CONVERGENCE OF OPERATOR SEMIGROUPS ASSOCIATED WITH GENERALISED ELLIPTIC FORMS

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Abstract. In a recent article, Arendt and ter Elst have shown that every sectorial form is in a natural way associated with the generator of an analytic strongly continuous semigroup, even if the form fails to be closable. As an intermediate step they have introduced so-called $j$-elliptic forms, which generalises the concept of elliptic forms in the sense of Lions. We push their analysis forward in that we discuss some perturbation and convergence results for semigroups associated with $j$-elliptic forms. In particular, we study convergence with respect to the trace norm or other Schatten norms. We apply our results to Laplace operators and Dirichlet-to-Neumann-type operators.

1. Introduction

The use of sesquilinear form in semigroup theory dates back to the works of Tosio Kato and Jacques-Louis Lions. A generalisation of Kato’s and Lions’ approach has been recently proposed by Wolfgang Arendt and Tom ter Elst [3]. Their method permits to treat differential operators on rough domains, strongly degenerate equations, Dirichlet-to-Neumann operators and Stokes-type equations with ease, cf. [3, 5]. In this article we consider only what they call the complete case, which corresponds to Lions’ forms, not their incomplete case, which corresponds to Kato’s approach. These two notions are different descriptions of the same ideas. In Section 2 we introduce $j$-elliptic forms and recall some basic facts which we need. We also prove that $j$-ellipticity is preserved under small perturbations and we also present a generalisation of the Courant’s minimax formula.

The study of convergence of sequences of $C_0$-semigroups goes back to the pioneering works on semigroup theory in the 1950s. In particular, convenient convergence criteria for semigroups associated with closed forms can be found in Kato’s book [27]. In Section 3 we establish criteria for $j$-elliptic forms that imply strong convergence of the associated semigroups. Such convergence results will in turn allow us to deduce convergence in stronger norms, for example Schatten norms. Our first result in this section is a Mosco-like convergence criterion for symmetric forms (Theorem 3.1).

The Schatten classes $L_p$ have been introduced in [39] by Robert Schatten and John von Neumann. For $p = 1$, one obtains the well-studied trace class. It became clear soon after the publication of [39] that trace class operators play an important rôle in spectral theory, perturbation theory and mathematical physics. An interesting account on the history of the development of the Schatten theory can be found in the introduction of [40]. Criteria for convergence of a sequence of operators with respect to Schatten norms have been investigated for a long time, see e.g. [44] and references therein. We translate a result due to Valentin A. Zagrebnov into the framework of $j$-elliptic forms, which gives a sufficient condition for convergence in Schatten norm. We then combine this with an interpolation result for Schatten class

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operators in order to prove convergence of semigroups as Schatten class operators into spaces of higher regularity, e.g. from $L^2(\Omega)$ into $H^k(\Omega)$ for any $k \in \mathbb{N}$.

Summarizing our main results, on $L^2(X)$, $X$ a finite measure space, the following holds:

Strong convergence implies trace norm (hence uniform) convergence of a family of self-adjoint contraction semigroups, provided that their generators all dominate the generator of an ultra-contractive semigroup; and this even as operators from $L^2(X)$ into a space of more regular functions.

This is made precise in Corollary 3.8 and the subsequent remark.

In Section 4 we present several applications for our theorems and ideas. More precisely, we study Schatten norm convergence of semigroups generated by Laplacians with varying Robin boundary conditions as well as trace norm convergence of semigroups generated by Dirichlet-to-Neumann-like operators with varying coefficients. We also compare the spectra of several self-adjoint operators based on our general version of the minimax formula.

2. Generalised elliptic forms

In this section we study $j$-elliptic forms. We start with some basic facts. For a broader introduction and proofs of the fundamental theorems we refer to [3].

Definition 2.1. Let $V$ and $H$ be Hilbert spaces and $j: V \to H$ a bounded linear map with dense range. A sesquilinear form $a: V \times V \to \mathbb{C}$ is called a $j$-elliptic form on $H$ with form domain $V$ if it is continuous as a function from $V \times V$ to $\mathbb{C}$ and there exist $\omega \in \mathbb{R}$ and $\mu > 0$ such that

$$\text{Re } a(u, u) - \omega \|j(u)\|^2_H \geq \mu \|u\|_V^2$$

for all $u \in V$.

The unique, densely defined, $m$-sectorial operator $A$ on $H$ given by

$$D(A) := \{x \in H : \exists u \in V, j(u) = x, \exists f \in H \text{ s.t. } a(u, v) = (f \mid j(v))_H \forall v \in V\}$$

$$Ax := f$$

is called the operator associated with $(a, j)$. We say that $(a, j)$ is associated to $A$ and also that $(a, j)$ is associated with the analytic $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $H$.

We say that $a$ is symmetric if $a(u, v) = \overline{a(v, u)}$ for all $u, v \in V$. In this case the associated operator $A$ and the semigroup $(e^{-tA})_{t \geq 0}$ are self-adjoint.

We say that $a$ is positive if $\alpha(u, u) \geq 0$ for all $u \in V$. By the polarisation identity every positive (or, more generally, every real-valued) sesquilinear form is symmetric.

If $j$ is injective, we can regard $V$ as a subspace of $H$, regarding $j$ as the embedding of $V$ into $H$. In this case the notion of a $j$-elliptic form $a$ introduced in Definition 2.1 coincides with Lions’ definition of elliptic forms. Thus we refer to this situation as classical, i.e., we say that $(a, j)$ is a classical form if $j$ is injective.

Remark 2.2. Let $a$ be a $j$-elliptic form. Then

$$V(a) := \{u \in V : a(u, v) = 0 \forall v \in \ker j\}$$

is a closed subspace of $V$, $j|_{V(a)}$ is injective and $V = V(a) \oplus \ker j$. In particular, $j(V(a)) = j(V)$ is a dense subspace of $H$ and $j|_{V(a)}$ is injective. The classical form $(a|_{V(a) \times V(a)}, j|_{V(a)})$ is associated to the same operator as $(a, j)$. This relation allows us to carry over many results about classical forms to $j$-elliptic forms, which is the basis of this section.
Remark 2.3. Remark 2.2 suggests that $a$ is associated with an $m$-sectorial operator if merely
\[ \text{Re} a(u, u) - \omega \| j(u) \|^2_H \geq \mu \| u \|^2_V \quad \text{for all } u \in V(a). \]
This is indeed true provided that we require $V = V(a) + \ker j$ in addition, cf. [3 Cor. 2.2].

We want to extend several classical results to $j$-elliptic forms. We begin with a generation result which is a translation of a celebrated by Michel Crouzeix on cosine function generators.

Proposition 2.4. Let $a$ be a $j$-elliptic form and denote by $A$ the associated operator. Assume that there exists $M \geq 0$ such that
\[ |\text{Im} a(u, u)| \leq M \| u \|_V \| j(u) \|_H \quad \text{for all } u \in V(a). \]
Then $-A$ generates a cosine operator function and hence a semigroup with analyticity angle of $\pi/2$.

Proof. For all $x \in D(A)$ with $\|x\|_H = 1$ there exists $u \in V(a)$ such that $j(u) = x$ and
\[ |\text{Im}(Ax|x)_H|^2 = |\text{Im} a(u, u)|^2 \leq M^2 \| u \|_V^2 \| j(u) \|_H^2 \]
\[ \leq \frac{M^2}{\mu} (\text{Re} a(u, u) - \omega \| j(u) \|_H^2) \| j(u) \|_H^2 \]
\[ = \frac{M^2}{\mu} (\text{Re}(Ax|x)_H - \omega). \]
Thus, the numerical range of $A$ is contained in a parabola and therefore $-A$ generates a cosine operator function by Crouzeix’ celebrated result [17]. Finally, every generator of a cosine function family generates a holomorphic semigroup of angle $\pi/2$ [6 Thm. 3.14.17].

The following perturbation results are analogous to two classical perturbation theorems for operators [19, 20], one relying on interpolation estimates, the other one on compactness.

Proposition 2.5. Let $a : V \times V \to \mathbb{C}$ be a $j$-elliptic form and let $H'$ be a subspace of $H$ containing $j(V)$. Let $H'$ carry its own norm $\| \cdot \|_{H'}$, for which it is a Banach space and is continuously embedded into $H$. Assume that there exist $\alpha \in [0, 1)$ and $M \geq 0$ such that
\[ \| j(u) \|_{H'} \leq M \| u \|_V \| j(u) \|_{H}^{1-\alpha} \quad \text{for all } u \in V. \]
Let $b : V \times V \to \mathbb{C}$ be a continuous sesquilinear form such that
\[ \text{Re} b(u, u) \geq -c \| u \|_V \| j(u) \|_{H'} \quad \text{for all } u \in V \]
for some $c \geq 0$. Then $a + b : V \times V \to \mathbb{C}$ is $j$-elliptic.

Proof. We apply Young’s inequality $\alpha \beta \leq \varepsilon \alpha^p + c_{\varepsilon,p}(\beta^{p-1})$, which is valid for every $p \in (1, \infty)$, every $\alpha, \beta \geq 0$, and every $\varepsilon > 0$ with some constant $c_{\varepsilon,p} \geq 0$. For $\varepsilon := 2^{1+\alpha} \mu$ we obtain that
\[ \text{Re} b(u, u) \geq -c \| u \|_V \| j(u) \|_{H'} \geq -cM \| u \|_V^{1+\alpha} \| j(u) \|_{H}^{1-\alpha} \]
\[ \geq -cM \varepsilon \| u \|_V^2 - cMc_{\varepsilon,p} \| j(u) \|_H^2 \]
for all $u \in V$. For $\varepsilon := \frac{\mu}{2cM}$ we thus obtain that
\[ \text{Re} a(u, u) + \text{Re} b(u, u) - (\omega - cMc_{\varepsilon,p}) \| j(u) \|_H^2 \geq \frac{\mu}{2} \| u \|_V^2 \]
for all $u \in V$, which is the claim. \qed
Remark 2.6. In the classical case Proposition 2.5 coincides with [33] Lemma 2.1.

For the second perturbation theorem we need the following simple lemma.

Lemma 2.7. Let $V$ be a reflexive Banach space, $T: V \to H$ an injective bounded linear operator into a Banach space $H$ and $S: V \to Z$ a compact linear operator into a Banach space $Z$. Then for every $\varepsilon > 0$ there exists $c_\varepsilon \geq 0$ such that

$$\|Su\|_Z \leq \varepsilon \|u\|_V + c_\varepsilon \|Tu\|_H$$

for all $u \in V$.

Proof. Assume to the contrary that there exist $\varepsilon_0 > 0$ and a sequence $(u_n)_{n \in \mathbb{N}} \subset V$ such that

$$\|Su_n\|_Z \geq \varepsilon_0 \|u_n\|_V + n \|Tu_n\|_H$$

for all $n \in \mathbb{N}$.

We can assume that $\|Su_n\|_Z = 1$ after rescaling. Passing to a subsequence we have $u_n \to u$ in $V$, hence $Tu_n \to Tu$ in $H$. Now $\|Tu_n\|_H = \frac{1}{n}$ implies that $Tu = 0$ and thus $u = 0$. Hence by compactness $\lim_{n \to \infty} Su_n = Su = 0$ in $Z$, contradicting $\|Su_n\|_Z = 1$.

The conclusion of the following perturbation result should be compared with Remark 2.3.

Proposition 2.8. Let $a$ be a $j$-elliptic form on $V$. Let $S$ be a compact operator from $V$ into a Banach space $Z$ and let $b_0: V \times Z \to \mathbb{C}$ be a bounded sesquilinear form. Define $b(u,v) := b_0(u, Sv)$ on $V \times V$. If $j$ is injective on $V(a + b)$, where $V(a + b)$ is defined as in (2.2), then there exist $\omega' \in \mathbb{R}$ and $\mu' > 0$ such that

$$\text{Re} a(u,u) + \text{Re} b(u,u) - \omega' \|j(u)\|^2_H \geq \mu' \|u\|^2_V$$

for all $u \in V(a + b)$.

Proof. Regarding $j$ as an injective operator on $V(a + b)$, from Lemma 2.7 we obtain that

$$\|Su\|_Z \leq \varepsilon \|u\|_V + c_\varepsilon \|j(u)\|_H$$

for all $u \in V(a + b)$. Hence

$$|b(u,u)| = |b_0(u, Su)| \leq c \|u\|_V \|Su\|_Z$$

$$\leq c \|u\|^2_V + c_\varepsilon \|u\|_V \|j(u)\|_H \leq \varepsilon c \|u\|^2_V + \varepsilon c_\varepsilon \|j(u)\|^2_H + \frac{ce\varepsilon}{4\delta} \|j(u)\|^2_H$$

for $u \in V(a + b)$ by Young’s inequality. If we first pick $\varepsilon > 0$ small enough and then $\delta > 0$, we easily obtain the claimed estimate from the $j$-ellipticity of $a$.

Strictly speaking, the preceding result is not quite a perturbation result because we leave the class of $j$-elliptic forms. It is, however, quite useful in situations where one cannot expect that a lower order perturbation preserves $j$-ellipticity, see [3] §4.4.2 for such an example.

We continue our investigation of $j$-elliptic forms with results about domination and convergence. It is well-known that domination of self-adjoint operators in terms of their resolvents can be expressed via their quadratic forms. One implication of this characterisation remains true for symmetric $j$-elliptic forms. The following proposition is a direct consequence of Remark 2.2 and [27] Thm. VI.2.21.

Proposition 2.9. Let $H$ be a Hilbert space, let $a_1$ be a symmetric $j_1$-elliptic form and let $a_2$ be a symmetric $j_2$-elliptic form, where $j_1: V_1 \to H$ and $j_2: V_2 \to H$. Let $A_i$ be the self-adjoint operator on $H$ which is associated with $a_i$, $i = 1, 2$. We say that $a_1$ lies above $a_2$ (and write $(a_1, j_1) \succeq (a_2, j_2)$) if

(1) $j_1(V_1) \subset j_2(V_2)$ and
(2) $a_1(u_1, u_1) \geq a_2(u_2, u_2)$ whenever $j_1(u_1) = j_2(u_2)$.

In this case $(\gamma + A_1)^{-1} \leq (\gamma + A_2)^{-1}$ in the sense of positive definite operators for all sufficiently large $\gamma \in \mathbb{R}$. 
We also give a result concerning the domination of the spectra in the case where reference spaces $H$ differ. The following is an easy consequence of the Courant–Fischer theorem for self-adjoint operators (or, rather, their quadratic forms) and Remark 2.7.

Lemma 2.10. Let $a$ be a symmetric $j$-elliptic form on a Hilbert space $H$ with form domain $V$ and associated operator $A$. If $j$ is compact, then the self-adjoint operator $A$ has compact resolvent, and we can order the eigenvalues of $A$ in increasing order, i.e.,

$$
\lambda_1(A) \leq \lambda_2(A) \leq \lambda_3(A) \leq \cdots \leq \lambda_n(A) \to \infty,
$$

taking into account multiplicities. In this case, the eigenvalues are given by the min-max principle

$$
\lambda_k(A) = \min_{E \subset V(a)} \frac{\max_{u \in E} a(u, u)}{\|j(u)\|^2_H},
$$
i.e., $E$ runs over the $k$-dimensional subspaces of $V(a)$.

The following theorem allows the comparison of operators on different spaces that have comparable $j$-elliptic forms.

Theorem 2.11. Let $V_1$, $V_2$, $H_1$, and $H_2$ be Hilbert spaces such that $V_2$ is a closed subspace of $V_1$, which is equipped with the norm of $V_1$. Let $a_1$ be a symmetric $j_1$-elliptic form, where $j_1 : V_1 \to H_1$ is compact, and let $a_2$ be a symmetric $j_2$-elliptic form, where $j_2 : V_2 \to H_2$ is bounded. Assume that $\ker j_1 \subset V_2$ and that

$$
\|j_1(u)\|_{H_1} \geq \|j_2(u)\|_{H_2} \quad \text{and} \quad a_1(u, u) \leq a_2(u, u) \quad \text{for all } u \in V_2.
$$

Then $j_2$ is compact and $\lambda_k(A_2) \leq \lambda_k(A_2)$ for all $k \in \mathbb{N}$, where $A_1$ and $A_2$ are the operators associated with $a_1$ and $a_2$ on $H_1$ and $H_2$, respectively.

Proof. Let $(u_n)$ be a bounded sequence in $V_2$. Then $(u_n)$ is a bounded sequence in $V_1$. Passing to a subsequence we can assume that $(j_1(u_n))$ converges in $H_1$. Since

$$
\|j_2(u_n) - j_2(u_m)\|_{H_2} \leq \|j_1(u_n) - j_1(u_m)\|_{H_1}
$$

by (2.5), this implies that $(j_2(u_n))$ is a Cauchy sequence in $H_2$, hence convergent. We have proved compactness of $j_2$.

For the spectral domination it suffices to consider the following three special cases:

(i) $a_2 = a_1|_{V_2} \times V_2$ and $j_2 = j_1|_{V_2}$; or
(ii) $V_1 = V_2$ and $j_1 = j_2$; or
(iii) $V_1 = V_2$ and $a_1 = a_2$.

In fact, once we have established the result in these situations, we obtain that

$$
\lambda_k(a_1, j_1) \leq \lambda_k(a_1|_{V_2} \times V_2, j_1|_{V_2}) \leq \lambda_k(a_2, j_1|_{V_2}) \leq \lambda_k(a_2, j_2) \quad (k \in \mathbb{N}).
$$

Here we have defined $\lambda_k(a, j) := \lambda_k(A)$ with $A$ associated to $(a, j)$ to keep the notation simple. It should be noted that $a_1|_{V_2} \times V_2$ is $j_1|_{V_2}$-elliptic since $V_2 \subset V_1$ and $a_2$ is $j_1|_{V_2}$-elliptic since $a_2(u, u) \geq a_1(u, u)$ on $V_2$.

So let us prove the theorem in those three cases.

(i) Assume that $a_2 = a_1|_{V_2} \times V_2$ and $j_2 = j_1|_{V_2}$. Since $\ker j_1 \subset V_2$, this implies that $\ker j_1 = \ker j_2$. Thus trivially $V(a_2) \subset V(a_1)$, see (2.2), implying that every subspace of $V(a_2)$ is a subspace of $V(a_1)$. Hence $\lambda_k(A_1) \leq \lambda_k(A_2)$ for all $k \in \mathbb{N}$ by Lemma 2.10.

(ii) Assume that $V_1 = V_2 := V$ and $j_1 = j_2 = j$. Let $k \in \mathbb{N}$ be arbitrary and fix a subspace $E_2$ of $V(a_2)$ with $\dim E_2 = k$ such that

$$
\lambda_k(A_2) = \max_{u \in E_2, u \neq 0} \frac{a_2(u, u)}{\|j(u)\|^2_H}.
$$
Then in particular
\begin{equation}
\lambda_k(A_2) \geq \max_{u \in E_2 \setminus \{0\}} \frac{a_1(u, u)}{\|f(u)\|^2}
\end{equation}
by \[(2.5)\]. Define
\[ E_1 \coloneqq \{ u \in V(a_1) : j(u) \in j(E_2) \}. \]

Since \( j \) is bijective from \( V(a_1) \) and \( V(a_2) \) to \( j(V) \), respectively, see Remark \ref{rem:Mosco_embedding}
we have \( \dim E_1 = k \), thus
\begin{equation}
\lambda_k(A_1) \leq \max_{u \in E_1 \setminus \{0\}} \frac{a_1(u, u)}{\|f(u)\|^2}
\end{equation}
by Lemma \ref{lem:comparison_principle}. In view of \[(2.6)\] and \[(2.7)\] the theorem is proved once we show that for every \( u \in E_1 \) there exists \( \tilde{u} \in E_2 \) such that \( a_1(u, u) \leq a_1(u, \tilde{u}) \) and \( j(u) = j(\tilde{u}) \).

Thus fix \( u \in E_1 \subset V(a_1) \). By definition of \( E_1 \) there exists \( \tilde{u} \in E_2 \) such that \( j(u) = j(\tilde{u}) \). By Remark \ref{rem:Mosco_embedding} there exist \( \tilde{u}_1 \in V(a_1) \) and \( \tilde{u}_2 \in \ker j \) such that \( \tilde{u} = \tilde{u}_1 + \tilde{u}_2 \), so in particular \( j(\tilde{u}) = j(\tilde{u}_1) \). Since \( j \) is injective on \( V(a_1) \), this implies that \( u = \tilde{u}_1 \). Hence
\[
a_1(\tilde{u}, \tilde{u}) = a_1(u + \tilde{u}_2, u + \tilde{u}_2) = a_1(u, u) + 2 \text{Re} \, a_1(u, \tilde{u}_2) + a_1(\tilde{u}_2, \tilde{u}_2) \geq a_1(u, u)
\]
since \( a_1(u, \tilde{u}_2) = 0 \) by definition of \( V(a_1) \) and \( a_1(\tilde{u}_2, \tilde{u}_2) \geq 0 \) by \[(2.1)\].

(iii) Assume that \( V_1 = V_2 \) and \( a_1 = a_2 = a \). From \[(2.5)\] we obtain that \( \ker j_1 \subset \ker j_2 \), which implies \( V(a_2) \subset V(a_1) \). Now we can proceed as in the first case. \( \square \)

For semigroups on \( L^2(\Omega) \) associated with classical forms, ultra-contractivity is well-known to be equivalent to an embedding of the form domain into \( L^q(\Omega) \) for \( q > 2 \), provided that the semigroup extends to a contractive semigroup on \( L^\infty(\Omega) \).

We translate this result into the language of \( j \)-elliptic forms, which will be useful in the subsequent sections when we study Gibbs semigroups.

**Proposition 2.12.** Let \( \Omega \) be a \( \sigma \)-finite measure space. Let \( a \) be a \( j \)-elliptic form on \( H \coloneqq L^2(\Omega) \) with form domain \( V \) and associated operator \( A \). Assume that there exists \( M \geq 0 \) such that \( \|e^{-tA}f\|_{L^\infty} \leq M \|f\|_{L^\infty} \) for all \( f \in L^\infty(\Omega) \cap L^2(\Omega) \) and all \( t \in [0, 1] \). Assume moreover that \( j(V) \subset L^{\frac{2d}{d-2}}(\Omega) \) for some \( d > 2 \). Then \( (e^{\alpha t})_{t \geq 0} \) is ultra-contractive, i.e., \( e^{-tA}L^2(\Omega) \subset L^\infty(\Omega) \) and
\[
\|e^{-tA}\|_{L^2(L^2, L^\infty)} \leq ct^{-\frac{d}{4}}, \quad t \in (0, 1],
\]
for some constant \( c > 0 \).

**Proof.** By the closed graph theorem \( j \) is bounded from \( V \) to \( L^{\frac{2d}{d-2}}(\Omega) \). Thus the result follows from Remark \ref{rem:Mosco_embedding} and \cite{32} Thm. 6.4]. \( \square \)

3. CONVERGENCE RESULTS

Several results in [3] are based on a convergence result [3 Thm. 3.9]. We extend this criterion in the case of symmetric forms. It is well-known that for symmetric classical forms the convergence in the sense of Mosco, see [32], is equivalent to strong convergence of the resolvents. In fact, this holds even in the nonlinear case and is typically stated only in that situation. We show how this criterion translates to \( j \)-elliptic forms.
Theorem 3.1. Let \((a_n, j_n)_{n \in \mathbb{N}}\) and \((a, j)\) be positive forms on a Hilbert space \(H\) with form domains \((V_n)_{n \in \mathbb{N}}\) and \(V\), respectively. We assume that \(a_n\) is \(j_n\)-elliptic for all \(n \in \mathbb{N}\) and \(a\) is \(j\)-elliptic. Then the following are equivalent.

(a) The sequence of operators \((-A_n)_{n \in \mathbb{N}}\) associated with \((a_n, j_n)_{n \in \mathbb{N}}\) converges to the operator \(-A\) associated with \((a, j)\) in the strong resolvent sense.

(b) The following conditions are satisfied:

(i) If \(u_n \in V_n\), \(j_n(u_n) \to x\) for some \(x \in H\) and \(\liminf_{n \to \infty} a_n(u_n, u_n) < \infty\), then there exists \(u \in V\) such that \(j(u) = x\) and \(\liminf_{n \to \infty} a_n(u_n, u_n) \geq a(u, u)\);

(ii) For all \(u \in V\) there exists a sequence \((u_n)_{n \in \mathbb{N}}\) with \(u_n \in V_n\) such that

\[
\lim_{n \to \infty} j_n(u_n) = j(u) \quad \text{and} \quad \liminf_{n \to \infty} a_n(u_n, u_n) \leq a(u, u).
\]

If these equivalent conditions are satisfied, we say that \((a_n, j_n)_{n \in \mathbb{N}}\) converges to \((a, j)\) in the sense of Mosco.

Proof. Define \(\phi_n(u) := a_n(u, u)\) for \(u \in V_n(a_n)\), and \(\phi_n(x) := \infty\) for \(x \in H \setminus j_n(V_n)\). Then \(\phi_n : H \to (-\infty, \infty]\) is well-defined, convex and lower semicontinuous, and \(-A_n\) is the subdifferential of \(\phi_n\). This follows from [3] Thm. 2.5] and the well-known correspondence between the linear and the non-linear theory of forms. Moreover,

\[
\phi_n(x) = \min \{a(u, u) : u \in V_n, j_n(u) = x\}
\]

by Remark 2.2. A similar statement holds for the functional \(\phi\), which we define analogously for \((a, j)\).

The two conditions in (b) are equivalent to

(I) \(x_n \to x\) implies that \(\liminf_{n \to \infty} \phi_n(x_n) \geq \phi(x)\);

(II) for all \(x \in H\) there exists \((x_n) \subset H\) such that

\[
\lim_{n \to \infty} x_n = x \quad \text{and} \quad \lim_{n \to \infty} \phi_n(x_n) = \phi(x).
\]

In fact, assume (i) and (ii). If \(\liminf_{n \to \infty} \phi_n(x_n) = \infty\) in (I), then there is nothing to show. Otherwise, (I) follows from (i) and (3.1). In (II), if \(x \not\in j(V)\), i.e., \(\phi(x) = \infty\), then by (I) any sequence \((x_n)\) in \(H\) such that \(\lim_{n \to \infty} x_n = x\) does the job. On the other hand, if \(x = j(u)\) for some \(u \in V(a)\), then (II) follows from (ii) and (I). On the contrary, if (I) and (II) are satisfied, then (i) and (ii) follow easily using (3.1).

We have shown that condition (b) is equivalent to Mosco-convergence of \(\phi_n\) to \(\phi\), which by [\#] Prop. 3.19 and Thm. 3.26\] is equivalent to strong resolvent convergence of the subdifferentials, i.e., to (a). \(\square\)

Remark 3.2. The implication from (b) to (a) in Theorem 3.1 remains valid for symmetric, but not necessarily positive forms provided that there exists \(\omega \leq 0\) such that \(a_n(u, u) - \omega \|j_n(u)\|^2_H \geq 0\) for all \(u \in V_n\) and \(a(u, u) - \omega \|j(u)\|^2_H \geq 0\) for all \(u \in V\). In fact, assume that the conditions in (b) are fulfilled. Lower semicontinuity of the norm in \(H\) yields that then also the positive forms \(\tilde{a}_n\) and \(\tilde{a}\) given by \(\tilde{a}_n(u, v) := a_n(u, v) - \omega \|j_n(u)\|_H^2\) and \(\tilde{a}(u, v) := a(u, v) - \omega \|j(u)\|_H^2\) satisfy the conditions in (b). Now the theorem implies that the associated operators \((-A_n - \omega)\) converge to \((-A - \omega)\) in the strong resolvent sense, which trivially implies (a).

Let \(H_1\) and \(H_2\) be separable Hilbert spaces. For \(p \in [1, \infty)\) the \(p\)-Schatten class is defined by

\[
L_p(H_1, H_2) := \{T \in \mathcal{K}(H_1, H_2) : \|T\|_{L_p} := \|(s_n)_{n \in \mathbb{N}}\|_{\ell^p} < \infty\},
\]

where \((s_n)_{n \in \mathbb{N}}\) is the sequence of \emph{singular values} of \(T\), i.e., the sequence of eigenvalues of \(|T| := (T^*T)^{1/2}\). Then \(\|\cdot\|_{L_p}\) is a complete norm on \(L_p(H_1, H_2)\), called the...
p-Schatten norm. The operators in \( \mathcal{L}_1(H_1, H_2) \) are also called trace class operators with the trace norm, and the operators in \( \mathcal{L}_2(H_1, H_2) \) are called Hilbert–Schmidt operators. If \( H_1 = H_2 = H \) we frequently write \( \mathcal{L}_p(H) \) instead of \( \mathcal{L}_p(H, H) \). For more information about the Schatten classes we refer to [23] [40].

We are mainly interested in semigroups consisting of Schatten class operators. The following definition goes back to Dietrich A. Uhlenbrock [43] and first appeared in applications in statistical mechanics. Nowadays, Gibbs semigroups are popular objects in mathematical physics.

**Definition 3.3.** Let \( H \) be a Hilbert space. A Gibbs semigroup is a \( C_0 \)-semigroup 
\( (T(t))_{t \geq 0} \) on \( H \) such that each operator \( T(t) \), \( t > 0 \), is of trace class.

**Remarks 3.4.** (1) Since \( \mathcal{L}_p(H) \cdot \mathcal{L}_q(H) \subset \mathcal{L}_r(H) \subset \mathcal{L}_{p'}(H) \) for \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \) and \( r < r' \), every semigroup \( (T(t))_{t \geq 0} \) for which there exists \( p \in [1, \infty) \) such that \( T(t) \in \mathcal{L}_p(H) \) for all \( t > 0 \) is a Gibbs semigroup.

(2) Let \( X \) be a finite measure space. It is known that each bounded linear operator \( T \) from \( L^2(X) \) to \( L^\infty(X) \) is a Hilbert–Schmidt operator [2] Thm. 1.6.2. In particular every ultra-contractive semigroup on \( L^2(X) \) is a Gibbs semigroup. Hence Proposition 2.12 provides a sufficient condition for the Gibbs property, which is sometimes easy to check.

(3) It seems to be difficult to characterise the Gibbs property in terms of the resolvent. If \( -A \) generates an analytic semigroup \( (T(t))_{t \geq 0} \) on \( H \) and \( (\lambda + A)^{-k} \in \mathcal{L}_p(H) \) for some \( k \in \mathbb{N} \), some \( \lambda \) in the resolvent set and some \( p \in [1, \infty) \), then \( (T(t))_{t \geq 0} \) is a Gibbs semigroup. In fact, since in that case the embedding \( D(A^k) \hookrightarrow H \) is of Schatten class and \( T(t) : H \to D(A^k) \) is bounded for \( t > 0 \), the ideal property implies that \( T(t) \in \mathcal{L}_p(H) \) for all \( t > 0 \).

But the converse fails. In fact, consider the diagonal operator \( A = D_A \) on \( \ell^2 \) and \( (T(t))_{t \geq 0} = (e^{-tA})_{t \geq 0} \), where \( \lambda_n := \log^2 n \). Then the eigenvalues \( e^{-t \log n} = n^{-t} \) of \( T(t) \) are summable for every \( t > 0 \), i.e., \( (T(t))_{t \geq 0} \) is a Gibbs semigroup, but the eigenvalues \( (\lambda + \log^2 n)^{-k} \) of \( (\lambda + A)^{-k} \) are not \( p \)-summable for any \( k \in \mathbb{N} \), \( p \in [1, \infty) \) and \( \lambda \) in the resolvent set.

(4) The square root of the above operator \( D_A \) yields also another interesting counterexample. It is known that for an analytic semigroup immediate compactness and eventual compactness are equivalent. However, the square root of \( D_A \) generates a semigroup whose eigenvalues \( e^{-t \log n} = n^{-t} \) are \( p \)-summable if and only if \( t > 1/p \). In particular, this self-adjoint semigroup is eventually Gibbs, but not immediately Gibbs.

(5) It is known that for a bounded domain \( \Omega \subset \mathbb{R}^d \) with the cone property the embedding of \( H^s(\Omega) \) into \( L^2(\Omega) \) is a Hilbert–Schmidt operator whenever \( 2k > d \), see [30], and in fact a \( p \)-Schatten class operator if \( pk > d \), see [23]. Under certain assumptions on the geometry, Maurin’s and Gramsch’s result have been extended to unbounded domains [15] [28]. In such situations, if \( A \) generates an analytic semigroup and \( D(A) \subset H^1(\Omega) \), then \( A \) generates a Gibbs semigroup. Observe that by [1] Thm. 6.54 and Rem. 6.55 there exist domains with infinite measure such that the embedding of \( H^1(\Omega) \) into \( L^2(\Omega) \) is in \( \mathcal{L}_p(H^1(\Omega), L^2(\Omega)) \) for some \( p \in [1, \infty) \). In this situation criterion (2) does not apply.

(6) The preceding criterion can also be useful for semigroups on Sobolev spaces \( H^s \) with index \( s \neq 0 \). For example, it allows us to prove that the semigroup generated by the Wentzell–Robin Laplacian on a smooth domain (see Section 4.7 for details) on \( H^s(\Omega) \) considered in [4] §2.9 and [21] is Gibbs. To be more precise, recall that the domain of the Wentzell–Robin-Laplacian is a subspace of \( H^{2,1}(\Omega) \). Thus, the semigroup generated by its part in \( V := \{ (u, u_{\partial\Omega}) : u \in H^1(\Omega) \} \) maps \( V \) to \( \{ (u, u_{\partial\Omega}) : u \in H^{2,1}(\Omega) \} \) for all \( t > 0 \).
By [25] Satz 1 the embedding $H^2(\Omega) \hookrightarrow H^1(\Omega)$ is a $p$-Schatten class operator for all $p > 2d$, hence so is any operator of the semigroup for $t > 0$, and by part (1) this semigroup is Gibbs. The same argument applies to general (also non-selfadjoint) elliptic operators with Wentzell-Robin or similar boundary conditions. On the other hand, part (2) does not yield the result in this case since the semigroup is not defined on an $L^2$-space.

We apply known result about convergence in Schatten norms, cf. [40, Chapter 2], to semigroups arising from $j$-elliptic forms. The following proposition is a direct consequence of Proposition 2.9 together with [44, Lemma, p.271]. Its conditions are often easy to check; we will give some examples later on.

**Theorem 3.5.** Let $H$ be a Hilbert space and let $(a_n, j_n)$, $(a, j)$ and $(b, j)$ be symmetric, sesquilinear forms that satisfy the conditions in Definition 2.1. We denote by $A_n$, $A$ and $B$ the associated self-adjoint operators. Assume that

(i) $(b, j) \leq (a_n, j_n)$ for all $n \in \mathbb{N}$ in the sense of Proposition 2.2,

(ii) $-B$ generates a Gibbs semigroup, and

(iii) $(A_n)$ converges to $A$ in the strong resolvent sense.

Then

$$\lim_{n \to \infty} e^{-tA_n} = e^{-tA} \quad \text{in } L^1(H)$$

for every $t > 0$.

**Remark 3.6.** Theorem 3.5 tells us that the existence of a dominating form implies trace norm convergence of the semigroup. This is remarkable because, even though form domination implies domination for the resolvents, it does in general not imply domination for the semigroups. In fact, for $A := \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and $B := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$ we have $0 \leq A \leq B$, but $e^{-B} \nleq e^{-A}$ in the sense of positive definiteness. The authors are grateful to Ulrich Groh (Tübingen) for pointing out this example.

Finally, we also consider convergence of semigroups in Schatten norms as operators between different Hilbert spaces. We obtain our main result as a consequence of an interpolation theorem for Schatten class operators. A criterion which enables us to check the trace norm convergence required in the following theorem was given in Theorem 3.5.

**Theorem 3.7.** Let $p \in [1, \infty)$. Let $(A_n), A$ be uniformly $m$-sectorial operators on $H$, i.e., $m$-sectorial operators with uniform constants, which generate Gibbs semigroups. Assume that there exists a subspace $\tilde{H}$ of $H$ such that

- $\tilde{H}$ is a Hilbert space,
- $\tilde{H}$ is compactly embedded in $H$, and
- there exists some $k \in \mathbb{N}$ such that $D(A_n^k) \subset \tilde{H}$ for all $n \in \mathbb{N}$ with uniform embedding constants.

If

$$\lim_{n \to \infty} e^{-tA_n} = e^{-tA} \quad \text{in } L^p(H)$$

for every $t > 0$, then

$$\lim_{n \to \infty} e^{-tA_n} = e^{-tA} \quad \text{in } L^q(H, H_\theta)$$

for every $t > 0$ and every $\theta \in (0, 1)$, where $q \in [1, \infty)$ is given by

$$\frac{1}{q} = \frac{\theta}{p} + (1 - \theta)$$

and where $H_\theta$ denotes the complex interpolation space $[\tilde{H}, H]_\theta$. 
Proof. We first show that $D(A^k)$ is continuously embedded into $\tilde{H}$. Fix $\lambda > 0$ so large that $\lambda + A_n$ is invertible with uniformly bounded inverse with respect to $n \in \mathbb{N}$. Take $u \in \tilde{H}$. Then the uniform constants in the $m$-sectoriality and the embeddings ensure that the sequence $((\lambda + A_n)^{-k}u)_{n \in \mathbb{N}}$ is bounded in $\tilde{H}$. Hence there exists a weakly convergent subsequence in $\tilde{H}$, which necessarily converges to $(\lambda + A)^{-k}u$ since the semigroups and hence the resolvents converge strongly by assumption. This proves that $D(A^k) \subset \tilde{H}$. Now the closed graph theorem yields $D(A^k) \hookrightarrow \tilde{H}$.

For every $t > 0$ and every $n \in \mathbb{N}$ the operator $e^{-tA_n} = e^{-\frac{t}{2}A_n}e^{-\frac{t}{2}A_n}$ is a composition of an operator in $L_1(\tilde{H})$ and an operator in $L(H, \tilde{H})$, both with uniformly estimable norms, compare (1) in Remarks 3.4. Hence $\sup_{n \in \mathbb{N}} \|e^{-tA_n}\|_{L_1(H, \tilde{H})} < \infty$ by the ideal property of the norm, and by a similar argument $e^{-tA}$ is in $L_1(\tilde{H}, \tilde{H})$ as well.

Now we obtain from an interpolation result for Schatten class operators [22] that
$$\|e^{-tA_n} - e^{-tA}\|_{L_q(H, H_\theta)} \leq C\|e^{-tA_n} - e^{-tA}\|_{L_1(H, \tilde{H})} \|e^{-tA_n} - e^{-tA}\|_{L_1(H, H_\theta)},$$
for some constant $C \geq 1$ since the fractional domain space considered in [22] coincides with $H_\theta$ up to equivalent norms. The first factor is bounded by the above considerations whereas the second factor converges to zero by assumption.

Let us combine several of our observations into a final result.

**Corollary 3.8.** Let $X$ be a finite measure space. Let $(a_n, j_n)$, $(a, j)$ and $(b, j')$ be positive elliptic forms on $L^2(X)$ in the sense of Definition 2.7 with form domain $\mathcal{V}$, and denote the associated self-adjoint operators by $A_n$, $A$ and $B$, respectively. Let $\tilde{H}$ be a dense subspace of $H$, which is a Hilbert space in its own right. Assume that

- $(a_n, j_n)$ converges to $(a, j)$ in the sense of Mosco;
- $(b, j') \lesssim (a_n, j_n)$ for all $n \in \mathbb{N}$;
- there exists $q > 2$ such that $j'(V) \subset L^q(X)$;
- for all $u \in V$ there exists $w \in V$ such that $|j(u) \wedge 1| \leq j(w)$ and $\text{Re}b(w, u - w) \geq 0$;
- $\tilde{H}$ is compactly embedded into $H$;
- $D(A_n^k) \subset H$ for some $k \in \mathbb{N}$ with an embedding constant that is uniform in $n \in \mathbb{N}$;

For arbitrary $\theta \in (0, 1)$ let $H_\theta$ denote the complex interpolation space $H_\theta = [\tilde{H}, H]^\theta$. Then $e^{-tA_n} \rightarrow e^{-tA}$ in the trace norm $L_1(H, H_\theta)$ and hence in particular in the operator norm $L(H, H_\theta)$.

**Proof.** By the invariance criterion for $j$-elliptic forms [3 Prop 2.9] the semigroup is $L^\infty(X)$-contractive, analogously to the situation in [36 Thm. 2.13]. Hence by Proposition 2.12 it is ultra-contractive and thus Gibbs by Remark 3.4. Since in addition $-A_n$ converges to $-A$ in the strong resolvent sense by Theorem 3.1 we obtain from Theorem 3.5 that $e^{-tA_n} \rightarrow e^{-tA}$ in $L_1(H)$. Now the assertion follows from Theorem 3.7.

**Remark 3.9.** The assumption of Corollary 3.8 that $D(A_n^k) \subset \tilde{H}$ with uniform embeddings is in particular satisfied for $\tilde{H} = V$ if the constants in the ellipticity estimate (2.1) of $(a_n, j_n)$ are uniform in $n$, for the semigroups $(e^{-tA_n})$ are bounded as operators from $\tilde{H}$ to $V$, uniformly in $n$.

We emphasise that in this special case Corollary 3.8 yields a convergence result for semigroups under assumptions solely on the associated forms, with no reference to the associated operators.
Remarks 3.10. Let us finally remark on the Gibbs property for other kinds of operator families.

1. Let $-A$ be a self-adjoint operator, hence the generator of a sine operator function $(S(t))_{t \in \mathbb{R}}$, cf. [11] § 3.15. It is known that $S(t)$ maps $H$ into $V$ for all $t \in \mathbb{R}$, where $V$ is the domain of the form associated with $A$. If the embedding of $V$ into $H$ is of $p$-Schatten class (e.g., $V$ a closed subspace of $H^1(0,1)$, $H = L^2(0,1)$ and $p > 1$, cf. Remark 3.4(5)), then $S(t)$ is of $p$-Schatten class for all $t \in \mathbb{R}$.

2. Unlike in the semigroup case, however, there exist sine operator functions

\[ S(t)x := \begin{cases} \sinh(\sqrt{\lambda_n}t) x_n & n \in \mathbb{N} \\
\sin((\frac{n}{2} + 2\pi[n^\alpha])t) x_n & n \in \mathbb{N}, \end{cases} \quad t \in \mathbb{R}, \ x \in \ell^2, \]

so $S(t) \in \mathcal{L}_p(\ell^2)$ for all $p > \alpha^{-1}$ and all $t \in \mathbb{R}$, but $S(1)$ is not in $\mathcal{L}_{\alpha^{-1}}(\ell^2)$.

3. On an infinite dimensional Hilbert space a cosine operator function can only be compact on an interval. In fact, a cosine operator function is given by

\[ \lambda_n := -\left(\frac{\pi}{2} + 2\pi[n^\alpha]\right)^2, \quad n \in \mathbb{N} \]

and $\lfloor x \rfloor$ denotes the greatest integer below $x$. Then the corresponding sine operator function is given by

\[ S(t)x := \begin{cases} \frac{\sinh(\sqrt{\lambda_n}t)}{\sqrt{\lambda_n}} x_n & n \in \mathbb{N} \\
\frac{\sin((\frac{n}{2} + 2\pi[n^\alpha])t)}{\frac{n}{2} + 2\pi[n^\alpha]} x_n & n \in \mathbb{N}, \end{cases} \quad t \in \mathbb{R}, \ x \in \ell^2, \]

which shows how our convergence results can be combined to treat semigroups.

4. Applications

4.1. Convergence of Laplacians with respect to higher regularity Schatten norms. We begin with an application of our result about Schatten convergence, which shows how our convergence results can be combined to treat semigroups generated by elliptic operators: starting with convergence in the strong resolvent sense we are able to obtain trace norm convergence with respect to Sobolev spaces of arbitrarily high order.

**Theorem 4.1.** Let $\Omega$ be a bounded open domain in $\mathbb{R}^d$ with $C^\infty$-boundary. Consider a sequence of Laplacians $\Delta_{k_n}$ with Robin boundary conditions

\[ \frac{\partial u}{\partial \nu} + k_n u = 0 \quad \text{on} \ \partial \Omega, \]

for constants $(k_n)_{n \in \mathbb{N}} \subset [0, \infty)$. If $(k_n)$ is a monotonically decreasing null sequence, then

\[ \lim_{n \to \infty} e^{t\Delta_{k_n}} = e^{t\Delta_N} \quad \text{in} \ \mathcal{L}_1(L^2(\Omega), H^1(\Omega)) \]

for every $t > 0$ and every $\ell \in \mathbb{N}$, where $\Delta_N$ denotes the Laplace operator on $\Omega$ with Neumann boundary conditions.

**Proof.** By [27] Thm. 8.3.11] the sequence $(\Delta_{k_n})$ converges to $\Delta_N$ in the strong resolvent sense. Let $(a_n)$ and $a_N$ be the elliptic classical forms associated with $-\Delta_{k_n}$ and $-\Delta_N$, respectively. Then $a_n \geq a_N$ in the sense of Proposition 2.9. Moreover, $\Delta_N$ generates a Gibbs semigroup, see (2) in Remarks 3.4 and use Corollary 2.17 and Theorem 6.4. Hence $e^{-t\Delta_{k_n}} \to e^{-t\Delta_N}$ in $\mathcal{L}_1(L^2(\Omega))$ by Theorem 3.5. Moreover, following the proofs of elliptic regularity, cf. [26, §2.5.1], one can see that $D(\Delta_{k_n})$ is uniformly embedded into $H^{2\ell}(\Omega)$ for every $\ell \in \mathbb{N}$. Applying Theorem 3.7 with $\theta = \frac{1}{2}$ we conclude that $e^{t\Delta_{k_n}} \to e^{t\Delta_N}$ in $\mathcal{L}_1(L^2(\Omega), H^\ell(\Omega))$ for every $t > 0$ and every $\ell \in \mathbb{N}$. □
Remark 4.2. Analogous arguments work for heat equations with the dynamic boundary conditions
\[ \frac{\partial u}{\partial t} = -\Delta u - k_n u \] on \( \partial \Omega, \)
which arise from forms as seen in [7]. This complements the results of [16], where the emphasis lies in obtaining sharp estimates for the rate of convergence with respect to the \( H^1 \)-operator norm.

4.2. Convergence of Laplacians with variable boundary conditions on exterior domains. The result in this section is somewhat special, since we prove Schatten norm convergence of diffusion semigroups \((T_n(t))_{t \geq 0}\) to a semigroup \((T(t))_{t \geq 0}\), all acting on spaces of functions on exterior domains with varying boundary conditions. As we will see, in this situation it is sometimes possible to obtain that \(T_n(t) - T(t) \to 0\) in \( L_p\) (for sufficiently large values of \( p \)) as \( n \to \infty \) even though the operators \( T_n(t)\) and \( T(t)\) are not individually in \( L_p\) and in fact not even compact. In particular, Theorem 3.5 does not apply here. Instead, our argument relies upon classical results on differences of differential operators first due to Mikhail Š. Birman [12] Thm. 3.8 and recently improved in [10]; only Theorem 3.1 is additionally needed. Such situations indeed appear frequently in mathematical physics, see for example [14, 15, 31].

Theorem 4.3. Let \( \Omega \subset \mathbb{R}^d, d \geq 3, \) be an exterior domain with smooth boundary and \( \Delta_\beta \) the Laplace operator on \( \Omega \) with Robin boundary condition
\[ \frac{\partial u}{\partial \nu} + \beta u = 0 \quad \text{on} \quad \partial \Omega. \]
If \((\beta_n)_{n \in \mathbb{N}}\) is a bounded sequence in \( L^\infty(\partial \Omega) \) and converges to a function \( \beta_0 \) almost everywhere, then
\[ \lim_{n \to \infty} (e^{t \Delta_\beta_n} - e^{t \Delta_\infty}) = 0 \quad \text{in} \quad L_p(L^2(\Omega)) \]
for every \( t > 0 \) and all \( p > \frac{d+1}{4}. \)

Proof. Let us first show that the operators are uniformly m-sectorial. Since the trace operator \( u \mapsto u|_{\partial \Omega} \) is compact from \( H^1(\Omega) \) to \( L^2(\partial \Omega) \), by Lemma 2.7 there exists \( c > 0 \) such that
\[ ||u||^2_{L^2(\partial \Omega)} \leq \frac{1}{2} ||\nabla u||^2_{L^2(\Omega)} + c||u||^2_{L^2(\Omega)} \]
for all \( u \in H^1(\Omega). \) This shows that the quadratic form \( q_\beta \) associated with \(-\Delta_\beta\), i.e.
\[ q_\beta(u) := \int_{\Omega} |\nabla u|^2 + \int_{\partial \Omega} \beta_n |u|^2 \]
for \( u \in H^1(\Omega), \) is semi-bounded for every essentially bounded function \( \beta \) and hence that \( \Delta_\beta \) generates a \( C_0 \)-semigroup on \( L^2(\Omega). \) More precisely,
\[ q_{\beta_n}(u) \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 - cm||u||^2_{L^2(\Omega)} \]
with \( m := \sup_{n \in \mathbb{N}} ||\beta_n||_{\infty}. \) This proves uniform \( m \)-sectoriality.

Next we show convergence in the strong resolvent sense. We write \( \tilde{q}_\beta(u) := q_\beta(u) + (cm + 1)||u||^2_{L^2(\Omega)} \) for simplicity and show that \( \tilde{q}_{\beta_n} \) converges to \( \tilde{q}_{\beta_0} + cm + 1 \) in the sense of Mosco. To this end, let \((u_n)\) be a sequence in \( H^1(\Omega) \) such that \( u_n \rightharpoonup u \) in \( L^2(\Omega) \) and \( \liminf_{n \to \infty} \tilde{q}_{\beta_n}(u_n) < \infty. \) Then \((u_n)\) is bounded in \( H^1(\Omega), \) hence passing to a subsequence we can assume that \( u_n \to u \) in \( H^1(\Omega). \) Thus
in particular \( u \in H^1(\Omega) \). Now by compactness \( u_n|_{\partial\Omega} \to u|_{\partial\Omega} \) in \( L^2(\partial\Omega) \), so in particular
\[
\int_{\partial\Omega} \beta_n|u_n|^2 \to \int_{\partial\Omega} \beta|u|^2.
\]
By weak lower semicontinuity of the norm of \( H^1(\Omega) \) this proves
\[
\tilde{q}_{\beta_0}(u) \leq \liminf_{n \to \infty} \tilde{q}_{\beta_0}(u_n).
\]
Moreover, for given \( u \in H^1(\Omega) \) we clearly have \( \tilde{q}_{\beta_0}(u) \to \tilde{q}_{\beta_0}(u) \) by Lebesgue’s theorem. We thus have shown that \( \Delta_{\beta_n} \to \Delta_{\beta_0} \) in the strong resolvent sense, see Theorem 3.1.

We now prove the convergence in Schatten norm. Since \(-\Delta - m \leq -\Delta_{\beta_n} \leq -\Delta_m \) in the form sense we have
\[
(cm + 1 - \Delta_{\beta_n})^{-1} \geq (cm + 1 - \Delta_{\beta_0})^{-1} \geq (cm + 1 - \Delta_m)^{-1}
\]
as self-adjoint operators [27, Thm. 2.21]. A similar assertion holds for \( \Delta_{\beta_0} \). Consequently,
\[
|(cm + 1 - \Delta_{\beta_n})^{-1} - (cm + 1 - \Delta_{\beta_0})^{-1}| \leq (cm + 1 - \Delta_m)^{-1} - (cm + 1 - \Delta_m)^{-1},
\]
where the right hand side is in \( L_p(L^2(\Omega)) \) for every \( p > \frac{d+1}{d-1} \) by [10] Cor. 3.6. This implies that \( (cm + 1 - \Delta_{\beta_n})^{-1} \) converges to \( (cm + 1 - \Delta_{\beta_0})^{-1} \) in \( L_p(L^2(\Omega)) \) as \( n \to \infty \), see [33] Prop. 2.1.

Moreover, as in the proof of [10] Thm. 3.5], for all \( \lambda \) and \( \mu \) in the sector \( \Sigma := \mathbb{C} \setminus \mathbb{R}_{\leq cm} \) we have
\[
(\mu - \Delta_{\beta_n})^{-1} - (\mu - \Delta_{\beta_0})^{-1} = (I + (\lambda - \mu)(\mu - \Delta_{\beta_0})^{-1})((\lambda - \Delta_{\beta_n})^{-1} - (\lambda - \Delta_{\beta_0})^{-1})(I + (\lambda - \mu)(\mu - \Delta_{\beta_0})^{-1}).
\]
In fact, this identity is certainly satisfied on \( V = H^1(\Omega) \) since \( \kappa - \Delta_{\beta_n} \) and \( \kappa - \Delta_{\beta_0} \) are isomorphisms from \( V \) to the dual space \( V' \). Thus the identity extends to \( L^2(\Omega) \) by denseness.

Picking \( \lambda = cm + 1 \) we obtain from the ideal property that
\[
(\mu - \Delta_{\beta_n})^{-1} - (\mu - \Delta_{\beta_0})^{-1} \in L_p(L^2(\Omega))
\]
for all \( \mu \in \Sigma \). More precisely we even obtain that on every sector smaller than \( \Sigma \) this sequence of differences is bounded and convergent in \( L_p(L^2(\Omega)) \) on compact subsets of \( \Sigma \), uniformly with respect to \( n \). Here we have used that the operators \( \Delta_{\beta_n} \) are uniformly \( m \)-sectorial. Hence the integral representation [6] (3.46)
\[
e^{t\Delta_{\beta_n}} - e^{t\Delta_{\beta_0}} = \frac{1}{2\pi i} \int_{\gamma} e^{-t\lambda} ((\lambda - \Delta_{\beta_n})^{-1} - (\lambda - \Delta_{\beta_0})^{-1}) d\lambda,
\]
shows that \( e^{t\Delta_{\beta_n}} \) converges to \( e^{t\Delta_{\beta_0}} \) in \( L_p(L^2(\Omega)) \) for every \( t > 0 \).

4.3. Coupled boundary conditions. In this subsection we consider convergence for systems of Laplacians with a certain coupled boundary conditions, which are motivated by quantum graphs. It seems that [27], Thm. VI.3.6] cannot be used to obtain strong convergence of the resolvents in this example, so we employ an approach developed by Olaf Post [37] instead, where we use the notation of [33].

**Theorem 4.4.** Let \( (Y_n)_{n \in \mathbb{N}} \) be a sequence of closed subspaces of \( \mathbb{C}^k \), \( k \in \mathbb{N} \). Let \( \Omega \subset \mathbb{R}^d \) be an exterior domain with smooth compact boundary and \( (\Delta_{Y_n})_{n \in \mathbb{N}} \) be a sequence of Laplacians on \( L^2(\Omega; \mathbb{C}^k) \) with boundary conditions
\[
u|_{\partial\Omega} \in Y_n \quad \text{and} \quad \frac{\partial u}{\partial \nu} \in Y_n^\perp \quad \text{a.e., } n \in \mathbb{N}.
\]
Assume that there exist a subspace $Y$ of $\mathbb{C}^k$ and a family $(J_j^n)_{n\in \mathbb{N}}$ of unitary operators on $H$ converging to the identity $I$ such that $J_j^n Y_n = Y$ for all $n \in \mathbb{N}$. Denote by $\Delta_Y$ the Laplacian with corresponding boundary conditions. Then
\[ \lim_{n \to \infty} e^{t\Delta_Y} n = e^{t\Delta_Y} \quad \text{in } L^p(L^2(\Omega; \mathbb{C}^k)) \]
for every $t > 0$ and all $p > \frac{d-1}{2}$.

Proof. We introduce elliptic forms $(a_n)_{n\in \mathbb{N}}$ and $a_0$ with form domains
\[ V_n := \{ f \in H^1(\Omega; \mathbb{C}^k) : f|_{\partial\Omega} \in Y_n \} \quad n \in \mathbb{N}, \]
\[ V := \{ f \in H^1(\Omega; \mathbb{C}^k) : f|_{\partial\Omega} \in Y \} \]
as in [13 §2]. These forms are symmetric, and accordingly the associated Laplacians $\Delta_{Y_n}$ and $\Delta_Y$ are self-adjoint operators on $H := L^2(\Omega; \mathbb{C}^k)$.

We set
\[ J^{j_n} := J^n_j \circ f \quad \text{and} \quad J^{j_n} f := J^n_j \circ f, \quad n \in \mathbb{N}. \]
Since these operators are unitary on $H = L^2(\Omega; \mathbb{C}^k)$ as well as from $V_n$ to $V$, it is easy to see that the assumptions in [34 Def. 2.3] are satisfied, and we deduce from [34 Prop. 3.4] that $\Delta_{Y_n}$ converges to $\Delta_Y$ in the norm resolvent sense.

Moreover, $-\Delta_{\mathbb{C}^k} \leq -\Delta_{Y_n} \leq -\Delta_{(0)}$ in the form sense and hence
\[ |(\lambda - \Delta_{Y_n})^{-1} - (\lambda - \Delta_{Y_0})^{-1}| \leq (\lambda - \Delta_{(0)})^{-1} - (\lambda - \Delta_{\mathbb{C}^k})^{-1} \]
by [27 Thm. 2.21]. Using that $\Delta_{(0)}$ and $\Delta_{\mathbb{C}^k}$ act as uncoupled copies of $k$ Dirichlet and Neumann Laplace operators, respectively, we obtain from Birman’s result [12 Thm. 3.8] that the operator on the right hand side is in $L^p(L^2(\Omega))$ for every $p > \frac{d-1}{2}$. The conclusion now follows as in Theorem [13].

Remark 4.5. We have formulated Theorems [4.3] and [4.4] in the case of Laplacians only for the sake of simplicity: in fact, both results can be extended to strongly elliptic operators with coefficients in $W^{1,\infty}(\Omega)$. Also, rougher and even non-compact boundaries can be allowed, leading to convergence only for larger $p$, cf. [12 Rem. 3.4, Thm. 3.8 and Thm. 5.2].

4.4. Dirichlet-to-Neumann-type operators. By showing that the Dirichlet-to-Neumann operator is associated with a $j$-elliptic form, Arendt and ter Elst have delivered a most interesting application of their theory. This is an instance where a non-injective $j$ appears in a natural way.

Consider an open bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, where $d \geq 2$, and let $V := H^1(\Omega)$ and $H := L^2(\partial\Omega)$. We consider the sesquilinear form $a$ defined by
\[ a(u, v) := \int_\Omega a\nabla u \cdot \nabla v, \quad u, v \in V, \]
where the matrix-valued coefficient $a \in L^\infty(\Omega; \mathbb{C}^{d \times d})$ is uniformly positive definite, i.e., for a.e. $x \in \Omega$ the matrix $a(x)$ is Hermitian and satisfies
\[ (a(x)\xi|\xi) \geq k_0|\xi|^2 \quad \text{for all } \xi \in \mathbb{C}^d \]
for some $k_0 > 0$. Let $j$ be the trace operator from $V$ to $H$. It can be checked as in [5 §4.4] that $a$ is a $j$-elliptic symmetric form, and more precisely
\[ a(u, u) - \omega\|j(u)\|_H^2 \geq \mu\|u\|_V^2 \quad \text{for all } u \in V \]
for some $\omega \in \mathbb{R}$ and $\mu > 0$. 
Let $\lambda^D_1(a)$ denote the smallest eigenvalue of the operator associated with the restriction of $a$ to to $H^1_0(\Omega)$, i.e.,

$$\lambda^D_1(a) := \inf_{u \in H^1_0(\Omega)} \frac{a(u, u)}{\|u\|_{L^2(\Omega)}^2}.$$ 

Let $\gamma_0 < \lambda^D_1(a)$ and fix a complex-valued function $\gamma \in L^q(\Omega)$, $q > \frac{4}{3}$, satisfying $\Re \gamma \geq -\gamma_0$. Then

$$b(u, v) := \int_\Omega \gamma u\overline{v}, \quad u, v \in V,$$

defines a bounded sesquilinear form on $V$ by the Hölder inequality and the Sobolev embedding theorem.

**Proposition 4.6.** Under the above assumptions, $a + b$ is $j$-elliptic.

**Proof.** By the variational characterisation of $\lambda^D_1$ we have for all $u \in H^1_0(\Omega)$ that

$$\Re b(u, u) \geq -\gamma_0 \int_\Omega |u|^2 \geq -\frac{\gamma_0}{\lambda^D_1(a)} a(u, u)$$

and hence

$$(4.1) \quad a(u, u) + \Re b(u, u) \geq \tilde{\eta} a(u, u) \geq \tilde{\eta} k_0 \int_\Omega |\nabla u|^2 \geq \eta \|u\|_{V}^2$$

for all $u \in H^1_0(\Omega)$, where $\tilde{\eta} := 1 - \frac{\gamma_0}{\lambda^D_1(a)} > 0$ and $\eta > 0$ depends on the first eigenvalue of the Dirichlet Laplacian on $\Omega$. Moreover, $H^1(\Omega)$ is compactly embedded into $L^q(\Omega)$ and $j$ is injective on $V(a)$, hence for all $u \in V(a)$ we have

$$\Re b(u, u) \geq -\gamma_0 \int_\Omega |u|^2 \geq \frac{\mu}{2} \|u\|_V^2 - c_\mu \|j(u)\|_H^2$$

for some $c_\mu \geq 0$ by Lemma 2.7 and hence

$$(4.2) \quad a(u, u) + \Re b(u, u) - (\omega - c_\mu) \|j(u)\|_H^2 \geq \frac{\mu}{2} \|u\|_V^2$$

for all $u \in V(a)$.

For $u \in H^1_0(\Omega)$ and $v \in V(a)$ we have $a(v, u) = a(u, v) = 0$ by definition of $V(a)$. Moreover, for every $\varepsilon > 0$ we have

$$[b(u, v)] + |b(u, v)| \leq c \|w\|_{L^p(\Omega)} \|v\|_{L^p(\Omega)} \leq \frac{c_\varepsilon}{2} \|u\|_{L^p(\Omega)}^2 + \frac{c_\varepsilon}{2} \|v\|_{L^p(\Omega)}^2$$

$$(4.3) \quad \leq \frac{c_\varepsilon}{2} \|u\|_V^2 + c_\varepsilon \|j(v)\|_H^2 + \varepsilon \|v\|_V^2$$

for some $p \in [2, \frac{2(d+1)}{d-2})$ and some $c, c_\varepsilon \geq 0$ by the integrability assumptions on $\gamma$, the Sobolev embeddings theorems and Lemma 2.7.

Since every $u \in V$ has a representation of the form $u = u_1 + u_2$ with $u_1 \in H^1_0(\Omega)$ and $u_2 \in V(a)$ by Remark 2.2, combining (4.1), (4.2) and (4.3), where in the latter we pick $\varepsilon > 0$ such that $\frac{p}{2} < \frac{3}{2}$ and $\varepsilon < \frac{\varepsilon_d}{4}$, we obtain that

$$a(u, u) + \Re b(u, u)$$

$$= a(u_1, u_1) + a(u_2, u_2) + \Re b(u_1, u_1) + \Re b(u_2, u_2) + \Re b(u_1, u_2) + \Re b(u_2, u_1)$$

$$\geq \eta \|u_1\|_V^2 + \frac{\mu}{2} \|u_2\|_V^2 + (\omega - c_\mu) \|j(u_2)\|_H^2 - \frac{\eta}{2} \|u_1\|_V^2 - \frac{\mu}{4} \|u_2\|_V^2 - c_\varepsilon \|j(u_2)\|_H^2$$

$$= \frac{\eta}{2} \|u_1\|_V^2 + \frac{\mu}{4} \|u_2\|_V^2 + \omega' \|j(u_2)\|_H^2$$

for some $\omega' \in \mathbb{R}$. Finally, since $V = H^1_0(\Omega) \oplus V(a)$ by Remark 2.2, the expression

$$\|u\|^2 := \|u_1\|_V^2 + \|u_2\|_V^2$$

defines an equivalent norm on $V$, which allows us to write the previous estimate as

$$a(u, u) + \Re b(u, u) - \omega' \|j(u)\|_H^2 \geq \mu' \|u\|_V^2$$

for all $u \in V$. 


for some $\mu'>0$. This is the $j$-ellipticity of $a+b$. \hfill \Box

Following [3] §4.4 one can check that the operator $-D_\alpha^n$ associated with $a+b$ is some Dirichlet-to-Neumann operator. More precisely, $\varphi \in L^2(\partial \Omega)$ is in $D(D_\alpha^n)$ if and only if there exists a (necessarily unique) weak solution of the inhomogeneous Dirichlet problem

\[
\begin{aligned}
\gamma u - \text{div}(\alpha \nabla u) &= 0, & x \in \Omega, \\
u(z) &= \varphi(z), & z \in \partial \Omega,
\end{aligned}
\]

and the weak conormal derivative $\frac{\partial u}{\partial \nu_\gamma}$ exists as an element of $L^2(\partial \Omega)$. In this case, $-D_\alpha^n u = \frac{\partial u}{\partial \nu_\gamma}$. The above considerations show that $D_\alpha^n$ generates an analytic semigroup. We formulate this as a theorem.

**Theorem 4.7.** Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with Lipschitz boundary, where $d \geq 2$. Let $\alpha \in L^\infty(\Omega; C^{d\times d})$ be uniformly positive definite, and let $\gamma \in L^q(\Omega; \mathbb{C})$, $q > \frac{d}{2}$, be such that $\text{Re } \gamma \geq -\gamma_0$ for some $\gamma_0 < \lambda_1^\Omega$. Then the operator $D_\alpha^n$ generates an analytic semigroup on $H = L^2(\partial \Omega)$, which is Gibbs if additionally $\gamma \geq 0$.

If $\Omega$, $\alpha$ and $\gamma$ are smooth, then the analyticity angle of this semigroup is $\frac{\pi}{2}$.

**Proof.** Let us prove the assertion on the analyticity angle. Let everything be smooth, so that in particular $\gamma$ is bounded. By Proposition 2.4 it suffices to check that

\[
M \|u\|_{H^1(\Omega)} \|u\|_{L^2(\partial \Omega)} \geq \left| \text{Im } \int_\Omega \gamma|u|^2 \right|
\]

for all $u \in V(a+b)$ holds for some $M \geq 0$, where $V(a+b)$ consists by definition of all $H^1$-functions that are weak solutions of (IDP) for some $\gamma$. Since $\text{Re } \gamma \geq -\gamma_0$ the only function $u \in H^1_0(\Omega)$ satisfying $\gamma u - \text{div}(\alpha \nabla u) = 0$ is $u = 0$. Hence by [29] Thm. 2.7.4] the trace operator is an isomorphism from

\[
\{u \in H^\frac{1}{2}(\Omega) : \gamma u - \text{div}(\alpha \nabla u) = 0\}
\]

onto $L^2(\partial \Omega)$. Accordingly, the estimate in Proposition 2.4 can be equivalently formulated as

\[
M \|u\|_{H^1(\Omega)} \|u\|_{H^\frac{1}{2}(\Omega)} \geq \left| \text{Im } \int_\Omega \gamma|u|^2 \right|
\]

for some possibly larger constant $M$. This is satisfied whenever $\gamma$ is bounded.

Assume now that $\gamma \geq 0$. Then by [3] Prop. 2.9] the Dirichlet-to-Neumann semigroup of Theorem 1.7 submarkovian, i.e., positive and $L^\infty(\partial \Omega)$-contractive, which is easily checked by a version of an invariance criterion due to Ouhabaz for $j$-elliptic forms [3] Prop. 2.9], see also [3] Prop. 3.7]. In this case Proposition 2.12 and the Sobolev embedding theorems for $\partial \Omega$ (see e.g. [3] Thm. 2.20]) yield in particular that the Dirichlet-to-Neumann semigroup is a Gibbs semigroup. \hfill \Box

**Remark 4.8.** For the last step of the preceding proof we only need that $\gamma \in L^\infty(\Omega)$. Hence one could suspect that for all such $\gamma$ the operator $D_\alpha^n$ generates a cosine operator function without any additional conditions on the smoothness of $\alpha$, $\Omega$ and $\gamma$. However, to extend the result to this situation we would need a generalisation of [29] Thm. 2.7.4 to rough domains and rough coefficients. A partial result into this direction is [23] Lemma. 3.1], where for the Laplace operator [29] Thm. 2.7.4] is extended to Lipschitz domains.

**Remark 4.9.** We regard the perturbation we are considering as interesting mainly because it cannot be expressed as a perturbation by an operator. In comparison, if for smooth $\Omega$ we consider the vaguely related sesquilinear form $b' : V \times V \to \mathbb{C}$
defined by \( b(u, v) := \int_{\partial \Omega} \beta u \overline{v} \) with \( \beta \in L^{d-1}(\partial \Omega) \), then \( a + b' \) is associated with \(-D_{a}^n - B\), where \( B \) is a bounded operator from \( D(D_{a}^n) \) to \( L^2(\Omega) \), and we can deal with it using perturbation theorems for generators.

**Remark 4.10.** The Gibbs property of the semigroup in Theorem 4.7 has been observed before by Zagrebnov \([15]\) Lemma 2.14. His sketch of the proof is based on a Weyl-type asymptotic result for the Dirichlet-to-Neumann operator \([45, \text{Prop 2.5}]\), which seems to require smoothness of the boundary. A complete proof is announced for a forthcoming (but not yet accessible) joint paper with Hassan Emamirad.

In the self-adjoint case we can also prove the following convergence result.

**Theorem 4.11.** Let \( \Omega \subset \mathbb{R}^d \) be an open bounded domain with Lipschitz boundary, where \( d \geq 2 \). Let \((\alpha_n)_{n \in \mathbb{N}} \subset L^\infty(\Omega; \mathbb{C}^{d \times d})\) be such that \( \alpha_n(x) \) is uniformly positive definite uniformly with respect to \( n \), i.e.,

\[
(\alpha_n(x)\xi(\xi) \geq k_0|\xi|^2 \quad \text{for a.e. } x \in \Omega, \forall n \in \mathbb{N} \text{ and all } \xi \in \mathbb{C}^d
\]

for some \( k_0 > 0 \). Let finally \((\gamma_n)_{n \in \mathbb{N}} \subset L^q(\Omega; \mathbb{C})\), \( q > \frac{d}{2} \), be such that \( \text{Re} \gamma_n \geq -\gamma_0 \) a.e. for some \( \gamma_0 < \inf_{n \in \mathbb{N}} \inf_{u \in H_0^2(\Omega)} \frac{\int_{\Omega} \alpha_n \nabla u \cdot \nabla u}{\| u \|_{L^2(\Omega)}^2} \).

If \( \lim_{n \to \infty} \gamma_n = \gamma \) and \( \lim_{n \to \infty} \alpha_n = \alpha \) almost everywhere, then

\[
\lim_{n \to \infty} e^{-tD_{\alpha_n}} = e^{-tD_{\alpha}} \quad \text{in } L^1(L^2(\partial \Omega))
\]

for every \( t > 0 \).

**Proof.** Define

\[
b(u, v) := k_0 \int_{\Omega} \nabla u \cdot \nabla v - \gamma_0 \int_{\Omega} uv
\]

for \( u, v \in H^1(\Omega) \), and let \( j \) be the trace operator from \( H^1(\Omega) \) to \( L^2(\partial \Omega) \). Then \((b, j) \leq (\alpha_n, j)\) in the sense of Proposition 2.9 for all \( n \in \mathbb{N} \), where \( a_n \) denotes the form associated with \(-D_{\alpha_n}^n\). Moreover, the semigroup associated with \((b, j)\) is a Gibbs semigroup by Remark 4.10. So in view of Theorem 4.3 it only remains to show that \( D_{\alpha_n}^n \to D_{\alpha}^n \) in the strong resolvent sense, for which we employ Remark 3.2.

Let \((u_n)\) be a sequence in \( H^1(\Omega) \) such that \( u_n|_{\partial \Omega} \to \varphi \) in \( L^2(\partial \Omega) \) and \( s := \liminf \alpha_n(u_n, u_n) < \infty \). Since the constants in the \( j_n\)-ellipticity of \( \alpha_n \) are uniform with respect to \( n \), the sequence \((u_n)\) is bounded in \( V = H^1(\Omega) \), and thus we may assume that \( u_n \to u \) in \( H^1(\Omega) \) for some \( u \in H^1(\Omega) \). Then by compactness \( \lim_{n \to \infty} u_n = u \) in \( L^2(\Omega) \) and moreover \( u_n|_{\partial \Omega} \to u|_{\partial \Omega} \) in \( L^2(\partial \Omega) \). From this and weak lower semicontinuity of the norm in \( H^1(\Omega) \) it follows immediately that \( a(u, u) \leq s \), where \( a \) denotes the form associated with \(-D_{\alpha}^n\). Moreover, if \( u \in H^1(\Omega) \), then clearly \( \alpha_n(u_n, u_n) \to a(u, u) \). Now the convergence follows from Remark 3.2. \( \square \)

### 4.5. Multiplicative perturbations of Laplacians

Let \( \Omega \subset \mathbb{R}^d \) be an open, bounded set. Let \( V = H_0^1(\Omega) \) and \( H = L^2(\Omega) \). Define \( a(u, v) := \int_{\Omega} \nabla u \overline{\nabla v} \) and \( j(u) := \frac{u}{m} \), where the real-valued function \( m \) on \( \Omega \) satisfies \( 0 < \varepsilon \leq m \leq M < \infty \) for some constants \( \varepsilon \) and \( M \). Then for the operator \(-A_m \) associated with \((a, j)\) we have \( u \in D(A_m) \) with \(-A_m u = j \) if and only if \( u \in H_0^1(\Omega) \) and \( \Delta u = \frac{j}{m} \) distributionally, i.e., at least symbolically, \( A_m = -m\Delta \) with Dirichlet boundary conditions.
Theorem 4.12. Let \((m_n)_{n \in \mathbb{N}}\) be a sequence of measurable functions from \(\Omega\) to \(\mathbb{R}\) such that \(0 < \varepsilon \leq m_n \leq M < \infty\) for all \(n \in \mathbb{N}\). If this sequence converges a.e. to a measurable function \(m : \Omega \to \mathbb{R}\), then
\[
\lim_{n \to \infty} e^{tm_n\Delta} = e^{tm\Delta} \quad \text{in } L^1_1(L^2(\Omega))
\]
for every \(t > 0\).

Proof. Comparing with the Gibbs semigroup generated by \(-\varepsilon \Delta\), we see as in the previous section that it suffices to prove convergence of \(-m_n\Delta\) to \(-m\Delta\) in the strong resolvent sense. So take a sequence \((u_n)\) in \(H^1(\Omega)\) such that \(m_n u_n \rightharpoonup mu\) in \(L^2(\Omega)\) and \(s := \lim \inf_{n \to \infty} a(u_n, u_n) < \infty\). Then \(u_n \rightharpoonup u\) in \(H^1(\Omega)\) after passing to a subsequence, and hence \(\lim_{n \to \infty} u_n = u\) by compact embedding, which shows in particular that \(u \in H^1(\Omega)\). The relation \(a(u, u) \leq s\) is obvious from weak lower semicontinuity of the norm in \(H^1(\Omega)\). On the contrary, if \(u \in H^1(\Omega)\), then \(m_n u \rightharpoonup mu\) in \(L^2(\Omega)\). Hence we obtain convergence in the strong resolvent sense from Theorem 3.1. \(\square\)

Remark 4.13. For every \(k \in \mathbb{N}\), the \(k\)th eigenvalue \(\lambda_k(A_m)\) of \(A_m\) is an increasing function of \(m\). More precisely, if \(m_1 \leq m_2\) almost everywhere, then \(\|u\|_{m_2}^2 \geq \|u\|_{m_1}^2\) for all \(u \in H^1_0(\Omega)\) and hence \(\lambda_k(A_{m_1}) \leq \lambda_k(A_{m_2})\) by Theorem 2.11. By way of the, the operators can in general not be compared in the sense of positive definiteness, as they are not self-adjoint on the same reference space, so the expression \(A_{m_1} \leq A_{m_2}\) is not defined and we have to resort to the eigenvalues if we wish to compare the operators in some way.

4.6. Comparison of self-adjoint elliptic operators. Let \(\Omega \subset \mathbb{R}^d\) be a bounded open domain with Lipschitz boundary. Let \(V := H^1(\Omega)\) and define
\[
a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v + \int_{\partial \Omega} \beta u \overline{v} d\sigma
\]
for a given real-valued function \(\beta \in L^\infty(\partial \Omega)\). We consider the operators \(j_1 : V \to L^2(\Omega), j_2 : V \to L^2(\Omega) \times L^2(\partial \Omega)\) and \(j_3 : V \to L^2(\partial \Omega)\) given by \(j_1(u) := u, j_2(u) := (u, u|_{\partial \Omega})\) and \(j_3(u) := u|_{\partial \Omega}\), respectively. Then \(a\) is a \(j_k\)-elliptic form and we denote the operator associated with \((a, j_k)\) by \(A_k, k = 1, 2, 3\). These operators are given by

\[
\begin{align*}
\text{for } u \in D(A_1), A_1 u = f & \iff \begin{cases}
-\Delta u = f \\
\frac{\partial u}{\partial \nu} + \beta u = 0
\end{cases} \\
\text{for } (u, u|_{\partial \Omega}) \in D(A_2), A_2(u, u|_{\partial \Omega}) = (f, g) & \iff \begin{cases}
-\Delta u = f \\
\frac{\partial u}{\partial \nu} + \beta u = g
\end{cases} \\
\text{for } \varphi \in D(A_3), A_3 \varphi = g & \iff \exists u \in H^1(\Omega) : \begin{cases}
-\Delta u = 0 \\
\frac{\partial u}{\partial \nu} = \varphi \\
\frac{\partial u}{\partial \nu} + \beta u = g
\end{cases}
\end{align*}
\]

where the Laplace operator and the normal derivative are understood in a weak sense, see [3, §4.4] for \(A_3\).

Now Theorem 2.11 yields that
\[
\lambda_k(A_2) \leq \lambda_k(A_1) \quad \text{and} \quad \lambda_k(A_2) \leq \lambda_k(A_3) \quad \text{for all } k \in \mathbb{N}.
\]

These results have also been obtained in [11] Thm. 4.2 and Thm. 4.3 by the same argument.
4.7. Convergence and non-convergence of Wentzell–Robin operators. Let \( \Omega \subset \mathbb{R}^d \) be a bounded Lipschitz domain and define \( V := \{(u, u|_{\partial \Omega}) : u \in H^1(\Omega)\} \) and \( H := L^2(\Omega) \times L^2(\partial \Omega) \). Then
\[
a((u, u|_{\partial \Omega}), (v, v|_{\partial \Omega})) := \int_{\Omega} \nabla u \cdot \nabla v
\]
defines a bounded sesquilinear form on \( V \). We consider the embeddings
\[
j_{\rho, \sigma}((u, u|_{\partial \Omega})) := (\rho u, \sigma u|_{\partial \Omega})
\]
of \( V \) into \( H \), where \( \sigma > 0 \) and \( \rho > 0 \) are constants. Then clearly \( a \) is a positive \( j_{\rho, \sigma} \)-elliptic form, and the associated operator \( A_{\rho, \sigma} \) is (at least on a formal level) given by
\[
A_{\rho, \sigma}(u, u|_{\partial \Omega}) = \left( -\frac{1}{\rho} \Delta u, \frac{1}{\sigma} \frac{\partial u}{\partial \nu} \right).
\]

**Theorem 4.14.** The operator \( A_{1, \sigma} \) converges to \( -\Delta_D \oplus 0 \) in the strong resolvent sense as \( \sigma \to \infty \), where \( \Delta_D \) denotes the Dirichlet Laplacian on \( L^2(\Omega) \).

*Proof.* The operator \( -\Delta_D \oplus 0 \) is associated with the \( j_D \)-elliptic form \( a_D \) given by \( a_D((u, g), (v, h)) := \int_{\Omega} \nabla u \nabla v \), where \( j_D : H^1(\Omega) \times L^2(\partial \Omega) \to H \) is given by \( j_D((u, g)) := (u, g) \).

Let \( \sigma_n \to \infty \) and let \( (u_n) \) be a sequence in \( V \) such that
\[
j_n(u_n) := (u_n, u_n|_{\partial \Omega}) \to (u, g)
\]
in \( H \) and \( s := \liminf \int_{\Omega} |\nabla u_n|^2 < \infty \). Passing to a subsequence we can assume that \( \int_{\Omega} |\nabla u_n|^2 \to s \). Then \( (u_n) \) is bounded in \( H^1(\Omega) \), and passing to further subsequence we can assume that \( u_n \to u \) in \( H^1(\Omega) \). Then in particular \( u_n|_{\partial \Omega} \to u|_{\partial \Omega} \) and \( \int_{\Omega} |\nabla u|^2 \leq s \). Moreover,
\[
u_n|_{\partial \Omega} = \frac{\sigma_n u_n|_{\partial \Omega}}{\sigma_n} \to 0
\]
since \( (\sigma_n u_n|_{\partial \Omega}) \) is bounded and \( \sigma_n \to \infty \), hence \( u|_{\partial \Omega} = 0 \). Thus \( (u, g) \in H^1(\Omega) \times L^2(\partial \Omega) \), \( j_D((u, g)) = (u, g) \) and \( \liminf \int_{\Omega} |\nabla u_n|^2 \geq \int_{\Omega} |\nabla u|^2 \). We have checked the first part of the characterisation in Theorem 4.11.

For the second part, let \( (u, g) \in H^1(\Omega) \times L^2(\partial \Omega) \) be fixed. Since \( \sigma_n \to \infty \), there exist \( v_n \in H^1(\Omega) \) satisfying \( v_n|_{\partial \Omega} \to g \) in \( L^2(\partial \Omega) \) and \( \frac{\sigma_n}{\sigma_n} \to 0 \) in \( H^1(\Omega) \). Define \( u_n := u + \frac{v_n}{\sigma_n} \). Then \( (u_n, u_n|_{\partial \Omega}) \in V \) and
\[
j_n((u_n, u_n|_{\partial \Omega})) = (u_n, v_n|_{\partial \Omega}) \to (u, g) = j_D((u, g))
\]
in \( H \). Moreover, \( \int_{\Omega} |\nabla u_n|^2 \to \int_{\Omega} |\nabla u|^2 \) since \( u_n \to u \) in \( H^1(\Omega) \).

As already emphasised, one advantage of our Mosco-type result is that it *characterises* convergence, meaning that it paves the road to *non-convergence* results as well. Given that the eigenvalue problem associated with the operator \( A_{\rho, \sigma} \) is
\[
\begin{cases}
\lambda \rho u = \Delta u & \text{in } \Omega, \\
\lambda \sigma u|_{\partial \Omega} = -\frac{\partial u}{\partial \nu} & \text{on } \partial \Omega,
\end{cases}
\]
while the eigenvalue problem associated with the Dirichlet-to-Neumann operator is
\[
\begin{cases}
0 = \Delta u & \text{in } \Omega, \\
\lambda u|_{\partial \Omega} = -\frac{\partial u}{\partial \nu} & \text{on } \partial \Omega,
\end{cases}
\]
the following may look surprising.

**Proposition 4.15.** The following assertions hold in the space \( L^2(\Omega) \times L^2(\partial \Omega) \).
(1) $A_{1,\sigma}$ does not converge to any closed operator in the weak resolvent sense as $\sigma \to 0$.

(2) $A_{p,1}$ does not converge to any closed operator in the weak resolvent sense as $p \to 0$.

Proof. In both cases, we follow the same strategy. Assume that the family of operators converges to a densely defined (necessarily self-adjoint) operator $B$ on $H := L^2(\Omega) \times L^2(\partial \Omega)$ in the weak resolvent sense. Then the operators converge even in the strong resolvent sense \cite{38, VIII.7}, and hence the quadratic forms $H$ operators converge to a densely defined (necessarily self-adjoint) operator.

By \cite{31, §1.1.15}, hence $\rho \nabla u \to 0$, i.e., $u = 0$. Hence the set of possible limits in (b.ii) of Theorem 3.1 is contained in the non-dense set $\{0\} \times L^2(\partial \Omega)$.

References

[1] R. Adams. *Sobolev Spaces*. Pure Appl. Math. Academic Press, New York, 1975.

[2] W. Arendt. Heat Kernels – Manuscript of the 9th Internet Seminar, 2006. Freely available at [http://www.uni-ulm.de/fileadmin/website_uni_ulm/mawi.inst.020/arendt/downloads/internetseminar.pdf](http://www.uni-ulm.de/fileadmin/website_uni_ulm/mawi.inst.020/arendt/downloads/internetseminar.pdf)

[3] W. Arendt and T. ter Elst. Sectorial forms and degenerate differential operators. arXiv:0812.3944, 2009.

[4] W. Arendt and T. ter Elst. The Dirichlet-to-Neumann operator on rough domains. arXiv:1005.0875v1, 2010.

[5] W. Arendt and T. ter Elst. From forms to semigroups. arXiv:1104.1013, 2011.

[6] W. Arendt, C. Batty, M. Hieber, and F. Neubrander. *Vector-Valued Laplace Transforms and Cauchy Problems*, volume 96 of Monographs in Mathematics. Birkhäuser, Basel, 2001.

[7] W. Arendt, G. Metafune, D. Pallara, and S. Romanelli. The Laplacian with Wentzell–Robin boundary conditions on spaces of continuous functions. *Semigroup Forum*, 67:247–261, 2003.

[8] H. Attouch. *Variational convergence for functions and operators*. Applicable Mathematics Series. Pitman, Boston, 1984.

[9] T. Aubin. *Nonlinear analysis on manifolds, Monge-Ampere equations*, volume 252 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1982.
[10] J. Behrndt, M. Langer, I. Lobanov, V. Lotoreichik, and I. Popov. A remark on Schatten-von Neumann properties of resolvent differences of generalized Robin Laplacians on bounded domains. *J. Math. Anal. Appl.*, 371:750–758, 2010.

[11] J. Below and G. François. Spectral asymptotics for the Laplacian under an eigenvalue dependent boundary condition. *Bull. Belg. Math. Soc. - Simon Stevin*, 12:505–519, 2005.

[12] M. Birman. Perturbations of the continuous spectrum of a singular elliptic operator by varying the boundary and the boundary conditions. In T. Suslina and D. Yafaev, editors, *Spectral theory of differential operators: M. Sh. Birman 80th anniversary collection*, volume 225 of *Adv. Math. Sciences*, pages 19–54. Amer. Math. Soc., 2008.

[13] S. Cardanobile and D. Mugnolo. Parabolic systems with coupled boundary conditions. *J. Differ. Equ.*, 247:1229–1248, 2009.

[14] T. Cheon, P. Exner, and O. Turek. Approximation of a general singular vertex coupling in quantum graphs. *Ann. Phys.*, 325:548–578, 2010.

[15] C. Clark. The Hilbert–Schmidt property for embedding maps between Sobolev spaces. *Can. J. Math.*, 18:1079–1084, 1966.

[16] G. Coclite, G. Goldstein, and J. Goldstein. Stability estimates for parabolic problems with Wentzell boundary conditions. *J. Differ. Equ.*, 245(9):2595–2626, 2008.

[17] M. Crouzeix. Operators with numerical range in a parabola. *Arch. Math.*, 82:517–527, 2004.

[18] M. Demuth, P. Stollmann, G. Stolz, and J. van Casteren. Trace norm estimates for products of integral operators and diffusion semigroups. *Integral Equations Oper. Theory*, 23:145–153, 1995.

[19] W. Desch and W. Schappacher. On relatively bounded perturbations of linear $C_0$-semigroups. *Ann. Sc. Norm. Super. Pisa, Cl. Sci.*, 11:327–341, 1984.

[20] W. Desch and W. Schappacher. Some perturbation results for analytic semigroups. *Math. Ann.*, 281:157–162, 1988.

[21] A. Favini, G. Goldstein, J. Goldstein, E. Obrecht, and S. Romanelli. The Laplacian with generalized Wentzell boundary conditions. In M. Iannelli and G. Lumer, editors, *Evolution Equations 2000: Applications to Physics, Industry, Life Sciences and Economics (Proceedings Levico Terme 2000)*, volume 55 of *Progress in Nonlinear Differential Equations*, pages 169–180, Basel, 2003. Birkhäuser.

[22] C. Gapaillard. Un resultat de compacité pour l’interpolation de couples hilbertiens. *C.R. Acad. Sc. Paris Sér. A*, 278:681–684, 1974.

[23] F. Gesztesy and M. Mitrea. A description of all self-adjoint extensions of the Laplacian and Krein-type resolvent formulas in nonsmooth domains. arXiv:0907.1750, 2009.

[24] I. Gohberg and M. Krein. *Introduction to the theory of linear nonselfadjoint operators*, volume 18 of *Transl. Math. Monographs*. Amer. Math. Soc., Providence, RI, 1969.

[25] B. Gramsch. Zum Einbettungssatz von Rellich bei Sobolevräumen. *Math. Z.*, 106:81–87, 1968.

[26] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman, Boston, 1985.

[27] T. Kato. *Perturbation Theory for Linear Operators*. Classics in Mathematics. Springer-Verlag, New York, 1995.

[28] H. König. Operator properties of Sobolev imbeddings over unbounded domains. *J. Funct. Anal.*, 24:32–51, 1977.
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[29] J. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications*, volume 181–183 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1972.

[30] K. Maurin. Abbildungen vom Hilbert–Schmidtschen Typus und ihre Anwendungen. *Math. Scand.*, 9:359–371, 1961.

[31] V. Maz’ya. *Sobolev Spaces*. Springer-Verlag, Berlin, 1985.

[32] U. Mosco. Approximation of the solutions of some variational inequalities. *Ann. Sc. Norm. Super. Pisa, Cl. Sci., III Serie*, 21:373–394, 1967.

[33] D. Mugnolo. A variational approach to strongly damped wave equations. In H. Amann et al., editor, *Functional Analysis and Evolution Equations – The Günter Lumer Volume*, pages 503–514. Birkhäuser, Basel, 2008.

[34] D. Mugnolo, R. Nittka, and O. Post. Convergence of sectorial operators on varying Hilbert spaces. arXiv:1007.3932, 2010.

[35] H. Neidhardt and V. A. Zagrebnov. The Trotter-Kato product formula for Gibbs semigroups. *Comm. Math. Phys.*, 131(2):333–346, 1990.

[36] E. Ouhabaz. *Analysis of Heat Equations on Domains*, volume 30 of *Lond. Math. Soc. Monograph Series*. Princeton Univ. Press, Princeton, 2005.

[37] O. Post. Spectral convergence of quasi-one-dimensional spaces. *Ann. Henri Poincaré*, 7:933–973, 2006.

[38] M. Reed and B. Simon. *Methods of modern mathematical physics. I*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, second edition, 1980. ISBN 0-12-585050-6. Functional analysis.

[39] R. Schatten and J. von Neumann. The cross-space of linear transformations. II. *Ann. Math.*, 47:608–630, 1946.

[40] B. Simon. *Trace ideals and their applications*, volume 120 of *Math. Surveys and Monographs*. Amer. Math. Soc., Providence, RI, 2005.

[41] P. Stollmann. Trace ideal properties of perturbed Dirichlet semigroups. In *Mathematical results in quantum mechanics (Proc. Blossin 1993)*, volume 70 of *Oper. Theory, Adv. Appl.*, pages 153–158, Basel, 1994. Birkhäuser.

[42] C. Travis and G. Webb. Compactness, regularity, and uniform continuity properties of strongly continuous cosine families. *Houston J. Math.*, 3:555–567, 1977.

[43] D. Uhlenbrock. Perturbation of statistical semigroups in quantum statistical mechanics. *J. Math. Phys.*, 12:2503, 1971.

[44] V. Zagrebnov. On the families of Gibbs semigroups. *Commun. Math. Phys.*, 76:269–276, 1980.

[45] V. A. Zagrebnov. From Laplacian transport to Dirichlet-to-Neumann (Gibbs) semigroups. *Zh. Mat. Fiz. Anal. Geom.*, 4:551–568, 2008.

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