Stability of Topological Black Holes

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Abstract

We explore the classical stability of topological black holes in $d$-dimensional anti-de Sitter spacetime, where the horizon is an Einstein manifold of negative curvature. According to the gauge invariant formalism of Ishibashi and Kodama, gravitational perturbations are classified as being of scalar, vector, or tensor type, depending on their transformation properties with respect to the horizon manifold. For the massless black hole, we show that the perturbation equations for all modes can be reduced to a simple scalar field equation. This equation is exactly solvable in terms of hypergeometric functions, thus allowing an exact analytic determination of potential gravitational instabilities. We establish a necessary and sufficient condition for stability, in terms of the eigenvalues $\lambda$ of the Lichnerowicz operator on the horizon manifold, namely $\lambda \geq -4(d - 2)$. For the case of negative mass black holes, we show that a sufficient condition for stability is given by $\lambda \geq -2(d - 3)$.

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1 Introduction

Black holes in anti-de Sitter space have been the subject of much recent attention, particularly in connection with the proposed correspondence between anti-de Sitter gravity and boundary conformal field theory (AdS/CFT) [1]-[3]. A class of static black holes solutions to $d$-dimensional anti-de Sitter gravity was constructed in [4], with the special property that the horizon $M^{d-2}$ is a $(d - 2)$-dimensional compact Einstein manifold of positive, zero, or negative curvature. Furthermore, for the case of negative curvature horizon, there is a class of black holes for which the mass parameter $M$ can assume both negative and zero values, $M_{\text{crit}} \leq M \leq 0$. While topological black holes allow one to study the boundary conformal field theory on spaces of the form $S^1 \times M^{d-2}$ [4], they are also interesting structures in their own right. It is of particular interest to explore their classical stability properties.

The response of a black hole metric to small perturbations has been the subject of investigation for many years. By analyzing the perturbation equations subject to certain boundary conditions, one can gain valuable insight into the structure of the black hole. In particular, one can address the question of the classical stability of the black hole. For the Schwarzschild black hole in four dimensions, this boundary value problem was analyzed quite some time ago. It was shown that metric perturbations could be described by either a scalar mode (the Zerilli mode [5, 6]) or a vector mode (the Regge-Wheeler mode [7]). Using the Zerilli and Regge-Wheeler equations, the stability was then established in [8]-[10]. Remarkably, a similar analysis for the higher-dimensional Schwarzschild black hole [11] was lacking until quite recently. In [12], it was shown that an additional tensor mode is present in dimensions greater than four. Following this, a complete gauge invariant formalism was developed by Ishibashi and Kodama [13], where the equations describing all gravitational perturbations of all higher-dimensional static black holes have been presented. This powerful formalism identifies three basic types of gravitational master field, depending on how the field transforms with respect to the horizon manifold. One has scalar and vector modes, and an additional tensor mode in dimensions greater than four. Moreover, the equations for these master fields have a standard form as a Schrödinger-type second order ordinary differential equation with a potential.

The Ishibashi-Kodama formalism has been used successfully to establish the stability of a wide array of charged and uncharged black holes in asymptotically flat, de Sitter, and anti-de Sitter space [14]-[17]. Generalizations of this formalism to include the effects of rotation have also been considered [18]. In particular, the stability of the asymptotically flat Schwarzschild black hole was firmly established in all dimensions, following earlier work in [12]. The key technique used in [14]-[17] is a so-called $S$-deformation of the potentials appearing in the master equations. This technique allows one to establish stability based on positivity of the corresponding deformed potentials. However, apart from the asymptotically flat Schwarzschild black hole, there is no other class of black holes for which stability has been established in all dimensions.

Our goal here is to investigate the stability properties of topological black holes with negative curvature horizon, for mass parameter in the range $M_{\text{crit}} \leq M \leq 0$. We first consider the massless topological black hole, and show that the master field equations for all modes
can be solved explicitly in all dimensions [19]. The solution can be written in terms of hypergeometric functions, and by imposing appropriate boundary conditions, we can analyze the stability question in full detail. Using this explicit solution, we are led directly to a necessary and sufficient condition for stability of the massless black hole. This calculation completes and extends earlier work in [12, 15]. We show that stability is determined solely by the spectrum of the Lichnerowicz operator on the horizon manifold. In particular, we conclude that massless topological black holes with either constant curvature (hyperbolic) horizons, or Einstein-Kähler horizons, are stable in all dimensions. Following this, we consider the case of negative mass black holes. Although the explicit solution of the master equations is not available, we are nevertheless able to establish a sufficient condition for stability. Using positivity of the gravitational potentials, as well as the $S$-deformation technique, allows us to derive a sufficient condition for stability in all dimensions. As an example, we show that negative mass black holes with Einstein-Kähler horizon are stable in all dimensions. It should be pointed out that, in most cases, the boundary conditions are dictated by the requirement of normalizability of the perturbation. As we shall see, we must then impose Dirichlet boundary conditions at both the horizon and infinity. However, there is additional freedom in the choice of boundary conditions for certain perturbations in dimensions four, five, and six, as pointed out in [20].

The plan of this paper is as follows. In section 2, we recall the essential features of topological black holes in anti-de Sitter space. In particular, we identify the massless and negative mass black holes with negative curvature horizon, which are the main focus of interest. In section 3, we present the basic equations in the Ishibashi-Kodama gauge invariant formalism for gravitational perturbations. In section 4, we demonstrate the unified form which these equations take for the case of the massless black hole. The explicit solution of the master equations is given and a necessary and sufficient condition for stability is derived. Section 5 deals with negative mass black holes, and we use positivity properties of the gravitational potentials to establish a sufficient condition for stability. We conclude in section 6 with a brief discussion of our results, and also highlight the issue of boundary conditions in dimensions four, five, and six.

## 2 Topological Black Holes in anti-de Sitter Space

In $d$-dimensional anti-de Sitter space, there is a class of topological black hole solutions to Einstein’s equations which has the property that the horizon $M^{d-2}$ is a $(d-2)$-dimensional compact Einstein space of positive, zero or negative curvature $k$ [4]. Topological black holes solutions in four dimensions were first constructed in [21]-[24]. The line element of the topological black hole is given by [4]

$$ds^2 = -f(r) \, dt^2 + f^{-1}(r) \, dr^2 + r^2 h_{ij}(x) \, dx^i dx^j,$$

where

$$f(r) = \left( k - \frac{\omega_d M}{r^{d-2}} + \frac{r^2}{l^2} \right),$$

for

$$2$$
and
\[ \omega_d = \frac{16\pi G}{(d-2)\text{Vol}(M^{d-2})}. \]  (3)

The parameter \( k \) can take the values \( k = 1, 0, -1 \). The volume of the horizon is denoted by \( \text{Vol}(M^{d-2}) = \int d^{d-2}x \sqrt{h} \). The parameter \( l \), with dimensions of length, is related to the cosmological constant \( \Lambda \) by \( \Lambda = -(d-1)(d-2)/2l^2 \), and \( \omega_d \) is inserted so that \( M \) has dimensions of inverse length.

It is straightforward to check that the metric (1) satisfies Einstein’s equations with negative cosmological constant, namely
\[ R_{\mu\nu} = -\frac{(d-1)}{l^2} g_{\mu\nu}, \]  (4)
provided that the horizon is an Einstein space
\[ R_{ij}(h) = k(d-3)h_{ij}. \]  (5)

Our interest here is in the negative curvature case with \( k = -1 \). An interesting subclass of black holes is then obtained by taking the horizon to be a manifold of constant curvature, i.e., a hyperbolic manifold. In this case, \( M^{d-2} = H^{d-2}/\Gamma \), where \( H^{d-2} \) is hyperbolic space and \( \Gamma \) is a suitable discrete subgroup of the isometry group of \( H^{d-2} \).

The mass parameter \( M \) can be expressed in terms of the location of the horizon \( r_+ \), as
\[ M = \frac{r_+^{d-3}}{\omega_d} \left( -1 + \frac{r_+^2}{l^2} \right). \]  (6)

Furthermore, the inverse Hawking temperature is given by [4]
\[ \beta = \frac{4\pi l^2 r_+}{(d-1)r_+^2 - (d-3)l^2}. \]  (7)

A very special feature which is present in the case of negative curvature horizon, is that the parameter \( M \) can assume negative values, as first discussed in [23, 25, 24, 26]. The requirement of positivity of temperature enforces an inequality on the value of \( r_+ \), namely that \( r_+ > r_{\text{crit}} \), where
\[ r_{\text{crit}} = \left( \frac{d-3}{d-1} \right)^{1/2} l. \]  (8)

The corresponding value of \( M \) is then given by (6),
\[ M_{\text{crit}} = -\left( \frac{2}{d-1} \right) \left( \frac{d-3}{d-1} \right)^{(d-3)/2} \frac{l^{d-3}}{\omega_d}. \]  (9)
Thus, when $k = -1$, there is a class of black holes with mass parameter $M$ in the range $M_{\text{crit}} \leq M \leq 0$. For $M = 0$, we note that the event horizon occurs at $r_+ = l$, while for $M_{\text{crit}} \leq M < 0$, we have $r_+ < l$. It should be noted that the $M = M_{\text{crit}}$ solution has a degenerate horizon at $r = r_{\text{crit}}$ with $f(r_{\text{crit}}) = f'(r_{\text{crit}}) = 0$. Although these extremal solutions do not strictly have an interpretation as black holes [23, 24], we can still incorporate them into the stability analysis that follows.

These topological black holes are interesting structures in their own right, and our goal here is to investigate their classical stability properties. However, they also assume an importance within the context of the AdS/CFT correspondence. In particular, they allow us to study the dual conformal field theory on spaces of the form $S^1 \times M^{d-2}$, where $M^{d-2}$ is an Einstein space of positive, zero, or negative curvature [4, 27].

### 3 Gravitational Perturbations

In order to check for the existence of unstable gravitational perturbations of a black hole, we first need to obtain the relevant equations which describe these perturbations. In the four-dimensional asymptotically flat case, this was achieved quite some time ago, resulting in the Zerilli equation [5, 6] and the Regge-Wheeler equation [7]. These equations were generalized to the anti-de Sitter case in [28, 29]. However, a general analysis of gravitational perturbations in higher dimensions was presented only recently by Ishibashi and Kodama [13]–[17]. The formalism developed by Ishibashi and Kodama is both powerful and elegant, and is based on the introduction of gauge invariant variables. These gauge invariant combinations are then described by master fields $\Phi$. In general, there are three types of gravitational perturbation; the scalar mode which is the analogue of the Zerilli mode in higher dimensions, the vector mode which is the analogue of the Regge-Wheeler mode, and an additional tensor mode which is present in dimensions greater than four [12]. As shown in [13], each perturbation is simply described in terms of a master field $\Phi$ which satisfies a Schrödinger-type second order ordinary differential equation.

The Ishibashi-Kodama equations have been used to successfully establish the stability of asymptotically flat Schwarzschild black holes in all dimensions [14], following an earlier analysis in [12]. However, for many other higher-dimensional black holes, the stability question is still an open issue, with only partial results available. Our aim here is to find several new examples where precise conditions for stability can be established in all dimensions. The stability of topological black holes with respect to scalar field couplings has been analyzed in [30, 31].

To begin, we write the master field as

$$\Phi(t, r, x^i) = \Phi(r)Y(x^i)e^{\omega t}. \quad (10)$$

The type of perturbation then depends on whether $Y$ transforms as a scalar, vector, or tensor with respect to the horizon manifold $M^{d-2}$. In all cases, however, the master equation takes
the simple form
\[
\left[ - \left( f \frac{d}{dr} \right)^2 + V \right] \Phi(r) = -\omega^2 \Phi(r),
\]
(11)
where the structure of the potential \( V \) depends on the gravitational mode under consideration. For the scalar mode, we have
\[
V_S(r) = \frac{fU(r)}{16r^2 H^2},
\]
(12)
where
\[
x = \frac{\omega_d M}{r^{d-3}}, \quad \mu = k_S^2 + (d - 2),
\]
\[
H = \mu + \frac{1}{2}(d - 2)(d - 1)x.
\]
(13)
In this case, \( Y \) transforms as a scalar, and is an eigenfunction of the scalar Laplacian on the horizon manifold, \( \nabla^2 Y = -k_S^2 Y \). The function \( U(r) \) is given by
\[
U(r) = \frac{f}{r^2} \left[ k_V^2 - 1 - \frac{(d - 2)(d - 4)}{4} + \frac{(d - 2)(d - 4)}{4} \frac{r^2}{l^2} - \frac{3(d - 2)^2 \omega_d M}{4r^{d-3}} \right],
\]
(15)
where \( \nabla^2 Y = -k_V^2 Y \). Finally, the tensor mode in dimension \( d > 4 \) has the potential
\[
V_T(r) = \frac{f}{r^2} \left[ \lambda + 2(d - 3) - \frac{(d - 2)(d - 4)}{4} + \frac{d(d - 2)}{4} \frac{r^2}{l^2} + \frac{(d - 2)^2 \omega_d M}{4r^{d-3}} \right],
\]
(16)
where \( \lambda \) is the eigenvalue of the Lichnerowicz operator on the horizon.

In order to investigate the stability properties of the black hole, it is useful to re-cast Eq. (11) in the form
\[
A \Phi = -\omega^2 \Phi,
\]
(17)
where
\[
A = -\frac{d^2}{dr^2} + V(r),
\]
(18)
and the tortoise coordinate \( r_\ast \) is defined by \( dr_\ast = \frac{dr}{r} \). Our task is to solve this equation subject to appropriate boundary conditions. In particular, unstable modes correspond to normalizable negative energy \( (\omega > 0) \) states of the Schrödinger operator \( A \). In order to ensure normalizability, in the sense that \([12, 14]\),

\[
1 = \int dr_\ast \Phi^* \Phi, \tag{19}
\]

we must impose boundary conditions both at the horizon and infinity. Near the horizon, normalizability demands that we impose a Dirichlet boundary condition \( \Phi \rightarrow 0 \) on the perturbation \([12, 14, 32]\). For large \( r \), we see that the perturbation must behave as \( \Phi \sim r^{\alpha/2} \) as \( r \rightarrow \infty \), with \( \alpha < 1 \). For dimensions \( d \geq 7 \), the latter requirement is only satisfied by imposing Dirichlet boundary conditions on the perturbation. However, for certain perturbations in dimensions four, five, and six, there is additional freedom in the choice of boundary conditions at infinity. In fact, as shown in \([20]\) for the case of pure anti-de Sitter space, there is a one-parameter family of self-adjoint extensions of the operator \( A \) in these cases. Each of these self-adjoint extensions comes equipped with a choice of boundary conditions at infinity. Since the asymptotic form of the metric for topological black holes \([1]\) is akin to the pure anti-de Sitter case, we observe a similar freedom in the choice of boundary conditions for these these perturbations. In the following, we will study the stability properties in these special cases with respect to a choice of Dirichlet boundary conditions at infinity.

### 4 Stability of the Massless Black Hole

The form of the potentials \([12, 15, 16]\) simplifies considerably for the massless topological black hole \( M = 0 \). The scalar potential is given by

\[
V_S = \frac{f}{r^2} \left[ Q_S - \frac{(d-2)(d-4)}{4} + \frac{(d-4)(d-6)}{4} \frac{r^2}{l^2} \right], \tag{20}
\]

where we have introduced the notation \( Q_S = k_S^2 \). The vector potential is

\[
V_V = \frac{f}{r^2} \left[ Q_V - \frac{(d-2)(d-4)}{4} + \frac{(d-2)(d-4)}{4} \frac{r^2}{l^2} \right], \tag{21}
\]

with \( Q_V = k_V^2 - 1 \). The tensor potential is

\[
V_T = \frac{f}{r^2} \left[ Q_T - \frac{(d-2)(d-4)}{4} + \frac{d(d-2)}{4} \frac{r^2}{l^2} \right], \tag{22}
\]

with \( Q_T = \lambda + 2(d-3) \).

Before solving the above equations, let us first examine the case of a scalar field \( \phi \) of mass \( m \) in the background of the massless black hole. The equation of motion for the scalar field is

\[
(\nabla^2 - m^2)\phi = 0. \tag{23}
\]
Choosing the ansatz
\[ \phi = \phi(r)Y(x^i)e^{\omega t}, \]  
(24)

brings the radial equation to the form \((11)\), where \(\Phi = r^{d-2}\phi\). The potential is given by
\[ V = \frac{f}{r^2} \left[ Q + f' \left( \frac{d-2}{2} \right) r + f \left( \frac{(d-2)(d-4)}{4} + m^2r^2 \right) \right], \]
(25)

where \(\nabla^2 Y = -QY\). Since the metric involves the function \(f = -1 + \frac{r^2}{l^2}\), the potential of the scalar field takes the particularly simple form
\[ V = \frac{f}{r^2} \left[ Q - \frac{(d-2)(d-4)}{4} + \left( \frac{d(d-2)}{4} + m^2l^2 \right) \frac{r^2}{l^2} \right]. \]
(26)

In \([33]\), this equation was shown to be exactly solvable in terms of hypergeometric functions.

We now observe that the gravitational potentials \((20)-(22)\) have precisely the same structure as the potential of the scalar field \((26)\), for various values of the mass parameter. We have

\[
\begin{align*}
\text{scalar mode :} & \quad m^2l^2 = -2(d-3), \\
\text{vector mode :} & \quad m^2l^2 = -(d-2), \\
\text{tensor mode :} & \quad m^2l^2 = 0,
\end{align*}
\]
(27)

with the value \(Q\) replaced by the appropriate value \(Q_s, Q_v, Q_T\). It should be noted that the simplicity of the potentials in this case is essentially due to the fact that the mass parameter of the black hole is set to zero.

Our aim now is to solve Eq. \((11)\), with potentials \((20)-(22)\), subject to the boundary conditions that \(\Phi \to 0\), at the horizon and at infinity. To proceed towards the solution of \((11)\), we change variables to
\[ z = 1 - \frac{l^2}{r^2}. \]
(28)

Thus, \(z = 0\) corresponds to the location of the horizon \(r = l\), while \(z = 1\) corresponds to \(r = \infty\). The master equation then becomes
\[ z(1-z)\frac{d^2\Phi}{dz^2} + \left(1 - \frac{3z}{2}\right) \frac{d\Phi}{dz} + \left[ \frac{A}{z} + B + \frac{C}{1-z} \right] \Phi = 0, \]
(29)

where
\[
\begin{align*}
A &= -\frac{\omega^2l^2}{4}, \\
B &= \frac{1}{4} \left( \frac{(d-2)(d-4)}{4} - Q \right), \\
C &= -\frac{1}{4} \left( m^2l^2 + \frac{d(d-2)}{4} \right).
\end{align*}
\]
(30)
We now define
\[ \Phi(z) = z^\alpha (1 - z)^\beta F(z). \]  (31)

The master equation then reduces to the standard form of the hypergeometric equation
\[ z(1 - z) \frac{d^2 F}{dz^2} + [c - (a + b + 1)z] \frac{dF}{dz} - abF = 0, \]  (32)
provided that
\[ \alpha = \pm \frac{\omega l}{2}, \]
\[ \beta = \frac{1}{4} \pm \frac{1}{4} \sqrt{(d - 1)^2 + 4m^2l^2}, \]  (33)
with the coefficients determined as followed
\[ a = \frac{1}{4} + \alpha + \beta + \frac{1}{4} \sqrt{(d - 3)^2 - 4Q}, \]
\[ b = \frac{1}{4} + \alpha + \beta - \frac{1}{4} \sqrt{(d - 3)^2 - 4Q}, \]
\[ c = 2\alpha + 1. \]  (34)

Without loss of generality, we can take
\[ \alpha = \frac{\omega l}{2}, \]
\[ \beta = \frac{1}{4} - \frac{1}{4} \sqrt{(d - 1)^2 + 4m^2l^2}. \]  (35)

In the neighbourhood of the horizon, the two linearly independent solutions of (32) are
\[ F(a, b, c, z) \] and \[ z^{1-c} F(a - c + 1, b - c + 1, 2 - c, z). \] With the choice (35), the solution which is regular (satisfying Dirichlet boundary conditions) at the horizon is then given by
\[ \Phi(z) = z^\alpha (1 - z)^\beta F(a, b, c, z). \]  (36)

Having imposed the Dirichlet boundary condition at the horizon, we can now analytically continue this solution to infinity. In general, the form of the solution near \( z = 1 \) is given by [34]
\[ \Phi = z^\alpha (1 - z)^\beta \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} F(a, b, a + b - c + 1, 1 - z) \]
\[ + z^\alpha (1 - z)^{\beta + c - a - b} \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} F(c - a, c - b, c - a - b + 1, 1 - z). \]  (37)

However, special care is needed when \( c - a - b \) is an integer. Therefore, we should examine the coefficients closely, case by case. As we have seen, the gravitational scalar mode corresponds to a scalar field with mass \( m^2l^2 = -2(d - 3) \), the gravitational vector mode corresponds to
a scalar field of mass $m^2 l^2 = -(d - 2)$, and the gravitational tensor mode corresponds to a massless scalar field. It will be useful to record the values of the coefficient $\beta$ given by (35) for the three gravitational modes, as follows:

$$
\begin{align*}
\beta_S &= \begin{cases} 0, & d = 4, \\ -\left(\frac{d-6}{4}\right), & d \geq 5, \end{cases} \\
\beta_V &= -\left(\frac{d-4}{4}\right), \quad d \geq 4, \\
\beta_T &= -\left(\frac{d-2}{4}\right), \quad d > 4, 
\end{align*}
$$

where the subscript on $\beta$ specifies the particular mode. From (34), we also note that $c - a - b = \frac{1}{2} - 2\beta$.

Let us consider first the case in four dimensions. The scalar and vector modes both have a value of $\beta = 0$, and there is no tensor mode in four dimensions. Here, $c - a - b = 1/2$, so the analytic continuation to $z = 1$ is given by (37). The master field then takes the form

$$
\Phi = z^a \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b, a + b - c + 1, 1 - z) + z^a (1 - z)^{1/2} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c - a, c - b, c - a - b + 1, 1 - z). 
$$

The second term above clearly vanishes at infinity. Furthermore, the perturbation satisfies a Dirichlet boundary condition at infinity, $\Phi = 0$ at $z = 1$, if the coefficients $(a, b, c)$ can be chosen so that the gamma functions in the denominator of the first term have a pole. Namely, we require

$$
c - a = -n, \quad \text{or} \quad c - b = -n, \quad (40)
$$

where $(n = 0, 1, 2, 3, ...)$). The existence, or otherwise, of unstable modes then depends on whether these conditions (40) can be implemented. This in turn depends solely on the eigenvalue of the corresponding Laplacian on the horizon manifold.

For the scalar mode, the coefficients in the hypergeometric function take the form

$$
\begin{align*}
a &= \frac{\omega l}{2} + \frac{1}{4} + \frac{1}{4} \sqrt{1 - 4k_S^2}, \\
b &= \frac{\omega l}{2} + \frac{1}{4} - \frac{1}{4} \sqrt{1 - 4k_S^2}, \\
c &= \omega l + 1, 
\end{align*}
$$

where $k_S^2 \geq 0$ is the eigenvalue of the scalar Laplacian on the horizon manifold. In particular, let us examine the condition $c - a = 0$. This can be written as

$$
\frac{\omega l}{2} = -\frac{3}{4} + \frac{1}{4} \sqrt{1 - 4k_S^2}, \quad (42)
$$
Thus, in order for an unstable mode with $\omega > 0$ to exist, we require the scalar Laplacian to have an eigenvalue satisfying the condition

$$k_S^2 < -2.$$  \hfill (43)

However, $k_S^2 \geq 0$ for arbitrary Einstein horizons \cite{15}, and thus we conclude that such an unstable scalar mode does not exist. Moreover, the constraint $k_S^2 \geq 0$ also ensures that the conditions $c - a = -n$ for $(n = 1, 2, 3, \ldots)$ and $c - b = -n$ for $(n = 0, 1, 2, \ldots)$ cannot be satisfied. We conclude that the massless black hole is stable against scalar perturbations, for arbitrary Einstein horizon.

For the vector modes, we have

$$
\begin{align*}
a &= \frac{\omega l}{2} + \frac{1}{4} + \frac{1}{4}\sqrt{1 - 4(k_V^2 - 1)}, \\
b &= \frac{\omega l}{2} + \frac{1}{4} - \frac{1}{4}\sqrt{1 - 4(k_V^2 - 1)}, \\
c &= \omega l + 1.
\end{align*}
$$  \hfill (44)

In this case, the condition $c - a = 0$ requires the vector Laplacian to have an eigenvalue satisfying

$$k_V^2 < -1.$$  \hfill (45)

For general Einstein horizons, $k_V^2 \geq 0$ \cite{15}, and we conclude that the massless black hole is stable against vector perturbations. Collecting the above results, we conclude that (with respect to a choice of Dirichlet boundary conditions at infinity) the massless topological black hole is stable in four dimensions, in agreement with the analysis in \cite{14}. As one can see from (39), $\Phi$ tends to a constant at infinity, and is therefore normalizable as it stands. However, as shown in \cite{20}, there is a one-parameter family of self-adjoint extensions of the operator $A$ in this case. Imposing a Dirichlet boundary condition at infinity then corresponds to a particular choice of self-adjoint extension. The stability properties with respect to the other self-adjoint extensions is a problem that warrants further investigation.

Next, we consider all even dimensions greater than four. From (38), we see that $\beta \leq 0$ for all perturbations; furthermore, $c - a - b$ is not an integer. Thus, the continuation to $z = 1$ is given by (37)

$$
\begin{align*}
\Phi &= z^\alpha (1 - z)^\beta \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} F(a, b, a + b - c + 1, 1 - z) \\
&\quad + z^\alpha (1 - z)^{\frac{1}{2} - \beta} \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} F(c - a, c - b, c - a - b + 1, 1 - z).
\end{align*}
$$  \hfill (46)

Since $\beta \leq 0$, the second term vanishes automatically at $z = 1$. The Dirichlet boundary condition at infinity can be imposed by choosing the coefficients to satisfy (40). First, we consider the
scalar modes. The coefficients are given by
\[ a = \frac{\omega l}{2} - \left( \frac{d - 7}{4} \right) + \frac{1}{4} \sqrt{(d - 3)^2 - 4k_S^2}, \]
\[ b = \frac{\omega l}{2} - \left( \frac{d - 7}{4} \right) - \frac{1}{4} \sqrt{(d - 3)^2 - 4k_S^2}, \]
\[ c = \omega l + 1. \]  
(47)

None of the conditions (40) can be satisfied unless
\[ k_S^2 < 0. \]  
(48)

However, since \( k_S^2 \geq 0 \) [15], the black hole is stable against scalar perturbations.

For the vector modes, we have
\[ a = \frac{\omega l}{2} - \left( \frac{d - 5}{4} \right) + \frac{1}{4} \sqrt{(d - 3)^2 - 4(k_V^2 - 1)}, \]
\[ b = \frac{\omega l}{2} - \left( \frac{d - 5}{4} \right) - \frac{1}{4} \sqrt{(d - 3)^2 - 4(k_V^2 - 1)}, \]
\[ c = \omega l + 1. \]  
(49)

In this case, none of the conditions (40) can be satisfied unless
\[ k_V^2 < -(d - 3). \]  
(50)

Since \( k_V^2 \geq 0 \) [15], there are no unstable vector perturbations.

For the tensor modes, we have
\[ a = \frac{\omega l}{2} - \left( \frac{d - 3}{4} \right) + \frac{1}{4} \sqrt{(d - 3)^2 - 4[\lambda + 2(d - 3)]}, \]
\[ b = \frac{\omega l}{2} - \left( \frac{d - 3}{4} \right) - \frac{1}{4} \sqrt{(d - 3)^2 - 4[\lambda + 2(d - 3)]}, \]
\[ c = \omega l + 1. \]  
(51)

In this case, the condition \( c - a = 0 \) can be satisfied if the Lichnerowicz operator has an eigenvalue
\[ \lambda < -4(d - 2). \]  
(52)

This result was obtained previously in [12]. Furthermore, if \( \lambda \geq -4(d - 2) \), then we conclude that the black hole is stable against tensor perturbations. For the scalar perturbation in six dimensions, we notice from (38) that \( \beta_S = 0 \), and thus the perturbation is normalizable as it stands. Thus, the choice of Dirichlet boundary conditions at infinity corresponds to a particular choice of self-adjoint extension [20].
Turning now to odd dimensions, let us first consider the case of $d = 5$. The subtlety here is that $c - a - b$ is an integer, and the analytic continuation to $z = 1$ contains logarithmically divergent terms. For the scalar perturbation, we have $\beta_S = 1/4$ and $c - a - b = 0$. The master field near $z = 1$ is then given by \[ \Phi = z^\alpha (1 - z)^{-1/4} \frac{\Gamma(a + b)}{\Gamma(a) \Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} [2\psi(n + 1) - \psi(a + n) - \psi(b + n) - \ln(1 - z)](1 - z)^n, \]

where $(a)_n = \Gamma(a + n)/\Gamma(a)$, and $\psi(z) = \Gamma'(z)/\Gamma(z)$. In this case, $\Phi$ automatically vanishes at infinity. However, as shown in [20], there is a one-parameter family of self-adjoint extensions of the corresponding operator $A$ for pure anti-de Sitter space. In order to proceed, we must then make a choice of self-adjoint extension, and thus make a corresponding choice of boundary conditions at infinity. We shall require $(1 - z)^{-1/4}\Phi$ to vanish at infinity. This is achieved by choosing $a = -n$ or $b = -n$. However, since, $c - a - b = 0$, this can be re-written as \[ (1 - b)^{-1/4}\Phi = z^\alpha (1 - z)^{-1/4} F(a, b, a + b + 1, z), \]

where, for $(m = 1, 2, 3, \ldots)$, we have

\[ F(a, b, a + b + m, z) = \frac{\Gamma(m) \Gamma(a + b + m)}{\Gamma(a + m) \Gamma(b + m)} \sum_{n=0}^{m-1} \frac{(a)_n (b)_n}{n!(1 - m)_n} (1 - z)^n - \frac{\Gamma(a + b + m)}{\Gamma(a) \Gamma(b)} (1 - z)^m \sum_{n=0}^{\infty} \frac{(a + m)_n (b + m)_n}{n!(n + m)!} \ln(1 - z) - \psi(n + 1) - \psi(n + m + 1) + \psi(a + n + m) + \psi(b + n + m)]. \]

The conditions for the existence of unstable vector modes are then given by $a + 1 = -n$ or $b + 1 = -n$. Note that these conditions also ensure the vanishing of the logarithmic terms in \[ F(a, b, a + b + m, z) \]. Since $c - a - b = 1$, we can equivalently write these conditions as $c - a = -n$ or $c - b = -n$, which have already been treated in [50]. Again, we conclude that no unstable vector modes exist.

Finally, the tensor modes have $\beta_T = -3/4$, with $c - a - b = 2$. The Dirichlet boundary condition at infinity is enforced by setting $a + 2 = -n$ or $b + 2 = -n$. Since this is equivalent to $c - a = -n$ or $c - b = -n$, we recover the constraint \[ \frac{1}{2} + \frac{1}{2} = \frac{1}{2} \]. The generalization to all odd dimensions follows suit, with the knowledge that $\beta < 0$ for all perturbations, and $c - a - b = m$, with $(m = 1, 2, 3, \ldots)$. Thus, the criteria for the existence of unstable modes can again be written in the form \[ (1 - b)^{-1/4}\Phi = z^\alpha (1 - z)^{-1/4} F(a, b, a + b + 1, z), \] and the conclusions are as in the even-dimensional case.
We can now collect these results to establish a necessary and sufficient condition for stability of the massless topological black hole. As we have seen, the massless black hole is stable against scalar and vector perturbations in all dimensions, for an arbitrary Einstein horizon. The non-trivial constraint on stability arises from the tensor modes. We can state that a necessary and sufficient condition for stability of the black hole is given by,

$$\lambda \geq -4(d-2) \Leftrightarrow \text{The Massless Black Hole is Stable.}$$

(56)

In [12], the tensor perturbations alone were analyzed, and the condition $\lambda < -4(d-2)$ was thus obtained as a sufficient condition for instability of the black hole. By obtaining the explicit solution for the scalar and vector modes, we have elevated the result of [12] to a necessary and sufficient condition for stability.

If we take the horizon manifold to have constant curvature, then it is given by hyperbolic space $H^{d-2}$, or a quotient $H^{d-2}/\Gamma$, where $\Gamma$ is a suitable discrete subgroup of the isometry group of hyperbolic space. For such manifolds, the spectrum of the Lichnerowicz operator is bounded below by $\lambda \geq -2(d-2)$ [12], which satisfies the condition of (56). We conclude that this class of massless topological black holes is stable. A second class of stable black holes is provided by taking the horizon to be a negative scalar curvature Einstein-Kähler manifold. In this case, the spectrum of the Lichnerowicz operator is bounded by $\lambda \geq -2(d-3)$ [12, 35], which again satisfies the condition (56). For general Einstein horizons, we have reduced the stability issue to a simple bound on the Lichnerowicz spectrum.

5 Stability of Black Holes with $M < 0$

Explicit solutions to the perturbation equations for black holes of non-zero mass are not available. In order to study the stability properties of such black holes, we appeal to general arguments based on positivity of the corresponding potentials in the perturbations equations. The requirement of positivity of a particular potential (scalar, vector, tensor), then provides a sufficient condition for stability of the black hole with respect to the corresponding scalar, vector, or tensor perturbation. For positive mass black holes $M > 0$ in dimensions $d > 4$, no conclusion regarding stability against scalar perturbations can be made from such positivity arguments [15]-[17]. However, as we have seen, there is a class of black holes with negative curvature horizon, for which the mass parameter $M$ can take a range of negative values, namely, $M_{\text{crit}} \leq M < 0$.

We recall that the basic Schrödinger equation is of the form (17). In order to establish stability of the black hole, we need to prove that $A$ can be extended to a positive definite self-adjoint operator. The expectation value of $A$ is given by [14]

$$\langle \Phi, A\Phi \rangle = -\left[ \Phi^* \frac{d\Phi}{dr^*} \right]_{\text{Boundary}} + \int dr^* \left( |\frac{d\Phi}{dr^*}|^2 + V |\Phi|^2 \right).$$

(57)

Thus, if boundary conditions are imposed which render the boundary term zero, and if $V \geq$
0, then we conclude that $A$ is positive definite. A more powerful approach is to use an $S$-deformation of the potential $V$ in the following way [14]. One defines a derivative operator

$$\tilde{D} = f \frac{d}{dr} + S,$$

where $S$ is some function of $r$. Then, we can write the expectation value in the form

$$(\Phi, A\Phi) = -[\Phi^* \tilde{D}\Phi]_{\text{Boundary}} + \int dr (|\tilde{D}\Phi|^2 + \tilde{V} |\Phi|^2),$$

where the deformed potential is now

$$\tilde{V} = V + f \frac{dS}{dr} - S^2.$$

The boundary term in (59) vanishes when Dirichlet boundary conditions on $\Phi$ are chosen. Thus, if Dirichlet boundary conditions are chosen, and if $\tilde{V} \geq 0$, we can conclude that $A$ is a positive definite operator. The asymptotic (large $r$) form of the negative mass black hole metric coincides with the metric of the massless black hole. Thus, we must indeed impose Dirichlet boundary conditions at infinity for all perturbations, except for certain perturbations in dimensions four, five, and six. In the following analysis, we shall adopt a choice of Dirichlet boundary conditions in these cases also.

Let us first consider the constraints that arise by demanding positivity of the potentials $V$ and $\tilde{V}$ for vector modes. From [15], we note that the potential can be written in the form

$$V_V(r) = \frac{f}{r^2} \left[ k_V^2 - 1 + \frac{(d-2)(d-4)}{4} f - (d-2)(d-1) \frac{\omega dM}{2^d - 3} \right].$$

(61)

For negative mass black holes, we clearly have a positive potential if

$$k_V^2 \geq 1.$$  

(62)

However, in this case, we can achieve a better bound by using the $S$-deformation technique, with

$$S = \frac{(d-2)f}{2r}.$$  

(63)

The deformed potential is then given by

$$\tilde{V}_V(r) = \frac{f}{r^2} [k_V^2 + (d-3)].$$

(64)

Thus, positivity of the deformed potential is guaranteed when

$$k_V^2 \geq -(d-3).$$  

(65)
Since the eigenvalues of the vector Laplacian on a generic Einstein space satisfy the condition \( k_{V}^{2} \geq 0 \), we have established that this bound is satisfied. Thus, \( \tilde{V}_{V} \geq 0 \). Hence, negative mass black holes are stable against vector perturbations for arbitrary Einstein horizons.

Moving on to the tensor modes, we note that the potential takes the form

\[
V_{T}(r) = \frac{f}{r^{2}} \left[ \lambda + [3(d - 2) - 2] + \frac{(d - 2)d}{4} f + (d - 2)(d - 1) \frac{\omega_{d}M}{2r^{d-3}} \right].
\]  

(66)

Since \( M < 0 \), the last term in the potential is negative. However, the most negative value that it can take is attained when

\[
\frac{\omega_{d}M}{r^{d-3}} = \frac{\omega_{d}M_{\text{crit}}}{r^{d-3}_{\text{crit}}} = - \frac{2}{d - 1}.
\]

(67)

Inserting this value into (66), we find that the potential is positive if the Lichnerowicz spectrum satisfies the bound

\[
\lambda \geq -2(d - 3).
\]

(68)

The \( S \)-deformed potential is this case can be obtained by taking \[14\]

\[
S = - (d - 2)f \frac{f}{2r}.
\]

(69)

This leads to a deformed potential of the form

\[
\tilde{V}_{T}(r) = \frac{f}{r^{2}} [\lambda + 2(d - 3)].
\]

(70)

Positivity of \( \tilde{V}_{T} \) then requires the same bound as in (68).

The analysis of the scalar mode is the most lengthy. In this case, we appeal only to the \( S \)-deformation of the potential, which can be achieved by choosing \[15\]

\[
S = \frac{f}{r^{\frac{d-2}{2}} - 1} H \frac{d}{dH} (r^{\frac{d-2}{2}} - 1 H).
\]

(71)

After a lengthy calculation, one finds the deformed potential to be

\[
\tilde{V}_{S}(r) = \frac{k_{S}^{2}f}{2r^{2} H} [2\mu - (d - 1)(d - 4)x],
\]

(72)

where

\[
H = k_{S}^{2} + (d - 2) + \frac{(d - 2)(d - 1)}{2} x.
\]

(73)

The key point now is that we are considering negative mass black holes, and thus \( x = \omega_{d}M/r^{d-3} < 0 \). However, once again, the most negative that \( x \) can become is given by (67), and
hence the value of $H$ is at least $k_S^2$. The eigenvalues of the scalar Laplacian on a generic Einstein manifold satisfy the condition $k_S^2 \geq 0$. Thus, for negative mass black holes, the deformed scalar potential is positive definite for generic Einstein horizons.

Collecting these results, we can present a sufficient condition for the stability of negative mass black holes with respect to all perturbations, namely

$$\lambda \geq -2(d-3) \Rightarrow \text{The } M < 0 \text{ Black Hole is Stable.} \quad (74)$$

We recall that the Lichnerowicz spectrum for negative scalar curvature Einstein-Kähler manifolds satisfies the bound $\lambda \geq -2(d-3)$. We thus conclude that all negative mass black holes with an horizon of this type are stable.

Incidentally, it is also useful to consider the constraints which arise from positivity of the potentials in the massless case. For vector modes, the deformed potential (64) again leads to the constraint (65). Thus, vector modes are stable in all dimensions. For tensor modes, positivity of the potential (66) with $M = 0$ actually leads to a stronger constraint that the deformed potential, namely $\lambda \geq -3(d-2) + 2$. For scalar modes, we take $S = (d-4)\frac{f^2}{2}\pi^2$, leading to a deformed potential $\tilde{V}_S = k_S^2 f^2$. Thus, scalar modes are stable in all dimensions. These results are consistent with the necessary and sufficient condition for stability that we derived earlier, based on the explicit solution of the equations.

6 Discussion

A complete investigation of the classical stability properties of all higher-dimensional black holes is an important problem. Until quite recently, the stability of the asymptotically flat Schwarzschild black hole in all dimensions was an open issue. However, following the analysis of tensor perturbations in [12], a gauge invariant formalism for all gravitational perturbations was established by Ishibashi and Kodama [13]. Armed with this elegant and powerful formalism, the Schwarzschild black hole was indeed shown to be stable in all dimensions [12, 14].

The purpose of the present paper has been to extend this analysis to other classes of black holes. We have focused, in particular, on topological black holes in anti-de Sitter space, for which the horizon is a negative curvature Einstein manifold. A special feature in this case is the presence of zero mass and negative mass black holes. For the zero mass black hole, we showed that the equations for scalar, vector, and tensor perturbations assumed a unified form, which was exactly solvable in terms of hypergeometric functions. We showed that the massless black hole is stable against scalar and vector perturbations in all dimensions. The only dangerous mode is therefore the tensor mode. However, using the exact solutions, we succeeded in deriving a necessary and sufficient condition for stability in all dimensions. This condition was expressed solely in terms of the spectrum of the Lichnerowicz operator on the horizon manifold. The analysis presented here thus elevated the results of [12] on tensor perturbations to a necessary and sufficient condition for stability. While the form of the Lichnerowicz spectrum on general Einstein spaces is not known, there are some examples where enough information is available to
establish stability. In particular, we proved that massless black holes with constant curvature (hyperbolic) horizons, or Einstein-Kähler horizons, are stable in all dimensions.

For negative mass black holes, an explicit solution to the perturbation equations is not available. Nevertheless, using the requirement of positivity of the gravitational potentials, along with the $S$-deformation technique, a sufficient condition for stability for all negative mass black holes was derived. As in the massless case, we showed that these black holes are stable against all scalar and vector perturbations, the only dangerous mode being the tensor mode. Again, the sufficient condition for stability is expressed in terms of the Lichnerowicz spectrum. In particular, we concluded that negative mass black holes with Einstein-Kähler horizons are stable in all dimensions.

In general, the choice of boundary conditions appropriate to the stability problem are determined by requiring normalizability of the perturbation. In dimensions $d \geq 7$, this leads to the requirement of Dirichlet boundary conditions both at the horizon and infinity. However, for certain perturbations in dimensions four, five, and six, there is a subtlety in the choice of boundary conditions at infinity. This follows from the fact that there is a one-parameter family of self-adjoint extensions of the perturbation operator in these cases [20], with a corresponding freedom in the choice of boundary conditions. In these special cases, we considered the stability issue with a choice of Dirichlet boundary condition. The stability properties with respect to the other possible choices warrants further investigation.

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