Stability of neutron stars in Horndeski theories with Gauss-Bonnet couplings

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I. INTRODUCTION

After the dawn of gravitational waves (GW) astronomy from a binary system of black holes (BHs) \cite{1}, a new observational probe of the physics on a strong gravitational background has begun. In particular, the GW170817 event \cite{2} allowed us to put constraints on the mass-radius relation of neutron stars (NSs) from their tidal deformation before the coalescence \cite{3}. After a merger of binaries, a compact object exhibits a damped sinusoidal oscillation with quasi-normal frequencies \cite{4,5}. Upcoming observational data of GWs will provide us further detailed information for new physics in strong gravity regimes in the vicinity of BHs and in the interior of NSs.

From cosmological observational data, we know that about 95% of the energy density of today’s Universe is dominated by mysterious components dubbed dark energy and dark matter \cite{6,7,8,9,10}. Since it is still challenging to explain their origins within the framework of General Relativity (GR) and Standard Model of particle physics, there is a motivation to introduce new degrees of freedom \cite{11,12,13,14,15,16,17,18}. A scalar field is one of the simplest candidates, and widely used to explain physical phenomena relevant to the dark sector of the Universe.

On a spherically symmetric background, an asymptotically-flat vacuum solution in GR is uniquely described by the Schwarzschild metric with the mass of a compact body. The background Schwarzschild geometry can be modified by introducing a new degree of freedom. For asymptotically-flat BHs, however, the property of the absence of a nontrivial scalar profile holds for a wide class of scalar-tensor theories—\textit{including} a canonical scalar field \cite{19,20,21,22,23,24,25}. An exceptional case is a scalar coupling with the GB curvature invariant \( \alpha R_\mathrm{GB}^2 \), where \( \alpha \) is a dimensionless coupling constant and \( R_\mathrm{GB} \) is a regular function of the scalar field \( \phi \).

This scalar-GB coupling belongs to a subclass of Horndeski theories \cite{26}, containing nonanalytic functions \( 2^{2,3,4,5} \). This includes Brans-Dicke theories \cite{60} and \( f(R) \) gravity \cite{61}, where the latter corresponds to a particular class of the former with a scalar potential \cite{62,63}. These nonminimally coupled theories belong to a subclass of non-shift-symmetric Horndeski theories.

In shift-symmetric subclass of Horndeski theories where the field equations of motion are invariant under the shift \( \phi \rightarrow \phi + c \), there is a no-hair argument of stars under several assumptions \cite{64} analogous to BHs discussed in Ref. \cite{26}. The assumptions are as follows:
(i) the scalar field and metrics are regular, static, and spherically symmetric with an asymptotically-flat spacetime geometry,

(ii) a canonical kinetic term $X$ is present in the action,

(iii) the action is analytic with regular coupling functions $G_{2,3,4,5}$.

Under these hypotheses, we end up with a no-hair solution $\phi = \text{constant}$.

One way of breaking the assumption (i) is to postulate a scalar field of the form $\phi = qt + \psi(r)$, where $q$ is a nonvanishing constant, $t$ and $r$ are time and radial coordinates, respectively. Originally, this type of field configuration was considered to search for BH solutions with a nontrivial profile of the scalar field in shift-symmetric Horndeski theories [65]. The analysis was further extended to relativistic stars in DHOST theories [66–70]. In this paper, we do not consider such a time-dependent background scalar field and focus on the case $q = 0$, i.e., the static scalar field.

If we break the assumption (ii), i.e., no canonical kinetic term in the action, it is known that the quartic nonminimal derivative coupling $G_4 \supset \mu_3 X$ gives rise to a solution endowed with a nontrivial scalar profile inside the star [71]. However, it was recognized that this solution with a nontrivial scalar profile is plagued by an angular Laplacian instability of even-parity perturbations for large multipoles around the surface of star [72, 73].

Finally, one can break the above assumption (iii) by introducing nonanalytic coupling functions. In the presence of $X$, the quintic-order coupling $G_5 = -\alpha X |X|$, which is equivalent to the linear scalar-GB coupling $\alpha \phi R_{\text{GB}}^2$, gives rise to NS solutions endowed with a nontrivial scalar profile [72]. This linear coupling can also accommodate a dilatonic coupling $\xi(\phi) = \alpha \phi R_{\text{GB}}^2$ in the limit $|\mu \phi| \ll 1$. For the same scalar-GB coupling, there are also asymptotically-flat hairy BHs [33–35] consistent with all the linear stability conditions against odd- and even-parity perturbations [73]. We note that BH solutions present for other nonanalytic functions in $G_{2,3,4}$ [76] are either unstable around the horizon or asymptotically non-flat [75, 77]. For the linear scalar-GB coupling, it is not yet clear whether NS solutions with a nontrivial scalar profile satisfy all the stability conditions against odd- and even-parity perturbations. For this purpose, we can exploit conditions for the absence of ghost/Laplacian instabilities recently derived in full Horndeski theories [73] (see also Refs. [78–80]). NSs can have a nontrivial scalar profile in more general GB couplings $\alpha \xi(\phi) R_{\text{GB}}^2$ with a canonical scalar kinetic term [11, 81–83]. This belongs to a subclass of non-shift-symmetric Horndeski theories. In this paper, for such general GB couplings, we will first study the background NS solutions and their linear stability both analytically and numerically. We show that the strength of dimensionless coupling constant $|\alpha|$ has an upper bound to ensure the existence of NSs with a nontrivial scalar profile free from instabilities around the center of star. Besides $\alpha \xi(\phi) R_{\text{GB}}^2$, we also incorporate regular coupling functions like $G_3 \supset \mu_3 X$, $G_4 \supset \mu_4 X$, and $G_4 \supset \lambda_4 \phi$ as the representative cases, and study their effects on the existence and stability of solutions. Provided the coupling constants are in certain ranges, NS solutions with a nontrivial scalar profile consistent with all the linear stability conditions are still present in such combined theories.

Moreover, there are also several other gravitational theories containing the GB term in the action. In so-called 4-dimensional-Einstein-GB (4DEGB) gravity [80], the contribution of the GB term in spacetime dimensions $D$ higher than 4 can be extracted by rescaling the GB coupling constant $\alpha \to \alpha/(D - 4)$. If we perform a Kaluza-Klein reduction on a flat internal space whose volume is characterized by the scalar field $\phi$, the effective 4DEGB theory after the rescaling of $\alpha$ belongs to a subclass of shift-symmetric Horndeski theories containing the linear GB coupling but without the canonical scalar kinetic term [87–88] (see also Ref. [89–90] for a conformal regularization equivalent to the Kaluza-Klein reduction). It is known that NS solutions with a nontrivial scalar profile are present in the regularized 4DEGB theory [91], but we will show that they are plagued by a strong coupling problem and Laplacian instability of even-parity perturbations. In theories given by the Lagrangian $R + F(R_{\text{GB}}^2)$ [92–95], where $F$ is a positive power-law function of $R_{\text{GB}}^2$, we will also show that a ghost instability and strong coupling at spatial infinity arise for NS solutions with a nontrivial scalar profile. In these 4DEGB and $F(R_{\text{GB}}^2)$ theories, there are no canonical kinetic terms in the action, which implies the unhealthy propagation of scalar field perturbations, as it also happens in derivative coupling theories without the canonical kinetic term [72, 73].

This paper is organized as follows. In Sec. II, we present the linear stability conditions for relativistic stars on the static and spherically symmetric background. In Sec. III, we derive solutions expanded around the center of star and at spatial infinity in the presence of scalar-GB couplings $\alpha \xi(\phi) R_{\text{GB}}^2$. We give a new theoretical bound on $\alpha$ for the existence of NS solutions with a nontrivial scalar profile free from instabilities and confirm it numerically for the linear scalar-GB coupling. In Sec. IV, we implement several regular coupling functions besides the scalar-GB coupling and explore the parameter space of coupling constants in which there are NS solutions with a nontrivial scalar profile without ghost or Laplacian instabilities. In Sec. V, we show the existence of a strong coupling problem and Laplacian instability for NSs with a nontrivial scalar profile arising in 4DEGB gravity. In Sec. VI, we prove that a non-vanishing scalar-field branch appearing in power-law $F(R_{\text{GB}}^2)$ models is plagued by ghost and strong coupling problems at large distances. Sec. VII is devoted to conclusions.
II. BACKGROUND EQUATIONS AND LINEAR STABILITY CONDITIONS

We study the existence and stability of NS solutions in Horndeski theories \cite{[99][100][103][104]}, whose action is given by

\[ S = \int d^4x \sqrt{-g} \mathcal{L}_H + S_m(g_{\mu\nu}, \Psi_m), \quad (2.1) \]

where \( g \) is a determinant of the metric tensor \( g_{\mu\nu} \), and

\[ \mathcal{L}_H = G_2(\phi, X) - G_3(\phi, X)\Box \phi + G_4(\phi, X) R + G_{4,X}(\phi, X) \left[ (\Box \phi)^2 - (\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi) \right] + G_5(\phi, X) G_{\mu\nu} \nabla^\mu \nabla^\nu \phi \]

\[ - \frac{1}{6} G_5, X(\phi, X) \left[ (\Box \phi)^3 - 3(\Box \phi)(\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi) + 2(\nabla^\mu \nabla_\alpha \phi)(\nabla^\nu \nabla_\beta \phi)(\nabla^\alpha \nabla^\beta \phi) \right], \quad (2.2) \]

where the coupling functions \( G_j (j = 2, 3, 4, 5) \) depend on the scalar field \( \phi \) and its kinetic term \( X = -q^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi / 2 \), with the covariant derivative operator \( \nabla_\mu \). We will use the notations \( \Box \phi \equiv \nabla^\mu \nabla_\mu \phi \) and \( G_{j,\phi} \equiv \partial G_j / \partial \phi \), \( G_{j,X} \equiv \partial G_j / \partial X \), \( G_{j,\phi X} \equiv \partial^2 G_j / \partial X \partial \phi \), and so on. The scalar field \( \phi \) and its derivatives are nonminimally coupled to the Ricci scalar \( R \) and Einstein tensor \( G_{\mu\nu} \) through the couplings \( G_4 \) and \( G_5 \), respectively. For the matter fields \( \Psi_m \) inside NSs, we consider a perfect fluid given by the energy-momentum tensor

\[ T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu}, \quad (2.3) \]

where \( \rho \) and \( P \) are the density and pressure, respectively, and \( u_\mu \) is the four-velocity of the fluid satisfying the normalization relation \( u_\mu u^\mu = -1 \). Assuming that the perfect fluid is minimally coupled to gravity, it obeys the continuity equation

\[ \nabla^\mu T_{\mu\nu} = 0. \quad (2.4) \]

In terms of the action approach, the perfect fluid can be described by a Schutz-Sorkin action \cite{[99][101]}. 

A. Background equations of motion

A static and spherically symmetric background is described by the line element

\[ ds^2 = -f(r)dt^2 + h^{-1}(r)dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right), \quad (2.5) \]

where \( f(r) \) and \( h(r) \) are functions of the radial coordinate \( r \). On this background, we consider the scalar field that depends only on the radial coordinate

\[ \phi = \phi(r), \quad (2.6) \]

together with the four-velocity of the fluid \( u^\mu = [f(r)^{-1/2}, 0, 0, 0] \). Then, the mixed energy-momentum tensor \( T^\mu_{\nu} \) has the following diagonal components

\[ T^\mu_{\nu} = \text{diag} \left[ -\rho(r), P(r), P(r), P(r) \right], \quad (2.7) \]

where \( \rho \) and \( P \) are functions of \( r \) alone. The continuity Eq. (2.4) gives

\[ P' + \frac{f'}{2f} (\rho + P) = 0, \quad (2.8) \]

where a prime represents the derivative with respect to \( r \).

The \((00), (11), (22)\) components of gravitational field equations of motion are

\[ \left( A_1 + \frac{A_2}{r} + \frac{A_3}{r^2} \right) \phi'' + \left( \frac{\phi'}{2h} A_1 + \frac{A_4}{r} + \frac{A_5}{r^2} \right) h' + A_6 + \frac{A_7}{r} + \frac{A_8}{r^2} = \rho, \quad (2.9) \]

\[ - \left( \frac{\phi'}{2h} A_1 + \frac{A_4}{r} + \frac{A_5}{r^2} \right) h f' + A_9 - \frac{2\phi'}{r} A_1 - \frac{1}{r^2} \left[ \frac{\phi'}{2h} A_2 + (h - 1) A_4 \right] = P, \quad (2.10) \]

\[ \left[ A_2 + \frac{(2h - 1) \phi' A_3 + 2h A_5}{h \phi' r} \right] \frac{f'}{4f} + A_1 + \frac{A_2}{2r} \phi'' + \frac{1}{4f} \left( 2h A_4 - \phi' A_2 + \frac{2h A_5 - \phi' A_3}{r} \right) \left( f'' - \frac{f'^2}{2f} \right) \]
In the presence of perfect fluids the stability conditions against odd- and even-parity perturbations were already derived metrics, scalar field, and perfect fluids into the odd- and even-parity modes was addressed in Ref. [73, 78–80, 104, 105].

where vector and tensor perturbations contain both odd and even modes. The decomposition of perturbations of \( l \delta j \) perturbation

\[
\frac{1}{r^2} \sqrt{\frac{h}{f}} \left( r^2 \sqrt{\frac{f}{h}} J' \right) + \mathcal{P}_\phi = 0, \tag{2.12}
\]

with

\[
J' = h\phi' \left[ G_{2,X} - \left( \frac{2}{r} + \frac{f'}{2f} \right) h\phi' G_{3,X} + 2 \left( 1 - \frac{h}{r^2} \frac{h f'}{r f} \right) G_{4,X} + 2 h\phi^2 \left( \frac{h}{r^2} + \frac{h f'}{r f} \right) G_{4,XX} \right. \\
- \left. \frac{f'}{2r^2 f} (1 - 3h) h\phi' G_{5,X} - \frac{f' h^3 \phi'^3}{2r^2 f} G_{5,XX} \right],
\]

\[
\mathcal{P}_\phi = G_{2,\phi} + \lambda_1 G_{3,\phi} + \lambda_2 G_{3,\phi \phi} + \lambda_3 G_{3,\phi X} + \lambda_4 G_{4,\phi} + \lambda_5 G_{4,\phi X} + \lambda_6 G_{4,\phi \phi X} + \lambda_7 G_{4,\phi XX} + \lambda_8 G_{5,\phi} + \lambda_9 G_{5,\phi \phi} + \lambda_{10} G_{5,\phi X} + \lambda_{11} G_{5,\phi \phi X} + \lambda_{12} G_{5,\phi XX},
\]

where \( \lambda_1 - \lambda_{12} \) are presented in Appendix A. This equation also follows by combining Eqs. [2.9]-[2.11]. Note that, in shift-symmetric Horndeski theories where the coupling functions \( \lambda_1 \) contain the \( X \) dependence alone, we have \( \mathcal{P}_\phi = 0 \). In this case, Eq. (2.12) gives the solution \( J' = (Q/r^2) \sqrt{h/f} \), where \( Q \) is a constant.

### B. Linear stability conditions

To study the linear stability of NS solutions, we consider metric perturbations \( h_{\mu \nu} \) on top of the background [2.5] besides perturbations of the scalar field and perfect fluid. Expanding perturbations on the background [2.5] in terms of the spherical harmonics of the unit two-sphere \( Y_{lm}(\theta, \varphi) \), one can decompose them into the two different sectors depending on the parity under the rotation along two-dimensional sphere [102, 103]. The odd- and even-parity perturbations have the parities \((-1)^{l+1}\) and \((-1)^l\), respectively. Any scalar perturbation has the even mode alone, whereas vector and tensor perturbations contain both odd and even modes. The decomposition of perturbations of metrics, scalar field, and perfect fluids into the odd- and even-parity modes was addressed in Ref. [73, 78, 80, 104, 105]. In the presence of perfect fluids the stability conditions against odd- and even-parity perturbations were already derived in Ref. [73], so we briefly summarize them in the following.

In the odd-parity sector, there is a dynamical perturbation \( \chi \) arising from the gravity sector besides a nondynamical perturbation \( \delta j \) related to the \((\theta, \varphi)\) components of fluid four velocity [73]. In the limit of large frequencies and multipoles \( l \), the no-ghost condition for the dynamical field \( \chi \) translates to

\[
\mathcal{G} \equiv 2G_4 + 2h\phi^2 G_{4,X} - h\phi^2 \left( G_{5,\phi} + \frac{f' h \phi' G_{5,X}}{2f} \right) > 0. \tag{2.15}
\]

Under this condition, the Laplacian instability along the radial and angular directions can be avoided for

\[
\mathcal{H} \equiv 2G_4 + 2h\phi^2 G_{4,X} - h\phi^2 G_{5,\phi} - \frac{h^2 \phi'^2 G_{5,X}}{r} > 0, \tag{2.16}
\]

\[
\mathcal{F} \equiv 2G_4 + h\phi^2 G_{5,\phi} - h\phi^2 \left( \frac{1}{2} h' \phi' + h\phi'' \right) G_{5,X} > 0, \tag{2.17}
\]

under which the squared propagation speeds \( c_\phi^2 = \mathcal{G}/\mathcal{F} \) and \( c_\Omega^2 = \mathcal{G}/\mathcal{H} \) are positive. The expressions [2.15]-[2.17] coincide with those originally derived in Ref. [73] in the absence of the perfect fluid.

In the even-parity sector, there are three dynamical perturbations: matter perturbation \( \delta \rho \), gravitational perturbations \( \psi \), and scalar-field perturbation \( \delta \phi \). We focus on the linear stability conditions of high radial and angular momentum modes. With the condition [2.16], there are no ghosts for even-parity perturbations if

\[
\rho + P > 0, \tag{2.18}
\]

\[
\mathcal{K} \equiv (2\mathcal{P}_1 - \mathcal{F}) h\mu^2 - 2\mathcal{H} r^4 (\rho + P) > 0, \tag{2.19}
\]

where

\[
\mathcal{P}_1 = \frac{h \mu}{2fr^2 \mathcal{H}} \left( \frac{fr^4 \mathcal{H}^4}{\mu^2 h} \right)' , \quad \mu = \frac{2(\phi' a_1 + r\sqrt{fr^2 \mathcal{H}})}{\sqrt{fr^2 \mathcal{H}}}. \tag{2.20}
\]
The definition of $a_1$ is given in Appendix B.

The radial propagation speed squared of $\psi$ is given by $c_{m}^2 = G/F$, which is the same as that of $\chi$. Thus the gravitational perturbations in the odd- and even-parity sectors propagate in the same manner along the radial direction. The radial Laplacian instability of $\delta \rho$ and $\delta \phi$ can be avoided for
\[ c_{m}^2 \equiv \frac{n\rho_{nn}}{\rho_n} > 0, \tag{2.21} \]
\[ c_{r}^2 \equiv \frac{2\phi/4r^2(\rho h)^{3/2}H_4(2\phi a_1 + r\sqrt{\rho h} H) - 2a_2^2f^3/2\sqrt{\rho} h G + (a_1 f' + 2c_2 f)^2 f h H^2}{f^{5/2}\sqrt{\rho} H K} > 0, \tag{2.22} \]
where $c_2$ and $c_4$ are given in Appendix B. The conditions (2.18) and (2.21) correspond to those in the perfect fluid sector. We will consider the fluid equation of state (EOS) satisfying these inequalities. We note that $c_{r}^2$ corresponds to the propagation speed squared of scalar-field perturbation $\delta \phi$.

Along the angular direction, the perfect fluid in the even-parity sector has the propagation speed squared same as $c_{m}^2 = n\rho_{nn}/\rho_n$. The angular Laplacian instabilities for $\psi$ and $\delta \phi$ are absent if
\[ c_{T\pm}^2 = -B_1 \pm \sqrt{B_1^2 - B_2} > 0, \tag{2.23} \]
where
\begin{align*}
B_1 &\equiv r^2 \sqrt{\rho h}[4h\beta_0\beta_1 + \beta_2 - 4\phi a_1 \beta_3 + r \sqrt{\rho h} G(H(\rho + P)] - 2f h G [r \sqrt{\rho h}(2\beta_1 - 4\rho h(G(\rho + P)] + 2\phi^2 a_1^2 P_1] \tag{2.24} \\
B_2 &\equiv -2r^2 \sqrt{\rho h}\beta_1(2f h G \beta_0 + r^2\beta_2) - r^4 \beta_3 - f h G(\rho f h G a_1 + 2r^3 \sqrt{\rho h} H) \tag{2.25}.
\end{align*}

The explicit forms of $\beta_0, \beta_1, \beta_2, \beta_3$ are presented in Appendix B. The stability conditions (2.23) are satisfied if $B_1^2 \geq B_2 > 0$, and $B_1 < 0$. In summary, we require that NS solutions with a nontrivial scalar profile should satisfy the inequalities (2.15), (2.16), (2.17), (2.19), (2.22), and (2.23) besides the fluid stability conditions (2.18) and (2.21).

### III. SCALAR-GAUSS-BONNET COUPLINGS

Let us first consider the Einstein-scalar-GB theory given by the action
\[ S = \int d^4 x \sqrt{-g} \left[ \frac{M_{P1}^2}{2} R + \eta X + \alpha \xi(\phi) R_{GB}^2 \right], \tag{3.1} \]
where $\eta$ is a constant, $M_{P1}$ is the reduced Planck mass, $\alpha$ is a dimensionless coupling, $\xi$ is a function of $\phi$, and $R_{GB}^2 = R^2 - 4R_{\alpha\beta} R_{\alpha\beta} + R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$, with $R_{\alpha\beta}$ and $R_{\alpha\beta\mu\nu}$ being the Ricci and Riemann tensors respectively. The action (3.1) belongs to a subclass of Horndeski theories with the coupling functions
\begin{align*}
G_2 &= \eta X + 8\alpha \xi^{(4)}(\phi) X^2(3 - \ln |X|), \quad G_3 = 4\alpha \xi^{(3)}(\phi) X(7 - 3 \ln |X|), \\
G_4 &= \frac{M_{P1}^2}{2} + 4\alpha \xi^{(2)}(\phi) X(2 - \ln |X|), \quad G_5 = -4\alpha \xi^{(1)}(\phi) \ln |X|, \tag{3.3}
\end{align*}
where $\xi^{(n)}(\phi) = d^n \xi(\phi)/d\phi^n$.

The background Eqs. (2.9), (2.10), and (2.12) reduce, respectively, to
\begin{align*}
h' &= \frac{2(1 - h)[M_{P1}^2 - 8\alpha h(\phi'' \xi_{,\phi} + \phi^2 \xi_{,\phi})] - r^2(2\rho + \eta h \phi^{(2)})}{2M_{P1}^2 r^2 + 8\alpha(1 - 3 h) \phi^2 \xi_{,\phi}}, \tag{3.4} \\
f' &= \frac{f[2(1 - h)M_{P1}^2 + r^2(2\rho + \eta h \phi^{(2)})]}{h[2M_{P1}^2 r + 8\alpha(1 - 3 h) \phi^2 \xi_{,\phi}]}, \tag{3.5} \\
\eta h \phi'' + \frac{\eta f \phi h'}{2rf} + \frac{f h'' + 4fh}{2rf} \phi^2 - \frac{2a_1 \xi_{,\phi}(h(1 - h)(2ff'' - f'^2) + ff'h'(1 - 3h)}{r^2 f^2} &= 0. \tag{3.6}
\end{align*}
The linear stability conditions (2.15)-(2.17) in the odd-parity sector translate to
\[ G = M_{P_1}^2 - \frac{4\alpha\xi_\phi\phi' f'h}{f} > 0, \]  
\[ H = M_{P_1}^2 - \frac{8\alpha\xi_\phi\phi'h}{r} > 0, \]  
\[ F = M_{P_1}^2 - 4\alpha \left( 2h\xi_\phi\phi'^2 + 2h\xi_\phi\phi'' + h'\xi_\phi\phi' \right) > 0. \]
In the limit of a small GB coupling \( \alpha \to 0 \), all of \( G, H, \) and \( F \) approach \( M_{P_1}^2 \), so the stability against odd-parity perturbations is ensured. As we will see later in Sec. III A, the leading-order term of \( \phi' \) around \( r = 0 \) is proportional to \( r \), so \( H \) does not diverge at the center of star.

On using Eqs. (3.4) and (3.5) to eliminate \( \rho \) and \( P \), the no-ghost condition (2.19) in the even-parity sector yields
\[ K = 2\eta h\phi'^2 r^2(M_{P_1}^2 r - 8\alpha h\xi_\phi\phi')^2 
+ 64\alpha^2 h(h - 1)\phi'^2 \xi_\phi^2 [M_{P_1}^2(2rh' - h + 1) - 4\alpha\xi_\phi\phi'h'(1 + 3h) + 8\alpha h(h - 1)(\xi_\phi\phi'' + \xi_\phi\phi'^2)] > 0. \] (3.10)
In the limit \( \alpha \to 0 \), we have \( K \to 2\eta h\phi'^2 M_{P_1}^4 r^4 \) and hence the ghost can be avoided for
\[ \eta > 0. \] (3.11)
In the following, we will focus on the case in which \( \eta \) is positive.

The condition (2.22) for the absence of the Laplacian instability along the radial direction reduces to
\[ c_{r3}^2 = \frac{2h\phi'^2[\eta f^2 r^2(M_{P_1}^2 r - 8\alpha h\xi_\phi\phi')^2 - 32\alpha^2(h - 1)\xi_\phi^2 [M_{P_1}^2(f(h - 1) - 2rf'h) + 4\alpha\xi_\phi\phi'f'h(1 + 3h)]]}{fK} > 0. \] (3.12)
Expanding \( c_{r3}^2 \) around \( \alpha = 0 \), we obtain
\[ c_{r3}^2 = 1 - \frac{64\xi_\phi^2(1 - h)(f'h - fh')}{\eta f M_{P_1}^2 r^3} \alpha^2 + O(\alpha^3), \] (3.13)
and hence \( c_{r3}^2 \to 1 \) as \( \alpha \to 0 \).

The squared angular propagation speeds of even-parity perturbations are complicated, but expanding \( c_{\Omega,\pm}^2 \) around \( \alpha = 0 \) leads to
\[ c_{\Omega,\pm}^2 = 1 + \frac{\xi_\phi[2\sqrt{\eta}fh(2f - r')r\phi' \pm \sqrt{2B}]}{\sqrt{\eta f^2 M_{P_1}^2 r^2}} |\alpha| + O(\alpha^2), \] (3.14)
where
\[ B = M_{P_1}^2[f'^2 h^2 + 2f^2(rh' - 2h + 2) - rf(2f''rh + rf'h' - 2f'h)]^2 + 2\eta f^2 h^2(rf' - 2f)^2 r^2 \phi'^2. \] (3.15)
Under the no-ghost condition \( \eta > 0 \), \( B \) is positive and hence \( c_{\Omega,\pm}^2 \) are the real values. In the limit \( \alpha \to 0 \), \( c_{\Omega,\pm}^2 \) approach 1.

From the above discussion, all the linear stability conditions should be consistently satisfied for \( \eta > 0 \) and \( |\alpha| \ll 1 \). We note, however, that the quantities like \( H, c_{r3}^2, c_{\Omega,\pm}^2 \) contain positive power-law terms of \( r \) in the denominators. To show the finiteness of these quantities at \( r = 0 \), we derive the solutions to \( f, h, \phi \) expanded around \( r = 0 \) in Sec. III A for the existence of hairy stars free from instabilities, we then put a limit on the coupling constant \( \alpha \).

A. Solutions expanded around \( r = 0 \) and their stability

Around the center of star, we impose the regular boundary conditions \( f(0) = f_c, h(0) = 1, \phi(0) = \phi_c, \rho(0) = \rho_c, P'(0) = P_c \) and \( f'(0) = h'(0) = \phi'(0) = \rho'(0) = P'(0) = 0 \). Then, around \( r = 0 \), the scalar field is expanded as
\[ \phi = \phi_c + \phi_2 r^2 + O(r^3), \] (3.16)
likewise for \( f, h, \rho, \) and \( P \). We can also expand the coupling function \( \xi(\phi) \) (and its \( \phi \) derivatives), as
\[ \xi(\phi) = \xi(\phi_c) + \sum_{n\geq 1} \xi^{(n)}(\phi_c) (\phi - \phi_c)^n \frac{n!}{n!}. \] (3.17)
On using the background Eqs. (3.4)-(3.6) with Eq. (2.8), the quantity \( \phi_2 \) in Eq. (3.16) obeys the following algebraic equation

\[
Y(x) = \frac{9x(1 - 16x)^3}{2[1 + 3w_c(1 - 16x)]} = \kappa \alpha^2,
\]

where

\[
x = \frac{\alpha \xi_\phi(\phi_c)}{M_{P_1}^2}, \quad \kappa = \frac{\rho_c^2 \xi_\phi(\phi_c)}{\eta M_{P_1}^2}, \quad w_c = \frac{P_c}{\rho_c}.
\]

(3.19)

We assume that the EOS parameter \( w_c \) is in the range \( w_c > 0 \). In the small-coupling limit \( |\alpha| \ll 1 \), we have \( 9x/[2(1 + 3w_c)] \approx \kappa \alpha^2 \) and hence

\[
\phi_2 = \frac{2\xi_\phi(\phi_c)\rho_c^2(1 + 3w_c)}{9\eta M_{P_1}^2} \alpha + \mathcal{O}(\alpha^3).
\]

(3.20)

Substituting the leading-order solution of Eq. (3.20) into the definition of \( x \) in Eq. (3.19), we have

\[
x > 0,
\]

(3.21)

under the condition \( \eta > 0 \). By the end of this section, we will not exploit the small \( \alpha \) expansion to discuss the linear stability of NSs.

The solutions to \( f, h, P \) expanded around \( r = 0 \) are given, respectively, by

\[
f = f_c + \frac{f_c M_{P_1}^2 \rho_c (1 + 3w_c) - 48\alpha \xi_\phi(\phi_c)\rho_c w_c \phi_2}{6 M_{P_1}^2 - 16\alpha \xi_\phi(\phi_c)\phi_2^2} r^2 + \mathcal{O}(r^3),
\]

\[
h = 1 - \frac{\rho_c}{3 M_{P_1}^2 - 16\alpha \xi_\phi(\phi_c)\phi_2} r^2 + \mathcal{O}(r^3),
\]

\[
P = P_c - \frac{\rho_c (1 + w_c) M_{P_1}^2 \rho_c (1 + 3w_c) - 48\alpha \xi_\phi(\phi_c)\rho_c w_c \phi_2}{12 M_{P_1}^2 - 16\alpha \xi_\phi(\phi_c)\phi_2^2} r^2 + \mathcal{O}(r^3).
\]

(3.22)

(3.23)

(3.24)

The scalar field is of the form (3.16) with \( \phi_2 \) satisfying the relation (3.18). On using these solutions, the quantities \( G, H, \) and \( F \) at \( r = 0 \) reduce to

\[
G(r = 0) = M_{P_1}^2, \quad H(r = 0) = F(r = 0) = M_{P_1}^2 (1 - 16x),
\]

(3.25)

where \( H \) is finite at \( r = 0 \) due to the property that \( \phi' = 2\phi x r \) at leading order. The squared propagation speeds of odd-parity perturbations along the radial and angular directions are given by

\[
c^2_r(r = 0) = c^2_{\Omega}(r = 0) = \frac{1}{1 - 16x}.
\]

(3.26)

Then, there are neither ghost nor Laplacian instabilities in the odd-parity sector if

\[
0 < x < \frac{1}{16},
\]

(3.27)

where we have also taken into account the condition (3.21). Both \( c^2_r(r = 0) \) and \( c^2_{\Omega}(r = 0) \) are larger than 1. In the limit that \( \alpha \to 0 \), we have \( x \to 0 \) and hence \( c^2_r(r = 0) = c^2_{\Omega}(r = 0) \to 1 \) as expected. In the subject of NSs, “superluminality” is sometimes argued as a sign of “acausality.” We emphasize that the propagation speed of scalar-field perturbations just fixes a causal boundary of the scalar field at each position, and the superluminal speeds do not mean acausality.

In the even-parity sector, the leading-order term of \( K \) expanded around \( r = 0 \) is proportional to \( r^6 \), i.e.,

\[
K = \frac{16\rho_c^2 M_{P_1}^2 [1 + 48x + 3w_c(1 - 16x)]}{9(1 - 16x)} r^6 + \mathcal{O}(r^7),
\]

(3.28)

where we used Eq. (3.18) to eliminate \( \kappa \alpha^2 \). Under the condition (3.27), the coefficient of \( r^6 \) in Eq. (3.28) is positive. The squared radial and angular propagation speeds of \( \delta \phi \) at \( r = 0 \) are given by

\[
c^2_{r3}(r = 0) = c^2_{\Omega3}(r = 0) = \frac{(1 + 3w_c)(1 - 64x) + 2304w_c x^2}{(1 - 16x)[1 + 48x + 3w_c(1 - 16x)]},
\]

(3.29)
whereas $c_{n+1}^2(r = 0)$ is equivalent to Eq. (3.26). Under the condition (3.27), the absence of Laplacian instability of $\delta \phi$ requires a positivity of the numerator of Eq. (3.29), so that

$$0 < x < x_m \equiv \frac{2(1 + 3w_c) - \sqrt{(1 + 3w_c)(4 + 3w_c)}}{144w_c}. \quad (3.30)$$

For $0 < w_c < \infty$ we have $1/64 < x_m < 1/48$, so $x_m$ is smaller than $1/16$. The function $Y(x)$ in Eq. (3.31) has a maximum value $Y(x_m)$ at $x = x_m$. Provided that $0 < \kappa \alpha^2 < Y(x)$, there are solutions to Eq. (3.18). This gives an upper bound on $|\alpha|$, as

$$|\alpha| < \frac{\sqrt{\eta} M_{\rm Pl}^3}{\rho_c \xi_c(\phi_c)} \sqrt{6(2\sqrt{1 + 3w_c} - \sqrt{4 + 3w_c})^{1/2}[\sqrt{(1 + 3w_c)(4 + 3w_c)} - 2 + 3w_c]^{3/2}} \cdot \frac{1}{216w_c^2(\sqrt{1 + 3w_c} + \sqrt{4 + 3w_c})^{1/2}}. \quad (3.31)$$

Among the two solutions of $\kappa \alpha^2 = Y(x)$, one of them ($x = x_1$) exists in $0 < x_1 < x_m$, while the other ($x = x_2$) is in the region $x_m < x_2 < 1/16$. The former is in the region (3.30) satisfying the condition of Laplacian stability. In summary, as long as $\alpha$ is in the range (3.31), there is a solution $x = x_1$ consistent with all the linear stability conditions at $r = 0$. The criterion (3.31) is also valid for nonrelativistic stars with $w_c \ll 1$. In this case, the scalar-GB coupling is constrained to be

$$|\alpha| < \frac{9\sqrt{6\eta} M_{\rm Pl}^3}{128\rho_c \xi_c(\phi_c)}, \quad \text{for} \quad w_c \to 0, \quad (3.32)$$

which can be applied to nonrelativistic objects such as Sun and Earth.

Let us consider the power-law scalar-GB coupling

$$\xi_c(\phi) = M_{\rm Pl}^{2-n} r_0^2 \phi^n, \quad (3.33)$$

where

$$r_0 = \sqrt{\frac{8\pi M_{\rm Pl}^2}{\rho_0}} = 89.664 \, \text{km}, \quad \rho_0 = m_n n_0 = 1.6749 \times 10^{14} \, \text{g} \cdot \text{cm}^{-3}. \quad (3.34)$$

Here, $m_n = 1.6749 \times 10^{-24} \, \text{g}$ is the neutron mass and $n_0 = 0.1 \, (\text{fm})^{-3}$ is the typical density of NSs. For this coupling, the bound (3.31) yields

$$|\alpha| < |\alpha_{\max}| = \frac{\sqrt{\eta} \rho_0}{8\pi n \rho_c} \left( \frac{\phi_c}{M_{\rm Pl}} \right)^{1-n} \frac{\sqrt{6(2\sqrt{1 + 3w_c} - \sqrt{4 + 3w_c})^{1/2}[\sqrt{(1 + 3w_c)(4 + 3w_c)} - 2 + 3w_c]^{3/2}}}{216w_c^2(\sqrt{1 + 3w_c} + \sqrt{4 + 3w_c})^{1/2}}. \quad (3.35)$$

![FIG. 1. Maximum value of $|\alpha|$ versus $w_c$ constrained by the bound (3.35) for the linear GB coupling ($n = 1$) with $\eta = 1$. The solid and dashed lines correspond to the upper limits $|\alpha_{\max}|$ for SLy and FPS EOSs, respectively, with the central density in the range $10^{-2} \rho_0 \leq \rho_c \leq 20 \rho_0$.](image)
For the linear GB coupling \((n = 1)\), the central density \(\rho_c\) and EOS parameter \(w_c\) determine the upper limit of \(|\alpha|\), without having the dependence of \(\phi_c\). In Fig. 1 we plot \(|\alpha_{\text{max}}|\) as a function of \(w_c\) for SLy (solid) and FPS (dashed) EOSs [106] in the range \(10^{-2} \leq \rho_c \leq 20\rho_0\). As \(\rho_c\) increases, \(w_c\) grows from nonrelativistic values of order \(10^{-3}\) to relativistic values of order \(10^{-1}\). We find that \(|\alpha_{\text{max}}|\) is a decreasing function of \(w_c\). For SLy EOS, the maximum mass of NS is reached around the central density \(\rho_c \approx 15\rho_0\) with the EOS parameter \(w_c \approx 0.5\). To realize such a maximum mass of NS, the scalar-GB coupling is constrained to be

\[|\alpha| < 3 \times 10^{-4}.\]  

(3.36)

For FPS EOS, we also obtain a similar upper bound. The existence of NS solutions with the central density \(\rho_c \gtrsim 15\rho_0\) gives the value of \(|\alpha_{\text{max}}|\) even smaller than \(3 \times 10^{-4}\).

The dilatonic GB coupling \(\alpha_{\text{GB}}\) in the regime \(\phi/M_{\text{Pl}} \ll 1\), which was discussed in Refs. [107] [108] with the unit \(M_{\text{Pl}} = 1/\sqrt{8\pi}\), is related to our linear scalar-GB coupling \(\alpha\) by \(\alpha_{\text{GB}} = \alpha M_{\text{Pl}}^2 = \alpha r_0^2/\sqrt{8\pi}\). Then, the theoretical bound (3.36) translates to

\[\sqrt{|\alpha_{\text{GB}}|} < 0.7 \text{ km}.\]  

(3.37)

This is tighter than the typical observational bounds \(\sqrt{|\alpha_{\text{GB}}|} < O(1)\) km derived from the GW measurements of NS-NS, BH-NS, and BH-BH binaries, see Table I of Ref. [108]. Thus, the existence of NS solutions consistent with the linear stability conditions gives a new theoretical upper bound on the dilatonic GB coupling.

**B. Solutions expanded at spatial infinity and their stability**

The surface of star is defined by the radius \(r_s\) at which the fluid pressure \(P\) vanishes. Outside the star \((r > r_s)\), we have \(\rho = 0 = P\) in the background Eqs. (2.9)-(2.12). Imposing the asymptotic flatness at spatial infinity, we can expand \(f, h, \phi\) in the forms \(f = 1 + \sum_{i=1}^4 \text{fi}/r^i, h = 1 + \sum_{i=1}^4 \text{hi}/r^i, \) and \(\phi = \phi_0 + \sum_{i=1}^4 \text{Oi}/r^i\). We also use the expansions of \(\xi(\phi)\) and its \(O\) derivatives analogous to Eq. (3.17), with the replacement \(\phi_c \rightarrow \phi_0\). The large-distance solutions consistent with the background Eqs. (3.4)-(3.6) are given by

\[f = 1 - 2M/M_{\text{Pl}}^2 + \frac{\eta M^2 \phi_1^2}{6M_{\text{Pl}}^2 r^3} + \frac{M \phi_1 [\eta M \phi_1 + 24\alpha \xi_\phi(\phi_0)]}{3M_{\text{Pl}}^2 r^4} + O(r^{-5}),\]  

(3.38)

\[h = 1 - 2M/M_{\text{Pl}}^2 + \frac{\eta \phi_1^2}{2M_{\text{Pl}}^2 r^3} + \frac{M \phi_1 [\eta M \phi_1 + 24\alpha \xi_\phi(\phi_0)]}{3M_{\text{Pl}}^2 r^4} + O(r^{-5}),\]  

(3.39)

\[\phi = \phi_0 + \frac{\phi_1}{r} + \frac{(16M^2 M_{\text{Pl}}^2 - 6\eta \phi_1^2)\phi_1}{12M_{\text{Pl}}^2 r^3} + \frac{M [6\eta M^2 M_{\text{Pl}}^2 \phi_1 - 8\alpha \xi_\phi(\phi_0)\phi_1^2 - 12\alpha M^2 M_{\text{Pl}}^2 \phi_0]}{3\eta M_{\text{Pl}}^2 r^4} + O(r^{-5}),\]  

(3.40)

where we set \(\text{fi} = -2M\). Then the quantities (2.15)-2.17 can be estimated as \(G = M_{\text{Pl}}^2 + 8\alpha M \xi_\phi(\phi_0)\phi_1^2 r^{-4} + O(r^{-5}),\) \(\mathcal{H} = M_{\text{Pl}}^2 + 8\alpha M \xi_\phi(\phi_0)\phi_1 r^{-3} + O(r^{-4}),\) and \(\mathcal{F} = M_{\text{Pl}}^2 - 16\alpha \xi_\phi(\phi_0)\phi_1^2 r^{-3} + O(r^{-4}),\) so the squared radial and angular propagation speeds reduce, respectively, to

\[c_r^2 = 1 + \frac{16\alpha \xi_\phi(\phi_0)\phi_1^2}{M_{\text{Pl}}^2 r^3} + O(r^{-4}), \quad c_\Omega^2 = 1 - \frac{8\alpha \xi_\phi(\phi_0)\phi_1^2}{M_{\text{Pl}}^2 r^3} + O(r^{-4}).\]  

(3.41)

As \(r \rightarrow \infty\), both \(c_r^2\) and \(c_\Omega^2\) approach 1.

In the even-parity sector, the quantity \(\mathcal{K}\) is expressed as

\[\mathcal{K} = 2\eta M_{\text{Pl}}^4 \phi_1^2 + \frac{4\eta M_{\text{Pl}}^4 M \phi_1^2}{r} + O(r^{-2}),\]  

(3.42)

whose positivity is ensured for \(\eta > 0\). The radial propagation speed squared (3.12) yields

\[c_r^2 = 1 - \frac{128\alpha^2 M \xi_\phi(\phi_0)^2}{M_{\text{Pl}}^4 r^2} + O(r^{-8}),\]  

(3.43)

which quickly approaches 1 at large distances even compared to \(c_r^2\) and \(c_\Omega^2\). For the angular propagation, we obtain

\[c_{\Omega}^2 = 1 \pm \frac{4\xi_\phi(\phi_0)[(\eta \phi_1^2 + 72M^2 M_{\text{Pl}}^2)^{1/2} \mp \sqrt{\eta \phi_1^2}]}{\sqrt{\eta M_{\text{Pl}}^2 r^3}} |\alpha| + O(r^{-4}).\]  

(3.44)

For \(r \gg r_s\) all the squared propagation speeds given above rapidly approach 1, so the sign of \(\alpha\) does not matter for the discussion of Laplacian instability. Provided that \(\eta > 0\), there are neither ghost nor Laplacian instabilities at spatial infinity.
C. Numerical solutions and stability conditions

The discussions in Secs. IIIA and IIIB show that, under the bound (3.31) with \( \eta > 0 \), there are NS solutions with a nontrivial scalar profile consistent with all the linear stability conditions around \( r = 0 \) and \( r \to \infty \). However, they do not necessarily guarantee the linear stability of solutions at intermediate distances, so we will numerically study whether neither ghost nor Laplacian instabilities appear at any radius \( r \).

For concreteness, we study the linear scalar-GB coupling given by the \( n = 1 \) case of Eq. (3.33). Since this corresponds to \( G_2 = \eta X, G_3 = 0, G_4 = M_p^2/2 \), and \( G_5 = -4\alpha M_p^2 r_0^2 \ln |X| \), it belongs to a subclass of shift-symmetric Horndeski theories. The same scalar-GB coupling can also accommodate the dilatonic coupling \( \xi(\phi) \propto e^{\epsilon_{\phi}} \) in the limit \( |\mu\phi| \ll 1 \). Since \( \mathcal{P}_\phi = 0 \) in the scalar-field Eq. (2.12), we have \( r^2 \sqrt{f'} h J' = Q = \text{constant} \) and hence

\[
\frac{h}{f} [\eta f r^2 \phi' + 4\alpha M_p^2 r_0^2 f'(h - 1)] = Q \sqrt{\frac{f}{h}}. \tag{3.45}
\]

To satisfy the boundary conditions of \( f, h \), and \( \phi' \) at \( r = 0 \), we require that \( Q = 0 \). Then, the field derivative can be expressed as

\[
\phi'(r) = -\frac{4\alpha M_p^2 r_0^2 f'(h - 1)}{\eta f r^2}. \tag{3.46}
\]

Substituting Eqs. (3.22) and (3.23) into Eq. (3.46) around \( r = 0 \), we have \( \phi'(r) \propto r \) as consistent with Eq. (3.16). At spatial infinity, using the expanded solutions (3.38) and (3.39) in Eq. (3.46) leads to \( \phi'(r) \propto r^{-5} \). Since the integration constant \( Q \) corresponds to the scalar charge, the choice of \( Q = 0 \) means that \( \phi_1 = 0 \) in the expansion of Eq. (3.40) and hence \( \phi = \phi_0 - 4\alpha M^2 \xi_0(\phi_0)/\eta r^4 \). In Eq. (3.42) the leading-order term of \( K \) vanishes, but it is replaced by \( K = 512\alpha^2 M^4 M_p^2 \xi_0(\phi_0)^2/\eta r^6 + \mathcal{O}(r^{-7}) \). Then, the no-ghost condition is satisfied for \( \eta > 0 \).

To perform the numerical integration, we introduce the following variables

\[
s = \ln \frac{r}{r_0}, \quad \mathcal{M}(r) = 4\pi r M_p^2 (1 - h), \quad m(r) = \frac{3M(r)}{4\pi r_0^3 \rho_0}, \quad y = \frac{\rho}{\rho_0}, \quad z = \frac{P}{\rho_0}. \tag{3.47}
\]

The ADM mass of star can be computed as

\[
M \equiv \mathcal{M}(r \to \infty) = 2.5435 \times 10^2 m_\infty M_\odot, \tag{3.48}
\]

where \( m_\infty \equiv m(r \to \infty) \) and \( M_\odot = 1.9884 \times 10^{33} \) g is the solar mass.

For the NS EOS, we exploit the analytic representation of SLy EOS parametrized by

\[
\xi = \log_{10}(\rho/g \cdot \text{cm}^{-3}), \quad \zeta = \log_{10}(P/\text{dyn} \cdot \text{cm}^{-2}), \tag{3.49}
\]

where the relation between \( \xi \) and \( \zeta \) is given in Ref. [100]. In terms of \( y \) and \( z \), we can express \( \xi \) and \( \zeta \) as

\[
\xi = \alpha_1 + \alpha_2 \ln y, \quad \zeta = \alpha_3 + \alpha_2 \ln z, \tag{3.50}
\]

where \( \alpha_1 = \ln(\rho_0/g \cdot \text{cm}^{-3})/\ln 10 \), \( \alpha_2 = (\ln 10)^{-1} \), and \( \alpha_3 = \ln(\rho_0/\text{dyn} \cdot \text{cm}^{-2})/\ln 10 \). Then, the EOS translates to

\[
z = \exp \left[ \frac{\zeta(\xi) - \alpha_3}{\alpha_2} \right], \tag{3.51}
\]

so that \( z \) is known by the analytic representation of \( \zeta(\xi) \). The continuity Eq. (2.8) gives the differential equation for \( y \), as

\[
\frac{dy}{ds} = -\frac{y(y + z)}{2z} \left( \frac{dz}{d\xi} \right)^{-1} \frac{1}{f} \frac{df}{ds}. \tag{3.52}
\]

We derive the differential equations of \( f \) and \( h \) by solving Eqs. (2.9), (2.11), and (2.12) for \( f'', h', \) and \( \phi'' \). We replace the first-order field derivative \( \phi' \) in the differential equations of \( f \) and \( h \) by using Eq. (3.46). In the left panel of Fig. 3, we plot \( \mathcal{M}, \phi', \) and \( P \) as a function of \( r/r_0 \) for \( \alpha = 2 \times 10^{-5}, \eta = 1, \) and \( \rho_\epsilon = 15\rho_0 \). We choose the boundary conditions at \( r/r_0 = 10^{-5} \) to be consistent with Eqs. (3.16) and (3.22)-(3.24). As we estimated in Sec. IIIA, the field derivative increases as \( \phi'(r) \propto r \) around \( r = 0 \), with \( \mathcal{M}(r) \propto r^3 \) and \( P(r) \) decreasing according to Eq. (3.24). The radius at which \( P \) vanishes is \( r_s = 0.114 r_0 \approx 10.22 \) km, which corresponds to the surface of star.
The field derivative starts to decrease around \( r = r_s \) and it has the dependence \( \phi'(r) \propto r^{-5} \) at large distances. The growth of mass function \( M(r) \) saturates around the surface of star and it quickly approaches the ADM mass \( M \). In the numerical simulation of Fig. 2 we have \( M = 2.003 M_\odot \), which is smaller than the corresponding mass \( 2.044 M_\odot \) in GR (\( \alpha = 0 \)). This suppressed ADM mass in comparison to GR is consistent with the result obtained for the dilatonic scalar-GB coupling \( \xi(\phi) \propto e^{\alpha \phi} \) in Ref. [21].

In the right panel of Fig. 2 we show \( c_r^2, c_\Omega^2, c_\Omega^2, c_{\Omega^+}, \) and \( c_{\Omega^-} \) versus \( r/r_0 \) for the same model parameters as those used in the left. As estimated from Eq. (3.26), the radial and angular propagation speeds in the odd-parity sector are superluminal in the regime \( r \ll r_s \). In the even-parity sector the evolution of \( c_{\Omega^+}^2 \) around \( r = 0 \) is similar to \( c_r^2 \) and \( c_\Omega^2 \), whereas \( c_{\Omega^+}^2 \) and \( c_{\Omega^-}^2 \) are in the subluminal range as estimated by Eq. (3.13). Around \( r = r_s \), \( c_{\Omega^+}^2 \) and \( c_{\Omega^-}^2 \) exhibit temporal increase and decrease, respectively, but they remain to be finite positive values without Laplacian instabilities. Outside the star, all the squared propagation speeds shown in Fig. 2 quickly approach 1, as consistent with the discussion in Sec. III B. Numerically, we have also confirmed that the no-ghost conditions of odd- and even-parity perturbations are satisfied at any distance \( r \).

The numerical simulation of Fig. 2 corresponds to \( \rho_0 = 15 \rho_0, \) \( w_0 = 0.48, \) \( \eta = 1, \) and \( \alpha = 1, \) so the bound (3.35) gives \( |\alpha| < 3.19 \times 10^{-4} \). For the coupling \( \alpha \) in this range, we numerically confirmed that all the linear stability conditions are satisfied at any distance \( r \). Since \( c_{\Omega^+}^2 \) and \( c_{\Omega^-}^2 \) are smallest at the center of NS, the upper limit of \( |\alpha| \) is determined by their values at \( r = 0 \) as we performed in Sec. III A. When \( \alpha < 0 \), the background solution and its linear stability are similar to those for the corresponding positive value \( |\alpha| \). If \( |\alpha| \) exceeds the upper limit \( |\alpha_{\text{max}}| \), there are Laplacian instabilities associated with negative values of \( c_{\Omega^+}^2 \) and \( c_{\Omega^-}^2 \) at \( r = 0 \). Thus, the theoretical bound (3.35) is sufficiently accurate for the estimation of maximum allowed values of \( |\alpha| \). We have also performed numerical simulations for the scalar-GB couplings (3.33) with different powers \( n \) and confirmed that, for \( \alpha \) in the range (3.35) with \( \eta > 0 \), there are hairy NS solutions consistent with all the linear stability conditions.

IV. SCALAR-GAUSS-BONNET AND REGULAR COUPLINGS

In this section, we study the existence and the linear stability of NS solutions with a nontrivial scalar profile in the presence of several regular couplings besides the scalar-GB coupling \( \alpha \xi(\phi) R_{\text{GB}}^2 \). We also take into account the Einstein-Hilbert term \( M_{\text{Pl}}^2 R/2 \) and the canonical kinetic term \( \eta X \) (with \( \eta > 0 \)) in the action. We exploit SLy EOS for the numerical analysis in this section.
A. Cubic Galileon and scalar-GB couplings

The cubic Galileon corresponds to the coupling function $G_3 \supset \mu_3 X$, where $\mu_3$ is a constant. Let us consider theories given by the action

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_{p1}^2}{2} R + \eta X + \alpha \xi(\phi) R_{GB}^2 + \mu_3 X \Box \phi \right].$$

(4.1)

We first derive the solutions expanded around $r = 0$ and then discuss the linear stability of them. In doing so, we use the approximation $|\alpha| \ll 1$ and pick up terms up to the order of $\alpha^2$. Then, the coefficient $\phi_2$ in the field expansion (3.16) is

$$\phi_2 = \frac{3\eta M_{p1}^6 - 3\rho_2^2(1 + 2w_c)\xi_\phi(\phi_c)\alpha^2}{24\mu_3 M_{p1}^6} \left[ 1 - \sqrt{\frac{32M_{p1}^2\rho_2^2(1 + 3w_c)\xi_\phi(\phi_c)\mu_3\alpha}{3\eta M_{p1}^6 - 3\rho_2^2(1 + 2w_c)\xi_\phi(\phi_c)\alpha^2}} \right] + O(\alpha^3),$$

(4.2)

where we have chosen the branch recovering Eq. (3.20) in the limit $\mu_3 \to 0$. We do not necessarily assume that the cubic Galileon coupling is of the same order as the scalar-GB coupling. The consistency of the small $\alpha$ expansion requires that

$$\xi_\phi(\phi_c)|\mu_3\alpha| \ll \frac{\eta^2 M_{p1}^4}{\rho_2^2(1 + 3w_c)}.$$

(4.3)

where we also assumed $\alpha^2 \rho_2^2 \xi_\phi(\phi_c) \ll \eta M_{p1}^6$. Up to the order of $r^2$, the metric components and fluid pressure are the same forms as Eqs. (3.22)-(3.24) with $\phi'(r) = 2\phi_2 r$. In the limit that $\alpha \to 0$, we have $\phi_2 = 0$ even for $\mu_3 \neq 0$. This shows that the cubic-order coupling alone does not give rise to NS solutions with $\phi'(r) \neq 0$. This is consistent with the no-hair argument of Ref. [64] for regular couplings in shift-symmetric Horndeski theories. In other words, the scalar-GB coupling is needed for the realization of NSs with a nontrivial scalar profile.

On using the above background solutions around $r = 0$, we find that $G$, $H$, and $F$ are of the same form as Eq. (3.25), where $x = \alpha \xi_\phi(\phi_c)\phi_2/M_{p1}^2$ and $\phi_2$ satisfying Eq. (4.2). Then, the absence of ghost/Laplacian instabilities in the odd-parity sector requires that $x < 1/16$. The squared propagation speeds $c_{\gamma\gamma}^2, c_{\Omega\Omega}^2$ and $c_{\Omega\alpha}^2$ at $r = 0$ are equivalent to $1/(1 - 16x)$. The other two squared propagation speeds at $r = 0$, which are relevant to the radial and angular propagations of $\delta \phi$, are modified to

$$c_{\gamma\gamma}^2(r = 0) = \frac{\eta - 8\mu_3 \phi_2}{\eta - 12\mu_3 \phi_2} - \frac{64\rho_2^2 \xi_\phi(\phi_c)(\eta(1 + w_c) - 2\mu_3 \phi_2(5 + 6w_c))}{3M_{p1}^2(\eta - 12\mu_3 \phi_2)^2} \alpha^2 + O(\alpha^4).$$

(4.4)

Expanding Eq. (4.2) with respect to $\alpha$ and using the leading-order solution $\phi_2 = 2\rho_2^2(1 + 3w_c)\xi_\phi(\phi_c)\alpha/(9\eta M_{p1}^6)$, the inequality (4.3) translates to $\mu_3 \phi_2 \ll 2\eta/9$. In the limit that $|\mu_3\alpha| \ll 1$, the first term on the right hand-side of Eq. (4.4) approaches 1. On the other hand, even if $|\alpha| \ll 1$, the product $|\mu_3 \phi_2|$ can be as large as the order $0.1\eta$ for a large coupling $|\mu_3|$. To avoid that the leading-order term of Eq. (4.4) becomes negative, we require the condition $\eta - 12\mu_3 \phi_2 > 0$. On using the solution (4.2) under the approximation $\alpha^2 \rho_2^2 \xi_\phi(\phi_c) \ll \eta M_{p1}^6$, this condition translates to

$$\xi_\phi(\phi_c)|\mu_3\alpha| < \frac{\eta^2 M_{p1}^4}{4\rho_2^2(1 + 3w_c)}.$$

(4.5)

Around $r = 0$, the leading-order contribution to (2.19) is proportional to $r^6$, such that

$$K = \frac{8M_{p1}^2(3M_{p1}^2(1 - 16x)^3 \phi_2^2(\eta - 12\mu_3 \phi_2) + 32\rho_2^2 x \phi_2^2)}{3(1 - 16x)} r^6 + O(r^7).$$

(4.6)

Here, we have not used the expansion with respect to $\alpha$. Provided that $x < 1/16$ and $\eta - 12\mu_3 \phi_2 > 0$, the leading-order term of $K$ is positive. From the above discussion, the linear stability of NSs around $r = 0$ is ensured under the condition (4.5) with $x < 1/16$.

At large distances, the solutions to $f$ and $h$ are the same forms as Eqs. (3.38) and (3.39) up to the order of $r^{-4}$. The solution to the scalar field is modified to

$$\phi = \phi_0 + \frac{\hat{\phi}_1}{r} + M_{p1}^2 \frac{16M_{p1}^2 - \eta \rho_2^2}{12M_{p1}^2 r^3} + \frac{3M_{p1}^2(4\eta M_{p1}^3 \phi_0 - 8M_{p1}^2 \xi_\phi(\phi_0)\alpha - \mu_3 (\rho_2^2 + 2\eta^2 M_{p1}^4)}{6\eta M_{p1}^4 r^4} + O(r^{-5}),$$

(4.7)
in which the cubic Galileon coupling appears at the order of \( r^{-4} \). On using these large-distance solutions, it follows that \( c_{1 r}^2, c_{2 r}^2, \) and \( K \) are the same as those given in Eqs. (3.41) and (3.42) up to the next-to-leading order. On the other hand, the other squared propagation speeds are modified to

\[
\begin{align*}
c_{1 r}^2 &= 1 + \frac{4\delta \mu_3}{\eta r^3} + \mathcal{O}(r^{-4}) , \\
c_{2 r}^2 &= 1 + \frac{16 \alpha^2 \eta r^2 (\phi_0)(7M^2M_{Pl}^2 + \eta \phi_0^2) + \mu_3M_{Pl}^2 \mathcal{M}_3^2 [\mu_3M_{Pl}^2 - 8\alpha \eta \xi_\phi(\phi_0)] + \dot{\phi}[\mu_3M_{Pl}^2 + 4\alpha \eta \xi_\phi(\phi_0)]}{\eta M_{Pl}^2 r^3} + \mathcal{O}(r^{-4}) ,
\end{align*}
\]

both of which approach 1 in the limit \( r \to \infty \).

To study the linear stability of NS solutions with a nontrivial scalar profile discussed above, we perform the numerical integration for the linear scalar-GB coupling \( \xi(\phi) = M_{Pl}^2 \phi^2 \phi \) in the presence of cubic Galileons. Since this belongs to a subclass of shift-symmetric Horndeski theories, we have \( J^r = (Q/r^2) \sqrt{f/h} \) from Eq. (2.12). The regularity at \( r = 0 \) leads to \( Q = 0 \) and hence

\[
\phi'(r) = \frac{\eta fr}{\mu_3 h (rf' + 4f)} \left[ 1 - \frac{8\tilde{\mu}_3 \alpha^2 f h (rf' + 4f) (h - 1)}{\eta^2 r^3 f'^2} \right] ,
\]

where \( \tilde{\mu}_3 \) is a dimensionless coupling defined by

\[
\tilde{\mu}_3 \equiv \frac{\mu_3 M_{Pl}^2}{r_0^2} .
\]

In the limit \( \tilde{\mu}_3 \to 0 \), the branch (4.10) smoothly approaches (3.46). In the following, we consider the positive scalar-GB coupling \( (\alpha > 0) \) and study the effect of cubic Galileons on the linear stability of NS.

From Eq. (4.2), the positive Galileon coupling \( \mu_3 > 0 \) leads to the enhancement of \( \phi'(r) \) around \( r = 0 \) in comparison to the case \( \mu_3 = 0 \). On the other hand, Eq. (4.5) gives the following upper bound

\[
\tilde{\mu}_3 < \tilde{\mu}_3^{\text{max}} = \frac{\eta^2}{256\pi^2 \alpha} \left( \frac{\rho_0}{\rho_c} \right)^2 \frac{1}{1 + 3 w_c} .
\]

Due to this limit, the enhancement of \( \phi'(r) \) induced by the positive \( \tilde{\mu}_3 \) is not so significant. With the model parameters \( \alpha = 2.0 \times 10^{-4}, \rho_c = 15\rho_0, \) \( w_c = 0.48, \) and \( \eta = 1, \) for example, we have \( \tilde{\mu}_3^{\text{max}} = 3.6 \times 10^{-3} \) from Eq. (4.12). In this case, we numerically find that the Galileon coupling needs to be in the range \( \mu_3 < 3.0 \times 10^{-3} \) to satisfy all the linear stability conditions at any distance \( r \). Thus, the condition (4.12) gives a good approximate criterion for the existence of NSs with a nontrivial scalar profile without instabilities. For \( \tilde{\mu}_3 = 2.5 \times 10^{-3} \), the ADM mass and radius of NS are \( M = 1.998M_\odot \) and \( r_s = 10.22 \) km, respectively. They are similar to the values \( M = 2.003M_\odot \) and \( r_s = 10.22 \) km derived for \( \mu_3 = 0 \). For increasing \( \mu_3 \), the squared propagation speeds \( c_{1 r}^2 \) and \( c_{2 r}^2 \) tend to be larger than those for \( \mu_3 = 0 \) because of the approach of the term \( \eta - 12\mu_3 \phi_2 \) to \( +0 \) in Eq. (4.4). We note that \( c_{1}^2, c_{2}^2 \), and \( c_{3}^2 \) are not strongly affected by a positive coupling \( \mu_3 \).

The negative value of \( \mu_3 \) is not bounded by Eq. (4.12). In the left panel of Fig. 3 we plot \( M \) and \( \phi' \) versus \( r/r_0 \) for \( \mu_3 = -0.1 \) as solid lines. In comparison to the case \( \mu_3 = 0 \), the field derivative is suppressed by the negative Galileon coupling. For \( \mu_3 = -0.1 \) the ADM mass is \( M = 2.026M_\odot \), which is larger than \( M = 2.003M_\odot \) derived for \( \mu_3 = 0 \). This increase of \( M \) is attributed to the suppression of \( \phi' \) induced by the negative \( \mu_3 \). In the right panel of Fig. 3 we show the five squared propagation speeds versus \( r/r_0 \) for \( \mu_3 = -0.1 \). They are different from 1 inside the star, but all of them are larger than 0. In comparison to the case \( \mu_3 = 0 \) shown in Fig. 2 there are more rapid temporal increase of \( c_{3}^2 \) and decrease of \( c_{1}^2 \) around the surface of star. For decreasing \( \mu_3 \), we find that these transient variations of \( c_{3}^2 \) and \( c_{1}^2 \) tend to be more significant. Numerically, we find that there is a lower limit of \( \mu_3 \) in general to avoid that \( c_{1}^2 \) becomes negative around \( r = r_s \). With the model parameters \( \alpha = 2 \times 10^{-4}, \eta = 1, \) and \( \rho_c = 15\rho_0, \) for example, this instability arises for \( \mu_3 < \mu_3^{\text{min}} = -0.22 \).

The above argument shows that, for the coupling \( \mu_3 \) between \( \mu_3^{\text{min}} \) and \( \mu_3^{\text{max}} \), there are NS solutions with a nontrivial scalar profile free from ghost/Laplacian instabilities. As \( \mu_3 \) approaches \( \mu_3^{\text{min}} \) or \( \mu_3^{\text{max}} \), the squared propagation speeds \( c_{1 r}^2 \) and \( c_{2 r}^2 \) of \( \delta \phi \) exhibit difference from those for \( \mu_3 = 0 \).
theory is given by the action
\[ \alpha \xi M \]
and dotted lines correspond to
\[ \alpha r \]
contribution to the scalar-field equation of motion around
\[ c M \]
by
\[ \frac{\alpha}{r} \]
scalar-GB couplings modifies this no-hair property.

The quartic derivative coupling alone does not give rise to asymptotically-flat NS solutions [64], but the presence of scalar-GB couplings modifies this no-hair property.

Then, the solutions expanded around \( r = 0 \) are given by

\[ f = f_c + \frac{f_c M_{\text{Pl}}^2 \rho_c (1 + 3 w_c) - 48 \alpha \xi \phi (\phi_c) \phi_2 (w_c \rho_c - 8 \mu_4 \phi_2^2)}{6 [M_{\text{Pl}}^2 - 16 \alpha \xi \phi (\phi_c) \phi_2^2]} r^2 + \mathcal{O}(r^3) , \]

\[ h = 1 - \frac{\rho_c + 24 \mu_4 \phi_2^2}{3 [M_{\text{Pl}}^2 - 16 \alpha \xi \phi (\phi_c) \phi_2^2]} r^2 + \mathcal{O}(r^3) , \]

\[ P = P_c - \frac{\rho_c (1 + w_c) [M_{\text{Pl}}^2 \rho_c (1 + 3 w_c) - 48 \alpha \xi \phi (\phi_c) \phi_2 (w_c \rho_c - 8 \mu_4 \phi_2^2)]}{12 [M_{\text{Pl}}^2 - 16 \alpha \xi \phi (\phi_c) \phi_2^2]} r^2 + \mathcal{O}(r^3) , \]

where \( \phi_2 \) is the coefficient appearing in the scalar-field expansion \( \phi = \phi_0 + \phi_2 r^2 + \cdots \). Since the leading-order contribution to the scalar-field equation of motion around \( r = 0 \) is highly nonlinear in \( \phi_2 \), it is impossible to solve it for any value \( \alpha \). So, we restrict ourselves to the case \( |\alpha| \ll 1 \). For \( |\alpha| \ll 1 \), \( \phi_2 \) satisfies the cubic-order equation

\[ \frac{9 M_{\text{Pl}}^2 (\eta M_{\text{Pl}}^2 + 16 \mu_4 \phi_2^2 - 2 w_c \rho_c \mu_4)}{2 \xi \phi (\phi_c) \rho_c (1 + 3 w_c) (\rho_c + 72 \mu_4 \phi_2^2)} \phi_2 = \alpha , \]

at linear order in \( \alpha \). In the limit \( \alpha \to 0 \) with a small derivative coupling constant \( \mu_4 \), the only solution to Eq. (4.17) is \( \phi_2 = 0 \). Hence the nonvanishing scalar-GB coupling \( \alpha \) is required to have the solution with a nontrivial scalar profile with \( \phi_2 \neq 0 \). On using these expanded solutions, the quantities associated with the stability of odd-parity

B. Quartic derivative and GB couplings

We proceed to a theory with the linear quartic derivative coupling \( G_4 \supset \mu_4 X \) besides the scalar-GB coupling \( \alpha \xi(\phi) R_{\text{GB}}^2 \). We note that the quartic coupling \( \mu_4 X \) in \( G_4 \) is equivalent to the quintic coupling \( -\mu_4 \phi \) in \( G_5 \) [50]. This theory is given by the action

\[ S = \int d^4x \sqrt{-g} \left[ \frac{M_{\text{Pl}}^2}{2} R + \eta X + \alpha \xi (\phi) R_{\text{GB}}^2 + \mu_4 X R + \mu_4 \left\{ (\square \phi)^2 - (\nabla_\mu \phi_0) (\nabla_\nu \phi_0) \right\} \right] . \]

The quartic derivative coupling alone does not give rise to asymptotically-flat NS solutions [64], but the presence of scalar-GB couplings modifies this no-hair property.
perturbations at $r = 0$ are of the same form as Eq. (3.25), where $x = \alpha \xi_{,\phi}(\phi)c/\mathcal{M}^2$ and $\phi_2$ satisfies the relation (4.17). The stability in the odd-parity sector is ensured for $x < 1/16$.

In the following, we will exploit the expansion with respect to the small coupling $\alpha$. Around $r = 0$, the no-ghost parameter of even-parity perturbations has the dependence

$$
\mathcal{K} = [8\phi_2^2\mathcal{M}^2_r (\eta\mathcal{M}^2_r + 144\phi_2^2\mu^2_4 + 2\rho_c\mu_4) - 256\phi_2^2\xi_{,\phi}(\phi_c) (\eta\mathcal{M}^2_r + 24\phi_2^2\mu^2_4 - \mu_4\rho_c) + \mathcal{O}(\alpha^2)] r^6 + \mathcal{O}(r^7). 
$$

(4.18)

To avoid ghosts for small $\alpha$, the leading-order term $8\phi_2^2\mathcal{M}^2_r (\eta\mathcal{M}^2_r + 144\phi_2^2\mu^2_4 + 2\rho_c\mu_4) r^6$ in $\mathcal{K}$ needs to be positive. This amounts to the condition

$$
\mu_4 > -\frac{\eta\mathcal{M}^2_r + 144\phi_2^2\mu^2_4}{2\rho_c},
$$

(4.19)

which indicates the existence of a lower bound on negative values of $\mu_4$. We note that $\phi_2$ depends on $\mu_4$ through Eq. (4.17). At $r = 0$, $c^2_{r1}$, as well as $c^2_2$ and $c^2_{\Omega}$, are of the same forms as Eq. (3.26), with $\phi_2$ satisfying Eq. (4.17). The other two squared propagation speeds at $r = 0$ are

$$
c^2_{r3}(r = 0) = c^2_{\Omega, -}(r = 0) = \frac{\eta\mathcal{M}^2_r + 48\phi_2^2\mu^2_4 - 2w_c\rho_c\mu_4}{\eta\mathcal{M}^2_r + 144\phi_2^2\mu^2_4 + 2\rho_c\mu_4} + \mathcal{O}(\alpha).
$$

(4.20)

Under the condition (4.19), the absence of Laplacian instability requires that

$$
\mu_4 < \frac{\eta\mathcal{M}^2_r + 48\phi_2^2\mu^2_4}{2w_c\rho_c},
$$

(4.21)

which indicates the existence of an upper bound on positive values of $\mu_4$.

The solutions to $f$ and $\phi$ expanded at large distances are of the same forms as Eqs. (3.38) and (3.40), respectively, while the solution to $h$ is

$$
h = 1 - \frac{2M}{r} + \frac{\eta\phi_1^2}{2\mathcal{M}^2_r r^2} + \frac{\eta\phi_1^2}{2\mathcal{M}^2_r r^2} + \frac{2\phi_1 [(M^2\eta - 3\mu_4)^{\phi_1} + 24\mu_4\xi_{,\phi}(\phi_0)]}{5\mathcal{M}^2_r r^4} + \mathcal{O}(r^{-5}).
$$

(4.22)

Up to next-to-leading order, the quantities $c^2_r$, $c^2_{\Omega}$, $\mathcal{K}$, and $c^2_{\Omega, -}$ are the same as those in Eqs. (3.41), (3.42), and (3.44), while the third radial propagation speed squared is modified to

$$
c^2_{r3} = 1 - \frac{2\mu_4 \phi_1^2}{\mathcal{M}^2_r r^4} + \mathcal{O}(r^{-5}).
$$

(4.23)

Provided that $\eta > 0$, the linear stability conditions are consistently satisfied at spatial infinity.

For concreteness, we consider the linear scalar-GB coupling $\xi(\phi) = M_{\mathcal{P}r}r^2\phi$ besides the quartic derivative coupling. Since this theory falls in a subclass of shift-symmetric Horndeski theories, we have $J^r = (Q/r^2)\sqrt{f/h}$. The constant $Q$ must be 0 to satisfy the boundary conditions at $r = 0$. Then, it follows that

$$
\phi'(r) = -\frac{4\alpha f^r (h - 1) M_{\mathcal{P}r} r^2}{\eta r^2 f - 2\mu_4 r_0^2[f^r h + f(h - 1)]},
$$

(4.24)

where

$$
\bar{\mu}_4 \equiv \frac{\mu_4}{r_0^2}.
$$

(4.25)

Substituting the expanded solutions (4.14) and (4.15) into Eq. (4.24), we find that the field derivative behaves as $\phi'(r) \propto r$ around $r = 0$. Substitution of Eqs. (3.38) and (4.22) into Eq. (4.24) gives $\phi'(r) = 16\alpha M^2 M_{\mathcal{P}r}r_0^2/(\eta r^3)$ at spatial infinity and hence $\phi_1 = 0$ in Eq. (3.40). Note that the condition for the regularity at the center, $Q = 0$, makes the leading scalar charge vanish at spatial infinity ($\phi_1 = 0$) in the expansion of Eq. (3.40).

For given values of $\alpha$, $\rho_c$, $w_c$, and $\eta$, $\phi_2$ is known from Eq. (4.17) as a function of $\mu_4$. Then, the minimum value of $\bar{\mu}_4$ can be found by Eq. (4.19). When $\alpha = 2 \times 10^{-4}$, $\rho_c = 15\rho_H$, $w_c = 0.48$, and $\eta = 1$, we have $\bar{\mu}_4 > -1.36 \times 10^{-3}$. Numerically, we find that the values of $c^2_r$ and $c^2_{\Omega, -}$ around $r = 0$ become negative for $\bar{\mu}_4 < -1.30 \times 10^{-3}$. Hence the condition (4.19) gives a good approximate criterion for the existence of NS solutions with a nontrivial scalar profile consistent with linear stability conditions. At the background level the negative coupling $\bar{\mu}_4$ leads to tiny suppression of $\phi'(r)$, so the ADM mass of NS is only slightly increased. For $\mu_4 = -1 \times 10^{-3}$ and $\alpha = 2 \times 10^{-4}$, we obtain
M = 2.008M⊙, which is close to the value M = 2.003M⊙ derived for ˜µ4 = 0 and α = 2 × 10^{-4}. As µ4 approaches the lower bound (4.19), the leading-order contributions to c_{r3}^2(r = 0) and c_{Ω−}^2(r = 0) become highly superluminal. Outside the star, we numerically confirm that all of the propagation speeds quickly approach 1.

The positive quartic coupling µ4 is constrained to be in the range satisfying (4.21). With the model parameters α = 2 × 10^{-4}, ρc = 15ρ0, wc = 0.48, and η = 1, the condition (4.21) is satisfied for any positive µ4. However, we need to caution that the next-to-leading order correction to Eq. (4.20) gives rise to a negative term of order −0.1. For ˜µ4 > 1 × 10^{-3}, we numerically find that c_{r3}^2 and c_{Ω−}^2 become negative around the center of star. Hence there is actually an upper bound of ˜µ4 to avoid the Laplacian instability of even-parity perturbations. The plots in Fig. 2 correspond to the coupling ˜µ4 = 9 × 10^{-4}, in which case c_{r3}^2 and c_{Ω−}^2 are as close as 0.1 in the central region of star. Unlike the case ˜µ4 = 0 shown in Fig. 2, c_{Ω−}^2 grows smoothly as a function of r toward the asymptotic value 1. The behavior of other squared propagation speeds c_{r3}^2, c_{Ω−}^2, and c_{Ω+}^2 is not much different from those seen in Fig. 2. As we observe in the left panel of Fig. 4, the field derivative inside the star is slightly enhanced in comparison to the case ˜µ4 = 0. The resulting ADM mass M = 1.995M⊙ is a bit smaller than the value M = 2.003M⊙ obtained for ˜µ4 = 0.

In summary, there are NS solutions with a nontrivial scalar profile free from ghost/Laplacian instabilities in certain ranges of the coupling ˜µ4. For ˜µ4 close to its lower limit, c_{r3}^2 and c_{Ω−}^2 are highly superluminal deep inside the star. For ˜µ4 close to its upper limit, c_{r3}^2 and c_{Ω−}^2 approach +0 around r = 0.

C. Nonminimal Ricci scalar and GB couplings

At the end of this section, we study the existence and stability of NS solutions in theories with nonminimal Ricci scalar and scalar-GB couplings. We incorporate a linear nonminimal coupling of the form λ4ϕR in G₄. This also accommodates the dilatonic coupling e^{λϕ}R in the regime |λϕ| ≪ 1. The action in such theories is given by

\[ S = \int d^4x \sqrt{-g} \left[ \frac{M_P^2}{2} R + \lambda_4 \phi R + \eta X + \alpha \xi(\phi) R^2 G_{\text{GB}} \right]. \]  

(4.26)

Provided that the scalar-GB coupling is suppressed on a weak gravitational background, the dimensionless coupling constant λ₄ = λ₄/Mₚ is constrained to be λ₄ ≤ 2.5 × 10^{-3} from Solar System experiments [13, 109]. On the strong gravitational background, we will study the effect of nonminimal coupling on the existence and the linear stability of NSs.
In the absence of the scalar-GB coupling, the squared propagation speeds are
\[ c_r^2 = c_\Omega^2 = c_{r3}^2 = c_{\Omega3}^2 = c_{\Omega-}^2 = 1, \quad \text{for} \quad \alpha = 0, \] (4.27)

at any distance \( r \). These values are the same as those in GR, but the nonminimal coupling can give rise to NS solutions with a nontrivial scalar profile with \( \phi'(r) \neq 0 \) even in the absence of the scalar-GB coupling.

Let us consider the case in which both nonminimal and scalar-GB couplings are present. The solutions expanded around \( r = 0 \) are given by
\[
\begin{align*}
    f &= f_c^\alpha + \frac{f_c}{6[M_{\Omega1}^2 + 2\lambda_4\phi_c]} \left[ \rho_c (1 + 3w_c) - 12\lambda_4\phi_2 - 48\alpha\xi,\phi(\phi_c)\phi_2(w_c \rho_c - 8\lambda_4\phi_2) \right] r^2 + O(r^3), \\
    h &= 1 - \frac{\rho_c + 12\lambda_4\phi_2}{3[M_{\Omega1}^2 + 2\lambda_4\phi_c - 16\alpha\xi,\phi(\phi_c)\phi_2]} r^2 + O(r^3), \\
    P &= P_c - \frac{\rho_c (1 + w_c) \left[ (M_{\Omega1}^2 + 2\lambda_4\phi_c) \right] \rho_\alpha (1 + 3w_c) - 12\lambda_4\phi_2 - 48\alpha\xi,\phi(\phi_c)\phi_2(w_c \rho_c - 8\lambda_4\phi_2) \right] r^2 + O(r^3),
\end{align*}
\] (4.28, 4.29, 4.30)

where \( \phi_2 \) appears as a coefficient in Eq. (3.16). Performing the expansion with respect to a small scalar-GB coupling \( \alpha, \phi_2 \) obeys the second-order algebraic equation
\[ \frac{3(M_{\Omega1}^2 + 2\lambda_4\phi_c) \left[ \eta (M_{\Omega1}^2 + 2\lambda_4\phi_c) + 2\lambda_4\phi_2 \left[ \rho_\alpha (1 + 3w_c) + 72\lambda_4\phi_2 (w_c \rho_c - 6\lambda_4\phi_2) \right] \right] = \alpha, \] (4.31)

up to linear order in \( \alpha \). Choosing the regular branch in the limit \( \alpha \to 0 \), we obtain
\[ \phi_\alpha (\alpha = 0) = \frac{\lambda_4 \rho_c (3w_c - 1)}{6[\eta (M_{\Omega1}^2 + 2\lambda_4\phi_c) + 4\lambda_4^2 \phi_c]} . \] (4.32)

Thus the nonminimal coupling \( \lambda_4 \neq 0 \) alone leads to a nonvanishing value \( \phi_\alpha (\alpha = 0) \neq 0 \) around the center of body. At \( r = 0 \), the absence of ghost/Laplacian instabilities for odd-parity perturbations requires that
\[ G(r = 0) = M_{\Omega1}^2 + 2\lambda_4\phi_c > 0, \] (4.33)
\[ H(r = 0) = F(r = 0) = M_{\Omega1}^2 + 2\lambda_4\phi_c - 16\alpha\xi,\phi(\phi_c)\phi_2 > 0, \] (4.34)

with the associated squared propagation speeds
\[ c_r^2 (r = 0) = c_\Omega^2 (r = 0) = \left[ 1 - \frac{16\alpha\xi,\phi(\phi_c)\phi_2}{M_{\Omega1}^2 + 2\lambda_4\phi_c} \right]^{-1} . \] (4.35)

Around \( r = 0 \), the no-ghost parameter \( K \) is in proportion to \( r^6 \) and hence
\[ K/r^6 = 8\phi_2^2 \left[ M_{\Omega1}^2 + 2\lambda_4\phi_c \right] \left[ \eta (M_{\Omega1}^2 + 2\lambda_4\phi_c) + 4\lambda_4^2 \phi_c \right] + O(\alpha), \] (4.36)

where we performed the expansion with respect to small \( \alpha \). Under the condition (4.33), the absence of ghosts in the limit \( \alpha \to 0 \) requires that
\[ \eta (M_{\Omega1}^2 + 2\lambda_4\phi_c) + 4\lambda_4^2 \phi_c > 0. \] (4.37)

Provided that \( \eta > 0 \) and \( \lambda_4\phi_c > 0 \), the conditions (4.33), (4.34), and (4.37) are automatically satisfied for \( \alpha \to 0 \). At \( r = 0 \), we also obtain
\[ c_{r3}^2 (r = 0) = c_{\Omega-}^2 (r = 0) = 1 - \frac{16\alpha\xi,\phi(\phi_c)\phi_2}{(M_{\Omega1}^2 + 2\lambda_4\phi_c) \eta (M_{\Omega1}^2 + 2\lambda_4\phi_c) + 6\lambda_4^2} + O(\alpha^2), \] (4.38)
\[ c_{\Omega+}^2 (r = 0) = 1 + \frac{16\alpha\xi,\phi(\phi_c)\phi_2}{M_{\Omega1}^2 + 2\lambda_4\phi_c} + O(\alpha^2). \] (4.39)

From Eqs. (4.35) and (4.39) we have \( c_r^2 (r = 0) = c_\Omega^2 (r = 0) = c_{\Omega+}^2 (r = 0) \) up to linear order in \( \alpha \). In the limit \( \alpha \to 0 \), all the above squared propagation speeds approach 1.

The solutions expanded at spatial infinity are given by
\[
f = 1 - \frac{2M}{r} + \frac{\lambda_4 \phi_1 (4\eta M M_{\Omega1}^2 + 8\eta M \lambda_4 \phi_0 + \eta \lambda_4 \phi_1 + 24M\lambda_4^2)}{[(M_{\Omega1}^2 + 2\lambda_4\phi_0) \eta (M_{\Omega1}^2 + 2\lambda_4\phi_0) + 6\lambda_4^2] r^2} + O(r^{-3}), \] (4.40)
\begin{equation}
\phi = \frac{\hat{\phi}_0}{r} + \frac{\hat{\phi}_1}{r} + \frac{\hat{\phi}_2}{r^2} + \frac{\hat{\phi}_3}{r^3} + \mathcal{O}(r^{-4}),
\end{equation}

(4.41)

\begin{equation}
\phi = \frac{\hat{\phi}_0}{r} + \frac{\hat{\phi}_1}{r} + \frac{\hat{\phi}_2}{r^2} + \frac{\hat{\phi}_3}{r^3} + \mathcal{O}(r^{-4}),
\end{equation}

(4.42)

where \( M, \hat{\phi}_0, \hat{\phi}_1 \) are integration constants. The coupling \( \alpha \) does not appear in Eqs. (4.40)-(4.42) up to the order of \( r^{-2} \), so the nonminimal coupling \( \lambda_4 \) provides larger contributions to \( f, h, \) and \( \phi \) far outside the star in comparison to the scalar-GB term. On using these large-distance solutions, the dominant contributions to \( g, h, \) and \( f \) are \( M_P^2 + 2\lambda_4 \hat{\phi}_0 \), so the linear stability conditions of odd-parity perturbations are satisfied if \( M_P^2 + 2\lambda_4 \hat{\phi}_0 > 0 \). As for \( c_r^2 \) and \( c_{\Omega}^2 \), the term \( M_P^2 r^3 \) appearing in the denominators of Eq. (3.41) is modified to \( (M_P^2 + 2\lambda_4 \hat{\phi}_0) r^3 \). In the even-parity sector, we have

\begin{equation}
\mathcal{K} = 2\hat{\phi}_1^2 \left( M_P^2 + 2\lambda_4 \hat{\phi}_0 \right) \left[ \eta(M_P^2 + 2\lambda_4 \hat{\phi}_0) + 6\lambda_4^2 \hat{\phi}_0 \right] + \mathcal{O}(r^{-1}),
\end{equation}

(4.43)

\begin{equation}
c_{r\Omega}^2 = 1 - \frac{96\alpha \xi_\phi(\phi_c) \lambda_4^2 \hat{\phi}_0}{(M_P^2 + 2\lambda_4 \hat{\phi}_0) \eta(M_P^2 + 2\lambda_4 \hat{\phi}_0) + 6\lambda_4^2 \hat{\phi}_0} + \mathcal{O}(r^{-4}).
\end{equation}

(4.44)

Under the condition \( M_P^2 + 2\lambda_4 \hat{\phi}_0 > 0 \), the ghosts are absent for \( \eta(M_P^2 + 2\lambda_4 \hat{\phi}_0) + 6\lambda_4^2 > 0 \). In comparison to Eq. (3.43), the nonminimal coupling gives rise to a term proportional to \( r^{-3} \) in \( c_{r\Omega}^2 - 1 \). The expressions of squared angular propagation speeds \( c_{\phi r}^2 \) are complicated, but they have the dependence \(|c_{\phi r}^2 - 1| \propto |\alpha|/r^3 \) under the small \( \alpha \) expansion.

**FIG. 5.** (Left) Mass function \( \mathcal{M} \) (normalized by the solar mass \( M_\odot \)) and field derivative \( \phi' \) (normalized by \( M_P/r_0 \)) versus \( r/r_0 \) for \( \lambda_4 = -2.5 \times 10^{-3}, \phi_c = M_P, \alpha = 2.5 \times 10^{-4}, \eta = 1, \) and \( \rho_c = 10\rho_0 \) (solid lines). The dashed lines correspond to the case \( \alpha = 0 \), while the other model parameters are unchanged. (Right) \( c_r^2, c_{\phi r}^2, c_{\phi 3}^2, c_{\phi 4}^2, \) and \( c_{\phi r}^2 \) versus \( r/r_0 \) for the same model parameters as those used for plotting solid lines in the left panel (i.e., both nonminimal and scalar-GB couplings are present).

In the left panel of Fig. 5 we plot \( \phi' \) versus \( r/r_0 \) for \( \alpha = 0, \lambda_4 = \lambda_4/M_P = -2.5 \times 10^{-3}, \phi_c = M_P, \eta = 1, \) and \( \rho_c = 10\rho_0 \) as a dashed line. The field derivative increases around \( r = 0 \) according to the relation \( \phi' \simeq 2\phi_2 r \), where \( \phi_2 \) is given by Eq. (4.32). In Fig. 5 we observe that \( \phi' \) starts to decrease around the surface of star and it joins the large-distance solution \( \phi' \simeq -\hat{\phi}_1/r^2 \) with \( \hat{\phi}_1 < 0 \). Note that breaking the shift symmetry leads to the leading scalar charge \( \hat{\phi}_1 \neq 0 \) even if the regular boundary conditions are imposed at the center. This hairy NS solution satisfies all the linear stability conditions, with the squared propagation speeds given by Eq. (4.27). In this case, the ADM mass
of NS is $M = 1.908M_\odot$ with the radius $r_s = 10.97$ km. They are almost similar to the values $M = 1.912M_\odot$ and $r_s = 11.00$ km in GR with the same central density $\rho_c = 10\rho_0$. Increasing the value of $|\lambda_4\phi_c|$ further, it is possible to realize NSs with a nontrivial scalar profile whose mass and radius exhibit notable difference from those in GR. If we take into account Solar System constraints, however, the coupling $\lambda_4$ should be less than the order $10^{-3}$. Hence we do not consider the case in which the product $|\lambda_4\phi_c|$ exceeds the order of $10^{-3}$.

If the scalar-GB coupling is present besides the nonminimal coupling, it is possible to realize NS solutions with a nontrivial scalar profile as well. The solid curves in the left panel of Fig. 5 correspond to the radial dependence of $\phi$ for $\alpha = 2.5 \times 10^{-4}$, $\lambda_4 = -2.5 \times 10^{-3}$, and $\phi_c = M_P$. Inside the NS, the field derivative is significantly enhanced in comparison to the case $\alpha = 0$. Around the surface of star $\phi'$ starts to decrease rapidly, but it enters the region with the radial dependence $\phi' \approx -\phi_1/r^2$ for $r \gtrsim r_0$. As we already mentioned, this latter property is attributed to the fact that the contribution to $\phi'$ from the nonminimal coupling dominates over that from the scalar-GB coupling at large distances. For $\alpha = 2.5 \times 10^{-4}$ the mass and radius of NS are found to be $M = 1.873M_\odot$ and $r_s = 10.93$ km, both of which are slightly smaller than those for $\alpha = 0$ mentioned above. In the right panel of Fig. 5 we show the five squared propagation speeds versus $r/r_0$ for $\alpha = 2.5 \times 10^{-4}$, $\lambda_4 = -2.5 \times 10^{-3}$, and $\phi_c = M_P$. In comparison to the case $\alpha = 0$ where all the propagation speeds are equivalent to 1, they are different from 1 deep inside the NS and approach 1 outside the star. For the model parameters used in Fig. 5 there are neither ghost nor Laplacian instabilities for NS solutions with a nontrivial scalar profile.

Provided that $|\lambda_4\phi_c| \lesssim 10^{-3}$, the scalar-GB coupling $|\alpha|$ larger than the order $10^{-5}$ gives the dominant contribution to $\phi'$ inside the star. Then, for $|\alpha| \gtrsim 10^{-5}$, the background NS solution and its linear stability are not much different from those for the scalar-GB coupling alone discussed in Sec. [II].

V. REGULARIZED 4D-EINSTEIN-GAUSS-BONNET GRAVITY

In this section, we study the linear stability of NS solutions with a nontrivial scalar profile in so-called “4DEGB gravity” arising from the reduction of higher-dimensional GB theory to 4 dimensions. If we consider the GB term $R_{GB}^2$ in spacetime dimensions $D$ higher than 4, the field equations of motion following from the Lagrangian $L = \sqrt{-g} \hat{\alpha}_{GB} R_{GB}^2$ vanish in 4 dimensions [110]. However, rescaling the GB coupling constant as $\hat{\alpha}_{GB} \rightarrow \alpha_{GB}/(D - 4)$ allows a possibility for extracting contributions of the higher-dimensional GB term [86]. Under such a rescaling, in the limit $D \rightarrow 4$, it is possible to construct a regularized 4-dimensional theory by adding a counter-term to eliminate divergent parts of the theory [89, 90]. The other equivalent procedure is to perform a Kaluza-Klein reduction of $D$-dimensional Einstein-GB gravity on a $(D - 4)$-dimensional maximally symmetric space with a vanishing spatial curvature [87, 88]. The size of such a maximally symmetric space is characterized by a scalar field $\phi$. The 4-dimensional action obtained from the Kaluza-Klein reduction of $D$-dimensional Einstein-GB theory belongs to a subclass of shift-symmetric Horndeski theories given by the coupling functions

$$G_2 = 8\alpha_{GB}X^2, \quad G_3 = 8\alpha_{GB}X, \quad G_4 = 1 + 4\alpha_{GB}X, \quad G_5 = 4\alpha_{GB} \ln |X|. \quad (5.1)$$

Notice that the standard kinetic term $X$ is absent in $G_2$. Throughout this section, we use the unit $M_P^2/2 = 1$.

From the scalar-field Eq. (2.12), we have $r^2\sqrt{f/h} J^r = Q$ constant and hence

$$4\sqrt{\frac{h}{f}} (f' + 2\phi f) \left[1 - h(1 + r\phi')^2\right] \alpha_{GB} = Q. \quad (5.2)$$

Due to the regularity conditions $f'(0) = 0$ and $\phi'(0) = 0$ at the center of star, it follows that $Q = 0$. To satisfy Eq. (5.2) with $Q = 0$ at any radius $r$, we require that $1 - h(1 + r\phi')^2 = 0$. The branch where $\phi(r)$ decreases at spatial infinity (satisfying the asymptotic flatness $h \rightarrow 1$ as $r \rightarrow \infty$) is given by

$$\phi' = \frac{1}{r} \left(\frac{1}{\sqrt{h}} - 1\right). \quad (5.3)$$

Taking the $r$ derivative of this equation and substituting $\phi''$ and Eq. (5.3) into Eqs. (2.9) and (2.10), we obtain

$$h' = -\frac{2(\phi - 1) \left[\phi' + (\phi - 1)\alpha_{GB}\right] + pr^4}{2\left[r^2 - 2(\phi - 1)\alpha_{GB}\right]}, \quad (5.4)$$

$$f' = -f \frac{2(\phi - 1) \left[\phi' + (\phi - 1)\alpha_{GB}\right] - pr^4}{2hr^2 \left[r^2 - 2(\phi - 1)\alpha_{GB}\right]} \quad (5.5)$$
Outside the star \((\rho = 0 = P)\), there is the following analytic solution \([87, 111]\)

\[
h = 1 + f = 1 + \frac{r^2}{2\alpha_{GB}} \left[ 1 - \sqrt{1 + \frac{8\alpha_{GB}M}{r^3}} \right], \tag{5.6}
\]

where \(M\) is an integration constant. At spatial infinity, this has the asymptotic behavior \(h = f = 1 - 2M/r + \mathcal{O}(r^{-4})\) and \(\phi' = M/r^2 + \mathcal{O}(r^{-3})\).

Inside the NS, the integrated solutions to \(h\) and \(f\) depend on the fluid EOS. For constant density \(\rho\), we have the following exact solution \([11]\)

\[
h = 1 - \zeta r^2, \tag{5.7}
\]

\[
f = \frac{(1 - \alpha_{GB} \zeta)^2}{4(1 + 2\alpha_{GB} \zeta)} \left[ \frac{3(1 + \alpha_{GB} \zeta)}{1 - \alpha_{GB} \zeta} \sqrt{1 - \zeta r^2} - \sqrt{1 - \zeta r^2} \right]^2, \tag{5.8}
\]

\[
P = \rho \frac{(1 - \alpha_{GB} \zeta) [\sqrt{1 - \zeta r^2} - \sqrt{1 - \zeta r^2}]}{3(1 + \alpha_{GB} \zeta) \sqrt{1 - \zeta r^2} - (1 - \alpha_{GB} \zeta) \sqrt{1 - \zeta r^2}} = \rho \left( \sqrt{\frac{1 - \zeta r^2}{f^2}} - 1 \right), \tag{5.9}
\]

where

\[
\zeta \equiv \frac{1}{2\alpha_{GB}} \left( \sqrt{1 + \frac{2}{3\alpha_{GB}}} - 1 \right). \tag{5.10}
\]

At the surface of star \((r = r_s)\), the fluid pressure \(\Box\) vanishes. Matching \(h\) and \(f\) with Eq. \(\Box\) at \(r = r_s\), there is the relation \(M = \rho r_s^3/12\). In the following, we will study the linear stability of NSs without assuming their EOSs.

The quantities associated with the linear stability against odd-parity perturbations are given by

\[
G = \frac{2\sqrt{h} r^4 - 2r^2(\sqrt{h} - 1)(Pr^2 + 2 - 2h + 4\sqrt{h})\alpha_{GB} - 4(\sqrt{h} + 1)(\sqrt{h} - 1)^4 \alpha_{GB}^2}{r^2 \sqrt{h} [r^2 + 2(1 - h)\alpha_{GB}]}, \tag{5.11}
\]

\[
F = \frac{2(r^4 + r^2[pr^2 + 2(h - 1)]\alpha_{GB} - 2(h - 1)^2 \alpha_{GB}^2)}{r^2 [r^2 + 2(1 - h)\alpha_{GB}]}, \tag{5.12}
\]

\[
H = 2 + \frac{4(1 - h)}{r^2} \alpha_{GB}. \tag{5.13}
\]

In the limit that \(\alpha_{GB} \to 0\) we have \(G = F = H = 2\), so the linear stability against odd-parity perturbations is ensured for the small GB coupling. On using the background Eqs. \([5.4]\) and \([5.5]\), it follows that

\[
\mathcal{K} = 0, \tag{5.14}
\]

at any radius \(r > 0\). Inside the star, the term \(2P_1 - F\) exactly cancels the contribution \(-2H^2 r^4 (\rho + P)\), while, outside the star, \(\mathcal{K} = (2P_1 - F)\mu^2 = 0\). The fact that \(\mathcal{K}\) vanishes everywhere, which is mostly attributed to the absence of a standard kinetic term in \(G_2\), is the signal of a strong coupling problem. Indeed, the squared radial propagation speed associated with the stability of scalar-field perturbation in the even-parity sector yields

\[
c_{r3}^2 \to \infty. \tag{5.15}
\]

This divergent property of \(c_{r3}^2\) arises from the existence of term \(\mathcal{K}\) in the denominator of Eq. \(2.22\). We note that the product \(\mathcal{K}c_{r3}^2\) is finite. Exploiting the exact solution \(\Box\) outside the star and performing the expansion with respect to \(1/r\) at spatial infinity, we obtain the dependence

\[
\mathcal{K}c_{r3}^2 = -\frac{256 M^3 \alpha_{GB}}{r^3} + \mathcal{O}(r^{-4}). \tag{5.16}
\]

For \(\alpha_{GB} < 0\), the leading-order contribution to \(\mathcal{K}c_{r3}^2\) is positive at large distances.

The quantities \(B_1\) and \(B_2\) also diverge, so this leads to the divergence of angular propagation speeds in the even-parity sector. Expanding the product \(\mathcal{K}B_2\) at spatial infinity, it follows that

\[
\mathcal{K}B_2 = \frac{128 M^3 \alpha_{GB}}{r^3} + \mathcal{O}(r^{-4}). \tag{5.17}
\]
Dividing Eq. (5.16) by Eq. (5.17), we obtain

\[ \frac{c_3^2}{B_2} = -2 + \mathcal{O}(r^{-1}). \]  

(5.18)

Since the signs of \( c_3^2 \) and \( B_2 \) are different from each other, either of the linear stability conditions (2.22) or (2.26) is violated at large distances. Thus, the NS solutions in 4DEGB gravity not only suffer from the strong coupling problem but also the Laplacian instability of even-parity perturbations. These problems also persist for hairy BHs given by the line element (5.6) present in 4DEGB theory. In the case of BHs, there is also the instability of even-parity perturbations in the vicinity of the event horizon [112].

VI. \( F(R_{GB}^2) \) GRAVITY

Finally, we study a modified GB gravity in which the 4-dimensional action contains an arbitrary function \( F \) of the GB term \( R_{GB}^2 \) besides the Einstein-Hilbert term. This theory is given by the action

\[ S = \int d^4x\sqrt{-g} \left[ \frac{M_{Pl}^2}{2} R + F(R_{GB}^2) \right], \]

which is equivalent to [50]

\[ S = \int d^4x\sqrt{-g} \left[ \frac{M_{Pl}^2}{2} R + \xi(\phi)R_{GB}^2 - V(\phi) \right], \]

(6.2)

where

\[ \phi \equiv R_{GB}^2, \quad \xi(\phi) \equiv F_{,\phi}, \quad V(\phi) \equiv \phi - F. \]

(6.3)

The action (6.2) belongs to a subclass of Horndeski theories given by the coupling functions

\[ G_2 = -V(\phi) + 8\xi^{(4)}(\phi)X^2(3 - \ln|X|), \quad G_3 = 4\xi^{(3)}(\phi)X(7 - 3 \ln|X|), \]

\[ G_4 = \frac{M_{Pl}^2}{2} + 4\xi^{(2)}(\phi)X(2 - \ln|X|), \quad G_5 = -4\xi^{(1)}(\phi)\ln|X|. \]

(6.4)

Hence there is no standard scalar kinetic term in \( F(R_{GB}^2) \) gravity.

Let us focus on the power-law \( F(R_{GB}^2) \) models given by

\[ F(R_{GB}^2) = \beta (R_{GB})^n, \]

(6.5)

where \( \beta \) and \( n \) are constants. We consider the positive integers \( n \) in the range \( n \geq 2 \). Then, the GB coupling function \( \xi(\phi) \) and scalar potential \( V(\phi) \) in the action (6.2) yield

\[ \xi(\phi) = \alpha n \frac{\sqrt{2}M_{Pl}^2}{2} \left( \frac{\phi}{M_{Pl}} \right)^{n-1}, \quad V(\phi) = \alpha (n-1) \frac{M_{Pl}^2}{2 \phi_0^2} \left( \frac{\phi}{M_{Pl}} \right)^n, \]

(6.6)

where \( \alpha \equiv 2r_0^2 - 4nM_{Pl}^{-2} \beta \) is a dimensionless coupling.

From the scalar-field Eq. (2.12), we obtain

\[ \alpha \phi^{n-2} \left\{ r^2 f^2 \phi - 2 \left[ f f' h'(3h - 1) + f^2 h(1 - h) + 2 f'' h(h - 1) \right] M_{Pl}^2 \phi_0^4 \right\} = 0. \]

(6.7)

For \( n = 2 \), there is only one branch characterized by

\[ \phi = 2 \frac{\left[ f f' h'(3h - 1) + f^2 h(1 - h) + 2 f'' h(h - 1) \right] M_{Pl}^2 \phi_0^4}{r^2 f^2}. \]

(6.8)

For \( n \geq 3 \), we also have the no-hair branch \( \phi = 0 \) besides (6.8).

Let us first consider the power \( n = 2 \). Around the center of star, the solutions consistent with the boundary conditions at \( r = 0 \) are

\[ f = f_c + \frac{f_c M_{Pl}^2 \phi_2(1 + 3 \omega_c) - M_{Pl} \phi_2^2 \alpha - 24(2 \omega_c - r_0^2 \phi_0^2 \phi_2^2 \alpha)(r_0^2 \phi_2^2 \alpha)}{6M_{Pl}^2 (M_{Pl} - 16r_0^2 \phi_2^2 \alpha)^2} r^2 + \mathcal{O}(r^3), \]

(6.9)
\[
\begin{align*}
\left( \frac{2}{6M_1 r_0^2 (M_1 - 16r_0^2 \phi_2)} \right)^2 + O(r^3),
\phi = \phi_c + \phi_2 r^2 + O(r^3),
\end{align*}
\]

Thus, the scalar field is divergent as \(\alpha \to 0\). In general, in the limit \(\alpha \to 0\), the same type of divergence can be also observed for \(n \geq 3\). This indicates a pathology of the interior solutions in the power-law \(f(R_{\text{GB}})\) model.

The solutions expanded far outside the star are
\[
\begin{align*}
f &= 1 - \frac{2M}{r} + \frac{1024M^3 r_0^6 \alpha}{r^9} + O(r^{-10}),
\end{align*}
\]

\[
\begin{align*}
h &= 1 - \frac{2M}{r} + \frac{4608M^3 r_0^6 \alpha}{r^9} + O(r^{-10}),
\end{align*}
\]

\[
\begin{align*}
\phi &= \frac{48M^2 M_1 r_0^4}{r^6} - \frac{1216512M^4 M_1 r_0^9 \alpha}{r^{14}} + O(r^{-15}).
\end{align*}
\]

The Schwarzschild metrics receive corrections from the coupling \(\alpha\) at the order of \(r^{-9}\). If there are NSs with a nontrivial scalar profile, the interior solutions (6.9)-(6.11) should be joined with the exterior solutions (6.15)-(6.17). On using the large-distance solutions (6.15)-(6.17), we obtain
\[
K = -\frac{63700992 M^6 M_1^6 r_0^{12} \alpha^2}{r^{16}} + O(r^{-17}).
\]

The leading-order term of \(K\) is negative, and hence there is a ghost instability issue at large distances. Moreover, since \(K\) is suppressed by a high power \(O(r^{-16})\), it quickly approaches 0 for increasing \(r\). Then the above solution also has a strong coupling problem in the asymptotic region. Note that in Ref. [48] the definition of \(K\) is \(2P_1 - \mathcal{F}\), in which case \(K \propto r^{-18}\) as consistent with Eq. (6.18). In Ref. [48] a small \(\alpha\) expansion was used for deriving the background BH solution, which means that our result (6.18) is more general.

For \(n \geq 3\), there is also the branch (6.8) of a nonvanishing scalar field. For this branch, the solutions in the vicinity of \(r = 0\) are similar to Eqs. (6.9)-(6.12) with some modifications of coefficients. At large distances, the leading-order solutions are Schwarzschild metric components as in Eqs. (6.15)-(6.17), in which case we obtain
\[
K = \frac{-3^{2n+1} \cdot 256^n M^2 (2n-1) M_1^2 n^2 (n-1)^2 r_0^{(4(n-1)-2) \alpha^2}}{r^{4(3n-2)}} + O(r^{-12n}).
\]

Thus the leading-order term of \(K\) is negative, with a rapid decrease of \(K\) toward 0 at large distances. To compute the quantity \(K\) above, we have not used the expansion with respect to a small coupling \(\alpha\). Hence the ghost instability and asymptotic strong coupling problem of NS solutions with a nontrivial scalar profile given by the branch (6.8) are generally present for an arbitrary nonvanishing coupling \(\alpha\).

**VII. CONCLUSIONS**

In gravitational theories with a coupling to GB curvature invariant \(R_{\text{GB}}^2\), we studied the existence and stability of NS solutions with a nontrivial profile of the scalar field on a static and spherically symmetric background. For this
purpose, we exploited conditions for avoiding ghost/Laplacian instabilities of odd- and even-parity perturbations with high radial and angular momentum modes [47]. These linear stability conditions in full Horndeski theories, which are summarized in Sec. I I, can be applied not only to BHs, i.e., the vacuum case, but also to NSs, i.e., the case with high radial and angular momentum modes [73]. These linear stability conditions in full Horndeski theories, which were derived for avoiding ghost/Laplacian instabilities of odd- and even-parity perturbations with coupling constant $\alpha$. For the linear scalar-GB coupling $\alpha \xi(\phi) R_{GB}^2$, this bound translates to $\sqrt{\alpha GB} < 0.7 \text{ km}$ to realize maximum masses of NSs for typical EOSs. This theoretical limit is stronger than those observationally constrained from the GW measurements emitted from binaries containing NSs. Numerically, we confirmed that the bound (3.31) is sufficiently accurate for ensuring the existence of NSs with a nontrivial scalar profile consistent with all the linear stability conditions. As we see in Fig. 2, the squared propagation speeds of odd- and even-parity perturbations deviate from 1 deep inside the NS, but all of them approach 1 outside the star. Note that the propagation speeds different from the speed of light do not imply any acausality and instead determine the causal boundaries locally at each position.

In Sec. IV we incorporated several regular Horndeski couplings besides the scalar-GB coupling $\alpha \xi(\phi) R_{GB}^2$. In the presence of a canonical kinetic term $\eta X$, the cubic-order derivative interaction of the scalar field $G_3 \supset \mu_3 X$ or the quartic-order derivative coupling $G_4 \supset \mu_4 X$ alone do not give rise to asymptotically-flat NS solutions with a nontrivial scalar profile [64]. We confirmed that the presence of the GB coupling is mandatory to obtain the NS solutions with a nontrivial profile of the scalar field.

In the case where the scalar-GB coupling is present, we showed the existence of NS solutions with a nontrivial scalar profile free from ghost/Laplacian instabilities for certain ranges of the coupling constants $\mu_3$ and $\mu_4$. As we see in Figs. 3 and 4, these new couplings do not lead to significant changes to the ADM mass of NSs, but the squared propagation speeds $c^2_3$ and $c^2_4$ inside the star are subject to modifications. We also showed that NS solutions with a nontrivial scalar profile exist for a linear nonminimal coupling $G_4 \supset \lambda_4 \phi$ alone. Under local gravity constraints on the coupling constant $\lambda_4$, however, the scalar-GB coupling gives dominant contributions to the background scalar-field profile as well as the linear stability of NSs for $|\alpha|$ close to its upper limit.

In Sec. V we addressed the linear stability of NS solutions with a nontrivial scalar profile in regularized 4DEGB gravity. After the Kaluza-Klein reduction of 4-dimensional Einstein-GB theory on a flat internal space, the resulting 4-dimensional action belongs to a subclass of shift-symmetric Horndeski theories. For the solution with a nontrivial scalar profile $\phi$, we showed that $\mathcal{K} = 0$ and $c^2_3 \rightarrow \infty$ at any radius $r$, and hence it is plagued by a strong coupling problem. Moreover the leading-order contribution to $c^2_3/B_2$ is $-2$ at large distances, so there is also the Laplacian instability for even-parity perturbations. Along with the fact that the BH solutions in regularized 4DEGB gravity are also unstable [112], there are no stable compact objects endowed with a nontrivial scalar profile in this theory.

In Sec. VI we considered the power-law $F(R_{GB}^2)$ models with the Einstein-Hilbert term, which are equivalent to the action (6.2) with $\xi(\phi)$ and $V(\phi)$ given by Eq. (6.6). For $n \geq 2$, there is a nonvanishing scalar-field branch characterized by Eq. (6.8). If NSs with a nontrivial scalar profile are present, the interior solutions should join the large-distance exterior solutions (6.15)–(6.17). On using the latter, we find that the leading-order term of $\mathcal{K}$ is negative. For increasing $r$, $\mathcal{K}$ rapidly decreases toward $-\mathcal{K}_0$ with a large negative power-law dependence on $r$. Hence the hairy branch (6.8) is excluded by the problems of ghost instability and strong coupling at spatial infinity. This is analogous to what was found for BHs in the same theory [48]. In both regularized 4DEGB theory and $F(R_{GB}^2)$ gravity, the instabilities of NSs with a nontrivial scalar profile arise from the unhealthy propagation of $\delta \phi$ associated with the absence of a canonical kinetic term. Moreover, we found that in power-law $F(R_{GB}^2)$ models, the interior solution of the scalar field suffers from the divergence in the zero-coupling limit, which indicates an intrinsic pathology.

In summary, we have shown that NSs with a nontrivial scalar profile consistent with the linear stability conditions are present for the scalar-GB coupling $\alpha \xi(\phi) R_{GB}^2$ besides regular Horndeski coupling functions. As we studied in Sec. IV C, nonminimal couplings with the Ricci scalar $G_4(\phi)R$ also give rise to NS solutions with a nontrivial scalar profile even without the scalar-GB coupling. The latter includes NSs with spontaneous scalarization, which can occur for a large nonminimal coupling constant of order unity [51]–[53]. It will be of interest to extend our linear stability analysis to such large nonminimal coupling regimes by taking the scalar-GB coupling into account.

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Appendix A: Coefficients in the background equations

The coefficients in Eqs. (2.9)-(2.11) are given by

\[ A_1 = -h^2(G_{3,X} - 2G_{4,\phi X})\phi'^2 - 2G_{4,\phi h}, \quad A_2 = 2h^3(2G_{4,XX} - G_{5,\phi X})\phi'^3 - 4h^2(G_{4,X} - G_{5,\phi})\phi', \]
\[ A_3 = -h^4G_{5,XX}\phi'^4 + h^2G_{5,X}(3h - 1)\phi'^2, \quad A_4 = h^2(2G_{4,XX} - G_{5,\phi X})\phi'^4 + h(3G_{5,\phi} - 4G_{4,X})\phi'^2 - 2G_4, \]
\[ A_5 = \frac{1}{2} \left[ G_{5,XX}h^3\phi'^2 - hG_{5,X}(5h - 1)\phi'^3 \right], \quad A_6 = h(G_{3,\phi} - 2G_{4,\phi h})\phi'^2 + G_2, \]
\[ A_7 = -2h^2(2G_{4,\phi X} - G_{5,\phi})\phi'^3 - 4G_{4,\phi h}\phi', \quad A_8 = G_{5,\phi X}h^3\phi'^4 - h(2G_{4,X} - G_{5,\phi}h - G_{5,\phi})\phi'^2 - 2G_4(h - 1), \]
\[ A_9 = -h(G_{2,X} - G_{3,\phi})\phi'^2 - 2G_2, \quad A_{10} = \frac{1}{2} G_{5,\phi X}h^3\phi'^4 - \frac{1}{2} h^2(2G_{4,XX} - G_{5,\phi})\phi'^2 - G_4h. \]  

(A1)

The coefficients in Eq. (2.14) are

\[ \lambda_1 = -\left( h' + \frac{4h}{r} + \frac{f'h}{r} \right) \phi' - 2h\phi'', \quad \lambda_2 = -h\phi'^2, \quad \lambda_3 = \frac{1}{2} h\phi'^2 \left( h'\phi' + 2h\phi'' \right), \]
\[ \lambda_4 = \frac{2}{r^2} \left[ 1 - h - rh' \right] + \frac{h f'^2}{2r^2} - \frac{r(2f''h + f'h')}{2fr} + 4f'h', \]
\[ \lambda_5 = h\phi' \left[ \frac{8h'}{r^2} - \frac{6h}{r} - \frac{f'^2h}{2r^2} + \frac{(f''h + 2r f'h')}{fr} \right] \phi' + 3h \left( \frac{f'}{r} + \frac{4}{r} \phi' \right), \]
\[ \lambda_6 = h^2\phi'^3 \left( r' + \frac{4}{r} \right), \quad \lambda_7 = -\frac{1}{2} h^2\phi'^3 \left( r' + \frac{4}{r} \right) \left( h'\phi' + 2h\phi'' \right), \]
\[ \lambda_8 = \frac{1}{r^2} \left[ h'(3h - 1)\phi' + 2h(h - 1)\phi'' \right] - \frac{f'^2h^2}{f^2r^2} \phi' + \frac{1}{fr^2} \left[ (2f''h + 3f'h)h^2\phi' + f'h(3rh' - 1)\phi' + 2f'h^2\phi'' \right], \]
\[ \lambda_9 = \frac{h\phi'^2}{f^2r} \left[ f(h - 1) + f'hr \right], \]
\[ \lambda_{10} = \frac{h^2\phi'^2}{2r^2} \left[ 10h^2\phi'' + h(7h'\phi' - 2\phi'') - h\phi' \right] + \frac{f'^2h^3\phi'^3}{2r^2f} - \frac{h^2\phi'^2}{2r^2f} \left[ (2f''h + 4f'h)\phi' + 10f'hr\phi'' + 7f'h'r\phi' \right], \]
\[ \lambda_{11} = -\frac{h^3\phi'^4}{fr^2} \left( r'f' + f \right), \quad \lambda_{12} = \frac{h^3\phi'^4}{2f^2r} \left( r'f' + f \right)(h'\phi' + 2h\phi''). \]  

(A2)

Appendix B: Coefficients in the perturbation equations

The quantities appearing in the linear stability conditions (2.19), (2.22) and (2.23) are

\[ a_1 = \sqrt{f} h \left\{ G_{4,\phi} + \frac{1}{2} h(G_{3,X} - 2G_{4,\phi X})\phi'^2 \right\} r^2 + 2h\phi' \left\{ G_{4,X} - G_{5,\phi} - \frac{1}{2} h(2G_{4,XX} - G_{5,\phi X})\phi'^2 \right\} r \]
\[ + \frac{1}{2} G_{5,XX}h^3\phi'^4 - \frac{1}{2} G_{5,X}(3h - 1)\phi'^2 \right\}, \]
\[ c_2 = \sqrt{f} h \left\{ \frac{1}{2} \left[ -\frac{1}{2} h(3G_{3,X} - 8G_{4,\phi X})\phi'^2 + \frac{1}{2} h^2(G_{4,XX} - 2G_{4,\phi X})\phi'^4 - G_{4,\phi} \right] r^2 \right. \]
\[ - \frac{h\phi'}{f} \left( \frac{1}{2} h^2(2G_{4,XX} - G_{5,\phi X})\phi'^4 - \frac{1}{2} h(12G_{4,XX} - 7G_{5,\phi X})\phi'^2 + 3(4G_{4,X} - G_{5,\phi}) \right) r \]
\[ + \frac{h\phi'^2}{4f} \left( G_{5,XX}h^3\phi'^4 - G_{5,X}(10h - 1)\phi'^2 + 3G_{5,X}(5h - 1) \right) \right\} f' \]
\[ + \phi' \left\{ \frac{1}{2} G_{2,X} - G_{3,\phi} - \frac{1}{2} h(G_{2,XX} - G_{3,\phi X})\phi'^2 \right\} r^2 \]
\[+2 \left\{ -\frac{1}{2} h(3G_{3,X} - 8G_{4,\phi X})\phi^2 + \frac{1}{2} h^2(G_{3,X} - 2G_{4,\phi XX})\phi'^4 - G_{4,\phi} \right\} r - \frac{1}{2} h^3(G_{4,XX} - G_{5,\phi XX})\phi^6 + \frac{1}{2} \{ 2(6h - 1) G_{4,XX} + (1 - 7h) G_{5,\phi XX} \} \phi'^3 - (3h - 1)(G_{4,X} - G_{5,\phi}) \phi' \},
\]
\[c_4 = \frac{1}{4} \sqrt{\frac{\phi'}{f}} \left\{ 2G_{4,X} - 2G_{5,\phi} - h(2G_{4,XX} - G_{5,\phi XX})\phi'^2 - \frac{h\phi'(3G_{5,X} - G_{5,XX}\phi'^2 h)}{r} \right\} f' + 4G_{4,\phi} + 2h(G_{3,X} - 2G_{4,\phi})\phi'^2 + \frac{4h(G_{4,X} - G_{5,\phi})\phi' - 2h^2(2G_{4,XX} - G_{5,\phi XX})\phi'^3}{r}, \tag{B1}\]

and
\[
\begin{align*}
\beta_0 &= \phi' a_1 + r \sqrt{f h} \mathcal{H}, \\
\beta_1 &= \frac{1}{2} \phi'^2 \sqrt{f h} (c_4 - \phi') \left( \sqrt{\frac{f h}{r}} \right) c_4 + \sqrt{\frac{f h}{r}} \mathcal{G}' + \frac{1}{h \phi'^2} \left( \frac{\phi'^{11/2}}{2h} \right) \mathcal{A}_1 + \frac{1}{r} \mathcal{A}_2 \\
&\quad + \frac{1}{r} \mathcal{A}_3 \\
\beta_2 &= \left[ \sqrt{\frac{f h}{r^2}} \left( h c_4 + \frac{1}{2} h c_4 - d_3 \right) \right] - \sqrt{\frac{f h}{r^2}} \mathcal{H} + \frac{\phi' f h \mathcal{G}}{r} \left( \frac{\mathcal{G}}{r} - \frac{\mathcal{H}}{r} + f' - \frac{1}{r} \right) a_1 - \frac{2}{r} (f h)^{3/2} f \mathcal{G} \mathcal{H}, \\
\beta_3 &= \frac{1}{4r} \sqrt{\frac{f h}{c_4} + \frac{3}{2} \sqrt{f h} \mathcal{G}} + \frac{f' \sqrt{f h}}{2h}\mathcal{H}, \\
\end{align*}\tag{B2}
\]

with
\[
\begin{align*}
e_4 &= \frac{1}{\phi'} e_4 - \frac{f'}{4f h \phi'^2} \left( \sqrt{\frac{f h}{r}} \right) - \frac{1}{2} \phi'^4 \mathcal{H} + \frac{1}{h \phi'^2} \left( \frac{\phi'}{2} \right) a_1 + \frac{1}{r} \mathcal{A}_4 \\
&\quad + \frac{1}{r} \mathcal{A}_5 \\
d_3 &= \frac{1}{2} \mathcal{A}_3 \left( \frac{\phi'^{3/2}}{2h} \right) + \frac{1}{r} \mathcal{A}_6 \\
&\quad + \frac{1}{r} \mathcal{A}_7 \\
d_5 &= \frac{1}{2} \mathcal{A}_8 \left( \frac{\phi'^{3/2}}{2h} \right) + \frac{1}{r} \mathcal{A}_9 \\
&\quad + \frac{1}{r} \mathcal{A}_{10}, \tag{B3}
\end{align*}\]

[1] B. P. Abbott et al. (LIGO Scientific, Virgo), *Phys. Rev. Lett.*, 116, 061102 (2016), arXiv:1602.03837 [gr-qc]
[2] B. P. Abbott et al., *Phys. Rev. Lett.*, 119, 161101 (2017), arXiv:1710.05832 [gr-qc]
[3] B. P. Abbott et al. (LIGO Scientific, Virgo), *Phys. Rev. Lett.*, 121, 161101 (2018), arXiv:1805.11581 [gr-qc]
[4] K. D. Kokkotas and B. G. Schmidt, *Living Rev. Rel.*, 2, 2 (1999), arXiv:gr-qc/9909058
[5] H.-P. Nollert, *Class. Quant. Grav.*, 16, R159 (1999)
[6] A. G. Riess et al. (Supernova Search Team), *Astron. J.*, 116, 1009 (1998), arXiv:astro-ph/9805201
[7] S. Perlmutter et al. (Supernova Cosmology Project), *Astrophys. J.*, 517, 565 (1999), arXiv:astro-ph/9812133
[8] D. N. Spergel et al. (WMAP), *Astrophys. J. Suppl.*, 148, 175 (2003), arXiv:astro-ph/0302209
[9] M. Tegmark et al. (SDSS), *Phys. Rev. D*, 69, 103501 (2004), arXiv:astro-ph/0310723
[10] P. R. Ade et al. (Planck), *Astron. Astrophys.*, 571, A16 (2014), arXiv:1303.5076 [astro-ph.CO]
[11] E. J. Copeland, M. Sami, and S. Tsujikawa, *Int. J. Mod. Phys. D*, 15, 1753 (2006), arXiv:hep-th/0603057
[12] A. Silvestri and M. Trodden, *Rept. Prog. Phys.*, 72, 096901 (2009), arXiv:0904.0024 [astro-ph.CO]
[13] A. De Felice and S. Tsujikawa, *Living Rev. Rel.*, 13, 3 (2010), arXiv:1002.4928 [gr-qc]
[14] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, *Phys. Rept.*, 513, 1 (2012), arXiv:1106.2476 [astro-ph.CO]
[15] A. Joyce, B. Jain, J. Khoury, and M. Trodden, *Phys. Rept.*, 568, 1 (2015), arXiv:1407.0059 [astro-ph.CO]
[16] K. Koyama, *Rept. Prog. Phys.*, 79, 046902 (2016), arXiv:1504.04623 [astro-ph.CO]
[17] L. Heisenberg, *Phys. Rept.*, 796, 1 (2019), arXiv:1807.01725 [gr-qc]
[70] T. Ikeda, A. Iyonaga, and T. Kobayashi, \textit{Phys. Rev. D} \textbf{104}, 104009 (2021), arXiv:2107.13804 [gr-qc].
[71] A. Cisterna, T. Delsate, and M. Rinaldi, \textit{Phys. Rev. D} \textbf{92}, 044050 (2015), arXiv:1504.05189 [gr-qc].
[72] R. Kase and T. Tsujikawa, \textit{JCAP} \textbf{01}, 008 (2021), arXiv:2008.13350 [gr-qc].
[73] R. Kase and T. Tsujikawa, \textit{Phys. Rev. D} \textbf{105}, 024059 (2022), arXiv:2110.12728 [gr-qc].
[74] A. Maselli, H. O. Silva, M. Minamitsuji, and E. Berti, \textit{Phys. Rev. D} \textbf{93}, 124056 (2016), arXiv:1603.04876 [gr-qc].
[75] M. Minamitsuji, K. Takahashi, and S. Tsujikawa, \textit{Phys. Rev. D} \textbf{105}, 104001 (2022), arXiv:2201.09687 [gr-qc].
[76] E. Babichev, C. Charmousis, and A. Lehébel, \textit{JCAP} \textbf{04}, 027 (2017), arXiv:1702.01938 [gr-qc].
[77] P. Cremolinelli, N. Loayza, F. Serra, E. Trincherini, and L. G. Trombetta, \textit{JHEP} \textbf{08}, 045 (2020), arXiv:2004.02893 [hep-th].
[78] T. Kobayashi, H. Motohashi, and T. Suyama, \textit{Phys. Rev. D} \textbf{85}, 084025 (2012), Erratum: \textit{Phys. Rev. D} \textbf{96}, 109903 (2017), arXiv:1202.4893 [gr-qc].
[79] T. Kobayashi, H. Motohashi, and T. Suyama, \textit{Phys. Rev. D} \textbf{89}, 084042 (2014), arXiv:1402.6740 [gr-qc].
[80] R. Kase, R. Kimura, S. Sato, and S. Tsujikawa, \textit{Phys. Rev. D} \textbf{102}, 084037 (2020), arXiv:2007.09864 [gr-qc].
[81] P. Pani, E. Berti, V. Cardoso, and J. Read, \textit{Phys. Rev. D} \textbf{84}, 104035 (2011), arXiv:1109.0928 [gr-qc].
[82] B. Kleihaus, J. Kunz, and S. Mojica, \textit{Phys. Rev. D} \textbf{90}, 061501 (2014), arXiv:1407.6884 [gr-qc].
[83] D. D. Doneva and S. S. Yazadjiev, \textit{JCAP} \textbf{04}, 011 (2018), arXiv:1712.03715 [gr-qc].
[84] J. L. Blázquez-Salcedo, L. M. González-Romero, J. Kunz, S. Mojica, and F. Navarro-Lerida, \textit{Phys. Rev. D} \textbf{93}, 024052 (2016), arXiv:1511.03960 [gr-qc].
[85] G. J. Olmo, D. Rubiera-Garcia, and A. Wojnar, \textit{Phys. Rept.} \textbf{876}, 1 (2020), arXiv:1912.05202 [gr-qc].
[86] D. Glavan and C. Lin, \textit{Phys. Rev. Lett.} \textbf{124}, 081301 (2020), arXiv:1905.03601 [gr-qc].
[87] H. Lu and Y. Pang, \textit{Phys. Lett. B} \textbf{809}, 135717 (2020), arXiv:2003.11552 [gr-qc].
[88] T. Kobayashi, \textit{JCAP} \textbf{07}, 013 (2020), arXiv:2003.12771 [gr-qc].
[89] P. G. S. Fernandes, P. Carrilho, T. Clifton, and D. J. Mulryne, \textit{Phys. Rev. D} \textbf{102}, 024025 (2020), arXiv:2004.08362 [gr-qc].
[90] R. A. Hennigar, D. Kubizňák, R. B. Mann, and C. Pollack, \textit{JHEP} \textbf{07}, 027 (2020), arXiv:2004.09472 [gr-qc].
[91] D. D. Doneva and S. S. Yazadjiev, \textit{JCAP} \textbf{05}, 024 (2021), arXiv:2006.10284 [gr-qc].
[92] S. Nojiri and S. D. Odintsov, \textit{Phys. Lett. B} \textbf{631}, 1 (2005), arXiv:hep-th/0508049.
[93] A. De Felice and M. Hindmarsh, \textit{Phys. Rev. D} \textbf{80}, 084027 (2009), arXiv:0905.3375 [astro-ph].
[94] B. Li, J. D. Barrow, and D. F. Mota, \textit{Phys. Rev. D} \textbf{76}, 044027 (2007), arXiv:0705.3795 [gr-qc].
[95] A. De Felice and S. Tsujikawa, \textit{Phys. Rev. D} \textbf{76}, 1 (2009), arXiv:0810.5712 [hep-th].
[96] A. De Felice and S. Tsujikawa, \textit{Phys. Rev. D} \textbf{84}, 064039 (2011), arXiv:1103.3260 [hep-th].
[97] C. Charmousis, X. Gao, D. A. Steer, and G. Zahariade, \textit{Phys. Rev. D} \textbf{84}, 064039 (2011), arXiv:1103.3260 [hep-th].
[98] C. Charmousis, E. J. Copeland, A. Padilla, and P. M. Saffin, \textit{Phys. Rev. Lett.} \textbf{108}, 051101 (2012), arXiv:1106.2000 [hep-th].
[99] B. F. Schutz and R. Sorkin, \textit{Annals Phys.} \textbf{107}, 1 (1977).