The Isospectral Dirac Operator on the 4-dimensional Orthogonal Quantum Sphere

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13th February 2008

Abstract

Equivariance under the action of $U_q(so(5))$ is used to compute the left regular and (chiral) spinorial representations of the algebra of the orthogonal quantum 4-sphere $S^4_q$. These representations are the constituents of a spectral triple on $S^4_q$ with a Dirac operator which is isospectral to the canonical one on the round sphere $S^4$ and which then gives $4^+$-summability. Non-triviality of the geometry is proved by pairing the associated Fredholm module with an ‘instanton’ projection. We also introduce a real structure which satisfies all required properties modulo smoothing operators.

Keywords: Noncommutative geometry, quantum group symmetries, quantum spheres, spectral triples, isospectral deformations.
1 Introduction

The recent constructions of spectral triples – with the consequent analysis of the corresponding spectral geometry – for the manifold of the quantum $SU(2)$ group in $[7, 5, 11, 12]$ and for its quantum homogeneous spaces (the Podleś spheres) in $[13, 10, 8, 9]$, have provided a number of examples showing that a marriage between noncommutative geometry and quantum groups theory is indeed possible. A common feature of most of these examples is that the dimension spectrum is the same as in the commutative ($q = 1$) limit. Furthermore, with the only known exception of the $0^+$-summable ‘exponential’ spectral triple on the standard Podleś sphere given in $[13]$, in order to have a real spectral triple one is forced to weaken the usual requirements that the real structure should satisfy.

It is then only natural to try and construct additional explicit examples wondering in particular if these properties are common to all quantum spaces or are rather coincidences which happen for low dimensional examples (all related to the quantum group $SU_q(2)$). In this paper we present an example in ‘dimension four’ given by a spectral triple on the orthogonal quantum sphere $S^4_q$ which is isospectral to the canonical spectral triple on the classical sphere with the round metric. There exists also a real structure which satisfies all required properties modulo an ideal of smoothing operators.

There are a few reasons why in dimension greater than or equal to four the orthogonal quantum sphere $S^4_q$ is most interesting to study. Firstly, all the relevant irreducible representations of the symmetry algebra $U_q(so(5))$ are known $[2]$ and both the algebra $\mathcal{A}(S^4_q)$ of polynomial functions as well as the modules of chiral spinors carry representations of $U_q(so(5))$ which are multiplicity free. Secondly, the spectrum of the Dirac operator $D$ for the round metric on the undeformed sphere $S^4$ is known $[20, 1]$. All this allows us to apply the already tested methods of isospectral deformations and to construct an $U_q(so(5))$-equivariant spectral triple on $S^4_q$.

The sphere $S^4_q$ could also be relevant for noncommutative physical models. In particular, on $S^4_q$ there is a canonical ‘instantonic vector bundle’ $[16]$ and the study of the noncommutative geometry of $S^4_q$ could be a first step for the construction of $SU_q(2)$ instantons on this space.

In Sect. 2 we recall all generalities about spectral triples that we need. We give also some properties of finitely generated projective modules over algebras having quantum group symmetries. The rest of the paper is organized as follows. Sections 3 and 4 are devoted to the symmetry Hopf algebra $U_q(so(5))$ and its fundamental $*$-algebra module, the orthogonal quantum sphere $S^4_q$. In Sect. 5 we describe the $\mathcal{A}(S^4_q)$-modules of chiral spinors over $S^4_q$. Sect. 6 is devoted to the left regular representation of the algebra $\mathcal{A}(S^4_q)$ of polynomial functions over $S^4_q$ and to the representations of $\mathcal{A}(S^4_q)$ which in the $q = 1$ limit correspond to the modules of chiral spinors. These representations are $U_q(so(5))$-equivariant, that is they correspond to representations of the crossed product algebra $\mathcal{A}(S^4_q) \rtimes U_q(so(5))$. In Section 7 we use the isospectral Dirac operator to construct a spectral triple on $S^4_q$; it will be $U_q(so(5))$-equivariant, regular, even and of metric dimension 4. We also prove that it is non-trivial by pairing the Fredholm module canonically associated to the spectral triple to an ‘instanton’ projection $e$. It turns out that the projection $e$ has charge 1, as in the classical case. In Sect. 8 we compute the part of the dimension spectrum contained in the right half plane $\{s \in \mathbb{C} | \text{Re } s > 2\}$, as well as
the top residue (which in the commutative case is proportional to the integral). This is done by quotienting by a suitable ideal of ‘infinitesimals’ $I$, which is larger than smoothing operators. At the moment we are unable to comment on the part of the dimension spectrum which is in the left half plane $\text{Re} s \leq 2$. Finally, in Sect. 9 we produce an equivariant real structure for which both the ‘commutant property’ and the ‘first order condition’ are satisfied modulo the ideal of smoothing operators; this is consonant with the cases of the manifold of $SU_q(2)$ in [11] and of Podleś spheres in [10, 9]. In fact, we also show that these conditions are much easier to handle modulo the ideal $I$.

2 Some useful preliminaries

In this section, we collect some basic notions concerning equivariant spectral triples. We also give some general properties of finitely generated projective modules over algebras having quantum group symmetries.

2.1 Generalities about Spectral Triples

We start with the notion of finite summable spectral triples [3].

**Definition 2.1.** A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is the datum of a complex associative unital $*$-algebra $\mathcal{A}$, a $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by bounded operators on a (separable) Hilbert space $\mathcal{H}$ and a self-adjoint (unbounded) operator $D = D^*$ such that,

- $(D + i)^{-1}$ is a compact operator;
- $[D, \pi(a)]$ is a bounded operator for all $a \in \mathcal{A}$.

We refer to $D$ as the ‘generalized’ Dirac operator, or the Dirac operator ‘tout court’ and for simplicity we assume that it is invertible. Usually, the representation symbol $\pi$ is removed when no risk of confusion arises.

With $n \in \mathbb{R}^+$, $D$ is called $n^+$-summable if the operator $(D^2 + 1)^{-1/2}$ is in the Dixmier ideal $\mathcal{L}^{n^+}(\mathcal{H})$. We shall also call $n$ the *metric dimension* of the spectral triple.

A spectral triple is called *even* if there exists a grading $\gamma$, i.e. a bounded operator satisfying $\gamma = \gamma^*$ and $\gamma^2 = 1$, such that the Dirac operator is odd and the algebra is even:

$$\gamma D + D \gamma = 0 \ , \quad a\gamma = \gamma a \ , \quad \forall \ a \in \mathcal{A} .$$

We recall from [6] a few analytic properties of spectral triples. To the unbounded operator $D$ on $\mathcal{H}$ one associates an unbounded derivation $\delta$ on $\mathcal{B}(\mathcal{H})$ by,

$$\delta(a) = [D|, a] ,$$

for all $a \in \mathcal{B}(\mathcal{H})$. A spectral triple is called *regular* if the following inclusion holds,

$$\mathcal{A} \cup [D, \mathcal{A}] \subset \bigcap_{j \in \mathbb{N}} \text{dom } \delta^j ,$$
and we refer to $\text{OP}^0 := \bigcap_{j \in \mathbb{N}} \text{dom} \delta^j$ as the ‘smooth domain’ of the operator $\delta$. For a regular spectral triple, the class $\Psi^0$ of pseudodifferential operators of order less than or equal to zero is defined as the algebra generated by $\bigcup_{k \in \mathbb{N}} \delta^k (A \cup \{D, A\})$. If the triple has finite metric dimension $n$, the ‘zeta-type’ function

$$\zeta_a(s) := \text{Tr}_H(a|D|^{-s})$$

associated to $a \in \Psi^0$ is defined (and holomorphic) for $s \in \mathbb{C}$ with $\text{Re} \ s > n$ and the following definition makes sense.

**Definition 2.2.** A spectral triple has dimension spectrum $\Sigma$ iff $\Sigma \subset \mathbb{C}$ is a countable set, for all $a \in \Psi^0$ the function $\zeta_a(s)$ extends to a meromorphic function on $\mathbb{C}$ with poles as unique singularities, and the union of such singularities is the set $\Sigma$.

If $\Sigma$ is made only of simple poles, the Wodzicki-type residue functional

$$\int T := \text{Res}_{s=0} \text{Tr}(T|D|^{-s})$$

is tracial on $\Psi^0$. We also recall the definition of ‘smoothing operators’ $\text{OP}^{-\infty}$,

$$\text{OP}^{-\infty} := \{ T \in \text{OP}^0 \, | \, |D|^k T \in \text{OP}^0, \, \forall \, k \in \mathbb{N} \} .$$

The class $\text{OP}^{-\infty}$ is a two-sided $*$-ideal in the $*$-algebra $\text{OP}^0$, is $\delta$-invariant and then in the smooth domain of $\delta$. If $T$ is a smoothing operator, $\zeta_T(s)$ is holomorphic on $\mathbb{C}$ and (2.1) vanishes. Thus, elements in $\text{OP}^{-\infty}$ can be neglected when computing the dimension spectrum and residue. Finally, we note that if the metric dimension is finite, rapid decay matrices – in a basis of eigenvectors for $D$ with eigenvalues in increasing order – are smoothing operators.

In analogy with the notion of spin manifold, one asks for the existence of a real structure $J$ on a spectral triple $(A, H, D)$. Motivated by the examples of real spectral triples on Podleś spheres [10, 9] and on $SU_q(2)$ [11], we use the following weakened definition of real structure.

**Definition 2.3.** A real structure is an antilinear isometry $J$ on $H$ such that $\forall \, a, b \in A$,

$$J^2 = \pm 1, \quad JD = \pm DJ, \quad [a, JbJ^{-1}] \subset \mathcal{I}, \quad [[D, a], JbJ^{-1}] \subset \mathcal{I}. $$

If the spectral triple is even with grading $\gamma$, we impose the further relation $J\gamma = \pm \gamma J$.

The signs ‘$\pm$’ are determined by the dimension of the geometry [4]. A real spectral triple of dimension 4 corresponds to the choices $J^2 = -1, JD = DJ$ and $J\gamma = \gamma J$.

The set $\mathcal{I}$ is a suitable two-sided ideal in the algebra $\text{OP}^0$ of ‘order zero’ operators which is made of ‘infinitesimals’. The original definition [4] corresponds to $\mathcal{I} = 0$; while in examples coming from quantum groups [10, 11, 9] one usually takes $\mathcal{I} = \text{OP}^{-\infty}$.

Let $F := \text{sign} \, D|D|^{-1}$ be the sign of $D$; if $(A, H, D)$ is a regular even spectral triple, the datum $(A, H, F, \gamma)$ is an even Fredholm module. We say that the Fredholm module is $p$-summable if $p \geq 1$ and, for all $a \in A$, $[F, a]$ belongs to the $p$-th Schatten-von Neumann ideal $\mathcal{L}^p(H)$.
of compact operators $T$ such that $|T|^p$ is of trace class. Associated with a $p$-summable even Fredholm module there are cyclic cocycles defined by

$$
\mathrm{ch}^F_n(a_0, \ldots, a_n) = \frac{1}{2n!} \Gamma\left(\frac{n}{2} + 1\right) \Tr(\gamma F[a_0] \cdots [F, a_n]),
$$

(2.2)

for all even integers $n \geq p - 1$. By composing it with a matrix trace, $\mathrm{ch}^F_n$ is canonically extended to matrices with entries in $A$. The pairing with elements $[e] \in K_0(A)$, given by $\mathrm{ch}^F_n(e, e, \ldots, e)$ build up to an integer-valued map $\mathrm{ch}^F([e])$ which depends only on the class $[e]$ and which yields the index of the Dirac operator $D$ twisted with the projection $e$ (for further details see [3]).

Finally, we turn now to symmetries; these will be implemented by an action of a Hopf $*$-algebra. Firstly, let $V$ be a dense linear subspace of a Hilbert space $H$ with inner product $\langle \cdot, \cdot \rangle$, and let $U$ be a $*$-algebra. An (unbounded) $*$-representation of $U$ on $V$ is a homomorphism $\lambda : U \to \text{End}(V)$ such that $\langle \lambda(h)v, w \rangle = \langle v, \lambda(h^*)w \rangle$ for all $v, w \in V$ and all $h \in U$. From now on, the symbol $\lambda$ will be omitted. Next, let $U = (U, \Delta, \epsilon, S)$ be a Hopf $*$-algebra and let $A$ be a left $U$-module $*$-algebra, i.e., there is a left action $\triangleright$ of $U$ on $A$ satisfying

$$
h \triangleright ab = (h(1) \triangleright a)(h(2) \triangleright b), \quad h \triangleright 1 = \epsilon(h)1, \quad h \triangleright a^* = \{S(h)^* \triangleright a\}^*,
$$

for all $h \in U$ and $a, b \in A$. As customary, $\Delta(h) = h(1) \otimes h(2)$.

A $*$-representation of $A$ on $V$ is called $U$-equivariant if there exists a $*$-representation of $U$ on $V$ such that, for all $h \in U$, $a \in A$ and $v \in V$, it happens that

$$
hav = (h(1) \triangleright a)h(2)v.
$$

Given $U$ and $A$ as above, the left crossed product $*$-algebra $A \rtimes U$ is defined as the $*$-algebra generated by the two $*$-subalgebras $A$ and $U$ with crossed commutation relations

$$
ha = (h(1) \triangleright a)h(2), \quad \forall h \in U, \ a \in A.
$$

Thus, $U$-equivariant $*$-representations of $A$ correspond to $*$-representations of $A \rtimes U$.

A linear operator $D$ defined on $V$ is said to be equivariant if it commutes with $U$, i.e.,

$$
Dhv = hDv
$$

(2.3)

for all $h \in U$ and $v \in V$. On the other hand, an antilinear operator $T$ defined on $V$ is called equivariant if it satisfies the relation

$$
Thv = S(h)^*Tv,
$$

(2.4)

for all $h \in U$ and $v \in V$, where $S$ denotes the antipode of $U$. Notice that if $T$ is an equivariant antilinear operator, its square $T^2$ is an equivariant linear operator, but $T^*T$ is not an equivariant linear operator unless $S^2 = 1$.

We use all these equivariance requirements in the following definition (see also [19]).

**Definition 2.4.** Let $U$ be a Hopf $*$-algebra and $A$ a left $U$-module $*$-algebra. A (real, even) spectral triple $(A, \mathcal{H}, D, \gamma, J)$ is called equivariant if $U$ is represented on a dense subspace $V$ of $\mathcal{H}$, $V \subset \text{dom} D$, the representation of $U$ commutes with the grading $\gamma$, the restriction of the representation of $A$ on $V$ is $U$-equivariant, the operator $D$ is equivariant and $J$ is the antunitary part of the polar decomposition of an equivariant antilinear operator.
2.2 Projective module description of equivariant representations

In order to construct the analogues of the modules of chiral spinors on the sphere $S_q^4$ we need some properties of finitely generated projective modules over algebras having quantum group symmetries.

Let $U$ be a Hopf $\ast$-algebra, $A$ be an $U$-module $\ast$-algebra and $\varphi : A \to \mathbb{C}$ be an invariant faithful state (i.e. $\varphi$ is linear, $\varphi(a^*a) > 0$ for all nonzero $a \in A$, and $\varphi(h \triangleright a) = \epsilon(h)\varphi(a) \forall a \in A$ and $h \in U$). Suppose also that there exists $\kappa \in \text{Aut}(A)$ such that the ‘twisted’ cyclicity

$$\varphi(ab) = \varphi(h \kappa(a))$$

holds for all $a, b \in A$. Instances of this situation are provided by subalgebras of compact quantum group algebras with $\varphi$ the Haar state and $\kappa$ the modular involution. KMS states in Thermal Quantum Field Theory provide additional examples. In particular, for the case $A = A(S_q^2)$ and $U = U_q(so(5))$, $\varphi$ comes from the Haar functional of $A(SO_q^2(5))$ and the modular automorphism is $\kappa(a) = K_8^6K_6^2 \triangleright a$ [15, Sect. 11.3.4].

For $N \in \mathbb{N}$, let $A^N := A \otimes \mathbb{C}^N$ be the linear space with elements $v = (v_1, \ldots, v_N), v_i \in A$, and $\mathbb{C}$-valued inner product given by

$$\langle v, w \rangle := \sum_{i=1}^{N} \varphi(v_i^*w_i). \quad (2.5)$$

Lemma 2.5. Let $\sigma : U \to \text{Mat}_N(\mathbb{C})$ be a $\ast$-representation. The formulæ:

$$(a.v)_i := av_i, \quad (h.v)_i := \sum_{j=1}^{N} (h_{(1)} \triangleright v_j)\sigma_{ij}(h_{(2)}), \quad (2.6)$$

for all $a, v \in A$ and $h \in U$ (and $i = 1, \ldots, N$), define a $\ast$-representation of the crossed product algebra $A \rtimes U$ on the linear space $A^N$.

Proof. The inner product allows us to define the adjoint of an element of $A \rtimes U$ in the representation on $A^N$. For $x \in \text{End}(A^N)$, its adjoint denoted with $x^\dagger$, is defined

$$\langle x^\dagger.v, w \rangle := \langle v, x.w \rangle, \quad \forall, v, w \in A^N.$$ 

Recall that being a $\ast$-representation means that $x^\dagger.v = x^*.v$ for any operator $x$ and any $v \in A^N$. The nontrivial part of the proof consists in showing that $h^\dagger.v = h^*.v$ for all $h \in U$ and $v \in A$. For $N > 1$ we are considering the Hopf tensor product of the $N = 1$ representation with a matrix representation that is a $\ast$-representation by hypothesis. Thus, it is enough to take $N = 1$.

The $U$-invariance of $\varphi$ implies:

$$\epsilon(h) \langle v, w \rangle = \varphi(h \triangleright (v^*w)) = \varphi((h_{(1)} \triangleright v^*)(h_{(2)} \triangleright w)).$$

But $h_{(1)} \triangleright v^* = \{S(h_{(1)})^* \triangleright v\}^*$ by definition of module $\ast$-algebra. Then,

$$\epsilon(h) \langle v, w \rangle = \langle S(h_{(1)})^*v, h_{(2)}.w \rangle = \langle v, S(h_{(1)})^*h_{(2)}.w \rangle.$$
We deduce that for all \( h \in U \) one has that
\[
S(h^{(1)})^{\dagger}h^{(2)} = \epsilon(h) .
\] (2.7)

Recall that the convolution product ‘\(*\)' for any \( F, G \in \text{End}(U) \) is defined by
\[
(F \ast G)(h) := F(h^{(1)})G(h^{(2)}) \quad \forall \ h \in U ;
\]
and \( (\text{End}(U), \ast) \) is an associative algebra with unity given by the endomorphism \( h \mapsto \epsilon(h)1_U \), with \( S \) a left and right inverse for \( \text{id}_U \) in \( (\text{End}(U), \ast) \), that is
\[
S \ast \text{id}_U = 1_U \epsilon = \text{id}_U \ast S .
\]

Let \( S' \in \text{End}(U) \) be the composition \( S' := \dagger \circ \ast \circ S \). Equation (2.7) implies that \( S' \) is a left inverse for \( \text{id}_U \):
\[
S' \ast \text{id}_U = 1_U \epsilon .
\]
Applying \( \ast S \) to the right of both members of this equation and using \( \text{id}_U \ast S = 1_U \epsilon \) we get
\[
S' = S \text{ as endomorphisms of } U , \text{ i.e. } S(h)^{\dagger} = S(h) \text{ for all } h \in U .
\]
This concludes the proof.

Now, let \( e = (e_{ij}) \in \text{Mat}_N(\mathcal{A}) \) be an \( N \times N \) matrix with entries \( e_{ij} \in \mathcal{A} \). Let \( \pi : \mathcal{A}^N \to \mathcal{A}^N \) be the (linear) endomorphism defined by:
\[
\pi(v)_j := \sum_{i=1}^N v_i e_{ij} ,
\] (2.8)
for all \( v \in \mathcal{A}^N \) and \( j = 1, \ldots, N \). Since \( \mathcal{A} \) is associative, left and right multiplication commute and \( \pi(av) = a \pi(v) \) for all \( a \in \mathcal{A} \) and \( v \in \mathcal{A}^N \). Thus we have the following lemma.

**Lemma 2.6.** The map \( \pi \) defined by \( (2.8) \) is an \( \mathcal{A} \)-module map.

Recall that an endomorphism \( p \) of an inner product space \( V \) is a projection (not necessarily orthogonal) if \( p \circ p = p \). A projection \( p \) is orthogonal if the image of \( p \) and \( \text{id}_V - p \) are orthogonal with respect to the inner product of \( V \), and this happens exactly when \( p^\dagger = p \).

A simple computation shows that the map \( \pi \) in \( (2.8) \) is a projection iff \( e^2 = e \), that is the matrix \( e \in \text{Mat}_N(\mathcal{A}) \) is an idempotent. Now we use the twisted-cyclicity of \( \varphi \) to deduce:
\[
\langle v, \pi^\dagger(w) \rangle = \langle \pi(v), w \rangle = \sum_{ij} \varphi(e_{ij}^* v_i^* w_j) = \sum_{ij} \varphi(v_i^* w_j \kappa(e_{ij}^*)) ,
\]
for all \( v, w \in \mathcal{A}^N \). Hence the adjoint \( \pi^\dagger \) of the endomorphism \( \pi \) is given by
\[
\pi^\dagger(w)_i = \sum_{j=1}^N w_j \kappa(e_{ij}^*) .
\]
Let \( e^* \) be the matrix with entries \( (e^*)_{jk} := e_{kj}^* \). We have the following lemma.
Lemma 2.7. The endomorphism $\pi$ in (2.8) is an orthogonal projection iff $e^2 = e = \kappa(e^*)$.

Next, we determine a sufficient condition for the endomorphism $\pi$ to be not only an $\mathcal{A}$-module map, but also an $\mathcal{U}$-module map.

Lemma 2.8. With $' t$ denoting transposition, if

$$h \triangleright e = \sigma(h_{(1)})^t e \sigma(S^{-1}(h_{(2)}))^t,$$

(2.9)

for all $h \in \mathcal{U}$, the endomorphism $\pi$ in (2.8) is an $\mathcal{U}$-module map.

Proof. Equation (2.9) can be rewritten as,

$$h \triangleright e_{ij} = \sum_{kl} \sigma_{kl}(h_{(1)}) e_{kl} \sigma_{ji}(S^{-1}(h_{(2)})�;

by using it into the definition (2.6) one checks that $\pi(h.v) = h.\pi(v)$ for all $h \in \mathcal{U}$ and $v \in \mathcal{A}^N$.

When Lemma 2.7 and Eq. (2.9) are satisfied, the orthogonal projections $\pi$ and $\pi^\perp = 1 - \pi$ split $\mathcal{A}^N$ into the orthogonal sum of two sub $*$-representations $\pi(\mathcal{A}^N)$ and $\pi^\perp(\mathcal{A}^N)$ of $\mathcal{A} \times \mathcal{U}$. The next lemma gives a (quite obvious) sufficient condition for $\pi(\mathcal{A}^N)$ and $\pi^\perp(\mathcal{A}^N)$ to be not equivalent as representations of $\mathcal{A}$. Recall that an isomorphism of $\mathcal{A}$-modules is an invertible $\mathcal{A}$-linear map, so isomorphic modules correspond to equivalent representations.

Lemma 2.9. Let $(\mathcal{A}, \mathcal{H}, F, \gamma)$ be an even Fredholm module over $\mathcal{A}$. If $\text{ch}^F([e]) \neq 0$, the $\mathcal{A}$-modules $\pi(\mathcal{A}^N)$ and $\pi^\perp(\mathcal{A}^N)$ are not equivalent.

Proof. The map $K_0(\mathcal{A}) \to \mathbb{Z}$, $[e] \mapsto \text{ch}^F([e])$ is an homomorphism. Suppose $\pi(\mathcal{A}^N)$ and $\pi^\perp(\mathcal{A}^N)$ are isomorphic $\mathcal{A}$-modules, then $[e] = [1 - e]$ and $\text{ch}^F([1 - e]) = \text{ch}^F([e])$. But from Eq. (2.2), $\text{ch}^F([1 - e]) = -\text{ch}^F([e])$ (since $[F, 1 - e] = -[F, e]$ and $n$ is even). Hence $\text{ch}^F([e]) = 0$, and this concludes the proof by contradiction.

3 The symmetry Hopf algebra $\dot{U}_q(so(5))$

Let $0 < q < 1$. We call $U_q(so(5))$ the real form of the Drinfeld-Jimbo deformation of $so(5, \mathbb{C})$, corresponding to the Euclidean signature $(+, +, +, +, +)$; it is a real form of the Hopf algebra called $\dot{U}_q(so(5, \mathbb{C}))$ in [15] Sect. 6.1.2. As a $*$-algebra, $U_q(so(5))$ is generated by $\{K_i = K_i^*, K_i^{-1}, E_i, F_i := E_i^*\}_{i=1,2}$ ($i \to 3 - i$ with respect to the notations of [15]), with relations:

$$[K_1, K_2] = 0, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$[E_i, F_j] = \delta_{ij} K_i^2 - K_i^{-2},$$

$$K_iE_jK_i^{-1} = q^i E_i, \quad K_iE_jK_i^{-1} = q^{-1}E_j \quad \text{if } i \neq j,$$

together with the ones obtained by conjugation and Serre relations, explicitly, given by

$$E_1E_2^2 - (q^2 + q^{-2})E_2E_1E_2 + E_2^3E_1 = 0,$$

$$E_1^3E_2 - (q^2 + 1 + q^{-2})(E_1^2E_2E_1 - E_1E_2E_1^2) - E_2E_1^3 = 0,$$

(3.1a)

(3.1b)
The Hopf algebra structure \((\Delta, \epsilon, S)\) of \(U_q(\mathfrak{so}(5))\) is given by:

\[
\begin{align*}
\Delta K_i &= K_i \otimes K_i, \\
\Delta E_i &= E_i \otimes K_i + K_i^{-1} \otimes E_i, \\
\epsilon(K_i) &= 1, \quad \epsilon(E_i) = 0, \\
S(K_i) &= K_i^{-1}, \quad S(E_i) = -q^i E_i.
\end{align*}
\]

For each non-negative \(n_1, n_2\) such that \(n_2 \in \frac{1}{2} \mathbb{N}\) and \(n_2 - n_1 \in \mathbb{N}\) there is an irreducible representation of \(U_q(\mathfrak{so}(5))\) whose representation space we denote \(V_{(n_1, n_2)}\). We call it “the representation with highest weight \((n_1, n_2)\)” since the highest weight vector is an eigenvector of \(K_1\) and \(K_1^{-1} K_2\) with eigenvalues \(q^{n_1}\) and \(q^{n_2}\), respectively.

Irreducible representations with highest weight \((0, l)\) and \((\frac{1}{2}, l)\) (the ones that we need explicitly) can be found in [2] and are recalled presently. Let us use the shorthand notation \(V_l := V_{(0,l)}\) if \(l \in \mathbb{N}\) and \(V_{\frac{l}{2}} := V_{(\frac{1}{2}, l)}\) if \(l \in \mathbb{N} + \frac{1}{2}\). The vector space \(V_l\), for all \(l \in \frac{1}{2} \mathbb{N}\), has orthonormal basis \([l, m_1, m_2; j]\), where the labels \((j, m_1, m_2)\) satisfy the following constraints. For \(l \in \mathbb{N}\):

\[
\begin{align*}
&j = 0, 1, \ldots, l, \quad j - |m_1| \in \mathbb{N}, \quad l - j - |m_2| \in 2\mathbb{N}, \\
&j = \frac{1}{2}, \frac{3}{2}, \ldots, l - 1, l, \quad j - |m_1| \in \mathbb{N}, \quad l + \frac{1}{2} - j - |m_2| \in \mathbb{N}.
\end{align*}
\]

Notice that for any admissible \((l, m_1, m_2, j)\) there exists a unique \(\epsilon \in \{0, \pm \frac{1}{2}\}\) such that \(l + \epsilon - j - m_2 \in 2\mathbb{N}\) (that is, \(\epsilon = 0\) if \(l \in \mathbb{N}\) and \(\epsilon = \frac{1}{2}(-1)^{\lfloor j \rfloor}(-1)^{j - m_2}\) if \(l \in \mathbb{N} + \frac{1}{2}\)). We shall need the coefficients,

\[
\begin{align*}
a_l(j, m_2) &= \frac{1}{2} \sqrt{\frac{[l - j - m_2 + \epsilon][l + j + m_2 + 3 + \epsilon]}{[2(j + |\epsilon|) + 1][2(j - |\epsilon|) + 3]}}, \\
b_l(j, m_2) &= 2|\epsilon| \sqrt{\frac{[l - \epsilon(2j + 1) - m_2 + 1][l - \epsilon(2j + 1) + m_2 + 2]}{[2j][2j + 2]}}, \\
c_l(j, m_2) &= \frac{(-1)^{2\epsilon}}{2} \sqrt{\frac{[l - j + m_2 + 2 - \epsilon][l + j - m_2 + 1 - \epsilon]}{[2(j + |\epsilon|) - 1][2(j - |\epsilon|) + 1]}},
\end{align*}
\]

where, as usual, \([z] := (q^z - q^{-z})/(q - q^{-1})\) denotes the \(q\)-analogue of \(z \in \mathbb{C}\).

The \(*\)-representation \(\sigma_1 : U_q(\mathfrak{so}(5)) \rightarrow \text{End}(V_l)\) is defined by the rules,

\[
\begin{align*}
\sigma_1(K_1) [l, m_1, m_2; j] &= q^{m_1} [l, m_1, m_2; j], \\
\sigma_1(K_2) [l, m_1, m_2; j] &= q^{m_2 - m_1} [l, m_1, m_2; j], \\
\sigma_1(E_1) [l, m_1, m_2; j] &= \sqrt{[j - m_1][j + m_1 + 1]} [l, m_1 + 1, m_2; j],
\end{align*}
\]

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\[
\sigma_l(E_2) | l, m_1, m_2; j \rangle = \sqrt{[j - m_1 + 1][j - m_1 + 2]} a_{l}(j, m_2) | l, m_1 - 1, m_2 + 1; j + 1 \rangle \\
+ \sqrt{[j + m_1][j - m_1 + 1]} b_{l}(j, m_2) | l, m_1 - 1, m_2 + 1; j \rangle \\
+ \sqrt{[j + m_1][j + m_1 - 1]} c_{l}(j, m_2) | l, m_1 - 1, m_2 + 1; j - 1 \rangle .
\]

When there is no risk of ambiguity the representation symbol \(\sigma_l\) will be suppressed.

For \(l \in \mathbb{N}\) the representation \(\sigma_l\) is real. That is, there is an antilinear map \(C : V_l \to V_l\), which satisfies \(C^2 = 1\) and \(C\sigma_l(h)C = \sigma_l(S(h)^*)\). This map is explicitly given by

\[
C | l, m_1, m_2; j \rangle := (-q)^{m_1}q^{3m_2} | l, -m_1, -m_2; j \rangle.
\]

The operator

\[
C_1 := q^{-1}K_1^2 + q K_1^{-2} + (q - q^{-1})^2 E_1 F_1,
\]

is a Casimir for the subalgebra generated by \((K_1, K_1^{-1}, E_1, F_1)\). For future reference, we note the action of \(C_1\) on a vector of \(V_l\), with \(l \in \frac{1}{2}\mathbb{N}\); it is

\[
C_1 | l, m_1, m_2; j \rangle = (q^{2j+1} + q^{-2j-1}) | l, m_1, m_2; j \rangle.
\]

4 The orthogonal quantum 4-Sphere

**Definition 4.1** ([18]). *We call orthogonal quantum 4-sphere the virtual space underlying the algebra \(A(S^4_q)\) generated by \(x_0 = x_0^*, x_1, x_i^* \) (with \(i = 1, 2\)), with commutation relations:

\[
x_i x_j = q^2 x_j x_i, \quad \forall \ 0 \leq i < j \leq 2,
\]

\[
x_i^* x_j = q^2 x_j x_i^*, \quad \forall \ i \neq j,
\]

\[
[x_1^*, x_1] = (1 - q^4)x_0^2,
\]

\[
[x_2^*, x_2] = x_1^* x_1 - q^4 x_1 x_1^*,
\]

\[
x_0^2 + x_1 x_1^* + x_2 x_2^* = 1.
\]

The original notations of Fadeev-Reshetikhin-Takhtadzhyan [18, Eq. (1.14)] can be obtained by defining \(x'_i := x_2^*, x'_2 := x_1^*, x'_3 := \sqrt{q(1 + q^2)} x_0, x'_4 := x_1, x'_5 := x_2\) and \(q' := q^2\). The notations in [10, Eq. (2.1)] can be obtained by the replacement \(x_i \mapsto x_i^*\) and \(q^2 \mapsto q^{-1}\).

In the next propositions we summarize some well known facts.

**Proposition 4.2.** *The algebra \(A(S^4_q)\) is an \(U_q(so(5))\)-module \(*\)-algebra for the action given by:

\[
K_i \triangleright x_i = qx_i, \quad i = 1, 2,
\]

\[
K_2 \triangleright x_1 = q^{-1} x_1,
\]

\[
E_1 \triangleright x_0 = q^{-1/2} x_1, \quad E_2 \triangleright x_1 = x_2,
\]

\[
F_1 \triangleright x_1 = q^{1/2} [2] x_0, \quad F_1 \triangleright x_0 = -q^{-3/2} x_1^*, \quad F_2 \triangleright x_2 = x_1,
\]

while \(K_i \triangleright x_j = x_j, E_i \triangleright x_j = 0\) and \(F_i \triangleright x_j = 0\) in all other cases.*
Notice that the action on the $x_i^*$’s is determined by compatibility with the involution:

$$K_i \triangleright a^* = \{K_i^{-1} \triangleright a\}^* , \quad E_i \triangleright a^* = \{-q F_i \triangleright a\}^* , \quad E_2 \triangleright a^* = \{-q^2 F_2 \triangleright a\}^* .$$

**Proof.** The bijective linear map from the linear span of $\{x_i, x_i^*\}$ to the representation space $V_i$ defined (modulo a global proportionality constant) by

$$x_2 \mapsto |0, 1; 0\rangle , \quad x_1 \mapsto |1, 0; 1\rangle , \quad x_0 \mapsto (q[2])^{-1/2} |0, 0; 1\rangle , \quad x_i^* \mapsto -q |1, 0; 1\rangle , \quad x_2^* \mapsto q^3 |0, -1; 0\rangle ,$$

is a unitary equivalence of $U_q(so(5))$-modules (here unitary means that the real structure $C$ on $V_i$ is implemented by the $*$ operation on $x_i$’s). This guarantees that the free $*$-algebra $C\langle x_i, x_i^*\rangle$ generated by $\{x_i, x_i^*\}$ is an $U_q(so(5))$-module $*$-algebra.

The degree $\leq 2$ polynomials generating the ideal which defines $A(S_q^4)$ span the real representations $V_0$ and $V_{(1,1)}$, inside the tensor product $V_1 \otimes V_1$. The quotient $*$-algebra of $C\langle x_i, x_i^*\rangle$ by this ideal, $A(S_q^4)\langle x_i, x_i^*\rangle$, is then an $U_q(so(5))$-module $*$-algebra. \[\Box\]

**Proposition 4.3.** There is an isomorphism $A(S_q^4) \simeq \bigoplus_{i \in \mathbb{N}} V_i$ of $U_q(so(5))$ left modules.

**Proof.** A linear basis for $A(S_q^4)$ is made of monomials $x_0^{n_0} x_1^{n_1} (x_2^*)^{n_2} x_2^* x_3^*$ with $n_0, n_1, n_2 \in \mathbb{N}, n_3 \in \mathbb{Z}$ and with the notation $x_2^{n_3} := (x_2^*)^{n_3}$ if $n_3 < 0$. Using this basis one proves that a weight vector of $A(S_q^4)$ is annihilated by both $E_1$ and $E_2$ if and only if it is of the form $x_2^l, l \in \mathbb{N}$. Thus, highest weight vectors are proportional to $x_2^I$ and the algebra decomposes as multiplicity free direct sum of highest weight representations with weights $(0, I)$. \[\Box\]

The algebra $A(S_q^4)$ has two inequivalent irreducible infinite dimensional representations. The representation space is the Hilbert space $\ell^2(\mathbb{N}^2)$ and the representations are given by

$$x_0 |k_1, k_2\rangle_\pm := \frac{1}{\sqrt{2}} (q^{k_1+k_2} |k_1, k_2\rangle_\pm ,$$

$$x_1 |k_1, k_2\rangle_\pm := q^{2k_1} \frac{1}{\sqrt{1-q^{4k_1+k_2}}} |k_1+1, k_2\rangle_\pm ,$$

$$x_2 |k_1, k_2\rangle_\pm := \frac{1}{\sqrt{1-q^{4k_2+k_1}}} |k_1, k_2+1\rangle_\pm .$$

(4.1)

The direct sum of these representations, with obvious grading $\gamma$ and operator $F$ given by $F |k_1, k_2\rangle_\pm := |k_1, k_2\rangle_{\pm}$, constitutes a 1-summable Fredholm module over $A(S_q^4)$.

In the sequel we shall need both the quantum space $SU_q(2)$ as well as the equatorial Podleś sphere, whose algebras are given in [21] and [17] respectively.

**Definition 4.4.** The algebra $A(SU_q(2))$ of polynomial functions on $SU_q(2)$ is the $*$-algebra generated by $\alpha, \beta$ and their adjoints, with relations:

$$\beta \alpha = q \alpha \beta, \quad \beta^* \alpha = q \alpha \beta^*, \quad [\beta, \beta^*] = 0, \quad \alpha^* + \beta^* = 1, \quad \alpha^* \alpha + q^2 \beta^* \beta = 1 .$$

We call equatorial Podleś sphere the virtual space underlying the $*$-algebra $A(S_q^2)$ generated by $A = A^*, B$ and $B^*$ with relations:

$$AB = q^2 BA, \quad BB^* + A^2 = 1, \quad B^* B + q^3 A^2 = 1 .$$

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Proposition 4.5. There is a $*$-algebra morphism $\varphi : \mathcal{A}(S^4_q) \to \mathcal{A}(SU_q(2)) \otimes \mathcal{A}(S^2_q)$ defined by:

$$\varphi(x_0) = -(\alpha \beta + \beta^* \alpha^*) \otimes A,$$

$$\varphi(x_1) = (-\alpha^2 + q(\beta^*)^2) \otimes A,$$

$$\varphi(x_2) = 1 \otimes B.$$  \hfill (4.2)

Proof. One proves by direct computation that the five elements $\varphi(x_i), \varphi(x_i)^*$ satisfy all the defining relations of $\mathcal{A}(S^4_q)$.

\[\square\]

5 The modules of chiral spinors

We apply the general theory of Sect. 2.2 to the case $\mathcal{A} = \mathcal{A}(S^4_q)$ and $\mathcal{U} = U_q(so(5))$. Recall that in this case $\kappa(a) = K^2_2K^q_2 \triangleright a$ is the modular automorphism. We shall use the notations of Sect. 3 for the irreducible representations $(V_i, \sigma_i)$ of $U_q(so(5))$.

By Proposition 4.3 we have the equivalence $\mathcal{A}(S^4_q) \simeq \bigoplus_{i \in \mathbb{N}} V_i$ as left $U_q(so(5))$-modules. Using Lemma 2.5 for $N = 1$, we deduce that on the vector space $\bigoplus_{i \in \mathbb{N}} V_i$ there exists at least one $*$-representation of the crossed product $\mathcal{A}(S^4_q) \rtimes U_q(so(5))$ that extends the $*$-representation $\bigoplus_{i \in \mathbb{N}} \sigma_i$ of $U_q(so(5))$.

Let $e \in \text{Mat}_4(\mathcal{A}(S^4_q))$ be the following idempotent:

$$e := \frac{1}{2} \begin{pmatrix}
1 + x_0 & q^3x_2 & -qx_1 & 0 \\
q^{-3}x_2 & 1 - q^2x_0 & 0 & q^3x_1 \\
-q^{-1}x^*_1 & 0 & 1 - q^2x_0 & q^3x_2 \\
0 & qx^*_1 & q^{-3}x_2 & 1 + q^4x_0
\end{pmatrix}. \hfill (5.1)$$

By direct computation one proves that $K^2_2K^q_2 \triangleright e^* = e = e^2$ and then, by Lemma 2.7, $e$ defines an orthogonal projection $\pi$, by equation (2.8), on the linear space $\mathcal{A}(S^4_q)^4$ with inner product (2.5).

Next, let $\sigma : U_q(so(5)) \to \text{Mat}_4(\mathbb{C})$ be the $*$-representation defined by

$$\sigma(K_1) = \begin{pmatrix}
q^{1/2} & 0 & 0 & 0 \\
0 & q^{1/2} & 0 & 0 \\
0 & 0 & q^{-1/2} & 0 \\
0 & 0 & 0 & q^{-1/2}
\end{pmatrix}, \quad \sigma(K_2) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & q^{1} & 0 & 0 \\
0 & 0 & q & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \hfill (5.2a)$$

$$\sigma(E_1) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \sigma(E_2) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \hfill (5.2b)$$

Again, by direct computation one proves that:

$$K_i \triangleright e = \sigma(K_i) \ e \sigma(K_i)^{-1}, \hfill (5.3a)$$

$$E_i \triangleright e = \sigma(F_i) \ e \sigma(K_i)^{-1} - q^{-1}\sigma(K_i)^{-1}e \sigma(F_i). \hfill (5.3b)$$

Since $\sigma(K_i) = (K_i)^t$ and $\sigma(F_i) = (E_i)^t$, we conclude that condition (2.9) is satisfied and that $\pi$ and $\pi^t = 1 - \pi$ project $\mathcal{A}(S^4_q)^4$ onto sub $*$-representations of $\mathcal{A}(S^4_q) \rtimes U_q(so(5))$.

We state the main proposition of this section.
Proposition 5.1. There exists two inequivalent representations of the crossed product algebra \( \mathcal{A}(S_q^4) \times U_q(\mathfrak{so}(5)) \) on \( \bigoplus_{l \in \mathbb{N} + \frac{1}{2}} V_l \) that extend the representation \( \bigoplus_{l \in \mathbb{N} + \frac{1}{2}} \sigma_l \) of \( U_q(\mathfrak{so}(5)) \).

The proof is in two steps. We first prove (in Lemma 5.2) that \( \pi(\mathcal{A}(S_q^4)^4) \) and \( \pi^\perp(\mathcal{A}(S_q^4)^4) \) are not equivalent as representations of the algebra \( \mathcal{A}(S_q^4) \). Then we prove (in Lemma 5.3) that as \( U_q(\mathfrak{so}(5)) \) representations they are both equivalent to \( \bigoplus_{l \in \mathbb{N} + \frac{1}{2}} V_l \).

Lemma 5.2. The idempotent e in (5.1) splits \( \mathcal{A}(S_q^4)^4 \) into two inequivalent \( * \)-representations of the crossed product algebra \( \mathcal{A}(S_q^4) \times U_q(\mathfrak{so}(5)) \).

Proof. To prove the statement we apply Lemma 2.3. We use the Fredholm module associated to the representation on \( \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N}) \) defined by Eq. (1.1). One has

\[
\text{ch}^F(e) = \frac{1}{2} \text{Tr}_{\ell^2(\mathbb{N}) \oplus C^*} (\gamma F[F,e]) = \frac{1}{4} (1 - q^2)^2 \text{Tr}_{\ell^2(\mathbb{N})} (\gamma F[F,x_0]) = (1 - q^2)^2 \sum_{k_1,k_2 \in \mathbb{N}} q^{2(k_1+k_2)} = 1.
\]

The statement of Proposition 5.1 follows from the obvious observation that if the two representations of the crossed product algebra were equivalent, their restrictions to representations of \( \mathcal{A}(S_q^4) \) would be equivalent too.

Lemma 5.3. \( \pi(\mathcal{A}(S_q^4)^4) \simeq \pi^\perp(\mathcal{A}(S_q^4)^4) \simeq \bigoplus_{l \in \mathbb{N} + \frac{1}{2}} V_l \) as \( U_q(\mathfrak{so}(5)) \) representations.

Proof. In this proof, \( \simeq \) always means equivalence of representations of \( U_q(\mathfrak{so}(5)) \).

Since \( \sigma \) in (5.2) is unitary equivalent to the spin representation \( V_{1/2} \), the representation of \( U_q(\mathfrak{so}(5)) \) on \( \mathcal{A}(S_q^4)^4 \) is the Hopf tensor product of the representation over \( \mathcal{A}(S_q^4) \) with the representation \( V_{1/2} \). From \( \mathcal{A}(S_q^4) \simeq \bigoplus_{l \in \mathbb{N}} V_l \) and from the decomposition \( V_l \otimes V_{1/2} \simeq V_{l-1/2} \oplus V_{l+1/2} \) for all \( l \in \{1, 2, 3, \ldots \} \), we deduce that \( \mathcal{A}(S_q^4)^4 \simeq \bigoplus_{l \in \mathbb{N} + \frac{1}{2}} (V_l \oplus V_l) \) and then,

\[
\pi(\mathcal{A}(S_q^4)^4) \simeq \bigoplus_{l \in \mathbb{N} + \frac{1}{2}} m_l^+ V_l, \quad \pi^\perp(\mathcal{A}(S_q^4)^4) \simeq \bigoplus_{l \in \mathbb{N} + \frac{1}{2}} m_l^- V_l,
\]

with multiplicities \( m_l^+ \) to be determined, such that \( m_l^+ + m_l^- = 2 \). For \( l \in \mathbb{N} + \frac{1}{2} \), the vectors

\[
v_l^\pm := x_2^{-l} (1 \pm x_0, \pm q^3 x_2, \mp qx_1, 0)
\]

are highest weight vectors, being annihilated by both \( E_1 \) and \( E_2 \), and have weight \( (\frac{1}{2}, l) \). Furthermore, \( v_l^+(1 - e) = v_l^- e = 0 \). Thus, \( v_l^+ \in \pi(\mathcal{A}(S_q^4)^4) \) and \( v_l^- \in \pi^\perp(\mathcal{A}(S_q^4)^4) \).

Then in both modules \( \pi(\mathcal{A}(S_q^4)^4) \) and \( \pi^\perp(\mathcal{A}(S_q^4)^4) \) each representation \( V_l \), \( l \in \mathbb{N} + \frac{1}{2} \), appears with multiplicity \( m_l^\pm \geq 1 \). Since \( m_l^+ + m_l^- = 2 \), we deduce that \( m_l^\pm = 1 \) for all \( l \in \mathbb{N} + \frac{1}{2} \).

6 Equivariant representations of \( \mathcal{A}(S_q^4) \)

Next, we construct \( U_q(\mathfrak{so}(5)) \)-equivariant representations of \( \mathcal{A}(S_q^4) \) which classically correspond to the left regular and chiral spinor representations. The representation spaces will be (the closure of) \( \bigoplus_{l \in \mathbb{N}} V_l \) and \( \bigoplus_{l \in \mathbb{N} + \frac{1}{2}} V_l \).
Equivariance of a representation means that it is a representation of the crossed product algebra $\mathcal{A}(S_q^4) \rtimes U_q(so(5))$. The latter is defined by the crossed relations $ha = (h(1) \triangleright a)h(2)$ for all $a \in \mathcal{A}(S_q^4)$ and $h \in U_q(so(5))$; explicitly, the relations between the generators read:

$$
\begin{align*}
K_1, x_0 &= 0 , & K_1x_1 &= qx_1K_1 , & K_1x_2 &= x_2K_1 , \\
K_2, x_0 &= 0 , & K_2x_1 &= q^{-1}x_1K_2 , & K_2x_2 &= qx_2K_2 , \\
E_1, x_0 &= q^{-1/2}x_1K_1 , & E_1x_1 &= q^{-1}x_1E_1 , & E_1x_2 &= x_2E_1 , \\
F_1, x_0 &= -q^{-1/2}K_1x_1^* , & F_1x_1 &= q^{-1}x_1F_1 + q^{1/2}[2]x_0K_1 , & F_1x_2 &= x_2F_1 , \\
E_2, x_0 &= 0 , & E_2x_1 &= qx_1E_2 + x_2K_2 , & E_2x_2 &= q^{-1}x_2E_2 , \\
F_2, x_0 &= 0 , & F_2x_1 &= qx_1F_2 , & F_2x_2 &= q^{-1}x_2F_2 + x_1K_2 .
\end{align*}
$$

In the previous section we proved that on $\bigoplus_{l \in \mathbb{Z}} V_l$ there is at least one equivariant representation, the left regular one, and that on $\bigoplus_{l \in \mathbb{N}+\frac{1}{2}} V_l$ there are at least two equivariant representations, corresponding to the projective modules $\mathcal{A}(S_q^4)^e$ and $\mathcal{A}(S_q^4)^l(1-e)$. In this section we’ll prove that on such spaces there are no other equivariant representations besides the ones just mentioned.

Let us denote with $|l, m_1, m_2; j\rangle$ the basis of the representation space $V_l$ of $U_q(so(5))$ as discussed in Sect. 3. From the first two lines of (6.1) we deduce that

$$
\begin{align*}
x_0 |l, m_1, m_2; j\rangle &= \sum_{l', j'} A_{l,j,l',j'}^{m_1,m_2} |l', m_1, m_2; j'\rangle , & (6.2a) \\
x_1 |l, m_1, m_2; j\rangle &= \sum_{l', j'} B_{l,j,l',j'}^{m_1,m_2} |l', m_1 + 1, m_2; j'\rangle , & (6.2b) \\
x_2 |l, m_1, m_2; j\rangle &= \sum_{l', j'} C_{l,j,l',j'}^{m_1,m_2} |l', m_1 + 1; j'\rangle , & (6.2c)
\end{align*}
$$

with coefficients to be determined. Notice that from the crossed relations

$$
\begin{align*}
x_1 |l, m_1, m_2; j\rangle &= (F_2x_2 - q^{-1}x_2F_2)K_2^{-1} |l, m_1, m_2; j\rangle , \\
x_0 |l, m_1, m_2; j\rangle &= q^{-1/2}[2]^{-1}(F_1x_1 - q^{-1}x_1F_1)K_1^{-1} |l, m_1, m_2; j\rangle ,
\end{align*}
$$

the matrix coefficients of $x_0$ and $x_1$ can be expressed in terms of the coefficients of $x_2$.

**Lemma 6.1.** Let $k \in \mathbb{N}$. The following formulae hold:

$$
\begin{align*}
F_1^k |l, m_1, m_2; j\rangle &= \left\{ \begin{array}{ll}
0 & \text{if } k > j + m_1 \\
\neq 0 & \text{if } k \leq j + m_1
\end{array} \right. , & (6.3a) \\
E_1^k |l, m_1, m_2; j\rangle &= \left\{ \begin{array}{ll}
0 & \text{if } k > j - m_1 \\
\neq 0 & \text{if } k \leq j - m_1
\end{array} \right. . & (6.3b)
\end{align*}
$$

**Proof.** By direct computation:

$$
F_1^k |l, m_1, m_2; j\rangle = \sqrt{(j + m_1)(j + m_1 - 1) \ldots (j + m_1 - k + 1)} \times
$$
The second square root is always different from zero since the $q$-analogues are in increasing order and $j - m_1 + 1 \geq 1$. In the first square root $q$-analogues are in decreasing order and are all different from zero if and only if $j + m_1 - k + 1 \geq 1$. This proves Eq. (6.3a).

In the same way one establishes (6.3b) by computing that

$$ E_1^k |l, m_1, m_2; j\rangle = \sqrt{[j - m_1][j - m_1 - 1] \ldots [j - m_1 - k + 1]} \times \sqrt{[j + m_1 + 1][j + m_1 + 2] \ldots [j + m_1 + k]} |l, m_1 + k, m_2; j\rangle. $$

Lemma 6.2. The coefficients in (6.2) satisfy:

$$ A_{j,j',l,l'}^{m_1,m_2} = B_{j,j',l,l'}^{m_1,m_2} = 0 \text{ if } |j - j'| > 1, \quad C_{j,j',l,l'}^{m_1,m_2} = 0 \text{ if } j' \neq j. $$

Proof. From (6.1), (6.3a) and (6.3b) we derive:

$$ E_1^{j - m_1 + 1} x_1 |l, m_1, m_2; j\rangle = q^{j + m_1 - 1} E_1^{j - m_1 + 1} |l, m_1, m_2; j\rangle = 0, $$

$$ F_1^{j' + m_1 + 2} x_1^* |l', m_1 + 1, m_2; j'\rangle = q^{j' + m_1 + 2} F_1^{j' + m_1 + 2} |l', m_1 + 1, m_2; j'\rangle = 0. $$

We expand the left hand sides and use the independence of the vectors $E_1^{j - m_1 + 1} |l', m_1 + 1, m_2; j'\rangle$ and $F_1^{j' + m_1 + 2} |l, m_1, m_2; j\rangle$ to arrive at the conditions:

$$ B_{j,j',l,l'}^{m_1,m_2} \left\{ E_1^{j - m_1 + 1} |l', m_1 + 1, m_2; j'\rangle \right\} = 0, $$

$$ B_{j,j',l,l'}^{m_1,m_2} \left\{ F_1^{j' + m_1 + 2} |l, m_1, m_2; j\rangle \right\} = 0. $$

By (6.3b) the graph parenthesis in the first line is different from zero if $j - m_1 + 1 \leq j' - m_1 - 1$, i.e. $B_{j,j',l,l'}^{m_1,m_2}$ must be zero if $j' \geq j + 2$. By (6.3a) the graph parenthesis in the second line is different from zero if $j' + m_1 + 2 \leq j + m_1$, i.e. $B_{j,j',l,l'}^{m_1,m_2}$ must be zero if $j' \leq j - 2$. This proves 1/3 of the statement

$$ B_{j,j',l,l'}^{m_1,m_2} = 0 \quad \forall \ j' \notin \{j - 1, j, j + 1\}. $$

A similar argument applies to $x_0$. From the coproduct of $E_1^n$ we deduce:

$$ E_1^n x_0 = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] (E_1^k \triangleright x_0) E_1^{n-k} K_1^k = x_0 E_1^n - [n] q^{-1/2} x_1 E_1^{n-1} K_1. $$

This implies that $E_1^{j - m_1 + 2} x_0 |l, m_1, m_2; j\rangle = 0$ and $F_1^{j' + m_1 + 2} x_0 |l', m_1, m_2; j\rangle = 0$. From these conditions we deduce that also $x_0$ shift $j$ by $\{0, \pm 1\}$ only.

Finally, let $C_1$ be the Casimir element in Eq. (3.4). Then $[C_1, x_2] = 0$ and from (3.5) we deduce that $x_2$ is diagonal on the index $j$.

Lemma 6.3. The coefficients in (6.2c) satisfy

$$ C_{j,j',l,l'}^{m_1,m_2} = 0 \text{ if } |l - l'| > 1 \text{ or if } |l - l'| = 0 \text{ and } l \in \mathbb{N}. $$
Proof. The elements \( \{x_i, x^*_i\} \) are a basis of the irreducible representation \( V_1 \). Covariance of the action tells that \( x_i \mid l, m_1, m_2; j \rangle \) and \( x^*_i \mid l, m_1, m_2; j \rangle \) are a basis of the tensor representation \( V_1 \otimes V_1 \). Equations (14–15) in Chapter 7 of [15] tell that \( V_1 \otimes V_1 \simeq V_{l-1} \oplus V_{l+1} \) if \( l \in \mathbb{N} \) and that \( V_1 \otimes V_1 \simeq V_{l-1} \oplus V_l \oplus V_{l+1} \) if \( l \in \mathbb{N} + \frac{1}{2} \) (with \( V_{-1} \) omitted if \( l - 1 < 0 \)). This Clebsh-Gordan decomposition tells that \( x_2 \mid l, m_1, m_2; j \rangle \) is in the linear span of the basis vectors \( \{l', m_1, m_2 + 1; j \rangle \) with \( l' - l = \pm 1 \) if \( l \in \mathbb{N} \) or with \( l' - l = 0, \pm 1 \) if \( l \in \mathbb{N} + \frac{1}{2} \). This concludes the proof of the lemma.

6.1 Computing the coefficients of \( x_2 \)

By Lemma 6.3 we have to consider only the cases \( j' = j \), \( |l' - l| \leq 1 \) if \( l \in \mathbb{N} + \frac{1}{2} \) or \( |l' - l| = 1 \) if \( l \in \mathbb{N} \). The condition \( [E_1, x_2] = 0 \) implies that \( C_{m_1, m_2}^{m_1, m_2} = C_{m_1, m_2}^{m_2, m_1} \) is independent on \( m_1 \). Equations \( (E_2 x_2 - q^{-1} x_2 E_2) \mid l, j, m_2; j \rangle = 0 \) and \( (F_2 x_2^* - q x_2^* F_2) \mid l', j, m_2 + 1; j \rangle = 0 \) imply, respectively:

\[
C_{j,l,l+1}^{m_2} \sqrt{|l' - j - m_2 - 1 + \epsilon'|[l' + j + m_2 + 4 + \epsilon']} = C_{j,l,l+1}^{m_2+1, m_2} q^{-1} \sqrt{|l - j - m_2 + \epsilon|}|l + j + m_2 + 3 + \epsilon|, \\
C_{j,l,l+1}^{m_2} \sqrt{|l + j - m_2 - 3 - \epsilon||l - j + m_2 - \epsilon|} = C_{j,l,l+1}^{m_2-1, m_2} q \sqrt{|l' - j - m_2 + 2 - \epsilon'|[l' - j - m_2 + 1 - \epsilon'|},
\]

with \( \epsilon, \epsilon' \in \{0, \pm \frac{1}{2}\} \) determined by the conditions \( l + \epsilon - j - m_2 \in 2\mathbb{N} \) and \( l' - \epsilon' - j - m_2 \in 2\mathbb{N} \). Notice that if \( l' - l \in 2\mathbb{N} + 1 \) then \( \epsilon' = \epsilon \), while if \( l' - l \in 2\mathbb{N} \) then \( \epsilon' = -\epsilon \). Looking at the cases \( l' - l = \pm 1 \), we deduce that

\[
q^{\frac{1}{2}(j+m_2)} \sqrt{|l + j + m_2 + 3 + \epsilon|} C_{j,l,l+1}^{m_2} \text{ and } q^{\frac{1}{2}(j+m_2)} \sqrt{|l - j - m_2 + \epsilon|} C_{j,l,l-1}^{m_2}
\]
depend on \( j + m_2 \) only through their parity (i.e. they depend only on the value of \( \epsilon \)). Similarly,

\[
q^{\frac{1}{2}(j-m_2)} \sqrt{|l - j + m_2 + 2 - \epsilon|} C_{j,l,l+1}^{m_2} \text{ and } q^{\frac{1}{2}(j-m_2)} \sqrt{|l + j - m_2 + 1 - \epsilon|} C_{j,l,l-1}^{m_2}
\]
depend on \( j - m_2 \) only through their parity. Combining these informations, we deduce that the following elements do not depend on the exact value of \( j, m_2 \), but only on the value of \( \epsilon \),

\[
q^{-m_2} \sqrt{|l + j + m_2 + 3 + \epsilon||l - j + m_2 + 2 - \epsilon|} C_{j,l,l+1}^{m_2} =: C_{l,l+1}^{m_2} \, (\epsilon), \\
q^{-m_2} \sqrt{|l - j - m_2 + \epsilon||l + j - m_2 + 1 - \epsilon|} C_{j,l,l-1}^{m_2} =: C_{l,l-1}^{m_2} \, (\epsilon).
\]

If \( l \in \mathbb{N} \) there are no other coefficients \( C_{j,l,l}^{m_2} \) to compute. If \( l \not\in \mathbb{N} \), we have to compute also \( C_{j,l,l}^{m_2} \). In this case \( \epsilon' = -\epsilon \) and we get:

\[
C_{j,l,l}^{m_2} \sqrt{|l - j - m_2 - 1 - \epsilon||l + j + m_2 + 4 - \epsilon|} = C_{j,l,l+1}^{m_2+1, m_2} q^{-1} \sqrt{|l - j - m_2 + \epsilon||l + j + m_2 + 3 + \epsilon|}, \\
C_{j,l,l}^{m_2} \sqrt{|l + j - m_2 - 3 - \epsilon||l - j + m_2 - \epsilon|} = C_{j,l,l+1}^{m_2-1, m_2} q \sqrt{|l' + j - m_2 + 2 - \epsilon'|[l' - j - m_2 + 1 - \epsilon|}.
\]

Again, looking at the two cases \( \epsilon = \pm \frac{1}{2} \) we deduce that

\[
q^{\frac{1}{2}(j+m_2)} \sqrt{|l + \frac{1}{2} - j - m_2|} C_{j,l,l}^{m_2} \text{ if } \epsilon = \frac{1}{2} \text{ and } q^{\frac{1}{2}(j+m_2)} \sqrt{|l + \frac{1}{2} + j + m_2 + 2|} C_{j,l,l}^{m_2} \text{ if } \epsilon = -\frac{1}{2}.
\]
do not depend on \( j + m_2 \) (this time \( \epsilon \) is fixed, so the parity of \( j + m_2 \) is fixed). Similarly,

\[
\frac{q^{\frac{1}{2}(j-m_2)}}{\sqrt{l + \frac{1}{2} - j + m_2 + 1}} C_{j,l,l}^{m_2} \quad \text{if} \quad \epsilon = \frac{1}{2} \quad \text{and} \quad \frac{q^{\frac{1}{2}(j-m_2)}}{\sqrt{l + \frac{1}{2} + j - m_2 + 1}} C_{j,l,l}^{m_2} \quad \text{if} \quad \epsilon = -\frac{1}{2}
\]
do not depend on \( j - m_2 \). Combining these informations, we deduce that the following element does not depend on the exact value of \( j, m_2 \), but only on the value of \( \epsilon \):

\[
q^{m_2} \sqrt{[l - 2\epsilon j - m_2 + 1 - \epsilon][l - 2\epsilon j + m_2 + 2 - \epsilon]} C_{j,l,l}^{m_2} =: C_{l,l}(\epsilon).
\]
The denominator of the left-hand side is just \([2j][2j + 2]b_l(j, m_2)\) with \( b_l \) the coefficient in Eq. (3.2b). The formula \( C_{j,l,l}^{m_2} = q^{m_2}[2j][2j + 2]b_l(j, m_2)C_{l,l}(\epsilon) \) is valid for all \( l \), since \( b_l(j, m_2) \) vanish for \( l \) integer.

Summarizing, we find that

\[
C_{j,l,l+1}^{m_2} = q^{m_2} \sqrt{l + j + m_2 + 3 + \epsilon} \sqrt{l - j + m_2 + 2 - \epsilon} C_{l,l+1}(\epsilon), \quad (6.4a)
\]

\[
C_{j,l,l}^{m_2} = q^{m_2}[2j][2j + 2]b_l(j, m_2) C_{l,l}(\epsilon), \quad (6.4b)
\]

\[
C_{j,l,l-1}^{m_2} = q^{m_2} \sqrt{l - j - m_2 + 2} \sqrt{l + j - m_2 + 1 - \epsilon} C_{l,l-1}(\epsilon), \quad (6.4c)
\]

with coefficients \( C_{l,l}(\epsilon) \) to be determined.

### 6.2 Computing the coefficients of \( x_1 \)

From Lemma 6.2 we have to consider only the three cases \( j' = j, j \pm 1 \). Using equation \( E_1x_1 = q^{-1}x_1E_1 \) we get,

\[
\frac{q^{-m_1}}{\sqrt{j + m_1 + 1}[j + m_1 + 2]} B_{j,j+1,l,l'}^{m_1,m_2} = \frac{q^{-m_1-1}}{\sqrt{j + m_1 + 2}[j + m_1 + 3]} B_{j,j+1,l,l'}^{m_1+1,m_2},
\]

\[
\frac{q^{-m_1}}{\sqrt{j - m_1}[j + m_1 + 1]} B_{j,j,l,l'}^{m_1,m_2} = \frac{q^{-m_1-1}}{\sqrt{j - m_1 - 1}[j + m_1 + 2]} B_{j,j,l,l'}^{m_1+1,m_2},
\]

\[
\frac{q^{-m_1}}{\sqrt{j - m_1}[j - m_1 - 1]} B_{j,j-1,l,l'}^{m_1,m_2} = \frac{q^{-m_1-1}}{\sqrt{j - m_1 - 1}[j - m_1 - 2]} B_{j,j-1,l,l'}^{m_1+1,m_2}.
\]

We see that the left hand sides of these three equations are independent of \( m_1 \), and call:

\[
B_{j,j+1,l,l'}^{m_1,m_2} =: q^{m_1} \sqrt{j + m_1 + 1}[j + m_1 + 2] B_{j,j+1,l,l'}^{m_2}, \quad (6.5a)
\]

\[
B_{j,j,l,l'}^{m_1,m_2} =: q^{m_1} \sqrt{j - m_1}[j + m_1 + 1] B_{j,j,l,l'}^{m_2}, \quad (6.5b)
\]

\[
B_{j,j-1,l,l'}^{m_1,m_2} =: q^{m_1} \sqrt{j - m_1}[j - m_1 - 1] B_{j,j-1,l,l'}^{m_2}. \quad (6.5c)
\]

Imposing the condition \( x_1K_2 = F_2x_2 - q^{-1}x_2F_2 \) on the subspace spanned by \([l, j, m_2; j]\) (so \( m_1 = j \) and \( B_{j,j,l,l'}^{m_1,m_2} = B_{j,j,l,l'}^{m_1+1,m_2} = 0 \) on this subspace) we get:

\[
q^{m_2} B_{j,j+1,l,l'}^{m_2} = c_{l'}(j + 1, m_2)C_{j,l,l'}^{m_2} - q^{-1}c_l(j + 1, m_2 - 1)C_{j+1,l,l'}^{m_2-1}.
\]
From this we deduce that, that since coefficients $C_{j,l'}^m$ vanish for $|l - l'| > 1$, also $B_{j,j+1,l,l'}^m$ is zero in these cases. In the remaining three cases $l' = l, l ± 1$, using equation (6.4) we get:

$$B_{j,j+1,l,l+1}^m = (-1)^{2m} q^{l-j+m_2-\epsilon} \sqrt{\frac{[l + j + m_2 + 3 + \epsilon][l + j - m_2 + 3 - \epsilon]}{[2(j + |\epsilon|) + 1][2(j - |\epsilon|) + 3]}} C_{l,l+1}(\epsilon) ,$$  
(6.6a)

$$B_{j,j, l,l}^m = -2c q^{2l-j+m_2-3+\epsilon} \sqrt{\frac{[l + 1/2 + j - 2\epsilon m_2 + 2][l + 1/2 - j + 2\epsilon m_2]}{[2j + 2]}} C_{l,l}(\epsilon) ,$$  
(6.6b)

$$B_{j,j+1,l,l-1}^m = (-1)^{2m+1} q^{l-j+m_2-3+\epsilon} \sqrt{\frac{[l - j + m_2 - \epsilon][l - j - m_2 + \epsilon]}{[2(j + |\epsilon|) + 1][2(j - |\epsilon|) + 3]}} C_{l,l-1}(\epsilon) .$$  
(6.6c)

Imposing $q x^†_1 K_2 = x^†_2 E_2 - q^{-1} E_2 x^*_2$ on the subspace spanned by $(l', j + 1, m_2; j - 1)$ (so $B_{j,j,l,l'}^{m,m_2} = B_{j,j+1,l,l'}^{m,m_2}$ vanishes on this subspace) we get:

$$q^{m_2} B_{j,j-1,l,l'}^{m_2} = a_l(j - 1, m_2) C_{j,l' , l}^m - q^{-1} a_l(j - 1, m_2 - 1) C_{j,l , l'}^{m-1} .$$

We deduce that $B_{j,j-1,l,l'}^{m_2}$ vanishes if $|l - l'| > 1$, while in the three remaining cases $l' = l, l ± 1$ using Eq. (6.4) we get:

$$B_{j,j-1,l,l+1}^{m_2} = q^{l+j+m_2+1+\epsilon} \sqrt{\frac{[l - j - m_2 + 2 + \epsilon][l - j + m_2 + 2 - \epsilon]}{[2(j + |\epsilon|) + 1][2(j - |\epsilon|) + 3]}} C_{l,l+1}(\epsilon) ,$$  
(6.7a)

$$B_{j,j, l,l}^{m_2} = -2c q^{-2l+j+m_2-3+\epsilon} \sqrt{\frac{[l + 1/2 + j + 2\epsilon m_2 + 1][l + 1/2 - j + 2\epsilon m_2 + 1]}{[2j]}} C_{l,l}(\epsilon) ,$$  
(6.7b)

$$B_{j,j-1,l,l-1}^{m_2} = -q^{-l+j+m_2-2-\epsilon} \sqrt{\frac{[l + j + m_2 + 1 + \epsilon][l - j + m_2 + 2 - \epsilon]}{[2(j + |\epsilon|) + 1][2(j - |\epsilon|) + 3]}} C_{l,l-1}(\epsilon) .$$  
(6.7c)

Moreover, the condition $\langle l', j, m_2; j | x^*_1 K_2 + q^{-1} x^*_2 F_2 - F^*_2 x^*_2 | l, j - 1, m_2; j \rangle = 0$ implies that

$$q^{m_2} B_{j,j,l,l'}^{m_2} = b_l(j, m_2) C_{j,l' , l}^{m_2} - q^{-1} b_l(j, m_2 - 1) C_{j,l , l'}^{m_2-1} .$$  
(6.8)

A further elaboration on these coefficients is postponed to after the following section.

### 6.3 Computing the coefficients of $x_0$

The condition $q^{1/2} [2] x_0 K_1 = F_1 x_1 - q^{-1} x_1 F_1$ implies:

$$q^{m_1+1/2} [2] A_{j,j', l,l'}^{m_1,m_2} = \sqrt{[j' - m_1][j' + m_1 + 1]} B_{j,j', l,l'}^{m_1,m_2} - q^{-1} \sqrt{[j + m_1][j - m_1 + 1]} B_{j,j', l,l'}^{m_1-1,m_2} .$$

In the three non-trivial cases $j' = j, 0, -1$, using (6.5), we get:

$$A_{j,j+1,l,l'}^{m_1,m_2} = q^{j+m_1-1/2} \sqrt{[j + m_1 + 1][j - m_1 + 1]} B_{j,j+1,l,l'}^{m_2} ,$$  
(6.9a)

$$A_{j,j,l,l'}^{m_1,m_2} = [2]^{-1} q^{-2} \sqrt{[j + m_1+1][j - m_1]} B_{j,j,l,l'}^{m_2} ,$$  
(6.9b)

$$A_{j,j-1,l,l'}^{m_1,m_2} = -q^{-j+m_1-1/2} \sqrt{[j + m_1][j - m_1]} B_{j,j-1,l,l'}^{m_2} .$$  
(6.9c)
The hermiticity condition $x_0 = x_0^*$ means that $A_{j,l,l'}^{m_1,m_2} = A_{j,l,l'}^{m_1,m_2}$ and $A_{j,l,l'}^{m_1,m_2} = A_{j,l,l'}^{m_1,m_2}$. Thus, from (6.9) it follows that:

$$B_{j+1,l,l'}^{m_2} = -q^{2j+2}B_{j,l,l'}^{m_2}, \quad B_{j,l,l'}^{m_2} = B_{j,l,l'}^{m_2}.$$

Using (6.6), the first equation turns out to be equivalent to the following conditions:

$$C_{l+1,l}(\epsilon) = (-1)^{2q}q^{2l+4}C_{l+1,l}(\epsilon), \quad C_{l,l}(\epsilon) = C_{l,l}(\epsilon). \quad (6.10a)$$

The second of equation together with (6.8) implies:

$$b_l(j, m_2)C_{j,l,l}^{m_2} - q^{-1}b_l(j, m_2 - 1)C_{j,l,l}^{m_2} = b_l(j, m_2)C_{j,l,l}^{m_2} - q^{-1}b_l(j, m_2 - 1)C_{j,l,l}^{m_2}.$$

That is, using (6.4):

$$C_{l,l+1}^{1}(\epsilon) = C_{l,l+1}(\epsilon), \quad C_{l,l}^{1}(\epsilon) = C_{l,l}(\epsilon). \quad (6.10b)$$

### 6.4 Again the coefficients of $x_1$

Now, using (6.10) together with (6.8) we are able to compute the last coefficients. Notice that from (6.2) the coefficients $b_l$ vanish if $\epsilon = 0$ (i.e. in the left regular representation), and then from (6.8) $B_{j,l,l'}^{m_2}$ vanish too if $\epsilon = 0$. Moreover, from Lemma 6.2, $B_{j,l,l'}^{m_2}$ vanish also if $|l-l'| > 1$. In the three cases $l' = l, l \pm 1$, using Eq. (6.4) we get:

$$B_{j,l,l}^{m_2} = 2|\epsilon|2|l+m_2+1+\epsilon(2j+1)|\sqrt{l+2\epsilon j - m_2 + 2 + \epsilon}[l - 2\epsilon j + m_2 + 2 - \epsilon]C_{l,l+1}(\epsilon),$$

$$B_{j,l,l}^{m_2} = \frac{2|\epsilon|2|[l+2\epsilon j - m_2 + 2 + \epsilon]}{[2j][2j+2]}\left\{(l-\epsilon(2j+1)-m_2+1)[l-\epsilon(2j+1)+m_2+2]+q^{-2}[l+\epsilon(2j+1)-m_2+2][l+\epsilon(2j+1)+m_2+1]\right\}$$

$$= -\frac{2|\epsilon|2|l+m_2+1+\epsilon(2j+1)|\sqrt{l+2\epsilon j - m_2 + 2 + \epsilon}[l - 2\epsilon j + m_2 + 1 - \epsilon]}{[2j][2j+2]}C_{l,l}(\epsilon),$$

$$B_{j,l,l-1}^{m_2} = -2|\epsilon|2|l+m_2-2-\epsilon(2j+1)|\sqrt{l+2\epsilon j - m_2 + 1 + \epsilon}[l - 2\epsilon j + m_2 + 1 - \epsilon]C_{l,l-1}(\epsilon).$$

We have inserted the factor $2|\epsilon|$, so that the expressions remain valid also when $\epsilon = 0$.

### 6.5 The condition on the radius

Orbits for $SO(5)$ are spheres of arbitrary radius, equivariance alone not imposing constraints on the radius. Similarly, for the quantum spheres one has to impose a constraint on the radius to determine the coefficients of the representation. In fact, this will determine $C_{l,l+1}(0)$, $C_{l,l+1}(1/2)$ and $C_{l,l}(1/2)$ only up to a phase. Different choices of the phases correspond to unitary equivalent representations and without losing generality we choose $C_{l,l}(\epsilon) \in \mathbb{R}$. A possible expression for the radius is $q^8x_0^2 + q^4x_1^4 + x_2^2$ which we constrain to be equal to 1. Let then,

$$r(l, m_1, m_2; j) := \langle l, m_1, m_2; j | (q^8x_0^2 + q^4x_1^4 + x_2^2) | l, m_1, m_2; j \rangle.$$
All these matrix coefficients must be 1. In particular, for \( l \in \mathbb{N} \) the condition \( r(l, 0; 0) = 1 \) implies (up to a phase) that

\[
C_{l,l+1}(0) = \frac{q^{-l-3/2}}{\sqrt{[2l+3][2l+5]}}.
\]

(6.11)

For \( l \in \mathbb{N} + \frac{1}{2} \) we first require that \( r(l, \frac{1}{2}, l; \frac{1}{2}) = r(l, -\frac{1}{2}, l; \frac{1}{2}) \) obtaining two possibilities:

\[
C_{l,l}(\frac{1}{2}) = \pm \frac{[2]q^{1/2}[2l+2]}{[2l+2][2l+4]} C_{l,l+1}(\frac{1}{2}).
\]

Then imposing \( r(l, \frac{1}{2}, l; \frac{1}{2}) = 1 \), yields (up to a phase)

\[
C_{l,l+1}(\frac{1}{2}) = \frac{q^{-l-3/2}}{[2l+4]},
\]

(6.12)

hence,

\[
C_{l,l}(\frac{1}{2}) = \pm \frac{q^{1/2}[2]}{[2l+2][2l+4]}.
\]

(6.13)

With these, all the coefficients are completely determined.

### 6.6 Explicit form of the representations

Let us recall what we know on the equivariant representations of the algebra \( \mathcal{A}(S^4_q) \).

By the decomposition \( \mathcal{A}(S^4_q) \simeq \bigoplus_{l \in \mathbb{N}} V_l \) into irreducible representations of \( U_q(so(5)) \), there exists (at least) one representation of \( \mathcal{A}(S^4_q) \times U_q(so(5)) \) on the vector space \( \bigoplus_{l \in \mathbb{N}} V_l \) extending the representation \( \bigoplus_{l \in \mathbb{N}} \sigma_l \) of \( U_q(so(5)) \). As we computed above, the equivariance uniquely determines (for \( l \in \mathbb{N} \), up to unitary equivalence) the matrix coefficients of the representation, whose expression is characterized by (6.11). On the other hand, by Proposition 5.1 there are (at least) two inequivalent representations of \( \mathcal{A}(S^4_q) \times U_q(so(5)) \) on the vector space \( \bigoplus_{l \in \mathbb{N} + \frac{1}{2}} V_l \) extending the representation \( \bigoplus_{l \in \mathbb{N} + \frac{1}{2}} \sigma_l \) of \( U_q(so(5)) \). These correspond, by Lemma 5.2 to the projective modules \( \mathcal{A}(S^4_q)^4 e \) and \( \mathcal{A}(S^4_q)^4 (1 - e) \), with \( e \) the idempotent in Eq. (5.1). The computation above (for \( l \in \mathbb{N} + \frac{1}{2} \)), which culminates in Eq. (6.13), tells us that there are only two possibilities for the matrix coefficients (up to unitary equivalence). Therefore, the two possible choices in (6.13) must correspond to the inequivalent representations associated with the projective modules \( \mathcal{A}(S^4_q)^4 e \) and \( \mathcal{A}(S^4_q)^4 (1 - e) \).

Let us summarize these results in the following two theorems, which correspond to the scalar (i.e. left regular) and chiral spinor representations, respectively.

**Theorem 6.4.** The vector space \( \mathcal{A}(S^4_q) \) has orthonormal basis \( |l, m_1, m_2; j\rangle \) with,

\[
l \in \mathbb{N}, \quad j = 0, 1, \ldots, l, \quad j - |m_1| \in \mathbb{N}, \quad l - j - |m_2| \in 2\mathbb{N}.
\]
We call $L^2(S^4_q)$ the Hilbert space completion of $\mathcal{A}(S^4_q)$. Modulo a unitary equivalence, the left regular representation is given by

\[
x_0 |l, m_1, m_2; j\rangle = A_{j,m_1} C^+_{l,j,m_2} |l + 1, m_1, m_2; j + 1\rangle \\
+ A_{j,m_1} C^-_{l,j,m_2} |l - 1, m_1, m_2; j + 1\rangle \\
+ A_{j-1,m_1} C^+_{l+1,j-1,m_2} |l + 1, m_1, m_2; j - 1\rangle \\
+ A_{j-1,m_1} C^-_{l+1,j-1,m_2} |l - 1, m_1, m_2; j - 1\rangle,
\]

\[
x_1 |l, m_1, m_2; j\rangle = B^+_{j,m_1} C^+_{l,j,m_2} |l + 1, m_1 + 1, m_2; j + 1\rangle \\
+ B^+_{j,m_1} C^-_{l,j,m_2} |l - 1, m_1 + 1, m_2; j + 1\rangle \\
+ B^-_{j,m_1} C^+_{l+1,j-1,m_2} |l + 1, m_1 + 1, m_2; j - 1\rangle \\
+ B^-_{j,m_1} C^-_{l+1,j-1,m_2} |l - 1, m_1 + 1, m_2; j - 1\rangle,
\]

\[
x_2 |l, m_1, m_2; j\rangle = D^+_{l,j,m_2} |l + 1, m_1, m_2 + 1; j\rangle \\
+ D^+_{l,j,m_2} |l - 1, m_1, m_2 + 1; j\rangle,
\]

with coefficients

\[
A_{j,m_1} = q^{m_1-1} \sqrt{\frac{[j + m_1 + 1][j - m_1 + 1]}{[2j + 1][2j + 3]}},
\]

\[
B^+_{j,m_1} = q^{-j+m_1-1/2} \sqrt{\frac{[j + m_1 + 1][j + m_1 + 2]}{[2j + 1][2j + 3]}},
\]

\[
B^-_{j,m_1} = -q^{j+m_1+1/2} \sqrt{\frac{[j - m_1][j - m_1 - 1]}{[2j - 1][2j + 1]}},
\]

and

\[
C^+_{l,j,m_2} = q^{m_2-1} \sqrt{\frac{[l + j + m_2 + 3][l + j - m_2 + 3]}{[2l + 3][2l + 5]}},
\]

\[
C^-_{l,j,m_2} = -q^{m_2-1} \sqrt{\frac{[l - j + m_2][l - j - m_2]}{[2l + 1][2l + 3]}},
\]

\[
D^+_{l,j,m_2} = q^{-l+m_2-3/2} \sqrt{\frac{[l + j + m_2 + 3][l - j + m_2 + 2]}{[2l + 3][2l + 5]}},
\]

\[
D^-_{l,j,m_2} = q^{l+m_2+3/2} \sqrt{\frac{[l - j - m_2][l + j - m_2 + 1]}{[2l + 1][2l + 3]}}.
\]

The two chiral spinorial representations (corresponding to the sign $\pm$ in Eq. (6.13)) are described in the following theorem.

**Theorem 6.5.** Let $\mathcal{H}_\pm$ be two Hilbert spaces with orthonormal basis $|l, m_1, m_2; j\rangle_\pm$, where

\[
l \in \mathbb{N} + \frac{1}{2}, \quad j = \frac{1}{2}, \frac{3}{2}, \ldots, l, \quad j - |m_1| \in \mathbb{N}, \quad l + \frac{1}{2} - j - |m_2| \in \mathbb{N}.
\]
Let $\epsilon = \pm \frac{1}{2}$ be defined by $l + \epsilon - j - m_2 \in 2\mathbb{N}$. On each space $\mathcal{H}_\pm$ there is an equivariant *-representation of $\mathcal{A}(S^3_q)$ defined by:

$$x_0 |l, m_1, m_2; j\rangle_\pm = A_{j,m_1}^+ C_{l,j,m_2}^+ |l + 1, m_1, m_2; j + 1\rangle_\pm + A_{j,m_1}^0 C_{l,j,m_2}^0 |l, m_1, m_2; j + 1\rangle_\pm + A_{j,m_1}^- C_{l,j,m_2}^- |l - 1, m_1, m_2; j + 1\rangle_\pm + A_{j,m_1}^0 H_{l,j,m_2}^+ |l + 1, m_1, m_2; j\rangle_\pm \\
+ A_{j,m_1}^0 H_{l,j,m_2}^- |l - 1, m_1, m_2; j\rangle_\pm + A_{j-1,m_1}^+ C_{l+1,j-1,m_2}^+ |l + 1, m_1, m_2; j - 1\rangle_\pm + A_{j-1,m_1}^0 C_{l+1,j-1,m_2}^0 |l, m_1, m_2; j - 1\rangle_\pm + A_{j-1,m_1}^- C_{l+1,j-1,m_2}^- |l - 1, m_1, m_2; j - 1\rangle_\pm ,$$

$$x_1 |l, m_1, m_2; j\rangle_\pm = B_{j,m_1}^+ C_{l,j,m_2}^+ |l + 1, m_1 + 1, m_2; j + 1\rangle_\pm + B_{j,m_1}^0 C_{l,j,m_2}^0 |l, m_1 + 1, m_2; j + 1\rangle_\pm + B_{j,m_1}^- C_{l,j,m_2}^- |l - 1, m_1 + 1, m_2; j + 1\rangle_\pm + B_{j,m_1}^0 H_{l,j,m_2}^+ |l + 1, m_1 + 1, m_2; j\rangle_\pm \\
+ B_{j,m_1}^0 H_{l,j,m_2}^- |l - 1, m_1 + 1, m_2; j\rangle_\pm + B_{j-1,m_1}^+ C_{l+1,j-1,m_2}^+ |l + 1, m_1 + 1, m_2; j - 1\rangle_\pm + B_{j-1,m_1}^0 C_{l+1,j-1,m_2}^0 |l, m_1 + 1, m_2; j - 1\rangle_\pm + B_{j-1,m_1}^- C_{l+1,j-1,m_2}^- |l - 1, m_1 + 1, m_2; j - 1\rangle_\pm ,$$

$$x_2 |l, m_1, m_2; j\rangle_\pm = D_{l,j,m_2}^+ |l + 1, m_1, m_2 + 1; j\rangle_\pm + D_{l,j,m_2}^0 |l, m_1, m_2 + 1; j\rangle_\pm + D_{l,j,m_2}^- |l - 1, m_1, m_2 + 1; j\rangle_\pm ,$$

with coefficients

$$A^+_{j,m_1} = q^{m_1 - 1} \frac{\sqrt{[j + m_1 + 1][j - m_1 + 1]}}{[2j + 2]},$$

$$A^0_{j,m_1} = q^{-2} \frac{2^{j+m_1+1}[2][j-m_1][2j]}{[2j][2j+2]},$$

$$B^+_{j,m_1} = q^{-j+m_1-1/2} \frac{\sqrt{[j + m_1 + 1][j + m_1 + 2]}}{[2j + 2]},$$

$$B^0_{j,m_1} = (1 + q^2)q^{m_1 - 1/2} \frac{\sqrt{[j - m_1][j + m_1 + 1]}}{[2j][2j + 2]},$$

$$B^-_{j,m_1} = -q^{j+m_1+1/2} \frac{\sqrt{[j - m_1][j - m_1 - 1]}}{[2j]} .$$
These two representations are inequivalent and correspond to the projective modules $\mathcal{A}(S_q^4) e$ and $\mathcal{A}(S_q^4) e(1 - e)$, with $e$ the idempotent in Eq. (5.1).

7 The Dirac operator on the orthogonal quantum 4-sphere

We start by constructing a non-trivial Fredholm module on the orthogonal quantum sphere (with different representations a non-trivial Fredholm module was already constructed in [16]).

**Proposition 7.1.** Consider the representations of $\mathcal{A}(S_q^4)$ on $\mathcal{H}_\pm$ given in Theorem 6.5. Then, the datum $(\mathcal{A}(S_q^4), \mathcal{H}, F, \gamma)$ is a 1-summable even Fredholm module, where $\mathcal{H} := \mathcal{H}_+ \oplus \mathcal{H}_-$, $\gamma$ is the natural grading and $F \in \mathcal{B}(\mathcal{H})$ is defined by

$$F [l, m_1, m_2; j]_{\pm} := [l, m_1, m_2; j]_{\mp} .$$

This Fredholm module is non-trivial. In particular,

$$\text{ch}^F([e]) := \frac{1}{2} \text{Tr}_{\mathcal{H} \otimes C^2}(\gamma F[F, P]) = 1 ,$$

with $e$ the idempotent defined by Eq. (5.1).

**Proof.** That $F = F^*$, $F^2 = 1$ and $\gamma F + F \gamma = 0$ is obvious. Then, it is enough to show that $[F, x_i] \in \mathcal{L}^1(\mathcal{H})$ for $i = 0, 1, 2$. From this and the Leibniz rule it follows that $[F, a]$ is trace class, and then compact, for all $a \in \mathcal{A}(S_q^4)$.

Now, notice that

$$[F, x_0] [l, m_1, m_2; j]_{\pm} = \mp 2A_{j,m_1}^+ C_{l,j,m_2}^0 [l, m_1, m_2; j + 1]_{\mp}$$
The sum over $m$ and in turn, using Eq. (7.2),
\[ i \equiv 2 \sum_{m} \pm A_{j,m}^{0} H_{l,m_{2}}^{0} |l, m_{1}, m_{2}; j\rangle_{\mp} \]
\[ \mp 2 A_{j-1,m_{1}}^{+} C_{l,j-1,m_{2}}^{0} |l, m_{1}, m_{2}; j-1\rangle_{\mp} , \]
\[ [F, x_{1}] |l, m_{1}, m_{2}; j\rangle_{\pm} = \mp 2 B_{j,m_{1}}^{+} C_{l,j,m_{2}}^{0} |l, m_{1}+1, m_{2}; j+1\rangle_{\mp} \]
\[ \pm 2 B_{j,m_{1}}^{0} H_{l,j,m_{2}}^{0} |l, m_{1}+1, m_{2}; j\rangle_{\mp} \]
\[ \mp 2 B_{j-1,m_{1}}^{0} C_{l,j-1,m_{2}}^{0} |l, m_{1}+1, m_{2}; j-1\rangle_{\mp} , \]
\[ [F, x_{2}] |l, m_{1}, m_{2}; j\rangle_{\pm} = \pm 2 D_{l,m_{2}}^{0} |l, m_{1}, m_{2}+1; j\rangle_{\mp} . \]

All the coefficients appearing in these equations are bounded by $q^{2l}$. Thus the commutators are trace class and this concludes the first part of the proof.

To prove non-triviality it is enough to prove (7.1). Substituting (5.1) into (7.1) yields
\[ \text{ch} F([e]) = \frac{(1-q^{2})^{2}}{4} \text{Tr}_{\mathcal{H}}(\gamma F[F;x_{0}]) . \]
and in turn, using Eq. (7.2),
\[ \text{ch} F([e]) = (1-q^{2})^{2} \sum_{l,j,m_{1},m_{2}} A_{j,m_{1}}^{0} H_{l,j,m_{2}}^{0} . \]

Summing over $m_{1}$ from $-j$ to $j$ we obtain that
\[ \text{ch} F([e]) = q^{-3}(1-q^{2})^{2} \sum_{l,j,m_{2}} \frac{l+\epsilon(2j+1)-m_{2}+2}[l+\epsilon(2j+1)+m_{2}+1]}{[2l+2][2l+4][2j][2j+2]} \times \]
\[ \times \sum_{m_{1}} \left\{ q^{2j+2} + q^{-2j} - [2]q^{2m_{1}+1} \right\} \]
\[ = \sum_{l,j} \frac{(2j+1)(q^{2j+1}+q^{-2j-1})-[2][2j+1]}{[2l+2][2l+4][2j][2j+2]} \times \]
\[ \times \sum_{m_{2}} \left\{ q^{2l+2}(2j+1)+3 + q^{-2l-2}(2j+1)-3 - q^{2m_{2}-1} - q^{-2m_{2}-1} \right\} . \]
The sum over $m_{2}$ requires additional care. For $\epsilon$ fixed, $l - \epsilon - j + m_{2} = 0, 2, 4, \ldots, 2(l - j)$. If we call $2i := l - \epsilon - j + m_{2}$ and sum first over $i = 0, 1, \ldots, l - j$ and then over $\epsilon = \pm 1/2$ we get:
\[ \text{ch} F([e]) = \sum_{l,j} \frac{(2j+1)(q^{2j+1}+q^{-2j-1})-[2][2j+1]}{[2l+2][2l+4][2j][2j+2]} \times \]
\[ \times \sum_{2\epsilon = \pm 1} \left\{ (l-j+1)(q^{2l+2(2j+1)+3} + q^{-2l-2(2j+1)-3}) - (q^{2\epsilon-1} + q^{-2\epsilon+1})[2]^{-1}[2(2l-j+1)] \right\} \]
\[ = \sum_{l,j} \frac{(2j+1)(q^{2j+1}+q^{-2j-1})-[2][2j+1]}{[2l+2][2l+4][2j][2j+2]} \times \]
\[ \times \left\{ (l-j+1)(q^{2l+3} + q^{-2l-3})(q^{2j+1} + q^{-2j-1}) - [2][2(l-j+1)] \right\} \]
\[ =: \sum_{l,j} f(l,j) =: f(q) . \]
We call $f_{ij}(q)$ the generic term of last series, explicitly written as
\[ f_{ij}(q) = (1 - q^2)^4 \frac{(2j + 1)(1 + q^{4j+2}) - \frac{1+q^2}{1-q^2}(1 - q^{4j+2})}{(1 - q^{4l+4})(1 - q^{4l+8}) (1 - q^{4j})(1 - q^{4j+4})} \times q^{2l-1} \left\{ (l - j + 1)(1 + q^{1+6})(1 + q^{4j+2}) - \frac{1+q^2}{1-q^2} q^2(q^{4j} - q^{4l-4}) \right\}, \]
and consider it as a function of $q \in [0, 1]$. Notice that each $f_{ij}(q)$ is a $C^\infty$ function of $q$ (they are rational functions whose denominators never vanish for $0 \leq q < 1$). From the inequality
\[ 0 \leq f_{ij}(q) \leq 4(2j + 1)q^{2l-1} \]
we deduce (using the Weierstrass M-test) that the series is absolutely (hence uniformly) convergent in each interval $[0, q_0] \subset [0, 1]$. Then, it converges to a function $f(q)$ which is continuous in $[0, 1]$. Being the index of a Fredholm operator, $f(q)$ is integer valued in $[0, 1]$; by continuity it is constant and can be computed in the limit $q \to 0$. In this limit we have $f_{ij}(q) = 2j(l - j + 1)q^{2l-1} + O(q^{2l})$. Thus, $f_{ij}(0) = \delta_{i,1/2}\delta_{j,1/2}$ and $\text{ch}^F([e]) = f(0) = 1$.

The next step is to define a spectral triple whose Fredholm module is the one described in Proposition 7.1.

**Proposition 7.2.** Let $D$ be the (unbounded) operator on $\mathcal{H} := \mathcal{H}_+ \oplus \mathcal{H}_-$ defined by
\[ D[l, m_1, m_2; j] := (l + \frac{3}{2}) |l, m_1, m_2; j\rangle. \]
Then, the datum $(\mathcal{A}(S_q^4), \mathcal{H}, D, \gamma)$ is a $U_q(\text{so}(5))$-equivariant regular even spectral triple of metric dimension 4.

**Remark:** The operator $D$ is isospectral to the classical Dirac operator on $S^4$ (whose spectrum has been computed in [20, 11]). When $q = 1$, this spectral triple becomes the canonical one associated to the spin structure of $S^4$.

**Proof.** Clearly the representation of the algebra is even, $D$ is odd, with compact resolvent and $4^\infty$-summable (being isospectral to the classical Dirac operator on $S^4$).

Let $\delta$ be the unbounded derivation on $\mathcal{B}(\mathcal{H})$ defined by $\delta(T) := [[D], T]$. Each generator of $\mathcal{A}(S_q^4)$ is the sum of a finite number of weighted shifts; each of these weighted shifts is a bounded operator (the coefficients are all bounded by 1) and is an eigenvector of $\delta$, i.e., if $T$ shifts the index $l$ by $k$, then $\delta(T) = kT$. Thus, such weighted shifts are not only bounded but also in the smooth domain of $\delta$, which we denote by $\text{OP}^0 := \bigcap_{j \in \mathbb{N}} \text{dom} \delta^j$. As a consequence $\mathcal{A}(S_q^4) \subset \text{OP}^0$.

Recall that $[F, x_i]$ has coefficients decaying faster than $q^l$; thus $[D][F, x_i]$ is a matrix of rapid decay. In particular, $[D][F, x_i] \in \text{OP}^{-\infty} \subset \text{OP}^0$. The identity
\[ [D, x_i] = \delta(x_i)F + [D][F, x_i], \quad (7.3) \]
tells us that $[D, x_i]$ is not only bounded but even in $\text{OP}^0$ – being the sum of two bounded operators contained in the $*$-algebra $\text{OP}^0$. Then, $D$ defines a spectral triple and such a spectral triple is regular.

Finally, since $D$ is proportional to the identity in any irreducible subrepresentation $V_l$ of $U_q(\text{so}(5))$, it commutes with all $h \in U_q(\text{so}(5))$ and it is equivariant. \qed

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As a preparation for the study of the dimension spectrum in Sect. 8, let us explicitly verify the 4-summability of \( D \). As one can easily check, the dimension of \( V_l \) is
\[
\dim V_l = \frac{2}{3}(l + \frac{3}{2})(l + \frac{3}{2})(l + \frac{1}{2}).
\]
From this we get
\[
\text{Tr}(|D|^{-s}) = \sum_{l \in \mathbb{N} + \frac{1}{2}} 2(l + \frac{3}{2})^{-s} \dim V_l = \frac{4}{3} \sum_{n=1}^{\infty} (n^2 - 1)n^{-s+1},
\]
where \( n = l + \frac{3}{2} \) (and we added the term with \( n = 1 \) since it is identically zero). The above series is convergent in the right half-plane \( \{ s \in \mathbb{C} \mid \text{Re} s > 4 \} \), thus \( D \) has metric dimension 4.

Notice that \( \text{Tr}(|D|^{-s}) \) has meromorphic extension on \( \mathbb{C} \) given by
\[
\text{Tr}(|D|^{-s}) = \frac{4}{3}\left\{ \zeta(s-3) - \zeta(s-1) \right\},
\]
where \( \zeta(s) \) is the Riemann zeta-function. We recall that \( \zeta(s) \) has a simple pole in \( s = 1 \) as unique singularity and that \( \text{Res}_{s=1} \zeta(s) = 1 \).

8. The dimension spectrum and residues

To compute the dimension spectrum we shall use a very simple representation of the algebra which differs – in a sense which will be clear in Proposition 8.3 – from the chiral ones by a suitable ideal of operators. This is the class of operators,
\[
\mathcal{I} := \left\{ T \in \text{OP}^0 \mid T|D|^{-p} \in \mathcal{L}^1(\mathcal{H}), \forall p > 2 \right\}.
\]

Lemma 8.1. The collection \( \mathcal{I} \) is a two-sided ideal in \( \text{OP}^0 \).

Proof. Clearly \( \mathcal{I} \) is a vector space: if \( T_1, T_2 \in \mathcal{I} \), that is \( T_1|D|^{-p} \in \mathcal{L}^1(\mathcal{H}), T_2|D|^{-p} \in \mathcal{L}^1(\mathcal{H}) \) for all \( p > 2 \), then \( T_1|D|^{-p} + T_2|D|^{-p} \in \mathcal{L}^1(\mathcal{H}) \) for all \( p > 2 \), which means \( T_1 + T_2 \in \mathcal{I} \).

That \( \mathcal{I} \) is a left ideal is straightforward. Since \( \mathcal{L}^1(\mathcal{H}) \) is a two-sided ideal in \( \mathcal{B}(\mathcal{H}) \), if \( T_1 \in \text{OP}^0 \) and \( T_2 \in \mathcal{I} \), for all \( p > 2 \) we have that \( T_1 \cdot T_2|D|^{-p} \) is the product of a bounded operator, \( T_1 \), with a trace class one, \( T_2|D|^{-p} \), thus it is of trace class, and \( T_1T_2 \in \mathcal{I} \).

From Appendix B of [6] for any \( p > 0 \), we know that the bounded operator \( |D|^{-p} \) maps \( \mathcal{H} \) to \( \mathcal{H}^p := \text{dom} |D|^p \), that \( T \in \text{OP}^0 \subset \text{op}^0 \) is a bounded operator \( \mathcal{H}^p \to \mathcal{H}^p \), and finally that \( |D|^p \) is bounded from \( \mathcal{H}^p \) to \( \mathcal{H} \). Thus, for \( T \in \text{OP}^0 \), the product \( |D|^p T |D|^{-p} \) is a bounded operator on \( \mathcal{H} \). Now, if \( T_1 \in \text{OP}^0 \) and \( T_2 \in \mathcal{I} \), for all \( p > 2 \) we can write \( T_2T_1|D|^{-p} = T_2|D|^{-p} \cdot |D|^p T_1 |D|^{-p} \) as the product of a bounded operator, \( |D|^p T_1 |D|^{-p} \), with a trace class one, \( T_2|D|^{-p} \); thus \( T_2T_1|D|^{-p} \) is of trace class so \( T_2T_1 \in \mathcal{I} \) and \( \mathcal{I} \) is also a right ideal.

Clearly, if \( T \) is of trace class, so is \( |D|^{-p} T \) for any positive \( p \), and \( \mathcal{L}^1(\mathcal{H}) \subset \mathcal{I} \). Since \( \text{OP}^{-\infty} \subset \mathcal{L}^1(\mathcal{H}) \), smoothing operators belong to \( \mathcal{I} \) as well. On the other hand, \( \mathcal{I} \) is strictly bigger than \( \mathcal{L}^1(\mathcal{H}) \); indeed, the operator \( L_q \in \mathcal{B}(\mathcal{H}) \), given by
\[
L_q |l, m_1, m_2; j\rangle_\pm := q^{j + \frac{1}{2}} |l, m_1, m_2; j\rangle_\pm,
\]
is not of trace class but belongs to \( \mathcal{I} \), by the following proposition.
Proposition 8.2. For any $s \in \mathbb{C}$ with $\text{Re } s > 2$ one has that

$$\zeta_L(s) := \sum_{l, j, m_1, m_2} (l + \frac{3}{2})^{-s} q^{l+\frac{j}{2}} = \frac{4q}{(1-q)^2} \left( \zeta(s-1) - \frac{1+q}{1-q} \zeta(s) \right) + \text{holomorphic function},$$

where $\zeta(s)$ is the Riemann zeta-function. In particular, this means that $L_q$ belongs to the ideal $\mathcal{I}$. Furthermore, since the series $\zeta_L(0)$ is divergent, $L_q$ is not of trace class.

Proof. Calling $n := l + \frac{3}{2}$, $k := j + \frac{1}{2}$, we have

$$\zeta_L(s) = 4 \sum_{n=2}^{\infty} n^{-s} \sum_{k=1}^{n-1} k(n-k)q^k,$$

We can sum starting from $n = 1$ and for $k = 0, \ldots, n$ (we simply add zero terms) to get

$$\zeta_L(s) = 4 \sum_{n=1}^{\infty} n^{-s} \left( nq\partial_q - (q\partial_q)^2 \right) \sum_{k=0}^{n} q^k = 4 \sum_{n=1}^{\infty} n^{-s} \left( nq\partial_q - (q\partial_q)^2 \right) \frac{1-q^{n+1}}{1-q}.$$

Terms decaying as $q^n$ give a holomorphic function of $s$, thus modulo holomorphic functions,

$$\zeta_L(s) \sim 4 \sum_{n=1}^{\infty} n^{-s} \left\{ n \frac{q}{(1-q)^2} - \frac{q(1+q)}{(1-q)^s} \right\}.$$

The last series is summable for all $s$ with $\text{Re } s > 2$, and its sum can be written in terms of the Riemann zeta-function as in the statement of the proposition. \hfill $\square$

8.1 An approximated representation

Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\|l, m_1, m_2; j\|_\pm$ labelled by,

$$l \in \frac{1}{2}\mathbb{Z}, \quad l + j \in \mathbb{Z}, \quad j + m_1 \in \mathbb{N}, \quad l + \frac{1}{2} - j + m_2 \in \mathbb{N}.$$

Let $I$ be the labelling set of the Hilbert space $\mathcal{H}_\pm$ as in Theorem 6.5, and given by

$$I := \left\{ (j, m_1, m_2, j) \mid l \in \mathbb{N} + \frac{1}{2}, \ j = \frac{1}{2}, \frac{3}{2}, \ldots, l, \ j - |m_1| \in \mathbb{N}, \ l + \frac{1}{2} - j - |m_2| \in \mathbb{N} \right\}.$$

Notice that $I$ is the subset of labels of $\mathcal{H}$ satisfying $l \in \mathbb{N} + \frac{1}{2}$, $m_1 \leq j \leq l$ and $m_2 \leq l + \frac{1}{2} - j$.

Define the inclusion $Q : \mathcal{H} \to \mathcal{H}$ and the adjoint projection $P : \mathcal{H}_\pm \to \mathcal{H}$ by,

$$Q \|l, m_1, m_2; j\|_\pm := \|l, m_1, m_2; j\|_\pm \quad \text{for all } (l, m_1, m_2, j) \in I,$$

$$P \|l, m_1, m_2; j\|_\pm := \left\{ \begin{array}{ll} \|l, m_1, m_2; j\|_\pm & \text{if } (l, m_1, m_2, j) \in I, \\ 0 & \text{otherwise}. \end{array} \right.$$ Clearly, $PQ = id_{\mathcal{H}}$. The Hilbert space $\mathcal{H}$ carries a bounded $*$-representation of the algebra $\mathcal{A}(SU_q(2)) \otimes \mathcal{A}(S^2_\mathbb{C})$ defined by,

$$\alpha \|l, m_1, m_2; j\|_\pm = \sqrt{1 - q^{2(j+m_1+1)}} \|l + \frac{1}{2}, m_1 + \frac{1}{2}, m_2; j + \frac{1}{2}\|_\pm,$$
\[ \beta \|l, m_1, m_2; j\|_\pm = q^{j+m_1} \|l + \frac{1}{2}, m_1 - \frac{1}{2}, m_2; j + \frac{1}{2}\|_\pm, \]

\[ A\|l, m_1, m_2; j\|_\pm = q^{-j+m_2-\epsilon} \|l, m_1, m_2; j\|_\pm, \]

\[ B\|l, m_1, m_2; j\|_\pm = \sqrt{1 - q^{2(l-j+m_2+2-\epsilon)}} \|l + 1, m_1, m_2 + 1; j\|_\pm, \]

where, as before, \( \epsilon := \frac{1}{2}(-1)^{j+\frac{1}{2}-j-m_2} \). Composition of such a representation with the algebra embedding \( \mathcal{A}(S_q^4) \hookrightarrow \mathcal{A}(SU_q(2)) \otimes \mathcal{A}(S_q^2) \) given in equation (1.2) results in a \( * \)-representation \( \pi : \mathcal{A}(S_q^4) \rightarrow \mathcal{B}(\mathcal{H}) \). The sandwich \( \tilde{\pi}(a) := P\pi(a)Q \) defines a \( * \)-linear map \( \tilde{\pi} : \mathcal{A}(S_q^4) \rightarrow \mathcal{B}(\mathcal{H}) \).

**Proposition 8.3.** With \( \mathcal{I} \) the class of operators defined in Eq. (8.1), one has that the difference \( a - \tilde{\pi}(a) \in \mathcal{I} \) for all \( a \in \mathcal{A}(S_q^4) \).

**Proof.** Define \( \hat{\mathcal{I}} \) as the collection of bounded operators \( T : \mathcal{H} \rightarrow \mathcal{H} \) such that \( |D|^{-p}T \) is trace class for all \( p > 2 \). Since trace class operators are a two-sided ideal in bounded operators, the space \( \hat{\mathcal{I}} \) is stable when multiplied from the right by bounded operators: \( T_1 \in \hat{\mathcal{I}} \) and \( T_2 \in \mathcal{B}(\mathcal{H}) \Rightarrow T_1T_2 \in \hat{\mathcal{I}} \).

Next, suppose that \( a, b \) satisfy \( a - \tilde{\pi}(a) \in \mathcal{I} \) and \( b - \tilde{\pi}(b) \in \mathcal{I} \) and consider the following algebraic identity:

\[ ab - \tilde{\pi}(ab) = a\{b - \tilde{\pi}(b)\} + \{aP - P\pi(a)\} \pi(b)Q. \]

Since \( \mathcal{I} \) is a two-sided ideal in \( \text{OP}^0 \), the first summand is in \( \mathcal{I} \). The stability of \( \hat{\mathcal{I}} \) discussed above implies that \( \{aP - P\pi(a)\} \pi(b) \in \hat{\mathcal{I}} \), but if \( T \in \hat{\mathcal{I}} \) clearly \( TQ \in \mathcal{I} \). Hence the second summand in \( \mathcal{I} \) too. Thus, \( ab - \tilde{\pi}(ab) \in \mathcal{I} \) whenever this property holds for each of the operators \( a, b \). We conclude that it is enough to show that \( a - \tilde{\pi}(a) \in \mathcal{I} \) when \( a \) is a generator of \( \mathcal{A}(S_q^4) \).

By Proposition 8.2, this amounts to prove that the matrix elements of \( a - \tilde{\pi}(a) \) are bounded in modulus by \( q^{j+\frac{1}{2}} \).

Let us have a close look at the coefficients of \( a \in \{x_i, x_i^*\} \) in Theorem 6.5. Firstly, \( A_{j,m_1}^+, B_{j,m_1}^+, B_{j,m_1}^-, q^{-2j} A_{j,m_1}^0 \) and \( q^{-2j} B_{j,m_1}^0 \) are uniformly bounded by a constant, as one can see by writing explicitly the \( q \)-analogues in their expressions, getting:

\[ A_{j,m_1}^+ = q^{j+m_1} (1 - q^{4j+4})^{-1} \sqrt{(1 - q^{2(j+m_1+1)}) (1 - q^{2(j-m_1+1)})}, \]

\[ q^{-2j} A_{j,m_1}^0 = (1 - q^2) (1 - q^j) (1 - q^{4j+4})^{-1} \sqrt{2 [q^{2(j+m_1)} - q^{4j+1} - q^{-1}]}, \]

\[ B_{j,m_1}^+ = (1 - q^{4j+4})^{-1} \sqrt{(1 - q^{2(j+m_1+1)}) (1 - q^{2(j+m_1+2)})}, \]

\[ q^{-2j} B_{j,m_1}^0 = (1 + q^2) q^{j+m_1+1} (1 - q^{4j})^{-1} (1 - q^{4j+4})^{-1} \sqrt{(1 - q^{2(j-m_1)}) (1 - q^{2(j+m_1+1)})}, \]

\[ B_{j,m_1}^- = -q^{2(j+m_1+1)} (1 - q^{4j})^{-1} \sqrt{(1 - q^{2(j-m_1)}) (1 - q^{2(j+m_1-1)})}. \]

Analogously, the coefficients \( q^{2j} H_{l,j,m_2}^0, C_{l,j,m_2}^0 \) and \( D_{l,j,m_2}^0 \) are seen to be bounded by \( q^l \). Thus, modulo rapid decay matrices (i.e. smoothing operators),

\[ x_0 |l, m_1, m_2; j\rangle \simeq A_{j,m_1}^+ C_{l,j,m_2}^+ |l + 1, m_1, m_2; j + 1\rangle + A_{j,m_1}^+ C_{l,j,m_2}^- |l - 1, m_1, m_2; j + 1\rangle + A_{j,m_1}^0 H_{l,j,m_2}^+ |l + 1, m_1, m_2; j\rangle. \]
\[ x_1 |l, m_1, m_2; j \rangle \approx B_{j,m_2}^+ C_{l,j,m_2}^+ |l + 1, m_1 + 1, m_2; j + 1 \rangle + B_{j,m_2}^+ H_{l,j,m_2}^+ |l + 1, m_1 + 1, m_2; j \rangle + B_{j,m_2}^+ C_{l+1,j,m_2}^- |l + 1, m_1 + 1, m_2; j - 1 \rangle, \tag{8.2a} \]
\[ x_2 |l, m_1, m_2; j \rangle \approx D_{l,m_2}^+ C_{l,j,m_2}^+ |l + 1, m_1, m_2 + 1; j \rangle + D_{l,m_2}^- |l - 1, m_1, m_2 + 1; j \rangle. \tag{8.2c} \]

Since modulo smoothing operators the representations are the same we are omitting the label ‘±’ in the vector basis. Furthermore, using the inequalities
\[ 0 \leq (1 - qu)^{-1} - 1 \leq q(1 - q)^{-1} u, \quad 0 \leq 1 - (1 - u)^{2} \leq u, \tag{8.3} \]
which are valid when \( 0 \leq u \leq 1 \), we prove that modulo terms bounded by \( q' \), one has
\[ C_{l,j,m_2}^+ \approx -q^{l-j+m_2-\epsilon} \sqrt{1 - q^{2(l-j+m_2+3+\epsilon)}}, \tag{8.4a} \]
\[ C_{l,j,m_2}^- \approx -q^{l+j+m_2+1+\epsilon} \sqrt{1 - q^{2(l-j+m_2-\epsilon)}}, \tag{8.4b} \]
\[ H_{l,j,m_2}^+ \approx q^{l+m_2+1} \sqrt{q^{2(2j+1)} - q^{2(l+m_2+2)}}, \tag{8.4c} \]
\[ D_{l,j,m_2}^+ \approx \sqrt{1 - q^{2(l+j+m_2+3+\epsilon)}} \sqrt{1 - q^{2(l-j+m_2+2-\epsilon)}}, \tag{8.4d} \]
\[ D_{l,j,m_2}^- \approx -q^{2(l+m_2)+3}. \tag{8.4e} \]

Up to now, we neglected only smoothing contributions (the above approximation will be needed when dealing with the real structure later on). We use again (8.3) to get a rougher approximation by neglecting terms bounded by \( q' \). This yields
\[ A_{j,m_1}^+ \approx \tilde{A}_{j,m_1}^+ := q^{j+m_1} \sqrt{1 - q^{2(j+m_1+1)}}, \tag{8.5a} \]
\[ A_{j,m_1}^0 H_{l,j,m_2}^+ \approx 0, \tag{8.5b} \]
\[ B_{j,m_1}^+ \approx \tilde{B}_{j,m_1}^+ := \sqrt{(1 - q^{2(j+m_1+1)})(1 - q^{2(j+m_1+2)})}, \tag{8.5c} \]
\[ B_{j,m_1}^0 H_{l,j,m_2}^+ \approx 0, \tag{8.5d} \]
\[ B_{j,m_1}^- \approx \tilde{B}_{j,m_1}^- := -q^{2(j+m_1)+1}, \tag{8.5e} \]
\[ C_{l,j,m_2}^+ \approx \tilde{C}_{l,j,m_2}^+ := -q^{l-j+m_2-\epsilon}, \tag{8.5f} \]
\[ C_{l,j,m_2}^- \approx 0, \tag{8.5g} \]
\[ D_{l,j,m_2}^+ \approx \tilde{D}_{l,j,m_2}^+ := \sqrt{1 - q^{2(l-j+m_2+2-\epsilon)}}, \tag{8.5h} \]
\[ D_{l,j,m_2}^- \approx 0. \tag{8.5i} \]
Proposition 8.4. which we are lacking at the moment.

S-classical points of poles of the zeta-functions. The top residue coincides with the integral on the subspace of the dimension spectrum in the left half plane Re $s > 2$. To study the part of the dimension spectrum in the left half plane Re $s \leq 2$ would require a less drastic approximation which we are lacking at the moment.

**Proof.** Let $\Psi^0$ be the $*$-algebra generated by $A(S_q^4)$, by $[D, A(S_q^4)]$ and by iterated applications of the derivation $\delta$. Let $A \subset \mathcal{A}(SU_q(2)) \otimes \mathcal{A}(S_q^2) \otimes \text{Mat}_2(\mathbb{C})$ be the $*$-algebra generated by $\alpha$, $\beta$, $\alpha^*$, $\beta^*$, $A$, $B$, $B^*$ and $F$. By Proposition 8.3 there is an inclusion $\mathcal{A}(S_q^4) \subset P\mathcal{A}Q + I$.

A linear basis for $A$ is given by,

$$T := \alpha^{k_1} \beta^{n_1} (\beta^*)^{n_2} A^{n_3} B^{k_2} F^h,$$

where $h \in \{0,1\}$, $n_i \in \mathbb{N}$, $k_i \in \mathbb{Z}$ and with the notation $\alpha^{k_1} := (\alpha^*)^{-k_1}$ if $k_1 < 0$ and $B^{k_2} := (B^*)^{-k_2}$ if $k_2 < 0$. For this operator,

$$\delta(PTQ) = \left( \frac{1}{2}(k_1 + n_1 - n_2) + k_2 \right) PTQ \quad \text{and} \quad [D, PTQ] = \delta(PTQ) F.$$

The observation that

$$-P(\alpha \beta + \beta^* \alpha^*) A Q = \hat{\pi}(x_0), \quad P(-\alpha^2 + q(\beta^*)^2) A Q = \hat{\pi}(x_1), \quad PBQ = \hat{\pi}(x_2),$$

concludes the proof. \qed

8.2 The dimension spectrum and the top residue

The approximation modulo $I$ allows considerable simplifications when getting information on the part of the dimension spectrum contained in the half plane Re $s > 2$. To study the part of the dimension spectrum in the left half plane Re $s \leq 2$ would require a less drastic approximation which we are lacking at the moment.

**Proposition 8.4.** In the region Re $s > 2$ the dimension spectrum $\Sigma$ of the spectral triple $(\mathcal{A}(S_q^4), \mathcal{H}, D, \gamma)$ given in Proposition 7.2 consists of the two points $\{3, 4\}$, which are simple poles of the zeta-functions. The top residue coincides with the integral on the subspace of classical points of $S_q^4$, that is

$$\int a |D|^{-4} = \frac{2}{3\pi} \int_0^{2\pi} \sigma(a)(\theta) d\theta,$$

with $\sigma : \mathcal{A}(S_q^4) \to \mathcal{A}(S^4)$ the $*$-algebra morphism defined by $\sigma(x_0) = \sigma(x_1) = 0$ and $\sigma(x_2) = u$, where $u$, given by $u(\theta) := e^{i\theta}$, is the unitary generator of $\mathcal{A}(S^4)$.

**Proof.** Let $\Psi^0$ be the $*$-algebra generated by $A(S_q^4)$, by $[D, A(S_q^4)]$ and by iterated applications of the derivation $\delta$. Let $A \subset \mathcal{A}(SU_q(2)) \otimes \mathcal{A}(S_q^2) \otimes \text{Mat}_2(\mathbb{C})$ be the $*$-algebra generated by $\alpha$, $\beta$, $\alpha^*$, $\beta^*$, $A$, $B$, $B^*$ and $F$. By Proposition 8.3 there is an inclusion $\mathcal{A}(S_q^4) \subset P\mathcal{A}Q + I$.

A linear basis for $A$ is given by,

$$T := \alpha^{k_1} \beta^{n_1} (\beta^*)^{n_2} A^{n_3} B^{k_2} F^h,$$
which has meromorphic extension on $\mathbb{C}$ and hence $\Psi^0 \subset P\mathfrak{A}I + \mathcal{I}$.

For the part of the dimension spectrum in the right half plane $\text{Re}\ s > 2$, we can neglect $\mathcal{I}$ and consider only the singularities of zeta-functions associated with elements in $P\mathfrak{A}I$. By linearity of the zeta-functions, it is enough to consider the generic basis element in Eq. (8.7).

Such a $T$ shifts $l$ by $\frac{1}{2}(k_1 + n_1 - n_2) + k_2$, $m_1$ by $\frac{1}{2}(k_1 - n_1 + n_2)$, $m_2$ by $k_2$, and flips the chirality if $h = 1$. Thus it is off-diagonal unless $h = k_i = 0$ and $n_1 = n_2$. The zeta-function associated with a bounded off-diagonal operator is identically zero in the half-plane $\text{Re}\ z > 4$, and so is its holomorphic extension to the entire complex plane. It remains to consider the cases $T = P(\beta \beta^*)^k A^n Q$, with $n, k \in \mathbb{N}$.

If $n$ and $k$ are both different from zero, one finds

$$
\zeta_T(s) = 2 \sum_{l,j,m_1,m_2} (l + \frac{3}{2})^{-s} q^{n(l-j+m_2-\epsilon)+2k(j+m_1)} = 2 \sum_{l,j,m_2} (l + \frac{3}{2})^{-s} q^{n(l-j+m_2-\epsilon)} \frac{1 - q^{2k(2j+1)}}{1 - q^{2k}}.
$$

For $\epsilon$ fixed, set $2i := l - \epsilon - j + m_2 \in \{0, 2, \ldots, 2(l-j)\}$. Then,

$$
\zeta_T(s) = 2 \sum_{l,j} (l + \frac{3}{2})^{-s} \frac{1 - q^{2k(2j+1)}}{1 - q^{2k}} \sum_{\epsilon = \pm 1/2} \sum_{i=0}^{l-j} q^{2ni} = 4 \sum_{l,j} (l + \frac{3}{2})^{-s} \frac{1 - q^{2k(2j+1)}}{1 - q^{2k}} \frac{1 - q^{2n(l-j+1)}}{1 - q^{2n}}
$$

which has meromorphic extension on $\mathbb{C}$ with simple pole in $s = \{1, 2\}$.

If $n = 0$ and $k \neq 0$,

$$
\zeta_T(s) = 4 \sum_{l,j} (l + \frac{3}{2})^{-s} (l - j + 1) \frac{1 - q^{2k(2j+1)}}{1 - q^{2k}}
$$

$$
= \frac{4}{1 - q^{2n}} \left( \frac{1}{2} \zeta(s-2) - \left( \frac{1}{2} + \frac{1}{1 - q^{2n}} \right) \zeta(s-1) + \frac{1 - q^{2n}}{(1 - q^{2n}) \log q^{2n}} \zeta(s) \right) + \text{hol. function},
$$

which has meromorphic extension on $\mathbb{C}$ with simple pole in $s = \{1, 2, 3\}$.

If $n \neq 0$ and $k = 0$,

$$
\zeta_T(s) = 4 \sum_{l,j} (l + \frac{3}{2})^{-s} (2j + 1) \frac{1 - q^{2n(l-j+1)}}{1 - q^{2n}}
$$

$$
= \frac{4}{1 - q^{2n}} \left\{ \zeta(s-2) - \left( 1 + \frac{2q^{2n}}{1 - q^{2n}} \right) \zeta(s-1) + \frac{2q^{2n}}{1 - q^{2n}} \left( 1 + \frac{q^{2n}}{(1 - q^{2n}) \log q^{2n}} \zeta(s) \right) \right\} + \text{hol. fun.},
$$

which has meromorphic extension on $\mathbb{C}$ with simple pole in $s = \{1, 2, 3\}$.

Finally, if both $n$ and $k$ are zero we get (cf. Eq. (7.3)),

$$
\zeta_T(s) = \frac{4}{3} \left\{ \zeta(s-3) - \zeta(s-1) \right\},
$$

and this is meromorphic with simple poles in $\{2, 4\}$. Thus, the part of the dimension spectrum in the region $\text{Re}\ s > 2$ consists at most of the two points $\{3, 4\}$ and both are simple poles.
Since we have considered the enlarged algebra $P\mathfrak{a}Q + I$, it suffices to prove that there exists an $a \in \Psi^0$ whose zeta-function is singular in both points $s = 3$ and $s = 4$. We take $a = x_2x_2^*$. From the definition
\[ \tilde{\pi}(x_2x_2^*)|l, m_1, m_2; j\rangle_\pm = (1 - q^{2(l-e-j+m_2)})|l, m_1, m_2; j\rangle_\pm . \]
Then, modulo functions that are holomorphic when $\text{Re}\, s > 2$, we have
\[ \zeta_{x_2x_2^*}(s) \sim \zeta_{\tilde{\pi}(x_2x_2^*)}(s) = \zeta_1(s) - 2 \sum_{l,j,m_2} (l + \frac{3}{2})^{-s} q^{2(l-e-j+m_2)} \sim \frac{4}{3} \zeta(s - 3) - \frac{4}{3} \zeta(s - 2) . \]
This proves the first part of the proposition, that is $\Sigma \cap \{\text{Re}\, s > 2\} = \{3, 4\}$.

The proof of Eq. (3.3) is based on the observation that the residue in $s = 4$ of $\zeta_T$, for $T$ a basis element of $P\mathfrak{a}Q$, is zero unless $T = 1$. That is, it depends only on the image of $T$ under the map sending $\beta, A$ and $F$ to 0 while $a \mapsto e^{i\varphi}$ and $B \mapsto e^{i\theta}$. Composing this map with $\tilde{\pi}$ we get the morphism $\sigma : \mathcal{A}(S^4) \to \mathcal{A}(S^1)$ of the proposition and that
\[ \int a|D|^{-4} \propto \int_0^{2\pi} \sigma(a) d\theta . \]
The equality $\int|D|^{-4} = \frac{4}{3}$ fixes the proportionality constant. \hfill \Box

9 Reality and first order conditions

Classically, if $(\mathcal{A}(M), \mathcal{H}, D, \gamma)$ is the canonical spectral triple associated with a 4-dimensional spin manifold $M$, there exists an antilinear isometry $J$ on $\mathcal{H}$, named the real structure, satisfying the following compatibility condition
\[ J^2 = -1 , \quad J\gamma = \gamma J , \quad JD = DJ . \tag{9.1} \]
There are also two additional conditions involving the coordinate algebra $\mathcal{A}(M)$:
\[ [a, JbJ^{-1}] = 0 , \quad [[D, a], JbJ^{-1}] = 0 , \quad \forall a, b \in \mathcal{A}(M) . \tag{9.2} \]
The real structure on $S^4$ is equivariant and equivariance is sufficient to determine $J$.

In the deformed situation one has to be careful on how to implement equivariance. Let us start with the working hypothesis that equivariance for $J$ is the requirement that it satisfies $Jh = S(h)^*J$ for all $h \in U_q(\text{so}(5))$. Then, consider the Casimir operator $C_1$ given in equation (3.4). This operator commutes with $J$ since $S(C_1)^* = C_1$ and from its expression, $C_1|l, m_1, m_2; j\rangle = (q^{2j+1} + q^{-2j-1})|l, m_1, m_2; j\rangle$, we conclude that $J$ leaves the index $j$ invariant. Compatibility with $\gamma$ and $D$ in Eq. (9.1) and equivariance with respect to $h = K_1$ and $h' = K_2$ yields
\[ J|l, m_1, m_2; j\rangle_\pm = c_\pm(l, m_1, m_2; j)|l, m_1, m_2; j\rangle_\pm , \]
with some constants $c_\pm$ to be determined. Equivariance with respect to $h = E_1$ implies
\[ c_\pm(l, m_1, m_2; j) = (-1)^{m_1+1/2} q^{m_1} c_\pm(l, m_2; j) . \]
For $h = E_2$, looking at the piece diagonal in $j$ we deduce that the dependence on $m_2$ is through a factor $q^{3m_2}$; and looking at the piece shifting $j$ by $±1$ we conclude that

$$c_±(l, m_1, m_2; j) = (-1)^{j+m_1}q^{m_1+3m_2}c_±(l) .$$

Such an operator $J$ cannot be antunitary unless $q = 1$. At $q = 1$ the antiunitarity condition requires that $c_±(l) ∈ U(1)$ and modulo a unitary equivalence we can choose $c_±(l) = i^{2l+1}$. In conclusion for $q = 1$ the operator

$$J |l, m_1, m_2; j⟩_± = i^{2l+1}(-1)^{j+m_1} |l, -m_1, -m_2; j⟩_± ,$$ (9.3)

is the real structure on $S^4$ (modulo a unitary equivalence).

For $q ≠ 1$ we keep (9.3) as the real structure and notice that conditions (9.1) are satisfied, but $J$ no longer satisfies the requirement $Jh = S(h)^*J$ for all $h ∈ U_q(so(5))$. Nevertheless, $J$ is the antiunitary part of an antilinear operator $T$ that has this property. The antilinear operator $T$ defined by

$$T |l, m_1, m_2; j⟩_± = i^{2l+1}(-1)^{j+m_1}q^{m_1+3m_2} |l, -m_1, -m_2; j⟩_± ,$$

has $J$ in (9.3) as the antiunitary part and it is equivariant, i.e. it is such that $Th = S(h)^*T$ for all $h ∈ U_q(so(5))$.

Next, we turn to the conditions (9.2). In parallel with the cases of the manifold of $SU_q(2)$ in [11] and of Podleś spheres in [10, 9], once again we need to modify them. For instance, the commutator $[x_2, Jx_2J]$ is not zero, as one can see by computing the matrix element

$$f(l, j, m_2) := ± ⟨l + 1, m_1, m_2; j |[x_2, Jx_2J]|l, m_1, m_2; j⟩_±$$

$$= D_{l+1,j,m_2−1}^0 D_{l,j−m_2}^+ − D_{l,j+m_2−1}^0 D_{l,j,m_2}^+ + D_{l+1,j,m_2−1} D_{l,j−m_2}^+ − D_{l,j+m_2−1} D_{l,j,m_2}^+ ,$$

which for $j = \frac{1}{2}$ and $m_2 = l$ is

$$f(l, \frac{1}{2}, l) = -q^{l+4}(1 - q^2)^2 [2] (q^{l+1} + q^{-l+1}) [2l+3] [l+1][l+2][l+3]/[2l+2][2l+4][2l+6] ≠ 0 .$$

It is relatively easy to prove that the two conditions are satisfied modulo the ideal $I$. It is much more cumbersome computationally to show that they are in fact satisfied modulo the smaller ideal of smoothing operators.

**Proposition 9.1.** Let $J$ be the antilinear isometry given by (9.3). Then,

$$[a, JbJ] ∈ I , \quad [[D, a], JbJ] ∈ I , \quad ∀ a, b ∈ A(S^4_q) .$$

**Proof.** We lift $J$ and $D$ to the Hilbert space $\hat{H}$ defined in Sect. [8.1] as follows:

$$\hat{J} |l, m_1, m_2; j⟩_± = i^{2l+1}(-1)^{j+m_1} |l, -m_1, -m_2; j⟩_± ,$$

$$\hat{D} |l, m_1, m_2; j⟩_± = (l + \frac{3}{2}) |l, m_1, m_2; j⟩_± .$$

Notice that $\hat{J}^2 = -1$ on $\hat{H}$ (thanks to the phase $i^{2l+1}$ that is irrelevant when restricted to $H$).
Let now \( \{\alpha, \beta, \alpha^*, \beta^*, A, B, B^*\} \) be the operators defined in Section 8.1 generators of the algebra \( \mathcal{A}(SU_q(2)) \otimes \mathcal{A}(S_q^4) \). Due to Proposition 8.3 it is enough to prove that for all pairs \((a, b)\) of such generators, the commutators \([a, \hat{J}b\hat{J}]\) and \([\hat{D}, a], \hat{J}b\hat{J}\) are weighted shifts with weight which are bounded by \( q^{2j} \). From

\[
[\hat{D}, a] = \frac{1}{2} \alpha \hat{F}, \quad [\hat{D}, \beta] = \frac{1}{2} \beta \hat{F}, \quad [\hat{D}, A] = 0, \quad [\hat{D}, B] = B \hat{F},
\]

the condition on \([\hat{D}, a], \hat{J}b\hat{J}\) follows from the same condition on \([a, \hat{J}b\hat{J}]\), and we have to compute only the latter commutators.

Since \([a, \hat{J}b\hat{J}] = -[a^*, \hat{J}b\hat{J}]^*\) and \([b, \hat{J}a\hat{J}] = J[a, \hat{J}b\hat{J}]\hat{J},\) we have to check the 16 combinations in the following table.

| \( b \setminus a \) | \( \alpha \) | \( \alpha^* \) | \( \beta \) | \( \beta^* \) | \( A \) | \( B \) | \( B^* \) |
|---|---|---|---|---|---|---|---|
| \( \alpha \) | \( \bullet \) | \( \times \) | \( \times \) | \( \bullet \) | \( \bullet \) | \( \bullet \) | \( \bullet \) |
| \( \beta \) | \( \bullet \) | \( \times \) | \( \bullet \) | \( \bullet \) | \( \bullet \) | \( \bullet \) | \( \bullet \) |
| \( B \) | \( \bullet \) | \( \bullet \) | \( \bullet \) | \( \bullet \) | \( \bullet \) | \( \bullet \) | \( \bullet \) |
| \( A \) | \( \bullet \) | \( \bullet \) | \( \bullet \) | \( \bullet \) | \( \bullet \) | \( \bullet \) | \( \bullet \) |

By direct computations one shows that bullets in the table correspond to vanishing commutators. On the other hand, the commutators corresponding to the crosses in the table are given on the subspace with \( j - |m_1| \in \mathbb{N} \), by

\[
\begin{align*}
[\beta^*, \hat{J}a\hat{J}] ||l, m_1, m_2; j\rangle \rangle_{\pm} &= q^{j+m_1} \left\{ \sqrt{1 - q^{2(j+m_1+1)}} - \sqrt{1 - q^{2(j-m_1)}} \right\} ||l, m_1, m_2; j\rangle \rangle_{\pm} \\
[\beta, \hat{J}a\hat{J}] ||l, m_1, m_2; j\rangle \rangle_{\pm} &= -[\beta^*, \hat{J}a\hat{J}] ||l+1, m_1-1, m_2; j+1\rangle \rangle_{\pm}, \\
[\beta^*, \hat{J}b\hat{J}] ||l, m_1, m_2; j\rangle \rangle_{\pm} &= -[2]q^{2j} ||l, m_1+1, m_2; j\rangle \rangle_{\pm}.
\end{align*}
\]

Since \( 1-u \leq \sqrt{1-u} \leq 1 \) for all \( u \in [0, 1] \), we have that

\[
0 \leq q^{j+m_1} \left\{ \sqrt{1 - q^{2(j-m_1+1)}} - \sqrt{1 - q^{2(j-m_1)}} \right\} \leq q^{j+m_1}(1 - 1 + q^{2(j-m_1)}) \leq q^{2j}.
\]

Then, all three non-zero commutators are weighted shifts with weights bounded by \( q^{2j} \). \( \square \)

**Proposition 9.2.** Let \( J \) be the antilinear isometry given by \((9.3)\). Then,

\[
[a, \hat{J}b\hat{J}] \in \text{OP}^{-\infty}, \quad [[D, a], \hat{J}b\hat{J}] \in \text{OP}^{-\infty}, \quad \forall a, b \in \mathcal{A}(S_q^4).
\]

**Proof.** By Leibniz rule, it is sufficient to prove the statement when \( a \) and \( b \) are generators of the algebra. By \((7.3)\), \([D, a] - \delta(a)F\) is a smoothing operator. Thus, it is enough to show that

\[
[a, \hat{J}b\hat{J}] \in \text{OP}^{-\infty}, \quad [\delta(a), \hat{J}b\hat{J}] \in \text{OP}^{-\infty}, \quad (9.4)
\]

for any pair \((a, b)\) of generators. From

\[
[b, \hat{J}a\hat{J}] = J[a, \hat{J}b\hat{J}]J, \quad [\delta(b), \hat{J}a\hat{J}] = -J[\delta(a), \hat{J}b\hat{J}]J + \delta([b, \hat{J}a\hat{J}]),
\]

\[34\]
it follows that if (9.4) is satisfied for a particular pair \((a, b)\), then it is satisfied for \((b, a)\) too. From

\[
[a^*, Jb^*] = [a, Jb]^* \quad \text{and} \quad [\delta(a^*), b^*] = [\delta(a), b]^*,
\]

we see that if (9.4) is satisfied for a pair \((a, b)\), then it is satisfied for \((a^*, b^*)\) too. With these symmetries we need to check only the following 9 cases out of 25:

\[
\begin{array}{c|ccccc}
\hline
b \setminus a & x_0 & x_1 & x_1^* & x_2 & x_2^* \\
\hline
x_0 & & & & & \\
x_1 & & & & & \\
x_2 & & & & & \\
\hline
\end{array}
\]

From Eqs. (8.2) and (8.3) we see that modulo smoothing operators

\[
\frac{1}{2}\{x_0 + \delta(x_0)\} \mid l, m_1, m_2; j \simeq \begin{cases} 
A_{j,m_1}^+ \hat{C}_{l,j,m_2}^+ \mid l + 1, m_1, m_2; j + 1 \mid \\
\quad + q^{-2j} A_{j,m_1}^0 \hat{H}_{l,j,m_2}^+ \mid l + 1, m_1, m_2; j \mid \\
\quad + A_{j-1,m_1}^+ \hat{C}_{l+1,j-1,m_2}^- \mid l + 1, m_1, m_2; j - 1 \mid , \\
\end{cases}
\]

\[
\frac{1}{2}\{x_0 - \delta(x_0)\} \mid l, m_1, m_2; j \simeq \begin{cases} 
A_{j,m_1}^+ \hat{C}_{l,j,m_2}^- \mid l - 1, m_1, m_2; j + 1 \mid \\
\quad + q^{-2j} A_{j,m_1}^0 \hat{H}_{l,j,m_2}^- \mid l - 1, m_1, m_2; j \mid \\
\quad + A_{j-1,m_1}^+ \hat{C}_{l+1,j-1,m_2}^+ \mid l - 1, m_1, m_2; j - 1 \mid , \\
\end{cases}
\]

\[
\frac{1}{2}\{x_1 + \delta(x_1)\} \mid l, m_1, m_2; j \simeq \begin{cases} 
B_{j,m_1}^+ \hat{C}_{l,j,m_2}^+ \mid l + 1, m_1 + 1, m_2; j + 1 \mid \\
\quad + q^{-2j} B_{j,m_1}^0 \hat{H}_{l,j,m_2}^+ \mid l + 1, m_1 + 1, m_2; j \mid \\
\quad + B_{j,m_1}^- \hat{C}_{l+1,j-1,m_2}^- \mid l + 1, m_1 + 1, m_2; j - 1 \mid , \\
\end{cases}
\]

\[
\frac{1}{2}\{x_1 - \delta(x_1)\} \mid l, m_1, m_2; j \simeq \begin{cases} 
B_{j,m_1}^+ \hat{C}_{l,j,m_2}^- \mid l - 1, m_1 + 1, m_2; j + 1 \mid \\
\quad + q^{-2j} B_{j,m_1}^0 \hat{H}_{l,j,m_2}^- \mid l - 1, m_1 + 1, m_2; j \mid \\
\quad + B_{j,m_1}^- \hat{C}_{l+1,j-1,m_2}^+ \mid l - 1, m_1 + 1, m_2; j - 1 \mid , \\
\end{cases}
\]

\[
\frac{1}{2}\{x_2 + \delta(x_2)\} \mid l, m_1, m_2; j \simeq \begin{cases} 
D_{l,j,m_2}^+ \mid l + 1, m_1, m_2 + 1; j \mid , \\
\end{cases}
\]

\[
\frac{1}{2}\{x_2 - \delta(x_2)\} \mid l, m_1, m_2; j \simeq \begin{cases} 
D_{l,j,m_2}^- \mid l - 1, m_1, m_2 + 1; j \mid , \\
\end{cases}
\]

where

\[
\hat{C}_{l,j,m_2}^+ = -q^{-j+m_2+\epsilon} \sqrt{1 - q^{2(l+j+m_2+3+\epsilon)}}, \quad \hat{C}_{l,j,m_2}^- = -q^{l+j+m_2+1+\epsilon} \sqrt{1 - q^{2(l-j+m_2+\epsilon)}},
\]

\[
\hat{H}_{l,j,m_2}^+ = q^{2j} q^{l+m_2+1} \sqrt{1 - q^{2(2j+1)} - q^{2(l+m_2+2)}}, \quad \hat{H}_{l,j,m_2}^- = \sqrt{1 - q^{2(l+j+m_2+3+\epsilon)} - q^{2(l-j+m_2+2+\epsilon)}},
\]

\[
\hat{D}_{l,j,m_2}^+ = \sqrt{1 - q^{2(l+j+m_2+3+\epsilon)} - q^{2(l-j+m_2+2+\epsilon)}}, \quad \hat{D}_{l,j,m_2}^- = -q^{2(l+m_2)+3}.
\]

We have divided the terms in three classes, which need to be analysed separately.
All terms $T$ which are not ‘boxed’ have coefficients which are uniformly bounded by $q^{l+m_2}$; since the conjugation with $J$ changes the sign of the labels $m_1, m_2$, for such $T$’s, the coefficients of $JTJ$ are uniformly bounded by $q^{l-m_2}$. They give products (and so commutators) with coefficients bounded by $q^{l+m_2}q^{l-m_2} = q^l$, and so (these products) are smoothing operators.

Analogously, the coefficients of single-boxed terms are bounded by $q^{l-j+m_2}$, and become smoothing when multiplied by the $J$-conjugated of non-boxed terms (as $q^{l-j+m_2}q^{l-m_2} \leq q^l$), and vice versa for the product of a non-boxed term with the $J$-conjugated of a single-boxed one ($q^{l+m_2}q^{-j-m_2} \leq q^l$).

Next we consider pairs of single-boxed terms. A closer look at the single-boxed terms in $x_0 \pm \delta(x_0)$ and $x_1 - \delta(x_1)$ (and then $x_1^* + \delta(x_1^*)$) shows that they have coefficients bounded by $q^{l+m_1+m_2}$, and become smoothing when multiplied by the $J$-conjugated of one of them ($q^{l+m_1+m_2}q^{-l-m_2} = q^2$). Last single-boxed term is the one in $x_1 + \delta(x_1)$ (and $x_1^* - \delta(x_1^*)$).

The relevant terms for the commutators involving them are

$$\frac{1}{2}[x_1 + \delta(x_1), Jx_0J] | l, m_1, m_2; j \rangle \sim \{ A_{l+j,1,m_1-1}^+ \dot{C}_{l+1,j+1,m_2} J, m_1 \dot{C}_{l,j,m_2} +$$

$$= A_{l+m_2} J, m_1 \dot{C}_{l+1,j+1,m_2} B_{j,m_1} \dot{C}_{l,j,m_2} +$$

$$+ \{ A_{l,m_1-1} J, m_1 \dot{C}_{l+1,j+1,m_2} B_{j,m_1} \dot{C}_{l,j,m_2} +$$

$$\} | l + 2, m_1 + 1, m_2; j + 2 \rangle \right.$$

$$\frac{1}{2}[x_1 + \delta(x_1), Jx_1J] | l, m_1, m_2; j \rangle \sim \{ B_{l+m_1-1} J, m_1 \dot{C}_{l+1,j+1,m_2} B_{j,m_1} \dot{C}_{l,j,m_2} +$$

$$= B_{l,m_1} J, m_1 \dot{C}_{l+1,j+1,m_2} B_{j,m_1} \dot{C}_{l,j,m_2} +$$

$$+ \{ B_{l,m_1-1} J, m_1 \dot{C}_{l+1,j+1,m_2} B_{j,m_1} \dot{C}_{l,j,m_2} +$$

$$\} | l + 2, m_1 + 1, m_2; j \rangle \right.$$

$$\frac{1}{2}[x_1^* + \delta(x_1^*), Jx_1J] | l, m_1, m_2; j \rangle \sim \{ B_{l+1,1,m_2-1} J, m_1 \dot{C}_{l+1,j+1,m_2} B_{j,m_1} \dot{C}_{l,j,m_2} +$$

$$= B_{l,j,m_1} J, m_1 \dot{C}_{l+1,j+1,m_2} B_{j,m_1} \dot{C}_{l,j,m_2} +$$

$$+ \{ B_{l+1,1,m_2-1} J, m_1 \dot{C}_{l+1,j+1,m_2} B_{j,m_1} \dot{C}_{l,j,m_2} +$$

$$\} | l + 2, m_1 - 1, m_2; j + 2 \rangle \right.$$

$$\frac{1}{2}[x_1^* + \delta(x_1^*), Jx_1J] | l, m_1, m_2; j \rangle \sim \{ B_{l+1,1,m_2-1} J, m_1 \dot{C}_{l+1,j+1,m_2} B_{j,m_1} \dot{C}_{l,j,m_2} +$$

$$= B_{l,j,m_1} J, m_1 \dot{C}_{l+1,j+1,m_2} B_{j,m_1} \dot{C}_{l,j,m_2} +$$

$$+ \{ B_{l+1,1,m_2-1} J, m_1 \dot{C}_{l+1,j+1,m_2} B_{j,m_1} \dot{C}_{l,j,m_2} +$$

$$\} | l + 2, m_1 - 1, m_2; j \rangle \right.$$
operators, we find
\[
\tilde{C}_{l,i,-m_2}^+ \tilde{C}_{l,i,+m_2}^+ \simeq \sqrt{q^{2(l-j)} - q^{2l-m_2+3\epsilon}} \sqrt{q^{2(l-j)} - q^{2l-m_2+3+2\epsilon}}
\]
\[\simeq \sqrt{q^{2(l-j)}} \sqrt{q^{2(l-j)}} = q^{2(l-j)}.\]

Using this we get
\[
\frac{1}{\tilde{z}}[x_1 + \delta(x_1), J_{x_0}J] |l, m_1, m_2; j\rangle
\]
\[\simeq q^{2(l-j)} \{ A_{j+1,-m_1+1}^+ B_{j,m_1}^+ - A_{j,-m_1}^+ B_{j+1,m_1}^+ \} |l + 2, m_1 + 1, m_2; j + 2\rangle
\]
\[+ q^{2(l-j)} \{ A_{j,-m_1+1}^+ B_{j,m_1}^+ - A_{j-1,-m_1}^+ B_{j-1,m_1}^+ \} |l, m_1 + 1, m_2; j\rangle,
\]
\[
\frac{1}{\tilde{z}}[x_1 + \delta(x_1), J_{x_1}J] |l, m_1, m_2; j\rangle
\]
\[\simeq q^{2(l-j)} \{ B_{j+1,-m_1+1}^+ B_{j,m_1}^+ - B_{j+1,-m_1-1}^+ B_{j,m_1}^+ \} |l + 2, m_1, m_2; j + 2\rangle
\]
\[+ q^{2(l-j)} \{ B_{j-1,-m_1+1}^+ B_{j-1,m_1}^+ - B_{j+1,-m_1-1}^+ B_{j-1,m_1}^+ \} |l, m_1, m_2; j\rangle,
\]
\[
\frac{1}{\tilde{z}}[x_1 - \delta(x_1), J_{x_1}J] |l, m_1, m_2; j\rangle
\]
\[\simeq q^{2(l-j)} \{ B_{j+2,m_1-2}^+ B_{j,m_1}^+ - B_{j+1,-m_1+1}^+ B_{j+1,m_1-1}^+ \} |l + 2, m_1 - 2, m_2; j + 2\rangle
\]
\[+ q^{2(l-j)} \{ B_{j+2,m_1-1}^+ B_{j+1,m_1-1}^+ - B_{j-1,-m_1+1}^+ B_{j-1,m_1-1}^+ \} |l - 2, m_1 - 2, m_2; j - 2\rangle.
\]

To prove that these commutators are smoothing we still need to check that the terms in braces are bounded by \(q^j\) (since \(q^{2(l-j)} q^j \leq q^j\) is of rapid decay). This is done by using Eqs. (8.5). For example the first two braces are identically zero, while the third one is
\[
B_{j+1,-m_1+1}^+ B_{j,-m_1}^+ - B_{j+1,-m_1-1}^+ B_{j,m_1}^+ = B_{j+1,-m_1+1}^+ B_{j,-m_1}^+ - B_{j+1,-m_1-1}^+ B_{j,m_1}^+ + O(q^j)
\]
\[= 0 + O(q^j).
\]

What remains to control are the commutators \([x^*_2 - \delta(x_2^*), JbJ]\) for \(b = x_0, x_1, x_2\) and the commutators \([x^*_2 - \delta(x_2^*), JbJ]\) for \(b = x_1, x_2\) (which involve the ‘doubly-boxed’ term).

The operators \(x_2\) and \(\delta(x_2)\) do not shift \(m_1, j\) and have coefficients independent on \(m_1\). Thus, any operator acting only on the label \(m_1\) and with coefficients depending only on \(m_1, j\), commutes with \(x_2\) and \(\delta(x_2)\) and so can be neglected. In particular, \(x_0\) and \(x_1\) can be written as sums of products of operators of this kind (commuting with \(x_2\) and \(\delta(x_2)\)) by operators \(y_i\)’s,
\[
y_1 |l, m_1, m_2; j\rangle := \tilde{C}_{l,j,m_2}^+ |l + 1, m_1, m_2; j + 1\rangle,
\]
\[
y_2 |l, m_1, m_2; j\rangle := \tilde{H}_{l,j,m_2}^+ |l + 1, m_1, m_2; j\rangle,
\]
\[
y_3 |l, m_1, m_2; j\rangle := \tilde{C}_{l,j,m_2}^- |l - 1, m_1, m_2; j + 1\rangle,
\]
and their adjoints. To prove that the commutators \([x^*_2 - \delta(x_2^*), JbJ]\), for \(b = x_0, x_1, x_2\), and \([x^*_2 - \delta(x_2^*), JbJ]\), for \(b = x_1, x_2\), are smoothing, is sufficient to establish the same for \(b = y_1, y_2, y_3\). For these operators we have
\[
\frac{1}{\tilde{z}}[x_2 + \delta(x_2), Jy_1J] |l, m_1, m_2; j\rangle
Now \( \hat{C}_{t+1,j,-m_2-1}^+ \hat{D}_{l,j,m_2}^+ - \hat{D}_{l+1,j+1,m_2}^+ \hat{C}_{t,j,-m_3}^+ \) \( |l + 2, m_1, m_2 + 1; j + 1) \)
\[ = \hat{C}_{l,j,-m_2}^+ \{ \hat{D}_{l,j,m_2}^+ - \hat{D}_{l+1,j+1,m_2}^+ \} \{ l + 2, m_1, m_2 + 1; j + 1 \} \) ,
\[
\frac{1}{2} [x^2 - \delta(x_2^*), J_{y_2} J] | l, m_1, m_2; j \)
\[ \approx \{ \hat{H}_{l+1,j,-m_2}^+ \hat{D}_{l+1,j,m_2}^+ - \hat{D}_{l,j,-m_2}^+ \hat{H}_{l+1,j,m_2}^+ \} \{ l + 2, m_1, m_2 + 1; j \} \]
\[ = \hat{H}_{l,j,-m_2}^+ \{ \hat{D}_{l,j,m_2}^+ - \hat{D}_{l+1,j,m_2}^+ \} \{ l + 2, m_1, m_2 + 1; j \} \) ,
\[
\frac{1}{2} [x^2 - \delta(x_2^*), J_{y_3} J] | l, m_1, m_2; j \)
\[ \approx \{ \hat{C}_{l+1,j,-m_2-1}^+ \hat{D}_{l,j,m_2}^+ - \hat{D}_{l+1,j+1,m_2}^+ \hat{C}_{l,j,-m_3}^+ \} \{ l, m_1, m_2 + 1; j + 1 \} \]
\[ = \hat{C}_{l,j,-m_2}^+ \{ \hat{D}_{l,j,m_2}^+ - \hat{D}_{l+1,j,m_2}^+ \} \{ l, m_1, m_2 + 1; j + 1 \} \) ,
\[
\frac{1}{2} [x^2 - \delta(x_2^*), J_{y_1} J] | l, m_1, m_2; j \)
\[ \approx \{ \hat{C}_{l+1,j,-m_2-1}^+ \hat{D}_{l,j,m_2}^+ - \hat{D}_{l+1,j+1,m_2}^+ \hat{C}_{l,j,-m_3}^+ \} \{ l, m_1, m_2 + 1; j + 1 \} \]
\[ = \hat{C}_{l,j,-m_2}^+ \{ \hat{D}_{l,j,m_2}^+ - \hat{D}_{l+1,j,m_2}^+ \} \{ l, m_1, m_2 + 1; j + 1 \} \) ,
\[
\frac{1}{2} [x^2 - \delta(x_2^*), J_{y_2} J] | l, m_1, m_2; j \)
\[ \approx \{ \hat{H}_{l+1,j,-m_2}^+ \hat{D}_{l,j,m_2}^+ - \hat{D}_{l+1,j+1,m_2}^+ \hat{H}_{l,j,-m_2}^+ \} \{ l, m_1, m_2 + 1; j \} \]
\[ = \hat{H}_{l,j,-m_2}^+ \{ \hat{D}_{l,j,m_2}^+ - \hat{D}_{l+1,j,m_2}^+ \} \{ l, m_1, m_2 + 1; j \} \) ,
\[
\frac{1}{2} [x^2 - \delta(x_2^*), J_{y_3} J] | l, m_1, m_2; j \)
\[ \approx \{ \hat{C}_{l+1,j,-m_2-1}^+ \hat{D}_{l,j,m_2}^+ - \hat{D}_{l+1,j+1,m_2}^+ \hat{C}_{l,j,-m_3}^+ \} \{ l - 2, m_1, m_2 + 1; j + 1 \} \]
\[ = \hat{C}_{l,j,-m_2}^+ \{ \hat{D}_{l,j,m_2}^+ - \hat{D}_{l+1,j,m_2}^+ \} \{ l - 2, m_1, m_2 + 1; j + 1 \} \) .

Now \( \hat{D}_{l+1,j,m_2}^+ - \hat{D}_{l,j,m_2}^+ \) is bounded by \( q^{l+j+m_2} \), and \( q^{l+j+m_2} \hat{C}_{l,j,-m_2}^- \) is bounded by \( q^{2l} \). Next, \( \hat{H}_{l,j,-m_2}^+ \) is bounded by \( q^{l+j-m_2} \), and \( q^{l+j-m_2} \hat{D}_{l,j,m_2}^- \simeq 1 \). This proves that all previous commutators are smoothing. For the \( g_1^i \)'s the same statement follows from the symmetry (9.15).

We have arrived at last two commutators. Modulo smoothing operators, the first one is
\[
\frac{1}{2} [x^2 - \delta(x_2^*), J_{y_2} J] | l, m_1, m_2; j \)
\[ \approx \{ \hat{D}_{l+1,j,-m_2}^- \hat{D}_{l,j,m_2}^- - \hat{D}_{l+1,j+1,m_2}^- \hat{D}_{l,j,-m_2}^- \} \{ l + 2, m_1, m_2; j \} \]
\[ + \{ \hat{D}_{l+1,j,-m_2}^+ \hat{D}_{l,j,m_2}^+ - \hat{D}_{l+1,j+1,m_2}^+ \hat{D}_{l,j,-m_2}^+ \} \{ l, m_1, m_2; j \} \]
\[ = \hat{D}_{l,j,-m_2} \{ \hat{D}_{l,j,m_2}^- - \hat{D}_{l+1,j,m_2}^- \} \{ l, m_1, m_2; j \} \) ,
where the second equality follows from the fact that both \( \hat{D}_{l,j,m_2}^+ \) and \( \hat{D}_{l,j,m_2}^- \) in (9.6) depend on \( l \) and \( m_2 \) only through their sum. For the same reason we have also that
\[
\frac{1}{2} [x^2 - \delta(x_2^*), J_{y_2} J] | l, m_1, m_2; j \)
\[ \approx \{ \hat{D}_{l+1,j,-m_2-1}^- \hat{D}_{l,j,m_2-1}^- - \hat{D}_{l+1,j+1,m_2-1}^- \hat{D}_{l,j,-m_2-1}^- \} \{ l, m_1, m_2 - 2; j \} \]
\[ + \{ \hat{D}_{l+1,j,-m_2+1}^- \hat{D}_{l,j,m_2+1}^- - \hat{D}_{l+1,j+1,m_2+1}^- \hat{D}_{l,j,-m_2+1}^- \} \{ l - 2, m_1, m_2 - 2; j \} \]
\[ = \hat{D}_{l,j,-m_2} \{ \hat{D}_{l,j,m_2-1}^- - \hat{D}_{l+1,j,m_2-1}^- \} | l - 2, m_1, m_2 - 2; j \) .

The final observation that \( \hat{D}_{l,j,-m_2}^- \hat{D}_{l+1,j,m_2-1}^- \simeq \hat{D}_{l,j,-m_2}^- \), for \( i = 0, 1, 2 \), gives that these commutators vanish modulo smoothing operators.
Acknowledgements

We are grateful to the referee whose remarks led to a much improved version of the paper. This work was partially supported by the ‘Italian project Cofin06 – Noncommutative geometry, quantum groups and applications’.

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