Spontaneous symmetry breaking in the linear sigma model
at finite chemical potential: One-loop corrections

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We investigate spontaneous symmetry breaking within the linear sigma model with the SU(2) × U(1) internal symmetry at finite chemical potential, which was suggested as a model for kaon condensation in the CFL phase of dense quark matter. One-loop corrections to the scalar field effective potential as well as its propagator are calculated. Particular attention is paid to the type-II Goldstone boson that appears in the Bose–Einstein condensed phase. Furthermore, we show that the type-I Goldstone boson — the superfluid phonon — is allowed to decay due to the nonlinearity of its dispersion relation at high momentum, and determine its decay width.

I. INTRODUCTION

One of the most striking and general consequences of spontaneous symmetry breaking is the existence of gapless modes, guaranteed by the celebrated Goldstone theorem [1, 2]. The number and properties of the Goldstone bosons are crucial for the low-energy dynamics (and the low-temperature thermodynamics) of the system. In particular, these soft modes play a major role in transport phenomena such as heat conductivity or viscosity.

It has been known for a long time that the physics of spontaneous symmetry breaking in Lorentz-noninvariant systems may be quite intricate. The fundamental result in this respect was achieved by Nielsen and Chadha [3]: They showed in a nonperturbative manner that, under certain technical assumptions, the energy of the Goldstone bosons stemming from the spontaneous symmetry breaking is, in the long-wavelength limit, proportional to some power of momentum. The Goldstone bosons are then classified as type-I, if this power is odd, and type-II, if it is even, respectively. The numbers of the Goldstone bosons of the respective types are related to the number of broken symmetry generators by the following inequality,

\[ \# \text{type-I GBs} + 2 \times \# \text{type-II GBs} \geq \# \text{broken generators}. \tag{1} \]

Eq. (1) shows that whenever there is a type-II Goldstone boson, the number of Goldstone bosons may be smaller than the number of broken generators. A profound example of a system where this happens is provided by the ferromagnet. In the past decade, however, several other systems with type-II Goldstone bosons have been studied, ranging from various high-density phases of QCD [4, 5, 6, 7] and the relativistic nuclear ferromagnet [8] to Bose–Einstein condensed atomic gases [8, 10, 11] and the general effect of relativistic vector condensation [12, 13].

In our recent paper [14] we investigated the Goldstone boson counting in a particular class of Lorentz-noninvariant systems — the relativistic linear sigma model at finite chemical potential. Based on previous partial results [5, 15] we clarified the connection of Goldstone boson counting with the possibility of nonzero densities of Noether charges and conjectured a general counting rule: Non-zero density of a commutator of two broken generators implies one type-II Goldstone boson with a quadratic dispersion law. As a consequence, the inequality (1) is saturated, up to exceptional cases like the phase transitions, where the phase velocity of a type-I Goldstone boson may vanish, thus making it an ‘accidental’ type-II one. Within the linear sigma model, we were able to give a detailed proof of these statements.

Our analysis was, however, purely classical — we worked all the time at the tree level. The present paper is intended to fill this gap. There are several reasons that make the inclusion of the loop corrections necessary. First, it is only at the one-loop level that the \( \lambda (\phi^\dagger \phi)^2 \) interaction plays a nontrivial role. Indeed, at the tree level, it has no effect on the spectrum in the normal phase, while in the Bose–Einstein condensed phase it merely serves to stabilize the scalar field potential. Second, the loop corrections may play a significant role, especially in the vicinity of the phase transition, where the classical analysis cannot be trusted. Third, it is well known that when the static part of the Lagrangian has higher symmetry than the full Lagrangian, the tree level is not sufficient to determine the spectrum (even qualitatively) correctly [16]. The quantum corrections are therefore necessary in order to check the saturation of the Nielsen–Chadha inequality [3].

For simplicity, we do not follow the general symmetry-breaking pattern of Ref. [14]. Instead, we analyze a particular model — the linear sigma model with the global SU(2) × U(1) symmetry. This model has been used to describe kaon condensation in the Color-Flavor-Locked phase of dense QCD [4, 5]. It is also a special case of

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a class of models of relativistic Bose–Einstein condensation investigated by Andersen [17].

The plan of the paper is as follows. Using the manifestly invariant formalism of generating functionals, we determine in Sec. II the one-loop effective potential and numerically find its minimum as a function of the chemical potential. In Sec. III we calculate the one-loop correction to the scalar field propagator. We demonstrate analytically the presence of gapless (Goldstone) poles and then compute numerically the corrections to the tree-level dispersion relations of the type-II Goldstone boson and its massive counterpart. In Sec. IV, we calculate the one-loop correction to the scalar field propagator. We demonstrate analytically the presence of gapless (Goldstone) poles and then compute numerically the corrections to the tree-level dispersion relations of the type-II Goldstone boson and its massive counterpart. In Sec. IV we calculate the one-loop correction to the scalar field propagator. We demonstrate analytically the presence of gapless (Goldstone) poles and then compute numerically the corrections to the tree-level dispersion relations of the type-II Goldstone boson and its massive counterpart. In Sec. IV we calculate the one-loop correction to the scalar field propagator. We demonstrate analytically the presence of gapless (Goldstone) poles and then compute numerically the corrections to the tree-level dispersion relations of the type-II Goldstone boson and its massive counterpart. In Sec. IV we calculate the one-loop correction to the scalar field propagator. We demonstrate analytically the presence of gapless (Goldstone) poles and then compute numerically the corrections to the tree-level dispersion relations of the type-II Goldstone boson and its massive counterpart.

II. EFFECTIVE POTENTIAL

We work with the model of Miransky and Shovkovy [4] and Schaefer et al. [5] which is defined by the Lagrangian

$$\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi - M^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2. \quad (2)$$

Here $\phi$ is a complex doublet of the global SU(2) symmetry and the chemical potential $\mu$, associated with the global U(1) symmetry (particle number), is included in the covariant derivative, $D_\mu \phi = (\partial_\mu - i\sigma_\mu \lambda) \phi$. This model has been analyzed, at the tree level, in detail in Refs. [4, 5], so we only summarize the main results for later reference.

When $\mu > M$ the static potential develops a nontrivial minimum and the scalar field condenses. As a result, the SU(2) $\times$ U(1) symmetry of the Lagrangian (2) breaks down to its U(1) subgroup [different from the original U(1)]. Associated with the three broken generators there are two Goldstone bosons, one type-I and one type-II. Their low-energy dispersion relations read

$$E = \sqrt{\frac{\mu^2 - M^2}{3\mu^2 - M^2} |p| \text{ and } E = \frac{p^2}{2\mu},} \quad (3)$$

respectively. In addition to the Goldstone bosons, there are two massive excitations with energy gaps $\sqrt{2(3\mu^2 - M^2)}$ and $2\mu$.

A. One-loop correction

Let us now proceed to the one-loop calculation. In order to account for the breaking of the U(1) symmetry associated with the particle number (or strangeness, in the context of kaon condensation) we introduce the formal Nambu doublet

$$\Phi_i = \begin{pmatrix} \phi_i \\ \phi_i^\dagger \end{pmatrix}. \quad [\text{The index } i \text{ refers to the doublet representation of the SU(2).}]$$

The one-loop effective action is then given by the textbook formula [20],

$$\Gamma_{1L}[\phi, \phi^\dagger] = S[\phi, \phi^\dagger] + \frac{i}{2} \log \det \Delta^{-1} \quad (4)$$

plus counterterms. Here $S[\phi, \phi^\dagger] = \int d^4x \mathcal{L}(x)$ is the classical action and $\Delta$ is the tree-level matrix propagator defined by

$$\Delta^{-1}_{ij}(x, y) = \frac{\delta^2 S}{\delta \Phi_i(x) \delta \Phi_j(y)}. \quad \text{Displaying explicitly the matrix structure in the Nambu space, it reads, in the momentum representation,}$$

$$\Delta^{-1}_{\mu}(p) = \begin{pmatrix} [(p_0 + \mu)^2 - p^2 - M^2 - 2\lambda \phi^\dagger \phi] \delta_{ij} - 2 \lambda \phi^\dagger_i \phi_j & -2 \lambda \phi_i \phi_j \\
-2 \lambda \phi_i \phi_j & [(p_0 - \mu)^2 - p^2 - M^2 - 2\lambda \phi^\dagger \phi] \delta_{ij} - 2 \lambda \phi^\dagger_i \phi_j \end{pmatrix}. \quad (5)$$

Here the classical field $\phi$ is already assumed to be constant. There is no lack of generality in this requirement as long as the vacuum is translationally invariant.

With the constant classical field $\phi$ we may evaluate the one-loop effective potential as

$$V_{1L} = V_{cl} - \frac{i}{2} \int \frac{d^4 k}{(2\pi)^4} \log \det \Delta^{-1}(k) \quad (6)$$

plus counterterms, where $V_{cl} = (M^2 - \mu^2)\phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$.

In this form, the effective potential is manifestly SU(2) $\times$ U(1) invariant. For detailed calculations it is, however, more convenient to fix the direction of $\phi$ in the SU(2) ‘flavor’ space. As usual, we set

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}. \quad (7)$$

With this choice, $\Delta^{-1}$ becomes diagonal in the flavor space and may easily be inverted, yielding
\[ \Delta_{11}(p) = \left( \left[ (p_0 + \mu)^2 - \mathbf{p}^2 - M^2 - \lambda v^2 \right]^{-1} 0 \right. \]

\[ \Delta_{22}(p) = \frac{1}{|p_0^2 - E_+^2(p)|} \left( \left[ (p_0 - \mu)^2 - \mathbf{p}^2 - M^2 - \lambda v^2 \right]^{-1} \left[ (p_0 + \mu)^2 - \mathbf{p}^2 - M^2 - 2\lambda v^2 \right] \right) \]

where

\[ E_+^2(p) = p^2 + \mu^2 + M^2 + 2\lambda v^2 \]
\[ \pm 4\mu^2(p^2 - \mu^2 + M^2 + \lambda v^2) + (2\mu^2 + \lambda v^2)^2. \]

By minimizing the classical potential \( V_{\text{cl}} \) we find that, at tree level, \( \lambda v^2 = \mu^2 - M^2 \), which yields the dispersion relations

\[ E^2 = p^2 + 3\mu^2 - M^2 \pm \sqrt{4\mu^2 p^2 + (3\mu^2 - M^2)^2}, \]

in accord with the result of Refs. [4, 5].

### B. Renormalization

The loop integral in Eq. (6) is divergent. To renormalize it, we add to the effective potential the counterterm

\[ V_{\text{counterterm}} = \delta v + \delta M^2 \phi^\dagger \phi + \delta \lambda (\phi^\dagger \phi)^2. \]

This term generates, by means of its functional derivatives, also the counterterm for the scalar propagator. The constant \( \delta v \) is irrelevant for our purposes as it just fixes the offset of the effective potential at \( \phi = 0 \). The counterterms in the effective potential are sufficient to absorb all infinities for, as is well known, there is no wave function renormalization in the scalar \( \phi^4 \) theory at one loop [27].

Actually, we shall not need to evaluate the effective potential at all. What we are interested in is just the vacuum expectation value of the scalar field, or \( v \). We shall be therefore solving the ‘gap equation’ \( \partial V_{1L}/\partial v = 0 \). Now since the effective potential is a function of \( v \), there is always the solution \( v = 0 \). In the Bose–Einstein condensed phase this is, however, not the only solution and not the global minimum. The nontrivial solution is found by means of \( \partial V_{1L}/\partial v = 0 \). Formulas (6) and (8) give

\[ 0 = \frac{\partial V_{1L}}{\partial v} = \frac{1}{2}(M^2 + \delta M^2 - \mu^2) + \frac{1}{2}(\lambda + \delta \lambda)v^2 \]
\[ + \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \left[ 2\lambda \Delta_{11}^{\phi^\dagger \phi} (k) - \frac{\partial}{\partial v} \det \Delta_{22}^{\phi^\dagger \phi} (k) \right]. \]

The momentum integral in Eq. (10) is only formal and implicitly involves a convenient regularization. In deriving Eq. (10) we already assumed that the regularization allows to change the sign of the integration variable so that the contribution of the particles circulating in the loop is the same as that of antiparticles. For the later calculation of the scalar field propagator it will also be useful to be able to shift the integration variable. In concrete computations we shall use the analytic integration over energy (frequency) in combination with dimensional regularization around \( d = 3 \) space dimensions [17].

Eq. (10) may be rewritten in a particularly useful way. With a decent use of Eq. (6) we find (for the sake of brevity, we temporarily drop out the arguments of the propagators as well as the integration measure)

\[ \frac{\partial}{\partial v^2} \det \Delta_{22}^{-1} = \frac{\partial}{\partial v^2} \left( \Delta_{22}^{-1} - 1 \phi^\dagger - \Delta_{22}^{-1} - 1 \phi^\dagger - \Delta_{22}^{-1} - 1 \phi^\dagger \right) \]
\[ = -2\lambda \left( \Delta_{22}^{-1} \phi^\dagger + \Delta_{22}^{-1} \phi^\dagger - \Delta_{22}^{-1} \phi^\dagger \right) \]

so that

\[ - \int \frac{\partial}{\partial v^2} \det \Delta_{22}^{-1} = 2\lambda \int \left( \Delta_{22}^{-1} \phi^\dagger + \Delta_{22}^{-1} \phi^\dagger + \Delta_{22}^{-1} \phi^\dagger \right) \]
\[ = 2\lambda \int \left( 2\Delta_{22}^{-1} \phi^\dagger + \Delta_{22}^{-1} \phi^\dagger \right). \]

Eq. (10) thus acquires the form

\[ \mu^2 - (M^2 + \delta M^2) - (\lambda + \delta \lambda)v^2 \]
\[ = 2\lambda \int \frac{d^4k}{(2\pi)^4} \left[ \Delta_{11}^{\phi^\dagger \phi} (k) + 2\Delta_{22}^{\phi^\dagger \phi} (k) + \Delta_{22}^{\phi^\dagger \phi} (k) \right]. \]

Even though this result has been derived by a not very transparent manipulation, it could have been expected: It is equivalent, up to an overall factor, to the requirement that the one-particle-irreducible tadpole contributions to the vacuum expectation value of \( \phi_2 \) vanish. For the diagrammatic representation of the individual terms see Appendix [3].

For the concrete calculations, we adopted the following renormalization conditions. The scalar mass was renormalized by subtracting the whole one-loop correction at \( \mu = 0 \) and \( v = 0 \) [see also Eq. (15) later], i.e.,

\[ \delta M^2 = -3\lambda \Lambda^{3-d} \int \frac{d^4k}{(2\pi)^d} \frac{1}{\sqrt{k^2 + M^2}}. \]

where the integral is regularized by minimal subtraction in \( d \) dimensions and \( \Lambda \) is the renormalization scale. This procedure guarantees that the parameter \( M \) keeps its
interpretation as the physical (pole) mass of the scalar at zero chemical potential.

The coupling constant $\lambda$ was renormalized by modified minimal subtraction (MS) so that

$$\delta \lambda = \frac{3 \lambda^2}{4 \pi^2} \left( \frac{2}{\epsilon} - \gamma + \log 4\pi \right),$$

where $\epsilon = 3 - d$ is the expansion parameter of dimensional regularization $[28]$. (The pole does not depend on the chemical potential and hence could be extracted from the analytically calculable loop integral at $\mu = 0$.) The renormalization scale $\Lambda$ was set equal to the characteristic scale of the system: we chose $\Lambda = M$ in the normal phase and $\Lambda = \mu$ in the Bose–Einstein condensed phase. Since the phase transition occurs at $\mu = M$ (see the following discussion), the subtraction point is a connected function of the chemical potential.

For sake of numerical solution the gap equation (11) was rewritten, with the help of Eq. (8) and upon the integration over frequencies, as

$$v^2 = \frac{\mu^2 - M^2}{\lambda} - \frac{2}{\lambda E_+(k) + E_-(k)} \int \frac{d^dk}{(2\pi)^d} \left\{ \frac{1}{\sqrt{k^2 + M^2 + \lambda v^2}} \right\}$$

$$+ \frac{2}{\lambda E_+(k) - E_-(k)} \left[ 1 + \frac{k^2 + M^2 + \frac{8}{3} \lambda v^2 - \mu^2}{E_+(k) E_-(k)} \right]$$

$$- \frac{\delta M^2 + \delta \lambda v^2}{\lambda}. \quad (13)$$

This equation was solved iteratively with the initial ansatz given by the tree-level value, $v_0 = \sqrt{\frac{\mu^2 - M^2}{\lambda}}$. To implement the MS scheme within the numerical computation, we subtracted the pole part in the form

$$\delta \lambda = - \frac{3 \lambda^2}{4 \pi^2} \log \frac{\Lambda^2}{M^2} + 3 \lambda^2 \frac{\Lambda^{3-d}}{\lambda^{3/2}}.$$

The integral here reproduces the pole and is, in fact, equal to the full one-loop correction to $\lambda$ at $\mu = 0$ and $v = 0$. Upon the subtraction of $\delta M^2$ and $\delta \lambda$, the integral in (13) is rendered finite and was evaluated approximately with a simple cutoff.

The numerical results are displayed in Fig. 1. We emphasize the fact that the phase transition to the Bose–Einstein condensed phase happens at the chemical potential equal to the renormalized scalar mass. This is of course not surprising from the physical point of view and in fact is an important check of consistency of our calculations.

This conclusion is also easily proved analytically. Just set $v = 0$ and then immediately observe that the right hand side of Eq. (11) is exactly canceled by the counterterm $\delta M^2$ i.e., $v = 0$ is a solution of Eq. (11) exactly for $\mu = M$.

Finally, note that the convergence of our iterative solution of the gap equation guarantees that the found solution is, as required, a (at least local) minimum of the effective potential.

FIG. 1: Relative increase of the vacuum expectation value $v$ with respect to the tree-level value $v_0 = \sqrt{\frac{\mu^2 - M^2}{\lambda}}$. The numerical data were obtained with the renormalized coupling $\lambda$ set to 1.

III. SCALAR PROPAGATOR

The one-loop-corrected propagator $D$ is obtained as a second functional derivative of the one-loop effective action $[4]$.

$$D_{ij}^{-1}(x,y) = \frac{\delta^2 \Gamma_{1L}}{\delta \Phi_i^\dagger(x) \delta \Phi_j^\dagger(y)}.$$

After differentiating the logarithm of the determinant we arrive at

$$D_{ij}^{-1}(x,y) = \Delta_{ij}^{-1}(x,y)$$

$$+ \frac{i}{2} \text{Tr} \left[ \Delta \frac{\delta^2 \Delta^{-1}}{\delta \Phi_i^\dagger(x) \delta \Phi_j^\dagger(y)} - \Delta \frac{\delta \Delta^{-1}}{\delta \Phi_i^\dagger(x)} \frac{\delta \Delta^{-1}}{\delta \Phi_j^\dagger(y)} \right]$$

plus counterterms. Here ‘Tr’ denotes the full functional trace, over both spacetime and internal degrees of freedom.

The derivatives of the inverse bare propagator $\Delta^{-1}$ are nothing but the cubic and the quartic interaction vertex. They are local and may be represented by the coupling matrices $T^{(i)}$ and $Q^{(ij)}$, defined as

$$\frac{\delta \Delta_{kl}^{-1}(s,t)}{\delta \Phi_i^\dagger(x)} = T_{kl}^{(i)} \delta^4(s-t) \delta^4(s-x)$$

$$\frac{\delta^2 \Delta_{kl}^{-1}(s,t)}{\delta \Phi_i^\dagger(x) \delta \Phi_j^\dagger(y)} = Q_{kl}^{(ij)} \delta^4(s-t) \delta^4(s-x) \delta^4(t-y).$$

The explicit formulas for the matrices $T^{(i)}$ and $Q^{(ij)}$ are listed in the Appendix A. With this notation, the inverse
A. Normal phase

When the chemical potential is small enough, the system is in the normal phase: \( v = 0 \) and the symmetry is not spontaneously broken. In this phase the one-loop correction to the propagator is particularly simple. The bare propagator \( \Delta \) is diagonal in both the flavor and the Nambu space, the only nonzero components being:

\[
\Delta^{\phi \phi^*}_{ij}(p) = [(p_0 + \mu)^2 - p^2 - M^2]^{-1},
\]

\[
\Delta^{\phi^* \phi}_{ij}(p) = [(p_0 - \mu)^2 - p^2 - M^2]^{-1}.
\]

Since in the normal phase there is no cubic coupling \( T^{(i)} \), the final formula for the one-loop propagator is easily found to be

\[
D_{ij,\phi\phi^*}^{-1}(p) = \left[ (p_0 + \mu)^2 - p^2 - M^2 + \delta M^2 \right] \delta_{ij} - 6i\lambda \delta_{ij} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k_0 + \mu)^2 - k^2 - M^2}.
\]

The other nonzero component of the propagator, \( D_{\phi,\phi^*} \), is related to Eq. (15) by the first of the identities,

\[
D_{\phi^* \phi}(p) = D_{\phi \phi^*}(-p),
\]

\[
D_{\phi \phi}(p) = D_{\phi^* \phi^*}(p) = D_{\phi,\phi^*}(-p).
\]

which follow from the very definition of the propagator.

Note that the loop correction in Eq. (15) is momentum-independent, as usual in a \( \phi^4 \) theory. The renormalization is therefore trivial and corresponds to a mere redefinition of the mass.

Since in dimensional regularization we may freely shift the integration variable, the loop integral in Eq. (15) is \( \mu \)-independent and thus is completely canceled by the mass counterterm \( \mu^2 \). Therefore, in the normal phase, the scalar field propagator gets no correction and the mass spectrum simply consists of two doubly degenerate levels at \( M \pm \mu \).

B. Bose–Einstein condensed phase

When the scalar condenses i.e., \( v > 0 \), we have to work with the full formulas (8). The matrix propagator \( D \) has altogether 16 components (four flavor times four Nambu). Fortunately, several of them are actually zero due to the clever choice of the vacuum, Eq. (4). With this choice, the unbroken flavor U(1)_Q symmetry is generated by the matrix \( Q = \frac{1}{2}(1 + \tau_3) \). This means that the upper flavor component of \( \phi \), \( \phi_1 \), carries the unbroken charge and the lower component \( \phi_2 \) does not. The charge conservation then immediately implies that the propagator is diagonal in the flavor space, i.e., \( D_{12}(p) = D_{21}(p) = 0 \). Moreover, the propagator of \( \phi_1 \) is diagonal in the Nambu space, that is, \( D_{11}^{\phi \phi^*}(p) \neq D_{11}^{\phi^* \phi}(p) = 0 \). [All these relations could, of course, also be demonstrated explicitly by a proper analysis of the formula (14)].

The remaining nonzero components of the propagator are strongly constrained by Eq. (16) so that only three of them are independent: \( D_{11}^{\phi \phi^*} \), \( D_{22}^{\phi \phi^*} \) and \( D_{22}^{\phi^* \phi^*} \).

1. Propagator of \( \phi_1 \)

The only independent component of \( D_{11} \) is \( D_{11}^{\phi \phi^*} \).

A straightforward application of Eq. (14) yields

\[
D_{11}^{\phi \phi^*}(p) = (p_0 + \mu)^2 - p^2 - (M^2 + \delta M^2) - (\lambda + \delta \lambda)v^2 - 2i\lambda \int \frac{d^4k}{(2\pi)^4} \left[ 2\Delta_{11}^{\phi \phi^*}(k) + \Delta_{22}^{\phi \phi^*}(k) \right] - 2i\lambda v^2 \int \frac{d^4k}{(2\pi)^4} \Delta_{11}^{\phi \phi^*}(p + k) \left[ \Delta_{22}^{\phi \phi^*}(k) + \Delta_{22}^{\phi^* \phi^*}(k) + \Delta_{22}^{\phi^* \phi}(k) + \Delta_{22}^{\phi \phi}(k) \right].
\]

(17)

The first integral represents the tadpoles and is momentum-independent. A qualitative difference in comparison with the normal phase comes in the second integral, which contains the contribution of the cubic vertices. The diagrammatic representation of all terms in the propagator is given in Appendix B.

The propagator \( D_{11}^{\phi \phi^*}(p) \) is expected to have a massless pole corresponding to the Goldstone boson. Indeed, a short glance at Eq. (8) reveals that at tree level, when \( \lambda v^2 = \mu^2 - M^2 \), there is a gapless pole in \( \Delta_{11}^{\phi \phi^*}(p) \). The
field $\phi_1$ annihilates a particle with dispersion relation $E = \sqrt{p^2 + \mu^2 - \mu}$, i.e., a type-II Goldstone boson. The other excitation, annihilated by $\phi_1^\dagger$, with a tree-level gap $2\mu$ is manifested as a pole in $D_{11}^{\phi^\dagger}\phi(p)$.

The analytic proof of the existence of a massless pole in $D_{11}^{\phi^\dagger}\phi(p)$ is provided in Appendix C. Here we just observe that once the relation $D_{11}^{-1\phi^\dagger}\phi(0) = 0$ is proved, we may subtract it from $D_{11}^{-1\phi^\dagger}\phi(p)$ to obtain a convenient expression

$$D_{11}^{-1\phi^\dagger}\phi(p) = p_0^2 + 2\mu p_0 - p^2 - 2i\lambda^2 v^2 \int \frac{d^4k}{(2\pi)^4} \left[ \Delta_{11}^{\phi^\dagger}(p + k) - \Delta_{11}^{\phi^\dagger}(k) \right]$$

$$\times \left[ \Delta_{22}^{\phi^\dagger}(k) + \Delta_{22}^{\phi^\dagger}(k) + \Delta_{22}^{\phi^\dagger}(k) + \Delta_{22}^{\phi^\dagger}(k) \right]. \quad (18)$$

Note that the propagator no longer depends explicitly on the renormalization counterterms. Also, the momentum-independent tadpole graphs disappeared and the remaining integral is finite and perfectly well defined even when the regularization is removed.

In order to establish the radiative corrections to the type-II Goldstone boson dispersion relation, we performed the integration over frequency and expanded the result in powers of the external momentum $p$, up to order $p_0$ and $p^2$, respectively. The final formula is rather cumbersome, so we do not write it out and instead report just the result of the numerical integration of the remaining momentum integral.

At the leading order of the momentum power expansion, the inverse propagator reads

$$D_{11}^{-1\phi^\dagger}(p) = 2\mu p_0 (1 + Z_1) - p^2 (1 + Z_2).$$

The coefficients $Z_1, Z_2$ determine the renormalization of both the wave function and the dispersion relation of the Goldstone boson. Their dependence on the chemical potential is displayed in Fig. 2.

Next we turn our attention to the massive partner of the type-II Goldstone boson. Analogously to Ref. [5], we shall calculate just the leading order of its dispersion relation, i.e., its gap. The corresponding propagator, $D_{11}^{\phi^\dagger}\phi$, is obtained from Eq. (18) via the relations (16). Next we set $p = 0$. However, the explicit form of the tree-level propagators, Eq. (8), reveals that

$$\Delta_{11}^{\phi^\dagger}(k_0 - 2\mu) = \Delta_{11}^{\phi^\dagger}(k_0, k)$$

so that the difference of propagators in the loop correction to $D_{11}^{-1\phi^\dagger}(-p_0, 0)$ exactly vanishes at $p_0 = 2\mu$, see Eq. (18). As a result, the renormalized gap is equal to its tree-level value $2\mu$ and receives no radiative corrections.

2. Propagator of $\phi_2$

In this case, Eq. (18) yields, after a reasonable application of the rules (16),

$$D_{22}^{-1\phi^\dagger}(p) = (p_0 + \mu)^2 - p^2 - (M^2 + \delta M^2) - 2(\lambda + \delta\lambda) v^2 - 2i\lambda \int \frac{d^4k}{(2\pi)^4} \left[ \Delta_{11}^{\phi^\dagger}(k) + 2\Delta_{22}^{\phi^\dagger}(k) \right]$$

$$- 2i\lambda^2 v^2 \int \frac{d^4k}{(2\pi)^4} \Delta_{11}^{\phi^\dagger}(k) \Delta_{11}^{\phi^\dagger}(p + k)$$

$$- 8i\lambda^2 v^2 \int \frac{d^4k}{(2\pi)^4} \left[ \Delta_{22}^{\phi^\dagger}(p + k) \left[ \Delta_{22}^{\phi^\dagger}(k) + \Delta_{22}^{\phi^\dagger}(k) + \Delta_{22}^{\phi^\dagger}(k) + \Delta_{22}^{\phi^\dagger}(k) \right] + \Delta_{22}^{\phi^\dagger}(p + k) \Delta_{22}^{\phi^\dagger}(k) \right], \quad (19)$$

$$D_{22}^{-1\phi^\dagger}(p) = -(\lambda + \delta\lambda) v^2 - 2i\lambda \int \frac{d^4k}{(2\pi)^4} \Delta_{11}^{\phi^\dagger}(k) - 2i\lambda^2 v^2 \int \frac{d^4k}{(2\pi)^4} \Delta_{11}^{\phi^\dagger}(k) \Delta_{11}^{\phi^\dagger}(p + k)$$

$$- 8i\lambda^2 v^2 \int \frac{d^4k}{(2\pi)^4} \left[ \Delta_{22}^{\phi^\dagger}(p + k) \left[ \Delta_{22}^{\phi^\dagger}(k) + \frac{1}{2} \Delta_{22}^{\phi^\dagger}(k) + \Delta_{22}^{\phi^\dagger}(k) + \Delta_{22}^{\phi^\dagger}(k) \right] + \Delta_{22}^{\phi^\dagger}(p + k) \Delta_{22}^{\phi^\dagger}(k) \right]. \quad (20)$$

Here the propagator is not diagonal in the Nambu space and the situation is thus more complicated. We cannot look for the zeros of the individual components of the inverse propagator. Instead, we have to solve the condition $\det D_{22}^{-1}(p) = 0$, which seems a rather formidable task in view of Eqs. (19) and (20).

Fortunately, we need not calculate the determinant explicitly, at least to prove the existence of a Goldstone boson. Recalling the relations (18), the determinant at zero momentum simplifies to

$$\det D_{22}^{-1}(0) = \left[D_{22}^{-1\phi^\dagger}(0)\right]^2 - \left[D_{22}^{-1\phi^\dagger}(0)\right]^2.$$
condensed weakly interacting Bose gas [21]. This in turn
found e.g. for the superfluid phonon in the Bose–Einstein
phonon to decay. (A similar, convex, dispersion law is
reaching consequences for it kinematically allows the
phonon) turns out to be convex. This fact has far-
dispersion relation of the type-I Goldstone boson (or, the
boson is proved once we verify the relation
In view of Eq. (5) we see that the existence of a Goldstone
with
The secular equation thus reduces to
\[ D^{-1\phi\phi}_{22}(0) = \pm D^{-1\phi\phi}_{22}(0) \].
In view of Eq. (5) we see that the existence of a Goldstone
boson is proved once we verify the relation
\[ D^{-1\phi\phi}_{22}(0) = D^{-1\phi\phi}_{22}(0). \] (21)
A detailed proof of this statement is provided in Ap-
pendix C 2.

The analysis of Eqs. 19 and 20 would yield the ra-
diative correction to the type-I Goldstone boson disper-
sion relation, i.e., the speed of sound in the Bose–Einstein
condensate. We do not perform the detailed calculation
here since our main concern was the type-II Goldstone
boson studied above. Moreover, such a straightforward
approach is unnecessarily complicated because the one-
loop correction to the speed of sound has been deter-
mined by Andersen using a different method [17]. In the
following, we shall instead concentrate on another spe-
cific property of the superfluid phonon in our system —
its decay.

IV. PHONON DECAY

Upon inspection of Eq. 9, the full tree-level dis-
ispersion relation of the type-I Goldstone boson (or, the
phonon) turns out to be convex. This fact has far-
reaching consequences for it kinematically allows the
phonon to decay. (A similar, convex, dispersion law is
found e.g. for the superfluid phonon in the Bose–Einstein
condensed weakly interacting Bose gas [21].) This in turn
strongly affects the transport properties of the system
such as the shear viscosity 22.

Here we calculate the two-particle decay rate of the
phonon at the leading order of perturbation theory. Such
a quantity could in principle be extracted, via the optical
theorem, from the imaginary part of the one-loop
propagator calculated in Sec. III B 2. This would, how-
ever, be rather complicated for two reasons. First, the
phonon is, in the propagator of \( \phi_2 \), entangled with
the massive ‘radial’ mode. Second, we would have to deal
with the full momentum dependence of the propagator;
the low-momentum limit studied before would not be su-
ficient for our purpose. Since the upcoming calculation
of the phonon decay rate involves a few rather nonstan-
dard pieces, we regard it instructive to perform it step
by step, using ordinary perturbation theory.

Before starting the calculation we briefly comment why
we have chosen to study such a particular process. First,
as has already been mentioned above, the low-energy
properties of the Goldstone bosons are essential for the
thermodynamics of the system at low temperature. Two
factors are determinant for the possibility of the decay of
the Goldstone bosons: The conservation of the unbroken
U(1)_Q charge and phase space (kinematic) restrictions.
The type-II Goldstone boson, \( G \), carries the unbroken
charge. Since at low energy it can only decay to gapless
particles, the only processes allowed by charge conserva-
tion are \( G \to G\pi, G \to G\pi\pi, \) etc. (Here and in the
following, \( \pi \) denotes the type-I Goldstone boson i.e., the
phonon.) Taking into account the dispersion relations
18, these decay modes, however, fail to satisfy the en-
ergy and momentum conservation simultaneously. Thus,
the type-II Goldstone boson is stable at low momentum.

On the other hand, the phonon may decay, by the pro-
cesses \( \pi \to \pi\pi, \pi \to \pi\pi\pi \) etc. Only the two processes ex-
plicitly written here receive nonzero contribution at the
first order of perturbation theory. As will become clear
later, the three-particle decay is suppressed, at least in
certain ranges of momentum and chemical potential. We
shall therefore calculate just the decay rate for the pro-
cess \( \pi \to \pi\pi \) and comment on the relevance of our result
afterwards.

A. Vacuum transition amplitudes

The decay rate cannot be straightforwardly deduced
from the Lagrangian 9 because it is expressed in terms
of the scalar field \( \phi \) rather than the physical modes \( \pi, G \)
and their massive counterparts. The problem is that the
kinetic term in the Lagrangian cannot be diagonalized at
the level of fields. Indeed, Eq. 8 shows that the di-
agonalizing unitary transformation is necessarily energy-
dependent. We therefore have to calculate the three-
point Green’s function of \( \phi \) and use the LSZ reduction
formula to extract the physical scattering amplitude.

Alternatively, we may directly calculate the \( S \)-matrix;
the couplings of \( \phi \) to the physical states follow from
Let $A(x)$ and $B(x)$ be two local bosonic operators. The general form of the Källén–Lehmann representation of the time-ordered Green’s function, $D^{AB}(x,y) = -i\langle 0|T\{A(x)B(y)\}|0\rangle$, reads, in the momentum representation,

$$D^{AB}(p,\omega) = (2\pi)^3 \sum_n \left\{ \frac{\langle 0|A(0)|n,p\rangle\langle n,p|B(0)|0\rangle}{\omega - E(p) + i\epsilon} - \frac{(0)|B(0)|n,-p\rangle\langle n,-p|A(0)|0\rangle}{\omega + E(p) - i\epsilon} \right\} ,$$

where $E(p)$ is the energy of the Hamiltonian eigenstate $|n,p\rangle$. The summation index $n$ is discrete for one-particle intermediate states, and continuous for multi-particle states. Also, the states are assumed to be normalized as $\langle n,p|m,k\rangle = \delta_{mn}\delta^3(p-k)$. Note that we use the general Källén–Lehmann representation (22) rather than the covariant form usual in literature [21].

The poles in the propagator of $\phi_2$ correspond to the dispersion relations given in Eq. (24). The contribution of the massless pole at $\omega = E_\pm(p) - i\epsilon$ is

$$\Delta^{pole}_{22}(p,\omega) = \frac{1}{2E_-(p)} \frac{1}{E^2_-(p) - E^2_+(p)} \frac{1}{\omega - E_-(p) + i\epsilon} \left( [E_-(p) - \mu]^2 - \mathbf{p}^2 - M^2 - 2\lambda v^2 \right) \right) .$$

This expression immediately yields the transition amplitudes

$$\langle 0|\phi_2(0)|\pi(p)\rangle = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_-(p)}} \frac{1}{\sqrt{E^2_-(p) - E^2_+(p)}} \left\{ -[E_-(p) - \mu]^2 + \mathbf{p}^2 + M^2 + 2\lambda v^2 \right\}^{1/2} ,$$

$$\langle 0|\phi'_2(0)|\pi(p)\rangle = -\frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_-(p)}} \frac{1}{\sqrt{E^2_-(p) - E^2_+(p)}} \left\{ -[E_-(p) + \mu]^2 + \mathbf{p}^2 + M^2 + 2\lambda v^2 \right\}^{1/2} .$$

The relative sign of the amplitudes follows from the non-diagonal part of the propagator. Note that the amplitudes are particularly simple in the limit $v \to 0$ (i.e., at the phase boundary) where $E_\pm(p) = \sqrt{\mathbf{p}^2 + M^2} \pm \mu = \epsilon_p \pm \mu$. It follows that, in this limit,

$$\langle 0|\phi_2(0)|\pi(p)\rangle = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\epsilon_p}} \cdot \langle 0|\phi'_2(0)|\pi(p)\rangle = 0 .$$

Thus the result typical for Lorentz-invariant theories is recovered in the normal phase.

At the first order of perturbation theory, the amplitude for the two-particle decay of $\pi$ follows from the Lagrangian (2) upon shifting $\phi_2$ by its vacuum expectation value, $v/\sqrt{2}$. The relevant cubic terms read

$$-\sqrt{2}\lambda v (\phi'_2 \phi_2 \phi'_2 + \phi'_2 \phi_2 \phi'_2) .$$

These two terms generate three diagrams each, contributing to the given decay process, that differ just by a permutation of the lines. The result for the $S$-matrix element is

$$S = -(2\pi)^4 \delta^4(p - k - q)iM \frac{1}{(2\pi)^{3/2}} ,$$

where $E(p)$ is the energy of the Hamiltonian eigenstate $|n,p\rangle$. The summation index $n$ is discrete for one-particle intermediate states, and continuous for multi-particle states. Also, the states are assumed to be normalized as $\langle n,p|m,k\rangle = \delta_{mn}\delta^3(p-k)$. Note that we use the general Källén–Lehmann representation (22) rather than the covariant form usual in literature [21].

B. Kinematics

Our analysis of the two-particle decay is further complicated by the fact that the Lorentz invariance is broken by the chemical potential. We therefore have to evaluate the decay rate as a function of the momentum of the
decaying particle. After the common trick of squaring the $S$-matrix in finite volume and performing the trivial three-momentum integration we arrive at the formula for the decay rate,

$$\Gamma = \frac{1}{2} \frac{1}{4\pi^2} \int k^2 \delta^3(k) |M|^2 \delta(E_-(p) - E_-(k) - E_-(q)).$$

The factor $\frac{1}{2}$ arises from the fact that the two particles in the final state are identical. The $\delta$-function fixes the moduli of the momenta $k$ and $q$ as a function of the scattering angle $\theta$, which measures the deflection of $k$ from the direction of $p$. The $|q|$ is given by $|q| = \sqrt{p^2 + k^2 - 2|p||k|\cos \theta}$ and $|k|$ is then fixed by energy conservation. Due to the form of the dispersion, Eq. (9), the solution of the energy conservation condition is, however, rather involved, so we work it out analytically only in a special case (see Sec. IV C). In the general case, the decay rate is computed numerically.

Using the chain rule to differentiate $E_-(q)$ in the $\delta$-function, we arrive at the final formula for the decay rate,

$$\Gamma = \frac{1}{4\pi} \int_{0}^{\theta_m} \sin \theta \, d\theta \, k^2(\theta)$$

$$\times \left| \frac{M^2}{E'_-(q(\theta)) + E'_+(q(\theta)) \frac{|k(\theta)| - |p|\cos \theta}{|q(\theta)|} \right|. \quad (26)$$

where $\theta_m$ is the maximum angle to which $\pi$ can decay at a given energy. Fig. 3 displays the expression $-E_-(p) + E_-(k) + E_-(q)$ as a function of $|k|$, for $\mu = 2M$ and $|p| = M$. The curves are labeled by the value of the angle $\theta$ in degrees. The critical angle for the chosen set of parameters is $\theta_m \approx 26.3$ deg.

C. Decay rate

The phonon decay rate may be determined analytically in the limit $v \to 0$. Strictly speaking, the following calculation is justified only for $p^2 \gg \lambda v^2$. However, later the analytic result will be checked by a direct numerical integration of the full formula (26). For the time being, we simply set $v = 0$ everywhere except, of course, the overall factor $v$ in the amplitude $M$.

Eq. (26) shows that only the first term survives in Eq. (26) so that

$$M = 2\sqrt{2}\lambda v \frac{1}{\sqrt{2\epsilon_p \sqrt{2\epsilon_k} \sqrt{2\epsilon_q}}}. \quad (27)$$

Setting $\mu = M$ in Eq. (9) gives $E_\pm(p) = \epsilon_p \pm M$, hence $E'_-(p) = |p|/\epsilon_p$ and Eq. (26) reduces to

$$\Gamma = \frac{\lambda^2 v^2}{4\pi \epsilon_p} \int_{0}^{\pi/2} \sin \theta \, d\theta \times \left| \frac{k^2(\theta)}{|k(\theta)| (\epsilon_k(\theta) + \epsilon_q(\theta)) - |p|\epsilon_k(\theta)\cos \theta}. \quad (28)$$

With the simplified dispersion relation, the energy and momentum conservation may easily be solved to yield

$$|k| = \frac{2M |p| E_+(p) \cos \theta}{E'_+(p) - p^2 \cos^2 \theta} \, E_-(k) = \frac{2M p^2 \cos^2 \theta}{E'_+(p) - p^2 \cos^2 \theta}. \quad (29)$$

The angular integration in Eq. (28) is then elementary and we arrive at the final formula for the decay rate,

$$\Gamma = \frac{\lambda^2 v^2}{4\pi} \frac{|p|}{\epsilon_p (\epsilon_p + M)}. \quad (30)$$
Such a dependence of the decay rate on the momentum of the initial phonon might have been expected: The decay rate tends to zero at both small and large momenta, where the dispersion relation becomes linear. A comparison with the exact numerical integration of Eq. (26) is displayed in Fig. 4.

V. CONCLUSIONS

In this paper we studied spontaneous symmetry breaking by Bose–Einstein condensation in scalar $\phi^4$ theory with two charged scalar flavors. Our primary goal was to refine our previous results [14] by including radiative corrections.

Within the one-loop calculation we verified that the Bose–Einstein condensation sets on at the chemical potential equal to the renormalized scalar mass (at $\mu = 0$). This assertion, though perfectly physical, may seem surprising within the approximation used since, as is well known [20], in massless $\phi^4$ theory the one-loop correction draws the scalar field expectation value off the origin to a nonzero value. The spontaneous symmetry breaking thus achieved, however, eventually turns out to be an artifact of the one-loop effective potential and disappears upon the renormalization group improvement.

Fortunately, such a fake vacuum apparently does not occur in our case for the phase transition point is exactly where it should be. This gives us hope that the renormalization group improvement, however important for critical phenomena, should not change the conclusions of this paper qualitatively.

In Sec. [14] we determined the radiative corrections to the dispersion relations of the type-II Goldstone boson and its massive counterpart. The numerical results suggest that the dependence of the renormalization constants $Z_1$ and $Z_2$ on the chemical potential saturates at large $\mu$. Since in this region the two massive modes in the model become very heavy, the asymptotic behavior of the type-II Goldstone boson dispersion relation should be governed by a low-energy effective theory for the Goldstone bosons, presumably analogous to that for the nonrelativistic ferromagnet [15, 23].

Furthermore, we showed that the massive counterpart of the type-II Goldstone boson has gap $2\mu$, which receives no radiative corrections. This assertion has a simple physical interpretation. Recall that $\mu$ is, originally, a chemical potential associated with the $U(1)$ symmetry of the Lagrangian [2]. Once this symmetry is spontaneously broken, $\mu$ may be reinterpreted as a chemical potential of the unbroken $U(1)_Q$. Since $\phi_1$ and $\phi_1^*$ carry the charges $Q = \pm 1$, the corresponding excitations differ by $2\mu$ in energy.

In a sense, this is a remnant of the Goldstone theorem: It guarantees nonrenormalization of the Goldstone boson mass to all orders in the loop expansion. Once the chemical potential is switched on, the excitation annihilated by $\phi_1^*$ can no longer be massless, but the Goldstone theorem still assures that its gap does not renormalize.

In Sec. [14] we determined the two-particle decay rate of the superfluid phonon. Such a quantity as well as the very possibility of the phonon decay may be important for the transport phenomena in the Color-Flavor-Locked quark matter with a kaon condensate [22]. As shown in Refs. [24, 25, 26], the CFL phase with a meson condensate appears to be energetically preferred to the simple CFL phase, thus making our results of a possible relevance for the phenomenology of compact stars.

To summarize, in this paper we investigated spontaneous symmetry breaking within the linear sigma model at the one-loop level. We verified the saturation of the Nielsen–Chadha inequality [14] and hence also the counting rule proposed (and proved at the tree level) in our previous work [14]. We thus fulfilled one part of the program outlined in Ref. [14]. The other, more ambitious part is to extend the results achieved so far to other relativistic systems with finite chemical potential. Our recent work shows that at least the type-II Goldstone bosons may be described, to a large extent, in a model-independent way. These results will be reported elsewhere.

APPENDIX A: CUBIC AND QUARTIC COUPLING MATRICES

In this appendix the explicit form of the coupling matrices $T^{(i)}$ and $Q^{(i)}$ is given. The indices $\phi$ and $\phi^*$ refer to the upper and lower components of the Nambu dou-

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FIG. 4: Decay rate for the process $\pi \rightarrow \pi\pi$ as a function of momentum (solid lines). The numerical data were obtained with $\lambda = 1$. The curves are labeled by the dimensionless parameter $\mu/M - 1$. The dashed line corresponds to the analytic result, Eq. (29). The decay rates for different values of the chemical potential are scaled by the factor $\lambda_\nu^2$ so that their convergence to the analytic result is comparable.
The propagators (8) are represented by the following lines.

\[ T_{k,lφ}^{(i)} = \begin{pmatrix} -2λ(δ_{il}φ_k + δ_{kl}φ_i) & 0 \\ -2λ(δ_{il}φ_k' + δ_{kl}φ_i') & -2λ(δ_{il}φ_i + δ_{kl}φ_i) \end{pmatrix}, \]

\[ T_{klφφ}^{(i)} = \begin{pmatrix} -2λ(δ_{ik}φ_l' + δ_{kl}φ_i') & -2λ(δ_{il}φ_k + δ_{ik}φ_l) \\ 0 & -2λ(δ_{il}φ_k' + δ_{kl}φ_i') \end{pmatrix}, \]

\[ Q_{k,lφφ}^{(i)} = \begin{pmatrix} -2λ(δ_{j,k}δ_{il} + δ_{kl}δ_{ij}) & 0 \\ 0 & -2λ(δ_{ik}δ_{jl} + δ_{kl}δ_{ij}) \end{pmatrix}, \]

\[ Q_{k,lφφ}^{(i)} = \begin{pmatrix} -2λ(δ_{ik}δ_{jl} + δ_{jk}δ_{il}) & 0 \\ 0 & -2λ(δ_{ik}δ_{jl} + δ_{jk}δ_{il}) \end{pmatrix}, \]

We start with the interaction vertices. These all come from the \(-λ(φ^1φ)^2\) term in the Lagrangian. Upon shifting \(φ_2\) by \(v/√2\), the interaction yields the following vertices,

\[ -2iλ, \quad -4iλ, \quad -2iλ, \quad -4iλ. \]

One-particle irreducible contributions to the vacuum expectation value of the scalar field \(φ_2\), see Eq. (11):

\[ \text{We have explicitly indicated the symmetry factor } \frac{1}{2} \text{ at the last graph. Note that in our formalism all such factors are automatically generated by the functional differentiation of the effective action.} \]

Next is the diagrammatic representation of the propagators. The loop graphs are in one-to-one correspondence with the terms in the formulas (17), (19) and (20).
APPENDIX C: PROOF OF THE EXISTENCE OF GAPLESS POLES

Here we provide a detail proof of the existence of massless poles in the propagators $D_{11}$ and $D_{22}$. The key ingredients are the formulas (17), (19) and (20), as well as the gap equation (11).

1. Pole in $D_{11}$

We are going to prove that $D_{11}^{-1\phi\phi^\dagger}(0) = 0$. Eq. (17) together with the expressions (8) for the bare propagators give

$$D_{22}^{-1\phi\phi^\dagger}(0) = \mu^2 - (M^2 + \delta M^2) - (\lambda + \delta \lambda) v^2 - 4i\lambda \int \frac{d^4k}{(2\pi)^4} \left( \frac{1}{(k_0 + \mu)^2 - \epsilon_k^2 - \epsilon_{\phi}^2 - \lambda v^2} - 2i\lambda \int \frac{d^4k}{(2\pi)^4} \left[ \frac{1}{(k_0 + \mu)^2 - \epsilon_k^2 - \epsilon_{\phi}^2 - \lambda v^2} \right] \right),$$

where we abbreviate $\epsilon_k = \sqrt{k^2 + M^2}$. Next we expand the gap equation (11), using the explicit form of the propagators (8),

$$\mu^2 - (M^2 + \delta M^2) - (\lambda + \delta \lambda) v^2 = 2i\lambda \int \frac{d^4k}{(2\pi)^4} \left( \frac{1}{(k_0 + \mu)^2 - \epsilon_k^2 - \epsilon_{\phi}^2 - \lambda v^2} + 4i\lambda \int \frac{d^4k}{(2\pi)^4} \left[ \frac{1}{(k_0 + \mu)^2 - \epsilon_k^2 - \epsilon_{\phi}^2 - \lambda v^2} \right] \right),$$

This expression is used to substitute for $\mu^2 - (M^2 + \delta M^2) - (\lambda + \delta \lambda) v^2$ in Eq. (C1), which finally yields

$$D_{11}^{-1\phi\phi^\dagger}(0) = -2i\lambda \int \frac{d^4k}{(2\pi)^4} \left( \frac{1}{(k_0 + \mu)^2 - \epsilon_k^2 - \epsilon_{\phi}^2 - \lambda v^2} - \frac{(k_0 - \mu)^2 - \epsilon_k^2 - 2\lambda v^2}{(k_0^2 - E_+^2(k))[(k_0^2 - E_+^2(k)]} \right) \left( 1 - \frac{\lambda v^2}{(k_0 + \mu)^2 - \epsilon_k^2 - 2\lambda v^2} \right).$$

Using the identity

$$\det \Delta_2(k) = [k_0^2 - E_+^2(k)][(k_0^2 - E_-^2(k)] = [(k_0 + \mu)^2 - \epsilon_k^2 - 2\lambda v^2][(k_0 - \mu)^2 - \epsilon_k^2 - 2\lambda v^2] - (\lambda v^2)^2,$$

which follows from Eq. (8), it is now straightforward to show that the integrand in the last expression vanishes.

2. Pole in $D_{22}$

In this case we are going to prove Eq. (21). We shall use a more symbolic method, without referring to the explicit form of the bare propagators (8). Upon a comparison of Eqs. (19) and (20) we can see that several terms immediately cancel out. After a mild rearrangement we are to prove the identity

$$\mu^2 - (M^2 + \delta M^2) - (\lambda + \delta \lambda) v^2 = 2i\lambda \int \left( \frac{\Delta_{11}^{\phi\phi^\dagger} + 2\Delta_{22}^{\phi\phi^\dagger} - \Delta_{22}^{\phi\phi^\dagger}}{\det \Delta_{22}} \right) + 4i\lambda^2 v^2 \int \left( \frac{\Delta_{22}^{\phi\phi^\dagger} \Delta_{22}^{\phi\phi^\dagger} - \Delta_{22}^{\phi\phi^\dagger} \Delta_{22}^{\phi\phi^\dagger}}{\det \Delta_{22}} \right).$$

For the sake of legibility we dropped out the integration measure and arguments of the propagators. Also, we used of the fact that $\Delta_{22}^{\phi\phi^\dagger} = \Delta_{22}^{\phi\phi^\dagger}$. On the right hand side of Eq. (C2), we rewrite $\lambda v^2 \det \Delta_{22}$ as $\lambda v^2 / \det \Delta_{22}^{-1} = \Delta_{22}^{\phi\phi^\dagger}$ and immediately arrive at an expression identical to the right hand side of Eq. (11). Eq. (C2) is thus proved and hence also the existence of a massless pole in the propagator $D_{22}$.

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Strictly speaking, this is only true for the normal phase.

In the Bose–Einstein condensed phase, there is a finite wave function renormalization induced by the second term in the integral in Eq. (14). This wave function renormalization is, in fact, responsible for the renormalization of the Goldstone boson dispersion relations.

This is of course a standard textbook result — see, for instance, Eq. (11.76) in Ref. [20]. It should be noted, however, that we use a slightly different regularization than usual, namely the dimensional regularization in three spatial dimensions.