Abstract. SPARQL is the W3C candidate recommendation query language for RDF. In this paper we address systematically the formal study of SPARQL, concentrating in its graph pattern facility. We consider for this study a fragment without literals and a simple version of filters which encompasses all the main issues yet is simple to formalize. We provide a compositional semantics, prove there are normal forms, prove complexity bounds, among others that the evaluation of SPARQL patterns is PSPACE-complete, compare our semantics to an alternative operational semantics, give simple and natural conditions when both semantics coincide and discuss optimizations procedures.

1 Introduction

The Resource Description Framework (RDF) [14] is a data model for representing information about World Wide Web resources. Jointly with its release in 1998 as Recommendation of the W3C, the natural problem of querying RDF data was raised. Since then, several designs and implementations of RDF query languages have been proposed (see [11] for a recent survey). In 2004 the RDF Data Access Working Group (part of the Semantic Web Activity) released a first public working draft of a query language for RDF, called SPARQL [16], whose specification does not include RDF Schema. Currently (April 2006) SPARQL is a W3C Candidate Recommendation.

Essentially, SPARQL is a graph-matching query language. Given a data source $D$, a query consists of a pattern which is matched against $D$, and the values obtained from this matching are processed to give the answer. The data source $D$ to be queried can be composed of multiple sources. A SPARQL query consists of three parts. The pattern matching part, which includes several interesting features of pattern matching of graphs, like optional parts, union of patterns, nesting, filtering (or restricting) values of possible matchings, and the possibility of choosing the data source to be matched by a pattern. The solution modifiers, which once the output of the pattern is ready (in the form of a table of values of variables), allows to modify these values applying classical operators like projection, distinct, order, limit, and offset. Finally, the output of a SPARQL query can be of different types: yes/no queries, selections of values of the variables which match the patterns, construction of new triples from these values, and descriptions about resources queries.
Although taken one by one the features of SPARQL are simple to describe and understand, it turns out that the combination of them makes SPARQL into a complex language, whose semantics is far from being understood. In fact, the semantics of SPARQL currently given in the document [16], as we show in this paper, does not cover all the complexities brought by the constructs involved in SPARQL, and includes ambiguities, gaps and features difficult to understand. The interpretations of the examples and the semantics of cases not covered in [16] are currently matter of long discussions in the W3C mailing lists.

The natural conclusion is that work on formalization of the semantics of SPARQL is needed. A formal approach to this subject is beneficial for several reasons, including to serve as a tool to identify and derive relations among the constructors, identify redundant and contradicting notions, and to study the complexity, expressiveness, and further natural database questions like rewriting and optimization. To the best of our knowledge, there is no work today addressing this formalization systematically. There are proposals addressing partial aspects of the semantics of some fragments of SPARQL. There are also works addressing formal issues of the semantics of query languages for RDF which can be of use for SPARQL. In fact, SPARQL shares several constructs with other proposals of query languages for RDF. In the related work section, we discuss these developments in more detail. None of these works, nevertheless, covers the problems posed by the core constructors of SPARQL from the syntactic, semantic, algorithmic and computational complexity point of view, which is the subject of this paper.

Contributions

An in depth analysis of the semantics benefits from abstracting some features, which although relevant, in a first stage tend to obscure the interplay of the basic constructors used in the language. One of our main goals was to isolate a core fragment of SPARQL simple enough to be the subject matter of a formal analysis, but which is expressive enough to capture the core complexities of the language. In this direction, we chose the graph pattern matching facility, which is additionally one of the most complex parts of the language. The fragment isolated consists of the grammar of patterns restricted to queries on one dataset (i.e. not considering the dataset graph pattern) over RDF without vocabulary of RDF Schema and literals. There are other two sources of abstractions which do not alter in essential ways SPARQL: we use set semantics as opposed to the bag semantics implied in the document of the W3C, and we avoid blanks in the syntax of patterns, because in our fragment can be replaced by variables [10, 5].

The contributions of this paper are:

- A streamlined version of the core fragment of SPARQL with precise Syntax and Semantics. A formal version of SPARQL helps clarifying cases where the current english-wording semantics gives little information, identify areas of problems and permits to propose solutions.
- We present a compositional semantics for patterns in SPARQL, prove that there is a notion of normal form for graph patterns in the fragment consid-
ered, and indicate optimization procedures and rules for the operators based on them.

- We give thorough analysis of the computational complexity of the fragment. Among other bounds, we prove that the complexity of evaluation of a general graph pattern in SPARQL is PSPACE-complete even if we not consider filter conditions.

- We formalize a natural procedural semantics which is implicitly used by developers. We compare these two semantics, the operational and the compositional mentioned above. We show that putting some slight and reasonable syntactic restrictions on the scope of variables, they coincide, thus isolating a natural fragment having a clear semantics and an efficient evaluation procedure.

1.1 Related Work

Works on the SPARQL semantics. A rich source on the intended semantics of the constructors of SPARQL are the discussions around W3C document [16], which is still in the stage of Candidate Recommendation. Nevertheless, systematic and comprehensive approaches to define the semantics are not present, and most of the discussion is based on use cases.

Cyganiak [4] presents a relational model of SPARQL. The author uses relational algebra operators (join, left outer join, projection, selection, etc.) to model SPARQL SELECT clauses. The central idea in [4] is to make a correspondence between SPARQL queries and relational algebra queries over a single relation $T(S, P, O)$. Indeed a translation system between SPARQL and SQL is outlined. The system needs extensive use of COALESCE and IS NULL operations to resemble SPARQL features. The relational algebra operators and their semantics in [4] are similar to our operators and have similar syntactic and semantic issues. With different motivations, but similar philosophy, Harris [12] presents an implementation of SPARQL queries in a relational database engine. He uses relational algebra operators similar to [4]. This line of work, which models the semantics of SPARQL based on the semantics of some relational operators, seems to be very influent in the decisions on the W3C semantics of SPARQL.

De Bruin et al. [5] address the definition of mapping for SPARQL from a logical point of view. It slightly differs from the definition in [16] on the issue of blank nodes. Although De Bruin et al.'s definition allows blank nodes in graph patterns, it is similar to our definition which does not allow blanks in patterns. In their approach, these blanks play the role of “non-distinguished” variables, that is, variables which are not presented in the answer.

Franconi and Tessaris [6], in an ongoing work on the semantics of SPARQL, formally define the solution for a basic graph pattern (an RDF graph with variables) as a set of partial functions. They also consider RDF datasets and several forms of RDF–entailment. Finally, they propose high level operators (Join, Optional, etc.) that take set of mappings and give set of mappings, but currently they do not have formal definitions for them, stating only their types, i.e., the domain and codomain.
Works on semantics of RDF query languages. There are several works on the semantics of RDF query languages which tangentially touch the issues addressed by SPARQL. Gutierrez et al. [10] discuss the basic issues of the semantics and complexity of a conjunctive query language for RDF with basic patterns which underlies the basic evaluation approach of SPARQL.

Haase et al. [11] present a comparison of functionalities of pre-SPARQL query languages, many of which served as inspiration for the constructs of SPARQL. There is, nevertheless, no formal semantics involved.

The idea of having an algebraic query language for RDF is not new. In fact, there are several proposals. Chen et al. [3] present a set of operators for manipulating RDF graphs, Frasincar et al. [7] study algebraic operators on the lines of the RQL query language, and Robertson [17] introduces an algebra of triadic relations for RDF. Although they evidence the power of having an algebraic approach to query RDF, the frameworks presented in each of these works makes not evident how to model with them the constructors of SPARQL.

Finally, Serfiotis et al. [19] study RDFS query fragments using a logical framework, presenting results on the classical database problems of containment and minimization of queries for a model of RDF/S. They concentrate on patterns using the RDF/S vocabulary of classes and properties in conjunctive queries, making the overlap with our fragment and approach almost empty.

Organization of the paper The rest of the paper is organized as follows. Section 2 presents a formalized algebraic syntax and a compositional semantics for SPARQL. Section 3 presents the complexity study of the fragment considered. Section 4 presents and in depth discussion of graph patterns not including the UNION operator. Finally, Section 5 presents some conclusions. Appendix A contains detailed proofs of all important results.

2 Syntax and Semantics of SPARQL

In this section, we give an algebraic formalization of the core fragment of SPARQL over simple RDF, that is, RDF without RDFS vocabulary and literal rules. This allows us to take a close look at the core components of the language and identify some of its fundamental properties (for details on RDF formalization see [10], or [15] for a complete reference including RDFS vocabulary).

Assume there are pairwise disjoint infinite sets $I$, $B$, and $L$ (IRIs, Blank nodes, and RDF literals, respectively). A triple $(v_1, v_2, v_3) \in (I \cup B) \times I \times (I \cup B \cup L)$ is called an RDF triple. In this tuple, $v_1$ is the subject, $v_2$ the predicate and $v_3$ the object. We denote by $T$ the union $I \cup B \cup L$. Assume additionally the existence of an infinite set $V$ of variables disjoint from the above sets.

Definition 1. An RDF graph [13] is a set of RDF triples. In our context, we refer to an RDF graph as an RDF dataset, or simply a dataset.
2.1 Syntax of SPARQL graph pattern expressions

In order to avoid ambiguities in the parsing, we present the syntax of SPARQL graph patterns in a more traditional algebraic way, using the binary operators UNION, AND and OPT, and FILTER. We fully parenthesize expressions and make explicit the left associativity of OPTIONAL and the precedence of AND over OPTIONAL implicit in [16].

A SPARQL graph pattern expression is defined recursively as follows:
1. A tuple from \((T \cup V) \times (I \cup V) \times (T \cup V)\) is a graph pattern (a triple pattern).
2. If \(P_1\) and \(P_2\) are graph patterns, then expressions \((P_1 \text{ AND } P_2)\), \((P_1 \text{ OPT } P_2)\), and \((P_1 \text{ UNION } P_2)\) are graph patterns.
3. If \(P\) is a graph pattern and \(R\) is a SPARQL built-in condition, then the expression \((P \text{ FILTER } R)\) is a graph pattern.

A SPARQL built-in condition is constructed using elements of the set \(V \cup T\) and constants, logical connectives \((\neg, \land, \lor)\), inequality symbols \((<, \leq, \geq, >)\), the equality symbol \(=\), unary predicates like \(\text{bound}, \text{isBlank}, \text{and isIRI}\), plus other features (see [16] for a complete list).

In this paper, we restrict to the fragment of filters where the built-in condition is a boolean combination of terms constructed by using \(=\) and \(\text{bound}\), that is:
1. If \(?X, ?Y \in V\) and \(c \in I \cup L\), then \(\text{bound}(?X), ?X = c\) and \(?X = ?Y\) are built-in conditions.
2. If \(R_1\) and \(R_2\) are built-in conditions, then \(\neg R_1\), \((R_1 \lor R_2)\) and \((R_1 \land R_2)\) are built-in conditions.

Additionally, we assume that for \((P \text{ FILTER } R)\) the condition \(\text{var}(R) \subseteq \text{var}(P)\) holds, where \(\text{var}(R)\) and \(\text{var}(P)\) are the sets of variables occurring in \(R\) and \(P\), respectively. Variables in \(R\) not occurring in \(P\) bring issues that are not computationally desirable. Consider the example of a built in condition \(R\) defined as \(?X = ?Y\) for two variables not occurring in \(P\). What should be the result of evaluating \((P \text{ FILTER } R)\)? We decide not to address this discussion here.

2.2 Semantics of SPARQL graph pattern expressions

To define the semantics of SPARQL graph pattern expressions, we need to introduce some terminology. A mapping \(\mu\) from \(V\) to \(T\) is a partial function \(\mu : V \rightarrow T\). Abusing notation, for a triple pattern \(t\) we denote by \(\mu(t)\) the triple obtained by replacing the variables in \(t\) according to \(\mu\). The domain of \(\mu\), \(\text{dom}(\mu)\), is the subset of \(V\) where \(\mu\) is defined. Two mappings \(\mu_1\) and \(\mu_2\) are compatible when for all \(x \in \text{dom}(\mu_1) \cap \text{dom}(\mu_2)\), it is the case that \(\mu_1(x) = \mu_2(x)\), i.e. when \(\mu_1 \cup \mu_2\) is also a mapping. Note that two mappings with disjoint domains are always compatible, and that the empty mapping (i.e. the mapping with empty domain) \(\emptyset\) is compatible with any other mapping. Let \(\Omega_1\) and \(\Omega_2\) be sets of mappings.

We define the join of, the union of and the difference between \(\Omega_1\) and \(\Omega_2\) as:
- \(\Omega_1 \Join \Omega_2 = \{\mu_1 \cup \mu_2 \mid \mu_1 \in \Omega_1, \mu_2 \in \Omega_2\\}\) are compatible mappings},
- \(\Omega_1 \cup \Omega_2 = \{\mu \mid \mu \in \Omega_1 \text{ or } \mu \in \Omega_2\}\),
- \(\Omega_1 \setminus \Omega_2 = \{\mu \in \Omega_1 \mid \text{ for all } \mu' \in \Omega_2, \mu \text{ and } \mu' \text{ are not compatible}\}\).
Based on the previous operators, we define the left outer-join as:

$$\Omega_1 \bowtie \Omega_2 = (\Omega_1 \bowtie \Omega_2) \cup (\Omega_1 \setminus \Omega_2).$$

We are ready to define the semantics of graph pattern expressions as a function $\llbracket \cdot \rrbracket_D$ which takes a pattern expression as input and returns a set of mappings. We follow the approach in [10] defining the semantics as the set of mappings that matches the dataset $D$. For simplicity, in this work we assume all datasets are already lean, i.e. (for simple RDF graphs) this means they do not have redundancies, which as is proved in [10], ensures that the property that for all patterns and datasets, if $D \equiv D'$ then $\llbracket P \rrbracket_D = \llbracket P \rrbracket_{D'}$. This issue is not discussed in [16].

**Definition 2.** Let $D$ be an RDF dataset over $T$, $t$ a triple pattern and $P_1, P_2$ graph patterns. Then the evaluation of a graph pattern over $D$, denoted by $\llbracket t \rrbracket_D$, is defined recursively as follows:

1. $\llbracket t \rrbracket_D = \{ \mu \mid \text{dom}(\mu) = \text{var}(t) \text{ and } \mu(t) \in D \}$, where var($t$) is the set of variables occurring in $t$.
2. $\llbracket (P_1 \ \text{AND} \ P_2) \rrbracket_D = \llbracket P_1 \rrbracket_D \bowtie \llbracket P_2 \rrbracket_D$.
3. $\llbracket (P_1 \ \text{OPT} \ P_2) \rrbracket_D = \llbracket P_1 \rrbracket_D \bowtie \llbracket P_2 \rrbracket_D$. 
4. $\llbracket (P_1 \ \text{UNION} \ P_2) \rrbracket_D = \llbracket P_1 \rrbracket_D \cup \llbracket P_2 \rrbracket_D$.

The semantics of FILTER expressions goes as follows. Given a mapping $\mu$ and a built-in condition $R$, we say that $\mu$ satisfies $R$, denoted by $\mu \models R$, if:

1. $R$ is bound(?$X$) and $?X \in \text{dom}(\mu)$;
2. $R$ is $?X = c$, $?X \in \text{dom}(\mu)$ and $\mu(?)X = c$;
3. $R$ is $?X = ?Y$, $?X \in \text{dom}(\mu)$, $?Y \in \text{dom}(\mu)$ and $\mu(?)X = \mu(?)Y$;
4. $R$ is $\neg R_1$, $R_1$ is a built-in condition, and it is not the case that $\mu \models R_1$;
5. $R$ is $(R_1 \ \text{OR} \ R_2)$, $R_1$ and $R_2$ are built-in conditions, and $\mu \models R_1$ or $\mu \models R_2$;
6. $R$ is $(R_1 \ \text{AND} \ R_2)$, $R_1$ and $R_2$ are built-in conditions, $\mu \models R_1$ and $\mu \models R_2$.

**Definition 3.** Given an RDF dataset $D$ and a FILTER expression $(P \ \text{FILTER} \ R)$, $\llbracket (P \ \text{FILTER} \ R) \rrbracket_D = \{ \mu \in \llbracket P \rrbracket_D \mid \mu \models R \}$.

**Example 1.** Consider the RDF dataset $D$:

$$D = \{(B_1, \text{name}, \text{paul}), (B_2, \text{name}, \text{john}), (B_3, \text{name}, \text{george}), (B_4, \text{name}, \text{gingo}), (B_4, \text{webPage}, \text{www.starr.edu}), (B_4, \text{phone}, 888-4537), (B_4, \text{email}, \text{ringo@acd.edu}), (B_2, \text{phone}, 777-3426), (B_3, \text{webPage}, \text{www.george.edu}), (B_4, \text{email}, \text{john@acd.edu})\}$$

The following are graph pattern expressions and their evaluations over $D$ according to the above semantics:

1. $P_1 = ((?A, \text{email}, ?E) \ \text{OPT} \ (?A, \text{webPage}, ?W))$. Then

$$\llbracket P_1 \rrbracket_D = \begin{array}{ccc}
\mu_1 & \mu_2 \\
B_2 & B_2 \\
\text{john@acd.edu} & \text{ringo@acd.edu} \\
\text{www.starr.edu} & \text{www.starr.edu}
\end{array}$$
(2) \( P_2 = ((?A, \text{name}, ?N) \text{OPT} (?A, \text{email}, ?E)) \text{OPT} (?A, \text{webPage}, ?W)). \)

Then

\[
\begin{array}{|c|c|c|}
\hline
A & ?N & ?E & ?W \\
\hline
\mu_1 & B_1 & paul & \\
\mu_2 & B_2 & john & john@acd.edu \\
\mu_3 & B_3 & george & www.george.edu \\
\mu_4 & B_4 & ringo & ringo@acd.edu \\
\hline
\end{array}
\]

(3) \( P_3 = ((?A, \text{name}, ?N) \text{OPT} ((?A, \text{email}, ?E) \text{OPT} (?A, \text{webPage}, ?W))). \)

Then

\[
\begin{array}{|c|c|c|}
\hline
A & ?N & ?E & ?W \\
\hline
\mu_1 & B_1 & paul & \\
\mu_2 & B_2 & john & john@acd.edu \\
\mu_3 & B_3 & george & www.george.edu \\
\mu_4 & B_4 & ringo & ringo@acd.edu \\
\hline
\end{array}
\]

Note the difference between \( [P_2]_D \) and \( [P_3]_D \). These two examples show that \( [(A \text{OPT} B) \text{OPT} C]_D \not= [(A \text{OPT} (B \text{OPT} C))_D \) in general.

(4) \( P_4 = ((?A, \text{name}, ?N) \text{AND} ((?A, \text{email}, ?E) \text{UNION} (?A, \text{webPage}, ?W))). \)

Then

\[
\begin{array}{|c|c|c|}
\hline
A & ?N & ?E & ?W \\
\hline
\mu_1 & B_2 & john & john@acd.edu \\
\mu_2 & B_3 & george & www.george.edu \\
\mu_4 & B_4 & ringo & www.starr.edu \\
\hline
\end{array}
\]

(5) \( P_5 = (((?A, \text{name}, ?N) \text{OPT} (?A, \text{phone}, ?P)) \text{FILTER } ?P = 777-3426). \)

Then

\[
\begin{array}{|c|c|}
\hline
A & ?N \\
\hline
\mu_1 & B_1 & paul & 777-3426 \\
\hline
\end{array}
\]

2.3 A simple normal form for graph patterns

We say that two graph pattern expressions \( P_1 \) and \( P_2 \) are equivalent, denoted by \( P_1 \equiv P_2 \), if \( [P_1]_D = [P_2]_D \) for every RDF dataset \( D \).

**Proposition 1.** Let \( P_1, P_2 \) and \( P_3 \) be graph pattern expressions and \( R \) a built-in condition. Then:

1. \( \text{AND and UNION are associative and commutative}. \)
2. \( (P_1 \text{ AND } (P_2 \text{ UNION } P_3)) \equiv ((P_1 \text{ AND } P_2) \text{ UNION } (P_1 \text{ AND } P_3)). \)
3. \( (P_1 \text{ OPT } (P_2 \text{ UNION } P_3)) \equiv ((P_1 \text{ OPT } P_2) \text{ UNION } (P_1 \text{ OPT } P_3)). \)
4. \( ((P_1 \text{ UNION } P_2) \text{ OPT } P_3) \equiv ((P_1 \text{ OPT } P_3) \text{ UNION } (P_2 \text{ OPT } P_3)). \)
5. \( ((P_1 \text{ UNION } P_2) \text{ FILTER } R) \equiv ((P_1 \text{ FILTER } R) \text{ UNION } (P_2 \text{ FILTER } R)). \)

The application of the above equivalences permits to translate any graph pattern into an equivalent one of the form:

\[
P_1 \text{ UNION } P_2 \text{ UNION } P_3 \text{ UNION } \ldots \text{ UNION } P_n,
\]

where each \( P_i \) (\( 1 \leq i \leq n \)) is a UNION-free expression. In Section 4, we study UNION-free graph pattern expressions.
3 Complexity of Evaluating Graph Pattern Expressions

A fundamental issue in any query language is the complexity of query evaluation and, in particular, what is the influence of each component of the language in this complexity. In this section, we address these issues for graph pattern expressions.

As it is customary when studying the complexity of the evaluation problem for a query language, we consider its associated decision problem. We denote this problem by EVALUATION and we define it as follows:

**INPUT**: An RDF dataset $D$, a graph pattern $P$ and a mapping $\mu$.

**QUESTION**: Is $\mu \in [[P]]_D$?

We start this study by considering the fragment consisting of graph pattern expressions constructed by using only AND and FILTER operators. This simple fragment is interesting as it does not use the two most complicated operators in SPARQL, namely UNION and OPT. Given an RDF dataset $D$, a graph pattern $P$ in this fragment and a mapping $\mu$, it is possible to efficiently check whether $\mu \in [[P]]_D$ by using the following algorithm. First, for each triple $t$ in $P$, verify whether $\mu(t) \in D$. If this is not the case, then return $false$. Otherwise, by using a bottom-up approach, verify whether the expression generated by instantiating the variables in $P$ according to $\mu$ satisfies the FILTER conditions in $P$. If this is the case, then return $true$, else return $false$. Thus, we conclude that:

**Theorem 1.** EVALUATION can be solved in time $O(|P| \cdot |D|)$ for graph pattern expressions constructed by using only AND and FILTER operators.

We continue this study by adding to the above fragment the UNION operator. It is important to notice that the inclusion of UNION in SPARQL is one of the most controversial issues in the definition of this language. In fact, in the W3C candidate recommendation for SPARQL [16], one can read the following: “The working group decided on this design and closed the disjunction issue without reaching consensus. The objection was that adding UNION would complicate implementation and discourage adoption”. In the following theorem, we show that indeed the inclusion of UNION operator makes the evaluation problem for SPARQL considerably harder:

**Theorem 2.** EVALUATION is NP-complete for graph pattern expressions constructed by using only AND, FILTER and UNION operators.

We conclude this study by adding to the above fragments the OPT operator. This operator is probably the most complicated in graph pattern expressions and, definitively, the most difficult to define. The following theorem shows that the evaluation problem becomes even harder if we include the OPT operator:

**Theorem 3.** EVALUATION is PSPACE-complete for graph pattern expressions.

It is worth mentioning that in the proof of Theorem 3, we actually show that EVALUATION remains PSPACE-complete if we consider expressions without FILTER conditions, showing that the main source of complexity in SPARQL comes from the combination of UNION and OPT operators.
When verifying whether $\mu \in \llbracket P \rrbracket_D$, it is natural to assume that the size of $P$ is considerably smaller than the size of $D$. This assumption is very common when studying the complexity of a query language. In fact, it is named data-complexity in the database literature [20] and it is defined as the complexity of the evaluation problem for a fixed query. More precisely, for the case of SPARQL, given a graph pattern expression $P$, the evaluation problem for $P$, denoted by $\text{Evaluation}(P)$, has as input an RDF dataset $D$ and a mapping $\mu$, and the problem is to verify whether $\mu \in \llbracket P \rrbracket_D$. From known results for the data-complexity of first-order logic [20], it is easy to deduce that:

**Theorem 4.** $\text{Evaluation}(P)$ is in LOGSPACE for every graph pattern expression $P$.

4 On the Semantics of UNION-free Pattern Expressions

The exact semantics of graph pattern expressions has been largely discussed on the mailing list of the W3C. There seems to be two main approaches proposed to compute answers to a graph pattern expression $P$. The first uses an operational semantics and consists essentially in the execution of a depth-first traversal of the parse tree of $P$ and the use of the intermediate results to avoid some computations. This approach is the one followed by ARQ [1] (a language developed by HPLabs) in the cases we test, and by the W3C when evaluating graph pattern expressions containing nested optionals [18]. For instance, the computation of the mappings satisfying $(A \text{ OPT } (B \text{ OPT } C))$ is done by first computing the mappings that match $A$, then checking which of these mappings match $B$, and for those who match $B$ checking whether they also match $C$ [18]. The second approach, compositional in spirit and the one we advocate here, extends classical conjunctive query evaluation [10] and is based on a bottom up evaluation of the parse tree, borrowing notions of relational algebra evaluation [4, 12] plus some additional features.

As expected, there are queries for which both approaches do not coincide (see Section 4.1 for examples). However, both semantics coincide in most of the “real-life” examples. For instance, for all the queries in the W3C candidate recommendation for SPARQL, both semantics coincide [16]. Thus, a natural question is what is the exact relationship between the two approaches mentioned above and, in particular, whether there is a “natural” condition under which both approaches coincide. In this section, we address these questions: Section 4.1 formally introduces the depth-first approach, discusses some issues concerning it, and presents queries for which the two semantics do not coincide; Section 4.2 identifies a natural and simple condition under which these two semantics are equivalent; Section 4.3 defines a normal form and simple optimization procedures for patterns satisfying the condition of Section 4.2.

Based on the results of Section 2.3, we concentrate in the critical fragment of UNION-free graph pattern expressions.
4.1 A depth-first approach to evaluate graph pattern expressions

As we mentioned earlier, one alternative to evaluate graph pattern expressions is based on a “greedy” approach that computes the mappings satisfying a graph pattern expression \( P \) by traversing the parse tree of \( P \) in a depth-first manner and using the intermediate results to avoid some computations. This evaluation includes at each stage three parameters: the dataset, the subtree pattern of \( P \) to be evaluated, and a set of mappings already collected. Formally, given an RDF dataset \( D \), the evaluation of pattern \( P \) with the set of mappings \( \Omega \), denoted by \( \text{Eval}_D(P, \Omega) \), is a recursive function defined as follows:

\[
\text{Eval}_D(P; \text{graph pattern expression}, \Omega; \text{set of mappings})
\]

\[
\begin{align*}
\text{if } \Omega = \emptyset & \text{ then return } (\emptyset) \\
\text{if } P \text{ is a triple pattern } t & \text{ then return } \Omega \triangleleft [[t]]_D \\
\text{if } P = (P_1 \text{ AND } P_2) & \text{ then return } \text{Eval}_D(P_2, \text{Eval}_D(P_1, \Omega)) \\
\text{if } P = (P_1 \text{ OPT } P_2) & \text{ then return } \text{Eval}_D(P_1, \Omega) \triangleright \text{Eval}_D(P_2, \text{Eval}_D(P_1, \Omega)) \\
\text{if } P = (P_1 \text{ FILTER } R) & \text{ then return } \{ \mu \in \text{Eval}_D(P_1, \Omega) \mid \mu \models R \}
\end{align*}
\]

Then, the evaluation of \( P \) against a dataset \( D \), which we denote simply by \( \text{Eval}_D(P) \), is \( \text{Eval}_D(P; \{ \mu_0 \}) \), where \( \mu_0 \) is the mapping with empty domain.

Example 2. Assume that \( P = (t_1 \text{ OPT } (t_2 \text{ OPT } t_3)) \), where \( t_1, t_2 \) and \( t_3 \) are triple patterns. To compute \( \text{Eval}_D(P) \), we invoke function \( \text{Eval}_D(P, \{ \mu_0 \}) \). This function in turn invokes function \( \text{Eval}_D(t_1, \{ \mu_0 \}) \), which returns \([[t_1]]_D \) since \( t_1 \) is a triple pattern and \([[t_1]]_D \triangleleft \{ \mu_0 \} = [[t_1]]_D \), and then it invokes \( \text{Eval}_D((t_2 \text{ OPT } t_3), [[t_1]]_D) \). As in the previous case, \( \text{Eval}_D((t_2 \text{ OPT } t_3), [[t_1]]_D) \) first invokes \( \text{Eval}_D(t_2, [[t_1]]_D) \), which returns \([[t_1]]_D \triangleleft [[t_2]]_D \) since \( t_2 \) is a triple pattern, and then it invokes \( \text{Eval}_D(t_3, [[t_1]]_D \triangleleft [[t_2]]_D) \). Since \( t_3 \) is a triple pattern, the latter invocation returns \([[t_1]]_D \triangleleft [[t_2]]_D \triangleleft [[t_3]]_D \). Thus, by the definition of \( \text{Eval}_D \) we have that \( \text{Eval}_D((t_2 \text{ OPT } t_3), [[t_1]]_D) \) returns \([[t_1]]_D \triangleleft ([[t_2]]_D \triangleleft ([[t_3]]_D)) \). Therefore, \( \text{Eval}_D(P) \) returns

\[
[[t_1]]_D \triangleleft ([[t_1]]_D \triangleleft ([[t_2]]_D \triangleleft ([[t_3]]_D)).
\]

Note that the previous result coincides with the evaluation algorithm proposed by the W3C for graph pattern \((t_1 \text{ OPT } (t_2 \text{ OPT } t_3)); [18]\). as we first compute the mappings that match \( t_1 \), then we check which of these mappings match \( t_2 \), and for those who match \( t_2 \) we check whether they also match \( t_3 \). Also note that the result of \( \text{Eval}_D(P) \) is not necessarily the same as \([[P]]_D \) since \([[t_1 \text{ OPT } (t_2 \text{ OPT } t_3)]]_D = [[t_1]]_D \triangleleft ([[[t_2]]_D \triangleleft ([[[t_3]]_D)). In Example 3 we actually show a case where the two semantics do not coincide.

Some issues on the depth-first approach There are two relevant issues to consider when using the depth-first approach to evaluate SPARQL queries. First, this approach is not compositional. For instance, the result of \( \text{Eval}_D(P) \) cannot in general be used to obtain the result of \( \text{Eval}_D((P' \text{ OPT } P)) \), or even the result of \( \text{Eval}_D((P' \text{ AND } P)) \), as \( \text{Eval}_D(P) \) results from the computation
of \( \text{Eval}_D(P, \{\mu_0\}) \) while \( \text{Eval}_D((P' \text{ OPT } P)) \) results from the computation of 
\[ \Omega = \text{Eval}_D(P', \{\mu_0\}) \] and \( \text{Eval}_D(P, \Omega) \). This can become a problem in cases of data integration where global answers are obtained by combining the results from several data sources; or when storing some pre–answered queries in order to obtain the results of more complex queries by composition. Second, under the depth-first approach some natural properties of widely used operators do not hold, which may confuse some users. For example, it is not always the case that 
\[ \text{Eval}_D((P_1 \text{ AND } P_2)) = \text{Eval}_D((P_2 \text{ AND } P_1)), \] violating the commutativity of the conjunction and making the result to depend on the order of the query.

Example 3. Let \( D \) be the RDF dataset shown in Example 1 and consider the pattern 
\[ P = ((?X, \text{name}, \text{paul}) \text{ OPT } ((?Y, \text{name}, \text{george}) \text{ OPT } (?X, \text{email}, ?Z))). \] 
Then \( [P]_D = \{ \{?X \rightarrow B_1\} \} \), that is, \( [P]_D \) contains only one mapping. On the other hand, following the recursive definition of \( \text{Eval}_D \) we obtain that \( \text{Eval}_D(P) = \{ \{?X \rightarrow B_1, ?Y \rightarrow B_3\} \} \), which is different from \( [P]_D \).

Example 4 (Not commutativity of AND). Let \( D \) be the RDF dataset in Example 1, 
\[ P_1 = ((?X, \text{name}, \text{paul}) \text{ AND } ((?Y, \text{name}, \text{george}) \text{ OPT } (?X, \text{email}, ?Z))) \] and 
\[ P_2 = (((?Y, \text{name}, \text{george}) \text{ OPT } (?X, \text{email}, ?Z)) \text{ AND } (?X, \text{name}, \text{paul})). \] 
Then \( \text{Eval}_D(P_1) = \{ \{?X \rightarrow B_1, ?Y \rightarrow B_3\} \} \) while \( \text{Eval}_D(P_2) = \emptyset \). Using the compositional semantics, we obtain \( [P_1]_D = [P_2]_D = \emptyset \).

Let us mention that ARQ [1] gives the same non-commutative evaluation.

4.2 A natural condition ensuring \( [P]_D = \text{Eval}_D(P) \)

If for a pattern \( P \) we have that \( [P]_D = \text{Eval}_D(P) \) for every RDF dataset \( D \), then we have the best of both worlds for \( P \) as the compositional approach gives a formal semantics to \( P \) while the depth-first approach gives an efficient way of evaluating it. Thus, it is desirable to identify natural syntactic conditions on \( P \) ensuring \( [P]_D = \text{Eval}_D(P) \). In this section, we introduce one such condition.

One of the most delicate issues in the definition of a semantics for graph pattern expressions is the semantics of OPT operator. A careful examination of the conflicting examples reveals a common pattern: A graph pattern \( P \) mentions an expression \( P' = (P_1 \text{ OPT } P_2) \) and a variable \(?X\) occurring both in \( P_2 \) and outside \( P' \) but not occurring in \( P_1 \). For instance, in the graph pattern expression shown in Example 3:

\[ P = ((?X, \text{name}, \text{paul}) \text{ OPT } ((?Y, \text{name}, \text{george}) \text{ OPT } (?X, \text{email}, ?Z))), \]

the variable \(?X\) occurs both in the optional part of the sub-pattern \( P' = ((?Y, \text{name}, \text{george}) \text{ OPT } (?X, \text{email}, ?Z)) \) and outside \( P' \) in the triple \((?X, \text{name}, \text{paul})\), but it is not mentioned in \((?Y, \text{name}, \text{george})\).

What is unnatural about graph pattern \( P \) is the fact that \((?X, \text{email}, ?Z)\) is giving optional information for \((?X, \text{name}, \text{paul})\) but in \( P \) appears as giving optional information for \((?Y, \text{name}, \text{george})\). In general, graph pattern expressions having the condition mentioned above are not natural. In fact, no queries in the W3C candidate recommendation for SPARQL [16] exhibit this condition. This motivates the following definition:
Definition 4. A graph pattern $P$ is well designed if for every occurrence of a sub-pattern $P' = (P_1 \text{ OPT } P_2)$ of $P$ and for every variable $?X$ occurring in $P$, the following condition holds:

if $?X$ occurs both in $P_2$ and outside $P'$, then it also occurs in $P_1$.

Graph pattern expressions that are not well designed are shown in Examples 3 and 4. For all these patterns, the two semantics differ. The next result shows a fundamental property of well-designed graph pattern expressions, and is a welcome surprise as a very simple restriction on graph patterns allows the users of SPARQL to alternatively use any of the two semantics shown in this section:

Theorem 5. Let $D$ be an RDF dataset and $P$ a well-designed graph pattern expression. Then $\text{Eval}_D(P) = [\text{P}]_D$.

4.3 Well-designed patterns and normalization

Due to the evident similarity between certain operators of SPARQL and relational algebra, a natural question is whether the classical results of normal forms and optimization in relational algebra are applicable in the SPARQL context. The answer is not straightforward, at least for the case of optional patterns and its relational counterpart, the left outer join. The classical results about outer join query reordering and optimization by Galindo-Legaria and Rosenthal [8] are not directly applicable in the SPARQL context because they assume constraints on the relational queries that are rarely found in SPARQL. The first and more problematic issue, is the assumption on predicates used for joining (outer joining) relations to be null-rejecting [8]. In SPARQL, those predicates are implicit in the variables that the graph patterns share and by the definition of compatible mappings they are never null-rejecting. In [8] the queries are also enforced not to contain Cartesian products, situation that occurs often in SPARQL when joining graph patterns that do not share variables. Thus, specific techniques must be developed in the SPARQL context.

In what follows we show that the property of a pattern being well designed has important consequences for the study normalization and optimization for a fragment of SPARQL queries. We will restrict in this section to graph patterns without FILTER.

We start with equivalences that hold between sub-patterns of well-designed graph patterns.

Proposition 2. Given a well-designed graph pattern $P$, if the left hand sides of the following equations are sub-patterns of $P$, then:

\[(P_1 \text{ AND } (P_2 \text{ OPT } P_3)) \equiv ((P_1 \text{ AND } P_2) \text{ OPT } P_3), \quad (2)\]
\[(((P_1 \text{ OPT } P_2) \text{ OPT } P_3) \equiv ((P_1 \text{ OPT } P_3) \text{ OPT } P_2). \quad (3)\]

Moreover, in both equivalences, if one replaces in $P$ the left hand side by the right hand side, then the resulting pattern is still well designed.
From this proposition plus associativity and commutativity of AND, it follows:

**Theorem 6.** Every well-designed graph pattern $P$ is equivalent to a pattern in the following normal form:

$$
(\cdots (t_1 \text{ AND } \cdots \text{ AND } t_k) \text{ OPT } O_1) \text{ OPT } O_2) \cdots \text{ OPT } O_n),
$$

where each $t_i$ is a triple pattern, $n \geq 0$ and each $O_j$ has the same form (4).

The proof of the theorem is based on term rewriting techniques. The next example shows the benefits of using the above normal form.

**Example 5.** Consider dataset $D$ of Example 1 and well-designed pattern $P = (((?X, \text{name}, ?Y) \text{ OPT } (?X, \text{email}, ?E)) \text{ AND } (?X, \text{phone}, 888-4537))$. The normalized form of $P$ is $P' = (((?X, \text{name}, ?Y) \text{ AND } (?X, \text{phone}, 888-4537)) \text{ OPT } (?X, \text{email}, ?E))$. The advantage of evaluating $P'$ over $P$ follows from a simple counting of maps.

Two examples of implicit use of the normal form. There are implementations (not ARQ[1]) that do not permit nested optionals, and when evaluating a pattern they first evaluate all patterns that are outside optionals and then extend the results with the matchings of patterns inside optionals. That is, they are implicitly using the normal form mentioned above. In [4], when evaluating a graph pattern with relational algebra, a similar assumption is made. First the join of all triple patterns is evaluated, and then the optional patterns are taken into account. Again, this is an implicit use of the normal form.

5 Conclusions

The query language SPARQL is in the process of standardization, and in this process the semantics of the language plays a key role. A formalization of a semantics will be beneficial on several grounds: help identify relationships among the constructors that stay hidden in the use cases, identify redundant and contradicting notions, study the expressiveness and complexity of the language, help in optimization, etc.

In this paper, we provided such a formal semantics for the graph pattern matching facility, which is the core of SPARQL. We isolated a fragment which is rich enough to present the main issues and favor a good formalization. We presented a formal semantics, made observations to the current syntax based on it, and proved several properties of it. We did a complexity analysis showing that unlimited used of OPT could lead to high complexity, namely PSPACE. We presented an alternative formal procedural semantics which closely resembles the one used by most developers. We proved that under simple syntactic restrictions both semantics are equivalent, thus having the advantages of a formal compositional semantics and the efficiency of a procedural semantics. Finally, we discussed optimization based on relational algebra and show limitations based
on features of SPARQL. On these lines, we presented optimizations based on normal forms.

Further work should concentrate on the extensions of these ideas to the whole language and particularly to the extension –that even the current specification of SPARQL lacks– to RDF Schema.

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A Proofs and Intermediate Results

A.1 Some technical results

**Lemma 1.** All the following equivalences hold:

1. If \( P \) is a graph pattern and \( R_1, R_2 \) are built-in conditions such that \( \text{var}(R_1) \subseteq \text{var}(P) \) and \( \text{var}(R_2) \subseteq \text{var}(P) \), then

\[
(P \text{ FILTER } R_1) \text{ FILTER } R_2 \equiv (P \text{ FILTER } (R_1 \land R_2)),
\]

\[
(P \text{ FILTER } (R_1 \lor R_2)) \equiv ((P \text{ FILTER } R_1) \text{ UNION } (P \text{ FILTER } R_2)).
\]

2. If \( P_1 \) and \( P_2 \) are conjunctions of triple patterns and \( R \) is a built-in condition such that \( \text{var}(R) \subseteq \text{var}(P_1) \), then

\[
((P_1 \text{ FILTER } R) \text{ AND } P_2) \equiv ((P_1 \text{ AND } P_2) \text{ FILTER } R).
\]

**Proof:** (1.1) Let \( D \) be an RDF database. Assume first that \( \mu \in \mathbb{D}((P \text{ FILTER } R_1) \text{ FILTER } R_2) \). Then \( \mu \in \mathbb{D}(P \text{ FILTER } R_1) \land \mu \models R_2 \). Thus, \( \mu \in \mathbb{D}P \land \mu \models R_1 \) and \( \mu \models R_2 \). Therefore, \( \mu \models (R_1 \land R_2) \) and, hence, we conclude that \( \mu \in \mathbb{D}(P \text{ FILTER } (R_1 \land R_2)) \). Now assume that \( \mu \in \mathbb{D}(P \text{ FILTER } (R_1 \land R_2)) \). Then \( \mu \in \mathbb{D}P \land \mu \models (R_1 \land R_2) \). Thus, \( \mu \in \mathbb{D}P \land \mu \models R_1 \) and \( \mu \models R_2 \). We conclude that \( \mu \in \mathbb{D}(P \text{ FILTER } (R_1 \land R_2)) \) and, therefore, given that \( \mu \models R_2 \), we have that \( \mu \in \mathbb{D}(P \text{ FILTER } R_1) \land \mu \models R_2 \).

(1.2) Given an RDF database \( D \), we have that:

\[
\mathbb{D}(P \text{ FILTER } (R_1 \lor R_2)) = \{ \mu \in \mathbb{D}P \mid \mu \models (R_1 \lor R_2) \}
\]

\[
= \{ \mu \in \mathbb{D}P \mid \mu \models R_1 \text{ or } \mu \models R_2 \}
\]

\[
= \{ \mu \in \mathbb{D}P \mid \mu \models R_1 \} \cup \{ \mu \in \mathbb{D}P \mid \mu \models R_2 \}
\]

\[
= \mathbb{D}(P \text{ FILTER } R_1) \cup \mathbb{D}(P \text{ FILTER } R_2).
\]

(2) Let \( D \) be an RDF database. Assume first that \( \mu \in \mathbb{D}(P_1 \text{ FILTER } R) \land \mu \models P_2 \). Then there exist \( \mu_1 \in \mathbb{D}(P_1 \text{ FILTER } R) \land \mu_2 \in \mathbb{D}P_2 \) such that \( \mu_1 \) and \( \mu_2 \) are compatible and \( \mu = \mu_1 \cup \mu_2 \). Since \( \mu_1 \in \mathbb{D}(P_1 \text{ FILTER } R) \), we have that \( \mu_1 \in \mathbb{D}P \) and \( \mu_1 \models R \). Given that \( P_1 \) is a conjunction of triple patterns and \( \text{var}(R) \subseteq \text{var}(P_1) \), we have that \( \mu_1(\exists X \in \text{var}(R)) \). Thus, given that \( \mu_1 \models R \) and \( \mu_1 \) is contained in \( \mu \), we conclude that \( \mu \models R \). Therefore, given that \( \mu_1 \in \mathbb{D}P \land \mu_2 \in \mathbb{D}P_2 \), we have that \( \mu = \mu_1 \cup \mu_2 \in \mathbb{D}(P_1 \text{ AND } P_2) \). and, hence, \( \mu \in \mathbb{D}(P_1 \text{ AND } P_2) \). Now assume that \( \mu \in \mathbb{D}(P_1 \text{ AND } P_2) \). Then \( \mu \models R \) and \( \mu \in \mathbb{D}(P_1 \text{ AND } P_2) \). and, therefore, there exist \( \mu_1 \in \mathbb{D}P_1 \) and \( \mu_2 \in \mathbb{D}P_2 \) such that \( \mu_1 \) and \( \mu_2 \) are compatible and \( \mu = \mu_1 \cup \mu_2 \). Given that \( P_1 \) and \( P_2 \) is a conjunction of triple patterns and \( \text{var}(R) \subseteq \text{var}(P_1) \), \( \text{var}(P_2) \), we have that \( \mu(\exists X \in \text{var}(R)) \). Moreover, given that \( P_1 \) is a conjunction of triple patterns and \( \text{var}(R) \subseteq \text{var}(P_1) \), we have that \( \mu_1(\exists X \in \text{var}(R)) \). This concludes the proof of the equivalence of \( ((P_1 \text{ FILTER } R) \text{ AND } P_2) \) and \( ((P_1 \text{ AND } P_2) \text{ FILTER } R) \).
Lemma 2. Let $P$ be a UNION-free graph pattern expression. Then we have that

$$(P \text{ AND } P) \equiv P.$$

Proof: Next we show by induction on the structure of $P$ that for every RDF database $D$ and pair of mappings $\mu_1, \mu_2 \in [P]_D$, if $\mu_1$ and $\mu_2$ are compatible, then $\mu_1 = \mu_2$. It is easy to see that this condition implies that $(P \text{ AND } P) \equiv P$.

If $P$ is a triple pattern, then the property trivially holds. Assume first that $P = (P_1 \text{ AND } P_2)$, where $P_1$ and $P_2$ satisfy the condition, that is, if $\xi, \zeta \in [P_i]_D$ $(i = 1, 2)$ and $\xi, \zeta$ are compatible, then $\xi = \zeta$. Let $\mu_1$ and $\mu_2$ be compatible mappings in $[P]_D$. Then there exist $\nu_1, \omega_1 \in [P_1]_D$ and $\nu_2, \omega_2 \in [P_2]_D$ such that $\mu_1 = \nu_1 \cup \omega_1$ and $\mu_2 = \nu_2 \cup \omega_2$. Given that $\mu_1$ and $\mu_2$ are compatible, we have that $\nu_1, \nu_2$ are compatible and $\omega_1, \omega_2$ are compatible. Thus, by induction hypothesis we have that $\nu_1 = \nu_2$ and $\omega_1 = \omega_2$ and, hence, $\mu_1 = \mu_2$. Second, assume that $P = (P_1 \text{ OPT } P_2)$, and let $\mu_1$ and $\mu_2$ be compatible mappings in $[P]_D$. We consider four cases.

1. If there exist $\nu_1, \omega_1 \in [P_1]_D$ and $\nu_2, \omega_2 \in [P_2]_D$ such that $\mu_1 = \nu_1 \cup \omega_1$ and $\mu_2 = \nu_2 \cup \omega_2$, then we conclude that $\mu_1 = \mu_2$ as in the case $P = (P_1 \text{ AND } P_2)$.

2. If $\mu_1, \mu_2 \in [P_1]_D$ and both are not compatible with any mapping in $[P_2]_D$, then by induction hypothesis we conclude that $\mu_1 = \mu_2$.

3. If $\mu_1 \in [P_1]_D$, $\mu_1$ is not compatible with any mapping in $[P_2]_D$, $\mu_2 = \nu_2 \cup \omega_2$, $\nu_2 \in [P_1]_D$ and $\omega_2 \in [P_2]_D$, then given that $\mu_1$ and $\mu_2$ are compatible, we have that $\mu_1$ and $\nu_2$ are compatible. Thus, by induction hypothesis we conclude that $\mu_1 = \nu_2$ and, therefore, $\mu_1$ is compatible with $\omega_2 \in [P_2]_D$, which contradicts our original assumption.

4. If $\mu_1 = \nu_1 \cup \omega_1$, $\nu_1 \in [P_1]_D$, $\omega_1 \in [P_2]_D$, $\mu_2 \in [P_1]_D$ and $\mu_2$ is not compatible with any mapping in $[P_2]_D$, then we obtain a contradiction as in the previous case.

Finally, assume that $P = (P_1 \text{ FILTER } R)$, where $P_1$ satisfy the condition. Let $\mu_1$ and $\mu_2$ be compatible mappings in $[P]_D$. Then $\mu_1 \in [P_1]_D$, $\mu_1 \models R$, $\mu_2 \in [P_2]_D$, $\mu_2 \models R$ and, thus, $\mu_1 = \mu_2$ by induction hypothesis. This concludes the proof of the lemma. \qed

A.2 Proof of Proposition 1

(1) Associative and commutative are consequences of the definitions of operators AND and UNION.

(2) To prove that $(P_1 \text{ AND } (P_2 \text{ UNION } P_3)) \equiv ((P_1 \text{ AND } P_2) \text{ UNION } (P_1 \text{ AND } P_3))$, we consider two cases. First, we show that for every RDF database $D$, we have that $[[P_1 \text{ AND } (P_2 \text{ UNION } P_3)]]_D \subseteq [[[P_1 \text{ AND } P_2] \text{ UNION } (P_1 \text{ AND } P_3)]]_D$. Assume that $D$ is an RDF database and that $\mu \in [[[P_1 \text{ AND } (P_2 \text{ UNION } P_3)]]_D$. Then there exists $\mu_1 \in [[[P_1 \text{ AND } P_2]]_D$ and $\mu_2 \in [[[P_2 \text{ UNION } P_3]]_D$ such that $\mu_1$ and $\mu_2$ are compatible and $\mu = \mu_1 \cup \mu_2$. If $\mu_2 \in [[[P_2]]_D$, then we have that $\mu = \mu_1 \cup \mu_2 \in [[[P_1 \text{ AND } P_2]]_D$ and, therefore, $\mu \in [[[P_1 \text{ AND } P_2] \text{ UNION } (P_1 \text{ AND } P_3)]]_D$. Analogously, if $\mu_2 \in [[[P_3]]_D$, then we have that $\mu = \mu_1 \cup \mu_2 \in [[[P_1 \text{ AND } P_3]]_D$ and, therefore, $\mu \in [[[P_1 \text{ AND } (P_2 \text{ UNION } P_3)]]_D$. Second, we prove that for every RDF database $D$, we have that $[[((P_1 \text{ AND } P_2) \text{ UNION } (P_1 \text{ AND } P_3))]_D \subseteq [[(P_1 \text{ AND } P_2) \text{ UNION } (P_1 \text{ AND } P_3)]_D$. Assume that $D$ is an RDF database and that $\mu \in [[[P_1 \text{ AND } P_2] \text{ UNION } (P_1 \text{ AND } P_3)]_D$. Then $\mu \in [[[P_1 \text{ AND } P_2]]_D$
or \( \mu \in [(P_1 \text{ AND } P_3)]_D \). If \( \mu \in [(P_1 \text{ AND } P_3)]_D \), then we conclude that there exists \( \mu_1 \in [P_1]_D \) and \( \mu_2 \in [P_2]_D \), such that \( \mu_1 \) and \( \mu_2 \) are compatible and \( \mu = \mu_1 \cup \mu_2 \). Since \( \mu_2 \in [P_2]_D \), we have that \( \mu_2 \in [(P_2 \text{ UNION } P_3)]_D \) and, hence, \( \mu = \mu_1 \cup \mu_2 \in [(P_1 \text{ AND } (P_2 \text{ UNION } P_3))]_D \). If \( \mu \in [(P_1 \text{ AND } P_3)]_D \), then we conclude that there exists \( \mu_1 \in [P_1]_D \) and \( \mu_3 \in [P_3]_D \) such that \( \mu_1 \) and \( \mu_3 \) are compatible and \( \mu = \mu_1 \cup \mu_3 \). Since \( \mu_3 \in [P_3]_D \), we have that \( \mu_3 \in [(P_2 \text{ UNION } P_3)]_D \) and, therefore, \( \mu = \mu_1 \cup \mu_3 \in [(P_1 \text{ AND } (P_2 \text{ UNION } P_3))]_D \). This concludes the proof of the equivalence of \((P_1 \text{ AND } (P_2 \text{ UNION } P_3)) \) and \(((P_1 \text{ AND } P_3) \text{ UNION } (P_1 \text{ AND } P_3)) \).

(3) To prove that \((P_1 \text{ OPT } (P_2 \text{ UNION } P_3)) \equiv ((P_1 \text{ OPT } P_2) \text{ UNION } (P_1 \text{ OPT } P_3)) \), we consider two cases. First, we show that for every RDF database \( D \), we have that \([(P_1 \text{ OPT } (P_2 \text{ UNION } P_3))]_D \subseteq [(P_1 \text{ OPT } P_2) \text{ UNION } (P_1 \text{ OPT } P_3))]_D \). Let \( D \) be an RDF database and assume that \( \mu \in [(P_1 \text{ OPT } (P_2 \text{ UNION } P_3))]_D \). Then there exists \( \mu_1 \in [P_1]_D \) such that either (a) there exists \( \mu_2 \in [(P_2 \text{ UNION } P_3)]_D \) such that \( \mu_1 \) and \( \mu_2 \) are compatible and \( \mu = \mu_1 \cup \mu_2 \), or (b) there is no \( \mu_2 \in [(P_2 \text{ UNION } P_3)]_D \) such that \( \mu_1 \) and \( \mu_2 \) are compatible and \( \mu = \mu_1 \). In case (a), if \( \mu_2 \in [P_2]_D \), then \( \mu = \mu_1 \cup \mu_2 \in [(P_1 \text{ OPT } P_2)]_D \), and if \( \mu_2 \in [P_3]_D \), then \( \mu = \mu_1 \cup \mu_2 \in [(P_1 \text{ OPT } P_3)]_D \). In both cases, we conclude that \( \mu \in [(P_1 \text{ OPT } P_2) \text{ UNION } (P_1 \text{ OPT } P_3))]_D \). In case (b), we have that there is no \( \mu_2 \in [P_2]_D \) such that \( \mu_1 \) and \( \mu_2 \) are compatible and, hence, \( \mu = \mu_1 \in [(P_1 \text{ OPT } P_2)]_D \). We conclude that \( \mu \in [(P_1 \text{ OPT } P_2) \text{ UNION } (P_1 \text{ OPT } P_3))]_D \). Second, we show that for every RDF database \( D \), we have that \([(P_1 \text{ OPT } P_2) \text{ UNION } (P_1 \text{ OPT } P_3))]_D \subseteq [(P_1 \text{ OPT } (P_2 \text{ UNION } P_3))]_D \). Let \( D \) be an RDF database and assume that \( \mu \in [(P_1 \text{ OPT } P_2) \text{ UNION } (P_1 \text{ OPT } P_3))]_D \). Then there exists \( \mu_1 \in [P_1]_D \) such that (a) there exists \( \mu_2 \in [P_2]_D \) such that \( \mu_1 \) and \( \mu_2 \) are compatible and \( \mu = \mu_1 \cup \mu_2 \), or (b) there exists \( \mu_3 \in [P_3]_D \) such that \( \mu_1 \) and \( \mu_3 \) are compatible and \( \mu = \mu_1 \cup \mu_3 \), or (c) \( \mu = \mu_1 \) and there is neither \( \mu_2 \in [P_2]_D \) compatible with \( \mu_1 \), nor \( \mu_3 \in [P_3]_D \) compatible with \( \mu_1 \). In case (a), given that \( \mu_2 \in [P_2]_D \), we have that \( \mu_2 \in [(P_2 \text{ UNION } P_3)]_D \) and, therefore, \( \mu = \mu_1 \cup \mu_2 \in [(P_1 \text{ OPT } (P_2 \text{ UNION } P_3))]_D \). In case (b), given that \( \mu_3 \in [P_3]_D \), we have that \( \mu_3 \in [(P_2 \text{ UNION } P_3)]_D \) and, therefore, \( \mu = \mu_1 \cup \mu_3 \in [(P_1 \text{ OPT } (P_2 \text{ UNION } P_3))]_D \). Finally, in case (c) we have that there is no \( \mu' \in [(P_2 \text{ UNION } P_3)]_D \) such that \( \mu_1 \) and \( \mu' \) are compatible and, therefore, \( \mu = \mu_1 \in [(P_1 \text{ OPT } (P_2 \text{ UNION } P_3))]_D \). This concludes the proof of the equivalence of \((P_1 \text{ OPT } (P_2 \text{ UNION } P_3)) \) and \(((P_1 \text{ OPT } P_2) \text{ UNION } (P_1 \text{ OPT } P_3)) \).

(4) To prove that \((P_1 \text{ UNION } P_2) \text{ OPT } P_3 \) \equiv \((P_1 \text{ OPT } P_3) \text{ UNION } (P_2 \text{ OPT } P_3)\), we consider two cases. First, we show that for every RDF database \( D \), we have that \([(P_1 \text{ UNION } P_2) \text{ OPT } P_3)]_D \subseteq [(P_1 \text{ OPT } P_3) \text{ UNION } (P_2 \text{ OPT } P_3)]_D \). Let \( D \) be an RDF database and assume that \( \mu \in [(P_1 \text{ UNION } P_2) \text{ OPT } P_3)]_D \). Then either (a) there exist \( \mu_1 \in [(P_1 \text{ UNION } P_2)]_D \) and \( \mu_2 \in [P_3]_D \) such that \( \mu_1 \) and \( \mu_2 \) are compatible and \( \mu = \mu_1 \cup \mu_2 \), or (b) \( \mu \in [(P_1 \text{ UNION } P_2)]_D \) and there is no \( \mu_3 \in [P_3]_D \) such that \( \mu_1 \) and \( \mu_3 \) are compatible. In case (a), if \( \mu_1 \in [P_1]_D \), then \( \mu = \mu_1 \cup \mu_2 \in [(P_1 \text{ OPT } P_3)]_D \). In case (a), if \( \mu_1 \in [P_1]_D \), then \( \mu = \mu_1 \cup \mu_2 \in [(P_1 \text{ OPT } P_3)]_D \). In case (a), if \( \mu_1 \in [P_1]_D \), then \( \mu = \mu_1 \cup \mu_2 \in [(P_1 \text{ OPT } P_3)]_D \). In case (a), if \( \mu_1 \in [P_1]_D \), then \( \mu = \mu_1 \cup \mu_2 \in [(P_1 \text{ OPT } P_3)]_D \). In
case (b), if \( \mu \in \mathcal{B}_3 \), then \( \mu \in \mathcal{B}_2 \) since \( \mu \) is not compatible with any \( \mu_3 \in \mathcal{B}_3 \). In any of the previous cases, we conclude that \( \mu \in \mathcal{B}_2 \) and \( \mathcal{B}_3 \). Second, we show that for every RDF database \( D \), we have that \( \mathcal{B}_2 \subseteq \mathcal{B}_3 \). Let \( D \) be a RDF database and assume that \( \mu \in \mathcal{B}_3 \). Without loss of generality, we assume that \( \mu \in \mathcal{B}_3 \). Then either (a) there exists \( \mu_1 \in \mathcal{B}_3 \) and \( \mu_2 \in \mathcal{B}_3 \) such that \( \mu_1 \) and \( \mu_2 \) are compatible and \( \mu = \mu_1 \cup \mu_2 \), or (b) \( \mu \in \mathcal{B}_3 \) and there is no \( \mu_3 \in \mathcal{B}_3 \) such that \( \mu \) and \( \mu_3 \) are compatible. In case (a), we have that \( \mu_1 \in \mathcal{B}_3 \) and hence, \( \mu = \mu_1 \cup \mu_2 \in \mathcal{B}_3 \). In case (b), we have that \( \mu \in \mathcal{B}_3 \) and, therefore, \( \mu \in \mathcal{B}_3 \) since \( \mu \) is not compatible with any \( \mu_3 \in \mathcal{B}_3 \). This concludes the proof of the equivalence of \((\text{P}_1 \cup \text{P}_2 \cup \text{P}_3)\) and \((\text{P}_1 \cup \text{P}_2)\).

(5) Clearly, for every RDF database \( D \) and built-in condition \( R \), we have that \( \{ \mu \in \mathcal{B}_D \mid \mu \models R \} \subseteq \{ \mu \in \mathcal{B}_D \mid \mu \models R \} \) and \( \{ \mu \in \mathcal{B}_D \mid \mu \models R \} \subseteq \{ \mu \in \mathcal{B}_D \mid \mu \models R \} \) since \( \mathcal{B}_D \subseteq \mathcal{B}_D \) and \( \mathcal{B}_D \subseteq \mathcal{B}_D \). Thus, we only need to show that for every RDF database \( D \) and built-in condition \( R \), it is the case that \( \mathcal{B}_D \subseteq \mathcal{B}_D \). Assume that \( \mu \in \mathcal{B}_D \). Then \( \mu \in \mathcal{B}_D \) and \( \mu \models R \). Thus, if \( \mu \in \mathcal{B}_D \), then \( \mu \in \mathcal{B}_D \), and if \( \mu \in \mathcal{B}_D \), then \( \mu \in \mathcal{B}_D \). Therefore, we conclude that \( \mu \in \mathcal{B}_D \).

A.3 Proof of Theorem 2

It is straightforward to prove that \text{Evaluation} is in NP for the case of graph pattern expressions constructed by using only AND, UNION and FILTER operators. To prove the NP-hardness of \text{Evaluation} for this case, we show how to reduce in polynomial time the satisfiability problem for propositional formulas in CNF (SAT-CNF) to our problem. An instance of SAT-CNF is a propositional formula \( \varphi \) of the form:

\[
C_1 \wedge \ldots \wedge C_n,
\]

where each \( C_i \) (\( i \in [1, n] \)) is a clause, that is, a disjunction of propositional variables and negations of propositional variables. Then the problem is to verify whether there exists a truth assignment satisfying \( \varphi \). It is known that SAT-CNF is NP-complete [9].

In the reduction from SAT-CNF, we use a fixed RDF database:

\[
D = \{(a, b, c)\}
\]

Assume that \( x_1, \ldots, x_m \) is the list of propositional variables mentioned in \( \varphi \). For each \( x_i \) (\( i \in [1, m] \)), we use SPARQL variables \( ?X_i, ?Y_i \) to represent \( x_i \) and \( \neg x_i \), respectively. Then for each clause \( C \) in \( \varphi \) of the form:

\[
x_{i1} \lor \cdots \lor x_{i_k} \lor x_{j1} \lor \cdots \lor x_{j_t},
\]

we get the following query:

\[
\text{SELECT} (a, b, c) \text{ WHERE } X_{i1} \lor \cdots \lor X_{i_k} \lor X_{j1} \lor \cdots \lor X_{j_t}.
\]
we define a graph pattern $P_C$ as:

$$((a, b, ?X_1) \ \text{UNION} \ \cdots \ \text{UNION} \ (a, b, ?X_k) \ \text{UNION} \ (a, b, ?Y_1) \ \text{UNION} \ \cdots \ \text{UNION} \ (a, b, ?Y_l)), $$

and we define a graph pattern $P_\varphi$ for $\varphi$ as:

$$(P \ \text{AND} \ ((P_{C_1} \ \text{AND} \ \cdots \ \text{AND} \ P_{C_n}) \ \text{FILTER} \ R)),$$

where:

$$P = ((a, b, ?X_1) \ \text{AND} \ \cdots \ \text{AND} \ (a, b, ?X_m) \ \text{AND} \ (a, b, ?Y_1) \ \text{AND} \ \cdots \ \text{AND} \ (a, b, ?Y_n)).$$

$$R = ((\neg \text{bound}(?X_1) \lor \neg \text{bound}(?Y_1)) \land \cdots \land (\neg \text{bound}(?X_m) \lor \neg \text{bound}(?Y_n))).$$

Let $\mu = \{?X_1 \rightarrow c, \ldots, ?X_m \rightarrow c, \ ?Y_1 \rightarrow c, \ldots, ?Y_m \rightarrow c\}$. Then it is straightforward to prove that $\varphi$ is satisfiable if and only if $\mu \in \|P_\varphi\|_D$.

### A.4 Proof of Theorem 3

Membership in PSPACE is a corollary of the membership in PSPACE of the evaluation problem for first-order logic [20].

To prove the PSPACE-hardness of EVALUATION for the case of graph pattern expressions not containing FILTER conditions, we show how to reduce in polynomial time the quantified boolean formula problem (QBF) to our problem. An instance of QBF is a quantified propositional formula $\varphi$ of the form:

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \forall x_3 \exists y_3 \cdots \forall x_m \exists y_m \ \psi,$$

where $\psi$ is a quantifier-free formula of the form $C_1 \land \ldots \land C_n$, with each $C_i$ ($i \in [1, n]$) being a disjunction of literals, that is, a disjunction of propositional variables and negations of propositional variables. Then the problem is to verify whether $\varphi$ is valid. It is known that QBF is PSPACE-complete [9].

In the reduction from QBF, we use a fixed RDF database:

$$D = \{(a, \text{tv}, 0), \ (a, \text{tv}, 1), \ (a, \text{false}, 0), \ (a, \text{true}, 1)\}.$$ 

Then for each clause $C$ in $\psi$ of the form

$$\left( \bigvee_{i=1}^k u_i \right) \lor \left( \bigvee_{j=1}^l \neg v_j \right),$$

we define a graph pattern $P_C$ as:

$$((a, \text{true}, ?U_1) \ \text{UNION} \ \cdots \ \text{UNION} \ (a, \text{true}, ?U_k) \ \text{UNION} \ (a, \text{false}, ?V_1) \ \text{UNION} \ \cdots \ \text{UNION} \ (a, \text{false}, ?V_l)), $$

and we define a graph pattern $P_\psi$ for $\psi$ as:

$$(P_{C_1} \ \text{AND} \ \cdots \ \text{AND} \ P_{C_n}).$$
It is easy to see that $\psi$ is satisfiable if and only if there exists a mapping $\mu \in \llbracket P_\psi \rrbracket_D$. In particular, for each mapping $\mu$, there exists a truth assignment $\sigma_\mu$ defined as $\sigma_\mu(x) = \mu(\langle X \rangle)$ for every variable $x$ in $\psi$, such that $\mu \in \llbracket P_\psi \rrbracket_D$ if and only if $\sigma_\mu$ satisfies $\psi$.

Now we explain how we represent quantified propositional formula $\varphi$ as a graph pattern expression $P_\varphi$. We use SPARQL variables $\langle X_1, \ldots, X_m \rangle$ and $\langle Y_1, \ldots, Y_m \rangle$ to represent propositional variables $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$, respectively, and we use SPARQL variables $\langle A_0, \ldots, A_m \rangle$, $\langle B_0, \ldots, B_m \rangle$ and operators $\text{OPT}$ and $\text{AND}$ to represent the quantifier sequence $\forall x_1, \ldots, \exists y_m$. More precisely, for every $i \in [1, m]$, we define graph pattern expressions $P_i$ and $Q_i$ as follows:

$$P_i := \langle \text{a, tv, } ?X_i \rangle \text{ AND } \cdots \text{ AND } \langle \text{a, tv, } ?X_1 \rangle \text{ AND}$$

$$\langle \text{a, false, } ?A_{i-1} \rangle \text{ AND } \langle \text{a, true, } ?A_i \rangle \rangle,$$

$$Q_i := \langle \text{a, tv, } ?X_i \rangle \text{ AND } \cdots \text{ AND } \langle \text{a, tv, } ?X_1 \rangle \text{ AND}$$

$$\langle \text{a, false, } ?B_{i-1} \rangle \text{ AND } \langle \text{a, true, } ?B_i \rangle \rangle,$$

and then we define $P_\varphi$ as:

$$(\langle \text{a, true, } ?B_0 \rangle \text{ OPT } \langle P_1 \text{ OPT } (Q_1 \text{ OPT } (P_2 \text{ OPT } (Q_2 \text{ OPT } (\cdots \text{ OPT } \langle P_m \text{ OPT } (Q_m \text{ AND } P_\varphi) \rangle)\cdots))\rangle)).$$

Next we show that we can use graph expression $P_\varphi$ to check whether $\varphi$ is valid. More precisely, we show that $\varphi$ is valid if and only if $\mu \in \llbracket P_\varphi \rrbracket_D$, where $\mu$ is a mapping such that $\text{dom}(\mu) = \{?B_0\}$ and $\mu(\langle X_1 \rangle) = 1$.

$(\Leftarrow)$ Assume that $\mu \in \llbracket P_\varphi \rrbracket_D$. It is easy to see that $\llbracket P_i \rrbracket_D = \{\mu_0, \mu_1\}$, where $\mu_0 = \{?X_1 \rightarrow 0, ?A_0 \rightarrow 0, ?A_1 \rightarrow 1\}$ and $\mu_1 = \{?X_1 \rightarrow 1, ?A_0 \rightarrow 0, ?A_1 \rightarrow 1\}$.

Then, given that these two mappings are compatible with $\mu$ and that $\mu \in \llbracket P_\varphi \rrbracket_D$, there exist mappings $\nu_0$ and $\nu_1$ in $\llbracket Q_i \rrbracket_D$ such that $\mu_0$, $\nu_0$ are compatible, $\mu_1$, $\nu_1$ are compatible and

$$\begin{align*}
\mu_0 \cup \nu_0 & \in [\llbracket P_1 \text{ OPT } (Q_1 \text{ OPT } (P_2 \text{ OPT } (Q_2 \text{ OPT } (\cdots \text{ OPT } \langle P_m \text{ OPT } (Q_m \text{ AND } P_\varphi) \rangle)\cdots))\rrbracket_D], & (5) \\
\mu_1 \cup \nu_1 & \in [\llbracket P_1 \text{ OPT } (Q_1 \text{ OPT } (P_2 \text{ OPT } (Q_2 \text{ OPT } (\cdots \text{ OPT } \langle P_m \text{ OPT } (Q_m \text{ AND } P_\varphi) \rangle)\cdots))\rrbracket_D]. & (6)
\end{align*}$$

We note that $\nu_0(\langle X_1 \rangle) = \mu_0(\langle X_1 \rangle) = 0$, $\nu_1(\langle X_1 \rangle) = \mu_1(\langle X_1 \rangle) = 0$ and $\nu_0(\langle Y_1 \rangle)$, $\nu_1(\langle Y_1 \rangle)$ are not necessarily distinct.

Since $P_1$ mentions triple $\langle \text{a, true, } ?A_1 \rangle$ and $P_2$ mentions triple $\langle \text{a, false, } ?A_1 \rangle$, there is no mapping in $\llbracket P_i \rrbracket_D$ compatible with some mapping in $\llbracket P_\varphi \rrbracket_D$. Furthermore, since $Q_1$ mentions $\langle \text{a, true, } ?B_1 \rangle$ and $Q_2$ mentions triple $\langle \text{a, false, } ?B_1 \rangle$, there is no mapping in $\llbracket Q_i \rrbracket_D$ compatible with some mapping in $\llbracket Q_\varphi \rrbracket_D$. Thus, given that (5) holds, for every mapping $\zeta$ in $\llbracket P_\varphi \rrbracket_D$, we have that if $\nu_0$ and $\zeta$ are compatible, then there exist $\xi$ in $\llbracket Q_\varphi \rrbracket_D$ such that $\zeta$ and $\xi$ are compatible and

$$\zeta \cup \xi \in [\llbracket P_2 \text{ OPT } (Q_2 \text{ OPT } (\cdots \text{ OPT } \langle P_m \text{ OPT } (Q_m \text{ AND } P_\varphi) \rangle)\cdots))\rrbracket_D].$$

There are two mappings in $\llbracket P_\varphi \rrbracket_D$ which are compatible with $\nu_0$:

$$\begin{align*}
\mu_{00} & = \{\langle X_1 \rightarrow 0, ?X_2 \rightarrow 0, ?Y_1 \rightarrow \nu_0(\langle Y_1 \rangle), ?A_1 \rightarrow 0, ?A_2 \rightarrow 1\}, \\
\mu_{01} & = \{\langle X_1 \rightarrow 0, ?X_2 \rightarrow 1, ?Y_1 \rightarrow \nu_0(\langle Y_1 \rangle), ?A_1 \rightarrow 0, ?A_2 \rightarrow 1\}.
\end{align*}$$
Thus, from the previous discussion we conclude that there exist mappings $\nu_{00}$ and $\nu_{01}$ such that $\mu_{00}$, $\nu_{00}$ are compatible, $\mu_{01}$, $\nu_{01}$ are compatible and

$$
\mu_{00} \cup \nu_{00} \in \mathcal{Q}_m \text{OPT} (Q_m \text{ OPT} (Q_m \text{ AND P}_m)),
\mu_{01} \cup \nu_{01} \in \mathcal{Q}_m \text{OPT} (Q_m \text{ OPT} (Q_m \text{ AND P}_m)).
$$

Similarly, there are two mapping in $\mathcal{Q}_m \text{OPT}$ which are compatible with $\nu_1$:

$$
\mu_{10} = \{?X_1 \rightarrow 1, ?X_2 \rightarrow 0, ?Y_1 \rightarrow \nu_1(?Y_1), ?A_1 \rightarrow 0, ?A_2 \rightarrow 1\},
\mu_{11} = \{?X_1 \rightarrow 1, ?X_2 \rightarrow 1, ?Y_1 \rightarrow \nu_1(?Y_1), ?A_1 \rightarrow 0, ?A_2 \rightarrow 1\}.
$$

Thus, given that (6) holds, we conclude that there exist mappings $\nu_{10}$ and $\nu_{11}$ such that $\mu_{10}$, $\nu_{10}$ are compatible, $\mu_{11}$, $\nu_{11}$ are compatible and

$$
\mu_{10} \cup \nu_{10} \in \mathcal{Q}_m \text{OPT} (Q_m \text{ OPT} (Q_m \text{ AND P}_m)),
\mu_{11} \cup \nu_{11} \in \mathcal{Q}_m \text{OPT} (Q_m \text{ OPT} (Q_m \text{ AND P}_m)).
$$

If we continue in this fashion, we conclude that for every $i \in [2, m-1]$ and $n_1 \cdots n_i \in \{0,1\}^i$, and for the following mappings in $\mathcal{Q}_{i+1} \text{OPT}$:

$$
\mu_{n_{i+1} \cdots n_i} = \{?X_i \rightarrow n_i, \ldots, ?X_i \rightarrow n_i, ?X_{i+1} \rightarrow 0, ?Y_i \rightarrow \nu_{n_{i+1} \cdots n_i}(?Y_i), ?A_i \rightarrow 0, ?A_i \rightarrow 1\},
$$

there exist mappings $\nu_{n_{i+1} \cdots n_i, 0}$ and $\nu_{n_{i+1} \cdots n_i, 1}$ in $\mathcal{Q}_{i+1} \text{OPT}$ such that $\mu_{n_{i+1} \cdots n_i, 0}$, $\nu_{n_{i+1} \cdots n_i, 0}$ are compatible, $\mu_{n_{i+1} \cdots n_i, 1}$, $\nu_{n_{i+1} \cdots n_i, 1}$ are compatible and

$$
\mu_{n_{i+1} \cdots n_i, 0} \cup \nu_{n_{i+1} \cdots n_i, 0} \in \mathcal{Q}_m \text{OPT} (Q_{i+1} \text{ OPT} (Q_m \text{ OPT} (Q_m \text{ AND P}_m))),
\mu_{n_{i+1} \cdots n_i, 1} \cup \nu_{n_{i+1} \cdots n_i, 1} \in \mathcal{Q}_m \text{OPT} (Q_{i+1} \text{ OPT} (Q_m \text{ OPT} (Q_m \text{ AND P}_m))).
$$

In particular, for every $n_1 \cdots n_m \in \{0,1\}^m$, given that $\nu_{n_1 \cdots n_m} \in \mathcal{Q}_m \text{OPT}$, $Q_m$ is a conjunction of triple patterns and $\text{var}(P_m) \subseteq \text{var}(Q_m)$, we conclude that $\nu_{n_1 \cdots n_m} \in \mathcal{Q}_m \text{OPT}$. Hence, if $\sigma_{n_1 \cdots n_m}$ is a truth assignment defined as $\sigma_{n_1 \cdots n_m}(x) = \nu_{n_1 \cdots n_m}(?x)$ for every variable $x$ in $\psi$, then $\sigma_{n_1 \cdots n_m}$ satisfies $\psi$. Thus, given that for every $n_1 \cdots n_m \in \{0,1\}^m$ we have that:

$$
\nu_1 \cdots n_m (?X_j) = \nu_1 \cdots n_m (?X_j) = \mu_1 \cdots n_m (?X_j) \quad i \in [1, m] \text{ and } j \in [1, i],
\nu_1 \cdots n_m (?Y_k) = \nu_1 \cdots n_m (?Y_k) = \mu_1 \cdots n_m (?Y_k) \quad i \in [1, m] \text{ and } k \in [1, i-1],
\nu_1 \cdots n_m (?Y_i) = \nu_1 \cdots n_m (?Y_i) = \mu_1 \cdots n_m (?Y_i) \quad i \in [1, m],
$$

we conclude that $\varphi$ is valid.

$(\Rightarrow)$ The proof that $\varphi$ is valid implies $\mu \in \mathcal{Q}_m \text{OPT}$ is similar to the previous proof.

### A.5 Proof of Theorem 5

To prove Theorem 5, we need some technical lemmas.

**Lemma 3.**
(1) Let \( \Omega_1, \Omega_2, \) and \( \Omega_3 \) be set of mappings, then \( \Omega_1 \times (\Omega_2 \setminus \Omega_3) \subseteq (\Omega_1 \times \Omega_2) \setminus \Omega_3. \)

(2) Let \( \Omega_1 \) and \( \Omega_2 \) be set of mappings, then \( \Omega_1 \setminus \Omega_2 = \Omega_1 \setminus (\Omega_1 \times \Omega_2). \)

(3) Let \( P_1, P_2 \) be UNION-free graph pattern expressions and \( \Omega_1, \Omega_2 \) set of mappings such that \( \Omega_1 \subseteq \llbracket P_1 \rrbracket_D \) and \( \Omega_2 \subseteq \llbracket P_2 \rrbracket_D \). Then \( \Omega_1 \not\Join (\Omega_1 \times \Omega_2) = \Omega_1 \not\Join \Omega_2. \)

Proof:

(1) Let \( \mu \in \Omega_1 \times (\Omega_2 \setminus \Omega_3) \) then \( \mu = \mu_1 \cup \mu_2 \) where \( \mu_1 \in \Omega_1 \), and \( \mu_2 \in \Omega_2 \setminus \Omega_3 \) with \( \mu_1 \) and \( \mu_2 \) compatible mappings. From \( \mu_2 \in \Omega_2 \setminus \Omega_3 \) we have that \( \mu_2 \in \Omega_2 \) and for every mapping \( \mu' \in \Omega_3 \), \( \mu_2 \) is not compatible with \( \mu' \). Note that since \( \mu_1 \) and \( \mu_2 \) are compatible mappings, then \( \mu = \mu_1 \cup \mu_2 \in \Omega_1 \times \Omega_2 \). Thus, given that \( \mu_2 \) is not compatible with any mapping \( \mu' \in \Omega_3 \), we conclude that \( \mu \) is not compatible with any mapping \( \mu' \in \Omega_3 \). Thus, \( \mu \in (\Omega_1 \times \Omega_2) \setminus \Omega_3. \)

(2) First we show that \( \Omega_1 \setminus \Omega_2 \subseteq \Omega_1 \setminus (\Omega_1 \times \Omega_2). \) Let \( \mu \in \Omega_1 \setminus \Omega_2. \) Then \( \mu \in \Omega_1 \) and for all \( \mu' \in \Omega_2 \), \( \mu \) is not compatible with \( \mu'. \) Let \( \mu'' \) be any mapping in \( \Omega_1 \in \Omega_2 \), then \( \mu'' = \mu_1 \cup \mu_2 \) with \( \mu_1 \in \Omega_1 \), \( \mu_2 \in \Omega_2 \) and then, since \( \mu \) is not compatible with \( \mu_2 \), necessarily \( \mu \) is not compatible with \( \mu'' \). Then \( \mu \) is not compatible with every \( \mu'' \in \Omega_1 \times \Omega_2 \), and finally \( \mu \in \Omega_1 \setminus (\Omega_1 \times \Omega_2). \) Now we show that \( \Omega_1 \setminus (\Omega_1 \times \Omega_2) \subseteq \Omega_1 \setminus \Omega_2. \) Let \( \mu \in \Omega_1 \setminus (\Omega_1 \times \Omega_2) \), then \( \mu \in \Omega_1 \) and for every \( \mu' \in \Omega_1 \times \Omega_2 \), \( \mu \) is not compatible with \( \mu'. \) Suppose that \( \mu \) is compatible with some \( \mu'' \in \Omega_2 \), then \( \mu \cup \mu'' \in \Omega_1 \times \Omega_2 \) and \( \mu \) is compatible with \( \mu \cup \mu'' \) which is a contradiction with the assumption that \( \mu \in \Omega_1 \setminus (\Omega_1 \times \Omega_2). \) Finally, \( \mu \in \Omega_1 \) is not compatible with any \( \mu'' \in \Omega_2 \) and then \( \mu \in \Omega_1 \setminus \Omega_2. \)

(3) By definition of \( \Join \), we have that \( \Omega_1 \not\Join (\Omega_1 \times \Omega_2) = (\Omega_1 \setminus (\Omega_1 \times \Omega_2)) \cup (\Omega_1 \setminus (\Omega_1 \times \Omega_2)). \) By associativity of AND, we have that \( \Omega_1 \setminus (\Omega_1 \times \Omega_2) = (\Omega_1 \setminus (\Omega_1 \times \Omega_2)). \) This is not equal to \( \Omega_1 \times \Omega_2 \) since \( \Omega_1 \times \Omega_1 = \Omega_1 \) by Lemma 2 and the fact that \( \Omega_1 \subseteq \llbracket P_1 \rrbracket_D \) and \( P_1 \) is a UNION-free expression. Furthermore, by property (2), we conclude that \( \Omega_1 \setminus (\Omega_1 \times \Omega_2) = \Omega_1 \setminus \Omega_2 \) and, therefore, \( \Omega_1 \not\Join (\Omega_1 \times \Omega_2) = (\Omega_1 \setminus (\Omega_1 \times \Omega_2)) \cup (\Omega_1 \setminus (\Omega_1 \times \Omega_2)) = (\Omega_1 \times \Omega_2) \cup (\Omega_1 \setminus \Omega_2) = \Omega_1 \not\Join \Omega_2. \)

\( \square \)

Lemma 4. Let \( P \) be a UNION-free graph pattern and \( ?X \in \text{var}(P) \) a variable of \( P. \) If there is a single occurrence of \( ?X \) that appear in \( P \) but in no right hand size of any OPT subpattern of \( P \), then \( ?X \in \text{dom}(\mu) \) for all \( \mu \in \llbracket P \rrbracket_D. \)

Proof: First note that the Lemma speaks of occurrence of a variable \( ?X \) and not of the variable itself. The intuition of this lemma is that, if an occurrence of \( ?X \) appear at least in one of the mandatory parts of \( P \), then the variable must be bounded in all the mappings of \( \llbracket P \rrbracket_D. \) The formal proof is by induction in the construction of the pattern.

(1) If \( P \) is a triple pattern and \( ?X \in \text{var}(P) \) then clearly \( ?X \in \text{dom}(\mu) \) for all \( \mu \in \llbracket P \rrbracket_D. \)

(2) Suppose \( P = (P_1 \text{ AND } P_2). \) Then if the occurrence of \( ?X \) that concern us is in \( P_1 \) then by induction hypothesis, \( ?X \in \text{dom}(\mu) \) for all \( \mu \in \llbracket P_1 \rrbracket_D \) and then \( ?X \in \text{dom}(\mu) \) for all \( \mu \in \llbracket (P_1 \text{ AND } P_2) \rrbracket_D. \) The case for \( P_2 \) is the same.

(3) Suppose \( P = (P_1 \text{ OPT } P_2), \) then the occurrence of \( ?X \) that concern us is necessarily in \( P_1. \) By induction hypothesis \( ?X \in \text{dom}(\mu) \) for all \( \mu \in \llbracket P_1 \rrbracket_D \) and then by the definition of OPT, \( ?X \in \text{dom}(\mu) \) for all \( \mu \in \llbracket (P_1 \text{ OPT } P_2) \rrbracket_D. \)

\( \square \)
Lemma 5. Let $D$ be an RDF database and $P$ a well-designed graph pattern expression. Assume that $P' = (P_1 \text{ AND } P_2)$ is a sub-pattern of $P$ and $?X$ is a variable such that $?X$ occurs in $P_2$ and $?X$ occurs in $P$ outside $P'$. Then $?X \in \text{dom}(\mu)$ for every $\mu \in \llbracket P_1 \rrbracket_D$.

Proof: Let $P' = (P_1 \text{ AND } P_2)$ be a subpattern of a well designed graph pattern $P$ such that $?X \in \text{var}(P_1)$ and $?X$ occurs outside $P'$. By the property of $P$ of being well designed, we have that $?X \in \text{var}(P_1)$. We concentrate now in subpatterns of $P_1$.

Note that because $?X \in \text{var}(P_2)$ and by the hypothesis of $P$ being well designed for every occurrence of $?X$ in the right hand size of an OPT subpattern of $P_1$ there is an occurrence of $?X$ in the left hand size of the same OPT subpattern. The last statement imply that there is necessarily an occurrence of $?X$ that is not at the right hand size of any of the OPT subpatterns of $P_1$, because if it were not the case $P_1$ would have an infinite number of occurrence of $?X$ (we would never stop applying the property of well designed pattern). Then applying Lemma 4 we obtain that for every $\mu \in \llbracket P_1 \rrbracket_D$, $?X \in \text{dom}(\mu)$, completing the proof. 

Lemma 6. Let $D$ be an RDF database and $P$ a well-designed graph pattern expression. Suppose that $P'$ is a sub-pattern of $P$ and $?X$ is a variable such that $?X$ occurs in $P'$ and $?X$ occurs in $P$ outside $P'$. Then $?X \in \text{dom}(\mu)$ for every $\mu \in \llbracket P' \rrbracket_D$.

Proof: By induction on $P'$.

(1) If $P'$ is a triple pattern $t$, then $?X \in \text{dom}(\mu)$ for every $\mu \in \llbracket t \rrbracket_D$.

(2) Let $P' = (P_1 \text{ AND } P_2)$. If $?X \in \text{var}(P_1)$, then by induction hypothesis $?X \in \text{dom}(\mu)$ for every $\mu \in \llbracket P_1 \rrbracket_D$, then $?X \in \text{dom}(\nu)$ for every $\nu \in \llbracket (P_1 \text{ AND } P_2) \rrbracket_D$. If $?X \in \text{var}(P_2)$ the proof is similar.

(3) Let $P' = (P_1 \text{ OPT } P_2)$. If $?X \in \text{var}(P_1)$ then by induction hypothesis $?X \in \text{dom}(\mu)$ for every $\mu \in \llbracket P_1 \rrbracket_D$, and then $?X \in \text{dom}(\nu)$ for every $\nu \in \llbracket (P_1 \text{ OPT } P_2) \rrbracket_D$. If $?X \in \text{var}(P_2)$, then given that $P$ is a well-designed graph pattern expression and $?X$ occurs in $P$ outside $P'$, we have that $?X \in \text{var}(P_1)$. We conclude that $?X \in \text{dom}(\nu)$ for every $\nu \in \llbracket (P_1 \text{ OPT } P_2) \rrbracket_D$ as in the previous case.

(4) Let $P' = (P_1 \text{ FILTER } R)$. Then $?X \in \text{var}(P_1)$ and, thus, by induction hypothesis $?X \in \text{dom}(\mu)$ for every $\mu \in \llbracket P_1 \rrbracket_D$. Now by definition $\llbracket (P_1 \text{ FILTER } R) \rrbracket_D \subseteq \llbracket P_1 \rrbracket_D$ and, therefore, $?X \in \text{dom}(\nu)$ for every $\nu \in \llbracket (P_1 \text{ FILTER } R) \rrbracket_D$.

Proof of Theorem 5: We will prove that during the execution of $\text{Eval}_D(\cdot)$, for every call $\text{Eval}_D(P, \Omega)$ it holds that $\text{Eval}_D(P, \Omega) = \Omega \bowtie \llbracket P \rrbracket_D$. This immediately implies that $\text{Eval}_D(P) = \llbracket P \rrbracket_D$ because $\text{Eval}_D(P) = \text{Eval}_D(P, \{\mu_0\})$.

The property trivially holds when $\Omega = \emptyset$ since $\text{Eval}_D(P, \emptyset) = \emptyset = \emptyset \bowtie \llbracket P \rrbracket_D$.

Thus, we assume that $\Omega \neq \emptyset$. Now the proof goes by induction on $P$.

- If $P$ is a triple pattern $t$, then $\text{Eval}_D(P, \Omega) = \Omega \bowtie \llbracket t \rrbracket_D$.
- Suppose that $P = (P_1 \text{ AND } P_2)$. Computing $\text{Eval}_D(P, \Omega)$ is equivalent to compute $\text{Eval}_D(P_2, \text{Eval}_D(P_1, \Omega))$ then by induction hypothesis, $\text{Eval}_D(P, \Omega) = \text{Eval}_D(P_2, \Omega \bowtie \llbracket P_1 \rrbracket_D) = \Omega \bowtie \llbracket P_1 \rrbracket_D \bowtie \llbracket P_2 \rrbracket_D = \Omega \bowtie \llbracket P_1 \text{ AND } P_2 \rrbracket_D$.
- Suppose that $P = (P_1 \text{ OPT } P_2)$. Computing $\text{Eval}_D(P, \Omega)$ is equivalent to compute $\text{Eval}_D(P_1, \Omega) \bowtie \text{Eval}_D(P_2, \text{Eval}_D(P_1, \Omega))$ and then by induction hypothesis $\text{Eval}_D(P, \Omega) = (\Omega \bowtie \llbracket P_1 \rrbracket_D) \bowtie (\Omega \bowtie \llbracket P_1 \rrbracket_D \bowtie \llbracket P_2 \rrbracket_D)$. Thus, we need to show that

$$ (\Omega \bowtie \llbracket P_1 \rrbracket_D) \bowtie (\Omega \bowtie \llbracket P_1 \rrbracket_D \bowtie \llbracket P_2 \rrbracket_D) = \Omega \bowtie (\llbracket P_1 \rrbracket_D \bowtie \llbracket P_2 \rrbracket_D). $$
First we show that \( \Omega \times ([P_1 \times P_2]) \subseteq (\Omega \times [P_1]) \times (\Omega \times [P_2]) \times \times \). Let \( \mu \in \Omega \times ([P_1 \times P_2]) \times \times \) then \( \mu = \mu_1 \cup \mu_2 \) where \( \mu_1 \in \Omega \), \( \mu_2 \in ([P_1 \times P_2]), \) and \( \mu_1, \mu_2 \) are compatible mappings. We consider two cases:

(a) \( \mu_2 \in [P_1 \times P_2]. \) Then \( \mu \in \Omega \times ([P_1 \times P_2]) \times \times \) and, hence, by commutativity and associativity of the AND operator and Lemma 2, we have that \( \mu \in (\Omega \times [P_1]) \times (\Omega \times [P_1]) \times \times \). By Lemma 3 (1) and, thus, \( \mu \in \Omega \times ([P_1 \times P_2]) \times \times \) (by Lemma 3 (2) and commutativity and associativity of the AND operator).

(b) \( \mu_2 \in [P_1 \times P_2]. \) Then \( \mu \in \Omega \times ([P_1 \times P_2]) \times \times \) (by Lemma 3 (1)) and, thus, \( \mu \in \Omega \times ([P_1 \times P_2]) \times \times \) (by Lemma 3 (2) and commutativity and associativity of the AND operator). We conclude that \( \mu \in (\Omega \times [P_1]) \times (\Omega \times [P_2]) \times \times \).

Now we show that \( \Omega \times ([P_1 \times P_2]) \times \times \omega = \omega \times \times \Omega \times ([P_1 \times P_2]) \times \times \).

By commutativity and associativity of the AND operator and Lemma 2, we have that \( \Omega \times ([P_1 \times P_2]) \times \times \omega = \omega \times \times \Omega \times ([P_1 \times P_2]) \times \times \).

By Lemma 3 (2), to show that \( \Omega \times ([P_1 \times P_2]) \times \times \omega = \omega \times \times \Omega \times ([P_1 \times P_2]) \times \times \) is equivalent to show that \( \Omega \times ([P_1 \times P_2]) \times \times \omega = \omega \times \times \Omega \times ([P_1 \times P_2]) \times \times \). Let \( \mu \in (\Omega \times [P_1]) \times \times \omega = \omega \times \times \Omega \times ([P_1 \times P_2]) \times \times \). By Lemma 3 (2) and commutativity and associativity of the AND operator. We conclude that \( \mu \in (\Omega \times [P_1]) \times (\Omega \times [P_2]) \times \times \).

Suppose that \( \mu_2 \) is not compatible with any \( \mu' \in [P_2], \mu_1 \) is compatible with \( \mu' \). Suppose that \( \mu_2 \) is not compatible with any \( \mu' \in [P_2], \mu_1 \) is compatible with \( \mu' \). Then \( \mu = \mu_1 \cup \mu_2 \) with \( \mu_1 \in \Omega \), \( \mu_2 \in [P_1 \times P_2], \) and \( \mu_1, \mu_2 \) compatible mappings. Furthermore, for every \( \mu' \in [P_2], \mu_1 \cup \mu_2 \) is not compatible with \( \mu' \). Suppose that \( \mu_2 \) is not compatible with any \( \mu' \in [P_2], \mu_1 \) is compatible with \( \mu' \). Then \( \mu = \mu_1 \cup \mu_2 \) with \( \mu_1 \in \Omega \), \( \mu_2 \in [P_1 \times P_2], \) and \( \mu_1, \mu_2 \) compatible mappings. Furthermore, for every \( \mu' \in [P_2], \mu_1 \cup \mu_2 \) is not compatible with \( \mu' \). Suppose that \( \mu_2 \) is not compatible with any \( \mu' \in [P_2], \mu_1 \) is compatible with \( \mu' \). Then there exists a variable \( ?X \in \text{dom}(\mu_1) \) such that \( ?X \in \text{dom}(\nu) \) and \( \mu_1(\mu_1) \neq \nu(\mu_1) \). Since \( \mu_2 \) is compatible with both \( \mu_1 \) and \( \nu \), we have that \( ?X \notin \text{dom}(\mu_2) \). This implies that \( ?X \notin \text{dom}(\mu_2) \). Then \( ?X \in \text{dom}(\nu) \) and there exists a mapping \( \omega = \mu_2 \in [P_1 \times P_2] \) such that \( ?X \notin \text{dom}(\omega) \), which contradicts Lemma 5.

This conclude the proof of the inclusion \( \Omega \times ([P_1 \times P_2]) \times \times \omega = \omega \times \times \Omega \times ([P_1 \times P_2]) \times \times \).

Suppose that \( \mu \in \Omega \times ([P_1 \times P_2]) \times \times \). By induction hypothesis this set is equal to \( \{ \mu \in \Omega \times [P_1 \times P_2] \mid \mu \models R \}. \) Thus, we need to show that this set is equal to \( \Omega \times \times \Omega \times \times \). First, assume that \( \nu \in \Omega \times \times \Omega \times \times \). Then \( \nu = \nu_1 \cup \nu_2 \) with \( \nu_1 \in \Omega, \nu_2 \in \Omega \times \times \Omega \times \times \times \). Since \( \nu_2 \in \Omega \times \times \Omega \times \times \times \), we have that \( \nu_2 \in [P_1 \times P_2] \) and \( \nu_2 \models R \). Next we show that \( \nu \models R \). By contradiction, assume that \( \nu \not\models R \). Then there exists a variable \( ?X \in \text{dom}(\nu) \) such that \( ?X \in \text{dom}(\nu) \) and, therefore, \( ?X \) occurs outside \( P \) since \( \nu_1 \in \Omega \). We conclude that \( ?X \) occurs in \( P \), \( ?X \) occurs outside \( P \) and there exists a mapping \( \omega = \nu_2 \in [P_1 \times P_2] \) such that \( ?X \notin \text{dom}(\omega) \), which contradicts Lemma 6. Thus, we conclude that \( \nu \models R \) and, therefore, \( \nu = \nu_1 \cup \nu_2 \in \{ \mu \in \Omega \times [P_1 \times P_2] \mid \mu \models R \}. \) Second, assume that \( \nu \in \{ \mu \in \Omega \times [P_1 \times P_2] \mid \mu \models R \}. \) Then \( \nu \models R \) and \( \nu = \nu_1 \cup \nu_2 \) with
To simplify the notation we will suppose that $\nu_1 \in \Omega$, $\nu_2 \in \llbracket P_1 \rrbracket_D$ and $\nu_1, \nu_2$ compatible mappings. Next we show that $\nu_2 \models R$. By contradiction, assume that $\nu_2 \not\models R$. Then given that $\nu \models R$ and $\nu = \nu_1 \cup \nu_2$, we have that there exists variable $?X \in \text{var}(R)$ such that $?X \in \text{dom}(\nu)$ but $?X \not\in \text{dom}(\nu_2)$. But this implies that $?X \in \text{dom}(\nu_1)$ and, therefore, $?X$ occurs outside $P_1$ since $\nu_1 \in \Omega$. We conclude that $?X$ occurs in $P_1$ since $\text{var}(R) \subseteq \text{var}(P_1)$, $?X$ occurs outside $P_1$ and there exists a mapping $\omega = \nu_2 \in \llbracket P_1 \rrbracket_D$ such that $?X \not\in \text{dom}(\omega)$, which contradicts Lemma 6. Thus, we conclude that $\nu = \nu_1 \cup \nu_2 \in \Omega \setminus \llbracket (P_1 \text{ FILTER } R) \rrbracket_D$. Hence, we deduce that $\nu = \nu_1 \cup \nu_2 \in \Omega \setminus \llbracket (P_1 \text{ FILTER } R) \rrbracket_D$. This concludes the proof of the theorem.

A.6 Proof of Proposition 2

First we show that for every subpattern $(P_1 \text{ AND (P_2 OPT P_3))}$ of a well designed pattern $P$, it holds that $(P_1 \text{ AND (P_2 OPT P_3))} \equiv ((P_1 \text{ AND P_2}) \text{ OPT P_3})$.

Proof: To simplify the notation we will suppose that $\mu_1 \in \llbracket P_1 \rrbracket_D$, $\mu_2 \in \llbracket P_2 \rrbracket_D$, and $\mu_3 \in \llbracket P_3 \rrbracket_D$.

- First $\llbracket (P_1 \text{ AND (P_2 OPT P_3))} \rrbracket_D \subseteq \llbracket ((P_1 \text{ AND P_2}) \text{ OPT P_3}) \rrbracket_D$. Let $\mu \in \llbracket (P_1 \text{ AND (P_2 OPT P_3))} \rrbracket_D = \llbracket P_1 \rrbracket_D \times (\llbracket P_2 \rrbracket_D \vartriangleright \llbracket P_3 \rrbracket_D)$. Then $\mu = \mu_1 \cup \mu'$ with $\mu_1$ and $\mu'$ compatible mappings, and $\mu' \in \llbracket P_2 \rrbracket_D \vartriangleright \llbracket P_3 \rrbracket_D$, depending on $\mu'$ there are two cases:
  - If $\mu' \in \llbracket P_2 \rrbracket_D \times \llbracket P_3 \rrbracket_D$ then $\mu \in \llbracket P_1 \rrbracket_D \times (\llbracket P_2 \rrbracket_D \times \llbracket P_3 \rrbracket_D)$, and then $\mu \in \llbracket (P_1 \text{ AND (P_2 OPT P_3))} \rrbracket_D$.
  - If $\mu' \in \llbracket P_2 \rrbracket_D \setminus \llbracket P_3 \rrbracket_D$ then $\mu' \in \llbracket P_2 \rrbracket_D$ and is incompatible with every $\mu_3 \in \llbracket P_3 \rrbracket_D$, then $\mu = \mu_1 \cup \mu'$ is incompatible with $\mu_3$ and then $\mu \in \llbracket (P_1 \text{ AND (P_2 OPT P_3))} \rrbracket_D$.

- Now $\llbracket ((P_1 \text{ AND P_2}) \text{ OPT P_3}) \rrbracket_D \subseteq \llbracket (P_1 \text{ AND (P_2 OPT P_3))} \rrbracket_D$. Let $u \in \llbracket ((P_1 \text{ AND P_2}) \text{ OPT P_3}) \rrbracket_D = \llbracket P_1 \rrbracket_D \times (\llbracket P_2 \rrbracket_D \vartriangleright \llbracket P_3 \rrbracket_D)$. There are two cases:
  - $\mu \in \llbracket P_1 \rrbracket_D \times (\llbracket P_2 \rrbracket_D \times \llbracket P_3 \rrbracket_D) = (\llbracket P_1 \rrbracket_D \times \llbracket P_3 \rrbracket_D) \times (\llbracket P_2 \rrbracket_D)$, then $\mu \in \llbracket ((P_1 \text{ AND P_2}) \text{ OPT P_3}) \rrbracket_D$.
  - $\mu \in \llbracket (P_1 \rrbracket_D \times \llbracket P_2 \rrbracket_D) \setminus \llbracket P_3 \rrbracket_D$, then $\mu = \mu_1 \cup \mu_2$ with $\mu_1$ and $\mu_2$ compatible mappings and for every $\mu_3$, $\mu_1 \cup \mu_2$ is incompatible with $\mu_3$. Suppose first that $\mu_2$ is incompatible with $\mu_3$, then $\mu_2 \in \llbracket P_2 \rrbracket_D \setminus \llbracket P_3 \rrbracket_D \subseteq \llbracket P_2 \rrbracket_D \vartriangleright \llbracket P_3 \rrbracket_D$ and then $\mu_1 \cup \mu_2 \in \llbracket P_1 \rrbracket_D \times (\llbracket P_2 \rrbracket_D \vartriangleright \llbracket P_3 \rrbracket_D) = \llbracket (P_1 \text{ AND (P_2 OPT P_3))} \rrbracket_D$. Suppose now that $\mu_1$ is incompatible with $\mu_3$, then there exists a variable $?X \in \text{dom}(\mu_1)$, $?X \in \text{dom}(\mu_3)$ such that $\mu_1(?X) \neq \mu_3(?X)$. This last statement imply that $?X \in \text{var}(P_1) \cap \text{var}(P_3)$ and then because $P$ is well designed by Lemma 5 we obtain $?X \in \text{dom}(\mu_2)$ and because $\mu_2$ is compatible with $\mu_1$ we have that $\mu_2(?X) \neq \mu_3(?X)$. Finally $\mu_2 \in \llbracket P_2 \rrbracket_D \setminus \llbracket P_3 \rrbracket_D \subseteq \llbracket P_2 \rrbracket_D \vartriangleright \llbracket P_3 \rrbracket_D$, and then $\mu = \mu_1 \cup \mu_2 \in \llbracket ((P_1 \text{ OPT P_2}) \text{ OPT P_3}) \rrbracket_D$.

Now we show that for every subpattern $(P_1 \text{ OPT P_2}) \text{ OPT P_3}$ of a well designed pattern $P$, it holds that $(P_1 \text{ OPT P_2}) \text{ OPT P_3) \equiv ((P_1 \text{ OPT P_2}) \text{ OPT P_3)}$.

Proof:

- First $\llbracket ((P \text{ OPT P_1}) \text{ OPT P_2}) \rrbracket_D \subseteq \llbracket ((P \text{ OPT P_2}) \text{ OPT P_3}) \rrbracket_D$. Let $\mu \in \llbracket (P \text{ OPT P_1}) \text{ OPT P_2}) \rrbracket_D = \llbracket P \rrbracket_D \vartriangleright \llbracket (P_1 \text{ OPT P_2}) \rrbracket_D$. Suppose that $\mu \in \llbracket (P_1 \text{ OPT P_2}) \rrbracket_D \vartriangleright \llbracket P_2 \rrbracket_D$, there are two cases:
• \( \mu \in ([P]_D \times [P_1]_D) \times [P_2]_D \subseteq ([P]_D \times [P_2]_D) \times [P_3]_D \subseteq (((P \text{ OPT } P_2) \text{ OPT } P_1)]_D \).
• \( \mu \in ([P]_D \setminus [P_1]_D) \times [P_2]_D \subseteq ([P]_D \times [P_2]_D) \setminus [P_3]_D \), by proposition 3 (1), then \( \mu \in (((P \text{ OPT } P_2) \text{ OPT } P_1)]_D \).

Suppose now that \( \mu \in ([P]_D \setminus [P_1]_D) \setminus [P_2]_D \). There are two cases: (i assume \( \mu' \in [P]_D, \mu_1 \in [P_1]_D, \mu_2 \in [P_2]_D \)).

• \( \mu \in ([P]_D \times [P_1]_D) \setminus [P_2]_D \), then \( \mu = \mu' \cup \mu_1 \) such mappings, and for every \( \mu_2, \mu' \cup \mu_1 \) is incompatible with \( \mu_2 \). If \( \mu' \) is incompatible with \( \mu_2 \) then \( \mu' \in [P]_D \setminus [P_2]_D \) and \( \mu' \cup \mu_1 \in ([P]_D \setminus [P_2]_D) \times [P_1]_D \) and then \( \mu \in (((P \text{ OPT } P_2) \text{ OPT } P_1)]_D \). Suppose that \( \mu_2 \) is incompatible with \( \mu_2 \), then there is some \( \lambda \) such that \( \mu_1(\lambda \lambda) \neq \mu_2(\lambda \lambda) \). Then \( \lambda \in \text{var}(P_1) \cap \text{var}(P_2) \) and because the whole pattern is well designed, by Lemma 5 we obtain that \( \mu_1 \neq \mu_2 \) and \( \mu' \) compatible with \( \mu_1 \) we obtain that \( \mu(\lambda \lambda) \neq \mu_2(\lambda \lambda) \), and then \( \mu' \) is incompatible with \( \mu_2 \). Then \( \mu \in (((P \text{ OPT } P_2) \text{ OPT } P_1)]_D \).

• \( \mu \in ([P]_D \times [P_1]_D) \times [P_2]_D \), then \( \mu \in [P]_D \) and is such that for all \( \mu_1 \) and for all \( \mu_2, \mu \) is incompatible with \( \mu_1 \) and \( \mu_2 \). Suppose that \( \mu \in ([P]_D \times [P_2]_D) \setminus [P_3]_D \), there are two cases:
  • \( \mu \in ([P]_D \times [P_2]_D) \times [P_1]_D \subseteq (((P \text{ OPT } P_1)]_D \setminus [P_2]_D \). Let \( \mu \in (((P \text{ OPT } P_1)]_D \setminus [P_2]_D \), then \( \mu \in (((P \text{ OPT } P_1)]_D \setminus [P_2]_D \) for \( (i \text{ assume } \mu' \in [P_1]_D, \mu_1 \in [P_2]_D, \mu_2 \in [P_2]_D \) ). Suppose that \( \mu \in (((P \text{ OPT } P_1)]_D \setminus [P_2]_D \setminus [P_3]_D \) by prop. 3 (1) and then \( \mu \in (((P \text{ OPT } P_1)]_D \setminus [P_2]_D \).

Suppose now that \( \mu \in ([P]_D \times [P_2]_D) \setminus [P_3]_D \), there are two cases:

• \( \mu \in ([P]_D \times [P_2]_D) \times [P_1]_D \), then \( \mu = \mu' \cup \mu_1 \) such mappings, and for every \( \mu_2, \mu' \cup \mu_1 \) is incompatible with \( \mu_1 \). If \( \mu' \) is incompatible with \( \mu_1 \) then \( \mu' \in [P]_D \setminus [P_1]_D \) and \( \mu' \cup \mu_2 \in ([P]_D \setminus [P_1]_D) \times [P_2]_D \) and then \( \mu \in (((P \text{ OPT } P_1)]_D \setminus [P_2]_D \). If \( \mu_2 \) is incompatible with \( \mu_1 \) then there exists a variable \( \lambda \in \text{var}(\mu_1) \cap \text{var}(\mu_2) \) such that \( \mu_1(\lambda \lambda) \neq \mu_2(\lambda \lambda) \). Then \( \lambda \in \text{var}(P_1) \cap \text{var}(P_2) \) and because the whole pattern is well designed, by Lemma 5 we obtain that \( \mu_1(\lambda \lambda) \neq \mu_2(\lambda \lambda) \) and then \( \mu' \) is incompatible with \( \mu_1 \). Then \( \mu \in (((P \text{ OPT } P_1)]_D \setminus [P_2]_D \).

To finish the proof we must show that replacing the respective equivalences do not affect the property of \( P \) of being well designed. Let \( (P_1 \text { AND } (P_2 \text { OPT } P_3)) \) be a subpattern of \( P \). Well designed says that, if a variable \( ?X \) occurs outside \( (P_2 \text{ OPT } P_3) \) and inside \( P_2 \) then it occurs in \( P_2 \). Suppose that this is the case and that \( ?X \) occurs outside \( (P_1 \text{ AND } (P_2 \text{ OPT } P_3)) \), then because \( ?X \) occurs in \( P_2 \) then \( ?X \) occurs in \( P_1 \text{ AND } P_3 \) and the pattern \( P' \) obtained from \( P \) by replacing \( (P_1 \text{ AND } (P_2 \text{ OPT } P_3)) \) by \( (P_1 \text{ AND } P_2 \text{ OPT } P_3) \) is well designed. Suppose now that \( ?X \) does not occur outside \( (P_1 \text{ AND } (P_2 \text{ OPT } P_3)) \) and then the pattern obtained from \( P \) is well designed.

The proof for \( P' = ((P_1 \text{ OPT } P_3) \text{ OPT } P_5) \) is similar. There are various cases for variables occurring inside \( P_2, P_3 \).
- ?X occurs in $P_2$ and in $P_3$,
- ?X occurs in $P_2$ and outside $P'$ but not in $P_3$,
- ?X occurs in $P_3$ and outside $P'$ but not in $P_2$,

in all cases because $P$ is well designed.

$?X$ occurs in $P_2$ and outside $P'$ but not in $P_3$,

$?X$ occurs in $P_3$ and outside $P'$ but not in $P_2$,

in all cases because $P$ is well designed.

**A.7 Proof of Theorem 6**

To prove Theorem 6 we use the following Lemma. In the Lemma we use rewriting concepts and results (see [2]).

**Lemma 7.** Let us consider the theory $E$ formed by the equations of associativity and commutativity for AND (Proposition 1), and equation

$((X \text{ OPT } Y) \text{ OPT } Z) \equiv ((X \text{ OPT } Z) \text{ OPT } Y)$

Then the rule

$$(X \text{ AND } (Y \text{ OPT } Z)) \rightarrow ((X \text{ AND } Y) \text{ OPT } Z) \quad (7)$$

is $E$-terminating and $E$-confluent in the set of well designed patterns, and hence has $E$-normal forms in the set of well designed patterns.

**Proof:**

1. First we prove that rule (7) is terminating. Consider the measure

$m(P) : \text{ number of OPT inside AND-trees in the parsing of } P$.

Then clearly the theory $E$ keeps $m(P)$ constant. Let $P'$ and $P''$ be the left and right hand side in rule (7) respectively; then $m(P') > m(P'')$. Hence successive application of rule (7) must terminate.

2. Now we prove that rule (7) is $E$-locally confluent. Note that the only critical pair (see [2]) is: $((P_1 \text{ OPT } P_2) \text{ AND } (P_3 \text{ OPT } P_4))$ Then it only left to check that both applications of rule (7)

$((P_1 \text{ OPT } P_2) \text{ AND } P_3) \text{ OPT } P_4)$

and

$((P_3 \text{ OPT } P_4) \text{ AND } P_1) \text{ OPT } P_2)$

can be rewritten to a common term using the axioms of $E$ and the rule (7):

$(((P_1 \text{ OPT } P_2) \text{ AND } P_3) \text{ OPT } P_4) \overset{E}{\equiv} (((P_3 \text{ AND } (P_3 \text{ OPT } P_4)) \text{ OPT } P_4) \text{ OPT } P_4)$

$\overset{C_1}{\equiv} (((P_3 \text{ AND } P_1) \text{ OPT } P_2) \text{ OPT } P_4) \text{ OPT } P_4)$

$((P_3 \text{ OPT } P_4) \text{ AND } P_1) \text{ OPT } P_2) \overset{E}{\equiv} (((P_1 \text{ AND } (P_3 \text{ OPT } P_4)) \text{ OPT } P_4) \text{ OPT } P_4)$

$\overset{C_1}{\equiv} (((P_3 \text{ AND } P_1) \text{ OPT } P_2) \text{ OPT } P_4) \text{ OPT } P_4)$

$\overset{E}{\equiv} (((P_1 \text{ AND } P_3) \text{ OPT } P_2) \text{ OPT } P_4)$

Theorem 6 follows from the existence of $E$ normal forms for rule (7), and the application of (7) and $E$ identities to well designed graph patterns.