A note on connectivity preserving splitting operation for matroids representable over $GF(p)$

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Abstract

The splitting operation on a $p$-matroid does not necessarily preserve connectivity. It is observed that there exists a single element extension of the splitting matroid which is connected. In this paper, we define the element splitting operation on a $p$-matroids which is a splitting operation followed by a single element extension. It is proved that the element splitting operation on connected $p$-matroid yields a connected $p$-matroid. We give a sufficient condition to yield Eulerian $p$-matroids from Eulerian $p$-matroids under the element splitting operation. A sufficient condition to obtain hamiltonian $p$-matroid by applying the element splitting operation on $p$-matroid is also provided.

Keywords: $p$-matroid; element splitting operation; Eulerian matroid; connected matroid; hamiltonian matroid; elementary lift.

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1 Introduction

We discuss loopless and coloopless $p$-matroids, by $p$-matroid we mean a vector matroid $M \cong M[A]$ for some matrix $A$ of size $m \times n$ over the field $F = GF(p)$, for prime $p$.

We denote the set of column labels of $M$ (viz. the ground set of $M$) by $E$, the set of circuits of $M$ by $C(M)$, and the set of independent sets of $M$ by $I(M)$. For undefined, standard terminology in graphs and matroids, see Oxley [12].

Malavadkar et al. [8] defined the splitting operation for $p$-matroids as:

Definition 1.1. Let $M \cong M[A]$ be a $p$-matroid on the ground set $E$, $\{a, b\} \subset E$, and $\alpha \neq 0$ in $GF(p)$. The matrix $A_{a,b}$ is constructed from $A$ by appending an extra row to $A$ which has coordinates equal to $\alpha$ in the columns corresponding to the elements $a$, $b$, and zero elsewhere. Define the splitting matroid $M_{a,b}$ to be the vector matroid $M[A_{a,b}]$. The transformation of $M$ to $M_{a,b}$ is called the splitting operation.

A circuit $C \in C(M)$ containing $\{a, b\}$ is said to be a $p$-circuit of $M$, if $C \in C(M_{a,b})$. And if $C$ is a circuit of $M$ containing either $a$ or $b$, but it is not a circuit of $M_{a,b}$, then
we say $C$ is an $np$-circuit of $M$. For $a, b \in E$, if the matroid $M$ contains no $np$-circuit then splitting operation on $M$ with respect to $a, b$ is called trivial splitting.

Note that the class of connected $p$-matroids is not closed under splitting operation.

**Example 1.2.** The vector matroid $M \cong M[A]$ represented by the matrix $A$ over the field $GF(3)$ is connected, whereas the splitting matroid $M_{1,4} \cong M[A_{1,4}]$ is not connected.

$$
A = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
A_{1,4} = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

It is interesting to see that the vector matroid $M'_{1,4} \cong M[A'_{1,4}]$, which is a single element extension of $M_{1,4}$, is connected.

$$
A'_{1,4} = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

This example motivates us to investigate the question: If $M$ is a connected $p$-matroid and $M_{a,b}$ is the splitting matroid of $M$, then does there exist a single element extension of the splitting matroid that is connected? In the next section, we answer this question by defining the element splitting operation on a $p$-matroid $M$ which is splitting operation on $M$ followed by a single element extension.

## 2 Element Splitting Operation

In this section, we define the element splitting operation on a $p$-matroid $M$ and characterize its circuits.

**Definition 2.1.** Let $M \cong M[A]$ be a $p$-matroid on the ground set $E$, $\{a, b\} \subset E$, and $M_{a,b}$ be the corresponding splitting matroid. Let the matrix $A_{a,b}$ represents $M_{a,b}$ on $GF(p)$. Construct the matrix $A'_{a,b}$ from $A_{a,b}$ by adding an extra column to $A_{a,b}$, labeled as $z$, which has the last coordinate equal to $\alpha \neq 0$ and the rest are equal to zero. Define
the element splitting matroid \( M'_{a,b} \) to be the vector matroid \( M[A'_{a,b}] \). The transformation of \( M \) to \( M'_{a,b} \) is called the element splitting operation.

Splitting and element splitting operations on binary matroids are closely studied in [7, 9, 10, 13, 14, 15, 16]. A matroid \( L \) is a lift of the matroid \( M \), if there exists a matroid \( N \), and \( X \subseteq E(N) \) such that \( N/X = M \) and \( N \setminus X = L \). If \( X \) is a singleton set, then \( L \) is called an elementary lift of \( M \). In the following result, Mundhe et al. [11] showed the equivalence of splitting matroid with elementary lift for binary matroids:

**Lemma 2.2.** Let \( M \) and \( L \) be binary matroids. Then \( L \) is an elementary lift of \( M \) if and only if \( L \) is isomorphic to \( M_T \) for some \( T \subseteq E(M) \).

Lemma 2.2 can be extended to \( p \)-matroids by using the similar arguments used to prove it in [11]. Thus a splitting matroid \( M_{a,b} \) of \( p \)-matroid \( M \) is an elementary lift of \( M \). In-depth study on lifted graphic matroid is done in [2, 3, 5].

**Remark 2.3.** \( \text{rank}(A) < \text{rank}(A'_{a,b}) = \text{rank}(A) + 1 \). If the rank functions of \( M \) and \( M'_{a,b} \) are denoted by \( r \) and \( r' \), respectively, then \( r(M) < r'(M'_{a,b}) = r(M) + 1 \).

Let \( C = \{v_1, v_2, \ldots, v_k\} \), where \( v_i, i = 1, 2, \ldots, k \) are column vectors of the matrix \( A \), be an \( np \)-circuit of \( M \) containing only \( a \). Assume \( v_1 = a \), without loss of generality. Then there exist non-zero scalars \( \alpha_1, \alpha_2, \ldots, \alpha_k \in GF(p) \) such that \( \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k \equiv 0 \pmod{p} \). Let \( \alpha_z \in GF(p) \) be such that \( \alpha_z + \alpha_1 \equiv 0 \pmod{p} \). Note that \( \alpha_z \neq 0 \). Then in the matrix \( A'_{a,b} \), we have \( \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k + \alpha_z z \equiv 0 \pmod{p} \). Therefore the set \( C \cup z = \{v_1, v_2, \ldots, v_k, z\} \) is a dependent set of \( M'_{a,b} \). If both \( a, b \in C \), then by the similar arguments, we can show that \( C \cup z \) is a dependent set of \( M'_{a,b} \).

In the next Lemma, we characterize the circuits of \( M'_{a,b} \) containing the element \( z \).

**Lemma 2.4.** Let \( C \) be a circuit of \( p \)-matroid \( M \). Then \( C \cup z \) is a circuit of \( M'_{a,b} \) if and only if \( C \) is an \( np \)-circuit of \( M \).

**Proof.** First assume that \( C \cup z \) is a circuit of \( M'_{a,b} \). If \( C \) is not an \( np \)-circuit of \( M \), then it is a \( p \)-circuit of \( M \), and hence it also is a circuit of \( M_{a,b} \) and \( M'_{a,b} \), as well. Thus we get a circuit \( C \) contained in \( C \cup z \), a contradiction.

Conversely, suppose \( C \) is an \( np \)-circuit of \( M \). Then \( C \) is an independent set of \( M'_{a,b} \). As noted earlier, \( C \cup z \) is a dependent set of \( M'_{a,b} \). On the contrary, assume that \( C \cup z \) is not a circuit of \( M'_{a,b} \), and \( C_1 \subseteq C \cup z \) be a circuit of \( M'_{a,b} \).

**Case 1:** \( z \notin C_1 \). Then \( C_1 \) is a circuit contained in \( C \), which is contradictory to the fact that \( C \) is independent in \( M'_{a,b} \).

**Case 2:** \( z \in C_1 \). Then \( C_1 \setminus z \) is a dependent set of \( M \) contained in the circuit \( C \) which is not possible. Thus we get \( C \cup z \) is a circuit of \( M'_{a,b} \).

We denote the collection of circuits described in Lemma 2.4 by \( C_z \).
Theorem 2.5. Let $M$ be a $p$-matroid on the ground set $E$ and $\{a,b\} \subset E$. Then $\mathcal{C}(M'_{a,b}) = \mathcal{C}(M_{a,b}) \cup \mathcal{C}_z$.

Proof. The inclusion $\mathcal{C}(M_{a,b}) \cup \mathcal{C}_z \subset \mathcal{C}(M'_{a,b})$ follows from the Definition [2.1] and Lemma [2.4]. For the other inclusion, let $C \in \mathcal{C}(M'_{a,b})$. If $z \notin C$, then $C \in \mathcal{C}(M_{a,b})$. Otherwise, $C \in \mathcal{C}_z$.

Example 2.6. Consider the matroid $R_8$, the vector matroid of the following matrix $A$ over field $GF(3)$.

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad A'_{3,5} = \begin{pmatrix} 1 & 0 & 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

For $a = 3, \ b = 5$ and $\alpha = 1$ the representation of element splitting matroid $M'_{3,5}$ over $GF(3)$ is given by the matrix $A'_{3,5}$. The collection of circuits of $M, M_{3,5}$ and $M'_{3,5}$ is given in the following table.

| Circuits of $M$ | Circuits of $M_{3,5}$ | Circuits of $M'_{3,5}$ |
|-----------------|----------------------|----------------------|
| $\{1,2,3,4,5\}$, $\{1,2,7,8\}, \{1,4,6,7\}$ | $\{1,2,3,4,5\}$, $\{1,2,7,8\}$, $\{1,4,6,7\}$ | $\{1,2,3,4,5\}$, $\{1,2,7,8\}$ |
| $\{2,4,6,8\},\{3,5,6,7,8\}$ | $\{2,4,6,8\},\{3,5,6,7,8\}$ | $\{2,4,6,8\},\{3,5,6,7,8\}$ |
| $\{1,2,3,5,6,7\}$, $\{1,2,3,5,6,8\}$ | $\{1,2,3,5,6,7\}$, $\{1,2,3,5,6,8\}$ | $\{1,2,3,5,6,7\}$, $\{1,2,3,5,6,8\}$ |
| $\{1,3,4,5,6,8\}$, $\{1,3,4,5,7,8\}$ | $\{1,3,4,5,6,8\}$, $\{1,3,4,5,7,8\}$ | $\{1,3,4,5,6,8\}$, $\{1,3,4,5,7,8\}$ |
| $\{2,3,4,5,6,7\}$, $\{2,3,4,5,7,8\}$ | $\{2,3,4,5,6,7\}$, $\{2,3,4,5,7,8\}$ | $\{2,3,4,5,6,7\}$, $\{2,3,4,5,7,8\}$ |
| $\{1,2,3,4,6\}$, $\{1,2,3,4,7\}$ | $\{1,2,3,4,6\}$, $\{1,2,3,4,7\}$ | $\{1,2,3,4,6\}$, $\{1,2,3,4,7\}$ |
| $\{1,2,3,4,8\}, \{1,2,5,6\}$ | $\{1,2,3,4,8\}, \{1,2,5,6\}$ | $\{1,2,3,4,8\}, \{1,2,5,6\}$ |
| $\{1,3,5,7\}, \{1,3,6,8\}$ | $\{1,3,5,7\}, \{1,3,6,8\}$ | $\{1,3,5,7\}, \{1,3,6,8\}$ |
| $\{1,4,5,8\}, \{1,5,6,7,8\}$ | $\{1,4,5,8\}, \{1,5,6,7,8\}$ | $\{1,4,5,8\}, \{1,5,6,7,8\}$ |
| $\{2,3,5,8\}, \{2,3,6,7\}$ | $\{2,3,5,8\}, \{2,3,6,7\}$ | $\{2,3,5,8\}, \{2,3,6,7\}$ |
| $\{2,4,5,7\}, \{2,5,6,7,8\}$ | $\{2,4,5,7\}, \{2,5,6,7,8\}$ | $\{2,4,5,7\}, \{2,5,6,7,8\}$ |
| $\{3,4,5,6\}, \{3,4,7,8\}$ | $\{3,4,5,6\}, \{3,4,7,8\}$ | $\{3,4,5,6\}, \{3,4,7,8\}$ |
| $\{4,5,6,7,8\}$ | $\{4,5,6,7,8\}$ | $\{4,5,6,7,8\}$ |
2.1 Independent sets, Bases and Rank function of $M'_{a,b}$

In this section, we describe independent sets, bases and rank function of $M'_{a,b}$. Denote the set $I_z = \{ I \cup z : I \in \mathcal{I}(M) \}$.

**Lemma 2.7.** Let $M \cong M[A]$ be a $p$-matroid with the ground set $E$ and $M'_{a,b}$ be its element splitting matroid. Then $\mathcal{I}(M'_{a,b}) = \mathcal{I}(M_{a,b}) \cup I_z$.

**Proof.** Notice that $\mathcal{I}(M_{a,b}) \cup I_z \subseteq \mathcal{I}(M'_{a,b})$. For other inclusion, assume $T \in \mathcal{I}(M'_{a,b})$. If $z \notin T$, then $T \in \mathcal{I}(M_{a,b})$. And if $z \in T$, then $T \setminus \{z\} \in \mathcal{I}(M_{a,b})$. That is $T = I \cup z$ for some $I \in \mathcal{I}(M_{a,b})$.

**Case 1:** $I \in \mathcal{I}(M)$. Then $T \in I_z$.

**Case 2:** $I = C \cup I'$ where $C$ is an $np$-circuit of $M$ and $I' \in \mathcal{I}(M)$. Then by Lemma 2.4, $C \cup z$ is a circuit of $M'_{a,b}$ contained in $T$, a contradiction.

**Lemma 2.8.** Let $M$ be a $p$-matroid and $\{a, b\} \subset E$. Then $\mathcal{B}(M'_{a,b}) = \mathcal{B}(M_{a,b}) \cup B_z$, where $B_z = \{ B \cup z : B \in \mathcal{B}(M) \}$.

**Proof.** It is easy to observe that $\mathcal{B}(M_{a,b}) \cup B_z \subseteq \mathcal{B}(M'_{a,b})$. Next assume that $B \in \mathcal{B}(M'_{a,b})$. Then $\text{rank}(B) = \text{rank}(M) + 1$. If $B$ contains $z$, then $B \setminus z$ is an independent set of $M_{a,b}$ of size $\text{rank}(M)$. Then by similar arguments given in the proof of Lemma 2.7, $B = I \cup z$, for some $I \in \mathcal{I}(M)$. Therefore $B \setminus z$ is a basis of $M$ and $B \in B_z$. If $z \notin B$, then $B$ is an independent set of size $\text{rank}(M) + 1$. Therefore $B \in \mathcal{B}(M_{a,b})$.

In the following lemma, we provide the rank function of $M'_{a,b}$ in terms of the rank function of $M$.

**Lemma 2.9.** Let $r$ and $r'$ be the rank functions of the matroids $M$ and $M'_{a,b}$, respectively. Suppose $S \subseteq E(M)$. Then $r'(S \cup z) = r(S) + 1$, and

$$r'(S) = r(S), \quad \text{if } S \text{ contains no np-circuit of } M; \text{ and}$$

$$= r(S) + 1, \quad \text{if } S \text{ contains an np-circuit of } M. \quad (1)$$

**Proof.** The equality $r'(S \cup z) = r(S) + 1$ follows from the definition. The proof of the Equation(1) is discussed in Corollary 2.13 of [8].
3 Connectivity of element splitting p-matroids

Let $M$ be a matroid having the ground set $E$, and $k$ be a positive integer. The $k$-separation of matroid $M$ is a partition $\{S,T\}$ of $E$ such that $|S|, |T| \geq k$ and $r(S) + r(T) - r(M) < k$. For an integer $n \geq 2$, we say $M$ is an $n$-connected if $M$ has no $k$-separation, where $1 \leq k \leq n - 1$.

In the following theorem, we provide a necessary and sufficient condition to preserve the connectedness of a $p$-matroid under element splitting operation.

**Theorem 3.1.** Let $M$ be a connected $p$-matroid on the ground set $E$. Then $M'_{a,b}$ is a connected $p$-matroid on the ground set $E \cup \{z\}$ if and only if $M_{a,b}$ is the splitting matroid obtained by applying non-trivial splitting operation on $M$.

**Proof.** First assume that $M'_{a,b}$ is a connected $p$-matroid on the ground set $E \cup \{z\}$. On the contrary, suppose $M_{a,b}$ is obtained by applying trivial splitting operation. Then $M$ contains no $np$ circuits with respect to the splitting by elements $a,b$. Now, let $S = \{z\}$ and $T = E$. Then $r'(S) + r'(T) - r'(M'_{a,b}) = 1 + r(E) - (r(M) + 1) = 0 < 1$ gives a 1-separation of $M'_{a,b}$, which is a contradiction.

For converse part, assume that $M_{a,b}$ is the splitting matroid obtained by applying non-trivial splitting operation on $M$. Suppose that, $M'_{a,b}$ is not connected. It means $M'_{a,b}$ has 1-separation, say $\{S,T\}$. Then $|S|, |T| \geq 1$ and

$$r'(S) + r'(T) - r'(M'_{a,b}) < 1.$$  

**(2)**

**Case 1:** Assume $S = \{z\}$. Then $T$ contains an $np$ circuit. Then Equation (2) gives, $1 + (1 + r'(T)) - r'(M) - 1 < 1 \implies r'(T) < r'(M)$, which is not possible.

**Case 2:** Assume $|S| \geq 2$, $z \in S$. If $T$ contains no $np$-circuit then Equation (2) yields, $(r(S \setminus z) + 1) + r(T) - r(M) - 1 < 1$, that is $r(S \setminus z) + r(T) - r(M) < 1$. Therefore $\{S \setminus z, T\}$ gives 1-separation of $M$, a contradiction. Further, if $T$ contains an $np$-circuit, then $r'(S) = r(S \setminus z) + 1$, $r'(T) = r(T) + 1$. By Equation (2) we get $(r(S \setminus z) + 1) + (r(T) + 1) - r(M) - 1 < 1$, which gives $r(S \setminus z) + r(T) - r(M) < 0$, which is not possible. So in either case such separation does not exist. Therefore $M'_{a,b}$ is connected. \[\square\]

In Example 2.6 the $p$-matroid $R_8 \cong M[A]$ and its element splitting $p$-matroid $M'_{3,5} \cong M[A'_{3,5}]$ both are connected. In the next result we give a necessary and sufficient condition to preserve 3-connectedness of a $p$-matroid under the element splitting operation.

**Theorem 3.2.** Let $M$ be a 3-connected $p$-matroid. Then $M'_{a,b}$ is 3-connected $p$-matroid if and only if for every $t \in E(M)$ there is an $np$-circuit of $M$ not containing $t$.

**Proof.** Let $M'_{a,b}$ be 3-connected $p$-matroid. On contrary, if there is an element $t \in E(M)$ contained in every $np$-circuit of $M$. Take $S = \{z,t\}$ and $T = E \setminus S$. Then
r'(S) + r'(T) - r'(M'_{a,b}) = r(\{t\}) + 1 + r(T) - r(M) - 1 = r(\{t\}) + r(T) - r(M) = 1 < 2.
Because, in this case, t \in cl(T) hence r(T) = r(M). That is \{S,T\} forms a 2-separation of M'_a,b, a contradiction.
For converse part suppose, for every t \in E(M) there is an np-circuit of M not containing t. On the contrary assume that M'_a,b is not a 3-connected matroid. Then there exists a k separation, for k \leq 2, of M'_a,b. By Theorem 3.1. k can not be equal to 1. For k = 2, let \{S,T\} be a 2-separation of M'_a,b. Then \{S,T\} is a partition of E \cup \{z\} such that |S|, |T| \geq 2 and
\r'(S) + r'(T) - r'(M'_{a,b}) < 2. (3)

Case 1 : Suppose S = \{z,t\}, t \in E(M). By hypothesis, T contains an np-circuit not containing t. Then Equation 3 gives, (r(\{t\}) + 1 + (1 + r(T)) - r(M) - 1 < 2 \implies r(t) + r(T) - r(M) < 1. Thus \{\{t\},T\} forms a 1-separation of M, which is a contradiction.

Case 2 : Suppose z \in S and |S| \geq 3. If T contains no np-circuit then Equation 3 yields (r(S \setminus z) + 1 + r(T) - r(M) - 1 < 2 \implies r(S \setminus z) + r(T) - r(M) < 2. Therefore \{S \setminus z, T\} gives a 2-separation of M, a contradiction.

Further, if T contains an np-circuit, then r'(S) = r(S \setminus z) + 1, r'(T) = r(T) + 1. By Equation 3 we get (r(S \setminus z) + 1) + (r(T) + 1) - r(M) - 1 < 2 \implies r(S \setminus z) + r(T) - r(M) < 1. Thus, \{S \setminus z, T\} gives a 1-separation of M, a contradiction. So in either case such partition does not exist. Therefore M'_a,b is 3-connected.

4 Applications

For Eulerian matroid M on the ground set E there exists disjoint circuits C_1, C_2, \ldots, C_k of M such that E = C_1 \cup C_2 \cup \ldots \cup C_k. We call the collection \{C_1, C_2, \ldots, C_k\} a circuit decomposition of M.

Let \{a,b\} \subset E. We say a circuit decomposition \tilde{C} = \{C_1, C_2, \ldots, C_k\} of M an ep-decomposition of M if it contains exactly one np-circuit with respect to the a,b splitting of M. In the next proposition, we give a sufficient condition to yield Eulerian p-matroids from Eulerian p-matroids after the element splitting operation.

Proposition 4.1. Let M be Eulerian p-matroid and a,b \in E. If M has an ep-decomposition, then M'_{a,b} is Eulerian p-matroid.

Proof. Let \tilde{C} = \{C_1, C_2, \ldots, C_k\} be an ep-decomposition of M and C_1 be an np-circuit in it. Then C_1 \cup z is a circuit of M'_{a,b}. Thus \{C_1 \cup z, C_2, \ldots, C_k\} is the desired circuit decomposition of M'_{a,b}.

Proposition 4.2. Let M'_{a,b} is Eulerian p-matroid and \tilde{C} = \{C_1, C_2, \ldots, C_k\} be a circuit
decomposition of $M'_{a,b}$. If $\tilde{C}$ contains no member which is a union of an $np$-circuit and an independent set of $M$, then $M$ is Eulerian and has an $ep$-decomposition.

Proof. Assume, without loss of generality, $z \in C_1$. Then $C_1 \in C_z$ and $C_1 \setminus z$ is an $np$-circuit of $M$. We will show $C_1 \setminus z$ contains both $a$ and $b$. On the contrary assume that $C_1 \setminus z$ contains only $a$. Then $b \in C_i$ for some $i \in \{2, 3, \ldots, k\}$. Since $C_i$ is also a circuit of $M'_{a,b}$ containing only $b$, by Theorem 2.10 of [8] it must be a union of an $np$-circuit and an independent set of $M$, which is a contradiction to the hypothesis. Therefore $C_1 \setminus z$ contains both $a$ and $b$ and the collection $\{C_1 \setminus z, C_2, \ldots, C_k\}$ forms an $ep$-decomposition of $M$.

In Example 2.6, the matroid $R_8$ is Eulerian with $ep$-decomposition $E = C_1 \cup C_2$, where $C_1 = \{2, 4, 6, 8\}$ is a $p$-circuit and $C_2 = \{1, 3, 5, 7\}$ is an $np$-circuit. An element splitting matroid $M'_{3,5}$ is also Eulerian with circuit decomposition $E \cup z = C_1 \cup (C_2 \cup z)$.

M. Borowiecki [1] defined hamiltonian matroid as a matroid containing a circuit of size $r(M) + 1$. This circuit is called the hamiltonian circuit of the matroid $M$. In the next corollary, we give a sufficient condition to yield hamiltonian matroid from hamiltonian matroid after the element splitting operation.

**Corollary 4.3.** If $M$ is hamiltonian matroid with an $np$-circuit of size $r(M) + 1$, then $M'_{a,b}$ is hamiltonian.

Proof. Let $C$ be an $np$-circuit of $M$ of size $r(M) + 1$. Then by Proposition 2.4, $C \cup z$ is a circuit in $M'_{a,b}$ of size $r(M) + 2$.

In Example 2.6, the matroid $R_8 \cong M[A]$ is hamiltonian and its element splitting matroid $M'_{3,5} \cong M[A'_{3,5}]$ is also hamiltonian.

Rota conjectured that the family of matroids that are representable over finite fields has only finitely many excluded minors [6]. For example, the 4-point line, $U_{2,4}$, is the only excluded minor for the class of binary matroids. In the following example, we demonstrate that there exist a splitting of the ternary matroid $U_{2,4}$, which yields a graphic matroid.

**Example 4.4.** Let the matrix $A$ represents the ternary matroid $U_{2,4}$ and the vector matroid of $A_{1,3}$ represents the splitting matroid $M[A_{1,3}]$.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}, \quad A_{1,3} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad A'_{1,3} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Observe that
• the splitting matroid $M[A_{1,3}]$ is binary and matrix $B = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ gives its binary representation.

• $A'_{1,3}/5 = U_{2,4}$.

However, the element splitting operation on $U_{2,4}$ does not give a binary matroid. With this observation, we propose the following question:

For a given ternary matroid $M$, does there always exist a pair of elements $\{a, b\}$ in $E(M)$ such that the splitting matroid $M_{a,b}$ is binary (graphic)?

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