POLY-MEROMORPHIC ITÔ–HERMITE FUNCTIONS ASSOCIATED WITH A SINGULAR POTENTIAL VECTOR ON THE PUNCTURED COMPLEX PLANE

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Abstract. We provide a theoretical study of a new family of orthogonal functions on the punctured complex plane solving the eigenvalue problems for some magnetic Laplacian perturbed by a singular vector potential with zero magnetic field modeling the Aharonov–Bohm effect. The functions are defined by their $\beta$-modified Rodrigues type formula and extend the poly-analytic Itô–Hermite polynomials to the poly-meromorphic setting. Mainly, we derive their different operational representations and give their explicit expressions in terms of special functions. Different generating functions and integral representations are obtained.

1. Introduction

The real Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}).$$

(1.1)

are first considered by Laplace [20] and next studied in details by Tchebeychev [6]. They play a crucial role in the one dimensional quantum harmonic oscillator [9, 21] and are used to provide analytic proofs for some combinatorial results [7]. Moreover, they have been shown to be useful in expanding the probability generating function of a generalized Poisson distribution [18]. A non-trivial two dimensional analog that is not the tensor product of the real Hermite polynomials is giving by the Itô–Hermite polynomials defined by the Rodrigues formula

$$H^\alpha_{m,n}(z, \overline{z}) = (-1)^{m+n} e^{\alpha|z|^2} \frac{\partial^{m+n}}{\partial z^m \partial \overline{z}^n} \left( e^{-\alpha|z|^2} \right), \quad \alpha > 0.$$  

(1.2)

Here $\partial/\partial z$ and $\partial/\partial \overline{z}$ denote the Wirtinger derivatives with respect to the variables $z$ and $\overline{z}$, respectively. The polynomials in (1.2) have been introduced by Itô within the framework of multiple Wiener integrals [17]. Since then, they have been extensively studied, namely in connection with various branches of engineering sciences, mathematics and physics [10–14, 16, 17, 23, 24]. They form an orthogonal complete system in the Hilbert space $L^{2,\alpha}(\mathbb{C}) = L^2(\mathbb{C}, e^{-\alpha|z|^2} \, dx \, dy)$ (see [10, 16]). Moreover, they provide a concrete description of the spectral analysis of the Landau Hamiltonian [19]

$$\Delta_\alpha = \left( i \frac{\partial}{\partial x} - 2\alpha y \right)^2 + \left( i \frac{\partial}{\partial y} + 2\alpha x \right)^2 ; \quad z = x + iy.$$  

(1.3)

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Different generalized class of these polynomials are suggested by special magnetic Schrödinger operators. The one associated with constant magnetic field acting on mixed planar automorphic functions attached to a given equivariant pair \[ \mathbf{G} \] is given by

\[
G_{\nu_{m,n}}(z, \zeta) = (-1)^{m+n} e^{\nu|z|^2 + \frac{i\beta}{2} \zeta \frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n}} (e^{-\nu|z|^2 - \frac{i\beta}{2} \zeta^2})
\]

and has been studied in sufficient detail in [10].

In the present paper, we identify a special class of orthogonal and poly-meromorphic functions generalized to previous one and associated with the second order differential operator

\[
\mathcal{D}_{\alpha,\beta} = \Delta_\alpha + S_{\alpha,\beta},
\]

which is essentially the Laplacian \( \Delta_\alpha \) in (1.3) perturbed by the first order differential operator \( S_{\alpha,\beta} \) explicitly given by

\[
S_{\alpha,\beta} = \frac{\beta}{|z|^2} \left( z \frac{\partial}{\partial z} - \zeta \frac{\partial}{\partial \zeta} \right) - \beta \left( 2\alpha - \frac{\beta}{|z|^2} \right)
\]

and is associated with the closed and singular potential vector

\[
\tilde{\theta}_{\beta}(z, \zeta) = -i\beta \frac{|z|^2}{|z|^2} (\zeta dz - z d\zeta),
\]

modeling an Aharonov–Bohm effect. Mainly, we aim to explore the role played by the injection of this singular potential in generating non-trivial orthogonal eigenstates within the factorization formalism. In fact, we provide an accurate study of the special functions

\[
\psi_{\alpha,\beta}^{n,m}(z, \zeta) := (-1)^n z^{-\beta} e^{\alpha|z|^2} \frac{\partial^n}{\partial z^n} (z^{\beta+m} e^{-\alpha|z|^2}),
\]

referred to as poly-meromorphic (or \( \beta \)-modified complex) Itô–Hermite functions. Here, \( \alpha \) and \( \beta \) are given fixed reals with \( \alpha > 0 \), and \( n \) and \( m \) are varying integers such that \( n = 0, 1, 2, \cdots \) and \( m > -\beta - 1 \). This leads to a special generalization of the complex Itô-Hermite polynomials in (1.2). The polynomial case, shown in Section 4 to correspond to \( m \geq n \) and \( \beta \geq 0 \), coincides with the polynomials \( Z_{\beta_{m,n}} \) considered by Ismail and Zeng in [15, Section 3]. More precisely, we are concerned with certain basic algebraic, analytic and spectral properties of \( \psi_{\alpha,\beta}^{n,m}(z, \zeta) \) defined by their Rodrigues formula (1.7). In particular, the interrelation of \( \psi_{\alpha,\beta}^{n,m}(z, \zeta) \) with to spacial functions such as the Itô–Hermite polynomials and the confluent hypergeometric functions are considered in Section 2. Their spectral realization as eigenfunctions of the magnetic Laplacian \( \mathcal{D}_{\alpha,\beta} \) as well as their regularity and their exact bi-order as poly-meromorphic functions on the complex plane are also studied in Sections 3 and 4, respectively. Associated generating functions are obtained in Section 5, and next employed to discuss some of their applications such as their integral representations (Section 6) and spacial attached integral transforms (Section 7). The fractional side as well as the associated functional spaces of Segal–Bargmann type will be introduced and studied in details in forthcoming papers.

2. Preliminary results

It is worth noticing that to avoid multiple-valuedness of the argument \( \arg(z) \) in \( z^\beta = e^{\beta \log(z)} \) involved in (1.7), we choose the principal branch for the logarithm. Notice also that the
functions \( \psi_{\alpha,\beta}^{n,m}(z, \bar{z}) \) satisfy \( \psi_{\alpha,\beta}^{n,m}(z, \bar{z}) = \psi_{\alpha,\beta}^{n,m}(\bar{z}, z) \) as well as the symmetry relation

\[
\alpha^m z^\beta \psi_{n+m, m-\beta}^{\alpha,\beta}(z, \bar{z}) = \alpha^{n+m} z^\beta \psi_{m,n}^{\alpha,\beta}(z, \bar{z}) \tag{2.1}
\]

valid for \( \beta \) being integer and for given non-negative integers \( m \) and \( n \) such that \( n \geq \max(0, -\beta) \). We have in addition

\[
\bar{z}^\beta \psi_{1,\beta}^{n,m}(z, \bar{z}) = z^\beta \psi_{m+\beta, m-\beta}^{1,\beta}(z, \bar{z})
\]

and

\[
\psi_{n,m}^{\alpha,\beta}(z, \bar{z}) = z^{-[\beta]} \psi_{n,m+\beta}^{\alpha,\beta}(z, \bar{z}),
\]

valid for any real \( \beta \) such that \( \beta + m > -1 \), where \( [\beta] \) denotes the integer part of \( \beta \). For the particular case of \( \beta = 1/2 \) we have

\[
\sqrt{\alpha|z|2^{m+1}} \psi_{m+m}^{\alpha,\beta}(z, \bar{z}) = H_{2m+1}(\sqrt{\alpha|z|}) = 2(-4)^m m!(\alpha|z|)^{1/2} L_{m}^{1/2}(\alpha|z|),
\]

where \( H_m(x) \) are the Hermite polynomials in (1.1) and \( L_k^\alpha \) denotes the generalized Laguerre polynomials. For arbitrary non-negative integer \( \beta \), the functions \( \psi_{m,m}^{\alpha,\beta}(z, \bar{z}) \) are closely connected to the complex Itô-Hermite polynomials in (1.2) on the punctured complex plane \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \),

\[
z^\beta \psi_{n,m}^{\alpha,\beta}(z, \bar{z}) = \psi_{n,m+\beta}^{\alpha,\beta}(z, \bar{z}) = \alpha^{n+m} \psi_{n,m}^{\alpha,\beta}(z, \bar{z}). \tag{2.2}
\]

The interrelation with these polynomials for arbitrary \( \beta \) can be obtained by specifying \( f \) in the Burchnall’s operational formula [11, Proposition 2.3]

\[
(-1)^n e^{\alpha|z|^2} \frac{\partial^n}{\partial z^n} \left( z^m e^{-\alpha|z|^2} f \right) = n! \sum_{k=0}^{n} (-1)^k \frac{\Gamma(\beta+1)}{k!(n-k)!} H_{m,n-k}^{\alpha}(z, \bar{z}) \frac{\partial^k f}{\partial z^k}.
\]

Thus, for \( f(z) = z^\beta \) we obtain

\[
\psi_{n,m}^{\alpha,\beta}(z, \bar{z}) = \frac{n!}{\alpha^m} \sum_{k=0}^{n} \frac{(-1)^k \Gamma(\beta+1)}{k!(n-k)!} z^{-k} H_{m,n-k}^{\alpha}(z, \bar{z}).
\]

For the explicit expression of \( \psi_{n,m}^{\alpha,\beta}(z, \bar{z}) \) (with \( m > -\beta - 1 \)), we claim that

\[
\psi_{n,m}^{\alpha,\beta}(z, \bar{z}) = \sum_{k=0}^{n \wedge^* b} c_{m,n,k}^\alpha z^{-k} \bar{z}^{n-k}, \tag{2.3}
\]

where the starred minimum \( m \wedge^* b \) is defined by the classical minimum \( m \wedge^* b = \min(m, b) \) when \( b \) is an integer and by \( m \wedge^* b = m \) otherwise. The involved constants \( c_{m,n,k}^\alpha \) stand for

\[
c_{m,n,k}^\alpha := \frac{(-1)^k n! \Gamma(\beta + m + 1)}{k!(n-k)! \Gamma(\beta + m + k - 1)} \alpha^{n-k}.
\]

The expression in (2.3) can be handled by applying the Leibniz formula to (1.7) keeping in mind the fact that

\[
z^{-(\beta+m-k)} \frac{\partial^k}{\partial z^k} (z^{\beta+m}) = \begin{cases} 0, & \beta = 0, 1, 2, \ldots, k > \beta + m, \\ \frac{\Gamma(\beta + m + 1)}{\Gamma(\beta + m + k - 1)}, & \text{otherwise}. \end{cases}
\]
Notice also that the monomials $z^q \bar{z}^p$ can be expressed in terms of the considered functions.
In fact, by rewriting them as a derivation of the Gaussian function $e^{-\alpha|z|^2}$, we get
\[
\alpha^p z^q \bar{z}^p = (-1)^p z^q e^{\alpha|z|^2} \frac{\partial^p}{\partial z^p} (e^{-\alpha|z|^2}) \\
= (-1)^p z^q e^{\alpha|z|^2} \frac{\partial^p}{\partial z^p} (z^{-\beta-n+p-q}z^{\beta+n-q} - e^{-\alpha|z|^2}) \\
= \sum_{n=0}^{p} \left( \begin{array}{c} p \\ n \end{array} \right) (-1)^{p-n} \frac{\partial^{p-n}}{\partial z^{p-n}} (z^{-\beta-n+p-q}z^q e^{\alpha|z|^2} \frac{\partial^n}{\partial z^n} (z^{\beta+n-q} - e^{-\alpha|z|^2})).
\]

The last equality holds making use of the Leibniz formula. Now, by means of the Rodrigues formulas \((1.7)\) with $q-p \geq 0$, it follows
\[
\alpha^p z^q \bar{z}^p = \sum_{n=0}^{p} \left( \begin{array}{c} p \\ n \end{array} \right) \frac{\Gamma(\beta + q)}{\Gamma(\beta + q + n - p)} \psi^{\alpha,\beta}_{n,n+q-p}(z, \bar{z})
\]
for every non-negative integers $p$ and $q$ with $p \leq q$.

The first few terms of $\psi^{\alpha,\beta}_{n,m}$ are given by $\psi^{\alpha,\beta}_{0,m}(z, \bar{z}) = z^m$ and $\psi^{\alpha,\beta}_{1,m}(z, \bar{z}) = z^{m-1} (\alpha z \bar{z} - (\beta + m))$ when $n = 0$ and $n = 1$ respectively, while for $n = 2$ we have
\[
\psi^{\alpha,\beta}_{2,m}(z, \bar{z}) = z^{m-2} (\alpha^2 z^2 \bar{z}^2 - 2\alpha(\beta + m)z \bar{z} + (\beta + m)(\beta + m - 1)).
\]

This reveals in particular that the $\psi^{\alpha,\beta}_{n,m}$ are no longer polynomials unless $\beta$ is a non-positive integer. This becomes clear from their hypergeometric representation in terms of the hypergeometric functions defined by the series
\[
pFq\left( \begin{array}{c} a_1, a_2, \ldots, a_p \\ c_1, c_2, \ldots, c_q \end{array} \middle| x \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \cdots (a_p)_n x^n}{(c_1)_n(c_2)_n \cdots (c_q)_n n!}
\]
provided that $c_\ell \neq 0, -1, -2, \ldots$ for $\ell = 1, 2, \ldots, q$. Indeed, from \((2.3)\) and making use of the classical identities on Gamma function and Pochhammer symbol, we get
\[
\psi^{\alpha,\beta}_{n,m}(z, \bar{z}) = \alpha^n z^m \bar{z}^n \sum_{k=0}^{n^\ast(m+\beta)} (-n)_k (-\beta - m)_k \frac{(-\alpha z \bar{z})^{-k}}{k!}
\]
\[
= \alpha^n z^m \bar{z}^n 2F_0\left( \begin{array}{c} -n, -\beta - m \\ - \end{array} \middle| \frac{1}{\alpha|z|^2} \right)
\]
\[
= \frac{(-1)^n \Gamma(\beta + m + 1)}{\Gamma(\beta + m + n - 1)} z^{m-n} \frac{\Gamma(\beta + m - n + 1)}{\Gamma(\beta + m + n - 1)} 1F_1\left( \begin{array}{c} -n \\ \beta + m - n + 1 \end{array} \middle| \alpha|z|^2 \right).
\]

Subsequently, by means of \((2.6)\) we obtain the expression of $\psi^{\alpha,\beta}_{n,m}$ in terms of the generalized Laguerre polynomials \([22, p. 240]\), to wit
\[
\psi^{\alpha,\beta}_{n,m}(z, \bar{z}) = (-1)^n n! z^{m-n} L^{m-n}_{\beta + m - n} (\alpha|z|^2).
\]

This can also be recovered starting from \((1.7)\) using the derivation formula \([22, p.241]\). Accordingly, with the use of \((2.7)\), it is straightforward to prove the orthogonality of $\psi^{\alpha,\beta}_{n,m}$ in the Hilbert space $L^2_\beta^{\alpha,\beta}(\mathbb{C}) := L^2(\mathbb{C}, d\mu_{\alpha,\beta})$ of square integrable functions with respect to the measure $d\mu_{\alpha,\beta}(z) = |z|^{2\beta} e^{-\alpha|z|^2} d\lambda(z)$, where $d\lambda(z) = dxdy$ denotes the Lebesgue measure on
the complex plane with \( z = x + iy, x, y \in \mathbb{R} \). More explicitly, we have
\[
\int_{\mathbb{C}} \psi_{n,m}^{\alpha,\beta}(z, \overline{z}) \frac{\alpha^{n}n!}{\alpha^{m+\beta+1}} \frac{\beta + m + 1}{\delta_{m,j} \delta_{n,k}} d\lambda(z) = \pi^{2}e^{-\alpha|z|^{2}}d\lambda(z) - \frac{\pi^{2}n!}{\alpha^{m+\beta+1}} \Gamma(\beta + m + 1) \delta_{m,j} \delta_{n,k}.
\]

We conclude these preliminaries by noticing that starting from (1.7), it is clear that
\[
z\psi_{n,m}^{\alpha,\beta}(z, \overline{z}) = \psi_{n,m+1}^{\alpha,\beta-1}(z, \overline{z}).
\]

Also, by rewriting \( \partial^{n+1} \) as \( \partial^{n} \partial \), one obtains the recurrence formula
\[
\alpha z\psi_{n,m}^{\alpha,\beta}(z, \overline{z}) = \psi_{n+1,m}^{\alpha,\beta}(z, \overline{z}) + (\beta + m)\psi_{n,m-1}^{\alpha,\beta}(z, \overline{z}).
\]

Additional recurrence formulas can be derived from the different known ones for the generalized Laguerre polynomials. For example from those in [22, p. 241] one obtains
1. \( \psi_{n,m+1}^{\alpha,\beta}(z, \overline{z}) = \psi_{n,m}^{\alpha,\beta}(z, \overline{z}) + n\psi_{n-1,m}^{\alpha,\beta}(z, \overline{z}) \)
2. \( \psi_{n+1,m+1}^{\alpha,\beta}(z, \overline{z}) = (\alpha|z|^{2} - [n + \beta + m + 1])\psi_{n,m}^{\alpha,\beta}(z, \overline{z}) - n(\beta + m)\psi_{n-1,m-1}^{\alpha,\beta}(z, \overline{z}) \)
3. \( \alpha z\psi_{n,m}^{\alpha,\beta}(z, \overline{z}) = (\alpha|z|^{2} - n)\psi_{n+1,m}^{\alpha,\beta}(z, \overline{z}) - n(\beta + m)\psi_{n-1,m}^{\alpha,\beta}(z, \overline{z}) \)
4. \( \psi_{n+1,m+1}^{\alpha,\beta}(z, \overline{z}) = (\alpha|z|^{2} - n - 1)\psi_{n,m}^{\alpha,\beta}(z, \overline{z}) - z(\beta + m)\psi_{n,m-1}^{\alpha,\beta}(z, \overline{z}) \)
5. \( (\beta + m - n + \alpha|z|^{2})\psi_{n,m}^{\alpha,\beta}(z, \overline{z}) = z(\beta + m)\psi_{n,m-1}^{\alpha,\beta}(z, \overline{z}) + \alpha z\psi_{n+1,m}^{\alpha,\beta}(z, \overline{z}). \)

3. Spectral realization

The result below shows that the functions \( \psi_{n,m}^{\alpha,\beta} \) are \( L^{2} \)-eigenfunctions of the perturbed magnetic Laplacian defined by
\[
\Delta_{\alpha,\beta} := -\frac{\partial^{2}}{\partial z \partial \overline{z}} + \left( \alpha - \frac{\beta}{|z|^{2}} \right) z \frac{\partial}{\partial \overline{z}}
\]
and the second order differential operator
\[
\tilde{\Delta}_{\alpha,\beta} := -\frac{\partial^{2}}{\partial z \partial \overline{z}} + \alpha z \frac{\partial}{\partial \overline{z}} - \frac{\beta}{z} \frac{\partial}{\partial z}.
\]

**Theorem 3.1.** The functions \( \psi_{n,m}^{\alpha,\beta} \) form an orthogonal system in \( L_{\beta}^{2}(\mathbb{C}) \) that solve the eigenvalue problems \( \Delta_{\alpha,\beta} = \alpha n \) and \( \tilde{\Delta}_{\alpha,\beta} = \alpha m \).

**Proof.** Notice first that the second order differential operator in (3.1) can be rewritten as
\[
\Delta_{\alpha,\beta} = A^{*\alpha,\beta}A = AA^{*\alpha,\beta} - \alpha,
\]
where \( A = \partial/\partial \overline{z} \) and \( A^{*\alpha,\beta} \) are the first order differential operators given by
\[
A^{*\alpha,\beta} = -[\rho_{\alpha,\beta}(z)]^{-1} \frac{\partial}{\partial z} \left( \rho_{\alpha,\beta}(z)f(z) \right).
\]
Here \( \rho_{\alpha,\beta}(z) := |z|^{2}e^{-\alpha|z|^{2}} \). Thus, using the commutation rule \( AA^{*\alpha,\beta} - A^{*\alpha,\beta}A = \alpha Id \), one proceeds by induction to get the identity \( A(A^{*\alpha,\beta})^{n+1} = (A^{*\alpha,\beta})^{n+1}A + \alpha(n + 1)(A^{*\alpha,\beta})^{n} \). Subsequently, we have
\[
\Delta_{\alpha,\beta}((A^{*\alpha,\beta})^{n}(g)) = (AA^{*\alpha,\beta} - \alpha)((A^{*\alpha,\beta})^{n}(g)) = \alpha n ((A^{*\alpha,\beta})^{n}(g)),
\]
for any $g \in \ker(A)$. Thus, by considering the case of the generic elements $g_m(z) = z^m$ with $m \in \mathbb{Z}$ for $z$ in the punctured complex plane, one deduces that the function

\[ \psi_n^{\alpha,\beta}(z, \overline{z}) := (A^* \alpha, \beta)^n(g_m) = (-1)^n[\rho_{\alpha, \beta}(z)]^{-1} \frac{\partial^n}{\partial z^n} (z^m \rho_{\alpha, \beta}(z)) \quad (3.4) \]

is an eigenfunction of $\Delta_{\alpha, \beta}$ with $n\alpha$ as a corresponding eigenvalue.

Now, to prove that \( \tilde{\Delta}_{\alpha, \beta} \psi_n^{\alpha,\beta} = \alpha m \psi_n^{\alpha,\beta} \), we make use of the partial raising operations

\[ -\left( \frac{\partial}{\partial z} - \alpha \overline{z} + \frac{\beta}{z} \right) \psi_n^{\alpha,\beta}(z, \overline{z}) = \psi_{n+1,m}^{\alpha,\beta}(z, \overline{z}) \quad (3.5) \]

and

\[ -\frac{1}{\alpha} \left( \frac{\partial}{\partial \overline{z}} - \alpha z \right) \psi_n^{\alpha,\beta}(z, \overline{z}) = \psi_{n,m+1}^{\alpha,\beta}(z, \overline{z}), \quad (3.6) \]

which are immediate by straightforward computation. On the other hand, from (2.8) and (3.5) one has

\[ \left( \frac{\partial}{\partial z} + \frac{\beta}{z} \right) \psi_{n,m}^{\alpha,\beta}(z, \overline{z}) = (\beta + m) \psi_{n,m-1}^{\alpha,\beta}(z, \overline{z}). \quad (3.7) \]

Accordingly, by combining (3.6) and (3.7), it follows

\[ \left( \frac{\partial}{\partial z} + \frac{\beta}{z} \right) \left( \frac{\partial}{\partial \overline{z}} - \alpha z \right) \psi_{n,m}^{\alpha,\beta}(z, \overline{z}) = -\alpha(\beta + m + 1) \psi_{n,m}^{\alpha,\beta}(z, \overline{z}). \quad (3.8) \]

This completes the proof by observing that the operator \( \tilde{\Delta}_{\alpha, \beta} \) in (3.2) can be factorized, up to an additive constant, as

\[ \tilde{\Delta}_{\alpha, \beta} = -\left( \frac{\partial}{\partial z} + \frac{\beta}{z} \right) \left( \frac{\partial}{\partial \overline{z}} - \alpha z \right) - \alpha(\beta + 1). \]

\[ \square \]

**Remark 3.2.** Let $E = z\partial/\partial z$ be the Euler derivative operator and $\overline{E} = \overline{z}\partial/\partial \overline{z}$ its complex conjugate. Then, the functions $\psi_n^{\alpha,\beta}$ satisfy

\[ (E - \overline{E}) \psi_n^{\alpha,\beta} = (m - n) \psi_n^{\alpha,\beta}, \]

which readily follows since \( \tilde{\Delta}_{\alpha, \beta} - \Delta_{\alpha, \beta} = \alpha(E - \overline{E}) \). This is the analog for $\psi_n^{\alpha,\beta}$ at arbitrary integer $m$ such that $m > -\beta - 1$ of the one obtained in [15].

**Remark 3.3.** The orthogonality of $\psi_n^{\alpha,\beta}$ in $L_2^\alpha(\mathbb{C})$ can be reproved by observing that the operator $A^* \alpha, \beta$ is in fact the adjoint of $A$ when acting on a densely domain in $L_2^\alpha(\mathbb{C})$. In
fact, we have
\[
\left\langle \psi^{\alpha,\beta}_{n,m}, \psi^{\alpha,\beta}_{n+p,q} \right\rangle_{\alpha,\beta} = \left\langle A(A^{*\alpha,\beta})^{n}(g_{m}), (A^{*\alpha,\beta})^{n+p-1}(g_{q}) \right\rangle_{\alpha,\beta}
\]
\[
= \left\langle (A^{*\alpha,\beta})^{n}A(g_{m}) + n\alpha(A^{*\alpha,\beta})^{n-1}(g_{m}), (A^{*\alpha,\beta})^{n+p-1}(g_{q}) \right\rangle_{\alpha,\beta}
\]
\[
= \alpha n\left\langle \psi^{\alpha,\beta}_{n-1,m}, \psi^{\alpha,\beta}_{n+p-1,q} \right\rangle_{\alpha,\beta}
\]
\[
= \alpha^{n}n!\left\langle \psi^{\alpha,\beta}_{0,m}, \psi^{\alpha,\beta}_{p,q} \right\rangle_{\alpha,\beta}
\]
\[
= \frac{\pi \alpha^{n}n!}{\alpha^{m+\beta+1}} \Gamma(\beta + m + 1) \delta_{0,p} \delta_{m,q}.
\]

Below, we prove that the considered functions are closely connected to the spectral analysis of a specific Schrödinger operator \( L = \nabla^*_{\theta} \nabla_{\theta} \) associated with a specific singular vector potential \( \theta \), where \( \nabla_{\theta} = d + i \text{ext}_{\theta} \) is the co-derivation operator acting on \( \Omega^\infty_{p,c}(\mathbb{C}) \), the space of smooth differential \( p \)-forms with compact support, and \( \text{ext}_{\theta}(\omega := \theta \wedge \omega) \) denotes the exterior multiplication by \( \theta \). The operator \( \nabla^*_{\theta} \) denotes its formal adjoint with respect to the Hermitian scalar product
\[
(\omega_{1}, \omega_{2})_{p} = \int_{\mathbb{C}} \omega_{1} \wedge * \omega_{2},
\]
for \( \omega_{1}, \omega_{2} \in \Omega^\infty_{p,c}(\mathbb{C}) \). Here \( * \) is the Hodge star operator on differential forms defined to satisfy \( *(f \omega) = \overline{f}(\omega) \) for scalar functions \( f \), and \( *(dz \wedge d\bar{z}) = 2i \). This readily follows for the metric \( ds^{2} \) being conformal to the Euclidean metric \( ds^{2}(z) = dz \otimes d\bar{z} \). Now, by considering the strong extensions of the differential operators \( d, \nabla \) and \( L \) initially defined on \( C^\infty_{0}(\mathbb{C}) = \Omega^\infty_{0,c}(\mathbb{C}) \), we can extend them to the whole \( L^{2} \)-Hilbert space as the closure of the \( L^{2} \)-norm with respect to \( (\omega_{1}, \omega_{2})_{0} \).

**Lemma 3.4.** For given reals \( \alpha \) and \( \beta \) such that \( \alpha > 0 \), we set \( \nabla_{\alpha,\beta} = d + i \text{ext}_{\theta_{\alpha,\beta}} \) and we let \( \theta_{\alpha,\beta} \) be the real-valued differential 1-form given by \( \theta_{\alpha,\beta} := -i(\partial - \bar{\partial}) \log(\rho_{\alpha,\beta}) \). Then, \( \nabla^*_{\alpha,\beta} \nabla_{\alpha,\beta} \) coincides with the second order differential operator given by
\[
D_{\alpha,\beta} = -\left\{ \frac{\partial^{2}}{\partial z \partial \bar{z}} + \left( \frac{\alpha - \beta}{\vert z \vert^{2}} \right) \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right) \right\} \left( \alpha - \beta \right)^{2} \vert z \vert^{2}.
\]

**Proof.** Notice first that the differential 1-form \( \theta_{\alpha,\beta} \) is explicitly given by \( \theta_{\alpha,\beta} = ik_{\beta}^{\alpha}(z) \left( \bar{z} dz - z d\bar{z} \right) \) with \( k_{\beta}^{\alpha}(z) := \alpha - \beta / \vert z \vert^{2} \). Straightforward computation making use of the well-known facts \( d^{*} = -*d* \) and \( (\text{ext}_{\theta_{\alpha,\beta}})^{*} = *(\text{ext}_{\theta_{\alpha,\beta}}) \) shows that for every smooth differential 1-form \( \omega = Adz + Bd\bar{z} \) we have
\[
d^{*} \left( Adz + Bd\bar{z} \right) = -2 \left( \frac{\partial A}{\partial \bar{z}} + \frac{\partial B}{\partial z} \right)
\]
and
\[
(\text{ext}_{\theta_{\alpha,\beta}})^{*} \left( Adz + Bd\bar{z} \right) = -2ik_{\beta}^{\alpha}(z) (zA - \bar{z}B).
\]
Therefore, by taking \( \omega = df = \partial f dz + \bar{\partial} f d\bar{z} \) in (3.11) one recovers the explicit expression of the Hodge–de Rham operator
\[
\nabla^{*}_{0,0} \nabla_{0,0} = \frac{1}{4} d^{*}d = -\frac{\partial^{2}}{\partial z \partial \bar{z}}.
\]
Moreover, the explicit differential expression of the operators $d^*(\text{ext}_{\alpha,\beta})$, $(\text{ext}_{\alpha,\beta})^*d$ and $(\text{ext}_{\alpha,\beta})^*(\text{ext}_{\alpha,\beta})^*$ are given respectively by
\[
\begin{align*}
d^*(\text{ext}_{\alpha,\beta})f &= 2ik^\alpha_\beta(z)(E - \bar{E})f, \\
(\text{ext}_{\alpha,\beta})^*df &= -2ik^\alpha_\beta(z)(E - \bar{E})f, \\
(\text{ext}_{\alpha,\beta})^*(\text{ext}_{\alpha,\beta})^*f &= 4k^\alpha_\beta(z)^2|z|^2f.
\end{align*}
\]
Subsequently, by expanding $\nabla^{\alpha,\beta}_\alpha \nabla_{\alpha,\beta}$ as
\[
\nabla^{\alpha,\beta}_\alpha \nabla_{\alpha,\beta} = \frac{1}{4}\{d^*d + i(d^*\text{ext}_{\alpha,\beta} - (\text{ext}_{\alpha,\beta})^*d) + (\text{ext}_{\alpha,\beta})^*(\text{ext}_{\alpha,\beta})\},
\]
and next making use of (3.13)-(3.16), we get its explicit expression given through $\mathcal{D}_{\alpha,\beta}$ in (3.10).

Remark 3.5. The second order differential operator $\mathcal{D}_{\alpha,\beta}$ in (3.10) is a magnetic Laplacian with a constant homogeneous magnetic field of magnitude $\alpha$ applied perpendicularly on the complex plane. Indeed, we have
\[
d\theta_{\alpha,\beta} = d\theta_\alpha = 2i\partial \overline{\partial} (\text{Log}(\rho_{\alpha,\beta})) = 2i\alpha dz \wedge d\overline{z},
\]
where $\theta_{\alpha,\beta} = \theta_\alpha + \overline{\theta}_\beta$ with $\overline{\theta}_\alpha = i\alpha (\overline{dz} - dz \overline{\overline{\overline{z}}})$ and $\overline{\theta}_\beta = -i\beta (dz - \overline{z}d\overline{\overline{z}})/|z|^2$. Moreover, the operator $\mathcal{D}_{\alpha,\beta}$ is essentially the classical Landau Hamiltonian in (1.3) perturbed by a first order differential operator associated with the potential 1-form $\tilde{\gamma}$ closed (with zero magnetic field), singular (at the origin) and modeling the Aharonov–Bohm effect.

Theorem 3.6. The functions $|z|^{2\beta} e^{-\alpha|z|^2}\psi_{n,m}^{2\alpha,2\beta}$ are $L^2$-eigenfunctions of the magnetic Laplacian $\nabla^{\alpha,\beta}_\alpha \nabla_{\alpha,\beta}$ with $\alpha(2n + 1)$ as associated eigenvalue.

Proof. The proof is immediate using Theorem 3.1, Lemma 3.4 and observing that the operators $\Delta_{2\alpha}$ in (3.1) and the magnetic Laplacian $\mathcal{D}_{\alpha,\beta}$ in (3.10) are unitary equivalent. More precisely, we have
\[
\rho_{\alpha,\beta} \left( \Delta_{2\alpha}^{2\beta} + \alpha \right) \left( (\rho_{\alpha,\beta})^{-1}f \right) = \mathcal{D}_{\alpha,\beta},
\]
which readily follows since
\[
\mathcal{D}_{\alpha,\beta} = B^{*-\alpha,-\beta} \circ A^{*\alpha,\beta} + \alpha = A^{*\alpha,\beta} \circ B^{*-\alpha,-\beta} - \alpha.
\]
Here $A^{*\alpha,\beta}$ is as in (3.3) and $B^{*\alpha,\beta}$ is the differential operator given by
\[
B^{*\alpha,\beta} f = [\rho_{\alpha,\beta}(z)]^{-1} \frac{\partial}{\partial z} (\rho_{\alpha,\beta}(z)f).
\]

4. Analytical side (Poly-meromorphy)

In this section, we discuss the regularity of $\psi_{n,m}^{\alpha,\beta}(z, \overline{z})$ as poly-meromorphic functions on the complex plane and we determine the “bi-order” of its unique pole. Recall first from [3, p 199] that a $n$-meromorphic function (or poly-meromorphic of order $n$) on an open set $U \subset \mathbb{C}$ is a complex-valued function for which there exist some meromorphic functions $\psi_k$; $k = 0, 1, \cdots, n - 1$ on $U$ such that
\[
f(z, \overline{z}) = \psi_0(z) + \overline{z}\psi_1(z) + \cdots + \overline{z}^{n-1}\psi_{n-1}(z).
\]
(4.1)
They are called simply $n$-analytic ($n$-poly-holomorphic) when the component functions are holomorphic in $U$, $\psi_k \in Hol(U)$. The latter ones can equivalently be defined as those satisfying the generalized Cauchy–Riemann equation $\partial^n/\partial \bar{z}^n = 0$. In order to give the exact statement of the main result of this section, we need first to precis e the notion of bi-order of a zero or a pole of a given poly-meromorphic function on $\mathbb{C}$. Thus, for a given non-constant $n$-analytic function $f$ on an open set $U \subseteq \mathbb{C}$, a point $z_0 \in U$ is said to be a zero of bi-order $(r, s)$, for given non-negative integers $r, s$ with $0 \leq r \leq n - 1$ and $(r, s) \neq (0, 0)$, if the following conditions are met

(a) $f$ can be rewritten as $f = \overline{z - z_0}^r g$ for certain $(n - s)$-analytic function $g = \sum_{k=0}^{n-s-1} (z - z_0)^k \phi_k,$ (4.2)

with $\phi_k \in Hol(U)$ and $\phi_0$ is not identically zero on $U$.

(b) $z_0$ is a zero of order $r$ for the constant component function $\phi_0$ in (4.2).

The first condition (a) is to say that $z_0$ is a zero of order $s$ for $f$ seen as a polynomial in $\bar{z}$. Notice also that the suggested definition is equivalent to have

$$f(z, \bar{z}) = (z - z_0)^r \left((z - z_0)^t \varphi_0 + \sum_{k=1}^{n-s-1} (z - z_0)^k \phi_k\right)$$

(4.3)

for $\varphi_0, \phi_j \in Hol(U)$ with $\varphi(z_0) \neq 0$. Notice here that $z_0$ does not need to be a zero of the holomorphic components $\phi_k; k = 1, 2, \cdots, n - s - 1$. However, for the particular case of $z_0$ being a common zero of $\phi$ the expression in (4.3) reduces to

$$f(z, \bar{z}) = (z - z_0)^r (z - z_0)^t g(z, \bar{z}),$$

for certain non-vanishing poly-analytic function $g$. This makes $z_0$ a zero of $f$ of bi-order $(r, s)$.

A point $z_0 \in U$ is said to be a pole of order $r$ ($r < 0$) for given $n$-poly-meromorphic function $f$ in (4.1) if $(z - z_0)^{|r|} f$ is a $n$-analytic function on $U$ and $r$ is the smallest negative integer satisfying this property. This is equivalent to $z_0$ being a pole for certain component meromorphic function $\psi_j$ with

$$r = \min\{\text{Ord}_p(z_0, \psi_j), j = 0, 1, \cdots, n - 1\},$$

where $\text{Ord}_p(z_0, \psi_j)$ is exactly the multiplicity of $z_0$ if it is a pole of $\psi_j$ and 0 otherwise. Such pole is said to be of bi-order $(r, s)$, if in addition (a) is satisfied. Accordingly, we denote by bi-Ord$(z_0; f)$ the bi-order of a point $z_0$ when is a zero or a pole of given $n$-poly-meromorphic function $f$.

**Theorem 4.1.** The functions $\psi^{\alpha, \beta}_{n,m}(z, \bar{z})$ are poly-meromorphic on $\mathbb{C}$. The origin is either a zero or a pole of bi-order

$$\text{bi-Ord}(0; \psi^{\alpha, \beta}_{n,m}) = (m - [n \wedge^* (\beta + m)], n - [n \wedge^* (\beta + m)]).$$

(4.4)

**Proof.** First of all, we point out that in view of (2.3), it is clear that the terms $\bar{z}^{n-k}$ are always regulars for every $k \leq n \wedge^* (m + \beta) \leq n$. The singularity of $\psi^{\alpha, \beta}_{n,m}(z, \bar{z})$ then lies in $z^{m-k}$ for $k \leq n \wedge^* (m + \beta)$. In particular, the functions $\psi^{\alpha, \beta}_{n,m}(z, \bar{z})$ are poly-holomorphic (since they are polynomials in $z$ and $\bar{z}$) if and only if $m \geq n \wedge^* (m + \beta)$. The latter condition is equivalent
to $\beta$ being a non-positive integer or $m \geq n$. In this case the expression of $\psi_{n,m}(z, \overline{z})$ reduces to

$$
\psi_{n,m}^{\alpha,\beta}(z, \overline{z}) = z^{m-[n\wedge^*(\beta+m)-n]+[n\wedge^*(\beta+m)]} R_{n,m;n\wedge^*(\beta+m)}^{\alpha,\beta}(z, \overline{z}),
$$

where the involved $R_{n,m;N}^{\alpha,\beta}(z, \overline{z})$ are the radial polynomials given by

$$
R_{n,m;N}^{\alpha,\beta}(z, \overline{z}) := \sum_{k=0}^{N} c_{m,n,k}^\beta |z|^{2([n\wedge^*(\beta+m)]-k)}.
$$

(4.5)

Subsequently, since $\beta + m + 1 > n \wedge^* (\beta + m)$ and then $c_{m,n,n\wedge^*(m+\beta)}^\alpha,\beta \neq 0$, the origin is a zero of $\psi_{n,m}^{\alpha,\beta}(z, \overline{z})$ whenever $\min(m, n) > n \wedge^* (\beta + m)$. Its bi-order is then

$$
\text{bi-Ord}(0; \psi_{n,m}^{\alpha,\beta}) = \begin{cases} 
(m-n, 0), & \text{if } \beta \in \mathbb{R} \setminus \mathbb{Z}^- \text{ or } m \geq n, \\
(-\beta, n - m - \beta), & \text{if } \beta \in \mathbb{Z}^-, \beta \neq 0 \text{ and } n > \beta + m.
\end{cases}
$$

To achieve the proof, it remains sufficient to discuss the case of $m < n \wedge^* (\beta + m)$ (i.e. $n > m > -\beta - 1$ and $\beta \notin \mathbb{Z}^-$). In this case we have

$$
\psi_{n,m}^{\alpha,\beta}(z, \overline{z}) = \sum_{k=0}^{m} c_{m,n,k}^{\alpha,\beta} z^{-k-n} + \sum_{k=m+1}^{n\wedge^*(\beta+m)} c_{m,n,k}^{\alpha,\beta} z^{-k-n-k} \\
= \overline{z}^{m-n} R_{n,m;m}^{\alpha,\beta}(z, \overline{z}) + \overline{z}^{n-[n\wedge^*(\beta+m)]/m-1} \sum_{j=0}^{n\wedge^*(\beta+m)-1} c_{m,n,m+1+j}^{\alpha,\beta} \overline{z}^{n\wedge^*(\beta+m)-m-1-j} \\
= \overline{z}^{m-n} R_{n,m;m}^{\alpha,\beta}(z, \overline{z}) + \frac{\overline{z}^{n-[n\wedge^*(\beta+m)]/m-1} S_{n,m;m\wedge^*(\beta+m)-m-1}^{\alpha,\beta}(z, \overline{z}),
$$

where $R_{n,m;N}^{\alpha,\beta}(z, \overline{z})$ is as in (4.5) with $N = m$, and

$$
S_{n,m;N}^{\alpha,\beta}(z, \overline{z}) = \sum_{j=0}^{n\wedge^*(\beta+m)-1} c_{m,n,m+1+j}^{\alpha,\beta} \overline{z}^{2([n\wedge^*(\beta+m)]-m-j-1)}.
$$

It convenient to mention here that both $R_{n,m;N}^{\alpha,\beta}(z, \overline{z})$ and $S_{n,m;N}^{\alpha,\beta}(z, \overline{z})$ are poly-analytic radial polynomials on the whole complex plane for which the origin is not a zero (for again $c_{m,n,m\wedge^*(\beta+m)}^{\alpha,\beta} \neq 0$). This proves that the functions $\psi_{n,m}^{\alpha,\beta}$ are purely poly-meromorphic functions with 0 as unique pole if and only if $\beta \notin \mathbb{Z}^-$ and $m < n$. The multiplicity of their unique pole is given by $\text{Ord}(0; \psi_{n,m}^{\alpha,\beta}) = m - [n \wedge^* (\beta + m)] < 0$ and then

$$
\text{bi-Ord}(0; \psi_{n,m}^{\alpha,\beta}) = \begin{cases} 
(m-n, 0), & \beta \in \mathbb{R} \setminus \mathbb{Z}, n > m, \\
(m-n, 0), & \beta \in \mathbb{Z}^+, \beta + m \geq n > m, \\
(-\beta, n - m - \beta), & \beta \in \mathbb{Z}^+, n \geq \beta + m > m, \\
(-\beta, n - m - \beta), & \beta \in \mathbb{Z}^-, n > m.
\end{cases}
$$

\[\square\]

Remark 4.2. The polynomial case, i.e., the restriction to the case of $m \geq n$ (with $\beta \geq 0$) leads to the class of polynomials $Z_{m,n}^{\beta}(z, w)$ introduced and studied by Ismail and Zeng \[15, \text{Section 3}\]. Some of the obtained results in the previous section generalize the one derived in [15, Section 3].
5. Generating functions

Notice first that using the relation of $\psi_{n,m}^{\alpha,\beta}$ to the generalized Laguerre polynomials and the generating function for the latter ones [5, 22], one obtains

$$\frac{(-1)^n}{n!} z^{j+n} \psi_{n,m}^{\alpha,\beta}(z, \overline{z}) = \sum_{k=0}^{n} \frac{(-1)^k z^k}{k!(n-k)!(j-(n-k))!} \psi_{k,j}^{\alpha,\beta}(z, \overline{z})$$

for all $j > \text{max}(-\beta, n)$. Moreover, by [22, p 242] with $\beta + k > -1$, we have

$$\sum_{n=0}^{\infty} \frac{t^n \psi_{n,m}^{\alpha,\beta}(z, \overline{z})}{n!(1+\beta+k)^n} = \left(\frac{wz}{1-t}\right)^{\beta+k+1} \frac{e^{\alpha(tz^2+|w|^2)}}{(1-t)^{\beta+1}} _0F_1 \left(\frac{-|\alpha|z^2}{t(1-t)^2}\right).$$

The next one is an analog of the standard one for the Itô–Hermite polynomials [11, p. 7] which appears as the special case when $\beta = 0$ and $\alpha = 1$.

**Theorem 5.1.** For any real $\beta > -1$, the functions $\psi_{n,m}^{\alpha,\beta}$ satisfy

$$\sum_{m,n=0}^{\infty} \frac{u^m v^n}{m! n!} \psi_{n,m}^{\alpha,\beta}(z, \overline{z}) = \left(1 - \frac{v}{z}\right)^\beta e^{|u+v|z}.$$

**Proof.** Starting from the left hand-side of (5.1) and inserting (1.7) and next interchanging the sum in $m$ and the $n$-th derivative, one obtains

$$\sum_{m,n=0}^{\infty} \frac{u^m v^n}{m! n!} \psi_{n,m}^{\alpha,\beta}(z, \overline{z}) = z^{-\beta} e^{\alpha z^2} \sum_{n=0}^{\infty} \frac{(-v)^n}{n!} \left(\sum_{k=0}^{n} \frac{(-1)^k z^k}{k!(n-k)!(j-(n-k))!}\right) \left|\varphi_{\beta}(x)\right|_{x=z}$$

$$= z^{-\beta} e^{\alpha z^2} \varphi_{\beta}(z - v),$$

with $\varphi_{\beta}(x) := z^\beta e^{-\alpha(z+u)x}$. The last equality follows using the translation operator of the Taylor series of the involved function and gives rise to the right hand-side of (5.1). \qed

The following results are partial generating functions for $\psi_{n,m}^{\alpha,\beta}$ (with fixed $n$ or $m$).

**Proposition 5.2.** For $\beta > -1$, we have

$$\sum_{n=0}^{\infty} \psi_{n,k}^{\alpha,\beta}(z, \overline{z}) \frac{v^n}{n!} = \frac{(z-v)^{k+\beta}}{z^\beta} e^{\alpha z}$$

as well as

$$\sum_{m=0}^{\infty} \frac{u^m}{m!} \psi_{n,m}^{(\alpha,\beta)}(z, \overline{z}) = \frac{(-1)^n}{z^n} L_{n}^{(\beta-n)}(\alpha|z|^2 - uz).$$

**Proof.** The first assertion can be handled starting from the Rodrigues formula (1.7) and next expanding $e^{(z-v)u}$ in the second right hand-side of (5.1). Indeed, the identity (5.2) immediately follows from Theorem 5.1 by identifying the obtained series in $u$.

For (5.3), we have

$$\sum_{m=0}^{\infty} \frac{u^m}{m!} \psi_{n,m}^{(\alpha,\beta)}(z, \overline{z}) = \frac{(-1)^n}{z^n} e^{\alpha|z|^2} \frac{\partial^n}{\partial z^n} (z^\beta e^{-\alpha|z|^2-zu})$$

$$= \frac{(-1)^n}{z^n} e^{zu} L_{n}^{(\beta-n)}(\alpha|z|^2 - uz).$$
The latter expression in terms of the generalized Laguerre polynomials is immediate by making the variable change \( x = \alpha |z|^2 - uz \).

The exact statement of the next result concerning a special generating function for the \( \psi_{\alpha,\beta}^{\alpha,\beta}(z, \bar{z}) \) makes appeal to the lower incomplete Gamma function defined by

\[
\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt, \: \Re(s) > 0. \tag{5.4}
\]

Its expansion series reads [22, p.337]

\[
\gamma(s, x) = e^{-x} \sum_{k=0}^{\infty} \frac{x^{k+s}}{(s)_{k+1}}. \tag{5.5}
\]

**Theorem 5.3.** For every \( \beta > 0 \) and \( |uv| < |uz| \), we have

\[
\sum_{\substack{m,n=0 \atop m \neq n}}^{\infty} \frac{u^m v^n}{(\beta + 1)_{m+n}!} \psi_{\alpha,\beta}^{\alpha,\beta}(z, \bar{z}) = \beta u^{-\beta} z^{-\beta} e^{u(z-v)+u\bar{z}v} \gamma(\beta, u(z - v)). \tag{5.6}
\]

**Proof.** The left hand-side in (5.6) can be expressed as

\[
\sum_{\substack{m,n=0 \atop m \neq n}}^{\infty} \frac{u^m v^n}{(\beta + 1)_{m+n}!} \psi_{\alpha,\beta}^{\alpha,\beta}(z, \bar{z}) = \beta e^{-\alpha |z|^2} \sum_{n=0}^{\infty} \frac{(-v)^n}{n!} \frac{\partial^n}{\partial z^n} \left( u^{-\beta} e^{-\alpha |z|^2} \left[ \sum_{\substack{m=0 \atop m \neq n}}^{\infty} \frac{(zu)^{m+n}}{(\beta)_{m+n}!} \right] \right). \]

This follows making use of (1.7) as well as the expansion (5.5). Moreover, we get

\[
\sum_{\substack{m,n=0 \atop m \neq n}}^{\infty} \frac{u^m v^n}{(\beta + 1)_{m+n}!} \psi_{\alpha,\beta}^{\alpha,\beta}(z, \bar{z}) = \frac{\beta}{z^\beta u^\beta} e^{z u + u \bar{z} v} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-v)^n (-z \bar{v})^{n-k} (-u)^k}{k!(n-k)!} (1 - \beta)_k \gamma(\beta - k, z u) = \frac{\beta}{(zu)^\beta} e^{zu + u \bar{z} v} \sum_{k=0}^{\infty} \frac{u^k v^k}{k!} (1 - \beta)_k \gamma(\beta - k, z u). \]

The second equality follows using the Leibniz formula combined with the derivative formula given for the lower incomplete Gamma function in [5, p. 21]. Finally, by means of the series formula in [5, p. 460] we arrive at the expression

\[
\sum_{\substack{m,n=0 \atop m \neq n}}^{\infty} \frac{u^m v^n}{(\beta + 1)_{m+n}!} \psi_{\alpha,\beta}^{\alpha,\beta}(z, \bar{z}) = \frac{\beta}{(zu)^\beta} \gamma(\beta, (z - u)v) e^{zu + u \bar{z} v - uv},
\]

valid for all \( z, v \in \mathbb{C} \) such that \( |v| < |z| \). \( \square \)

**Corollary 5.4.** Let \( u, v \) and \( z \) be complex numbers such that \( z \neq 0, |z| > |v| \) and \( \Re(u(z - v)) > 0. \) Then, for every \( \beta > 0 \) we have

\[
\sum_{\substack{m,n=0 \atop m \neq n}}^{\infty} \frac{u^m v^n}{(\beta + 1)_{m+n}!} \psi_{\alpha,\beta}^{\alpha,\beta}(z, \bar{z}) = \left( 1 - \frac{v}{z} \right)^\beta e^{u \bar{z} v} \frac{1}{\beta + 1} F_1 \left( \frac{1}{\beta + 1} \bigg| u(z - v) \right) F_1.
\]
**Proof.** This can be handled by means of Theorem 5.3 and the hypergeometric representation of the lower incomplete Gamma function [22, p.337]

\[
\gamma(s, x) = \frac{e^{-x}}{s} x^s \frac{1}{s+1} F_1 \left( \frac{1}{s+1} \left| x \right| \right), \Re(x) > 0.
\] (5.7)

**Remark 5.5.** The generating function in (5.6) can be rewritten in terms of the upper incomplete Gamma function [22, p.337]

\[
\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt
\] (5.8)

since \(\gamma(s, x) = \Gamma(s) - \Gamma(s, x)\). Indeed, for \(\beta\) being a positive integer we have

\[
\sum_{m,n=0}^{+\infty} \psi^{\alpha,\beta}_{n,m}(z, \z) \frac{u^m v^n}{(\beta + 1)m!} = \frac{\beta (\Gamma(\beta) - \Gamma(\beta, u(z-v)))}{(zu)^\beta} e^{\alpha \z v + u(z-v)}. \quad (5.9)
\]

**Remark 5.6.** As immediate consequence of Theorem 5.3, one can prove that the partial generating function in (5.6) remains valid for \((v, z)\) in a special region of \(\mathbb{C} \times \mathbb{C}\).

### 6. Integral representations

The aim below is to derive some integral representations for the considered poly-meromorphic Itô–Hermite functions \(\psi^{\alpha,\beta}_{n,m}(z, \z)\). The first one involves the Bessel function of order \(\nu > -1\) of the first kind defined by [5, p. 675],

\[
J_\nu(z) := \frac{1}{\Gamma(\nu + 1)} \left( \frac{z}{2} \right)^\nu \sum_{m=0}^{+\infty} \frac{\nu^m}{m!} J_{\nu + m}(2\sqrt{\alpha}|z|x)e^{-x^2} dt.
\] (6.1)

**Proposition 6.1.** For fixed real \(\beta\) and integers \(n, m\) such that \(n = 0, 1, \cdots\) and \(\beta + m - n > -1\), we have

\[
\psi^{\alpha,\beta}_{n,m}(z, \z) = (-1)^n \frac{z^{m-n} e^{\alpha |z|^2}}{(\sqrt{\alpha}|z|)^{\beta+m-n}} \int_0^{+\infty} x^{n+m+\beta+1} J_{\beta+m-n}(2\sqrt{\alpha}|z|x)e^{-x^2} dt.
\] (6.1)

**Proof.** Making use of the close connection of \(\psi^{\alpha,\beta}_{n,m}(z, \z)\) to the Laguerre polynomials combined with their integral representation in terms of the Bessel function [22, p. 243]

\[
\mathcal{L}^{(n)}(x) = \frac{x^{n/2} e^x}{n!} \int_0^{+\infty} e^{-t} t^{n+\beta/2} J_\beta(2\sqrt{x}t) dt,
\]

valid for \(n = 0, 1, 2, \cdots\), and \(n+\mu > -1\) with \(x\) being a real positive number, the expression (2.7) of \(\psi^{\alpha,\beta}_{n,m}\) implies

\[
\psi^{\alpha,\beta}_{n,m}(z, \z) = (-1)^n \frac{z^{m-n} e^{\alpha |z|^2}}{(\sqrt{\alpha}|z|)^{\beta+m-n}} \int_0^{+\infty} e^{-t} t^{n+m+\beta/2} J_{\beta+m-n}(2|z|\sqrt{\alpha t}) dt,
\] (6.2)

for every integer \(m\) such that \(\beta + m > -1\). Finally, the change of variable \(t = x^2\) infers the expression in (6.1).

The next integral representations are obtained by means of the generating functions (5.1) and (5.2).
Proposition 6.2. The integral representation
\[ \psi_{n,m}^{(\alpha,\beta)} (z, \overline{z}) = \frac{1}{\pi^{2|\beta|}} \int_{C^2} u^m v^n (z - \overline{v})^\beta \ e^{-(|u|^2 + |v|^2 + \alpha |z| + \alpha |w|)} \ d\lambda(u,v) \] (6.3)
holds for every $\beta > -1$. Moreover, we have
\[ \psi_{n,k}^{\alpha,\beta} (z, \overline{z}) = \frac{1}{z^{2|\beta|}} \int_{C} u^m \ e^{-(z - \overline{v})^\beta + k e^{-(v - \alpha \xi)}} \ d\lambda(v). \] (6.4)

Proof. Thanks to $\psi_{m,n}^{\alpha,\beta} (z, \overline{z}) = \psi_{m,n}^{\alpha,\beta} (z, \overline{z})$, we can rewrite the generating function (5.1) in the following equivalent form
\[ \sum_{m,n=0}^{+\infty} \frac{m^\alpha n^\beta}{m!n!} \psi_{n,m}^{\alpha,\beta} (z, \overline{z}) = \left( 1 - \frac{\overline{v}}{z} \right)^\beta e^{\pi z + \alpha \overline{w} - w}. \] (6.5)

Next, by multiplying the both sides by the monomials in $u$ and $v$ and integrating on the whole two-dimensional complex space endowed with the Gaussian measure, it follows
\[ \int_{C^2} \left( 1 - \frac{\overline{v}}{z} \right)^\beta u^m v^n e^{-(|u|^2 + |v|^2 + \alpha |z| + \alpha |w|)} \ d\lambda(u,v) = \pi^2 \psi_{n,m}^{\alpha,\beta} (z, \overline{z}), \]
which leads to (6.3). Analogously, one gets (6.4) starting from (5.2).

The generating function in Theorem 5.1 can be also employed to establish the following integral representation.

Proposition 6.3. Let $\beta$ be an integer such that $\beta + m \geq 0$. Then, we have
\[ \psi_{n,m}^{\alpha,\beta} (z, \overline{z}) = \frac{(-1)^{m+\beta} n^{\alpha+1}}{\pi^{2|\beta|}} \int_{C} \xi^n \overline{\xi}^{m+\beta} e^{-\alpha(|\xi|^2 - |z|^2 + \alpha |w|)} \ d\lambda(\xi). \] (6.6)

Proof. Note that making use of the $2d$ fractional Fourier transform (1.2) introduced in [4] one obtains the following integral formula
\[ e^{\alpha zw} = \frac{\alpha}{\pi} \int_{C} e^{-\alpha|\xi|^2 + \alpha(\xi z + \overline{\xi} w)} \ d\lambda(\xi) \] (6.7)
for every complex numbers $z, w$ and real $\alpha > 0$. It can also be viewed as a reproducing property for the reproducing kernel of the Segal–Bargmann space. Next, by rewriting the generating function in (5.2) in the following equivalent form
\[ \sum_{n=0}^{+\infty} \psi_{n,k-\beta}^{\alpha,\beta} (z, \overline{z}) \frac{v^n}{n!} = \frac{(-1)^{k+\beta}}{\alpha^{k+\beta}} \frac{\partial^k}{\partial z^k} \left( e^{\alpha(v - z)|\overline{w}|} \right) \] (6.8)
and making appeal to the formula (6.7), it follows
\[ \sum_{n=0}^{+\infty} \psi_{n,k-\beta}^{\alpha,\beta} (z, \overline{z}) \frac{v^n}{n!} = \frac{(-1)^{k+\beta}}{\alpha^{k+\beta}} \frac{\partial^k}{\partial z^k} \left( e^{\alpha(v - z)|\overline{w}|} \right) \]
and
\[ = \sum_{n=0}^{+\infty} \frac{v^n}{n!} \left( \frac{(-1)^{k+\beta}}{\alpha^{k+\beta}} \frac{\partial^k}{\partial z^k} \left( e^{\alpha(v - z)|\overline{w}|} \right) \right). \]
The result in (6.6) is then immediate by identification.

Remark 6.4. The identity (6.6) can be reproved starting from (2.2) and making use of the classical integral representation of the Itô–Hermite polynomials in [12].
7. Applications

7.1. New integral formula for the generalized Laguerre polynomials. Using the obtained results one can derive new interesting integral formulas for the generalized Laguerre polynomials. Thus, we claim the following.

Theorem 7.1. The integral identity

\[ L_n^{(\beta-n)}(\alpha|z|^2 - uz) = \frac{(-1)^n z^n}{n! \pi} \int_C \overline{\psi}^n(z-v)^{\beta} e^{\alpha \overline{\sigma} - uv \overline{\sigma} + |v|^2} d\lambda(v) \]  

(7.1)

holds for \( \beta > -1 \).

Proof. From (5.3), one has

\[ \sum_{m=0}^{+\infty} \frac{u^m}{m!} \psi_{k,m}(z, \overline{z}) = \frac{1}{\pi} \left( \sum_{m,n=0}^{+\infty} \frac{u^m v^n}{m! n!} \psi_{n,m}^\alpha(z, \overline{z}), v^k \right) \]

\[ = \frac{e^{uz}}{\pi z^\beta} \int_C \overline{\sigma}^k(z-v)^{\beta} e^{\alpha \overline{\sigma} - uv \overline{\sigma} + |v|^2} d\lambda(v). \]

Accordingly, the proof of (7.1) readily follows from (5.3). \( \square \)

Remark 7.2. As particular case we get

\[ L_n^{(\beta-n)}(z) = \frac{(-1)^n z^n}{n! \pi} \int_C \overline{\psi}^n(z-v)^{\beta} e^{-v(\sigma-1)} d\lambda(v) \]

(7.2)

for any \( z \in \mathbb{C} \) by specifying \( \alpha = 0 \) and \( u = -1 \). Also, by taking \( \alpha = 1 \) and \( u = 0 \), we get

\[ L_n^{(\beta-n)}(|z|^2) = \frac{(-1)^n z^n}{n! \pi} \int_C \overline{\psi}^n(z-v)^{\beta} e^{\sigma} e^{-|v|^2} d\lambda(v). \]

(7.3)

7.2. Associated integral transforms. The orthogonality property of the functions \( \psi_{n,m}^{\alpha,\beta} \) suggest the consideration of two special functional spaces \( \mathcal{F}^2_{\alpha,n}(\mathbb{C}) \) and \( \overline{\mathcal{F}}^2_{\alpha,n}(\mathbb{C}) \) of poly-meromorphic or anti-poly-meromorphic functions on the punctured complex plane in \( L^2_{\beta}(\mathbb{C}) \) for fixed non-negative integers \( m \) and \( n \) with \( m + \beta > -1 \). These spaces are spanned by \( \psi_{n,j}^{\alpha,\beta}, j \geq [-\beta] \), and \( \psi_{k,m}^{\alpha,\beta}, k = 0, 1, 2, \ldots \), respectively. Moreover, they can be seen as the poly-meromorphic analogs of the true poly-analytic (and anti-poly-analytic) Bargmann spaces \([1, 25]\) defined as specific closed subspace in ker \( (\partial^{n+1}/\partial \overline{\sigma}^{n+1}) \cap L^2_{\beta}(\mathbb{C}) \), and realized also as \( L^2 \)-eigenspace \( \mathcal{F}^2_{\alpha,n}(\mathbb{C}) = \ker(\Delta_{\alpha} - \alpha n) \) associated with the \( n \)-th Landau levels of the self-adjoint magnetic Laplacian

\[ \Delta_{\alpha} = \Delta_{\alpha,\beta} = -\frac{\partial^2}{\partial z \partial \overline{z}} + \alpha \frac{\partial}{\partial \overline{z}} \]

acting on \( L^2_{\beta}(\mathbb{C}) \) (see \([2, 24]\)).

Next, we show that \( \overline{\mathcal{F}}^2_{\alpha,n}(\mathbb{C}) = \text{Span}\{\psi_{k,m}^{\alpha,\beta}, k = 0, 1, 2, \ldots\} \) can be realized as the image of the classical Segal–Bargmann space \( \mathcal{F}^2_{\alpha}(\mathbb{C}) = Hol(\mathbb{C}) \cap L^2_{\beta}(\mathbb{C}) \) of holomorphic functions in the Hilbert space of the Gaussian functions, \( L^2_{\beta}(\mathbb{C}) := L^2(\mathbb{C}, e^{-|x|^2} d\lambda) \), by means of the specific integral transform

\[ B^\alpha_{\beta,m} f(z) := \frac{\alpha}{\pi} \left( \frac{\alpha^{\beta+m}}{\Gamma(\beta + m + 1)} \right)^{1/2} z^m \int_C \left( 1 - \frac{\overline{w}}{z} \right)^{\beta+m} e^{-\alpha \overline{w}(w-\overline{z})} f(w) d\lambda(w), \]

(7.4)
provided that the integral exists. Here \( m \) is a fixed integer such that \( m > -\beta - 1 \) for given \( \beta > -1 \). In fact, the transform \( B_m^{\alpha,\beta} \) is well defined and maps the orthonormal basis \( e_n^{\alpha}(z) = (\alpha^{n+1}/\pi n!)^{1/2}z^n \) of \( F^{2,\alpha}(\mathbb{C}) \) to an orthonormal basis of \( \tilde{F}^{2,\alpha}_{\beta,m}(\mathbb{C}) \). More precisely, we have

\[
B_m^{\alpha,\beta}(e_n)(z) = \left( \frac{\alpha^{\beta+m+1}}{\pi \alpha^n \Gamma(\beta+m+1)n!} \right)^{1/2} \psi_{n,m}^{\alpha,\beta}(z,\overline{z}).
\]

This follows by observing that the integral kernel of the transform \( B_m^{\alpha,\beta} \) in (7.4) is the generating function in (5.2). Its inverse is given by

\[
(B_m^{\alpha,\beta})^{-1}(f)(w) = \frac{\alpha}{\pi} \left( \frac{\alpha^{\beta+m}}{\Gamma(\beta+m+1)} \right)^{1/2} \int_{\mathbb{C}} \frac{(z-w)^{\beta+m}}{\overline{z}^\beta} e^{-\alpha(\overline{w}-w)z} f(z)d\lambda(z). \tag{7.5}
\]

It is worth noticing that for the particular case of \( \beta = 0 \), the corresponding transform \( B_m^{\alpha,0} \) reduces further to the one considered in [4, Remark 2.13] mapping unitarily the Segal-Bargmann space to the true anti-poly-analytic Bargmann spaces \( \tilde{F}^{2,\alpha}_{\beta,m}(\mathbb{C}) \).

Similarly, associated with the kernel function on \( \mathbb{C} \times \mathbb{C} \) given through the partial generating function in (5.3),

\[
s_n^{(\alpha,\beta)}(u, z) := \frac{(-1)^n n!}{z^n} e^{uz} L_n^{(\beta-n)}(\alpha |z|^2 - u z), \tag{7.6}
\]

we consider the integral transform

\[
S_n^{\alpha,\beta} f(z) := \frac{(-1)^n n!}{z^n} \int_{\mathbb{C}} L_n^{(\beta-n)}(\alpha |z|^2 - u z) e^{-u(\overline{w}-w)z} f(u)d\lambda(u). \tag{7.7}
\]

The image of \( F^{2,\alpha}(\mathbb{C}) \) by \( S_n^{\alpha,\beta} \) for arbitrary \( \beta > -1 \) is the closed subspace of \( L_{\beta,n}^{2,\alpha}(\mathbb{C}) \) spanned by \( \psi_{n,j}^{\alpha,\beta} \) for varying \( j = 0, 1, 2, \cdots \),

\[
S_n^{\alpha,\beta}(F^{2,\alpha}(\mathbb{C})) = \text{Span}\{\psi_{n,j}^{\alpha,\beta}; j = 0, 1, 2, \cdots\} L_{\beta,n}^{2,\alpha}(\mathbb{C}) =: \tilde{F}_{\beta,n}^{2,\alpha}(\mathbb{C}).
\]

For \( \beta > 0 \), it reduces to the restricted \( \beta \)-poly-meromorphic space \( \tilde{F}_{\beta,n}^{2,\alpha}(\mathbb{C}) \) which is strictly contained in \( F_{\beta,n}^{2,\alpha}(\mathbb{C}) := \text{Span}\{\psi_{n,j}^{\alpha,\beta}; j = [-\beta], [-\beta] + 1, \cdots\} L_{\beta,n}^{2,\alpha}(\mathbb{C}) \). For \(-1 < \beta \leq 0\), this is exactly the poly-meromorphic space \( S_n^{\alpha,\beta}(F^{2,\alpha}(\mathbb{C})) = \tilde{F}_{\beta,n}^{2,\alpha}(\mathbb{C}) = F_{\beta,n}^{2,\alpha}(\mathbb{C}) \).

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