Footprints of a Broad $\sigma(600)$ in Weak-Interaction Processes

A.D. Polosa and N. A. Törnqvist

*Physics Department, POB 9, FIN–00014, University of Helsinki, Finland*

M. D. Scadron

*Physics Department, University of Arizona, Tucson, AZ 85721, USA*

V. Elias

*Department of Applied Mathematics, The University of Western Ontario, London, ON N6A 5B7, Canada*

Abstract

We explore how chiral-symmetry constraints on weak-interaction matrix elements point toward the existence of an intermediate-state $\sigma$ in several different weak-interaction processes. Particular attention is directed toward recent evidence for a $\sigma$ within three-body nonleptonic weak decays.

PACS numbers: 11.30.Rd, 11.40.Ha, 14.40.Cs

HIP-2000-22/TH

The existence of a broad scalar-isoscalar $\pi\pi$ resonance near 600 MeV has long been controversial. After having been absent for many years from the listings of the Particle Data Group (PDG) in the Reviews of Particle Physics, this resonance has reappeared under the name “$f_0(400 \pm 1200)$ or $\sigma$” (hereafter called $\sigma(600)$) after reanalysis by several groups of the available data on the scalar nonet and $\pi\pi$ phase shifts. Today an increasing number of theoretical and experimental analyses point toward the existence of this important meson
A salient feature of such analyses is chiral symmetry and its soft breaking. Essentially all groups who have recently analyzed the strong-interaction scalar meson data within a chiral framework (together with unitarity, analyticity, flavour symmetry etc.) seem to require the existence of the $\sigma(600)$. The simplest model in which $\sigma$ occurs is the linear sigma model ($L\sigma M$), a model which implements chiral symmetry for scalar and pseudoscalar mesons (and unconfined quarks) together with flavour symmetry. Below we shall appeal to several applications of this model to weak decays, as naive quark models without chiral-symmetry constraints are unsuccessful in describing properties of the lightest scalars. Similarly, the nonlinear sigma model (and chiral-perturbation-theory approaches which follow from it) can be understood as an $m_\sigma \to \infty \sigma$-model limit, a limit appropriate at very low energies but inappropriate for processes whose momenta are comparable to the $\sigma$-mass.

The large width of the $\sigma(600)$ (over 300 MeV) has led many to argue that its nature as a resonance is obscured, and that the $\pi\pi$ phase shift in the 500 $-$ 900 MeV region may arise solely from the large contributions of crossed channel diagrams \[2\]. However, the duality between $s$- and $t$-channel exchanges (Regge poles), which has been well-established for more than two decades, indicates that strong crossed-channel effects always appear in conjunction with a resonance.

In this paper we shall briefly review the evidence for the $\sigma(600)$ in weak interaction decays. Although this evidence, like that from strong interactions, is largely indirect and contingent upon the theoretical framework for incorporating chiral symmetry, we nevertheless find that analyses of several different processes point toward the existence of the $\sigma(600)$, as predicted by $L\sigma M$ physics. Thus direct experimental proof of the $\sigma$’s existence is becoming more and more pressing. The recent data analysis \[7\] on $D \to \sigma\pi \to 3\pi$ from the E791

\[1\]For example, Sannino and Schechter [5] utilise chiral-perturbation-theory to predict the initial behaviour of the $\pi\pi$-scattering amplitude for the sub-400 MeV region, but in subsequent work with Harada [4] see clear evidence for a $\sigma$-resonance past that region.
experiment at Fermilab, as discussed in the final section of this paper, appears to provide more direct weak-interaction evidence for a physical $\sigma$ resonance.

\textit{L$\sigma$M and ($\pi^+, K^+\rightarrow e^+\nu\gamma$ Semileptonic Weak Decays}

Let us first review some L$\sigma$M evidence for a $\sigma$ within the $\pi^+\rightarrow e^+\nu\gamma$ decay. The $uud$ quark triangle graph of Fig. 1(a) is known [8] to predict a value of unity for the structure-dependent axial-to-vector form factor ratio at $q^2 = 0$:

$$\gamma_q = \frac{F_A^q(0)}{F_V^q(0)} = 1. \quad (1)$$

Within a L$\sigma$M framework for chiral symmetry breaking, one also must consider triangle graphs involving mesons [9], as in Fig. 1(b), in which case the net $q^2 = 0$ form factor ratio is reduced to

$$\gamma_{L\sigma M} = \frac{F_{A}^{L\sigma M}(0)}{F_{V}^{L\sigma M}(0)} = 1 - \frac{1}{3} = \frac{2}{3}. \quad (2)$$

The $q^2 = 0$ limit of the axial-to-vector form factor ratio has also been extracted from experimental data [1]:

$$\gamma_{PDG} = \frac{F_A(0)}{F_V(0)} = \frac{0.0116 \pm 0.0016}{0.017 \pm 0.008} = 0.68 \pm 0.33. \quad (3)$$

The agreement between (2) and the central value of (3) suggests the inclusion of $\pi$ and $\sigma$ within the low-energy effective theory, although the empirical range (3) cannot be said to exclude the meson-free prediction (1).

In a similar manner, at $q^2 = 0$ the $SU(3)_f$ L$\sigma$M applied to $K^+\rightarrow e^+\nu\gamma$ has been shown to predict [4] that

$$|F_V^K(0) + F_A^K(0)| \approx (0.22 + 0.09) \text{ GeV}^{-1} = 0.31 \text{ GeV}^{-1}. \quad (4)$$

Once again the L$\sigma$M meson loops reduce $F_A$ with respect to $F_V$ because of the relative minus sign between quark and meson loops. The result (4) is obtained using L$\sigma$M tree-level scalar masses $m_\sigma = 680$ MeV, $m_\kappa = 850$ MeV away from values characterizing the chiral limit. In any case, the present PDG [1] average value for the $q^2 = 0 K^+\rightarrow e^+\nu\gamma$ form-factor sum is.
\[ |F_V^K(0) + F_A^K(0)| = (0.148 \pm 0.010) m_K^{-1} = (0.30 \pm 0.02) \text{ GeV}^{-1}, \] (5)

in excellent agreement with (4).

\[ \sigma\text{-Sensitive Matrix Elements and } K_S \rightarrow \pi \pi \text{ Decays.} \]

Since it is well known [1] that \( K_S \rightarrow (\pi^+\pi^-, \pi^0\pi^0) \) decay amplitudes are overwhelmingly (95\%) in the \( \Delta I = \frac{1}{2} \) channel, we shall focus entirely on the \( \Delta I = \frac{1}{2} \) contribution anticipated from intermediate \( \sigma \)-state effects. From the perspective of the \( s \)-channel, the scalar \( I = 0 \) \( \sigma(600) \) meson (nonperturbative) pole of Fig. 2(a) supports the \( \Delta I = \frac{1}{2} \) rule [10] because the matrix element \( \langle \sigma|H_w|K_S \rangle \) probes only the \( \Delta I = \frac{1}{2} \) component of the parity violating weak Hamiltonian density \( H_{\nu w} \). Invoking the L\( \sigma \)M vertex [11] \( \langle 2\pi|\sigma \rangle = \frac{m^2}{2f_\pi} \) and assuming that \( \Gamma(\sigma) \simeq m_\sigma \) [12], one can estimate the \( K_S \rightarrow 2\pi^0 \) amplitude of Fig. 2(a) to be [13,14]

\[ |\langle 2\pi^0|H_{\nu w}^p|K_S \rangle| \simeq 3.45 \times 10^{-8} \text{ GeV}^2. \] (8)

The larger \( K_S \rightarrow \pi^+\pi^- \) amplitude \( |\langle \pi^+\pi^-|H_{\nu w}^p|K_S \rangle| = 39.10 \pm 0.01 \times 10^{-8} \text{ GeV} \), analogous to (7), yields a slightly larger estimate.

A corresponding estimate of the matrix element \( \langle \pi|H_{\nu w}^p|K_L \rangle \) can be extracted from the \( K_S \rightarrow \pi\pi \) process in the dual \( t \)-channel via the \( K_S \) tadpole graph depicted in Fig. 2(b). The chiral relation \( [Q + Q_5, H_w] = 0 \) for an \( H_w \) built up from \( V - A \) currents generates a PCAC-consistent \( K_{2\pi^0} \) amplitude in the chiral limit [15-17]:

\[ |\langle 2\pi^0|H_{\nu w}^p|K_S \rangle| \simeq |\langle 0|H_{\nu w}^p|K_S \rangle \langle K_S 2\pi^0|K_S \rangle| \frac{1}{m_{K_S}^2} = \frac{1}{f_\pi} |\langle \pi^0|H_{\nu w}^p|K_L \rangle|, \] (9)
where the required tadpole-PCAC transition $|\langle 0|H_{\pi}^{\nu}|K_S\rangle| = |2f_\pi\langle \pi^0|H_{\pi}^{pc}|K_L\rangle|$ appears in the middle term of (9), and where $\langle K_S2\pi^0|K_S\rangle = m_{K_S}^2/2f_\pi^2$ is a matrix-element estimate from chiral strong-interaction physics \cite{18}. The right hand sides of (6) and (9) thus appear to be chirally related:

$$|\langle \sigma|H_{\pi}^{\nu}|K_S\rangle| \cong |\langle \pi^0|H_{\pi}^{pc}|K_L\rangle|.$$  \hspace{1cm} (10)

Note that this chiral-symmetry relation (10) follows either from a broad light $\sigma$ or from a narrow heavier $\sigma$; both the former $[\Gamma_\sigma \simeq m_\sigma, m_K^2 - m_\sigma^2 \ll m_\sigma^2]$ and the latter $m_\sigma^2 >> \{m_K^2, m_\sigma \Gamma_\sigma\}$ case lead to the final result of (6). Moreover, if one regards equality within (10) as a fundamental constraint from chiral symmetry, one can reason backward from (10) to require via (6) that $\sqrt{(m_K^2 - m_\sigma^2)^2 + m_\sigma^2 \Gamma_\sigma^2} \simeq m_\sigma^2$, in which case

$$\Gamma_\sigma \simeq \frac{m_K}{m_\sigma} \sqrt{2m_\sigma^2 - m_K^2}. \hspace{1cm} (11)$$

The above relation suggests that $\sigma$ is broad even if its mass is at the high end of the 400-1200 MeV empirical range \cite{1} – we see from (11) that $\Gamma_\sigma \simeq 660$ MeV when $m_\sigma = 1$ GeV. Note also that $\Gamma_\sigma = 570$ MeV when $m_\sigma = 600$ MeV, consistent with estimates suggested in \cite{12}.

$\sigma$-Sensitive Matrix Elements and Radiative Neutral Kaon Decays

For the weak radiative kaon decay $K_L \to 2\gamma$, the dominant $\pi^0$ pole graph of Fig. 3(a) generates the amplitude

$$\langle 2\gamma|H_{\pi}^{pc}|K_L\rangle = \langle 2\gamma|\pi^0\rangle \left(\frac{1}{(m_K^2 - m_\sigma^2)}\langle \pi^0|H_{\pi}^{pc}|K_L\rangle = F_{K_L\gamma\gamma}\epsilon^\mu\epsilon^\nu(\epsilon_{\mu\nu\alpha\beta}k^\alpha k^\beta). \hspace{1cm} (12)$$

The near-cancellation of possible additional $\eta', \eta \to 2\gamma$ pole terms \cite{10,19} is discussed in the Appendix to this paper.

Factoring out the Levi-Civita covariant from the $\pi^0 \to 2\gamma$ amplitude, \cite{1} one extracts

\footnote{One obtains the $\pi^0\gamma\gamma$ amplitude $\alpha/f_\pi = 0.025$ GeV$^{-1}$ via PCAC arguments involving the AVV anomaly or alternatively, via the $\Lambda\pi\pi$ PVV quark loop \cite{20,21}. The measured $\pi^0\gamma\gamma$ amplitude is $0.025 \pm 0.001$ GeV$^{-1}$.}
\[ |F_{KL\gamma\gamma}| \cong \left( \frac{\alpha}{\pi f_\pi} \right) \frac{1}{(m_{KL}^2 - m_{\sigma}^2)} \langle \pi^0 | H^{pc}_w | K_L \rangle = (3.49 \pm 0.05) \times 10^{-9} \text{ GeV}^{-1} \] 

(13)

from the lifetime \( \tau_{KL} = (5.17 \pm 0.04) \times 10^{-8} \text{ sec} \) and the branching ratio \[ B(K_L \rightarrow 2\gamma) = (5.86 \pm 0.15) \times 10^{-4} \]. Solving (13) for the \( K_L \rightarrow \pi^0 \) weak transition, one obtains

\[ |\langle \pi^0 | H^{pc}_w | K_L \rangle| = (3.20 \pm 0.04) \times 10^{-8} \text{ GeV}^2. \] 

(14)

If \( \eta \) and \( \eta' \) pole contributions to \( \langle 2\gamma | H^{pc}_w | K_L \rangle \) are taken into consideration, we see from (A.12) of the Appendix that the central value of \( |\langle \pi^0 | H^{pc}_w | K_L \rangle| \) in (14) will increase by a multiplicative factor of 1/0.90 to 3.56 \( \times 10^{-8} \text{ GeV}^2 \). In any case, the estimate (14) is quite close to its chiral-partner amplitude (8), providing further phenomenological support for the relation (10) anticipated from chiral symmetry.

The analogue \( \sigma(600) L\sigma M \) pole contribution to \( K_S \rightarrow 2\gamma \) in Fig. 3(b) yields

\[ \langle 2\gamma | H^{pv}_w | K_S \rangle = \langle 2\gamma | \sigma \rangle \frac{1}{m_{KS}^2 - m_\sigma^2 + i m_\sigma \Gamma_\sigma} \langle \sigma | H^{pv}_w | K_S \rangle = F_{KS\gamma\gamma} \epsilon^\mu \epsilon^\nu (k_\mu k_\nu' - k \cdot k' g_{\mu\nu}). \] 

(15)

Recall that \( |1/(m_{KL}^2 - m_\sigma^2 + i m_\sigma \Gamma_\sigma)| \approx 1/m_\sigma^2 \) for consistency of (6) with the chiral symmetry relation (10). This constraint enables one to extract the strength of the \( \sigma \rightarrow \gamma\gamma \) amplitude within (15) from measurable quantities:

\[ < 2\gamma | \sigma > \equiv F_{\sigma\gamma\gamma} \epsilon^\mu \epsilon^\nu (k_\mu k_\nu' - k \cdot k' g_{\mu\nu}), \] 

(16)

\[ |F_{\sigma\gamma\gamma}| \cong \left| \frac{F_{KS\gamma\gamma} m_\sigma^2}{< \pi^0 | H^{pc}_w | K_L >} \right| = (5.5 \pm 1.6) \times 10^{-2} \text{ GeV}^{-1}. \] 

(17)

The final numerical value in (17) is obtained for \( m_\sigma = 600 \pm 50 \text{ MeV} \) from (14) [or from (8) via (10)], \( |< \pi^0 | H^{pc}_w | K_L >| = 3.56 \times 10^{-8} \text{ GeV}^2 \), as discussed above, and the observed \( K_S \rightarrow \gamma\gamma \)

---

A model independent (but non-chiral) estimate of \( \langle \pi^0 | H_w | K_L \rangle = \sqrt{2} \langle \pi^0 | H_w | K^0 \rangle \), as obtained from the meson self-energy type graphs of Fig. 4, yields

\[ |\langle \pi^0 | H_w | K_L \rangle| = \left| \frac{G_{F} \alpha_c}{\sqrt{2}} (m_D^2 - m_K^2) \int^\Lambda \frac{d^4 p}{(2\pi)^4} \frac{p^2}{(p^2-m_D^2)(p^2-m_K^2)} \right| \approx 3.5 \times 10^{-8} \text{ GeV}^2 \text{ for the UV cutoff } \Lambda \approx m_D = 1.87 \text{ GeV}. \]
amplitude $|F_{K_S\gamma\gamma}| = (5.4 \pm 1.0) \times 10^{-9}$ GeV$^{-1}$ [4]. The result (17) leads to a $\sigma(600) \rightarrow \gamma\gamma$ width of $m_\sigma^3 |F_{\sigma\gamma\gamma}|^2/64\pi = (3.2 \pm 2.2)$ keV, which is compatible with a $3.8 \pm 1.5$ keV estimate [22] deduced from $\gamma\gamma \rightarrow \pi\pi$ data.

The estimate (17) is particularly useful for the weak decay $K_L \rightarrow \pi^0\gamma\gamma$, which is anticipated (within a $\sigma\sigma M$ context) to be dominated by the $\sigma(600)$ pole graph of Fig. 5. We see from (17) that

$$|F_{\sigma\gamma\gamma}| = (2.20 \pm 0.64) \frac{\alpha}{\pi f_\pi}. \tag{18}$$

The Figure 5 process has already been utilized [14] to predict $\Gamma(K_L \rightarrow \pi^0\gamma\gamma) \approx 1.7 \times 10^{-23}$ GeV with $F_{\sigma\gamma\gamma}$ assumed equal to $2\alpha/\pi f_\pi$. The corresponding prediction rescaled via (18) is

$$\Gamma(K_L \rightarrow \pi^0\gamma\gamma) = \left(2.1^{+1.3}_{-1.1}\right) \times 10^{-23} \text{ GeV}. \tag{19}$$

This prediction is insensitive to three-body amplitudes (such as $K \rightarrow 3\pi$) involving two amplitudes which partially subtract, due to chiral symmetry, and is consistent with the measured rate [1, 23]

$$\Gamma(K_L \rightarrow \pi^0\gamma\gamma) = \frac{h}{2\pi \tau_{K_L}}(1.68 \pm 0.07 \pm 0.08) \times 10^{-6} = (2.15 \pm 0.14) \times 10^{-23} \text{ GeV}. \tag{20}$$

This rate, originally obtained from the $\sigma\sigma M$ Lagrangian [1], is useful for testing PCAC. The ratio of the rates $K_S^{pv} \rightarrow \gamma\gamma$ and $K_L^{pc} \rightarrow \pi^0\gamma\gamma$, as obtained from the $\sigma$-pole graphs of Figs. 3b and 5, permits a cancellation of common $\langle \sigma|H_w^{pv}|K_S\rangle F_{\sigma\gamma\gamma}$ amplitude scales within each rate. To see this, first note that the two amplitudes are related via a PCAC reduction of the $\pi^0$ state [24]:

$$|\langle \gamma\gamma\pi^0|H_w^{pc}|K_L\rangle| = \frac{1}{f_\pi}|\langle \gamma\gamma|[Q_s^3, H_w^{pc}]|K_L\rangle| = \frac{1}{2f_\pi}|\langle \gamma\gamma|H_w^{pv}|K_S\rangle|. \tag{21}$$

On the other hand the $\sigma$-pole graph of Fig. 5 provides a direct determination of the $K_L \rightarrow \pi^0\gamma\gamma$ rate. Using the three body phase space integral of ref. [14] for $m_\sigma = 640$ MeV, we find

$$\Gamma(K_L \rightarrow \pi^0\gamma\gamma) = (1.7 \times 10^{-4} \text{ GeV}^4)|\langle \pi^0|H_w^{pc}|K_L\rangle|^2 \frac{|F_{\sigma\gamma\gamma}|^2}{64m_K^34\pi^3}, \tag{22}$$
whereas the $K_S \rightarrow \gamma\gamma$ amplitude in eq. (15) corresponds to the rate

$$\Gamma(K_S \rightarrow \gamma\gamma) = \frac{m_K^3}{64\pi} |\langle \gamma\gamma | H^{pv}_w | K_S \rangle|^2.$$  

(23)

Applying PCAC to $\langle \pi^0 \sigma | H^{pc}_w | K_L \rangle$ in (22) as in (21), and again utilizing the $\sigma$-pole graph of Fig. 3b in eq. (23), we obtain the following ratio of rates:

$$\frac{\Gamma(K_L \rightarrow \pi^0\gamma\gamma)}{\Gamma(K_S \rightarrow \gamma\gamma)} = \frac{m_\sigma \Gamma_\sigma}{m_K^6} \left( \frac{1.7 \times 10^{-4} \text{GeV}^4}{4\pi f_\pi^2} \right) = 11.6 \times 10^{-4},$$

(24)

for $f_\pi \approx 93\text{MeV}$, $m_\sigma = 640$ MeV and, via eq. (17), $\Gamma_\sigma = 588$ MeV. Note the elimination of the amplitude factor $\langle \sigma | H^{pv}_w | K_S \rangle$ $F_{\sigma\gamma\gamma}$ within the ratio (24). This ratio is also seen to be compatible with data [1],

$$\frac{\Gamma(K_L \rightarrow \pi^0\gamma\gamma)}{\Gamma(K_S \rightarrow \gamma\gamma)} = \frac{b}{2\pi\tau_L} \left( \frac{1.68 \pm 0.10}{2.4 \pm 0.9} \right) \times 10^{-6} = (12.1 \pm 4.6) \times 10^{-4},$$

(25)

given the measured lifetimes $\tau_L = (5.17 \pm 0.04) \times 10^{-8}$ sec, $\tau_S = (0.8935 \pm 0.0008) \times 10^{-10}$ sec. In effect the near-equivalence of (24) and (25), modulo the large error in the latter result, provides confirmation for the PCAC reduction employed in (21).

_Evidence for $\sigma$ in Three-Body Non-Leptonic Weak Decays_

Consider the kaon three-body decay $K^+ \rightarrow \pi^+\sigma \rightarrow \pi^+\pi\pi$, for which the intermediate $\sigma(600)$ is virtual. The (model-dependent) dynamical graphs of Fig. 6, which partially cancel due to chiral symmetry, predict an analogue $K_L \rightarrow 3\pi^0$ decay rate $\Gamma \sim 3 \times 10^{-18} \text{GeV}$ [13,14] in rough agreement with data. However, using (7) and PCAC consistency [17], the $K^+ \rightarrow 3\pi$ decay amplitude

$$|\langle \pi^+\pi^-\pi^+ | H_w | K^+ \rangle| = \frac{1}{2f_\pi} |\langle \pi^+\pi^- | H_w | K_S \rangle| \approx 1.94 \times 10^{-6},$$

(26)

is in excellent agreement with the experimental amplitude [1]

$$A_K = |\langle \pi^+\pi^-\pi^+ | H_w | K^+ \rangle|_{\text{exp}} = (1.93 \pm 0.01) \times 10^{-6}$$

(27)

obtained via the following three-body phase space integral [17]:

8
\[ \Gamma(K^+ \rightarrow \pi^+\pi^-\pi^+) = \left(\frac{1}{8\pi m}\right)^3 |A_K|^2 \int_{4\mu^2}^{(m-\mu)^2} ds \left[ \frac{(s-4\mu^2)(s-(m+\mu)^2)(s-(m-\mu)^2)}{s} \right]^{1/2} \]

\[ = I_K |A_K|^2 = (2.97 \pm 0.03) \times 10^{-18} \text{ GeV}, \quad (28) \]

with \(m, \mu\) being the kaon and pion masses respectively, and with \(I_K = 0.798 \times 10^{-6} \text{ GeV}\).

Note that we have used current-algebra/PCAC consistency, as evident from the agreement between (26) and (27), to infer that the \(\sigma\) and \(\pi\) mesons in Fig. 6 are chiral partners. In effect, we have factored \(|A_K|^2\) from the integral in eq. (28), treating \(|A_K|\) as independent of \(s\) because the underlying \(\sigma\)-pole graph in Fig. 6 is virtual in the transition \(K^+ \rightarrow \pi^+\sigma \rightarrow \pi^+\pi\pi\).

Evidence for a non-virtual \(\sigma\) may be extracted by relating the non-resonant fraction of the \(D^+ \rightarrow \pi^+\pi^+\pi^-\) three body decay to the known two body \(|\langle \pi^+\pi^- | H_w | D^0 \rangle|\) matrix element via analyses which either do or do not include an appreciable \(\pi^+\sigma\)-resonant contribution. The former case corresponds to a fit obtained by the E791 Collaboration [7] with an apparent \(\sigma\) mass of \(478^{+24}_{-23} \pm 17 \text{ MeV}\) and width of \(324^{+42}_{-40} \pm 21 \text{ MeV}\). The latter case is implicit within the PDG estimate for the non-resonant (NR) contribution to the \(D^+ \rightarrow \pi^+\pi^-\pi^+\) rate [1]:

\[ \Gamma_{\text{PDG}}^{\text{(NR)}}(D^+ \rightarrow \pi^+\pi^-\pi^+) = \frac{\hbar}{2\pi \tau_{D^+}} (2.2 \pm 0.4) \times 10^{-3} = (1.37 \pm 0.25) \times 10^{-15} \text{ GeV}. \quad (29) \]

This corresponds to \((2.2 \pm 0.4)/(3.6 \pm 0.4) = (61 \pm 18)\%\) of the total \(D^+ \rightarrow \pi^+\pi^+\pi^-\) decay rate. Since the \(\rho^0\pi^+\) resonant-channel branching fraction is \((1.05 \pm 0.31)/(3.6 \pm 0.4) = (29 \pm 12)\%\), the PDG non-resonant contribution \((29)\) is essentially the difference between the full \(D^+ \rightarrow \pi^+\pi^+\pi^-\) rate and the \(D^+ \rightarrow \pi^+\rho^0 \rightarrow \pi^+\pi^+\pi^-\) resonant sub-rate; the \(D^+ \rightarrow \pi^+\sigma \rightarrow \pi^+\pi^+\pi^-\) resonant sub-rate is at best assumed by the PDG to be a secondary contribution within the 10\% remaining for resonance sub-rates other than \(\rho\). Assuming the rate \((29)\) is truly due to non-resonant virtual intermediate states (which presumably are independent of the squared energy variable \(s\)), we find that

\[ \Gamma_{\text{PDG}}^{\text{(NR)}} = J_{D^+} \left| \langle \pi^+\pi^- \pi^+ | H_w | D^+ \rangle \right|_{\text{NR}}^2, \quad (30) \]
where the constant squared amplitude has been factored out from the three body phase space integral (analogous to (28)) whose numerical value \( J_{D^+} = 48.8 \times 10^{-6} \) GeV. Substitution of eq. (29) into (30) then predicts the non-resonant amplitude

\[
|\langle \pi^+ \pi^+ \pi^- | H_w | D^+ \rangle_{NR}| = (5.3 \pm 0.5) \times 10^{-6}, \tag{31}
\]

which leads, via PCAC (as in (24), (26)), to the prediction

\[
|\langle \pi^+ \pi^- | H_w | D^0 \rangle| \approx \sqrt{2} f_\pi |\langle \pi^+ \pi^+ \pi^- | H_w | D^+ \rangle_{NR}| = (7.0 \pm 0.6) \times 10^{-7} \text{ GeV}. \tag{32}
\]

Comparison of this prediction to the empirical value for this matrix element

\[
|\langle \pi^+ \pi^- | H_w | D^0 \rangle| = m_{D^0} \sqrt{\frac{8\pi \Gamma(D^0 \to \pi^+ \pi^-)}{q}} = (4.8 \pm 0.2) \times 10^{-7} \text{ GeV}, \tag{33}
\]

(where \( q = 922 \) MeV and \( \Gamma(D^0 \to \pi^+ \pi^-) \approx 24 \times 10^{-16} \) GeV [1]) suggests that the PDG value (29) for the non-resonant \( D^+ \to \pi^+ \pi^+ \pi^- \) rate may be too large. Such would be the case if it failed to take into account a \( D^+ \to \pi^+ \sigma \to \pi^+ \pi^+ \pi^- \) resonant contribution.

As noted above, such a contribution appears to be evident in the recent E791 Collaboration measurements of the \( D^+ \to \pi^+ \pi^+ \pi^- \) rate, leading to a fit in which 46.3% of the rate occurs via \( D^+ \to \pi^+ \sigma \), and that 33.6% of the rate occurs via \( D^+ \to \pi^+ \rho^0 \). This latter branching fraction is consistent with the \( (29 \pm 12)\% \) Particle Data Group branching fraction for the \( \pi^+ \rho^0 \) resonant channel. Hence, the net effect of the new E791 data is essentially to reduce the non-resonant fraction of \( D^+ \to \pi^+ \pi^+ \pi^- \) from its PDG value \( (61 \pm 18)\% \) to no more than 20% \( (= 1 - (33.6\%)_{\pi^+ \rho^0} - (46.3\%)_{\pi^+ \sigma}) \). Using this upper bound, the non-resonant rate is sharply reduced from (29) to

\[
\Gamma_{(NR \text{ no } \sigma)} = 0.20 \times \frac{\hbar}{2\pi \tau_{D^+}} B(D^+ \to \pi^+ \pi^+ \pi^-) = (4.48 \pm 0.5) \times 10^{-16} \text{ GeV}, \tag{34}
\]

as obtained from the PDG branching ratio \( B(D^+ \to \pi^+ \pi^+ \pi^-) = (3.6 \pm 0.4) \times 10^{-3} \). Eq. (34) leads via PCAC to a prediction for the \( D^0 \to \pi^+ \pi^- \) matrix element that appears to be consistent with the experimental value (33),

\[
\langle \pi^+ \pi^- | H_w | D^0 \rangle \approx \sqrt{2} f_\pi \sqrt{\frac{\Gamma_{(NR \text{ no } \sigma)}}{J_{D^+}}} = (4.0 \pm 0.2) \times 10^{-7} \text{ GeV}. \tag{35}
\]
The reasonable agreement between eqs. (33) and (35) is evidence for the production of a physical isoscalar \( \sigma(600) \) meson within the \( D^+ \to \pi^+\pi^+\pi^- \) weak decay.

This agreement is subject to two experimental caveats: 1) the non-resonant rate (34) may in fact have other resonant processes buried within it besides the dominant \( D^+ \to \pi^+\sigma \) and \( D^+ \to \pi^+\rho^0 \) resonant channels, and 2) the PDG branching fraction quoted above for the full \( D^+ \to \pi^+\pi^+\pi^- \) rate may in fact be too large. This second possibility is suggested by the E791 Collaboration’s new measurements of \( \Gamma(D^+ \to \pi^+\pi^+\pi^-)/\Gamma(D^+ \to \pi^+\pi^+K^-) = 0.0311 \pm 0.018_{-0.0026}^{+0.0016} \). The \( D^+ \to \pi^+\pi^+K^- \) branching fraction \( (9.0 \pm 0.6)\% \) [1] suggests a lowering of \( \mathcal{B}(D^+ \to \pi^+\pi^+\pi^-) \) to \( (2.96^{+0.48}_{-0.57}) \times 10^{-3} \), a reduction of \( \sim 20\% \) from the PDG value \( (3.6 \pm 0.4) \times 10^{-3} \). Such a 20\% reduction in the \( D^+ \to \pi^+\pi^+\pi^- \) branching fraction would reduce the prediction of (35) by 10\%.

ACKNOWLEDGMENTS

ADP and NAT acknowledge support from EU-TMR programme, contract CT98-0169 and ADP is grateful to M.L. Mangano for his kind hospitality at CERN. VE is grateful for research support from the Natural Sciences and Engineering Research Council of Canada and for hospitality from The University of Arizona. MDS acknowledges prior work with A. Bramon, R. E. Karlsen and S.R. Choudhury.

APPENDIX: \( \eta^- \) AND \( \eta'^- \)-POLE CONTRIBUTIONS TO \( K_L \to \gamma\gamma \)

If the \( K_L \to \gamma\gamma \) amplitude of Fig. 3a is augmented by contributions from \( \eta \) and \( \eta' \) poles, the \( \pi^0, \eta, \) and \( \eta' \) contributions to the amplitude are respectively given by

\[ 4 \text{The E791 collaboration central value estimate for } \Gamma_{(NR)}(D^+ \to \pi^+\pi^+\pi^-) \text{ is only 7.8\%, due to an additional contribution from resonances other than } \rho \text{ and } \sigma \text{. Using this estimate, the eq. (35) matrix element is reduced to } 2.5 \times 10^{-7} \text{ GeV}. \]
\[ M_{\pi^o} = <2\gamma|\pi^o> \frac{1}{m_{K_L}^2 - m_{\pi^o}^2} <\pi^o|H_{w}^{pc}|K_L>, \]  
\[ (A.1) \]

\[ M_{\eta} = <2\gamma|\eta> \frac{1}{m_{K_L}^2 - m_{\eta}^2} <\eta|H_{w}^{pc}|K_L>, \]  
\[ (A.2) \]

\[ M_{\eta'} = <2\gamma|\eta'> \frac{1}{m_{K_L}^2 - m_{\eta'}^2} <\eta'|H_{w}^{pc}|K_L>. \]  
\[ (A.3) \]

To find the relative contributions of these matrix elements, we first note that

\[ <\eta|H_w|K_L> = \cos\theta_p <\eta_8|H_w|K_L> - \sin\theta_p <\eta_0|H_w|K_L>, \]  
\[ (A.4) \]

\[ <\eta'|H_w|K_L> = \sin\theta_p <\eta_8|H_w|K_L> + \cos\theta_p <\eta_0|H_w|K_L>, \]  
\[ (A.5) \]

where the pseudoscalar mixing angle \( \theta_p = -12.9^\circ \) \[25,26\]. If the relative sizes of transitions from \( K_L \) to nonstrange pseudoscalar-nonet states is scaled to the \( U(3) \) structure constants \[ i.e., ( <\pi^o|H_{w}^{pc}|K_L>: <\eta_8|H_{w}^{pc}|K_L>: <\eta_0|H_{w}^{pc}|K_L> ) = (d_{366} : d_{866} : d_{066}) = (-\frac{1}{2} : -\frac{1}{2\sqrt{3}} : \sqrt{\frac{2}{3}}) \], we then find that

\[ <\eta|H_w|K_L> = 0.198 <\pi^o|H_w|K_L>, \]  
\[ (A.6) \]

\[ <\eta'|H_w|K_L> = -1.72 <\pi^o|H_w|K_L>. \]  
\[ (A.7) \]

Using the matrix elements \[26\]

\[ <2\gamma|\pi^o> = 0.0250 \text{ GeV}^{-1}, \]

\[ <2\gamma|\eta> = 0.0255 \text{ GeV}^{-1}, \]

\[ <2\gamma|\eta'> = 0.0335 \text{ GeV}^{-1} \]  
\[ (A.8) \]

[Levi-Civita covariants have been factored out of (A.8)], we then find from (A.1-3) that the matrix elements for \( \pi^o, \eta, \) and \( \eta' \) pole contributions to \( K_L \to 2\gamma \) are respectively given by

\[ M_{\pi^o} = (0.109 \text{ GeV}^{-3}) <\pi^o|H_{w}^{pc}|K_L>, \]  
\[ (A.9) \]
\[ M_\eta = (-0.0975 \text{ GeV}^{-3}) \langle \pi^0 | H_{w\eta}^{pc} | K_L \rangle, \quad (A.10) \]

\[ M_{\eta'} = (+0.0861 \text{ GeV}^{-3}) \langle \pi^0 | H_{w\eta'}^{pc} | K_L \rangle. \quad (A.11) \]

Consequently, there is a near cancellation of \( \eta \) and \( \eta' \) pole contributions in the matrix-element sum:

\[ M_{\pi^0} + M_\eta + M_{\eta'} = (0.0976 \text{ GeV}^{-3}) \langle \pi^0 | H_{w\pi^0}^{pc} | K_L \rangle = (0.90) M_{\pi^0}. \quad (A.12) \]
REFERENCES

[1] C.Caso et al. (The Particle Data Group) Eur.Phys. J C3, 1 (1998).

[2] N.A. Törnqvist and M. Roos, Phys. Rev. Lett. 76, 1575 (1996); N.A. Törnqvist Zeit. Physik C68, 647 (1995); S. Ishida et al. Prog. Theor. Phys. 95, 745 (1995); R. Kaminski et al. Phys. Rev. D50, 3145 (1994); V. Elias and M. D. Scadron, Phys. Rev. Lett. 53, 1129 (1984); R. Delbourgo and M.D. Scadron, Phys. Rev. Lett. 48, 379 (1982) and Mod. Phys. Lett. A10, 251 (1995).

[3] E. van Beveren et al. Zeit. Phys. C30, 616 (1986); J.E. Augustin et al. Nucl. Phys. B320, 1 (1989); J.A. Oller et al. Nucl. Phys. A620, 438 (1997); R. Kaminski et al. Z. Phys. C74, 79 (1997) and Eur. Phys. J. C9, 141 (1999); M.P. Locher et al. Eur. Phys. J. C4, 317 (1998); D. Black et al. Phys. Rev. D58, 054012 (1998); R. Delbourgo and M. Scadron, Int. J. Mod. Phys. A13, 657(1998); K. Igi and K. Hikasa, Phys. Rev. D59, 034005 (1999); J.A. Oller and E. Oset, Phys. Rev. D59, 074001 (1999) and Phys. Rev. D60, 074023 (1999); M. Scadron, Eur. Phys. J. C6, 141 (1999); N.A. Törnqvist Eur. Phys. J. C11, 359 (1999); T. Hannah, Phys. Rev. D60, 017502 (1999); CLEO Collaboration, Phys. Rev. D61 (2000) 012002.

[4] M. Harada, F. Sannino and J. Schechter, Phys. Rev. D54, 1991 (1996).

[5] F. Sannino and J. Schechter, Phys. Rev. D52, 96 (1995).

[6] N.Isgur and J. Speth, Phys. Rev. Lett. 77, 2332 (1996); N.A. Törnqvist and M. Roos, ibid. 77, 2333 (1996).

[7] E. M. Aitala et al. (E791 collaboration) Experimental evidence for a light and broad scalar resonance in \( D^+ \rightarrow \pi^-\pi^+\pi^+ \) decay, Phys. Rev. Lett. 86, 770 (2001).

[8] See e.g. N.Paver and M.D. Scadron, Nuovo Cimento A78, 159 (1983); Ll- Ametller, C. Ayala and A. Bramon, Phys.Rev D29, 916 (1984).

[9] A. Bramon and M.D. Scadron, Europhys. Lett. 19, 663 (1992); A. Bramon, R.E. Karlsen
and M.D. Scadron, Mod. Phys. Lett. A8, 97 (1993); also see P. Pascual and R. Tarrach, Nucl. Phys. B146, 509 (1978) and S. B. Gerasimov, Sov. J. Nucl. Phys. 29, 259 (1979).

For a current-algebra estimate of $F_A/F_V = 0.6$, see T. Das, V. S. Mathur and S. Okubo, Phys. Rev. Lett. 19, 859 (1967).

[10] T. Marozuni, C.S. Lim and A.I. Sanda, Phys. Rev. Lett. 65, 404 (1990).

[11] M. Gell-Mann and M. Levy, Nuovo Cimento 16, 705 (1960); also see V. de Alfaro, S. Fubini, G. Furlan and C. Rossetti in Currents in Hadron Physics (North Holland, 1973), chap. 5.

[12] S. Weinberg, Phys. Rev. Lett. 65, 1177 (1990); M.D. Scadron, Mod. Phys. Lett. A7, 497 (1992); P. Ko and S. Rudaz, Phys. Rev. D50, 6877 (1994).

[13] R.E. Karlsen and M. D. Scadron, Mod. Phys. Lett. A6, 543 (1991).

[14] R.E. Karlsen and M.D. Scadron, Nuovo Cimento A106, 113 (1993).

[15] R.E. Karlsen and M.D. Scadron, Phys. Rev. D44, 2192 (1991).

[16] G. Eilam and M.D. Scadron, Phys. Rev. D31, 2263 (1985); S.R. Choudhury and M. D. Scadron, Nuovo Cimento A108, 289 (1995).

[17] R.E. Karlsen and M.D. Scadron, Phys. Rev. D45, 4108 (1992); D45, 4113 (1992).

[18] S. Weinberg, Phys. Rev. Lett 17, 616 (1966); H. Osborn, Nucl. Phys. B15, 501 (1970).

[19] M.D. Scadron, Repts. on Prog. Phys. 44, 213 (1981).

[20] J. Steinberger, Phys. Rev. 76, 1180 (1949).

[21] A. S. Deakin, V. Elias and M.D. Scadron, Mod. Phys. Lett. A9, 955 (1994).

[22] M. Boglione and M. R. Pennington, Eur. Phys. J. C9, 11 (1999).

[23] KTeV Collab. [A. Alavi-Harati et al.], Phys. Rev. Lett. 83, 917 (1999).
[24] A. Della Selva, A. De Rujula, and M. Mateev, Phys. Lett. B24, 468 (1967).

[25] H. F. Jones and M. D. Scadron, Nucl. Phys. B155, 409 (1979); M. D. Scadron, Phys. Rev. D29, 2076 (1984); for a review, see T. Feldmann, Int. J. Mod. Phys. A15, 159 (2000).

[26] R. Delbourgo, Dongsheng Liu and M. D. Scadron, Int. J. Mod. Phys. A14, 4331 (1999).
FIGURES

FIG. 1. LσM quark (a) and meson (b) loops for $\pi^+ \to e^+\nu\gamma$ decay.

FIG. 2. $K_S \to \pi\Delta I = \frac{1}{2}$ tree graphs in the $s$-channel (a) or in the $t$-channel (b).

FIG. 3. $K_L \to 2\gamma$ decay dominated by a $\pi^0$ pole (a), $K_S \to 2\gamma$ decay dominated by a $\sigma$ pole (b).
FIG. 4. Self energy-type $W$ graphs for the $K^0 \rightarrow \pi^0$ weak transition.

FIG. 5. $K_L \rightarrow \pi^0 \gamma \gamma$ weak decay via $\sigma$ pole.

FIG. 6. $K^+ \rightarrow \pi^+ \pi \pi$ decay via chirally subtracted $\sigma$ and $\pi^+$ pole graphs.