Anisotropic (2+1)-d growth and Gaussian limits of $q$-Whittaker processes

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A model of growth in $d = 2$ dimension is the KPZ equation

$$\frac{\partial h}{\partial t} = \nu \Delta h + Q(\nabla h) + \eta$$

for some quadratic form $Q$ ($\eta$: local noise term)

Two cases behaving quite differently:

(a) **Isotropic KPZ:** $\text{sign}(Q) = (+1, +1)$ or $\text{sign}(Q) = (-1, -1)$
   - Fluctuations grow as $t^\chi$ for some $\chi \approx 0.240...$ (numerics)
   - No analytic results

(b) **Anisotropic KPZ:** $\text{sign}(Q) = (+1, -1)$ (including 0 as well)
Prediction from physics: for anisotropic KPZ, the non-linearity becomes irrelevant for the fluctuations

For $Q = 0$, we have the Edwards-Wilkinson equation

$$\frac{\partial h}{\partial t} = \nu \Delta h + \eta$$

for which

$$\text{Var}(h(\xi t, t)) \sim \ln(t), \quad t \to \infty$$

Logarithmic fluctuations shown in a special model

The model we consider in this talk is in the anisotropic KPZ framework
A model of interacting particles in the anisotropic KPZ growth

State space: interlacing particles $x_k^n \in \mathbb{Z}$, $1 \leq k \leq n \leq N$ satisfy

$$x_1^{n+1} < x_1^n \leq x_2^{n+1} < x_2^n \leq \ldots < x_n^n \leq x_{n+1}^{n+1}.$$
Packed initial condition: \( x^n_k = -n + k \)

Dynamics: particles try to jump to its right with rate 1. The jump of a particle \((k, n)\):
- is blocked by \((k, n - 1)\): if \( x^n_k = x^{n-1}_k \),
- pushes \((k + 1, n + 1)\) if \( x^n_k = x^{n+1}_{k+1} \).

Height function \( h \):

\[
\{ h(x, n) \geq a \} = \{ x^n_a \geq x \}
\]
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- is blocked by $(k, n - 1)$: if $x_k^n = x_k^{n-1}$,
- pushes $(k + 1, n + 1)$ if $x_k^n = x_k^{n+1}$.

Height function $h$:

$$\{ h(x, n) \geq a \} = \{ x_a^n \geq x \}$$
Key property for the analysis: the model is expressed in terms of a Schur process and has determinantal correlations

Theorem

Consider any $N$ (distinct) triples $(x_j, n_j, t_j)$ such that

$$t_1 \leq t_2 \leq \ldots \leq t_N, \quad n_1 \geq n_2 \geq \ldots \geq n_N.$$  

Then,

$$\mathbb{P}(\text{at each } (x_j, n_j, t_j), j = 1, \ldots, N, \text{ there exists a particle})$$

$$= \det [K(x_i, n_i, t_i; x_j, n_j, t_j)]_{1 \leq i, j \leq N}$$

for an explicit kernel $K$.  

Intro Determinantal $q$-Whittaker
2+1 dim particles: speed of growth

- Height function \( h = h(x, n) \)
- \( u_x = \text{Slope along the } x\text{-direction} \)
- \( u_n = \text{Slope along the } n\text{-direction} \)

⇒ Speed of growth \( v \) is given by

\[
v(u_x, u_n) = -\frac{\sin(\pi u_x) \sin(\pi u_n)}{\pi \sin(\pi (u_x + u_n))}
\]

- Using correlations at different times \text{Borodin,Ferrari’08}
- Using the definition from the dynamics \text{Chhita,Ferrari’15}

Is the model in the anisotropic KPZ class?

\[
\det(\text{Hessian}(v)) = -4\pi^2 \frac{\sin(\pi u_x)^2 \sin(\pi u_n)^2}{\sin(\pi (u_x + u_n))^4} < 0
\]

⇒ Yes.
Macroscopic parametrization

\[ n = \eta L, \quad x = -\eta L + \nu L, \quad t = \tau L \]

Random region is

\[ D = \{(\nu, \eta, \tau) \in \mathbb{R}_+^3, |\sqrt{\tau} - \sqrt{\eta}| < \sqrt{\nu} < \sqrt{\tau} + \sqrt{\eta}\} \]

Kenyon’s map \( \Omega : D \rightarrow \mathbb{H} = \{z \in \mathbb{C} | \text{Im}(z) > 0\} \)

\( (\pi \nu / \pi, \pi \eta / \pi, \pi \tau / \pi) \) are the frequencies of the three types of lozenge tilings.  

Kenyon’04
Macroscopic parametrization

\[ n = \eta L, \quad x = -\eta L + \nu L, \quad t = \tau L \]

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\[ \mathcal{D} = \{ (\nu, \eta, \tau) \in \mathbb{R}_+^3, |\sqrt{\tau} - \sqrt{\eta}| < \sqrt{\nu} < \sqrt{\tau} + \sqrt{\eta} \} \]

Kenyon’s map \( \Omega : \mathcal{D} \rightarrow \mathbb{H} = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \} \)

\((\pi_\nu / \pi, \pi_\eta / \pi, \pi_\tau / \pi)\) are the frequencies of the three types of lozenge tilings. 

Kenyon’04
Consider any (disjoint) $N$ triples $\kappa_j = (\nu_j - \eta_j, \eta_j, \tau_j)$, with $(\nu_j, \eta_j, \tau_j) \in \mathcal{D}$, 

$$\tau_1 \leq \tau_2 \leq \ldots \leq \tau_N \quad \eta_1 \geq \eta_2 \geq \ldots \geq \eta_N.$$ 

Set $H_L(\kappa) := \sqrt{\pi} \left( h(\kappa L) - \mathbb{E}(h(\kappa L)) \right)$. Then, 

$$\lim_{L \to \infty} \mathbb{E}(H_L(\kappa_1) \cdots H_L(\kappa_N)) = \begin{cases} 0, & \text{odd } N, \\ \sum_{\text{pairings } \sigma} \prod_{j=1}^{N/2} G(\Omega_{\sigma(2j-1)}, \Omega_{\sigma(2j)}), & \text{even } N, \end{cases}$$

with $G(z, w) = -(2\pi)^{-1} \ln |(z - w)/(z - \bar{w})|$ is the Green function of the Laplacian on $\mathbb{H}$ with Dirichlet boundary conditions.
Conjecture

The same result holds without the requirement

\[ \tau_1 \leq \tau_2 \leq \ldots \leq \tau_N, \quad \eta_1 \geq \eta_2 \geq \ldots \geq \eta_N \]

provided all the \( \Omega(\kappa_j) \)'s are distinct.

Reason: the space-time directions with constant \( \Omega \) are the "characteristic lines", along which we expect the decorrelation to be much slower than along any space-time directions.

Slow decorrelations phenomena: true for the \( d = 1 \) KPZ

Ferrari’08; Corwin, Ferrari, Péché’10
Is there a model in which one can prove existence of the slow decorrelation?

Can we study the full space-time covariance?
Discrete time versions leads to "more natural" random tiling measure, like the Aztec diamond or lozenge tilings

Nordenstam’08; Borodin, Gorin’08; Nordenstam, Young’11

At fixed time one has a non-intersecting line ensemble description: it is a Schur processes (Okounkov, Reshetikin’03), as in the Aztec diamond case

Johansson’03

The corresponding Markov dynamics on Schur process generalizes the one coming from the shuffling algorithm for Aztec diamond (Elkies, Kuperbert, Larsen, Propp’92)

Borodin, Ferrari’15

These measures and the corresponding Markov dynamics can also be described in terms of dimer models on Rail Yard graphs studied in

Betea, Boutilier, Bouttier, Chapuy, Corteel, Ramassamy’14–’15
Not all random tiling fits in the Schur process framework. E.g., periodic Aztec \cite{Chhita_Young_2013, Chhita_Johansson_2014}

With periodicity one can get a "gas phase" macroscopically flat. Proof of the Airy point field at the edge of the gas phase was recently obtained. \cite{Beffara_Chhita_Johansson_2016}
- Same state space and initial condition as in the model discussed above
  - State space: interlacing particles $x^n_k \in \mathbb{Z}$, $1 \leq k \leq n \leq N$ satisfy
    \[
    x_{n+1}^1 < x_1^n \leq x_{n+1}^2 < x_2^n \leq \ldots < x_n^n \leq x_{n+1}^n.
    \]
  - Packed initial condition: $x_k^n = -n + k$
- Dynamics: fix $q \in (0, 1)$. The jump rate of particle $(k, n)$ is given by
  \[
  \frac{(1 - qx_{k-1}^n - x_k^n - 1)(1 - qx_k^n - x_{k-1}^n)}{1 - qx_k^n - x_{k-1}^n}
  \]
- The $q = 0$ is the determinantal model studied before
• **Key property for the analysis:** the $q$-moments have explicit expressions.

\[
\mathbb{E}\left[ \prod_{i=1}^{m} q^{x_{1}^{n_{i}}(t)+\cdots+x_{r_{i}}^{n_{i}}(t)} \right]
\]

\[
= \prod_{i=1}^{m} \frac{1}{(2\pi i)^{r_{i}r_{i}!}} \int \cdots \int \prod_{1 \leq i < j \leq m} \prod_{k=1}^{r_{i}} \prod_{\ell=1}^{r_{j}} \frac{q(z_{i,k} - z_{j,\ell})}{z_{i,k} - qz_{j,\ell}} z_{i,k}^{-r_{i}} \prod_{1 \leq k < \ell \leq r_{i}} (z_{i,k} - z_{i,\ell})^{2} \times \prod_{i=1}^{m} \frac{(-1)^{r_{i}(r_{i}+1)}}{2} \prod_{1 \leq k \leq \ell \leq r_{i}} (z_{i,k} - z_{i,\ell})^{2} \prod_{k=1}^{r_{i}} \frac{e^{z_{i,k}(q-1)t}}{(1 - z_{i,k})^{n_{i}}} \, dz_{i,k}.
\]

(The domain of integrations are nested contours)
Some known results on the $q$-Whittaker process

- **Tracy-Widom distribution** for fixed $q$, let $\kappa > 1/(1 - q)$. Then as $N \to \infty$,
  \[ \frac{x_1^N(\kappa N) - c_1(\kappa)N}{c_2(\kappa)N^{1/3}} \overset{(d)}{=} \xi_{\text{GUE}} \]

  - *Ferrari, Veto’13; Barraquand’14*

- **Polymer models**: for $q = e^{-\varepsilon}$ and $t = \tau/\varepsilon^2$, as $\varepsilon \to 0$,
  \[ x_1^N(t) \overset{(d)}{=} \frac{\tau}{\varepsilon^2} - N \frac{\ln(1/\varepsilon)}{\varepsilon} - \frac{\ln Z(\tau, N)}{\varepsilon} \]

  where $Z(\tau, N)$ is the partition function of the semidiscrete directed polymer model (*O’Connell, Yor’01*).

  - *Borodin, Corwin’11*

- This polymer model was further used to get distribution laws of the solution of the KPZ equation

  - *Borodin, Corwin, Ferrari’12; +Veto’14*
In our work we consider shorter time scales:

\[ q = e^{-\varepsilon}, \quad t = \tau / \varepsilon. \]

Simulation of \( q \)-Whittaker particle system with \( N = 20 \) particles, \( q = e^{-\varepsilon} \) and \( \varepsilon = 0.01 \). The centered and diffusively scaled particle process \( x^{(n)}_k (\tau / \varepsilon) \) is plotted for \( \tau = 1 \).
In our work we consider shorter time scales:

\[ q = e^{-\varepsilon}, \quad t = \tau / \varepsilon. \]

Simulation of \( q \)-Whittaker particle system with \( N = 20 \) particles, \( q = e^{-\varepsilon} \) and \( \varepsilon = 0.01 \). The centered and diffusively scaled particle process \( x_{k}^{(n)}(\tau / \varepsilon) \) is plotted for \( \tau = 10 \).
Law of large numbers

\[
\lim_{\varepsilon \to 0} \varepsilon^{-1} x_k^\varepsilon (\varepsilon^{-1} \tau) = y_k^n (\tau)
\]

where

\[
e^{-\left( y_1^n(\tau) + \cdots + y_r^n(\tau) \right)} = \int \cdots \int \mathcal{F}_\tau (n, r; z_1, \ldots, z_r) \, d z_1 \cdots d z_r
\]

with

\[
\mathcal{F}_\tau (n, r; z_1, \ldots, z_r) = \frac{(-1)^{r(r+1)/2} \prod_{1 \leq k < \ell \leq r} (z_k - z_\ell)^2}{(2\pi i)^r r! \prod_{k=1}^r (z_k)^r} \prod_{k=1}^r \frac{e^{-z_k \tau}}{(1 - z_k)^n}
\]
Law of large numbers

\[ \lim_{\varepsilon \to 0} \varepsilon^{-1} x_k^n(\varepsilon^{-1} \tau) = y_k^n(\tau) \]

where

\[ e^{-\left(y_1^{(n)}(\tau) + \cdots + y_r^{(n)}(\tau)\right)} = e^{-\tau r} \det \left[ G_{r,\tau}(n + 1 - r + j - i) \right]_{i,j=1}^r \]

with \( G \) being polynomials given by

\[ G_{r,\tau}(m) = \sum_{i \geq 0} \frac{\tau^i(r)^{m-i-1}}{i!(m-i-1)!}. \]
Gaussian fluctuations

For any fixed $N$, as $\varepsilon \to 0$,

$$x_k^n(\tau/\varepsilon) = \varepsilon^{-1} y_k^n(\tau) + \varepsilon^{-1/2} \xi_k^n(\tau)$$

where $\{\xi_k^n(\tau), 1 \leq k \leq n \leq N\}$ is a centered Gaussian process with explicit covariance (written in terms of contour integrals).

We would like to study the large time and large-$N$ limit.
Large time simplification allowing large-$N$ asymptotic analysis

For any fixed $T > 0$, the limit

$$
\zeta^n_k(T) = \lim_{L \to \infty} L^{-1/2} \xi^n_k(LT)
$$

exists and $\zeta = \{\zeta^n_k(T), 1 \leq k \leq n \leq N\}$ is a centered Gaussian process with covariance given through the following formula. For $n_1 \geq n_2$,

$$
\text{Cov} \left( \zeta^{n_1}_{r_1}(T) + \cdots + \zeta^{n_1}_{r_1}(T); \zeta^{n_2}_{r_2}(T) + \cdots + \zeta^{n_2}_{r_2}(T) \right) = \int \int \frac{r_1 r_2}{z_1 - w_1} \left( \prod_{1 \leq i < j \leq r_1} (z_j - z_i)^2 \prod_{m=1}^{r_1} \frac{e^{T z_m}}{(z_m)^n_1} dz_m \right) \left( \prod_{1 \leq i < j \leq r_2} (w_j - w_i)^2 \prod_{m=1}^{r_2} \frac{e^{T w_m}}{(w_m)^n_2} dw_m \right)
$$

$$
\left( \int \prod_{1 \leq i < j \leq r_1} (z_j - z_i)^2 \prod_{m=1}^{r_1} \frac{e^{T z_m}}{(z_m)^n_1} dz_m \right) \left( \int \prod_{1 \leq i < j \leq r_2} (w_j - w_i)^2 \prod_{m=1}^{r_2} \frac{e^{T w_m}}{(w_m)^n_2} dw_m \right)
$$
Using random matrix type algebra we rewrite

\[
\text{Cov}\left(\zeta_{r_1}^{n_1}(T), \zeta_{r_2}^{n_2}(T)\right) = \oint_{\Gamma_0} \frac{dw}{2\pi i} \oint_{\Gamma_0,w} \frac{dz}{2\pi i} \frac{1}{z - w} \frac{e^{Tz}e^{Tw}}{z^{n_1}w^{n_2}} \frac{(p_{r_1-1}^{n_1}(z))^2}{\langle p_{r_1-1}^{n_1}, p_{r_1-1}^{n_1} \rangle_{n_1}} \frac{(p_{r_2-1}^{n_2}(w))^2}{\langle p_{r_2-1}^{n_2}, p_{r_2-1}^{n_2} \rangle_{n_2}}
\]

where \( p_r^n \) are orthogonal polynomials with respect to the scalar product \( \langle f, g \rangle_n = \frac{1}{2\pi i} \oint_{\Gamma_0} f(z)g(z) \frac{e^{Tz}}{z^n} dz \). Finally we have

\[
\text{Cov}\left(\zeta_{r_1}^{n_1}(T = 1), \zeta_{r_2}^{n_2}(T = 1)\right) = \frac{1}{(2\pi i)^2} \oint_{\Gamma_0} dw \oint_{\Gamma_0,w} dz \frac{1}{z - w} 
\times \left( \int_0^\infty dx (w - x)^{r_2-1} x^{n_2-r_2} e^{-x} \frac{1}{2\pi i} \oint_{\Gamma_w} du \frac{e^u}{(w - u)^{r_2} u^{n_2-r_2+1}} \right)
\times \left( \int_0^\infty dy (z - y)^{r_1-1} y^{n_1-r_1} e^{-y} \frac{1}{2\pi i} \oint_{\Gamma_z} dv \frac{e^v}{(z - v)^{r_1} v^{n_1-r_1+1}} \right).
\]
Asymptotic covariance in the bulk

Let us denote

\[ \Omega(c, b) = c(1 - 2b + 2i \sqrt{b(1 - b)}) \].

Take any \( a, b \in (0, 1) \), \( d > 0 \) and \( c \in (0, d] \) such that \( \Omega(c, b) \neq \Omega(d, a) \).

Then, the large \( N \) limit of the covariance is given by

\[
\lim_{N \to \infty} N \text{Cov} \left( \zeta^N_{adN}(T = 1), \zeta^N_{bcN}(T = 1) \right) = \frac{16}{(2\pi i)^2} \int_{\Omega(c, b)} dW \int_{\Omega(d, a)} dZ \frac{1}{Z - W} \\
\times \frac{1}{\sqrt{(W - \Omega(c, b))(W - \overline{\Omega}(c, b))} \sqrt{(Z - \Omega(d, a))(Z - \overline{\Omega}(d, a))}}
\]
Slow decorrelation

Take any \( a, b \in (0, 1), \ d > 0 \) and \( c \in (0, d] \). Then for any \( T > 1 \),

\[
\lim_{N \to \infty} N \text{Cov} \left( \zeta^{dN_T}_{adN_T}(T), \zeta^{cN}_{bcN}(T = 1) \right)
= \lim_{N \to \infty} N \text{Cov} \left( \zeta^{dN}_{adN}(T = 1), \zeta^{cN}_{bcN}(T = 1) \right).
\]

Here we looked at space points \( \mathcal{O}(N) \) away. On which scale one has non-trivial time-time correlations?
Short distance behavior

\[
\lim_{N \to \infty} N \text{Cov}\left( \zeta^{dN}_{adN}(T = 1), \zeta^{cN}_{bcN}(T = 1) \right)
= -4 \frac{\ln(|\Omega(d, a) - \Omega(c, b)|)}{\pi \sqrt{\text{Im}\Omega(d, a)} \sqrt{\text{Im}\Omega(c, b)}} + O(1)
\]

as \(|\Omega(d, a) - \Omega(c, b)| \to 0\).
Time correlation at $O(\sqrt{N})$ from the characteristic lines

Fix $d > 0$, $a \in (0, 1)$, $T > S > 0$. For $\eta = (\eta_1, \eta_2)$ let

$$\zeta(T, \eta; N) = N^{1/2}\zeta^n_k(T)$$

such that

$$n = dNT + \left(\eta_1\sqrt{(1-a)d} + \eta_2\sqrt{ad}\right)\sqrt{NT},$$

$$k = adNT + \eta_2\sqrt{ad}\sqrt{NT}.$$
Time correlation at $O(\sqrt{N})$ from the characteristic lines

Fix $d > 0$, $a \in (0, 1)$, $T > S > 0$.

For $\eta = (\eta_1, \eta_2)$ let $\zeta(T, \eta; N) = N^{1/2} \zeta_n(T)$

$$n = dNT + \left( \eta_1 \sqrt{(1 - a)d} + \eta_2 \sqrt{ad} \right) \sqrt{NT},$$

$$k = adNT + \eta_2 \sqrt{ad} \sqrt{NT}.$$

Then for $\eta, \lambda, \mu, \nu \in \mathbb{R}^2$ (all different),

$$\lim_{N \to \infty} \text{Cov} \left( \zeta(T, \eta; N) - \zeta(T, \lambda; N), \zeta(S, \mu; N) - \zeta(S, \nu; N) \right)$$

$$= \frac{S}{\pi d \sqrt{a(1 - a)}} \left( G_{\tau}(|\eta - \mu|) - G_{\tau}(|\eta - \nu|) - G_{\tau}(|\lambda - \mu|) + G_{\tau}(|\lambda - \nu|) \right)$$

where $\tau = (T - S)/T$ and $G_{\tau}(r) = -\ln(r^2) - \Gamma(0, r^2/2\tau)$
Relation with Edwards-Wilkinson and GFF

The space-time covariance at $O(\sqrt{N})$ away from the characteristic is the ones of the covariance of the Edwards-Wilkinson equation

\[ \partial_t u(x, t) = \frac{1}{2} \Delta u(x, t) + \xi(x, t) \]

with $\xi$ space-time white noise and GFF initial conditions
Asymptotic analysis - key idea

\[
\text{Cov}
\left(
\zeta^N_{aN}(1), \zeta^N_{bcN}(1)
\right)
= \frac{N}{(2\pi i)^2} \oint_{\Gamma_0} dW \oint_{\Gamma_{0,W}} dZ \frac{1}{Z - W}
\times \left(\int_0^\infty dX \frac{e^{NF(c,b,W,X)}}{X - W} \frac{1}{2\pi i} \oint_{\Gamma_0} dU e^{NG(c,b,W,U)} \right)
\times \left(\int_0^\infty dY \frac{e^{NF(1,a,Z,Y)}}{Y - Z} \frac{1}{2\pi i} \oint_{\Gamma_0} dV e^{NG(1,a,Z,V)} \right),
\]

where

\[
F(c, b, W, X) = bc \ln(X - W) + (1 - b)c \ln(X) - X,
\]
\[
G(c, b, W, U) = -bc \ln(U) - (1 - b)c \ln(W + U) + U + W.
\]
Cov\left(\zeta_{aN}(1), \zeta_{bcN}(1)\right) = \frac{N}{(2\pi i)^2} \oint_{\Gamma_0} dW \oint_{\Gamma_{0,W}} dZ \frac{1}{Z - W} \\
\times \left( \int_{0}^{\infty} dX \frac{e^{NF(c,b,W,X)}}{X - W} \frac{1}{2\pi i} \oint_{\Gamma_0} dU \frac{e^{NG(c,b,W,U)}}{W + U} \right) \\
\times \left( \int_{0}^{\infty} dY \frac{e^{NF(1,a,Z,Y)}}{Y - Z} \frac{1}{2\pi i} \oint_{\Gamma_0} dV \frac{e^{NG(1,a,Z,V)}}{Z + V} \right),

Key idea: Study \( H(W, X, U) = F(c, b, W, X) + G(c, b, W, U) \).
\( W, Z \) are slow varying variables, \( X, U, Y, V \) are fast varying variables.

(1) Integrate first \( X, U, Y, V \).
(2) Integrate after that \( W, Z \).

Difficulty: After (1), there are mixed-terms which are bounded only along some non-explicitly known paths for \( W, Z \). The mixed terms vanish only after (2).