Impact of a moon on the evolution of a planet’s rotation axis: a non-resonant case

O. M. Podvigina · P. S. Krasilnikov

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Abstract
We investigate how the temporal evolution of the rotation axis of a hypothetical exo-Earth is affected by the presence of a satellite, an exo-Moon. Namely, we study analytically and numerically how the range of the nutation angle of an exo-Earth changes if an exo-Moon is added to a system comprised of an exo-Sun, the exo-Earth and exoplanets. We say that the impact of an exo-Moon is stabilising if upon including the exo-Moon the range of the nutation angle decreases, and destabilising otherwise. The problem is considered in a general set-up. The exo-Earth is supposed to be rigid, axially symmetric and almost spherical, the difference between the largest and the smallest principal moments of inertia being a small parameter of the problem. Assuming the orbits of the celestial bodies to be quasi-periodic, we apply time averaging over fast variables associated with order one frequencies to study rotation of the exo-Earth at times large relative the respective periods. Non-resonant frequencies are assumed. For a system comprised of the exo-Sun and exoplanets in the absence of small orbital frequencies, the system is integrable, which allows to calculate the range of the nutation angle as a function of initial conditions. Using these expressions, we identify a class of systems for which we prove analytically that the impact of the exo-Moon is stabilising and a class where it is destabilising. Namely, if the orbits of the planets are circular and their orbital planes coincide then the impact is destabilising. The impact is stabilising if the angle between orbital planes of the exo-Moon and the exo-Earth vanishes. We also investigate numerically how the impact of the exo-Moon in a particular system comprised of a star and two planets varies on modifying parameters of the orbits of the exo-Moon and the second planet, and the initial nutation angle.

Keywords Nutation angle · Obliquity · Exoplanet · Averaging · Hamiltonian dynamics

O. M. Podvigina
olgap@mitp.ru

1 Institute of Earthquake Prediction Theory and Mathematical Geophysics, Russian Academy of Sciences, 84/32 Profsoyuznaya St, Moscow, Russian Federation 117997

2 Moscow Aviation Institute, 4 Volokolamskoe Av., Moscow, Russian Federation 125993
1 Introduction

Rotation of a rigid body, planet or a satellite, due to the torques exerted by other bodies is a classical problem of celestial mechanics. The recently increased interest in this problem is related to the dependence of a planet’s climate on the temporal evolution of its rotation axis and the fact that stable benign climate is favourable for the development of advanced life (Armstrong et al. 2014; Cowan et al. 2012; Ferreira et al. 2014; Heller et al. 2011; Kilic et al. 2017; Spiegel et al. 2009). For instance, the obliquity of the Earth, i.e. the orientation of its rotation axis relative the orbit, varies just between 22.1° and 24.5° and its orbital eccentricity between 0 and 0.06, but even such small variations result in the occurrence of glacial/interglacial cycles accompanied by substantial changes of average temperatures (Kasting and Catling 2003; Milankovitch 1941).

Numerical simulations of Laskar and co-authors of the evolution of the Earth’s obliquity with and without the Moon (Laskar and Robutel 1993a; Laskar et al. 1993b, c) revealed that the Moon has a stabilising effect. If in general the presence of a heavy moon stabilises the obliquity of a planet, then the presence of a moon may be useful for identifying possibly habitable planets. No compelling evidence was found that the observed exoplanets possess exomoons (Kipping et al. 2013a, b). Therefore, if a planet is life-bearing only when it is accompanied by a large moon, this requirement significantly reduces the chances for intelligent life to develop.

Most studies of the influence of a satellite on the evolution of a planet’s obliquity have focused on a particular case of the Earth–Moon system. Numerical integration of the equations of precession indicates that while the obliquity of the moonless Earth would vary chaotically from 0° to 85° (Néron de Surgy and Laskar 1997; Laskar and Robutel 1993a; Laskar et al. 1993b, c), in the presence of the Moon the window of initial values of obliquity resulting in chaotic behaviour decreases to between 60° to 90°, and outside this window variations of obliquity are much smaller. However, numerical results of Lissauer et al. (2011) imply that the timescales involved are very large. In their simulations of moonless Earth’s obliquity over up to 4 Gyr, the difference between its maximal and minimal values did not exceed 10°, implying the conclusion that “A large moon thus does not seem to be needed to stabilise the obliquity of an Earth-like planet on timescales relevant to the development of advanced life”. This conjecture was supported in Li and Batygin (2014) by analytical estimates of the characteristic Lyapunov exponents and the chaotic diffusion rate, their smallness implying that “the stochastic change in Earth’s obliquity is sufficiently slow to not preclude long-time habitability”.

The evolution of the obliquity of a planet not influenced by a moon was studied in many papers, e.g. in Correia et al. (2003), Correia and Laskar (2003), Hamilton and Ward (2004), Krasilnikov and Amelin (2018), Laskar et al. (2004), Ward and Hamilton (2004) for Solar system planets and in Quarles et al. (2020), Shan and Li (2018) for exoplanets. In such systems, the obliquity often undergoes substantial variations in the course of temporal evolution, thus supporting the conjecture on the stabilising influence of a moon. Studies of a planet with a moon, rather than the Earth, are very limited in number. The findings of Quillen et al. (2018) indicate that inclusion of a moon can possibly have stabilising or destabilising effects.

Here we investigate the influence of a heavy satellite on the rotation of a planet in a general set-up. We study numerically and analytically the behaviour of the nutation angle on large timescales in a planetary system comprised of a star, planets and a satellite, orbiting one of the planets. We call this planet and its satellite exo-Earth and exo-Moon, respectively. The exo-Earth is an axially symmetric rigid body, and the difference between its largest and
The smallest principal moments of inertia is a small parameter. Other celestial bodies are assumed to be point masses. The planets move along quasi-periodic orbits with prescribed frequencies \((\omega_1, \ldots, \omega_K)\). The orbit of the exo-Moon keeps a constant inclination to the ecliptic and is involved in two types of slow precessional motion, nodal and apsidal, with frequencies \(\sigma_n\) and \(\sigma_a\), respectively. The frequencies \(\omega = (\omega, \omega_1, \ldots, \omega_K)\), where \(\omega\) is the frequency of the exo-Earth’s rotation, are order one and non-resonant.

The nutation angle is one of the three Euler’s angles relating two coordinate systems: one with the origin at the centre of mass of the exo-Earth and the coordinate axes coinciding with the exo-Earth’s principal axes, and the other one with the same origin and coordinate axes parallel to those of a steady inertial reference frame. The angle of nutation is the angle between the normal to a fixed ecliptic and the exo-Earth symmetry axis.

Following the approach of Krasilnikov and Podvigina (2018), Podvigina and Krasilnikov (2020) (see also Markeev and Krasilnikov 1981), we study the temporal evolution of the rotation axis of the exo-Earth under the torques due to other bodies by expanding the Hamiltonian describing rotation of the exo-Earth in a power series in the small parameter and applying averaging over fast variables related to the motions associated with the order one frequencies \(\omega\). Averaging over one or several fast variables is often employed (Beletskii 1966, 1972; Bouquillon et al. 2003; Correia 2015; Lhotka 2017; Peale 1969; Saillenfest et al. 2019; Ward 1975) in the studies of rotation of celestial bodies. It may reduce equations for the temporal evolution of the rotation axis of a planet or a satellite into an integrable problem, see, for example, Beletskii (1975), Krasil’nikov and Zaharova (1993), Krasilnikov and Podvigina (2018) and Podvigina and Krasilnikov (2020).

Our work differs from many other studies of the rotation of a planet in that our equations involve nutation angle, i.e. the angle between a principal axis of the planet and the normal to a non-moving plane, while typically the Hamilton equations involve obliquity, i.e. the angle between the rotation axis of a planet and the normal to its orbital plane varying in time. Consequently, the equations derived in Sect. 2 differ from those, for example, in Néron de Surgy and Laskar (1997) or Li and Batygin (2014), Laskar and Robutel (1993a), Laskar et al. (1993b, c), Laskar et al. (2004) and Lissauer et al. (2011). Note that the nutation angle was employed in the studies of rotation of a planet or a satellite in the classical works of Tisserand (1889), Smart (1953), Beletskii (1966, 1972, 1975) and in more recent papers (Aslanov 2017; Krasilnikov 2015, 1990; Krasil’nikov and Zaharova 1993; Markeev and Krasilnikov 1981).

Our study is also distinct in that the torques from all celestial bodies are involved in the Hamiltonian, while usually only the torque from the (exo-)Sun is present. To be more precise, consider a simple system comprised of an exo-Sun and two planets, an exo-Earth and an exo-Jupiter. Without the exo-Jupiter, the orbit of the exo-Earth is a Keplerian ellipse and the rotation of the exo-Earth is accompanied by a uniform precession. If the rotation axis coincides with a principal axis, then the nutation angle coincides with the obliquity and does not change in time. Including the exo-Jupiter into the system affects the rotation of the exo-Earth in two ways. The direct impact is that the torque from the exo-Jupiter acting on the exo-Earth changes the motion of its rotation axis. The indirect impact is that the mutual attraction between the exo-Jupiter and the exo-Earth perturbs their orbits, thus perturbing the torque exerted by the exo-Sun on the exo-Earth.

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1 Each frequency involved in the motion of celestial bodies can be assigned to one of two sets as follows. Some of the frequencies (such as the frequency of rotation of the Earth around its axis or the frequencies of the planetary orbital motions) are relatively large, and some (such as the frequency of precession of the Moon’s orbit or the frequencies related to perturbations of the Earth’s motion due to the motion of other planets) are relatively small (secular frequencies). In our asymptotic analysis, upon appropriately rescaling time, we treat frequencies belonging to the first set as order one frequencies.
Denote by $a_E$ and $a_J$ the semi-major axes of the orbits of the exo-Earth and the exo-Jupiter, respectively, orbiting the exo-Sun. We may presume that if $|a_E - a_J| \ll a_E$ then the torque from the exo-Jupiter can become comparable with the torque from the exo-Sun, implying that the direct impact is larger than the indirect one, since the torque from the exo-Jupiter is likely to be larger than a perturbation of the torque from the exo-Sun. (For example, Krasilnikov and Amelin (2018) have shown that the direct impact from Jupiter on the rotation of Saturn prevails over the indirect impact from any Solar system planet.) By contrast, if $a_E \ll a_J$, we may expect the indirect impact to be larger than the direct one.

In the system studied in Sect. 3, the motion of planets is quasi-periodic with frequencies that are non-resonant and order one. In this case, the averaged Hamilton equations for the rotation of the exo-Earth derived in Sect. 2 are integrable. This is also in contrast with many papers exploring the rotation of planets, where numerical results indicate that the temporal evolution of the rotation axis is chaotic (e.g. Néron de Surgy and Laskar 1997 or Li and Batygin 2014; Laskar and Robutel 1993a; Laskar et al. 1993b,c, 2004; Lissauer et al. 2011). Chaos is absent for two reasons: there are no resonances, and equations are averaged over fast variables related to motions associated with all frequencies, except those describing the evolution of the exo-Moon’s orbit: our intent is to study a system as simple as possible. The approach enables us to take into account resonances between order one frequencies (see, for example, Krasilnikov 2015, 1990), and also include several small frequencies.

Our formal definition of the stabilising and destabilising influence of the exo-Moon on the rotation of the exo-Earth is based on comparison of the range of nutation angle,

$$
\Delta(I_0, h_0) = \sup_{-\infty < t < \infty} I(t, I_0, h_0) - \inf_{-\infty < t < \infty} I(t, I_0, h_0),
$$

in the moonless system and in the system with the exo-Moon. Here $I_0$ and $h_0$ are the initial inclination and longitude of the spin axis, and $I(t, I_0, h_0)$ is the angle of nutation at time $t$ for this initial data. If the range $\Delta$ decreases upon including the exo-Moon, we call the impact of the exo-Moon stabilising, and otherwise, we call the impact destabilising.

The paper has the following structure:

In Sect. 2, we recall the Hamilton equations for rotation of a rigid body and describe averaging. The averaged equations involve six functions $D_j(t)$ which are constant in the moonless system and become time-periodic upon including the exo-Moon. We simulate their evolutions for the given masses and orbits of the celestial bodies. In Sect. 3, we calculate analytically the range of the nutation angle in a system comprised of an exo-Sun and exoplanets, exploiting the smallness of the planets’ masses relative to the mass of the exo-Sun. The main result of this section is Eq. (28) that gives the range as a function of the initial nutation angle. In Sect. 4, we analyse the impact of the exo-Moon using the results of Sect. 3. We present examples of systems, for which we prove analytically that the impact of the exo-Moon is stabilising or destabilising (Sects. 4.1 and 4.2, respectively). We continue by considering a simple system comprised of a star and two planets, an exo-Sun, an exo-Earth and an exo-Jupiter, the orbits of the planets being Keplerian ellipses (see Fig. 1). We investigate numerically how adding an exo-Moon modifies the range of nutation angle depending on the eccentricities, semi-major axes and inclinations of the exo-Jupiter’s and exo-Moon’s orbits. In Sect. 5, we compare the temporal behaviour of the exo-Earth’s rotation axis described by the original and averaged Hamilton equations for the system studied in Sect. 4.3. In Sect. 6, we study the impact of a hypothetical exo-Moon on rotation on a planet in the system 7 Canis Majoris. Finally, we briefly summarise our results and indicate possible directions for further studies.
2 Equations of motion

In this section, we derive equations governing the rotation of the exo-Earth on large timescales that will be used in Sects. 3–6 for investigating the impact of the exo-Moon. The orbits of celestial bodies are quasi-periodic with the respective frequencies divided into two groups: the order one frequencies that are non-resonant and other frequencies that typically are small (secular frequencies). By averaging over the fast variables associated with these non-resonant frequencies, we obtain a Hamiltonian which involves six quasi-periodic functions $D_j(t)$, the frequencies of $D_j(t)$ being the small frequencies of the original system. The procedure can be applied for investigation of the rotation of a planet (or any other rigid body), as long as it is almost spherical and the order one frequencies over which the averaging is performed are non-resonant.

This approach was proposed in Markeev and Krasilnikov (1981), where the rotation of a satellite in a three-body system was studied. It was also applied in Krasil’nikov and Zaharova (1993) and Podvigina and Krasilnikov (2020) to investigate the rotation of a planet in a system comprised of a star and several planets. In these papers, the motions of celestial bodies were quasi-periodic involving order one frequencies only, and thus, the functions $D_j$ were time-independent constants.

Let a system under investigation involve $N$ celestial bodies, other than the exo-Earth. Each of $D_j(t)$, $1 \leq j \leq 6$, can be expressed as $D_j(t) = \sum_{1 \leq n \leq N} D^{(n)}_j(t)$, where $D^{(n)}_j$ are calculated from the torque exerted by the $n$th body on the exo-Earth. In Sect. 2.3, we calculate the functions $D^{(n)}_j$ for the exo-Sun provided the orbit of the exo-Earth is a Keplerian ellipse, and for the exo-Moon, provided its orbit is a Keplerian ellipse with a constant inclination to the ecliptic, which is involved in two types of slow precessional motion, nodal and apsidal, with frequencies $\sigma_n$ and $\sigma_a$. The functions $D^{(n)}_j$ for the exo-Sun are time-independent and those for the exo-Moon depend on $\sigma_n t$ only.

2.1 Hamilton equations

Denote by $OXYZ$ a non-moving inertial reference frame, by $M\xi\eta\zeta$ the coordinate system whose origin is at the centre of mass of the exo-Earth and axes are parallel to those of the $OXYZ$, and by $Mxyz$ the coordinate system with the same origin and coordinate axes
coinciding with the exo-Earth’s principal axes. We assume that $M_z$ is the axis associated with the maximum moment of inertia. (The coordinate systems $OXYZ$ and $M_ξ η ζ$ are shown in Fig. 1.)

To investigate the rotation of the exo-Earth, we employ the Andoyer variables (Andoyer 1923), for which, following Kinoshita (1977), we use the set of canonical variables $(G, H, L, g, h, l)$, where

- $G$ is the magnitude of the exo-Earth angular momentum vector $L$,
- $H$ is the $Z$-component of $L$,
- $L$ is the $z$-component of $L$,
- $g$ is the angle between the intersections of the plane $Σ$ with the planes $M_ξ η$ and $Mxy$,
- $h$ is the angle between the axis $Mξ$ and the intersection of the planes $Σ$ and $Mξ η$,
- $l$ is the angle between the axis $Mx$ and the intersection of the planes $Σ$ and $Mxy$ and $Σ$ is the equatorial plane orthogonal to $L$. The respective Hamilton equations for the rotating rigid exo-Earth are

\[
\frac{d}{dt}(g, h, l) = \frac{\partial H}{\partial (G, H, L)}, \quad \frac{d}{dt}(G, H, L) = -\frac{\partial H}{\partial (g, h, l)},
\]

where the Hamiltonian is

\[
H = \frac{G^2 - L^2}{2} \left( \frac{\sin^2 l}{A} + \frac{\cos^2 l}{B} \right) + \frac{L^2}{2C} + \sum_{n=1}^{N} V_n,
\]

$A ≤ B < C$ are the principal moments of inertia of the exo-Earth and $V_n$ is the potential energy of the gravitational interaction with the $n$th celestial body, $N$ being the number of celestial bodies in the system, other than the exo-Earth. Assuming that the radius of the planet is small compared to the distance between celestial bodies, we preserve the leading-order part of the potential energy only (Beletskii 1966, 1975), namely

\[
V_n = \frac{3\mu_n}{2R_n^3} (Aγ_{n,1}^2 + Bγ_{n,2}^2 + Cγ_{n,3}^2), \quad \mu_n = fm_n,
\]

where $f$ is the universal gravitation constant, $m_n$ the mass of the body, $R_n$ its geocentric distance, and $(γ_{n,1}, γ_{n,2}, γ_{n,3})$ are the direction cosines of the directional vector $R_n = (R_nX, R_nY, R_nZ)$ from the geocentre to the $n$th body in the coordinate system $Mxyz$.

The planets of the Solar system are almost spherical, hence it is natural to assume that for the exo-Earth this also holds true. The moments of inertia can be expressed as

\[
A = J_0 + \epsilon A_1, \quad B = J_0 + \epsilon B_1, \quad C = J_0 + \epsilon C_1,
\]

where $\epsilon/J_0 ≪ 1$ and $J_0 = (A + B + C)/3$ is the mean moment of inertia of the exo-Earth. Therefore, the Hamiltonian (3), (4) takes the form

\[
H = \frac{G^2}{2J_0} + \epsilon \mathcal{H}_1 + o(\epsilon),
\]

where, by (4),

\[
\mathcal{H}_1 = -\frac{1}{2J_0^2} \left[ L^2C_1 + (G^2 - L^2) \left( \frac{\sin^2 l}{A_1} + \frac{\cos^2 l}{B_1} \right) \right] + \frac{3}{2} \sum_{n=1}^{N} \frac{\mu_n}{R_n^3} (A_1γ_{n,1}^2 + B_1γ_{n,2}^2 + C_1γ_{n,3}^2).
\]
For \( \varepsilon = 0 \), Eqs. (2) and (5) imply that \( h, l, G, H \) and \( L \) do not change in time and
\[ g = g_0 + \Omega_g t, \]
where \( \Omega_g = G/J_0 \), i.e. the rotation of the exo-Earth is a uniform precession about its angular momentum axis. For a small non-vanishing \( \varepsilon \), the variables \( h, l, G, H \) and \( L \) are slow, while \( g \) is a fast one.

The rotation of the coordinate system \( M_{xyz} \) relative to \( M_{\xi\eta\zeta} \) can be described by three Euler’s angles. The angle between the axes \( M_z \) and \( M_\zeta \) is called the nutation angle. If the vector of the angular momentum is oriented along the positive direction of \( M_z \), which is the case considered in Sects. 3 and 4, then the nutation angle is
\[ I = \arccos \left( \frac{H}{G} \right). \]

In general, the nutation angle \( I \) is different from the obliquity, i.e. the angle between \( L \) and the normal to the exo-Earth’s (osculating) orbital plane. Since the temporal evolution of the orbital plane is known, the temporal evolution of the obliquity can be calculated from that of the nutation angle. The difference between the range of obliquity and the range of nutation angle does not exceed \( 2\gamma_{\text{max}} \), where \( \gamma_{\text{max}} \) is the exact upper bound for the angle between the exo-Earth’s orbital plane and the plane \( OXY \).

If the orbital plane of the exo-Earth is time-independent and coincides with the plane \( OXY \) of the inertial reference frame, then the nutation angle and obliquity coincide. In particular, this is the case for the system considered in Sect. 4, where the orbit of the exo-Earth is a Keplerian ellipse that belongs to the plane \( OXY \).

### 2.2 Averaging

Let \( \omega = (\omega, \omega_1, \ldots, \omega_K) \) be \( K + 1 \) prescribed order one frequencies of the motion of the considered \( N + 1 \) celestial bodies, i.e. any coordinate \( Q(t) \) (where \( Q \) stands for \( X, Y, Z, X_n, Y_n \) or \( Z_n \)) can be expressed as
\[ Q(t) = \sum_{s=(s, s_1, \ldots, s_K), \ 0<|s|<\infty} q_s e^{i(s\cdot\omega)t}. \]

In the case of several fast frequencies, we can either employ the so-called general averaging (Krasilnikov 2015; Sanders and Verhust 1985; Volosov 1962b, a; Zheligovsky 2011), or, following (Krasil’nikov and Zaharova 1993; Markeev and Krasilnikov 1981), introduce the fast variables
\[ \theta = \omega t, \quad \theta_k = \omega_k t, \quad 1 \leq k \leq K, \]
and define the average of a function \( F \) as
\[ \bar{F} = \frac{1}{(2\pi)^K} \int_0^{2\pi} \cdots \int_0^{2\pi} F \ d\theta \ d\theta_1 \ldots d\theta_K. \]

In the absence of resonances between \( \omega, \omega_k \) and \( G/J_0 \),
\[ \mathcal{H} \approx \frac{G^2}{2J_0} + \varepsilon \mathcal{H}_1, \]
where, by (4)–(8), the mean Hamiltonian is
\[ \mathcal{H}_1 = \mathcal{F}(G, L, I) \mathcal{G}(G, H, h) + \mathcal{C}. \]
Here

\[ \mathcal{F}(G, L, I) = (B_1 - A_1) \left( \frac{1}{3} - \cos^2 I \sin^2 I \right) - (C_1 - A_1) \left( \frac{2}{3} - \sin^2 J \right), \]

\[ \mathcal{G}(G, H, h) = \frac{G^2}{2} + \frac{9}{4} \left( \frac{1}{3} - \cos^2 h \sin^2 I \right) \sum_{n=1}^{N} \mu_n (D_{nY^2} - D_{nX^2}) \]

\[ - \left( \frac{2}{3} - \sin^2 I \right) \sum_{n=1}^{N} \mu_n (D_{nZ^2} - D_{nX^2}) + \sin(2h) \sin^2 I \sum_{n=1}^{N} \mu_n D_{nXY} \]

\[ - \sin(2I) \left( \sin h \sum_{n=1}^{N} \mu_n D_{nXZ} - \cos h \sum_{n=1}^{N} \mu_n D_{nYZ} \right) \]

\[ C = \frac{1}{2} \left( J_0 \sum_{n=1}^{N} \mu_n D_{nR} - G^2 \right), \]

\[ \cos I = \frac{H}{G}, \quad \cos J = \frac{L}{G} \]

and

\[ D_{n\rho^2} = \frac{1}{(2\pi)^{K+1}} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{R^2_{n\rho}}{R^5_n} d\theta_1 \cdots d\theta_K, \]

\[ D_{n\rho^v} = \frac{1}{(2\pi)^{K+1}} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{R_{n\rho} R_{n^v}}{R^5_n} d\theta_1 \cdots d\theta_K, \]

\[ D_{nR} = \frac{1}{(2\pi)^{K+1}} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{1}{R^3_n} d\theta_1 \cdots d\theta_K, \] (12)

where \( \rho \) and \( v \) denote \( X, Y \) or \( Z \).

Up to the order \( \varepsilon^2 \) terms, the first equation in (11) reduces to

\[ \mathcal{F}(G, L, I) = \frac{2J_0^2}{G^2} \left[ \frac{G^2 - L^2}{2} \left( \frac{\sin^2 I}{A_1} + \frac{\cos^2 I}{B_1} \right) + \frac{L^2}{2C_1} \right] - \frac{J_0^2}{2\varepsilon C_1} + \frac{A_1 + B_1 - 2C_1}{3}. \] (13)

The second equation in (11) takes the form

\[ \mathcal{G}(G, I, h) = \frac{G^2}{2J_0^2} + \sin^2 I (-D_1 \sin^2 h - D_2 \cos^2 h + D_3 + D_4 \sin(2h)) \]

\[ - \sin 2I (D_5 \sin h - D_6 \cos h). \] (14)

Here,

\[ D_j = \sum_{n=1}^{N} D_{j}^{(n)}, \quad 1 \leq j \leq 6, \]

\[ D_{1}^{(n)} = \mu_n D_{nX^2}, \quad D_{2}^{(n)} = \mu_n D_{nY^2}, \quad D_{3}^{(n)} = \mu_n D_{nZ^2}, \]

\[ D_{4}^{(n)} = \mu_n D_{nXY}, \quad D_{5}^{(n)} = \mu_n D_{nXZ}, \quad D_{6}^{(n)} = \mu_n D_{nYZ}, \] (15)

and \( D_{n\rho v} \) are given by (12).

**Remark 1**  If only non-resonant order one frequencies are present in the motion of the celestial bodies, then the functions \( D_j \) are time-independent constants and the system is integrable due to the existence of a complete set of independent first integrals, \( G = \text{const}, \mathcal{F}(G, L, I) = \text{const} \) and \( \mathcal{G}(G, H, h) = \text{const} \) (Markeev and Krasilnikov 1981; Krasil’nikov and Zaharova 1993). Up to factors and terms that are constant, Eq. (13) coincides with the Hamiltonian of the Euler–Poinset motion. Therefore, the rotation of the exo-Earth about its angular momentum...
axis is of the Euler–Poisot type. By considering the equation $G = \text{const}$, the dependence of $I$ on $h$ was studied \textit{ibid}. We recall these results in Sect. 3.

Below we consider only the case, where the exo-Earth is axially symmetric and the rotation axis coincides with the symmetry axis of the body, which implies $A_1 = B_1$ and $J = 0$. Under these assumptions the angular momentum $L = (L_\xi, L_\eta, L_\zeta)$ becomes

\begin{align*}
L_\xi &= G \sin h \sin I,
L_\eta &= -G \cos h \sin I,
L_\zeta &= G \cos I.
\end{align*}

Axial symmetry of the exo-Earth implies that $\partial \mathcal{H}/\partial g = 0$, i.e. that $G$ does not vary in time. By (2) and (9)–(11), the evolution of the angles $h$ and $I$ satisfies the ODEs

\begin{align*}
\frac{d h}{d t} &= \frac{3 \epsilon}{2} (C_1 - A_1) \frac{1}{G \sin I} \frac{\partial \tilde{G}}{\partial I},
\frac{d I}{d t} &= -\frac{3 \epsilon}{2} (C_1 - A_1) \frac{1}{G \sin I} \frac{\partial \tilde{G}}{\partial h}, \tag{18}
\end{align*}

where $\tilde{G}(I, h) = G(G, I, h) - G^2/2J_0^2$ (see (14)) is

\begin{align*}
\tilde{G}(I, h) &= \sin^2 I (-D_1 \sin^2 h - D_2 \cos^2 h + D_3 + D_4 \sin(2h)) \\
&\quad - \sin 2I (D_5 \sin h - D_6 \cos h). \tag{19}
\end{align*}

In sums (15), the terms $D_j^{(n)}$, $1 \leq j \leq 6$, originate from the gravitation interaction of the exo-Earth with the $n$th celestial body. We label the bodies as follows: the first one is the exo-Sun, the second is the exo-Moon and the numbers from three to $n$ are assigned to exoplanets other than the exo-Earth.

### 2.3 Calculation of functions $D_j$ for the exo-Moon and exo-Sun

The functions $D_j$ related to the planets, in general, can be found only numerically. We have assumed that the orbit of the exo-Moon is a Keplerian ellipse with a constant inclination to the ecliptic undergoing two types of precessional motion with the respective frequencies $\sigma_\alpha$ and $\sigma_\omega$. (The nodal precession is the precession of the exo-Moon’s orbital plane, and the apsidal one is the rotation of the exo-Moon’s orbit within the plane.) In this case, the functions $D_j^{(2)}$ related to the exo-Moon can be found analytically and we evaluate them in this subsection.

When the orbit of the exo-Earth around the exo-Sun is a Keplerian ellipse, its elliptic orbit satisfies the equations

\begin{align*}
X &= \frac{a_E(1 - e_E^2) \cos \nu_E}{1 + e_E \cos \nu_E}, \\
Y &= \frac{a_E(1 - e_E^2) \sin \nu_E}{1 + e_E \cos \nu_E}, \\
Z &= 0, \tag{20}
\end{align*}

where $a_E$, $e_E$ and $\nu_E$ are the semi-major axis, eccentricity and true anomaly of the exo-Earth. Following the averaging procedure discussed in Podviga and Krasilnikov (2020), we introduce the fast variable $\theta_E = \omega_E t$, where $\theta_E$ is the mean anomaly of the exo-Earth’s orbits, that is related to the true anomaly as follows:

\begin{align*}
\frac{d \theta_E}{d \nu_E} &= \frac{(1 - e_E^2)^{3/2}}{(1 + e_E \cos \nu_E)^2}.
\end{align*}

Since $X_1(t) = Y_1(t) = Z_1(t) = 0$, the functions are:

\begin{align*}
D_1^{(1)} = D_2^{(1)} = \frac{\mu S}{2a_E^3(1 - e_E^2)^{3/2}}, \\
D_3^{(1)} = D_4^{(1)} = D_5^{(1)} = D_6^{(1)} = 0. \tag{21}
\end{align*}
The orbit of the exo-Moon is an ellipse with the exo-Earth located in one of the focuses. The inclination \( i \) of the lunar orbit to the ecliptic plane does not change in time. The longitude of the ascending node and the argument of periapsis evolve as

\[
\Omega = \Omega_0 + \sigma_n t, \quad \vartheta = \vartheta_0 + \sigma_d t.
\]

For calculating the functions \( D_j^{(2)} \), we recall that (see, for example, Balk 1965)

\[
\begin{align*}
R_{2X} &= X - X_2 = \left[ \cos \Omega \cos \vartheta - \sin \Omega \cos i \sin \vartheta \right] \xi' - \left[ \cos \Omega \sin \vartheta + \sin \Omega \cos i \cos \vartheta \right] \eta' + \sin \Omega \sin i \zeta' \\
R_{2Y} &= Y - Y_2 = \left[ \sin \Omega \cos \vartheta + \cos \Omega \cos i \sin \vartheta \right] \xi' + \left[ - \sin \Omega \sin \vartheta + \cos \Omega \cos i \cos \vartheta \right] \eta' - \cos \Omega \sin i \zeta' \\
R_{2Z} &= Z - Z_2 = \sin i \sin \vartheta \xi' + \sin i \cos \vartheta \eta' + \cos i \zeta',
\end{align*}
\]

where \((\xi', \eta', \zeta')\) are the coordinates of the exo-Moon in the coordinate system \( M\xi', \eta', \zeta' \) related to the Moon’s orbit: the origin is in the centre of mass of the exo-Earth, the positive \( M\xi' \) axis points to the perigee of the exo-Moon’s orbit, the axis \( M\eta' \) belongs to the orbit plane and is obtained by rotating the \( M\xi' \) axis by \( \pi/2 \) in the direction of the Moon’s motion, and the axis \( M\zeta' \) is orthogonal to the orbit and its direction is such that we obtain a right-handed coordinate system. The coordinates of a point in the orbit satisfy the equations

\[
\begin{align*}
\xi' &= \frac{a_2(1 - e_2^2)}{1 + e_2 \cos v_2} \cos v_2, \quad \eta' = \frac{a_2(1 - e_2^2)}{1 + e_2 \cos v_2} \sin v_2, \quad \zeta' = 0,
\end{align*}
\]

where \(a_2, e_2\) and \(v_2\) are the semi-major axis, eccentricity and true anomaly of the exo-Moon orbiting the exo-Earth. Therefore,

\[
\begin{align*}
R_{2X} &= \frac{a_2(1 - e_2^2)}{1 + e_2 \cos v_2} \left[ \cos \Omega \cos(\vartheta + v_2) - \sin \Omega \cos i \sin(\vartheta + v_2) \right] \\
R_{2Y} &= \frac{a_2(1 - e_2^2)}{1 + e_2 \cos v_2} \left[ \sin \Omega \cos(\vartheta + v_2) + \cos \Omega \cos i \sin(\vartheta + v_2) \right] \\
R_{2Z} &= \frac{a_2(1 - e_2^2)}{1 + e_2 \cos v_2} \sin i \sin(\vartheta + v_2).
\end{align*}
\]

Substituting (23) into (12) and (16), we obtain

\[
\begin{align*}
D_j^{(2)} &= \Xi \left[ \cos^3 \Omega \sin^2 i + \cos^2 i \right], \quad D_j^{(2)} = \Xi \left[ - \cos^2 \Omega \sin^2 i + 1 \right], \quad D_3^{(2)} = \Xi \sin^2 i \\
D_4^{(2)} &= \Xi \sin \Omega \cos \Omega \sin^2 i, \quad D_5^{(2)} = - \Xi \sin \Omega \sin i \cos i, \quad D_6^{(2)} = \Xi \cos \Omega \sin i \cos i,
\end{align*}
\]

where \(\Omega = \Omega_0 + \sigma_n t\) and

\[
\Xi = \frac{f m_2}{2a_2^3(1 - e_2^2)^{3/2}}.
\]

### 3 Planetary system, comprised of the exo-Sun, exo-Earth and exoplanets.

Evolution of the nutation angle in the system considered in this section was studied in a general set-up in Podvigina and Krasilnikov (2020). In this section, we present an approximation of the range of the nutation angle. The mass of the star is assumed to be much larger than the mass of any planet. Details of the derivation are given in “Appendix A”.
Under the assumptions of this paper, in a system comprised of the exo-Sun and exoplanets only the evolution of the angles \( h \) and \( I \) satisfies the ODEs (18), (19), and \( D_j \) for \( j = 1, \ldots, 6 \) are time-independent constants. Equation (19) is invariant under the symmetry \( (I, h) \rightarrow (\pi - I, h + \pi) \).

The right-hand side of (19) can be expressed as

\[
\tilde{\mathcal{G}}(I, h) = \sin^2 I(-D + \alpha \cos 2h') + \beta \sin 2I \cos(h' + \gamma),
\]

where

\[
D = \frac{D_1 + D_2 - 2D_3}{2}, \quad \alpha = \left(\frac{(D_1 - D_2)^2}{4} + D_4^2\right)^{1/2}, \quad \beta = (D_5^2 + D_6^2)^{1/2},
\]

\( h' = h + s(D_1 - D_2, -2D_4)/2 \), \( \gamma = s(D_6, D_5) - s(D_1 - D_2, -2D_4)/2 \) and the function \( s(\cdot, \cdot) \) is defined as follows:

\[
s(x, y) = \begin{cases} 
\arctan(y/x), & x > 0 \\
\arctan(y/x) + \pi, & x < 0 \\
\pi/2, & x = 0, \ y \geq 0 \\
3\pi/2, & x = 0, \ y < 0 
\end{cases}
\]

In view of (15), (16) and (27), smallness of the planets’ masses relative to the mass of the exo-Sun implies \( D \gg \max(|\alpha|, |\beta|) \).

Steady states of system (18), (26) are (see “Appendix A”)

\[
I \approx 0; \quad I \approx \pi; \quad I \approx \pi/2, \quad h' \approx 0, \pi/2, \pi, 3\pi/2;
\]

\( I \approx \pi/2, h' \approx 0 \) and \( I \approx \pi/2, h' \approx \pi \) are saddles, and all the rest are centres. The saddles are connected by heteroclinic trajectories that divide the celestial sphere into four regions, each comprised of a centre and closed trajectories around it, see Fig. 2a. We call polar the regions including the steady states \( I \approx 0 \) and \( \pi \), and equatorial the ones including the steady states with \( I \approx \pi/2 \).

In Appendix A, we approximate \( \Delta(I_0, h_0) \) defined by (1), where \( h_0 = \pi/2 \) (shown by grey line in Fig. 2a), and consider the interval \( 0 \leq I_0 \leq \pi/2 \). The range \( \Delta \) is a continuous function of \( I_0 \) and \( h_0 \) inside a region, and it is discontinuous at a boundary. The expressions that we derive depend on the initial conditions; namely, we consider three possibilities for \( I_0 \) when the initial condition \((I_0, h_0)\) belongs to the polar region (cases (i)–(iii)), and one possibility when \((I_0, h_0)\) is in the equatorial region (case (iv)):

| Case | \( I_0 \) |
|------|-----------|
| (i)  | \( D^{1/2} |\sin I_0| < \beta^{1/2}, \ |\sin I_0| < |\cos I_0| \) |
| (ii) | \( D^{1/2} \sin^2 I_0 > \max(\alpha, \beta^{1/2} |\sin I_0|) \) |
| (iii) | \( D^{1/2} \sin^2 I_0 < \alpha^{1/2}, \ |\sin I_0| > |\cos I_0|, \ |I_0 - \frac{\pi}{2}| > (2\alpha)^{1/2} D^{-1/2} \) |
| (iv) | \( |I_0 - \frac{\pi}{2}| < (2\alpha)^{1/2} D^{-1/2} \) |
For \( D = 1, \alpha = 0.001, \beta = 0.002 \) and \( \gamma = \pi / 8 \), the motion of \( L(17) \) on the celestial sphere according to (18), (19) (a) and \( \Delta(I_0, \pi / 2) \) as a function of \( I_0 \) computed by integrating equations (18), (19) (black line) or using approximations (28) (grey area) (b). The meridian \( h = \pi / 2 \) is shown by a grey line on the sphere.

Fig. 3 Dependence of \( I \) on \( h \) for a moonless planetary system (blue line) and for the system with an exo-Moon included (grey line). The parameters of the planetary system are: \( m_S = 1, a_E = 1, e_E = 0, m_J = 0.05, a_J = 1.5, e_J = 0.1, i = \pi / 64 \) (a) and \( i = \pi / 8 \) (b). The initial condition is \( (I_0, h_0) = (\pi / 8, 0) \)

The ranges \( \Delta(I_0, h_0) \) are:

| Case | \( \Delta(I_0, h_0) \) |
|------|------------------|
| (i)  | \( \Delta \approx \frac{2\beta}{D} \) |
| (ii) | \( \frac{D \cos I_0}{\max(|\alpha| \sin I_0, 2|\beta| \cos I_0)} < \Delta \leq \frac{|\alpha| \sin I_0 + 2|\beta| \cos I_0}{D \cos I_0} \) |
|      | or \( \Delta \approx \frac{\chi(I_0, \alpha, \beta, \gamma)}{D \cos I_0} \) |
| (iii)| \( \Delta \approx \frac{D \sin 2I_0 - \beta \cos 2I_0 \cos \gamma + ((-D \sin 2I_0 + \beta \cos 2I_0 \cos \gamma)^2 + 8D\alpha)^{1/2}}{2D} \) |
| (iv) | \( \Delta \approx 2|\pi / 2 - I_0| \), |

where the function \( \chi(I_0, \alpha, \beta, \gamma) \) is defined by (44) and (45). The approximations (28) are in a good agreement with \( \Delta(I_0, \pi / 2) \) calculated by integrating the Eqs. (18), (26), see Fig. 2.
4 Planetary system, comprised of the exo-Sun, exo-Earth, exo-Moon and exoplanets

In this section, we study how the range of the nutation angle changes when we add an exo-Moon to the system considered in the previous section. As follows from Fig. 3, introducing an exo-Moon may result in a decrease or increase of the range. In Sect. 4.1, we prove analytically that for certain systems the impact of the exo-Moon is stabilising, and in Sect. 4.2, we prove that for some systems it is destabilising. In Sect. 4.3, we study numerically the impact of the exo-Moon in a particular system comprised of an exo-Sun, exo-Earth and an exoplanet for varying orbital parameters of the exo-Moon and exoplanet.

4.1 A stabilising moon

Let $\Delta^P(I_0, h)$ denote the range of $I$ in the moonless system considered in Sect. 3 and $\Delta^{P+M}(I_0, h)$ the range in the system with an added exo-Moon. Denote by $D_{j}^{P+M}$ the functions in Eqs. (18), (19) in the system equipped with the exo-Moon and by $D_{j}^{P}$ the functions for the moonless system. In agreement with (15), $D_{j}^{P+M} = D_{j}^{P} + D_{j}^{(2)}$, $j = 1, \ldots, 6$. We assume that the orbit of the exo-Moon belongs to the ecliptic, i.e. $i = 0$, and then the functions $D_{j}^{(2)}$ (see (24)) satisfy

$$D_{1}^{(2)} = D_{2}^{(2)} > 0, \quad D_{3}^{(2)} = D_{4}^{(2)} = D_{5}^{(2)} = D_{6}^{(2)} = 0,$$

where $D_{1}^{(2)}$ is time-independent.

Therefore, results of Sect. 3 can be applied to the modified system. Moreover, $D_{j}^{P+M} > D_{j}^{P}$, $\alpha_{P+M}^{P} = \alpha_{P}$ and $\beta_{P+M}^{P} = \beta_{P}$ (see (27)), where the upper indices refer to the system with the added exo-Moon or to the original system. If $I_0$ satisfies (i) or (ii) in (28), then the respective expressions for $\Delta$ involve $D$ in the denominator only, resulting in $\Delta^{P+M}(I_0, h_0) < \Delta_{P}(I_0, h_0)$, unless $I_0$ is close to $\pi/2$.

4.2 A destabilising moon

Consider a moonless system where the orbits of all planets including the exo-Earth are circular and belong to the equatorial plane. In such a system, the functions $D_{i} = D_{i}^{P}$ satisfy

$$D_{1}^{P} = D_{2}^{P} > 0, \quad D_{3}^{P} = D_{4}^{P} = D_{5}^{P} = D_{6}^{P} = 0.$$

The rotation of the exo-Earth reduces to a regular precession about the axis orthogonal to its orbital plane. (See section 5 in Podvigina and Krasilnikov 2020.) Therefore, $\Delta^{P}(I_0, h)$ vanishes for any initial condition. After the exo-Moon is added into the system, the, respectively, modified functions $D_{j}^{P+M}$ become nonzero and we expect that the range of the nutation angle is positive, except for some special initial conditions. Since $\Delta^{P+M} > \Delta^{P} = 0$, the impact of the exo-Moon is destabilising.

---

2 We assume that in the moonless system the range does not vanish for all initial conditions, because in this case the idea of stabilisation does not make sense. If this happens, $\Delta^{P+M}(I_0, h_0) = \Delta^{P}(I_0, h_0)D^{P}/D^{P+M} = 0.$
4.3 Numerical study of the impact

In this subsection, we investigate how an added exo-Moon affects the range of the nutation angle in a simple system composed of the exo-Sun, exo-Earth and a planet that we call exo-Jupiter (see Fig. 1). We assume that the exo-Sun is the origin of the OXYZ coordinate system and the orbit of the exo-Earth belongs to the OXY plane. The orbits of the exo-Earth and exo-Jupiter are Keplerian ellipses, whose semi-major axes are $a_E$ and $a_J$ and the eccentricities $e_E$ and $e_J$, respectively, and the angle between the orbital planes is $\gamma_J$. The orbital planes intersect along the axis OY, and the major axes of both ellipses are orthogonal to OY.

Upon the canonical change of variables discussed in “Appendix B”, the equations of motion (18), (19) reduce to

$$\frac{dh}{dt} = \frac{\rho}{\sin I} \frac{\partial G'}{\partial t}, \quad \frac{dI}{dt} = -\frac{\rho}{\sin I} \frac{\partial G'}{\partial h},$$

$$G'(I, h) = \sin^2 I \left(-D_1^P \sin^2 h - D_2^P \cos^2 h + D_3^P + D_4^P \sin(2h)\right)$$

$$+ \Xi \left(\frac{1}{2} \sin^2 i - \cos^2 i\right) + \frac{\Xi}{2} \sin^2 i$$

$$- \sin 2I \left(D_5^P \sin h - D_6^P \cos h + \frac{\Xi}{2} \sin 2i \cos h\right) + \sigma \cos I,$$

where

$$\rho = \frac{3\varepsilon(C_1 - A_1)}{2G} \text{ and } \sigma = \frac{\sigma_n}{\rho}.$$

Equation (32) indicates that the contribution of the exo-Moon into the motion of the rotation axis is determined by $i$, $\Xi$ and $\sigma$.

Due to the integrability of the moonless system, the range of nutation angle can be computed from Eqs. (18) and (19) following the approach outlined in section 4 of Podvigina and Krasilnikov (2020). In such a system, the coefficients $D_j$ are time-independent constants $D_j = D_j^{(1)} + D_j^{(3)}$, where $D_j^{(1)}$ for the exo-Sun are computed from (21) and $D_j^{(3)}$ for the exo-Jupiter from (16). The geometry of the exo-Earth’s and exo-Jupiter’s orbits implies that $D_4 = D_6 = 0$. The extreme values of $I(h)$ over $h$ are achieved when $\partial G'/\partial h = 0$, i.e.

$$(D_1 - D_2) \sin h \cos h \sin I - D_5 \cos h \cos I = 0.$$

Therefore, there exist two or four extrema along a particular trajectory. Two of them take place at $h = \pi/2$ and $3\pi/2$. For a trajectory through $(I_0, h_0)$, the respective values of $I$ are found from

$$h = \pi/2 : (D_3 - D_1) \sin^2 I - D_5 \sin(2I) + S = 0,$$

$$h = 3\pi/2 : (D_3 - D_1) \sin^2 I + D_5 \sin(2I) + S = 0,$$

$$S = (D_1 \sin^2 h_0 + D_2 \cos^2 h_0 - D_3) \sin^2 I_0 + D_5 \sin h_0 \sin(2I_0).$$

The other two extrema, whenever they exist, are found by solving the system

$$(D_1 - D_2) \sin h \sin I - D_5 \cos I = 0,$$

$$(-D_1 \sin^2 h - D_2 \cos^2 h + D_3) \sin^2 I - D_5 \sin h \sin(2I) + S = 0.$$

Finally, the range is calculated from (1). The range for the complete system is found by numerical integration of Eqs. (31), (32). Time integration is performed using a fourth order Runge–Kutta scheme.
We have performed two series of computations. In the first one, we have started by computing the range of the nutation angle for the system considered in Sect. 4.1, where the angle $i$ between the exo-Moon’s orbit and the ecliptic vanishes implying that the impact is stabilising except for $I_0$ near $\pi/2$. This is confirmed by numerical simulations shown in Fig. 4a. Subsequent computations have been performed for increasing $i$. For small values of $i$, the impact has been expected to remain stabilising. These expectations have been confirmed numerically, see Fig. 4b. As the angle is increased, for $I_0$ near $\pi/4$ and $3\pi/4$ the difference $\Delta P^{+M} - \Delta P$ becomes positive and the difference grows with $i$, see Fig. 4c, d.

The system considered in Sect. 4.2, in which planets’ orbits are circular and orbital planes coincide, has been used to start the second series. Since $\Delta P^{+M}(I_0, h_0) = 0$ for any initial condition in the moonless system, the impact of the exo-Moon is destabilising (see Fig. 5a). On increasing the eccentricity and inclination of the exo-Jupiter’s orbit, the impact changes to stabilising, as shown in Fig. 5b–d. Note that the impact is more destabilising (or less stabilising) near $I_0 = \pi/4$ or $3\pi/4$ then at the poles or near the equator on both Figs. 4 and 5. Such dependence of $\Delta P^{+M} - \Delta P$ on $I_0$ might be a peculiarity of two considered systems, or it may be of a general type.
Fig. 5 Dependence of $\Delta(I_0, 3\pi/2)$ on $I_0$ in the moonless system (blue line) and in the system with an included exo-Moon (grey line). The parameters are: $m_S = 1, a_E = 1, e_E = 0, m_J = 0.05, a_J = 1.5, \sigma = 10, \Xi = 0.5, \rho = 1$ and $i = \pi/128$ and $e_J = 0, \gamma_J = 0$ (a), $e_J = 0.005, \gamma_J = \pi/128$ (b), $e_J = 0.01, \gamma_J = \pi/64$ (c), $e_J = 0.02, \gamma_J = \pi/32$ (d). In plate a the range of nutation angle in the moonless system vanishes identically

5 Comparison of the original and averaged equations

In this section, we compare solutions to the original equations for the temporal evolution of the rotation axis of the exo-Earth (2), (3) with the averaged ones (18), (19) for the planetary system comprised of the exo-Sun, exo-Earth, exo-Jupiter and exo-Moon considered in Sect. 4.3. Under the assumption that the exo-Earth is axially symmetric and the angular momentum axis coincides with the figure axis, Eqs. (2), (3) reduce to

$$\frac{dh}{dt} = -\frac{1}{G \sin I} \frac{\partial \mathcal{H}}{\partial I}, \quad \frac{dI}{dt} = \frac{1}{G \sin I} \frac{\partial \mathcal{H}}{\partial h}, \quad \text{where} \quad \mathcal{H} = \frac{G^2}{2C} + V$$

(33)

and $G$ is a constant that we set equal to one. The potential energy $V = V_S + V_M + V_J$ is found from (4).

We employ the following values for the masses and orbits of celestial bodies:

$$\mu_S = 1, \mu_J/\mu_S = 0.1, a_E = 1, a_J = 1.5, e_E = e_J = 0, \gamma_J = \pi/4, \epsilon = 0.00025, C_1 - A_1 = 1, \mu_E/\mu_S = 10^{-8}, \mu_M/\mu_S = 2 \cdot 10^{-11}, a_M = 2 \cdot 10^{-4}, e_M = 0.2$$

(34)
Fig. 6 Nutation angle $I$ as a function of time for the original and the averaged equations for masses and orbits of celestial bodies listed in (34), initial conditions $I = \pi/4$, $h = \pi/2$ and and the inclination of the exo-Moon’s orbit $i = \pi/50$ (a) and $i = \pi/200$ (b). The plots for the original equations are drawn by black lines, for the averaged ones by the grey lines.

The precession frequencies $\sigma_n$ and $\sigma_a$ for a circular orbit of the exo-Moon are Smart (1953)

$$\sigma_n = -\frac{3}{4} \frac{\omega_E^2}{\omega_M} \quad \text{and} \quad \sigma_a = \frac{3}{4} \frac{\omega_E^2}{\omega_M};$$

although the orbit is non-circular, we use these formulas to calculate the precession frequencies. Since $\omega_E = \sqrt{(\mu_S/a_E^3)}$ and $\omega_M = \sqrt{(\mu_E/a_M^3)}$, for the masses and semi-major axes listed in (34) we have that $\sigma_n = -0.021$ and $\sigma_a = 0.021$. We employ two values for the inclination of the exo-Moon orbital plane, $i = \pi/50$ and $i = \pi/200$. From (12), (15) and (16), the coefficients $D_i^P = D_i^{(1)} + D_i^{(3)}$ of the averaged equations are then

$$D_1^P = 0.51076, \quad D_2^P = 0.52471, \quad D_3^P = 0.01977, \quad D_4^P = 0, \quad D_5^P = 0.00648, \quad D_6^P = 0.$$  

The coefficients for the exo-Moon are given by (25) with $\Xi = 1.33$.

The evolution of $I(t)$ for both the original and averaged equations is shown in Fig. 6, the panels (a) and (b) corresponding to different values of $i$. Notice the similarity of the original and the averaged solutions. In the plots, the oscillations with a period close to 300 are induced by the evolution of the exo-Moon orbit. Note that the amplitude of these oscillations with $i = \pi/50$ is larger than with $i = \pi/200$.

6 A hypothetical exo-Moon in the system 7 Canis Majoris

So far, no discovery of moons in other planetary systems has been confirmed. The satellites orbiting the planets of the Solar system do not satisfy the requirement about the absence of resonances. Consequently, as an example of application of our results, we explore the impact of a hypothetical exo-Moon on the rotation of a known exoplanet. As a study case, we consider the system 7 Canis Majoris, comprised of two planets orbiting a star. According to exoplanet catalog, the masses and orbits of the planets, the exo-Earth and exo-Jupiter, are characterised by the following relations:

$$\mu_J/\mu_S = 0.0014, \quad \mu_E/\mu_S = 0.00065, \quad a_E = 2.1, \quad a_J = 1.8, \quad e_E = 0.08, \quad e_J = 0.06.$$  

(35)
Fig. 7 Range $\Delta$ as a function of the inclination of the exo-Moon’s orbital plane in the moonless system (black line) and in the system involving the exo-Moon (grey line). The masses and orbits of celestial bodies are listed in (35) and (36). The initial conditions are $I = \pi/16$, $h = \pi/2$ (a) and $I = \pi/4$, $h = \pi/2$ (b).

We assume that the planets’ orbits are Keplerian ellipses, the orbit of the exo-Earth belongs to the plane $OXY$ and its semi-major axis belongs to the axis $OX$. The orbital plane of the exo-Jupiter intersects with the plane $OXY$ along the axis $OX$, and the angle between the orbital planes is $\gamma_J = \pi/50$. For a hypothetical exo-Moon, the following mass and orbital elements are assumed:

$$\mu_M/\mu_S = 1.5 \cdot 10^{-7}, \ a_M = 0.01, \ e_M = 0.$$  \hfill (36)

The inclination of the exo-Moon’s orbital plane varies between 0 and $\pi/50$. We consider two sets of initial conditions, $(I_0, h_0) = (\pi/16, \pi/2)$ and $(\pi/4, \pi/2)$. The range $\Delta$ found by numerical integration of Eqs. (18) and (19) is shown in Fig. 7, where the black line indicates the range in the moonless system. In agreement with findings of Sect. 4.1, for zero inclination the impact is stabilising. It changes to a destabilising one as the inclination of the orbit increases.

7 Conclusion

We have studied the impact of a massive satellite on the evolution of the angle of nutation of a hypothetical exoplanet (an exo-Earth) at large times under the conditions that the orbital motions of celestial bodies affecting the rotation of the exoplanet are quasi-periodic, the relevant frequencies are non-resonant order one and the orbit of the satellite is a Keplerian ellipse in a plane that keeps a constant angle with the ecliptic while precessing at a prescribed angular velocity $\sigma_n$. A rigid axially symmetric exo-Earth is assumed.

We have followed the approach of Podvigina and Krasilnikov (2020), where the evolution of the rotation axis of a planet in a system comprised of stars and planets was studied by applying time averaging over several fast variables associated with non-resonant respective frequencies. At large times, the evolution is governed by a Hamiltonian involving six functions which can be calculated for prescribed masses and orbits of the celestial bodies. If the satellite is absent, the functions are constant, and when the satellite is included, they become time-periodic with the frequency $\sigma_n$.

In the moonless system (Podvigina and Krasilnikov 2020), the Hamilton equations for the evolution of the rotation axis are integrable. Using the essential smallness of the celestial
bodies’ masses relative that of the exo-Sun, we have derived an approximation for the range of
the nutation angle as a function of initial conditions and the six constants involved in the
Hamiltonian.

In general, the full system is non-integrable analytically and the range of the angle of
nutation can only be found numerically. However, for some special cases it has proven
possible to demonstrate that the influence of the exo-Moon is stabilising or destabilising.
Namely, it is stabilising if the orbital plane of the exo-Moon coincides with the ecliptic. If
orbits of all planets are circular and their orbital planes coincide, the influence of the exo-
Moon is destabilising. Both findings hold true for almost all initial values of the nutation
angle, with the exception of the ones close to \( \pi/2 \), i.e. for the rotation axis of the exo-Earth
close to the ecliptic.

Simulations for a system, comprised of an exo-Sun, exo-Earth and exo-Jupiter supple-
mented by an exo-Moon, confirm the stabilising effect of the exo-Moon for a zero angle
between the orbital planes of the exo-Moon and the exo-Earth. The stabilising effect persists
for small nonzero angles. On further increasing the angle, for obliquities around \( \pi/4 \) and
\( 3\pi/4 \), the impact switches to the destabilising one. Similarly, the impact of the exo-Moon
changes from destabilising to the stabilising one on increasing the eccentricity of exo-Jupiter’s
orbit and the angle between the orbital planes of the exo-Earth and the exo-Jupiter.

In this paper, we have considered only the direct influence of an exo-Moon, namely, the
torque from the exo-Moon affecting the rotation of the exo-Earth. The indirect influence,
caused by the modified torque from the exo-Sun due to the alternation of exo-Earth’s orbit
by the added exo-Moon, can be investigated by similar techniques.

Another possible continuation is to study planetary systems with resonances. Since they are
common in the Solar system (Molchanov 1968; Murray and Dermott 1999), we expect them
to occur in other planetary systems, which is also confirmed by simulations in Millholland
and Batygin (2019), Millholland and Laughlin (2019). The presence of resonances changes
drastically the behaviour of the averaged system (Arnold et al. 2006; Henrard and Lemaitre
1983), because one additional slowly changing variable must be introduced per resonance.
We can conjecture that when some of the exo-Moon frequencies, \( \sigma_n \), \( \sigma_a \) or \( \omega_2 \), are in resonance
with some frequencies of the planetary motions, the impact of the exo-Moon is destabilising
due to enlargement of the phase space of the averaged system. In the higher-dimensional
phase space, a chaotic behaviour of the trajectories may set in, resulting in the increase of
the range of the nutation angle. The destabilising effect of an exo-Moon can be related to
mean-motion resonances, as revealed, for example, in the studies of Quillen et al. (2018) or
Millholland and Laughlin (2019).

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Data Availability The data that support the findings of this study will be available by request.

A Calculation of \( \Delta \) in a planetary system, comprised of the exo-Sun,
exo-Earth and exoplanets

Here we give a detailed derivation of approximations (28) for the range of the nutation angle
from the Eqs. (18), (26), (27) for the temporal evolution of the angles \( I \) and \( h \).
Steady states of system (18) satisfy \( d I / d t = d h / d t = 0 \). Therefore, due to (26) they can be found from the equations

\[
\begin{align*}
\sin 2I(-D + \alpha \cos 2h') + \beta \cos 2I \cos(h' + \gamma) &= 0, \\
-\sin^2 I \alpha \sin 2h' - \beta \sin 2I \sin(h' + \gamma) &= 0.
\end{align*}
\]  

(37)

Since the mass of the exo-Sun is much larger than the masses of planets, due to (15), (16) and (27) we have that \( D \gg \max(|\alpha|, |\beta|) \). Therefore, the first equation in (37) implies \( \sin 2I \approx 0 \). Hence, the steady states are

\[ I \approx 0; \quad I \approx \pi; \quad I \approx \pi/2, \quad h' \approx 0, \pi/2, \pi, 3\pi/2, \]

where \((I \approx \pi/2, h' \approx 0)\) and \((I \approx \pi/2, h' \approx \pi)\) are saddles and the other ones are centres. The saddles are connected by heteroclinic trajectories that divide the celestial sphere into four regions, each comprised of a centre and closed trajectories around it, see Fig. 2a.

Since \( d I / d h' = d I / dt (d h' / d t)^{-1} \), the extrema of \( I(h') \) take place at

\[ -\sin^2 I \alpha \sin 2h' - \beta \sin 2I \sin(h' + \gamma) = 0. \]

(38)

Below we evaluate \( \Delta(I_0, h_0) \) defined by (1), where \( h_0 = \pi/2 \) (shown by grey line in Fig. 2a) and consider \( 0 \leq I_0 \leq \pi/2 \). Since \( D \) is large compared with \( \alpha \) and \( \beta \), the nutation angle \( I(h') \) is close to \( I_0 \) and we can write \( I(h') = I_0 + I_1(h') \). Therefore,

\[
\sin^2 I \approx \sin^2 I_0 + I_1 \sin 2I_0 + I_1^2 \cos 2I_0, \quad \sin 2I \approx \sin 2I_0 + 2I_1 \cos 2I_0.
\]

Since \( \tilde{\mathcal{G}}(I, h') \) is a constant on trajectories, substituting the above expressions into (26) we obtain a quadratic equation on \( I_1(h') \)

\[
I_1^2(-D \cos 2I_0) + I_1(-D \sin 2I_0 + 2\beta \cos 2I_0 \cos(h' + \gamma)) + \alpha \sin^2 I_0(\cos 2h' - 1) + \beta \sin 2I_0(\cos(h' + \gamma) + \sin \gamma) = 0.
\]

(39)

The range \( \Delta \) is a continuous function of \( I_0 \) and \( h_0 \) inside a region and is discontinuous at a boundary. For a heteroclinic trajectory through \( (\pi/2, 0) \), we have

\[ \tilde{\mathcal{G}}(I, h') = (-D + \alpha), \]

hence trajectories with the initial conditions \((I_0, \pi/2)\) such that

\[
\sin^2 I_0(-D - \alpha) + \beta \sin 2I_0 \sin \gamma < -D + \alpha
\]

(40)

belong to the equatorial regions, while the other ones to the polar regions. Since \( D \gg \max(\alpha, \beta) \), the inequality (40) may be simplified to

\[
|I_0 - \frac{\pi}{2}| < \left( \frac{2\alpha}{D} \right)^{1/2},
\]

(41)

i.e. the trajectories through \((I_0, \pi/2)\) belong to an equatorial region if \( \pi/2 - \delta_{het} < I_0 < \pi/2 + \delta_{het} \), where \( \delta_{het} = (2\alpha/D)^{1/2} \), and to a polar one otherwise.

To solve Eq. (39) we consider three possibilities for \( I_0 \) if the initial condition \((I_0, h_0)\) belongs to the polar region:

(i) \( D^{1/2} |\sin I_0| < \beta^{1/2}, |\sin I_0| < |\cos I_0| \);

(ii) \( D^{1/2} \sin^2 I_0 > \max(\alpha, \beta^{1/2} |\sin I_0|) \);

(iii) \( D^{1/2} \sin^2 I_0 < \alpha^{1/2}, |\sin I_0| > |\cos I_0|, |I_0 - \frac{\pi}{2}| > (2\alpha)^{1/2}D^{-1/2} \).

and the equatorial region:

(iv) \( |I_0 - \frac{\pi}{2}| < (2\alpha)^{1/2}D^{-1/2} \).
Since $D \gg \beta$, in case (i) we have that $I_0$ is close to 0 or $\pi$. Therefore, from (38) the extrema of $I(h')$ take place at $\sin(h' + \gamma) \approx 0$. Substituting $h' + \gamma = 0$ and $\pi$ into (39), solving the quadratic equation and subtracting the root at $h' + \gamma = 0$ from the one at $h' + \gamma = \pi$ we find that

$$\Delta \approx \frac{2\beta}{D}. \quad (42)$$

In case (ii) in Eq. (39), the quadratic term can be neglected and the remaining linear equation can be easily solved for any value of $h'$. We cannot derive from (38) the particular value of $h'$ where the maxima and minima take place; hence, we give upper and lower bound for $\Delta$ (they differ less than a factor 2):

$$\frac{\max(|\alpha| \sin I_0, 2|\beta| \cos I_0)}{D \cos I_0} \leq \Delta \leq \frac{|\alpha| \sin I_0 + 2|\beta| \cos I_0}{D \cos I_0}. \quad (43)$$

Alternatively, we introduce the function

$$\chi(I_0, \alpha, \beta, \gamma) = \max_{0 \leq h \leq 2\pi} f(I_0, \alpha, \beta, \gamma, h) - \min_{0 \leq h \leq 2\pi} f(I_0, \alpha, \beta, \gamma, h) \quad (44)$$

where

$$f(I_0, \alpha, \beta, \gamma, h) = \alpha \sin^2 I_0 \cos 2h + \beta \sin 2I_0 \cos(h + \gamma). \quad (45)$$

Then

$$\Delta(I_0, h_0) \approx \frac{\chi(I_0, \alpha, \beta, \gamma)}{D \cos I_0}. \quad (46)$$

As we noted $D \gg \alpha$, therefore in case (iii) we have that $I_0 \approx \pi/2$. Hence (see (38)), the minima of $I(h')$ take place at $h' = 0$ and $\pi$ and the maxima at $h' = \pi/2$ and $3\pi/2$. Solving (39) we obtain that

$$\Delta \approx \frac{D \sin 2I_0 - \beta \cos 2I_0 \cos \gamma + (\beta D \sin 2I_0 + \beta \cos 2I_0 \cos \gamma)^2 + 8D\alpha)^{1/2}}{2D} \quad (47)$$

In case (iv) a trajectory twice intersects the meridian $h' = \pi/2$, at the intersection points $I$ takes the maximum and minimum values, $I_{\max}$ and $I_{\min}$, for this particular trajectory (see (38)). Moreover, (39) implies that $I_{\max} - \pi/2 \approx \pi/2 - I_{\min}$. Hence,

$$\Delta \approx 2|\pi/2 - I_0|. \quad (48)$$

Overall, for $0 \leq I_0 \leq \pi/2$ we have

| case | $I_0$ |
|------|--------|
| (i)  | $D^{1/2}|\sin I_0| < \beta^{1/2}$ |
| (ii) | $D^{1/2} \sin^2 I_0 > \max(\alpha, \beta^{1/2} | \sin I_0|)$ |
| (iii)| $D^{1/2} \sin^2 I_0 < \alpha^{1/2}$, $|I_0 - \pi \gamma| < (2\alpha)^{1/2}(D')^{-1/2}$ |
| (iv) | $|I_0 - \pi \gamma| < (2\alpha)^{1/2}D^{-1/2}$ |

$$\Delta \approx \frac{2\beta}{D}. \quad (42)$$

or inequality (43)

Remark 2 We have calculated $\Delta(I_0, h_0)$ for $h_0 = \pi/2$ only. Unless $I_0$ is close to $\pi/2$, the dependence of $\Delta$ on $h_0$ is marginal, as illustrated by Fig. 2a. By contrast, near $I_0 = \pi/2$ the range essentially depends on $h_0$. For example, for an initial condition corresponding to a saddle steady state the range vanishes, while for a slightly perturbed initial condition one can get $\Delta$’s close to $2\delta_{het}$ (inside the equatorial region) or $\delta_{het}$ (inside the polar region), see in Fig. 2a. Investigation of the dependence of $\Delta(I_0, h_0)$ on $h_0$ for $I_0$ near $\pi/2$, which can be carried out similarly, is left for future studies.
B The canonical change of variables employed in Sect. 4.3

Let the functions $D_j(t)$ in the averaged Eqs. (18), (19) be represented as the sums

$$D_j = D_j^s + D_j^{(2)}$$

where $D_j^{(1)}$ (21) result from the torque from the Sun, $D_j^{(2)}$ (24) from the Moon and $D_j^{(3)}, \ldots, D_j^{(n)}$ from the planets. As discussed in Sect. 2, the terms $D_j^s$ are time-independent.

The mean Hamiltonian takes the form

$$H = -\frac{3\varepsilon}{2}(C_1 - A_1)[\sin^2 I - (D_1^s + D_2^{(2)}) \sin^2 h + (D_1^s + D_2^{(2)}) \cos^2 h + (D_3^s + D_4^{(2)}) \sin(2h)]$$

Substitution of (24) and (29) into (50) followed by a series of algebraic transformations yield

$$\mathcal{H}_1 = -\frac{3\varepsilon}{2}(C_1 - A_1)[(D_1^s \sin^2 h - D_2^s \cos^2 h + D_3^s + D_4^s \sin(2h) + (D_5^s \sin h - D_6^s \cos h + \frac{\varepsilon}{2} \sin 2i \cos(\Omega - h)) \sin 2I]$$

Using the generating function $F_2(h, H') = H'(h - \Omega_0 - \sigma_n t)$, we obtain that in the canonical coordinates $(H', h') = (H, h - \Omega_0 - \sigma_n t)$ the Hamiltonian (51) takes the form

$$\mathcal{H}'_1 = \mathcal{H}_1 + \frac{\partial F_2}{\partial t} = \mathcal{H}_1 - \sigma_n H'.$$

Therefore, (10), (11) imply that

$$\frac{d h'}{d t} = \frac{\rho}{\sin I} \frac{\partial G'}{\partial I}, \quad \frac{d I}{d t} = -\frac{\rho}{\sin I} \frac{\partial G'}{\partial h'},$$

where

$$G'(I, h') = \sin^2 I - (D_1^s D_2^s \cos^2 h' + D_3^s + D_4^s \sin(2h') + (D_5^s \sin h' - D_6^s \cos h' + \frac{\varepsilon}{2} \sin 2i \cos h') + \frac{\varepsilon}{\rho} \cos I$$

and $\rho = 3\varepsilon(C_1 - A_1)/2G$. In the new variables $(H, h')$, we have that $G' = const$ along the trajectories. Note that the Eq. (53) is invariant under the symmetry $h' \rightarrow -h'$

$$G'(I, h') = G'(I, -h').$$

The equation is also invariant after the transformation $(\sigma_n, I, h') \rightarrow (-\sigma_n, \pi - I, h' + \pi)$, therefore without the loss of generality we consider non-negative $\sigma_n$’s only.

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