Elliptic Weight Functions and Elliptic $q$-KZ Equation

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Abstract

By using representation theory of the elliptic quantum group $U_{q,p}(\widehat{sl}_N)$, we present a systematic method of deriving the weight functions. The resultant $sl_N$ type elliptic weight functions are new and give elliptic and dynamical analogues of those obtained in the rational and the trigonometric cases. We then discuss some basic properties of the elliptic weight functions. We also present an explicit formula for formal elliptic hypergeometric integral solution to the face type, i.e. dynamical, elliptic $q$-KZ equation.

1 Introduction

The weight functions are one of the main parts in hypergeometric integral solutions to the $q$-KZ equations. See for example [44]. Recently new interests in the weight functions have been developing. Among others Gorbounov, Rimányi, Tarasov and Varchenko [16] have succeeded to identify the rational weight functions with the stable envelopes associated with the torus equivariant cohomology of the cotangent bundle to the partial flag variety. The stable envelopes were introduced by Maulik and Okounkov [33, 39] in more general setting associated with the equivariant cohomology of Nakajima’s quiver variety [36]. They are maps from the equivariant cohomology of the fixed point set of the torus action to the equivariant cohomology of the variety and play an important role in a formulation of a geometric representation theory of quantum groups on the equivariant cohomology.

This identification has been extended to the equivariant $K$-theory case in [40] as well as to the dynamical version of the equivariant cohomology [41] and equivariant elliptic cohomology [14] cases both associated with the cotangent bundles to the Grassmannians. It has also been succeeded to construct a geometric representation of the Yangian $Y(gl_N)$ [10], the affine quantum group $U_q(\widehat{gl}_N)$ [10], and the rational dynamical quantum group $E_y(gl_2)$ [41] and the elliptic dynamical quantum group $E_{\tau,y}(gl_2)$ [14] on the corresponding equivariant cohomology, $K$-theory, dynamical version of the cohomology and elliptic cohomology, respectively. There it is essential...
to consider the finite-dimensional representations of the quantum groups on the Gelfand-Tsetlin basis of the tensor product of the vector representations. They are lifted to the geometric representations via the correspondence between the weight functions and the stable envelopes. It is also remarkable that the elliptic stable envelopes introduced by Aganagic and Okounkov [1] are the dynamical ones, where the Kähler variables play a role of the dynamical parameters.

The purpose of this paper and subsequent papers is to extend these constructions to the higher rank elliptic quantum group $U_{q,p}(\hat{\mathfrak{sl}}_N)$ [20, 25, 30], which is an elliptic and dynamical analogue of the Drinfeld’s new realization of $U_q(\hat{\mathfrak{sl}}_N)$ [4] and is isomorphic to the central extension of Felder’s elliptic quantum group [10, 30]. We expect that the elliptic weight functions of the $\mathfrak{sl}_N$-type can be identified with the elliptic stable envelopes associated with the torus equivariant elliptic cohomology of the cotangent bundles to the partial flag variety. Such identification should allow us to formulate a geometric action of $U_{q,p}(\hat{\mathfrak{sl}}_N)$ on the equivariant elliptic cohomology. Recently, Felder, Rimányi and Varchenko has accomplished such study in the $\hat{\mathfrak{sl}}_2$ case [14].

In this paper we discuss the elliptic weight functions of type $\mathfrak{sl}_N$ and study their properties. We give a systematic derivation of the elliptic weight functions by using the vertex operators of $U_{q,p}(\hat{\mathfrak{sl}}_N)$ [20, 23, 24]. For the $\hat{\mathfrak{sl}}_2$ case we use the level-$k$ representation of $U_{q,p}(\hat{\mathfrak{sl}}_2)$, and the resultant elliptic weight functions coincide with those obtained in [12, 44]. We discuss this case in a separate paper. We here concentrate on the higher rank level-1 representation and obtain a new result. The resultant elliptic weight functions are described by using the partitions of $[1,n]$ in a combinatorial way and give elliptic and dynamical analogues to those obtained by Mimachi and Noumi [34, 35], Tarasov and Varchenko [40, 43]. The same method can be applied to the trigonometric case too by using the vertex operators of $U_q(\hat{\mathfrak{sl}}_N)$ and allows us to derive the trigonometric weight functions in [32, 44] for $\hat{\mathfrak{sl}}_2$ case and in [31, 35, 40] for $\mathfrak{sl}_N$ case. See also [18, 37, 43]. In the sequent paper we will discuss the finite-dimensional representations of $U_{q,p}(\hat{\mathfrak{sl}}_N)$ on the Gelfand-Tsetlin basis and their geometric interpretations.

Our derivation has the following advantages.

1) It makes a representation theoretical meaning of the combinatorial structure of the weight functions as well as of the partial flag variety transparent. See 4.1.

2) It makes the transition property of the weight function manifest. See 5.2.

3) It allows us to derive the shuffle algebra structure of the space of the weight functions. See 5.5 & Appendix C.

As a byproduct we also give a new formula for formal elliptic hypergeometric integral solution to the face type elliptic $q$-KZ equation [11, 15].
A part of the results has been presented in the workshops “Classical and Quantum integrable systems”, July 11-15, 2016, EIMI, St.Petersburg, “Recent Advances in Quantum Integrable Systems”, August 22-26, 2016, Univ.of Geneva and “Elliptic Hypergeometric Functions in Combinatorics, Integrable Systems and Physics”, March 20-24, 2017, ESI, Vienna.

This paper is organized as follows. In Section 2 we prepare some notations including the elliptic dynamical $R$ matrices. In Section 3 we review a construction of the vertex operators of the $U_{q,p}(\widehat{\mathfrak{sl}}_N)$-modules. In particular we provide a free field realization of the vertex operators for the level 1 representation. Section 4 is devoted to a derivation of the elliptic weight functions of the $\mathfrak{sl}_N$-type by using the vertex operators. In Section 5 we discuss some basic properties of the elliptic weight functions such as the triangular property, transition property, orthogonality, quasi-periodicity and the shuffle algebra structures. In Section 6, we give a formal elliptic hypergeometric integral solution to the face type elliptic $q$-KZ equation. In Appendix A we summarize some basic facts on the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_N)$. Appendix B is a summary on the dynamical representations of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ both the finite and infinite-dimensional cases. In Appendix C we provide a proof of Proposition 5.10 including a derivation of the shuffle algebra structure of the space of the weight functions.

While preparing this manuscript, we become aware of the paper by Rimányi, V.Tarasov and A.Varchenko [42] which has some overlap with the content of the present paper.

2 Preliminaries

In this section we prepare the notation to be used in the text. Throughout this paper $q$ are generic complex numbers satisfying $|q| < 1$ unless otherwise specified.

2.1 The commutative algebra $H$

Let $A = (a_{ij})$ $(0 \leq i, j \leq N - 1)$ be the generalized Cartan matrix of $\widehat{\mathfrak{sl}}_N = \widehat{\mathfrak{sl}}(N, \mathbb{C})$ [21]. Let $\mathfrak{h} = \mathfrak{h} \oplus C d$, $\mathfrak{h} = \mathfrak{h} \oplus C c$, $\mathfrak{h} = \oplus_{i=1}^{N-1} C h_i$ be the Cartan subalgebra of $\mathfrak{sl}_N$. Define $\delta, \Lambda_0, \alpha_i, \tilde{\Lambda}_i$ $(1 \leq i \leq N - 1) \in \mathfrak{h}^*$ by

$$<\alpha_i, h_j> = a_{ij}, \quad <\delta, d> = 1 = <\Lambda_0, c>, \quad <\tilde{\Lambda}_i, h_j> = \delta_{i,j} \quad (2.1)$$

the other pairings are 0. We set $\mathfrak{h}^* = \oplus_{i=1}^{N-1} C \tilde{\Lambda}_i$, $\mathfrak{h}^* = \mathfrak{h}^* \oplus C \Lambda_0$, $Q = \oplus_{i=1}^{N-1} \mathbb{Z} \alpha_i$ and $P = \oplus_{i=1}^{N-1} \mathbb{Z} \tilde{\Lambda}_i$. Let $\{e_j \ (1 \leq j \leq N)\}$ be an orthonormal basis in $\mathbb{R}^N$ with the inner product $(e_j, e_k) = \delta_{j,k}$. We set $\tilde{e}_j = e_j - \sum_{k=1}^{N} e_k/N$. We realize the simple roots by $\alpha_j = e_j - e_{j+1}$ $(1 \leq j \leq N - 1)$ and the fundamental weights by $\tilde{\Lambda}_j = \tilde{e}_1 + \cdots + \tilde{e}_j$ $(1 \leq j \leq N - 1)$. We define
We use the following notations.

### 2.2 Infinite products

Abbreviate $P$ denote the dual space of $H$ and $< Q_r |$.

We regard $\mathfrak{g} \oplus \mathfrak{g}^*$ as the Heisenberg algebra by

$$[h_{\epsilon_j}, \epsilon_k] = (\epsilon_j, \epsilon_k), \quad [h_{\epsilon_j}, h_{\epsilon_k}] = 0 = [\epsilon_j, \epsilon_k].$$

(2.2)

In particular, we have $[h_j, \alpha_k] = a_{jk}$. We also set $h^j = h_{\lambda_j}$.

Let $\{P_{\alpha}, Q_{\beta}\}$ ($\alpha, \beta \in \mathfrak{h}^*$) be the Heisenberg algebra defined by the commutation relations

$$[P_{\epsilon_j}, Q_{\epsilon_k}] = (\epsilon_j, \epsilon_k), \quad [P_{\epsilon_j}, P_{\epsilon_k}] = 0 = [Q_{\epsilon_j}, Q_{\epsilon_k}],$$

where $P_{\alpha} = \sum_j c_j P_j$ for $\alpha = \sum_j c_j \epsilon_j$. We set $P_{\mathfrak{h}} = \bigoplus_{j=1}^{\infty} \mathbb{C} P_j$, $Q_{\mathfrak{h}} = \bigoplus_{j=1}^{\infty} \mathbb{C} Q_j$ $P_j = P_{\alpha_j}, P_j^\dagger = P_{\lambda_j}$ and $Q_j = Q_{\alpha_j}, Q_j^* = Q_{\lambda_j}$.

For the abelian group $R_Q = \bigoplus_{j=1}^{\infty} \mathbb{Z} Q_{\alpha_j}$, we denote by $\mathbb{C}[R_Q]$ the group algebra over $\mathbb{C}$ of $R_Q$. We denote by $e^{q\alpha}$ the element of $\mathbb{C}[R_Q]$ corresponding to $Q_{\alpha} \in R_Q$.

We define the commutative algebra $H$ by $H = \mathfrak{g} \oplus P_{\mathfrak{h}} = \sum_j \mathbb{C}(P_{\epsilon_j} + h_{\epsilon_j}) + \sum_j \mathbb{C} P_{\epsilon_j} + \mathbb{C}c$. We denote the dual space of $H$ by $H^* = \mathfrak{g}^* \oplus Q_{\mathfrak{h}}^*$. We define the paring by $\langle \alpha, \beta \rangle$, $< Q_{\alpha}, P_{\beta} >= (\alpha, \beta)$ and $< Q_{\alpha}, h_{\beta} >=< Q_{\alpha}, c >$. $< Q_{\alpha}, d >= 0 =< \alpha, P_{\beta} >=< \delta, P_{\beta} >=< \lambda_0, P_{\beta} >$. We often abbreviate $P_{\epsilon_j} + h_{\epsilon_j}$ as $(P + h)_{\epsilon_j}$ and use $(P + h)_{j,k} = (P + h)_{\epsilon_j} - (P + h)_{\epsilon_k}, P_{j,k} = P_{\epsilon_j} - P_{\epsilon_k}, h_{j,k} = h_{\epsilon_j} - h_{\epsilon_k}$ etc.

We define $\mathbb{F} = \mathcal{M}_{H^*}$ to be the field of meromorphic functions on $H^*$.

### 2.3 Theta functions

Let $r$ be a generic positive real number. We set $p = q^{2r}$. In general we consider the level $k \in \mathbb{R}$ representation of the elliptic quantum group $U_{q,p}(\mathfrak{g})$. See Appendix 3. In that case we assume $r^* = r - k > 0$ and set $p^* = q^{2r^*}$.

$$|q| < 1, |t| < 1, |p| < 1.$$
We use the following Jacobi’s odd theta functions.

\[ [u] = q^{\frac{2}{\tau} u - u} \Theta_p(z), \quad [u]^* = q^{\frac{2}{\tau} u - u} \Theta_p^*(z), \tag{2.4} \]

\[ [u + r] = -[u], \quad [u + r \tau] = -e^{-\pi i r} e^{-2 \pi i u / [u]}, \tag{2.5} \]

\[ [u + r^*] = -[u]^*, \quad [u + r^* \tau^*] = -e^{-\pi i r^*} e^{-2 \pi i u / [u]^*}, \tag{2.6} \]

where \( z = q^{2u}, p = e^{-2 \pi i / \tau}, p^* = e^{-2 \pi i / \tau^*}. \)

### 2.4 The elliptic dynamical \( R \)-matrix of type \( \widehat{sl}_N \)

Let \( \widehat{V}_z \) be the \( N \)-dimensional dynamical evaluation representation of \( U_{q,p}(\widehat{sl}_N) \) given in Sec \[19\] and \( \{ v_\mu \ (\mu = 1, \cdots, N) \} \) be its basis. We consider the following elliptic dynamical \( R \)-matrix \( R^+(z_1/z_2, \Pi) \in \text{End}_C(\widehat{V}_{z_1} \otimes \widehat{V}_{z_2}) \) given by

\[ R^+(z, \Pi) = \rho^+(z) R(z, \Pi), \tag{2.6} \]

\[ \overline{R}(z, \Pi) = \sum_{j=1}^{N} E_{j,j} \otimes E_{j,j} + \sum_{1 \leq j_1 < j_2 \leq N} (b(u, (P + h)j_1, j_2)E_{j_1, j_1} \otimes E_{j_2, j_2} + \tilde{b}(z)E_{j_2, j_2} \otimes E_{j_1, j_1} + c(u, (P + h)j_1, j_2)E_{j_1, j_2} \otimes E_{j_2, j_1} + \tilde{c}(u, (P + h)j_1, j_2)E_{j_2, j_1} \otimes E_{j_1, j_2}), \tag{2.7} \]

where \( z = q^{2u}, \Pi_{j,k} = q^{2(P+h)j,k}, \rho^+(z) = q^{\frac{\pi i}{\tau} z \frac{\pi i}{\tau + 1}} p^+(z), \quad \tilde{\rho}^+(z) = \frac{q^{2N} q^{-2} z \{q^2 z\} \{p q^2 z\}}{\{q^2 z\} \{z\} \{pq^2 z\}}, \)

\[ b(u, s) = \frac{[s + 1][s - 1][u]}{[s] [u + 1]}, \quad \tilde{b}(u) = \frac{[u]}{[u + 1]}, \quad \tilde{c}(u, s) = \frac{[1][s + u]}{[s][u + 1]} \]

and \( \{z\} = (z;p,q^{2N})_\infty. \) This \( R \) matrix is gauge equivalent to Jimbo-Miwa-Okado’s \( A_{N-1}^{(1)} \) face type Boltzmann weight \( \text{[17]} \) and can be obtained by taking the vector representation of the universal elliptic dynamical \( R \) matrix derived in \( \text{[19]}. \) See \( \text{[20]}. \)

We also use \( R^{*+}(z, \Pi^*) \) with \( \Pi^*_{j,k} = q^{2P_{j,k}}, \) which is obtained from \( R^+(z, \Pi) \) by the following replacement.

\[ R^{*+}(z, \Pi^*) = R^+(z, \Pi) |_{p \rightarrow p^*, r \rightarrow r^*, [u] \rightarrow [u]^*, P + h \rightarrow P^*}. \]

The \( R^+(z, q^{2s}) \) satisfies the dynamical Yang-Baxter equation

\[ R^{+(12)}(z_1/z_2, q^{2(s+h^{(3)})})R^{+(13)}(z_1/z_3, q^{2s})R^{+(23)}(z_2/z_3, q^{2(s+h^{(1)})}) = R^{+(23)}(z_2/z_3, q^{2s})R^{+(13)}(z_1/z_3, q^{2(s+h^{(2)})})R^{+(12)}(z_1/z_2, q^{2s}). \tag{2.10} \]

where \( q^{2h^{(l)}_{j,k}} \) acts on the \( l \)-th tensor space \( \widehat{V}_{z_1} \) by \( q^{2h^{(l)}_{j,k} \mu} = q^{2 \epsilon_{\mu,j,k}} v_\mu, \) and the unitarity

\[ R^+(z, q^{2s}) R^{(21)}(z^{-1}, q^{2s}) = \text{id}_{\widehat{V}_z \otimes \widehat{V}_z}. \tag{2.11} \]
3 Vertex Operators of $U_{q,p}(\widehat{\mathfrak{sl}_N})$

In this section we summarize the known facts on the type I and II vertex operators of the $U_{q,p}(\widehat{\mathfrak{sl}_N})$-modules obtained in [20, 23].

3.1 Definition

Let $\hat{V}_z$ be as in Sec. 2.4 and $\hat{V}(\lambda, \nu)$ denote the irreducible level-$k$ highest weight $U_{q,p}(\widehat{\mathfrak{sl}_N})$-module with highest weight $(\lambda, \nu)$ in Definition B.4. The level-1 case is given in Theorem B.7. The type I $\Phi(z)$ and the type II $\Psi^*(z)$ vertex operators are the intertwiners of the $U_{q,p}(\widehat{\mathfrak{sl}_N})$-modules of the form

$$\Phi(z) : \hat{V}(\lambda, \nu) \rightarrow \hat{V}_z \otimes \hat{V}(\lambda', \nu),$$

$$\Psi^*(z) : \hat{V}(\lambda, \nu) \otimes \hat{V}_z \rightarrow \hat{V}(\lambda, \nu'),$$

where $\lambda, \lambda' \in \mathfrak{h}^*$, $\nu, \nu' \in H^*$. The vertex operators satisfy the intertwining relations with respect to the comultiplication $\Delta$ given in Sec. A.6

$$\Delta(x)\Phi(z) = \Phi(z)x,$$  \quad (3.3)

$$x\Psi^*(z) = \Psi^*(z)\Delta(x), \quad \forall x \in U_{q,p}(\widehat{\mathfrak{sl}_N}).$$  \quad (3.4)

By using the $L$ operator $\hat{L}^+(z)$ given in [A.34] the main part of the intertwining relations can be re-expressed as follows [27]:

$$\left(\text{id} \otimes \Phi(z_2) \right) \hat{L}^+(z_1) = R^{+(12)}(z_1/z_2, \Pi) \hat{L}^+\otimes^{(13)}(z_1) (\text{id} \otimes \Phi(z_2)), \quad (3.5)$$

$$\hat{L}^+(z_1)\Psi^*\otimes^{(23)}(z_2) = \Psi^*\otimes\hat{L}^+\otimes^{(12)}(z_1) R^{*+(13)}(z_1/z_2, \Pi^* q^{-2(h^{(1)} + h^{(3)})}). \quad (3.6)$$

The relation (3.5) (resp. (3.6)) should be understood on $\hat{V}_{z_1} \otimes \hat{V}(\lambda, \nu)$ (resp. $\hat{V}_{z_1} \otimes \hat{V}(\lambda, \nu) \otimes \hat{V}_{z_2}$).

We define the components of the vertex operators by

$$\Phi(zq^{-1})u = \sum_{\mu=1}^{N} v_{\mu} \otimes \Phi_{\mu}(z) u, \quad \Psi^*(zq^{-1})(u \otimes v_{\mu}) = \Psi^*_{\mu}(z) u \quad \forall u \in \hat{V}(\lambda, \nu). \quad (3.7)$$

Let $F_j(z), E_j(z), K_j^+(z)$ be the elliptic currents in Definition A.9 and let $F_{j,l}^+(z), E_{l,j}^+(z)$ be the half currents in Definition A.12. From the intertwining relations (3.5)- (3.6) one can deduce the following relations as the sufficient conditions [23].

Proposition 3.1. For the type I,

$$\Phi_{\mu}(z_2) = F_{\mu,N}^+(pq^{-1}z_2) \Phi_{N}(z_2) \quad (1 \leq \mu \leq N-1), \quad (3.8)$$
and

\[ \Phi_N(z_2)K^+_N(z_1) = \rho^+(qz_1/z_2)K^+_N(z_1)\Phi_N(z_2), \]
\[ [\Phi_N(z), P_{k,l}] = 0, \quad [\Phi_N(z), E_j(w)] = 0, \quad (1 \leq l \leq N, 1 \leq j \leq N - 1), \]
\[ [\Phi_N(z), (P + h)_{k,l}] = -\delta_{l,N}\Phi_N(z), \quad (k < l) \]
\[ F_{N-1}(z_1)\Phi_N(z_2) = \left[\frac{u_1 - u_2 + \frac{1}{2}}{u_1 - u_2 - \frac{1}{2}}\right] \Phi_N(z_2)F_{N-1}(z_1), \]
\[ F_j(z_1)\Phi_N(z_2) = \Phi_N(z_2)F_j(z_1) \quad (1 \leq j \leq N - 2). \]

For the type II,

\[ \Psi^*_\mu(z_2) = \Psi^*_N(z_2)E^+_N,\mu(\rho^* z_2) \quad (1 \leq \mu \leq N - 1), \]

and

\[ K^+_N(z_1)\Psi^*_N(z_2) = \Psi^*_N(z_2)K^+_N(z_1)\rho^{\ast\ast}(q^{1-k}z_1/z_2), \]
\[ [\Psi^*_N(z_2), (P + h)_{k,l}] = 0, \quad [\Psi^*_N(z_2), F_j(z_1)] = 0 \quad (1 \leq l \leq N, 1 \leq j \leq N - 1), \]
\[ [\Psi^*_N(z), P_{k,l}] = -\delta_{l,N}\Psi^*_N(z) \quad (k < l), \]
\[ E_{N-1}(z_1)\Psi^*_N(z_2) = \left[\frac{u_1 - u_2 - \frac{1}{2}}{u_1 - u_2 + \frac{1}{2}}\right] \Psi^*_N(z_2)E_{N-1}(z_1), \]
\[ E_j(z_1)\Psi^*_N(z_2) = \Psi^*_N(z_2)E_j(z_1) \quad (1 \leq j \leq N - 2). \]

### 3.2 The level-1 vertex operators and commutation relations

For the level-1 representation (Sec.B.3), we have a free field realization of the vertex operators.

**Theorem 3.2.** [23] Let us assume \(|p| < |z| < 1\). Let \(\Lambda_a \ (a = 0, 1, \cdots, N-1)\) be the fundamental weights of \(\hat{sl}_N\). The components of the type I and the type II vertex operators, \(\Phi_\mu(z) : \hat{V}(\Lambda_a + \nu, \nu) \to \hat{V}(\Lambda_a - \mu + \nu, \nu)\) and \(\Psi^*_\mu(z) : \hat{V}(\Lambda_a + \nu, \nu) \to \hat{V}(\Lambda_a + \nu, \nu - \mu)\), are realized as follows.

\[ \Phi_N(z) = \exp \left\{ \sum_{m \neq 0} (q^m - q^{-m})e_{m}^{N} z^{-m} \right\} : e^{-\hat{e}_N(-z)}h_N z^{\frac{1}{2}}(P + h)_{\mu}, \]
\[ \Phi_\mu(z) = F^{+}_{\mu,N}(q^{-1}pz)\Phi_N(z) \]
\[ = a_{\mu,N} \int_{2\pi i t_m}^{N-1} \prod_{m=\mu}^{N-1} \frac{dt_m}{2\pi i t_m} \Phi_N(z)F_{N-1}(t_{N-1})F_{N-2}(t_{N-2}) \cdots F_{\mu}(t_{\mu})\varphi_\mu(z, t_\mu, \cdots, t_{N-1}; \Pi), \]
\[ (1 \leq \mu \leq N - 1), \]
\[ \Psi^*_N(z) = \exp \left\{ -\sum_{m \neq 0} (q^m - q^{-m})e_{m}^{N} z^{-m} \right\} : e^{\hat{e}_N(-z)}h_N z^{-\frac{1}{2}}P_N, \]
\[ (3.22) \]
Here we set

\[ \Psi_\mu^* (z) = \Psi_N(z) E_{N,\mu}^+ (p^* z) \]

\[ = a_{\mu,N} \int_{T^{N-\mu}} \prod_{m=\mu}^{N-1} \frac{dt_m}{2\pi i t_m} \Psi_N(z) E_{N-1}(t_{N-1}) E_{N-2}(t_{N-2}) \cdots E_{\mu}(t_{\mu}) \varphi_\mu^*(z, t_{\mu}, \cdots, t_{N-1}; \Pi^*), \]

(1 \leq \mu \leq N - 1) \hspace{1cm} (3.23)

where \( \mathcal{E}_{m}^+ \) and \( \mathcal{E}_{m}^{++} = 1 - \frac{p^* m q m}{1 - p^* m q m} \mathcal{E}_{m}^+ \) are the orthonormal basis type elliptic bosons of level 1 given in (A.17). We also set \( \tilde{\mathcal{E}}_N = \tilde{\mathcal{E}}_N + \mathcal{E}_N \) as in Sec B.3 \( \Pi = \{ \Pi_{\mu,m} (m = \mu + 1, \cdots, N) \}, \)

\[ \Pi^* = \{ \Pi_{\mu,m}^* (m = \mu + 1, \cdots, N) \}, \]

\[ \varphi_\mu(z, t_{\mu}, \cdots, t_{N-1}; \Pi) = \prod_{m=\mu}^{N-1} \frac{[v_{m+1} - v_m + (P + h)_{\mu,m+1} - \frac{1}{2}]^2}{[v_{m+1} - v_m + \frac{1}{2}] [(P + h)_{\mu,m+1} + \frac{1}{2}]} , \]

(3.24)

\[ \varphi_\mu^*(z, t_{\mu}, \cdots, t_{N-1}; \Pi^*) = \prod_{m=\mu}^{N-1} \frac{[v_{m+1} - v_m - P_{\mu,m+1} + \frac{1}{2}]^* [1]^*}{[v_{m+1} - v_m - \frac{1}{2}]^* [P_{\mu,m+1} - 1]^*} \]

(3.25)

with \( z = q^{2n}, t_m = q^{2vm}, v_N = u \) and

\[ T^{N-\mu} = \{ t \in \mathbb{C}^{N-\mu} \mid |t_\mu| = \cdots = |t_{N-1}| = 1 \}. \]

Theorem 3.3. \[ 23 \] The free field realizations of the type I \( \Phi_\mu(z) \) and the type II \( \Psi_\mu^*(z) \) vertex operators satisfy the following commutation relations:

\[ \Phi_{\mu_2}(z_2) \Phi_{\mu_1}(z_1) = \sum_{\mu_1', \mu_2' = 1}^{N} R(z_1/2, z_2, \Pi)_{\mu_1' \mu_2} \Phi_{\mu_1'}(z_1) \Phi_{\mu_2'}(z_2), \]

(3.26)

\[ \Psi_{\mu_1}^*(z_1) \Psi_{\mu_2}^*(u_2) = \sum_{\mu_1', \mu_2' = 1}^{N} \Psi_{\mu_2'}^*(z_2) \Psi_{\mu_1'}^*(z_1) R^*(z_1/2, \Pi^*)_{\mu_1' \mu_2'}, \]

(3.27)

\[ \Phi_{\mu}(z_1) \Psi_{\nu}^*(z_2) = \chi(z_1/2) \Psi_{\nu}^*(z_2) \Phi_{\mu}(z_1). \]

(3.28)

Here we set

\[ R(z, \Pi) = \mu(z) \overline{R}(z, \Pi), \hspace{0.5cm} R^*(z, \Pi^*) = \mu^*(z) \overline{R}^*(z, \Pi^*) \]

(3.29)

with

\[ \mu(z) = z^{-\frac{N-1}{N}} \cdot \frac{pq^{2N} q^{-2} z}{z} \cdots \frac{p^2 z}{z} \cdots \frac{q^{2N} z}{z}, \]

(3.30)

\[ \mu^*(z) = z^{-\frac{N-1}{N}} \cdot \frac{q^{2N} q^{-2} z^*}{z} \cdots \frac{p^* z}{z} \cdots \frac{q^{2N} z^*}{z}, \]

(3.31)

and

\[ \chi(z) = z^{-\frac{N-1}{N}} \frac{\Theta_{q^{2N}}(q^z)}{\Theta_{q^{2N}}(q/z)}. \]

(3.32)
4 Elliptic Weight Functions

In this section we present a simple derivation of the elliptic weight functions of type $\mathfrak{sl}_N$ by using the vertex operators prepared in the last section. The method can also be applied to the rational and trigonometric cases if one has an appropriate realization of the vertex operators.

Let us consider the $U_{q,p}(\mathfrak{sl}_N)$ level-1 type I vertex operators $\Phi_{\mu}(z)$ $(\mu = 1, \cdots, N)$ in Theorem 3.2 and their $n$-point composition $\phi(z_1, \cdots, z_n) = \Phi(z_1) \circ \cdots \circ \Phi(z_n) : \hat{V}(\Lambda_\mu + \nu, \nu) \to \hat{V}_{z_1} \otimes \cdots \otimes \hat{V}_n \otimes \hat{V}(\Lambda_{a'} + \nu, \nu)$, where $a' = (0N - 1N - 2 \cdots 21)N(a)$ is the cyclic permutation of $a$. We have

$$\phi(z_1, \cdots, z_n) = \sum_{\mu_1, \cdots, \mu_n = 1}^N \nu_{\mu_n} \otimes \cdots \otimes \nu_{\mu_1} \otimes \phi_{\mu_1 \cdots \mu_n}(z_1, \cdots, z_n),$$

$$\phi_{\mu_1 \cdots \mu_n}(z_1, \cdots, z_n) = \Phi_{\mu_1}(z_1) \cdots \Phi_{\mu_n}(z_n).$$

4.1 Combinatorial notations

For a given $n$-point operator $\phi_{\mu_1 \cdots \mu_n}(z_1, \cdots, z_n)$, it is convenient to introduce the following combinatorial notations. Let $[1, n] = \{1, \cdots, n\}$. Define the index set $I_l := \{i \in [1, n] \mid \mu_i = l\}$ ($l = 1, \cdots, N$) and set $\lambda_l := |I_l|$, $\lambda := (\lambda_1, \cdots, \lambda_N)$. Then $I = (I_1, \cdots, I_N)$ is a partition of $[1, n]$, i.e.

$$I_1 \cup \cdots \cup I_N = [1, n], \quad I_k \cap I_l = \emptyset \quad (k \neq l).$$

We often denote thus obtained partition $I$ by $I_{\mu_1 \cdots \mu_n}$ and conversely the $n$-point operator by $\phi_I(z_1, \cdots, z_n)$. Let $N = \{m \in \mathbb{Z} \mid m \geq 0\}$. For $\lambda = (\lambda_1, \cdots, \lambda_N) \in N^N$, let $I_\lambda$ be the set of all partitions $I = (I_1, \cdots, I_N)$ satisfying $|I_l| = \lambda_l$ ($l = 1, \cdots, N$). Note that for all $I \in I_\lambda$ the $n$-point operators $\phi_I(z_1, \cdots, z_n)$ have the same weight $-\sum_{j=1}^n \epsilon_{\mu_j}$. We call $\sum_{j=1}^n \epsilon_{\mu_j}$, the weight associated with $\lambda$. We also set $\lambda^{(I)} := \lambda_1 + \cdots + \lambda_l$, $I^{(I)} := I_1 \cup \cdots \cup I_l$ and let $\lambda^{(I)} := \{t_1^{(I)} < \cdots < t_\lambda^{(I)}\}$. It is also important to specify the numbering of the arguments of the elliptic currents $F_I$’s appearing in the realization of the vertex operators $\Phi_{\mu}(z)$. For $l = 1, \cdots, N - 1$, we assign the argument $i_k^{(I)}$ to the elliptic current $F_I$ attached to the $i_k^{(I)}$-th vertex operator.

Example 1. Let us consider the $N = 3$, $n = 5$, $\lambda = (2, 2, 1)$ case. For example, the 5-point operator

$$\Phi_2(z_1)\Phi_1(z_2)\Phi_3(z_3)\Phi_1(z_4)\Phi_2(z_5) \quad (4.1)$$

gives a partition $I = (I_1 = \{2, 4\}, I_2 = \{1, 5\}, I_3 = \{3\})$. Hence $I^{(1)} = \{2, 4\}$, $I^{(2)} = \{1, 2, 4, 5\}$, $I^{(3)} = \{1, 2, 3, 4, 5\}$. In particular, $t_1^{(2)} = t_1^{(3)} = 1$, $t_1^{(1)} = i_2^{(2)} = i_3^{(3)} = 2$, $i_3^{(3)} = 3$, $i_3^{(1)} = i_3^{(2)} = 3$. 

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\(i_4^{(3)} = 4, i_4^{(2)} = i_5^{(3)} = 5\). Then from Theorem 3.2 we obtain the following realization of the vertex operators in (4.1).

\[
\Phi_2(z_1) = a_{2,3} \int_T \frac{dt_1^{(2)}}{2\pi it_1^{(2)}} \Phi_3(z_1) F_2(t_1^{(2)}) \varphi_2(z_1, t_1^{(2)}; \Pi), \\
\Phi_1(z_2) = a_{1,3} \int_T \frac{dt_2^{(2)}}{2\pi it_2^{(2)}} \frac{dt_1^{(1)}}{2\pi it_1^{(1)}} \Phi_3(z_2) F_2(t_2^{(2)}) F_1(t_1^{(1)}) \varphi_1(z_2, t_2^{(2)}, t_1^{(1)}; \Pi), \\
\Phi_1(z_4) = a_{1,3} \int_T \frac{dt_3^{(2)}}{2\pi it_3^{(2)}} \frac{dt_2^{(1)}}{2\pi it_2^{(1)}} \Phi_3(z_4) F_2(t_3^{(2)}) F_1(t_2^{(1)}) \varphi_1(z_4, t_3^{(2)}, t_2^{(1)}; \Pi), \\
\Phi_2(z_5) = a_{2,3} \int_T \frac{dt_4^{(2)}}{2\pi it_4^{(2)}} \Phi_3(z_5) F_2(t_4^{(2)}) \varphi_2(z_5, t_4^{(2)}; \Pi).
\]

**Remark.** [16, 40] For a partition \(I\) and associated assignment of the variables \(t_a^{(l)}\), it is convenient to make a \(n \times N\) table of the variables \(z_j\) and \(t_a^{(l)}\) (\(j = 1, \ldots, n, l = 1, \ldots, N - 1, a = 1, \ldots, \lambda^{(l)}\)) obtained by the following rule: put \(t_a^{(l)}\) into the \((i_a^{(l)}, l)\) box (i.e. a box located at the \(i_a^{(l)}\)-th row and the \(l\)-th column) and put \(z_1, \ldots, z_n\) into the \(N\)-th column from top to bottom.

**Example 2.** The case given in Example 1, we obtain the following table.

|   | \(t_1^{(2)}\) | \(t_2^{(2)}\) | \(z_1\) |
|---|----------------|----------------|--------|
| \(t_1^{(1)}\) | \(t_2^{(2)}\) | \(z_2\) |
| \(t_2^{(1)}\) | \(t_3^{(2)}\) | \(z_4\) |
| \(t_4^{(2)}\) | \(t_5^{(2)}\) | \(z_5\) |

**Remark.** [16, 35, 40] For \(\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{N}^N\), the above notation is well fit for a parametrization of the partial flag variety \(\mathcal{F}_\lambda\) consisting of \(0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_N = \mathbb{C}^n\) with \(\dim \mathcal{F}_l / \mathcal{F}_{l-1} = \lambda_l\). Our assignment hence gives a representation theoretical meaning to such parametrization.

### 4.2 Derivation

Substituting the expressions of the vertex operators (3.21) into the \(n\)-point operator, we obtain

\[
\phi_{\mu_1, \ldots, \mu_n}(z_1, \ldots, z_n) = \int_{T^{N-n}} \Phi_N(z_1) F_{N-1}(t_1^{(N-1)}; \cdots) F_{\mu_1}(t_1^{(\mu_1)}) \varphi_{\mu_1}(z_1, t_1^{(\mu_1)}; \cdots, t_1^{(\mu_1)}; \{\Pi_{\mu_1}, m\}) \cdots \\
\cdots \int_{T^{N-n}} \Phi_N(z_n) F_{N-1}(t_\lambda^{(N-1)}; \cdots) F_{\mu_n}(t_\lambda^{(\mu_n)}) \varphi_{\mu_n}(z_n, t_\lambda^{(\mu_n)}; \cdots, t_\lambda^{(\mu_n)}; \{\Pi_{\mu_n}, m\}),
\]

(4.2)
where we set
\[ dt_1 = a_{\mu_1,N} \prod_{j=\mu_1}^{N-1} \frac{dt^{(j)}}{2\pi i t_1^{(j)}}, \quad dt_n = a_{\mu_n,N} \prod_{j=\mu_n}^{N-1} \frac{dt^{(j)}}{2\pi i t_1^{(j)}}, \quad \text{etc.} \]

We then divide the integrand into two parts, the operator part \( \Phi(t,z) \) and the kinematical factor part \( \omega_{\mu_1 \cdots \mu_n}(t,z,\Pi) \). The operator part consists of the normal ordered bare vertex operators \( \Phi_N(z) \)'s, the index-wise normal ordered elliptic currents \( F_l(t) \) \((l=1,\cdots,N-1)\) and the symmetric part of the OPE coefficients. We put \( \Phi_N(z) \)'s and \( F_l(t) \)'s \((l=1,\cdots,N-1)\) in the definite ordering specified below. The kinematical factor part consists of all \( \varphi_{\mu}(z,t_{\mu},\cdots;\Pi) \)'s, all factors arising from the exchange relations between \( \Phi_N(z) \)'s and \( F_{N-1}(t) \)'s as well as among \( F_l(t) \)'s \((l=1,\cdots,N-1)\) and of all the non-symmetric part of the OPE coefficients.

The following procedure yields the weight functions, which are natural elliptic and dynamical analogues of the trigonometric ones obtained in \([34,35,40]\), as the kinematical factor part. The procedure consists of the following 4 steps.

1. **Move all** \( \varphi_{\mu} \)'s **to the right end.** Then the dynamical parameters in \( \varphi_{\mu} \) get shift following the exchange relation
   \[ \Pi_{\nu,m}\Phi_{\mu}(z) = \Phi_{\mu}(z)\Pi_{\nu,m}q^{-2\delta_{\mu,\nu},h_{\nu,m}}. \]

2. **Move all** the elliptic currents \( F_l(t) \)'s **to the right of all** the bare vertex operators \( \Phi_N(z) \)'s and arrange the order of all \( F_l(t) \)'s by collecting them into \( N-1 \) groups by their indices. Put these \( N-1 \) groups in the decreasing order. Then one gets appropriate factors by the exchange relations \([A.47]\) and \([8.12]\).

3. **Take normal ordering of all** \( \Phi_N(z) \)'s and each group of \( F_l(t) \)'s **having the same indices** \((l=1,\cdots,N-1)\). Then one gets appropriate factors following the rule
   \[ \Phi_N(z_1) \cdots \Phi_N(z_m) =: \Phi_N(z_1) \cdots \Phi_N(z_m) : \prod_{1 \leq k < l \leq m} < \Phi_N(z_k)\Phi_N(z_l) >, \]
   \[ F_l(t_1) \cdots F_l(t_n) =: F_l(t_1) \cdots F_l(t_n) : \prod_{1 \leq a < b \leq n} < F_l(t_a)F_l(t_b) >. \]

The OPE coefficients are given as follows.

\[
< \Phi_N(z_k)\Phi_N(z_l) > = \frac{\{q^2 z_l / z_k\} \{pq^2N q^2 z_l / z_k\}}{\{pq^2N z_l / z_k\}} = (-)^{(N-1)/2} r^r(N-1)/r N \frac{\Gamma(q^2 z_l / z_k; p, q^{2N})}{\Gamma(q^{2N} z_l / z_k; p, q^{2N})} < \Phi_N(z_k)\Phi_N(z_l) >_{sym},
\]

\[
< F_l(t_a)F_l(t_b) > = q^{2(r-1)/r} \frac{(q^2 t_b / t_a; p)_{\infty} (t_b / t_a; p)_{\infty}}{(pt_b / t_a; p)_{\infty} (pq^{-2}t_b / t_a; p)_{\infty}} = \frac{[v_a - v_b - 1]}{[v_a - v_b]} < F_l(t_a)F_l(t_b) >_{sym}.
\]
with the symmetric parts

\[
< F_l(t_a) F_l(t_b) >^{Sym} = -(q t_a t_b)^{1-1/\rho} \frac{(t_a/t_b; p)\infty (t_b/t_a; p)\infty}{(pq^{-2}t_a/t_b; p)\infty (pq^{-2}t_b/t_a; p)\infty},
\]

\[
< \Phi_N(z_k) \Phi_N(z_l) >^{Sym} = \frac{(q^2 z_k/z_l, q^2 z_l/z_k; p, q^{2N})\infty}{(q^{2N} z_k/z_l, q^{2N} z_l/z_k; p, q^{2N})}\infty.
\]

4. For each \( l \in \{1, \ldots, N-1\} \), symmetrize the integration variables \( t_1^{(l)}, \ldots, t_{\lambda(l)}^{(l)} \). We denote this procedure by Sym_{t(l)}.

Applying the above procedure to (4.12), we obtain the following.

\[
\phi_{\mu_1 \cdots \mu_n} (z_1, \ldots, z_n) = \int_{T^{N-1}} dt_1 \cdots dt_n \Phi_N(z_1) F_{N-1}(t_1^{(N-1)}) \cdots \Phi_N(z_n) F_{N-1}(t_{\lambda(N-1)}^{(N-1)}) \cdots \Phi_N(t_{\mu_n}^{(l)})
\]

\[
\times \varphi_{\mu_1}(z_1, t_1^{(N-1)}, \ldots, t_{\mu_1}^{(l)}; \{\Pi_{\mu_1, m}^{2n} \Sigma_{k=2}^{n} \epsilon_{\mu_k, \lambda_{\mu_1, \mu_1}^{(l)}}\}) \cdots \varphi_{\mu_n}(z_n, t_{\lambda(N-1)}^{(l)}; \lambda_{\mu}^{l}, \ldots, t_{\mu_n}^{(l)}; \{\Pi_{\mu_n, m}\})
\]

\[
= \int_{T^M} dt : \Phi_N(z_1) \cdots \Phi_N(z_n) : F_{N-1}(t_1^{(N-1)}) \cdots F_{N-1}(t_{\lambda(N-1)}^{(N-1)}) : \cdots : F_1(t_1^{(l)}) \cdots F_1(t_{\lambda_1^{(l)}})
\]

\[
\times \prod_{1 \leq k \leq l \leq n} < \Phi_N(z_k) \Phi_N(z_l) >^{N-1} \prod_{l=1}^{N-1} \prod_{1 \leq a < b \leq \lambda(l)} < F_l(t_a^{(l)}) F_l(t_b^{(l)}) >^{N-1}
\]

\[
\times \prod_{l=1}^{N-2} \prod_{a=1}^{\lambda(l)} \prod_{b \leq \lambda(l)} \prod_{t_1^{(l)}}^{(l)} \frac{[v^{(l+1)}_b - v^{(l)}_a - 1/2]}{[v^{(l+1)}_b - v^{(l)}_a + 1/2]} \prod_{a=1}^{n} \prod_{l=2}^{\lambda(l)} \frac{[u^{(l+1)}_a - v^{(l)}_a - 1/2]}{[u^{(l+1)}_a - v^{(l)}_a + 1/2]}
\]

\[
\times \varphi_{\mu_1}(z_1, t_1^{(N-1)}, \ldots, t_{\mu_1}^{(l)}; \{\Pi_{\mu_1, m}^{2n} \Sigma_{k=2}^{n} \epsilon_{\mu_k, \lambda_{\mu_1, \mu_1}^{(l)}}\}) \cdots \varphi_{\mu_n}(z_n, t_{\lambda(N-1)}^{(l)}; \lambda_{\mu}^{l}, \ldots, t_{\mu_n}^{(l)}; \{\Pi_{\mu_n, m}\}).
\]

\[
= \int_{T^M} dt : \Phi_N(z_1) \cdots \Phi_N(z_n) : F_{N-1}(t_1^{(N-1)}) \cdots F_{N-1}(t_{\lambda(N-1)}^{(N-1)}) : \cdots : F_1(t_1^{(l)}) \cdots F_1(t_{\lambda_1^{(l)}})
\]

\[
\times \prod_{1 \leq k \leq l \leq n} < \Phi_N(z_k) \Phi_N(z_l) >^{N-1} \prod_{l=1}^{N-1} \prod_{1 \leq a < b \leq \lambda(l)} < F_l(t_a^{(l)}) F_l(t_b^{(l)}) >^{N-1}
\]

\[
\times \text{Sym}_{t(l)} \cdots \text{Sym}_{t(N-1)} U(t, z, \Pi),
\]

where we set \( M = \sum_{l=1}^{N-1} \lambda(l) = \sum_{l=1}^{N-1} (N-l) \lambda_1 \),

\[
dt = \prod_{j=1}^{n} a_{\mu_j, N} \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda(l)} \frac{t_a^{(l)}}{2\pi i t_a^{(l)}},
\]
\[ \bar{U}_l(t, z, \Pi) = \prod_{l=1}^{N-1} \prod_{1 \leq a < b \leq \lambda(l)} \frac{[v_a^{(l)} - v_b^{(l)}]}{[v_a^{(l)} - v_b^{(l)}]} \times \prod_{l=1}^{N-2} \prod_{a=1}^{\lambda(l)} \frac{\chi^{(l+1)}}{[v_a^{(l+1)} - v_a^{(l)}]} \times \prod_{b=1}^{\lambda(l)+1} \frac{\chi^{(l+1)}}{[v_b^{(l+1)} - v_b^{(l)}]} \times \prod_{k=2}^{\lambda(l)+1} \frac{u_k - v_a^{(N-1)} - 1/2}{u_k - v_a^{(N-1)} + 1/2} \times \prod_{a=1}^{\lambda(l)+1} \frac{u_k - v_a^{(N-1)} - 1/2}{u_k - v_a^{(N-1)} + 1/2} \times \varphi_{\mu_1}(z_1, t_1^{(N-1)}, \ldots, t_1^{(\mu_1)}; \{\mu_1, m_q^{-2} \sum_{k=2}^{\Lambda_\mu_1} h_{\mu_1, m_k}^{(1)}\}) \cdots \varphi_{\mu_n}(z_n, t_n^{(N-1)}, \ldots, t_n^{(\mu_n)}; \{\mu_n, m\}) \]

with \( t = (t_1^{(1)}, \ldots, t_1^{(\mu_1)}, \ldots, t_1^{(N-1)}), z = (z_1, \ldots, z_n), z_k = q^{2uk} (k = 1, \ldots, n), t_a^{(l)} = q^{2v_a^{(l)}} (l = 1, \ldots, N - 1, a = 1, \ldots, \lambda(l)). \)

Substituting (3.24) into \( \bar{U}_l(t, z, \Pi) \), we obtain

**Proposition 4.1.**

\[ \bar{U}_l(t, z, \Pi) = \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda(l)} \left( \frac{[v_a^{(l+1)} - v_a^{(l)}]}{[v_a^{(l+1)} - v_a^{(l)}]} + \frac{1}{2} \left( P + h \right) \mu_a, t+1 - C_{\mu_a, t+1} - \frac{1}{2} \right) \times \prod_{b=1}^{\lambda(l)+1} \frac{[v_b^{(l+1)} - v_a^{(l)}]}{[v_b^{(l+1)} - v_a^{(l)}]} \times \prod_{b=1}^{\lambda(l)+1} \frac{[v_b^{(l+1)} - v_b^{(l)}]}{[v_b^{(l+1)} - v_b^{(l)}]} \times \prod_{a=1}^{\lambda(l)+1} \frac{u_k - v_a^{(N-1)} - 1/2}{u_k - v_a^{(N-1)} + 1/2} \times \prod_{a=1}^{\lambda(l)+1} \frac{u_k - v_a^{(N-1)} - 1/2}{u_k - v_a^{(N-1)} + 1/2} \right), \]

where we set \( v_a^{(N)} = u_s \) and \( C_{\mu_a, t+1}(s) := \sum_{j=s+1}^{n} < e_{\mu_j}, h_{\mu_a, t+1} >. \)

We thus obtain the following formula.

**Theorem 4.2.** For \( |p| = |z_1|, \ldots, |z_n| < 1 \), we have

\[ \phi_{\mu_1 \cdots \mu_n}(z_1, \ldots, z_n) = \oint_{TM} \frac{dt}{2\pi i} \Phi(t, z) \omega_{\mu_1 \cdots \mu_n}(t, z, \Pi), \]

where we set

\[ \Phi(t, z) = \Phi_N(z_1) \cdots \Phi_N(z_n) \cdots F_{N-1}(t_1^{(N-1)}) \cdots F_{N-1}(t_1^{(\mu_1)}); \ldots; F_1(t_1^{(1)}); \cdots \]

\[ \times \prod_{1 \leq k < l \leq n} \Phi_N(z_k) \Phi_N(z_l) > Sym \prod_{l=1}^{N-1} \prod_{1 \leq a < b \leq \lambda(l)} < F_l(t_a^{(l)}); F_l(t_b^{(l)}) > Sym, \]

\[ \omega_{\mu_1 \cdots \mu_n}(t, z, \Pi) = \mu_+^+(z) \tilde{W}_l(t, z, \Pi), \]

\[ \mu_+^+(z) = \prod_{1 \leq k < l \leq n} (-1)^{(N-1)/2} \zeta_k^{r(N-1)/rN} \Gamma(q^{2N} z_k/z_l; p, q^{2N}) \]

\[ \tilde{W}_l(t, z, \Pi) = \text{Sym}_{(1)} \cdots \text{Sym}_{(N-1)} \tilde{U}_l(t, z, \Pi). \]

Note that \( \tilde{U}(t, z) \) is a symmetric function in \( z_1, \ldots, z_n \) as well as in \( t_1^{(l)}, \ldots, t_l^{(l)} \) for each \( l \in \{1, \ldots, N - 1\} \). The function \( \tilde{W}_l(t, z, \Pi) \) is the weight function of type \( \text{sl}_N \), which is an elliptic and dynamical analogue of the trigonometric one in [34][35][40].
Remark. In our derivation, the weight functions are determined modulo adding co-boundary terms associated with the multiple integral. In [4,6] we have resolved this ambiguity by requiring that their trigonometric limit coincide with the known results in [34, 35, 40]. An alternative and more intrinsic way to fix this ambiguity is provably to require the triangular property in Proposition 5.1. We apply the same argument to the proof of Propositions 5.2 and C.2.

For a partition \( I = (I_1, \ldots, I_N) \in \mathcal{I}_\lambda \), let \( I_k = \{i_{k,1} < \cdots < i_{k,\lambda_k}\} \) \( (k = 1, \cdots, N) \). The dynamical shift term \( C_{\mu_s, l+1} \) appearing in \( \tilde{U}_I(t, z, \Pi) \) has the following combinatorial expression.

**Proposition 4.3.**

\[
C_{\mu_s, l+1}(s) = \begin{cases} 
\lambda_{\mu_s} - \lambda_{l+1} - \tilde{s} + m_{\mu_s, l+1}(s) - 1 & \text{if } s < i_{l+1, \lambda_{l+1}} \\
\lambda_{\mu_s} - \tilde{s} & \text{if } s > i_{l+1, \lambda_{l+1}}
\end{cases}
\]

where for \( s \in [1, n] \) we define \( \tilde{s} \) by \( i_{\mu_s, \tilde{s}} = s \) and \( m_{\mu_s, l+1}(s) \) by

\[
m_{\mu_s, l+1}(s) = \min\{1 \leq j \leq \lambda_{l+1} \mid s < i_{l+1,j}\} \text{ for } s < i_{l+1, \lambda_{l+1}}.
\]

**Proof.** Note by definition \( \mu_s \leq l \) and

\[
\sum_{j=s+1}^{n} \langle \varepsilon_{\mu_j}, h_{\mu_s, l+1} \rangle = \sum_{j=s+1}^{n} (\delta_{\mu_j, \mu_s} - \delta_{\mu_j, l+1}).
\]

Let \( I_{\mu_s} = \{i_{\mu_s, 1} < \cdots < s = i_{\mu_s, \tilde{s}} < \cdots < i_{\mu_s, \lambda_{\mu_s}}\} \); then \( \sum_{j=s+1}^{n} \delta_{\mu_j, \mu_s} = \lambda_{\mu_s} - \tilde{s} \). If \( s < i_{l+1, \lambda_{l+1}} \) there exists \( m_{\mu_s, l+1}(s) \) so that \( s < i_{l+1,j} \in I_{l+1} \) \( (m_{\mu_s, l+1}(s) \leq j \leq \lambda_{l+1}) \). Hence \( \sum_{j=s+1}^{n} \delta_{\mu_j, l+1} = \lambda_{l+1} - m_{\mu_s, l+1}(s) + 1 \). If \( s > i_{l+1, \lambda_{l+1}} \) then \( \sum_{j=s+1}^{n} \delta_{\mu_j, l+1} = 0. \)

4.3 Entire function version

Let us set

\[
H_{\lambda}(t, z) := \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda(l)} \prod_{b=1}^{\lambda(l)+1} \left[ v_b^{l+1} - v_a^l + \frac{1}{2} \right] .
\]

(4.7)

The following gives an entire function version of \( \tilde{W}_I \)

\[
W_I(t, z, \Pi) = H_{\lambda}(t, z) \tilde{W}_I(t, z, \Pi) = \text{Sym}_{l(1)} \cdots \text{Sym}_{l(N-1)} U_I(t, z, \Pi),
\]

(4.8)

where

\[
U_I(t, z, \Pi) = \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda(l)} \left( \frac{v_b^{l+1} - v_a^l + (P + h)_{\mu_s, l+1} - C_{\mu_s, l+1}(s) - \frac{1}{2}}{[P + h]_{\mu_s, l+1} - C_{\mu_s, l+1}(s)} \right)_{i_b^l = 1}^{i_b^{l+1} = 1} \cdot \\
\times \prod_{b=1}^{\lambda(l)+1} \left[ v_b^{l+1} - v_a^l - \frac{1}{2} \right] \prod_{b=1}^{\lambda(l)+1} \left[ v_b^{l+1} - v_a^l + \frac{1}{2} \right] \prod_{b=a+1}^{\lambda(l)+1} \left[ v_b^{l+1} - v_a^l + 1 \right] .
\]

(4.9)
In the trigonometric \((p \to 0)\) and non-dynamical (neglecting the factors depending on \(P + h\)) limit the \(W_I\) coincides with those discussed in [40] by making the shift \(v^{(l)} \mapsto v^{(l)} + l/2\). See Sec 4. Note however that in this limit our \(\tilde{W}_I\) are slightly different from those in (6.2) [40] due to the difference between our \(H_\lambda(t,z)\) and \(E(t,h)\) in (6.1) from [40].

4.4 Combinatorial description

One can determine each factor in \(U_I(t,z,\Pi)\) by using the following one to one correspondence to a pair of variables \((t_a^{(l)}, t_b^{(l+1)})\) in a table introduced in Sec 4.1 [16, 40]. As above we set \(t_a^{(N)} = z_a\).

- for each pair \((t_a^{(l)}, t_b^{(l+1)})\) \((l = 1, \cdots, N - 1)\) in the same row (i.e. the \(i_a^{(l)} = i_b^{(l+1)}\)-th row), assign a factor
\[
\left[ \frac{v_b^{(l+1)} - v_a^{(l)}}{(P + h)_{\mu_s,t+1} - C_{\mu_s,t+1}(s)} - \frac{1}{2} \right]
\]

where \(s\) is the index of \(z_a\) located in the same row \((i_a^{(l)} = i_b^{(l+1)} = i_s^{(N)}).\)

- for each pair \((t_a^{(l)}, t_b^{(l+1)})\) \((l = 1, \cdots, N - 1)\), where \(t_b^{(l+1)}\) is located in the lower row than \(t_a^{(l)}\) (i.e. \(i_a^{(l)} < i_b^{(l+1)}\)), assign
\[
\left[ v_b^{(l+1)} - v_a^{(l)} - \frac{1}{2} \right].
\]

- for each pair \((t_a^{(l)}, t_b^{(l+1)})\) \((l = 1, \cdots, N - 1)\), where \(t_b^{(l+1)}\) is located in the upper row than \(t_a^{(l)}\) (i.e. \(i_a^{(l)} > i_b^{(l+1)}\)), assign
\[
\left[ v_b^{(l+1)} - v_a^{(l)} + \frac{1}{2} \right].
\]

- for each pair \((t_a^{(l)}, t_b^{(l)})\) \((1 \leq a < b \leq \lambda^{(l)})\) in the \(l\)-th column \((l = 1, \cdots, N - 1)\), assign
\[
\left[ \frac{v_b^{(l)} - v_a^{(l)} + 1}{v_b^{(l)} - v_a^{(l)}} \right].
\]

**Example 3.** For the case \(N = 3, n = 4, I = \{I_1 = \{2\}, I_2 = \{1,4\}, I_3 = \{3\}\} = I_{2132} \)

| \(t_1^{(1)}\) | \(t_2^{(2)}\) | \(z_1\) |
| \(t_1^{(2)}\) | \(t_2^{(1)}\) | \(z_2\) |
| \(z_3\) |
| \(t_3^{(2)}\) | \(z_4\) |
we obtain

\[ U_1(t, z, \Pi) = \frac{[v_2^{(2)} - v_1^{(1)} + (P + h)_{1,2} - C_{1,2}(2) - 1/2][1]}{[(P + h)_{1,2} - C_{1,2}(2)]} \left[ u_1 - v_1^{(2)} + (P + h)_{2,3} - C_{2,3}(1) - 1/2][1] \right] \
\times \frac{[u_2 - v_2^{(2)} + (P + h)_{1,3} - C_{1,3}(2) - 1/2][1]}{[(P + h)_{1,3} - C_{1,3}(2)]} \left[ u_4 - v_3^{(2)} + (P + h)_{2,3} - C_{2,3}(4) - 1/2][1] \right] \
\frac{[v_3^{(2)} - v_1^{(1)} - 1/2][u_2 - v_1^{(2)} - 1/2][u_3 - v_1^{(2)} - 1/2][u_4 - v_1^{(2)} - 1/2][u_3 - v_2^{(2)} - 1/2][u_4 - v_2^{(2)} - 1/2]}{[v_3^{(2)} - v_2^{(2)} + 1/2][u_1 - v_3^{(2)} + 1/2][u_2 - v_3^{(2)} + 1/2][u_3 - v_3^{(2)} + 1/2][u_4 - v_3^{(2)} + 1/2]} \
\prod_{1 \leq a < b \leq 3} \frac{[v_b^{(2)} - v_a^{(2)} + 1]}{[v_b^{(2)} - v_a^{(2)} + 1]} .
\]

Furthermore, in order to compute \( C_{\mu, t+1}(s) \) by Proposition 4.3, it is convenient to draw a \( n \times N \) table of the elements in \( I_l = \{i_{l,1} < \cdots < i_{l,N}\} \) \((l = 1, \cdots, N)\) obtained by the rule: put \( i_{l,a} \) into the \((i_{l,a}, l)\) box.

**Example 4.** For the case in Example 3, we have

|   |   |
|---|---|
| 2 | i_2,1 |
| i_1,1 |   |
|   | i_3,1 |
|   | i_2,2 |

For \( C_{1,2}(2) \), we have \( s = i_{1,1} = 2 < i_{2,2} = 4, \) \( m_{1,2}(2) = 2. \) Hence

\[ C_{1,2}(2) = 1 - 2 - 1 + 2 - 1 = -1 = \sum_{j=3,4} < \tilde{\epsilon}_{\mu_j}, h_{1,2} > . \]

For \( C_{2,3}(1) \), we have \( s = i_{2,1} = 1 < i_{3,1} = 3, \) \( m_{2,3}(1) = 1. \) Hence

\[ C_{2,3}(1) = 2 - 1 - 1 + 1 - 1 = 0 = \sum_{j=2,3,4} < \tilde{\epsilon}_{\mu_j}, h_{2,3} > . \]

For \( C_{1,3}(2) \), we have \( s = i_{1,1} = 2 < i_{3,1} = 3, \) \( m_{1,3}(2) = 1. \) Hence

\[ C_{1,3}(2) = 1 - 1 - 1 + 1 - 1 = -1 = \sum_{j=3,4} < \tilde{\epsilon}_{\mu_j}, h_{1,3} > . \]

For \( C_{2,3}(4) \), we have \( s = i_{2,2} = 4 > i_{3,1} = 3. \) Hence

\[ C_{2,3}(4) = 2 - 2 = 0. \]
5 Properties of the Elliptic Weight Functions

In this section we discuss some basic properties of the elliptic weight functions. We consider the following weight functions obtained from those in the last section by \( v_{a}^{(l)} \mapsto v_{a}^{(l)} + l/2 \) (\( l = 1, \ldots, N, a = 1, \ldots, \lambda^{(l)} \)).

\[
W_{I}(t, z, \Pi) = \text{Sym}_{t(1)} \cdots \text{Sym}_{t(N-1)} U_{I}(t, z, \Pi),
\]

\[
U_{I}(t, z, \Pi) = \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda^{(l)}} \left( \frac{v_{b}^{(l+1)} - v_{a}^{(l)} + (P + h)_{\mu_{s}, l+1} - C_{\mu_{s}, l+1}(s)}{[P + h)_{\mu_{s}, l+1} - C_{\mu_{s}, l+1}(s)]} \right) \prod_{b=a+1}^{\lambda^{(l)}} \frac{v_{b}^{(l)} - v_{a}^{(l)} + 1}{v_{b}^{(l)} - v_{a}^{(l)}}.
\]

Accordingly using

\[
H_{\lambda}(t, z) := \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda^{(l)}} \prod_{b=1}^{\lambda^{(l+1)}} \left[ v_{b}^{(l+1)} - v_{a}^{(l)} + 1 \right],
\]

the meromorphic function version of \( W_{I} \) is given by

\[
\widetilde{W}_{I}(t, z, \Pi) = \frac{W_{I}(t, z, \Pi)}{H_{\lambda}(t, z)}.
\]

5.1 Triangular property

For \( I, J \in \mathcal{I}_{\lambda} \), let \( I^{(l)} = \{i_{1}^{(l)} < \cdots < i_{\lambda^{(l)}}^{(l)}\} \) and \( J^{(l)} = \{j_{1}^{(l)} < \cdots < j_{\lambda^{(l)}}^{(l)}\} \) (\( l = 1, \ldots, N \)). Define a partial ordering \( \leq \) by

\[
I \leq J \iff i_{a}^{(l)} \leq j_{a}^{(l)} \quad \forall l, a.
\]

Let us denote by \( t = z_{I} \) the specialization \( t_{a}^{(l)} = z_{i_{a}^{(l)}}^{(l)} \) (\( l = 1, \cdots, N - 1, a = 1, \cdots, \lambda^{(l)} \)). The weight function has the following triangular property.

Proposition 5.1. For \( I, J \in \mathcal{I}_{\lambda} \),

1. \( \widetilde{W}_{J}(z_{I}, z, \Pi) = 0 \) unless \( I \leq J \).

2. 

\[
\widetilde{W}_{I}(z_{I}, z, \Pi) = \prod_{1 \leq k < l \leq N} \prod_{a \in I_{k}} \prod_{b \in I_{l}} \frac{[u_{b} - u_{a}]}{[u_{b} - u_{a} + 1]}.
\]
Proposition 5.2. Transition property

Proof. (1) follows from the same argument as in \[40\] Lemma 6.2. (2) follows from

\[
W_I(z_I, z, \Pi) = \prod_{1 \leq k < l \leq N} \left\{ \prod_{a \in I_k} \left( \prod_{b \in I_k} [u_b - u_a + 1] \prod_{b \in I_l} [u_b - u_a] \prod_{b \in I_l} [u_b - u_a + 1] \right) \right\} \\
\times \prod_{1 \leq k < l \leq N - 1} \left\{ \prod_{a \in I_k} \prod_{b \in I_l} [u_b - u_a + 1][u_a - u_b + 1] \right\}.
\]

For \( \sigma \in \mathfrak{S}_n \), let us denote \( \sigma^{-1}(I) = I_{\mu_\sigma(1)} \cdots \mu_{\sigma(n)} \) and \( \sigma(z) = (z_{\sigma(1)}, \cdots, z_{\sigma(n)}) \). Following \[40\], let us set \( \tilde{W}_{\sigma,I}(t, z, \Pi) = \tilde{W}_{\sigma^{-1}(I)}(t, \sigma(z), \Pi) \) and \( \tilde{W}_{\sigma,J}(t, z, \Pi) = \tilde{W}_I(t, z, \Pi) \). Let us consider the matrix \( \tilde{W}_{\sigma}(z, \Pi) \), whose \((I, J)\) element is given by \( \tilde{W}_{\sigma,J}(z_I, z, \Pi) \) \((I, J \in \mathfrak{I}_\lambda)\). We put the matrix elements in the decreasing order with respect to \( \leq \). Then Proposition 5.1 yields that the matrix \( \tilde{W}_{id}(z, \Pi) \) is lower triangular, whereas \( \tilde{W}_{\sigma_0}(z, \Pi) \) with \( \sigma_0 \) being the longest element in \( \mathfrak{S}_n \) is upper triangular. In particular, for generic \( u_a \) \((a = 1, \cdots, n)\), \( \tilde{W}_{\sigma}(z, \Pi) \) is invertible.

5.2 Transition property

Proposition 5.2. Let \( I = I_{\mu_1} \cdots \mu_{i+1} \cdots \mu_n \in \mathfrak{I}_\lambda \).

\[
\tilde{W}_{\lambda-I_{\mu_1} \cdots \mu_{i+1} \cdots \mu_n}(t, \cdots, z_{i+1}, z_i, \cdots, \Pi) = \sum_{\mu'_1, \mu'_{i+1}} \mathcal{R}(z_i/z_{i+1}, \Pi q^{2 \sum_{j=1}^n \langle \xi_{\mu'_j}, h \rangle})_{\mu'_1, \mu'_{i+1}} \tilde{W}_{I_{\mu'_1} \cdots \mu'_{i+1}}(t, \cdots, z_i, z_{i+1}, \cdots, \Pi). \quad (5.5)
\]

Proof. From Theorem 4.2 we have

\[
\phi_{\mu_{i+1} \mu_i \cdots}(z_i, z_{i+1}, \cdots) = \int_{T^M} dt \tilde{\Phi}(t, z) \omega_{\mu_{i+1} \mu_i \cdots}(t, \cdots, z_i, z_{i+1}, \cdots; \Pi)
\]

where we used the symmetry property of \( \tilde{\Phi}(t, z) \) under any permutations of \( z_1, \cdots, z_n \). Using the exchange relation (3.26) in the left hand side, we obtain

\[
\phi_{\mu_{i+1} \mu_i \cdots}(z_i, z_{i+1}, \cdots) = \sum_{\mu'_1, \mu'_{i+1}} R(z_i/z_{i+1}, \Pi q^{2 \sum_{j=1}^{i-1} \langle \xi_{\mu'_j}, h \rangle})_{\mu'_1, \mu'_{i+1}} \phi_{\mu'_1 \mu'_{i+1} \cdots}(z_i, z_{i+1}, \cdots) \\
= \int_{T^M} dt \tilde{\Phi}(t, z) \sum_{\mu'_1, \mu'_{i+1}} R(z_i/z_{i+1}, \Pi q^{2 \sum_{j=1}^n \langle \xi_{\mu'_j}, h \rangle})_{\mu'_1, \mu'_{i+1}} \omega_{\mu'_1 \mu'_{i+1} \cdots}(t, \cdots, z_i, z_{i+1}, \cdots; \Pi).
\]
Note that $\mu(z)$ in the $R$ matrix is related to $\mu^+(z)$ by

$$\mu(z_i/z_{i+1}) = \mu^+(\cdots, z_{i+1}, z_i, \cdots) / \mu^+(\cdots, z_i, z_{i+1}, \cdots).$$

Comparing the integrand we obtain the desired relation.

Let $s_i = (i \ i + 1) \in S_n(i = 1, \cdots, n - 1)$ denote the adjacent transpositions. Let us set

- $R^{(s_i, \text{id})}(z, \Pi)^t_I = R(z_i/z_{i+1}, \Pi q^{2 \sum_{j=1}^{i-1} \langle i, j \rangle}) \mu_i \mu_{i+1}$,
- $R^{(s_i, \text{id})}(z, \Pi)^{s_i(I)}_I = R(z_i/z_{i+1}, \Pi q^{2 \sum_{j=1}^{i-1} \langle i, j \rangle}) \mu_i \mu_{i+1}$.

Then one can rewrite (5.5) as

$$\tilde{W}_{s_i, I}(t, z, \Pi) = \left\{ \begin{array}{ll}
\tilde{W}_I(t, z, \Pi) & \text{if } s_i(I) = I \\
R^{(s_i, \text{id})}(z, \Pi q^{-2 \sum_{j=1}^{i-1} \langle i, j \rangle})_{I}^{t} \tilde{W}_{\text{id}, I}(t, z, \Pi) & \text{if } s_i(I) \neq I
\end{array} \right..$$

In general, let us consider the $n$-point operator

$$\phi_{\sigma^{-1}(1)}(\sigma(z)) = \Phi_{\mu_{\sigma(1)}}(z_{\sigma(1)}) \cdots \Phi_{\mu_{\sigma(n)}}(z_{\sigma(n)}).$$

By using the exchange relation repeatedly we obtain

$$\phi_{\sigma^{-1}(1)}(\sigma(z)) = \sum_{I'} \mathcal{R}^{(\sigma, \sigma')}(z, \Pi)_{I}^{t'} \phi_{\sigma'^{-1}(1)}(\sigma'(z))$$

where $\mathcal{R}^{(\sigma, \sigma')}(z, \Pi)^t_I$ denotes a coefficient given by a certain sum of products of the $R$ matrix elements in (2.9). Then by the same argument as in the proof of Proposition 5.2 we obtain

$$\tilde{W}_{\sigma, I}(t, z, \Pi) = \sum_{I'} \mathcal{R}^{(\sigma, \sigma')}(z, \Pi q^{-2 \sum_{j=1}^{n} \langle i, j \rangle})_{I}^{t'} \tilde{W}_{\sigma', I'}(t, z, \Pi). \quad (5.6)$$

Let us define the matrix $\mathcal{R}^{(\sigma, \sigma')}(z, \Pi)$ by

$$\mathcal{R}^{(\sigma, \sigma')}(z, \Pi) = \left( \mathcal{R}^{(\sigma, \sigma')}(z, \Pi)^t_I \right)_{I, J \in L_\lambda}.$$

Then we can rewrite (5.6)

$$\tilde{W}_\sigma(z, \Pi) = \tilde{W}_{\sigma'}(z, \Pi) \left( \mathcal{R}^{(\sigma, \sigma')}(z, \Pi q^{-2 \sum_{j=1}^{n} \langle i, j \rangle}) \right)^{-1}.$$

or

$$\tilde{W}_{\sigma'}(z, \Pi)^{-1} \tilde{W}_{\sigma}(z, \Pi) = \mathcal{R}^{(\sigma, \sigma')}(z, \Pi q^{-2 \sum_{j=1}^{n} \langle i, j \rangle}). \quad (5.8)$$

By taking the transposition and the shift $\Pi \mapsto \Pi q^{2 \sum_{j=1}^{n} \langle i, j \rangle}$, we obtain

$$\mathcal{R}^{(\sigma, \sigma')}(z, \Pi q^{-2 \sum_{j=1}^{n} \langle i, j \rangle})^{-1} = \mathcal{R}^{(\sigma, \sigma')}(z, \Pi).$$

In addition, we have
Proposition 5.3.

\[ t^i R^{(\sigma, \sigma')} (z, \Pi q^{-2} \sum_{j=1}^n \epsilon_{\mu_j, h}) = R^{(\sigma, \sigma')} (z, \Pi^{-1}) \tag{5.10} \]

Proof. It is enough to show the case that \( R^{(\sigma, \sigma')} (z, \Pi) \) is given by

\[ R^{(23)} (z_2 / z_3, \Pi) R^{(13)} (z_1 / z_3, \Pi q^{2h(2)}) R^{(12)} (z_1 / z_2, \Pi). \]

The desired equality follows from the properties of the \( R \) matrix \((2.6)\) such as \( t^i R (z, \Pi) = R (z, \Pi^{-1}) \) and \( R (z, \Pi q^{2(h(1) + h(2))}) = R (z, \Pi) \) as well as the dynamical Yang-Baxter equation \((2.10)\).

\[ \square \]

5.3 Orthogonality

This is an elliptic and dynamical analogue of the same property given in [40].

Proposition 5.4. For \( J, K \in \mathcal{I}_\lambda \),

\[ \sum_{I \in \mathcal{I}_\lambda} W_J (z_I, z, \Pi q^{-2} \sum_{j=1}^n \epsilon_{\mu_j, h}) W_{\sigma_0(K)} (z_I, \sigma_0(z), \Pi) \frac{Q(z_I) R(z_I) S(z_I)^2}{Q(z_I)} = \delta_{J,K}, \]

where \( \sum_{j=1}^n \bar{\epsilon}_{\mu_j} \) is the weight associated with \( \lambda \) (Sec. 4.1), and

\[ Q(z_I) = \prod_{1 \leq k < l \leq N} \prod_{a \in I_k, b \in I_l} [u_b - u_a + 1], \]

\[ R(z_I) = \prod_{1 \leq k < l \leq N} \prod_{a \in I_k, b \in I_l} [u_b - u_a], \]

\[ S(z_I) = \prod_{1 \leq k < l \leq N} \prod_{a, b \in I_k} [u_a - u_b + 1] \prod_{1 \leq k < l \leq N} \prod_{a \in I_k, b \in I_l} [u_a - u_b + 1]. \]

Proof. The proof is parallel to the one in [40] except for the dynamical shift. From \((5.8), (5.9)\) and \((5.10)\) with \( \sigma = \sigma_0 \) and \( \sigma' = \text{id} \), we have

\[ \widehat{W}_{\text{id}} (z, \Pi) W_{\sigma_0} (z, \Pi) = t^i \widehat{W}_{\sigma_0} (z, \Pi q^{-2} \sum_{j=1}^n \epsilon_{\mu_j, h}) \left( t^i \widehat{W}_{\text{id}} (z, \Pi^{-1} q^{2} \sum_{j=1}^n \epsilon_{\mu_j, h}) \right)^{-1}. \]

Hence

\[ \widehat{W}_{\text{id}} (z, \Pi) t^i \widehat{W}_{\sigma_0} (z, \Pi q^{-2} \sum_{j=1}^n \epsilon_{\mu_j, h}) = \widehat{W}_{\sigma_0} (z, \Pi) t^i \widehat{W}_{\text{id}} (z, \Pi^{-1} q^{2} \sum_{j=1}^n \epsilon_{\mu_j, h}). \]

Since the LHS is a lower triangular matrix and the RHS is an upper triangular matrix, this must be a diagonal matrix. Let us denote it by \( S \). It’s diagonal entries are obtained from Proposition \((5.1) (2)\) as

\[ S_{II} = \widehat{W}_{I} (z_I, z, \Pi) \widehat{W}_{\sigma_0(I)} (z_I, \sigma_0(z), \Pi^{-1} q^{2} \sum_{j=1}^n \epsilon_{\mu_j, h}) = \frac{R(z_I)}{Q(z_I)}. \]

20
We then obtain
\[ t \hat{W}_{id}(z, \Pi^{-1} q^{2 \sum_{j=1}^{n} \langle \epsilon_{\mu_{j}}, h \rangle}) S^{-1} \hat{W}_{\sigma_{0}}(z, \Pi) = \text{id} \]

Taking the \((J, K)\) component of this relation, we obtain
\[ \sum_{I \in I_{\lambda}} \hat{W}_{J}(z_{I}, z, \Pi^{-1} q^{2 \sum_{j=1}^{n} \langle \epsilon_{\mu_{j}}, h \rangle}) \frac{Q(z_{I})}{R(z_{I})} \hat{W}_{\sigma_{0}(K)}(z_{I}, \sigma_{0}(z), \Pi) = \delta_{J, K} \]

Furthermore noting
\[ \hat{W}_{J}(z_{I}, z, \Pi) = W_{J}(z_{I}, z, \Pi) \]
\[ \forall J \in I_{\lambda} \]
and from the proof of Proposition 5.1
\[ H_{\lambda}(z_{I}, z) = H_{\lambda}(z_{I}, \sigma_{0}(z)) = Q(z_{I})S(z_{I}) \]
we obtain the desired result.

**Remark.** In [42], the following function is used instead of our \(H_{\lambda}(t, z)\).
\[ E_{\lambda}(t, z) = \prod_{l=1}^{N-1} \prod_{a=1}^{\chi_{(l)}} \prod_{b=1}^{\lambda(l)} [v_{b}^{(l)} - v_{a}^{(l)} + 1]. \]
The specialization \(E_{\lambda}(z_{I}, z)\) coincides with our \(S(z_{I})\).

### 5.4 Quasi-periodicity

Remember that we set \(t_{a}^{(l)} = q^{2v_{a}^{(l)}}, z_{k} = q^{2u_{k}}\) and \(\Pi_{j,k} = q^{2(P+h)j,k}\). Note that \(t_{a}^{(l)} \rightarrow pt_{a}^{(l)} \Leftrightarrow v_{a}^{(l)} \rightarrow v_{a}^{(l)} + r\) and \(t_{a}^{(l)} \rightarrow e^{-2\pi i} t_{a}^{(l)} \Leftrightarrow v_{a}^{(l)} \rightarrow v_{a}^{(l)} + r\tau\). From (2.5) and Proposition 4.3 we obtain the following statement.

**Proposition 5.5.** For \(I \in I_{\lambda}\), the weight functions \(W_{I}(t, z, \Pi)\) have the following quasi-periodicity.
\[
W_{I}(\cdots, pt_{a}^{(l)}, \cdots, z, \Pi) = (-1)^{\lambda^{(l+1)} + \lambda^{(l-1)}} W_{I}(\cdots, t_{a}^{(l)}, \cdots, z, \Pi),
\]
\[
W_{I}(\cdots, e^{-2\pi i} t_{a}^{(l)}, \cdots, z, \Pi) = (-e^{-\pi i r})^{\lambda^{(l+1)} + \lambda^{(l-1)}}
\]
\[
\times \exp \left\{ \frac{2\pi i}{r} \left( (\lambda^{(l+1)} + \lambda^{(l-1)})v_{a}^{(l)} - \sum_{b=1}^{\lambda(l+1)} v_{b}^{(l+1)} - \sum_{c=1}^{\lambda(l-1)} v_{c}^{(l-1)} - (P+h)_{l,l+1} - \lambda_{l+1} \right) \right\}
\]
\[
\times W_{I}(\cdots, t_{a}^{(l)}, \cdots, z, \Pi) \quad (1 \leq a \leq \lambda^{(l)}, 1 \leq l \leq N - 1).
\]
Furthermore noting

\[ H_\lambda(\ldots, pt_a^{(l)}, \ldots, z) = (-1)^{\lambda(l+1)+\lambda(l-1)} H_\lambda(\ldots, t^{(l)}a, \ldots, z), \]

\[ H_\lambda(\ldots, e^{-2\pi i t_a^{(l)}} \ldots, z) = (-e^{-2\pi i r})^{\lambda(l+1)+\lambda(l-1)} \times \exp \left\{ -\frac{2\pi i}{r} \left( (\lambda(l+1) + \lambda(l-1)) t_a^{(l)} - \sum_{b=1}^{\lambda(l+1)} t_b^{(l)} - \sum_{c=1}^{\lambda(l-1)} t_c^{(l)} - \lambda_{l+1} - \lambda_l \right) \right\} \]

\[ \times \lambda l_{a}^{(l)} \lambda l_{a}^{(l)}, \lambda \lambda^{(l)} \lambda \lambda^{(l)}, \ldots, z) \]

we obtain

**Proposition 5.6.** The meromorphic weight functions \( \tilde{W}_I(t, z, \Pi) \) have the following quasi-periodicity.

\[ \tilde{W}_I(\ldots, pt_a^{(l)}, \ldots, z, \Pi) = \tilde{W}_I(\ldots, t_a^{(l)}, \ldots, z, \Pi), \]

\[ \tilde{W}_I(\ldots, e^{-2\pi i t_a^{(l)}} \ldots, z, \Pi) = \exp \left\{ -\frac{2\pi i}{r} ((P + h)_{l,l+1} - \lambda_l) \right\} \tilde{W}_I(\ldots, t_a^{(l)}, \ldots, z, \Pi) \]

\((l = 1, \ldots, N - 1, a = 1, \ldots, \lambda(l))\).

For \( \lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{N}^N, |\lambda| = n \), let \( z^{(n)} = (z_1, \cdots, z_n) \in (\mathbb{C}^*)^n \). For \( I = I_{\mu_1 \cdots \mu_n} \in \mathcal{I}_\lambda \), we denote by \( \Pi_I \) a set of dynamical parameters \( \{ \Pi_{\mu_k,j} = q_{(P + h)_{k,j}} (k = 1, \cdots, n, j = \mu_k + 1, \cdots, N) \} \), where \((P + h)_{j,k} \in \mathbb{C}/r\mathbb{Z} \) \((1 \leq j < k \leq N)\), and set \( \Pi_\lambda = \cup_{I \in \mathcal{I}_\lambda} \Pi_I \).

**Definition 5.7.** For \( \lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{N}^N, |\lambda| = n \), we define \( \mathcal{M}_\lambda^{(n)}(z^{(n)}, \Pi_\lambda) \) to be the space of meromorphic functions \( F(t; z, \Pi) \) of \( M = \sum_{l=1}^{N-1} (N - l)\lambda_l \) variables \( t = (t_1^{(1)}, \cdots, t_{\lambda_1}^{(1)}), \cdots, t_1^{(N-1)}, \cdots, t_{\lambda(N-1)}^{(N-1)}) \) such that

1. \( F(t; z, \Pi) \) is symmetric in \( t_1^{(l)}, \cdots, t_{\lambda(l)}^{(l)} \) for each \( l \in \{1, \cdots, N - 1\} \).
2. \( F(t; z, \Pi) \) has the quasi-periodicity

\[ F(\ldots, pt_a^{(l)}, \ldots, z, \Pi) = F(t; z, \Pi), \]

\[ F(\ldots, e^{-2\pi i t_a^{(l)}} \ldots, z, \Pi) = \exp \left\{ -\frac{2\pi i}{r} ((P + h)_{l,l+1} - \lambda_l) \right\} F(t; z, \Pi) \]

\((l = 1, \ldots, N - 1, a = 1, \cdots, \lambda(l))\).

Let us consider the subspace \( \mathcal{M}_\lambda^{\pm(n)}(z^{(n)}, \Pi_\lambda) \) := Span_\mathbb{C}\{ \tilde{W}_I(t, z, \Pi) \ (I \in \mathcal{I}_\lambda) \} \) of \( \mathcal{M}_\lambda^{(n)}(z, \Pi_\lambda) \).

From Proposition 5.1, we obtain

**Proposition 5.8.** \( \dim_\mathbb{C} \mathcal{M}_\lambda^{\pm(n)}(z^{(n)}, \Pi_\lambda) = \frac{n!}{\lambda_1! \cdots \lambda_N!} \).
Remark. For $\lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{N}^N$, let $x = t(x_1^{(1)}, \cdots, x_1^{(N-1)}, x_2^{(1)}, \cdots, x_2^{(N-1)}, \cdots, x_N^{(1)}, \cdots, x_N^{(N-1)}) \in \mathbb{C}^M$. From Proposition 5.5, one can deduce a symmetric integral $M \times M$ matrix $N$ and a vector $\xi \in (\mathbb{C}/\mathbb{Z})^M$, which yield the following quadratic form $N(x) = t^x Nx$ and the linear form $\xi(x) = t^x \xi$.

\[
N(x) = \sum_{l=1}^{N-2} \sum_{a=1}^{\lambda^{(l)}} \sum_{b=1}^{\lambda^{(l+1)}} (x_a^{(l)} - x_b^{(l+1)})^2 + \sum_{a=1}^{\lambda^{(N-1)}} (x_a^{(N-1)})^2,
\]

\[
\xi(x) = -\sum_{l=1}^{\lambda^{(l)}} \sum_{a=1}^{\lambda^{(l)}} x_a^{(l)}((P + h)_{l,l+1} + \lambda_{l+1}) - \sum_{a=1}^{\lambda^{(N-1)}} \sum_{k=1}^{n} x_a^{(N-1)} u_k.
\]

Then by Appel-Humbert theorem [31], a pair $(N, \xi)$ characterizes a line bundle $\mathcal{L}(N, \xi) : (\mathbb{C}^M \times \mathbb{C})/\Lambda^M \to \mathbb{C}^M$, where $\Lambda = r\mathbb{Z} + r\mathbb{Z}_\tau$, with action

\[
\omega \cdot (x, \eta) = (x + \omega, e_{\omega}(x)\eta), \quad \omega \in \Lambda^M, \ x \in \mathbb{C}^M, \ \eta \in \mathbb{C},
\]

and cocycle

\[
e_{nr+mt\tau}(x) = (-1)^{t^x Nn}(-e^{i\pi\tau})^{t^x Nm} e^{\frac{2\pi i}{m}(nx + \xi)}, \quad n, m \in \mathbb{Z}^M.
\]

Moreover $\Theta^+_{t}(z, \Pi_{\lambda}) := \text{Span}_{\mathbb{C}}\{ W_{I}(t, z, \Pi) \ (I \in \mathcal{I}_{\lambda}) \}$ is a space of sections of $\mathcal{L}(N, \xi)$. See for example, [14].

### 5.5 Shuffle algebra structure

Consider a graded $\mathbb{C}$-vector space

\[
\mathcal{M}(z, \Pi) = \bigoplus_{n \in \mathbb{N}} \bigoplus_{\lambda \in \mathbb{N}^N} \mathcal{M}_{\lambda}^{(n)}(z^{(n)}, \Pi_{\lambda})
\]

with $\mathcal{M}_{(0, \ldots, 0)}^{(0)}(z^{(0)}, \Pi) = \mathbb{C}1$.

**Definition 5.9.** For $F(t; z^{(m)}, \Pi_I) \in \mathcal{M}_{\lambda}^{(m)}(z^{(m)}, \Pi_{\lambda})$, $G(t'; z'^{(n)}, \Pi_{I'}) \in \mathcal{M}^{(n)}_{\lambda'}(z'^{(n)}, \Pi_{\lambda'})$, we define the bilinear product $*$ on $\mathcal{M}(z, \Pi)$ by

\[
(F \star G)(t_1^{(1)}, \cdots, t_1^{(1)}_{\lambda_1 + \cdots + \lambda^{(l)'}}, \cdots, t_1^{(N-1)}, \cdots, t_1^{(N-1)}_{\lambda^{(N-1)} + \cdots + \lambda^{(N-1)}'}; z_1, \cdots, z_{m+n}, \Pi_{I+I'})
\]

\[
:= \frac{1}{\prod_{l=1}^{N-1} \Lambda^{(l)!}} \text{Sym}(1) \cdots \text{Sym}^{(N-1)} \left[ F(t, z, \Pi_I q^{-2 \sum_{j=1}^{n} \xi_j h_j} G(t', z', \Pi_{I'}) \Xi(t, t', z, z') \right],
\]

where $I' = I_{\mu_1', \cdots, \mu_n'}$ and

\[
\Xi(t, t', z, z') = \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda^{(l+1)}} \left( \prod_{b=1}^{\lambda^{(l)}} \frac{[v_{b}^{(l+1)} - v_{a}^{(l)}]}{[v_{b}^{(l+1)} - v_{a}^{(l)} + 1]} \prod_{c=1}^{\lambda^{(l')}} \frac{[v_{c}^{(l)} - v_{a}^{(l)}]}{[v_{c}^{(l)} - v_{a}^{(l)} + 1]} \right).
\]

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In the LHS of (5.11), we set \( t_{\lambda}^{(l)} + a := t_{a}^{(l)} \) \((a = 1, \cdots, \lambda^{(l)})\), \( z_{m+k} := z'_{k} \) \((k = 1, \cdots, n)\) and \( \Pi_{I+I'} = \{ \Pi_{\mu_{k,j}} (k = 1, \cdots, m + n, j = \mu_{k} + 1, \cdots, N) \} \), where \( \Pi_{\mu_{m+k,j}} := \Pi'_{\mu_{k,j}} \) \((k = 1, \cdots, n, j = \mu_{k}' + 1, \cdots, N)\).

This endows \( \mathcal{M}(z, \Pi) \) with a structure of an associative unital algebra with the unit 1. In [14], a \( \mathfrak{sl}_2 \) version of the \( \star \)-product is given.

Let us consider the subspace of \( \mathcal{M}(z, \Pi) \).

\[
\mathcal{M}^{+}(z, \Pi) = \bigoplus_{n \in \mathbb{N}} \bigoplus_{\lambda \leq N \atop |\lambda| = n} \mathcal{M}_{\lambda}^{+}(z^{(n)}, \Pi_{\lambda}).
\]

From (5.1)-(5.2) it turns out that all the elements in \( \mathcal{M}^{+}(z, \Pi) \) satisfy the following pole and wheel conditions. For \( F(t; z, \Pi) \in \mathcal{M}_{\lambda}^{+}(z^{(n)}, \Pi_{\lambda})\),

1) there exists an entire function \( f(t; z, \Pi) \in \Theta_{\lambda}^{+}(z^{(n)}, \Pi_{\lambda}) \) such that

\[
F(t; z, \Pi) = \frac{f(t; z, \Pi)}{H_{\lambda}(t, z)}
\]

where \( H_{\lambda}(t, z) \) is given in (5.3).

2) \( f(t; z, \Pi) = 0 \) once \( t_{a}^{(l)}/t_{c}^{(l+\varepsilon)} = q^{2\varepsilon} \) and \( t_{c}^{(l+\varepsilon)}/t_{b}^{(l)} = 1 \) for some \( l, \varepsilon, a, b, c \), where \( \varepsilon \in \{ \pm 1 \} \), \( l = 1, \cdots, N, a, b = 1, \cdots, \lambda^{(l)} \), \( c = 1, \cdots, \lambda^{(l+\varepsilon)} \) and \( t_{a}^{(N)} = z_{a} \).

**Proposition 5.10.** The subspace \( \mathcal{M}^{+}(z, \Pi) \subset \mathcal{M}(z, \Pi) \) is \( \star \)-closed.

A proof is given in Appendix C.

**Remark.** The subalgebra \( (\mathcal{M}^{+}(z, \Pi), \star) \) is an \( A_{N-1} \)-type elliptic and dynamical analogue of the shuffle algebra studied in [9, 38], where the \( A_{N-1}^{(1)} \)-type was discussed. \( (\mathcal{M}^{+}(z, \Pi), \star) \) is also similar to the elliptic algebra discussed in [8]. We will discuss the implications of the algebra \( (\mathcal{M}^{+}(z, \Pi), \star) \) in a separate paper.

### 6 Elliptic q-KZ Equation

In this section we consider traces of the \( n \)-point operators \( \phi_{\mu_{1}, \cdots, \mu_{n}}(z_{1}, \cdots, z_{n}) \) in Sec[4] and show that they satisfy the face type elliptic q-KZ equation derived in [11, 15]. Evaluating the traces explicitly we obtain formal elliptic hypergeometric integral solutions to the equation.
6.1 Trace of the Vertex Operators

Let us consider the following trace of the $n$-point operator.

$$F^a(z_1, \cdots, z_n; \Pi) = \text{tr}_{F_{a,\nu}}(q^{-\nu a} \Phi(z_1) \cdots \Phi(z_n))$$

$$= \sum_{\mu_1, \cdots, \mu_n} v_{\mu_n} \cdots v_{\mu_1} F^a_{\mu_1 \cdots \mu_n}(z_1, \cdots, z_n; \Pi)$$

$$F^a_{\mu_1 \cdots \mu_n}(z_1, \cdots, z_n; \Pi) = \text{tr}_{F_{a,\nu}}(q^{-\nu a} \Phi_{\mu_1}(z_1) \cdots \Phi_{\mu_n}(z_n))$$

where $F_{a,\nu} = F_{a,\nu}(\xi, \eta) \ (a = 0, 1, \cdots, N-1)$ are given in (B.1). To make the trace well-defined, the total weight of the operators in side of the trace must be zero

$$\sum_{j=1}^n \epsilon_{\mu_j} = 0.$$

Since $\sum_{j=1}^n \epsilon_j = 0$, this condition is equivalent to $\lambda_1 = \lambda_2 = \cdots = \lambda_N$ for $I = I_{\mu_1 \cdots \mu_n} \in \mathcal{I}$.  

Lemma 6.1.

1. $F^a_{\mu_1 \mu_2 \cdots \mu_n}(z_1, z_2, \cdots, q^k z_n; \Pi) = F^a'_{\mu_1 \mu_2 \cdots \mu_{n-1}}(z_n, z_1, \cdots, z_{n-1}; \Pi q^{-2<\epsilon_{\mu_n}, h>})$,
2. $F^a_{\mu_1 \cdots \mu_i \cdots \mu_n}(\cdots, z_i, z_{i+1}, \cdots; \Pi) = \sum_{\mu_i' \mu_{i+1}' \cdots} R_i(\frac{z_{i+1}}{z_i}, \Pi q^{2\sum_{j=1}^{i-1} <\epsilon_{\mu_j}, h>}) \mu_{i+1}' \mu_i' F^a_{\mu_1 \cdots \mu_i' \cdots \mu_n}(\cdots, z_{i+1}, z_i, \cdots; \Pi)$,

where $a'$ denotes the cyclic permutation of $a$ by $(0 N-1 N-2 \cdots 21) \in \mathfrak{S}_N$.

Proof. (1) follows from the cyclic property of trace, $\Pi \Phi_{\mu_j}(z) = \Phi_{\mu_j}(z) \Pi q^{-2<\epsilon_{\mu_j}, h>}$, the zero weight condition $\sum_{j=1}^n \epsilon_{\mu_j} = 0$, $\Phi_{\mu}(q^k z) = q^{-\nu a} \Phi_{\mu}(z) q^{\nu a}$ and $\Phi(z) : F_{a,\nu}(\xi, \eta) \rightarrow \hat{V}_z \otimes F_{a',\nu}(\xi, \eta)$.

(2) follows from the exchange relation of the vertex operators (3.26).

By using the properties (1) and (2), we obtain the following statement.

Theorem 6.2. $F^a_{\mu_1 \cdots \mu_n}(z_1, \cdots, z_n; \Pi)$ satisfies the face type elliptic $q$-$KZ$ equation

$$F^a(z_1, \cdots, q^k z_1, \cdots, z_n; \Pi)$$

$$= R^{(i+1)}(\frac{q^{-k} z_{i+1}}{z_i}, \Pi q^{2\sum_{j=1}^{i-1} h^{(k)}}) \cdots R^{(n)}(\frac{q^{-k} z_n}{z_i}, \Pi q^{2\sum_{j=1}^{n-1} h^{(k)}})$$

$$\times \Gamma_1 R^{(1)}(\frac{z_1}{z_i}, \Pi) \cdots R^{(i-1)}(\frac{z_{i-1}}{z_i}, \Pi q^{2\sum_{j=1}^{i-2} h^{(k)}}) F^{a'}(z_1, \cdots, z_i, \cdots, z_n; \Pi).$$

Here $\Gamma_i$ denotes the shift operator

$$\Gamma_i f(\cdot; \Pi) = f(\cdot; \Pi q^{-2<\epsilon_{\mu_i}, h>})$$

if $q^{2h^{(i)}} f(\cdot; \Pi) = q^{2<\epsilon_{\mu_i}, h>} f(\cdot; \Pi)$ for $f(\cdot; \Pi) = \sum_{\mu_n} \tilde{v}_{\mu_n} \cdots \tilde{v}_{\mu_1} \otimes f_{\mu_1 \cdots \mu_n}(\cdot; \Pi).$
6.2 Evaluation of the trace

We now apply the formula in Theorem 4.2 to the n-point operator \( \Phi_{\mu_1}(z_1) \cdots \Phi_{\mu_n}(z_n) \), and evaluate the trace \( \Phi(t, z) := \text{tr}_{\mathcal{F}_{\alpha, \nu}} \left( q^{-\kappa d} \Phi(t, z) \right) \) explicitly. We obtain the following result.

**Theorem 6.3.** For \(|p| < |z_1|, \ldots, |z_n| < 1\), we obtain

\[
F_{\mu_1 \cdots \mu_n}^n(z_1, \ldots, z_n; \Pi) = \int_{T_M} dt \frac{\Phi(t, z) \omega_{\mu_1 \cdots \mu_n}(t, z, \Pi)}{q^{(N-1)\lambda(t) + (P+h)\alpha_1 - 1}}
\]

\[
\times \prod_{k=1}^n z_k^{-\lambda(N-1) - h\lambda_N + \frac{1}{2}((P+h)\lambda_N + \lambda(N-1))} \prod_{1 \leq k \neq l \leq n} \frac{\Gamma(q^{2}z_k/z_l; p, q^\kappa, q^{2N})}{\Gamma(q^{2N}z_k/z_l; p, q^\kappa, q^{2N})}
\]

\[
\times \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda(l)} \left( \frac{(t_1^{(l)})^\lambda - t_1^{(l)} + \frac{1}{2}((P+h)\alpha_1 - \lambda(l))} {\frac{\lambda(l) + 1}{2} \prod_{b=1} \frac{\Gamma(q^{2N}t_a^{(l)}/t_b^{(l)}; p, q^\kappa)}{\Gamma(q^{2N}t_a^{(l)}/t_b^{(l)}; p, q^\kappa)}} \right)
\]

\[
\times \prod_{1 \leq a < b \leq \lambda(l)} \frac{\Gamma(p^\kappa t_a^{(l)}/t_b^{(l)}; p, q^\kappa)}{\Gamma(t_a^{(l)}/t_b^{(l)}; p, q^\kappa)}
\]

(6.4)

where

\[
C_n = (-)^n \lambda(N-1) - q^{\frac{1}{2}} \sum_{l=1}^{\lambda(l) - 1}
\]

\[
\times \frac{(p^\kappa q^\kappa)^{\lambda(\lambda_N - 1)}}{(q^2)^{\lambda(\lambda_N - 1)}} \prod_{l=1} \frac{\lambda(l) + 1}{2} \prod_{b=1} \frac{\Gamma(q^{2N}p, q^\kappa, q^{2N})}{\Gamma(q^{2N}p, q^\kappa, q^{2N})}
\]

Proof. In \( \Phi(t, z) \), taking normal ordering further among \( \Phi_{\lambda_N} \)'s and \( F_{N-1} \)'s as well as among \( F_{l+1} \)'s and \( F_{l} \)'s \((l = 1, \ldots, N - 1)\), we obtain

\[
\Phi(t, z) = : \Phi_N(z_1) \cdots \Phi_N(z_n) F_{N-1}^{(N-1)}(t_1^{(N-1)}) \cdots F_{N-1}^{(N-1)}(t_1^{(N-1)}) \cdots F_{l+1}(t_1^{(l+1)}) \cdots F_{l+1}(t_1^{(l+1)}) :
\]

\[
\times \prod_{1 \leq l < m \leq n} \Phi_N(z_l) \Phi_N(z_m) > \frac{\text{Sym}}{\prod_{l=1}^{N-1} \prod_{1 \leq a < b \leq \lambda(l)}} \frac{\Gamma(q^{2N}p, q^\kappa, q^{2N})}{\Gamma(q^{2N}p, q^\kappa, q^{2N})} \prod_{l=1}^{N-2} \lambda(l+1) \lambda(l)
\]

\[
\times \prod_{l=1}^{n} \prod_{a=1}^{\lambda(l-1)} \Phi_N(z_l) F_{N-1}^{(N-1)}(t_a^{(N-1)}) > \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda(l)} \prod_{b=1}^{\lambda(l)} \Phi_{l+1}^{(l+1)}(t_a^{(l+1)}) F_l^{(l)}(t_a^{(l)}) >
\]

where

\[
< \Phi_N(z) F_{N-1}(t) > = -z^{-(r-1)/r} \frac{(pq^{-1}t/z; p)_\infty}{(qt/z; p)_\infty},
\]

\[
< F_{l+1}(t_a) F_l(t_b) > = t_a^{-(r-1)/r} \frac{(pq^{-1}t_b/t_a; p)_\infty}{(qt_b/t_a; p)_\infty}.
\]

Then using the formulas in Theorems 4.2 and 4.7, the trace of the normal ordered operator \( : \Phi_N(z_1) \cdots F_l^{(l)}(t_a^{(l)}) : \) in \( \Phi(t, z) \) can be evaluated, for example, by the coherent state method. Combining the result with all the OPE coefficients in \( \Phi(t, z) \) we obtain the desired result. \( \square \)
Remark. The integrand of (6.3) is a single valued function of $t_1^{(1)}, \cdots, t_{\lambda(N-1)}^{(N-1)}$ due to Proposition 5.6.

Remark. As discussed in [44] for the trigonometric \( \hat{sl}_2 \) case, one can specify the cycles of the integral (6.3) by further inserting the other elliptic weight functions \( \omega_\kappa^I(t, z, \Upsilon) \), which are the same as \( \omega_\lambda(t, z, \Pi) \) in (4.4) except for replacing \( p \) by \( q^\kappa \), i.e. replacing all the theta functions with elliptic nome \( p \) by those with \( q^\kappa \), and \( \Pi_{j,k} \) by the other dynamical parameters \( \Upsilon_{j,k} \). Then we have the following elliptic hypergeometric paring for \( I, J \in \mathcal{I}_\lambda \)

\[
I(\omega_\kappa^I, \omega_\kappa^J) = \oint_{\mathcal{T}M} dt \Phi(t, z) \omega_\kappa^I(t, z, \Upsilon) \omega_\kappa^J(t, z, \Pi).
\]

(6.5)

We will discuss the details of such integral in elsewhere.

Remark. The elliptic algebra \( U_{q,p}(\hat{sl}_N) \) at the level 1 is deeply related to the deformed \( \mathcal{W} \)-algebra \( \mathcal{W}_{q,t}(sl_N) \) in [3,7] : identifying \( p, p^* = pq^2 - 2 \) in \( U_{q,p}(\hat{sl}_N) \) with \( q, t \) in \( \mathcal{W}_{q,t}(sl_N) \), respectively, the elliptic currents \( F_j(z), E_j(z) \) \( (j = 1, \cdots, N - 1) \) can be identified with the screening currents \( S_j^+(z), S_j^-(z) \), respectively \[20][25], the bare type I vertex \( \Phi_N(z) \) and the type II vertex \( \Psi^*_N(z) \) with certain deformed primary fields in \( \mathcal{W}_{q,t}(sl_N) \) \[29\] as well as the space \( \mathcal{F}_{a,\nu} \) with the Verma module of \( \mathcal{W}_{q,t}(sl_N) \) \[6\]. Hence \( \Phi(t, z) \) can be regarded as an (elliptic) correlation function of \( \mathcal{W}_{q,t}(sl_N) \). Recently, Aganagic, Frenkel and Okounkov \[2\] proposed the notion of quantum \( q \)-Langlands duality, which states the correspondence between the solutions to the trigonometric \( q \)-KZ equation associated with \( U_q(\hat{g}) \) and the correlation functions of \( \mathcal{W}_{q,t}(L g) \) (\( L g \) is the Langlands dual Lie algebra of \( g \)). If one identifies the elliptic weight functions \( \omega_{\mu_1,\cdots,\mu_n}(t, z, \Pi) \) with the stable envelopes, the formula (6.3) seems to give an elliptic analogue of the formulas, for example, (5.11) in \[2\]. In this sense the quantum \( q \)-Langlands duality seems to become more transparent in the elliptic algebra level. We will discuss this point in a separate paper.

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A Elliptic Algebra $U_{q,p}(\hat{\mathfrak{sl}}_N)$

In this appendix, we summarize some basic facts on the elliptic algebra $U_{q,p}(\hat{\mathfrak{sl}}_N)$.

A.1 Definition

In this subsection, let $q = e^h \in \mathbb{C}[[h]]$ and $p$ be an indeterminate.

Definition A.1. [6] The elliptic algebra $U_{q,p}(\hat{\mathfrak{sl}}_N)$ is a topological algebra over $\mathbb{F}[[p]]$ generated by $e_{j,m}, f_{j,m}, \alpha_{j,n}, K_j^\pm$, $(1 \leq j \leq N - 1, m \in \mathbb{Z}, n \in \mathbb{Z}_{\neq 0})$, $d$ and the central element $c$. We assume $K_j^\pm$ are invertible and set

$$ e_j(z) = \sum_{m \in \mathbb{Z}} e_{j,m}z^{-m}, \quad f_j(z) = \sum_{m \in \mathbb{Z}} f_{j,m}z^{-m}, $$

$$ \psi_j^+(q^{1/2}z) = K_j^+ \exp\left(-(q - q^{-1}) \sum_{n>0} \frac{\alpha_{j,n}}{1 - p^n} z^{n}\right) \exp\left((q - q^{-1}) \sum_{n>0} \frac{p^n \alpha_{j,n}}{1 - p^n} z^{-n}\right), $$

$$ \psi_j^-(q^{1/2}z) = K_j^- \exp\left(-(q - q^{-1}) \sum_{n>0} \frac{p^n \alpha_{j,n}}{1 - p^n} z^{-n}\right) \exp\left((q - q^{-1}) \sum_{n>0} \frac{\alpha_{j,n}}{1 - p^n} z^{n}\right). $$

We call $e_j(z), f_j(z), \psi_j^\pm(z)$ the elliptic currents. The defining relations are as follows. For $g(P), g(P + h) \in \mathcal{M}_{H^*},$

$$ g(P + h)e_j(z) = e_j(z)g(P + h), \quad g(P)e_j(z) = e_j(z)g(P - < Q_{\alpha_j}, P >), \quad (A.1) $$

$$ g(P + h)f_j(z) = f_j(z)g(P + h - < \alpha_j, P >), \quad g(P)f_j(z) = f_j(z)g(P), \quad (A.2) $$

$$ [g(P), \alpha_{i,m}] = [g(P + h), \alpha_{i,n}] = 0, \quad (A.3) $$

$$ g(P)K_j^\pm = K_j^\pm g(P - < Q_{\alpha_j}, P >), \quad (A.4) $$

$$ g(P + h)K_j^\pm = K_j^\pm g(P + h - < Q_{\alpha_j}, P >), \quad (A.5) $$

$$ [d, g(P + h)] = [d, g(P)] = 0, \quad (A.6) $$

$$ [d, \alpha_{j,n}] = n \alpha_{j,n}, \quad [d, e_j(z)] = -z \frac{\partial}{\partial z} e_j(z), \quad [d, f_j(z)] = -z \frac{\partial}{\partial z} f_j(z), \quad (A.7) $$

$$ K_j^\pm e_j(z) = q^{1-\alpha_{i,j}} e_j(z) K_j^\pm, \quad K_j^\pm f_j(z) = q^{1-\alpha_{i,j}} f_j(z) K_j^\pm, \quad (A.8) $$

$$ [\alpha_{i,m}, \alpha_{j,n}] = \delta_{m+n,0} \frac{[a_{ij}m]q[cm]q}{m} \frac{1 - p^m}{1 - p^m q^{-cm}}, \quad (A.9) $$

$$ [\alpha_{i,m}, e_j(z)] = \frac{[a_{ij}m]q}{m} \frac{1 - p^m}{1 - p^m q^{-cm}} zm e_j(z), \quad (A.10) $$

$$ [\alpha_{i,m}, f_j(z)] = -\frac{[a_{ij}m]q}{m} zm f_j(z), \quad (A.11) $$

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Let us define the orthonormal basis type elliptic bosons \( E_{m}^{j} \) \((1 \leq j \leq N, m \in \mathbb{Z}_{\neq 0})\) by

\[
E_{m}^{j} = q^{jm} C_m \frac{1}{q-q^{-1}} \left( -q^{-Nm} \sum_{k=1}^{j-1} [km]_{q} \alpha_{k,m} + \sum_{k=j}^{N-1} [(N-k)m]_{q} \alpha_{k,m} \right) \quad (A.17)
\]

with

\[
C_m = \frac{1}{|m|_{q}^2 |Nm|_{q}}.
\]

One can show the following relations.
Proposition A.2.

\[ \sum_{j=1}^{N} q^{(j-1)m} \mathcal{E}_m^j = 0, \] (A.18)

\[ [\mathcal{E}_m^j, \mathcal{E}_n^j] = \delta_{m+n,0} \frac{[cm]_q[nm]_q[(N-1)m]_q}{m(q - q^{-1})^2[Nm]_q} \frac{1 - p^m}{1 - p^m q^{cm}}, \] (A.19)

\[ [\mathcal{E}_m^j, \mathcal{E}_n^k] = -\delta_{m+n,0} q^{(\text{sgn}(k-j)N-k-j)m} \frac{[cm]_q}{m(q^m - q^{-m})^2[Nm]_q} \frac{1 - p^m}{1 - p^m q^{cm}}, \] (A.20)

where

\[ \text{sgn}(l - j) = \begin{cases} + & (l > j) \\ 0 & (l = j) \\ - & (l < j). \end{cases} \]

Note also Proposition A.3.

\[ \alpha_{j,m} = [m]_q^2 (q - q^{-1}) (\mathcal{E}_m^j - q^{-m} \mathcal{E}_m^{j+1}). \] (A.21)

Furthermore if we set

\[ A_m^j = (q^m - q^{-m}) \sum_{k=1}^{j} q^{(k-j-1)m} \mathcal{E}_m^k, \]

then we have

\[ [\alpha_{i,m}, A_m^j] = -\delta_{i,j} \delta_{m+n,0} \frac{[cm]_q}{m} \frac{1 - p^m}{1 - p^m q^{cm}} \quad (1 \leq i, j \leq N - 1) \]

Hence we call \( A_m^j \) the fundamental weight type bosons.

A.3 The elliptic currents \( k_j(z) \)

Let us set

\[ \psi_j(z) = : \exp \left\{ (q - q^{-1}) \sum_{m \neq 0} \frac{\alpha_{j,m} p^m z^{-m}}{1 - p^m} \right\} :. \] (A.22)

Here : : denotes the normal ordering defined by

\[ : \alpha_{j,m} \alpha_{k,n} : = \begin{cases} \alpha_{j,m} \alpha_{k,n} & \text{if } m \leq n \\ \alpha_{k,n} \alpha_{j,m} & \text{if } m > n \end{cases} \]

for \( 1 \leq j, k \leq N - 1 \). Then the elliptic currents \( \psi_j^+(z) \) in Definition A.1 can be written as

\[ \psi_j^+(q^{-\frac{c}{2}}z) = K_j^+ \psi_j(z), \quad \psi_j^-(q^{-\frac{c}{2}}z) = K_j^- \psi_j(pq^{-c}z). \] (A.23)
Let us introduce the new elliptic currents \( k_j(z) \) \( (j \in I \cup \{ N \}) \) associated with \( \mathcal{E}_m \) by

\[
k_j(z) = \exp \left\{ \sum_{m \neq 0} \frac{[m]_q^2 (q - q^{-1})^2}{1 - p^m} p^m \mathcal{E}_m z^{-m} \right\}.
\]

(A.24)

Then from Proposition A.3 we obtain the following relations.

**Proposition A.4.**

\[
\psi_j(z) = \rho k_j(z) k_{j+1}(q z)^{-1},
\]

where

\[
\rho = \frac{(p;p)_\infty (p^* q^2; p^*)_\infty}{(p^*; p^*)_\infty (p q^2; p)_\infty}.
\]

(A.26)

In addition, from Proposition A.2 we obtain the following commutation relations.

**Theorem A.5.**

\[
k_j(z_1)k_j(z_2) = \frac{\tilde{\rho}^+(z)}{\rho^+(z)} k_j(z_2)k_j(z_1), \quad (1 \leq j \leq N),
\]

\[
k_j(q^j z_1)k_j(q^k z_2) = \frac{\tilde{\rho}^+(z)}{\rho^+(z)} \frac{\Theta_{p^*}(q^{-2} z) \Theta_p(z)}{\Theta_{p^*}(q^2 z) \Theta_p(q^{-2} z)} k_j(q^k z_2) k_j(q^j z_1), \quad (1 \leq j < k \leq N),
\]

where \( z = z_1/z_2 \), \( \tilde{\rho}^+(z) \) is given in (2.28) and \( \rho^+(z) = \rho^+(z)|_{p \rightarrow p^*}. \)

**Proposition A.6.**

\[
k_j(z_1) e_j(z_2) = \frac{\Theta_{p^*}(q^{-c} z)}{\Theta_{p^*}(q^{-c-1} z)} e_j(z_2) k_j(z_1), \quad (1 \leq j \leq N),
\]

\[
k_j(z_1) e_{j-1}(z_2) = \frac{\Theta_{p^*}(q^{-c-1} z)}{\Theta_{p^*}(q^{-c} z)} e_{j-1}(z_2) k_j(z_1), \quad (2 \leq j \leq N),
\]

\[
k_j(z_1) e_k(z_2) = e_k(z_2) k_j(z_1), \quad (k \neq j, j - 1),
\]

\[
k_j(z_1) f_j(z_2) = \frac{\Theta_p(q^{-2} z)}{\Theta_p(z)} f_j(z_2) k_j(z_1), \quad (1 \leq j \leq N),
\]

\[
k_j(z_1) f_{j-1}(z_2) = \frac{\Theta_p(q z)}{\Theta_p(q^{-1} z)} f_{j-1}(z_2) k_j(z_1), \quad (2 \leq j \leq N),
\]

\[
k_j(z_1) f_k(z_2) = f_k(z_2) k_j(z_1), \quad (k \neq j, j - 1).
\]

### A.4 Modified elliptic currents

The \( R \) matrix (2.7) is gauge equivalent to Jimbo-Miwa-Okado’s \( A^{(1)}_{N-1} \) face type Boltzmann weight (17) and conveniently expressed by using Jacobi’s theta function (2.24). However one drawback is that Jacobi’s theta is accompanied by the fractional power of \( z \). In order to deal with this one needs to introduce the following modifications of the elliptic currents [23].
Definition A.7. We introduce the new generators $e^{\pm \zeta_j}$ $(1 \leq j \leq N)$ satisfying

\begin{align}
E_j(z) &= e_j(z) e^{\zeta_j}(q^{N-j}z)^{-\delta_{j,k+1}-\delta_{j,k-1}} e^{\alpha_k} e^{Q_{\alpha_j}}, \\
F_j(z) &= f_j(z) e^{-\zeta_j}(q^{N-j}z)^{(P+h)\alpha_j-1} e^{\alpha_k} e^{Q_{\alpha_j}}, \\
K_j^+(z) &= k_{j+1}(q^{j-N+1}z) e^{-Q_{\alpha_j}} q^{-h_{j_1}(q^{-r+1}z)^{-\frac{1}{2}}} (P_{\alpha_j-1})^{-\frac{1}{2}} ((P+h)\alpha_j-1), \\
K_j^-(z) &= K_j^+ (pq^{-c}z).
\end{align}

for $1 \leq j \leq N$. We also set

\begin{align}
H_j^\pm(z) &= qK_j^\pm (q^{N-j-1}q^{2}z) K_{j+1}^\pm (q^{N-j-1}q^{2}z)^{-1} \\
&= q\psi_j^\pm(z)(K_j^\pm)^{-1} e^{-Q_{\alpha_j}} q^{h_{j_1}(q^{-r+1}z)^{-\frac{1}{2}}} (P_{\alpha_j-1})^{-\frac{1}{2}} ((P+h)\alpha_j-1), \\
&= \psi_j^\pm(z)(K_j^\pm)^{-1} e^{-Q_{\alpha_j}} q^{h_{j_1}(q^{-r+1}z)^{-\frac{1}{2}}} (P_{\alpha_j-1})^{-\frac{1}{2}} ((P+h)\alpha_j-1),
\end{align}

\begin{align}
\hat{d} &= d + \frac{1}{2r} \sum_{j=1}^{N} (P_j + 2) P^j - \frac{1}{2r} \sum_{j=1}^{N} ((P + h)j + 2)(P + h)^j.
\end{align}

Then the defining relations (A.7)-(A.10) of $U_{q,p}(\hat{sl}_N)$ can be rewritten as follows in the sense of analytic continuation.
Proposition A.10.

\[ [h_i, \alpha_{j,n}] = 0, \quad [h_i, E_j(z)] = a_{ij} E_j(z), \quad [h_i, F_j(z)] = -a_{ij} F_j(z), \quad (A.40) \]

\[ [\vec{d}, h_i] = 0, \quad [\vec{d}, \alpha_{i,n}] = n \alpha_{i,n}, \quad (A.41) \]

\[ [\vec{d}, E_i(z)] = \left( -z \frac{\partial}{\partial z} + \frac{1}{r^*} \right) E_i(z), \quad [\vec{d}, F_i(z)] = \left( -z \frac{\partial}{\partial z} + \frac{1}{r} \right) F_i(z), \quad (A.42) \]

\[ [\alpha_{i,m}, \alpha_{j,n}] = \delta_{m+n,0} \frac{[aij]mq[cm]q}{m} \frac{1 - pm}{1 - pmq} z^m, \quad (A.43) \]

\[ [\alpha_{i,m}, E_j(z)] = \frac{[aij]mq}{m} \frac{1 - pm}{1 - pmq} z^m E_j(z), \quad (A.44) \]

\[ [\alpha_{i,m}, F_j(z)] = -\frac{[aij]mq}{m} z^m F_j(z), \quad (A.45) \]

\[ [u - v - \frac{a_{ij}}{2}]^* E_i(z) E_j(w) = [u - v + \frac{a_{ij}}{2}]^* E_j(w) E_i(z), \quad (A.46) \]

\[ [u - v + \frac{a_{ij}}{2}] F_i(z) F_j(v) = [u - v - \frac{a_{ij}}{2}] F_j(w) F_i(z), \quad (A.47) \]

\[ [E_i(z), F_j(w)] = \frac{\delta_{ij}}{q - q^{-1}} \left( \delta(q^{-c}z) H_i^-(q^{c/2}w) - \delta(q^c/z) H_i^+(q^{-c/2}w) \right), \quad (A.48) \]

\[ z_1^{\frac{1}{p} \frac{pq - z_2}{q z_2 / z_1; p}} \left\{ \begin{array}{l} \left( z_2 / z_1 \right) F_i(z_1) F_i(z_2) \\ \left( z_2 / z_1 \right) F_i(z_1) F_i(z_2) \\ \left( z_2 / z_1 \right) F_i(z_1) F_i(z_2) \end{array} \right\} + (z_1 \leftrightarrow z_2) = 0, \quad (A.49) \]

\[ z_1^{\frac{1}{p} \frac{pq - z_2}{q z_2 / z_1; p}} \left\{ \begin{array}{l} \left( z_2 / z_1 \right) F_i(z_1) F_i(z_2) \\ \left( z_2 / z_1 \right) F_i(z_1) F_i(z_2) \\ \left( z_2 / z_1 \right) F_i(z_1) F_i(z_2) \end{array} \right\} + (z_1 \leftrightarrow z_2) = 0 \quad (|i - j| = 1). \quad (A.50) \]

In addition, one can rewrite the formulas in Theorem [A.5] and Proposition [A.6] as follows.
Proposition A.11. 

\[ K_j^+(z_1)K_j^+(z_2) = \frac{\rho^+(z_1/z_2)}{\rho^+(z_1/z_2)}K_j^+(z_2)K_j^+(z_1), \]

\[ K_j^+(z_1)K_i^+(z_2) = \frac{\rho^+(z_1/z_2)}{\rho^+(z_1/z_2)}[u_1 - u_2 - 1]^* [u_1 - u_2] K_i^+(z_2)K_j^+(z_1) \quad (1 \leq j < l \leq N), \]

\[ K_j^+(z_1)E_j(z_2) = \left[ u_1 - u_2 + \frac{i-N+1}{2} \right]^* E_j(z_2)K_j^+(z_1) \quad (1 \leq j \leq N), \]

\[ K_{j+1}^+(z_1)E_j(z_2) = \left[ u_1 - u_2 + \frac{i-N+1-c}{2} \right]^* E_j(z_2)K_{j+1}^+(z_1) \quad (1 \leq j \leq N - 1), \]

\[ K_i^+(z_1)E_j(z_2) = E_j(z_2)K_i^+(z_1) \quad (l \neq j, j + 1), \]

\[ K_j^+(z_1)F_j(z_2) = \left[ u_1 - u_2 + \frac{i-N+1}{2} \right] F_j(z_2)K_j^+(z_1) \quad (1 \leq j \leq N), \]

\[ K_{j+1}^+(z_1)F_j(z_2) = \left[ u_1 - u_2 + \frac{i-N+1+1}{2} \right] F_j(z_2)K_{j+1}^+(z_1) \quad (1 \leq j \leq N - 1), \]

\[ K_i^+(z_1)F_j(z_2) = F_j(z_2)K_i^+(z_1) \quad (l \neq j, j + 1). \]

A.5 The half currents and the $L$-operators

We define the half currents of $U_{q,p}(\hat{sl}_N)$ as follows.

**Definition A.12.** Let us assume $|p| < |z| < 1$. We set

\[ F_{j,t}^+(z) = a_{j,t} \int_{\mathbb{T}^{l-1}} \prod_{m=j}^{l-1} \frac{dt_m}{2\pi i t_m} F_{l-1}(t_{l-1})F_{l-2}(t_{l-2}) \cdots F_j(t_j) \]

\[ \times \left[ u - v_{l-1} + (P + h)_{j,t} + \frac{l-N}{2} - 1 \right][1]\left[ u - v_{l-1} + \frac{l-N}{2} \right][P_{j,t} + h_{j,t} - 1] \prod_{m=j}^{l-2} \left[ v_{m+1} - v_m + (P + h)_{j,m+1} + \frac{1}{2} \right][1]\left[ v_{m+1} - v_m + \frac{1}{2} \right][P_{j,m+1} + h_{j,m+1}]. \]  \hspace{1cm} (A.51)

\[ E_{l,j}^+(z) = \alpha_{j,t}^* \int_{\mathbb{T}^{l-1}} \prod_{m=j}^{l-1} \frac{dt_m}{2\pi i t_m} E_{l-1}(t_{l-1})E_{l-2}(t_{l-2}) \cdots E_j(t_j) \]

\[ \times \left[ u - v_{l-1} - P_{j,t} + \frac{l-N}{2} - 1 \right][1]\left[ u - v_{l-1} + \frac{l-N}{2} \right][P_{j,t} - 1] \prod_{m=j}^{l-2} \left[ v_{m+1} - v_m - P_{j,m+1} + \frac{1}{2} \right][1]\left[ v_{m+1} - v_m + \frac{1}{2} \right][P_{j,m+1} - 1]. \]  \hspace{1cm} (A.52)

where $z = q^{2a}$, $t_a = q^{2\alpha_a}$ ($a = j, j + 1, \ldots, l - 1$) and

\[ \mathbb{T}^{l-j} = \{ t \in \mathbb{C}^{l-j} \mid |t_j| = \cdots = |t_{l-1}| = 1 \}. \]

The constants $a_{j,t}$ and $\alpha_{j,t}^*$ are chosen to satisfy

\[ -\frac{q}{q - q^{-1}} a_{j,t}^* \alpha_{j,t}^* [1] = 1. \]  \hspace{1cm} (A.53)
We call $F^+_{j,l}(z), E^+_{j,l}(z), (1 \leq j < l \leq N)$ and $K^+_j(z) (j = 1, \cdots, N)$ the half currents.

**Definition A.13.** By using the half currents, we define the $L$-operator $\hat{L}^+(z) \in \text{End}(\mathbb{C}^N) \otimes U_q,p(\hat{\mathfrak{sl}}_N)$ as follows.

\[
\hat{L}^+(z) = \begin{pmatrix}
1 & F^+_1(z) & \cdots & F^+_N(z) \\
0 & 1 & \cdots & F^+_N(z) \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & F^+_N(z) \\
0 & \cdots & \cdots & 1
\end{pmatrix}
\times
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
E^+_2(z) & 1 & \cdots & \vdots \\
E^+_3(z) & E^+_2(z) & \ddots & \vdots \\
\vdots & \vdots & \ddots & 1 \\
E^+_N(z) & E^+_N(z) & \cdots & E^+_N(z)
\end{pmatrix}
\]

We conjecture that the $L$-operator satisfies the following dynamical $RLL$-relation \[19\].

**Conjecture A.14.** \[23\]

\[
R^{+(12)}(z_1/z_2, \Pi)\hat{L}^+(z_1)\hat{L}^+(z_2) = \hat{L}^+(z_2)\hat{L}^+(z_1)R^{+(12)}(z_1/z_2, \Pi^*). \tag{A.55}
\]

**A.6 The $H$-Hopf algebroid $U_q,p(\hat{\mathfrak{sl}}_N)$**

Let $\mathcal{A}$ be a complex associative algebra, $\mathcal{H}$ be a finite dimensional commutative subalgebra of $\mathcal{A}$, and $\mathcal{M}_{\mathcal{H}^*}$ be the field of meromorphic functions on $\mathcal{H}^*$ the dual space of $\mathcal{H}$.

**Definition A.15 ($\mathcal{H}$-algebra \[5\]).** An $\mathcal{H}$-algebra is an associative algebra $\mathcal{A}$ with 1, which is bigraded over $\mathcal{H}^*$, $\mathcal{A} = \bigoplus_{\alpha, \beta \in \mathcal{H}^*} \mathcal{A}_{\alpha, \beta}$, and equipped with two algebra embeddings $\mu_1, \mu_r : \mathcal{M}_{\mathcal{H}^*} \to \mathcal{A}_{0,0}$ (the left and right moment maps), such that

\[
\mu_1(\hat{f})a = a\mu_1(T_\alpha \hat{f}), \quad \mu_r(\hat{f})a = a\mu_r(T_\beta \hat{f}), \quad a \in \mathcal{A}_{\alpha, \beta}, \quad \hat{f} \in \mathcal{M}_{\mathcal{H}^*},
\]

where $T_\alpha$ denotes the automorphism $(T_\alpha \hat{f})(\lambda) = \hat{f}(\lambda + \alpha)$ of $\mathcal{M}_{\mathcal{H}^*}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{H}$-algebras. The tensor product $\mathcal{A} \otimes \mathcal{B}$ is the $\mathcal{H}^*$-bigraded vector space with

\[
(\mathcal{A} \otimes \mathcal{B})_{\alpha, \beta} = \bigoplus_{\gamma \in \mathcal{H}^*} (\mathcal{A}_{\alpha, \gamma} \otimes \mathcal{M}_{\mathcal{H}^*} \mathcal{B}_{\gamma, \beta}),
\]

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where \( \otimes_{\mathcal{M}_H} \) denotes the usual tensor product modulo the following relation.

\[
\mu_r^A(\hat{f})a \otimes b = a \otimes \mu^B_\hat{f}(\hat{f})b, \quad a \in A, b \in B, \hat{f} \in \mathcal{M}_{H^*}.
\] (A.56)

The tensor product \( \mathcal{A} \hat{\otimes} \mathcal{B} \) is again an \( H \)-algebra with the multiplication \((a \otimes b)(c \otimes d) = ac \otimes bd\) and the moment maps

\[
\mu_l^{\mathcal{A} \hat{\otimes} \mathcal{B}} = \mu_l^A \otimes 1, \quad \mu_r^{\mathcal{A} \hat{\otimes} \mathcal{B}} = 1 \otimes \mu_r^B.
\]

**Proposition A.16.** \([28, 30]\) \( U = U_{q, p}(\hat{\mathfrak{sl}_N}) \) is an \( H \)-algebra by

\[
U = \bigoplus_{\alpha, \beta \in H^*} U_{\alpha, \beta}
\]

\[
U_{\alpha, \beta} = \{ x \in U \mid q^{P + h}xq^{-(P + h)} = q^{q_{\alpha, P + h}x}, \quad q^{P^*xq^{-P}} = q^{q_{\beta, P^*}x} \forall P + h, P \in H \}
\]

and \( \mu_l, \mu_r : \mathbb{F} \rightarrow U_{0, 0} \) defined by

\[
\mu_l(\hat{f}) = f(P + h, p) \in \mathbb{F}[[p]], \quad \mu_r(\hat{f}) = f(P, p^*) \in \mathbb{F}[[p]].
\]

We regard \( T_\alpha = e^\alpha \in \mathbb{C}[\mathbb{R}_Q] \) as the shift operator \( \mathcal{M}_{H^*} \rightarrow \mathcal{M}_{H^*} \)

\[
(T_\alpha \hat{f}) = e^\alpha f(P, p^*)e^{-\alpha} = f(P + <\alpha, P >, p^*).
\]

Hereafter we abbreviate \( f(P + h, p) \) and \( f(P, p^*) \) as \( f(P + h) \) and \( f^*(P) \), respectively.

We also consider the \( H \)-algebra of the shift operators

\[
\mathcal{D} = \{ \sum_\alpha \hat{f}_\alpha T_\alpha \mid \hat{f}_\alpha \in \mathcal{M}_{H^*}, \alpha \in \mathbb{R}_Q \}
\]

\[
\mathcal{D}_{\alpha, \alpha} = \{ \hat{f}T_{-\alpha} \}, \quad \mathcal{D}_{\alpha, \beta} = 0 \ (\alpha \neq \beta),
\]

\[
\mu_l^\mathcal{D}(\hat{f}) = \mu_r^\mathcal{D}(\hat{f}) = \hat{f}T_0 \quad \hat{f} \in \mathcal{M}_{H^*}.
\]

Then we have the \( H \)-algebra isomorphism

\[
U \cong U \hat{\otimes} \mathcal{D} \cong \mathcal{D} \hat{\otimes} U.
\] (A.57)

We define two \( H \)-algebra homomorphisms, the co-unit \( \varepsilon : U \rightarrow \mathcal{D} \) and the co-multiplication
\( \Delta : U \to U \otimes U \) as well as the algebra antihomomorphism \( S : U \to U \) by
\[
\varepsilon(\hat{L}_{ij}^+(z)) = \delta_{i,j} T_{Q_i}, \quad (n \in \mathbb{Z}), \quad \varepsilon(e^Q) = e^Q, \quad (A.58)
\]
\[
\varepsilon(\mu_l(\hat{f})) = \varepsilon(\mu_r(\hat{f})) = \hat{f} T_0, \quad (A.59)
\]
\[
\Delta(\hat{L}_{ij}^+(z)) = \sum_k \hat{L}_{i,k}^+(z) \otimes \hat{L}_{k,j}^+(z), \quad (A.60)
\]
\[
\Delta(e^Q) = e^Q \otimes e^Q, \quad (A.61)
\]
\[
\Delta(\mu_l(\hat{f})) = \mu_l(\hat{f}) \otimes 1, \quad \Delta(\mu_r(\hat{f})) = 1 \otimes \mu_r(\hat{f}), \quad (A.62)
\]
\[
S(\hat{L}_{ij}^+(z)) = (\hat{L}^+ (z)^{-1})_{ij}, \quad (A.63)
\]
\[
S(e^Q) = e^{-Q}, \quad S(\mu_r(\hat{f})) = \mu_l(\hat{f}), \quad S(\mu_l(\hat{f})) = \mu_r(\hat{f}). \quad (A.64)
\]

Then the set \( (U_{q,p}(\hat{\mathfrak{sl}}_N), H, \mathcal{M}_{H^*}, \mu_l, \mu_r, \Delta, \varepsilon, S) \) becomes an \( H \)-Hopf algebroid \([5, 22, 28, 30]\).

### B Representations

#### B.1 Dynamical representations

Let us consider a vector space \( \hat{V} \) over \( F = \mathcal{M}_{H^*} \), which is \( H \)-diagonalizable, i.e.
\[
\hat{V} = \bigoplus_{\lambda, \nu \in H^*} \hat{V}_{\lambda, \nu}, \quad \hat{V}_{\lambda, \nu} = \{ v \in \hat{V} \mid q^{\nu + h} \cdot v = q^{<\lambda, P + h>} v, \quad q^P \cdot v = q^{<\nu, P>} v \quad \forall P + h, P \in H \}.
\]

Let us define the \( H \)-algebra \( \mathcal{D}_{H, \hat{\mathcal{V}}} \) of the \( \mathbb{C} \)-linear operators on \( \hat{V} \) by
\[
\mathcal{D}_{H, \hat{\mathcal{V}}} = \bigoplus_{\alpha, \beta \in H^*} (\mathcal{D}_{H, \hat{\mathcal{V}}})_{\alpha, \beta},
\]
\[
(\mathcal{D}_{H, \hat{\mathcal{V}}})_{\alpha, \beta} = \left\{ X \in \text{End}_\mathbb{C} \hat{V} \mid \begin{array}{l}
\quad f(P + h) X = X f(P + h + <\alpha, P + h>), \\
\quad f(P) X = X f(P + <\beta, P>), \\
\quad f(P), f(P + h) \in \mathbb{F}, \quad X \cdot \hat{V}_{\lambda, \mu} \subseteq \hat{V}_{\lambda + \alpha, \mu + \beta}
\end{array} \right\},
\]
\[
\mu^\mathcal{D}_{H, \hat{\mathcal{V}}} \hat{f} v = f(<\lambda, P + h>, P)v, \quad \mu^{\mathcal{D}_{H, \hat{\mathcal{V}}}}_r \hat{f} v = f(<\nu, P>, P^*v), \quad \hat{f} \in \mathbb{F}, \quad v \in \hat{V}_{\lambda, \nu}.
\]

**Definition B.1.** \([5, 22, 25]\) We define a dynamical representation of \( U_{q,p}(\hat{\mathfrak{sl}}_N) \) on \( \hat{V} \) to be an \( H \)-algebra homomorphism \( \pi : U_{q,p}(\hat{\mathfrak{sl}}_N) \to \mathcal{D}_{H, \hat{\mathcal{V}}} \). By the action of \( U_{q,p}(\hat{\mathfrak{sl}}_N) \) we regard \( \hat{V} \) as a \( U_{q,p}(\hat{\mathfrak{sl}}_N) \)-module.

**Definition B.2.** For \( k \in \mathbb{C} \), we say that a \( U_{q,p}(\hat{\mathfrak{sl}}_N) \)-module has level \( k \) if \( c \) act as the scalar \( k \) on it.

**Definition B.3.** Let \( \mathcal{H}, \mathcal{N}^+, \mathcal{N}^- \) be the subalgebras of \( U_{q,p}(\hat{\mathfrak{sl}}_N) \) generated by \( c, d, K^+_i \) (\( i \in I \)), by \( \alpha_i, (i \in I, n \in \mathbb{Z}_{>0}) \), \( e_i, (i \in I, n \in \mathbb{Z}_{>0}) \) \( f_{i,n} \) (\( i \in I, n \in \mathbb{Z}_{>0} \)) and by \( \alpha_i, (i \in I, n \in \mathbb{Z}_{>0}) \), \( e_i, (i \in I, n \in \mathbb{Z}_{>0}) \), \( f_{i,n} \) (\( i \in I, n \in \mathbb{Z}_{>0} \)), respectively.
**Definition B.4.** For $k \in \mathbb{C}$, $\lambda \in h^*$ and $\nu \in H^*$, a (dynamical) $U_{q,p}(\hat{sl}_N)$-module $\hat{V}(\lambda, \nu)$ is called the level-$k$ highest weight module with the highest weight $(\lambda, \nu)$, if there exists a vector $v \in \hat{V}(\lambda, \nu)$ such that

$$\hat{V}(\lambda, \nu) = U_{q,p}(\hat{sl}_N) \cdot v, \quad N_+ \cdot v = 0,$$

$$c \cdot v = kv, \quad f(P) \cdot v = f(<\nu, P>)v, \quad f(P + h) \cdot v = f(<\lambda, P + h>)v.$$

**B.2 The $N$-dimensional dynamical evaluation representation**

Let $\hat{V} = \bigoplus_{\mu=1}^{N} \mathbb{F}v_{\mu} \otimes 1$ and set $\hat{V}_z = \hat{V}[z, z^{-1}]$. Let $e^{Q_\alpha} \in \mathbb{C}[R_Q]$ act on $f(P)_{\beta}v \otimes 1$ by $e^{Q_\alpha}(f(P)_{\beta}v \otimes 1) = f(P_{\beta} - (\alpha, \beta))v \otimes 1$.

**Theorem B.5.** Let $E_{j,k}$ ($1 \leq j, k \leq N$) denote the matrix units such that $E_{j,k}v_{\mu} = \delta_{k,\mu}v_j$. The following gives the $N$-dimensional dynamical evaluation representation of $U_{q,p}(\hat{sl}_N)$ on $\hat{V}_z$.

$$\pi_z(c) = 0, \quad \pi_z(d) = -z \frac{d}{dz},$$

$$\pi_z(a_{j,m}) = \frac{\lfloor m \rfloor}{m} (q^{j-N+1}z^n)(q^{-m}E_{j,j} - q^mE_{j+1,j+1}),$$

$$\pi_z(e_{j}(w)) = \frac{\frac{pq^2; p}{p} \infty}{(p; p) \infty} E_{j,j+1} \delta (q^{j-N+1}z/w) e^{-Q_{\alpha_j}},$$

$$\pi_z(f_{j}(w)) = \frac{\frac{pq^{-2}; p}{p} \infty}{(p; p) \infty} E_{j+1,j} \delta (q^{j-N+1}z/w),$$

$$\pi_z(\psi^*_j(w, p)) = q^{-\pi(h_j)} e^{-Q_{\alpha_j}} \frac{\varTheta_p(q^{-j-N+1+2\pi(h_j)w})}{\varTheta_p(q^{-j-N+1+w/z})} \quad (1 \leq j \leq N - 1),$$

$$\pi_z(\psi_j(w, p)) = q^{\pi(h_j)} e^{-Q_{\alpha_j}} \frac{\varTheta_p(q^{j-N+1-2\pi(h_j)w})}{\varTheta_p(q^{j-N+1+w/z})}.$$

Here $\pi(h_j) = E_{j,j} - E_{j+1,j+1}$.

Combining the formulas in Definition [A.9] [A.12] [A.13] and Theorem [B.5] we obtain

**Corollary B.6.**

$$\pi_z(\hat{L}^+_k(w))_{j,k} = R^+(w/z, \Pi^*)_{ij}.$$  

**B.3 The level-1 representation**

Next we consider level-1 ($c = 1$) representation of $U_{q,p}(\hat{sl}_N)$. We mainly follow the work [6].

It is convenient to extend the root lattice $Q$ by adding the elements $\zeta_j$ ($j = 1, \cdots, N - 1$) in Definition [A.7]. Let us set $\hat{\alpha}_j = \alpha_j + \zeta_j$ and consider $\hat{Q} = \oplus_j \mathbb{Z} \hat{\alpha}_j$. We define the extended group algebra $\mathbb{C}[\hat{Q}]$ with assuming the following central extension.

$$e^{\alpha_i}e^{\alpha_j} = (-1)^{(\alpha_i, \alpha_j)}e^{\alpha_j}e^{\alpha_i}.$$
For $\omega = \sum_j c_j \zeta_j \in \mathfrak{h}^*$, we also set $\zeta_\omega = \sum_j c_j \zeta_j$ and $\tilde{\omega} = \omega + \zeta_\omega$. Set also $\Lambda_0 = \Lambda_0$.

Let $\Lambda_0$ and $\Lambda_n = \Lambda_0 + \Lambda_\alpha$ ($a = 1, \ldots, N-1$) be the fundamental weights of $\mathfrak{sl}_N$. For generic $\nu \in \mathfrak{h}^*$, we set

$$\tilde{V}(\Lambda_a + \nu, \nu) = \mathbb{F} \otimes \mathbb{C} (\mathcal{F}_{a,1} \otimes e^{\Lambda_a} \mathbb{C}[\tilde{Q}] \otimes e^{Q_\omega} \mathbb{C}[\mathcal{R}_Q],$$

where $\mathcal{F}_{a,1} = \mathbb{C}\{\alpha_{j,-m} \mid j = 1, \ldots, N-1, m \in \mathbb{N}_{>0}\}$. Then we have the following decomposition.

$$\tilde{V}(\Lambda_a + \nu, \nu) = \bigoplus_{\xi,\eta \in \mathcal{Q}} \mathcal{F}_{a,\nu}(\xi, \eta),$$

where

$$\mathcal{F}_{a,\nu}(\xi, \eta) = \mathbb{F} \otimes \mathbb{C} (\mathcal{F}_{a,1} \otimes e^{\Lambda_a + \xi}) \otimes e^{Q_\nu + \eta}. \quad (B.1)$$

**Theorem B.7.** The spaces $\tilde{V}(\Lambda_a + \nu, \nu)$ ($a = 0, \ldots, N$) give the level-1 irreducible $U_{q,p}(\mathfrak{sl}_N)$-modules with the highest weight $(\Lambda_a + \nu, \nu)$, where the highest weight vectors are given by $1 \otimes e^{\Lambda_a} \otimes e^{Q_\nu}$. The action of the elliptic currents is given by

$$E_j(z) = : \exp \left\{ -\sum_{n \neq 0} \frac{1}{[n]_q} \alpha_{j,n} z^{-n} \right\} \cdot e^{\tilde{\Lambda}_j} e^{-Q_{\alpha_j} z^{h_{\alpha_j}+1}(q^{N-j}z)^{-\frac{p_{\alpha_j}^{-1}}{r}}}, \quad (B.2)$$

$$F_j(z) = : \exp \left\{ \sum_{n \neq 0} \frac{1}{[n]_q} \alpha'_{j,n} z^{-n} \right\} \cdot e^{-\tilde{\Lambda}_j} z^{-h_{\alpha_j}+1}(q^{N-j}z)^{-\frac{(P+h)\alpha_j^{-1}}{r}}, \quad (B.3)$$

$$(1 \leq j \leq N-1)$$

together with $H_j^\pm(z), K_j^\pm(z)$ in Sec. A.4 and

$$\hat{d} = d + \frac{1}{2p} \sum_{j=1}^N (P_j + 2)P_j - \frac{1}{2p} \sum_{j=1}^N ((P + h)j + 2)(P + h)^j,$$

$$d = -\frac{1}{2} \sum_{j=1}^{N-1} h_j h_j - \sum_{j=1}^{N-1} \sum_{m \in \mathbb{Z}_{>0}} \frac{m^2}{[m]} \frac{1 - p^m}{1 - p^m} q^m \alpha_{j,-m} A_j^m.$$  

In (B.3) we set $\alpha'_{j,n} = \frac{1 - p^m}{1 - p^n} q^n \alpha_{j,n}$.

**C Proof of Proposition 5.10**

Let $\lambda, \lambda' \in \mathbb{N}^N, |\lambda| = m, |\lambda'| = n$ and consider $I = I_{\mu_1 \cdots \mu_m} = (I_1, \ldots, I_N) \in \mathcal{I}_\lambda$ and $I' = I_{\mu'_1 \cdots \mu'_m} = (I'_1, \ldots, I'_N) \in \mathcal{I}_{\lambda'}$. For each $I'_l = \{i'_l, \cdots, i'_{l,\lambda'_l}\}$ ($l = 1, \cdots, N$), let us set $\tilde{I}'_l =$
Their composition gives the $\Pi$ where

$$I = \Pi_{\mu_1, \ldots, \mu_{m+n}}(z_1, \ldots, z_{m+n}) = \phi_{\mu_1, \ldots, \mu_m}(z_1, \ldots, z_m)\phi_{\mu'_1, \ldots, \mu'_n}(z'_1, \ldots, z'_n),$$

where we set $z_{m+k} := z'_k$ and $\mu_{m+k} := \mu'_k (k = 1, \ldots, n)$.

On the other hand, from Theorem 4.2 we have

$$\phi_{\mu_1, \ldots, \mu_m}(z_1, \ldots, z_m) = \int_{\mathbb{T}^m} dt \quad \tilde{\Phi}(t, z)\omega_{\mu_1, \ldots, \mu_m}(t, z, \Pi_I),$$

$$\phi_{\mu'_1, \ldots, \mu'_n}(z'_1, \ldots, z'_n) = \int_{\mathbb{T}^m} dt' \quad \tilde{\Phi}(t', z')\omega_{\mu'_1, \ldots, \mu'_n}(t', z', \Pi_{I'})$$

and

$$\phi_{\mu_1, \ldots, \mu_{m+n}}(z_1, \ldots, z_{m+n}) = \int_{\mathbb{T}^{m+n}} dt \quad \tilde{\Phi}(t, z \cup z')\omega_{\mu_1, \ldots, \mu_{m+n}}(t, z \cup z', \Pi).$$

Here we set $z = (z_1, \ldots, z_m)$, $z' = (z'_1, \ldots, z'_n)$, $M = \sum_{j=1}^{N-1} (N - j)\lambda_j$, $M' = \sum_{j=1}^{N-1} (N - j)\lambda'_j$, $t = (t^{(1)}_1, \ldots, t^{(1)}_{\lambda_1}, \ldots, t^{(1)}_{\lambda_{N-1}})$, $t' = (t'^{(1)}_1, \ldots, t'^{(1)}_{\lambda^{(1)}_{N-1}}, \ldots, t'^{(1)}_{\lambda^{(N-1)}})$ and $\tilde{t} = (\tilde{t}^{(1)}_1, \ldots, \tilde{t}^{(1)}_{\lambda^{(1)}_{N-1}}, \ldots, \tilde{t}^{(1)}_{\lambda^{(N-1)}})$. Theorem 4.2.1 and $\Pi = \{\Pi_{k, j} (k = 1, \ldots, m + n, j = 1, \ldots, N)\}$.

Substituting (C.2)-(C.4) into (C.1), one can obtain a relation among the weight functions $\omega_{\mu_1, \ldots, \mu_m} (t, z, \Pi_I)$, $\omega_{\mu'_1, \ldots, \mu'_n} (t', z', \Pi_{I'})$ and $\omega_{\mu_1, \ldots, \mu_{m+n}} (t, z \cup z', \Pi)$.

**Definition C.1.** For the weight functions $\omega_{\mu_1, \ldots, \mu_m} (t, z, \Pi)$ and $\omega_{\mu'_1, \ldots, \mu'_n} (t', z', \Pi')$, we define the $*$-product as follows.

$$(\omega_{\mu_1, \ldots, \mu_m} * \omega_{\mu'_1, \ldots, \mu'_n})(t \cup t', z \cup z', \Pi_{I+I'})$$

is defined by

$$\tilde{\Xi}(t, t', z, z') = \mu_{m,n}(z, z') \prod_{l=1}^{N-1} \prod_{a=1}^{\lambda^{(l)}} (z^{(l+1)}_{a} - v^{(l)}_{a} - \frac{1}{2}) \prod_{b=1}^{\lambda^{(l)}} (v^{(l)}_{b} - v^{(l)}_{a} + \frac{1}{2}) \prod_{c=1}^{\lambda^{(l)}} (v^{(l)}_{c} - v^{(l)}_{a})$$

where $\Pi_{I+I'} = \{\Pi_{k, j} (k = 1, \ldots, m + n, j = 1, \ldots, N)\}$ with $\Pi_{\mu_{m+k}, j} = \Pi_{k, j} (k = 1, \ldots, n, j = 1, \ldots, N)$, and

$$\tilde{\Xi}(t, t', z, z') = \mu_{m,n}(z, z') \prod_{l=1}^{n} \prod_{a=1}^{\lambda^{(l)}_{+}} \frac{\Gamma(q^{2} z^{(l)}_{a}/z_{k}; p, q^{2N})}{\Gamma(q^{2} z^{(l)}_{a}/z_{k}; p, q^{2N})},$$

$$\mu_{m,n}(z, z') = \prod_{k=1}^{n} \prod_{l=1}^{N-1} \frac{\Gamma(q^{2} z^{(l)}_{a}/z_{k}; p, q^{2N})}{\Gamma(q^{2} z^{(l)}_{a}/z_{k}; p, q^{2N})}.$$
for \( l = 1, \ldots, N - 1 \) and \( \Pi_{\ell_2} = \tilde{\Pi} \), we obtain
\[
(\omega_{\mu_1 \cdots \mu_n} \star \omega_{\mu'_1 \cdots \mu'_n})(t \cup t', z \cup z', \Pi_{\ell_2}) = \omega_{\mu_1 \cdots \mu_{m+n}}(\tilde{t}, z \cup z', \tilde{\Pi}).
\]

**Proof.** Substituting (\ref{C2}) and (\ref{C.3}) into the right hand side of (\ref{C.1}), we obtain
\[
\phi_{\mu_1 \cdots \mu_{m+n}}(z_1, \ldots, z_{m+n}) = \int_{T^M} dt \int_{T^{M'}} dt' \Phi(t, z) \omega_{\mu_1 \cdots \mu_n}(t, z, \Pi_{\ell_2}) \omega_{\mu'_1 \cdots \mu'_n}(t', z', \Pi_{\ell_2}')
\]
\[
= \int_{T^M} dt \int_{T^{M'}} dt' \Phi(t, z) \omega_{\mu_1 \cdots \mu_n}(t, z, \Pi_{\ell_2}) \omega_{\mu'_1 \cdots \mu'_n}(t', z', \Pi_{\ell_2}').
\]
Furthermore from (\ref{A.47}) and (\ref{3.12}) we obtain
\[
\Phi(t, z) \Phi(t', z') = Y(t, t', z, z') \Xi(t, t', z, z')
\]
where
\[
Y(t, t', z, z') =: \Phi_N(z_1) \cdots \Phi_N(z_{m+n}) \cdot F_{N-1}(t_1^{(N-1)}) \cdots F_{N-1}(t_N^{(N-1)}),
\]
\[
\Rightarrow \prod_{1 \leq k < l \leq m} \Phi_N(z_k) \Phi_N(z_l) > \text{Sym} \prod_{1 \leq k < l \leq n} \Phi_N(z_k') \Phi_N(z_l') > \text{Sym} \prod_{k=1}^m \prod_{l=1}^n \Phi_N(z_k) \Phi_N(z_l') > \text{Sym}.
\]
\[
\prod_{l=1}^{N-1} \left( \prod_{1 \leq a < b \leq \lambda(l)} \Phi(t_a^{(l)}) \Phi(t_b^{(l)}) > \text{Sym} \prod_{1 \leq a < b \leq \lambda(l)} \Phi(t_a^{(l)}) \Phi(t_b^{(l)}) > \text{Sym} \prod_{a=1}^\lambda(l) \prod_{b=1}^{\lambda(l)} \Phi(t_a^{(l)}) \Phi(t_b^{(l)}) > \text{Sym} \right).
\]

Then it turns out that \( Y(t, t', z, z') \) coincides with \( \Phi(\tilde{t}, z \cup z') \) under the identification (\ref{C.3}).

Hence we have
\[
\phi_{\mu_1 \cdots \mu_{m+n}}(z_1, \ldots, z_{m+n}) = \int_{T^M \times T^{M'}} dt \int_{T^{M'}} dt' \Phi(\tilde{t}, z \cup z') \omega_{\mu_1 \cdots \mu_n}(t, z, \Pi_{\ell_2} q^{-2 \sum_{j=1}^n \epsilon_j^{(t)} h}) \omega_{\mu'_1 \cdots \mu'_n}(t', z', \Pi_{\ell_2}').
\]
\[
= \int_{T^M \times T^{M'}} dt \int_{T^{M'}} dt' \Phi(\tilde{t}, z \cup z') \omega_{\mu_1 \cdots \mu_n}(t, z, \Pi_{\ell_2} q^{-2 \sum_{j=1}^n \epsilon_j^{(t)} h}) \omega_{\mu'_1 \cdots \mu'_n}(t', z', \Pi_{\ell_2}').
\]

The last equality follows from the symmetry of \( \Phi(\tilde{t}, z \cup z') \) under the action of \( \sigma \in \mathfrak{S}_{\lambda(1)} \times \cdots \times \mathfrak{S}_{\lambda(N-1)} \) on \( \tilde{t} \). Comparing this with (\ref{C.4}) we obtain the desired result. \( \square \)

Proposition 5.10 is a direct consequence of this proposition.
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