Induced (N,0) supergravity as a constrained Osp(N\|2) WZWN model and its effective action

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ABSTRACT

A chiral (N,0) supergravity theory in d=2 dimensions for any N and its induced action can be obtained by constraining the currents of an Osp(N\|2) WZWN model. The underlying symmetry algebras are the nonlinear SO(N) superconformal algebras of Knizhnik and Bershadsky. The case N = 3 is worked out in detail. We show that by adding quantum corrections to the classical transformation rules, the gauge algebra on gauge fields and currents closes. Integrability conditions on Ward identities are derived. The effective action is computed at one loop. It is finite, and can be obtained from the induced action by rescaling the central charge and fields by finite Z factors.

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1 Introduction.

Recently, it has been conjectured, and verified at the one-loop level, that gauge theories in two-dimensional Euclidean space have remarkable renormalizability properties. Namely, their full effective actions are free from any divergences, and obtained from their induced actions merely by rescaling the coupling constants and the fields by constant $Z$-factors, and these $Z$-factors are power series in the inverse of the coupling constants. The induced actions can be obtained by first coupling matter to external gauge fields in a minimal way, and then integrating out the matter. In other words, the induced actions are the set of 1PI diagrams with propagating matter and external gauge fields. Since the gauge symmetry of these classical actions is broken at the quantum level by anomalies, a quadratic term and vertices involving these fields develop, so that loops with propagating gauge fields can be constructed. The induced actions are nonlocal.

Both the induced actions and the effective actions are finite (i.e., no divergences are generated by the loops) and both depend only on the gauge fields, but whereas the former are due to diagrams with matter loops and external gauge fields, the latter are due to diagrams with gauge fields both in the loops and as external fields. In local quantum field theories loops produce nonlocalities, needed for unitarity, and there is no way that the complete nonlocal effective action can be obtained from the local quantum action by constant rescalings. However, in the class of theories we are going to consider, the nonlocalities due to gauge loops are exactly of the same form as the nonlocalities present in the induced actions, and this allows the effective actions to be obtained from the induced actions in the manner described above.

To date, three models are known with these properties:

(i) Yang-Mills theory, whose induced action is the WZWN model, but in a purely two-dimensional nonlocal formulation [1].

(ii) ordinary gravity, whose induced action is Polyakov’s chiral gravity [1,2].

(iii) $W_3$ gravity, based on the nonlinear $W_3$ algebra of Zamolodchikov. Its induced and (one-loop) effective actions were obtained in refs. [3,4,5] and contain a spin 2 gauge field $h$ and a spin 3 gauge field $b$.

To this list we shall add a fourth example: gauge theories based on the nonlinear SO(N) algebras of Knizhnik [6] and Bershadsky [7]. The interest of these models is that they lead to chiral supergravities for any $N$. They lie between the models in (ii) and (iii): they are based on a nonlinear algebra (as $W_3$ gravity) but contain no higher spins (as Polyakov’s gravity). By a nonlinear algebra we mean a set of operators whose operator product expansion (OPE) contains terms quadratic (or higher) in the operators. In the $W_3$ algebra, one finds in the OPE of two spin 3 currents two terms bilinear in the stress tensor. In the SO(N) models one finds in the OPE of two spin 3/2 (supersymmetry) currents a term bilinear in the SO(N) Kac-Moody currents. The corresponding induced actions describe the gauge fields $h$ (graviton), $\psi^i$ (gravitini), and $\omega^a$ (SO(N) gauge vector fields).
It may be useful to be a bit more explicit about these models. Therefore we briefly summarize the main results for the first three models.

(i) **Yang-Mills theory.** If one couples a chiral gauge field \( A_+(x^+, x^-) = A^a_+ T_a \) (with \([T_a, T_b] = f_{ab}^c T_c\)) to chiral matter fermions, and integrates over these fermions (i.e., evaluates all one-loop diagrams with internal fermions and any number of external \( A_+ \) fields), one finds the following result for the induced action \([3]\) \[ S_{\text{ind}}[k, A_+] = k S_{\text{ind}}[A_+] \]

\[
S_{\text{ind}}[A_+] = -\frac{1}{2\pi\chi} \sum_{n=0}^{\infty} \frac{1}{2+n} \text{tr} \int d^2 x A_+ \left[ \frac{1}{\partial_+} A_+, \cdots \left[ \frac{1}{\partial_+} A_+, \frac{\partial_-}{\partial_+} A_+ \right] \cdots \right] \overset{n \text{ times}}{=} \frac{1}{2\pi} \int d^2 x \left[ \frac{1}{2} A^a_+ \frac{\partial_-}{\partial_+} A^-_a + \frac{1}{3} f_{abc} A^a_+ \frac{1}{\partial_-} A^b_+ \frac{\partial_-}{\partial_+} A^c_+ + \cdots \right] \tag{1.1} \]

Under \( \delta A^a_+ = \partial_- \eta^a - f^a_{bc} A^b_+ \eta^c \) the induced action varies into \( \frac{k}{2\pi} \int d^2 x A^a_+ \partial_- \eta^a \). Hence, writing \( \delta S = (\delta S/\delta A) \delta A \) it satisfies the following Ward identity

\[
\partial_+ u^-_a - \partial_- A^a_+ - f^a_{bc} A^b_+ u^-_c = 0 \tag{1.2} \]

where \( u^-_a \) are the currents, suitably normalized,

\[
u^-_a = \frac{2\pi \delta S_{\text{ind}}}{k \delta A^a_+} = \frac{\partial_-}{\partial_+} A^a_+ + \cdots \tag{1.3} \]

In fact, any matter system with Kač-Moody current \( J_a(x) \) satisfying the OPE

\[
J_a(x)J_b(y) = \frac{k}{2} \frac{g_{ab}}{(x^- - y^-)^2} + f^c_{ab} J_c(y) \frac{1}{x^- - y^-} \tag{1.4} \]

leads to this induced action if one adds to the matter action the following minimal coupling

\[
S_{\text{int}} = \frac{1}{\pi} \int d^2 x A^a_+(x) J_a(x) \tag{1.5} \]

For example, a WZWN model itself is such a matter system, although in this case the induced action receives contributions from arbitrarily many matter loops.

The *effective* action, i.e., the sum of all 1PI graphs computed from (1.1) with internal and external \( A_+ \) lines, is conjectured \([1,8]\) to be given by

\[
S_{\text{eff}}[A] = Z_k k S_{\text{ind}}[A] \tag{1.6} \]

where the \( Z \) factors are constant, and power series in the inverse of the level parameter \( k \) of the Kač-Moody current \([8]\) \[ Z_k = 1 + \frac{2\tilde{\hbar}}{k} \quad ; \quad Z_A = \left( 1 + \frac{\tilde{\hbar}}{k} \right)^{-1} \tag{1.7} \]

1 Conventions for (super)algebras: \((-1)^p f^a_{pq} f^{pq} = -g_{ab} \tilde{\hbar} \), \( str T_a T_b = -\chi g_{ab} \). Raising and lowering is done with \( g_{ab} \) and its inverse \( g^{ab} \); \( \tilde{\hbar} \) is the dual Coxeter number.

2 In ref. [1] the result for \( Z_A \) differs from our result by a factor of 2 in front of \( \tilde{\hbar} \).
and $\tilde{h}$ is the dual Coxeter number. To one loop this conjecture has been verified (see below).

(ii) Polyakov gravity. If one couples the component $h \equiv h_{++}$ of the gravitational field to the stress tensor of matter, and integrates over matter, one finds the action studied by Polyakov, Zamolodchikov, and Knizhnik [2]

$$S_{\text{ind}}[c, h] = c S_{\text{ind}}^{(0)}[h]$$

with

$$S_{\text{ind}}^{(0)}[h] = -\frac{1}{24\pi} \int d^2x (\partial^2 h) \frac{1}{1 - \frac{1}{\partial_+ h} \partial_- h}$$

$$= -\frac{1}{24\pi} \int d^2x \left[ \frac{h \partial_+^3 h - h (\partial_+^2 h)^2}{2} - \left( \frac{h \partial_+^2 h}{\partial_+ h} \right) \partial_- \left( \frac{h \partial_+^2 h}{\partial_+ h} \right) + \cdots \right]$$

This result holds whenever the OPE for $T \equiv T_{--}$ has the standard form

$$T(x)T(y) = \frac{c}{2}(x^- - y^-)^4 + 2\frac{T(y)}{(x^- - y^-)^2} + \frac{T'(y)}{x^- - y^-}$$

The effective action has been conjectured [9] to be given by

$$S_{\text{eff}}[c, h] = Z_c c S_{\text{ind}}^{(0)}[h]$$

where

$$Z_c = \frac{6}{c} k_c = 1 - \frac{25}{c} + \cdots ; \quad Z_h = \frac{k_c + 2}{k_c} = 1 + \frac{12}{c} + \cdots$$

$$c - 13 = 6[(k_c + 2) + (k_c + 2)^{-1}]$$

$$k_c + 2 = -\frac{1}{12} [13 - c + \sqrt{(c - 1)(c - 25)}]$$

for $c < 0$ (1.12)

(iii) $W_3$ gravity. Coupling the chiral gauge fields $h \equiv h_{++}$ with spin 2 and $b \equiv b_{++}$ with spin 3 to matter currents $T_{--}$ and $W_{--}$ which satisfy the exact $W_3$ algebra of Zamolodchikov with central charge $c$, one finds that the induced action is a power series in $1/c$ [4]

$$S_{\text{ind}}[c, h, b] = \sum_{j=0}^{\infty} c^{-j-j} S_{\text{ind}}^{(j)}[h, b]$$

$$S_{\text{ind}}^{(0)}[h, b] = -\frac{1}{24\pi} \int d^2x h \partial_+^3 h - \frac{1}{720\pi} \int d^2x b \partial_+^5 b + \cdots$$

The effective action, due to $h$ and $b$ loops with vertices from all $S_{\text{ind}}^{(j)}$, is conjectured to be obtained by just rescaling $c S_{\text{ind}}^{(0)}$ [5]

$$S_{\text{eff}}[c, h, b] = Z_c c S_{\text{ind}}^{(0)}[Z_h h, Z_b b]$$

(1.14)
where
\[
Z_c = \frac{24}{c} k_c = 1 - \frac{122}{c} + \cdots; \quad Z_h = \frac{k_c + 3}{k_c} = 1 + \frac{72}{c} + \cdots
\]
\[
Z_b = \frac{(k_c + 3)^{3/2}\sqrt{30\gamma}}{2k_c} = 1 + \frac{224}{5c} + \cdots
\]
\[
c - 50 = 24 \left[ (k_c + 3) + (k_c + 3)^{-1} \right]
\]
\[
k_c + 3 = -\frac{1}{48} \left[ 50 - c + \sqrt{(c - 2)(c - 98)} \right] \quad \text{for} \quad c < 0 \quad (1.15)
\]

Here \( \beta = 16(22 + 5c)^{-1} \) is a constant which appears in the OPE of \( W_{-\cdots}(x) \) and \( W_{-\cdots}(y) \) in front of the terms with two \( T_{-\cdots} \) operators.

Before closing this introduction we review how one computes the one-loop contribution to the effective action since by this method we shall determine the \( Z \)-factors. We shall do this for Yang-Mills theory.

For Yang-Mills theory, one starts from the Ward identity in eq. (1.2). (It states that \( u_- = \partial_- gg^{-1} \) and \( A_+ = \partial_+ gg^{-1} \) for some \( g \), and if one were to make this substitution for \( A_+ \) in \( S_{\text{ind}}[A_+] \), one would recover the WZW model in a form where the usual local 3-dimensional WZW term is written as a nonlocal 2-dimensional term. However, we do not make this substitution and keep working with \( A_+ \), because only in terms of \( A_+ \) do we have the remarkable renormalizability.) By differentiating the Ward identity one finds a relation of the form
\[
M^a_b(x) \left( \frac{\partial u^b_- (x)}{\partial A^c_+(y)} \right) = N^a_c(x) \delta(x - y) \quad (1.16)
\]
with
\[
M^a_b = \delta^a_b \partial_+ + f^a_{bc} A^c_+ \\
N^a_b = \delta^a_b \partial_- + f^a_{bc} u^c_-
\quad (1.17)
\]

Since the one-loop contributions are given by the determinant of the matrix \( \partial u(x)/\partial A(y) \), it suffices to evaluate \( \det M \) and \( \det N \). Both \( M \) and \( N \) are local matrix-operators, and we obtain their determinants by using “ghosts” \( b_a, c^a \) for \( M \), and \( B_a, C^a \) for \( N \), i.e.,
\[
\det M = \int dc \, db \, \exp[b_a M^a_b c^b] \\
\det N = \int dB \, \exp[B_a N^a_b C^b] \quad (1.18)
\]

To actually evaluate these determinants, one introduces the notation
\[
\det M = \exp \left[ b_a j^a_{b c} \right] \quad , \quad j^a_{b c} = f^a_{b c} A^c_+ \quad (1.19)
\]
where the expectation value is taken with respect to the free ghost actions, and uses
\[
\langle c^a(x) b_b(y) \rangle = -\delta^a_b \frac{1}{\partial_+} \delta^2(x - y) \quad (1.20)
\]
One finds then
\[ \det M = 1 - \frac{1}{2\pi} \text{tr} \int j \frac{\partial}{\partial j} + \frac{1}{3\pi} \text{tr} \int j \left[ \frac{1}{2} \frac{\partial}{\partial j} \frac{\partial}{\partial j} \right] + \cdots \] (1.21)

Evaluating the traces one finds
\[ \frac{1}{2} \ln \det M = \hbar \left[ \frac{1}{4\pi} \int A^a_+ \frac{\partial}{\partial a} A^a_+ + \frac{1}{6\pi} f_{abc} \int A^a_+ \left( \frac{1}{2} A^b_+ \frac{\partial}{\partial a} A^c_+ + \cdots \right) \right] \] (1.22)

In other words, \( \frac{1}{2} \ln \det M \) is proportional to the induced action
\[ \frac{1}{2} \ln \det M = \hbar \tilde{S}^{(0)}_{\text{ind}}[A_+] \] (1.23)

To prove this to all orders in \( A_+ \), one may use the Ward identity for (1.18) \[1,8\].

For \( N \) one finds a slightly different result. Of course \( \det N \) is the same functional as \( \det M \), but with \( j^a_b \) replaced by \( J^a_b = f^a_{bc} u^c_- \), and with \( \partial_- \) and \( \partial_+ \) interchanged. Hence
\[ \frac{1}{2} \ln \det N = \hbar \tilde{S}^{(0)}_{\text{ind}}[u_-] \] (1.24)

where we denote by \( \tilde{S}^{(0)}_{\text{ind}}[u_-] \) the induced action with \( A_+ \) replaced by \( u_- \) and \( \partial_+ \) and \( \partial_- \) interchanged. (Also this result holds to all orders as can be shown \[1,8\] by using the Ward identity for \( \det N \) in (1.18).) However, if we substitute for \( u_-^a \) its dependence on \( A_+ \)
\[ u_-^a = \frac{\partial_-}{\partial a} A^a_+ + \frac{1}{3} f^a_{bc} \left\{ \left( \frac{1}{2} \frac{\partial}{\partial a} A^b_+ \right) \left( \frac{\partial_-}{\partial a} A^c_+ \right) + \frac{1}{\partial_+} \left( A^b_+ \frac{\partial_-}{\partial a} A^c_+ \right) \right\} + \cdots \] (1.25)

we find
\[ \frac{1}{2} \ln \det N = \hbar \left[ \frac{1}{4\pi} \int A^a_+ \frac{\partial}{\partial a} A^a_+ + \frac{1}{3\pi} f_{abc} \int A^a_+ \left( \frac{1}{2} A^b_+ \frac{\partial}{\partial a} A^c_+ + \cdots \right) \right] \] (1.26)

To this order at least, \( \tilde{S}^{(0)}_{\text{ind}}[u_-(A_+)] \) is related to \( S^{(0)}_{\text{ind}}[A_+] \) by a Legendre transform \[1\]
\[ S^{(0)}_{\text{ind}}[A_+] + \tilde{S}^{(0)}_{\text{ind}}[u_-(A_+)] = \frac{1}{2\pi} \int d^2 x A^a_+ u_--a \] (1.27)

This is easy to check; \( Au \sim A\delta S_{\text{ind}}/\delta A \) counts the number of \( A \)-fields in \( S_{\text{ind}} \), and multiplies the terms with \( 2,3,\ldots \) \( A_+ \) fields in \( S^{(0)}_{\text{ind}} \) by \( 2,3,\ldots \). Therefore the complete one-loop contribution to the effective action is given by
\[ S^{1-\text{loop}}_{\text{eff}} = \frac{1}{2} \ln \det M - \frac{1}{2} \ln \det N = 2\hbar S^{(0)}_{\text{ind}} - \frac{\hbar}{2\pi} \int d^2 x A^a_+ u_-a \] (1.28)

In other words, the one-loop contributions replace \( k \) in front of the induced action by \( 2\hbar \) and scale each field \( A_+ \) with a factor \( -\hbar \). This implies
\[ S_{\text{ind}}[A_+] + S^{1-\text{loop}}_{\text{eff}}[A_+] = (k + 2\hbar) S^{(0)}_{\text{ind}} \left[ (1 - \frac{\hbar}{k}) A_+ \right] \] (1.29)

The all-loop result is conjectured \[8\] to be obtained by replacing \( 1 - \hbar/k \) by \( (1 + \hbar/k)^{-1} \).
The SO(N) extended superconformal algebras.

These algebras [6,7] contain a current of dimension 2 (the stress tensor $T(z)$), $N$ currents of dimension 3/2 (the supersymmetry currents $Q^i(z)$, with $i = 1, \ldots, N$), and $\frac{1}{2}N(N-1)$ currents of dimension 1 (the Kač-Moody currents $J^a(z)$ for SO(N)). We shall concentrate on SO(3), because it is the simplest interesting case. (For $N=2$ the algebra becomes linear.) The OPE for these currents is as expected: $T(z)T(w)$ was given in eq. (1.10), and $Q^i, J^a$ are primary:

$$
\begin{align*}
T(z)Q^i(w) &= \frac{3}{2} \frac{Q^i(w)}{(z-w)^2} + \frac{Q^j(w)}{z-w} \\
T(z)J^a(w) &= \frac{J^a(w)}{(z-w)^2} + \frac{J^a(w)}{z-w}
\end{align*}
$$

(2.1)

Furthermore, the Kač-Moody currents act on themselves and on $Q^i(z)$ as SO(3) generators

$$
\begin{align*}
J^a(z)J^b(w) &= -\sigma \frac{\delta^{ab}}{(z-w)^2} + \epsilon^{abc} \frac{J^c(w)}{z-w} \\
J^a(z)Q^i(w) &= \epsilon^{aij} \frac{Q^j(w)}{z-w}
\end{align*}
$$

(2.2)

However, the $QQ$ OPE contains a nonlinear term

$$
\begin{align*}
Q^i(z)Q^j(w) &= B \frac{\delta^{ij}}{(z-w)^3} - K \epsilon^{ija} \frac{J^a(w)}{(z-w)^2} - \frac{K}{2} \epsilon^{ija} \frac{J^a'(w)}{z-w} \\
&\quad + \delta^{ij} \frac{2T(w)}{z-w} + 2\gamma \frac{J^i(w)}{z-w}
\end{align*}
$$

(2.3)

where $J_{ij}$ is the normal-ordered product of two Kač-Moody currents, symmetrized in the indices

$$
\begin{align*}
J_{ij} &= \frac{1}{2} : J_i J_j : + \frac{1}{2} : J_j J_i : \equiv : J_i J_j :
\\
: J_i J_j : (w) &= \frac{1}{2\pi i} \oint \frac{dx}{x-w} J_i(x) J_j(w)
\end{align*}
$$

(2.4)

There is only one independent central charge, which we choose to be $\sigma$, as the Jacobi identities require the following relations

$$
\begin{align*}
c &= \frac{1}{2} (6\sigma - 1) , \quad K = \frac{2\sigma - 1}{\sigma} \\
B &= K\sigma = 2\sigma - 1 , \quad \gamma = \frac{1}{2\sigma}
\end{align*}
$$

(2.5)

To check these results one may compute, for example,

$$
\begin{align*}
\langle J_a(z)Q_i(x)Q_j(y) \rangle &= \langle J_a(z)Q_i(x)Q_j(y) \rangle + \langle Q_i(x)J_a(z)Q_j(y) \rangle \\
&= \langle Q_i(x)J_a(z)Q_j(y) \rangle + \langle J_a(z)Q_i(x)Q_j(y) \rangle
\end{align*}
$$

(2.6)
The first way of contracting yields
\[
B\epsilon_{aij}\left(\frac{1}{x'-y'}\right)^{\frac{1}{2}} \left(\frac{1}{z'-x'} - \frac{1}{z'-y'}\right)
\]  
(2.7)
while the second way yields
\[
B\epsilon_{aij}\left(\frac{1}{z'-x'}\right)^{\frac{1}{2}} \left(\frac{1}{z'-y'}\right)^{\frac{1}{2}} + \sigma K\epsilon_{aij}\left(\frac{1}{(x'-y')^2(z'-y')^2} + \frac{1}{(x'-y')(z'-y')^3}\right)
\]
(2.8)
Clearly \(\sigma K = B\).  

Decomposing the currents in modes, the SO(N) algebras are of the general form
\[
[H_i, H_j] = f_{ij}^k H_k + h_{ij} I
\]
\[
[H_i, S_\alpha] = f_{i\alpha}^\beta S_\beta + f_{i\alpha}^j H_j
\]
\[
[S_\alpha, S_\beta] = f_{i\alpha}^i H_i + f_{i\alpha}^\gamma S_\gamma + V_{i\beta}^{ij} : H_i H_j : + h_{i\beta} I
\]
(2.9)
where we can, without loss of generality, take the constants \(V_{i\beta}^{ij}\) to be symmetric in \(ij\). Indeed, if \(H_i = \{J_a\}\) and \(S_\alpha = \{T, Q_i\}\), one obtains this structure. Another division of generators with this structure is \(H_i = \{J_a, T\}\) and \(S_\alpha = \{Q_i\}\). For algebras of this kind, a nilpotent BRST operator exists, provided the central charges \(h_{ab}\) and the structure constants satisfy a relation of the form [10]
\[
h_{ab} \sim F_{ac}^d F_{db}^{c}
\]
(2.10)
where the index \(a\) denotes both \(\alpha\) and \(i\). The \(F_{ac}^d, F_{db}^{c}\) are equal to the classical \(f_{ac}^d\), plus, for \(f_{i\beta}^j\), a correction term of the form \(f_{i\alpha}^\gamma V_{\beta}^{ij}\). The BRST charge reads then [10]
\[
Q = c^a T_a - \frac{1}{2}c^c F_{ab}^{c}c^a b - \frac{1}{2}V_{i\beta}^{ij} T_i \bar{c}_j c^\alpha c^\beta - \frac{1}{24}V_{i\beta}^{ij} V_{\gamma\delta}^{kl} F_{ik}^{m} c^\gamma c^\delta + V_{i\beta}^{ij} \bar{c}_i c^\gamma c^\delta - \frac{1}{2}V_{i\beta}^{ij} \bar{c}_i c^\gamma c^\delta
\]
(2.11)
(In the last term one may clearly replace \(F_{ik}^{m}\) by \(f_{ik}^{m}\).)

For the SO(N) algebras the condition (2.10) is satisfied and the structure constants become multiplicatively renormalized, namely such that in the structure constants for \(QQ \sim J\) the factor \(K\) is replaced by \(\frac{1}{2}\), provided
\[
\sigma = 6 - 2N \quad , \quad B = 16 - 6N \quad , \quad c = N^2 - 12N + 26
\]
(2.12)
For \(N=3\) (our case), however, one finds that \(\sigma = 0\), hence \(V_{i\beta}^{ij}\) which is proportional to the constant \(\gamma^2 = \frac{1}{2}\sigma^{-1}\), becomes singular. First multiplying \(Q\) for general \(N\) by \(\gamma^{-2}\) and then taking \(N=3\) leaves only the last term in \(Q\), which is trivially nilpotent. At \(N=3\) also \(K \to -\infty\), so that presumably no unitary irreducible representations exist. In any event, we are considering general values of \(\sigma\) and these issues are of no concern for us.

\(^3\)In ref. [6], a factor 1/2 is missing in the \(QQ \sim J\) term, while in ref. [7] this term has an incorrect sign.
3 The Ward identities for the SO(3) induced action.

We define the induced action for the gauge fields \( h, \psi^i, \omega^a \), by

\[
e^{S_{\text{ind}}[\sigma,h,\psi,\omega]} = \langle e^{S_{\text{int}}} \rangle
\]

\[
S_{\text{int}} = -\frac{1}{\pi} \int d^2x(hT + \psi^i Q_i + \omega^a J_a)
\]  \hspace{1cm} (3.1)

Assuming that \( \langle T \rangle = \langle Q_i \rangle = \langle J_a \rangle = 0 \), and expanding the exponential, we can use the OPE given in the previous sections to determine \( S_{\text{ind}} \). For example, the kinetic terms are found to be

\[
S_{\text{kin}}^{\text{ind}} = -\frac{c}{24\pi} \int d^2x h \frac{\partial^3}{\partial^+} h - \frac{B}{4\pi} \int d^2x \psi^i \frac{\partial^2}{\partial^+} \psi_i + \frac{\sigma}{2\pi} \int d^2x \omega^a \frac{\partial^-}{\partial^+} \omega_a
\]  \hspace{1cm} (3.2)

In general, the induced action is an infinite series in inverse powers of the independent central charge \( \sigma \). As in the \( W_3 \) gravity case we write

\[
S_{\text{ind}}(\sigma,h,\psi,\omega) = \sum_{j=0}^{\infty} \sigma^{1-j} S^{(j)}_{\text{ind}}[h,\psi,\omega]
\]  \hspace{1cm} (3.3)

The Ward identities for the induced action can be obtained by varying \( \exp[S_{\text{ind}}] \) under the leading terms in the variations of the gauge fields

\[
\delta h = \partial_+ \epsilon + \cdots, \quad \delta \psi^i = \partial_+ \eta^i + \cdots, \quad \delta \omega^a = \partial_+ \lambda^a + \cdots
\]  \hspace{1cm} (3.4)

and then finding extra terms in the transformation laws such that only the minimal anomalies remain, plus terms due to the nonlinearity of the algebra. The minimal anomalies, which correspond to the central terms in the OPE, are obtained by substituting the above variations into (3.1) and retaining only the terms quadratic in operators; hence the result is the same as obtained by substituting (3.4) directly into (3.2)

\[
\text{Minimal anomaly} = -\frac{c}{12\pi} \int d^2x d_+^3 \epsilon - \frac{B}{2\pi} \int d^2x \psi^i \frac{\partial^2}{\partial^+} \eta^i + \frac{\sigma}{\pi} \int d^2x \omega^a \frac{\partial^-}{\partial^+} \lambda^a
\]  \hspace{1cm} (3.5)

For the local \( \epsilon \) symmetry one finds that under \( \delta h = \partial_+ \epsilon \)

\[
\delta S_{\text{ind}} \exp[S_{\text{ind}}] = \langle -\frac{1}{\pi} \int d^2x \partial_+ \epsilon T e^{S_{\text{int}}} \rangle
\]

\[
= \langle -\frac{1}{\pi} \int d^2x \partial_+ \epsilon T (\underbrace{-\frac{1}{\pi} \int d^2y (hT + \psi^i Q_i + \omega^a J_a) e^{S_{\text{int}}}}_{\text{Central term from } \langle TT \rangle}) \rangle
\]  \hspace{1cm} (3.6)

The central term from \( \langle TT \rangle \) yields the minimal \( \epsilon \)-anomaly, while all other terms in the OPE are linear in operators, and are cancelled by suitable extra \( \delta h, \delta \psi^i \), and \( \delta \omega^a \). One finds then that

\[
\delta(\epsilon) S_{\text{ind}} = -\frac{c}{12\pi} \int d^2x d_+^3 \epsilon
\]  \hspace{1cm} (3.7)
\[ \delta(\epsilon)h = \partial_+ \epsilon - h \partial_- \epsilon + \epsilon \partial_- h \]
\[ \delta(\epsilon)\psi^i = -\frac{1}{2} \sigma^i \partial_- \epsilon + \epsilon \partial_- \psi^i \]
\[ \delta(\epsilon)\omega^a = \epsilon \partial_- \omega \] (3.8)

Introducing suitably normalized currents by
\[ u = - \frac{12 \pi}{c} \frac{\delta}{\delta h} S_{\text{ind}} = \frac{\partial_+^2}{\partial_+} h + \cdots \]
\[ q_i = - \frac{2 \pi}{B} \frac{\delta}{\delta \psi^i} S_{\text{ind}} = \frac{\partial_+^2}{\partial_+} \psi_i + \cdots \]
\[ v_a = \frac{\pi}{\sigma} \frac{\delta}{\delta \omega^a} S_{\text{ind}} = \frac{\partial_-}{\partial_+} \omega_a + \cdots \] (3.9)

we find, as in (1.2), the Ward identity for \( \epsilon \) symmetry
\[ \partial_+ u = D_1 h + \frac{3B}{c} (3 \psi^i q'_i + \psi q'_i) - \frac{12 \pi}{c} \omega'_a v_a \] (3.10)
where
\[ D_1 \equiv \partial_+^2 + 2 u \partial_- + u' \] (3.11)

We have introduced the notation \( u' = \partial_- u \), \( q'_i = \partial_- q_i \), etc.

For local supersymmetry we will encounter \( J^{eff}_{ab}(x) \equiv \langle J_{ab}(x) \exp S_{\text{int}} \rangle \). The OPE for \( T(x) J_{ab}(y) \) contains a central term \( (TJ \text{ contains } J, \text{ and } JJ \text{ contains a central term}) \), but \( Q^i J_{ab} \) has of course no central term, and also \( J_a J_{bc} \) is without central term (see below). By direct evaluation one finds
\[ T(z) : J_a J_b : (w) = -\sigma \frac{\delta_{ab}}{(z-w)^4} + \epsilon_{abc} \frac{J^c(w)}{(z-w)^3} \]
\[ + 2 \frac{J_{ab}(w)}{(z-w)^2} + \frac{J'_{ab}(w)}{z-w} \] (3.12)

Since \( J_{ab} \) is symmetric in \( ab \), one finds
\[ T(z) J_{ab}(w) = -\sigma \frac{\delta_{ab}}{(z-w)^4} + 2 \frac{J_{ab}(w)}{(z-w)^2} + \frac{J'_{ab}(w)}{z-w} \] (3.13)

Therefore we redefine \( J_{ab} \) by adding a term with \( T \) such that the redefined operator, denoted by \( \Lambda_{ab} \), has no central term in the OPE with \( S_{\text{int}} \)
\[ \Lambda_{ab}(z) = J_{ab}(z) + \frac{2 \sigma}{c} \delta_{ab} T(z) \] (3.14)

Another result we shall need is the OPE for \( J_a(z) \) and \( J_{bc}(w) \). Before symmetrizing on \( bc \) one gets
\[ J_a(z) : J_b J_c : (w) = -\sigma \epsilon_{abc} \frac{1}{(z-w)^3} + \epsilon_{abf} \epsilon_{ced} \frac{J^e(w)}{(z-w)^2} \]
\[ - \sigma \frac{\delta_{ab} J_c(w) + \delta_{ac} J_b(w)}{(z-w)^2} + \frac{\epsilon_{abf} : J_d J_c : (w) + \epsilon_{acd} : J_d J_b : (w)}{z-w} \] (3.15)
Symmetrizing in $b, c$ one finds

$$J_a(z)J_{bc}(w) = \frac{(-\sigma + \frac{1}{2})[\delta_{ac}J_b(w) + \delta_{ab}J_c(w)] - \delta_{bc}J_a(w)}{(z^- - w^-)^2} + \frac{\epsilon_{abcd}J_{dc}(w) + \epsilon_{acdb}J_d(w)}{z^- - w^-}$$

which is clearly without central charge.

We can therefore compute the terms in $\Lambda_{ab}^{eff}$ which are quadratic in fields

$$\Lambda_{ab}^{eff}(\text{quadr})(z) = \langle (J_{ab} + \frac{2\sigma}{c}\delta_{ab}T)(z) \frac{1}{2!\pi^2} \int d^2x(hT + \psi^i Q_i) \int d^2y(hT + \psi^j Q_j + \omega^b J_b) \rangle$$

(3.16)

We will be interested in the leading terms for $\sigma \rightarrow \infty$. Now, $T\Lambda$ contains terms with $T$ and $\Lambda$, see (3.13), but only the former contribute to $\Lambda_{ab}^{eff}(\text{quadr})$ since $\langle AT \rangle = 0$, and they are of order $\sigma$. From $Q_i\Lambda_{ab}$ one gets terms with $Q_i$ and $J_a$; but their contribution to (3.17) is of order $B$, i.e., of order $\sigma$. The leading terms in $\sigma$ only come from the OPE of $J^a(x)\Lambda_{ab}(z)$ because it contains terms of the form $\sigma J$, see (3.15), and $\langle J^a(x)J^b(y) \rangle$ is itself again of order $\sigma$. In fact

$$\Lambda_{ab}^{eff}(\text{quadr}, \sigma \rightarrow \infty) = \frac{1}{2\pi^2} \int \int d^2x d^2y \langle J_c(x)\Lambda_{ab}(z)J_d(y) \rangle \omega^c(x)\omega^d(y)$$

(3.17)

because the other contraction $\langle \Lambda_{ab}(z)J_c(x)J_d(y) \rangle$ vanishes. One finds

$$\langle J_c(x)\Lambda_{ab}(z)J_d(y) \rangle = \sigma^2 \delta_{ac}\delta_{bd} + \delta_{bc}\delta_{ad}$$

(3.18)

(3.19)

Therefore, to second order in $\omega^a$,

$$\Lambda_{ab}^{eff}(z) = \sigma^2 v_a(z)v_b(z) + O(\sigma)$$

(3.20)

One can, actually, prove this result to all orders in the fields in $\nu_a(z)$ by a Ward identity [11]. Hence, the terms of order $\sigma^2$ are exact.

With these preparations done, we can return to the supersymmetry Ward identity. Under $\delta \psi^i = \partial_+ \eta^i$ one has

$$\delta(\exp S_{\text{ind}}) = \left\{ -\frac{1}{\pi} \int d^2x \partial_+ \eta^i Q_i \left( -\frac{1}{\pi} \int d^2y (hT + \psi^j Q_j + \omega^a J_a) \right) e^{S_{\text{tot}}} \right\}$$

(3.21)

The central term in $QQ$ yields the minimal anomaly, while all terms linear in currents are eliminated by suitable extra terms in the transformation laws, but a nonlinear term $QQ \sim \Lambda_{ab}$ remains. One finds then

$$\delta(\eta)S_{\text{ind}} = \int d^2x \left( \frac{B}{2\pi} \partial_+^2 \eta^i \psi_i + \frac{2\gamma}{\pi} \eta^i \psi^j \Lambda_{ij}^{eff} \right)$$

(3.22)
under
\[\delta(\eta)h = 2\eta^i \psi_i - \frac{4\gamma\sigma}{c} \eta^i \psi_i\]
\[\delta(\eta)\psi^i = \partial_+ \eta^i - h \partial_- \eta^i + \frac{1}{2} \eta^j \partial_- h - \epsilon_{ij} \eta_j \omega_a\]
\[\delta(\eta)\omega^a = -\frac{K}{2} \epsilon^{ija} (\eta_i \psi_j - \eta_i \psi'_j)\]  
(3.23)

(At this point we already note that we could modify the constant in front of the last term in (3.22) by adding a nonlinear term \(\delta_{\text{nonlinear}} \omega \sim \psi \eta v\) in (3.23). In the next section we shall fix this ambiguity such that the local gauge algebra closes; as a result the sign in front of the last term in (3.22) changes.) Extracting the supersymmetry parameters \(\eta^i\) one is left with
\[\left(\partial_+ - \frac{3}{2} h' - h \partial_-\right) q^i - \frac{c}{6 B} \left(2 - \frac{4\gamma\sigma}{c}\right) \psi^i u + \epsilon^{imj} \omega_a q_j + \epsilon^{ija} (2 \psi_j' + \psi_j \partial_-) v_a = \partial_- \psi^i + \frac{4\gamma}{B} \psi \omega^i \Lambda_{ij}^{\text{eff}}\]  
(3.24)

Finally, the SO(3) Ward identity is derived without any complications of nonlinear terms. One finds that
\[\delta(\lambda) S_{\text{ind}} = \frac{\sigma}{\pi} \int d^2 x \partial_- \lambda^a \omega_a\]  
(3.25)

under
\[\delta(\lambda)h = 0\]
\[\delta(\lambda)\psi^i = \epsilon^{aj} \lambda_b \psi_j\]
\[\delta(\lambda)\omega^a = \partial_+ \lambda^a - h \partial_- \lambda^a - \epsilon^{abc} \lambda_b \omega_c\]  
(3.26)

Extracting \(\lambda^a\) we get
\[\partial_+ v^a - \epsilon^{aj} \psi_i q_j - h' v^a - h \partial_- v^a + \epsilon^{abc} \omega_b v_c = \partial_- \omega^a\]  
(3.27)

For our purpose we need the terms of leading order in \(\sigma\) in the three Ward identities. This means that only \(S_{\text{ind}}^{(0)}\) contributes to the currents while only the \(\sigma^2\) term in (3.20) survives. One has, in fact,
\[\left(\partial_+ - h \partial_- - 2 h'\right) u - 2 (3 \psi_i' q^i + \psi^i q'_i) + 4 \omega_a v^a = \partial^2 h\]
\[\left(\partial_+ - h \partial_- - \frac{3}{2} h'\right) q^i - \epsilon^{aj} \omega_a q_j - \frac{1}{2} \psi^i u + \epsilon^{ija} (2 \psi_j' + \psi_j \partial_-) v^a - \psi_j v^i v^i = \partial^2 \psi^i\]
\[\left(\partial_+ - h \partial_- - h'\right) v^a + \epsilon^{abc} \omega_b v_c - \epsilon^{aj} \psi_i q_j = \partial_- \omega^a\]  
(3.28)

4 Local gauge algebras and the Ward identities for \(\sigma \to \infty\).

In the previous section we obtained the Ward identities for the induced action \(S_{\text{ind}}\). For general central charge \(\sigma\), the nonlinearity of the algebra leads to \(\Lambda_{ij}^{\text{eff}}\) which is
a nonlocal functional depending on the fields $h, \psi^i, \omega^a$, and the currents $u, q^i, v^a$ (the latter are normalized to $\partial^2/h, \partial^2/\psi^i, \partial^2/\omega^a$, by $\sigma$-independent rescalings). However, for $\sigma \to \infty$, we found local Ward identities for the leading part of the induced action, $S_{\text{ind}}^{(0)}$. It is these local Ward identities from which we will obtain the one-loop contributions to the effective action, and it is obviously important to have a check on their correctness. In addition, we want to establish a connection between these Ward identities and the gauging of nonlinear algebras, for which a general formalism was constructed in ref. [12]. In fact, we will see that at the quantum level the anomalies add quantum corrections to the transformation laws of the currents so that the nonclosure terms in the classical gauge algebra are eliminated.

Let us begin by emphasizing that the Ward identities are a property of an induced action, not of particular transformation laws. However, we can derive them by choosing certain transformation rules for the gauge fields, and then varying the induced action under these particular transformation rules. In our case we determined the transformation rules for the fields of the form $\delta(f\text{ield}) = \partial_+(\text{parameter}) + (\text{field}) \times (\text{parameter})$ by removing terms in the OPE which are linear in operators (see section 3). The left-over, e.g. the right-hand-sides in (3.7) or (3.22) is the anomaly. Since the currents are the Euler-Lagrange equations of the induced action, it is clear that also terms in the Ward identities which are quadratic in currents can be (partly or completely) removed from the anomaly by adding terms to the transformation rules of the gauge fields of the form $\delta(f\text{ield, extra}) = (\text{field}) \times (\text{current}) \times (\text{parameter})$. In fact, one needs such terms if one requires that the currents transform as given by the OPE: $\delta u = \langle \epsilon(x)T(x)dxT(y)\exp S_{\text{ind}} \rangle$, etc.

It is clear, from the fact that for example $u \sim \langle T \exp S_{\text{ind}} \rangle$ and $T$ is holomorphic, that the currents transform only into expressions involving $\partial_-$ derivatives, but no $\partial_+$ derivatives. From this observation one can immediately read off the transformation laws of the currents from the Ward identities. Namely, by varying the Ward identities one obtains expressions of the form $\partial_+(\delta \text{ current}) + \text{more} = 0$, and only the variations of the gauge fields produce further $\partial_+$ derivatives. One can pull all these $\partial_+$ derivatives in front of the whole term in which they appear, because the extra terms one produces in this way are of the form $\partial_+(\text{current})$ which can be rewritten in terms of $\partial_-$ derivatives by using the Ward identities. As an example consider the term $-h\partial_- u$ in the first Ward identity in (3.28). It varies into $-\partial_+ [\epsilon \partial_- u] + \epsilon \partial_- [\partial_+ u]$ under $\epsilon$ symmetry, and $\partial_+ u$ can be replaced by $h\partial_- u + \cdots$. In this way we find the following transformation rules for the currents

$$
\begin{align*}
\delta u &= \epsilon u' + 2\epsilon' u + 6\eta_i q^i + 2\eta_i' q_i - 4\chi^a_v v^a + \partial^2 \epsilon \\
\delta q^i &= \epsilon q^i + \frac{3}{2} \epsilon' q^i - \epsilon' q^i + \frac{1}{2} \eta^i u - \epsilon' \eta^i (2\eta_j' + \eta_j \partial_- v) + \eta_j v^j v^i + \partial^2 \eta^i \\
\delta v^a &= \epsilon v^a + \epsilon' v^a - \epsilon v^a + \epsilon' v^a - \epsilon^{abc} \lambda_b v_c + \eta_i q^i + \partial_- \eta^a 
\end{align*}
$$

(4.1)

Note that these results hold only for $\sigma \to \infty$ because we used the $\sigma \to \infty$ Ward identities, but we could also obtain the results for finite $\sigma$ by using the same ideas.

Note also that in this derivation we only used the leading term in the gauge field transformation laws (the $\partial_+ (\text{parameter})$ part) and the possibility of extra terms in
the gauge field transformation laws involving currents is still completely left open. For further use we draw the reader's attention to the one nonminimal term in δq^i, and to the minimal anomalies ∂^2 ϵ, ∂^2 η^i, and ∂_λ^a.

We have, in fact, used in obtaining (4.1) that the Ward identities in (3.28) can be written in terms of supercovariant derivatives as

\[(D_+ j)_A = η_{AB} \partial_+ ^{B} φ^B, \quad A, B = 1, 2, 3\] (4.2)

where \(j_1 = u, j_2 = q, j_3 = v_a\), and \(φ^1 = h, φ^2 = ψ^i, φ^3 = ω^a\), while \(η_{AB} = δ_{AB}\). By supercovariance of \(D_+\) we mean that the variation of \(D_+ j\) is independent of \(∂_+ ξ^A\) if \(ξ^A\) are the local parameters. This allows us to determine \(δj_A\).

We could now deduce the transformation rules of the gauge fields by requiring that they, together with the current laws given above, leave the Ward identities invariant. However, there is a simpler, more general and more elegant method, and that is to note that given a nonlinear algebra of the form

\[[ ˆT_A, ˆT_B] = ˆT_C f^{C}_{AB} \] (4.3)

one can gauge it. One can then derive the following transformation rules of the gauge fields \(h^A_μ\) and "auxiliary fields" \(T_A\)

\[δh^A_μ = ∂_μ ϵ^A + f^{A}_{BC} h^C_μ ϵ^B + T_D V^{DA}_{BC} h^C_μ ϵ^B\]

\[δT_A = T_C (f^{C}_{AB} + \frac{1}{2} T_D V^{DC}_{AB}) ϵ^B\] (4.4)

and find the following results [12]:

(i) the gauge commutator on gauge fields closes up to a covariant derivative

\[[δ(ϵ_1), δ(ϵ_2)]h^A_μ = δ(ϵ_3)h^A_μ - D_μ T_D V^{DA}_{BC} ϵ^C_1 ϵ^B_2\] (4.5)

where the covariant derivative of \(T_A\) is given by

\[D_μ T_A = ∂_μ T_A - T_C f^{C}_{AB} h^B_μ - \frac{1}{2} T_D V^{DC}_{AB} h^B_μ\] (4.6)

Note the factor \(\frac{1}{2}\) in \(δT_A\) and \(D_μ T_A\).

(ii) The covariant derivatives are really covariant: they transform in the coadjoint representation, defined by

\[δD_μ T_A = D_μ T_C f^{C}_{AB} ϵ^B\] (4.7)

where \(f^{C}_{AB}\) are field-dependent structure constants

\[f^{C}_{AB} = f^{C}_{AB} + T_D V^{DC}_{AB}\] (4.8)

(iii) The gauge commutator on the auxiliary fields \(T_A\) closes

\[[δ(ϵ_1), δ(ϵ_2)]T_A = δ(ϵ_3)T_A\]

\[ϵ^C_3 = f^{C}_{AB} ϵ^B_1 ϵ^A_2\] (4.9)
curvatures are defined by
\[ [D_\mu, D_\nu] T_A = - T_C (f^C_{AB} + \frac{1}{2} T_D V^D_{AB}) R^{B}_{\mu\nu} \] (4.10)
and transform as follows
\[ \delta R^A_{\mu\nu} = \tilde{f}^A_{BC} R^C_{\mu\nu} \epsilon^B + D_\mu T_D V^D_{BC} h^C_{\nu} \epsilon^B - \mu \leftrightarrow \nu \] (4.11)
They satisfy the Bianchi identities
\[ D_{[\mu} R^A_{\nu\rho]} = 0 \] (4.12)

We shall now make contact with our Ward identities by the following observations:
(i) the auxiliary fields \( T_A \) are identified with the currents \( j_A \) in the Ward identities.

(ii) the formalism for gauging a nonlinear algebra holds for classical algebras. The Ward identities, however, contain a minimal anomaly term (the term \( \partial^{4-B} \phi^B \) in eq. (4.2)), and this term leads to a quantum correction \( \delta_q T_A \).

(iii) the covariant derivatives \( D_\mu T_A \) are the Ward identities except for the minimal anomaly.

In fact, using our rule of extracting \( \partial_\tau \) derivatives from the variations of the Ward identities, we already found the extra quantum terms: \( \partial^3 \epsilon \) in \( \delta u \), \( \partial^2 \eta^i \) in \( \delta q^i \), and \( \partial_\tau \lambda^a \) in \( \delta v^a \), see (4.1). Since these terms are field-independent, the gauge commutator on the currents \( j_A \) must still close (it indeed does, see below). However, also on the gauge fields the gauge commutator now closes because the quantum term in \( \delta T_A \) precisely cancels the covariant derivative in eq. (4.5). In other words, the minimal anomaly completes the Ward identity so that the gauge commutator closes on fields and currents! Finally, since there is a nonlinear term in \( \delta T_A \) (namely the term \( \delta q_t \sim \eta^j v_j v_i \)), we predict one nonlinear term in the gauge field laws as well
\[ \delta^{\text{nonlinear}} \omega^a = (\psi^a \eta^b + \psi^b \eta^a) v_b \] (4.13)
With this, we have checked that the gauge algebra on fields and currents closes. The commutator of two supersymmetries is, as always, the most interesting,
\[ [\delta(\eta), \delta(\eta)] = \delta(\epsilon = 2 \eta^i \eta'^i) + \delta(\hat{\lambda}^a) \]
\[ \hat{\lambda}^a = \epsilon^{a i j} (\eta_{2 i} \eta'^{j} - \eta_{1 i} \eta'^{j}) + (\eta'^{a} \eta^b - \eta^a \eta'^{b}) v_b \] (4.14)
The \( \epsilon \)-term in \( \hat{\lambda}^a \) is due to the structure constants \( V \) in (4.3). The rest of the gauge algebra is as expected
\[ [\delta(\eta), \delta(\epsilon)] = \delta(\hat{\eta}_i = \epsilon \eta'_i - \frac{1}{2} \epsilon' \eta_i) \]
\[ [\delta(\eta), \delta(\lambda)] = \delta(\hat{\eta}' = \epsilon' \eta' k) \]
\[ [\delta(\epsilon), \delta(\epsilon)] = \delta(\hat{\epsilon}' = \epsilon'_2 \eta' - \epsilon'_1 \eta) \]
\[ [\delta(\lambda), \delta(\epsilon)] = \delta(\hat{\lambda}' = \epsilon' \lambda) \]
\[ [\delta(\lambda_1), \delta(\lambda_2)] = \delta(\hat{\lambda}_a = 2 \epsilon_{abc} \lambda'_1 \lambda'_2) \] (4.15)
The extra variation (4.13) gives an extra contribution to the anomaly in (3.22) which is of the same form as the last term. In fact, the net result of including this variation is to change the sign of the nonlinear anomaly in (3.22).

5 The SO(N) theories from constrained WZWN models.

The SO(N) models are based on nonlinear superconformal algebras which contain the same set of generators as one encounters in the linear superalgebras Osp(N|2). It points to a close relation between the nonlinear SO(N) theories and WZWN models based on Osp(N|2). In fact, it has been shown in ref. [3] that Polyakov gravity or \( W_3 \) gravity can be obtained from Sl(2,R) or Sl(3,R) induced Yang-Mills theory by imposing constraints on the currents. This suggests imposing constraints on the currents \( u \equiv u^a T_a \) of induced Yang-Mills theory with \( T_a \) the generators of Osp(N|2) (see also ref. [13]). The constraints for Osp(1|2) which produce (1,0) supergravity were already given in [3]. Our aim is to obtain the Ward identities (3.28) from the Ward identities of Yang-Mills by imposing constraints on the Yang-Mills currents and by making suitable identifications between the Yang-Mills gauge fields and currents and those of SO(N) supergravity. We will give the details for the case \( N=3 \).

Since we need all SO(3) connections \( \omega^a \) in the nonlinear theory, we cannot put constraints in the SO(3) sector of \( u \). Hence, if the SO(3) models can be obtained at all by imposing constraints on Osp(3|2), these constraints must be of the following form

\[
u^a T_a = \begin{pmatrix}0 & u^0 & u^1 & u^2 & u^3 \\
1 & 0 & 0 & 0 & 0 \\
0 & -u^0 & u^{11} & u^{12} & u^{13} \\
0 & -u^1 & u^{21} & u^{22} & u^{23} \\
0 & -u^2 & u^{31} & u^{32} & u^{33} \end{pmatrix}
\]

(5.1)

with \( u^{ij} = \epsilon^{ija} u^a \). The Yang-Mills fields \( A = A^a T_a \) are not a priori constrained

\[
A^a T_a = \begin{pmatrix}A^0 & A^1 & A^2 & A^3 \\
-A^0 & A^0 & A^0 & A^0 \\
A^{-1} & -A^{-1} & A^{-1} & A^{-1} \\
-A^{-2} & A^{-2} & A^{-2} & A^{-2} \\
-A^{-3} & A^{-3} & A^{-3} & A^{-3} \end{pmatrix}
\]

(5.2)

with \( A^{ij} = \epsilon^{ija} A^a \). However, substituting these expressions for \( u \) and \( A \) into the Ward identity of the induced Osp(3|2) Yang-Mills theory, cf. (1.2)

\[
\partial_+ u = \partial_- A + [A,u]
\]

(5.3)

all components of \( A \) are expressed in terms of the following fields: \( A^0, A^{-i}, A^a \). For example, from \( \partial_+ u^0 = 0 \) one finds that

\[
A^0 = \frac{1}{2} (A^0)'
\]

(5.4)
while from $\partial_+ u^0 = 0$ one obtains

$$A^\# = -\frac{1}{2}(A^=)'' + u^\# A^= + u^{+i} A^{-i}$$

(5.5)

The constraint $u^{-i} = 0$ leads to

$$A^{+i} = (A^{-i})' + u^{+i} A^= + \epsilon^{ijk} u^j A^{-k}$$

(5.6)

Substituting these relations into the remaining Ward identities for $u^\#$, $u^{+i}$ and $u^a$ one finds the following results

$$\partial_+ u^\# = -\frac{1}{2}(A^=)'' + (u^\# A^= + u^{+i} A^{-i})' + (A^=)' u^\#
-2[(A^{-i})' + u^{+i} A^= + \epsilon^{ijk} u^j A^{-k}] u^{+i}
\partial_+ u^{+i} = (A^{-i})'' + (u^{+i} A^= + \epsilon^{ijk} u^j A^{-k})' - A^{-i} u^{\#} - \epsilon^{ijk} A^j u^{+k}
+ \frac{1}{2}(A^=)' u^{+i} - \epsilon^{ijk} [(A^{-i})' + u^{+j} A^=] u^k + u^i u^k A^{-k} - A^{-i} u^k u^k
\partial_+ u^{ij} = (A^{ij})' + (A^{-i} u^{+j} - A^{-j} u^{+i}) + \epsilon^{kli} \epsilon^{lmn} (A^k u^m - A^m u^k)$$

(5.7)

It is clear that one can choose the scales such that $A^= A^a = \omega^a + \alpha h v^a$. For the currents, $u^{+i} = q^i$ and $u^a = v^a$ are the only possibilities, but $u^\# = -\frac{1}{2} u + \beta v^a v_a$ is possible. Substituting these identifications into (5.7), we find that with

$$u^\# = -\frac{1}{2} u - v^a v^a$$
$$A^a = \omega^a + h v^a$$

(5.8)

one reproduces indeed the Ward identities in (3.28). In particular, the nonlinear term in the $\partial_+ q^i$ Ward identity is due to the term $\epsilon^{ijk} A^{+j} u^k$ after substituting $A^{+j} = \epsilon^{pq} u^p A^{-q} + \cdots$. The reason there are no further nonlinear terms in the Ward identities is that terms with $q^i q^j$ or $\epsilon^{ibc} v^b v^c$ are produced, which obviously cancel, while the redefinitions in (5.8) remove some nonlinear terms.

One can now substitute the constraints on the currents into the transformation laws of the currents. The latter follow from $u = \partial_- g^{-1} g$ and read $\delta u = \partial_- \eta + [\eta, u]$. In this way one finds that only the parameters $\eta^-, \eta^{-i}$ and $\eta^a$ are unconstrained, and furthermore one recovers (4.1), including the minimal anomaly terms. Repeating the analysis for the gauge fields, one finds from $\delta A = \partial_+ \eta + [\eta, A]$ and the relations between the parameters which we mentioned above, that the gauge fields transform as given in section 3 and (4.13).

It is also possible to obtain $\sigma \to \infty$ part of the induced action, $S^{(0)}_{ind}(h, \psi, \omega)$, in closed form from the WZW action by using the identifications between the currents. Before doing so, we must first discuss a subtlety concerning the indices of the currents. In section 3 all currents have lower indices because the Ward identities
follow from $\delta \phi^A j_A = \phi^A O_{AB} \partial \xi^B$, where $j_A \sim (\delta/\delta \phi^A) S$. Even in the term nonlinear in currents we identified $\frac{\partial}{\partial u^a} \omega^a$ with the current $\delta S/\delta u^a = j_a$. On the other hand, in this section all currents have upper indices. Hence, the identifications we made were really as follows:

$$A^\alpha = h; \quad A^{-i} = \psi^i; \quad A^a = \omega^a + h \delta^{ab} v_b$$

$$u^{+i} = \delta_{ij} q_j; \quad u^a = \delta^{ab} v_b; \quad u^\pm = -\frac{1}{2} u - \delta^{ab} v_a v_b \quad (5.9)$$

The numerical constants $\delta^{ij}$ and $\delta_{ij}$ were introduced by the OPE given in section 2.

Consider now the relation $u^\pm = -\frac{1}{2} u - \delta^{ab} v_a v_b$. We can substitute the definition of the currents using $u^\pm = g^{\pm} = u_-$. Recall that

$$u_- = 2\pi \frac{\delta S^{(0)}_{\text{ind}}(A)}{\delta \varphi}; \quad u = -4\pi \frac{\delta S^{(0)}_{\text{ind}}(h, \psi, \omega)}{\delta h}$$

$$q_i = -\pi \frac{\delta}{\delta \psi^i} S^{(0)}_{\text{ind}}(h, \psi, \omega); \quad v_a = \pi \frac{\delta S^{(0)}_{\text{ind}}(h, \psi, \omega)}{\delta \omega^a} \quad (5.10)$$

We obtain then

$$\left( \frac{\delta}{\delta h} - \frac{1}{2} \delta^{ab} v_b \frac{\delta}{\delta \omega^b} \right) S^{(0)}_{\text{ind}}(h, \psi, \omega) = g^{\pm \pm} = \left( \frac{\delta S^{(0)}_{\text{ind}}(A)}{\delta A^\pm} \right) \quad (5.11)$$

where the bar on the right hand side indicates that one should express all fields $A^a$ in terms of the unconstrained final fields $H = \{ A^\alpha = h, A^{-i} = \psi^i, A^a = \omega^a + h \delta^{ab} v_b \}$ after one has performed the variation with respect to $A^\alpha$. If we invert the order of variation and imposing constraints we get extra terms, due to the chain rule. Namely, we find the following equality

$$\left( \frac{\delta S^{(0)}_{\text{ind}}[A]}{\delta A^\alpha} \right) = \left( \frac{\delta S^{(0)}_{\text{ind}}[A(H)]}{\delta h} \right)$$

$$- \left( \frac{\delta S^{(0)}_{\text{ind}}[A]}{\delta A^\pm} \right) \frac{\delta A^\pm(H)}{\delta h} - \left( \frac{\delta S^{(0)}_{\text{ind}}[A]}{\delta A^a} \right) \frac{\delta A^a(H)}{\delta h} \quad (5.12)$$

There are no extra terms coming from varying $A^0$ or $A^{+i}$ with respect to $h$ because $u^0$ and $u^{-i}$ vanish due to the constraints. Moreover, using that $A^a = \omega^a + h \delta^{ab} v_b$, $u^\alpha = 1$, and $u_a = g_{ab} v_b$, we see that

$$\left( \frac{\delta}{\delta h} - \frac{1}{2} \delta^{ab} v_b \frac{\delta}{\delta \omega^b} \right) S^{(0)}_{\text{ind}}(h, \psi, \omega) = g^{\pm \pm} = \frac{\delta}{\delta h} S^{(0)}_{\text{ind}}(A[H])$$

$$- \frac{1}{2\pi} \int d^2 x \frac{\delta A^\pm}{\delta h} - g^{\pm \pm} g_{ab} \frac{1}{2\pi} \int d^2 x u^b \frac{\delta}{\delta h} (hv_a) \quad (5.13)$$

Since $g^{\pm \pm} = g_{ab} = +2\delta_{ab}$ (the positive sign is due to the supertrace), both sides can be written as a $\delta/(\delta h)$ derivative (the second term on the left hand side cancels against the extra piece from the last term on the right hand side), and we conclude that

$$S^{(0)}_{\text{ind}} = g^{\pm \pm} = S^{(0)}_{\text{ind}}(A[H]) - \frac{1}{2\pi} \int d^2 x (A^\pm + h \delta^{ab} v_a v_b) \quad (5.14)$$
The total derivative \(-\frac{1}{2}(A^\pm)''\) in \(A^\pm\), eq. (5.5), can be dropped, and replacing \(v_a\) and \(v_b\) by \(u^a\) and \(u^b\), we obtain the induced action of (N,0) supergravity in terms of the action and currents of the corresponding WZWN model.

As a check, we may repeat the derivation by using \(u^+ = \delta^{ij} q_j\) or \(u^a = \delta^{ab} v_b\). In the former case one finds

\[
\frac{\delta}{\delta \psi^i} S^{(0)}_{\text{ind}}(h, \psi, \omega) = \left( \frac{-2g^{+i,-i}}{g^{\pm,-}} \right) \left[ g^{\pm,-} S^{(0)}_{\text{ind}}(A[H]) \right]
\]

\[
- \frac{1}{2\pi} \int d^2 x \frac{\delta}{\delta \psi^a} A^\pm - \frac{1}{2\pi} (g^{\pm,-} g_{ab}) \int d^2 x v_b \frac{\delta}{\delta \psi^a} (hv_a)
\]

which agrees with (5.14) since \(g^{+i,-i}/g^{\pm,-} = -\frac{1}{2}\) (using the fact that \(\text{str} T^+_i T^+_i = 2 = -\text{str} T^-_i T^-_i\) and \(\text{str} T^+_\pm T^-_\mp = 1\)). In the latter case we may note that differentiation of \(S^{(0)}_{\text{ind}}\) with respect to \(A^b\) and then imposing constraints, differs from \((\delta S^{(0)}_{\text{ind}}/\delta c)(\delta A^c/\delta \omega^b)\) by a term proportional to \(u^c \delta \frac{\delta}{\delta \omega^b} hv_c = \frac{1}{2} \delta \frac{\delta}{\delta \omega^b} (hv_c v_d g_{cd})\). We get then

\[
S^{(0)}_{\text{ind}}(h, \psi, \omega) = \left( \frac{2g^{aa}}{g^{\pm,-}} \right) g^{\pm,-} S^{(0)}_{\text{ind}}(A[H])
\]

\[
- \frac{1}{\pi} \left( \frac{g^{aa}}{g^{\pm,-}} \right) \int d^2 x A^\pm - g^{aa} g_{cd} \frac{1}{2\pi} \int d^2 x hv_c v_d
\]

(with no summation over \(a\) in \(g^{aa}\)), which again agrees with (5.14) if we use that

\[
g^{aa}/g^{\pm,-} = \frac{1}{2}.
\]

Summarizing, the \(\sigma \to \infty\) part of the action for induced supergravity is equal to the WZWN action in which the constraints have been inserted, plus correction terms which are due to the noncommutativity of varying and imposing constraints, and which depend on the currents of the WZWN model in which the constraints have been substituted.

The method of putting constraints on the currents of a WZWN model and obtaining Ward identities for a model based on a nonlinear superalgebra can be pulled back to the algebraic level. We expect that the \(\text{U}(n)\) nonlinear superalgebra given in refs. [6,7] can be obtained in the same manner from the corresponding linear \(\text{SU}(n|2)\) superalgebras, but note that for these models no BRST charge seems to exist [10].

6 Computation of one-loop contributions to the effective action.

As explained in the introduction and in ref. [5], we obtain the one-loop contributions to the effective action by taking the determinants of the matrices \(M\) and \(N\), obtained by differentiating the \(\sigma \to \infty\) Ward identities with respect to the gauge fields. The
matrix $M$ is the covariant derivative

$$
M = \begin{pmatrix}
\nabla_+^2 & 4\omega' & -6\psi' - 2\psi \partial_-

0 & \nabla_+^1 \delta^a_b + \epsilon_{cb}^a \omega^c & \epsilon_{jk}^a \psi^k

-\frac{1}{2} \psi i & \epsilon_b^{ik}(2\psi_k' + \psi_k \partial_-) - \delta^i_b \psi^k v_k - \nu^i \psi_b & \nabla_+^2 \delta^i_j - \epsilon^i_j \omega^a
\end{pmatrix}
$$

(6.1)

where $\nabla_+^i = \partial_+ - h\partial_- - j\hbar'$.

The matrix $N_b^a$ is its dual in the sense of the introduction

$$
N = \begin{pmatrix}
\partial^2 + 2u\partial_- + u' & -4v_b \partial_- & -6q_j \partial_- - 2q_j'

v^a \partial_- + (v^a)' & \partial_- \delta^a_b - \epsilon_{bc}^a v^c & -\epsilon_{jk}^a q^k

(v^i)' + \frac{3}{2} q^i \partial_- & \epsilon_b^{ik} q_k & \left(\partial_-^2 + \frac{u}{2}\right) \delta^i_j + \nu^i v_j - \epsilon^i_j (2v^a \partial_- + (v^a)')
\end{pmatrix}
$$

(6.2)

The results for the $hh$, $\psi i \psi i$ and $\omega^a \omega^a$ self-energies are given below in (6.3). In each case one gets one contribution from $\frac{1}{2} \ln \text{sdet} M$ by evaluating a self-energy graph with two vertices, each linear in the external field and bilinear in internal ghosts, and another contribution from $-\frac{1}{2} \ln \text{sdet} N$, by evaluating the same graphs, but now with vertices linear in currents. For the currents we take the leading terms (which are linear in fields). In the loop one finds the anticommuting antighosts-ghosts $b_1, c^i, b_2a, c^{2a}$ and $B_1, C^1, B_2a, C^{2a}$, corresponding to the bosonic fields and the commuting pair $b_3i, c^{3i}$ and $B_3i, C^{3i}$ corresponding to the fermions. (We have ordered the rows and columns of $M$ and $N$ such that the fermions are in the last row and column.) We obtain

$$
\langle hh \rangle \quad \langle \omega^a \omega_a \rangle \quad \langle \psi i \psi i \rangle
$$

$$
\frac{1}{2} \ln \text{sdet} M : \quad -\frac{1}{48\pi} \hbar \frac{\partial^3}{\partial_t^3} \hbar \quad 0 \quad -\frac{1}{4p} \psi i \frac{\partial^2}{\partial_t^2} \psi i
$$

$$
-\frac{1}{8\pi} \frac{\partial^3}{\partial_t^3} \hbar \quad \frac{1}{2p} \omega^a \frac{\partial}{\partial_+} \omega_a \quad -\frac{1}{2\pi} \psi i \frac{\partial^2}{\partial_t^2} \psi i
$$

(6.3)

We comment briefly on these results. For the graviton self-energy $\langle hh \rangle$, the numerical factor in the $M$ contribution is $\frac{1}{4}(j^2 - j + \frac{1}{6})$, summed over $j = 2, \frac{3}{2}, 1$, and is a factor $-1/26$ smaller than the pure gravity case. (The sign is negative because the supersymmetry ghosts win.) The $N$ contribution is $\frac{-1}{2} j(2j + 1)(2j + 2)$ for spin $j + 1$. Hence it only comes from coordinate and supersymmetry ghosts, with contributions $-\frac{1}{2}$ and $3 \times (\frac{1}{8})$ respectively, giving a result which is a factor $1/4$ smaller than in pure gravity. The reason the Yang-Mills ghosts do not contribute to $N$ is that on dimensional grounds a coupling to the source $u$ is not possible. For the Yang-Mills self-energies, there is no contribution from $M$, because the 3 pure supersymmetry ghost loops cancel the 3 pure Yang-Mills ghost loops, while no mixed
loops with off-diagonal vertices can be constructed due to the triangular structure of the $\omega^a$ couplings in $M$. For $N$, the coordinate ghosts yield a vanishing contribution, while the Yang-Mills ghost loop yields a factor $-\frac{1}{2}$, and the supersymmetry ghost loop a factor $+1$. Finally, the gravitino selfenergy $\langle \psi^i \psi_i \rangle$ receives one $M$ contribution $\frac{5}{4}$ from the mixed $b_3, c_3, b_1, c_1$ loop, and another $M$ contribution $-\frac{3}{2}$ from the mixed $b_3, c_3, b_2, c_2$ loop. The $N$ contributions come from similar loops and are given by $-\frac{3}{2}$ and $+1$ respectively.

We can, in fact, easily find the one-loop contributions to the effective action with any number of $h$ fields

$$S_{1-loop}^{1-loop}(all \ h) = -\frac{1}{26}[-13S_{ind}^{(0)}(h)] + \frac{1}{4}[12S_{ind}^{(0)}(u)]$$

(6.4)

where

$$S_{ind}^{(0)}(h) = -\frac{1}{24\pi} \int h \frac{\partial^3}{\partial_+ h} + \cdots$$

(6.5)

is the Polyakov action in (1.8) and (1.9), and where

$$\bar{S}_{ind}^{(0)}(u) = -\frac{1}{24\pi} \int u \frac{\partial^3}{\partial_- u} + \cdots$$

(6.6)

is its dual in the sense of section 1. They are related by a Legendre transformation

$$S_{ind}^{(0)}(h) + \bar{S}_{ind}^{(0)}(u(h)) = -\frac{1}{12\pi} \int d^2 x h u(h)$$

(6.7)

Hence, in the $h$ sector we find

$$S_{eff}^{1-loop}(all \ h) = -\frac{5}{2}S_{ind}^{(0)}(h) - \frac{1}{4\pi} \int d^2 x h u$$

(6.8)

The complete one-loop effective action is obtained by an overall rescaling of the central charge $\sigma$ by a factor $Z_{\sigma}$ and rescalings of the gauge fields:

$$\sigma S_{ind}^{(0)}(h, \psi, \omega) + S_{eff}^{1-loop}(h, \psi, \omega) = Z_{\sigma} S_{ind}^{(0)}(Z_h h, Z_{\psi} \psi, Z_{\omega} \omega)$$

(6.9)

From the result in (6.8), plus those for the two-point functions, we deduce that at the one-loop order

$$Z_{\sigma} = 1 - \frac{5}{2\sigma}, \quad Z_h = 1 + \frac{11}{6\sigma}$$

$$Z_{\omega} = 1 + \frac{7}{4\sigma}, \quad Z_{\psi} = 1 + \frac{2}{\sigma}$$

(6.10)

We have repeated the calculations performed in this section for the $N=1$ and $N=2$ cases. In the $N=1$ case one simply drops the $\omega^a$ and $v^a$ fields and the index $i$ on the $\psi$ field from the $M$ and $N$ matrices, while in the $N=2$ case one has a single $\omega, v$ pair, with $\epsilon^{aij} \rightarrow \epsilon^{ij}, i, j = 1, 2$. In both cases the results are (rigidly) supersymmetric, i.e. all the fields are rescaled by a common wave-function renormalization factor, a feature which is lost in the $N=3$ case. To understand this, we are at present making a general study of rigid symmetries in models such as these.
7 Conclusions.

We have shown that by imposing constraints on the currents of a WZWN model based on the linear superalgebra Osp(N|2) one obtains the Ward identities for the induced action based on the nonlinear SO(N) superconformal algebra of Knizhnik [6] and Bershadsky [7] in the limit of large central charge. One can also find the induced action in closed form; it is nonlocal, and contains a one-component graviton, $N$ chiral gravitinos, and $\frac{1}{2}N(N-1)$ chiral Yang-Mills fields, and is a (N,0) supergravity theory in d=2 dimensions. We also computed the one-loop corrections to the self-energies of these gauge fields, and to the Green’s functions with $n$ external gravitons. They are finite, and the effective action is obtained from the induced action by a rescaling of the fields and central charge. These results are quite similar to those for $W_3$ gravity performed in ref. [3,4,5].

We emphasize that the simple relation between the effective and induced actions seems to be a consequence of working in chiral gauge, and is not so apparent when one imposes constraints on WZWN models which lead to Toda-like actions [14]. In the absence of an all-order rigorous proof of the conjectured relation between the induced action and the effective action, and its all-loop finiteness, it would be useful to calculate the two-loop corrections for our model or one of the models mentioned in the introduction. The Feynman rules and regularization of higher loops in these nonlocal chiral field theories has been discussed in ref. [15], but the usual aspects of local quantum field theory do not apply, so that many issues remain to be settled (see [15] for a detailed discussion).

We have shown that the Ward identities for an induced action in the limit that the central charge tends to infinity are of the form ”covariant derivative of current = minimal anomaly”. This led to an interesting extension of the classical gauging of algebras to the quantum level. Namely, in the classical gauging of nonlinear algebras, there appears for each gauge field a corresponding auxiliary field [12], but the local gauge algebra only closes on these auxiliary fields, whereas on the gauge fields one finds extra terms proportional to the covariant derivatives of the auxiliary fields. By identifying these auxiliary fields with the currents and adding the minimal anomalies as quantum corrections to the classical transformation rules of the currents, one obtains an extra term in the gauge commutator on the gauge fields, which cancels the covariant derivative of the current, so that the classical nonclosure turns into quantum closure. The reason for this remarkable cancellation is quite general (it was already found to hold for $W_3$ gravity [3]): the covariant derivative of the auxiliary fields is the Ward identity minus the minimal anomaly, since the auxiliary fields are nothing else but the currents. This quantum improvement of a classical imperfect theory suggests further interesting possibilities to which we hope to return.

Because of the presence of the factor 1/2 in the covariant derivative of the currents $D_\mu T_A$ in (4.6), but absence of a corresponding factor 1/2 in the gauge field variations $\delta h^A_\mu$ in (4.4), the response of the induced action under these variations $\delta h^A_\mu$ is the minimal anomaly minus the product of the nonlinear terms in the Ward identity times the gauge parameter. As observed in [3], if one would halve the nonlinear terms in $\delta h^A_\mu$ one would completely cancel the nonlinear terms in the anomaly,
but as we have explained here, only the $\delta h^A_\mu$ in (4.4) which came from the gauging of nonlinear algebras will lead to a closed gauge algebra.

A final comment concerns the integrability conditions of the differentiated Ward identities. Since $\delta u_{-a}/\delta A^b_\mu = M^{-1}c^b_a$ in (1.16) is symmetric in $a, b$ by virtue of the definition of the currents $u_{-a}$, the integrability conditions read $M^{-1}N = NT M^{-1,T}$, where the derivatives in $N^T$ and $M^{-1,T}$ act to the left. Partially integrating them, one obtains operators $N^t$ and $M^{-1,t}$ in which all derivatives act to the right. Hence one obtains the conditions

$$MN^t - NM^t = 0 \quad (7.1)$$

For induced Yang-Mills theory, $M = D_+(A) = -M^t$ and $N = D_-(u) = -N^t$, so that one obtains $[D_+(A), D_-(u)] = R_{+,-} = 0$, which is the well-known parallelizability of the WZWN model. For Polyakov gravity $N = -N^t = D_1 = \partial^2 + 2u\partial_+ + u'$, but $M = \nabla^2_+$ and $M^t = -\nabla^{-1}_+$, where $\nabla^j_+ = \partial_+ - h\partial_+ - jh'$. The integrability conditions now yield (with $D_+u \equiv \nabla^2_+ u$)

$$2(D_+ u - h'''\partial_+ + (D_+ u - h'''')' = 0 \quad (7.2)$$

which is indeed satisfied as long as the Ward identity $D_+ u - h''' = 0$ is satisfied. More generally, for nonlinear (super)algebras, the consistency conditions are obtained by replacing each current $T_B$ in the matrix $N$ by the corresponding Ward identity. Hence, using (4.6)

$$[D_+ T_C - (Anomaly)_{+,-}] \tilde{f}^C_{AB} = 0 \quad (7.3)$$

where $\tilde{f}^C_{AB} = f^C_{AB} + T_D V^{DC}_{AB}$ and $(Anomaly)_{+,-}$ is the minimal anomaly. In practice (7.1) yields a good check on the Ward identities. One may also view it as a quantum curvature for chiral gauge theories, which replaces the classical curvature $R^A_{\mu\nu} = \partial_\mu h^A_\nu - \partial_\nu h^A_\mu + \tilde{f}^A_{BC} h^B_\mu h^C_\nu$ of nonchiral gauge theories proposed in [12].

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