TRAVELING WAVES IN A THREE SPECIES
COMPETITION-COOPERATION SYSTEM

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ABSTRACT. This paper studies the traveling wave solutions to a three species
competition cooperation system. The existence of the traveling waves is investi-
gated via monotone iteration method. The upper and lower solutions come
from either the waves of KPP equation or those of certain Lotka Volterra
system. We also derive the asymptotics and uniqueness of the wave solutions.
The results are then applied to a Lotka Volterra system with spatially averaged
and temporally delayed competition.

1. INTRODUCTION

We study the traveling wave solutions of the following three species competition
cooperation system

\[
\begin{align*}
&u_t = u_{xx} + u(1 - u - a_1 w), \\
v_t = v_{xx} + rv(1 - a_2 u - v), \\
w_t = w_{xx} + \frac{1}{\tau}(v - w),
\end{align*}
\]

where \( u(x, t), v(x, t) \) and \( w(x, t) \) stand for the population densities of the three
different species, \( a_i > 0 \) is interaction constant, \( i = 1, 2 \) and \( r > 0 \) \((-\frac{1}{2} < 0, \) respectively) is the relative intrinsic growth rate of the species \( v \) \((w, \) respectively). Aside from the intra-specific competitions, system (1.1) describes the relation that the species \( w \) competes with \( u \) and \( u \) competes with \( v \), while \( v \) cooperates with \( w \).

The purpose of our study is of two folds: there are less results (see \[17, 9\]) on
the traveling wave solutions to the three species systems even for one with simple
form as \( \text{(1.1)} \); on the other hand we would like to extend the results of \[4\] from the
tempo-spatial delayed KPP (Kolmogorov, Petrovsky and Piscounov) equation to
the following Lotka Volterra competition system with spatial temporal delay

\[
\begin{align*}
&u_t = u_{xx} + u(1 - u - a_1 g \ast v), \\
v_t = v_{xx} + rv(1 - a_2 u - v),
\end{align*}
\]

where the function \( g \ast v = \int_{-\infty}^{+\infty} \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}} e^{-\frac{(t-s)}{\tau}} v(s) ds dy \) represents the tempo-spatial delay of the response of species \( v \) to \( u \), here \( g(x, t) = \frac{1}{\sqrt{4\pi t e^{-\frac{x^2}{4t}}} e^{-\frac{t}{\tau}}} \), and satisfies

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Cooperation, Existence.
Noting that if \( g \ast v \) is replaced by \( v \) in (1.2), we recover the classical Lotka Volterra competition system, and fruitful results have been devoted in the study of traveling waves \([3, 5, 8, 7, 11, 12, 6, 19, 20]\) arising from it. It is interesting to see the long term effect of introducing spatio-temporal delay to the competition. The temporal delay accounts for the time once consumed resource the dominated species needs to wait for its re-growth, and the spatial averaging accounts for the fact that individuals are moving around and have therefore not been at the same point in space at different times in their history (see [1]).

On setting (1.4)

\[
w(x, t) = \frac{1}{\tau} \int_{-\infty}^{+\infty} \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{4(t-s)}} e^{-\frac{(\tau-c)(s-t)}{\tau}} v(s) ds dy,
\]

we easily verify that \( w(x, t) \) satisfies the following equation

(1.5)

\[
w_t = w_{xx} + \frac{1}{\tau} (v - w).
\]

System (1.2) is now recasted into (1.1). The existence of the traveling wave solution of (1.1) is equivalent to that of (1.2).

Throughout the paper we make the following assumptions:

(1.6) Condition H1: \( 0 < a_1 < 1 < a_2 \)

and either

(1.7) Condition H2a: \( r(a_2 - 1) < (1 - a_1) \)

or

(1.8) Condition H2b: \( r(a_2 - 1) \geq (1 - a_1) \geq r(a_1a_2 - 1) \)

(1.9) Condition H3: \( a_2 > 0 \) is suitably large.

Under the condition (1.6), system (1.1) admits three constant equilibria: \((0, 0, 0)\), \((0, 1, 1)\) and \((1, 0, 0)\), and the first two are unstable while the third is stable. We are interested in finding the traveling wave solutions of (1.1) connecting \((0, 1, 1)\) with \((1, 0, 0)\). Via transformation (1.4), the existence of the traveling was solutions of (1.1) connecting \((0, 1, 1)\) to \((1, 0, 0)\) is equivalent to that of (1.2) connecting \((0, 1)\) with \((1, 0)\).

A traveling wave solution for (1.1) has the form \((u(x, t), v(x, t), w(x, t)) = (u(x + ct), v(x + ct), w(x + ct)) = (u(\xi), v(\xi), w(\xi)), \xi = x + ct, \) and satisfies the system

(1.10)

\[
\begin{align*}
    u_{\xi\xi} - cu_{\xi} + u(1 - u - a_1w) &= 0, \\
    v_{\xi\xi} - cv_{\xi} + rv(1 - a_2u - v) &= 0, \\
    w_{\xi\xi} - cw_{\xi} + \frac{1}{\tau}(v - w) &= 0, \\
    (u, v, w)(-\infty) &= (0, 1, 1), \quad (w, v, w)(+\infty) = (1, 0, 0).
\end{align*}
\]
For the convenience of later study, we change (1.10) into monotone (1.19). Let \( \bar{u} = u, \bar{v} = 1 - v \) and \( \bar{w} = 1 - w \), and drop the bars on \( u, v \) and \( w \), we arrive at

\[
\begin{aligned}
&u_\xi - cu_\xi + u(1 - a_1 - u + a_1 w) = 0, \\
v_\xi - cv_\xi + r(1 - v)(a_2 u - v) = 0, \\
w_\xi - cw_\xi + \frac{1}{r}(v - w) = 0,
\end{aligned}
\]

(1.11)

Conditions (1.7) and (1.8) stipulate that wave solution is either below or above the plane \( u = v \), and this is reflected in the construction of the upper and lower solutions in section 2.

**Definition 1.** A \( C^2(\mathbb{R}) \times C^2(\mathbb{R}) \) function \( (\bar{u}(\xi), \bar{v}(\xi), \bar{w}(\xi)), \ \xi \in \mathbb{R} \) is an upper solution of (1.11) if it satisfies the inequalities

\[
\begin{aligned}
u_\xi - cu_\xi + u(1 - a_1 - u + a_1 w) \leq 0, \\
v_\xi - cv_\xi + r(1 - v)(a_2 u - v) \leq 0, \\
w_\xi - cw_\xi + \frac{1}{r}(v - w) \leq 0,
\end{aligned}
\]

(1.12)

and the boundary conditions

\[
\begin{pmatrix}
u \\
w
\end{pmatrix}(-\infty) \geq \begin{pmatrix}0 \\
0\end{pmatrix}, \quad \begin{pmatrix}u \\
v \\
w
\end{pmatrix}(+\infty) \geq \begin{pmatrix}1 \\
1\end{pmatrix}.
\]

(1.13)

A lower solution of (1.11) is defined in a similar way by reversing the inequalities in (1.12) and (1.13).

**Remark 2.** The minimum of two upper solutions is still an upper solution, i.e. suppose \( W_1(\xi) = (w_1, w_2, w_3)(\xi) \) and \( W(\xi) = (\bar{w}_1, \bar{w}_2, \bar{w}_3)(\xi), \ \xi \in \mathbb{R} \) are two upper solutions, then \( V(\xi) = (\min(w_1(\xi), \bar{w}_1(\xi)), \min(w_2(\xi), \bar{w}_2(\xi)), \min(w_3(\xi), \bar{w}_3(\xi))) \) is also an upper solution. For the lower solutions, we have similar observation except that the minimum is replaced by maximum.

Since (1.11) is a monotone system, the monotone iteration method (20) is ready to apply once the orderness of the upper and lower solutions is established. The key to the monotone iteration is to identify a pair of ordered upper and lower solutions to (1.11) (20 3). There are two methods to set up the upper and lower solutions. The first one was used in [3, 8, 20], which consists of a pair non-smooth upper and lower solutions, and the similar idea was later successfully generalized to handle local and nonlocal equations, the second one is based on a pair of smooth upper and lower solutions from known equations, and the method was applied in [15] for a general form of two species Lotka Volterra competition system and in [6] for a model system arising from game theory. We will use the ideas of the second method to set up the upper and lower solutions for (1.11). See section 2 for details.

The other interesting aspects of the traveling front solutions are the minimal wave speed, the uniqueness, the asymptotics and the stability. The minimal wave speed is also referred to as the critical wave speed, below which there will be no monotonic traveling waves. Also the traveling waves with the critical speed behaves differently at \(-\infty\), see [15]. We will use a generalized version of sliding domain method (see [2]) to show the uniqueness of the front solution corresponding to each speed.
This paper is organized as follows: In section 2 we gather the necessary information about the KPP and Lotka Volterra waves, in particular we derive the asymptotics of the Lotka Volterra waves at $-\infty$ which is the key in setting up the upper and lower solutions for (1.11); In section 3 we show the existence of the wave solutions for (1.11) and further derive their properties such as strict monotonicity, the uniqueness and the asymptotics.

2. PROPERTIES OF WAVES FOR KPP AND A TWO SPECIES LOTKA VOLterra COMPETITION EQUATIONS

In this section, we introduce properties of the wave solutions to KPP equation and to a two species Lotka Volterra competition system, which will be a key ingredient in the construction of the upper and lower solutions for system (1.11). For the rest of the paper the inequality between two vectors is component-wise.

The construction of the smooth upper and lower solution pairs for system (1.11) is based on the known results on the KPP equations and the recent results on a rescaled Lotka Volterra system. It seems that the asymptotics of the Lotka Volterra waves derived in this section is new.

Consider the following form of the KPP equation:

\[
\begin{align*}
\omega'' - c \omega' + f(\omega) &= 0, \\
\omega(-\infty) &= 0, \quad \omega(+\infty) = b,
\end{align*}
\] (2.1)

where $f \in C^2([0, b])$ and $f > 0$ on the open interval $(0, b)$ with $f(0) = f(b) = 0$, $f'(0) = \bar{a} > 0$ and $f'(b) = -b_1 < 0$. We first recall the following result ([18]):

**Lemma 3.** Corresponding to every $c \geq 2\sqrt{\bar{a}}$, system (2.1) has a unique (up to a translation of the origin) monotonically increasing traveling wave solution $\omega(\xi)$ for $\xi \in \mathbb{R}$. The traveling wave solution $\omega(\xi)$, $\xi \in \mathbb{R}$ has the following asymptotic behaviors:

For the wave solution with non-critical speed $c > 2\sqrt{\bar{a}}$, we have

\[
\omega(\xi) = a_\omega e^{\frac{\sqrt{c^2 - 4\bar{a}} \xi}{2}} + o(e^{\frac{\sqrt{c^2 - 4\bar{a}} \xi}{2}}) \quad \text{as} \quad \xi \to -\infty,
\] (2.2)

\[
\omega(\xi) = b - b_\omega e^{\frac{-\sqrt{c^2 - 4\bar{a}} \xi}{2}} + o(e^{\frac{-\sqrt{c^2 - 4\bar{a}} \xi}{2}}) \quad \text{as} \quad \xi \to +\infty,
\] (2.3)

where $a_\omega$ and $b_\omega$ are positive constants.

For the wave with critical speed $c = 2\sqrt{\bar{a}}$, we have

\[
\omega(\xi) = (a_c + d_c \xi)e^{\sqrt{\bar{a}} \xi} + o(\xi e^{\sqrt{\bar{a}} \xi}) \quad \text{as} \quad \xi \to -\infty,
\] (2.4)

\[
\omega(\xi) = b - b_c e^{(\sqrt{\bar{a}} - \sqrt{\bar{a} + b_1}) \xi} + o(e^{(\sqrt{\bar{a}} - \sqrt{\bar{a} + b_1}) \xi}) \quad \text{as} \quad \xi \to +\infty,
\] (2.5)

where the constant $d_c$ is negative, $b_c$ is positive, and $a_c \in \mathbb{R}$.

We also need the existence and asymptotics (at $-\infty$) of the solutions of the following rescaled version of Lotka Volterra system:
Let the parameters satisfy either $H_1, H_{2a}$ or $H_1, H_{2b}$, then for each \( \xi \in \mathbb{R} \) the solution has the following asymptotical properties: for \( c > 2\sqrt{1-a_1} \), the solution \((u(\xi), v(\xi))\) satisfies, as \( \xi \to -\infty \),

\[
\begin{pmatrix}
u(\xi)
\end{pmatrix} = 
\begin{pmatrix}
u(0)
\end{pmatrix} e^{\frac{-\xi}{\sqrt{c^2-4(1-a_1)}}} + o(e^{\frac{-\xi}{\sqrt{c^2-4(1-a_1)}}}),
\]

while for \( c^* = 2\sqrt{1-a_1} \), the solution \((u(\xi), v(\xi))\) satisfies

\[
\begin{pmatrix}
u(\xi)
\end{pmatrix} = 
\begin{pmatrix}
u(0)
\end{pmatrix} e^{\frac{-\xi}{\sqrt{c^2-4(1-a_1)}}} e^{\frac{-\xi}{\sqrt{1-a_1}}} + o(e^{\frac{-\xi}{\sqrt{1-a_1}}}),
\]

as \( \xi \to -\infty \), where \( A_1, A_2 > 0 \), \( A_{11c}, A_{12c}, A_{21c}, A_{22c} \in \mathbb{R} \) and \( A_{11c}, A_{22c} < 0 \).

\textbf{Proof.} The existence of the waves under conditions $H_1$-$H_{2a}$, or $H_1, H_{2b}$ is contained in [16], and also refer to [15] for the existence and the asymptotics of the waves in a more general form of a two species competition system under conditions $H_1$-$H_{2a}$.

From now on we will concentrate on system \((2.6)\) under conditions $H_1$ and $H_{2b}$. The authors in [16] proved the existence of the monotone solutions for \( c > 2\sqrt{1-a_1} \) by showing that \((2.6)\) is linearly determinate. However such method does not bring us the crucial information on the asymptotics of the wave solutions that is needed in section [3]. We show that the traveling wave as derived in [16] is actually squeezed by the lower and upper solutions of \((2.6)\) constructed below. Noting the upper and lower solutions differ from that in [8].

We first set up the lower solution for \((2.6)\).

For a fixed \( c > 2\sqrt{1-a_1} \), let \( \underline{u}(\xi), \xi \in \mathbb{R} \) be a corresponding solution of the following KPP equation

\[
\begin{align*}
u_{\xi} - cu_{\xi} + u(1-a_1-u+a_1w) &= 0, \\
u_{\xi} - cu_{\xi} + r(1-v)(a_2u - v) &= 0, \\
(\xi, v)(-\infty) &= (0, 0), \quad (\xi, v)(+\infty) = (1, 1).
\end{align*}
\]

The authors in [16] proved the existence of the monotone solutions for a more general form of a two species competition system under conditions $H_1$-$H_{2a}$.

\textbf{Lemma 4.} Let the parameters satisfy either $H_1, H_{2a}$ or $H_1, H_{2b}$, then for each \( c \geq 2\sqrt{1-a_1} \), system \((2.6)\) has a unique (up to a shift of the origin) strictly monotonic solution, and for \( 0 < c < 2\sqrt{1-a_1} \), \((2.6)\) does not have monotonic solution. At $-\infty$ the solution has the following asymptotical properties:

For \( c > 2\sqrt{1-a_1} \), the solution \((u(\xi), v(\xi))\) satisfies, as \( \xi \to -\infty \);

\[
\begin{pmatrix}
u(\xi)
\end{pmatrix} = 
\begin{pmatrix}
u(0)
\end{pmatrix} e^{\frac{-\xi}{\sqrt{c^2-4(1-a_1)}}} + o(e^{\frac{-\xi}{\sqrt{c^2-4(1-a_1)}}}),
\]

while for \( c^* = 2\sqrt{1-a_1} \), the solution \((u(\xi), v(\xi))\) satisfies

\[
\begin{pmatrix}
u(\xi)
\end{pmatrix} = 
\begin{pmatrix}
u(0)
\end{pmatrix} e^{\frac{-\xi}{\sqrt{c^2-4(1-a_1)}}} e^{\frac{-\xi}{\sqrt{1-a_1}}} + o(e^{\frac{-\xi}{\sqrt{1-a_1}}}),
\]

as \( \xi \to -\infty \), where \( A_1, A_2 > 0 \), \( A_{11c}, A_{12c}, A_{21c}, A_{22c} \in \mathbb{R} \) and \( A_{11c}, A_{22c} < 0 \).
It is straightforward to verify the following claim.

Claim A. Under conditions H1 and H2b, for each fixed \( c \geq 2 \sqrt{1 - a_1} \), the smooth function \((u, v)(\xi) = (u, \hat{u})(\xi), \xi \in \mathbb{R}\) defines a lower solution for (2.6).

We next set up a super solution for (2.6). Choosing a small number \( l \) such that

\[
0 < l \leq \frac{1 - a_1 - r(a_1 a_2 - 1)}{1 + r - a_1}.
\]

Let \( c \geq 2 \sqrt{1 - a_1} \) be fixed and \( \hat{u}(\xi) \) be a solution of the following modified KPP equation:

\[
\begin{align*}
  u'' - cu' + (1 - a_1)u(1 - \hat{u} + a_1 \hat{v}) &= 0, \\
  u(-\infty) &= 0, \\
  u(\infty) &= 1.
\end{align*}
\]

Condition H2b (see 1.8) implies that \( \frac{1 - a_1 - r(a_1 a_2 - 1)}{1 + r - a_1} < 1 - a_1 \) we therefore have \( \frac{1 - a_1}{1 - l} > 1 \).

Setting

\[
(\bar{u}, \bar{v})(\xi) = \begin{cases} 
(\hat{u}(\xi), \frac{1 - l}{1 - a_1} \hat{u}(\xi)), & \text{if } \hat{u}(\xi) \leq \frac{a_1}{1 - l} ; \\
(\hat{u}(\xi), 1), & \text{if } \frac{a_1}{1 - l} \leq \hat{u}(\xi) \leq 1 ; \\
(1, 1), & \text{if } \hat{u}(\xi) \geq 1.
\end{cases}
\]

Claim B. Assume conditions in Claim A and let \( l \) satisfy (2.11), then (2.13) defines an upper solution for (2.6).

Proof of claim B. It is easy to see that \((\bar{u}, \bar{v})(-\infty) = (0, 0), (\bar{u}, \bar{v})(\infty) = (1, 1)\), and \((u, v)(\xi) = (1, 1), \xi \in \mathbb{R}\) solves the first two equations of (2.6). We next verify that \((u, v)(\xi) = (\hat{u}(\xi), \frac{1 - l}{1 - a_1} \hat{u}(\xi))\) is also an upper solution for (2.6).

For the \( u \) component we have

\[
\begin{align*}
  \hat{u}'' - c\hat{u}' + \hat{u}(1 - a_1 - \hat{u} + a_1 \hat{v}) &= \hat{u}'' - c\hat{u}' + \hat{u}(1 - a_1 - \hat{u} + (1 - l)\hat{u}) \\
  &= \hat{u}'' - c\hat{u}' + (1 - a_1)\hat{u}(1 - r \frac{1 - l}{1 - a_1} \hat{u}) \\
  &\quad - (1 - a_1)\hat{u}(1 - r \frac{1 - l}{1 - a_1} \hat{u}) + \hat{u}(1 - a_1 - \hat{u} + a_1 \hat{v}) \\
  &= \hat{u}[1 - a_1 - \hat{u} + (1 - l)\hat{u} - (1 - a_1)((1 - \frac{1}{1 - a_1} \hat{u}))] = 0,
\end{align*}
\]
Ordered upper and lower solutions under conditions H1 and H2

To construct the upper-solution for the system (1.11) in this case, we begin with

$$\dot{u}_\xi - c\dot{u}_\xi + r(1 - \dot{v})(a_2\dot{u} - \dot{v}) = \frac{1 - l}{a_1} \dot{u}'' - c\frac{1 - l}{a_1} \dot{u}' + r(1 - \frac{1 - l}{a_1} \dot{u})(a_2\dot{u} - \frac{1 - l}{a_1} \dot{u})$$

By the monotone iteration scheme (20), for each fixed $c$ we take wave solution $(2.15)$

$$\dot{u}\in R \cap a_2(\frac{a_2a_1}{1 - l} - 1) - (1 - a_1)\leq 0$$

and either

$$l - r(a_2 - \frac{1 - l}{a_1}) \leq 0,$$

or

$$r(\frac{a_2a_1}{1 - l} - 1) - (1 - a_1) + \dot{u}(l - r(a_2 - \frac{1 - l}{a_1})) \leq 0 \text{ for all } \xi \in R.$$
the following form of KPP system:

\[
\begin{cases}
\ddot{u}'' - c \dot{u}' + (1 - a_1) \dot{u}(1 - \dot{u}) = 0, \\
\tilde{v}(-\infty) = 0, \quad \tilde{v}(+\infty) = 1,
\end{cases}
\]

where relating to (2.1), \( f(\tilde{v}) = (1 - a_1) \dot{u}(1 - \dot{u}) > 0 \) for \( \dot{u} \in (0, 1) \). \( f(0) = f(1) = 0 \), \( f'(0) = (1 - a_1) > 0 \) and \( f'(1) = -(1 - a_1) < 0 \). According to Lemma 3, for each fixed \( c \geq 2 \sqrt{1 - a_1} \), system (3.1) has a unique (up to a translation of the origin) traveling wave solution \( \bar{u}(\xi) \) satisfying the given boundary conditions. Define

\[
\begin{pmatrix}
\bar{u}(\xi) \\
\bar{v}(\xi) \\
\bar{w}(\xi)
\end{pmatrix}
= 
\begin{pmatrix}
\tilde{u}(\xi) \\
\tilde{v}(\xi) \\
\tilde{w}(\xi)
\end{pmatrix}, \quad \xi \in \mathbb{R},
\]

we have the following result,

**Lemma 5.** Assume the conditions H1 and H2a, for each fixed \( c \geq 2 \sqrt{1 - a_1} \), (3.2) is a smooth upper solution for system (1.11).

**Proof.** On the boundary, one has \((\bar{u}, \bar{v}, \bar{w})(-\infty) = (0, 0, 0), (\bar{u}, \bar{v}, \bar{w})^T(+\infty) = (1, 1, 1)\).

As for the \( u \) component, we have

\[
\begin{align*}
\ddot{u}'' - c \dot{u}' + \bar{u}(1 - a_1 - \bar{u} + a_1 \bar{w}) \\
= \ddot{u}'' - c \dot{u}' + \bar{u}(1 - a_1 - \bar{u}) \\
= \ddot{u}'' - c \dot{u}' + (1 - a_1) \bar{u}(1 - \bar{u}) \\
= 0,
\end{align*}
\]

and for the \( v \) component

\[
\begin{align*}
\ddot{v}'' - c \dot{v}' + r(1 - \ddot{v})(a_2 \ddot{u} - \dot{v}) \\
= \ddot{u}'' - c \dot{u}' + r(1 - \ddot{u})(a_2 \ddot{u} - \dot{u}) \\
+ (1 - a_1) \ddot{u}(1 - \ddot{u}) - (1 - a_1) \ddot{u}(1 - \ddot{u}) \\
= |r(a_2 - 1) - (1 - a_1)| \ddot{u}(1 - \ddot{u}) \leq 0
\end{align*}
\]

due to the condition H2a.

As for the \( w \) component,

\[
\begin{align*}
\ddot{w}'' - c \dot{w}' + \frac{1}{\tau}(\ddot{v} - \dot{w}) \\
= \ddot{u}'' - c \dot{u}' + (1 - a_1) \ddot{u}(1 - \ddot{u}) - (1 - a_1) \ddot{u}(1 - \ddot{u}) \\
= -(1 - a_1) \ddot{u}(1 - \ddot{u}) \leq 0.
\end{align*}
\]

Thus the conclusion follows. \( \square \)
We next construct the lower solution pair for system (1.11). For any small but fixed number $l$ with

\begin{equation}
0 \leq l < \frac{ra_2}{1 - a_1 + r},
\end{equation}

we choose a number $\bar{l}$ such that

\begin{equation}
0 \leq \bar{l} \leq \frac{l}{(1 - a_1)\tau + 1}.
\end{equation}

We begin with yet another KPP system:

\begin{equation}
\begin{cases}
\hat{u}'' - c\hat{u}' + (1 - a_1)\hat{u}(1 - a_1)\hat{u} = 0, \\
\hat{u}(-\infty) = 0, \quad \hat{u}(+\infty) = \frac{1 - a_1}{1 + l} < 1.
\end{cases}
\end{equation}

Corresponding to the notions in Lemma 3, $f(\hat{u}) = (1 - a_1)\hat{u}(1 - a_1)\hat{u} > 0$ for $\hat{u} \in (0, \frac{1 - a_1}{1 + l})$. $f(0) = f(\frac{1 - a_1}{1 + l}) = 0$, $f'(0) = 1 - a_1 > 0$, and $f'(\frac{1 - a_1}{1 + l}) < 0$.

For each $c \geq 2\sqrt{1 - a_1}$ let $\underline{u}(\xi), \xi \in \mathbb{R}$ be a solution of (3.5) and let

\begin{equation}
\begin{pmatrix} u(\xi) \\ v(\xi) \\ w(\xi) \end{pmatrix} = \begin{pmatrix} l \bar{u}(\xi) \\ 1 \bar{u}(\xi) \end{pmatrix}, \quad \xi \in \mathbb{R},
\end{equation}

we have

**Lemma 6.** For each $c \geq 2\sqrt{1 - a_1}$, (3.6) is a smooth lower solution of system (1.11).

**Proof.** On the boundary, one has

\[
\begin{pmatrix} u(-\infty) \\ v(-\infty) \\ w(-\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} u(+\infty) \\ v(+\infty) \\ w(+\infty) \end{pmatrix} = \begin{pmatrix} \frac{1 - a_1}{1 + l} \\ l \frac{1 - a_1}{1 + l} \\ l \frac{1 - a_1}{1 + l} \end{pmatrix} < \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.
\]
Furthermore, for the $u$ component,

$$u'' - cu' + u(1 - a_1 - u + a_1u)$$

$$= u'' - cu' + u(1 - a_1 - u + a_1\bar{u})$$

$$+(1 - a_1)u(1 - \frac{1 + l}{1 - a_1}u) - (1 - a_1)u(1 - \frac{1 + l}{1 - a_1}u)$$

$$= u[1 - a_1 - u + a_1\bar{u} - (1 - a_1) + (1 + l)\bar{u}]$$

$$= u^2(a_1\bar{l} + l)$$

$$> 0,$$

and for the $v$ component, we have

$$v'' - cv' + r(1 - v)(a_2u - v)$$

$$= l[u'' - cu' + \frac{r}{l}(1 - l\bar{u})(a_2\bar{u} - l\bar{u})$$

$$+(1 - a_1)u(1 - \frac{1 + l}{1 - a_1}u) - (1 - a_1)u(1 - \frac{1 + l}{1 - a_1}u)$$

$$= l\bar{u}\{\frac{r}{l}a_2 - r - (1 - a_1) + [(1 + l) - r(a_2 - l)]\bar{u}\} \geq 0$$

because of condition (3.5).

As for the $w$ component we have

$$w'' - cw' + \frac{1}{r}(v - w)$$

$$= \bar{w}'' - c\bar{w}' + \frac{1}{r}(l\bar{u} - \bar{w})$$

$$= \bar{w}l\frac{1}{rt}(l - \bar{l}) - (1 - a_1)(1 - \frac{1 + l}{1 - a_1}\bar{w})$$

$$= \bar{w}l\frac{1}{rt}(l - \bar{l}) - (1 - a_1) + (1 + l)\bar{u}] > 0,$$

due to the choice of $l$ and $\bar{l}$.

The conclusion of the lemma follows. □

**Remark 7.** 1. For any fixed $c \geq 2\sqrt{1 - a_1}$ and $(\bar{u}, \bar{v}, \bar{w})^T(\xi), (u, v, w)^T(\xi), \xi \in \mathbb{R}$ the respectively upper and lower solutions defined in (3.2) and (3.4), we have relation $(\bar{u}, \bar{v}, \bar{w})^T(\xi) > (u, v, w)^T(\xi)$ for $\xi \in \mathbb{R}$. In fact $u(\xi) = \frac{1}{1 + l}\bar{u}(\xi) < \bar{u}(\xi), \xi \in \mathbb{R}$ where $u(\xi)$ and $\bar{u}(\xi)$ are the solutions of (3.5) and (3.1) respectively, it then follows $\bar{w}(\xi) = \bar{u}(\xi) < \bar{u}(\xi) = \bar{w}(\xi).$
2. The construction of lower solution in Lemma 5 also applies to the case H1-H2b, where we simply require the condition $0 < l < \frac{ra_2}{1 - a_1 + r}$ to be replaced by $0 \leq l < \min\{\frac{ra_2}{1 - a_1 + r}, 1\}$.

3.2. Ordered upper and lower solutions under conditions H1, H2b and H3. The following estimates of the solutions of system (2.6) is needed in the construction of the upper and lower solutions.

Lemma 8. Assume the conditions of Lemma 4. Let $(u(\xi), v(\xi)), \xi \in \mathbb{R}$, be a solution of (2.6) for a fixed $c \geq 2\sqrt{1 - a_1}$, then there exists a constant $a_2^{*} > 0$ such that for $a_2 \geq a_2^{*}$, $a_2 u(\xi) \geq v(\xi)$ for $\xi \in \mathbb{R}$.

Proof. Noting that $u(\xi)$ and $v(\xi)$ have the exactly same (up to the first order) exponential decay rate at $-\infty$, and at $+\infty$ we have $a_2 u(\xi) > v(\xi)$. The rest of the proof follows easily. □

We now set the upper and lower solution pairs for system (1.11).

Lemma 9. Let the parameters satisfy H1 (1.6) and H2b (1.8), then (3.6) consists of a lower solution for (1.11).

Proof. Similar to the proof of Lemma 6 so we skip it. □

We next set up the upper solution for (1.11).

For each fixed $c \geq 2\sqrt{1 - a_1}$ let $(\hat{u}, \hat{v}) (\xi)$ be the correspondingly the unique solution of (3.41), and we define the function

\[
(\bar{u}, \bar{v}, \bar{w})(\xi) = (\hat{u}, \hat{v}, \hat{v})(\xi), \quad \xi \in \mathbb{R}
\]

We then have

Lemma 10. Let the parameters satisfy H1 (1.6), H2b (1.8) and H3 (1.9), and $a_2^{*}$ be given in Lemma 8, then for $a_2 \geq a_2^{*}$ (3.7) defines an upper solution of (1.11).

Proof. The proof follows easily from the fact that $(\hat{u}, \hat{v})(\xi), \xi \in \mathbb{R}$ solves (2.6) and Lemma 8. □

We next show that such constructed upper and lower solutions are ordered. The following generalized version of sliding domain method ([2]) is needed.

Proposition 11. Let two $C^2$ vector functions $\bar{U}(\xi) = (\bar{u}_1(\xi), \bar{u}_2(\xi), ..., \bar{u}_n(\xi))$ and $\bar{U}(\xi) = (\bar{u}_1(\xi), \bar{u}_2(\xi), ..., \bar{u}_n(\xi))$ satisfy the following inequalities:

\[
D\bar{U}'' - c\bar{U}' + F(U) \leq 0 \leq D\bar{U}'' - c\bar{U}' + F(U) \quad \text{for } \xi \in [-N, N]
\]

and

\[
\bar{U}(-N) < \bar{U}(\xi) \quad \text{for } \xi \in (-N, N],
\]

\[
\bar{U}(\xi) < \bar{U}(N) \quad \text{for } \xi \in (-N, N),
\]

where $D$ is a diagonal matrix with positive entries $D_{ii}, i = 1, 2...n$, $F(U) = (F_1(U), ..., F_n(U))$ is $C^1$ with respect to its components and $\frac{\partial F_i}{\partial u_j} \geq 0$ for $i \neq j, i, j = 1, 2...n$, then
(3.11) \[ \underline{U}(\xi) \leq \bar{U}(\xi), \quad \xi \in [-N, N]. \]

**Proof.** We adapt the proof of [2]. Shift \( \bar{U}(\xi) \) to the left, for \( 0 \leq \mu \leq 2N \), consider \( \bar{U}^{\mu}(\xi) := \bar{U}(\xi + \mu) \) on the interval \((-N - \mu, N - \mu)\). On both ends of the interval, by (3.9) and (3.10), we have

\[
(3.12) \quad \underline{U}(\xi) < \bar{U}^{\mu}(\xi).
\]

Starting from \( \mu = 2N \), decreasing \( \mu \), for every \( \mu \) in \( 0 < \mu < 2N \), the inequality (3.12) is true on the end points of the respective interval. On decreasing \( \mu \), suppose that there is a first \( \mu \) with \( 0 < \mu < 2N \) such that

\[ \underline{U}(\xi) \leq \bar{U}^{\mu}(\xi) \quad \xi \in (-N - \mu, N - \mu) \]

and there is one component, for example the \( i \)-th, such that the equality holds on a point \( \xi_i \) inside the interval. Let \( W(\xi) = (w_1(\xi), w_2(\xi), ..., w_n(\xi)) = \bar{U}^r(\xi) - \underline{U}(\xi) \), then \( w_i(\xi), i = 1, 2, ..., n \) satisfies

\[ \begin{cases}
D_i w''_i - cw'_i + \frac{\partial F}{\partial u_i} w_i \leq D_i w''_i - cw'_i + \sum_{j=1}^{n} \frac{\partial F}{\partial u_j} w_j \\
w_i(\xi_1) = 0, \ w_j(\xi) \geq 0 \text{ for } \xi \in [-N - \mu, N - \mu],
\end{cases} \]

the Maximum principle further implies that \( w_i \equiv 0 \) for \( \xi \in [-N - \mu, N - \mu] \), but this is in contradiction with (3.12) on the boundary points \( \xi = -N - \mu \) and \( \xi = N - \mu \). So we can decrease \( \mu \) all the way to zero. This proves the conclusion. \( \square \)

We next shift the upper solution obtained in Lemma 10 far to the left to achieve the orderness between the upper and lower solutions.

**Lemma 12.** Let \( c \geq 2\sqrt{1 - a_1} \) be fixed and \((\bar{u}, \bar{v}, \bar{w})(\xi), (\underline{u}, \underline{v}, \underline{w})(\xi)\) the corresponding upper and lower solutions derived in Lemma 10 and Lemma 8 then there exists a \( \eta_0 \geq 0 \) such that for all \( \eta \geq \eta_0 \) we have

\[ (\bar{u}, \bar{v}, \bar{w})(\xi + \eta) > (\underline{u}, \underline{v}, \underline{w})(\xi), \quad \xi \in \mathbb{R}. \]

**Proof.** On the boundary, we have \((\bar{u}, \bar{v}, \bar{w})(\xi) \to (1, 1, 1)\) and \((\underline{u}, \underline{v}, \underline{w})(\xi) \to (\frac{1-a_1}{1+a_1}, \frac{1-a_1}{1+a_1}, \frac{1+a_1}{1+a_1}) < (1,1,1)\) as \( \xi \to +\infty \), hence there exists a sufficiently large \( N_1 > 0 \) such that for any \( \eta \geq 0 \),

\[ (\bar{u}, \bar{v}, \bar{w})(\xi + \eta) > (\underline{u}, \underline{v}, \underline{w})(\xi), \quad \xi \in [N_1, +\infty). \]

While at \( \xi = -\infty \), we have the asymptotics of the upper and lower solutions:

For \( c > 2\sqrt{1-a_1} \),

\[ (3.13) \quad (\bar{u}, \bar{v}, \bar{w})(\xi + \eta) > (\underline{u}, \underline{v}, \underline{w})(\xi), \quad \xi \in [N_1, +\infty). \]

(3.14) \[ \begin{pmatrix}
\bar{u}(\xi) \\
\bar{v}(\xi) \\
\bar{w}(\xi)
\end{pmatrix} =
\begin{pmatrix}
A_1 \\
A_2 \\
A_2
\end{pmatrix}
\begin{pmatrix}
e^{-\sqrt{\frac{c}{2}} \xi} (1 - a_1) \\
e^{-\sqrt{\frac{c}{2}} \xi} (1 + a_1) \\
e^{-\sqrt{\frac{c}{2}} \xi} (1 + a_1)
\end{pmatrix} + o(e^{-\sqrt{\frac{c}{2}} \xi} (1 + a_1))
\]

and
\begin{equation}
\begin{pmatrix}
\bar{u}(\xi) \\
\bar{v}(\xi) \\
\bar{w}(\xi)
\end{pmatrix} =
\begin{pmatrix}
A_{1u} \\
A_{2u} \\
A_{3u}
\end{pmatrix}
\begin{pmatrix}
e^{-\sqrt{c^2-4(1-a_1)^2}\xi} \\
1 \\
0
\end{pmatrix}
+ o\left(e^{-\sqrt{c^2-4(1-a_1)^2}\xi}\right);
\end{equation}

While for \( c = 2\sqrt{1-a_1} \)
\begin{equation}
\begin{pmatrix}
\bar{u}(\xi) \\
\bar{v}(\xi) \\
\bar{w}(\xi)
\end{pmatrix} =
\begin{pmatrix}
A_{11} + B_{11}\xi \\
A_{21} + B_{21}\xi \\
A_{31} + B_{31}\xi
\end{pmatrix}
\begin{pmatrix}
e^{\sqrt{1-a_1}\xi} \\
e^{\sqrt{1-a_1}\xi} \\
0
\end{pmatrix}
+ o\left(e^{\sqrt{1-a_1}\xi}\right).
\end{equation}

Then it is easy to see that there exists a \( \eta_0 \geq 0 \) such that for any \( \eta \geq \eta_0 \), we have \(-N_2 < 0\) and the relation

\begin{equation}
(u, \bar{v}, \bar{w})(\xi + \eta) > (\bar{u}, \bar{v}, \bar{w})(\xi), \quad \xi \in (-\infty, -N_2]
\end{equation}

holds for \( c \geq 2\sqrt{1-a_1} \). We may further adjust \( N_1 \) and \( N_2 \) such that \( (3.13) \) and \( (3.18) \) hold on the interval \([-N_2, \infty)\) and \((-\infty, N]\) respectively for some large \( N > 0 \).

While on the interval \([-N, N]\), \((\bar{u}, \bar{v}, \bar{w})(\xi + \eta)\) and \((\bar{u}, \bar{v}, \bar{w})(\xi)\) satisfy the conditions of Proposition 11, we therefore have

\begin{equation}
(u, \bar{v}, \bar{w})(\xi + \eta) \geq (\bar{u}, \bar{v}, \bar{w})(\xi), \quad \xi \in [-N, N].
\end{equation}

The conclusion of the lemma follows from \( (3.13), (3.18)\) and \( (3.19)\).

In the sequel we still write the shifted upper solution as \((\bar{u}, \bar{v}, \bar{w})(\xi), \xi \in \mathbb{R}\).

3.3. **Monotone waves and their asymptotics.** With such constructed ordered upper and lower solution pairs, we now have

**Theorem 13.** Assume either the conditions H1 and H2a or H1, H2b and H3, then for every \( c \geq 2\sqrt{1-a_1} \), system \([1.11]\) has a unique (up to a translation of the origin) traveling wave solution. The traveling wave solution is strictly increasing on \( \mathbb{R} \) and has the following asymptotic properties:

1. Corresponding to the wave speed \( c > 2\sqrt{1-a_1} \),

\begin{equation}
\begin{pmatrix}
u(\xi) \\
w(\xi)
\end{pmatrix} =
\begin{pmatrix}
A_1 \\
A_2 \\
A_3
\end{pmatrix}
\begin{pmatrix}
e^{-\sqrt{c^2-4(1-a_1)^2}\xi} \\
1 \\
0
\end{pmatrix}
+ o\left(e^{-\sqrt{c^2-4(1-a_1)^2}\xi}\right)
\end{equation}

as \( \xi \to -\infty \);
while Corresponding to the wave speed $c = 2\sqrt{1-a_1}$, we have

\[
\begin{pmatrix}
  u(\xi) \\
  v(\xi) \\
  w(\xi)
\end{pmatrix} = \begin{pmatrix}
  (A_{11c} + A_{12c}\xi) \\
  (A_{21c} + A_{22c}\xi) \\
  (A_{31c} + A_{32c}\xi)
\end{pmatrix} e^{\sqrt{1-a_1}\xi} + o(\xi e^{\sqrt{1-a_1}\xi})
\]

as $\xi \to -\infty$.

For any speed $c \geq 2\sqrt{1-a_1}$, we have

\[
\begin{pmatrix}
  u(\xi) \\
  v(\xi) \\
  w(\xi)
\end{pmatrix} = \begin{pmatrix}
  1 \\
  1 \\
  1
\end{pmatrix} - \tilde{M}(\xi) + o(\tilde{M}(\xi)) \quad \xi \to +\infty,
\]

where

\[
\tilde{M}(\xi) = s_1 \begin{pmatrix}
  1 \\
  0 \\
  0
\end{pmatrix} e^{c e^{-\sqrt{c^2+4a_1^2}\xi} + s_2 \begin{pmatrix}
  \frac{a_1}{\tau r(1-a_2) + 1} \\
  \tau r(1-a_2) + 1 \\
  1
\end{pmatrix} e^{c e^{-\sqrt{c^2+4(a_2-1)^2}\xi}} + s_3 \begin{pmatrix}
  1 \\
  0 \\
  1
\end{pmatrix} e^{c e^{-\sqrt{c^2+4(1-a_2)^2}\xi}},
\]

and the real numbers $s_i$, $i = 1, 2, 3$ are not all zeros and be such that $-\tilde{M}(\xi) + o(\tilde{M}(\xi)) < 0$ for $\xi > 0$ large enough. The speed $c = 2\sqrt{1-a_1}$ is the minimal wave speed in the sense that below which there is no monotone waves of (1.11).

Proof. Starting from the upper and lower solution pairs obtained in section 3.1 and section 3.2 and using the monotone iteration scheme provided in [20, 3], we obtain the existence of the solution $(u(\xi), v(\xi), w(\xi))$ to (1.11) for every fixed $c \geq 2\sqrt{1-a_1}$. The solution satisfies

\[
\begin{pmatrix}
  u(\xi) \\
  v(\xi) \\
  w(\xi)
\end{pmatrix} \leq \begin{pmatrix}
  \bar{u}(\xi) \\
  \bar{v}(\xi) \\
  \bar{w}(\xi)
\end{pmatrix} \quad \xi \in \mathbb{R}.
\]

Lemma 3 and Lemma 4 imply that the upper- and the lower-solutions as derived in section 3.1 and section 3.2 have the same asymptotic rates at $-\infty$. Then (3.20) and (3.21) then follow from Lemmas 3 and 4.
To derive the asymptotic decay rate of the traveling wave solutions at $+\infty$, we let $c \geq 2\sqrt{1-a_1}$ and
\begin{equation}
U(\xi) := (u(\xi), v(\xi), w(\xi)) \quad \xi \in \mathbb{R}
\end{equation}
be the corresponding traveling wave solution of (1.11) generated from the monotone iteration. We differentiate (1.11) with respect to $\xi$, and note that $U'(\xi) := (w_1, w_2, w_3)^T(\xi)$ satisfies
\begin{equation}
(w_1)_\xi - c(w_1) \xi + A_{11}(u, v, w)w_1 + A_{12}(u, v, w)w_2 + A_{13}(u, v, w)w_3 = 0,
\end{equation}
\begin{equation}
(w_2)_\xi - c(w_2) \xi + A_{21}(u, v, w)w_1 + A_{22}(u, v, w)w_2 + A_{23}(u, v, w)w_3 = 0,
\end{equation}
\begin{equation}
(w_3)_\xi - c(w_3) \xi + A_{31}(u, v, w)w_1 + A_{32}(u, v, w)w_2 + A_{33}(u, v, w)w_3 = 0
\end{equation}
where
\[
\begin{pmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{pmatrix} = \begin{pmatrix}
1 - a_1 - 2u + a_1w & 0 & a_1u \\
a_2v(1-v) & -r - 2a_2rv + 2rv & 0 \\
0 & \frac{1}{r} & -\frac{1}{r}
\end{pmatrix}.
\]

The limit system of (3.24), (3.25) and (3.26) at $\xi = +\infty$ is
\begin{equation}
\begin{cases}
(\psi_1)_\xi - c(\psi_1) \xi - \psi_1 + a_1\psi_3 = 0, \\
(\psi_2)_\xi - c(\psi_2) \xi + r(1-a_2)\psi_2 = 0, \\
(\psi_3)_\xi - c(\psi_3) \xi + \frac{1}{r}\psi_2 - \frac{1}{r}\psi_3 = 0.
\end{cases}
\end{equation}

It is easy to see that system (3.28) admits exponential dichotomy (34). Since the traveling wave solution $(u(\xi), v(\xi), w(\xi))$ converge monotonically to a constant limit as $\xi \to \pm\infty$, the derivative of the traveling wave solution satisfies $(w_1(\pm\infty), w_2(\pm\infty), w_3(\pm\infty)) = (0, 0, 0)$ (20, p.658 Lemma 3.2). Hence we are only interested in finding bounded solutions of (3.28) at $+\infty$.

Introducing transformation $\Psi = PY$ by
\begin{equation}
\begin{pmatrix}
\psi_1(\xi) \\
\psi_2(\xi) \\
\psi_3(\xi)
\end{pmatrix} = \begin{pmatrix}
1 & a_1 & a_1 \\
0 & \tau r(1-a_2) + 1 & 0 \\
0 & 1 & 1
\end{pmatrix} \begin{pmatrix}
y_1(\xi) \\
y_2(\xi) \\
y_3(\xi)
\end{pmatrix} \equiv P \begin{pmatrix}
y_1(\xi) \\
y_2(\xi) \\
y_3(\xi)
\end{pmatrix}
\end{equation}
we can decouple (3.28) into the following equivalent system:
\begin{equation}
\begin{cases}
(y_1)_\xi - c(y_1) \xi - y_1 = 0, \\
(y_2)_\xi - c(y_2) \xi + r(1-a_2)y_2 = 0, \\
(y_3)_\xi - c(y_3) \xi - \frac{1}{r}y_3 = 0,
\end{cases}
\end{equation}
and find its bounded solutions at $+\infty$ explicitely. In fact, for some nonzero constants $d_1, d_2, d_3$, we have
(3.31) \[ y_1(\xi) = d_1 e^{-\sqrt{\frac{\tau r a_2 - 1}{2}} \xi}, \quad y_2(\xi) = d_2 e^{\sqrt{\frac{\tau r a_2 - 1}{2}} \xi}, \quad y_3(\xi) = d_3 e^{-\sqrt{\frac{\tau r a_2 + 1}{2}} \xi}. \]

Transforming back to \( \Psi \) we have

(3.32) \[ \begin{pmatrix} \psi_1(\xi) \\ \psi_2(\xi) \\ \psi_3(\xi) \end{pmatrix} = P \begin{pmatrix} y_1(\xi) \\ y_2(\xi) \\ y_3(\xi) \end{pmatrix} = \begin{pmatrix} y_1(\xi) + \frac{\tau}{1-a_2} y_2(\xi) + \frac{\tau r}{r-1} y_3(\xi) \\ (\tau r - \tau r a_2 + 1)y_2(\xi) \\ y_2(\xi) + y_3(\xi) \end{pmatrix}, \]

Hence we have (3.22) on integrating (3.32).

We next show the strict monotonicity of the traveling wave solutions, which will be a key ingredient in locating the eigenvalues of the linearized operator about the traveling wave in a separate study. By the monotone iteration process (see [20]), the traveling wave solution \( U(\xi) \) is increasing for \( \xi \in \mathbb{R} \), it then follows that \((w_1(\xi), w_2(\xi), w_3(\xi))^T = U(\xi) \geq 0 \) and satisfies (3.25), (3.26) and (3.27). The monotonicity of system (1.11) and the Maximum Principle imply that \((w_1, w_2, w_3)^T(\xi) > 0 \) for \( \xi \in \mathbb{R} \). This concludes the strict monotonicity of the traveling wave solutions.

On the uniqueness of the traveling wave solution for every \( c \geq 2 \sqrt{1-a_1} \), we only prove the conclusion for traveling wave solutions with asymptotic rates given in (3.20) and (3.22), since other case can be proved similarly. Let \( U_1(\xi) = (u_1, v_1, w_1)^T \) and \( U_2(\xi) = (u_2, v_2, w_2)^T \) be two traveling wave solutions of system (1.11) with the same speed \( c > 2 \sqrt{1-a_1} \). There exist positive constants \( A_i, B_i, i = 1, 2, 3, 4 \) and a large number \( N > 0 \) such that for \( \xi < -N \),

(3.33) \[ U_1(\xi) = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} e^{c \sqrt{\frac{\tau r a_2 - 1}{2}} \xi} + o(e^{c \sqrt{\frac{\tau r a_2 - 1}{2}} \xi}), \]

(3.34) \[ U_2(\xi) = \begin{pmatrix} A_4 \\ A_5 \\ A_6 \end{pmatrix} e^{c \sqrt{\frac{\tau r a_2 - 1}{2}} \xi} + o(e^{c \sqrt{\frac{\tau r a_2 - 1}{2}} \xi}); \]

and for \( \xi > N \),

(3.35) \[ U_1(\xi) = \begin{pmatrix} 1 - B_1 e^{\mu_1 \xi} \\ 1 - B_2 e^{\mu_2 \xi} \\ 1 - B_3 e^{\mu_3 \xi} \end{pmatrix} + \begin{pmatrix} o(e^{\mu_1 \xi}) \\ o(e^{\mu_2 \xi}) \\ o(e^{\mu_3 \xi}) \end{pmatrix}, \]

(3.36) \[ U_2(\xi) = \begin{pmatrix} 1 - B_1 e^{\mu_1 \xi} \\ 1 - B_2 e^{\mu_2 \xi} \\ 1 - B_3 e^{\mu_3 \xi} \end{pmatrix} + \begin{pmatrix} o(e^{\mu_1 \xi}) \\ o(e^{\mu_2 \xi}) \\ o(e^{\mu_3 \xi}) \end{pmatrix}. \]
where \( \mu \) is one of the elements in the set \( \{ c_{-\sqrt{c^2+4(1-a)}} \}, \{ c_{-\sqrt{c^2+4/\tau}} \}, \) and \( B_i, B_2 \) are positive numbers, \( i = 1, 2, 3 \). The traveling wave solutions of system (1.11) are translation invariant, thus for any \( \theta > 0 \), \( U_1^{\theta}(\xi) := U_1(\xi + \theta) \) is also a traveling wave solution of (1.11). By (3.33) and (3.35), the solution \( U_1(\xi + \theta) \) has the asymptotics
\[
U_1^{\theta}(\xi) = \begin{cases} 
A_1 e^{c_{-\sqrt{c^2+4(1-a)}} \theta} e^{c_{-\sqrt{c^2+4(1-a)}} \xi} \\
A_2 e^{c_{-\sqrt{c^2+4(1-a)}} \theta} e^{c_{-\sqrt{c^2+4(1-a)}} \xi} \\
A_3 e^{c_{-\sqrt{c^2+4(1-a)}} \theta} e^{c_{-\sqrt{c^2+4(1-a)}} \xi}
\end{cases} + o(e^{c_{-\sqrt{c^2+4}} \xi})
\]
for \( \xi \leq -N \);
\[
U_1^{\theta}(\xi) = \begin{cases} 
1 - B_1 e^{\mu \theta} e^{\mu \xi} \\
1 - B_2 e^{\mu \theta} e^{\mu \xi} \\
1 - B_3 e^{\mu \theta} e^{\mu \xi}
\end{cases} + \begin{cases} 
\theta(\xi) \\
\theta(\xi) \\
\theta(\xi)
\end{cases}
\]
for \( \xi \geq N \).
Choosing \( \theta > 0 \) large enough such that
\[
A_1 e^{c_{-\sqrt{c^2+4(1-a)}} \theta} > A_3 + i, \quad i = 1, 2, 3
\]
then one has
\[
\bar{B}_i e^{\mu \theta} < B_i, \quad i = 1, 2, 3
\]
for \( \xi \in (\xi, -N) \cup [N, \infty) \). We now consider system (1.3) on \( [\xi, N] \). We can verify that \( U_1^{\theta}(\xi) \) satisfies all the conditions of Proposition 11 hence we have \( U_1^{\theta}(\xi) \geq U_2(\xi) \) for \( \xi \in [-N, N] \) Further applying the Maximum Principle and noting that \( U_1^{\theta}(\pm N) > U_2(\pm N) \), we have \( U_1^{\theta}(\xi) > U_2(\xi) \) for \( \xi \in [-N, N] \).
Consequently we have \( U_1^{\theta}(\xi) > U_2(\xi) \) on \( \mathbb{R} \).
Now, decrease \( \theta \) until one of the following situations happens.
1. There exists a \( \bar{\theta} \geq 0 \), such that \( U_1^{\bar{\theta}}(\xi) \equiv U_2(\xi) \). In this case we have finished the proof.
2. There exists a \( \bar{\theta} \geq 0 \) and \( \xi_1 \in \mathbb{R} \), such that one of the components of \( U_1^{\bar{\theta}} \) and \( U_2 \) are equal there; and for all \( \xi \in \mathbb{R} \), we have \( U_1^{\theta}(\xi) \geq U_2(\xi) \). On applying the Maximum Principle on \( \mathbb{R} \) for that component, we find \( U_1^{\theta} \) and \( U_2 \) must be identical on that component. To fix ideas, we suppose that the component is the first component. Then \( U_1^{\theta} - U_2 \) satisfies (3.24), (3.26) and (3.27). Plugging \( w_1 \equiv 0 \) into (3.26), we find that there is at least one \( \xi_\theta \) such that \( w_2(\xi_\theta) = 0 \). Then by applying maximum principle to (3.26), we have \( w_2(\xi) \equiv 0 \) for \( \xi \in \mathbb{R} \). Similarly we also find \( w_3(\xi) \equiv 0, \xi \in \mathbb{R} \). We have then returned to case 1.
Hence, in either situation, there exists a \( \bar{\theta} \geq 0 \), such that \( U_1^{\bar{\theta}}(\xi) \equiv U_2(\xi) \).
for all $\xi \in \mathbb{R}$.

The nonexistence of the monotone traveling waves for (1.11) comes from the fact that all its solutions are oscillatory for $c \leq 2\sqrt{1-a_1}$. □

Concerning the wave solutions of system (1.2) we immediately have

**Corollary 14.** Assume the conditions $H1$ (1.6) and $H2a$ (1.7) or conditions $H1$ (1.6), $H2b$ (1.8) and $H3$ (1.9). Then for each $c \geq 2\sqrt{1-a_1}$ and all $\tau > 0$ system (1.2) has a unique monotonic traveling wave solution connecting the equilibrium $(0, 1)$ with $(1, 0)$.

The conclusion of the Corollary says the delay does not change the course of the traveling waves, but it may change the asymptotic behaviors of the wave solutions at $+\infty$.

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