The noncommutative replica approach

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Abstract

$p$–Adic and non–commutative analysis are applied to describe phase transitions in disordered systems. In the noncommutative replica approach we replicate the disorder instead of the system degrees of freedom. The noncommutative replica symmetry breaking is formulated using the language of noncommutative analysis. This allows to derive the ultrametric space of states which is postulated in the standard replica approach.

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1 Introduction

In the present paper we apply $p$–adic and non–commutative analysis to describe disordered systems. For introduction to $p$–adic analysis see [1], [2]. $p$–Adic analysis and $p$–adic mathematical physics attract great interest, see [1]–[7]. For instance, $p$–adic models in string theory were introduced, see [6], and $p$–adic quantum mechanics [7] was investigated. $p$–Adic analysis was applied to investigate the spontaneous breaking of the replica symmetry, cf. [8], [9]. Similar approach was used in [10], [11].

We investigate disordered models such as the Sherrington–Kirkpatrick model. Disorder in such models is described by large stochastic matrices (the real symmetric matrices with matrix elements being independent Gaussian stochastic variables). The standard approaches to describe phase transition in such systems are the replica approach and the Parisi ultrametric anzats of breaking of replica symmetry [12]–[15].

In the present paper we propose a new approach to describe phase transitions in disordered systems based on non–commutative analysis. We construct an analogue of the replica procedure in the framework of non–commutative geometry and show that this procedure naturally implies introduction of $p$–adic geometry.

To obtain our results we apply the following sequence of actions.

First, by the Wigner theorem the Sherrington–Kirkpatrick model in the high temperature regime is described by the quantum Boltzmann algebra in the free Fock representation. Note that the high temperature regime here corresponds to the vacuum (zero temperature) state of the quantum Boltzmann algebra. Then, the transition from 1 to $p$ degrees of freedom is a morphism of quantum probability spaces. We call this transformation the Non–commutative Replica Procedure.

Second, we describe phase transition by putting the quantum Boltzmann algebra into the state with non–zero expectation of annihilators. To construct such a state we use the free coherent states and the correspondent $p$–adic representation of the Cuntz algebra. This representation acts on functions on the quantum line which is equivalent to $p$–adic disk.

The free coherent states are eigenvectors of the linear combination of annihilation operators from the quantum Boltzmann algebra, or solutions $\Psi$ of the following equation of the quantum line

$$(A - \lambda)\Psi = 0$$

where $A$ is the linear combination of quantum Boltzmann annihilators. In [23] it was shown that the space of free coherent states is isomorphous to the state of generalized functions on a $p$–adic disk. This means that the quantum line [1] is equivalent to a $p$–adic disk (and $\Psi$ in (1) corresponds to a generalized function supported at the quantum line).

Third, we conjecture that the $k$–th correlator of the SK model is equal to the expectation of the $k$–th degree of the order parameter $Q = A + A^\dagger$, $A = \frac{1}{\sqrt{p}} \sum_{i=0}^{p-1} A_i$ in the $p$–adic representation. This means that the corresponding correlators will be given by $p$–adic integrals.

The structure of the paper is as follows.

In Section 2 we remind the Sherrington–Kirkpatrick model and introduce the noncommutative replica procedure.

In Section 3 we put the Wigner theorem about the limits of large stochastic matrices.

In Section 4 we discuss the quantum Boltzmann and Cuntz algebras.

In Section 5 we describe the isomorphism of the free coherent states and generalized functions on $p$–adic disk.
In Section 6 we discuss the \( p \)-adic representation of the Cuntz algebra and its relation to the free coherent states.

In Section 7 we discuss the relation of phase transitions in quantum systems and coherent states.

In Section 8 we formulate a conjecture describing the state of disordered system (say of the SK model) after the phase transition.

In Section 9 we discuss the results of the present paper.

2 The SK model and replicas

The typical disordered model is the SK (Sherrington–Kirkpatrick) model with the Hamiltonian

\[
H = - \sum_{i<j} J_{ij} \sigma^i \sigma^j
\]  

(2)

where the summation runs over the spins \( \sigma_j \) in the lattice \( Z^d \) taking values \( \pm 1 \) and \( J_{ij} \) is the random matrix of interactions with the matrix elements which are independent Gaussian random variables with the probability distribution

\[
P[J_{ij}] = \prod_{i<j} \exp \left( -\frac{NJ_{ij}^2}{2} \right)
\]  

(3)

Let us note that the stochastic matrix \( J_{ij} \) is an example of a quenched disorder: to calculate the expectation of an observable we have to use some typical realization of the stochastic matrix \( J_{ij} \). To find the state one uses the assumption of self–averaging: to calculate the expectation value of the observable first we have to take the average over the spin degrees of freedom and then take the average over the Gaussian stochastic variables \( J_{ij} \). The statistic sum takes the form

\[
Z = \int \sum_{\{\sigma_i\}} \exp \left( \beta \sum_{i<j} J_{ij} \sigma_i \sigma_j - \frac{N}{2} \sum_{i<j} J_{ij}^2 \right) \prod_{i<j} dJ_{ij}
\]

The well known approach to investigation of the Sherrington–Kirkpatrick model and other disordered systems is the replica method \[12\]–\[15\], where the spontaneous breaking of the replica symmetry introduced by Parisi leads to introduction of an ultrametric space. In the simplest case this space coincides with a subset of the field \( Q_p \) of \( p \)-adic numbers and corresponding Parisi matrix is equal to the Vladimirov operator of \( p \)-adic fractional derivation \[8\], \[9\]. In the replica approach one introduces \( n \) identical replicas of the original system. This implies the following expression for the statistic sum

\[
Z_n = \int \sum_{\{\sigma^a_i\}} \exp \left( \beta \sum_{a=1}^n \sum_{i<j} J_{ij} a_i a_j - \frac{N}{2} \sum_{i<j} J_{ij}^2 \right) \prod_{i<j} dJ_{ij}
\]  

(4)

where \( \beta \) is the inverse temperature. The transformation

\[
Z \mapsto Z_n
\]
is called the replica procedure \[12\]. The spontaneous breaking of the replica symmetry \[13\] is the anzats giving an approximation of this statistical sum.

In the present paper we develop a different approach to describe phase transition in disordered system, based on noncommutative and $p$–adic analysis. The starting point of this approach is to replicate the disorder instead of the system degrees of freedom.

First, let us note that in the high temperature regime $\beta \to 0$, when we do not expect the phase transition, one can neglect the spin part in the statistic sum and the Gibbs state of the model will be given by (3). In the thermodynamic limit $N \to \infty$ is described by the Wigner theorem, see the next section.

Second, let us instead of replication of the system as in (4), replicate the disorder. Consider $p$ independent copies $J_{ij}^{(a)}$ of the random matrix $J_{ij}$ and modify the statistic sum in the following way

$$Z(p) = \int \sum_{\{\sigma_i\}} \exp \left( \frac{\beta}{\sqrt{p}} \sum_{a=0}^{p-1} \sum_{i<j} J_{ij}^{(a)} \sigma_i \sigma_j - \frac{N}{2} \sum_{a=0}^{p-1} \sum_{i<j} J_{ij}^{(a)2} \right) \prod_{a=0}^{p-1} \prod_{i<j} dJ_{ij}^{(a)}$$

We call the transformation $Z \mapsto Z(p)$ the noncommutative replica procedure. In this procedure $p = n^{-1}$ and the well known $n \to 0$ limit of the replica approach becomes $p \to \infty$ limit. Since in the high temperature regime the state of the disordered system is described by the Wigner theorem, for high temperatures the statistic sum above will give an equivalent description of the SK model. In the thermodynamic $N \to \infty$ limit, in the high temperature regime, the state defined by the statistic sum (5) will be described by the noncommutative extension of the Wigner theorem and will be give by the Fock state on the quantum Boltzmann algebra

$$A_i A_j^\dagger = \delta_{ij}$$

This observation explains the word noncommutative (replica procedure).

For lower temperatures we will obtain the phase transition in the described system, after which the state on the quantum Boltzmann algebra will become non–Fock. In the present paper we introduce the corresponding state and show, using the results of \[24\]–\[26\], that the introduced state is related to $p$–adic geometry, which is the simplest example of the ultrametric geometry. This observation is in agreement with the results obtained in the replica symmetry breaking approach \[14\].

### 3 Large random matrices and the Wigner theorem

In order to obtain the deeper insight, let us discuss in details the high temperature limit, in which the SK model is exactly solvable.

Let $X = (X_{ij})_{i,j=1}^N$ be an ensemble of real symmetric $N \times N$ matrices with distribution given by

$$\langle f(X) \rangle_N = \frac{1}{Z_N} \int \text{tr} f \left( \frac{X}{N} \right) e^{-\frac{1}{2} \text{tr} X^2} \prod_{i \leq j} dX_{ij}, \quad (6)$$

$$Z_N = \int e^{-\frac{1}{2} \text{tr} X^2} \prod_{i \leq j} dX_{ij}$$
The limits
\[
\lim_{N \to \infty} \langle X^k \rangle_N = (\Omega, Q^k \Omega) \quad ; \quad k = 1, 2, \ldots \quad (7)
\]
eexist. The limiting object \( Q \) in (7) is a quantum random variable which is called the master field. It is defined as
\[
Q = A^\dagger + A \quad (8)
\]
where \( A^\dagger \) and \( A \) are free creation and annihilation operators, i.e. they do not satisfy the usual canonical commutation relations but the following
\[
AA^\dagger = 1 \quad (9)
\]
The vector \( \Omega \) in (7) is a vacuum vector in the free Fock space,
\[
A\Omega = 0
\]
The free Fock space is constructed starting from \( A^\dagger \) and \( \Omega \), by the usual procedure but only by using the operators \( A \) and \( A^\dagger \) satisfying (9).
The expectation value in (7) is known to be
\[
(\Omega, Q^k \Omega) = \int_R \lambda^k w(\lambda) d\lambda \quad (10)
\]
where \( w(\lambda) \) is the Wigner semicircle density,
\[
w(\lambda) = \frac{1}{2\pi} \sqrt{4 - \lambda^2} \quad , \quad |\lambda| \leq 2 \quad (11)
\]
and \( w(\lambda) = 0 \) for \( |\lambda| \geq 2 \). The Wigner semicircle distribution in noncommutative probability plays the role of the Gaussian distribution in classical probability.

This form of the Wigner theorem was presented in [22]. Discuss now a generalization of the Wigner theorem.

Consider the space \( P(N, \mathbb{R}) \) of polynomials of symmetric \( N \times N \) matrices over real numbers. Consider the state over \( P(N, \mathbb{R}) \) (the same as in the Wigner theorem):
\[
\langle f(X) \rangle_N = \frac{1}{Z_N} \int \text{tr} f \left( \frac{X}{N} \right) e^{-\frac{1}{2} \text{tr} X^2} \prod_{i \leq j} dX_{ij}
\]
Introduce \( p \) independent copies of the space \( P(N, \mathbb{R}) \) and the tensor degree \( P(N, \mathbb{R})^\otimes p \). Introduce the state on \( P(N, \mathbb{R})^\otimes p \) in the following way
\[
\langle f \left( X^{(1)}, \ldots, X^{(p)} \right) \rangle_N = \frac{1}{Z_N} \int \text{tr} f \left( \frac{X^{(1)}}{N}, \ldots, \frac{X^{(p)}}{N} \right) e^{-\frac{1}{2} \sum_{k=1}^p \text{tr} X^2_k} \prod_{k=1}^p \prod_{i \leq j} dX_{ij}^{(k)} \quad (12)
\]
\[
Z_N = \langle 1 \rangle_N
\]
The following result was presented in [17] and was used in the \( N \to \infty \) limit of the matrix model, see [18]–[22].
Theorem 1  The limits (12) exist for each polynomial f and are equal to
\[
\lim_{N \to \infty} \langle f(X^{(1)}, \ldots, X^{(p)}) \rangle_N = (\Omega, f(Q_1, \ldots, Q_p)\Omega)
\]
where
\[
Q_i = A_i + A_i^\dagger
\]
and \(A_i\) are the quantum Boltzmann annihilators, satisfying
\[
A_i A_j^\dagger = \delta_{ij}
\]
and \(\Omega\) is the vacuum in the free Fock space:
\[
A_i \Omega = 0
\]

Idea of the Proof  Discuss first the sketch of the proof of the Wigner theorem.

The integral in the Wigner theorem is
\[
\langle f(X) \rangle_N = \frac{1}{Z_N} \int \text{tr} f \left( \frac{X}{N} \right) e^{-\frac{1}{2} \text{tr} X^2} \prod_{i \leq j} dX_{ij}
\]

The following integral is equal to the Fock expectation of the degree of the sum of Bose creator and annihilator:
\[
\frac{1}{\sqrt{2\pi}} \int x^k e^{-\frac{1}{2}x^2} dx = \langle (a + a^\dagger)^k \rangle; \quad [a, a^\dagger] = 1, \quad \langle a \rangle = 0 \quad (13)
\]
For the matrix \(X = (X_{ij})\) one get
\[
\langle X^k \rangle_N = \frac{1}{Z_N} \int \text{tr} \left( \frac{X}{N} \right)^k e^{-\frac{1}{2} \text{tr} X^2} \prod_{i \leq j} dX_{ij} = \frac{N^{-k}}{Z_N} \int X_{i_1i_2} X_{i_2i_3} \ldots X_{i_{k-1}i_k} e^{-\frac{1}{2} \text{tr} X^2} \prod_{i \leq j} dX_{ij} \quad (14)
\]
where we assume the summation over \(i_j\).

Using (13) and the Wick theorem, we see that the half–planar diagrams gives contribution of order \(N^k\) into the integral above, and the crossing diagrams give contribution of lower order, see \[22\] for details. In the large \(N\) limit only half–planar diagrams survive.

Since the limits of the correlation functions coincide with the correlations on the quantum Boltzmann algebra, this implies that the \(N \to \infty\) limits of the correlators are reproduced by the Fock state on the quantum Boltzmann algebra:
\[
\lim_{N \to \infty} \langle X^k \rangle_N = \langle (A + A^\dagger)^k \rangle, \quad AA^\dagger = 1, \quad \langle A \rangle = 0
\]

In the statement of theorem 1 we have the polynomials which are linear combinations of the monomials of the form
\[
f(X^{(1)}, \ldots, X^{(p)}) = X^{(j_1)} \ldots X^{(j_k)}
\]
We have the state on the monomial
\[ \langle X^{(j_1)} \ldots X^{(j_k)} \rangle_N = \frac{N^{-k}}{Z_N} \int X^{(j_1)}_{i_1 i_2} \ldots X^{(j_k)}_{i_k i_1} e^{-\frac{1}{2} \sum_{k=1}^p \text{tr} X^{(k)}_i \prod_{k=1}^p dX^{(k)}_{ij}} \]

Using the same arguments which were used to prove the Wigner theorem, we obtain the proof of theorem 1.

Consider the following linear map
\[ \Delta : P(N, \mathbb{R}) \rightarrow P(N, \mathbb{R})^\otimes p \]
\[ X \mapsto \frac{1}{\sqrt{p}} (X \otimes 1 \otimes 1 \ldots \otimes 1 + 1 \otimes X \otimes 1 \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots \otimes 1 \otimes X) \quad (15) \]
or equivalently
\[ X \mapsto \frac{1}{\sqrt{p}} \sum_{i=1}^p X_i \]
where \( X_i \) belongs to the \( i \)-th component of the tensor product. This map coincides with the map used in the central limit theorem. For example, for the gaussian random variables \( X \) the map (13) is an embedding of probability spaces (all the correlation functions are invariant). For large random matrices, in the thermodynamic (large \( N \)) limit, the central limit theorem becomes the free central limit theorem, see [17]. For instance, the Wigner state will be invariant under (15). We formulate the following:

**Theorem 2**  
*The map (14) with the state (12), where we put
\[ f \left( X^{(1)}, \ldots, X^{(p)} \right) = f \left( \Delta X \right) \]
is an embedding of algebraic probability spaces. In the limit \( N \to \infty \) this embedding becomes the \( * \)-homomorphism of the quantum Boltzmann algebra with one degree of freedom maps into the quantum Boltzmann algebra with \( p \) degrees of freedom defined according to the formula
\[ A \mapsto \frac{1}{\sqrt{p}} \sum_{i=1}^p A_i, \quad A^\dagger \mapsto \frac{1}{\sqrt{p}} \sum_{i=1}^p A_i^\dagger \quad (16) \]
and the Fock state \( \langle \Omega, x\Omega \rangle \) will map onto the Fock state \( \langle \Omega, x\Omega \rangle_p \) in the quantum Boltzmann Fock space with \( p \) degrees of freedom
\[ \langle \Omega, x\Omega \rangle \mapsto \langle \Omega, x\Omega \rangle_p \quad (17) \]
The map (16), (17) conserves all the correlators.*

**Proof**  
Follows from the Wick theorem and the analysis of the correlators.

The substitute (15) we call the noncommutative replica procedure for high temperature.

The embedding of the quantum probability spaces we understand in the following way. The morphism of quantum probability spaces \((A, \phi)\) and \((B, \psi)\), see [12], is the morphism of algebras
\[ f : A \rightarrow B \]
such that the correlators are invariant:

\[ \psi(f(a)) = \phi(a) \]

We call the morphism \( f \) an embedding, if it is an embedding as the map of algebras.

Summing up, we see that the Wigner theorem allows to compute the thermodynamic limit of the SK model in the high temperature regime. We introduce the noncommutative replica procedure \( (15) \) and show, that in the high temperature regime it realizes an equivalent description of the SK model (by theorems 1 and 2).

In the present paper we investigate phase transitions in the SK model. Before the phase transition the noncommutative replica procedure \( (15) \) is embedding of quantum probability spaces. After the phase transition the noncommutative replica procedure will give rise to non trivial state of the quantum Boltzmann algebra, which we will approximate using the free coherent states.

4 The quantum Boltzmann and Cuntz algebras

In the present section we discuss the quantum Boltzmann and the Cuntz algebras.

The quantum Boltzmann algebra arises in the limit of large stochastic matrices \([16]\), was used in the free probability \([17]\) and describes the quantum system in the stochastic approximation \([27]–[31]\). It is defined as follows. The quantum Boltzmann, or free commutation relation algebra (or FCR-algebra) over the Hilbert space \( H \) is the unital involutive algebra with generators (which are called free, or quantum Boltzmann, creation and annihilation operators) \( A^\dagger(f), A(f), f \in H \), and relation

\[ A(f)A^\dagger(g) = (f,g). \]

The FCR-algebra have a natural representation in the free Fock space. Free (or Boltzmannian) Fock space \( F \) over a Hilbert space \( H \) is the completion of the tensor algebra

\[ F = \oplus_{n=0}^{\infty} H^\otimes n. \]

Creation and annihilation operators are defined in the following way:

\[ A^\dagger(f_1 \otimes \ldots \otimes f_n) = f \otimes f_1 \otimes \ldots \otimes f_n \]

\[ A(f_1 \otimes \ldots \otimes f_n) = \langle f,f_1 \rangle f_2 \otimes \ldots \otimes f_n \]

where \( \langle f,g \rangle \) is the scalar product in the Hilbert space \( H \). Scalar product in the free Fock space is defined by the standard construction of the direct sum of tensor products of Euclidean spaces.

In the case when \( H \) is the \( p \)-dimensional complex Euclidean space we have \( p \) creation operators \( A_i^\dagger, i = 0, \ldots, p - 1 \); \( p \) annihilation operators \( A_i, i = 0, \ldots, p - 1 \) with the relations

\[ A_iA_j^\dagger = \delta_{ij}. \]  

(18)

and the vacuum vector \( \Omega \) in the free Fock space satisfies

\[ A_i\Omega = 0. \]  

(19)

Let us consider an orthonormal basis in the free Fock space of the form

\[ e_I = A_I^\dagger\Omega; \]
here the multiindex \( I = i_0 \ldots i_{k-1}, \) \( i_j \in \{0, \ldots, p - 1\} \) and \( A_I^\dagger = A_{i_{k-1}}^\dagger \ldots A_{i_0}^\dagger. \)

In the considered basis the Fock representation of FCR-algebra has the form

\[
A_I^\dagger e_I = e_{IR}.
\]

One also defines (in the same Hilbert space) the Antifock representation of FCR-algebra

\[
A_I^\dagger e_I = e_{iI}.
\]

Fock and Antifock representation of FCR-algebra are unitarily equivalent.

We will also use the following factor–algebra of the quantum Boltzmann algebra. The Cuntz algebra (with \( p \) degrees of freedom) is the algebra with involution which is generated by \( p \) creation operators \( A_i^\dagger, i = 0, \ldots, p - 1 \); \( p \) annihilation operators \( A_i, i = 0, \ldots, p - 1 \) with the relations

\[
A_i A_j^\dagger = \delta_{ij};
\]

\[
\sum_{i=0}^{p-1} A_i A_i^\dagger = 1.
\]

5 The free coherent states

Since the free coherent states [24], [25] are related to \( p \)-adic numbers, let us make here a brief review of \( p \)-adic analysis. The field \( \mathbb{Q}_p \) of \( p \)-adic numbers is the completion of the field of rational numbers \( \mathbb{Q} \) with respect to the \( p \)-adic norm on \( \mathbb{Q} \). This norm is defined in the following way. An arbitrary rational number \( x \) can be written in the form \( x = p^\gamma \frac{m}{n} \) with \( m \) and \( n \) not divisible by \( p \). The \( p \)-adic norm of the rational number \( x = p^\gamma \frac{m}{n} \) is equal to \( |x|_p = p^{-\gamma} \).

The most interesting property of the field of \( p \)-adic numbers is ultrametricity. This means that \( \mathbb{Q}_p \) obeys the strong triangle inequality

\[
|x + y|_p \leq \max(|x|_p, |y|_p).
\]

We will consider disks in \( \mathbb{Q}_p \) of the form \( \{x \in \mathbb{Q}_p : |x - x_0|_p \leq p^{-k}\} \). For example, the ring \( \mathbb{Z}_p \) of integer \( p \)-adic numbers is the disk \( \{x \in \mathbb{Q}_p : |x|_p \leq 1\} \) which is the completion of integers with the \( p \)-adic norm. The main properties of disks in arbitrary ultrametric space are the following:

1. Every point of a disk is the center of this disk.
2. Two disks either do not intersect or one of these disks contains the other.

Discuss in short the results of [24], [25], [26].

The free coherent states (or shortly FCS) were introduced in [24], [25] as the formal eigenvectors of the annihilation operator \( A = \frac{1}{\sqrt{p}} \sum_{i=0}^{p-1} A_i \) in the free Fock space \( \mathcal{F} \) for some eigenvalue \( \frac{\lambda}{\sqrt{p}} \),

\[
A \Psi = \frac{\lambda}{\sqrt{p}} \Psi.
\]

(22)

The formal solution of (22) is

\[
\Psi = \sum_I \lambda^{I^I} \Psi_I A_I^\dagger \Omega.
\]

(23)
Here the multiindex $I = i_0 \ldots i_{k-1}$, $i_j \in \{0, \ldots, p-1\}$ and

$$A_I^j = A_{i_{k-1}}^i \ldots A_{i_0}^i$$

(24)

$\Psi_I$ are complex numbers which satisfy

$$\Psi_I = \sum_{i=0}^{p-1} \Psi_{I_i}.$$  

(25)

The summation in the formula (23) runs on all sequences $I$ with finite length. The length of the sequence $I$ is denoted by $|I|$ (for instance in the formula above $|I| = k$). The formal series (23) defines the functional with a dense domain in the free Fock space. For instance the domain of each free coherent state for $\lambda \in (0, \sqrt{p})$ contains the dense space $X$ introduced below.

We define the free coherent state $X_I$ of the form

$$X_I = \sum_{k=0}^{\infty} \lambda^k \left( \frac{1}{p} \sum_{i=0}^{p-1} A_i^j \right)^k \lambda^{|I|} A_I^j \Omega + \sum_{l=1}^{\infty} \lambda^{-l} \left( \sum_{i=0}^{p-1} A_i^j \right)^l \lambda^{|I|} A_I^j \Omega$$

(26)

The sum on $l$ in fact contains $|I|$ terms. For $\lambda \in (0, \sqrt{p})$ the coherent state $X_I$ lies in the Hilbert space (the correspondent functional is bounded).

We denote by $X$ the linear span of free coherent states of the form (26) and by $X'$ we denote the space of all the free coherent states (given by (23)).

The following definitions and theorems were proposed in [24], [25], [26].

**Definition 3** We define the renormalized pairing of the spaces $X$ and $X'$ as follows:

$$(\Psi, \Phi) = \lim_{\lambda \to \sqrt{p}^{-0}} \left( 1 - \frac{\lambda^2}{p} \right) \langle \Psi, \Phi \rangle$$

(27)

Here $\Psi \in X'$, $\Phi \in X$.

Note that the coherent states $\Psi, \Phi$ defined by (23), (26) depend on $\lambda$ and the product $(\Psi, \Phi)$ does not.

**Definition 4** We denote $\tilde{F}$ the completion of the space $X$ of the free coherent states with respect to the norm defined by the renormalized scalar product.

The space $\tilde{F}$ is a Hilbert space with respect to the renormalized scalar product.

**Theorem 5** The space of the free coherent states

$$X \xrightarrow{i} \tilde{F} \xrightarrow{j} X'$$

(28)

is a rigged Hilbert space.

Define the characteristic functions of $p$–adic disks

$$\theta_k(x - x_0) = \theta(p^k|x - x_0|_p); \quad \theta(t) = 0, t > 1; \quad \theta(t) = 1, t \leq 1.$$  

(29)
Here \( x, x_0 \in \mathbb{Z}_p \) lie in the ring of integer \( p \)-adic numbers and the function \( \theta_k(x - x_0) \) equals to 1 on the disk \( D(x_0, p^{-k}) \) of radius \( p^{-k} \) with the center in \( x_0 \) and equals to 0 outside this disk.

We compare the rigged Hilbert spaces of the free coherent states (28) and of generalized functions over \( p \)-adic disk:

\[
\begin{align*}
D(\mathbb{Z}_p) & \xrightarrow{j'} \mathbb{L}^2(\mathbb{Z}_p) & \xrightarrow{j} \mathbb{D}'(\mathbb{Z}_p)
\end{align*}
\]

**Theorem 6**  
The map \( \phi \) defined by

\[
\phi : \ X_I \mapsto p^{|I|} \theta_{|I|}(x - I);
\]

extends to an isomorphism \( \phi \) of the rigged Hilbert spaces:

\[
\begin{align*}
\begin{array}{c}
X \\
\downarrow \phi
\end{array} & \xrightarrow{i} \begin{array}{c}
\tilde{\mathcal{F}} \\
\downarrow \tilde{\phi}
\end{array} & \xrightarrow{j} \begin{array}{c}
X' \\
\downarrow \phi'
\end{array}
\end{align*}
\]

between the rigged Hilbert space of the free coherent states (with the pairing given by the renormalized scalar product) and the rigged Hilbert space of generalized functions over \( p \)-adic disk.

Here \( I \) are \( p \)-adic numbers corresponding to multiindices: for multiindex \( I = i_0 \ldots i_{k-1} \) corresponding \( p \)-adic number (which we denote by the same symbol) equals to

\[
I = \sum_{j=0}^{k-1} i_j p^j
\]

Definition (22) of the space of FCS:

\[
\left( A - \frac{\lambda}{\sqrt{p}} \right) \Psi = 0
\]

may be interpreted as the equation of the noncommutative (or quantum) plane \( A = \frac{\lambda}{\sqrt{p}} \). The free coherent state \( \Psi \) in this equation corresponds to a generalized function on a non–commutative space (with non–commutative coordinates \( A_i, A_i^\dagger \)) with support on the non–commutative plane \( A = \frac{\lambda}{\sqrt{p}} \) for \( A = \frac{\lambda}{\sqrt{p}} \sum_{i=0}^{p-1} A_i \).

The theorem above means that the space of generalized functions over the non–commutative plane is isomorphic as a rigged Hilbert space to the space of generalized functions over a \( p \)-adic disk, or roughly speaking the non–commutative plane is equivalent to a \( p \)-adic disk.

Let us note that

\[
\frac{\lambda}{\sqrt{p}} = 1
\]

is the maximal possible value of \( \lambda \) (the threshold). For \( \frac{\lambda}{\sqrt{p}} > 1 \) any vector (23) has an infinite norm and therefore does not lie in the Hilbert space. We see that ultrametricity arise for the maximal value of the order parameter.

Note that if we consider the maximal eigenvalue \( \frac{\lambda}{\sqrt{p}} = 1 \) and take \( A = \frac{\lambda}{\sqrt{p}} = 1 \), then for \( Q = A + A^\dagger \) we obtain \( Q = 2 \). The Wigner semicircle distribution for \( Q \) is supported in the interval \([-2, 2]\). We see again that the limit \( \lambda \to \sqrt{p} - 0 \) corresponds to the order parameter \( Q \) tending to the maximum.
6 The \( p \)-adic representation of the Cuntz algebra

In the present section we construct the representation of the Cuntz algebra in the space \( L^2(Z_p) \) of quadratically integrable functions on a \( p \)-adic disk. We will call this representation the \( p \)-adic representation. Equivalent representations (without application of \( p \)-adic analysis) were considered in [33].

Let us define the following operators in \( L^2(Z_p) \)

\[
A_i^\dagger \xi(x) = \sqrt{p} \theta_1(x - i) \xi(\frac{1}{p} x); \quad (30)
\]

\[
A_i \xi(x) = \frac{1}{\sqrt{p}} \xi(i + px). \quad (31)
\]

Here

\[
[x] = x - x(\mod 1)
\]

for \( x \in Q_p \) is the integer part of \( x \). \( \theta_1(x - i) \) is an indicator (or characteristic function) of the \( p \)-adic disk with the center in \( i \) and the radius \( p^{-1} \).

We have the following

**Theorem 7** The operators \( A_i^\dagger \) and \( A_i \) defined by (30) and (31) are mutually adjoint and satisfy the relations of the Cuntz algebra (18), (21):

\[
A_i A_j^\dagger = \delta_{ij},
\]

\[
\sum_{i=0}^{p-1} A_i^\dagger A_i = 1.
\]

Let us define the linear functional \( \langle \cdot \rangle \) on the Boltzmann algebra as follows

\[
\langle A_i^\dagger A_j \rangle = p^{-\frac{1}{2}(|I|+|J|)} \quad (32)
\]

With \( A_i^\dagger \) defined by (24) and \( A_I \) adjoint.

In the present section we prove that the considered in the previous section the \( p \)-adic representation of the Boltzmann algebra is unitary equivalent to the GNS representation generated by the state \( \langle \cdot \rangle \).

**Theorem 8**

1) The functional \( \langle \cdot \rangle \) is a state.

2) In the corresponding GNS representation the condition (21) is satisfied.

3) The corresponding GNS representation is unitarily equivalent to the representation realized in the space of (quadratically integrable) functions on \( p \)-adic disk by the formula (31):

\[
A_i \xi(x) = \frac{1}{\sqrt{p}} \xi(i + px). \]
We will construct the $p$–adic representation of the Cuntz algebra by regularization of action of operators from the quantum Boltzmann algebra on the free coherent states.

Let us introduce some notations. Free coherent state (FCS) $\Phi$ is given by

$$\Phi = \sum_{I} \Phi_{I} \lambda^{I|} A_{I}^\dagger \Omega.$$ 

Here $\Omega$ is the vacuum vector in the free Fock space, $\Phi_{I}$ satisfies (25). We will use the notation

$$\Phi = A_{\Phi}^\dagger \Omega.$$ 

Let us introduce the representation $T$ of the Cuntz algebra in the space of FCS.

**Theorem 9** The regularizations $T_i = T(A_i^\dagger)$, $T_i = T(A_i)$ of the right shift operators on FCS

$$T_i \Phi = \frac{1}{\sqrt{p}} \left( \lambda A_{\Phi}^i A_{i}^\dagger + \Phi_0 \right) \Omega;$$

$$T_i \Phi = \sqrt{p} \sum_{I} \Phi_{Ii} \lambda^{I|} A_{I}^\dagger \Omega = \sqrt{p} \lambda^{-1} \left( A_{i} A_{\Phi}^\dagger \right)^{op} \Omega.$$ 

where

$$\left( A_{i_{k-1}}^\dagger ... A_{i_0}^\dagger \right)^{op} = A_{i_0}^\dagger ... A_{i_{k-1}}^\dagger.$$ 

map the space of FCS into itself and define the representation of the Cuntz algebra. Moreover, the representation of the Cuntz algebra defined by the operators $T_i$, $T_i^\dagger$ is unitarily equivalent to the $p$–adic representation.

**Theorem 10** $p$–Adic representation is the GNS representation generated by the state on the space of FCS

$$\langle X \rangle = (1, X 1)_{L^2(Z_p)} = (\hat{1}, AF(X) \hat{1})$$

Here

$$\hat{1} = \sum_{k=0}^{\infty} \lambda^k \left( \frac{1}{p} \sum_{i=0}^{p-1} A_i^\dagger \right)^{k} \Omega$$

is the coherent state corresponding to the indicator of $Z_p$ and $AF$ is the antifock representation.

### 7 Phase transition and coherent states

In the present paper we consider phase transition in non–commutative (quantum) system with quantum Boltzmann statistics.

Let us remind some properties of phase transitions. To describe phase transitions in quantum systems we use quantum (or algebraic) probability space — the algebra of observables $A$ with a state (or expectation) $\langle \cdot \rangle$ on this algebra. In the Gibbs case this state is equal to

$$\langle X \rangle = \operatorname{tr} e^{-\beta H} X$$

where $\operatorname{tr}$ is a trace on the algebra of observables, $H$ is an element of the algebra of observables (an Hamiltonian). This state depends on the control parameter $\beta$ (inverse temperature). For
different physical models we can have different control parameters. For example, for the models of lasers the typical control parameter is the intensity of the pumping field.

Using state $|3\rangle$ we construct the statistical sum

$$Z = \langle 1 \rangle = \text{tr} e^{-\beta H}$$

the free energy

$$F = -\beta^{-1} \ln Z$$

and other thermodynamical potentials.

If the dependence of thermodynamical potential on $\beta$ is non–smooth, we say that for such $\beta$ we have phase transition. For phase transitions we have an effect that the expectation of certain observables, called the order parameters, which are equal to zero for the temperature above the phase transition becomes non–zero after the phase transition. The typical example is the spontaneous magnetization.

To describe such effects as spontaneous symmetry breaking at the phase transition N.N.Bo
goliubov introduced the concept of quasiexpectation $|32\rangle$. We say that the state $\langle \cdot \rangle_1$ is a quasi-
expectation for the expectation $\langle \cdot \rangle$ if there exists an operator $H_1$ such that

$$\langle X \rangle_1 = \lim_{\mu \to 0} \lim_{N \to \infty} \text{tr} e^{-\beta (H + \mu H_1)} X$$

where the limit $N \to \infty$ is the thermodynamic limit. Note that the limits above do not commute, which makes the definition of quasiexpectation nontrivial.

Examples of phase transitions in quantum mechanical systems are the Bose condensation and lasers. For these cases the algebra of observables $A$ is the algebra of canonical commutation relations, or the CCR algebra (of creation and annihilation operators) with the relation

$$[a(k), a^\dagger(k')] = \delta(k - k')$$

Let us consider the situation when one of the annihilators (say $a = a(0)$) is the order parameter and due to the phase transition the expectation of it becomes non–zero:

$$\langle a \rangle = \text{const} \neq 0$$

To describe this effect one introduces the coherent states $\psi$ which are the eigenstates of the annihilation operator

$$a \psi = \lambda \psi$$

and considers representation of the CCR algebra generated from the state $\langle \psi, \cdot \psi \rangle$.

In the present paper we extend the outlined above procedure to the algebra with quantum Boltzmann relations and apply it to describe the phase transition in disordered system. This will allow us to develop the approach to phase transitions in disordered systems which will be an alternative to the replica symmetry breaking approach.

## 8 The noncommutative replica symmetry breaking

In the standard replica approach the phase transition in disordered system is described with the help of the replica symmetry breaking $|13\rangle$. In the noncommutative replica approach
the phase transition will be described by noncommutative replica symmetry breaking procedure, which involves the free coherent states. Since the space of FCS (free coherent states) and the space of generalized functions on $p$–adic disk are equivalent, this allows us to derive the ultrametric space of states which is postulated in the standard replica approach.

For the high temperature regime the state of the Sherrington–Kirkpatrick model is described by the expectations of large matrices which according to the Wigner theorem give rise to the vacuum expectation on the quantum Boltzmann algebra in the free Fock space. The procedure described by the theorems 1 and 2 allows to describe this state equivalently using the Fock state over the quantum Boltzmann algebra with $p$ degrees of freedom. In the high temperature regime the expectation of the order parameter is equal to zero:

$$\langle A \rangle = 0, \quad A = \frac{1}{\sqrt{p}} \sum_{i=0}^{p-1} A_i$$

which corresponds to the noncommutative replica symmetry.

After the phase transition in a system with the quantum Boltzmann statistics, analogously with the Bose condensation and models of lasers (see the discussion of the previous section), the state will have the property

$$\langle A \rangle = \text{const} \neq 0$$

which we call the noncommutative replica symmetry breaking condition.

The states satisfying (37) are related to the free coherent states. The space of the free coherent states is isomorphous to the space of generalized functions over $p$–adic disk (theorem 6), see [24], [25], [26], and the correspondent representation of the quantum Boltzmann algebra is realized in $L^2(Z_p)$ (see the theorems 9 and 10) and is described by theorem 7. This representation satisfies (37) and is not unitarily equivalent to the Fock representation. Similar representations were considered by Bratteli, Yorgensen et al [33] (without the application of $p$–adic analysis).

Let us note that in the noncommutative replica approach the operator $Q = A + A^\dagger$ is the analogue of the Parisi matrix, and (37) corresponds to nonzero expectation of the Edwards–Anderson order parameter.

Let us formulate the following conjecture, describing the state of the disordered system after the phase transition.

**Conjecture 11**  
The phase transition in the Sherrington–Kirkpatrick model (and also in a wide class of disordered models) is described by the state \((32)\) on the quantum Boltzmann algebra generated by annihilators $A_i$, $i = 0,\ldots,p-1$. The correspondent representation of the quantum Boltzmann algebra is given by (30)–(31). The order parameter $Q$ we take to be equal to $Q = A + A^\dagger$, where $A$ is the following linear combination of the quantum Boltzmann annihilators:

$$A = \frac{1}{\sqrt{p}} \sum_{i=0}^{p-1} A_i$$

In the noncommutative replica approach the order parameter $Q$ arise from the limit of large stochastic matrix $J_{ij}$. We did not introduce any replica matrices.

The ultrametric picture of the state describing the phase transition in disordered system, which is sometimes discussed as a result of infinite number of phase transitions, in the noncommutative
replica approach follows from one, but non–commutative, phase transition (the phase transition is non–commutative, or quantum, when the order parameter is an operator).

The introduced conjecture allows us to compute the correlation functions of the SK model, which are defined as follows:

\[
q^{(1)} = \frac{1}{N} \sum_i \langle \sigma_i \rangle^2 \\
q^{(2)} = \frac{1}{N^2} \sum_{i,j} \langle \sigma_i \sigma_j \rangle^2
\]

and correspondingly

\[
q^{(k)} = \frac{1}{N^k} \sum_{i_1 \ldots i_k} \langle \sigma_{i_1} \ldots \sigma_{i_k} \rangle^2
\]

The standard replica approach gives for the correlators above the following expression, see [14]

\[
q^{(k)} = \lim_{n \to 0} \frac{2}{n(n-1)} \sum_{a<b} [Q_{ab}]^k
\]

where \( Q_{ab} \) is the Parisi matrix.

Introduce the matrix \( Q^{(k)} \) with the matrix elements

\[
Q^{(k)}_{ab} = [Q_{ab}]^k
\]

For instance \( Q^{(1)} \) is the Parisi matrix.

Formula (38) means that (up to renormalization and the limit \( n \to 0 \)), the order parameter is given by

\[
q^{(k)} = (\hat{1}, Q^{(k)} \hat{1})
\]

where \( \hat{1} \) is the vector with all the coordinates equal to one.

The noncommutative replica approach suggests the following conjecture.

**Conjecture 12** The correlator \( q^{(k)} \) in the noncommutative replica approach is approximated by the following matrix element in the renormalized square product

\[
q^{(k)} = (\hat{1}, (cQ)^k \hat{1})
\]

where \( \hat{1} \) is the free coherent state corresponding to the indicator of unit \( p \)–adic disk,

\[
Q = A + A^\dagger, \quad A = \frac{1}{\sqrt{p}} \sum_{i=0}^{p-1} A_i
\]

and \( c \) is a constant.

By (30), (31) expression (40) reduces to

\[
q^{(k)} = (2c)^k \int_{Z_p} d\mu(x) = (2c)^k \mu(V)
\]

where \( V \) in the considered case is the unit disk (with the measure 1). In principle we may consider an equivalent representation acting on functions on a disk of different measure. We see that in this approach the correlators of the SK model reduce to \( p \)–adic integrals.
9 Conclusion

In the present paper we develop the noncommutative replica approach to phase transitions in disordered systems. Summing up, this approach looks as follows.

First, by the Wigner theorem, the Sherrington–Kirkpatrick model in the high temperature regime is described by the quantum Boltzmann algebra in the free Fock representation. Note that the high temperature regime here corresponds to the vacuum (zero temperature) state of the quantum Boltzmann algebra. Then, we apply the transformation from 1 to \( p \) degrees of freedom which is a morphism of quantum probability spaces. We call this transformation the Non–commutative Replica Procedure. In the noncommutative replica approach we replace the statistic sum \( Z \) of the disordered system by the noncommutative replica \( Z^{(p)} \) of the statistic sum, given by (3).

Second, we describe phase transition by putting the quantum Boltzmann algebra into the state with non–zero expectation of the linear combination of the annihilators \( \langle A \rangle \neq 0, A = \frac{1}{\sqrt{p}} \sum_{i=0}^{p-1} A_i \). We call the condition \( \langle A \rangle \neq 0 \) the condition of breaking of the noncommutative replica symmetry. To construct such a state we use the free coherent states and the correspondent \( p \)–adic representation of the Cuntz algebra. This representation acts on functions on the quantum line which is equivalent to \( p \)–adic disk.

Third, we conjecture that the \( k \)–th correlator of the SK model is equal to the expectation of the \( k \)–th degree of the order parameter \( Q = A + A^\dagger, A = \frac{1}{\sqrt{p}} \sum_{i=0}^{p-1} A_i \) in the \( p \)–adic representation. This means that the corresponding correlators will be given by \( p \)–adic integrals.

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