Ordering effect of Coulomb interaction in ballistic double-ring systems

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We study a model of two concentric onedimensional rings with incommensurate areas \(A_1\) and \(A_2\), in a constant magnetic field. The two rings are coupled by a nonhomogeneous inter-ring tunneling amplitude, which makes the one-particle spectrum chaotic. For noninteracting particles the energy of the many-body ground-state and the first excited state exhibit random fluctuations characterized by the Wigner-Dyson statistics. In contrast, we show that the electron-electron interaction orders the magnetic field dependence of these quantities, forcing them to become periodic functions, with period \(\propto 1/(A_1 + A_2)\). In such a strongly correlated system the only possible source of disorder comes from charge fluctuations, which can be controlled by a tunable inter-ring gate voltage.

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\[ H = -t \sum_{i,p} e^{i2\pi \varphi_p/N} c_{i,p}^\dagger c_{i+1,p} - \sum_i t_{12}(i) c_{i,1}^\dagger c_{i,2} + H.c. + \sum_{i,j,p,p'} V(|r_{i,p} - r_{j,p'}|) n_{i,p} n_{j,p'} \] (1)

The operators \(c_{i,p}^\dagger\) and \(c_{i,p}\) create and destroy a spinless electron at site \(i\) of ring \(p\). \((i,j = 1, \ldots, N; p = 1, 2)\) are site indexes and \(p, p' = 1, 2\) are ring indexes. The term proportional to \(t\) describes the intra-ring hopping, affected by the magnetic field. The site-dependent amplitude \(t_{12}(i)\) represents inter-ring tunneling. The last term represents the translationally invariant e-e interaction.

We first discuss the case of noninteracting particles. If the two rings are decoupled \((t_{12}(i) = 0)\) the single-particle spectrum is given by \(E_p(k_i) = -2t \cos[(k_i + 2\pi \varphi_p/N)]\). In \(N = 1\) and \(a_p = 1\) the flux enclosed by ring \(p\) is \(\phi_0\). Throughout this paper we will consider two incommensurate areas \(a_1 = (\sqrt{5}) - 1/2 \approx 0.618\) and \(a_2 = 1.5\). In that case the dependence of the single particle spectrum on \(H\) will be effectively random, with very complicated level crossings. Suppose now that the two rings are coupled by a tunneling amplitude \(t_{12}(i)\), small compared to the mean level spacing for each single ring, \(\Delta = 8t/N\). To first order the tunneling will couple states of different rings. If \(t_{12}(i)\) is also site independent, the spectrum will hardly be modified and, in particular, level crossings will not disappear since they always involve levels of different momenta which are not coupled by a homogeneous matrix element. However, if \(t_{12}(i)\) is not homogeneous, states of different momenta can be coupled, causing the lifting of the degeneracies. Such level repulsion is one of the essential characteristics of the Wigner-Dyson statistics. The single-particle energies can be made interact more strongly by increasing the magnitude of \(t_{12}(i)\). We expect that when the level splitting becomes of the order of \(\Delta\), a fully chaotic spectrum will develop.

An outstanding issue in the physics of mesoscopic systems is the interplay between disorder and chaos, and electron-electron (e-e) interaction. Experimentally this problem is relevant for the anomalously large persistent currents in disordered metallic rings \(1\), magnetoconductance fluctuations through ballistic quantum dots \(2\), Coulomb peak height fluctuations \(3\) and peak spacings fluctuations \(4\) in disordered quantum dots. Interaction is expected to play a crucial role in few-electron systems of low dimensionality; the way in which it affects quantum interference in phase coherent mesoscopic systems is presently under intense investigation.

With this motivation in mind, we consider here a system of two concentric ballistic rings with incommensurate areas \(A_1\) and \(A_2\), in a perpendicular constant magnetic field \(B\). The phase accumulated going around each ring is \(\varphi_p = BA_p/\phi_0\), \(p = 1, 2\), where \(\phi_0 = \hbar/2e\) is the flux quantum. We will assume that the two rings are coupled by a site-dependent tunneling matrix element. A particle will pick up random phases during its complicated motion. Such a breaking of the translational invariance can make the system nonintegrable and the one-particle spectrum chaotic, even in absence of e-e interaction.

Our aim is to investigate how strong interactions affect the energy fluctuations of the many-body ground-state and the first excited states as a function of the applied magnetic field. We will show that the interaction orders the flux dependence of the lowest energy levels, which in the noninteracting case are random functions described by the Wigner-Dyson statistics. In contrast to the case of a clean single-channel ring \(5\), in the two-ring geometry interaction plays an essential role and modifies qualitatively the period and the amplitude of the persistent currents.

We choose a lattice model of two concentric one-dimensional rings with \(N\) sites per ring, described by the following second quantized Hamiltonian

\[ \mathcal{H} = -t \sum_{i,p} e^{i2\pi \varphi_p/N} c_{i,p}^\dagger c_{i+1,p} - \sum_i t_{12}(i) c_{i,1}^\dagger c_{i,2} + H.c. + \sum_{i,j,p,p'} V(|r_{i,p} - r_{j,p'}|) n_{i,p} n_{j,p'} \]
In order to show that the one-particle spectrum is chaotic we study its level statistics. Statistical properties are obtained by performing a spectral average over many energy levels for fixed $H$ and a subsequent average over $H$. We consider the probability distribution of the nearest-neighbor level spacings, $P(s)$, which we compute numerically by diagonalizing exactly a system of two rings, both having $N = 11$ sites, coupled by $t_{12}(i) = -t/2 * \cos^2(3\pi i/2N)$. For each value of $H$ the corresponding energy spectrum is first unfolded to ensure that the mean level spacing is unity $\frac{1}{H}$. The distribution $P(s)$ is then computed and finally averaged over many values of $H$. In Fig. 1 we plot $P(s)$ averaged over 500 values of $H$ between 0 and 5. The numerical curve is in good agreement with the Wigner surmise of the orthogonal ensemble $\frac{1}{H}$ characterizing the Wigner-Dyson statistics.

Now we might ask what are the consequences of such a spectrum on the properties of the noninteracting many-body system. The $H$ dependence of an experimentally relevant quantity such as the total magnetic moment, $m(H) = \partial E^N_0(H)/\partial H$, where $E^N_0$ is the ground-state energy for a $N$ particle system, is just a very irregular function of $H$, see Fig 2(a). However the chaotic nature of the one-particle spectrum reveals itself in the statistical fluctuations of the many-body ground-state and quasiparticle excitations as function of $H$ and $N$. We are interested in two quantities: a) The distribution of the first excitation energy above the Fermi energy, $\delta E^N = E^N_1 - E^N_0$; b) The distribution of ground-state inverse compressibility: $\kappa^N = \partial^2 E^N_0 / \partial N^2 \sim E^{N+1}_0 - 2E^N_0 + E^{N-1}_0$. Here $E^N_1$ is the energy of the first excited state above the Fermi energy. It is obvious that for noninteracting particles at zero temperature $\delta E$ and $\kappa$ are the same and they are equal to $\epsilon_{N+1} - \epsilon_N$, where $\epsilon_N$ is the $N^{th}$ single-particle energy. Thus the fluctuations of $\delta E$ and $\kappa$, averaged over the Fermi energy (namely over $N$) and $H$, scale with the $\Delta$ and their distribution is given by the same $P(s)$ of the one-particle spectrum which, for our coupled-ring system, is given by the Wigner surmise.

How is this complex scenario modified by strong e-e interactions? To answer this question we have resorted to exact numerical calculations on small systems, using the Lanczos algorithm to compute the fluctuations of many-body ground state and the first low-lying excitations as a function of $E_F$ and $H$. We have considered systems with $N = 11$ sites per ring and $N = 4, 5, 6, 7, 8$ particles. The number of electrons in each ring, $N_p$, is not a good quantum number if the rings are coupled. The expectation values $\langle N_p \rangle$ will however satisfy the constraint $\langle N_1 \rangle + \langle N_2 \rangle = N$. We will first discuss the case of a short range interaction, coupling two nearest-neighbor sites $(i, p)$ and $(i + 1, p)$ in the same ring with matrix element $V_{11}$, and opposite sites $(i, 1)$ and $(i, 2)$ in different rings with matrix element $V_{12}$. The relevant energy parameter, indicating the relative strength of the interaction, is $V_{pp}/t$. It is interesting to see how $m(H)$ is modified. A plot of $m(H)$ vs $H$ for a system of $N = 8$ particles interacting with $V_{11} = 6.5t$, $V_{12} = 6.3t$ is shown in Fig. 2a. The effect of the interaction is visibly spectacular. We can see that the aperiodic $H$-dependence of $m(H)$ of the noninteracting case is replaced by a perfectly periodic function, oscillating with period $H_0 = 1/(a_1 + a_2)$, exactly as we would have for a single ring with area $a_{tot} = (a_1 + a_2)$. The periodic $m(H)$ develops smoothly with increasing interaction and eventually it becomes independent of the interaction strength. The amplitude of $m(H)$ in the strongly interacting regime is approximately one half of the amplitude of the noninteracting case.

FIG. 1. Probability distribution, $P(s)$, of the one-particle energy-level spacings for a system of two rings, coupled by the tunneling amplitude $t_{12}(i) = -t/2 * \cos^2(3\pi i/2N)$. Both rings have 11 sites; their area is $a_1 = [\sqrt(5) - 1]/2 \approx 0.618$ and $a_2 = 1.5$ respectively. The probability density $P(s)$ is obtained by performing a spectral average over the energies and an average over the flux. The dashed line represents the Wigner surmise plotted as a comparison.

FIG. 2. (a) Magnetic moment $m(H)$ vs $H$, and (b), (c) four lowest many-body energies vs $H$ for the two-ring system of Fig. 1. The total number of particles is $N = 8$. (a) The periodic solid line is the interacting case. The period is $H_0 = 1/(a_1 + a_2)$; the irregular dashed line is the noninteracting case. (b) Noninteracting case. (c) Interacting case.
Moreover, not only is the ground-state energy a periodic function of $H$, but so are the first low-lying excited states, as we show in Fig. 2(c). In particular the low energy sector of the many-body spectrum is identical to the single-particle spectrum in one ring. Similar results are obtained for other values of even $N$. By computing the ground-state wavefunction we can also evaluate the particle density in each ring. The calculation shows that in the strongly interacting case $(N_1) \approx (N_2) \approx N/2$ for any value of $H$. For odd $N$ the interaction again forces the system to develop a periodic low-energy spectrum with the same period, $1/(a_1 + a_2)$. The main difference with respect to even $N$ is that the $H$ dependence of the spectrum is shifted by $H_0/2$.

Strong intra-ring interaction is known to generate Wigner-crystal-like ground states in each single ring. It has also been shown that inter-ring interaction can cause a nondissipative Coulomb drag that affects qualitatively the persistent currents in each ring. Our exact results suggest that strong inter-ring interaction glues together the two stiff Wigner crystals created by the intra-ring interaction in each ring, resulting in a rigid structure that rotates like a single solid body under the effect of the magnetic field. Based on this picture, we can see why the oscillation period of the total magnetic moment of such a structure is $1/(a_1 + a_2)$. Indeed the Hamiltonian of the two frozen crystals glued together, each containing $N/2$ particles, is found to be

$$\mathcal{H} = \frac{\hbar^2}{2I} \left[ -i \frac{\partial}{\partial \theta} - \frac{N}{2} \frac{H}{2}(a_1 + a_2) \right]$$

where $-i \hbar \frac{\partial}{\partial \theta}$ is the canonical momentum conjugated to $\theta$, the rotation angle of the solid around an axis passing through the center of the rings. $I = m_e \frac{N}{2}(a_1 + a_2)$ is the moment of inertia. In zero flux, the eigenstates $\Psi_n(\theta) = \exp(i n \theta)$ must satisfy the boundary condition $\Psi_n(\theta) = \Psi_n(\theta + 2\pi/(N/2))$ due to the ordered crystal structure. This immediately implies the announced $1/(a_1 + a_2)$ periodicity as a function of $H$.

We have also studied the case of an unscreened long-range interaction coupling all the sites with matrix element $V(|\mathbf{r}_{i,p} - \mathbf{r}_{j,p'}|) = V_c R_1 / |\mathbf{r}_{i,p} - \mathbf{r}_{j,p'}| = V_c \sqrt{\alpha_1} \left[ a_p + a_{p'} - 2 \sqrt{a_p a_{p'}} \cos(2\pi(i - j)/N) \right]^{1/2}$. The dimensionless interaction constant is $\alpha = V_c R_1 / \hbar v_F = (V_c/t)[N/(4\pi \sin(k_F))]$. In this case we have to face first a new occurrence: in the strongly interacting regime $(\alpha \approx 5)$ where spectrum ordering should set in, the electronic configuration corresponding to the minimum energy is always the one with the inner ring empty and all the particles located in the outer ring. The corresponding low-energy sector of the spectrum is obviously a periodic function of $H$ with period $1/a_2$. To by-pass this unwanted situation we can add a term in the Hamiltonian in the form of a potential difference between the rings, e.g. provided by a gate voltage: $H_g = \sum_{i,p} \epsilon_p n_{i,p}$. Typically a voltage difference $V_g = \epsilon_1 - \epsilon_2 \approx -V_c/2$ ensures an almost equal population in the two rings. In Fig. 3 we plot $m(H)$ for $N = 6$ particles interacting with a Coulomb force of intensity $V_c = 10t$, for two values of the voltage $V_g$. Let us consider first zero inter-ring tunneling (solid line) \[\square\]. We can see that, for $V_g = -0.75 V_c$, $m(H)$ is perfectly periodic with the expected period $1/(a_1 + a_2)$. On the other hand imperfections and small glitches in the periodic pattern appear for $V_g = -0.5 V_c$. In fact the calculation of the ring occupancy shows that while for $V_g = -0.75 V_c$ the electronic configuration $(N_1 = 3, N_2 = 3)$ is stable for all values of $H$, when $V_g = 0.5 V_c$ the two configurations $(N_1 = 3, N_2 = 3)$ and $(N_1 = 2, N_2 = 4)$ alternate in the ground state. The aperiodic jumps in Fig. 3(a) correspond to a switch from one configuration to another as $H$ varies. It is instructive to look at the $H$ dependence of the lowest four energy levels, plotted in Fig. 4, and their corresponding ring density. Fig. 4(a) shows that when $V_g = -0.5 V_c$ these four levels are often two by two degenerate, the degeneracy being caused by the presence of the two electronic configurations $(N_1 = 3, N_2 = 3)$ and $(N_1 = 2, N_2 = 4)$ mentioned above. The gate voltage is not yet strong enough to enforce a fixed electronic configuration in the two rings. As a result the levels are not completely periodic and they are sensitive to a inter-ring tunneling– see dashed line of Fig 3(a). However if we increase $V_g$ up to $0.75 V_c$, the configuration $(N_1 = 3, N_2 = 3)$ is locked in the first three states. It is only starting from the fourth level that charge redistribution, with the appearance of the $(N_1 = 2, N_2 = 4)$ configuration at some values of $H$, is again possible. The important consequence is the ordering of the first three levels, which is robust against a nonhomogeneous tunneling between the rings, as shown in Fig. 4(c) and Fig. 3(b).

![FIG. 3. Magnetic moment vs $H$ for the system of Fig. 1. There are $N = 6$ particles interacting with a long-range Coulomb potential of intensity $V_c = 10t$. The periodic solid line represents the case of $t_{12}(i) = 0$. The oscillating period is $1/(a_1 + a_2)$. The dashed line represents the case of $t_{12}(i) = -t/2 \cos(2\pi i/N)$. A voltage difference $V_g$ is applied between the two rings. a) $V_g = -0.5 V_c$. b) $V_g = -0.75 V_c$.](image)
The example of 6 particles is representative of what should happen in a real mesoscopic system with \( N_1 \sim N_2 \sim 50 \), where gate and flux-induced-charge redistribution between the rings amounts to small changes in their electron concentration. If the gate voltage is strong enough to enforce an almost equal population and prevent charge rearrangement between the rings as \( H \) varies, then the e-e Coulomb interaction can fully develop the ordering effect that we have discussed. Disorder in this strongly correlated system can only be induced by charge redistribution, which is controllable by tuning \( V_g \). Our calculations show that spectrum ordering is already well supported from NFR and TFR is gratefully acknowledged.

\[
E^N_0 (H) = \frac{V_c}{C} N^2 + \Delta_N H^2 , \tag{3}
\]
\[
E^N_1 (H) = \frac{V_c}{C} N^2 + \Delta_N (H_0 - H)^2 , \tag{4}
\]
for even \( N \), while the expression for odd \( N \) are obtained shifting \( H \) by \( H_0/2 \). Here \( \Delta_N \approx \Delta (N/N) \) is a slowly increasing function of \( N \) and \( C \) is a constant. Thus for fixed \( N \) and \( H \) we obtain,

\[
\delta E^N (H) = \Delta_N (H_0^2 - 2H_0 H) , \tag{5}
\]
\[
\kappa^N (H) = 2 \frac{V_c}{C} - \Delta_N (H_0^2/2 - 2H_0 H) \tag{6}
\]
Notice that in the interacting case the fluctuations of \( \kappa \) scale with the Coulomb energy. To obtain the probability distribution of \( \delta E \) and \( \kappa \), for each value of \( H \) we first need to unfold the sequence of \( \delta E^N \) and \( \kappa^N \) obtained upon varying \( N \), to take into account the monotic change of \( \Delta_N \). Once this is done, no fluctuations around the average are left and the distributions of \( \delta E \) and \( \kappa \) are just delta functions centered at \( s = 1 \) in rescaled units. Since this result is valid for any value of \( H \), the average over \( H \) does not add anything. This is in sharp contrast to the case of noninteracting particles, where we saw that the two distributions were given by the Wigner surmise.

In conclusion, we have shown that e-e interaction orders the flux dependence of the low energy spectrum of a two-ring system, suppressing the irregular behavior of the free particle case. The fluctuations of the groundstate and first excited state are no longer described by the Wigner-Dyson statistics. When strong e-e correlations are present, disorder can only be induced by charge redistribution controllable through an external gate.

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\[\text{FIG. 4. Energy vs } H \text{ of the ground state and the first three excited states for the system of Fig. 3. (a) The two rings are decoupled } t_{12} = 0 \text{ and the voltage difference } V_g \text{ between the rings is } -0.5 \text{ of the Coulomb interaction intensity, } V_c = 10t. \] (b) \( t_{12} = 0 \) and \( V_g = -0.75 V_c \). (c) \( t_{12}(t) = -t/2 \cos^2(2\pi t/N) \) and \( V_g = -0.75 V_c \).

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\[\text{[11] Among all the configurations } (N_1, N_2) , \text{ satisfying the condition } N_1 + N_2 = N , \text{ the one of minimum energy is taken. Such } (N_1, N_2) \text{ depends, in general, on } H.\]