From quantum trajectories to classical orbits

T.A. Brun,
Physics Department, Queen Mary and Westfield College, University of London,
London E1 4NS, England

N. Gisin,
Group of Applied Physics, University of Geneva, 1211 Geneva 4, Switzerland

P.F. O’Mahony, M. Rigo
Mathematics Department, Royal Holloway College, University of London,
Egham, Surrey TW20 0EX, England
(April 1, 2022)

Abstract

Recently it has been shown that the evolution of open quantum systems may be “unraveled” into individual “trajectories,” providing powerful numerical and conceptual tools. In this letter we use quantum trajectories to study mesoscopic systems and their classical limit. We show that in this limit, Quantum Jump (QJ) trajectories approach a diffusive limit very similar to the Quantum State Diffusion (QSD) unraveling. The latter follows classical trajectories in the classical limit. Hence, both unravelings show the rise of classical orbits. This is true for both regular and chaotic systems (which exhibit strange attractors).

03.65.Bz, 03.65.Sq
I. INTRODUCTION

Quantum mechanics is nonlocal. Classical mechanics is local. Though it is widely believed that quantum mechanics is the more fundamental theory, attempts to describe classical phenomena by quantum equations are fraught with difficulties. Not only do the calculations become extremely cumbersome, but they are conceptually more difficult.

There is no particular consensus on what it means to cross the “quantum $\rightarrow$ classical” border. Many criteria have been suggested: rapid decay of macroscopic superpositions, localization in phase space, approach to coherent states, decoherence, near-determinism, positivity of the Wigner distribution, and non-violation of the Bell inequalities [1–5]. In fact, it is rarely necessary to choose; as a rule, all of these are satisfied for macroscopic systems. Both modern experiments and the emerging field of nanotechnology, however, are increasingly challenging this divide. As we probe the mesoscopic region, where both quantum and classical effects are important, it behooves us to have a better idea of what “classical” means. This is not only of theoretical interest; a better understanding should make it possible to produce more efficient numerical models of mesoscopic systems.

In this letter, we show how the use of quantum trajectories, both continuous and discontinuous, illuminates the intermediate scales where neither a purely quantum nor a purely classical description is practical. These models not only simplify computation (one of their major original motivations), but describe when and how quantum nonlocality disappears in the classical limit. We illustrate our results by plotting regular and chaotic quantum trajectories, and show the limit where they recover classical regular and chaotic orbits.

The Schrödinger equation is the basic dynamical law of nonrelativistic physics. However, strictly speaking, it applies only to the entire Universe as the only truly closed system. All other systems are open. It is well known that the environment is crucial for the emergence of classical features in a quantum system. The best-known example is the case of a measurement, leading to sharp values of some physical quantity. More generically, environments induce decoherence, which is closely related to the rise of classical properties [1–3]. Consequently, we shall concentrate on open quantum systems in the Markovian (time local) limit.

Markovian open quantum systems are usually described by a master equation:

$$\dot{\rho} = -i[H, \rho] + \sum_m \left( L^\dagger_m \rho L_m - \frac{1}{2} \{ L^\dagger_m L_m, \rho \} \right)$$  

where $\rho$ is the density matrix for the system, $H$ its Hamiltonian, and the linear operators $L_m$ describe the effects of the environment. However, equation (1) is not entirely satisfactory for our purpose. It describes only mean values, computed over both quantum and classical probabilities. Hence, density matrices by themselves do not tell us which features can be described classically and which require a quantum description [4].

Let us illustrates this distinction for the mean values over a density matrix $\rho$ of the position operator $q$ and its square $q^2$. These values give little idea of the actual degree of localization of the particle. A large value of the spread $(Tr(q^2 \rho) - Tr(q \rho)^2)^{\frac{1}{2}}$ could correspond either to delocalized particles (resulting from, e.g., the spreading of a single wavepacket) or to localized particles whose position is classically uncertain (resulting from, e.g., Brownian
motion). The evolution of localized particles can be efficiently computed with classical or semi-classical models, while delocalized particles necessarily require a less efficient but more complete quantum description.

In the latter case, generally, there is nothing which corresponds to classical phase space trajectories. This is one of the chief difficulties in characterizing quantum chaos in a way similar to that used in classical dynamics.

By unraveling the evolution of the density operator, one obtains, as we shall see, classical mixtures (i.e., classical probabilities) of quantum pure states (i.e., quantum probabilities). This allows one to distinguish quantum from classical and, at the same time, provides a powerful tool for practical computations.

II. QUANTUM STATE DIFFUSION AND QUANTUM JUMPS

In such an unraveling, one describes the system in terms of a normalized pure state $|\psi(t)\rangle$ which follows a stochastic “trajectory” in Hilbert space. By averaging the pure state projector $|\psi\rangle\langle\psi|$ over all possible trajectories with appropriate weights, one reproduces the density operator $\rho = M(|\psi\rangle\langle\psi|)$. This is analogous classically to replacing the Fokker-Planck equation for probability densities with a stochastic Langevin equation for single trajectories.

Unfortunately, unlike the case of classical Brownian motion, the unraveling of the master equation (1) is not unique. Thus, there is some ambiguity in how one separates classical and quantum uncertainties, related to the ambiguity in identifying density matrices with quantum ensembles. In this section we consider two well known unravelings of (1).

In quantum state diffusion (QSD), the (Itô) stochastic evolution equation for the normalized state vector $|\psi(t)\rangle$ reads:

$$|d\psi(t)\rangle = -iH|\psi(t)\rangle dt - \frac{1}{2} \sum_j (L_j^\dagger L_j - 2\langle L_j^\dagger\psi L_j + |\langle L_j\psi|^2|\psi(t)\rangle dt$$

$$+ \sum_j (L_j - \langle L_j\psi\rangle)|\psi(t)\rangle d\xi_j$$ (2)

where the “noises” $d\xi_j$ are complex-valued Wiener processes of zero mean $M(d\xi_j) = 0$ and correlations $M(d\xi_j d\xi_k) = \delta_{jk} dt$. This equation describes a continuous non-differentiable evolution similar to the familiar diffusive paths of a classical Brownian particle, but in Hilbert space instead of real space. QSD is the only continuous unraveling which satisfies the same symmetry properties as the master equation itself [1].

Our second example is the quantum jumps (QJ) unraveling, which is closely related to photon counting. However, it can be defined for any Lindblad master equation [8,9]. The stochastic increment for the wave-function is

$$|d\psi(t)\rangle = -iH|\psi(t)\rangle dt - \frac{1}{2} \sum_j (L_j^\dagger L_j - 2\langle L_j^\dagger\psi L_j + |\langle L_j\psi|^2|\psi(t)\rangle dt$$

$$+ \sum_j \left( \frac{L_j|\psi(t)\rangle}{\sqrt{\langle L_j^\dagger L_j\psi\rangle}} - |\psi(t)\rangle \right) dN_j$$ (3)

The discrete Poissonian noises $dN_j$ assume the values 0 or 1. Most of the time $dN_j = 0$ and the evolution is continuous and differentiable. However, whenever $dN_j = 1$ there is a “jump”
to the state $L_j |\psi(t)\rangle / \sqrt{\langle L_j^\dagger L_j \rangle_t}$. The $dN_j$ processes have mean values $M_{|\psi\rangle} (dN_j) = \langle L_j^\dagger L_j \rangle_{|\psi\rangle} dt$ and correlations $dN_j dN_k = \delta_{jk} dN_j$. This means, essentially, that jumps occur randomly with an average rate $\langle L_j^\dagger L_j \rangle_{|\psi\rangle}$.

Let us illustrate these two unravelings for a simple example: the damped harmonic oscillator at finite temperature. $H = \omega a^\dagger a$, $L_1 = \sqrt{\bar{n}\gamma} a^\dagger$ and $L_2 = \sqrt{\bar{n}+1}\gamma a$ where $\bar{n}$ is the thermal equilibrium mean photon number, $\bar{n} = \langle a^\dagger a \rangle_\rho$, and $\gamma$ is the inverse relaxation time. For QSD one can show that any initial states tends to a coherent state: $|\psi(t)\rangle \rightarrow |\alpha_t\rangle$, where $a|\alpha_t\rangle = \alpha_t|\alpha_t\rangle$ and $\alpha_t = (\langle q \rangle_\psi + i \langle p \rangle_\psi ) / \sqrt{2}$. Furthermore, the evolution of $\alpha_t$ is governed by a classical equation,

$$d\alpha_t = -i\omega \alpha_t dt - \frac{\gamma}{2} \alpha_t dt + \sqrt{\bar{n}\gamma} d\xi_t$$

Hence, for this example at least, the QSD equation fully describes how the environment localizes the quantum state down to a minimum Gaussian wavepacket, and how this wavepacket follows a classical trajectory.

In [7,10,11] it was argued that this is quite general, and not peculiar to the harmonic oscillator. In [12] the QSD equation was applied to the Kaotic Anharmonic Oscillator (KAOS) system; in this case localization takes place despite competition with the delocalizing nonlinear Hamiltonian. In general, when the system remains small (in the sense that it does not explore much of the phase space in units of $\hbar$), the QSD trajectory presents no definite structure. However, when the parameters are such that the system explores a larger portion of phase space, the quantum trajectories exhibit a clear structure which approaches the classical strange attractor.

Similar behavior has been demonstrated for other chaotic systems, including the weak link capacitor and quantum kicked rotor [4], and the forced damped Duffing oscillator [13]. Once again, as we approach the classical limit, the structure of the strange attractor begins to appear.

The case of QJ is quite different. No result analogous to that for QSD holds in the harmonic oscillator case. However, in [3] the same KAOS system was studied using the QJ equation, and again the classical strange attractor was observed whenever the system explored enough of phase space.

This result was very puzzling at the time. We can now show that this behavior is also generic for QJ. In particular, we shall see that whenever the system is far from the origin of phase space (i.e., the harmonic oscillator ground state) its dynamics become very similar to QSD.

### III. LOCALIZATION OF QJ

Consider a system in a state $|\psi(t)\rangle$ with a Hamiltonian $H$ and environment operators $L_1 = \sqrt{\gamma_1} a$ and $L_2 = \sqrt{\gamma_2} a^\dagger$. Define

$$|\phi(t)\rangle = D(-\alpha_t)|\psi(t)\rangle,$$

where $\alpha_t = \langle a \rangle_{\psi(t)}$ and $D(\alpha) = \exp\{\alpha a^\dagger - \alpha^* a\}$ is the displacement operator. $|\phi(t)\rangle$ is the state $|\psi(t)\rangle$ displaced such that the mean position and momentum vanish: $\langle \phi(t)|a|\phi(t)\rangle = 0$. 
Note that $|\alpha_t|$ measures the phase space “distance” of the state $|\psi(t)\rangle$ from the origin $|0\rangle$. Let $\Delta \alpha_t$ measure the width of the state $\psi_t$:

$$\Delta \alpha_t^2 \equiv \langle a^\dagger a \rangle_\psi - \langle a^\dagger \rangle_\psi \langle a \rangle_\psi = \langle \phi(t)|a^\dagger a|\phi(t)\rangle.$$  \hspace{1cm} (6)

Now consider the following limit. When the oscillator is well localized relative to its distance from the origin, so that $|\alpha_t| >> \Delta \alpha_t$, then the rate of jumps $\langle L^\dagger L \rangle_t \approx |\alpha_t|^2$ is large, while the size of the jumps are small: $\langle \psi(t)|(a^\dagger - a^\dagger_\ast)(a - \alpha_t)|\psi(t)\rangle = \Delta \alpha_t$, and similarly for $a^\dagger$. Hence, when the energy of the system is large relative to $\hbar$, the “jumpy” evolution approaches closer and closer to a diffusion process like QSD, describing localized wavepackets following classical trajectories.

Let us develop the QJ equation (3) to first order in $\Delta \alpha/|\alpha|$. To remove irrelevant phases, we use the 1-dimensional projector $P_t \equiv |\psi(t)\rangle\langle \psi(t)|$. From equation (3) one obtains:

$$dP_t = -i[H, P_t]dt + \sum_j \left( -\frac{1}{2}\{L^\dagger_j L_j, P_t\} + \langle L^\dagger_j L_j \rangle_\psi P_t \right) dt$$

$$+ \left( \frac{L_j P_j L_j^\dagger}{\langle L_j^\dagger L_j \rangle_\psi} - P_t \right) dN_j.$$ \hspace{1cm} (7)

Using $a|\psi\rangle = \alpha|\psi\rangle + D(\alpha)a|\phi\rangle$ and $a P_t a^\dagger = |\alpha_t|^2 P_t + \alpha^\ast_t (a - \alpha_t) P_t + O(\Delta \alpha^2/|\alpha|^2)$, one deduces:

$$\frac{a P_t a^\dagger}{\langle a^\dagger a \rangle_t} = P_t + \frac{a - \alpha_t}{\alpha_t} P_t + \frac{a^\dagger - \alpha^\ast_t}{\alpha^\ast_t} P_t + O\left( \frac{\Delta \alpha^2}{|\alpha|^2} \right),$$ \hspace{1cm} (8)

and similarly for $a^\dagger P_t a$. (These are our only environment operators.) Inserting this into (7) yields

$$dP_t \approx -i[H, P_t]dt + \sum_j \left( L_j P_j L_j^\dagger - \frac{1}{2}\{L^\dagger_j L_j, P_t\} \right) dt$$

$$+ (L_j - \langle L_j \rangle_\psi) P_t \frac{\langle L^\dagger_j \rangle}{\langle L_j \rangle} dW_j + P_t (L_j^\dagger - \langle L_j \rangle_\psi) \frac{\langle L_j \rangle}{\langle L_j \rangle} dW_j,$$ \hspace{1cm} (9)

where we have made the approximation

$$\frac{\langle L^\dagger_j \rangle}{\sqrt{\langle L_j^\dagger L_j \rangle}} \left( \frac{dN_j}{\sqrt{\langle L_j^\dagger L_j \rangle}} - \frac{\langle L^\dagger_j L_j \rangle dt}{\langle L_j \rangle} \right) \approx \frac{\langle L^\dagger_j \rangle}{\langle L_j \rangle} dW_j,$$ \hspace{1cm} (10)

with the $dW_j$ standard real Wiener processes ($dW_j dW_j = \delta_{jj} dt$). Hence,

$$|d\psi(t)\rangle \approx -iH|\psi(t)\rangle dt - \frac{1}{2} \sum_j (L^\dagger_j L_j - 2\langle L^\dagger_j \rangle_\psi L_j + |\langle L_j \rangle_\psi|^2)|\psi(t)\rangle dt$$

$$+ \sum_j (L_j - \langle L_j \rangle_\psi)|\psi(t)\rangle \frac{\langle L^\dagger_j \rangle}{\langle L_j \rangle} dW_j.$$ \hspace{1cm} (11)

5
Note that this equation (11) preserves the norm of $|\psi(t)\rangle$ and recovers (2) in the mean: $ho(t) = M(|\psi(t)\rangle\langle \psi(t)|)$ at all times. It is almost identical to the QSD equation (2), except that the noise is a real Wiener process multiplied by a phase which depends on the phase space position of the oscillator. This is the only choice of phase such that the state vector remains normalized in the diffusive limit. The QSD equation was chosen to be independent of the choice of phase. The solutions of equations (2) and (11) are very similar, as we shall see.

We can see this by going to the classical limit and plotting the trajectories for the QJ equation. We have done this for an extension of the thermal model presented in the previous section. Note how in this case the jumps occur very frequently as we move away from the origin of phase space. (See figure 1.) This also works for the chaotic cases already studied with QSD; in the QJ case as well, we can see the emergence of classical structure. (See figure 2.)

A similar derivation can be done for any choice of environment operators $L$ which are linear in $a$ and $a^\dagger$ (e.g., $x$ and $p$).

IV. CONCLUSION

We have seen how the use of stochastic evolution equations for pure states provides the ability to draw useful pictures of elementary quantum phenomena. These same equations provide powerful tools for practical numerical computations; in particular, one can exploit the existence of localized solutions to reduce the numerical difficulty of solving the equations, using the technique of moving bases [14–17] to produce more efficient computer simulations. Quantum unravelings have long suggested the possibility of separating quantum and classical uncertainties for open systems; however, the ambiguity in the choice of unraveling has prevented any conclusions from being drawn as to the exact meaning of this separation.

We now see that, as one approaches the classical limit, it should be possible to make this separation in a similar way for different unravelings. This suggests that, in a sense, the details don’t matter: there is a single classical limit towards which a broad class of different “quantum trajectory” techniques all tend. In treating physical systems in the mesoscopic regime, this should make it possible to determine unambiguously which characteristics may be given classical and which quantum descriptions. This, in turn, may contribute greatly to the numerical solution of practical problems.

We should also stress the limitations of this approach. In the first place, it depends on the “system/environment split,” which will always include a certain ambiguity. While the exact boundary between system and environment is not important for macroscopic systems, for a microscopic or mesoscopic system it can be crucial. Second, we have limited ourselves to the Markovian approximation. This is mainly because little is known about the more general case. We expect that similar considerations apply to non-Markovian systems.

We would like to thank Lajos Diósi, Francesca Mota-Furtado, Ian Percival and Rüdiger Schack for useful conversations. This research was funded in part by the UK EPSRC, the EU Human Capital and Mobility Programme, and the Swiss National Science Foundation.
REFERENCES

[1] E. Joos and H.D. Zeh, Z. Phys. B, 59, 223 (1985).
[2] W.H. Zurek, Physics Today, 36 (October 1991).
[3] R. Griffiths, J. Stat. Phys., 36, 219 (1984); R. Omnès, Rev. Mod. Phys., 64, 339 (1992); H.F. Dowker and J.J. Halliwell, Phys. Rev. D, 46, 1580 (1992); M. Gell-Mann and J.B. Hartle, Phys. Rev. D, 47, 3345 (1993).
[4] T.P. Spiller, J.F. Ralph, T.D. Clark, R.J. Prance and H. Prance, J. Low Temp. Phys., 101, 1037 (1995).
[5] M. Rigo and N. Gisin, Quantum and Semiclass. Optics 8(1), 255, (1996).
[6] M. Rigo, G. Alber, F. Mota-Furtado and P.F. O’Mahony, submitted to Phys. Rev. A.
[7] N. Gisin and I.C. Percival, J. Phys. A, 25, 5677 (1992); J. Phys. A, 26, 2233 (1993); J. Phys. A, 26, 2245 (1993).
[8] J. Dalibard, Y. Castin, and K. Mølmer, Phys. Rev. Lett., 68, 580 (1992); see also J. Opt. Soc. Am., 10, 524 (1993).
[9] H.J. Carmichael, An Open Systems Approach to Quantum Optics, Lecture Notes in Physics m18, Springer (Berlin 1993).
[10] I.C. Percival, J. Phys. A, 27, 1003 (1994).
[11] J.J. Halliwell and A. Zoupas, Phys. Rev. D, 52, 7294 (1995).
[12] T.P. Spiller and J.F. Ralph, Phys. Lett. A, 194, 235, (1994).
[13] T.A. Brun, J. Phys. A, 29, 2077 (1995).
[14] R. Schack, T. Brun and I.C. Percival, J. Phys. A, 28, 5401 (1995).
[15] T. Steimle, G. Alber and I.C. Percival, J. Phys. A, 28, L491 (1995).
[16] M. Holland, S. Marksteiner, P. Marte and P. Zoller, Phys. Rev. Lett., 76, 3683 (1996).
[17] R. Schack and T. Brun, quant-ph/9608004, submitted to Computer Physics Communications.

Figure 1. The harmonic oscillator with $\omega = 1$ at finite temperature, the equilibrium displaced by a constant force $\beta$; curves are plotted for the values $\beta = 1, 4, 10$ and average thermal excitation $\bar{n} = 0.2$. Each jump is marked; as the oscillator is displaced further from the origin, the jumps become frequent and small in effect, illustrating the transition from a “jumpy” trajectory to a diffusion process.

Figure 2. The Poincaré section of the forced damped Duffing oscillator in the chaotic regime. The value $\beta$ gives the scale of the system, with $\beta \to \infty$ the classical limit; this system classically exhibits dissipative chaos, and has already been investigated in this limit using QSD (see [13]). We keep $\hbar = 1$ constant. As the scale increases for $\beta = 1, 4, 10$, the structure of the strange attractor clearly emerges, showing the diffusive limit of quantum jumps. This is similar to the behavior observed for QSD. The classical result is included for comparison.
QJ evolutions for driving = 1, 4 and 10
\beta = 1

\beta = 4

\beta = 10

Classical limit (\beta \to \infty)