Multiplet Classification of Reducible Verma Modules over the $G_2$ Algebra

V.K. Dobrev
Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria; dobrev@inrne.bas.bg

Abstract. In the present paper we continue the project of systematic construction of invariant differential operators on the example of the non-compact algebra $G_2(2)$ which is split real form of $G_2$. We give the classification of reducible Verma modules $G_2$. We give also the singular vectors between these modules, thus setting the stage for construction of the invariant differential operators over $G_2(2)$.

Dedicated to I.E. Segal (1918-1998) in commemoration of the centenary of his birth. The author remembers with great pleasure the talk he gave at Segal’s seminar at MIT in 1975.

1. Introduction
Invariant differential operators play very important role in the description of physical symmetries. The general scheme for constructing these operators was given some time ago [1]. In recent papers [2,3] we started the systematic explicit construction of the invariant differential operators.

The first task in the construction is to make the multiplet classification of the reducible Verma modules over the algebra in consideration following [4]. Such classification provides the weights of embeddings between the Verma modules via the singular vectors, and thus, by [1], the weights of the invariant differential operators.

We have done the multiplet classification for many real non-compact algebras, first from the class of algebras that have discrete series representations, see [5]. In the present paper we focus on the complex algebra $G_2$ and on its split real form algebra $G_2(2)$. We present the multiplet classification of the reducible of Verma modules over $G_2$. We give also the singular vectors between these modules. By the scheme of [1] these explicit expressions produce the invariant differential operators.

This paper is a sequel of [2] and [3] and due to the lack of space we refer to these papers for motivations and [5] for extensive list of literature on the subject. For other approaches and applications of $G_2$, see, e.g., [6].
2. Preliminaries

2.1. Lie algebra

We start with the complex Lie algebra $G^C = G_2$. We use the standard definition of $G^C$ given in terms of the Chevalley generators $X^\pm_i$, $H_i$, $i=1,2$, by the relations:

$$[H_j, H_k] = 0, \quad [H_j, X^\pm_k] = \pm a_{jk} X^\pm_k, \quad [X^+_j, X^-_k] = \delta_{jk} H_j,$$

$$\sum_{m=0}^{n} (-1)^m \binom{n}{m} \left(X_j^+ \right)^m \left(X_k^- \right)^{n-m} = 0, \quad j \neq k, \quad n = 1 - a_{jk},$$

where

$$(a_{jk}) = (\alpha_j^\vee, \alpha_k) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

is the Cartan matrix of $G^C$, $\alpha_j^\vee \equiv \frac{2\alpha_j}{(\alpha_j, \alpha_j)}$ is the co-root of $\alpha_j$, $(\cdot, \cdot)$ is the scalar product of the roots, so that the nonzero products between the simple roots are: $(\alpha_1, \alpha_1) = 6$, $(\alpha_2, \alpha_2) = 2$, $(\alpha_1, \alpha_2) = -3$. The elements $H_i$ span the Cartan subalgebra $H$ of $G^C$, while the elements $X_i^\pm$ generate the subalgebras $G^\pm$. We shall use the standard triangular decomposition

$$G^C = G_+ \otimes H \oplus G_-, \quad G_\pm \equiv \bigoplus_{\alpha \in \Delta^\pm} G_\alpha,$$

where $\Delta^+$, $\Delta^-$, are the sets of positive, negative, roots, resp. Explicitly we have:

$$\Delta^+ = \{ \alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2 \}$$

Thus, $G_2$ is 14–dimensional ($14 = |\Delta| + \text{rank } G_2$).

For the simple roots we may choose in terms of ortho-normal basis $\varepsilon_1, \varepsilon_2, \varepsilon_3$:

$$\alpha_1 = \varepsilon_1 + \varepsilon_3 - 2\varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3$$

For future reference we introduce notation for the non-simple roots:

$$\alpha_3 \equiv \alpha_1 + \alpha_2 = \varepsilon_1 - \varepsilon_2, \quad \alpha_4 \equiv \alpha_1 + 2\alpha_2 = \varepsilon_1 - 3\varepsilon_3,$$

$$\alpha_5 \equiv \alpha_1 + 3\alpha_2 = \varepsilon_1 + \varepsilon_2 - 2\varepsilon_3, \quad \alpha_6 \equiv 2\alpha_1 + 3\alpha_2 = 2\varepsilon_1 - \varepsilon_2 - \varepsilon_3$$

With the chosen normalization the roots $\alpha_1$, $\alpha_5 = \alpha_1 + 3\alpha_2$, $\alpha_6 = 2\alpha_1 + 3\alpha_2$ have length 6, while $\alpha_2$, $\alpha_3 = \alpha_1 + \alpha_2$, $\alpha_4 = \alpha_1 + 2\alpha_2$ have length 2. The dual roots are:

$$\alpha_1^\vee = \alpha_1/3, \quad \alpha_2^\vee = \alpha_2, \quad \alpha_3^\vee = \left(\alpha_1 + \alpha_2\right)^\vee = \alpha_1 + \alpha_2 = 3\alpha_1^\vee + \alpha_2^\vee,$$

$$\alpha_4^\vee = \left(\alpha_1 + 2\alpha_2\right)^\vee = \alpha_1 + 2\alpha_2 = 3\alpha_1^\vee + 2\alpha_2^\vee,$$

$$\alpha_5^\vee = \left(\alpha_1 + 3\alpha_2\right)^\vee = \left(\alpha_1 + 3\alpha_2\right)/3 = \alpha_1^\vee + \alpha_2^\vee,$$

$$\alpha_6^\vee = \left(2\alpha_1 + 3\alpha_2\right)^\vee = \left(2\alpha_1 + 3\alpha_2\right)/3 = 2\alpha_1^\vee + 2\alpha_2^\vee$$

Note that the roots $\alpha_2, \alpha_3, \alpha_4$ form the $sl(3)$ root system, the first two being the two simple roots. The roots $\alpha_1, \alpha_5, \alpha_6$ also form the $sl(3)$ root system, the standard normalization being achieved after rescaling each root by factor $\sqrt{3}$. Note also the cases: $(\alpha_1, \alpha_4) = 0$, $(\alpha_2, \alpha_6) = 0$, $(\alpha_3, \alpha_5) = 0$.

The Weyl group $W(G_2)$ of $G_2$ is the dihedral group of order 12. This follows from the fact that $(s_1 s_2)^6 = 1$, where $s_1, s_2$ are the two simple reflections. Using the general formula:

$$s_\alpha(\lambda) = \lambda - \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha,$$

$$\lambda \in H^*, \quad \alpha \in \Delta, \quad \alpha^\vee \equiv \frac{2}{\langle \alpha, \alpha \rangle} \alpha,$$
we have the following action of the simple reflections: 

\[ s_i \equiv s_{\alpha_i}, \]

\[ s_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = (-\alpha_1, \alpha_3, \alpha_2, \alpha_4, \alpha_6, \alpha_5) \tag{9} \]

\[ s_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = (\alpha_5, -\alpha_2, \alpha_4, \alpha_3, \alpha_1, \alpha_6) \tag{10} \]

The 12 elements of \( W(G_2) \) are given in terms of the simple reflections as follows:

\[ W(G_2) = \{1, s_1, s_2, s_1s_2, s_2s_1, s_2s_1s_2, s_2s_1s_2s_1, \]

\[ s_1s_2s_1s_2s_1, s_2s_1s_2s_1s_2, s_1s_2s_1s_2s_1s_2 = s_2s_1s_2s_1s_2s_1 \} \tag{11} \]

Note the expressions for reflections corresponding to non-simple roots:

\[ s_{\alpha_3} = s_1s_2s_1, \quad s_{\alpha_4} = s_2s_1s_2s_1s_2, \quad s_{\alpha_5} = s_2s_1s_2, \quad s_{\alpha_6} = s_1s_2s_1s_2s_1 \tag{12} \]

Let us denote the root space vector of \( G_\alpha \) by \( X_\alpha \), or more explicitly: \( X^\pm_k \equiv X_{\pm \alpha_k} \), \( k = 1, ..., 6 \). To give the full Cartan-Weyl basis we need to define also \( X^\pm_k \), \( k = 3, ..., 6 \), for which we follow [1]:

\[ X_3^\pm = \pm[X_1^\pm, X_2^\pm], \quad X_4^\pm = \pm[X_2^\pm, X_3^\pm], \tag{13} \]

\[ X_5^\pm = \pm \frac{1}{\sqrt{3}}[X_4^\pm, X_2^\pm], \quad X_6^\pm = \pm[X_1^\pm, X_5^\pm] \]

Then we have for \( H_k \equiv [X^+_k, X^-_k] \):

\[ H_3 = 3H_1 + H_2, \quad H_4 = 3H_1 + 2H_2, \quad H_5 = H_1 + H_2, \quad H_6 = 2H_1 + H_2 \tag{14} \]

(compare with (7)).

Note that for \( H_k \) also holds:

\[ \lambda(H_k) = (\lambda, \alpha_k^\vee), \quad \forall \lambda \in \mathcal{H}^*, \quad k = 1, ..., 6. \tag{15} \]

2.2. Structure theory of the real form

The split real form of \( G_2 \) is denoted as \( G'_2 \), sometimes as \( G_{2(2)} \). This real form has quaternionic discrete series representations. We can use the same basis (but over \( \mathbb{R} \)) and the same root system.

The Iwasawa decomposition of the real split form \( G \equiv G'_2 \), is:

\[ G = K \oplus A_0 \oplus N_0, \tag{16} \]

the Cartan decomposition is:

\[ G = K \oplus Q, \tag{17} \]

where we use: the maximal compact subgroup \( K \cong su(2) \oplus su(2) \), \( \dim_{\mathbb{R}} Q = 8 \), \( \dim_{\mathbb{R}} A_0 = 2 \), \( N_0 = N_0^+ \), or \( N_0 = N_0^- \cong N_0^+ \), \( \dim_{\mathbb{R}} N_0^- = 6 \).

Since \( G \) is maximally split, then the centralizer \( A_0 \) of \( A_0 \) in \( K \) is zero, thus, the minimal parabolic \( P_0 \) and the corresponding Bruhat decomposition are:

\[ P_0 = A_0 \oplus N_0, \quad G = A_0 \oplus N_0^+ \oplus N_0^- \tag{18} \]

The importance of the parabolic subgroups comes from the fact that the representations induced from them generate all (admissible) irreducible representations of the group under consideration [7–9].
3. Verma modules and singular vectors

Let us recall that a Verma module $V^\Lambda$ is defined as the highest weight module over $G^C$ with highest weight $\Lambda \in H^*$ and highest weight vector $v_0 \in V^\Lambda$, induced from the one-dimensional representation $V_0 \cong C v_0$ of $U(B)$, where $B = H \oplus G_+$ is a Borel subalgebra of $G^C$, such that:

\begin{align*}
X v_0 &= 0, \quad \forall X \in G_+ \\
H v_0 &= \Lambda(H) v_0, \quad \forall H \in H
\end{align*}

(19)

Verma modules are generically irreducible. A Verma module $V^\Lambda$ is reducible [10] iff there exists a root $\beta \in \Delta^+$ and $m \in \mathbb{N}$ such that:

\begin{align*}
(\Lambda + \rho, \beta^\vee) &= m
\end{align*}

(20)

holds, where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. If (20) holds then the reducible Verma module $V^\Lambda$ contains an invariant submodule which is also a Verma module $V^{\Lambda'}$ with shifted weight $\Lambda' = \Lambda - m\beta$. This statement is equivalent to the fact that $V^\Lambda$ contains a singular vector $v_s \in V^\Lambda$, such that $v_s \neq \xi v_0, \ (0 \neq \xi \in C)$, and:

\begin{align*}
X v_s &= 0, \quad \forall X \in G_+ \\
H v_s &= \Lambda'(H) v_s, \quad \forall H \in H
\end{align*}

(21)

The general reducibility conditions (20) for $V^\Lambda$ spelled out for the six positive roots in our situation are:

\begin{align*}
m_1 &= m_1(\Lambda) \equiv \Lambda(H_1) + 1 \in \mathbb{N} , \quad (22a) \\
m_2 &= m_2(\Lambda) \equiv \Lambda(H_2) + 1 \in \mathbb{N} , \quad (22b) \\
m_3 &= m_3(\Lambda) \equiv \Lambda(H_3) + 4 = 3m_1 + m_2 \in \mathbb{N} , \quad (22c) \\
m_4 &= m_4(\Lambda) \equiv \Lambda(H_4) + 5 = 3m_1 + 2m_2 \in \mathbb{N} , \quad (22d) \\
m_5 &= m_5(\Lambda) \equiv \Lambda(H_5) + 2 = m_1 + m_2 \in \mathbb{N} , \quad (22e) \\
m_6 &= m_6(\Lambda) \equiv \Lambda(H_6) + 3 = 2m_1 + m_2 \in \mathbb{N} . \quad (22f)
\end{align*}
The singular vectors corresponding to these cases are:

\[ v^{a_1,m_1} = (X_1)^{m_1} v_0, \quad m_1 \in \mathbb{N} , \]  
\[ v^{a_2,m_2} = (X_2)^{m_2} v_0, \quad m_2 \in \mathbb{N} , \]  
\[ v^{a_3,m_3} = \sum_{k=0}^{m_3} g_{ak} X_1^{-m_3-k} X_2^{-m_3} v_0, \quad m_3 \in \mathbb{N} , \]  
\[ v^{a_4,m_4} = \sum_{k=0}^{m_4} \sum_{i=0}^{2m_4-i} g_{aik} X_1^{-i} X_2^{-m_4-i-k} v_0 , \quad m_4 \in \mathbb{N} , \]  
\[ v^{a_5,m_5} = \sum_{k=0}^{3m_5} g_{ak} X_2^{-3m_5-k} X_1^{-m_5} v_0, \quad m_5 \in \mathbb{N} , \]  
\[ v^{a_6,m_6} = \sum_{k=0}^{3m_6} \sum_{i=0}^{2m_6-i} g_{aik} X_1^{-i} X_2^{-3m_6-k} v_0 , \quad m_6 \in \mathbb{N} . \]  

(Note that in each of the six cases (23) only the relevant \( m_j \) must be a natural number (as displayed).) Formulæ (23a,b) are general for any simple root \([11],[1]\), while (23c,d) were given first in [12].

Certainly, (20) may be fulfilled for several positive roots, even for all of them if (20) is fulfilled for the two simple roots.

4. Classification of \( G_2 \) Verma modules

Here we classify the Verma modules over \( G^C = G_2 \). This also provides the classification of the \( P_0 \)-induced ERs since the restricted Weyl group \( W(\mathcal{G}, \mathcal{A}_0) \) related to the minimal parabolic subalgebra \( \mathcal{P}_0 = \mathcal{A}_0 \mathcal{V}_0 \), cf. (18), is isomorphic to the Weyl group \( W(\mathcal{G}^C, \mathcal{A}_0^C) \) (since \( \mathcal{G} \) is maximally split).

The classification is done as follows. We group the reducible Verma modules related by nontrivial embeddings in sets called multiplets [1,4]. These multiplets may be depicted as a connected graph, the vertices of which correspond to the reducible Verma modules and the lines between the vertices correspond to the embeddings. The explicit parametrization of the multiplets and of their Verma modules is important for understanding of the situation.

The classification can be summarized as follows. There are four main types of multiplets of reducible Verma modules:

- type \( F_{m_1,m_2} \), \((m_1,m_2) \in \mathbb{N} \);
- type \( N \) with six subtypes: \( N_k, k = 1, \ldots, 6 \);
- type \( L_m \), with two subtypes \( L_m^k, k = 1, 2, (m \in \mathbb{N}) \);
- type \( S_{j,k} \), \((m_j,m_k) \in \mathbb{N}, 1 \leq j < k \leq 6, (j,k) \neq (1,2) \).

Multiplets of type \( F_{m_1,m_2} \) are parametrized by two natural numbers \( m_1, m_2 \). They are given in Fig. 1A and Fig. 1B where we have given the multiplets in two ways: in Fig. 1A the Verma
modules are given by their highest weights, while on Fig. 1B they are given by the two Dynkin labels. In Fig. 1A we have indicated w.r.t. which reflection is the embedding, on Fig. 1B we have given the weight $m\beta$ of the embedding. All Verma modules of these multiplets, except $V^{\Lambda'}$, are reducible. Note that only embeddings which are not compositions of other embeddings are given on the Figures.

![Fig. 1A and 1B](image)

We note also some additional relations using notation from Fig. 1:

\[
\begin{align*}
\Lambda^1 &= \Lambda - m_1\alpha_1, & \Lambda^2 &= \Lambda - m_2\alpha_2, & \Lambda^{121} &= \Lambda - m_3\alpha_3, & \Lambda^{212} &= \Lambda - m_5\alpha_5, \\
\Lambda^{12121} &= \Lambda - m_6\alpha_6, & \Lambda^{21212} &= \Lambda - m_4\alpha_4
\end{align*}
\]

Multiplets of type $\mathcal{N}$ are given as follows. Fix $k = 1, \ldots, 6$ to fix a subtype $\mathcal{N}_k$. Then the multiplets of this subtype are parametrized by the natural number $m_k$ and are given as follows:

\[
V^{\Lambda_k} \rightarrow V^{\Lambda_k - m_k\alpha_k}, \quad m_k(\Lambda_k) = m_k \in \mathbb{N}, \quad m_j(\Lambda_k) \notin \mathbb{N}, \quad j \neq k.
\]

Note that we are using the convention that the arrows point to the embedded modules. The modules $V^{\Lambda_k - m_k\alpha_k}$ are irreducible.

For the multiplets of type $\mathcal{L}$ there are two subtypes $\mathcal{L}_k$, $k = 1, 2$ each parametrized by a natural number $m$ and given as follows.

The multiplets of subtype $\mathcal{L}_1$ are given on Fig. 2 below. In each multiplet there are six Verma modules which we give by the Dynkin labels. We also give the weights of the singular vectors between the Verma modules. The last module on the right is irreducible.

![Fig. 2](image)

The multiplets of subtype $\mathcal{L}_2$ are given on Fig. 3 below. They are similar to subtype $\mathcal{L}_1$, e.g., also here the last module on the right is irreducible.
For the lack of space we give only some examples of the multiplets of type $S_{j,k}$.

Multiplets of type $S_{1,4}$ are parametrized by two natural numbers $m_1, m_4$ so that $m_2 = \frac{1}{2}(m_4 - 3m_1) \notin \mathbb{Z}$; then also $m_3, m_5, m_6 \notin \mathbb{Z}$. They are given in the Fig. 4 below, where as above we have given the multiplet in two ways, and again the parametrizing numbers $m_1, m_4$ are related to the Verma module $V^\Lambda^s$ on the top: $m_k = m_k(\Lambda^s)$, $k = 1, 4$. The Verma modules of these multiplets, except $V^\Lambda''$, are reducible and their weights are given explicitly as follows:

$$\Lambda^s, \quad m_k = m_k(\Lambda^s) \in \mathbb{N}, \quad k = 1, 4, \quad \Lambda^s_k = \Lambda^s - m_k\alpha_k, \quad k = 1, 4.$$ \hspace{1cm} (26)

The weights of the irreducible modules $V^{\Lambda'}$ are: $\Lambda' = \Lambda^s - m_1\alpha_1 - m_4\alpha_4$.

Multiplets of type $S_{2,6}$ are parametrized by two natural numbers $m_2, m_6$, so that $m_1 = \frac{1}{2}(m_6 - m_2) \notin \mathbb{Z}$. Here we have two subcases: a) $3m_1 = \frac{3}{2}(m_6 - m_2) \notin \mathbb{Z}$, then also $m_3, m_4, m_5 \notin \mathbb{Z}$; b) $3m_1 = \frac{3}{2}(m_6 - m_2) \in \mathbb{N}$, then $m_3, m_4 \in \mathbb{N}, m_5 \notin \mathbb{Z}$.

Subcase a) is given in the Fig. 6 below, and we omit comments since this case is similar to Fig. 5.

Subcase b) contains eight Verma modules as given in Fig. 6 below. Here we use a mixture of notation since the Dynkin labels may be seen from comparing Fig. 1 and Fig. 2. Here there are two irreducible modules: $\Lambda'$ and $\Lambda^{21212}$ (the second one since $m_1 \notin \mathbb{N}$).
Multiplets of type $S_{2,4}$ are parametrized by two natural numbers $m_2, m_4$, so that $m_1 = \frac{1}{4}(m_4 - 2m_2) \notin \mathbb{N}$, while $m_3 = m_4 - m_2 \in \mathbb{N}$, $m_5, m_6 \notin \mathbb{N}$. These multiplets are given in the Fig. 7 below:

Note that this multiplet is the standard $sl(3)$ multiplet with $\alpha_2, \alpha_3$ playing the role of the two simple roots. Incidentally, it is a submultiplet of the previous case.

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