Boundedness of fractional maximal operator and its commutators on generalized Orlicz-Morrey spaces

Vagif S. Guliyev\(^a\), Fatih Deringoz\(^a\)

\(^a\)Department of Mathematics, Ahi Evran University, Kirsehir, Turkey
\(^b\)Institute of Mathematics and Mechanics, Baku, Azerbaijan

Abstract

We consider generalized Orlicz-Morrey spaces \(M_{\Phi,\psi}(\mathbb{R}^n)\) including their weak versions \(WM_{\Phi,\psi}(\mathbb{R}^n)\). We find the sufficient conditions on the pairs \((\varphi_1, \varphi_2)\) and \((\Phi, \Psi)\) which ensures the boundedness of the fractional maximal operator \(M_{\alpha}\) from \(M_{\Phi,\varphi_1}(\mathbb{R}^n)\) to \(M_{\Psi,\varphi_2}(\mathbb{R}^n)\) and from \(M_{\Phi,\varphi_1}(\mathbb{R}^n)\) to \(WM_{\Psi,\varphi_2}(\mathbb{R}^n)\). As applications of those results, the boundedness of the commutators of the fractional maximal operator \(M_{b,\alpha}\) with \(b \in BMO(\mathbb{R}^n)\) on the spaces \(M_{\Phi,\varphi}(\mathbb{R}^n)\) is also obtained. In all the cases the conditions for the boundedness are given in terms of supremal-type inequalities on weights \(\varphi(x,r)\), which do not assume any assumption on monotonicity of \(\varphi(x,r)\) on \(r\).

AMS Mathematics Subject Classification: 42B20, 42B25, 42B35; 46E30
Key words: generalized Orlicz-Morrey space; fractional maximal operator; commutator, BMO

1 Introduction

Boundedness of classical operators of the Real analysis, such as the maximal operator, fractional maximal operator, Riesz potential and the singular integral operators etc, have been extensively investigated in various function spaces. Results on weak and strong type inequalities for operators of this kind in Lebesgue

\(^1\) The research of V. Guliyev and F. Deringoz were partially supported by the grant of Ahi Evran University Scientific Research Projects (PYO.FEN.4003.13.003) and (PYO.FEN.4003-2.13.007).
E-mail addresses: vagif@guliyev.com (V.S. Guliyev), fderingoz@ahievran.edu.tr (F. Deringoz).
spaces are classical and can be found for example in [3, 38, 39]. These boundedness extended to several function spaces which are generalizations of $L_p$-spaces, for example, Orlicz spaces, Morrey spaces, Lorentz spaces, Herz spaces, etc.

Orlicz spaces, introduced in [33, 34], are generalizations of Lebesgue spaces $L_p$. They are useful tools in harmonic analysis and its applications. For example, the Hardy-Littlewood maximal operator is bounded on $L_p$ for $1 < p < \infty$, but not on $L_1$. Using Orlicz spaces, we can investigate the boundedness of the maximal operator near $p = 1$ more precisely (see [6, 21, 22]).

On the other hand, Morrey spaces were introduced in [29] to estimate solutions of partial differential equations, and studied by many authors.

Let $f \in L_{\text{loc}}^1(\mathbb{R}^n)$. The fractional maximal operator $M_\alpha$ and the Riesz potential operator $I_\alpha$ are defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)|dy, \quad 0 \leq \alpha < n,$$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}dy, \quad 0 < \alpha < n.$$

If $\alpha = 0$, then $M \equiv M_0$ is the Hardy-Littlewood maximal operator.

The operator $M_\alpha$ is of weak type $(p, np/(n - \alpha p))$ if $1 \leq p \leq n/\alpha$ and of strong type $(p, np/(n - \alpha p))$ if $1 < p \leq n/\alpha$. Also the operator $I_\alpha$ is of weak type $(p, np/(n - \alpha p))$ if $1 \leq p < n/\alpha$ and of strong type $(p, np/(n - \alpha p))$ if $1 < p < n/\alpha$.

The boundedness of $M_\alpha$ and $I_\alpha$ from Orlicz space $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ was studied by Cianchi [6]. For boundedness of $M_\alpha$ and $I_\alpha$ on Morrey spaces $M_{p,\lambda}(\mathbb{R}^n)$, see Peetre (Spanne) [35], Adams [1].

The definition of generalized Orlicz-Morrey spaces introduced in [9] and used here is different from that of Sawano et al. [37] and Nakai [31, 32].

In [9], the boundedness of the maximal operator $M$ and the Calderón-Zygmund operator $T$ from one generalized Orlicz-Morrey space $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2}(\mathbb{R}^n)$ and from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to the weak space $WM_{\Phi,\varphi_2}(\mathbb{R}^n)$ was proved (see, also [19]). Also in [18] the authors prove the boundedness of the Riesz potential operator $I_\alpha$ and its commutator $[b, I_\alpha]$ from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2}(\mathbb{R}^n)$ and from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_{\Phi,\varphi_2}(\mathbb{R}^n)$.

The main purpose of this paper is to find sufficient conditions on the general Young functions $\Phi, \Psi$ and the functions $\varphi_1, \varphi_2$ which ensure the boundedness of $M_\alpha$ from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2}(\mathbb{R}^n)$, from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_{\Phi,\varphi_2}(\mathbb{R}^n)$ and in the case $b \in BMO$ the boundedness of the commutator of the fractional maximal operator $M_{b,\alpha}$ from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2}(\mathbb{R}^n)$.

In the next section we recall the definitions of Orlicz and Morrey spaces and give the definition of generalized Orlicz-Morrey spaces in Section 3. In Section 4 and Section 5 the results on boundedness of $M_\alpha$ and its commutator operator $M_{b,\alpha}$ is obtained.
By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant $C$ independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that $A$ and $B$ are equivalent.

2 Some preliminaries on Orlicz and Morrey spaces

In the study of local properties of solutions to partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $M_{p,\lambda}(\mathbb{R}^n)$ play an important role, see [12]. Introduced by C. Morrey [29] in 1938, they are defined by the norm

$$\|f\|_{M_{p,\lambda}} := \sup_{x, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,r))},$$

where $0 \leq \lambda \leq n$, $1 \leq p < \infty$. Here and everywhere in the sequel $B(x, r)$ stands for the ball in $\mathbb{R}^n$ of radius $r$ centered at $x$. Let $|B(x, r)|$ be the Lebesgue measure of the ball $B(x, r)$ and $|B(x, r)| = v_n r^n$, where $v_n = |B(0,1)|$.

Note that $M_{p,0} = L^p(\mathbb{R}^n)$ and $M_{p,n} = L^\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where $\Theta$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $W M_{p,\lambda} \equiv W M_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in W L^1_{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{W M_{p,\lambda}} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{WL^p(B(x,r))} < \infty,$$

where $WL^p(B(x,r))$ denotes the weak $L^p$-space.

We refer in particular to [25] for the classical Morrey spaces.

We recall the definition of Young functions.

**Definition 2.1.** A function $\Phi : [0, +\infty) \to [0, \infty]$ is called a Young function if $\Phi$ is convex, left-continuous, $\lim_{r \to +0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r \to +\infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0, +\infty)$ such that $\Phi(s) = +\infty$, then $\Phi(r) = +\infty$ for $r \geq s$.

Let $\mathcal{Y}$ be the set of all Young functions $\Phi$ such that

$$0 < \Phi(r) < +\infty \quad \text{for} \quad 0 < r < +\infty$$

If $\Phi \in \mathcal{Y}$, then $\Phi$ is absolutely continuous on every closed interval in $[0, +\infty)$ and bijective from $[0, +\infty)$ to itself.

**Definition 2.2.** (Orlicz Space). For a Young function $\Phi$, the set

$$L_{\Phi}(\mathbb{R}^n) = \left\{ f \in L^1_{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|) dx < +\infty \text{ for some } k > 0 \right\}$$
is called Orlicz space. If $\Phi(r) = r^p$, $1 \leq p < \infty$, then $L_\Phi(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. If $\Phi(r) = 0$, $(0 \leq r \leq 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L_\Phi(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$. The space $L^\text{loc}_\Phi(\mathbb{R}^n)$ endowed with the natural topology is defined as the set of all functions $f$ such that $f\chi_B \in L_\Phi(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$. We refer to the books\[23, 24, 36\]for the theory of Orlicz Spaces.

Note that, $L_\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm
\[
\|f\|_{L_\Phi} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\},
\]
so that
\[
\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\|f\|_{L_\Phi}}\right) dx \leq 1.
\]
For a measurable set $\Omega \subset \mathbb{R}^n$, a measurable function $f$ and $t > 0$, let
\[
m(\Omega, f, t) = |\{x \in \Omega : |f(x)| > t\}|.
\]
In the case $\Omega = \mathbb{R}^n$, we shortly denote it by $m(f, t)$.

**Definition 2.3.** The weak Orlicz space
\[
WL_\Phi(\mathbb{R}^n) := \{f \in L^\text{loc}_1(\mathbb{R}^n) : \|f\|_{WL_\Phi} < +\infty\}
\]
is defined by the norm
\[
\|f\|_{WL_\Phi} = \inf \left\{ \lambda > 0 : \sup_{t \geq 0} \Phi(t)m\left(\frac{f}{\lambda}, t\right) \leq 1 \right\}.
\]
For a Young function $\Phi$ and $0 \leq s \leq +\infty$, let
\[
\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\} \quad (\inf\emptyset = +\infty).
\]
If $\Phi \in \mathcal{Y}$, then $\Phi^{-1}$ is the usual inverse function of $\Phi$. We note that
\[
\Phi(\Phi^{-1}(r)) \leq r \leq \Phi^{-1}(\Phi(r)) \quad \text{for } 0 \leq r < +\infty. \quad (2.1)
\]
A Young function $\Phi$ is said to satisfy the $\Delta_2$-condition, denoted by $\Phi \in \Delta_2$, if
\[
\Phi(2r) \leq k\Phi(r) \quad \text{for } r > 0
\]
for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function $\Phi$ is said to satisfy the $\nabla_2$-condition, denoted also by $\Phi \in \nabla_2$, if
\[
\Phi(r) \leq \frac{1}{2k}\Phi(kr), \quad r \geq 0,
\]
for some $k > 1$. The function $\Phi(r) = r$ satisfies the $\Delta_2$-condition but does not satisfy the $\nabla_2$-condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both the
conditions. The function \( \Phi(r) = e^r - r - 1 \) satisfies the \( \nabla_2 \)-condition but does not satisfy the \( \Delta_2 \)-condition.

For a Young function \( \Phi \), the complementary function \( \tilde{\Phi}(r) \) is defined by

\[
\tilde{\Phi}(r) = \left\{ \begin{array}{ll}
\sup\{rs - \Phi(s) : s \in [0, \infty) \} &, r \in [0, \infty) \\
+\infty &, r = +\infty.
\end{array} \right.
\]

The complementary function \( \tilde{\Phi} \) is also a Young function and \( \tilde{\Phi} = \Phi \). If \( \Phi(r) = r \), then \( \tilde{\Phi}(r) = 0 \) for \( 0 \leq r \leq 1 \) and \( \tilde{\Phi}(r) = +\infty \) for \( r > 1 \). If \( 1 < p < \infty \), \( 1/p + 1/p' = 1 \) and \( \Phi(r) = r^p/p \), then \( \tilde{\Phi}(r) = r^{p'/p'} \). If \( \Phi(r) = e^r - r - 1 \), then \( \tilde{\Phi}(r) = (1 + r) \log(1 + r) - r \). Note that \( \Phi \in \nabla_2 \) if and only if \( \tilde{\Phi} \in \Delta_2 \). It is known that

\[
r \leq \Phi^{-1}(r) - 1(r) \leq 2r \quad \text{for } r \geq 0. \tag{2.2}
\]

Note that Young functions satisfy the properties

\[
\Phi(\alpha t) \leq \alpha \Phi(t)
\]

for all \( 0 \leq \alpha \leq 1 \) and \( 0 \leq t < \infty \), and

\[
\Phi(\beta t) \geq \beta \Phi(t)
\]

for all \( \beta > 1 \) and \( 0 \leq t < \infty \).

The following analogue of the H"{o}lder inequality is known, see [40].

**Theorem 2.4.** [40] For a Young function \( \Phi \) and its complementary function \( \tilde{\Phi} \), the following inequality is valid

\[
\|fg\|_{L^1(\mathbb{R}^n)} \leq 2\|f\|_{L^{\Phi}(\mathbb{R}^n)}\|g\|_{L^{\tilde{\Phi}}(\mathbb{R}^n)}.
\]

The following lemma is valid.

**Lemma 2.5.** [3, 27] Let \( \Phi \) be a Young function and \( B \) a set in \( \mathbb{R}^n \) with finite Lebesgue measure. Then

\[
\left\| \chi_B \right\|_{W_{L^{\Phi}(\mathbb{R}^n)}} = \left\| \chi_B \right\|_{L^{\Phi}(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}(|B|^{-1})}.
\]

In the next sections where we prove our main estimates, we use the following lemma, which follows from Theorem 2.4, Lemma 2.5 and the inequality (2.2).

**Lemma 2.6.** For a Young function \( \Phi \) and \( B = B(x, r) \), the following inequality is valid

\[
\|f\|_{L^1(B)} \leq 2|B|\Phi^{-1}(|B|^{-1}) \|f\|_{L^{\Phi}(B)}.
\]
Necessary and sufficient conditions on \((\Phi, \Psi)\) for the boundedness of \(M_\alpha\) and \(I_\alpha\) from Orlicz spaces \(L_\Phi(\mathbb{R}^n)\) to \(L_\Psi(\mathbb{R}^n)\) and \(L_\Phi(\mathbb{R}^n)\) to \(W L_\Psi(\mathbb{R}^n)\) have been obtained in [6, Theorem 1 and 2]. In the statement of the theorems, \(\Psi_p\) is the Young function associated with the Young function \(\Psi\) and \(p \in (1, \infty]\) whose Young conjugate is given by

\[
\tilde{\Psi}_p(s) = \int_0^s r^{p'-1}(B_p^{-1}(r^{p'}))^p dr,
\]  

(2.3)

where

\[
B_p(s) = \int_0^s \frac{\Psi(t)}{t^{1+p'}} dt
\]

and \(p'\), the Holder conjugate of \(p\), equals either \(p/(p - 1)\) or 1, according to whether \(p < \infty\) or \(p = \infty\) and \(\Phi_p\) denotes the Young function defined by

\[
\Phi_p(s) = \int_0^s r^{p'-1}(A_p^{-1}(r^{p'}))^p dr,
\]  

(2.4)

where

\[
A_p(s) = \int_0^s \frac{\tilde{\Phi}(t)}{t^{1+p'}} dt.
\]

Recall that, if \(\Phi\) and \(\Psi\) are functions from \([0, \infty)\) into \([0, \infty]\), then \(\Psi\) is said to dominate \(\Phi\) globally if a positive constant \(c\) exists such that \(\Phi(s) \leq \Psi(cs)\) for all \(s \geq 0\).

**Theorem 2.7.** [6]  
(i) The fractional maximal operator \(M_\alpha\) is bounded from \(L_\Phi(\mathbb{R}^n)\) to \(W L_\Psi(\mathbb{R}^n)\) if and only if

\[
\Phi\ \text{dominates globally the function } Q,
\]

(2.5)

whose inverse is given by

\[
Q^{-1}(r) = r^{\alpha/n}\Psi^{-1}(r).
\]

(ii) The fractional maximal operator \(M_\alpha\) is bounded from \(L_\Phi(\mathbb{R}^n)\) to \(L_\Psi(\mathbb{R}^n)\) if and only if

\[
\int_0^1 \frac{\Psi(t)}{t^{1+n/(n-\alpha)}} dt < \infty \and \Phi\ \text{dominates globally the function } \Psi_{n/\alpha}.
\]

(2.6)

**Theorem 2.8.** [6] Let \(0 < \alpha < n\). Let \(\Phi\) and \(\Psi\) Young functions and let \(\Phi_{n/\alpha}\) and \(\Psi_{n/\alpha}\) be the Young functions defined as in (2.4) and (2.3), respectively. Then

(i) The Riesz potential \(I_\alpha\) is bounded from \(L_\Phi(\mathbb{R}^n)\) to \(W L_\Psi(\mathbb{R}^n)\) if and only if

\[
\int_0^1 \tilde{\Phi}(t)/t^{1+n/(n-\alpha)} dt < \infty \and \Phi_{n/\alpha}\ \text{dominates } \Psi \text{ globally.}
\]

(2.7)
(ii) The Riesz potential $I_\alpha$ is bounded from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ if and only if
\[
\int_0^1 \tilde{\Phi}(t)/t^{1+n/(n-\alpha)} dt < \infty, \quad \int_0^1 \Psi(t)/t^{1+n/(n-\alpha)} dt < \infty,
\]
$\Phi$ dominates $\Psi_{n/\alpha}$ globally and $\Phi_{n/\alpha}$ dominates $\Psi$ globally. \hfill (2.8)

3 Generalized Orlicz-Morrey Spaces

Definition 3.1. (Orlicz-Morrey Space). For a Young function $\Phi$ and $0 \leq \lambda \leq n$, we denote by $M_{\Phi,\lambda}(\mathbb{R}^n)$ the Orlicz-Morrey space, the space of all functions $f \in L_{\Phi}^{\text{loc}}(\mathbb{R}^n)$ with finite quasinorm
\[
\|f\|_{M_{\Phi,\lambda}} = \sup_{x \in \mathbb{R}^n, r > 0} \Phi^{-1}(r^{-\lambda}) \|f\|_{L_\Phi(B(x,r))}.
\]

Note that $M_{\Phi,0} = L_\Phi(\mathbb{R}^n)$ and if $\Phi(r) = r^p$, $1 \leq p < \infty$, then $M_{\Phi,\lambda}(\mathbb{R}^n) = M_{p,\lambda}(\mathbb{R}^n)$.

We also denote by $WM_{\Phi,\lambda}(\mathbb{R}^n)$ the weak Orlicz-Morrey space of all functions $f \in WL_{\Phi}^{\text{loc}}(\mathbb{R}^n)$ for which
\[
\|f\|_{WM_{\Phi,\lambda}} = \sup_{x \in \mathbb{R}^n, r > 0} \Phi^{-1}(r^{-\lambda}) \|f\|_{WL_\Phi(B(x,r))} < \infty,
\]

where $WL_\Phi(B(x,r))$ denotes the weak $L_\Phi$-space of measurable functions $f$ for which
\[
\|f\|_{WL_\Phi(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_\Phi(\mathbb{R}^n)}.
\]

Definition 3.2. (generalized Orlicz-Morrey Space) Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $\Phi$ any Young function. We denote by $M_{\Phi,\varphi}(\mathbb{R}^n)$ the generalized Orlicz-Morrey space, the space of all functions $f \in L_{\Phi}^{\text{loc}}(\mathbb{R}^n)$ with finite quasinorm
\[
\|f\|_{M_{\Phi,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x,r)^{-1}\Phi^{-1}(r^{-\lambda}) \|f\|_{L_\Phi(B(x,r))}.
\]

Also by $WM_{\Phi,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Orlicz-Morrey space of all functions $f \in WL_{\Phi}^{\text{loc}}(\mathbb{R}^n)$ for which
\[
\|f\|_{WM_{\Phi,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x,r)^{-1}\Phi^{-1}(r^{-\lambda}) \|f\|_{WL_\Phi(B(x,r))} < \infty.
\]
According to this definition, we recover the spaces $M_{\Phi,\lambda}$ and $WM_{\Phi,\lambda}$ under the choice $\varphi(x, r) = \Phi^{-1}\left(\frac{r-n}{r-\lambda}\right)$:

$$M_{\Phi,\lambda} = M_{\Phi,\varphi}\big|_{\varphi(x,r) = \Phi^{-1}\left(\frac{r-n}{r-\lambda}\right)}; \quad WM_{\Phi,\lambda} = WM_{\Phi,\varphi}\big|_{\varphi(x,r) = \Phi^{-1}\left(\frac{r-n}{r-\lambda}\right)}.$$  

According to this definition, we recover the generalized Morrey spaces $M_{p,\varphi}$ and weak generalized Morrey spaces $WM_{p,\varphi}$ under the choice $\Phi(r) = r^p$, $1 \leq p < \infty$:

$$M_{p,\varphi} = M_{\Phi,\varphi}\big|_{\Phi(r) = r^p}, \quad WM_{p,\varphi} = WM_{\Phi,\varphi}\big|_{\Phi(r) = r^p}.$$  

Sufficient conditions on $\varphi$ for the boundedness of $M_{\alpha}$ and $I_{\alpha}$ in generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ have been obtained in [4, 13, 14, 15, 17, 20, 28, 30].

4  Boundedness of the fractional maximal operator in the spaces $M_{\Phi,\varphi}(\mathbb{R}^n)$

In this section sufficient conditions on the pairs $(\varphi_1, \varphi_2)$ and $(\Phi, \Psi)$ for the boundedness of $M_{\alpha}$ from one generalized Orlicz-Morrey spaces $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to another $M_{\Psi,\varphi_2}(\mathbb{R}^n)$ and from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to the weak space $WM_{\Psi,\varphi_2}(\mathbb{R}^n)$ have been obtained. At first we recall some supremal inequalities which we use at the proof of our main theorem.

Let $v$ be a weight. We denote by $L_{\infty,v}(0, \infty)$ the space of all functions $g(t)$, $t > 0$ with finite norm

$$\|g\|_{L_{\infty,v}(0, \infty)} = \sup_{t > 0} v(t)|g(t)|$$

and $L_{\infty}(0, \infty) \equiv L_{\infty,1}(0, \infty)$. Let $\mathfrak{M}(0, \infty)$ be the set of all Lebesgue-measurable functions on $(0, \infty)$ and $\mathfrak{M}^+(0, \infty)$ its subset of all non-negative functions on $(0, \infty)$. We denote by $\mathfrak{M}^+(0, \infty; \uparrow)$ the cone of all functions in $\mathfrak{M}^+(0, \infty)$ which are non-decreasing on $(0, \infty)$ and

$$\mathcal{A} = \left\{ \varphi \in \mathfrak{M}^+(0, \infty; \uparrow) : \lim_{t \to 0^+} \varphi(t) = 0 \right\}.$$  

Let $u$ be a continuous and non-negative function on $(0, \infty)$. We define the supremal operator $S_u$ on $g \in \mathfrak{M}(0, \infty)$ by

$$(S_u g)(t) := \|u g\|_{L_{\infty}(t, \infty)}, \quad t \in (0, \infty).$$

The following theorem was proved in [4].
Theorem 4.1. Let $v_1, v_2$ be non-negative measurable functions satisfying $0 < \|v_1\|_{L_{\infty}(t, \infty)} < \infty$ for any $t > 0$ and let $u$ be a continuous non-negative function on $(0, \infty)$. Then the operator $S_u$ is bounded from $L_{\infty, v_1}(0, \infty)$ to $L_{\infty, v_2}(0, \infty)$ on the cone $A$ if and only if

$$\left\| v_2 S_u \left( \frac{1}{\|v_1\|_{L_{\infty}(t, \infty)}} \right) \right\|_{L_{\infty}(0, \infty)} < \infty. \quad (4.1)$$

For the Riesz potential the following local estimate was proved in [18].

Lemma 4.2. Let $0 < \alpha < n$, $\Phi$ and $\Psi$ Young functions, $f \in L_{\Phi}^\text{loc}(\mathbb{R}^n)$ and $B = B(x_0, r)$. If $(\Phi, \Psi)$ satisfy the conditions (2.8), then

$$\| I_\alpha f \|_{L_\Phi(B)} \lesssim \frac{1}{\Psi^{-1}(r^{-n})} \int_{2r}^\infty \|f\|_{L_\Phi(B(x_0, t))} \Psi^{-1}(t^{-n}) \frac{dt}{t}. \quad (4.2)$$

If $(\Phi, \Psi)$ satisfy the conditions (2.7), then

$$\| I_\alpha f \|_{W L_\Phi(B)} \lesssim \frac{1}{\Psi^{-1}(r^{-n})} \int_{2r}^\infty \|f\|_{L_\Phi(B(x_0, t))} \Psi^{-1}(t^{-n}) \frac{dt}{t}. \quad (4.3)$$

For the fractional maximal operator the following local estimate is valid.

Lemma 4.3. Let $\Phi$ and $\Psi$ Young functions, $f \in L_{\Phi}^\text{loc}(\mathbb{R}^n)$ and $B = B(x, r)$. If $(\Phi, \Psi)$ satisfy the conditions (2.5), then

$$\| M_\alpha f \|_{W L_\Phi(B)} \lesssim \|f\|_{L_\Phi(B(x, 2r))} + \frac{1}{\Psi^{-1}(r^{-n})} \sup_{t > 2r} t^{-\alpha} \|f\|_{L_1(B(x, t))}. \quad (4.4)$$

If $(\Phi, \Psi)$ satisfy the conditions (2.6), then

$$\| M_\alpha f \|_{L_\Phi(B)} \lesssim \|f\|_{L_\Phi(B(x, 2r))} + \frac{1}{\Psi^{-1}(r^{-n})} \sup_{t > 2r} t^{-\alpha} \|f\|_{L_1(B(x, t))}. \quad (4.5)$$

Proof. Let $(\Phi, \Psi)$ satisfy the conditions (2.6). We put $f = f_1 + f_2$, where $f_1 = f \chi_{B(x, 2r)}$ and $f_2 = f \chi_{B(x, 2r)}$. Then we get

$$\| M_\alpha f \|_{L_\Phi(B)} \leq \| M_\alpha f_1 \|_{L_\Phi(B)} + \| M_\alpha f_2 \|_{L_\Phi(B)}.$$

By the boundedness of the operator $M_\alpha$ from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ (see Theorem 2.7) we have

$$\| M_\alpha f_1 \|_{L_\Psi(B)} \lesssim \|f\|_{L_\Phi(B(x, 2r))}.$$

Let $y$ be an arbitrary point from $B$. If $B(y, t) \cap B(x, 2r) \neq \emptyset$, then $t > r$. Indeed, if $z \in B(y, t) \cap B(x, 2r)$, then $t > |y-z| \geq |x-z|-|x-y| > 2r-r = r$.

On the other hand, $B(y, t) \cap B(x, 2r) \subseteq B(x, 2t)$. Indeed, if $z \in B(y, t) \cap B(x, 2r)$, then we get $|x-z| \leq |y-z| + |x-y| < t + r < 2t$.
Hence
\[
M^\alpha f_2(y) = \sup_{t > 0} \frac{1}{|B(y,t)|^{1-\frac{n}{\alpha}}} \int_{B(y,t) \cap B(x,2r)} |f(z)|dz
\leq 2^{n-\alpha} \sup_{t > r} \frac{1}{|B(x,2t)|^{1-\frac{n}{\alpha}}} \int_{B(x,2t)} |f(z)|dz
= 2^{n-\alpha} \sup_{t > 2r} \frac{1}{|B(x,t)|^{1-\frac{n}{\alpha}}} \int_{B(x,t)} |f(z)|dz.
\]
Therefore, for all \( y \in B \) we have
\[
M^\alpha f_2(y) \leq 2^{n-\alpha} \sup_{t > 2r} \frac{1}{|B(x,t)|^{1-\frac{n}{\alpha}}} \int_{B(x,t)} |f(z)|dz. \tag{4.6}
\]
Thus
\[
\|M^\alpha f\|_{L^\psi(B)} \lesssim \|f\|_{L^\phi(B(x,2r))} + \frac{1}{\Psi^{-1}(r-n)} \left( \sup_{t > 2r} \frac{1}{|B(x,t)|^{1-\frac{n}{\alpha}}} \int_{B(x,t)} |f(z)|dz \right).
\]
Let now \( \Phi \) dominates globally the function \( Q \). It is obvious that
\[
\|M^\alpha f\|_{\mathcal{W} L^\psi(B)} \lesssim \|M^\alpha f_1\|_{\mathcal{W} L^\psi(B)} + \|M^\alpha f_2\|_{\mathcal{W} L^\psi(B)}
\]
for every ball \( B = B(x,r) \).

By the boundedness of the operator \( M^\alpha \) from \( L^\phi\) to \( \mathcal{W} L^\psi \), provided by Theorem 2.7, we have
\[
\|M^\alpha f_1\|_{\mathcal{W} L^\psi(B)} \lesssim \|f\|_{L^\phi(B(x,2r))}.
\]
Then by (4.6) we get the inequality (4.4).

\[ \square \]

**Lemma 4.4.** Let \( \Phi \) and \( \Psi \) Young functions, \( f \in L^\text{loc}_\Phi(\mathbb{R}^n) \) and \( B = B(x,r) \).

If \( (\Phi, \Psi) \) satisfy the conditions (2.5), then
\[
\|M^\alpha f\|_{L^\psi(B)} \lesssim \frac{1}{\Psi^{-1}(r-n)} \sup_{t > 2r} \Psi^{-1}(t^{-n}) \|f\|_{L^\phi(B(x,t))}. \tag{4.7}
\]

If \( (\Phi, \Psi) \) satisfy the conditions (2.6), then
\[
\|M^\alpha f\|_{L^\psi(B)} \lesssim \frac{1}{\Psi^{-1}(r-n)} \sup_{t > 2r} \Psi^{-1}(t^{-n}) \|f\|_{L^\phi(B(x,t))}. \tag{4.8}
\]

**Proof.** Suppose that the condition (2.6) satisfied. Denote
\[
\mathcal{M}_1 := \frac{1}{\Psi^{-1}(r-n)} \left( \sup_{t > 2r} \frac{1}{|B(x,t)|^{1-\frac{n}{\alpha}}} \int_{B(x,t)} |f(z)|dz \right),
\]
\[
\mathcal{M}_2 := \|f\|_{L^\phi(B(x,2r))}.
\]

10
By Lemma 2.6 we get
\[ M_1 \lesssim \frac{1}{\Psi^{-1}(r^{-n})} \sup_{t>2r} t^{\alpha} \Phi^{-1}(t^{-n}) \|f\|_{L_\Phi(B(x,t))}. \]

On the other hand, the conditions (2.6) implies the condition (2.5). Since from Theorem 2.7
\[ (2.6) \Rightarrow M_\alpha \text{ strong type } (\Phi, \Psi) \Rightarrow M_\alpha \text{ weak type } (\Phi, \Psi) \Rightarrow (2.5). \]
The condition (2.5) is equivalent the condition \( \Phi^{-1}(t) \lesssim t^{\frac{n}{p}} \Psi^{-1}(t) \). Indeed,
\[ Q^{-1}(t) = \inf\{r \geq 0 : Q(r) > t\} \geq \inf\{r \geq 0 : \Phi(Cr) > t\} = \frac{1}{C} \inf\{Cr \geq 0 : \Phi(Cr) > t\} = \frac{1}{C} \Phi^{-1}(t). \]

So we arrive
\[ M_1 \lesssim \frac{1}{\Psi^{-1}(r^{-n})} \sup_{t>2r} \Psi^{-1}(t^{-n}) \|f\|_{L_\Phi(B(x,t))}. \]

On the other hand
\[ \frac{1}{\Psi^{-1}(r^{-n})} \sup_{t>2r} \Psi^{-1}(t^{-n}) \|f\|_{L_\Phi(B(x,t))} \gtrsim \frac{1}{\Psi^{-1}(r^{-n})} \sup_{t>2r} \Psi^{-1}(t^{-n}) \|f\|_{L_\Phi(B(x,2r))} \approx M_2. \] (4.9)

Since \( \|M_\alpha f\|_{L_\Psi(B)} \leq M_1 + M_2 \) by Lemma 4.3, we arrive at (4.8).

Suppose that the condition (2.5) satisfied. The inequality (4.7) directly follows from (4.4).

If we take \( \Phi(t) = t^p, \Psi(t) = t^q, 1 \leq p, q < \infty \) at Lemma 4.4 we get following estimates which was proved at (2.6).

**Corollary 4.5.** Let \( f \in L_p^{\text{loc}}(\mathbb{R}^n), 0 \leq \alpha < n, 1 \leq p < \infty, 1/p - 1/q = \alpha/n. \) Then
\[ \|M_\alpha f\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \sup_{t>2r} t^{-\frac{n}{q}} \|f\|_{L_p(B(x_0,t))}, \quad 1 < p \leq q < \infty \]

and
\[ \|M_\alpha f\|_{W_{L_q(B(x_0,r))}} \lesssim r^{\frac{n}{q}} \sup_{t>2r} t^{-\frac{n}{q}} \|f\|_{L_1(B(x_0,t))}, \quad 1 \leq q < \infty. \]
Theorem 4.6. Let $0 \leq \alpha < n$ and the functions $(\varphi_1, \varphi_2)$ and $(\Phi, \Psi)$ satisfy the condition
\[
\sup_{r < t < \infty} \frac{\varphi_1(x, s)}{\Phi^{-1}(s^{-\alpha})} \leq C \varphi_2(x, r), \tag{4.10}
\]
where $C$ does not depend on $x$ and $r$. Then for the conditions (2.5), the fractional maximal operator $M_\alpha$ is bounded from $M_{\Phi, \varphi_1}(\mathbb{R}^n)$ to $M_{\Phi, \varphi_1}(\mathbb{R}^n)$ and for the conditions (2.5), it is bounded from $M_{\Phi, \varphi_1}(\mathbb{R}^n)$ to $W M_{\Phi, \varphi_2}(\mathbb{R}^n)$.

Proof. By Lemma 4.4 and Theorem 4.1 we get
\[
\|M_\alpha f\|_{M_{\Phi, \varphi_2}} \lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r)^{-1} \sup_{t > r} \varphi_1(x, r)^{-1} \Phi^{-1}(r^{-n}) \|f\|_{L\Phi(B(x,t))},
\]
if (2.6) satisfied and
\[
\|M_\alpha f\|_{W M_{\Phi, \varphi_2}} \lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r)^{-1} \sup_{t > r} \varphi_1(x, r)^{-1} \Phi^{-1}(r^{-n}) \|f\|_{L\Phi(B(x,t))},
\]
if (2.5) satisfied.

Note that analogue of the Theorem 4.6 for the Riesz potential proved in [18] as follows.

Theorem 4.7. Let $0 < \alpha < n$ and the functions $(\varphi_1, \varphi_2)$ and $(\Phi, \Psi)$ satisfy the condition
\[
\int_{r}^{\infty} \essinf_{t < s < \infty} \frac{\varphi_1(x, s)}{\Phi^{-1}(s^{-\alpha})} \Psi^{-1}(t^{-n}) \frac{dt}{t} \leq C \varphi_2(x, r), \tag{4.11}
\]
where $C$ does not depend on $x$ and $r$. Then for the conditions (2.8), $I_\alpha$ is bounded from $M_{\Phi, \varphi_1}(\mathbb{R}^n)$ to $M_{\Phi, \varphi_2}(\mathbb{R}^n)$ and for the conditions (2.7), $I_\alpha$ is bounded from $M_{\Phi, \varphi_1}(\mathbb{R}^n)$ to $W M_{\Phi, \varphi_2}(\mathbb{R}^n)$.

Remark 4.8. The condition (4.10) is weaker than (4.11). Indeed, (4.11) implies (4.10):
\[
\varphi_2(x, r) \geq \int_{r}^{\infty} \essinf_{t < s < \infty} \frac{\varphi_1(x, s)}{\Phi^{-1}(s^{-\alpha})} \Psi^{-1}(t^{-n}) \frac{dt}{t} = \int_{s}^{\infty} \essinf_{t < s < \infty} \frac{\varphi_1(x, t)}{\Phi^{-1}(t^{-\alpha})} \Psi^{-1}(t^{-n}) \frac{dt}{t} \approx \essinf_{s < t < \infty} \frac{\varphi_1(x, t)}{\Phi^{-1}(t^{-\alpha})} \Psi^{-1}(t^{-n}) \frac{dt}{t}.
\]
where we took $s \in (r, \infty)$, so that

$$\sup_{s>r} \inf_{s<r<\infty} \frac{\varphi_1(x, \tau)}{\Phi^{-1}(\tau^{-n})} \Psi^{-1}(s^{-n}) \lesssim \varphi_2(x, r).$$

On the other hand the functions $\varphi_1(x, t) = \frac{\Phi^{-1}(t^{-n})}{\psi^{-1}(t^{-n})}$ and $\varphi_2(x, t) = 1$ satisfy the condition \eqref{4.10}, but do not satisfy the condition \eqref{4.11}.

Consider the case $\alpha = 0$ and $\Phi = \Psi$. In this case condition \eqref{2.6} satisfied by any Young function and condition \eqref{2.6} satisfied if and only if $\Phi \in \nabla_2$ (see \cite{6, 22} for details). Therefore we get the following corollary for Hardy-Littlewood maximal operator which was proved in \cite{9}.

**Corollary 4.9.** Let the functions $\varphi_1, \varphi_2$ and $\Phi$ satisfy the condition

$$\sup_{r<t<\infty} \Phi^{-1}(t^{-n}) \inf_{t<s<\infty} \frac{\varphi_1(x, s)}{\Phi^{-1}(s^{-n})} \leq C \varphi_2(x, r), \quad (4.12)$$

where $C$ does not depend on $x$ and $r$. Then for $\Phi \in \nabla_2$, the maximal operator $M$ is bounded from $M_{\Phi, \varphi_1}(\mathbb{R}^n)$ to $M_{\Phi, \varphi_2}(\mathbb{R}^n)$ and for every Young function $\Phi$, it is bounded from $M_{\Phi, \varphi_1}(\mathbb{R}^n)$ to $WM_{\Phi, \varphi_2}(\mathbb{R}^n)$.

If we take $\Phi(t) = t^p$, $\Psi(t) = t^q$, $1 \leq p, q < \infty$ at Theorem 4.6 we get the Spanne-Guliyev type result which was proved in \cite{20}.

**Corollary 4.10.** Let $0 \leq \alpha < n$, $1 \leq p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and $(\varphi_1, \varphi_2)$ satisfy the condition

$$\sup_{r<t<\infty} \inf_{t<s<\infty} \frac{\varphi_1(x, s)}{t^{\frac{n}{q}}} \lesssim \frac{\varphi_2(x, r)}{t^{\frac{n}{q}}}, \quad (4.13)$$

where $C$ does not depend on $x$ and $r$. Then for $p > 1$, $M_\alpha$ is bounded from $M_{p, \varphi_1}(\mathbb{R}^n)$ to $M_{q, \varphi_2}(\mathbb{R}^n)$ and for $p = 1$, it is bounded from $M_{1, \varphi_1}(\mathbb{R}^n)$ to $WM_{q, \varphi_2}(\mathbb{R}^n)$.

In the case $\varphi_1(x, r) = \frac{\Phi^{-1}(r^{-n})}{\phi^{-1}(r^{-\lambda_1})}$, $\varphi_2(x, r) = \frac{\Psi^{-1}(r^{-n})}{\psi^{-1}(r^{-\lambda_2})}$ from Theorem 4.6 we get the following Spanne type theorem for the boundedness of the fractional maximal operator on Orlicz-Morrey spaces.

**Corollary 4.11.** Let $0 \leq \alpha < n$, $\Phi$ and $\Psi$ be Young functions, $0 \leq \lambda_1, \lambda_2 < n$ and $(\Phi, \Psi)$ satisfy the condition

$$\sup_{r<t<\infty} \frac{\Psi^{-1}(t^{-n})}{\Phi^{-1}(t^{-\lambda_1})} \leq C \frac{\Psi^{-1}(r^{-n})}{\Psi^{-1}(r^{-\lambda_2})}, \quad (4.14)$$

where $C$ does not depend on $r$. Then for the conditions \eqref{2.6}, $M_\alpha$ is bounded from $M_{\Phi, \lambda_1}(\mathbb{R}^n)$ to $M_{\Phi, \lambda_2}(\mathbb{R}^n)$ and for the conditions \eqref{2.5}, $M_\alpha$ is bounded from $M_{\Phi, \lambda_1}(\mathbb{R}^n)$ to $WM_{\Phi, \lambda_2}(\mathbb{R}^n)$.
Remark 4.12. If we take Φ(t) = t^p, Ψ(t) = t^q, 1 ≤ p, q < ∞ at Corollary 4.11 we get Spanne type boundedness of M_α, i.e. if 0 ≤ α < n, 1 < p < \frac{n}{α}, 0 < \lambda < n−αp, \frac{1}{p} − \frac{1}{q} = \frac{α}{n} and \frac{λ}{p} = \frac{µ}{q}, then for p > 1 M_α is bounded from M_{p,λ}(\mathbb{R}^n) to M_{q,µ}(\mathbb{R}^n) and for p = 1, M_α is bounded from M_{1,λ}(\mathbb{R}^n) to W M_{q,µ}(\mathbb{R}^n).

5 Commutators of the fractional maximal operator in the spaces M_{Φ,ϕ}

The theory of commutator was originally studied by Coifman, Rochberg and Weiss in [7]. Since then, many authors have been interested in studying this theory.

We recall the definition of the space of BMO(\mathbb{R}^n).

Definition 5.1. Suppose that f ∈ L_1^{loc}(\mathbb{R}^n), let

∥f∥_* = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y) − f_{B(x, r)}| dy < \infty,

where

f_{B(x, r)} = \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) dy.

Define

BMO(\mathbb{R}^n) = \{ f \in L_1^{loc}(\mathbb{R}^n) : ∥f∥_* < \infty \}.

Modulo constants, the space BMO(\mathbb{R}^n) is a Banach space with respect to the norm ∥·∥_*.

Remark 5.2. (1) The John–Nirenberg inequality: there are constants C_1, C_2 > 0, such that for all f ∈ BMO(\mathbb{R}^n) and β > 0

∥\{ x \in B : |f(x) − f_B| > β \}∥ ≤ C_1 |B|e^{-C_2β/∥f∥_*}, \forall B \subset \mathbb{R}^n.

(2) The John–Nirenberg inequality implies that

∥f∥_* ≈ \sup_{x \in \mathbb{R}^n, r > 0} \left( \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y) − f_{B(x, r)}|^p dy \right)^{\frac{1}{p}} (5.1)

for 1 < p < ∞.

(3) Let f ∈ BMO(\mathbb{R}^n). Then there is a constant C > 0 such that

|f_{B(x, r)} − f_{B(x, t)}| ≤ C∥f∥_* \ln \frac{t}{r} \quad \text{for}\quad 0 < 2r < t, (5.2)

where C is independent of f, x, r and t.
Definition 5.3. A Young function $\Phi$ is said to be of upper type $p$ (resp. lower type $p$) for some $p \in [0, \infty)$, if there exists a positive constant $C$ such that, for all $t \in [1, \infty)$(resp. $t \in [0, 1]$) and $s \in [0, \infty)$,

$$\Phi(st) \leq C t^p \Phi(s).$$

Remark 5.4. We know that if $\Phi$ is lower type $p_0$ and upper type $p_1$ with $1 < p_0 \leq p_1 < \infty$, then $\Phi \in \Delta_2 \cap \nabla_2$. Conversely if $\Phi \in \Delta_2 \cap \nabla_2$, then $\Phi$ is lower type $p_0$ and upper type $p_1$ with $1 < p_0 \leq p_1 < \infty$ (see [23]).

Lemma 5.5. [26] Let $\Phi$ be a Young function which is lower type $p_0$ and upper type $p_1$ with $0 < p_0 \leq p_1 < \infty$. Let $\tilde{C}$ be a positive constant. Then there exists a positive constant $C$ such that for any ball $B$ of $\mathbb{R}^n$ and $\mu \in (0, \infty)$

$$\int_B \Phi \left( \frac{|f(x)|}{\mu} \right) dx \leq \tilde{C}$$

implies that $\|f\|_{L^\Phi(B)} \leq C \mu$.

Lemma 5.6. Let $f \in BMO(\mathbb{R}^n)$ and $\Phi$ be a Young function. Let $\Phi$ is lower type $p_0$ and upper type $p_1$ with $1 < p_0 \leq p_1 < \infty$, then

$$\|f\|_* \approx \sup_{x \in \mathbb{R}^n, r > 0} \Phi^{-1}(r^{-n}) \|f(\cdot) - f_{B(x,r)}\|_{L^\Phi(B(x,r))}.$$

Proof. By Hölder’s inequality, we have

$$\|f\|_* \lesssim \sup_{x \in \mathbb{R}^n, r > 0} \Phi^{-1}(r^{-n}) \|f(\cdot) - f_{B(x,r)}\|_{L^\Phi(B(x,r))}.$$

Now we show that

$$\sup_{x \in \mathbb{R}^n, r > 0} \Phi^{-1}(r^{-n}) \|f(\cdot) - f_{B(x,r)}\|_{L^\Phi(B(x,r))} \lesssim \|f\|_*.$$

Without loss of generality, we may assume that $\|f\|_* = 1$; otherwise, we replace $f$ by $f/\|f\|_*$. By the fact that $\Phi$ is lower type $p_0$ and upper type $p_1$ and (2.1) it follows that

$$\int_{B(x,r)} \Phi \left( \frac{|f(y) - f_{B(x,r)}|}{\|f\|_*} \Phi^{-1}(|B(x,r)|^{-1}) \right) dy$$

$$= \int_{B(x,r)} \Phi \left( |f(y) - f_{B(x,r)}| \Phi^{-1}(|B(x,r)|^{-1}) \right) dy$$

$$\lesssim \frac{1}{|B(x,r)|} \int_{B(x,r)} \left[ |f(y) - f_{B(x,r)}|^{p_0} + |f(y) - f_{B(x,r)}|^{p_1} \right] dy \lesssim 1.$$

By Lemma 5.5 we get the desired result. $\square$
Definition 5.7. Let $\Phi$ be a Young function. Let

$$a_\Phi := \inf_{t \in (0, \infty)} \frac{t\Phi'(t)}{\Phi(t)}, \quad b_\Phi := \sup_{t \in (0, \infty)} \frac{t\Phi'(t)}{\Phi(t)}.$$ 

Remark 5.8. It is known that $\Phi \in \Delta_2 \cap \nabla_2$ if and only if $1 < a_\Phi \leq b_\Phi < \infty$ (See [24]).

Remark 5.9. Remark 5.8 and Remark 5.4 show us that a Young function $\Phi$ is lower type $p_0$ and upper type $p_1$ with $1 < p_0 \leq p_1 < \infty$ if and only if $1 < a_\Phi \leq b_\Phi < \infty$.

The commutators generated by $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and the operators $M_\alpha$ and $I_\alpha$ are defined by

$$M_{b,\alpha}(f)(x) = \sup_{t > 0} |B(x, t)|^{-1+\frac{\alpha}{n}} \int_{B(x, t)} |b(x) - b(y)||f(y)|dy, \quad (5.3)$$

$$[b, I_\alpha]f(x) = \int_{\mathbb{R}^n} \frac{b(x) - b(y)}{|x - y|^{n-\alpha}} f(y)dy, \quad (5.4)$$

$$|b, I_\alpha|f(x) := \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x - y|^{n-\alpha}} f(y)dy, \quad (5.5)$$

respectively.

The known boundedness statements for the commutator operators $[b, I_\alpha]$ and $|b, I_\alpha|$ in Orlicz spaces run as follows.

Theorem 5.10. [11] Let $0 < \alpha < n$ and $b \in \text{BMO}(\mathbb{R}^n)$. Let $\Phi$ be a Young function and $\Psi$ defined by its inverse $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$. If $1 < a_\Phi \leq b_\Phi < \infty$ and $1 < a_\Psi \leq b_\Psi < \infty$, then $[b, I_\alpha]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Remark 5.11. Note that, the operator $|b, I_\alpha|$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ under the conditions of Theorem 5.10.

In [10] it was proved that the commutator of the Hardy-Littlewood maximal operator $M_b$ with $b \in \text{BMO}(\mathbb{R}^n)$, is bounded in $L^\Phi(\mathbb{R}^n)$ for any Young function $\Phi$ with $1 < a_\Phi \leq b_\Phi < \infty$. This result together with the well known inequality $M_{\alpha,b}(f)(x) \lesssim |b, I_\alpha|(f)(x)$ and Remark 5.11 imply the following theorem.

Theorem 5.12. Let $0 \leq \alpha < n$ and $b, \Phi$ and $\Psi$ the same as in Theorem 5.10. If $1 < a_\Phi \leq b_\Phi < \infty$ and $1 < a_\Psi \leq b_\Psi < \infty$, then $M_{b,\alpha}$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

The following lemma is valid.
Lemma 5.13. Let $0 \leq \alpha < n$ and $b \in BMO(\mathbb{R}^n)$. Let $\Phi$ be a Young function and $\Psi$ defined, via its inverse, by setting, for all $t \in (0, \infty)$, $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$ and $1 < a_\Phi \leq b_\Psi < \infty$ and $1 < a_\Phi \leq b_\Psi < \infty$, then the inequality

$$\|M_{b, \alpha}f\|_{L^\Psi(B(x_0, r))} \lesssim \|b\|_* \sup_{t > 0} \frac{1}{\Psi^{-1}(t-n)} \int_{B(x_0, t)} |b(y) - b(x)|f(y)|dy$$

holds for any ball $B(x_0, r)$ and for all $f \in L^\Psi_{loc}(\mathbb{R}^n)$.

Proof. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at $x_0$ and of radius $r$. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\{2B\}}$. Hence

$$\|M_{b, \alpha}f\|_{L^\Psi(B)} \leq \|M_{b, \alpha}f_1\|_{L^\Psi(B)} + \|M_{b, \alpha}f_2\|_{L^\Psi(B)}.$$  

From the boundedness of $M_{b, \alpha}$ from $L^\Psi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ it follows that

$$\|M_{b, \alpha}f_1\|_{L^\Psi(B)} \leq \|M_{b, \alpha}f_1\|_{L^\Psi(\mathbb{R}^n)} \lesssim \|b\|_* \|f_1\|_{L^\Psi(\mathbb{R}^n)} = \|b\|_* \|f\|_{L^\Psi(2B)}.$$

For $x \in B$ we have

$$M_{b, \alpha}f_2(x) = \sup_{t > 0} \frac{1}{|B(x, t)|^{1-\frac{n}{\alpha}}} \int_{B(x, t)} |b(y) - b(x)|f_2(y)|dy$$

$$= \sup_{t > 0} \frac{1}{|B(x, t)|^{1-\frac{n}{\alpha}}} \int_{B(x, t) \cap \{2B\}} |b(y) - b(x)|f(y)|dy.$$

Let $x$ be an arbitrary point from $B$. If $B(x, t) \cap \{2B\} \neq \emptyset$, then $t > r$. Indeed, if $y \in B(x, t) \cap \{2B\}$, then $t > |x - y| = |x_0 - y| - |x_0 - x| > 2r - r = r$.

On the other hand, $B(x, t) \cap \{2B\} \subset B(x_0, 2t)$. Indeed, if $y \in B(x, t) \cap \{2B\}$, then we get $|x_0 - y| \leq |x - y| + |x_0 - x| < t + r < 2t$.

Hence

$$M_{b, \alpha}f_2(x) \leq \sup_{t > r} \frac{1}{|B(x, t)|^{1-\frac{n}{\alpha}}} \int_{B(x, t) \cap \{2B\}} |b(y) - b(x)|f(y)|dy$$

$$\leq 2^{n-\alpha} \sup_{t > r} \frac{1}{|B(x_0, 2t)|^{1-\frac{n}{\alpha}}} \int_{B(x_0, 2t)} |b(y) - b(x)|f(y)|dy$$

$$= 2^{n-\alpha} \sup_{t > 2r} \frac{1}{|B(x_0, t)|^{1-\frac{n}{\alpha}}} \int_{B(x_0, t)} |b(y) - b(x)|f(y)|dy.$$

Therefore, for all $x \in B$ we have

$$M_{b, \alpha}f_2(x) \leq 2^{n-\alpha} \sup_{t > 2r} \frac{1}{|B(x_0, t)|^{1-\frac{n}{\alpha}}} \int_{B(x_0, t)} |b(y) - b(x)|f(y)|dy. \quad (5.6)$$
Then
\[
\|M_{b,\alpha} f_2\|_{L_\Psi(B)} \lesssim \left( \sup_{t > 2r} \frac{1}{|B(x_0, t)|^{1 - \frac{n}{m}}} \int_{B(x_0, t)} |b(y) - b(\cdot)| |f(y)| dy \right)_{L_\Psi(B)}
\]

Then
\[
\begin{align*}
\|M_{b,\alpha} f_2\|_{L_\Psi(B)} & \lesssim \left( \sup_{t > 2r} \frac{1}{|B(x_0, t)|^{1 - \frac{n}{m}}} \int_{B(x_0, t)} |b(y) - b_B| |f(y)| dy \right)_{L_\Psi(B)} \\
& \quad + \left( \sup_{t > 2r} \frac{1}{|B(x_0, t)|^{1 - \frac{n}{m}}} \int_{B(x_0, t)} |b(\cdot) - B| |f(y)| dy \right)_{L_\Psi(B)} \\
& = J_1 + J_2.
\end{align*}
\]

Let us estimate $J_1$.
\[
J_1 = \frac{1}{\Psi^{-1}(r^{-n})} \sup_{t > 2r} \frac{1}{|B(x_0, t)|^{1 - \frac{n}{m}}} \int_{B(x_0, t)} |b(y) - b_{B(x_0, t)}| |f(y)| dy
\]

Applying Hölder’s inequality, by Lemma 5.6 and (5.2) we get
\[
J_1 \leq \frac{1}{\Psi^{-1}(r^{-n})} \sup_{t > 2r} t^{\alpha-n} \int_{B(x_0, t)} |b(y) - b_{B(x_0, t)}| |f(y)| dy
\]

In order to estimate $J_2$ note that
\[
J_2 \approx \|b(\cdot) - B_B\|_{L_\Psi(B)} \sup_{t > 2r} t^{\alpha-n} \int_{B(x_0, t)} |f(y)| dy
\]

Summing up $J_1$ and $J_2$ we get
\[
\|M_{b,\alpha} f_2\|_{L_\Psi(B)} \lesssim \|b\|_{*} \frac{1}{\Psi^{-1}(r^{-n})} \sup_{t > 2r} \Psi^{-1}(t^{-n}) \left( 1 + \ln \frac{t}{r} \right) \|f\|_{L_\Psi(B(x_0, t))}.
\]
Finally,
\[ \| M_{b,\alpha}f \|_{L^q(B)} \leq \| b \|_1 \| f \|_{L^q(2B)} + \| b \|_1 \frac{1}{\Psi^{-1}((r-\alpha)/n)} \sup_{t>2r} \Psi^{-1}(t^{-n}) \left( 1 + \ln \frac{t}{r} \right) \| f \|_{L^q(B(x_0,t))}, \]
and the statement of Lemma 5.13 follows by (4.9).

**Theorem 5.14.** Let \( 0 \leq \alpha < n \) and \( b \in BMO(\mathbb{R}^n) \). Let also \( \Phi \) be a Young function and \( \Psi \) defined, via its inverse, by setting, for all \( t \in (0, \infty) \), \( \Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n} \) and \( 1 < a_\Psi \leq b_\Psi < \infty \) and \( 1 < a_\Phi \leq b_\Phi < \infty \), \((\varphi_1, \varphi_2)\) and \((\Phi, \Psi)\) satisfy the condition
\[ \sup_{r<t<\infty} \left( 1 + \ln \frac{t}{r} \right) \Phi^{-1}(t^{-n}) \essinf_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}(s^{-n})} \leq C \varphi_2(x,r), \tag{5.8} \]
where \( C \) does not depend on \( x \) and \( r \). Then the operator \( M_{b,\alpha} \) is bounded from \( M_{\Phi,\varphi_1}(\mathbb{R}^n) \) to \( M_{\Phi,\varphi_2}(\mathbb{R}^n) \). Moreover
\[ \| M_{b,\alpha}f \|_{M_{\Phi,\varphi_2}} \leq \| b \|_* \| f \|_{M_{\Phi,\varphi_1}}. \]

**Proof.** The statement of Theorem 5.14 follows by Lemma 5.13 and Theorem 4.1 in the same manner as in the proof of Theorem 4.6.

If we take \( \alpha = 0 \) at Theorem 5.14 we get the following new result for the commutator of Hardy-Littlewood maximal operator \( M_b \).

**Corollary 5.15.** Let \( b \in BMO(\mathbb{R}^n) \), \( \Phi \) be a Young function with \( 1 < a_\Phi \leq b_\Phi < \infty \), \((\varphi_1, \varphi_2)\) and \( \Phi \) satisfy the condition
\[ \sup_{r<t<\infty} \left( 1 + \ln \frac{t}{r} \right) \Phi^{-1}(t^{-n}) \essinf_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}(s^{-n})} \leq C \varphi_2(x,r), \tag{5.9} \]
where \( C \) does not depend on \( x \) and \( r \). Then the operator \( M_b \) is bounded from \( M_{\Phi,\varphi_1}(\mathbb{R}^n) \) to \( M_{\Phi,\varphi_2}(\mathbb{R}^n) \). Moreover
\[ \| M_b f \|_{M_{\Phi,\varphi_2}} \leq \| b \|_* \| f \|_{M_{\Phi,\varphi_1}}. \]

Note that analogue of the Theorem 5.14 for the commutator of the Riesz potential \([b, I_\alpha]\) proved in [18] as follows.

**Theorem 5.16.** Let \( 0 < \alpha < n \) and \( b \in BMO(\mathbb{R}^n) \). Let \( \Phi \) be a Young function and \( \Psi \) defined, via its inverse, by setting, for all \( t \in (0, \infty) \), \( \Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n} \) and \( 1 < a_\Phi \leq b_\Phi < \infty \) and \( 1 < a_\Psi \leq b_\Psi < \infty \), \((\varphi_1, \varphi_2)\) and \((\Phi, \Psi)\) satisfy the condition
\[ \int_r^\infty \left( 1 + \ln \frac{t}{r} \right) \essinf_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}(s^{-n})} \Psi^{-1}(t^{-n}) \frac{dt}{t} \leq C \varphi_2(x,r), \tag{5.10} \]
where \( C \) does not depend on \( x \) and \( r \).

Then the operator \([b, I_\alpha]\) is bounded from \( M_{\Phi,\varphi_1}(\mathbb{R}^n) \) to \( M_{\Phi,\varphi_2}(\mathbb{R}^n) \). Moreover
\[ \| [b, I_\alpha] f \|_{M_{\Phi,\varphi_2}} \leq \| b \|_* \| f \|_{M_{\Phi,\varphi_1}}. \]
Remark 5.17. The condition (5.8) is weaker than (5.10). See Remark 4.8 for details.

If we take $\Phi(t) = t^p$, $\Psi(t) = t^q$, $1 < p, q < \infty$ at Theorem 5.14 we get the Spanne-Guliyev type result which was proved at [20].

Corollary 5.18. Let $0 \leq \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and $(\varphi_1, \varphi_2)$ satisfy the condition

$$
\sup_{r < t < \infty} \left(1 + \ln \frac{t}{r}\right) \essinf_{t < s < \infty} \frac{\varphi_1(x, s) s^{\frac{n}{r}}}{t^{\frac{n}{q}}} \leq C \varphi_2(x, r),
$$

(5.11)

where $C$ does not depend on $x$ and $r$. Then $M_{b, \alpha}$ is bounded from $M_{p, \varphi_1} (\mathbb{R}^n)$ to $M_{q, \varphi_2} (\mathbb{R}^n)$.

In the case $\varphi_1(x, r) = \Phi^{-1}\left(\frac{r^{-\alpha}}{\Phi^{-1}(r^{-\lambda_1})}\right)$, $\varphi_2(x, r) = \Psi^{-1}\left(\frac{r^{-\lambda_2}}{\Psi^{-1}(r^{-\lambda_2})}\right)$ from Theorem 5.14 we get the following Spanne type theorem for the boundedness of the operator $M_{b, \alpha}$ on Orlicz-Morrey spaces.

Corollary 5.19. Let $0 \leq \alpha < n$, $0 \leq \lambda_1, \lambda_2 < n$ and $b \in BMO(\mathbb{R}^n)$. Let also $\Phi$ be a Young function and $\Psi$ defined, via its inverse, by setting, for all $t \in (0, \infty)$, $\Psi^{-1}(t) := \Phi^{-1}(t^{-\alpha/n})$, $1 < a_\Phi \leq b_\Phi < \infty$, $1 < a_\Psi \leq b_\Psi < \infty$ and $(\Phi, \Psi)$ satisfy the condition

$$
\sup_{r < t < \infty} \left(1 + \ln \frac{t}{r}\right) \frac{\Psi^{-1}(t^{-\alpha/n})}{\Phi^{-1}(t^{-\alpha/n})} \leq C \frac{\Psi^{-1}(r^{-\lambda_2})}{\Phi^{-1}(r^{-\lambda_1})},
$$

(5.12)

where $C$ does not depend on $r$. Then $M_{b, \alpha}$ is bounded from $M_{\varphi_1, \lambda_1} (\mathbb{R}^n)$ to $M_{\varphi_2, \lambda_2} (\mathbb{R}^n)$.

Remark 5.20. If we take $\Phi(t) = t^p$, $\Psi(t) = t^q$, $1 \leq p, q < \infty$ at Corollary 5.19 we get Spanne type boundedness of $M_{b, \alpha}$, i.e. if $0 \leq \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\frac{1}{p} = \frac{\lambda}{q}$, then for $p > 1$ $M_{b, \alpha}$ is bounded from $M_{p, \lambda} (\mathbb{R}^n)$ to $M_{q, \mu} (\mathbb{R}^n)$ and for $p = 1$, $M_{b, \alpha}$ is bounded from $M_{1, \lambda} (\mathbb{R}^n)$ to $W M_{q, \mu} (\mathbb{R}^n)$.

References

[1] Adams, D. R.: A note on Riesz potentials. Duke Math. J. 42, 765-778 (1975).

[2] Akbulut, A., Guliyev, V.S., Mustafayev, R.: On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces. Math. Bohem. 137(1), 27-43 (2012).

[3] Bennett, C., Sharpley, R.: Interpolation of operators. Academic Press, Boston, (1988).
[4] Burenkov, V., Gogatishvili, A., Guliyev, V.S., Mustafayev, R.: Boundedness of the fractional maximal operator in local Morrey-type spaces. Complex Var. Elliptic Equ. \textbf{55} (8-10), 739-758 (2010).

[5] Chanillo, S.: A note on commutators. Indiana Univ. Math. J. \textbf{31}, 716 (1982).

[6] Cianchi, A.: Strong and weak type inequalities for some classical operators in Orlicz spaces. J. London Math. Soc. \textbf{60}(2) no. 1, 187-202 (1999).

[7] Coifman, R.R., Rochberg, R., Weiss,G.: Factorization theorems for Hardy spaces in several variables. Ann. of Math. (2) vol \textbf{103} no.3, 611-635 (1976).

[8] Garcia-Cuerva, J., Harboure, E., Segovia, C., Torrea, J. L.: Weighted norm inequalities for commutators of strongly singular integrals, Indiana Univ. Math. J. \textbf{40} no. 4, 13971420 (1991).

[9] Deringoz, F., Guliyev, V.S., Samko, S.: Boundedness of maximal and singular operators on generalized Orlicz-Morrey spaces. Operator Theory: Advances and Applications Vol. \textbf{242}, 139-158 (2014).

[10] Fu, X., Yang, D., Yuan, W.: Boundedness on Orlicz spaces for multilinear commutators of Calderón-Zygmund operators on non-homogeneous spaces. Taiwanese J. Math. \textbf{16}, 2203-2238 (2012).

[11] Fu, X., Yang, D., Yuan, W.: Generalized Fractional Integrals and Their Commutators over Non-homogeneous Metric Measure Spaces. arXiv: 1308.5877 [math.CA].

[12] Giaquinta, M.: Multiple integrals in the calculus of variations and nonlinear elliptic systems. Princeton Univ. Press, Princeton, NJ, (1983).

[13] Guliyev, V.S.: Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$. Doctor's degree dissertation, Mat. Inst. Steklov, Moscow, 329 pp. (in Russian) (1994)

[14] Guliyev, V.S.: Function spaces, Integral Operators and Two Weighted Inequalities on Homogeneous Groups. Some Applications. Casioglu, Baku, 332 pp. (in Russian) (1999).

[15] Guliyev, V.S.: Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces. J. Inequal. Appl. Art. ID 503948 (2009).

[16] Guliyev, V.S., Aliyev, S.S., Karaman, T., Shukurov, P.S.: Boundedness of sublinear operators and commutators on generalized Morrey Space. Int. Eq. Op. Theory. \textbf{71} (3), 327-355 (2011).
[17] Guliyev, V.S., Hasanov, J., Samko, S.: Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces. Math. Scand. 197(2), 285-304 (2010).

[18] Guliyev, V.S., Deringoz, F.: On the Riesz potential and its commutators on generalized Orlicz-Morrey spaces. J. Funct. Spaces. Volume 2014, Article ID 617414, 11 pages (2014).

[19] Guliyev, V.S., Deringoz, F., Hasanov, J.J.: $\Phi$-admissible singular operators and their commutators on vanishing generalized Orlicz-Morrey spaces. J. Inequal. Appl. 2014:143, 18 pp. (2014).

[20] Guliyev, V.S., Shukurov, P. S.: On the Boundedness of the Fractional Maximal Operator, Riesz Potential and Their Commutators in Generalized Morrey Spaces. Operator Theory: Advances and Applications Vol. 229, 175-194 (2013).

[21] Kita, H.: On maximal functions in Orlicz spaces. Proc. Amer. Math. Soc. 124, 3019-3025 (1996).

[22] Kita, H.: On Hardy-Littlewood maximal functions in Orlicz spaces. Math. Nachr. 183, 135-155 (1997).

[23] Kokilashvili, V., Krbec, M. M.: Weighted Inequalities in Lorentz and Orlicz Spaces. World Scientific, Singapore, (1991).

[24] Krasnoselskii, M.A., Rutickii, Ya. B.: Convex Functions and Orlicz Spaces. English translation P. Noordhoff Ltd., Groningen, (1961).

[25] Kufner, A., John O., Fučík, S.: Function Spaces. Noordhoff International Publishing: Leyden, Publishing House Czechoslovak Academy of Sciences: Prague, (1977).

[26] Ky, L. D.: New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators. arXiv: 1103.3757 [math.CA]

[27] Liu, P.D., Wang, M.F.: Weak Orlicz spaces: Some basic properties and their applications to harmonic analysis. Science China Mathematics 56 (4), 789-802 (2013).

[28] Mizuhara, T.: Boundedness of some classical operators on generalized Morrey spaces. Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer - Verlag, Tokyo, 183-189 (1991).

[29] Morrey, C.B.: On the solutions of quasi-linear elliptic partial differential equations. Trans. Amer. Math. Soc. 43, 126-166 (1938).
[30] Nakai, E.: Hardy-Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces. Math. Nachr. 166, 95-103 (1994).

[31] Nakai, E.: Generalized fractional integrals on Orlicz-Morrey spaces. in: Banach and Function Spaces (Kitakyushu, 2003), Yokohama Publ., Yokohama, 323-333 (2004).

[32] Nakai, E.: Orlicz-Morrey spaces and the Hardy-Littlewood maximal function. Studia Math. 188 no. 3, 193-221 (2008).

[33] Orlicz, W.: Über eine gewisse Klasse von Räumen vom Typus B. Bull. Acad. Polon. A, 207-220 (1932). ; reprinted in: Collected Papers, PWN, Warszawa, 217-230 (1988).

[34] Orlicz, W.: Über Räume ($L^M$). Bull. Acad. Polon. A, 93-107 (1936). ; reprinted in: Collected Papers, PWN, Warszawa, 345-359 (1988).

[35] Peetre, J.: On the theory of $M_{p,\lambda}$. J. Funct. Anal. 4, 71-87 (1969).

[36] Rao, M. M., Ren, Z. D.: Theory of Orlicz Spaces. M. Dekker, Inc., New York, (1991).

[37] Sawano, Y., Sugano, S., Tanaka, H.: Orlicz-Morrey spaces and fractional operators. Potential Anal. 36 no. 4, 517-556 (2012).

[38] Stein, E.M.: Singular integrals and differentiability of functions. Princeton University Press, Princeton, NJ, (1970).

[39] Torchinsky, A.: Real Variable Methods in Harmonic Analysis. Pure and Applied Math. 123, Academic Press, New York, (1986).

[40] Weiss, G.: A note on Orlicz spaces. Portugal Math. 15, 35-47 (1956).