How to Classify Reflexive Gorenstein Cones

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ABSTRACT

Two of my collaborations with Max Kreuzer involved classification problems related to string vacua. In 1992 we found all 10,839 classes of polynomials that lead to Landau-Ginzburg models with c=9 (Klemm and Schimmrigk also did this); 7,555 of them are related to Calabi-Yau hypersurfaces. Later we found all 473,800,776 reflexive polytopes in four dimensions; these give rise to Calabi-Yau hypersurfaces in toric varieties. The missing piece – toric constructions that need not be hypersurfaces – are the reflexive Gorenstein cones introduced by Batyrev and Borisov. I explain what they are, how they define the data for Witten’s gauged linear sigma model, and how one can modify our classification ideas to apply to them. I also present results on the first and possibly most interesting step, the classification of certain basic weights systems, and discuss limitations to a complete classification.
1 Introduction

1.1 Landau–Ginzburg models

I first met Max in 1987, when he was in the last stages of his doctorate studies at the Institute for Theoretical Physics of the TU Vienna, and I was just starting mine. We were then both working in quantum field theory, and Max already displayed his well known capacity for compressing essential information; in particular, he had one piece of paper containing all the formulas one would ever need for computing certain Feynman diagrams. Max then went on to postdoctoral positions at Hanover and Santa Barbara. When he came back to Vienna, I had finished my thesis and was looking for something new to work on. Max, who had started to collaborate with Rolf Schimmrigk in Santa Barbara, invited me to join a project related to string compactifications. Since neither of us knew much string theory at the time (but both of us had math diplomas), our strategy was to isolate a mathematical nucleus from an important topic, namely orbifolds of N=2 superconformal field theories (SCFTs) of Landau–Ginzburg type that can be used for string compactifications. Such a model requires a potential that is a quasihomogeneous function $f(\phi_i)$ of the fields with an isolated singularity at the origin:

$$f(\lambda^q \phi_i) = \lambda f(\phi_i), \quad \frac{\partial f}{\partial \phi_i} = 0 \forall i \Rightarrow \phi_i = 0 \forall i.$$  (1)

This gives rise to a superconformal field theory whose anomaly $c$ is determined by the positive rational numbers $q_i$ (the ‘weights’) via $c/3 = \sum_{i=1}^{N} (1 - 2q_i)$. In a first paper with Rolf [1] we considered symmetries of some known models of that type, but then we turned to the classification problem. This had been treated for very simple cases in a book by Arnold et al. [2]. For realistic string compactifications one requires $c = 9$, i.e. $\sum_{i=1}^{N} (1 - 2q_i) = 3$. Well known examples include

$$f = \phi_1^5 + \ldots + \phi_5^5 \quad \text{and} \quad f = \phi_1^3 + \ldots + \phi_6^3.$$  (2)

At that time it was known that the cases $N = 4, 5$ were related to Calabi–Yau manifolds — the functions could be used to define hypersurfaces in weighted projective spaces — whereas the cases $N = 6, 7, 8, 9$ were not (at least, not directly); however, the precise form of the relationship between Landau–Ginzburg models and Calabi–Yaus was unclear. Max and I extended the approach of [2] and used the resulting algorithm [3] to find 10,838 sets of $q_i$ that fit the criteria. In the meantime Klemm and Schimmrigk had done a more thorough search of the literature which allowed them to attack the problem directly, and while we were in the last stages of writing up our results they published theirs [4] which contained precisely one model more.
By taking another look at some candidates that we had rejected we managed to find that model and arrived at the same set of 10,839 [5]. One thing that was noticeable from these results was the fact that mirror symmetry was not complete within this class of models.

1.2 The gauged linear sigma model

The question about the precise relationship between Landau–Ginzburg models and Calabi–Yaus was settled quite beautifully by Witten [6]. He introduced a (2,2)–supersymmetric gauged linear sigma model in two dimensions which contained chiral superfields $\Phi_i$ (with component fields $\phi_i$, $F_i$) and vector superfields $V_a$ (with components $\sigma_a$, $D_a$). The theory’s superpotential $W(\Phi)$ is invariant under $\Phi_i \to e^{-iQ_{i,a}\lambda^a}\Phi_i$, and there are real parameters $r_a$ coming from the Fayet–Iliopoulos terms. It turns out that the model determines the $D$- and $F$-terms as

$$D_a = -e^2_a(\sum_i Q_{i,a}|\phi_i|^2 - r_a), \quad F_i = \frac{\partial W}{\partial \phi_i},$$

and requires minimization of the potential

$$U = \sum_a \frac{1}{2e^2_a}D_a^2 + \sum_i |F_i|^2 + 2\sum_{i,a} |\sigma_a|^2|\phi_i|^2Q_{i,a}^2. \quad (4)$$

The behaviour of the theory depends crucially on the values of the $r_a$ as the following classic example demonstrates.

Example 1. Consider the case of just one vector field and six chiral superfields with charges $Q_0 = -5$, $Q_1 = \ldots = Q_5 = 1$, and a superpotential $W = \Phi_0P^5(\Phi_1, \ldots, \phi_5)$, where $P^5$ is a polynomial of degree 5 that is non-degenerate, i.e. obeys eq. (1). Then

$$D \sim r + 5|\phi_0|^2 - \sum_{i=1}^5 |\phi_i|^2, \quad |F_0|^2 = |P^5|^2, \quad \sum_{i=1}^5 |F_i|^2 = |\phi_0|^2 \sum_{i=1}^5 \left|\frac{\partial P^5}{\partial \phi_i}\right|^2,$$

and we can distinguish the following cases.

$r >> 0$: Then $D^2 \to \min$ implies $(\phi_1, \ldots, \phi_5) \neq (0, \ldots, 0)$, so by non-degeneracy of $P^5$ we need $\phi_0 = 0$ to minimize $\sum_{i=1}^5 |F_i|^2$. The ground state is located at

$$\{(\phi_1, \ldots, \phi_5) : \sum_{i=1}^5 |\phi_i|^2 = \sqrt{r}, P^5(\phi_i) = 0\}/U(1),$$

which is just the symplectic quotient description of a quintic hypersurface in $\mathbb{P}^4$, i.e. the standard Calabi–Yau example.
\( r << 0 \): Then \( D^2 \to \min \) requires \( \phi_0 \neq 0 \), implying \( \frac{\partial P_5}{\partial \phi_0} = 0 \) and therefore \( \phi_1 = \ldots = \phi_5 = 0 \). The \( U(1) \) gauge symmetry may be used to fix \( \phi_0 = \sqrt{-r/5} \), leaving a residual \( \mathbb{Z}_5 \) symmetry. The resulting model is just a \( \mathbb{Z}_5 \) orbifold of a Landau–Ginzburg model with potential \( P_5 \), i.e. one of the 10,839 models we had classified.

By introducing more than one gauge field and more than one analogue of \( \phi_0 \) one can easily build models that correspond to complete intersection Calabi–Yaus.

### 1.3 Toric constructions

Around that time Victor Batyrev [7] introduced a construction that was manifestly mirror symmetric in the following sense. Given a dual pair of lattices \( M \simeq \mathbb{Z}^d \), \( N = \text{Hom}(M, \mathbb{Z}) \) and their real extensions \( M_\mathbb{R} \simeq \mathbb{R}^d \), \( N_\mathbb{R} \simeq \mathbb{R}^d \), one defines a lattice polytope \( \Delta \subset M_\mathbb{R} \) as a polytope with vertices in \( M \), and a reflexive polytope as a lattice polytope \( \Delta \ni 0 \subset M_\mathbb{R} \) whose dual

\[
\Delta^* = \{ y \in N_\mathbb{R} : \langle y, x \rangle + 1 \geq 0 \quad \forall x \in \Delta \} \tag{5}
\]

is also a lattice polytope. To any triangulation of the surface of \( \Delta^* \) one can assign the toric variety \( \mathcal{V} \) that is determined by the corresponding fan, with homogeneous coordinates \( z_i \) that are in one to one correspondence with the nonzero lattice points \( y_i \) of \( \Delta^* \). Then every lattice point \( x_j \) of \( \Delta \) determines a monomial \( M_j = \prod_i z_i^{\langle y_i, x_j \rangle + 1} \), and the generic polynomial consisting of these monomials defines a Calabi–Yau hypersurface \( \mathcal{C} \subset \mathcal{V} \). The Hodge numbers of \( \mathcal{C} \) can be computed directly from the structure of \( \Delta \) and it turns out that the exchange \( (M, \Delta) \leftrightarrow (N, \Delta^*) \) effects precisely the flip of the Hodge diamond that is associated with mirror symmetry.

Borisov [8] generalized this construction to complete intersections \( \mathcal{C} \) in toric varieties. The main idea is to generalize the duality of eq. (5) to sets of polytopes \( \nabla_i \subset N_\mathbb{R} \), \( \Delta_j \subset M_\mathbb{R} \) for \( i, j \in \{1, \ldots, \text{codim} \mathcal{C} \} \) via

\[
\langle y, x \rangle + \delta_{ij} \geq 0 \quad \forall \ y \in \nabla_i, \ x \in \Delta_j; \tag{6}
\]

the fan for \( \mathcal{V} \) is given by a triangulation of \( \text{Conv}(\{\nabla_i\}) \) which turns out to be reflexive. At this point it is not clear how this is related to the gauged linear sigma model; in particular, fields like \( \phi_0 \) have no analogues in the toric coordinates, and Landau–Ginzburg models without Calabi–Yau interpretation are missing. This situation changed with two papers by Batyrev and Borisov who introduced reflexive Gorenstein cones [9] and a formula for the corresponding ‘stringy Hodge numbers’ [10] that displays exactly the type of combinatorial duality required by mirror symmetry. We postpone precise definitions to the next section and just mention here that the data of these cones can be used to define gauged linear sigma models.
1.4 The classification of reflexive polyhedra

Given all these developments it was clear that the answer to the ‘missing mirror problem’ lay in the realm of gauged linear sigma models and toric geometry. At that time reflexive polytopes were classified only in dimensions up to two (there are 16 reflexive polygons), and no algorithm for a classification in higher dimensions was known. In the autumn of 1995 Max in I were both in Vienna again and started to work on a general algorithm. We realized that the inversion of inclusion relations by duality, \( \Delta \subseteq \tilde{\Delta} \Leftrightarrow \Delta^* \supseteq \tilde{\Delta}^* \), has the following implication. If we find a set \( S \) of polytopes such that every reflexive polytope contains at least one member of \( S \), then every reflexive polytope must be contained in one of the duals of the members of \( S \). One can choose \( S \) as a set of polytopes \( \nabla_{\min} \) that are minimal in the sense that 0 is in the interior of \( \nabla_{\min} \) but not in the interior of the convex hull of any subset of the set of vertices of \( \nabla_{\min} \). We proved that every such \( \nabla_{\min} \) is either a simplex or the convex hull of lower dimensional simplices in a specific way [11]. In two dimensions the only possibilities are triangles and parallelograms; in 3d examples would include tetrahedra, octahedra, egyptian pyramids, etc. Any simplex involved in such a construction determines a weight system \( \{ q_i \} \) via \( 0 = \sum_i q_i V_i \) where the \( V_i \) are the vertices of the simplex. In order to play a role for the classification of reflexive polytopes a minimal polytope must satisfy \( 0 \in \text{int}(\text{conv}(\nabla_{\min}^* \cap M)) \). This condition restricts the admissible weight systems to a finite set; a procedure for obtaining them in arbitrary dimensions and the results in up to four dimensions were presented in [12]. Combining these with all possible combinatorial structures of minimal polytopes led to a complete list of minimal polytopes. By considering all subpolytopes of polytopes in the dual list of maximal objects we could find all 4,319 reflexive polytopes in three dimensions [13] and all 473,800,776 in four dimensions [14]. A thorough description of the complete algorithm in its final form can be found in [15].

This project was at the limit of what could be achieved with the computers that were available to us, so we required extremely efficient routines for handling lattice polytopes. After some polishing these routines were published as the package PALP [16] which is still being updated every now and then. An up-to-date manual of the current version can be found in this volume [17].

1.5 Structure of the paper

From everything discussed so far it is clear that the piece that is missing from our classification results is the case of toric constructions that need not correspond to Calabi–Yau hypersurfaces; in other words it is the Batyrev/Borisov
construction of reflexive Gorenstein cones. This is what the rest of this paper will be about. In the following section some of the essential definitions are given. In section 3 the classification problem is analysed in the spirit of [11, 12, 15]. Section 4 describes the classification of the relevant new weight systems, and section 5 discusses the further steps that could be taken.

2 Some definitions

A Gorenstein cone $σ$ is a cone in $M_\mathbb{R}$ with generators $V_1, \ldots, V_k \in M$ satisfying $\langle V_i, n_σ \rangle = 1$ for some element $n_σ \in N$. The support $\Delta_σ$ of $σ$ is the polytope $\text{Conv}(\{V_1, \ldots, V_k\})$ in the hyperplane $\langle x, n_σ \rangle = 1$ in $M_\mathbb{R}$.

A reflexive Gorenstein cone $σ$ is a Gorenstein cone whose dual $σ^\vee = \{y \in N_\mathbb{R} : \langle y, x \rangle \geq 0 \ \forall x \in σ\}$ is also Gorenstein, i.e. there exists an $m_σ \in M$ such that $\langle m_σ, W_i \rangle = 1$ for all generators $W_i$ of $σ^\vee$; the integer $r = \langle m_σ, n_σ \rangle$ is called the index of $σ$. If $σ$ is reflexive with index $r$ then $r\Delta_σ$ is a reflexive polytope [9].

A reflexive Gorenstein cone $σ$ of index $r$ is called split if $M \simeq \mathbb{Z}^k \oplus \tilde{M}$ and $σ$ is generated by $(e_1, \Delta_1), \ldots, (e_k, \Delta_k)$ where the $e_i$ form a basis of $\mathbb{Z}^k$ and the $\Delta_i$ are lattice polytopes in $M_\mathbb{R}$. This implies $k \leq r$; $σ$ is called completely split if $k = r$.

If both $σ$ and $σ^\vee$ are completely split (the latter with a basis $\{f_i\}$ for $\mathbb{Z}^r$ and polytopes $\nabla_i \subset \tilde{N}_\mathbb{R}$) it can be shown [18] that one can choose the bases $\{e_i\}$ and $\{f_i\}$ dual to each other. Then the duality of the cones is equivalent to eq. (6) which is the defining property of a nef-partition [8].

The cartesian product $σ_1 \times σ_2 \subset M_{1,\mathbb{R}} \oplus M_{2,\mathbb{R}}$ of two reflexive Gorenstein cones is again a reflexive Gorenstein cone, with dimension $d = d_1 + d_2$, index $r = r_1 + r_2$ and dual cone $σ_1^\vee \times σ_2^\vee \subset N_{1,\mathbb{R}} \oplus N_{2,\mathbb{R}}$.

Given a reflexive pair $(σ, σ^\vee)$ of Gorenstein cones and denoting by $\{x_j\}$ ($\{y_i\}$) the set of lattice points in the support of $σ$ ($σ^\vee$) and by $l_a : \sum_i Q_{i,a} y_i = 0$ a basis for the set of linear relations among the $y_i$, one can define a gauged linear sigma model by introducing

- a chiral superfield $Φ_i$ for every $y_i$,
- a gauge field $V_a$ for every $l_a$,
- the charges $Q_{i,a}$ as the coefficients of the $l_a$,
- a monomial $M_j = \prod_i Φ_i^{(y_i,x_j)}$ for every $x_j$.  

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3 Analysis of the classification problem

Let us fix \( n \in N \) and \( m \in M \) with \( \langle m, n \rangle = r \). The main ideas of [11, 12, 15] can be adapted as follows. We say that a Gorenstein cone \( \sigma \subset M_{\mathbb{R}} \) with \( n_{\sigma} = n \) has the IP (for ‘interior point’) property if \( m \) is in the interior of \( \sigma \). This is equivalent to \( m/r \) being in the interior of the support \( \Delta_{\sigma} \). With an analogous definition of the IP property for a cone \( \rho \subset N_{\mathbb{R}} \) we call \( \rho \) minimal if it has the IP property, but if no cone generated by a proper subset of the set of generators of \( \rho \) has it. The support \( \nabla_{\rho} \) of \( \rho \) is a minimal polytope in the sense of [11], characterized by the fact that the set \( \{ V_1, \ldots, V_{d+k} \} \) of its vertices is the union of \( k \geq 1 \) subsets (possibly overlapping) such that each of them determines a simplex (of lower dimension unless \( k = 1 \)) with the interior point (here, \( n/r \)) in its relative interior. This implies that \( n \) lies in the interior of the cone generated by the vertices \( V_i \) of such a simplex, so there exists a uniquely defined set of positive rational numbers \( q_i \) such that \( \sum q_i V_i = n \); acting with \( m \) on this equation we see that \( \sum q_i = r \). We call the \( q_i \) the weight system associated with the simplex; if \( k > 1 \) the collection of weight systems is referred to as a combined weight system or CWS.

In the case \( k = 1 \) where \( \rho \) itself is simplicial we have an identification of \( \rho \subset N_{\mathbb{R}} \) with \( \mathbb{R}_{\geq 0}^d \subset \mathbb{R}^d \) via \( V_i \leftrightarrow e_i \). The corresponding identification of dual spaces implies \( m \leftrightarrow (1, \ldots, 1) \). Up to now we have not specified the lattice \( N \). Given \( n \) and the generators \( V_i \), clearly the coarsest possible lattice \( N_{\text{coarsest}} \) is the one generated by these vectors, corresponding to the lattice in \( \mathbb{R}^d \) generated by \( e_1, \ldots, e_d \) and \( \mathbf{q} = (q_1, \ldots, q_d) \). The lattice \( M_{\text{finest}} \) dual to \( N_{\text{coarsest}} \) is then determined by the isomorphism

\[
M_{\text{finest}} \simeq \{(x_1, \ldots, x_d) : x_i \in \mathbb{Z}, \sum x_i q_i \in \mathbb{Z} \}.
\]

Let us now define \( \sigma(\rho) \) as the cone over

\[
\text{Conv}(\rho^\vee \cap \{ x \in M : \langle x, n \rangle = 1 \}),
\]

and \( \sigma(\mathbf{q}) \) as \( \sigma(\rho) \) for the case \( M = M_{\text{finest}} \). We say that \( \mathbf{q} \) has the IP property if \( \sigma(\mathbf{q}) \) has it, i.e. if \( (1, \ldots, 1) \) is interior to the cone over \( \{(x_1, \ldots, x_d) : x_i \in \mathbb{Z}_{\geq 0}, \sum x_i q_i = 1 \} \). This is equivalent to \( (1/r, \ldots, 1/r) \in \text{int}(\Delta_{\mathbf{q}}) \) with

\[
\Delta_{\mathbf{q}} = \text{Conv}(\{(x_1, \ldots, x_d) : x_i \in \mathbb{Z}_{\geq 0}, \sum x_i q_i = 1 \}).
\] (7)

We note that this does not rely on \( r \) being integer, allowing us to talk about IPWSs (‘IP weight systems’) for \( (d, r) \) with rational \( r \).

The cartesian product of cones has an analogue in the fact that if \( \mathbf{q}^{(1)}, \mathbf{q}^{(2)} \) are IPWSs for \( (d_1, r_1) \) and \( (d_2, r_2) \), respectively, then \( \mathbf{q} = (\mathbf{q}^{(1)}, \mathbf{q}^{(2)}) \) is an IPWS for \( (d_1 + d_2, r_1 + r_2) \). Note, however, that generically \( M_{\text{finest}}(\mathbf{q}) \) is
finer than $M_{\text{finest}}(q^{(1)}) \oplus M_{\text{finest}}(q^{(2)})$.

**Lemma 1.** Assume that $(q_1, \ldots, q_d)$ form a $(d, r)$–IPWS. Then

a) every $q_i$ obeys $q_i \leq 1$;

b) if $q_d = 1$ then $(q_1, \ldots, q_{d-1})$ form a $(d - 1, r - 1)$–IPWS;

c) if $q_d = 1/2$ then $(q_1, \ldots, q_{d-1})$ form a $(d - 1, r - 1/2)$–IPWS;

d) if $q_d - 1 + q_d = 1$ then $(q_1, \ldots, q_{d-2})$ form a $(d - 2, r - 1)$–IPWS, and $q_{d-1} = q_d = 1/2$ or $q_{d-1}$ and $q_d$ can be written as nonnegative integer linear combinations of $q_1, \ldots, q_{d-2}$.

**Proof.**

a) If $q_i > 1$ then $x_i = 0$ in $\Delta_q$, so $(1/r, \ldots, 1/r)$ is not in the interior.

b), c) Here $\Delta_q$ is the pyramid over $\Delta_{(q_1, \ldots, q_{d-1})}$ with apex the point $e_d$ or $2e_d$ which has the IP property if and only if $\Delta_{(q_1, \ldots, q_{d-1})}$ has it.

d) The case $q_{d-1} = q_d = 1/2$ can be reduced to case c), so let us assume $q_{d-1} = 1 - q_d > 1/2$, $q_d < 1/2$. If we denote by $\lambda$ the largest integer satisfying $\lambda q_d \leq 1$, then $\Delta_q$ is the convex hull of

$$\Delta_1 \cup \{e_{d-1} + e_d\} \cup \{e_{d-1} + \Delta_{q_d}\} \cup \bigcup_{\mu=1}^{\lambda} \{\mu e_d + \Delta_{1-\mu q_d}\}$$

where we have written $\Delta_y$ for Conv($\{(x_1, \ldots, x_{d-2}, 0, 0) : x_i \in \mathbb{Z}_{\geq 0}, x_1 q_1 + \ldots x_{d-2} q_{d-2} = y\}$). If $q_d$ could not be written as a nonnegative integer linear combination of $q_1, \ldots, q_{d-2}$ then $\Delta_{q_d}$ would be empty and every point of $\Delta_q$ would satisfy $x_d \geq x_{d-1}$. But then $(1/r, \ldots, 1/r)$ would lie at the boundary of $\Delta_q$, thus violating the IP assumption. Similarly, if all $\Delta_{1-\mu q_d}$ were empty we would have the same type of contradiction via $x_{d-1} \geq x_d$, so at least one of the $\Delta_{1-\mu q_d}$ must be non-empty, but then $\Delta_{1-q_d} \supseteq \Delta_{1-\mu q_d} + (\mu - 1) \Delta_{q_d}$ implies that $\Delta_{1-q_d}$ must also be non-empty, hence $1 - q_d = q_{d-1}$ is a nonnegative linear combination of $q_1, \ldots, q_{d-2}$.

Finally, let us assume that $(q_1, \ldots, q_{d-2})$ does not form a $(d - 2, r - 1)$–IPWS. Then there is some hyperplane through 0 and $(1/(r-1), \ldots, 1/(r-1))$ such that all of $\Delta_1$ lies on the same side of it: $a_1 x_1 + \ldots + a_d x_d = 0$ for all $x \in \Delta_1$, with $a_i$ satisfying $a_1 + \ldots + a_d = 0$. As the point $x = \ldots = x_d = 1/r$ lies in the same hyperplane, the IP property can hold for $\Delta_q$ only if there is at least one point with $a_1 x_1 + \ldots + a_d x_d < 0$. If this point pertains to $\Delta_{q_d}$, denote by $c$ the maximal value for which $a_1 x_1 + \ldots + a_d x_d = -c$. Then $\Delta_1 \supseteq \Delta_{1-\mu q_d} + \mu \Delta_{q_d}$ implies $a_1 x_1 + \ldots + a_d x_d = \mu c$ for all $x \in \Delta_{1-\mu q_d}$ and inspection of the components of $\Delta_q$ shows that they all obey

$$a_1 x_1 + \ldots + a_d x_d + cx_d - cx_d \geq 0,$$

thereby violating the IP condition for $\Delta_q$. Similarly, if one or more of the $\Delta_{1-\mu q_d}$ contain points with $a_1 x_1 + \ldots + a_d x_d < 0$, we choose $c$ to be the maximal value for which $a_1 x_1 + \ldots + a_d x_d = -\mu c$. Then $\Delta_1 \supseteq$
$\Delta_{1-q_d} + \mu \Delta_{q_d}$ implies $a_1 x_1 + \ldots + a_{d-2} x_{d-2} \geq c$ for all $x \in \Delta_{q_d}$ and all components of $\Delta_q$ obey

$$a_1 x_1 + \ldots + a_{d-2} x_{d-2} - cx_{d-1} + cx_d \geq 0,$$

again violating the IP condition for $\Delta_q$. 

Note, however: $q_d - 1 + q_d = 1$ does not imply that one of these two repeats one of the other weights as the IPWS (111126) shows (the notation $(n_1 \ldots n_d)[k]$ means $q_i = n_i/k$; $q_d > 1/2$ does not imply $1 - q_d \in \{q_1, \ldots, q_{d-1}\}$ as demonstrated by the IPWS (111114)[6].

Motivated by the lemma we shall refer to a weight system as basic if it contains no weights $q_i \in \{1/2, 1\}$ and no $q_i, q_j$ with $q_i + q_j = 1$. For such a weight system, any $x$ satisfying $\sum x_i q_i = 1$ must obey $\sum x_i > 2$.

What happens if $\rho$ is not simplicial, i.e. $\nabla \rho$ consists of more than one simplex? Then one embeds each of the $k > 1$ simplices $S_i$ into $\mathbb{R}^{d_i} \subset \mathbb{R}^{d+k-1}$, where $\mathbb{R}^{d_i}$ is the subspace spanned by the $e_j$ corresponding to the $d_i$ vertices of $S_i$; the interior points $q^{(i)}$ of the resulting simplicial cones are identified and on gets

$$N \simeq (\mathbb{Z}^{d+k-1} \oplus \mathbb{Z}^{q^{(1)}} \ldots \oplus \mathbb{Z}^{q^{(k)}})/\{a_{ij}(q^{(i)} - q^{(j)}) : a_{ij} \in \mathbb{Z}\}.$$ 

On the $M$ lattice side one now has $k$ equations of the type $\sum x_i q_i = 1$ in $\mathbb{Z}^{d+k-1}_{\geq 0}$. In particular, if the simplices all have distinct vertices, one starts with the cartesian product of cones in $N$ and projects along the differences of the $n_i, i \in \{1, \ldots, k\}$; in $M$ this results in the support $\Delta_\sigma$ being the product of the supports $\Delta_{q_i}, i \in \{1, \ldots, k\}$.

Given these preparations the following algorithm for the classification of reflexive Gorenstein cones in dimension $d$ with index $r$ emerges.

1. Find all basic IPWSs for $d' \in \{0, 1, \ldots, d\}$, $r' \in \{0, 1/2, 1, \ldots, r\}$ with $r - r' \leq d - d'$.

2. Extend the results of the first step by weights $1, 1/2$ and $(q, 1 - q)$ to get all IPWSs with index $r$ and dimension $d' \leq d$.

3. Determine all possible structures of minimal polytopes in dimension $d - 1$.

4. Combine the last two steps to get all $d$-dimensional minimal cones.

5. Determine all subcones on all sublattices of $M_{\text{finest}}$. 

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4 Classification of basic weight systems

The classification of basic IPWSs relies on the algorithm of [12]. In order to find all $q_i$’s satisfying $(1/r, \ldots, 1/r) \in \text{int}(\Delta_q)$ with $\Delta_q$ determined by eq. (7) one uses the fact that $q_i$ is determined by a set of linearly independent $x_i$’s satisfying $\sum x_i q_i = 1$. The classification proceeds by successively choosing such $x_i$, starting with $x_i^{(0)} = (1/r, \ldots, 1/r)$ and continuing with lattice points $x_i^{(1)}, \ldots, x_i^{(k)}$. Every choice of a new $x_i$ restricts the set of allowed $q_i$’s. Given $x_i^{(0)}, \ldots, x_i^{(k)}$ one can choose any $\tilde{q}$ compatible with them and check whether it has the IP property. A further $q \neq \tilde{q}$ can have the IP property only if $k+1 < d$ and $\Delta_q$ contains points on both sides of the hyperplane $\sum x_i \tilde{q}_i = 1$. In particular, such a $q$ must be compatible with one of the finitely many lattice points obeying $x_i \leq 0$ for all $i$ and $\sum x_i \tilde{q}_i < 1$. For every choice of $x_i^{(k+1)}$ among these one should then continue in the same way.

Example 2. $d = 2, r = 1/2$: $x_i^{(0)} = (2, 2)$ is compatible with $\tilde{q} = (1/4, 1/4)$, which has the IP property. Any further $q$ must allow at least one integer point with $x_1 + x_2 < 4$. Up to permutation of coordinates the only possibilities are $x_i^{(1)} = (3, 0)$ which leads to $q = (1/6, 1/3)$, and $x_i^{(1)} = (2, 1)$ which does not result in a positive weight system.

Lemma 2. If $d = 3r$ there is precisely one basic IPWS $(1/3, \ldots, 1/3)$, and for $d < 3r$ there is no basic IPWS.

Proof. Let us assume $d \leq 3r$. The point $(1/r, \ldots, 1/r)$ is compatible with $\tilde{q} = (r/d, \ldots, r/d)$, which has the IP property if $d = 3r$. Any other $q$ must admit at least one point $x$ such that $1 > \sum x_i \tilde{q}_i \geq (\sum x_i)/3$, i.e. $\sum x_i \leq 2$, which is not consistent with a basic IPWS. □

The cases covered neither by example 2 nor by lemma 2 require the use of a computer. PALP [16] contains an implementation of the algorithm of [12] that works reasonably well for $r \leq 1$ and $d \leq 5$. In order to get a program that is fast enough even for the case $r = 3, d = 8$ the corresponding routines had to be rewritten completely. In particular, the present implementation takes into account some of the symmetry coming from permutations of the coordinates. At every choice of $x_i^{(k)}$ in the recursive construction the program computes the vertices of the $(d-k-1)$-dimensional polytope in $q$-space that is determined by $q_i \geq 0$ and $\sum x_i^{(j)} q_i = 1$ for $j \in \{0, \ldots, k\}$. This can be done efficiently by using the $(d-k)$-dimensional polytope of the previous recursive step. $\tilde{q}$ is chosen as the average of the vertices of the $q$-space polytope.

This program was used to determine all basic IPWSs for $r \leq 3$ and $d \leq 9$. The complete lists can be found at the website [19]. The results are
Table 1: Numbers of basic IPWSs for given values of $r$ vs. $d_{CY} = d - 2r$

| $r \backslash d_{CY}$ | 0   | 1   | 2   | 3   |
|-----------------------|-----|-----|-----|-----|
| 0                     | 1   | 0   | 0   | 0   |
| 1/2                   | 0   | 2   | 48  | 97,036 |
| 1                     | 0   | 1   | 47  | 86,990 |
| 3/2                   | 0   | 0   | 28  | 168,107 |
| 2                     | 0   | 0   | 1   | 34,256 |
| 5/2                   | 0   | 0   | 0   | 6,066 |
| 3                     | 0   | 0   | 0   | 1   |

summarized in table 1 which shows the numbers of basic IPWSs for given index $r$ and $d - 2r$. Following [18] we call the latter ‘Calabi–Yau dimension’ $d_{CY}$; in the case of a complete splitting of the cone it is indeed the dimension of a complete intersection Calabi–Yau variety defined by the corresponding nef-partition, and for any cone leading to a sensible superconformal field theory it is $c/3$ where $c$ is the conformal anomaly.

The first entry is the empty IPWS for $d = r = 0$ which is required as a starting point for the construction of IPWSs containing only weights $1/2$ or $1$.

For $d_{CY} = 1$ there are the three basic weight systems $(1/4, 1/4), (1/6, 1/3)$ and $(1/3, 1/3, 1/3)$ from example 2 and lemma 2.

For $d_{CY} = 2$ there are 48 basic IPWSs with $r = 1/2$ and 47 with $r = 1$. Together they determine precisely the well known 95 weight systems for weighted $\mathbb{P}^4$’s that have K3 hypersurfaces [20, 21]; as weight systems for reflexive polytopes they were determined in [12]. In addition there are 28 basic IPWSs with $r = 3/2$ as well as the IPWS $(1/3, \ldots, 1/3)$ for $r = 2$. These 29 additional basic IPWSs are again identical with the ones relevant to Landau–Ginzburg type SCFTs as determined in [5]; each of them gives rise to a reflexive Gorenstein cone.

Finally, for $d_{CY} = 3$ there are the 184,026 weight systems with $r \leq 1$ relevant to Calabi–Yau hypersurfaces in toric varieties [12], which contain the 7,555 weight systems relevant to weighted projective spaces [4, 5] as a small subset. In addition there are 168, 107 + 34, 256 + 6, 066 + 1 = 208, 430 IPWSs with $r > 1$ which are new (except for 3, 284 Landau–Ginzburg weights [4, 5]). These weight systems can be the starting points for constructing codimension 2 and 3 Calabi–Yau threefolds in toric varieties as well as N=2 SCFTs with $c = 9$. While each of the 184,026 weight systems with $r \leq 1$ determines a reflexive polytope (hence a reflexive Gorenstein cone) as shown already in [12], among the Gorenstein cones determined by IPWSs with $r > 1$ only 112, 817 + 18, 962 + 1, 321 + 1 = 133, 101 of 208, 430 are reflexive; nevertheless
the others are relevant to the classification because they may contain reflexive subcones. For the reflexive cases the ‘stringy Hodge numbers’ of [10] as computed by PALP 2.1 [17] are also listed at the website [19]. The pairs of Hodge numbers all seem to be in the range that is well known from the earlier classifications.

5 Further steps of the algorithm

We shall now illustrate further steps of the algorithm presented at the end of section 3 for some of the smallest \((d, r)\)-pairs. The case of \(d_{\text{CY}} = 1\) corresponds to \((d, r) \in \{(3, 1), (5, 2), (7, 3), \ldots\}\).

5.1 \(d = 3, \ r = 1\)

1. According to the previous section, the relevant basic IPWSs are
   \(d' = 0, \ r' = 0: ()\);
   \(d' = 2, \ r' = \frac{1}{2}: (1/4, 1/4), (1/6, 1/3)\);
   \(d' = 3, \ r' = 1: (1/3, 1/3, 1/3)\).

2. These give rise to the \(r = 1\) IPWSs
   \(d' = 2: (1/2, 1/2)\);
   \(d' = 3: (1/3, 1/3, 1/3), (1/4, 1/4, 1/2), (1/6, 1/3, 1/2)\).

3. A 2–dimensional minimal polytope is a triangle or a rhomboid [11].

4. A minimal cone is determined by one of the weight systems
   \((1/3, 1/3, 1/3), (1/4, 1/4, 1/2), (1/6, 1/3, 1/2)\) or the CWS
   \((1/2, 1/2, 0, 0; 0, 0, 1/2, 1/2)\).

5. All reflexive subcones correspond to all reflexive subpolytopes of the corresponding support polytopes (3 triangles and a square); these are the well known 16 reflexive polygons.

5.2 \(d = 5, \ r = 2\)

1. The relevant basic IPWSs are
   \(d' = 0, \ r' = 0: ()\);
   \(d' = 2, \ r' = \frac{1}{2}: (1/4, 1/4), (1/6, 1/3)\);
   \(d' = 3, \ r' = \frac{1}{2}: 48\) basic IPWSs (cf. table 1);
   \(d' = 3, \ r' = 1: (1/3, 1/3, 1/3)\);
   \(d' = 4, \ r' = 1: 47\) basic IPWSs (cf. table 1).
2. These give rise to the $r = 2$ IPWSs
   \[d' = 2: (1, 1);\]
   \[d' = 3: (1/2, 1/2, 1);\]
   \[d' = 4: (1/2, 1/2, 1/2, 1/2), (1/3, 1/3, 1/3, 1), (1/4, 1/4, 1/2, 1), (1/6, 1/3, 1/2, 1);\]
   \[d' = 5: (1/4, 1/4, 1/2, 1/2, 1), (1/2, 1/2, 1/2, 1), (1/3, 1/3, 1/2, 1/2), (1/4, 1/4, 1/4, 1/2, 3/4), (1/6, 1/6, 1/3, 1/2, 5/6), (1/2, 1/2, 1/2, 1/2, 1/2), (1/3, 1/3, 1/2, 1/2, 1/2), (1/3, 1/3, 1/3, 1/3, 1/2, 1/2, 1/2), (1/3, 1/3, 1/3, 1/3, 1/2, 1/2, 1/4, 1/4, 1/2, 2/3), (1/3, 1/3, 1/3, 1/3, 1/3, 1/2, 1/2, 1/2, 2/3), 48 IPWSs of the type $(q_1, q_2, q_3, 1/2, 1)$, 47 IPWSs of the type $(q_1, q_2, q_3, q_4, 1)$. \]

3. The 4-dimensional minimal polytopes were classified in [11].

4. 5. These steps would require the use of a computer and have not yet been performed.

5.3 Other cases

The next case with $d_{CY} = 1$ is $d = 7, r = 3$. Here already the first step of the algorithm involves the 184,026 weight systems that were used in the classification of reflexive polytopes in four dimensions, as well as as the 28 basic IPWSs for $d' = 5, r' = 3/2$. In addition it requires an analysis of the possible structures of minimal polytopes in dimensions up to 6. This should not be too hard, but one should be aware of the fact that a description of a minimal polytope in terms of IP simplices need not be unique, as pointed out already in [11].

From what we have seen it is clear that for any fixed value of $d_{CY}$ the complete classification problem gets harder for rising $r$. In particular the lists for $d_{CY} = 3$ contain weight systems of the type $(q_1, \ldots, q_6, 1)$ for $r = 2$ and of the type $(q_1, \ldots, q_7, 1, 1)$ for $r = 3$. In the cases where classifications have been completed it turns out that there are more weight systems for reflexive $(d - 1)$–polytopes than there are reflexive $d$–polytopes, so while it is conceivable that $(d = 5, r = 1/2)$ and $(d = 6, r = 1)$ might be within the range of present computer power, $(d = 7, r = 1)$ is definitely impossible.

However, one would not expect all reflexive Gorenstein cones to lead to sensible SCFTs. For example, consider a cone $\sigma$ whose support is a height one pyramid, which is equivalent to $\sigma = \sigma_b \times \sigma_1$ where $\sigma_b$ is the cone over the base of the pyramid and $\sigma_1$ is the unique one dimensional cone; this case leads to trivial $E_{\text{string}}$ [18]. Now $\sigma(q)$ with $q = (\tilde{q}, 1)$ is of this type, and any of its subcones with the IP property is also of this type because all lattice points are in the base or the apex; hence the apex of the pyramid cannot be dropped without violating the IP property. Therefore one can
omit cones defined by single weight systems containing a weight of 1 from the list of cones serving as starting points for step (5) of the classification procedure. This implies that in addition to the basic weight systems of table 1 only \((d = 5, r = 1/2)\) and \((d = 6, r = 1)\) are required for a classification of relevant CWS for \(d_{\text{CY}} \leq 3, r \leq 3\). More generally one might use the fact that \(E_{\text{string}}\) is multiplicative under taking cartesian products of cones [22]; the case above is a special case of this since \(E_{\text{string}} = 0\) for the one dimensional cone (actually \(E_{\text{string}} = 0\) whenever \(d_{\text{CY}} < 0\) [22]).

A further reduction of the number of relevant (C)WS may come from the following consideration related to the gauged linear sigma model. If the superpotential contains quadratic terms then its derivatives \(F_i\) (cf. eq. 3) have linear terms that can be used to eliminate (‘integrate out’ in physicists’ language) fields by replacing them by the expressions determined by \(F_i = 0\). In this way one can argue for the following simplifications: a support polytope that is a height 2 pyramid over a height 2 pyramid can be reduced to the base, implying that a weight system \((q, 1/2, 1/2)\) is equivalent to just \((q)\); the product of two height one pyramids can be reduced to the product of the bases, implying the equivalence \((q, 1, 0, 0, 0, 0, 0, 0, 1) \sim (q, 0, 0, q)\) of CWS; a weight system \((q, q, 1 - q)\) should be equivalent to \((q)\). While these considerations certainly need to be put on a firmer footing, they seem to be confirmed ‘experimentally’ as the following lines of PALP output (version 2.1 is required, see [17]) indicate.

\[
\begin{align*}
4 & 1 1 1 1 0 0 2 0 0 0 0 1 1 M:105 8 N:7 6 H:2,86 [-168] \\
4 & 1 1 1 1 4 0 0 0 2 0 0 0 0 0 1 1 2 M:144 15 N:10 8 H:2 86 [-168] \\
3 & 1 1 1 0 0 0 3 0 0 0 1 1 1 M:100 9 N:7 6 H:2,83 [-162] \\
3 & 1 1 1 3 0 0 0 0 3 0 0 0 0 1 1 1 3 M:121 16 N:10 8 H:2 83 [-162] \\
6 & 1 1 1 1 2 3 3 M:181 7 N:7 7 H:1 103 [-204] \\
5 & 1 1 1 1 1 4 M:258 12 N:8 8 H:1 101 [-200] \\
\end{align*}
\]

However, one should not draw the conclusion that only basic weight systems are relevant: for example, the CWS \((1, 1, 0, 0, 0, 0, 0, 0, 1/2, 1/2, 1/2, 1/2)\) corresponds to the perfectly sensible case of two quadrics in \(\mathbb{P}^3\).

Finally let us discuss what can be done in the future. Extending the basic weight systems with \(q \geq 1/2\)-weights is completely straightforward but only interesting once we also combine several weight systems into CWS, which should not be too hard, either. The classification of \((d = 5, r = 1/2)\) and \((d = 6, r = 1)\) basic weight systems probably is the most interesting step that may still be achieved, in particular since these same weight systems also give rise to Calabi–Yau fourfolds. In principle this could be done with the existing algorithm. In practice it is very unlikely that it would produce results within a reasonable computation time. One would probably need to work very hard on further elimination of redundancies, on parallelizing the computation and
on obtaining the necessary computer power. This would require someone with great skills in understanding the problem, programming, and organizing resources; in other words, someone like Max Kreuzer.

References

[1] M. Kreuzer, R. Schimmrigk and H. Skarke, Abelian Landau-Ginzburg Orbifolds and Mirror Symmetry, Nucl. Phys. B472 (1992) 61, hep-th/9112047.

[2] V.I.Arnold, S.M.Gusein-Zade and A.N.Varchenko, Singularities of Differentiable Maps, Vol. I, Birkhäuser 1985.

[3] M. Kreuzer, H. Skarke, On the Classification of Quasihomogeneous Functions, Commun. Math. Phys. 150 (1992) 137, hep-th/9202039.

[4] A. Klemm, R. Schimmrigk, Landau–Ginzburg String Vacua, Nucl. Phys. B411 (1994) 559, hep-th/9204060.

[5] M. Kreuzer, H. Skarke, No mirror symmetry in Landau-Ginzburg spectra!, Nucl. Phys. B388 (1992) 113, hep-th/9205004.

[6] E. Witten, Phases of N=2 theories in two dimensions, Nucl. Phys. B403 (1993) 159, hep-th/9301042.

[7] V.V. Batyrev, Dual Polyhedra and Mirror Symmetry for Calabi–Yau Hypersurfaces in Toric Varieties, J. Alg. Geom. 3 (1994) 493, alg-geom/9310003.

[8] Lev Borisov, Towards the Mirror Symmetry for Calabi-Yau Complete intersections in Gorenstein Toric Fano Varieties, alg-geom/9310001.

[9] V.V. Batyrev, L.A. Borisov, Dual Cones and Mirror Symmetry for Generalized Calabi-Yau Manifolds, Mirror symmetry II (eds. B. Greene, S. T. Yau) 71-86, alg-geom/9402002.

[10] V.V. Batyrev, L.A. Borisov, Mirror Duality and string-theoretic Hodge numbers, Invent. Meth. 126 (1996) 183, alg-geom/9509009.

[11] M. Kreuzer, H. Skarke, On the Classification of Reflexive Polyhedra, Commun. Math. Phys. 185 (1997) 495, hep-th/9512204.

[12] H. Skarke, Weight Systems for Toric Calabi–Yau Varieties and Reflexivity of Newton Polyhedra, Mod. Phys. Lett. A11 (1996) 1637, alg-geom/9603007.

[13] M. Kreuzer, H. Skarke, Classification of Reflexive Polyhedra in Three Dimensions, Adv. Theor. Math. Phys. 2 (1998) 847, hep-th/9805190.

[14] M. Kreuzer, H. Skarke, Complete Classification of Reflexive Polyhedra in Four Dimensions, Adv. Theor. Math. Phys. 4 (2000) no. 6, hep-th/0002240.

[15] M. Kreuzer, H. Skarke, Reflexive polyhedra, weights and toric Calabi-Yau fibrations, Rev. Math. Phys. 14 (2002) 343, math.AG/0001106.
[16] M. Kreuzer, H. Skarke, *PALP: A Package for Analyzing Lattice Polytopes with Applications to Toric Geometry*, Comput.Phys.Commun. 157 (2004) 87, math.SC/0204356.

[17] A. Braun, J. Knapp, E. Scheidegger, H. Skarke and N.-O. Walliser, *PALP: a User Manual*, to be published in ‘Strings, Gauge Fields, and the Geometry Behind - The Legacy of Maximilian Kreuzer’ (World Scientific).

[18] V.V. Batyrev, B. Nill, *Combinatorial aspects of mirror symmetry*, math/0703456.

[19] M. Kreuzer, H. Skarke, 
http://hep.itp.tuwien.ac.at/~kreuzer/CY.html.

[20] M. Reid, *Canonical 3-folds*, Proc. Alg. Geom. Anger 1979, Sijthoff and Nordhoff, 273.

[21] A. R. Fletcher, *Working with complete intersections*, Bonn preprint MPI/89–35 (1989).

[22] B. Nill, J. Schepers, *Gorenstein polytopes and their stringy E-functions*, Math. Ann. (to appear), DOI: 10.1007/s00208-012-0792-2, arXiv:1005.5158.