The gaussian propagator formalism and the determination of the leading Regge trajectory for $\phi^3$ field theory

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Abstract

As the number of loops goes to infinity Feynman $\alpha$-parameters undergo a fixing mechanism which entails a gaussian representation for propagators in scalar field theories. Here, we describe this mechanism in the fullest detail. The fixed values are in fact mean-values which can be determined via consistency conditions. The consistency conditions imply that one $\alpha$-parameter is integrated in the usual way and the dependence of the mean-values of the other $\alpha$-parameters on it must be determined. Here we present a method for doing this exactly which requires the solution of an equation system. We present an analytic solution for this equation system in the case of the ladder-graph topology. The Regge behaviour is obtained in a simple way as well as an analytic expression for the leading Regge trajectory. Then, the consistency equations for the two (in the ladder case) independent $\alpha$-parameters mean-values are solved numerically. Agreement with previous determinations of the intercept $\alpha(0)$ is obtained for $\alpha(0) \gtrsim 0.3$. However, we are able to calculate $\alpha(t/m^2)$ for $-3.6 \lesssim t/m^2 \lesssim 1.8$ and find that it is close to linear.

We consider the massless limit of the theory and find that the $\alpha$-parameters mean-values and the trajectory $\alpha(t)$ have limits which are independent of the mass, a phenomenon which also occurs for renormalizable theories via the renormalization group equations.

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1 Introduction

Methods for calculating in field theory are something of paramount importance, allowing in particular to understand the dynamical structure of interactions in particle physics. Some are non-perturbative like instantons [1] and lattices [2]. Instantons help to elucidate the vacuum structure and lattices up to now are confined to calculating masses and couplings. So we can consider that, in the near future, those ways of calculating will be restricted to very specific sectors of field theories. Recently, the holomorphic unraveling [3] of the properties of supersymmetric QCD has aroused interest. But, of course, one of the most important unattained goals is finding a way of dealing with QCD itself. Due to its asymptotic freedom, perturbation is useful when high momenta transfers are involved. However, we would also like to have a control on QCD when couplings are not small as is the case in soft physics. In some cases, the leading log approximation has been used, for instance in the calculation of the Pomeron trajectory [4]. But we are far from something satisfactory because the sub-leading level, if we want to take it into account, forces an enormous amount of work to be done, if only possible. Another method has been advocated, some years ago [5], using string theory to derive one-loop multi-gluons amplitudes taking the limit where $\alpha'$, the Regge slope or the inverse string tension, tends to zero. One of its most salient features is that this technique can be translated into a set of rules which requires no knowledge of string theory. Most strikingly, $\phi^3$-like diagrams only [5] are required to evaluate the kinematic factors. Indeed, each kinematic factor is the loop-integral that one would expect from a $\phi^3$ zero mass scalar field theory, expressed in terms of the well-known Feynman $\alpha$-parameters, with a factor taking into account the specificity of QCD. This factor called a reduced kinematic factor takes into account external polarizations and on-shell external momenta. At the one-loop level it contains only terms linear in the $\alpha$-parameters or containing no $\alpha$-parameter.

Calculations were made explicitly for one-loop $\phi^3$ $n$-gluons amplitudes but the field theory correspondence with string theory is expected to hold for an arbitrary number of loops with three gluon vertices.

Here, we shall expose a method for calculating Feynman graphs with an infinite number of loops for scalar massive $\phi^3$ field theory that we will extend to the massless case. Multiplying the integrand by the appropriate reduced kinematic factor then
yields an extension of our method for calculating amplitudes with an infinite number of loops in QCD. In fact, were not for the additional $\alpha$-parameter dependence of the reduced kinematic factor, all the calculations made here for the Regge behaviour and the determination of the Regge trajectory would be valid for the same objects in QCD.

We start from an expression [7] of 1-particle, 1 vertex-irreducible, Euclidean Feynman graphs amplitudes in terms of the well-known $\alpha$-parameters. We separate an overall scale which is integrated over separately. This integration controls the divergence of the amplitudes. The rescaled $\alpha$-parameters are then integrated over by using the mean-value theorem. When the number of $\alpha$’s tends to infinity, which is precisely the limit we are interested in here, the mean-value theorem has an important consequence: the mean-values $\bar{\alpha}_i$ of the $\alpha$-parameters have to be of order $1/I$ if $I$ is the number of propagators (and $\alpha$-parameters) of the graph. Moreover, each $\bar{\alpha}_i$ can be determined by a consistency equation obtained by equating the value of the amplitude obtained by using the mean-value theorem for all $I$ $\alpha$-parameters and the value of the same amplitude obtained by integrating with the mean-value theorem over $I-1$ $\alpha_j$-parameters, $j \neq i$, and integrating in the usual way over the last ($\alpha_i$) $\alpha$-parameter. Of course, in principle there are as many consistency equations to solve as there are $\alpha$-parameters. However, when some symmetry exists in the topology of the graph, the number of consistency equations can be greatly reduced. The simplest topology that one can imagine with an infinite number of loops is the ladder-graph topology in which only two independent $\bar{\alpha}_i$’s survive. This is, of course, a very interesting topology because it leads to Reggeisation [8, 9] when the invariant $s \to \infty$. In the present article we deal with the two consistency equations for $\bar{\alpha}_-$ and $\bar{\alpha}_+$, which are respectively the mean-value for the $\alpha$-parameter of the central rungs of the ladder and the mean-value for the $\alpha$-parameter of the side rungs. The formalism we develop here gives the amplitude under a compact form [10] where no integration is left-over. This compactness is very useful. Knowing the $\bar{\alpha}_i$’s, we simply plug in their values in the amplitude expression to get the amplitude’s value. In this way, we got a compact expression [11] for the leading Regge trajectory which is analytic as a function of $\bar{\alpha}_+$ and $\bar{\alpha}_-$. The consistency equations, when solved, also give for $\bar{\alpha}_+$ and $\bar{\alpha}_-$ analytic functions of $t$, the other invariant, $\gamma$, the coupling.
constant, and \( m \) the mass of the theory.

The other important consequence of compactness is that we can easily derive the massless limit of the theory. In the expression for the amplitude, the mass \( m \) only appears in one place in the combination

\[
Q_G(P,\{\tilde{\alpha}\}) + h_0 \, m^2
\]

where \( h_0 \) is the sum of the rescaled \( \alpha_i \)'s, \( Q_G \) is the ratio of two homogeneous polynomials of degree \( L + 1 \) and \( L \) in the \( \bar{\alpha}_i \)'s, \( L \) being the number of loops of the diagram considered. The polynomial in the numerator of \( Q_G \) also depends linearly on \( s \) and \( t \) (or in general any invariant) via a form quadratic in the external momenta \( P_j \). So, letting \( m \to 0 \) and obtaining the resulting amplitude is rather trivial! Of course, we have to introduce a mass in order to avoid infrared problems in the definition of the Feynman graph amplitudes, but once this is done, one can calculate everything as a function of \( m \) and let \( m \) tend to zero.

This is what we will do here for the Regge trajectory which has an \( m \to 0 \) limit independent of \( m \) as well as the consistency equations for \( \bar{\alpha}_+ \) and \( \bar{\alpha}_- \). We remark that this phenomenon is close in spirit to what happens in the renormalization group \([12]\) when the mass dependence also disappears when momenta are taken to go to infinity. This possibility of having a massless limit independent of the mass was already contained in our first paper \([10]\) proving the exactness of the Gaussian representation for propagators and where a compact expression for amplitudes was already given. (Replacing \( \alpha_i \) by \( \bar{\alpha}_i \) gives a Gaussian representation for the propagator \( i \)).

In section 2 we give the basics of the \( \alpha \)-parameter representation for Euclidean scalar massive \( \phi^3 \) field theory. The mean-value theorem allows us to give the amplitude for any diagram in a compact form where no integration is left over. The integration over an overall scale for the \( \alpha_i \)'s is done separately and controls the possible ultraviolet divergences via a gamma function factor \( \Gamma(I - dL/2) \) where \( d \) is the dimension of space-time. A consistency equation for \( \bar{\alpha}_i \) is obtained by using the mean-value theorem for all \( \alpha_j, \, j \neq i \) and integrating normally on \( \alpha_i \), thus giving an expression for the diagram amplitude where \( \bar{\alpha}_i \) does not appear. Consistency requires that this expression is equal to the expression where the mean-value theorem
is used for all $I$ $\alpha$-parameters, including $\alpha_i$.

Because of the constraint $\sum_\ell \alpha_\ell = h_0$ the phase-space allowed for each $\alpha_\ell$ is of order $h_0/I$. Then, assuming $\bar{\alpha}_j = O(h_0/I)$, the consistency equation naturally introduces a scale $\Lambda^{-1}$ where $\Lambda$ is proportional to $(I - dL/2)/h_0$ such that we expect $\bar{\alpha}_i \Lambda$ to be of order one.

In the consistency equation a parameter $\mu_i$, defined in (1.2) below, which is of degree one in the $\bar{\alpha}_j$’s, appears added to $\alpha_i$ and $\bar{\alpha}_i$. Because $\mu_i$ is of degree one in the $\bar{\alpha}_j$’s we also expect $\mu_i$ to be of order $h_0/I$. If we assume this, then the consistency equation gives $\bar{\alpha}_i = O(h_0/I)$ proving consistency. It is shown that if $\mu_i \Lambda \to 0$, i.e. if $\mu_i$ is decreasing faster than $(h_0/I)$, the consistency equation entails that $\bar{\alpha}_i/\mu_i \to \infty$ breaking the expected proportionality between $\bar{\alpha}_i$ and $\mu_i$. Assuming $\mu_\ell \Lambda \to \infty$ for a non-infinitesimal fraction of the $\mu_\ell$’s is also forbidden because this would violate the constraint $\sum_\ell \bar{\alpha}_\ell = h_0$. Therefore, only when $\mu_i \Lambda = O(1)$ is consistency achieved, thereby showing that only $\bar{\alpha}_i \Lambda = O(1)$ or $\bar{\alpha}_i = O(h_0/I)$ is consistent. It is also proved that even if some $\bar{\alpha}_j$ are assumed to be decreasing faster than $h_0/I$ the consistency equation still gives that $\bar{\alpha}_i$ should be $O(h_0/I)$. In the Appendix a proof is given, independent of the precedent arguments, that $\mu_i$ cannot decrease faster than $h_0/I$ using a continuous fraction expression for $\mu_i$ as a function of the $\bar{\alpha}_j$’s.

In section 3 we specialize to infinite ladder-diagrams and calculate the sum of their amplitudes through a saddle-point estimate. We find that the value of $L$ at the saddle-point is proportional to $\ell n(s/s_0)$ where $s_0$ is some finite scale. This is a very natural result expected in any parton-like or multiperipheral model, giving a multiplicity proportional to the rapidity. An explicit analytic expression for the leading Regge trajectory is obtained giving a function of $\gamma$, $m$, $\bar{\alpha}_+$ and $\bar{\alpha}_-$ and $\Lambda$, $\bar{\alpha}_+$ being the mean-value of the $\alpha$-parameters along the sides of the ladder and $\bar{\alpha}_-$ the mean-value of the $\alpha$-parameters of the central rungs. The $m \to 0$ limit of the Regge trajectory is examined and is shown to be independent of $m$ if $\bar{\alpha}_+, \bar{\alpha}_-$ and $\Lambda$ also have $m$-independent limits, which will be proved in section 9.

We discuss the dependence of $\bar{\alpha}_j$ on $\alpha_i$ when $\alpha_i$ is left free to vary. $Q_G(P, \{\bar{\alpha}\})$ also varies because $\bar{\alpha}_j \neq \alpha_i$ when $\alpha_i$ varies. These two dependencies on $\alpha_i$ were not taken into account in the consistency equation written in section 2, which therefore was a simplified version of the true consistency equation. These dependencies
introduce correction factors which will be calculated in the next sections.

In section 4 we derive the explicit dependence of \( Q_G(P, \{ \bar{\alpha}_j \}_{j \neq i}, \alpha_i) \) on \( \alpha_i \) distinguishing the two cases where \( \alpha_i \) is an \( \alpha_+ \)-parameter or an \( \alpha_- \)-parameter. That dependence introduces a factor \( H_i(\alpha_i - \bar{\alpha}_i) \) in the integrand of the consistency equation which is explicitly evaluated in both cases, + and −.

In section 5 the dependence of \( P_G(\{ \bar{\alpha} \}) \) on \( \alpha_i \) is examined for the factor \( b_i(\{ \alpha \}_{j \neq i}) \) if \( P_G(\{ I \}) \) is a homogeneous polynomial of degree \( L \) in the \( \alpha_i \)'s written when \( \alpha_i \) varies

\[
P_G(\{ \bar{\alpha}_j \}_{j \neq i}, \alpha_i) = b_i(\{ \bar{\alpha} \}_{j \neq i})(\mu_i + \alpha_i)
\]

(1.2)

The polynomial \( P_G(\{ \bar{\alpha} \}) \) intervenes explicitly in the compact expression for the diagram amplitude. In the consistency equation written in section 2 only the \( (\mu_i + \alpha_i) \) part of \( P_G(\{ \bar{\alpha} \}) \) depends on \( \alpha_i \). As only relative variations are useful the variable \( \text{Log} P(\{ \bar{\alpha} \}) \) is used and assimilated to \( \text{Log} b_i(\{ \bar{\alpha} \}_{j \neq i}) \) both quantities differing only infinitesimally in their variations. The \( \bar{\alpha}_j(\alpha_i) \) dependence then introduces an additional factor \( H_0(\alpha_i - \bar{\alpha}_i) \) in the integrand of the consistency equation.

Then, a general method for finding \( \bar{\alpha}_j(\alpha_i) \) is given which relies on the evaluation of a particular \( \bar{\alpha}_\ell \) as a function of \( \alpha_i \). For that purpose the \( \alpha_\ell \) variable must also be left free to vary in order to write a consistency equation for \( \bar{\alpha}_\ell \). Then, if terms proportional to \( \alpha_i \alpha_\ell \) are neglected the consistency equation written by letting \( \alpha_i \) and \( \alpha_\ell \) vary together factorizes into two independent consistency equations which are each identical to the one written by letting only one variable \( \alpha_i \) or \( \alpha_\ell \) vary. Then, we examine the source of the \( \alpha_i \alpha_\ell \) terms and give an explicit expression for a factor containing all of them in the consistency equation where both \( \alpha_i \) and \( \alpha_\ell \) are free to vary. This factor implies that the scale \( \Lambda \) is “renormalized” into a scale \( \Lambda_\ell^R = \Lambda + d/2 LE_\ell^0 + \Lambda_\ell(\alpha_i) \) for the \( \alpha \)-parameter mean-value \( \bar{\alpha}_\ell \), where \( E_\ell^0 \) is some constant and \( \Lambda_\ell(\alpha_i) \) a term of order one (and therefore infinitesimal with respect to \( \Lambda \) and \( d/2LE_\ell^0 \) containing the dependence of the scale on \( \alpha_i \). The constant \( E_\ell^0 \) comes from the factor \( H_0(\alpha_i - \bar{\alpha}_i) \) discussed above. If, however, the \( \alpha_i \alpha_\ell \) terms were neglected these would be no \( \bar{\alpha}_\ell(\alpha_i) \) or \( \bar{\alpha}_j(\alpha_i) \) dependence and \( E_\ell^0 \) would be zero. In fact, now, the problem is to derive \( E_\ell^0 \) knowing \( \Lambda_\ell(\alpha_i) \).

In section 6 we tackle the problem of determining \( E_\ell^0 \) and \( \Lambda_\ell(\alpha_i) \). As \( \alpha_i \) varies \( \Lambda_\ell^R \) will vary and so does \( \bar{\alpha}_\ell \). Differentiating the consistency condition written for
\( \bar{\alpha}_\ell \) allows to write a relation tying the variation of \( \Lambda_R^\ell \) to that of \( \bar{\alpha}_\ell \)

\[
\frac{\delta \bar{\alpha}_\ell}{\bar{\alpha}_\ell} = A_\ell \frac{\delta \Lambda_R^\ell}{\Lambda_R^\ell}
\]  

(1.3)

where \( A_\ell \) is calculated as a function of several quantities (\( \ell \) can be of the + or \( - \) kind): \( y, \delta \Lambda_R^\ell/\Lambda_R^\ell, \delta \Lambda_R^\ell/\Lambda_R^\ell, \), \( z_\pm, \zeta_\pm \) and \( \varepsilon_\pm \) which we now briefly describe. \( y \) is \((2\bar{\alpha}_\mu/\bar{\alpha}_\ell)^{1/2}\), \( z_\ell \) is proportional to \( \partial \log (J_{\ell}^{-1}H_\ell \bar{H}_0)/\partial (\bar{\alpha}_\ell \Lambda_R^\ell_0) \), \( \zeta_\ell \) is proportional to \( \partial \log (J_{\ell}^{-1}H_\ell \bar{H}_0)/\partial (\mu_\ell \Lambda_R^\ell) \), \( \varepsilon_\ell \) is equal to one minus a weighted “mean-value” of \((\mu_\ell + \bar{\alpha}_\ell)/(\mu_\ell + \alpha_\ell)\), this mean-value being obtained by inserting \((\mu_\ell + \bar{\alpha}_\ell)/(\mu_\ell + \alpha_\ell)\) into the integrand of the integral of the consistency equation for \( \bar{\alpha}_\ell \) (the consistency equation is written as one minus the integral equal zero). \( J_\ell \) is a Jacobian allowing to change the integration variable from \( \alpha_\ell \) to \( \alpha_\ell \Lambda_R^\ell \). \( \bar{H}_0 \) is the factor in \( H_0(\alpha_\ell - \bar{\alpha}_\ell) \) which does not depend on \( \alpha_\ell \) and it is equal to \( \exp(d/2 E_0^\ell \bar{E}_0^\ell) \). \( H_\ell \) is the factor described in section 4 which takes into account the effect on the consistency integral of the variation of \( Q_G(P, \{\bar{\alpha}\}_{j\neq \ell}, \alpha_\ell) \). Equation (1.3) is essential as it will allow us to determine the dependence of \( \bar{\alpha}_\ell \) on \( \alpha_\ell \) and find \( E_0^\ell \).

The determination of \( \Lambda_\ell(\alpha_i) \) is made by regrouping all terms in the consistency equation proportional to \( \alpha_i \alpha_\ell \) and writing them down as

\[
\alpha_\ell \Lambda_\ell(\alpha_i) + \alpha_i \Lambda_i(\alpha_\ell) = F_i^\ell(\alpha_i, \alpha_\ell)
\]

(1.4)

The decomposition (1.4) is not unique, even if we take into account possible symmetry constraints and demand moreover that \( \Lambda_\ell(\alpha_i) \) should be the same, \( \alpha_\ell \) taking the value \( \bar{\alpha}_+ \) or \( \bar{\alpha}_- \). We are led to introduce five parameters in order to take into account this non-uniqueness. However, two constraints are imposed on them by requiring \( \Lambda_\ell(\bar{\alpha}_i) \) to be independent of \( i \) being + or –.

We introduce the variables

\[
x_{\ell i} = \bar{\alpha}_\ell^{-1} d\bar{\alpha}_\ell/d\alpha_i
\]

(1.5)

and the equation (1.2) can be expressed as a system of equations for these variables because

\[
d\Lambda_R^\ell/d\alpha_i = d\Lambda/d\alpha_i + d/2 L d E_0^\ell/d\alpha_i + d\Lambda_\ell(\alpha_i)/d\alpha_i
\]

(1.6)
where $d\Lambda/d\alpha_i$ is shown to be proportional to $x_{+i}$, and $E^0_\ell$ is shown to be a linear combination of $x_{-\ell}$ and $x_{+\ell}$. Then, $d\Lambda_\ell(\alpha_i)/d\alpha_i$ plays the role of the inhomogeneous term determined from the $\alpha_i\alpha_\ell$ terms. So we get a system of four equations for the four variables $x_{++}$, $x_{+-}$, $x_{-+}$ and $x_{-}$.

This system is solved and two compatibility conditions emerge,

\[ dE^0_+/dy = 0 \quad ; \quad dE^0_-/dy = 0 \]  

(1.7)

$E^0_+$ and $E^0_-$ are explicitly determined.

In section 7, explicit expressions for the quantities $J_\ell$, $z_\ell$, $\zeta_\ell$, $\delta \Lambda_\ell(\alpha_i)$ and $d\varepsilon_\ell/dy$ ($\ell = \pm$) are given. They all enter in the consistency equations determining $\bar{\alpha}_+$ and $\bar{\alpha}_-$. Then, explicit expressions for $\delta \Lambda_\ell^R(\alpha_i)/\Lambda_\ell^R(\alpha_i)$ are also given, these quantities entering in the definition of $A_\ell$ in (1.3).

In section 8, we study the consistency of the assumption $\bar{\alpha}_j = O(h_0/I)$ in the realm of the complete consistency conditions for $\bar{\alpha}_+$ and $\bar{\alpha}_-$. This implies replacing $\Lambda$ by $\Lambda^R$ and taking into account the effect of the factors $J_\ell^{-1}$, $H_\ell(\alpha_i - \bar{\alpha}_i)$ and $\tilde{H}_0(\bar{\alpha}_i)$. This was not done in section 2 for simplicity and pedagogical reasons.

We conclude that this change consists in introducing in the integrand a bounded function $B_i(x)$, $x$ being the integration variable and therefore that the consistency arguments developed in section 2 still hold. Then, $\bar{\alpha}_i$ should be $O(h_0/I)$.

In section 9, we examine the massless limit of the consistency equations. We find that the quantities appearing in them have definite $m$ independent limits and therefore that $\bar{\alpha}_i$ has a massless limit which is independent of $m$. This entails that the whole scheme is also $m$-independent when $m \to 0$.

In section 10, we present numerical solutions of the consistency equations and the compatibility conditions. The trajectory $\alpha(t/m^2)$ is determined for $-3.6 < t/m^2 < 1.8$ (in Minkowski space). The numerical solutions are difficult to obtain, due to chaotic effects because many quantities are determined through calculations loops. Nevertheless, we obtain results with a reasonable precision using a very performant minimization algorithm. The results show an agreement for the trajectory intercept $\alpha(0)$ with previous determinations (which were assuming massless fields for the central rungs of the ladder) for the range $0.3 \lesssim \alpha(0) \lesssim 1.6$. Previously no result was, to our knowledge, available at $t \neq 0$. We find, with an improvement over a
previous determination [11], that the leading trajectory $\alpha(t/m^2)$ is compatible with linearity in the range where we could determine it.

Section 11 will be the conclusion.

2 Basics of the $\alpha$-parameter representation

Let us start with a 1-line irreducible, 1-vertex irreducible graph $G$ with $L$ loops, $I$ propagators in $d$ dimensions and let us give its amplitude $F_G$ in Euclidean space when the coupling is equal to $-1$. We have [7]

$$F_G = (4\pi)^{-dL/2} h_0 \int_0^{h_0} d\alpha_i \delta \left( h_0 - \sum_i \alpha_i \right)$$

$$[P_G(\{\alpha\})]^{-d/2} \int_0^{\infty} d\lambda \lambda^{d/2} L \exp \left\{ -\lambda \left[ Q_G(P, \{\alpha\}) + m^2 h_0 \right] \right\} \tag{2.1}$$

where $P_G(\alpha)$ is a homogeneous polynomial of degree $L$ defined as

$$P_G(\{\alpha\}) = \sum_T \prod_{\ell \not\subset T} \alpha_\ell \tag{2.2}$$

with a sum over all spanning trees $T$ of $G$. We recall that a spanning tree has to be incident with all vertices of $G$. We have

$$Q_G(P, \{\alpha\}) = [P_G(\{\alpha\})]^{-1} \sum_C s_C \prod_{\ell \subset C} \alpha_\ell \tag{2.3}$$

where the sum runs over all cuts $C$ of $L + 1$ lines that divide $G$ into two connected parts $G_1(C)$ and $G_2(C)$ with

$$s_C = \left( \sum_{v \in G_1(C)} P_v \right)^2 = \left( \sum_{v \in G_2(C)} P_v \right)^2 \tag{2.4}$$

the external momenta $P_v$ being associated with external lines of $G$. As a cut $C$ is the complement of a spanning tree $T$ plus one propagator, $\prod_{\ell \in T} \alpha_\ell$ in (2.3) is of degree $L + 1$. Integrating over $\lambda$ in (2.1) gives a convergent integral for $\phi^3$ when $d < 6$. The divergence at $d = 6$ is consistent with the fact that $\phi^3$ is renormalizable for this value of $d$. In what follows we will choose $d < 6$ and in fact most of time $d = 4$. Using the mean-value theorem for the $I$ variables $\alpha_i$ in (2.1) we obtain, integrating also on $\lambda$,
\[ F_G = (4\pi)^{-dL/2} h_0 \left[ P_G(\{\bar{\alpha}\})\right]^{-d/2} \left[ Q_G(P, \{\bar{\alpha}\}) + m^2 h_0 \right]^{-(I-d/2) L} \cdot \Gamma(I - d/2 L - 1) \]

the factor \( h_0^{L-1}/(I - 1) \) representing the phase-space volume for \( I \) variables \( \alpha_i \). \( F_G \) is then given under a compact form where no integral subsists. We remark that \( h_0 \) is a free parameter representing the sum of all \( \bar{\alpha}_i \)'s because of the constraint \( h_0 = \sum_i \alpha_i \) in (2.1) and \( F_G \) should not depend on it.

Our next step will consist in giving a way of determining \( \bar{\alpha}_i \). This will be done through a consistency equation which is obtained by using the mean-value theorem for \( I - 1 \) variables \( \alpha_j \), letting \( \alpha_i \) unintegrated. For that purpose we need to isolate the dependence on \( \alpha_i \) of \( P_G(\{\alpha\}) \) and \( Q_G(P, \{\bar{\alpha}\}) \). Therefore we write:

\[ P_G(\{\alpha\}) = a_i + b_i \alpha_i = b_i (a_i/b_i + \alpha_i) \quad (2.6a) \]

\[ \sum_C s_C \prod_{i \in C} \alpha_i = d_i + e_i \alpha_i = e_i (d_i/e_i + \alpha_i) \quad (2.6b) \]

where \( a_i \) and \( b_i \) are respectively polynomials of degree \( L \) and \( L - 1 \) in the \( \bar{\alpha}_j \)'s, \( d_i \) and \( e_i \) being polynomials of degree \( L + 1 \) and \( L \) in the \( \bar{\alpha}_j \)'s. We demonstrated also that

\[ a_i/b_i = d_i/e_i \quad (2.7) \]

should be equal up to terms vanishing as \( I \to \infty \). Therefore we get

\[ Q_G(P, \{\bar{\alpha}\}) = e_i/b_i \quad (2.8) \]

which does not depend anymore explicitly on \( \alpha_i \). \( Q_G(P, \{\bar{\alpha}\}) \) is then homogeneous to one power of \( \bar{\alpha}_j \). (In \( Q_G(P, \{\bar{\alpha}\}) \) every \( \alpha_j \) is replaced by \( \bar{\alpha}_j \)).

The equality (2.7) can be understood by saying that the cutting of a tree \( T \) far from a propagator \( i \) will bring up the same factor for trees going through \( i \) and trees not going through \( i \). That is, for infinitely large graphs there is a factorization of
spanning trees on domains which are far apart on $G$ in terms of the minimum number of propagators separating them. Now, due to the constraint given by $\delta(h_0 - \sum_i \alpha_i)$,

$$h_0 - \alpha_i = \sum_{j \neq i} \bar{\alpha}_j$$

(2.9)

when taking the mean-values of $I-1$ variables $\alpha_j$, which gives a rescaling of all $\bar{\alpha}_j$’s when $\alpha_i$ varies. Then, (2.9) let us think that we can write

$$\bar{\alpha}_j = O\left[\frac{(h_0 - \alpha_i)}{(I-1)}\right].$$

(2.10)

This is consistent with the fact that the phase-space for $(I-1)$ variables can be written

$$(h_0 - \alpha_i)^{I-2}/(I-2)! \sim [e(h_0 - \alpha_i)/I]^{I-1}$$

(2.11)

which leaves a phase-space of order $h_0/I$ for each $\bar{\alpha}_j$. Taking into account (2.10) we can express $a_i$, $b_i$ and $Q_G(P, \{\bar{\alpha}\})$ as

$$a_i = a_{i0} \left[\frac{(h_0 - \alpha_i)}{(I-1)}\right]^L$$

(2.12a)

$$b_i = b_{i0} \left[\frac{(h_0 - \alpha_i)}{(I-1)}\right]^{L-1}$$

(2.12b)

$$Q_G(P, \{\bar{\alpha}\}) = Q_G_0(h_0 - \alpha_i)/(I-1).$$

(2.12c)

In a first approximation we will consider $a_{i0}$, $b_{i0}$ and $Q_G_0$ constant as $\alpha_i$ differs from $\bar{\alpha}_i$. (In section 5 we shall take them varying with $\alpha_i$, but here we want to present the matter in a first approximation as simply as possible). Consequently, we have as $I, L \to \infty$
\[ F_G = (4\pi)^{-dL/2} h_0 \int_0^{h_0} d\alpha_i \left[ (h_0 - \alpha_i)^{I-2}/(I-2)! \right] b_i^{d/2} (a_i/b_i + \alpha_i)^{-d/2} \]

\[ \Gamma(I - dL/2) \left[ Q_{G0}(h_0 - \alpha_i)/(I - 1) + h_0 m^2 \right]^{-(I - dL/2)} \]

\[ = (4\pi)^{-dL/2} h_0 \Gamma(I - dL/2) \left[ h_0^{I-2}/(I - 2)! \right] \left[ b_{G0}(h_0/I)^{L-1} \right]^{-d/2}. \]

\[ \exp \left\{ -dL/(2I) - (1 - dL/(2I)) \left[ Q_{G0}(h_0/I)/ \left[ Q_{G0}(h_0/I) + h_0 m^2 \right] \right] \right\} \]

\[ \int_0^{h_0} d\alpha_i (a_i/b_i + \alpha_i)^{-d/2} \exp \left[ -(I - dL/2)(1 - \beta)\alpha_i/h_0 \right] \]

(2.13)

where \(1 - \beta\) is defined as

\[ 1 - \beta \equiv \left[ 1 + Q_{G0}(h_0/I)/(h_0 m^2) \right]^{-1} = \left[ 1 + Q_{G}(P, \{\bar{\alpha}\})/h_0 m^2 \right]^{-1} \]

(2.14)

\[ P_G(\{\bar{\alpha}\}) \text{ in } (2.3) \text{ is } P_G(\{\alpha\}) \text{ with every } \alpha_j \text{ replaced by } \bar{\alpha}_j \text{ and } \]

\[ P_G(\{\bar{\alpha}\}) = (a_i/b_i + \bar{\alpha}_i) b_{G0}(h_0/I)^{L-1} \]

(2.15)

because \(\bar{\alpha}_j = O(h_0/I)\) when mean-values of \(I\) variables are taken. Equating expression (2.5) and (2.13) for \(F_G\) we get

\[ (\mu_i + \bar{\alpha}_i)^{-d/2} = \exp \left[ -dL/(2I) - (1 - dL/(2I))\beta \right] \]

\[ [(I - 1)/h_0] \int_0^{h_0} d\alpha_i (\mu_i + \alpha_i)^{-d/2} \exp(-\alpha_i \Lambda) \]

(2.16)

with \(\Lambda\) and \(\mu_i\) defined as

\[ \Lambda \equiv (I - dL/2)(1 - \beta)/h_0 \]

(2.17)

\[ \mu_i \equiv a_i/b_i \]

(2.18)

Now, depending on the size of \(\mu_i\), let us see what \(\bar{\alpha}_i\) should be. \(\mu_i\) is homogeneous to one power of \(\bar{\alpha}_j\) and therefore, according to our guess (2.10), we should have \(\mu_i \Lambda = O(1)\) or
\[ \mu_i = O(h_0/I) \]  

(2.19)

In the Appendix we give an argument for \( \mu_i \) satisfying (2.19) (the ratio \( \mu_i/\bar{\alpha}_i \) is, in fact, the ratio of the sum of weights of spanning trees going through \( i \) to the sum of weights of spanning trees not going through \( i \), any propagator \( j \) being weighted by \( \bar{\alpha}_j^{-1} \)).

Then, writing the integral in (2.16) as

\[ \Lambda^{-1} \int_{O}^{\infty} d(\alpha_i \Lambda) \exp(-\alpha_i \Lambda) \left( \mu_i + (\alpha_i \Lambda)/\Lambda \right)^{-d/2} \]

\[ = \Lambda^{-1+d/2} \exp(\mu_i \Lambda) \int_{\mu_i \Lambda}^{\infty} dx \exp(-x)x^{-d/2} \]

we obtain

\[ (\mu_i + \bar{\alpha}_i)^{-d/2} = \exp \left[ -dL/(2I) - (1 - dL/(2I)) \beta \right] \]

\[ [(I - 1)/(h_0 \Lambda)] \Lambda^{d/2} \exp(\mu_i \Lambda) \int_{\mu_i \Lambda}^{\infty} dx \exp(-x)x^{-d/2} \]  

(2.21)

where

\[ \beta = \left[ Q_G/(h_0 m^2) \right] / \left[ 1 + Q_G/(h_0 m^2) \right] \]  

(2.22)

is \( O(1) \). Then, the only factor which is not bounded by a constant on the right-hand side of (2.21) is \( \Lambda^{d/2} \) and thus

\[ \mu_i + \bar{\alpha}_i = O(\Lambda^{-1}) \]  

(2.23)

which means that \( \bar{\alpha}_i \) should also be \( O(\Lambda^{-1}) \), demonstrating the consistency of our assumption (2.10).

We can go further and ask what happens when we assume that \( \mu_i \Lambda \to 0 \). Then, the integral in (2.21) is dominated by its contribution near its lower boundary and behaves like

\[ (\mu_i \Lambda)^{1-d/2} \exp(-\mu_i \Lambda) \]  

(2.24)

which leads to

\[ (\mu_i + \bar{\alpha}_i)^{-d/2} \sim \mu_i \bar{I} \mu_i^{-d/2} \]  

(2.25)
and because $\mu_i I \to 0$, we have

$$\bar{\alpha}_i / \mu_i \to \infty \quad .$$

(2.26)

This is inconsistent because $\mu_i$ is homogeneous to one power of $\bar{\alpha}_j$. Remark that our conclusion stays unchanged if we multiply the integrand in (2.21) by some function $B_i(x)$ with a bounded variation. This remark will be later useful when we will include the variations of $\bar{\alpha}_j(\alpha_i)$ and $Q_G(P,\{\bar{\alpha}_j\}_{j\neq i}\alpha_i)$, not due to the constraint (2.9), this inclusion leading to such a factor.

The only case left to see is $\mu_i \Lambda \to \infty$. We eliminate this possibility straight away if it is to hold true for any $i$ because it would mean $\bar{\alpha}_i I \to \infty$ and therefore

$$\sum_{i=1}^{I} \bar{\alpha}_i \to \infty$$

(2.27)

if all $\bar{\alpha}_i$'s have this behaviour, which is incompatible with the constraint

$$\sum_{i=1}^{I} \bar{\alpha}_i = h_0 \quad .$$

(2.28)

We now consider the most general case where an $\bar{\alpha}_i$ can have the following behaviour

$$\bar{\alpha}_{i\delta} = C_\delta(h_0/I)^{1+\delta}$$

(2.29)

where $\bar{\alpha}_{i\delta}$ denotes such an $\bar{\alpha}_i$ and $\delta$ is real with $\delta \geq -1$. Then, (2.28) can be written

$$\sum_{\delta} n_\delta C_\delta(h_0/I)^{1+\delta} = h_0 \quad ,$$

(2.30)

and the relation

$$\sum_{\delta} n_\delta = I$$

(2.31)

takes care of the fact that there are $I$ propagators. For $\delta > 0$, $\bar{\alpha}_{i\delta}$ is then decreasing faster than $(h_0/I)$, $n_\delta(h_0/I)^{1+\delta}$ is tending to zero, and the sum of the corresponding $\bar{\alpha}_i$'s contributes infinitiesmally to the sum in (2.30). For $\delta < 0$, $\bar{\alpha}_{i\delta}$ is decreasing slower than $(h_0/I)$, and therefore this case corresponds to $\bar{\alpha}_i \Lambda, \mu_i \Lambda \to \infty$ and (2.30) implies
\( n_\delta \leq (h_0/C_\delta)(I/h_0)^{1+\delta} \), \hspace{1cm} (2.32)

i.e. \( n_\delta/I \) is infinitesimal. Taking into account the fact that \( \delta > 0 \) is inconsistent (we will see below that this is even forbidden if \( \delta \neq 0 \) is allowed from the start), the sum in (2.31) is saturated by \( n_\delta \) with \( \delta = 0 \).

Now, let us consider some monomial in (2.2) which can be written

\[
\prod_{\ell \notin T} \bar{\alpha}_\ell = \exp \left( -\sum_{\ell \notin T} \log \bar{\alpha}_\ell \right). \hspace{1cm} (2.33)
\]

Looking at the contribution to the sum in (2.33) coming from \( \bar{\alpha}_\ell \)'s with \( \delta < 0 \), we get, \( \delta_- \) corresponding to an \( \bar{\alpha}_{i\delta} \) with \( \delta < 0 \),

\[
\log \left( \prod_{\delta_-} \bar{\alpha}_{i\delta} \right) = -n_{\delta_-} < 1 + \delta_- > \log I \hspace{1cm} (2.34)
\]

neglecting non-leading terms, \( < 1 + \delta_- > \) being the average of \( (1 + \delta) \) for \( \delta < 0 \) and \( n_{\delta_-} \) the total number of them. We see that (2.34) is infinitesimal with respect to the contribution of \( \bar{\alpha}_i \)'s with \( \delta = 0 \), which is

\[
- \left( L - n_{\delta_-} \right) \log I \hspace{1cm} (2.35)
\]

with \( L \) being a constant times \( I \). We therefore conclude that \( \log P(\{\bar{\alpha}\}) \) is built up from \( \bar{\alpha}_i \)'s with \( \delta = 0 \) up to a possible relatively infinitesimal contribution from \( \bar{\alpha}_i \)'s with \( \delta < 0 \).

Starting from (2.29) we have assumed that \( \bar{\alpha}_j \) can have a different behaviour than that of (2.10) where only the case \( \delta = 0 \) was considered at first. Now, we would like to extend the argument made for excluding \( \delta < 0 \) to \( \delta > 0 \).

Repeating an argument made before [10] we will conclude that consistency is achieved only for all \( \bar{\alpha}_i \)'s having the behaviour (2.10).

Let us restate this argument (in a somewhat modified form).

I - If some \( \bar{\alpha}_j \)'s decrease faster than \( O(h_0/I) \), i.e. if \( \delta > 0 \) for some of them, this would either

i) not change the behaviour

\[
b_i = b_0 \left( (h_0 - \alpha_i)/I \right)^{L-1}
\]
and then, as we have seen previously, $\bar{\alpha}_i$ is $O(h_0/I)$ for any $i$.

ii) or change $b_i$ such that $b_i$ would decrease faster than $[(h_0 - \alpha_i)/I]^{d-1}$. Then, two cases are to be evaluated

a) $(h_0 - \alpha_i)^{(1-d/2L)(1-\beta)}$ is replaced by $(h_0 - \alpha_i)^{M_a}$ with $M_a = O(I)$. Then, $\bar{\alpha}_i$ has still to be $O(h_0/I)$, for any $i$ and this contradicts our assumption $\bar{\alpha}_j I \to 0$.

b) $(h_0 - \alpha_i)^{(1-d/2L)(1-\beta)}$ is replaced by $(h_0 - \alpha_i)^{M_b}$ with $M_b < O(I)$. Then, $\Lambda$ is replaced by $M_b/h_0$ in (2.21). If we assume $\mu_i M_b = O(1)$, then (2.21) gives us

$$[(\mu_i + \bar{\alpha}_i) M_b]^{-d/2} \sim I/M_b$$

which leads to a contradiction because the left-hand side is $O(1)$ and the right-hand side is infinite. If we assume $\mu_i M_b \to \infty$, then (2.21) gives us

$$[(\mu_i + \bar{\alpha}_i) M_b]^{-d/2} \sim (I/M_b)(\mu_i M_b)^{-d/2}$$

which, again, is inconsistent with $(I/M_b) \to \infty$.

If we assume $\mu_i M_b \to 0$, we get the relation (2.22) and because $\mu_i$ cannot decrease faster than $1/I$, the only possibility left is $\mu_i = O(1/I)$ which entails $\bar{\alpha}_i = O(1/I)$ and (2.10) is recovered.

Of course, we also have to take into account the change on $\beta$ produced by the possible altering of some $\bar{\alpha}_i$’s decreasing faster than $O(h_0/I)$. We could have instead of (2.12c)

$$Q_G(P, \{\bar{\alpha}\}) = Q_G_0 [(h_0 - \alpha_i)/(I - 1)]^{1+\delta_Q}$$

with $\delta_Q > 0$. However, we see that this will make $Q_G(P, \{\bar{\alpha}\})$ tend to zero and $\beta$ will be tending to zero too. So the reasoning made above remains unchanged and the decrease of $\bar{\alpha}_i$ faster than $1/I$ remains forbidden.

Finally, we can rewrite (2.21) under the following form

$$(\mu_i \Lambda + \bar{\alpha}_i \Lambda)^{-d/2} = \exp \left[ -dL/(2I) - (1 - dL(2I))\beta \right]$$

$$[(I - 1)/(h_0 \Lambda)] \int_0^\infty dx (\mu_i \Lambda + x)^{-d/2} \exp(-x)$$

which enables a numerical resolution, $\mu_i \Lambda$, $\bar{\alpha}_i \Lambda$, $dL/(2I)$, $(I - 1)/\Lambda$ and $\beta$ being $O(1)$. For each independent $\bar{\alpha}_i$ then exists such an equation. So, in principle, we
have a complete perturbative solution of scalar massive \( \phi^3 \) field theory by solving such equations. As we will see in the end of the next section and section 9 the massless case can also be solved in the same way.

3 Regge behaviour, the leading Regge trajectory and its \( m \to 0 \) limit

A - Regge behaviour

One may wonder how the formalism we have described in the last section can yield the Regge behaviour for appropriate graphs. Previous work made some thirty years ago [8, 14] will serve as a check of ours ideas. In fact we will see that using the ladder graphs we can get the Regge behaviour in an easy but curious way. The leading Regge trajectory will also be easy to write down, even for arbitrary argument \( t/m^2 \) which was not the case in the approaches using the Bethe-Salpeter equation or the multiperipheral model.

So let us take ladder graphs of massive scalar \( \phi^3 \) field theory. For those graphs, due to the symmetry existing for central propagators and side propagators of the ladder, we only have [15] two independent \( \bar{\alpha}_i \) left that we call \( \bar{\alpha}_+ \) for the mean-value of the \( \alpha \)-parameters on the sides and \( \bar{\alpha}_- \) for the mean-value of the \( \alpha \)-parameters in the center. (See Fig. 1). Defining

\[
y \equiv (2\bar{\alpha}_+/\bar{\alpha}_-)^{1/2}
\]

we get [15], neglecting all terms which vanish as \( L \to \infty \),

\[
P_G(\{\bar{\alpha}\}) = (\bar{\alpha}_-)^L \exp(yL) f(g)
\]

\[
f(y) = 1/2 \quad y(1 + y^{-1})^2
\]

and

\[
Q_G(P, \{\bar{\alpha}\}) = t/2 \ L \ \bar{\alpha}_+ + \bar{\alpha}_- \ s \exp(-yL)[f(g)]^{-1}.
\]
Reinstating the coupling constant dependence through the factor \((-\gamma)^{2L+2}\), we get, for \(\phi^3\) at \(d = 4\) and therefore replacing \(I - d/2L\) by \(L + 1\),

\[
F_G = \left(\frac{e^2}{\sqrt{3}}\right)^2 \left[-\gamma/(mf(y))\right]^2 \left[-\gamma e/(m4\pi 3\sqrt{3})\right]^{2L} \\
\frac{[\exp(-y)/g_-]^{2L}}{(1 - \beta)^{3L+1}}
\]

with \(g_- \equiv \bar{\alpha}_- \Lambda\). Now, we can sum over \(L\) the ladder amplitudes (3.4) and find the saddle-point equation

\[
2\ln C^{st} + 3\ln(1 - \beta) + (3L + 1) [y + 1/(L + 1)] bs/(1 - \beta) = 0
\]

(3.5)

with the following definitions

\[
C^{st} \equiv \left[-\gamma e/(m4\pi 3\sqrt{3})\right] \left[\exp(-y)/g_-\right]
\]

(3.6a)

\[
\beta = a + bs
\]

(3.6b)

\[
a = t/2 \bar{\alpha}_+ \Lambda/m^2
\]

(3.6c)

\[
bs = s \left[g_-/(m^2 f(y))\right] \exp(-yL)/(L + 1).
\]

(3.6d)

We see immediately that as \(s \to \infty\), there is no solution of (3.3) for finite \(L\), which is satisfactory. \(bs\) being constant also brings no solution and \(bs \to \infty\) gives \(F_G\) behaving like \((-bs)^L\) which is exploding. The only possibility left is \(bs\) tending to zero. Then, (3.3) becomes

\[
(1 - a) \left[2/3 \ln C^{st} + \ln(1 - a)\right] + Ly \ bs = 0
\]

(3.7)

which has the solution \(L_{sp}\) at the saddle point with

\[
L_{sp} = (1/y) \ln \left(s/s_0\right)
\]

(3.8a)

with
\[-(1-a)\left[\frac{2}{3} \ln C + \ln(1-a)\right] = (s_0/m^2)y g_+/f(y) \quad . \quad (3.8b)\]

The relation \((3.8a)\) is a very natural one meaning that the dominant ladders have a length proportional to the “rapidity” \(\ln(s/s_0)\). The same phenomenon appears in the multiperipheral model \([9]\) and in the parton model \([10]\). Putting this value of \(L\) in \((3.4)\) we immediately obtain the leading Regge trajectory

\[\alpha(t) = y^{-1}\left[2 \ln C + 3 \ln(1-\beta)\right] \quad , \quad (3.9)\]

which is a simple analytic expression.

**B - The \(m \to 0\) limit**

Let us now examine the case where \(m^2 \to 0\) in order to obtain massless \(\phi^3\) results. The first thing to note is that \(1-\beta \to 0\) as \(m^2 \to 0\) due to the fact that by definition

\[1-\beta \equiv \left[1 + Q_G(P, \{\bar{\alpha}\})/h_0m^2\right]^{-1} \quad . \quad (3.10)\]

We expect, looking at \((2.5)\), that \(F_G\) becomes independent of \(m\) as \(m \to 0\). Of course the introduction of a mass is necessary in order to avoid infrared problems in the definition of Feynman integrals but the final result for the amplitude \(F_G\) and all physical quantities should be that they are well defined and independent of \(m\) as the mass \(m\) tends to zero.

We will first verify that the trajectory \(\alpha(t)\) obtained in \((3.9)\) is indeed independent of \(m\) as \(m\) tends to zero. Looking at the definition \((3.6a)\) of the \(C^s\) we get

\[\alpha(t) = y^{-1}\left\{2 \left[\ln \left(e/(4\pi3\sqrt{3})\right) - y + \ln(-\gamma/m) - \ln(\bar{\alpha}_-\Lambda)\right] + 3\ln(1-\beta)\right\} \quad . \quad (3.11)\]

Because \(\Lambda = (I - d/2 L)(1-\beta)/h_0\) (see \((2.17)\)) we have

\[\ln(\bar{\alpha}_-\Lambda) = \ln(1-\beta) + \text{terms independent of } m \quad , \quad (3.12)\]
so the sum of terms dependent on \( m \) in the bracket of (3.11) is

\[
2 \left[ -\ell n m - \ell n(1 - \beta) \right] + 3\ell n(1 - \beta) = -\ell n m^2 + \ell n(1 - \beta) \quad (3.13)
\]

which is independent of \( m \) in the limit \( m \to 0 \) as can be readily deduced from (3.10). Of course, there could be some indirect dependence on \( m \) through \( \bar{\alpha}_- \) or \( y = (2\bar{\alpha}_+ / \bar{\alpha}_-)^{1/2} \) but we will see later on in section 9 that the equations determining \( \bar{\alpha}_+ \) and \( \bar{\alpha}_- \) are indeed independent of \( m \) as \( m \) tends to zero, making the transition to QCD possible for what concerns this limit.

C - Introduction to the determination of \( \bar{\alpha}_+ \) and \( \bar{\alpha}_- \)

We now turn to the task of determining the values of \( \bar{\alpha}_+ \) and \( \bar{\alpha}_- \) in the case of the ladder graphs. Some refinements of the consistency equations (2.16) or (2.21) are necessary in order to really be able to calculate the leading Regge trajectory. In fact, until now only the dependence on \( \alpha_i \) induced by the \( \delta \)-function is \( \delta(h_0 - \sum_i \alpha_i) \) has been taken into account. Taking a careful look at quantities which may have additional dependence on \( \alpha_i \) when it is varied as in (2.16) and (2.21) we find two kinds of dependencies.

The first one is due to the fact that \( Q_G(P, \{\bar{\alpha}\}) \), where all \( \alpha_\ell \)'s have been replaced by their mean-values is not exactly equal to \( Q_G(P, \{\bar{\alpha}\}) \), which is \( Q_G(P, \{\alpha\}) \) where \( (I - 1) \) variables \( \alpha_\ell \) have been replaced by \( \bar{\alpha}_\ell \) and where \( \alpha_i \) is left free to vary. We will examine in the next section the corrections which must be added to (2.16) and (2.21) in order to take into account this phenomenon.

The second one is due to the dependency of the mean-values \( \bar{\alpha}_\ell \) on \( \alpha_i \). The relative variation \( \delta\bar{\alpha}_\ell / \bar{\alpha}_\ell \) of one mean-value will be found to be of order \( 1/L \). However, as we are dealing with powers \( (\bar{\alpha}_\ell)^L \), the final outcome will be a sort of “renormalization” of the constants \( \bar{\alpha}_+ \Lambda, \bar{\alpha}_- \Lambda \). The definition of the scale \( \Lambda \) will also be affected and \( \Lambda \) itself will also be renormalized. The details of this renormalization will be exposed in section 5.
4 The $Q_G(P, \{\bar{a}\})_i$ dependence on $\alpha_i$

First, let us give the way the expression for $P_G(\{\bar{a}\})$ is obtained. In Fig. 2a we displayed a ladder with $L$ loops where $p$ central propagators are removed and $L - p$ side propagators removed giving rise to a monomial $(\bar{a}_-)^p(\bar{a}_+)^{L-p}$. In the topology displayed in Fig. 2a we have $L - p$ “cells” of lengths $\ell_1, \ell_2, \ldots, \ell_{L-p}$ where a cell of length $\ell_i$ is obtained by removing $\ell_i - 1$ center propagators. For each cell a side-propagator has to be removed in order to obtain “open cells”, i.e. no loop left. The remaining propagators then form a spanning tree on the ladder graph. The two other topologies displayed in Fig. 2b and Fig. 2c correspond to one and two end-cells being opened by removing a center-propagator instead of a side-propagator.

Then, the expression for $P_{G}^{(a)}(\{\bar{a}\})$ which is the part of $P_G(\{\bar{a}\})$ corresponding to the topology displayed in Fig. 2a is

$$P_{G}^{(a)}(\{\bar{a}\}) = \sum_{p=0}^{L-1} (\bar{a}_-)^p(\bar{a}_+)^{L-p} \sum_{\ell_1+\cdots+\ell_{L-p}=L} 2^{L-p} \ell_1 \cdots \ell_{L-p}$$

where the factors $2\ell_i$ comes from the fact that there are $2\ell_i$ side-propagators along a cell which can be removed in order to open it.

In order to obtain $\sum_{c} s_{c} \prod_{\ell=0}^{L} \bar{a}_{\ell}$ in (2.3) from $P_G(\{\bar{a}\})$ one has to remove one more propagator. To select the part with $s_c = t$ one has to remove a second side-propagator on the opposite side of the first one along a cell. The $s_c = s$ part is obtained by removing the $L + 1$ center-propagators and no side-propagator. So, we have

$$Q_{G}^{(a)}(P, \{\bar{a}\}) = \left[(1/2)\bar{a}_+ t L P_{G}^{(a)}(\{\bar{a}\}) + s \bar{a}_{L+1}^{-1} \right] P_{G}^{-1}(\{\bar{a}\}) \quad ,$$

the factor $L$ coming from $L$ side-propagators which can be removed, and the factor $1/2$ in order not to double-count the cuts obtained. Of course $Q_G(P, \{\bar{a}\})$ is the sum of the three contributions obtained from cutting $P_{G}^{(a)}(\{\bar{a}\})$, $P_{G}^{(b)}(\{\bar{a}\})$ and $P_{G}^{(c)}(\{\bar{a}\})$. (We have integrated in $Q_{G}^{(a)}(P, \{\bar{a}\})$ the term proportional to $s$). We now have to replace one $\bar{a}_+$ by $\alpha_{i+}$ or one $\bar{a}_-$ by $\alpha_{i-}$ to compute the dependence of $Q_G(P, \{\bar{a}\})_i$ on either $\alpha_{i+}$ or $\alpha_{i-}$ when one of them is left free to vary.
A - The $\alpha_{i_+}$ dependence

So let us consider

$$P_G^{(a)}(\{\bar{\alpha}\})_{i_+} = a_{i_+} + b_{i_+} \alpha_{i_+} \quad (4.3)$$

where $a_{i_+}$ and $b_{i_+}$ are polynomials of mean-values $\bar{\alpha}_j, j \neq i_+$. We know that the propagator $i_+$ is along one cell which has a length $\ell$. Therefore, the cutting of all other cells is unaffected and we get for these cells a term contributing to $Q_G^{(a)}(P,\{\bar{\alpha}\})_{i_+}$ which is

$$t/2 \bar{\alpha}_+ (L - \ell) \; P_G^{(a)}(\{\bar{\alpha}\})_{i_+} \quad (4.4)$$

reminiscent of the first term in (4.2). Let us now consider the cutting of the cell which contains $i_+$ (see fig. 3). The term $b_{i_+} \alpha_{i_+}$ has $i_+$ as the propagator removed (which is not the case of fig. 3 where $i_+$ has been chosen on the opposite side of the removed propagator) and therefore cutting the cell on the other side, we have $\ell$ possibilities which gives the term

$$t/2 \bar{\alpha}_+ \ell \; b_{i_+} \alpha_{i_+} \quad (4.5)$$

However when considering $a_{i_+}, i_+$ can be any of the $2\ell - 1$ side-propagators which are still part of the open cell in Fig. 3. If $i_+$ is on the same side as the cut propagator it cannot be removed because this would isolate a part of the cell where no external line is attached (more exactly this gives a zero contribution because $\sum P_v = 0$ is that case in (2.4)). Then, we get two terms

$$t/2 \bar{\alpha}_+ \ell \; [(\ell - 1)/(2\ell - 1)] a_{i_+} \quad (4.6a)$$

$$t/2 [(\ell - 1)\bar{\alpha}_+ + \alpha_{i_+}] [\ell/(2\ell - 1)] a_{i_+} \quad (4.6b)$$

the first one corresponding to $i_+$ on the side of the already cut propagator and the second one (corresponding to Fig. 3) where $i_+$ is on the opposite side. Summing (4.4), (4.5), (4.6a) and (4.6b) we get
\[
t/2 \ L \ \bar{\alpha}_+ \ P_G^{(a)}(\{\bar{\alpha}\}) + t/2 [\ell/(2\ell - 1)] \ (\alpha_{i+} - \bar{\alpha}_+)a_{i+}.
\]

(4.7)

Defining:

\[
\mu_+ \equiv a_{i+}/b_{i+}
\]

(4.8)

we get

\[
Q_G^{(a)}(P, \{\bar{\alpha}\})_{i+} = Q_G^{(a)}(P, \{\bar{\alpha}\}) + \varepsilon_{+}^{(a)}
\]

(4.9a)

\[
\varepsilon_{+} = t/2 < \ell/(2\ell - 1) > \ [\mu_+/(\mu_+ + \alpha_{i+})]
\]

(4.9b)

For the other topologies in Fig. 2b and Fig. 2c the same reasoning provides us with the same expressions replacing (a) by (b) and (c). \( \mu_+ \) keeps the same value up to terms of order \( 1/L \) because the topologies differ only on one or two cells and because there are an infinite number of them. \( < \ell/(2\ell - 1) > \) is the mean-value of \( \ell/(2\ell - 1) \), integrating over all \( \ell \)'s. Then,

\[
Q_G(P, \{\bar{\alpha}\})_{i+} = Q_G(P, \{\bar{\alpha}\}) + \varepsilon_{+}
\]

(4.10a)

\[
\varepsilon_{+} = t/2 < \ell/(2\ell - 1) > (\alpha_{i+} - \bar{\alpha}_+)/\mu_+ + \alpha_{i+} \)
\]

(4.10b)

In a recent letter [15] we gave an estimate for \( < 1/\ell > \).

\[
< 1/\ell > = \mu_-/(\mu_- + \bar{\alpha}_-) = y
\]

(4.11a)

\[
< 1/(2\ell) > = \bar{\alpha}_+/(\mu_+ + \bar{\alpha}_+) = y/2
\]

(4.11b)

Writing \( \ell/(2\ell - 1) \) as \( (2 - 1/\ell)^{-1} \) and replacing \( < (2 - 1/\ell)^{-1} > \) with \( (2 - < 1/\ell >)^{-1} \) we get a crude estimate for this average which is \( (2 - y)^{-1} \). We used such an estimation in our numerical solutions of equations.
Now, $\varepsilon_\pm$ is of order $1/L$ relative to $Q_G(P, \{\bar{\alpha}\}) + h_0 m^2$, but as this quantity is elevated to a power $-(I - d/2L)$ in (2.3) and in (2.13) we get a finite correction factor. With $\Lambda$ given in (2.17) we get a factor

$$H_{i_+} (\alpha_{i_+} - \bar{\alpha}_+ ) = \exp \left( -\varepsilon_+ \Lambda / m^2 \right)$$

$$= \exp \left\{ -t/2 < \ell / (2\ell - 1) > \mu_+ (\Lambda / m^2) (\alpha_{i_+} - \bar{\alpha}_+) / (\mu_+ + \alpha_{i_+}) \right\} \quad (4.12)$$

which must be inserted in the integrand of the right-hand side of (2.16) when calculating $\bar{\alpha}_+$.

**B - The $\alpha_{i_-}$ dependence**

We are now interested in the case where one $\alpha_{i_-}$ Feynman parameter of a central propagator is left free to vary. Then,

$$P_G(\{\bar{\alpha}\})_{i_-} = a_{i_-} + b_{i_-} \alpha_{i_-} \quad (4.13)$$

where $a_{i_-}$ and $b_{i_-}$ are polynomials of mean-values $\bar{\alpha}_j$, $j \neq i_-$. Correspondingly, $Q_G(P, \{\bar{\alpha}\})_{i_-}$ being the expression for $Q_G(P, \{\bar{\alpha}\})$ when $\alpha_{i_-}$ is free, we have

$$Q_G(P, \{\bar{\alpha}\})_{i_-} = (t/2) L \bar{\alpha}_+ + s \bar{\alpha}_{i_-} P_G^{-1}(\{\bar{\alpha}\})_{i_-} \quad (4.14)$$

and we see that its $\alpha_{i_-}$ dependence is confined to the term containing $s$. We can therefore write

$$Q_G(P, \{\bar{\alpha}\})_{i_-} = Q_G(P, \{\bar{\alpha}\}) + s \bar{\alpha}_{i_-} P_G^{-1}(\{\bar{\alpha}\}) \left[ \alpha_{i_-} \left( P_G(\{\bar{\alpha}\})_{i_-} / P_G(\{\bar{\alpha}\}) \right)^{-1} - \bar{\alpha}_- \right]. \quad (4.15)$$

Defining

$$\mu_- \equiv a_{i_-} / b_{i_-} \quad , \quad (4.16)$$

$$\alpha_{i_-} \left( P_G(\{\bar{\alpha}\})_{i_-} / P_G(\{\bar{\alpha}\}) \right)^{-1} - \bar{\alpha}_- = \alpha_{i_-} (\mu_- + \bar{\alpha}_- ) / (\mu_- + \alpha_{i_-} ) - \bar{\alpha}_-$$

$$= \mu_- \left( \alpha_{i_-} - \bar{\alpha}_- \right) / (\mu_- + \alpha_{i_-} ) \quad . \quad (4.17)$$
On the other hand

\[ s \, \bar{\alpha}_L^\dagger \, P_G^{-1}(\{\bar{\alpha}\}) = s \exp(-yL)[f(y)]^{-1} \]  

(4.18)

and if we are at the saddle point \( L = L_{sp} \), \( s \exp(-yL) = s_0 \) and (3.8b) gives

\[ s_0[f(y)]^{-1} = \left[ m^2/(yg_-) \right] \left\{ -(1-a) \left[ 2/3 \ln C^{st} + \ln(1-a) \right] \right\} . \]  

(4.19)

So,

\[ Q_{G}(P, \{\bar{\alpha}\})_{i-} = Q_{G}(P, \{\bar{\alpha}\}) + \varepsilon_- \]  

(4.20a)

with

\[ \varepsilon_- = s_0[f(y)]^{-1} \left( \alpha_{i-} - \bar{\alpha}_- \right) \mu_-/ \left( \mu_- + \alpha_{i-} \right) . \]  

(4.20b)

Again, \( Q_G(P, \{\bar{\alpha}\})_{i-} \) appears in

\[ \left[ Q_{G}(P, \{\bar{\alpha}\})_{i-} + h_0 m^2 \right]^{-\left(1-d/2L\right)} \]  

(4.21)

with for \( d = 4 \) and \( \phi^5 \), \( I - d/2 \) \( L = L + 1 \). So with \( \Lambda = (I - d/2L)(1-\beta)/h_0 \) we get a factor

\[ H_{i-}(\alpha_{i-} - \bar{\alpha}_-) = \exp(-\varepsilon_- \Lambda/m^2) \]

\[ = \exp \left\{ y^{-1}(1-a) \left[ 2/3 \ln C^{st} + \ln(1-a) \right] \left( \mu_- / \bar{\alpha}_- \right)(\alpha_{i-} - \bar{\alpha}_-)/(\mu_- + \alpha_{i-}) \right\} \]  

(4.22)

which must be inserted in the integrand of the right-hand side of (2.16) when calculating \( \bar{\alpha}_- \). In the following the notation \( H_{i}(\alpha_i - \bar{\alpha}_i) \) will designate either \( H_{i+}(\alpha_{i+} - \bar{\alpha}_+) \) or \( H_{i-}(\alpha_{i-} - \bar{\alpha}_-) \) depending on \( i \) being a side- or center-propagator.

5 The \( \bar{\alpha}_\ell(\alpha_i) \) dependence: introduction

\( P_G(\{\bar{\alpha}\}) \) is a polynomial of degree \( L \) in the mean-values \( \bar{\alpha}_\ell \). Therefore, an infinitesimal variation \( \delta \bar{\alpha}_\ell(\alpha_i) \) as \( \alpha_i \) varies can result in a finite correction factor.
In the same way, the variation of $\bar{\alpha}_+$ in $Q_G(P, \{\bar{\alpha}\}) = t/2 \ L \ \bar{\alpha}_+$ (up to terms vanishing as $1/L$) could matter because $Q_G(P, \bar{\alpha}) + h_0m^2$ is elevated to the power $-(I - d/2L) = -(L + 1)$. In this section and the following ones we will be trying to evaluate the dependence of $\bar{\alpha}_j$ on $\alpha_{i_+}$ or $\alpha_{i_-}$ ($\alpha_{i_-}$ being a center-propagator variable and $\alpha_{i_+}$ a side-propagator variable) when one of them is free to vary, with of course $j \neq \alpha_{i_+}$ or $\alpha_{i_-}$. This will be a somewhat lengthy task, at least to be exposed clearly, so we divided it into several steps.

The first one is, given $\bar{\alpha}_{j-}(\alpha_i)$ and $\bar{\alpha}_{j+}(\alpha_i)$ (we will thereafter use the notation $\bar{\alpha}_{-}(\alpha_i)$ for $\bar{\alpha}_{j-}(\alpha_i)$ and $\bar{\alpha}_{+}(\alpha_i)$ for $\bar{\alpha}_{j+}(\alpha_i)$), determinating what is the factor which modifies the integrand of the right-hand side of the consistency equation (2.16).

The second step is exposing the method we followed to calculate $\bar{\alpha}_{-}(\alpha_i)$ and $\bar{\alpha}_{+}(\alpha_i)$. In this section we deal with these two first steps. The actual determination of $\bar{\alpha}_{+}(\alpha_i)$ and $\bar{\alpha}_{-}(\alpha_i)$ will be exposed in the next section.

A - General form of the correction factor

We start with the factor

$$[b_i(\{\bar{\alpha}\})]^{-d/2} \equiv \left[ P_G(\{\bar{\alpha}\})/ (\mu_i + \bar{\alpha}_i) \right]^{-d/2} \quad (5.1)$$

which appears as $[b_i(h_0/I)^{L-1}]^{-d/2}$ in (2.13). In the notation used in (2.13) $b_{i0}$ is a constant. Here $b_i(\{\bar{\alpha}\})$ is $b_{i0}$ multiplied by $(h_0/I)^{L-1}$. Because $\delta \bar{\alpha}_j/\bar{\alpha}_j$ will be infinitesimal when the $\alpha_i$ variation is $O(h_0/I)$, we will consider that $\log b_i(\{\bar{\alpha}\})$ and $\log P_G(\{\bar{\alpha}\})$ have the same variation. Then, as $L \to \infty$ (see (3.2a))

$$\log P(\{\bar{\alpha}\}) = L \left[ \log \bar{\alpha}_- + (2\bar{\alpha}_+/\bar{\alpha}_-)^{1/2} \right] + \log f(y) \quad . \quad (5.2)$$

Taking the variation and neglecting $df(y)$ because it is of order $(1/L)$ with respect to the other terms

$$d \log P_G(\{\bar{\alpha}\}) = L \left[ d\bar{\alpha}_-/\bar{\alpha}_- + 1/2 \ (2\bar{\alpha}_+/\bar{\alpha}_-)^{1/2} d\bar{\alpha}_+/\bar{\alpha}_+ - 1/2 \ (2\bar{\alpha}_+/\bar{\alpha}_-)^{1/2} d\bar{\alpha}_-/\bar{\alpha}_- \right]$$

$$= L \left[ (1 - y/2) d\bar{\alpha}_-/\alpha_+ + y/2 \ d\bar{\alpha}_+/\alpha_+ \right] \quad (5.3)$$

which is equal to $db_i(\{\bar{\alpha}\})$ up to $O(1/L)$ terms. Next comes the variation of
\[ \log [Q_G(P, \{\bar{\alpha}\}) + h_0 m^2]^{-(I - d/2L)} \] which is (the term proportional to \(\bar{\alpha}_-\) in \(Q_G(P, \{\bar{\alpha}\})\)) is \(O(h_0/I)\) at the saddle-point, see (5.3)

\[ d \log \left[ t/2 L \bar{\alpha}_+ + h_0 m^2 \right]^{-(I - d/2L)} = -(I - d/2L) t/2 (\Lambda/m^2) d\bar{\alpha}_+ . \quad (5.4) \]

Because \(I - d/2L = L + 1\), we can define

\[ \eta \equiv (2/d) t/2 \bar{\alpha}_+ \Lambda/m^2 \quad (5.5) \]

so that we get the correction factor

\[ H_0(\alpha_i - \bar{\alpha}_i) = \exp \left\{ -d/2L \left[ (1 - y/2) d\bar{\alpha}_- / \bar{\alpha}_- + (y/2 + \eta) d\bar{\alpha}_+ / \bar{\alpha}_+ \right] \right\} \]

\[ \equiv \exp \left[ -d/2L \ E_0(\alpha_i - \bar{\alpha}_i) \right] \quad (5.6) \]

where terms of order \((1/L)\) relative to \(\log H_0(\alpha_i - \bar{\alpha}_i)\) have been neglected in the argument of the exponential.

**B - General method for determining \(\bar{\alpha}_\ell(\alpha_i)\)**

The first thing we will do is to leave unintegrated two variables \(\alpha_\ell\) and \(\alpha_i\) instead of zero or one until now. Then, we will be able to see how \(\alpha_i\) influences the consistency equation for \(\bar{\alpha}_\ell\). With the two variables \(\alpha_i\) and \(\alpha_\ell\) left free we will have to take into account the \(\delta\)-function constraint \(\delta(h_0 - \sum \alpha_i)\) through the replacement of \(h_0\) (in the case when no variable was left free) by \(h_0 - (\alpha_i + \alpha_\ell)\). Then, we have to evaluate \(P_G(\{\bar{\alpha}\})_{i,\ell}\) when the mean-values have been taken on \(I - 2\) variables \(j \neq i, \ell\). We write

\[ P_G(\{\bar{\alpha}\})_{i,\ell} = a_i(\alpha_\ell) + b_i(\alpha_\ell)\alpha_i \quad (5.7a) \]

\[ a_i(\alpha_\ell) = a_{i1} + a_{i2} \alpha_\ell \quad (5.7b) \]

\[ b_i(\alpha_\ell) = b_{i1} + b_{i2} \alpha_\ell \quad (5.7c) \]
giving

\[ P_G(\{\bar{\alpha}\})_{i,\ell} = (a_{i2} + b_{i2}\alpha_i) \left[ \frac{(a_{i1} + b_{i1}\alpha_i)}{(a_{i2} + b_{i2}\alpha_i) + \alpha_\ell} \right] . \]  

(5.8)

Now, if \( i \) and \( \ell \) are infinitely far away along the ladder, the value of the ratio \( a_i/b_i \) will be independent of \( \ell \). This is the factorization phenomenon \cite{13}, \cite{13} for infinite graphs. So, we will be able to write

\[ (a_{i2}/b_{i2})_\pm = (a_{i1}/b_{i1})_\pm = \mu_{i\pm} \]  

(5.9)

where + or − refers to \( i \) being a side- or center-propagator. (Here, of course \( i_1 \) and \( i_2 \) refer to the same propagator \( i \) but for trees going through \( \ell \) or not). So we get

\[ P_G(\{\bar{\alpha}\})_{i,\ell} = (a_{i2} + b_{i2}\alpha_i) \left( b_{i1}/b_{i2} + \alpha_\ell \right) . \]  

(5.10)

Invoking again the factorization phenomenon

\[ (a_{i1}/a_{i2})_\pm = (b_{i1}/b_{i2})_\pm = \mu_{\ell\pm} \]  

(5.11)

but with the role of \( i \) and \( \ell \) interchanged (here + or − refers to \( \ell \) being a side- or center-propagator) we finally get

\[ P_G(\{\bar{\alpha}\})_{i,\ell} = b_{i2} \left( \mu_i + \alpha_i \right) \left( \mu_\ell + \alpha_\ell \right) \]  

(5.12)

provided \( i \) and \( \ell \) are infinitely far away along the ladder. However, as this the case which dominates when \( i \) and \( \ell \) are arbitrary, we will take \( (5.12) \) to be valid in general, making so an error of order \( 1/L \). As for \( \bar{\alpha}_j(\alpha_i) \) we will also simplify our notation and write

\[ \mu_{i\pm} = \mu_{\ell\pm} = \mu_\pm \]  

(5.13)

which is justified for symmetry reasons. \( \mu_i \) or \( \mu_\ell \) will be used if the nature of \( i \) or \( \ell \) is not specified. Now, keeping only terms proportional to a constant as \( L \to \infty \), it is easy to proceed as for the obtention of \( (2.16) \) and get
\[ 1 = \kappa^2 (I - 1)(I - 2)/h_0^2 \]

\[ \int_0^{a_i} d\alpha_i [(\mu_i + \bar{\alpha}_i) / (\mu_i + \alpha_i)]^{-d/2} \exp(-\alpha_i \Lambda) H_i(\alpha_i - \bar{\alpha}_i) H_0(\alpha_i - \bar{\alpha}_i) \]

\[ \int_0^{a_\ell} d\alpha_\ell [(\mu_\ell + \bar{\alpha}_\ell) / (\mu_\ell + \alpha_\ell)]^{-d/2} \exp(-\alpha_\ell \Lambda) H_\ell(\alpha_\ell - \bar{\alpha}_\ell) H_0(\alpha_\ell - \bar{\alpha}_\ell) \]

\[ \kappa = \exp[-dL/(2I) - (1 - dL/(2I))\beta] \]  \quad (5.14a)

Using (2.16), with the factor \( H_i(\alpha_i - \bar{\alpha}_i)H_0(\alpha_i - \bar{\alpha}_i) \) included (taking into account the corrections evaluated in section 4 and in A of this section) we see that (5.14a) would then simply imply two independent consistency equations for \( \alpha_i \) and \( \alpha_\ell \). So we can say that at the “leading order” there is a decoupling of \( \alpha_i \) and \( \alpha_\ell \). The coupling of \( \alpha_i \) and \( \alpha_\ell \) which arises when \( \alpha_i \alpha_\ell \) terms appear are only visible when we go to the next order, i.e. when we take into account terms of order \( 1/L \). This \( (\alpha_i, \alpha_\ell) \) coupling only at the next-to-leading order is of course welcome because it means that the derivative \( d\bar{\alpha}_\ell(\alpha_i)/d\alpha_i \) is of order \( 1/L \), so that powers \( \bar{\alpha}_\ell \) will only give rise to a finite correction factor. Then, to leading order, because of \( d\bar{\alpha}_\ell(\alpha_i)/d\alpha_i = 0 \) we have \( H_0(\alpha_i - \bar{\alpha}_i) = 1 \). Next-to-leading order will give \( d\bar{\alpha}_\ell(\alpha_i) \) of order \( 1/L \), i.e. \( d\bar{\alpha}_-/\bar{\alpha}_- \) and \( d\bar{\alpha}_+/\bar{\alpha}_+ \) in (5.6) will be \( O(1) \) and \( H_0(\alpha_i - \bar{\alpha}_i) \) will contribute to a “renormalization” of the scale \( \Lambda \).

Let us now examine the source of these \( \alpha_i \alpha_\ell \) terms. We recall that the \( \delta \)-function. \( \delta(h_0 - \sum \alpha_i) \) implies a replacement of \( \bar{\alpha}_j \) by \( \bar{\alpha}_j[1 - (\alpha_i + \alpha_\ell)/h_0] \) in \( b_{i2}^{-d/2} \) and as \( \log (1 - x) = -x - x^2/2 \) we get the replacements

\[ (b_{i2})^{-d/2} \rightarrow (b_{i2})^{-d/2} \exp[d/2(L - 2)(\alpha_i + \alpha_\ell)/h_0] \exp[d/2(L - 2)\alpha_i\alpha_\ell/h_0^2] \]

\[ (5.15a) \]

\[ h_0^{-3} \rightarrow h_0^{-3} \exp[-(I - 3)(\alpha_i + \alpha_\ell)/h_0] \exp[-(I - 3)\alpha_i\alpha_\ell/h_0^2] \]

\[ (5.15b) \]

\[ \left\{ 1 + \frac{Q_G(P, \{\bar{\alpha}\})/h_0 m^2}{(1 - (\alpha_i + \alpha_\ell)/h_0)^{-d/2}} \right\}^{-l-d/2} \rightarrow \]

\[ \left[ 1 + \frac{Q_G(P, \{\bar{\alpha}\})/h_0 m^2}{(I - d/2 L)^\beta(\alpha_i + \alpha_\ell)/h_0} \right]^{-l-d/2} \exp [(I - d/2 L)\beta(\alpha_i + \alpha_\ell)/h_0] \exp [(I - d/2 L)\beta^2\alpha_i\alpha_\ell/h_0^2] \]

\[ (5.15c) \]
Also, because (remember that \( \mu_i \) is of degree 1 in \( \bar{\alpha}_j \))

\[
\mu_i (1 - (\alpha_i + \alpha_\ell)/h_0) + \alpha_i = (\mu_i + \alpha_i) \{1 - [(\alpha_i + \alpha_\ell)/h_0] \mu_i/(\mu_i + \alpha_i)\} \tag{5.16}
\]

we get the replacement

\[
[(\mu_i + \alpha_i) (\mu_\ell + \alpha_\ell)]^{-d/2} \to
[(\mu_i + \alpha_i) (\mu_\ell + \alpha_\ell)]^{-d/2} \exp \left\{ d/2 \left[ (\alpha_i + \alpha_\ell)/h_0 \right] \left[ \mu_i/ (\mu_i + \alpha_i) + \mu_\ell/ (\mu_\ell + \alpha_\ell) \right] \right\} . \tag{5.17}
\]

The replacement \( \bar{\alpha}_j \to \bar{\alpha}_j(1 - 2/I) \) implied because we take the mean-value over \((I - 2)\) variables instead of \( I \) provides us with the factor \( \kappa^2 \) in (5.14a). Gathering all terms containing \( \alpha_i \alpha_\ell \) we get the factor

\[
\exp \left\{ -(I - dL/2)(1 - \beta^2) \alpha_i \alpha_\ell/h_0^2 \right\} + d/2 \left[ (\alpha_i/h_0) \mu_i/ (\mu_i + \alpha_i) + (\alpha_\ell/h_0) \mu_i/ (\mu_\ell + \alpha_\ell) \right] \right\} . \tag{5.18}
\]

The terms linear in \( \alpha_i \) or \( \alpha_\ell \) give \( \exp(-\alpha_\ell \Lambda) \) and \( \exp(-\alpha_i \Lambda) \) in (5.14a). We will interpret the terms in the exponential in (5.17) as additional contributions to \( \alpha_\ell \Lambda \) and \( \alpha_i \Lambda \), that we define as \( \alpha_\ell \Lambda_\ell(\alpha_i) \) and \( \alpha_i \Lambda_i(\alpha_\ell) \) such that:

\[
\alpha_\ell \Lambda_\ell(\alpha_i) + \alpha_i \Lambda_i(\alpha_\ell) = (I - d/2 L)(1 - \beta^2) \alpha_i \alpha_\ell/h_0^2
\]

\[
- d/2 \left[ (\alpha_\ell/h_0) \mu_i/ (\mu_i + \alpha_i) + (\alpha_i/h_0) \mu_\ell/ (\mu_\ell + \alpha_\ell) \right] \right\} . \tag{5.19}
\]

We therefore get a “renormalization” of the scale \( \Lambda \) which is dependent on \( \alpha_i \), the integration variable. We note that another contribution to this renormalization comes from \( H_0(\alpha_i - \bar{\alpha}_i) \). To the extent where \( E_0(\alpha_i - \bar{\alpha}_i) \) can be linearized, which happens to be justified by the fact that \( \alpha_i - \bar{\alpha}_i \) is infinitesimal, writing

\[
E_0(\alpha_i - \bar{\alpha}_i) = E_i^0 \alpha_i - E_i^0 \bar{\alpha}_i \tag{5.20}
\]

\( E_i^0 \) being some constant, we can introduce a renormalized \( \Lambda \),

\[
\Lambda_i^R = \Lambda + d/2 L E_i^0 + \Lambda_i(\alpha_\ell) \tag{5.21}
\]

Then, defining
\[ \tilde{H}_0(\bar{\alpha}_i) \equiv \exp\left(\frac{d}{2} L E_i^0 \bar{\alpha}_i\right), \quad (5.22) \]

the consistency equation for \( \bar{\alpha}_i \) now reads

\[ 1 = (I/h_0)\kappa \int_0^{h_0} d\alpha_i \left[ (\mu_i + \bar{\alpha}_i) / (\mu_i + \alpha_i) \right]^{d/2} \exp \left(-\alpha_i \Lambda_i^R\right) H_i(\alpha_i - \bar{\alpha}_i) \tilde{H}_0(\bar{\alpha}_i). \quad (5.23) \]

And, of course, the same equation holds for \( \bar{\alpha}_\ell \) interchanging \( i \) and \( \ell \). Our following task will be the determination of \( \Lambda_i(\alpha_\ell) \) and \( E_i^0 \). This is done in the next section.

6 Determination of \( \bar{\alpha}_\ell(\alpha_i) \)

In order to get \( \bar{\alpha}_\ell(\alpha_i) \) we have several steps to accomplish. First, we have to write down an equation relating the variation of \( \bar{\alpha}_\ell(\alpha_i) \) with that of \( \Lambda_\ell^R(\alpha_i) \) as \( \alpha_i \) varies. We use the consistency equation for \( \bar{\alpha}_\ell \) under the form

\[ \left[ (\mu_\ell + \bar{\alpha}_\ell) \Lambda_\ell^R \right]^{-d/2} = (I/h_0)\Lambda_\ell^{R-1} \kappa \cdot \int_0^\infty dx \left( \mu_\ell \Lambda_\ell^R + x \right)^{-d/2} \exp(-x)J_\ell^{-1} H_\ell(\alpha_\ell - \bar{\alpha}_\ell) \tilde{H}_0(\bar{\alpha}_\ell) \quad (6.1a) \]

with

\[ J_\ell = 1 + \left( \alpha_\ell / \Lambda_\ell^R \right) \partial \Lambda_\ell^R / \partial \alpha_\ell \quad (6.1b) \]

and obtain a relation between \( \delta \bar{\alpha}_\ell, \delta \mu_\ell \) and \( \delta \Lambda_\ell^R \) by differentiating \((6.1a)\). This will give us the two coefficients \( A_\ell \) defined by

\[ \delta \bar{\alpha}_\ell / \bar{\alpha}_\ell \equiv A_\ell \delta \Lambda_\ell^R / \Lambda_\ell^R. \quad (6.2) \]

Secondo, we determine \( \Lambda_\ell(\alpha_i) \) with the help of \((5.19)\). \( \delta \Lambda_\ell^R \) will then be expressed as a function of \( \delta \bar{\alpha}_+ \) and \( \delta \bar{\alpha}_- \) using \((5.21)\) and \((5.6)\). Then, \((6.2)\) will give a system of equations for \( d\bar{\alpha}_-/d\alpha_\ell \) and \( d\bar{\alpha}_+/d\alpha_\ell \). Finally, this system will be solved and \( \bar{\alpha}_-(\alpha_\ell) \) and \( \bar{\alpha}_+(\alpha_\ell) \) determined as well as \( H_0(\bar{\alpha}_\ell), \alpha_\ell \) being either \( \alpha_\ell^+ \) or \( \alpha_\ell^- \), i.e. a side- or center-variable (\( \bar{\alpha}_\ell(\alpha_{i+}) \) and \( \bar{\alpha}_\ell(\alpha_{i-}) \) will be different functions). We start with the
determination of $A_\ell$.

**A - Determination of $A_\ell$**

Let us first define $z_\ell$, $\zeta_\ell$ and $\varepsilon_\ell$ such that

\[ z_\ell = (2/d) \left( \mu_\ell + \bar{\alpha}_\ell \right) \Lambda_\ell^R \partial \log \left( J_\ell^{-1} H_\ell \tilde{H}_0 \right) / \partial (\bar{\alpha}_\ell \Lambda_\ell^R) \quad (6.3a) \]

\[ \zeta_\ell = (2/d) \left( \mu_\ell + \bar{\alpha}_\ell \right) \Lambda_\ell^R \partial \log \left( J_\ell^{-1} H_\ell \tilde{H}_0 \right) / \partial (\mu_\ell \Lambda_\ell^R) \quad (6.3b) \]

\[ \varepsilon_\ell = 1 - \frac{1}{2} \left[ (\mu_\ell + \bar{\alpha}_\ell) \Lambda_\ell^R \right]^{-d/2} \left( \beta/\Lambda_\ell^R \right) I_{\ell} \quad (6.3c) \]

We remind us that $\beta = t/2 \tilde{a}_+ \Lambda/m^2$ from (3.6b) and $bs \to 0$. If (6.1a) is written

\[ \left[ (\mu_\ell + \bar{\alpha}_\ell) \Lambda_\ell^R \right]^{-d/2} = I_{\ell} \quad (6.4) \]

we obtain, taking variations,

\[ -d/2 \left[ (\mu_\ell + \bar{\alpha}_\ell) \Lambda_\ell^R \right]^{-d/2} \delta \left[ (\mu_\ell + \bar{\alpha}_\ell) \Lambda_\ell^R \right] = - \left[ (1 - dL/(2I) - (1 - dL/(2I))\beta \right] \left[ (1 - dL/(2I)) \beta \delta \tilde{a}_+ / \tilde{a}_+ + \delta \Lambda_\ell^R / \Lambda_\ell^R \right] I_{\ell} \]

\[ + d/2 \left[ z_\ell \delta \left( \bar{\alpha}_\ell \Lambda_\ell^R \right) \right] \left[ (\mu_\ell + \bar{\alpha}_\ell) \Lambda_\ell^R \right]^{-d/2-1} I_{\ell} \]

\[ - d/2 \delta \left( \mu_\ell \Lambda_\ell^R \right) (1 - \varepsilon_\ell) \left[ (\mu_\ell + \bar{\alpha}_\ell) \Lambda_\ell^R \right]^{-d/2-1} \quad (6.5) \]

or replacing $dL/(2I)$ by $2/3$, $d/2$ by $2$ and multiplying by $(2/d)\left[ (\mu_\ell + \bar{\alpha}_\ell) \Lambda_\ell^R \right]^{d/2+1}$,

\[ (1 + z_\ell) \delta \left( \bar{\alpha}_\ell \Lambda_\ell^R \right) + (\varepsilon_\ell + \zeta_\ell) \delta \left( \mu_\ell \Lambda_\ell^R \right) = \frac{1}{2} \left[ (\mu_\ell + \bar{\alpha}_\ell) \Lambda_\ell^R \right] \left( \beta/3 \delta \tilde{a}_+ / \tilde{a}_+ + \delta \Lambda_\ell^R / \Lambda_\ell^R \right) \quad (6.6) \]

or
This provides us with two equations. Two more equations come from the relations (derived in [15])

\[
1 + \bar{\alpha}_-/\mu_- = y^{-1} \tag{6.8a}
\]

\[
1 + \mu_+/ar{\alpha}_+ = 2y^{-1} \tag{6.8b}
\]

and a fifth equation can be derived from \( y = (2\bar{\alpha}_+/\bar{\alpha}_-)^{1/2} \). So that five equations arise for \( \delta\bar{\alpha}_+, \delta\mu_+, \delta\bar{\alpha}_-, \delta\mu_- \) and \( \delta y \) which can be expressed as functions of \( \delta\Lambda_+^R \) and \( \delta\Lambda_-^R \). The solution of this five equations system is easy to obtain. Defining \( a, b, c, d, \Delta_-, \Delta_+ \) as

\[
a \equiv 1/2 \ (\varepsilon_- + \zeta_-) [y/(1-y)] - \beta/6 \tag{6.9a}
\]

\[
b \equiv (1 + z_-)(1-y) + (\varepsilon_- + \zeta_-) y[1 - 1/2/(1-y)] \tag{6.9b}
\]

\[
c \equiv (1 + z_+)y/2 - \beta/6 + (\varepsilon_+ + \zeta_+) (1 - y/2) [1 - 1/(2-y)] \tag{6.9c}
\]

\[
d \equiv (\varepsilon_+ + \zeta_+) (1 - y/2)/(2-y) \tag{6.9d}
\]

\[
\Delta_- \equiv [1/2 - (1 + z_-)(1-y) - (\varepsilon_- + \zeta_-) y] \delta\Lambda_+^R/\Lambda_-^R \tag{6.9e}
\]

\[
\Delta_+ \equiv [1/2 - (1 + z_+)y/2 - (\varepsilon_+ + \zeta_+) (1 - y/2)] \delta\Lambda_+^R/\Lambda_+^R \tag{6.9f}
\]

the result is

\[
A_+ = [1/2 - (1 + z_+)y/2 - (\varepsilon_+ + \zeta_+) (1 - y/2)] [d (\Delta_-/\Delta_+) - b] /(ad - bc) \tag{6.10a}
\]
$A_- = \left[1/2 - (1 + z_-)(1 - y) - (\varepsilon_- + \zeta_-) y \right] \left[a(\Delta_+ / \Delta_-) - c\right] / (ad - bc)$ . (6.10b)

The next step is the calculation of $\Lambda_\ell(\alpha_i)$.

**B - Determination of $\Lambda_\ell(\alpha_i)$**

Let us rewrite (5.19) under the form

$$\alpha_\ell \Lambda_\ell(\alpha_i) + \alpha_i \Lambda_i(\alpha_\ell) = F_{i\ell}(\alpha_i, \alpha_\ell)$$

(6.11a)

$$F_{i\ell} \equiv (I - d/2 \ L)(1 - \beta^2)\alpha_i \alpha_\ell / h_0^2$$

$$- d/2 \left[ (\alpha_\ell / h_0) \mu_i / (\mu_i + \alpha_i) + (\alpha_i / h_0) \mu_\ell / (\mu_\ell + \alpha_\ell) \right] .$$

(6.11b)

If $i$ and $\ell$ are both side-propagators or center-propagators, a symmetry exists exchanging $i$ and $\ell$, i.e. $\Lambda_\ell$ and $\Lambda_i$ should be the same function. In particular taking $\alpha_i$ and $\alpha_\ell$ at their common mean-value $\bar{\alpha}_+$ or $\bar{\alpha}_-$ we get

$$\Lambda_{\ell+}(\bar{\alpha}_{i+}) = \Lambda_{i+}(\bar{\alpha}_{\ell+})$$

(6.12a)

$$\Lambda_{\ell-}(\bar{\alpha}_{i-}) = \Lambda_{i-}(\bar{\alpha}_{\ell-})$$

(6.12b)

with

$$\bar{\alpha}_{i+} = \bar{\alpha}_{\ell+} = \bar{\alpha}_+ ; \quad \bar{\alpha}_{i-} = \bar{\alpha}_{\ell-} = \bar{\alpha}_- .$$

(6.12c)

The second constraint we shall impose is that we want the equation for $\bar{\alpha}_\ell$ to be the same whatever $i$ is, i.e. $i_+$ or $i_-$. This can be approximately realized by demanding

$$\Lambda_{\ell+}(\bar{\alpha}_{i+}) = \Lambda_{\ell+}(\bar{\alpha}_{i_-})$$

(6.13a)

$$\Lambda_{\ell-}(\bar{\alpha}_{i+}) = \Lambda_{\ell-}(\bar{\alpha}_{i_-})$$

(6.13b)

the same being also true exchanging $i$ and $\ell$. Of course, a priori, it could be that $\bar{\alpha}_\ell$ would be the same using two different equations for it, but that would be some
sort of a miracle. So we feel safer imposing (6.13), which moreover will be easily implementable.

Let us start by writing $\mu_\ell/(\mu_\ell + \alpha_\ell)$ under the form

$$\frac{\mu_\ell}{(\mu_\ell + \alpha_\ell)} = \frac{(\mu_\ell/(\mu_\ell + \bar{\alpha}_\ell))[1 - (\alpha_\ell - \bar{\alpha}_\ell)/(\mu_\ell + \alpha_\ell)]}{(\mu_\ell + \bar{\alpha}_\ell)}, \quad (6.14)$$

$\mu_i/(\mu_i + \alpha_i)$ being also treated in the same way. Plugging this into (6.11b) we get $h_0 F_{i\ell}$ under the form

$$h_0 F_{i\ell} = \Lambda(1 + \beta)\alpha_i \alpha_\ell - (d/2) \{\alpha_i [1 - \alpha_\ell/ (\mu_\ell + \alpha_\ell)] \mu_\ell/ (\mu_\ell + \bar{\alpha}_\ell)$$

$$+ \alpha_i [\bar{\alpha}_\ell/ (\mu_\ell + \alpha_\ell)] \mu_\ell/ (\mu_\ell + \bar{\alpha}_\ell) + \alpha_\ell [1 - \alpha_i/ (\mu_i + \alpha_i)] \mu_i/ (\mu_i + \bar{\alpha}_i)$$

$$+ \alpha_\ell [\bar{\alpha}_i/ (\mu_i + \alpha_i)] \mu_i/ (\mu_i + \bar{\alpha}_i)\}$$

$$= \alpha_i \alpha_\ell \{\Lambda(1 + \beta) + d/2 \left[ (\mu_\ell + \alpha_\ell)^{-1} \mu_\ell/ (\mu_\ell + \bar{\alpha}_\ell) + (\mu_i + \alpha_i)^{-1} \mu_i/ (\mu_i + \bar{\alpha}_i) \right]$$

$$- \alpha_i (d/2) [1 + \bar{\alpha}_i/ (\mu_i + \alpha_i)] \mu_i/ (\mu_i + \bar{\alpha}_i)$$

$$- \alpha_\ell (d/2) [1 + \bar{\alpha}_i/ (\mu_i + \alpha_i)] \mu_i/ (\mu_i + \bar{\alpha}_i)\} \quad (6.15)$$

which has the advantage that in every term a factor $\alpha_i$ or $\alpha_\ell$ exists and that every factor $(\mu_\ell + \alpha_\ell)^{-1}$ is multiplied by $\mu_\ell/\mu_\ell + \bar{\alpha}_\ell$ which is less than one, thus minimizing the $\alpha_\ell$ variation of such a term (the same being of course true for $(\mu_i + \alpha_i)^{-1}$).

Using (6.11a) we are now able to write

$$h_0 \Lambda_i(\alpha_\ell) = x_{i\ell} \Lambda(1 + \beta) \alpha_\ell + a_{i\ell}(d/2) \left[ \alpha_\ell/ (\mu_\ell + \alpha_\ell) \right] \mu_\ell/ (\mu_\ell + \bar{\alpha}_\ell)$$

$$+ b_{i\ell}(d/2) \left[ \alpha_\ell/ (\mu_i + \alpha_i) \right] \mu_i/ (\mu_i + \bar{\alpha}_i)$$

$$- (d/2) [1 + \bar{\alpha}_i/ (\mu_i + \alpha_i)] \mu_i/ (\mu_i + \bar{\alpha}_i) \quad (6.16a)$$

$$h_0 \Lambda_\ell(\alpha_i) = (1 - x_{i\ell}) \Lambda(1 + \beta) \alpha_i + (1 - a_{i\ell})(d/2) \left[ \alpha_i/ (\mu_\ell + \alpha_\ell) \right] \mu_\ell/ (\mu_\ell + \bar{\alpha}_\ell)$$

$$+ (1 - b_{i\ell})(d/2) \left[ \alpha_i/ (\mu_i + \alpha_i) \right] \mu_i/ (\mu_i + \bar{\alpha}_i)$$

$$- (d/2) [1 + \bar{\alpha}_i/ (\mu_i + \alpha_i)] \mu_i/ (\mu_i + \bar{\alpha}_i) \quad (6.16b)$$

If $i$ and $\ell$ are both + or −, the $i \leftrightarrow \ell$ symmetry imposes

$$1 - x_{i\ell} = x_{i\ell} \quad (6.17a)$$
\[ a_{i\ell} = 1 - b_{i\ell} \quad (6.17b) \]
\[ b_{i\ell} = 1 - a_{i\ell} \quad (6.17c) \]

which leads to (we recall that \( \mu_-/(\mu_- + \bar{\alpha}_-) = y, \mu_+/\mu_+ + \bar{\alpha}_+ = 1 - y/2, y = (2\bar{\alpha}_+\bar{\alpha}_-)^{1/2} \))

\[
h_0\Lambda_{i+}(\alpha_{i+}) = \frac{1}{2}\Lambda(1 + \beta)\alpha_{i+} + (d/2) \left[ a_{i+}\alpha_{i+}/(\mu_+ + \alpha_{i+}) + b_{i+}\alpha_{i+}/(\mu_+ + \alpha_{i+}) - (1 + \alpha_+/(\mu_+ + \alpha_{i+})) \right] (1 - y/2) \quad (6.18a)
\]
\[
h_0\Lambda_{i-}(\alpha_{i-}) = \frac{1}{2}\Lambda(1 + \beta)\alpha_{i-} + (d/2) \left[ a_{i-}\alpha_{i-}/(\mu_- + \alpha_{i-}) + b_{i-}\alpha_{i-}/(\mu_- + \alpha_{i-}) - (1 + \alpha_-/(\mu_- + \alpha_{i-})) \right] y \quad (6.19a)
\]
\[
h_0\Lambda_{i+}(\alpha_{i+}) = \frac{1}{2}\Lambda(1 + \beta)\alpha_{i+} + (d/2) \left[ (1 - a_{i+})\alpha_{i+}/(\mu_+ + \alpha_{i+}) + (1 - b_{i+})\alpha_{i+}/(\mu_+ + \alpha_{i+}) - (1 + \alpha_+/(\mu_+ + \alpha_{i+})) \right] (1 - y/2) \quad (6.18b)
\]
\[
h_0\Lambda_{i-}(\alpha_{i-}) = \frac{1}{2}\Lambda(1 + \beta)\alpha_{i-} + (d/2) \left[ (1 - a_{i-})\alpha_{i-}/(\mu_- + \alpha_{i-}) + (1 - b_{i-})\alpha_{i-}/(\mu_- + \alpha_{i-}) - (1 + \alpha_-/(\mu_- + \alpha_{i-})) \right] y \quad (6.19b)
\]

for the \( \ell = i = + \) case and

\[
h_0\Lambda_{i+}(\alpha_{i+}) = \frac{1}{2}\Lambda(1 + \beta)\alpha_{i+} + (d/2) \left[ a_{i+}\alpha_{i+}/(\mu_+ + \alpha_{i+}) + b_{i+}\alpha_{i+}/(\mu_+ + \alpha_{i+}) - (1 + \alpha_+/(\mu_+ + \alpha_{i+})) \right] y \quad (6.19a)
\]
\[
h_0\Lambda_{i-}(\alpha_{i-}) = \frac{1}{2}\Lambda(1 + \beta)\alpha_{i-} + (d/2) \left[ (1 - a_{i-})\alpha_{i-}/(\mu_- + \alpha_{i-}) + (1 - b_{i-})\alpha_{i-}/(\mu_- + \alpha_{i-}) - (1 + \alpha_-/(\mu_- + \alpha_{i-})) \right] y \quad (6.19b)
\]

for the \( \ell = i = - \) case. We have taken the notation \( a_{i+} = 1 - a_{i+}, b_{i+} = 1 - b_{i+}, a_{i-} = 1 - a_{i-}, b_{i-} = 1 - b_{i-} \). Then, \((6.17b)\) and \((6.17c)\) give

\[
a_{i+} + b_{i+} = 1 \quad (6.20a)
\]
\[
a_{i-} + b_{i-} = 1 \quad . \quad (6.20b)
\]

For \( \ell = + \) and \( i = - \), we get

\[
h_0\Lambda_{i+}(\alpha_{i+}) = x\Lambda(1 + \beta)\alpha_{i+} + (d/2) \left[ u(1 - y/2)\alpha_{i+}/(\mu_+ + \alpha_{i+}) + z y \alpha_{i-}/(\mu_- + \alpha_{i-}) - y(1 + \alpha_+/(\mu_+ + \alpha_{i+})) \right] \quad (6.21a)
\]
\[ h_0 \Lambda_{\epsilon_+}(\alpha_{\ell_+}) = (1 - x) \Lambda(1 + \beta)\alpha_{\ell_+} + (d/2) \left[ (1 - u)(1 - y/2)\alpha_{\ell_+}/(\mu_+ + \alpha_{\ell_+}) + (1 - z)y\alpha_{\ell_+}/(\mu_- + \alpha_{\ell_-}) - (1 - y/2)(1 + \bar{\alpha}_+/(\mu_+ + \alpha_{\ell_+})) \right] \cos(6.21b) \]

and also the same equations exchanging \( i \) and \( \ell \) for \( \ell = - \) and \( i = + \). Here we have taken the notations \( x = 1 - x_{-+} = x_{+-} \), \( u = 1 - a_{-+} = a_{+-} \), \( z = 1 - b_{-+} = b_{+-} \).

Now, if we want to satisfy the constraint (6.13a), \( \Lambda_{\ell_+}(\bar{\alpha}_{i_+}) = \Lambda_{\ell_-}(\bar{\alpha}_{i_-}) \), we have

\[ \frac{1}{2} \Lambda(1 + \beta)\alpha_{\ell_+} + (d/2)(1 - y/2) \left[ a_{+}\bar{\alpha}_{i_+}/(\mu_+ + \alpha_{\ell_+}) + b_{+}y/2 - (1 + y/2) \right] = x\Lambda(1 + \beta)\alpha_{\ell_-} + (d/2) \left[ u(1 - y/2)\bar{\alpha}_{i_-}/(\mu_+ + \alpha_{\ell_+}) + z\bar{\alpha}_{i_-}/(\mu_- + \alpha_{\ell_-}) - y(1 + 1 - y) \right] \] (6.22a)

and for the constraint (6.13b), \( \Lambda_{\ell_-}(\bar{\alpha}_{i_+}) = \Lambda_{\ell_-}(\bar{\alpha}_{i_-}) \)

\[ \frac{1}{2} \Lambda(1 + \beta)\alpha_{\ell_-} + (d/2)y \left[ a_{-}\bar{\alpha}_{i_-}/(\mu_- + \alpha_{\ell_-}) + b_{-}(1 - y) - (1 + 1 - y) \right] = (1 - x)\Lambda(1 + \beta)\alpha_{\ell_-} + (d/2) \left[ (1 - u)(1 - y/2)y/2 + (1 - z)y\bar{\alpha}_{i_+}/(\mu_- + \alpha_{\ell_-}) - (1 - y/2)(1 + y/2) \right] \] (6.22b)

The effect of integrating over \( \alpha_{\ell_+} \) or \( \alpha_{\ell_-} \) will be taken into account by replacing \((\mu_+ + \alpha_{\ell_+})^{-1}\) by \((1 - x_+)(\mu_+ + \bar{\alpha}_{\ell_+})^{-1}\) and \((\mu_- + \alpha_{\ell_-})^{-1}\) by \((1 - x_-)(\mu_- + \bar{\alpha}_{\ell_-})^{-1}\), \( x_+ \) and \( x_- \) being two parameters to be determined. Then, the constraints (6.22a) and (6.22b) read

\[ (2/d)(x - 1/2 \bar{\alpha}_+ / \bar{\alpha}_- )\bar{\alpha}_- \Lambda(1 + \beta) + (1 - y/2)(1 + a_{+}x_{+}y/2) + u(\bar{\alpha}_- / \bar{\alpha}_+)(1 - x_{+})(1 - y/2)y/2 - zy - (1 - z)(2 - y) = 0 \] (6.23a)

\[ (2/d)[(1 - x)\bar{\alpha}_+ / \bar{\alpha}_- - 1/2] \bar{\alpha}_- \Lambda(1 + \beta) + y[1 + a_{-}x_{-}(1 - y)] + (1 - z)(\bar{\alpha}_+ / \bar{\alpha}_-)(1 - x_{-})y(1 - y) - (1 - u)(1 - y/2) - u(1 - y/2)(1 + y/2) = 0 \] (6.23b)

Solving the constraints we can express \( u \) and \( z \) as a function of \( x, a_+, a_-, x_+ \) and \( x_- \).

In practice, we keep the constraints under the form (6.23) because, then, numerical calculations are much more stable. Having explicit expressions for \( \Lambda_{\ell}(\alpha_i) \) we can
proceed further and calculate $\delta \bar{\alpha}_\ell(\alpha_i)$.

C - Solving equations for $\delta \bar{\alpha}_\ell(\alpha_i)$

Let us recall that we were looking for the variation of $\Lambda^R_\ell$ given by

$$
\Lambda^R_\ell = \Lambda + d/2 \ L \ E^0_\ell + \Lambda_\ell(\alpha_i)
$$

which is merely rewriting (5.21) with an interchange of $i$ and $\ell$. Let us now, as a preliminary task, evaluate the variation of $\Lambda$, the unrenormalized scale, with respect to $\bar{\alpha}^+ (\alpha_i)$.

(2.17), (3.6b), (3.6c) (and the limit $b_5 \to 0$) give

$$
d\Lambda/d\alpha_i = - [(I - dL/2)/h_0] \ d\bar{\beta}/d\alpha_i
$$

which leads to

$$
\Lambda^{-1}d\Lambda/d\alpha_i = - \beta \ \bar{\alpha}_+^{-1} \ d\bar{\alpha}_+/d\alpha_i
$$

Defining the variable $x_{\ell i}$ through

$$
x_{\ell i} \equiv \bar{\alpha}_\ell^{-1} \ d\bar{\alpha}_\ell/d\alpha_i
$$

we get (see (6.2))

$$
x_{\ell i} = A_{\ell i} \ \Lambda^R_\ell^{-1} \left( d\Lambda/d\alpha_i + (d/2) L \ dE^0_\ell/d\alpha_i + d\Lambda_\ell(\alpha_i)/d\alpha_i \right)
$$

where $A_{\ell i}$ is $A_{\ell}(\alpha_i)$, the dependence on $\alpha_i$ coming from the dependence of $\Delta_-$ and $\Delta_+$ on $\delta \Lambda^R_\ell/\Lambda^R_\ell$ and $\delta \Lambda^R_\ell/\Lambda^R_\ell$ respectively (see (6.9c) and (5.11)). Defining ($c_{\ell i}$ and $d_{\ell i}$ being read on (6.18), (6.19) and (6.21))

$$
c_{\ell i} + d_{\ell i}/[(\mu_i + \alpha_i)L] \equiv (\Lambda^R_\ell)^{-1} \ d\Lambda_\ell(\alpha_i)/d\alpha_i
$$

$$
z_{\ell R} \equiv \Lambda/\Lambda^R_\ell
$$
and using (5.6), (5.20) through
\[ E_0^\ell = \left[(1 - y/2)\bar{\alpha}^{-1}_- d\bar{\alpha}_- / d\alpha_\ell + (y/2 + \eta)\bar{\alpha}^{-1}_+ d\bar{\alpha}_+ / d\alpha_\ell\right], \quad (6.30) \]
we get from (6.28) the system of equations
\[
\left(\beta z_{R_+} + A^{-1}_+\right)x_{++} = c_{++} + d_{++} / [(\mu_+ + \alpha_+)L] \\
+ d/2 (L/\Lambda^R) d/d\alpha_+ [(1 - y/2)x_{+-} + (y/2 + \eta)x_{++}] 
\]
\[
(6.31a) \\
(\beta z_{R_+} + A^{-1}_+\right)x_{+-} = c_{+-} + d_{+-} / [(\mu_- + \alpha_-)L] \\
+ d/2 (L/\Lambda^R) d/d\alpha_- [(1 - y/2)x_{-+} + (y/2 + \eta)x_{++}] 
\]
\[
(6.31b) \\
A^{-1}_- x_{-+} + \beta z_{R_-} x_{++} = c_{-+} + d_{-+} / [(\mu_+ + \alpha_+)L] \\
+ d/2 (L/\Lambda^R) d/d\alpha_+ [(1 - y/2)x_{-+} + (y/2 + \eta)x_{++}] 
\]
\[
(6.31c) \\
A^{-1}_- x_{-+} + \beta z_{R_-} x_{++} = c_{--} + d_{--} / [(\mu_- + \alpha_-)L] \\
+ d/2 (L/\Lambda^R) d/d\alpha_- [(1 - y/2)x_{--} + (y/2 + \eta)x_{++}] 
\]
\[
(6.31d) \\
(6.32a) \\
(6.32b) \\
(6.32c) \\
(6.32d)
\]
We will now write \(x_{\ell i}\) as a power series in \([(\mu_i + \alpha_i)L]^{-1}\), coefficients depending only on \(y\). This is because all dimensionless quantities may be written as a function of \(y\) and \((\mu_i + \alpha_i)L\). Limiting ourselves to the constant term and the first power in \([(\mu_i + \alpha_i)L]^{-1}\) (which will be sufficient as we show next) we write
\[
x_{\pm \ell} = a^0_{\pm \ell} + a_{\pm \ell} / [(\mu_+ + \alpha_+)L] \quad (6.32a) \\
x_{\pm \ell} = b^0_{\pm \ell} + b_{\pm \ell} / [(\mu_- + \alpha_-)L] \quad (6.32b)
\]
where the first expression occurs when \(\alpha_i\) is \(+\) variable and the second when \(\alpha_i\) is \(-\) variable. We also have two useful relations stemming from \(y = (2\bar{\alpha}_+ / \bar{\alpha}_-)^{1/2}\)
\[
dy / d\alpha_+ = (x_{++} - x_{-+}) y/2 \quad (6.33a)
\]
\[
\frac{dy}{d\alpha_-} = (x_+ - x_-) y/2 \quad .
\] (6.33b)

We now make the observation that

\[
d/d\alpha_i \left[(\mu_i + \alpha_i)L\right]^{-1} = -L \left[(\mu_i + \alpha_i)L\right]^{-2}
\] (6.34)

diverges as \( L \to \infty \). Therefore, the coefficients of diverging terms should be \( O(1/L) \) in order to cancel the divergence. This entails the constraints

\[
(1 - y/2)a_{-\ell} + (y/2 + \eta)a_{+\ell} = O(1/L)
\] (6.35a)

\[
(1 - y/2)b_{-\ell} + (y/2 + \eta)b_{+\ell} = O(1/L)
\] (6.35b)

easily derived from (6.31).

Let us take the derivative in (6.31a)

\[
d/d\alpha_+ \left[(1 - y/2)x_+ + (y/2 + \eta)x_+\right] = (x_+ - x_-) y/2.
\]

\[
\left\{\frac{1}{(\mu_+ + \alpha_+)L} \frac{d}{dy} \left[(1 - y/2)a_{-+} + (y/2 + \eta)a_{++}\right]
\right.
\]

\[
+ \frac{d}{dy} \left[(1 - y/2)a_{0-} + (y/2 + \eta)a_{0+}\right]\right\} + \text{h.o.}
\] (6.36)

where h.o. means higher order terms \( \sim [L(\mu_+ + \alpha_+)]^{-n}, \ n > 1 \). Now, due to the constraints (6.35), we see that the first term in the bracket of the right-hand side of (6.36) vanishes. This property also holds for higher-order coefficients because of constraints similar to (6.35) acting for higher-orders as it is easy to verify. Keeping track of powers of \( [L(\mu_+ + \alpha_+)]^{-1} \) equal to zero and one, one gets the following constraints obtained by using (6.36) in (6.31), identifying coefficients for a given power of \( [L(\mu_+ + \alpha_+)]^{-1} \) and introducing \( E^0_{i\ell} \equiv E^0_{i\ell}(\alpha_i) \),

\[
\left(\beta z_{R_+} + A_{++}^{-1}\right) a_{++}^0 = c_{++} + \left[dyL/(4\Lambda_R^+)\right]\left(a_{++}^0 - a_{+-}^0\right) dE_{++}^0/dy
\] (6.37a)

\[
\left(\beta z_{R_+} + A_{+-}^{-1}\right) b_{+-}^0 = c_{+-} + \left[dyL/(4\Lambda_R^+)\right]\left(b_{+-}^0 - b_{-+}^0\right) dE_{+-}^0/dy
\] (6.37b)

\[
A_{--}^{-1} a_{--}^0 + \beta z_{R_+} a_{++}^0 = c_{--} + \left[dyL/(4\Lambda_R^+)\right]\left(a_{++}^0 - a_{--}^0\right) dE_{--}^0/dy
\] (6.37c)
\[ A_{-1}^0 b_{-}^0 + \beta z_R b_{+}^0 = c_{-} + \left[ dyL/(4\Lambda^R) \right] \left( b_{+}^0 - b_{-}^0 \right) dE_{-}/dy \]  

(6.37d)

i.e. four equations determining \( a_{+++}^0, a_{+-}^0, b_{+-}^0, b_{-}^- \) with

\[ E_{+++}^0 = (1 - y/2)a_{+++}^0 + (y/2 + \eta)a_{+++}^0 \]  

(6.38a)

\[ E_{+-}^0 = (1 - y/2)b_{+-}^0 + (y/2 + \eta)b_{+-}^0 \]  

(6.38b)

\[ E_{-+}^0 = (1 - y/2)a_{-+}^0 + (y/2 + \eta)a_{-+}^0 \]  

(6.38c)

\[ E_{--}^0 = (1 - y/2)b_{--}^0 + (y/2 + \eta)b_{--}^0 \]  

(6.38d)

\( b_{+-}^0, b_{++}^0, a_{++}^0, a_{+-}^0 \) are undetermined but this is welcome because one should have

\[ E_{+++}^0 = E_{+-}^0 = E_{+}^0 \]  

(6.39a)

\[ E_{--}^0 = E_{-+}^0 = E_{-}^0 \]  

(6.39b)

i.e. \( E_0^0 \) should be independent of \( \alpha_i \). \( \Lambda^R \) does not depend on any other \( \alpha \) when all \( \alpha \)'s, \( i \neq \ell \), are integrated over). Therefore, \( E_+^0 \) and \( E_-^0 \) are given by (6.38a) and (6.38d) respectively, which express \( E_+^0 \) and \( E_-^0 \) as a function of \( (a_{++}^0, a_{+++}^0) \) and \( (b_{--}^0, b_{+-}^0) \) respectively. One remarkable thing is that only zero-order coefficients enter in the expression of \( E_0^0 \) because it is proportional to the linear combination \((1 - y/2)x_{-\ell} + (y/2 + \eta)x_{+\ell} \). The knowledge of higher order coefficients is then unnecessary for our purpose. However, a constraint is visible, looking at (6.31b) and (6.31c), which demands that the order of derivation should not matter, namely

\[ d/d\alpha_-(1 - y/2)x_{-+} + (y/2 + \eta)x_{++} \]

\[ d/d\alpha_+(1 - y/2)x_{-} + (y/2 + \eta)x_{+} \]  

(6.40)

or

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\[ \Lambda_+^R \left\{ \left( \beta z_{R_+} + A_{++++}^{-1} \right) \left( b_+^0 + b_+ / [(\mu_- + \alpha_-) L] \right) - c_+ - d_+ / [(\mu_- + \alpha_-) L] \right\} \\
= \Lambda_+^R \left\{ \left( A_{++}^{-1} a_{++}^0 + \beta z_{R_+} a_{++}^0 \right) + \left( A_{++}^{-1} a_{++} - \beta z_{R_+} a_{++} \right) / [(\mu_+ + \alpha_+) L] \\
- c_+ - d_+ / [(\mu_- + \alpha_-) L] \right\}. \tag{6.41} \]

Identifying to zero the coefficients of \([(\mu_- + \alpha_-) L]^{-1}\) and \([(\mu_+ + \alpha_+) L]^{-1}\) and balancing constant coefficients we get

\[ \Lambda_+^R \left[ \left( \beta z_{R_+} + A_{+++}^{-1} \right) b_+ - c_+ \right] \\
= \Lambda_+^R \left[ \left( \beta z_{R_+} a_{++}^0 + A_{++}^{-1} a_{++}^0 \right) - c_+ \right] \tag{6.42a} \]

\[ \left( \beta z_{R_-} a_{+++} + A_{+++}^{-1} a_{+++} \right) = d_+ \tag{6.42b} \]

Looking at (6.37b) and (6.37c), we see that (6.42a) is equivalent to

\[ \left( b_+ - b_- \right) dE_0^+/dy = \left( a_{+++}^0 - a_{+++}^0 \right) dE_-^+/dy \tag{6.43} \]

On the other hand, (6.31b) and (6.31c) give, looking at coefficients of \([(\mu_+ + \alpha_-) L]^{-1}\) and \([(\mu_- + \alpha_-) L]^{-1}\) respectively,

\[ \left( \beta z_{R_+} + A_{+++}^{-1} \right) b_+ = d_+ + [dyL/(4\Lambda_{R_+})] (b_+ - b_-) dE_+^0/dy \tag{6.44a} \]

\[ \beta z_{R_-} a_{+++} + A_{+++}^{-1} a_{+++} = d_+ + [dyL/(4\Lambda_{R_+})] (a_{+++} - a_{+++}) dE_0^+/dy \tag{6.44b} \]

which combined with (6.42b) and (6.42c) respectively give

\[ (b_+ - b_-) dE_+^0/dy = 0 \tag{6.45a} \]

\[ (a_{+++} - a_{+++}) dE_0^-/dy = 0 \tag{6.45b} \]
In general (6.43) and (6.45a), (6.45b) give three constraints. However, two constraints only are sufficients if

\[ dE_0^+/dy = 0 \]  \hspace{1cm} (6.46a)

\[ dE_0^-/dy = 0 \]  \hspace{1cm} (6.46b)

As this is the minimum-constraint choice, we will stick to (6.46a) and (6.46b) as the two constraints to be imposed. Using them in (6.37) gives the solution

\[ a_{++}^0 = c_{++} \left( \beta z_R + A_{++}^{-1} \right)^{-1} \]  \hspace{1cm} (6.47a)

\[ a_{--}^0 = A_{--} \left( c_{--} - \beta z_R a_{++}^0 \right) \]  \hspace{1cm} (6.47b)

\[ b_{++}^0 = c_{++} \left( \beta z_R + A_{++}^{-1} \right)^{-1} \]  \hspace{1cm} (6.47c)

\[ b_{--}^0 = A_{--} \left( c_{--} - \beta z_R b_{++}^0 \right) \]  \hspace{1cm} (6.47d)

So, finally, we have two equations for \( \bar{\alpha}_- \) and \( \bar{\alpha}_+ \), four constraints (6.23a), (6.23b), (6.46a), (6.46b) and the seven parameters \( x, u, z, a_+, a_-, x_+, x_- \). Another fifth constraint will come from the fact that \( I \sum_{i=1}^{I} \bar{\alpha}_i = h_0 \). This will be discussed in section 10 where the numerical solutions will be presented. We still need some work to be done before tackling the numerical solution of our equations. In particular we have to give explicit expressions for \( J_\ell, z_\ell, \zeta_\ell, \delta \Lambda_\ell^R(\alpha_i) \) and the derivative \( d\varepsilon_\ell/dy \) which enters in the expression of \( dE_\ell/dy \). All this will be done in the next section.

7 Explicit expressions for \( J_\ell, z_\ell, \zeta_\ell, \delta \Lambda_\ell, d\varepsilon_\ell/dy \)

We begin with the calculation of the Jacobian \( J_\ell = 1 + (\alpha_\ell/\Lambda_\ell^R) d\Lambda_\ell^R/d\alpha_\ell \) and therefore we need

\[ \Lambda_\ell^R = [(I - dL/2)/h_0] (1 - \beta) + d/2 L E_\ell^0 \]  \hspace{1cm} (7.1)
which is the expression for $\Lambda^R_{\ell}$ when all $\alpha_j$ ($j \neq \ell$) have been integrated with the mean-value theorem and therefore where the $\Lambda_{\ell}(\alpha_i)$ term present in (6.24) is omitted. Now, the expression for $E^0_{\ell}$ given by (6.38) is a function of $a^0_{-\ell}$ and $a^0_{+\ell}$, which themselves are functions of $c_{j\ell}$. We therefore need explicit expressions for $c_{j\ell}$ which will be obtained through the definition (6.29a) by (6.18), (6.19) and (6.21).

We get

\begin{align}
    h_0c_{++} &= 1/2 (1 + \beta)z_{R+} + d/2 a_+ (1 - y/2)/ [(\mu_+ + \alpha_{\ell+})\Lambda^R_{+}] \\ 
    h_0c_{+-} &= x(1 + \beta)z_{R+} + d/2 u(1 - y/2)/ [(\mu_+ + \alpha_{\ell+})\Lambda^R_{+}] \\ 
    h_0c_{-+} &= (1 - x)(1 + \beta)z_{R-} + d/2 (1 - z)y/ [(\mu_- + \alpha_{\ell-})\Lambda^R_-] \\ 
    h_0c_{--} &= 1/2 (1 + \beta)z_{R-} + d/2 a_- y/ [(\mu_- + \alpha_{\ell-})\Lambda^R_-].
\end{align}

(7.2a, 7.2b, 7.2c, 7.2d)

We are now ready to calculate $J_\ell$.

**A - Explicit expressions for $J_\ell$ and $\zeta_\ell$**

Taking derivatives and neglecting in a first approximation $\partial z_{R\ell}/\partial \alpha_{\ell}$ we get

\begin{align}
    \partial (h_0c_{++})/\partial \alpha_+ &= -d/2 \Lambda^R_+ a_+(1 - y/2)/ [(\mu_+ + \alpha_{\ell+})\Lambda^R_+]^2 \\ 
    \partial (h_0c_{-+})/\partial \alpha_+ &= 0 \\ 
    \partial (h_0c_{+-})/\partial \alpha_- &= 0 \\ 
    \partial (h_0c_{--})/\partial \alpha_- &= -d/2 \Lambda^R_- a_- y/ [(\mu_- + \alpha_{\ell-})\Lambda^R_-]^2.
\end{align}

(7.3a, 7.3b, 7.3c, 7.3d)

Using (6.38), (6.47) and

\[ L/h_0 = [L/(I - dL/2)] z_{R\ell} \Lambda^R_{\ell} / (1 - \beta) \]

(7.4)
we obtain

\[
J_+ = \left[1 - (d/2)^2 \left[ L/(I - dL/2) \right] \left[ z_{R_+} / (1 - \beta) \right] \right] (y/2 + \eta) \cdot \\
\left( \beta z_{R_+} + A_{++}^{-1} \right)^{-1} - (1 - y/2) A_{+}\beta z_{R_+} \left[ a_+ \left( 1 - y/2 \right) \alpha_+ \Lambda_+^R / \left[ (\mu_+ + \alpha_+) \Lambda_+^R \right]^2 \right] 
\]

(7.5a)

\[
J_- = \left[1 - (d/2)^2 \left[ L/(I - dL/2) \right] \left[ z_{R_-} / (1 - \beta) \right] \right] (1 - y/2) \cdot \\
A_{--} \alpha_- \Lambda_-^R / \left[ (\mu_- + \alpha_-) \Lambda_-^R \right]^2 \right] 
\]

(7.5b)

We remark that \( J_\ell \) is a function of \( \mu_\ell \Lambda_\ell^R \) and not of \( \bar{\alpha}_\ell \Lambda_\ell^R \). Because (neglecting \( \partial z_{R_\ell} / \partial (\mu_\ell \Lambda_\ell^R) \)) we will see that \( H_0(\bar{\alpha}_\ell) \) only depends on \( \bar{\alpha}_\ell \Lambda_\ell^R \), we will get \( \zeta_\ell \) from

\[
\zeta_\ell = 2/d \left( \mu_\ell + \bar{\alpha}_\ell \right) \Lambda_\ell^R \partial \log \left[ J_\ell^{-1} H_\ell(\alpha_\ell - \bar{\alpha}_\ell) \right] / \partial \left( \mu_\ell \Lambda_\ell^R \right) .
\]

(7.6)

However, looking at the expressions (4.12) and (4.22) for \( H_{i_+}(\alpha_{i_+} - \bar{\alpha}_{i_+}) \) and \( H_{i_-}(\alpha_{i_-} - \bar{\alpha}_{i_-}) \) respectively we see that

\[
\partial \log \left[ H_{\ell_+} (\alpha_+ - \bar{\alpha}_+) \right] / \partial \left( \mu_+ \Lambda_+^R \right) \sim \alpha_+ - \bar{\alpha}_+ 
\]

(7.7a)

\[
\partial \log \left[ H_{\ell_-} (\alpha_- - \bar{\alpha}_-) \right] / \partial \left( \mu_- \Lambda_-^R \right) \sim \alpha_- - \bar{\alpha}_- 
\]

(7.7b)

In a first approximation where the mean-values of the left-hand side is taken, we will neglect these contributions to \( \zeta_\ell \), so that finally we will have

\[
\zeta_\ell = 2/d \left( \mu_\ell + \bar{\alpha}_\ell \right) \Lambda_\ell^R \partial \log \left[ J_\ell^{-1} \right] / \partial \left( \mu_\ell \Lambda_\ell^R \right)
\]

(7.8)

which is easily obtained from (7.5).

**B - Explicit expression for** \( z_\ell \)

Looking at (5.22), we have (replacing \( i \) by \( \ell \)
\[
\log \tilde{H}_0(\bar{\alpha}_\ell) = \frac{d}{2} L \mathcal{E}_\ell^0 \bar{\alpha}_\ell \\
= \left(\frac{d}{2} L \mathcal{E}_\ell^0/\Lambda^R_\ell\right) \bar{\alpha}_\ell \Lambda^R_\ell \\
= (1 - z_{R_\ell}) \bar{\alpha}_\ell \Lambda^R_\ell \\
\tag{7.9}
\]

and therefore, neglecting \(\partial z_{R_\ell}/\partial (\bar{\alpha}_\ell \Lambda^R_\ell)\) in a first approximation,

\[
\partial \log \tilde{H}_0(\bar{\alpha}_\ell)/\partial (\bar{\alpha}_\ell \Lambda^R_\ell) = 1 - z_{R_\ell} \\
\tag{7.10}
\]

On the other hand looking at (7.5) we see that \(J_\ell\) does not depend on \(\bar{\alpha}_\ell \Lambda^R_\ell\) either than through \(z_{R_\ell}\). Neglecting again \(\partial z_{R_\ell}/\partial (\bar{\alpha}_\ell \Lambda^R_\ell)\) we conclude that the factor \(J_\ell^{-1}\) does not contribute to \(z_\ell\). Then, we have contributions from \(H_\ell(\alpha_\ell - \bar{\alpha}_\ell)\). First,

\[
\partial \log H_\ell+/\partial (\bar{\alpha}_+ \Lambda^R_+) = -t/2m^2(2 - y)^{-1}\mu_+ \Lambda/ \left[(\mu_+ + \alpha_+) \Lambda^R_+\right] \tag{7.11}
\]

where we have used \(<(2 - 1/\ell)^{-1}>(2 - y)^{-1}\). Taking the mean-value of (7.11) we get for our numerical resolution

\[
\partial \log H_\ell+/\partial (\bar{\alpha}_+ \Lambda^R_+) = -t/4m^2 z_{R_+}(1 - x_+) \\
= -t/4m^2 z_{R_+}(1 - x_+) \\
\tag{7.12}
\]

Now for the contribution of \(H_{\ell-}(\alpha_- - \bar{\alpha}_-)\), we get

\[
\partial \log H_{\ell-}/\partial (\bar{\alpha}_- \Lambda^R_-) = y^{-1}(1 - \beta) \left[2/3 \ell n C^st + \ell n(1 - \beta)\right] \cdot \\
(-\alpha_- \Lambda^R_-)^{-1} \{(\mu_-/\bar{\alpha}_-) [(\alpha_- - \bar{\alpha}_-) / (\mu_- + \bar{\alpha}_-)] - y\} \\
\tag{7.13}
\]

and, again, taking the mean-value to facilitate the numerical resolution

\[
\partial \log H_{\ell-}/\partial (\bar{\alpha}_- \Lambda^R_-) = -(1 - \beta) \left[2/3 \ell n C^st + \ell n(1 - \beta)\right] / (\bar{\alpha}_- \Lambda^R_-) \\
\tag{7.14}
\]

We now have \(z_\ell\) by adding the contribution obtained from \(\tilde{H}_0(\bar{\alpha}_\ell)\) and \(H_\ell(\alpha_\ell - \bar{\alpha}_\ell)\) and multiplying by \((2/d)(\mu_\ell + \bar{\alpha}_\ell)\Lambda^R_\ell\).
C - Explicit expressions for $\delta \Lambda^R_\ell(\alpha_i)/\Lambda^R_\ell(\alpha_i)$

$A_{\ell i}$ being a function of

$$\left[\delta \Lambda^R_+(\alpha_i)/\Lambda^R_+(\alpha_i)\right] / \left[\delta \Lambda^R_-(\alpha_i)/\Lambda^R_-(\alpha_i)\right]$$

(7.15)

(see (6.9a) and (6.9f)) through the ratio $(\Delta_+ / \Delta_-)$ (see (6.10a) and (6.10b)), we provide explicit expressions for $(\Lambda^R_\ell)^{-1}\delta \Lambda^R_\ell / \delta \alpha_i$ which are read from (6.18), (6.19) and (6.21). We get for $\alpha_i = \bar{\alpha}_i$, starting with $\Lambda^R_\ell^{-1} \delta \Lambda^R_\ell / \delta \alpha_i$

$$h_0 \Lambda^R_+^{-1} \delta \Lambda^R_+(\alpha_+) / \delta \alpha_+ = 1/2 \ z_{R+}(1 + \beta) +$$

$$(d/2) \left\{ a_+(1 - x_+) + [1 - a_+(1 - y/2)] \right\} (1 - y/2) / \left[ (\mu_+ + \tilde{\alpha}_+) \Lambda^R_+ \right]$$

(7.16a)

$$h_0 \Lambda^R_-^{-1} \delta \Lambda^R_-(\alpha_+) / \delta \alpha_+ = (1 - x)z_{R-}(1 + \beta) +$$

$$(d/2) \left\{ (1 - z)(1 - x)z_{R-} / \left[ (\mu_- + \tilde{\alpha}_-) \Lambda^R_- \right] \right\}$$

$$+ (1 - y/2) \left( z_{R-} / z_{R+} \right) \left[ 1 - u(1 - y/2) \right] / \left[ (\mu_+ + \tilde{\alpha}_+) \Lambda^R_+ \right]$$

(7.16b)

$$h_0 \Lambda^R_+^{-1} \delta \Lambda^R_-(\alpha_-) / \delta \alpha_- = x \ z_{R+}(1 + \beta) +$$

$$(d/2) \left\{ u(1 - x_+)(1 - y/2) / \left[ (\mu_+ + \tilde{\alpha}_+) \Lambda^R_+ \right] \right\}$$

$$+ y \left( z_{R+} / z_{R-} \right) \left[ 1 - (1 - z) \right] / \left[ (\mu_- + \tilde{\alpha}_-) \Lambda^R_- \right]$$

(7.16c)

$$h_0 \Lambda^R_-^{-1} \delta \Lambda^R_-(\alpha_-) / \delta \alpha_- = 1/2 \ z_{R-}(1 + \beta) +$$

$$(d/2) \left\{ a_-(1 - x_-) + (1 - a_- y) \right\} \ y / \left[ (\mu_- + \tilde{\alpha}_-) \Lambda^R_- \right]$$

(7.16d)

Here the meaning of $\delta$ is a difference operator,

$$\delta \Lambda_\ell(\alpha_i) = \Lambda_\ell(\alpha_i) - \Lambda_\ell(\bar{\alpha}_i)$$  (7.17a)

$$\delta \alpha_i = \alpha_i - \bar{\alpha}_i$$  (7.17b)
and \((\mu_i + \alpha_i)^{-1}\) is replaced by its average \(\mu_i + \bar{\alpha}_i\) in order to take into account the integration through the mean-value theorem of the variable \(\alpha_i\). We have not finished our calculation of \(\delta \Lambda^R_\ell(\alpha_i)/\Lambda^R_\ell(\alpha_i)\) because we need to add \(\delta \Lambda/\Lambda^R_\ell\) in

\[
\Lambda^R_\ell^{-1}\delta \Lambda^R_\ell/\delta \alpha_i = \Lambda^R_\ell^{-1} \left( \delta \Lambda/\delta \alpha_i + \delta \Lambda_\ell(\alpha_i)/\delta \alpha_i \right) .
\] (7.18)

In fact we know from (6.26) that

\[
\Lambda^R_\ell^{-1}\delta \Lambda/\delta \alpha_i = -\beta z_{R_\ell} x_{+i} .
\] (7.19)

Looking at (6.32a) and (6.32b) we note that we need \(a_{++}\) and \(b_{+-}\) in order to know the values of \(x_{++}\) and \(x_{+-}\) in (7.19). The equations (6.31a) and (6.31b) give, identifying first power coefficients and with (6.45)

\[
a_{++} = d_{++} \left( \beta z_{R_+} + A^{-1}_{++} \right)^{-1}
\] (7.20a)

\[
b_{+-} = d_{+-} \left( \beta z_{R_+} + A^{-1}_{+-} \right)^{-1} .
\] (7.20b)

We have already \(a^0_{++}\) and \(b^0_{+-}\) from (6.47a) and (6.47c) and so we have, using (6.29a),

\[
\Lambda^R_+^{-1}\delta \Lambda/\delta \alpha_+ = -\beta z_{R_+} \left( \beta z_{R_+} + A^{-1}_{++} \right) \cdot \Lambda^R_+^{-1}\delta \Lambda_+ (\alpha_+)/\delta \alpha_+ .
\] (7.21a)

\[
\Lambda^R_-^{-1}\delta \Lambda/\delta \alpha_- = -\beta z_{R_-} \left( \beta z_{R_+} + A^{-1}_{+-} \right) \cdot \Lambda^R_+^{-1}\delta \Lambda_+ (\alpha_-)/\delta \alpha_- .
\] (7.21b)

Finally, defining (obtained from (7.16))

\[
R_\pm \equiv \left[ \delta \Lambda_+ (\alpha_\pm)/\Lambda^R_+ (\alpha_\pm) \right] / \left[ \delta \Lambda_- (\alpha_\pm)/\Lambda^R_- (\alpha_\pm) \right]
\] (7.22)

we get

\[
\left( \Lambda^R_+^{-1}\delta \Lambda^R_+/\delta \alpha_+ \right) / \left( \Lambda^R_-^{-1}\delta \Lambda^R_-/\delta \alpha_+ \right) = \left[ -\beta z_{R_-} A_{++} + \left( 1 + \beta z_{R_+} A_{++} \right) R_+^{-1} \right]^{-1}
\] (7.23a)
\[
\left( \Lambda^{-1}_- \frac{\delta \Lambda^R_+}{\delta \alpha_-} \right) / \left( \Lambda^{-1}_- \frac{\delta \Lambda^R_-}{\delta \alpha_-} \right) = \left[ -\beta z_{R-} A_{+-} + \left( 1 + \beta z_{R+} A_{+-} \right) R^{-1}_- \right]^{-1}
\]

(7.23b)

ratios which are needed in order to obtain \( A_{\ell i} \) through (6.10) and (6.9).

**D - Explicit expression for \( d\varepsilon_\ell /dy \)**

From the definition (6.3c) of \( \varepsilon_\ell \) we deduce the partial derivatives \( \partial\varepsilon_\ell /\partial \mu_\ell \) and \( \partial\varepsilon_\ell /\partial \bar{\alpha}_\ell \), denoting \( A_\ell \) and \( B_\ell \) by

\[
B_\ell \equiv (\Lambda^R_\ell)^{-1} \partial \varepsilon_\ell /\partial \mu_\ell = (\varepsilon_\ell - 1) \left[ d/2 + 1 + d/2 \tilde{\zeta}_\ell \right] / \left[ (\mu_\ell + \bar{\alpha}_\ell) \Lambda^R_\ell \right] - (d/2 + 1)(\eta_\ell - 1)/ \left[ (\mu_\ell + \bar{\alpha}_\ell) \Lambda^R_\ell \right]
\]

(7.24a)

\[
A_\ell \equiv (\Lambda^R_\ell)^{-1} \partial \varepsilon_\ell /\partial \bar{\alpha}_\ell = (\varepsilon_\ell - 1) \left[ d/2 + 1 + d/2 \bar{z}_\ell \right] / \left[ (\mu_\ell + \bar{\alpha}_\ell) \Lambda^R_\ell \right]
\]

(7.24b)

with \( \tilde{\zeta}_\ell \) and \( \bar{z}_\ell \) being the mean-values of \( \zeta_\ell \) and \( z_\ell \) and \( \eta_\ell \) being defined through

\[
\eta_\ell = 1 - (I/h_0)(\Lambda^R_\ell)^{-1} \exp \left[ -dL/(2I) - (1 - dL/(2I))\beta \right] \left[ (\mu_\ell + \bar{\alpha}_\ell) \Lambda^R_\ell \right]^{d/2+2} \int_0^\infty dx \left( \mu_\ell \Lambda^R_\ell + x \right)^{-d/2-2} \exp(-x) \cdot J^{-1}_\ell H_\ell (\alpha_\ell - \bar{\alpha}_\ell) \tilde{H}_0(\bar{\alpha}_\ell).
\]

(7.25)

Then, we get \( d\varepsilon_\ell /dy \) through

\[
(\Lambda^R_\ell)^{-1} d\varepsilon_\ell /dy = (A_\ell + B_\ell) \left( \partial \mu_\ell /\partial \bar{\alpha}_\ell \right) \partial \bar{\alpha}_\ell /dy.
\]

(7.26)

The next step consists in obtaining \( d\mu_\ell /d\bar{\alpha}_\ell \) and \( d\bar{\alpha}_\ell /dy \). This can be done by first writing (6.7) as

\[
d\mu_\ell (\varepsilon_\ell + \zeta_\ell) + d\bar{\alpha}_\ell (1 - z_\ell) - (\beta/6) (\mu_\ell + \bar{\alpha}_\ell) d\bar{\alpha}_+ /\bar{\alpha}_+ = [1/2 (\mu_\ell + \bar{\alpha}_\ell) - (1 + z_\ell) \bar{\alpha}_\ell - (\varepsilon_\ell + \zeta_\ell) \mu_\ell] d\Lambda^R_\ell /\Lambda^R_\ell.
\]

(7.27)

From (6.26) we know that
\[
\Lambda^{-1}d\Lambda = -\beta \bar{\alpha}^{-1}_+ d\bar{\alpha}_+ .
\]

Because \( \Lambda^R = \Lambda + d/2 \ L E^0_\ell \) and \( dE^0_\ell /dy = 0 \) (see (6.21) and (6.46)), we get

\[
d\Lambda^R_\ell = d\Lambda
\]

and, therefore,

\[
\Lambda^{-1}_\ell d\Lambda^R_\ell /d\bar{\alpha}_+ = - \left( \Lambda / \Lambda^R_\ell \right) \beta / \bar{\alpha}_+
\]

\[
= -z_{R_\ell} \beta \bar{\alpha}^{-1}_+. \quad (7.30)
\]

Then, putting (7.30) into the right-hand side of (7.27), one gets two equations

\[
d\mu_\ell \left[ \varepsilon_\ell + \zeta_\ell \right] + d\bar{\alpha}_\ell (1 + z_\ell) - (\beta / 6) (\mu_\ell + \bar{\alpha}_\ell) d\bar{\alpha}_\ell /\bar{\alpha}_+ =
\]

\[
- z_{R_\ell} \beta \left( d\bar{\alpha}_+/\bar{\alpha}_+ \right) \left[ 1/2 (\mu_\ell + \bar{\alpha}_\ell) - (1 + \bar{\varepsilon}_\ell) \bar{\alpha}_\ell - (\varepsilon_\ell + \zeta_\ell) \mu_\ell \right] \quad (7.31)
\]

which with the equations \( y^{-1} = 1 + \bar{\alpha}_- /\mu_- \) and \( 2y^{-1} = 1 + \mu_+ /\bar{\alpha}_+ \) will allow to determine \( d\mu_\ell /d\bar{\alpha}_\ell \) and \( d\mu_\ell /d\bar{\alpha}_\ell \). We have

\[
dy = (a_\ell + b_\ell \ d\mu_\ell /d\bar{\alpha}_\ell) \ d\bar{\alpha}_\ell \quad (7.32)
\]

with

\[
a_- = -y^2 /\mu_-
\]

\[
b_- = y^2 \bar{\alpha}_- /\mu^2_-
\]

\[
a_+ = 1/2 \ y^2 \mu_+ /\bar{\alpha}^2_+
\]

\[
b_+ = -1/2 \ y^2 /\bar{\alpha}_+
\]

and therefore:
\[ d\varepsilon_\ell/dy = \Lambda^R_\ell \left( A_\ell + B_\ell \frac{d\mu_\ell}{d\bar{\alpha}_\ell} \right) / \left( a_\ell + b_\ell \frac{d\mu_\ell}{d\bar{\alpha}_\ell} \right) . \]  \quad (7.34)

From (7.31) we get

\[ d\mu_+ / d\bar{\alpha}_+ = (\varepsilon_+ + \zeta_+)^{-1} \left\{ -(1 + z_+) + \beta / 3 \ y^{-1} \right. \]
\[ - z_{R+} \beta \left[ y^{-1} - (1 + z_+ - (\varepsilon_+ + \zeta_+) (2y^{-1} - 1) \right] \} \]  \quad . (7.35)

For the \( - \) case we have to work a little bit in order to obtain \( d\mu_- / d\bar{\alpha}_- \). First, we note that because \( y^2 = 2\bar{\alpha}_+/\bar{\alpha}_- \) we have

\[ d\bar{\alpha}_+ / \bar{\alpha}_+ = d\bar{\alpha}_- / \bar{\alpha}_- + 2dy/y \]  \quad (7.36)

and that (7.31) gives us for \( \ell = - \)

\[ d\mu_- (\varepsilon_- + \zeta_-) + d\bar{\alpha}_-(1 + z_-) = \]
\[ d\bar{\alpha}_+ / \bar{\alpha}_+ \left\{ (\beta/6)(\mu_- + \bar{\alpha}_-) - z_{R-} \beta \left[ 1/2 (\mu_- + \bar{\alpha}_-) - (1 + z_-)\bar{\alpha}_- \right. \right. \]
\[ - (\varepsilon_- + \zeta_-) \mu_- \} \} . \]  \quad (7.37)

Together with (7.32) taken for \( \ell = - \) we then have the system of equations

\[ d\mu_-(\varepsilon_- + \zeta_-) + d\bar{\alpha}_-(1 + z_-) - \{}/\bar{\alpha}_- = 2dy/y \} \] \quad (7.38a)

\[ d\mu_- \bar{\alpha}_-/\mu_-^2 - d\bar{\alpha}_- \mu_-^{-1} = y^{-2}dy \] \quad (7.38b)

where \( \} \} \) denotes the quantity between brackets on the right-hand side of (7.37). Then, solving (7.38) gives

\[ d\mu_- = (dy/y) \left[ \frac{2\{}/\mu_- + y^{-1} \left((1 + z_-) - \{}/\bar{\alpha}_- \right) \right] / \Delta \] \quad (7.39a)

\[ d\bar{\alpha}_- = (dy/y) \left[ \frac{2(\bar{\alpha}_-/\mu_-^2)\{ - (\varepsilon_- + \zeta_-) y^{-1} \} \right] / \Delta \] \quad (7.39b)

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with

\[
\Delta = (\varepsilon_- + \zeta_-)/\mu_- + (\bar{\alpha}_- / \mu^2_-) [ (1 + z_-) - \{ \} / \bar{\alpha}_- ]
\]  
\[
(7.39c)
\]

and therefore

\[
d\mu_- / d\bar{\alpha}_- = \left\{ 2c_- + y^{-1} [ (1 + z_-) - c_- y/(1 - y)] \right\} / \left[ 2(y^{-1} - 1)c_- - (\varepsilon_- + \zeta_-) y^{-1} \right]
\]
\[
(7.40a)
\]

with

\[
c_- = \{ \} / \mu_- = \beta \left\{ y^{-1}/6 - z_{R_-} \left[ y^{-1}/2 - (1 + z_-)(y^{-1} - 1) - (\varepsilon_- + \zeta_-) \right] \right\}.
\]
\[
(7.40b)
\]

Now, \((7.32)\) gives \((\ell = +)\)

\[
\left( \Lambda^R_+ d\bar{\alpha}_+/dy \right)^{-1} = 1/2 \left[ y^2 / (\bar{\alpha}_+ \Lambda^R_+) \right] \left( 2y^{-1} - 1 - d\mu_+ / d\bar{\alpha}_+ \right).
\]
\[
(7.41)
\]

Taking \(A_+\) and \(B_+\) from \((7.24)\) and putting them into \((7.34)\), we get with the help of \((7.41)\)

\[
d\varepsilon_+ / dy = y^{-1} \left\{ (d/2 + 1 + d/2\bar{\zeta}_+)(\varepsilon_+ - 1) + \left[ (d/2 + 1)(\varepsilon_+ - \eta_+) + d/2 \bar{\zeta}_+ \right] d\mu_+ / d\bar{\alpha}_+ \right\} / (2y^{-1} - 1 - d\mu_+ / d\bar{\alpha}_+).
\]
\[
(7.42)
\]

Again, \((7.32)\) for \(\ell = -\) gives

\[
\left( \Lambda^R_- d\bar{\alpha}_-/dy \right)^{-1} = \left[ y^2 / (\mu_- \Lambda^R_-) \right] \left[ -1 + (y^{-1} - 1)d\mu_- / d\bar{\alpha}_- \right]
\]
\[
(7.43)
\]

and taking \(A_-\) and \(B_-\) from \((7.24)\), putting them into \((7.34)\), taking into account \((7.43)\), we get

\[
d\varepsilon_- / dy = y^{-1} \left\{ (d/2 + 1 + d/2\bar{\zeta}_-)(\varepsilon_- - 1) + \left[ (d/2 + 1)(\varepsilon_- - \eta_-) + d/2 \bar{\zeta}_- \right] d\mu_- / d\bar{\alpha}_- \right\} / [-1 + (y^{-1} - 1)d\mu_- / d\bar{\alpha}_-]
\]
\[
(7.44)
\]

which completes our evaluation of quantities used in the solution of \((6.1a)\).
8 Consistency of $\alpha_i = O(h_0/I)$

We have concluded in section 2 that $\bar{\alpha}_i$ and $\mu_i$ should be proportional to $h_0/I$ in order to achieve consistency. This was done, however, neglecting the dependence $\bar{\alpha}_j(\alpha_i)$ which renormalizes $\Lambda$ into $\Lambda_i^R$ and the factors $H_i(\alpha_i - \bar{\alpha}_i)$. So one can ask what happens to this consistency when all these changes are taken into account. The effects of taking into account the $\bar{\alpha}_j(\alpha_i)$ dependence in the equation determining $\bar{\alpha}_i$ is translated into

a) changing $\Lambda$ into $\Lambda_i^R = \Lambda + d/2 LE_i^0$

b) introducing the factors $J_\ell^{-1}$, $H_\ell(\alpha_\ell - \bar{\alpha}_\ell)$ and $\tilde{H}_0(\bar{\alpha}_\ell)$ in the integrand (see (5.1a)).

Now, if we look at the $d/2 LE_i^0$ part of $\Lambda_i^R$, we see that $E_i^0$ is proportional to $1/h_0$ times a function of the variables $\beta$, $z_{R_i}$, $y$, $\mu_i\Lambda_i^R$, $\bar{\alpha}_i\Lambda_i^R$. (The $\alpha_i\Lambda_i^R$ dependence is transmutated into a $\bar{\alpha}_i\Lambda_i^R$ dependence by taking the mean-value of $E_i^0$ as a function of $\alpha_i\Lambda_i^R$. This is the origin of the parameters $x_\ell$ which appear when replacing $(\mu_\ell + \alpha_\ell)^{-1}$ by $(1 - x_\ell)(\mu_\ell + \bar{\alpha}_\ell)^{-1}$ in the expression for $\Lambda_i^R \delta\Lambda_\ell/\delta\alpha_i$ which in turn appears in $\Delta_+/\Delta_-$, which itself is appearing in $A_{\ell\pm}$. See the equations (7.18), (6.9) and (6.10)). $\beta$ being proportional to $\bar{\alpha}_i\Lambda$ and $z_{R_i}$ being the ratio $\Lambda/\Lambda_i^R$, we can therefore consider $E_i^0$ as a constant when $I, L \to \infty$ and be perfectly consistent with the assumption $\bar{\alpha}_i = O(h_0/I)$.

For $\tilde{H}_0(\bar{\alpha}_\ell) = (1 - z_{R_i})\bar{\alpha}_\ell\Lambda_i^R$ the above reasoning leads to $\tilde{H}_0(\bar{\alpha}_\ell)$ being constant. Looking at the expressions (7.5a) and (7.5b) for $J_+$ and $J_-$ we see that the varying factors are $\alpha_+\Lambda_i^R/[(\mu_+ + \alpha_+)\Lambda_i^R]_+^2$ and $\alpha_-\Lambda_i^R/[(\mu_- + \alpha_-)\Lambda_i^R]_+^2$ which have limited variations for any value of $\alpha_+\Lambda_i^R$ or $\alpha_-\Lambda_i^R$ respectively. Then, we conclude that $J_+$ and $J_-$ have limited variations too. Of course, for specific values of $\alpha_\ell\Lambda_i^R$, $J_\ell^{-1}$ can have a pole and then the integrand can become infinite. However, for reasonable values of $\alpha_\ell$ this pole does not exist for real values of $\alpha_\ell\Lambda_i^R$ as the numerical equations resolution show.

There are also factors $H_i(\bar{\alpha}_i_\pm - \bar{\alpha}_\mp)$ to consider which comes from the fact that when $\alpha_i \neq \bar{\alpha}_i$, $Q_G(P, \{\bar{\alpha}_j\}_{j \neq i}, \alpha_i)$ varies. However, looking at (1.12) and (1.22) we easily conclude that these factors too have bounded variations as $\alpha_i_+$ or $\alpha_i_-$ varies.

Then, we can write the equation (5.1a) determining $\alpha_\ell$ (or $\bar{\alpha}_i$)
\[
(\mu_i + \bar{\alpha}_i)^{-d/2} = \left[ \frac{I}{(h_0\Lambda_i^R)} \right] \kappa \Lambda_i^{Rd/2} \exp(\mu_i \Lambda_i^R) \int_{\mu_i \Lambda_i^R}^{\infty} dx \exp(-x) x^{-d/2} \ B_i(x)
\]

where \( B_i(x) \) has a bounded variation. This equation is the equation (2.21) with \( \Lambda_i^R \) replacing \( \Lambda \) and \( B_i(x) \) multiplying the integrand. As \( B_i(x) \) has a bounded variation and \( \Lambda_i^R \) is proportional to \( \Lambda \) everything we have said in section 2 concerning consistency is still true here and therefore the consistency of the assumption \( \bar{\alpha}_j = O(h_0/I) \) is established in the general case where the variations of \( \bar{\alpha}_j(\alpha_i) \) and \( Q_G(P, \{\bar{\alpha}_j\}_{j \neq i}, \alpha_i) \) are taken into account.

### 9 The \( m \rightarrow 0 \) limit of the consistency equations

For the determination of \( \bar{\alpha}_i \) we have to solve the consistency equation (5.23) or (6.1a). We are interested in showing that when the mass \( m \) tends to zero, this equation becomes independent of \( m \). This is done most easily by considering the form (5.23) of the equation where the variable change \( \alpha_i \Lambda_i^R \rightarrow x \) has not been done as in (6.1a). So let us rewrite it

\[
1 = \left( \frac{I}{h_0} \right) \kappa \int_0^{h_0} d\alpha_i \left[ (\mu_i + \bar{\alpha}_i)/(\mu_i + \alpha_i) \right]^{d/2} \ \exp(-\alpha_i \Lambda_i^R) H_i(\alpha_i - \bar{\alpha}_\ell) \widetilde{H}_0(\bar{\alpha}_i)
\]

with

\[
\kappa = \exp \left[ -dL/(2I) - (1 - dL/(2I))\beta \right]
\]

\[
\widetilde{H}_0(\bar{\alpha}_i) = \exp \left( d/2 \ L \ E^0_i \bar{\alpha}_i \right)
\]

\[
H_{i+}(\alpha_{i+} - \bar{\alpha}_+) = \exp \left( -\varepsilon_+ \Lambda/m^2 \right)
\]

\[
H_{i-}(\alpha_{i-} - \bar{\alpha}_-) = \exp \left( -\varepsilon_- \Lambda/m^2 \right)
\]

and

\[54\]
\[ \varepsilon_+ = \frac{t}{2} L \bar{\alpha}_+ < \ell/(2\ell - 1) > (\alpha_{i_+} - \bar{\alpha}_+)\mu_+/(\mu_+ + \alpha_{i_+}) \]  

(9.2a)

\[ \varepsilon_- = \left[ (\alpha_{i_-} - \bar{\alpha}_{i_-})\mu_-/(\mu_- + \alpha_{i_-}) \right] \cdot \left( m^2/(\bar{\alpha}_{i_-} \Lambda) \right)^{-1} \left\{ -(1 - a) \left[ \frac{2}{3} \ell n \, C^u + \ell n(1 - a) \right] \right\} \]  

(9.2b)

where the expressions for \( H_{i_+}(\alpha_{i_+} - \bar{\alpha}_+) \) and \( H_{i_-}(\alpha_{i_-} - \bar{\alpha}_{i_-}) \) have been taken from section 4 and \( \tilde{H}_0(\bar{\alpha}_i) \) from (5.22). We have seen in section 3 that as \( m \to 0 \)

\[(1 - \beta)/m^2 \to h_0/Q_G(P, \{\bar{\alpha}\}) \]  

(9.3)

which is independent of \( m \) and so

\[ \Lambda/m^2 = (I - d/2 \, L)(1 - \beta)/(h_0 m^2) \]  

(9.4)

is also independent of \( m \) in that limit. Because \( \varepsilon_+ \) is independent of \( m \) we also conclude that \( H_{i_+}(\alpha_{i_+} - \bar{\alpha}_+) \) is independent of \( m \) as \( m \to 0 \). \( \varepsilon_- \) depends on \( m \) only through \( m^2/\Lambda \) and is also independent of \( m \) as \( m \to 0 \), and so is \( H_{i_-}(\alpha_{i_-} - \bar{\alpha}_{i_-}) \) for the same reason. As \( m \to 0 \), \( \beta \to 1 \) and therefore \( \kappa \) tends to \( \exp(-e) \), a constant independent of \( m \). Remains \( \tilde{H}_0(\bar{\alpha}_i) \) which a priori could depend on \( m \) through \( E^0_i \) and \( \exp(-\alpha_i \Lambda_i^R) \) which also depends on \( E^0_i \) because

\[ \Lambda_i^R = (I - d/2L)(1 - \beta)/h_0 + d/2 \, L \, E^0_i \]  

(9.5)

\( \Lambda_i^R \) can only depend on \( m \) through \( E^0_i \) because, as \( \beta \to 1 \) the first term in (9.3) becomes negligible compared to the second one. This also entails that \( z_{R_i} \to 0 \).

We have (see (6.38))

\[ E^0_+ = (1 - y/2)a^0_{++} + (y/2 + \eta)a^0_{++} \]  

(9.6a)

\[ E^0_- = (1 - y/2)b^0_{--} + (y/2 + \eta)b^0_{+-} \]  

(9.6b)

where the coefficients \( a^0_{++}, a^0_{+-}, b^0_{--}, b^0_{+-} \) are given by (6.47). First, as (see (5.5))
\[ \eta = (2/d) \frac{t/2}{\bar{\alpha}_+ \Lambda/m^2} , \]  

(9.7)

\( \eta \) has a limit independent of \( m^2 \) as \( m \to 0 \). Now let us rewrite the equations (6.47)

\[ a^0_{++} = c_{++} \left( \beta z_{R+} + A^{-1}_{++} \right)^{-1} \]  

(9.8a)

\[ a^0_{-+} = A_{-+} \left( c_{-+} - \beta z_{R-} a^0_{++} \right) \]  

(9.8b)

\[ b^0_{+-} = c_{+-} \left( \beta z_{R+} + A^{-1}_{++} \right)^{-1} \]  

(9.8c)

\[ b^0_{-} = A_{-} \left( c_{-} - \beta z_{R-} b^0_{+-} \right) \]  

(9.8d)

which lead us to look after the expressions of \( c_{i\ell} \) and \( A_{i\ell} \). Looking at (7.2) we see that \( c_{i\ell} \) only depends on \( m \) through \( \Lambda^R_\ell \) as \( m \to 0 \). Looking at (6.10) defining \( A_+ \) and \( A_- \), we see that these quantities depend on \( \delta \Lambda^R_\ell / \Lambda^R_\ell \), \( \varepsilon_\ell \) and \( z_\ell \) (see (6.9)) and \( \beta \) (see (6.9a) and (6.9c)).

However, once more, looking at (7.16) and (7.23) we see that, as \( m \to 0 \), \( \delta \Lambda^R_\ell / \Lambda^R_\ell \) only depends on \( m \) through \( \Lambda^R_\ell \). Finally, \( \varepsilon_\ell \) and \( z_\ell \) also have the same property (see (6.3) and the expressions for \( J_\ell \) in (7.5)). So \( E^0_+ \) and \( E^0_- \) are determined as a function of themselves only in the limit \( m \to 0 \) and therefore do not depend on \( m \) in that limit.

This completes our verification that the consistency equations determining \( \bar{\alpha}_i \) are indeed independent of \( m \) as \( m \) tends to zero as they should.

10 The numerical evaluation of the leading Regge trajectory

The consistency equations can be solved numerically when all quantities appearing in them are \( O(1) \). Here, we deal with the massive case, deferring the massless case to a later study. In (6.14) we have

\[ (I/h_0) \Lambda_i^{-1} = \left[ I/(I - d/2 L) \right] (1 - \beta)^{-1} z_{R_i} \]  

(10.1)
and therefore the consistency equation for \( \bar{\alpha}_i \) takes the following form (for \( \phi^3 \) and \( d = 4 \))

\[
1 = z_{R_i} \exp(-2/3 - \beta/3)/(1 - \beta) \int_0^\infty \left[ \mu_i \Lambda_i^R + \bar{\alpha}_i \Lambda_i^R \right]/(\mu_i \Lambda_i^R + x)^2 \exp(-x)J_i^{-1}H_i(\alpha_i - \bar{\alpha}_i)\tilde{H}_0(\bar{\alpha}_i)
\]

(10.2)

where \( \mu_i \Lambda_i^R \) and \( \bar{\alpha}_i \Lambda_i^R \) are finite unknown quantities. In the ladder case, there remain two consistency equations and four unknown quantities \( \mu_- \Lambda_-^R, \bar{\alpha}_- \Lambda_-^R, \mu_+ \Lambda_+^R, \bar{\alpha}_+ \Lambda_+^R \), which, however are not independent. We have

\[
\mu_- \Lambda_-^R/ \left( \mu_- \Lambda_-^R + \bar{\alpha}_- \Lambda_-^R \right) = y
\]

(10.3a)

\[
\bar{\alpha}_+ \Lambda_+^R/ \left( \mu_+ \Lambda_+^R + \bar{\alpha}_+ \Lambda_+^R \right) = \frac{y}{2}
\]

(10.3b)

with

\[
y^2 = 2\bar{\alpha}_+/\bar{\alpha}_- = \left( 2\bar{\alpha}_+ \Lambda_+/ (\bar{\alpha}_- \Lambda_-) \right) \Lambda_-/\Lambda_+
\]

\[
= \left( 2\bar{\alpha}_+ \Lambda_+/ (\bar{\alpha}_- \Lambda_-) \right) z_{R_+}/z_{R_-}
\]

(10.4)

which gives five relations for five unknowns.

We recall that (see (6.29))

\[
 z_{R_i} = \Lambda/\Lambda_i^R = (1 - \beta)/(1 - \beta + d/2 h_0 E_i^0)
\]

(10.5)

and (see (3.6))

\[
\beta = [t/(2m^2)]\bar{\alpha}_+ \Lambda = [t/(2m^2)]z_{R_+} \bar{\alpha}_+ \Lambda_+^R.
\]

(10.6)

Moreover, there are parameters appearing in the decomposition of \( F_{i\ell} \) in (6.11) which are

\[
x , u , z , a_+ , a_- , x_+ , x_-
\]

(10.7)
with the four constraints (6.23a), (6.23b) and (6.46a), (6.46b). Let us also remind
that \( x_+ \) and \( x_- \) are introduced because, in order to simplify the calculations, we
replace in the expression of \( E_0^i \), i.e. in \( c_i, A_i, z_{R_i}, (\mu_i + \alpha_i)^{-1} \) by \((1 - x_i)(\mu_i + \bar{\alpha}_i)^{-1}\).

Now, another constraint comes from the relation \( \sum_{i=1}^{L} \bar{\alpha}_i = h_0 \) which takes the form
in the ladder case

\[
2L\bar{\alpha}_+ + (L + 1)\bar{\alpha}_- = h_0 \tag{10.8}
\]
as \( L + 1 = \Lambda h_0/(1 - \beta) \), this is converted into (neglecting 1 in front of \( L \))

\[
2\bar{\alpha}_+ \Lambda + \bar{\alpha}_- \Lambda = (1 - \beta)
\]
or

\[
2\bar{\alpha}_+ \Lambda^R z_{R_+} + \bar{\alpha}_- \Lambda^R z_{R_-} = (1 - \beta) \tag{10.9}
\]
which we use to obtain \( z_{\Lambda_+} \) as a function of \( z_{R_-} \). This is the fifth constraint.

The equations (10.3) and (10.4) are used to eliminate \( \bar{\alpha}_- \Lambda^R, \bar{\alpha}_+ \Lambda^R, \mu_+ \Lambda^R \) as
free parameters and keep \( \mu_- \Lambda^R \) and \( y \) as the free ones. Then, we have got nine
unknowns

\[
\mu_- \Lambda^R, \ y, \ x, \ z, \ a_+, \ a_-, \ x_+, \ x_-
\]
(10.10)

together with the two consistency conditions (10.2) and five constraints (6.23), (6.46)
and (10.9), i.e. seven equations. In practice, \( \mu_- \Lambda^R / \bar{\alpha}_- \Lambda^R \) will be large (\( \gtrsim 10 \)) and
the mean-value of \( (\mu_- + \alpha_-)^{-1} \) will be very close to \( (\mu_- + \bar{\alpha}_-)^{-1} \). So \( x_- \) will be close
to zero. So, we take \( x_- \) to be zero and we are left with 8 parameters instead of 9.

Of course, we have to have also

\[
0 < x_+ < 1 \tag{10.11}
\]
which can be considered as an eighth constraint. In practice, \( x_+ \) will be close to
1/2.

The procedure we take to solve the systems of equations is to add the absolute
values of sides which have to be zero and minimize their weighted sum. Problems
occur because we get a chaotic behaviour of this sum. This is easily understood
because several quantities are expressed as functions of themselves. So we have to
make calculational loops and verify that output values are the same as input values.
This is first done for $z_R-$. Then, the $z_R-$ loop is inserted into another calculational
loop where the value of the left-hand side of (10.2) is compared with one for $i = -$. 
Again, this loop is contained in a last loop where the left-hand side of (10.2) for $i = +$ is compared with one. This gives a total of three loops in a “Russian doll”
configuration. No wonder that we may encounter some chaotic behaviour! In order
to cope with this phenomenon we have devised a minimization algorithm [17] which
does not use any gradient approach. It is more in the Monte-Carlo spirit but much
more efficient. Its main feature is the construction of a cube in $n$ dimensions, if
there are $n$ parameters, i.e. to calculate two values of a particular parameter for
any other parameter value. So, we have $2^n$ values of the function to calculate. We
take the minimum of these $2^n$ values to construct around it another cube, but with
a side being reduced with respect to the former cube.

We found that this algorithm is much more powerful than a well-known minimiza-
tion program known as MINUIT [18], widely used by experimentalists for instance.

The results for the trajectory $\alpha(t/m^2)$ are contained in fig. 4 and fig. 5. In
fig. 4 we have taken two values of the coupling constant $\gamma$ such that $ln\gamma_m =
ln(\gamma e/(m4\pi3\sqrt{3})$ is equal to $-0.1$ and $0$. The obtained intercept are, roughly $0.25$
and $0.47$ respectively. Calculations [14] using the Bethe-Salpeter approach give an
intercept (assuming a zero mass for the central-rung fields)

$$\alpha(0) = -3/2 + \sqrt{1/4 + \gamma^2/(16\pi^2m^2)}$$

(10.12)
corresponding to values $\simeq 0.300$ and $0.475$ for the same values of $\gamma$ as quoted
above. The intercept that we calculated is compared with (10.12) in the range
$-0.4 \leq ln \gamma_m < .5$ in fig. 5. Agreement is obtained for $\alpha(t/m^2) \gtrsim 0.3$.

The fact that for $ln \gamma_m \lesssim 0.1$ our calculated intercept is lower than that given
by (10.12) can easily be explained. We know that when $\gamma \to 0$ the finite ladders give
the dominating contribution to the scattering amplitude. What we see on Fig. 5 is
that the finite ladders still dominate for $ln \gamma_m \lesssim 0.1$ and when $\gamma_m$ grows larger the
saddle-point contribution of infinite ladders takes over.

In table 1, we report the values of $\mu$-$\Lambda$, $y$, $x$, $u$, $z$, $a_+$, $x_+$ and $\alpha(t/m^2)$ for
\ln \gamma_m = 0 \text{ and } -3.6 \leq t/m^2 \leq 2.0. \text{ We remark that } \alpha(t/m^2) \text{ is compatible with a linear function of } t/m^2. \text{ Such linear fit made with the eye are drawn on fig. 4. This result is new and in an improvement over a previous [11] determination of } \alpha(t/m^2) \text{ with the same method where we could not have a real result for } t/m^2 > 0.8. \text{ This change is due to the correction of some errors among which was the omission of the term (7.19) in } (\Lambda_\ell^R)^{-1} \delta \Lambda_\ell^R / \delta \alpha_i. \text{ So, apparently we can go further out in } t/m^2 \text{ range. However, this takes more computer time and this is the reason why we limited ourselves to the present range. Let us note that the loop over } z_{R-} \text{ is made twice, that for the } -\text{ consistency equation four times, but that for the } +\text{ consistency equation is made fifteen times in order to get a reasonable safety in convergence. As } |t/m^2| \text{ grows it becomes more and more difficult to get precise results.}

Let us digress a little bit on the linear property. This is what would be expected if the infinite number of loops part of } \phi^3 \text{ was equivalent to a string theory. Of course, this argument is not new [19]. We even found [20] that a local Polyakov lagrangian could be deduced (with some weak logarithmic corrections) from the planar } \phi^3 \text{ graphs with an infinite density of vertices. So we would expect linear Regge trajectories for this sector of } \phi^3. \text{ Our present work is an indication that this may be true indeed.}

11 Conclusion

We have shown that the infinite loop limit in scalar field theories can be accessible to practical calculation. Of course, in general, we have an infinite system of consistency equations if no symmetry appears in the topology of the considered Feynman graphs. However, in the ladder case, only two of them survive, allowing us to calculate the leading Regge trajectory, which, our results show, may be linear.

Consistency equations are obtained by using the mean-value theorem for all } \alpha\text{-parameters and for all but one, } \alpha_i, \text{ the one for which usual integration is needed in order to determine its mean-value } \bar{\alpha}_i.

So doing, a crucial parameter } \mu_i \text{ appears to be related to the ratio of the sum of weighted spanning trees going through } i \text{ to the sum of weighted spanning trees not going through } i, \text{ which is in fact, } \mu_i/\bar{\alpha}_i. \text{ This parameter } \mu_i \text{ represents the } local
topological properties of the graph. In the ladder case the ratio $\mu_i/\bar{\alpha}_i$ is easily determined for each kind of propagators, belonging to the sides of the ladder or central. As $\mu_i$ is homogeneous to one power of $\alpha$-parameter it should have a priori the same behaviour as a function of $I$ as $I$ tends to infinity, i.e. it should behave like $O(h_0/I)$. We proved that $\mu_i$ is $O(h_0/I)$ for all propagators except for an infinitesimal proportion of them. In fact, once the behaviour of $\mu_i$ is known, that of $\bar{\alpha}_i$ is determined by the consistency equation and when $\mu_i$ is $O(h_0/I)$, $\bar{\alpha}_i$ has been shown to have the same behaviour as expected for homogeneity reasons. In that respect the Gaussian propagator representation is therefore wholly consistent. A scale $\Lambda$ proportional to $I/h_0$ was also introduced making $\bar{\alpha}_i\Lambda$ a constant as $I$ tends to infinity. Because $\bar{\alpha}_i\Lambda$ and $\mu_i\Lambda$ are constant, they appear in practical numerical computations rather than $\bar{\alpha}_i$ and $\mu_i$. When the variation of $\bar{\alpha}_j(\alpha_i)$ is taken into account a renormalization of $\Lambda$ into $\Lambda_j^R = \Lambda + d/2 E_j^0 L$ occurs where $E_j^0$ is some constant, leaving $\Lambda_j^R$ also proportional to $I/h_0$. In the ladder case, $E_j^0$ has been explicitly determined. In fact, $E_j^0$ is the result of the interaction of $\alpha_j$ and $\alpha_i$ through terms proportional to $\alpha_i\alpha_j$ in the consistency equations making $d\bar{\alpha}_j(\alpha_i)/d\alpha_i$ of order $1/I$. A finite renormalization effect occurs because there are $I$ propagators. It has been shown that this renormalization leaves unchanged the consistency of the scheme. We expect the renormalization procedure developed for the ladder topology to be only slightly modified in the general topology case as the procedure used for the ladder topology can be readily used in the general topology case. What has to be provided in order to have a complete resolution of the general case is the ratio $\mu_i/\bar{\alpha}_i$, which is of local nature on the graph. However, dealing with sums over graphs amplitudes instead of individual graph amplitude could be the way for treating the general case, $\mu_i/\bar{\alpha}_i$ then taking an average value for sums of graphs amplitudes. With this averaging procedure only one consistency equation would be needed, simplifying somewhat the scheme. Therefore, we expect the road to be open to a complete solution of massive scalar $\phi^3$ field theory, using the Gaussian propagator formalism. We have seen that the massless limit, being independent of the mass, is also tractable in our scheme. This opens the road to QCD if the reduced kernel can be found for the multi-loop case. Therefore, finding this reduced kernel will be one of our priorities in the near future.
A - Appendix

Definitions

i) Let us define the contraction of a propagator by the fusion of its two end-vertices.

ii) We define the contraction of a loop by the contraction of all propagators belonging to the loop.

If we draw loops on a surface we can define an interior and an exterior for a loop.

iii) An elementary loop or mesh contains all the propagators on the lines joining its vertices if these lines are drawn on the closed interior of the loop. The boundary of the interior of the loop is then the loop itself. In other words, there are no propagators on the open interior of an elementary loop \( \square \).

So, from now on we will consider graphs ordered by a topological expansion as the consideration of elementary loops will take a primordial importance. We consider first the effect of the contraction of an elementary loop \( \mathcal{L} \) on a graph \( G \) containing the propagator \( i \). All spanning trees on \( G \) can be constructed by cutting open all loops of \( G \). In particular, if the propagator \( i \) is cut we have cut \( \mathcal{L} \) open at \( i \).

We begin by considering the spanning trees on \( G \) with \( \mathcal{L} \) cut once. Then, all vertices of \( \mathcal{L} \) are connected. It follows that if we contract \( \mathcal{L} \), we have, after this contraction, the spanning trees of \( G \) which are constructed from \( \mathcal{L} \) cut once becoming spanning trees of \( G_{\mathcal{L}_c} \) where \( G_{\mathcal{L}_c} \) means \( G \) with \( \mathcal{L} \) contracted. This is because having contracted some connected piece of a tree, the result of this contraction is still a tree. So, for \( \mathcal{L} \) cut once, all spanning trees of \( G \) can be constructed by constructing first all spanning trees on \( G_{\mathcal{L}_c} \) and then, return on \( G \) (that is decontracting \( \mathcal{L} \)) and cut \( \mathcal{L} \) at some propagator. The net result is that we have built the spanning trees on \( G \) with \( \mathcal{L} \) cut one in two independent steps. Then, the ratio \( \mu_i/\bar{\alpha}_i \) for these spanning trees is simply
\[ \frac{\mu_i}{\bar{\alpha}_i} = \sum_{j \in \mathcal{L}, j \neq i} \frac{\bar{\alpha}_j}{\bar{\alpha}_i} \]  
(A.1)

and, therefore

\[ \mu_i + \bar{\alpha}_i = \sum_{j \in \mathcal{L}} \bar{\alpha}_j \]  
(A.2)

Looking at the consistency equation (2.16), we see that the left-hand side is exactly the same for all \( \bar{\alpha}_j \)'s belonging to \( \mathcal{L} \) if we were only considering the spanning trees built from \( \mathcal{L} \) cut once. So, we have a symmetry between all \( \bar{\alpha}_j \)'s belonging to \( \mathcal{L} \) which reduces \( n_{\mathcal{L}} \) consistency equations to one with the constraint

\[ \frac{\mu_i}{\bar{\alpha}_i} = n_{\mathcal{L}} - 1 \]  
(A.3)

if \( n_{\mathcal{L}} \) is the total number of propagators of \( \mathcal{L} \).

We note that a finite ratio \( \frac{\mu_i}{\bar{\alpha}_i} \) implies that \( \mu_i I \) cannot tend to zero because otherwise \( \mu_i \Lambda \rightarrow 0 \) and \( \bar{\alpha}_i/\mu_i \rightarrow \infty \) as deduced in (2.20).

Now, we argue that for most propagators \( n_{\mathcal{L}} \) is finite. Indeed, let us start the construction of \( G \) with one loop incident with all external lines (this is possible if \( G \) is 1-line irreducible). If we keep \( G \) planar and add loops to it, each time a loop is added, three propagators are added. When the number of loops is infinite with respect to the number of external lines, only an infinitesimal proportion of the propagators will be part of only an elementary loop with an infinite number of propagators. So, for almost all propagators \( n_{\mathcal{L}} \) will be finite if the number of loops of \( G \) is sufficiently high.

We now examine the effect of taking into account the topologies of the spanning trees on \( G \) where \( \mathcal{L} \) is cut more than once.

In order to keep having a tree when cutting \( \mathcal{L} \) twice we consider a loop \( \mathcal{L}_1 \) having some propagator \( j \) in common with \( \mathcal{L} \). So, cutting \( \mathcal{L} \) at \( i \) and \( j \) we have a tree on \( \mathcal{L} \cup \mathcal{L}_1 \). All the spanning trees on \( G \) having a spanning tree on \( \mathcal{L} \cup \mathcal{L}_1 \) are built by taking all the spanning trees on \( G_{(\mathcal{L} \cup \mathcal{L}_1)} \), (i.e. \( G \) where \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are contracted) and combining them with all spanning trees on \( \mathcal{L} \cup \mathcal{L}_1 \). This can be done because contraction preserves the tree topology. We can continue the process by considering all spanning trees on \( \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2 \) where \( \mathcal{L}_2 \) is a third elementary loop. Then, a
total of three propagators will be cut on them. Again, the factorization will be
at work for all spanning trees on $G$ having a spanning tree on $\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2$. The
factorization will continue to work out in the same way taking an arbitrary number
of connected elementary loops $\mathcal{L}_1$, $\mathcal{L}_2$, $\cdots$, $\mathcal{L}_n$ on $G$. This will allow $\mathcal{L}$ to be cut an
arbitrary number of times. Of course, an arbitrary spanning tree on $G$ can have
disconnected sub-trees on $\mathcal{L} \cup \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n$. In order to recover all topologies we have
to let $n$ tending to infinity until $G$ is completely covered by $\mathcal{L} \cup \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n$. However,
we have a systematic way of constructing spanning trees on $G$ starting from the one
loop topology with the important property that factorization will continue to work
out when we add an arbitrary number of loops.

Now, let us find a general expression for the total weight $W_{\mathcal{L} \cup \cdots \cup \mathcal{L}_n}$ of spanning
trees on $\mathcal{L} \cup \cdots \cup \mathcal{L}_n$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\cdots$, $\mathcal{L}_n$ being a propagator-connected set of elementary
loops. For $\mathcal{L}$ alone

$$W_\mathcal{L} = \sum_{\ell \in \mathcal{L}} \bar{\alpha}_\ell$$

and for $\mathcal{L} \cup \mathcal{L}_1$

$$W_{\mathcal{L} \cup \mathcal{L}_1} = \left( \sum_{\ell \in \mathcal{L}} \bar{\alpha}_\ell \right) \left( \sum_{k \in \mathcal{L}_1} \bar{\alpha}_k \right) - \bar{\alpha}_j^2$$

where $j$ is the propagator common to $\mathcal{L}$ and $\mathcal{L}_1$ ($j \in \mathcal{L} \cap \mathcal{L}_1$). We remark that
when a loop $\mathcal{L}_i$ is added we can multiply the weight of the set of loops to which
it is connected by $\sum_{k_i \in \mathcal{L}_i} \bar{\alpha}_k$, but we have to subtract the terms which contains $\bar{\alpha}_j^2$
if $j$ is a propagator common to $\mathcal{L}_i$ and the set of other loops. The reason is that
we cannot cut twice the same propagator. However, the first term in (A.5) is the
manifestation of the factorization property. Furthermore, adding a loop, we cannot
cut that loop more than once on propagators not belonging to other loops because
this would create disconnected sub-trees on the set of connected loops considered.
This is why we only have a first power polynomial for each loop. These remarks
help enormously writing down the weight of spanning trees on any number of loops.
For three loops $\mathcal{L}$, $\mathcal{L}_1$ and $\mathcal{L}_2$ we have

$$W_{\mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2} = \left( \sum_{\ell \in \mathcal{L}} \bar{\alpha}_\ell \right) W_{\mathcal{L}_1 \cup \mathcal{L}_2} - \bar{\alpha}_{j_1}^2 W_{\mathcal{L}_2} - \bar{\alpha}_{j_2}^2 W_{\mathcal{L}_1}$$
where \( j_1 \) and \( j_2 \) are propagators of \( \mathcal{L} \) common to \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) respectively. If, for instance, \( \mathcal{L}_2 \) has no propagator in common with \( \mathcal{L} \), then the third term in (A.6) disappears. We can generalize to \( \mathcal{L} \cup \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n \), and taking the notation where \( \mathcal{L}_1 \cup \cdots \hat{\mathcal{L}}_k \cdots \cup \mathcal{L}_n \) means that propagators of \( \mathcal{L}_k \) not belonging to \( \mathcal{L}_1, \cdots, \mathcal{L}_{k-1}, \mathcal{L}_{k+1}, \cdots, \mathcal{L}_n \) are suppressed.

has been omitted from \( \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n \), we get

\[
W_{\mathcal{L}_1 \cup \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n} = \left( \sum_{\ell \in \mathcal{L}} \bar{\alpha}_\ell \right) W_{\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n} - \sum_k \bar{\alpha}_{j_k}^2 W_{\mathcal{L}_1 \cup \cdots \cdots \cup \mathcal{L}_n} \quad (A.7a)
\]

\[
j_h \in \mathcal{L} \cap \mathcal{L}_h \
\]

This can easily be understood because if there is a spanning tree on \( \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n \) its restriction on \( \mathcal{L}_1 \cup \cdots \hat{\mathcal{L}}_k \cdots \cup \mathcal{L}_n \) is still a spanning tree if \( \mathcal{L}_1 \cup \cdots \hat{\mathcal{L}}_k \cdots \cup \mathcal{L}_n \) is connected or a set of spanning trees if \( \mathcal{L}_1 \cup \cdots \hat{\mathcal{L}}_k \cdots \cup \mathcal{L}_n \) happens to be disconnected.

Reciprocally, if there is a spanning-tree on \( \mathcal{L}_1 \cup \cdots \hat{\mathcal{L}}_k \cdots \cup \mathcal{L}_n \) and if we add all the propagators of \( \mathcal{L}_k \) not already in \( \mathcal{L}_1 \cup \cdots \hat{\mathcal{L}}_k \cdots \cup \mathcal{L}_n \), except for \( j_k \), we get a spanning tree on \( \mathcal{L}_1 \cup \cdots \mathcal{L}_n \).

We recall that the propagator \( i \) on \( \mathcal{L} \) is never shared with another loop of \( \mathcal{L}_1, \mathcal{L}_2, \cdots \mathcal{L}_n \) in this construction. Then, we can write

\[
W_{\mathcal{L} \cup \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n} = \bar{\alpha}_i W_{\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n} + \sum_{\ell_k} \bar{\alpha}_{\ell_k} \left( W_{\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n} - \bar{\alpha}_{\ell_k} W_{\mathcal{L}_1 \cup \cdots \hat{\mathcal{L}}_k \cdots \cup \mathcal{L}_n} \right) \quad (A.8)
\]

with \( \ell_k \in \mathcal{L}, \ell_k \neq i \), taking the convention that

\[
W_{\mathcal{L}_1 \cup \cdots \hat{\mathcal{L}}_k \cdots \cup \mathcal{L}_n} = 0 \quad (A.9)
\]

whenever no loop \( \mathcal{L}_k \) shares the propagator \( \ell_k \) with \( \mathcal{L} \). It follows from (A.8) that \( \mu_i \) can be written (see (5.9) for its definition)

\[
\mu_i = \sum_{\ell_k} \bar{\alpha}_{\ell_k} \left( 1 - \bar{\alpha}_{\ell_k} W_{\mathcal{L}_1 \cup \cdots \hat{\mathcal{L}}_k \cdots \cup \mathcal{L}_n} / W_{\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n} \right) \quad (A.10a)
\]

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\[ \ell_k \neq i, \quad \ell_k \in \mathcal{L} \cap \mathcal{L}_k \quad \text{if } \mathcal{L}_k \text{ exists} \]
\[ \ell_k \in \mathcal{L} \quad \text{if } \mathcal{L}_k \text{ does not exist} \]  \hspace{1cm} (A.10b)

However, \( W_{\mathcal{L}_1 \cup \ldots \cup \mathcal{L}_n} \) can also be expressed as

\[
W_{\mathcal{L}_1 \cup \ldots \cup \mathcal{L}_n} = \tilde{\alpha}_{\ell_k} W_{\mathcal{L}_1 \cup \ldots \cup \mathcal{L}_k \cup \mathcal{L}_n} \\
+ \sum_{k_m} \tilde{\alpha}_{k_m} \left( W_{\mathcal{L}_1 \cup \ldots \cup \mathcal{L}_k \cup \mathcal{L}_n} - \tilde{\alpha}_{k_m} W_{\mathcal{L}_1 \cup \ldots \cup \mathcal{L}_k \cup \mathcal{L}_m \cup \mathcal{L}_n} \right) \]  \hspace{1cm} (A.11a)

\[ k_m \neq \ell_k, \quad k_m \in \mathcal{L}_k \cap \mathcal{L}_m \quad \text{if } \mathcal{L}_m \text{ exists} \]
\[ k_m \in \mathcal{L}_k \quad \text{if } \mathcal{L}_m \text{ does not exist} \]  \hspace{1cm} (A.11b)

with the same convention that

\[ W_{\mathcal{L}_1 \cup \ldots \cup \mathcal{L}_k \cup \mathcal{L}_m \cup \mathcal{L}_n} = 0 \]

whenever \( k_m \) is not shared by \( \mathcal{L}_k \) with any other loop of \( \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_n \).

This leads us to write \( \mu_i \) as

\[
\mu_i = \sum_{\ell_k} \tilde{\alpha}_{\ell_k} \left\{ 1 - \varepsilon_{\ell_k} \tilde{\alpha}_{\ell_k} / \left[ \tilde{\alpha}_{\ell_k} + \sum_{k_m} \tilde{\alpha}_{k_m} \left( 1 - \tilde{\alpha}_{k_m} W_{\mathcal{L}_1 \cup \ldots \cup \mathcal{L}_k \cup \mathcal{L}_m \cup \mathcal{L}_n} \right) \right] \right\} \]  \hspace{1cm} (A.12a)

\[ \varepsilon_{\ell_k} = 1 \quad \text{if} \quad W_{\mathcal{L}_1 \cup \ldots \cup \mathcal{L}_k \cup \mathcal{L}_n} \neq 0 \]
\[ \varepsilon_{\ell_k} = 0 \quad \text{if} \quad W_{\mathcal{L}_1 \cup \ldots \cup \mathcal{L}_k \cup \mathcal{L}_n} = 0 \]  \hspace{1cm} (A.12b)

introducing a continued fraction representation for \( \mu_i \). Let us now see how we can make \( \mu_i \) vanish faster than \( 1/I \).

i) The most immediate way to make \( \mu_i \) tends to zero faster than \( 1/I \) is to assume that every \( \tilde{\alpha}_{\ell_k} \) in (A.11) is decreasing faster than \( 1/I \). However note that if \( \tilde{\alpha}_{\ell} \) is zero for some propagator \( \ell \) it will not contribute either to \( P_G(\{\tilde{\alpha}\}) \) nor to \( Q_G(P,\{\tilde{\alpha}\}) \), see (2.2) and (2.3)), so that we can consider that \( \ell \) has been erased from \( G \). If we assume that the \( \tilde{\alpha}_{k_m} \)'s in (A.11) are \( O(1/I) \) we then have
\[\bar{\alpha}_\ell k / \bar{\alpha}_{km} \to 0 \quad (A.13)\]

and we can proceed as if the propagators \(\ell_k\) did not exist, this amounting to replace the loop \(\mathcal{L}\) by a loop \(\mathcal{L}'\) made of all propagators belonging to the loops \(\mathcal{L}_k\) having a propagator in common with \(\mathcal{L}\) except for the common propagators \(\ell_k\). We then have to write \(\mu_i\) with \(\mathcal{L}\) replaced by \(\mathcal{L}'\) which leads to an expression different from (A.12a) and we have to redo this reasoning again if we want to make \(\mu_i\) vanish faster than \(1/I\).

ii) The second way to make \(\mu_i\) vanish faster than \(1/I\) is to assume that for every propagator \(k_m\)

\[\bar{\alpha}_{km} / \bar{\alpha}_\ell k \to 0 \quad (A.14)\]

According to what we said in i) this would make the propagators \(k_m\) disappear and every loop \(\mathcal{L}_k\) would fuse with a neighbouring loop \(\mathcal{L}_m\) (erasing their common propagator) giving a loop \(\mathcal{L}'_k\) instead. Again, (A.12) would be modified by such a change and \(\mu_i\) would be \(O(1/I)\) unless the reasoning is repeated.

So, we see that in order to make \(\mu_i\) vanish faster than \(1/I\) we have to repeat the loop cancellation mechanism forever. This leads to have all \(\bar{\alpha}_j\)'s to be vanishing faster than \(1/I\) which is forbidden by the constraint (2.9). We therefore conclude that \(\mu_i\) must be \(O(1/I)\).

**Remarks**

i) In order to be able to use the cancellation mechanism, the sum of the weights of the spanning trees having a factor \(\bar{\alpha}_\ell k\) should be negligible with respect to the sum of the weights of the spanning trees not having this factor on \(\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n\). So, instead of (A.13), a priori, the real condition would be, see (A.11a),

\[\bar{\alpha}_\ell k W_{\mathcal{L}_1 \cup \cdots \hat{\mathcal{L}}_k \cdots \cup \mathcal{L}_n} / \sum_{k_m} \bar{\alpha}_{km} \left( W_{\mathcal{L}_1 \cup \cdots \hat{\mathcal{L}}_k \cdots \cup \mathcal{L}_n} - \bar{\alpha}_{km} W_{\mathcal{L}_1 \cup \cdots \hat{\mathcal{L}}_k \cdots \hat{\mathcal{L}}_m \cdots \cup \mathcal{L}_n} \right) \to 0 \quad (A.15)\]

which is equivalent to (A.13) provided there is at least one \(k_m\) such that

\[1 - \bar{\alpha}_{km} W_{\mathcal{L}_1 \cup \cdots \hat{\mathcal{L}}_k \cdots \hat{\mathcal{L}}_m \cdots \cup \mathcal{L}_n} / W_{\mathcal{L}_1 \cup \cdots \hat{\mathcal{L}}_k \cdots \mathcal{L}_n} \neq 0 \quad (A.16)\]
However, breaking (A.16) would mean that in

\[
W_{L_1 \cup \ldots \hat{L}_k \ldots \cup L_n} = \bar{\alpha}_{km} W_{L_1 \cup \ldots \hat{L}_k \ldots \cup L_n} \\
+ \sum_{m_{\rho}} \bar{\alpha}_{m_{\rho}} \left( W_{L_1 \cup \ldots \hat{L}_k \ldots \cup L_n} - \bar{\alpha}_{m_{\rho}} W_{L_1 \cup \ldots \hat{L}_k \ldots \hat{L}_m \ldots \cup L_n} \right)
\]

we should have for every \( m_{\rho} \), either

\[
\bar{\alpha}_{m_{\rho}} / \bar{\alpha}_{km} \to 0
\]

leading to a cancellation of \( m_{\rho} \) or an analog of (A.16) leading to a cancellation at a further step of the reasoning. So, indeed, (A.13) is sufficient to induce the cancellation of \( \ell_k \).

ii) Concerning (A.14), we can look at (A.11a) and note that if \( \bar{\alpha}_{km} / \bar{\alpha}_{\ell_k} \to 0 \), then

\[
\bar{\alpha}_{km} \left( W_{L_1 \cup \ldots \hat{L}_k \ldots \cup L_n} - \bar{\alpha}_{km} W_{L_1 \cup \ldots \hat{L}_k \ldots \hat{L}_m \ldots \cup L_n} \right) / \\
\bar{\alpha}_{km} W_{L_1 \cup \ldots \hat{L}_k \ldots \cup L_n} \to 0
\]

because

\[
W_{L_1 \cup \ldots \hat{L}_k \ldots \cup L_n} > W_{L_1 \cup \ldots \hat{L}_k \ldots \cup L_n} - \bar{\alpha}_{km} W_{L_1 \cup \ldots \hat{L}_k \ldots \hat{L}_m \ldots \cup L_n} > 0.
\]

So, in the same way, (A.14) is sufficient to have \( k_m \) cancelled (of course the same also apply to (A.18) and the cancellation of \( m_{\rho} \)).
Table 1

\(a_- = x_- = 0\)

| \(t/m^2\) | \(\mu_\Lambda\) | \(y\)     | \(x\)   | \(u\)   | \(z\)   | \(a_+\) | \(x_+\) | \(\alpha(t/m^2)\) |
|-----------|------------|---------|-------|-------|-------|-------|-------|-----------------|
| 2.0       | 11.01      | 0.92796 | 1.223 | 1.269 | 0.639 | -0.311 | 0.584 | 0.50            |
| 1.5       | 10.93      | 0.92794 | 1.210 | 1.262 | 0.637 | -0.298 | 0.588 | 0.438           |
| 1.0       | 10.82      | 0.92779 | 1.179 | 1.263 | 0.639 | -0.276 | 0.588 | 0.423           |
| 0.5       | 10.80      | 0.92778 | 1.155 | 1.262 | 0.640 | -0.264 | 0.590 | 0.321           |
| 0.0       | 10.69      | 0.92793 | 1.140 | 1.267 | 0.640 | -0.250 | 0.590 | 0.240           |
| -0.5      | 10.58      | 0.92776 | 1.116 | 1.270 | 0.633 | -0.231 | 0.590 | 0.163           |
| -1.0      | 10.48      | 0.92783 | 1.082 | 1.261 | 0.647 | -0.210 | 0.585 | 0.079           |
| -1.5      | 10.40      | 0.92782 | 1.064 | 1.266 | 0.644 | -0.196 | 0.586 | 0.012           |
| -2.0      | 10.32      | 0.92777 | 1.041 | 1.267 | 0.638 | -0.176 | 0.586 | -0.020          |
| -2.5      | 10.19      | 0.92773 | 1.017 | 1.266 | 0.645 | -0.158 | 0.586 | -0.108          |
| -3.0      | 10.05      | 0.92772 | 0.996 | 1.264 | 0.638 | -0.143 | 0.587 | -0.170          |
| -3.5      | 10.04      | 0.92786 | 0.963 | 1.271 | 0.640 | -0.126 | 0.586 | -0.195          |
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Figure Captions

Fig. 1:
A ladder graph is shown. Central-propagators are weighted by $\alpha_-$ and side-propagators by $\alpha_+$. $t$ is the invariant equal to the sum squared of momenta entering at one end of the ladder. $s$ is the large invariant when Regge behaviour is obtained. It is equal to the sum squared of momenta entering at one side of the ladder (here, one side is up and the other down).

Fig. 2:
We display the three kinds of topology obtained by removing propagators in order to obtain spanning trees on the ladder. When a central propagator is removed a factor $\bar{\alpha}_-$ is obtained and when a side propagator is removed a factor $\bar{\alpha}_+$ is obtained. Cells of lengths $\ell_1, \ell_2, \cdots, \ell_{L-p}$ are formed. Each cell has propagators on its border. Removed central propagators are shown as dashed lines, removed side-propagators are simply cancelled. In a) the end-cells are opened on the sides of the ladder. In b) one end-cell is opened at one end of the ladder. In c) both end-cells are opened at the ends of the ladder.

Fig. 3:
A cell of length $\ell$ is displayed as well as a propagator $i_+$ having a weight $\alpha_{i_+}$ (and not $\bar{\alpha}_+$. In this configuration the removed propagator on the down-side of the cell brings up a factor $\bar{\alpha}_+$. When $i_+$ is on the opposite side of the removed propagator it can be removed too, bringing up a factor $\alpha_{i_+}$. When $i_+$ is on the same side as the removed propagator it cannot be removed because some propagators would be isolated from the rest of the ladder without being attached to an external line.

Fig. 4:
We display the Regge trajectory $\alpha(t/m^2)$ for $\ell n \gamma_m$ equal to -0.1 (lower line and squares) and to 0 (upper line and losanges). Both lines are straight-lines, which are parallel and give a good fit to the computed data. The error bars show the dispersion given by repeating the calculations several times with different starting values for the parameters. We remark that the dispersion grows for uncreasing values of $|t/m^2|$. Also displayed is the axis $t = 0$ and two full circles corresponding
to \( \alpha(0) \) given by (10.12).

**Fig. 5:**

The intercept \( \alpha(0) \) is shown for \( \ell n \gamma_m \) ranging from -0.4 to 0.5. The squares are given by (10.12) and the crosses are the result of our calculations. Agreement is observed for \( \alpha(0) \gtrsim 0.3 \). Error bars for crosses give the dispersion of the calculations.
Fig. 1
Fig. 2
Fig. 4
Fig. 5