The de Finetti problem with unknown competition

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Abstract

We consider a resource extraction problem which extends the classical de Finetti problem for a Wiener process to include the case when a competitor, who is equipped with the possibility to extract all the remaining resources in one piece, may exist; we interpret this unknown competition as the agent being subject to possible fraud. This situation is modelled as a controller-and-stopper non-zero-sum stochastic game with incomplete information. In order to allow the fraudster to hide his existence, we consider strategies where his action time is randomised. Under these conditions, we provide a Nash equilibrium which is fully described in terms of the corresponding single-player de Finetti problem. In this equilibrium, the agent and the fraudster use singular strategies in such a way that a two-dimensional process, which represents available resources and the filtering estimate of active competition, reflects in a specific direction along a given boundary.

1 Introduction

In the classical single-player de Finetti problem for a Wiener process, the value of a limited resource evolves, in the absence of extraction, as

\[ Y_t = x + \mu t + \sigma W_t, \]

where \( \mu \) and \( \sigma \) are positive constants and \( W \) is a standard Brownian motion. The de Finetti problem – also known as the dividend problem – then consists of maximising

\[ \mathbb{E} \left[ \int_0^{\tau_0} e^{-rt} dD_t \right] \]

over all adapted, non-decreasing, and right-continuous processes \( D \) with \( D_{0-} = 0 \), where \( \tau_0 := \inf\{t \geq 0 : Y_t - D_t \leq 0\} \) is the extinction time (or bankruptcy time). It is well-known (see, e.g., Asmussen and Taksar [1] and Jeanblanc and Shiryaev [12]) that the optimal strategy \( \tilde{D} \) is given by \( \tilde{D}_t = \sup_{0 \leq s \leq t} (Y_s - B)^+ \), where \( (x)^+ := \max\{x, 0\} \) and \( B \) is a constant that can be calculated explicitly.

In the current article, we study the de Finetti problem under the threat of unknown competition. We interpret this unknown competition as the agent, who exerts the control \( D \) to extract from the source \( Y \), being subject to possible fraud. More precisely, we include the possibility that a fraudster exists, with the capacity to extract all the remaining resources at once at a
random time $\gamma$. We use a Bernoulli random variable $\theta$ to model whether the fraudster exists ($\theta = 1$) or not ($\theta = 0$) and we consider maximisation of
\[
E \left[ \int_0^{\tau_0 \wedge \hat{\gamma}} e^{-rt} dD_t \right]
\]
over controls $D$ as above and where $\hat{\gamma} := \gamma 1_{\{\theta=1\}} + \infty 1_{\{\theta=0\}}$. At the same time, the fraudster seeks to choose $\gamma$ to optimise the expected payoff
\[
E \left[ e^{-r(\tau_0 \wedge \gamma)} X^{D}_{\tau_0 \wedge \gamma} \right],
\]
where $X^D = Y - D$ represents the remaining resources after extraction.

The above game is a controller-and-stopper non-zero-sum stochastic game and we extend the stream of literature on stochastic games of control and stopping: Karatzas and Sudderth [13] studied three stochastic games of classical control and stopping for a linear diffusion. Karatzas and Zamfirescu [14] developed a martingale approach for studying zero-sum stochastic games combining classical controls and stopping in a non-Markovian framework. Bayraktar and Huang [3] studied multidimensional controller-and-stopper zero-sum stochastic games in finite horizon. Kwon and Zhang [15] investigated a stochastic game combining singular control and stopping. Hernandez-Hernandez et al. [10] studied a zero-sum game between a singular stochastic controller and a discretionary stopper. Bovo et al. [4] applied PDE methods to study variational inequalities on unbounded domains for zero-sum games between a singular stochastic controller and a discretionary stopper. De Angelis and Ferrari [7] established a connection between a class of two-player non-zero-sum games of optimal stopping and certain two-player non-zero-sum games of singular control.

In contrast to most of the literature on stochastic games of control and stopping, which studies zero-sum games, we formulate and solve a non-zero-sum game. Moreover, a relevant feature that distinguishes our game from the works mentioned above is incomplete information. In our framework, incomplete information stems from the fact that the existence of the fraudster is uncertain. Since the fraudster is equipped with a binary stopping control, inference about the existence of the fraudster is based on observations of the events $\{\hat{\gamma} \leq t\}$. In fact, the strategies that we consider are based on observations/calculations of the two-dimensional process $(X, \Pi)$, where $X = X^D = Y - D$ is observed and represents the value of resources after extraction, and $\Pi$ it calculated, corresponding to the adjusted belief of active competition, i.e., the conditional probability that $\theta = 1$ given that stopping has not yet occurred, see Section 3.2.

Remarkably, this controller-and-stopper non-zero-sum game with incomplete information has an equilibrium which can be described explicitly. The equilibrium is derived using the Ansatz that the equilibrium value for the controller is $(1 - p)V(x)$, where $p$ is the initial probability of active competition, and $V$ is the value in the single-player de Finetti problem. In this equilibrium the controller extracts resources and the fraudster stops at a randomised stopping time, specified in terms of a generalised intensity, in such a way that the corresponding two-dimensional process $(X, \Pi)$ reflects obliquely at a given monotone boundary $x = b(p)$. To construct this two-dimensional reflected process, including a carefully specified reflection direction, we use the notion of perturbed Brownian motion (see, e.g., Carmona et al. [5] and Perman and Werner [16]).

Our paper is the third in a series of papers investigating the role of uncertain competition in stochastic games. This strand of research was initiated by De Angelis and Ekström [6] in which the term “ghost” was also introduced to represent the players that may not exist. In [6] an optimal stopping game in which both players are uncertain of the existence of the opponent was studied. Next, Ekström et al. [8] proposed and studied a ghost game in a setting related to fraud detection and so called “salami slicing” fraudulence. As in the current paper, a controller-and-stopper non-zero-sum game of ghost type is studied in [8], but with the “ghost” role inverted. More precisely, in [8] the controller is a ghost whereas in the current paper the stopper is a
ghost. In [6], the ghost has also a stopping control and a similar Ansatz as above was shown to hold, namely, an equilibrium with equilibrium value $(1 − p)V$ is obtained, where again $p$ is the probability of competition and $V$ is the value in the corresponding single-player game. Similar observations can be made also in non-dynamic auction games with unknown competition, see Hirschleifer and Riley [11, pages 386-389]. On the other hand, in the setting of [8] with a ghost controller, such an Ansatz was not used, but instead an equilibrium was obtained using variational methods. In view of this, a rule-of-thumb seems to be that the equilibrium value in the case of a ghost game where the ghost is equipped with a stopping control is given by $(1 − p)V$, where $V$ is the value in the corresponding single-player game. A precise formulation and verification of such a claim remains to be found.

The paper is organized as follows. In Section 2 we provide the precise game formulation of the de Finetti problem under unknown competition. In Section 3 we review the standard single-player de Finetti problem and we provide properties of its game version that should hold in equilibrium using heuristic arguments. Section 4 uses the notion of perturbed Brownian motion to construct the candidate equilibrium. Our main result Theorem 11, in which the candidate equilibrium is verified, is presented in Section 5. Finally, Section 6 illustrates our findings with a numerical study.

2 Problem set-up

We begin by setting the mathematical stage necessary for our analysis. Throughout the paper, we let $(Ω, ℱ, ℙ)$ be a complete probability space on which a standard Brownian motion $W$, a Bernoulli random variable $θ$ with $ℙ(θ = 1) = 1 − ℙ(θ = 0) = p ∈ [0, 1]$ and a Uniform-(0, 1) random variable $U$ are defined. Moreover, $W$, $θ$ and $U$ are assumed to be independent.

We consider a stochastic game between Player 1 and Player 2 in which both players seek to maximise certain quantities to be specified below. Let $Y$ be a Brownian motion with drift given by

$$Y_t = x + μt + σW_t,$$

where the initial condition satisfies $x ≥ 0$ and $μ$ and $σ$ are given positive constants. Denote by $ℙ^W = (ℱ^W_t)_{0 ≤ t < ∞}$ the augmentation of the filtration generated by the Brownian motion $W$; this filtration will represent the information that Player 1 (the “controller”) is equipped with.

**Definition 1** (Admissible controls for Player 1). An admissible control for Player 1 is a non-decreasing, right-continuous, $ℱ^W$-adapted processes $D = (D_t)_{t ≥ 0}$ satisfying $D_0 = 0$ and $D_t ≤ Y_t$ for every $t ∈ [0, ∞)$. We denote by $𝒜_1$ the set of admissible controls for Player 1.

For any strategy $D ∈ 𝒜_1$, let $X = X^D := Y − D$ and define

$$τ^X_0 := \inf\{t ≥ 0 : X_t ≤ 0\}. \quad (1)$$

To simplify the notation, we will often omit the superscript and simply write $X$ instead of $X^D$ and $τ_0$ instead of $τ^X_0$.

In order to let Player 2 (the “fraudster”) hide his existence, he will be equipped with randomized stopping times. To define the strategies of Player 2, we denote by $𝒫$ the Skorokhod space of cadlag paths on $[0, ∞)$.

**Definition 2** (Admissible controls for Player 2). An admissible control $Γ = (Γ_t(X))_{t ≥ 0}$ for Player 2 is a mapping $(t, X) → Γ_t(X)$ from $[0−, ∞) × 𝒫$ into $[0, 1]$ which is progressively measurable for the canonical filtration on $𝒫$, non-decreasing and right-continuous in $t$, and satisfying $Γ_0−(X) = 0$. We denote by $𝒜_2$ the set of admissible controls for Player 2.

Given a pair of admissible strategies $(D, Γ) ∈ 𝒜_1 × 𝒜_2$, we define a randomized stopping time $γ$ as

$$γ := γ^Γ := \inf\{t ≥ 0 : Γ_t(X^D) > U\}. \quad (2)$$
where we recall that $U$ is a random variable which is Unif$(0,1)$-distributed and independent of $\theta$ and $W$. In accordance with the notation for $X = X^D$, we will often omit the superscript and simply write $\gamma$ instead of $\gamma^\Gamma$.

**Remark 3.** We note that Player 2 selects a universal map $\Gamma$ that he will apply to any given path of $X = Y - D$ to generate his randomized stopping time $\gamma = \gamma^\Gamma$ in (2). In this way, Player 2 is equipped with feed-back controls, and we will obtain a Markovian game structure.

Given a fixed discount rate $r > 0$ and a pair $(D, \Gamma) \in A_1 \times A_2$, we define the payoffs for Player 1 and Player 2 as

$$J_1(x, p, D, \Gamma) := E \left[ \int_{\tau_0}^{\tau_0 \wedge \hat{\gamma}} e^{-rt} dD_t \right]$$

and

$$J_2(x, p, D, \Gamma) := E \left[ e^{-rt(\tau_0 \wedge \gamma)} X_{\tau_0 \wedge \gamma} \right],$$

respectively, where $\tau_0 = \tau^X_0$ and $\gamma = \gamma^\Gamma$ are defined as in (1)-(2), and

$$\hat{\gamma} := \begin{cases} \gamma & \text{if } \theta = 1 \\ \infty & \text{if } \theta = 0. \end{cases}$$

The integral in (3) is interpreted in the Lebesgue-Stieltjes sense, with

$$\int_0^{\tau_0 \wedge \hat{\gamma}} e^{-rt} dD_t := \int_{[0, \tau_0 \wedge \hat{\gamma}]} e^{-rt} dD_t.$$

The inclusion of the lower limit 0 of integration thus accounts for the contribution to Player 1 from an initial push $dD_0 > 0$.

Each player seeks to maximise their respective profit, and we are looking for a Nash equilibrium to this non-zero-sum game in the sense of the following definition.

**Definition 4.** A pair $(D^*, \Gamma^*) \in A_1 \times A_2$ is a Nash equilibrium (NE) if

$$J_1(x, p, D^*, \Gamma^*) \geq J_1(x, p, D, \Gamma^*)$$

and

$$J_2(x, p, D^*, \Gamma^*) \geq J_2(x, p, D^*, \Gamma)$$

for any pair $(D, \Gamma) \in A_1 \times A_2$.

**Remark 5.** Note that it is a consequence of the game set-up that Player 1 has precedence over Player 2 in the sense that if a lump sum $dD_t > 0$ is paid out at the same time $t = \hat{\gamma}$ as Player 2 stops, then Player 1 receives the lump sum, whereas Player 2 receives the reduced amount $Y_t - D_t$. Consequently, since Player 1 may choose a strategy with $D_0 = x$, for any Nash equilibrium $(D^*, \Gamma^*) \in A_1 \times A_2$ we must have

$$J_1(x, p, D^*, \Gamma^*) \geq \sup_{D \in A_1} J_1(x, p, D, \Gamma^*) \geq x.$$

**Proposition 6.** For a given pair $(D, \Gamma) \in A_1 \times A_2$, we have

$$J_1(x, p, D, \Gamma) = E \left[ \int_0^{\tau_0} e^{-rt(1 - p\Gamma_t \wedge \gamma)} dD_t \right],$$

where $\Gamma_t := \Gamma_t(Y - D)$.
Proof. By conditioning we have

\[
J_1(x, p, D, \Gamma) = \mathbb{E} \left[ \int_0^{\tau_0 \wedge \hat{\gamma}} e^{-rt} dD_t \right] = \mathbb{E} \left[ \int_0^{\tau_0} e^{-rt} 1_{\{t \leq \hat{\gamma}\}} dD_t \right] \\
= \mathbb{E} \left[ \int_0^{\tau_0} e^{-rt} \mathbb{P}(t \leq \hat{\gamma}|F_t^W) dD_t \right].
\]

Since

\[
\{\Gamma_t^- < U\} \subseteq \{t \leq \gamma\} \subseteq \{\Gamma_t^- \leq U\},
\]

we have that

\[
\mathbb{P}(t \leq \hat{\gamma}|F_t^W) = 1 - p + p\mathbb{P}(t \leq \gamma|F_t^W) = 1 - p\Gamma_t^-,
\]

so

\[
J_1(x, p, D, \Gamma) = \mathbb{E} \left[ \int_0^{\tau_0} e^{-rt}(1 - p\Gamma_t^-) dD_t \right].
\]

Remark 7. Notice that for Player 2 we have chosen to maximise his expected payoff when he is active, i.e., when \( \theta = 1 \). Alternatively, one could set Player 2 to maximise

\[
\hat{J}_2(x, p, D, \Gamma) := \mathbb{E} \left[ \theta e^{-r(\tau_0 \wedge \hat{\gamma})} X_{\tau_0 \wedge \hat{\gamma}} \right].
\]

The formulations for \( J_2 \) and \( \hat{J}_2 \) have the following interpretations. Imagine that before the game starts, at time \( t = 0^- \), neither player knows \( \theta \) and that the value of \( \theta \) will be revealed to Player 2 at time \( t = 0^- \). Then, \( J_2 \) is the expected payoff for Player 2 at time \( t = 0^- \), whereas \( \hat{J}_2 \) is the expected payoff at time \( t = 0 \) when \( \theta = 1 \). These games are referred to as the ex-ante version of the game and the interim version of the game, respectively (see [2, 9] for classical theory of games under incomplete information). Also notice that the two formulations are equivalent as by independence one obtains \( \hat{J}_2(x, p, D, \Gamma) = pJ_2(x, p, D, \Gamma) \) and so the second inequality in Definition 4 can be equivalently replaced by \( \hat{J}_2(x, p, D^*, \Gamma^*) \geq \hat{J}_2(x, p, D^*, \Gamma) \) for \( p > 0 \).

3 Background material and heuristics

3.1 The single-player de Finetti problem

Note that if \( p = 0 \), then Player 1 acts under no competition and thus faces the standard de Finetti problem for which the value function

\[
V(x) := \sup_{D \in A_1} \mathbb{E} \left[ \int_0^{\tau_0} e^{-rt} dD_t \right]
\]

and the optimal strategy \( \tilde{D} \) are well known (see, e.g., [12]). To describe this solution in more detail, let \( \psi \) be the unique increasing solution of

\[
\mathcal{L}\psi(x) = 0, \quad x \geq 0,
\]

with \( \psi(0) = 0 \) and \( \psi'(0) = 1 \), where \( \mathcal{L} \) denotes the differential operator

\[
\mathcal{L} := \frac{\sigma^2}{2} \partial_x^2 + \mu \partial_x - r.
\]

More explicitly,

\[
\psi(x) = \frac{e^{\zeta_2 x} - e^{\zeta_1 x}}{\zeta_2 - \zeta_1},
\]

5
where $\zeta_i, i = 1, 2$ are the solutions of the quadratic equation

$$\zeta^2 + \frac{2\mu}{\sigma^2} \zeta - \frac{2r}{\sigma^2} = 0$$

with $\zeta_1 < 0 < \zeta_2$. Setting

$$B := \frac{\ln(\zeta_1^2) - \ln(\zeta_2^2)}{\zeta_2 - \zeta_1},$$

we have that $\psi$ is concave on $[0, B]$ and convex on $(B, \infty)$, and

$$V(x) = \begin{cases} \frac{\psi(x)}{\psi(B)}, & x \leq B, \\ x - B + V(B), & x > B. \end{cases}$$

Moreover,

$$\tilde{D}_t = \sup_{s \in [0,t]} (Y_s - B)^+$$

is an optimal strategy in (5), i.e.,

$$V(x) = \mathbb{E} \left[ \int_0^{\tilde{\tau}_0} e^{-rt} d\tilde{D}_t \right],$$

where $\tilde{X} := X^{\tilde{D}}$ and $\tilde{\tau}_0 := \tau_0^{\tilde{X}}$. We also remark that $(\tilde{X}, \tilde{D})$ is the solution of a Skorokhod reflection problem with reflection at the barrier $B$.

### 3.2 Adjusted beliefs

We now return to our version of the game including a ghost feature as described in Section 2. At the beginning of the game, from the perspective of Player 1 there is active competition (i.e., $\theta = 1$) with probability $p$. As time passes, and if no stopping occurs, Player 1’s conditional probability of competition $\Pi$ will decrease. More precisely, at time $t \geq 0$, assuming that the strategy pair $(D, \Gamma) \in A_1 \times A_2$ is played, we have

$$\Pi_t = \Pi_t^\Gamma := \mathbb{P}(\theta = 1|F_t^W, \hat{\gamma} > t) = \frac{\mathbb{P}(\theta = 1, \hat{\gamma} > t|F_t^W)}{\mathbb{P}(\hat{\gamma} > t|F_t^W)} = \frac{p \mathbb{P}(\gamma > t|F_t^W)}{(1 - p) + p \mathbb{P}(\gamma > t|F_t^W)} = \frac{p(1 - \Gamma_t(X^D))}{1 - p \Gamma_t(X^D)}$$

since $\mathbb{P}(\gamma > t|F_t^W) = 1 - \mathbb{P}(U \leq \Gamma_t|F_t^W) = 1 - \Gamma_t$ for $\Gamma = \Gamma(X^D)$. Moreover, since the initial probability of the event $\{\theta = 1\}$ is $p$, we also have $\Pi_{0-} := p$. Also note that solving for $\Gamma_t$ in the equation above gives

$$\Gamma_t = \Gamma_t^\Pi = \frac{p - \Pi_t}{p(1 - \Pi_t)},$$

so there is a bijection between $\Pi$ and $\Gamma$.

### 3.3 Heuristics

Since

$$J_1(x, p, D, \Gamma) = \mathbb{E} \left[ \int_0^{\tilde{\tau}_0} e^{-rt(1 - p\Gamma_t - )dD_t} \right] \leq \mathbb{E} \left[ \int_0^{\tilde{\tau}_0} e^{-rt} d\tilde{D}_t \right] \leq V(x)$$

for any strategy pair $(D, \Gamma) \in A_1 \times A_2$, it is clear that the risk of competition decreases the value from the perspective of Player 1. On the other hand, to obtain a lower bound, let $\tilde{D}$ denote the optimal control of the single-player de Finetti problem, see (10). Then,

$$J_1(x, p, \tilde{D}, \Gamma) = \mathbb{E} \left[ \int_0^{\tilde{\tau}_0} e^{-rt(1 - p\Gamma_t - )d\tilde{D}_t} \right] \geq (1 - p) \mathbb{E} \left[ \int_0^{\tilde{\tau}_0} e^{-rt} d\tilde{D}_t \right] = (1 - p)V(x)$$
for any $\Gamma \in \mathcal{A}_2$. It is thus clear that
\[(1 - p)V(x) \leq J_1(x, p, D^*, \Gamma^*) \leq V(x)\] if $(D^*, \Gamma^*) \in \mathcal{A}_1 \times \mathcal{A}_2$ is a Nash equilibrium.

In this section we will provide heuristic arguments to obtain a candidate Nash equilibrium $(D^*, \Gamma^*) \in \mathcal{A}_1 \times \mathcal{A}_2$. To do that, we make the Ansatz that

(a) there exists a non-increasing continuous boundary $p = c(x)$ such that the overall effect of the equilibrium strategy $(D^*, \Gamma^*) \in \mathcal{A}_1 \times \mathcal{A}_2$ amounts to reflection of the two-dimensional process $(X^*, \Pi^*) = (Y - D^*, \Pi^D)$ along this boundary;

(b) the corresponding equilibrium value $v$ of Player 1 satisfies
\[v(x, p) = (1 - p)V(x), \quad \text{for } p \leq c(x).\] (14)

Note that by the bijection between $\Gamma$ and $\Pi$ we have that $\Pi^* = \Pi^*(X^D)$ for every $D \in \mathcal{A}_1$ and to obtain the reflection of $(X^*, \Pi^*)$ along the monotone boundary $c$ we need that
\[\Pi^* = \Pi^*(X^D) = p \land c\left(\sup_{0 \leq s \leq t} X^D_s\right), \quad \text{for } t \geq 0.\] (15)

With a slight abuse of notation, $\Pi^*$ will be used to indicate both $\Pi^*(X^D)$ and $\Pi^*(X^*)$ but this will be clear from the context as it will depend on whether Player 1 plays an arbitrary admissible strategy $D \in \mathcal{A}_1$ or the equilibrium strategy $D^*$.

Notice also that the Ansatz (14) coincides with the lower bound in (13), and is thus of the same type as the equilibrium obtained in the ghost Dynkin game studied in [6].

Given this Ansatz, we further need to determine

(i) the boundary $c$;

(ii) the direction of reflection when the process $(X^*, \Pi^*)$ is at the boundary;

(iii) the strategy pair $(D^*, \Gamma^*)$ corresponding to the reflected process $(X^*, \Pi^*)$.

(iv) the strategy for starting points $(x, p)$ with $p > c(x)$;

We do this below, and then the candidate Nash equilibrium that we produce is verified in Section 5. Notice that we will not discuss item (iv) here as it is not relevant at this stage, but it will be considered in Theorem 11.

First, let us consider a starting point $(x, p) \in [0, \infty) \times (0, 1)$ with $p \leq c(x)$, and recall that we expect in equilibrium that
\[(X^*_t, \Pi^*_t) = \left(Y_t - D^*_t, p \land c\left(\sup_{0 \leq s \leq t} (Y_s - D^*_s)\right)\right),\] for $D^* \in \mathcal{A}_1$ to be specified. Since $c$ is assumed to be continuous and non-increasing, we see that
\[p \land c\left(\sup_{0 \leq s \leq t} (Y_s - D_s)\right) \leq c(Y_t - D_t)\] for any choice $D \in \mathcal{A}_1$. By construction, $\Pi^*$ is continuous and we have
\[\Gamma^*_t = \frac{p - \Pi^*_t}{p(1 - \Pi^*_t)}\]
and
\[d\Pi^*_t = -\frac{1}{1 - \Gamma^*_t} \Pi^*_t (1 - \Pi^*_t) d\Gamma^*_t\] (17)
on \{t \geq 0: \Gamma_t^* < 1\}, cf. (11) and (12).

Note that by the dynamic programming principle one would expect that the process \(M = M^D\) given by
\[
M_t := \int_0^{t \wedge \gamma^*} e^{-rs} dD_s + e^{-rt} v(X_t, \Pi_t^*) \mathbb{1}_{\{t < \gamma^*\}}
\]
is an \(\mathbb{F}^{W, \gamma^*}\)-martingale if \(D = D^* \in \mathcal{A}_1\) is an optimal response to \(\Gamma^* \in \mathcal{A}_2\), and an \(\mathbb{F}^{W, \gamma^*}\)-supermartingale if \(D \in \mathcal{A}_1\) is any admissible response. Here, \(\mathbb{F}^{W, \gamma^*} = (\mathbb{F}^{W, \gamma^*}_{\{t \leq \gamma^*\}})_{0 \leq t < \infty}\) is the smallest right-continuous filtration to which \(W\) and \(\mathbb{1}_{\{t \geq \gamma^*\}}\) are adapted, augmented with the \(\mathbb{P}\)-null sets of \(\Omega\). Moreover, by conditioning (cf. Proposition 6), \(M\) is an \(\mathbb{F}^{W, \gamma^*}\)-(super)martingale if and only if
\[
\dot{M}_t := \int_0^t e^{-rs} \left(1 - p \Gamma_s^* \right) dD_s + e^{-rt} (1 - p \Gamma_t^*) v(X_t, \Pi_t^*)
\]
is an \(\mathbb{F}^W\)-(super)martingale.

Thus, by an application of Ito’s formula, we see that when Player 2 plays the equilibrium strategy \(\Gamma^*\) and \((X^*, \Pi^*)\) is at the boundary we need that
\[
(1 - v_x) dD_t^* - \frac{\Pi_t^*}{1 - \Gamma_t^*} ((1 - \Pi_t^*) v_p + v) d\Gamma_t^* = 0 \quad \text{(optimality)};
\]
whereas, when Player 2 plays the equilibrium strategy \(\Gamma^*\) and Player 1 plays any admissible strategy \(D \in \mathcal{A}_1\), we need that
\[
(1 - v_x) dD_t - \frac{\Pi_t^*}{1 - \Gamma_t^*} ((1 - \Pi_t^*) v_p + v) d\Gamma_t^* \leq 0 \quad \text{(suboptimality)},
\]
We stress that \(\Pi^*\) here stands for \(\Pi^*(X^*)\) in the optimality condition and \(\Pi^*(X^D)\) in the suboptimality condition. Note that we obtain from (14) and (16) that
\[
(1 - p) v_p(x, p) + v(x, p) = 0
\]
when \(p \leq c(x)\). Thus, to satisfy the optimality condition we need to have \(v_x(x, p) = 1\) at the boundary, and consequently the boundary \(p = c(x)\) should be defined by
\[
(1 - c(x)) V'(x) = 1
\]
for \(x \in [0, B]\) where \(B\) is as specified in (8). Hence, for \(x \in [0, B]\) we have
\[
c(x) = \frac{V'(x) - 1}{V'(x)}; \quad (18)
\]
from which it follows immediately that \(c(B) = 0, c'(x) < 0, \text{ and } c'(x) \to 0\) as \(x \not\to B\) by (9). Let \(\hat{p} := (V'(0) - 1)/V'(0)\). Then \(c: [0, B] \to [0, \hat{p}]\) is a continuous strictly decreasing bijection and we denote its inverse by \(b: [0, \hat{p}] \to [0, B]\). From here on, we will refer to \(b\) (instead of \(c\)) as the boundary when it is more convenient to do so. By convention, we also extend \(b\) and \(c\) by continuity and define \(b(p) = 0\) for every \(p \in (\hat{p}, 1]\), and \(c(x) = 0\) for \(x \in (B, \infty)\).

Moreover, notice that since \(\Pi^* \leq c(X^D)\), for every admissible strategy \(D \in \mathcal{A}_1\), we also have that
\[
v_x(X^D, \Pi_t^*) = (1 - \Pi_t^*) V'(X^D_t) \geq (1 - c(X^D_t)) V'(X^D_t) = 1,
\]
so that the suboptimality condition is verified as well.

Since Player 2 in equilibrium only stops at time points when \((X^*, \Pi^*)\) is at the boundary, we expect his equilibrium value \(u\) to be of the form \(u(x, p) = g(p) \psi(x)\), for some function \(g\), and to satisfy the condition \(u(b(p), p) = b(p)\). Consequently,
\[
u(x, p) = b(p) \frac{\psi(x)}{\psi(b(p))} \quad (19)
\]
for \( x \leq b(p) \). Furthermore, by the indifference principle for equilibria in randomised strategies, the process 
\[
N_t = e^{-rt}u(X_t^*, \Pi_t^*)
\]
should be a martingale when Player 1 plays the equilibrium strategy \( D^* \). After applying Ito’s formula this yields 
\[
-u_x dD_t^* + u_p d\Pi_t^* = 0 \tag{20}
\]
on the boundary, so the reflection direction of \( (X^*, \Pi^*) \) needs to be \( (u_p, -u_x) \).

We now show how to construct the candidate Nash equilibrium \( (D^*, \Gamma^*) \) so that the corresponding process \( (X^*, \Pi^*) \) reflects along the boundary \( c \) in the direction \( (u_p, -u_x) \). To do that, we first specify \( \Gamma^* \) by setting
\[
\Gamma_t^*(X_{D^*}) = p - \Pi_t^*(X_{D^*}) \tag{12},
\]
for \( t \geq 0 \), where \( \Pi_t^* = \Pi_t^*(X_{D^*}) = p \wedge c(\sup_{0 \leq s \leq t} (X_s^D)) \) for an arbitrary strategy \( D \in \mathcal{A}_1 \). The process \( (X^D, \Pi^*) \) then reflects at the boundary but the direction of reflection is, for an arbitrary strategy \( D \in \mathcal{A}_1 \), not necessarily equal to \( (u_p, -u_x) \).

One should only push in \( X = Y - D \) when the process is at its current maximum (after the first time it hits the boundary). Therefore, one would expect to choose \( D^* \) so as to satisfy
\[
dD_t^* = \lambda(X_t^*) dX_t^*,
\]
where \( X_t^* := b(p) \vee \sup_{0 \leq s \leq t} X_s^* \) and \( X^* = Y - D^* \), for some function \( \lambda \) to be determined. Moreover, from (15) we have that, when Player 1 plays the equilibrium strategy \( D^*, \Pi_t^* = c(X_t^*) \), so (20) gives
\[
\lambda(x) = \frac{c'(x)u_p(x, c(x))}{u_x(x, c(x))}. \tag{21}
\]
Using (19), we then get
\[
u_x(x, c(x)) = \frac{\psi'(x)}{\psi(x)} x
\]
and
\[
u_p(x, c(x)) = \frac{\psi(x) - x\psi'(x)}{\psi(x)c'(x)},
\]
so
\[
\lambda(x) = \frac{\psi(x) - x\psi'(x)}{x\psi'(x)}. \tag{22}
\]
and since \( \psi \) is concave on \([0, B]\), we have \( \lambda \geq 0 \) on \((0, B]\).

In the next section we study in detail the solvability of the equation
\[
X_t^* = Y_t - \int_0^t \lambda(X_s^*) dX_s^*
\]
using the notion of *perturbed Brownian motion*, which will allow us to obtain the equilibrium strategy \( D^* \) for Player 1.

### 4 A perturbed Brownian motion with drift

To construct the equilibrium strategy \( D^* \) for Player 1 we will use the notion of perturbed Brownian motion. Here we provide what is needed for the study of our problem, and refer to [5], [16] and the references therein for further details on such processes. First, define \( \Lambda : [b(p), B] \to [0, \infty) \) by
\[
\Lambda(x) := \int_{b(p)}^x \lambda(y) dy, \tag{23}
\]
where
\[ \lambda(x) = \frac{\psi(x)}{x \psi'(x)} - 1 \]
as in (22). Since \( \lambda \geq 0 \) on \((0, B]\), we note that \( \Lambda \) is increasing. Note also that \( \lambda(x) \) is a bounded function for \( x \in [0, B] \) so \( \Lambda \) is well-defined. For \( x \leq b(p) \) we now consider the equation
\[ X_t = Y_t - \Lambda(\bar{X}_t), \quad t \in [0, \tau_B], \tag{24} \]
where \( Y_t = x + \mu t + \sigma W_t \), \( \bar{X}_t := b(p) \vee \sup_{0 \leq s \leq t} X_s \), and \( \tau_B = \tau^X_B := \inf\{t \geq 0 : X_t \geq B\} \). The process \( X \) is then a perturbed Brownian motion with drift.

To construct a solution of (24), let
\[ \bar{Y}_t := b(p) \vee \sup_{0 \leq s \leq t} Y_s. \tag{25} \]
Define the function \( f : [b(p), \infty) \to [b(p), B] \) by the relations
\begin{align*}
\Lambda(f(y)) + f(y) &= y, \quad y \in [b(p), \Lambda(B) + B], \tag{26} \\
f(y) &= B, \quad y > \Lambda(B) + B,
\end{align*}
i.e., \( f \) is the inverse of the increasing function \( x \mapsto y := \Lambda(x) + x \) for \( y \in [b(p), \Lambda(B) + B] \) and then extended constantly for \( y > \Lambda(B) + B \). Now define
\[ X_t := Y_t - \bar{Y}_t + f(\bar{Y}_t). \tag{27} \]

**Proposition 8.** Assume that \( x \leq b(p) \). Then the process \( X \) in (27) solves equation (24).

**Proof.** Let \( t \in [0, \tau_B] \). Since \( \bar{X}_t := b(p) \vee \sup_{s \leq t} X_s \) we obtain, from (27), that \( \bar{X}_t = f(\bar{Y}_t) \) as \( f(b(p)) = b(p) \). Consequently \( \tau_B = \inf\{t \geq 0 : Y_t \geq \Lambda(B) + B\} \) and so, by (26), we have \( f(\bar{Y}_t) - \bar{Y}_t = -\Lambda(f(\bar{Y}_t)) \). This leads to
\[ X_t = Y_t - \Lambda(\bar{X}_t), \]
which proves the claim. \( \square \)

**Remark 9.** The set-up in (24) of a perturbed Brownian motion is slightly more general than what is used in most literature on perturbed Brownian motions; in fact, the typical choice of perturbation used in the literature is linear, corresponding to a linear function \( \Lambda \) in (24). On the other hand, we only deal with one-sided perturbation, in which case the solution can be constructed explicitly as in (27) above. It is straightforward to check that the argument for pathwise uniqueness of solutions of (24), cf. [5, Proposition 2.1], carries over to our setting.

**Remark 10.** The function \( f \) defined in (26) is constructed in such a way that the process \( X_t = Y_t - \bar{Y}_t + f(\bar{Y}_t) \) is a perturbed Brownian motion with drift for \( t \in [0, \tau_B] \) (as proved in Proposition 8) and it is the Skorokhod reflection of the process \( Y_t \) at the barrier \( B \) for \( t \in (\tau_B, \infty) \). Indeed, for \( t \in (\tau_B, \infty) \), we have
\begin{align*}
X_t &= Y_t - \bar{Y}_t + f(\bar{Y}_t) = Y_t - \bar{Y}_t + B \\
&= Y_t - \sup_{s \in [0, t]} (Y_s - B) = Y_t - \sup_{s \in [0, t]} (Y_s - B)^+, \tag{28}
\end{align*}
i.e., we have \( X_t = X^D_t \) for \( t \in (\tau_B, \infty) \) where \( \hat{D} \) is defined as in (10).
5 Main result

In this section, we state and prove our main result: an explicit Nash equilibrium for our game. To do that, let us fix \( (x, p) \in [0, \infty) \times [0, 1] \) and recall that \( Y \) is given by

\[
Y_t = x + \mu t + \sigma W_t.
\]

First, define a new process \( Y^\wedge \) by

\[
Y^\wedge_t := x \wedge b(p) + \mu t + \sigma W_t = Y_t - (x - b(p))^+,
\]

so that \( Y^\wedge \) starts below the boundary \( b(p) \). Then define \( Y^\wedge \) as in (25) but with \( Y^\wedge \) instead of \( Y \), i.e.,

\[
\bar{Y}^\wedge_t := b(p) \vee \sup_{0 \leq s \leq t} Y_s^\wedge.
\]

Also, recall the definitions of \( \Lambda: [b(p), B] \to [0, \infty) \) in (23) and \( f: [b(p), \infty) \to [b(p), B] \) in (26), and define \( D^* \in \mathcal{A}_1 \) by \( D^* = 0 \) and

\[
D^*_t := (x - b(p))^+ + Y^\wedge_t - f(\bar{Y}^\wedge_t), \quad t \geq 0.
\]

Setting

\[
X^*_t := Y_t - D^*_t,
\]

Proposition 8 applied with \( Y^\wedge \) in place of \( Y \) yields

\[
X^*_t = Y^\wedge_t - \bar{Y}^\wedge_t + f(\bar{Y}^\wedge_t) = Y^\wedge_t - \Lambda(\bar{X}^*_t), \quad t \in [0, \tau_B^*],
\]

where \( \tau_B^* = \tau_B^X := \inf\{t \geq 0 : X^*_t \geq B\} \). Note that by construction we have \( dD^*_t = \Lambda(X^*_t)d\bar{X}^*_t \) for \( t \in (0, \tau_B^* \)

Moreover, for a given path \( X = X^D \in \mathcal{D} \) (with \( D \in \mathcal{A}_1 \)), define \( Z^* = Z^*(X) \) by \( Z^*_0 := p \) and

\[
Z^*_t := p \wedge c(\sup_{0 \leq s \leq t} X_s), \quad t \geq 0
\]

(cf. (15)), and define \( \Gamma^* \in \mathcal{A}_2 \) by

\[
\Gamma^*_t(X) := \begin{cases} 
1_{(t \geq \tau^\theta)}(p), & p = 0, \\
\frac{p - Z^*_t}{p(1 - Z^*_t)}, & p > 0,
\end{cases}
\]

where we recall that \( \tau^\theta := \inf\{t \geq 0 : X^*_t \geq B\} \).

**Theorem 11.** Let \( (x, p) \in [0, \infty) \times [0, 1] \). The pair \( (D^*, \Gamma^*) \) defined above is a NE for the stochastic game (3)-(4), with equilibrium values

\[
J_1(x, p, D^*, \Gamma^*) = v(x, p) := \begin{cases} 
(1 - p)V(x), & x \leq b(p), \\
(1 - p)V(b(p)) + x - b(p), & x > b(p),
\end{cases}
\]

\[
J_2(x, p, D^*, \Gamma^*) = u(x, p) := \begin{cases} 
b(p)\psi(x)/\psi(b(p)), & x \leq b(p), \\
b(p), & x > b(p),
\end{cases}
\]

(with the understanding that \( b(p)\psi(x)/\psi(b(p)) = 0 \) for \( x = 0 \) also when \( b(p) = 0 \)). Here, \( V \) is the value of the single-player de Finetti problem given in (9), and \( \psi \) is given by (7).

**Proof. Step 1.** We first prove that \( D^* \) is an optimal response to \( \Gamma^* \). Let \( D \in \mathcal{A}_1 \) be an arbitrary strategy for Player 1 and set \( X := Y - D \). Let \( Z^* \) be defined as in (31) and \( \Gamma^*_t := \Gamma^*_t(X) \) as in (32) accordingly.
Namely, the optimization problem for Player 1 degenerates into the single-player de Finetti problem, and $D^*$ coincides with its optimal solution $\bar{D}$, as highlighted in Remark 10. Hence, also $v(x, 0) = J_1(x, 0, D^*, \Gamma^*) \geq J_1(x, 0, D, \Gamma^*)$ for every $D \in A_1$.

If $x = 0$, then $J_1(0, p, D, \Gamma^*) = 0$ for every $p \in [0, 1]$, $D \in A_1$ and so, in particular, $v(0, p) = J_1(0, p, D^*, \Gamma^*)$ for every $p \in [0, 1]$.

Now let $p \in (0, 1]$ and let us first consider $0 < x \leq b(p)$ (note that this implies that $p \in (0, \hat{p})$ as $b(p) = 0$ for every $p \in [\hat{p}, 1]$). By (32), we have

$$Z_t^* = \frac{p(1 - \Gamma_t^*)}{1 - p \Gamma_t^*}, \quad t \geq 0.$$ 

Since $Z^*$ and $\Gamma^*$ are continuous and of finite variation, we obtain

$$dZ_t^* = -\frac{p(1 - Z_t^*)}{1 - p \Gamma_t^*} d\Gamma_t^*, \quad t \geq 0.$$ 

Now define

$$\tilde{v}(x, p) := (1 - p)V(x) \in C^2([0, \infty) \times [0, 1]).$$

By setting $\tau := \tau_0 \wedge T$ with $T \geq 0$ and applying Itô’s formula to $e^{-rt} (1 - p \Gamma_t^*) \tilde{v}(X_t, Z_t^*)$, we have that

$$e^{-rt} (1 - p \Gamma_t^*) \tilde{v}(X_t, Z_t^*) = \tilde{v}(x, p) + \int_0^\tau e^{-rt} (1 - p \Gamma_t^*) \mathcal{L} \tilde{v}(X_{t-}, Z_t^*) \, dt$$

$$- \int_0^\tau e^{-rt} (1 - p \Gamma_t^*) \tilde{v}_x(X_{t-}, Z_t^*) \, dD_t^c$$

$$+ \int_0^\tau \sigma e^{-rt} (1 - p \Gamma_t^*) \tilde{v}_x(X_{t-}, Z_t^*) \, dW_t$$

$$- \int_0^\tau e^{-rt} p [(1 - Z_t^*) \tilde{v}_p(X_{t-}, Z_t^*) + \tilde{v}(X_{t-}, Z_t^*)] \, d\Gamma_t^*$$

$$+ \sum_{0 \leq t \leq \tau} e^{-rt} (1 - p \Gamma_t^*) \left( \tilde{v}(X_t, Z_t^*) - \tilde{v}(X_{t-}, Z_t^*) \right),$$

where $\mathcal{L}$ is defined as in (6) and $D^c$ denotes the continuous part of $D$. Notice that $\tilde{v}(x, p) = v(x, p)$ for $x \leq b(p)$ and that by definition of $\tilde{v}$, we have for every $t > 0$

$$\mathcal{L} \tilde{v}(X_{t-}, Z_t^*) = 0 \quad \text{and} \quad (1 - Z_t^*) \tilde{v}_p(X_{t-}, Z_t^*) + \tilde{v}(X_{t-}, Z_t^*) = 0.$$ 

Hence, equation (33) becomes

$$v(x, p) = e^{-rt} (1 - p \Gamma_t^*) \tilde{v}(X_t, Z_t^*) + \int_0^\tau e^{-rt} (1 - p \Gamma_t^*) \tilde{v}_x(X_{t-}, Z_t^*) \, dD_t^c$$

$$- \int_0^\tau \sigma e^{-rt} (1 - p \Gamma_t^*) \tilde{v}_x(X_{t-}, Z_t^*) \, dW_t$$

$$- \sum_{0 \leq t \leq \tau} e^{-rt} (1 - p \Gamma_t^*) \left( \tilde{v}(X_t, Z_t^*) - \tilde{v}(X_{t-}, Z_t^*) \right).$$

For the summation term we have by the mean value theorem that

$$\sum_{0 \leq t \leq \tau} e^{-rt} (1 - p \Gamma_t^*) \left( \tilde{v}(X_t, Z_t^*) - \tilde{v}(X_{t-}, Z_t^*) \right) = - \sum_{0 \leq t \leq \tau} e^{-rt} (1 - p \Gamma_t^*) \tilde{v}_x(\xi_t, Z_t^*) \Delta D_t$$

(35)
where \( \xi_t \in (X_{t-}, X_t) \) and \( \Delta D_t := D_t - D_{t-} \). By plugging (35) into (34), and using that \( \tilde{v} \geq 0 \) and \( \tilde{v}_x \geq 1 \), we obtain

\[
v(x, p) \geq \int_0^T e^{-rt} (1 - p \Gamma^*_t) \, dD_t - \int_0^T \sigma e^{-rt} (1 - p \Gamma^*_t) \tilde{v}_x(X_{t-}, Z_t^*) \, dW_t. \tag{36}
\]

Let

\[ O := \{(x, p) \in [0, \infty) \times [0, 1] : x \leq b(p)\} \cup ((B, \infty) \times \{0\}) \tag{37} \]

and note that \( (X_{t-}, Z_t^*) \in O \) for every \( t \geq 0 \) (by construction of \( Z_t \)) and that \( \tilde{v}_x \) is bounded on \( O \) \( (\tilde{v}_x(x, p) = 1 \text{ for } (x, p) \in (B, \infty) \times \{0\}) \). Thus, the stochastic integral above is a martingale and by an application of the optional sampling theorem we have that

\[
\tilde{v}(x, p) \geq \mathbb{E} \left[ \int_0^{T_0 \wedge T} e^{-rt} (1 - p \Gamma^*_t) \, dD_t \right].
\]

Letting \( T \to \infty \) yields, by the monotone convergence theorem,

\[
v(x, p) \geq \mathbb{E} \left[ \int_0^{T_0} e^{-rt} (1 - p \Gamma^*_t) \, dD_t \right] = \mathbb{E} \left[ \int_0^{T_0} e^{-rt} (1 - p \Gamma^*_t) \, dD_t \right] = J_1(x, p, D, \Gamma^*) \]

for every \( D \in D_c \), with the last equality follows by Proposition 6.

Now notice that \( D^*_t \) defined in (29) is continuous for every \( t \geq 0 \), when \( x \leq b(p) \), and that the same holds for \( X^*_t := X_{D^*_t} \). Let \( \tau^*_0 := X^*_0 \), then equation (34) for \( D = D^* \) and \( \tau^* := \tau^*_0 \wedge T \) becomes

\[
v(x, p) = e^{-r \tau^*} (1 - p \Gamma^*_0) \tilde{v}(X^*_0, Z_{\tau^*}) + \int_0^{\tau^*} e^{-r \tau} (1 - p \Gamma^*_\tau) \tilde{v}_x(X^*_\tau, Z^*_\tau) \, dD^*_\tau \]

\[- \int_0^{\tau^*} \sigma e^{-r \tau} (1 - p \Gamma^*_\tau) \tilde{v}_x(X^*_\tau, Z^*_\tau) \, dW_\tau]

\[= e^{-r \tau^*} (1 - p \Gamma^*_0) \tilde{v}(X^*_0, Z_{\tau^*}) + \int_0^{\tau^*} e^{-r \tau} (1 - p \Gamma^*_\tau) \, dD^*_\tau \]

\[- \int_0^{\tau^*} \sigma e^{-r \tau} (1 - p \Gamma^*_\tau) \tilde{v}_x(X^*_\tau, Z^*_\tau) \, dW_\tau,
\]

where the last equality holds since \( \tilde{v}_x(x, p) = 1 \) if \( x \geq b(p) \) and \( dD^*_t = 0 \) if \( X^*_t < b(Z^*_t) \). Hence, again by taking expected values, we obtain

\[
v(x, p) = \mathbb{E} \left[ e^{-r(\tau^*_0 \wedge T)} (1 - p \Gamma^*_0) \tilde{v}(X^*_0, Z_{\tau^*_0 \wedge T}) + \int_0^{\tau^*_0 \wedge T} e^{-r \tau} (1 - p \Gamma^*_\tau) \, dD^*_\tau \right]
\]

\[ \rightarrow \mathbb{E} \left[ e^{-rT} (1 - p \Gamma^*_T) \right] \]

as \( T \to \infty \) by dominated convergence (the first term tends to 0 since \( \tilde{v}(X^*_0, Z_{\tau^*_0}) = 0 \)). Thus, we have proved that

\[ J_1(x, p, D^*, \Gamma^*) = v(x, p) \geq \sup_{D \in A_1} J_1(x, p, D, \Gamma^*), \quad \forall (x, p) \in O. \]

Let us now consider \( (x, p) \in ([0, \infty) \times [0, 1]) \setminus O =: O^c \), i.e., \( x > b(p) \) with \( p \neq 0 \). Then,

\[ v(x, p) = v(b(p), p) + x - b(p) = J_1(b(p), p, D^*, \Gamma^*) + x - b(p) = J_1(x, p, D^*, \Gamma^*). \]

Thus, we are left to prove that also in this case

\[ J_1(x, p, D^*, \Gamma^*) \geq J_1(x, p, D, \Gamma^*), \quad \forall D \in A_1. \]
For $(x, p) \in \mathcal{O}$, let the admissible strategy $D \in \mathcal{A}_1$ have an initial jump $\Delta D_0 = x - y$ with either $b(p) \leq y \leq x$ or $0 \leq y < b(p)$. In the former case, by definition (32) of $\Gamma^*$, we have that

$$J_1(x, p, D, \Gamma^*) = (1 - \Gamma_0^*)J_1(b(q), q, D, \Gamma^*) + x - y = \frac{q(1-p)}{p} V(b(q)) + x - y,$$

where $q := c(y) \leq p$ (and hence $y = b(q)$). Since $V$ is concave with $V'(b(p)) = 1/(1-p)$, then

$$J_1(x, p, D, \Gamma^*) \leq \frac{q(1-p)}{p} \left( V(b(p)) + \frac{y - b(p)}{1-p} \right) + x - y$$

$$= \frac{q}{p} \left( (1-p)V(b(p)) + y - b(p) \right) + x - y$$

$$\leq (1-p)V(b(p)) + x - b(p) = J_1(x, p, D^*, \Gamma^*).$$

If instead $0 \leq y < b(p)$, then by a similar argument

$$J_1(x, p, D, \Gamma^*) = J_1(y, p, D, \Gamma^*) + x - y = (1-p)V(y) + x - y$$

$$\leq (1-p)V(b(p)) + x - b(p) = J_1(x, p, D^*, \Gamma^*).$$

This concludes Step 1, i.e., shows that the strategy $D^*$ is an optimal response to $\Gamma^*$.

**Step 2.** We now prove that $\Gamma^*$ is an optimal response to $D^*$. Recall that

$$u(x, p) := \begin{cases} b(p) \frac{\psi(x)}{\psi(b(p))}, & x \leq b(p), \\ b(p), & x > b(p), \end{cases}$$

set $X^* := X^{D^*}$ with $D^*$ defined in (29), $\tau^*_0 := \tau^*_0 X^*$, and let

$$Z^*_t := p \land \epsilon \left( \sup_{0 \leq s \leq t} X^*_s \right), \quad t \geq 0,$$

$$Z^*_{0-} := p,$$

as in (31) with $D = D^*$.

Let $p \in [0,1]$ and assume $x \leq b(p)$. If $p \in [\hat{p}, 1]$, then $b(p) = 0$ and so $x = 0$ and the strategy $\Gamma \in \mathcal{A}_2$ is irrelevant since the game stops immediately. It hence suffices to check $p \in [0, \hat{p})$. For notational convenience we treat the case $p = 0$ separately at the end and assume first $p \in (0, \hat{p})$. Note that $X^*_t \leq b(Z^*_t)$ for every $t \geq 0$ and that $Z^*_t$, $D^*_t$ and $X^*_t$ are continuous for every $t \geq 0$. Define

$$\tilde{u}(x, p) := b(p) \frac{\psi(x)}{\psi(b(p))} \in C^2([0, \infty) \times (0, \hat{p}).$$

and let $\tau$ be any $\mathbb{F}^W$-stopping time s.t. $\tau \leq \tau^*_B$ a.s., where $\tau^*_B = \inf \{ t \geq 0 : X^*_t \geq B \}$. Define $\tau^* = \tau^*_B : = \tau^*_0 \land \tau^*_B - \epsilon \land \tau \land T$ for $T, \epsilon \geq 0$ arbitrary and note that $Z^*_t > 0$ for $t \in [0, \tau^*)$. By applying Ito’s formula to $e^{-r \tau} \tilde{u}(X^*_t, Z^*_t)$ we obtain

$$e^{-r \tau} \tilde{u}(X^*_t, Z^*_t) = \tilde{u}(x, p) + \int_0^\tau e^{-rs} \mathcal{L} \tilde{u}(X^*_s, Z^*_s) \, ds - \int_0^\tau e^{-rs} \tilde{u}_x(X^*_s, Z^*_s) \, dD_s^* + \int_0^\tau e^{-rs} \tilde{u}_x(X^*_s, Z^*_s) \, dW_s + \int_0^\tau e^{-rs} \tilde{u}_p(X^*_s, Z^*_s) \, dZ^*_s.$$

By definition of $\tilde{u}$, we have that $\mathcal{L} \tilde{u}(X^*_s, Z^*_s) = 0$ for every $0 \leq s \leq \tau^*$ and by construction of $D^*$ and $Z^*$ (recall (30)), we obtain

$$\int_0^\tau e^{-rs} \tilde{u}_p(X^*_s, Z^*_s) \, dZ^*_s - \int_0^\tau e^{-rs} \tilde{u}_x(X^*_s, Z^*_s) \, dD^*_s$$

$$= \int_0^\tau e^{-rs} \left( \tilde{u}_p(X^*_s, Z^*_s) c'(X^*_s) - \tilde{u}_x(X^*_s, Z^*_s) \lambda(X^*_s) \right) \, dX^*_s = 0$$

(38)
where the last equality holds by definition of $\lambda$ in (21). Hence,

$$e^{-\tau r^*} \tilde{u}(X^*_t, Z^*_t) = \tilde{u}(x, p) + \int_0^\tau \sigma e^{-\tau s} \tilde{u}_x(X^*_s, Z^*_s) \, dW_s. \tag{39}$$

Since $\tilde{u}_x$ is bounded on $\{(x, p) : x \leq b(p)\}$, the stochastic integral in (39) is a martingale. Since $X^*$ and $Z^*$ are continuous, applying the optional sampling theorem and using dominated convergence yields

$$\tilde{u}(x, p) = \mathbb{E}\left[e^{-\tau r^*} \tilde{u}(X^*_\tau, Z^*_\tau) \right] \to \mathbb{E}\left[e^{-\tau r^*} \tilde{u}(X^*_{\tau_0}, Z^*_{\tau_0}) \right],$$

as $T \to \infty$ and $\varepsilon \to 0$, so

$$\tilde{u}(x, p) = \mathbb{E}\left[e^{-\tau r^*} \tilde{u}(X^*_{\tau_0}, Z^*_{\tau_0}) \right] \tag{40}$$

for any $\mathbb{F}^W$-stopping time $\tau \leq \tau_B$ a.s. Now, for any $\Gamma \in A_2$, define the $\mathbb{F}^W$-stopping times

$$\gamma(\rho) := \inf\{t \geq 0 : \Gamma_t(X^*) > \rho\}, \quad \rho \in [0, 1),$$

and let $\gamma_B(\rho) := \gamma(\rho) \land \tau_B^\rho \leq \tau_B^\rho$. Since $\tilde{u} = u$ on $\{(x, p) : x \leq b(p)\}$, equality (40) for $\tau = \gamma_B(\rho)$ reads

$$u(x, p) = \mathbb{E}\left[e^{-\tau r^*} u(X^*_{\tau_0}, Z^*_{\tau_0}) \right], \quad \rho \in [0, 1).$$

Thus,

$$u(x, p) = \int_0^1 \mathbb{E}\left[e^{-\tau r^*} X^*_{\tau_0} \right] \text{d}\rho \tag{41}$$

where the inequality holds because $\psi(x)$ is concave for $x \leq B$ with $\psi(0) = 0$.

Last, we note that

$$e^{-\tau r^*} X^*_{\tau_0} \geq e^{-\tau r^*} X^*_{\tau_0} \text{ a.s.} \tag{42}$$

since $X^*_t \leq B$ for all $t > 0$ and $r > 0$ and thus

$$u(x, p) \geq \int_0^1 \mathbb{E}\left[e^{-\tau r^*} X^*_{\tau_0} \right] \text{d}\rho = J(x, p, D^*, \Gamma).$$

If $\Gamma = \Gamma^*$, then by (32) we have that $\gamma^*(\rho) \leq \tau_B^\rho$ for every $\rho \in [0, 1)$, where

$$\gamma^*(\rho) := \inf\{t \geq 0 : \Gamma_t^*(X^*) > \rho\}, \quad \rho \in [0, 1)$$

and thus the inequality in (42) is an equality in this case. Moreover, $\Gamma^*_t$ only increases when $Z^*_t$ increases and $Z^* = Z^*_t := p \land \sup_{0 \leq s \leq t} X^*_s$ so

$$u(X^*_{\tau_0}, Z^*_{\tau_0}) = b(c(\tilde{X}^*_{\tau_0}, \gamma^*(\rho))) = X^*_{\tau_0} \text{ in (41). Thus all the inequalities above become equalities and}$$

$$u(x, p) = J_2(x, p, D^*, \Gamma^*). \tag{43}$$
If $p = 0$, we have $u(x, 0) = \tilde{u}(x, 0) = b(0) \psi(x) \psi(B) = B \psi(x) \psi(B)$ and $Z_t^* = 0$ for all $t \geq 0$. Applying Ito’s formula to $e^{-rt}u(X_t^*, 0)$ between 0 and $\tau_0 \wedge \tau_B^*$ and using the properties of $\psi(x)$ gives

$$e^{-r(t_0 \wedge \tau)} \tilde{u}(X_{\tau_0 \wedge \tau}, 0) = \tilde{u}(x, 0) - \int_{0}^{\tau_0 \wedge \tau} e^{-rs} \tilde{u}_x(X_s^*, 0)dD_s^* + \int_{0}^{\tau_0 \wedge \tau} e^{-rs} \sigma \tilde{u}_x(X_s^*, 0)dW_s.$$  

Taking expected value and arguing as above thus gives

$$u(x, 0) = E[e^{-r(t_0 \wedge \tau)}u(X_{t_0 \wedge \tau}, 0)] = e^{-r(t_0 \wedge \tau)}X_{t_0 \wedge \tau} = J_2(x, 0, D^*, \Gamma^*)$$

and

$$u(x, 0) = \int_{0}^{1} E[e^{-r(t_0 \wedge \gamma_B)}u(X_{t_0 \wedge \gamma_B}, 0)]d\rho \geq \int_{0}^{1} E[e^{-r(t_0 \wedge \gamma_B)}X_{t_0 \wedge \gamma_B}]d\rho \geq J_2(x, p, D^*, \Gamma)$$

where we again have used convexity of $\psi$ and the fact that any stopping time $\gamma(\rho) > \tau_B^*$ yields a lower payoff that $\tau_B^*$.

The above treats the case $x \leq b(p)$ so let us finalize the proof by considering $x > b(p)$. We have, for every $\Gamma \in A_2$, that

$$u(x, p) = u(b(p), p) \geq J_2(b(p), p, D^*, \Gamma) = J_2(x, p, D^*, \Gamma),$$

where the last equality holds by the precedence of Player 1 over Player 2 and since $D_0^* = x - b(p)$ for $x > b(p)$. Similarly, we obtain

$$u(x, p) = u(b(p), p) = J_2(b(p), p, D^*, \Gamma^*) = J_2(x, p, D^*, \Gamma^*).$$

Hence, $\Gamma^*$ is an optimal response to $D^*$. Together with Step 1, this implies that $(D^*, \Gamma^*)$ is a NE and that the equilibrium values are $v$ and $u$, respectively. This concludes the proof. \qed

**Remark 12.** It is a remarkable feature of the equilibrium strategy $(D^*, \Gamma^*)$ that it allows the process $\Pi^*$ to reach 0 in finite time, thereby completely ruling out the possibility that a fraudster exists if he did not stop the game yet. Indeed, let $x \leq b(p)$, then we have

$$X_t^* = Y_t - \bar{Y}_t + f(\bar{Y}_t)$$

and thus $\bar{X}_t^* = f(\bar{Y}_t)$ where $f$ is an increasing bounded function such that $f(x) = B$ for all $x \geq \Lambda(B) + B$. Consequently, $\Pi_t^* = p \land c(\bar{X}_t^*) = p \land c(f(\bar{Y}_t)) = c(B) = 0$ for all $t \geq \tau_B = \inf\{s \geq 0 : Y_s \geq \Lambda(B) + B\},$

the first time the unrestricted Brownian motion (with drift) $Y$ reaches $\Lambda(B) + B$ (which is finite a.s.).

### 6 A numerical example

To provide the reader with further intuition, we conclude by looking at some numerical experiments. Throughout the section, we consider parameters $\mu = 0.03$, $\sigma = 0.12$, and $r = 0.01$. The optimal strategy $\hat{D}$ in the single-player de Finetti problem given by (10) then amounts to reflection at $B \approx 1.12$.

Note that whereas the qualitative form of the single-player strategy de Finetti problem is fixed, the nature of the NE strategy for Player 1 varies depending on the value of $p \in [0, 1]$. To be more precise, if Player 1 is certain that no fraudster exists, i.e., if $p = 0$, then the problem degenerates into the standard single-player de Finetti problem and the optimal strategy is $\hat{D}$.
Figure 1: The boundary \( b(p) \) and the direction of reflection for the equilibrium process \((X^*, \Pi^*)\).

(and Player 2 would stop as soon as \( X \) hits \( B \)). On the other hand, if Player 1 has sufficient evidence of the existence of a fraudster, i.e., if \( p \in [\hat{p}, 1] \) where \( \hat{p} := (V'(0) - 1)/V'(0) \), then the agent extracts the whole resource immediately and the game terminates at \( t = 0 \). The most interesting scenario is when \( p \in (0, \hat{p}) \). In this case, the NE described in Theorem 11 amounts to a (possible) initial lump sum extraction of size \((x - b(p))^+\), and then continuous extraction as to reflect the two-dimensional process \((X^*, \Pi^*)\) along the boundary \( b \), with reflection in the prescribed direction \((u_p, -u_x)\). Figures 2 and 3 are derived with initial values \( p_0 = 0.8 \cdot \hat{p} \approx 0.72 \) and \( x_0 = \frac{b(p_0)}{2} \approx 0.13 \), putting us in the last of the three cases above.

Figure 1 shows the boundary \( p \mapsto b(p) \) (or equivalently \( x \mapsto c(x) \)) together with the direction of reflection of the equilibrium process \((X^*, \Pi^*)\). Note that \( b(0) = B \) and \( b(\hat{p}) = 0 \). Figures 2 and 3 show a simulated path of the equilibrium process \((X^*, \Pi^*)\) and the corresponding processes \(\Pi^*, \Gamma^*, \) and \(D^*\), respectively. Flat portions of \(\Gamma^*, \Pi^*, \) and \(D^*\) correspond to \(X^*\) being strictly below the boundary \(b(\Pi^*)\). Note also that in Figure 2, the process \(\Pi^*\) reaches 0 in finite time, ruling out the existence of a fraudster playing the equilibrium strategy if he did not stop yet, see Remark 12.
Figure 2: A simulated path of $(\Pi^*, X^*)$ reflected along the boundary $p \mapsto b(p)$.

Figure 3: Auxiliary processes $\Pi^*$ (dashed), $\Gamma^*$ (dash-dot), and $D^*$ (dotted).
References

[1] S. Asmussen and M. Taksar. Controlled diffusion models for optimal dividend pay-out. *Insurance Math. Econom.*, 20(1):1–15, 1997.

[2] R. J. Aumann, M. Maschler, and R. E. Stearns. *Repeated games with incomplete information*. MIT press, 1995.

[3] E. Bayraktar and Y.-J. Huang. On the multidimensional controller-and-stopper games. *SIAM J. Control Optim.*, 51(2):1263–1297, 2013.

[4] A. Bovo, T. De Angelis, and E. Issoglio. Variational inequalities on unbounded domains for zero-sum singular-controller vs. stopper games. *arXiv preprint arXiv:2203.06247*, 2022.

[5] P. Carmona, F. Petit, and M. Yor. Beta variables as times spent in $[0, \infty]$ by certain perturbed Brownian motions. *J. London Math. Soc. (2)*, 58(1):239–256, 1998.

[6] T. De Angelis and E. Ekström. Playing with ghosts in a Dynkin game. *Stochastic Process. Appl.*, 130(10):6133–6156, 2020.

[7] T. De Angelis and G. Ferrari. Stochastic nonzero-sum games: a new connection between singular control and optimal stopping. *Adv. in Appl. Probab.*, 50(2):347–372, 2018.

[8] E. Ekström, K. Lindensjö, and M. Olofsson. How to detect a salami slicer: a stochastic controller-and-stopper game with unknown competition. *arXiv preprint arXiv:2010.03619*, to appear in *SIAM J. Control Optim.*, 2021.

[9] J. C. Harsanyi. Games with incomplete information played by “Bayesian” players. I. The basic model. *Management Sci.*, 14:159–182, 1967.

[10] D. Hernandez-Hernandez, R. S. Simon, and M. Zervos. A zero-sum game between a singular stochastic controller and a discretionary stopper. *Ann. Appl. Probab.*, 25(1):46–80, 2015.

[11] J. Hirshleifer and J. Riley. *The Analytics of Uncertainty and Information*. Cambridge University Press, 1992.

[12] M. Jeanblanc and A. N. Shiryaev. Optimization of the flow of dividends. *Uspekhi Matematicheskikh Nauk*, 50(2):25–46, 1995.

[13] I. Karatzas and W. Sudderth. Stochastic games of control and stopping for a linear diffusion. In *Random Walk, Sequential Analysis And Related Topics: A Festschrift in Honor of Yuan-Shih Chow*, pages 100–117. World Scientific, 2006.

[14] I. Karatzas and I.-M. Zamfirescu. Martingale approach to stochastic differential games of control and stopping. *Ann. Probab.*, 36(4):1495–1527, 2008.

[15] H. D. Kwon and H. Zhang. Game of singular stochastic control and strategic exit. *Math. Oper. Res.*, 40(4):869–887, 2015.

[16] M. Perman and W. Werner. Perturbed Brownian motions. *Probab. Theory Related Fields*, 108(3):357–383, 1997.