On the Number of Perfect Triangles with a Fixed Angle

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Abstract

Richard Guy asked the following question: can we find a triangle with rational sides, medians and area? Such a triangle is called a perfect triangle and no example has been found to date. It is widely believed that such a triangle does not exist. Here we use the setup of Solymosi and de Zeeuw about rational distance sets contained in an algebraic curve, to show that for any angle $0 < \theta < \pi$, the number of perfect triangles with an angle $\theta$ is finite.

Keywords
Perfect triangle · Rational distance set · Algebraic curves

Mathematics Subject Classification 14Hxx · 11D09

1 Introduction

A median of a triangle is a line segment joining a vertex to the midpoint of the opposite side. Finding a triangle with rational sides, medians, and area was asked as an open problem by Richard Guy [6, D21]. Such a triangle is called a perfect triangle. Various research has been done towards this question, but to date the problem remains unsolved (see, e.g., [9] for a survey). If we do not require the area to be rational, there are infinitely many solutions. Euler gave a parametrization of such ‘rational triangles’, in which all three medians were rational, see [3], however there are examples of triangles with three integer sides and three integer medians that are not given by the Euler parametrization. Buchholz [3] showed that every rational triangle with rational medians corresponds to a point on a one parameter elliptic curve. In the same vein Buchholz and Rathbun [4] have shown the existence of infinitely many Heron triangles with two rational medians, where a Heron triangle is a triangle that has side lengths and area that are all rationals.

A related, but slightly different problem is the Erdős–Ulam problem. We say that a subset $S \subseteq \mathbb{R}^2$ is a rational distance set if the distance between any two points in $S$
is a rational number. In 1945 Ulam posed the following question, based on a result of Anning–Erdős [1]. See [6, Prob. D20].

**Question** (Erdős–Ulam) Is there a rational distance set \( S \) in the plane \( \mathbb{R}^2 \) that is dense for the Euclidean topology?

Solymosi and de Zeeuw [18] used Faltings’ Theorem to show that a rational distance set contained in a real algebraic curve contains finitely many points, unless the curve has a component which is either a line or a circle. Furthermore, if a line (resp. circle) contains infinitely many points of a rational distance set, then it contains all but at most 4 (resp. 3) points of the set.

Although this problem is still open, there are several conditional proofs that show that the answer to the Erdős–Ulam question is no. Shaffaf [15] and Tao [19] independently used the weak Lang conjecture to give a negative answer to this question. Pasten [13] also proved that the abc conjecture implies a negative solution to the Erdős–Ulam problem.

In the same circle of ideas, the weak Lang conjecture was used [12] to show that if \( S \) is a rational distance set of \( \mathbb{R}^2 \) that intersects any line in only finitely many points, then there is a uniform bound on the cardinality of the intersection of \( S \) with any line. Recently, Ascher et al. [2] considered rational distance sets \( S \subset \mathbb{R}^2 \) such that no line contains all but at most four points of \( S \), and no circle contains all but at most three points of \( S \). They showed by assuming the weak Lang conjecture that there exists a uniform bound on the cardinality of such sets \( S \).

Along the same lines, a *perfect cuboid* is a rectangular box with all sides, face diagonals, and main diagonals being integers. In [11] van Luijk used the weak Lang conjecture to show that the set of perfect cuboids is very small, where small means that it is not Zariski-dense in its projective parameter space. Luca [10] has shown, the existence of a perfect cuboid is equivalent to the existence of a *perfect square triangle*, where a perfect square triangle is a triangle whose sides are perfect squares and whose angle bisectors are integers.

In this paper we use the setup of Solymosi and de Zeeuw [18], to analyse the following problem. Fix an angle \( \theta \) and consider Heron triangles with two rational medians such that \( \theta \) is one the angles of triangle. If we remove the constrain on the angle, we get infinitely many such triangles. Instead, in our setting:

**Theorem 1.1** Given \( 0 < \theta < \pi \) where \( \theta \neq \pi/2 \), there are, up to similarity, finitely many Heron triangles with an angle \( \theta \) and with two rational medians.

**Corollary 1.2** Given \( 0 < \theta < \pi \), there are, up to similarity, finitely many perfect triangles with an angle \( \theta \). In particular, there is no perfect right triangle.

### 2 Preliminaries on Genera of Curves

Given an affine algebraic curve in \( \mathbb{R}^2 \), defined by a polynomial \( f \in \mathbb{K}[x,y] \) (\( \mathbb{K} \) is a subfield of \( \mathbb{R} \)) one can consider its projective closure, which is a projective algebraic curve, by taking the zero set of the homogenisation of \( f \). This curve in \( \mathbb{P}^2_\mathbb{R} \) then extends to \( \mathbb{P}^2_\mathbb{C} \), by taking the complex zero set of the homogenised polynomial. In particular,
when we consider the genus of a curve, we are talking about complex projective algebraic curves.

To a given irreducible projective curve $X$ over complex numbers $\mathbb{C}$ we associate two invariants. One is the geometric genus $g(X)$, and the other one is the arithmetic genus $p_a(X)$. In particular, it is known that if $X$ is a smooth curve, then $p_a(X) = g(X)$. For more details on these notions we refer the reader to [21, Tag 0BYE]. In particular, if $X$ is a reducible curve with components $D_1, \ldots, D_m$, then we have

$$p_a(X) = \sum_{k=1}^{m} p_a(D_k) + \sum_{i \neq j} D_i \cdot D_j - (m - 1). \quad (1)$$

In this paper, by genus of a curve we will mean the geometric genus, unless otherwise specified.

The main ingredients in our proof are the following theorem of Faltings [5] and the Riemann–Hurwitz formula [17, Thm. 5.9].

**Theorem 2.1** (Faltings) Let $K$ be a number field. If $X$ is an algebraic curve over $K$ of genus $g \geq 2$, then the set $X(K)$ of $K$-rational points is finite.

**Theorem 2.2** (Riemann–Hurwitz) Let $\phi: X_1 \to X_2$ be a non-constant separable map of curves. Then

$$2g_1 - 2 \geq (2g_2 - 2) \deg \phi + \sum_{p \in X_1} (e_p - 1),$$

where $g_i$ is the genus of $X_i$ and $e_p$ is the ramification index of $\phi$ at $p$.

We will make use of the following result from [8].

**Lemma 2.3** Let $Y_1$ and $Y_2$ be two smooth curves of genus at most 1. Let $Y \subset Y_1 \times Y_2$ be an irreducible curve such that the two projections restricted to $Y$ are either birational or 2:1 maps to $Y_1$, resp. $Y_2$. Then $g(Y) \leq 5$, with equality only if $g(Y_1) = g(Y_2) = 1$ and both projections being 2:1.

### 3 Proof of Theorem 1.1

Fix an angle $\theta \neq \pi/2$. Let $a$, $b$ denote the side lengths of a Heron triangle $\Delta$ such that the angle between these two sides is $\theta$. Without loss of generality we may assume the

![Fig. 1 A triangle with side lengths $a$, $b$, and 1, and median lengths $m_1$, $m_2$, and $m_3$.](image)
side opposite $\theta$ has length 1. Let $\lambda = \cos \theta$ and $\alpha = \sin \theta$. By the law of cosines we have

$$1 = a^2 + b^2 - 2\lambda ab.$$ 

The rationality of the area of $\Delta$ and law of cosines guarantee that both $\lambda$ and $\alpha$ are rational numbers. Let $X_0$ be the ellipse defined by

$$G(x, y) = 1 - x^2 - y^2 + 2\lambda xy.$$ 

Since $\Delta$ is a Heron triangle, two of its medians ($m_1$ and $m_2$) are rational. We have

$$4m_1^2 = 2a^2 + 2b^2 - 1, \quad 4m_2^2 = 2a^2 + 2 - b^2.$$ 

On the other hand, $G(a, b) = 0$. So for every Heron triangle as above we obtain a rational point $(a, b, m_1, m_2)$ on the curve $X_{12}$ in $\mathbb{R}^4$, given by

$$G(x, y) = 0, \quad 4t_1^2 - 2x^2 - 2y^2 + 1 = 0, \quad 4t_2^2 - 2x^2 - 2 + y^2 = 0.$$ 

We shall show that the genus of $X_{12}$ is strictly bigger than 1. To do that consider the curves

$$X_1 = \{(x, y, t_1) : G(x, y) = 0, \ 4t_1^2 - 2x^2 - 2y^2 + 1 = 0\},$$

$$X_2 = \{(x, y, t_2) : G(x, y) = 0, \ 4t_2^2 - 2x^2 - 2 + y^2 = 0\}.$$ 

By considering the Jacobian matrix of $X_1$ we can see $X_1$ is a smooth curve (even in the projective space $\mathbb{P}^3$), hence the geometric genus of $X_1$ is equal to the arithmetic genus of $X_1$. Now we show that the geometric genus of $X_1$ is 1. Consider the projection map $\pi_1 : X_1 \to X_0$ given by

$$(x, y, t_1) \mapsto (x, y).$$ 

The preimage of each point $(x, y) \in X_0$ contains two points $(x, y, \pm t_1)$ in $X_1$ except when $t_1 = 0$. Hence $\pi_1$ is a map of degree 2. By applying the Riemann–Hurwitz formula we can bound the genus of $X_1$ from below. In particular,

$$2g(X_1) - 2 \geq (2g(X_0) - 2) \deg \pi_1 + \sum_{p \in X_1} (e_p - 1).$$ 

Since the genus of $X_0$ is zero (it is a conic), we have

$$g(X_1) \geq -1 + \frac{1}{2} \sum_{p \in X_1} (e_p - 1).$$ 

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1 In general the conic $ax^2 + bxy + cy^2 + dx + ey + f = 0$ is an ellipse if $b^2 - 4ac < 0$. In our situation $b = 2\lambda = 2 \cos \theta$, where $0 < \theta < \pi$ and $\theta \neq \pi/2$ and $d = e = 0$. 

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So to get $g(X_1) \geq 1$, we need to show that the projection $\pi_1$ has at least three ramification points. The potential ramification points correspond to the preimages of the intersection of $X_0 = V(G(x, y))$ with the conic $2x^2 + 2y^2 - 1 = 0$, where by Bézout’s Theorem there are 4 such points, counting with multiplicities.

$$2x^2 + 2y^2 - 1 = 0, \quad x^2 + y^2 - 1 - 2\lambda xy = 0.$$ 

By computing the discriminant we can see that this circle and ellipse intersect at 4 distinct points. Therefore, we get 4 ramification points. Thus Riemann–Hurwitz implies that the genus of $X_1$ is at least 1. On the other hand, $X_1$ is a smooth space curve of degree 4, so its genus is at most 1. Hence $g(X_1) = 1$.

**Claim: $X_1$ is Irreducible**

The proof is by contradiction. Suppose that $X_1$ is a reducible curve, and $D_1, \ldots, D_m$ are its irreducible components, then by (1) we know that the arithmetic genus $p_a(X_1)$ is

$$p_a(X_1) = \sum_{k=1}^{m} p_a(D_k) + \sum_{i \neq j} D_i \cdot D_j - (m - 1),$$

where $D_i \cdot D_j$ is the intersection of the components $D_i$ and $D_j$. On the other hand, we have seen that $X_1$ is smooth, hence its geometric genus is equal to the arithmetic genus. Moreover, its irreducible components do not intersect. Hence $p_a(X_1) = g(X_1) = 1$, which implies that the number of irreducible components of $X_1$ is at most two. However, the degree of $X_1$ is 4, thus if $X_1$ is reducible then it must be the union of an elliptic curve $E$ and a line $l$ that does not intersect $E$. Therefore,

$$p_a(X_1) = p_a(E) + p_a(l) - 1,$$

and this is a contradiction. Hence, $X_1$ is irreducible. A similar argument implies that $X_2$ is also irreducible.

**Claim: $X_{12}$ is an Irreducible Curve**

Consider two $2:1$ projection maps $\pi_1: X_1 \to X_0$ and $\pi_2: X_2 \to X_0$ defined by $\pi_1((x, y, t_1)) = (x, y)$ and $\pi_2((x, y, t_2)) = (x, y)$ respectively. The curve $X_{12}$ is given as follows,

$$X_{12} := \{(p_1, p_2) \in X_1 \times X_2 : \pi_1(p_1) = \pi_2(p_2)\}.$$ 

$X_{12}$ is the fiber product of $X_1$ and $X_2$. By [7, Thm. 3.3, p. 86], since $X_1$ and $X_2$ are irreducible, $X_{12}$ is irreducible, unless the two projection maps $\pi_1$ and $\pi_2$ have some branching points in common. The branching points of $\pi_1$ are in the form $(x_i, y_i), \ldots,$
where $x_i$ and $y_i$ satisfy
\[
x_i^2 = \frac{\lambda + \sqrt{\lambda^2 - 1}}{4\lambda}, \quad y_i^2 = \frac{1}{4\lambda(\lambda + \sqrt{\lambda^2 - 1})} \quad \text{for } i = 1, 2,
\]
\[
x_3^2 = \frac{\lambda - \sqrt{\lambda^2 - 1}}{4\lambda}, \quad y_3^2 = \frac{1}{4\lambda(\lambda - \sqrt{\lambda^2 - 1})} \quad \text{for } i = 3, 4.
\]
None of them is a branching point of $\pi_2$. In particular, this implies that $X_{12}$ is a smooth curve.\footnote{https://math.stackexchange.com/questions/1479139/fiber-products-of-curves.} Now by applying Lemma 2.3 (see [8, Lem. 3]) $X_{12}$ is an irreducible curve with genus 5. Indeed, the two projection maps $\phi_1$ and $\phi_2$ from $X_{12}$ to $X_1$ and $X_2$ respectively, defined by
\[
\phi_1(x, y, t_1, t_2) = (x, y, t_1) \quad \text{and} \quad \phi_2(x, y, , t_1, t_2) = (x, y, t_2)
\]
are 2:1 maps. \qed

\section{4 Proof of Corollary 1.2}

If $\theta \neq \pi/2$, then it is a consequence of Theorem 1.1. So we just deal with the case $\theta = \pi/2$. The proof of this case can be found in [20]; we report it just for self-containedness.

Suppose that $\Delta$ is a right triangle with rational sides; we show that at least one of its medians has irrational length. Let $c$ be the length of the hypotenuse of $\Delta$ and let $a$ and $b$ be the lengths of the sides adjacent to the right angle. Moreover, we may assume that $(a, b, c)$ is a primitive Pythagorean triple. Therefore, there exist two positive integers $m$ and $n$, with $m > n > 0$ and $m$ and $n$ coprime and of opposite parity, such that
\[
a = 2mn, \quad b = m^2 - n^2, \quad c = m^2 + n^2.
\]
Let $m_1$, $m_2$, and $m_3$ denote the median lengths of $\Delta$. It is known that $m_2 = c/2$. By the formulae expressing medians in terms of sides, we have
\[
4m_1^2 = 2a^2 + 2b^2 - 1, \quad 4m_2^2 = 2b^2 + 2 - a^2.
\]
Hence $m_3 = \sqrt{m^4 + n^4 - m^2n^2}/2$. Notice that $m_3$ is a rational number if and only if $m^4 + n^4 - m^2n^2$ is a perfect square of an integer. However, by [16, Thm. 7, p. 74], the equation $z^2 = x^4 + y^4 - x^2y^2$ has no positive integer solutions apart from the trivial ones: $x = y, z = x^2$. Therefore, $m_3$ has irrational length.

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