On Communication Compression for Distributed Optimization on Heterogeneous Data

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Abstract
Lossy gradient compression, with either unbiased or biased compressors, has become a key tool to avoid the communication bottleneck in centrally coordinated distributed training of machine learning models. We analyze the performance of two standard and general types of methods: (i) distributed quantized SGD (D-QSGD) with arbitrary unbiased quantizers and (ii) distributed SGD with error-feedback and biased compressors (D-EF-SGD) in the heterogeneous (non-iid) data setting.

Our results indicate that D-EF-SGD is much less affected than D-QSGD by non-iid data, but both methods can suffer a slowdown if data-skewness is high. We propose two alternatives that are not (or much less) affected by heterogeneous data distributions: a new method that is only applicable to strongly convex problems, and we point out a more general approach that is applicable to linear compressors.

1 Introduction
We consider the distributed optimization problem

$$f^* := \min_{x \in \mathbb{R}^d} f(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x),$$

where the objective function $f : \mathbb{R}^d \to \mathbb{R}$ is split among $n$ terms $f_i : \mathbb{R}^d \to \mathbb{R}, i \in [n]$, that are distributed among $n$ nodes. We assume that $f$ is $L$-smooth and that we have access to unbiased gradient oracles with $\sigma^2$-bounded variance for each $f_i, i \in [n]$. We study the heterogeneous setting and allow skewed data distributions on the nodes. We quantify the data-dissimilarity by a parameter $\zeta^2 \geq 0$.

Synchronous parallel SGD and variants thereof (e.g. Duchi et al., 2011; Kingma and Ba, 2015) are among the most popular optimization algorithms in machine- and deep-learning (Bottou, 2010). Because the number of parameters in neural networks can we very large, the time required to share the gradients across workers limits the scalability of deep learning training (Seide et al., 2014; Strom, 2015). To address this bottleneck, lossy gradient compression techniques have been proposed as a solution, for instance (Seide et al., 2014; Alistarh et al., 2017; Wen et al., 2017; Bernstein et al., 2018).

Whilst many empirical works highlighted the importance of data-adaptive compressors (Lin et al., 2018; Alistarh et al., 2018; Wangni et al., 2019; Vogels et al., 2019) that adapt to the local data distribution on the nodes, many theoretical analyses did often not consider the heterogeneous setting so far. In this note, we refine the analyses in (Alistarh et al., 2017; Cordonnier, 2018) and show how two commonly used training schemes are impacted by (potentially) non-iid data distributions on the nodes.

We consider two classes of methods: distributed methods with (i) unbiased gradient compressors (denoted as D-QSGD in the following), with QSGD (Alistarh et al., 2017), Terngrad (Wen et al., 2017) and signSGD (Bernstein et al., 2018) as a few representative members, and distributed methods with (ii) biased

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compressors and error-feedback (denoted by D-EF-SGD), such as proposed in (Seide et al., 2014; Alistarh et al., 2018; Stich et al., 2018).

As our first contribution, we tighten the existing analyses of these two types of methods and provide analyses for general non-convex, convex and strongly-convex problems. Exemplary, for instance for the case of $\mu$-strongly convex functions, we show that these methods converge as

$$
\begin{align*}
\text{D-QSGD type :} & & \hat{O}\left(\frac{\sigma^2 + \zeta^2}{\mu n \delta} + \left(\frac{L}{\mu} + \frac{L}{\delta n}\right) \log \frac{1}{\epsilon}\right) \\
\text{D-EF-SGD type :} & & \hat{O}\left(\frac{\sigma^2}{\mu n} + \frac{(\sigma + \zeta/\sqrt{\delta})\sqrt{L}}{\mu \sqrt{\delta} \epsilon} + \frac{L}{\mu \delta} \log \frac{1}{\epsilon}\right)
\end{align*}
$$

where here $0 < \delta \leq 1$ is a parameter measuring the compression quality ($\delta = 1$ meaning no compression and recovering the standard SGD convergence rates). In the presence of high stochastic noise, $\sigma^2 \gg 1$, D-QSGD methods suffer from a linear slow-down with respect to the the compression quality $\delta$, whereas for D-EF-SGD methods the first term is not affected by $\delta$. This characteristic performance difference has been discussed in prior work (e.g. Stich et al., 2018; Stich and Karimireddy, 2019) and we here show in addition that D-EF-SGD methods are also less sensitive to data-skewness.

In a slightly stronger setting, under the additional assumption that the local functions $f_i$ are smooth and convex, Mishchenko et al. (2019) proposed the DIANA framework that is even less sensitive to data-skewness. In particular, DIANA converges linearly in the special case when $\sigma^2 = 0$, in contrast to the two methods introduced above. Building on their techniques, we propose a new method (D-EF-SGD with bias correction) that converges as

$$
\text{D-EF-SGD with bias correction :} & & \hat{O}\left(\frac{\sigma^2}{\mu n} + \frac{\sigma \sqrt{L}}{\mu \sqrt{\delta} \epsilon} + \left(\frac{1}{\delta^2} + \frac{L}{\mu \delta}\right) \log \frac{1}{\epsilon}\right)
$$

and the rate depends only poly-logarithmically on the data-dissimilarity parameter $\zeta^2$ (hidden in the $\hat{O}(\cdot)$ notation). However, this technique gives only an improvement in the strongly-convex case but not in general on non-convex problems.

We further point out an important observation, that when using linear compressors the convergence rate

$$
\text{D-EF-SGD with linear compressors :} & & \hat{O}\left(\frac{\sigma^2}{\mu n} + \frac{L}{\mu \delta} \log \frac{1}{\epsilon}\right)
$$

can be obtained, which does not depend on the data-skewness. Whilst this approach requires additional restriction on the amenable compressors, it does not require additional assumptions on the regularity of the objective function and works also for the convex and non-convex case.

## 2 Related Work

Communication compression is an established approach to alleviate the communication bottleneck in parallel optimization for deep-learning and a variety of different compressors have been proposed and studied (Seide et al., 2014; Alistarh et al., 2017; Aji and Heafield, 2017; Wen et al., 2017; Zhang et al., 2017; Bernstein et al., 2018; Wangni et al., 2018). It has been demonstrated that the application of these methods is not limited to parallel SGD implementations alone, but can be combined e.g. with variance reduction (Künstner, 2017) or with communication over arbitrary network topologies (Tang et al., 2018). The analyses of D-QSGD in (Alistarh et al., 2017) for the stochastic case ($\sigma^2 > 0$), and in (Khirirat et al., 2018) for the deterministic ($\sigma^2 = 0$) case the most closely related works, which both did not consider the data-dissimilarity parameter in their analysis.

The observed practical successes of error-feedback mechanisms (that compensate compression errors), such as in (Seide et al., 2014), could be theoretically explained in (Stich et al., 2018; Alistarh et al., 2018;...
Karimireddy et al., 2019; Stich and Karimireddy, 2019). Error-feedback mechanism have been successfully applied for different compressors (Ivkin et al., 2019; Vogels et al., 2019) or different settings, such as decentralized (Koloskova et al., 2019; Tang et al., 2019; Koloskova et al., 2020a) or federated learning (Rothchild et al., 2020). The first analyses for the multiple worker case were given in (Alistarh et al., 2018; Cordonnier, 2018) and refined in (Beznosikov et al., 2020) by considering the heterogeneous setting. Our results improve over these prior works as we explain in more detail below.

The DIANA method was proposed by Mishchenko et al. (2019) to address distributed training with communication compression for problems with non-smooth regularizers. In this work we follow closely the analysis presented in (Horváth et al., 2019).

Whilst for centralized parallel SGD the data-dissimilarity between the local objective functions does not affect the performance of SGD (Bottou et al., 2018), it has been observed for instance in federated learning (where methods are allowed to perform several local gradient steps before synchronization) or in decentralized optimization (where methods typically only use imperfect synchronization in each round), data-skewness heavily impacts the performance of most standard training schemes (cf. Hsieh et al., 2020; Karimireddy et al., 2020; Koloskova et al., 2020b; Li et al., 2020; Woodworth et al., 2020).

3 Assumptions

We now list the main assumptions on the optimization problem (1). For simplicity and the ease of presentation, we focus here on the most common standard assumptions, but the analyses could be tightened for many special cases, following techniques developed in other works.

3.1 Regularity assumptions

For all our results, we assume $L$-smoothness of $f$:

$$\|\nabla f(y) - \nabla f(x)\| \leq L\|y - x\|, \quad \forall x, y \in \mathbb{R}^d. \quad (2)$$

For some results we further require that each $f_i$, $i \in [n]$ is $L$-smooth. This assumption could for instance be relaxed by considering different smoothness constants $L_i$, $i \in [n]$.

Sometimes we require $\mu$-strong convexity of $f$ (or just convexity for $\mu = 0$):

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2}\|x - y\|^2, \quad \forall x, y \in \mathbb{R}^d. \quad (3)$$

For some results we require in addition each $f_i$, $i \in [n]$ to be convex. The convexity assumption can for most of the results be relaxed to star-convexity (cf. Stich and Karimireddy, 2019) instead, or by assuming the Polyak-Łojasiewicz condition (cf. Karimi et al., 2016).

3.2 Assumption on noise

We assume that we have access to stochastic gradient oracles $g^i(x): \mathbb{R}^d \to \mathbb{R}^d$ for each component $f_i$, $i \in [n]$. For simplicity we only consider the instructive case of uniformly bounded noise:

$$g^i(x) = \nabla f_i(x) + \xi^i, \quad \mathbb{E}_{\xi^i}\|\xi^i\|^2 = 0_d, \quad \mathbb{E}_{\xi^i}\|\xi^i\|^2 \leq \sigma^2, \quad \forall x \in \mathbb{R}^d, i \in [n]. \quad (4)$$

With techniques introduced in other works, one can for instance extend the analysis to variations when the noise is assumed to scale with the squared norm of the gradient (cf. Bottou et al., 2018; Stich, 2019), or function suboptimality gap (cf. Khale et al., 2020). Under additional structural assumptions, for instance assuming that each $f_i$ is $L$-smooth, or that each stochastic gradient is the gradient of a smooth function $g^i(x) = \nabla F(x, \xi^i)$, additional tightening of the results can be obtained (such as for instance replacing $\sigma^2$ in the rates by a bound on the noise $\sigma^2_\star$ at the optimum $x_\star$ only).
3.3 Gradient Dissimilarity

In this work we consider the heterogeneous data setting and allow the functions \( f_i, i \in [n] \) to be different on each node. We measure dissimilarity by two constants \( \zeta^2 \geq 0, Z \geq 1 \) that bound the variance across the \( n \) nodes:

\[
\frac{1}{n} \sum_{i=1}^{n} \| \nabla f_i(x) \|^2 \leq \zeta^2 + Z^2 \| \nabla f(x) \|^2. \tag{5}
\]

This is similar to the assumption in (Koloskova et al., 2020b). For the special case of \( Z = 1 \) this matches the notions in related works (such as Mishchenko et al., 2019; Vogels et al., 2020), but allowing \( Z \geq 1 \) is slightly more general. Whilst in principle we could also allow \( Z \in [0, 1] \) (at the expense of a larger \( \zeta^2 \)), we note that \( \| \nabla f(x) \|^2 \leq \frac{1}{n} \sum_{i=1}^{n} \| \nabla f_i(x) \|^2 \) and hence only \( Z \geq 1 \) allows for scale-free bounds in general (for instance, imposing \( Z = 0 \) would imply an uniform bound on the gradient norms; an assumption which we do want to avoid here). When assuming smoothness and convexity, it is often natural to measure dissimilarity only at the optimum \( x_* \), denoted by a constant \( \zeta^2 \).

3.4 Compressors

We introduce two notions of compressors that have become popular in the literature. To better distinguish them in this manuscript, we will use slightly different terms and parameters to denote them.

A \( \delta \)-compressor (cf. Stich et al., 2018) is a mapping \( C_\delta : \mathbb{R}^d \to \mathbb{R}^d \), with the property

\[
E_{C_\delta} \| C_\delta(x) - x \|^2 \leq (1 - \delta) \| x \|^2, \quad \forall x \in \mathbb{R}^d. \tag{6}
\]

A \( \omega \)-quantizer (cf. Alistarh et al., 2017; Künstner, 2017), is a mapping \( Q_\omega : \mathbb{R}^d \to \mathbb{R}^d \), with the property

\[
E_{Q_\omega} Q_\omega(x) = x, \quad E_{Q_\omega} \| Q_\omega(x) \|^2 \leq (1 + \omega) \| x \|^2, \quad \forall x \in \mathbb{R}^d. \tag{7}
\]

Any \( \omega \)-quantizer can be rescaled to satisfy (6): \( \frac{1}{1 + \omega} Q_\omega \) is a \( \delta = \frac{1}{1 + \omega} \) compressor.

These notions do not guarantee a ‘compression’ in the classical sense\(^{1}\), but have been proven to be useful abstractions for the theoretical analysis of communication efficient SGD algorithms. Intuitively, we can assume that many compressors used in practice require approximately a \( \delta \)-fraction (or \( \frac{1}{1 + \omega} \) fraction, respectively) less bits compared to sending the full vector \( x \in \mathbb{R}^d \), but this is not a rigorous statement.

An important (and illustrative) class of quantizers (or compressors) are sketching operators. As a guiding example, consider a linear sketch \( S_V : \mathbb{R}^d \to \mathbb{R}^d \) of the form:

\[
S_V(x) := V (V^\top V)^{-1} V^\top x \tag{8}
\]

for a matrix \( V \in \mathbb{R}^{d \times p} \), \( p \geq 1 \). For instance, for \( V = e_i \), a standard unit vector, this recovers random sparsification (when \( e_i \) is chosen uniformly at random) or top-1 sparsification, when the index \( i \) is chosen to match with the element of \( x \) with largest magnitude. Both these operators are \( \delta = \frac{1}{n} \) compressors, and the rescaled \( n \cdot S_{e_i}(x) \) operator is a \( \omega = n - 1 \) quantizer for a random choice of \( e_i \), but not for the (biased) top-1 selection. These sketches can be (approximately) encoded only using \( \mathcal{O}(p \log(d) + B) \) bits at most, where \( B \) denotes the bit length of a floating point number. These statements can be made more rigorous, but are not in the central focus here.

Popular sketching operators are for instance top-\( k \) compressors (Aji and Heafield, 2017; Alistarh et al., 2018; Stich and Karimireddy, 2019), linear sketches (Konečný et al., 2016), count-sketches (Ivkin et al., 2019; Rothchild et al., 2020) and low-rank projections (Vogels et al., 2019).

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\(^{1}\)For instance, \( x \mapsto (1 - \delta)x \) is a \( \delta \)-compressor, and \( x \mapsto \{ (1 - \sqrt{\omega})x, (1 + \sqrt{\omega})x \} \) with equal probability is a \( \omega \)-quantizer.
4 Distributed QSGD and Distributed EF-SGD

In this section we derive new and improved convergence rates for the baseline algorithms D-QSGD and D-EF-SGD, tightening prior results in the literature. For instance, the analysis of D-QSGD in (Alistarh et al., 2017) assumed a uniform bound on the gradient norms, \( E\|g_i\|^2 \leq G^2, \forall i \in [n] \). This assumption can hide effects of non-iid data distributions across the nodes (as \( \|\nabla f_i(x)\|^2 \leq G^2 \) is bounded). With our more general assumptions we are able to disentangle the two effects of the stochastic noise and the data-dissimilarity.

All results, also for the following sections, are listed in Table 1 for reference. In the main body of the text we only list the results for strongly convex functions for conciseness (and we do neither optimize nor compare constants in all the rates). All proofs can be found in the appendix.

4.1 D-QSGD, Algorithm 1

Whilst variations of quantized SGD have been discussed in many early works or for special cases, a thorough theoretical discussion was provided in (Alistarh et al., 2017), which popularized quantized SGD methods for efficient optimization in machine learning. Whilst their analysis required a uniform bound on the gradients, \( E\|g_i\|^2 \leq G^2 \), we do not require this assumption here. Khirirat et al. (2018) only study the case when \( \sigma^2 = 0 \) for a subset of loss functions that we consider here.

**Theorem 1** (D-QSGD). Let \( f: \mathbb{R}^d \to \mathbb{R} \) be \( \mu \)-strongly convex and \( L \)-smooth. Then there exists a stepsize \( \gamma \leq \frac{1}{\mu L(1+2Z/\delta n)} \) such that after at most

\[
    T = \mathcal{O}\left( \frac{\sigma^2(1+\omega) + \zeta^2 \omega}{\mu \epsilon} + \frac{L(1+Z^2\omega/n)}{\mu} \right)
\]

iterations of Algorithm 1 it holds \( E f(x_{\text{out}}) - f^* \leq \epsilon \), where \( x_{\text{out}} = x_T \) denotes an iterate \( x_T \in \{x_0, \ldots, x_{T-1}\} \), chosen at random with probability proportional to \( (1 - \mu \gamma)^{-t} \).

**Remark 2.** We here state all convergence results for \( x_{\text{out}} \) chosen to be a random iterate. For convex functions this also implies convergence in function value of a weighted average of the iterates.

**Remark 3.** Assuming \( \sigma^2 \leq G^2, \zeta^2 \leq G^2 \) for a constant \( G^2 \), we recover the \( \mathcal{O}\left( \frac{\sigma^2(1+\omega)}{\mu \epsilon} \right) \) leading term derived in (Alistarh et al., 2017).

4.2 D-EF-SGD, Algorithm 2

Next, we consider distributed SGD with error-feedback. Whilst the first analysis was presented in (Stich et al., 2018) only for the case \( n = 1 \) and extended to \( n > 1 \) in (Cordonnier, 2018), both works assumed a uniform bound on the gradient norms. This assumption was revoked later in (Stich and Karimireddy, 2019) for \( n = 1 \) and in (Beznosikov et al., 2020) for \( n > 1 \). Our analysis improves over (Beznosikov et al., 2020, Theorem 15) in various aspects, for instance their result shows a dependence on \( \mathcal{O}\left( \frac{\sigma^2(1+\omega)}{\mu \epsilon} + \frac{\zeta^2}{\mu \delta \epsilon} \right) \) under the additional assumption that each \( f_i \) is smooth and strongly convex, whilst we improve the respective terms to \( \mathcal{O}\left( \frac{\sigma^2}{\mu \epsilon} + \frac{\zeta^2}{\mu \delta \epsilon} \right) \) here under weaker assumptions, i.e. showing a linear speedup in \( n \) for the leading term and a weaker dependency on \( \zeta^2 \) (though \( \zeta^2 \leq \zeta^2 \)).

**Theorem 4** (D-EF-SGD). Let \( f: \mathbb{R}^d \to \mathbb{R} \) be \( \mu \)-strongly convex and \( L \)-smooth. Then there exists a stepsize \( \gamma \leq \frac{1}{14L(1+Z/\delta)} \) such that after at most

\[
    \mathcal{O}\left( \frac{\sigma^2}{\mu \epsilon} + \left( \frac{L(\sigma^2 + \zeta^2 / \delta)}{\mu \delta \epsilon} \right)^{1/2} + \frac{L(1+Z/\delta)}{\mu} \right)
\]

iterations of Algorithm 2 it holds \( E f(x_{\text{out}}) - f^* \leq \epsilon \), where \( x_{\text{out}} = x_T \) denotes an iterate \( x_T \in \{x_0, \ldots, x_{T-1}\} \), chosen at random with probability proportional to \( (1 - \min\{\frac{\sigma^2}{\mu \epsilon}, \frac{\zeta^2}{\mu \delta \epsilon}\})^{-t} \).
Table 1: Summary of the convergence results for $L$-smooth functions (with additional assumptions per column). $R_0^2 \geq \|x_0 - x_*\|^2$, $F_0 \geq f(x_0) - f^*$. 

| Algorithm | compressor | $\mu$-strongly convex$^a$ | convex$^b$ |
|-----------|------------|-----------------|-----------|
| D-QSGD    | $Q_\omega$ | $O\left(\frac{\sigma^2(1+\omega) + 1 + \frac{L^2(1+\omega)}{\mu n}}{\mu n} \right) \cdot R_0^2 \cdot L F_0$ | $O\left(\frac{\sigma^2(1+\omega) + 1 + \frac{L^2(1+\omega)}{\mu n}}{\mu n} \right) \cdot R_0^2 \cdot L F_0$ |
| D-EF-SGD  | $c_i$      | $O\left(\frac{\sigma^2(1+\omega) + 1 + \frac{L^2(1+\omega)}{\mu n}}{\mu n} \right) \cdot R_0^2 \cdot L F_0$ | $O\left(\frac{\sigma^2(1+\omega) + 1 + \frac{L^2(1+\omega)}{\mu n}}{\mu n} \right) \cdot R_0^2 \cdot L F_0$ |
| DIANA$^d$ | $Q_\omega$ | $O\left(\frac{\sigma^2(1+\omega) + 1 + \frac{L^2(1+\omega)}{\mu n}}{\mu n} \right) \cdot R_0^2 \cdot L F_0$ | $O\left(\frac{\sigma^2(1+\omega) + 1 + \frac{L^2(1+\omega)}{\mu n}}{\mu n} \right) \cdot R_0^2 \cdot L F_0$ |
| D-EF b-cor$^{d,e}$ | $c_i$, $Q_\omega$ | $O\left(\frac{\sigma^2(1+\omega) + 1 + \frac{L^2(1+\omega)}{\mu n}}{\mu n} \right) \cdot R_0^2 \cdot L F_0$ | $O\left(\frac{\sigma^2(1+\omega) + 1 + \frac{L^2(1+\omega)}{\mu n}}{\mu n} \right) \cdot R_0^2 \cdot L F_0$ |
| D-QSGD    | $Q_\omega$ linear | $O\left(\frac{\sigma^2(1+\omega) + 1 + \frac{L^2(1+\omega)}{\mu n}}{\mu n} \right) \cdot R_0^2 \cdot L F_0$ | $O\left(\frac{\sigma^2(1+\omega) + 1 + \frac{L^2(1+\omega)}{\mu n}}{\mu n} \right) \cdot R_0^2 \cdot L F_0$ |
| D-EF-SGD  | $c_i$ linear | $O\left(\frac{\sigma^2(1+\omega) + 1 + \frac{L^2(1+\omega)}{\mu n}}{\mu n} \right) \cdot R_0^2 \cdot L F_0$ | $O\left(\frac{\sigma^2(1+\omega) + 1 + \frac{L^2(1+\omega)}{\mu n}}{\mu n} \right) \cdot R_0^2 \cdot L F_0$ |

$^a$ Convergence $E[f(x_{out})] - f^* \leq \epsilon$, where $x_{out}$ is a random iterate, chosen with exponentially increasing probability in $t$.

$^b$ Convergence $E[f(x_{out})] - f^* \leq \epsilon$, where $x_{out}$ is a uniformly at random chosen iterate.

$^c$ Convergence $E[\|\nabla f(x_{out})\|^2] \leq \epsilon$, where $x_{out}$ is a uniformly at random chosen iterate.

$^d$ Require each $f_i$ to be $L$-smooth. For the first two columns require each $f_i$ to be convex. $\zeta^2 \geq \zeta^2 \ := \frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(x_i)\|^2$.

$^e$ For the choice $\beta = \delta$.

4.3 Discussion

With stochastic noise. In the presence of stochastic noise $\sigma^2 > 0$, the first term is dominating in the rates when $\epsilon \to 0$. Due to mini-batching, this term decreases linearly in $n$ for both methods. We see that D-QSGD without error-feedback suffers from a linear slow-down in $(1 + \omega)$, $O\left(\frac{\sigma^2(1+\omega)}{\mu n} \right)$, whereas in D-EF-SGD the term $O\left(\frac{\sigma^2(1+\omega)}{\mu n} \right)$ is not affected by $\delta$. These characteristic effects and benefits of error compensation have been discussed in many prior works (cf. Stich et al., 2018; Karimireddy et al., 2019).

Without stochastic noise. For the special case when $\sigma^2 = 0$, we observe that both D-QSGD and D-EF-SGD only converge sublinearly, at rates $O\left(\frac{\sigma^2}{\mu n} \right)$ and $O\left(\frac{\sigma^2}{\mu n} \right)$, respectively. Despite that the parameter $\zeta$ can be zero for many applications (for instance for overparametrized optimization problems), these results show that data-dissimilarity impose additional challenges to optimization schemes with communication compression.

Qualitatively, the effects of the data-dissimilarity parameter on the convergence rate is similar as for local update methods that perform several local steps between communication rounds (Koloskova et al., 2020b). This might just be a consequence of the (similar) proof techniques but might hint to an intrinsic limitation of the two approaches discussed in this section.

5 Bias Correction for Improving Data-Depencence

In this section, we discuss a technique proposed in (Mishchenko et al., 2019) that allows to improve the algorithms dependence on the data-dissimilarity parameter for strongly convex problems. However, this technique requires slightly stronger assumptions, such as smoothness of each $f_i$, $i \in [n]$ and convexity.

5.1 DIANA, Algorithm 3

Mishchenko et al. (2019) introduced DIANA, an alternative to D-QSGD that allows to solve constrained optimization problems with quantized communication. Whilst this is one key application of DIANA, we focus here on the benefits this method can offer for unconstrained optimization with communication compression.

A key mechanism in DIANA (Algorithm 3) is that it maintains a sequence of auxiliary variables $h_i^\omega$ on each node $i \in [n]$, with the property $h_i^\omega \rightarrow \nabla f_i(x_\ast)$ when $x_\ast \rightarrow x_\ast$. These variables can be used to design compression operators with smaller variance: instead of compressing $Q_\omega(g_i^\ast)$ as D-QSGD, DIANA uses the quantizer $h_i^\omega + Q_\omega(g_i^\ast - h_i^\omega)$ instead in each round. This is still an unbiased quantizer, but the variance can
The data-dissimilarity

Remark 6. The data-dissimilarity $\zeta^2$ appears only in poly-logarithmic factors in the convergence rate (11) and is thus hidden in the $\tilde{O}()$ notation.

Theorem 5 (DIANA). Let $f : \mathbb{R}^d \to \mathbb{R}$ be $\mu$-strongly convex and $L$-smooth and assume in addition that each $f_i : \mathbb{R}^d \to \mathbb{R}$ is $L$-smooth and convex. Then there exists a stepsize $\gamma \leq \frac{1}{\mu(1+\omega/n)}$ such that for $\alpha = \frac{1}{1+\omega}$, after at most

$$T = \tilde{O}\left(\frac{\sigma^2(1+\omega)}{\mu \epsilon} + \omega + \frac{L(1+\omega/n)}{\mu}\right)$$

iterations of Algorithm 3 it holds $\mathbb{E}f(x_{out}) - f^* \leq \epsilon$, where $x_{out} = x_t$ denotes an iterate $x_t \in \{x_0, \ldots, x_{T-1}\}$, chosen at random with probability proportional to $\left(1 - \min\left\{\mu_{\gamma}, \frac{a}{\mu}\right\}\right)^{-t}$. 

Figure 1: Algorithms with broadcast. All algorithms require coordination with a central parameter server, that broadcasts the updated parameter $x_t$ to the working nodes in each iteration. Quantization operators $Q_\omega$ are assumed to be independent of $t$ and $i$.
5.2 D-EF-SGD with bias correction, Algorithm 4

Whilst DIANA is much less affected by non-iid data than D-QSGD, it still suffers from the linear slow-down in \((1 + \omega)\) in the presence of stochastic noise. In this section we show that by applying error-feedback we can obtain a new algorithm with the optimal \(O(\frac{1}{\mu n})\) dependence on \(\sigma^2\).

D-EF-SGD (Algorithm 2) maintains local error correction terms \(e_i^t\) on each node \(i \in [n]\), however, \(e_i^t \neq 0_d\) in general, even when \(x_i \rightarrow x_*\). This causes the appearance of the \(\zeta\) term in the rate. We adapt a similar bias correction mechanism as in DIANA and propose to compress \(C_\delta(e_i^t + g_i^t - h_i^t)\) instead, where \(h_i^t\) should converge to \(\nabla f_i(x_*)\) and can for instance be update as proposed in DIANA. The resulting scheme is stated in Algorithm 4.

**Theorem 7** (D-EF-SGD with bias correction). Let \(f: \mathbb{R}^d \rightarrow \mathbb{R}\) be \(\mu\)-strongly convex and \(L\)-smooth and assume in addition that each \(f_i: \mathbb{R}^d \rightarrow \mathbb{R}\) is \(L\)-smooth and convex and \(\beta \leq 1\). Then there exists a stepsize \(\gamma \leq \frac{1}{\beta L}\) such that after at most

\[
T = O\left(\frac{\sigma^2}{\mu n} + \left(\frac{a^2 L(1 - \delta)}{\mu^2 \delta \epsilon}\right)^{1/2} + \left(\frac{a^2 \beta L(1 - \delta)}{\mu^2 \delta^2 \epsilon}\right)^{1/2} + \frac{1 + \omega}{\beta} + \frac{L}{\mu \delta}\right)
\]

iterations of Algorithm 4 it holds \(E f(x_{\text{out}}) - f^* \leq \epsilon\), where \(x_{\text{out}} = x_t\) denotes an iterate \(x_t \in \{x_0, \ldots, x_{T-1}\}\), chosen at random with probability proportional to \((1 - \min\{\frac{\mu}{2, \frac{\mu}{2}, \frac{\mu}{2}\})^{-1}i\).

**Remark 8.** When \(\sigma^2 > 0\) or when \((1 + \omega) \leq \frac{L}{\mu}\), then the choice \(\beta = \delta\) gives asymptotically the best complexity. When \(\sigma^2 = 0\), choosing \(\beta = 1\) gives the best linear convergence. In Table 1 we list the result for the choice \(\beta = \delta\), as we mostly focus on noisy stochastic problems in our discussion.

5.3 Discussion

**Linear convergence without stochastic noise.** Without stochastic noise (\(\sigma^2 = 0\)), both algorithms presented in this section converge linearly on strongly-convex problems. For comparable choices of \(\delta \approx \frac{1}{L}\), and \(Z = 1\), the linear convergence rate of DIANA is better as the method can benefit from mini-batching effects. The speedup in \(n\) in the \(O\left(\frac{L(1 + \omega/n)}{\beta}\right)\) term stems from the fact that the quantization operators are independent on each node. In contrast, biased compressors cannot benefit from such effects and the \(O\left(\frac{1}{\mu n}\right)\) term has the best possible dependence on the compression parameter \(\delta\) that cannot be improved in general (cf. Stich and Karimireddy, 2019). Both algorithms depend linearly on the condition number \(\frac{1}{n}\). This dependence could be improved with acceleration techniques (cf. Lin et al., 2015).

**Dependence on data-dissimilarity.** Whilst the convergence results on strongly-convex functions show that both DIANA and bias corrected D-EF-SGD only depend polylogarithmic on the data-dissimilarity parameter \(\zeta\), a closer inspection of the results in Table 1 reveals that unfortunately both methods still depend on \(\zeta\) without the convexity assumptions.

We conjecture that some partial improvements can obtained for non-convex problems, for instance by extending the analysis to non-convex problems with additional PL condition. However, the current results seem to indicate that a fundamental different technique is required to remove the dependence on the data-dissimilarity parameter \(\zeta\) from the convergence rates entirely.

5.4 Convergence Proof for Bias Corrected D-EF-SGD

Our proposed algorithm is a straightforward combination of D-EF-SGD with a feature of DIANA, and a convergence proof can be derived from techniques and tools developed in earlier work (Stich and Karimireddy, 2019; Horváth et al., 2019). As a technical novelty, we here present a novel proof technique for general error-feedback SGD algorithms by introducing a Lyapunov function instead of the unrolling technique used
in (Stich and Karimireddy, 2019). Moreover, we also need a slight strengthening of one of the lemmas in (Horváth et al., 2019) to show that the choice $\beta < 1$ gives an improvement in the convergence rate.

We give the convergence proof for the strongly convex case in the main text, all other proofs are given in the appendix. Define $X_t := \mathbb{E}||\tilde{x}_t - x^*||^2$, $F_t := \mathbb{E} f(x_t) - f^*$. $E_t := \frac{1}{n} \sum_{i \in [n]} \mathbb{E} ||e_i^t||^2$, and $H_t := \frac{1}{n} \sum_{i \in [n]} \mathbb{E} ||h_i^t - \hat{h}_i^t||^2$ for $h_i^t := \nabla f_i(x_t), h^* := \nabla f(x^*)$ and the virtual sequence

$$\tilde{x}_0 := x_0, \quad \tilde{x}_{t+1} := \tilde{x}_t - \frac{\gamma}{n} \sum_{i \in [n]} g_i^t. \quad (12)$$

We note:

$$x_{t+1} - \tilde{x}_{t+1} = \frac{\gamma}{n} \sum_{i \in [n]} e_i^t. \quad (13)$$

A decent lemma for convex functions. First, we borrow a standard lemma for the analysis of error-feedback algorithms (for the proof see also Lemma 18 in the appendix).

Lemma 9 (Stich and Karimireddy (2019, Lemma 7)). Let $f$ be $L$-smooth and $\mu$-convex. If the stepsize $\gamma \leq \frac{1}{4L}$, then it holds for the iterates of Algorithms 2 and 4:

$$X_{t+1} \leq \left(1 - \frac{\gamma \mu}{2}\right) X_t - \frac{\gamma}{2} F_t + \frac{\gamma^2 \sigma^2}{n} + 3L\gamma^4 E_t. \quad (14)$$

Bound on the error. Next, we derive a recursive bound on $E_t$.

Lemma 10. It holds

$$E_{t+1} \leq \left(1 - \frac{\delta}{2}\right) E_t + \frac{4(1-\delta)}{\delta} (2LF_t + H_t) + (1-\delta)\sigma^2. \quad (15)$$

Proof. By using the definition $e_i^t + g_i^t - h_i^t - \hat{h}_i^t$, we obtain:

$$\mathbb{E}_{\xi^t; c_i} ||e_i^t||^2 = \mathbb{E}_{\xi^t; c_i} ||e_i^t + g_i^t - h_i^t - \hat{h}_i^t||^2$$

$$\leq (1-\delta) \mathbb{E}_{\xi^t} ||e_i^t + g_i^t - h_i^t||^2$$

$$\overset{(6)}{=} (1-\delta) \mathbb{E}_{\xi^t} ||e_i^t + \nabla f_i(x_t) + \xi_i^t - h_i^t||^2$$

$$\overset{(20)}{=} (1-\delta) ||e_i^t + \nabla f_i(x_t) - h_i^t||^2 + (1-\delta) \mathbb{E}_{\xi^t} ||\xi_i^t||^2$$

$$\overset{(4)}{\leq} (1-\delta) ||e_i^t + \nabla f_i(x_t) - h_i^t||^2 + (1-\delta) \sigma^2$$

$$\overset{(24)}{\leq} (1-\delta/2)||e_i^t||^2 + \frac{2(1-\delta)}{\delta} ||\nabla f_i(x_t) - h_i^t||^2 + (1-\delta) \sigma^2. \quad (16)$$

Using smoothness (and convexity) of $f_i(x)$, we observe

$$||\nabla f_i(x_t) - h_i^t||^2 \overset{(23)}{\leq} 2||\nabla f_i(x_t) - h_i^t||^2 + 2||h_i^t - h_i^t||^2$$

$$\overset{(25)}{\leq} 4L(f_i(x_t) - f_i(x^*) + \langle \nabla f_i(x^*), x_t - x^* \rangle) + 2||h_i^t - h_i^t||^2$$

The claim now follows by summing and averaging over $i \in [n]$. □
Estimate $H$. The next lemma tightens (Horváth et al., 2019, Lemma 2) (with $\alpha^2$ instead of only $\alpha$ in the last term).

**Lemma 11.** Let $h^i_t$ be updated with an unbiased quantizer $Q_\omega$, $\alpha \leq \frac{1}{\sqrt{\omega}+\tau}$, and stepsize $\alpha$. Then

$$H_{t+1} \leq (1-\alpha)H_t + 2\alpha LF_t + \alpha^2(1+\omega)\sigma^2. \quad (17)$$

**Proof.** Closely following (Horváth et al., 2019) we observe

$$
\mathbb{E}_{\xi_t, Q_\omega} \| h^i_{t+1} - h^i_t \|^2 = \| h^i_t - h^i_t \|^2 + 2\alpha \left\langle \mathbb{E}_{\xi_t, Q_\omega} \Delta t, h^i_t - h_t^i \right\rangle + \alpha^2 \mathbb{E}_{\xi_t, Q_\omega} \| \Delta t \|^2
\leq \| h^i_t - h^i_t \|^2 + 2\alpha \left\langle \nabla f_i(x_t) - h^i_t, h^i_t - h^i_t \right\rangle + \alpha^2(1+\omega)\mathbb{E}_{\xi_t} \| g^i_t - h^i_t \|^2
\quad + \alpha^2(1+\omega) \left( \| \nabla f_i(x_t) - h^i_t \|^2 + \mathbb{E}_{\xi_t} \| \xi_t \|^2 \right)
\leq \| h^i_t - h^i_t \|^2 + 2\alpha \left\langle \nabla f_i(x_t) - h^i_t, h^i_t - h^i_t \right\rangle + \alpha \| \nabla f_i(x_t) - h^i_t \|^2 + \alpha^2(1+\omega)\sigma^2
= (1-\alpha) \| h^i_t - h^i_t \|^2 + \alpha \| \nabla f_i(x_t) - h^i_t \|^2 + \alpha^2(1+\omega)\sigma^2 \quad (18)
$$

where we used the equality $2 \langle a, b \rangle + \| b \|^2 = \| a + b \|^2 - \| a \|^2$ for vectors $a, b \in \mathbb{R}^d$ for the last estimate.

The claim follows with (26).

We can summarize the statements of Lemmas 9–11 in the following descent lemma.

**Lemma 12** (Lyapunov function). Let $f$ be $L$-smooth, $\mu$-convex and each $f_i : \mathbb{R}^d \to \mathbb{R}$ convex and $L$-smooth, the stepsize $\gamma \leq \frac{\mu}{4L}$ and $\alpha = \frac{\beta}{1+\omega}$ with a parameter $\beta \leq 1$. Then

$$\Psi_{t+1} \leq (1-c)\Psi_t - \frac{F_t}{4} + \gamma^2 \frac{\sigma^2}{n} + \frac{L(1-\delta)\sigma^2}{\delta} \quad (19)$$

for $\Psi_t := X_t + aE_t + bH_t$ with $a = \frac{12\gamma^2 L}{3\delta}$ and $b = \frac{8a(1-\delta)}{\alpha\mu}$ and $c = \min\{\frac{1}{2}, \frac{\gamma^2}{\mu}, \frac{\gamma}{L} \}$.

**Proof.** Observe that it holds $\left(1 - \frac{\gamma}{2} + \frac{3\gamma^2 L}{n}\right) \leq (1 - \frac{\gamma}{4})$ and $\left(1 - \alpha + \frac{4\alpha(1-\delta)}{b\delta}\right) \leq (1 - \frac{\gamma}{4})$ by the choice of $a, b$. Therefore

$$
\Psi_{t+1} = X_{t+1} + aE_{t+1} + bH_{t+1}
\leq \left(1 - \frac{\gamma}{2}\right) X_t + a \left(1 - \frac{\gamma}{2} + \frac{3\gamma^2 L}{n}\right) E_t + b \left(1 - \alpha + \frac{4\alpha(1-\delta)}{b\delta}\right) H_t
\quad + \left(\frac{8a(1-\delta) L}{\delta} + 2abL - \frac{\gamma}{2}\right) F_t + \left(\frac{\gamma^2}{n} + a(1-\delta) + \alpha^2 b(1+\omega)\right) \sigma^2$$

$$\leq (1-c)\Psi_t - \frac{F_t}{4} + \gamma^2 \frac{\sigma^2}{n} + \frac{12L(1-\delta)\sigma^2}{\delta} + \frac{96L(1-\delta)\sigma^2}{\delta^2} \quad (14), (15), (17)
$$

where we used the choice of the parameters. For the $F_t$ terms:

$$\frac{8a(1-\delta) L}{\delta} + 2abL - \frac{\gamma}{2} = \frac{96\gamma^3 L^2 (1-\delta)}{\delta^2} + \frac{192L^2 \gamma^3 (1-\delta)}{\delta^2} - \frac{\gamma}{2} \leq -\frac{\gamma}{4},$$

for $\gamma \leq \frac{\delta}{34L}, \alpha \leq \frac{1}{1+\omega}$. And for the $\sigma^2$ term, with $\alpha = \frac{\delta}{1+\omega}$,

$$\frac{\gamma^2}{n} + a(1-\delta) + \alpha^2 b(1+\omega) = \frac{\gamma^2}{n} + \frac{12\gamma^3 L (1-\delta)}{\delta} + \frac{96\gamma^3 L (1-\delta)}{\delta^2} \quad (20)$$

**Proof of Theorem 7.** Lemma 12, together with Lemma 25 and Remark 26 show the claim.
6 Avoiding Data-Dependent Rates with Linear Compressors

Before concluding this note, we like to remark that with a very simple modification the data dependent parameter $\zeta^2$ can entirely be removed from the convergence rates in D-QSGD and D-EF-SGD. This is possible while leaving the algorithms unchanged, but instead we propose to restrict the class of admissible quantization (or compression) operators.

The main component in the convergence proofs was to estimate the variance (for D-QSGD) and the bound on the memory $E_t$ (for D-EF-SGD). For certain classes of compressors these bounds can significantly be improved. As one example, we here highlight linear compressors for which it holds $E_{Q_\omega} \bigl\| \frac{1}{n} \sum_{i=1}^n Q_\omega(g^i_t) \bigr\|^2 \leq (1 + \omega) \frac{1}{n} \sum_{i=1}^n \| g^i_t \|^2$. With this property it is immediate to see that the proof of D-QSGD boils down to the $n = 1$ worker case, and the data-dependent terms disappear (similarly for error-feedback algorithms with compressors).

For example, consider linear sketching operators $S_{\omega_t}$, defined in (8), with a sketching matrix $V_t$ that can change over iterations $t$, but is identical on all $n$ nodes at every $t$. Then it holds

$$\left\| \frac{1}{n} \sum_{i \in [n]} S_{\omega_t}(g^i_t - e^i_t) \right\|^2 = \left\| S_{\omega_t}(g_t - e_t) - e_t \right\|^2 \leq (1 - \delta) \| g_t - e_t \|^2,$$

where here $e_t := \frac{1}{n} \sum_{i \in [n]} e^i_t$. An analogous observation holds for rescaled (unbiased) sketching operators. We summarize the consequences of this observation in Table 1.

The benefits given by linear compressors have been exploited in some recent works (such as Rothchild et al., 2020; Vogels et al., 2020). We believe—given the benefits of the much improved convergence rates and possibility to use efficient all-reduce implementations—the small overhead of synchronizing the compressors can be beneficial in many practical settings, especially for distributed optimization in data-centers. For instance, (pseudo-)random projections can be implemented with the help of a shared random seed without overhead, and certain data-adaptive protocols can also be implemented without a central coordinator. However, for optimization in federated learning scenarios, where communication is extremely limited and all-reduce not available, or when the data distribution is very different on each node, then the optimal trade-off between linear and locally adaptive compressors still remains to be studied in detail.
7 Conclusion

In this work we derive new and improved converge rates for D-QSGD and D-EF-SGD. Our derivations reveal that both methods can suffer a slow-down in the case of heavily skewed data-distributions on the nodes. Whilst this slow-down can be linear in the data-dissimilarity parameter for D-QSGD, it is much less severe for D-EF-SGD, where the data-distribution does not impact the asymptotically dominating terms in the convergence rate. We further present a new bias corrected variant of D-EF-SGD that is even more mildly affected by data-skewness on strongly convex problems (similar to DIANA, while maintaining the optimal stochastic terms as in vanilla D-EF-SGD). Furthermore, we point out that when using linear compressors, this slow-down can entirely be avoided for all considered classes of smooth optimization problems. Whilst this small fix might be an interesting avenue for practical applications, it remains an open theoretical problem data-dependence of the convergence rates can be achieved for general compressors and problem classes.

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A Technical Lemmas

We list a few technical lemmas that are helpful in the proofs below:

- For a random variable $X$:
  $$E\|X - E X\|^2 = E\|X\|^2 - \|E X\|^2$$  \hspace{1cm} (20)

- For pairwise independent random variables $X_1, \ldots, X_k$:
  $$E\left\| \sum_{i \in [k]} X_i - E X_i \right\|^2 = \sum_{i \in [k]} E\|X_i - E X_i\|^2$$  \hspace{1cm} (21)

- In contrast, for any arbitrary $k$ vectors $a_1, \ldots, a_k \in \mathbb{R}^d$:
  $$\left\| \sum_{i \in [k]} a_i \right\|^2 \leq k \sum_{i \in [k]} \|a_i\|^2$$  \hspace{1cm} (22)

- For any vectors $a, b \in \mathbb{R}^d$ and $\eta > 0$:
  $$\|a + b\|^2 \leq (1 + \eta)\|a\|^2 + (1 + 1/\eta)\|b\|^2$$  \hspace{1cm} (23)

- As a consequence, we will be often using the inequality
  $$(1 - \delta)\|a + b\|^2 \leq (1 - \delta/2)\|a\|^2 + \frac{2(1 - \delta)}{\delta}\|b\|^2$$  \hspace{1cm} (24)

  for $\delta \in (0, 1]$. This follows from (23) with $\eta = \frac{\delta}{2(1 - \delta)}$.

- For $L$-smooth and convex functions we have the inequality:
  $$\frac{1}{2L}\|\nabla f(x) - \nabla f(y)\|^2 \leq f(y) - f(x) + \langle \nabla f(x), y - x \rangle , \quad \forall x, y \in \mathbb{R}^d$$  \hspace{1cm} (25)

- It is also useful to note: for convex and smooth $f$, with $h_i := \nabla f(x_*)$, $h_* = \nabla f(x_*)$:
  $$\frac{1}{n} \sum_{i \in [n]} E_{\xi_i} \|g_i - h_i\|^2 \overset{(4)}{\leq} \frac{1}{n} \sum_{i \in [n]} \|\nabla f_i(x_i) - h_i\|^2 + \sigma^2$$

  $$\overset{(25)}{\leq} 2L \sum_{i \in [n]} \left( f_i(x_i) - f_i(x^*) + \langle \nabla f_i(x^*), x_i - x^* \rangle \right) + \sigma^2$$

  $$= 2L(f(x_i) - f^*) + \sigma^2$$  \hspace{1cm} (26)

B D-QSGD

In this section we prove Theorem 1. The iterations of D-QSGD can be written as

$$x_{t+1} := x_t - \gamma \hat{g}_t , \quad \text{where} \quad \hat{g}_t = \frac{1}{n} \sum_{i=1}^n Q_\omega(g_i^t(x_i)) ,$$

and $Q_\omega$ are (independent) $\omega$-quantizers and $g_i^t, i \in [n]$ are unbiased gradient estimators $g_i^t(x) := \nabla f_i(x) + \xi_i^t$ on each worker $i \in [n]$. With the observation that the update in each iteration is an unbiased estimator of the gradient, $E_{\xi_i Q_\omega} \hat{g}_t = \nabla f(x)$, the convergence proof follows directly from standard SGD analyses with the proper upper bound of the variance of the $\hat{g}_t$ estimator.
Lemma 13 (Variance of D-QSGD update). For \( \hat{g}(x) := \frac{1}{n} \sum_{i=1}^{n} Q_\omega(g_i(x)) \) with independent \( \omega \)-quantizers \( Q_\omega \) and unbiased gradient estimators with \( \sigma^2 \)-bounded variance, it holds \( \mathbb{E}_{\xi, \omega} \| \hat{g}(x) - \nabla f(x) \|^2 \leq \frac{\omega Z^2}{n} \| \nabla f(x) \|^2 + \frac{\omega \zeta^2}{n} + \frac{\sigma^2(1 + \omega)}{n} \), \( \forall x \in \mathbb{R}^d \).

Proof. We derive:

\[
\mathbb{E}_{\xi, \omega} \| \hat{g}(x) - \nabla f(x) \|^2 = \mathbb{E}_{\xi, \omega} \left\| \frac{1}{n} \sum_{i=1}^{n} (Q(g_i(x)) - \nabla f_i(x)) \right\|^2
\]

\[
\overset{(21)}{=} \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E}_{\xi_i, \omega_i} \| Q(g_i(x)) - \nabla f_i(x) \|^2
\]

\[
\overset{(20)}{=} \frac{1}{n^2} \sum_{i=1}^{n} \left( \mathbb{E}_{\xi, \omega} \| Q(g_i(x)) \|^2 - \| \nabla f_i(x) \|^2 \right)
\]

\[
\overset{(7)}{=} \frac{1}{n^2} \sum_{i=1}^{n} \left( (1 + \omega) \mathbb{E}_{\xi} \| g_i(x) \|^2 - \| \nabla f_i(x) \|^2 \right)
\]

\[
\overset{(4)}{\leq} \frac{\omega}{n^2} \sum_{i=1}^{n} \| \nabla f_i(x) \|^2 + \frac{\sigma^2(1 + \omega)}{n}
\]

\[
\overset{(5)}{\leq} \frac{\omega Z^2}{n} \| \nabla f(x) \|^2 + \frac{\omega \zeta^2}{n} + \frac{\sigma^2(1 + \omega)}{n} \quad \qed
\]

Lemma 14 (Decent Lemma for D-QSGD). For iterates \( x_t \) defined as in QSGD, and \( \gamma \leq \frac{1}{2L(1 + 2Z^2\omega/n)} \) it holds for \( L \)-smooth functions

\[
F_{t+1} \leq F_t - \gamma \frac{G_t}{2} + L\gamma^2 \frac{\sigma^2(1 + \omega) + \zeta^2 \omega}{2n},
\]

and if the function is in addition \( \mu \)-convex:

\[
X_{t+1} \leq (1 - \mu \gamma) X_t - \gamma F_t + \gamma^2 \frac{\sigma^2(1 + \omega) + \zeta^2 \omega}{n},
\]

with \( X_t := \mathbb{E} \| x_t - x_* \|^2 \), \( F_t := \mathbb{E} f(x_t) - f^* \), \( G_t := \mathbb{E} \| \nabla f(x_t) \|^2 \).

Proof. For convex functions, it holds

\[
X_{t+1} = X_t - 2\gamma \langle \mathbb{E}_\xi g_t, x_t - x_* \rangle + \gamma^2 \mathbb{E}_\xi \| g_t \|^2
\]

\[
\overset{(3),(20)}{\leq} (1 - \mu \gamma) X_t - 2\gamma F_t + \gamma^2 \left( \mathbb{E}_\xi \| g_t - \nabla f(x_t) \|^2 + \| \nabla f(x_t) \|^2 \right)
\]

\[
\overset{(27)}{\leq} (1 - \mu \gamma) X_t - 2\gamma F_t + \gamma^2 \left( 1 + \frac{Z^2\omega}{n} \right) \| \nabla f(x_t) \|^2 + \gamma^2 \frac{\sigma^2(1 + \omega) + \zeta^2 \omega}{n},
\]

and the claim follows with \( \mathbb{E} \| \nabla f(x_t) \|^2 \leq 2LF_t \) and the choice of \( \gamma \). For smooth functions,

\[
F_{t+1} \leq F_t - \gamma G_t + \frac{L\gamma^2}{2} \left( \mathbb{E} \| \hat{g}(x_t) - \nabla f(x_t) \|^2 + \| \nabla f(x_t) \|^2 \right)
\]

and again Lemma 13 together with \( \gamma \leq \frac{1}{2L(1 + 2Z^2\omega/n)} \) show the claim. \( \square \)

Now the convergence proof for strongly convex functions follows with Lemma 25 and with Lemma 27 for convex and only smooth functions.
C DIANA

In this section we prove Theorem 5. For the strongly convex case this proof is a direct copy of the template from (Mishchenko et al., 2019; Horváth et al., 2019).

For the proof, it will be useful to define
\[ \hat{g}_t := \frac{1}{n} \sum_{i \in [n]} (h_i^t + \Delta_{t+1}) , \quad g_t := \frac{1}{n} \sum_{i \in [n]} g_i^t , \]
and to observe \( E_{t} \hat{g}_t = \nabla f(x_t) \). We need a few observations:

**Lemma 15 (Horváth et al. (2019, Lemma 1 & 2)).** Let each \( f_i \) be convex. Then, for \( h_* = \nabla f(x^*) \), \( h_i^t := \nabla f_i(x_*), F_t = Ef(x_t) - f^* \) and \( H_t = \frac{1}{n} \sum_{i \in [n]} E \| h_i^t - h_i^* \|^2 \),
\[ E \| \hat{g}_t - h_* \|^2 \leq 2L \left( 1 + \frac{2\omega}{n} \right) F_t + \frac{(1 + \omega)\sigma^2}{n} + \frac{2\omega}{n} H_t , \] and for \( \alpha \leq \frac{1}{1+\omega} \)
\[ H_{t+1} \leq (1 - \alpha) H_t + \alpha (2L F_t + \sigma^2) . \] Without convexity assumption, we can only derive the following weaker statements:

**Lemma 16 (Without convexity).** It holds
\[ E \| \hat{g}_t - \nabla f(x_t) \|^2 \leq \frac{2\omega Z^2}{n} \| \nabla f(x) \|^2 + \frac{\omega \zeta^2}{n} + \frac{\sigma^2 (1 + \omega)}{n} + \frac{\omega}{n} H_t . \]

For \( \alpha \leq 1 \) and \( H_i^t := \frac{1}{n} \sum_{i \in [n]} E \| h_i^t \|^2 \) it holds
\[ H_{i+1}^t \leq (1 - \alpha) H_i^t + \alpha (\zeta^2 + Z^2 G_i) + \alpha^2 \sigma^2 (1 + \omega) . \]

**Proof.** The proof of the first claim follows along the same lines as the proof of Lemma 13, for the proof of the second claim we refer to the comments below Lemma 22.

Next, we state a decent lemma. In contrast to (Mishchenko et al., 2019; Horváth et al., 2019) that considered more general proximal updates, we consider here unconstrained optimization and present a simplified result.

**Lemma 17 (Lyapunov function).** Let \( f \) be \( L \)-smooth and the stepsize \( \gamma \leq \frac{1}{2L(1 + 8\omega/n)} \) and \( \alpha = \frac{1}{1+\omega} \). Then
\[ \Xi_{t+1} \leq \Xi_t - \gamma G_t + L \gamma^2 \frac{\sigma^2 (2 + \omega) + 2 \zeta^2 \omega}{2n} , \]
for \( b = \frac{\gamma^2 L \omega}{2n} \) and \( \Xi_t := F_t + b H_{i}^t \), with \( F_t = Ef(x_t) - f^* \), \( H_i^t := \frac{1}{n} \sum_{i \in [n]} E \| h_i^t \|^2 \).

If in addition \( f \) \( \mu \)-strongly convex and each \( f_i \) convex and \( L \)-smooth, and stepsize \( \gamma \leq \frac{1}{2L(1 + 8\omega/n)} \), then it holds
\[ \Psi_{t+1} \leq \left( 1 - \min \left\{ \mu \gamma, \frac{\alpha}{2} \right\} \right) \Psi_t - \frac{\gamma F_t}{2} + \frac{5\gamma^2 (1 + \omega) \sigma^2}{n} , \]
for \( \Psi_{i} := X_t + a H_t \), with \( a = \frac{\gamma^2 \omega}{n} \), \( X_t = E \| x_t - x^* \|^2 \) and \( F_t = Ef(x_t) - f^* \) and \( H_t = \frac{1}{n} \sum_{i \in [n]} E \| h_i^t - h_i^* \|^2 \) as before.
Proof. We follow the usual template, and start with the convex case:

\[
\begin{align*}
E_{\xi_t, Q_t} \| x_{t+1} - x^* \|^2 &= \| x_t - x^* \|^2 - 2\gamma \langle E_{\xi_t, Q_t} \hat{g}_t, x_t - x^* \rangle + \gamma^2 E_{\xi_t, Q_t} \| \hat{g}_t \|^2 \\
&= \| x_t - x^* \|^2 - 2\gamma \langle \nabla f(x_t), x_t - x^* \rangle + \gamma^2 E_{\xi_t, Q_t} \| \hat{g}_t - h_t \|^2 \\
&\leq (1 - \gamma) \| x_t - x^* \|^2 - 2\gamma (f(x_t) - f^*) + \gamma^2 E_{\xi_t, Q_t} \| \hat{g}_t - h_t \|^2.
\end{align*}
\]

By Lemma 15, and taking full expectation

\[
X_{t+1} \overset{(30)}{\leq} (1 - \mu \gamma) X_t - \gamma \left( 2L\gamma \left( 1 + \frac{2\omega}{n} \right) - 2 \right) F_t + \frac{\gamma^2 (1 + \omega) \sigma^2}{n} + \frac{2\gamma^2 \omega}{n} H_t
\]

with the choice of the stepsize, \( \gamma \leq \frac{1}{4L(1 + 2\omega/n)} \).

We now combine the bound in (32) with the estimate on \( H_t \) provided in Lemma 15. With the observation that for the chosen \( a = \frac{4\gamma^2 \omega}{n} \) it holds \( (1 + \frac{2\gamma^2 \omega}{n}) (1 - \alpha) \leq (1 - \alpha/2) \) and we obtain

\[
\Psi_{t+1} = X_{t+1} + aH_{t+1}
\]

\[
\overset{(17),(32)}{\leq} (1 - \mu \gamma) X_t + a \left( 1 + \frac{2\gamma^2 \omega}{an} \right) (1 - \alpha) H_t + (2a\alpha L - \gamma) F_t + \left( \frac{\gamma^2}{n} + a\alpha^2 \right) \sigma^2 (1 + \omega)
\]

\[
\leq (1 - c) \Psi_t + (2a\alpha L - \gamma) F_t + \left( \frac{\gamma^2}{n} + a\alpha^2 \right) \sigma^2 (1 + \omega),
\]

for \( c = \min\{\mu \gamma, \frac{\alpha}{2}\} \). By the choice of \( a \), we can simplify the terms in the two brackets:

\[
2a\alpha L - \gamma = \frac{8\gamma^2 L\omega}{n} - \gamma \leq -\gamma/2,
\]

as \( \gamma \leq \frac{1}{16L\omega/n} \). Furthermore

\[
\frac{\gamma^2}{n} + a\alpha^2 = \frac{\gamma^2}{n} (1 + 4\omega \alpha) \leq \frac{5\gamma^2}{n}
\]

as \( \alpha \leq \frac{1}{4\omega} \). Combining these bounds proves the claim.

Finally, without convexity, we start with the smoothness inequality, and derive similarly as in Lemma 14

\[
F_{t+1} \leq F_t - \gamma \frac{G_t}{2} + L\gamma^2 \sigma^2 (1 + \omega) + c^2 \omega \frac{2}{2n} + \gamma^2 L\omega H_t'.
\]

Therefore, together with Lemma 23,

\[
\Xi_{t+1} \leq F_t + b \left( 1 - \alpha + \frac{\gamma^2 L\omega}{2b} \right) H_t' + a b \left( \sigma^2 - \frac{\gamma^2}{2} \right) G_t + a b c^2 + a^2 b \sigma^2 (1 + \omega) + L\gamma^2 \sigma^2 (1 + \omega) + c^2 \omega \frac{2}{2n}
\]

\[
\leq \Xi_t + \frac{\gamma}{4} \left( \frac{4\gamma^2 \omega^2}{n} - 2 \right) G_t + L\gamma^2 \sigma^2 (2 + \omega) + 2c^2 \omega \frac{2}{2n}
\]

for \( b = \frac{\gamma^2 L\omega}{2bn} \) and \( \alpha = \frac{1}{4\omega} \).

The proof of Theorem 5 follows now from Lemma 17 with the help of the tools provided in Lemma 25 below for the strongly convex case, and with Lemma 27 for the non-convex case. For the convex (\( \mu = 0 \))
case, note that unrolling the expression from Lemma 17 (as in Lemma 27) gives

\[ F_t = O\left( \frac{\Psi_0}{\gamma} + \gamma \frac{\sigma^2(1 + \omega)}{n} \right) \]

by the observation \( \Psi_0 := X_0 + aH_0 \) with \( aH_0 \leq \frac{4\gamma^2\omega}{\alpha n} \). The proof of the claim now follows by the same steps as in Lemma 27.

### D D-EF-SGD.

In this section we prove Theorem 4. We follow closely (Stich and Karimireddy, 2019) and define a virtual sequence

\[ \tilde{x}_0 := x_0, \quad \tilde{x}_{t+1} := \tilde{x}_t - \frac{\gamma}{n} \sum_{i \in [n]} g^i_t, \]

similar as in the main text in (12). Further, we will be using the notation \( X_t := E\|x_t - x^*\|^2, F_t := Ef(x_t) - f^* \), \( G_t := E\|\nabla f(x_t)\|^2, E_t := \frac{1}{n} \sum_{i \in [n]} E\|e^i_t\|^2 \).

**Lemma 18** (Stich and Karimireddy (2019)). Let \( f \) be \( L \)-smooth. If the stepsize \( \gamma \leq \frac{1}{4L} \), then it holds for the iterates of Algorithm 2 and 4:

\[ F_{t+1} \leq F_t - \frac{\gamma}{4} G_t + \frac{\gamma^2 L \sigma^2}{2n} + \frac{\gamma^3 L^2}{2} E_t \tag{33} \]

and if \( f \) is in addition \( \mu \)-convex,

\[ X_{t+1} \leq \left( 1 - \frac{\gamma \mu}{2} \right) X_t - \frac{\gamma}{2} F_t + \frac{\gamma^2 \sigma^2}{n} + 3\gamma^3 LE_t. \tag{34} \]

**Proof.** First, we observe that the update applied to the virtual sequence in (12) is an unbiased estimator of \( \nabla f(x_t) \), with variance:

\[ E_x \left\| \frac{1}{n} \sum_{i \in [n]} g^i_t - \nabla f(x_t) \right\|^2 \leq \frac{\sigma^2}{n}. \tag{4} \]

With this observation, the proof follows directly from (Stich and Karimireddy, 2019) with the observation that

\[ \left\| \tilde{x}_t - x_t \right\|^2 \leq \left\| \frac{\gamma}{n} \sum_{i \in [n]} e^i_t \right\|^2 \leq \frac{\gamma^2}{n} \sum_{i \in [n]} \|e^i_t\|^2. \tag{13} \]

**Lemma 19.** It holds

\[ E_{t+1} \leq \left( 1 - \frac{\delta}{2} \right) E_t + \frac{2(1 - \delta)}{\delta} \left( \zeta^2 + Z^2 G_t \right) + (1 - \delta) \sigma^2. \tag{35} \]

**Proof.** From (Stich and Karimireddy, 2019) it follows

\[ E_{\xi, \epsilon} \left\| e^i_{t+1} \right\|^2 \leq \left( 1 - \frac{\delta}{2} \right) \|e^i_t\|^2 + \frac{2(1 - \delta)}{\delta} \left\| \nabla f_i(x_t) \right\|^2 + (1 - \delta) \sigma^2, \]

The claim now follows by summing and averaging over \( i \in [n] \).
For the convergence proof, we can now either follow (Stich and Karimireddy, 2019) again, or for a slightly simpler proof, we can combine Lemmas 18 and Lemma 19 together:

**Lemma 20** (Lyapunov function). Let $f$ be $L$-smooth and $\gamma \leq \frac{\delta}{4L(1+\delta)}$. Then it holds

$$\Xi_{t+1} \leq \Xi_{t} - \frac{\gamma}{8} G_t + \gamma^2 \frac{L\sigma^2}{2n} + \gamma^3 \left( \frac{2L^2\varsigma^2}{\delta^2} + \frac{L^2\sigma^2}{\delta} \right)$$

for $\Xi_{t} := F_{t} + bE_{t}$, for $b = \frac{2L^2}{\delta}$, and

$$\Psi_{t+1} \leq (1 - \min\{\gamma\mu / 2, \delta / 4\})\Psi_{t} - \frac{\gamma}{4} G_t + \gamma^2 \frac{\sigma^2}{n} + \gamma^3 \left( \frac{24L^2\varsigma^2}{\delta^2} + \frac{12L^2\sigma^2}{\delta} \right)$$

for $\Psi_{t} := X_{t} + aE_{t}$, with $a = \frac{12\gamma^3 L}{\delta}$.

**Proof.** For convex functions, we first note that $G_t \leq 2LF_t$. Now

$$\Psi_{t+1} \leq \left(1 - \frac{\gamma\mu}{2}\right)X_t + a \left(1 - \frac{\delta}{2} + \frac{3\gamma^3 L}{a}\right) + \left( \frac{4a(1-\delta)LZ^2}{\delta} - \frac{\gamma}{2} \right) F_t$$

$$+ \frac{2a(1-\delta)}{\delta} \varsigma^2 + \frac{\gamma^2 \sigma^2}{n} + a(1-\delta)\sigma^2$$

$$\leq (1 - c)\Psi_{t} + \gamma \left( \frac{19\gamma^2 L^2 Z^2}{\delta^2} - 2 \right) F_t + \gamma^2 \frac{\sigma^2}{n} + \gamma^3 \left( \frac{24L^2\varsigma^2}{\delta^2} + \frac{12L^2\sigma^2}{\delta} \right)$$

with the choice $a = \frac{12\gamma^3 L}{\delta}$, $c = \min\{\gamma\mu / 2, \delta / 4\}$. Now the claim follows with $\gamma \leq \frac{\delta}{14LZ}$.

For smooth functions,

$$\Xi_{t+1} \leq \left(33\right)^{(33),(35)} F_t + a \left(1 - \frac{\delta}{2} + \frac{\gamma^3 L^2}{2b} \right) E_t + \left( \frac{2bZ^2}{\delta} - \frac{\gamma}{4} \right) G_t + \frac{2b\varsigma^2}{\delta} + b\sigma^2 + \frac{L\sigma^2}{2n}$$

$$\leq \Xi_{t} + \gamma \left( \frac{16\gamma^2 Z^2}{\delta^2} - 2 \right) G_t + \gamma^2 \frac{L\sigma^2}{2n} + \gamma^3 \left( \frac{2L^2\varsigma^2}{\delta^2} + \frac{L^2\sigma^2}{\delta} \right)$$

for $b = \frac{2L^2}{\delta}$.

As in the previous sections, the claims of the theorem follow from Lemma 20 together with Lemmas 25 and 27 below.

### E D-EF-SGD with bias correction

In this section we give the remaining proofs for the convergence of D-EF-SGD with bias correction for the convex and non-convex case.

**Convex case.** In the case when $\mu = 0$, Lemma 12 still applies. After unrolling the recursion (as in Lemma 27), we obtain that

$$F_t = O \left( \frac{\Psi_0}{\gamma} + \frac{\sigma^2}{n} + \gamma^2 \frac{1 + \beta/\delta}{\delta} L(1-\delta)\sigma^2 \right)$$

$$= O \left( \frac{X_0}{\gamma} + \frac{\sigma^2}{n} + \gamma^2 \sigma^2 + \frac{\varsigma^2}{\delta} L(1-\delta) \right)$$

when plugging in $\Psi_0 = X_0 + aE_0 + bH_0$, $H_0 \leq \varsigma^2$ and $\beta = \delta$. Now the result follows by tuning $\gamma$ as in Lemma 27.
Non-convex case. In the case when $f$ is only to be assumed $L$ smooth, the proof follows immediately by slight adaptations from tools that we have already developed above:

**Lemma 21** (Stich and Karimireddy (2019, Lemma 8)). Let $f$ be $L$-smooth. If the stepsize $\gamma \leq \frac{1}{2L}$, then it holds for the iterates of Algorithm:

$$F_{t+1} \leq F_t - \frac{\gamma}{4} G_t + \frac{\gamma^2 L \sigma^2}{2n} + \frac{\gamma^3 L^2}{2} E_t.$$  \hspace{1cm} (36)

**Proof.** (Stich and Karimireddy, 2019, Lemma 8) yields the result when resorting to the same observations as outline in the proof of Lemma 9 above.

With the dissimilarity assumption, we can derive a new version of Lemma 10 without the need of the convexity assumption.

**Lemma 22.** Let $f$ satisfy bounded dissimilarity (5) for $\zeta^2$, $Z^2 \geq 0$. Then

$$E_{t+1} \leq \left(1 - \frac{\delta}{2}\right) E_t + \frac{4(1 - \delta)}{\delta} \left(\zeta^2 + ZG_t + H'_t\right) + (1 - \delta)\sigma^2,$$  \hspace{1cm} (37)

for $H'_t := \frac{1}{n} \sum_{i \in [n]} E\|\nabla f_i(x_t)\|^2$.

**Proof.** This readily follows from the proof of Lemma 10. It remains to note that

$$\frac{1}{n} \sum_{i \in [n]} \|\nabla f_i(x_t) - h_i^t\|^2 \leq \frac{2}{n} \sum_{i \in [n]} \|\nabla f_i(x_t)\|^2 + 2H'_t \leq 2\zeta^2 + 2Z^2\|\nabla f(x_t)\|^2 + 2H'_t,$$

and to plug this estimate into (16).

Lastly, we show that $H'_t$ follows a similar recursion as $H_t$:

**Lemma 23.** Let $f$ satisfy bounded dissimilarity (5) for $\zeta^2$, $Z^2 \geq 0$ and let $h_i^t$ be updated with an unbiased quantizer $Q_\omega$, and stepsize $\alpha \leq \frac{1}{1 + \omega}$. Then

$$H'_{t+1} \leq (1 - \alpha)H'_t + \alpha(\zeta^2 + Z^2G_t) + \alpha^2(1 + \omega)\sigma^2.$$  \hspace{1cm} (38)

**Proof.** For the proof we note that the derivations in the proof of Lemma 11 hold for arbitrary choice of $h_i^t$, especially the choice $h_i^t = 0_d$, up to equation (18). The claim now follows with (5).

We can summarize the statements of Lemmas 21–23 the following descent lemma.

**Lemma 24** (Lyapunov function). Let $f$ be $L$-smooth and the stepsize $\gamma \leq \frac{1}{2L(1 + 4Z)}$. Then for $\beta = \delta$ it holds

$$\Psi_{t+1} \leq \Psi_t - \frac{G_t}{8} + \frac{\gamma^2 L \sigma^2}{2n} + \frac{\gamma^3 8L^2(1 - \delta)(\delta\sigma^2 + \zeta^2)}{\delta^2},$$  \hspace{1cm} (39)

for $\Psi_t := F_t + aE_t + bH'_t$ with $a = \frac{\gamma^2 L^2}{8}$ and $b = \frac{4a(1 - \delta)}{\delta}$.

Now the claimed convergence bound from Table 1 follows by Lemma 27, with the observation $\Psi_0 = F_0 + aE_0 + bH_0 \leq F_0$ for the chosen initialization.
F Summation Lemmas

In this section we repeat a few useful lemmas that have been (only slightly) adapted from other works.

Lemma 25 (Based on Appendix A.2 of Koloskova et al. (2020b)). Let \((r_t)_{t \geq 0}\) and \((s_t)_{t \geq 0}\) be sequences of positive numbers satisfying

\[
    r_{t+1} \leq (1 - \min\{\gamma A, F\})r_t - B\gamma s_t + C\gamma^2 + D\gamma^3,
\]

for some positive constants \(A, B > 0, C, D \geq 0\), and for constant step-sizes \(0 < \gamma \leq \frac{1}{E}\), for \(E \geq 0\), and for parameter \(0 < F \leq 1\). Then there exists a constant stepsize \(\gamma \leq \frac{1}{E}\) such that

\[
    \frac{B}{W_T} \sum_{t=0}^{T} w_t s_t + \min\{A, \frac{F}{\gamma}\} r_{t+1} \leq r_0 \left( E + \frac{A}{F} \right) \exp \left[ -\min\left\{ \frac{A}{E}, F \right\} (T + 1) \right] + 2C \ln \tau + \frac{D \ln^2 \tau}{A(T + 1)^2}
\]

for \(w_t := (1 - \min\{\gamma A, F\})^{-(t+1)}\), \(W_T := \sum_{t=0}^{T} w_t\) and

\[
    \tau = \max \left\{ \exp[1], \min \left\{ \frac{A^2 r_0(T+1)^2}{C}, \frac{A^3 r_0(T+1)^3}{D} \right\} \right\}
\]

Remark 26. Lemma 25 establishes a bound of the order

\[
    \mathcal{O}\left( r_0 \left( E + \frac{A}{F} \right) \exp \left[ -\min\left\{ \frac{A}{E}, F \right\} T \right] + \frac{C}{AT} + \frac{D}{A^2 T^2} \right),
\]

that decreases with \(T\). To ensure that this expression is less than \(\epsilon\),

\[
    T = \tilde{O}\left( \frac{C}{A\epsilon} + \frac{\sqrt{D}}{A^\sqrt{\epsilon}} + \frac{1}{F} \log \frac{1}{\epsilon} + \frac{E}{A} \log \frac{1}{\epsilon} \right) = \tilde{O}\left( \frac{C}{A\epsilon} + \frac{\sqrt{D}}{A^\sqrt{\epsilon}} + \frac{1}{F} + \frac{E}{A} \right)
\]

steps are sufficient.

Proof of Lemma 25. After rearranging and multiplying (40) by \(w_t\) we obtain

\[
    Bw_t s_t \leq \left( \frac{1 - \min\{\gamma A, F\}}{\gamma} \right) w_t r_t - \frac{w_t r_{t+1}}{\gamma} + \gamma C + \gamma^2 D.
\]

Observing that that \(w_t(1 - \min\{\gamma A, F\}) = w_{t-1}\) we obtain a telescoping sum,

\[
    \frac{B}{W_t} \sum_{t=0}^{T} w_t s_t \leq \left( \frac{1 - \min\{\gamma A, F\}}{\gamma} \right) w_0 r_0 - \frac{w_T r_{T+1}}{\gamma W_T} + \gamma C + \gamma^2 D = \frac{r_0}{\gamma W_T} - \frac{w_T r_{T+1}}{\gamma W_T} + \gamma C + \gamma^2 D.
\]

Using that \(W_T = w_T \sum_{t=0}^{T} (1 - \min\{\gamma A, F\})^t \leq \frac{w_T}{\min\{\gamma A, F\}}\) and \(W_T \geq w_T = (1 - \min\{\gamma A, F\})^{-(T+1)}\) we can simplify

\[
    B \sum_{t=0}^{T} w_t s_t + \min\{A, F/\gamma\} r_{T+1} \leq \left( \frac{1 - \min\{\gamma A, F\}}{\gamma} \right)^{T+1} r_0 + \gamma C + \gamma^2 D \leq \frac{r_0}{\gamma} \exp \left[ -\min\{\gamma A, F\}(T + 1) \right] + \gamma C + \gamma^2 D =: \Psi_T
\]

Now the lemma follows by tuning \(\gamma\) in the same way as in (Stich, 2019, Lemma 2) (slightly more carefully):
• If $\frac{\ln \tau}{A(T+1)} \leq \frac{1}{E}$ then we choose $\gamma = \frac{\ln \tau}{A(T+1)}$. With observing $\ln \tau \geq 1$ we obtain that

$$
\Psi_T \leq \frac{1}{\ln \tau} \max \left\{ \frac{C}{A(T+1)} \cdot \frac{D}{A^2(T+1)^2} \right\} + r_0 \frac{A}{F} \exp \left[ -F(T+1) \right] + \frac{C \ln \tau}{A(T+1)} + \frac{D \ln^2 \tau}{A^2(T+1)^2}
$$

Case: $\min(\gamma, F) = \gamma A$

$$\leq r_0 \frac{A}{F} \exp \left[ -\min \left\{ \frac{A}{E}, F \right\} (T+1) \right] + \frac{2C \ln \tau}{A(T+1)} + \frac{2D \ln^2 \tau}{A^2(T+1)^2}.$$

• If otherwise $\frac{1}{E} \leq \frac{\ln \tau}{A(T+1)}$ and we pick $\gamma = \frac{1}{E}$ and get that

$$
\Psi_T \leq r_0 E \exp \left[ -\min \left\{ \frac{A}{E}, F \right\} (T+1) \right] + \frac{C}{E} + \frac{D}{E^2}
$$

$$\leq r_0 E \exp \left[ -\min \left\{ \frac{A}{E}, F \right\} (T+1) \right] + \frac{C \ln \tau}{A(T+1)} + \frac{D \ln^2 \tau}{A^2(T+1)^2}.
$$

\[\square\]

Lemma 27. Let $(r_t)_{t \geq 0}$ and $(s_t)_{t \geq 0}$ be sequences of positive numbers satisfying

$$r_{t+1} \leq r_t - B \gamma s_t + C \gamma^2 + D \gamma^3,$$

for some positive constants $B > 0$, $C, D \geq 0$ and step-sizes $0 < \gamma \leq \frac{1}{E}$, for $E \geq 0$. Then there exists a constant stepsize $\gamma \leq \frac{1}{E}$ such that

$$\frac{B}{T+1} \sum_{t=0}^{T} s_t \leq \frac{E r_0}{T+1} + 2D^{1/3} \left( \frac{r_0}{T+1} \right)^{2/3} + 2 \left( \frac{Cr_0}{T+1} \right)^{1/2}.
$$

(43)

Remark 28. To ensure that the right hand side in (43) is less than $\epsilon > 0$,

$$T = O \left( \frac{C}{\epsilon^2} + \frac{\sqrt{D}}{\epsilon^{3/2}} + \frac{E}{\epsilon} \right) \cdot r_0
$$

steps are sufficient.

Proof of Lemma 27. Rearranging and dividing by $\gamma > 0$ gives

$$B s_t \leq \frac{r_t}{\gamma} - \frac{r_{t+1}}{\gamma} + C \gamma + D \gamma^2
$$

and summing from $t = 0$ to $T$ yields

$$\frac{B}{T+1} \sum_{t=0}^{T} s_t \leq \frac{r_0}{\gamma(T+1)} + C \gamma + D \gamma^2.
$$

Now the claim follows by choosing $\gamma = \min \left\{ \frac{1}{E}, \left( \frac{r_0}{C(T+1)} \right)^{1/2}, \left( \frac{r_0}{D(T+1)} \right)^{1/3} \right\}$. See for instance (Koloskova et al., 2020b, Lemma 15).