Characters and patterns of communities in networks

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A community can be seen as a group of vertices with strong cohesion among themselves and weak cohesion between each other. Community structure is one of the most remarkable features of many complex networks. There are various kinds of algorithms for detecting communities. However it is widely open for the question: what can we do with the communities? In this paper, we propose some new notions to characterize and analyze the communities. The new notions are general characters of the communities or local structures of networks. At first, we introduce the notions of internal dominating set and external dominating set of a community. We show that most communities in real networks have a small internal dominating set and a small external dominating set, and that the internal dominating set of a community keeps much of the information of the community. Secondly, based on the notions of the internal dominating set and the external dominating set, we define an internal slope (ISlope, for short) and an external slope (ESlope, for short) to measure the internal heterogeneity and external heterogeneity of a community respectively. We show that the internal slope (ISlope) of a community largely determines the structure of the community, that most communities in real networks are heterogeneous, meaning that most of the communities have a core/periphery structure, and that both ISlopes and ESlopes (reflecting the structure of communities) of all the communities of a network approximately follow a normal distribution. Therefore typical values of both ISlopes and ESlopes of all the communities of a given network are in a narrow interval, and there is only a small number of communities having ISlopes or ESlopes out of the range of typical values of the ISlopes and ESlopes of the network. Finally, we show that all the communities of the real networks we studied, have a three degree separation phenomenon, that is, the average distance of communities is approximately 3, implying a general property of true communities for many real networks, and that good community finding algorithms find communities that amplify clustering coefficients of the networks, for many real networks.

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Additional Key Words and Phrases: community, internal dominating set, external dominating set, internal slope, external slope

1. INTRODUCTION

Real networks differ from random graphs in the way that they are organized with a high level of order. Such an organization results to remarkable common phenomena of real networks, for instance: the heavy tail degree distributions, the high clustering coefficients and the small average distances etc [Barabasi and Albert 1999] [Watts and Strogatz 1998]. In addition, another remarkable common feature in various networks is the community structure. Community is an important notion to disclose the structure of networks, playing the role in bridging the local vertices and the global network. On one hand, we could extract communities from a network to study its internal structure and its relationship with the rest.
of the network from the local point of view. On the one hand, we could take each community as a unit of the network, to illustrate the connecting patterns of different communities of real networks through the distributions of different properties of communities from the global point of view [De Nooy et al. 2011].

Massive work has been devoted to the study of communities, including the main definitions of the community problem, algorithms developing for finding communities, comparison and tests of different algorithms etc. [Fortunato 2010]. Leskovec et al. [Leskovec et al. 2009] analyzed community structures in large real networks and tried to find the “best” communities at various sizes. They showed that the “best” communities seem to be characterized by size of 100. The distribution of sizes of communities has also been studied, showing that in some cases, they have the skewed distribution [Clauset et al. 2004; Newman 2004b]. The small community phenomenon was introduced recently, that is, there are models, classical or new, such that networks from the models are rich in small communities, that is, quality communities of small sizes [Li and Peng 2011; Li and Peng 2012], for which the mechanism is homophily.

Intuitively speaking, a community of a network can be interpreted as a relatively independent and stable unit of the network, and the rich communities of a network are taken as the local structures of the network. This suggests fundamental questions such as: What can we do with the communities? Are there some characters of all the communities of a network? What information of the network can we extract from the communities? What characters of communities (largely) determine the local patterns of the network? What are the relationships between the found communities and the true communities? These questions are widely open in the current state of the art. This motivates the research in the present paper. For this, we investigate the following: (1) How to extract central nodes from a community? (2) How to extract useful information from the communities? (3) How do communities interact with each other? (4) How to measure the heterogeneity of a community? (5) What general properties do the communities (found by a reasonably good algorithm) have?

By using a variant of the local spectral partitioning algorithm [Andersen et al. 2006], we find rich communities in real networks. These networks include collaboration networks, citation networks, email networks and one benchmark network [Girvan and Newman 2002]. In collaboration network a node denotes a scientist and an edge indicates that the two scientists have coauthored a paper. In the citation networks a node denotes a paper in some fields and an edge between two papers indicates that at least one paper has cited the other. Communities in this networks may correspond to different research groups or research themes. Two email networks are also used in our study, in which each node corresponds to an email address and an edge between nodes i and j represents i sending at least one message to j or j sending at least one message to i. A well known benchmark network of American college football teams compiled by Girvan and Newman [Girvan and Newman 2002] is also used. Nodes of the network represent teams and an edge between two nodes represents that the corresponding two teams play against each other. The network contains 12 true communities, which correspond to 12 different conferences that the teams belong to. All networks above have good community structures so that they are good candidates for investigating the characters and connecting patterns of local structures of networks.

We organize the paper as follows. In section 2, we propose the notions of internal dominating ratio and external dominating ratio to measure the importance of a subset of a community. Then we give the definition of internal dominating set (IDS) and external dominating set (EDS). In section 3, we verify that, the internal dominating set of a community is

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1All the data in this paper can be found from the websites: [http://snap.standford.edu](http://snap.standford.edu) or [http://www-personal.umich.edu/~mejn/netdata](http://www-personal.umich.edu/~mejn/netdata) and we only consider the corresponding undirected graphs.
much more smaller than the community and keeps largely the information of the community. In section 4, we define internal slope (ISlope) and external slope (ESlope) of a community to measure the internal heterogeneity and the external heterogeneity of the community, respectively. We analyze the relationship between the structure and the ISlopes and give the distributions of the ISlopes and the ESlopes of all the communities of the real networks. In Section 5, we analyze more general properties like average distances, diameters and clustering coefficients of all the communities for each of the networks. Finally, in section 6, we summarize the conclusions of the paper.

2. INTERNAL AND EXTERNAL DOMINATING SETS

Table I. Statistics of real networks. All the results are calculated by averaging the corresponding properties of all the communities. The IDR and EDR are the ratios of centrality of 5-IDS and 5-EDS; the IDN and EDN are the sizes of 0.8-IDS and 0.8-EDS.

| Network     | IDR  | EDR  | IDN  | EDN  | ISlope | ESlope |
|-------------|------|------|------|------|--------|--------|
| football    | 0.99 | 0.61 | 2.6  | 9.3  | 0.19   | 0.37   |
| cit_hepth   | 0.75 | 0.49 | 12   | 32   | 0.41   | 0.54   |
| cit_hepgh   | 0.74 | 0.49 | 12   | 36   | 0.5    | 0.58   |
| col_acstroph| 0.93 | 0.79 | 3.7  | 8.1  | 0.36   | 0.65   |
| col_econdmat| 0.85 | 0.79 | 9.6  | 16   | 0.42   | 0.66   |
| col_grqc    | 0.94 | 0.91 | 3.1  | 3.9  | 0.37   | 0.67   |
| col_hepth   | 0.69 | 0.64 | 23   | 17   | 0.38   | 0.64   |
| col_hepgh   | 0.8  | 0.7  | 11   | 16   | 0.38   | 0.64   |
| email_euron | 0.93 | 0.86 | 3.6  | 7.8  | 0.55   | 0.68   |
| email_euall | 0.98 | 0.95 | 1.7  | 2.4  | 0.92   | 0.89   |

Given a community of a network, we may want to extract a small set of nodes that are more central to the community than the rest of nodes in the community. Taking the citation network for an example, we are interested in a small number, 10 say, of important papers that are central to the whole community which usually includes hundreds of papers. In this case, we would hope that with the short list of key papers, we will not lose any essential information of the whole community. This analysis of centrality has been studied for the whole networks, for example, it was shown that a small fraction of nodes accumulates a large proportion of links in the networks [Newman 2004a], and that only 20% of most-linked authors in Economics account for about 60% of all the links [Goyal et al. 2006]. So there are indeed some nodes taking the central position in networks. We believe that similar centrality phenomena occurs in true communities of many real networks, and that the main goal of community finding algorithms is to find the true communities of the networks. The question is: what can we say about the centrality of the communities found by our algorithms? This would be the first step to understand the relationship between the true communities and the communities found by algorithms.

Some centrality measures, initially introduced in social studies, could be used, for instance, the degree centrality, the closeness centrality, and the betweenness centrality etc [Freeman 1979]. These measures assume a relationship between the structural position and influential power in group processes [Bavelas 1948], and are developed and widely used in the literature [Nicosia et al. 2012]. The mechanism behind this idea is that the centrality of a vertex could be predicted from its position and the network structure in which it was embedded as well as from its own characteristics [Rogers 1974]. Except for these centrality measures, vertices could also be classified according to their roles within their communities. Guimerà and Amaral decide the role of a vertex by a within-module degree $z_i$ and a participation ratio $P_i$ and distinguish seven roles that vertices can play, based on the values of the pair $(z, P)$ [Guimera and Amaral 2005].
In this section, we propose the notion of internal and external dominating sets of a community by modifying the notion of the dominating set. The dominating set problem is classical in graph algorithms: Given a graph $G = (V, E)$, we say that a set $S \subseteq V$ is a dominating set if every node $v \in V$ is either an element of $S$ or adjacent to an element of $S$. The dominating number is the number of vertices in a smallest dominating set for $G$ [Allan and Laskar 1978; Haynes et al. 1998].

For a community, we distinguish two roles that nodes can play in a community, as an internal role and an external role, measured by links within and outside of the community respectively. For a subset of a given community, its internal dominating ratio (IDR, for short) is defined as follows.

Let $C$ be a community, $S$ be a subset of $C$, $N(S, C)$ be the neighbors of $S$ within community $C$. Then we define the internal dominating ratio of $S$ in $C$, written by IDR, as follows:

$$IDR(S) = \frac{|S \cup N(S, C)|}{|C \cup N(C, C)|} = \frac{|S \cup N(S, C)|}{|C|}$$

The dominating ratio has been used previously to measure the social centrality in social networks [Freeman 1979]. Our internal dominating ratio (IDR) measures the importance of a group of nodes in a community, and thus it can be seen as a general format of degree centrality of communities.

Following the definition above, we consider two problems: 1) when given a number $k$ (usually small), we want to find a subset $S$ of size $k$ with max $\{IDR\}$, in which case, we call this subset a $k$-IDS; 2) when given a real number $p$ in $[0, 1]$, we want to find a subset $S$ whose IDR is bigger than $p$ with the minimum number of nodes, in which case, we call this subset a $p$-IDS.

Similarly to IDR, we give the definition of external dominating ratio (EDR). Let $C$ be a community, $S$ be a subset of $C$, $N(S, \bar{C})$ be the neighbors of node $v$ that are outside of $C$. Then the external dominating ratio (EDR) of $S$ in $C$ is defined as follows:

$$EDR(S) = \frac{|N(S, \bar{C})|}{|N(C, \bar{C})|}$$

We also give the notations $k$-EDS and $p$-EDS similarly. Figure 1 is an example of the IDR's and EDR's. From the definitions, we notice that we are not using the notion of classic...
ALGORITHM 1: Finding $p$-{IDS}

**Input:** Graph $G$, community $C$, and a real number $p \in [0, 1]$.

**Output:** The $p$-internal dominating set $S$.

Let $G_C$ be the induced subgraph of vertices $C$ from $G$. Set $S = \emptyset$;

repeat

Let $v$ be a node in $C \setminus S$ such that $v$ has the maximal number of neighbors in $C \setminus (S \cup N(S))$, where $N(S)$ is the neighbors of $S$ in $G_C$;

$S \leftarrow S \cup \{v\}$;

until IDR($S$) $\leq p$;

ALGORITHM 2: Finding $p$-{EDS}

**Input:** Graph $G$, community $C$, and a real number $p \in [0, 1]$.

**Output:** The $p$-external dominating set $S$.

Set $S = \emptyset$, $N(S, \bar{C}) = \emptyset$, where $N(S, \bar{C})$ is the number of neighbors of $S$ outside of $C$;

repeat

Let $v$ be a node in $C \setminus S$ such that $v$ has the maximal number of neighbors in $N(C, \bar{C}) \setminus N(S, \bar{C})$;

$S \leftarrow S \cup \{v\}$;

$N(S, \bar{C}) \leftarrow N(S, \bar{C}) \cup N(v, \bar{C})$;

until EDR($S$) $\leq p$;

dominating set [Allan and Laskar 1978] [Haynes et al. 1998], instead, we introduce two parameters $k$ and $p$ to define the general format of dominating sets. We emphasize that the classification are based on nodes positions in a community. By definition, it is conceivable that nodes in the IDS are more important for the function and stability of the community, and that nodes in the EDS mainly take charge of the communication between the community and the nodes outside of the community.

The dominating problem is an NP-complete decision problem [Haynes et al. 1998]. Here we introduce a simple greedy algorithm to find the $p$-IDS and $p$-EDS, where $G$ is a graph, $C$ is a community and $p$ is a real number in $[0, 1]$.

Given a number $p$ between 0 and 1, we could find the $p$-{IDS} and $p$-{EDS} by using the above algorithms. Similarly when given a small number $k$, we could calculate the $k$-{IDS} and $k$-{EDS} by using the same algorithm with slight modification of the terminating condition. In our experiment, we set $k = 5$ when calculating the $k$-{IDS} and the $k$-{EDS}, and set $p = 0.8$ when calculating the $p$-{IDS} and $p$-{EDS}, see Table I for details.

From Table I we observe that only five nodes could dominate most of the members of the communities from both internal and external sides, that the internal dominating ratios of 5 internally central nodes are larger than the external dominating ratio of 5 externally central nodes, for each of the networks, that external connecting patterns of the communities are more decentralizing than that of the internal connecting patterns, for each of the networks, that it only needs at most 10 nodes to internally dominate at least 80% of the whole community, that it needs at most 32 nodes to externally dominate 80% of the outgoing links of the communities, and that external dominating numbers are larger than the internal dominating numbers for all communities and for all the networks.

In summary, we have that most communities have a small internal dominating set, and a small external dominating set, which is slightly larger than the internal dominating set of the corresponding communities, on the average, for all the networks.
ALGORITHM 3: Predicting keywords using internal dominating set

**Input:** Graph $G$, community $C$, and keyword dictionary $Dic$

**Output:** Papers with predicting keywords

Calculate $p$-IDS or $k$-IDS of $C$;
Suppose that $L = \{k_1, k_2, \cdots, k_i\}$ are listed keywords from the IDS with descending order according to their popularity in $C$. For a given paper $P$ in $C$ whose keywords are not listed in the network, for each $j \leq i$, if $k_j$ appears in either the title or the abstract of paper $P$, we say that $k_j$ is a predicted and confirmed keyword of $P$;

3. EXTRACTING LOCAL INFORMATION

In the last section, we verify that most communities have a small internal dominating set, and a small external dominating set. The questions are: How much information of a community is preserved in the dominating set of the community? How to extract essential information of a community from the small dominating sets?

In this section, we verify that the internal dominating sets (IDSs) indeed preserve essential information of the communities. We verify this result by predicting and confirming keywords of papers in a citation network.

We say that a paper has keywords, if its authors have explicitly list its keywords, and does not have keywords, otherwise.

Keywords of papers play an important role in information retrieval. In many citation networks, there is a huge number of papers whose keywords are not listed by their authors, which is an obstacle for people to sufficiently use the network.

In the citation-hepth networks, there are about $27,770$ papers, in which only $10\%$ or so have keywords. Predicting and confirming the missing keywords for the other papers are obviously significant for information retrieval.

Given a community $C$ in a citation network, we predict and confirm keywords for papers in $C$ by the following procedure.

We choose parameter $p = 0.8$, run the algorithm on the citation network, and report the results in Table III. The first column of Table III presents the number of keywords we used for the prediction and confirmation for each communities, that is, the length $i$ of $L$ in the algorithm, the second column of the table are numbers of papers whose keywords have been predicted and confirmed corresponding to different lengths of $L$ in the first column.

From Table III, taking the first row of the table for example, we know that if we use the most popular 5 keywords appearing in the IDS of each of the communities, then there are $13,283$ papers in the network whose keywords are predicted and confirmed. As the number of keywords used in the algorithm, i.e., the lengths of $L$ in the algorithm, becomes larger, we can predict and confirm keywords for more papers, that is up to $14,691$ papers. The results show that the IDS is much smaller than the corresponding community and that the IDS preserves much information of the corresponding community. From the experiment, it is conceivable that in practical applications, it is sound to recommend the IDS of a community instead of the whole community which is usually much larger. The result above is unexpectedly good. We believe that this property may hold for many other networks other than citation networks, that is, the internal dominating set of a community keep essential information of the community. More importantly, the essential information of the internal dominating set of a community can be easily extracted.

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2We implement the verification for just one citation network, because this is the only available network in which titles, abstract of papers, and keywords of a small number of papers are included. Most networks have a topological structure with nodes and edges only.

ACM Journal Name, Vol. V, No. N, Article A, Publication date: December 2012.
Table II. Using 0.8-IDS to predict keywords in citation network hepth

| Keyword Number | Predicted Paper Number |
|----------------|------------------------|
| 5              | 13283                  |
| 10             | 13906                  |
| 15             | 14375                  |
| 20             | 14592                  |
| 25             | 14641                  |
| 30             | 14647                  |
| 35             | 14654                  |
| 40             | 14691                  |
| 45             | 14691                  |
| 50             | 14691                  |

4. INTERNAL AND EXTERNAL SLOPES

In the last section, we show that most communities have a small IDS and a small EDS, and that the small IDS of a community preserves much information of the community.

In this section, we will show that the IDS and EDS of a community usually take the central positions in the community with low degree nodes around them, so that the community forms a core/periphery structure.

Intuitively speaking, if all nodes in a community have equal position, i.e., the regular graph or a random graph, then they are homogeneous; if nodes in a community form a core/periphery structure, i.e., the star-like graphs, then they are heterogeneous. Our main question is: How do the IDS and EDS of a community reflect the homogeneity or the heterogeneity of the community?

Before answering this question, we look at the power law distribution. It was shown that most networks follow a power law distribution [Barabási and Albert 1999], meaning that the number of nodes of degree k is proportional to $k^{-\beta}$. A power law distribution of power exponent $\beta$, which is typically lying in the range $2 < \beta < 3$, measures the heterogeneity of a network. However it is nontrivial to estimate the exponent $\beta$, especially for small networks, and not all networks follow the power law distribution [Clauset et al. 2009]. Most communities are small, although they may have heavy tail degree distributions, it is not clear whether they have power law distributions. More seriously, even if the communities have power law distributions, fluctuations caused by the small sizes of communities may make the result inaccurate, and the number of communities is large, it is hard to characterize the power law distributions of all the communities. Therefore the power exponent $\beta$ is not suitable to measuring the heterogeneity of all the communities of a network. Another measure is to notice the relationship between the number of dominating set and the degree distribution. In fact, it was shown that the more heterogeneous the degree distribution of a network is, the smaller the number of dominating set is [Nacher and Akutsu 2012]. This suggests that the internal and external dominating sets are closely related to the heterogeneity of the communities.

We now measure the heterogeneity of communities by the internal and external dominating sets of communities. See figure 2(a) in which case the community is homogeneous. All members of the community have equal position, and any single node could dominate the whole community. From the dominating number, we could not know the heterogeneity of the community. So the dominating set itself is insufficient to measure the homogeneity and heterogeneity of a community. To solve this problem, we use the internal dominating ratio (IDR) of the internal dominating set (IDS), together with the expectation internal dominating ratio (IDR) of random selection of nodes of the same size as that of the IDS.

We define the internal slope (ISlope, for short) and external slope (ESlope, for short) of a community to measure the internal and external heterogeneity (or the core/periphery structure) of the community. Intuitively, the ISlope of a community is to measure the
Let $C$ be a community, $p \in [0, 1]$ be a real number. Suppose that $K$ is the size of the $p$-IDS of $C$, that $S$ be the $p$-IDS of $C$, and that $\mathcal{V} = \{V_1, V_2, \ldots, V_M\}$ is the set of all subsets of $C$ of size $K$. Then define the internal slope of $C$, written by $\text{ISlope}(C)$ as follows:

$$\text{ISlope}(C) = \text{IDR}(S) - \frac{\sum_{X \in \mathcal{V}} \text{IDR}(X)}{|\mathcal{V}|} \tag{3}$$

The ISlope of a community represents the difference between the internal dominating ratio of the most central nodes and the expectation internal dominating ratio of random choices of nodes of the same size. It measures the homogeneity and heterogeneity (core/periphery structure) of the community from the internal point of view. We extract some communities of real networks found by our algorithm in Figure 2. From these figures we can observe that the ISlopes and ESlopes of the communities largely reflect the homogeneity and the heterogeneity of the corresponding communities.

By observing Figure 2, we know that the structures of communities are closely related to the corresponding ISlopes of the communities. In particular, in Figure 2(a), all nodes have equal position and a single node could dominate the whole community; in Figure 2(b), there are some central nodes with periphery nodes around; in Figure 2(c), the central position...
of one node is more obvious, and the structure is a star-like graph; in Figure 2(d), the community is a star graph with a hub in the center, and the ISlope of the community is very near 1. Notice that a star graph is the most heterogeneous community, in which the hub in its center is the most important node. In summary, we observe that the smaller the ISlope of a community is, the more homogeneous a community is, and that on the contrary, the larger the ISlope of a community is, the more heterogeneous a community is, and that the ISlope of a community roughly reflects the pattern or structure of the community.

Similarly to ISlope, we define the *external slope of a community* (ESlope) to measure the external heterogeneity of the community. By using the ESlope of a community, we are able to examine the pattern that nodes in a community connect nodes outside of the community. Whether or not nodes in a community connect the rest of the community through a small number of representatives or evenly through most members.

It has been shown that in a collaboration network, most people in the network (theme, or topic) contact people in the network through just one or two of their best-connected collaborators [Newman 2004a; Newman et al. 2001].

Our results show that such a funneling pattern of connections from a community to outside of the community is very popular in all the communities of a network, for a wide range of real networks.

Fig. 3. Real communities to illustrate ESlope. All of them are from collaboration grqc network. In each figure, red nodes come from the same community.
Let $C$ be a community, $p \in [0, 1]$ be a real number. Suppose that $K$ is the size of a $p$-EDS of $C$, that $\mathcal{V} = \{V_1, V_2, \cdots, V_m\}$ is the set of all subsets of $C$ of size $K$. Then we define the \textit{external slope of $C$} (ESlope($C$)) as follows:

$$\text{ESlope}(C) = \text{EDR}(\text{EDS}) - \frac{\sum_{Y \in \mathcal{V}} \text{EDR}(Y)}{|\mathcal{V}|}$$

The ESlope of a community represents the difference between the external dominating ratio of the most central nodes and the expectation external dominating ratio of random selection of nodes of the same size.

Figure 3 illustrates different connecting patterns of communities with different ESlopes. In these figures, we also keep the neighbors and the neighbors of neighbors of the community to highlight their connecting patterns. In figure 3(a), all members have equal position to connect with nodes outside of the community. Some nodes only have internal links, while others have both external and internal links in figure 3(b). Also, some nodes play the role of bridge in linking nodes in and outside of its community in figure 3(c). At last, figure 3(d) shows a community in which only one node is the bridge. All other members communicate with the outside world through this node. The ESlope indeed identifies different connecting patterns of how communities connect with each other.

Table I gives the average ISlopes and ESlopes of all the communities of various networks. Except for the football and the email_euall, all other networks have similar ISlopes and ESlopes with ESlopes larger than ISlopes, on the average. ISlope and ESlope of a community quantify the core/periphery structure of the community. Our results indicate that such structures are universal in real networks and that real networks tend to avoid communities of either regular or star-like graphs and have structures with ISlopes and ESlopes in some fixed interval, that is, the ISlopes are roughly in $[0.35, 0.55]$ and the ESlopes in $[0.5, 0.7]$.

These results pose a question that why networks tend to have such structures. We try to explain these as follows: For a community, it is possible that some key nodes are essential to its formation and evolution. On one hand, it is unusual to have a community with all members having equal position for a long period of time. On the other hand, the key nodes of a community should not be too strong or too weak since otherwise, the community structure may be fragile. It is intuitive that if the central nodes of a community breakdown, then the community structure would not exist any more. Therefore too big ISlopes or ESlopes and too small ISlopes or ESlopes will both go ill with the evolution of communities. The structures of typical communities of a real network may be a compromise between the effectiveness and robustness of the communities. We conjecture that the ESlopes may largely determine the evolution of communities, which needs to be further investigated (in our ongoing project).

Besides the average values, we also report the distributions of the ISlopes and ESlopes in figure 4 and figure 5 of all the communities of the real networks. Figure 6 and figure 7 are the corresponding cumulative distribution. By observing these figures, we know that:

— Most communities have a core/periphery structure, with a small core in central positions and some low degree nodes in the periphery.
— The ISlopes largely determine the structure of the communities.
— There are indeed some typical thresholds at which the distribution curve decreases sharply in most networks.
— The typical values of ESlopes are more obvious than that of the ISlopes in the citation and collaboration networks, in which, the ESlopes of most communities lie in a very narrow interval.
— Communities of the email-euall network have much larger ISlopes and ESlopes in general.
— The ISlopes and ESlopes of all the communities of the citation and collaboration networks approximately follow a normal distribution.
Fig. 4. Distribution of communities’ ISlope

Table III. Statistics of communities. APL represents average path length, D represents diameter, CCC represents community clustering coefficient and NCC represents the network clustering coefficient. All the results except NCC are calculated by averaging the corresponding property of all communities.

| Network  | APL | D   | CCC | NCC |
|----------|-----|-----|-----|-----|
| football | 1.8 | 3.2 | 0.6 | 0.41|
| cit_hepH | 2.9 | 7   | 0.36| 0.12|
| cit_heppH| 2.7 | 6.7 | 0.29| 0.15|
| col_astroph| 2.2 | 4.6 | 0.74| 0.82|
| col_cndmat| 2.7 | 5.4 | 0.53| 0.26|
| col_grqc  | 2.4 | 4.7 | 0.51| 0.63|
| col_hepH  | 3.3 | 7.2 | 0.39| 0.28|
| email_enron| 2.8 | 5.9 | 0.65| 0.66|
| email_euall| 2.2 | 4.1 | 0.39| 0.085|
| email_enron| 2.3 | 3.5 | 0.0019| 0.0042|

5. MORE GENERAL PROPERTIES

In the last section, we show that the internal slope (ISlope) of a community basically determines the structure of the community. In this section, we study more general properties of the communities. In particular, we consider the average distances, average diameters and average clustering coefficients of all the communities in each of the real networks, for which the results are given in Table III.
The distance between two nodes is defined as the number of “hops” in the network one needs to move from one given node to another [Newman 2004a]. Usually people are interested in the average distances of the whole network [Milgram 1967; Newman 2001b; Newman et al. 2001; Travers and Milgram 1969], showing that most real networks have very short average distances. In this section, we consider the average distance between two nodes within a community, which represents the number of “hops” one needs to move from one node to another only through members of the same community.

From Table III we have that, the communities of each network have a small average distance. In particular, the average distance of all the communities of the collaboration network hepth reaches 3.3, which is the largest value of the average distances of all the communities for all the networks studied in this paper. Besides, we also give the average diameter of communities. The average diameter of all the communities for each of the networks is between 3.2 and 7.2. This experiment suggests a conjecture that: there is a three degree separation property of (true) communities for many real networks. The conjecture calls for further investigation, which may provide useful information for understanding both true communities and communities found by various algorithms.

Clustering coefficient (or transitivity) has been a well studied property for networks [Newman 2001a; Newman 2001b; Watts and Strogatz 1998]. It refers to the phenomenon that the existence of ties between nodes $A$ and $B$ and between nodes $B$ and $C$ implies a tie...
between $A$ and $C$. Given a graph $G$, the clustering coefficient of $G$ is defined by:

$$C = \frac{3 \times \text{number of triangles on the graph}}{\text{number of connected triples of vertices}}$$ (5)

From Table I, we observe that most communities of the networks have very large clustering coefficients except for that of the email_euall network, and that most small communities found by our algorithm have larger clustering coefficients than that of the corresponding original graphs.

However, in the collaboration network grqc, the clustering coefficient of the original graph is 0.63, but many small communities we found have smaller clustering coefficients. In fact, communities with clustering coefficients less than 0.6 take up more than 74% of the communities in this network. To explain this phenomenon, we count the triangles in the original graph and its communities respectively. In the original graph, there are 1,350,014 triangles in all. If we divide the communities into two groups, so that the first group consists of the ones having clustering coefficients larger than 0.6, and the second group consists of the rest of the communities, then we discover that communities in the first group have 3,306 triangles on the average, while communities in the second group contain 60 triangles on the average. If we divide communities by clustering coefficient 0.8 as above, then the average numbers of triangles appear in the communities in the first and the second classes are 5,027 and 147 respectively. Therefore the triangles are unevenly distributed in communities with a small number of communities containing most of triangles of the network. The high
clustering coefficients are mainly caused by the small group of communities which contain much larger number of triangles.

From table III, we observe that clustering coefficients of communities vary among different types of networks. Communities in collaboration networks have higher clustering coefficients than that of citation and email networks. In the collaboration networks, two authors having common collaborators are more likely to collaborate with each other in the future. In the citation networks, an author citing a paper, tends to cite the references of the paper, especially when the references are from the same topic. This explains the reason why collaboration networks and citation networks have higher clustering coefficients.

Email networks have different patterns. Communities in email_enron network have average clustering coefficient 0.39, at the same time, the origin graph has clustering coefficient only 0.085. In this case, the communities found by our algorithm largely amplify the clustering coefficients of the network. This means that although the network has a small clustering coefficient, there are also significantly many local structures of the network showing strong cohesion among themselves. However communities in email_euall network has the lowest clustering coefficient (only 0.0019). Both its origin and communities have very small clustering coefficients. In this case, most communities in this network are very similar to star-like graphs which have clustering coefficients near 0. This local structure of the network is very much different from other networks.

Fig. 7. Cumulative distribution of communities’ ESlope
6. CONCLUSIONS
In this paper, we propose a methodology to characterize and analyze the local structures and information of real networks, which includes new notions of internal dominating set, external dominating set, internal slope and external slope of a community, and analysis of the distributions of internal and external slopes, average distances, diameters, and clustering coefficients of all communities for each of the real networks.

We implement experiments of our method on five collaboration networks, two citation networks, two email networks and one benchmark network.

The experiments show that: 1) The notions of internal dominating ratio, external dominating ratio, internal slope and external slope and clustering coefficients are essential characteristics to understand the patterns and information of the communities of a real network. 2) Different networks have different local structures (or patterns). 3) Most communities of a real network have a small internal dominating set and a small external dominating set, although the communities may still very large. 4) The small dominating set of a community keeps much of the information of the community and more importantly the information of a community can be extracted from the internal dominating set of the community. 5) Both internal and external slopes of all the communities of a network approximately follow a normal distribution for most real networks. This means that typical communities of the networks have both ISlopes and ESlopes in some small intervals, so that the communities have similar patterns. 6) The internal slope (ISlope) of a community basically determines the structure of the community. 7) The result that communities have average distances less than or equal to 3 implies a general conjecture that there is a 3 degree separation phenomenon of true communities of most real networks. 8) Normally, communities amplify the clustering coefficients of the corresponding network. 9) If a reasonably good algorithm fails to find communities that amplify clustering coefficients of the network, then the communities explore special structures of the network.

The discoveries above are significant in both understanding the structures of networks, and in practical applications. Most communities in real networks are not regular or star-like graphs, but they usually appear with some central nodes with periphery around forming a core/periphery structure. Such structure favors the evolution of communities. A small set of nodes lead to the formation and evolution of the communities. Our results also indicate that in real communities, a single node could rarely take absolute central position as in star-like graphs, due to the reason that such structures are highly unstable. Our analysis provides some intuitive pictures of the rich communities of a network.

In best of our knowledge, this is the first time we can rigorously analyze the characteristics and patterns, and extract information of the communities of a real network, although there are already a huge number of community detection algorithms in the literature. The significance of the research are three folds: 1) To understand the local structures and connecting patterns of a network. 2) To extract useful information from the communities of a network. 3) To help to judge the community finding algorithms.

Our future project (in progress) is to understand the roles of the small internal and external dominating sets in the formation and evolution of communities, and to understand the mechanisms of the patterns of the communities.

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