Collective multipole-like signatures of entanglement in symmetric $N$-qubit systems

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A cogent theory of collective multipole-like quantum correlations in symmetric multiqubit states is presented by employing $SO(3)$ irreducible spherical tensor representation. An arbitrary bipartite division of this system leads to a family of inequalities to detect entanglement involving averages of these tensors expressed in terms of the total system angular momentum operator. Implications of this theory to the quantum nature of multipole-like correlations of all orders in the Dicke states are deduced. A selected set of examples illustrate these collective tests. Such tests detect entanglement in macroscopic atomic ensembles, where individual atoms are not accessible.

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Correlated macroscopic atomic ensembles offer promising possibilities in low-noise spectroscopy, high precision interferometry, and in the implementation of quantum information protocols. Experimental characterization of entanglement - which is the key ingredient in these applications - has attracted considerable attention. Main difficulty in analyzing such composite systems with large number, $N$, of particles is the corresponding exponential size of the Hilbert space. So, major effort is focused on exploring inseparability status of special classes of quantum states, confined to smaller subspaces of the Hilbert space due to symmetry requirements. For example, a macroscopic atomic ensemble of $N$ two level atoms is a collective system of $N$ spin-$\frac{1}{2}$ systems (qubits), in a $2^N$ dimensional Hilbert space $\mathcal{H} = (c^{2N})^{\otimes N}$. However, when the dynamics of the atomic ensemble is governed by collective operations - which do not address the atoms individually - the atoms within the system are completely symmetric with respect to interchange. This class of permutation symmetric states, labeled by total spin $J = \frac{N}{2}$ (maximal value in the addition of $N$ spin-$\frac{1}{2}$ angular momenta), is restricted to a $N+1$ dimensional subspace $\mathcal{H}_{\text{Sym}} = \text{Sym}(c^{2N})^{\otimes N}$ (here ‘Sym’ denotes symmetrization). The eigen states $|\langle J = N/2, M \rangle; -N/2 \leq M \leq N/2 \rangle$ of total angular momentum operator $\vec{J} = \frac{1}{2} \sum_{i=1}^{N} \sigma_i$, where $\sigma_i$ denotes Pauli spin operator of the $i^{th}$ atom, span the space. A much simpler analysis of inseparability in symmetric atomic ensembles becomes possible in this space. As individual atoms are not accessible in the macroscopic ensemble, only collective measurements are feasible and any test of entanglement requiring individual control of atoms cannot be implemented experimentally. For example, spin squeezing, i.e., reduction of quantum fluctuations in one of the spin component orthogonal to the mean spin direction below the fundamental noise limit $N/4$ is an important collective signature of entanglement in symmetric $N$ qubit systems, a consequence of two-qubit pairwise entanglement. Recently, necessary and sufficient conditions for pairwise entanglement have been formulated in terms of negativity of intergroup covariance matrix. These are equivalent to the generalized spin squeezing inequalities (for two qubit entanglement) involving collective first and second order moments of total angular momentum operator. Further, genuine three particle entanglement in symmetric multiqubit systems is shown to obey inequalities involving bulk observables up to third order in total angular momentum $\vec{J}$.

In this paper, a family of sufficient conditions to detect entanglement of atoms in a macroscopic ensemble, through collective measurements on the system, is derived. These are formulated in terms of covariance matrix condition involving averages of $SO(3)$ irreducible tensor operators $\gamma^K_{\vec{Q}\vec{M}}(N)$ of rank $K = 1, 2, . . . , N$, constructed from the total angular momentum operator $\vec{J}$ and are identified via an arbitrary bipartite split of the symmetric states of $N$ atoms. The physical significance of these operators are that they express $K = 1$, dipole-like; $K = 2$, quadrupole-like correlations etc., among the multiqubits. The advantages of this procedure are mainly two-fold: one, the simplicity in dealing with a large variety of correlations in multiqubit systems, and two, the entanglement conditions expressed in terms of experimentally observable signatures associated with the correlations among irreducible tensor operators. This elegant formalism enlarges the scope of the covariance matrix condition beyond those given in [9, 11]. The significance of this new approach is substantiated through illustrative examples.

The $SO(3)$ spherical tensor operators $\gamma^K_{\vec{Q}\vec{M}}(N)$ are constructed such that their matrix elements in the basis $|\langle N/2, M \rangle; -N/2 \leq M \leq N/2 \rangle$ are given by $\langle N/2, M' | \gamma^K_{\vec{Q}\vec{M}}(N) | N/2, M \rangle = \sqrt{2K+1} C(N/2K N/2; \vec{Q}_{\vec{M}}')$, in terms of the Clebsch Gordan (CG) coefficients. These irreducible tensors are orthogonal, $\text{Tr}(\gamma^K_{\vec{Q}\vec{M}}(N) \gamma^Q_{\vec{Q}'\vec{M}'}(N')) = (N+1) \delta_{K', K} \delta_{Q, Q'}$, and the set $\langle \gamma^K_{\vec{Q}\vec{M}}(N) : K = 0, 1, 2, . . . , N, -K \leq Q \leq K \rangle$ forms a linearly independent basis of operators in the Hilbert space of spin $J = N/2$ states. Thus a useful representation for the density operator of symmetric $N$-qubits is given in terms...
of these operators by

$$\dot{\rho}(N) = \frac{1}{(N+1)} \sum_{K=0}^{N} K N t^K_N(\tau^N_Q(\lambda)) \tau^K_Q(N).$$  (1)

It is completely specified by \((N+1)^2 - 1\) irreducible tensor moments,

$$t^K_N(\lambda) = \text{Tr}[\rho_{\text{sym}}(\tau^N_Q(\lambda))] = (-1)^Q t^K_{Q,N}(\lambda),$$  (2)

with \(t^0_N(\lambda) = 1\) because \(\text{Tr}\dot{\rho} = 1\). An important composition law appropriate for examining multiple correlations between two constituent symmetric parts of this \(N\) particle system, characterized by angular momenta \(j_1 = N_1/2\), and \(j_2 = N_2/2\) (with \(N = N_1 + N_2\), of the ensemble is constructed in terms of direct product of spherical tensors \(\tau^\kappa_q(N_1) \otimes \tau^\kappa_q(N_2)\):

$$\dot{\rho}(N_1, N_2) = \frac{1}{(N_1+1)(N_2+1)} \sum_{K=0}^{N_1} \sum_{Q=0}^{N_2} (\tau^\kappa_q(N_1) \otimes \tau^\kappa_q(N_2)) \tau^K_{Q,N}(N_1, N_2).$$  (3)

where \(\tau^\kappa_q(N_1, N_2) = \text{Tr}[\rho_{\text{sym}}(\tau^N_Q(\kappa)) \otimes \tau^N_Q(\kappa)] = (-1)^{\kappa + q} \tau^\kappa_q(N_1, N_2); \kappa = 0, 1, \ldots, N_1, \kappa' = 0, 1, \ldots, N_2; -\kappa \leq \kappa' \leq \kappa', q \leq \kappa\). Equations (11) and (3) represent the \(N\)-qubit symmetric system in two equivalent ways and thus the tensor parameters appearing therein are related as will be shown presently. These relations form the central core of the theory presented here.

By taking trace over \(N_2\) particles from the composite system (using the representation \(3\)), we obtain the density matrix of \(N_1\) particles. We find that the tensor parameters \(\tau^\kappa_q(N_1)\) characterizing the \(N_1\) subsystem are given by,

$$t^\kappa_q(N_1) = \text{Tr}\left[\tau^\kappa_q(N_1)\right] = t^0_{\kappa q}(N_1, N_2); \kappa = 0, 1, \ldots, N_1.$$  (4)

Similar consideration for \(N_2\)-subsystem leads to \(t^\kappa_q(N_2) = t^0_{\kappa q}(N_1, N_2)\). The second set of relations connecting the tensor parameters of the bipartite system given by \(3\) with those of \(1\), are obtained by using the orthogonality property of the tensor operators:

$$\tau^\kappa_q(N_1, N_2) = \sum_{K, Q} \tau^\kappa_q(N_1) \otimes \tau^\kappa_q(N_2) \tau^K_{Q,N}(N_1, N_2).$$  (5)

where \(|\kappa - \kappa'| \leq K \leq \kappa + \kappa'; -K \leq Q \leq K\), and \(\tau^\kappa_q(N_1, N_2)\) are found to be

$$\tau^\kappa_q(N_1, N_2) = \text{Tr}\left[\tau^\kappa_q(N_1) \otimes \tau^\kappa_q(N_2) \tau^K_{Q,N}(N_1, N_2)\right].$$

Here \(\{\cdot\} = \text{Tr}\{\cdot\}\) denotes the Wigner-9j symbol \([12, 14]\) and \(|\alpha| = \sqrt{2} + 1\). Using the properties of the 9j-symbols and the CG coefficients \([12, 14]\) for the special values \(\kappa = 0, q = 0\) in \(5\) and \(6\), we obtain

$$t^0_{\kappa q}(N_1, N_2) = \mathcal{P}_\kappa(N_1, N_2) t^\kappa_q(N_1),$$  (7)

where \(\mathcal{P}_\kappa(N_1, N_2) = \frac{N_1!}{(N_1+1)!} \frac{N_2!}{(N_2+1)!} \frac{1}{(\kappa + 1)! k!}; \kappa = 0, 1, 2, \ldots, N_1\). By replacing \(N_2 \rightarrow N_2 - N_1\) (where \(N_2 \geq N_1\)) in both sides of \(7\) and using \(1\), we obtain an important equivalent relation

$$t^\kappa_q(N_1, N_2) = \mathcal{P}_\kappa(N_1, N_2 - N_1) t^\kappa_q(N_2).$$  (8)

The product tensor parameters \(t^\kappa_q(N_1, N_2)\) of the bipartite system exhibit a similar relationship with the corresponding coefficients \(t^\kappa_q(\kappa, \kappa')\) of the \(\kappa + \kappa'\) subsystem. This follows by expressing \(t^K_{Q,N}(\lambda)\) of the composite system in the RHS of \(5\) in terms of \(\kappa + \kappa'\) subsystem parameters \(t^K_{Q,N}(\kappa + \kappa')\) (obtained by substituting \(N_1 = \kappa + \kappa', N_2 = N - (\kappa + \kappa')\) in \(5\)). Alternately, we can also relate \(t^\kappa_q(N_1, N_2)\) to \(t^\kappa_q(\kappa + \kappa')\) by choosing \(N_1 = \kappa, N_2 = \kappa'\) in \(5\). Comparing the resulting equations for \(t^\kappa_q(N_1, N_2)\) and \(t^\kappa_q(\kappa, \kappa')\) and using explicit expressions for the associated 9j-symbols \([12]\) we obtain, after some algebraic manipulation,

$$t^\kappa_q(N_1, N_2) = f(N_1, \kappa) f(N_2, \kappa') t^\kappa_q(\kappa, \kappa').$$  (9)

$$f(N, \kappa) = \sqrt{\left(\frac{(N+1)(2N+1)!}{(2\kappa + 1)!(N-\kappa)!}\right)}, \alpha = 1, 2.$$

Equations \(10, 11\) and \(12\) prove to be significant in identifying collective signatures of entanglement in symmetric atomic ensembles, obtained through an analysis of the bipartite representation \(3\).

Consider a set of \(2(2 \kappa + 1)\) operators \(A^{(\kappa)} = \tau^0_q(N_1) \otimes I_{N_2} = B^{(\kappa)} = I_{N_1} \otimes \tau^0_q(N_2)\) where \(I_{N_i}\) is the identity operator). Arranging them as a column \(\xi^{(\kappa)}(\alpha)\) (corresponding row of operators being \(\xi^{(\kappa)}(\alpha) = (A^{(\kappa)}_\alpha, B^{(\kappa)}_\alpha)\)), define the \(2\kappa\)th order covariance matrix for the symmetric system as

$$V^{(2\kappa)} = \frac{1}{2} \left(\Delta \xi^{(\kappa)}, \Delta \xi^{(\kappa)}\right),$$  (10)

where \(\Delta \xi^{(\kappa)} = \xi^{(\kappa)} - \xi^{(\kappa)}\) and \(\Delta \xi^{(\kappa)} = \Delta \xi^{(\kappa)}, \Delta \xi^{(\kappa)} = \Delta \xi^{(\kappa)}\). \(V^{(2\kappa)}\) exhibits a \((2\kappa + 1) \times (2\kappa + 1)\) block matrix form, \(V^{(2\kappa)} = \begin{pmatrix} A^{(2\kappa)}(N_1, N_2) & C^{(2\kappa)}(N_1, N_2) \\ C^{(2\kappa)}(N_1, N_2) & B^{(2\kappa)}(N_2) \end{pmatrix}\).

The diagonal blocks \(A^{(2\kappa)}(N_1, N_2)\) correspond to multipole correlations among the intra-group tensors and the off-diagonal block \(C^{(2\kappa)}(N_1, N_2)\) comprises of inter-group multipole correlations. Explicitly,

$$C^{(2\kappa)}(N_1, N_2) = (-1)^{\kappa'} \left[\xi^{\kappa' - \kappa}(N_1, N_2) - \tau^0_q(N_1, N_2) N^\kappa_{\kappa' q}(N_1, N_2)\right].$$  (11)

Here we focus on the \((2\kappa + 1) \times (2\kappa + 1)\) hermitian cross-correlation matrix \(C^{(2\kappa)}\) and prove the following theorem:

**Theorem:** The cross-correlation matrix \(C^{(2\kappa)}(N_1, N_2)\) of a given rank \(\kappa\) associated with any partition \(N_1, N_2\) of an \(N\)-qubit symmetric system is necessarily positive semidefinite for all separable symmetric \(N\)-qubit bipartite states. The sign of the corresponding matrix \(C^{(2\kappa)}(\kappa, \kappa)\), associated with the \(2\kappa\) atom reduced system with equal partition, suffices to determine that of \(C^{(2\kappa)}(N_1, N_2)\), irrespective of the partitioning.

**Proof:** In the product representation \(3\) with an arbitrary partition \((N_1, N_2)\), a separable symmetric \(N\)-qubit state has the following structure:

$$\hat{\rho}_{\text{sep}} = \sum_{w} p_w \hat{\rho}(N_1) \otimes \hat{\rho}(N_2), \quad 0 \leq p_w \leq 1; \sum_{w} p_w = 1.$$  (12)
where \( \hat{\rho}_w(N_i) \) denotes the density matrices of the subensemble of \( N_i \) qubits and is expressible as (11), in terms of \( \tau_q^w(N_i) \):
\[
\hat{\rho}_w(N_i) = \frac{1}{N_i} \sum_{w} \sum_{q} \tau_q^{w-1}(N_i) \tau_q^{*}(N_i,w).
\]

In a separable symmetric state (12) we have,
\[
\tau_q^v(N_1, N_2) = \text{Tr} \left( \hat{\rho}_{sep} [\tau_q^v(N_2) \otimes \tau_q^v(N_2)] \right)
= \sum_{w} \text{Tr} \left( \hat{\rho}_w(N_1) \tau_q^v(N_1) \right) \text{Tr} \left( \hat{\rho}_w(N_2) \tau_q^v(N_2) \right)
= \sum_{w} \tau_q^v(N_1, w) \tau_q^v(N_2, w).
\]

Without any loss of generality, we may assume that \( N_2 \geq N_1 \). Now, employing (11) we express \( \tau_q^v(N_2, w) = [\mathcal{P}_w(N_1, N_2 - N_1)]^{-1} \tau_q^v(N_1, w) \), which leads to an interesting form for the matrix \( C^{(2e)}(N_1, N_2) \) of (11) in a separable symmetric state:
\[
C^{(2e)}(N_1, N_2) = [\mathcal{P}_w(N_1, N_2 - N_1)]^{-1} C^{(2e)}(N_1, N_2)
\]
(14)

Consider an hermitian quadratic form:
\[
Q^x = \nabla^2 C^{(2e)}(N_1, N_2) \nabla^2 = \sum_{q, q'} C^{(2e)}(N_1, N_2) X_q^{\alpha} X_q'^{\beta},
\]
(15)
with \( X^\alpha \in \mathbb{R}^{(2N+1)} \) being an arbitrary real column vector, whose spherical components are (12), are denoted by \( X_q^\alpha = (-1)^q X_q^{\alpha} \). In a separable symmetric state (12) it is readily seen that \( C_{q, q'}^{(2e)} = [\mathcal{P}_w(N_1, N_2 - N_1)]^{-1} C_{q, q'}^{(2e)} \geq 0 \), with
\[
C_{q, q'}^{(2e)} = X^\alpha C^{(2e)}(N_1, N_2) X^\beta = \sum_{w} \sum_{(q, q')} \sum_{q} X_q^\alpha \tau_q^v(N_1, w) \tau_q^v(N_2, w) X_q'^{\beta},
\]
(16)

This proves the first part of our theorem.

The second part of the theorem follows from (9), leading to the result
\[
C^{(2e)}(N_1, N_2) = f(N_1, \kappa) f(N_2, \kappa) C^{(2e)}(\kappa, \kappa),
\]
(17)
i.e., the covariance matrix \( C^{(2e)}(N_1, N_2) \) of an arbitrary symmetric system is proportional (with an overall positive multiplication factor) to that associated with the equal partitioning of a 2\( \kappa \) qubit reduced system. (Note that the covariance matrix \( C^{(2e)}(N_1, N_2) \) could be related to its reduced system counterpart \( C^{(2e)}(n_1, n_2) \), which is then seen to be proportional to \( C^{(2e)}(n_1, n_1) \); \( n_1 \leq n_2 \). Further, \( C^{(2e)}(n_1, n_1) \) may be related to \( C^{(2e)}(n, n) \); \( n < n_1 \) etc. This can go down the whole up to \( C^{(2e)}(\kappa, \kappa) \). Hence, the positivity (negativity) of \( C^{(2e)}(N_1, N_2) \) has its origin in the 2\( \kappa \)-qubit covariance matrix \( C^{(2e)}(\kappa, \kappa) \), with equal partition (\( \kappa, \kappa \)).

Thus, for any arbitrary (pure or mixed) symmetric ensemble of qubits, \( C^{(2e)}(\kappa, \kappa) < 0 \), for various orders \( \kappa = 1, 2, \ldots, 2\kappa \leq N \), serves as a sufficient condition of entanglement and leads to a family of inseparability conditions associated with the quantum correlations between inter-group tensor operators i.e., \( (\Delta A_i^{(\kappa)} \Delta B_i^{(\kappa)}) \). As the product tensor parameters \( t_q^{v,w}(\kappa, \kappa) \) and \( t_q^{v,0}(\kappa, \kappa) \), \( \Phi_q^{v,0}(\kappa, \kappa) \), specifying the covariance matrix \( C^{(2e)}(\kappa, \kappa) \) - are given in terms of collective tensor moments \( t_q^{v}(N) \), (see (6), (8) and (9)), the negativity of the covariance matrix \( C^{(2e)}(\kappa, \kappa) \) is readily expressed in terms of averages of symmetrized homogeneous 2\( \kappa \)th order polynomials [13] of \( f \), thus leading to a family of multipole-like collective signatures of entanglement.

We now show that the spin squeezing inequality [3, 10] is a consequence of the intergroup dipole correlations (for \( \kappa = 1 \)) viz., \( C^{(2)}(N_1, N_2) < 0 \). In the 3 \( \times \) 3 matrix \( C^{(2)}(N_1, N_2) \) we substitute \( t_q^{v}(N_1, N_2); \kappa, \kappa' = 0, 1 \) in terms of total system collective parameters \( t_q^{v}(N) \) (see (15)) using explicit values [12] for Wigner 9j symbols and CG coefficients. Following this by a unitary transformation corresponding to a change from spherical basis [12] \( e_\mu \), \( \mu = \pm 1, 0 \) to Cartesian basis \( e_i \), \( i = x, y, z \) leads to
\[
U C^{(2)}(N_1, N_2) U^\dagger = A \left[ -\frac{N}{2} I + V + \frac{1}{\sqrt{N}} S S^T \right]
\]
(17)
\[
\text{where } I \text{ denotes the } 3 \times 3 \text{ identity matrix; } V_{\alpha, \beta} = \frac{1}{2} \left( (J_{\alpha} J_\beta + J_\beta J_\alpha) - (J_\alpha J_\beta) ; S_\alpha = (J_\alpha) \text{ and } A = \frac{\sqrt{2}}{2} \right. \sqrt{\left. \frac{N}{2} \right)}.
\]

Hence, \( C^{(2)}(N_1, N_2) < 0 \iff U C^{(2)}(N_1, N_2) U^\dagger < 0 \iff V + \frac{1}{\sqrt{N}} S S^T < 0 \), which is the known spin squeezing inequality [9, 10], deduced here from any bipartite division \((N_1, N_2)\). (The covariance matrix condition of (11) coincides with that given here only for dipole correlations.)

An important application of our theorem concerns the Dicke states [15] \( |\varphi, M; \rangle \). These states provide an excellent set of physically relevant multiatom symmetric states for illustrating other types of multipole correlations. The collective tensor moments for the Dicke states are \( t_q^{v}(N) = |\varphi, M; \rangle \langle \varphi, M; | \) \( (n, n) \) (obtained from (15)) implying that \( C^{(2e)}(N_1, N_2) \) matrix is diagonal (see the definition (11)). Except for \( |\varphi, \pm M; \rangle \), which are product states, the covariance matrix \( C^{(2e)}(\kappa, \kappa) \) of each of the Dicke states is negative, for all orders \( \kappa \), showing the quantum nature of multipole-like correlations of various orders. Recently [16], an experimental scheme to reconstruct the spin-excitation number distribution of the collective spin states (i.e., tomographic reconstruction of the diagonal elements of the density matrix in the Dicke basis) of macroscopic ensembles containing \( \sim 10^{10} \) atoms, with low mean spin excitations, has been proposed. Implementation of such schemes would enable experimental detection of collective quantum multipole-like correlations in macroscopic assembly of entangled atoms.

As an illustration of our method, we consider symmetric mixed states of the form
\[
\hat{\rho}_i = \frac{(1-x)}{(N+1)} I + x |\phi_i; \rangle \langle \phi_i|; \ 0 \leq x \leq 1,
\]
(18)
where \( x \) denotes \( (N+1) \times (N+1) \) unit matrix; the states \( |\phi_i; \rangle \), for \( i = 1, 2, 3 \), are given by \( |\phi_1; \rangle = \left( \frac{\hat{x}}{\sqrt{2}} - \frac{\hat{z}}{\sqrt{2}} \right) \), \( |\phi_2; \rangle = \left( \frac{\hat{z}}{\sqrt{2}} + \frac{\hat{x}}{\sqrt{2}} \right) \) and \( |\phi_3; \rangle = \left( \frac{\hat{x}}{\sqrt{2}} + \frac{\hat{z}}{\sqrt{2}} \right) \). We find the values of \( x \), using the condition \( C^{(2e)}(N_1, N_2) < 0 \) as a function of number of atoms, for which \( \hat{\rho}_0 \) of (18) are inseparable. For the state \( \hat{\rho}_3 \) we find that \( C^{(2e)}(\kappa, \kappa) \) are all positive for \( 2\kappa < N \) and the highest order co-variance matrix \( C^{(N)}(N/2, N/2) < 0 \).
scopic ensembles of symmetric atoms. An important consequence is that the Dicke states exhibit quantum multipole-like correlations of all orders. Moreover, our approach is directly applicable to characterize entanglement in spatially separated bipartite symmetric atomic ensembles, for example, two macroscopic gas samples of cesium atoms [17].

These results are presented in graphical form in Fig. 1. From Fig. 1(a) and 1(b) it is clear that dipole ($\kappa = 1$) quantum correlations lead to $x_{\text{min}} \to 1$ for large $N$ values, implying that the mixed states $\rho_i$; $i = 1,2$ are separable throughout the range $0 \leq x < 1$ in this limit. However, higher order multipole correlations are more effective in revealing that these states are indeed entangled over a larger domain of $x$. The range of inseparability is sensitive to the difference $N_{1} \sim N_0$ of the number of atoms in ground and excited states (which is $N - 1$ in $|\phi_1\rangle$ and zero in $|\phi_2\rangle$). Further, from Fig. 1(c) we find that highest order quantum correlations lead to $x_{\text{min}} \to 0$ in the large $N$ limit, implying that all the three mixed states $\rho_i$, for $i = 1,2,3$, are entangled in the range $0 < x \leq 1$, when $N \to \infty$.

In conclusion, we have shown here that SO(3) irreducible tensor representation provides a powerful method to investigate inseparability in symmetric multiquit systems. A family of sufficient conditions of inseparability to detect multipole-like collective quantum correlations derived here may be useful for experimental characterization of entanglement in macroscopic atomic ensembles. These conditions are generalizations of the ones obtained in[11]. More specifically, instead of the Cartesian tensor product observables of Ref. [11], SO(3) irreducible tensor observables are shown here to characterize any bipartite division $(N_1,N_2)$ of symmetric $N$-qubit systems. The two techniques serve different purposes. The first considers groups of qubits to investigate the intergroup entanglement, whereas the second describes them in terms of spherical tensors involving collective $N$-qubit angular momenta. The latter sheds light on entanglement among dipole-like, etc. multiphoton correlations, which may be physically observable [16] in macroscopic ensembles of symmetric atoms. An important

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