Free quantum fields in 4D and Calabi-Yau spaces.

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(Dated: May 12, 2017)

We develop general counting formulae for primary fields in free four dimensional (4D) scalar conformal field theory (CFT). Using a duality map between primary operators in scalar field theory and multi-variable polynomial functions subject to differential constraints, we identify a sector of holomorphic primary fields corresponding to polynomial functions on a class of permutation orbifolds. These orbifolds have palindromic Hilbert series, which indicates they are Calabi-Yau. We construct the top-dimensional holomorphic form expected from the Calabi-Yau property. This sector includes and extends previous constructions of infinite families of primary fields. We sketch the generalization of these results to free 4D vector and matrix CFTs.

I. INTRODUCTION

In [1] we started a program of describing the discrete combinatoric data of four dimensional conformal field theories (CFT4) using the framework of SO(4,2) invariant 2D topological field theory (TFT2). TFT2 associates theories (CFT4) using the framework of combinatoric data of four dimensional conformal field states to circles and the operator product expansion of the 4D CFT determines amplitudes for 3-holed spheres. We described how the associativity conditions of 2D TFT are satisfied by the correlators of free scalar CFT4. We initiated the investigation of SO(4,2) invariant TFT2 as an approach to perturbative field theory in [2], making contact with the equivariant interpretation of conformal Feynman integrals in mathematical work [3].

Using a duality between primary fields and multi-variable polynomial functions subject to differential constraints, we identify a sector of holomorphic primary fields corresponding to polynomial functions on the Calabi-Yau orbifold

\[ (\mathbb{C}^2)^n / (\mathbb{C} \times S_n) \]

which can also be written as

\[ (\mathbb{C}^n / \mathbb{C} \times \mathbb{C}^n / \mathbb{C}) / S_n \]

where \( n \) is the number of elementary fields \( \phi \). A generalization of our discussion to the \( O(N) \) vector model shows that the holomorphic singlet primary fields correspond to functions on the Calabi-Yau orbifold

\[ (\mathbb{C}^2)^{2n} / (\mathbb{C}^2 \times S_n[S_2]) = (\mathbb{C}^{2n} / \mathbb{C} \times \mathbb{C}^{2n} / \mathbb{C}) / S_n[S_2] \]

where \( S_n[S_2] \) is a wreath product subgroup of \( S_{2n} \). For the \( S \) matrix model in which \( \phi \) transforms in the adjoint of \( U(N) \), we find holomorphic primaries corresponding to polynomial functions on

\[ ((\mathbb{C}^2)^n \times S_n) / (\mathbb{C}^2 \times S_n) \]

which can also be written as

\[ (\mathbb{C}^n / \mathbb{C} \times \mathbb{C}^n / \mathbb{C}) / S_n \]

II. MULTI-VARIABLE POLYNOMIAL

(MANY-BODY) REPRESENTATION OF SO(4,2)

In radial quantization, the scalar field has a mode expansion given by

\[ \phi(x_\mu) = \sum_{l=0}^{\infty} \sum_{m \in V_l} a_{l,m}^{\dagger} Y_{l,m}(x) + \sum_{l=0}^{\infty} \sum_{m \in V_l} a_{l,m}|x|^{-2} Y_{l,m}(x') \]

where \( V_l \) is the representation of \( SO(4) \) corresponding to symmetric traceless tensors of rank \( l \). The index \( m \) runs over a basis for this vector space. Acting on the vacuum state
transformations and translations are ranges from 1 to \( n \) of the elementary field \( \phi \) which is, by definition, annihilated by the \( a_{\mu, \nu} \) with a local operator \( \partial_\mu \cdots \partial_\nu \phi \) and taking the limit \( x \to 0 \), we get a state. Taking the dual of this state and pairing with \( \phi(x)[0] \) we get a polynomial. Thus there is a map

\[
\partial_\mu_1 \cdots \partial_\mu_k \phi \leftrightarrow P_{\mu_1} \cdots P_{\mu_k} \cdot 1
\]

where \( P_\mu = x^2 \partial_\mu - 2x_\mu x.\partial - 2x_\mu \)

The scalar field itself maps to 1. The free field satisfies the equation of motion \( \partial_\mu \partial_\mu \phi = 0 \). Correspondingly \( P_\mu P_\mu = x^2 \partial^2 \) annihilates 1. When considering operators constructed using \( n \) fields, we have a representation of the conformal group on polynomials in variables \( x^I_\mu \) where \( I \) ranges from 1 to \( n \). The generators for special conformal transformations and translations are

\[
K_\mu = \sum_{I=1}^n x^I_\mu \partial x^I_\mu
\]

\[
P_\mu = \sum_{I=1}^n \left( x^I_\rho x^I_\sigma \partial x^I_\mu - 2x^I_\rho x^I_\sigma \partial x^I_\mu - 2x^I_\mu \right)
\]

The remaining generators are determined by the \( \text{so}(4,2) \) algebra. The \( x^I_\mu \) can be considered as the coordinates of \( n \) particles. The construction of primaries using \( n \) copies of the elementary field \( \phi \) is therefore mapped to a many-body quantum mechanics problem with \( n \) particles.

Tracelessness can be implemented using variables \( z \cdot x^I = z^\mu x^I_\mu \) with null \( z^\mu \cdot z^\mu = 0 \). Any polynomial in \( z \cdot x^I \) gives a traceless symmetric polynomial in \( x^I_\mu \) after the \( z^\mu \)'s are stripped away. The translation between polynomials and operators is

\[
(z \cdot \partial)^k \phi \leftrightarrow (-1)^k 2^k k! (z \cdot x)^k
\]

This construction is not general: there are primaries that are not symmetric in their indices and so can’t be represented as a polynomial in \( z \cdot x \). For the general discussion, introduce projectors from symmetric tensors to traceless symmetric tensors. For example, for tensors of rank 2 and 3 we have

\[
S^{\alpha \beta}_{\mu \nu} = \delta^\alpha_\mu \delta^\beta_\nu - \frac{1}{4} \delta_{\mu \nu} \delta^{\alpha \beta}
\]

\[
S^{\alpha \beta \gamma}_{\mu \nu \rho} = \delta^\alpha_\mu \delta^\beta_\nu \delta^\gamma_\rho - \frac{1}{6} (\delta_{\mu \rho} \delta^{\alpha \beta} \delta^\gamma_\nu + \delta_{\mu \nu} \delta^{\alpha \beta} \delta^\gamma_\rho + \delta_{\nu \rho} \delta^{\alpha \beta} \delta^\gamma_\mu + \delta^{\alpha \beta} \delta^\gamma_\mu \delta_{\nu \rho})
\]

We recognize that these are projectors in the Brauer algebra of tensor space operators which commute with \( \text{SO}(4,2) \) algebra.

\[
S^{(2)} = \frac{1}{2} - \frac{C_{12}}{4}
\]

\[
S^{(3)} = 1 - \frac{1}{6} (C_{12} + C_{13} + C_{23})
\]

They satisfy

\[
(S^{(n)})^2 P_n = S^{(n)} P_n
\]

where

\[
P_n = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma
\]

The projector property along with the property that they start with 1 completely determines these elements of the Brauer algebra. In general

\[
P_{\mu_1} \cdots P_{\mu_n} \cdot 1 = (-1)^n 2^n n! (S^{(n)})_{\mu_1 \cdots \mu_n} x_{\nu_1} \cdots x_{\nu_n}
\]

The multiplication \( (15) \) is in the Brauer algebra, where loops are assigned the value of 4. The factor on the RHS above is obtained by deriving an obvious recursion formula. Note that the term \( x^2 \partial_\mu \) in \( P_\mu \) does not raise the rank of the tensor. The other two terms contribute to the recursion.

States at dimension \( n + k \) in \( V^\otimes n \) correspond to polynomials in \( x^I_\mu \) of degree \( k \). Primaries at dimension \( n + k \) are degree \( k \) polynomials \( \Psi(x^I_\mu) \) with the conditions

\[
K_\mu \Psi(x^I_\mu) = \sum_I \partial x^I_\mu \Psi(x^I_\mu) = 0
\]

\[
L_I \Psi(x^I_\mu) = \sum_\mu \partial x^I_\mu \partial x^I_\mu \Psi(x^I_\mu) = 0
\]

\[
\Psi(x^I_\mu) = \Psi(x^{(I)}_\mu)
\]

The first condition says the special conformal generators annihilate a primary operator. The second condition implements the free scalar equation of motion. The last condition imposes \( S_n \) invariance, to implement bosonic statistics of the scalar field.

We find it useful to employ the complex coordinates

\[
z = x_1 + ix_2 \quad w = x_3 + ix_4
\]

\[
\bar{z} = x_1 - ix_2 \quad \bar{w} = x_3 - ix_4
\]

which have the following \( (j^3_L, j^3_R) \) charge assignments

\[
z \leftrightarrow (\frac{1}{2}, \frac{1}{2}) \quad \bar{z} \leftrightarrow (\frac{1}{2}, -\frac{1}{2})
\]

\[
w \leftrightarrow (\frac{1}{2}, -\frac{1}{2}) \quad \bar{w} \leftrightarrow (-\frac{1}{2}, \frac{1}{2})
\]

This amounts to choosing an isomorphism between \( \mathbb{R}^4 \) and \( \mathbb{C} \times \mathbb{C} \). We will construct a class of primaries corresponding to holomorphic polynomial functions on

\[
\mathbb{C}^2 / (\mathbb{C}^2 \times S_n)
\]

III. COUNTING WITH \( \text{SO}(4,2) \) CHARACTERS

The number \( N_{[\Delta, j_1, j_2]} \) of primary operators, of dimension \( \Delta \) and spin \( (j_1, j_2) \) built out of \( n \) scalar fields \( \phi \) is obtained by expanding the generating function

\[
G_n(s, x, y) = \sum_{m, j_1, j_2} N_{[m, j_1, j_2]} s^m x^{j_1} y^{j_2}
\]
The generating function is given by (take $n \geq 3$ to avoid complications associated to null states)

$$G_n(s, x, y) = \left[ (1 - \frac{1}{x})(1 - \frac{1}{y}) Z_n(s, x, y)(1 - s \sqrt{xy}) \right](1 - s \sqrt{\frac{y}{x}})(1 - s \sqrt{\frac{x}{y}}) \geq 0 \quad (21)$$

where $Z_n(s, x, y)$ is defined by

$$\prod_{q=0}^{\infty} \prod_{a=-\frac{1}{2}}^{\frac{1}{2}} \prod_{s=-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1 - ts^{q+1} x^ay^b} = \sum_{n=0}^{\infty} t^n Z_n(s, x, y) \quad (22)$$

This is obtained by constructing the character for the symmetric product of $n$ copies of the representation of the scalar field, and decomposing into $SO(4, 2)$ irreps [12, 13].

We can specialize this counting formula. Consider the leading twist fields, with $[\Delta, j_1, j_2] = [n + q, \frac{q}{2}, \frac{q}{2}]$. This is a complete spin multiplet. The highest spin primary corresponds to a polynomial in $z$. For counting these primaries, the general formulas given above reduce to

$$G_n^s(s, x, y) = [Z_n^s(s, x, y)(1 - s \sqrt{xy})] \quad (23)$$

where

$$\prod_{q=0}^{\infty} \prod_{a=-\frac{1}{2}}^{\frac{1}{2}} \prod_{s=-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1 - ts^{q+1} x^ay^b} = \sum_{n=0}^{\infty} t^n Z_n^s(s, x, y) \quad (24)$$

Using the simplified formulas we have

$$G_n^{\text{max}}(s) = \frac{s^n}{(1 - s^2)(1 - s^3) \cdots (1 - s^n)} \quad (25)$$

Note the close connection to multiplicities of $V_{\Lambda=n+k}^{SL(2)} \otimes V_{[n]}^S$, which is the coefficient of $q^k$ in

$$\prod_{i=2}^{n} \frac{1}{1 - q^i} \quad (26)$$

The result was also recently obtained in [14].

A more general counting involves polynomials of $z_t$ and $w^I_\mu$. We denote these as exponential primaries, since they have $(s, j_L, j_R) = (n + q, \frac{q}{2}, j_R)$. In this case

$$G_n^{z, w}(s, x, y) = \left[ \left( 1 - \frac{1}{y} \right) (1 - s \sqrt{xy}) \right](1 - s \sqrt{x/y}) Z_n(s, x, y) \quad (27)$$

where

$$\prod_{q=0}^{\infty} \prod_{m=0}^{q} \frac{1}{1 - ts^{q+1} x^ay^b} = \sum_{n=0}^{\infty} t^n Z_n^{z, w}(s, x, y) \quad (28)$$

As explained in more detail in Section 4,

$$Z_n^{z, w}(s, x, y) = s^n \sum_{\Lambda_1} Z_{SH}(s \sqrt{xy}, \Lambda_1) Z_{SH}(s \sqrt{x/y}, \Lambda_1) \quad (29)$$

where $\Lambda_1$ is a partition of $n$, and $Z_{SH}(q)$ is given in [33]. Using these formulae, one finds, for $n = 3$

$$Z_3^{z, w} = \frac{s^3}{(1 - s^2)(1 - s^3)(1 - s^4)} \quad (30)$$

Computing $Z_4(s, x, y)$ in the same way we find

$$G_4^{z, w} = \frac{s^4 R(s, x, y)}{D(s, x, y)} \quad (30)$$

where

$$R(s, x, y) = 1 + s^3 x^{\frac{3}{2}} (1 + s^3 x^{3}) (1 - s^3 x^{3}) (1 - s^4 x^{2}) \quad (30)$$

Similar constructions with the pairs $(z, w), (\bar{z}, w), (z, \bar{w})$ are possible.

**IV. COUNTING AND CONSTRUCTION WITH SYMMETRIC GROUPS**

The counting formulas derived in section [11] can be used to construct families of primary operators. The coordinates $x^I_\mu, I = 1, \ldots, n$ admit a natural action of $S_n$. To satisfy the first of [10], build $n - 1$ translation invariant “relative coordinates” given by the successive differences $X^I_n = x^I_n - x^I_{n+1}$. Using the complex coordinates $z, w$ on $\mathbb{R}^4 = \mathbb{C}^2$, we have $(z, w)$ on $(\mathbb{R}^4)^n = (\mathbb{C}^2)^n$. These differences span the $S_n$ irrep labeled by hook Young diagram with row lengths $[n-1, 1]$. A more convenient basis which connects with Young’s orthonormal representation is useful for computations (see [13] for this basis). Using complex variables we have $Z^{a}, \bar{Z}^{a}, Z(a), W(a)$, which each transform in the irrep $[n-1, 1] \equiv V_H$. Products $Z^{a_1} Z^{a_2} \cdots Z^{a_k}$ are in the $V_H^{k}$ tensor product representation of $S_n$. Any polynomial in the hook variables automatically obeys the first two constraints of [10]. This follows since the Laplacian in the second equation is

$$\left( \partial^2 \partial_{z(t)} \partial_{\bar{z}(t)} + \partial^2 \partial_{w(t)} \partial_{\bar{w}(t)} \right) \psi = 0 \quad (31)$$

The only thing left is to project to the $S_n$ invariant subspace of $V_H^{\otimes k}$. The matrices representing the $k$-fold tensor product are

$$(\Gamma_k^n(\sigma))_{a_1, \ldots, a_k; b_1, \ldots, b_k} = \Gamma_{a_1, b_1}(\sigma) \cdots \Gamma_{a_k, b_k}(\sigma)$$
where $\Gamma_{a,b}(\sigma)$ are matrices representing $S_n$ in an orthogonal basis of $[n - 1, 1]$. We can project to the invariants by averaging over the group

$$P_{a_1a_2\cdots a_k} = \frac{1}{n!} \sum_{\sigma \in S_n} (\Gamma_{a,b}(\sigma))_{a_1a_2\cdots a_k,b_1b_2\cdots b_k} Z^{(b_1)} \cdots Z^{(b_k)}$$

The above expression gives $\sum \hat{n}_i P_i(z)$ where $\hat{n}_i$ are unit vectors and $P_i(z_1, \cdots, z_n)$ are the polynomials we want.

By considering all possible degrees $k \in \{0, 1, 2, \cdots \}$ we have a ring. These primaries have a ring structure, since they obey a stronger linear version of the Laplacian condition, which means that a product of solutions is also a solution to the constraints. The counting formula gives the Hilbert series for holomorphic functions on $(\mathbb{C}^n/\mathbb{C})/S_n$. The quotient by $\mathbb{C}$ is effected by the first of which sets the centre of mass momentum of the many body wavefunction to zero. The orbifold by $S_n$ is the symmetry condition in $(\mathbb{C}^n/\mathbb{C})/S_n$. Using properties of Hilbert series, it follows that the ring at hand has $n - 1$ generators, whose form is outlined in the Appendix A.

The construction is easily extended to polynomials of holomorphic coordinates $z^I$ and $w^I$. Use hook variables $Z^{(a)}, W^{(a)}$. The products $Z^{(a_1)} \cdots Z^{(a_k)} W^{(a_{k+1})} \cdots W^{(a_{k+l})}$ belong to a subspace of the representation $V_H^{(a)} \otimes V_H^{(b)}$ of $S_n$, which will characterize in terms of representation theory. Consider the expansions in terms of $S_n \times S_k$ irreps

$$V_H^{(a)} = \bigoplus_{A \in S_n \times S_k} V_A^{(a)} \otimes V^{(a)}_{\text{Com}(S_n \times S_k)}$$

$$V_H^{(b)} = \bigoplus_{A' \in S_n \times S_k} V_A^{(b)} \otimes V^{(b)}_{\text{Com}(S_n \times S_k)}$$

Multiplicities are given by dimensions of irreps of the commutants $\text{Com}(S_n \times S_k)$ in $V_H^{(a)}$. Since the $Z$ and $W$ variables are commuting, the monomials belong to the trivial irreps $A = [k] \otimes A = [l]$ of $S_k \times S_l$. To satisfy the third constraint, project to $S_n$ invariants in $V_H^{(a)} \otimes V_H^{(b)}$. This constrains $A = A'$. So the number of $S_k \times S_l \times S_n$ invariants is

$$\sum_{A \in S_n \times S_k} \text{Mult}(A, [k]; S_n \times S_k) \text{ Mult}(A, [l]; S_n \times S_l)$$

The expansions are explained further and used in the construction of BPS states in [15]. The generating functions for these multiplicities are derived in [15]. $\text{Mult}(A, [k]; S_n \times S_k) \equiv Z_{SH}^k$ is the coefficient of $q^k$ in

$$Z_{SH}(q; A) = (1 - q) \sum_{c_I} \frac{1}{z} \prod_{b} \left(1 - q^{b_b}\right)$$

$$= \sum_{k} q^k Z_{SH}^k(A)$$

Here $c_i$ is the length of the $i$th column in $A_1$, $b$ runs over boxes in the Young diagram $A$, and $h_b$ is the hook length of the box $b$. Thus, for the number of primaries constructed from $z_i, w_i$ we get

$$\sum_{A \in S_n \times S_k} Z_{SH}^k(A_1) Z_{SH}^l(A_1)$$

These are primaries of weight $n + k + l$, with $(J_1^I, J_2^{\bar{I}}) = (k^{\bar{I}}, l^I)$. We can also show directly that $Z_n(s, x, y)$ in (24) is a sum over irreps $A_1$ of $S_n$ as above. Thus

$$Z_n(s, x, y) = \sum_{A \in S_n} Z_{SH}^k(A_1) Z_{SH}^l(A_1) s^{n+k+l} x^{\frac{k}{2}} y^{\frac{l}{2}}$$

$$= s^n \sum_{A \in S_n} Z_{SH}(q = s \sqrt{x/y}, A_1) Z_{SH}(q = s \sqrt{x/y}, A_1)$$

(35)

Using the generating function given, we get the rational expressions for $Z_3(s, x, y), Z_4(s, x, y)$ used in section III by explicitly doing the sum over $A_1$.

This structure in the counting problem provides an explicit construction formula. First, decompose the $z$ and $w$ polynomials into definite $S_n$ irreps. The projector onto $r$ from the tensor product of $k$ copies of the hook is

$$P_{a_1 \cdots a_k, b_1 \cdots b_k} = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_n(\sigma)(\Gamma_q^k(\sigma))_{a_1 \cdots a_k, b_1 \cdots b_k}$$

We also need the projection onto the symmetric irrep

$$P_{a_1 \cdots a_n, b_1 \cdots b_n} = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_n(\sigma)(\Gamma_q^k(\sigma))_{a_1 \cdots a_n, b_1 \cdots b_n}$$

(36)

Using these two projectors, the polynomials constructed using these hook variables are

$$\sum_{A} P_A(z^I, w^I) \hat{n}_A^{a_1 \cdots a_k, b_1 \cdots b_l} = P_{c_1, c_2, \cdots, c_{k+l}} \times P_{a_1 \cdots a_k, b_1 \cdots b_l} \times \cdots \times \cdots$$

(37)

which imply the Laplacian conditions. As a result, taking all possible $k, l$, we have a space of solutions to the constraints which forms a ring due to the Leibniz rule for products of functions. This is the polynomial ring of holomorphic functions for

$$C^2/((C^2 \times S_n)$$

Using generalities about Hilbert series for algebraic varieties (see [16, 17] for applications in the context of moduli spaces of SUSY gauge theories), we see from (29) that for $n = 3$ the polynomials $P_A(z, w)$ are a finitely generated polynomial ring with 3 generators. The explicit constructions described above allow us to identify the generators $(z_{ij} \equiv z_i - z_j)$

$$(z_{12})^{2k} + (z_{13})^{2k} + (z_{23})^{2k} \leftrightarrow (s^2xy)^k$$

$$(z_{13} + z_{23})^{k}(z_{31} + z_{21})^{k}(z_{12} + z_{32})^{k} \leftrightarrow (s^3\sqrt{x^2y^2})^k$$

(38)
orbifolds the adjoint. The relevant geometries are the Calabi-Yau and 

\[ (s^4x^2)^k \quad (39) \]

This is explained in more detail in the forthcoming paper [13].

The Hilbert series associated to the counting of primary fields ensures a palindromic property of the numerators. This can be verified for \( Z_3(s, x, y), Z_4(s, x, y) \). A general property of the numerators

\[ Q_n(s, x, y) = \sum_{k=0}^{D} a_k(x, y)s^k \quad (40) \]

is that \( a_{D-k}(x, y) = a_k(x, y) \). A direct proof using the combinatoric expressions like equation (28) in terms of symmetric group representation theory data, is given in [13]. The theorem of Stanley [13] suggests that these orbifolds are Calabi-Yau. This can be explicitly demonstrated by constructing the top form and verifying that it is nowhere vanishing [13].

The above argument starting from counting to motivate a construction of the primary operators and then an associated Calabi-Yau geometry goes through when the single scalar is generalized to the \( O(N) \) vector model and to the free \( U(N) \) gauge theory with \( \phi \) a matrix in the adjoint. The relevant geometries are the Calabi-Yau orbifolds

\[(\mathbb{C}^2)^n/((\mathbb{C}^2 \times S_n)[S_2]) \quad (41)\]

and

\[((\mathbb{C}^2)^n \times S_n)/((\mathbb{C}^2 \times S_n) \quad (42)\]

respectively. It is fascinating that non-trivial properties of the combinatorics of primary fields in free four dimensional conformal field theory is related to the geometry of Calabi-Yau orbifolds [29], [11] and [12].

ACKNOWLEDGMENTS

This work of RdMK, PR and RR is supported by the South African Research Chairs Initiative of the Department of Science and Technology and National Research Foundation as well as funds received from the National Institute for Theoretical Physics (NITheP). SR is supported by the STFC consolidated grant ST/L000415/1 String Theory, Gauge Theory & Duality and a Visiting Professorship at the University of the Witwatersrand, funded by a Simons Foundation grant held at the Mandelstam Institute for Theoretical Physics.

Appendix A: Appendix: Leading Twist Generators

The counting formula (25) demonstrates that the leading twist primaries form a ring generated by \( n-1 \) generators. These generators are given by constructing the \( n-1 \) possible independent \( S_n \) invariants out of the hook variables, which are given by [13]

\[ X^{(a)} = \frac{1}{\sqrt{a(a+1)}}(x_1 + \cdots + x^a - ax^{a+1}) \quad (A1) \]

For example, for \( n = 2 \) fields the polynomials are generated by \( (z_1 - z_2)^2 \). The polynomials corresponding to primaries are

\[ (z_1 - z_2)^{2k} \quad (A2) \]

Using (A1) it is easy to see that (these primaries vanish if \( s \) is odd)

\[ O_s = (z_1 - z_2)^s \leftrightarrow \frac{s!}{2} \sum_{k=0}^{s} (-1)^k \frac{(k!)^2}{(s-k)!^2} \partial_x^{-k} \phi \partial_x^k \phi \quad (A3) \]

reproducing the higher spin currents, given for example in [13]. For \( n = 3 \) fields the ring of polynomials that correspond to primary operators is generated by

\[ (z_1 - z_2)^2 + (z_1 - z_3)^2 + (z_2 - z_3)^2 \quad (A4) \]

and

\[ (z_1 + z_2 - 2z_3)(z_3 + z_2 - 2z_1)(z_1 + z_3 - 2z_2) \quad (A5) \]

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