Relativistic Aberration: The Transformation of a Propagating Field

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The fields used to describe the influence of masses and electric charges are generally accepted to emanate at the speed of light from their sources. To obtain these fields for a moving particle which are consistent with special relativity, transformations should be applied to the emanating field in the particle’s rest frame to acquire the field propagating in the particle’s primed inertial frame. These transformations are of its coordinates, and also the velocity vectors of the field’s propagation, which consequently lead to a change in the field’s direction and as a result its density and strength. Yet, an electric charge’s field propagation velocities are neglected and not currently transformed. This omission has consequences when it comes to transforming the field from the source’s rest frame to the source’s primed frame, as this primed frame’s field would then be inconsistent with the calculated retarded field’s directions and consequently its strengths in this primed frame. Here the retarded field of a moving point particle will be derived and from this the requirement of the velocity transforms and the aberrational effects will be shown. Hence, the full consistent inertial frame transform of the emanating field will be given. This work shows the general theoretical basis and legitimacy of the aberrational and time dilation effects on the field’s density and thus strength and acts as an argument for their implementation on all luminally emanating fields.

I. INTRODUCTION

In astrophysics, when describing the observed luminosity from a relativistic plasma jet, the transformation of the radiated light’s velocities, known as relativistic aberration or relativistic beaming, is used [1][2]. These aberrational effects are required to explain the luminosity of the radiated light (flux of photons) from different points in the plasma, as depending on the plasma’s relative velocity at a position there is a different weighting on the intensity of the number of photons received by an observer. However, coordinate transformations are not included in derivations, hence the requirement to apply the effects of aberration on the flux and a derivation of the frame transforms have not been shown.

In general, it is accepted that the electric field and its information propagate from a particle at the speed of light. However, when it comes to the relativistic transformations of electric fields, the literature applies the Lorentz transforms to the field’s coordinates without transforming the field’s propagation velocity [3][6]. If the electric field propagates at the speed of light then relativity requires the velocity vector of this propagation at each coordinate to also be transformed. This transformation does not change the luminal speed of the propagation but does change the propagation direction.

Here, the retarded field of a moving particle will be derived. From this, we will show the requirement of applying both the aberrational effects (due to velocity transforms) on the field’s propagation, which result in a change of the field’s direction and density, and the time dilation effects on the field’s density. This work shows the theoretical basis and legitimacy of both the aberrational and time dilation effects and acts as an argument for their implementation on all luminally emanating fields.

II. THE RETARDED FIELD

In the primed frame of a particle $q$, it is taken that the particle is moving in the positive Z-Axis only with velocity $V_{q} = (0, 0, v_{q})$, with it currently (primed time $t' = 0$) positioned at the origin. If the field emanates from the particle at luminal speed $c$. Then the field that is currently propagating through a general primed coordinate $R' = (x', y', z')$, is the field that was emanated from the particle when it was at a previous time and position, which can be calculated from its trajectory. These corresponding coordinates are referred to as the retarded time $T'_{ret}$ (the time when the field had emanated from the particle to reach $R'$ at time $t' = 0$) and the particle’s retarded position $P'_{ret} = (0, 0, v_{q}T'_{ret})$, corresponding to this retarded time. Hence the retarded field displacement that the field has propagated along is

$$\mathbf{R'}_{ret} = \mathbf{R'} - \mathbf{P'}_{ret} = \begin{pmatrix} x' \\ y' \\ z' - v_{q}T'_{ret} \end{pmatrix}. \quad (1)$$

A visual of these values are shown in figure [1] below. The magnitude of the retarded field displacement is equal to the distance the field, propagating at the speed of light, travels in the corresponding retarded time. This gives

$$(cT'_{ret})^{2} = ||\mathbf{R'}_{ret}||^{2} = (x'^{2} + y'^{2} + z'^{2}) + v_{q}^{2}T'^{2}_{ret} - 2v_{q}z'T'_{ret}, \quad (2)$$

rearranging this, we get the quadratic

$$T'^{2}_{ret} + \left(2\frac{v_{q}z'}{c^{2}}\right)T'_{ret} - \frac{\gamma^{2}}{c^{2}}(x'^{2} + y'^{2} + z'^{2}) = 0, \quad (3)$$

where $\gamma = (1 - v_{q}^{2}/c^{2})^{-1/2}$. Now taking the solution for the past time, and making use of the identity

$$\gamma^{2} = 1 + \frac{v_{q}^{2}}{c^{2}}, \quad (4)$$

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we get the result
\[
T'_{\text{ret}} = -\gamma^2 \frac{v_q}{c^2} z' - \sqrt{\left(-\gamma^2 \frac{v_q}{c^2} z'\right)^2 + \frac{\gamma^2}{c^2} (x'^2 + y'^2 + z'^2)}
\]
\[
= -\gamma^2 \frac{v_q}{c^2} z' - \frac{\gamma}{c} \sqrt{x'^2 + y'^2 + \left(1 + \frac{\gamma^2}{2} \frac{v_q^2}{c^2}\right) z'^2}
\]
\[
= -\gamma^2 \frac{v_q}{c^2} z' - \frac{\gamma}{c} \left|\mathbf{R}\right|
\]
(5)

In the final step, we required \(t' = 0\) for all coordinates, and hence used the Lorentz transform of the spatial coordinates; \(R = (x, y, z) = (x', y', \gamma z')\), to get \(\left|\mathbf{R}\right|\).

With this we can rewrite the Z-component of equation (11) as
\[
z' - v_q T'_{\text{ret}} = \gamma \left(z + \frac{v_q}{c} \cdot \left|\mathbf{R}\right| \right)
\]
(6)
giving the retarded field displacement vector to be
\[
\mathbf{R'}_{\text{ret}} = \left(\frac{x}{\gamma \left(z + \frac{v_q}{c} \cdot \left|\mathbf{R}\right| \right)} \gamma \left(z + \frac{v_q}{c} \cdot \left|\mathbf{R}\right| \right), \frac{y}{\gamma \left(z + \frac{v_q}{c} \cdot \left|\mathbf{R}\right| \right)} \gamma \left(z + \frac{v_q}{c} \cdot \left|\mathbf{R}\right| \right), \frac{z}{\gamma \left(z + \frac{v_q}{c} \cdot \left|\mathbf{R}\right| \right)} \gamma \left(z + \frac{v_q}{c} \cdot \left|\mathbf{R}\right| \right)\right).
\]
(7)

Since the field propagates along this in the primed frame, the unit vector of the primed propagation velocity at any general primed coordinate can be worked out to be
\[
\hat{U}' = \mathbf{R'}_{\text{ret}} = \frac{1}{A} \left(\frac{x}{\gamma \left(z + \frac{v_q}{c} \cdot \left|\mathbf{R}\right| \right)} \gamma \left(z + \frac{v_q}{c} \cdot \left|\mathbf{R}\right| \right), \frac{y}{\gamma \left(z + \frac{v_q}{c} \cdot \left|\mathbf{R}\right| \right)} \gamma \left(z + \frac{v_q}{c} \cdot \left|\mathbf{R}\right| \right), \frac{z}{\gamma \left(z + \frac{v_q}{c} \cdot \left|\mathbf{R}\right| \right)} \gamma \left(z + \frac{v_q}{c} \cdot \left|\mathbf{R}\right| \right)\right),
\]
(8)

where the factor
\[
A = \gamma \left(1 + \frac{v_q}{c} \frac{z}{\left|\mathbf{R}\right|} \right)
\]
(9)

Now taking the magnitude of the retarded field displacement from equation (7) and using equation (4), we have
\[
\left|\mathbf{R'}_{\text{ret}}\right|^2 = x^2 + y^2 + \gamma^2 \left(z^2 + \frac{v_q^2}{c^2} \left|\mathbf{R}\right|^2 + 2 \frac{v_q}{c} z \left|\mathbf{R}\right|\right)
\]
\[
= \gamma^2 \left|\mathbf{R}\right|^2 + \frac{v_q^2}{c^2} \gamma^2 z^2 + 2 \frac{v_q}{c} \gamma^2 z \left|\mathbf{R}\right|
\]
\[
= \gamma^2 \left(\left|\mathbf{R}\right| + \frac{v_q}{c} z\right)^2
\]
\[
= \gamma^2 \left(1 + \frac{v_q}{c} \frac{z}{\left|\mathbf{R}\right|}\right)^2 \left|\mathbf{R}\right|^2
\]
\[
= \hat{A}^2 \left|\mathbf{R}\right|^2.
\]
(10)

Since the luminal speed of the propagation will be same in both frames we have
\[
c = \frac{\left|\mathbf{R}\right|}{T_R} = \frac{\left|\mathbf{R'}_{\text{ret}}\right|}{T'_{\text{ret}}}
\]
\[
= \hat{A} \left|\mathbf{R}\right| \frac{T'_{\text{ret}}}{T_{\text{ret}}}
\]
\[
T'_{\text{ret}} = \hat{A} T_R,
\]
(11)

where \(T_R\) is the proper (particles rest frame) time the field takes to propagate to \(\mathbf{R}\) from the particle. Time and length are stretched in the primed frame along \(\mathbf{R'}_{\text{ret}}\) relative to the proper frame along \(\mathbf{R}\) by a factor \(\hat{A}\), leading to the relative radial density of the field in the primed frame to that of the proper frame, which we will refer to as the radial field strength weighting, given as
\[
W_\rho = \frac{1}{\hat{A}}.
\]
(12)

### III. ABERRATION

The propagation of the field in the proper frame, with the particle at rest, is taken as evenly distributed with its strength spherically symmetrical, and with the magnitude of its velocity equal to the speed of light \(c\) in all directions, as shown in the left graph of figure (2) below. If the field propagates, then a velocity transform of this propagation at all coordinates should also be performed in order to change inertial frame. The luminal velocity of a field emanating from a particle in its own proper frame, can be written in spherical polar coordinates as
\[
U = \left(\frac{U_x}{U_y}, \frac{U_y}{U_z}\right) = c \left(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta\right),
\]
(13)
\[ U' = \frac{1}{\gamma} \left( \frac{\gamma - 1}{\|V\|^2} (U \cdot V) - \gamma \right) V \]

\[ \gamma = \frac{1}{\sqrt{1 - \frac{v_q}{c} \cos \theta}} \]

where \( \theta \in [0, \pi] \) and \( \phi \in [0, 2\pi] \) are the angles from the Z-Axis and X-axis respectively. It is taken that the primed frame is moving as it did in the previous chapter, with velocity \( V = (0, 0, -v_q) \) relative to the particles proper frame. Therefore we only require a transformation of the Z-component to get the primed propagation velocity. Using the equation for the relativistic transform of a general velocity vector between two inertial frames, we have

\[ \cos \theta' = \frac{U'_z}{\|U'\|} = \frac{R'_{retz}}{\|R'_{ret}\|} = \frac{\cos \theta + \frac{v_q}{c}}{1 + \frac{v_q}{c} \cos \theta}. \] (16)

This is the relativistic aberration formula \(^7\), it shows how the field’s propagation direction transforms, it can be used to give \( \tilde{A} \) in primed terms, by rearranging this for \( \cos \theta \) and substituting into equation (15), leading to

\[ \tilde{A} = \frac{1}{\gamma} \left( 1 - \frac{v_q}{c} \cos \theta' \right) = \frac{1}{\gamma} \left( 1 - \frac{v_q}{c} \frac{R'_{retz}}{\|R'_{ret}\|} \right). \] (17)

A. The Field Strength’s Dependence on Angular Density

The proper frame’s differential solid angle element

\[ d\Omega = \sin \theta \sin \phi \, d\theta \, d\phi, \]

encompasses a certain amount of the field, this is the same amount of the field that is encompassed by the coinciding aberrated differential solid angle

\[ d\Omega' = \sin \theta' \sin \phi' \, d\theta' \, d\phi'. \] (19)

We can calculate this element by differentiating both sides of equation (16) with respect to \( \theta \) \(^8\), which gives

\[ \sin \theta' \, d\theta' = \frac{1 - \frac{v_q^2}{c^2} \cos^2 \theta}{\left( 1 + \frac{v_q}{c} \cos \theta \right)^2} \sin \theta \, d\theta = \frac{1}{\tilde{A}^2} \sin \theta \, d\theta. \] (20)

Using this and \( d\phi' = d\phi \) (as the angle \( \phi \) is always perpendicular to the motion of the particle and hence unaffected...
by transformation) we have the solid angle in the primed frame given as

$$d\Omega' = \frac{1}{A^2} \sin \theta d\theta d\phi.$$  \hspace{1cm} (21)

The relative primed field strength at a given angle is taken as being proportional to the amount of the field per solid angle in the primed frame relative to that in the proper frame, referred to here as the aberrational field strength weighting, given as

$$W_\Omega = \frac{d\Omega}{d\Omega'} = A^2.$$  \hspace{1cm} (22)

IV. THE FIELD TRANSFORMATION

In the particle’s proper frame the field strength’s magnitude \(f\), obeys the inverse square law, Giving

$$f = \frac{k}{||R||^2},$$  \hspace{1cm} (23)

where \(k\) is a constant for the particle. In the primed frame the field strength at a coordinate will be equal to the inverse square of the retarded field displacement with constant \(k\) and the field strength weightings from equations (12) and (22) applied

$$f' = W_\rho W_\Omega \frac{k}{||R'_{\text{ret}}||^2} = \bar{A} \frac{k}{||R'_{\text{ret}}||^2}.$$  \hspace{1cm} (24)

It can be noted that both field strength weightings together give \(\bar{A}\), which is the same as the generalised Doppler factor. Now using this equation of the field strength and the direction of the field’s propagation which is the unit vector for the retarded field displacement, we have the primed frame’s field described by the vector

$$f' = f'\hat{R}'_{\text{ret}} = \bar{A} \frac{kR'_{\text{ret}}}{||R'_{\text{ret}}||^3}.$$  \hspace{1cm} (25)

V. DISCUSSION

If the electric field propagates at the speed of light and special relativity is applied, then the aberrational and time dilation effects are required, and hence the electric field will be described differently than previous relativistic transforms, described by equation (25).

One consequence of aberration is that if a particle approaches the speed of light, then every proper propagation direction of the field from the particle would, in the primed frame, approach the direction of the particle’s motion, the retarded position of the particle would also tend to negative infinity. Given this and that the particle’s motion is approaching the speed of the lumi- nal propagation, then there would be no field outside of where the particle is located. Therefore if a photon did have a emanating field itself, it would be expected that a field would not exist outside the space that the photon occupies.

The aberrational weighting from the transformations of a field in this paper can also be applied to the average flux of photons in all directions from a moving radiating source (instead of the field strength) and might provide another way of explaining the distribution of synchrotron radiation, where the Liénard – Wiechert field equations which include similar factors have been used to explain the phenomena [9].

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