AN EXISTENCE RESULT ON TWO-ORBIT MANIPLEXES

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Abstract. A maniplex of rank $n$ is a connected, $n$-valent, edge-coloured graph that generalises abstract polytopes and maps. If the automorphism group of a maniplex $\mathcal{M}$ partitions the vertex-set of $\mathcal{M}$ into $k$ distinct orbits, we say that $\mathcal{M}$ is a $k$-orbit $n$-maniplex. The symmetry type graph of $\mathcal{M}$ is the quotient pregraph obtained by contracting every orbit into a single vertex. Symmetry type graphs of maniplexes satisfy a series of very specific properties. The question arises whether any pregraph of order $k$ satisfying these properties is the symmetry type graph of some $k$-orbit maniplex. We answer the question when $k = 2$.

1. Introduction

A maniplex $\mathcal{M}$ of rank $n$ can be defined as an $n$-regular properly edge-coloured graph satisfying some additional properties (see Section 2). It generalises a map on a surface to higher ranks, and at the same time is a relaxation of the notion of abstract polytope. Maniplexes were first introduced in [17], and later used for example in [5] and [7].

The symmetry of $\mathcal{M}$ is measured by the number of orbits under the automorphism group, where by an automorphism of $\mathcal{M}$ we mean a color-preserving automorphism of the graph $\mathcal{M}$. If this number is $k$ then $\mathcal{M}$ is said to be a $k$-orbit maniplex. Under this notion, the most symmetric maniplexes are those where $k = 1$, called regular.

The symmetry type graph of a maniplex $\mathcal{M}$ is the quotient of $\mathcal{M}$ by its automorphism group. For instance, the symmetry type graph of a regular maniplex of rank $n$ consists of one vertex and $n$ semi-edges, one of each colour. For a fixed $k \geq 2$, there may be several non-isomorphic candidates for symmetry type graphs of $k$-orbit maniplexes.

For example, when $k = 2$ the possible symmetry type graphs of maniplexes of rank $n$ consist of two vertices, $\ell \geq 1$ edges between them, and $n - \ell$ semi-edges at each of the vertices. Since there are $2^n - 1$ such properly $n$-coloured two-vertex graphs, there are at most $2^n - 1$ symmetry type graphs of 2-orbit maniplexes of rank $n$. Motivated by the terminology from the theories of maps and abstract polytopes, we call a 2-orbit maniplex of rank $n$ whose symmetry type graph consists of two vertices and $n$ edges between them chiral.

The existence of a chiral maniplex for each rank $n$ is a consequence of a result proved in [15], which settled affirmatively a twenty-year-old question about existence of chiral abstract polytopes of every rank $n \geq 3$.

In this paper, we extend this consequence by showing that indeed each of the $2^n - 1$ properly edge-coloured $n$-regular two-vertex graphs is the symmetry type graph of a 2-orbit maniplex (see Theorem 3).
As mentioned above, the maniplexes constructed in \[15\] also satisfy the necessary conditions to be abstract polytopes. This was proven using a vast background available on the automorphism groups of chiral polytopes. Nowadays there is still no analogous background for 2-orbit polytopes. Therefore, determining whether the maniplexes constructed here are abstract polytopes or not would require much more work, and is left out of this paper.

In Sections 2 and 3 we recall basic concepts of maniplexes and polytopes. The polytope \(2^M\) is used in Section 4 to construct a family of polytopes that are the start point of our main construction. Then in Section 5 we relate maniplexes with the theory of graph coverings, and use this viewpoint to construct 2-orbit \((n+1)\)-maniplexes in Section 6.

2. Basic notions

2.1. Maniplexes. A maniplex can be defined in several equivalent ways, one of them being as follows. For a positive integer \(n\), an \(n\)-maniplex \(M\) is an ordered pair \((\mathcal{F}(M),\{r_0,r_1,...,r_{n-1}\})\) where \(\mathcal{F}(M)\) is a non-empty set and \(\{r_0,r_1,...,r_{n-1}\}\) is a set of fixed-point free involutory permutations of \(\mathcal{F}(M)\) satisfying the following three conditions: first, the permutation group \(\langle r_0,r_1,...,r_{n-1}\rangle\) is transitive on \(\mathcal{F}\); second, for every \(\Phi \in \mathcal{F}\), its images under \(r_i\) and \(r_j\) are different if \(i \neq j\); and third, for every \(i,j \in \{0,...,n-1\}\) such that \(|i-j| \geq 2\), the permutations \(r_i\) and \(r_j\) commute. The group \(\langle r_0,r_1,...,r_{n-1}\rangle\) is then called the monodromy group of \(M\) (also called the connection group) and is denoted \(\text{Mon}(M)\). We shall call flags the elements of \(\mathcal{F}\), denote the image of a flag \(\Phi\) under \(r_i\) as \(\Phi^i\) and say that \(\Phi\) and \(\Phi^i\) are \(i\)-adjacent.

We can visualize an \(n\)-maniplex \((\mathcal{F},\{r_0,r_1,...,r_{n-1}\})\) as a simple connected graph with vertex-set \(\mathcal{F}\) and edge-set \(\{\Phi,\Phi^i\} : \Phi \in \mathcal{F}, i \in \{0,...,n-1\}\) together with a proper edge-colouring with the \(i\)th chromatic class consisting of edges \(\Phi,\Phi^i\). Observe that the commutativity condition on the permutations \(r_i\) implies that the induced subgraph with edges of colours \(i\) and \(j\), with \(|i-j| \geq 2\), is a union of disjoint 4-cycles.

Conversely, each connected \(n\)-regular graph with vertex-set \(V\) and a proper edge colouring with chromatic classes \(\{R_0,...,R_{n-1}\}\) satisfying the above condition on induced subgraphs arises from an \(n\)-maniplex \((V,\{r_0,r_1,...,n-1\})\), where \(\Phi^i\) is the unique vertex such that \(\{\Phi,\Phi^i\} \in R_i\). In this sense, an \(n\)-maniplex can be defined as a graph with an appropriate edge colouring. In what follows we will use both points of view interchangeably, as convenient.

Henceforth, let \(M = (\mathcal{F},\{r_0,r_1,...,r_{n-1}\})\) be an \(n\)-maniplex.

For each \(i \in \{0,1,...,n-1\}\) we define the set of \(i\)-faces of \(M\) as the set of connected components of \(M\) after the removal of all \(i\)-coloured edges. Equivalently, we may define an \(i\)-face as an orbit of the group \(\langle r_j | j \in \{0,...,n-1\}\setminus\{i\}\rangle\).

We may associate each \(i\)-face \(F\) of \(M\) with an \(i\)-maniplex by identifying two flags of \(F\) whenever they are joined by a \(j\)-coloured edge, with \(j \geq i\). Similarly as it is done with abstract polytopes, the \((n-1)\)-faces of \(M\) will be called the facets of \(M\).

Let \(p_i\) be the order of \(r_{i-1}r_i\) for \(i \in \{1,...,n-1\}\). Then the Schl"afli type of \(M\) is \(\{p_1,...,p_{n-1}\}\). Whenever \(\text{Aut}(M)\) acts transitively on the pairs \((F_{i-2},F_{i+1})\) consisting of an \((i-2)\)-face \(F_{i-2}\) and an \((i+1)\)-face \(F_{i+1}\) incident to \(F_{i-2}\), \(p_i\) indicates the number of \(i\)-faces incident simultaneously with \(F_{i-2}\) and \(F_{i+1}\). When \(\text{Aut}(M)\) does not possess this transitivity property then \(p_i\) is the least common
multiple of these numbers of \(i\)-faces, when taking all possible pairs \((F_{i-2}, F_{i+1})\). A regular map with \(p\)-gonal faces and degree \(q\) has Schl"{a}fli type \(\{p, q\}\).

An automorphism of the maniplex \(\mathcal{M}\) is a permutation of \(\mathcal{F}\) that commutes with each \(r_i, i \in \{0, \ldots, n - 1\}\), and can thus be viewed as a colour-preserving automorphism of the associated coloured graph. Or equivalently, an automorphism of \(\mathcal{M}\) is a permutation of flags of \(\mathcal{M}\) that maps each pair of \(i\)-adjacent flags to a pair of \(i\)-adjacent flags (this is stated for future reference in Lemma 1 below). As \(\mathcal{M}\) is connected, we see that the action on \(\mathcal{F}\) of its automorphism group, denoted \(\text{Aut}(\mathcal{M})\), is semiregular. Moreover, every automorphism is completely determined by its action on a single flag.

**Lemma 1.** Let \(\mathcal{M}\) be an \(n\)-maniplex and let \(G\) be its automorphism group. Let \(\Phi\) and \(\Phi^i\) be two \(i\)-adjacent flags of \(\mathcal{M}\) and let \(\Phi^G\) and \((\Phi^i)^G\) be their respective orbits under \(G\). Then, for every flag \(\Psi \in \Phi^G\) we have \(\Psi^i \in (\Phi^i)^G\).

Recall that if \(\text{Aut}(\mathcal{M})\) has \(k\) orbits of flags, we say that \(\mathcal{M}\) is a \(k\)-orbit maniplex. A 1-orbit maniplex is called also a regular maniplex, and if \(\mathcal{M}\) has two orbits of flags in such a way that adjacent flags belong to different orbits, the maniplex \(\mathcal{M}\) is said to be chiral.

Suppose \(\Phi\) is a fixed flag of a regular \(n\)-maniplex \(\mathcal{M}\). Define \(\rho_i\) as the unique automorphism of \(\mathcal{M}\) such that \(\Phi \rho_i = \Phi^i\). The following lemma is then a direct consequence of the commutativity of the action of \(\text{Mon}(\mathcal{M})\) and \(\text{Aut}(\mathcal{M})\).

**Lemma 2.** Let \(\Phi\) be a fixed flag of \(\mathcal{M}\) and let \(\alpha \in \text{Aut}(\mathcal{M})\) be such that \(\Phi^\alpha = \Phi^{r_1, r_2, \ldots, r_k}\). Then, \(\alpha = \rho_{\mu_1} \rho_{\mu_2} \cdots \rho_{\mu_k}\).

It follows from the previous lemma and the fact that \(\mathcal{M}\) is connected that \(\text{Aut}(\mathcal{M}) = \langle \rho_0, \rho_1, \ldots, \rho_{n-1} \rangle\). We call \(\{\rho_0, \ldots, \rho_{n-1}\}\) the standard generators of \(\text{Aut}(\mathcal{M})\) relative to the base flag \(\Phi\).

The dual of an \(n\)-maniplex \(\mathcal{M} = (\mathcal{F}, \{r_0, r_1, \ldots, r_{n-1}\})\), denoted \(\mathcal{M}^\ast\), is the \(n\)-maniplex defined as \(\mathcal{M}^\ast = (\mathcal{F}, \{r_0', r_1', \ldots, r_{n-1}'\})\), where \(r_i' = r_n - i - 1\). It is straightforward to see that the dual is well defined and that \((\mathcal{M}^\ast)^\ast = \mathcal{M}\). Moreover, \(\text{Aut}(\mathcal{M}) \cong \text{Aut}(\mathcal{M}^\ast)\).

### 2.2. Polytopes

Here we briefly summarise some facts about abstract polytopes needed in this paper. We refer the reader to [12] for further information. An abstract \(n\)-polytope is a partially ordered set \(\mathcal{P}\) with a monotone function \(\rho : \mathcal{P} \to \{-1, 0, \ldots, n\}\). Elements of \(\mathcal{P}\) of rank \(i\) will be called \(i\)-faces and maximal chains in \(\mathcal{P}\) will be called flags. Two flags are said to be \(i\)-adjacent if they differ only in their \(i\)-face. We will also ask of \(\mathcal{P}\) to satisfy the following four conditions:

1. \(\mathcal{P}\) has a unique least face \(F_{-1}\) and a unique greatest face \(F_n\).
2. All flags in \(\mathcal{P}\) have length \(n + 2\).
3. For every \(i \in \{0, \ldots, n - 1\}\) and every two faces \(F\) and \(G\), of ranks \(i - 1\) and \(i + 1\) respectively, such that \(F < G\), there are exactly two \(i\)-faces \(H_1\) and \(H_2\) satisfying \(F < H_1, H_2 < G\). This is generally called the diamond condition.
4. If \(\Phi\) and \(\Psi\) are two flags, then there exists a sequence of successively adjacent flags \(\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi\) such that \(\Phi \cap \Psi \subset \Phi_i\), for all \(i \in \{0, \ldots, k\}\).

We may associate each \(i\)-face \(F\) of \(\mathcal{P}\) with the poset \(\{G \in \mathcal{P} : G \leq F\}\), which satisfy all the conditions to be an abstract polytope. In that sense, every \(i\)-face
of $\mathcal{P}$ is an abstract $i$-polytope in its own right. As is customary, 0-faces will be called vertices, 1-faces will be called edges, 2-faces will simply be called faces and $(n-1)$-faces will be called facets.

Let $\mathcal{P}$ be an abstract $n$-polytope and let $\mathcal{F}$ be the set of all its flags. If $\Phi \in \mathcal{F}$, then it follows easily from the diamond condition that there is a unique flag, denoted $\Phi^i$, that is $i$-adjacent to $\Phi$. For each $i \in \{0, \ldots, n-1\}$, let $r_i$ be the permutation on $\mathcal{F}$ that sends each flag to its $i$-adjacent flag, that is $\Phi^{r_i} = \Phi^i$. Define the monodromy group (also called the connection group) of $\mathcal{P}$ as $\text{Mon}(\mathcal{P}) = \langle r_0, r_1, \ldots, r_{n-1} \rangle$ and note that, since $\mathcal{P}$ satisfies the diamond condition, each generator $r_i$ is an involution. Furthermore, $r_i r_j = r_j r_i$ if $|i - j| \geq 2$. It follows from Condition (4) above that the monodromy group of an abstract polytope acts transitively on its flags. This allows us to define an $n$-maniplex $\mathcal{M}_\mathcal{P} = (\mathcal{F}, \{r_0, r_1, \ldots, r_{n-1}\})$ associated to $\mathcal{P}$. In this sense, an $n$-polytope can also be viewed as an $n$-maniplex. The converse, however, is not true: there are maniplexes that are not associated to any polytope (see, for example, [17]).

An automorphism of the polytope $\mathcal{P}$ is an order-preserving permutation of $\mathcal{P}$, and the group of all automorphisms of $\mathcal{P}$ is denoted by $\text{Aut}(\mathcal{P})$. Since all the flags of $\mathcal{P}$ have the same length, $\text{Aut}(\mathcal{P})$ induces a faithful action on $\mathcal{F}$. Moreover, every $\alpha \in \text{Aut}(\mathcal{P})$ maps $i$-adjacent flags to $i$-adjacent flags, for all $i \in \{0, \ldots, n-1\}$. It follows that the actions of $\text{Mon}(\mathcal{P})$ and $\text{Aut}(\mathcal{P})$ on $\mathcal{F}$ commute. We say that an abstract polytope $\mathcal{P}$ is regular if its automorphism group is transitive on flags and we say that $\mathcal{P}$ is chiral if $\text{Aut}(\mathcal{P})$ has two orbits of flags and adjacent flags always belong to different orbits.

Note that the notion of the monodromy group, the automorphism group, the regularity and the chirality of a polytope coincide with those of the associated maniplex.

2.3. Symmetry Type Graphs. Intuitively speaking, the symmetry type graph of an $n$-maniplex $\mathcal{M}$ is the edge-coloured graph whose vertices are orbits of $\text{Aut}(\mathcal{M})$ (in its action on the flags of $\mathcal{M}$) and with two orbits $A$ and $B$ adjacent by an edge of colour $i$ whenever there is a flag $\varphi \in A$ and a flag $\psi \in B$ that are $i$-adjacent in $\mathcal{M}$. However, this definition requires the symmetry type graph to have edges that connect an orbit to itself. For that reason we define the following notion of a pregraph (which is widely used in the context of graph quotients and covers; see for example [10]).

A pregraph is an ordered 4-tuple $(V, D; \text{beg}, \text{inv})$ where $V$ and $D \neq \emptyset$ are disjoint finite sets of vertices and darts, respectively, $\text{beg}: D \to V$ is a mapping which assigns to each dart $x$ its initial vertex $\text{beg} x$, and $\text{inv}: D \to D$ is an involution which interchanges every dart $x$ with its inverse dart, also denoted by $x^{-1}$. The neighbourhood of a vertex $v$ is defined as the set of darts that have $v$ for its initial vertex and the valence of $v$ is the cardinality of the neighbourhood.

The orbits of $\text{inv}$ are called edges. The edge containing a dart $x$ is called a semi-edge if $x^{-1} = x$, a loop if $x^{-1} \neq x$ while $\text{beg}(x^{-1}) = \text{beg} x$, and is called a link otherwise. The endvertices of an edge are the initial vertices of the darts contained in the edge. Two links are parallel if they have the same endvertices.

A pregraph with no semi-edges, no loops and no parallel links can clearly be viewed as a (simple) graph. Conversely, every simple graph given in terms of its vertex-set $V$ and edge-set $E$ can be viewed as the graph $(V, D; \text{beg}, \text{inv})$, where
$D = \{(u, v) \mid uv \in E\}$, $\text{inv}(u, v) = (v, u)$ and $\text{beg}(u, v) = u$ for any $(u, v) \in D$. The mapping from $\Gamma$ to $\Gamma/G$ is called the quotient projection with respect to $G$.

An automorphism of a pregraph $\Gamma = (V, D; \text{beg}, \text{inv})$ is a permutation on $D \cup V$ that preserves both $V$ and $D$ and commutes with $\text{beg}$ and $\text{inv}$. The group of all automorphisms of $\Gamma$ is denoted by $\text{Aut}(\Gamma)$.

Given a pregraph $\Gamma = (V, D; \text{beg}, \text{inv})$ and $G \leq \text{Aut}(\Gamma)$, one can define the quotient pregraph $\Gamma/G = (V/G, D/G, \text{beg}_G, \text{inv}_G)$, where $V/G$ and $D/G$ are the sets of all orbits of $G$ on $V$ and $D$, respectively, and $\text{beg}_G: D/G \to V/G$ and $\text{inv}: D/G \to D/G$ are defined in such a way that $\text{beg}_G(x^G) = \text{beg}(x)^G$ and $\text{inv}_G(x^G) = \text{inv}(x)^G$ for every $x \in D$.

Let $\mathcal{M}$ be a $k$-orbit $n$-maniplex viewed as an edge-coloured graph, in fact, as a coloured pregraph $\Gamma_M$ where each dart is coloured by the same colour as the underlying edge. Let $G$ be the group of colour preserving automorphism of $\Gamma_M$. The symmetry type graph $T(\mathcal{M})$ of $\mathcal{M}$ is then defined as the pregraph $\Gamma/G$, together with the colouring of the darts of $\Gamma/G$ where a dart $x^G$ is coloured by the same colour as the dart $x$ of $\Gamma_M$. To see that this colouring is well defined, consider a dart $x$ in $\Gamma_M$ and note that, since the elements of $G$ are colour-preserving, every $y \in x^G$ must have the same colour as $x$, and so the colour of $x^G$ does not depend on its representative. Furthermore, Lemma 4 guarantees that, for each vertex $v^G$ of $T_M$, there is exactly one dart of each colour having $v^G$ as its initial vertex. It follows that, for each $i \in \{0, 1, ..., n - 1\}$, there is exactly one edge of colour $i$ incident to each vertex of $T(\mathcal{M})$. In particular, this proves that the symmetry type graph of a maniplex cannot have loops (and thus every edge is either a link or a semi-edge).

Let $i, j \in \{0, ..., n - 1\}$ be such that $|i - j| \geq 2$. Since every walk of length 4 in $\Gamma(\mathcal{M})$ of alternating colours $i$ and $j$ is necessarily a 4-cycle, we have that any walk of length 4 in $T(\mathcal{M})$ of alternating colours $i$ and $j$ must be a closed walk. We thus have that each connected component of the subgraph of $T(\mathcal{M})$ induced by edges (which could be links or semi-edges) of colour $i$ and $j$ is isomorphic to one of the following pregraphs.

![Possible quotients of 2-coloured 4-cycles](image)

The symmetry type graph of a regular $n$-maniplex consist of a single vertex with $n$ semi-edges incident to it. Regular abstract polytopes have been widely studied ([12] for details) and constructions of regular polytopes abound.

The symmetry type graph $\Gamma$ of a 2-orbit $n$-maniplex, is a connected $n$-valent, loopless pregraph of order 2. Furthermore, the edges of $\Gamma$ are coloured in such a way that each vertex has either an edge or a semi-edge of each colour in $\{0, 1, ..., n - 1\}$ incident to it. If $\mathcal{I}$ is the set of colours appearing in the semi-edges of $\Gamma$, then $\Gamma$ is denoted by $2^n_{\mathcal{I}}$ (see Figure 2.2 for two examples). Following [8] and [9], a maniplex having $2^n_{\mathcal{I}}$ as its symmetry type graph is said to be of type $2^n_{\mathcal{I}}$.

We are now ready to state our main theorem. Its proof is given in Section 6.
Theorem 3. For every $n \geq 3$ and every proper subset $\mathcal{I}$ of $\{0, \ldots, n-1\}$ there exists an $n$-maniplex of type $2_{\mathcal{I}}^n$.

Most of the works on polytopes and maniplexes has been devoted to regular ones \[12\]. The most studied class of non-regular polytopes and maniplexes are the chiral ones. Its existence in ranks 3 and 4 was established in the 20-th century (see \[3\] and \[2\]), and in ranks 5 and higher only in the 21-th century (see \[1\] and \[15\]). Other classes of polytopes and maniplexes have been seldom studied \[13\]. The study of the particular case of 2-orbit polytopes and maniplexes started in \[8\] and \[9\].

There were several attempts to show existence or non-existence of chiral polytopes of all ranks $n \geq 3$ in the years between 1991 (where the basic theory of chiral polytopes was established in \[16\]) and 2010 (when the existence was finally proven in \[15\]), and all faced great difficulties, particularly when ensuring that the constructed object is not regular. In view of this, it sounds a great task to determine if for every $n \geq 3$ and every $n$-regular pregraph $\Gamma$ satisfying the restrictions illustrated in Figure 2.1 there is a maniplex with symmetry type $\Gamma$.

A more manageable question is whether every graph $2^\emptyset_{\mathcal{I}}$ is the symmetry type graph of some 2-orbit maniplex, for each $n \geq 3$ and each $\mathcal{I} \subseteq \{0, 1, \ldots, n-1\}$. Since chiral polytopes exist for all ranks equal or greater than 3, every pregraph $2^\emptyset_{\mathcal{I}}$ (one with no semi-edges) is the symmetry type graph of some (polytopal) maniplex.

We extend the work of the first author \[15\] by providing a construction that yields, for each $n \geq 3$ and each $\mathcal{I} \subset \{0, 1, \ldots, n-1\}$, an $n$-maniplex of type $2^\emptyset_{\mathcal{I}}$.

The solution of this problem only for pregraphs with two vertices may seem limited. However, the technique developed here may prove useful when attacking this problem for pregraphs with more vertices. We believe that the lack of symmetry on pregraphs (for instance, not being vertex-transitive) may be used to produce maniplexes that gain no extra symmetry.

Incidentally, it is known that every $n$-regular pregraph with 3 vertices satisfying the restrictions illustrated in Figure 2.1 is the symmetry type graph of a 3-orbit $n$-polytope \[4\]. This is thanks to some hereditary property of the pregraphs with three vertices \[5\]. Then the present paper establishes that every $n$-regular pregraph with at most 3 vertices satisfying the restrictions illustrated in Figure 2.1 is the symmetry type graph of a maniplex.

3. Bi-colourings consistent with $\mathcal{I}$

Given an $n$-maniplex $\mathcal{M}$ and $\mathcal{I} \subseteq \{0, \ldots, n-1\}$, a bi-colouring of $\mathcal{M}$ consistent with $\mathcal{I}$ is a colouring of $\mathcal{F}(\mathcal{M})$ with colours black and white such that $i$-adjacent flags have the same colour if and only if $i \in \mathcal{I}$. In the context of maps by denoting $\mathcal{J} := \{0, 1, 2\} \setminus \mathcal{I}$, such a bi-colouring is called an $\mathcal{J}$-colouring in \[11\].
For example, every 2-maniplex is bipartite as a coloured graph, and hence it admits a bi-colouring consistent with ∅. In general, orientable maniplexes (that is, maniplexes that are bipartite as coloured graphs) admit a bi-colouring consistent with ∅. On the other hand, a triangle does not admit a bi-colouring consistent with {0}, since that would be equivalent to a proper 2-colouring of the edges of the triangle.

The following proposition relates maniplexes that have bi-coulorings consistent to \( I \) with cycles in the maniplex, in a similar way as bipartite graphs are related to the non-existence of odd cycles.

**Proposition 4.** Let \( I \subseteq \{0, \ldots, n-1\} \) and \( \mathcal{M} \) be an \( n \)-maniplex. Then \( \mathcal{M} \) admits a bi-colouring consistent with \( I \) if and only if there is no cycle in \( \mathcal{M} \) (as a graph) having an odd number of edges whose labels are not in \( I \).

**Proof.** A cycle in \( \mathcal{M} \) having an odd number of edges whose labels are not in \( I \) cannot be bi-coloured in such a way that adjacent flags have the same colour only when \( i \in I \). Hence, if \( \mathcal{M} \) contains such a cycle then it cannot admit a bi-colouring consistent with \( I \).

Conversely, if all cycles in \( \mathcal{M} \) have an even number of edges whose labels are not in \( I \), it is possible to assign a bi-colouring consistent with \( \mathcal{M} \) by colouring black any flag \( \Phi \), and also any flag that can be reached from \( \Phi \) by a path having an even number of edges whose labels are not in \( I \). The remaining flags, namely those that can be reached from \( \Phi \) by a path having an odd number of edges with labels that are not in \( I \), are coloured white. This colouring is well-defined since otherwise it would be possible to construct a cycle in \( \mathcal{M} \) with an odd number of edges having labels not in \( I \). Furthermore, this colouring is clearly consistent with \( I \).

When having a presentation for the automorphism group of a regular maniplex \( \mathcal{M} \), in terms of the standard generators, it is easy to verify if \( \mathcal{M} \) admits a bi-colouring consistent with some \( I \). This is shown in the following proposition.

**Proposition 5.** Let \( \mathcal{M} \) be a regular maniplex of rank \( n \), let \( \Phi \) be a flag of \( \mathcal{M} \), let \( \text{Aut}(\mathcal{M}) = \langle \rho_0, \ldots, \rho_{n-1} \mid R \rangle \) be a presentation of \( \text{Aut}(\mathcal{M}) \) in terms of the standard generators \( \rho_0, \ldots, \rho_{n-1} \) (with respect to the base flag \( \Phi \)), and let \( I \subseteq \{0, \ldots, n-1\} \). Suppose that for every \( i \notin I \) every relator in \( R \) contains an even number of symbols \( \rho_i \). Then \( \mathcal{M} \) admits a bi-colouring consistent with \( I \).

**Proof.** Assume to the contrary that \( \mathcal{M} \) does not admit a bi-colouring consistent with \( I \). Then, by Proposition 4 there is a closed cycle that has an odd number of edges whose labels are not in \( I \). Since the standard generators \( \rho_i \) were chosen with respect to the base flag \( \Phi \), this can be rephrased as

\[
\Phi = \Phi^{r_{i_1}r_{i_2} \cdots r_{i_k}} = \Phi^{\rho_{i_k} \cdots \rho_{i_1}}
\]

for some \( r_{i_1}, \ldots, r_{i_k} \in \{r_0, \ldots, r_{n-1}\} \), where \( r_{i_j} \notin I \) for an odd number of \( j \)'s. It follows that \( \rho_{i_k} \cdots \rho_{i_1} \) is a relator for \( \text{Aut}(\mathcal{M}) \) with an odd number of symbols \( \rho_i \) with \( i \notin I \). Such a relator must be a product of conjugates of the relators in \( R \), all of which have an even number of symbols \( \rho_i \) with \( i \notin I \). This contradicts our hypothesis. \( \square \)
Remark 6. Proposition 5 requires a condition on one particular set $\mathcal{R}$ of relators for $\text{Aut}(\mathcal{M})$. However, if this condition holds for one such set then it holds for every possible set of relators. Namely, if $w$ is a word consisting of symbols in $\{\rho_0, \ldots, \rho_{n-1}\}$ that equals 1 in $\text{Aut}(\mathcal{M}) = \langle \rho_0, \ldots, \rho_{n-1} \mid \mathcal{R} \rangle$ then it is contained in the normal closure $N$ in the group $\langle \rho_0, \ldots, \rho_{n-1} \rangle$ of the group generated by all conjugates of words in $\mathcal{R}$. It is now obvious that if every relator contains an even number of symbols $\rho_i$ then so does every element in $N$.

4. Maniople $\hat{2}^M$

In this section we recall the construction of the structure $\hat{2}^M$. It was initially introduced by Danzer in [6] in the context of incidence complexes, which are combinatorial structures more general than abstract polytopes. This construction has appeared several times when studying regular polytopes and was generalised to maniplies in [7].

Let $\mathcal{M} = (\mathcal{F}, \{r_0, \ldots, r_{n-1}\})$ be an $n$-maniople with set of facets $\mathcal{S}$, $|\mathcal{S}| \geq 2$. For $F \in \mathcal{S}$ let $\chi_F : \mathcal{S} \to \mathbb{Z}_2$ be the characteristic function of $F$ (that is, the function from $\mathcal{S}$ to $\mathbb{Z}$ assigning 1 to $F$ and 0 to all other facets in $\mathcal{S}$). Furthermore, for a flag $\Phi$ of $\mathcal{M}$ let $F(\Phi)$ denote the facet of $\mathcal{M}$ containing $\Phi$.

We now define the $(n+1)$-maniople $\hat{2}^M = (\mathcal{F}(\hat{2}^M), \{r'_0, \ldots, r'_n\})$ as follows:

\begin{align*}
(4.1) \quad \mathcal{F}(\hat{2}^M) &= \mathcal{F}(\mathcal{M}) \times \mathbb{Z}_2^|\mathcal{S}|, \\
(4.2) \quad (\Phi, x)^{r'_i} &= (\Phi^{r'_i}, x) \text{ for } i < n, \\
(4.3) \quad (\Phi, x)^{r'_n} &= (\Phi, x + \chi_{F(\Phi)}).
\end{align*}

Note that $x + \chi_{F(\Phi)}$ in the equation above is the function in $\mathbb{Z}_2^|\mathcal{S}|$ that differs from $x$ only when evaluated in the facet $F(\Phi)$.

The construction of the maniople $\hat{2}^M$ from $\mathcal{M}$ is illustrated in Figure 4.1. On the left-hand side, we take $\mathcal{M}$ to be the square with a set of facets $\mathcal{S} = \{F_1, F_2, F_3, F_4\}$ and each facet $F_i$ containing flags $\Psi_i$ and $\Phi_i = \Psi_i^0$. The right-hand side depicts the maniople $\hat{2}^M$ as a map on the torus, having $|\mathbb{Z}_2^|\mathcal{S}|$ copies of $\mathcal{M}$ as facets, each labelled with an element of $\mathbb{Z}_2^|\mathcal{S}|$, written as a vector. The glueing of the facets is given by (4.3). For instance, facet $(0,1,1,0)$ is glued to $(0,1,1,1)$ through the edge with flags indexed 4 because $(0,1,1,0) + \chi_{F_4} = (0,1,1,1)$.

For $\gamma \in \text{Aut}(\mathcal{M})$ and $x \in \mathbb{Z}_2^{|\mathcal{S}|}$, let the mapping $x^\gamma : \mathcal{S} \to \mathbb{Z}_2$ be defined by $x^\gamma = \gamma^{-1} \circ x$ and observe that this defines an embedding of $\text{Aut}(\mathcal{M})$ to the automorphism group of the elementary abelian 2-group $\mathbb{Z}_2^{|\mathcal{S}|}$. This allows us to define the wreath product

\begin{equation}
\mathbb{Z}_2 \wr \text{Aut}(\mathcal{M}) = \mathbb{Z}_2^{|\mathcal{S}|} \rtimes \text{Aut}(\mathcal{M}).
\end{equation}

Furthermore, for $\gamma \in \text{Aut}(\mathcal{M})$ and $y \in \mathbb{Z}_2^{|\mathcal{S}|}$, let $\tilde{\gamma}$ and $\tilde{y}$ be the permutations of $\mathcal{F}(\hat{2}^M)$ defined by

\begin{align*}
(4.5) \quad (\Phi, x)^{\tilde{\gamma}} &= (\Phi^{\gamma}, x) \\
(4.6) \quad (\Phi, x)^{\tilde{y}} &= (\Phi, x + y).
\end{align*}

Let $\Phi_0$ be the base flag of $\mathcal{M}$ and $\rho_0, \ldots, \rho_{n-1}$ the standard generators of $\text{Aut}(\mathcal{M})$ with respect to $\Phi_0$. Let $F_0$ be the base facet of $\mathcal{M}$ (that is, the facet containing $\Phi_0$). We choose the base flag of $\hat{2}^M$ to be $(\Phi_0, \overline{0})$, where $\overline{0} \in \mathbb{Z}_2^{|\mathcal{S}|}$ is the 0-constant function.
The following proposition summarises Section 6 of [7]. Item (6) was proven in [14].

**Proposition 7.** Let $M$ be a regular maniplex with at least 2 facets and Schläfli type $\{p_1, \ldots, p_{n-1}\}$. Then the following hold:

1. if $\gamma \in \text{Aut}(M)$ then $\bar{\gamma} \in \text{Aut}(\hat{2}M)$, and if $y \in \mathbb{Z}_2^S$ then $\bar{y} \in \text{Aut}(\hat{2}M);$  
2. $\text{Aut}(\hat{2}M)$ equals the wreath product $\mathbb{Z}_2 \wr \text{Aut}(M)$ defined in (4.4) and is generated by all $\bar{\gamma}$ for $\gamma \in \text{Aut}(M)$ and all $\bar{y}$ for $y \in \mathbb{Z}_2^S$;  
3. the stabiliser of the base facet of $\hat{2}M$ equals $\langle \bar{\rho}_0, \ldots, \bar{\rho}_{n-1} \rangle$ and there exists an isomorphism from $\text{Aut}(M)$ to $\langle \bar{\rho}_0, \ldots, \bar{\rho}_{n-1} \rangle$ mapping $\rho_i$ to $\bar{\rho}_i$ for every $i \in \{0, \ldots, n-1\};$  
4. the standard generators of $\text{Aut}(\hat{2}M)$ with respect to the base flag $(\Phi_0, \bar{0})$ are $\bar{\rho}_0, \ldots, \bar{\rho}_{n-1}, \chi_{F_0}$ where $\chi_{F_0}$ is the characteristic function of the base facet $F_0$;  
5. $\hat{2}M$ has Schläfli type $\{p_1, \ldots, p_{n-1}, 4\}$;  
6. if $M$ is a polytope then so is $2^M$.

**Remark 8.** All items of Proposition 6 hold if $M$ has only one facet, except for the fifth one. In this situation $\hat{2}M$ coincides with the trivial maniplex over $M$ defined in [7, Section 2.1] and the last entry of the Schläfli type is 2.

### 4.1. Symmetry type graph of $\hat{2}M$

Given a maniplex $M$, the next two lemmas describe transitivity properties of $\text{Aut}(\hat{2}M)$. Throughout this subsection the Schläfli type of $2^M$ plays no role, and Remark 8 allows us to state these results for arbitrary maniplexes.

**Lemma 9.** The automorphism group of the maniplex $\hat{2}M$ acts transitively on the set of facets of $\hat{2}M$. 

![Figure 4.1. The construction of $\hat{2}M$ from a square $M$](image)
Proof. Since a facet of $\hat{2}^\mathcal{M}$ is an orbit of a flag under the group $\langle r_0, \ldots, r_{n-1} \rangle$, from (4.2) it follows that every facet of $\hat{2}^\mathcal{M}$ can be described as

$$\{(\Phi, y_0) : \Phi \in \mathcal{F}(\mathcal{M})\}$$

for some $y_0 \in \mathbb{Z}_2^S$. Then the elements $\tilde{y} \in \text{Aut}(\hat{2}^\mathcal{M})$ act transitively on the facets of $\hat{2}^\mathcal{M}$ (see Proposition 7 (1)). □

Lemma 10. If $\mathcal{M}$ is a $k$-orbit maniplex then so is $\hat{2}^\mathcal{M}$.

Proof. Since all facets of $\hat{2}^\mathcal{M}$ are isomorphic to $\mathcal{M}$, the number of flag-orbits of $\hat{2}^\mathcal{M}$ is at least $k$. On the other hand, the transitivity of $\text{Aut}(\hat{2}^\mathcal{M})$ on the set of facets and the automorphisms $\tilde{y}$'s imply that $\hat{2}^\mathcal{M}$ has precisely $k$ flag-orbits. □

The symmetry type graph of the maniplex $\hat{2}^\mathcal{M}$ can be easily described in terms of that of $\mathcal{M}$.

Proposition 11. Let $\mathcal{M}$ be an $n$-maniplex. Then the symmetry type graph $T(\hat{2}^\mathcal{M})$ is obtained from $T(\mathcal{M})$ by adding a semi-edge labelled $n$ at every vertex.

Proof. Any flag $\Phi$ is mapped to $\Phi^n$ by the automorphism $\chi_{F(\Phi)}$, implying that any two $n$-adjacent flags belong to the same flag-orbit. It follows that there is a semi-edge labelled $n$ at every vertex of $T(\hat{2}^\mathcal{M})$.

By Lemma 10, $\mathcal{M}$ and $\hat{2}^\mathcal{M}$ have the same number of flag-orbits. Then, by Lemma 9, every flag-orbit has a representative in every facet. Since the facets of $\hat{2}^\mathcal{M}$ are isomorphic to $\mathcal{M}$, the flag-orbits of two flags $\Phi_1$ and $\Phi_2$ of $\mathcal{M}$ are $i$-adjacent if and only if the flag orbits of the flags $(\Phi_1, y)$ and $(\Phi_2, y)$ of $\hat{2}^\mathcal{M}$ are $i$-adjacent, for any $y \in \mathbb{Z}_2^S$. It follows that the graph obtained from $T(\hat{2}^\mathcal{M})$ by removing all edges labelled $n$ is isomorphic to $T(\mathcal{M})$ as a labelled graph. This finishes the proof. □

Corollary 12. Let $\mathcal{M}$ be an $n$-maniplex with at least two facets and let $\mathcal{G}$ be an $(n+1)$-regular graph with edges labelled in $\{0, \ldots, n\}$ that can be obtained from $T(\mathcal{M})$ by one of the two following procedures:

1. Add semi-edges labelled $n$ at every vertex.
2. Add 1 to the label of every edge and add a semi-edge labelled 0 at every vertex.

Then $\mathcal{G}$ is the symmetry type graph of some maniplex $\mathcal{M}'$.

Proof. The symmetry type of the maniplexes $\hat{2}^\mathcal{M}$ and $(\hat{2}^\mathcal{M}')^*$, where $\mathcal{M}'$ is the dual maniplex defined in 3.1, are $\mathcal{G}$ for the first and second procedures, respectively. □

Similar arguments to the ones in this section can be used to prove Corollary 12 using the trivial maniplex over $\mathcal{M}$ defined in [7] instead of $\hat{2}^\mathcal{M}$.

4.2. The polytopes $\mathcal{M}_n$. We now use the above construction to define recursively a family of polytopes.

Definition 13. Let $\mathcal{M}_1$ be the unique regular polytope with rank 1. Then, for $n \geq 2$ let $\mathcal{M}_n$ be the regular polytope $\hat{2}^\mathcal{M}_{n-1}$. It is easy to see that $\mathcal{M}_2$ is isomorphic to the square, and $\mathcal{M}_3$ to the toroidal map $\{4, 4\}_{(4,0)}$.

The family of polytopes $\{\mathcal{M}_n\}_{n \in \mathbb{N}}$ has several interesting properties. One of them is given in the following proposition.
Proposition 14. For every \( n \in \mathbb{N} \) and \( I \subseteq \{0, \ldots, n-1\} \) the polytope \( \mathcal{M}_n \) admits a bi-colouring of flags consistent with \( I \).

Proof. Let \( G_n = \text{Aut}(\mathcal{M}_n) \) and let

\[
G_n = \langle \rho_0, \ldots, \rho_{n-1} \mid \mathcal{R} \rangle
\]

be a presentation of \( G_n \) in terms of standard generators. We will show that for every \( i \in \{0, \ldots, n-1\} \) and every word \( w \in \mathcal{R} \) the symbol \( \rho_i \) appears an even number of times in \( w \). Proposition 5 will then yield the result. By Remark 6 it suffices to prove this condition for any set of relators \( \mathcal{R} \) for which (4.7) holds.

The proof of the above fact is by induction on \( n \). For \( n = 1 \) we may choose a presentation of \( G_1 \) to be \( G_1 = \langle \rho_0 \mid \rho_0^2 \rangle \), which clearly satisfies the condition above.

Now assume that the claim holds for \( G_{n-1} \). Let

\[
G_{n-1} = \langle \rho_0, \ldots, \rho_{n-2} \mid \mathcal{R}' \rangle
\]

be a presentation for \( G_{n-1} \) in terms of standard generators. Let \( \mathcal{S} \) be the set of facets of \( \mathcal{M}_{n-1} \) and fix a base facet \( F_0 \in \mathcal{S} \). By Proposition 7 (2), \( G_n = \mathbb{Z}_2^S \rtimes G_{n-1} \) where \( G_{n-1} \) acts upon the elementary abelian group \( \mathbb{Z}_2^S \) according to the rule \( x^\gamma = \gamma^{-1} \circ x \) for every \( x \in \mathbb{Z}_2^S \) and \( \gamma \in G_{n-1} \). Furthermore, let \( e_0 = \chi_{F_0} \in \mathbb{Z}_2^S \) be the characteristic function of the base facet \( F_0 \).

Recall that the standard generators of \( G_n \) are \( \tilde{\rho}_0, \ldots, \tilde{\rho}_{n-2}, \tilde{e}_0 \) and that the mapping \( \gamma \mapsto \tilde{\gamma} \) in an embedding of \( G_{n-1} \) to \( G_n \).

Let \( w \) be a word in the symbols \( \{\tilde{\rho}_0, \ldots, \tilde{\rho}_{n-2}, \tilde{e}_0\} \) that evaluates to 1 in \( G_n \). Then \( w \) can be written as

\[
w = \tilde{\gamma}_1 x_1 \tilde{\gamma}_2 x_2 \cdots x_{k-1} \tilde{\gamma}_k
\]

where \( \tilde{\gamma}_i \)'s are words in \( \{\tilde{\rho}_0, \ldots, \tilde{\rho}_{n-2}\} \) and \( x_i = \tilde{e}_0^s \) for some integer \( s \). Observe that in \( G_n \) the word \( w \) evaluates to

\[
\tilde{\gamma}_1 \tilde{\gamma}_2 \cdots \tilde{\gamma}_k \, \tilde{x}_1 \tilde{\gamma}_2 \cdots \tilde{\gamma}_k \, \tilde{x}_2 \cdots \tilde{\gamma}_k \cdots \, \tilde{x}_{k-1} \tilde{\gamma}_k.
\]

Hence, in \( G_n \) the following holds:

\[
\tilde{\gamma}_1 \tilde{\gamma}_2 \cdots \tilde{\gamma}_k \, \tilde{x}_1 \tilde{\gamma}_2 \cdots \tilde{\gamma}_k \, \tilde{x}_2 \cdots \tilde{\gamma}_k \cdots \, \tilde{x}_{k-1} \tilde{\gamma}_k = 1,
\]

which implies that

\[
\tilde{\gamma}_1 \tilde{\gamma}_2 \cdots \tilde{\gamma}_k = 1 \quad \text{and} \quad \tilde{x}_1 \tilde{\gamma}_2 \cdots \tilde{\gamma}_k \cdots \tilde{x}_{k-1} = 1.
\]

But then

\[
\gamma_1 \gamma_2 \cdots \gamma_k
\]

is a trivial element of \( G_{n-1} \), and by induction hypothesis, for every \( i \in \{0, \ldots, n-2\} \) each symbol \( \rho_i \) appears an even number of times in (4.9). Consequently, each symbol \( \tilde{\rho}_i \) appears an even number of times in \( \tilde{\gamma}_1 \tilde{\gamma}_2 \cdots \tilde{\gamma}_k \). Moreover, since each \( x_j \) is a power of \( \tilde{e}_0 \), the symbols \( \tilde{\rho}_i \) also appear an even number of times in the expression \( \tilde{x}_1 \tilde{\gamma}_2 \cdots \tilde{\gamma}_k \cdots \tilde{x}_{k-1} \). Now let \( \mathcal{S} \) be the set of indices \( i \) for which \( s_i \) is odd. Since \( \tilde{e}_0^2 = 1 \) and since \( \tilde{x}_1 \tilde{\gamma}_2 \cdots \tilde{\gamma}_k \cdots \tilde{x}_{k-1} = 1 \), we see that

\[
\prod_{i \in \mathcal{S}} \tilde{\gamma}_i = 1.
\]

However, a factor \( \tilde{e}_0^{\tilde{\gamma}_i-1} \), \( i \in \mathcal{S} \), is a characteristic function of the facet \( F_0^{\tilde{\gamma}_i-1} \), implying that the size of \( \mathcal{S} \) is even. This shows that \( \tilde{e}_0 \) appears an even number of times in the expression \( \tilde{x}_1 \tilde{\gamma}_2 \cdots \tilde{\gamma}_k \cdots \tilde{x}_{k-1} \). \( \square \)
The following lemmas show that $\mathcal{M}_m$ satisfies some technical conditions required in Section 6 for the main construction.

**Lemma 15.** Let $\mathcal{M}$ be a regular $n$-maniplex, and let $\mathcal{S}_0$ be a set of facets of $\mathcal{M}$ such that $\mathcal{S}_0$ is not invariant under any non-trivial element of $\text{Aut}(\mathcal{M})$. Then there exists an involutory element $\eta \in \text{Mon}(2^\mathcal{M})$ such that it maps every two distinct flags of $2^\mathcal{M}$ in the same facet to flags in distinct facets.

**Proof.** We choose the base facet $F_0$ of $\mathcal{M}$ in such a way that $F_0 \in \mathcal{S}_0$. For every $F \in \mathcal{S}_0$ we choose a flag $\Phi_F$, and we let $\Phi_0 = \Phi_{F_0}$ be the base flag. For each flag $\Phi_F$ let $\omega_F$ be the element in $\text{Mon}(\mathcal{M})$ that maps $\Phi_0$ to $\Phi_F$, and let $\mathcal{W} = \{\omega_F : F \in \mathcal{S}_0\}$.

Denote the standard generators of $\mathcal{M}$ by $r_0, \ldots, r_{n-1}$ and let the standard generators $r'_0, \ldots, r'_{n-1}$ of $2^\mathcal{M}$ be as in [4.2] and [4.3]. Note that for every $\omega \in \mathcal{W}$, the conjugate $(r'_n)^\omega$ maps a flag $(\Phi, x)$ of $2^\mathcal{M}$ to $(\Phi, x + \chi_{F}(\phi_{\omega}))$. Let $\eta = \prod_{\omega \in \mathcal{W}} (r'_n)^\omega \in \text{Mon}(2^\mathcal{M})$, that is,

$$\eta : (\Phi, x) \mapsto (\Phi, x + \sum_{\omega \in \mathcal{W}} \chi_{F}(\phi_{\omega})).$$

Clearly, $\eta$ is an involution.

Let $\Phi, \Psi$ be two distinct flags in the same facet of $2^\mathcal{M}$. By regularity of $2^\mathcal{M}$ we may assume that $\Phi = \Phi_0$, and $\Psi$ can be expressed as $\Phi^\nu$ for some non-trivial $\nu \in \langle r'_0, \ldots, r'_{n-1} \rangle$. It suffices to show that $\Phi^0_0$ and $\Phi^\nu_0$ belong to distinct facets, that is, there exists no $\nu' \in \langle r'_0, \ldots, r'_{n-1} \rangle$ such that $\Phi^0_0 = \Phi^\nu_0$. Since the action of the monodromy group of $2^\mathcal{M}$ is regular on the flags, this is equivalent to saying that $\nu^0 \notin \langle r'_0, \ldots, r'_{n-1} \rangle$.

We know that

$$
(\phi, x)^{\nu^0} = (\phi, x + \sum_{\omega \in \mathcal{W}} \chi_{F}(\phi_{\omega}))^{\nu^0} = (\phi^\nu, x + \sum_{\omega \in \mathcal{W}} \chi_{F}(\phi_{\omega}))^{\eta}
$$

(4.10)

$$
= (\phi^\nu, x + \sum_{\omega \in \mathcal{W}} (\chi_{F}(\phi_{\omega}) + \chi_{F}(\phi_{\omega}))).
$$

(4.11)

From the definition of the standard generators of $\text{Mon}(2^\mathcal{M})$, the element $\nu^0$ belongs to $\langle r'_0, r'_1, \ldots, r'_{n-1} \rangle$ if and only if $\sum_{\omega \in \mathcal{W}} (\chi_{F}(\phi_{\omega}) + \chi_{F}(\phi_{\omega}))$ is trivial.

Write $\nu = r'_{i_1} r'_{i_2} \cdots r'_{i_k}$ and let $\{\rho_0, \rho_1, \ldots, \rho_{n-1}\}$ be the standard generators of $\text{Aut}(\mathcal{M})$. Then

$$
\phi^{\nu^0} = \phi^{r'_{i_1} r'_{i_2} \cdots r'_{i_k}} = (\phi^{\rho_{i_k} \cdots \rho_{i_1}})^{\omega} = \phi^{\omega\rho_{i_k} \cdots \rho_{i_1}}
$$

and therefore $F(\phi^{\nu^0}) = F(\phi^{\omega} \rho_{i_k} \cdots \rho_{i_1}) = F(\phi^{\omega}) \rho_{i_k} \cdots \rho_{i_1}$.

For every $\omega \in \mathcal{W}$ the facet $F(\phi_{\omega})$ belongs to $\mathcal{S}_0$, then the expression in 4.11 can be written as

$$
(\phi^\nu, x + \sum_{F \in \mathcal{S}_0} \chi_{F} + \sum_{F \in \mathcal{S}_0 \rho_{i_k} \cdots \rho_{i_1}} \chi_{F}).
$$

By hypothesis, $\mathcal{S}_0 \rho_{i_k} \cdots \rho_{i_1} \neq \mathcal{S}_0$, and therefore $\sum_{\omega \in \mathcal{W}} (\chi_{F}(\phi_{\omega}) + \chi_{F}(\phi_{\omega}))$ is non-trivial and $\nu^0$ does not belong in $\langle r'_0, r'_1, \ldots, r'_{n-1} \rangle$. □

**Lemma 16.** Let $\mathcal{M}$ be a regular $m$-maniplex defined in Definition 13. Then there exists a set of facets $\mathcal{S}_0$ of $\mathcal{M}$ such that $\mathcal{S}_0$ is not invariant under any non-trivial element of $\text{Aut}(\mathcal{M})$. 


Figure 4.2. The toroidal map \{4,4\}_{(4,0)}. The set of shaded facets is fixed only by the identity

Proof. The proof is by induction on \(m\). If \(m = 3\), then \(\mathcal{M}\) is isomorphic to the toroidal map \{4,4\}_{(4,0)} shown in Figure 4.2, viewed as a 3-maniplex. The set \(S_0\) can then be chosen to be the set of grey faces in the figure. The reader can convince herself that \(S_0\) is fixed by no non-trivial automorphism of the map \(\mathcal{M}\).

Now let \(n\) be an integer, \(n \geq 3\), and suppose that the lemma holds with \(m = n\). We will show that it then also holds for \(m = n + 1\). Let \(\mathcal{M} = \mathcal{M}_n\), so that \(\mathcal{M}_m = \mathcal{M}_{n+1} = \hat{2}\mathcal{M}\). Further, let \(S\) be the set of facets of \(\mathcal{M}\) and let \(S_0 \subset S\) be a set of facets fixed by no non-trivial automorphism of \(\mathcal{M}\).

Recall that a facet \(F\) of an \((n+1)\)-maniplex determined by the monodromy group \(\langle r_0, \ldots, r_n \rangle\) equals an orbit of \(\langle r_0', \ldots, r_{n-1}' \rangle\) acting on the set of flags of the maniplex.

Further, recall that the set of flags of \(\hat{2}\mathcal{M}\) is \(\mathcal{F}(\hat{2}\mathcal{M}) \times \mathbb{Z}_2^S\) (see (4.1)) and that the monodromy group of \(\hat{2}\mathcal{M}\) equals \(\langle r_0', \ldots, r_n' \rangle\) where \((\Phi, x)^{r_i'} = (\Phi^{r_i}, x)\) for \(i < n\) (see (4.2)) and \(r_n'\) is as in (4.3).

In particular, a typical facet of \(\hat{2}\mathcal{M}\) equals the set

\[
F_x = \{(\Phi, x) : \Phi \in \mathcal{F}(\mathcal{M})\}
\]

where \(x\) is an arbitrary element of \(\mathbb{Z}_2^S\).

We will now define a set of facets \(S'_0\) of \(\hat{2}\mathcal{M}\) and prove that it is fixed by no non-trivial automorphism of \(\hat{2}\mathcal{M}\):

\[
S'_0 = \{F_{x_H} : H \in S_0\} \cup \{F_0\}.
\]

(recall that \(0\) denotes the constant 0 function).

We claim that the facet \(F_0\) is the only facet in \(S'_0\) such that for each \(H \in S_0\) it contains a flag which is \(n\)-adjacent to some flag in \(F_{x_H}\). This will then imply that every automorphism of \(\hat{2}\mathcal{M}\) that preserves \(S'_0\) fixes the facet \(F_0\).

To prove the claim, note that if \(\Phi\) is a flag in \(H\) then \((\Phi, 0)^{r_n}\) belongs to \(F_{x_H}\), while \((\Phi, 0)\) belongs to \(F_0\). Now let \(H, H' \in S\). Then a flag in \(F_{x_H}\) is of the form \((\Phi, \chi_H)\), while a flag in \(F_{x_{H'}}\) is of the form \((\Phi', \chi_{H'})\) for some \(\Phi \in H\) and \(\Phi' \in H'\). Since \(\chi_H\) and \(\chi_{H'}\) differ in precisely two values, these two flags cannot be \(n\)-adjacent in \(\hat{2}\mathcal{M}\). This completes the proof of our claim that every automorphism of \(\hat{2}\mathcal{M}\) that preserves \(S'_0\) fixes the facet \(F_0\).
By part (2) of Proposition 7, an automorphism of $\hat{\mathcal{M}}^2$ can be written uniquely as a product $\tilde{y}\tilde{\gamma}$ for some $\gamma \in \text{Aut}(\mathcal{M})$ and $y \in \mathbb{Z}_2^2$, as defined in (4.5) and (4.6); that is

$$(\Phi, x)^\tilde{y} = (\Phi^{\gamma}, x^{\gamma}) \quad \text{and} \quad (\Phi, x)^\tilde{y} = (\Phi, x + y).$$

Suppose that $\tilde{y}\tilde{\gamma}$ preserves $S_0$. We have shown that then $\tilde{y}\tilde{\gamma}$ fixes $F_0$. Note that $\tilde{0}\tilde{\gamma} = 0$ and hence

$$F_0^\tilde{y} = \{(\Phi^{\gamma}, 0^\gamma) : \Phi \in \mathcal{F}(\mathcal{M})\} = F_0.$$ 

This implies that $\tilde{y}$ fixes $F_0$, which clearly implies that $y = 0$ and thus $\tilde{y}$ is trivial. Further, observe that $(F_{xH})^{\tilde{y}} = F_{xH^{\gamma}}$,

$$(F_{xH})^{\tilde{y}} = \{(\Phi, xH) : \Phi \in \mathcal{F}(\mathcal{M})\}^{\tilde{y}}$$

implies that

$$S_0^0 = S_0^{\tilde{y}} = \{(F_{xH})^{\tilde{y}} : H \in S_0\} \cup \{F_0^\tilde{y}\} = \{F_{xH^{\gamma}} : H \in S_0\} \cup \{F_0\}.$$ 

By the inductive hypothesis $\text{id}$ is the only automorphism of $\mathcal{M}$ that preserves $S_0$, hence $\gamma = \text{id}$ and the lemma follows.

\[\square\]

5. Voltage construction

In this section we briefly summarise a few facts about covering projections and voltage assignments that are significant for the proof of our main theorem. We refer the reader to [10] for further background on this topic. We then relate this theory with the notion of a 2-orbit maniplex.

Let $\Gamma = (D, V; \text{beg}, \text{inv})$ and $\Gamma' = (D', V'; \text{beg}', \text{inv}')$ be two pregraphs. A morphism of pregraphs, $f: \Gamma \to \Gamma'$, is a function $f: V \cup D \to V' \cup D'$ such that $f(V) \subseteq V'$, $f(D) \subseteq D'$, $f \circ \text{beg} = \text{beg}' \circ f$ and $f \circ \text{inv} = \text{inv}' \circ f$. A graph morphism is an epimorphism (automorphism) if it is a surjection (bijection, respectively). Furthermore, $f$ is a covering projection provided that it is an epimorphism and if for every $v \in V$ the restriction of $f$ onto the neighbourhood of $v$ is a bijection. If $f$ is a covering projection, then the group of automorphisms, that for every $x \in V' \cup D'$ preserve the preimage $f^{-1}(x)$ setwise, is called the group of covering transformations. If, in addition, the graph $\Gamma$ is connected and the group of covering transformations is transitive on each such preimage, then the covering projection $f$ is said to be regular and the graph $\Gamma$ is then called a regular cover of $\Gamma'$.

Note that if $H \leq \text{Aut}(\Gamma)$ acts semiregularly on $V(\Gamma)$, then the resulting quotient projection from $\Gamma$ to $\Gamma/H$ is a covering projection.

Let $\Gamma' = (D', V', \text{beg}', \text{inv}')$ be an arbitrary connected pregraph, let $N$ be a group and let $\zeta: D' \to N$ be a mapping (called a voltage assignment) satisfying the condition $\zeta(x) = \zeta(\text{inv'}x) = 1$ for every $x \in D'$. Then Cov$(\Gamma', \zeta)$ is the graph with $D' \times N$ and $V' \times N$ as the sets of darts and vertices, respectively, and the functions $\text{beg}$ and $\text{inv}$ defined by $\text{beg}(x,a) = (\text{beg}'x, a)$ and $\text{inv}(x,a) = (\text{inv'}x, \zeta(x)a)$. If, for a voltage assignment $\zeta: D(\Gamma') \to N$, there exists a spanning tree $T$ in $\Gamma'$ such that $\zeta(x)$ is the trivial element of $N$ for every dart in $T$, then we say that $\zeta$ is normalized; note that in this case Cov$(\Gamma', \zeta)$ is connected if and only if the image of $\zeta$ generates $N$. 

Two-Orbit Manifolds
It is well known that every regular cover of a given (pre)graph $\Gamma'$ is isomorphic to $\text{Cov}(\Gamma', \zeta)$ for some normalised voltage assignment $\zeta$. To be precise, let $\varphi : \Gamma \to \Gamma'$ be a regular covering projection and let $T$ be a spanning tree in $\Gamma'$. Then there exists a voltage assignment $\zeta : D(\Gamma') \to N$ which is normalised with respect to $T$, such that $\Gamma \cong \text{Cov}(\Gamma', \zeta)$.

We define a walk of length $n$ in a pregraph as a sequence $(x_1, x_2, \ldots, x_n)$ of darts such that $\text{beg}(x_{i+1}) = \text{beg}(x_i^{-1})$ for all $i \in \{1, \ldots, n-1\}$. A walk is said to be reduced if it contains no subsequence of the form $x_i, x_i^{-1}$. From this point on all walks will be reduced and will be referred to simply as walks.

If $\zeta : D(\Gamma) \to G$ is a voltage assignment, then we define the net voltage of a walk $(x_1, x_2, \ldots, x_n)$ in $\Gamma$ as the product $\zeta(x_1)\zeta(x_2)\ldots\zeta(x_n)$ in the voltage group. We will slightly abuse notation and use $\zeta(W)$ to denote the net voltage of a walk $W$.

For a vertex $u$, the local voltage group $\text{Loc}(G, u)$ is the group of net voltages of all closed walks based at $u$. The fiber of $u$ under a covering projection $\varphi : \text{Cov}(\Gamma, \zeta)$ will be denoted by $\text{fib}_u$.

Let us now move to the special situation of symmetry type graphs $\Gamma$ on two points introduced in Section 2. Recall that the darts of such a pregraph $\Gamma$ are coloured in such a way that each vertex has either a link or a semi-edge of each colour in $\{0, 1, \ldots, n-1\}$ incident to it (see figure 2.3).

Let $\Gamma$ be the pregraph $2^n_2$ and let $u$ and $v$ be its vertices. Denote by $u_i$ the unique $i$-coloured dart having initial vertex $u$. Define $v_i$ similarly. Observe that $\text{inv}(u_i) = u_i$ whenever $i \in I$ and $\text{inv}(u_i) = v_i$ otherwise. The following two propositions give sufficient conditions on a voltage assignment $\zeta$ for $\Gamma$ to lift into an $n$-maniplex.

**Proposition 17.** Let $n \geq 3$ be an integer, let $I \subseteq \{2, \ldots, n-1\}$, and let $\Gamma$ be the pregraph $2^n_2$. Let $G$ be a group with distinct non-trivial generators $\{z, y_2, y_3, \ldots, y_{n-1}\}$ such that

\begin{align*}
(5.1) & \quad y_i^2 = 1_G \\
(5.2) & \quad z^{y_i} = \begin{cases} 
  z & \text{if } i \geq 3 \text{ and } i \in I \\
  z^{-1} & \text{if } i \geq 3 \text{ and } i \notin I
\end{cases} \\
(5.3) & \quad [y_i, y_j] = 1_G \text{ if } |i - j| > 1
\end{align*}

If $\zeta : D(\Gamma) \to G$ is the voltage assignment satisfying $\zeta(u_0) = 1_G$, $\zeta(u_1) = z$ and $\zeta(u_i) = \zeta(v_i) = y_i$ for $i \geq 2$, then $\text{Cov}(\Gamma, \zeta)$ is a maniplex.

**Proof.** Let $\mathcal{M} := \text{Cov}(\Gamma, \zeta)$. Now, in $\mathcal{M}$, colour every dart in the fiber of $x$ with the same colour $x$ has in $\Gamma$. Note that $\zeta$ is normalised with respect to the spanning tree induced by the dart $u_0$, and therefore $\mathcal{M}$ is connected. Moreover, it is straightforward to see that $\mathcal{M}$ is an $n$-valent simple graph. It remains to show that the subgraph induced by colour $i$ and $j$ is a union of squares whenever $|i - j| \geq 2$. Clearly, a closed walk in $\Gamma$ lifts to a closed walk in $\mathcal{M}$ if and only if its net voltage is $1_G$. By observing $\Gamma$, it is not too difficult to notice that any walk of length 4 in $\mathcal{M}$ that traces darts of only two colours $i$ and $j$ with $|i - j| \geq 2$ necessarily ends and starts in vertices belonging to the same fibre. By taking a closer look at $\Gamma$, we also see that any such walk $W$ has net voltage $1_G$; for example: if $i \geq 2$ and $j = 0$, the net voltage of $W$ is either $y_i(1_G)y_j(1_G)$ or $y_i^{-1}(1_G)y_j^{-1}(1_G)$. From [5.1] we get that this equals $1_G$ in both cases. Further, suppose that $i \geq 3$ and $j = 1$. If $i \in I$, then the net voltage of $W$ is either $y_i z y_i z^{-1}$ or $y_i z^{-1} y_i z$. From [5.2] and the fact that $y_i$ is an involution, we have that $y_i z y_i z^{-1} = (y_i z y_i) z^{-1} = z z^{-1} = 1_G$ and $y_i z^{-1} y_i z = (y_i z^{-1} y_i) z = z^{-1} z = 1_G$. Then there exists a voltage assignment $\zeta : D(\Gamma') \to N$ which is normalised with respect to $T$, such that $\Gamma \cong \text{Cov}(\Gamma', \zeta)$.
Now, if \( i \not\in \mathcal{I} \), then \( W \) has net voltage \( y_i z^{-1} y_i z^{-1} \) or \( y_i z y_i z \). Again, using \ref{5.1} and \ref{5.2} we get \( y_i z^{-1} y_i z^{-1} = (y_i z^{-1} y_i) z^{-1} = z z^{-1} = 1_G \) and \( y_i z y_i z = z^{-1} z = 1_G \). Finally, if \( i, j \geq 2 \), the result follows from \ref{5.1} and \ref{5.3}. Since every closed walk of length 4 in a simple graph is a cycle, this implies that every walk of length 4 in \( \mathcal{M} \) with alternating colour \( i \) and \( j \), where \( |i - j| \geq 2 \), is a 4-cycle.

**Proposition 18.** Let \( n \geq 3 \) be an integer, let \( \mathcal{I} \subseteq \{1, \ldots, n - 1\} \) with \( 1 \in \mathcal{I} \), and let \( \Gamma \) be the pregraph \( 2^2_n \). Let \( G \) be a group with generators \( \{z, z', y_2, y_3, \ldots, y_{n-1}\} \) such that
\[
\begin{align*}
  z^2 &= (z')^2 = y_i^2 = 1_G \\
  z^{y_i} &= \begin{cases} 
    z & \text{if } i \geq 3 \text{ and } i \in \mathcal{I} \\
    z' & \text{if } i \geq 3 \text{ and } i \not\in \mathcal{I}
  \end{cases} \\
  (z')^{y_i} &= \begin{cases} 
    z' & \text{if } i \geq 3 \text{ and } i \in \mathcal{I} \\
    z & \text{if } i \geq 3 \text{ and } i \not\in \mathcal{I}
  \end{cases} \\
  [y_i, y_j] &= 1_G \text{ if } |i - j| > 1
\end{align*}
\]
If \( \zeta : D(\Gamma) \to G \) is the voltage assignment satisfying \( \zeta(u_0) = 1_G, \zeta(u_1) = z, \zeta(v_1) = z' \) and \( \zeta(u_i) = \zeta(v_i) = y_i \) for \( i \geq 2 \), then \( \text{Cov}(\Gamma, \zeta) \) is a maniplex.

In Figure 5.1 we show examples of the voltage assignments described in Propositions 17 and 18.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure51.png}
\caption{Voltage assignment for \( 2^7_{\{2,5,6\}} \) and \( 2^7_{\{1,5,6\}} \).}
\end{figure}

**Proof.** The proof is almost identical to the proof of Proposition 17 except for the case when \( i \geq 3 \) and \( j = 1 \) which, as the reader can verify, follows from \ref{5.5} \ref{5.6} and the fact that both \( z \) and \( z' \) are involutions. \( \square \)

Suppose now that \( G \) satisfies the relations \ref{5.1} \ref{5.2} \ref{5.3} in Proposition 17 (if \( 1 \not\in \mathcal{I} \)) or relations \ref{5.4} \ref{5.5} \ref{5.6} and \ref{5.7} in Proposition 18 (if \( 1 \in \mathcal{I} \)), so that \( \text{Cov}(G, \zeta) \) is a maniplex. Observe that \( G \) acts on the vertices of \( \text{Cov}(\Gamma, \zeta) \) by right multiplication on the second coordinate. This is, for every \( g \in G \), we can define a mapping \( f_g \) given by \( (u, h) \mapsto (u, hg) \). It is straightforward to see that this mapping is a covering transformation and, furthermore, that it preserves the colour of the darts of \( \text{Cov}(G, \zeta) \). That is, each \( f_g \) is an automorphism of the maniplex \( \text{Cov}(G, \zeta) \). This means that all the vertices within a fibre belong to the same orbit under the action of the (maniplex) automorphism group, and so \( \text{Cov}(G, \zeta) \) is an \( n \)-maniplex with at most 2 orbits. If no automorphism of \( \text{Cov}(G, \zeta) \) preserving the colouring (that is, automorphisms of the resulting maniplex) maps a vertex in \( \text{fib}_u \) to a vertex in \( \text{fib}_v \), then \( \text{Cov}(G, \zeta) \) is a 2-orbit \( n \)-maniplex and \( \Gamma \) is its symmetry type graph.
The non-existence of such an automorphism, however, will be somewhat intricate to prove.

The group $G$ of automorphisms of $\text{Cov}(G, \zeta)$ induced by right multiplication by elements of $G$ has index at most 2 in the full maniplex automorphism group $\text{Aut}(\text{Cov}(G, \zeta))$. Hence the orbits (on vertices) of $G$, which correspond precisely to the fibres of $u$ and $v$, are blocks of imprimitivity. It follows that, if an automorphism $\tau$ maps one vertex in $\text{fib}_u$ into a vertex in $\text{fib}_v$, then it must map every vertex in $\text{fib}_u$ to a vertex in $\text{fib}_v$. This is, $\tau$ must be a lift of $R$, the unique colour preserving automorphism of $\Gamma$ that interchanges $u$ and $v$. Thus, to show that $\text{Cov}(G, \zeta)$ is not a regular maniplex it suffices to show that $R$ does not lift. In light of the results shown in [10], we have that $R$ has a lift if and only $R$ maps isomorphically the group of closed walks based at $u$ into the group of closed walks based at $v$. The latter, in turn, implies that there is a induced group isomorphism $R^\#_i$ mapping the local voltage group $\text{Loc}(G, u)$ into $\text{Loc}(G, v)$ and satisfying $R^\#_i(\zeta(W)) = \zeta(R(W))$, for every closed walk $W$ based at $u$. Observe that, in the case of the pregraph $2^2_2$ with a voltage assignment like in Propositions 17 or 18, we have $G \cong \text{Loc}(G, u) \cong \text{Loc}(G, v)$. Hence, we have the following two propositions

**Proposition 19.** Let $\Gamma$ be the pregraph $2^2_2$ with $1 \notin I$ and let $\zeta : D(\Gamma) \to G$ be the voltage assignment described in Proposition 17. Let $R$ be the colour preserving automorphism interchanging the two vertices of $\Gamma$. If no automorphism of $G$ inverts $z$ while fixing $y_i$ for $i \in \{2, ..., n - 1\}$, then $R$ does not lift into an automorphism of $\text{Cov}(\Gamma, \zeta)$.

**Proposition 20.** Let $\Gamma$ be the pregraph $2^2_2$ with $1 \in I$ and let $\zeta : D(\Gamma) \to G$ be the voltage assignment described in Proposition 18. Let $R$ be the colour preserving automorphism interchanging the two vertices of $\Gamma$. If no automorphism of $G$ interchanges $z$ and $z'$ while fixing $y_i$ for $i \in \{2, ..., n - 1\}$, then $R$ does not lift into an automorphism of $\text{Cov}(\Gamma, \zeta)$.

6. Construction

In this section we construct 2-orbit $(n+1)$-maniplexes from regular $n$-maniplexes. In Theorems 29 and 31 we show that for any $I \subseteq \{1, ..., n\}$ there is a maniplex of type $2^2_{n+1}$. This, together with Corollary 12 proves Theorem 3.

Let $\Gamma$ be the pregraph $2^2_{n+1}$ and let $\mathcal{M}$ be a regular $n$-polytope admitting a bi-colouring (white and black) of flags consistent with $I$ (that is, $\Phi$ and $\Phi'$ have the same colour if and only if $i \in I$). Let $\text{Mon}_I(\mathcal{M})$ be the subgroup of $\text{Mon}(\mathcal{M})$ preserving the colours of the bi-colouring and assume there exists a facet $F_0$ and $\eta \in \text{Mon}_I(\mathcal{M})$ such that $\eta^2 = 1$ and that for every two flags $\Phi$ and $\Psi$ in $F$, $\Phi^n$ and $\Psi^n$ are in distinct facets of $\mathcal{M}$. Now, choose an arbitrary white flag $\Phi_0$ in $F_0$ and define it to be the base flag of $F_0$. Let $\Phi_1 = \Phi_0^n$ be the base flag of the facet $F_1$ containing $\Phi_1$. Let $X = F(\mathcal{M}) \times \mathbb{Z}_{2k}$ and $X_w$ the subset of $X$ consisting of pairs where the first element is a white flag.

We will provide a construction whose output is a permutation group acting on the set $X_w$. This group will serve as a voltage group to lift $\Gamma$ into a $(n+1)$-maniplex of type $2^2_{n+1}$.

The following two Remarks are a reminder of some of the contents of Section 2.
Remark 21. If $\mathcal{M}$ is an $n$-maniplex and $r_0, r_1, \ldots, r_{n-1}$ are the standard generators of $\text{Mon}(\mathcal{M})$ then
\[
\begin{align*}
\quad r_i^2 & = 1, \\
[r_i, r_j] & = 1 \text{ if } |i - j| > 1.
\end{align*}
\]

Remark 22. If $\Phi$ is a flag of a maniplex $\mathcal{M}$, then $\Phi$ and $\Phi^{r_0}$ belong to the same facet of $\mathcal{M}$.

Lemma 23. With the notation above, the following hold:

1. If $\Phi \in F_0$, then $\Phi^\eta \notin F_0$;
2. If $\Phi$ and $\Psi$ are distinct flags contained in the same facet of $\mathcal{M}$, then $\Phi^\eta$ and $\Psi^\eta$ are in distinct facets of $\mathcal{M}$;
3. If $\Phi$ and $\Psi$ are distinct white flags contained in the same facet of $\mathcal{M}$, then $\Phi^{r_0\eta^0}$ and $\Psi^{r_0\eta^0}$ are in distinct facets of $\mathcal{M}$.
4. If $\Phi$ is a white flag in $F_0$ other than $\Phi_0$, then $\Phi^{r_0\eta^0}$ is neither in $F_0$ nor in $F_1$.

Proof. In order to prove part (1), suppose that $\Phi^\eta \in F_0$ for some $\Phi \in F_0$ and let $\Psi$ be an arbitrary flag in $F_0$ other than $\Phi$. Since $\mathcal{M}$ is a regular polytope, there exists an automorphism $\gamma$ of $\mathcal{M}$ mapping $\Phi$ to $\Psi$. Note that $\gamma$ then preserves the facet $F_0$. Since the actions of $\text{Mon}(\mathcal{M})$ and $\text{Aut}(\mathcal{M})$ commute, it follows that $\Psi^\eta = (\Phi^\gamma)^\eta = (\Phi^\eta)^\gamma$. Since $\Phi^\eta \in F_0$ and since $\gamma$ preserves $F_0$, it follows that $\Psi^\eta \in F_0$. This contradicts the assumptions on $\eta$.

Observe that part (2) follows directly from the regularity of $\mathcal{M}$.

To prove part (3), observe that by Remark 22 and part (2), it follows that $\Phi^{r_0\eta}$ and $\Psi^{r_0\eta}$ are in distinct facets, and again by Remark 22 the result follows.

Finally, to prove part (4), observe that by part (1) and Remark 22, $\Phi^{r_0\eta^0} \notin F_0$. Further, since $\Phi^{r_0}$ is a black flag in $F_0$, it is not equal to $\Phi_0$, and by the definition of $\eta$, $\Phi_0^\eta$ and $\Phi^{r_0\eta}$ are in different facets of $\mathcal{M}$. In particular, $\Phi^{r_0\eta}$ is not in $F_1$, and by Remark 22 neither is $\Phi^{r_0\eta^0}$.

We will now fix a base flag in each facet of $\mathcal{M}$. For an arbitrary white flag $\Phi$ of $F_0$, let $\Phi^{r_0\eta^0}$ be the base flag of the facet that contains it. Observe that all base flags determined so far are white, and that by Lemma 23 no facet is given more than one base flag, in this way. For each remaining facet, choose its base flag to be an arbitrary white flag.

Let $\{r_0, r_1, \ldots, r_{n-1}\}$ be the distinguished generators of $\text{Mon}(\mathcal{M})$. We extend the action of $r_i$ to $X$ by letting $(\Phi, \ell)^{r_i} = (\Phi^{r_i}, \ell)$.

For each facet $F$, let $\rho^0_0$ be the automorphism of $F$ sending its distinguished base flag into its 0-neighbour, and define $\tilde{\rho}_0$ as the permutation on $\mathcal{F}(\mathcal{M})$ that acts like $\rho^0_0$ on the flags of each facet $F$. Note that the actions of $\tilde{\rho}_0$ and $r_0$ commute within each facet of $\mathcal{M}$. We extend the action of $\tilde{\rho}_0$ to $X$ by letting $(\Phi, \ell)^{\tilde{\rho}_0} = (\Phi^{\tilde{\rho}_0}, \ell)$ and therefore $r_0$ and $\tilde{\rho}_0$ commute as permutations of $X$.

Now let $y_n$ be the permutation of $X$ satisfying:

\[
(\Phi, \ell)^{y_n} = \begin{cases} 
(\Phi^{\tilde{\rho}_0^{r_0}}, \ell + 1) & \text{if } \ell \text{ odd and } \Phi \in F_0, \text{ or } \ell \text{ even and } \Phi \in F_1; \\
(\Phi^{\tilde{\rho}_0^{r_0}}, \ell - 1) & \text{if } \ell \text{ even and } \Phi \in F_0, \text{ or } \ell \text{ odd and } \Phi \in F_1; \\
(\Phi^{\tilde{\rho}_0^{r_0}}, \ell) & \text{otherwise.}
\end{cases}
\]

Since $r_0$ and $\tilde{\rho}_0$ map white flags to black flags, $y_n$ permutes $X_w$. 
In the following paragraphs we will define a set of generators for a voltage group $G \subset S_{X_w}$ for $\Gamma$ and then prove that it has some desirable properties. Namely, that it agrees with the hypothesis of Propositions 17 and 19 when $1 \notin I$, or those of Propositions 18 and 20 when $1 \in I$. We will produce two slightly different constructions yielding two different groups: one when $1 \in I$ and another when $1 \notin I$.

6.1. Suppose $1 \notin I$. Define:

$$z = r_0r_1,$$

$$y_i = \begin{cases} r_i & \text{if } i \in I, \\ r_ir_0 & \text{if } i \notin I. \end{cases}$$

for $i \in \{2, \ldots, n-1\}$.

Note that $z$ and $y_i$ are in $\text{Mon}_\Gamma(\mathcal{M})$ and hence can be seen as permutations of $X_w$.

We define $G = \langle z, y_2, \ldots, y_n \rangle$ and note that it acts on $X_w$.

**Lemma 24.** The group $G$ just defined together with its generators $z, y_2, \ldots, y_n$ satisfy relations (5.1), (5.2) and (5.3).

**Proof.** For $i \in \{2, \ldots, n-1\}$, the relations (5.1) and (5.3) follow from Remark 21. To prove (5.1) for $i = n$, observe that $y_n^2 = 1$ follows directly from the definition of $y_n$. On the other hand, if $i < n - 1$ then $\Phi$ and $\Phi^r_i$ belong to the same facet for every flag $\Phi$, and (5.3) follows from Remark 21 and the fact that $\rho_0$ commutes with $r_i$ for all $i \in \{0, \ldots, n-2\}$.

Let us now prove relation (5.2). Suppose first that $i \in \{3, \ldots, n-1\}$ and recall that then $r_i$ commutes with $r_0$ and $r_1$. If $i \in I$, then $z^y = r_0r_1r_i = r_0r_1r_0^2 = z$. If $i \notin I$, then $z^y = r_0r_1r_i r_0 = r_1r_0r_1^2 = z^{-1}$. In order to prove the relation (5.2) for $i = n$, recall that for any flag $\Phi$ the flags $\Phi^\rho$ and $\Phi$ belong to the same facet. The fact that $z^y = z^{-1}$ follows directly from the definition of $y_n$ and the fact that $r_0r_1 = z^{-1}$. \qed

In what follows we will define an element $\mu$ of $G$ and we will show that, if an automorphism $\tau \in \text{Aut}(G)$ fixes all $y_i$’s while inverting $z$, then $\mu$ and its image under $\tau$ cannot have the same order. Thus proving that such an automorphism cannot exists.

Let $\mu = (y_n)^2$. Now let $\ell$ be an arbitrary odd element of $\mathbb{Z}_{2k}$. Then:

$$(\Phi_0, \ell)^\mu = (\Phi_0, \ell)^{y_n^\ell y_n} = (\Phi_1, \ell)^{y_n y_n} = (\Phi_1, \ell - 1)^{y_n}$$

Since $\Phi_1$ is the base flag of the facet $F_1$, the permutation $r_0 \rho_0$ fixes it. Therefore, the above equals:

$$(\Phi_1, \ell - 1)^{y_n} = (\Phi_1, \ell - 1)^{y_n} = (\Phi_0, \ell - 1)^{y_n} = (\Phi_0, \ell - 2)^{y_n} = (\Phi_0, \ell - 2),$$

as illustrated in Figure 6.1. By applying this argument $k$ times, we get the following lemma. The orbit of $\langle \mu \rangle$ containing $\langle \Phi_0, 1 \rangle$ consists of all the elements of the form $\langle \Phi_0, \ell \rangle$ for $\ell$ odd and thus:

**Lemma 25.** The orbit of $\langle \mu \rangle$ containing $\langle \Phi_0, 1 \rangle$ has size $k$.

We want to show that no automorphism of $G = \langle z, y_2, y_3, \ldots, y_{n-1}, y_n \rangle$ fixes all generators $y_i$ while inverting $z$. Suppose one such automorphism, $\tau$, exists. Observe
that the action of $\tau$ upon the subgroup $H = \langle y_2, y_3, \ldots, y_{n-1}, z \rangle \leq G$ coincides with the conjugation by $r_0$. In particular, since $\eta \in H$ and $y_n^\tau = y_n$, we see that

\begin{equation}
\mu^\tau = ((\eta y_n^\tau)^2)^\tau = (\eta y_n)^2 = (r_0 \eta y_n)^2.
\end{equation}

For $\ell \in \mathbb{Z}_{2k}$ let $X_\ell = \{(\Phi, \ell) : \Phi \in F(M)\}$. The aim of the following three lemmas is to show that an orbit of every element $(\Phi, \ell) \in X_\ell$ under $(\mu^\tau)$ is contained either in $X_\ell \cup X_{\ell+1}$ or in $X_\ell \cup X_{\ell-1}$. This will then imply that for a sufficiently large $k$, the order of $\mu$ is larger than the order of $\mu^\tau$, contradicting our assumption on $\tau$.

**Lemma 26.** Let $\Phi$ be a white flag in $F_0$. Then $\Phi^\eta$ and $\Phi^{r_0 \eta r_0}$ are fixed by $\tilde{\rho}_0 r_0$.

**Proof.** The flags $\Phi^\eta$ and $\Phi^{r_0 \eta}$ are base flags of some facets of $M$, and therefore $\tilde{\rho}_0$ acts like $r_0$ on each of them. Then $\Phi^\eta r_0 = \Phi^\eta$, and since $\tilde{\rho}_0$ and $r_0$ commute, $(\Phi^{r_0 \eta r_0})_{\tilde{\rho}_0 r_0} = \Phi^{r_0 \eta r_0} = \Phi^{r_0 \eta \eta}$.

**Lemma 27.** Let $\Phi$ be a white flag, let $\ell \in \mathbb{Z}_{2k}$. Let $(\Psi_1, \ell_1) = (\Phi, \ell)^{r_0 \eta \rho_0 r_0 \eta r_0}$ and let $(\Psi_2, \ell_2) = (\Phi, \ell)^{r_0 \eta r_0}$. If $\Psi_1$ or $\Psi_2$ is in $F_0$, then the orbit of $(\Phi, \ell)$ under $\langle \mu^\tau \rangle$ is of length 2.

**Proof.** Suppose first that $\Psi_1$ is in $F_0$. Let $(\Psi, \ell') = (\Psi_1, \ell_1)^{y_n}$ and observe that $\Psi \in F_0$ and that $(\Psi, \ell') = (\Phi, \ell')^{\mu^\tau}$. We will now show that the orbit of $(\Psi, \ell')$ under $(\mu^\tau)$ has length 2, implying that so does the orbit of $(\Phi, \ell)$.

By part (4) of Lemma 23 it follows that $\Psi^{r_0 \eta r_0}$ is neither in $F_0$ nor in $F_1$. Therefore $(\Psi^{r_0 \eta r_0}, \ell')^{y_n} = (\Psi^{r_0 \eta \rho_0 r_0}, \ell')$, and by Lemma 26, this equals $(\Psi^{r_0 \eta r_0}, \ell')$. Hence

\begin{equation}
(\Psi, \ell')^{\mu^\tau} = (\Psi^{r_0 \eta r_0})(y_n^{r_0 \eta r_0 y_n} = (\Psi^{r_0 \eta r_0}, \ell')^{r_0 \eta r_0 y_n} = (\Psi^{r_0 \eta r_0}, \ell')^{y_n}\eta
\end{equation}

By the definition of $y_n$, the latter equals $(\Psi, \ell' + 1)$ if $\ell'$ is odd, and $(\Psi, \ell' - 1)$ if $\ell'$ is even. By applying $\mu^\tau$ to $(\Psi, \ell')^{\mu^\tau}$, we thus obtain $(\Psi, \ell)$. This shows that the orbit $(\Psi, \ell)$ under the action of $(\mu^\tau)$ contains precisely two elements.

Suppose now that $\Psi_2$ is in $F_0$ and recall that $(\Psi_2, \ell_2) = (\Phi^{r_0 \eta r_0}, \ell)$. Then $(\Phi^{r_0 \eta r_0})^{r_0 \eta r_0} = \Phi$ is the base flag of some facet of $M$, and by part (4) of Lemma 23, $\Phi$ belongs to neither $F_0$ nor $F_1$.

Therefore

\begin{equation}
(\Phi^{r_0 \eta r_0}, \ell)^{y_n} = (\Phi^{r_0 \eta r_0}(\tilde{\rho}_0 r_0), \ell + (-1)^\tau)
\end{equation}
where $\epsilon$ is 0 or 1 depending on whether $\ell$ is odd or even, respectively. Then
\[(\Phi, \ell)^{\mu^\tau} = (\Phi, \ell)^{(r_0 \rho \sigma_0 y_n)^2} = (\Phi, \ell)^{r_0 \rho \sigma_0 y_n} = (\Phi, \ell)^{\rho \sigma_0 (\rho_0 \sigma_0)} (r_0 \rho \sigma_0), \ell + (-1)^\epsilon y_n\]

Since $\Phi^{r_0 \rho_0} \in F_0$ and since $\rho_0$ preserves the set of flags of $F_0$, we see that $\Phi^{r_0 \rho_0} (\rho_0 \sigma_0) \in F_0$ and hence $\Phi^{r_0 \rho_0 (\rho_0 \sigma_0)} (r_0 \rho \sigma_0)$ is the base flag of some facet in $\mathcal{M}$, which by part (4) of Lemma 23 is neither $F_0$ nor $F_1$. By the definition of $y_n$, Lemma 26 and the computation above, it follows that
\[(\Phi, \ell)^{\mu^\tau} = (\Phi, \ell)^{(r_0 \rho \sigma_0 (\rho_0 \sigma_0)) (r_0 \rho \sigma_0), \ell + (-1)^\epsilon}.

Let us now apply $\mu^\tau$ one more time. Note that in the computation below we will make use of formula (6.3) and the fact that $y_n$ is an involution.
\[(\Phi, \ell)^{(\mu^\tau)^2} = (\Phi, \ell)^{(r_0 \rho \sigma_0 (\rho_0 \sigma_0)) (r_0 \rho \sigma_0), \ell + (-1)^\epsilon} (r_0 \rho \sigma_0 y_n)^2 = (\Phi, \ell)^{(r_0 \rho \sigma_0 (\rho_0 \sigma_0)) (r_0 \rho \sigma_0), \ell + (-1)^\epsilon y_n} = (\Phi, \ell)^{y_n}.

Recall that $\Phi$ is the base flag of a facet that is neither $F_0$ nor $F_1$. The definition of $y_n$ and Lemma 26 then imply that $(\Phi, \ell)^{y_n} = (\Phi, \ell)$. In particular, the orbit of $(\Phi, \ell)$ under $\langle \mu^\tau \rangle$ again consists of two elements only.

**Lemma 28.** For an element $\ell \in \mathbb{Z}_{2k}$ let $t$ be such that $\ell \in \{2t, 2t+1\}$. Then:

1. If $(\Phi, \ell)^{\mu^\tau} \notin X_{2t} \cup X_{2t+1}$, then either $\Phi^{r_0 \rho_0}$ or $\Phi^{r_0 \rho_0 (r_0 \rho_0)}$ is in $F_0$.
2. If the orbit of $(\Phi, \ell)$ under $\langle \mu^\tau \rangle$ is not contained in $X_{2t} \cup X_{2t+1}$, then its length is 2.

**Proof.** Part (1) follows directly from formula (6.2) and the definition of $y_n$. To prove part (2), suppose that the orbit of $(\Phi, \ell)$ under $\langle \mu^\tau \rangle$ is of length at least 3 and is not contained in $X_{2t} \cup X_{2t+1}$. Let $i$ be the smallest integer such that $(\Phi, \ell)^{(\mu^\tau)^{i+1}} \notin X_{2t} \cup X_{2t+1}$. Let $(\Psi, \ell') = (\Phi, \ell)^{(\mu^\tau)^{i}}$. Then $\ell' \in \{2t, 2t+1\}$ and by part (1), either $\Psi^{r_0 \rho_0}$ or $\Psi^{r_0 \rho_0 (r_0 \rho_0)}$ is in $F_0$. By Lemma 27, the orbit of $(\Psi, \ell')$ under $\langle \mu^\tau \rangle$ is of length 2, and thus so is the orbit of $(\Phi, \ell)$.

**Theorem 29.** The group $G = \langle z, y_2, \ldots, y_n \rangle$ defined as above satisfies relations (5.1), (5.2), (5.3) and for a sufficiently large $k$ (in the definition of $X$) it admits no automorphism that inverts $z$ while fixing $y_i$ for $i \in \{2, \ldots, n\}$.

**Proof.** Lemma 24 shows that $G$ satisfies the desired relations. Now suppose there exists an automorphism $\tau$ of $G$ that inverts $z$ while fixing $y_i$ for $i \in \{2, \ldots, n\}$. It follows from Lemma 28 that the orbit of any pair $(\Phi, \ell)$ under $\langle \mu^\tau \rangle$ is of size at most 2$|\mathcal{F}_w(\mathcal{M})|$, where $\mathcal{F}_w(\mathcal{M})$ denotes the set of white flags of $\mathcal{M}$. Recall that the orbit of $(\Phi, 1)$ under $\langle \mu \rangle$ is of size $k$. By choosing $k > 2|\mathcal{F}_w(\mathcal{M})|$ in the definition of $X$, we ensure that the order of $\mu$ is necessarily larger than the order of $\mu^\tau$, contradicting that $\tau$ is an automorphism of $G$. The result follows.

**6.2.** Suppose $1 \in \mathcal{I}$. Define:

$$
\begin{align*}
z &= r_1, \\
z' &= r_0 r_1 r_0, \\
y_i &= \begin{cases} r_i & \text{if } i \in \mathcal{I}, \\ r_1 r_0 & \text{if } i \notin \mathcal{I} \end{cases} \quad \text{for } i \in \{2, \ldots, n-1\},
\end{align*}
$$
Note that $z$, $z'$ and $y_i$ are in $\text{Mon}_I(M)$ and hence can be seen as permutations of $X_w$. Define $G = \langle z, z', y_2, \ldots, y_n \rangle$ and note that it acts on $X_w$.

**Lemma 30.** The group $G$ just defined together with its generators $z, z', y_2, \ldots, y_n$ satisfy relations (5.4), (5.5), (5.6) and (5.7).

**Proof.** From remark 21 we have that $z^2 = (z')^2 = y_i = 1$, when $i < n$. That $y_n = 1$ follows from the definition. Hence, condition (5.4) holds.

To show that condition (5.5) is satisfied first suppose $i \in I$. Then $z^y_i = y_i z y_i = r_i r_1 r_i = r_1 = z$. Now, if $i \notin I$ and $i < n$, then $z^y_i = r_i r_0 r_1 r_i r_0 = r_0 r_1 r_0 = z^r_0$. Finally, note that $\tilde{\rho}_0$ commutes with $r_0$ and $r_1$. Thus $z^{\rho}_n = z'$.

Let us now prove relation (5.6). Suppose $i \in I$, so $(z')^{y_i} = r_i r_0 r_1 r_i = r_i r_i r_0 r_i = z'$. If $i \notin I$ and $i < n$, this is just equivalent to condition (5.5) when $i < n$. Once more, since $\tilde{\rho}_0$ commutes with each $r_i$, $i \in \{0, 1\}$, we have that $(z')^{y_i} = z$. Condition (5.6) is therefore satisfied.

Now, if $i, j < n$, from Remark 21 we have that $[y_i, y_j] = 1$. On the other hand, if $i = n$, condition (5.7) follows from the fact that $\tilde{\rho}_0$ commutes with $r_i$ for all $i \in \{0, \ldots, n - 2\}$. □

Let $\mu = (\eta y_n)^2$, just as in the previous case.

We want to show that no automorphism of $G = \langle z, z', y_2, y_3, \ldots, y_{n-1}, y_n \rangle$ fixes all generators $y_i$ while interchanging $z$ and $z'$. Suppose one such automorphism, $\tau$, exists. Again, the action of $\tau$ upon the subgroup $H = \langle y_2, y_3, \ldots, y_{n-1}, z, z' \rangle \leq G$ coincides with the conjugation by $r_0$. Notice that Lemmas 26, 27 and 28 do not depend on $z$ nor $z'$, but only on the definitions of $y_n, \eta$ and the fact that $\tau$ acts like conjugation by $r_0$. Therefore, Lemmas 26, 27 and 28 also hold when $1 \in I$. We thus have the following theorem.

**Theorem 31.** The group $G = \langle z, z', y_2, \ldots, y_n \rangle$ defined as above satisfies relations (5.4), (5.5), (5.6) and (5.7), and for a sufficiently large $k$ (in the definition of $X$) it admits no automorphism that interchanges $z$ and $z'$ while fixing $y_i$, for $i \in \{2, \ldots, n\}$.

**Proof.** Lemma 30 shows that $G$ satisfies the desired relations. The proof that no automorphism of $G$ fixes $y_i$, $i \in \{2, \ldots, n\}$, while interchanging $z$ and $z'$ is identical to that of Theorem 29. □

We are finally ready to prove Theorem 3. The proof is by induction over $n$, where the base case $n = 3$ is known to be true (see for example 19).

Assume that the result is true for certain $n$ and let $\Gamma$ be the pregraph $22^{n+1}$. The result follows from Corollary 12 if $0 \in I$, and from Theorems 29 and 31 if $0 \notin I$.

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