Indeterministic Quantum Gravity and Cosmology

VII. Dynamical Passage through Singularities: Black Hole and Naked Singularity, Big Crunch and Big Bang

Vladimir S. MASHKEVICH

Institute of Physics, National academy of sciences of Ukraine
252028 Kiev, Ukraine

Abstract

This paper is a continuation of the papers [1-6]. The aim of the paper is to incorporate singularities—both local (black hole and naked singularity) and global (big bang and big crunch)—into the dynamics of indeterministic quantum gravity and cosmology. The question is whether a singularity is dynamically passable, i.e., whether a dynamical process which ends with a singularity may be extended beyond the latter. The answer is yes. A local singularity is trivially passable, while the passableness for a global singularity may invoke CPT transformation. The passableness of the singularities implies pulsating black holes and the oscillating universe. For the local singularity, the escape effect takes place: In a vicinity of the singularity, quantum matter leaves the gravitational potential well.
... put thyself into the trick of singularity.

William Shakespeare

Introduction

In literature, there is a great body of information on singularities in gravity and cosmology, so that we shall restrict our consideration of them to the theory being developed in this series of papers. To wit, the aim of the present paper is to incorporate the singularities into the dynamics of indeterministic quantum gravity and cosmology (IQGC). What is at issue is the question whether a singularity is dynamically passable, i.e., whether a dynamical process which ends with the singularity may be extended beyond the latter. The answer is yes.

There are local singularities (black hole and naked singularity) and global ones (big crunch and big bang). For any singularity, dynamical passableness means that the singularity exists only instantly in time, so that a gravitational contraction resulting in the singularity is immediately followed by an expansion. The expansion in its turn is followed by the next contraction and so on, so that we have either a pulsating local object or the oscillating universe.

A singularity manifests itself both in a static aspect and in a dynamical one. The former is that the metric $g$ relating to the singularity is degenerate, the latter is that $g$ as a function of time cannot be $C^\infty$. It is the dynamical aspect that we are mainly interested in. Indeed $g$ may be $C^1$ or $C$.

There are singularities of two types: tempered and strong ones. A tempered singularity is $C^1$, and in this case a state vector $\Psi$ of matter and energy $\varepsilon$ exist. A strong singularity is $C$, and is that case $\Psi$ and $\varepsilon$ do not exist ($\varepsilon = \infty$). The tempered singularity is trivially passable. The passableness of the strong singularity invokes $CPT$ transformation. A local singularity is tempered. A global singularity (big crunch followed by big bang) may be both tempered and strong, depending on the pressure of matter.

For a local singularity, the escape effect takes place: In a vicinity of the singularity, quantum matter leaves the gravitational potential well before the well results in the singularity. The effect is impossible in (semiclassical) general relativity because of the constraint equation (which does not hold in IQGC).

1 SINGULARITIES AND PASSABLENESSE

1.1 Future-singular point

Let $P_{(-\theta,0)}$ be a deterministic process (from here on see [5]) and $g_t, \Psi_t, \varepsilon_t, t \in (-\theta,0)$, be determined by $P$. The point $t = 0$ is a future-singular point for the process iff $g_{-0} \equiv \lim_{t \to -0} g_t$ exists and is degenerate, $\dot{g}_{-0} \equiv (\partial g/\partial t)_{-0}$ exists, and a finite $\ddot{g}_{-0}$ does not exists. The point $t = 0$ is temperately singular iff $\Psi_{-0}$ and $\varepsilon_{-0}$ exist for a solution to

$$H_t \Psi_t = \varepsilon_t \Psi_t; \quad \text{(1.1.1)}$$

the point $t = 0$ is strongly singular iff $\Psi_{-0}$ and $\varepsilon_{-0}$ do not exist ($\varepsilon_{-0} = \infty$).
1.2 Past-singular point

A past-singular point \( t = 0 \) is defined straightforwardly by the replacements

\[ \mathcal{P}_{(-\theta,0)} \to \mathcal{P}_{(0,\theta)}, \quad -0 \to +0. \]  

(1.2.1)

1.3 Passable singular point

Let \( \mathcal{P}_{(-\theta,-0)} \) be a deterministic process and \( t = 0 \) be a future-singular point for it. The point is passable iff there exists a deterministic process \( \mathcal{P}_{(0,\theta^+)} \), such that \( t = 0 \) is a past-singular point for it and:

- for a tempered singularity

\[ \omega_{-0} = \lim_{t \to +0} \mathcal{P}_{(0,\theta^+)}(t) = \lim_{t \to -0} \mathcal{P}_{(-\theta,-0)}(t) = \omega_{-0}; \]  

(1.3.1)

- for a strong singularity

\[ \theta^+ = \theta^- \equiv \theta, \quad \mathcal{P}_{(0,\theta)}(t) = CPT \mathcal{P}_{(-\theta,0)}(-t) \]  

(1.3.2)

(as to \( CPT \) in IQGC see [6]).

In IQGC, for any given \( \theta^+ \), \( \mathcal{P}_{(0,\theta^+)} \) should be unique.

We shall say that a passable tempered (respectively strong) singularity is trivially (respectively \( CPT \)) passable.

The passableness for a past-singular point is defined in a similar manner.

1.4 Local and global singularity

For a local singular point \( t = 0 \), the metric \( g \) is degenerate at a single point of spacetime manifold \( M \). For a global singular point \( t = 0 \), the metric is degenerate at \( t = 0 \) on the unit 3-sphere, i.e., in the subset

\[ \{0\} \times S^3 \subset M \]  

(1.4.1)

of \( M \) (see [6]).

2 Black hole and naked singularity

2.1 Equation of motion

In the case of black hole or naked singularity, a singularity is local. To solve the problem of passableness for the singularity, it suffices to consider the simplest case: a gravitational collapse of a spherically symmetric ball of dust with uniform density [7,8].

The interior metric in comoving coordinates is

\[ g = dt^2 - R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\varphi^2 \right]. \]  

(2.1.1)
The singularity relates to the center of the ball, i.e., to \( r = 0 \). The projected Einstein equation \([2,3]\) gives
\[
G_{ij} = 8\pi\kappa T_{ij} \Rightarrow k + 2R\ddot{R} + \dot{R}^2 = -8\pi\kappa p R^2,
\]
so that for \( p = 0 \) we have
\[
k + 2R\ddot{R} + \dot{R}^2 = 0,
\]
whence
\[
\frac{d}{dR}[R(k + \dot{R}^2)] = 0, \quad R(k + \dot{R}^2) = C.
\]
The initial conditions are
\[
R(-\tau/2) = 1, \quad \dot{R}(-\tau/2),
\]
so that \( C = k \) and we obtain
\[
R(k + \dot{R}^2) = k.
\]
Matching the interior and exterior solutions leads to the value
\[
k = \frac{2\kappa M}{a^3},
\]
where \( a \) is the ball radius in the comoving coordinate system,
\[
M = \frac{4\pi}{3} \rho(0)a^3
\]
is the ball mass, and \( \rho(0) \) is the initial matter density.

An important point is that since the pressure on the ball surface is always zero, the energy constancy condition
\[
\varepsilon = M = \text{const}
\]
holds even for a nonzero internal pressure.

### 2.2 Metric singularity

The metric (2.1.1) depends on
\[
q = R^2
\]
rather than on \( R \) itself. Thus a metric singularity should be treated in terms of \( q \). Equation (2.1.3) and the initial conditions (2.1.3) take the form
\[
q^2 + 4kq - 4kq^{1/2} = 0, \quad q(-\tau/2) = 1, \quad \dot{q}(-\tau/2) = 0.
\]

For \( q \to 0 \) we have
\[
q^2 - 4kq^{1/2} = 0,
\]
so that
\[
\frac{dq}{dt} = \pm 2k^{1/2}q^{1/4},
\]
the minus (respectively plus) sign corresponding to a contracting (respectively expanding) ball. Choosing \( \tau \) in (2.2.3) so that
\[ q(0) = 0, \] (2.2.6)
we obtain for \( t \to 0 \)
\[ q = \left( \frac{3}{2} \right)^{4/3} k^{2/3} t^{4/3}, \] (2.2.7)
\[ \dot{q} = 2 \left( \frac{3}{2} \right)^{1/3} k^{2/3} t^{1/3}, \] (2.2.8)
\[ \ddot{q} = \left( \frac{2}{3} \right)^{2/3} k^{2/3} t^{-2/3}. \] (2.2.9)

Equation (2.2.6) implies that the metric (2.1.1) is degenerate, i.e., singular at \( t = 0 \).

### 2.3 Tempered singularity and trivial passableness

Equations (2.2.7), (2.2.8) are valid both for \( t \leq 0 \) and for \( t > 0 \); equation (2.2.9) is valid both for \( t < 0 \) and for \( t > 0 \). Thus equations (2.2.7)-(2.2.9), (2.1.9) imply that the singularity considered is a tempered one: In accordance with eq. (1.3.1), \( g, \dot{g}, \) and \( \varepsilon \) are continuous at \( t = 0 \), and we may assume that \( \Psi \) is continuous as well. So that the state
\[ \omega = (g, \dot{g}, \Psi) \] (2.3.1)
is continuous at \( t = 0 \). The metric is \( C^1 \). Such a singularity is trivially passable.

### 2.4 Pulsating black hole

It is obvious what is the time evolution of the system considered. The time \( \tau/2 \) is given by [8]
\[ \frac{\tau}{2} = \frac{\pi}{2 \sqrt{k}} = \frac{\pi}{2} \left( \frac{3}{8 \pi \kappa \rho(0)} \right)^{1/2}. \] (2.4.1)
We have in terms of \( R \):
\[ R(-\tau/2) = 1, \ \dot{R}(-\tau/2) = 0; \quad R(0) = 0, \ \dot{R}(0) = 0; \quad R(\tau/2) = 1, \ \dot{R}(\tau/2) = 0; \quad \text{and so on}. \] (2.4.2)
Thus the ball, or the relating black hole pulsates with the period \( \tau \).

It must be emphasized that the case in point is a pulsating black hole rather than a black hole and a white one that are periodically interconverted.

### 2.5 Escape effect

Now we should like to take into account the quantum nature of matter. To do this we consider the motion of a quantum particle in a gravitational potential well generated by the dust ball.

From eqs. (2.1.3), (2.1.6), (2.1.7) it follows
\[ \frac{d^2 y}{dt^2} = -\frac{\kappa M}{y^2}, \quad y = aR. \] (2.5.1)
An analogous equation holds \[9\] for a quantity

\[\eta = aR_\eta, \quad R_\eta < R,\]  

i.e.,

\[\frac{d^2\eta}{dt^2} = -\frac{\kappa M_\eta}{\eta^2}\]  

(2.5.3)

where

\[M_\eta = M(\eta/y)^3\]  

(2.5.4)

is the mass of the ball with radius \(\eta\). Equations (2.5.3),(2.5.4) imply that the potential energy of a particle with mass \(m\) is

\[U(\eta) = \begin{cases} -u + (\gamma/2)\eta^2 & \text{for } \eta \leq y \\ -\kappa Mm/\eta & \text{for } \eta \geq y. \end{cases}\]  

(2.5.5)

From the conditions

\[U\big|_{y+0} = U\big|_{y-0}, \quad \frac{dU}{d\eta}\big|_{y+0} = \frac{dU}{d\eta}\big|_{y-0}\]  

(2.5.6)

it follows

\[U(\eta) = \frac{\kappa Mm}{2y^3}(\eta^2 - 3y^2) \quad \text{for } \eta \leq y.\]  

(2.5.7)

We find for the average value inside the ball

\[\langle U \rangle = \frac{1}{y^3/3} \int_0^y d\eta \eta^2 U(\eta) = -\frac{6\kappa Mm}{5y} \cong -\frac{\kappa Mm}{y}.\]  

(2.5.8)

Thus we may consider the particle in a well of radius \(y\) and depth

\[U_0(y) = \frac{\kappa Mm}{y}.\]  

(2.5.9)

The criterion for the existence of at least one discrete level is

\[U_0y^2 > \frac{\pi^2}{8m} \cong \frac{1}{m},\]  

(2.5.10)

which is valid both for the nonrelativistic and the relativistic case \[10\]. Thus for

\[y < \frac{1}{\kappa Mm_{\text{max}}}, \quad m_{\text{max}} < M,\]  

(2.5.11)

there is no matter inside the ball. We call this the escape effect. Needless to say, the effect is quantum.

### 2.6 Violation of the constraint equation of general relativity

The 0\(\mu\)-components of the Einstein equation give

\[G_{0\mu} = 8\pi\kappa T_{0\mu} \Rightarrow \dot{R}^2 + k = \frac{8\pi\kappa}{3} \rho(t)R^2,\]  

(2.6.1)

which is the constraint equation for the problem considered. After the escape, eq.(2.6.1) reduces to

\[\dot{R}^2 + k = 0,\]  

(2.6.2)

which contradicts eq.(2.1.6). This result shows once again that generally only the projected Einstein equation is fulfilled.
2.7 Naked singularity

For a naked singularity, eq.(2.1.6) is replaced by [11]

\[ \dot{R}^2 = \frac{F(r)}{R} + f(r), \]

so that the principal results do not change.

3 Big cram: big crunch and big bang

3.1 The Robertson-Walker spacetime and the cosmic-length universe

We consider a global, or cosmic singularity for the cosmic-length universe [3,4]—a model based on the Robertson-Walker spacetime.

The Robertson-Walker metric is of the form

\[ g = dt^2 - R^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2d\theta^2 + r^2 \sin^2 \theta \, d\varphi^2 \right\}, \quad k = -1, 0, 1. \]  

We are interested in the closed universe, \( k = 1 \). The projected Einstein equation gives

\[ G_{ij} = 8\pi\kappa T_{ij} \Rightarrow 2\ddot{R}R + \dot{R}^2 + k = -8\pi\kappa pR^2. \]

The equation of motion for matter is

\[ T^0_{\ nu} = 0, \]

or

\[ d\varepsilon = -pdV, \]

where

\[ \varepsilon = \rho V, \]

\[ V = 2\pi^2 R^3. \]

Eqs.(3.1.2),(3.1.4) lead to

\[ R\ddot{R}^2 + kR - \frac{8\pi\kappa}{3}\rho R^3 = L = \text{const} \]

where \( L \) is the cosmic length [4].

3.2 Equation of state

The equation of state for matter is

\[ p = \gamma\rho, \quad \gamma = \gamma(V), \quad -1 \leq \gamma \leq \frac{1}{3}. \]
From eqs. 3.1.4, 3.2.1 it follows
\[ \frac{d\rho}{\rho} = -\frac{1 + \gamma(V)}{V} dV, \]  
whence
\[ \frac{\rho V}{\rho_0 V_0} \exp \left\{ \int_{V_0}^V \frac{\gamma(V')}{V'} dV' \right\} = 1. \]  
(3.2.3)

With singularity in mind, we put
\[ V_0 \to +0, \quad V \to +0, \quad \gamma(V') = \gamma(0) \]  
(3.2.4)
and obtain
\[ \rho V^{1+\gamma(0)} = \text{const}, \]  
(3.2.5)
which is the equation of state of matter for small \( V \).

### 3.3 Equation of motion for metric

From here on we shall use the equation of state of matter (3.2.3). We have
\[ \varepsilon = \frac{\beta}{R^{3\gamma(0)}}, \quad \beta = \text{const}. \]  
(3.3.1)

From eqs. 3.1.7, 3.1.5, 3.1.6, 3.3.1 we obtain
\[ R\ddot{R}^2 + kR = \frac{B}{R^{3\gamma(0)}} + L \]  
(3.3.2)
where
\[ B = \frac{4\kappa\beta}{3\pi}. \]  
(3.3.3)

The metric (3.1.1) depends on
\[ q = R^2. \]  
(3.3.4)
We obtain in terms of \( q \)
\[ \varepsilon = \frac{\beta}{q^{3\gamma(0)/2}} \]  
(3.3.5)
and the equation of motion for the metric:
\[ \dot{q}^2 + 4kq = 4q^{1/2} \left\{ \frac{B}{q^{3\gamma(0)/2}} + L \right\}. \]  
(3.3.6)

### 3.4 Singularity

In the neighborhood of the singularity \( q = 0 \), eq. 3.3.6 reduces to
\[ \dot{q}^2 = 4q^{1/2} \left\{ \frac{B}{q^{3\gamma(0)/2}} + L \right\}. \]  
(3.4.1)

Let \( q = 0 \) correspond to \( t = 0 \).
3.5 Nonpositive pressure: tempered singularity and trivial passage

Let

\[ -1 \leq \gamma(0) \leq 0 \]  

(3.5.1)

(the values \(-1\) and \(0\) relate to the false vacuum and dust respectively). Then eq. (3.4.1) reduces to

\[ \dot{q}^2 = 4LBq^{1/2} \]  

(3.5.2)

where

\[ L_B = \begin{cases} L & \text{for } \gamma(0) < 0 \\ L + B & \text{for } \gamma(0) = 0, \end{cases} \]  

(3.5.3)

so that

\[ \frac{dq}{dt} = \mp 2L_B^{1/2} q^{1/4}. \]  

(3.5.4)

Taking into account eqs. (2.2.5)-(2.2.9), we obtain

\[ q = \left(\frac{3}{2}\right)^{4/3} L_B^{2/3} t^{4/3}, \]  

(3.5.5)

\[ \dot{q} = 2 \left(\frac{3}{2}\right)^{1/3} L_B^{2/3} t^{1/3}, \]  

(3.5.6)

\[ \ddot{q} = \left(\frac{2}{3}\right)^{2/3} L_B^{2/3} t^{-2/3}. \]  

(3.5.7)

Eq. (3.3.5) results in

\[ \varepsilon = \beta \left[ \frac{3}{2} \right]^2 L_B^{[\gamma(0)]} t^{2[\gamma(0)]}. \]  

(3.5.8)

Thus the singularity is tempered and the metric is \(C^1\). The passage of the singularity is trivial.

3.6 Positive pressure: strong singularity and CPT passage

Let

\[ 0 < \gamma(0) \leq \frac{1}{3} \]  

(3.6.1)

(the value \(1/3\) relates to relativistic matter). From eq. (3.3.5) it follows

\[ \lim_{q \to 0} \varepsilon = \infty, \]  

(3.6.2)

so that the singularity is strong. Eq. (3.4.1) reduces to

\[ \dot{q}^2 = 4Bq^{[1-3\gamma(0)]/2}, \]  

(3.6.3)

whence

\[ \frac{dq}{dt} = \mp 2B^{1/2} q^{\alpha} \]  

(3.6.4)
where
\[ \alpha = \frac{1 - 3\gamma(0)}{4}, \quad 0 \leq \alpha < \frac{1}{4} \]  
(3.6.5)

We obtain for \( \alpha > 0 \):

\[ q = wt^{1/(1-\alpha)}, \]  
(3.6.6)
\[ \dot{q} = \frac{1}{1-\alpha} wt^{\alpha/(1-\alpha)}, \]  
(3.6.7)
\[ \ddot{q} = \frac{\alpha}{(1-\alpha)^2} wt^{-(1-2\alpha)/(1-\alpha)}, \]  
(3.6.8)

where

\[ w = [2B^{1/2}(1-\alpha)]^{1/(1-\alpha)}; \]  
(3.6.9)

the metric is \( C^1 \).

We have for \( \alpha = 0 \):

\[ q = 2B^{1/2} \left\vert t \right\vert, \]  
(3.6.10)
\[ \dot{q} = \begin{cases} -2B^{1/2} & \text{for } t < 0 \\ 2B^{1/2} & \text{for } t > 0, \end{cases} \]  
(3.6.11)
\[ \ddot{q} = 4B^{1/2}\delta(t); \]  
(3.6.12)

the metric is \( C \).

Thus, for a positive pressure, the cosmic singularity is strong and the passage is a \( CPT \) one.

### 3.7 Quantum matter

Now let us take into account the quantum nature of matter. In the case of the closed universe with radius \( R \), we have for the momentum of a quantum particle

\[ p \geq \frac{1}{R} \]  
(3.7.1)

so that on account of

\[ E^2 = m^2 + p^2, \]  
(3.7.2)

we obtain

\[ E > \frac{1}{R} = \frac{1}{q^{1/2}}. \]  
(3.7.3)

Therefore, in view of eq.(3.3.3), if there is some matter in the vicinity of the big cram,

\[ \gamma(0) = \frac{1}{3} \]  
(3.7.4)

holds.

Thus there are two possibilities for the global singularity:

\[ \gamma(0) = -1, \quad \text{the false vacuum}; \]  
(3.7.5)
\[ \gamma(0) = \frac{1}{3}, \quad \text{some relativistic matter.} \quad (3.7.6) \]

In the case of the false vacuum, we obtain from eqs. (3.5.3), (3.5.5)-(3.5.8):

\[
q = \left( \frac{3}{2} \right)^{4/3} L^{2/3} t^{4/3}, \quad (3.7.7)
\]
\[
\dot{q} = 2 \left( \frac{3}{2} \right)^{1/3} L^{2/3} t^{1/3}, \quad (3.7.8)
\]
\[
\ddot{q} = \left( \frac{2}{3} \right)^{2/3} L^{2/3} t^{-2/3}, \quad (3.7.9)
\]

and

\[
\epsilon = \beta \left( \frac{3}{2} \right)^{2} L t^{2}; \quad (3.7.10)
\]

the singularity is tempered, the metric is \( C^{1} \), and the passage is trivial.

In the case of relativistic matter, we obtain from eqs. (3.6.10)-(3.6.12), (3.3.5), (3.3.3):

\[
q = 2B^{1/2} \mid t \mid, \quad (3.7.11)
\]
\[
\dot{q} = 2B^{1/2}[ -\theta(-t) + \theta(t) ], \quad (3.7.12)
\]
\[
\ddot{q} = 4B^{1/2} \delta(t), \quad (3.7.13)
\]

and

\[
\epsilon = \frac{(3\pi)^{1/4} \beta^{3/4}}{2\kappa^{1/4}} \frac{1}{\mid t \mid^{1/2}}; \quad (3.7.14)
\]

the singularity is strong, the metric is \( C \), and the passage is a \( CPT \) one.

### 3.8 The oscillating universe

The passableness for the cosmic singularity implies the oscillating universe (for some discussion, see [1]). We have for the universe:

... expansion, contraction, big cram (big crunch followed by big bang), expansion ...

\[(3.8.1)\]

### References

[1] Vladimir S. Mashkevich, *Indeterministic Quantum Gravity* (gr-qc/9409010, 1994).

[2] Vladimir S. Mashkevich, *Indeterministic Quantum Gravity II. Refinements and Developments* (gr-qc/9505034, 1995).

[3] Vladimir S. Mashkevich, *Indeterministic Quantum Gravity III. Gravidynamics versus Geometrodynamics: Revision of the Einstein Equation* (gr-qc/9603022, 1996).
[4] Vladimir S. Mashkevich, *Indeterministic Quantum Gravity IV. The Cosmic-length Universe and the Problem of the Missing Dark Matter* (gr-qc/9609035, 1996).

[5] Vladimir S. Mashkevich, *Indeterministic Quantum Gravity V. Dynamics and Arrow of Time* (gr-qc/9609046, 1996).

[6] Vladimir S. Mashkevich, *Indeterministic Quantum Gravity and Cosmology VI. Predynamical Geometry of Spacetime Manifold, Supplementary Conditions for Metric, and CPT* (gr-qc/9704033, 1997).

[7] Ronald Adler, Maurice Bazin, Menachem Schiffer, *Introduction to General Relativity* (McGraw-Hill, Book Company, New York etc., 1975).

[8] Steven Weinberg, *Gravitation and Cosmology* (John Wiley and Sons, Inc., New York etc., 1972).

[9] Alan P. Lightman, William H. Press, Richard H. Price, Saul A. Teukolsky, *Problem Book in Relativity and Gravitation* (Princeton University Press, Princeton, New Jersey, 1975).

[10] V.M. Galickiy, B.M. Karnakov, V.I. Kogan, *Problems in Quantum Mechanics* (Nauka, Moscow, 1981, in Russian).

[11] S. Jhingan and P.S. Joshi, *The Structure of Singularity in Spherical Inhomogeneous Dust Collapse* (gr-qc/9701016, 1997).