ON MINIMAX NONPARAMETRIC ESTIMATION OF SIGNAL IN GAUSSIAN NOISE

MIKHAIL ERMAKOV, IPME RAS, RUSSIA

Abstract. For the problem of nonparametric estimation of signal in Gaussian noise we point out the strong asymptotically minimax estimators on maxisets for linear estimators (see [10, 18]). It turns out that the order of rates of convergence of Pinsker estimator on this maxisets is worse than the order of rates of convergence for the class of linear estimators considered on this maxisets. We show that balls in Sobolev spaces are maxisets for Pinsker estimators.

1. Introduction

For nonparametric estimation problem of a signal in Gaussian white noise optimal rates of convergence of estimators has been explored for a wide range of functional spaces and for a completely different setups (see [4, 15, 8, 20] and references therein). The strong asymptotically minimax estimators are known only if a priori information is provided that a signal belongs to ellipsoid in $L_2$ [13, 17, 8, 20, 16], balls in $L_\infty$ [1, 3, 12, 14] or for bodies in Besov spaces [8]. The paper goal is to pay attention that strong asymptotically minimax estimators can be obtained for other sets of functions. For trigonometric orthogonal system of functions the definition of these sets coincides with the definition of a ball in Besov space $B_{2\infty}^\alpha$ for some norm. We shall denote these sets as $B(\alpha, P_0)$ with $\alpha > 0$ and $P_0 > 0$.

The balls $B_{2\infty}^\alpha(P_0)$ have remarkable properties in nonparametric estimation. This sets carry a rather reasonable information on a signal smoothness. These sets are the sets having a given rates of convergence for the most wellknown linear nonparametric estimators [10, 11].

For linear statistical estimators these sets are the largest sets with a given rate of convergence [15].

Nonparametric estimation of solutions of linear ill-posed inverse problems in Gaussian noise for the sets $B(\alpha, P_0)$ has been also explored earlier in econometrics [9].

The arising strong asymptotically minimax estimators are penalized maximum likelihood estimators for some quadratic penalty function [5, 21]. Thus we obtain that likelihood estimation with quadratic penalty function is optimal not only in Bayes sense but in the minimax sense as well. These asymptotically minimax estimators are also trigonometric spline estimators [8, 20, 21]. The results can be also interpreted as a solution of inverse problem. For Bayes estimators and maximum penalized estimators one needs to find the largest sets such that these estimators are asymptotically minimax on these sets.

The nonasymptotic setup is also explored. In this setup we show that our estimator is minimax for the class of all linear estimators.

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The results can be easily modified on the case of minimax estimation of linear ill-posed problem. For this setup minimax estimator can be treated as some version of Tikhonov regularization algorithm \cite{18}.

We show that the order of rates of convergence of Pinsker estimator on \(B(\alpha, P_0)\) is worse than the order of rates of convergence for the asymptotically minimax estimators on this maxisets, and the balls in Sobolev spaces are maxisets for Pinsker estimators.

The results will be provided in terms of sequence model. Let us observe a random sequence \(y = \{y_j\}_{j=1}^{\infty}\),

\[ y_j = x_j + \epsilon \sigma_j \xi_j, \quad \epsilon > 0, \quad 1 \leq j < \infty \]

where \(\sigma_j > 0\) are known constants and \(\xi_j, 1 \leq j < \infty\), are independent Gaussian random variables, \(E\xi_j = 0\) and \(E\xi_j^2 = 1\).

The problem is to estimate the parameter \(x = \{x_j\}_{j=1}^{\infty}\).

Denote \(\sigma = \{\sigma_j\}_{j=1}^{\infty}\) and \(\xi = \{\xi_j\}_{j=1}^{\infty}\).

For the estimation with fixed \(\epsilon > 0\) minimax estimators will be established if a priori information is provided in the following form

\[ x \in B(a, P_0) = \left\{ x = \{x_j\}_{j=1}^{\infty} : \sup_k a_k^{-1} \sum_{j=k}^{\infty} x_j^2 \leq P_0 \right\} \tag{1.1} \]

where \(a = \{a_k\}_{k=1}^{\infty}\) and \(a_k > 0\) is decreasing sequence.

For asymptotically minimax estimation we shall consider the more narrow class of sets \(B(\alpha, P_0) = B(\tilde{a}, P_0)\) with \(\tilde{a} = \{k^{-2\alpha}\}, \alpha > 0\). The analysis of the proof shows that the results can be extended on another sequences \(a_k\). However this requires more accurate reasoning. For trigonometric orthogonal system of function \(\sup_k a_k^{-1} \sum_{j=k}^{\infty} x_j^2\) can be considered as some norm in Besov space \(B_{2\infty}^0\). For Besov bodies in \(B_{2\infty}^0\) generated wavelets asymptotically minimax estimators one can find in Johnstone \cite{8}. For this setup another extremal problem arises.

There are numerous research on strong adaptive asymptotically minimax estimation \cite{8, 20}. The results on adaptive estimation in Pinsker model \cite{8, 20} are easily carried over on paper setup for asymptotically minimax estimation on the sets \(B(\alpha, P_0)\).

Below we remind the definition of maxisets.

For estimator \(\hat{x}_\epsilon\), for the loss function \(||\hat{x}_\epsilon - x||^2\), for rates of convergence \(\epsilon^\gamma, \gamma > 0\), and for the constant \(C > 0\), the maxiset is

\[ MS(\hat{x}_\epsilon, \gamma)(C) = \{ x : \sup_{\epsilon} \epsilon^{-2\gamma} E_x ||\hat{x}_\epsilon - x||^2 < C \}. \]

Here \(\|x\|\) denotes norm of vector \(x = \{x_j\}_{j=1}^{\infty}\) in Hilbert space,

\[ \|x\|^2 = \sum_{j=1}^{\infty} x_j^2. \]

In what follows we shall denote letters \(c, C\) positive constants and let \(a_\epsilon \approx b_\epsilon\) imply \(c < a_\epsilon/b_\epsilon < C\) for all \(\epsilon > 0\).

\section{Main Results}

We say that linear estimator \(\hat{x}_\epsilon = \{\hat{x}_{j\lambda}\}_{j=1}^{\infty}\) is minimax in the class of linear estimators \(\hat{x}_{\lambda} = \{\hat{x}_{j\lambda}\}_{j=1}^{\infty}, \hat{x}_{j\lambda} = \lambda_j y_j, \lambda_j \in R^1, 1 \leq j < \infty, \lambda = \{\lambda_j\}_{j=1}^{\infty}\), if

\[ R_{\epsilon} = \sup_{x \in B} E_x ||\hat{x}_\epsilon - x||^2 = \inf_{\lambda} \sup_{x \in B} E_x ||\hat{x}_{\lambda} - x||^2. \tag{2.1} \]
We say that the estimator $\hat{x}_\epsilon$ is asymptotically minimax if

$$R_\epsilon = \sup_{x \in B(\alpha, P_0)} E_x ||\hat{x}_\epsilon - x||^2 = \inf_{\hat{x} \in \Psi} \sup_{x \in B(\alpha, P_0)} E_x ||\hat{x}_\epsilon - x||^2(1 + o(1))$$  \hspace{1cm} (2.2)

as $\epsilon \to 0$. Here $\Psi$ is the set of all estimators.

The minimax estimator in the class of linear estimators will be established if the following assumptions hold.

**A1** There is $c > 0$ such that $c < \sigma_j^2 < \infty$ for all $j$.

**A2.** For all $j > 1$

$$\frac{\sigma_j^2(a_j - a_{j+1})}{\sigma_{j-1}(a_j - a_{j+1})} > 1.$$  \hspace{1cm} (2.3)

This implies that sequence $\sigma_j^2(a_j - a_{j+1})$ is strictly increasing.

**Theorem 2.1.** Assume **A1, A2.** Then the linear estimator $\hat{x}_\lambda$ with

$$\lambda_j = \frac{P_0(a_j - a_{j+1})}{P_0(a_j - a_{j+1}) + \epsilon^2 \sigma_j^2}. \hspace{1cm} (2.4)$$

is minimax on the set of all linear estimators.

The minimax risk equals

$$R_\epsilon = \epsilon^2 \sum_{j=1}^{\infty} \frac{P_0 \sigma_j^2(a_j - a_{j+1})}{P_j(a_j - a_{j+1}) + \epsilon^2 \sigma_j^2}.$$  \hspace{1cm} (2.5)

**Remark 2.1.** The estimator $\hat{x}_\lambda$ is maximum penalized likelihood estimator with quadratic penalty function

$$P_0^{-1} \sum_{j=1}^{\infty} (a_j - a_{j+1})^{-1} \sigma_j^2 x_j^2$$

and Bayes estimator with a priori measure corresponding independent Gaussian coordinates $x_j$, $E x_j = 0$ and $E x_j^2 = P_0(a_j - a_{j+1})$, $1 \leq j < \infty$.

**Remark 2.2.** Theorem holds also for a finite number of observations $y_1, \ldots, y_k$ with $k < \infty$.

In Theorem **2.1A** we replace A2 more simple assumption.

**B1.** For all $j > j_0$

$$\frac{\sigma_j^2(a_j - a_{j+1})}{\sigma_{j-1}(a_j - a_{j+1})} > 1.$$  \hspace{1cm} (2.6)

This implies that sequence $\sigma_j^2(a_j - a_{j+1})$ is strictly increasing.

**Theorem 2.2.** Assume **A1, B1.** Then the linear estimator $\hat{x}_\lambda$ with

$$\lambda_j = \frac{2\alpha P_0 j^{-2\alpha - 1}}{2\alpha P_0 j^{-2\alpha - 1} + \epsilon^2 \sigma_j^2}. \hspace{1cm} (2.7)$$

is asymptotically minimax on the set of all estimators.

The asymptotically minimax risk equals

$$R_\epsilon = \epsilon^2 \sum_{j=1}^{\infty} \frac{2\alpha P_0 j^{-2\alpha - 1} \sigma_j^2}{2\alpha P_0 j^{-2\alpha - 1} + \epsilon^2 \sigma_j^2}(1 + o(1)).$$  \hspace{1cm} (2.8)

**Remark 2.3.** The estimator $\hat{x}_\lambda$ is maximum penalized likelihood estimator with quadratic penalty function

$$(2\alpha P_0)^{-1} \sum_{j=1}^{\infty} j^{1+2\alpha} \sigma_j^2 x_j^2$$

and Bayes estimator with a priori measure corresponding independent Gaussian coordinates $x_j$, $E x_j = 0$ and $E x_j^2 = 2\alpha P_0 j^{-2\alpha - 1}$, $1 \leq j < \infty$. 


Theorems 2.1 and 2.2 are extended easily on linear ill-posed inverse problem setup. The maxisets for for linear ill-posed inverse problems has been studied Loubes and Rivoirard [9].

Suppose we observe a random vector

\[ y = Rx + \epsilon \xi \]

with linear self-adjoint operator \( R : H \to H \) in a separable Hilbert space \( H \). Other notations are the same as in previous setup.

Suppose the linear operator \( R \) admits singular value decomposition (see [20, 8, 7, 9]) with eigenvalues \( r_j, 1 \leq j < \infty \). Then we can consider this setup in the following form.

We observe random vector

\[ z_j = r_j x_j + \sigma_j \xi_j, \quad 1 \leq j < \infty. \]

Suppose \( \xi_j, 1 \leq j < \infty \), are i.i.d. Gaussian r.v.’s, \( E\xi_j = 0 \), \( E\xi_j^2 = 1 \). The problems of estimation of \( x = \{x_j\}_{j=1}^{\infty} \) are the same. Dividing on \( r_j \), we obtain the setup of signal estimation.

Below two asymptotics of minimax risks for linear ill-posed inverse problems are provided.

**Example 2.1.** Let \( \alpha > 0, \gamma > 0 \). Let \( |r_j| = C j^{-\gamma} (1 + o(1)) \) and \( \sigma_j = 1, 1 \leq j < \infty \). Then

\[
R_\epsilon = \epsilon^{\frac{4\alpha}{\gamma 2\alpha + 2\gamma}} \pi \frac{2\alpha + 1}{2\alpha} C^{\frac{2\alpha + 2\gamma}{2\gamma + 2\alpha}} (1 + o(1)). \tag{2.9}
\]

**Example 2.2.** Let \( \alpha > 0, \gamma > 0, B > 0, \kappa \in \mathbb{R}^1 \). Let \( |r_j| = C j^{-\kappa} \exp\{-B j^\gamma\} \) and \( \sigma_j = 1, 1 \leq j < \infty \). Then

\[
R_\epsilon = P_0 B^{2\alpha/\gamma} \log \epsilon^{-2\alpha/\gamma} (1 + o(1)). \tag{2.10}
\]

Note that these asymptotics coincide with the asymptotics of risks of corresponding Bayes estimators.

Johnstone (Th 3.10, Ch3, [8]) has provided the comparison of strong asymptotics of minimax risks for trigonometric spline estimators and Pinsker estimators if unknown signal belongs to a ball in Sobolev space. The trigonometric spline estimators are strong asymptotically minimax estimators on maxisets \( B(\alpha, P_0) \). Thus we can consider this result as a comparison of risk asymptotics for strong asymptotically minimax estimators on maxisets \( B(\alpha, P_0) \) and Pinsker estimators. Below we provide similar comparison, if a priori information is provided, that unknown signal belongs to maxiset \( B(\alpha, P_0) \).

Pinsker estimator \( \hat{\theta}_\mu = \{\hat{\theta}_{j\epsilon}\}_{j=1}^{\infty} \) is linear estimator

\[ \hat{\theta}_{j\epsilon} = \lambda_{j\epsilon} y_j \]

with

\[ \lambda_{j\epsilon} = (1 - \mu b_j)_+ \]

where \( b_j = j^\beta, \beta > 0 \), and parameter \( \mu \) is defined by equation

\[
\epsilon^2 \sum_{j=1}^{\infty} b_j^2 ((\mu b_j)^{-1} - 1)_+ = P.
\]

Pinsker estimator is asymptotically minimax on ellipsoids

\[
S(\beta, P) = \left\{ x : \sum_{j=1}^{\infty} b_j^2 x_j^2 \leq P, x = \{x_j\}_{j=1}^{\infty} \right\},
\]

\[
Pinsker estimator is asymptotically minimax on ellipsoids
\]
with \( b = \{b_j\}_1^{\infty} \) and \( P > 0 \).

Denote
\[
R_c(\alpha, \beta) = \inf_{\mu} \sup_{\theta \in B(\alpha, P_0)} E_\theta[|\tilde{\theta}_c - \theta|^2].
\]

Denote
\[
C = \frac{2\alpha^2}{(1 + \alpha)(1 + 2\alpha)}
\]

**Theorem 2.3.** Let \( 0 < \alpha < \beta \). Then
\[
R_c(\alpha, \beta) = C^{\frac{2\beta}{1+2\beta}} C_1^{\frac{1}{1+2\beta}} \left( (2\alpha)^{-\frac{2\alpha}{1+2\alpha}} + (2\alpha)^{\frac{1}{1+2\alpha}} \right) \epsilon^{\frac{4\alpha}{1+2\alpha}} \tag{2.11}
\]
with \( C_1 = \frac{\beta}{\beta - \alpha} P_0 \).

Let \( \alpha > \beta > 0 \). Then
\[
R_c(\alpha, \beta) = C^{\frac{2\beta}{1+2\beta}} C_1^{\frac{1}{1+2\beta}} \left( (2\beta)^{-\frac{2\beta}{1+2\beta}} + (2\beta)^{\frac{1}{1+2\beta}} \right) \epsilon^{\frac{4\beta}{1+2\beta}} \tag{2.12}
\]
with
\[
C_1 = \sum_{j=1}^{\infty} j^{2\beta} (j^{-2\alpha} - (j+1)^{-2\alpha})
\]
If \( \alpha = \beta \), then
\[
R_c(\alpha, \beta) = \left( (2\alpha^2)^{\frac{1}{1+2\alpha}} + 2^{-\frac{2\alpha}{1+2\alpha}} \alpha^{\frac{1-2\alpha}{1+2\alpha}} P_0^{\frac{1}{1+2\alpha}} \epsilon^{\frac{4\alpha}{1+2\alpha}} \right) (1 + 2\alpha)^{-\frac{1}{1+2\alpha}} P_0^{\frac{1}{1+2\alpha}} \epsilon^{\frac{4\alpha}{1+2\alpha}} |2 \ln \epsilon|^{\frac{1}{1+2\alpha}} \tag{2.13}
\]

The most interest represents the comparison of risks of Pinsker estimator and asymptotically minimax estimators on maxisets if \( \alpha = \beta \). For this setup we compare the risks of estimators on the sets having almost the same smoothness. We see that the risks of Pinsker estimators have additional logarithmic term in asymptotic. Pinsker estimators do not belong to the class of linear estimators having the maxisets \( B(\alpha, P_0) \). It turns out that the balls in Sobolev space \( S(\beta, P) \) are maxisets for Pinsker estimators.

**Theorem 2.4.** There exists \( C > 0 \) such that, for all \( \epsilon > 0 \),
\[
R_c(\beta, x) = \epsilon^{-\frac{4\beta}{1+2\beta}} \inf_{\mu} E_x[|\tilde{x}_c - x|^2] < C < \infty, \tag{2.14}
\]
if and only if \( x \) belongs to Sobolev space
\[
S^\beta = \left\{ x : \sum_{j=1}^{\infty} b_j^2 x_j^2 < \infty, \quad x = \{x_j\}_1^{\infty} \right\}.
\]

In the theory of linear ill-posed inverse problems one of the most widespread assumption is that the solution \( x \) satisfies a source condition [2]
\[
x \in \{ x : x = Bu, \|u\| \leq 1, u \in H \},
\]
where \( B \) is linear self-conjugate compact operator. This implies that the solution \( x \) belongs to ellipsoid. Theorems 2.3 and 2.4 show that optimal linear solution on such sets can have worse rates of convergence on more wider sets then other linear estimators.
3. Proof of Theorems

3.1. Proof of Theorem 2.1

We begin with the proof of lower bound. Denote \( \theta_j^2 = P_0(a_j - a_{j+1}), \theta = \{\theta_j\}_{j=1}^\infty. \)

We have

\[
\inf_{\lambda} \sup_{x \in B} E_x||\hat{x}_\lambda - x||^2 \geq \inf_{\lambda} E_\theta||\bar{x}_\lambda - \theta||^2 = \epsilon^2 \sum_{j=1}^\infty \frac{\theta_j^2 a_j^2}{\theta_j^2 + \epsilon^2 \sigma_j^2}
\]

(3.1)

and infimum is attained for

\[
\lambda_j = \frac{\theta_j^2}{\theta_j^2 + \epsilon^2 \sigma_j^2} = \frac{P_0(a_j - a_{j+1})}{P_0(a_j - a_{j+1}) + \epsilon^2 \sigma_j^2}.
\]

Proof of upper bound is based on the following reasoning. Let \( x = \{x_j\}_{j=1}^\infty \in B. \) For all \( k \) denote

\[
u_k = a_k^{-1} \sum_{j=k}^\infty x_j^2.
\]

Then \( x_k^2 = a_k u_k - a_{k+1} u_{k+1}. \)

For the sequence of \( \lambda_j \) defined in Theorem 2.1 we have

\[
E_x \sum_{j=1}^\infty (\lambda_j y_j - x_j)^2 = \epsilon^2 \sum_{j=1}^\infty \lambda_j^2 \sigma_j^2 + \sum_{j=1}^\infty (1 - \lambda_j)^2 x_j^2
\]

\[
= \epsilon^2 \sum_{j=1}^\infty \lambda_j^2 \sigma_j^2 + \sum_{j=1}^\infty (\theta_j^2 \sigma_j^{-1} \epsilon^{-2} + 1)^{-1} (a_j u_j - a_{j+1} u_{j+1})
\]

\[
= \epsilon^2 \sum_{j=1}^\infty \lambda_j^2 \sigma_j^2 + (\theta_1^2 \sigma_1^{-1} \epsilon^{-2} + 1)^{-2} u_1
\]

\[
- \sum_{j=2}^{\infty} u_j a_j ((\theta_{j-1}^2 \sigma_{j-1}^{-1} \epsilon^{-2} + 1)^{-2} - (\theta_j^2 \sigma_j^{-1} \epsilon^{-2} + 1)^{-2})
\]

(3.2)

By A2, the last addendums in the right hand-side of (3.2) are negative. Therefore the supremum of right hand-side of (3.2) is attained for \( u_j = P_0, 1 \leq j < \infty. \) This completes the proof of Theorem 2.1.

3.2. Proof of Theorem 2.2

The upper bound follows from Theorem 2.1 below the proof of lower bound will be provided. This proof has a lot of common features with the proof of lower bound in Pinsker Theorem [8, 17, 20].

Fix values \( \delta_1, 0 < \delta_1 < 1, \) and \( \delta, 0 < \delta < P_0. \) Define a family of natural numbers \( k_r, \epsilon > 0, \) such that \( \epsilon^{-2} \sigma_k^2 \rightarrow 2r P_0 k_r^{-2r-1} = 1 + o(1) \) as \( \epsilon \rightarrow 0. \) Define sequence \( \eta = \{\eta_j\}_{j=1}^\infty \) of Gaussian i.i.d.r.v.’s \( \eta_j = \eta_j(\delta), E[\eta_j] = 0, \) \( \text{Var}[\eta_j] = (P_0 - \delta)(2r)^{-1} j^{-2r-1}, \) if \( \delta k_r \leq j \leq \delta_1 k_r, \) and \( \eta_j = 0 \) if \( j < \delta_1 k_r \) or \( j > \delta_1^{-1} k_r. \)

Denote \( \mu \) the probability measure of random vector \( \eta. \) Define \( \bar{x} \) Bayes estimator with a priory measure \( \mu. \)

Define the conditional probability measure \( \nu_\delta \) of random vector \( \eta \) given \( \eta \in B(\alpha, P_0). \) Define \( \bar{x} \) Bayes estimator of \( x \) with a priori measure \( \nu_\delta. \) Denote \( \theta \) the random variable having probability measure \( \nu_\delta. \)

For any estimator \( \hat{x} \) we have

\[
\sup_{x \in B_\infty^2} E_x||\hat{x} - x||^2 \geq E_\nu_\delta E_{\theta}||\bar{x} - \theta||^2
\]

\[
\geq E_\mu E_{\nu_\delta}||\bar{x} - \eta||^2 - E_\mu E_{\nu_\delta}||\bar{x} - \eta||^2, \eta \notin B(\alpha, P_0))P_\mu^{-1}(\eta \in B(\alpha, P_0).
\]

(3.3)

We have

\[
E_\mu E_{\nu_\delta}||\bar{x} - \eta||^2 = I(P_0 - \delta)(1 + o(1)),
\]

(3.4)
where

\[ I(P_0 - \delta) = \epsilon^2 \sum_{j=l_1}^{l_2} \frac{\sigma_j^2}{1 + (2\alpha (P_0 - \delta_1))^{-1} \epsilon^2 \sigma_j^2 j^{2\alpha + 1}} \]

with \( l_1 = [\delta_1 k_\epsilon] \) and \( l_2 = [\delta_1^{-1} k_\epsilon] \). Here \([a] \) denotes whole part of a number \( a \in \mathbb{R}^1 \).

Since

\[ ||\bar{x}||^2 \leq \sup_{x \in B_{\Sigma^\infty}} ||x||^2 \leq P_0, \]

we have

\[ E_\mu E_\eta(||\bar{x} - \eta||^2, \eta \notin B(\alpha, P_0)) \leq 2E_\mu E_\eta(||\bar{x}||^2 + ||\eta||^2, \eta \notin B(\alpha, P_0)) \]

\[ \leq 2P_0 P_\mu(\eta \notin B(\alpha, P_0)) + \sum_{j=l_1}^{l_2} (E_\mu \eta_j^4)^{1/2} P_\mu^{1/2}(\eta \notin B(\alpha, P_0)). \]  

(3.5)

Since \( E_\mu[\eta_j^4] \leq Cj^{-2(\alpha - 2)} \), we have

\[ \sum_{j=l_1}^{l_2} (E_\mu \eta_j^4)^{1/2} \leq C\delta_1^{-\alpha} k_\epsilon^{-2\alpha}. \]  

(3.6)

It remains to estimate

\[ P_\mu(\eta \notin B_{2\infty}^\epsilon) = P \left( \max_{1 \leq i \leq l_2} i^{2\alpha} \sum_{j=1}^{l_2} \eta_j^2 - P_0(1 - \delta_1/2) > P_0 \delta_1/2 \right) \leq \sum_{i=l_1}^{l_2} J_i \]  

(3.7)

with

\[ J_i = P \left( i^{2\alpha} \sum_{j=1}^{l_2} \eta_j^2 - P_0(1 - \delta/2) > P_0 \delta/2 \right) \]

To estimate \( J_i \) we implement the following Proposition [6].

**Proposition 3.1.** Let \( \xi = \{\xi_i\}^1_{i=1} \) be Gaussian random vector with i.i.d.r.v.’s \( \xi_i \), \( E\xi_i = 0, E\xi_i^2 = 1 \). Let \( A \) be \( l \times l \) matrix and \( \Sigma = A^T A \). Then

\[ P(||A\xi||^2 > \text{tr}(\Sigma) + 2\sqrt{\text{tr}(\Sigma^2) l} + 2||\Sigma||l) \leq \exp(-t). \]  

(3.8)

Here \( \text{tr}(\Sigma) \) denote the trace of matrix \( \Sigma \).

Define matrix \( \Sigma = \{\sigma_{ij}\}_{i,j=1}^{l_2} \) with \( \sigma_{jj} = j^{-2\alpha - 1}i^{2\alpha} \frac{P_0 - \delta}{\alpha(4\alpha + 1)} \) and \( \sigma_{lj} = 0 \) if \( l \neq j \).

Then

\[ 2\sqrt{\text{tr}(\Sigma^2) l} + 2||\Sigma||l = \frac{P_0 - \delta}{\alpha(4\alpha + 1)} \sqrt{i^{-1}t(1 + o(1)) + i^{-1}l \leq V_i(t)} \]  

(3.9)

We put \( t = k_\epsilon^{1/2} \). Then \( V_i(t) < Ck_\epsilon^{1/2} l_1 \leq i \leq l_2 \) and implementing (3.8) we have

\[ J_i \leq \exp(-k_\epsilon^{-1/2}) \]  

(3.10)

and therefore

\[ \sum_{j=l_1}^{l_2} J_i \leq \delta_1^{-1} k_\epsilon \exp(-k_\epsilon^{1/2}) \]  

(3.11)

To complete the proof it remains to estimate \( R_\epsilon - I(P_0 - \delta) \).

By straightforward estimation, it is easy to verify that

\[ |I(P_0) - I(P_0 - \delta)| < C\delta I(P_0) \]  

(3.12)
We have
\[
e^2 \sum_{j=1}^{l} \frac{\sigma_j^2}{1 + 2\alpha P_0^{-1} e^{2\sigma_j^2 j^{2\alpha+1}}} \leq e^2 \sum_{j=1}^{l} \sigma_j^2
\]
(3.13)

We have
\[
e^2 \sum_{j=1}^{l} \frac{\sigma_j^2}{1 + (2\alpha P_0)^{-1} e^{2\sigma_j^2 j^{2\alpha+1}}} \leq e^2 \sum_{j=1}^{l} j^{2\alpha-1}
\]
(3.14)

Now (3.12)-(3.14) imply that \( R_e = I(P_0 - \delta) \to 0 \) for some \( \delta = \delta(\epsilon) \to 0 \) and \( \delta_1 = \delta_1(\epsilon) \to 0 \) as \( \epsilon \to 0 \).

3.3. Proof of Theorem 2.3. The reasoning is based on the following Lemma.

Lemma 3.1.
\[
\sup_{x \in B(\alpha, P_0)} E_x[|\tilde{x}_e - x|^2] = E_{\theta_e}[|\tilde{x}_e - \theta_e|^2]
\]
(3.15)
with \( \theta_e = \{\theta_k\}_{k=1}^\infty, \theta_k = P_0(a_k - a_{k+1}) \).

Proof of Lemma 3.1. Denote \( u_k = a_k^{-1} \sum_{j=k}^\infty \theta_j^2 \). Then
\[
\theta_k^2 = a_k u_k - a_{k+1} u_{k+1}.
\]

Denote \( l = [\mu^{-1/\beta}] \).

We have
\[
E_x[|\tilde{x}_e - x|^2] = \mu^2 \sum_{j=1}^{l} b_j^2 x_j^2 + \sum_{j=l+1}^{\infty} x_j^2 + \epsilon^2 \sum_{j=1}^{l} \lambda_j^2 \geq J_1 + J_2 + J_3
\]
(3.16)
respectively.

We have
\[
J_1 + J_2 = \mu^2 \sum_{j=1}^{l} b_j^2 (a_j u_j - a_{j+1} u_{j+1}) + a_{l+1} u_{l+1}
\]
\[
= \mu^2 a_l b_l^2 u_l^2 - \mu^2 a_{l+1} b_{l+1}^2 u_{l+1}^2
\]
\[
+ \mu^2 \sum_{j=2}^{l} a_j u_j (b_j^2 - b_{j-1}^2) + a_{l+1} u_{l+1}.
\]
(3.17)

The maximum of right-hand side of (3.17) is attained for \( u_j = P_0, 1 \leq j < \infty \) with \( x_j^2 = P_0(a_j - a_{j+1}) \).

By straightforward calculations, we get \( J_3 = C e^2 \).

If \( \beta > \alpha \), we get
\[
J_1 + J_2 = \frac{\beta}{\beta - \alpha} l^{-2\alpha} (1 + o(1))
\]

If \( \alpha > \beta \), we get
\[
J_1 + J_2 = P_0 l^{-2\alpha} C (1 + o(1))
\]

If \( \alpha = \beta \), we get
\[
J_1 + J_2 = \alpha P_0 l^{-2\alpha} \ln l
\]

Minimizing \( J_1 + J_2 + J_3 \) with respect to \( l \), we get Theorem 2.3.
3.4. Proof of Theorem 2.4. It suffices to prove necessary conditions. We have

\[ E_{x} ||\tilde{x}_{\epsilon} - x||^2 = \epsilon^2 \sum_{j=1}^{l} (1 - l^{-\beta} j^\beta) + l^{-2\beta} \sum_{j=1}^{l} j^{2\beta} x_j^2 + \sum_{j=l}^{\infty} x_j^2 \geq C\epsilon^2 l + l^{-2\beta} \sum_{j=1}^{l} j^{2\beta} x_j^2 = J_\epsilon(l, x). \]  

(3.18)

It easy to see that, if

\[ \sum_{j=1}^{l} j^{2\beta} x_j^2 \to \infty \quad \text{as} \quad l \to \infty, \]  

(3.19)

then

\[ \lim_{\epsilon \to 0} \epsilon^{-\frac{2\beta}{1+2\beta}} \inf_{l} J_\epsilon(l, x) = \infty \]  

(3.20)

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