All Entanglements in a Multipartite System

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Abstract

For a multipartite system, we sort out all possible entanglements, each of which is among a set of subsystems. Each entanglement can be measured by a generalized relative entropy of entanglement, which is conserved on average under reversible local operations and classical communication (LOCC) defined for all the parties. Then we derive a series of inequalities of different entanglements that have to be satisfied by any pure state which can be generated by reversible LOCC from the set of all GHZ-like states.

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1. Introduction

Entanglement is an essential quantum feature which lies at the heart of quantum mechanics [1] and is also a crucial resource of quantum information processing [2]. Many recent results in characterizing entanglement follow the idea that entanglement is not changed by any local unitary transformation, and on average cannot be increased by local operations and classical communication (LOCC) [3, 4, 5], hereby called LOCC non-increasing principle. In the bi-partite case, the entanglement is just between the two parties, though there may be different measures for a mixed state. It is known that $p$ copies of a bi-partite pure state $\psi$, whose von Neumann entropy of either party is $E(\psi)$, can be transformed by reversible LOCC into $pE(\psi)$ copies of EPR singlet states asymptotically, i.e. in the limit of large $p$ [3]. Thus $E(\psi)$ is the measure of the bi-partite entanglement in the pure state $\psi$. One can use the term LOCC equivalence to refer to the transformation under reversible LOCC, with certain number of copies for each state involved in the transformation. Thus for a bi-partite system, any entangled pure state is LOCC asymptotically equivalent to EPR state.

For a multipartite system, how to characterize the nature of entanglement has remained quite open, with many interesting questions, for example, how many different kinds of entanglement there exist and how different entangled pure states are transformed into each other by reversible LOCC. First of all, what does a $N$-partite entangled state precisely mean?

Let us refer to the state $g(1 \cdots n) \equiv \frac{1}{\sqrt{2}}(|0_1\rangle|0_2\rangle \cdots |0_n\rangle + |1_1\rangle|1_2\rangle \cdots |1_n\rangle)$ as $n$-GHZ state or a GHZ-like state, where the subscripts represent the different parties. Hence 2-GHZ state is just the EPR state. Recently it was shown that 4-GHZ state is not LOCC asymptotically equivalent to any combination of the six different EPR pairs and that two different 3-GHZ states is not exactly LOCC equivalent to three EPR pairs [6]. It was also shown that $N$-GHZ is not LOCC asymptotically equivalent to any combination of $k$-GHZ states, for all $k < N$ [7].

An interesting concept is the reversible LOCC entanglement generating set (RLEGS) [6]. For a given multipartite system, a set of different states is a RLEGS if a combination of these states, each entering with certain copies, can be transformed, under reversible LOCC, into certain copies of an arbitrary entangled pure state. The numbers of copies of the states concerned could be arbitrarily large. For example, EPR state comprises a RLEGS for a bi-partite system. An interesting question is that for a system of $N > 2$ parties, whether all kinds of GHZ-like states, each shared by a subset of the parties, is a RLEGS. A negative answer is known for $N = 4$ [8].

For a given $N$-partite system, let us define a proper subset of parties as a generalized parties (GP). A “proper subset” means that the number of elements in the subset is less than $N$. The Hilbert space of each GP is the tensor product of those of the parties belonging to it. The notion of LOCC can be generalized to the generalized LOCC (GLOCC), which means operations that are local with respect to the GPs and communication among the GPs. We shall sort out all different entanglements in a $N$-partite system, each corresponding to a different set of at least two GPs. Thus a $N$-partite entangled state is a $N$-partite state in which at least one of the entanglements is nonzero. The LOCC-nonincreasing principle appropriately formulated for a $N$-partite system leads to the conservation of any entanglement involving all the $N$ parties (see below for precise meaning), under reversible LOCC.
This approach reproduces the conclusion about the existence of the so-called true \( N \)-partite entanglement, with which a state cannot be transformed, by using reversible LOCC, into states in which only \( k < N \) parties are entangled \([7]\). Moreover, it is identified as the entanglement among all the \( N \) parties (see below for precise meaning). For each entanglement, we introduce a generalized relative entropy (GRE) as a measure. GRE of each entanglement is on average conserved under reversible LOCC. Based on these conservation laws, we consider an arbitrary \( N \)-partite pure state, with certain copies, which can be generated, under reversible LOCC, from the set of all GHZ-like states. We obtain a series of inequalities of different entanglements, which must be satisfied by any such state.

2. Entanglement among a set of generalized parties

Entanglement is a kind of correlation, among at least two objects. Thus we can consider all possible entanglements in a \( N \)-partite system, by associating an entanglement with each set of at least two GPs, called the entanglement among, or of, a set of GPs. When we say “a set of GPs”, we always mean a set of nonoverlapping GPs, i.e. there is no party shared by different GPs. We use a number, sometimes put in a bracket, to represent each party, a sequence of numbers in a bracket to represent the GP consisting of these parties, and a succession of brackets as a set of GPs. For example, for a system of three parties (1), (2) and (3), there are entanglements of (1)(2), (1)(3), (2)(3), (1)(23), (2)(13), (3)(12) and (1)(2)(3), respectively. For a \( N \)-partite system, the total number of different sets of at least two GPs is

\[
\sum_{n=2}^{N} \frac{N!}{n!(N-n)!} \sum_{m=2}^{n} P(n,m),
\]

where \( P(n,m) = \sum_{n_1,\ldots,n_m} \frac{n!}{n_1!\cdots n_m!} \), with \( n_1 + \cdots + n_m = n \), is the number of different partitions of \( n \) parties into \( m \geq 2 \) GPs, denoted as \((1^{n_1})(2^{n_2})\cdots(m^{n_m})\), where \( q^i \) denotes the \( i \)-th party in the \( q \)-th GP. Note that in this terminology, the amount of an entanglement could be zero.

Now we introduce generalized local operation and generalized LOCC (GLOCC). A generalized local operation is an operation on a GP. It is local with respect to the GP, but may be nonlocal with respect to the parties. One knows that a process of LOCC transforms the state as \( \rho \rightarrow \sum_k L_k \rho L_k^\dagger \), where \( L_k = \otimes_{i=1}^n l_k^i \), \( l_k^i \) is a local operation on party \( i \), with \( \sum_k l_k^i l_k^i \leq 1 \). Similarly, a process of GLOCC, on \( m \) GPs corresponding to a partition of \( N \) parties, transform the state as \( \rho \rightarrow \sum_j G_j \rho G_j^\dagger \), where \( G_j = \otimes_{q=1}^m g_j^q \), \( g_j^q \) is a generalized local operation on GP \( q \), with \( \sum_j g_j^q g_j^q \leq 1 \).

A local operation on a party is also local with respect to any GP to which the party belongs, while communication among all the parties is also communication among all the GPs corresponding any partition of the parties. Therefore, a process of LOCC is also a process of GLOCC corresponding to any partition of the parties to GPs. \( L_k = \otimes_{i=1}^N l_k^i \) can also be written as a product of generalized local operations on different GPs under any partition, i.e. \( L_k = \otimes_{q=1}^m g_k^q \), where \( g_k^q = \otimes_{i \in l_k^i} l_k^i \). However, a generalized local operation on a GP may not be a product of local operations on parties belonging to this GP.

A LOCC process for \( N \) parties may not be a LOCC process on a proper subset of the parties, i.e. \( n \) of the \( N \) parties, \( n < N \), while a LOCC process for a proper subset of \( N \) parties is still a LOCC process for the \( N \) parties. If one treat the system as two parts \( A \) and \( B \), then an operation \( L_k \) can be written as \( L_k^A \otimes L_k^B \), where \( L_k^A = \otimes_{i \in A} l_k^i \), \( L_k^B = \otimes_{n \in B} l_k^i \).
Under a process of LOCC on the total system, the reduced density matrix of $A$ evolves as $\rho_A \to \text{tr}_B (\sum L_k \rho L_k^\dagger) = \sum L_k^A \text{tr}_B (L_k^B \rho L_k^B) L_k^A \leq \sum L_k^A \rho_A L_k^A$. Similar is the situation of GLOCC.

It is straightforward to obtain the following. (i) A generalized local operation on a GP does not change the reduced density matrix of any other nonoverlapping GP. (ii) GLOCC among a set of GPs do not change the reduced density matrix of any other GP nonoverlapping with them. (iii) The von Neumann entropy of a GP is non-increasing under GLOCC for any set of GPs including the concerned one.

There may be various relations among different entanglements. After all, each entanglement is a function of the same state. Consider one copy of a $N$-qubit pure state, which can be specified by $2^{N+1} - 2$ real parameters, up to a global phase. For sufficiently large $N$, the number of entanglements is larger than the number of parameters, thus there must be relations among different entanglements. The relations depend on the state. However, there are some qualitative relations independent of the state, as reflected in the features of the set of the separable states in defining the GRE. For example, if a subsystem of a GP is entangled with a subsystem of another GP, these two GPs must be entangled. One can observe that it is impossible to have a state for which only one of the entanglements is nonvanishing, otherwise it would be straightforward to obtain a RLEG.

3. Entanglements involving all $n$ parties and entanglement among all $n$ parties

For $n$ parties $(1), (2), \cdots, (n)$, we refer to the entanglement of $(1)(2)\cdots(n)$ as the entanglement among all the $n$ parties. In other words, for this entanglement, each party is a GP in the definition of the entanglement.

If the set of GPs, in the definition of an entanglement, ultimately consists of all $n$ parties, we refer to the entanglement as “an entanglement involving all the $n$ parties”. Obviously, the first example is just the entanglement among all the $n$ parties. Other examples include $(12)(3\cdots n), (12)(345)(6\cdots n)$, etc.

In a $n$-partite Schmidt decomposable pure state $\sum a_k |\eta_k(1)) \cdots |\eta_k(n))$, where $|\eta_k(i))$ represents the $k$-th basis state of party $i$, the only nonzero entanglements are those involving all the $n$ parties. Tracing out any number of parties leads to a completely separable state. A $n$-GHZ state is an example of $n$-partite Schmidt decomposable state.

4. LOCC non-increasing principle and its consequences

The LOCC non-increasing principle for a $N$-partite system, as the straightforward generalization of $N = 2$ case, can be formulated as the following.

**LOCC non-increasing principle**—For a $N$-partite system, on average the entanglement among all the $N$ parties is invariant under local unitary transformations and cannot be increased by LOCC for all these $N$ parties.

By “coarse graining”, i.e. replacing the parties as GPs while LOCC as GLOCC, the above principle also implies the following. (i) A generalized local unitary transformation on a GP does not change any entanglement among a set of GPs one of which is this GP, and does not change any entanglement among any other GP nonoverlapping with this GP. (ii) GLOCC among a set of GP does not increase the entanglement among all these GPs.
These results, together with the fact that a process of LOCC is also a process of GLOCC with respect to any partition, lead to the conclusion that LOCC on a multipartite system does not increase any entanglement involving all the parties.

This deduction is like that of renormalization group: making coarse graining and then finding the fixed points. Here coarse graining is regarding a set of parties as a basic unit, i.e., partitioning a set of parties into a GP. In doing so, a process of GLOCC, previously corresponding to a certain partition, may not be a GLOCC process corresponding to the coarse graining made. However, any process of LOCC is still a valid process of GLOCC corresponding to the coarse graining made. Hence a process of LOCC is a fixed point under any coarse graining.

Therefore we have obtained the following conclusion.

**Theorem 1:** Each entanglement involving all the parties is on average conserved in a reversible process of LOCC for all these parties.

Consider two pure states of \( N \) parties, which are partitioned into \( m \) GPs. By generalizing a theorem in [6], one knows that if for the two pure states, each GP has a same von Neumann entropy, then these two states are either GLOCC incomparable or equivalent under the generalized local unitary transformations, with respect to the given partition. From this, we obtain the following.

**Theorem 2:** If two \( N \)-partite pure states are LOCC equivalent, then they are equivalent under the generalized local unitary transformations with respect to any partition.

**Proof:** As a special case of Theorem 1, LOCC equivalence implies that for these two states, any GP has a same entropy, which measures the bi-partite entanglement between this GP and the rest parties. Thus these two states are either equivalent under the generalized local unitary transformations, or GLOCC incomparable, with respect to any partition. The latter possibility can be excluded by the fact that a LOCC process is a also process of GLOCC corresponding to any partition of parties into GPs.

The so-called *true* \( n \)-partite entanglement can be identified as nothing but the entanglement among all the \( n \) parties, in the following way. First the true \( n \)-partite entanglement must involve all the \( n \) parties. Then consider an entanglement involving all the \( n \) parties but is not the one among all the \( n \) parties. Such an entanglement can be contributed by a state shared by less than \( n \) parties, if the state is shared among all the GPs. For example, the entanglement between GPs (12) and (3 \( \cdots \) \( n \)) is contributed by a bi-partite entangled state shared between parties (2) and (3). Indeed, a process of LOCC for (2) and (3) is also a process of GLOCC for (12) and (3 \( \cdots \) \( n \)). On the other hand, when each of the \( n \) parties is a GP, GLOCC is just LOCC. Therefore among all the entanglements involving all the \( n \) parties, only the entanglement among all the \( n \) parties is the true \( n \)-partite entanglement, according to its definition. Theorem 1 guarantees the LOCC inequivalence of different entanglements involving all the parties (each of them is conserved respectively under reversible LOCC defined for these \( n \) parties, hence there is not a situation that one of them is transformed into another). A corollary is that \( N \)-GHZ state, with nonzero true \( N \)-partite entanglement, is not LOCC equivalent to any repertoire of \( k \)-GHZ states for all \( k < N \).

Note that for a \( N \)-partite system, there exists a true \( n \)-partite entanglement for any \( n \) parties, \( 2 \leq n \leq N \).
4. Generalized relative entropy of entanglement

For a $N$-partite system, consider $n \leq N$ parties, with the (reduced) density matrix $\rho$. The generalized relative entropy of entanglement (GRE) defined for the entanglement among $m$ GPs $(1^1 \cdots 1^{n_1}), (2^1 \cdots 2^{n_2}), \ldots$, and $(m^1, \ldots, m^{n_m})$, with $n_1 + \cdots + n_m = n$, is

$$E'(\rho) = \min_{\sigma} Tr(\rho \ln \rho - \rho \ln\sigma),$$

where the separable density matrix $\sigma$ is of the form

$$\sigma = \sum_{s} \sum_{k_s} p_{k_s}^{s} \otimes \alpha_s^{(s)} \sigma_{k_s}^{(s)}$$  \hspace{1cm} (2)

where $s$ denotes each different partition of the $m$ GPs further into various sets of GPs numbered as $\alpha_s$ (each partition $s$ represents a way of separation), $k_s$ denotes the decomposition of density matrix under the partition $s$, $\sigma_{k_s}^{(s)}$ is in the Hilbert space of $\alpha_s$ and can be required to be pure. For a given $\sigma$, $\sigma$ is separable with respect to a certain partition of GPs into sets of GPs (of course, some separable states can be separated in more than one way), i.e. $p_{k_s}^{s} \neq 0$ only for $s = s_0$. Hence $\sum_{k_{s_0}} p_{k_{s_0}}^{s_0} = 1$ while $p_{k_s}^{s} = 0$ for $s \neq s_0$. In other words, the set of separable states is the union of all sets of the states separable with respect to different partitions of the $m$ GPs. The reason to consider the separability of all different ways is to exclude the entanglements among not all the $m$ GPs.

Take a density matrix of three parties 1, 2 and 3 as an illustration. For the GRE of $(1)(2)(3)$, $\sigma = \sum_{k_1} p_{k_1}^{(1)} \sigma_{k_1}^{(1)} \otimes \sigma_{k_1}^{(2)} \otimes \sigma_{k_1}^{(3)}$ or $\sigma = \sum_{k_2} p_{k_2}^{(12)} \sigma_{k_2}^{(12)} \otimes \sigma_{k_2}^{(3)}$ or $\sigma = \sum_{k_3} p_{k_3}^{(13)} \sigma_{k_3}^{(13)} \otimes \sigma_{k_3}^{(2)}$ or $\sigma = \sum_{k_4} p_{k_4}^{(23)} \sigma_{k_4}^{(23)} \otimes \sigma_{k_4}^{(1)}$. For the GRE of $(12)(3)$, $\sigma = \sum_{k} p_{k}^{(12)} \sigma_{k}^{(12)} \otimes \sigma_{k}^{(3)}$. For the GRE of $(1)(2)$, $\sigma = \sum_{k} p_{k}^{(1)} \sigma_{k}^{(1)} \otimes \sigma_{k}^{(2)}$.

For convenience, let us normalize the GRE of $n$-parties as

$$E(\rho) = E'(\rho)/E'_0,$$

where $E'_0$ is for $n$-GHZ state, such that $E = 1$ for $n$-GHZ state. Clearly the GRE satisfies the LOCC non-increasing principle.

The GRE of each entanglement in a $N$-partite pure state is on average conserved under reversible LOCC process. For an entanglement involving all the parties, this has been stated in Theorem 1. An entanglement not involving all the parties can be increased under LOCC for all the $N$ parties, but its GRE is still conserved on average under reversible LOCC, as a result of [7], seen as follows. Consider the GRE for an entanglement involving $n$ parties out of the $N$ parties, $2 \leq n < N$. Any one of the $n$ parties can act as the Bob in [7], while any one of the other $N-n$ parties can act as the Alice. Then a straightforward use of the proof in [7] shows that the average increase of the GRE of any entanglement involving the $n$ parties is not larger than the average decrease of the von Neumann entropy of the GP consisting of these $n$ parties.

Obviously, subadditivity and monotonicity [3] holds for GRE, since they do not depend on the details of the separable density matrices in the definition of the relative entropy of entanglement. Subadditivity means $E(\rho_a \otimes \rho_b) \leq E(\rho_a) + E(\rho_b)$ for any $\rho_a$ and $\rho_b$. Monotonicity means $E(\rho_a \otimes \rho_b) \geq E(\rho_a)$ if $E(\rho_b) = 0$. The combination of these two properties means $E(\rho_a \otimes \rho_b) = E(\rho_a) + E(\rho_b)$ if $E(\rho_b) = 0$ [3]. Besides, for an entanglement between a GP and its complementary GP, i.e. the rest of system, the GRE reduces to a von Neumann entropy and is thus additive.
5. Constraints on any state which can be reversibly generated by LOCC from the set of all GHZ-like states

A GHZ-like state is a sort of “canonical” entangled state, hence it is interesting to consider the set of all GHZ-like states for \( N \) parties, denoted as \( \mathcal{G}(N) \). Each of these GHZ-like states is shared by a different set of parties. For \( N \) parties, there are altogether \( M_{ghz} = 2^N - N - 1 \) GHZ-like states.

Suppose a \( N \)-partite state can be reversibly generated by LOCC from \( \mathcal{G}(N) \). This means that \( c_\psi \) copies of \(|\psi\rangle\) can be converted, by using reversible LOCC, into a set of \( c_z \) copies of state \(|z\rangle\), with each \(|z\rangle\) belonging to \( \mathcal{G}(N) \). \( c_\psi \neq 0 \), \( c_z \) could be arbitrary non-negative integers, and not all \( c_z \)-s are zero. In the following, we shall give some necessary conditions for \(|\psi\rangle\). If an arbitrary \( N \)-partite entangled pure state can be reversibly generated by LOCC from \( \mathcal{G}(N) \), then \( \mathcal{G}(N) \) is a RLEGS. Hence if any of the necessary conditions about \(|\psi\rangle\) can be violated by any \( N \)-partite entangled state, then \( \mathcal{G}(N) \) is not a RLEGS.

The conservation of GRE for each entanglement implies

\[
E_\alpha(\otimes_z |z\rangle^{\otimes c_z}) = E_\alpha(|\psi\rangle^{\otimes c_\psi}),
\]

where \( \alpha \) denotes the entanglement corresponding to each different set of GPs, \( E_\alpha(\phi) \) represents the corresponding GRE possessed by state \( \phi \). Subadditivity of GRE implies

\[
E_\alpha(\otimes_z |z\rangle^{\otimes c_z}) \leq \sum_z c_z E_\alpha(|z\rangle),
\]

and

\[
E_\alpha(|\psi\rangle^{\otimes c_\psi}) \leq c_\psi E_\alpha(|\psi\rangle),
\]

for any entanglement \( \alpha \).

It has recently been known that in general, relative entropy is not additive \[9\]. Thus (3) cannot be reduced a systems of equations of \( \{c_z\} \) and \( c_\psi \), which would be easy to handle. However, some of the entanglements are additive, leading to some simplification.

First we note that for the combination of GHZ-like states, \( \otimes_z |z\rangle^{\otimes c_z} \), an entanglement is only contributed by those GHZ-like states only involving a subset of each of the GPs in defining the entanglement.

Then we consider the entanglement between two GPs, referred to as a bi-GP entanglement, in \( \otimes_z |z\rangle^{\otimes c_z} \). If a bi-GP entanglement is contributed by a certain GHZ-like state, it must be from the entanglement between two complementary parts in this GHZ-like state, and thus must be a von Neumann entropy, which is additive. On the other hand, trivially if a GHZ state is only held by one of the two GPs, it does not contribute this bi-GP entanglement. For example, to calculate the amount of entanglement between party (1) and (2) in a GHZ-like state, one first traces out any party which is neither (1) nor (2), and then consider the reduced density matrix shared by party (1) and (2). Thus the entanglement between party (1) and (2) equals 1 in state \( g(12) \), but is zero in any other GHZ-like state, e.g. \( g(13) \), \( g(23) \), \( g(123) \), etc. As another example, the entanglement between GP (12) and (3) equals 1 in \( g(123) \), \( g(23) \) and \( g(13) \), while is zero in any other GHZ-like state, e.g. \( g(12) \), \( g(1234) \), \( g(234) \), etc.

Therefore for any GHZ-like state, any bi-GP entanglement either vanishes or is a von Neumann entropy of the reduced density matrix of either of the two GPs. Consequently any bi-GP entanglement is additive for \( \otimes_z |z\rangle^{\otimes c_z} \).
where the subscript $bG$ as a von Neumann entropy, is additive. There are $M$ involving all the parties involving all the parties. The subscript $x_N$ the parties. Therefore one obtains

$$E_{bG}(\otimes z|z)^{\otimes c_z} = \sum z c_z E_{bG}(|z\rangle),$$

where the subscript $bG$ represents bi-GP entanglements.

Combining Eq. (4) and Eq. (5), one obtains

$$\sum x_z E_{bG}(|z\rangle) \leq E_{bG}(|\psi\rangle),$$

where $x_z \equiv c_z/c_\psi$. The total number of different bi-GP entanglements is $M_{bG} = \sum_{n=1}^{N-1} \sum_{n_2=1}^{N-n_1} \frac{N!}{n_1!n_2!(N-n_1-n_2)!}$, which is larger than the total number of GHZ-like states $M_{ghz}$.

Moreover, for pure states, e.g. copies of $|\psi\rangle$ on RHS of Eq. (3), a bi-GP entanglement involving all the parties, i.e. the entanglement between a GP and its complementary GP, as a von Neumann entropy, is additive. There are $M_{bGa} = 2^{N-1} - 1$ bi-GP entanglements involving all the parties. The subscript $bGa$ represents bi-GP entanglements involving all the parties. Therefore one obtains $M_{bGa}$ equations,

$$\sum x_z E_{bGa}(|z\rangle) = E_{bGa}(|\psi\rangle),$$

in addition to $M_{bG} - M_{bGa}$ inequalities given by Eq. (7).

In terms of the GHZ-like states with nonzero amounts of the entanglement in consideration, Eq. (7) reduces to

$$\sum x_g(i_1 \cdots j_i i_{i+1} \cdots j_k) \leq E_{(i_1 \cdots i_r)(i_{r+1} \cdots i_n)}(|\psi\rangle)$$

where $2 \leq n \leq N, 1 \leq r \leq n - 1, (i_1 \cdots i_r, i_{r+1} \cdots i_n)$ represents any $n$ parties. The summation is over all such GPs $(j_1 \cdots j_i j_{i+1} \cdots j_k)$ in which $(j_1 \cdots j_i)$ is a subset of GP $(i_1 \cdots i_r)$ while $(j_{i+1} \cdots j_k)$ is a subset of GP $(i_{r+1} \cdots i_n)$. Eq. (8) reduces to

$$\sum x_g(i_1 \cdots j_i j_{i+1} \cdots j_k) = E_{(i_1 \cdots i_r)(i_{r+1} \cdots i_N)}(|\psi\rangle)$$

for any $1 \leq r \leq N - 1$.

Besides the bi-GP entanglements, which we have considered so far, it is also useful to note that in $\otimes z|z)^{\otimes c_z}$, the entanglement among all $n$ parties, $2 \leq n \leq N$, i.e. the entanglement of $(i_1 \cdots i_n)$, where each $i_j$ is a party, which we have called a true $n$-partite entanglement, is only contributed by $g(i_1 \cdots i_n)$, with the amount equal to 1. Therefore

$$E_{(i_1 \cdots i_n)(\otimes z|z)^{\otimes c_z})} = c_{g(i_1 \cdots i_n)},$$

which, combined with the GRE subadditivity (3), leads to

$$x_{g(i_1 \cdots i_n)} \leq E_{(i_1 \cdots i_n)}(|\psi\rangle),$$

which represents $N - 1$ inequalities, each of which is between the relative number of copies of a GHZ-like state and a corresponding entanglement in $|\psi\rangle$. The case of $n = 2$ is already contained in Eq. (7). So in addition to $M_{bGa}$ equations and $M_{bG} - M_{bGa}$ inequalities concerning the bi-GP entanglements, we obtain extra $N - 2$ inequalities concerning entanglement among more than two parties, with each party as a GP.
Combining Eq. (12) with Eq. (10), one thus obtains inequalities for different entanglements in $|\psi\rangle$, which can be generally written as

$$E_{(i_1 \cdots i_r)(i_{r+1} \cdots N)}(|\psi\rangle) \leq \sum E_{(j_1 \cdots j_l)(j_{l+1} \cdots j_k)}(|\psi\rangle),$$

for all $1 \leq r \leq N-1$, where the summation is over all such entanglements for which $(j_1 \cdots j_l)$ is a subset of GP $(i_1 \cdots i_r)$ while $(j_{l+1} \cdots j_k)$ is a subset of GP $(i_{r+1} \cdots N)$. For example, in the case of $N = 3$, Eq. (13) represents

$$E_{(12)(3)}(|\psi\rangle) \leq E_{(1)(3)}(|\psi\rangle) + E_{(2)(3)}(|\psi\rangle) + E_{(1)(2)(3)}(|\psi\rangle),$$

and the other two inequalities with the party labels cycled. Without the appearance of numbers of copies of different states, inequalities represented by Eq. (13) are entirely the properties of $|\psi\rangle$.

As the primary result of this section, Eq. (13) gives constraints for any state which can be generated from $\mathcal{G}(N)$ by reversible LOCC. Any violation of the constraints Eq. (13), or any inconsistency of any subset of the inequalities of $\{x_z\}$ given by Eq. (9), Eq. (10) and Eq. (12), would disprove that $\mathcal{G}(N)$ is a RLEGS. For this purpose, one could also reduce some inequalities concerning bi-GP entanglements not involving all the parties to equalities by setting these entanglements in $|\psi\rangle$ to zero, and then look for a counterexample for which the system of equations of $\{x_z\}$ has no solutions, which would signal the frustration in those constraints on the relation between different entanglements. However, we have not yet found a counterexample.

### 6. Summary

To summarize, we sort out all entanglements in a $N$-partite system, where $N \geq 2$. by associating an entanglement with each different set of at least two GPs. The notion of local operation is extended to generalized local operation on a GP. For each different set of GPs, there is a different kind of GLOCC. A process of LOCC is a special process of GLOCC for any partition of these parties into GPs. The LOCC non-increasing principle implies that each entanglement involving all the parties is invariant under local unitary transformations and is non-increasing under LOCC for all the parties. The so-called true $n$-partite entanglement is identified as the entanglement among all the $n$ parties, with each party as a GP. A GRE is introduced as a measure for each entanglement. A GRE is subadditive and monotonic. GRE of each entanglement is on average conserved in a reversible LOCC process. Then we consider a combination of GHZ-like states states, with each of which entering with a certain copies. For this combination, additivity holds for the GRE of each bi-GP entanglement and the GRE of true $n$-partite entanglements, $2 \leq n \leq N$. Consequently, if a number of copies of a pure state is generated from the set of all GHZ-like states, $\mathcal{G}(N)$, the number of copies of different states concerned have to satisfy a series of inequalities. Moreover, for the entanglement between a GP and its complementary GP, the inequalities reduce to equalities. Besides, a true $n$-partite entanglement is only contributed by the corresponding $n$-GHZ state. These lead to a series of inequalities between different entanglements of any state which can be generated from $\mathcal{G}(N)$ under reversible LOCC. Any violation of these relations would indicate that $\mathcal{G}(N)$ is not a RLEGS.

Finally we mention that one can measure each different entanglement by determining how many copies of one generalized GHZ-like state $|\psi\rangle$ shared by a set of GPs, can be dis-
tilled from how many copies of the concerned multipartite state, by using the corresponding GLOCC.

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Note added: It is claimed in [11] that for three parties, the set of GHZ and EPR states is not a RLEGs.

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