Abstract. In this paper we study the the Gauss image problem, which is a generalization of the Aleksandrov problem in convex geometry. By considering a geometric flow involving Gauss curvature and functions of normal vectors and radial vectors, we obtain the existence of smooth solutions to this problem.

1. Introduction

Let $K \subset \mathbb{R}^n$ be a convex body which contains the origin in its interior and $x \in \partial K$ be a boundary point, then the normal cone of $K$ at $x$ is defined by

$$
\mathcal{N}(K, x) = \{v \in S^{n-1} : \langle y - x, v \rangle \leq 0 \text{ for all } y \in K\},
$$

where $\langle y - x, v \rangle$ denotes the standard inner product of $y - x$ and $v$ in $\mathbb{R}^n$. For $\omega \subset S^{n-1}$, the radial Gauss image of $\omega$ is defined by

$$
\alpha_K(\omega) = \bigcup_{x \in \rho_K(\omega)} \mathcal{N}(K, x) \subset S^{n-1},
$$

where $\rho_K : S^{n-1} \to \partial K$ is the radial function of $\partial K$ (see Section 2 for the definition). Recently, Boroczky, Lutwak, Yang, Zhang and Zhao [4] proposed the Gauss image problem which link two given submeasures via the radial Gauss image of a convex body:

**The Gauss image problem.** Suppose $\lambda$ is a submeasure defined on the Lebesgue measurable subsets of $S^{n-1}$, and $\mu$ is a Borel submeasure on $S^{n-1}$. What are the necessary and sufficient conditions, on $\lambda$ and $\mu$, so that there exists a convex body $K$ such that

$$
\lambda(\alpha_K(\cdot)) = \mu
$$
on the Borel subsets of $S^{n-1}$? And if such a body exists, to what extent is it unique?

When $\lambda$ is spherical Lebesgue measure, the Gauss image problem is just the classical Aleksandrov problem. It is necessary to contrast the Gauss image problem with the various Minkowski problems and dual Minkowski...
problems that have been extensively studied, see [7, 12, 15, 29, 32, 33, 38, 39, 40, 41, 42, 44, 48, 49] for the \( L_p \)-Minkowski problem, [6, 23, 25, 26, 36, 46, 47] for the dual Minkowski problem, [5, 10, 11, 27, 28, 35, 43] for the \( L_p \) dual Minkowski problem, [3, 21, 24, 31] for the Orlicz Minkowski problem, [17, 19, 37] for the dual Orlicz Minkowski problem. In the Gauss image problem, a pair of submeasures is given and it is asked if there exists a convex body “linking” them via its radial Gauss image. However, in a Minkowski problem, only one measure is given, and the question asks if this measure is a specific geometric measure of a convex body.

To state the solutions to the Gauss image problem in [4]. We introduce some concepts (see [4] for details). If \( \omega \subset S^{n-1} \) is contained in a closed hemisphere, then the polar set \( \omega^* \) is defined by
\[
\omega^* = \{ v \in S^{n-1} : \langle u, v \rangle \leq 0 \text{ for all } u \in \omega \}.
\]

**Definition 1.** Two Borel measures \( \mu \) and \( \lambda \) on \( S^{n-1} \) are called Aleksandrov related if
\[
\lambda(S^{n-1}) = \mu(S^{n-1}) > \lambda(\omega^*) + \mu(\omega)
\]
for any compact, spherically convex set \( \omega \in S^{n-1} \).

Note that \( \lambda(S^{n-1}) = \mu(S^{n-1}) \) is obvious for a solution to (1). The following existence result for solutions to the Gauss image problem was proved in [4].

**Theorem 1.** Suppose \( \lambda, \mu \) are Borel measures on \( S^{n-1} \) and \( \lambda \) is absolutely continuous. If \( \mu \) and \( \lambda \) are Aleksandrov related, then there exists a body \( K \) containing the origin in its interior such that
\[
\lambda(\alpha_K(\cdot)) = \mu(\cdot).
\]

Note that for the special case in which \( \mu \) is a measure that has a density with respect to the spherical Lebesgue measure, say \( f \), and \( \lambda \) is a measure that has a density with respect to the spherical Lebesgue measure, say \( g \). In this case, \( \mu \) and \( \lambda \) on \( S^{n-1} \) are Aleksandrov related if
\[
\int_{S^{n-1}} f = \int_{S^{n-1}} g > \int_{\omega} f + \int_{\omega^*} g
\]
for any compact, spherically convex set \( \omega \in S^{n-1} \). Moreover, the geometric problem (1) is the equation of Monge-Ampère type
\[
g \left( \frac{\nabla h + hx}{|\nabla h + hx|} \right) |\nabla h + hx|^{-n} h \det(\nabla^2 h + hI) = f \quad \text{on} \quad S^{n-1},
\]
where \( h \) is the support function of the polar body \( K^* \) of \( K \) which is defined as
\[
K^* = \{ x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K \}.
\]
Here \( \nabla \) is the covariant derivative with respect to an orthonormal frame on \( S^{n-1} \), \( I \) is the unit matrix of order \( n-1 \), and \( \nabla h(x) + h(x)x \) is just the point on \( \partial K^* \) whose outer unit normal vector is \( x \in S^{n-1} \).

In this paper we study the existence of smooth solutions to the equation (1). We obtain the following existence result.
Theorem 2. Suppose that $f$ and $g$ are two positive smooth functions on $\mathbb{S}^{n-1}$. If $f$ and $g$ satisfy the condition (3), then there exists a smooth solution to the equation (4).

The proof of Theorem 2 is inspired by [36], where the existence of smooth solutions to the Aleksandrov and dual Minkowski problem was obtained by studying a generalized Gauss curvature flow. In fact, the Gauss curvature flow and its various generalizations have been extensively studied by many scholars; see for example [1, 2, 8, 9, 10, 14, 16, 20, 22, 30, 37, 45] and the references therein.

Let $M_0$ be a smooth, closed, uniformly convex hypersurface in $\mathbb{R}^n$, which contains the origin in its interior and given by a smooth embedding $X_0 : \mathbb{S}^{n-1} \to \mathbb{R}^n$. We consider a family of closed hypersurfaces $\{M_t\}$ given by $M_t = X(\mathbb{S}^{n-1}, t)$, where $X : \mathbb{S}^{n-1} \times [0, T) \to \mathbb{R}^n$ is a smooth map satisfying the following initial value problem:

\[
\begin{aligned}
\frac{\partial X}{\partial t}(x, t) &= -\frac{f(\nu)}{g(X/|X|)}|X|^n K \nu + X, \\
X(x, 0) &= X_0(x).
\end{aligned}
\]  

(5)

Here $\nu$ is the unit outer normal vector of the hypersurface $M_t$ at the point $X(x, t)$, $K$ is the Gauss curvature of $M_t$ at $X(x, t)$, and $T$ is the maximal time for which the solution exists. We obtain the long-time existence and convergence of the flow (5).

Theorem 3. Suppose $f$ and $g$ satisfy the assumptions of Theorem 2. Let $M_0$ be a smooth, closed, uniformly convex hypersurface in $\mathbb{R}^n$, which contains the origin in its interior. Then, the flow (5) has a unique smooth solution, which exists for all time $t > 0$. Moreover, when $t \to \infty$, a subsequence of $M_t = X(\mathbb{S}^{n-1}, t)$ converges in $C^\infty$ to a smooth, closed, uniformly convex hypersurface, whose support function is a smooth solution to the equation (4).

This paper is organized as follows. In section 2, we give some basic knowledge about convex hypersurfaces and the flow (5). In section 3, more properties of the flow (5) will be proved, based on which we can obtain the uniform lower and upper bounds of support functions of $\{M_t\}$ via delicate analyses. In the last section, the long-time existence and convergence of the flow (5) will be proved, which completes the proofs of our theorems.

2. Preliminaries

2.1. Basic properties of convex hypersurfaces. We first recall some basic properties of convex hypersurfaces in $\mathbb{R}^n$; see [15] for details. Let $M$ be a smooth, closed, uniformly convex hypersurface in $\mathbb{R}^n$ enclosing the origin. The support function $h$ of $M$ is defined as

\[
h(x) := \max_{y \in M} \langle y, x \rangle, \quad \forall x \in \mathbb{S}^{n-1},
\]  

(6)
where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$.

The convex hypersurface $M$ can be recovered by its support function $h$. In fact, writing the Gauss map of $M$ as $\nu_M$, we parametrize $M$ by $X : S^{n-1} \to M$ which is given as

$$X(x) = \nu_M^{-1}(x), \quad \forall x \in S^{n-1}.$$  

Note that $x$ is the unit outer normal vector of $M$ at $X(x)$. On the other hand, one can easily check that the maximum in the definition (6) is attained at $y = \nu_M^{-1}(x)$, namely

$$h(x) = \langle x, X(x) \rangle, \quad \forall x \in S^{n-1}.$$  

Let $e_{ij}$ be the standard metric of the unit sphere $S^{n-1}$, and $\nabla$ be the corresponding connection on $S^{n-1}$. Then, it is easy to check that

$$X(x) = \nabla h(x) + h(x)x, \quad \forall x \in S^{n-1}.$$  

By differentiating (7) twice, the second fundamental form $A_{ij}$ of $M$ can be also computed in terms of the support function:

$$A_{ij} = \nabla_i \nabla_j h + he_{ij},$$

where $\nabla_i \nabla_j$ denotes the second order covariant derivative with respect to $e_{ij}$. The induced metric matrix $g_{ij}$ of $M$ can be derived by Weingarten’s formula:

$$e_{ij} = \langle \nabla_i x, \nabla_j x \rangle = A_{ik} A_{lj} g_{kl},$$

The principal radii of curvature are eigenvalues of the matrix $b_{ij} = A_{ik} A_{lj} g_{kl}$. When considering a smooth local orthonormal frame on $S^{n-1}$, by virtue of (9) and (10), we have

$$b_{ij} = A_{ij} = \nabla_i \nabla_j h + h \delta_{ij}.$$  

In particular, the Gauss curvature of $M$ at $X(x)$ is given by

$$K(x) = \frac{1}{\det(\nabla_i \nabla_j h + h \delta_{ij})}.$$  

We shall use $b^{ij}$ to denote the inverse matrix of $b_{ij}$.

The radial function $\rho$ of the convex hypersurface $M$ is defined as

$$\rho(u) := \max \{ \lambda > 0 : \lambda u \in M \}, \quad \forall u \in S^{n-1}.$$  

Note that $\rho(u)u \in M$. The Gauss map $\nu_M$ can be computed as

$$\nu_M(\rho(u)u) = \frac{\rho(u)u - \nabla \rho}{\sqrt{\rho^2 + |\nabla \rho|^2}}.$$  

If we connect $u$ and $x$ through the following equality:

$$\rho(u)u = X(x) = \nabla h(x) + h(x)x = \nabla h(x),$$

where $\nabla$ is the standard connection of $\mathbb{R}^n$, then we have the following relations

$$x = \frac{\rho(u)u - \nabla \rho}{\sqrt{\rho^2 + |\nabla \rho|^2}}, \quad u = \frac{\nabla h + h(x)x}{\sqrt{|\nabla h|^2 + h^2}}.$$
2.2. Geometric flow and its associated functional. Recalling the evolution equation of $X(x,t)$ in the geometric flow (5), and using similar computations as in [45], we obtain the evolution equation of the corresponding support function $h(x,t)$:

\[
\frac{\partial h}{\partial t}(x,t) = -f(x)\rho^n(u)K(x,t) + h(x,t) \quad \text{in} \quad S^{n-1} \times (0, T).
\]

Since $M_t$ can be recovered by $h(\cdot, t)$, the flow (15) is equivalent to the original flow (5).

Denote the radial function of $M_t$ by $\rho(u, t)$. For any $t$, let $u$ and $x$ be related through the following equality:

\[
\rho(u, t)u = \nabla h(x, t) = \nabla h(x, t) + h(x, t)x.
\]

Therefore, $x$ can be expressed as $x = x(u, t)$. By a direction computation (see [10]), we have

\[
\frac{1}{\rho(u, t)} \frac{\partial}{\partial t} \rho(u, t) = \frac{1}{h(x, t)} \frac{\partial}{\partial t} h(x, t).
\]

Now by virtue of (15) and (16), we obtain the evolution equation of $\rho(u, t)$:

\[
\frac{\partial \rho}{\partial t}(u, t) = -\frac{f(x)\rho^{n+1}(u, t)}{g(u)h(x, t)}K(x, t) + \rho(u, t) \quad \text{in} \quad S^{n-1} \times (0, T),
\]

where $x = x(u, t)$ is the unit outer normal vector of $M_t$ at the point $\rho(u, t)u$.

Consider the following functional:

\[
J(t) = \int_{S^{n-1}} f(x) \log h(x, t) \, dx - \int_{S^{n-1}} g(u) \log \rho(u, t) \, du, \quad t \geq 0,
\]

which will turn out to be monotonic along the flow (15).

**Lemma 1.** $J(t)$ is non-increasing along the flow (15). Namely \( \frac{d}{dt} J(t) \leq 0 \), and the equality holds if and only if $M_t$ satisfies the elliptic equation (4).

**Proof.** Using (14) and (16), we have

\[
\frac{d}{dt} J(t) = \int_{S^{n-1}} f \frac{\partial h}{h} \, dx - \int_{S^{n-1}} g \frac{\partial \rho}{\rho} \, du
\]

\[
= \int_{S^{n-1}} \frac{\partial h}{h} (f - \frac{gh}{K\rho^n}) \, dx
\]

\[
= \int_{S^{n-1}} \frac{1}{h} (f - \frac{gh}{K\rho^n})(h - \frac{fK\rho^n}{g}) \, dx
\]

\[
= -\int_{S^{n-1}} \frac{1}{ghK\rho^n}(gh - fK\rho^n)^2 \, dx
\]

\[
\leq 0.
\]
Clearly \( \frac{d}{dt} J(t) = 0 \) if and only if
\[
g h = f K \rho^n.
\]
Namely \( M_t \) satisfies (14). This completes the proof. \( \square \)

**Lemma 2.** Assume that
\[
\int_{S^{n-1}} f(x) dx = \int_{S^{n-1}} g(u) du,
\]
then the log-volume of \( M_t \)
\[
V_g(M_t) = \int_{S^{n-1}} g(u) \log \rho(u, t) du,
\]
(20) remain unchanged under the flow (15).

**Proof.** Using (14) and (17), we have
\[
\frac{d}{dt} V_g(M_t) = \int_{S^{n-1}} \frac{g}{\rho} \partial_t \rho du
= \int_{S^{n-1}} \frac{g}{\rho} (\rho - f \frac{n+1}{g h} K) du
= \int_{S^{n-1}} g(u) du - \int_{S^{n-1}} f(x) \frac{n K}{h} du
= \int_{S^{n-1}} g(u) du - \int_{S^{n-1}} f(x) dx
= 0.
\]
So, we complete the proof. \( \square \)

3. Uniform bounds of support functions

In this section, we will derive uniformly positive lower and upper bounds of support functions along the flow (5). Our idea comes from the proof of Lemma 3.2 in [36].

**Lemma 3.** Suppose \( f \) and \( g \) satisfy the assumptions in Theorem 2. Let \( X(\cdot, t) \) be a strictly convex solution to the flow (5) which encloses the origin for \( t \in [0, T) \), then exists a positive constant \( C \) depending only on \( M_0, f \) and \( g \), such that for every \( t \in [0, T) \),
\[
1/C \leq h(\cdot, t) \leq C \quad \text{on} \quad S^{n-1},
\]
(21) and
\[
1/C \leq \rho(\cdot, t) \leq C \quad \text{on} \quad S^{n-1}.
\]
(22)

**Proof.** Since \( f \) and \( g \) satisfy the condition (3), the measure \( \mu \) with the density \( f \), and the measure \( \lambda \) with the density \( g \) are Aleksandrov related. Then, using Theorem 1 there exists a body \( N^* \) containing the origin in its interior
satisfies the equation (1). Let $N$ be the polar dual of $N^*$. Choosing the constants $s_1 > s_0 > 0$ such that

$$N_0 = s_0 N \subset K_0 \subset s_1 N = N_1.$$ 

Let $r_0$ and $r_1$ be respectively the radial functions of $N_0$ and $N_1$. Clearly, $sN$ is a stationary solution to (5) in the generalised sense. We firstly prove that

$$K_t \subset N_1$$

for all $t > 0$ by a contradiction. Otherwise, there exists a first time $t_0 > 0$ such that

$$\sup_{u \in S^{n-1}} \frac{\rho(u, t_0)}{r_1(u)} = 1.$$ 

Set

$$P = M_{t_0} \cap N_1,$$

which can be a point or a closed set. Clearly, the unit normal vector of $M_{t_0}$ coincide with that of $N_1$ for any $p \in P$. Namely, $\nu_{M_{t_0}}(p) = \nu_{N_1}(p)$ for any $p \in P$. Moreover, replacing $r_1$ by $(1 + a)r_1$ for a small constant $a$, we may assume that

$$\frac{\partial}{\partial t} \rho(u, t) > 0 \quad \text{on} \quad P \times \{t_0\}$$

and also in a neighbourhood of $P \times \{t_0\}$. There exists sufficiently small constants $\epsilon, \delta > 0$ such that

$$\frac{\partial}{\partial t} \rho(u, t) > \delta$$

for all $u \in E = \{\xi \in S^{n-1} : \rho(\xi, t) > (1 - \epsilon)r_1(\xi)\}$. Since $\nu_{M_{t_0}}(p) = \nu_{N_1}(p)$ for any $p \in P$, making $\epsilon$ small again, we have $\nu_{N_1}(u) \approx \nu_{M_{t_0}}(u)$ for $u \in E$. Thus, using the equation (17), the Gauss curvature of $M_{t_0}$ satisfies

$$K(M_{t_0}) < \frac{(\rho(u, t_0) - \delta)g(u)}{f(\nu_{M_{t_0}})\rho^{n+1}(u, t_0)} < \frac{(r_1(u, t_0) - \delta)g(u)}{f(\nu_{N_1})(r_1(u, t_0) - \epsilon)^{n+1}} < \frac{1}{(1 - \epsilon)^n f(\nu_{N_1})(r_1(u, t_0))^n} < K((1 - \epsilon)N_1).$$

Namely, the Gauss curvature of $M_{t_0}$ is strictly small than that of $(1 - \epsilon)N_1$ for all $\xi \in E$. Applying the comparison principle for generalised solutions to the elliptic Monge-Ampère equation (see Theorem 1.4.6 in [18]) to the functions $\rho(u, t_0)$ and $(1 - \epsilon)r_1(u)$, we reach a contradiction. Similarly, we can prove that $N_0 \subset M_t$ for all $t > 0$. \qed
From Theorems 2 and 3 in [4], we know that if \( f \) and \( g \) are even functions satisfying
\[
\int_{S^{n-1}} f(x) dx = \int_{S^{n-1}} g(u) du,
\]
then \( f \) and \( g \) satisfy the condition (23). In this case, if \( M_0 \) is origin-symmetric, we can give a proof of Lemma 3 without using Theorem 1.

**Lemma 4.** Suppose that \( M_0 \) is origin-symmetric, \( f \) and \( g \) are two smooth, positive even functions satisfying the condition (23), then the conclusions in Lemma 3 hold true.

**Proof.** Note that \( M_0 \) is origin-symmetric, \( f \) and \( g \) are even functions, thus \( M_t \) is origin-symmetric and \( h(x, t) \) is an even function. For fixed \( t \in [0, T) \), assume \( h(x, t) \) attains its maximum at \( x_t \). Since \( h(x, t) \) is an even function, we have by the definition of the support function (6)
\[
(24) \quad h(x, t) \geq |\langle x, x_t \rangle| h(x_t, t).
\]
By Lemmas 1 and 2
\[
\int_{S^{n-1}} f(x) \log h(x, 0) dx \geq \int_{S^{n-1}} f(x) \log h(x, t) dx \geq \int_{S^{n-1}} f(x) \log |\langle x, x_t \rangle| h(x_t, t) dx \geq C \log h(x_t, t) - C,
\]
which implies that
\[
\max_{S^{n-1} \times [0, T]} h(x, t) \leq C
\]
for some positive constant \( C \).
The positive lower bound of \( h \) will be proved by contradiction. Let \( \{t_k\} \subset [0, T) \) be a sequence such that
\[
\min_{S^{n-1}} h(\cdot, t_k) \to 0 \quad \text{as} \quad k \to \infty.
\]
Let \( K_t \) be the convex body enclosed by \( M_t \). By Blaschke selection theorem, there is a sequence in \( \{K_{t_k}\} \), which is still denoted by \( \{K_{t_k}\} \), such that
\[
K_{t_k} \to \tilde{K} \quad \text{as} \quad k \to +\infty.
\]
Since \( K_{t_k} \) is an origin-symmetric convex body, \( \tilde{K} \) is also origin-symmetric. Then
\[
\min_{S^{n-1}} h_{\tilde{K}} = \lim_{k \to +\infty} \min_{S^{n-1}} h_{K_{t_k}} = 0.
\]
It follows that \( \tilde{K} \) is contained in a hyperplane in \( \mathbb{R}^n \). Then
\[
\rho_{\tilde{K}} = 0, \quad \text{a.e. in} \ S^{n-1}.
\]
Using Lemma 2, we have for any $\epsilon > 0$

$$V_g(M_0) = V_g(M_{t_k}) \leq \lim_{k \to +\infty} \int_{S^{n-1}} g(u) \log[\rho(u, t_k) + \epsilon] du$$

$$= \int_{S^{n-1}} g(u) \log \epsilon du$$

$$= C \log \epsilon \to -\infty \quad \text{as} \quad \epsilon \to 0,$$

which is a contradiction. Then

$$\min_{S^{n-1} \times [0,T]} h(x, t) \geq C$$

for some positive constant $C$. So we complete the proof. \(\Box\)

Due to the convexity of $M_t$, Lemma 3 also implies the gradient estimates of $h(\cdot, t)$ and $\rho(\cdot, t)$.

**Lemma 5.** Let $X(\cdot, t)$ be a strictly convex solution to the flow (5) which encloses the origin for $t \in [0, T)$, then we have

$$|\nabla h(x, t)| \leq C, \quad \forall (x, t) \in S^{n-1} \times [0, T),$$

$$|\nabla \rho(u, t)| \leq C, \quad \forall (u, t) \in S^{n-1} \times [0, T),$$

where $C$ is a positive constant depending only on the constant in Lemma 3.

**Proof.** By virtue of (12), we have

$$\rho^2 = |\nabla h|^2 + h^2 \geq |\nabla h|^2.$$ 

By (7), (12) and (13), we have

$$h = \frac{\rho^2}{\sqrt{\rho^2 + |\nabla \rho|^2}} \leq \frac{\rho^2}{|\nabla \rho|}.$$ 

Using the two inequalities (25) and (26), the estimates of this lemma now follows directly from Lemma 3. \(\Box\)

4. **Uniform bounds for principal curvatures**

In this section, we continue to establish uniform upper and lower bounds for principal curvatures. These estimates can be obtained by considering proper auxiliary functions, see [10, 13, 36, 37] for similar techniques. First, we need the following lemma.

**Lemma 6.** Given two positive constants $r < R$. For the function

$$G(y) = \frac{|y|^n}{g(\frac{y}{|y|})}, \quad y = (y^1, ..., y^n) \in A(r, R) = \{y \in \mathbb{R}^n : r < |y| < R\},$$

we have

$$\|G\|_{C^k(A(r, R))} \leq C_k \|g\|_{C^k(S^{n-1})},$$
where \( k = 0, 1, 2 \), and \( C_k \) is a positive constant depending only on \( n, r, R \), \( \| g \|_{C^k(\mathbb{S}^{n-1})} \), and \( \min_{\mathbb{S}^{n-1}} g \).

**Proof.** Denote \( \partial_i = \frac{\partial}{\partial y_i} \) for \( 1 \leq i \leq n \). It is clearly seen that

\[
\nabla_i |y|^n = n|y|^{n-2}y^i \quad \text{and} \quad \nabla_i g = \frac{1}{|y|^3} (\nabla g, |y|^2 \partial_i - yy^i).
\]

Thus,

\[
\nabla_i G(y) = \frac{n|y|^{n-2}y^i}{g} - \frac{|y|^{n-3} (\nabla g, |y|^2 \partial_i - yy^i)}{g^2}.
\]

It follows consequently

\[
\| G \|_{C^1(A(r,R))} \leq C_1 \| g \|_{C^1(\mathbb{S}^n)}.
\]

Moreover, we have

\[
\nabla_j \nabla_i |y|^n = n(n-2)|y|^{n-4}y^jy^i + n|y|^{n-2} \delta_{ij}
\]

and

\[
\nabla_j \nabla_i g = \frac{1}{|y|^3} \nabla^2 g \left( |y|^2 \partial_i - yy^i, |y|^2 \partial_j - yy^j \right) + \frac{1}{|y|^3} \left( \nabla g, 2y^j \partial_i - y^i \partial_j \right) - \frac{3y^j}{|y|^5} \left( \nabla g, |y|^2 \partial_i - yy^i \right).
\]

Note that

\[
\nabla_j \nabla_i G = \frac{1}{g} \nabla_j \nabla_i |y|^n - |y|^n \frac{\nabla_j \nabla_i g}{g^2} + 2|y|^n \frac{\nabla_i \nabla_j g}{g^3} - 2 \nabla_i (|y|^n) \frac{\nabla_j g}{g}.
\]

Thus, we get

\[
\| G \|_{C^2(A(r,R))} \leq C_2 \| g \|_{C^2(\mathbb{S}^n)}
\]

in view of

\[
\nabla^2 g(e_i, e_j) = \nabla^2 g(e_i, e_j) - \langle \nabla g, \frac{y}{|y|} \rangle \delta_{ij}
\]

for a local orthonormal frame \( \{ e_1, \cdots, e_{n-1} \} \) on \( \mathbb{S}^{n-1} \). So, our proof is completed.

If we have proved Lemma 6, we can derive the uniform upper bound of the Gauss curvature of \( M_t \) and the uniform lower bound for principal curvatures by similar calculations which have been done in [13]. In the rest of this section, we take a local orthonormal frame \( \{ e_1, \cdots, e_{n-1} \} \) on \( \mathbb{S}^{n-1} \) such that the standard metric on \( \mathbb{S}^{n-1} \) is \( \{ \delta_{ij} \} \). And double indices always mean to sum from 1 to \( n - 1 \).

**Lemma 7.** Let \( X(\cdot, t) \) be a strictly convex solution to the flow \([5]\) which encloses the origin for \( t \in [0, T) \), then we have

\[
K(x, t) \leq C, \quad \forall (x, t) \in \mathbb{S}^{n-1} \times [0, T),
\]
where $C$ is a positive constant depending only on the constants in Lemmas 3 and 5.

Proof. Set

$$Q(x, t) = -\partial_t h(x, t) + h(x, t) - \varepsilon_0 = \frac{f(x) \rho^n(u)}{(h - \varepsilon_0)g(u)} K(x, t),$$

where

$$\varepsilon_0 = \frac{1}{2} \inf_{S^n \times [0, T]} h(x, t)$$

and the second equality follows from (15). For each $t \in [0, T)$, assume $Q(\cdot, t)$ attains its maximum at some point $x_t \in S^{n-1}$. At $(x_t, t)$, we can obtain

(27) \quad 0 = Q_{i} = -\partial_t h_i + h_i + \frac{\partial h - h}{h - \varepsilon_0} h_i,

and

(28) \quad 0 \geq Q_{ij} = -\partial_t h_{ij} + h_{ij} + \frac{(\partial h - h)h_{ij}}{(h - \varepsilon_0)^2},

where (27) is used in (28). Recall that $b_{ij} = h_{ij} + h\delta_{ij}$, and $b^{ij}$ is its inverse matrix. Using the inequality (28), it yields

$$\partial_t b_{ij} = \partial_t h_{ij} + \partial_t h\delta_{ij} \geq h_{ij} + \frac{\partial h - h}{h - \varepsilon_0} h_{ij} + \partial_t h\delta_{ij} = b_{ij} - Q(b_{ij} - \varepsilon_0\delta_{ij}).$$

Noticing that $K = 1/\det(b_{ij})$, we have

$$\partial_t K = -K b^{ij} \partial_t b_{ij} \leq -K b^{ij} [b_{ij} - Q(b_{ij} - \varepsilon_0\delta_{ij})] = -K [(n - 1)(1 - Q) + Q\varepsilon_0 \text{tr}(b^{ij})].$$

Note that

(30) \quad Q(x, t) = \frac{f(x) \rho^n(u)}{(h - \varepsilon_0)g(u)} K(x, t),

we derive by Lemma 3

(31) \quad \frac{1}{C_1} Q(x, t) \leq K(x, t) \leq C_1 Q(x, t),

where $C_1$ is a positive constant depending only on the constant $C$ in Lemma 3 and the upper and lower bounds of $f, g$ on $S^{n-1}$. Combining Lemma 3 and the inequalities (29) and (31), we have

$$\partial_t K \leq (n - 1)K Q - (n - 1)\varepsilon_0 QK\frac{n}{n-1} \leq C_2 Q^2 - C_3 Q\frac{2n-4}{n-1},$$

where $C_2$ and $C_3$ are positive constants depending only on the constants in Lemmas 3 and 5.
where the inequality
\[ \frac{1}{n-1} \text{tr}(b^{ij}) \geq \det(b_{ij})^{\frac{1}{n-1}} = K^{\frac{1}{n-1}} \]
is used. Here \( C_2, C_3 \) are positive constants depending only on \( n, \varepsilon_0 \) and \( C_1 \).

By the definition of \( Q \) and (27),
\[ \partial_t h_i = (1 - Q)h_i. \]  
Thus, we have
\[ \partial_t (\nabla h + hx) = \partial_t (h_ie_i + hx) \]
\[ = \partial_t h_ie_i + (\partial_t h)x \]
\[ = (1 - Q)h_ie_i + (h - (h - \varepsilon_0)Q)x \]
\[ = (1 - Q)(\nabla h + hx) + \varepsilon_0 Qx. \]

So, we can say that
\[ \partial_t G(\nabla h + hx) = \langle \nabla G, \partial_t (\nabla h + hx) \rangle \]
\[ = (1 - Q)\langle \nabla G, \nabla h + hx \rangle + \varepsilon_0 Q \langle \nabla G, x \rangle \]
\[ \leq (1 - Q)|\nabla G| |\nabla h + hx| + \varepsilon_0 Q |\nabla G| \]
\[ \leq C_4(1 - Q)|\nabla h + hx| + C_4\varepsilon_0 Q, \]
where we know by Lemma 6 that \( C_4 \) is a positive constant depending on \( n, \varepsilon_0 \), \( C_1 \) and \( \|g\|_{C^1(S_n-1)} \) and \( \min g \).

Thus,
\[ \frac{\partial}{\partial t} \left[ G(\nabla h + hx) \right] \]
\[ = \frac{\partial_t G}{h - \varepsilon_0} - \frac{G \partial h}{(h - \varepsilon_0)^2} \]
\[ \leq (1 - Q)\frac{|\nabla G| |\nabla h| + \varepsilon_0 Q |\nabla G|}{h - \varepsilon_0} + \frac{(1 - Q)h + \varepsilon_0 Q |G|}{(h - \varepsilon_0)^2} \]
\[ \leq C_5 + C_5Q, \]
where \( C_5 \) is a positive constant depending only on \( \varepsilon_0, C_4 \) and the constant \( C \) in Lemmas 3 and 5.

By virtue of (30), (32) and (36), we have at \((x_t, t)\)
\[ \partial_t Q = f \partial_t \left[ \frac{G}{h - \varepsilon_0} \right] + f \frac{G}{h - \varepsilon_0} \partial_t K \]
\[ \leq \max_{S_n-1} f \cdot C_5(1 + Q)C_1 Q + \frac{Q}{K} (C_2 Q^2 - C_5 Q^{\frac{2n-1}{n-1}}) \]
\[ \leq \max_{S_n-1} f \cdot C_1 C_5 Q(1 + Q) + C_1 C_2 Q^2 - C_1^{-1} C_3 Q^{\frac{2n-1}{n-1}} \]
\[ \leq C Q + C Q^2 - C Q^{\frac{2n-1}{n-1}}, \]
where we have used (31) to obtain the third inequality and \( C \) is a positive constant depending only on \( \max_{S^{n-1}} f, C_1, C_2, C_3 \) and \( C_5 \). Thus, whenever \( Q(x, t) \) is greater than some constant which is independent of \( t \), we have
\[
\partial_t Q < 0,
\]
which implies that \( Q \) has an uniform upper bound. By (31), \( K \) has a uniform upper bound. \( \square \)

**Lemma 8.** Let \( X(\cdot, t) \) be a strictly convex solution to the flow (5) which encloses the origin for \( t \in [0, T) \), then for the principal curvatures \( \kappa_i(x, t) \) of \( M_t \), we have
\[
\kappa_i(x, t) \geq C, \quad \forall (x, t) \in S^{n-1} \times [0, T), \quad \forall i = 1, \ldots, n-1,
\]
where \( C \) is a positive constant depending only on the constant in Lemmas 3 and 5.

**Proof.** Set
\[
\tilde{\Lambda}(x, t) = \log \lambda_{\max}(b_{ij}) - A \log h + B|\nabla h|^2, \quad \forall (x, t) \in S^{n-1} \times [0, T),
\]
where \( b_{ij} = h_{ij} + h \delta_{ij} \) as before, \( \lambda_{\max}(b_{ij}) \) denotes the maximal eigenvalue of the matrix \( (b_{ij}) \), and \( A \) and \( B \) are positive constants to be chosen later.

For any fixed \( T' \in (0, T) \), assume \( \max_{S^{n-1} \times [0, T']} \tilde{\Lambda}(x, t) \) is attained at some point \((x_0, t_0) \in S^{n-1} \times [0, T']\). By choosing a suitable orthonormal frame, we may assume
\[
\{b_{ij}(x_0, t_0)\} \text{ is diagonal and } \lambda_{\max}(b_{ij})(x_0, t_0) = b_{11}(x_0, t_0).
\]
Thus, the new function defined on \( S^{n-1} \times [0, T'] \)
\[
\Lambda(x, t) = \log b_{11} - A \log h + B|\nabla h|^2
\]
also attains its maximum at \((x_0, t_0)\). Thus, we have at \((x_0, t_0)\)
\[
0 = \Lambda_i = b^{11}b_{11;i} - A \frac{h_i}{h} + 2B \sum_k h_k h_{ki}, \quad (38)
\]
and
\[
0 \geq \Lambda_{ij} = b^{11}b_{11;ij} - (b^{11})^2 b_{11;j}b_{11;i} - A \left( \frac{h_{ij}}{h} - \frac{h_i h_j}{h^2} \right) + 2B \sum_k (h_k h_{ki} + h_k h_{kij}), \quad (39)
\]
where \((b^{ij})\) is the inverse of the matrix \((b_{ij})\). Without loss of generality, we can assume \( t_0 > 0 \). Then, we get at \((x_0, t_0)\)
\[
0 \leq \partial_t \Lambda = b^{11}(\partial_t h_{11} + \partial_t h) - A \frac{\partial_t h}{h} + 2B \sum_k h_k \partial_t h_k. \quad (40)
\]
From the equation (15), we know
\[
\log(h - \partial_t h) = \log K(x, t) + \log f(x)G(\nabla h + hx). \quad (41)
\]
Set

\[ w(x,t) = \log \left[ f(x)G(\nabla h + hx) \right], \]

where

\[ G(\nabla h + hx) = \frac{|\nabla h + hx|^n}{g\left(\frac{\nabla h + hx}{|\nabla h + hx|}\right)}. \]

Differentiating (41) gives

\[ (42) \quad h_{k} - \partial_{t} h_{k} = -b^{ji}b_{ij;k} + w_{k}, \]

and

\[ (43) \quad \frac{h_{11} - \partial_{t} h_{11}}{h - \partial_{t} h} = \frac{(h_{1} - \partial_{t} h_{1})^2}{(h - \partial_{t} h)^2} - b^{ji}b_{ij;11} + b^{jj}(b_{ij;1})^2 + w_{11}. \]

Multiplying both sides of (43) by \(-b^{11}\), it yields

\[ (44) \quad b^{11}h_{11} - b^{11}h_{11} \leq b^{11}b^{jj}b_{ij;11} - b^{11}b^{ij}(b_{ij;1})^2 - b^{11}w_{11} \]

\[ \leq b^{11}b^{ii}b_{11;ii} - b^{11}b^{ii}b_{11;11} - \sum b^{ii} \]

\[ + b^{11}(n - 1 - w_{11}), \]

where we use the Ricci identity \(b_{ii;11} = b_{11;ii} - b_{11} + b_{ii}\). We know that \(b^{ij}A_{ij} \leq 0\) from (39) which implies at \((x_{0}, t_{0})\)

\[ b^{11}b^{ii}b_{11;ii} - (b^{11})^2b^{ii}(b_{11;ii})^2 \leq Ab^{ii}\left(\frac{h_{ii}}{h} - \frac{h_{i}^2}{h^2}\right) - 2B \sum b^{ii}(h_{ki} + h_{kk}h_{kii}). \]

Thus,

\[ (45) \quad b^{11}b^{ii}b_{11;ii} - (b^{11})^2b^{ii}(b_{11;ii})^2 \leq \frac{(n - 1)A}{h} - A \sum b^{ii} - Ab^{ii}h_{i}^2 + 4(n - 1)Bh - 2B \sum b^{ii} - 2Bh^2 \sum b^{ii} \]

\[ - 2Bb^{ii}h_{k}b_{i;k} + 2Bb^{ii}h_{i}^2, \]

where we use the following equalities

\[ b^{ii}h_{ii} = b^{ii}(b_{ii} - h) = n - 1 - h \sum b^{ii}, \]

\[ \sum_{k} b^{ii}h_{ki}^2 = b^{ii}h_{ii}^2 = b^{ii}(b_{ii}^2 - 2hb_{ii} + h^2) = -2(n - 1)h + \sum b_{ii} + h^2 \sum b^{ii}, \]

\[ \sum_{k} b^{ii}h_{k}h_{kii} = \sum_{k} b^{ii}h_{k}(b_{ki;i} - h_{i}\delta_{ki}) = \sum_{k} b^{ii}h_{k}b_{i;k} - b^{ii}h_{i}^2. \]
Here the fact that $b_{ij:k}$ is symmetric in all indices is used to get the third equality above. Inserting the inequality (45) into (44), we obtain that

\[
\frac{b_{11}^1}{h - \partial_t h} \leq \frac{(n-1)A}{h} - (A + 2Bh^2 + 1) \sum_i b_{ii} - \frac{A - 2Bh^2}{h^2} b_{ii}^2h_i^2 + 4(n-1)Bh - 2B \sum_i b_{ii}
\]

\[
- 2B \sum_k b_{ii}h_{ik} b_{ik} + b_{11}^1(n - 1 - w_{11}).
\]

Using (42), we get

\[
\frac{2B \sum h_k \partial_t h_k}{h - \partial_t h} = \frac{2B|\nabla h|^2}{h - \partial_t h} + 2B \sum_k b_{ii}h_k b_{ii} - 2B \langle \nabla h, \nabla w \rangle.
\]

Now dividing (46) by $h - \partial_t h$ gives

\[
0 \leq \frac{b_{11}^1(\partial_th_{11} - h_{11} + b_{11} - h + \partial_t h)}{h - \partial_t h} - \frac{A\partial_t h}{h(h - \partial_t h)} + \frac{2B \sum_k h_k \partial_t h_k}{h - \partial_t h}
\]

\[
= \frac{b_{11}^1(\partial_th_{11} - h_{11})}{h - \partial_t h} + \frac{2B \sum_k h_k \partial_t h_k}{h - \partial_t h} - b_{11} + \frac{A - 1}{h - \partial_t h},
\]

which together with (46) and (47) implies that

\[
0 \leq \frac{nA}{h} - (A + 2Bh^2 + 1) \sum_i b_{ii} - \frac{A - 2Bh^2}{h^2} \sum_i b_{ii}^2h_i^2
\]

\[
+ 4(n-1)Bh - 2B \sum_i b_{ii} + b_{11}(n - 2 - w_{11})
\]

\[
- 2B \langle \nabla h, \nabla w \rangle - \frac{A - 1}{h - \partial_t h}.
\]

Now we choose $A = n + 2BC^2$, where $C$ is the constant in Lemma 3, we can obtain

\[
(A - n + 3) \sum_i b_{ii} + 2B \sum_i b_{ii} \leq C_1(A + B) - b_{11}^1 w_{11} - 2B \langle \nabla h, \nabla w \rangle,
\]

where $C_1$ is a positive constant depending only on $n$ and the constant $C$ in Lemma 3

A direct calculation gives

\[
\nabla_i G(\nabla h + hx) = \langle \nabla G, \nabla_i \nabla h \rangle = \langle \nabla G, e_k \rangle b_{ik},
\]

and

\[
\nabla_i \nabla_j G(\nabla h + hx) = \nabla^2 G(\nabla_j(\nabla h)), \nabla_i(\nabla h) \rangle + \langle \nabla G, \nabla_i \nabla_j(\nabla h) \rangle
\]

\[
= \nabla^2 G\left( \sum_k b_{ik}e_k, \sum_l b_{lj}e_l \right) - \langle \nabla G, x \rangle b_{ij} + \sum_k \langle \nabla G, e_k \rangle b_{ij;k}.
\]
Thus, we have

\[-2B\langle \nabla h, \nabla w \rangle = -2B \sum_k h_k \left( \frac{f_k}{f} + \frac{\sum_l (\nabla G, e_l) b_{kl}}{G} \right) \]

\[\leq C_2 B + \sum_k 2B h_k (\nabla G, e_k) b_{kk},\]

where \(C_2\) is a positive constant depending only on the constants \(C\) in Lemma \(\text{H}\) \(\|f\|_{C^1(\mathbb{S}^{n-1})}\) and \(\min f\). Moreover, we have by Lemma \(\text{H}\)

\[-w_{11} = \frac{f_1^2}{f^2} - \frac{f_{11}}{f} + \frac{[\langle \nabla G, e_1 \rangle b_{11}]^2}{G^2} \]

\[-\frac{1}{G} \left[ \nabla^2 G(1, e_1) b_{11} - \langle \nabla G, x \rangle b_{11} + \langle \nabla G, e_1 \rangle b_{11,1} \right] \]

\[\leq C_3 (1 + b_{11} + b_{11}^2) + \frac{\langle \nabla G, e_1 \rangle b_{11,1}}{G},\]

where \(C_3\) is a positive constant depending only on the constants \(C\) in Lemmas \(\text{H}\) and \(\text{I}\) \(\|f\|_{C^2(\mathbb{S}^{n-1})}, \|g\|_{C^2(\mathbb{S}^{n-1})}, \min f\) and \(\min g\). Therefore, combining the two inequalities above, we have

\[-b_{11}w_{11} - 2B\langle \nabla h, \nabla w \rangle \]

\[\leq C_3 (b_{11}^4 + 1 + b_{11}) + C_2 B + \sum_k \frac{\langle \nabla G, e_k \rangle}{G} (b_{11} b_{11,k} + 2B h_k h_{kk}) \]

\[= C_3 (b_{11}^4 + 1 + b_{11}) + C_2 B + \sum_k \frac{\langle \nabla G, e_k \rangle}{G} \cdot A h_k \]

\[\leq C_3 (b_{11}^4 + 1 + b_{11}) + C_2 B + C_4 A,\]

where we have used the equality \(\text{[38]}\), and \(C_4\) is a positive constant depending only on the constants \(C\) in Lemma \(\text{I}\) and \(n\). Inserting \(\text{[49]}\) into \(\text{[48]}\), we have

\[(A - n) \sum_i b_{ii}^2 + 2B \sum_i b_{ii} \leq (C_1 + C_4) A + (C_1 + C_2) B + C_3 (b_{11}^4 + 1 + b_{11}),\]

which together with \(A = n + 2BC^2\) implies that

\[(2BC^2 - C_3) \sum_i b_{ii}^2 + (2B - C_3) \sum_i b_{ii} \]

\[\leq (C_1 + C_4) (n + 2BC^2) + (C_1 + C_2) B + C_3.\]

If we choose \(B = \max \left\{ \frac{C_3}{2}, \frac{C_4}{2C} \right\}\), we see that \(b_{11}(x_0, t_0)\) is bounded from above by a positive constant depending only on \(n, C_1, C_2, C_3\) and \(C_4\). Using Lemmas \(\text{H}\) \(A(x_0, t_0)\) is bounded from above by a positive constant depending only on \(n, C_1, C_2, C_3\) and \(C_4\). Thus, we prove the conclusion of this lemma, by noticing that \(T^\prime\) can be any number in \((0, T)\). \(\square\)
5. Existence of solutions

In this section, we will complete the proof of Theorem 3. Combining Lemma 7 and Lemma 8, we see that the principal curvatures of $M_t$ has uniform positive upper and lower bounds. This together with Lemmas 3 and 5 implies that the evolution equation (15) is uniformly parabolic on any finite time interval. Thus, using Krylov-Safonov estimates [34] and Schauder estimates of the parabolic equations, we can say that the smooth solution of (15) exists for all time. And by these estimates again, a subsequence of $M_t$ converges in $C^\infty$ to a positive, smooth, uniformly convex hypersurface $M_\infty$ in $\mathbb{R}^n$. Now to complete the proof of Theorem 3, it remains to check the support function of $M_\infty$ satisfies Eq. (4).

Let $\tilde{h}$ be the support function of $M_\infty$. We need to prove that $\tilde{h}$ is a solution to the following equation

$$g\left(\frac{\nabla h + hx}{\nabla h + hx}\right)|\nabla h + hx|^{-n}h \det(\nabla^2 h + hI) = f \text{ on } S^{n-1}.$$  \hfill (50)

By Lemma 11, $J'(t) \leq 0$ for any $t > 0$. Since

$$\int_0^t [-J'(t)] \, dt = J(0) - J(t) \leq C \int_{S^{n-1}} f \, dx,$$

we get

$$\int_0^\infty [-J'(t)] \, dt \leq C \int_{S^{n-1}} f \, dx.$$

Thus, there exists a subsequence of times $t_j \to \infty$ such that

$$-J'(t_j) \to 0 \text{ as } t_j \to \infty.$$

Thus, we obtain using (19)

$$\int_{S^{n-1}} \frac{1}{ghK\rho^n}(g\tilde{h} - f\tilde{K}\rho^n)^2 \, dx = 0,$$

where $\tilde{K}$ is the Gauss curvature of $M_\infty$. It implies that

$$g(u) \frac{\tilde{h}(x)}{\tilde{K}\rho^n(u)} = f(x) \text{ on } S^{n-1},$$

which means $\tilde{h}$ is a solution to the equation (50).

References

[1] B. Andrews, Monotone quantities and unique limits for evolving convex hypersurfaces, Internat. Math. Res. Notices, (1997), pp. 1001–1031.
[2] B. Andrews, P. Guan and L. Ni, Flow by powers of the Gauss curvature, Adv. Math., 299 (2016), pp. 174–201.
[3] G. Bianchi, K. J. Böröczky and A. Colesanti, The Orlicz version of the $L_p$ Minkowski problem for $-n < p < 0$, Adv. in Appl. Math., 111 (2019), p. 101937.
[4] K. J. Böröczky, E. Lutwak, D. Yang, G. Y. Zhang and Y. M. Zhao, The Gauss image problem, Communications on Pure and Applied Mathematics, Vol. LXXIII, 1406-1452 (2020), pp. 1046–1452.
K. J. Böröczky and F. Fodor, *The $L_p$ dual Minkowski problem for $p > 1$ and $q > 0*", J. Differential Equations, 266 (2019), pp. 7980–8033.

K. J. Böröczky, M. Henk and H. Pollehn, *Subspace concentration of dual curvature measures of symmetric convex bodies*, J. Differential Geom., 109 (2018), pp. 411–429.

K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang, *The logarithmic Minkowski problem*, J. Amer. Math. Soc., 26 (2013), pp. 831–852.

S. Brendle, K. Choi and P. Daskalopoulos, *Asymptotic behavior of flows by powers of the Gaussian curvature*, Acta Math., 219 (2017), pp. 1–16.

P. Bryan, M. N. Ivaki and J. Scheuer, *A unified flow approach to smooth, even $L_p$-Minkowski problems*, Anal. PDE, 12 (2019), pp. 259–280.

C. Chen, Y. Huang and Y. Zhao, *Smooth solutions to the $L_p$ dual Minkowski problem*, Math. Ann., 373 (2019), pp. 953–976.

H. Chen, S. Chen and Q.-R. Li, *Variations of a class of Monge-Ampère type functionals and their applications*. Accepted by Anal. PDE.

S. Chen, Q.-R. Li and G. Zhu, *The logarithmic Minkowski problem for non-symmetric measures*, Trans. Amer. Math. Soc., 371 (2019), pp. 2623–2641.

L. Chen, Y. Liu, J. Lu and N. Xiang, *Existence of smooth even solutions to the dual Orlicz-Minkowski problem Authors*, arXiv:2005.02639.

K.-S. Chou and X.-J. Wang, *A logarithmic Gauss curvature flow and the Minkowski problem*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 17 (2000), pp. 733–751.

R. J. Gardner, D. Hug, W. Weil, S. Xing and D. Ye, *General volumes in the Orlicz-Brunn-Minkowski theory and a related Minkowski problem I*, Calc. Var. Partial Differential Equations, 58 (2019), pp. Paper No. 12, 35.

C. Gutiérrez, *The Monge-Ampère Equation*, Second Edition, Part of the Progress in Nonlinear Differential Equations and Their Applications book series (PNLDE, volume 44).

R. J. Gardner, D. Hug, S. Xing and D. Ye, *General volumes in the Orlicz-Brunn-Minkowski theory and a related Minkowski problem II*, Calc. Var. Partial Differential Equations, 59 (2020), pp. Paper No. 15, 33.

C. Haberl, E. Lutwak, D. Yang and G. Zhang, *The even Orlicz Minkowski problem*, Adv. Math., 323 (2018), pp. 114–141.

Q. Huang and B. He, *On the Orlicz Minkowski problem for polytopes*, Discrete Comput. Geom., 48 (2012), pp. 281–297.

Y. Huang and Y. Jiang, *Variational characterization for the planar dual Minkowski problem*, J. Funct. Anal., 277 (2019), pp. 2209–2236.

Y. Huang, E. Lutwak, D. Yang and G. Zhang, *Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems*, Acta Math., 216 (2016), pp. 325–388.

C. Haberl, E. Lutwak, D. Yang and G. Zhang, *The even Orlicz Minkowski problem*, Adv. Math., 224 (2010), pp. 2485–2510.

R. S. Hamilton, *Remarks on the entropy and Harnack estimates for the Gauss curvature flow*, Comm. Anal. Geom., 2 (1994), pp. 155–165.

M. Henk and H. Pollehn, *Necessary subspace concentration conditions for the even dual Minkowski problem*, Adv. Math., 323 (2018), pp. 114–141.

R. J. Gardner, D. Hug, S. Xing and D. Ye, *General volumes in the Orlicz-Brunn-Minkowski theory and a related Minkowski problem II*, Calc. Var. Partial Differential Equations, 59 (2020), pp. Paper No. 15, 33.

C. Gerhardt, *Non-scale-invariant inverse curvature flows in Euclidean space*, Calc. Var. Partial Differential Equations, 49 (2014), pp. 471–489.

C. Haberl, E. Lutwak, D. Yang and G. Zhang, *The even Orlicz Minkowski problem*, Adv. Math., 224 (2010), pp. 2485–2510.

R. S. Hamilton, *Remarks on the entropy and Harnack estimates for the Gauss curvature flow*, Comm. Anal. Geom., 2 (1994), pp. 155–165.

M. Henk and H. Pollehn, *Necessary subspace concentration conditions for the even dual Minkowski problem*, Adv. Math., 323 (2018), pp. 114–141.

Q. Huang and B. He, *On the Orlicz Minkowski problem for polytopes*, Discrete Comput. Geom., 48 (2012), pp. 281–297.

Y. Huang and Y. Jiang, *Variational characterization for the planar dual Minkowski problem*, J. Funct. Anal., 277 (2019), pp. 2209–2236.

Y. Huang, E. Lutwak, D. Yang and G. Zhang, *Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems*, Acta Math., 216 (2016), pp. 325–388.

C. Gerhardt, *Non-scale-invariant inverse curvature flows in Euclidean space*, Calc. Var. Partial Differential Equations, 49 (2014), pp. 471–489.
[29] D. Hug, E. Lutwak, D. Yang and G. Zhang, On the L_p Minkowski problem for polytopes, Discrete Comput. Geom., 33 (2005), pp. 699–715.
[30] M. N. Ivaki, Deforming a hypersurface by Gauss curvature and support function, J. Funct. Anal., 271 (2016), pp. 2133–2165.
[31] H. Jian and J. Lu, Existence of solutions to the Orlicz-Minkowski problem, Adv. Math., 344 (2019), pp. 262–288.
[32] H. Jian, J. Lu and X.-J. Wang, A priori estimates and existence of solutions to the prescribed centroaffine curvature problem, J. Funct. Anal., 274 (2018), pp. 826–862.
[33] H. Jian, J. Lu and G. Zhu, Mirror symmetric solutions to the centro-affine Minkowski problem, Calc. Var. Partial Differential Equations, 55 (2016), pp. Art. 41, 22 pp.
[34] N. V. Krylov and M. V. Safonov, A property of the solutions of parabolic equations with measurable coefficients, Izv. Akad. Nauk SSSR Ser. Mat., 44 (1980), pp. 161–175, 239.
[35] Q.-R. Li, J. Liu and J. Lu, Non-uniqueness of solutions to the L_p dual Minkowski problem, Preprint.
[36] Q.-R. Li, W. Sheng and X.-J. Wang, Flow by Gauss curvature to the Aleksandrov and dual Minkowski problems, J. Eur. Math. Soc. (JEMS), 22 (2020), pp. 893–923.
[37] Y. Liu and J. Lu, A flow method for the dual Orlicz-Minkowski problem, Trans. Amer. Math. Soc., 373 (2020), pp. 5833–5853.
[38] J. Lu, Nonexistence of maximizers for the functional of the centroaffine Minkowski problem, Sci. China Math., 61 (2018), pp. 511–516.
[39] , A remark on rotationally symmetric solutions to the centroaffine Minkowski problem, J. Differential Equations, 266 (2019), pp. 4394–4431.
[40] J. Lu and X.-J. Wang, Rotationally symmetric solutions to the L_p-Minkowski problem, J. Differential Equations, 254 (2013), pp. 983–1005.
[41] E. Lutwak, The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem, J. Differential Geom., 38 (1993), pp. 131–150.
[42] E. Lutwak, D. Yang and G. Zhang, On the L_p-Minkowski problem, Trans. Amer. Math. Soc., 356 (2004), pp. 4359–4370.
[43] L_p dual curvature measures, Adv. Math., 329 (2018), pp. 85–132.
[44] A. Stancu, The discrete planar L_0-Minkowski problem, Adv. Math., 167 (2002), pp. 160–174.
[45] J. Urbas, An expansion of convex hypersurfaces, J. Differential Geom., 33 (1991), pp. 91–125.
[46] Y. Zhao, The dual Minkowski problem for negative indices, Calc. Var. Partial Differential Equations, 56 (2017), p. 56:18.
[47] , Existence of solutions to the even dual Minkowski problem, J. Differential Geom., 110 (2018), pp. 543–572.
[48] G. Zhu, The logarithmic Minkowski problem for polytopes, Adv. Math., 262 (2014), pp. 909–931.
[49] , The centro-affine Minkowski problem for polytopes, J. Differential Geom., 101 (2015), pp. 159–174.
Faculty of Mathematics and Statistics, Hubei Key Laboratory of Applied Mathematics, Hubei University, Wuhan 430062, P.R. China
Email address: chernli@163.com

Faculty of Mathematics and Statistics, Hubei Key Laboratory of Applied Mathematics, Hubei University, Wuhan 430062, P.R. China
Email address: wudi19950106@126.com

Faculty of Mathematics and Statistics, Hubei Key Laboratory of Applied Mathematics, Hubei University, Wuhan 430062, P.R. China
Email address: nixiang@hubu.edu.cn