2D Yang-Mills Theory
and
Topological Field Theory

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Contribution to the Proceedings of the International Congress of Mathematicians 1994. We review recent developments in the physics and mathematics of Yang-Mills theory in two dimensional spacetimes.

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1. Introduction

Two-dimensional Yang-Mills theory ($YM_2$) is often dismissed as a trivial system. In fact it is very rich mathematically and might be the source of some important lessons physically.

Mathematically $YM_2$ has served as a tool for the study of the topology of the moduli spaces of flat connections on surfaces [2, 26, 33, 40]. Moreover, recent work has shown that it contains much information about the topology of Hurwitz spaces - moduli spaces of coverings of surfaces by surfaces.

Physically, $YM_2$ is important because it is the first example of a nonabelian gauge theory which can be reformulated as a string theory. Such a reformulation offers one of the few ways in which analytic results could be obtained for strongly coupled gauge theories. Motivations for a string reformulation include experimental “approximate duality” of strong interaction amplitudes, weak coupling expansions [35], strong coupling expansions [38] and loop equations [30]. The evidence is suggestive but far from conclusive. In [20] D. Gross proposed the search for a string formulation of Yang-Mills theory using the exact results of $YM_4$. This program has enjoyed some success. A successful outcome for $YM_4$ would have profound consequences, both mathematical and physical.

In order to describe the string interpretation of $YM_2$ properly we will be led to a subject of broader significance: the construction of cohomological field theory (CohFT). This is reviewed in section 6.

2. Exact Solution of $YM_2$

Let $\Sigma_T$ be a closed 2-surface equipped with Euclidean metric. Let $G$ be a compact Lie group with Lie algebra $\mathfrak{g}$, $P \to \Sigma_T$ a principal $G$-bundle, $\mathcal{G}(P) = Aut(P)$, $\mathcal{A}(P)$ = the space of connections on $P$. The action for $YM_2$ is the $\mathcal{G}(P)$-invariant function on $\mathcal{A}(P)$ defined by: $I_{YM} = \frac{1}{4e^2} \int_{\Sigma_T} \text{Tr}(F \wedge *F)$; $F = dA + A^2$, $* = $ Hodge dual, $e^2 = $ gauge coupling. $I_{YM}$ is equivalent to a theory with action: $I(\phi, A) = -\frac{1}{2} \int_{\Sigma_T} i\text{Tr}(\phi F) + \frac{1}{2} e^2 \mu \text{Tr} \phi^2$; $\phi \in \Omega^0(M; \mathfrak{g})$, $\mu = *1$, and Tr is normalized as in [39]: $\frac{1}{8\pi^2} \text{Tr} F^2$ represents the fundamental class of $H^4(B\tilde{G}; \mathbb{Z})$, where $\tilde{G}$ is the universal cover of $G$. Various definitions of the quantum theory will differ by a renormalization ambiguity $\Delta I = \alpha_1 \int \frac{R}{4\pi} + \alpha_2 e^2 \int \mu$. Equivalence to the theory $I(\phi, A)$ shows that $YM_2$ is $\text{SDiff}(\Sigma_T)$ invariant (no gluons!) and that amplitudes are functions only of the topology of $\Sigma_T$ and $e^2 a$, where $a = \int \mu$. 

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The Hilbert space $\mathcal{H}_G$ is the space of class functions $L^2(G)^{Ad(G)}$ and has a natural basis given by unitary irreps: $\mathcal{H}_G = \oplus_R \mathbb{C} \otimes | R \rangle$. The Hamiltonian is essentially the quadratic Casimir: $C_2 + \alpha_2$. The amplitudes are nicely summarized using standard ideas from topological field theory. Let $\mathcal{S}$ be the tensor category of oriented surfaces with area: $\text{Obj}(\mathcal{S}) = \text{disjoint oriented circles}$, $\text{Mor}(\mathcal{S}) = \text{oriented cobordisms}$, then:

**Theorem 2.1:** $YM_2$ amplitudes provide a representation of the geometric category $\mathcal{S}$. The state associated to the cap of area $a$ is:

$$e^{\alpha_1} \sum_R \text{dimRe}^{-e^2 a (C_2(R) + \alpha_2)} | R \rangle$$

The morphism associated to the tube is

$$\sum_R e^{-e^2 a (C_2(R) + \alpha_2)} | R \rangle \langle R |$$

and the trinion with two ingoing and one outgoing circle is:

$$e^{-\alpha_1} \sum_R (\text{dimR})^{-1} e^{-e^2 a (C_2(R) + \alpha_2)} | R \rangle \langle R | \otimes | R \rangle$$

**Proof:** The heat kernel defines a renormalization-group invariant plaquette action ♠

**Corollary:** On a closed oriented surface $\Sigma_T$ of area $a$ and genus $p$ the partition function is

$$Z(e^2 a, p, G) = e^{\alpha_1 (2 - 2p)} \sum_R (\text{dimR})^{2 - 2p} e^{-e^2 a (C_2(R) + \alpha_2)} \tag{2.1}$$

These considerations go back to [29]. A clear exposition is in [39].

3. $YM_2$ and the moduli space of flat bundles

At $e^2 a = 0$ the action $I(\phi, A)$ defines a topological field theory “of Schwarz type” [8]. In [39] Witten applied $YM_2$ to the study of the topology of the space of flat $G$-connections on $\Sigma_T$: $\mathcal{M} \equiv \mathcal{M}(F = 0; \Sigma_T, P) = \{ A \in \mathcal{A}(P): F(A) = 0 \}/G(P)$.

Witten’s first result is that, for appropriate choice of $\alpha_1$, $Z$ computes the symplectic volume of $\mathcal{M}$ [33]:

$$Z(0, p, G) = \frac{1}{\#Z(G)} \int_{\mathcal{M}} \exp \omega = \frac{1}{\#Z(G)} \text{vol}(\mathcal{M}) \tag{3.1}$$

1 We take a topologically trivial $P$ for simplicity. $\mathcal{M}$ then has singularities, but the results extend to the case of twisted $P$, where $\mathcal{M}$ can be smooth [3].
where $Z(G)$ is the center, and $\omega$ is the symplectic form on $\mathcal{M}$ inherited from the 2-form on $\mathcal{A}$: $\omega(\delta A_1, \delta A_2) = \frac{1}{4\pi} \int_\Sigma \text{Tr}(\delta A_1 \wedge \delta A_2)$. The argument uses a careful application of Faddeev-Popov gauge fixing and the triviality of analytic torsion on oriented two-spheres. The result extends to the unorientable case, and the constant $\alpha_1$ can be evaluated by a direct computation of the Reidemeister torsion.

According to [39], (3.1) is the large $k$ limit of the Verlinde formula [36]. Let $S_{R'R'}(k)$ be the modular transformation matrix for the characters of integrable highest weight modules $R \in P^+_k$ of the affine Lie algebra $\mathfrak{g}^{(1)}$ under $\tau \rightarrow -1/\tau$ [24]. At $e^2 a = 0$ we have: $Z = \lim_{k \rightarrow \infty} e^{\alpha_1 \chi(\Sigma T)} \sum_{R \in P^+_k} \left( \frac{S_{00}}{S_{0R}(k)} \right)^{2p-2}$ where 0 denotes the basic representation. On the other hand, we may choose a complex structure $J$ on $\Sigma T$ inducing a holomorphic line bundle $L \rightarrow \mathcal{M}$ with $c_1(L) = \omega$, and apply the Verlinde formula to get: $\lim_{k \rightarrow \infty} k^{-n} \sum_{R \in P^+_k} \left( 1 + \frac{1}{S_{0R}(k)} \right)^{2p-2} = \lim_{k \rightarrow \infty} k^{-n} \dim H^0(\Sigma T; \mathcal{L}^{\otimes k}) = \lim_{k \rightarrow \infty} k^{-n} \langle e^{k\alpha_1(\mathcal{L})} \text{Td} \mathcal{M}, \mathcal{M} \rangle = \text{vol} \mathcal{M}$, where $n = \frac{1}{2} \dim \mathcal{M}$. Using [24] one recovers (3.1) with $e^{\alpha_1} = (2\pi)^{\dim G}/(\sqrt{|P/L| \text{vol} G}) = \left( \prod_{\alpha > 0} 2\pi (\alpha, \rho) \right)/\sqrt{|P/L|}$, $P$ is the weight lattice, $L$ the long root lattice, and $\rho$ the Weyl vector. The fact that the trinion is diagonal in the sum over representations is the large $k$ limit of Verlinde’s diagonalization of fusion rules.

Witten’s second result [40] gives the asymptotics of (2.1) for $e^2 a \rightarrow 0$ (set $a = 1$):

$$Z(e^2, p, G) \sim \frac{1}{\#Z(G)} \int_{\mathcal{M}} e^{\omega + e \Theta} + O(e^{-c/e^2})$$

(3.2)

$e^2 = 2\pi^2 \epsilon$, $\alpha_2 = (\rho, \rho)$, and $c$ is a constant. $\Theta \in H^4(\mathcal{M}; \mathbb{Q})$ is - roughly - the characteristic class obtained from $c_2(Q)$ where $Q \rightarrow \Sigma T \times \mathcal{M}^{irr}$, is the universal flat $G$-bundle. $\Theta$ is best thought of in terms of the $G(P)$-equivariant cohomology of $\{ A \in \mathcal{A}(P) : F(A) = 0 \}$.

In the Cartan model it is represented by $\frac{1}{8\pi^2} \text{Tr} \phi^2$. The “physical argument” for (3.2) proceeds by writing the path integral as:

$$Z(e^2, p, G)$$

$$= \frac{1}{\text{vol} G} \int d\phi dA d\psi \exp \left\{ \left[ \frac{i}{4\pi^2} \int_\Sigma \text{Tr}(\phi F - \frac{1}{2} \phi \wedge \psi) \right] + \left[ \epsilon \int_\Sigma \mu \frac{1}{8\pi^2} \text{Tr} \phi^2 \right] \right\}$$

(3.3)

\(^2\) first proved, using conformal field theoretic techniques, in [32].

\(^3\) Precise definitions are in [3].
where $\psi$ are the odd generators of the functions on the superspace $\Pi T A$ and $d A d \psi$ is the Berezin measure. This path integral is the $t \to 0$ limit of the partition function of a cohomological field theory whose $Q$-exact action is $\Delta I = tQ \int \mu \text{Tr} \psi^\alpha D_\alpha f$; $f = * F$, $Q$ is the Cartan model differential for $G$-equivariant cohomology of $A$. The partition function is $t$-independent and localizes on the classical solutions of Yang-Mills. A clever argument maps the theory at $t \to \infty$ to “$D = 2$ Donaldson theory” and establishes the result. From a mathematical perspective the first term in the action of (3.3) is the $G$-equivariant extension of the moment map on $A$, the integral over $A, \psi$ defines an equivariant differential form in $\Omega^*_G(A)$, and the integral over $\phi$ defines equivariant integration of such forms. When the argument is applied to finite dimensional integrals it leads to a rigorous result, namely the nonabelian localization theorem for equivariant integration of equivariant differential forms [40][23].

4. Large $N$ Limit: the Hilbert Space

The large $N$ limit of $YM_2$ amplitudes is defined by taking $N \to \infty$ asymptotics for gauge group $G = SU(N)$, holding $e^2aN \equiv \frac{1}{2} \lambda$ fixed. It is instructive to consider first the Hilbert space of the theory. In the large $N$ limit the statespace can be described by the conformal field theory (CFT) of free fermions [31,13,14]. Bosonization then provides the key to a geometrical reformulation in terms of coverings [18].

Nonrelativistic free fermions on $S^1$ enter the theory since class functions on $SU(N)$ can be mapped to totally antisymmetric functions on the maximal torus. The Slater determinants of $N$-body wavefunctions give the numerators of the Weyl character formula. The Fermi sea corresponds to the trivial representation with one-body states $\psi(\theta) = e^{in\theta}$ occupied for $| n | \leq \frac{1}{2}(N - 1)$. In the representation basis the Hilbert space is: $H_{SU(N)} = \bigoplus_{n \geq 0} \bigoplus_{Y \in Y_n^{(N)}} C^* | R(Y) \rangle$; $Y_n=$the set of Young diagrams with $n$ boxes, $Y_n^{(N)}$ is the subset of diagrams with $\leq N$ rows, $R(Y)$ is the $SU(N)$ representation corresponding to $Y \in Y_n^{(N)}$. The naive $N \to \infty$ limit of $H_{SU(N)}$ is $H^+ = \bigoplus_{n \geq 0} \bigoplus_{Y \in Y_n} C^* | Y \rangle$. The space $H^+$ is related to the state space of a $c=1$ CFT. Excitations of energy $\ll N$ around the Fermi level $n_F = \frac{1}{2}(N - 1)$ are described using the zero-charge sector $H_{bc}^{Q=0}$ of a “$\lambda = 1/2$ bc CFT” [16], where $Q = \oint_S bc$. The point of this reformulation is that one can apply the well-known bosonization theorem which relates the “representation basis” to the “conjugacy class basis.” Focusing on
one Fermi level we define fermionic oscillators \{b_n, c_n\} = \delta_{n+m,0}, a Heisenberg algebra [\alpha_n, \alpha_m] = n\delta_{n+m,0} related by \alpha_n = \sum b_{n-m}c_m, and compare, at level \(L_0 = n\), the fermionic basis: \(|Y(h_1,\ldots,h_s)) = c_{-h_1+1-\frac{1}{2}} \cdots c_{-h_s+s-\frac{1}{2}} b_{-v_1+1-\frac{1}{2}} \cdots b_{-v_s+s-\frac{1}{2}} | 0\rangle\)

where \(Y(h_1,\ldots,h_s) \in Y_n\) is a Young diagram with row lengths \(h_i\), with the bosonic basis \(|\vec{k}\rangle \equiv \prod_{j=1}^{\infty} (\alpha_{-j})^k_j | 0\rangle\) where \(\vec{k} = (k_1, k_2, \ldots)\) is a tuple of nonnegative integers, almost all 0. \(\vec{k}\) specifies a partition of \(n = \sum jk_j\) and a conjugacy class \(C(\vec{k}) \subset S_n\). The fermi/bose overlap is given by the characters of the symmetric group representation \(r(Y)\):

\[\langle \vec{k} | Y \rangle = \frac{1}{n!} \chi(r(Y))(C(\vec{k})).\]

When applying the above well-known technology to \(YM_2\) one finds a crucial subtlety \cite{L3}: \(H^+\) is not the appropriate limit for \(YM_2\). At \(N \leq \infty\) there are two Fermi levels \(n_F = \pm \frac{1}{2}(N - 1)\); excitations around these different levels are related to tensor products of \(N, \bar{N}\) representations, respectively. In the large \(N\) limit one must consider representations occurring in the decomposition of tensor products \(R \otimes \tilde{S}\) where \(R, S\) are associated with Young diagrams with \(n \ll N\) boxes, and \(\tilde{S}\) is the conjugate representation. That is, the correct limit for \(YM_2\) is \(H_{SU(N)}^+ \otimes H_{SU(N)}^-\). The two \(bc\) systems are naturally interpreted as left- and right-moving sectors of a \(c = 1\) CFT.

Gross and Taylor provided an elegant interpretation of the \(N \to \infty\) \(YM_2\) Hilbert space in terms of covering maps \cite{L3}. The one-body string Hilbert space is identified with the group algebra \(\mathbb{C}[\pi_1(S^1)]\). The state \(|\vec{k}\rangle \in H^+\) is identified with a state in the Fock space of strings defined by \(k_j\) \(j\)-fold oriented coverings \(S^1 \to S^1\). The structure of the statespace \(H^+ \otimes H^-\) has a natural geometrical interpretation in terms of string states \(|\vec{k}\rangle \otimes |\vec{l}\rangle\): \(\vec{k}, \vec{l}\) describe orientation preserving/reversing coverings.

5. \(1/N\) Expansion of Amplitudes

The \(1/N\) expansion of \(YM_2\) has a very interesting interpretation in terms of the mathematics of covering spaces of \(\Sigma_T\). Heuristically the worldsheet swept out by a \(j\)-fold cover \(S^1 \to S^1\) defines a \(j\)-fold cover of a cylinder by a cylinder. Moreover, the Hamiltonian \(H = C_2\) is not diagonal in the string basis. One finds a cubic interaction term describing the branched cover of a cylinder by a trinion \cite{L1} \cite{L3} \cite{L4}.

To state a more precise relation we define the chiral partition function to be:

\[Z^+(\lambda, p) \equiv \sum_{n \geq 0} \sum_{Y \in \mathcal{Y}_n} (\dim \mathcal{R}(Y))^2 2^p e^{-\lambda C_2(\mathcal{R}(Y))/2N}.\]

\(Z^+\) exists as an asymptotic expansion in \(1/N\). The \(1/N\) expansion is related to topological invariants of Hurwitz spaces. To define these let \(H(n, B, p, L)\) stand for the equivalence classes of connected
branched coverings of $\Sigma_T$ of degree $n$, branching number $B$ and $L$ branch points. If $C_L(\Sigma_T) \equiv \{(z_1, \ldots , z_L) \in \Sigma_T^n | z_i \in \Sigma_T, z_i \neq z_j\}/S_L$, then $H(n, B, p, L) \to C_L(\Sigma_T)$ is an unbranched cover with discrete fiber above $S \in C_L$ given by the equivalence classes of homomorphisms $\pi_1(\Sigma_T - S, y_0) \to S_n$, $y_0 \notin S$ \[17\]. Let $H(h, p) \equiv \Pi_{n,B \geq 0} \Pi_{L=0} H(n, B, p, L)$ where the union on $n, B$ is taken consistent with the Riemann-Hurwitz relation: $2h - 2 = n(2p - 2) + B$. We define the orbifold Euler characters of Hurwitz spaces by the formula $\chi_{\text{orb}}\{H(h, p)\} \equiv \sum'_{n,B \geq 0} \sum_{L=0}^B \chi(C_L(\Sigma_T)) \sum_{\pi_0(H(n, B, p, L))} |Aut_f|^{-1}$.

**Theorem 5.1** \([18] + [10]\). For $p > 1$:

$$Z^+(0, p)^{N \to \infty} \sim \exp\left[\sum_{h=0}^{\infty} \left(\frac{1}{N}\right)^{2h-2} \chi_{\text{orb}}\{H(h, p)\}\right].$$

**Proof:** In \([18]\) Gross and Taylor used Schur-Weyl reciprocity to write $SU(N)$ representation-theoretic objects in terms of symmetric groups. A key step was the introduction of an element of the group algebra $\Omega_n = \sum_{v \in S_n} \left(\frac{1}{N}\right)^{n-K_v} v \in \mathfrak{C}[S_n]$ where $K_v$ is the number of cycles in $v$. $\Omega_n$ is invertible for $N > n$ and satisfies: $(\dimr(Y))^m = \frac{(N^{\dimr(Y)})^m \chi_{\text{orb}}\{\Omega_n^m\}}{\dimr(Y)^m}$ for all integers $m$. Gross and Taylor showed that:

$$Z^+(\lambda, p) \sim \sum_{n,i,t,h=0}^{\infty} e^{-n\lambda/2} (-1)^i \frac{(\lambda)^{i+t+h}}{i!t!h!} \left(\frac{1}{N}\right)^{n(2p-2) + 2h + i + 2t} n^h (n^2 - n)^t 2^{t+h} \sum_{p_1, \ldots , p_t \in T_{2,n}} \sum_{s_1, t_1, \ldots , s_p, t_p \in S_n} \left[ \frac{1}{n!} \delta(p_1 \cdots p_t \Omega_n^{2-2p} \prod_{j=1}^p s_j t_j s_j^{-1} t_j^{-1}) \right]. \quad (5.1)$$

$T_{2,n} \subset S_n$ is the conjugacy class of transpositions, and $\delta$ acts on an element of the group algebra by evaluation at $1$. $\delta$ is nonvanishing when its argument defines a homomorphism $\psi : \pi_1(\Sigma_T - S, y_0) \to S_n$ for some subset $S \subset \Sigma_T$. One now uses Riemann’s theorem identifying equivalence classes of degree $n$ branched covers branched at $S \in C_L(\Sigma_T)$ with equivalence classes of homomorphisms $\psi : \pi_1(\Sigma_T - S, y_0) \to S_n$ to interpret \((5.1)\) as a sum over branched covers. Expanding the $\Omega^{-1}$ points to obtain the coefficients of the $1/N$ expansion gives the orbifold Euler characters of Hurwitz spaces $\spadesuit$.

The significance of this theorem is that it relates $YM_2$ to CohFT. To see this note that branched covers are related to holomorphic maps. Indeed, let $\tilde{\mathcal{M}}(\Sigma_w, \Sigma_T) = C^\infty(\Sigma_w, \Sigma_T) \times \text{Met}(\Sigma_w)$; $\text{Met}(\Sigma_w)$ is the space of smooth Riemannian metrics on a 2-surface $\Sigma_w$ of genus $h$. The moduli space of holomorphic maps is $\text{Hol}(\Sigma_w, \Sigma_T) \equiv \{(f, g) \in \tilde{\mathcal{M}}(\Sigma_w, \Sigma_T) : df \epsilon(g) = J df\} / (\text{Diff} + \kappa \text{Weyl}(\Sigma_w))$; $\epsilon(g)$ is the complex structure on $\Sigma_w$ inherited from $g$, $\text{Weyl}(\Sigma_w)$ is the group of local conformal rescalings acting on $\text{Met}(\Sigma_w)$. The definition of
orbifold Euler character above is thus natural since the action by Diff\(^+(\Sigma_w)\) on \(\tilde{\mathcal{M}}\) has fixed points at maps with automorphism: \(\chi_{\text{orb}}(H(h,p)) = \chi_{\text{orb}}(\text{Hol}(\Sigma_w, \Sigma_T))\). As we explain in the next section, CohFT partition functions are Euler characters of vector bundles over moduli spaces.

Theorem 5.1 has been extended in many directions to cover other correlation functions of \(YM_2\)\(^\dagger\)\(^\dagger\). The results are not yet complete but are all in harmony with the identification of \(YM_2\) as a CohFT. Wilson loop amplitudes are accounted for by Hurwitz spaces for coverings \(\Sigma_w \rightarrow \Sigma_T\) by manifolds with boundary.\(^\dagger\) A formula analogous to (5.1) for the full, nonchiral theory has been given in \([18]\). The proof is not as rigorous as one might wish, but we do not doubt the result. The analog of theorem 5.1 involves “coupled covers”\(^\dagger\). A coupled cover \(f : \Sigma_w \rightarrow \Sigma_T\) of Riemann surfaces is a map such that on the normalization of \(N(\Sigma_w) = N^+(\Sigma_w) \coprod N^-(\Sigma_w)\) along the double points \(\{Q_1, \ldots, Q_d\}\) of \(\Sigma_w\), \(N(f) = f^+ \coprod f^-\) where \(f^+ : N^+(\Sigma_w) \rightarrow \Sigma_T\) is holomorphic and \(f^- : N^-(\Sigma_w) \rightarrow \Sigma_T\) is antiholomorphic and \(\forall i\), ramification indices match: \(\text{Ram}(f^+, Q^+_i) = \text{Ram}(f^-, Q^-_i)\).

One may define a “coupled Hurwitz space” \(\mathcal{CH}(\Sigma_w, \Sigma_T)\) along the lines of the purely holomorphic theory. The \(1/N\) expansion of the partition function again generates the Euler characters of \(\mathcal{CH}(\Sigma_w, \Sigma_T)\), if coupled covers with ramified double points receive a weighting factor \(\prod_Q \text{Ram}(f^+, Q^+)\) in the calculation of the Euler characteristic\(^\dagger\). Put differently, the proper definition of “coupled Hurwitz space” involves a covering of the naive moduli space of coupled covers. This point has not been properly understood from the string viewpoint.

Finally, the results need to be extended to the case of nonzero area. When \(\lambda \neq 0\) the expansion (5.1) and its nonchiral analog have the form: \(Z^+(\lambda, p) = \sum_{h \geq 0} (1/N)^{2h-2} Z^+_{h,p}(\lambda) = \sum_{n \geq 0} e^{-n\lambda/2} Z^+_{n,h,p}(\lambda)\). For \(p > 1\), \(Z^+_{n,h,p}(\lambda)\) is polynomial in \(\lambda\), of degree at most \((2h-2) - n(2p-2) = B\). The string interpretation described below shows that these polynomials are related to intersection numbers in \(H(h, p)\). For \(p = 1\), \(Z^+_{h,1}(\lambda)\) are infinite sums which can be calculated using the relation to CFT described above\(^\dagger\). These functions may be expressed in terms of Eisenstein series and hence satisfy modular properties in \(\tau = i\lambda/(4\pi)\). For example: \(Z^+_{1,1} = e^{\lambda/48} \eta(i\lambda/(4\pi))\) (\(\eta\) is the Dedekind function)\(^\dagger\). The modularity in the coupling constant might be an example of the phenomenon of “S-duality” which is currently under intensive investigation.

\(^4\) The SDiff(\(\Sigma_T\)) invariance of \(YM_2\) implies that Wilson loop averages define infinitely many invariants of immersions \(S^1 \rightarrow \Sigma_T\).
in other theories. For the case of a sphere: $Z_{0,0}(\lambda)$ has finite radius of convergence. At $\lambda = \pi^2$ there is a third order phase transition (=discontinuity in the third derivative of the free energy) \[13\]. The existence of such large N phase transitions might present a serious obstacle to a string formulation of higher-dimensional Yang-Mills.

The $\lambda$-dependence of (5.1) has been interpreted geometrically in \[18\]. Contributions with $h, t > 0$ are related to degenerate $\Sigma_w$. In the framework of topological string theory the $h > 0$ contributions are probably related to the phenomenon of bubbling \[3\].

6. Cohomological Field Theory

CohFT is the study of intersection theory on moduli spaces using quantum field theory. Reviews include \[11,8,9,11\]. The following discussion is a summary of the point of view explained at length in \[11\]. In physics the moduli spaces are presented as $\mathcal{M} = \{ f \in \mathcal{C} : Df = 0 \}/G$ where $\mathcal{C}$ is a space of fields, $D$ is a differential operator, and $G$ is a group of local transformations. The action is an exact form in a model for the $G$-equivariant cohomology of a vector bundle over $\mathcal{C}$. The path integral localizes to the fixed points of the differential $Q$ of equivariant cohomology.

More precisely, the following construction of CohFT actions can be extracted from the literature \[11,15,16,18,25\]. We begin with the basic data:

1.) $\mathcal{E} \to \mathcal{C}$, a vector bundle over field space which is a sum of three factors: $\mathcal{E} = \Pi \mathcal{E}_{\text{loc}} \oplus \mathcal{E}_{\text{proj}} \oplus \Pi \mathcal{E}_{g.f.}$ (the $\Pi$ means the fiber is considered odd).

2.) $G$-invariant metrics on $\mathcal{C}$ and $\mathcal{E}$.

3.) a $G$-equivariant section $s : \mathcal{C} \to \mathcal{E}_{\text{loc}}$, a $G$-equivariant connection $\nabla s = ds + \theta s \in \Omega^1(\mathcal{C}; \mathcal{E}_{\text{loc}})$, and a $G$-nonequivariant section $\mathcal{F} : \mathcal{C} \to \mathcal{E}_{g.f.}$ whose zeros determine local cross-sections for $\mathcal{C} \to \mathcal{C}/G$.

The observables and action are best formulated using the “BRST model” of $G$-equivariant cohomology \[33,25,34\]. To any Lie algebra $\mathfrak{g}$ there is an associated differential graded Lie algebra (DGLA) $\mathfrak{g}[\theta] \equiv \mathfrak{g} \otimes \Lambda^* \theta; \theta^2 = 0, \deg \theta = -1, \deg \mathfrak{g} = 0, \partial \theta = 1$. Moreover, if $M$ is a superspace with a $\mathfrak{g}$-action then $\Omega^*(M)$ is a differential graded $\mathfrak{g}[\theta]$ module, with $X \in \mathfrak{g} \to \mathcal{L}_X, X \otimes \theta \to \iota_X$. In our case $\mathfrak{g} \to \text{Lie}(G)$ and $M$ is the total space of $\mathcal{E}$. The BRST complex is $\hat{\mathcal{E}} \equiv \Lambda^* \Sigma \text{(Lie}(G)[\theta])^* \otimes \Omega^*(\mathcal{E})$ where $\Sigma$ is the suspension, increasing grading by 1. The differential on the complex is $Q = (d_\mathcal{E} + \partial') + d_{C.E.}$, where $\partial'$ is dual to $\partial$ and $d_{C.E.}$ is the Chevalley-Eilenberg differential for the DGLA $\text{Lie}(G)[\theta]$. 

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acting on $\Omega^*({\mathcal{E}})$. Physical observables $\hat{O}_i$ are $Q$-cohomology classes of the “basic” ($\text{Lie}(G)$-relative) subcomplex and correspond to basic forms $O_i \in \Omega^*(C)$ which descend and restrict to cohomology classes $\omega_i \in H^*(M)$.

The Lagrangian is $I = Q\Psi$, the gauge fermion is a sum of three terms: $\Psi = \Psi_{\text{loc}} + \Psi_{\text{proj}} + \Psi_{\text{g.f.}}$ for localization, projection, and gauge-fixing, respectively. Denoting antighosts (= generators of the functions on the fibers of $E$) by $\rho + \theta\pi, \lambda + \theta\eta, \bar{c} + \theta\bar{\pi}$, of degrees $-1, -2, -1$, respectively, and taking, for definiteness, $E_{\text{g.f.}} |_f = E_{\text{proj}} |_f = \text{Lie}(G)$ we have:

$$
\begin{align*}
\Psi_{\text{loc}} &= -i (\rho, s) - (\rho, \theta \cdot \rho)\xi^*_\text{loc} + \frac{1}{2} (\rho, \pi)\xi^*_\text{loc} \\
\Psi_{\text{proj}} &= i (\lambda, C^t)_{\text{Lie}(G)} \\
\Psi_{\text{g.f.}} &= \langle \bar{c}, F[A] \rangle - \langle \bar{c}, \bar{\pi} \rangle_{\text{Lie}(G)}
\end{align*}
$$

(6.1)

where $C^t = (dR_f)^t \in \Omega^1(C; \text{Lie}(G))$, is obtained, using the metrics, from the right $G$ action through $f, R_f : G \to C$.

The main result of the theory is a path-integral representation for intersection numbers on $M$ as correlation functions in the cohomological field theory:

$$
\int_{\hat{E}} \hat{\mu}_{e^{-I}} \hat{O}_1 \cdots \hat{O}_k = \int_{M = Z(s)/G} \chi(\text{cok}(\Phi)/G) \wedge \omega_1 \wedge \cdots \wedge \omega_k \tag{6.2}
$$

where $\hat{\mu}$ is the Berezin measure on $\hat{E}$ and $\Phi = \nabla s \oplus C^t \in \Omega^1(C; V \oplus \text{Lie}(G))$ is Fredholm with $T_M \cong \text{ker} \Phi/G$. The argument for (6.2) may be sketched as follows. The equations $Df = 0$ define the vanishing locus of a cross-section $s(f) = Df \in \Gamma[\mathcal{E}_{\text{loc}} \to C]$. Using the data of a metric and connection $\nabla$ on a vector bundle $E$, one constructs the Mathai-Quillen representative $\Phi(E, \nabla)$ of the Thom class of $E$ [28]. This construction can be applied - formally - in infinite dimensions to write the Thom class for $\mathcal{E}_{\text{loc}}/G$. When pulled back by a section $s : C/G \to \mathcal{E}_{\text{loc}}/G$, $s^*(\Phi(\mathcal{E}_{\text{loc}}/G, \nabla))$ is Poincaré dual to the zero locus $Z(s) = \mathcal{Z}(s)/G$. The natural connection on $\mathcal{E}_{\text{loc}}/G$ is nonlocal in spacetime. In order to find a useful field-theoretic representation of the integral over $C/G$ one uses the “projection gauge fermion” $\Psi_{\text{proj}}$ to rewrite the expression as an integral over $C$. Finally, one must divide by the volume of the gauge group $\text{vol}G$, necessitating the introduction of $\Psi_{\text{g.f.}}$. The “extra” factor of $\chi(\text{cok}(\Phi)/G)$ follows from a general topological argument [12] or from a careful evaluation of the measure near the $Q$-fixedpoints $\bullet$.

Two remarks are in order: First, the factor $\chi(\text{cok}(\Phi)/G)$ is crucial in studies of mirror symmetry [1] [43] and is also crucial to the formulation of the $YM_2$ string. Second, the formula (6.2) ignores the (important) singularities in $M$. We conclude with four examples:
1. Donaldson Theory: Let $P \to M$ be a principal $G$-bundle over a 4-fold $M$. $\mathcal{C} = \mathcal{A}(P)$, $\mathcal{G} = \text{Aut}(P)$, $s(A) = F_+ \in \mathcal{E}_{\text{loc}} = \mathcal{A} \times \Omega^{2,+}(M; \mathbb{g})$, $\text{cok}(\mathbf{0}) = \{0\}$ (at irreducible connections). Observables are $\int_\gamma \Phi^*(\xi); \gamma \in H_*(M), \xi \in H^*(BG), \Phi : (P \times \mathcal{A}(P))/(G \times \mathcal{G}(P)) \to BG$ is the classifying map of the $G$-bundle $Q \to A/G \times M$ of Atiyah-Singer. The formula becomes Witten’s path integral representation of the Donaldson polynomials.

2. Topological $\sigma$ Model, $T\sigma(X)$: $X$ is a compact, almost Kähler manifold with almost complex structure $J$. $\mathcal{C} = \text{Map}(\Sigma_w, X)$. $\Sigma_w$ has complex structure $\epsilon$ and $s(f) = df + Jdf \epsilon \in \mathcal{E}_{\text{loc}, \epsilon} = \Gamma(T^*\Sigma \otimes f^*TX)$. Choosing a natural connection on $\mathcal{E}_{\text{loc}}$ one finds $\text{cok}(\mathbf{0}) \cong H^1(\Sigma, f^*(TX))$. Observables are the Gromov-Witten classes: $\int_\gamma \Phi^*(\xi); \gamma \in H_*(\Sigma_w), \xi \in H^*(X), \Phi : \Sigma_w \times C \to X$ is the universal map.

3. Topological String Theory, $TS(X)$: $X$ is a compact, Kähler manifold, $\mathcal{F} = (f, h) \in \mathcal{C} = \mathcal{M}(\Sigma_w, X) = \text{Map}(\Sigma_w, X) \times \text{Met}(\Sigma_w), \mathcal{G} = \text{Diff}^+(\Sigma_w), s(\mathcal{F}) = (R(h)+1, df + Jdf \epsilon(h)), \mathcal{E}_{\text{loc}} \subset L(\Sigma_w, X)$ is a vector bundle with fiber $\mathcal{E}_{\text{loc}} \oplus \text{Lie}(\mathcal{G})$. The section is: $s(\mathcal{F}, \mathcal{F}) = (s(\mathcal{F}), \mathcal{O}^\dagger(\mathcal{F}))$, and, by construction, the partition function is: $Z = \chi_{\text{orb}}(\text{Hol}(\Sigma_w, X))$, so $\mathcal{E}\sigma(X)$ is the string theory of (chiral) $YM_2$. The area dependence is obtained by perturbing the action by $\Delta I = \frac{1}{2} \int f^*k$ where $k$ is the Kähler class of $\Sigma_T$ (this is only partially proven). A similar construction reproduces the nonchiral amplitudes but introduces an action which is fourth-order in derivatives and requires further investigation. An alternative proposal for a string interpretation of $YM_2$ was made in [22]. This approach certainly deserves further study.

4. Euler $\sigma$ Model, $E\sigma(X)$: $X$ is compact, Kähler. If, in $TS(X)$, $\text{cok}(\mathbf{0}) = \{0\}, \mathcal{E}\sigma(X)$ computes the Euler character of $\text{Hol}(\Sigma_w, X)$. The fieldspace $\mathcal{C} \to \mathcal{M}(\Sigma_w, X)$ is a vector bundle with fiber $\mathcal{E}_{\text{loc}} \oplus \text{Lie}(\mathcal{G})$. The section is: $s(\mathcal{F}, \mathcal{F}) = (s(\mathcal{F}), \mathcal{O}^\dagger(\mathcal{F}))$, and, by construction, the partition function is: $Z = \chi_{\text{orb}}(\text{Hol}(\Sigma_w, X))$, so $\mathcal{E}\sigma(X)$ is the string theory of (chiral) $YM_2$. The area dependence is obtained by perturbing the action by $\Delta I = \frac{1}{2} \int f^*k$ where $k$ is the Kähler class of $\Sigma_T$ (this is only partially proven). A similar construction reproduces the nonchiral amplitudes but introduces an action which is fourth-order in derivatives and requires further investigation. An alternative proposal for a string interpretation of $YM_2$ was made in [22]. This approach certainly deserves further study.

7. Application and a Guess

The original motivation for the program of Gross was to find a string interpretation of $YM_4$. Have we made any progress towards this end? The answer is not clear at present. We offer one suggestion here in the form of a guess.

Combining (6.2) with the $1/N$ asymptotics of the $YM_2$ partition function we expect an intriguing relation between intersection theory on $\mathcal{M}(F = 0, \Sigma_T)$ for $G = SU(N)$ and

---

5 To make this statement rigorous one must (a.) take care of the singularities in $\mathcal{M}$ and (b.) ensure that the corrections $\sim O(e^{-2Nc'/\lambda})$ from (6.2) are not overwhelmed by the “entropy of unstable solutions” [21]. The absence of phase transitions as a function of $\lambda$ for $G > 1$ suggests that, for $G > 1$, these terms are indeed $\sim O(e^{-Nc'})$ for some constant $c'$. 

10
the moduli spaces of holomorphic maps $\Sigma_w \to \Sigma_T$:

$$\left\langle \exp \left[ \omega + \frac{\lambda}{4\pi^2N} \Theta \right] \right\rangle \overset{N \to \infty}{\sim} C_N \sum_{h \geq 0} \left( \frac{1}{N} \right)^{2h-2} \sum_{d \geq 0} e^{-\frac{d}{N}\lambda} P_d(\lambda) \chi_{\text{orb}}(\mathcal{H}(\Sigma_w, \Sigma_T, d))$$

(7.1)

$$C_N = Ne^{\alpha_1(2-2p)-\lambda\alpha_2/(2N)}$$, $\mathcal{H}(\Sigma_w, \Sigma_T, d)$ is the coupled Hurwitz space for maps of total degree $d$, and $P_d$ is a polynomial with $P_d(0) = 1$.

Now, the string theory of $YM_2$ does have a natural extension to four-dimensional target spaces: $I = I(\mathcal{E}\sigma(X))$ for $X$ a compact Kähler 4-fold. Let $e^\alpha$ be a basis of $H_2(X, \mathbb{Z})$ with Poincaré dual basis $k_\alpha$. The action may be perturbed by $\Delta I = t^\alpha \int f^*k_\alpha$. Defining degrees $d_\alpha$ by: $f(\Sigma) = \sum d_\alpha e^\alpha \in H_2(X, \mathbb{Z})$, the partition function of the theory should have the form $Z(\mathcal{E}\sigma(X)) \sim \sum_{h \geq 0} \kappa^{2h-2} \sum_{d_\alpha \geq 0} e^{-t^\alpha d_\alpha} P_d(t^\alpha) \chi(\mathcal{H}(\Sigma_w; X; d_\alpha))$, more or less by construction, where $\kappa$ is a string coupling constant and $P_d(t^\alpha)$ is a polynomial whose value at zero is one. Our guess is that a formula analogous to (7.1) holds in four dimensions, and that the asymptotic expansion of $Z(\mathcal{E}\sigma(X))$ in $\kappa$ is closely related to the large $N$ asymptotics of intersection numbers of the classes $O_2^{(2)}(e_\alpha) = \int_{e_\alpha} c_2(\mathcal{Q})$ on the moduli space of antiselfdual instantons on $X$: $\langle e^{r^\alpha O_2^{(2)}(e_\alpha)} \rangle_{\mathcal{M}_+(X; SU(N))}$ where $\kappa \sim 1/N$ and $r^\alpha$ are analytic functions of the $t^\alpha$.

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