A parabolic equation on domains with random boundaries

Duong Thanh Pham∗ Thanh Tran†‡

Abstract

A heat equation with uncertain domains is thoroughly investigated. Statistical moments of the solution is approximated by the counterparts of the shape derivative. A rigorous proof for the existence of the shape derivative is presented. Boundary integral equation methods are used to compute statistical moments of the shape derivative.

1 Introduction

Parabolic partial differential equations arise in a wide family of science, including heat diffusion, ocean acoustic propagation, physical and mathematical systems with a time variable, and processes having behaviour of heat diffusion through a solid. A typical example of parabolic partial differential equations is the heat equation that describes distribution of heat in a given region over time. Provided that the problem parameters are known exactly, nowadays powerful computers together with advanced numerical schemes are capable of producing highly accurate deterministic numerical solutions.

However, in reality problem parameters are prone to uncertainty for many reasons. First, the parameters are often obtained through inexact measurements due to imperfect measurement devices. Second, the parameters are approximated from a large but finite number of system samples; this approximation can be incomplete or stochastic. Finally, mathematical problems are themselves only approximations of the actual processes. Under these circumstances, numerical results of a finite number of deterministic simulations with a finite set of problem parameters are of limited use. An important paradigm, becoming rapidly popular over the past years, see e.g. [1, 2, 5, 6, 8, 9, 10, 11, 12, 16, 17] and the references therein, is to treat the lack of knowledge via modelling uncertain parameters as random fields.

In this paper we consider the following initial-boundary value problem

\[
\begin{align*}
\frac{\partial u}{\partial t}(\omega) - \Delta u(\omega) &= f & \text{in } Q_T(\omega) := (0, T) \times U(\omega), \\
Bu(\omega) &= 0 & \text{on } \Sigma_T(\omega) := (0, T) \times \partial U(\omega) \\
u(\omega)|_{t=0} &= g & \text{in } U(\omega)
\end{align*}
\]

where \(Bu(\omega) = 0\) indicates either the Dirichlet boundary condition

\[
u(t, x; \omega) = 0, \quad (t, x) \in (0, T) \times \partial U(\omega),
\]

or the Neumann boundary condition

\[
\frac{\partial u(t, x; \omega)}{\partial n} = 0, \quad (t, x) \in (0, T) \times \partial U(\omega).
\]
Here, the domain $U(\omega)$ and so does its boundary $\partial U(\omega)$ depend on a "random event" $\omega \in \Omega$, where $(\Omega, \Sigma, \mathbb{P})$ is a generic complete probability space. In this paper, we shall estimate probabilistic properties of the solution perturbation $u(\omega) - \mathbb{E}[u]$. We postpone until the next section a precise description of the random domain $U^\epsilon(\omega)$ and random boundary $\Gamma^\epsilon(\omega)$.

In this article, we develop a deterministic method for numerical solution to the problems \([1.1a] - [1.1c]\) with either \([1.2]\) or \([1.3]\), respectively. In this model, the spatial domain on which the problem is stated depends on the "random event" $\omega$ and the parameter $\epsilon > 0$ controlling the amplitude of the perturbation. Thus, the solution depends on $\omega$ and $\epsilon$ and is denoted by $u^\epsilon(\omega)$. The case $\epsilon = 0$ corresponds to the zero perturbation and the solution is denoted by $u^0$. In the paper, we shall estimate probabilistic properties of the solution perturbation $u^\epsilon(\omega) - u^0$ when the perturbation amplitude is small.

2 Preliminaries

2.1 Sobolev spaces

In this subsection we introduce the function spaces needed for the forthcoming analysis. Let $U$ be a bounded domain in $\mathbb{R}^3$. The Sobolev space $H^1(U)$ is defined, as usual, as the space of all distributions which together with their first order partial derivatives are square integrable. The corresponding norm $\| \cdot \|_{H^1(U)}$ is defined by

$$
\| v \|_{H^1(U)} := \left( \int_U \left( |v(x)|^2 + |\nabla v(x)|^2 \right) \, dx \right)^{1/2}.
$$

The space $H^1_0(U)$ is the space of all functions in $H^1(U)$ vanishing on the boundary $\partial U$ of $U$. The following Friedrich–Poincare inequality (see e.g. [19, Page 61]) will be frequently used in this paper.

**Lemma 2.1.** In the Sobolev space $H^1_0(U)$, the seminorm

$$
|v|_{H^1(U)} := \left( \int_U \left( |\nabla v(x)|^2 \right) \, dx \right)^{1/2}
$$

is a norm and it is equivalent to the norm given by (2.1).

We denote by $H^{-1}(U)$ the dual space of $H^1_0(U)$ with the norm

$$
\| v \|_{H^{-1}(U)} := \sup_{w \in H^1_0(U), w \neq 0} \frac{\langle v, w \rangle}{\|w\|_{H^1(U)}}, \quad v \in H^{-1}(U).
$$

The Sobolev space $H^{1/2}(\partial U)$ is defined by

$$
H^{1/2}(\partial U) = \{ g : \partial U \to \mathbb{R} \mid g = v \text{ on } \partial U \text{ (in the trace sense) for some } v \in H^1(U) \}
$$

and equipped with the following norm

$$
\| g \|_{H^{1/2}(\partial U)} := \inf \{ \| v \|_{H^1(U)} : v \in H^1(U) \text{ and } g = v|_{\partial U} \}.
$$

The dual space of $H^{1/2}(\partial U)$ is denoted by $H^{-1/2}(\partial U)$.

In the study of parabolic PDEs, it is important to identify functions $v : [a, b] \times U \to \mathbb{R}$ which maps from $[a, b]$ into a Banach space. Let $X$ denote a Banach space with the norm $\| \cdot \|_X$. The space $L^p(a, b; X)$ is the space of all functions $v : [a, b] \to X$ so that $v(t) \in X$ for almost all $t \in [a, b]$. The $L^p(a, b; X)$-norm of $v$ is given by

$$
\| v \|_{L^p(a, b; X)} = \left\{ \begin{array}{ll}
\left( \int_a^b \| v(t) \|_X^p \, dt \right)^{1/p} & \text{if } 1 \leq p \leq \infty \\
\text{ess sup} \| v(t) \|_X & \text{if } p = \infty.
\end{array} \right.
$$

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In this paper, we often work on the space \( L^p(a, b; X) \) where \( p = 2 \). The space \( H^1(a, b; X) \) is a subspace of \( L^2(a, b; X) \) consisting all functions \( v : [a, b] \times X \to \mathbb{R} \) satisfying \( \partial v/\partial t \in L^2(a, b; X) \). The corresponding norm is defined by

\[
\|v\|_{H^1(a, b; X)} = \left( \int_a^b \left[ \|v(t)\|_X^2 + \left\| \frac{\partial v(t)}{\partial t} \right\|_X^2 \right] dt \right)^{1/2}.
\]

The space \( C([a, b]; X) \) consists of continuous function \( v : [a, b] \to X \). The \( C([a, b]; X) \)-norm is given by

\[
\|v\|_{C([a, b]; X)} = \max_{t \in [a, b]} \|v(t)\|_X.
\]

We note here that the spaces \( L^p(a, b; X) \), \( H^1(a, b; X) \) and \( C([a, b]); X) \) are Banach spaces for all \( p \geq 1 \).

### 2.2 Bochner spaces

Throughout this paper we denote by \( (\Omega, \Sigma, \mathbb{P}) \) a generic complete probability space. Let \( X \) be a Banach space. For any \( 1 \leq k \leq \infty \), the Bochner space \( \mathcal{L}^k(\Omega, X) \) is defined as usual by

\[
\mathcal{L}^k(\Omega, X) := \left\{ v : \Omega \to X, \text{ measurable : } \|v\|_{\mathcal{L}^k(\Omega, X)} < \infty \right\}
\]

with the norm

\[
\|v\|_{\mathcal{L}^k(\Omega, X)} := \begin{cases} \left( \int_\Omega \|v(\omega)\|_X^k \, d\mathbb{P}(\omega) \right)^{1/k}, & 1 \leq k < \infty, \\ \text{esssup}_{\omega \in \Omega} \|v(\omega)\|_X, & k = \infty. \end{cases}
\]

The elements of \( \mathcal{L}^k(\Omega, X) \) are called random fields. We remark that for a part of the subsequent analysis we may restrict to the special case when \( X \) is a Hilbert space. In particular, when \( X_1 \) and \( X_2 \) are two separable Hilbert spaces, their tensor product \( X_1 \otimes X_2 \) is a separable Hilbert space with the natural inner product extended by linearity from \( \langle v \otimes a, w \otimes b \rangle_{X_1 \otimes X_2} = \langle v, w \rangle_{X_1}(a, b)_{X_2} \), cf. e.g. [13 p. 20], [3] Definition 12.3.2, p.298]. In this paper we work with \( k \)-fold tensor products of Hilbert spaces

\[
X^{(k)} := X \otimes \cdots \otimes X.
\]

with the natural inner product satisfying \( \langle v_1 \otimes \cdots \otimes v_k, w_1 \otimes \cdots \otimes w_k \rangle_{X^{(k)}} = \langle v_1, w_1 \rangle_X \cdots \langle v_k, w_k \rangle_X \).

**Definition 2.2.** For a random field \( v \in \mathcal{L}^k(\Omega, X) \), its \( k \)-order moment \( \mathcal{M}^k[v] \) is an element of \( X^{(k)} \) defined by

\[
\mathcal{M}^k[v] := \int_\Omega \left( v(\omega) \otimes \cdots \otimes v(\omega) \right) \, d\mathbb{P}(\omega).
\]

In the case \( k = 1 \), the statistical moment \( \mathcal{M}^1[v] \) coincides with the mean value of \( v \) and is denoted by \( \mathbb{E}[v] \). If \( k \geq 2 \), the statistical moment \( \mathcal{M}^k[v] \) is the \( k \)-point autocorrelation function of \( v \). The quantity \( \mathcal{M}^k[v - \mathbb{E}[v]] \) is termed the \( k \)-th central moment of \( v \). We distinguish in particular second order moments: the correlation and covariance defined by

\[
\text{Cor}[v] := \mathcal{M}^2[v] \quad \text{and} \quad \text{Covar}[v] := \mathcal{M}^2[v - \mathbb{E}[v]].
\]

In this paper we work with \( X \) being Sobolev spaces of real-valued functions defined on a domain \( U \subseteq \mathbb{R}^3 \) yielding, in particular, the representation

\[
\text{Cor}[v](x, y) := \int_\Omega v(x, \omega) v(y, \omega) \, d\mathbb{P}(\omega), \quad x, y \in U.
\]

We observe that \( \text{Cor}[v] \) is defined on the Cartesian product \( U \times U \). Similarly, \( \mathcal{M}^k[v] \) is defined on the \( k \)-fold Cartesian product \( U \times \cdots \times U \). Here, the dimension of the underlying domain grows rapidly with increasing moment order \( k \).
2.3 Random domains

In this subsection, we describe the random domain and its boundary on which the initial-boundary value problem (1.1) is stated. Let $U^0$ be a fixed bounded domain in $\mathbb{R}^n$, $n = 2, 3$. Then the boundary $\Gamma^0 := \partial U^0$ is a closed manifold in $\mathbb{R}^n$. We assume that $\Gamma^0 \in C^{1,1}$ so that the outward normal vector $n^0$ to $\Gamma^0$ satisfies $n^0 \in C^{0,1}(\Gamma^0)$. Suppose that $\kappa \in \mathcal{L}^k(\Omega, C^{0,1}(\Gamma^0))$ is a random field, i.e., for almost any realization $\omega \in \Omega$, we have $\kappa(\cdot, \omega) \in C^{0,1}(\Gamma^0)$. For some sufficiently small, nonnegative $\epsilon$, we consider a family of random closed surfaces of the form

$$\Gamma^\epsilon(\omega) = \{ x + \epsilon \kappa(x, \omega)n^0(x) : x \in \Gamma^0 \}, \quad \omega \in \Omega. \quad (2.11)$$

The bounded domain which is surrounded by $\Gamma^\epsilon(\omega)$ is denoted by $U^\epsilon(\omega)$. Here, the uncertainty is represented by the uncertainty in $\kappa(\cdot, \omega)$. We assume further that the random perturbation amplitude $\kappa(x, \omega)$ is centered, i.e.,

$$\mathbb{E}[\kappa(x, \cdot)] = 0 \quad \forall x \in \Gamma^0, \quad (2.12)$$

and $\kappa$ is uniformly bounded, i.e., there exist bounded domain $U_-$ and $U_+$ satisfying

$$U_- \subset U^\epsilon(\omega) \subset U_+ \quad \forall \omega \in \Omega, \quad \forall \epsilon \leq \epsilon_0, \quad (2.13)$$

for some sufficiently small and positive $\epsilon_0$. Due to (2.12), the mean random boundary satisfies

$$\mathbb{E}[\Gamma^\epsilon] = \{ x + \epsilon \mathbb{E}[\kappa]n^0(x) : x \in \Gamma^0 \} = \Gamma^0$$

and $\text{Covar}[\kappa] = \text{Cor}[\kappa]$. We consider initial-boundary value problem on random domains $U^\epsilon(\omega)$,

$$u^\epsilon_t(\omega) - \Delta u^\epsilon(\omega) = f \quad \text{in } Q_T^\epsilon(\omega) := (0, T) \times U^\epsilon(\omega)$$
$$Bu^\epsilon(\omega) = 0 \quad \text{on } \sigma_T^\epsilon(\omega) := (0, T) \times \Gamma^\epsilon(\omega)$$
$$u^\epsilon(\omega)|_{t=0} = g \quad \text{in } U^\epsilon(\omega). \quad (2.14)$$

The randomnesses in the domain $U^\epsilon(\omega)$ and its boundary result in randomness of the solution $u^\epsilon(\cdot, \omega)$. Here, the solution operator $u^\epsilon(\cdot, \omega) = \text{Sol}(U^\epsilon(\omega))$ is nonlinear. Thus, linearisation by using shape calculus is in demand. In this process, existence of a shape derivative of the solutions of deterministic perturbed problems has to be clarified. The shape derivative will then be used to approximate statistical moments of the solution.

3 Shape calculus

3.1 Deterministic perturbed domains

In this section, we aim to prove the existence of shape derivative of the solution $u^\epsilon$ of (2.14), which will then be used in linearisation development of the solution $u^\epsilon$ with respect to the perturbed domain $U^\epsilon$. In this section, we temporarily stay away from randomness and only work on deterministic perturbed domains. Let $U^0$ be a fixed bounded domain in $\mathbb{R}^n$, $n = 2, 3$. Assume that the boundary $\Gamma^0 := \partial U^0$ is a closed manifold in $\mathbb{R}^n$ satisfying $\Gamma^0 \in C^{1,1}$. Let $\kappa \in C^{0,1}(\Gamma^0)$. For any $\epsilon \in (0, \epsilon_0)$, where $\epsilon_0$ is some sufficiently small positive number, we consider a family of deterministic closed surfaces of the form

$$\Gamma^\epsilon = \{ x + \epsilon V(x) : x \in \Gamma^0 \}. \quad (3.1)$$

The bounded domain surrounded by $\Gamma^\epsilon$ is denoted by $U^\epsilon$. Analogously to (2.13), we assume that $V$ is uniformly bounded, i.e., there exist bounded domain $U_-$ and $U_+$ satisfying

$$U_- \Subset U^\epsilon \Subset U_+ \quad \forall \epsilon \leq \epsilon_0, \quad (3.2)$$
where \( U^\epsilon \Subset U_+ \) means that the closures of all \( U^\epsilon \) are proper subsets of \( U_+ \) for all \( \epsilon \leq \epsilon_0 \). Following [18], we define a mapping \( T^\epsilon : U_+ \to U_+ \) which transforms \( \Gamma^0 \) into \( \Gamma^\epsilon \) and \( U^0 \) into \( U^\epsilon \), respectively, by

\[
T^\epsilon(x) := x + \epsilon \tilde{V}(x), \quad x \in U_+,
\]

where \( \tilde{V} \) is any smoothness-preserving extensions of \( V \). Without loss of generality, we may assume that

\[
\text{supp}(\tilde{V}) := \{x \in U_+ : V(x) \neq 0\}
\]

is a proper subset of \( U_+ \). Denoting by

\[
U^\star := \text{supp}(\tilde{V}),
\]

the set \( U^\star \) is a compact subset in \( U_+ \). In this paper, we require in particular that \( \tilde{V} \in W^{1,\infty}(U_+) \). For the ease of notation we also use \( V \) for its extension in the rest of the paper. In [18], \( V \) is called the velocity field of the mapping \( T^\epsilon \). In the present paper, for any function \( v \) defined on the \([0, T] \times U^\epsilon \), we denote

\[
v \circ T^\epsilon(t, x) := v(t, T^\epsilon(x)), \quad (t, x) \in [0, T] \times U^0
\]

for notational convenience.

In the subsequent analysis, for any 3 by 3 matrix \( M(x) \) whose entries are functionals of \( x \in U_+ \subset \mathbb{R}^3 \), we denote

\[
\|M(\cdot)\|_{L^p(U)} := \max_{i,j=1,2,3} \{\|M_{i,j}(\cdot)\|_{L^p(U)}\}, \quad 1 \leq p \leq \infty,
\]

where \( M_{i,j} \) are components of \( M \). In this section, we assume that \( T^\epsilon \) is defined by (3.3) where \( \tilde{V} \in W^{1,\infty}(U_+) \), and denote its Jacobian matrix and Jacobian determinant by \( J_{T^\epsilon}(\cdot) \) and \( \gamma(\epsilon, \cdot) \), respectively. It can be proved that a function \( v \) belongs to \( H^1(U^\epsilon) \) (\( H^1_0(U^\epsilon) \) or \( L^2(U^\epsilon) \), resp.) if and only if \( v \circ T^\epsilon \) belongs to \( H^1(U^0) \) (\( H^1_0(U^0) \) or \( L^2(U^0) \), resp.) and there hold

\[
\|v \circ T^\epsilon\|_{H^1(U^0)} \simeq \|v\|_{H^1(U^\epsilon)}, \quad v \in H^1(U^\epsilon),
\]

\[
\|v \circ T^\epsilon\|_{L^2(U^0)} \simeq \|v\|_{L^2(U^\epsilon)}, \quad v \in L^2(U^\epsilon).
\]

The following lemmas which will be frequently used in the rest of this section state some important properties of the transformation \( T^\epsilon \).

**Lemma 3.1.** Assume that \( V \in W^{1,\infty}(U_+) \). The Jacobian determinant \( \gamma(\epsilon, \cdot) \) of the transformation \( T^\epsilon \) satisfies

\[
\lim_{\epsilon \to 0} \|\gamma(\epsilon, \cdot) - 1\|_{L^\infty(U_+)} = 0,
\]

and

\[
\lim_{\epsilon \to 0} \left\| \frac{\gamma(\epsilon, \cdot) - 1}{\epsilon} - \text{div} V \right\|_{L^\infty(U_+)} = 0.
\]

**Proof.** Recalling (3.3), we denote \( V(x) := (V_1(x), V_2(x), V_3(x))^T \). The Jacobian matrix and the Jacobian determinant of \( T^\epsilon \) are given by

\[
J_{T^\epsilon}(x) = \begin{bmatrix}
1 + \epsilon \frac{\partial V_1(x)}{\partial x_1} & \epsilon \frac{\partial V_1(x)}{\partial x_2} & \epsilon \frac{\partial V_1(x)}{\partial x_3} \\
\epsilon \frac{\partial V_2(x)}{\partial x_1} & 1 + \epsilon \frac{\partial V_2(x)}{\partial x_2} & \epsilon \frac{\partial V_2(x)}{\partial x_3} \\
\epsilon \frac{\partial V_3(x)}{\partial x_1} & \epsilon \frac{\partial V_3(x)}{\partial x_2} & 1 + \epsilon \frac{\partial V_3(x)}{\partial x_3}
\end{bmatrix}
\]
and
\[
\gamma(\epsilon, \mathbf{x}) = 1 + \epsilon \left( \sum_{k=1}^{3} \frac{\partial V_k(x)}{\partial x_k} + \epsilon^2 \left( \sum_{k,l=1}^{3} \frac{\partial V_k(x)}{\partial x_k} \frac{\partial V_l(x)}{\partial x_l} - \frac{\partial V_l(x)}{\partial x_l} \frac{\partial V_k(x)}{\partial x_k} \right) \right) \\
+ \epsilon^3 \left| \sum_{i,j,k=1}^{3} \text{sign}(i,j,k) \frac{\partial V_i(x)}{\partial x_i} \frac{\partial V_j(x)}{\partial x_j} \frac{\partial V_k(x)}{\partial x_k} \right|
\]
\[
=: 1 + \epsilon \gamma_1(x) + \epsilon^2 \gamma_2(x) + \epsilon^3 \gamma_3(x).
\] (3.9)

Here \(\text{sign}(i,j,k)\) denotes the sign of the permutation \((i,j,k)\). The entries \(A_{ij}(\epsilon, \mathbf{x})\), \(i,j = 1,2,3\), of the matrix \(A(\epsilon, \mathbf{x})\) are given by
\[
A_{ij}(\epsilon, \mathbf{x}) = \gamma(\epsilon, \mathbf{x})^{-1} \left( \delta_{ij} + \sum_{n=1}^{4} \epsilon^n h_{ijn}(\mathbf{x}) \right),
\] (3.10)

where \(h_{ijn}\) is a polynomial of partial derivatives of \(V\) and \(\delta_{ij}\) is the Kronecker delta. Since \(V \in W^{1,\infty}(U_+)\), we deduce
\[
\gamma_n, h_{ijn} \in L^\infty(U_+) \cap L^2(U_+), \quad i,j = 1,2,3 \text{ and } n = 1,\ldots, 4,
\]
\[
\lim_{\epsilon \to 0} \|\gamma(\epsilon, \cdot)\|_{L^n(U_+)} > 0,
\] (3.11)

where \(\gamma_1, \gamma_2, \gamma_3\) are defined by (3.9) and \(\gamma_4 := 0\) for notational convenience later. In particular, for sufficiently small \(\epsilon > 0\), there holds
\[
\gamma(\epsilon, \mathbf{x}) = 1 + \epsilon \gamma_1(x) + \epsilon^2 \gamma_2(x) + \epsilon^3 \gamma_3(x) \geq c > 0 \quad \forall \mathbf{x} \in U_+.
\] (3.12)

We then have
\[
\lim_{\epsilon \to \infty} \|\gamma(\epsilon, \cdot) - 1\|_{L^n(U_+)} = \lim_{\epsilon \to \infty} \epsilon \left\| \gamma_1 + \epsilon \gamma_2 + \epsilon^2 \gamma_3 \right\|_{L^n(U_+)} = 0,
\]
noting (3.11). Furthermore, it follows from (3.12) and (3.9) that
\[
\lim_{\epsilon \to 0} \left\| \frac{\gamma(\epsilon, \cdot) - 1}{\epsilon} - \text{div} V \right\|_{L^n(U_+)} = \epsilon \|\gamma_2 + \epsilon \gamma_3\|_{L^n(U_+)}.
\]

Letting \(\epsilon\) go to zero and noting (3.11), we obtain (3.7), completing the proof of the lemma.

\[\square\]

\textbf{Lemma 3.2.} Assume that \(V \in W^{1,\infty}(U_+)\). Consider \(A(\epsilon, \cdot) := \gamma(\epsilon, \cdot)J_{\mathcal{T}}^{-1}J_{\mathcal{T}}^\top\), where \(J_{\mathcal{T}}^\top\) is the transpose of \(J_{\mathcal{T}}\). Then there hold
\[
\lim_{\epsilon \to 0} \|A(\epsilon, \cdot) - I\|_{L^n(U_+)} = 0
\] (3.13)

and
\[
\lim_{\epsilon \to 0} \left\| \frac{A(\epsilon, \cdot) - I}{\epsilon} - A'(0, \cdot) \right\|_{L^n(U_+)} = 0.
\] (3.14)

Here, \(A'(0, \cdot)\) is the Gâteaux derivative of \(A(\epsilon, \cdot)\) at \(\epsilon = 0\), namely
\[
A'(0, \mathbf{x}) = \lim_{\epsilon \to 0} \frac{A(\epsilon, \mathbf{x}) - I(\mathbf{x})}{\epsilon}, \quad \mathbf{x} \in U_+.
\]

\textbf{Proof.} The proof of the lemma can be done in the same manner as the proof of [4] Lemma 3.1. \[\square\]
Lemma 3.3. For any function \( v \in L^2([0, T] \times U^*) \), there holds
\[
\lim_{\epsilon \to 0} \| v \bar{T}^\epsilon(x, \cdot) - v \|_{L^2(0, T; L^2(U^*))} = 0. \quad (3.15)
\]

Proof. We then have
\[
\int_0^T \| (\gamma(\epsilon, \cdot) - 1) v \bar{T}^\epsilon(\tau) \|^2_{L^2(U^0)} d\tau \leq \| \gamma(\epsilon, \cdot) - 1 \|^2_{L^\infty(U^0)} \int_0^T \| v \bar{T}^\epsilon(\tau) \|^2_{L^2(U^0)} d\tau
\leq C \epsilon^2 \int_0^T \| v \bar{T}^\epsilon(\tau) \|^2_{L^2(U^0)} d\tau. \quad (3.16)
\]
Using the change of variables \( y = T^\epsilon(x) \) and noting (3.11), we have
\[
\| v \bar{T}^\epsilon(\tau) \|^2_{L^2(U^0)} = \int_{U^*} |v(\tau, y)|^2 \left( \gamma(\epsilon, (T^\epsilon)^{-1}(y)) \right)^{-1} d\gamma \leq C \| v(\tau) \|^2_{L^2(U^*)} \quad \forall \tau \in (0, T).
\]

Here, the constant \( C \) is independent of \( \tau \). Thus,
\[
\int_0^T \| v \bar{T}^\epsilon(\tau) \|^2_{L^2(U^0)} d\tau \leq \int_0^T \| v(\tau) \|^2_{L^2(U^*)} d\tau \leq \int_0^T \| v(\tau) \|^2_{L^2(U^*)} d\tau.
\]
This together with (3.16) implies
\[
\int_0^T \| (\gamma(\epsilon, \cdot) - 1) v \bar{T}^\epsilon(\tau) \|^2_{L^2(U^0)} d\tau \leq \epsilon^2 \int_0^T \| v(\tau) \|^2_{L^2(U^*)} d\tau.
\]
Therefore,
\[
\lim_{\epsilon \to 0} \int_0^T \| (\gamma(\epsilon, x) - 1)v \bar{T}^\epsilon(\tau) \|^2_{L^2(U^0)} d\tau = 0. \quad (3.17)
\]
Assume that \( v \) belongs to \( C([0, T] \times U^*) \). Then \( v \) is continuous and thus uniformly continuous on the compact set \([0, T] \times U^* \). Furthermore, since \( \bar{\kappa} \in W^{1, \infty}(U^*) \), there holds
\[
\lim_{\epsilon \to 0} \| T^\epsilon(x) - x \|_{L^\infty(U^*)} = 0.
\]
Thus,
\[
\lim_{\epsilon \to 0} \| v \bar{T}^\epsilon - v \|_{L^\infty([0, T] \times U^*)} = 0.
\]
Noting (3.4), the difference \( v \bar{T}^\epsilon - v \) vanishes outside \( U^* \) and thus
\[
\lim_{\epsilon \to 0} \int_0^T \| (v \bar{T}^\epsilon - v)(\tau) \|^2_{L^2(U^*)} d\tau = \lim_{\epsilon \to 0} \int_0^T \| (v \bar{T}^\epsilon - v)(\tau) \|^2_{L^2(U^*)} d\tau = 0.
\]
By using a density argument we deduce that
\[
\lim_{\epsilon \to 0} \int_0^T \| (v \bar{T}^\epsilon - v)(\tau) \|^2_{L^2(U^*)} d\tau = 0 \quad \forall v \in L^2([0, T] \times U^*).
\]
This together with (3.17) (by using the triangle inequality) yields the desired equality. \( \square \)

Lemma 3.4. For any function \( v \in H^1(U^*) \), there hold
\[
\lim_{\epsilon \to 0} \left\| \frac{v \bar{T}^\epsilon - v}{\epsilon} - \nabla v \right\|_{L^2(U^*)} = 0, \quad (3.18)
\]
\[
\lim_{\epsilon \to 0} \left\| \frac{\gamma(\epsilon, \cdot)(v \bar{T}^\epsilon) - v}{\epsilon} - \text{div} \left( vV \right) \right\|_{L^2(U^*)} = 0. \quad (3.19)
\]
Proof. First of all, the equality (3.18) can be proved by using the density argument in which we shall prove (3.18) for an arbitrary function \( v \in C^\infty(U_+) \) and because of the density of \( C^\infty(U_+) \) in \( H^1(U_+) \), the equality is also true for all functions in \( H^1(U_+) \). Indeed, let \( v \in C^\infty(U_+) \). Applying the mean value theorem, for any \( x \in U^* \), there exists a \( \theta x \in (0, 1) \) such that

\[
 v(T^\epsilon(x)) - v(x) = (T^\epsilon(x) - x) \cdot \nabla v(\theta x T^\epsilon(x)) + (1 - \theta x)x.
\]

This gives, noting (3.3),

\[
\frac{v(T^\epsilon(x)) - v(x)}{\epsilon} - V(x) \cdot \nabla v(x) = V(x) \cdot \nabla \left[ v(\theta x T^\epsilon(x)) + (1 - \theta x)x - v(x) \right].
\]

(3.20)

Since \( \tilde{\kappa} \in W^{1,\infty}(U^*) \), \( \lim_{\epsilon \to 0} \|T^\epsilon(x) - x\|_{L^\infty(U^*)} = 0 \). If \( v \in C^\infty(U^*) \), its partial derivatives are uniformly continuous in \( U^* \). Thus

\[
\lim_{\epsilon \to 0} \|\nabla \left[ v(\theta x T^\epsilon(x)) + (1 - \theta x)x - v(x) \right]\|_{L^\infty(U^*)} = 0.
\]

This together with (3.20) implies

\[
\lim_{\epsilon \to 0} \left\|\frac{v(\epsilon T^\epsilon(x)) - v(x)}{\epsilon} - V \cdot \nabla v\right\|_{L^2(U_+)} = \lim_{\epsilon \to 0} \left\|\frac{v(\epsilon T^\epsilon(x)) - v(x)}{\epsilon} - V \cdot \nabla v\right\|_{L^2(U^*)} = 0.
\]

(3.21)

We next apply the triangle inequality to obtain

\[
\left\|\frac{(\gamma(\epsilon, \cdot)(\epsilon^{\epsilon T^\epsilon(x)}) - v \div (v V)}{\epsilon} - \epsilon \epsilon \epsilon \div (v V)\right\|_{L^2(U_+)} \leq \left\|\frac{(\gamma(\epsilon, \cdot) - 1}{\epsilon}(\epsilon^{\epsilon T^\epsilon(x)}) - v \div (v V)\right\|_{L^2(U_+)} + \left\|\frac{v(\epsilon T^\epsilon(x)) - v(x)}{\epsilon} - V \cdot \nabla v\right\|_{L^2(U^*)}.
\]

(3.22)

Recall from (3.9) that \( \gamma_1 = \div V \). It follows from (3.12) that

\[
\frac{(\gamma(\epsilon, \cdot) - 1}{\epsilon}(v \epsilon T^\epsilon(x)) - v \div (v V) = \gamma_1(v \epsilon T^\epsilon(x) - v) + \epsilon_1 + \epsilon_3(v \epsilon T^\epsilon(x)).
\]

Employing the density argument as in proof of Lemma 3.3 and noting 3.11, we obtain

\[
\lim_{\epsilon \to 0} \|\gamma_1(v \epsilon T^\epsilon(x) - v)\|_{L^2(U_+)} = 0.
\]

Noting (3.11), we deduce

\[
\lim_{\epsilon \to 0} \|\epsilon_1(v_1 + \epsilon_3)(v \epsilon T^\epsilon(x))\|_{L^2(U_+)} = 0.
\]

Hence,

\[
\lim_{\epsilon \to 0} \left\|\frac{(\gamma(\epsilon, \cdot) - 1}{\epsilon}(v \epsilon T^\epsilon(x)) - v \div (v V)\right\|_{L^2(U_+)} = 0.
\]

(3.23)

The equality (3.18) can be derived from (3.21)–(3.23). This completes the proof of the lemma.

\[\square\]

Lemma 3.5. Let \( v \in L^2(0, T; H^2(U_+)) \cap H^1(0, T; L^2(U_+)) \). There hold

\[
\lim_{\epsilon \to 0} \left\|\frac{\epsilon^{\epsilon T^\epsilon(x)} - v}{\epsilon} - V \cdot \nabla v\right\|_{L^2(0, T; L^2(U_+))} = 0,
\]

(3.24)

and

\[
\lim_{\epsilon \to 0} \left\|\frac{(\gamma(\epsilon, \cdot) - 1}{\epsilon}(v \epsilon T^\epsilon(x)) - v \div (v V)\right\|_{L^2(0, T; L^2(U_+))} = 0.
\]

(3.25)

Proof. This proof can be obtained by employing similar arguments as used in the proof of Lemma 3.4, noting that \( v \in C(0, T; H^1(U_+)) \) ([7] Theorem 4)).

\[\square\]
3.2 Material and shape derivatives

Definition 3.6. For any sufficiently small \( \epsilon \), let \( v^\epsilon \) be an element in \( H^1(U^\epsilon) \). The material derivative of \( v^\epsilon \), denoted by \( \dot{v} \), is defined by

\[
\dot{v} := \lim_{\epsilon \to 0} \frac{v^\epsilon \circ T^\epsilon - v^0}{\epsilon},
\]

if the limit exists in the corresponding space \( H^1(U^0) \). The shape derivative of \( v^\epsilon \) is defined by

\[
v' = \dot{v} - \nabla v^0 \cdot V.
\]

Lemma 3.7. If \( v' \) is a shape derivative of \( v^\epsilon \in H^1(U^\epsilon) \), then for any compact set \( K \subseteq U^0 \) we have

\[
v' = \lim_{\epsilon \to 0} \frac{v^\epsilon - v^0}{\epsilon} \quad \text{in} \quad H^1(K).
\]

Proof. Given \( K \subseteq U^0 \), there exists an \( \epsilon_0 > 0 \) such that \( K \subseteq U^\epsilon \) for all \( 0 \leq \epsilon \leq \epsilon_0 \). We denote by \( T(\epsilon, x) := T^\epsilon(x), \quad \forall (\epsilon, x) \in [0, \epsilon_0] \times \mathbb{R}^3 \).

We also denote by \( \tilde{v}(\epsilon, x) := v^\epsilon(x) \) for any \( 0 \leq \epsilon \leq \epsilon_0 \) and \( x \in U^\epsilon \). By the definition of material derivative, we have

\[
\dot{v} = \frac{\partial}{\partial \epsilon} \tilde{v}(\epsilon, T(\epsilon, \cdot)) \bigg|_{\epsilon=0}, \quad \text{in} \quad H^1(K).
\]

Applying the chain rule, we obtain

\[
\dot{v} = \frac{\partial \tilde{v}(0, \cdot)}{\partial \epsilon} + \nabla \tilde{v}(0, \cdot) \cdot \frac{\partial T(0, \cdot)}{\partial \epsilon} = \frac{\partial \tilde{v}(0, \cdot)}{\partial \epsilon} + \nabla v^0 \cdot V, \quad \text{in} \quad H^1(K).
\]

This implies

\[
v' = \frac{\partial \tilde{v}(0, \cdot)}{\partial \epsilon} = \lim_{\epsilon \to 0} \frac{v^\epsilon - v^0}{\epsilon} \quad \text{in} \quad H^1(K).
\]

\[\blacksquare\]

Remark 3.8. The limit in the above lemma does not hold in \( H^1(U^0) \) since, in general, \( v^\epsilon \) is not defined in \( U^0 \).

Similar definitions can be introduced for vector functions \( \mathbf{v} \). The following lemmas state some useful properties of material and shape derivatives which will be used frequently in the remainder of the paper.

Lemma 3.9. Let \( \dot{v}, \dot{w} \) be material derivatives, and \( v', w' \) be shape derivatives of \( v^\epsilon, w^\epsilon \) in \( H^1(U^\epsilon) \), \( \epsilon \geq 0 \), respectively. Then the following statements are true.

(i) The material and shape derivatives of the product \( v^\epsilon w^\epsilon \) are \( \dot{v} w^\epsilon + v^\epsilon \dot{w} \) and \( v' w^0 + v^0 w' \), respectively.

(ii) The material and shape derivatives of the quotient \( v^\epsilon / w^\epsilon \) are \( (\dot{v} w^0 - v^\epsilon \dot{w}^0) / (w^0)^2 \) and \( (v' w^0 - v^0 w') / (w^0)^2 \), respectively, provided that all the fractions are well-defined.

(iii) If \( v^\epsilon = v \) for all \( \epsilon \geq 0 \), then \( \dot{v} = \nabla v^0 \cdot V = \nabla v \cdot V \) and \( \dot{v}' = 0 \).
In this paper, we assume that the sequences

\[ \{ J_1(U^\epsilon) \} \]

and

\[ \{ J_2(U^\epsilon) \} \]

where the bilinear form

\[ a \]

is defined by (18, pages 113, 116).

Lemma 3.10. The material and shape derivatives of the normal field \( n^\epsilon \) are given by

\[ \dot{n} = n' = -\nabla_{\Gamma^0} \kappa. \]

3.3 Shape derivative for Dirichlet conditions

In this subsection, existence of the shape derivative of the solution to heat equation with Dirichlet condition will be clarified. We consider the perturbed initial-boundary value problem

\[ u^\epsilon_t - \Delta u^\epsilon = f^\epsilon \quad \text{in} \quad Q^\epsilon_T := (0, T) \times U^\epsilon \]
\[ u^\epsilon = 0 \quad \text{on} \quad \Sigma^\epsilon_T := (0, T) \times \Gamma^\epsilon \]
\[ u^\epsilon|_{t=0} = g^\epsilon \quad \text{in} \quad U^\epsilon. \]

Meanwhile, the reference initial-boundary value problem on the reference domain \( U^0 \) is given by

\[ u^0_t - \Delta u^0 = f^0 \quad \text{in} \quad Q^0_T \]
\[ u^0 = 0 \quad \text{on} \quad \Sigma^0_T \]
\[ u^0|_{t=0} = g^0 \quad \text{in} \quad U^0. \]

The weak formulation of (3.29) reads as follows: given \( f^\epsilon \in L^2(Q^\epsilon_T) \) and \( g^\epsilon \in L^2(U^\epsilon) \), find \( u^\epsilon \in L^2([0, T]; H^1_0(U^\epsilon)) \cap C^0([0, T]; L^2(U^\epsilon)) \) such that

\[ \begin{cases} \frac{d}{dt} \langle u^\epsilon(t), v \rangle_{L^2(U^\epsilon)} + a(u^\epsilon(t), v; U^\epsilon) = \langle f^\epsilon(t), v \rangle_{L^2(U^\epsilon)} & \forall v \in H^1_0(U^\epsilon) \\ u^\epsilon(0) = g^\epsilon, \end{cases} \]

where the bilinear form \( a(\cdot, \cdot; U) \) associated with a domain \( U \) is defined by

\[ a(v, w; U) := \int_U \nabla v \cdot \nabla w \, dx, \quad v, w \in H^1(U). \]

In this paper, we assume that the sequences \( \{ f^\epsilon \}_{0 < \epsilon \leq \epsilon_0} \) and \( \{ g^\epsilon \}_{0 < \epsilon \leq \epsilon_0} \) satisfy

\[ \lim_{\epsilon \to 0} \| f^\epsilon \circ T^\epsilon - f^0 \|_{L^2(0, T; L^2(U^0))} = 0, \]
\[ \lim_{\epsilon \to 0} \| g^\epsilon \circ T^\epsilon - g^0 \|_{H^1(U^0)} = 0. \]
The above equations suggest that the \( \| f^\epsilon \|_{L^2(0,T;L^2(U_\epsilon^0))} \) and \( \| g^\epsilon \|_{H^1(U_\epsilon^0)} \) are bounded for sufficiently small \( \epsilon \). This together with the assumption that \( V \in W^{1,\infty}(U_+) \) implies that for sufficiently small \( \epsilon \), there hold

\[
\| f^\epsilon \|_{L^2(0,T;L^2(U_\epsilon^0))} \leq M_1, \quad \| f^\epsilon \circ T^\epsilon \|_{L^2(0,T;L^2(U_\epsilon^0))} \leq M_1, \\
\| g^\epsilon \|_{H^1(U_\epsilon^0)} \leq M_2, \quad \| g^\epsilon \circ T^\epsilon \|_{H^1(U_\epsilon^0)} \leq M_2,
\]

where \( M_1 \) and \( M_2 \) are two positive constants depending on \( f^0 \) and \( g^0 \).

The following lemmas (see [15, Page 366]) state the unique existence of the weak solution to the initial-boundary value problem with the Dirichlet boundary condition.

**Lemma 3.11.** Given \( f^\epsilon \in L^2(Q_T^\epsilon) \) and \( g^\epsilon \in L^2(U_\epsilon^0) \), there exists a unique solution \( u^\epsilon \in L^2(0,T;H_0^1(U_\epsilon^0)) \cap C^0([0,T]; L^2(U_\epsilon)) \) to (3.31). Moreover, \( \partial u^\epsilon / \partial t \in L^2(0,T;H^{-1}(U_\epsilon)) \) and the energy estimate

\[
\| u^\epsilon(t) \|^2_{L^2(U_\epsilon)} + \int_0^t \| u^\epsilon(\tau) \|^2_{H^{-1}(U_\epsilon)} \, d\tau \lesssim \| g^\epsilon \|^2_{L^2(U_\epsilon)} + \int_0^t \| f^\epsilon(\tau) \|^2_{L^2(U_\epsilon)} \, d\tau
\]

holds for each \( t \in [0,T] \). Here, the constant implicitly included in the above inequality is independent of \( T \).

**Lemma 3.12.** Given \( f^\epsilon \in L^2(Q_T^\epsilon) \) and \( g^\epsilon \in H_0^1(U_\epsilon^0) \) for all sufficiently small \( \epsilon > 0 \), the solution \( u^\epsilon \) to (3.31) belongs to \( L^\infty(0,T;H_0^1(U_\epsilon^0)) \cap H^1(0,T;L^2(U_\epsilon)) \) and satisfies

\[
\sup_{t \in (0,T)} \| u^\epsilon(t) \|^2_{H^{-1}(U_\epsilon)} + \int_0^T \| u^\epsilon(t) \|^2_{L^2(U_\epsilon)} \, dt \lesssim \| g^\epsilon \|^2_{H^{-1}(U_\epsilon)} + \int_0^T \| f^\epsilon(t) \|^2_{L^2(U_\epsilon)} \, dt,
\]

where the constant implicitly included in the above inequality is independent of \( T \).

The results in Lemmas [3.11] and [3.12] and (3.31) yield the following lemma.

**Lemma 3.13.** Assume that the conditions (3.33) and the assumptions in Lemma 3.12 are satisfied. There hold

\[
\sup_{t \in (0,T)} \| u^\epsilon(t) \|^2_{L^2(U_\epsilon^0)} + \| u^\epsilon \|^2_{L^2(0,T;H^1(U_\epsilon^0))} \lesssim M_3^2,
\]

(3.37a)

\[
\sup_{t \in (0,T)} \| u^\epsilon(t) \|^2_{H^1(U_\epsilon^0)} + \| u^\epsilon \|^2_{L^2(0,T;L^2(U_\epsilon))} \lesssim M_4^2,
\]

(3.37b)

where \( M_3 \) and \( M_4 \) are positive constants depending on \( f^0 \) and \( g^0 \).

In this paper, we shall frequently use the following results.

**Lemma 3.14.** Let \( v^\epsilon, w^\epsilon \in L^2(0,T;L^2(U)) \) satisfy

\[
\lim_{\epsilon \to 0} \| v^\epsilon \|_{L^2(0,T;L^2(U))} = 0 \quad \text{and} \quad \| w^\epsilon \|_{L^2(0,T;L^2(U))} \leq M \quad \text{for sufficiently small} \ \epsilon.
\]

There holds

\[
\lim_{\epsilon \to 0} \int_0^t \int_U v^\epsilon(\tau,x) w^\epsilon(\tau,x) \, dx \, d\tau = 0 \quad \forall t \in [0,T].
\]

**Proof.** We have

\[
\left| \int_0^t \int_U v^\epsilon(\tau,x) w^\epsilon(\tau,x) \, dx \, d\tau \right| \leq \| v^\epsilon \|_{L^2(0,T;L^2(U))} \| w^\epsilon \|_{L^2(0,T;L^2(U))} \leq M \| v^\epsilon \|_{L^2(0,T;L^2(U))}.
\]

Letting \( \epsilon \) go to zero, we obtain the desired equality.
Lemma 3.15. Let $v^\epsilon \in L^\infty(U)$ and $w^\epsilon \in L^2(0, T; L^2(U))$ satisfy
\[
\lim_{\epsilon \to 0} \|v^\epsilon\|_{L^\infty(U)} = 0 \quad \text{and} \quad \|w^\epsilon\|_{L^2(0, T; L^2(U))} \leq M \quad \text{for sufficiently small } \epsilon.
\]
There holds
\[
\lim_{\epsilon \to 0} \int_0^t \int_U v^\epsilon(x) w^\epsilon(\tau, x) \, dx \, d\tau = 0 \quad \forall t \in [0, T].
\]
Proof. The proof can be done in the same manner as the proof in Lemma 3.14. □

Lemma 3.16. Let $v^\epsilon, v^0 \in L^\infty(U)$ and $w^\epsilon, w^0 \in L^2(0, T; L^2(U))$ satisfy
\[
\lim_{\epsilon \to 0} \left\| v^\epsilon - v^0 \right\|_{L^\infty(U)} = 0 \quad \text{and} \quad \lim_{\epsilon \to 0} \left\| w^\epsilon - w^0 \right\|_{L^2(0, T; L^2(U))} = 0.
\]
There holds
\[
\lim_{\epsilon \to 0} \left\| v^\epsilon w^\epsilon - v^0 w^0 \right\|_{L^2(0, T; L^2(U))} = 0.
\]
Proof. Applying the triangle inequality, we have
\[
\left\| v^\epsilon w^\epsilon - v^0 w^0 \right\|_{L^2(0, T; L^2(U))} \leq \left\| (v^\epsilon - v^0) \right\|_{L^2(0, T; L^2(U))}^2 + \left\| w^\epsilon - w^0 \right\|_{L^2(0, T; L^2(U))}^2.
\]
Letting $\epsilon$ go to zero and noting (3.38), we obtain (3.39). □

The following lemma states the convergence of $u^\epsilon$ and its derivative $u_1^\epsilon$ to $u^0$ and $u_0^0$, respectively.

Lemma 3.17. Assume that the assumptions in Lemma 3.15 are satisfied. Let $u^\epsilon$ and $u^0$ be solutions to (3.29) and (3.30), respectively. There holds
\[
\lim_{\epsilon \to 0} \sup_{t \in [0, T]} \left\| u^\epsilon \circ T^\epsilon - u^0 \right\|_{L^2(U^0)} = 0,
\]
\[
\lim_{\epsilon \to 0} \left\| u_1^\epsilon \circ T^\epsilon - u_0^0 \right\|_{L^2(0, T; H^{-1}(U^0))} = 0,
\]
\[
\lim_{\epsilon \to 0} \left\| u_1^\epsilon \circ T^\epsilon - u_0^0 \right\|_{L^2(0, T; H^{-1}(U^0))} = 0.
\]
Proof. The first equation in (3.31) gives
\[
\int_{U^\epsilon} u_1^\epsilon(t, x) v^\epsilon(x) \, dx + \int_{U^\epsilon} \nabla u^\epsilon(t, x) \cdot \nabla v^\epsilon(x) \, dx = \int_{U^\epsilon} f^\epsilon(t, x) v^\epsilon(x) \, dx \quad v^\epsilon \in C_0^\infty(U^\epsilon)
\]
for $0 \leq t \leq T$. In particular,
\[
\int_{U^0} u_1^0(t, x) w(x) \, dx + \int_{U^0} \nabla u^0(t, x) \cdot \nabla w(x) \, dx = \int_{U^0} f^0(t, x) w(x) \, dx \quad \forall w \in C_0^\infty(U^0).
\]
Using the change of variables $x = T^\epsilon(y)$ in (3.41), we have
\[
\int_{U^0} u_1^\epsilon \circ T^\epsilon(t, y) v^\epsilon \circ T^\epsilon(y) \gamma(\epsilon, y) \, dy + \int_{U^0} \left[ \nabla (v^\epsilon \circ T^\epsilon)(y) \right]^T A(\epsilon, y) \nabla (u_1^\epsilon \circ T^\epsilon)(t, y) \, dy
\]
\[
= \int_{U^0} f^\epsilon \circ T^\epsilon(t, y) v^\epsilon \circ T^\epsilon(y) \gamma(\epsilon, y) \, dy.
\]
Since $C_0^\infty(U^\epsilon)$ is dense in $H_0^1(U^\epsilon)$, this is true for all $v^\epsilon \in H_0^1(U^\epsilon)$. Thus, for all $w \in H_0^1(U^\epsilon)$ there holds

$$\int_{U^\epsilon} u^\epsilon \hat{\mathcal{T}}^\epsilon(t, y) w(y) \gamma(\epsilon, y) \, dy + \int_{U^\epsilon} \left[ \nabla w(y) \right]^T A(\epsilon, y) \nabla (u^\epsilon \hat{\mathcal{T}}^\epsilon)(t, y) \, dy = \int_{U^\epsilon} f^\epsilon \hat{\mathcal{T}}^\epsilon(t, y) w(y) \gamma(\epsilon, y) \, dy. \quad (3.43)$$

Subtracting (3.42) from (3.43) we deduce

$$\int_{U^\epsilon} (u^\epsilon \hat{\mathcal{T}}^\epsilon - u^0)(t, y) w(y) \, dy + \int_{U^\epsilon} \left[ \nabla w(y) \right]^T \nabla (u^\epsilon \hat{\mathcal{T}}^\epsilon - u^0)(t, y) \, dy = - \int_{U^\epsilon} u^\epsilon \hat{\mathcal{T}}^\epsilon(t, y) w(y) \gamma(\epsilon, y) - 1) \, dy$$

$$- \int_{U^\epsilon} \left[ \nabla (u^\epsilon \hat{\mathcal{T}}^\epsilon - u^0)(t, y) \right]^T (A(\epsilon, y) - I) \nabla (u^\epsilon \hat{\mathcal{T}}^\epsilon)(t, y) \, dy + \int_{U^\epsilon} \left[ f^\epsilon \hat{\mathcal{T}}^\epsilon(t, y) \gamma(\epsilon, y) - f^0(t, y) \right] w(y) \, dy. \quad (3.44)$$

For any $t \in [0, T]$, we choose in (3.44) $w(y) = (u^\epsilon \hat{\mathcal{T}}^\epsilon - u^0)(t, y)$ to obtain

$$\frac{1}{2} \int_{U^\epsilon} \left[ (u^\epsilon \hat{\mathcal{T}}^\epsilon - u^0)(t, y) \right]^2 \, dy + \int_{U^\epsilon} \left\| \nabla (u^\epsilon \hat{\mathcal{T}}^\epsilon - u^0)(t, y) \right\|^2 \, dy$$

$$= - \int_{U^\epsilon} u^\epsilon \hat{\mathcal{T}}^\epsilon(t, y) \left( (u^\epsilon \hat{\mathcal{T}}^\epsilon - u^0)(t, y) \gamma(\epsilon, y) - 1 \right) \, dy$$

$$- \int_{U^\epsilon} \left[ \nabla (u^\epsilon \hat{\mathcal{T}}^\epsilon - u^0)(t, y) \right]^T (A(\epsilon, y) - I) \nabla (u^\epsilon \hat{\mathcal{T}}^\epsilon)(t, y) \, dy + \int_{U^\epsilon} \left[ f^\epsilon \hat{\mathcal{T}}^\epsilon(t, y) \gamma(\epsilon, y) - f^0(t, y) \right] \, dy. \quad (3.45)$$

Integrating both sides of (3.45) over $[0, t]$ for any $t \in [0, T]$ and noting the initial conditions $u^\epsilon \hat{\mathcal{T}}^\epsilon(0, y) = g^\epsilon \hat{\mathcal{T}}^\epsilon(y)$ and $u^0(0, y) = g^0(y)$ for all $y \in U^\epsilon$, we obtain

$$\frac{1}{2} \int_{U^\epsilon} \left[ (u^\epsilon \hat{\mathcal{T}}^\epsilon - u^0)(t, y) \right]^2 \, dy + \int_0^t \int_{U^\epsilon} \left\| \nabla (u^\epsilon \hat{\mathcal{T}}^\epsilon - u^0)(\tau, y) \right\|^2 \, dy \, d\tau$$

$$= \frac{1}{2} \int_{U^\epsilon} \left[ g^\epsilon \hat{\mathcal{T}}^\epsilon(y) - g^0(y) \right]^2 \, dy$$

$$- \int_0^t \int_{U^\epsilon} u^\epsilon \hat{\mathcal{T}}^\epsilon(\tau, y) \left( (u^\epsilon \hat{\mathcal{T}}^\epsilon - u^0)(\tau, y) \gamma(\epsilon, y) - 1 \right) \, dy \, d\tau$$

$$- \int_0^t \int_{U^\epsilon} \left[ \nabla (u^\epsilon \hat{\mathcal{T}}^\epsilon - u^0)(\tau, y) \right]^T (A(\epsilon, y) - I) \nabla (u^\epsilon \hat{\mathcal{T}}^\epsilon)(\tau, y) \, dy \, d\tau + \int_0^t \int_{U^\epsilon} \left[ f^\epsilon \hat{\mathcal{T}}^\epsilon(\tau, y) \gamma(\epsilon, y) - f^0(\tau, y) \right] \, dy \, d\tau. \quad (3.46)$$

Applying Lemma 2.1 and the triangle inequality, we derive

$$\left\| (u^\epsilon \hat{\mathcal{T}}^\epsilon - u^0)(t) \right\|^2_{L^2(U^\epsilon)} + \int_0^t \left\| (u^\epsilon \hat{\mathcal{T}}^\epsilon - u^0)(\tau) \right\|^2_{H^1(U^\epsilon)} \, d\tau \lesssim \left\| g^\epsilon \hat{\mathcal{T}}^\epsilon - g^0 \right\|^2_{L^2(U^\epsilon)}$$

$$+ \left\| \int_0^t \int_{U^\epsilon} u^\epsilon \hat{\mathcal{T}}^\epsilon(\tau, y) \left( (u^\epsilon \hat{\mathcal{T}}^\epsilon - u^0)(\tau, y) \gamma(\epsilon, y) - 1 \right) \, dy \, d\tau \right\|$$

$$+ \left\| \int_0^t \int_{U^\epsilon} \left[ \nabla (u^\epsilon \hat{\mathcal{T}}^\epsilon - u^0)(\tau, y) \right]^T (A(\epsilon, y) - I) \nabla (u^\epsilon \hat{\mathcal{T}}^\epsilon)(\tau, y) \, dy \, d\tau \right\|$$

$$+ \left\| \int_0^t \int_{U^\epsilon} \left[ f^\epsilon \hat{\mathcal{T}}^\epsilon(\tau, y) \gamma(\epsilon, y) - f^0(\tau, y) \right] \, dy \, d\tau \right\|. \quad (3.47)$$
Noting (3.33a), the result in Lemma 3.1 and applying triangle inequality we deduce
\[
\lim_{\epsilon \to 0} \left\| f'(\gamma(\epsilon, \cdot) - f^0) \right\|_{L^2(0,T;L^2(U^0))} = 0. \tag{3.48}
\]
Letting \(\epsilon\) go to zero, noting (3.33b), (3.6), (3.13), (3.48), (3.37) and applying Lemmas 3.14 and 3.15, we obtain
\[
\begin{align*}
\lim_{\epsilon \to 0} & \sup_{t \in [0,T]} \left\| (u^\epsilon \gamma - u^0)(t) \right\|_{L^2(U^0)} = 0 \\
\lim_{\epsilon \to 0} & \left\| u^\epsilon \gamma - u^0 \right\|_{L^2(0,T;H^1(U^0))} = 0. \tag{3.49}
\end{align*}
\]
We shall next prove that \(\lim_{\epsilon \to 0} \left\| u^\epsilon \gamma - u^0 \right\|_{L^2(0,T;H^{-1}(U^0))} = 0\). From (3.44), for any \(\tau \in [0,T]\) we have
\[
\begin{align*}
\int_{U^0} \left[ u^\epsilon \gamma(t, y) - u^0(t, y) \right] w(y) \, dy & \leq \left\| u^\epsilon \gamma - u^0 \right\|_{H^1(U^0)} + \left\| (u^\epsilon \gamma - u^0)(\tau) \right\|_{H^{-1}(U^0)} \\
& \leq \left\| u^\epsilon \gamma - u^0 \right\|_{H^1(U^0)} + \left\| (u^\epsilon \gamma - u^0)(\tau) \right\|_{H^{-1}(U^0)} \\
& \leq \left\| A(\epsilon, \cdot) - I \right\|_{L^\infty(U^0)} \left\| u^\epsilon \gamma - u^0 \right\|_{H^1(U^0)} + \left\| (u^\epsilon \gamma - u^0)(\tau) \right\|_{H^1(U^0)} \\
& \leq \left\| A(\epsilon, \cdot) - I \right\|_{L^\infty(U^0)} \left\| u^\epsilon \gamma - u^0 \right\|_{H^1(U^0)} + \left\| (u^\epsilon \gamma - u^0)(\tau) \right\|_{H^1(U^0)}.
\end{align*}
\]
This is true for all \(w \in H^1_0(U^0)\). Thus, for any \(\tau \in [0,T]\), there holds
\[
\begin{align*}
\left\| (u^\epsilon \gamma - u^0)(\tau) \right\|_{H^{-1}(U^0)} & \leq \left\| u^\epsilon \gamma - u^0 \right\|_{H^1(U^0)} + \left\| (u^\epsilon \gamma - u^0)(\tau) \right\|_{H^{-1}(U^0)} \\
& \leq \left\| A(\epsilon, \cdot) - I \right\|_{L^\infty(U^0)} \left\| u^\epsilon \gamma - u^0 \right\|_{H^1(U^0)} + \left\| (u^\epsilon \gamma - u^0)(\tau) \right\|_{H^1(U^0)}.
\end{align*}
\]
Squaring up both sides, using the Cauchy–Schwarz inequality and then integrating over \([0,T]\), we obtain
\[
\begin{align*}
\left\| u^\epsilon \gamma - u^0 \right\|^2_{L^2(0,T;H^{-1}(U^0))} & \leq \left\| u^\epsilon \gamma - u^0 \right\|^2_{L^2(0,T;H^1(U^0))} + \left\| (u^\epsilon \gamma - u^0)(\tau) \right\|^2_{H^{-1}(U^0)} \\
& \leq \left\| A(\epsilon, \cdot) - I \right\|^2_{L^\infty(U^0)} \left\| u^\epsilon \gamma - u^0 \right\|^2_{L^2(0,T;H^1(U^0))} + \left\| (u^\epsilon \gamma - u^0)(\tau) \right\|^2_{H^{-1}(U^0)}.
\end{align*}
\]
It follows from (3.49), (3.6), (3.13), (3.37) and (3.38) that
\[
\lim_{\epsilon \to 0} \left\| (u^\epsilon \gamma - u^0)(\tau) \right\|_{L^2(0,T;H^{-1}(U^0))} = 0,
\]
finishing the proof of the theorem. \(\square\)
Lemma 3.18. Let $f^\epsilon \in L^2(Q_T^\epsilon)$ satisfy
\[
\lim_{\epsilon \to 0} \left\| \frac{f^\epsilon \partial T^\epsilon - f^0}{\epsilon} - \nabla f^0 \cdot V \right\|_{L^2(0,T;L^2(U^0))} = 0. \tag{3.52}
\]
Then, there holds
\[
\lim_{\epsilon \to 0} \left\| \frac{f^\epsilon \partial T^\epsilon \gamma(\epsilon, \cdot) - f^0}{\epsilon} - \text{div} (V f^0) \right\|_{L^2(0,T;L^2(U^0))} = 0. \tag{3.53}
\]

Proof. We first note that if (3.52) is satisfied, there holds
\[
\lim_{\epsilon \to 0} \left\| f^\epsilon \partial T^\epsilon - f^0 \right\|_{L^2(0,T;L^2(U^0))} = 0. \tag{3.54}
\]
The triangle inequality gives
\[
\left\| \frac{f^\epsilon \partial T^\epsilon \gamma(\epsilon, \cdot) - f^0}{\epsilon} - \text{div} (V f^0) \right\|_{L^2(0,T;L^2(U^0))} \lesssim \left\| \frac{\gamma(\epsilon, \cdot) - 1}{\epsilon} (f^\epsilon \partial T^\epsilon - f^0 \text{div} V) \right\|_{L^2(0,T;L^2(U^0))} + \left\| \frac{f^\epsilon \partial T^\epsilon - f^0}{\epsilon} - V \cdot \nabla f^0 \right\|_{L^2(0,T;L^2(U^0))}.
\]
Applying Lemma 3.15 and noting (3.54) and (3.7), we prove that the first norm on the right hand side of the above inequality converges to zero when $\epsilon$ goes to zero. This together with the assumption (3.52) yields (3.53).

\[\square\]

Lemma 3.19. Assume that $f^\epsilon \in L^2(Q_T^\epsilon)$ and $g^\epsilon \in H_0^2(U^\epsilon)$ for all sufficiently small $\epsilon \geq 0$. Let $u^\epsilon$ and $u^0$ be solutions to (3.29) and (3.30), respectively. There hold
\[
\lim_{\epsilon \to 0} \left\| \frac{u^\epsilon \partial T^\epsilon \gamma(\epsilon, \cdot) - 1}{\epsilon} - u^0 \text{div} V \right\|_{L^2(0,T;H^{-1}(U^0))} = 0 \tag{3.55}
\]
\[
\lim_{\epsilon \to 0} \left\| \frac{A(\epsilon, \cdot) - I}{\epsilon} \nabla (u^\epsilon \partial T^\epsilon) - A'(0, \cdot) \nabla u^0 \right\|_{L^2(0,T;L^2(U^0))} = 0 \tag{3.56}
\]

Proof. The triangle inequality gives
\[
\left\| \frac{u^\epsilon \partial T^\epsilon \gamma(\epsilon, \cdot) - 1}{\epsilon} - u^0 \text{div} V \right\|^2_{L^2(0,T;H^{-1}(U^0))} \lesssim \left\| \frac{u^\epsilon \partial T^\epsilon - u^0}{\epsilon} \gamma(\epsilon, \cdot) - 1 \right\|^2_{L^2(0,T;H^{-1}(U^0))} + \left\| \frac{u^0}{\epsilon} \left( \frac{\gamma(\epsilon, \cdot) - 1}{\epsilon} - \text{div} V \right) \right\|^2_{L^2(0,T;H^{-1}(U^0))}. \tag{3.57}
\]
Using the duality argument and noting (3.12), for every $\tau \in [0, T]$ we have
\[
\left\| \frac{u^\epsilon \partial T^\epsilon - u^0}{\epsilon} (\tau) \gamma(\epsilon, \cdot) - 1 \right\|_{H^{-1}(U^0)} = \sup_{w \in H_0^1(U^0)} \left\| \frac{u^\epsilon \partial T^\epsilon - u^0}{\epsilon} (\tau, y) \gamma(\epsilon, y) - 1 \right\|_{H^{-1}(U^0)} w(y) dy \leq \left\| \gamma(\epsilon, y) - 1 \right\|_{L^\infty(U^0)} \sup_{w \in H_0^1(U^0)} \left\| \frac{u^\epsilon \partial T^\epsilon - u^0}{\epsilon} (\tau, y) w(y) dy \right\|_{H^{-1}(U^0)} \leq \left\| \gamma + \epsilon \gamma_2 + \epsilon^2 \gamma_3 \right\|_{L^\infty(U^0)} \left\| (u^\epsilon \partial T^\epsilon - u^0) (\tau) \right\|_{H^{-1}(U^0)}.
\]
This implies
\[ \left\| \left( u^\varepsilon \circ T^\varepsilon - u^0 \right) \frac{\gamma(\varepsilon, \cdot) - 1}{\varepsilon} \right\|_{L^2(0,T;H^{-1}(U^0))} \leq \| \gamma_1 + \varepsilon \gamma_2 + \varepsilon^2 \gamma_3 \|_{L^\infty(U^0)} \left\| u^\varepsilon \circ T^\varepsilon - u^0 \right\|_{L^2(0,T;H^{-1}(U^0))}. \] (3.58)

Similar arguments give
\[ \left\| u^0 \left( \frac{\gamma(\varepsilon, \cdot) - 1}{\varepsilon} - \text{div } V \right) \right\|_{L^2(0,T;H^{-1}(U^0))} \leq \left\| \frac{\gamma(\varepsilon, \cdot) - 1}{\varepsilon} - \text{div } V \right\|_{L^\infty(U^0)} \left\| u^0 \right\|_{L^2(0,T;H^{-1}(U^0))}. \] (3.59)

It follows from (357) – (359), (3.40) and (3.7) that
\[ \lim_{\varepsilon \to 0} \left\| u^\varepsilon \circ T^\varepsilon \gamma(\varepsilon, \cdot) - 1 - u^0 \right\|_{L^2(0,T;H^{-1}(U^0))} \text{div } V = 0, \]
proving (3.55).

Applying the triangle inequality again, we have
\[ \left\| \frac{A(\varepsilon, \cdot) - I}{\varepsilon} \nabla (u^\varepsilon \circ T^\varepsilon) - A'(0, \cdot) \nabla u^0 \right\|_{L^2(0,T;L^2(U^0))}^2 \leq \left\| \frac{A(\varepsilon, \cdot) - I}{\varepsilon} \nabla (u^\varepsilon \circ T^\varepsilon) - u^0 \right\|_{L^2(0,T;L^2(U^0))}^2 + \left\| \frac{A(\varepsilon, \cdot) - I}{\varepsilon} - A'(0, \cdot) \right\|_{L^\infty(U^0)} \left\| \nabla u^0 \right\|_{L^2(0,T;L^2(U^0))}. \] (3.60)

For every \( \tau \in [0,T] \), we have
\[ \left\| \frac{A(\varepsilon, \cdot) - I}{\varepsilon} \nabla (u^\varepsilon \circ T^\varepsilon) - u^0(\tau) \right\|_{L^2(U^0)} \leq \left\| \frac{A(\varepsilon, \cdot) - I}{\varepsilon} \right\|_{L^\infty(U^0)} \left\| \nabla (u^\varepsilon \circ T^\varepsilon) - u^0(\tau) \right\|_{L^2(U^0)} \leq \left\| \frac{A(\varepsilon, \cdot) - I}{\varepsilon} \right\|_{L^\infty(U^0)} \left\| (u^\varepsilon \circ T^\varepsilon) - u^0(\tau) \right\|_{H^1(U^0)}, \] (3.61)
and
\[ \left\| \left( \frac{A(\varepsilon, \cdot) - I}{\varepsilon} - A'(0, \cdot) \right) \nabla u^0(\tau) \right\|_{L^2(U^0)} \leq \left\| \frac{A(\varepsilon, \cdot) - I}{\varepsilon} - A'(0, \cdot) \right\|_{L^\infty(U^0)} \left\| \nabla u^0(\tau) \right\|_{L^2(U^0)} \leq \left\| \frac{A(\varepsilon, \cdot) - I}{\varepsilon} - A'(0, \cdot) \right\|_{L^\infty(U^0)} \left\| u^0(\tau) \right\|_{H^1(U^0)}. \] (3.62)

Note that the inequalities (3.61) and (3.62) are true for every \( \tau \in [0,T] \). This together with (3.60) implies
\[ \left\| \frac{A(\varepsilon, \cdot) - I}{\varepsilon} \nabla (u^\varepsilon \circ T^\varepsilon) - A'(0, \cdot) \nabla u^0 \right\|_{L^2(0,T;L^2(U^0))}^2 \leq \left\| \frac{A(\varepsilon, \cdot) - I}{\varepsilon} \right\|_{L^\infty(U^0)}^2 \left\| u^\varepsilon \circ T^\varepsilon - u^0 \right\|_{L^2(0,T;H^1(U^0))}^2 + \left\| \frac{A(\varepsilon, \cdot) - I}{\varepsilon} - A'(0, \cdot) \right\|_{L^\infty(U^0)}^2 \left\| u^0 \right\|_{L^2(0,T;H^1(U^0))}^2. \]

Letting \( \varepsilon \) go to zero on both sides and noting (3.14) and (3.40b), we obtain (3.56), finishing the proof of the lemma. \( \Box \)
Lemma 3.20. Assume that $f^\epsilon \in L^2(Q_T^\epsilon)$ and $g^\epsilon \in H_0^2(U^\epsilon)$ for all sufficiently small $\epsilon > 0$. Assume further that

$$\lim_{\epsilon \to 0} \left\| \frac{f^\epsilon \circ T^\epsilon - f^0}{\epsilon} - \nabla f^0 \cdot V \right\|_{L^2(0,T;L^2(U^0))} = 0$$

(3.63)

and

$$\lim_{\epsilon \to 0} \left\| \frac{g^\epsilon \circ T^\epsilon - g^0}{\epsilon} - \nabla g^0 \cdot V \right\|_{L^2(U^0)} = 0.$$  

(3.64)

Let $u^\epsilon$ and $u^0$ be solutions to (3.29) and (3.30), respectively. The material derivative of $u^\epsilon$ exists and is the solution of the following problem: Find $z \in L^2(0,T;H_0^1(U^0))$ satisfying

$$z_t(t,y) - \Delta z(t,y) = - u^0_0(t,y) \text{ div } V(y) - \nabla \cdot (A'(0,y) \nabla u^0_0(t,y))$$

$$+ \text{ div } \left( V(y) f^0(t,y) \right), \quad (t,y) \in [0,T] \times U^0$$

(3.65)

$$z(t,y) = 0, \quad (t,y) \in [0,T] \times \Gamma^0$$

$$z(0,y) = \nabla g^0(y) \cdot V(y), \quad y \in U^0.$$  

Proof. First, since $f^\epsilon \in L^2(Q_T^\epsilon)$ and $g^\epsilon \in H_0^2(U^\epsilon)$ for all $\epsilon \geq 0$, by Lemma 3.12 the solution $u^\epsilon$ of (3.31) exists and $u^\epsilon \in L^\infty(0,T;H_0^1(U^\epsilon)) \cap H^1(0,T;L^2(U^\epsilon))$ for any $\epsilon \geq 0$. In particular, when $\epsilon = 0$, the solution $u^0$ belongs to $L^\infty(0,T;H_0^1(U^0)) \cap H^1(0,T;L^2(U^0))$. Note also that $V \in W^{1,\infty}(U^0)$. The unique existence of the solution $z$ of (3.63) is assured by using [15, Theorem 11.1.1 and Remark 11.1.1]. Furthermore, the corresponding weak formulation is: Find $z \in L^2(0,T;H_0^1(U^0))$ satisfying

$$\int_{U^0} z_t(t,y) w(y) \, dy + \int_{U^0} \nabla [w(y)]^T \nabla z(t,y) \, dy = - \int_{U^0} u^0_0(t,y) w(y) \, dy$$

$$- \int_{U^0} \nabla \cdot (A'(0,y) \nabla u^0_0(t,y)) w(y) \, dy$$

$$+ \int_{U^0} \text{ div } \left( V(y) f^0(t,y) \right) w(y) \, dy \quad \forall w \in H_0^1(U^0).$$

(3.66)

Here, the second integral on the right hand side can be rewritten by using the Divergence Theorem

$$\int_{U^0} \nabla \cdot (A'(0,y) \nabla u^0_0(t,y)) w(y) \, dy = - \int_{U^0} \left[ \nabla w(y) \right]^T A'(0,y) \nabla u^0_0(t,y) \, dy,$$

noting that $w$ vanishes on the boundary of $U^0$. On the other hand, by dividing both sides of (3.44) by $\epsilon$ we have

$$\int_{U^0} \frac{(u^\epsilon_0 \circ T^\epsilon - u^0_0)(t,y)}{\epsilon} w(y) \, dy + \int_{U^0} \left[ \nabla w(y) \right]^T \nabla \left( \frac{u^\epsilon_0 \circ T^\epsilon - u^0_0}{\epsilon}(t,y) \right) \, dy$$

$$= - \int_{U^0} u^\epsilon_0 \circ T^\epsilon(t,y) w(y) \frac{g(\epsilon,y) - 1}{\epsilon} \, dy - \int_{U^0} \left[ \nabla w(y) \right]^T \frac{A(\epsilon,y) - I}{\epsilon} \nabla (u^\epsilon_0 \circ T^\epsilon)(t,y) \, dy$$

$$+ \int_{U^0} \frac{f^\epsilon \circ T^\epsilon(t,y)}{\epsilon} \gamma(\epsilon,y) - f^0(t,y) \, dy \quad \forall w \in H_0^1(U^0).$$

(3.67)
Subtracting (3.66) from (3.67), we have

\[
\int_{U^0} \left[ \frac{(u^t_\epsilon \cdot \nabla \epsilon - u^0_\epsilon)(t, y)}{\epsilon} - z(t, y) \right] w(y) \, dy \\
+ \int_{U^0} \left[ \nabla w(y) \right]^T \nabla \left[ \frac{(u^t_\epsilon \cdot \nabla \epsilon - u^0_\epsilon)(t, y)}{\epsilon} - z(t, y) \right] \, dy \\
= - \int_{U^0} w(y) \left[ u^t_\epsilon \cdot \nabla \epsilon \left( t, y \right) \right] \frac{\gamma(\epsilon, y) - 1}{\epsilon} - u^0_\epsilon(t, y) \, dy \\
- \int_{U^0} \nabla w(y) \left[ A(\epsilon, y) - 1 \right] \nabla (u^t_\epsilon \cdot \nabla \epsilon)(t, y) - A'(0, y) \nabla u^0_\epsilon(t, y) \, dy \\
+ \int_{U^0} \left[ f^t_\epsilon \cdot \nabla \epsilon(t, y) \right] \frac{\gamma(\epsilon, y) - 1}{\epsilon} - f^0_\epsilon(t, y) - \text{div} \left( V(y) f^0_\epsilon(t, y) \right) \, dy \\
\forall w \in H^1_0(U^0).
\]

(3.68)

For each \( t \in (0, T) \), substituting \( w(y) = (u^t_\epsilon \cdot \nabla \epsilon - u^0_\epsilon)(t, y)/\epsilon - z(t, y) \) into (3.68) and then integrating both sides over \([0, t]\), we obtain

\[
\frac{1}{2} \int_{U^0} \left[ \frac{u^t_\epsilon \cdot \nabla \epsilon(t, y) - u^0_\epsilon(t, y)}{\epsilon} - z(t, y) \right]^2 \, dy + \int_{t_0}^t \int_{U^0} \left\| \left[ \frac{(u^t_\epsilon \cdot \nabla \epsilon - u^0_\epsilon)(\tau)}{\epsilon} - z(\tau) \right] \right\|_{L^2(U^0)}^2 \, d\tau \\
= \frac{1}{2} \int_{U^0} \left[ \frac{g^t_\epsilon \cdot \nabla \epsilon(t, y) - g^0_\epsilon(t, y)}{\epsilon} - \nabla g^0_\epsilon(y) \cdot V(y) \right]^2 \, dy \\
- \int_{t_0}^t \int_{U^0} \left[ \frac{(u^t_\epsilon \cdot \nabla \epsilon - u^0_\epsilon)(\tau, y)}{\epsilon} - z(\tau, y) \right] \, dy \\
\times \left[ \frac{u^t_\epsilon \cdot \nabla \epsilon(\tau, y) \gamma(\epsilon, y) - 1}{\epsilon} - u^0_\epsilon(\tau, y) \, \text{div} \, V(y) \right] \, dy \\
- \int_{t_0}^t \int_{U^0} \left[ \frac{A(\epsilon, y) - 1}{\epsilon} \nabla (u^t_\epsilon \cdot \nabla \epsilon)(\tau, y) - A'(0, y) \nabla u^0_\epsilon(\tau, y) \right] \, dy \\
+ \int_{t_0}^t \int_{U^0} \left[ \frac{f^t_\epsilon \cdot \nabla \epsilon(\tau, y) \gamma(\epsilon, y) - f^0_\epsilon(\tau, y)}{\epsilon} - \text{div} \, \left( V(y) f^0_\epsilon(\tau, y) \right) \right] \\
\times \left[ \frac{(u^t_\epsilon \cdot \nabla \epsilon - u^0_\epsilon)(\tau, y)}{\epsilon} - z(\tau, y) \right] \, dy \\
\quad \quad =: B_1 + B_2 + B_3 + B_4.
\]

(3.69)

Considering the second integral on the right hand side of (3.69), the duality and Cauchy inequalities give

\[
|B_2| \leq \int_{t_0}^t \left\| \frac{(u^t_\epsilon \cdot \nabla \epsilon - u^0_\epsilon)(\tau)}{\epsilon} - z(\tau) \right\|_{H^1(U^0)} \left\| \frac{u^t_\epsilon \cdot \nabla \epsilon(\tau)}{\epsilon} \gamma(\epsilon, \cdot) - 1 - u^0_\epsilon(\tau) \, \text{div} \, V \right\|_{H^{-1}(U^0)} \, d\tau \\
\leq \alpha \int_{t_0}^t \left\| \frac{(u^t_\epsilon \cdot \nabla \epsilon - u^0_\epsilon)(\tau)}{\epsilon} - z(\tau) \right\|_{H^1(U^0)}^2 \, d\tau \\
+ \frac{1}{4\alpha} \int_{t_0}^t \left\| \frac{u^t_\epsilon \cdot \nabla \epsilon(\tau)}{\epsilon} \gamma(\epsilon, \cdot) - 1 - u^0_\epsilon(\tau) \, \text{div} \, V \right\|_{H^{-1}(U^0)}^2 \, d\tau
\]

(3.70)

for any \( \alpha > 0 \). Similarly, for any positive numbers \( \beta \) and \( \eta \), we have

\[
|B_3| \leq \beta \int_{t_0}^t \left\| \nabla \left[ \frac{(u^t_\epsilon \cdot \nabla \epsilon - u^0_\epsilon)(\tau)}{\epsilon} - z(\tau) \right] \right\|_{L^2(U^0)}^2 \, d\tau \\
+ \frac{1}{4\beta} \int_{t_0}^t \left\| A(\epsilon, \cdot) - 1 \right\|_{L^2(U^0)} \left\| \nabla (u^t_\epsilon \cdot \nabla \epsilon)(\tau) - A'(0, \cdot) \nabla u^0_\epsilon(\tau) \right\|_{L^2(U^0)}^2 \, d\tau
\]

(3.71)
and
\[
|B_4| \leq \frac{1}{4\eta} \int_0^t \left\| \frac{f^*\mathcal{T}^e(\tau) \gamma(\epsilon, \cdot) - f^0(\tau)}{\epsilon} - \text{div} \left( V f^0(\tau) \right) \right\|_{L^2(U^0)}^2 d\tau \\
+ \eta \int_0^t \left\| \frac{(u^* \mathcal{T}^e - u^0)(\tau)}{\epsilon} - z(\tau) \right\|_{L^2(U^0)}^2 d\tau.
\] (3.72)

From (3.69)–(3.72) and noting that \( \|v\|_{L^2(U^0)} \leq \|v\|_{H^1(U^0)} \) for all \( v \in H^1(U^0) \), we obtain
\[
\frac{1}{2} \left\| \frac{u^* \mathcal{T}^e(t) - u^0(t)}{\epsilon} - z(t) \right\|_{L^2(U^0)}^2 + \int_0^t \left\| \nabla \left( \frac{(u^* \mathcal{T}^e - u^0)(\tau)}{\epsilon} - z(\tau) \right) \right\|_{L^2(U^0)}^2 d\tau
\leq \frac{1}{2} \left\| \frac{g^* \mathcal{T}^e - g^0}{\epsilon} - \nabla g^0 \cdot V \right\|_{L^2(U^0)}^2 + \alpha \int_0^t \left\| \frac{(u^* \mathcal{T}^e - u^0)(\tau)}{\epsilon} - z(\tau) \right\|_{H^1(U^0)}^2 d\tau
\]
\[
+ \frac{1}{4\alpha} \int_0^t \left\| \frac{\gamma(\epsilon, \cdot) - 1}{\epsilon} - u^0(\tau) \right\|_{H^{-1(U^0)}}^2 d\tau
\]
\[
+ \beta \int_0^t \left\| \frac{A(\epsilon, \cdot) - I}{\epsilon} \nabla (u^* \mathcal{T}^e)(\tau) - A(0, \cdot) \nabla u^0(\tau) \right\|_{L^2(U^0)}^2 d\tau
\]
\[
+ \frac{1}{4\beta} \int_0^t \left\| \frac{f^* \mathcal{T}^e(\tau) \gamma(\epsilon, \cdot) - f^0(\tau)}{\epsilon} - \text{div} \left( V f^0(\tau) \right) \right\|_{L^2(U^0)}^2 d\tau
\]
\[
+ \eta \int_0^t \left\| \frac{(u^* \mathcal{T}^e - u^0)(\tau)}{\epsilon} - z(\tau) \right\|_{H^1(U^0)}^2 d\tau.
\] (3.73)

The Poincaré Inequality gives
\[
\sqrt{C_1} \left\| \frac{(u^* \mathcal{T}^e - u^0)(\tau)}{\epsilon} - z(\tau) \right\|_{H^1(U^0)} \leq \left\| \nabla \left( \frac{(u^* \mathcal{T}^e - u^0)(\tau)}{\epsilon} - z(\tau) \right) \right\|_{L^2(U^0)}
\] (3.74)
for some \( C_1 \in (0, 1) \). This together with (3.73) implies
\[
\frac{1}{2} \left\| \frac{u^* \mathcal{T}^e(t) - u^0(t)}{\epsilon} - z(t) \right\|_{L^2(U^0)}^2 + (C_1 \alpha - \beta - \eta) \int_0^t \left\| \frac{(u^* \mathcal{T}^e - u^0)(\tau)}{\epsilon} - z(\tau) \right\|_{H^1(U^0)}^2 d\tau
\leq \frac{1}{2} \left\| \frac{g^* \mathcal{T}^e - g^0}{\epsilon} - \nabla g^0 \cdot V \right\|_{L^2(U^0)}^2 + \frac{1}{4\alpha} \int_0^t \left\| \frac{\gamma(\epsilon, \cdot) - 1}{\epsilon} - u^0(\tau) \right\|_{H^{-1(U^0)}}^2 d\tau
\]
\[
+ \frac{1}{4\beta} \int_0^t \left\| \frac{A(\epsilon, \cdot) - I}{\epsilon} \nabla (u^* \mathcal{T}^e)(\tau) - A(0, \cdot) \nabla u^0(\tau) \right\|_{L^2(U^0)}^2 d\tau
\]
\[
+ \frac{1}{4\eta} \int_0^t \left\| \frac{f^* \mathcal{T}^e(\tau) \gamma(\epsilon, \cdot) - f^0(\tau)}{\epsilon} - \text{div} \left( V f^0(\tau) \right) \right\|_{L^2(U^0)}^2 d\tau
\] (3.75)
for all positive numbers $\alpha, \beta$ and $\eta$. We can choose $\alpha = \beta = \eta = C_1/4$ to obtain
\[
\left\| u^\epsilon \delta T^\epsilon(t) - u^0(t) \right\|_{L^2(U^0)}^2 + \int_0^t \left| \frac{u^\epsilon \delta T^\epsilon - u^0}{\epsilon} \right|_{L^1(U^0)}^2 + \int_0^t \left| (u^\epsilon \delta T^\epsilon - u^0)(\tau) \right|_{L^1(U^0)}^2 d\tau 
\]
\[
\lesssim \left| \frac{g^\epsilon \delta T^\epsilon - g^0}{\epsilon} - \nabla g^0 \cdot V \right|_{L^2(U^0)}^2 + \int_0^t \left| u^\epsilon \delta T^\epsilon(\tau) \gamma(\epsilon, \cdot) - 1 \right|_{L^2(U^0)}^2 - u^0(\tau) \right|_{H^{-1}(U^0)}^2 d\tau 
\]
\[
+ \int_0^t \left| A(\epsilon, \cdot) - \frac{1}{\epsilon} \nabla (u^\epsilon \delta T^\epsilon)(\tau) - A'(0, \cdot) \nabla u^0(\tau) \right|_{L^2(U^0)}^2 + \int_0^t \left| f^\epsilon \delta T^\epsilon(\tau) \gamma(\epsilon, \cdot) - f^0(\tau) \right|_{L^2(U^0)}^2 + \nabla (V f(\tau)) d\tau. 
\]
(3.76)

Letting $\epsilon \to 0$ and noting [3.18] and Lemma [3.19] we obtain
\[
\lim_{\epsilon \to 0} \left\| \frac{u^\epsilon \delta T^\epsilon - u^0}{\epsilon} - z \right\|_{L^\infty(0, T; L^2(U^0))} = 0, 
\]
and
\[
\lim_{\epsilon \to 0} \left\| \frac{u^\epsilon \delta T^\epsilon - u^0}{\epsilon} - z \right\|_{L^2(0, T; H^1(U^0))} = 0. 
\]
(3.77)

Noting [3.68] and employing the duality argument, we obtain
\[
\left\| \frac{u^\epsilon \delta T^\epsilon - u^0}{\epsilon} - z_t \right\|_{L^2(0, T; H^{-1}(U^0))} \leq \left\| \frac{u^\epsilon \delta T^\epsilon - u^0}{\epsilon} - z \right\|_{L^2(0, T; H^1(U^0))} + \left\| \frac{u^\epsilon \delta T^\epsilon(\gamma(\epsilon, \cdot) - 1}{\epsilon} - u^0 \right\|_{L^2(0, T; H^{-1}(U^0))} 
\]
\[
+ \frac{|A(\epsilon, \cdot) - I}{\epsilon} \nabla (u^\epsilon \delta T^\epsilon) - A'(0, \cdot) \nabla u^0 \right|_{L^2(0, T; L^2(U^0))} + \frac{|f^\epsilon \delta T^\epsilon \gamma(\epsilon, \cdot) - f^0}{\epsilon} - \nabla (V f(\tau)) \right|_{L^2(0, T; L^2(U^0))}. 
\]
(3.78)

This together with [3.77] and the results in Lemma [3.19] yields
\[
\lim_{\epsilon \to 0} \left\| \frac{u^\epsilon \delta T^\epsilon - u^0}{\epsilon} - z_t \right\|_{L^2(0, T; H^{-1}(U^0))} = 0, 
\]
(3.79)
finishing the proof of this lemma.

Lemma [3.20] assures the existence of the material derivative of the solution $u^\epsilon$ of the perturbed problem [3.20] and thus assures the existence of the corresponding shape derivative. Furthermore, the shape derivative turns out to be the solution of a heat equation.

**Theorem 3.21.** Under the assumptions of Lemma 3.20, the shape derivative $u'$ of $u^\epsilon$ exists and is the weak solution of the following heat problem
\[
u'_t(t, x) - \Delta u'(t, x) = 0, \quad (t, x) \in (0, T) \times U^0 
\]
(3.80a)
\[
u'(t, x) = -\nabla u^0(t, x) \cdot V(x), \quad (t, x) \in (0, T) \times \Gamma^0 
\]
(3.80b)
\[
u'(0, x) = 0, \quad x \in U^0. 
\]
(3.80c)
Proof. Let \( v \) be an arbitrary function in \( C_0^\infty(U^0) \). Then there is a sufficiently small \( \epsilon_0 > 0 \) such that for all \( \epsilon < \epsilon_0, \ v \in C_0^\infty(U^\epsilon) \). Multiplying both sides of the first equation in (3.29) with \( v \) and then integrating over \( U^\epsilon \), using the Green’s identity, we obtain

\[
\int_{U^\epsilon} u_\epsilon'(t, x) v(x) \, dx - \int_{U^\epsilon} u'(t, x) \Delta v(x) \, dx = \int_{U^\epsilon} f(t, x) v(x) \, dx. \tag{3.81}
\]

Integrating both sides of (3.81) over \([0, t]\), we obtain

\[
\int_{U^0} u'(t, x) v(x) \, dx - \int_0^t \int_{U^0} u'(\tau, x) \Delta v(x) \, dx \, d\tau = \int_0^t \int_{U^0} f(\tau, x) v(x) \, dx \, d\tau,
\]

noting (3.80b). We then differentiate both sides of the above equation over \( \epsilon \) to obtain

\[
\int_{U^0} u'(t, x) v(x) \, dx + \int_{\Gamma^0} [u_0(t, x) - g(x)] v(x) \, \langle V(x), n^0(x) \rangle \, d\sigma_x
\]

\[
- \int_0^t \int_{U^0} u'(\tau, x) \Delta v(x) \, dx \, d\tau - \int_0^t \int_{U^0} u_0'(\tau, x) \Delta v(x) \, \langle V(x), n^0(x) \rangle \, d\sigma_x \, d\tau
\]

\[
= \int_0^t \int_{\Gamma^0} f(\tau, x) v(x) \, \langle V(x), n^0(x) \rangle \, d\sigma_x \, d\tau.
\]

Noting that \( v \in C_0^\infty(U^0) \), there holds

\[
\int_{U^0} u'(t, x) v(x) \, dx - \int_0^t \int_{U^0} u'(\tau, x) \Delta v(x) \, dx \, d\tau = 0 \quad \forall v \in C_0^\infty(U^0). \tag{3.82}
\]

Rewriting the first integral in form of integral over \([0, t]\) and applying the Green’s identity again for the second integral we arrive at

\[
\int_0^t \int_{U^0} u_0'(\tau, x) v(x) \, dx \, d\tau - \int_0^t \int_{U^0} \Delta u'(\tau, x) v(x) \, dx \, d\tau + \int_{U^0} u'(0, x) v(x) = 0. \tag{3.83}
\]

It follows from the definition of shape derivatives, equations (3.65) and (3.30c) that

\[
u'(0, x) = u(0, x) - \nabla u^0(0, x) \cdot V(x)
\]

\[
= \nabla g(x) \cdot V(x) - \nabla u^0(0, x) \cdot V(x) = 0 \quad \forall x \in U^0. \tag{3.84}
\]

This together with (3.83) yields

\[
\int_0^t \int_{U^0} u_0'(\tau, x) v(x) \, dx \, d\tau - \int_0^t \int_{U^0} \Delta u'(\tau, x) v(x) \, dx \, d\tau = 0 \quad \forall v \in H_0^1(U^0). \tag{3.85}
\]

The condition on boundary surface of the material derivative (see (3.65) and the definition of shape derivative) gives

\[
u'(t, x) = -\nabla u^0(t, x) \cdot V(x), \quad (t, x) \in (0, T) \times \Gamma^0.
\tag{3.86}
\]

From (3.84)–(3.86), the shape derivative is the solution to the following problem

\[
\begin{align*}
u_t'(t, x) - \Delta u'(t, x) &= 0, \quad (t, x) \in U^0 \times (0, T) \tag{3.87a} \\
u'(t, x) &= -\nabla u^0(t, x) \cdot V(x), \quad (t, x) \in (0, T) \times \Gamma^0 \tag{3.87b} \\
u'(0, x) &= 0, \quad x \in U^0, \tag{3.87c}
\end{align*}
\]

finishing the proof of the theorem. □
4 Boundary reduction

Let \( r \) and \( s \) be nonnegative real numbers. For \( U \) an open set in \( \mathbb{R}^n \) and for \( 0 < T < \infty \), we denote \( Q_T := (0, T) \times U \) and \( \Sigma_T := (0, T) \times \Gamma \) where \( \Sigma \) is the boundary of \( U \). The space \( H^{r,s}(Q_T) \) is defined by
\[
H^{r,s}(Q_T) = L^2(0, T; H^r(U)) \cap H^s(0, T; L^2(U)). \tag{4.1}
\]
This is a Hilbert space with the following norm
\[
\|v\|_{H^{r,s}(Q_T)} = \left( \int_0^T \|v(t)\|_{H^r(U)}^2 \, dt + \|v\|_{H^s(0,T;L^2(U))}^2 \right)^{1/2}.
\]
The space \( H^{r,s}(\Sigma_T) \) is analogously defined with the corresponding norm
\[
\|v\|_{H^{r,s}(\Sigma_T)} = \left( \int_0^T \|v(t)\|_{H^r(\Gamma)}^2 \, dt + \|v\|_{H^s(0,T;L^2(\Gamma))}^2 \right)^{1/2}.
\]
The following property whose proof can be found in [14, Proposition 2.1] states the interpolation property of the time-varying Sobolev spaces \( H^{r,s}(Q_T) \) and \( H^{r,s}(\Sigma_T) \).

**Proposition 4.1.** For \( r_1, r_2, s_1, s_2 \geq 0 \) and \( \theta \in [0, 1] \), there hold
\[
[H^{r_1,s_1}(Q_T), H^{r_2,s_2}(Q_T)]_{\theta} = H^{r_1+\theta r_2, s_1+\theta s_2}(Q_T),
[H^{r_1,s_1}(\Sigma_T), H^{r_2,s_2}(\Sigma_T)]_{\theta} = H^{r_1+\theta r_2, s_1+\theta s_2}(\Sigma_T). \tag{4.2}
\]

The space \( H^{r,s}(Q_T) \) turns out to be the space of restrictions to \( Q_T \) of the functions in \( H^{r,s}(\mathbb{R}; U) \), equipped with the obvious quotient norm. Here, the space \( H^{r,s}(\mathbb{R}; U) \) is defined by
\[
H^{r,s}(\mathbb{R}; U) := L^2(\mathbb{R}; H^r(U)) \cap H^s(\mathbb{R}; L^2(U))
\]
with the natural norm defined on these spaces of Hilbert space valued distributions. If
\[
\tilde{v}(\tau, x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-it\tau} v(t, x) \, dt
\]
is the Fourier transform of \( u \) w.r.t. the time variable, we have
\[
\|v\|_{H^{r,s}(\mathbb{R}; \mathbb{R}^n)}^2 = \int_{\mathbb{R}} \left( \|\tilde{v}(\cdot, \tau)\|_{H^r(U)}^2 + (1 + |\tau|^2)^s \|\tilde{v}(\tau)\|_{L^2(U)}^2 \right) \, d\tau.
\]

For \( r, s \leq 0 \), the space \( H^{r,s}(Q_T) \), \( H^{r,s}(\Sigma_T) \) and \( H^{r,s}(\mathbb{R}; U) \) are defined to be the dualities of \( H^{-r,-s}(Q_T) \), \( H^{-r,-s}(\Sigma_T) \) and \( H^{-r,-s}(\mathbb{R}; U) \), respectively. We also need the following subspaces
\[
\tilde{H}^{r,s}(Q_T) := \{ v \in H^{r,s}((-\infty, T); U) \mid u(t, x) = 0 \quad \text{for} \quad t < 0 \}
\]
and
\[
\mathcal{V}(Q_T) = L^2((0, T); H^1(U)) \cap H^1((0, T); H^{-1}(U)).
\]
In this section, boundary integral equation methods will then be used to compute statistical moments of the shape derivative (see (3.57)). We recall here some required boundary integral operators.

Let \( v \in \tilde{H}^{1,1/2}(\mathbb{R}_+; U) \) be given and \( t_0 \in \mathbb{R} \) be arbitrary. Define the time-reversal map \( \lambda_{t_0} \) by
\[
\lambda_{t_0} v(t, x) := v(t_0 - t, x).
\]
Then \( \lambda_{t_0} v \in H^{1,1/2}_0((-\infty, t_0]; U) \). Let
\[
G(t, x) := (4\pi t)^{-3/2} e^{-\frac{|x|^2}{4t}} \vartheta(t)
\]
be the fundamental solution of the heat equation, where \( \vartheta(t) = 1/2(1+\text{sign}t) \) is the Heaviside function.

Denoting \( \mathcal{G}(t, x) = G(t, x_0 - x) \), then \( \mathcal{G} \in H^{1,1/2}(\mathbb{R}_+; U) \) and \( \vartheta(t, x) - \Delta \vartheta(t, x) = 0 \) for \( x \neq x_0 \).

For \( (t_0, x_0) \in Q_T := (0, T) \times U \), the single layer potential is defined by

\[
K_0(v)(t_0, x_0) := \int_0^T \int_U v(x_0, t) \gamma_0(\lambda_{t_0} G) \, dx \, dt
= \int_0^T \int_U v(t, x) \gamma_0 \left( [4\pi(t_0 - t)]^{-3/2} e^{-\frac{|x-x_0|^2}{4(t_0-t)} \vartheta(t_0-t)} \right) \, dx \, dt.
\] (4.3)

and the double layer potential is defined by

\[
K_1(v)(t_0, x_0) := \int_0^T \int_U v(x_0, t) \gamma_1(\lambda_{t_0} G) \, dx \, dt
= \int_0^T \int_U v(t, x) \gamma_1 \left( [4\pi(t_0 - t)]^{-3/2} e^{-\frac{|x-x_0|^2}{4(t_0-t)} \vartheta(t_0-t)} \right) \, dx \, dt.
\] (4.4)

The boundary integral operators are defined as follows:

\[
(\mathcal{V}\psi)(t, x) := \lim_{t^0 \ni x \to x} (K_0 \psi)(t, x), \quad x \in \Gamma^0,
\]

\[
(\mathcal{N}\psi)(t, x) := \frac{1}{2} \left( \lim_{t^0 \ni x \to x} \nabla_x (K_0 \psi) \cdot n_x + \lim_{t^0 \ni x \to x} \nabla_x (K_0 \psi) \cdot n_x \right),
\]

\[
(\mathcal{K}w)(t, x) := \frac{1}{2} \left( \lim_{t^0 \ni x \to x} (K_1 \psi)(t, x) + \lim_{t^0 \ni x \to x} (K_1 \psi)(t, x) \right),
\]

\[
(\mathcal{W}w)(t, x) := - \lim_{t^0 \ni x \to x} (K_1 w)(t, x),
\]

for \( \psi \in H^{-1/2,-1/4}((0, T); \Gamma^0) \) and \( w \in H^{1/2,1/4}((0, T); \Gamma^0) \).

The unique solution of (3.87) can be represented

(a) as \( u' = K_0 \psi - K_1(-\nabla u^0 \cdot V) \), where \( \psi \) is the unique solution of the first kind integral equation

\[
\mathcal{V}\psi = \left( \frac{1}{2} I + \mathcal{K} \right) (-\nabla u^0 \cdot V).
\] (4.5)

(b) as \( u' = K_0 \psi - K_1(-\nabla_{\Gamma^0} u^0 \cdot V) \), where \( \psi \) is the unique solution of the second kind integral equation

\[
\left( \frac{1}{2} I - \mathcal{N} \right) \psi = \mathcal{W}(-\nabla u^0 \cdot V).
\] (4.6)

(c) as \( u' = K_0 \psi \), where \( \psi \) is the unique solution of the first kind integral equation

\[
\mathcal{V}\psi = -\nabla u^0 \cdot V.
\] (4.7)

(d) as \( u = K_1 w \), where \( w \) is the unique solution of the second kind integral equation

\[
\left( \frac{1}{2} I - \mathcal{K} \right) w = \nabla u^0 \cdot V.
\] (4.8)

We may use the representations (4.5)–(4.8) to compute the statistical moments of the shape derivative (3.80). As a model, we shall present the boundary integral equation method to compute statistical moments of the solution to the problem (2.14) by using (4.7). Taking the randomness of the domain into account. The random shape derivative of the solutions to (2.14) satisfies the following problem

\[
u'_t(t, x; \omega) - \Delta u'_t(t, x; \omega) = 0, \quad (t, x) \in U^0 \times (0, T)
\]

\[
u'_t(t, x; \omega) = -\nabla u'_t(t, x) \cdot V(x; \omega), \quad (t, x) \in (0, T) \times \Gamma^0
\]

\[
u'_0(x; \omega) = 0, \quad x \in U^0.
\] (4.9c)
Following (4.7), the solution \( u'(t, x; \omega) \) can be represented as

\[
u'(t, x; \omega) = (K_0 \psi(\omega)) (t, x), \quad (t, x) \in (0, T) \times U_0,
\]

where \( \psi(\omega) \) is the solution of

\[
\mathcal{V} \psi(\omega) = -\nabla u_0 \cdot V(\cdot; \omega).
\]

Tensorising and integrating (4.10) yield

\[
\mathcal{M}^k[u'] = (K_0)^{(k)} \mathcal{M}^k[\psi] \in \bigotimes_{i=1}^k \tilde{H}^{1,1/2}(Q_T^0; \partial_t - \Delta),
\]

where \( \mathcal{M}^k[\psi] \in \bigotimes_{i=1}^k H^{-1/2,-1/4}(\Sigma_0^0) \) is the solution of the following equation

\[
\mathcal{V}^{(k)} \mathcal{M}^k[\psi] = (-1)^k \mathcal{M}^k[\kappa] \bigotimes_{i=1}^k (\nabla u_0 \cdot n^0) \in \bigotimes_{i=1}^k H^{1/2,1/4}(\Sigma_T^0).
\]

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