VERBALLY CLOSED SUBGROUPS
OF FREE SOLVABLE GROUPS

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We establish a series of results on verbally closed and l-verbally closed subgroups of free solvable groups; here l is a natural number and the concept of l-verbal closedness is a generalization of the concept of verbal closedness corresponding to the value l = 1. Under certain assumptions, these subgroups turn out to be retracts and, consequently, are algebraically closed.

INTRODUCTION

We prove a series of statements on verbally closed and l-verbally closed subgroups of free solvable groups. Under certain assumptions, these subgroups turn out to be retracts and, consequently, are algebraically closed.

Algebraically closed objects play an important role in modern algebra. We recall basic definitions. Let $\mathcal{K}$ be a class of algebraic structures in a language $\mathcal{L}$. A structure $A \in \mathcal{K}$ is said to be algebraically closed in $\mathcal{K}$ if, for any positive $\exists$-sentence $\varphi(x_1, \ldots, x_n)$ in $\mathcal{L}$ with constants from $A$, the fact that $\varphi(x_1, \ldots, x_n)$ is true in some structure $B \in \mathcal{K}$ containing $A$ as a substructure implies that $\varphi(x_1, \ldots, x_n)$ is true in $A$. General properties of algebraically closed structures can be found, for instance, in [1, 2].

An interesting particular case of the notion of algebraic closedness appears when as a $\mathcal{K}$ we take the class sub($B$) of all substructures of a given structure $B$. Moreover, if as the class $\mathcal{K}$ we

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take a pair \( \{ A, B \} \), where \( A \) is a substructure of \( B \), then we can speak of \( A \) being algebraically closed in \( B \).

Now let \( L \) be a language of a group-theoretic signature and \( K \) some class of groups. In this case the above concepts can be worded in purely algebraic terms. For groups \( H \) and \( G \), we write \( H \leq G \) if \( H \) is a subgroup of \( G \), and we call \( G \) an extension of \( H \). By an equation \( w(x_1, \ldots, x_n; H) = h \), \( h \in H \), in variables (unknowns) \( x_1, \ldots, x_n \) with constants from \( H \) we mean the specified expression in which \( w(x_1, \ldots, x_n; H) \) is a group word in \( x_1, \ldots, x_n \) and elements of \( H \). We say that an equation \( w(x_1, \ldots, x_n; H) = h \) has a solution (is solvable) in a group \( H \), or, more generally, in an extension \( G \geq H \), if there exists a tuple of elements \( g_1, \ldots, g_n \) in \( H \) or respectively in \( G \) such that substituting \( g_1, \ldots, g_n \) for \( x_1, \ldots, x_n \) yields a true equality \( w(g_1, \ldots, g_n; H) = h \). An equation of the form \( w(x_1, \ldots, x_n) = h \) in which the left part does not depend on constants but depends only on variables is said to be split. The notions of a system of equations and of a solution to a system of equations are also introduced naturally.

The concept of being algebraically closed for a subgroup in a group is rephrased as follows.

**Definition 1.** Let \( G \) be a group. A subgroup \( H \leq G \) is algebraically closed in \( G \) if every finite system of equations with constants from \( H \) that is solvable in \( G \) has a solution in \( H \).

**Definition 2** [3]. Let \( G \) be a group. A subgroup \( H \leq G \) is verbally closed in \( G \) if, for every group word \( w(x_1, \ldots, x_n) \) in independent variables \( x_1, \ldots, x_n \) without constants and for an arbitrary element \( h \in H \), the split equation

\[
w(x_1, \ldots, x_n) = h
\]  

(1)
solvable in \( G \) is solvable in \( H \).

The above concept is weaker than algebraic closedness. Nevertheless, it is connected with the concept of a retract, which is even stronger than being algebraically closed.

**Definition 3.** A retract of a group \( G \) is a subgroup \( H \leq G \) for which there exists an endomorphism \( \varphi : G \to H \) identical on \( H \). The endomorphism \( \varphi \) is called a retraction.

Let \( F_r \) be a free group of rank \( r \). Its obvious retracts are free factors. For \( r \geq 2 \), however, \( F_r \) also admits retracts that are not free factors. For instance, an element of the form \( g = f_1 u \) in a free group \( F_r \) with basis \( \{ f_1, \ldots, f_r \} \), where \( u \) belongs to the normal closure of the elements \( f_2, \ldots, f_r \), generates a cyclic retract \( R \). A retraction, in this case, is an endomorphism \( \rho : F_r \to R \) defined by a mapping \( f_1 \mapsto g, f_j \mapsto 1 \) for \( j = 2, \ldots, r \), while a cyclic subgroup generated by an element \( g \) will be a free factor of the group \( F_r \) iff \( g \) is a primitive element, i.e., an element of some basis of \( F_r \). If \( r = 2 \) and \( u \) belongs to the factor subgroup of \( F_2 \), then an element \( g = f_1 u \) is primitive iff it is conjugate to \( f_1 \). This follows from the classical result due to Nielsen which says that an automorphism of the group \( F_2 \) that is identical modulo the factor subgroup is inner. For instance, an element \( g = f_1 f_2^{-1} f_1^{-1} f_2 f_1 \) generates in \( F_2 \) a retract, but not a free factor.

**Remark 1.** If a subgroup \( H \) of a group \( G \) is a retract, then \( H \) is algebraically closed in \( G \). In fact, in order to derive a solution for the equation \( w(x_1, \ldots, x_n; H) = h \) in \( H \) from the solution \( x_i = g_i \),
For any natural number $l$, a subgroup $H \leq G$ is \textit{l-verbally closed} in $G$ if any system consisting of $l$ split equations in an arbitrary number $n$ of unknowns,

$$w_i(x_1, \ldots, x_n) = h_i, \quad h_i \in H, \quad i = 1, \ldots, l,$$

which has a solution in $G$ is solvable in $H$.

For $l = 1$, the above definition yields verbal closedness.

**Remark 2.** The property of a subgroup $H$ being $l$-verbally closed in $G$ does not generally follow from the verbal closedness of $H$ in $G$. In fact, every equation $w(x_1, \ldots, x_n; H) = h$, $h \in H$, is equivalent to a system of split equations which can be obtained by replacing constants in this equation with new variables, simultaneously writing into the system the split equations of replacement. If $H$ is not algebraically closed in $G$, then it is not $l$-verbally closed for some $l$. As noted in Remark 1, there exist verbally but not algebraically closed subgroups of some groups. These are not $l$-verbally closed for sufficiently large $l$.

In what follows, $g^f = fgf^{-1}$ will stand for conjugation, and by $[f, g] = fgf^{-1}g^{-1}$ we denote a commutator for elements $f$ and $g$ of an arbitrary group $G$. The derived subgroup of a group
$G$ is denoted by $G'$. A member of the lower central series of $G$ with number $i$ is denoted $\gamma_i G$. In this case $\gamma_1G = G$ and $\gamma_2G = G'$. A member of the derived series of $G$ with number $i$ is denoted $G^{(i)}$, in which case $G^{(0)} = G$ and $G^{(1)} = G'$. We write $\langle g_1, \ldots, g_k \rangle$ for a subgroup generated by elements $g_1, \ldots, g_k$ of some group $G$. For the varieties of Abelian groups and of nilpotent groups of nilpotency class at most $c$, we use the designations $\mathfrak{A}$ and $\mathfrak{N}_c$ respectively. By $\mathfrak{A}^d$ we denote the variety of all solvable groups of derived length at most $d$. For arbitrary varieties $\mathfrak{B}_i$, $i = 1, \ldots, k$, by writing $\prod_{i=1}^k \mathfrak{B}_i$ we mean their product. The designation $F_r(\mathfrak{B})$ is used for a free group of rank $r$ in $\mathfrak{B}$. For free groups of some varieties, we adopt special notation: for example, $A_r = F_r(\mathfrak{A})$, $N_{r,c} = F_r(\mathfrak{N}_c)$, and $S_{r,d} = F_r(\mathfrak{A}^d)$.

Let $H$ be a finitely generated subgroup of $G$. Then $r_{ab}(H)$ denotes the rank, i.e., the least number of generators of an Abelian group $HG'/G'$. Let $V$ be a variety of groups. If $H$ is a verbally closed subgroup of $G$, then $H \cap V(G) = V(H)$. In particular, $H \cap G' = H'$. Therefore, $H/H' \cong HG'/G'$. From this, we readily obtain

**PROPOSITION 1.** Let $G = \langle g_1, \ldots, g_k, \ldots \rangle$ be a solvable group and $H$ its subgroup for which $r_{ab}(H) = 0$, i.e., $H \leq G'$. If $H$ is verbally closed in $G$, then $H$ is a trivial subgroup.

**Proof.** Indeed, $H = H \cap G' = H'$. Hence $H = 1$. $\square$

We formulate basic results of the paper.

**THEOREM 1.** Let $H$ be a finitely generated verbally closed subgroup of a free solvable group $S_{r,d}$. If $r_{ab}(H) = 1$, then $H$ is a retract.

**THEOREM 2.** Let $H$ be a finitely generated verbally closed subgroup of a free solvable group $S_{r,d}$. If $r_{ab}(H) = r$, then $H = S_{r,d}$.

**THEOREM 3.** Every 2-generated verbally closed subgroup $H$ of a free solvable group $S_{r,d}$ is a retract.

**THEOREM 4.** For $l$ arbitrary, every finitely generated $l$-verbally closed subgroup $H$ of a free metabelian group $M_r$ for which $r_{ab}(H) = l + 1$ is a retract.

1. PRELIMINARY INFORMATION AND RESULTS

Recall that a set $\{g_1, \ldots, g_n\}$ of elements of a group $G$ is called a test set if every endomorphism $\varphi : G \to G$ fixing each of the elements $g_i$, $i = 1, \ldots, n$, is an automorphism of the group $G$. An element $g \in G$ is referred to as a test element if $\{g\}$ is a test set. The least of cardinalities of test sets is called the test rank of $G$ and is denoted by $tr(G)$. For available results on test sets and ranks of solvable groups, see [14-19].

Let $F_r$ be a free group of rank $r$ and $V$ its normal subgroup. We call a test set $\{g_1, \ldots, g_n\}$ of a group $G = F_r/V'$ a strongly test set if every endomorphism $\varphi$ leaving each element of this set fixed is an inner automorphism corresponding to conjugation by an element of $V/V'$.

Recall that a variety of the form $\mathfrak{N}_{c_1} \ldots \mathfrak{N}_{c_m}$, $m \geq 1$, is said to be polynilpotent of class $\bar{c} = (c_1, \ldots, c_m)$. Consider a polynilpotent variety $\mathfrak{N}_{\bar{c}} = \mathfrak{N}_{c_1} \ldots \mathfrak{N}_{c_m}$, $m \geq 1$, of class $\bar{c} =$ 256
(2, c_1, \ldots, c_m). Note that for \( \bar{c} = (2, 2, \ldots, 2) \), we obtain a variety \( \mathfrak{A}^{m+1} \). A consequence of the theorems and their proofs in [16, 17] is

**Lemma 1.** A free group \( F_r(\mathfrak{N}_c) \) of rank \( r \geq 2 \) in a variety \( \mathfrak{N}_c, m \geq 1 \), has a strongly test set consisting of \((r-1)\) elements.

**Proof.** Let \( F_r/V \) be a free group of rank \( r \) in a polynilpotent variety \( \mathfrak{N}_{c_1} \cdots \mathfrak{N}_{c_m}, m \geq 1 \). The main result of [16] was computing test rank of a free solvable group. There, in fact, something more was done: namely, the test rank of a group \( F_r/V' \), including the test rank of a free group in a polynilpotent variety \( \mathfrak{A}\mathfrak{N}_{c_1} \cdots \mathfrak{N}_{c_m} \), was computed. Moreover, the proof of the main theorem in [16] shows how to construct a test set \( \{g_1, \ldots, g_{r-1}\} \) for a group \( F_r/V' \) that possesses the following extra property: if some endomorphism \( \varphi \) leaves every element in this set fixed, then \( \varphi \) is an inner automorphism acting identically on a subgroup \( V/V' \). In [17], it was stated that the specified inner automorphism is induced by an element of \( V/V' \), i.e., \( \{g_1, \ldots, g_{r-1}\} \) is a strongly test set. □

**Lemma 2.** Let \( G = F_r/V' = F_r(\mathfrak{A}\mathfrak{N}_{c_1} \cdots \mathfrak{N}_{c_m}) \), \( m \geq 1 \), \( r \geq 2 \), \( \{z_1, \ldots, z_r\} \) be a basis for \( G \), \( H \) be a subgroup of \( G \) isomorphic to \( F_n(\mathfrak{A}\mathfrak{N}_{c_1} \cdots \mathfrak{N}_{c_m}) \) for some \( n \geq 2 \), and \( T = \{t_1(z_1, \ldots, z_r), \ldots, t_{n-1}(z_1, \ldots, z_r)\} \) be a strongly test set in \( H \). Suppose that

\[
t_i(h_1, \ldots, h_r) = t_i(z_1, \ldots, z_r), \quad i = 1, \ldots, n-1,
\]

for some elements \( h_1, \ldots, h_r \in H \). Then the mapping

\[
z_j \mapsto h_j, \quad j = 1, \ldots, r,
\]

extends to a retraction \( G \to H \).

**Proof.** We extend the mapping \( z_j \mapsto h_j, j = 1, \ldots, r \), to an endomorphism \( \varphi : G \to H \). Let \( \varphi|_H \) be the restriction of \( \varphi \) to \( H \). This restriction is an endomorphism of the group \( H \).

We have

\[
\varphi|_H(t_i(z_1, \ldots, z_r)) = \varphi(t_i(z_1, \ldots, z_r)) = t_i(\varphi(z_1), \ldots, \varphi(z_r)) = t_i(h_1, \ldots, h_r) = t_i(z_1, \ldots, z_r), \quad i = 1, \ldots, n-1.
\]

Hence \( \varphi|_H \) is an inner automorphism \( \alpha_a \) of a subgroup \( H \) induced by an element \( a \in V/V' \). Consider \( \psi = \varphi_{a^{-1}} \varphi \), where \( \varphi_{a^{-1}} \) is an inner automorphism of \( G \) induced by an element \( a^{-1} \). For \( h \in H \), we obtain

\[
\psi(h) = \varphi_{a^{-1}} \varphi(h) = \varphi_{a^{-1}}(h^a) = h,
\]

and for \( g \in G \), we have

\[
\psi(g) = (\varphi(g))^{a^{-1}} \in H.
\]

Hence \( \psi \) is the desired retraction and \( H \) is a retract. □

Lemmas 2 and 3 can be combined to yield the following:

**Proposition 2.** Let a subgroup \( H \) of a group \( F_r(\mathfrak{A}\mathfrak{N}_{c_1} \cdots \mathfrak{N}_{c_i}) \) be generated by \((m+1)\) elements and let the rank of a group \( H_{ab} = H/H' \) be equal to \( m+1 \). If \( H \) is \( m \)-verbally closed, then it is a retract.
**LEMMA 3** [9]. Let $N$ be a verbal subgroup of $G$. If $H$ is a verbally closed subgroup of $G$, then its image $H_N = HN/N$ is verbally closed in $G_N = G/N$. In particular, if the factor group $G/G'$ is a free Abelian group of finite rank, then the image $\hat{H}$ of its verbally closed subgroup $H$ in $G/G'$ is a direct factor.

**LEMMA 4.** Let $G = S_{r,d}$ be a free solvable group of derived length $d \geq 2$ with basis $z_1, \ldots, z_r$, $r \geq 2$, and let $H$ be a verbally closed subgroup of $G$ generated by elements $c_1 z_1, \ldots, c_r z_r, c_{r+1} \ldots$, where all $c_i$ are in $G^{(d-1)}$. Then $H^{(d-1)}$ is a $\mathbb{Z}[H/H^{(d-1)}]$-closed subgroup of $G^{(d-1)}$, i.e., if $c^\alpha \in H^{(d-1)}$ holds for some $c \in G^{(d-1)}$ and some $0 \neq \alpha \in \mathbb{Z}[G/G^{(d-1)}]$, then $c \in H^{(d-1)}$.

**Proof.** Let $A = G/G^{(d-1)}$ and $B = H/H^{(d-1)}$. Note that $G^{(d-1)} \cap H = H^{(d-1)}$.

Let $1 \neq c^\alpha \in H^{(d-1)}$. We write out the element $c$ via the basis, i.e., set $c = c(z_1, \ldots, z_r)$. The element $\alpha$ is written in the form of a finite sum $\alpha = \Sigma m_p \overline{h}_p$, where $m_p \in \mathbb{Z}$ and $\overline{h}_p \in B$. Let $a_i = z_i G^{(d-1)}$. Thus each element $\overline{h}_p$ belongs to a subgroup generated by elements $a_1, \ldots, a_r$, i.e., $\alpha = \alpha(a_1, \ldots, a_r)$.

Consider an equation in unknowns $x_1, \ldots, x_r$:

\[
\begin{align*}
\underbrace{c(x_1, \ldots, x_r)}_{(1)} & \in \mathbb{Z}^m [1-\overline{p}_1] \cdots [1-\overline{p}_r, a_1, \ldots, a_r] \in \mathbb{Z}^m [1-\overline{p}_1] \cdots [1-\overline{p}_r, a_1, \ldots, a_r] \\
& = c(z_1, \ldots, z_r)(1-a_1^{-m_1})(1-a_2^{-m_2}) \cdots (1-a_r^{-m_r}) \alpha(a_1, \ldots, a_r),
\end{align*}
\]

where $m_i$ are integers. The equation is solvable in $G$. Hence it has a solution $x_i = h_i$ in $H$. Thus

\[
\begin{align*}
\underbrace{c(h_1, \ldots, h_r)}_{(2)} & \in \mathbb{Z}^m [1-\overline{p}_1] \cdots [1-\overline{p}_r, a_1, \ldots, a_r] \in \mathbb{Z}^m [1-\overline{p}_1] \cdots [1-\overline{p}_r, a_1, \ldots, a_r] \\
& = c(z_1, \ldots, z_r)(1-a_1^{-m_1})(1-a_2^{-m_2}) \cdots (1-a_r^{-m_r}) \alpha(a_1, \ldots, a_r).
\end{align*}
\]

Here $\overline{g}$ denotes the image of an element $g \in G$ in the group $A$.

On the group ring $\mathbb{Z}[G]$ we define left Fox derivatives $\partial_j(g)$ with values in the ring $\mathbb{Z}[A]$. (For more detailed information about Fox derivatives, see [18, 20-22].) Note that for $\alpha \in \mathbb{Z}[A]$ and $b \in G^{(d-1)}$,

\[
\partial_j(b^\alpha) = \alpha \partial_j(b).
\]

Computing the $j$th derivative in the left and right parts of (2), we obtain

\[
\begin{align*}
(1 - \overline{h}_1)(1 - \overline{h}_1^{-m_1}) \cdots (1 - \overline{h}_r)(1 - \overline{h}_r^{-m_r}) \alpha(\overline{h}_1, \ldots, \overline{h}_r) \partial_j(c(h_1, \ldots, h_r)) & = (1 - a_1^{-m_1})(1 - a_2^{-m_2}) \cdots (1 - a_r^{-m_r}) \alpha(a_1, \ldots, a_r) \partial_j(c).
\end{align*}
\]

(3)

Let $\Delta$ be a fundamental ideal of the ring $\mathbb{Z}[A]$. On $\Delta$, a valuation $\omega$ is defined [23]. Namely, for an element $u \in \Delta$, we put $\omega(u) = n$ if $u \in \Delta^m \setminus \Delta^{n+1}$.

We show that all elements $h_i$ do not lie in the derived subgroup $G'$. The element $c = c(z_1, \ldots, z_r)$ is distinct from 1. Therefore, $\partial_j(c) \neq 0$ for some $j$. We choose $j$ so that the norm of a derivative $\partial_j(c)$ be minimal among all the norms $\omega(\partial_1(c)), \ldots, \omega(\partial_r(c))$. We will assume that $j = 1$. Compare the norms for the left and right parts in (3). Using a derivation rule for a composite function yields

\[
\partial_1(c(h_1, \ldots, h_r)) = \partial_1(c)[\overline{h}_1, \ldots, \overline{h}_r] \partial_1(h_1) + \ldots + \partial_r(c)[\overline{h}_1, \ldots, \overline{h}_r] \partial_1(h_r),
\]

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where $\partial_i(c)[h_1, \ldots, h_r]$ is the result of substituting elements $\overline{h}_q$ for $z_q$ in the derivative $\partial_i(c)$. By the choice of $j$, $\omega(\partial_i(c(h_1, \ldots, h_r))) \leq \omega(\partial_1(c))$. If at least one element $h_i$ were in $G'$, then the norm of the left part of (3) would be smaller than that of the right part, since $\omega(1-a_i) = 1 < \omega(1-\overline{h}_i)$ if $h_i \in G'$.

In [16], the following statement was proved. Let $S$ be a free polynilpotent group of rank $r \geq 2$ with basis $\{s_1, \ldots, s_r\}$, let $\alpha$ be a nonzero element of the ring $\mathbb{Z}[S]$, and let $p_i$ be the lowest $s_i$-degree of an element $\alpha$ and $q_i$ the highest $s_i$-degree of that element. Suppose that an integer $m$ is greater than the sum

$$\sum_{i=1}^{r}(|p_i| + |q_i|).$$

For an element $y_1$ of $G$ which does not lie in the commutator subgroup $S'$, the equality

$$(1-s_i^m)\alpha = (1-y_1^m)\beta \tag{4}$$

is possible for some $\beta \in \mathbb{Z}[S]$ iff $y_1 = s_1$ or $y_1 = s_1^{-1}$.

We choose the required numbers $m_i$. We see that $\overline{h}_1 = a_1^{1\pm 1}$. The ring $\mathbb{Z}[B]$ has no zero divisors. Notice that the expression $(1-a_i^m)(1-a_i^{-m})$ is not changed when $a_i$ is replaced by $a_i^{-1}$. Therefore, if we cancel (3) one by one by all $(1-a_i^m)(1-a_i^{-m})$ we obtain $\overline{h}_i = a_i^{1\pm 1}$ for all $i = 1, \ldots, r$. Then $\partial_i(c(h_1, \ldots, h_r)) = \partial_i(c)$. This implies that

$$c = c(h_1, \ldots, h_r) \in H^{(d-1)}. \quad \square$$

### 2. PROOFS OF THE BASIC RESULTS

**Proof** of Theorem 1. Put $G = S_{r,d}$. Since $H/H' \cong HG'/G'$, $H/H'$ is a cyclic group. The group $H$ is approximated by nilpotent groups. Then $H$ is an Abelian group; hence it is cyclic. The subgroup $H$ has no intersection with the derived subgroup $G'$ because $H \cap G' = H'$. Therefore, $H$ embeds in a free Abelian group $G/G'$ of finite rank. By Lemma 3, $H$ is a retract of $G/G'$ and is distinguished in $G/G'$ by a direct factor. Hence $H$ is a retract of $G$. \( \square \)

**Proof** of Theorem 2. We use induction on $d$. The case $d = 1$ is obvious. Let $G = S_{r,d}$, $C = G^{(d-1)}$, and $A = G/C \cong S_{r,d-1}$. We suppose that $r, d \geq 2$. Based on the induction hypothesis and Lemma 3, we choose a basis $\{z_1, \ldots, z_r\}$ for $G$ so that

$$H = \langle z_1 c_1, \ldots, z_r c_r, c_{r+1}, \ldots, c_m \rangle,$$

where $c_1, \ldots, c_r, c_{r+1}, \ldots, c_m \in C$. It suffices to verify that every element $c \in C$ is in $H^{(d-1)}$.

An Abelian group $C$ is a right $\mathbb{Z}[A]$-module if the action of an element $a$ of $A$ on $c \in C$ is defined to be $a^{-1}ca$. Elements $a_i = z_i C$, $i = 1, \ldots, r$, constitute a basis for the group $A$. Let $T$ be
a free right \(\mathbb{Z}[A]\)-module with basis \(\{t_1, \ldots, t_r\}\). We use the Magnus embedding of \(G\) into the split extension

\[
M = \begin{pmatrix}
A & 0 \\
T & 1
\end{pmatrix}.
\]

For information on a Magnus embedding, see [20]. Under the embedding, \(z_i\) is mapped into the matrix

\[
\begin{pmatrix}
a_i & 0 \\
t_i & 1
\end{pmatrix}.
\]

The image of \(C\) in \(M\) is a submodule of \(T\) which is identified with \(C\). It is well known that an element \(t_1u_1 + \ldots t_ru_r, u_i \in \mathbb{Z}[A]\), belongs to \(C\) iff

\[
(a_1 - 1)u_1 + \ldots + (a_r - 1)u_r = 0.
\]

The image of \(C\) in \(M\) is a submodule of \(T\) which consists of elements satisfying (5). For it, we keep the notation \(C\).

It is well known that the ring \(\mathbb{Z}[A]\) satisfies the right Ore condition [24]; i.e., for any nonzero elements \(\alpha\) and \(\beta\) of \(\mathbb{Z}[A]\), there are \(\mu\) and \(\nu\) in \(\mathbb{Z}[A]\) such that \(\alpha\mu = \beta\nu \neq 0\). Furthermore, this ring has no zero divisors. Therefore, \(\mathbb{Z}[A]\) embeds in a right division ring of quotients, \(P\). Hence the module \(T\) embeds in a linear space \(V\) of dimension \(r\) over the division ring \(P\). From (5), it follows that the dimension of a subspace \(C^\beta\) generated by \(C\) in \(V\) equals \(r - 1\).

Consider some nonidentity elements \(w_i = w_i(z_1, z_i), i = 2, \ldots, r\), of \(C\). Using just the definition of a Magnus embedding, we can easily deduce that the image of an element \(w_i\) in the group \(M\) (in fact in the module \(T\)) has the form \(t_1\gamma_1 + t_i\gamma_i\), with \(\gamma_1\) and \(\gamma_i\) not being equal to zero. That \(t_1\gamma_1 + t_i\gamma_i, i = 2, \ldots, r\), are independent in \(T\) is obvious. We verify that \(w_i(z_1c_1, z_i c_i), i = 2, \ldots, r\), likewise are independent over the ring \(\mathbb{Z}[A]\).

Denote by \(G_1\) a subgroup generated by elements \(z_1c_1, \ldots, z_r c_r\). By Baumslag’s theorem [25], the mapping

\[
\varphi : z_i \mapsto z_i c_i, \ i = 1, \ldots, r,
\]

extends to an isomorphism between \(G\) and \(G_1\), and then to an isomorphism between groups \(C\) and \(G_1^{(d-1)}\) as well as between rings \(\mathbb{Z}[A]\) and \(\mathbb{Z}[G_1/G_1^{(d-1)}]\). Moreover, a \(\mathbb{Z}[A]\)-module \(C\) is isomorphic to a \(\mathbb{Z}[G_1^{(d-1)}]\)-module \(G_1^{(d-1)}\). All the isomorphisms will be denoted by \(\varphi\).

We verify that \(w_i(z_1c_1, z_i c_i), i = 2, \ldots, r\), are independent over \(\mathbb{Z}[A]\).

Assume to the contrary that the elements mentioned are dependent over \(\mathbb{Z}[A]\):

\[
w_2(z_1c_1, z_2 c_2)^{\beta_2(z_1, \ldots, z_r)} \ldots w_r(z_1c_1, z_r c_r)^{\beta_r(z_1, \ldots, z_r)} = 1
\]

for some \(\beta_i \in \mathbb{Z}[A]\) which are not simultaneously equal to zero. Consequently,

\[
w_2(z_1c_1, z_2 c_2)^{\beta_2(z_1c_1, \ldots, z_r c_r)} \ldots w_r(z_1c_1, z_r c_r)^{\beta_r(z_1c_1, \ldots, z_r c_r)} = 1.
\]
If we apply an isomorphism $\varphi^{-1}$ we are led to a contradiction with $w_2, \ldots, w_r$ being independent over $\mathbb{Z}[A]$.

Now we verify that the elements $w_i(z_1c_1, z_ic_i)$, $i = 2, \ldots, r$, are independent not only over $\mathbb{Z}[A]$ but also over $P$. It is well known that elements of the division ring $P$ are equivalence classes of the set $\mathbb{Z}[A] \times \mathbb{Z}[A]$, usually denoted by $\alpha\beta^{-1}$ for $0 \neq \beta, \alpha \in \mathbb{Z}[A]$. Suppose that there exists a nontrivial linear combination over $P$ which equals zero:

$$w_2(z_1c_1, z_2c_2)\alpha_2\delta_2^{-1} + \ldots + w_r(z_1c_1, z_rc_r)\alpha_r\delta_r^{-1} = 0.$$  

Since $\mathbb{Z}[A]$ is an Ore ring, every finite set of its elements has a right common multiple. We find the right common multiple for elements $\delta_2, \ldots, \delta_r$. Denote it by $\delta$. By definition, $0 \neq \delta = \delta_i\gamma_i$, $i = 2, \ldots, r$, $\gamma_i \in \mathbb{Z}[A]$. Hence

$$w_2(z_1c_1, z_2c_2)\alpha_2\gamma_2 + \ldots + w_r(z_1c_1, z_rc_r)\alpha_r\gamma_r = 0,$$

which is a contradiction with $w_i(z_1c_1, z_ic_i)$, $i = 2, \ldots, r$, being independent over $\mathbb{Z}[A]$. Thus the elements $w_2(z_1c_1, z_2c_2), \ldots, w_r(z_1c_1, z_rc_r)$ are independent over $P$.

Therefore, $P$ contains elements $0 \neq \sigma, \sigma_1, \ldots, \sigma_{r-1}$ such that

$$c'^r = w_2(z_1c_1, z_2c_2)\sigma^1 \ldots w_r(z_1c_1, z_rc_r)\sigma^{r-1}.$$  

Multiplying, if necessary, all $\sigma, \sigma_1, \ldots, \sigma_{r-1}$ by a suitable element $u \in \mathbb{Z}[A]$, we can ensure that $\sigma, \sigma_1, \ldots, \sigma_{r-1} \in \mathbb{Z}[A]$. Hence $c' \in H^{(d-1)}$. It follows from Lemma 4 that $c \in H^{(d-1)}$. \(\square\)

**Proof** of Theorem 3. Let $G = S_{r,d}$. If $r_{ab}(H) = 0$, then $H = 1$ by Proposition 1, which contradicts the hypothesis. If $r_{ab}(H) = 1$, then $H$ is cyclic by the proof of Theorem 1, which is again a contradiction with the hypothesis. Hence $r_{ab}(H) = 2$. By Baumslag’s theorem [25], the group $H$ is isomorphic to $S_{2d}$. In view of Lemma 1, it contains a strongly test singleton set. By virtue of Lemma 2, the subgroup $H$ is a retract. \(\square\)

The **proof** of Theorem 4. First we argue to state two auxiliary results.

**Lemma 5.** Let $R$ be a factorial ring. Suppose that a nonzero element $u \in R$ has $n_0$ factors in its decomposition as a product of prime elements. Let $p_1, \ldots, p_s$ be a tuple of pairwise nonassociated prime elements of $R$. With it we associate a tuple of natural numbers $n_1, \ldots, n_s$ such that

$$n_1 > n_0, n_2 > n_0 + n_1, \ldots, n_s > n_0 + n_1 + \ldots + n_{s-1}.$$  

Then, for any tuple of noninvertible elements $q_1, \ldots, q_s$ and for a nonzero element $v$ of $R$, the equality

$$u \prod_{i=1}^{s} p_i^{n_i} = v \prod_{i=1}^{s} q_i^{n_i}$$  

implies that the element $p_i$ is associated with $q_i$ for any $i \in \{1, \ldots, s\}$.
implies the associativity of \( p \). Generated by the image \( \mu \), which contradicts the assumption on the number of prime divisors in the decomposition of \( p \) by \( i \). Element \( f \) in \( G \) is expressed as a group word \( f = \langle a_1, \ldots, a_r \rangle \), and \( H = \langle h_1, \ldots, h_{l+1}, h_{l+2}, \ldots, h_{l+t} \rangle \), where \( h_i \equiv z_i (\text{mod} \ G') \), \( i = 1, \ldots, l+1 \), and \( h_{l+2}, \ldots, h_{l+t} \in H' \), \( l \geq 1 \). Denote by \( \tilde{h} \) the image of an element \( f \) in \( G \). Then \( \tilde{h}_i = a_i \), for \( i = 1, \ldots, l+1 \), and \( \tilde{h}_i = 1 \) for \( i = l+2, \ldots, l+t \). The group \( G' \) is endowed with the structure of a right \( ZA \)-module, which, in compliance with the Magnus embedding, embeds in a free \( ZA \)-module \( T \) of rank \( r \) with basis \( t_1, \ldots, t_r \).

Let \( [h_1, h_2] = \sum_{i=1}^{r} t_i b_i(a), b_i(a) \in ZA \), be the canonical representation of an element \( [h_1, h_2] \) in \( T \).

Lemma 5 will be applied as follows. Given the element \( [h_1, h_2] \), we define \( n_0 \) to be the least number of prime factors in decompositions of its coordinates \( b_i(a) \), \( i = 1, \ldots, r \). As prime elements \( p_i, i = 1, \ldots, s \), in \( R = ZA \) figuring in that lemma we take \( p_1 = a_1 - 1, \ldots, p_{l+1} = a_{l+1} - 1, p_{l+2} = a_1 - 2, \ldots, p_{2(l+1)} = a_{l+1} - 2 \). In this case \( s = 2(l+1) \). Choose \( n_i, i = 1, \ldots, n_0 \), as in Lemma 5.

Consider the following system of equations written in the language of modules:

\[
\begin{align*}
[h_1(x), h_2(x)](h_1(\bar{x}) - 2)^{n_1+2} \cdots (h_{l+1}(\bar{x}) - 2)^{n_2(l+1)} &= [h_1, h_2](a_1 - 1)^{n_1} \cdots (a_{l+1} - 1)^{n_{l+1}} \\
(a_1 - 2)^{n_{l+2}} \cdots (a_{l+1} - 2)^{n_{2(l+1)}}
\end{align*}
\]  
(7)
The given system of equations has a solution in the group $G$; so it is solvable in $H$: $x_i = f_i$, $i = 1, \ldots, r$. Consider an endomorphism $\sigma : G \to H$ defined by a mapping $z_i \mapsto f_i$, $i = 1, \ldots, r$. By Lemma 5, for every $i = 1, \ldots, l + 1$, the element $\sigma(h_i) - 1$ is associated with $a_i - 1$; hence either $\sigma(h_i) = a_i$ or $\sigma(h_i) = a_i^{-1}$. The latter case is impossible since the element $\sigma(h_i) - 2 = a_i^{-1} - 2 = a_i^{-1}(1 - 2a_i)$ is not associated with $a_i - 2$.

Thus $\sigma(h_i) = a_i$. Then the equality $\sigma([h_1, h_i]) = [h_1, h_i]$ holds for every $i = 2, \ldots, l + 1$. Let $\sigma(h_i) = h_1v_i$, $v_i \in H'$, $i = 1, \ldots, l + 1$. In the language of modules, $\sigma([h_1, h_i]) = [h_1, h_i]$ is equivalent to $v_1(a_1 - 1) = v_i(a_1 - 1)$. Hence $v_1 = w(a_1 - 1)$ and $v_i = w(a_i - 1)$. Then $w \in G'$. If we correct the endomorphism $\sigma$ by multiplying by an inner automorphism of conjugation by $w$ we obtain an endomorphism $\tau : G \to G$ acting identically on $h_1, \ldots, h_{l+1}$. Since $\tau(G)$ is in $H$, it follows by Lemma 6 that $\tau$ is a retraction of $G$ on $H$. □

3. PROBLEMS

We formulate several natural problems.

**Problem 1.** Is it true that every finitely generated verbally closed subgroup $H$ of a free solvable group $S_{r,d}$, $r \geq 2$, $d \geq 2$, is a retract?

**Problem 2.** Is it true that a subgroup $H$ of a free solvable group $S_{r,d}$ of derived length $d \geq 3$ which satisfies the condition that $r_{ab}(H) = 2$ is verbally closed iff it is a retract of $S_{r,d}$?

A positive solution to Problem 2 strengthens Theorem 3.

**Problem 3.** Is it true that every finitely generated subgroup $H$ of a free metabelian group $M_r$ of finite rank $r$ is verbally closed iff it is a retract of $M_r$?

A positive solution to Problem 3 strengthens Theorem 4.

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REFERENCES

1. W. Hodges, *Model Theory*, Enc. Math. Appl., 42, Cambridge Univ. Press, Cambridge (1993).
2. W. R. Scott, “Algebraically closed groups,” *Proc. Am. Math. Soc.*, 2, 118-121 (1951).
3. A. Myasnikov and V. Roman’kov, “Verbally closed subgroups of free groups,” *J. Group Theory*, 17, No. 1, 29-40 (2014).
4. A. M. Mazhuga, “Verbally closed subgroups,” Cand. Sci. Dissertation, Moscow State Univ., Moscow (2018).
5. G. M. Bergman, “Supports of derivations, free factorizations and ranks of fixed subgroups in free groups,” Trans. Am. Math. Soc., 351, No. 4, 1531-1550 (1999).

6. Unsolved Problems in Group Theory, The Kourovka Notebook, No. 19, Institute of Mathematics SO RAN, Novosibirsk (2018); http://www.math.nsc.ru/~alglog/19tkt.pdf.

7. G. Baumslag, A. G. Myasnikov, and V. Shpilrain, “Open problems in combinatorial group theory. Second edition,” in Contemp. Math., 296, Am. Math. Soc., Providence, RI (2002), pp. 1-38.

8. I. Snopce, S. Tanushevski, and P. Zalesskii, “Retracts of free groups and a question of Bergman,” arXiv:1902.02378 [math.GR].

9. V. A. Roman’kov and N. G. Khisamiev, “Verbally and existentially closed subgroups of free nilpotent groups,” Algebra and Logic, 52, No. 4, 336-351 (2013).

10. A. M. Mazhuga, “On free decompositions of verbally closed subgroups in free products of finite groups,” J. Group Theory, 20, No. 5, 971-986 (2017).

11. A. M. Mazhuga, “Strongly verbally closed groups,” J. Alg., 493, 171-184 (2018).

12. A. A. Klyachko and A. M. Mazhuga, “Verbally closed virtually free subgroups,” Mat. Sb., 209, No. 6, 75-82 (2018).

13. A. A. Klyachko, A. M. Mazhuga, and V. Yu. Miroshnichenko, “Virtually free finite-normal-subgroup-free groups are strongly verbally closed,” J. Alg., 510, 319-330 (2018).

14. E. I. Timoshenko, “Test elements and test rank for a free metabelian group,” Sib. Math. J., 41, No. 6, 1200-1204 (2000).

15. V. A. Roman’kov, “Test elements for free solvable groups of rank 2,” Algebra and Logic, 40, No. 2, 106-111 (2001).

16. E. I. Timoshenko, “Computing test rank for a free solvable group,” Algebra and Logic, 45, No. 4, 254-260 (2006).

17. Ch. K. Gupta and E. I. Timoshenko, “Test rank for some free polynilpotent groups,” Algebra and Logic, 42, No. 1, 20-28 (2003).

18. E. I. Timoshenko, Endomorphisms and Universal Theories of Soluble Groups [in Russian], NGTU, Novosibirsk (2011).

19. C. K. Gupta, V. A. Roman’kov, and E. I. Timoshenko, “Test ranks of free nilpotent groups,” Comm. Alg., 33, 1627-1634 (2005).

20. V. N. Remeslennikov and V. G. Sokolov, “Some properties of the Magnus embedding,” Algebra and Logic, 9, No. 5, 342-349 (1970).

21. R. Crowell and R. Fox, Introduction to Knot Theory, Springer, New York (1963).
22. V. A. Roman'kov, *Essays in Algebra and Cryptology. Solvable Groups*, Dostoevsky Omsk State University, Omsk (2018).

23. C. K. Gupta and N. S. Romanovski, “On torsion in factors of polynilpotent series of a group with a single relation,” *Int. J. Alg. Comp.*, 14, No. 4, 513-523 (2004).

24. J. Lewin, “A note on zero divisors in group-rings,” *Proc. Am. Math. Soc.*, 31, No. 2, 357-359 (1972).

25. G. Baumslag, “Some subgroup theorems for free $v$-groups,” *Trans. Am. Math. Soc.*, 108, 516-525 (1963).