Pointwise approximation of functions by matrix operators of their Fourier series with $r$-differences of the entries

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Abstract. We extend the results of Xh.Z. Krasniqi [Acta Comment. Univ. Tartu. Math., 17:89–101, 2013] and the authors [Acta Comment. Univ. Tartu. Math., 13:11–24, 2019] to the case where in the measures of estimations $r$-differences of the entries are used.

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1 Introduction

Let $L^p$ ($1 \leq p < \infty$) be the class of all $2\pi$-periodic real-valued functions integrable in the Lebesgue sense with $p$th power over $Q = [-\pi, \pi]$ with the norm

$$\|f\| = \|f(\cdot)\|_{L^p} = \left( \int_Q |f(t)|^p \, dt \right)^{1/p}.$$ 

Consider the trigonometric Fourier series

$$S f(x) := a_0(f) + \sum_{\nu=1}^{\infty} (a_{\nu}(f) \cos \nu x + b_{\nu}(f) \sin \nu x)$$

with partial sums $S_k f$ and the conjugate one

$$\tilde{S} f(x) := \sum_{\nu=1}^{\infty} (a_{\nu}(f) \sin \nu x - b_{\nu}(f) \cos \nu x)$$
with partial sums $\widetilde{S}_k f$. We know that if $f \in L^1$, then

$$\widetilde{f}(x) := -\frac{1}{\pi} \int_0^\pi \psi_x(t) \frac{1}{2} \cot \frac{t}{2} \, dt = \lim_{\epsilon \rightarrow 0^+} \widetilde{f}(x, \epsilon),$$

where

$$\widetilde{f}(x, \epsilon) := -\frac{1}{\pi} \int_0^\pi \psi_x(t) \frac{1}{2} \cot \frac{t}{2} \, dt \quad \text{with} \quad \psi_x(t) := f(x + t) - f(x - t),$$

exists for almost all $x$ [10, Chap. 4, Thm. (3-1)].

Let $A := (a_{n,k})$ be an infinite matrix of real numbers such that

$$a_{n,k} \geq 0 \quad \text{when} \quad k, n = 0, 1, 2, \ldots, \quad \lim_{n \to \infty} a_{n,k} = 0, \quad \text{and} \quad \sum_{k=0}^\infty a_{n,k} = 1,$$

but $A^0 := (a_{n,k})$, where

$$a_{n,k} = 0 \quad \text{when} \quad k > n = 0, 1, 2, \ldots, \quad \text{and} \quad \sum_{k=0}^n a_{n,k} = 1.$$

We will use the notations

$$A_{n,r} = \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}|, \quad A^0_{n,r} = \sum_{k=0}^n |a_{n,k} - a_{n,k+r}|$$

for $r \in \mathbb{N}$, and

$$\left( \frac{T_{n,A} f(x)}{\widetilde{T}_{n,A} f(x)} \right) := \sum_{k=0}^\infty a_{n,k} \left( \frac{S_k f(x)}{\widetilde{S}_k f(x)} \right) \quad (n = 0, 1, 2, \ldots)$$

for the $A$-transformation of $S_k f$ or $\widetilde{S} f$, respectively.

In this paper, we study the upper bounds of $|T_{n,A} f(x) - f(x)|$, $|\widetilde{T}_{n,A} f(x) - \widetilde{f}(x)|$, and $|\widetilde{T}_{n,A} f(x) - \widetilde{f}(x, \epsilon)|$ by functions of modulus of continuity type, that is, nondecreasing continuous functions such that $\omega(0) = 0$ and $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$ for all $0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 2\pi$. We also consider functions from the following subclasses of $L^p$:

$$L^p(\omega)_\beta = \{ f \in L^p : \omega_\beta f(\delta)_{L^p} = O(\omega(\delta)) \quad \text{when} \quad \delta \in [0, 2\pi] \text{ and } \beta \geq 0 \},$$

$$L^p(\bar{\omega})_\beta = \{ f \in L^p : \bar{\omega}_\beta f(\delta)_{L^p} = O(\bar{\omega}(\delta)) \quad \text{when} \quad \delta \in [0, 2\pi] \text{ and } \beta \geq 0 \},$$

where $\omega$ and $\bar{\omega}$ are the functions of modulus of continuity type, and

$$\omega_\beta f(\delta)_{L^p} := \sup_{0 \leq |t| \leq \delta} \left\{ \left| \sin \frac{rt}{2} \right|^{\beta} \left\| \varphi(t) \right\|_{L^p} \right\} \quad \text{with} \quad \varphi(t) := f(x + t) + f(x - t) - 2f(x),$$

$$\bar{\omega}_\beta f(\delta)_{L^p} := \sup_{0 \leq |t| \leq \delta} \left\{ \left| \sin \frac{rt}{2} \right|^{\beta} \left\| \psi(t) \right\|_{L^p} \right\},$$

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where $r \in \mathbb{N}$. It is clear that for $\beta > \alpha \geq 0$,
\[
\tilde{\omega}_\beta f(\delta)_{L^p} \leq \tilde{\omega}_\alpha f(\delta)_{L^p} \quad \text{and} \quad \omega_\beta f(\delta)_{L^p} \leq \omega_\alpha f(\delta)_{L^p},
\]
and we easily see that $\tilde{\omega}_0 f(\cdot)_{L^p} = \omega f(\cdot)_{L^p}$ and $\omega_0 f(\cdot)_{L^p} = \omega f(\cdot)_{L^p}$ are the classical moduli of continuity.

The above deviations were estimated in [1] and generalized in [2] as follows.

**Theorem A.** (See [2, p. 97, Thm. 10].) Let $f \in L^p(\omega_\beta)$ ($p > 1$) with $\beta < 1 - 1/p$, and let $\omega$ satisfy
\[
\left\{ \int_{\pi/(n+1)}^{\pi} \left( \frac{t^{-\gamma}|\varphi_x(t)|}{\omega(t)} \right)^p \sin^{\beta_p} t \, dt \right\}^{1/p} = O_x((n+1)^\gamma), \tag{1.1}
\]
and
\[
\left\{ \int_0^{\pi/(n+1)} \left( \frac{\varphi_x(t)}{\omega(t)} \right)^p \sin^{\beta_p} t \, dt \right\}^{1/p} = O_x((n+1)^{-1}), \tag{1.2}
\]
with $0 < \gamma < \beta + 1/p$, where $q = p(p-1)^{-1}$. Then
\[
|T_{n,A^p} f(x) - f(x)| = O_x \left( (n+1)^{\beta+1/p+1} A_{n,1}^p \omega \left( \frac{\pi}{n+1} \right) \right).
\]

**Theorem B.** (See [2, p. 95, Thm. 8].) If $f \in L^p(\tilde{\omega}_\beta)$ ($p > 1$) with $\beta < 1 - 1/p$, where $\tilde{\omega}$ satisfies the conditions
\[
\left\{ \int_{\pi/(n+1)}^{\pi} \left( \frac{t^{-\gamma}|\varphi_x(t)|}{\tilde{\omega}(t)} \right)^p \sin^{\beta_p} t \, dt \right\}^{1/p} = O_x((n+1)^\gamma), \tag{1.4}
\]
and
\[
\left\{ \int_0^{\pi/(n+1)} \left( \frac{t|\varphi_x(t)|}{\tilde{\omega}(t)} \right)^p \sin^{\beta_p} t \, dt \right\}^{1/p} = O_x((n+1)^{-1}), \tag{1.5}
\]
with $0 < \gamma < \beta + 1/p$, then
\[
|\tilde{T}_{n,A^p} f(x) - \tilde{f}(x, \frac{\pi}{n+1})| = O_x \left( (n+1)^{\beta+1/p+1} A_{n,1}^p \tilde{\omega} \left( \frac{\pi}{n+1} \right) \right).
\]

**Theorem C.** (See [2, p. 97, Thm. 9].) If $f \in L^p(\tilde{\omega}_\beta)$ ($p > 1$) with $\beta < 1 - 1/p$, where $\tilde{\omega}$ satisfies condition (1.4),
\[
\left\{ \int_0^{\pi/(n+1)} \left( \frac{|\varphi_x(t)|}{\tilde{\omega}(t)} \right)^p \sin^{\beta_p} t \, dt \right\}^{1/p} = O_x((n+1)^{-1/p}),
\]
with $0 < \gamma < \beta + 1/p$. Then
\[
|\tilde{T}_{n,A^p} f(x) - \tilde{f}(x, \frac{\pi}{n+1})| = O_x \left( (n+1)^{\beta+1/p+1} A_{n,1}^p \tilde{\omega} \left( \frac{\pi}{n+1} \right) \right).
\]
and
\[ \left\{ \frac{\pi}{r(n+1)} \right\}^{\frac{q}{r}} \int_0^{\pi} \left( \frac{\bar{\omega}(t)}{t \sin^{\beta+1/2} \frac{\pi}{2}} \right)^q dt \right\}^{1/q} = O((n+1)^{\beta+1/p} \bar{\omega} \left( \frac{\pi}{n+1} \right)) \]

with \( 0 < \gamma < \beta + 1/p \), where \( q = p(p-1)^{-1} \), then
\[ |\widetilde{A}_{n,r} f(x) - \widetilde{f}(x)| = O_{x} \left( (n+1)^{\beta+1/p} \right) A_{n,1}^p \left( \frac{\pi}{n+1} \right). \]

In our theorems, we generalize Theorems A–C using \( A_{n,r} \) with \( r \in \mathbb{N} \) instead of \( A_{n,1} \).

In the paper, \( \sum_{k=a}^{b} = 0 \) when \( a > b \).

## 2 Statement of the results

At the beginning, we present an estimate of the quantity \( |\widetilde{T}_{n,A} f(x) - f(x)| \). Next, we will estimate the quantities \( |\widetilde{T}_{n,A} f(x) - f(x)| \) and \( |\widetilde{T}_{n,A} f(x) - f(x, \epsilon)| \). Finally, we will formulate some remarks and corollaries.

**Theorem 1.** Suppose that \( f \in L^p \ (p \geq 1) \) and a function \( \omega \) of modulus of continuity type satisfy, for \( r \in \mathbb{N} \), the conditions
\[
\left\{ \frac{\pi}{r(n+1)} \right\}^{\frac{q}{r}} \int_0^{\pi} \left( \frac{\omega(t)}{t \sin^{\beta+1/2} \frac{\pi}{2}} \right)^q dt \right\}^{1/q} = O((n+1)^{\beta+1/p} \omega \left( \frac{\pi}{n+1} \right)) \quad (p > 1),
\]
\[
\text{ess sup}_{0 < t < \frac{\pi}{r(n+1)}} \left\{ \frac{\omega(t)}{t \sin^{\beta+1/2} \frac{\pi}{2}} \right\} = O((n+1)^{\beta+1} \omega \left( \frac{\pi}{n+1} \right)) \quad (p = 1, \beta = 0),
\]

where \( q = p(p-1)^{-1} \),
\[
\left\{ \frac{2m\pi}{r(n+1)} \right\}^{\frac{q}{r(n+1)}} \int_{2m\pi/r}^{2m\pi} \left( \frac{\omega(t)}{t \sin^{\beta+1/2} \frac{\pi}{2}} \right)^q dt \right\}^{1/q} = O((n+1)^{-1/p}),
\]
\[
\left\{ \frac{2m\pi + \pi}{r(n+1)} \right\}^{\frac{q}{r(n+1)}} \int_{2m\pi + \pi/r}^{2m\pi + \pi} \left( \frac{\omega(t)}{t \sin^{\beta+1/2} \frac{\pi}{2}} \right)^q dt \right\}^{1/q} = O((n+1)^{\gamma} \quad (0 < \gamma < \beta + \frac{1}{p}),
\]

where \( m \in \{0, \ldots, \lfloor r/2 \rfloor \} \) when \( r \) is odd or \( m \in \{0, \ldots, \lfloor r/2 \rfloor - 1 \} \) when \( r \) is even. Moreover, let \( \omega \) satisfy, for natural \( r \geq 2 \), the conditions
\[
\left\{ \frac{2(m+1)\pi}{r(n+1)} \right\}^{\frac{q}{r(n+1)}} \int_{2(m+1)\pi/r - \pi}^{2(m+1)\pi/r} \left( \frac{\omega(t)}{t \sin^{\beta+1/2} \frac{\pi}{2}} \right)^q dt \right\}^{1/q} = O((n+1)^{-1/p}),
\]
\[
\left\{ \frac{2(m+1)\pi - \pi}{r(n+1)} \right\}^{\frac{q}{r(n+1)}} \int_{2(m+1)\pi - \pi}^{2(m+1)\pi} \left( \frac{\omega(t)}{t \sin^{\beta+1/2} \frac{\pi}{2}} \right)^q dt \right\}^{1/q} = O((n+1)^{\gamma} \quad (0 < \gamma < \beta + \frac{1}{p}),
\]

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where \( m \in \{0, \ldots, \lfloor r/2 \rfloor - 1\} \). If a matrix \( A \) is such that

\[
\left[ \sum_{l=0}^{n} \sum_{k=l}^{r+l-1} a_{n,k} \right]^{-1} = O(1)
\]

(2.7)

for \( r \in \mathbb{N} \), then

\[
|T_{n,A}f(x) - f(x)| = O_{\omega}( (n+1)^{\beta+1/p+1} A_{n,r} \omega \left( \frac{\pi}{n+1} \right) ).
\]

Theorem 2. Let \( f \in L^p \) (\( p \geq 1 \)), \( \beta < 1 - 1/p \) when \( p > 1 \) or \( \beta = 0 \) when \( p = 1 \), and let a function \( \tilde{\omega} \) of modulus of continuity type satisfy: for \( r \in \mathbb{N} \), the conditions

\[
\left\{ \int_0^{\pi/(r(n+1))} \left( \frac{|\psi_x(t)|}{\tilde{\omega}(t)} \right)^p \sin^{\beta_p} \frac{t}{2} \, dt \right\}^{1/p} = O_{\omega}( (n+1)^{-1} ), \quad (p > 1),
\]

(2.8)

and for natural \( r \geq 2 \), the conditions

\[
\left\{ \int_{2m\pi/r}^{2m\pi/r+\pi/(r(n+1))} \left( \frac{|\psi_x(t)|}{\tilde{\omega}(t)} \right)^p \sin^{\beta_p} \frac{rt}{2} \, dt \right\}^{1/p} = O_{\omega}( (n+1)^{\gamma} ) \quad (0 < \gamma < \beta + \frac{1}{p}),
\]

(2.9)

where \( q = p(p-1)^{-1} \),

\[
\left\{ \int_{2m\pi/r}^{2m\pi/r+\pi/(r(n+1))} \left( \frac{|\psi_x(t)|}{\tilde{\omega}(t)} \right)^p \sin^{\beta_p} \frac{rt}{2} \, dt \right\}^{1/p} = O_{\omega}( (n+1)^{-1/p} ),
\]

(2.10)

where \( m \in \{0, \ldots, \lfloor r/2 \rfloor \} \) when \( r \) is odd or \( m \in \{0, \ldots, \lfloor r/2 \rfloor - 1 \} \) when \( r \) is even. Moreover, let \( \tilde{\omega} \) satisfy, for natural \( r \geq 2 \), the conditions

\[
\left\{ \int_{2(m+1)\pi/r}^{2(m+1)\pi/r-\pi/(r(n+1))} \left( \frac{|\psi_x(t)|}{\tilde{\omega}(t)} \right)^p \sin^{\beta_p} \frac{rt}{2} \, dt \right\}^{1/p} = O_{\omega}( (n+1)^{-1/p} ),
\]

(2.11)

\[
\left\{ \int_{2(m+1)\pi/r-\pi/(r(n+1))}^{2(m+1)\pi/r} \left( \frac{|\psi_x(t)|}{\tilde{\omega}(t)} \right)^p \sin^{\beta_p} \frac{rt}{2} \, dt \right\}^{1/p} = O_{\omega}( (n+1)^{\gamma} ) \quad (0 < \gamma < \beta + \frac{1}{p}),
\]

(2.12)

where

\[
\tilde{\omega}(t) = \omega(t) \left( \frac{t}{r} \right)^{m\pi/r - 1} \left( 1 + \beta \frac{t}{r} \right)^{\beta \gamma},
\]

(2.13)
where \( m \in \{0, \ldots, [r/2] - 1\} \). If a matrix \( A \) is such that (2.7) is satisfied and

\[
\sum_{k=0}^{\infty} (k+1)^2 a_{n,k} = O((n+1)^2)
\]

for \( r \in \mathbb{N} \), then

\[
\left| \tilde{T}_{n,A} f(x) - \tilde{f} \left( x, \frac{\pi}{n+1} \right) \right| = O_x \left( (n+1)^{\beta+1/p+1} A_{n,r} \tilde{\omega} \left( \frac{\pi}{n+1} \right) \right).
\]

**Theorem 3.** Suppose that \( f \in L^p \) (\( p \geq 1 \)) and a function \( \tilde{\omega} \) of modulus of continuity type satisfy, for \( r \in \mathbb{N} \), conditions (2.9), (2.10) or (2.11), and (2.12), where \( m \in \{0, \ldots, [r/2] \} \) when \( r \) is odd or \( m \in \{0, \ldots, [r/2] - 1 \} \) when \( r \) is even. Moreover, let \( \tilde{\omega} \) satisfy, for natural \( r \geq 2 \), conditions (2.13) and (2.14), where \( m \in \{0, \ldots, [r/2] - 1 \} \). If a matrix \( A \) is such that (2.7) is true for \( r \in \mathbb{N} \), then

\[
\left| \tilde{T}_{n,A} f(x) - \tilde{f}(x) \right| = O_x \left( (n+1)^{\beta+1/p+1} A_{n,r} \tilde{\omega} \left( \frac{\pi}{n+1} \right) \right).
\]

In the proofs of our results, we will consider the case \( p > 1 \) only. The case \( p = 1 \) can be examined analogously under terminology from conditions (2.1) or (2.2), respectively.

**Remark 1.** If, instead of (2.4) and (2.6), we consider, respectively, the more natural conditions

\[
\left\{ \begin{array}{l}
\left( \frac{|\varphi_x(t)|}{\omega(t)(t - 2m\pi r)} \right)^p \left| \sin \frac{rt}{2} \right|^{\beta p} dt^{1/p} = O_x ((n+1)^{\gamma-1/p}), \\
\left( \frac{|\varphi_x(t)|}{\omega(t)(2(m+1)\pi r - t)} \right)^p \left| \sin \frac{rt}{2} \right|^{\beta p} dt^{1/p} = O_x ((n+1)^{\gamma-1/p})
\end{array} \right.
\]

for \( \gamma \in (1/p, 1/p + \beta) \), where \( \beta > 0 \), and analogously for \( \tilde{\omega} \) and \( \psi \), then our estimates take the forms

\[
\left| T_{n,A} f(x) - f(x) \right| = O_x \left( (n+1)^{\beta+1} A_{n,r} \omega \left( \frac{\pi}{n+1} \right) \right),
\]

\[
\left| \tilde{T}_{n,A} f(x) - \tilde{f} \left( x, \frac{\pi}{n+1} \right) \right| = O_x \left( (n+1)^{\beta+1} A_{n,r} \tilde{\omega} \left( \frac{\pi}{n+1} \right) \right),
\]

\[
\left| \tilde{T}_{n,A} f(x) - \tilde{f}(x) \right| = O_x \left( (n+1)^{\beta+1} A_{n,r} \tilde{\omega} \left( \frac{\pi}{n+1} \right) \right).
\]

These considerations are natural because in the case of norm approximation the new and old conditions always hold with \( \| \varphi_x(t) \|_{L^p} \) instead of \( |\varphi_x(t)| \) and with \( \| \psi_x(t) \|_{L^p} \) instead of \( |\psi_x(t)| \), for \( f \in L^p(\omega)_{\beta} \) and \( f \in L^p(\tilde{\omega})_{\beta} \), respectively.

**Remark 2.** Note that in the case \( r = 1 \), conditions (2.1)–(2.6) in Theorem 1 reduce to (1.1)–(1.3) and conditions (2.8)–(2.14) in Theorem 2 reduce to (1.4)–(1.5). We have a similar situation in the case of Theorem 3.

**Remark 3.** Note that for \( r = 1 \), if in the proof of Theorem 1 we use the estimate \( |D_{k,1}^\alpha(t)| \leq k + 1/2 \) from Lemma 2, then we additionally need the condition

\[
\sum_{k=0}^{\infty} (k+1)^2 a_{n,k} = O(n+1).
\]
In this case, we can apply the weaker conditions

\[
\left\{ \int_0^{\pi/(n+1)} \left( \frac{t|\varphi_x(t)|}{\omega(t)} \right)^p \sin \frac{t}{2} \frac{\beta_p}{\omega(t)} \right\}^{1/p} = O_x((n+1)^{-1-1/p})
\]

instead of condition (2.3).

**Remark 4.** Note that our extra conditions (2.7), (2.15), and (2.16) for a lower triangular infinite matrix \( A \) always hold.

**Corollary 1.** Under Remark 2 and the obvious inequality

\[
A_{n,r} \leq rA_{n,1} \quad \text{for } r \in \mathbb{N},
\]

our results improve and generalize the mentioned Theorems A and C without the assumption \( \beta < 1 - 1/p \) and Theorem B of Krasniqi [2].

**Remark 5.** In the case \( r = 1 \) the deviation \( T_{n,A^*} f - f \) was examined by many authors. Mishra, Khatri, Mishra, and Deepmala [4, 7], Mishra [5], and Mishra and Mishra [3] considered the mentioned deviation with some conditions on the quantity \( A_{n,1} \) and on the entries of the matrix \( A^* \). In a particular case the Nörlund and Riesz means were considered. Iterations of some methods of summability in such investigations were used in [4, 6] and [3, 5]. The earlier estimates were very often obtained for functions from \( L^p(\omega)^\beta \) where \( \omega(\delta) = O(\delta^{\alpha}) \) with \( \alpha \in (0, 1] \), and in the conjugate case.

**Example 1.** Let a matrix \( A^* \) is such that

\[
A_{n,1}^* = O(a_{n,0}) \quad \text{and} \quad (n+1)a_{n,0} = O(1).
\]

If \( f \in L^p(\omega)^\beta[L^p(\tilde{\omega})^\beta] \) with \( \beta \geq 0 \) and \( p \geq 1 \) in the particular case \( r = 1 \) and \( \omega(\delta)|\tilde{\omega}(\delta)| = O(\delta^{\alpha}) \) with \( \alpha \in (0, 1] \), then our Theorems 1 and 3 with Remarks 2 and 4 give the norm estimates

\[
\|T_{n,A^*} f(\cdot) - f(\cdot)\|_{L^p} = O(n^{-\alpha+1/p+\beta}),
\]

\[
\|\tilde{T}_{n,A^*} f(\cdot) - f(\cdot)\|_{L^p} = O(n^{-\alpha+1/p+\beta})
\]

and Remark 1, the following slightly better ones with \( \beta > 0 \):

\[
\|T_{n,A^*} f(\cdot) - f(\cdot)\|_{L^p} = O(n^{-\alpha+\beta}),
\]

\[
\|\tilde{T}_{n,A^*} f(\cdot) - f(\cdot)\|_{L^p} = O(n^{-\alpha+\beta}).
\]

The reader can find the presented estimates in, for example, [3, 5] and [7].

**Remark 6.** Note that instead of \( L^p(\omega)^\beta \) and \( L^p(\tilde{\omega})^\beta \), we can consider other subclasses of \( L^p \) generated by any function of the continuity modulus type, for example, by a function \( \omega_x \) such that

\[
\omega_x(f,\delta) = \sup_{|t| \leq \delta} |\varphi_x(t)| \leq \omega_x(\delta) \quad \text{or} \quad \omega_x(f,\delta) = \frac{1}{\delta} \int_0^\delta |\varphi_x(t)| \, dt \leq \omega_x(\delta),
\]

and in the conjugate case as well.
3 Auxiliary results

We begin this section by some notations from [9] and [10, Chap. II, Sect. 5]. Let for \( r = \pm 1, \pm 2, \ldots, \)

\[
D_{k,r}^0(t) = \frac{\sin \left(\frac{(2k+r)t}{2}\right)}{2 \sin \frac{rt}{2}}, \quad \bar{D}_{k,r}^0(t) = \frac{\cos \left(\frac{(2k+r)t}{2}\right)}{2 \sin \frac{rt}{2}}, \quad \text{and} \quad \bar{D}_{k,r}(t) = \frac{\cos \frac{rt}{2} - \cos \frac{(2k+r)t}{2}}{2 \sin \frac{rt}{2}}.
\]

By [10] it is clear that

\[
S_k f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) D_{k,1}^0(t) \, dt, \quad \bar{S}_k f(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) \bar{D}_{k,1}^0(t) \, dt,
\]

and

\[
T_{n,A} f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) \sum_{k=0}^{\infty} a_{n,k} D_{k,1}^0(t) \, dt,
\]

\[
\bar{T}_{n,A} f(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) \sum_{k=0}^{\infty} a_{n,k} \bar{D}_{k,1}^0(t) \, dt.
\]

Hence

\[
T_{n,A} f(x) - f(x) = \frac{1}{\pi} \int_{0}^{\pi} \varphi_x(t) \sum_{k=0}^{\infty} a_{n,k} D_{k,1}^0(t) \, dt,
\]

\[
\bar{T}_{n,A} f(x) - \bar{f}(x) = \frac{1}{\pi} \int_{0}^{\pi} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \bar{D}_{k,1}^0(t) \, dt,
\]

and

\[
\bar{T}_{n,A} f(x) - \bar{f} \left( x, \frac{\pi}{r(n+1)} \right) = -\frac{1}{\pi} \int_{0}^{\pi/\left( r(n+1) \right)} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \bar{D}_{k,1}^0(t) \, dt
\]

\[
+ \frac{1}{\pi} \int_{\pi/\left( r(n+1) \right)}^{\pi} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \bar{D}_{k,1}^0(t) \, dt.
\]

At the beginning, we present a very useful property of functions of the continuity modulus type.

**Lemma 1.** (See [10].) A function \( \omega \) of modulus of continuity type on the interval \([0, 2\pi]\) satisfies the following condition:

\[
\delta_2^{-1} \omega(\delta_2) \leq 2 \delta_1^{-1} \omega(\delta_1) \quad \text{for} \quad \delta_2 > \delta_1 > 0.
\]

Next, we present some known estimates.

**Lemma 2.** (See [10].) If \( 0 < |t| \leq \pi \), then for \( k \in \mathbb{N} \), \( |D_{k,1}^0(t)| \leq \pi/(2|t|) \), \( |\bar{D}_{k,1}^0(t)| \leq \pi/(2|t|) \), and \( |\bar{D}_{k,1}(t)| \leq \pi/|t| \), and, for real \( t \), we have

\[
|D_{k,1}^0(t)| \leq k + \frac{1}{2}, \quad |\bar{D}_{k,1}(t)| \leq \frac{1}{2} k(k + 1)|t|, \quad |\bar{D}_{k,1}(t)| \leq k + 1.
\]
Lemma 3. (See [8, 9].) Let \( m, n, r \in \mathbb{N}, l \in \mathbb{Z}, \) and \( (a_n) \subset \mathbb{C}. \) If \( t \neq 2l\pi/r, \) then for all \( m \geq n, \)

\[
\begin{align*}
\sum_{k=n}^m a_k \sin kt &= - \sum_{k=n}^m (a_k - a_{k+r}) \tilde{D}_{k,r}^o(t) + \sum_{k=m+1}^{m+r} a_k \tilde{D}_{k,-r}^o(t) - \sum_{k=n}^{n+r-1} a_k \tilde{D}_{k,-r}^o(t), \\
\sum_{k=n}^m a_k \cos kt &= \sum_{k=n}^m (a_k - a_{k+r}) \tilde{D}_{k,r}^o(t) - \sum_{k=m+1}^{m+r} a_k \tilde{D}_{k,-r}^o(t) + \sum_{k=n}^{n+r-1} a_k \tilde{D}_{k,-r}^o(t).
\end{align*}
\]

We additionally need two estimates as a consequence of Lemma 3.

Lemma 4. Let \( r \in \mathbb{N}, l \in \mathbb{Z}, \) and \( (a_{n,k}) \subset \mathbb{C}. \) If \( t \neq 2l\pi/r, \) then

\[
\left| \sum_{k=0}^\infty a_{n,k} D_{k,1}^o(t) \right| \leq \frac{1}{2|\sin \frac{t}{2} \sin \frac{r}{2}|} \left( A_{n,r} + \sum_{k=0}^{r-1} a_{n,k} \right) \leq \frac{1}{|\sin \frac{t}{2} \sin \frac{r}{2}|} A_{n,r}.
\]

Proof. By Lemma 3

\[
\begin{align*}
\sum_{k=0}^\infty a_{n,k} D_{k,1}^o(t) &= \sum_{k=0}^\infty a_{n,k} \frac{\sin (2k+1)t}{2 \sin \frac{t}{2}} = \frac{1}{2 \sin \frac{t}{2}} \left( \sum_{k=0}^\infty a_{n,k} \sin kt \cos \frac{t}{2} + \sum_{k=0}^\infty a_{n,k} \cos kt \sin \frac{t}{2} \right) \\
&= \frac{\cos \frac{t}{2}}{2 \sin \frac{t}{2}} \left( - \sum_{k=0}^\infty (a_{n,k} - a_{n,k+r}) \tilde{D}_{k,r}^o(t) - \sum_{k=0}^{r-1} a_{n,k} \tilde{D}_{k,-r}^o(t) \right) \\
&\quad + \frac{1}{2} \left( \sum_{k=0}^\infty (a_{n,k} - a_{n,k+r}) \tilde{D}_{k,r}^o(t) + \sum_{k=0}^{r-1} a_{n,k} \tilde{D}_{k,-r}^o(t) \right),
\end{align*}
\]

and our lemma is proved. \( \square \)

Lemma 5. Let \( r \in \mathbb{N}, l \in \mathbb{Z}, \) and \( (a_{n,k}) \subset \mathbb{C}. \) If \( t \neq 2l\pi/r, \) then

\[
\left| \sum_{k=0}^\infty a_{n,k} \tilde{D}_{k,1}^o(t) \right| \leq \frac{1}{2|\sin \frac{t}{2} \sin \frac{r}{2}|} \left( A_{n,r} + \sum_{k=0}^{r-1} a_{n,k} \right) \leq \frac{1}{|\sin \frac{t}{2} \sin \frac{r}{2}|} A_{n,r}.
\]

Proof. By Lemma 3

\[
\begin{align*}
\sum_{k=0}^\infty a_{n,k} \tilde{D}_{k,1}^o(t) &= \sum_{k=0}^\infty a_{n,k} \frac{\cos (2k+1)t}{2 \sin \frac{t}{2}} = \frac{1}{2 \sin \frac{t}{2}} \left( \sum_{k=0}^\infty a_{n,k} \cos kt \cos \frac{t}{2} - \sum_{k=0}^\infty a_{n,k} \sin kt \sin \frac{t}{2} \right) \\
&= \frac{\cos \frac{t}{2}}{2 \sin \frac{t}{2}} \left( \sum_{k=0}^\infty (a_{n,k} - a_{n,k+r}) D_{k,r}^o(t) + \sum_{k=0}^{r-1} a_{n,k} D_{k,-r}^o(t) \right) \\
&\quad - \frac{1}{2} \left( \sum_{k=0}^\infty (a_{n,k} - a_{n,k+r}) \tilde{D}_{k,r}^o(t) - \sum_{k=0}^{r-1} a_{n,k} \tilde{D}_{k,-r}^o(t) \right),
\end{align*}
\]

and thus our proof is complete. \( \square \)

We also need some special conditions, which follow from those mentioned earlier.
Lemma 6. If condition (2.1) holds with \( q = p(p - 1)^{-1} \) and natural \( r \geq 2 \), then for any function \( \omega \) of modulus of continuity type, we have

\[
\left\{ \int_{2(m+1)\pi/r}^{2(m+1)\pi/(r(n+1))} \left( \frac{\omega(t)}{t \sin \frac{rt}{2} \beta} \right)^q \, dt \right\}^{1/q} = O_n \left( (n + 1)^{\beta+1/p} \omega \left( \frac{\pi}{n + 1} \right) \right),
\]

where \( m \in \{0, \ldots, [r/2] - 1\} \).

Proof. By substitution \( t = 2(m + 1)\pi/r - u \) we obtain

\[
\left\{ \int_{2(m+1)\pi/r - \pi/(r(n+1))}^{\pi/(r(n+1))} \left( \frac{\omega(t)}{t \sin \frac{rt}{2} \beta} \right)^q \, dt \right\}^{1/q} = \left\{ \int_{0}^{\pi/(r(n+1))} \left( \frac{\omega(2(m+1)\pi/r - u)}{\sin \frac{2(m+1)\pi}{r} - u} \right)^q \, du \right\}^{1/q} \leq \left\{ \int_{0}^{\pi/(r(n+1))} \left( \frac{\omega(u)}{u \sin \frac{\pi u}{2} \beta} \right)^q \, du \right\}^{1/q}.
\]

Hence our estimate follows by (2.1).

Lemma 7. If condition (2.1) holds with \( q = p(p - 1)^{-1} \) and natural \( r \), then for any function \( \omega \) of modulus of continuity type, we have

\[
\left\{ \int_{2m\pi/r}^{2m\pi/(r(n+1))} \left( \frac{\omega(t)}{t \sin \frac{rt}{2} \beta} \right)^q \, dt \right\}^{1/q} = O_n \left( (n + 1)^{\beta+1/p} \omega \left( \frac{\pi}{n + 1} \right) \right),
\]

where \( m \in \{0, \ldots, [r/2]\} \).

Proof. By substitution \( t = 2m\pi/r + u \), analogously to the previous proof, we obtain

\[
\left\{ \int_{2m\pi/r + \pi/(r(n+1))}^{2m\pi/(r(n+1))} \left( \frac{\omega(t)}{t \sin \frac{rt}{2} \beta} \right)^q \, dt \right\}^{1/q} = \left\{ \int_{0}^{\pi/(r(n+1))} \left( \frac{\omega(2m\pi/r + u)}{2m\pi/r + u} \sin \frac{2m\pi}{r} + u \right)^q \, du \right\}^{1/q}.
\]

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and we have the desired estimate. □

4 Proofs of the results

4.1 Proof of Theorem 1

It is clear that for an odd \( r \),

\[
T_{n,A} f(x) - f(x) = I_1(x) + I_2(x)
\]

and for an even \( r \),

\[
T_{n,A} f(x) - f(x) = I_1'(x) + I_2(x).
\]

Then

\[
|T_{n,A} f(x) - f(x)| \leq |I_1(x)| + |I_1'(x)| + |I_2(x)|.
\]

and by Lemmas 2 and 4

\[
|I_1(x)| \leq \frac{1}{\pi} \sum_{m=0}^{[r/2]} \int_{2m\pi/r}^{2m\pi/r+\pi/r} |\varphi_x(t)| \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}^0(t) \right| dt
\]

\[
= \frac{1}{\pi} \sum_{m=0}^{[r/2]} \left( \int_{2m\pi/r}^{2m\pi/r+\pi/(r+1)} + \int_{2m\pi/r+\pi/r}^{2m\pi/r+\pi/(r+1)} \right) |\varphi_x(t)| \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}^0(t) \right| dt
\]

\[
\leq \frac{1}{2} \sum_{m=0}^{[r/2]} \int_{2m\pi/r}^{2m\pi/r+\pi/(r+1)} \left| \frac{\varphi_x(t)}{t} \right| dt + \frac{1}{\pi} A_{n,r} \sum_{m=0}^{[r/2]} \int_{2m\pi/r+\pi/(r+1)}^{2m\pi/r+\pi/r} \left| \frac{\varphi_x(t)}{\sin \frac{t}{2} \sin \frac{\pi t}{2}} \right| dt.
\]
Using the estimates \(|\sin(t/2)| \geq |t|/\pi\) for \(t \in [0, \pi]\) and \(|\sin(rt/2)| \geq rt/\pi - 2m\) for \(t \in [2m\pi/r, 2m\pi/r + \pi/r]\), where \(m \in \{0, \ldots, [r/2]\}\), we obtain

\[
|I_1(x)| \leq \frac{1}{2} \sum_{m=0}^{[r/2]} \int_{2m\pi/r}^{2m\pi/r+\pi/(r(n+1))} \frac{|\varphi_r(t)|}{t} \, dt + A_{n,r} \sum_{m=0}^{[r/2]} \int_{2m\pi/r+\pi/(r(n+1))}^{2m\pi/r+\pi/r} \frac{|\varphi_r(t)|}{t^{(\frac{r}{2\pi} - 2m)}} \, dt \leq \frac{1}{2} \sum_{m=0}^{[r/2]} \int_{2m\pi/r}^{2m\pi/r+\pi/(r(n+1))} \left( \frac{|\varphi_r(t)|}{\omega(t)} \right) ^\beta \left( \sin \frac{rt}{2} \right) ^p \, dt \left[ \int_{2m\pi/r}^{2m\pi/r+\pi/(r(n+1))} \left( \frac{\omega(t)}{t^{(\frac{r}{2\pi} - 2m)}} \frac{t}{t^{(\frac{r}{2\pi} - 2m)}} \right) ^q \, dt \right]^{1/q} + \frac{\pi}{r} A_{n,r} \sum_{m=0}^{[r/2]} \int_{2m\pi/r+\pi/(r(n+1))}^{2m\pi/r+\pi/r} \left( \frac{\omega(t)(t - \frac{2m\pi}{r})^\gamma}{t(t - \frac{2m\pi}{r})^{\frac{t}{2\pi} - \frac{r}{2}}} \right) ^q \, dt \left[ \int_{2m\pi/r+\pi/(r(n+1))}^{2m\pi/r+\pi/r} \left( \frac{\omega(t)(t - \frac{2m\pi}{r})^\gamma}{t(t - \frac{2m\pi}{r})^{\frac{t}{2\pi} - \frac{r}{2}}} \right) ^q \, dt \right]^{1/q},
\]

and by (2.3) and (2.1) with Lemma 7 and (2.4) we have

\[
|I_1| = O_x(1) \sum_{m=0}^{[r/2]} (n + 1)^{-1/p} (n + 1)^{\beta + 1/p} \omega \left( \frac{\pi}{n + 1} \right) + O_x(1) A_{n,r} \sum_{m=0}^{[r/2]} (n + 1)^{\gamma} \left[ \int_{2m\pi/r+\pi/(r(n+1))}^{2m\pi/r+\pi/r} \left( \frac{\omega(t)(t - \frac{2m\pi}{r})^\gamma}{t(t - \frac{2m\pi}{r})^{\frac{t}{2\pi} - \frac{r}{2}}} \right) ^q \, dt \right]^{1/q}.
\]

Since

\[
\left[ \int_{2m\pi/r+\pi/(r(n+1))}^{2m\pi/r+\pi/r} \left( \frac{\omega(t)(t - \frac{2m\pi}{r})^\gamma}{t(t - \frac{2m\pi}{r})^{\frac{t}{2\pi} - \frac{r}{2}}} \right) ^q \, dt \right]^{1/q} \leq \left[ \int_{\pi/(r(n+1))}^{\pi/r} \left( \omega \left( \frac{\pi}{r(n+1)} \right) \right) ^q \right]^{1/q} \leq \left[ \int_{\pi/(r(n+1))}^{\pi/r} \left( \omega \left( \frac{\pi}{r(n+1)} \right) \right) ^q \, dt \right]^{1/q},
\]

we obtain

\[
O \left( (n + 1)^{\omega \left( \frac{\pi}{n + 1} \right)} \left( \frac{\pi}{r(n + 1)} \right)^{(\gamma - 1 - \beta + 1/q)} = O \left( (n + 1)^{1 - \gamma - \beta + 1/p} \omega \left( \frac{\pi}{n + 1} \right) \right).
\]
for $0 < \gamma < \beta + 1/p$, we have

$$|I_1(x)| = O_x(1) \sum_{m=0}^{\lfloor r/2 \rfloor} (n+1)^{-1/p} (n+1)^{\beta+1/p} \omega \left( \frac{\pi}{n+1} \right)$$

$$+ O_x(1) A_{n,r} \sum_{m=0}^{\lfloor r/2 \rfloor} (n+1)^{\gamma} (n+1)^{1-\gamma+\beta+1/p} \omega \left( \frac{\pi}{n+1} \right)$$

$$= O_x \left( (n+1)^{\beta \omega} \left( \frac{\pi}{n+1} \right) \right) + O_x \left( (n+1)^{1+\beta+1/p} A_{n,r} \omega \left( \frac{\pi}{n+1} \right) \right).$$

Analogously,

$$|I'_1(x)| = O_x(1) \sum_{m=0}^{\lfloor r/2 \rfloor-1} (n+1)^{-1/p} (n+1)^{\beta+1/p} \omega \left( \frac{\pi}{n+1} \right)$$

$$+ O_x(1) A_{n,r} \sum_{m=0}^{\lfloor r/2 \rfloor-1} (n+1)^{\gamma} (n+1)^{1-\gamma+\beta+1/p} \omega \left( \frac{\pi}{n+1} \right)$$

$$= O_x \left( (n+1)^{\beta \omega} \left( \frac{\pi}{n+1} \right) \right) + O_x \left( (n+1)^{1+\beta+1/p} A_{n,r} \omega \left( \frac{\pi}{n+1} \right) \right).$$

Similarly,

$$|I_2(x)| \leq \frac{1}{\pi} \sum_{m=0}^{\lfloor r/2 \rfloor-1} \int_{2m\pi/r - \pi/r}^{2(m+1)\pi/r} |\varphi_\omega(t)| \sum_{k=0}^{\infty} a_{n,k} D^2_{k,1}(t) \, dt$$

$$= \frac{1}{\pi} \sum_{m=0}^{\lfloor r/2 \rfloor-1} \left( \int_{2(m+1)\pi/r - \pi/\pi(r(n+1))}^{2(m+1)\pi/r} + \int_{2(m+1)\pi/r - \pi/(n+1))}^{2(m+1)\pi/r} \right) |\varphi_\omega(t)| \sum_{k=0}^{\infty} a_{n,k} D^2_{k,1}(t) \, dt$$

$$\leq \frac{1}{\pi} \sum_{m=0}^{\lfloor r/2 \rfloor-1} \int_{2(m+1)\pi/r - \pi/r}^{2(m+1)\pi/r} \frac{\left| \varphi_\omega(t) \right|}{\sin \frac{t}{2} \sin \frac{\pi}{2}} A_{n,r} \, dt + \frac{1}{\pi} \sum_{m=0}^{\lfloor r/2 \rfloor-1} \int_{2(m+1)\pi/r - \pi/(r(n+1))}^{2(m+1)\pi/r} \frac{\varphi_\omega(t)}{t} \, dt,$$

and by the estimates $|\sin(t/2)| \geq |t|/\pi$ for $t \in [0, \pi]$ and $|\sin(rt/2)| \geq 2(m+1)-rt/\pi$ for $t \in [2(m+1)\pi/r - \pi/r, 2(m+1)\pi/r - \pi/(r(n+1))]$, we get

$$|I_2(x)| \leq A_{n,r} \sum_{m=0}^{\lfloor r/2 \rfloor-1} \int_{2(m+1)\pi/r - \pi/r}^{2(m+1)\pi/r} \left| \frac{\varphi_\omega(t)}{t} \right| \, dt + \frac{1}{\pi} \sum_{m=0}^{\lfloor r/2 \rfloor-1} \int_{2(m+1)\pi/r - \pi/(r(n+1))}^{2(m+1)\pi/r} \left| \frac{\varphi_\omega(t)}{t} \right| \, dt$$

$$\leq \frac{\pi}{r} A_{n,r} \sum_{m=0}^{\lfloor r/2 \rfloor-1} \left[ \int_{2(m+1)\pi/r - \pi/r}^{2(m+1)\pi/r} \left( \frac{\left| \varphi_\omega(t) \right|}{\omega(t)(2(m+1)\pi/r - t)^\gamma} \right)^p \sin \frac{rt}{2} \right]^{1/p}.$$
\[
\left| I_2(x) \right| \leq \frac{\pi}{r} A_{n,r} \sum_{m=0}^{\lfloor r/2 \rfloor - 1} O_x \left( (n+1)^{\gamma} \right) \left[ \int_{2(m+1)\pi/r - \pi/((n+1))}^{2(m+1)\pi/r} \left( \frac{\omega(t)\left(\frac{2(m+1)\pi}{r} - t\right)^\gamma}{t\left(\frac{2(m+1)\pi}{r} - t\right)\sin \frac{rt}{2}} \right)^q dt \right]^{1/q} \\
+ \frac{1}{\pi} \sum_{m=0}^{\lfloor r/2 \rfloor - 1} O_x \left( (n+1)^{-1/p} \right) O_x \left( (n+1)^{\beta+1/p} \omega \left( \frac{\pi}{n+1} \right) \right).
\]

Since
\[
\left[ \int_{2(m+1)\pi/r - \pi/((n+1))}^{2(m+1)\pi/r} \left( \frac{\omega(t)\left(\frac{2(m+1)\pi}{r} - t\right)^\gamma}{t\left(\frac{2(m+1)\pi}{r} - t\right)\sin \frac{rt}{2}} \right)^q dt \right]^{1/q} \\
= \left[ \int_{\pi/((n+1))}^{\pi/r} \left( \frac{\omega(t)\left(\frac{2(m+1)\pi}{r} - t\right)^\gamma}{t\left(\frac{2(m+1)\pi}{r} - t\right)\sin \frac{rt}{2}} \right)^q dt \right]^{1/q} \\
\leq \left[ \int_{\pi/((n+1))}^{\pi/r} \left( \frac{2\omega(t)\gamma}{t^2\sin \frac{rt}{2}} \right)^q dt \right]^{1/q} = O \left( (n+1)^{\omega \left( \frac{\pi}{n+1} \right)} \left( \frac{\pi}{r(n+1)} \right)^{(\gamma-1)-1/q} \right) = O \left( (n+1)^{1-\gamma+\beta+1/p} \omega \left( \frac{\pi}{n+1} \right) \right)
\]

for \(0 < \gamma < \beta + 1/p\), we have
\[
\left| I_2 \right| = O_x(1) A_{n,r} \sum_{m=0}^{\lfloor r/2 \rfloor - 1} \left( n+1 \right)^{\gamma} \left( n+1 \right)^{1-\gamma+\beta+1/p} \omega \left( \frac{\pi}{n+1} \right) \\
+ O_x(1) \sum_{m=0}^{\lfloor r/2 \rfloor - 1} \left( n+1 \right)^{-1/p} \left( n+1 \right)^{\beta+1/p} \omega \left( \frac{\pi}{n+1} \right)
\]

\[
= O_x \left( (n+1)^{1+\beta+1/p} A_{n,r} \omega \left( \frac{\pi}{n+1} \right) \right) + O_x \left( (n+1)^{\beta} \omega \left( \frac{\pi}{n+1} \right) \right).
\]
Finally, note that applying condition (2.7), we have
\[
\left[(n + 1)A_{n,r}\right]^{-1} = \left[\sum_{l=0}^{n} A_{n,r}\right]^{-1} \leq \left[\sum_{l=0}^{n} \sum_{k=l}^{\infty} \left|a_{n,k} - a_{n,k+r}\right|\right]^{-1} = \left[\sum_{l=0}^{n} \sum_{k=l}^{r+l-1} a_{n,k}\right]^{-1} = O(1),
\]
and our proof is complete.

4.2 Proof of Theorem 2

It is clear that for an odd r,
\[
\tilde{T}_{n,A} f(x) - f\left(x, \frac{\pi}{r(n+1)}\right) = \frac{1}{\pi} \int_{0}^{\pi/(r(n+1))} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \tilde{D}_{k,1}(t) \, dt + \frac{1}{\pi/r} \int_{0}^{\pi/(r(n+1))} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \tilde{D}_{k,1}^0(t) \, dt
\]
\[
+ \frac{1}{\pi} \sum_{m=1}^{[r/2]} \int_{2m\pi/r}^{2m\pi/r+r/2} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \tilde{D}_{k,1}^0(t) \, dt + \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \int_{2m\pi/r}^{2(m+1)\pi/r} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \tilde{D}_{k,1}^0(t) \, dt
\]
\[
= J_1(x) + J_2(x) + J_3(x) + J_4(x)
\]
and for an even r,
\[
\tilde{T}_{n,A} f(x) - f\left(x, \frac{\pi}{r(n+1)}\right) = -\frac{1}{\pi} \int_{0}^{\pi/(r(n+1))} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \tilde{D}_{k,1}(t) \, dt + \frac{1}{\pi/r} \int_{0}^{\pi/(r(n+1))} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \tilde{D}_{k,1}^0(t) \, dt
\]
\[
+ \frac{1}{\pi} \sum_{m=1}^{[r/2]-1} \int_{2m\pi/r}^{2m\pi/r+r/2} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \tilde{D}_{k,1}^0(t) \, dt + \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \int_{2m\pi/r}^{2(m+1)\pi/r} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \tilde{D}_{k,1}^0(t) \, dt
\]
\[
= J_1(x) + J_2(x) + J_3(x) + J_4(x).
\]

Then
\[
\left|\tilde{T}_{n,A} f(x) - f\left(x, \frac{\pi}{r(n+1)}\right)\right| \leq |J_1(x)| + |J_2(x)| + |J_3(x)| + |J_3'(x)| + |J_4(x)|,
\]
and by Lemma 2
\[
|J_1(x)| \leq \frac{1}{\pi} \int_{0}^{\pi/(r(n+1))} \left|\psi_x(t)\right| \left|\sum_{k=0}^{\infty} a_{n,k} \tilde{D}_{k,1}(t)\right| \, dt \leq \frac{1}{2\pi} \int_{0}^{\pi/(r(n+1))} \left|\psi_x(t)\right| \sum_{k=0}^{\infty} a_{n,k} \frac{1}{2} k(k+1) |t| \, dt.
\]
whence by (2.15) and (2.8)
\[ |J_1(x)| \leq O((n + 1)^2) \int_0^{\pi/(r(n+1))} \left| \psi_x(t) \right| |t| \, dt \]
\[ \leq O((n + 1)^2) \left\{ \int_0^{\pi/(r(n+1))} \left( \frac{t |\psi_x(t)|}{\overline{\omega}(t)} \right)^p \sin^{\beta p} \frac{rt}{2} \, dt \right\}^{1/p} \left\{ \int_0^{\pi/(r(n+1))} \left( \frac{\overline{\omega}(t)}{\sin^{\beta} \frac{rt}{2}} \right)^q \, dt \right\}^{1/q} \]
\[ \leq O((n + 1)^2) O_x((n + 1)^{-1}) \overline{\omega}\left( \frac{\pi}{r(n+1)} \right) \left\{ \int_0^{\pi/(r(n+1))} \left( \frac{\pi}{rt} \right)^{\beta q} \, dt \right\}^{1/q} \]
\[ = O_x(n + 1) \overline{\omega}\left( \frac{\pi}{r(n+1)} \right) \left( \frac{\pi}{r(n+1)} \right)^{1/q - \beta} = O_x((n + 1)^{\beta + 1/p}) \overline{\omega}\left( \frac{\pi}{n + 1} \right) \]
for $\beta < 1 - 1/p$. Next, by Lemmas 2 and 5
\[ |J_2(x)| + |J_3(x)| + |J'_3(x)| \]
\[ \leq \frac{1}{\pi} \left( \sum_{m=1}^{[r/2]} \int_{2m\pi/r}^{2m\pi/r + \pi/(r(n+1))} |\psi_x(t)| \left| \sum_{k=0}^{\infty} a_{n,k} \overline{D}_{k,1}(t) \right| \, dt \right) + \frac{1}{\pi} \int_{2m\pi/r}^{2m\pi/r + \pi/(r(n+1))} |\psi_x(t)| \left| \sum_{m=0}^{[r/2]} \int_{2m\pi/r + \pi/(r(n+1))}^{2m\pi/r + \pi/r} \frac{\psi_x(t)}{t} \, dt \right| \, dt \]
\[ \leq \frac{1}{2} \sum_{m=1}^{[r/2]} \int_{2m\pi/r}^{2m\pi/r + \pi/(r(n+1))} \frac{|\psi_x(t)|}{t} \, dt + \frac{1}{\pi} \sum_{m=0}^{[r/2]} \int_{2m\pi/r + \pi/(r(n+1))}^{2m\pi/r + \pi/r} \frac{|\psi_x(t)|}{t} \, dt \]
Using the estimates $|\sin t/2| \geq |t|/\pi$ for $t \in [0, \pi]$ and $|\sin rt/2| \geq rt/\pi - 2m$ for $t \in [2m\pi/r, 2m\pi/r + \pi/r]$, where $m \in \{0, \ldots, [r/2]\}$, we obtain
\[ |J_2(x)| + |J_3(x)| + |J'_3(x)| \]
\[ \leq \frac{1}{2} \sum_{m=1}^{[r/2]} \int_{2m\pi/r}^{2m\pi/r + \pi/(r(n+1))} \frac{|\psi_x(t)|}{t} \, dt + A_{n,r} \sum_{m=0}^{[r/2]} \int_{2m\pi/r + \pi/(r(n+1))}^{2m\pi/r + \pi/r} \frac{|\psi_x(t)|}{t} \, dt \]
\[ = \frac{1}{2} \sum_{m=1}^{[r/2]} \int_{2m\pi/r}^{2m\pi/r + \pi/(r(n+1))} \frac{|\psi_x(t)|}{t} \, dt + A_{n,r} \sum_{m=0}^{[r/2]} \int_{2m\pi/r + \pi/(r(n+1))}^{2m\pi/r + \pi/r} \frac{|\psi_x(t)|}{t} \, dt \]
\[ = \frac{1}{2} \sum_{m=1}^{[r/2]} \int_{2m\pi/r}^{2m\pi/r + \pi/(r(n+1))} \left( \frac{|\psi_x(t)|}{\overline{\omega}(t)} \right)^p \sin^{\beta p} \frac{rt}{2} \, dt \left\{ \int_{2m\pi/r}^{2m\pi/r + \pi/(r(n+1))} \left( \frac{\overline{\omega}(t)}{t \sin^{\beta} \frac{rt}{2}} \right)^q \, dt \right\}^{1/q} \]
Lith. Math. J., 60(4):494–512, 2020.
Further, analogously as in the previous proof, by Lemma 1, (2.12), (2.9), and (2.10) with Lemma 7 we have

\[ |J_2(x)| + |J_3(x)| + |J_3'(x)| \]

\[ = O_x(1) \sum_{m=1}^{[r/2]} (n+1)^{-1/p} (n+1)^{\beta + 1/p} \phi \left( \frac{\pi}{n+1} \right) \]

\[ + O_x(1)A_{n,r} \sum_{m=0}^{[r/2]} (n+1)^\gamma \left[ \int_{2m\pi/r + \pi/(r(n+1))}^{2m\pi/r + \pi/r} \left( \frac{\tilde{\omega}(t)(t - \frac{2m\pi}{r})}{t(\frac{rt}{\pi} - 2m)\sin \frac{rt}{2}} \right)^q dt \right]^{1/q} \]

\[ = O_x(1)(n+1)^{\beta} \phi \left( \frac{\pi}{n+1} \right) + O_x(1)A_{n,r} \sum_{m=0}^{[r/2]} (n+1)^\gamma (n+1)^{1-\gamma + \beta + 1/p} \phi \left( \frac{\pi}{n+1} \right) \]

\[ = O_x(n+1)^{\beta} \phi \left( \frac{\pi}{n+1} \right) + O_x(n+1)^{1+\beta + 1/p} A_{n,r} \phi \left( \frac{\pi}{n+1} \right). \]

Similarly, by Lemmas 2, 5 and the estimates \(|\sin(t/2)| > |t|/\pi\) for \(t \in [0, \pi]\) and \(|\sin(rt/2)| \geq 2(m+1) - rt/\pi\) for \(t \in [2(m+1)\pi/r - \pi/r, 2(m+1)\pi/r - \pi/(r(n+1))]\), where \(m \in \{0, \ldots, [r/2] - 1\}\), we get

\[ |J_4(x)| \leq \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \left| \int_{2m\pi/r + \pi/r}^{2(m+1)\pi/r} \left| \psi_x(t) \right| \sum_{k=0}^{\infty} a_{n,k} \tilde{D}_{k,1}^2(t) \right| dt \]

\[ = \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \left( \int_{2m\pi/r + \pi/r}^{2(m+1)\pi/r} + \int_{2(m+1)\pi/r - \pi/(r(n+1))}^{2(m+1)\pi/r} \right) \left| \psi_x(t) \right| \sum_{k=0}^{\infty} a_{n,k} \tilde{D}_{k,1}^2(t) \left| dt \right| \]

\[ \leq \frac{1}{2} \sum_{m=0}^{[r/2]-1} \int_{2m\pi/r + \pi/r}^{2(m+1)\pi/r} \left| \psi_x(t) \right| \left| \frac{t}{\pi} \right| \left| \sin \frac{rt}{2} \right| dt + \frac{1}{\pi} A_{n,r} \sum_{m=0}^{[r/2]-1} \left( \int_{2m\pi/r + \pi/r}^{2(m+1)\pi/r - \pi/(r(n+1))} \left| \psi_x(t) \right| \left| \frac{t}{\pi} \right| \left| \sin \frac{rt}{2} \right| dt \right) \]

\[ \leq \frac{1}{2} \sum_{m=0}^{[r/2]-1} \int_{2(m+1)\pi/r - \pi/(r(n+1))}^{2(m+1)\pi/r} \left| \psi_x(t) \right| \left| \frac{t}{\pi} \right| \left| \sin \frac{rt}{2} \right| dt + A_{n,r} \sum_{m=0}^{[r/2]-1} \left( \int_{2(m+1)\pi/r - \pi/(r(n+1))}^{2(m+1)\pi/r - \pi/r} \left| \psi_x(t) \right| \left| \frac{t}{\pi} \right| \left| \sin \frac{rt}{2} \right| dt \right) \]

\[ \leq \frac{1}{2} \sum_{m=0}^{[r/2]-1} \left[ \int_{2(m+1)\pi/r - \pi/(r(n+1))}^{2(m+1)\pi/r} \left( \left| \psi_x(t) \right| \left| \frac{t}{\pi} \right| \left| \sin \frac{rt}{2} \right| \right)^p \right]^{1/p} \]
Analogously, as in the proofs of Theorems 1 and 2, we consider an odd
4.3 Proof of Theorem 3
and for an even

\[
T_{n,Af}(x) - \bar{f}(x)
= \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \int_{2m\pi/r}^{2m\pi/r+\pi/r} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \tilde{D}_{k,1}^o(t) \, dt
+ \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \int_{2m\pi/r+\pi/r}^{2(m+1)\pi/r} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \tilde{D}_{k,1}^o(t) \, dt
= J_3''(x) + J_4(x).
\]
Hence

\[ | \tilde{T}_{n,A} f(x) - \tilde{f}(x) | \leq | J''_3(x) | + | J'''_3(x) | + | J_4(x) |. \]

From the previous proof we have

\[ | J_4(x) | = O_x \left( (n + 1)^{\frac{\beta}{n+1}} \left( \frac{\pi}{n+1} \right) \right) + O_x \left( (n + 1)^{1+\frac{\beta+1/p}{n+1}} A_{n,r} \left( \frac{\pi}{n+1} \right) \right) \]

for \(0 < \gamma < \beta + 1/p\). Further, we can observe that the quantities \( J''_3(x) \) and \( J'''_3(x) \) are similar to the quantities \( | J'_3(x) | \) and \( | J_3'(x) | \) from the previous proof; the differences are in the ranges of \( m \) only. Therefore we immediately obtain the same estimates of these terms. Thus our proof is complete.

Conclusion. In our theorems, we considered a very general class of matrices defining the means \( T_{n,A} f \) in particular cases that appear in the papers cited. We constructed the measures of such approximations by quantities based on \( r \)-differences of the entries of \( A \). The degrees of approximations obtained in some cases (see Remark 1) are better than those obtained in the earlier results.

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