SUB-EXPONENTIALLY MANY 3-COLORINGS OF TRIANGLE-FREE PLANAR GRAPHS

Arash Asadi
Luke Postle\textsuperscript{1}
Robin Thomas\textsuperscript{2}

School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332-0160, USA

ABSTRACT

Thomassen conjectured that every triangle-free planar graph on \( n \) vertices has exponentially many 3-colorings, and proved that it has at least \( 2^{n^{1/12}/20000} \) distinct 3-colorings. We show that it has at least \( 2^{\sqrt{n/362}} \) distinct 3-colorings.

April 2010, revised 8 July 2010. An extended abstract of this paper appeared in [1].

\textsuperscript{1}Partially supported by an NSF Graduate Research Fellowship.
\textsuperscript{2}Partially supported by NSF under Grant No. DMS-0701077.
1 Introduction

All graphs in this paper are finite, and have no loops or multiple edges. Our terminology is standard, and may be found in [2] or [3]. In particular, cycles and paths have no repeated vertices. The following is a well-known theorem of Grötzsch [7].

Theorem 1.1 Every triangle-free planar graph is 3-colorable.

Theorem 1.1 has been the subject of extensive research. Thomassen [11] gave several short proofs [11, 12, 13] of Grötzsch’s theorem and extended it to projective planar and toroidal graphs. The theorem does not extend verbatim to any non-planar surface, but Thomassen proved that every graph of girth at least five embedded in the projective plane or the torus is 3-colorable. Gimbel and Thomassen [6] found an elegant characterization of 3-colorability for triangle-free projective planar graphs. There does not seem to be a corresponding counterpart for other surfaces, but Kráľ’ and Thomas [9] found a characterization of 3-colorability for toroidal and Klein bottle graphs that are embedded with all faces even. It was an open question for a while whether a 3-coloring of a triangle-free planar graph can be found in linear time. First Kowalik [8] designed an almost linear time algorithm, and then a linear-time algorithm was found by Dvořák, Kawarabayashi and Thomas in [4]. For a general surface Σ, Dvořák, Kráľ’ and Thomas [5] found a linear-time algorithm to decide whether a triangle-free graph in Σ is 3-colorable.

In this paper we study how many 3-colorings a triangle-free planar graph must have. Thomassen conjectured in [15] that exponentially many:

Conjecture 1.2 There exists an absolute constant $c > 0$, such that if $G$ is a triangle-free planar graph on $n$ vertices, then $G$ has at least $2^{cn}$ distinct 3-colorings.

Thomassen gave a short proof of this conjecture under the additional hypothesis that $G$ has girth at least five. We use that argument in Lemma 2.3 below; Thomassen’s original proof may be recovered by taking $F$ to be the set of all facial cycles. Thomassen [15] then extended this result by showing that every planar graph of girth at least five has exponentially many list-colorings for every list assignment that gives each vertex a list of size at least three. For triangle-free graphs Thomassen [15] proved a weaker version of Conjecture 1.2, namely that every triangle-free planar graph on $n$ vertices has at least $2^{n^{7/12}/20000}$ distinct 3-colorings. Our main result is the following improvement.

Theorem 1.3 Every triangle-free planar graph on $n$ vertices has at least $2^{\sqrt{n}/362}$ distinct 3-colorings.

In closely related work Thomassen [14] proved that every (not necessarily triangle-free) planar graph has exponentially many list colorings provided every vertex has at least five available colors.
Our paper is organized as follows. In the next section we investigate non-crossing families of 5-cycles, and reduce Theorem 1.3 to Lemma 2.4, which states that if a triangle-free planar graph has \( k \) nested 5-cycles, then it has at least \( 2^{k/12} \) 3-colorings. The rest of the paper is devoted to a proof of Lemma 2.4, which we complete in Section 4. In Section 3 we prove an auxiliary result stating that some entries in the product of certain matrices grow exponentially in the number of matrices.

We end this section by stating two useful theorems of Thomassen [11].

**Theorem 1.4** Let \( G \) be a plane graph of girth at least five and \( C = v_1v_2\ldots v_k \) be an induced facial cycle of \( G \) of length \( k \leq 9 \). Then a 3-coloring \( \Phi \) of \( C \) extends to a 3-coloring of \( G \), unless \( k = 9 \) and there exists a vertex \( v \in V(G) - V(C) \) such that \( v \) is adjacent to three vertices of \( C \) that received three different colors under \( \Phi \).

**Theorem 1.5** Let \( G \) be a triangle-free plane graph with facial cycle \( C \) of length at most five. Then every 3-coloring of \( C \) extends to a 3-coloring of \( G \).

We would like to acknowledge that an extended abstract of this paper appeared in [1].

## 2 Laminar Families of 5-Cycles

First we define some terminology. Let \( A \) and \( B \) be two subsets of \( \mathbb{R}^2 \). We say that \( A \) and \( B \) cross if \( A \cap B, A \cap B^c, A^c \cap B, A^c \cap B^c \) are all non-null. Then we say that a family \( \mathcal{F} \) of subsets of \( \mathbb{R}^2 \) is laminar if for every two sets \( A, B \in \mathcal{F} \), \( A \) and \( B \) do not cross. Now let \( G \) be a plane graph and \( C \) be a cycle in \( G \). Then we let \( \text{Int}(C) \) denote the bounded component of \( \mathbb{R}^2 - C \) and \( \text{Ext}(C) \) denote the unbounded component of \( \mathbb{R}^2 - C \). Now we say that a family \( \mathcal{F} \) of cycles of \( G \) is laminar if the corresponding family of sets \( \bigcup_{C \in \mathcal{F}} \text{Int}(C) \) is laminar. We call a family \( \mathcal{F} \) of cycles an antichain if \( \text{Int}(C_1) \cap \text{Int}(C_2) = \emptyset \) for every distinct \( C_1, C_2 \in \mathcal{F} \), and we call it a chain if for every two cycles \( C_1, C_2 \in \mathcal{F} \), either \( \text{Int}(C_1) \subseteq \text{Int}(C_2) \) or \( \text{Int}(C_2) \subseteq \text{Int}(C_1) \).

Let \( G \) be a triangle-free plane graph, and let \( v \in V(G) \). We define \( G_v \) to be the graph obtained from \( G \) by deleting \( v \), identifying all the neighbors of \( v \) to one vertex, and deleting resulting parallel edges. We also let \( D_k(G) \) denote the set of vertices of \( G \) with degree at most \( k \).

**Lemma 2.1** If \( G \) is a triangle-free plane graph and \( k \geq 0 \) is an integer, then either

(i) there exists \( v \in D_k(G) \) such that \( G_v \) is triangle-free or,

(ii) there exists a laminar family \( \mathcal{F} \) of 5-cycles such that every \( v \in D_k(G) \) belongs to some member of \( \mathcal{F} \).
Proof. We proceed by induction on the number of vertices of $G$. Suppose condition (i) does not hold. Notice that if $v \in V(G)$ and $G_v$ is not triangle-free, this implies, since $G$ is triangle-free, that $v$ is in a 5-cycle in $G$. Hence if condition (i) does not hold, every $v \in D_k(G)$ must be in a 5-cycle in $G$.

Now suppose there does not exist a separating 5-cycle in $G$. Then we let $F$ be the set of all 5-cycles in $G$. The second condition then holds since the absence of separating cycles implies that $F$ is laminar.

Thus we may assume that there exists a 5-cycle $C$ that separates $G$ into two triangle-free plane graphs $G_1$ and $G_2$, where both $G_1$ and $G_2$ include $C$. By induction, the lemma holds for $G_1$ and $G_2$. Suppose that both $G_1$ and $G_2$ satisfy condition (ii) with laminar families $F_1$ and $F_2$, respectively. Then let $F = F_1 \cup F_2$. Note that $F$ is laminar. Now $G$ satisfies condition (ii) since every $v \in D_k(G)$ is contained in either $D_k(G_1)$ or $D_k(G_2)$. Thus we may assume without loss of generality that $G_1$ satisfies condition (i). That is, there exists $v \in D_k(G_1)$ such that $(G_1)_v$ is triangle-free. This implies that $v$ is not in a 5-cycle in $G_1$. In particular, $v \notin V(C)$, and hence $v \in D_k(G)$. Yet since $G_v$ is not triangle-free by assumption, $v$ must be in a 5-cycle in $G$, say $C'$. It follows that $C'$ intersects $C$. Since $G$ is triangle-free, $C$ and $C'$ intersect in exactly two vertices $u_1$ and $u_2$. Now the path from $u_1$ to $u_2$ along $C'$ that includes $v$ must have $t$ edges, where $t \in \{2, 3\}$. But then there is another path from $u_1$ to $u_2$ along $C$ with $5 - t$ edges. Hence $v$ is in a 5-cycle in $G_1$, a contradiction. □

Lemma 2.2 If $G$ is a triangle-free plane graph on $n$ vertices, then either

(i) there exists $v \in D_k(G)$ such that $G_v$ is triangle-free, or

(ii) $G$ has an antichain $F$ of 5-cycles such that $|F| \geq \sqrt{\frac{(k-3)n}{10(k-1)}}$, or

(iii) $G$ has a chain $F$ of 5-cycles such that $|F| \geq \sqrt{\frac{2(k-3)n}{5(k-1)}}$.

Proof. Since $G$ is triangle-free and planar, it satisfies $2|V(G)| \geq |E(G)|$. We may assume that (i) does not hold and hence every vertex of $G$ has degree at least two. It follows that

$$4|V(G)| \geq 2|E(G)| = \sum_{v \in V(G)} \deg(v) \geq (k + 1) (|V(G)| - |D_k(G)|) + 2|D_k(G)|,$$

and hence $|D_k(G)| \geq \frac{k-3}{k-1}|V(G)|$. Since (i) does not hold, we deduce from Lemma 2.1 that there exists a laminar family of 5-cycles $\mathcal{G}$ of size at least $|D_k(G)|/5 \geq \frac{k-3}{5(k-1)}n$. By Dilworth’s theorem applied to the partial order on $\mathcal{G}$ defined by $Int(C_1) \subseteq Int(C_2)$ we deduce that $\mathcal{G}$ has either an antichain of size at least $\sqrt{|\mathcal{G}|}/2$, in which case condition (ii) holds, or a chain of size at least $\sqrt{2|\mathcal{G}|}$, in which case condition (iii) holds. □
Lemma 2.3 Let $G$ be a triangle-free plane graph. If $G$ has an antichain $\mathcal{F}$ of 5-cycles, then $G$ has at least $2^{\lfloor |\mathcal{F}|/6 \rfloor}$ distinct 3-colorings.

Proof. Let $G'$ be obtained from $G$ by deleting the vertices in $\bigcup_{C \in \mathcal{F}} \text{Int}(C)$. Now $G'$ has at least $|\mathcal{F}|$ facial 5-cycles. By Euler's formula $|E(G')| \leq 2|V(G')| - |\mathcal{F}|/2$. By Theorem 1.1 the graph $G'$ has a 3-coloring $\Phi$. For $i, j \in \{1, 2, 3\}$ with $i < j$ let $G_{ij}$ denote the subgraph of $G$ induced by the vertices colored $i$ or $j$. Since $\sum_{i < j}(|V(G_{ij})| - |E(G_{ij})|) = 2|V(G')| - |E(G')| \geq |\mathcal{F}|/2$, there exist $i, j \in \{1, 2, 3\}$ such that $i < j$ and $G_{ij}$ has at least $|\mathcal{F}|/6$ components. But then there are at least $2^{\lfloor |\mathcal{F}|/6 \rfloor}$ distinct 3-colorings of $G'$ since switching the colors on any subset of the components of $G_{ij}$ gives rise to a distinct coloring of $G'$. Furthermore, every 3-coloring of $G'$ extends to a 3-coloring of $G$ by Theorem 1.5. □

Lemma 2.4 Let $G$ be a triangle-free plane graph. If $G$ has a chain $\mathcal{F}$ of 5-cycles, then $G$ has at least $24 \cdot 2^{\lfloor |\mathcal{F}|/12 \rfloor}$ distinct 3-colorings.

We will prove Lemma 2.4 in Section 4, but now we deduce the main theorem from it.

Proof of Theorem 1.3, assuming Lemma 2.4. We proceed by induction on the number of vertices. If $n \leq 362$, then the conclusion clearly holds. We may therefore assume that $n \geq 363$ and that the theorem holds for all graphs on fewer than $n$ vertices. If there exists $v \in D_{363}(G)$ such that the graph $G_v$ (defined prior to Lemma 2.1) is triangle-free, then by induction $G_v$ has at least $2\sqrt{(n - \deg(v))/362}$ distinct 3-colorings. Hence $G$ has at least $2 \times 2\sqrt{(n - \deg(v))/362}$ distinct 3-colorings, which is greater than $2\sqrt{n/362}$ since $\deg(v) \leq 363$. So we may assume by Lemma 2.2 applied to $k = 363$ that $G$ has either an antichain of 5-cycles of size at least $\sqrt{36n/362}$, in which case the theorem holds by Lemma 2.3; or a chain of 5-cycles of size at least $\sqrt{720n/1810}$, in which case the theorem holds by Lemma 2.4. □

3 Two matrix lemmas

Let the matrices $A_1, A_2$ be defined by

$$A_1 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0
\end{bmatrix}.$$

By a cyclic permutation matrix we mean a permutation matrix such that the corresponding permutation is cyclic. Let $A$ and $B$ be two $5 \times 5$ matrices. We say that $A$ majorizes $B$ if every entry in $A$ is greater than or equal to the corresponding entry of $B$. We say that $A$ dominates $B$ if there exist cyclic permutation matrices $P, Q$ such that $A$ majorizes $PBQ$.  

5

Lemma 3.1 Let \( n \geq 0 \) be an integer, and let \( M_1, \ldots, M_n \) be \( 5 \times 5 \) matrices such that each dominates the matrix \( A_1 \). Then \( M_1 M_2 \ldots M_n \) dominates the matrix \( A_1^{\lceil \frac{n}{5} \rceil} \).

**Proof.** We may assume that \( M_1 M_2 \ldots M_n = A_1 P_1 A_1 P_2 \ldots A_1 P_n \), where \( P_1, \ldots, P_n \) are cyclic permutation matrices. Let \( m = \lceil \frac{n}{5} \rceil \). Since there are only five \( 5 \times 5 \) cyclic permutation matrices and the product of two cyclic permutation matrices is a cyclic permutation matrix, we deduce that there exist integers \( 1 \leq i_1 < i_2 < \cdots < i_m \leq n \) such that the matrices \( P_{i_j} P_{i_j+1} \ldots P_{i_j+m-1} \) are equal for all \( j = 1, \ldots, m \). It follows that for all \( j = 1, \ldots, m-1 \) the matrix \( P_{i_j} P_{i_j+1} \ldots P_{i_j+m-1} \) is the identity matrix, and hence the matrix \( B_j = P_{i_j} A_1 P_{i_j+1} \ldots A_1 P_{i_j+m-1} \) majorizes the identity matrix. Let \( B_0 = A_1 P_1 A_1 P_2 \ldots A_1 P_{i_1-1} \) and \( B_m = P_{i_m} A_1 P_{i_m+1} \ldots A_1 P_n \). Then \( M_1 M_2 \ldots M_n = B_0 A_1 B_1 A_2 \ldots A_1 B_{m-1} A_1 B_m \).

Now, \( B_0 \) majorizes some cyclic permutation, and so does \( B_m \), and \( A_1 B_1 A_2 \ldots A_1 B_{m-1} A_1 \) majorizes \( A_1^m \), as desired. \( \square \)

We denote the vector of all ones by \( \mathbf{1} \).

Lemma 3.2 Let \( n \geq 2 \) be an integer, and let \( M_1, M_2, \ldots, M_{n-1} \) be \( 5 \times 5 \) matrices with non-negative entries such that each of them dominates \( A_1 \) or \( A_2 \). Let \( M = M_1 M_2 \cdots M_{n-1} \), and let \( \mathbf{1}^T M = (x_0, x_1, \ldots, x_4) \). Then there exist four distinct indices \( 0 \leq i, j, k, l \leq 4 \) such that \( \min \{ x_i, x_j \} \cdot \min \{ x_k, x_l \} \geq 2^{n/6} \).

**Proof.** We prove the lemma by induction on \( n \). If \( n = 2 \) then we may assume that \( M_1 = A_1 \) or \( M_1 = A_2 \), and hence \( x_0, x_1 \geq 2, x_2, x_3, x_4 \geq 1 \), and hence the lemma holds. We may therefore assume that \( n \geq 3 \), and that the lemma holds for all smaller values of \( n \). If \( M_i \) dominates \( A_1 \) for all \( i = 1, 2, \ldots, n-1 \), then by Lemma 3.1 the product \( M_1 \cdots M_{n-1} \) dominates \( A_1^{\lceil n/5 \rceil} \). Thus there exist distinct indices \( i, j = 0, 1, \ldots, 4 \) such that \( \min \{ x_i, x_j \} \geq 2^{\lceil n/5 \rceil} \). Let \( k, l \in \{ 0, 1, \ldots, 4 \} - \{ i, j \} \) be distinct. Then \( x_k, x_l \geq 1 \), and hence the indices \( i, j, k, l \) satisfy the conclusion of the lemma. This completes the case when each \( M_i \) dominates \( A_1 \).

So we may select the largest integer \( p \in \{ 1, 2, \ldots, n-1 \} \) such that \( M_p \) dominates \( A_2 \).

Without loss of generality we may assume that \( M_p = A_2 \). Let \( \mathbf{1}^T (M_1 \cdots M_{p-1}) = (y_0, \ldots, y_4) \).

Since \( p < n \), the induction hypothesis implies that there exist four distinct indices \( 0 \leq i, j, k, l \leq 4 \) such that \( \min \{ y_i, y_j \} \cdot \min \{ y_k, y_l \} \geq 2^{p/6} \). Without loss of generality we may assume that \( 0 \leq i \leq 1 \).

We first dispose of the case \( p = n - 1 \). Then \( x = (x_0, \ldots, x_4) = (y_0, \ldots, y_4) M_p \).

Thus \( x_0 = x_1 \geq y_k + y_l \geq 2 \min \{ y_k, y_l \} \) and \( x_2 = x_3 = x_4 \geq y_i \). Therefore,

\[
\min \{ x_0, x_1 \} \cdot \min \{ x_2, x_3 \} \geq 2 \min \{ y_i, y_j \} \cdot \min \{ y_k, y_l \} \geq 2 \cdot 2^{(n-1)/6} \geq 2^{n/6},
\]

as desired. This completes the case \( p = n - 1 \).
We may therefore assume that \( p < n - 1 \). The choice of \( p \) implies that \( M_{p+1}, M_{p+2}, \ldots, M_{n-1} \) all dominate \( A_1 \). Let \( B = M_{p+1}M_{p+2} \ldots M_{n-1} \) and let \( u = \sum_{i=0}^{4} y_i \) and let \( v = y_0 + y_1 \). Thus \((x_0, \ldots, x_4) = (y_0, \ldots, y_4)M_pB = (u, u, v, v, v)B\).

By Lemma 3.1, \( B \) dominates \( A_1^{\lceil (n-p-1)/5 \rceil} \). Thus there exist distinct indices \( s, t \in \{0, 1, \ldots, 4\} \) and distinct indices \( s', t' \in \{0, 1, \ldots, 4\} \) such that \( b := \min\{B_{ss'}, B_{st'}, B_{ts'}, B_{tt'}\} \geq 2^{\lceil (n-p-1)/5 \rceil - 1} \). It follows that \( x_{s'}, x_{t'} \geq 2bv \), and if \( \{s, t\} \cap \{0, 1\} \neq \emptyset \), then \( x_{s'}, x_{t'} \geq bu \).

Recall that the matrix \( B \) dominates \( A_1^{\lceil (n-p-1)/5 \rceil} \). If \( \{s, t\} \cap \{0, 1\} = \emptyset \), then there exist distinct indices \( r, r' \in \{0, 1, \ldots, 4\} \) such that \( B_{1r}, B_{2r'} \geq 1 \), and hence \( x_r, x_{r'} \geq u \).

If \( \{s, t\} \cap \{0, 1\} \neq \emptyset \), then we select \( r, r' \in \{0, 1, \ldots, 4\} \) arbitarily; in that case \( x_r, x_{r'} \geq v \).

The results of the previous two paragraphs imply that \( \min\{x_{s'}, x_{t'}\} \cdot \min\{x_r, x_{r'}\} \geq bvu \). But \( v \geq y_i \geq \min\{y_i, y_j\} \) and \( u \geq y_k + y_l \geq 2\min\{y_k, y_l\} \). We conclude that

\[
\min\{x_{s'}, x_{t'}\} \cdot \min\{x_r, x_{r'}\} \geq bvu \geq 2b\min\{y_i, y_j\} \cdot \min\{y_k, y_l\} \geq 2^{p/6 + [(n-p-1)/5]} \geq 2^{n/6},
\]

as desired. \( \square \)

4 Chains of 5-Cycles

In order to prove Lemma 2.4, we will first characterize how the 3-colorings of an outer 5-cycle of a plane graph \( G \) extend to the 3-colorings of another 5-cycle. If \( C \) is a 5-cycle in a graph \( G \) and \( \Phi \) a 3-coloring of \( C \), then there exists a unique vertex \( v \in V(C) \) such that \( v \) is the only vertex of \( C \) colored \( \Phi(v) \). We call such a vertex the special vertex of \( C \) for \( \Phi \). Let \( G \) be a triangle-free plane graph and \( C_1, C_2 \) be 5-cycles in \( G \) such that \( C_1 \neq C_2 \) and \( \text{Int}(C_2) \subseteq \text{Int}(C_1) \). Let us choose a fixed orientation of the plane, and let \( C_1 := u_1 \ldots u_5, C_2 := v_1 \ldots v_5 \) be both numbered in clockwise order. Then we define a color transition matrix \( M \) of \( G \) with respect to \( C_1 \) and \( C_2 \) as follows. Let \( G' \) be the subgraph of \( G \) consisting of all the vertices and edges of \( G \) drawn in the closed annulus bounded by \( C_1 \cup C_2 \). We let \( M_{ij} \) equal one sixth the number of 3-colorings \( \Phi \) of \( G' \) such that \( u_i \) is the special vertex of \( C_1 \) for \( \Phi \) and \( v_j \) is the special vertex of \( C_2 \) for \( \Phi \). The following lemma is straightforward.

**Lemma 4.1** Let \( G \) be a triangle-free graph and \( \mathcal{F} = \{C_1, \ldots, C_n\} \) be a family of 5-cycles such that \( \text{Int}(C_i) \supseteq \text{Int}(C_j) \) if \( 1 \leq i < j \leq n \). Let \( M_i \) be a color transition matrix of \( G \) with respect to \( C_i \) and \( C_{i+1} \). Then \( M_1M_2 \ldots M_{n-1} \) is a color transition matrix of \( G \) with respect to \( C_1 \) and \( C_n \).

Let us recall that the matrices \( A_1, A_2 \) were defined at the beginning of Section 3.

**Lemma 4.2** Let \( G \) be a graph isomorphic to one of the graphs shown in Fig. 1. If \( C_1, C_2 \) are the two cycles of \( G \) shown in Fig. 1, and \( M \) is a color transition matrix of \( G \) with respect to \( C_1 \) and \( C_2 \), then \( M \) dominates either \( A_1 \) or \( A_2 \).
Proof. Let $M_i$ be a color transition matrix of $B_i$ with respect to $C_1, C_2$, where $1 \leq i \leq 4$. Determining the various valid colorings of the $B_i$'s gives the following matrices up to cyclic permutations of rows and columns:

$$M_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$ 

Therefore, $M_1, M_3$ and $M_4$ dominate $A_1$ and $M_2$ dominates $A_2$. □

In the rest of the paper, we call the graphs shown in Figure 1 basic graphs.

Definition 4.3 Let $G$ be a triangle-free plane graph and $C_1, C_2$ be two distinct 5-cycles in $G$. Let $i \in \{1, 2\}$, and let $C_i = v_1w_1w_2w_3w_4$. Further suppose that there exist two vertices of degree three $w_5, w_6 \in V(G) - V(C_i)$ and three facial 5-cycles distinct from $C_1$ and $C_2$: $v_1v_4w_6w_3w_4, v_1v_3w_5w_2w_1$, and $v_2w_5w_2w_3w_6$. Finally suppose that either $v_1$ has degree four and does not belong to $C_3-i$, or that $v_2$ has degree three and does not belong to $C_3-i$. Then we say that $G$ has an $H$-structure around $C_i$, or simply an $H$-structure. An illustration is shown in Fig. 2.

We denote the $5 \times 5$ matrix of all ones by $J$. If $G$ is a graph and $X$ is a vertex or a set of vertices of $G$, then we denote by $G\setminus X$ the graph obtained from $G$ by deleting $X$.

Lemma 4.4 Let $G$ be a triangle-free plane graph and let $C_1, C_2$ be two distinct 5-cycles in $G$. If $G$ has an $H$-structure and every 4-cycle and every 6-cycle in $G$ separates $C_1$ from $C_2$, then every color transition matrix of $G$ with respect to $C_1, C_2$ dominates the matrix $J$.

Proof. Let $G$ have an $H$-structure around $C_2$ with its vertices labeled as in Definition 4.3. Let $W := \{w_1, w_2, \ldots, w_6\}$. We will need the following claim.
(1) Every 3-coloring of $C_1$ can be extended to a 3-coloring of $G \setminus W$ such that $v_1$ and $v_2$ are colored the same.

Let us first deduce the lemma from the claim. It is straightforward to verify that for every $v \in V(C_2)$, every 3-coloring of $G \setminus W$ in which $v_1$ and $v_2$ are colored the same color can be extended to a 3-coloring $\Phi$ of $G$ such that $v$ is the special vertex of $C$ for $\Phi$. This and (1) imply the conclusion of the lemma.

Thus it remains to prove (1). To that end assume first that $v_1$ has degree four and does not belong to $C_1$. Let $G'$ be the graph obtained from $G \setminus W \setminus v_1$ by adding the edges $v_2v_3$ and $v_2v_4$. We claim that $G'$ is triangle-free. Indeed, otherwise either $v_2$ and $v_3$ have a common neighbor, or $v_2$ and $v_4$ have a common neighbor in $G \setminus W \setminus v_1$. From the symmetry we may assume that $v_2$ and $v_3$ have a common neighbor $z \not\in W \cup \{v_1\}$. But then either $zv_3w_5v_2$ is a 4-cycle in $G$ which does not separate $C_1$ from $C_2$ or $zv_2w_6v_4v_1v_3$ is a 6-cycle in $G$ which does not separate $C_1$ from $C_2$, a contradiction in either case. Thus $G'$ is triangle-free, and hence every 3-coloring of $C_1$ extends to a 3-coloring $\Phi$ of $G'$ by Theorem 1.5. By letting $\Phi(v_1) := \Phi(v_2)$ we obtain a coloring desired for (1).

We may therefore assume that $v_2$ has degree three and does not belong to $C_1$. Let $v_5 \neq w_5, w_6$ be the third neighbor of $v_2$. Notice that $v_5 \neq v_1$ as then there would be a 4-cycle which does not separate $C_1$ from $C_2$. Moreover, $v_5 \neq v_3, v_4$ as $G$ is triangle-free. Now let $G_1$ be the graph obtained from $G \setminus W \setminus v_2$ by identifying $v_3$ and $v_5$. Similarly, let $G_2$ be the graph obtained from $G \setminus W \setminus v_2$ by identifying $v_4$ and $v_5$.

Now we claim that at least one of the graphs $G_1, G_2$ is triangle-free. To prove this claim, suppose that $G_1$ and $G_2$ are not triangle-free. Since $G_1$ is not triangle-free, there exists a path $v_5z_1z_2v_3$. Moreover, the 6-cycle $v_5z_1z_2v_3w_5v_2$ must separate $C_1$ and $C_2$. Furthermore, $z_1 \neq v_4$ as otherwise $v_5v_2w_6v_4$ is a 4-cycle not separating $C_1$ from $C_2$. Also $z_1 \neq v_1$ and $z_2 \neq v_4$ since $G$ is triangle-free. Similarly $z_2 \neq v_1$ as otherwise $v_1v_4w_6v_2v_5z_1$ is a 6-cycle not separating $C_1$ from $C_2$.

Since $G_2$ is not triangle-free, there exists a path $v_5z'_1z'_2v_4$. By a similar reasoning as for $G_1$, we find that $z'_1, z'_2$ are distinct from $v_4$ and $v_1$ and that the 6-cycle $v_5z'_1z'_2v_4w_6v_2$ must separate $C_1$ and $C_2$. Since $G$ is a plane graph, this implies that $\{z_1, z_2\} \cap \{z'_1, z'_2\} \neq \emptyset$. Notice
that \( z_1 \neq z'_1 \) as \( G \) is simple and both 6-cycles described above separate \( C_1 \) from \( C_2 \). If \( z'_1 = z_2 \) or if \( z_1 = z'_2 \), then \( G \) has a triangle, a contradiction. Thus \( z_2 = z'_2 \). But then \( v_4z_2v_3v_1 \) is a 4-cycle which does not separate \( C_1 \) from \( C_2 \), a contradiction. This proves our claim that one of \( G_1, G_2 \), say \( G' \), is triangle-free.

Since \( G' \) is triangle-free, every 3-coloring of \( C_1 \) extends to a 3-coloring \( \Phi \) of \( G' \) by Theorem 1.5. Since one neighbor of \( v_1 \) has the same color as \( v_5 \), we may set \( \Phi(v_2) := \Phi(v_1) \) and thus obtain a coloring desired for (1). This completes the proof of (1), and hence the proof of the lemma. \( \square \)

**Definition 4.5** Let \( G \) be a triangle-free plane graph, let \( C_1, C_2 \) be two 5-cycles in \( G \), and let \( f \) be a face of \( G \) bounded by a 5-cycle \( C_3 \), where \( C_1, C_2, C_3 \) are pairwise distinct. We say that \( f \) is a **good face** if one of the following conditions hold:

1. At least four vertices of \( C_3 \) have degree three and \( E(C_3) \cap (E(C_1) \cup E(C_2)) = \emptyset \), or
2. all five vertices of \( C_3 \) have degree three and either \( E(C_3) \cap E(C_1) = \emptyset \) or \( E(C_3) \cap E(C_2) = \emptyset \).

If the first condition holds, then we say that \( f \) is a **good face of the first kind**, and if the second condition holds, then we say that \( f \) is a **good face of the second kind**.

**Lemma 4.6** Let \( G \) be a triangle-free plane graph and \( C_1, C_2 \) be two distinct 5-cycles in \( G \). Assume that all vertices of \( G \) of degree two are on \( C_1 \) and \( C_2 \); for every integer \( k \in \{4, 6, 7\} \) every \( k \)-cycle in \( G \) separates \( C_1 \) from \( C_2 \); every 5-cycle of \( G \) bounds a face; and every face in \( G \) is bounded by a cycle of length five. If \( G \) has a good face, then there exists a triangle-free graph \( G' \) and two 5-cycles \( C'_1 \neq C'_2 \) in \( G' \) such that \( |V(G')| < |V(G)| \) and every color transition matrix of \( G \) with respect to \( C_1 \) and \( C_2 \) dominates some color transition matrix of \( G' \) with respect to \( C'_1 \) and \( C'_2 \).

**Proof.** We say that a cycle \( C \) in \( G \) is an **important cycle** if \( C = C_1 \); or \( C = C_2 \); or \( C \) has length four, six or seven; or \( C \) has length nine and no vertex of \( G \) has three or more neighbors on \( C \). Let us assume for a moment that some important cycle \( C \) does not separate \( C_1 \) from \( C_2 \). Then \( C \) has length nine by hypothesis, and hence no vertex of \( G \) has three or more neighbors on \( C \). Let \( G' \) be the subgraph of \( G \) obtained by deleting all vertices and edges drawn in the face of \( C \) that is disjoint from \( C_1 \) and \( C_2 \). Then \( |V(G')| < |V(G)| \), because every face of \( G \) is bounded by a 5-cycle. By Theorem 1.4 every 3-coloring of \( G' \) extends to a 3-coloring of \( G \), and hence \( G' \) satisfies the conclusion of the lemma. Thus we may assume that every important cycle in \( G \) separates \( C_1 \) from \( C_2 \). It follows that

\((*)\) for every subgraph \( H \) of \( G \), at most two facial cycles of \( H \) are important.
Good face of the first kind. Let $f$ be a good face of the first kind bounded by the cycle $C_3 := v_1, \ldots, v_5$ where $v_1, \ldots, v_4$ are vertices of degree three on $C_3$. For $i = 1, 2, 3, 4$ let $w_i$ be the neighbor of $v_i$ which is not on $C_3$. Let $S = \{v_1, v_2, \ldots, v_5, w_1, w_2, w_3, w_4\}$. These vertices are pairwise distinct, because $G$ is triangle-free and has no separating 5-cycles. Similarly, no $v_i$ is adjacent to $w_j$ for $i \neq j$. Finally, we claim that $w_i$ is not adjacent to $w_j$ for $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$. Indeed, this follows similarly if $j = i + 2$, or $i = 1$ and $j = 4$, and so we may assume that $i \in \{1, 2, 3\}$ and $j = i + 1$. But $w_i$ and $w_{i+1}$ have a common neighbor, because the path $w_i v_i v_{i+1} w_{i+1}$ is a subpath of a facial cycle (of length five), and hence $w_i$ and $w_j$ are not adjacent, because $G$ is triangle-free. Thus we have shown that

(0) the vertices of $S$ are pairwise distinct, and the only edges of $G$ with both ends in $S$ are the edges of $C_3$ and the edges $v_i w_i$ for $i = 1, 2, 3, 4$.

Let $G_1$ be the graph obtained from $G$ by deleting the vertices $v_1, \ldots, v_4$, identifying $w_2$ with $w_3$, and adding the edge $w_1 w_4$. Let $G_2$ be the graph obtained from $G$ by deleting the vertices $v_1, \ldots, v_4$, identifying $w_3$ with $v_5$, and identifying $w_1$ with $w_2$. Let $G_3$ be the graph obtained from $G$ by deleting the vertices $v_1, \ldots, v_4$, identifying $w_2$ with $v_5$, and identifying $w_3$ with $w_4$.

We will prove that at least one of the graphs $G_1, G_2$ or $G_3$ is triangle-free. The lemma then follows, for if $G'$ is one of the above three graphs that is triangle-free, then we may assume that $C_1$ and $C_2$ are 5-cycles in $G'$. It is well-known [7] that every 3-coloring of $G'$ can be converted to a 3-coloring of $G$, and hence every color transition matrix of $G'$ with respect to $C_1$ and $C_2$ dominates some color transition matrix of $G$ with respect to $C_1$ and $C_2$, as desired.

Thus it remains to prove the claim that one of $G_1, G_2, G_3$ is triangle-free. To that end we may assume the contrary. From the fact that $G_1$ is not triangle-free, we deduce that there exist vertices $z_1, z_2 \in V(G) - \{v_1, v_2, v_3, v_4\}$ such that either

(1a) $w_1 z_1 w_4 v_4 v_5 v_1$ is a cycle in $G$

or

(1b) $w_2 z_1 z_2 v_3 v_2$ is a cycle in $G$.

In either case, let $C_4$ denote the corresponding cycle. Similarly, the fact that $G_2$ is not triangle-free implies that there exist vertices $x_1, x_2 \in V(G) - \{v_1, v_2, v_3, v_4\}$ such that either

(2a) $w_1 x_1 x_2 w_2 v_2 v_1$ is a cycle in $G$

or

(2b) $v_5 x_1 x_2 w_3 v_3 v_4$ is a cycle in $G$.

In either case, let $C_5$ denote the corresponding cycle. Finally, the fact that $G_3$ is not triangle-free implies that there exist vertices $x_3, x_4 \in V(G) - \{v_1, v_2, v_3, v_4\}$ such that either
(3a) $w_3x_3x_4w_4v_4v_3$ is a cycle in $G$

or

(3b) $v_5x_3x_4w_2v_2v_1$ is a cycle in $G$.

In either case, let $C_6$ denote the corresponding cycle. From (0) we deduce that

(4) \( x_1, x_2, x_3, x_4, z, z_1, z_2 \notin S \).

If (2a) and (3a) hold, then in both cases (1a) and (1b) the subgraph $C_3 \cup C_4 \cup C_5$ has at least three important faces, contrary to (\ast\ast). (This requires some checking. For instance, in case (1b) it is possible that $z_1 = x_4$ or $z_2 = x_1$.) Thus from the symmetry we may assume that (2b) holds.

Assume now that (1a) holds. Then by planarity and (4) either $z_1 = x_1$ or $z_1 = x_2$. In the former case the subgraph of $G$ induced by \{v_1, v_4, v_5, w_1, w_4, x_1\} has three important faces, contrary to (\ast\ast), and in the latter case the 5-cycles $v_5x_1x_2w_4v_4$ and $v_4w_4x_2w_3v_3$ are facial and not equal to $C_1$ or $C_2$ (because each shares an edge with $C_3$), and hence $w_4$ has degree two, contrary to hypothesis. We conclude that (1a) does not hold.

Thus (1b) and (2b) hold. By considering the subgraph $C_3 \cup C_4 \cup C_5$ of $G$ we deduce from (\ast\ast) that $v_5v_1v_2w_2z_1z_2w_3x_2x_1$ is a cycle of length nine, and some vertex of $G$ has at least three neighbors on it. That vertex is $w_1$, and hence $w_1$ has degree three and is adjacent to $v_1, z_1, x_2$. Since every face of $G$ is bounded by a 5-cycle, there is a vertex $u$ such that $w_2v_2v_3w_3u$ is a facial 5-cycle. Then $w_2z_1z_2w_3u$ is also a 5-cycle, and hence also bounds a face. It follows that $u$ has degree two, and so one of the incident faces is $C_1$ or $C_2$; but $C_3$ shares no edges with $C_1$ or $C_2$, and hence we may assume that $C_2 = w_2z_1z_2w_3u$. It follows that $G$ has an $H$-structure around $C_2$, where the degree three vertex $v_1$ plays the role of the vertex $v_2$ from the definition of $H$-structure. Thus the subgraph $C_1 \cup C_2$ of $G$ satisfies the conclusion of the lemma by Lemma 4.4.

Good face of the second kind. Let $f$ now be a good face of the second kind bounded by the cycle $C_3 := v_1v_2\ldots v_5$, where each $v_i$ has degree three. For $i = 1, 2, \ldots, 5$ let $w_i$ be the neighbor of $v_i$ which is not on $C_3$, and let $W := \{v_1, v_2, \ldots, v_5, w_1, w_2, \ldots, w_5\}$. We have the following analogue of (0):

(5) the vertices of $W$ are pairwise distinct, and the only edges of $G$ with both ends in $W$ are the edges of $C_3$ and the edges $v_iw_i$ for $i = 1, 2, 3, 4, 5$.

We may assume that $G$ does not have a good face of the first kind, and hence we may assume from the symmetry that $C_2$ shares an edge with $C_3$. Thus we may assume without loss of generality that $C_2 := v_4v_5w_5zw_4$ for some vertex $z \in V(G)$. Let $G_1$ be the graph obtained from $G \setminus \{v_1, v_2, v_3, v_4\}$ by identifying $w_3$ and $v_5$, and identifying $w_1$ with $w_2$. Let $G_2$ be the graph obtained from $G \setminus \{v_5, v_1, v_2, v_3\}$ by identifying $w_1$ with $v_4$, and identifying $w_2$ with $w_3$. 

12
We will prove below that one of $G_1, G_2$ is triangle-free, but let us first deduce the lemma from this assertion. From the symmetry we may assume that $G_1$ is triangle-free. Let $x$ be the fifth vertex in the facial cycle $w_4v_4v_3w_3x$ of $G$, and let $y$ be the vertex of $G_1$ obtained by identifying $w_3$ and $v_5$. Then $C_1$ is a cycle of $G_1$, and let $C'_2 = w_5zw_4yx$.

Now every 3-coloring of $G_1$ extends to a 3-coloring of $G$ by coloring $v_4$ the same as $x$ and then coloring $v_3$. The vertices $w_1$ and $w_2$ are colored the same, and $v_5$ and $v_3$ are colored differently. It follows that this coloring can be extended to $v_1$ and $v_2$, as desired. Thus every color transition matrix of $G$ with respect to $C_1$ and $C_2$ dominates some color transition matrix of $G_1$ respect to $C_1$ and $C'_2$ and Lemma 4.6 holds.

It remains to prove the claim that at least one of $G_1$ or $G_2$ is triangle-free. To that end we may assume the contrary. Since $G_1$ is not triangle free, there exist vertices $x_1, x_2 \in V(G) \setminus \{v_1, v_2, v_3, v_4\}$ such that either

(6a) $w_1x_1x_2w_2v_2v_1$ is a cycle in $G$

or

(6b) $w_3x_1x_2v_5v_4v_3$ is a cycle in $G$.

In either case, let $D_1$ denote the corresponding cycle. Since $G_2$ is not triangle free, there exist vertices $y_1, y_2 \in V(G) \setminus \{v_1, v_2, v_3, v_5\}$ such that either

(7a) $w_3y_1y_2w_2v_2v_3$ is a cycle in $G$

or

(7b) $w_1y_1y_2v_4v_5v_1$ is a cycle in $G$.

In either case, let $D_2$ denote the corresponding cycle. It follows from (5) that

(8) $z, x_1, x_2, y_1, y_2 \notin W$.

If (6a) and (7a) hold, then the graph $H := C_2 \cup C_3 \cup D_1 \cup D_2$ has at least three important faces, contrary to (*). Next we show that if (7b) holds, then $w_1$ is adjacent to $z$. To that end assume that (7b) holds. Since $v_4$ has degree three it follows that $y_2 = w_4$. If $y_1 \neq z$, then $H$ has at least three important faces, contrary to (*). Thus $y_1 = z$ and hence $w_1$ is adjacent to $z$ if (7b) holds. Similarly, if (6b) holds, then $w_3$ is adjacent to $z$. We conclude that if (6b) and (7b) hold, then $z$ is adjacent to $w_1$ and $w_3$, and it follows that $G$ has an $H$-structure around $C_2$, where the degree three vertex $v_2$ plays the role of the vertex $v_2$ from the definition of $H$-structure. Therefore, the subgraph $C_1 \cup C_2$ of $G$ satisfies the conclusion of the lemma by Lemma 4.4.

Finally, by symmetry we may assume that (6a) and (7b) hold, and that (6b) does not. In particular, $w_3$ is not adjacent to $z$. Then, as we have shown above, $w_1$ is adjacent to $z$. It follows that $z \notin \{x_1, x_2\}$, and hence $D := w_1zw_4v_4v_3v_2w_2x_2x_1$ is a cycle of length nine. We deduce from (*) that some vertex of $G$ has three neighbors on $D$. This vertex must be $w_3$, and its three neighbors are $v_3, z, x_2$, contrary to the fact that $w_3$ is not adjacent to $z$. 

13
This completes the proof of the fact that one of the graphs $G_1, G_2$ is triangle-free, and hence completes the proof of the lemma. □

Let us recall that the matrices $A_1, A_2$ were defined at the beginning of Section 3.

**Lemma 4.7** Let $G$ be a triangle-free plane graph and $C_1, C_2$ be two distinct 5-cycles in $G$. Then every color transition matrix of $G$ with respect to $C_1$ and $C_2$ dominates either $A_1$ or $A_2$.

**Proof.** We use an argument similar to the proof of Grötzsch’s Theorem given in [13]. Let us assume for a contradiction that the lemma is false, and choose a counterexample $G$ with cycles $C_1$ and $C_2$ with $|V(G)|$ minimum. Let $M$ be a color transition matrix of $G$ with respect to $C_1$ and $C_2$.

1. **If a cycle $C$ in $G$ of length at most seven does not bound a face, then it separates $C_1$ and $C_2$.**

To prove (1) suppose for a contradiction that a cycle $C$ of length at most seven is not facial and does not separate $C_1$ from $C_2$. Then some component $J$ of $G \setminus V(C)$ is disjoint from $C_1 \cup C_2$, and hence every 3-coloring of $G \setminus V(J)$ extends to $G$ by Theorem 1.5. Thus $M$ is a color transition matrix of $G \setminus V(J)$ with respect to $C_1$ and $C_2$, and hence $M$ dominates $A_1$ or $A_2$ by the minimality of $G$, a contradiction. This proves (1).

2. **$G$ is 2-connected.**

To prove (2) we may assume that $G$ is not 2-connected. If $C_1$ and $C_2$ belong to the same block $B$ of $G$, then $M$ is a color transition matrix of $B$ and we obtain contradiction as above. If $C_1$ and $C_2$ are in different blocks, then $M$ dominates the matrix of all ones, as is easily seen, a contradiction. This proves (2).

3. **Every vertex of $G$ of degree two belongs to $C_1 \cup C_2$.**

Claim (3) follows similarly by deleting a vertex of degree two not in $C_1 \cup C_2$.

4. **Every 5-cycle in $G$ bounds a face.**

To prove (4) let $C$ be a 5-cycle in $G$ that does not bound a face. By (1) it separates $C_1$ from $C_2$. Let $M_1$ be a color transition matrix of $G$ with respect to $C_1$ and $C$, and let $M_2$ be a color transition matrix of $G$ with respect to $C$ and $C_2$. By Lemma 4.1 the matrix $M_1M_2$ is a color transition matrix of $G$ with respect to $C_1$ and $C_2$. By the minimality of $G$ the matrices $M_1$ and $M_2$ dominate $A_i$ and $A_j$, respectively, where $i, j \in \{1, 2\}$. It follows that $M$ dominates $A_iA_j$. Notice that $A_1^2, A_1A_2$ and $A_2A_1$ dominate $A_1$ and $A_2^2$ dominates $A_2$, and so $M$ dominates $A_1$ or $A_2$, a contradiction. This proves (4).

5. **$G$ has no facial 4-cycle.**
To prove (5) suppose for a contradiction that \( C := v_1v_2v_3v_4 \) is a facial 4-cycle in \( G \). Let \( G_1 \) be the graph obtained from \( G \) identifying \( v_1 \) and \( v_3 \) and let \( G_2 \) be the graph obtained from \( G \) by identifying \( v_2 \) and \( v_4 \). At least one of the graphs \( G_1, G_2 \) is a triangle-free plane graph. From the symmetry we may assume that \( G_1 \) is triangle-free. Let \( C_1', C_2' \) be the cycles in \( G_1 \) that correspond to \( C_1 \) and \( C_2 \), respectively. As every 3-coloring of \( G_1 \) extends to a 3-coloring of \( G \), a color transition matrix of \( G \) with respect to \( C_1, C_2 \) dominates a color transition matrix of \( G_1 \) with respect to \( C_1', C_2' \). If \( C_1' \neq C_2' \), then \( G_1 \) satisfies the hypotheses of lemma 4.7, and so we obtain contradiction to the minimality of \( G \). Thus \( C_1' = C_2' \). Now \( G \) must be isomorphic to the basic graph \( B_1 \). Then by Lemma 4.2 a color transition matrix of \( G \) with respect to \( C_1, C_2 \) dominates \( A_1 \), a contradiction. This proves (5).

(6) \( G \) has no facial cycle of length six or more.

To prove (6) suppose for a contradiction that \( C := v_1v_2 \ldots v_k \) is a facial cycle in \( G \) of length \( k \geq 6 \). Let \( G_1 \) be the graph obtained from \( G \) identifying \( v_1 \) and \( v_3 \) and let \( G_2 \) be the graph obtained from \( G \) by identifying \( v_2 \) and \( v_4 \). If \( G_1 \) is triangle-free, let \( G' = G_1 \). If \( G_1 \) is not triangle-free, then there exists a path \( v_1u_1u_2v_3 \) in \( G \). Since \( v_1v_2v_3u_2u_1 \) is not a separating 5-cycle, it must be facial. Hence \( v_2 \) is degree two in \( G \). This implies that \( G_2 \) is a triangle-free plane graph, for otherwise there exists a path \( v_2v_1w_1v_4 \) in \( G \), in which case \( v_1v_2v_3v_4w_1 \) is a separating 5-cycle, a contradiction. In this case let \( G' = G_2 \).

Let \( C_1', C_2' \) be the cycles in \( G' \) that correspond to \( C_1 \) and \( C_2 \), respectively. Moreover, the cycles cannot be equal as there are at least three faces in \( G' \). As every 3-coloring of \( G' \) extends to a 3-coloring of \( G \), a color transition matrix of \( G \) with respect to \( C_1, C_2 \) dominates a color transition matrix of \( G' \) with respect to \( C_1', C_2' \), contrary to the minimality of \( G \). This proves (6).

(7) Every cycle in \( G \) of length four, six or seven separates \( C_1 \) from \( C_2 \).

Claim (7) follows immediately from (1), (5) and (6).

It follows from (3), (4) and (7) that \( G \) satisfies the hypotheses of Lemma 4.6. In particular, every facial cycle in \( G \) has length exactly five. Let us recall that good faces were defined in Definition 4.5. Let \( f_1 \) and \( f_2 \) be the faces bounded by \( C_1 \) and \( C_2 \), respectively. Thus \( f_1, f_2 \) are never good. We may assume that

(8) \( G \) has no good face,

because otherwise the lemma follows from Lemma 4.6.

Now we use a standard discharging argument. Let the charge of a vertex \( v \) be \( ch(v) = 4 - \text{deg}(v) \) and the charge of a face \( f \) be \( ch(f) = 4 - |f| \). Then by Euler’s formula the sum of the charges of all vertices and faces is 8. Now we discharge the vertices as follows. Suppose \( v \) is a vertex of \( G \). If the degree of \( v \) is at least three, distribute the charge of it uniformly over the faces incident with it. Thus if \( v \) has degree \( d \geq 5 \), it will receive \( 1/d \) from each
If the degree of $v$ is two, $v$ must be on $C_1$ or $C_2$. If $v$ is incident with both $f_1$ and $f_2$ then distribute the charge of $v$ uniformly over $f_1$ and $f_2$. Otherwise, let $f_3 \not\in \{f_1, f_2\}$ be the other face incident with $v$. In this case let $v$ send $+5/3$ to $f_i$ and $+1/3$ to $f_3$. We denote the new charge of a face $f$ by $ch'(f)$. The new charge of every vertex is zero. Let us recall that every face of $G$ is bounded by a 5-cycle. The discharging rules imply that for every face $f \not\in \{f_1, f_2\}$ of $G$:

(9) if $f$ is incident with five vertices of degree at most three, then $ch'(f) = 2/3$; otherwise $ch'(f) \leq 1/3$,

(10) $ch'(f) > 0$ if and only if $f$ is incident with at least four vertices of degree at most three.

Let $F_1, F_2$ be the set of faces other than $f_1$ and $f_2$ which are adjacent to $f_1$ and $f_2$, respectively. Since the sum of the new charges of all faces is 8, we have either

- $N_1 = ch'(f_1) + \sum_{f \in F_1 - F_2} ch'(f) + \frac{1}{2} \sum_{f \in F_1 \cap F_2} ch'(f) \geq +4$, or
- $N_2 = ch'(f_2) + \sum_{f \in F_2 - F_1} ch'(f) + \frac{1}{2} \sum_{f \in F_1 \cap F_2} ch'(f) \geq +4$, or
- there exists a face $f_3 \neq f_1, f_2$ which is not adjacent to $f_1$ or $f_2$, such that $ch'(f_3) > 0$.

The last case does not happen, because the face $f_3$ would be good by (9), contrary to (8). By the symmetry between $f_1$ and $f_2$ we may therefore assume that $N_2 \geq 4$. Let $C_2 := v_1v_2\ldots v_5$.

(11) At least two vertices of $C_2$ have degree two.

To prove (11) we may assume for a contradiction that $C_2$ has at most one vertex of degree two. Thus $ch'(f_2) \leq 2$. If for every face $f \in F_2$ either $f \in F_1$ or $ch'(f) \leq +1/3$, then $N_2 \geq 4$ implies $|F_2| = 5$. But then $C_2$ has no vertex of degree two, implying $ch'(f_2) \leq 2/3$, and hence $N_2 \leq 8/3$, a contradiction. Thus there exists a face $f \in F_2 - F_1$ such that $ch'(f_3) > 1/3$. But then $f$ is good by (9), contrary to (8). This proves (11).

(12) If $C_2$ has exactly two vertices of degree two, then they are not consecutive on $C_2$.

To prove (12) we may assume for a contradiction that $v_1$ and $v_2$ are the only vertices of degree two on $C_2$. Since all of the faces of $G$ are bounded by 5-cycles, there exists a vertex $w$ such that $w$ is adjacent to $v_3$ and $v_5$. Since the 4-cycle $wv_3v_4v_5$ separates $C_1$ from $C_2$, we deduce that $C_3 := v_1v_5wv_3v_2$ is facial 5-cycle.

Suppose that the degree of $v_3$ or of $v_5$ is three. Since there must be a facial 5-cycle incident with this vertex, $v_4$ and $w$, there must exist a path $v_4xyw$. However, the 5-cycles $v_4xwv_3$ and $v_4xywv_5$ must be facial. Hence $G$ is isomorphic to $B_2$ and Lemma 4.7 follows from Lemma 4.2.

Now we may assume that $v_3$ and $v_5$ have degree at least four. Thus, $ch'(f_2) \leq +8/3$. Moreover, $|F_2| = 3$. Notice that every face $f$ in $F_2$ has a vertex of degree at least four so that $ch'(f) \leq 1/3$. Hence, $N_2 \leq 11/3$, a contradiction. This proves (12).
(13) $C_2$ has at least three vertices of degree two.

To prove (13) we may assume by (11) and (12) that $C_2$ has exactly two vertices of degree two, and that they are not consecutive. Thus we may assume that $v_1$ and $v_3$ are the vertices of degree two on $C_2$. First suppose that $v_2$ has degree three and let $z \neq v_1, v_3$ be a neighbor of $v_2$. Since $v_1$ and $v_3$ have degree two, there exist facial 5-cycles $v_5v_1v_2zw_1$ and $v_4v_3v_2zw_2$. Moreover, $w_1 \neq w_2$ since $G$ is triangle-free. But then $zw_1v_5v_4w_2$ is a 5-cycle, and so it is $C_1$ by (4). Thus $G$ is isomorphic to the basic graph $B_3$ and Lemma 4.7 follows from Lemma 4.2.

So we may assume that $v_2$ has degree at least four. Thus, $ch'(f_2) \leq +3$. Let $f_3 \neq f_2$ be the face incident with $v_1$, $f_4 \neq f_2$ be the face incident with $v_2$, and let $f_5 \neq f_2$ be the face incident with the edge $v_4v_5$. Since the degree of $v_2$ is at least four, $ch'(f_3), ch'(f_4) \leq +1/3$. If $\text{deg}(v_2) \geq 5$, then $ch'(f_2) \leq +3 - 1/5$ and $ch'(f_3), ch'(f_4) \leq +1/3 - 1/5$. Thus $N_2 \leq 13/3 - 3/5 < 4$, a contradiction.

So we may assume that $\text{deg}(v_2) = 4$. Now if $v_5$ has degree at least four, then $ch'(f_2) \leq 8/3$ and $ch'(f_3) \leq 0$. In that case, $N_2 \leq +11/3$, a contradiction. Thus $v_5$ has degree three. Similarly we find that $v_4$ has degree three. Let $y_1$ be the neighbor of $v_5$ not on $C_2$ and let $y_2$ be the neighbor of $v_4$ not on $C_2$. Note that $y_1 \neq y_2$. Now $f_5$ must be incident with $y_1$ and $y_2$. If $y_1$ has degree at least four, then $ch'(f_3) \leq 0$ and $ch'(f_5) \leq +1/3$. In that case, $N_2 \leq +11/3$, a contradiction. Thus $y_1$ has degree three. Similarly we find that $y_2$ also has degree three. Let $z_1$ be the neighbor of $y_1$ incident with $f_3 = v_5v_1v_2z_1y_1$ and let $z_2$ be the neighbor of $y_2$ incident with $f_4 = v_4v_3v_2z_2y_2$. Finally, let $z_3$ be the common neighbor of $y_1$ and $y_2$ incident with $f_5 = v_4v_5y_1z_3y_2$. Thus $G$ has an $H$-structure and Lemma 4.7 follows from Lemma 4.4. This proves (13).

(14) If $C_2$ has exactly three vertices of degree two, then they are not consecutive on $C_2$.

To prove (14) we may assume for a contradiction that $v_1, v_2$ and $v_3$ have degree two, and $v_4, v_5$ have degree at least three. Let $G' = G \setminus \{v_1, v_2, v_3\}$. Notice that $v_1, v_2$ and $v_3$ do not belong to $C_1$ (because $C_1 \neq C_2$), so $V(C_1) \subseteq V(G')$. Obviously for every $1 \leq i \leq 5$ and any 3-coloring $\Phi$ of $G'$, we can extend $\Phi$ to a 3-coloring of $G$ such that $v_i$ is the special vertex on $C_2$ for that coloring. Since by Theorem 1.4 any 3-coloring of $C_1$ can be extended to a 3-coloring of $G'$, a color transition matrix of $G$ with respect to $C_1$ and $C_2$ dominates the matrix $J$. This proves (14).

We are now ready to complete the proof of the lemma. By (2) and (13) there are exactly three vertices of degree two on $C_2$, and by (14) we may assume that they are $v_1, v_2$ and $v_4$. Let $f_3 = v_1v_2v_3z_1v_5$ be the face distinct from $f_2$ that is incident with $v_1$ and $v_2$ and let $f_4 = v_3v_4v_5z_2z_3$ be the face distinct from $f_2$ incident with $v_4$. Note that $z_1 \neq z_2, z_3$ as $G$ is triangle-free. Since the 5-cycle $v_3z_1v_5z_2z_3$ does not separate $C_1$ from $C_2$, it must be $C_1$. Hence $G$ is isomorphic to the basic graph $B_4$ and Lemma 4.7 follows from Lemma 4.2. □
Proof of Lemma 2.4. Suppose $n = |\mathcal{F}| \geq 2$ and let $C_{1}, C_{2}, \ldots, C_{n}$ be the elements of $\mathcal{F}$ such that $\text{Int}(C_{i}) \supseteq \text{Int}(C_{j})$ if and only if $1 \leq i < j \leq n$. For $i = 1, 2, \ldots, n - 1$ let $M_{i}$ be a color transition matrix of $G$ with respect to $C_{i}, C_{i+1}$. Lemma 4.1 implies that $M = M_{1}M_{2}\ldots M_{n-1}$ is a color transition matrix of $G$ with respect to $C_{1}, C_{n}$. Hence the number of 3-colorings of $G$ is at least six times $1^{T}M1$. For all $1 \leq i \leq n - 1$, Lemma 4.2 implies that $M_{i}$ dominates either $A_{1}$ or $A_{2}$, the matrices defined in Section 3. It follows from Lemma 3.2 that the number of 3-colorings of $G$ is at least $24 \cdot 2^{n/12}$, as desired. □

References

[1] A. Asadi, L. Postle and R. Thomas, Sub-exponentially many 3-colorings of triangle-free planar graphs, Electronic Notes in Discrete Mathematics 34 (2009), 81–87.

[2] B. Bollobás, Modern Graph Theory. Springer-Verlag Heidelberg, New York, 1998.

[3] R. Diestel, Graph Theory. Springer-Verlag Heidelberg, New York, 2005.

[4] Z. Dvořák, K. Kawarabayashi and R. Thomas, Three-coloring triangle-free planar graphs in linear time, Proceedings of the twentieth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), New York, NY (2009), 1176–1182.

[5] Z. Dvořák, D. Král’ and R. Thomas, Three-coloring triangle-free graphs on surfaces, Proceedings of the twentieth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), New York, NY (2009), 120–129.

[6] J. Gimbel and C. Thomassen, Coloring graphs with fixed genus and girth, Trans. Amer. Math. Soc. 349 (1997), 4555–4564.

[7] H. Grötzsch, Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe 8 (1959), 109–120.

[8] L. Kowalik, Fast 3-colorings triangle-free planar graphs, ESA 2004, Lecture Notes in Comput. Sci., 3221:436–447, 2004.

[9] D. Král’ and R. Thomas, Coloring even-faced graphs in the torus and the Klein bottle, Combinatorica 28 (2008), 325–341.

[10] C. Thomassen, Every planar graph is 5-choosable, J. Combin. Theory Ser. B 62 (1994), 180–181.

[11] C. Thomassen, Grötzsch’s 3-color theorem and its counterparts for the torus and the projective plane, J. Combin. Theory Ser. B 62 (1994), 268–279.
[12] C. Thomassen, 3-list coloring planar graphs of girth 5, *J. Combin. Theory Ser. B* 64 (1995), 101–107.

[13] C. Thomassen, A short list color proof of Grotzsch’s theorem, *J. Combin. Theory Ser. B* 88 (2003), 189–192.

[14] C. Thomassen, Exponentially many 5-list-colorings of planar graphs, *J. Combin. Theory Ser. B* 97 (2007), 571–583.

[15] C. Thomassen, Many 3-colorings of triangle-free planar graphs, *J. Combin. Theory Ser. B* 97 (2007), 334–349.

This material is based upon work supported by the National Science Foundation. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.