A SECOND LOOK AT THE TORIC H-POLYNOMIAL OF A CUBICAL COMPLEX

GÁBOR HETYEI

Abstract. We provide an explicit formula for the toric \( h \)-contribution of each cubical shelling component, and a new combinatorial model to prove Clara Chan’s result on the non-negativity of these contributions. Our model allows for a variant of the Gessel-Shapiro result on the \( g \)-polynomial of the cubical lattice, this variant may be shown by simple inclusion-exclusion. We establish an isomorphism between our model and Chan’s model and provide a reinterpretation in terms of noncrossing partitions. By discovering another variant of the Gessel-Shapiro result in the work of Denise and Simion, we find evidence that the toric \( h \)-polynomials of cubes are related to the Morgan-Voyce polynomials via Viennot’s combinatorial theory of orthogonal polynomials.

Introduction

The toric \( h \)-polynomial of a lower Eulerian partially ordered set, introduced by Stanley \cite[§2]{stanley}, is a highly sophisticated invariant. As noted by Stanley himself \cite[p. 193]{stanley}, it “seems quite difficult to compute,” “without using the laborious defining recurrence”. Yet in those few cases where a combinatorial interpretation was found, it lead to deep enumerative results. An excellent example of this is Chan’s work \cite{chan} who showed that the toric \( h \)-polynomial of a shellable cubical complex has nonnegative coefficients, by finding a set of highly nontrivial enumeration problems for which the number of objects counted turns out to be exactly the contribution of a cubical shelling component to a given coefficient in the toric \( h \)-polynomial. She later extended her results in collaboration with Billera and Liu \cite{billera} showing that the toric \( h \)-polynomial of a star-shellable cubical complex is the \( h \)-polynomial of a shellable simplicial complex. It seems that very little has been done in the area ever since.

The main purpose of this paper is to convince the reader that the toric \( h \)-polynomials of shellable cubical complexes are worth a “second look”. There are other combinatorial models for Chan’s results, some of them linked to her model via non-trivial bijections, others may be derived in the future from an unexpected and unstated appearance of the toric \( g \)-polynomial of a cubical lattice in the work of Denise and Simion \cite{denise}. The explicit formulas we find for the contribution of each shelling component to the toric \( h \)-polynomial of a shellable cubical complex link the study of these polynomials to the combinatorics of

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Catalan numbers. It is not the first time that such a connection appears in the literature: the formulas of Bayer and Ehrenborg [1, Theorems 4.1 and 4.2] expressing the toric $h$-polynomial of an Eulerian poset in terms of its flag $h$-vector and cd-index abound in Catalan numbers, and same holds for the formula of Billera and Brenti [2, Theorem 3.3] expressing the Kazhdan-Lusztig polynomial of any Bruhat interval in any Coxeter group in terms of the complete cd-index.

Our paper is structured as follows. After the Preliminaries, in Section 2 we use Chan’s recursions [5] to build explicit formulas for the contribution of each shelling component to the toric $h$-polynomial of a shellable cubical complex. These formulas are meant to serve as “yardstick”, against which present or future combinatorial models can be measured.

In Section 3 we introduce a new combinatorial model, counting the total weight of postorder trees which are weighted according to the numbers of “first-born” leaf children, not “first-born” leaf children and “first-born” non-leaf children in the tree. The greatest simplification occurs when we prove the variant of the Gessel-Shapiro result [11, Ex. 3.71g] on the toric $g$-polynomial of a cubical lattice. The original variant counts forks in a plane tree, and the easiest proof on record involves solving a quadratic equation for a generating function, whereas in our model we count “first-born” leaves in plane trees and there is a proof essentially based on inclusion-exclusion. In our model, all three types of “special vertices” are equally simply defined, whereas the third type of “special vertices” in Chan’s original model [5] has a little more technical definition.

Our model is closely related to Chan’s original model [5], and a bijection is outlined in Section 4. This calls for transforming a preorder tree into a binary tree, reflecting the binary tree in the plane, transforming it back to a plane tree, whose mirror image is a postorder tree. Under this correspondence forks in the postorder tree correspond to first-born leaf children of a nonroot parent, and the other two classes of special vertices also turn out to be bijective equivalents of the classes in [5].

The postorder tree model is easily transformed into a noncrossing partition model, and the results of Chan [5] have a rephrasing in terms of noncrossing partitions, where we need to consider a statistics of nonsingleton blocks, singleton blocks, and elements that are not the minimum in their block, nor are they the maximum in a block that does not contain 1. This is shown in Section 5. We also note that the Gessel-Shapiro result has a variant in the work of Denise and Simion [6]. This observation allows us to present a new recursion formula for the toric $g$-polynomials of cubical lattices, and establish a relation between these polynomials and the Morgan-Voyce polynomials via Viennot’s combinatorial theory of orthogonal polynomials [16]. This connection deserves further study in a future work.

Hopefully our “second look” will not be the last look at these remarkable polynomials, and in the concluding Section 6 we collect some of the questions raised by the present work.
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1. Preliminaries

1.1. The toric \(h\)-polynomial of a lower Eulerian poset and a polyhedral complex. A partially ordered poset is graded, if it has a unique minimum element \(\hat{0}\), a unique maximum element \(\hat{1}\) and a rank function \(\rho\). A graded partially ordered poset is Eulerian if in any open interval \((x, y)\) the number of elements at odd ranks equals the number of elements at even ranks. Given an Eulerian poset \(\hat{P}\), let us denote by \(P\) the poset obtained by removing the maximum element \(\hat{1}\) from \(\hat{P}\). Stanley [10, §2] defines the toric \(h\)-vector of \(P\) by first defining the polynomials \(f(P, x)\) and \(g(P, x)\) by the following intertwined recurrence:

(a) \(f(\emptyset, x) = g(\emptyset, x) = 1\)
(b) If \(\hat{P}\) has rank \(d + 1 \geq 1\) and if \(f(P, x) = k_0 + k_1 x + \cdots\) then
\[
g(P, x) = \sum_{i=0}^{\lfloor d/2 \rfloor} (k_i - k_{i-1})x^i
\]
(we set \(k_{-1} = 0\)),
(c) If \(\hat{P}\) has rank \(d + 1 \geq 1\) then
\[
f(P, x) = \sum_{t \in P} g(\hat{0}, t, x)(x - 1)^{d - \rho(t)}.
\]

If \(f(P, x) = k_0 + k_1 x + \cdots + k_d x^d\), then he sets \(h_i := k_{d-i}\) and calls the vector \((h_0, \ldots, h_d)\) the \(h\)-vector of \(P\). Since, by [10] Theorem 2.4, any Eulerian poset \(\hat{P}\) of rank \(d + 1\) satisfies \(h_i = h_{d-i}\), we may refer to \(f(P, x)\) as the toric \(h\)-polynomial of \(P\), in which the coefficient of \(x^i\) is \(h_i\).

A finite set \(P\) is lower Eulerian if it has a unique minimum element \(\hat{0}\), and every interval of the form \([\hat{0}, t]\) in it is Eulerian. Thus the polynomials \(f([\hat{0}, t], x)\) \(g([\hat{0}, t], x)\) are still defined for all \(t \in P\). Stanley [10] Eq. (18)] extends the definition of \(f(P, x)\) to a lower Eulerian poset \(P\) by
\[
f(P, x) = \sum_{t \in P} g([\hat{0}, t], x)(x - 1)^{d - \rho(t)},
\]
where \(d\) is the maximum chain length in \(P\) and \(\rho(t)\) is the length of a maximal chain in \([\hat{0}, t]\).
An important example of a lower Eulerian poset is the face poset \( P(\mathcal{P}) \) of a polyhedral complex \( \mathcal{P} \), ordered by inclusion. Given a \( d \)-dimensional polyhedral complex \( \mathcal{P} \), for each face \( F \in \mathcal{P} \) the half open interval \([\emptyset, F) \subset P(\mathcal{P})\) is the face lattice of the boundary complex \( \partial F \) of \( F \). Introducing \( \overline{h}(\mathcal{P}, x) \) as a new notation for the \( h \)-polynomial \( f(P(\mathcal{P}), x) \) and \( g(\mathcal{P}, x) \) as a shorthand for for \( g(P(\mathcal{P}), x) \) we obtain the following adaptation of Stanley’s original definition to polyhedral complexes:

\[
(1) \quad \overline{h}(\emptyset, x) = g(\emptyset, x) = 1.
(2) \quad \text{If } \dim \mathcal{P} = d \geq 0 \text{ then } \overline{h}(\mathcal{P}, x) = \sum_{F \in \mathcal{P}} g(\partial F, x)(x - 1)^{d - \dim F}.
(3) \quad \text{If } \dim \mathcal{P} = d \geq 0 \text{ then } g(\mathcal{P}, x) = \sum_{i=0}^{\lfloor (d+1)/2 \rfloor} k_i x^i \quad \text{where } k_i \text{ is the coefficient of } x^i \text{ in } \overline{h}(\mathcal{P}, x).
\]

This adaptation may be found in the paper [4] of Billera, Chan, and Liu. They then define the toric \( h \)-vector \( \overline{h}(\mathcal{P}) = (h_0, \ldots, h_{d+1}) \) of the polyhedral complex \( \mathcal{P} \) by the formula

\[
h(\mathcal{P}, x) = \sum_{i=0}^{d+1} h_i x^i = x^{d+1} \overline{h}(\mathcal{P}, 1/x),
\]

and the toric \( g \)-vector of \( \mathcal{P} \) by \( g(\mathcal{P}) = (h_0, h_1 - h_0, \ldots, h_m - h_{m-1}) \) where \( m = \lfloor (d+1)/2 \rfloor \).

As noted above, Stanley [10, Theorem 2.4] has shown that for an Eulerian poset \( \hat{\mathcal{P}} \) of rank \( d+1 \geq 1 \) the polynomial \( f(\hat{\mathcal{P}}, x) \) satisfies the generalized Dehn-Sommerville equations

\[
(1) \quad x^d f(\hat{\mathcal{P}}, 1/x) = f(\hat{\mathcal{P}}, x).
\]

Let now \( \hat{Q} \) be the poset obtained from \( \hat{\mathcal{P}} \) by adding a new maximum element. (Thus \( Q = \hat{\mathcal{P}} \).) Using (c) of Stanley’s definition we obtain

\[
f(Q, x) = \sum_{t \in \mathcal{P}} g(\emptyset, t, x)(x - 1)^{d+1 - \rho(t)} + g(\mathcal{P}, x)
\]

from which, using (1) is not hard to derive the equation

\[
(2) \quad x^{d+1} f(Q, 1/x) = g(\mathcal{P}, x).
\]

For polyhedral complexes this equation has the following consequence.

**Corollary 1.1.** Let \( \mathcal{P} \) be the complex of all faces of a convex polytope, and \( \partial \mathcal{P} \) its boundary complex. Then we have

\[
h(\mathcal{P}, x) = g(\partial \mathcal{P}, x).
\]

In particular, the \( h \)-polynomial of a cube is the \( g \)-polynomial of its boundary, as this was noted in [3, Lemma 3.1]. Note that the \( g \)-polynomial of the boundary of a \( d \)-dimensional cube (as a \( g \)-polynomial of a polyhedral complex) is the same as the \( g \)-polynomial of \( L_d \) (as the \( g \)-polynomial of a poset) where \( L_d \) is the face lattice of a \( d \)-dimensional cube.
1.2. Results on the toric $h$-vector of a cubical complex. The first result on the toric $g$- and $h$-polynomials of a cubical complex is due to Gessel and it may be found in Stanley’s seminal paper [10, Proposition 2.6]. This states that for the face lattice $\hat{L}_d$ of a $d$-dimensional cube we have

$$g(L_d, x) = \sum_{k=0}^{\lfloor d/2 \rfloor} \frac{1}{d-k+1} \binom{d}{k} \binom{2d-2k}{d} (x-1)^k.$$  

By Corollary 1.1 this is also the $h$-polynomial of the $d$-dimensional cube. A combinatorial interpretation of the right hand side of (3) is due to Shapiro [11, Ex. 3.71g] stating that the coefficient of $x^i$ in $g(L_d, x)$ is the number of plane trees on $d+1$ vertices with $i$ forks, where a fork is a vertex having more than one child. A proof that involves solving a quadratic equation for a generating function is given in [5, Proposition 2].

A complete description of the $h$-polynomial of a shellable cubical complex was given by Clara Chan [5]. The general notion of shellability was used by topologists for a long time. The fact that the boundary complex of any convex polytope is shellable was shown by Bruggeser and Mani [4]. A $(d-1)$-dimensional cubical complex is shellable exactly when there is an ordering $(F_1, \ldots, F_r)$ of its facets such that for $t > 1$ the intersection $F_t \cap (F_1 \cap \cdots \cap F_{t-1})$ is a union of $(d-2)$-faces homeomorphic to a ball or sphere. As observed by Clara Chan [5] this is equivalent to stating that there is a pair $(i, j)$ such that $F_t \cap (F_1 \cap \cdots \cap F_{t-1})$ is the union of $i$ antipodally unpaired $(d-2)$-faces and $j$ antipodally paired $(d-2)$-faces. Here $0 \leq i \leq d-1$ and if $i = 0$ then $j = d-1$. (An explanation this observation may be found in [7, Lemma 3.2].) We call $(i, j)$ the type of the cubical shelling component $(i, j)$.

The main result of [5] that each cubical shelling component contributes a polynomial with positive coefficient to the toric $h$-polynomial. Thus the toric $h$-vector of a shellable cubical complex has non-negative entries. Later Billera, Chan, and Liu [3] proved that for star-shellable cubical complexes (these are cubical complexes such that each shelling component has type $(i, 0)$ for some $i$) the toric $h$-vector is the $h$-vector of a shellable simplicial complex. The proof depends on using the combinatorial description of the toric $h$-polynomial contributions that may be found in [5].

2. Explicit formulas for the toric $h$-contribution of the cubical shelling components

Using the recursions given in Chan’s paper [5] we compute an explicit formula for the contribution of each cubical shelling component to the toric $h$-polynomial of a shellable cubical complex. Our main result in this section is the following.

**Theorem 2.1.** Let $F_t$ be a facet of type $(i, j)$ in the shelling of a $(d-1)$-dimensional cubical complex. Then adding $F_t$ to $F_1 \cup \cdots \cup F_{t-1}$ changes the polynomial $f(P, x)$ of the
face poset $P$ by

$$f_d(i, j, x) := \sum_{k=0}^{d-1} C_{d-1-k}(1-x)^k x^{d-k} \sum_{s=0}^{j} \binom{j}{s} \left(\frac{1}{d-i+j-1-k-s}\right)$$

if $j \neq d-1$, and by

$$f_d(0, d-1, x) = \sum_{k=0}^{d-1} C_{d-1-k} \left(\frac{d-1-k}{k}\right) (x-1)^k$$

if $i = 0$ and $j = d-1$.

Here $C_n = \binom{2n}{n}/(n+1)$ is a Catalan number.

Proof. Let us verify the statement first for $(i, j) = (0, 0)$, that is, for the contribution of the first facet in the shelling. For this we want to prove

$$f_d(0, 0, x) := \sum_{k=0}^{d-1} C_{d-1-k} \left(\frac{d-1-k}{k}\right) (1-x)^k x^{d-k}.$$  \hfill (6)

As noted in the proof of Lemma 1 in [5], as a consequence of (2) we have

$$f_d(0, 0, x) = x^d g(L_{d-1}, x^{-1}).$$

Using this observation, equation (6) may be obtained by rewriting (3) as

$$g(L_d, x) = \sum_{k=0}^{\lfloor d/2 \rfloor} \frac{1}{d-k+1} \binom{d}{d-k} \binom{2d-2k}{d} (x-1)^k$$

$$= \sum_{k=0}^{\lfloor d/2 \rfloor} \frac{1}{d-k+1} \binom{2d-2k}{d-k} \binom{d-k}{k},$$

yielding

$$g(L_d, x) = \sum_{k=0}^{\lfloor d/2 \rfloor} C_{d-k} \left(\frac{d-k}{k}\right) (x-1)^k. \hfill (7)$$

Next, using induction on $i$, we show that

$$f_d(i, 0, x) := \sum_{k=0}^{d-1} C_{d-1-k} \left(\frac{d+i-1-k}{k}\right) (1-x)^k x^{d-k}. \hfill (8)$$

holds for all $i \geq 0$. For $i = 0$ this is (6) above. According to the proof of Lemma 2 in [5], the polynomials $f_d(i, 0, x)$ satisfy the recursion formula

$$f_d(i, 0, x) = f_d(i-1, 0, x) - (x-1)f_{d-1}(i-1, 0, x).$$
Thus, using the induction hypothesis for \( i - 1 \), we obtain

\[
\begin{align*}
  f_d(i, 0, x) &= \sum_{k=0}^{d-1} C_{d-1-k} \binom{d + i - 2 - k}{k} (1 - x)^k x^{d-k} \\
  &\quad + (1 - x) \sum_{k=0}^{d-2} C_{d-2-k} \binom{d + i - 3 - k}{k} (1 - x)^k x^{d-1-k}.
\end{align*}
\]

Shifting the value \( k \) by one in the second sum yields

\[
\begin{align*}
  f_d(i, 0, x) &= \sum_{k=0}^{d-1} C_{d-1-k} \binom{d + i - 2 - k}{k} (1 - x)^k x^{d-k} \\
  &\quad + \sum_{k=1}^{d-1} C_{d-1-k} \binom{d + i - 2 - k}{k - 1} (1 - x)^k x^{d-k}.
\end{align*}
\]

Equation (8) is now an immediate consequence of

\[
\binom{d + i - 2 - k}{k} + \binom{d + i - 2 - k}{k - 1} = \binom{d + i - 1 - k}{k}.
\]

To show the statement for \( i > 0 \) and all \( j \geq 0 \) we use induction on \( j \). The basis of the induction is (8). Using the recursion formula

\[
f_d(i, j, x) = f_d(i, j - 1, x) - 2(x - 1)f_{d-1}(i, j - 1, x),
\]

which may be found in the proof of Lemma 3 in [5], and the induction hypothesis for \( j - 1 \) we obtain

\[
\begin{align*}
  f_d(i, j, x) &= \sum_{k=0}^{d-1} C_{d-1-k}(1 - x)^k x^{d-k} \sum_{s=0}^{j-1} \binom{j - 1}{s} \binom{d + i + j - 2 - k - s}{k - s} \\
  &\quad + 2(1 - x) \sum_{k=0}^{d-2} C_{d-2-k}(1 - x)^k x^{d-1-k} \sum_{s=0}^{j-1} \binom{j - 1}{s} \binom{d + i + j - 3 - k - s}{k - s}.
\end{align*}
\]

Shifting the value of \( k \) by one in the second sum yields

\[
\begin{align*}
  f_d(i, j, x) &= \sum_{k=0}^{d-1} C_{d-1-k}(1 - x)^k x^{d-k} \sum_{s=0}^{j-1} \binom{j - 1}{s} \binom{d + i + j - 2 - k - s}{k - s} \\
  &\quad + 2 \sum_{k=1}^{d-1} C_{d-1-k}(1 - x)^k x^{d-k} \sum_{s=0}^{j-1} \binom{j - 1}{s} \binom{d + i + j - 2 - k - s}{k - 1 - s}.
\end{align*}
\]

Since

\[
\begin{align*}
  \binom{d + i + j - 2 - k - s}{k - s} + 2 \binom{d + i + j - 2 - k - s}{k - 1 - s} &= \\
  \binom{d + i + j - 1 - k - s}{k - s} + \binom{d + i + j - 2 - k - s}{k - 1 - s},
\end{align*}
\]
we may write
\[
f_d(i, j, x) = \sum_{k=0}^{d-1} C_{d-1-k}(1-x)^k x^{d-k} \sum_{s=0}^{j-1} \binom{j-1}{s} \binom{d+i+j-1-k-s}{k-s}.
\]
+ \sum_{k=0}^{d-1} C_{d-1-k}(1-x)^k x^{d-k} \sum_{s=0}^{j-1} \binom{j-1}{s} \binom{d+i+j-2-k-s}{k-1-s}.
\]

Shifting the value of \(s\) by one in the second sum yields
\[
f_d(i, j, x) = \sum_{k=0}^{d-1} C_{d-1-k}(1-x)^k x^{d-k} \sum_{s=0}^{j-1} \binom{j-1}{s} \binom{d+i+j-1-k-s}{k-s}.
\]
+ \sum_{k=0}^{d-1} C_{d-1-k}(1-x)^k x^{d-k} \sum_{s=1}^{j} \binom{j-1}{s-1} \binom{d+i+j-1-k-s}{k-s}.
\]

The statement follows now from
\[
\binom{j-1}{s} + \binom{j-1}{s-1} = \binom{j}{s}.
\]

Consider finally the case when \(i = 0\) and \(j = d-1\). As noted in [5], we have
\[
f_d(0, d-1, x) = g(L_{d-1}, x),
\]
thus (5) follows from (7). □

Since the maximum rank that occurs in the face poset of a \((d-1)\)-dimensional cubical complex is \(d\), Theorem 2.1 may be rephrased for the toric \(h\)-polynomial, as defined in [3], as follows.

**Corollary 2.2.** Let \(F_t\) be a facet of type \((i, j)\) in the shelling of a \((d-1)\)-dimensional cubical complex. Then adding \(F_t\) to \(F_1 \cup \cdots \cup F_{t-1}\) changes the toric \(h\)-polynomial of the cubical complex by
\[
h_d(i, j, x) := \sum_{k=0}^{d-1} C_{d-1-k}(x-1)^k \sum_{s=0}^{j} \binom{j}{s} \binom{d+i+j-1-k-s}{k-s}
\]
if \(j \neq d-1\), and by
\[
h_d(0, d-1, x) = \sum_{k=0}^{d-1} C_{d-1-k} \left(\binom{d-1-k}{k}\right) x^{d-k}(1-x)^k
\]
if \(i = 0\) and \(j = d-1\).
3. A new plane tree enumeration model

In this section we develop new plane tree model with enumeration problems having the toric $h$-contributions of the cubical shelling components as their answer. The model is bijectively related to Clara Chan’s model, the bijection will be outlined in Section 4. Our interest in this model is due to the fact that the formulas have relatively simple direct proofs, taking advantage of the explicit formulas given in Section 2 and that the definition of the third class of special vertices we use is less technical. In our model leftmost children will play a special role, and we will sometimes refer to them as first-born children.

Because of “historic reasons” (a bijective connection to notions introduced by Clara Chan) and because of the case of the shelling components of type $(0, 0)$ we first introduce the following notions:

**Definition 3.1.** We define three types of special vertices in a plane tree as follows.

- A vertex $v$ is a type (0) special vertex if $v$ is a leaf, $v$ is the leftmost child of its parent, and the parent of $v$ is not the root.
- A vertex $v$ is a type (1) special vertex if $v$ is a leaf and it is either the leftmost child of the root, or the parent of $v$ is not the root and $v$ is not the leftmost child of its parent.
- A vertex $v$ is a type (2) special vertex if $v$ is not a leaf, and it is the leftmost child of its parent.

The distinction between first-born leaf children of nonroot parents (type (0)) and the first-born leaf child of the root (type (1)) seems artificial but inevitable when one wants to state Lemma 3.3 below. However, to prove our enumeration statement for shelling components of type $(i, j)$ where $i > 0$, we will be more comfortable using the following, more straightforward variant of Definition 3.1.

**Definition 3.2.** We define three types of special vertices in a plane tree as follows.

- A vertex $v$ in is a type [0] special vertex if $v$ is a leaf, and $v$ is the leftmost child of its parent.
- A vertex $v$ is a type [1] special vertex if $v$ is a leaf and it is not the leftmost child of its parent.
- A vertex $v$ is a type [2] special vertex if $v$ is not a leaf, and it is the leftmost child of its parent.

Both definitions define disjoint sets of special vertices, and the definition of type (2) is the same as the definition of type [2]. For those trees where the leftmost child of the root is not a leaf, the two definitions coincide.

We have the following variant of the Gessel-Shapiro result.

**Lemma 3.3.** The coefficient of $x^k$ in $g(L_d, x)$ is the number of plane trees on $d+1$ vertices with exactly $k$ type (0) special vertices.
Proof. The statement may be rephrased as follows: \( g(L_d, x) \) is the total weight of all plane trees on \( d + 1 \) vertices, where the weight of each tree is the product of the weight of its vertices, each type (0) special vertex has weight \( x \), all other vertices have weight one.

Replacing \( x \) with \( x + 1 \) yields the following equivalent statement: the coefficient \( x^k \) in \( g(L_d, x + 1) \) is the number of plane trees on \( d + 1 \) vertices with exactly \( k \) marked type (0) special vertices. In fact, replacing each \( x \) with \( x + 1 \) corresponds to choosing whether to mark or not to mark each type (0) special vertex.

Substituting \( x+1 \) into \( x \) in (7) we obtain the formula

\[
g(L_d, x+1) = \sum_{k=0}^{\lfloor d/2 \rfloor} C_{d-k} \binom{d-k}{k} x^k.
\]

Using this formula, the last equivalent rephrasing of the lemma is easy to show. In fact, every plane tree \( T \) with \( k \) marked type (0) vertices may be uniquely given by determining the plane tree \( T' \) on \( d + 1 - k \) vertices obtained by deleting the marked leaves, and the \( k \)-element subset \( S \) of nonroot vertices of \( T' \) that are parents of a marked vertex. Note that no two marked vertices have the same parent, since each is the leftmost child of its parent. Given a plane tree \( T' \) on \( d + 1 - k \) vertices we may select any \( k \)-element subset of its nonroot vertices as the parents which obtain a new leftmost and marked child. Therefore there are \( C_{d-k} \binom{d-k}{k} \) plane trees with \( k \) marked type (0) special vertices. \( \square \)

Using Theorem 2.1 we obtain the following consequences of Lemma 3.3.

**Corollary 3.4.** The coefficient of \( x^{d-k} \) in \( f_d(0, 0, x) \) is the number of plane trees on \( d \) vertices with \( k \) type (0) special vertices.

**Corollary 3.5.** The coefficient of \( x^k \) in \( f_d(0, d - 1, x) \) is the number of plane trees on \( (d + 1) \) vertices with \( k \) type (0) special vertices.

Just like in the work of Chan [5], we describe the coefficient of \( x^{d-k} \) in \( f_d(i, j, x) \) for other values of \( i \) and \( j \) by introducing a labeling of the vertices. In [5] the preorder labeling was used for this purpose. We will use the postorder labeling. This may be defined recursively as follows. First remove the root, and label the resulting trees in postorder, left to right. Then we label the root with the highest number. An example is shown in Figure 1. The meaning of the bold edges will be explained in Section 5.

For each positive integer \( n \) let us fix an algebraic operation \( F_n \) with \( n \) variables. Let us use “Polish notation”, that is, write \( a_1a_2 \ldots a_nF_n \) instead of \( F_n(a_1, \ldots, a_n) \). There is an obvious bijection between plane trees on \( d \) vertices and polynomial expressions of length \( d \), using a single variable \( x \) and the operations \( F_1, F_2, \ldots \). In fact, we can think of leaves as nodes representing the variable \( x \), and of each non-leaf with \( n \) children as an operation \( F_n \). Each operation represented by a non-leaf takes the outputs calculated at its children (listed left to right) as its input. Under this bijection, the postorder tree in Figure 1 corresponds to the polynomial \( xxF_2xxF_2F_1F_2 \). Each letter \( u \), except for the last one, is substituted into a variable of the operation associated to its parent node, which we call
its *parent operation*. The root in the tree corresponds to the last letter in the expression, we will thus call the operation represented by the last letter the *root operation*. In the rest of the paper we will call the polynomial expression assigned to a given plane tree $T$ by the bijection described above the *associated polynomial expression* and denote it by $P(T)$.

In $P(T)$, type (0) special vertices correspond to all letters $x$ that are substituted into the first variable of their parent operation, provided the parent operation is not the root operation. Type (1) special vertices correspond to all letters $x$ that are either substituted into the first variable of the root operation, or into a non-first variable of a nonroot operation. Type (2) special vertices correspond to operation symbols which are substituted into the first variable of their parent operation. The correspondence for type [0] and type [1] vertices is similar but easier.

Consider plane trees as Hasse diagrams of partial orders with each parent being greater than its child. Removing a non-maximum element from a partial order corresponds to removing a nonroot vertex $v$ in the plane tree, and turning the children of $v$ (if any) into children of the parent of $v$. Let us define the *removal of $v$* in such a way that it also preserves the plane order: when removing $v$, replace $v$ in the ordered list of the children of the parent of $v$ with the ordered list of the children of $v$. This removal operation results in a smaller plane tree.

The effect of the removal of a single vertex from a plane tree $T$ on $P(T)$ may be very easily described. In fact, if the removed vertex has $c \geq 0$ children and $s \geq 0$ siblings then in $P(T)$ we need to remove the letter associated to the removed vertex and replace its parent operation $F_{s+1}$ with $F_{s+c}$.

**Proposition 3.6.** Let $T$ be a plane tree and \{$v_1, \ldots, v_k$\} be a subset of its nonroot vertices. Then removing these vertices in any order $v_{\pi(1)}, \ldots, v_{\pi(k)}$ results in the same plane tree. Here $\pi$ is any permutation of \{1, 2, $\ldots$, $k$\}.

**Proof.** The removal operation does not change the induced partial order on the remaining vertices. Thus the resulting tree will always induce the same partial order on the remaining
vertices. The plane order does not change either since in the associated polynomial expressions the order of the variables associated to the remaining vertices does not change under the removal operation.

**Definition 3.7.** We say that the plane tree $T$ is obtained from a plane tree $T'$ by inserting a vertex $v$ if $v$ is a nonroot vertex of $T$ and removing $v$ from $T$ yields $T'$.

Whereas the removal of a vertex is uniquely defined, there may be more than one way to insert a nonroot vertex $v$ into a plane tree such that we obtain a plane tree. The following two insertion lemmas describe two situations where the insertion at a prescribed postorder position may be done in one and only one way, if we want to create a special vertex of a certain type. The first lemma states that there is a unique way to insert a type (1) special vertex at a given position, and this will not change the special properties of the vertices that precede in postorder.

**Lemma 3.8 (First insertion lemma).** Let $T'$ be a plane tree on $d$ vertices and $p$ an integer satisfying $1 \leq p \leq d$. Then there is a unique plane tree $T$ on $d + 1$ vertices such that the vertex $u$ whose postorder label is $p$ in $T$ is a type (1) special vertex and removing this vertex from $T$ yields $T'$. Furthermore, for any $\tau \in \{0, 1, 2\}$ a nonroot vertex $v \neq u$ whose postorder label is at most $p-1$ (in either $T'$ or $T$) is a type $(\tau)$ special vertex in $T'$ if and only if it has the same property in $T$.

**Proof.** As noted before Proposition 3.6 after removing the $p$-th letter from $P(T)$, the resulting word must differ at at most one position from $P(T')$. The letter that is removed from $P(T)$ must be a variable $x$. This letter corresponds to a type (1) special vertex if and only if either $p = 1$ and the letter is the first input of the root operation or $p > 1$ and the letter at the $(p-1)$-th place belongs to the same parent operation. Either way there is exactly one way to change a $P(T')$ to obtain a $P(T)$ such that $T$ and $T'$ have the required relation: we must insert a new $p$-th letter $x$ and, for $p = 1$ we must increase the number of variables of the root operation by one, whereas for $p > 1$ we must increase the number of variables of the parent of the letter at the $(p-1)$-th position by one.

For example if $p = 1$ and $P(T') = xxxxF_3F_1$ then the only way to insert a new first leaf as a type (1) special vertex is to create the plane tree defined by $P(T) = xxxxF_3F_2$. If $p = 4$ and $P(T') = xxF_2xxF_3xF_2$ then we must have $P(T) = xxF_2xxxF_4xF_2$.

To prove the last part of the Lemma, observe first that the vertices $v \neq u$ having the same postorder label in $T'$ and in $T$ are exactly the vertices whose postorder label is at most $p-1$. We are inserting $u$ as a leaf, and the parent of $u$ is either the root, or a vertex that already has at least one child (the $(p-1)$-th vertex in postorder). Thus a nonroot vertex $v \neq u$ is a leaf in $T'$ if and only if it is a leaf in $T$. Similarly $v \neq u$ is a root child in $T'$ if and only if it is a root child in $T$. If $p \neq 1$ then $u$ is not inserted as the first child, thus any $v \neq u$ is the first child of its parent in $T'$ if and only if the same holds in $T$. Therefore in the case when $p \neq 1$, we have the stronger statement that any nonroot vertex $v \neq u$ is a type $(\tau)$ special vertex in $T'$ if and only if it has the same property in $T$. Finally if $p = 1$ then the last part of the Lemma is trivially true, since no nonroot vertex $v \neq u$ has postorder label less than 1. \qed
Corollary 3.9. Let $T'$ be a plane tree on $d$ vertices and $p$ an integer satisfying $2 \leq p \leq d$. Then there is a unique plane tree $T$ on $d+1$ vertices such that the vertex $u$ whose postorder label is $p$ in $T$ is a type [1] special vertex and removing this vertex from $T$ yields $T'$. Furthermore, for any $\tau \in \{0,1,2\}$ a nonroot vertex $v \neq u$ whose postorder label is at most $p-1$ (in either $T'$ or $T$) is a type $[\tau]$ special vertex in $T'$ if and only if it has the same property in $T$.

In fact, for $p \geq 2$ the vertex we are about to insert can not be the leftmost leaf whose parent is the root. Thus the unique possibility of inserting a type (1) special vertex is equivalent to the unique possibility of inserting a type special [1] vertex. For the preceding vertices, a vertex is a type ($\tau$) special vertex if and only if it is a type $[\tau]$ special vertex, except for the first vertex in postorder, if it is a leaf. Being such a vertex is preserved by the insertion or removal of $u$.

The second insertion lemma states that there is a unique way to insert a type (2) (that is, type [2]) special vertex at a prescribed position $p \geq 2$. Alas it is not possible to state that this insertion has to leave the special types ($\tau$) where $\tau \in \{0,1,2\}$ unchanged for the preceding vertices, because of the following example. Assume we are given a tree $T'$ such that the first vertex $v_1$ in postorder is a leaf and the child of the root. As we will see in the proof of Lemma 3.10, the only way to insert a type (2) special vertex as the second vertex, is by making this new vertex a root child and a parent of $v_1$. In the new tree, $v_1$ is not a type (1) special vertex any more but a type (0) special vertex. However, this complication may be eliminated by making the statement about type $[\tau]$ vertices.

Lemma 3.10 (Second insertion lemma). Let $T'$ be a plane tree on $d$ vertices and $p$ an integer satisfying $2 \leq p \leq d$. Then there is a unique plane tree $T$ on $d+1$ vertices such that the vertex whose postorder label is $p$ in $T$ is a type [2] special vertex and removing this vertex from $T$ yields $T'$. Furthermore, for any $\tau \in \{0,1,2\}$ if nonroot vertex $v \neq u$ whose postorder label is at most $p-1$ (in either $T'$ or $T$) is a type $[\tau]$ special vertex in $T'$ if and only if it has the same property in $T$.

Proof. In analogy to the proof of Lemma 3.8 the first part of the statement is easily shown by considering $P(T)$ and $P(T')$. Given a polynomial expression $P(T')$, now we have to insert an operation $F_c$ as the new $p$-th letter, and the $(p-1)$-th letter $L$ must have the inserted operation as its parent in $P(T)$. The index $c$ of $F_c$ is determined by the fact that $L$ must be the $c$-th child of its parent $L'$ in $P(T')$. Finally, inserting $F_c$ at the $p$-th position decreases the number of variables of $L'$ by $c-1$. For example, if $p = 4$ and $P(T') = xF_1 xxF_3$ then the third letter $x$ is the second child of its parent operation $F_3$. The only way to get $P(T)$ is to insert an $F_2$ as a new fourth letter and to change the last $F_3$ to $F_2$. Thus we must have $P(T) = xF_1 xF_2 xF_2$.

To prove the last part of the Lemma, observe that the vertices $v \neq u$ whose postorder label is at most $p-1$ do not lose any of their children nor do they lose any preceding sibling at the insertion of $u$. \hfill $\square$
Repeated use of the two insertion lemmas yields the following.

**Proposition 3.11.** Let $T'$ be a plane tree on $d+1-k$ vertices, $\{p_1, \ldots, p_k\} \subseteq \{1, 2, \ldots, d\}$ and $(\tau_1, \ldots, \tau_k) \in \{1, 2\}^k$. Then there is a unique plane tree $T$ on $d+1$ vertices such that for each $s \in \{1, 2, \ldots, k\}$ the vertex $v_s$ of $T'$ whose postorder label is $p_s$ is a type $[\tau_s]$ special vertex, and removing the vertices $v_1, \ldots, v_k$ from $T'$ yields $T'$.

**Proof.** Note that by Proposition 3.6 the order of removing the vertices $v_1, \ldots, v_k$ is irrelevant. Without loss of generality we may assume $p_1 < \cdots < p_k$. We proceed by induction on $k$. For $k = 1$ and $\tau_1 = 1$ the statement is identical with Corollary 3.9 whereas setting $k = 1$ and $\tau_1 = 2$ makes the statement identical with Lemma 3.10. Assume the statement is true for $k-1$. Let $T'$ be a plane tree on $d+1-k$ vertices, $\{p_1, \ldots, p_k\} \subseteq \{1, 2, \ldots, d\}$ and $(\tau_1, \ldots, \tau_k) \in \{1, 2\}^k$. If there is any tree $T$ satisfying the stated conditions, the tree $T''$ obtained by inserting $v_k$ into $T$ must be a tree on $d$ vertices such that for each $s \in \{1, 2, \ldots, k-1\}$ the vertex $v_s$ of $T''$ whose postorder label is $p_s$ is a type $[\tau_s]$ special vertex, and removing the vertices $v_1, \ldots, v_{k-1}$ from $T''$ yields $T'$. By our induction hypothesis, $T''$ may be obtained from $T'$ by inserting the vertices $v_1, \ldots, v_{k-1}$ appropriately. Given $T''$ there is a unique way to insert $v_k$ as a type $(\tau_k)$ special vertex because of the already shown $k = 1$ case of this Proposition. Finally note that by Corollary 3.9 and Lemma 3.10 inserting $v_k$ can not change the property of being type $[\tau_s]$ special of any preceding vertex $v_s$. \hfill $\square$

The main result of this section is the following.

**Theorem 3.12.** Let $(i, j)$ be the type of a shelling component in a $(d-1)$-dimensional cubical complex and assume $j < d-1$. Then the coefficient of $x^{d-m}$ in $f_d(i, j, x)$ is the number of plane trees on $d$ vertices with exactly $m$ vertices $v$ having (exactly) one of the following properties:

- $v$ is a type $(0)$ special vertex;
- $v$ is a type $(1)$ special vertex whose label in postorder is at most $i$ or at least $d-j$;
- $v$ is a type $(2)$ special vertex whose label in postorder is at least $d-j$.

For $i = 0$, requiring $j < d-1$ forces $(i, j) = (0, 0)$, and Theorem 3.12 becomes Corollary 3.4. For $i > 0$, Theorem 3.12 becomes equivalent to the following statement, which seems easier to prove.

**Theorem 3.13.** Let $(i, j)$ be the type of a shelling component in a $(d-1)$-dimensional cubical complex and assume $j < d-1$. Then the coefficient of $x^{d-m}$ in $f_d(i, j, x)$ is the number of plane trees on $d$ vertices with exactly $m$ vertices $v$ having (exactly) one of the following properties:

- $v$ is a type $[0]$ special vertex;
- $v$ is a type $[1]$ special vertex whose label in postorder is at most $i$ or at least $d-j$;
- $v$ is a type $[2]$ special vertex whose label in postorder is at least $d-j$. 
Proof. By equation (4) we have

\[ (11) \quad [x^{d-m}]f_d(i,j,x) = \sum_{k=m}^{d-1} (-1)^{k-m} C_{d-1-k} \binom{k}{k-m} \sum_{s=0}^{j} \binom{j}{s} \binom{d+i+j-1-k-s}{k-s}. \]

Thus it is sufficient to show that the number of plane trees on \(d\) vertices with \(k\) marked special vertices having the properties listed in the Theorem is

\[ C_{d-1-k} \sum_{s=0}^{j} \binom{j}{s} \binom{d+i+j-1-k-s}{k-s}. \]

The statement will then follow from (11) by inclusion-exclusion.

Consider a plane tree \(T\) with with \(k\) marked special vertices. Let us remove the marked vertices (the order of removing them does not matter by Proposition 3.6). We will be left with a plane tree \(T'\) on \(d-k\) vertices, which is one of \(C_{d-1-k}\) possible trees. Thus it suffices to show that for each plane tree \(T'\) on \(d-k\) vertices there are

\[ \sum_{s=0}^{j} \binom{j}{s} \binom{d+i+j-1-k-s}{k-s} \]

plane trees \(T\) with \(k\) marked special vertices that yield \(T'\) after removing the marked vertices. Assume exactly \(k_r\) of the marked vertices have type \([\tau]\) for \(t = 0, 1, 2\). (Thus \(k = k_0 + k_1 + k_2\).) It is sufficient to show that for any fixed triplet \((k_0, k_1, k_2)\) of natural numbers satisfying \(k_0 + k_1 + k_2 = k\), there are

\[ \binom{d-k}{k_0} \binom{j}{k_2} \binom{i-1+j-k_2}{k_1} \]

plane trees \(T\) with \(k_r\) marked type \([\tau]\) special vertices with the stated properties such that removing the marked special vertices results in \(T'\). The statement then follows from

\[ \sum_{k_0+k_1+k_2=k} \binom{d-k}{k_0} \binom{j}{k_2} \binom{i-1+j-k_2}{k_1} = \sum_{s=0}^{j} \binom{j}{s} \binom{d+i+j-1-k-s}{k-s}. \]

Let us reinsert first the marked type \([0]\) special vertices into \(T'\) and then the other marked special vertices. (We are allowed to fix such an order by Proposition 3.6.) In analogy to the proof of Lemma 3.3 there are exactly \((d-k_0)\) ways to insert \(k_0\) marked type \([0]\) vertices into \(T'\), to obtain a plane tree \(T''\) with \(d-k+k_0\) vertices containing \(k_0\) marked type \([0]\) special vertices such that the removal of the marked vertices from \(T''\) results in \(T'\). Thus we are left to show that there are exactly

\[ \binom{j}{k_2} \binom{i+j-k_2}{k_1} \]

ways to insert marked type \([1]\) and type \([2]\) special vertices to obtain a plane tree \(T\) satisfying our requirements. By the stated restrictions, there are \(j\) positions \((d-j, \ldots, d-1)\) where \(T\) may contain a marked type \([2]\) special vertex. Once we selected these positions, there are \(i-1+j-k_2\) positions \((2, \ldots, i)\) and the ones not yet selected from \((d-j, \ldots, d-1)\) where \(T\) may contain a marked type \([1]\) special vertex. Once we select the positions where we want to have our marked type \([1]\) and type \([2]\) vertices, by Proposition 3.11 there is
4. CONNECTION BETWEEN THE TWO PLANE TREE ENUMERATION MODELS

In this section we outline a bijection between our model and Chan’s model \cite{5}, showing that the proofs of the present paper could be directly translated into proofs of Chan’s results and vice versa.

We begin with a preorder plane tree from Chan’s paper \cite{5}, shown in Figure 2. It is well known that plane trees on \(d\) vertices are in bijection with Catalan paths of length \(2d\), and a bijection may be given by walking around the tree in counterclockwise order, keeping very close to its edges, and recording an “up” (or +) for each step when along the nearest edge our move represents moving away from the root, and a “down” (or –) otherwise. Thus the plane tree shown in Figure 2 corresponds to the sequence ++−−+++−++. Let us think now of each + as “opening a parenthesis” and of each – as “closing a parenthesis”. We obtain a parenthesization of the product \(x_1 \cdots x_d\) which may be represented by a binary tree as shown in the right hand side of Figure 2. Let us reflect the binary tree about a vertical axis, we then obtain a binary tree with decreasing labels, such as the one in Figure 3. Using the bijection described above in the opposite direction, we obtain the mirror image of a postorder tree. This postorder tree is the planar mirror image of the one shown in Figure 1.

**Proposition 4.1.** The sequence of transformations described above establishes a bijection between preorder trees and postorder trees.

**Proof.** When walking around a preorder tree, the steps marked + represent visiting a vertex that was nor visited before, whereas the steps marked – represent “backtracking”. When we reflect the binary tree in the plane, the associated parenthesization, along with the associated sequence of + and – steps go into their mirror image. When walking...
around a mirrored postorder tree, the steps marked $+$ represent again visiting a vertex that was nor visited before, whereas the steps marked $-$ represent “backtracking”. The only difference is that this time we visit the highest labeled vertex first and we number the vertices in decreasing order along the way. It is not hard to complete the proof using these observations, the details are left to the reader. □

Proposition 4.2. The bijection described above maps the set of forks in a preorder tree onto the set of type $(0)$ special vertices in the corresponding postorder tree.

Proof. A vertex $v$ is a fork in the preorder tree if and only if $v$ and its parent are both left children in the associated binary tree. In fact, a vertex is a leaf if and only if it corresponds to a right child in the binary tree, thus $v$ corresponding to a left child is equivalent to saying that $v$ has children. If we follow the path from $v$ to its parent, grandparent and so on in the binary tree, the first time we reach a right child is the time when we have to “backtrack” in the preorder tree. Up until that point the descendants of the visited ancestors in the binary tree are also the descendants of $v$, and the leftmost descendants of the children of the visited vertices in the binary tree are the children of $v$ in the preorder tree.

Similarly, $v$ is a leaf and a rightmost child of a nonroot vertex in a mirrored postorder tree if and only if $v$ is a right child of its parent, and the parent is a right child of its parent in the corresponding binary tree. In fact, as noted above, $v$ being a right child in the binary tree is equivalent to $v$ being a leaf in the corresponding plane tree, whereas the parent of $v$ being a right child is equivalent to saying that $v$ is the last child of its parent, and the parent of $v$ is not the root. □

Remark 4.3. As a consequence we obtain that the Gessel-Shapiro result [11, Ex. 3.71g], shown by Chan [5] by solving a quadratic equation for a generating function is a “mirror image” of Lemma 3.3, which was shown essentially by inclusion-exclusion. The difference lays perhaps in the fact, that it is easy to visualize how to “attach a first-born marked leaf” to any vertex, whereas “inserting a marked fork” seems to be less intuitive. However, the
diligent reader should be able to transform the leaf insertion process into a fork insertion process, by tracing back along the reverse of the transformation described in this section.

We conclude this section by noting that the special vertices marked $1',\ldots,i'$ in Chan’s work [5] correspond exactly to our type (1) special vertices, whereas the special vertices marked $1'',2'',\ldots,j''$ correspond exactly to our type (2) special vertices. The proof of these facts is left to the reader as an exercise.

5. Noncrossing partitions and a surprising coincidence

A partition of $\{1,2,\ldots,d\}$ is noncrossing if for any four elements $a < b < c < d$ the following condition is satisfied: if $a, c$ are in the same class and $b, d$ are in the same class then $a, b, c, d$ are in the same class. The number of noncrossing partitions of $\{1,2,\ldots,d\}$ is the Catalan number $C_d$, the same as the number of plane trees on $d + 1$ vertices. An explicit bijection may be given as follows.

**Definition 5.1.** Let $u$ and $v$ be nonroot vertices in a plane tree. We say that $v$ is a favorite ancestor of $u$ if $v$ is an ancestor of $u$ and the unique path from $v$ to $u$ involves at each step selecting the leftmost child. Under these circumstances, we also call $u$ a favorite descendant of $v$.

The choice of the terminology is motivated by the fact that in many societies special importance is attached to the first-born child, and many parents feel special affection towards their youngest. Such choices correspond to always favoring the leftmost (or rightmost) child in the family tree. Being a favorite ancestor-favorite descendant pair is a transitive relation, its symmetrization is an equivalence relation. Let us call this equivalence being “favorite relatives”. The equivalence classes are singletons or paths in the tree. The bold edges and vertices marked in bold in Figure 1 represent the equivalence classes of this equivalence relation. After labeling the vertices in postorder, the equivalence classes become the noncrossing partition $13/2/467/5$.

**Proposition 5.2.** The “favorite relative” equivalence classes of nonroot vertices of a postorder tree on $\{1,2,\ldots,n+1\}$ form a noncrossing partition. Associating to each postorder tree the noncrossing partition of its “favorite relative” equivalence classes is a bijection between plane trees on $\{1,\ldots,n+1\}$ and noncrossing partitions of $\{1,2,\ldots,n\}$.

**Proof.** Let us verify first that we obtain a noncrossing partition. Assume the nonroot vertices $a, b, c, d$ satisfy $a < b < c < d$ in postorder, $a$ and $c$ are favorite relatives and $b$ and $d$ are favorite relatives. By the nature of the postorder labeling, $c$ is then an ancestor of $a$, and it must also be an ancestor of $b$ since all vertices preceding $c$ in postorder that are not descendants of $c$ precede all descendants of $c$ as well, and $b$ does not precede $a$. This means that $c$ belongs to the unique path from $d$ to $b$. This path involves only choosing the leftmost child in each step, so $b$ is a favorite descendant of $c$, and $a, b, c$ belong to the same equivalence class. This class also contains $d$ since $b$ is equivalent to $d$. 
For the converse it is sufficient to show that each noncrossing partition of \( \{1, 2, \ldots, n\} \) arises as the collection of equivalence classes of favorite relatives thus the map is onto. We know that the two sets (plane trees on \( \{1, 2, \ldots, n+1\} \) and non-crossing partitions of \( \{1, 2, \ldots, n\} \)) have the same cardinality, so an onto map between them is a bijection. Consider thus a noncrossing partition \( \pi \) on \( \{1, 2, \ldots, n\} \). Represent the vertices 1, 2, \ldots, \( n+1 \) as numbers on the number line. For each block \( \{a_1, \ldots, a_k\} \in \pi \), where \( a_1 < \cdots < a_k \), create a path \( a_1 - a_2 - \cdots - a_k \). Represent the edges \( a_i - a_{i+1} \) as upper semicircles, as they are shown with continuous lines in Figure 4. For each \( i < k \) define the parent of \( a_i \) as to be \( a_{i+1} \) The fact that \( \pi \) is a noncrossing partition is equivalent to saying that no two semicircles introduced up to this point intersect in an interior point. If \( a \) is the maximum element of a block, define its parent as the smallest \( b > a \) such that the block containing \( b \) contains a \( c < a \). If no such \( b \) exists then select the root to be the parent of \( a \). Represent each edge \( a - b \) also with an upper semicircle as the ones shown with dashed lines in Figure 4. Even after adding the semicircles with dashed lines, the arcs will not cross at an interior point. In fact, assume by way of contradiction that adding an arc \( a - b \) where \( a \) is the maximum element of a block and \( b \) is its parent, as defined above, creates a crossing at an interior point. If the arc \( a - b \) intersects an earlier \( a' - b' \) in such a way, then these satisfy \( a' < a < b' < b \). By the definition of the selection of \( b' \), there is a \( c' < a' \) such that \( c' \) and \( b' \) belong to the same block. But then \( b \) is not the smallest element above \( a \) whose block contains an element preceding \( a \). This contradiction shows that adding a new dashed arc never creates an intersection of dashed arcs at an interior point. We are left only with the possibility that the arc \( a - b \) intersects an arc \( a'_i - a'_{i+1} \) where \( a'_i \) and \( a'_{i+1} \) are subsequent elements in a block of \( \pi \). Then we have either \( a < a'_i < b < a'_{i+1} \) or \( a'_i < a < a'_{i+1} < b \). In the first case \( c < a'_i < b < a'_{i+1} \) implies that \( c, a'_i, a'_{i+1} \) and \( b \) all belong to the same block of \( \pi \), and we should have chosen \( a'_i \) or an even smaller number to be the parent of \( a \). In the second case, by \( a'_i < a < a'_{i+1} \), the number \( a'_{i+1} \) or an even smaller number should have been chosen as the parent of \( a \), instead of \( b \). We obtained a contradiction in all cases.

We obtain a rooted tree, since each vertex has a uniquely defined parent, and every edge connects some vertex to its parent. For each vertex, the parent is larger than the vertex itself. Choosing the above representation we obtain a plane tree, since the arcs do not cross at an interior point. It is easy to show that each bold edge connects the parent to its “leftmost” (in the picture: “uppermost”) child. Moreover, an arc to a smaller child

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**Figure 4.** Postorder tree generated by a noncrossing partition
passes “to the left” (in the picture: “above”) of an arc to a larger child of the same parent. Thus the tree has postorder labeling, and for any vertex \( v \) the smallest labeled child of \( v \) is the one that is in the same equivalence class. (The minimum elements of the equivalence classes are leaves.) Thus the collection of equivalence classes of favorite relatives is \( \pi \). □

**Remark 5.3.** Using the bijection between preorder and postorder trees introduced in Section 4 we may transform Proposition 5.2 into the following result. Consider two non-root vertices in a preorder tree equivalent if they have the same parent. The equivalence classes associated to any preorder tree form a noncrossing partition and associating to each preorder tree the noncrossing partition of these equivalence classes defines a bijection between plane trees and noncrossing partitions. This result is mentioned in [12, Ex. 5.35a]. The first proof on record is due to Kreweras [8]. Under this correspondence, the nonsingleton blocks of a noncrossing partition correspond to the set of all children of a fork in the preorder tree. These blocks play the essential role in the work of Billera, Chan, and Liu [3] when they construct a simplicial complex whose \( h \)-vector is the toric \( h \)-vector of a cube.

It is easy to verify that under the correspondence introduced above, type (0) special vertices correspond exactly to the minimum elements of the nonsingleton blocks in the corresponding noncrossing partition. Thus we obtain the following variant of the Gessel-Shapiro result and Lemma 3.3.

**Lemma 5.4.** The coefficient of \( x^k \) in \( g(L_d, x) \) is the number of noncrossing partitions on \( \{1, \ldots, d\} \) with exactly \( k \) nonsingleton blocks.

Similarly, type (1) special vertices correspond to singleton blocks. In fact, \( a \) forms a singleton block exactly when \( a \) has no descendants at all (thus it is a leaf) and it is not the “favorite child” of its parent, or its parent is the root. Type (2) special vertices correspond to elements that are not the minimal element in their block, and not the maximal either unless the block contains 1. In fact, being in a nonsingleton block and not being the minimum element of the block is equivalent to having descendants, whereas not being the maximum element in the block is equivalent to being the favorite child of a nonroot parent. Finally, the leftmost child of the root is the maximum element of the block containing 1. Therefore we obtain the following rephrasing of Theorem 3.12.

**Theorem 5.5.** Let \((i, j)\) be the type of a shelling component in a \((d - 1)\)-dimensional cubical complex and assume \( j < d - 1 \). Then the coefficient of \( x^{d-m} \) in \( f_d(i, j, x) \) is the number of noncrossing partitions on \( \{1, 2, \ldots, d-1\} \) with exactly \( m \) elements \( v \) having (exactly) one of the following properties:

- \( v \) is the minimum element of a nonsingleton block;
- \( v \) is at most \( i \) or at least \( d - j \) and \( \{v\} \) is a singleton block;
- \( v \) is at least \( d - j \), \( v \) is not the minimum element of its block, and if \( v \) is the maximum element of its block then the block contains 1.

The enumeration problems associated to the results in this section are reminiscent of the problems discussed in the work of Yano and Yoshida [17], and Denise and Simion [6].
Perhaps it is thus not surprising that the toric $g$-polynomial of the $d$-cube explicitly appears in the work of Denise and Simion \[1\]. What is perhaps surprising that this “coincidence” does not seem to have been noted earlier. In fact, Corollary 3.8 in \[1\] states the following:

**Corollary 5.6 (Denise-Simion).** Let $M(n)$ be the set of colored Motzkin paths which start at the origin and end at the point $(n,0)$, that is, lattice paths whose steps are $(+1, +1)$ (North-East), $(+1, −1)$ (South-East), or $(+1, 0)$ (horizontal), and in which a horizontal step is colored red if it is at zero abscissa and is colored either red or blue if it is at a positive abscissa. To each such path $p$ we associate its number, $s(p)$, of occurrences of two consecutive steps of the form $\text{(NE, red)}$ or $\text{(blue, red)}$ or $\text{(blue, SE)}$ or $\text{(NE, SE)}$. Then

$$\sum_{p \in M(n)} t^{s(p)} = \sum_{j \geq 0} (-1)^j (1-t)^j \binom{n-j}{j} C_{n-j}.$$  

As a consequence of equation (7), the above cited result is also a variant of the Gessel-Shapiro result, since it states

$$\sum_{p \in M(n)} t^{s(p)} = g(L_n, t).$$

Corollary 5.6 is a consequence of results on the polynomials $P_n(t)$, which may be defined by the relation

$$g(L_n, t) = (1-t)^{n+1} + t \cdot P_{n+1}(t).$$

The polynomials $P_n(t)$ are related to counting noncrossing partitions, weighted according to the number of their filler points. Denise and Simion \[3\] define a filler of a noncrossing partition $\pi$ as a point $i \in \{2, \ldots, n\}$ such that either $i-1$ and $i$ are in the same block and $i$ is the largest element in its block or $i$ forms a singleton block and $i-1$ is not the largest element of its block. Introducing $m(\pi)$ for the number of filler points of the noncrossing partition $\pi$, according to the proof of \[3\] Lemma 3.3, the polynomial $P_k(t)$ is the total weight of all noncrossing partitions $\pi$ of $\{1, \ldots, k-1\}$ where the weight of $\pi$ is

$$w(\pi) = \begin{cases} \frac{1-(1-t)^k}{t^{m(\pi)-1}} & \text{if } \pi = 1/2/\ldots/k-1, \\ 1 & \text{otherwise.} \end{cases}$$

The proof of this fact is very similar to our proof of our Lemma 3.3. It begins with stating that one may start with an arbitrary noncrossing partition of $k-1$ elements (counted by a Catalan number) and then insert $j$ additional points in any of $\binom{k-1-j}{j}$ ways. The $j$ additional points inserted will be the (marked) filler points, the rest of the proof is slightly different only because a different weight function is used. That said, from a combinatorial perspective, inserting a filler after a prescribed element in a non-crossing partition is an operation that behaves exactly the same way as attaching a first-born leaf to a nonroot vertex. Thus, in analogy of Lemma 3.3 we have the following statement.

**Lemma 5.7 (Denise-Simion).** The coefficient of $x^k$ in $g(L_d, x)$ is the number of noncrossing partitions on $\{1, 2, \ldots, d\}$ with $k$ fillers.
Lemma 5.7 appears in the implicit form of \( \sum_{p \in M(n)} t^{s(p)} = \sum_{\pi \in NC(n)} t^{m(\pi)} \) in the proof of [6, Corollary 3.8]. It is also worth noting that the proof of [6, Remark 3.4], involving the description of all non-crossing partitions with exactly one filler point involuntarily outlines a “blueprint” for a new construction of a simplicial complex whose \( h \)-polynomial is the toric \( h \)-polynomial of a cube. By working out small examples it is easy to verify that this simplicial complex is not isomorphic in general to the one introduced by Billera, Chan, and Liu [3, Theorem 3.2].

We may obtain a new recursion formula for the polynomials \( g(L_n, t) \) by observing a consequence of [6, Lemma 3.3]. This states that the polynomials \( P_k(t) \) satisfy the following recursion.

\[
P_k(t) = (1 - t)^{k - 1} + \sum_{i=1}^{k-1} ((1 - t)^i P_{k-i}(t) + tP_i(t)P_{k-i}(t)).
\]

Using (13) we may show the following.

**Proposition 5.8.** Introducing \( g_n(t) := g(L_n, t) \) we have the following recursion.

\[
g_k(t) = (1 - t)^k + \sum_{i=1}^{k} g_{i-1}(t) \left( g_{k-i}(t) - (1 - t)^{n-i} \right).
\]

**Proof.** (13) implies

\[
P_n(t) = \frac{g_{n-1}(t) - (1 - t)^n}{t}.
\]

Substituting this into (14) we obtain that \( \frac{g_{k-1}(t) - (1 - t)^k}{t} \) is the sum of \( (1 - t)^{k-1} \) and of

\[
\sum_{i=1}^{k-1} \left( (1 - t)^i \frac{g_{k-i-1}(t) - (1 - t)^{k-i}}{t} + (g_{i-1}(t) - (1 - t)^{n-i}) \right).
\]

Multiplying both sides by \( t \) and adding \( (1 - t)^k \) to both sides yields

\[
g_{k-1}(t) = (1 - t)^k + t(1 - t)^{k-1}
\]

\[
+ \sum_{i=1}^{k-1} ((1 - t)^i (g_{k-i}(t) - (1 - t)^{k-i}) + (g_{i-1}(t) - (1 - t)^{n-i}) (g_{k-i-1}(t) - (1 - t)^{k-i}))
\]

\[
= (1 - t)^k + t(1 - t)^{k-1} + \sum_{i=1}^{k-1} (1 - t)^i g_{k-i-1}(t) - (k - 1)(1 - t)^k
\]

\[
+ \sum_{i=1}^{k-1} g_{i-1} g_{k-i-1} - \sum_{i=1}^{k-1} (1 - t)^i (g_{k-i-1}(t) - \sum_{i=1}^{k-1} g_{i-1}(t)(1 - t)^{k-i} + (k - 1)(1 - t)^k.
\]

Canceling the terms \( \sum_{i=1}^{k-1} (1 - t)^i g_{k-i-1}(t) \) and \( (k - 1)(1 - t)^k \) which also appear with a negative sign and shifting the index \( k \) up by one yields the stated identity. \( \square \)
We conclude this section by observing that Corollary 5.6 links the study of the cubical toric $h$-polynomials to the combinatorial theory of orthogonal polynomials developed by Viennot [16]. This theory uses weighted Motzkin paths to find the moment functional of an orthogonal polynomial sequence given by a recursion formula. Recall that a moment functional is a linear map $f : \mathbb{K}[x] \rightarrow \mathbb{K}$ from a polynomial ring to its field of scalars, and it is uniquely defined by the moments $f(x^n)$. A sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ is orthogonal with respect to the moment functional $f$ if $f(p_k(x) \cdot p_l(x)) = 0$ whenever $k \neq l$. The following statement is a consequence of \cite{16} Proposition 17.

**Theorem 5.9** (Viennot). If an orthogonal polynomial sequence $\{p_n(x)\}_{0}^{\infty}$ is defined by the recursion formula
\[
p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_np_{n-1}(x) \quad \text{for } n \geq 1,
\]
subject to the initial conditions $p_0(x) = 1$ and $p_1(x) = x - b_0$, then the sequence $\{p_n(x)\}_{n=0}^{\infty}$ is orthogonal with respect to the moment functional $f : \mathbb{K}[x] \rightarrow \mathbb{K}$ where $f(x^n)$ is the total weight of all Motzkin paths from $(0,0)$ to $(n,0)$ such that each NE step has weight 1, each horizontal step at abscissa $i$ has weight $b_i$ and each SE step starting at abscissa $i$ has weight $\lambda_i$.

Thus Corollary 5.6 suggests that the polynomials $g(L_n, x)$ are related to the orthogonal polynomial sequence $\{p_n(x)\}_{n=0}^{\infty}$, defined by the recursion formula
\[
p_{n+1}(x) = (x - 2)p_n(x) - p_{n-1}(x) \quad \text{for } n \geq 1,
\]
subject to the initial conditions $p_0(x) = 1$ and $p_1(x) = x - 1$. Introducing
\[
b_n(x) := (-1)^np_n(-x),
\]
we obtain the recursion formula
\[
b_{n+1}(x) = (-1)^{n+1}p_{n+1}(x) = (-1)^{n+1}((-x - 2)p_n(-x) - p_{n-1}(-x))
\]
\[
= (x + 2)b_n(x) - b_{n-1}(x) \quad \text{for } n \geq 1,
\]
subject to the initial conditions $b_0(x) = 1$ and $b_1(x) = x + 1$. These are exactly the Morgan-Voyce polynomials which are widely studied. They first appeared in the study of electrical networks \cite{9}, some of the other early references include Swamy’s work \cite{14} and \cite{15}, the latest publication on them appeared in 2008 \cite{13}. By Corollary 5.6, the linear map defined by $x^n \mapsto g(L_n, x)$ may be considered as a polynomial generalization of the moment functional of the Morgan-Voyce polynomials.

6. Concluding remarks

While giving a few modest answers, this paper raises many questions. The first ones are inspired by the results of Section 4. What we can observe, that by applying a sequence of involutions (taking mirror images of appropriate objects after and before applying a less trivial bijection and its inverse) we obtain a different model, in which the the definitions of the special vertices become overall a little less technical. There are many other ways to transform Chan’s model, but when we take a “random” transformation, we
usually end up with highly technical definitions for our special vertices. In this sense, the transformation presented in Section 4 appears to be a “lucky guess” (actually it was not, the new model was built directly, inspired by the formulas in Section 2, the bijection was found later). Is there a way to recode the original model into an even simpler one? Or is it better to start with a completely different model, inspired by some other form of the formulas in Section 2? At a more philosophical level, it seems that “easy” combinatorial transformations take one statistics of plane trees into another statistic of plane trees. Analogous results for permutation statistics have been produced for over a hundred year, perhaps it is worthwhile to take a similar systematic approach to the study of various statistics on plane trees.

We have a strong reason to suspect that the variant of the Gessel-Shapiro result that may be found in the work of Denise and Simion [6] will not be shown equivalent via some bijection that is similar to the one presented in Section 4. As it was mentioned after Lemma 5.7, it is possible to construct a new simplicial complex verifying the Billera-Chan-Liu result [3, Theorem 3.2] stating that the toric $h$-polynomial of a cube is the $h$-polynomial of a simplicial complex. However, the resulting simplicial complex will not be isomorphic to the one that can be found in [3]. For the postorder tree model, obviously, one ends up constructing essentially the same simplicial complex (associated to the same noncrossing partitions). Thus it may be worthwhile to generalize the Denise-Simion colored Motzkin path enumeration problem for $f_d(0, 0, x)$ to questions whose answers are given by the polynomials $f_d(i, j, x)$. This seems feasible since, as it was noted above, essentially the same insertion technique may be used to prove Lemma 5.7 as the one used to show Lemma 3.3. One only needs to come up with the appropriate analog of type (1) and type (2) special elements such that the analogues of Lemma 3.8 and Lemma 3.10 become valid.

Obviously it would be great to use one of the new variants (presented or to be worked out) to extend the results of Billera, Chan, and Liu [3] to all shellable cubical complexes. This seems hard, because after careful reading of [3] we will discover that the construction does not only depend on constructing shellable simplicial complexes associated to cubical shelling components but one also changes the underlying cubical complex. No further generalization seems possible there without modifying or generalizing that rearranging of cubes.

Finally the most interesting and enigmatic question is raised by the connection between the polynomials $g(L_n, x)$ and the Morgan-Voyce polynomials. Is it a coincidence or is there a deeper reason? Is there a way to relate other $g$-polynomials to orthogonal polynomials? Is there a way to extend the correspondence to all polynomials $f_d(i, j, x)$? If nothing else, these questions certainly make a future “third look” at the toric $h$-polynomials of cubical complexes worthwhile.

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Department of Mathematics and Statistics, UNC Charlotte, Charlotte, NC 28223

E-mail address: ghetyei@uncc.edu