EXISTENCE AND ASYMPTOTIC BEHAVIOR OF NON-NORMAL CONFORMAL METRICS ON $\mathbb{R}^4$ WITH SIGN-CHANGING $Q$-CURVATURE

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Abstract. We consider the following prescribed $Q$-curvature problem
\[
\begin{aligned}
\Delta^2 u &= (1 - |x|^p)e^{4u}, \quad \text{on } \mathbb{R}^4 \\
\Lambda &= \int_{\mathbb{R}^4} (1 - |x|^p)e^{4u}dx < \infty.
\end{aligned}
\]
We show that for every polynomial $P$ of degree 2 such that $\lim_{|x| \to +\infty} P = -\infty$, and for every $\Lambda \in (0, \Lambda_{\text{ sph}})$, there exists at least one solution to problem (1) which assume the form $u = w + P$, where $w$ behaves logarithmically at infinity. Conversely, we prove that all solutions to (1) have the form $v + P$, where
\[
v(x) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left( \frac{|y|}{|x-y|} \right) (1 - |y|^p)e^{4y} dy
\]
and $P$ is a polynomial of degree at most 2 bounded from above. Moreover, if $u$ is a solution to (1), it has the following asymptotic behavior
\[
u(x) = -\frac{\Lambda}{8\pi^2} \log |x| + P + o(\log |x|), \quad \text{as } |x| \to +\infty.
\]
As a consequence, we give a geometric characterization of solutions in terms of the scalar curvature at infinity of the associated conformal metric $e^{2u}|dx|^2$.

Keywords $Q$-curvature · Asymptotic behavior · Conformal metrics

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1. Introduction

Let $M$ be a 4-dimensional Riemannian manifold endowed with a metric $g$. If $\text{Ric}_g$ denotes the Ricci tensor of $(M, g)$, $R_g$ the scalar curvature and $\Delta_g$ the Laplace-Beltrami operator, the $Q$-curvature $Q^4_g$ and the Paneitz operator $P^4_g$ are defined as follows
\[
Q^4_g := -\frac{1}{6} (\Delta_g R_g - R_g^2 + 3|Ric_g|^2)
\]
\[
P^4_g(f) := \Delta_g^2 f + \text{div} \left( \frac{2}{3} R_g g - 2 \text{Ric}_g \right) df, \quad \text{for } f \in C^\infty(M).
\]
These definitions have been introduced by T. Branson and B. Órsted [3] and S. Paneitz [19], and then generalized to higher order $Q$-curvatures $Q^m_g$ and Paneitz operators $P^m_g$ on a $2m$-dimensional manifold (see e.g. [1, 2, 8, 9, 12]). The Paneitz operator is a sort of higher order Laplace-Beltrami operator and the $Q$-curvature can be seen as the higher order counterpart of the Gaussian curvature, this is also pointed out by the fact that in dimension 2 we have $P^2_g = -\Delta_g$ and $Q^2_g = K_g$. The study of Paneitz operator and $Q$-curvature has gained a lot of attention in conformal geometry due to their covariant properties. The total $Q$-curvature is a global conformal invariant, namely if $M$ is closed...
and \( g_u = e^{2u} g \), we have
\[
\int_M Q^2_{g_u} d\text{vol}_{g_u} = \int_M Q^2_g d\text{vol}_g
\]
moreover, this integral gives information on the topology of the manifold (see the Gauss-Bonnet-Chen’s Theorem [7]). Recall also that the Paneitz operator is conformally invariant, this means that if we consider the conformal metric \( g_u = e^{2u} g \), we have
\[
P^2_{g_u} = e^{-2mu} P^2_g
\]
and it satisfies a generalised version of the Gauss equation
\[
P^2_g u + Q^2_g = Q^2_{g_u} e^{2mu}.
\]
On \( \mathbb{R}^2m \) with the standard Euclidean metric \( |dx|^2 \) we have
\[
P^2_{|dx|^2} = (\Delta)^m m
\]
and
\[
Q^2_{|dx|^2} \equiv 0.
\]
A classical problem in conformal geometry is the “problem of prescribing \( Q \)-curvature”, that is finding whether a given function \( K \) on a manifold \((M, g)\), can be the \( Q \)-curvature of a conformal metric \( g_u = e^{2u} g \). Therefore, the question concerns the set of solutions to the following equation
\[
P^2_{g_u} u + Q^2_{g_u} = K e^{2mu}
\]
where \( K = Q^2_{g_u} \) is the prescribed \( Q \)-curvature of the metric \( g_u \). If we consider the Euclidean space \( \mathbb{R}^2m \) endowed with the standard Euclidean metric \( |dx|^2 \), equation (2) becomes
\[
(\Delta)^m u = K e^{2mu}, \quad \text{in} \; \mathbb{R}^{2m}.
\]
Due to its geometric meaning, the constant \( Q \)-curvature case for equation (3) has been extensively studied (see e.g. [4, 5, 6, 14, 16, 17, 20] and the references within). The problem which we address, namely the case of non-normal metrics with sign-changing prescribed \( Q \)-curvature having power-like growth, is (to the best of our knowledge) still unexplored.

Let \( p > 0 \) be fixed, we consider the prescribed \( Q \)-curvature equation
\[
\Delta^2 u = (1 - |x|^p) e^{4u}, \quad \text{in} \; \mathbb{R}^4
\]
under the assumption
\[
\Lambda := \int_{\mathbb{R}^4} (1 - |x|^p) e^{4u} dx < \infty.
\]
Geometrically, this means that if \( u \) is a solution to (4)-(5), then the metric \( g_u := e^{2u} |dx|^2 \), which is conformal to the Euclidean metric on \( \mathbb{R}^4 \), has \( Q \)-curvature equal to \( 1 - |x|^p \) and finite total \( Q \)-curvature \( \Lambda \). In what follows, we will always assume that \( (1 - |x|^p) e^{4u} \in L^1(\mathbb{R}^4) \). Let \( u \) be a solution to (4)-(5), we define \( v \) as
\[
v(x) := \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left( \frac{|y|}{|x-y|} \right) (1 - |y|^p) e^{4u} dy.
\]

**Definition 1.1** (Normal and non-normal solutions). We call a solution \( u \) to (4)-(5) normal if there exists a constant \( c \in \mathbb{R} \) such that \( u \) solves the following integral equation
\[
u(x) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left( \frac{|y|}{|x-y|} \right) (1 - |y|^p) e^{4u} dy + c.
\]
All other solutions to problem (4)-(5) are called non-normal.
For what concerns normal solutions to (4)-(5), recently, A. Hyder and L. Martinazzi [15] proved some existence and non-existence results. In particular, among other things, they showed that problem (4)-(5) admits normal solutions if and only if \( p \in (0, 4) \) and \( \Lambda_{s,p} \leq \Lambda < \Lambda_{sph} \) where \( \Lambda_{s,p} := \left( 1 + \frac{p}{4} \right) 8\pi^2 \) and \( \Lambda_{sph} := 16\pi^2 \). Moreover, every normal solution has the following asymptotic behavior

\[
\lim_{|x| \to \infty} u(x) = -\frac{\Lambda}{8\pi^2} \log |x| + C + O(|x|^{-\alpha}), \quad \text{as } |x| \to \infty
\]

for every \( \alpha \in [0, 1] \) such that \( \alpha < \frac{\Lambda - \Lambda_{s,p}}{2\pi^2} \).

Motivated by the above results, we study the properties of more general solutions (not necessarily normal) to problem (4)-(5). Although for \( p \geq 4 \), problem (4)-(5) admits no normal solutions, we prove that non-normal solutions do exist. To this end, we consider all polynomials \( P \) of degree 2 such that

\[
\lim_{|x| \to \infty} P(x) = -\infty
\]

and define the set

\[
\mathcal{P}_2 := \{ P \text{ polynomial in } \mathbb{R}^4 \mid \deg P = 2 \quad \text{and} \quad \lim_{|x| \to \infty} P(x) = -\infty \}.
\]

By means of a result of A. Chang and W. Chen (see Theorem 2.1 in [4] where, under suitable assumptions on the curvature \( K \), using a variational approach, they prove existence of at least one solution to equation \((-\Delta)^{n/2} u = K(x)e^{nu} \) in \( \mathbb{R}^n \) we prove the following theorem, which extends to the non-normal case existence results in [15].

**Theorem 1.1.** Let \( p > 0 \) be fixed. Then for any \( P \in \mathcal{P}_2 \) and for every \( \Lambda \in (0, \Lambda_{sph}) \), there exists at least one solution to problem (4)-(5) of the form

\[
u = w + P
\]

where

\[
w(x) = -\frac{\Lambda}{8\pi^2} \log |x| + C + o(1), \quad \text{as } |x| \to +\infty.
\]

We shall prove the following classification result.

**Theorem 1.2.** Let \( u \) be a solution to (4)-(5) and \( v \) as defined in (6). Then there exists an upper-bounded polynomial \( P \) of degree at most 2 such that

\[
u = v + P.
\]

Moreover, \( u \) has the following asymptotic behavior

\[
u(x) = -\frac{\Lambda}{8\pi^2} \log |x| + P(x) + o(\log |x|), \quad \text{as } |x| \to \infty.
\] (7)

Note that \( P \) being upper bounded means that \( P \) has even degree, and since \( P \) has degree at most 2, this implies that \( P \) could only have degree 2 or be constant. For this reason, we can rephrase Theorem 1.2 by saying that all solutions to problem (4)-(5) have the form \( v + P \), where \( v \) behaves logarithmically at infinity and \( P \) is an upper bounded polynomial of degree 2 if the solution is non-normal, whereas \( P \) is constant if the solution is normal.

**Remark 1.3.** We can observe that the function \( w \) of Theorem 1.1 and the function \( v \) of Theorem 1.2 differ by a constant.

In order to prove Theorem 1.2, we obtain some suitable upper and lower estimates for the function \( v \) (see Lemma 3.1 and Proposition 4.4 below). First, we prove the lower estimate (11). To this end, we need to overcome some difficulties compared to previous works, due to the fact that the \( Q \)-curvature is not constant and it changes sign (compare
e.g., to the proof of Lemma 2.4 in [16]). The lower estimate (11) will be crucial to obtain a Liouville-type theorem (see Theorem 3.4), also in this case the proof is quite delicate because the estimate for $-v$ contains the singular integral

$$A(x) := \frac{1}{8\pi^2} \int_{B_1(x)} \log(|x-y|)(1-|y|^p)e^{4u}dy.$$ 

Then we show that the polynomial $P$ is upper-bounded (see Proposition 4.2). To do this, we take advantage of a useful result of Gorin and the fact that, since $A(x) \in L^1$, there must exist a set $X_0$ of finite measure such that $-A(x) \geq -C$ in $\mathbb{R}^4 \setminus X_0$. For what concern the proof of Proposition 4.4, we take some ideas from the proof of Lemma 13 in [17] and the one of Lemma 2.4 in [16], but our case is more challenging. This is due to the fact that we lack of a good estimate for $\int_{\mathbb{R}^4 \setminus B_R} v^+ dx$ which indeed is necessary in [17], and we do not know a priori the sign of $\Delta u$, which is essential to apply an Harnack inequality as in [16].

Finally, we characterize solutions in terms of the behavior at infinity of the scalar curvature of the associated conformal metric.

**Theorem 1.4.** Let $u$ be a solution to problem (4)-(5) and set $g_u = e^{2u}|dx|^2$. If $u$ is a normal solution, then

$$\lim_{|x| \to +\infty} R_{g_u} = 0.$$ 

If $u$ is a non-normal solution, we get

$$\liminf_{|x| \to +\infty} R_{g_u} = -\infty.$$ 

Our result proves that normal solutions have a quite different geometry compared to that of non-normal solutions. Moreover, it also points out that, for this problem, the $Q$-curvature and the scalar curvature are independent of each other.

**Open problem** Can we find non-normal solutions to (4)-(5) with arbitrarily large but finite total $Q$-curvature $\Lambda$? In the constant $Q$-curvature case, C.-S. Lin [16] (see also [4]) proved that all solutions to

$$\begin{cases}
\Delta^2 u = 6e^{4u} & \text{in } \mathbb{R}^4 \\
e^{4u} \in L^1(\mathbb{R}^4)
\end{cases}$$

satisfy $V \in (0, \text{vol}(S^4)]$, but his approach is no longer applicable in our case, in fact when we have a polynomial $P$ we need $Q(x) = (1-|x|^p)e^{4P}$ to be radially decreasing, which is not true in general. Since $x \cdot \nabla Q(x)$ does not have a fixed sign, using methods from [18] or [13], it would be interesting to see whether there exist solutions to (4)-(5) with total $Q$-curvature $\Lambda \geq \Lambda_{\text{sph}}$.

2. **Existence of solutions**

In this section, we take advantage of a result of A. Chang and W. Chen (see Theorem 2.1 in [4]). Using a variational approach in a Sobolev space defined on a conical singular manifold, they prove the existence of at least one solution to equation

$$(-\Delta) \tilde{w} = K(x)e^{nw} \quad \text{in } \mathbb{R}^n,$$

in even dimensions, assuming that $K$ is positive somewhere and for some $s > 0$, $K(x) = O\left(\frac{1}{|x|^s}\right)$ near infinity.
Proof of Theorem 1.1. Let us fix $P \in \mathcal{P}_2$ and $\Lambda \in (0, \Lambda_{sph})$. By Theorem 2.1 in [4] and its proof (refer also to Section 7 in [15]) setting $K(x) := (1 - |x|^p)e^{4P}$ and $\mu := 1 - \frac{\Lambda}{\Lambda_{sph}} \in (0, 1)$, one can find at least one solution $w$ to equation
\[
\Delta^2 w = K(x) e^{4w}, \quad \text{in } \mathbb{R}^4
\] such that
\[
\int_{\mathbb{R}^4} K(x) e^{4w} dx = (1 - \mu)\Lambda_{sph} = \Lambda.
\]
It follows immediately that $u := w + P$ is the desired solution to problem (4)-(5). More precisely, $w$ is of the form
\[
w = \tilde{w} \circ \Pi^{-1} + (1 - \mu)w_0
\]
where $w_0 = \log \left( \frac{2}{1 + |x|^2} \right)$, $\Pi : S^4 \to \mathbb{R}^4$ denotes the stereographic projection, $\tilde{w} = \bar{w} + C$, where $\tilde{w}$ minimize a certain functional which takes values in a Sobolev space defined on a conical singular manifold, and $C$ is a suitable constant such that
\[
\int K(x) e^{4\tilde{w}} dV = (1 - \mu)\Lambda_{sph}
\]
where the corresponding volume element is $dV = e^{4(1-\mu)w_0} dx$. We have
\[
P^4 g_0 \tilde{w} + 6(1 - \mu) = (K \circ \Pi)e^{-4\mu(w_0 \circ \Pi)}e^{4\tilde{w}},
\]
from this identity, with the same argument as in Section 7 of [15], we obtain $\tilde{w} \in C^{3,\alpha}(S^4)$ for $\alpha \in (0, 1)$. In particular, $\tilde{w}$ in continuous at the South pole $S = (0,0,0,0,-1)$, which implies
\[
w(x) = (1 - \mu)w_0(x) + \tilde{w}(S) + o(1) = -\frac{\Lambda}{8\pi^2} \log |x| + C + o(1), \quad \text{as } |x| \to \infty.
\]

Remark 2.1. If $P \in \mathcal{P}_2$ is a radially symmetric polynomial, there exists at least one non-normal radial solution to problem (4)-(5) of the form $u = w + P$. This follows from the fact that we can minimize the previous functional over radial functions and obtain $\tilde{w}$ radially symmetric.

3. ASYMPTOTIC BEHAVIOR

In all this section, let $u$ be a solution to problem (4)-(5), we define
\[
v(x) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log \left( \frac{|y|}{|x-y|} \right) (1 - |y|^p)e^{4u} dy.
\] Obviously, we have $\Delta^2 v(x) = (1 - |x|^p)e^{4u}$ in $\mathbb{R}^4$.

Lemma 3.1. For $|x| \geq 4$, there exists a constant $C$ such that
\[
v(x) \leq -\frac{\Lambda}{8\pi^2} \log |x| + C.
\]
Proof. For \(|x| \geq 4\), we decompose \(\mathbb{R}^4 = B_1(0) \cup A_1 \cup A_2 \cup A_3\) where \(A_1 = \{y \mid |y-x| \leq |x|/2\}\), \(A_2 = \{y \mid 1 \leq |y| \leq 2\}\) and \(A_3 = \mathbb{R}^4 \setminus (A_1 \cup A_2 \cup B_1)\). For \(y \in B_1\) we have

\[
\log \left( \frac{|y|}{|x-y|} \right) \leq -\log |x| + C
\]

hence

\[
\int_{B_1} \log \left( \frac{|y|}{|x-y|} \right) (1-|y|^p)e^{4u}dy \leq (-\log |x| + C) \int_{B_1} (1-|y|^p)e^{4u}dy.
\]

For \(y \in A_1\) we have \(\log \left( \frac{|y|}{|x-y|} \right) \geq 0\) hence the integral over \(A_1\) is negative. For what concern \(A_2\) we have

\[
\int_{A_2} \log \left( \frac{|y|}{|x-y|} \right) (1-|y|^p)e^{4u}dy = \int_{A_2} \log(|y|) (1-|y|^p)e^{4u}dy - \int_{A_2} \log(|x-y|)(1-|y|^p)e^{4u}dy
\]

using the fact that the first integral is non positive, we get

\[
\leq - \int_{A_2} \log(|x-y|)(1-|y|^p)e^{4u}dy \leq -\log |x| \int_{A_2} (1-|y|^p)e^{4u}dy + C
\]

where in the last inequality we used that for \(y \in A_2\) \(\log |x-y| \leq \log |x| + C\). For \(y \in A_3\) since \(|x-y| \leq |x| + |y| \leq |x||y|\) we have

\[
\log \left( \frac{|y|}{|x-y|} \right) \geq -\log |x|
\]

and since in this case \(1-|y|^p < 0\) we get

\[
\log \left( \frac{|y|}{|x-y|} \right) (1-|y|^p) \leq -\log(|x|) (1-|y|^p)
\]

and hence

\[
\int_{A_3} \log \left( \frac{|y|}{|x-y|} \right) (1-|y|^p)e^{4u}dy \leq -\log(|x|) \int_{A_3} (1-|y|^p)e^{4u}dy.
\]

Summing up, we finally obtain

\[
v(x) \leq -\frac{1}{8\pi^2} \log(|x|) \int_{A_3^c} (1-|y|^p)e^{4u}dy + C,
\]

since \(\int_{A_3^c} (1-|y|^p)e^{4u}dy \geq \Lambda\) we have

\[
v(x) \leq -\frac{\Lambda}{8\pi^2} \log |x| + C.
\]

Lemma 3.2. For any \(\varepsilon > 0\), there exists \(R = R(\varepsilon) > 0\) such that for \(|x| \geq R\)

\[
v(x) \geq -\frac{1}{8\pi^2}(\Lambda + 5\varepsilon) \log |x| - \frac{1}{8\pi^2} \int_{B_1(x)} \log(|x-y|)(1-|y|^p)e^{4u}dy.
\]

Proof. We can choose \(R_0 = R_0(\varepsilon) > 1\) such that

\[
\int_{B_{R_0}} (1-|y|^p)e^{4u}dy \leq \Lambda + \varepsilon.
\]

Let us take \(R > 2R_0\) and assume that \(|x| \geq R\), we can decompose

\[
\mathbb{R}^4 = B_{R_0}(0) \cup A_1 \cup A_2
\]
where
\[ A_1 := \{ y \in \mathbb{R}^4 : |y - x| \leq |x|/2 \} \]
\[ A_2 := \{ y \in \mathbb{R}^4 : |y - x| > |x|/2, \ |y| \geq R_0 \}. \]

For $|x| \geq R$ and $|y| \leq R_0$ we have
\[
\log \left( \frac{|y|}{|x - y|} \right) \leq -\log |x| + C < 0
\]

hence, we get
\[
\int_{B_{R_0} \setminus B_1} \log \left( \frac{|y|}{|x - y|} \right) (1 - |y|^p)e^{4u} dy
\]
\[
\geq (-\log |x| + C) \int_{B_{R_0} \setminus B_1} (1 - |y|^p)e^{4u} dy
\]
\[
\geq (-\log |x| + C) \int_{B_{R_0}} (1 - |y|^p)e^{4u} dy
\]
\[
\geq (-\log |x| + C)(\Lambda + \varepsilon) \geq -(\Lambda + \varepsilon) \log |x|
\]

where we used the fact that $\int_{B_{R_0} \setminus B_1} (1 - |y|^p)e^{4u} dy \leq \int_{B_{R_0}} (1 - |y|^p)e^{4u} dy$. Concerning the integral over $B_1$ we have
\[
\int_{B_1} \log \left( \frac{|x - y|}{|y|} \right) (1 - |y|^p)e^{4u} dy \leq \int_{B_1} \log \left( \frac{|y|}{|y|} \right) e^{4u} dy
\]
\[
= \int_{B_1} \log \left( \frac{1}{|y|} \right) e^{4u} dy + \int_{B_1} \log(|x - y|)e^{4u} dy \leq C
\]

using Holder’s inequality. Therefore, we obtain
\[
\int_{B_{R_0}} \log \left( \frac{|y|}{|x - y|} \right) (1 - |y|^p)e^{4u} dy \geq -(\Lambda + \varepsilon) \log |x| - C. \quad (12)
\]

We observe that $\log |x - y| \geq 0$ for $y \notin B_1(x)$, $\log |y| \leq \log (2|x|)$ for $y \in A_1$, $\int_{A_1} (1 - |y|^p)e^{4u} dy \geq -\varepsilon$ and $\log (2|x|) \leq 2\log |x|$ for $|x| \geq R$, hence we get
\[
\int_{A_1} \log \left( \frac{|y|}{|x - y|} \right) (1 - |y|^p)e^{4u} dy = \int_{A_1} \log(|y|) (1 - |y|^p)e^{4u} dy - \int_{A_1} \log(|x - y|) (1 - |y|^p)e^{4u} dy
\]
\[
\geq \log (2|x|) \int_{A_1} (1 - |y|^p)e^{4u} dy - \int_{B_1(x)} \log(|x - y|) (1 - |y|^p)e^{4u} dy
\]
\[
\geq 2\log (|x|) \int_{A_1} (1 - |y|^p)e^{4u} dy - \int_{B_1(x)} \log(|x - y|) (1 - |y|^p)e^{4u} dy
\]
\[
\geq -2\varepsilon \log |x| - \int_{B_1(x)} \log(|x - y|) (1 - |y|^p)e^{4u} dy. \quad (13)
\]

For $y \in A_2$, in the case $|y| \leq 2|x|$ we have $\frac{|y|}{|x - y|} \leq 4$, while in the case $|y| \geq 2|x|$ we get $\frac{|y|}{|x - y|} \leq 2$, so when $y \in A_2$ we have the estimate
\[
\log \left( \frac{|y|}{|x - y|} \right) \leq \log 4,
\]
hence using that \( \int_{A_2} (1 - |y|^p)e^{4u} dy \geq -\varepsilon \)

\[
\int_{A_2} \log \left( \frac{|y|}{|x - y|} \right) (1 - |y|^p)e^{4u} dy \geq \log(4) \int_{A_2} (1 - |y|^p)e^{4u} dy \geq -\varepsilon \log 4. \tag{14}
\]

Putting together (12), (13) and (14), possibly taking \( R \) larger we get

\[
v(x) \geq -\frac{1}{8\pi^2}(\Lambda + 5\varepsilon) \log |x| + \frac{1}{8\pi^2} \int_{B_1(x)} \log \left( \frac{1}{|x - y|} \right) (1 - |y|^p)e^{4u} dy.
\]

\[
\square
\]

From (11) changing signs, it follows that for any \( \varepsilon > 0 \) there is \( R > 0 \) such that for \( |x| \geq R \)

\[-v(x) \leq \frac{1}{8\pi^2}(\Lambda + 5\varepsilon) \log |x| + \frac{1}{8\pi^2} \int_{B_1(x)} \log(|x - y|)(1 - |y|^p)e^{4u} dy. \tag{15}\]

3.1. A Liouville type-theorem. To prove a Liouville-type theorem (see Theorem 3.4 below) we will need the following useful result.

**Lemma 3.3.** Let \( u \) be a measurable function such that \( (1 - |y|^p)e^{4u} \in L^1(\mathbb{R}^4) \). Then for any \( x \in \mathbb{R}^4 \)

\[
\int_{B_r(x)} u^+ dy \to 0, \quad \text{as } r \to +\infty.
\]

**Proof.** Let \( x \in \mathbb{R}^4 \) be fixed, using the fact that \( 4u^+ \leq e^{4u} \) we get

\[
4 \int_{B_r(x)} u^+ dy \leq \int_{B_r(x)} e^{4u} dy = \frac{C}{r^4} \int_{B_r(x)} \frac{1}{1 - |y|^p} (1 - |y|^p)e^{4u} dy. \tag{16}
\]

Observing that for \( y \in B_r(x) \) we have \( |y| \leq r + |x| \) and \( |y| \geq r - |x| \), we obtain the following inequalities

\[
\frac{1}{1 - |y|^p} \leq \frac{1}{1 - (|x| + r)^p},
\]

and

\[
\frac{1}{1 - |y|^p} \geq \frac{1}{1 - (|x| - r)^p},
\]

by the means of them we get

\[
4 \int_{B_r(x)} u^+ dy \leq \frac{C}{r^4} \left[ \frac{1}{1 - (|x| + r)^p} \int_{B_r(x) \cap B_1} (1 - |y|^p)e^{4u} dy + \frac{1}{1 - (|x| - r)^p} \int_{B_r(x) \cap B_1^c} (1 - |y|^p)e^{4u} dy \right]
\]

since \( \int_{B_r(x)} (1 - |y|^p)e^{4u} dy < \infty \), we can estimate (16) with \( O(r^{-p-4}) \) as \( r \to \infty \). The claim follows as \( r \to +\infty \) since by assumption \( p > 0 \). \( \square \)

We are now in position to prove the following Liouville-type theorem, which will be crucial to prove that \( u - v \) is a polynomial of degree at most 2.

**Theorem 3.4.** Consider \( h : \mathbb{R}^4 \to \mathbb{R} \) such that

\[
\Delta^2 h = 0 \quad \text{and} \quad h \leq u - v.
\]

Assume that \( (1 - |y|^p)e^{4u} \in L^1(\mathbb{R}^4) \), \( v \in L^1_{loc}(\mathbb{R}^4) \) and further that (15) holds. Then \( h \) is a polynomial of degree at most 2.
Proof. We take some ideas from the proof of Theorem 6 in [17], but this proof is more delicate since our estimate for \(-v\) contains the singular integral
\[
A(x) := \frac{1}{8\pi^2} \int_{B_1(x)} \log(|x-y|)(1 - |y|^p)e^{4u} dy.
\] (17)
By elliptic estimates for biharmonic functions (see Proposition 4 in [17]) for any \(x \in \mathbb{R}^4\) we have
\[
|D^3 h(x)| \leq \frac{C}{r^3} \int_{B_r(x)} |h(y)| dy = -\frac{C}{r^3} \int_{B_r(x)} h(y) dy + \frac{2C}{r^3} \int_{B_r(x)} h^+ dy.
\] (18)
From Pizzetti’s formula (refer e.g. to [17]) we have
\[
\int_{B_r(x)} h(y) dy = O(r^2), \quad \text{as } r \to \infty.
\] (19)
In order to estimate the term \(\frac{2C}{r^3} \int_{B_r(x)} h^+ dy\), we observe that
\[
\int_{B_r(x)} h^+ dy \leq \int_{B_r(x)} u^+ dy + C \int_{B_r(x)} (-v)^+ dy,
\]
thanks to Lemma 3.3 the term \(\int_{B_R(x)} u^+ dy \to 0\). Using Tonelli’s theorem, we can prove that \(A \in L^1(\mathbb{R}^4)\) as follows
\[
\int_{\mathbb{R}^4 \setminus B_R} |A(x)| dx = \int_{\mathbb{R}^4 \setminus B_R} \left| \frac{1}{8\pi^2} \int_{B_1(x)} \log(|x-y|)(1 - |y|^p)e^{4u} dy \right| dx
\]
\[
\leq \int_{\mathbb{R}^4} \frac{1}{8\pi^2} \int_{B_1(x)} \log|x-y|\frac{1}{1 - |y|^p}e^{4u} dx dy dx
\]
\[
= C \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \chi_{|x-y|<1} \log|x-y|\frac{1}{1 - |y|^p}e^{4u} dy dx
\]
\[
= C \int_{\mathbb{R}^4} |1 - |y|^p|e^{4u} \int_{B_1(y)} \log\left(\frac{1}{|x-y|}\right) dx dy
\]
\[
= C \int_{\mathbb{R}^4} |1 - |y|^p|e^{4u} dy < \infty
\]
since \(A\) is continuous, we have \(\int_{B_R} |A(x)| dx \leq C (\Lambda + 5\varepsilon) \log|x| + A(x) \geq 0\), we get
\[
(-v)^+ \leq \frac{1}{8\pi^2} (\Lambda + 5\varepsilon) \log|x| + A(x)
\]
for \(x \in \mathbb{R}^4 \setminus B_R\). Taking into account that \(A(x) \in L^1\) we obtain
\[
\int_{B_r(x)} (-v)^+ dy \leq C \int_{B_r(x)} \log(|y| + 1) dy + \int_{B_r(x)} A(y) dy
\]
\[
\leq C \log r + \frac{C}{r^4}
\]
(if \(y \in B_R(0)\) the previous estimate for \((-v)^+\) does not hold, we can overcome this problem since \(v \in L^1_{loc}(\mathbb{R}^4)\)). Hence,
\[
\int_{B_r(x)} h^+ dy \leq \int_{B_r(x)} u^+ dy + \frac{C}{r^4} + C \log r.
\] (20)
From (19) and (20), we get that all terms in (18) go to 0 as \(r \to \infty\), hence we obtain \(D^3 h \equiv 0\). □
4. Proof of Theorem 1.2

In order to prove that all solutions to problem (4)-(5) have the form $v + P$, where $v$ behaves logarithmically at infinity and $P$ is an upper-bounded polynomial of degree at most 2, we proceed by steps.

**Theorem 4.1.** Let $u$ be a solution to problem (4)-(5) such that $(1 - |y|^p)e^{4u} \in L^1(\mathbb{R}^4)$ and $v$ as in (9). Then $u = v + P$ where $P$ is a polynomial of degree at most 2.

**Proof.** Consider $P = u - v$, we have $\Delta^2 P = 0$. From (15) using Theorem 3.4 we can conclude. □

Let us prove that the polynomial $P$ is upper-bounded.

**Proposition 4.2.** Let $P$ be the polynomial of Theorem 4.1. Then

$$
\sup_{x \in \mathbb{R}^4} P(x) < +\infty.
$$

**Proof.** Following [17] we define

$$
f(r) := \sup_{\partial B_r} P
$$

and assume by contradiction that $\sup_{\mathbb{R}^4} P = +\infty$. From Theorem 3.1 in [11] it must exist $s > 0$ such that

$$
\lim_{|x| \to +\infty} \frac{f(r)}{r^s} = +\infty.
$$

Moreover, $P$ is a polynomial of degree at most 2, hence $|\nabla P(x)| \leq c|x|$ for $|x|$ large. Since $A(x)$ (see definition (17)) belongs to $L^1(\mathbb{R}^4)$, there must exist a set $X_0$ of finite measure such that

$$
-A(x) \geq -C, \quad \text{in } \mathbb{R}^4 \setminus X_0.
$$

Using this and Lemma 3.1, we get that there is $R > 2$ such that for every $r \geq R$ we can find $x_r$ with $|x_r| = r$ such that

$$
u(y) = v(y) + P(y) \geq r^s, \quad \text{for } |y - x_r| \leq \frac{1}{r} \quad \text{and } y \in \mathbb{R}^4 \setminus X_0.
$$

Note that we can chose the set $X_0$ with arbitrary small measure, hence we do not have problems. We consider

$$
-\int_{B_1^c} (1 - |y|^p)e^{4u} dy = \int_{B_1^c} (|y|^p - 1)e^{4u} dy \geq \int_{B_R^c \setminus X_0} (|y|^p - 1)e^{4u} dy
$$

$$
\geq \int_{B_R^c \setminus X_0} e^{4u} dy \geq C \int_{+\infty}^R \int_{(\partial B_r \cap B_{1/r}(x_r)) \setminus X_0} e^{4u} d\sigma dr
$$

$$
\geq C \int_{R}^{+\infty} \int_{(\partial B_r \cap B_{1/r}(x_r)) \setminus X_0} e^{4r^s} d\sigma dr \geq C \int_{R}^{+\infty} e^{4r^s} \frac{r^3}{r^3} dr = +\infty.
$$

which is absurd since $(1 - |y|^p)e^{4u} \in L^1(\mathbb{R}^4)$. □

The fact that $P$ is upper-bounded implies that $P$ has even degree, hence or $P$ has degree 2 or is constant, moreover, after a change of coordinate by an orthogonal transformation, we can write $P(x) = \sum_{i=1}^4 (a_ix_i^2 + b_ix_i) + c_0$ where $a_i \leq 0$. 
Lemma 4.3. Suppose $u$ is a solution to (4)-(5). Then $\Delta u(x)$ can be represented by

$$\Delta u(x) = -\frac{1}{4\pi^2} \int_{\mathbb{R}^4} \frac{1}{|x-y|^2} (1-|y|^p)e^{4u}dy - C_1,$$

where $C_1$ is a nonnegative constant.

Proof. Since, by Theorem 4.1, $Harnack$ inequality as in \[ \Delta \]
we show that \( \tau \) where

$$0 \varepsilon > R > \Delta$$
we get that \( (1 \) and by assumption

$$\text{singular integral in (11) is over } B \text{ one of Lemma 2.4 in [17] later. In addition, we lack of a good estimate for } \Delta u, \text{ which is fundamental to apply an Harnack inequality as in [16]. In what follows, } C \text{ denotes a generic constant which may change from line to line and also within the same line. From Lemma 3.2 for any } \varepsilon > 0 \text{ there exists } R > 0 \text{ such that for } |x| \geq R \text{ satisfies}

$$v(x) \geq -\frac{1}{8\pi^2}(\Lambda + 6\varepsilon) \log |x|.
$$

Moreover, we have

$$\lim_{|x| \to +\infty} \Delta v(x) = 0.$$

Proposition 4.4. Let $u$ be a solution to (4)-(5) and $v$ as defined in (9). Then given any $\varepsilon > 0$, there exists $R = R(\varepsilon)$ such that for $|x| \geq R$, $v(x)$ satisfies

$$v(x) \geq -\frac{1}{8\pi^2}(\Lambda + 6\varepsilon) \log |x|.$$

Proof. First we prove (22). We take advantage of the proof of Lemma 13 in [17] and the one of Lemma 2.4 in [16], but our case is more challenging. This is due to the fact that the singular integral in (11) is over $B_1(x)$ and we need a radius $\tau \in (0,1)$ which can be fixed later. In addition, we lack of a good estimate for $\int_{\mathbb{R}^4 \setminus B_R} v^+ dx$, which indeed is necessary in [17], and we do not know a priori the sign of $\Delta u$, which is fundamental to apply an Harnack inequality as in [16]. In what follows, $C$ denotes a generic constant which may change from line to line and also within the same line. From Lemma 3.2 for any $\varepsilon > 0$ there exists $R > 0$ such that for $|x| \geq R$

$$v(x) \geq -\frac{1}{8\pi^2}(\Lambda + 5\varepsilon) \log |x| - \frac{1}{8\pi^2} \int_{B_1(x)} \log(|x-y|)(1-|y|^p)e^{4u}dy.$$

Notice that a priori the second term on the right-hand side may be very little, so we want to estimate its absolute value. We observe that

$$\left| \int_{B_1(x)} \log \left(\frac{1}{|x-y|}\right) (1-|y|^p)e^{4u}dy \right| \leq \int_{B_1(x)} \log \left(\frac{1}{|x-y|}\right) |1-|y|^p| e^{4u}dy$$

$$= \int_{B_1(x) \setminus B_{\tau}(x)} \log \left(\frac{1}{|x-y|}\right) |1-|y|^p| e^{4u}dy + \int_{B_{\tau}(x)} \log \left(\frac{1}{|x-y|}\right) |1-|y|^p| e^{4u}dy$$

where $\tau \in (0,1)$ will be fixed later. Since $\log \left(\frac{1}{|x-y|}\right) \in (0, -\log \tau)$ for $y \in B_1(x) \setminus B_{\tau}(x)$ and by assumption $(1-|y|^p)e^{4u} \in L^1(\mathbb{R}^4)$, we have

$$\int_{B_1(x) \setminus B_{\tau}(x)} \log \left(\frac{1}{|x-y|}\right) |1-|y|^p| e^{4u}dy < C.$$
where $f$ is a constant which depends on $\omega$. Let $h := \Delta h = 0$ on $\partial B_4(x)$. Fix $0 < \varepsilon_0 < 1$, we can choose $R_0 > 6$ sufficiently large such that

$$\int_{B_4(x)} |1 - |y|^p| e^{4u} dy \leq \varepsilon_0$$

for $|x| \geq R_0$. Let $h$ be the solution of

$$\begin{cases}
\Delta^2 h = f & \text{on } B_4(x) \\
h = \Delta h = 0 & \text{on } \partial B_4(x)
\end{cases}$$

where $f(y) := (1 - |y|^p) e^{4u}$, then by Theorem 7 in [17] (see also Lemma 2.3 [16]) for any $k \in \left(0, \frac{8\pi^2}{\|f\|_{L^1(B_4(x))}} \right)$, we have $e^{4k|h|} \in L^1(B_4(x))$, namely

$$\int_{B_4(x)} e^{4k|h|} dy \leq C$$

where $C$ is a constant which depends on $k$ but is independent from $x$. For $y \in B_4(x)$ define $q(y) := u(y) - h(y)$, then $q$ satisfies

$$\begin{cases}
\Delta^2 q = 0 & \text{on } B_4(x) \\
\Delta q = \Delta u & \text{on } \partial B_4(x)
\end{cases}$$

Integrating equation $\Delta^2 u = (1 - |y|^p) e^{4u}$ on $B_\rho(x)$ we get

$$\int_{\partial B_\rho(x)} \frac{\partial}{\partial r} (\Delta u) d\sigma = \int_{B_\rho(x)} (1 - |y|^p) e^{4u} dy.$$ 

Dividing by $\omega_4 \rho^3$ and integrating with respect to $\rho$ from 0 to $R$ (we assume $R < 5$), using Fubini’s theorem, we obtain

$$\int_0^R \frac{1}{\omega_4 \rho^3} \int_{\partial B_\rho(x)} \frac{\partial}{\partial r} (\Delta u) d\sigma d\rho = \int_{\partial B_R(x)} \Delta u d\sigma - \Delta u(x)$$

and similarly

$$\int_0^R \frac{1}{\omega_4 \rho^3} \int_{B_\rho(x)} (1 - |y|^p) e^{4u} dy d\rho = \frac{1}{4\pi^2} \int_{B_R(x)} (1 - |y|^p) e^{4u} \left[ \frac{1}{|x-y|^2} - \frac{1}{R^2} \right] dy.$$ 

Hence

$$\int_{\partial B_R(x)} \Delta u d\sigma = \Delta u(x) + \frac{1}{4\pi^2} \int_{B_R(x)} (1 - |y|^p) e^{4u} \left[ \frac{1}{|x-y|^2} - \frac{1}{R^2} \right] dy.$$ 

by means of identity (21) we get

$$-\int_{\partial B_R(x)} \Delta u d\sigma = \frac{1}{4\pi^2} \int_{|x-y| \geq R} \frac{1 - |y|^p}{|x-y|^2} e^{4u} dy + \frac{1}{4\pi^2 R^2} \int_{B_R(x)} (1 - |y|^p) e^{4u} dy + C.$$
If we take \( R = 4 \) we have
\[
- \int_{\partial B_4(x)} \Delta u \, d\sigma \leq C.
\]
Let \( G \) be the Green’s function for the operator \(-\Delta\) on \( B_4(x)\), namely
\[
-\Delta G = \delta_x, \quad G = 0 \text{ on } \partial B_4(x),
\]
we have
\[
-\Delta q(x) = - \int_{\partial B_4(x)} \frac{\partial G}{\partial n} \Delta u \, dS = - \int_{\partial B_4(x)} c_0 \Delta u \, dS \leq C
\]
since \( c_0 \) is a positive constant there exist some \( \tau \in (0, 2) \) such that if \( \xi \in B_{2\tau}(x) \) and \( G_\xi \) is the Green’s function defined by
\[
-\Delta G_\xi = \delta_\xi, \quad G_\xi = 0 \text{ on } \partial B_4(x),
\]
then
\[
0 \leq \frac{\partial G_\xi(\eta)}{\partial r} \leq C, \quad \text{for } \eta \in \partial B_4(x), \quad r := \frac{\eta - x}{4}
\]
and as before we get
\[
- \Delta q(y) \leq C \quad \text{on } B_{2\tau}(x).
\]
Define \( \tilde{q}(y) := -\Delta q(y) \), obviously \( q \) satisfies
\[
\begin{cases}
\Delta q(y) = -\tilde{q}(y) & \text{on } B_4(x) \\
q = u & \text{on } \partial B_4(x)
\end{cases}
\]
hence by elliptic estimates for any \( k > 1 \) and \( \sigma > 2 \)
\[
\sup_{B_r(x)} q \leq c(p, \sigma) (\|q^+\|_{L^k(B_{2\tau}(x))} + \|\tilde{q}\|_{L^\sigma(B_{2\tau}(x))}).
\]
From (26) we get \( \|\tilde{q}\|_{L^\sigma(B_{2\tau}(x))} \leq C \). Since \( q = u - h \), it follows that \( q^+(y) \leq u^+(y) + |h(y)| \) for \( y \in B_4(x) \) and hence
\[
\int_{B_{2\tau}(x)} (q^+)^k \leq C \int_{B_4(x)} e^{2q^+} \leq C \left( \int_{B_{2\tau}(x)} e^{4u^+} \right)^{1/2} \left( \int_{B_{2\tau}(x)} e^{4|h|} \right)^{1/2}.
\]
Note that
\[
e^{4u^+} \leq 1 + e^{4u} \leq 1 + |1 - |y||^p \ e^{4u}, \quad \text{for } |y| \geq 2^{1/p}.
\]
(27)
Since \( B_{2\tau}(x) \subset B_{2^{1/p}}^c \) (eventually choosing \( R_0 \) greater) from (24) we get
\[
\int_{B_{2\tau}(x)} e^{4u^+} \, dy \leq C
\]
finally, together with (25), we obtain
\[
\|q^+\|_{L^k(B_{2\tau}(x))} \leq C.
\]
In this way we have shown that
\[
\sup_{B_r(x)} q \leq C,
\]
hence for \( y \in B_r(x) \) we have
\[
u(y) = q(y) + h(y) \leq C + |h(y)|,
\]
from which we get
\[
\int_{B_r(x)} e^{16u} \, dy \leq C \int_{B_r(x)} e^{16|h|} \, dy < C.
\]
Scalar curvature is given by the following formula (refer to Theorem 1.2 in [1] to the metric \( g \)) and (22) follows at once.

Now we prove (23). Differentiating we have

\[
\Delta v(x) = -\frac{1}{4\pi^2} \int_{\mathbb{R}^4} \frac{1}{|x-y|^2} (1 - |y|^p) e^{4u} dy.
\]

For any \( \sigma > 0 \), by dominated convergence we get

\[
\int_{\mathbb{R}^4 \setminus B_\sigma(x)} \frac{(1 - |y|^p) e^{4u}}{|x-y|^2} dy \to 0, \quad \text{as} \quad |x| \to +\infty.
\]

By Holder’s inequality we get

\[
\int_{B_\sigma(x)} \frac{(1 - |y|^p) e^{4u}}{|x-y|^2} dy \leq \left( \int_{B_\sigma(x)} \frac{(1 - |y|^p)^{k}}{|x-y|^{2k}} dy \right)^{1/k} \left( \int_{B_\sigma(x)} e^{4k'u} dy \right)^{1/k'}
\]

if \( \sigma \) is small enough, by (28) we can conclude. \( \square \)

**Proof of Theorem 1.2.** It follows from Lemma 3.1, Theorem 4.1, Proposition 4.2 and Proposition 4.4. \( \square \)

**Corollary 4.5.** Any solution \( u \) to (4)-(5) is bounded from above.

**Proof.** The solution \( u \) is continuous and \( u = v + P \). Moreover from (10) we have that \( v(x) \leq C \) on \( B_1^c \) and from Proposition 4.2 we have \( \sup_{\mathbb{R}^4} P(x) < +\infty \). \( \square \)

## 5. SCALAR CURVATURE

In this section we prove Theorem 1.4. First, recall that if \( u \) is a solution to (4)-(5), then the metric \( g_u = e^{2u}g_{\mathbb{R}^4} \) is conformal to the flat metric on \( \mathbb{R}^4 \) and has \( Q \)-curvature equal to \( 1 - |x|^p \). If we consider a metric \( g_u \) conformal to a flat metric, the conformal change of scalar curvature is given by the following formula

\[
R_{g_u} = -6e^{-2u} (\Delta u + |\nabla u|^2).
\]

In the case when \( u \) is a normal solution to problem (4)-(5), A. Hyder and L. Martinazzi (refer to Theorem 1.2 in [15]) proved that for \( \ell = 1, 2, 3 \)

\[
|\nabla^{\ell} u(x)| = O(|x|^{-\ell}), \quad \text{as} \quad |x| \to +\infty.
\]

Moreover, from the fact that \( |\Delta u(x)| \to 0 \) as \( |x| \to +\infty \) and \( \Delta^2 u \leq 0 \) for \( |x| \geq 1 \), we get \( \Delta u(x) \to 0 \) as \( |x| \to +\infty \). Hence, if \( u \) normal solution, we have

\[
\lim_{|x|\to+\infty} R_{g_u} = 0.
\]

In the case when \( u = v + p \) non-normal solution, we have proved that the polynomial \( p \) has degree 2, hence \( \Delta p \) is constant and \( \text{deg} |\nabla p|^2 = 2 \). In this way we obtain that

\[
\limsup_{|x|\to+\infty} (\Delta p + |\nabla p|^2) = +\infty.
\]

Differentiating (9) we get

\[
\nabla v(x) = -\frac{1}{8\pi^2} \int_{\mathbb{R}^4} \frac{x-y}{|x-y|^2} (1 - |y|^p) e^{4u} dy.
\]
hence $|\nabla v(x)| \to 0$ as $|x| \to +\infty$; from Proposition 4.4 we get $\lim_{|x| \to +\infty} \Delta v(x) = 0$. Then we obtain

$$\limsup_{|x| \to +\infty} (\Delta u + |\nabla u|^2) = \limsup_{|x| \to +\infty} (\Delta p + |\nabla p|^2) = +\infty$$

and hence since $e^{-2u} > 0$ we get

$$\liminf_{|x| \to +\infty} R_{qu} = -\infty.$$

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