Locally-Optimized Reweighted Belief Propagation for Decoding LDPC Codes with Finite-Length

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Abstract—In practice, LDPC codes are decoded using message passing methods. These methods offer good performance but tend to converge slowly and sometimes fail to converge and to decode the desired codewords correctly. Recently, tree-reweighted message passing methods have been modified to improve the convergence speed at little or no additional complexity cost. This paper extends this line of work and proposes a new class of locally optimized reweighting strategies, which are suitable for both regular and irregular LDPC codes. The proposed decoding algorithm first splits the factor graph into subgraphs and subsequently performs a local optimization of reweighting parameters. Simulations show that the proposed decoding algorithm significantly outperforms the standard message passing and existing reweighting techniques.

Index Terms—LDPC codes, belief propagation algorithms, decoding techniques.

I. INTRODUCTION

Low density parity check (LDPC) codes are among the most important capacity-approaching error-correcting codes [1]. They have a long history, deep theoretical underpinning, and have found practical application in multiple standards [2]. The main power of LDPC codes lies in their pseudo-random nature, assuring good properties for code, and the availability of low-complexity decoding algorithms. Recently, a great deal of research has been devoted to the design of LDPC codes with short to moderate block lengths, which correspond to most of the application scenarios of these codes in wireless standards [2].

Decoding is commonly based on iterative message-passing methods, allowing local parallel computations. While message-passing decoding leads to good performance in terms of bit error rate (BER), it suffers from a number of drawbacks: (i) convergence to a codeword can take many iterations, especially with low signal to noise ratios (SNR); (ii) convergence to a codeword is not guaranteed; (iii) LDPC code design is guided by the decoding algorithm, constraining codes to have large girths. In this context, the occurrence of short cycles and stopping sets makes a significant impact on the performance of LDPC codes, and requires the development of novel decoding strategies that mitigate these drawbacks.

Different approaches have been considered to deal with these issues. The most prominent approach is linear programming (LP) based decoding, which, through a relaxation, formulates the decoding problem as an LP, and has a maximum likelihood certificate property [3]. LP decoders suffer from high complexity (exponential in the check node degree), unless further relaxations are employed [4]. Another line of investigation, which aims to improve performance while still maintaining the message passing nature of the decoder, is that of tree-reweighted message passing decoding. Based on tree-reweighted belief propagation [5], decoding reverts to a tractable convex optimization problem, iteratively computing beliefs and factor appearance probabilities (FAPs). These concepts were applied to LDPC decoding in [6], [7] where the FAPs were optimized in an offline procedure, subject to additional constraints: in [6], the FAPs were constrained to be constant, while in [7], the FAPs were constrained to take on two possible values. In both cases, gains with respect to standard message passing decoding were observed.

In this paper, we continue this latter line of research, and explicitly optimize the FAPs off-line, without the constraints from [6], [7]. This allows more freedom in the decoding algorithm without additional online computational complexity. We propose a locally-optimized reweighting belief propagation (LOW-BP) decoding algorithm that first splits the factor graph corresponding to the code into subgraphs and then performs local optimization of the reweighting parameters. The proposed LOW-BP algorithm can mitigate the effects of short cycles and stopping sets in factor graphs by applying a reweighting strategy per subgraph. The LOW-BP algorithm is evaluated for regular and irregular LDPC codes.

This paper is structured as follows. Section II briefly describes the LDPC system model. Section III reviews reweighting strategies and existing algorithms. Section IV is dedicated to a detailed description of the proposed LOW-BP algorithm, whereas Section V presents and discusses the simulation results. Section VI draws conclusions from the work.
II. LDPC SYSTEM MODEL

We consider a rate $K/N$ code, with parity check matrix $H$, and a corresponding set of codewords $C$. Note that $x \in C$ if and only if $Hx = 0$. Assuming binary phase-shift keying and transmission over an additive white Gaussian noise (AWGN) channel, the received data are described by

$$y = 2x - 1 + n,$$  

where $n$ is a sequence of $N$ independently identically distributed (i.i.d.) AWGN samples with variance $\sigma^2$, and $x \in C$ is the transmitted codeword. Given $y$, the aim of the iterative decoder is to recover $x$ in an iterative fashion until either $Hx = 0$ or the maximum number of decoding iterations is reached. Iterative decoding can be interpreted as message passing on a suitable factor graph, and is often implemented using belief propagation (BP), or a variation thereof.

The factor graph corresponding to our model, $G(V, E)$, include the check and variable nodes $V = V_c \cup V_s$, as well as a set of edges, $E \subseteq V_c \times V_s$, such that an edge connecting the check node $c_i$ and the variable node $s_j$ exists in the factor graph only if the entry $h_{ij}$ of the parity-check matrix $H$ equals 1. The decoding process can be interpreted as message passing on a suitable factor graph, and is often implemented using belief propagation (BP), or a variation thereof.

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In general, reweighting (TRW)-BP algorithms for cyclic graphs, which include the check and variable nodes $V_c \cup V_s$ and only if

$$m = \text{arg max}_{m} \text{success rate}.$$  

where $b(\cdot)$ denotes the so-called belief, $H(b_n)$ is the entropy of the belief of the $n$-th variable, $\phi_n(x_n)$ and $\psi_m(x_m)$ are the potential function and the compatibility function, respectively, which are defined depending on the application. Notice that if we use $C_m$ to denote the $m$-th node’s cluster $(m = 1, \ldots, M)$, then $b_{C_m}(x_{C_m})$ denotes joint belief and $\mathcal{I}_{C_m}(b_{C_m})$ represents joint mutual information term [8]. The optimization with respect to $(b, \rho)$ starts with a fixed $\rho^{(k)}$, then solves this for the stationary points of $\mathcal{F}(b, \rho^{(k)})$ via TRW-BP. Next, for a fixed belief vector $b$, minimizing the function $\mathcal{F}(b, \rho^{(k)})$ with respect to $\rho^{(k)}$ results in an updated $\rho^{(k+1)}$. This algorithm keeps running recursively until the belief converges. Observe that standard BP corresponds to the sub-optimal and generally invalid choice $\rho = 1$. The work reported in [5] is not directly applicable to problems such as the decoding of LDPC codes, and only derives message-passing rules for graphs with pairwise interactions.

In [6], [9], the uniformly reweighted (URW)-BP algorithm extends pairwise factorizations to higher order interactions and reduces a series of globally optimized parameters $\rho \in [0, 1]^M$ to a simple constant $\rho_n \in [0, 1]$. Additionally, the FAPs are generalized to edge appearance probabilities (EAPs) so that the problem size is significantly reduced. Another reweighting strategy is referred to as variable FAP (VFAP)-BP [7] that aims to select $\rho$ on the basis of the cycle distribution of the graph. However, neither URW-BP nor VFAP-BP optimizes the values of $\rho$ explicitly.

IV. PROPOSED LOW-BP ALGORITHM FOR DECODING LDPC CODES

In this section, we describe LOW-BP, which explicitly optimizes the reweighting parameter vector $\rho = [\rho_1, \rho_2, \ldots, \rho_M]$. By allowing optimization over smaller subgraphs, LOW-BP is able to trade off complexity vs. performance. LOW-BP comprises an offline phase, during which, for a fixed SNR and a fixed code, the best choice of $\rho$ is determined, as outlined in Table I. The online phase of LOW-BP occurs during real-time decoding, when optimized $\rho$ is used in the reweighted message passing decoding algorithm.

A. Offline Phase of LOW-BP

In the offline phase, we transform the factor graph into a set of $T \geq 1$ subgraphs and then locally optimize the reweighting parameter vector $\rho_t$ for each subgraph, where $t = 1, 2, \ldots, T$. Note that when $T > 1$ the dimension of $\rho_t$ depends on the size of the $t$-th subgraph. The optimization turns out to be significantly less complex when more subgraphs are considered, hence there is a need for a flexible method to decompose the original factor graph into subgraphs. We apply the progressive-edge growth (PEG) technique [10] to this end.

Construction of T Subgraphs: PEG expands $G(V, E)$ into $T$ subgraphs $G_t(V_t, E_t)$, where $V_t \subseteq V$ and $E_t \subseteq E$, respectively. This method is straightforward to use if the LDPC code was designed by PEG, or its variations [12], [13], but is not limited to such designs. We consider a disjoint strategy or a re-appearance (RA) strategy, to apply
but a high probability of the existence of very short cycles within subgraphs; (ii) given $d_{\text{max}}$, whether all the nodes of $V$ are included in the expanded subgraphs. Let us denote the degree of a variable node $s_j$ by $w_{s_j}$, and define $N_{s_j}^d$ as the neighborhood of $s_j$ at current expansion level $d$, as well as $\bar{N}_{s_j}^d$ being the complement of $N_{s_j}^d$. To generate the $t$-th subgraph $G_t(V_t, E_t)$ based on $G(V, E)$, the PEG expansion is detailed in Algorithm 1. In the case of the RA strategy, $V_t$, the set of candidate nodes of $G_t(V_t, E_t)$, is always initialized as $V$, for each of the $T$ expansions. On the other hand, $V_t$ is the complement set of $V_{t-1}$ if the disjoint strategy is applied, so that the size of subgraph $G_t(V_t, E_t)$ decreases as $t$ increases. Furthermore, some of the subgraphs, such as $G_T(V_T, E_T)$ or $G_{T-1}(V_{T-1}, E_{T-1})$, may be a acyclic (i.e., a tree), with corresponding reweighting factors $\rho_t = 1$, which complies with the observations in 5, 6. Compared to the greedy search algorithm in 10, 12, our PEG-based expansion stops as soon as every member of $V_t$ has been visited. The number of edges incident to $s_j$ might be less than $w_{s_j}$ as some short cycles are excluded from the subgraphs so as to guarantee that the local girth of each subgraph is always larger than the global girth of the original graph.

**Optimization of FAPs**: After obtaining $T$ subgraphs, we introduce $L = \{L_1, L_2, \ldots, L_T\}$, where $L_t$ is the number of check nodes (possibly with duplicates) in the $t$-th subgraph. In case of disagreement on a value $\rho_M$, for the $m$-th check node, choose the value offering the best performance; offline: optimization of $\rho$, for the $t$-th subgraph

2: Initialize $\rho^{(0)}_t$ to an allowable value;
3: For each subgraph, calculate the beliefs $b(x_t)$ and the mutual information term $I_t = [I_{t,1}, I_{t,2}, \ldots, I_{t,L_t}]$ for the $m$-th check node, choose the value offering the best performance;
4: With $b(x_t)$ and $I_t$ obtained from step 3, update $\rho_t^{(t)}$ to $\rho^{(t+1)}_t$ using the conditional gradient method (detailed in the Appendix);
5: Repeat steps 3–4 until $\rho_t$ converges for each subgraph;

Offline: choice of $\rho = [\rho_1, \rho_2, \ldots, \rho_M]$ for decoding

6: For all $T$ subgraphs, collect $\rho_1, \rho_2, \ldots, \rho_T$. In case of disagreement on a value $\rho_M$ for the $m$-th check node, choose the value offering the best performance;

**Online: real-time decoding**

7: Use reweighted message passing decoding (5)–(7) with optimized $\rho = [\rho_1, \rho_2, \ldots, \rho_M]$ during actual data transmission.

### PEG. The disjoint strategy prohibits duplicates of check nodes in all subgraphs, while the RA strategy allows check nodes to appear more than once over $T$ subgraphs. In general, the number of subgraphs $T$ depends on: (i) a pre-set maximum expansion level $d_{\text{max}}$, as a large $d_{\text{max}}$ results in a small $T$ but a high probability of the existence of very short cycles
variable node \( s_n \) to the \( m \)-th check node \( c_m \) is given by
\[
\Psi_{nm} = \lambda_{ch,n} + \sum_{m' \in \mathcal{N}(n) \setminus m} \rho_{m'} \Lambda_{m'n} - (1 - \rho_m) \Lambda_{mn}, \quad (5)
\]
where \( \lambda_{ch,n} = \log(p(y_n|x_n = 1)/p(y_n|x_n = 0)) = 2y_n/\sigma^2 \), \( m' \in \mathcal{N}(n) \setminus m \) is the neighboring set of check nodes of \( s_n \) except \( c_m \). The quantity \( \Lambda_{mn} \) denotes messages sent from \( c_m \) to \( s_n \) in previous decoding iteration, then for check nodes \( c_m \) we update \( \Lambda_{mn} \) as:
\[
\Lambda_{mn} = f_{\Psi}(\{\rho_{m'}\Psi_{mn'}\}_{m' \in \mathcal{N}(n) \setminus m}) - (1 - \rho_m) \Psi_{nm}, \quad (6)
\]
where \( f_{\Psi}(\cdot) \) denotes the standard BP message passing rule to compute an LLR message from check node \( c_m \) to variable node \( s_n \). The function \( f_{\Psi}(\cdot) \) can be implemented by using the well-known hyperbolic tangent expressions [7], or the numerically more stable Jacobian logarithm [16, 17]. Upon convergence, we have the belief \( \lambda_{\text{belief},n} \) with respect to \( x_n \) given by
\[
\lambda_{\text{belief},n} = \lambda_{ch,n} + \sum_{m \in \mathcal{N}(n)} \rho_m \Lambda_{mn}. \quad (7)
\]
It should be noted that in (5)–(7), the standard BP algorithm corresponds to \( \rho_m = 1, \forall m \). The receiver utilizes the above message-passing rules and does not update \( \rho \) as long as the channel conditions are unchanged.

V. SIMULATION RESULTS

In this section, we show the numerical results obtained from applying the proposed LOW-BP algorithm to the decoding of regular and irregular LDPC codes with short block lengths, over the additive white Gaussian noise (AWGN) channel. The regular code has block length \( N = 500 \) and rate \( R = 1/2 \), with constant column weight \( w_s = 4 \) and row weight \( w_c = 6 \). The irregular code has the same block length and rate, but a variable node degree distribution \( \lambda(x) = 0.21 \times x^5 + 0.25 \times x^3 + 0.25 \times x^2 + 0.29 \times x \) and a constant check node degree of 5. For the sake of numerical stability and data storage, all messages are represented as LLRs, and the Jacobian logarithm [16, 17] is used to compute the messages passed from check nodes to variable nodes. In the offline phase of LOW-BP, 1000 codewords known to the receiver are transmitted so as to optimize \( \rho = [\rho_1, \rho_2, \ldots, \rho_M] \). We allowed up to 60 decoding iterations in the online phase.

Fig. 1 illustrates the distribution of the reweighting parameters for regular codes and irregular codes, at an SNR of 2 dB. It is clear that the optimized \( \rho \) of irregular codes is widely distributed over the range of \([0.6, 0.9]\), while the \( \rho \)-distribution for regular codes is more concentrated within a smaller range \([0.8, 0.85]\). This observation is congruent with the findings in [5], [6], which state that for symmetric graphs, the optimal reweighting parameters should be more or less uniform.

In Fig. 2 and Fig. 3 the BER performance with different
values of $T$, with RA and disjoint selection, are compared to the standard BP algorithm, for regular and irregular codes, respectively. We observe a performance gain of up to 0.4 dB over the standard BP algorithm by using the proposed LOW-BP method. For regular code, all check nodes are visited once in $T = 9$ subgraphs with the disjoint selection, while with RA selection $T = 25$ subgraphs are generated (maximum 60 recursions), where some check nodes are revisited. For irregular code, the disjoint selection generates $T = 12$, meanwhile the RA selection gives $T = 30$ (maximum 100 recursions). When using disjoint selection, $\rho$ converges to a set of stable values after a number of recursions that varies from one subgraph to another. Notice that in both figures, $T = 1$ is a special case that corresponds to Wainwright et al.’s optimal solution from [5]. For $T = 1$, to improve convergence in the offline phase, we initialized $\rho$ from URW-BP [6] for regular code and from VFAP-BP [7] for irregular code. Normally, around 800 recursions we required to converge for regular code and 2700 recursions for irregular code.

A comparison with existing reweighted methods is shown in Fig. 4 and Fig. 5 for regular and irregular codes, respectively. The algorithms considered are URW-BP from [6], VFAP-BP from [7], and the proposed LOW-BP algorithm (with $T = 1$). For regular code, we observe that URW-BP and VFAP-BP outperform standard BP. LOW-BP is able to provide further improvements. For irregular code, the optimal constant value of the FAP in URW-BP is $\rho = 1$, so that BP and URW-BP coincide. VFAP-BP provides a small performance gain, while LOW-BP again outperforms BP by up to 0.4 dB. We clearly see that explicit optimization of $\rho$ leads to non-trivial performance gains.

### VI. Conclusion

We have proposed a locally-optimized reweighting belief propagation (LOW-BP) algorithm for decoding finite-length LDPC codes. The proposed algorithm has been compared to previously reported reweighted belief propagation algorithms and has demonstrated superior performance for the scenarios considered. LOW-BP comprises an offline and an online stage. The online stage relies on standard tree-reweighted belief propagation, while the offline stage involves an optimization problem over subgraphs of the original factor graph. By increasing the number of subgraphs, the offline stage converges faster and exhibits less complexity. Reducing the number of subgraphs will lead to improved BER performance, albeit at an additional delay and complexity cost during the offline stage. LOW-BP is especially well suited to decoding of short to moderate LDPC codes and is a promising choice for applications that require a reduced number of decoding iterations. Future avenues of research include fast adaptation of the offline stage to time-varying channel conditions.

### Appendix: Details of Conditional Gradient

Our goal is to minimize (4) with respect to the column vector $\rho_t$, for a specific subgraph $G_t(V_t, E_t)$. Dropping terms that do not depend on $\rho_t$, we find the following optimization problem with $\bar{I}_t = [I_{t,1} I_{t,2} \ldots I_{t,T}]^T$, where $(\cdot)^T$ denotes the transpose:

$$\text{minimize} \quad -\rho_t^T \bar{I}_t$$

$$\text{s.t.} \quad \rho_t \in \mathbb{T}(G_t(V_t, E_t)),$$

where $\mathbb{T}(G_t(V_t, E_t))$ is the set of all valid FAPs over the subgraph $G_t(V_t, E_t)$ and $I_{t,t}$ is a mutual information term, which depends on $\rho_t^{(r)}$, the previous value of $\rho_t$. We will denote the objective function by $f(\rho_t) = -\rho_t^T \bar{I}_t$ and use the conditional gradient method to update $\rho_t$, similar to [5]. In the conditional gradient method, we first linearize the objective around the current value $\rho_t^{(r)}$:

$$f_{\text{lin}}(\rho_t) = f(\rho_t^{(r)}) + \nabla_{\rho_t} f(\rho_t^{(r)})(\rho_t - \rho_t^{(r)}),$$

in which $\nabla_{\rho_t} f(\rho_t^{(r)}) = -\bar{I}_t$. Secondly, we minimize $f_{\text{lin}}(\rho_t)$ with respect to $\rho_t$, denoting the minimizer by $\rho_t^{(r+1)}$ and $z_t^{(r+1)}$ =
\[
\max\left(f\left(\rho_t^r, z_t^r\right), z_t^0 = -\infty\right). \quad \text{Finally, } \rho_t^r \text{ is updated to } \rho_t^{r+1} \text{ as}
\]
\[
\rho_t^{r+1} = \rho_t^r + \alpha(\rho_t^* - \rho_t^r), \quad (9)
\]
in which \(\alpha\) is chosen as
\[
\arg\min_{\alpha \in [0,1]} f(\rho_t^r + \alpha(\rho_t^* - \rho_t^r)). \quad (10)
\]
At every iteration, \(f(\rho_t^r)\) is an upper bound on the optimized objective, while \(z_t^{r+1}\) is a lower bound.

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