Time-dependent Inclusions and Sweeping Processes in Contact Mechanics

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Abstract

We consider a class of time-dependent inclusions in Hilbert spaces for which we state and prove an existence and uniqueness result. The proof is based on arguments of variational inequalities, convex analysis and fixed point theory. Then we use this result to prove the unique weak solvability of a new class of Moreau’s sweeping processes with constraints in velocity. Our results are useful in the study of mathematical models which describe the quasistatic evolution of deformable bodies in contact with an obstacle. To provide some examples we consider three viscoelastic contact problems which lead to time-dependent inclusions and sweeping processes in which the unknowns are the displacement and the velocity fields, respectively. Then we apply our abstract results in order to prove the unique weak solvability of the corresponding contact problems.

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1 Introduction

Contact phenomena with deformable bodies arise in a large variety of industrial settings and engineering applications. Their classical formulation leads to challenging nonlinear boundary value problems in which the unknowns are the displacement and the stress field. Most of these problems include unilateral constraints and represent free boundary problems. For this reason, their mathematical analysis is done by using the so-called weak formulation which, usually, is expressed in terms of variational or hemivariational inequalities in which the unknown is the displacement or the velocity field. Comprehensive reference in the field are [4, 6, 7, 10, 11, 19, 20, 21] and, more recently, [22].

An important number of problems arising in Mechanics, Physics and Engineering Science leads to mathematical models expressed in terms of nonlinear time-dependent inclusions. For this reason the mathematical literature dedicated to this field is extensive and the progress made in the last decades is impressive. It concerns both results on the existence, uniqueness, regularity and behavior of the solution for various classes of inclusions as well as results on the numerical approaches to the solution of the corresponding problems. Variational and hemivariational inequalities represent a class of nonlinear inclusions that are associated with the subdifferential in the sense of convex analysis and the Clarke subdifferential operator, respectively. They have made the object of various books and surveys, see [9, 13, 18, 20, 21, 22], for instance.

The notion of “sweeping process” was introduced by Jean Jacques Moreau in early seventies, in connexion with the study of displacement-tractions problems for elastic-plastic materials, see [14, 15, 16, 17]. There, the treatment of both theoretical and numerical aspects of sweeping processes have been developed and their applications in unilateral mechanics were illustrated. Since the pioneering works of Moreau, several extensions and generalizations have been considered in literature for which various existence and uniqueness results have been provided. References on the field are [2, 3] and, more recently [1].

The aim of this paper is two folds. The first one to introduce a new class of time-dependent inclusions and sweeping processes and to study their unique solvability. Here, the novelty arises in the special structure of the problems we consider, which are governed by two nonlinear operators, possible history-dependent, and are defined on a time interval which could be either bounded or unbounded. Moreover, one of the operators appears in the set of constraints. The second aim is to illustrate the use of these results in the study of mathematical models arising in Contact Mechanics. In contrast with the standard variational formulations considered in the literature, the contact models we consider here lead to time-dependent inclusions and sweeping processes, which represents the second trait of novelty of this paper.

The paper is structured as follows. In Section 2 we introduce the notation we use and the preliminaries of convex analysis and nonlinear analysis we need in the
rest of the paper. They include an existence and uniqueness result for elliptic vari-
ational inequalities and a fixed point result for almost history-dependent operators,
among others. In Section 3 we introduce the time-dependent inclusions and prove their
unique solvability, Theorem 3.3. Then, in Section 4 we introduce the sweeping pro-
cesses we are interested in and prove an existence and uniqueness result, Theorem 4.1.
Finally, in Sections 5 and 6 we illustrate the use of our abstract results in the study of
three contact models with viscoelastic materials, both in the frictionless and frictional
case. In this way we provide an example of cross fertilization between models and
applications, in one hand, and the nonsmooth analysis, on the other hand.

2 Preliminaries

Most of the material presented in this section is standard. Therefore, we introduce it
without proofs and restrict ourselves to mention that details on the definitions and
statements below can be found in the monographs [5, 8, 12, 13] as well as in the paper
[1].

Elements of convex analysis. Everywhere in this paper $X$ will represent a real
Hilbert space with the inner product $(\cdot, \cdot)_X$ and the associated norm $\| \cdot \|_X$. Moreover,
we denote by $0_X$ the zero element of $X$ and by $2^X$ the set of parts of $X$.
Assume that $J : X \to ] - \infty, +\infty]$ is a convex lower semicontinuous function such
that $J \not\equiv \infty$, i.e., $J$ is proper. The effective domain of $J$ is the set $\text{Dom}(J)$ defined by
$$\text{Dom}(J) = \{ u \in X : J(u) < +\infty \}.$$ The subdifferential of $J$ (in the sense of convex analysis) is the multivalued operator
$\partial J : X \to 2^X$ defined by
$$\partial J(u) = \{ \xi \in X : J(v) - J(u) \geq (\xi, v - u)_X \quad \forall v \in X \}.$$ An element $\xi \in \partial J(u)$ (if any) is called a subgradient of $J$ in $u$. We recall that
if $u \notin \text{Dom}(J)$ then $\partial J(u) = \emptyset$. For the above function $J$, its Legendre-Fenchel
conjugate is defined as $J^* : X \to ] - \infty, +\infty]$, $J^*(u^*) = \sup_{u \in X} \left( (u^*, u)_X - J(u) \right)$. Moreover, the following equivalence holds.
$$u^* \in \partial J(u) \iff u \in \partial J^*(u^*).$$

Let $C \subset X$ be a nonempty closed convex subset. The function $I_C$ defined by
$$I_C(x) = \begin{cases} 
0 & \text{if } x \in C, \\
+\infty & \text{if } x \notin C 
\end{cases}$$
is called the indicator function of $C$. Using (2.1) it follows that the subdifferential of \( I_C \) is the multivalued operator \( \partial I_C : X \to 2^X \) defined by

\[
\partial I_C(u) = \begin{cases} 
\{ \xi \in X : (\xi, v-u)_X \leq 0 \ \forall v \in C \} & \text{if } u \in C, \\
\emptyset & \text{if } u \notin C.
\end{cases}
\] (2.3)

As usual in the convex analysis, we denote the subdifferential of the function \( I_C \) by \( N_C \), i.e., \( \partial I_C = N_C \). For a given \( u \in C \), the set \( \partial I_C(u) = N_C(u) \subset X \) represents the set of outward normals of the convex set at the point \( u \in C \). Moreover, it is easy to check that

\[
N_C(-u) = -N_{-C}(u) \quad \forall \ u \text{ such that } u \in -C
\] (2.4)

\[
N_C(u + v) = N_{-v}(u) \quad \forall \ u, v \text{ such that } u + v \in C.
\] (2.5)

**Variational inequalities.** We recall that an operator \( A : X \to X \) is said to be strongly monotone if there exists \( m_A > 0 \) such that

\[
(Au - Av, u - v)_X \geq m_A \|u - v\|_X^2 \quad \forall \ u, v \in X.
\] (2.6)

The operator \( A \) is Lipschitz continuous if there exists a constant \( L_A > 0 \) such that

\[
\|Au - Av\|_X \leq L_A \|u - v\|_X \quad \forall \ u, v \in X.
\] (2.7)

A function \( j : K \subset X \to \mathbb{R} \) is said to be lower semicontinuous (l.s.c.) at \( u \in K \) if

\[
\liminf_{n \to \infty} j(u_n) \geq j(u)
\] (2.8)

for each sequence \( \{u_n\} \subset K \) converging to \( u \) in \( X \). The function \( j \) is lower semicontinuous (l.s.c.) if it is lower semicontinuous at every point \( u \in K \). We now recall a classical result in the study of variational inequalities.

**Theorem 2.1.** Let \( X \) be a Hilbert space and assume that \( K \) is a nonempty closed convex subset of \( X \), \( A : X \to X \) is a strongly monotone Lipschitz continuous operator and \( j : K \to \mathbb{R} \) is a convex lower semicontinuous function. Then, for each \( f \in X \), there exists a unique solution of the variational inequality

\[
u \in K, \quad (Au, v-u)_X + j(v) - j(u) \geq (f, v-u)_X \quad \forall \ v \in K.
\] (2.9)

Theorem 2.1 will be used in Section 3 to prove the unique solvability of our nonlinear inclusion. Its proof is based on the Banach fixed point argument and could be found in [21], for instance.

**History and almost history-dependent operators.** Everywhere below \( I \) will denote either a bounded interval of the form \([0, T]\) with \( T > 0 \), or the unbounded
interval $\mathbb{R}_+ = [0, +\infty)$. For a normed space $(Y, \| \cdot \|_Y)$ we denote by $C(I; Y)$ the space of continuous functions defined on $I$ with values in $Y$, that is,

$$C(I; Y) = \{ v: I \to Y \mid v \text{ is continuous} \}.$$ 

The case $I = [0, T]$ leads to the space $C([0, T]; Y)$ which is a normed space equipped with the norm

$$\|v\|_{C([0, T]; Y)} = \max_{t \in [0, T]} \|v(t)\|_Y.$$

If $Y$ is a Banach space, then $C([0, T]; Y)$ is a Banach space, too. The case $I = \mathbb{R}_+$ leads to the space $C(\mathbb{R}_+; Y)$. If $Y$ is a Banach space then $C(\mathbb{R}_+; Y)$ can be organized in a canonical way as a Fréchet space, i.e., a complete metric space in which the corresponding topology is induced by a countable family of seminorms. For a subset $K \subset Y$ we still use the symbol $C(I; K)$ for the set of continuous functions defined on $I$ with values on $K$.

We also denote by $C^1(I; Y)$ the space of continuously differentiable functions on $I$ with values in $Y$ and, we note that $v \in C^1(I; Y)$ if and only if $v \in C(I; Y)$ and $\dot{v} \in C(I; Y)$ where, here and below, $\dot{v}$ represents the derivative of the function $v$. Moreover, for a subset $K \subset Y$, we denote by $C^1(I; K)$ the set of continuously differentiable functions on $I$ with values in $K$. For a function $v \in C^1(I; Y)$, the equality below will be used in various places of this manuscript:

$$v(t) = \int_0^t \dot{v}(s) \, ds + v(0) \text{ for all } t \in I.$$

Two important classes of operators defined on the space of continuous functions are provided by the following definition.

**Definition 2.2.** Assume that $(Y, \| \cdot \|_Y)$ and $(Z, \| \cdot \|_Z)$ are normed spaces. An operator $S: C(I; Y) \to C(I; Z)$ is called:

a) history-dependent (h.d.), if for any compact set $J \subset I$, there exists $L^S_J > 0$ such that

$$\|S u_1(t) - S u_2(t)\|_Z \leq L^S_J \int_0^t \|u_1(s) - u_2(s)\|_Y \, ds$$

for all $u_1, u_2 \in C(I; Y)$, $t \in J$.

b) almost history-dependent (a.h.d.), if for any compact set $J \subset I$, there exists $l^S_J \in [0, 1)$ and $L^S_J > 0$ such that

$$\|S u_1(t) - S u_2(t)\|_Z \leq l^S_J \|u_1(t) - u_2(t)\|_Y$$

$$+ L^S_J \int_0^t \|u_1(s) - u_2(s)\|_Y \, ds$$

for all $u_1, u_2 \in C(I; Y)$, $t \in J$. 

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Note that here and below, when no confusion arises, we use the shorthand notation $S_u(t)$ to represent the value of the function $S_u$ at the point $t$, i.e., $S_u(t) = (S_u)(t)$. It follows from the previous definition that any h.d. operator is an a.h.d. operator. History-dependent and almost history-dependent operators arise in Contact Mechanics and Nonlinear Analysis. They have important fixed point properties which are very useful to prove the solvability of various classes of nonlinear equations and variational inequalities.

**Theorem 2.3.** Let $Y$ be a Banach space and let $\Lambda : C(I; Y) \to C(I; Y)$ be an almost history-dependent operator. Then, $\Lambda$ has a unique fixed point, i.e., there exists a unique element $\eta^* \in C(I; Y)$ such that $\Lambda \eta^* = \eta^*$.

A proof of Theorem 2.3 can be found in [22, p. 41–45]. There, the main properties of history-dependent and almost history-dependent operators are stated and proved, together with various examples and applications.

**Function spaces.** Let $d \in \{1, 2, 3\}$ and denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order $d$. The zero element of the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$ will be denoted by $0$. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

\[
\begin{align*}
\mathbf{u} \cdot \mathbf{v} &= u_i v_i, \\
\|\mathbf{v}\| &= (\mathbf{v} \cdot \mathbf{v})^{1/2} \quad \forall \mathbf{u} = (u_i), \mathbf{v} = (v_i) \in \mathbb{R}^d, \\
\mathbf{\sigma} \cdot \mathbf{\tau} &= \sigma_{ij} \tau_{ij}, \\
\|\mathbf{\tau}\| &= (\mathbf{\tau} \cdot \mathbf{\tau})^{1/2} \quad \forall \mathbf{\sigma} = (\sigma_{ij}), \mathbf{\tau} = (\tau_{ij}) \in \mathbb{S}^d,
\end{align*}
\]

where the indices $i, j$ run between 1 and $d$ and, unless stated otherwise, the summation convention over repeated indices is used.

Consider now a bounded domain $\Omega \subset \mathbb{R}^d$ with a Lipschitz continuous boundary $\Gamma$ and let $\Gamma_1$ be a measurable part of $\Gamma$ such that $\operatorname{meas}(\Gamma_1) > 0$. In Sections 5 and 6 of this paper we use the standard notation for Sobolev and Lebesgue spaces associated to a bounded domain $\Omega \subset \mathbb{R}^d (d = 1, 2, 3)$, with a Lipschitz continuous boundary $\Gamma$. In particular, we use the spaces $L^2(\Omega)^d$, $L^2(\Gamma_1)^d$, $L^2(\Gamma_2)^d$, $L^2(\Gamma_3)^d$ and $H^1(\Omega)^d$, endowed with their canonical inner products and associated norms. Moreover, for an element $\mathbf{v} \in H^1(\Omega)^d$ we usually write $\mathbf{v}$ for the trace $\gamma \mathbf{v} \in L^2(\Gamma)^d$ of $\mathbf{v}$ to $\Gamma$. In addition, we consider the following spaces:

\[
\begin{align*}
V &= \{ \mathbf{v} \in H^1(\Omega)^d : \mathbf{v} = 0 \text{ on } \Gamma_1 \}, \\
Q &= \{ \mathbf{\sigma} = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \}.
\end{align*}
\]

The spaces $V$ and $Q$ are real Hilbert spaces endowed with the canonical inner products given by

\[
\begin{align*}
(\mathbf{u}, \mathbf{v})_V &= \int_{\Omega} \mathbf{\varepsilon}(\mathbf{u}) \cdot \mathbf{\varepsilon}(\mathbf{v}) \, dx, \\
(\mathbf{\sigma}, \mathbf{\tau})_Q &= \int_{\Omega} \mathbf{\sigma} \cdot \mathbf{\tau} \, dx. 
\end{align*}
\] (2.12)

Here and below $\mathbf{\varepsilon}$ represents the deformation operator, that is

\[
\mathbf{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} (u_{i,j} + u_{j,i}),
\]
the index that follows a comma denoting the partial derivative with respect to the corresponding component of the spatial variable $x$, i.e., $u_{ij} = \partial u_i / \partial x_j$. The associated norms on these spaces are denoted by $\| \cdot \|_V$ and $\| \cdot \|_Q$, respectively. Recall that the completeness of the space $V$ follows from the assumption $\text{meas}(\Gamma_1) > 0$ which allows the use of Korn’s inequality. Let $\nu = (\nu_i)$ be the outward unit normal at $\Gamma$. For any element $v \in V$, we denote by $v_\nu$ and $v_\tau$ its normal and tangential components on $\Gamma$ given by $v_\nu = v \cdot \nu$ and $v_\tau = v - v_\nu \nu$, respectively. In addition, we recall that the Sobolev trace theorem yields
\[ \|v\|_{L^2(\Gamma_3)^d} \leq c_0 \|v\|_V \quad \text{for all } v \in V, \]
$c_0$ being a positive constant which depends on $\Omega$, $\Gamma_1$ and $\Gamma_3$.

Next, for a regular stress function $\sigma : \Omega \to \mathbb{S}^d$, the following Green’s formula holds:
\[ \int_{\Omega} \sigma \cdot \varepsilon(v) \, dx + \int_{\Omega} \text{Div } \sigma \cdot v \, dx = \int_{\Gamma} \sigma \nu \cdot v \, da \quad \text{for all } v \in H^1(\Omega)^d. \]
(2.14)
Here and below in this paper Div denotes the divergence operator, i.e., $\text{Div } \sigma = (\sigma_{ij,j})$.

Finally, we introduce the space of fourth order tensors defined by
\[ Q_\infty = \{ \mathcal{E} = (e_{ijkl}) \mid e_{ijkl} = e_{jikl} = e_{klij} \in L^\infty(\Omega), \ 1 \leq i, j, k, l \leq d \}. \]
(2.15)
It is a Banach space endowed with the norm
\[ \|\mathcal{E}\|_{Q_\infty} = \max_{0 \leq i,j,k,l \leq d} \|e_{ijkl}\|_{L^\infty(\Omega)}. \]
Moreover it is easy to see that
\[ \|\mathcal{E}\mathcal{T}\|_Q \leq d \|\mathcal{E}\|_{Q_\infty} \|\mathcal{T}\|_Q \quad \text{for all } \mathcal{E} \in Q_\infty, \ \mathcal{T} \in Q. \]
(2.16)
This inequality will be repeatedly used in Sections 5 and 6 to provide the history-dependent feature of the relaxation tensors.

3 Time-dependent inclusions

In this section we state and prove existence and uniqueness results for time-dependent inclusions with nonlinear operators and, in particular, with history-dependent operators. The functional framework is the following: besides the Hilbert space $X$ we consider a real Hilbert space $Y$ endowed with the inner product $(\cdot, \cdot)_Y$ and the associated norm $\| \cdot \|_Y$. We denote by $Y \times X$ the product space of $Y$ and $X$, endowed with the inner product product $(\cdot, \cdot)_{Y \times X}$ and the associated norm $\| \cdot \|_{Y \times X}$. Moreover, we assume the following.

$(K) \quad K \subset X$ is a nonempty closed convex cone (and, therefore, $0_X \in K$).
Example 3.1. Consider the operator $\mathcal{R} : C(I; X) \to C(I; X)$ defined by
\[
\mathcal{R} u(t) = e^t u(t) + \int_0^t s u(s) \, ds \quad \text{for all } u \in C(I; X), \ t \in I.
\]
Then it is easy to see that $\mathcal{R}$ satisfies condition $(\mathcal{R})$ with
\[
\mathcal{R}_J^r = \max_{t \in J} e^t \quad \text{and} \quad \mathcal{R}_J^l = \max_{t \in J} t.
\]
In addition, note that $\mathcal{R}$ is not an almost history-dependent operator.
Example 3.2. Let \( j : Y \times K \to \mathbb{R} \) be the function defined by \( j(\eta, v) = p(\eta)q(v) \), where \( p : Y \to \mathbb{R} \) and \( q : K \to \mathbb{R} \). Assume that \( p \) is a Lipschitz continuous function with Lipschitz constant \( L_1 \) and \( q \) is a convex positively homogeneous Lipschitz continuous function with Lipschitz constant \( L_2 \). Then, it is easy to see that \( j \) satisfies condition \((j)\) with \( \alpha_j = L_1 L_2 \).

We now extend the function \( j \) from \( Y \times K \) to the whole product space \( Y \times X \) by introducing the function \( J : Y \times X \to (-\infty, +\infty] \) defined by

\[
J(\eta, v) = \begin{cases} j(\eta, v) & \text{if } v \in K, \\ +\infty & \text{if } v \notin K \end{cases} \quad \forall \eta \in Y.
\]

(3.1)

Using assumptions \((K)\) and \((j)\) it is easy to see that for any \( \eta \in Y \), \( J(\eta, \cdot) \) is proper, positively homogenous, convex, lower semicontinuous and, moreover, \( J(\eta, 0_X) = 0 \). Denote by \( C(\eta) \) the subdifferential of \( J(\eta, \cdot) \) in \( 0_X \), i.e.,

\[
C(\eta) = \partial J(\eta, 0_X) = \{ \xi \in X : J(\eta, v) \geq (\xi, v)_X \ \forall v \in X \}
\]

(3.2)

and, for any \( t \in I \), let

\[
C(\eta, t) = f(t) - C(\eta).
\]

(3.3)

Note that, using assumptions \((K)\), \((j)\) and \((f)\) it follows that for any \( \eta \in X \) and \( t \in I \) the set \( C(\eta, t) \) is a nonempty closed convex subset of \( X \).

With these notation, the inclusion problem we consider in this section is the following.

**Problem 1.** Find a function \( u : I \to X \) such that

\[
-u(t) \in N_{C(R_u(t), t)}(Au(t) + S_u(t)) \quad \forall t \in I.
\]

(3.4)

In the study of Problem 1 we have the following existence and uniqueness result.

**Theorem 3.3.** Assume \((K)-(f)\) and, moreover, assume that for any compact set \( J \subset I \) the following smallness assumption holds:

\[
(\alpha_j + 1)(l_F^R + l_F^S) < m_A.
\]

(3.5)

Then, Problem 1 has a unique solution with regularity \( u \in C(I; K) \).

Before providing the proof of Theorem 3.3 we start with a preliminary result which will repeatedly used in Sections 5 and 6 of this paper.

**Lemma 3.4.** Let \( X, Y \) be Hilbert spaces and assume that \((K)\) and \((j)(a)\) hold. Moreover, let \( f : I \to X \), \( \eta \in Y \), \( u, z \in X \), \( t \in I \) and let \( J, C(\eta), C(\eta, t) \) be given by \((3.1)\), \((3.2)\) and \((3.3)\), respectively. Then, the following equivalence holds:

\[
u \in K, \quad j(\eta, v) - j(\eta, u) \geq (f(t) - z, v - u)_X \ \forall v \in K \iff -u \in N_{C(\eta, t)}(z).
\]

(3.6)
Proof. Using (3.1) and the definition of the subdifferential have the equivalences
\[ u \in K, \quad j(\eta, v) - j(\eta, u) \geq (f(t) - z, v - u)_X \quad \forall v \in K \]
\[ \iff J(\eta, v) - J(\eta, u) \geq (f(t) - z, v - u)_X \quad \forall v \in X \]
\[ \iff f(t) - z \in \partial J(\eta, u) \]
and, therefore, (2.2) yields
\[ u \in K, \quad j(\eta, v) - j(\eta, u) \geq (f(t) - z, v - u)_X \quad \forall v \in K \tag{3.7} \]
\[ \iff u \in \partial J^*(\eta, f(t) - z). \tag{3.8} \]

On the other hand, assumption (j) guarantees that \( J(\eta, \cdot) : X \to (-\infty, +\infty] \) is positively homogenous with \( J(\eta, 0_X) = 0 \) and, therefore, \( J(\eta, \cdot) = \partial J^*(\eta, \cdot) \) which implies that \( J^*(\eta, \cdot) \) is \( C(\eta)(\cdot) \)–\( \theta \). It follows from here that \( \partial J^*(\eta, \cdot) = N_C(\eta)(\cdot) \). We use this equality to see that
\[ u \in \partial J^*(\eta, f(t) - z) \iff u \in N_C(\eta)(f(t) - z). \tag{3.9} \]

Finally, using (2.4) and (2.5) we deduce that
\[ u \in N_C(\eta)(f(t) - z) = N_{C(\eta)}(f(t) - z) \iff -u \in N_{f(t) - C(\eta)}(z) \tag{3.10} \]

We now combine the equivalences (3.7)–(3.9), then we use notation (3.3) to deduce that (3.6) holds, which concludes the proof.

We now return back to the proof of Theorem 3.3 which is carried out in several steps that we describe in what follows. To this end, everywhere below we assume that \((K)\)–(f) and (3.3) hold. The first step of the proof is the following.

Lemma 3.5. For any \( \theta = (\eta, \xi) \in C(I; Y \times X) \) there exists a unique function \( u_\theta \in C(I; K) \) such that
\[ -u_\theta(t) \in N_{C(\eta(t), t)}(Au_\theta(t) + \xi(t)) \quad \forall t \in I. \tag{3.11} \]

Moreover, if \( u_i \in C(I; K) \) represents the solution of inclusion (3.11) for \( \theta_i = (\xi_i, \eta_i) \in C(I; Y \times X), i = 1, 2, \) then
\[ \|u_1(t) - u_2(t)\|_X \leq \frac{1}{m_A} (\alpha_j \|\eta_1(t) - \eta_2(t)\|_Y + \|\xi_1(t) - \xi_2(t)\|_X) \quad \forall t \in I. \tag{3.12} \]

Proof. Let \( \theta = (\eta, \xi) \in C(I; Y \times X) \). We use Lemma 3.4 to see that the time-dependent inclusion (3.11) is equivalent with the problem of finding a function \( u_\theta : I \to X \) such that
\[ u_\theta(t) \in K, \quad j(\eta(t), v) - j(\eta(t), u_\theta(t)) \geq (f(t) - Au_\theta(t) - \xi(t), v - u_\theta(t))_X \tag{3.13} \]
\[ \forall v \in K, \ t \in I. \]
We claim that this time-dependent variational inequality has a unique solution \( u_\theta \in C(I; K) \). To this end we consider an arbitrary element \( t \in I \) be fixed. Then, using assumptions \((K), (A), (j)\) it follows from Theorem 2.1 that there exists a unique element \( u_\theta(t) \) which solves (3.12). Now, let us prove that the map \( t \mapsto u_\theta(t) : I \to K \) is continuous. For this, consider \( t_1, t_2 \in I \) and, for the sake of simplicity in writing, denote \( \eta(t_i) = \eta_i, \xi(t_i) = \xi_i, u_\theta(t_i) = u_i, f(t_i) = f_i \) for \( i = 1, 2 \). Using (3.12) we obtain

\[
\begin{align*}
u_1 &\in K, \quad j(\eta_1, v) - j(\eta_1, u_1) \geq (f_1 - Au_1 - \xi_1, v - u_1)_X \quad \forall v \in K, \quad (3.13) \\
u_2 &\in K, \quad j(\eta_2, v) - j(\eta_2, u_2) \geq (f_2 - Au_2 - \xi_2, v - u_2)_X \quad \forall v \in K. \quad (3.14)
\end{align*}
\]

Taking \( v = u_2 \) in (3.13), \( v = u_1 \) in (3.14) and adding the resulting inequalities yields

\[
(Au_1 - Au_2, u_1 - u_2)_X
\]

\[
\leq j(\eta_1, u_2) - j(\eta_1, u_1) + j(\eta_2, u_1) - j(\eta_2, u_2)
\]

\[
+ (\xi_1 - \xi_2, u_1 - u_2)_X + (f_1 - f_2, u_1 - u_2)_X.
\]

Then, using assumptions \((A) \) and \((j)(b)\), we obtain

\[
m_A \|u_1 - u_2\|_X \leq \alpha_j \|\eta_1 - \eta_2\|_Y + \|\xi_1 - \xi_2\|_X + \|f_1 - f_2\|_X. \quad (3.16)
\]

Inequality (3.16) combined with assumption \((f)\) implies that \( t \mapsto u_\theta(t) : I \to K \) is a continuous function. This concludes the existence part of the claim. The uniqueness part is a direct consequence of the uniqueness of the solution \( u_\theta(t) \) to the inequality (3.12), at each \( t \in I \), guaranteed by Theorem 2.1.

Assume now that if \( u_i \in C(I; K) \) represents the solution of inequality (3.12) for \( \theta_i = (\xi_i, \eta_i) \in C(I; Y \times X) \), \( i = 1, 2 \). Then, arguments similar to those used in the proof of inequality (3.16) show that (3.11) holds. Lemma 3.5 is now a direct conclusion of the equivalence between inclusion (3.10) and the inequality (3.12), as already mentioned at the beginning of the proof.

Next, we consider the operator \( \Lambda : C(I; Y \times X) \to C(I; Y \times X) \) defined by

\[
\Lambda \theta = (R u_\theta, S u_\theta) \quad \forall \theta \in C(I; Y \times X). \quad (3.17)
\]

We have the following result.

**Lemma 3.6.** The operator \( \Lambda \) has a unique fixed point \( \theta^* = (\eta^*, \xi^*) \in C(I; Y \times X) \).

**Proof.** Let \( \theta_1 = (\eta_1, \xi_1), \theta_2 = (\eta_2, \xi_2) \in C(I; Y \times X) \) and denote by \( u_i \) the solution of the variational inequality (3.12) for \( \theta = \theta_i \), i.e., \( u_i = u_{\theta_i} \), \( i = 1, 2 \). Let \( J \) be a compact subset of \( I \) and \( t \in J \). Then, using (3.17) and assumptions \((R) \) and \((S) \) on the operators \( R \) and \( S \) yields

\[
\|\Lambda \theta_1(t) - \Lambda \theta_2(t)\|_{Y \times X} \leq \|R u_1(t) - R u_2(t)\|_Y + \|S u_1(t) - S u_2(t)\|_X
\]

\[
\leq (L_R^J + L_S^J)\|u_1(t) - u_2(t)\|_X + (L_R^J + L_S^J) \int_0^t \|u_1(s) - u_2(s)\|_X ds.
\]

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This inequality combined with inequality (3.11) and the elementary inequalities \( \| \eta \|_Y \leq \| \theta \|_{Y \times X}, \| \xi \|_X \leq \| \theta \|_Y \times X \), valid for all \( \theta = (\eta, \xi) \in Y \times X \), implies that

\[
\| \Lambda \theta_1 (t) - \Lambda \theta_2 (t) \|_{Y \times X} \leq \frac{(\alpha_j + 1)(l^R_J + l^S_J)}{m_A} \| \theta_1 (t) - \theta_2 (t) \|_{Y \times X} + \frac{(\alpha_j + 1)(L^R_J + L^S_J)}{m_A} \int_0^t \| \theta_1 (s) - \theta_2 (s) \|_{Y \times X} \, ds.
\]

We now use the smallness assumption (3.5) to obtain that the operator \( \Lambda \) is an almost history-dependent operator, see Definition 2.2 (b). Finally, we apply Theorem 2.3 to conclude the proof of the lemma.

We are now in a position to provide the proof of Theorem 3.3.

Proof. Let \( \theta^* = (\eta^*, \xi^*) \in C(I; Y \times X) \) be the fixed point of the operator \( \Lambda \) and let \( u^* = u_{\theta^*} \in C(I; K) \) be the solution of the intermediate problem (3.10) for \( \theta = \theta^* \). Then, using equality \( \theta^* = \Lambda \theta^* \) we find that \( \eta^* = R u^* \) and \( \xi^* = S u^* \). We now use these equalities in (3.10) to see that \( u^* \) is a solution to Problem 1. This proves the existence part in Theorem 3.3. The uniqueness part is a consequence of the uniqueness of the fixed point of the operator \( \Lambda \), guaranteed by Lemma 3.6.

We end this sections with some consequence of Theorem 3.3 which are relevant for the applications we present in Section 5 of this paper.

**Corollary 3.7.** Assume \((K), (A), (j), (f)\) and, moreover, assume that \(R : C(I; X) \to C(I; Y)\) and \(S : C(I; X) \to C(I; X)\) are history-dependent operators. Then, Problem 1 has a unique solution with regularity \( u \in C(I; K) \).

Proof. Definition 2.2 (a) shows that in this case conditions \((R)\) and \((S)\) are satisfied with \(l^R_J = l^S_J = 0\) and, therefore, the smallness condition (3.5) is satisfied. Corollary 3.7 is now a direct consequence of Theorem 3.3.

**Corollary 3.8.** Assume \((K), (A), (f)\) and, moreover, assume that \(S : C(I; X) \to C(I; X)\) is a history-dependent operator. In addition, assume that \(j\) satisfies condition \((j)\) with \(Y = X\) and

\[
\alpha_j + 1 < m_A. \tag{3.18}
\]

Then, there exists a unique function \( u \in C(I; K) \) such that

\[
- u(t) \in N_{C(u(t), t)}(Au(t) + Su(t)) \quad \forall t \in I. \tag{3.19}
\]

Proof. We take \(Ru = u\) for all \( u \in C(I; X) \). Then, using Definition 2.2 (a) we see that in this case conditions \((R)\) and \((S)\) are satisfied with \(l^R_J = 1\) and \(l^S_J = 0\), respectively. Therefore, (3.18) implies that the smallness condition (3.5) holds, too. Corollary 3.8 is now a direct consequence of Theorem 3.3.
We now consider the particular case when the function \( j : K \to \mathbb{R} \). In this case we define the function \( J : X \to (-\infty, +\infty] \) and the sets \( C, C(t) \subseteq H \) by equalities

\[
J(v) = \begin{cases} 
  j(v) & \text{if } v \in K, \\
  +\infty & \text{if } v \notin K,
\end{cases}
\]

\[ C = \partial J(0_X), \quad C(t) = f(t) - C \quad \forall t \in I. \tag{3.21} \]

With these notation, we have the following result which, clearly, represents a direct consequence of Theorem 3.3.

**Corollary 3.9.** Assume \((K), (A), (f)\) and, moreover, assume that \( S : C(I; X) \to C(I; X) \) is a history-dependent operator. In addition, assume that \( j : K \to \mathbb{R} \) is a convex positively homogenous Lipschitz continuous function. Then, there exists a unique function \( u \in C(I; K) \) such that

\[-u(t) \in N_{C(I)}(Au(t) + Su(t)) \quad \forall t \in I.\]

Corollary 3.9 will be used in Section 4 in the study of a frictionless unilateral contact problem.

### 4 Sweeping processes

In this section we use Theorem 3.3 and its consequences in order to obtain existence and uniqueness results for several sweeping processes. To this end, besides the data \( K, A, R, S, j \) and \( f \) introduced in the previous section, we consider an operator \( B \) and an initial data \( u_0 \) such that

\[(B) \quad B : X \to X \text{ is a Lipschitz continuous operator.}\]

\[(u_0) \quad u_0 \in X.\]

We start by considering the following sweeping process.

**Problem 2.** Find a function \( u : I \to X \) such that

\[
-\dot{u}(t) \in N_{C(\mathbb{R}\dot{u}(t),t)}(A\dot{u}(t) + Bu(t) + S\dot{u}(t)) \quad \forall t \in I, \tag{4.1}
\]

\[ u(0) = u_0. \tag{4.2} \]

Our first result in this section is the following.

**Theorem 4.1.** Assume \((K)-(f), (B), (u_0)\) and, moreover, assume that \((3.5)\) holds. Then, Problem 2 has a unique solution with regularity \( u \in C^1(I; X) \) and \( \dot{u} \in C(I; K) \).
Proof. We introduce the operator \( \tilde{S} : C(I; X) \to C(I; X) \) defined by

\[
\tilde{S}v(t) = B \left( \int_0^t v(s) \, ds + u_0 \right) + S v(t)
\]

(4.3)

for all \( t \in I, \, v \in C(I; X) \), then we consider the auxiliary problem of finding a function \( v : I \to X \) such that

\[
- v(t) \in N_{C(Rv(t),t)}(Av(t) + \tilde{S}v(t)) \quad \forall t \in I.
\]

(4.4)

Let \( L_B \) be the Lipschitz constant of the operator \( B \). We use assumptions (\( S \)) and (\( B \)) to see that for any compact set \( J \subset I \), any functions \( v_1, v_2 \in C(I; X) \) and any \( t \in I \), the inequality below holds:

\[
\| \tilde{S}v_1(t) - \tilde{S}v_2(t) \|_X \leq l^S_J \| v_1(t) - v_2(t) \|_X \\
+ (L_B + L^S_J) \int_0^t \| v_1(s) - v_2(s) \|_X \, ds.
\]

It follows from here that the operator \( \tilde{S} \) satisfies condition (\( S \)) with \( l^S_J = l^S_J \). Therefore, we are in a position to apply Theorem 3.3 in order to obtain the existence of a unique function \( v \in C(I; K) \) which satisfies the time-dependent inclusion (4.4). Denote by \( u : I \to X \) the function defined by

\[
u(t) = \int_0^t v(s) + u_0 \quad \forall t \in I.
\]

(4.5)

Then, (4.3)–(4.5) and assumption (\( u_0 \)) imply that \( u \) is a solution of Problem 2 with regularity \( u \in C^1(I; X) \) and \( \dot{u} \in C(I; K) \). This proves the existence part of the theorem. The uniqueness part follows from the unique solvability of the auxiliary problem (4.4), guaranteed by Theorem 3.3.

Theorem 4.1 can be used in the study of various versions of sweeping process of the form (4.1) and (4.2). We provide below some consequence of this theorem in the study of three relevant examples.

Corollary 4.2. Assume (\( K \)), (\( A \)), (\( j \)), (\( f \)), (\( B \)), (\( u_0 \)) and, moreover, assume that \( R : C(I; X) \to C(I; Y) \) and \( S : C(I; X) \to C(I; X) \) are history-dependent operators. Then, Problem 2 has a unique solution with regularity \( u \in C^1(I; X) \) and \( \dot{u} \in C(I; K) \).

Proof. Definition 2.2 (a) shows that in this case conditions (\( R \)) and (\( S \)) are satisfied with \( l^R_J = l^S_J = 0 \) and, therefore, the smallness condition (3.5) is satisfied. Corollary 3.9 is now a direct consequence of Theorem 4.1.
Corollary 4.3. Assume \((K), (A), (f), (B), (u_0)\) and, moreover, assume that \(S : C(I; X) \rightarrow C(I; X)\) is a history-dependent operator. In addition, assume that \(j\) satisfies condition \((j)\) with \(Y = X\). Then, there exists a unique function \(u \in C^1(I; X)\) such that

\[
-\dot{u}(t) \in N_{C(u(t), t)}(A \dot{u}(t) + Bu(t) + Su(t)) \quad \forall t \in I,
\]

\[u(0) = u_0.\]

Moreover, \(\dot{u} \in C(I; K)\).

Proof. Consider the operator \(R : C(I; X) \rightarrow C(I; X)\) defined by equality

\[
Rv(t) = \int_0^t v(s) \, ds + u_0 \quad \forall v \in C(I; V), \ t \in I.
\]

Then, using Definition 2.2 (a) we see that in this case conditions \((R)\) and \((S)\) are satisfied with \(l^R_j = 0\) and \(l^S_j = 0\), respectively. Therefore, the smallness condition \((3.5)\) is satisfied. Moreover, \(R \dot{u} = u\) for all \(u \in C(I; X)\). Corollary 4.3 is now a direct consequence of Corollary 4.2. \(\Box\)

Corollary 4.4. Assume \((K), (A), (f), (B), (u_0)\), and, moreover, assume that \(S : C(I; X) \rightarrow C(I; X)\) is a history-dependent operator. In addition, assume that \(j\) satisfies condition \((j)\) with \(Y = X\). Then, there exists a unique function \(u \in C(I; K)\) such that

\[
-\dot{u}(t) \in N_{C(u(t), t)}(A \dot{u}(t) + Bu(t) + Su(t)) \quad \forall t \in I,
\]

\[u(0) = u_0.\]

Moreover, \(\dot{u} \in C(I; K)\).

Proof. Consider the operator \(\tilde{S} : C(I; X) \rightarrow C(I; X)\) defined by equality

\[
\tilde{S}v(t) = S\left(\int_0^t v(s) \, ds + u_0\right) \quad \forall v \in C(I; V), \ t \in I.
\]

Then, using Definition 2.2 (a) it is easy to see that \(\tilde{S}\) is a history-dependent operator and, moreover, \(\tilde{S} \dot{u} = Su\) for all \(u \in C^1(I; X)\). Corollary 4.4 is now a direct consequence of Corollary 4.3. \(\Box\)

5 Two frictionless contact problems

The physical setting, already considered in many papers and surveys, can be resumed as follows. A deformable body occupies, in its reference configuration, a bounded
domain $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$), with a Lipschitz continuous boundary $\Gamma$, divided into three measurable disjoint parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, such that $\text{meas}(\Gamma_1) > 0$. The body is fixed on $\Gamma_1$, is acted upon by given surface tractions on $\Gamma_2$, and is in contact with an obstacle on $\Gamma_3$. The equilibrium of the body in this physical setting can be described by various mathematical models, obtained by using different mechanical assumptions. The first contact model we consider in this section is based on specific constitutive law and interface boundary conditions which will be described below. Its statement is as follows.

**Problem 3.** Find a displacement field $u : \Omega \times I \to \mathbb{R}^d$ and a stress field $\sigma : \Omega \times I \to S^d$ such that

\begin{align*}
\sigma(t) &= A\varepsilon(u(t)) + \int_0^t B(t-s)\varepsilon(u(s)) \, ds \quad \text{in } \Omega, \quad (5.1) \\
\text{Div } \sigma(t) + f_0(t) &= 0 \quad \text{in } \Omega, \quad (5.2) \\
u(t) &= 0 \quad \text{on } \Gamma_1, \quad (5.3) \\
\sigma(t)\nu &= f_2(t) \quad \text{on } \Gamma_2, \quad (5.4) \\
-F\left(\int_0^t u_{\nu}^+(s) \, ds\right) &\leq \sigma_{\nu}(t) \leq 0, \quad \text{on } \Gamma_3, \quad (5.5) \\
-\sigma_{\nu}(t) &= \begin{cases} 
0 & \text{if } u_{\nu}(t) < 0, \\
F\left(\int_0^t u_{\nu}^+(s) \, ds\right) & \text{if } u_{\nu}(t) > 0,
\end{cases} \quad \text{on } \Gamma_3, \quad (5.6)
\end{align*}

for all $t \in I$.

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $x \in \Omega \cup \Gamma$. Moreover, we use the notation introduced in Section 2 and, in addition, $\sigma_{\nu}$ and $\sigma_{\tau}$ denote the normal and tangential stress on $\Gamma$, that is $\sigma_{\nu} = (\sigma\nu) \cdot \nu$ and $\sigma_{\tau} = \sigma\nu - \sigma_{\nu}\nu$. We now provide a short description of the equations and boundary conditions in Problem 3.

First, equation (5.1) represents the constitutive law in which $A$ is the elasticity operator, assumed to be nonlinear, and $B$ represents the relaxation tensor. Next, equation (5.2) is the equation of equilibrium in which $f_0$ represents the density of the body forces, assumed to be time-dependent. Condition (5.3) represents the displacement boundary condition which shows that the body is fixed on the part $\Gamma_1$ of its boundary, during the process. Condition (5.4) represents the traction condition which shows that surface tractions of density $f_2$, assumed to be time-dependent, act...
on $\Gamma_2$. Condition (5.5) models the contact with a rigid-deformable body with memory effects. Here $F$ is a positive function and $r^+$ represents the positive part of $r$, i.e., $r^+ = \max \{r, 0\}$. Details on this condition can be found in [22, Ch.9]. Finally, condition (5.6) represents the frictionless contact condition. It shows that the friction force, $\sigma$, vanishes during the process. This is an idealization of the process, since even completely lubricated surfaces generate shear resistance to tangential motion. However, this condition is a sufficiently good approximation of the reality in some situations, especially when the contact surfaces are lubricated.

In the study of the mechanical problem (5.1)–(5.6) we assume that the elasticity operator $A$ satisfies the following conditions.

\[ \begin{align*}
(a) & \quad A : \Omega \times S^d \to S^d. \\
(b) & \quad \text{There exists } L_A > 0 \text{ such that} \\
& \quad \|A(x, \epsilon_1) - A(x, \epsilon_2)\| \leq L_A \|\epsilon_1 - \epsilon_2\| \\
& \quad \forall \epsilon_1, \epsilon_2 \in S^d, \text{ a.e. } x \in \Omega. \\
(c) & \quad \text{There exists } m_A > 0 \text{ such that} \\
& \quad (A(x, \epsilon_1) - A(x, \epsilon_2)) \cdot (\epsilon_1 - \epsilon_2) \geq m_A \|\epsilon_1 - \epsilon_2\|^2 \\
& \quad \forall \epsilon_1, \epsilon_2 \in S^d, \text{ a.e. } x \in \Omega. \\
(d) & \quad \text{The mapping } x \mapsto A(x, \epsilon) \text{ is measurable on } \Omega, \\
& \quad \text{for any } \epsilon \in S^d. \\
(e) & \quad A(x, 0) = 0 \text{ a.e. } x \in \Omega.
\end{align*} \]

We also assume that the relaxation tensor $B$ and the densities of body forces and surface tractions are such that

\[ \begin{align*}
B & \in C(I; Q_{\infty}). \\
f_0 & \in C(I; L^2(\Omega^d)). \\
f_2 & \in C(I; L^2(\Gamma_2^d)).
\end{align*} \]

Finally, the memory surface function $F$ satisfies:

\[ \begin{align*}
(a) & \quad \text{There exists } L_F > 0 \text{ such that} \\
& \quad |F(x, r_1) - F(x, r_2)| \leq L_F |r_1 - r_2| \\
& \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3, \\
(b) & \quad \text{The mapping } x \mapsto F(x, r) \text{ is measurable on } \Gamma_3 \text{ for any } r \in \mathbb{R}, \\
(c) & \quad F(x, 0) = 0 \text{ a.e. } x \in \Gamma_3.
\end{align*} \]

We now turn to the variational formulation of Problem 3 and, to this end, we assume in what follows that $(u, \sigma)$ represents a couple of regular functions which
satisfies (5.1)–(5.6). Then, using standard arguments based on the Green formula (2.14) we find that

\[
\int_{\Omega} \sigma(t) \cdot (\varepsilon(v) - \varepsilon(u(t))) \, dx \\
+ \int_{\Gamma_3} F\left( \int_0^t u^+_\nu(s) \, ds \right) (v^+_\nu - u^+_\nu(t)) \, da + \\
\geq \int_{\Omega} f_0(t) \cdot (v - u(t)) \, dx + \int_{\Gamma_2} f_2(t) \cdot (v - u(t)) \, da
\]

for all \( v \in V \) and every \( t \in I \). Recall that here and in the rest of the paper we use the function spaces \( V \) and \( Q \) introduced in Section 2. We now consider the operators

\[ A: V \to V, \quad R: C(I;V) \to C(I,L^2(\Gamma_3)), \quad S: C(I;V) \to C(I;V), \]

the functional \( j: L^2(\Gamma_3) \times V \to \mathbb{R} \) and the function \( f: I \to V \) defined by

\[
(Au,v)_V = \int_{\Omega} A\varepsilon(u) \cdot \varepsilon(v) \, dx \quad \text{for all } u, v \in V, \tag{5.13}
\]

\[
R u(t) = F\left( \int_0^t u^+_\nu(s) \, ds \right) \quad \text{for all } u \in C(I;V), \tag{5.14}
\]

\[
(Su(t),\varepsilon(v))_V = (\int_0^t B(t-s)\varepsilon(u(s)) \, ds, \varepsilon(v))_Q \tag{5.15}
\]

\[
\text{for all } u, v \in C(I;V),
\]

\[
j(\eta, v) = \int_{\Gamma_3} \eta v^+_\nu \, da \quad \text{for all } \eta \in L^2(\Gamma_3), \ v \in V, \tag{5.16}
\]

\[
(f(t),v)_V = \int_{\Omega} f_0(t) \cdot v \, dx + \int_{\Gamma_2} f_2(t) \cdot v \, da \quad \text{for all } v \in V, \ t \in I. \tag{5.17}
\]

We now substitute equation (5.1) in (5.12), then we use notation (5.13)–(5.17) to see that

\[
j(Ru(t),v) - j(Ru(t),u(t)) \geq (f(t) - Au(t) - Su(t), v - u(t))_V \tag{5.18}
\]

for all \( v \in V \) and every \( t \in I \). Let

\[
C(\eta) = \partial j(\eta,0), \quad C(\eta,t) = f(t) - C(\eta) \quad \text{for all } \eta \in L^2(\Gamma_3), \ t \in I.
\]

We take \( X = V, \ K = V \) and note that in this case condition \((K)\) is satisfied. Moreover, taking \( Y = L^2(\Gamma_3) \) and using the trace inequality (2.13) it is easy to see that condition \((j)(a)\) is satisfied, too. Therefore, from inequality (5.18) and Lemma 3.1 with \( J = j \), we derive the following variational formulation of Problem 2.

**Problem 4.** Find a displacement field \( u: I \to V \) such that

\[
- u(t) \in N_{C(Ru(t),t)}(Au(t) + Su(t)) \quad \forall t \in I. \tag{5.19}
\]
In the study of Problem 4 we have the following existence and uniqueness result.

**Theorem 5.1.** Assume that (5.7)–(5.11) hold. Then Problem 4 has a unique solution \( u \in C(I; V) \).

**Proof.** We use Corollary 3.7 on the spaces \( X = V, Y = L^2(\Gamma_3) \), with \( K = V \). As already mentioned, assumptions \((K)\) and \((j)(a)\) are obviously satisfied. Moreover, using (5.16) and (2.13) it is easy to see that for any \( \eta_1, \eta_2 \in L^2(\Gamma_3) \) and any \( u_1, u_2 \in V \) we have

\[
    j(\eta_1, u_2) - j(\eta_1, u_1) - j(\eta_2, u_1) \leq c_0 \| \eta_1 - \eta_2 \|_{L^2(\Gamma_3)} \| u_1 - u_2 \|_V,
\]

which implies that function \( j \) satisfies condition \((j)(b)\) with \( \alpha_j = c_0 \). On the other hand, assumption (5.7) implies that for any \( u, v \in V \) the inequalities below hold:

\[
    (Au - Av, u - v)_V \geq m_A \| u - v \|_V^2,
\]

\[
    \| Au - Av \|_V \leq L_A \| u - v \|_V.
\]

We conclude from here that condition \((A)\) is satisfied. Next, we use assumptions (5.11), (5.8) and inequalities (2.13), (2.16) to see that for any compact \( J \), any functions \( u_1, u_2 \) and any \( t \in J \) we have

\[
    \| Ru_1(t) - Ru_2(t) \|_{L^2(\Gamma_3)} \leq c_0 L_F \int_0^t \| u_1(s) - u_2(s) \|_Y ds,
\]

\[
    \| Su_1(t) - Su_2(t) \|_V \leq d \max_{s \in J} \| B(s) \|_{Q_\infty} \int_0^t \| u_1(s) - u_2(s) \|_V ds,
\]

which prove that the operators \( R \) and \( S \) are history-dependent operators. Finally, the regularities (5.9) and (5.10) imply that \( f \in C(I; V) \) and, therefore, condition \((f)\) holds, too. Theorem 5.1 is now direct consequence of Corollary 3.7. 

A second viscoelastic contact problem for which the abstract results provided in Section 3 work is the Signorini frictionless contact problem, which models the contact with a perfectly rigid foundation. The statement of this problem is the following.

**Problem 5.** Find a displacement field \( u: \Omega \times I \rightarrow \mathbb{R}^d \) and a stress field \( \sigma: \Omega \times I \rightarrow \mathbb{S}^d \) such that (5.1)–(5.4), (5.6) hold for all \( t \in I \) and, moreover,

\[
    u_\nu(t) \leq 0, \; \sigma_\nu(t) \leq 0, \; \sigma_\nu(t) u_\nu(t) = 0 \quad (5.20)
\]

for all \( t \in I \).

We assume conditions (5.7)–(5.10) and use notation (5.13), (5.15) and (5.17). Moreover, we consider the set \( U \) and the function \( j: U \rightarrow \mathbb{R} \) defined by

\[
    U = \{ v \in V : v_\nu \leq 0 \text{ a.e. on } \Gamma_3 \}, \quad (5.21)
\]

\[
    j(v) = 0 \quad \forall v \in U. \quad (5.22)
\]
Note that in this case the function \( j \) does not depend on the first variable and, therefore, using notations (3.20), (3.21) with \( X = V \), \( K = U \) we deduce that \( J = I_U \), \( C = N_U(0_V) \) and \( C(t) = f(t) - N_U(0_V) \) for all \( t \in I \). Then, using arguments similar to those used in the study of Problem 3, based on the Green formula and Lemma 3.4 we derive the following variational formulation of Problem 5.

**Problem 6.** Find a displacement field \( u : I \to V \) such that
\[
- u(t) \in N_{C(t)}(Au(t) + Su(t)) \quad \forall t \in I.
\]
(5.23)

In the study of Problem 6 we have the following existence and uniqueness result.

**Theorem 5.2.** Assume that (5.7)–(5.10) hold. Then Problem 6 has a unique solution \( u \in C(I; U) \).

**Proof.** The proof of Theorem 5.2 is a direct consequence of Corollary 3.9. It is based on arguments similar to those used in the proof of Theorem 5.1 and, for this reason, we skip the details.

6 A frictional viscoelastic contact problem

For the model we consider in this section the contact is frictional. As a consequence, its variational formulation leads to a sweeping process in which the unknown is the displacement field. The model is formulated as follows.

**Problem 7.** Find a displacement field \( u : \Omega \times I \to \mathbb{R}^d \) and a stress field \( \sigma : \Omega \times I \to \mathbb{S}^d \) such that
\[
\sigma(t) = A\varepsilon(\dot{u}(t)) + \mathcal{E}(\varepsilon(u(t)) + \int_0^t B(t - s)\varepsilon(\dot{u}(s)) ds \quad \text{in } \Omega,
\]
\[
\text{Div } \sigma(t) + f_0(t) = 0 \quad \text{in } \Omega,
\]
\[
u(t) = 0 \quad \text{on } \Gamma_1,
\]
\[
\sigma(t)\nu = f_2(t) \quad \text{on } \Gamma_2,
\]
\[
u(t) = 0 \quad \text{on } \Gamma_3
\]
\[
\left\|
\begin{array}{c}
\sigma_r(t) \\
-\sigma_r(t)
\end{array}
\right\| \leq F\left( \int_0^t \left\|
\begin{array}{c}
\dot{u}_r(t) \\
\dot{u}_r(t)
\end{array}
\right\| ds \right), \quad \text{on } \Gamma_3
\]
\[
-\sigma_r(t) = F\left( \int_0^t \left\|
\begin{array}{c}
\dot{u}_r(t) \\
\dot{u}_r(t)
\end{array}
\right\| ds \right) \frac{\dot{u}_r(t)}{\left\|
\begin{array}{c}
\dot{u}_r(t) \\
\dot{u}_r(t)
\end{array}
\right\|} \quad \text{if } \dot{u}_r(t) \neq 0
\]
for all \( t \in I \) and, moreover,
\[
u(0) = u_0 \quad \text{in } \Omega.
\]
(6.7)
The equations and boundary conditions in Problem 7 have a similar meaning to those in Problems 3 and 5 studied in the previous section. Note that (6.1) represents the constitutive law in which now \( A \) represents the viscosity operator, \( E \) is the elasticity operator and, again, \( B \) represents the relaxation tensor. Condition (6.5) represents the bilateral contact condition; it shows that there is no separation between the body and the foundation, during the process. Condition (6.6) represents a total-slip version of Coulomb’s law of dry friction. Here \( F \) denotes the friction bound and the quantity

\[
T(x, t) = \int_0^t \|\dot{u}_\tau(x, s)\| ds
\]

represents the total slip-rate in the point \( x \in \Gamma_3 \), at the time moment \( t \in I \). Considering a friction bound \( F \) which depends on the total slip rate describes the rearrangement of the contact surfaces during the sliding process. Finally, condition (6.6) represents the initial condition in which \( u_0 \) denotes a given initial displacement field.

The weak solution of the mechanical problem (5.1)–(5.6) will be sought in the space

\[
V_1 = \{ v \in V : v_\nu = 0 \text{ on } \Gamma_3 \}.
\]

Note that \( V_1 \) is a closed subspace of the space \( V \) and, therefore, is a Hilbert space equipped with the inner product \((\cdot, \cdot)_V\) and the associated norm \(\| \cdot \|_V\).

In the study of the mechanical problem (6.1)–(6.7) we assume that the viscosity operator \( A \) and the relaxation tensor satisfy conditions (5.7) and (5.8), respectively. Moreover, the density of applied forces and the friction bound are such that (5.9), (5.10) and (5.11), hold. Finally, for the elasticity operator and the initial displacement we assume that

\[
\begin{align*}
(a) & \quad E : \Omega \times S^d \to S^d. \\
(b) & \quad \text{There exists } L_E > 0 \text{ such that } \\
& \quad \|E(x, \varepsilon_1) - E(x, \varepsilon_2)\| \leq L_E \|\varepsilon_1 - \varepsilon_2\| \\
& \quad \forall \varepsilon_1, \varepsilon_2 \in S^d, \text{ a.e. } x \in \Omega. \\
(c) & \quad \text{The mapping } x \mapsto E(x, \varepsilon) \text{ is measurable on } \Omega, \\
& \quad \text{for any } \varepsilon \in S^d. \\
(d) & \quad E(x, 0) = 0 \text{ a.e. } x \in \Omega.
\end{align*}
\]

(6.8)

(6.9)

We now turn to the variational formulation of Problem 7 and, to this end, we assume in what follows that \((u, \sigma)\) represents a couple of regular functions which satisfies (6.1)–(6.7). Then, using standard arguments based on the Green formula (2.14) we find that
\begin{equation}
\int_{\Omega} \sigma(t) \cdot (\varepsilon(v) - \varepsilon(\dot{u}(t)))
\end{equation}

\begin{equation}
+ \int_{\Gamma_3} F \left( \int_0^t \| \dot{u}_r(s) \| ds \right) (\| v_r(s) \| - \| \dot{u}_r(s) \|) \, da
\end{equation}

\begin{equation}
\geq \int_{\Omega} f_0(t) \cdot (v - \dot{u}(t)) \, dx + \int_{\Gamma_2} f_2(t) \cdot (v - \dot{u}(t)) \, da
\end{equation}

for all \( v \in V_1 \) and every \( t \in I \). We now introduce the operators \( A : V_1 \to V_1, \) \( B : V_1 \to V_1, \) \( \mathcal{R} : C(I; V_1) \to C(I; L^2(\Gamma_3)), \) \( S : C(I; V_1) \to C(I; V_1), \) the functional \( j : L^2(\Gamma_3) \times V_1 \to \mathbb{R} \) and the function \( f : I \to V_1 \) defined by

\begin{equation}
(Au, v)_V = \int_{\Omega} A\varepsilon(u) \cdot \varepsilon(v) \, dx \quad \text{for all } u, v \in V_1,
\end{equation}

\begin{equation}
(Bu, v)_V = \int_{\Omega} B\varepsilon(u) \cdot \varepsilon(v) \, dx \quad \text{for all } u, v \in V_1,
\end{equation}

\begin{equation}
\mathcal{R} u(t) = F \left( \int_0^t \| u_r(t) \| ds \right) \quad \text{for all } u \in C(I; V_1),
\end{equation}

\begin{equation}
(Su(t), v)_V = (\int_0^t B(t - s)\varepsilon(u(s)) \, ds, \varepsilon(v))_Q
\end{equation}

\begin{equation}
\text{for all } u, v \in C(I; V_1),
\end{equation}

\begin{equation}
j(\eta, v) = \int_{\Gamma_3} \eta \| v_r(t) \| \, da \quad \text{for all } \eta \in L^2(\Gamma_3), \ v \in V_1,
\end{equation}

\begin{equation}
(f(t), v)_V = \int_{\Omega} f_0(t) \cdot v \, dx + \int_{\Gamma_2} f_2(t) \cdot v \, da \quad \text{for all } v \in V_1, \ t \in I.
\end{equation}

We now substitute equation (6.11) in (6.10), then we use notation (6.11) - (6.16) to see that

\begin{equation}
j(\mathcal{R}\dot{u}(t), v) - j(\mathcal{R}\dot{u}(t), \dot{u}(t)) \geq (f(t) - A\dot{u}(t) - Bu(t) - S\dot{u}(t), v - \dot{u}(t))_V
\end{equation}

for all \( v \in V_1 \) and \( t \in I \). Let

\begin{equation}
C(\eta) = \partial j(\eta, 0_V), \quad C(\eta, t) = f(t) - C(\eta) \quad \text{for all } \eta \in L^2(\Gamma_3), \ t \in I.
\end{equation}

Take \( X = V_1, \ K = V_1 \) and note that in this case conditions (K) and \( (j)(a) \) are satisfied, the later one being the consequence of the trace inequality (2.13). Then, using inequality (6.17), Lemma 3.4 with \( J = j \) and the initial condition (6.7), we derive the following variational formulation of Problem 7.

**Problem 8.** Find a displacement field \( u : I \to V_1 \) such that

\begin{equation}
-\dot{u}(t) \in N_{C(\mathcal{R}\dot{u}(t), t)}(A\dot{u}(t) + Bu(t) + S\dot{u}(t)) \quad \forall t \in I,
\end{equation}

\begin{equation}
u(0) = u_0.
\end{equation}
In the study of Problem 8 we have the following existence and uniqueness result.

**Theorem 6.1.** Assume that \((5.7)-(5.11), (6.8), (6.9)\) hold. Then Problem 8 has a unique solution \(u \in C^1(I; V_1)\).

**Proof.** We use Corollary 4.2 on the spaces \(X = V_1, Y = L^2(\Gamma_3)\), with \(K = V_1\). As already mentioned, assumptions \((K)\) and \((j)(a)\) are obviously satisfied. Moreover, it follows from arguments similar to those in the proof of Theorem 5.1 that assumptions \((j)(b), (A), (f)\) hold too, and the operators \(R\) and \(S\) are history-dependent operators. In addition, assumptions \((6.8)\) and \((6.9)\) guarantee that conditions \((B)\) and \((u_0)\) are satisfied. It follows from above that we are in a position to apply Corollary 4.2 to conclude the proof. \(\Box\)

7  Concluding remarks

Using tools from convex analysis and fixed points theory, we obtained existence and uniqueness results for a class of time-dependent inclusions in Hilbert spaces. These results were used to provide the unique solvability of a new class of Moreau’s first order sweeping processes with constraints in velocity. Our results are of interest in the study of quasistatic mathematical models of contact with deformable bodies. Two frictionless and a frictional viscoelastic contact problems were introduced in order to illustrate these abstract results. Nevertheless, several questions and problems still remain open and need to be investigated in the future. One of these questions is the following: is the smallness condition \((3.5)\) an intrinsic condition in the study of Problem 1 or it is only a mathematical tool? An open problem is to extend our results in the case when the data has an \(L^p\)-regularity, with \(p \in [1, +\infty]\). Note that, in this case, there is a need to replace the fixed point Theorem 2.3 with an appropriate \(L^p\)-version. The study of second-order evolutionary sweeping processes would be a valuable extension of the result of this paper. In addition, problems related to the optimal control of time-dependent inclusions and sweeping processes of the form \((3.4)\) and \((4.1)\), respectively, represent a topic which deserves to be addressed in the future. All these issues would open the way to important applications in Contact Mechanics.

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