Exact dynamical structure factor of the degenerate Haldane-Shastry model

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Recently much interest has been focused on magnetic systems with orbital degeneracy \( SU(4) \). In the case of two-fold orbital degeneracy, the total degeneracy per site becomes \( 4 (= 2 \times 2) \), and the simplest model to realize this situation in one dimension is the spin chain with \( SU(4) \) symmetry. The static property of the \( SU(4) \) spin chain has been studied mainly by numerical methods. It has been reported that the spin correlation has a period of four unit cells, and that the asymptotic decay has a power-law exponent different from unity \( \frac{1}{2} \). Such exponent has also been derived by use of conformal field theory \( \mathcal{K} \). In view of this situation, one can naturally ask how the dynamical property depends on the number of internal degrees of freedom. Experimental investigations of orbitally degenerate quasi-one-dimensional magnetic compounds such as \( \text{NaV}_2\text{O}_5 \) \( \mathcal{K} \) are being performed with increasing accuracy. Hence it is useful to clarify the difference from systems without orbital degeneracy not only for static properties, but also for dynamic ones.

In this Letter, we derive exact analytic formula for the dynamical structure factor \( S(q, \omega) \) of the \( SU(2) \) spin chain at zero temperature, and provide intuitive explanations. The analytic result is checked numerically for finite systems.

The dynamical structure factor \( S(q, \omega) \) of the \( K \)-component \( (K = 2, 3, 4) \) spin chain with the \( 1/r^2 \) exchange is derived exactly at zero temperature for arbitrary size of the system. The result is interpreted in terms of a free quasi-particle picture which is generalization of the spinon picture in the \( SU(2) \) case; the excited states consist of quasi-particles each of which is characterized by a number \( \beta \) of \( K \)-component \( \text{SU}(4) \) exchange \( 2 \mathbf{S}_i \cdot \mathbf{S}_j + 1/2 \). This model is a generalization of the Haldane-Shastry (HS) model \( \mathcal{K} \) for the \( SU(2) \) chain, and hence is called the \( SU(K) \) HS model in the following.

In the original HS model, the spinons form an ideal spin \( 1/2 \) “semion” gas \( \mathcal{K} \) obeying the fractional exclusion statistics \( \mathcal{K} \). The dynamical structure factor \( S(q, \omega) \) of the \( SU(2) \) HS model has a remarkably simple structure in terms of the spinon picture: only two spinons contribute to \( S(q, \omega) \). Since the semionic statistics is applicable only to the case of \( SU(2) \), one has to take more general fractional statistics in order to apply a quasi-particle description.

To derive the exact formula for the dynamical structure factor, we use the \( U(K) \) spin Calogero-Sutherland (CS) model \( \mathcal{K} \) as an auxiliary. The Hamiltonian of the \( U(K) \) spin CS model is given by

\[
H_{\text{spin-CS}} = \frac{1}{2} \sum_{i=1}^{N} \sum_{1 \leq i < j \leq N} J_{ij} P_{ij} + \frac{\pi}{L} \sum_{1 \leq i < j \leq N} \beta (\beta + P_{ij}) \frac{\beta + P_{ij}}{2L(x_i - x_j)},
\]

where \( \beta > 0 \) is the coupling parameter and \( L \) is the size of system. This continuous model is more tractable than the \( SU(K) \) HS model, because the eigenfunctions of the model have been explicitly constructed \( \mathcal{K} \). We take the strong coupling limit \( \beta \to \infty \) of the \( U(K) \) spin CS model. Then particles crystallize with the lattice parameter \( L/N \) which is taken as the unit of length. Then we are left with the center of mass motion, the lattice vibration and the dynamics of the internal degrees of freedom which is called the “color”. The color dynamics is equivalent to the dynamics of the \( SU(K) \) HS model. The freezing trick described above was firstly introduced by Polychronakos \( \mathcal{K} \), and has been applied to thermodynamics of lattice models \( \mathcal{K} \). The present Letter is the first application of the freezing trick to dynamical quantities.
In the $U(2)$ spin CS model, Uglov has derived the exact formula of the dynamical spin-density correlation function with a finite number of particles \([7]\). We shall first extend his result to the case of $K \geq 3$, and then take the strong coupling limit. In doing so we have to make correspondence between physical quantities defined in the continuum and discrete spaces. Let us define the following operator in the continuum space:

$$X_q^{\delta} = \frac{1}{\sqrt{L}} \sum_{j=1}^{N} X_j^{\delta} e^{-iqx_j}, \quad (4)$$

where the momentum $q$ takes values $2\pi n/L$ with $n$ an arbitrary integer. We first derive the dynamical structure factor in the continuum model defined by

$$S^{(\gamma\delta)}(q, \omega; \beta) = \sum_{\alpha} \langle \{\alpha\} | X_q^{\delta} | 0 \rangle^2 \delta(\omega - E_{\alpha} + E_0), \quad (5)$$

where \{\{\alpha\}\} is the normalized complete basis of the system with eigenvalues \{\{E_{\alpha}\}\}, and $0\rangle$ is the groundstate. We assume that $N$ is an integer multiple of $K$ so that the groundstate is a nondegenerate singlet.

In the strong coupling limit the coordinate $x_j$ in Eq. (3) is written as $x_j = R_j + u_j$, where $R_j = j$ is a lattice point, and $u_j$ describes the lattice vibration. Typically for the uniform motion of the lattice described by $u_j = \text{const}$, we may regard $u_j$ as a small quantity. In fact the density response can be shown to be smaller than the spin response by $O(\beta^{-1})$. Then the dynamical structure factor of the $SU(K)$ HS model is given simply by the strong coupling limit of Eq. (3) provided that one restricts $q$ in the range of the first Brillouin zone: $|q| \leq \pi$.

The dynamical structure factor (3) can be derived in a manner analogous to the case of $K = 2$ \([17]\). However, the following observations are necessary for generalization. First, each excited state relevant to Eq. (3) transforms as one of the weight vectors for the adjoint representation of $SU(K)$. This observation allows us to find the selection rule for the $SU(K)$ spin. In the $SU(2)$ case, this selection rule is reduced to the simple fact that excited states relevant to Eq. (3) are spin-triplet states. Second, in order to derive the matrix element in Eq. (3) we find a convenient set of operators given by

$$J_a = \frac{1}{\sqrt{L}} \sum_q \left( \sum_{b=1}^{a} X_q^{bK-a+b} + \sum_{b=1}^{K-a} X_q^{a+b} \right) \quad (6)$$

with $a = 1, \cdots, K - 1$. More details of calculation will be presented elsewhere.

In order to give the formula of $S^{(\gamma\delta)}(q, \omega)$, we fix some notations for partitions \([17]\). Let $\Lambda_N^{(K)}$ be the set of all partitions whose length are less than $N$ and the largest entry is less than $K + 1$. Namely we have $\Lambda_N^{(K)} = \{ \lambda (= (\lambda_1, \lambda_2, \cdots, \lambda_N)) | K \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0 \}$. For a partition $\lambda$, we define subsets by $C_K(\lambda) = \{(i, j) \in \lambda | j - i \equiv 0 \text{ mod } K \}$ and $H_K(\lambda) = \{(i, j) \in \lambda | \lambda_i + \lambda_j' - i - j + 1 \equiv 0 \text{ mod } K \}$.

Next we define the concept type of a partition. The type of a given partition $\lambda$ contains sufficient information for determining the sets $C_K(\lambda)$ and $H_K(\lambda)$ explicitly. We introduce a reductive transformation $\tau$ on the set of all partitions as follows \([22]\).

(i) If there exist $K$ rows or $K$ columns which have same number of boxes in a partition, remove those rows or columns;

(ii) Apply the reduction (i) repeatedly until the newly generated partition is no longer reducible.

We then determine a subset $A_N^{(K)}$ of $\Lambda_N^{(K)}$ as the image of $\tau$, i.e., $A_N^{(K)} = \tau(\Lambda_N^{(K)})$. For any partition $\lambda \in \Lambda_N^{(K)}$, we say that $\nu$ is of the type $\lambda$ if $\nu = \tau(\lambda)$. The total number of types increases from 3 in the case of $SU(2)$ to 25 in $SU(3)$, and to 252 in $SU(4)$. For a box $s = (i, j) \in \lambda$, the numbers $l_{\lambda}(s) = \lambda_j' - i$ and $l'_{\lambda}(s) = i - j$ are called the leg-length and coleg-length, respectively. For any subset $\nu \subseteq \lambda$, the order $|\nu|$ is defined as the number of boxes in $\nu$.

Now we give the exact formula for the dynamical structure factor $S^{(\gamma\delta)}(q, \omega)$ for a finite size of the system with $\gamma \neq \delta$. We obtain

$$S^{(\gamma\delta)}(q, \omega) = \sum_{\lambda} \langle F^{(K)}_{\lambda} \rangle^2 \delta(\omega - E_{\lambda}), \quad (7)$$

where the primed summation is restricted so as to satisfy the momentum conservation $q = 2\pi|C_K(\lambda)|/N$, and the color selection rule for $\lambda \in \Lambda_N^{(K)}$. The latter is conveniently implemented by introducing a subset $A_N^{(K; \gamma\delta)}$ of $A_N^{(K)}$, and decomposes the summation over $\lambda$ by each type $\nu = \tau(\lambda)$ such that $\nu \in A_N^{(K; \gamma\delta)}$. For example, in the case of $K = 3$, we have $A_N^{(3; 21)} = \{(2, 1, 1), (3, 3, 1), (3, 3, 2, 2)\}$ and $A_N^{(3; 13)} = \{(1), (2, 2), (3, 2, 1, 1)\}$.

In Eq. (7) the excitation energy is given by

$$E_{\lambda} = \frac{J}{4} \left(\frac{2\pi}{N}\right)^2 \left[(N - 1)|C_K(\lambda)| - 2 \sum_{s \in C_K(\lambda)} l'_{\lambda}(s) \right], \quad (8)$$

and the squared form factor by

$$|F^{(K)}_{\lambda}|^2 = \frac{1}{N} \prod_{s \in H_K(\lambda) \backslash \{(1, 1)\}} l'_{\lambda}(s)^2 \prod_{s \in C_K(\lambda)} \frac{N - l'_{\lambda}(s) - 1}{N - l'_{\lambda}(s) + 1} \prod_{s \in C_K(\lambda)} \frac{N - l'_{\lambda}(s) - 1}{N - l'_{\lambda}(s) + 1}. \quad (9)$$

Since we consider the case of zero external magnetic fields, the $SU(K)$ symmetry demands that $S^{(\gamma\delta)}(q, \omega)$ is actually independent of $(\gamma, \delta)$ as long as $\gamma \neq \delta$. We can prove this fact using the expressions (3)-(5). In the particular case of $K = 2$, our formulae (3)-(5) generalize the known one \([13]\) to arbitrary size of the system. We have checked the validity of Eqs. (7)-(9) with $K = 2$ and 3 by comparing with the numerical result for $N \leq 24$ and...
$N \leq 15$, respectively. The numerical result is obtained
via exact diagonalization and the recursion method. The
agreement is excellent in both cases of $K = 2$ and 3. In
Fig. 2, we present the result for $K = 3$ and $N = 15$.

We now consider the quasi-particle interpretation of
the color selection rule. For labelling the excited states
relevant to $S^{(\gamma \delta)}(q, \omega)$, it is more convenient to use
the conjugate partition $X' = (\lambda'_1, \cdots, \lambda'_K) \in \Lambda^K_N$
instead of $\lambda \in \Lambda^K_N$ Each $\lambda'_i$ has the information
on the momentum and $SU(K)$ spin of a quasi-particle.
We call this quasi-particle a spinon following the $SU(2)$ case. The spinon
is considered to be an object possessing the $SU(K)$
spins. Here the $SU(K)$ spin means the spin $K - 1$ eigenvalues
$(s_1, \cdots, s_{K-1})$ of a set of operators $(h^1, \cdots, h^{K-1})$
where $h^\gamma$ is defined by $h^\gamma = \sum_{i=1}^{N} (X_i^\gamma - X_i^{\gamma+1})/2$
for $\gamma = 1, \cdots, K - 1$. For the $SU(2)$ case, this definition
gives the z-component of spin. The $SU(K)$ spin
of the spinon with $\lambda'_i$ is specified by a certain condition
on the pair $(i, \lambda'_i)$. Since the condition for general $K$
is rather complicated, we give an example in the case of $K = 3$. The $SU(3)$ spin for $\lambda'_i$ is assigned as follows: It is $(0, 1/2)$ if $(i, \lambda'_i) = (0, 0), (1, 1), (2, 2)$ mod 3; $(1/2, -1/2)$
if $(i, \lambda'_i) = (0, 1), (1, 2), (2, 0)$ mod 3; and $(-1/2, 0)$ if
$(i, \lambda'_i) = (0, 2), (1, 0), (2, 1)$ mod 3. It is important to
note that a spinon transforms as a weight vector of the
fundamental representation $K$ of $SU(K)$.

From the formulae (7)-(9), we can conclude that relevant
excited states for the $SU(K)$ HS model consist of K
spinons. Moreover the conditions on the type of excited states lead to an important consequence: $K$-spinon ex-
cited states have $K - 1$ different $SU(K)$ spins. That is,
the excited states contain $K - 1$ species of quasi-particles.
For instance, in the case of $K = 3$ excited states relevant
to $S^{(13)}(q, \omega)$ consist of three spinons with $SU(3)$ spins
$(0, 1/2), (0, 1/2)$ and $(1/2, -1/2)$. For the $SU(2)$ case, we
recover the well-known fact that only two spinons with
the same spin contribute to $S(q, \omega)$ [13].

The $K$-spinon excitation belongs to the tensor represen-
tation $K^{\otimes K}$ of $SU(K)$. This representation contains
the adjoint representation as an irreducible component.
From the condition stated above for the $K$-spinon excita-
tion, we see that the $K$-spinon excitation transforms as one of the weight vectors for the adjoint representation.
This is consistent with the condition for the $SU(K)$ spins
of the excited states which are relevant to $S^{(\gamma \delta)}(q, \omega)$.

Now we present the thermodynamic limit of the for-
mae (7)-(9). Performing a procedure similar to that
in Refs. 23,24, for $K = 2, 3$ and 4, we obtain the final
result as follows:

$$S(q, \omega) = A_K \sum_{1 \leq a < b \leq K} \prod_{i=1}^{K} \int_{-1}^{1} dk_i |F_{ab}^{(K)}(k)|^2 \delta(\omega - \epsilon(k)), \quad (10)$$

where $\epsilon(k) = [\pi v_s/(2K)] \sum_{i=1}^{K} (1 - k_i^2)$ with $v_s = J/\pi/2,$ and
$p(k) = (\pi/K) \sum_{i=1}^{K} k_i$. In the above formula, $A_K$ is a normalization constant given by

$$A_K = \frac{2^K \pi}{K^3(K-1)} \prod_{j=1}^{K} \frac{\Gamma((K-1)/K)}{\Gamma(j/K)^2}, \quad (11)$$

and the form factor is given by

$$F_{ab}^{(K)}(k) = \frac{|k_a - k_b|^{g_K} \prod_{1 \leq i < j \leq K_i \neq i} |k_i - k_j|^{g_K' \delta}}{\prod_{i=1}^{K} (1 - k_i^2)^{(1-g_K)/2}} \quad (12)$$

with $g_K = (K-1)/K$ and $g_K' = -1/K$. Since the formula
(10) for $S^{(\gamma \delta)}(q, \omega)$ does not depend on the pair
$\gamma, \delta$ with $\gamma \neq \delta$, we have omitted the superscript. Un-
fortunately our exact result is inconsistent with a conjec-
ture proposed several years ago [23].

For $K = 2$, the formula reproduces the result of Haldane-Zirnbauer [3] which was obtained by a com-
pletely different method. Notice that the second product in the numerator of Eq. (12) is absent in the $SU(2)$ case.

We can derive static structure factor $S(q)$ by integrat-
ing over $\omega$ in Eq. (10). In the low energy limit we recover
the results of Ref. 1 which are obtained by conformal
field theory. By analyzing the form factor for $q \sim 2k_F$
with $k_F = \pi/K$, we can show that it has the asymptotic
form

$$S(q) \sim a_1 |q - 2k_F|^{\alpha_1 - 1} \quad (13)$$

with the exponent $\alpha_1 = 2-2/K$ and a non-universal con-
stant $a_1$. In the real space the spin (or color) correlation
decays as $b_1 \cos(2k_F x)|x|^{-\alpha_1}$ with a certain coefficient
$b_1$. Similar analysis shows that there are also weaker
singularities around $q = 2k_F l$ for $l = 2, \cdots, K - 1$ with
exponents $\alpha_l = 2l(1 - l/K)$. These $K - 1$ singularities
are similar to $K - 1$ gapless bosonic modes [3].

The spinon interpretation of the formula (10) goes as follows. As in the case of finite systems, the excited
states for $S(q, \omega)$ in the thermodynamic limit consist of $K$
spinons with $K - 1$ different $SU(K)$ spins. This fact
means, as in the low energy limit, the dynamics of the
$SU(K)$ HS model can be described by $K - 1$ species of
quasi-particles. This simple structure reflects the Yang-
gian symmetry of the $SU(K)$ HS model [26]. In the form
factor (12), the factor $|k_a - k_b|^{g_K}$ represents the interac-
tional interactions of spinons with the same $SU(K)$ spin, while the factor $|k_i - k_j|^{g_K'}$ represents those of spinons
with different $SU(K)$ spins. We refer to Ref. [7] for
more detailed explanation of statistical interactions. It
will be interesting to consider the relation between our
results and the exclusion statistics in conformal field
theory discussed in Refs. 25,26,27.

The support of $S(q, \omega)$ represents the region in the
momentum-frequency plane where $S(q, \omega)$ takes the non-
zero value. We see that the support of $S(q, \omega)$ as deter-
ned from the formula (10) is compact, i.e., there is no
intensity outside of the finite area. In the $SU(3)$ case,
for example, the support is determined as:
\[ \omega \leq \frac{v_s}{(2\pi)}[q(2\pi - q) = \epsilon^{(q)}(q)] \text{ for } 0 \leq q \leq 2\pi; \]
\[ \omega \geq \frac{3v_s}{(2\pi)}[q(2\pi - q) = \epsilon^{(q)}(q)] \text{ for } 0 \leq q \leq 2\pi/3; \]
\[ \omega \geq \frac{3v_s}{(2\pi)}[q - 2\pi/3)(4\pi/3 - q) \text{ for } 2\pi/3 \leq q \leq 4\pi/3; \]
\[ \omega \geq \frac{3v_s}{(2\pi)}[q - 4\pi/3)(2\pi - q) \text{ for } 4\pi/3 \leq q \leq 2\pi. \]

For a general value of \( K \), there are \( K \) lower boundaries as given by
\[ \epsilon^{(l)}_j(q) \equiv \left[ K v_s/(2\pi) \right](q - 2\pi j/K)[2\pi(j + 1)/K - q], \]
for \( 2\pi j/K \leq q \leq 2\pi(j + 1)/K \) with \( j = 0, 1, \ldots, K - 1 \).

The behavior of \( S(q, \omega) \) near the boundaries of the support is derived for general \( K \) as follows. We can show that there is a stepwise discontinuity at the upper boundary \( \omega = \epsilon^{(q)}(q) \). On the other hand, there are divergent singularities at the lower boundaries \( \omega = \epsilon^{(l)}_0(q) \) and \( \omega = \epsilon^{(l)}_{K-1}(q) \). Here \( S(q, \omega) \) diverges by the power law with the exponent \(-1/K\). At the other lower boundaries \( \omega = \epsilon^{(l)}_j(q) \) with \( j \neq 0, K - 1 \), \( S(q, \omega) \) has threshold singularities but no divergence. As in the \( SU(2) \) case \([29,31]\), we expect that the divergences at two of the lower boundaries occur also in the \( SU(K) \) Heisenberg model with the nearest-neighbor exchange.

In conclusion, we have derived the exact formulae \([5,6]\) and \([10]\) for \( S(q, \omega) \) of the \( SU(K) \) HS model for arbitrary size of the system at zero temperature. Our exact result of \( S(q, \omega) \) for \( K \leq 4 \) is likely to be valid for larger \( K \) as well. We have also clarified the quasi-particle picture of the spin dynamics. The relevant excited states consist of \( K \) spinons with \( K - 1 \) different \( SU(K) \) spins.

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FIG. 1. Numerical result of the dynamical structure factor \( S(q, \omega) \) in the case of \( K = 3 \) and \( N = 15 \). The vertical and horizontal axis represent the rescaled energy and momentum, respectively. The intensity is proportional to the area of the circle. The solid lines are the dispersion lines of the elementary excitations in the thermodynamic limit. The analytic results are in excellent agreement with numerical ones, and are not distinguishable from the latter.
