An interacting particle system for the front of an epidemic advancing through a susceptible population

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Abstract

We propose an interacting particle system with a moving boundary to model the spread of an epidemic. Each individual in a given population starts from some level of shielding, given by a non-negative real number, and this level then evolves over time according to diffusive dynamics driven by independent Brownian motions. If the level of shielding gets too low, individuals will find themselves in 'at-risk' situations, as captured by proximity to a lower moving boundary, which we call the ‘advancing front’ of the epidemic. Specifically, local time is accumulated along this boundary and, while individuals are initially reflected, infection may eventually occur with a likelihood depending on the accumulated local time as well as the intrinsic infectiousness of the disease and the current contagiousness within the population. Our main technical contribution is to give a rigorous formulation of this model in the form of a well-posed interacting particle system. Moreover, we exploit the precise construction of the system to establish an important martingale property of the infected proportion and present a result on its limiting behaviour if the size of the population is sent to infinity.

1 Introduction

Consider a given population of size \( n \) and let \( I_t \) denote the infected proportion accumulated up to time \( t \), let \( C_t \) denote the currently infectious proportion at time \( t \), and \( S_t \) denote the susceptible population at time \( t \), with \( S_t + I_t = 1 \).

In the classical SIR model, the evolution of these quantities is deterministic and governed by the so-called basic reproduction number \( R_0 \). This number is defined as the (average) total number of new cases caused by a single infectious individual in a population where everyone is susceptible, and it is decomposed as

\[
R_0 := \beta \bar{d} = \theta \bar{c} \bar{d},
\]

where \( \theta \) is the intrinsic transmissibility of the disease, \( \bar{c} \) is the rate of contact in the population, and \( \bar{d} \) is the duration of infectiousness. The SIR model then amounts to saying that \( I \) evolves as

\[
\frac{d}{dt} I_t = \beta S_tC_t, \quad \frac{d}{dt} C_t = \frac{d}{dt} I_t - \bar{d}^{-1} C_t,
\]

where the term \( \bar{d}^{-1} C_t \) accounts for recovery from infection at rate \( \bar{d}^{-1} \). Over the duration of an infection, we see that the SIR model produces a total amount of new infections

\[
n(I_{t+\bar{d}} - I_t) \approx n \beta S_tC_t \bar{d} = R_0 S_t \cdot n C_t,
\]

where \( R_t := R_0 S_t \) is called the effective reproduction number at time \( t \).
Taking away some of the focus from the basic reproduction number in \((1)\), it can be instructive to instead decompose the amount of new infections as

\[
n(I_{t+d} - I_t) \approx n\beta S_t C_t \bar{\delta} = \frac{nS_t}{\theta C_t} \cdot \theta C_t \cdot \bar{\delta} \tag{2}
\]

Here we stress that ‘contact’ just refers to contact with another individual, not necessarily contact with a currently infectious individual, which is why \(C_t\) enters in the effective rate of infection given contact. In our model, we wish to change perspective from thinking about a general constant rate of contact, \(\bar{c}\), across the population, to instead thinking about a changing rate at which susceptibles bring themselves in at-risk situations (being in places and behaving in ways that make infection possible). Being at-risk at a given instant means that infection is a real possibility at that time, but it may or may not happen. Replacing \(I\) and \(C\) with their stochastic counterparts \(I^n\) and \(C^n\) from our model, the analogous version of the decomposition \((2)\) then takes the form

\[
n\mathbb{E}[I^n_{t+h} - I^n_t \mid \mathcal{F}^n_t] \approx \mathbb{E}\left[ \sum_{i=1}^n \gamma(t, C^n_i) \cdot \frac{\ell^i_{(t+h)\wedge \tau^i} - \ell^i_{t \wedge \tau^i}}{\theta C^n_i} \mid \mathcal{F}^n_t \right] \tag{3}
\]

where \(\tau^i\) is the infection time of the \(i^{th}\) individual and \((\mathcal{F}^n_t)_{t \geq 0}\) is a suitable filtration to which all the processes are adapted. For any \(h > 0\), the increment \(\ell^i_{(t+h)\wedge \tau^i} - \ell^i_{t \wedge \tau^i}\) captures the magnitude of the at-risk situations that the \(i^{th}\) susceptible individual has found him or herself in during a given time interval \([t, t + h]\). The details of what we mean by this are presented in Section \(1.1\) below: in short, \(\ell^i\) is the local time of the \(i^{th}\) individual along what we term the ‘advancing front’ of the epidemic, \(A^n\) (defined in \((6)\) below).

Before turning to the finer details of our model, we wish to briefly discuss how one can think of the effective reproduction number for our model vis-a-vis the SIR model. In the SIR model, when accounting for new infections in terms of the decomposition \((2)\), we see that the effective reproduction number \(R_t = R_0 S_t\) corresponds to

\[
(n(I_{t+d} - I_t) \text{ keeping fixed } C_{+t} \equiv \frac{1}{n}) \approx nS_t \cdot \theta \cdot \frac{1}{n} \cdot \bar{\delta} = R_t \tag{4}
\]

In view of \((3)\), the analogous expression for the effective reproduction number at time \(t\) in our model is

\[
(n\mathbb{E}[I^n_{t+d} - I^n_t \mid \mathcal{F}^n_t] \text{ keeping fixed } C^n_{+t} \equiv \frac{1}{n}) \approx \mathbb{E}\left[ \sum_{i=1}^n \gamma(t, \frac{1}{n}) (\ell^i_{(t+h)\wedge \tau^i} - \ell^i_{t \wedge \tau^i}) \mid \mathcal{F}^n_t \right], \tag{5}
\]

where the advancing front \(A^n_{+t}\), along which the local times accumulate, is also taken to move only in accordance with \(C^n_{+t} \equiv \frac{1}{n}\). To be precise, what we mean by keeping \(C^n_{+t}\) fixed at \(1/n\) is that all further infections (generated by one initial infection) are not allowed to impact the effective rate of infection \(\gamma\) and the advancing front \(A^n\). As our model is stochastic, the conditional expectation is needed, but we can refer to the same interpretations as in \((3)\) to see that each term within the expectation can be compared with the deterministic and constant quantities on the right-hand side of \((4)\).

### 1.1 The key mechanisms of the particle system

The starting point of our analysis is to represent each individual in the population by an initial level of shielding \(X^{i,n}_0\). This level of shielding is a summary measure of all relevant characteristics...
such as inherent susceptibility to the disease, lifestyle, precautionary measures, and spatial distance from areas where infection is possible. Over time, the level of shielding \( X_{t}^{i,n} \) evolves as a stochastic differential equation driven by Brownian motion.

As the epidemic progresses, it is the individuals with lower levels of shielding that will become infected first. We model this by a moving boundary \( A_{t}^{n} \) which represents the level of shielding above which one is presently safe from being at-risk of infection. Since the level-of-shielding is a stochastic process, there is randomness involved in whether an individual is about to reach this level. We refer to \( A_{t}^{n} \) as the advancing front of the epidemic and take it to be of the form

\[
A_{t}^{n} = a_{0} + \alpha \int_{t-d}^{t} \varrho(t - s) I_{s}^{n} \, ds
\]

for given constants \( a_{0}, \alpha, \bar{d} \geq 0 \) and a given infection-to-recovery kernel \( \varrho \geq 0 \) with \( \text{supp}(\varrho) = [0, \bar{d}] \) and \( \| \varrho \|_{L^1} = 1 \). Here \( I_{t}^{n} \) is the accumulated infected proportion over the period \([0, t]\), defined by

\[
I_{t}^{n} = \frac{1}{n} \sum_{i=1}^{n} 1_{[0, t]}(\tau^{i}),
\]

where \( \tau^{i} \) is the infection time of the \( i^{th} \) individual. We will soon return to discuss the mechanism by which infection may or may not occur given that an individual is at-risk, but right now we only wish to highlight the following: in the definition of the advancing front (6), the infection-to-recovery kernel \( \varrho \) models how the impact of an infected individual starts increasing some time after the actual infection happens, then it slows down, and finally stops affecting the system altogether after \( \bar{d} \) units of time.

Considering the recent COVID-19 pandemic, an example of a sensible infection-to-recovery kernel could be a Weibull distribution with negligible mass after \( \bar{d} = 15 \) days, cut off at that point and normalised. This would be in line with what is typically used to model incubation time and the severity of contagiousness during the infected period for COVID-19 infections in the literature.

As the epidemic progresses, we assume that higher levels of shielding are required to keep individuals away from risk, because the disease is increasing its reach. This needs not just be geographical, at a country-wide scale or similar, but it is also intended to capture the increased reach within cities and communities where the disease continues to linger. Of course, one could also allow for an entirely different notion of spatial reach, if one wishes to model epidemic spread within non-human species. The increasing reach of the disease is captured by the front \( A_{t}^{n} \) being non-decreasing in time. For example, once the disease has spread out to certain areas of a country, we assume that it always remains possible to become infected when visiting these areas, whereas before the epidemic had spread its reach to these areas the probability of infection was null. Naturally, the disease may cease to increase its reach any further for some period, effectively staying dormant, but this does not mean that it has retracted its reach. Rather, it means that the current level of contagiousness has decreased to a low level: this feature is part of what we discuss next.

Since the advancing front is part of our mechanism for determining infections, it is clear that (6) presents a first nonlinearity of the system. Moreover, the infection times themselves will feed into the mechanism for infection through a separate channel, namely the effective rate of infection given that an individual is at-risk. This will depend critically on the current level of contagiousness within the population, in the sense of how large a proportion of the population is currently infected and what stage of the infection each individual is currently at (as modelled by the infection-to-recovery kernel). Whenever a given path of \( X_{t}^{i,n} \) reaches the moving boundary \( A_{t}^{n} \), then we say that the individual \( i \) is at-risk of infection, and we measure the magnitude of this risk in a given period of time by the corresponding increment of the local time of \( X_{t}^{i,n} \) along the boundary. An individual
who is at-risk (meaning that $X^{i,n}_{t}$ is at the boundary) may or may not be infected. Informally, we model this by asking that, conditionally on a realisation of the dynamics of each individual, the probability of infection in $[t, t + h]$, given non-infection up to time $t$, is

$$\gamma(t, C^n_t) \cdot (\ell^{i,n}_{t+h} - \ell^{i,n}_t) + o(\ell^{i,n}_{t+h} - \ell^{i,n}_t),$$

where we have used little-O notation as $h \downarrow 0$. Here $C^n_t$ is the current level of contagiousness in the population captured by

$$C^n_t = \int_{t-d}^t \varrho(t - s)(I^n_s - I^n_{s-d})\, ds.$$

As we discussed in relation to (3), the value $\gamma(t, C^n_t)$ is the effective rate of infection given that an individual is presently at-risk, while the local-time increment $\ell^{i,n}_{t+h} - \ell^{i,n}_t$ quantifies the magnitude of at-risk situations for the $i$th individual in the interval of time $[t, t + h]$. Concerning $C^n_t$, we stress that this captures both how large a proportion of the population is currently infected and how contagious these individuals are (in terms of how far into the course of the disease they are).

In Section 2, we confirm that one can indeed formulate a well-posed particle system which evolves according to the above mechanisms, up to some technical adjustments. The precise statement is given in Theorem 2.3. We note here that it is the property (12) in Theorem 2.3 that clarifies the precise sense in which (7) holds, and we note that it is Theorem 2.5 which justifies (3) and (5).

We conclude this section by discussing some interesting extensions. Firstly, as it is presented here, our model is intended for the short or medium term, meaning a period for which it is acceptable to perform predictions without worrying about re-infection of previously infected individuals. If one is interested in a longer term model, then this could be addressed by allowing for re-insertion of infected individuals at some later point in time when their immunity has waned. Secondly, it could be interesting to study interventions in the system. It may be possible to study this dynamically, by formulating suitable stochastic control problems where individuals and or a government may control the drift $b$ in the dynamics for the level of shielding or the size of the effective rate of infection $\gamma$. In this way, it may be possible to capture effects of different forms of interventions.

Figure 1: A first wave of infections amongst individuals with low levels of shielding, followed by a dormant period turning into a gradual uptick in infections amongst individuals with higher levels of shielding. The vertical axis represents individual levels of shielding. The horizontal axis is time. The red curve shows the advancing front. The green curve shows the (accumulated) infected proportion.
1.2 Related literature

In the existing literature, the interacting particle system that is closest to the one we study here is the following system considered in [Bar20]: it consists of $n$ independent Brownian motions (in one dimension) reflected off a moving boundary, which moves at a speed proportional to the sum of the local times of each Brownian motion along this boundary. Another interesting work in this direction is [BBF18a], which studies a (one-dimensional) SDE reflected off a moving boundary which is a function of the local time of the SDE along the boundary itself (see also [BBF18b] for a financial application of this). These works, however, consider particles that are globally reflected, thus making the analysis quite different from what is needed to handle the notion of infection in our system, which is what drives the moving boundary and determines the effective rate of infection. Still, one can make a precise connection between a version of our system and a suitable extension of the particle system from [Bar20], as we briefly discuss in Remark 1 in the next section.

A rather different, but still closely related line of work is the analysis of particle systems approximating the one-dimensional Stefan problem for the freezing of a super-cooled liquid, see e.g. [HLS19, LS21, NS19, NS20, CRSF]. Unlike our model, these works consider particle systems with immediate absorption upon first reaching the boundary. While not concerned with particle systems, [BS22], and later also [HM22], have recently studied a version of the aforementioned Stefan problem with kinetic under-cooling, using that it can be analysed in terms of the law of a single Brownian motion reflecting off a moving boundary proportional to the loss of mass. The loss of mass comes from the Brownian motion being absorbed after a scalar multiple of its local time surpasses an independent exponential random variable, a mechanism known as elastic killing.

Finally, we wish to stress that there are many interesting works in the PDE literature aimed at modelling the spread of an epidemic, see for example [Had16], [DL15], and references therein. We mention the two former works in particular because they represent well the PDE perspective on approaches similar to ours, where the front of an epidemic is modelled through the free boundary of a one-dimensional PDE capturing the density of a susceptible population. However, we note that both works impose Stefan conditions at the free boundary to model the evolution of the front.
From a PDE point of view, our model is more akin to a Robin type boundary condition at the free boundary with a convective velocity that depends nonlinearly on the system, due to the way in which our effective rate of infection is specified. In a companion paper [FS22], we study the corresponding free boundary problem that characterises the mean-field limit of our particle system, as the size of the population tends to infinity, but we do not discuss this any further here.

2 The interacting particle system: main results

In this section, we first give a precise formulation of the particle system discussed above and confirm its well-posedness (Theorem 2.3). Next, we then present a result on how one can work with the laws of each particle in a more tractable way (Proposition 2.4). Finally, we establish a martingale property of the infected proportion and give a result on its asymptotic behaviour as \( n \to \infty \) (Theorem 2.5). The two latter results are crucial for our analysis of the mean-field problem arising from this system which we study in a companion paper [FS22].

We place the following assumptions on the coefficients already discussed in Section 1.1 as well as on the drift and diffusive coefficients, \( b \) and \( \sigma \), for the stochastic dynamics of the individual levels of shielding.

Assumption 2.1 (Structural conditions). Firstly, we assume that \( b(t, x) \) is Lipschitz continuous in \( x \) and locally integrable in \( t \), and we assume that \( \sigma(t, x) \) is Lipschitz continuous jointly in \( (t, x) \) bounded, and non-degenerate (i.e., bounded away from zero). Secondly, we assume that \( \gamma(t, y) \) is continuous jointly in \( (t, y) \). Finally, we assume that the infection-to-recovery kernel \( \varrho \) is in \( L^1(\mathbb{R}) \) with \( \text{supp}(\varrho) = [0, \bar{d}] \), for a given duration \( \bar{d} > 0 \), and \( \| \varrho \|_{L^1(0, \bar{d})} = 1 \).

In terms of the starting points and the random inputs, we assume the following.

Assumption 2.2 (Exogenous inputs to the particle system). Let a family of starting points \( \{X_i^0\}_{n=1}^\infty \) be given, for all \( n \geq 1 \), where each \( X_i^0 \) is a fixed starting point or random variable with a given distribution \( \pi_i^0 \) on \( (a_0, \infty) \) such that \( \int_{a_0}^\infty e^{\delta_i x^2} d\pi_i^0(x) < \infty \) for some \( \delta_i > 0 \). Let \( \{B_j^n\}_{n,j=1} \) be a family of independent standard Brownian motions, independent of the initial conditions. Finally, let \( \{\chi^{j,(k)}_{i,n}\}_{i,j,n,k=0} \) be a given collection of independent standard Exponential random variables, independent of the Brownian motions and the starting points.

As in [BN02] and [Bar20], we define the local time \( \ell^h \) of a real-valued continuous semimartingale \( X \) along a real-valued continuous curve \( t \mapsto h(t) \) as

\[
\ell^h_t(X) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{[h(s), h(s)+\varepsilon)}(X_s) \, d\lfloor X \rfloor_s, \tag{9}
\]

where \( \lfloor X \rfloor_t \) is the quadratic variation of \( X \) at time \( t \).

Given the exogenous inputs from Assumption 2.2, we define, for every \( n \geq 1 \), a corresponding filtration \( (\mathcal{F}^n_t)_{t \geq 0} \) by

\[
\mathcal{F}^n_t := \sigma(X^j_{0,n}, B^j_s, \{\chi^{j,(k)}_{i,n}\}_{i,k=0} : s \in [0, t], j = 1, \ldots, n). \tag{10}
\]

We can then state our result on the existence and uniqueness of the desired particle system.

Theorem 2.3 (Well-posedness of the particle system). Let Assumptions 2.1 and 2.2 be satisfied, and denote by \( \ell^h_t \) the local time of \( X^{i,n}_t \) along the advancing front \( t \mapsto A^n_t \) from (6), where both are
defined by the system (11) below. Then the particle system
\[
\begin{cases}
    dX^{i,n}_t = b(t, X^{i,n}_t) \, dt + \sigma(t, X^{i,n}_t) \, dB^i_t + d\ell^{i,n}_t, & t \in [0, \tau^i), \\
    A^n_t = a_0 + \alpha \int_0^t g(t-s) I^n_s \, ds, & t \geq 0, \\
    I^n_t = \frac{1}{n} \sum_{j=1}^n \mathbb{1}_{[0,t]}(\tau^j), & t \geq 0, \\
    X^{i,n}_t = 1, & t \geq \tau^i,
\end{cases}
\]
has a unique $\mathcal{F}^n_t$-adapted solution $X^n_t = (X^{1,n}_t, \ldots, X^{n,n}_t)$, living in $([A^n_t, \infty) \cup \uparrow)^\times n$, with initial conditions $X^{i,n}_0 = X_0^i$ such that, for each particle indexed by $i \in \{1, \ldots, n\}$, its infection time $\tau^i = \inf\{t \geq 0 : X_t^{i,n} = 1\}$ satisfies
\[
\mathbb{P}(\tau^i \leq t \mid \mathcal{F}^{i,n}_t) = 1 - \exp\left\{- \int_0^t \gamma(s, \tilde{\gamma}^{i,n}(s)) \, d\tilde{\gamma}^{i,n}_s \right\}, \quad \text{for all } t \geq 0,
\]
for the reduced-information filtration
\[
\mathbf{\hat{F}}^{i,n}_t := \sigma((X^{i}_0, B^i_0), (X^{j}_0, B^j_0, \chi^{j,(k)})_{k=1}^n) : s \in [0, t], j \in \{1, \ldots, n\} \setminus \{i\}.
\]
Here each $\tilde{\gamma}^{i,n}$ in (12) denotes the local time of the $i$th particle along the corresponding advancing front $\hat{A}^{n,(-i)}_t$ in the artificial particle system $\hat{X}^{n,(-i)}_t = (\hat{X}^{1,n,(-i)}_t, \ldots, \hat{X}^{n,n,(-i)}_t)$ for which the dynamics and interactions are specified as in the true particle system $\hat{X}^n_t$, except that $\hat{X}^{i,n,(-i)}_t$ is fully reflected without the possibility of infection and has no affect on the other particles indexed by $j \neq i$. Correspondingly, $\hat{\gamma}^{i,n}(s)$ denotes the current contagiousness in this system, defined as in (8), but with $I^n_t$ replaced by the infected proportion $I^{n,(-i)}_t$ for the artificial system $\hat{X}^{n,(-i)}_t$.

The proof of this theorem is the subject of Section 3. The precise formulation of each artificial particle system $\hat{X}^{n,(-i)}_t$, for $i = 1, \ldots, n$, is given in (36) of Proposition 3.4. In terms of properly formulating and establishing the above result, the main intricacies are related to, firstly, the fact that the infection times $\tau^i$ both affect the system dynamics and the effective rate of infection $\gamma$ when at the boundary, and, secondly, the need to ensure the appropriate conditional law of each $\tau^i$ with respect to a suitable filtration so that $\gamma$ has the desired interpretation and so that we can deduce the results that follow in the remainder of this section.

For what follows, it will be useful to decompose the reduced-information filtrations $\mathbf{\hat{F}}^{i,n}_t$ from (13) as
\[
\mathbf{\hat{F}}^{i,n}_t = \mathbf{G}^{i,n}_t \vee \sigma(B^i_r, X^i_0 : r \leq t)
\]
where
\[
\mathbf{G}^{i,n}_t := \sigma((X^{j}_0, B^j_r, \chi^{j,(k)}) : j \neq i, k \leq n, r \leq s), \quad \text{for } i = 1, \ldots, n.
\]
Indeed, the next proposition highlights the first crucial consequence of the construction of the particle system, namely that, for a tagged particle $X^{i,n}_t$, it is, in a suitable sense, possible to ‘freeze’ its interactions with the other particles, by conditioning on $\mathbf{G}^{i,n}_t$. In this way, one can isolate the regularising effect of the Brownian motion for probabilistic estimates. Moreover, the proposition highlights how smoothness of the infection-to-recovery kernel allows one to transfer the problem to the analysis of drifted Brownian motions on a fixed domain. The latter opens up for the use of Girsanov arguments and could also be beneficial for numerical implementations by avoiding discretisation of the multiplicative noise.

**Proposition 2.4** (Drifted Brownian motion in the frame of the moving boundary). Fix an arbitrary index $i \in \{1, \ldots, n\}$ and let $\hat{A}^{n,(-i)}_t$ denote the advancing front for the artificial system $\hat{X}^{n,(-i)}_t$,
been infected at any point up to the present time. Provided the infection-to-recovery kernel \( \varrho \) is absolutely continuous, there is a random and time-dependent bijection \( \Upsilon^{i,n}(t, \omega, \cdot) : [0, \infty) \to [0, \infty) \), adapted to \( \mathcal{G}^{i,n} \), such that, almost surely,

\[
P(X^{i,n}_t \in (a, b) \mid \mathcal{G}^{i,n}_t) = P(Z^{i,n}_t \in (\Upsilon^{i,n}(t, a - \hat{A}^{n(-i)}_t), \Upsilon^{i,n}(t, b - \hat{A}^{n(-i)}_t)), t < \tau^{i,n}_Z \mid \mathcal{G}^{i,n}_t),
\]

for all \((a, b) \subseteq \mathbb{R} \) and all \( t \in [0, T] \), where each \( Z^{i,n} \) is a drifted Brownian motion on the positive half-line with reflection at zero. Specifically,

\[
dZ^{i,n}_t = \tilde{b}^{i,n}(t, \omega, Z^{i,n}_t) \, dt + dB^{i,n}_t + d\ell_0^0(Z^{i,n}_t),
\]

where \( \ell_0^0(Z^{i,n}) \) is the local time of \( Z^{i,n} \) at zero, for a jointly measurable drift function \( \tilde{b}^{i,n} \) satisfying the linear growth bound (12) such that \((t, \omega) \mapsto \tilde{b}^{i,n}(t, \omega, z) \) is \( \mathcal{G}^{i,n} \)-adapted, and where the random time \( \tau^{i,n}_Z \) is defined by

\[
\tau^{i,n}_Z := \inf \left\{ t \geq 0 : \int_0^t \sigma(r, \hat{A}^{n(-i)}_r) \gamma(r, \hat{C}^{n(-i)}_r) \, d\ell^0_r(Z^{i,n}_r) > \chi^{i,n} \right\}
\]

for a standard exponential random variable \( \chi^{i,n} \) independent of \( \hat{F}^{i,n} \). Moreover, each Brownian motion \( B^{i,n} \) is independent of \( \mathcal{G}^{i,n}_t \).

The above Proposition 2.4 and the below Theorem 2.5 serve as the starting points for a companion paper \[FS22\], where we show converge to a unique mean-field limiting characterised by a deterministic free boundary problem. Furthermore, we note that the idea of ‘freezing’ the interactions, in the above sense, will also be an important ingredient in the proof of Theorem 2.5.

The contributions of our final main result, Theorem 2.5, are twofold. Firstly, we show that the difference between the infection proportion \( I^n_t \) and the local-time term

\[
V^n_t := \frac{1}{n} \sum_{i=1}^n \int_0^t \mathbb{1}_{s<\tau^i} \gamma(s, C^n_s) \, d\ell^i_s
\]

is a martingale for a suitable filtration, where \( C^n \) is the current level of contagiousness defined by \[5\]. This result justifies the decompositions (3) and (5) for the new number of infections and the effective reproduction number that were discussed in the introduction. Secondly, we provide an asymptotic result, highlighting that the infected proportion \( I^n_t \) at time \( t \) in fact becomes equivalent to \( V^n_t \), in a suitably uniform way, as the population size \( n \) tends to infinity.

**Theorem 2.5** (Martingale property and limiting behaviour of the infected proportion). There is a subfiltration \( (\hat{F}^{i,n}_t)_{t \geq 0} \) of \( (\hat{F}^{n}_t)_{t \geq 0} \) for which the difference \( I^n - V^n \) is a martingale. Furthermore, as \( n \to \infty \), this difference \( I^n - V^n \) vanishes uniformly on compact time intervals in \( L^2(\mathbb{P}) \) and hence also uniformly on compacts in probability.

The first part of the theorem relies on (12) and the particular construction of the particle system in Section 3 to show the martingale property with respect to a suitable filtration \( (\hat{F}^{i,n}_t)_{t \geq 0} \) for which the infection times are not measurable, but it is known, for each particle, whether or not it has been infected at any point up to the present time \( t \). For the second part of the theorem, we first show that, with probability one, no two particles can become infected at the same time, which then allows us to deduce the convergence from the martingale property. We implement this in Section 3.

**Remark 1** (Brownian particles colliding with a Newtonian barrier). The above Theorem 2.5 allows us to draw an interesting connection to the recent paper \[Bar20\]. Motivated by a model proposed
where \( \tilde{\tau} \) with the moving boundary replaced by \( \tilde{\tau} \) system is constructed in [Bar20, Prop. 2.10]. It could be of interest to formulate a variant of (19) where the moving boundary is instead replaced by \( \bar{\tau} \).

\( \text{in [Kni01], [Bar20] studies a system of } n \text{ Brownian motions reflecting off a Newtonian barrier, where the latter means that the barrier has momentum by way of experiencing an impulse away from the particles upon collision: these collisions create an additional velocity negatively proportional to the empirical average of the local times at each point in time. The particle system takes the form}

\[
\left\{ \begin{array}{l}
\frac{dX_{i,n}^t}{dt} = dB_{i,n}^t + d\ell_{i,n}^t, \\
\frac{d}{dt}Y_{i,n}^t = U_{i,n}^t, \quad U_{i,n}^t := u - \frac{\kappa}{n} \sum_{i=1}^n \ell_{i,n}^t,
\end{array} \right. \quad t \in [0,T],
\]

(19)

where \( \ell_{i,n} \) denotes the local time of \( X_{i,n} \) along the moving boundary \( \tilde{Y}^n \). A strong solution to this system is constructed in [Bar20] Prop. 2.10]. It could be of interest to formulate a variant of (19) with the moving boundary replaced by \( \tilde{Y}^n = \int_0^t U_s^t ds \) for \( \tilde{U}^n = u - \frac{1}{n} \sum_{i=1}^n \int_0^t 1_{s<\tilde{\tau}^i}(s, \tilde{U}^n_s) d\ell_{i,n}^t \), where \( \tilde{\tau}^i \) represents a time after which the \( i^{th} \) particle ceases to exert an influence on the barrier with \( \tilde{\tau}^i \) satisfying the corresponding version of (12). Our results allow us to also consider another variant where the moving boundary is instead replaced by \( \bar{Y}^n = \int_0^t U_s^t ds \) for \( \bar{U}^n = u - \frac{1}{n} \sum_{i=1}^n 1_{[0,\bar{\tau}^i]}(t) \) with each \( \bar{\tau}^i \) satisfying (12) for this system. The conclusions of Theorem 2.5 then imply that these two particle systems will behave almost identically for very large \( n \). We note that the moving boundary for our actual particle system is of a different form, more similar to having \( \bar{\tau}^i \) proportional to \( \tilde{U}^n \) (subject to the infection-to-recovery kernel), and also with opposite sign, so that the boundary is pulled into the population, hence having an exacerbated effect, rather than being pushed away.

3 Construction of the particle system

In this section, we construct a particle system \( X^n \) with \( X_{i,n}^t : \Omega \to [A^n, \infty) \cup \{1\} \) for all \( t \geq 0 \) and every \( i = 1, \ldots, n \) so that the requirements of Theorem 2.3 are satisfied, namely that the particles \( X_{i,n}^t \) evolve according to the desired dynamics (11) and that the infection times \( \tau^i \) have the desired conditional laws, as specified by (12)–(13). In order to satisfy (12)–(13), we shall need to introduce an auxiliary system \( X^{(n,-i)} \), for each \( i = 1, \ldots, n \), which is constructed analogously to \( X^n \) in all aspects but one: the \( i^{th} \) particle \( X_{i,n,-i} \) is fully reflected with no possibility of infection, so it continues to reflect off the moving boundary (corresponding to this system) and it does not assert any influence on the dynamics or the infection times of the other particles.

The construction of \( X^n \) and each of the auxiliary systems \( X^{(n,-i)} \) will go through a sequence of basic intermediate systems \( X^{n,(k)} \). The main idea behind their construction is as follows. Let \( \zeta^{(k)} \) denote the \( k^{th} \) time a particle in the system gets infected. We partition \([0, \infty) \) in \( n+1 \) time segments, each describing the particle dynamics between one infection time \( \zeta^{(k)} \) and the next \( \zeta^{(k+1)} \). The final interval is then from \( \zeta^{(n)} \) to infinity, since there are only \( n \) particles. In each of these random time intervals, the interactions of the system are determined by what took place in the previous time interval, so it becomes straightforward that ones has existence of a unique strong solution for each intermediate system. However, the intricacies lie in making sure that everything adds up correctly and that the constructions actually make it feasible to achieve not only the desired dynamics (11), but also the conditional laws (12)–(13). The latter is not obvious, as one needs to work around the effect that each \( \tau^i \) itself has on the system’s dynamics and the rates of infection. Naturally, the trajectories of \( X_{i,n}^t \) and their individual infection times \( \tau^i \) will be constructed through a suitable concatenation of the \( n+1 \) intermediate systems. The constructions of the intermediate systems are outlined in Section 3.1 and the concatenations are then implemented in Section 3.3.
3.1 Recursive construction of intermediate systems

We now describe the base step of our construction, Step 1, the iterative steps, Step $k$, for $k = 2, \ldots, n$, and the final step, Step $n + 1$. In addition to the actual particle system, we shall also be interested in a reflected version of the particle system, where infection times and interactions are as in the true particle system, but particles continue with fully reflected dynamics indefinitely after infection. We distinguish the intermediate constructions for these two systems by the notation $X^{n,k}$ for the true system and $\hat{X}^{n,k}$ for the aforementioned one.

Finally, we shall be wanting to keep track of the fact that, for $t < \tau_i$, the moving boundary is not affected by the infection time $\tau_i$, so in order to determine the infection time for the $i$th particle, we can work with a different moving boundary where its own infection is not counted. We make this precise in Step $k$ of the construction of the intermediate systems by having different moving boundaries for different classes of particles. This will still lead to a solution of the desired system, since we are only interested in the dynamics of each particle up until its infection time.

Remark 2 (Notation). The variables and parameters related to the $k$th step of the construction will be denoted with an added superscript $(k)$. We will use boldface symbols when dealing with vectors.

Step 1. Define the particle system $\hat{X}^{n,1}_t = (\hat{X}^{1,n,1}_t, \ldots, \hat{X}^{n,n,1}_t) \in [A^{1,n,1}_t, \infty) \times \cdots \times [A^{n,n,1}_t, \infty)$ as follows:

$$\begin{cases}
   \frac{d\hat{X}^{i,n,1}_t}{dt} = b^{(1)}(t, \hat{X}^{i,n,1}_t) - \sigma^{(1)}(t, \hat{X}^{i,n,1}_t) \frac{d\ell^{A^{i,n,1}}_t}{dt} + \frac{d\ell^{B^{i,n,1}}_t}{dt}, & t \geq 0, \\
   \hat{X}^{i,n,1}_0 = a_0, & t \geq 0, \\
   \hat{X}^{i,n,1}_0 = X^{i}_0,
\end{cases}$$

for $i = 1, \ldots, n$, where $X^{i}_0 \in (a_0, \infty)$ for $i = 1, \ldots, n$ and $a_0 \in \mathbb{R}_{\geq 0}$ are the (given) initial conditions for $X^{i,n}_t$ and $\ell^{A^{i,n}}_t$; $b^{(1)}(t, x) := b(t, x)$ and $\sigma^{(1)}(t, x) := \sigma(t, x)$ for any $t \geq 0$ and $x \in \mathbb{R}$; and $B^{i,n}_t = B^{i}_t$ for $i = 1, \ldots, n$. For shortness, we denote by $A^{1,n,1}_t = (A^{1,1}_t, \ldots, A^{n,n}_t)$ and $B^{1}_t = (B^{1}_t, \ldots, B^{n}_t)$.

For all $i$, we now introduce elastic absorption at $A^{i,n,1}_t$ at rate $\gamma^{i,n,1}(t) := \gamma(t, 0)$ for all $i = 1, \ldots, n$. Let $\{\gamma^{j,i,1}\}_{i,j=1}^n$ be a family of iid exponential random variables, independent from all the components of $\hat{X}^{i,n,1}_t$, $A^{n,1}_t$ and $B^{1}_t$. Define the ‘abstract infection times’

$$\varsigma^{i,1} := \inf \left\{ t \geq 0 : \int_0^t \gamma^{j,i,1}(s) \frac{d\ell^{a_0}_s}{dt} (\hat{X}^{i,n,1}_t) \geq \gamma^{j,1} \right\},$$

so that the actual ‘time of the first infection’ $\varsigma^{(1)}$ is given by the random time

$$\varsigma^{(1)} := \min \left\{ \varsigma^{i,1} : i = 1, \ldots, n \right\}.$$

At the time $\varsigma^{(1)} = \varsigma^{(1),1}$, for some random index $j^{(1)} \in \{1, \ldots, n\}$, the trajectory of particle $j^{(1)}$ is at the boundary $a_0$ and gets infected.

From here, we introduce a new particle system $X^{n,1}_t = (X^{1,n,1}_t, \ldots, X^{n,n,1}_t) \in [A^{n,1}_t, \infty) \cup \uparrow$ defined as

$$\begin{cases}
   X^{i,n,1}_t = \hat{X}^{i,n,1}_t, & t \geq 0, \ i = 1, \ldots, n, \ i \neq j^{(1)}, \\
   X^{j^{(1)},n,1}_t = \hat{X}^{i,n,1}_t, & t \in [0, \varsigma^{(1)}), \\
   X^{j^{(1)},n,1}_t = \uparrow, & t \geq \varsigma^{(1)}.
\end{cases}$$

Running the particle system (22) from time 0 to $\varsigma^{(1)}$ gives us the dynamics of the desired particle system (11) up to but excluding the first infection time. As soon as the first infection happens, the
Moreover, note that the definition of \( J^1 \) is the random index of the first particle to get infected so \( J^1 = \tau^1 \). We also define
\[
\hat{C}_i = 0.
\]

**Step \( k \).** Let \( J^{(k-1)} = \{ j^{(m)} : m = 1, \ldots, k-1 \} \) denote the random set of indices for the particles that have been infected up to step \( k-1 \). Consider the particle system \( \hat{X}_{i}^{n(k)} \in [A_{i}^{n(k)}, \infty) \) with dynamics
\[
d\hat{X}_{i}^{n(k)}(t) = b(i)(t, \hat{X}_{i}^{n(k)}(t)) dt + \sigma(i)(t, \hat{X}_{i}^{n(k)}(t)) dB_{i}^{n(k)}(t) + d\tilde{C}_{i}^{n(k)}(\hat{X}_{i}^{n(k)}(t)), \forall i,
\]

\[
\begin{align*}
A_{i}^{n(k)} &= A_{i}^{n(k-1)} + \alpha \sum_{j=1}^{k-2} \int_{\zeta(k-1)}^{t+\zeta(j)} \theta(t + \zeta(k-1) - s) \, ds + \frac{\alpha}{n} \int_{0}^{t} \theta(t - s) \, ds, \quad i \neq J^{(k-1)}, \\
A_{i}^{n(k)} &= A_{i}^{n(k-1)} + \alpha \sum_{j=1, j \neq m}^{k-2} \int_{\zeta(k-1)}^{t+\zeta(j)} \theta(t + \zeta(k-1) - s) \, ds + \frac{\alpha}{n} \int_{0}^{t} \theta(t - s) \, ds, \quad i = j^{(m)} \in J^{(k-2)}, \\
A_{i}^{n(k)} &= A_{i}^{n(k-1)} + \alpha \sum_{j=1}^{k-2} \int_{\zeta(k-1)}^{t+\zeta(j)} \theta(t + \zeta(k-1) - s) \, ds, \quad i = J^{(k-1)}, \\
\hat{C}_{i}^{n(k)} &= \hat{C}_{i}^{n(k-1)}, \forall i,
\end{align*}
\]

for all \( t \geq 0 \), where the drift, the diffusion coefficient, and the Brownian motions are defined as
\[
b^{(i)}(t, x) := b(t + \zeta(k-1), x), \quad \sigma^{(i)}(t, x) := \sigma(t + \zeta(k-1), x),
\]

\[
B_{i}^{n(k)} := B_{i}^{n(k-1)}(t + \zeta(k-1) - t), \quad \forall t \geq 0, i = 1, \ldots, n, \quad \text{and} \quad x \in \mathbb{R}.
\]

Note that we have taken as initial conditions for \( X_{i}^{n(k-1)} \) and \( A_{i}^{n(k-1)} \) the values of \( X_{i}^{n(k-1)} \) and \( A_{i}^{n(k-1)} \) respectively at the time of the first infection in the previous construction step, \( \zeta(k-1) \). Moreover, note that the definition of \( A_{i}^{n(k)} \) differs depending on whether particle \( i \) has or has not been infected in one of the previous steps of the construction. This is to account for the fact that the infection of particle \( i \) should not affect the movement of the boundary \( A_{i}^{n(k)} \), while still affecting the movement of \( A_{j}^{n(k)} \) for \( j \neq i \).

To construct our mechanism for infection we need to introduce a quantity encoding the current level of contagiousness among the population. We denote this by \( C_{i}^{n(k)} \) and let it be defined by
\[
C_{i}^{n(k)} := \frac{1}{\alpha} \left( A_{i}^{n(k)} - A_{i}^{n(k)}(t) \chi(d \leq t) - \sum_{j=0}^{k-1} A_{i}^{n(j+1)}(t + \zeta(j) - d) \chi(\zeta(j) \leq t + \zeta(k-1) - d < \zeta(j+1)) \right),
\]

where we recall that \( d \) is the duration of the infection (expressed in units of time).

Take the infection rate at the boundary \( t \mapsto A_{i}^{n(k)} \) to be given by
\[
\gamma^{i}(t) := \gamma \left( t + \zeta(k-1), C_{i}^{n(k)} \right) \quad \forall i = 1, \ldots, n.
\]

Let \( \{ \chi^{j(l)} \}_{j=1}^{n} \) be a family of \( n \) iid exponential random variables such that \( \chi^{j(l)}(k) \perp \chi^{l(m)} \) for all \( j, l \in \{1, \ldots, n\} \) and all \( m \leq k - 1 \); moreover, let \( \chi^{j(l)} \perp \hat{X}_{i}^{n(m)}, A_{i}^{n(m)}, B_{i}^{n(m)} \) for all \( j = 1, \ldots, n \).
and for all $m \leq k$ (where $\perp$ followed by a vector indicates independence from all the components of the vector). Define $\zeta^{i,k}$ for $i \in \{1, \ldots, n\}$ as follows:

$$\zeta^{i,(k)} := \inf \left\{ t \geq 0 : \int_0^t \gamma^{i,(k)}(s) \, d\ell^{i,n,(k)}_s(\hat{X}^{i,n,(k)}) \geq \chi^{i,(k)} \right\}.$$  \hspace{1cm} (28)

We further define:

$$\zeta^{(k)} := \min \left\{ \zeta^{i,(k)} : i \notin J^{(k-1)} \right\}.$$  \hspace{1cm} (29)

Then at the random time $\zeta^{(k)}$ particle $j^{(k)} \notin J^{(k-1)}$ gets infected.

We conclude step $k$ of our construction with a few further definitions. First, recalling that particles $j^{(m)}$ for $m = 1, \ldots, k-1$ are infected from the beginning, the system $X^{n,(k)}_t = (X^{1,n,(k)}_t, \ldots, X^{n,n,(k)}_t) \in [A^{n,(k)}_t, \infty) \cup \uparrow$ is given by:

$$\begin{align*}
X^{i,n,(k)}_t &= X^{i,n,(k)}_t, & t \geq 0, i \notin J^{(k-1)}, i \neq j^{(k)}, \\
X^{i,n,(k)}_t &= \uparrow, & t \geq 0, i \in J^{(k-1)}, \\
X^{j^{(k)},n,(k)}_t &= X^{j^{(k)},n,(k)}_t, & t \in [0, \zeta^{(k)}), \\
X^{j^{(k)},n,(k)}_t &= \uparrow, & t \geq \zeta^{(k)}.
\end{align*}$$  \hspace{1cm} (30)

Secondly, we define the sum of all previous infection times recursively as

$$\zeta^{(k)} := \zeta^{(k-1)} + \zeta^{(k)} = \sum_{m=1}^k \zeta^{(m)}.$$  \hspace{1cm} (31)

**Step $n + 1$.** After repeating step $k$ for $k = 2, \ldots, n$, we are left with a system in which all particles have been infected. That is, we simply set $X^{n,(n+1)}_t := \uparrow^{n}$ for $t \geq \zeta^{(n)}$. The globally reflected system $\hat{X}^{n,(n+1)}_t \in [A^{n,(n+1)}_t, \infty]$ is defined exactly as in the previous steps of the construction, with $b^{(n+1)}(t, x)$, $\sigma^{(n+1)}(t, x)$ and $B^{i,(n+1)}_t$ analogous to (24) and (25); however, there is no need to construct the other variables ($C^{i,n,(n+1)}_t$, $\gamma^{i,n,(n+1)}$, etc.) since no further infection can occur.

### 3.2 Global-in-time reflection and dismissed infection

We will shortly see how to use systems $X^{n,(k)}_t$ (and $\hat{X}^{n,(k)}_t$) to construct the dynamics $X^i_t$ of the particle system $[\Pi]$. On the other hand, as we have already anticipated in the statement of Thm. 2.3, we will need further auxiliary particle systems $\{\hat{X}^{i,(n-i)}_t\}_{i=1}^n$ to understand expression (12) for the conditional probability distribution of the infection times $\tau^i$ for $i = 1, \ldots, n$. These new ‘artificial’ particle systems $\hat{X}^{i,(n-i)}_t$ are also constructed piecewise, although only in $n$ steps instead of $n + 1$. This is due to the fact that particle $i$ cannot be infected in system $\hat{X}^{i,(n-i)}_t$ (and bears no effect on the dynamics of the other particles), and therefore there is one fewer infection time that we need to account for during the construction. For each particle $i = 1, \ldots, n$ and each step $k = 1, \ldots, n$, we denote by $\hat{X}^{i,(n-i),(k)}_t$ the intermediate systems for the construction of $\hat{X}^{i,(n-i)}_t$. We make their characterization more precise, and clarify the difference between $X^{n,(k)}_t$, $\hat{X}^{n,(k)}_t$ and $\hat{X}^{n,(n-i),(k)}_t$ in what follows.

#### 3.2.1 Reflecting infected particles

It is clear from our construction that the dynamics of particle $i$ in the systems $X^{n,(k)}_t$ and $\hat{X}^{n,(k)}_t$ are equivalent up until its infection time $\tau^i$. After infection, particle $i$ is moved to the ‘infected state’
† in the systems $X^n_{s}(k)$, while it goes back to its reflected dynamics in $\hat{X}^n_{s}(k)$. Then, the systems $\hat{X}^n_{s}(k)$ are artificial particle systems with global-in-time diffusive dynamics, where the interaction between the particles is still accounted for in the movement of the boundary.

For clarity in the statements that follow, we introduce one more definition: for each $i = 1, \ldots, n$, let $\xi^i_k$ denote the first time a particle different from $i$ gets infected in the ‘true’ particle systems $X^n_{s}(k)$. That is, if particle $i$ is the $m$th particle to get infected (for a given trajectory, $i = j(m)$ in above construction), then

$$\xi^i_l = \varsigma^{(l)} \quad \text{for } l = 1, \ldots, m - 1, \quad \text{and} \quad \xi^i_l = \varsigma^{(l+1)} \quad \text{for } l = m, \ldots, n. \quad (31)$$

### 3.2.2 Dismissing infection of a tagged particle

Fix $i \in \{1, \ldots, n\}$ and assume that particle $i$ cannot be infected. We construct $\hat{X}^n_{s}(-i)$ by defining the intermediate systems $\hat{X}^n_{s}(-i),(k)$ for $k = 1, \ldots, n$, in a recursive way analogous to how we defined the systems $X^n_{s}(k)$. The only difference is that we now partition $[0, \infty)$ into $n$ time segments, with start- and end-points the random times $\xi^i_k$ for $k = 1, \ldots, n - 1$ instead of $\varsigma^i_k$. The $\xi^i_k$ indicate the $k$th time a particle gets infected, in a system where particle $i$ cannot get infected (compare with $\xi^i_l$ in (31) above). In other words, let $\xi^{i,(0)} = 0$ and define the first time of infection as

$$\hat{\xi}^{i,(1)} = \xi^{i,(1)} = \min\{\hat{\varsigma}^{j,(1)} : j = 1, \ldots, n, j \neq i\},$$

with $\xi^{i,(1)} = \hat{\xi}^{i,(1)}$ for some random index $j^{i,(1)} \in \{1, \ldots, n\} \setminus \{i\}$, and each $\hat{\varsigma}^{j,(1)}$ is defined in (21). Running system (20) up to time $\hat{\xi}^{i,(1)}$ and then moving particle $j^{i,(1)}$ to the state $\dagger$ (analogously to (22), but with $\hat{\xi}^{i,(1)}$ instead of $\varsigma^{i,(1)}$) takes care of the construction of $\hat{X}^n_{s}(-i),(1)$.

The next steps of the construction are just as similar; one needs to be careful however that we can only define $\hat{\xi}^{i,(k)}$ in terms of $\{\hat{\varsigma}^{j,(k)}\}_{j=1}^n$ up until infection of particle $i$ happens in the ‘true’ systems $X^n_{s}(k)$. That is, when $i = j^{(m)}$ in the construction of $X^n_{s}(k)$, the systems $X^n_{s}(k)$ and $\hat{X}^n_{s}(-i),(k)$ start diverging, and from step $m + 1$ it will be necessary to define new potential infection times $\{\hat{\varsigma}^{j,(k)}\}$ that for $k \geq m + 1$ are different from $\{\hat{\varsigma}^{j,(k)}\}$. With the convention that $\{\hat{\varsigma}^{j,(k)}\}$ are equal to $\{\hat{\varsigma}^{j,(k)}\}$ up until $k = m$, and defined analogously to (28) but using instead contagiousness levels $\hat{\varsigma}^{n,-i,(k)}$ appropriate for the current system (more on this shortly), we let

$$\hat{\xi}^{i,(k)} = \min\{\hat{\varsigma}^{j,(k)} : j = 1, \ldots, n, j \neq i\}, \quad (32)$$

and let $\xi^{i,(k)} = \xi^{i,(k-1)} + \xi^{i,(k),k}$ (note that $\xi^{i,(k)} = \xi^i_k$ for all $k \leq m - 1$). Then for $k = 2, \ldots, n - 1$ we define $\hat{X}^n_{s}(-i),(k)$ analogously to (30), with $\hat{\xi}^{i,(k)}$ and $j^{i,(k)}$ (the index of the $k$th particle different from $i$ to be infected) taking the place of $\hat{\varsigma}^{i,(k)}$ and $j^{i,(k)}$. At the $n$th step of the construction, we are left with particle $i$ still having reflected diffusive dynamics, and all the other particles removed to state $\dagger$. Note that the moving boundary in the systems $\hat{X}^n_{s}(-i),(k)$ does not feel the effect of the infection of particle $i$, so it is essentially different from $A^n_{s}(k)$ (or at least, it differs from it starting from construction step $m + 1$). We denote the moving boundary for $\hat{X}^n_{s}(-i),(k)$ by $\hat{A}^{n,-i,(k)}$ and remark that it is defined as in (23) but with $\hat{\xi}^{i,(j)}$ and $\xi^{i,(j)}$ taking the place of $\hat{\varsigma}^{(j)}$ and $\varsigma^{(j)}$ as appropriate. The changes in the boundary affect also a change in the levels of contagiousness $C^{j,n,(k)}$ defined in (26). We let $\hat{C}^{j,n,-i,(k)}$ be defined analogously to (26), again replacing $A^{j,n,(k)}$ with $\hat{A}^{j,n,-i,(k)}$ and $\varsigma^{(j)}$ with $\xi^{(j)}$ appropriately.
3.3 Well-posedness of the particle system

We are now ready to put together all the different pieces in order to rigorously construct a solution to the particle system satisfying the formulation in Theorem 2.3.

**Definition 3.1** (Piecewise construction). Let \( \{\Psi^{(k)}_t\}_{k=1}^{n+1} \) be one of the collections of processes \( \{\hat{X}^{i,n,(k)}_{\cdot}\}_{k=1}^{n} \) or \( \{X^{j,n,(k)}_{\cdot}\}_{k=1}^{n} \), or \( \{A^{j,n,(k)}_{\cdot}\}_{k=1}^{n} \), each of them defined as part of the above recursive construction. For all \( t \geq 0 \), we construct their piecewise concatenation \( \Psi_t \) by

\[
\Psi_t := \Psi_t^{(1)} \mathbb{1}_{\{0,\xi^{(1)}(t)\}} + \sum_{k=2}^{n} \Psi^{(k)}_{t-\xi^{(k-1)}(t)} \mathbb{1}_{\{\xi^{(k-1)}(t),\xi^{(k)}(t)\}} + \Psi^{(n+1)}_{t} \mathbb{1}_{\{t \geq \xi^{(n)}(t)\}}.
\]

(33)

Similarly, for all \( i = 1, \ldots, n \), let \( \{\Psi^{(-i),(k)}_t\}_{k=1}^{n} \) be one of the collections of processes \( \{\hat{X}^{j,n,(-i),(k)}_{\cdot}\}_{k=1}^{n} \) or \( \{A^{j,n,(-i),(k)}_{\cdot}\}_{k=1}^{n} \) from Sec. 3.2 and recall the random times \( \xi^{(-i),(k)}(t) \) defined from (32). For all \( t \geq 0 \), we construct their concatenation \( \Psi^{(-i),(k)}_t \) by

\[
\Psi^{(-i),(k)}_t := \Psi^{(-i),(1)}_t \mathbb{1}_{\{0,\xi^{(-i),(1)}(t)\}} + \sum_{k=2}^{n-1} \Psi^{(-i),(k)}_{t-\xi^{(-i),(k-1)}(t)} \mathbb{1}_{\{\xi^{(-i),(k-1)}(t),\xi^{(-i),(k)}(t)\}} + \Psi^{(-i),(n)}_{t-\xi^{(-i),(n-1)}(t)} \mathbb{1}_{\{t \geq \xi^{(-i),(n-1)}(t)\}}.
\]

(34)

We start by considering the system where every particle continues to be reflected upon infection (instead of being moved to the state \( \dagger \)) and where the infection of a given particle does not affect its own moving boundary.

**Proposition 3.2** (Global-in-time reflected trajectories). The processes \( \hat{X}^n_t = (\hat{X}^{1,n}_t, \ldots, \hat{X}^{n,n}_t) \in [A^n_t, \infty) \) and \( A^n_t = (A^{1,n}_t, \ldots, A^{n,n}_t) \in \mathbb{R}^n_{\geq 0} \) constructed according to (33) in Definition 3.1 satisfy the system of equations

\[
\begin{align*}
\text{d}X_{t}^{i,n} &= b(t, X_{t}^{i,n}) \text{d}t + \sigma(t, X_{t}^{i,n}) \text{d}B_{t}^{i} + \text{d}\ell_{t}^{i,n} (\hat{X}_{t}^{i,n}), & t \geq 0, \ i = 1, \ldots, n, \\
A_{t}^{i,n} &= a_0 + \alpha \int_{0}^{t} g(t-s) I_{s}^{i,n} \text{d}s, & t \geq 0, \\
I_{t}^{i,n} &= \frac{1}{n} \sum_{j=1}^{n} \mathbb{1}_{[0,t]}(\xi_{j}^{i,n}), & t \geq 0,
\end{align*}
\]

(35)

where \( \{\xi_{j}^{i,n}\}_{j=1}^{n} \) is defined in (31), and \( \ell^{A^{i,n}}(\hat{X}^{i,n}) \) is the local time of \( \hat{X}^{i,n} \) along \( A^{i,n} \) in accordance with (9).

**Proof.** See Appendix.

\[\square\]

At first, it may seem natural to seek to use this system for the formulation of (12), replacing the filtration (13) with the filtration generated by the trajectories of this reflected system. However, in order to succeed with the proof of Proposition 3.5 below, where (12) is verified, it is not enough to only ‘remove’ the direct impact of \( \tau^i \) on the boundary of the \( i^{th} \) particle itself, as is done in (35). There remains a problematic indirect effect through its impact on the other particles, and so, for each particle, we shall have to consider another system where the infection of that tagged particle is not at all a possibility. Before turning to this, we first confirm that our recursive construction leads to the correct form of the dynamics for the true particle system that we are interested in. The proof of this amounts to little more than a reformulation of the previous proposition.

**Proposition 3.3** (True trajectories with infection). Let the processes \( X_t^n = (X_1^{1,n}, \ldots, X_n^{n,n}) \) be given by (33) in Definition 3.1 and define \( \tau^i := \inf\{t \geq 0 : X_{t}^{i,n} = \dagger\} \) for all \( i = 1, \ldots, n \). Let also \( I_{t}^{i,n} := \frac{1}{n} \sum_{j=1}^{n} \mathbb{1}_{[0,t]}(\tau^{j}) \) and \( A_{t}^{i,n} := a_0 + \alpha \int_{0}^{t} g(t-s) I_{s}^{i,n} \text{d}s \) for all \( t \geq 0 \). Then the triple \( (X^n, I^n, A^n) \) satisfies the system of equations (11) from Theorem 2.3.
Proof. For each $i = 1, \ldots, n$, the construction of the intermediate systems in Section 3.1 gives that, \( \omega \) by \( \omega \), we have \( X_{i,n}^i = X_s^{i,n} \) for all \( s \in [0, \tau^i) \), where \( X_s^{i,n} \) satisfies (35). On the other hand, we have \( X_{i,n}^i = \top \) for all \( s \in [\tau^i, \infty) \). It also holds by construction that, for each \( i = 1, \ldots, n \), we have \( \tau^i = \varsigma^{(k)} \) for a unique random index \( k \in \{1, \ldots, n\} \). Moreover, on the event \( \{ t < \tau^i \} \), we then have \( \xi^i_j = \varsigma^{(j)} \) for \( j = 1, \ldots, k-1 \) (with each \( \varsigma^{(j)} \) equal to \( \tau^{i(j)} \) for a unique random \( i(j) \neq k \)), while \( \xi^i_j > t \) for \( j = k, \ldots, n \). Consequently, we in fact have \( I^i_t = \frac{1}{n} \sum_{j=1}^n \mathbf{1}_{[0,t]}(\tau^j) = \frac{1}{n} \sum_{j=1}^{k-1} \mathbf{1}_{[0,t]}(\xi^i_j) = I^n_t \) pathwise for \( t \in [0, \tau^i) \), for every \( i = 1, \ldots, n \). It then also follows that we have \( A_t^i = A_t^n \) pathwise for \( t \in [0, \tau^i) \), for each \( i = 1, \ldots, n \). Therefore, we can conclude that, on \([0, \tau^i)\), the local time in \( \ell^{A_t^n}(X_{i,n}^i) \) from (35) agrees pathwise with the local time of \( X_{i,n}^i \) along \( A_t^n \). This confirms that (11) is indeed satisfied and so the proof is complete.

In order to address the second part of Theorem 2.3 we need to also confirm the construction of the artificial systems \( \hat{X}_{i,n}^{(i)} \) for any tagged particle \( i = 1, \ldots, n \).

**Proposition 3.4** (Artificial trajectories dismissing infection of a tagged particle). Let the processes \( \hat{X}_{i,n}^{(i)} = (\hat{X}_{i,n}^{(i),1}, \ldots, \hat{X}_{i,n}^{(i),n}(i)) \) be given by (34) in Definition 3.1 and define \( \tau^{i,(i)} := \inf\{ t \geq 0 : \hat{X}_{i,n}^{(i),i}(i) = \top \} \) for all \( j = 1, \ldots, n \) with \( j \neq i \). Then the triple \( (\hat{X}_{i,n}^{(i)}, \hat{H}_{i,n}^{(i),i}, \hat{A}_{i,n}^{(i)}) \) is the unique solution to the system

\[
\begin{aligned}
\frac{dX_{i,n}^{(i)}}{dt} &= b(t, \hat{X}_{i,n}^{(i)}) dt + \sigma(t, \hat{X}_{i,n}^{(i)}) dW_t + d\ell^n_{i,n}, \\
\frac{dX_{i,n}^{(i)}}{dt} &= b(t, \hat{X}_{i,n}^{(i)}) dt + \sigma(t, \hat{X}_{i,n}^{(i)}) dW_t + d\ell^n_i, \\
A_{i,n}^{(i)} := a_0 + \alpha \int_0^t \varrho(t-s) \hat{H}_{i,n}^{(i),i} ds, \\
H_{i,n}^{(i)} := \frac{1}{n} \sum_{j=1}^{n-1} \mathbf{1}_{[0,t]}(\tau^{i,(i),j}), \\
X_{i,n}^{(i)} := \top,
\end{aligned}
\]

adapted to the reduced-information filtration \( \hat{F}_{i,n}^{(i)} \) defined in (13), where for all \( j = 1, \ldots, n \), \( \hat{H}_{i,n}^{(i),j} \) denotes the local time of particle \( \hat{X}_{i,n}^{(i),j} \) along the boundary \( \hat{A}_{i,n}^{(i)} \). Recall also \( \hat{v}_{i,n}^{(i)} = \hat{v}_{i,n}^{(i)} \) with the notation from Theorem 2.3.

**Proof.** Similar to the proofs of Propositions 3.2 and 3.3 relying instead on the construction of the intermediate systems \( \hat{X}_{i,n}^{(i),(k)} \) in Section 3.2.2.

Using Proposition 3.4, we can finally verify that the infection times \( \tau^i \) from Proposition 3.3 satisfy the required property (12).

**Proposition 3.5.** The infection times \( \tau^i = \inf\{ t \geq 0 : X_{i,n}^i = \top \} \) from Proposition 3.3 satisfy

\[
\mathbb{P}(\tau^i \leq t \mid \hat{F}_{i,n}^{(i)}) = 1 - \exp\left\{ - \int_0^t \gamma(s, \hat{v}_{i,n}^{(i)}) d\hat{v}_{i,n}^{(i)} \right\}, \quad \forall t \geq 0,
\]

for \( i = 1, \ldots, n \), where \( \hat{F}_{i,n}^{(i)} \) is defined in (13), \( \hat{v}_{i,n}^{(i)} \) is the local time of the process \( \hat{X}_{i,n}^{(i)} \) along the boundary \( \hat{A}_{i,n}^{(i)} \) from Proposition 3.4, and \( \hat{C}_{i,n}^{(i)} := \int_0^t \varrho(t-s)(I^n_{s,i}^{(i)} - I_{s-i}^{(i,d)} ds). \)

**Proof.** Fix an arbitrary index \( i \in \{1, \ldots, n\} \), and recall the random times \( \xi^{(i),k,(k)} \), for \( k = 1, \ldots, n-1 \) defined from (32) in Section 3.2.2. We split up the probability that the \( i \)th particle has not yet been infected at time \( t \) according to these random times, which yields

\[
\mathbb{P}(\tau^i > t \mid \hat{F}_{i,n}^{(i)}) = \sum_{k=1}^{n-1} \mathbb{P}\left( \{ \tau^i > t \} \cap \{ t \in [\xi^{(i),k,(k)}, \xi^{(i),k,(k)}) \} \mid \hat{F}_{i,n}^{(i)} \right) + \mathbb{P}\left( \{ \tau^i > t \} \cap \{ t \geq \xi^{(i),n-1} \} \mid \hat{F}_{i,n}^{(i)} \right).
\]
Now fix a time \( t \geq 0 \) and consider the event \( E_k := \{ \tau^i > t \} \cap \{ t \in [\xi^{(-i),(k-1)}, \xi^{(-i),(k)}) \} \), for any given \( k \in \{1, \ldots, n-1\} \). On this event, we have that particle \( i \) is not yet infected, and we have that an exponential clock has rung in precisely the first \( k-1 \) intermediate systems in the construction of \( \tilde{X}^{i,(-i)}_t \) from Proposition 3.4. Observe that the \( i^{th} \) particle not being infected at such a time means that, in the construction of the true particle system, this particle has not been moved to the infected state in any of the first \( j = 0, \ldots, k-2 \) iterations by the time \( \zeta^{(j)} \) from (29). More precisely, we can write the event \( E_k \) equivalently as

\[
\bigcap_{j=1}^{k-1} \left\{ X^{i,n,(j)} \neq \dagger \right\} \bigcap \left\{ X^{i,n,(k)}_{t-\zeta^{(k-1)}} \neq \dagger \right\} \bigcap \left\{ t \in [\xi^{(-i),(k-1)}, \xi^{(-i),(k)}) \right\} \bigcap \left\{ \xi^{(-i),(k-1)} = \zeta^{(k-1)} \right\},
\]

where the last event in the expression above highlights that \( t - \zeta^{(k-1)} \geq 0 \). In fact, we can see that, on the event \( E_k \), we must have \( \zeta^{(j)} = \tilde{\zeta}^{(-i),(j)} \) for all \( j = 1, \ldots, k-1 \). For every \( \omega \in E_k \), we use the construction \( \tilde{X}^{i,n,(j)}_t \) of \( X^{i,n,(j)}_t \) and the definition (32) of \( \tilde{\zeta}^{(-i),(j)} \), to get rid of the overlap between the first \( k \) events in (38) and instead rewrite the full intersection as

\[
E_k = \bigcap_{j=1}^{k-1} \int_0^{\tilde{\zeta}^{(-i),(j)}(s)} \gamma^{i,(j)}(s) \, d\tilde{\ell}^{i,n,(j)}(s) < \chi^{i,(j)} \bigcap \left\{ t \in [\xi^{(-i),(k-1)}, \xi^{(-i),(k)}) \right\} \bigcap \left\{ t \in [\xi^{(-i),(k-1)}, \xi^{(-i),(k)}) \right\} \bigcap \left\{ t \in [\xi^{(-i),(k-1)}, \xi^{(-i),(k)}) \right\} \bigcap \left\{ t \in [\xi^{(-i),(k-1)}, \xi^{(-i),(k)}) \right\}.
\]

Recalling the details of the recursive construction, we can identify a conditionally independent structure with the event \( \{ t \in [\xi^{(-i),(k-1)}, \xi^{(-i),(k)}) \} \) being \( F^{i,n}_{t} \)-measurable while the other events are conditionally independent given \( F^{i,n}_{t} \). Combining this with Lemma A.2 (where we recall that \( \zeta^{(j)} = \tilde{\zeta}^{(-i),(j)} \) for all \( j = 1, \ldots, k-1 \), for all \( \omega \in E_k \)), we therefore obtain that

\[
P \left( E_k \mid F^{i,n}_{t} \right) = \prod_{j=1}^{k-1} \mathbb{P} \left( \int_{\xi^{(-i),(j-1)}}^{\tilde{\zeta}^{(-i),(j)}} \gamma(s, \tilde{C}^{m,(-i)}_s) \, d\tilde{\ell}^{i,n}_s < \chi^{i,(j)} \mid F^{i,n}_{t} \right) \cdot \mathbb{P} \left( \int_{\xi^{(-i),(k-1)}}^{t} \gamma(s, \tilde{C}^{m,(-i)}_s) \, d\tilde{\ell}^{i,n}_s < \chi^{i,(k)} \mid F^{i,n}_{t} \right) \cdot \mathbb{1} \{ t \in [\xi^{(-i),(k-1)}, \xi^{(-i),(k)}) \},
\]

for \( \tilde{C}^{m,(-i)} \) and \( \tilde{\ell}^{i,n} \) as in the statement of the proposition. By construction, the local time \( \tilde{\ell}^{i,n} = \tilde{\ell}^{i,n,(-i)} \) and the current level of contagion \( \tilde{C}^{m,(-i)} \) are both adapted processes for the filtration \( (F^{i,n}_{t})_{t \geq 0} \). Likewise, for any \( s \leq t \), the event \( \{ t \in [\xi^{(-i),(k-1)}, \xi^{(-i),(k)}) \right\} \) and the events \( \{ \xi^{(-i),(j-1)} \leq s < \xi^{(-i),(j)} \}, \) for \( j = 1, \ldots, k-1 \), are elements of \( F^{i,n}_{t} \). Since the exponential random variables \( \chi^{i,(j)} \), for \( j = 1, \ldots, k \), were chosen to be independent from the random inputs generating \( F^{i,n}_{t} \), we thus arrive at

\[
P \left( E_k \mid F^{i,n}_{t} \right) = \prod_{j=1}^{k-1} \exp \left\{ - \int_{\xi^{(-i),(j-1)}}^{\tilde{\zeta}^{(-i),(j)}} \gamma(s, \tilde{C}^{m,(-i)}_s) \, d\tilde{\ell}^{i,n}_s \right\} \cdot \exp \left\{ - \int_{\xi^{(-i),(k-1)}}^{t} \gamma(s, \tilde{C}^{m,(-i)}_s) \, d\tilde{\ell}^{i,n}_s \right\} \cdot \mathbb{1} \{ t \in [\xi^{(-i),(k-1)}, \xi^{(-i),(k)}) \},
\]

\[
= \exp \left\{ - \int_{0}^{t} \gamma(s, \tilde{C}^{m,(-i)}_s) \, d\tilde{\ell}^{i,n}_s \right\} \cdot \mathbb{1} \{ t \in [\xi^{(-i),(k-1)}, \xi^{(-i),(k)}) \},
\]

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for \( k = 0, \ldots, n - 1 \). Using this expression in (37), and noting that an analogous argument applies to the final event \( E_n = \{ \tau^i > t \} \cap \{ t \geq \xi^{-1}(i, (n-1)) \} \), we can finally conclude that

\[
\mathbb{P}(\tau^i > t \mid \hat{F}_t^{i,n}) = \exp \left\{ - \int_0^t \gamma(s, \hat{c}_s^{n,(-i)}) \, d\hat{\xi}^{i,n}_s \right\}.
\]

Since \( i \in \{1, \ldots, n\} \) and \( t \geq 0 \) were arbitrary, the proof is complete. \( \square \)

4 The local time under a re-scaling of the state space

In this section we first present some general results on how the local time of a continuous semi-martingale behaves under re-scaling of the state space (Section 4.1) and how a special case of this allows one to relate the analysis of a wide class of one-dimensional reflected diffusions to the simpler study of reflected Brownian motion on a half-line (Section 4.1). We have kept the results of this section quite general, as it is our hope that the arguments could be of more general interest. In Section 4.3 we apply these results to give a short proof of Proposition 2.4.

4.1 Re-scaling the state space and its effect on local times

Let \( X_t \) be a real-valued continuous semi-martingale with respect to a given filtration \( \mathcal{F}_t \), and consider a random map \( (t, \omega, x) \mapsto \Upsilon(t, \omega, x) \) such that \( (t, \omega) \mapsto \Upsilon(t, \omega, x) \) is adapted to \( \mathcal{F}_t \) for all \( x \in \mathbb{R} \). Suppose also that \( \Upsilon(t, \omega, \cdot) : \mathbb{R} \to \mathbb{R} \) is a homeomorphism with \( \lambda' = \Upsilon(t, \omega, \lambda) \) for all \( t \in [0, T] \) and \( \omega \in \Omega \), for two given points \( \lambda, \lambda' \in \mathbb{R} \). This map can then be viewed as a random and time-dependent re-scaling of the state space on either side of \( \lambda \), mapping \( \lambda \) to a given point \( \lambda' \). We are interested in characterising how the local time of \( X \) at \( \lambda \) behaves under such a transformation.

Throughout this subsection, we place the following assumptions on the random map \( \Upsilon(t, \omega, x) \). These assumptions are all natural in view of the application for which we need a re-scaling of the state space in Section 4.2 below. We choose to take \( \Upsilon \) strictly increasing in \( x \), but of course it could also be taken strictly decreasing.

**Assumption 4.1 (The re-scaling map \( \Upsilon \)).** Fix \( T > 0 \). In addition to the adaptedness, the re-scaling map \( \Upsilon : [0, T] \times \Omega \times \mathbb{R} \to \mathbb{R} \) is required to satisfy the following regularity properties:

(a) Each map \( x \mapsto \Upsilon(t, \omega, x) \) is strictly increasing with \( \Upsilon(t, \omega, \lambda) = \lambda' \), therefore invertible with strictly increasing inverse denoted \( x \mapsto \Upsilon^{-1}(t, \omega, x) \) and satisfying \( \Upsilon^{-1}(t, \omega, \lambda') = \lambda \).

(b) The map \( x \mapsto \Upsilon(t, \omega, x) \) is a difference of two convex (or concave) functions.

(c) For all \( \omega \in \Omega \) and \( t \in [0, T] \), the map \( x \mapsto \Upsilon(t, \omega, x) \) is continuously differentiable on \( [\lambda, \lambda + \delta) \), for some \( \delta > 0 \), where the right-derivative \( \partial^+_x \Upsilon(t, \omega, \lambda) \) is used at \( x = \lambda \).

(d) Uniformly in \( \omega \in \Omega \) and \( t \in [0, T] \), we have \( c \leq \partial^+_x \Upsilon(t, \omega, \lambda) \leq C \) and \( c \leq \partial_x \Upsilon(t, \omega, x) \leq C \) for all \( x \in (\lambda, \lambda + \delta) \), for a small enough \( \delta > 0 \) and given constants \( c, C > 0 \).

(e) Uniformly in \( \omega \in \Omega \) and \( x \in (\lambda, \lambda + \delta) \), for some \( \delta > 0 \), the map \( t \mapsto \partial_x \Upsilon(t, \omega, x) \) is Lipschitz.

With these assumptions, we obtain the following result on the local time after re-scaling.

**Proposition 4.2 (Local time under re-scaling of the state space).** Let \( X \) be a continuous semi-martingale and let \( \Upsilon \) be a random and time-dependent re-scaling map satisfying Assumption 4.1.
Then the local time of $X$ at $\lambda$ is related to the local time of $\Upsilon_t(\omega) := \Upsilon(t, \omega, X_t(\omega))$ at $\lambda'$ by the expression

$$
\ell^\lambda_t(\Upsilon)(\omega) = \int_0^t \partial^+ \Upsilon(s, \omega, \lambda)d(\ell^\lambda_s(X))(\omega),
$$

for all $t \geq 0$, almost surely.

For clarity of presentation, we postpone the proof to Section 4.4 and proceed directly to the specific application for which we need this result in the present paper. We note that, in the case where $\Upsilon$ does not depend on time, the proof drastically simplifies to become a straightforward application of the occupation time formula with a change of variables, and an analogous statement for this case can e.g. be found in [RY99, Ex. 1.23, Ch. VI]. With time-dependency, it still feels natural to view the statement as a consequence of the occupation time formula, but several technical hurdles must be overcome. We are under no illusions that this type of result should not already be part of the folklore of stochastic analysis, but looking in the literature we were unable to find a proof or a statement, so we give a detailed proof in Section 4.4.

### 4.2 Lamperti transformation of reflected diffusions

Consider a reflected diffusion $X_t$ on the positive half-line with dynamics

$$
\text{d}X_t = b(t, \omega, X_t)\text{d}t + \sigma(t, \omega, X_t)\text{d}W_t + \text{d}\ell^0_0(X),
$$

(39)
on a given filtered probability space. Throughout this section, we make the following assumptions on the coefficients of (39). Firstly, the functions $b$ and $\sigma$ are jointly measurable in $(t, \omega, x)$, and, for each $x$, adapted in $(t, \omega)$. Secondly, $\sigma$ is weakly differentiable in $t$ and $x$, and there exist constants $c, C, K_1, K_2 > 0$ such that, for all $t \in [0, T]$, $x \in \mathbb{R}^+$ and $\omega \in \Omega$, we have $c \leq \sigma(t, \omega, x) \leq C$, $|\partial_x \sigma(t, \omega, x)| \leq K_1$, and $|\partial_t \sigma(t, \omega, x)| \leq K_2$, for the precise representatives of the weak derivatives. Thirdly, the weak derivatives $\partial_t \sigma$ and $\partial_x \sigma$ are jointly measurable in $(t, \omega, x)$ and adapted in $(t, \omega)$ for each $x$. Finally, the drift $b$ is of at most linear growth in $x$, uniformly in $(t, \omega)$.

By analogy with the usual Lamperti transformation for diffusions on the real line (see e.g. [Luschgy & Pages, Sect. 3]), we define the random map

$$
\Upsilon(t, \omega, y) := \int_0^y \frac{1}{\sigma(t, \omega, x)}\text{d}x.
$$

(40)

Provided $(t, \omega) \mapsto \sigma(t, \omega, x)$ is adapted for the given filtration, which we assume, it is clear that each $(t, \omega) \mapsto \Upsilon(t, \omega, y)$ is adapted to this filtration. Naturally, one could consider other base points than 0 in (40), but we stick with 0 here.

**Lemma 4.3.** In addition to the requisite adaptedness, the map (40) satisfies all the properties (a), (b), (c), (d), and (e) of Assumption 4.1.

**Proof.** The adapted was addressed just above. Properties (a)–(d) follow immediately from the definition of $\Upsilon$ together with the non-degeneracy and boundedness of $\sigma(t, \omega, x)$. For the remaining property (d), we can observe that

$$
|\partial_x \Upsilon(t, x) - \partial_x \Upsilon(s, x)| = \left| \frac{1}{\sigma(t, x)} - \frac{1}{\sigma(s, x)} \right| \leq \frac{|\sigma(s, y) - \sigma(t, y)|}{\sigma(s, y)\sigma(t, y)} \leq \frac{K_2}{c^2}|t - s|,
$$

so we have the desired Lipschitzness. 

\[\square\]
We can now give the main result of this subsection, showing how the rescaling of the state-space by the Lamperti transformation (40) affects the local time of the reflected diffusion at the origin.

**Proposition 4.4.** Let $X$ be given by (39) and let $\Upsilon$ be given by (40). Then $\Upsilon_t := \Upsilon(t, X_t)$ defines another reflected diffusion on $\mathbb{R}^+$ with dynamics

$$d\Upsilon_t = \tilde{b}(t, \omega, \Upsilon_t) \, dt + dW_t + d\ell_t^0(\Upsilon),$$

where the modified drift $\tilde{b}$ is jointly measurable, adapted in $(t, \omega)$, and satisfies

$$\sup_{(t, \omega) \in [0, T] \times \Omega} |\tilde{b}(t, \omega, x)| \leq \tilde{C}(1 + x) \quad \text{for all} \quad x \geq 0$$

for a given constant $\tilde{C} > 0$.

**Proof.** By definition, we have $\Upsilon_t(\omega) = \Upsilon(t, X_t(\omega))(\omega)$, where $\Upsilon$ is given by (40). By the assumptions on $\sigma$, we can then apply the generalized Itô’s formula [RY99, Ex. 3.12, Ch. 4] (also generalized to weak differentiability as in [Kry09, Thm. 10.1, Ch. 2]) for the adapted function $(t, \omega, x) \mapsto \Upsilon(t, \omega, x)$. This is readily seen to yield

$$\Upsilon_t(\omega) - \Upsilon_0(\omega) = \int_0^t \tilde{b}(s, \omega, X_s(\omega)) \, ds + W_t(\omega) + \int_0^t \frac{1}{\sigma(s, \omega, X_s(\omega))} \, d(\ell_s^0(X)(\omega)), \quad (43)$$

where we have introduced the jointly measurable (and adapted) function

$$\tilde{b}(t, \omega, x) := -\int_0^x \frac{\partial \sigma(t, \omega, y)}{\sigma(t, \omega, y)^2} \, dy + \frac{b(t, \omega, x)}{\sigma(t, \omega, x)} - \frac{1}{2} \partial_x \sigma(t, \omega, x).$$

Since $t \mapsto \ell_t^0(X)$ is carried by the set $\{t \geq 0 : X_t = 0\}$ [RY96, Prop. 1.3], the final integral in (43) equals

$$\int_0^t \frac{1}{\sigma(s, \omega, 0)} \, d(\ell_s^0(X)(\omega)).$$

By Lemma 4.3, we can apply Proposition 4.2 to see that this integral equals $\ell_t^0(\Upsilon)$. Next, letting $\Upsilon^{-1}(t, x)(\omega)$ denote the well-defined inverse of $x \mapsto \Upsilon(t, x)(\omega)$ on $\mathbb{R}^+$, we can define

$$\tilde{b}(t, \omega, x) := \tilde{b}(t, \omega, \Upsilon^{-1}(t, x)(\omega))$$

to find that (a version of) $\Upsilon$ has dynamics of the desired form (41). Finally, the linear growth bound (42) for the new drift $\tilde{b}(t, \omega, x)$, defined by the above expression, follows from Lemma 4.3 together with the linear growth assumption on the original drift $b(t, x, \omega)$ uniformly in $t$ and $\omega$. \qed

It should be pointed out that, working with reflected diffusions as we do here, one could give a different proof of Proposition 4.4. Instead of relying on the general result Proposition 4.2, we could base an alternative argument on an application of Tanaka’s formulation, utilising that the local time is already part of the dynamics for the reflected diffusion. Indeed, in (43), one can note that $\Upsilon_t = |\Upsilon_t|$, since $X_t$ remains non-negative at all times, and so Tanaka’s formula can be applied to the absolute value, which makes the local time of $\Upsilon_t$ appear also on the left-hand side of (43). Using the properties of $\ell_0(X)$ and cancelling terms, one then ends up concluding from the resulting equality and the properties of our coefficients that the local time of $\Upsilon_t$ must be of the desired form. Nonetheless, we prefer to rely on Proposition 4.2 as it feels like a natural result that one would expect to hold, and, once we have it, the above proof of Proposition 4.4 becomes nothing more than an application of Itô’s formula, as we believe it should be seen. Moreover, we stress that the arguments in the proof of Proposition 4.2 are useful, also on their own, in the companion paper [PS22], as part of identifying the correct boundary condition for the limiting mean-field problem.
4.3 Brownian motion in the frame of the moving boundary

In this section, we return to the specific setting of our particle system and use Proposition 4.4 to give a proof of Proposition 2.4.

Proof of Proposition 2.4. By the construction of the particle system in Section 3 and recalling the definition $\tau^i = \inf\{t > 0 : X^{i,n}_t = \hat{a}\}$, we can check that $X^{i,n}_t \in (a, b)$ if and only if we have $\hat{X}^{i,n,(-i)} \in (a, b)$ and $t < \tau^i$, since $X^{i,n}_t(\omega) = \hat{X}^{i,n,(-i)}(\omega)$ for all $\omega \in \{t < \tau^i\}$. Moreover, $\mathcal{G}_t^{i,n} \subseteq \tilde{\mathcal{F}}_t^{i,n}$, so we can then write

$$\mathbb{P}(X^{i,n}_t \in (a, b) \mid \mathcal{G}_t^{i,n}) = \mathbb{E}[1_{\{\hat{X}^{i,n,(-i)} \in (a, b)\}} \mathbb{P}(t < \tau^i \mid \tilde{\mathcal{F}}_t^{i,n}) \mid \mathcal{G}_t^{i,n}].$$

Viewing each particle $\hat{X}^{i,n,(-i)}$ in the frame of its moving boundary $\hat{A}^{n,(-i)}$, we obtain the dynamics

$$Y^{i,n}_t = Y^{i,n}_0 + \int_0^t b(t, Y^{i,n}_s + \hat{A}^{n,(-i)}(s))ds + \int_0^t \sigma(t, Y^{i,n}_s + \hat{A}^{n,(-i)}(s))dB_s + \ell^0_t(Y^{i,n}_s)$$

with $Y^{i,n}_t := \hat{X}^{i,n,(-i)} - \hat{A}^{n,(-i)}$, so that $Y^{i,n}_t$ is a reflected diffusion on the positive half-line. From here, the idea is to re-scale the state-space according to the Lamperti transformation

$$Υ^{i,n}(t, y)(\omega) := \mathcal{T}^{i,n}(t, ω, y) := \int_0^y \frac{1}{σ(t, x + \hat{A}^{n,(-i)}(ω))} dx$$

Since we are assuming $σ$ to be absolutely continuous, properties of mollification give that each $\hat{A}^{n,(-i)}$ is absolutely continuous with a finite $L^1$-derivative on any given interval $[0, T]$. This ensures that $(t, ω, x) \mapsto σ(t, x + \hat{A}^{n,(-i)}(ω))$ is such that Lemma 4.3 holds for the above definition of $Υ^{i,n}$ and so we can apply Proposition 4.4 to see that

$$Z^{i,n}_t(ω) := Υ^{i,n}(t, ω, \hat{X}^{i,n,(-i)}(ω) - \hat{A}^{n,(-i)}(ω))$$

has the desired dynamics (16). Next, $Υ^{i,n}(t, \cdot)$ is strictly increasing, so the image $Υ^{i,n}_t(t, (x, y))$ is equal to the open interval from $Υ^{i,n}_t(t, x)$ to $Υ^{i,n}_t(t, y)$ for any $x \leq y$. Thus, the event that $\hat{X}^{i,n,(-i)}$ lies in the open interval $(a, b)$ is equivalent to $Z^{i,n}_t$ lying in the open interval from $Υ^{i,n}_t(t, a - \hat{A}^{n,(-i)}(ω))$ to $Υ^{i,n}_t(t, b - \hat{A}^{n,(-i)}(ω))$, and hence

$$\mathbb{P}(X^{i,n}_t \in (a, b) \mid \mathcal{G}_t^{i,n}) = \mathbb{E}[1_{\{Z^{i,n}_t(Υ^{i,n}_t(t, a - \hat{A}^{n,(-i)}(ω)), \hat{Y}^{i,n}_t(t, b - \hat{A}^{n,(-i)}(ω)))\}} \mathbb{P}(t < \tau^i \mid \tilde{\mathcal{F}}_t^{i,n}) \mid \mathcal{G}_t^{i,n}].$$

Finally, Proposition 4.2 gives $d\hat{Z}^{i,n}_t = σ(r, \hat{A}^{n,(-i)}_t) d\ell^0_t(Z^{i,n}_t)$, so the definition of $\tau^{i,n}_Z$ in (17) yields

$$\mathbb{P}(t < \tau^{i,n}_Z \mid \mathcal{G}_t^{i,n} \vee \sigma(Z^{i,n}_s : s \in [0, t])) = e^{-\int_0^t σ(r, \hat{A}^{n,(-i)}_t)σ(r, \hat{A}^{n,(-i)}_t) d\ell^0_t(Z^{i,n}_t)} = e^{-\int_0^t γ(r, \hat{A}^{n,(-i)}_t) d\ell^0_t(Z^{i,n}_t)},$$

where the last term agrees with $\mathbb{P}(t < \tau^i \mid \hat{F}_t^{i,n})$ by virtue of (12). Inserting this into (44), we have arrived at the desired expression (15) with $Z^{i,n}$ and $\tau^{i,n}_Z$ satisfying the correct properties. Thus, the proof is complete.

4.4 Local times under re-scaling of the state-space: proof of Proposition 4.2

In this section, we give a proof of Proposition 4.2. To simplify the presentation, we first single out an auxiliary lemma.
Lemma 4.5. Let \( \Upsilon \) satisfy Assumption 4.1 and, for every \( \varepsilon > 0 \), let \( \varphi_\varepsilon \in C_0^\infty \) be a mollifier supported on \([0, \varepsilon]\) which smoothly approximates the Dirac mass at 0. For all \( \varepsilon > 0 \), the function \( t \mapsto \varphi_\varepsilon(\Upsilon(t, \omega, x))(\partial_x \Upsilon(t, \omega, x))^2 \) is then Lipschitz continuous, and the Lipschitz constant can be taken to be proportional to \( \varepsilon^{-1} \) uniformly in \( x \in \mathbb{R}^+ \) and \( \omega \in \Omega \).

Proof. To keep the notation simple, we suppress the dependence on \( \omega \) throughout. For each \( \varepsilon > 0 \), we have \( \varphi_\varepsilon \in C_0^\infty \) supported on \([0, \varepsilon]\) with \( \|\varphi_\varepsilon\|_{L^1} = 1 \). Furthermore, we can assume without loss of generality that there are constants \( k_1, k_2 > 0 \) such that \( \|\varphi_\varepsilon\|_\infty \leq k_1 \varepsilon^{-1} \) and \( \|\partial_\nu \varphi_\varepsilon\|_\infty \leq k_2 \varepsilon^{-2} \) for all \( \varepsilon > 0 \). For every \( s \geq 0 \), it follows from (a) and (d) in Assumption 4.1 along with the fundamental theorem of calculus that \( \Upsilon(s, x) = \int_0^x \partial_y \Upsilon(s, y) \, dy \geq cx \) for small enough \( x \geq 0 \), and the map is also increasing, so \( \varphi_\varepsilon(\Upsilon(t, x)) = \varphi_\varepsilon(\Upsilon(t, x)) \mathbb{1}_{\{0 \leq x \leq \varepsilon/c\}} \) for \( \varepsilon > 0 \) small enough. Given this, and applying the triangle inequality, we get

\[
|\varphi_\varepsilon(\Upsilon(t, x))(\partial_x \Upsilon(t, x))^2 - \varphi_\varepsilon(\Upsilon(s, x))(\partial_x \Upsilon(s, x))^2| \\
\leq (\partial_x \Upsilon(s, x))^2 |\varphi_\varepsilon(\Upsilon(t, x)) - \varphi_\varepsilon(\Upsilon(s, x))| \mathbb{1}_{\{0 \leq x \leq \varepsilon/c\}} \\
+ \varphi_\varepsilon(\Upsilon(s, x))|(|\partial_x \Upsilon(t, x)|^2 - (\partial_x \Upsilon(s, x))^2| \mathbb{1}_{\{0 \leq x \leq \varepsilon/c\}}.
\]

(45)

We now look for bounds for each of the two terms on the right-side. By the fundamental theorem of calculus for \( \varphi_\varepsilon \) and Assumption 4.1(e) for \( \Upsilon \), we get

\[
|\varphi_\varepsilon(\Upsilon(t, x)) - \varphi_\varepsilon(\Upsilon(s, x))| = \left| \int_{\Upsilon(s, x)}^{\Upsilon(t, x)} \partial_y \varphi_\varepsilon(y) \, dy \right| \leq k_2 \varepsilon^{-2} |\Upsilon(t, x) - \Upsilon(s, x)|
\]

Since \( \Upsilon(t, 0) = \Upsilon(s, 0) \), the fundamental theorem of calculus and Assumption 4.1(e) then gives

\[
|\varphi_\varepsilon(\Upsilon(t, x)) - \varphi_\varepsilon(\Upsilon(s, x))| \leq k_2 \varepsilon^{-2} \int_0^x |\partial_y \Upsilon(t, y) - \partial_y \Upsilon(s, y)| \, dy \leq k_2 \varepsilon^{-2} Cx |t - s|,
\]

(46)

for \( x \in [0, C\varepsilon] \) with \( \varepsilon > 0 \) small enough. Moreover, \( \partial_x \Upsilon(s, x) \leq Cx \) for all \( x \in [0, C\varepsilon] \) with \( \varepsilon > 0 \) small enough, by Assumption 4.1(d). Turning to the second term on the right-hand side of (45), we have \( \|\varphi_\varepsilon\|_\infty \leq k_1 \varepsilon^{-1} \) and

\[
|(|\partial_x \Upsilon(t, x)|^2 - (\partial_x \Upsilon(s, x))^2| = (\partial_x \Upsilon(t, x) + \partial_x \Upsilon(s, x))|\partial_x \Upsilon(t, x) - \partial_x \Upsilon(s, x)| \leq 2C^2 |t - s|,
\]

by (d) and (e) of Assumption 4.1, for all \( x \) small enough. Using this inequality and (46) for the right-hand side of (45), we are left with the Lipschitz estimate

\[
\left|\varphi_\varepsilon(\Upsilon(t, x)) \frac{1}{\sigma(t, x)^2} - \varphi_\varepsilon(\Upsilon(s, x)) \frac{1}{\sigma(s, x)^2}\right| \leq (\tilde{C}_1 \varepsilon^{-2} x + \tilde{C}_2 \varepsilon^{-1}) |t - s| \mathbb{1}_{\{0 \leq x \leq \varepsilon/c\}}
\]

for given constants \( \tilde{C}_1, \tilde{C}_2 > 0 \). Since the indicator function restricts to \( x \leq \varepsilon/c \), the claim follows. \( \square \)

We now return to the proof of Proposition 4.2.

Proof of Proposition 4.2. Instead of \( \lambda' = \Upsilon(t, \omega, \lambda) \), there is no loss of generality in considering \( \lambda = \Upsilon(t, \omega, \lambda) \), as only a linear shift is involved. For notational simplicity, we will furthermore take \( \lambda = 0 \), and we also suppress the dependence of \( \Upsilon \) on \( \omega \) throughout.

Let us begin the proof by fixing a family of mollifiers \( \{\varphi_\varepsilon\}_{\varepsilon \in \mathbb{R}} \) smoothly approximating the Dirac mass at 0, as in the proof of Lemma 4.5. By Assumption 4.1(b)–(e), \( \Upsilon_t \) is again a semimartingale with \( [\Upsilon]_t = \int_0^t (\partial_x \Upsilon(s, X_s))^2 \, d[X]_s \), for all \( t \geq 0 \), by the Meyer–Itô formula [Pro05 Ch. IV, Thm. 70]. By the standard occupation time formula [RY99 Cor. 1.6, Ch. 6] with positive Borel function \( \varphi_\varepsilon \), and
using the almost sure right-continuity of the local time in $x$ for continuous semi-martingales [RY99 Ch. 6, Thm. 1.7], we therefore have that

$$\ell^0_t(\mathcal{Y}) = \lim_{\varepsilon \downarrow 0} \int_0^t \varepsilon \varphi_\varepsilon(\mathcal{Y}_s) \, d[\mathcal{Y}]_s = \lim_{\varepsilon \downarrow 0} \int_0^t \varepsilon \varphi_\varepsilon(\mathcal{Y}(s, X_s))(\partial_x \mathcal{Y}(s, X_s))^2 \, d[X]_s,$$

for all $t \geq 0$ (almost surely). Next, the generalised occupation time formula for Borel functions on $[0, \infty) \times \Omega \times \mathbb{R}$ [RY99 Ex. 1.13, Ch. 6] then gives that

$$\ell^0_t(\mathcal{Y}) = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^+} \int_0^t \varphi_\varepsilon(\mathcal{Y}(s, y))(\partial_x \mathcal{Y}(s, y))^2 \, d\ell^y_s(X) \, dy$$

for all $t \geq 0$ (almost surely). From here, the objective of the proof is to show that the right-hand side above in fact converges to the desired expression for all $t$ and subtracting the Riemann sums for the Riemann-Stieltjes integrals of its approximating sums along arbitrary partitions $\{\pi_n = \{s_0, s_1, \ldots, s_n\}\}_{n \in \mathbb{N}}$ of $[0, t]$. Adding and subtracting the Riemann sums for the Riemann-Stieltjes integrals $\int_0^t \partial_x^+ \mathcal{Y}(s, 0) \, d\ell^0_s(X)$ and $\int_0^t \varphi_\varepsilon(\mathcal{Y}(s, y))(\partial_x \mathcal{Y}(s, y))^2 \, d\ell^y_s(X)$, we can write down the following equality

$$\int_0^t \partial_x^+ \mathcal{Y}(s, 0) \, d\ell^0_s(X) = \int_{\mathbb{R}^+} \int_0^t \varphi_\varepsilon(\mathcal{Y}(s, y))(\partial_x \mathcal{Y}(s, y))^2 \, d\ell^y_s(X) \, dy + A_1(n) + A_2(n, \varepsilon) + A_3(n, \varepsilon),$$

where we have defined

$$A_1(n) := \int_0^t \partial_x^+ \mathcal{Y}(s, 0) \, d\ell^0_s(X) - \sum_{i=0}^{n-1} \partial_x^+ \mathcal{Y}(s_i, 0)(\ell^0_{s_{i+1}} - \ell^0_{s_i}),$$

$$A_2(n, \varepsilon) := \int_{\mathbb{R}^+} \sum_{i=0}^{n-1} \varphi_\varepsilon(\mathcal{Y}(s_i, y))(\partial_x \mathcal{Y}(s_i, y))^2(\ell^y_{s_{i+1}} - \ell^y_{s_i}) \, dy$$

$$- \int_{\mathbb{R}^+} \int_0^t \varphi_\varepsilon(\mathcal{Y}(s, y))(\partial_x \mathcal{Y}(s, y))^2 \, d\ell^y_s(X) \, dy,$$

$$A_3(n, \varepsilon) := \sum_{i=0}^{n-1} \partial_x^+ \mathcal{Y}(s_i, 0)(\ell^0_{s_{i+1}} - \ell^0_{s_i}) - \int_{\mathbb{R}^+} \sum_{i=0}^{n-1} \varphi_\varepsilon(\mathcal{Y}(s_i, y))(\partial_x \mathcal{Y}(s_i, y))^2(\ell^y_{s_{i+1}} - \ell^y_{s_i}) \, dy.$$

Let $\delta > 0$ be given. By definition of the Riemann-Stieltjes integral, there exists $N_1(\delta) \in \mathbb{N}$ such that for all $n > N_1(\delta)$ we have

$$|A_1(n)| = \left| \int_0^t \partial_x^+ \mathcal{Y}(s, 0) \, d\ell^0_s(X) - \sum_{i=0}^{n-1} \partial_x^+ \mathcal{Y}(t_i, 0)(\ell^0_{t_{i+1}}(X) - \ell^0_{t_i}(X)) \right| \leq \delta.$$  

(48)
Now consider $|A_2(n, \varepsilon)|$. Using linearity of the integrals, we can rewrite

$$A_2(n, \varepsilon) = \int_{\mathbb{R}^+} \left( \sum_{i=0}^{n-1} \varphi_\varepsilon(\Upsilon(s_i, y))(\partial_x \Upsilon(s_i, y))^2(\ell^y_{s_{i+1}} - \ell^y_{s_i}) \right.$$

$$- \int_0^t \varphi_\varepsilon(\Upsilon(s, y))(\partial_x \Upsilon(s, y))^2 \, d\ell^y_s(X) \bigg) \, dy$$

$$= \int_{\mathbb{R}^+} \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \varphi_\varepsilon(\Upsilon(s, y))(\partial_x \Upsilon(s, y))^2 \, d\ell^y_s(X)$$

$$- \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \varphi_\varepsilon(\Upsilon(s, y))(\partial_x \Upsilon(s, y))^2 \, d\ell^y_s(X) \, dy.$$ 

Since $\ell^y_t$ is an increasing process, $\text{TV}_{[0,t]}(\ell^y) = \ell^y_t$, and so we have

$$|A_2(n, \varepsilon)| \leq \int_{\mathbb{R}^+} \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} \left| \varphi_\varepsilon(\Upsilon(s, y))(\partial_x \Upsilon(s, y))^2 - \varphi_\varepsilon(\Upsilon(s, y))(\partial_x \Upsilon(s, y))^2 \right| \, d\ell^y_s(X) \, dy.$$ 

Applying Lemma 4.5 and recalling from its proof that $\varphi_\varepsilon(\Upsilon(s, y))$ is only supported on $y \in [0, \varepsilon/c]$, we get

$$|A_2(n, \varepsilon)| \leq \int_{\mathbb{R}^+} \int_0^{\varepsilon/c} \left( \int_0^t s \, d\ell^y_s(X) - \int_0^t \sum_{i=0}^{n-1} s_i \mathbbm{1}_{[s_i, s_{i+1}]}(s) \, d\ell^y_s(X) \right) \, dy$$

$$\leq \tilde{C} \varepsilon^{-1} \int_0^{\varepsilon/c} \sup_{y \in \mathbb{R}^+} \left( \sup_{s \in [0,t]} \left| s - \sum_{i=0}^{n-1} s_i \mathbbm{1}_{[s_i, s_{i+1}]}(s) \right| \right) \, dy.$$ 

From Barlow–Yor’s BDG type inequality for local times [RY99, Ch. XI, Thm. 2.4] (see also [BY82, Page 199] for the case of semimartingales), we know that the expectation of $L := \sup_{y \in \mathbb{R}^+} \ell^y_t$ is finite. In particular, $L$ is finite almost surely. Moreover, we can always find a partition $\pi_n$ of $[0, t]$ fine enough that $\sup_{s \in [0,t]} \left| s - \sum_{i=0}^{n-1} s_i \mathbbm{1}_{[s_i, s_{i+1}]}(s) \right|$ is as small as we like. In other words, for $\delta > 0$, there exists $N_2(\delta)$ such that

$$\sup_{s \in [0,t]} \left| s - \sum_{i=0}^{n-1} s_i \mathbbm{1}_{[s_i, s_{i+1}]}(s) \right| \leq \delta, \quad \forall n > N_2(\delta).$$ 

Hence we conclude that, for all $n > N_2(\delta)$,

$$|A_2(n, \varepsilon)| \leq \tilde{C} \varepsilon^{-1} L \delta,$$

and we know the right-hand side is almost-surely finite.

Finally, consider $|A_3(n, \varepsilon)|$. Applying the change of variables $z = \Upsilon(s_i, y)$, and using (a) and (c) from Assumption 4.1 as well as the inverse function theorem, we get

$$A_3(n, \varepsilon) = \sum_{i=0}^{n-1} \left( \partial_x^* \Upsilon(s_i, 0)(\ell^0_{s_{i+1}} - \ell^0_{s_i}) - \int_{\mathbb{R}^+} \varphi_\varepsilon(z) \partial_x \Upsilon(s_i, \Upsilon^{-1}(s_{i+1}, z))(\ell^0_{s_{i+1}} - \ell^0_{s_i}) \, dz \right).$$
Recalling that \( \varphi_\varepsilon \) integrates to 1 and is supported only \([0, \varepsilon]\), we can then estimate
\[
|A_3(n, \varepsilon)| = \left| \int_0^\varepsilon \varphi_\varepsilon(z) \sum_{i=0}^{n-1} \left( \partial_x^+ \Upsilon(s_i, 0)(\ell^{0}_i - \ell^{0}_{i+1}) - \partial_x \Upsilon(s_i, \Upsilon^{-1}(s_i, z))(\ell^{0}_{i+1} - \ell^{0}_{i}) \right) \, dz \right|
\]
\[
\leq \int_0^\varepsilon \varphi_\varepsilon(z) \sum_{i=0}^{n-1} \left| \partial_x \Upsilon(s_i, \Upsilon^{-1}(s_i, 0))\ell^{0}_{i+1} - \partial_x \Upsilon(s_i, \Upsilon^{-1}(s_i, z))\ell^{0}_{i} \right| \, dz
\]
\[
+ \int_0^\varepsilon \varphi_\varepsilon(z) \sum_{i=0}^{n-1} \left| \partial_x \Upsilon(s_i, \Upsilon^{-1}(s_i, 0))\ell^{0}_{i+1} - \partial_x \Upsilon(s_i, \Upsilon^{-1}(s_i, z))\ell^{0}_{i} \right| \, dz
\]

Since \( y \mapsto \ell^0_t(Y) \) is cadlag, and \( z \mapsto \Upsilon^{-1}(s, z) \) is continuous and strictly increasing, by Assumption 4.1(a), we have that \( z \mapsto \ell^0_t(X) \) is right-continuous. Likewise, \( z \mapsto \partial_x \Upsilon(s, \Upsilon^{-1}(s, z)) \) will be right-continuous on a right-neighborhood of zero, by Assumption 4.1(c). In particular, their product is right continuous, for all small enough \( z \), so, given \( \delta/n \), there exists \( \beta(\delta) \) such that
\[
|z| \leq \beta(\delta) \implies |\partial_x \Upsilon(s, \Upsilon^{-1}(s, z))\ell^0_t(X) - \partial_x \Upsilon(s, \Upsilon^{-1}(s, 0))\ell^0_t(X)| \leq \delta/n,
\]
for all \( t \geq 0 \). It then follows that, for all \( \varepsilon \in [0, \beta(\delta)] \), we have
\[
|A_3(n, \varepsilon)| \leq 2\delta, \quad \forall n \in \mathbb{N}.
\] (50)

From the expression (47) and the bounds (48), (49) and (50) for \( A_1, A_2, \) and \( A_3 \), on the right-hand side, we conclude that for all \( \delta > 0 \) and all \( n \in \mathbb{N} \) such that \( n \geq N(\delta) = \max\{N_1(\delta), N_2(\delta)\} \), there exists \( \beta(\delta) > 0 \) such that
\[
\left| \int_0^t \partial_x^+ \Upsilon(s, 0) \, d\ell^0_s(X) - \int_0^t \int_0^\varepsilon \varphi_\varepsilon(Y(s, y))(\partial_x \Upsilon(s, y))^2 \, d\ell^0_s(X) \, dy \right| \leq (3 + \tilde{CC}L)\delta \quad \forall \varepsilon \in [0, \beta(\delta)].
\] (51)

Taking the limit as \( \delta \to 0 \), and noting that \( \varepsilon \to 0 \) at the same time, we conclude that the right-hand side of (4.4) converges to the intended limit (almost surely). This completes the proof. \( \square \)

5 Key properties of the infected proportion

The purpose of the present section is to give the proof of Theorem 2.5. The proof itself is given in Section 5.2, but the main work lies in verifying suitable martingale properties of the infected proportion, which is the subject of the first subsection.

5.1 Martingale properties of the infected proportion

Recall that, by Theorem 2.3, the infection times \( \tau^i \) satisfy
\[
\mathbb{P}(\tau^i \geq s \mid \tilde{\mathcal{F}}^{i,n}_t) = 1 - e^{-\int_0^s \gamma(r, \tilde{C}^{i,n}_{r'}(-1)) \, dr}, \quad \text{for} \quad s \leq t,
\] (52)
for each \( i = 1, \ldots, n \), where we recall that \( \tilde{\mathcal{F}}^{i,n} \) denotes the reduced-information filtration (13), which is given by
\[
\tilde{\mathcal{F}}^{i,n}_t = \sigma((X^i_0, B^i_s), (X^j_0, B^j_s, \chi^{j,(k)}_{k=1})_s : s \in [0, t], j \in \{1, \ldots, n\} \setminus \{i\})
\] (53)
for each $i = 1, \ldots, n$. In addition to this, we can consider also the filtration $\mathcal{F}^{i,n}$, given by

$$\mathcal{F}^{i,n}_t := \sigma\left(\{s < \tau^{i}\} : s \in [0, t]\right), \quad \text{for} \quad t \geq 0,$$

which reveals the infection status of particle $i$ up to time $t$ and nothing else. We shall then be interested in the filtration $\mathcal{F}^{i,n}$ obtained by adding this information to $\mathcal{F}^{i,n}$, that is, by defining

$$\mathcal{F}^{i,n}_t := \mathcal{F}^{i,n}_t \vee \mathcal{F}^{i,n}_t, \quad \text{for} \quad t \geq 0,$$

noting that the original filtration $\mathcal{F}^{i,n}$, on its own, does not reveal anything about the timing of particle $i$’s infection (and neither does it give away any of the other infection times in the true particle system, up to the given time $t$, although it does contain some information about this).

We shall first need two auxiliary lemmas, which relate the larger filtration $\mathcal{F}^{i,n}$ to the smaller filtration $\mathcal{F}^{i,n}$ for certain conditional expectations involving the infection times. These auxiliary results have analogues in the theory of hazard processes in credit risk theory (see e.g. [YJC99], Ch. 7), but the setting here is quite different, so it is worth working out the proofs in detail. In particular, we are dealing with a particle system and we have to work with the local times. Moreover, the arguments must rely on the precise construction of the particle system and the corresponding artificial system that enters into through $\hat{\mathcal{L}}^{i,n}$ and $\hat{\mathcal{C}}^{i,n}$.

**Lemma 5.1.** Fix an arbitrary $n \geq 0$ and $i \in \{1, \ldots, n\}$, and let $\mathcal{F}^{i,n}$ and $\mathcal{F}^{i,n}$ be the filtrations defined in (55) and (56), respectively. For any random variable $Y \in L^1(\Omega, \mathbb{P})$, defined on the same probability space as the particle system, we have

$$\mathbb{E}[Y \mathbf{1}_{s < \tau^{i}} | \mathcal{F}^{i,n}_s] = e^{\int_0^s \gamma(r, \hat{C}_r^{n}(-i)) \, dr} \mathbb{E}[Y \mathbf{1}_{s < \tau^{i}} | \hat{\mathcal{F}}^{i,n}_s] \mathbf{1}_{s < \tau^{i}},$$

for all times $s \geq 0$.

**Proof.** Let $n \geq 0$ and $i \in \{1, \ldots, n\}$ be given. Firstly, the event $\{s < \tau^{i}\}$ is an element of $\mathcal{F}^{i,n}_s$, so we of course have

$$\mathbb{E}[Y \mathbf{1}_{s < \tau^{i}} | \mathcal{F}^{i,n}_s] = \mathbb{E}[Y \mathbf{1}_{s < \tau^{i}} | \mathcal{F}^{i,n}_s] \mathbf{1}_{s < \tau^{i}}.$$

Next, noting that the $\sigma$-algebra $\mathcal{F}^{i,n}_s$ is generated by events of the form $A \cap \mathcal{F}^{i,n}_s$ for $A \in \mathcal{F}^{i,n}_s$ and $\mathcal{E} \in \mathcal{F}^{i,n}_s$, and noting that any such intersection satisfies $A \cap \mathcal{E} \cap \{s < \tau^{i}\} = \emptyset$ or $A \cap \mathcal{E} \cap \{s < \tau^{i}\} = A \cap \{s < \tau^{i}\}$, we deduce that the restriction $\mathbb{E}[Y \mathbf{1}_{s < \tau^{i}} | \mathcal{F}^{i,n}_s] | \{s < \tau^{i}\}$ is measurable for the restricted $\sigma$-algebra $\mathcal{F}^{i,n}_s | \{s < \tau^{i}\}$. From this, it is easy to verify that $\mathbb{E}[Y \mathbf{1}_{s < \tau^{i}} | \mathcal{F}^{i,n}_s] | \{s < \tau^{i}\}$ satisfies the definition of $\mathbb{E}[Y | \mathcal{F}^{i,n}_s] | \{s < \tau^{i}\}$, and hence we get

$$\mathbb{E}[Y \mathbf{1}_{s < \tau^{i}} | \mathcal{F}^{i,n}_s] = \mathbb{E}[Y | \{s < \tau^{i}\}] \mathbf{1}_{s < \tau^{i}} = \mathbb{E}[Y | \hat{\mathcal{F}}^{i,n}_s] \mathbf{1}_{s < \tau^{i}} = e^{\int_0^s \gamma(r, \hat{C}_r^{n}(-i)) \, dr} \mathbb{E}[Y \mathbf{1}_{s < \tau^{i}} | \hat{\mathcal{F}}^{i,n}_s] \mathbf{1}_{s < \tau^{i}},$$

where the last equality comes from (52). This completes the proof. $\square$

The above result takes a particular form if the random variable is the result of stopping a stochastic process at the infection time of a given particle.

**Lemma 5.2.** Fix an arbitrary choice of $n \geq 1$ and $i \in \{1, \ldots, n\}$, and let $s \geq 0$ be given. Let also $\mathcal{F}^{i,n}$ and $\mathcal{F}^{i,n}$ be given by (55) and (56), respectively. If $(Y_t)_{t \in [s, \infty)}$ is a càdlàg process adapted to the filtration $(\mathcal{F}^{i,n}_t)_{t \in [s, \infty)}$ and satisfying $\mathbb{E}[\sup_{t \in [s, \infty)} |Y_t|] \leq \infty$, then we have

$$\mathbb{E}[Y_r \mathbf{1}_{s < \tau^{i}} | \mathcal{F}^{i,n}_s] = \mathbb{E}\left[\int_s^\infty Y_r \gamma(r, \hat{C}_r^{n}(-i)) e^{-\int_s^r \gamma(\theta, \hat{C}_\theta^{n}(-i)) \, d\theta} \, dr \mathbf{1}_{s < \tau^{i}} | \hat{\mathcal{F}}^{i,n}_s\right] \mathbf{1}_{s < \tau^{i}}.$$

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Proof. By Lemma 6.1 applied to the random variable \( Y_{ts} \in L^1(\Omega, \mathbb{P}) \), we have

\[
E[Y_{ts} \mathbb{1}_{s<r} \mid \mathcal{F}_s^{i,n}] = e^{-\int_0^r \gamma(r, C_r^{n,(-i)}) \, d\hat{\theta}} E[Y_{ts} \mathbb{1}_{s<r} \mid \mathcal{F}_s^{i,n}] \mathbb{1}_{s<r}.
\] (56)

Now, using left-continuity and adaptedness to \( \mathcal{F}_s^{i,n} \), we can approximate \( Y \) by simple left-continuous processes of the form

\[
Y_s^{(k)} := \sum_{l=1}^k A_l^{(k)} \mathbb{1}_{s_l < s \leq s_{l+1}},
\]

where \( A_l^{(k)} \) is \( \mathcal{F}_{s_l} \)-measurable, for \( l = 1, \ldots, k \). For each \( k \geq 1 \), the tower law gives

\[
E\left[ \mathbb{1}_{s<r} Y^{(k)}_s \mid \mathcal{F}_s^{i,n} \right] = \sum_{l=1}^k E\left[ A_l^{(k)} \mathbb{1}_{s_l < r} \mathbb{1}_{s < s_l < s_{l+1}} \mid \mathcal{F}_s^{i,n} \right] \mathcal{F}_s^{i,n},
\]

and we can then observe that the inner expectations can be expressed as

\[
E\left[ \mathbb{1}_{s<r} \mathbb{1}_{s < s_l < s_{l+1}} \mid \mathcal{F}_s^{i,n} \right] = \int_{s_l}^{s_{l+1}} \mathbb{1}_{s<r} \, d\mathbb{P}(r \leq s \mid \mathcal{F}_s^{i,n}).
\]

Clearly, the range of integration is a null set when \( r < s_l \lor s \), while for \( r \geq s_l \lor s \) we can use the tower law to see that

\[
\mathbb{P}(r \leq s \mid \mathcal{F}_{s_l \lor s}) = \mathbb{E}[\mathbb{P}(r \leq s \mid \mathcal{F}_r^{i,n}) \mid \mathcal{F}_{s_l \lor s}] = \mathbb{E}[\mathbb{E}(1 - e^{\int_0^r \gamma(r, C_r^{n,(-i)}) \, d\hat{\theta}} \mid \mathcal{F}_s^{i,n})],
\]

by (52). Recalling again that \( A_l \) is \( \mathcal{F}_{s_l} \)-measurable, we thus arrive at

\[
E\left[ \mathbb{1}_{s<r} Y^{(k)}_s \mid \mathcal{F}_s^{i,n} \right] = \sum_{l=1}^k E\left[ A_l^{(k)} \int_s^{\infty} \mathbb{1}_{s_l < s \leq s_{l+1}} d(1 - e^{\int_0^r \gamma(r, C_r^{n,(-i)}) \, d\hat{\theta}}) \mid \mathcal{F}_s^{i,n} \right]
\]

\[
= E\left[ \int_s^{\infty} Y^{(k)}_s \gamma(r, C_r^{n,(-i)}) e^{-\int_0^r \gamma(r, C_r^{n,(-i)}) \, d\hat{\theta}} \, d\hat{\theta} \mid \mathcal{F}_s^{i,n} \right].
\]

Combining this with (56), it follows from \( E[\sup_{t \in [s, \infty]} |Y_t|] \leq \infty \) that we can apply dominated convergence to complete the proof, by sending \( k \to \infty \).

Based on the preceding observations, we can now verify the desired martingale property of the infected proportion. To this end, as in (54), shall need the \( \sigma \)-algebras \( \mathcal{I}_t^n \) generated by all the events \( \{s < \tau^j \} \) for \( s \leq t \) and \( j \in \{0, \ldots, n\} \), namely

\[
\mathcal{I}_t^n = \sigma(\{s < \tau^j : s \in [0, t], j \in \{0, \ldots, n\}\}).
\]

Let \( \mathcal{D}_t^n \) be the \( \sigma \)-algebra generated by the inputs driving the dynamics of every particle, namely

\[
\mathcal{D}_t^n := \sigma(X_0^j, B_s^j : s \in [0, t], j \in \{0, \ldots, n\}).
\]

Then we are interested in the combined filtration

\[
\mathcal{F}_t^n := \mathcal{D}_t^n \lor \mathcal{I}_t^n,
\] (57)

which collects all the information needed to reconstruct the true particle system up to time \( t \), without peeking into the future neither for the reflected dynamics nor for the realisation of the infection times.
Proposition 5.3 (Martingale property). For each $n \geq 1$, the stochastic process $(M^n_t)_{t \geq 0}$ defined by

$M^n_t := I^n_t - V^n_t, \quad V^n_t := \frac{1}{n} \sum_{i=1}^{n} \int_0^{t} 1_{s < \tau^i} \gamma(s, I^n_s) \, d\ell^n_{s,i},$

is a martingale with respect to the filtration $(\mathcal{F}^n_t)_{t \geq 0}$ from (57).

Proof. Notice first that $M^n$ is indeed adapted to $(\mathcal{F}^n_t)_{t \geq 0}$, since (i) the individual events $\{s < \tau^i\}$, for $s \leq t$, are in $\mathcal{F}^n_t$ by virtue of $\mathcal{I}^n_t$, and (ii) combining these events with the information in $\mathcal{F}^n_t$ is sufficient to reconstruct the full particle system with infection up to time $t$, so the processes $\ell^n_{i,n}$ and $I^n$ are also adapted, for $i = 1, \ldots, n$. Next, fixing any $i \in \{0, \ldots, n\}$, we can note that the particle system in Section 3, we can note that $\hat{\theta}$ and $\hat{\gamma}$ in fact also contains all the events $\{s < \tau^i\}$, for $s \in [0, t]$ and $j \in \{1, \ldots, n\}$, so we deduce that $\mathcal{F}^n_s$ is contained in $\mathcal{F}^i_{s,n}$ for each $i = 1, \ldots, n$. Write

$V^n_t = \frac{1}{n} \sum_{i=1}^{n} C^n_{i,t}, \quad C^n_{i,t} := \int_0^{t \wedge \tau^i} \gamma(s, I^n_s) \, d\ell^n_{s,i} \quad \text{for} \quad i = 1, \ldots, n,$

we therefore have

$$E[M^n_t \mid \mathcal{F}^n_s] = E\left[\frac{1}{n} \sum_{i=1}^{n} E[1_{t \geq \tau^i} - V^n_t \mid \mathcal{F}^i_{s,n}] \mid \mathcal{F}^n_s\right]$$

(58)

for all times $s, t \geq 0$. Now fix an arbitrary pair of times $s < t$. By writing

$$E[1_{t \geq \tau^i} - 1_{s \geq \tau^i} \mid \mathcal{F}^i_{s,n}] = P(s < \tau^i \leq t \mid \mathcal{F}^i_{s,n}) = E[1_{s < \tau^i} 1_{t \geq \tau^i} \mid \mathcal{F}^i_{s,n}],$$

we see that Lemma 5.1 gives

$E[1_{t \geq \tau^i} - 1_{s \geq \tau^i} \mid \mathcal{F}^i_{s,n}] = 1_{s < \tau^i} e^{\int_0^s \gamma(r, C^n_{\tau^i-1}(r)) \, d\ell^n_{s,i}} P(s < \tau^i, t \geq \tau^i \mid \mathcal{F}^i_{s,n}) = 1_{s < \tau^i} e^{\int_0^s \gamma(r, C^n_{\tau^i-1}(r)) \, d\ell^n_{s,i}} (P(t \geq \tau^i \mid \mathcal{F}^i_{s,n}) - P(s \geq \tau^i \mid \mathcal{F}^i_{s,n}))$

From here, we can then use the distributional properties of $\tau^i$ from (52) to get

$$P(t \geq \tau^i \mid \mathcal{F}^i_{s,n}) = E[P(t \geq \tau^i \mid \mathcal{F}^i_{s,n}) \mid \mathcal{F}^i_{s,n}] = 1 - E[e^{-\int_0^t \gamma(r, C^n_{\tau^i-1}(r)) \, d\ell^n_{s,i}} \mid \mathcal{F}^i_{s,n}]),$$

and so we arrive at

$$E[1_{t \geq \tau^i} - 1_{s \geq \tau^i} \mid \mathcal{F}^i_{s,n}] = 1_{s < \tau^i} e^{\int_0^s \gamma(r, C^n_{\tau^i-1}(r)) \, d\ell^n_{s,i}} \left( E[e^{-\int_0^t \gamma(r, C^n_{\tau^i-1}(r)) \, d\ell^n_{s,i}} \mid \mathcal{F}^i_{s,n}] + e^{-\int_0^s \gamma(r, C^n_{\tau^i-1}(r)) \, d\ell^n_{s,i}} \right)$$

(59)

In addition to the times $s$ and $t$, fix also an arbitrary index $i \in \{1, \ldots, n\}$. From the construction of the particle system in Section 3, we can note that $\hat{C}^{n,-1}_{\theta} = C^n_{\theta}$ and $\hat{\theta}_n = \ell^n_\theta$ on the event $\{\theta < \tau^i\}$, and hence we can write

$$V^n_t - V^n_s = \int_0^{t \wedge \tau^i} \gamma(\theta, \hat{C}^{n,-1}_{\theta}) \, d\hat{\theta}_n - \int_0^{s \wedge \tau^i} \gamma(\theta, \hat{C}^{n,-1}_{\theta}) \, d\hat{\theta}_n = \int_0^{t \wedge \tau^i} \gamma(\theta, \hat{C}^{n,-1}_{\theta}) \, d\hat{\theta}_n.$$
we can then apply Lemma \ref{lem:conv2} in order to deduce that

\[
\mathbb{E}[V^i_t - V^i_s | \mathcal{F}^i_t] = 1_{s < t} \mathbb{E} \left[ \int_s^t \left( \int_s^{t \wedge r} \gamma(\theta, \hat{C}^n_{\theta}(-i)) \, d\tilde{t}^i_\theta \right) (r, \hat{C}^n_r(-i)) e^{-\int_s^r \gamma(\theta, \hat{C}^n_{\theta}(-i)) \, d\tilde{t}^i_\theta} \, d\tilde{t}^i_r | \mathcal{F}^i_s \right].
\]

(60)

By making repeated use of the chain rule and the integration by parts formula for Lebesgue–Stieltjes integrals, we get the string of equalities

\[
1 - e^{-\int_s^t \gamma_r \, dr} = - \int_s^t e^{-\int_s^{r} \gamma(\theta, \hat{C}^n_{\theta}(-i)) \, d\tilde{t}^i_\theta} \left( \int_s^{r} \gamma(\theta, \hat{C}^n_{\theta}(-i)) \, d\tilde{t}^i_\theta \right)
\]

\[
= e^{-\int_s^t \gamma(\theta, \hat{C}^n_{\theta}(-i)) \, d\tilde{t}^i_\theta} \left( \int_s^{t} \gamma(\theta, \hat{C}^n_{\theta}(-i)) \, d\tilde{t}^i_\theta \right)
\]

\[
= \int_s^t \left( \int_s^{t} \gamma(\theta, \hat{C}^n_{\theta}(-i)) \, d\tilde{t}^i_\theta \right) e^{-\int_s^r \gamma(\theta, \hat{C}^n_{\theta}(-i)) \, d\tilde{t}^i_\theta} \gamma(r, \hat{C}^n_r(-i)) \, d\tilde{t}^i_r
\]

and so (60) simplifies to

\[
\mathbb{E}[V^i_t - V^i_s | \mathcal{F}^i_s] = 1_{s < t} \left( 1 - \mathbb{E} \left[ e^{-\int_s^t \gamma(r, \hat{C}^n_r(-i)) \, d\tilde{t}^i_r} | \mathcal{F}^i_s \right] \right).
\]

Combined with (59), we have therefore shown that

\[
\mathbb{E}[V^i_t - V^i_s | \mathcal{F}^i_s] = 1_{s \geq t^i} - V^i_s.
\]

Finally, since \( i \) was arbitrary, we can plug this back into (58) to see that

\[
\mathbb{E}[M^i_t | \mathcal{F}^i_s] = \mathbb{E}[M^i_s | \mathcal{F}^i_s] = M^i_s,
\]

using also that the sum \( M^i_t \) is \( \mathcal{F}^i_t \)-measurable. As \( s < t \) were arbitrary, this confirms that \((M^i_t)_{t \geq 0}\) is indeed a martingale in the filtration \((\mathcal{F}^i_t)_{t \geq 0}\). \( \square \)

5.2 Limiting behaviour as the population size tends to infinity

To give a proof of Theorem \ref{thm:conv2}, we first need an auxiliary result that rules out the eventuality of two particles getting infected at the same time.

**Lemma 5.4 (Distinct infection times).** For any pair of indices \( i \neq j \), we have

\[
\mathbb{P}(\tau^i = \tau^j) = 0,
\]

**Proof.** Fix any pair of indices \( i \neq j \). By analogy with the construction of the artificial systems \( \hat{X}^n_{\theta}(-i) \) and \( \hat{X}^n_{\theta}(-j) \) in Section 3.2, we can consider the artificial system \( \hat{X}^n_{\theta}(-i,-j) \), where everything is as in the true particle system, except that neither \( i \) nor \( j \) can be become infected but instead are globally reflected. Let \( \hat{A}^n_{\theta}(-i,-j) \) and \( \hat{C}^n_{\theta}(-i,-j) \) denote the advancing front and the current
contagiousness in this system. Similarly, let $\hat{\ell}_{i,n,(-i,j)}$ and $\hat{\ell}_{j,n,(-i,j)}$ denote the local times of $X_{i,n,(-i,j)}$ and $X_{j,n,(-i,j)}$ along $A_{n,(-i,j)}$. Let also $G^{(i,j),n}$ be defined by analogy with (14). It suffices to show $\mathbb{P}(\tau^i = \tau^j < t) = \mathbb{E}[\mathbb{P}(\tau^i = \tau^j < t \mid G^{(i,j),n})]$ for all $t \geq 0$. Now, by arguments analogous to Proposition 2.4 and those leading to (11) in Theorem 2.3, using the construction of the particle system, we can deduce that $\mathbb{P}(\tau^i = \tau^j < t \mid G^{(i,j),n})$ is less than

$$
\mathbb{P}\left(\int_0^{\bar{T} \wedge t} \gamma(r, C^{(i,j)}_r) \, d\hat{\ell}_{i,n,(-i,j)} = \chi^i \mid G^{(i,j),n}\right), \tag{61}
$$

where

$$\bar{T} := \min\left\{ t \geq 0 : \int_0^t \gamma(r, \hat{C}_{i,n,(-i,j)}) \, d\hat{\ell}_{i,n,(-i,j)} \geq \chi^j \right\},$$

for two independent exponential random variables $\chi^i$ and $\chi^j$ that are also independent of $G^{(i,j),n}$. Moreover, we can ensure $\chi^i$ is independent of $(X_{j,0}^n, B^j)$ and likewise for $j$ and $i$ interchanged. We can now start by observing that $\hat{C}_{r,n,(-i,j)}$, $\hat{\ell}_{r,n,(-i,j)}$, and $\hat{\ell}_{r,n,(-i,j)}$ are adapted to $G^{(i,j),n}$. Hence the integral on the left-hand side in (61) is fixed given $G^{(i,j),n} \vee \sigma(\chi^j, B^j, X_{j,0}^n : r \leq t)$ while $\chi^i$ is an exponential random variable independently of this, and so the conditional probability (61) must be zero. This completes the proof.

Using this lemma, the proof of Theorem 2.5 can now be deduced from the martingale property that we established in Proposition 5.3

**Proof of Theorem 2.5.** By Proposition 5.3, the difference $I^n_t - V^n_t$ is a martingale in the filtration $(\mathcal{F}^n_t)_{t \geq 0}$ from (57). Since the local times are continuous and of finite variation, we have that the quadratic variation is

$$[I^n - V^n]_t = \sum_{0 \leq s \leq t} (\Delta I^n_s)^2 = \frac{1}{n^2} \sum_{i,j=1}^n \mathbb{1}_{[0,t]}(\tau^i) \mathbb{1}_{[0,t]}(\tau^j).$$

Applying the Burkholder–Davis–Gundy inequality, it therefore follows that, for every $t \geq 0$,

$$\mathbb{E}\left[\sup_{s \leq t} |I^n_s - V^n_s|^2\right] \leq C \frac{n}{n^2} \sum_{i,j=1}^n \mathbb{P}(\tau^i = \tau^j \leq t), \quad \text{for all } n \geq 1,$$

for a fixed constant $C > 0$. In view of Lemma 5.4, the off-diagonal terms in the sum are all zero, so we get

$$\mathbb{E}\left[\sup_{s \leq t} |I^n_s - V^n_s|^2\right] \leq C \frac{n}{n}, \quad \text{for all } n \geq 1.$$

In particular, Markov’s inequality gives that there is also convergence to zero uniformly on compacts in probability. This completes the proof.

**A Appendix**

For the notation used in this appendix we refer to Section 3 and the objects defined therein. The main objective of this appendix is proving Proposition 3.2.
Proof of Proposition 3.2. Recall the definitions of $X_t^{i,n,k}$ for $k = 1, \ldots, n+1$ in Section 3.1. According to Definition 3.1, we rewrite their dynamics in integral form, so that for $t \geq 0$ we have

$$X_t^{i,n} = \left(X_0^{i,n} + \int_0^t b(s, X_s^{i,n}) \, ds + \int_0^t \sigma(s, X_s^{i,n}) \, dB_s^{i,n}(1) \right) 1_{t \in [0,\varsigma(1))} + \sum_{k=2}^{n} \left( \int_0^{t-\varsigma(k-1)} b(k, X_s^{i,n}) \, ds + \int_0^{t-\varsigma(k-1)} \sigma(k, X_s^{i,n}) \, dB_s^{i,n}(k) \right) 1_{t \in [\varsigma(k-1),\varsigma(k))} + \left( X_0^{i,n,n+1} + \int_0^{t-\varsigma(n)} b(n+1, X_s^{i,n,n+1}) \, ds + \int_0^{t-\varsigma(n)} \sigma(n+1, X_s^{i,n,n+1}) \, dB_s^{i,n}(n+1) \right) 1_{t \geq \varsigma(n)} + \int_0^t \ell_t^{A^{i,n}(n+1)}(X_t^{i,n,1}) 1_{t \in [0,\varsigma(1))} + \sum_{k=2}^{n} \int_0^{t-\varsigma(k-1)} \ell_t^{A^{i,n}(k)}(X_t^{i,n,k}) 1_{t \in [\varsigma(k-1),\varsigma(k))} + \int_0^t \ell_t^{A^{i,n,n+1}}(X_t^{i,n,n+1}) 1_{t \geq \varsigma(n)}. \quad (62)$$

From the definitions of $X_0^{i,n,k}, \varsigma^{(k)}$ and $\varsigma^{(k)}$ (recalling that $\varsigma^{(0)} = 0$), we see that

$$X_0^{i,n,k} = X_0^{i,n,k-1} = X_0^{i,n} + \int_0^{\varsigma^{(k-1)}} dX_s^{i,n} = X_0^{i,n} + \sum_{m=1}^{k-1} \int_0^{\varsigma^{(m)}} b^{(m)}(s - \varsigma^{(m-1)}, X_s^{i,n,m}) \, ds
$$

$$+ \int_0^{\varsigma^{(m)}} \sigma^{(m)}(s - \varsigma^{(m-1)}, X_s^{i,n,m}) \, dB_s^{i,n,m}(s - \varsigma^{(m-1)}) + \ell_t^{A^{i,n,m}(m)}(X_t^{i,n,m}), \quad (63)$$

where we have applied the change of variables $s \mapsto s + \varsigma^{(m-1)}$ to get the last equality. By (24), we have that $b^{(m)}(s - \varsigma^{(m-1)}, Y_s) = b(s, Y_s)$ and $\sigma^{(m)}(s - \varsigma^{(m-1)}, Y_s) = \sigma(s, Y_s)$. Moreover, by (25) the stochastic increments $B_{s_{j+1}} - B_{s_j}$ are equal to $B_{s_{j+1}}^{i,n} - B_{s_j}^{i,n}$ for any discretization $\{s_j\}$ of the time-interval $(\varsigma^{(m-1)}, \varsigma^{(m)})$. Finally, noting that $X_s^{i,n} 1_{s \in [\varsigma^{(m-1)}, \varsigma^{(m)})} = X_{s - \varsigma^{(m-1)}}^{i,n}$ by (23), we can simplify (63) as

$$X_0^{i,n,k} = X_0^{i,n} + \int_0^{\varsigma^{(k-1)}} b(s, X_s^{i,n}) \, ds + \int_0^{\varsigma^{(k-1)}} \sigma(s, X_s^{i,n}) \, dB_s^{i,n} + \sum_{m=1}^{k-1} \ell_t^{A^{i,n,m}(m)}(X_t^{i,n,m}). \quad (64)$$

By similar computations, and using (64) into (62), we finally have:

$$X_t^{i,n} = X_0^{i,n} + \int_0^t b(s, X_s^{i,n}) \, ds + \int_0^t \sigma(s, X_s^{i,n}) \, dB_s^{i,n} + \ell_t^{A^{i,n}(1)}(X_t^{i,n,1}) 1_{t \in [0,\varsigma(1))}$$

$$+ \sum_{k=2}^{n} \left( \int_0^{t-\varsigma(k-1)} \ell_t^{A^{i,n}(k)}(X_t^{i,n,k}) \right) 1_{t \in [\varsigma(k-1),\varsigma(k))}$$

$$+ \left( \sum_{m=1}^{n} \ell_t^{A^{i,n,m}}(X_t^{i,n,m}) + \ell_t^{A^{i,n,n+1}}(X_t^{i,n,n+1}) \right) 1_{t \geq \varsigma(n)}.$$
We now move on to showing that the evolution equation of the process $A_{t}^{i,n}$ constructed according to (33) satisfies the required equation in (35). Recalling the notation used in Section 3.1, we have that $j^{(k)}$ denotes the index of the particle infected at the $k^{th}$ step of the construction. Since by the end of our construction all $n$ particles have been infected, we can assume without loss of generality that $i = j^{(m)}$ for some $m \in \{1, \ldots, n\}$. We compute $A_{t-\zeta^{(k-1)}}^{i,n(k)}$ for $t \in [\zeta^{(k-1)}, \zeta^{(k)}]$ distinguishing three cases: $k = 1, \ldots, m$, $k = m + 1$ and $k = m + 2, \ldots, n + 1$.

For $k = 1, \ldots, m$ one can easily check by induction that

$$A_{t-\zeta^{(k-1)}}^{i,n(k)} = a_{0} + \frac{\alpha}{n} \sum_{j=1}^{k-1} \int_{\zeta^{(j)}}^{t} g(t-s) \, ds, \quad \text{for} \ t \in [\zeta^{(k-1)}, \zeta^{(k)}],$$

using the change of variables $s \mapsto s + \zeta^{(k-1)}$ and $s \mapsto s + \zeta^{(k-1)}$ when appropriate.

For the case $k = m + 1$, we have that

$$A_{t-\zeta^{(m)}}^{i,n(m+1)} = A_{t-\zeta^{(m)}}^{i,n(m)} + \frac{\alpha}{n} \sum_{j=1}^{m-1} \int_{\zeta^{(j)}}^{t+\zeta^{(j)}-\zeta^{(m)}} g(t-s) \, ds$$

$$= a_{0} + \frac{\alpha}{n} \sum_{j=1}^{m-1} \left( \int_{\zeta^{(j)}}^{\zeta^{(m)}} g(s) \, ds + \int_{\zeta^{(j)}}^{t+\zeta^{(j)}-\zeta^{(m)}} g(t) \, ds \right)$$

$$= a_{0} + \frac{\alpha}{n} \sum_{j=1}^{m-1} \int_{\zeta^{(j)}}^{t} g(t-s) \, ds, \quad \text{for} \ t \in [\zeta^{(m)}, \zeta^{(m+1)}],$$

where we have applied the change of variables $s \mapsto s + \zeta^{(m)}$ to get the last equality. Finally, for $k = m + 2, \ldots, n$ again by induction we can show that

$$A_{t-\zeta^{(k-1)}}^{i,n(k)} = a_{0} + \frac{\alpha}{n} \sum_{j=1}^{m-1} \int_{\zeta^{(j)}}^{t} g(t-s) \, ds + \frac{\alpha}{n} \sum_{j=m+1}^{k-1} \int_{\zeta^{(j)}}^{t} g(t-s) \, ds, \quad \text{for} \ t \in [\zeta^{(k-1)}, \zeta^{(k)}],$$

and similarly $A_{t-\zeta^{(m)}}^{i,n(n+1)} = a_{0} + \frac{\alpha}{n} \sum_{j \neq m}^{n} \int_{\zeta^{(j)}}^{t} g(t-s) \, ds$ for $t \geq \zeta^{(n)}$.

Recalling (31) for the definition of the random times $\xi_{k}^{i}$, we have that $\xi_{k}^{i} = \zeta^{(k)}$ for $k = 1, \ldots, m-1$ and $\xi_{k}^{i} = \zeta^{(k+1)}$ for $k = m, \ldots, n-1$. Then for all $k = 1, \ldots, n$ we can express $A_{t-\zeta^{(k-1)}}^{i,n(k)}$ for $t \in [\zeta^{(k-1)}, \zeta^{(k)}]$ in terms of $\{\xi_{k}^{i}\}_{k}$ as

$$A_{t-\zeta^{(k-1)}}^{i,n(k)} = a_{0} + \frac{1}{n} \sum_{j=1}^{n} \int_{\xi_{j}^{i}}^{t} g(t-s) \mathbb{1}_{s \geq \zeta^{(j)}} \, ds = a_{0} + \frac{1}{n} \sum_{j=1}^{n-1} \int_{\xi_{j}^{i}}^{t} g(t-s) \mathbb{1}_{s \geq \xi_{j}^{i}} \, ds,$$

and similarly for $A_{t-\zeta^{(m)}}^{i,n(n+1)}$. Then by (33) we have that, for $t \geq 0$,

$$A_{t}^{i,n} = a_{0} + \frac{1}{n} \sum_{j=1}^{n-1} \int_{0}^{t} g(t-s) \mathbb{1}_{s \geq \xi_{j}^{i}} \, ds = a_{0} + \int_{0}^{t} g(t-s) \mathcal{I}_{s}^{i,n} \, ds,$$

where we let the process $\mathcal{I}_{t}^{i,n}$ be defined as $\mathcal{I}_{t}^{i,n} = \frac{1}{n} \sum_{j=1}^{n-1} \mathbb{1}_{[0,t]}(\xi_{j}^{i})$ for $t \geq 0$. This concludes the proof of Proposition 3.2.

For the above proof, we relied on the following observations, which we prove next.
Lemma A.1. For $k = 2, \ldots, n + 1$, and $t \in \lbrack \varsigma^{(k-1)}, \varsigma^{(k)} \rbrack$, 
\[
\sum_{m=1}^{k-1} \ell_{\varsigma^{(m)}}^{A_i^{n,(m)}}(\hat{X}^{i,n,(m)}) + \ell_{t-\varsigma^{(k-1)}}^{A_i^{n,(k)}}(\hat{X}^{i,n,(k)}) = \ell_t^{A_i^{n,n}}(\hat{X}^{i,n}), \quad \forall i = 1, \ldots, n.
\]

Corollary A.1.1. For $t < \varsigma^{(1)}$,
\[
\ell_t^{A_i^{n,1}}(\hat{X}^{i,n,(1)}) = \ell_t^{A_i^{n,n}}(\hat{X}^{i,n}), \quad \forall i = 1, \ldots, n.
\]
For $t \geq \varsigma^{(n)}$,
\[
\sum_{m=1}^{n} \ell_{\varsigma^{(m)}}^{A_i^{n,(m)}}(\hat{X}^{i,n,(m)}) + \ell_{t-\varsigma^{(n)}}^{A_i^{n,(n+1)}}(\hat{X}^{i,n,(n+1)}) = \ell_t^{A_i^{n,n}}(\hat{X}^{i,n}), \quad \forall i = 1, \ldots, n.
\]

Proof of Lemma A.1. Consider $\ell_{t-\varsigma^{(k-1)}}^{A_i^{n,(k)}}(\hat{X}^{i,n,(k)})$ for $t \in \lbrack \varsigma^{(k-1)}, \varsigma^{(k)} \rbrack$. According to definition (9),
\[
\ell_{t-\varsigma^{(k-1)}}^{A_i^{n,(k)}}(\hat{X}^{i,n,(k)}) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{0}^{t-\varsigma^{(k-1)}} \mathbbm{1}_{[A_s^{i,n,(k)},A_s^{i,n,(k)}+\varepsilon]}(\hat{X}^{i,n,(k)}) \left( \frac{1}{2} (\sigma^{(k)}(s,\hat{X}^{i,n,(k)}) - \hat{X}^{i,n,(k)}) \right) ds
\]
where we have applied the usual change of variables $s \mapsto s + \varsigma^{(k-1)}$ and used that $\hat{X}^{i,n,(k)} = \hat{X}^{i,n,1} \mathbbm{1}_{s \in \lbrack \varsigma^{(k-1)}, \varsigma^{(k)} \rbrack}$ and $A_s^{i,n,(k)} = A_s^{i,n,1} \mathbbm{1}_{s \in \lbrack \varsigma^{(k-1)}, \varsigma^{(k)} \rbrack}$ for $s \in \lbrack \varsigma^{(k-1)}, \varsigma^{(k)} \rbrack$ by Definition 3.1. Similarly, for $m = 1, \ldots, k-1$, we have that
\[
\ell_{\varsigma^{(m)}}^{A_i^{n,(m)}}(\hat{X}^{i,n,(m)}) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\varsigma^{(m-1)}}^{\varsigma^{(m)}} \mathbbm{1}_{[A_s^{i,n,(m)},A_s^{i,n,(m)}+\varepsilon]}(\hat{X}^{i,n,(m)}) d\langle \hat{X}^{i,n} \rangle_s.
\]
Summing these expressions together, we conclude that
\[
\sum_{m=1}^{k-1} \ell_{\varsigma^{(m)}}^{A_i^{n,(m)}}(\hat{X}^{i,n,(m)}) + \ell_{t-\varsigma^{(k-1)}}^{A_i^{n,(k)}}(\hat{X}^{i,n,(k)}) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{0}^{t} \mathbbm{1}_{[A_s^{i,n,(k)},A_s^{i,n,(k)}+\varepsilon]}(\hat{X}^{i,n,(k)}) d\langle \hat{X}^{i,n} \rangle_s = \ell_t^{A_i^{n,n}}(\hat{X}^{i,n}),
\]
for all $t \in \lbrack \varsigma^{(k-1)}, \varsigma^{(k)} \rbrack$.

\[\square\]

3.3 Finally, we used the following lemma in the proof of Proposition 3.5 in Section 3.

Lemma A.2. For all $i = 1, \ldots, n$ and $k = 1, \ldots, n$,
\[
\int_{0}^{\varsigma^{(k)}} \gamma^{i,(k)}(s) d\ell_{\varsigma^{(k)}}^{A_i^{n,(k)}}(\hat{X}^{i,n,(k)}) = \int_{0}^{\varsigma^{(k)}} \gamma^{i,(k)}(s, C_{s}^{n,(k)}) d\ell_{s}^{A_i^{n,(k)}}(\hat{X}^{i,n}).
\]

Proof. Define a partition $P_n$ of $\lbrack \varsigma^{(k-1)}, \varsigma^{(k)} \rbrack$; applying the change of variables $s \mapsto s + \varsigma^{(k-1)}$ and writing down the random Stieltjes integral as an infinite sum, we have:
\[
\int_{0}^{\varsigma^{(k)}} \gamma^{i,(k)}(s) d\ell_{s}^{A_i^{n,(k)}}(\hat{X}^{i,n,(k)}) = \int_{0}^{\varsigma^{(k)}} \gamma^{i,(k)}(s) d\ell_{s-\varsigma^{(k-1)}}^{A_i^{n,(k)}}(\hat{X}^{i,n,(k)})
\]
\[
= \lim_{n \rightarrow \infty} \sum_{j \in P_n} \gamma^{i,(k)}(s_j - \varsigma^{(k-1)}) \left( \ell_{s_j-\varsigma^{(k-1)}}^{A_i^{n,(k)}}(X^{i,n,(k)}) - \ell_{s_j-\varsigma^{(k-1)}}^{A_i^{n,(k)}}(X^{i,n,(k)}) \right).
\]
By similar computations to those in the proof of Lemma A.1 we have that
\begin{align*}
\ell^{A_{i,n},(k)}_{s_{j+1}-\varsigma(k-1)}(X^{i,n,(k)}) - \ell^{A_{i,n},(k)}_{s_{j}-\varsigma(k-1)}(X^{i,n,(k)}) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{s_{j}}^{s_{j+1}} 1_{[A_{i,n},A_{i,n}+\varepsilon]}(\hat{X}^{i,n}) \, d\langle \hat{X}^{i,n} \rangle_s \\
&= \ell^{A_{i,n}}_{s_{j+1}}(\hat{X}^{i,n}) - \ell^{A_{i,n}}_{s_{j}}(\hat{X}^{i,n}),
\end{align*}
so that
\begin{align*}
\int_{0}^{\varsigma^{(k)}} \gamma^{i,(k)}(s) \, d\ell^{A_{i,n},(k)}_{s}(\hat{X}^{i,n,(k)}) &= \int_{\varsigma^{(k-1)}}^{\varsigma^{(k)}} \gamma^{i,(k)}(s - \varsigma(k-1)) \, d\ell^{A_{i,n}}_{s}(\hat{X}^{i,n}),
\end{align*}
and by definition (27) of $\gamma^{i,(k)}(t)$ and $C_{i}^{<(k)}$ when $t \in [\varsigma(k-1), \varsigma^{(k)})$, the claim follows. \qed
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