Componentwise linearity of ideals arising from graphs

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Abstract. Let $G$ be a simple undirected graph on $n$ vertices. Francisco and Van Tuyl have shown that if $G$ is chordal, then $\bigcap\{x_i, x_j\} \in E_G \langle x_i, x_j \rangle$ is componentwise linear. A natural question that arises is for which $t_{ij} > 1$ the ideal $\bigcap\{x_i, x_j\} \in E_G \langle x_i, x_j \rangle^{t_{ij}}$ is componentwise linear, if $G$ is chordal. In this report we show that $\bigcap\{x_i, x_j\} \in E_G \langle x_i, x_j \rangle^t$ is componentwise linear for all $n \geq 3$ and positive $t$, if $G$ is a complete graph. We give also an example where $G$ is chordal, but the intersection ideal is not componentwise linear for any $t > 1$.

1. Introduction

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Let $G$ be a simple graph on $n$ vertices, $E_G$ the edge set of $G$ and $V_G$ the vertex set of $G$. Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring over a field $k$. The edge ideal of $G$ is the quadratic squarefree monomial ideal $I(G) = \langle \{x_i x_j\} \mid \{x_i, x_j\} \in E_G \rangle \subset R$. Then we define the squarefree Alexander dual of $I(G)$ as $I(G)^\vee = \bigcap\{x_i, x_j\} \in E_G \langle x_i, x_j \rangle$. Calling $I(G)^\vee$ the squarefree Alexander dual of $I(G)$ is natural since $I(G)^\vee$ is the Stanley–Reisner ideal of the simplicial complex $\Delta^\vee$, that is, the Alexander dual simplicial complex of $\Delta$. Here $\Delta$ is the simplicial complex, the Stanley-Reisner ideal of which is $I(G)$.

In [HH] Herzog and Hibi give the following definition. Given a graded ideal $I \subset R$, we denote by $I_{(d)}$ the ideal generated by the
elements of degree $d$ that belong to $I$. Then we say that a (graded) ideal $I \subset R$ is componentwise linear if $I_{(d)}$ has a linear resolution for all $d$.

If the graph $G$ is chordal, that is, every cycle of length $m \geq 4$ in $G$ has a chord, then it is proved by Francisco and Van Tuyl [FvT1] that $I(G)^V$ is componentwise linear. (The authors then use the result to show that all chordal graphs are sequentially Cohen-Macaulay.)

In this report we examine componentwise linearity of ideals arising from complete graphs and of the form $\bigcap_{\{x_i,x_j\} \in E_G} \langle x_i, x_j \rangle^t$.

2. Intersections for complete graphs

Let $K_n$ be a complete graph on $n$ vertices, that is, $\{x_i, x_j\} \in E_{K_n}$ for all $1 \leq i \neq j \leq n$. We write $K_n^{(t)} = \bigcap_{\{x_i,x_j\} \in E_{K_n}} \langle x_i, x_j \rangle^t$. We will show that the ideal $K_n^{(t)}$ is componentwise linear for all $n \geq 3$ and $t \geq 1$. Recall that a vertex cover of a graph $G$ is a subset $A \subset V_G$ such that every edge of $G$ is incident to at least one vertex of $A$. One can show that $I(G)^V = \langle x_{i_1} \cdots x_{i_k} \mid \{x_{i_1}, \ldots, x_{i_k}\} \text{ a vertex cover of } G \rangle$. A $t$-vertex cover (or a vertex cover of order $t$) of $G$ is a vector $a = (a_1, \ldots, a_n)$ with $a_i \in \mathbb{N}$ such that $a_i + a_j \geq t$ for all $\{x_i, x_j\} \in E_G$.

In the proof of our main result Theorem 2.3, we use the following definition and proposition.

**Definition 2.1.** A monomial ideal $I$ is said to have linear quotients, if for some degree ordering of the minimal generators $f_1, \ldots, f_r$ and all $k > 1$, the colon ideals $\langle f_1, \ldots, f_{k-1} \rangle : f_k$ are generated by a subset of $\{x_1, \ldots, x_n\}$.

**Proposition 2.2** (Proposition 2.6 in [FvT2] and Lemma 4.1 in [CH]). If $I$ is a homogeneous ideal with linear quotients, then $I$ is componentwise linear.

**Theorem 2.3.** The ideal $K_n^{(t)}$ is componentwise linear for all $n \geq 3$ and $t \geq 1$.

**Proof.** For calculating an explicit generating system of $K_n^{(t)}$ we will use $t$-vertex covers. Pick any monomial $m$ in the generating set of $K_n^{(t)}$ and, for some $k$ and $l$, consider the greatest exponents $t_k$ and $t_l$ such that $x_k^t x_l^l$ is a factor in $m$. As $m$ is contained in $\langle x_k, x_l \rangle^t$ we must have $t_k + t_l \geq t$. Hence, $K_n^{(t)}$ is generated by the monomials of the form $x^a$, where $a$ is an $t$-cover of $K_n$. That is, the sum of the two lowest exponents in every (monomial) generator of $K_n^{(t)}$ is at least $t$.

First we assume that $t = 2m + 1$ is odd. Using the degree lexicographic ordering $x_1 \prec x_2 \prec \cdots \prec x_n$ on the the minimal generators we
get

\[ K_n^{(t)} = K_n^{(2m+1)} = \langle x_1^m \prod_{i \neq 1} x_i^{m+1}, \ldots, x_n^m \prod_{i \neq n} x_i^{m+1}, \]
\[ x_1^{m-1} \prod_{i \neq 1} x_i^{m+2}, \ldots, x_n^{m-1} \prod_{i \neq n} x_i^{m+2}, \]
\[ \vdots \]
\[ \prod_{i \neq 1} x_i^{2m+1}, \ldots, \prod_{i \neq n} x_i^{2m+1} \rangle. \]

This ordering of the minimal generators satisfies the condition in Definition 2.1. Hence, \( K_n^{(t)} \) has linear quotients and is componentwise linear by Proposition 2.2.

If \( t = 2m \) is even, then the degree lexicographic ordering yields the sequence

\[ K_n^{(t)} = K_n^{(2m)} = \langle \prod_{i=1}^{2m} x_i^m, x_1^{m-1} \prod_{i \neq 1} x_i^{m+1}, \ldots, x_n^{m-1} \prod_{i \neq n} x_i^{m+1}, \]
\[ x_1^{m-2} \prod_{i \neq 1} x_i^{m+2}, \ldots, x_n^{m-2} \prod_{i \neq n} x_i^{m+2}, \]
\[ \vdots \]
\[ \prod_{i \neq 1} x_i^{2m}, \ldots, \prod_{i \neq n} x_i^{2m} \rangle, \]

which also satisfies the condition in Definition 2.1 and the same result follows.

**Example 2.4.**

\[ K_{12}^{(5)} = \langle \left\{ x_j^2 \prod_{i \neq j} x_i^3 \right\}, \left\{ x_j \prod_{i \neq j} x_i^4 \right\}, \left\{ \prod_{i \neq j} x_i^5 \right\} \rangle \]

and

\[ K_5^{(6)} = \langle \prod_{i=1}^5 x_i^3, \left\{ x_j^2 \prod_{i \neq j} x_i^4 \right\}, \left\{ x_j \prod_{i \neq j} x_i^5 \right\}, \left\{ \prod_{i \neq j} x_i^6 \right\} \rangle. \]

**Remark 2.5.** A monomial ideal is called polymatroidal if it is generated in one degree and its minimal generators satisfy a certain ”exchange condition”. In [HT] Herzog and Takayama show that polymatroidal ideals have linear resolutions. Later Francisco and van Tuyl [FvT2] proved that some families of ideals \( I \) are componentwise linear showing in their Theorem 3.1 that \( I_{(d)} \) are polymatroidal for all \( d \).

The ideals \( K_n^{(t)} \) are also polymatroidal, but the proof using the same techniques as in the proof of Theorem 3.1 in [FvT2] is rather tedious and takes a few pages.
3. A counterexample

There exists a chordal graph $G$ such that $\bigcap_{(x_i, x_j) \in E_G} \langle x_i, x_j \rangle^t$ is not componentwise linear for any $t > 1$.

**Proof.** Let $G$ be the chordal graph

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and denote the intersection $\langle a, b \rangle^t \cap \langle a, c \rangle^t \cap \langle b, c \rangle^t \cap \langle b, d \rangle^t$ by $I_4^{(t)}$. We have

$$I_4^{(1)} = \bigcap_{\{i, j\} \in E_G} \langle i, j \rangle = \langle bc \rangle + \langle abd, acd \rangle$$

and

$$I_4^{(2)} = \bigcap_{\{i, j\} \in E_G} \langle i, j \rangle = \langle b^2c^2, abcd \rangle + \langle a^2b^2d^2, a^2c^2d^2 \rangle.$$

Arguing in the same way as for $K_n^{(t)}$ we see that the minimal generating set consists of generators of exactly degree $2t$ and generators of higher degrees:

- If $t_a \leq \lfloor \frac{t}{2} \rfloor$ then $t_b = t - t_a = t_c$ (the sum $t_b + t_c \geq t$ automatically) and $t_d = t - t_b = t - t_c = t_a$. We get the set of minimal generators of degree $2t$:
  $$\{a^i(bc)^{t-i}d^i\}_{0 \leq i \leq \lfloor \frac{t}{2} \rfloor}.$$

- If $t_a > \lfloor \frac{t}{2} \rfloor$, then either $t_b = t - t_a$ and $t_c = t - t_b = t_a$, or $t_c = t - t_a$ and $t_b = t_a$. Further $t_d = t_a$. The set of minimal generators we get in this way is equal to
  $$\{(acd)^i(b^t-i)\}_{\lfloor \frac{t}{2} \rfloor < i \leq t} \cup \{(acd)^i(b^t-i)\}_{\lfloor \frac{t}{2} \rfloor < i \leq t}.$$

The generators in this set are of degree at least $(2t + 1)$ for odd $t$ and of degree at least $(2t + 2)$ for even $t$.

Now consider the minimal free resolution $F$ of $(I_4^{(t)})_{(2t)}$. Since $F$ is contained in any free resolution $G$ of $(I_4^{(t)})_{(2t)}$ we have that if $F_1$ (the component of $F$ in homological degree 1) has a non-zero component in a certain degree, then so does $G_1$. Let $G$ be the Taylor resolution of $(I_4^{(t)})_{(2t)}$. The degrees in which $G_1$ has nonzero components come from
least common multiples of pairs of minimal generators of \((I_4^{(t)})_{(2t)}\). By considering the above description of the minimal generators in degree \(2t\), one sees that \(G_1\) has non-zero components only in degrees strictly larger than \(2t + 1\). Thus \(F\) cannot be a linear resolution and, hence, \(I_4^{(t)}\) is not componentwise linear.

\[\square\]

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