Essential paths space on ADE $SU(3)$ graphs: A geometric approach

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Abstract

Using triangular sequences of vertices as an analogue for a backtracking step on a path in $SU(3)$, we offer a geometric understanding of the path creation and annihilation operators in terms of creation and annihilation of triangular sequences of vertices. We find that our implementation of these operators yields the expected mathematical properties as applied to essential and non essential paths. We prove that the space of paths of a given length can be decomposed in a way similar to that shown previously for $SU(2)$, which is to say that one can write a path of a given length as a direct sum of sub-spaces which are orthogonal subspaces constructed by recurrent applications of the path creation operator on subspaces of essential paths of shorter length. We propose an algorithm that shows explicitly how a given path can be obtained as an iterated and ordered application of the creation operator on a path belonging to the kernel of the annihilation operator.

1 Introduction.

The ADE classification of $\hat{su}(N)$ WZW Conformal Field Theories (CFT) modular invariants revealed the tip of the iceberg of a rich and interesting algebraic structure, which is still being explored today. Historically the identification that led to the ADE classification was first established for $N = 2$ by Cappelli, Itzykson and Zuber in [1]. Later, Terry Gannon [2] provided the full list of partition functions associated to $\hat{su}(3)$, and the list of graphs generalizing the ADE classification was constructed by Di Francesco and Zuber [3] and by A. Ocneanu [4].

Roughly speaking, the clasification was originally established by the straightforward identification of the labels of the partition function (expressed in term of the characters of the chiral algebra) with the Coxeter exponents of the $ADE$ (generalized) Dynkin graphs. In addition the fusion coefficients of the theory, obtained through the Verlinde formula [5] from the modular data can completely be encoded in the graphic representation associated to the partition function. The interrelation thus found was purely structural and therefore required further formal elucidation in order to be properly justified.

The formal structure that lies behind this classification has been developed over time [6], [7], [8], [9], [10], [11] and can be summarized as follows: To every modular invariant of a 2D RCFT with chiral algebra $\hat{sl}(N)$ at level $k$ one can associate a special kind
of quantum groupoid\footnote{In \cite{14} aspects of the general \(su(N)\) case were discussed, and graphs families for \(N = 2, 3, 4\) are shown. In \cite{12} the algebra of quantum symmetries is determined for the three excepcionals of \(su(4)\) at levels 4, 6 and 8.} (weak Hopf algebra) \(B\) constructed from the combinatorial data associated to a especial pair of graphs \(\{G, A(G)\}\). \(G\) being an \(ADE\) generalized Coxeter-Dynkyn graph \cite{3, 4, 13} (when \(N = 2\) this means an \(A, D\) or \(E\) Dynkin graph) with generalized Coxeter number \(\kappa = k + N\) and \(A(G)\) an \(A_k\) graph of the same generalized Coxeter-Dynkyn system, with the same generalized Coxeter number. \(A(G)\) is called the Fusion Graph of \(G\). The vertices of \(G\) and \(A(G)\) form vector spaces, whose elements, called irreducible quantum dimensions can be added and multiplied much like the irreducible representations of groups. It turns out that the vector space spanned by the vertices of the graph \(G\) is a module over the graph \(A(G)\), and the module action is defined by a set of structure constants \(F_{n,a,b}\) that encode the so called admissible triangles, diffusion graphs or even essential paths on the graph \[14], [15], [16].

Given this structure, one finds that the vector space \(B\) can be generated by taking the set of graded endomorphisms of admissible triangles as a basis. In this particular basis it is natural to define an associative product that enriches \(B\) making it an associative, unital and semisimple algebra. The vertices of the \(A(G)\) graph, that we label \(k, m, n, \ldots\), encode the decomposition in simple blocks of the algebra \(B\). This construction naturally endowes the dual space \(\hat{B}\) with an structure of coassociative and counital coalgebra. This is not the whole story, the particular set of graphs we are dealing with, allows the construction of an additional and, in general different, associative product operation that turns \(\hat{B}\) into an associative, unital semisimple algebra. In this new algebra the simple blocks are different from those of the original algebra and are labelled \(x, y, z, \ldots\). These blocks can be encoded in a third graph called the Ocneanu graph (\(Oc(G)\)). Duality so ensures that the original algebra \(B\) is imbued with a structure of counital, coasociative coalgebra. The two structures of algebra and coalgebra in both spaces so constructed are compatible, resulting in a pair of semisimple bialgebras generated by duality. They actually form a pair of Quantum Groupoids or weak Hopf algebra after defining a suitable antipode.

Being semi-simple, both algebras \(B\) and \(\hat{B}\) have two separate sets of characters associated to the corresponding simple blocks. These characters are defined as generalizations of the irreducible characters of a Highest Weight Module, and they are elements of the dual space. Their action on a vector of the corresponding space is defined as the trace of the projector on a simple block (of type \(m\) or \(x\) given the case) acting on the vector \[14], [17]. In this way characters of the simple blocks \(m\) of \(B\) are elements of \(\hat{B}\) and characters of the simple blocks \(x\) of \(\hat{B}\) are elements of \(B\). The character algebras, giving the generalised decompositions of tensor product representations as direct sum of irreps, are called the Fusion Algebra and the Ocneanu Algebra of \(G\). These algebras are encoded as Graph Algebras \[14], [17\] in the previously mentioned Fusion Graph and Ocneanu Graph. A detailed description of this structure including several examples can be found in \[14\].

An equivalent description of the above structure can also be given in terms of fusion categories and module categories, but here we have chosen not to follow this path, the interested reader may consult \[18], [19\] and references therein.

In order to recover the connection with 2D RCFT we have to consider additional properties of \(B\). As mentioned before, \(G\) is a module over \(A(G)\), in the same way that \(G\) is a module over \(Oc(G)\) and \(Oc(G)\) is a bimodule on \(A(G)\) \(mxn = \sum_y(W_{xy})_{m,n}y\). The structure constants of this bimodule define the set of Toric Matrices \(W_{xy}\). The matrix \(W_{00}\) is modular invariant and gives the partition function of the corresponding RCFT with chiral algebra \(\hat{sl}(N)\) in terms of the Virasoro characters. The remaining matrices can be physically interpreted as the partition functions of theories with twisted boundary...
conditions and defect lines [20]. The fusion rules of the theory are given, as expected, by the Fusion Graph.

A reverse technique has also been developed. In many cases the graph \( G \) or its simple blocks are unknown but the modular invariant \( W_{00} \) is known. In these cases, algorithms like the Modular Splitting Method [21] allow one to obtain the corresponding graphs \( G \) and \( Oc(G) \) from the modular invariant \( W_{00} \).

In practice, the products on \( \mathcal{B} \) and its dual are naturally defined through a pair of basis which are not dual to each other. We call these basis the ”vertical” and ”horizontal” basis [14], [17]. As a consequence of this, the coproduct in \( \mathcal{B} \) is naturally defined in a basis that is not the same as the one that gives the straightforward definition of the product on \( \mathcal{B} \). This in turn represents a computational complication that should not be underestimated, even more so as this problem is repeated for the coproduct in the dual space. The change of basis between the vertical and horizontal basis of the bialgebra is given by the set of Ocneanu cells (a kind of generalized 6j-symbols) [14]. The explicit calculation of these cells can be, computationally speaking, extremely demanding, primarily because dimensionality increases rapidly once one gets past the first simplest cases. Only for a few examples [22], [23] the complete Hopf algebra has been computed.

A particular basis for \( \mathcal{B} \) can be readily obtain as endomorphisms of a especial type of paths on the graph \( G \) called Essential paths. The natural basis of \( \mathcal{B} \) is the set of diffusion graphs \((anb) = \begin{array}{c} a \\ n \\ b \end{array} \) which is to be read as an admissible triangle encoding the occurrence of the irrep \( b \) in the decomosition of the module action \( na = \sum_{b} \mathcal{F}_{n,a,b} b \). Equivalently the triad \((anb)\) can be read as an Essential Path from the vertex \( a \) to the vertex \( b \) on \( G \), with length \( n \) on \( A(G) \). The basis of \( \mathcal{B} \) is then expressed as double admissible triangles or, equivalently as the set of graded endomorphisms of Essential paths \( \mathcal{B} = \text{gen} \oplus \sum_{n} \text{End}(E_{n}) \), where \( E_{n} \) is the vector sub-space of all the Essential paths of length \( n \). Formally the Essential Paths are obtained as the center of the action of certain annihilation operator \( C \) acting on the entire space of paths \( \mathcal{P} = \bigoplus_{n} \mathcal{P}_{n} \).

In [24] it is shown that for any simple bioriented graph there exists a quantum grupoid \( \mathcal{B} \). The quantum grupoid is obtained directly from the properties of the essential paths subspace without having to calculate Ocneanu cells. The key ingredient is the decomposi-tion of the space of paths as a direct sum of sub-spaces which are: either the subspace of essential paths of length \( n \), or orthogonal subspaces constructed by recurrent applications of the corresponding creation operator \( C^{\dagger} \) on subspaces of essential paths of shorter length. This decomposition and the corresponding orthogonal projectors, are sufficient to compute the quantum grupoid. In the cited reference two examples are are presented, the ADE graph \( A_{3} \) and the affine graph \( A_{2} \).

Concerning the especific properties of the subset of essential paths for the \( SU(3) \) familly not much is actually known, apart from a general idea of how the operators \( C \) and \( C^{\dagger} \) must behave [18], their precise definitions, properties and, very important, clear and distinct interpretations for their action on paths have been lacking.

In order to reproduce the results obtained by Trinchero in [24] we need first to master the action of the creation and annihilation operators on the family of \( SU(3) \) graphs. The behavior is very different compared to \( SU(2) \), mainly for two reasons: \( SU(3) \) graphs are oriented, and even more, each bialgebra \( \mathcal{B} \) is associated not just to one graph \( G \) but to a pair of graphs \( G \) and \( \bar{G} \) conjugate to one another. The paths on these graphs actually occur in 2D and the number of possible connections of vertices (and thus of paths connecting two arbitrary vertices) are larger and this creates issues that need to be explored. In particular the concept of a backtracking path, which is key in many aspects
of this construction, is not as straightforward as it is in $SU(2)$.

In the broadest of senses, the rules that creation an annihilation operators must satisfy are: the creation operator acting on the $i$-th vertex $C_i$ should insert a new vertex after the $i$-th vertex of the path in such a way that the edges thus produced are single-stepped backtracking edges connecting the $i$-th vertex to its nearest neighbors. $C_i$ should eliminate a backtracking segment passing through the $i$-th vertex on a given path.

In $SU(2)$ a backtracking path is one constructed in such a way that the sequence of vertices goes from the $n$-th vertex to one of its neighbors and back. Given the difficulty of defining what backtracking may mean for paths over an oriented graph in 2D, the construction of these two operators for $SU(3)$ requires some consideration of the geometric structure of the graph. Also the combined action of the two operators $e_i = \frac{1}{\beta} C_i^\dagger C_i$ must define a set of Jones’ projectors, with all the relevant relations between them satisfied. We must obtain in this way a path realization of the well known Jones–Temperley–Lieb algebra modified for $SU(3)$. In [18] a basic description of the operator $C_i$ and the unitary operator $C_i^\dagger C_i$ is given in terms the values of the Ocneanu triangular cells. In the same work and in [19] the values of this cells have been calculated for almost the complete list of graphs of $SU(3)$ family. This opened the door for the operational definition of the creation and annihilation operators in $SU(3)$, as they act on the triangles of the graph and require the values of the triangular cells.

In this work we present a proposal to extend reference [24] by explicitly implementing the path creation and annihilation operators $C$ and $C^\dagger$ for $SU(3)$ $ADE$ graphs. Our implementation is based on a simple geometric understanding of two notions, namely, backtracking on a path and essential path. We find that our implementation of $C$ and $C^\dagger$ satisfies the expected mathematical properties i.e. they create and annihilate backtracks on paths according to the geometric intuition, they satisfy the $SU(3)$ Temperley-Lieb algebra, and finally, they give the list of essential paths as provided by the admissible triangles. We prove that the space of paths of a given length can be decomposed as a direct sum just as happens in the the $SU(2)$ case. We also show that there is still a problem concerning degenerate triangles. Apart from this technicality our results seem satisfactory and we have no doubts that we shall be able to present rigorous proofs of all aspects of the formulation in the near future, including the construction of the bialgebra directly from the decomposition of the space of paths.

## 2 Paths on $SU(3)$ graph.

Let us begin by describing some features of any type $A$ graph belonging to the $SU(3)$ family, such a graph encodes the fusion rules of the irreps of the Fusion (Verlinde) algebra of some RCFT at level $k$. The dimensions of the irreps of the algebra can be read from the entries (on certain normalization) of the Perron-Frobenius eigenvector associated to the adjacency matrix of the graph, the highest-valued, real, non degenerate Perron-Frobenius eigenvalue is called $\beta$. As an extension of what happens in $SU(2)$, the generalized Coxeter number for this graph is read from the level as $\kappa = k + 3$ and it turns out that $\beta$ and $\kappa$ are related through the formula

$$\beta = 1 + 2 \cos(2\pi/\kappa)$$

The normalized eigenvector associated with the biggest eigenvalue $\beta$ is called the dimension vector, which is normalized by setting to 1 its smallest entry. The vertices of the graph are labelled by the so called triangular coordinates $\lambda = (\lambda_1, \lambda_2)$ [13], see figure [1]. The quantum dimension of a vertex is given by the q-analog of the classical for-
mula for dimensions of $SU(3)$ irreps, usual numbers being replaced by quantum numbers:

$$q\text{dim}(\lambda) = \left(\frac{1}{2\pi}\right)^2 q^{\lambda_1 + 1} q^{\lambda_2 + 1} q^{\lambda_1 + \lambda_2 + 2 + \lambda_1 \lambda_2},$$

where $q = \exp(i\pi \kappa)$ is a root of unity

and $[n]_q = q^n - q^{-n}$. With this normalization, the quantum dimension of the unit vertex is 1, while the number $\beta$ itself is the quantum dimension of the fundamental vertices, also called generators, $\sigma = (1, 0)$ and $\bar{\sigma} = (0, 1)$.

Let us now consider any ADE graph $G$ of the $SU(3)$ family. Its adjacency matrix has a Perron Frobenius eigenvalue and its related unique eigenvector. This eigenvalue defines a generalized Coxeter number for $G$ through formula 1 and this time the level of $G$ is defined from the generalized Coxeter number as $k = \kappa - 3$. As in the case of an $A$ graph, there is a unit vertex defined from the eigenvector corresponding to $\beta$ as described above. The components of the properly normalized eigenvector define the quantum dimensions of the corresponding vertices. When there is only one arrow leaving (and going to) the unit vertex, the quantum dimensions of its two neighbours are both equal to $\beta$ these neighbours are the generators of the graph and are referred to as $\sigma$ and $\bar{\sigma}$.

Now, the vector space spanned by the vertices is a module over the fusion algebra of the $A$ graph having the same Coxeter number as $G$, this module action is of upmost importance since it defines the admissible triangles. Note that when $G$ equals $A$ the graph $A$ acts on itself generating an associative algebra and it is said that $A$ graphs have self fusion. Some $G$ graphs may also show self fusion [13].

At this point we must note that the structure we have just described is associated not to one graph, but to two almost identical oriented graphs which are conjugate to each other, here conjugation is represented by the reversal of the direction of the arrows in the graph. Each of these graphs is generated by iterated applications of both generators $\sigma$ or $\bar{\sigma}$ and thus, the two graphs must be understood as a single object. This last notion cannot be overemphasized enough as the theory forces one to consider both a graph and its conjugate simultaneously.

Even though these attributes are common to all ADE graphs, they are not enough to determine them, nevertheless, for the purposes of this work we shall not go into much further detail. We limit ourselves to saying that when one has a $G$ graph, it is a good ADE $SU(3)$ graph if one is able to find an $A$ graph of the $SU(3)$ family, such that the pair $(A, G)$ with all their enriched algebraic structure defines a proper module category. These constructions are physically relevant in several instances. For example, given any pair of ADE graphs $(G, A)$ of the $SU(3)$ family with the same generalized Coxeter number $\kappa$, this pair yields the partition function of a $\hat{su}(3)_k$ RCFT.

To begin the study of our subject we must introduce the notion of a path, we will do it in two equivalent ways. We first define a elementary path as a sequence of vertices connected by arrows which may belong to either one graph or its conjugate. This definition implies that both graphs must be simultaneously considered since a given sequence of vertices can in general contain vertices connected by edges that follow any direction of the arrows (and thus can be understood as elements of the graph $G$ or its conjugate). A path can also be defined as a series of consecutive (oriented) edges on any of the graphs. As usual, the length of a path is the total number of edges or the number of its vertices minus 1.

Formally, an elementary path is built through a succession of actions of either generator, $\sigma$ and $\bar{\sigma}$. Therefore in $SU(3)$ it is natural to classify elementary paths with an ordered pair of integers $(\alpha, \beta)$ that counts the amount of edges generated by $\sigma$ and $\bar{\sigma}$. Consequently, the length of a path is given by $n = (\alpha + \beta)$. This labeling scheme implies that paths of the same length can have differing values for both indices and can therefore
be of different type.

As an illustrative example, let us consider the $A$-type $SU(3)$ graph of figure [1] and two paths on it given by the sequences $(133)$ and $(138)$. These paths rove through three vertices using two edges and have therefore length 2, however the edges in these paths are different: in the first case both edges follow the directions of the arrows of the graph and are thus of length $(2,0)$ while the second has one edge following the arrows and another one in the opposite direction and therefore is of length $(1,1)$.

The set of all paths we are considering can be naturally extended to be a finite dimensional vector space over the complex numbers. The inner product vector space of paths $\mathcal{P}$ is defined by saying that elementary paths provide a orthonormal basis of this space. This space is naturally graded in the following way:

$$\mathcal{P} = \bigoplus_n \mathcal{P}^{(n)} = \bigoplus_n \bigoplus_{\alpha+\beta=n} \mathcal{P}_{(\alpha,\beta)}.$$  

In the case of $SU(2)$ a particular subspace $\mathcal{E}$ is constituted by the so called essential paths [6]. All graphs of the $SU(2)$ family share some properties: being simply laced and lacking loops (i.e. no vertex is connected to itself). For the sake of simplicity, let us momentarily consider a linear $SU(2)$ graph, i.e. a type $A$ graph. Over such a graph an elementary path of length one is either a step to the ”right” or a step to the ”left”, a sequence of a right step followed by a left step (or viceversa) is obviously a length two path containing a backtrack (a step going back to the initial point), given this idea, it is clear that any connected path on a type $A$ graph can have a backtrack, i.e. a sequence with a structure $(\ldots, a, b, a, \ldots)$. Even in the case of a graph for which vertices are not arranged in a “pearl necklace” like pattern (e.g. $E$ type graphs), in the $SU(2)$ family, we find that the notion of a backtracking path is extremely simple: any path that goes through the same vertex twice visiting in the interim one of its immediate neighbors must have an even number of steps, half of them going in one direction and an equal number in the opposite direction. Now that we have a geometric intuition of a backtracking path we can roughly define an essential path on a $SU(2)$ graph as a path having no backtracks.

Sadly even for $SU(2)$ the geometric intuition can only take us so far, so a formal definition is badly needed. Previous works [6, 15, 24] introduced the path creation and annihilation operators ($C_i$ and $C_i^\dagger$) acting on the $i$-th vertex of a given path, which allow a clear and unambiguous definition of essential paths that works for $SU(2)$. In terms of these operators, the set of all essential paths is defined as the kernel of all annihilation operators, i.e. given these operators a path $\eta \in \mathcal{P}$ is essential if and only if:

$$C_i \eta = 0 \quad \forall i.$$  

In $SU(3)$ these operators acting on paths should provide us with a straightforward definition of essential paths identical to the one found for $SU(2)$ and they should have
SU(3) analogues to the following properties of the SU(2) operators: (a) The kernel of the annihilation operators $C_i$ must provide the full set of essential paths; this set matching the one defined from the module action of $A(G)$ on $G$. (b) The operators $C_i, C_i^\dagger$ must provide essential paths with the intuitive behavior described above, and finally (c) The object $e_i = \frac{1}{\beta} C_i^\dagger C_i$ should fulfill the SU(3) version of the Temperley–Lieb algebra.

It is of note that, if we want to extend this idea to the SU(3) family, we once again find that we must consider both the original graph and its conjugate.

2.1 Creation and annihilation operators

For the SU(2) family one knows that the action of the annihilation and creation operators yields a non zero result whenever they act on a path by means of the removal of a backtracking or the addition of a vertex that generates a backtracing respectively. This provides us with a clue from which to start building our list of requirements for the SU(3) creation and annihilation operators. It should be clear from this that we will need to define what we mean by a backtracking path in a SU(3) graph.

The ADE SU(3) family naturally lives on a 2D lattice, the vertices of the graphs are sub sets of the Weyl alcove of $su(3)$ at level $k$. The elements of this family are therefore characterized by graphs based on triangular cells, this property means that there are two geometrically natural ways of producing a round trip starting and ending on the same vertex. The first one implies going to one of the vertex nearest neighbors and back (exactly as in the SU(2) case). The second one requires a round trip through two additional vertices that, alongside the original one, form a closed triangular cell. This behavior makes the definition of essential paths far less intuitive.

Note that a back and forth backtrack can be thought of as a sort of round trip over a deformed triangle and is a degenerate version of the more general definition that we will describe below.

Let us now take a path as a sequence of consecutive vertices $\eta = v_0v_1 \ldots v_{i-1}v_iv_{i+1} \ldots v_n$. We should be careful to parse the meaning of the indices: in here we note the vertex in the $i$-th position of the path as $v_i$ and this should not be confused with the label of the $i$-th vertex in the graph, i.e. its associated quantum label.

We have already mentioned that in SU(2) the annihilation operator eliminates backtrakings by removing a vertex and pasting together the remaining vertices of the path. In SU(3) we expect the same behaviour with the appropriate definition of backtrakings. Assume that the set of vertices in the path $v_{i-1}v_iv_{i+1}$ form a triangle and that the the edges connecting the $v_{i-1}v_i$ and $v_iv_{i+1}$ vertices, when treaded in the way prescribed by the sequence, follow the direction of the arrows in the graph (or its conjugate) then, when acted upon by the annihilation operator on the $i$-eth vertex, the edge that connects the $v_{i-1}v_{i+1}$ vertices (once again treaded following the prescription) is one following the arrows of the conjugate graph (or of the original one).

It is with the previous discussion in mind that we now write the annihilation operator:

$$C_i(\eta) = \frac{T_{i-1,i,i+1}}{\sqrt{(i-1)(i+1)}}v_0v_1 \ldots v_{i-1}v_iv_{i+1} \ldots v_n$$

(4)

Formally, the prefactor that was $\sqrt{\mu_{i+1}/\mu_{i-1}}$ for a single SU(2) elementary backtrack is now replaced by an SU(3) elementary triangle $v_{i-1}v_iv_{i+1}$. In this context, the appearance of the $\frac{T_{i-1,i,i+1}}{\sqrt{(i-1)(i+1)}}$ prefactor is required in order to satisfy the Temperley-Lieb algebra, as we will show. Since the values of the triangular cells have already been calculated in the literature the action of the operator is straightforward for all triangular sequences save for
Figure 2: This figure graphically illustrates the action of annihilation operator on a section of path (overlying red arrow) that passes through three consecutive vertices forming a triangle \((v_i v_{i+1} v_{i+2})\) in the graph, i.e., a nondegenerate backtrack. Note that the operator removes one of the vertices and connects the remaining two through one shorter path.

the back and forth backtracking that, as we said above, are associated with degenerate triangles. This means that a detailed study of these backtracking paths lies beyond the scope of this work.

Let us now turn our attention to the path creation operator. In \(SU(2)\), its action is already well known: the operator introduces vertex in the immediate neighborhood of the \(i\)-th vertex in such a way that the resulting edges produce backtracking paths. In light of this, we would expect that an adequate creation operator for \(SU(3)\) would introduce vertices in the neighborhood of the \(i\)-th vertex so as to create edges that result in what we have found already to be backtracking in \(SU(3)\), that is, sequences of vertices producing triangles. As in the case of the annihilation operator the edges introduced by the action of the creation operator follow the orientation of the conjugate graph of the original edge.

Taking this into account, we now write the expression for the creation operator:

\[
C_i^\dagger(\eta) = \sum_{b.n.n. n} \frac{T_{i-1,b,i}}{\sqrt{|i-1|i}|i}} v_0 v_1 \ldots v_{i-1} v_b v_i \ldots v_n
\]

Finally, one can clearly see that given this definition of the creation operator, the second type of intuitive backtracking paths we discussed above is explained as the action of the creation operator on a back and forth backtracking path. It is therefore a path of length 3 containing two backtracks in contrast with all other cases that are of length 2.

The next step is to check that the definitions for the creation and annihilation operators satisfy the Temperley-Lieb algebra. The relations defining the Temperley-Lieb algebra for \(SU(3)\) are well known and have already been explored in the literature, see for example[25]:

\[
U_i^2 = \beta U_i, \tag{6}
\]

\[
U_i U_j = U_j U_i, \quad |i-j| > 1, \tag{7}
\]

\[
F_i = U_i U_{i+1} U_i - U_i = U_{i+1} U_i U_{i+1} - U_{i+1}, \tag{8}
\]

\[
(U_i - U_{i+2} U_{i+1} U_i + U_{i+1})(U_{i+1} U_{i+2} U_{i+1} - U_{i+1}) = 0 \tag{9}
\]

According to our general plan, we will now discuss this algebra under the light of a geometric interpretation, and leave the fully detailed proofs for an appendix.

The action of the \(U_i\) operator is to initially remove vertex \(i\) and (whenever this removal does not result in a null path) then add a new vertex in such a way that two paths are created: the original path \(\eta\) and a second path in which the \(i\)-th vertex is replaced by another one which, together with those in the original sequence \(v_{i-1} v_i v_{i+1}\), form a rhombus.
with the edge connecting the $i-1$ and $i+1$ vertices as a diagonal, i.e. a pair of elementary triangles that share the aforementioned edge.

The first relation of the Temperley-Lieb algebra formula (6) involves a squared application of $U_i$, since the second action acts on the exact same vertex in the sequence, the net result ends up being the exact same task of substraction and addition of the very same vertices as the first action, the sole difference between both actions being the coefficient of the resulting paths.

We now examine the next relation, namely, the restricted commutatation relation for the Hecke operators (7). Geometrically the iterated application of them is as simple as the previous case, furthermore, the restriction on the distance of the $i$ and $j$ indices becomes clear since a separation more than or equal to two vertices means that the action of each of the operators will not influence that of the other and since $U_i : \mathcal{P}_n \to \mathcal{P}_n$ the length of the resulting path after the action of the first operator is exactly the same as the original path $\eta$. Finally, with a separation smaller than that prescribed by the Hecke algebra we find that the triangular cells have vertices in common which break the commutation.

Relation (8) proves quite interesting since it provides compelling geometric insights for some paths. Whenever the path is not a closed triangle (i.e. that is one containing a sequence $v_{i-1}v_i v_{i+1}v_{i-1}$) the relation is trivially satisfied, which is to say the rhombii produced by either sequence of $U$ operators is exactly the same. However, if the path contains a closed triangular sequence we are naturally led to a definition for the $F_i$ operator previously studied in [25,18]:

\[
F_i(\eta) = \sum_{b',b''} T_{i-1,i+1} \frac{T_{i-1,b',b''}}{[i-1][i-1]} v_0 v_1 \ldots v_{i-1} v_{i+1} v_{i-1} \ldots v_n
\]

\[
= \sum_{b',b''} T_{i+1,i+1} \frac{T_{i,b',b''}}{[i-1][i-1]} v_0 v_1 \ldots v_{i-1} v_{i+1} v_{i-1} \ldots v_n = F_{i+1}(\eta)
\]
These $F_i$ operators are rather thought-provoking, indeed, our creation and annihilation operators were inspired by the $SU(2)$ idea of a backtracking path and replaced it with a triangular sequence of vertices. However, up to this point we have not made any mention of paths that have closed triangular paths, that is, paths in which the sequence of vertices define all the three edges of a triangle. They have finally showed up in the Temperley-Lieb algebra, closed triangular paths can be interpreted as resulting from a special case of rhombii that have been “folded up” onto themselves. One can produce a path that exhibits this behaviour by acting with the creation operator on a path that has a sequence of vertices with a naïve $SU(2)$-style backtrack. In this way we find that all straightforward reinterpretations of backtracking paths in $SU(3)$ (i.e. a path going from a vertex to a neighbor and back, and one taking a triangular round trip from a vertex to itself) are all accounted for and ruled out as the relevant ideas in the definition of the creation and annihilation operators.

The final relation in the (6)-(9) series, which defines the Temperley-Lieb algebra for $SU(3)$, has already been shown to be satisfied whenever the following simpler relation is fulfilled (cf. [25] Lemma 4.1): $F_i F_{i+1} F_i = \beta^2 F_i$. If a path lacks any closed triangular paths the expression is trivial as is expected. Once again we invite the reader to consult the appendix for the explicit proof of this property.

The final step is to check that indeed the kernel of the annihilation operator corresponds to the list of essential paths obtained through the module action $A(\mathcal{G}) \times \mathcal{G} \rightarrow \mathcal{G}$. The module action defines the admissible triangles $(anb)$ which as we explained before, point to the existence of an essential path going from vertex $a$ to vertex $b$ and having length given by the label $n = (\alpha, \beta)$ of the $A(\mathcal{G})$ graph, the decomposition of this path as a linear combination of elementary paths can be obtained, in principle, through the action of the annihilation operator $C$. The way to verify this consists on showing that for any admissible triangle there exists at least one linear combination (up to gauge fixing and given values for the set of triangular cells) of elementary paths that begin at vertex $a$ and end at vertex $b$ all of which have length $n$ and which gets mapped to zero under the action of $C$. A way to give a definite proof of this is through exhaustion, i.e. by performing the above calculation for each triangle in each graph.

### 2.2 Examples

$G = A_2$ Let us consider $A$ type graphs in $SU(3)$. As these graphs exhibit self-fusion, an explicit calculation of the fusion product table is straightforward and is displayed below for the $A_2$ graph.

| $\wedge$ | 1 | 3 | 6 | $\overline{3}$ | $\overline{6}$ | 8 |
|----------|---|---|---|-------------|-------------|---|
| 1        | 1 | 3 | 6 | $\overline{3}$ | $\overline{6}$ | 8 |
| 3        | 3 | $\overline{3} + 6$ | 8 | $1 + 3$ | $\overline{3} + \overline{6}$ | 8 |
| 6        | 6 | $\overline{6}$ | 8 | 3 | 1 | $\overline{3}$ |
| $\overline{3}$ | $\overline{3}$ | 1 | $\overline{8}$ | 3 | $\overline{6} + 3$ | 8 | $6 + \overline{3}$ |
| $\overline{6}$ | $\overline{6}$ | $\overline{3}$ | 1 | 8 | 6 | 3 |
| 8        | 8 | $\overline{6} + 3$ | $\overline{3}$ | $6 + \overline{3}$ | 3 | 1 | $+ 8$ |

Table 1: $A_2$ multiplication table. In this case because $G = A_2$ is its own $A$ type graph, it is endowed with self-fusion and then the multiplication coincides with the module action.

Table 1 allows for the construction of the space of admissible triangles, wherein the triplet $(n, k, m)$ will be admissible if the $m$-eth representation appears in the table as the
decomposition in irreducible sums of the tensor product of $n \otimes k$. Additionally, the table encodes the essential paths of all admissible sizes over the graph, as was shown in [24].

As we discussed in section 2 one must consider paths containing edges corresponding to the actions of the generators of both graphs ($\sigma$ and $\overline{\sigma}$) and we can interpret the triangular coordinates as the number of times each generator must act in order to construct the associated essential path. This allows us to sort the essential paths in terms of their length (see the leftmost column in the table [2]), or equivalently in terms of the combination of generators required to build the essential path (see the rightmost column):

| Path size | Essential paths | Generators |
|-----------|----------------|------------|
| (0, 0)    | (1), (3), (6), (6), (6), 8 | $\sigma^0$ |
| (1, 0)    | (13), (33), (36), (31), (38), (68), (63), (86), (83) | $\sigma^1$ |
| (0, 1)    | (13), (33), (36), (31), (38), (68), (63), (86), (83) | $\overline{\sigma}^1$ |
| (2, 0)    | $\sqrt{2}[338], (383) - \sqrt{2}[313], \sqrt{2}[383], (368) - \sqrt{2}[383]$ | $\sigma^2$ |
| (0, 2)    | $\sqrt{2}[338], (383) - \sqrt{2}[313], \sqrt{2}[383], (368) - \sqrt{2}[383]$ | $\overline{\sigma}^2$ |
| (1, 1)    | $a(313) + b(333) + c(383) + d(363), a(313) + b(333) + c(383) + d(363)$ | $\sigma \overline{\sigma}, \overline{\sigma} \sigma$ |

Table 2: List of essential paths ordered in terms of their length and the number of generators that produce them.

We must now verify that the kernel of the annihilation operators corresponds to the table above. For a path of size zero (i.e. a vertex) the removal of a vertex results in the complete elimination of the path.

In the case of a size 1 path the action of our operator removes one vertex and then would attempt to connect the remaining path to the next one in the sequence (since the annihilation operator eliminates the middle vertex of a triangular sequence), but since there is no subsequent vertex in the sequence the operator once again results in zero. Therefore only paths of length 2 are the first ones that could render a non trivial result.

We can distinguish three cases: first those in which the sequence of vertices does not result in a triangular sequence (e.g. (136), (631), (686) and their counterparts in the adjoint graph). For these paths the elimination of the middle vertex yields a disjoint path and is thus annihilated by our operator. The second case is that of paths that are indeed triangular sequences (e.g. (368), (338), (383), (313), (863), (833) and their counterparts). These ones are not annihilated by themselves but one can find a linear combination that indeed results in an essential path, as we see in the table [2].

The final case is that of the 1 + 1 size paths that start and end in the same vertex. Since there is a maximum length for essential paths we will not list as essential paths of greater length. This can be understood by noticing that the triple action of either
generator \( \sigma \) or \( \overline{\sigma} \) gives a reducible representation containing 1 and 8 (e.g., \( (3 \cdot 3) \cdot 3 = (3 + 6) \cdot 3 = 3 \cdot 3 + 6 \cdot 3 = 1 + 8 + 8 \), where symmetry ensures a similar result for all corners of the graph), in geometrical terms this means that a path with three edges following the arrows renders one of three cases: first, a closed triangular path, and two instances of paths that can be reduced to paths of length (1, 1). However, in \( SU(3) \), one can act with either \( \sigma \) or \( \overline{\sigma} \), which implies that paths longer than the essential path of maximal length, and belonging in the kernel of the annihilation operator, can be produced. These are explicitly the paths \( (1386), (6831), (6336) \) and the equivalent three generated by conjugation. As a consequence of the above the preceding list of essential paths is complete only if one takes as an essential path one that is given by the set of admissible triangles. Paths of greater size can be produced using the concatenation product acting on elementary paths.

We now shift our attention to the \( E_5 \) graph. The graph is a level 5 path and its fusion algebra is given by the action of an \( A_5 \) graph.

We shall verify that our chosen definition of the path creation and annihilation operators is useful in \( SU(3) \) graphs of a type different than the simple \( A \)-type case.

In order to save space, we avoid the fusion product table and the module product table (which can be found in [21, 13]). A similar procedure as the one employed for the previous example \( A_2 \in SU(3) \) yields the essential paths of which we provide the full list up to length \( n = 2 \) and a few examples of length \( n = 0 + 3 \):

In table 3 we have listed families of paths: due to the symmetries of the graph one can find sets of paths that share the same structure and are related by rotations and reflections, therefore the index \( i \) in the table run from 0 to 5 and all operations are \((\text{mod} \ 6)\). The triangular cells are \( \nu_0 = -\nu_1 = - (29 + \frac{11}{4} \sqrt{2})^{1/4} \) and \( \mu = (5 + \frac{7}{2} \sqrt{2})^{1/4} \) and have been calculated explicitly in [13]. Using conjugation over \( E_5 \), that amounts to symmetry with respect to the axis joining vertices \( 1_0 \) and \( 1_3 \) \((1_0 = 1_0, \overline{1_5} = 1_1, 1_4 = 1_2, \overline{1_3} = 1_3, \overline{2_0} = 2_3, \overline{2_1} = 2_2 \) and \( \overline{2_5} = 2_4 \)) one can find the list of the essential paths complementary to those listed here (i.e. those of length \((0, 2)\) and \((3, 0)\)).

The coefficients \( a \) to \( c \) appearing in the list of paths of length \((1, 1)\) cannot be determined at this point because they contain the collapsed triangular cells that, as we said above, are unknown. Finally, the coefficients \( A \) to \( C \) in the list of paths of length \((2, 0)\) point to the fact that for paths with the above starting and ending points, the fused adjacency matrices signal the existence of two essential paths that are to be constructed.
Table 3: Partial list of essential paths for $E_6$, sorted by length and number of generators.

| Path size | Essential paths                                | Generators |
|-----------|-----------------------------------------------|------------|
| (0, 0)    | $(1_i), (2_i)$                                | $\sigma^0$ |
| (1, 0)    | $(1, 2_{i+1}), (2, 2_{i+1}), (2, 2_{i+4}), (2, 2_{i+6})$ | $\sigma^1$ |
| (0, 1)    | $(1, 2_{i+2}), (2, 2_{i+2}), (2, 2_{i+5}), (2, 2_{i+6})$ | $\sigma^1$ |
|           | $(1, 2_{i+1}2_i), (1, 2_{i+2}2_i), (1, 2_{i+2}2_{i+3})$ |            |
| (1, 1)    | $(1, 2_{i+1}2_{i+3}), a(2, 2_{i+1}2_{i+3}) + b(2, 2_{i+1}2_{i+6}) + c(2, 2_{i+1}2_{i+3})$ | $\sigma, \sigma, \sigma$ |
|           | $(1, 2_{i+1}2_{i+5}), (1, 2_{i+1}2_{i+6}), (1, 2_{i+1}2_{i+6})$, |            |
| (2, 0)    | $(2, 2_{i+1}2_{i+2}) − \frac{\sigma}{\mu}(2, 2_{i+1}2_{i+2}), A(2, 2_{i+1}2_{i+2}) + B(2, 2_{i+1}2_{i+2}) + C(2, 2_{i+1}2_{i+2})$ | $\sigma^2$ |
|           | $(1, 2_{i+2}2_{i+1}1_{i+4}), (1, 2_{i+2}2_{i+1}2_{i})$ |            |
| (0, 3)    | $(1, 2_{i+2}2_{i+4}2_{i}), \frac{\sigma}{\mu}(1, 2_{i+2}2_{i+1}2_{i})$ | $\sigma^1$ |

with appropriate combinations of those listed above in such a way that the coefficients are not unique.

We recall now that the definition of the annihilation operator implies that any path for which three consecutive vertices form a triangle must not be essential since the application of the annihilation operator to the middle vertex of the triangle forming triplet results in a non zero path. As we have seen before, this is analogous to the backtracking paths in $SU(2)$. We now want to present the explicit action of the $C_i$ operator on a sample of paths:

The path $(1, 2_{i+1}2_{i+2})$ should be essential given our previous discussion, thus:

$$C_1(1, 2_{i+1}2_{i+2}) = \frac{T_{1, 2_{i+1}2_{i+2}}}{\sqrt{[1_i][1_2]}} (1, 2_{i+1}2_{i+2}) = 0 \quad (11)$$

The result is easy to reach since one can see from the values of the adjacency matrices or Figure [3] that the resulting path is disjoint. For another, slightly longer path $(n = (2, 1))$, $(1, 2_{i+1}2_{i+2}1_{i+1})$:

$$C_1(1, 2_{i+1}2_{i+2}1_{i+1}) = \frac{T_{1, 2_{i+1}2_{i+2}1_{i+1}}}{\sqrt{[1_i][1_2][1_3]}} (1, 2_{i+1}2_{i+1}2_{i+2}1_{i+3}) = 0 \quad (12)$$

$$C_2(1, 2_{i+1}2_{i+2}1_{i+1}) = \frac{T_{2, 2_{i+1}2_{i+2}1_{i+1}}}{\sqrt{[2_i][1_1]}} (1, 2_{i+1}2_{i+2}1_{i+1}) = 0 \quad (13)$$

Both paths then, belong to the kernel of the annihilation operators and we can thus claim that, for a given path length, some linear combination comprising the above paths conforms the basis of the space of essential paths.

For a path that we can easily tell is not essential $(1, 2_{i+1}2_{i+2})$ we get:

$$C_2(1, 2_{i+1}2_{i+2}) = \frac{T_{2, 2_{i+1}2_{i+2}}}{\sqrt{[2_i][2_2]}} (1, 2_{i+1}2_{i+2}) = 0 \quad (14)$$

Which is to be expected given that the triplet $2_{i+1}2_{i+2}$ forms a triangle in the graph and thus results in a non essential path.
3 Decomposition of the space of paths

Having defined our path creation and annihilation operators and now that we grasp their interpretation in terms of paths we can now move forward to verify that it is possible to decompose the space of paths of a given length in a manner analogous to:

$$\mathcal{P}_n = \bigoplus_{i \leq n-2} C_i^t(\mathcal{E}_{n-2}) \bigoplus_{i_1 < i_2 \leq n-2} C_{i_2}^t C_{i_1}(E_{n-4}) \bigoplus \cdots \bigoplus_{i_1 < i_2 \cdots < i_{[n/2]} \leq n-2} C_{i_{[n/2]}}^t C_{i_{[n/2]-1}} \cdots C_{i_1}(E_{[1/2]})$$

(15)

Here the space $C_i^t(\mathcal{E}_n)$ is the subspace of $\mathcal{P}_{n+1}$ generated by all elementary paths appearing in the action of the creation operator on elements of $\mathcal{E}_n$. Geometrically this decomposition implies that the space of paths of length $n$ is composed completely by the subspace of essential paths of length $n$ plus subspaces of essential paths of shorter length over which one has acted with iterations of the creation operator in all allowed vertices. In this manner, all paths of length $n$ can be obtained as a combination of elements of all these subspaces. In the process of proving the above, Trincherio introduces an algorithm for the decomposition of a given path $\eta$ (cf. [24] equation [5.7]).

In order to prove that such a decomposition of the space of paths of length $n$ connecting two vertices exists we will proceed through induction. We start by defining a family of operators that create backtracking segments acting on elementary paths (of which our creation operator will be an example). If $\eta = v_0 v_1 \ldots v_{i-1} v_i v_{i+1} \ldots v_n$ is an elementary path, then $c_i^t(\eta) = \sum b \beta_b v_0 v_1 \ldots v_{i-1} v_b v_i v_{i+1} \ldots v_n$. This definition allows us to select a particular elementary path with a backtracking segment in the result of the action of the creation operator on a given path $s$ (or a selection thereof) using appropriate $\beta$ coefficients. Clearly, the spaces $C_i^t(\mathcal{E}_n)$ and $c_i^t(\mathcal{E}_n)$ are equal once we allow for all possible values of the $\beta$ coefficients. Since the only relevant fact for us at this point is to have a way to create non essential elementary paths of length $n+1$, that is, elements of the base of paths of length $n+1$ which are, by definition, orthonormal to the elementary path $\eta$ (of length $n$) from which we obtained them.

Let $\eta_{ab}^{(n)}$ be a path of length $n$ connecting vertices $a$ and $b$. Since elementary paths provide an orthonormal basis of the space of paths we can write any path of a given length as

$$\eta_{ab}^{(n)} = \sum_i \alpha_i e_{ab}^{(n)} i, \quad (16)$$

where $\{e_{ab}^{(n)}\}$ is the basis of the space of elementary paths of length $n$ connecting vertex $a$ to vertex $b$. For elementary paths one can define the concatenation product • such that for two elementary paths $\eta = v_0 v_1 \ldots v_i$, $\eta' = v'_j v'_1 \ldots v'_p$ one gets $\eta • \eta' = \delta_{v_i v'_j} v_0 v_1 \ldots v_i v'_1 \ldots v'_p$. This allows us to split an elementary path $e_{ab}^{(n)} = e_{ac}^{(n-l)} • e_{cb}^{(l)}$ in such a way that $C_k(e_{ac}^{(l)}) = 0$ for all $k > n - l$ and $C_k(e_{ac}^{(n-l)}) \neq 0$ for at least one $k \leq n - l - 2$.

Given the decomposition of elementary paths in equation (16) we can then rewrite the $e_{ac}^{(n-l)}$ subpath, which is elementary but non-essential as one resulting from the action of the creation operator defined above with suitable values for the $\beta$ coefficients, thus $e_{ac}^{(n-l)} = c_k(e_{ac}^{(n-l-1)})$. This path of length $n - l - 2$ is now such that $c_k(e_{ac}^{(n-l-1)}) \neq 0$ for some $k' < n - l - 3$ since we have removed the leftmost “backtracking” from the sequence. We can now replicate this procedure on this new elementary path by cutting off the rightmost essential subpath using the concatenation product: $\eta_{ac}^{(n-l-1)} = e_{af}^{(n-l-1-m)} • e_{fc}^{(m)}$, now with $C_k(e_{af}^{(l)}) = 0$ for all $k > n - l - m - 3$ and $C_k(e_{ac}^{(n-l-1)}) \neq 0$ for at least one

2Note that this is not the same as the space generated by the paths in the image of $C_i^t(\mathcal{E})$. 14
\[ k \leq n - l - m - 2. \] This implies that our initial elementary subpath can be written as

\[ e^{(n-l)}_{ac} = c^\dagger_k \left( c^\dagger_{k'} \left( e^{(n-l-m-1)}_{af} \right) \bullet e^{(m)}_{fc} \right), \]

which in turn makes our original elementary path \( e^{(n)}_{ab} \):

\[ e^{(n)}_{ab} = \left( c^\dagger_k \left( c^\dagger_{k'} \left( \ldots c^\dagger_{k^{(s)}} c^{(r)} \right) \bullet e^{(s')}_{tp_1} \right) \bullet c^{(s')}_{p_1p_2} \ldots \right) \bullet e^{(l)}_{cb}. \]

We can repeat this procedure until we reach a rightmost subpath that is the result of the application of the creation operator on a path of size 0 (i.e., a vertex) or 1. This results in an expression for an elementary path of length \( n \) connecting \( a \) to \( b \) (and thus for a general path of the same length connecting the same vertices) as a sequence of ordered applications of the creation operator on an initial path in the kernel of the annihilation operator which is then concatenated with a series of essential subpaths, explicitly:

\[ e^{(n)}_{ab} = \left( c^\dagger_k \left( c^\dagger_{k'} \left( \ldots c^\dagger_{k^{(s)}} c^{(r)} \right) \bullet e^{(s')}_{tp_1} \right) \bullet c^{(s')}_{p_1p_2} \ldots \right) \bullet e^{(l)}_{cb}. \]

Where all the \( e_{ij} \) are subpaths in the kernel of \( C_i \) and in particular the innermost in the above equation can be of length 0 or 1 at the very smallest. As we can see this sequence of applications of the \( c^\dagger \) operators is unique since the ordering in our procedure results in only one such sequence for a given path and since the original path remains one of length \( n \) we can conclude that any path of length \( n \) connecting two vertices can be written as above. Since spaces \( C^i(\mathcal{E}_n) \) and \( c^i\mathcal{E}_n \) are equal the above procedure leads us to a natural decomposition for the space of paths of a given length connecting two vertices:

\[ P^{(n)}_{ab} = \mathcal{E}^{(n)}_{ab} \bigoplus_{i \leq n-2} c^i \left( \mathcal{E}^{(n-1)}_{ab} \bigoplus_{i_1 < i_2 \leq n-2} c^i_{12} c^{(n-2)}_{12} \bigoplus \ldots \right) \bigoplus_{i_1 < i_2 \cdots < c^{(n/2)}_{1} \leq n-2} c^i_{12} \cdots c^i_{12} \left( \mathcal{E}^{(1|0)}_{ab} \right). \]

What the above decomposition does not provide, is an explicit way of writing a given path in terms of a linear combination of elementary paths of equal length or of shorter length on which one has acted with a specific ordering of creation operators. It is with this in mind that we propose an algorithm that takes a path and explicitly deconstructs it in elements of the subspaces described above. A similar algorithm has already been fully explored for \( SU(2) \) in [21] equation 5.7. Once again, a geometric description of Trinchero’s algorithm may be enlightening: the non-essential path \( \eta \) must be such that beyond the \( i \)-th vertex it behaves like an essential path up until the last two vertices (i.e., there is no backtracking). The algorithm then takes the path and eliminates the backtracking “kink” in the \( i \)-th vertex and creates kinks further down the way of the path multiplied by some coefficients, thus “running” the kinks throughout the remaining length of the path. The task then of those coefficients is to ensure the cancelation of the spurious paths created by the algorithm in such a way that only two paths remain: the original path \( \eta \) and a path with a backtracking “kink” in the \( i - 1 \)-th vertex that is to be eliminated with an extra path \( \xi^{(i)} \) (which is precisely a path with a “kink” in the \( i - 1 \)-th vertex and placed “by hand” to ensure the cancelation.)

Our proposal is specifically for \( SU(3) \), the difference hinges upon the fact that graphs in \( SU(3) \) are oriented and belong in the \( SU(3) \) weight lattice: let \( \eta \) now be a path of length \( n \) such that \( C_i(\eta) \neq 0 \) for some \( i \) and \( C_j(\eta) = 0 \ \forall j \) such that \( i < j < n - 2 \) and \( \xi^{(i)} \) a path satisfying \( C_j(\xi^{(i)}) = 0 \ \forall j \) such that \( i \leq j < n - 2 \). One can decompose these paths using:
\[\eta = \sum_{k=i}^{n-2} \alpha_k (C_k^\dagger C_{k-1} C_k \ldots C_{i+1}^\dagger C_{i+1} C_i)(\eta) + \xi^{(i)} \quad (21)\]

\[\eta = \sum_{k=i}^{n-2} \alpha_k \prod_{m=i}^{k} (C_m^\dagger C_m(\eta)) + \xi^{(i)} \quad (22)\]

There are some differences with the algorithm for SU(2), however we can see that this procedure actually performs an analogous task as it “runs” pairs of edges that form the sides of a triangle in SU(3) (i.e. edges that survive the application of \(c_i\)) which serve as the backtracking kinks in a SU(3) path.

We should explain the action of our proposed algorithm on a path of length \(n\). If the path has no triangular sequences then it either is an essential path or is constructed through a concatenation of paths that is in itself in the kernel of the annihilation operator. This then means that the path belongs to the first subspace of the decomposition (15).

For a non-essential path we can ensure that at least one sequence of vertices \(v_{i-1} v_i v_{i+1}\) that is triangular and thus when acted upon by the annihilation operator first and the creation operator second, both times on the \(i\)-th vertex, we get not just the original path (up to a constant) but we also get another path due to the action of the creation operator (cf. Figure 3). The second iteration of the algorithm now requires we act upon the \(i+1\)-th vertex of these resulting paths. If the sequence of vertices that result from the first step is not triangular, that is if the sequences \(v_{i} v_{i+1} v_{i+2}\) and \(v_{b} v_{i+1} v_{i+2}\), then the algorithm stops and the path is decomposed in such a way that the \(\xi\) path, if necessary, cancels out the additional path created by the previous step. This means that the path belongs at least to the second subspace of \(P_n\), that is it acts like an essential path of length \(n-1\) on which one has acted with one creation operator in the \(i\)-th position. If the sequence is triangular however, this results in the triangular sequence moving one step down the length of the path since by hypothesis the path’s last triangular sequence did not involve the \(i+2\)-th vertex. We can see then that this procedure can be repeated and its result is to run the triangular “kink” from the \(i\)-th position towards the end of the sequence of vertices, stopping (at most) before reaching the \(n-2\)-th vertex of the path.

Once this is completed we could go back in the path to the next triangular sequence, now in a position \(j < i\) and act on this new sequence of triangular vertices. That is to say we can have the algorithm run the last triangular sequence of a path to its last possible position, then go back further in the path to the next triangular sequence and repeating the procedure, eventually ending with triangular sequences separated by at most 2 vertices. This in turn means that the resulting path would belong to a subspace of \(P_n\) in which one has acted over essential paths of length \(n-2\) with two creation operators in two positions separated by 2 vertices. From such an algorithm then the desired decomposition of the space of paths of a given size thus follows.

In order to prove the above one needs to show that there always exists a set of \(\alpha_k\) coefficients that allows one to write any path \(\eta\) as above. Although a formal proof is left for a future work, the arguments above and the examples below make us confident that our algorithm is a step in the right direction.

### 3.1 Examples

**A₂ graph:** We will now present a list of examples for the decomposition of elementary, non essential paths in the \(A_2\) graph of SU(3) starting from length zero up to length four. The list is fully developed for lengths zero through three and we provide an example for a path of length four.
For paths of length $n = 0 + 0$, $0 + 1$ and $1 + 0$ all paths are essential and the decomposition is trivial.

For paths of length $n = 2 + 0$ we find the following non essential, elementary paths: $(13\overline{3})$, $(\overline{3}31)$, $(\overline{3}38)$, $(368)$, $(\overline{3}13)$, $(\overline{3}8\overline{6})$, $(\overline{3}83)$, $(683)$, $(8\overline{6}3)$, $(83\overline{6})$, $(83\overline{3})$, $(\overline{6}38)$.

One can see that some of these paths appear in the list of essential paths in linear combinations but they are not, by themselves, essential. The decomposition of some of these paths as a direct sum is:

$$ (13\overline{3}) = \sqrt{[1][3]} C^1_{13}(\overline{13}) $$

$$ (\overline{3}31) = \sqrt{[3][1]} C^1_{31}(31) $$

$$ (\overline{3}13) = \alpha C^1_{13}(\overline{3}3) + \xi = \alpha C^1_{13}(\overline{3}3) + A((\overline{3}83) - \sqrt{\beta}(\overline{3}13)) $$

$$ = \alpha \left( T_{\overline{3}13} (\overline{3}13) + \frac{T_{\overline{3}83}}{\sqrt{[3][3]}} (\overline{3}83) \right) + A((\overline{3}83) - \sqrt{\beta}(\overline{3}13)) $$

Here $\alpha$ is the constant in equation (22) and $A$ is a normalization factor that ensures that the $\xi$ path is capable of performing the appropriate cancelations. The previous calculation then implies:

$$ \left( A + \alpha \frac{T_{\overline{3}83}}{\sqrt{[3][3]}} \right) = 0 $$

$$ \left( \alpha \frac{T_{\overline{3}13}}{\sqrt{[3][3]}} + \sqrt{\beta} \frac{T_{\overline{3}83}}{\sqrt{[3][3]}} \right) = 1. $$

It is important to note that the path $\xi$ that was added above is the path that has the triangular “kinks” in the correct location and also is the essential path connecting both vertices 3 and \overline{3}. As we can see the path $(\overline{3}13)$ has been decomposed in an essential path of the length of the original path and a path of shorter length on which one has acted upon with the creation operator. Please note that for paths obtained using conjugation and the $Z_3$ symmetry of the graph from this example one will find similar results with an analogous procedure.

For length $n = 3 + 0$ the possible cases for non essential paths can be obtained by rotations of the following set of paths:

$(1368)$, $(13\overline{3}8)$, $(13\overline{3}1)$, $(3683)$, $(36\overline{8}6)$, $(\overline{3}313)$, $(\overline{3}383)$, $(\overline{3}38\overline{6})$

We can decompose some of these paths with our proposal in the following way:

$$ (1368) = \frac{1}{\beta} C^1_{138} + \sqrt{\beta} C^1_{2}(138) $$

$$ (13\overline{3}8) = \frac{1}{\sqrt{\beta}} C^1_{13}(\overline{3}8) $$

$$ (36\overline{8}6) = \frac{1}{\sqrt{\beta}} \left( C^1_{1}(\overline{3}8\overline{6}) - \frac{1}{\beta} C^1_{2}(336) \right) $$

$$ (13\overline{3}1) = \frac{1}{\sqrt{\beta} T_{131}} C^1_{2}(C^1_{0}(1)) $$
The decomposition of this last path, since it requires a definition of the deformed triangular cells cannot be calculated completely at this point.

For the elementary and non-essential path \( \eta = (13686) \) we find (this path is clearly non essential since the sequence (368) is a triangular sequence in the order \( v_1 = 3, v_2 = 6, v_3 = 8 \) and therefore gives a non zero result when acted upon with the annihilation operator):

\[
C_2(\eta) = \frac{T_{368}}{\sqrt{3[8]}} 13\overline{86} \tag{32}
\]

\[
C_2^\dagger C_2(\eta) = \frac{T_{368}}{\sqrt{3[8]}} \left( \frac{T_{38\overline{6}}}{\sqrt{3[8]}} 133\overline{86} + \frac{T_{368}}{\sqrt{3[8]}} 13\overline{6} \right) \tag{33}
\]

Here our algorithm removes the triangular sequence (368) and then adds a vertex that produces a linear combination of two elementary paths: one is the original path (13686) and the other, the path (13\overline{38}). This last path has now a triangular sequence in the \( v_2 = 3, v_3 = 8, v_4 = 6 \), which has been displaced by one position in the order of the sequence of vertices of the elementary path.

Taking the result of this calculation, that is, and applying \( C_3^\dagger C_3 \) to it we get:

\[
C_3(\eta') = \frac{T_{368}T_{3\overline{8}}T_{36\overline{8}}}{\sqrt{3[8]3[8]3[8]}} 13\overline{86} \tag{34}
\]

\[
C_3^\dagger C_3(\eta') = \frac{T_{368}T_{3\overline{8}}T_{36\overline{8}}}{\sqrt{3[8]3[8]3[8]3[8]}} \left( \frac{T_{38\overline{6}}}{\sqrt{3[8]}} 133\overline{86} \right) \tag{35}
\]

We can get the values for the \( \alpha_k \) constants by cancelling the appropriate terms, getting then

\[
\alpha_1 = \frac{[3][8]}{|T_{368}|^2}, \quad \alpha_2 = -\frac{[3][3][8][6]}{|T_{368}|^2|T_{3\overline{8}}|^2}
\]

We can see that the iterated use of creation and annihilation operators allows us to obtain an arbitrary path in this example.

\( E_5 \) graph: We will now present a list of examples for the decomposition of elementary, non essential paths in the \( E_5 \) graph of \( SU(3) \) starting from length zero up to length three. The list is exhaustively calculated for lengths zero through two and we provide two examples of paths of length three.

Once again, for paths of length \( n = (0,0), (0,1) \) and \( (1,0) \) all paths are essential and the decomposition is trivial.

For paths of length \( n = (2,0) \) we find the following non essential, elementary paths:

\[
(1,2_{i+1}2_{i+2}) \quad (2,1_i2_{i+4}2_{i+5}) \quad (2,2_i2_{i+4}2_{i+5})
\]
\[
(2,2_{i+1}2_{i+2}) \quad (2,2_{i+1}2_{i+5}) \quad (2,2_{i+1}2_{i+2})
\]

For paths of length \( n = (1,1) \) we find either essential paths or elementary back and forth paths that require the use of the deformed triangular cells discussed above. Since at this point, their values have yet to be calculated, we will not deal with them here.

For paths of length \( n = (0,3) \) the list of non essential, elementary paths follows:

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\[
\begin{align*}
(1,2_{t+2}1_{i+1}2_{i+3}) & \quad (1,2_{i+2}2_{i+1}2_{i}) \quad (1,2_{i+2}2_{i+1}2_{i+3}) \\
(1,2_{i+2}2_{i+1}2_{i}) & \quad (1,2_{i+2}2_{i+1}2_{i+3}) \quad (2_{i+3}2_{i+1}1_{i}) \\
(2_{i+5}2_{i+4}1_{i+3}) & \quad (2_{i+5}2_{i+4}1_{i+3}) \quad (2_{i+5}2_{i+1}2_{i+3}) \\
(2_{i+5}2_{i+4}2_{i+3}) & \quad (2_{i+5}2_{i+4}2_{i+3}) \quad (2_{i+2}2_{i+1}2_{i+3}) \\
(2_{i+5}2_{i+1}1_{i}) & \quad (2_{i+5}2_{i+1}1_{i}) \quad (2_{i+2}2_{i+1}2_{i+3})
\end{align*}
\]

Let us show a few selected examples of the workings of our proposal for a selection of families of paths. First for a family of paths of length \( n = (2,0) \):

\[
(1,2_{i+1}2_{i+2}) = \frac{\sqrt{1_{i}2_{i+2}}}{T_{1_{i},2_{i+1}2_{i+2}}} C_{1}^{1}(1,2_{i+2})
\]

For a family of paths of length \( n = (0,3) \) we find:

\[
\begin{align*}
(1,2_{i+2}2_{i+4}2_{i}) = \alpha C_{2}^{1}(1,2_{i+2}2_{i}) + \xi \\
&= \alpha \left( \frac{T_{2_{i+2}2_{i+4}2_{i}}}{\sqrt{2_{i+2}[2_{i}]}}, \frac{T_{2_{i+2}2_{i+2}2_{i}}}{\sqrt{2_{i+2}[2_{i}]}}, 1_{i}, 2_{i} \right) + \\
&+ A \left( (1,2_{i+2}2_{i+4}2_{i}) - (-1)^{i^{2}} \frac{\nu_{0}}{\mu} (1,2_{i+2}2_{i+1}2_{i}) \right) + \\
&+ \left( \frac{\alpha T_{2_{i+2}2_{i+4}2_{i}}}{\sqrt{2_{i+2}[2_{i}]}} + A \right) (1,2_{i+2}2_{i+4}2_{i}) + \\
&+ \left( \frac{\alpha T_{2_{i+2}2_{i+4}2_{i}}}{\sqrt{2_{i+2}[2_{i}]}} - (-1)^{i^{2}} \frac{\nu_{0}}{\mu} A \right) (1,2_{i+2}2_{i+1}2_{i})
\end{align*}
\]

Which leads us to values for both constants:

\[
\begin{align*}
\left( \frac{\alpha T_{2_{i+2}2_{i+4}2_{i}}}{\sqrt{2_{i+2}[2_{i}]}} + A \right) &= 1 \\
\left( \frac{\alpha T_{2_{i+2}2_{i+4}2_{i}}}{\sqrt{2_{i+2}[2_{i}]}} - (-1)^{i^{2}} \frac{\nu_{0}}{\mu} A \right) &= 0
\end{align*}
\]

If we consider the elementary and non essential path \( \eta = (1_{3}2_{4}1_{2}2_{3}2_{2}) \) we find, for the first steps of the decomposition:

\[
C_{2}^{1}C_{2}(\eta) = \frac{T_{2_{i+2}2_{i}}}{\sqrt{[2_{i}][2_{i}]}} \left( \frac{T_{2_{i+2}2_{i}}}{\sqrt{[2_{i}][2_{i}]}} 1_{3}2_{4}1_{2}2_{3}2_{2} + \frac{T_{2_{i+2}2_{i}}}{\sqrt{[2_{i}][2_{i}]}} 1_{3}2_{4}2_{2}2_{3}2_{2} \right)
\]

\[
C_{3}^{1}C_{2}(\eta) = \frac{T_{2_{i+2}2_{i}}}{\sqrt{[2_{i}][2_{i}]}} \left( \frac{T_{2_{i+2}2_{i}}}{\sqrt{[2_{i}][2_{i}]}} 1_{3}2_{4}1_{2}2_{3}2_{2} + \frac{T_{2_{i+2}2_{i}}}{\sqrt{[2_{i}][2_{i}]}} 1_{3}2_{4}2_{2}2_{3}2_{2} \right) = 0
\]

We find in this path that the terms for the path \( 1_{3}2_{4}2_{2}2_{3}2_{2} \) do not cancel out. This is due to the particular geometry of the \( \mathcal{E}_{5} \) but we can recover the decomposition of the path keeping in mind that there is a \( \xi \) path that allows for the desired cancelation, thus:
\[ \alpha_1 = \frac{[2_4][2_3]}{|T_{2_412_3}|}, \quad \xi = -\alpha_1 \frac{T_{2_412_3}}{\sqrt{|2_4|2_3}} \frac{T_{2_22_3}}{\sqrt{|2_4|2_3}} 1_42_42_22_32_2 \]

A note of caution is in order on the use of the \( \xi \) path. In [24], the geometric necessity for the \( \xi \) path is clear: in some paths the application of the creation operator in the first step of the algorithm results in the creation of two paths, one that presents a backtracking in the \( i - 1 \)th vertex and the original path. The remaining iterations of the algorithm produce paths that “move” the backtracks towards the end of the path and ensure the cancelation of all paths thus created. Since for these paths the algorithm creates a path in which the backtrack has been moved back one step, one requires a cancelling path with the properties indicated in Trinchero’s work, i.e. \( \xi^{(i)} \) satisfying \( c_j(\xi^{(i)}) = 0 \) \( \forall j \) such that \( i - 1 < j < n - 2 \). Since our \( SU(3) \) graphs are now embedded in a two dimensional lattice there is no natural ordering of vertices, that is, there is no way to claim that a vertex comes before or after another one unambiguously which leads to the problem of the definition of backtracking paths. There is however a way out: to relax the definition of the \( \xi \) path in such a way that it is capable of ensuring the required cancelation, that is, that it presents a triangular path in the same position of the original path. In future works we shall elucidate the decomposition of the space of paths over \( SU(3) \) graphs and we will provide proof for this algorithm and the properties of the \( \xi \) path.

4 Discussion

Using triangular sequences of vertices as an analogue for a backtracking path in \( SU(3) \) we have provide here a geometrical approach that is both straightforward and natural given the building blocks of the family of \( SU(3) \) graphs.

This geometric understanding of a backtracking path in \( SU(3) \) leads us to grasp the meaning of the path creation and annihilation operators in terms of the creation and annihilation of triangular sequences of vertices. We also have found that our implementation of these operators yields the expected mathematical properties as applied to essential and non essential paths, in particular they fulfill the relations for the Temperley–Lieb algebra for \( SU(3) \). In addition we have shown, through the explicit calculation of some examples for both \( A \) type and \( E \) type graphs, that our definition of the annihilation operator and our interpretation in terms of triangular sequences, yields not just the desired results for essential and non essential paths but also sets up interesting questions that could allow for a generalization of this work for the remaining members of the \( SU(3) \) family, that is to say, the \( D \) series and the conjugate graphs. Another interesting result that follows from our implementation of the creation and annihilation operators is related to the labeling of the length of a given path: since in \( SU(3) \) graphs have oriented edges one must take into consideration two graphs with opposing orientations of the edges. These differing orientations are related to the existence of two generators \( \sigma \) and \( \overline{\sigma} \) that we use here to provide a formal interpretation of the length of a path. Since a path of a given length is built with edges of either orientation one can understand the length of a path as the sum of the number of edges that follow the direction of the arrows (and thus related with the \( \sigma \) generator) and those following the opposite direction (and related to \( \overline{\sigma} \)). Since to our knowledge there has been no labeling that explicitly laid out the fact that these graphs inhabit a two dimensional lattice (either through the use of “triangular coordinates” or of triality) we believe that this interpretation in terms of the generators should be rather useful.

An important result is our decomposition of the space of paths: we have found that
the space of paths of a given length can be decomposed in a way similar to that shown previously for $SU(2)$, which is to say that one can obtain a path of a given length by taking a path in the kernel of the annihilation operator and then acting upon it with an ordered sequence of creation operators until one obtains the desired path. This decomposition allows us to propose an algorithm that shows explicitly how a given path can be obtained as an interated and ordered application of the creation operator on a path belonging to the kernel of the annihilation operator. We not only propose said algorithm but we perform the calculation successfully for a number of selected paths for $A$ type and $E$ type graphs. Even though a formal proof of the algorithm is left for a future work, our calculations give us confidence that our suggestion is correct.

We have found here that even though there is a complete construction of the bialgebra for $SU(2)$ in terms of paths and there have been quite a few works laying the mathematically formal groundwork for extending this formalism to $SU(3)$ graphs, the paths formulation of the bialgebra for $SU(3)$ graphs still requires some work in order to provide a complete alternative to the calculation of Ocneanu cells in the construction of the double triangle algebra. As we have seen, a clear understanding of the path creation and annihilation operators for these graphs in geometrical terms and their properties has led us to find some interesting and important definitions and calculations that still require elucidation.

If one is to extend the program laid forth in $[24]$ to $SU(3)$, once one builds the essential paths, then the next step is to restrict the (now graded) space of paths to that of the essential paths. For this, Trinchero defines a projection that enforces the restriction, thus permitting a proper definition of the space of endomorphisms of essential paths (that, as is already known and have shown in detail here, are related to the double triangle algebra). However in $SU(3)$ a problem arises that demands our attention: In $SU(2)$ one finds that in $A$-type graphs there are no essential paths of length greater than the maximum allowed by the generalized Coxeter number, however in even the simplest cases of $SU(3)$ we see that this is not the case. Finding paths in the kernel space of the annihilation operator that have length greater than the maximum allowed (and that are therefore produced by the use of the concatenation product) implies that a careful parsing of the space of essential paths must be made before one endeavors to propose a projection. This means that reaching the analogue of the double triangle algebra for paths (and therefore the complete realization of the weak Hopf algebra of double triangles) is not straightforward.

Up to this point in the literature there has been no need of discussion of the existence of what we have called collapsed or degenerated triangular cells. However, as we have found repeatedly throughout this work both in formal definitions and in the explicit calculations for $A_2$ and $E_5$ graphs, in order to fully understand and explicitly obtain the complete set of essential paths for a given graph one must first come to grips with the idea of backtracking paths, specifically back and forth paths and the collapsed triangular cells that they imply. One can not just equate them to zero since it leads to issues in the process of constructing the set of essential paths of length $(1, 1)$. This can be easily seen by inspecting the tables of essential paths of that length. Furthermore, one can easily see that, since the conjugate graphs of $SU(3)$ are not constructed through combinations of triangles, the idea of collapsed triangular cells can provide us with a natural framework to study their bialgebra in the path formalism. How these collapsed cells can be constructed and their application for some $SU(3)$ graphs will be the subject of a future work.

Another important, albeit straightforward, avenue of further research is the extension of this work for multiply connected graphs such as the $D$ series or the $E$ series beyond $E_5$. Since there has been work already explicitly calculating the properties of triangular cells in these cases we believe that natural extensions of the annihilation and creation operators
will provide little issues. However the problem of collapsed triangular cells remains and should be treated carefully. In a related but more involved subject the investigation of the properties of combinations of the collapsed cells require a revision of the so called “small pocket” and “big pocket” equations in order to accommodate these new cases. In addition these modifications must preserve the algebraic properties of the creation and annihilation operators, that is any definition for the collapsed triangular cells must preserve the fact that the creation and annihilation operators are a representation of the Temperley-Lieb algebra.

A The Hecke and Temperley–Lieb algebras

An important set of properties that our operators must fulfill are those associated with the Temperley–Lieb algebra for SU(3). The relations that define the Temperley–Lieb algebra are well known and have already been explored in the literature [25]:

\[ U_i^2 = \beta U_i, \]
\[ U_i U_j = U_j U_i, \quad |i - j| > 1, \]
\[ U_i U_{i+1} U_i - U_i = U_{i+1} U_i U_{i+1} - U_{i+1}, \]
\[ (U_i - U_{i+2} U_i + U_{i+1})(U_{i+1} U_{i+2} U_i - U_{i+1}) = 0 \]

Therefore we should show that our operators can satisfy these relations. The proofs are rather straightforward but the geometrical interpretation of these relations should prove rather interesting.

\[
U_i^2(\eta) = \sum_{b,b'} \frac{T_{i-1,b,i+1} T_{i-1,b,i+1} T_{i-1,b,i+1} T_{i-1,b,i+1}}{[i-1][i+1][i+1]} \ v_0 v_1 \ldots v_{i-1} v_i v_{i+1} \ldots v_n \]
\[
= \sum_{b'} \frac{T_{i-1,b,i+1} T_{i-1,b,i+1}}{[i-1][i+1]} \left( \sum_{b} \frac{[T_{i-1,b,i+1}^2]}{[i-1][i+1]} \right) \ v_0 v_1 \ldots v_{i-1} v_i v_{i+1} \ldots v_n \]
\[
= \beta \sum_{b'} \frac{T_{i-1,b,i+1} T_{i-1,b,i+1}}{[i-1][i+1]} \ v_0 v_1 \ldots v_{i-1} v_i v_{i+1} \ldots v_n \]
\[
= \beta \ U_i(\eta) \]

In here we have used the “small pocket equation” for a graph with single edges.

Geometrically, this operation can be easily understood since the action of an \( U_i \) operator is to initially remove a vertex and (whenever this removal does not result in a null path) then add a vertex in such a way that two paths are created: the original path \( \eta \) and a new path in which the \( i \)-th vertex is replaced by a new vertex which, together with those in the original sequence \( v_{i-1} v_i v_{i+1} \), forms a rhombus with the edge connecting the \( i - 1 \) and \( i + 1 \) vertices as a diagonal, i.e. a pair of elementary triangles that share the aforementioned edge. The second application of the \( U_i \) operator, since it acts on the exact same vertex in the sequence, ends up performing the exact same task of substraction and addition of the very same vertices as before, with the sole difference between both actions being the coefficient of the resulting paths.

On now for the restricted commutation relation for the Hecke operators. Geometrically their application as simple as the above case, furthermore, the restriction on the distance of the \( i \) and \( j \) indices becomes clear since a separation more than or equal to two vertices means that the action of each of the operators will not influence that of the other and
since $U_i : P_n \to P_n$ the length of the resulting path after the action of the first operator is exactly the same as the original path $\eta$. Finally, with a separation smaller than that prescribed by the Hecke algebra we find that the triangular cells have vertices in common which break the commutation. Thus we find (for $i > j + 1$):

$$
U_i U_j(\eta) = \left( \sum_c T_{j-1,j,j+1} \frac{T_{j-1,c,j+1}}{[j-1][j+1]} \right) \left( \sum_b T_{i-1,i,i+1} \frac{T_{i-1,b,i+1}}{[i-1][i+1]} \right) v_0 v_1 \ldots v_{i-1} v_c v_{j+1} \ldots v_n = U_j U_i(\eta)
$$

$$
= \sum_{T\in P_n} \left( \sum_c T_{j-1,j,j+1} \frac{T_{j-1,c,j+1}}{[j-1][j+1]} \right) \left( \sum_b T_{i-1,i,i+1} \frac{T_{i-1,b,i+1}}{[i-1][i+1]} \right) v_0 v_1 \ldots v_{i-1} v_c v_{j+1} \ldots v_n
$$

The third relation of the Hecke algebra will provide interesting results that will be addressed below, we should then first study the action of $U_i U_{i+1} U_i$:

$$
U_i U_{i+1} U_i(\eta) = \sum_{b,b',b''} T_{i+1,i+1,b} \frac{T_{i+1,b,i+2} T_{i+1,b',i+2} T_{i+1,b''}}{[i+2][b+i+2][i+b']} v_0 v_1 \ldots v_{i-1} v_c v_{i+2} \ldots v_n
$$

$$
= \sum_{b',b''} \left( \sum_b T_{i+1,i+1,b} \frac{T_{i+1,b,i+2} T_{i+1,b',i+2} T_{i+1,b''}}{[i+2][b+i+2][i+b']} \right) v_0 v_1 \ldots v_{i-1} v_c v_{i+2} \ldots v_n
$$

$$
= \sum_{b'} \sum_{b''} T_{i+1,i+1,b} \frac{T_{i+1,b,i+2} T_{i+1,b',i+2} T_{i+1,b''}}{[i+2][b+i+2][i+b']} v_0 v_1 \ldots v_{i-1} v_c v_{i+2} \ldots v_n
$$

$$
+ \sum_{b''} T_{i+1,i+1,b+1} \frac{T_{i+1,b,i+2} T_{i+1,b',i+2} T_{i+1,b''}}{[i+2][b+i+2][i+b']} v_0 v_1 \ldots v_{i-1} v_c v_{i+2} \ldots v_n = U_i(\eta) + F_i(\eta)
$$

$$
U_{i+1} U_i U_{i+1}(\eta) = \sum_{b,b',b''} T_{i+1,i+1,b} \frac{T_{i+1,b,i+2} T_{i+1,b',i+2} T_{i+1,b''}}{[i+2][b+i+2][i+b']} v_0 v_1 \ldots v_{i-1} v_c v_{i+2} \ldots v_n
$$

$$
= \sum_{b',b''} \left( \sum_b T_{i+1,i+1,b} \frac{T_{i+1,b,i+2} T_{i+1,b',i+2} T_{i+1,b''}}{[i+2][b+i+2][i+b']} \right) v_0 v_1 \ldots v_{i-1} v_c v_{i+2} \ldots v_n
$$

$$
= \sum_{b'} \sum_{b''} T_{i+1,i+1,b} \frac{T_{i+1,b,i+2} T_{i+1,b',i+2} T_{i+1,b''}}{[i+2][b+i+2][i+b']} v_0 v_1 \ldots v_{i-1} v_c v_{i+2} \ldots v_n
$$

$$
+ \sum_{b''} T_{i+1,i+1,b+1} \frac{T_{i+1,b,i+2} T_{i+1,b',i+2} T_{i+1,b''}}{[i+2][b+i+2][i+b']} v_0 v_1 \ldots v_{i-1} v_c v_{i+2} \ldots v_n = U_{i+1}(\eta) + F_i(\eta)
$$

(50)
From these equations we get a definition for the $F_i$ operator previously studied in [24, 25]:

$$F_i(\eta) = \sum_{b',b''} T_{i-1,i,i+1,b',b''} \frac{T_{i-1,b''b_1}}{[i-1][i-1]} v_0 v_1 \cdots v_{i-1} v_{i} v_{i+1} \cdots v_n$$

$$= \sum_{b',b''} T_{i-1,i,i+1,b',b''} \frac{T_{i-1,b''b_1}}{[i-1][i-1]} v_0 v_1 \cdots v_{i-1} v_{i} v_{i+1} \cdots v_n = F_{i+1}(\eta)$$  \hspace{1cm} (51)

These $F_i$ operators are rather interesting: as we have mentioned before, our creation and annihilation operators take into account the $SU(2)$ idea of a backtracking path and replace it with a triangular sequence of vertices, however up to this point we have made only passing mention of closed triangular paths, that is, paths in which the sequence of vertices define all the three edges of a triangle. We find here that the $F_i$ provide a non zero result only when operating on closed triangular paths. Given these definitions we find that the third Hecke algebra relation is satisfied:

$$F_i = U_i U_{i+1} U_i - U_i = U_{i+1} U_i U_{i+1} - U_{i+1}$$  \hspace{1cm} (52)

The final relation in the [22, 25] series, which defines the Temperley-Lieb algebra for $SU(3)$, has already been shown to be satisfied whenever the following simpler relation is fulfilled (cf. [25] Lemma 4.1): $F_i F_{i+1} F_i = \beta^2 F_i$, so it should suffice to prove that our definition satisfies this last relation. Through explicit calculation we find:

$$F_i F_{i+1} F_i(\eta) = \sum_{b_1 b_2 b_3 b_4 b_5 b_6} \left( \frac{T_{i-1,i,i+1}}{[i-1][i-1]} \right) \left( \frac{T_{i-1,b_2b_1}}{[b_2][b_1]} \right) \times$$

$$\times \left( \frac{T_{i-1,b_2b_3}}{[i-1][i-1]} \right) \delta_{i-1,i+3} \delta_{b_2,b_3} \delta_{b_2,b_3} v_0 v_1 \cdots v_{i-1} v_{i} v_{i+1} v_{i+2} \cdots v_n$$

$$= \sum_{b_1 b_2 b_3 b_4 b_5 b_6} \left( \frac{T_{i-1,i,i+1}}{[i-1][i-1]} \right) \left( \frac{|T_{i-1,i,i+1,b_1}|^2}{[i-1][i-1]} \right) \left( \frac{|T_{i-1,i,i+3,b_4}|^2}{[i-1][i-1]} \right) \times$$

$$\times v_0 v_1 \cdots v_{i-1} v_{i} v_{i+1} v_{i+2} \cdots v_n$$

$$= \beta^2 \left( \frac{|T_{i-1,i,i+1}|^2}{[i-1][i-1]} \right) \left( \frac{|T_{i-1,i,i+3,b_4}|^2}{[i-1][i-1]} \right) \times$$

$$\times v_0 v_1 \cdots v_{i-1} v_{i} v_{i+1} v_{i+2} \cdots v_n = \beta^2 F_i(\eta)$$  \hspace{1cm} (53)

We thus find that our creation and annihilation operators not just comply with relations analogous to those proposed by Trinchero for $SU(2)$ paths but they also provide us with $U_i$ operators that are a representation of the Hecke algebra, as is to be expected. Furthermore in addition to the definitions we have provided with a geometric interpretation for the action of the creation and annihilation operators, the Hecke algebra operators and the $F_i$ operators in terms of triangular sequences of vertices (or closed triangular sequences) in paths over $SU(3)$ graphs.

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