Coefficient Analysis and Regression for Calculating Perimeter of Ellipse

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Abstract: Coefficient for computing ellipse perimeter is analyzed deeply in this article, relevant characteristics and numeric results are achieved too. A polynomial function to compute ellipse perimeter is obtained by regression, which may be applied widely, and has higher accuracy verified by C programming via computer.

Keywords: Coefficient; Regression; Polynomial; Ellipse; Perimeter.

1. Introduction

Elliptical objects or geometric shapes of ellipse exist widely in real world. For example, the orbit of a planet revolving presents an elliptic shape, and objects seen in daily life, such as fruit, egg and stadium, are elliptical too. The elliptical shape may be analyzed for scientific research. There is no accurate elementary mathematical formula to compute ellipse perimeter, but a numerical method by division infinitely then sum can be used through programming in computer. Calculus application is an effective way to analyze perimeter of elliptical shape.

As shown in Figure 1, an ellipse can be expressed by an equation in system of rectangular or polar coordinate, whose analytic equation is as follows.

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b > 0)
\]

or

\[
x = a \cos \theta \\
y = b \sin \theta \\
(0 \leq \theta \leq 2\pi)
\]

An ellipse is known when half major axis (i.e. a) and half minor axis (i.e. b) are given by its equation. We use \( I \) to denote ellipse with half major axis and half minor axis, and to denote its perimeter in this article for short.

The formula used to calculate ellipse perimeter may be derived from calculus\(^{(1)}\).
Symbol $e$ in formula (1) is ellipse eccentricity, and is expressed as follows.

$$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \frac{c}{a}$$

Formula (1) was put forward by Xiang Mingda\(^3\), and formula (2) is another form\(^2\).

Formula (1) and (2) are important foundation to demonstrate other formulas for calculating ellipse perimeter. Several approximation formulas were given in references\(^5-7\).

2. Algorithm

2.1 Formula derivation

Formula (1) and (2) can be transformed to (3) and (4) respectively.

$$LE_{(a,b)} = 2\pi a \left[ 1 - \sum_{n=1}^{\infty} \left( \frac{(2n-1)!!}{(2n)!!} \frac{e^{2n}}{2n-1} \right) \right]$$

$$= 2\pi a \left[ 1 - \left( \frac{1}{2} \right)^2 \frac{e^2}{1} - \left( \frac{1}{2} \cdot \frac{3}{4} \right)^2 \frac{e^4}{3} - \left[ \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \right]^2 \frac{e^6}{5} - \left[ \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \right]^2 \frac{e^8}{7} - \cdots \right]$$

Or

$$LE_{(a,b)} = \pi \cdot (a + b) \left[ 1 + \left( \frac{1}{2} \right)^2 \frac{(a-b)^2}{a+b} + \left( \frac{1}{2} \cdot \frac{3}{4} \right)^2 \frac{(a-b)^4}{a+b} + \left( \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \right)^2 \frac{(a-b)^6}{a+b} + \left( \frac{5!}{8!} \right)^2 \frac{(a-b)^8}{a+b} + \cdots \right]$$

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Formula (1) and (2) are important foundation to demonstrate other formulas for calculating ellipse perimeter. Several approximation formulas were given in references\(^5-7\).
We can see that $LE(a,b)$ is less than $2\pi a$ according to formula (3), and is greater than $\pi(a+b)$ according to formula (4).

Therefore, inequality about $LE(a,b)$ is acquired as follows.

\[
\pi(a+b) < LE(a,b) < 2\pi a \tag{5}
\]

The upper bound ($2\pi a$) and lower bound ($4a$) of $LE(a,b)$ have been given by references [4,8].

According to formula (3), the perimeter of $E(a,b)$ may be divided into two parts, one is $2\pi a$, the other is $2\pi b$. It is obvious that this part is only relevant to ratio ($b/a$), which is represented by coefficient $\xi$, thus formula (6) is derived.

\[
LE(a,b) = 2\pi a \times \xi \tag{6}
\]

According to formulas (3), (4), and (6), a conclusion is drawn that for two ellipses $E_1(a_1,b_1)$ and $E_2(a_2,b_2)$, if $b_1/a_1$ is equal to $b_2/a_2$ (with the same ratio ($b/a$), then $LE_1/LE_2$ is equal to $a_1/a_2$, and is equal to $b_1/b_2$ too.

For a known ellipse $E(a,b)$, parameter $a$ and $b$ are known too. To calculate the known ellipse's perimeter may be divided into three steps:

1. Step 1: To calculate $2\pi a$.
2. Step 2: To calculate coefficient $\xi$.
3. Step 3: To multiply $2\pi a$ by $\xi$ to get $LE(a,b)$.

Let $x$ denote ratio ($b/a$), then coefficient $\xi$ is the function of $x$, i.e.

\[
\xi = 1 - \left(\frac{1}{2}\right)^2 \frac{1}{1} \left(1-x^2\right) - \left(\frac{1}{2} \cdot \frac{3}{4} \right)^2 \frac{1}{3} \left(1-x^2\right)^3 - \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \right)^2 \frac{1}{5} \left(1-x^2\right)^5 - \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \right)^2 \frac{1}{7} \left(1-x^2\right)^7 - \ldots \tag{7}
\]

Variable $x$ ranges from 0 to 1 in this formula, because $b$ is less than $a$.

**2.2 Method to calculate coefficient $\xi$**
(1) Calculating directly via formula
Coefficient $\xi$ for a known ellipse can be computed directly by formula (1) or (2) via programming in computer.

(2) Method of dividing infinitely then summing

![Figure 2. Radian length approximation.](image)

Illustrated as Figure 2, for the first quadrant which needs to be focused in rectangular coordinate system, the radian length only within the focused quadrant would be analyzed, entire radian length of ellipse is 4 times that in focused part.

Dividing the perimeter in the first quadrant into N segments with the same $\Delta x$, the radian length between two continuous points $P_n$ and $P_{n+1}$ is approximated to linear length of the same two points. i.e.

$$ l_{(n,n+1)} \approx \sqrt{\Delta x^2 + \Delta y^2} $$

The elliptical perimeter can be expressed approximately by summing N segments of linear length.

$$ LE_{(a,b)} = 4 \sum_{n=1}^{N} \sqrt{(X_{n+1} - X_n)^2 + (Y_{n+1} - Y_n)^2} $$

More accurate perimeter of ellipse can be computed when N is greater enough. i.e.

$$ LE_{(a,b)} = 4 \sum_{n=1}^{N} \sqrt{(X_{n+1} - X_n)^2 + (Y_{n+1} - Y_n)^2} $$

2.3 Numeric result

By C programming in computer, the function of calculating approximately each part of radius is as follows according to formula (8).

```c
double StepL (double Xs, double Xe)
{
    double Ys;
    double Ye;
    Ys = b * sqrt(1 - (Xs / a) * (Xs / a));
    Ye = b * sqrt(1 - (Xe / a) * (Xe / a));
    return sqrt((Xs - Xe) * (Xs - Xe) + (Ys - Ye) * (Ys - Ye));
}
```

The more approximate to true perimeter the numeric result is, the larger the N is, when N segments are summed to get approximate ellipse perimeter.

The coefficient $\xi$ can be acquired by dividing the perimeter of the calculated ellipse by $2\pi a$. Table 1 shows numeric results calculated in C programming.

| $b/a$ | $\xi$ | $b/a$ | $\xi$ |
|------|-------|------|-------|
| 1    | .999999999997405 | 0.5 | .770982212589911 |
| 0.99 | .995006281414267 | 0.49 | .76698515296749 |
| 0.98 | .990025252683616 | 0.48 | .763019760160741 |
| 0.97 | .985057107424125 | 0.47 | .759086765772421 |
| 0.96 | .9801020434701 | 0.46 | .755186927036827 |
| 0.95 | .975160262993841 | 0.45 | .751321027989181 |
| 0.94 | .970231972630767 | 0.44 | .74748988036065 |
| 0.93 | .965317383610957 | 0.43 | .743694327400211 |
A curve has been drawn as Figure 3 according to numerical results shown in Table 1. The curve shows that the coefficient $\xi$ increases as increase of the ratio $(b/a)$. Coefficient $\xi$ approaches 1 as ratio $(b/a)$ approaches 1, and approaches $2/\pi$ as 0.

| $b/a$ | Data | $\xi$ |
|-------|------|-------|
| 0.92  | 0.960416711894757 | 0.42 |
| 0.91  | 0.955301783131377 | 0.41 |
| 0.9   | 0.950658008714452 | 0.4  |
| 0.89  | 0.945800434115315 | 0.39 |
| 0.88  | 0.940957690858662 | 0.38 |
| 0.87  | 0.936130207777777 | 0.37 |
| 0.86  | 0.93137671365347 | 0.36 |
| 0.85  | 0.926520895951119 | 0.35 |
| 0.84  | 0.92139953885294 | 0.34 |
| 0.83  | 0.916975110730265 | 0.33 |
| 0.82  | 0.9122663845931 | 0.32 |
| 0.81  | 0.907494815663719 | 0.31 |
| 0.8   | 0.902779927769093 | 0.3 |
| 0.79  | 0.898082267259694 | 0.29 |
| 0.78  | 0.8934021391361 | 0.28 |
| 0.77  | 0.888739835042624 | 0.27 |
| 0.76  | 0.884095685766587 | 0.26 |
| 0.75  | 0.879470092391113 | 0.25 |
| 0.74  | 0.874863136969934 | 0.24 |
| 0.73  | 0.870275409088495 | 0.23 |
| 0.72  | 0.865707174655475 | 0.22 |
| 0.71  | 0.861158791977865 | 0.21 |
| 0.7   | 0.85663062894221 | 0.2 |
| 0.69  | 0.85212306360238 | 0.19 |
| 0.68  | 0.847636483333721 | 0.18 |
| 0.67  | 0.84317128763406 | 0.17 |
| 0.66  | 0.838727886102083 | 0.16 |
| 0.65  | 0.8343067000668799 | 0.15 |
| 0.64  | 0.829908162784723 | 0.14 |
| 0.63  | 0.825532719900742 | 0.13 |
| 0.62  | 0.821180829933675 | 0.12 |
| 0.61  | 0.816852964785591 | 0.11 |
| 0.6   | 0.812549610277981 | 0.1 |
| 0.59  | 0.808271266717129 | 0.09 |
| 0.58  | 0.8040184449487659 | 0.08 |
| 0.57  | 0.79979168968105 | 0.07 |
| 0.56  | 0.795591534756544 | 0.06 |
| 0.55  | 0.791418549239867 | 0.05 |
| 0.54  | 0.787273315462401 | 0.04 |
| 0.53  | 0.783156433411393 | 0.03 |
| 0.52  | 0.77906852620621 | 0.02 |
| 0.51  | 0.775010231676932 | 0.01 |

Table 1. Numeric results
2.4 Numeric result regression

To calculate ellipse perimeter, the corresponding coefficient $\xi$ should be known first as in Table 1 by looking up according to ratio $(b/a)$. Because data amount in Table 1 is limited, linear interpolation may be used to get coefficient $\xi$ in operation. Therefore, calculating ellipse perimeter will be made easy if numerical results are regressed to approximate coefficient function of ratio $(b/a)$.

The curve shown as Figure 3 demonstrates that nonlinear regression should be done to get a coefficient function. Polynomial regression algorithm is adopted to acquire the expected approximate function of coefficient in this article. By regression the polynomial coefficient function is achieved as follows.

$$
\xi(x) = +0.6364709512x^8 + 0.01843013801x^7 + 1.0428285380x^6 - 2.37339457900x^5 + 4.8619297740x^4 - 7.0548436700x^3 + 6.4571120740x^2 - 3.31047081900x^1 + 0.7215915918x^0
$$

This can be seen that the coefficient function is a polynomial of degree 8, where $x$ denotes ratio $(b/a)$, $\xi$ is the corresponding coefficient.

Hence, to calculate perimeter of a known ellipse can be done conveniently as steps by the analytic formula achieved.

1. To calculate ratio $(b/a)$.
2. To calculate coefficient by ratio $(b/a)$ via formula (8).
3. To calculate ellipse perimeter by formula (6).

For example, for a known stadium with shape of ellipse in which major axis is 600 meters, and minor axis is 400 meters, its perimeter calculation is as follows.

Step 1: Calculating ratio $(b/a)$ $b/a = 200/300 = 0.666666666666667$.

Step 2: Calculating coefficient according to ratio $(b/a)$ by formula (8).

$\xi(X) = 0.841685412491346$

Step 3: Calculating perimeter.

The perimeter equates $2 \times \pi \times 300 \times 0.841685412491346 = 1586.5396251$ meters.

2.5 Accuracy verification of coefficient polynomial

Accuracy of perimeter calculated by formula (8) depends on the accuracy of coefficient. The coefficient accuracy is analyzed and verified as follows.

More accurate coefficient reference is needed to evaluate coefficient computed by the polynomial. Coefficient values which accuracy better than $10^{-9}$ are selected to evaluate accuracy of the coefficient polynomial.

$$
E_{\text{average}} = \frac{1}{N} \sum_{i=1}^{N} \left| \frac{\xi_{\text{ref}} - \xi_{\text{polynomial}}}{\xi_{\text{ref}}} \right|
$$

Where $\xi_{\text{ref}}$ is coefficient reference, and $\xi_{\text{polynomial}}$ is the coefficient to be evaluated, which is got by the polynomial.

One hundred values were used to compute average accuracy, its result is $0.0000081848$. The numerical result of
Coefficient accuracy is shown as Table 2.

| ξtn       | ξpn       | Error (abs)  |
|-----------|-----------|--------------|
| 0.999999999997405 | 1.00001330201 | 1.33020126   |
| 0.995006281414267 | 0.99500970285677 | 0.280286814  |
| 0.990025252836166 | 0.990021845217375 | 0.344180094  |
| 0.985057104241250 | 0.985056965395550 | 0.650813493  |
| 0.9801020434701 | 0.980094899257182 | 0.728925418  |
| 0.975160262938418 | 0.975139085550508 | 0.651630199  |
| 0.970231972630767 | 0.970227336914453 | 0.477794635  |
| 0.965317383610957 | 0.965314933247049 | 0.253840234  |
| 0.960416711894757 | 0.960416536911519 | 0.211142952  |
| 0.955530178313137 | 0.955532195847767 | 0.561145846  |
| 0.950658087144520 | 0.950661880681647 | 0.407293386  |
| 0.945800341153154 | 0.945805741435162 | 0.361148546  |
| 0.940957690858662 | 0.940963960005534 | 0.666251707  |
| 0.936130020777773 | 0.936136663578510 | 0.720581536  |
| 0.931317671365347 | 0.931324428994232 | 0.725598697  |
| 0.926520895951119 | 0.92652746651842 | 0.68435242   |
| 0.921739953885294 | 0.921745541148703 | 0.606164828  |
| 0.916975110730265 | 0.916979651288483 | 0.495167008  |
| 0.912226638459310 | 0.912229927743679 | 0.36057754   |
| 0.90749415663719 | 0.9074967288838 | 0.210819307  |
| 0.902779927769093 | 0.902780417151872 | 0.054208425  |
| 0.898082267259694 | 0.898081356878757 | 0.101369537  |
| 0.893402133913610 | 0.893399911855749 | 0.248718665  |
| 0.888739835045624 | 0.888736444219940 | 0.381531936  |
| 0.884095685765587 | 0.884091331359465 | 0.494535943  |
| 0.879470009293113 | 0.879464867775429 | 0.583585981  |
| 0.874863136969933 | 0.87485748783833 | 0.64571604   |
| 0.870275409088495 | 0.870269498621012 | 0.679149086  |
| 0.865707174655475 | 0.865701529522964 | 0.683271744  |
| 0.861158791977865 | 0.86115320574933 | 0.658578068  |
| 0.856630628942210 | 0.856625323705676 | 0.606587188  |
| 0.852123063360238 | 0.852118549336185 | 0.529738514  |
| 0.847663483337210 | 0.847632827737493 | 0.431269335  |
| 0.843171287634060 | 0.843168630984974 | 0.31507822   |
| 0.838728786102083 | 0.838726329602599 | 0.185578602  |
| 0.834060700066879 | 0.834063033874060 | 0.047546001  |
| 0.829908162784723 | 0.829908943210402 | 0.094037595  |
| 0.825532719900742 | 0.82553465279217 | 0.23413682  |
| 0.821180299336750 | 0.821183850492937 | 0.367823462  |
| 0.816582964785591 | 0.816586970651814 | 0.490402361  |
| 0.812549610277981 | 0.812554465790793 | 0.59756509   |
| 0.808271266711290 | 0.808276807446010 | 0.68503631   |
| 0.804018449876659 | 0.804024887368650 | 0.751021202  |
| 0.799791689681010 | 0.799798021029077 | 0.791624633  |
| 0.795951347506544 | 0.795957944033674 | 0.80559896   |
| 0.791418549239867 | 0.791424817768778 | 0.79206242   |
| 0.787273315462401 | 0.787279227890022 | 0.751000638  |
| 0.783156434341139 | 0.783161785488301 | 0.683279474  |
| 0.779068526206261 | 0.77907312766147 | 0.590635497  |
| 0.775010231676932 | 0.775013917968107 | 0.475644195  |
| 0.770982212589911 | 0.770984846774531 | 0.341660696  |
| 0.766985125986749 | 0.766986631509163 | 0.192770669  |
| 0.763019760160741 | 0.763020016841255 | 0.03364087   |
| 0.759086765772421 | 0.759085774803907 | 0.130547463  |
Table 2. Numerical result of coefficient accuracy

Average Error (abs): 0.818479891
Note: All errors have been multiplied by $10^5$

3. Conclusion
This article demonstrates that the coefficient $\xi$ is the key factor to calculate ellipse perimeter, and it is the function of ratio (b/a).

A variety of coefficients as to ratio (b/a) is calculated with amount of 100 pairs with ratio step 0.01. The numerical results are regressed non-linearly to an approximate coefficient polynomial. The average accuracy of the polynomial is
better than 8.2×10⁻⁶, which may be applied to common engineering design.

The polynomial described in this article may be used to compute by a calculator, or by programming in computer.

Some instruments may need to calculate ellipse perimeter for engineering applications. The coefficient polynomial acquired in this article contains only operations of add and multiplication, without complex operations, such as Power and Log, and has higher accuracy. Therefore, it may be suitable to be embedded into applications of instruments. The coefficient polynomial can be programmed easily also to a function library, and embedded into application software.

The issue needed to be researched in future is to combine theoretical analysis and mathematical operation in computer, to achieve more accurate formulas with better practicality.

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