d e f}, \frac{q a}{b c d f}, \frac{q a}{b c d e f}, \frac{q a}{b c d e f}, \frac{q a}{b c d e f}, \frac{q a}{b c d e f}, \frac{q a}{b c d e f} \right)_{\infty}
\] (4.60)
and setting

\[ \hat{B} = B \frac{\Theta(t; p)}{\Theta(-b_1 b_3 t; p)(-b_1 b_3)^{\log t / \log p}}. \]  

(4.61)

(3.42) is rewritten as

\[ F = AF_1 + \hat{B} \hat{F}_2, \]

\[ F_1 = 8 W_7 \left( -b_3^{-2} b_5; p^{1/2} b_3, -p^{1/2} b_3, -b_1 b_3 b_5, b_3 b_5 i, -b_1 b_3 b_5^{1-1}; p, p b_1^{-1} b_3^{-1} b_5^{-1} \right), \]

\[ \hat{F}_2 = \frac{(-b_3^{-1} i, -p^{1/2} b_1^{-1} b_3^{-1} b_5^{-1} i, p^{1/2} b_1^{-1} b_3^{-1} b_5^{-1} i, -p^{1/2} b_1^{-1} b_3^{-1} b_5^{-1} i, -p^{1/2} b_1^{-1} b_3^{-1} b_5^{-1} i; p)_{\infty}}{(b_1 b_5 i, -b_3 i, p^{1/2} i, -p^{1/2} b_1^{-1} b_3^{-1} b_5^{-1} i, -p^{1/2} b_1^{-1} b_3^{-1} b_5^{-1} i; p)_{\infty}} \times \frac{\Theta(-b_1 b_3 t; p)}{\Theta(t; p)} \]

\[ \times 8 W_7 \left( -b_1^{-1} b_3^{-1} b_5^{-1} i; -p b_1^{-1} b_3^{-1} b_5^{-1}, -p^{1/2} b_3^{-1}, p^{1/2} b_3^{-1}, b_3^{-1} b_5^{-1} i, -p b_1^{-1} b_3^{-1} b_5^{-1}; p, b_3 b_5 i \right). \]  

(4.62)

Then (4.66) is obtained from (4.62) by taking a term-by-term limit and using Proposition 4.7.

5 Proof of Lemma 4.5 and 4.6

In this section, we give the proof of Lemma 4.5 and 4.6. Taking the continuous limit term-by-term in the hypergeometric series in (3.14) does not yield any meaningful result. In this case, we use the saddle point method [9, 34, 45]. Consider the complex integral

\[ I(s) = \int_C e^{-g(z)/e} f(z) \, dz, \]  

(5.1)

where \( f(z) \) and \( g(z) \) are analytic functions. The point \( z = z_0 \) satisfying \( g'(z_0) = 0 \) and the direction

\[ \arg(z - z_0) = \frac{\pi}{2} - \frac{1}{2} \arg \left( \frac{g''(z_0)}{e^2} \right), \]  

(5.2)

are called a saddle point and a steepest descent direction, respectively. When the path \( C \) passes in the steepest descent direction, \( |e^{-g(z)/e} f(z)| \) reaches its peak at the saddle point. Therefore \( I(s) \) may be evaluated only around the saddle point when \( \epsilon \to 0 \). For the asymptotic expansions of the \( q \)-shifted factorials as \( q \to 1^- \), we use Proposition 4.7.

We here fix the branch of log and fractional power functions as

\[ \log z = \log |z| + i \arg z \quad (0 \leq \arg z < 2\pi), \]  

(5.3)

\[ z^{m/n} = e^{(m/n) \log z} = |z|^{m/n} e^{(m/n) \arg z} \quad (m \in \mathbb{Z}, n \in \mathbb{Z}_{>0}, 0 \leq \arg z < 2\pi), \]  

(5.4)

where \( z \in \mathbb{C}^* \). Note that \( \log(XY) = \log X + \log Y \) and \( \log(X/Y) = \log X - \log Y \) are valid only mod \( 2\pi i \).
5.1 Proof of Lemma 4.5

We consider the continuous limit of

\[ A_1 = 2\varphi_1 \left( \frac{a_2 z^2}{-p}; p, pa_2^{-1}t \right), \tag{5.5} \]

with

\[ t = 2ie^{i\epsilon - 3\alpha e^2/2}, \quad a_2 = ie^{-\epsilon e^2/2}, \quad p = e^{-\epsilon}, \quad \epsilon \to \pm 0. \tag{5.6} \]

We note that in taking the limit, the sign of \( \epsilon \) is chosen according to the value of \( s \) as shown later. By using Heine’s transformation [20]

\[ 2\varphi_1 \left( \frac{a, 0}{c}; q, z \right) = \frac{(az; q)_\infty}{(c, z; q)_\infty} 1\varphi_1 \left( \frac{z}{az}; q, c \right), \tag{5.7} \]

\( A_1 \) is rewritten as

\[ A_1 = \frac{(pa_2 t; p)_\infty}{(-p, pa_2^{-1}t; p)_\infty} 1\varphi_1 \left( \frac{pa_2^{-1}t}{pa_2 t}; p, -p \right). \tag{5.8} \]

We next prepare a suitable integral representation for \( A_1 \).

**Lemma 5.1** It holds that

\[ 1\varphi_1 \left( \frac{a}{c}; q, x \right) = \frac{(a; q)_\infty(q; q)_\infty}{2\pi i(c; q)_\infty} \int_C z^{-\log x/\log q(cz^{-1}; q)_\infty} \frac{dz}{(az^{-1}; q)_\infty(z; q)_\infty} \tag{5.9} \]

where the path \( C \) runs from \(-i\infty \) to \( i\infty \) so that the poles of \( 1/(z; q)_\infty \) lie to the right of the path and the other poles lie to the left of the path (as shown in Figure 1).

**Proof.** We derive (5.9) according to the method in [5, 34, 44]. We consider the integral

\[ \int_{C_N^0} z^{-\log x/\log q(cz^{-1}; q)_\infty} \frac{dz}{(az^{-1}; q)_\infty(z; q)_\infty} \tag{5.10} \]

where the contour \( C_N^0 \) is a large clockwise-oriented semicircle of radius \( q^{-N-1/2} \) with center at the origin and circle round only part of the poles of \( 1/(z; q)_\infty \) (as shown in Figure 2). By the residue theorem, we have

\[ \frac{1}{2\pi i} \int_{C_N^0} z^{-\log x/\log q(cz^{-1}; q)_\infty} \frac{dz}{(az^{-1}; q)_\infty(z; q)_\infty} = \frac{(c; q)_\infty}{(a; q)_\infty(q; q)_\infty} \sum_{n=0}^N \frac{(a; q)_n}{(c; q)_n(q; q)_n} (-1)^n q^{n(n-1)/2} x^n. \tag{5.11} \]

Therefore we obtain

\[ 1\varphi_1 \left( \frac{a}{c}; q, x \right) = \frac{(a; q)_\infty(q; q)_\infty}{2\pi i(c; q)_\infty} \lim_{N \to \infty} \int_{C_N^0} z^{-\log x/\log q(cz^{-1}; q)_\infty} \frac{dz}{(az^{-1}; q)_\infty(z; q)_\infty} \tag{5.12} \]

In order to estimate the contribution from

\[ C_N^1 = \left\{ z = q^{-N-1/2} e^{it} : |t| < \frac{\pi}{2} \right\}, \tag{5.13} \]
we need a bound on the integrand for large $|z|$

$$
\sup_{|z|=q^{-N-1/2}} \left| \frac{z^{-\log x/\log q}}{(z; q)_\infty} \right| \leq \left| x^{N+1/2} e^{-\pi^2/\log q} \frac{(q^{-1/2}; q)_\infty \prod_{n=1}^N (1 - q^{-1/2-n})}{(q^{-1/2}; q)_\infty} \right| = O(q^{N^2/2}), \tag{5.14}
$$

as it is dominated by the product in the denominator. From

$$
\left| \int_{C_N^1} \frac{z^{-\log x/\log q}(cz^{-1}; q)_\infty}{(az^{-1}; q)_\infty(z; q)_\infty} \, dz \right| \leq \pi q^{-N-1/2} \sup_{|z|=q^{-N-1/2}} \left| \frac{z^{-\log x/\log q}(cz^{-1}; q)_\infty}{(az^{-1}; q)_\infty(z; q)_\infty} \right| = O(q^{N^2/2}), \tag{5.15}
$$

and (5.12), we have completed the proof. 

\[ \square \]

Figure 1. Path of integration $C$

Figure 2. contour of integration $C_N^0$

By using Lemma 5.1, it holds that

$$
A_1 = \frac{(p; p)_\infty}{2\pi i(-p; p)_\infty} \int_{C_1} \frac{(pa_2tz^{-1}; p)_\infty e^{\pi i \log z/\log p}}{z(pa_2^{-1}t^{-1}; p)_\infty(z; p)_\infty} \, dz, \tag{5.16}
$$

where the path $C_1$ runs from $-i\infty$ to $i\infty$ so that the poles of $1/(z; p)_\infty$ lie to the right of the path and the other poles lie to the left of the path. We divide the integral path $C_1$ into $L_1$ which runs from $-i\infty$ to $i\infty$ so that all poles lie to the right of the path and $C_1$ which is an anticlockwise-oriented contour and encircles all poles except for the poles of $1/(z; p)_\infty$ (as shown in Figure 3).

We now evaluate $A_1$ at $\epsilon \to 0$ by using the saddle point method.

(1) Path $L_1$. Set

$$
I_1(s) = \int_{L_1} \frac{(pa_2tz^{-1}; p)_\infty e^{\pi i \log z/\log p}}{z(pa_2^{-1}t^{-1}; p)_\infty(z; p)_\infty} \, dz. \tag{5.17}
$$
Using Proposition 4.7, we rewrite $I_1(s)$ as

$$I_1(s) = \int_{L_1} e^{-g_1(z)/\epsilon^2} f_1(z)[1 + O(\epsilon)] \, dz,$$  \hspace{1cm} (5.18)

where

$$g_1(z) = Li_2 \left( -\frac{2}{z} \right) - Li_2 \left( \frac{2}{z} \right) + \pi i \log z + i \epsilon \log \left( \frac{z - 2}{z + 2} \right), \tag{5.19}$$

$$f_1(z) = e^{-2\epsilon^2 \frac{z}{(z^2 - 4)}} (z - 2)^{1/2 - \alpha} (z + 2)^{-1/2 - 2\alpha} (1 - z)^{-1/2} z^{\alpha - 1}. \tag{5.20}$$

By the identity of dilogarithm [12]

$$Li_2(z^{-1}) = -Li_2(z) - \frac{(\log z)^2}{2} - \pi i \log z + \frac{\pi^2}{3}, \tag{5.21}$$

g_1(z) is rewritten as

$$g_1(z) = Li_2 \left( \frac{z}{2} \right) - Li_2 \left( -\frac{z}{2} \right) - Li_2(z) + \pi i \log 2 + \frac{3\pi^2}{2} + i \epsilon \log \left( \frac{z - 2}{z + 2} \right) \tag{5.22}$$

$$= \pi i \log 2 + \frac{3\pi^2}{2} + \pi s \epsilon - i \epsilon \log \left( \frac{z^2}{4} + O(z^3) \right) (|z| < 1). \tag{5.23}$$
Let us first find a zero point of \( g'(z) \) (saddle point). Differentiating (5.22) by \( z \) and using (4.29), we obtain
\[
g'(z) = \frac{1}{z} \log \left( 1 + \frac{z^2}{z-2} \right) + \frac{4i\epsilon z}{z^2 - 4} = -i\epsilon - \frac{z}{2} - \frac{1 + i\epsilon}{4} z^2 + O(\epsilon^3) \quad (|z| < 1). \tag{5.25}
\]
We note here that only \( z = 0, z = \pm 2, z = \infty \) can be saddle points at \( \epsilon \to 0 \).

**Lemma 5.2** There exists only one saddle point in the domain \( |z| \leq 1/2 \).

**Proof.** Set
\[
A(z) = -\frac{4i\epsilon z}{z^2 - 4}. \tag{5.26}
\]
Functions \( g'(z) \) and \( A(z) \) are regular in \( |z| \leq 1/2 \). Furthermore, since
\[
|A(z)| << 1, \tag{5.27}
\]
it is obvious that
\[
|g'(z)| > |A(z)|, \tag{5.28}
\]
on the circle \( |z| = 1/2 \). By Rouché’s theorem, the numbers of zeros of \( A(z) + g'(z) = \log \left( 1 + \frac{z^2}{z-2} \right)/z \) are equal to those of \( g'(z) \). Therefore we have completed the proof. \( \square \)

From Lemma 5.2, the saddle point \( z_1 \) is determined by assuming the series expansion \( z = \sum_{k=1}^{\infty} C_k \epsilon^k \) in (5.25) as
\[
z_1 = -2i\epsilon + 2s^2\epsilon^2 + O(\epsilon^3). \tag{5.29}
\]
Since
\[
g''(z_1) = -\frac{1}{2} + O(\epsilon), \tag{5.30}
\]
the steepest descent direction is given as
\[
\arg(z - z_1) = \pi - \frac{1}{2} \arg \left( \frac{1 + O(\epsilon)}{2\epsilon^2} \right) - \frac{\pi}{2}. \tag{5.31}
\]
In order that \( L_1 \) passes in the steepest descent direction, it is necessary from the configuration of the paths (see Figure 3) that
\[
\text{Re}(z_1) = 2\text{Im}(s)\epsilon + 2 \left( \text{Re}(s)^2 - \text{Im}(s)^2 \right) \epsilon^2 + O(\epsilon^3) < 0. \tag{5.32}
\]
Therefore we obtain the following condition
\[
\epsilon > 0 \quad (\text{when Im}(s) < 0), \tag{5.33}
\]
\[
\epsilon < 0 \quad (\text{when Im}(s) > 0). \tag{5.34}
\]
Now, we consider an approximation of the integrand of \( I_1(s) \). Since the integrand of \( I_1(s) \) is evaluated only around the saddle point \( z = z_1 \), we assume that \( z \) is in the set
\[
D = \left\{ z = \epsilon^k A(\epsilon) \mid k \in \mathbb{R}_{>0}, \ 0 < |A(0)| < \infty \text{ or } A(\epsilon) = 0 \right\}. \tag{5.35}
\]
For simplicity, we first change the variable $z$ to $\hat{z}$ so that the saddle point is given as $\hat{z} = -2i\epsilon$. From (5.30), we set $\hat{z}$ so as to satisfy

$$g_1'(z)dz = -\frac{1}{2}(\hat{z} + 2i\epsilon)d\hat{z}.$$  (5.36)

Moreover, from (5.23) and to calculate the integration by substitution from $z$ to $\hat{z}$ simply, we also require

$$g_1(z) = -\frac{z^2}{4} - i\epsilon\hat{z}^2 + \pi i \log 2 + \frac{3\pi^2}{2} + \pi s + 2i\epsilon s^3 + \epsilon^4,$$  (5.37)

that is,

$$\hat{z}^2 + 4i\epsilon\hat{z} - 8i\epsilon s^3 - 4\epsilon^4 = 4i\epsilon s + z^2 + O(\epsilon^3).$$  (5.38)

We now add a condition between $z$ and $\hat{z}$ so that the correspondence $z \leftrightarrow \hat{z}$ is 1 : 1. To this end, we fix the branch at the point $\hat{z} = \hat{w}_0$ corresponding to $z = 0$. From (5.38) $\hat{w}_0$ satisfies

$$\hat{w}_0^2 + 4i\epsilon\hat{w}_0 - 8i\epsilon s^3 - 4\epsilon^4 = (\hat{w}_0 - 2\epsilon^2)(\hat{w}_0 + 4i\epsilon + 2\epsilon^2) = 0.$$  (5.39)

We fix the branch by choosing $\hat{w}_0 = 2\epsilon^2$, that is, from (5.38) $\hat{z}$ is expressed by $z$ as

$$\hat{z} = -2i\epsilon + 2i\epsilon \sqrt{1 - \frac{2i\epsilon}{s} - \frac{\epsilon^2}{s^2} - \frac{\hat{z}^2 + 4i\epsilon s + O(\epsilon^3)}{4s^2\epsilon^2}}.$$  (5.40)

Furthermore, $z$ can be also expressed by $\hat{z}$. Noticing that $z = e^{\hat{z}}A(\epsilon) \in D$, we obtain from (5.38)

$$z = -2i\epsilon \pm 2i\epsilon \sqrt{1 + \frac{2i\epsilon}{s} + \frac{\epsilon^2}{s^2} - \frac{\hat{z}^2 + 4i\epsilon s + O(\epsilon^3) + (A(\epsilon)^3\epsilon^3)}{4s^2\epsilon^2}}.$$  (5.41)

Since $z = 0$ corresponds to $\hat{z} = \hat{w}_0$, it holds that

$$z = -2i\epsilon + 2i\epsilon \sqrt{1 + \frac{2i\epsilon}{s} + \frac{\epsilon^2}{s^2} - \frac{\hat{z}^2 + 4i\epsilon s + O(\epsilon^3) + (A(\epsilon)^3\epsilon^3)}{4s^2\epsilon^2}}.$$  (5.42)

For later convenience, we consider the point $z = w_0$ corresponding to $\hat{z} = 0$. By the assumption, $w_0$ is given in the form

$$w_0 = e^{k_0}A_0(\epsilon),$$  (5.43)

where $k_0 \in \mathbb{R}_{>0}$ and $0 < |A_0(0)| < \infty$. From (5.38), we have

$$e^{2k_0}A_0^2 + 4i\epsilon e^{k_0+1}A_0 + 8i\epsilon s^3 + 4\epsilon^4 = O(\epsilon^{3k_0}).$$  (5.44)

Comparing the degree of $\epsilon$ of (5.44), we obtain $k_0 = 1$ or $k_0 = 2$. From (5.42), it holds that

$$w_0 = -2i\epsilon + 2i\epsilon \sqrt{1 + O(\epsilon)} = O(\epsilon^2),$$  (5.45)

which indicates

$$k_0 = 2.$$  (5.46)

Therefore we can determine $w_0$ from (5.44) as

$$w_0 = -2\epsilon^2 + O(\epsilon^3).$$  (5.47)
We proceed to the approximation of integrand of $I_1(s)$ by changing the variable from $z$ to $\hat{z}$. We expand $f_1(z)$ in the form

$$ f_1(z) \frac{dz}{d\hat{z}} = \sum_{k=0}^{\infty} C_k (\hat{z} - \hat{\omega}_0)^{k+\rho}.$$  

(5.48)

Since the characteristic exponent of $f_1(z)$ at $z = 0$ ($\hat{z} = \hat{\omega}_0 = 2e^2$) is $\alpha - 1$ (see (5.20)), it holds that

$$ \rho = \alpha - 1.$$  

(5.49)

From (5.18), (5.37) and (5.48), we obtain

$$ I_1(s) = e^{(\pi i \log 2 + \pi^2/2)/e^2 - \pi s/e} \int_{L_1} \sum_{k=0}^{\infty} C_k e^{(\hat{z}^2 + 4i\epsilon \hat{z})} (\hat{z} - 2e^2)^{k+\rho-1} [1 + O(\epsilon)] d\hat{z}$$

(5.50)

$$ = e^{(\pi i \log 2 + \pi^2/2)/e^2 - \pi s/e} C_0 \int_{\hat{L}_1} e^{(\hat{z}^2 + 4i\epsilon \hat{z})} (\hat{z} - 2e^2)^{\rho-1} [1 + O(\epsilon)] d\hat{z}.$$  

(5.51)

In the derivation of (5.51), we note that since $f_1(z)$ does not diverge as $\epsilon \to 0$, $C_k$ also does not diverge. $C_0$ in (5.51) can be determined in the following manner. From (5.36) and (5.48), we have

$$ - \frac{f_1(z)(\hat{z} + 2i\epsilon)}{2g'_1(z)} = \sum_{k=0}^{\infty} C_k (\hat{z} - 2e^2)^{k+\rho-1}.$$  

(5.52)

Substituting $z = w_0$ ($\hat{z} = 0$) in (5.52), we get

$$ C_0 = (-1)^{\alpha + 1/2} 2^{\alpha}.$$  

(5.53)

In order to adjust to the integral representation of the Weber function, we introduce the variable $u$ by $\hat{z} = 21^2 i e u$. Then we obtain

$$ I_1(s) = (-1)^{\alpha/2} 2^{\alpha} e^{(\pi i \log 2 + \pi^2/2)/e^2 - \pi s/e} C_0 \int_{\hat{L}_1} e^{-u^2/2 - 2^{1/2} su^2} u^{\alpha-1} [1 + O(\epsilon)] du$$

(5.54)

$$ = (-1)^{3\alpha/2 + 1/2} 2^{-\alpha/2} e^{(\pi i \log 2 + \pi^2/2)/e^2 - \pi s/e} \int_{\hat{L}_1} e^{-u^2/2 - 21^{1/2} su^2} u^{\alpha-1} [1 + O(\epsilon)] du,$$  

(5.55)

where the path of integration $\hat{L}_1$ runs from $-\infty$ to $\infty$ when $\text{Im}(s) < 0$ and $\infty$ to $-\infty$ when $\text{Im}(s) > 0$ so that $u = 0$ lies to the right of the path.

2) Contour $\hat{C}_1$. Set

$$ I_2(s) = \int_{\hat{C}_1} \frac{(pa z^{-1}; q)_\infty e^{\pi i \log z / \log p}}{z(a z^{-1}; q)_\infty (z; q)_\infty} d\hat{z}.$$  

(5.56)

To evaluate $I_2(s)$, we prepare the following lemma:

Lemma 5.3 It holds that

$$ \int_C \frac{(c z^{-1}; q)_\infty e^{\pi i \log z / \log p}}{z(a z^{-1}; q)_\infty (z; q)_\infty} d\hat{z} = \frac{2\pi i (ac^{-1}; q)_\infty e^{\pi i \log a / \log q}}{(q; q)_\infty (a; q)_\infty} 2\varphi_1 \left( a, qac^{-1}; 0; q, -\frac{c}{a} \right),$$

(5.57)

where $C$ is an anticlockwise-oriented contour and encircles all poles except for the poles of $1/(z; q)_\infty$. 

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\textbf{Proof.} We consider the integral
\[
\oint_{C_N} \frac{(cz^{-1}; q)_{\infty} e^{\pi i \log z / \log q}}{z(z^{-1}; q)_{\infty}(z; q)_{\infty}} \, dz,
\]
where $C_N$ is an anticlockwise-oriented contour and encircles the poles of $1/(az^{-1}; q)_N$. By the residue theorem, we obtain
\[
\frac{1}{2\pi i} \oint_{C_N} \frac{(cz^{-1}; q)_{\infty} e^{\pi i \log z / \log q}}{z(z^{-1}; q)_{\infty}(z; q)_{\infty}} \, dz = \frac{(ac^{-1}; q)_{\infty} e^{\pi i \log a / \log q}}{(q; q)_{\infty}(a; q)_{\infty}} \sum_{k=0}^{N-1} \frac{(a; q)(qac^{-1}; q)_k}{(q; q)_k} \left( -\frac{c}{a} \right)^k.
\]
Then the statement follows from the limit of $N \to \infty$. □

By using Lemma 5.3, $I_2(s)$ is rewritten as
\[
I_2(s) = \left. \frac{2\pi i(a_2^{-2}; p)_{\infty} e^{\pi i \log(pa_2^{-1}) / \log p}}{(p; p)_{\infty}(pa_2^{-1} t; p)_{\infty}} \right|_{2\varphi_1 (pa_2^{-1} t, pa_2^{-2}; 0, -a_2^2)}.
\]
Finally by using Heine’s transformation \(^{[20]}\)
\[
2\varphi_1 \left( a, b ; q, z \right) = \frac{(az, b; q)_{\infty}}{(z; q)_{\infty}} 2\varphi_1 \left( z, 0 ; a z; q, b \right),
\]
we obtain
\[
I_2(s) = \left. \frac{2\pi i(a_2^{-2}, -p; p)_{\infty} e^{\pi i \log(pa_2^{-1}) / \log p}}{(-a_2^2, p; p)_{\infty}(-1)^{\log t / \log p}} \right|_{2\varphi_1 \left( -a_2^2, 0 ; -p, pa_2^{-1} t \right)}.
\]
From (5.16), (5.55) and (5.62), we have
\[
A_1 = \frac{(p; p)_{\infty}}{2\pi i(-p; p)_{\infty}} \left( I_1(s) + I_2(s) \right)
= (1)^{3a/2} a^{-1/2} (\pi^2 / 4)^{a/2} e^{-1} \int_{L_1} e^{-u^2 / 2 - 1 / 2} u^{a-1} \left[ 1 + O(\varepsilon) \right] \, du
+ \frac{(a_2^{-2}; p)_{\infty} e^{\pi i \log(pa_2^{-1}) / \log p}}{(-a_2^2; p)_{\infty}(-1)^{\log t / \log p}} 2\varphi_1 \left( -a_2^2, 0 ; -p, pa_2^{-1} t \right),
\]
which proves Lemma 4.5.

\subsection{5.2 Proof of Lemma 4.6}

We consider the continuous limit of
\[
A_2 = 2\varphi_1 \left( -a_2^2, 0 ; p, pa_2^{-1} t \right),
\]
with (5.6). By using Heine’s transformation \(^{[20]}\)
\[
2\varphi_1 \left( a, 0 ; c, z \right) = \frac{1}{(z; q)_{\infty}} 1\varphi_1 \left( a^{-1} c ; c, q, az \right),
\]
(5.66)
$A_2$ is rewritten as

$$A_2 = \frac{1}{(pa_2^{-1} t; p)_\infty} \varphi_1 \left( pa_2^{-2} - p, -pa_2 t \right). \quad (5.67)$$

We prepare a suitable integral representation for $A_2$.

**Lemma 5.4** It holds that

$$\varphi_1 \left( \frac{a}{c}; q, x \right) = \frac{(a; q)_\infty(q; q)_\infty}{2\pi i(c; q)_\infty} \int_C \frac{z^{-\log \frac{x}{\log z} \log q(cz^{-1}; q)_\infty}}{(az^{-1}; q)_\infty(z; q)_\infty} \, dz, \quad (5.68)$$

where the path $C$ runs from $-\infty$ to $+\infty$ so that the poles of $1/(z; q)_\infty$ lie to the right of the path and the other poles lie to the left of it.

The Lemma 5.4 can be proved in a similar manner to Lemma 5.1. Although Lemma 5.1 and Lemma 5.4 differ only in the integration path, it is appropriate to apply Lemma 5.4 in order to choose the integration path in the steepest descent direction. By using Lemma 5.4, it holds that

$$A_2 = \frac{(pa_2^{-2}; p)_\infty(p; p)_\infty}{2\pi i(-p; p)_\infty(pa_2^{-1} t; p)_\infty} \int_{C_2} \frac{z^{-\log(-pa_2 t)/\log(-pz^{-1}; p)_\infty}}{(pa_2^{-2}z^{-1}; p)_\infty(z; p)_\infty} \, dz, \quad (5.69)$$

where the path $C_2$ runs from $-\infty$ to $+\infty$ so that the poles of $1/(z; p)_\infty$ lie to the right of the path and the other poles lie to the left of it. We divide the path $C_2$ into $L_2$ which runs from $-\infty$ to $+\infty$ so that all poles lie to the right of the path and $\hat{C}_2$ which is an anticlockwise-oriented contour and encircles all poles except for the poles of $1/(z; p)_\infty$ (as shown in Figure 4).

![Figure 4. The paths of integration $L_2$ and $\hat{C}_2$](image)
Further, by applying Heine's transformation
\[
I_3(s) = \int_{L_2} e^{-\log(-pa_2z) \log z/ \log p (-p^{-1}z); p)_{\infty}} (pa_2^{-2}z^{-1}; \infty(z; p)_{\infty}) \, dz.
\] (5.70)

In a similar manner to \( I_1(s) \), one can show that
\[
I_3(s) = (-1)^{-3a+1} 2^{a/2} e^{-a+1} e^{-2\pi i/2} e^{\log(-p^{-1}z)/e^{2}} e^{-\pi i/\varepsilon} \int_{L_2} e^{-i\varepsilon/2 - 2i/3 s} u^{-\sigma}[1 + O(\varepsilon)] \, du,
\] (5.71)

under the assumption
\[
\varepsilon > 0 \quad \text{(when Re}(s) < 0),
\]
\[
\varepsilon < 0 \quad \text{(when Re}(s) > 0).
\] (5.72) (5.73)

Here the path \( \hat{L}_2 \) runs from \(-\infty\) to \(+\infty\) when Re\( (s) > 0 \) and \(+\infty\) to \(-\infty\) when Re\( (s) < 0 \) so that \( u = 0 \) lies to the right of the path.

(2) Contour \( \hat{C}_2 \). Set
\[
I_4(s) = \int_{\hat{C}_2} e^{-\log(-pa_2z) \log z/ \log p (-p^{-1}z); p)_{\infty}} z^{-2} (pa_2^{-2}z^{-1}; p)_{\infty}(z; p)_{\infty}) \, dz.
\] (5.74)

In the following, we transform the integral in (5.74) into another suitable integral, then evaluate its asymptotic behavior. More precisely, we first identify \( I_4(s) \) with appropriate basic hypergeometric function, then apply Heine's transformation and reconstruct the integral representation of it. We next evaluate its asymptotic behavior.

**Lemma 5.5** It holds that
\[
\int_{C} e^{-\log(-pa_2z) \log z/ \log p (-p^{-1}z); p)_{\infty}} d\zeta
= \frac{2\pi (a_2^2; p)_{\infty}}{a_2^2 \Gamma(p; p)_{\infty}(pa_2^{-2}; p)_{\infty}} \, _2\Phi_1 \left( \begin{array}{c} pa_2^{-2}, -pa_2^{-2} \\ 0, p, \frac{a_2}{t} \end{array} \right),
\] (5.75)

where \( C \) is an anticlockwise-oriented contour and encircles all poles except for the poles of \( 1/(z; p)_{\infty} \).

Lemma 5.5 can be proved in a similar manner to Lemma 5.3. Then \( I_4(s) \) is rewritten by using Lemma 5.5 as
\[
I_4(s) = \frac{2\pi (a_2^2; p)_{\infty}}{a_2^2 \Gamma(p; p)_{\infty}(pa_2^{-2}; p)_{\infty}} 2\Phi_1 \left( \begin{array}{c} pa_2^{-2}, -pa_2^{-2} \\ 0, p, \frac{a_2}{t} \end{array} \right).
\] (5.76)

Further, by applying Heine’s transformation [20]
\[
2\Phi_1 \left( \begin{array}{c} a, b, q, z \\ 0, q, z \end{array} \right) = \frac{(bz; q)_{\infty}}{(z; q)_{\infty}} \Psi \left( \begin{array}{c} b z, q, a z \\ b z, q, a z \end{array} \right),
\] (5.77)
and using Lemma 5.4, we obtain

\[ I_4(s) = \frac{(-a_2^2, -pa_2^{-2}, p)_{\infty}}{a_2^2 t(pa_2^{-2}, a_2^{-1} t; p)_{\infty}} \int_{C_3} e^{-\log(pa_2^{-1} t^{-1}) \log z / \log p} \frac{-pa_2^{-1} t^{-1} z^{-1}; p_{\infty}}{(-pa_2^{-2} z^{-1}; p_{\infty}(z; p)_{\infty})} \, dz, \]  

where the path \( C_3 \) runs from \(-\infty\) to \(+\infty\) so that the poles of \( 1/(z; p)_{\infty} \) lie to the right of the path and the other poles lie to the left of it. We finally evaluate the integral of the right hand side of (5.77). To this end, we divide the path \( C_3 \) into \( L_3 \) which runs from \(-\infty\) to \(+\infty\) so that all poles lie to the right of the path and \( \hat{C}_3 \) which is an anticlockwise-oriented contour and encircles all poles except for the poles of \( 1/(z; p)_{\infty} \).

Set

\[ I_4^{(1)}(s) = \int_{L_3} \frac{e^{-\log(pa_2^{-1} t^{-1}) \log z / \log p} (-pa_2^{-1} t^{-1} z^{-1}; p_{\infty})}{(-pa_2^{-2} z^{-1}; p_{\infty}(z; p)_{\infty})} \, dz, \]  

\[ I_4^{(2)}(s) = \int_{\hat{C}_3} \frac{e^{-\log(pa_2^{-1} t^{-1}) \log z / \log p} (-pa_2^{-1} t^{-1} z^{-1}; p_{\infty})}{(-pa_2^{-2} z^{-1}; p_{\infty}(z; p)_{\infty})} \, dz. \]

In a similar manner to \( I_1(s) \), the asymptotic behavior of \( I_4^{(1)}(s) \) as \( \epsilon \to 0 \) can be evaluated as

\[ I_4^{(1)}(s) = \left( -1 \right)^{1/2 + a} 2^{-3a/2} 2^{-1/2 - 2} e^{\frac{r^2}{4} + is \log 2 / 2} \int_{L_2} e^{-u^2 / 2 - 2 \log u} u^{-a} \left[ 1 + O(\epsilon) \right] \, du. \]

\( I_4^{(2)}(s) \) can be expressed in terms of the basic hypergeometric function. In fact, we have the following lemma which is proved in a similar manner to Lemma 5.3:

**Lemma 5.6** It holds that

\[ \int_C \frac{e^{-\log(pa_2^{-1} t^{-1}) \log z / \log p} (-pa_2^{-1} t^{-1} z^{-1}; p_{\infty})}{(-pa_2^{-2} z^{-1}; p_{\infty}(z; p)_{\infty})} \, dz = -\frac{2\pi ia(z_2 t^{-1}; p_{\infty}) e^{\log(pa_2^{-1} t^{-1}) \log(-a_2^{-2}) / \log p}}{a_2(-pa_2^{-2}, p; p_{\infty})} 2\phi_1 \left( pa_2^{-1} t, -pa_2^{-2} \atop 0 \right), \]

where \( C \) is an anticlockwise-oriented contour and encircles all poles except for the poles of \( 1/(z; p)_{\infty} \).

Then by using Lemma 5.6, \( I_4^{(2)}(s) \) yields

\[ I_4^{(2)}(s) = -\frac{2\pi ia(z_2 t^{-1}; p_{\infty}) e^{\log(pa_2^{-1} t^{-1}) \log(-a_2^{-2}) / \log p}}{a_2(-pa_2^{-2}, p; p_{\infty})} 2\phi_1 \left( pa_2^{-1} t, -pa_2^{-2} \atop 0 \right), \]

which is rewritten by applying Heine’s transformation (5.61) as

\[ I_4^{(2)}(s) = -\frac{2\pi ia(z_2 t^{-1}, -p, pa_2^{-1} t; p_{\infty}) e^{\log(pa_2^{-1} t^{-1}) \log(-a_2^{-2}) / \log p}}{a_2(-pa_2^{-2}, p, a_2^{-1}; p_{\infty})} 2\phi_1 \left( a_2^2, 0 \atop -p \atop p, pa_2^{-1} \right). \]
Therefore we finally obtain
\[
A_2 = \frac{(pa_2^{-2}; p)_\infty(p; p)_\infty}{2\pi i(-p; p)_\infty(pa_2^{-1}t; p)_\infty}\left(I_3(s) - \frac{(-a_2^2, -pa_2^{-2}; p)_\infty e^{2\log(-pa_2t)\log a_2/\log p}}{a_2^2(t(pa_2^{-2}, a_2t^{-1}; p)_\infty}(I_4^{(1)}(s) + I_4^{(2)}(s))\right)
\]
\[
= (-1)^{-4a^2 - 1/2+\alpha/2} \pi^{-1/2} e^{-a} e^{2\pi i \log 2/\epsilon^2}\int_L e^{-i\theta^2/2 - 21/2isu} \epsilon^{-\alpha}[1 + O(\epsilon)]du
\]
\[
+ \frac{(-a_2^2; p)_\infty e^{(2\log(-pa_2)\log a_2 - \log(pa_2^{-1})\log(-a_2^{-2}))/\log p}}{a_2^4(a_2^2; p)_\infty}(-1)^{\log t/\log p} _2\phi_1\left(\begin{array}{c} a^2, 0 \\ -p^{1/2} \end{array} \right),
\]
where we have used Proposition 4.7. This proves Lemma 4.6.

6 Concluding remarks

In this paper, we have constructed the hypergeometric solutions to the symmetric $q$-Painlevé equations of the types $(A_2 + A_1)^{(1)}$, $A_4^{(1)}$, $D_5^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$, and discussed their continuous limits. In particular, we have shown that the hypergeometric function appearing in the solution to the symmetric $q$-P($A_4^{(1)}$) actually reduces to the Weber function by applying the saddle point method to its integral representation.

Before closing, we give a remark on a $q$-Painlevé equation of the type $A_4^{(1)}$. The following $q$-difference equation
\[
\frac{g}{g'} = \frac{(f + t^{-1})(f + \alpha t^{-1})}{1 + \gamma f}, \quad \frac{f}{f'} = \frac{(g + q^{1/2} \alpha \beta t^{-1})(g + q^{1/2} \beta^{-1} t^{-1})}{1 + \gamma^{-1} g},
\]
is usually referred to as a $q$-Painlevé equation of type $A_4^{(1)}$. Equation (6.1) also describes a translation on the root lattice of type $A_4^{(1)}$, but its direction is different from that of (2.5). It may be interesting to note that the symmetric $q$-Painlevé equation obtained from (6.1) by the projective reduction
\[
\frac{X}{X'} = \frac{(X + t^{-1})(X + \alpha t^{-1})}{1 + X},
\]
has no hypergeometric solution (see [26]) which is a sharp contrast to (2.6).

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References

[1] E. Brézin and V.A. Kazakov, Exactly solvable theories of closed strings, Phys. Lett. B 236 (1990) 144–150.

[2] M.R. Douglas and S.H. Shenker, Strings in less than one dimension, Nucl. Phys. B 335 (1990) 635–654.
[3] A.S. Fokas, A.R. Its and A.V. Kitaev, The isomonodromy approach to matrix models in 2D quantum gravity, Comm. Math. Phys. 147 (1992) 395–430.

[4] B. Grammaticos, F.W. Nijhoff, V. Papageorgiou, A. Ramani and J. Satsuma, Linearization and solutions of the discrete Painlevé III equation, Phys. Lett. A 185 (1994) 446–452.

[5] G. Gasper and M. Rahman, Basic Hypergeometric Series (Second edition), Encyclopedia of Mathematics and Its Applications 96 (Cambridge University Press, Cambridge, 2004).

[6] B. Grammaticos and A. Ramani, On a novel $q$-discrete analogue of the Painlevé VI equation, Phys. Lett. A 257 (1999) 288–292.

[7] B. Grammaticos and A. Ramani, Discrete Painlevé equations: a review, Lecture Notes in Physics 644 (2004) 245–321.

[8] B. Grammaticos, A. Ramani and V. Papageorgiou, Do integrable mappings have the Painlevé property?, Phys. Rev. Lett. 67 (1991) 1825–1828.

[9] T. Hamamoto, K. Kajiwara and N.S. Witte, Hypergeometric solutions to the $q$-Painlevé equation of type $(A_1 + A_1')^{(1)}$, Int. Math. Res. Not. 2006 (2006) Article ID 84619.

[10] M.E.H. Ismail and M. Rahman, The associated Askey-Wilson polynomials, Trans. Amer. Math. Soc. 328 (1991) 201–237.

[11] M. Jimbo and H. Sakai, A $q$-analog of the sixth Painlevé equation, Lett. Math. Phys. 38 (1996) 145–154.

[12] A.N. Kirillov, Dilogarithm identities, Progr. Theoret. Phys. Suppl. 118 (1995) 61–142.

[13] K. Kajiwara, The discrete Painlevé II equation and the classical special functions, in Symmetries and integrability of difference equations, eds. by P. Clarkson and F.W. Nijhoff, London Math. Soc. Lecture Note Ser. 255 (Cambridge University Press, Cambridge, 1999) 217–227.

[14] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada, Hypergeometric solutions to the $q$-Painlevé equations, Int. Math. Res. Not. 2004 (2004) 2497–2521.

[15] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada, Construction of hypergeometric solutions to the $q$-Painlevé equations, Int. Math. Res. Not. 2005 (2005) 1441–1463.

[16] K. Kajiwara, N. Nakazono and T. Tsuda, Projective reduction of the discrete Painlevé system of type $(A_2 + A_1)^{(1)}$, Int. Math. Res. Not. 2011 (2011) 930–966.

[17] K. Kajiwara, Y. Ohta and J. Satsuma, Casorati determinant solutions for the discrete Painlevé III equation, J. Math. Phys. 36 (1995) 4162–4174.

[18] K. Kajiwara, Y. Ohta, J. Satsuma, B. Grammaticos and A. Ramani, Casorati determinant solutions for the discrete Painlevé-II equation, J. Phys. A: Math. Gen. 27 (1994) 915–922.

[19] K. Kajiwara and J. Satsuma, $q$-difference version of the two-dimensional Toda lattice equation, J. Phys. Soc. Japan 60 (1991) 3986–3989.
[20] R. Koekoek, P.A. Lesky and R.F. Swarttouw, Hypergeometric orthogonal polynomials and their \(q\)-analogues, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010.

[21] M.D. Kruskal, K.M. Tamizhmani, B. Grammaticos and A. Ramani, Asymmetric discrete Painlevé equations, Regul. Chaotic Dyn. 5 (2000) 273–280.

[22] T. Masuda, Classical transcendental solutions of the Painlevé equations and their degeneration, Tohoku Math. J. 56 (2004) 467–490.

[23] R.J. McIntosh, Some asymptotic formulae for \(q\)-shifted factorials, Ramanujan J. 3 (1999) 205–214.

[24] M. Murata, H. Sakai and J. Yoneda, Riccati solutions of discrete Painlevé equations with Weyl group symmetry of type \(E_8^{(1)}\), J. Math. Phys. 44 (2003) 1396–1414.

[25] N. Nakazono, Hypergeometric \(\tau\) Functions of the \(q\)-Painlevé Systems of Type \((A_2 + A_1)^{(1)}\), SIGMA Symmetry Integrability Geom. Methods Appl. 6 (2010).

[26] N. Nakazono and S. Nishioka, Solutions to a \(q\)-analog of the Painlevé III equation of type \(D_4^{(1)}\), Funkcial. Ekvac. (in press).

[27] S. Nakao, K. Kajiwara and D. Takahashi, On the Multiplicative \(dP_\mathfrak{II}\) and its Ultradiscretization, Kyushu University Ouyorikigakuenkyushukai Houkoku, 9ME-S2, Evolutionary Advancement in Theories of Soliton (1998) 125–130 (In Japanese).

[28] M. Noumi, Painlevé equations through symmetry (American Mathematical Society, Providence, 2004).

[29] Y. Ohta, A. Ramani and B. Grammaticos, An affine Weyl group approach to the eight-parameter discrete Painlevé equation. Symmetries and integrability of difference equations, J. Phys. A: Math. Gen. 34 (2001) 10523–10532.

[30] Y. Ohta, Self-dual structure of the discrete Painlevé equations, RIMS Kokyuroku 1098 (1999) 130–137 (in Japanese).

[31] K. Okamoto, Studies on the Painlevé equations. III. Second and Fourth Painlevé equation, \(P_{\mathfrak{II}}\) and \(P_{\mathfrak{IV}}\), Math. Ann. 275 (1986) 221–255.

[32] K. Okamoto, Studies on the Painlevé equations. II. Fifth Painlevé equation \(P_{\mathfrak{V}}\), Japan. J. Math. 13 (1987) 47–76.

[33] V. Periwal and D. Shevitz, Unitary-matrix models as exactly solvable string theories, Phys. Rev. Lett. 64 (1990) 1326–1329.

[34] T. Prellberg, Uniform \(q\)-series asymptotics for staircase polygons, J. Phys. A 28 (1995) 1289–1304.

[35] G.R.W. Quispel, J.A.G Roberts and C.J. Thompson, Integrable mappings and soliton equations, Phys. Lett. A 126 (1988) 419–421.
[36] G.R.W. Quispel, J.A.G Roberts and C.J. Thompson, Integrable mappings and soliton equations II, Physica D 34 (1989) 183–192.

[37] A. Ramani and B. Grammaticos, Discrete Painlevé equations: coalescences, limits and degeneracies, Physica A 228 (1996) 150–159.

[38] A. Ramani, B. Grammaticos and J. Hietarinta, Discrete versions of the Painlevé equations, Phys. Rev. Lett. 67 (1991) 1829–1832.

[39] A. Ramani, B. Grammaticos, T. Tamizhmani and K. M. Tamizhmani, Special function solutions of the discrete Painlevé equations. Advances in difference equations, III, Comput. Math. Appl. 42 (2001) 603–614.

[40] A. Ramani, Y. Ohta, J. Satsuma and B. Grammaticos, Self-duality and schlesinger chains for the asymmetric d-P II and q-P III equations, Comm. Math. Phys. 192 (1998) 67–76.

[41] H. Sakai, Casorati determinant solutions for the q-difference sixth Painlevé equation, Nonlinearity 11 (1998) 823–833.

[42] H. Sakai, Rational surfaces associated with affine root systems and geometry of the Painlevé equations, Comm. Math. Phys. 220 (2001) 165–229.

[43] K.M. Tamizhmani, B. Grammaticos, A.S. Carstea and A. Ramani, The q-discrete Painlevé IV equations and their properties, Regul. Chaotic Dyn. 9 (2004) 13–20.

[44] G.N. Watson, The continuation of functions defined by generalized hypergeometric series, Trans. Camb. Phil. Soc 21 (1910) 281–299.

[45] R. Wong, Asymptotic approximations of integrals, (Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001).