FREE LÉVY PROCESSES ON DUAL GROUPS

UWE FRANZ

ABSTRACT. We give a short introduction to the theory of Lévy processes on dual groups. As examples we consider Lévy processes with additive increments and Lévy processes on the dual affine group.

1. Introduction

Lévy processes play a fundamental rôle in probability theory and have many important applications in other areas such as statistics, financial mathematics, functional analysis or mathematical physics, as well.

In quantum probability they first appeared in a model for the laser in [Wal84]. This lead to the theory of Lévy processes on involutive bialgebras, cf. [ASW88, Sch93, FS99]. The increments of these Lévy processes are independent in the sense of tensor independence, which is a straightforward generalisation of the notion of independence used in classical probability theory. However, in quantum probability there exist also other notions of independence like, e.g., freeness [VDN92], see Paragraph 2.2. In order to formulate a general theory of Lévy processes for these independences, the *-bialgebras or quantum groups have to be replaced by the dual groups introduced in [Voi87], see [Sch95b, BGS99, Fra01a, Fra01b].

In this paper we give an introduction to the theory of Lévy processes on dual groups, which avoids most of the algebraic prerequisites. In particular, we will not define dual groups, but only consider two examples, namely tensor and free Lévy processes with additive increments and tensor and free Lévy processes on the dual affine group. Our approach is similar to rewriting the definition of classical Lie group-valued Lévy processes in terms of a coordinate system, see Definitions 3.1, 3.2, 4.1, and 4.3.

Quantum Lévy processes play an important rôle in the theory of continuous measurement, cf. [Hol01], and in the theory of dilations, where they describe the evolution of a big system or heat bath, which is coupled to the small system whose evolution one wants to describe.

Additive free Lévy processes where first studied in [GSS92], and more recently in [Bia98, Ans01a, Ans01b, BNT01a, BNT01b].

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2. Preliminaries

In this section we introduce the basic notions and definitions that we will use. For more detailed introductions to quantum probability see, e.g., [Par92, Bia93, Mey95, Hol01].

2.1. Quantum probability. Non-commutative probability or quantum probability is motivated by the statistical interpretation of quantum mechanics where an operator is interpreted as an analog of a random variable. The role of the classical probability space is played by a (pre-)Hilbert space $\mathcal{H}$ and the measure is replaced by a unit vector $\Omega \in \mathcal{H}$ called state vector.

In this paper we will mean by a (real) quantum random variable $X$ on $(\mathcal{H}, \Omega)$ a (symmetric) linear operator on the pre-Hilbert space $\mathcal{H}$, which has an adjoint, i.e. for which there exists a linear operator $X^*$, such that

$$\langle u, Xv \rangle = \langle X^* u, v \rangle$$

for all $u, v \in \mathcal{H}$. Its law (w.r.t. the state vector $\Omega$) is the functional $\phi_X : \mathbb{C}[x] \to \mathbb{C}$ on the algebra $\mathbb{C}[x]$ of polynomials in one variable defined by

$$\phi_X(x^k) = \langle \Omega, X^k \Omega \rangle,$$

for $k \in \mathbb{N}$. If $X$ is symmetric, then there exists a (possibly non-unique) probability measure $\mu$ on $\mathbb{R}$ such that

$$\phi_X(x^k) = \int_{\mathbb{R}} x^k d\mu.$$

Let $X$ be a classical $\mathbb{R}$- or $\mathbb{C}$-valued random variable with finite moments on some probability space $(M, \mathcal{M}, P)$. It becomes a quantum random variable on $\mathcal{H} = L^\infty(M, \mathcal{M}, P)$, if we let it act on the bounded functions on $M$ by multiplication, $L^\infty(M) \ni f \mapsto Xf \in L^\infty(M)$ with $Xf(m) = X(m)f(m)$ for $m \in M$. If we take the constant function $\Omega(m) = 1$ for all $m \in M$ for the state vector, then we recover the classical distribution of $X$, i.e.,

$$\langle \Omega, X^k \Omega \rangle = \int_M X^k dP = \mathbb{E}(X^k),$$

for $k \in \mathbb{N}$. If $X$ is $\mathbb{R}$-valued, then it is also real as quantum random variable, i.e. symmetric.

A (real) quantum random vector on $(\mathcal{H}, \Omega)$ is an $n$-tuple $X = (X_1, \ldots, X_n)$ of (real) quantum random variables on $(\mathcal{H}, \Omega)$. Its law is the functional $\phi_X : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}$ on the algebra of non-commutative polynomials $\mathbb{C}[x_1, \ldots, x_n]$ in $n$ variables defined by

$$\phi_X(x_{i_1}^{k_1} \cdots x_{i_r}^{k_r}) = \langle \Omega, X_{i_1}^{k_1} \cdots X_{i_r}^{k_r} \Omega \rangle,$$

for all $i_1, \ldots, i_r \in \{1, \ldots, n\}$, $k_1, \ldots, k_r \in \mathbb{N}$.

A (real) operator process is an indexed family $(X_i)_{i \in I}$ on $(\mathcal{H}, \Omega)$ of (real) quantum random variables or (real) quantum random vectors on $(\mathcal{H}, \Omega)$. We will call
two operator processes \((X_i)_{i \in I}\) and \((Y_i)_{i \in I}\) equivalent, if they have the same joint moments, i.e. if
\[
\langle \Omega, X_{i_1}^{k_1} \cdots X_{i_r}^{k_r} \Omega \rangle = \langle \Omega, Y_{i_1}^{k_1} \cdots Y_{i_r}^{k_r} \Omega \rangle
\]
for all \(i_1, \ldots, i_r \in I\) and all \(k_1, \ldots, k_r \in \mathbb{N}\).

2.2. Freeness and Independence. Let now \(A_1, \ldots, A_k\) be algebras of adjointable linear operators on some pre-Hilbert space \(\mathcal{H}\), closed under taking adjoints and containing the identity operator 1.

2.1. Definition. \(A_1, \ldots, A_k\) are called tensor independent (w.r.t. to the state vector \(\Omega\)), if
(i) for all \(1 \leq i, j \leq k\) and all \(X \in A_i\) and \(Y \in A_j\), we have
\[
[X, Y] := XY - YX = 0,
\]
(ii) and for all \(X_1 \in A_1, \ldots, X_k \in A_k\) we have
\[
\langle \Omega, X_1 \cdots X_k \Omega \rangle = \langle \Omega, X_1 \Omega \rangle \cdots \langle \Omega, X_k \Omega \rangle.
\]

This definition is the natural analogue of the notion of independence in classical probability to our setting. It is also the one used in quantum physics when one speaks of independent observables. But in quantum probability there exist other, inequivalent notions of independence.

2.2. Definition. \(A_1, \ldots, A_k\) are called free, if for all \(1 \leq i_1, \ldots, i_r \leq k\) with \(i_1 \neq i_2 \neq \cdots \neq i_r\) (i.e. neighboring indices are different) and all \(X_1 \in A_{i_1}, \ldots, X_r \in A_{i_r}\) with
\[
\langle \Omega, X_1 \rangle = \cdots = \langle \Omega, X_r \rangle = 0,
\]
we have
\[
\langle \Omega, X_1 \cdots X_r \Omega \rangle = 0.
\]

Quantum random variables or quantum random vectors \(X, Y, Z, \ldots\) are called tensor independent or free, iff the unital \(*\)-algebras they generate are tensor independent or free.

2.3. Remark. These definitions allow to compute arbitrary joint moments of tensor independent or free random variables from their marginal distributions.

For the free case this computation can be done recursively on the order of the moment be expanding
\[
0 = \varphi \left( (X_1 - \varphi(X_1)1) \cdots (X_r - \varphi(X_r)1) \right),
\]
where we wrote \(\varphi(\cdot)\) instead of \(\langle \Omega, \cdot \rangle\) for the expectation.

Let \(X_1\) and \(X_2\) be two free quantum random variables, then one obtains in this way
\[
0 = \varphi \left( (X_1 - \varphi(X_1)1)(X_2 - \varphi(X_2)1) \right) = \varphi(X_1 X_2) - \varphi(X_1)\varphi(X_2),
\]
and therefore
\[ \varphi(X_1X_2) = \varphi(X_1)\varphi(X_2). \]
as for tensor independent quantum random variables or independent classical random variables. But for higher moments the formulas are different, one gets, e.g.,
\[ \varphi(X_1X_2X_1X_2) = \varphi(X_1^2)\varphi(X_2)^2 + \varphi(X_1)^2\varphi(X_2^2) - \varphi(X_1)^2\varphi(X_2)^2. \]
This formula can also be used to show that there exist no non-trivial examples of commuting free quantum random variables. If \( X_1 \) and \( X_2 \) commute, then we would get
\[ \varphi(X_1X_2X_1X_2) = \varphi(X_1^2)\varphi(X_2^2) = \varphi(X_1^2)\varphi(X_2^2), \]
since \( X_1^2 \) and \( X_2^2 \) are also free. Therefore
\[ \varphi(X_1 - \varphi(X_1)1)\varphi(X_2 - \varphi(X_2)1) = 0, \]
i.e. at least one of the two quantum random variables has a trivial distribution.

2.4. Remark. Besides tensor independence and freeness there exist other notions of independence that are used in quantum probability. In a series of papers \[ \text{[Sch95a, Spe97, BGS99, Mur01, Mur02]} \] it was shown that there exist exactly five “universal” notions of independence satisfying a natural set of axioms. Besides tensor independence and freeness these are boolean, monotone, and anti-monotone independence. In \[ \text{[Fra01b]} \] the boolean, monotone, and anti-monotone independence were reduced to tensor independence. If this is also possible for freeness is still an open problem.

3. Additive Lévy Processes

3.1. Definition. An operator process \( (X_t)_{t \geq 0} \) on \( (\mathcal{H}, \Omega) \) is called an additive tensor Lévy process (w.r.t. \( \Omega \)), if the increments
\[ X_{st} := X_t - X_s, \]
are
(i) tensor independent, i.e. the quantum random variables \( X_{s_1t_1}, \ldots, X_{s_rt_r} \) are tensor independent for all \( 0 \leq s_1 \leq t_1 \leq s_2 \leq \cdots \leq s_r \leq t_r \),
(ii) stationary, i.e. the law of an increment depends only on \( t-s \), and
(iii) weakly continuous, i.e. \( \lim_{t \searrow s} \langle \Omega, X_{st}^k\Omega \rangle = 0 \) for \( k = 1, 2, \ldots. \)

Replacing tensor independence by another universal notion of independence we can define the corresponding other classes of Lévy processes. E.g., for freeness we get the following definition.
3.2. Definition. An operator process \((X_t)_{t \geq 0}\) on \((\mathcal{H}, \Omega)\) is called an additive free Lévy process (w.r.t. \(\Omega\)), if the increments
\[
X_{st} := X_t - X_s,
\]
are
(i') free, i.e. the quantum random variables \(X_{s_1 t_1}, \ldots, X_{s_r t_r}\) are free for all \(0 \leq s_1 \leq t_1 \leq s_2 \leq \cdots \leq s_r \leq t_r\),
and satisfy conditions (ii) and (iii) of Definition 3.1.

For each of these notions of independence one can define a Fock space and creation, annihilation, and conservation or gauge operators on this Fock space.

For example the (algebraic) free Fock space \(\mathcal{H} = \mathcal{F}(\mathfrak{h})\) over a (pre-)Hilbert space \(\mathfrak{h}\) is defined as
\[
\mathcal{F}(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} \mathfrak{h}^\otimes n
\]
where \(\mathfrak{h}^\otimes 0 = \mathbb{C}\). The vector \(\Omega = 1 + 0 + \cdots\) is called the vacuum vector. For a vector \(u \in \mathfrak{h}\) we can define the creation operator \(a^+(u)\) and the annihilation operator \(a^-(u)\) by
\[
a^+(u) u_1 \otimes \cdots u_k = u \otimes u_1 \otimes \cdots \otimes u_k,\\
a^-(u) u_1 \otimes \cdots u_k = \langle u, u_1 \rangle u_2 \otimes \cdots \otimes u_k.
\]
These operators are mutually adjoint, on the vacuum vector they act as \(a^+(u) \Omega = u\) and \(a^-(u) \Omega = 0\).

The conservation operator \(\Lambda(X)\) of some linear operator \(X\) on \(\mathfrak{h}\) is defined by
\[
\Lambda(X) u_1 \otimes \cdots u_k = (X u_1) \otimes u_2 \otimes \cdots \otimes u_k
\]
and \(\Lambda(X) \Omega = 0\). It satisfies \(\Lambda(X)^* = \Lambda(X^*)\).

Glockner, Schürmann, and Speicher have shown that every additive free Lévy process can be realized as a linear combination of these three operators and time.

3.3. Theorem. [GSS92] Let \((X_t)_{t \geq 0}\) be an additive free Lévy process. Then there exists a pre-Hilbert space \(\mathfrak{k}\), a linear operator \(T\) on \(\mathfrak{k}\), vectors \(u, v \in \mathfrak{k}\), and a scalar \(\lambda \in \mathbb{C}\) such that \((X_t)_{t \geq 0}\) is equivalent to the operator process \((X'_t)_{t \geq 0}\) on the free Fock space \(\mathcal{F}_T(L^2(\mathbb{R}_+, \mathfrak{k}))\) over \(\mathfrak{h} = L^2(\mathbb{R}_+, \mathfrak{k}) \cong L^2(\mathbb{R}_+) \otimes \mathfrak{k}\) defined by
\[
X'_t = \Lambda(\chi_{[0,t]} \otimes T) + a^+(\chi_{[0,t]} \otimes u) + a^-(\chi_{[0,t]} \otimes v) + t \lambda \mathbf{1}
\]
for \(t \geq 0\). Furthermore, if we require that \(\mathfrak{k}\) is spanned by \(\{T^k u, T^k v | k = 0, 1, \ldots \}\), then \(\mathfrak{k}\), \(T\), \(u\), \(v\), and \(\lambda\) are unique up to unitary equivalence.

\((X'_t)_{t \geq 0}\) is symmetric, if and only if \(T^* = T\), \(u = v\) and \(\lambda \in \mathbb{R}\) in the unique minimal tuple.
3.4. Remark. Analogous results hold for the other universal independences. For tensor independence see [Sch91b], for the boolean [BG01], and [Fra01b] for the monotone case. Note that in the boolean and in the monotone case the time process has to be modified.

The five independences can also be used to define convolutions for compactly supported measures. Let \( \mu_1 \) and \( \mu_2 \) be two compactly supported probability measures on \( \mathbb{R} \) and choose two independent real quantum random variables \( X_1 \) and \( X_2 \) on some pre-Hilbert space \( \mathcal{H} \) such that

\[
\langle \Omega, X_i^k \Omega \rangle = \int_{\mathbb{R}} x^k \, d\mu_i
\]

for all \( k \in \mathbb{N} \) and \( i = 1, 2 \) and some unit vector \( \Omega \in \mathcal{H} \). The operator \( X_1 + X_2 \) is again symmetric and bounded, therefore there exists a unique compactly supported probability measure \( \mu \) such that

\[
\int_{\mathbb{R}} x^k \, d\mu = \langle \Omega, (X_1 + X_2)^k \Omega \rangle
\]

for all \( k \in \mathbb{N} \)

It is always possible to construct such a pair and the law of \( X_1 + X_2 \) depends only on the laws of \( X_1 \) and \( X_2 \) and the notion of independence that has been chosen.

If \( X_1 \) and \( X_2 \) are tensor independent, then the measure \( \mu \) obtained in this way is the usual additive convolution of \( \mu_1 \) and \( \mu_2 \). If \( X_1 \) and \( X_2 \) are free, then \( \mu \) is the free additive convolution of \( \mu_1 \) and \( \mu_2 \).

These convolutions can actually be defined for arbitrary probability measures.

It is possible to show that infinitely divisible measures can be embedded into continuous convolution semigroups in all five cases and that furthermore there exists a Lévy process for every continuous convolution semigroup. This shows that in all five cases the infinitely divisible measures on \( \mathbb{R} \) (which are characterized by their moments) can be classified by tuples \((\mathfrak{r}, T, u, \lambda)\) consisting of a pre-Hilbert space \( \mathfrak{r} \), a symmetric operator \( T \) on \( \mathfrak{r} \), a vector \( u \in \mathfrak{r} \), and a real number \( \lambda \).

3.5. Corollary. There exist bijections (up to moment uniqueness) between the five classes of infinitely divisible measures with finite moments.

3.6. Remark. The bijection between the usual infinitely divisible measures and the freely infinitely divisible measures is known under the name Pata-Bercovici bijection, cf. [BP99], it actually extends to all infinitely divisible measures, not just those characterized by their moments, and has many useful properties, cf. [BNT01a] and the references therein.

For example, the Bercovici-Pata bijection is a homomorphism between the usual infinitely divisible measures and the freely infinitely divisible measures and their respective convolutions. This is not the case for the bijection between usual
ininitely divisible measures and the monotone infinitely divisible measures, because due to the non-commutativity of the monotone convolution this is impossible. For the Lévy-Khintchine formula for the boolean and monotone case, see [SW97, Mur00].

3.7. Definition. Let \((X_t)_{t \geq 0}\) be a real additive Lévy process for one of the five universal independences.

If there exists a tuple \((k, T, u, \lambda)\) for \((X_t)_{t \geq 0}\) with \(T = 0\), then \((X_t)_{t \geq 0}\) is called Gaussian.

If there exists a tuple \((k, T, u, \lambda)\) for \((X_t)_{t \geq 0}\) and vector \(\omega \in k\) such that \(u = T\omega\) and \(\lambda = \langle \omega, T\omega \rangle\), then \((X_t)_{t \geq 0}\) is called a compound Poisson process.

If \((X_t)_{t \geq 0}\) is Gaussian, then the unique minimal tuple associated to it by Theorem 3.3 has the form \((C, 0, z, \lambda)\) and \((X_t)_{t \geq 0}\) can be realized as a sum of creation, annihilation and time only, with no conservation part.

3.8. Example. Let \((X_t)_{t \geq 0}\) be a classical compound Poisson process with Lévy measure \(\mu\), i.e. with characteristic function

\[
\mathbb{E}(e^{iuX_t}) = \exp\left(t \int_{\mathbb{R}\setminus\{0\}} (e^{ix} - 1) d\mu(x)\right).
\]

We assume that \(\mu\) has finite moments, then \((X_t)_{t \geq 0}\) is an additive tensor Lévy process in the sense of Definition 3.1. We can define a tuple \((k, T, u, \lambda)\) for \((X_t)_{t \geq 0}\) by Theorem 3.3 as follows. For the pre-Hilbert space \(k\) we take the space of polynomials

\[
k = \text{span}\{x^k; k = 0, 1, 2, \ldots\},
\]

considered as a subspace of the Hilbert space \(L^2(\mathbb{R}, \mu)\), i.e., with the inner product

\[
\langle x^k, x^\ell \rangle = \int_{\mathbb{R}} x^{k+\ell} d\mu(x),
\]

and divided by the the nullspace of this inner product, if \(\mu\) is finitely supported. The operator \(T\) is multiplication by \(x\), i.e., \(Tx^k = x^{k+1}\), the vector \(u\) is the function \(f(x) = x\), and the scalar \(\lambda\) is the first moment of \(\mu\), i.e., \(\lambda = \int_{\mathbb{R}} x d\mu(x)\).

Taking for \(\omega\) the constant function 1, we see that \((X_t)_{t \geq 0}\) is also a compound Poisson process in sense of Definition 3.7.

Theorem 3.3 can also be used to give an Itô-Lévy-type decomposition of additive quantum Lévy processes.

3.9. Corollary. Let \((X_t)_{t \geq 0}\) be a real additive Lévy process for one of the five universal independences. Then \((X_t)_{t \geq 0}\) can be realized as a sum of a Gaussian Lévy process \((X^G_t)_{t \geq 0}\) and a “jump” part \((X^P_t)_{t \geq 0}\), which can be approximated by (compensated) compound Poisson processes.
Proof. We only briefly outline the proof.

Let \((\mathfrak{t}, T, u, \lambda)\) be a tuple for \((X_t)_{t \geq 0}\). Since \(T\) is symmetric, we can decompose the closure of \(\mathfrak{t}\) into a direct sum of the closures of the kernel of \(T\) and the image of \(T\). Let \(u = u_0 + u_1\) with \(u_0 \in \ker T\) and \(u_1 \in \im T\).

If \(u_1\) is actually in the image of \(T\), then there exists a vector \(\omega \in k\) with \(T\omega = u_1\) and \((X_t)_{t \geq 0}\) is equivalent to the sum of the Gaussian Lévy process \((X_t^G)_{t \geq 0}\) with the tuple \((\mathbb{C}, 0, u_0, \lambda - \langle \omega, T\omega \rangle)\) and the compound Poisson process \((X_t^P)_{t \geq 0}\) with the tuple \((\mathfrak{t}, T, u_1, \langle \omega, T\omega \rangle)\).

If \(u_1\) is not in the image of \(T\), then we take a sequence \(\omega_n \in k\) such that \(\lim T\omega_n = u_1\) and define the “jump” part by

\[
X_t^P = \Lambda(\chi_{[0,t]} \otimes T) + a^+(\chi_{[0,t]} \otimes u_1) + a^-(\chi_{[0,t]} \otimes u_1)
\]

\[
= \lim_{n \to \infty} \left( \chi_{[0,t]} \otimes \Lambda(T) + a^+ \left( \chi_{[0,t]} \otimes (T\omega_n) \right) + a^- \left( \chi_{[0,t]} \otimes (T\omega_n) \right) \right),
\]

i.e. as the limit of compensated compound Poisson processes. The Gaussian part is then determined by the tuple \((\mathbb{C}, 0, u_0, \lambda)\).

3.10. Remark. Using the spectral representation \(T = \int_\mathbb{R} x dP_x\) of the closure of \(T\), the “jump” part can be written as an integral over the “jump” sizes.

However, note that the Itô-Lévy-type decomposition gives a decomposition into a continuous Gaussian part and a jump part only in the tensor case. In the other cases the classical processes that can be associated to the corresponding Gaussian processes do not have continuous paths, see, e.g., [Bia98].

Using different methods and not assuming the existence of moments, Barndorff-Nielsen and Thorbjørnson [BNT01b] have also obtained an Itô-Lévy decomposition for additive free Lévy processes.

4. Lévy Processes on the (Dual) Affine Group

Recall that the affine group can be defined as the group of matrices

\[
\text{Aff} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0, b \in \mathbb{R} \right\}.
\]

The calculation

\[
\begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1b_2 + b_1 \\ 0 & 1 \end{pmatrix}
\]

shows that the group multiplication takes the form

\[
A(g_1g_2) = A(g_1)A(g_2), \quad B(g_1g_2) = A(g_1)B(g_2) + B(g_1),
\]

for the coordinates \(A, B\) defined by

\[
A \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = a, \quad B \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = b.
\]
We define tensor Lévy processes on the dual affine group in term of increments. Since the increments are tensor independent for disjoint time intervals, they commute, and we can write the products in the multiplication formula in any order we like.

4.1. Definition. An operator process \(((A_{st}, B_{st})_{0 \leq s \leq t})\) on \((\mathcal{H}, \Omega)\) is called a (left) tensor Lévy process on the dual affine group (w.r.t. \(\Omega\)), if the following four conditions are satisfied.

(i) (Increment property) For all \(0 \leq s \leq t \leq u\),
\[
A_{su} = A_{tu}A_{st},
B_{su} = A_{tu}B_{st} + B_{tu}.
\]

(ii) (Independence) The increments \((A_{s_1t_1}, B_{s_1t_1}), \ldots, (A_{s_rt_r}, B_{s_rt_r})\) are tensor independent for all \(0 \leq s_1 \leq t_1 \leq s_2 \leq \cdots \leq s_r \leq t_r\).

(iii) (Stationarity) The law of \((A_{st}, B_{st})\) depends only on \(t - s\).

(iv) (Weak continuity) For all \(k_1, \ldots, k_r, \ell_1, \ldots, \ell_r \in \mathbb{N}\), we have
\[
\lim_{t \searrow s} \langle \Omega, A_{st}^{k_1}B_{st}^{\ell_1} \cdots A_{st}^{k_r}B_{st}^{\ell_r} \Omega \rangle = \begin{cases} 1 & \text{if } \ell_1 + \cdots \ell_r = 0, \\ 0 & \text{if } \ell_1 + \cdots \ell_r > 0. \end{cases}
\]

Every Lévy process with values in the semi-group
\[
\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R} \right\}
\]
with finite moments gives an example of a tensor Lévy process on the dual affine group in the sense of our definition, since we didn’t impose that the \(A_{st}\) are strictly positive or invertible.

4.2. Example. There are also examples in which \(A_{st}\) and \(B_{st}\) do not commute and which do not correspond to classical Lévy processes. E.g., the quantum Azéma martingale \cite{Eme89, Par90, Sch91a} with parameter \(q \in \mathbb{R}\) defined by the quantum stochastic differential equations
\[
dA_{st} = A_{st}d\Lambda_t(q-1),
 dB_{st} = B_{st}d\Lambda_t(q-1) + da^+_t(1) + da^-_t(1),
\]
on the Bose Fock space \(\mathcal{F}_T(L^2(\mathbb{R}_+))\), with initial conditions
\[
A_{ss} = \text{id},
B_{ss} = 0,
\]
defines a tensor Lévy process on the dual affine group. For \(q = 1\), we have \(A_{st} = \text{id}\) for all \(0 \leq s \leq t\) and in the vacuum state \((B_{st})\) is equivalent to classical Brownian motion. For \(q \neq 1\), \(A_{st}\) and \(B_{st}\) do not commute and \((B_{st})\) is equivalent to the classical Azéma martingale with parameter \(c = q - 1\).
When we want to define free Lévy processes, different orders of the products in the multiplication formula will lead to different classes of Lévy processes. The choice in the definition proposed here is motivated by the fact that if $B_{st}$ and $B_{tu}$ are symmetric, then $B_{su}$ is also symmetric.

4.3. Definition. An operator process $\left( (a_{st}, B_{st}) \right)_{0 \leq s \leq t}$ on $(\mathcal{H}, \Omega)$ is called a (left) free Lévy process on the dual affine group (w.r.t. $\Omega$), if the conditions

(i') (Increment property) For all $0 \leq s \leq t \leq u$,

$$a_{su} = a_{tu}a_{st},$$

$$B_{su} = a_{tu}B_{st}a_{st}^* + B_{st}.$$  

(ii') (Independence) The increments $(a_{s_1t_1}, B_{s_1t_1}), \ldots, (a_{s_rt_r}, B_{s_rt_r})$ are free for all $0 \leq s_1 \leq t_1 \leq s_2 \leq \cdots \leq s_r \leq t_r$.

and conditions (iii) and (iv) from the previous definition are satisfied (with $A_{st} = a_{st}a_{st}^*$).

Note that with this definition $A_{st} = a_{st}a_{st}^*$ is automatically positive.

4.4. Example. Let $\gamma \in \mathbb{C}$. The operator process $\left( (a_{st}, B_{st}) \right)_{0 \leq s \leq t}$ defined by the quantum stochastic equations

$$d a_{st} = d\Lambda_t(\gamma - 1)a_{st},$$

$$d B_{st} = d\Lambda_t(\gamma - 1)B_{st} + B_{st}d\Lambda_t(\overline{\gamma} - 1) + da_{st}^+(1) + da_{st}^-(1),$$

on the free Fock space $\mathcal{F}_F(L^2(\mathbb{R}_+))$, with initial conditions

$$a_{ss} = \text{id},$$

$$B_{ss} = 0,$$

defines a free Lévy process on the dual affine group. For $\gamma = 1$, we get $a_{st} = \text{id}$ and $(B_{st})$ is equal to the free Brownian motion,

$$B_{st} = a^+(\chi_{[s,t]}) + a^-(\chi_{[s,t]}).$$

For general $\gamma \in \mathbb{C}$, the process $(B_{st})$ can be considered as a free analog of the (quantum) Azéma martingale with parameter $q = |\gamma|^2$.

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