Loop unrolling of UCA models: distance labeling

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Abstract

A proper circular-arc (PCA) model is a pair \( M = (C, A) \) where \( C \) is a circle and \( A \) is a family of inclusion-free arcs on \( C \) whose extremes are pairwise different. The model \( M \) represents a digraph \( D \) that has one vertex \( v(A) \) for each \( A \in A \) and one edge \( v(A) \to v(B) \) for each pair of arcs \( A, B \in A(M) \) such that the beginning point of \( B \) belongs to \( A \). For \( k \geq 0 \), the \( k \)-th power \( D^k \) of \( D \) has the same vertices as \( D \) and \( v(A) \to v(B) \) is an edge of \( D^k \) when \( A \neq B \) and the distance from \( v(A) \) to \( v(B) \) in \( D \) is at most \( k \). A unit circular-arc (UCA) model is a PCA model \( U = (C, A) \) in which all the arcs have the same length \( \ell + 1 \). If \( \ell \), the length \( c \) of \( C \), and the extremes of the arcs of \( A \) are integer, then \( U \) is a \((c, \ell + 1)\)-CA model. For \( i \geq 0 \), the model \( i \times U \) of \( U \) is obtained by replacing each arc \((s, s + \ell + 1)\) with the arc \((s, s + i\ell + 1)\). If \( U \) represents a digraph \( D \), then \( U \) is \( k \)-multiplicative when \( i \times U \) represents \( D^i \) for every \( 0 \leq i \leq k \). In this article we design a linear time algorithm to decide if a PCA model \( M \) is equivalent to a \( k \)-multiplicative UCA model when \( k \) is given as input. The algorithm either outputs a \( k \)-multiplicative UCA model \( U \) equivalent to \( M \) or a negative certificate that can be authenticated in linear time. Our main technical tool is a new characterization of those PCA models that are equivalent to \( k \)-multiplicative UCA models. For \( k = 1 \), this characterization yields a new algorithm for the classical representation problem that is simpler than the previously known algorithms.

keywords: multiplicative UCA models, distance labeling, powers of UCA models, representation problem

1 Introduction

The last decade saw an increasing amount of research on numerical representation problems for unit circular-arc (UCA) models and some of its subclasses [6, 7, 15, 17, 16, 26, 27]. In these problems we are given a proper circular-arc (PCA) model \( M \) and we have to find a UCA model \( U \), related to \( M \), that satisfies certain numerical constraints. The paradigmatic example is the classical representation problem in which a
UCA model $U$ equivalent to an input PCA model $M$ has to be computed. The equivalence of $M$ and $U$ means that the endpoints of $U$ must appear in the same circular order as those of $M$.

In this article we consider a generalization of the classical representation problem. In a nutshell, given a PCA model $M$ and $k \geq 0$, the goal is to find a UCA model $U$ whose “multiplication” $i \times U$ is equivalent to the “power” $M^i$ of $M$ for every $0 \leq i \leq k$. Here, $i \times U$ is obtained from $U$ by lengthening each arc to have length $i \times \ell + 1$, where $\ell + 1$ is the length of the arcs in $U$. On the other hand, $M^i$ is a PCA model whose intersection graph is the $i$-th power of the intersection graph of $M$. To formally state the problem we require some terminology that will be used throughout the article.

### 1.1 Statement of the problem

In this work, the term arc refers to open circular arcs. For points $s \neq t$ of a circle $C$, we write $(s, t)$ to denote the arc of $C$ that goes from $s$ to $t$ in a clockwise traversal of $C$. Each arc $A = (s, t)$ of $C$ with extremes $s$ and $t$ is described by its beginning point $s(A) = s$ and its ending point $t(A) = t$. We write $|A| = |s, t|$ and $|C|$ to denote the lengths of $A$ and $C$, respectively. We assume that every circle $C$ has a special point 0 such that $p = [0, p]$ for every point $p \in C$. Thus, $p < q$ if and only if $p$ appears before $q$ in a clockwise traversal of $C$ from 0. For arcs $A_1$ and $A_2$ of $C$, we write $A_1 < A_2$ to mean that $s(A_1) < s(A_2)$. We classify the arcs of $C$ as being external or internal according to whether $A \cup \{t(A)\}$ contains 0 or not, respectively. In other words, $A$ is external when $t(A) < s(A)$.

A proper circular-arc (PCA) model (Fig. 1) is a pair $M = (C, A)$ where $C$ is a circle and $A$ is a family of inclusion-free arcs on $C$, no two of which share an extreme. We write $C(M) = C$ and $A(M) = A$ to denote the circle and the family of arcs of $M$, respectively. The extremes of $M$ are those extremes of the arcs in $A$. Say that $M$ and a PCA model $M'$ are equivalent if there exists a bijection $f: A(M) \rightarrow A(M')$ such that $e(A) < e'(B)$ if and only if $e(f(A)) < e'(f(B))$, for $e, e' \in \{s, t\}$. Colloquially, $M$ and $M'$ are equivalent if their extremes appear in the same order, regarding $f$, when their circles are traversed clockwise from their respective 0 points.

A unit circular-arc (UCA) model is a PCA model $M$ whose arcs all have the same length $\ell$. If every extreme of $M$ is integer, then we refer to $M$ as being a $(|C|, \ell)$-CA model.

![Figure 1](image-url)

Figure 1: From left to right: a PCA model $M$ with arcs $A_0 < \ldots < A_4$; a $(48, 13)$-CA model $U$ equivalent to $M$; the 13-IG model obtained by removing the external arc of $U$ (the circle represents $\mathbb{R}$); the digraph $D(M)$. 
model (Fig. 1). Note that if $M$ is a PCA model with no external arcs, then we can remove a segment $([C(M)] - \varepsilon, 0)$ from $C(M)$ to obtain a line $L$, without removing points of the arcs of $M$. Replacing $L$ with the real line (of infinite length), we obtain a new representation of $M$ where each arc corresponds to an interval of the real line. Conversely, any family of intervals on the real line can be transformed into arcs of a circle by pasting together two points of the line that bound all the intervals. To keep a uniform terminology for both PCA and proper intervals models, in this work we say that $I$ is a proper interval (PIG) or unit interval (UIG) model to mean that $(\mathbb{R}, I)$ is a PCA or UCA model with no external arcs, respectively, where the real line $\mathbb{R}$ is thought of as a circle with infinite length (Fig. 1). Moreover, instead of stating that $(\mathbb{R}, I)$ is an $(\infty, \ell)$-CA model, we simply state that $I$ is an $\ell$-IG model.

Every PCA model $M$ defines a digraph $D(M)$ that has a vertex $v(A)$ for each $A \in A(M)$ where $v(A) \rightarrow v(B)$ is a directed edge $(A, B \in A(M))$ if and only if $s(B) \in A$ (Fig. 1). In the underlying graph $G(M)$ of $D(M)$, $v(A)$ and $v(B)$ are adjacent if and only if $A \cap B \neq \emptyset$. A (di)graph $G$ is a proper circular-arc (PCA) (di)graph represented by $M$ when $G$ is isomorphic to $G(M)$. Unit circular-arc (UCA), proper interval (PIG) and unit interval (UIG) (di)graphs are defined analogously. Because of the circular nature of $M$, the distance between $v(A)$ and $v(B)$ in $G(M)$ is the minimum of the distances in $D(M)$ from $v(A)$ to $v(B)$ and from $v(B)$ to $v(A)$ (e.g. [10, Lemmas 5 and 6]). Thus, to determine the distance between two vertices of a graph represented by a PCA model $M$, it suffices to find the distances of their respective vertices in $D(M)$. And, as $D(M)$ is implicitly encoded by $M$, we can work directly with $M$.

Let $M$ be a PCA model with arcs $A_0 < \ldots < A_{n-1}$. The arc $A_0$ is called the initial arc of $M$. Any sequence $L = A_i, A_{i+1}, \ldots, A_{i+k}$, with subindices modulo $n$, is said to be contiguous. The arcs $A_i$ and $A_{i+k}$ are the leftmost and rightmost arcs of $L$, respectively. For $0 \leq i < n$, define:

- $N^{-}[A_i]$ as the contiguous sequence of arcs with ending point in $A_i \cup \{t(A_i)\}$ that has $A_i$ as its rightmost arc,
- $N^{+}[A_i]$ as the contiguous sequence of arcs with beginning point in $A_i \cup \{s(A_i)\}$ that has $A_i$ as its leftmost arc,
- $F_l(A_i)$ as the leftmost arc in $N^{-}[A_i]$ and $F_r(A_i)$ as the rightmost arc in $N^{+}[A_i]$,
- $L(A_i) = A_{i-1}$ and $R(A_i) = A_{i+1}$ (modulo $n$),
- $H_l(A_i)$ as the unique arc $A$ such that $F_l(A) = A_i$ and $F_l \circ R(A) \neq A_i$; if $A$ does not exist, then $H_l(A_i) = \bot$, and
- $H_r(A_i)$ as the unique arc $A$ such that $F_l(A) = A_i$ and $F_l \circ L(A) \neq A_i$; if $A$ does not exist, then $H_r(A_i) = \bot$.

In Fig. 1, $N^{+}[A_1] = N^{-}[A_3] = A_1, A_2, A_3, R(A_1) = L(A_3) = A_2, F_l(A_1) = F_l(A_4) = A_3, H_l(A_1) = H_l(A_3) = A_2$, and $H_r(A_2) = H_r(A_2) = \bot$. Note that $v(A_i) \rightarrow v(A_j)$ is a directed edge of $D(M)$ if and only if $i \neq j$ and $A_i \in N^{-}[A_j]$, which happens if and only if $i \neq j$ and $A_j \in N^{+}[A_i]$. Therefore, $N^{-}[A_i]$ and $N^{+}[A_i]$ represent the in and out closed neighborhoods of $v(A_i)$ in $D(M)$, respectively.
For a (di)graph $D$ with vertex set $V(D)$, its $k$-th power $D^k$ is the (di)graph with vertex set $V(D)$ such that $v \to w$ is a (directed) edge of $D^k$ if and only if $v \neq w$ and the distance from $v$ to $w$ in $D$ is at most $k$. Let $D^k(M) = (D(M))^k$. The known fact that $D^k(M)$ is a PCA digraph can be proved with the following construction. For $k \geq 0$, $A \in A(M)$, and $f \in \{L, R, F_l, F_r, H_l, H_r\}$, let $f^0(A) = A$ and $f^{k+1}(A) = f \circ f^k(A)$, where $f^{k+1}(A) = \perp$ if $f^k(A) = \perp$. The $k$-th power of $A$ is the arc $A^k = (s(A), s(F^k_l(A)) + \varepsilon(x + 1))$, where $x$ is the number of arcs $B$ with $s(A) \in B$ and $F^k(B) = F^k_l(A)$, and $\varepsilon < n^{-1}$ is small enough so that $t(A') \notin R \circ F^i_l(A)$. Note that $A^k$ and $B^k$ share no extremes for $A, B \in A$. The $k$-th power of $M$ is the pair $M^k = (C(M), \{A^k \mid A \in A(M)\})$; see Fig. 2.

Define the wraparound value $\omega(M)$ of $M$ to be the minimum $\omega > 1$ for which there is an arc $A \in A(M)$ such that $A^\omega \subset A$. For the sake of notation, we usually omit the parameter $M$ of $\omega$. Note that $\omega$ is well defined unless $M$ is a PLG model, in which case we let $\omega = n$. Although $M^k$ is defined for proving that $D^k(M)$ is PCA, the statement $D(M^k) = D^k(M)$ is false when $k \geq \omega$ and $M$ is not PLG. Indeed, as $A^\omega \subset A$ for some arc $A$, $A^\omega$ intersects fewer arcs than $A$, whereas $v(A)$ has more neighbors in $D^\omega(M)$ than in $D(M)$. The reason why this happens is that $A^{\omega-1} \cup F^i_l(A)$ covers the circle. To fix this issue it can be observed that $D^\omega(M)$ is a complete digraph, thus it suffices to define $A^k = (s(A), s(A) - \varepsilon)$ when $k \geq \omega$. In this article we are concerned with the model, thus we avoid this approach. Nevertheless, the following well-known theorem holds.

**Theorem 1 ([8]).** Let $M$ be a PCA model. If $0 \leq k < \omega$, then $M^k$ is a PCA model that represents $D^k(M)$; otherwise, $D^k(M)$ is a complete digraph.

If we store $M^k$ for every $k < \omega$, then we can efficiently answer any distance query in $G(M)$. Our goal, however, is to define one UCA model $M$ to answer these queries efficiently. Let $M$ be a $(c, \ell + 1)$-CA model. For $i \geq 0$ and $A \in A(M)$, define the $i$-multiple of $A$ as the arc $i \times A = (s(A), s(A) + il + 1 \mod c)$. The $i$-multiple of $M$ is the pair $i \times M = (C(M), \{i \times A \mid A \in A(M)\})$; Fig. 2. For $k \geq 0$, we say that $M$ is $k$-multiplicative when $i \times M$ is a UCA model equivalent to $M'$ for every $0 \leq i \leq k$ (and thus it represents $D^i(M)$ for $i$ up to $k$). We remark that $i \times M$ is a UCA model unless two arcs have a common extreme. To avoid this possibility, say that a $(c, \ell + 1)$-CA model $M$ is even when $c$, $\ell$, and all the beginning points of the arcs in $A(M)$ are even.
It is not hard to see that \( i \times \mathbf{M} \) is an even UCA model when \( \mathbf{M} \) is even. There is no loss of generality in restricting the study to even models, as every \( (c, \ell) \)-CA model \( \mathbf{M} \) can be transformed into an equivalent \( (2c, 2\ell + 1) \)-CA model by replacing every arc \( A \) by the arc \( (2s(A), 2\ell(A) + 1) \). Note that \( O(1) \) time is enough to decide if the distance from \( v(A) \) to \( v(B) \) in \( D(\mathbf{M}) \) is \( i \leq k \) when a \( k \)-multiplicative \( (c, \ell + 1) \)-CA model \( \mathbf{M} \) is given, as it suffices to check that \( s(A), s(B), t(i \times A) = s(A) + i\ell + 1 \mod c \) appear in this order in a clockwise traversal of \( C(\mathbf{M}) \).

In this article we study the \( k \)-MULT problem. Given a PCA model \( \mathbf{M} \) and \( 0 \leq k < \omega \), the goal of \( k \)-MULT is to determine if \( \mathbf{M} \) is equivalent to a \( k \)-multiplicative model. If affirmative, a certifying algorithm outputs a \( k \)-multiplicative model \( \mathbf{U} \) equivalent to \( \mathbf{M} \); otherwise, it outputs a negative certificate. The problem is trivial when \( k = 0 \) because \( \mathbf{M} \) is 0-multiplicative. For this reason, we restrict our attention to the case \( k > 0 \) in which \( \mathbf{U} \) must be UCA.

### 1.2 Motivation for the problem

Every PIG model \( \mathbf{I} \) yields a metric \( d_I \) on \( V(G(\mathbf{I})) \) where \( d_I(v(A), v(B)) = |s(A) - s(B)| \) for every \( A, B \in \mathbf{I} \). Among all the PIG representations of \( G = G(\mathbf{I}) \), those that are UIG provide a better notion of nearness, as the vertices adjacent in \( G \) are nearer than those non-adjacent. Indeed, if \( \mathbf{U} \) is an \( \ell \)-IG model and \( d_G(v(A), v(B)) \leq 1 < d_G(v(X), v(Y)) \), then \( d_U(v(A), v(B)) < \ell < d_U(v(X), v(Y)) \). This feature is one of the main reasons why (some notion equivalent to) UIG models are introduced in many different theoretical frameworks, including uniform arrays \([9, 11]\), semiorders \([18, 24]\) and indifference graphs \([23]\). As argued by \([11]\), \( d_U \) reflects the natural idea that among all the pair of adjacent vertices of \( G \), some are nearer than others. This is important in Goodman’s work about the topology of quality, as large gaps in \( d_U \) may suggest that some qualia are yet undiscovered. However, when greater distances on \( G \) are considered, the main feature of UIG models is lost: there are UIG models \( \mathbf{U} \) with \( d_G(v(A), v(B)) \leq k < d_G(v(X), v(Y)) \) and \( d_U(v(A), v(B)) > d_U(v(X), v(Y)) \). Instead, if \( \mathbf{U} \) is \((\ell + 1)\)-IG and \( \infty \)-multiplicative, then \( d_U(v(A), v(B)) < k\ell + 1 < d_U(v(X), v(Y)) \). Figure 3 depicts the situation for general PCA models.

\([23]\) “PIG=UIG” theorem states that every PIG model is equivalent to a UIG model. The classical representation problem \textsc{RepUIG} asks to find a UIG model \( \mathbf{U} \) equivalent to an input PIG model \( \mathbf{I} \). By definition, \( \mathbf{U} \) is a UIG model if and only if \( \mathbf{U} \) is \( 1 \)-multiplicative. Thus, \textsc{RepUIG} is simply the restriction of \( \textsc{1-MULT} \) to PIG inputs. There are many algorithms to solve \textsc{RepUIG}, at least three of which run in linear time \([5, 17, 19]\). It is not hard to prove that the UIG models produced by the linear time algorithms by \([5]\) and \([17]\) are \( \infty \)-multiplicative. (An implicit proof for the algorithm by \([5]\) follows from \([10]\); see also Theorem 8.) The following generalization of Roberts’ “PIG=UIG” theorem is obtained.

**Theorem 2.** Every PIG model \( \mathbf{I} \) is equivalent to some \( \infty \)-multiplicative UIG model. Furthermore, an \( \infty \)-multiplicative UIG model equivalent \( \mathbf{I} \) can be computed in linear time.

The problem \( k \)-MULT shares a strong relation to the **distance labeling** problem for circular-arc graphs. The latter problem asks to assign a label \( L(v) \) to each vertex \( v \) of a
graph $G$ in such a way that the adjacency between $v$ and $w$ in $G^k$ can be determined from $L(v)$ and $L(w)$ alone, i.e., $d_G(v, w) = f(L(v), L(w))$ for some function $f$. The primary goal is to minimize the number of bits required by each label $L(v)$, the secondary goal is to minimize the time required by $f$, and the third goal is to minimize the time required to compute $L$ from $G$. If $U$ is an $\ell$-multiplicative $(\ell + 1)$-IG model representing $G = G(U)$, then we can assign the label $L(v(I)) = s(I)$ for every $I \in U$ because $d_G(v, w) = \lceil(L(v) - L(w))/\ell \rceil$. When $U$ is produced using the algorithm by [5], each interval has a length $n$ whereas each beginning point is a number in $[1, n^2]$, thus each label requires at most $2\lceil \log(n) \rceil$ bits which is asymptotically optimal. Moreover, $d_G(v, w)$ can be computed in $O(1)$ time, whereas $L$ can be computed in linear time. Essentially, this is the labeling scheme proposed by [10] for PIG graphs, even though they do not mention that the generated labels are the possible beginning points of a UIG model. [10] also show that this scheme can be applied to solve the labeling problem for the general class of circular-arc graphs. However, contrary to our goal in this article, the labels generated for UCA graphs have little to do with the UCA models representing them.

Theorem 2 yields an $O(n)$ time algorithm to find a UIG model $U$, equivalent to an input PIG model $M$, that implicitly encodes a UIG model $i \times U$ of $G^i(M)$ for every $i \geq 0$. There is no hope in finding a similar algorithm when the input $M$ is UCA because $G^i(M)$ need not be UCA. By Theorem 1, this implies the well known fact that PCA and UCA are different classes of graphs [29]. The classical representation problem REP asks to determine if an input PCA model $M$ is equivalent to some UCA model. A UCA model equivalent to $M$ or a negative certificate should be given as well. Observe that a PCA model is UCA if and only if it is 1-multiplicative. Thus, $k$-MULT is a natural generalization of REP = 1-MULT that asks for a UCA model $U$, if existing, to implicitly encode the UCA model $i \times U$ of $G^i(M)$ for every $0 \leq i < k$. The problem 1-MULT can be solved in linear time using any of the algorithms for REP [14, 16, 27]. As far as our knowledge extends, no efficient algorithms are known to solve $k$-MULT for $k > 1$. 

Figure 3: From left to right: a PCA model $M$, a UCA model $U_1$ equivalent to $M$, and a 6-multiplicative UCA model $U_6$ equivalent to $M$. In $M$, $s(A_2) - s(A_0) = 7 < 18 = s(A_7) - s(A_8)$ and $d(v(A_0), v(A_2)) = 2 > 1 = d(v(A_7), v(A_8))$. In $U_1$, $s(A_4) - s(A_0) = 25 < 30 = s(A_7) - s(A_4)$ and $d(v(A_0), v(A_4)) = 4 > 3 = d(v(A_4), v(A_7))$. In turn, $U_6$ preserves the proportions for every $k < \omega = 7$. 

6
1.3 Brief history of the problems

The problem $k$-MULT is a generalization of REP that, in turn, is a generalization of REPUIG. One of the earliest references to REPUIG was given by [11] in the 1940’s, predating the current definition of UIG graphs. Since then, several algorithms to solve REPUIG were developed, many of which run in linear time (e.g. [5, 17, 19]). Regarding REP, Goodman states that no adequate rules to transform a PCA model into an equivalent UCA model are known. Of course, such general rules do not exist because some PCA graphs are not UCA. [29] characterized those PCA graphs that are not UCA by showing a family of forbidden induced subgraphs. His proof yields an effective method to transform a PCA model $M$ into an equivalent UCA model $U$. The first linear time algorithm to solve REP was given by [16]. Their algorithm outputs a UCA model $U$ equivalent to the input PCA model $M$ when the output is yes, but it fails to provide a negative certificate when the output is no. A different algorithm to find such a negative certificate was developed by [14], who left open the problem of finding a unified certifying algorithm for REP; such an algorithm was given by [26, 27].

From a technical point of view, our manuscript can be thought of as the sixth on a series of articles that deal with REPUIG and REP. The series started when [20] proved that every PIG model $I$ is equivalent to a minimal UIG model. Although Pirlot’s work is not of an algorithmic nature, his results yield an $O(n^2 \log n)$ time algorithm to solve the minimal representation problem. As part of his work, Pirlot shows that the problem of computing an $\ell$-IG model equivalent to $I$, when $\ell$ is given, can be modeled with a system $S_\ell$ having $O(n)$ difference constraints. A solution to $S_\ell$, if existing, can be found in $O(n^2)$ time by running a shortest path algorithm on its weighted constraint graph $S(\ell)$ (see Theorem 3). As every PIG graph is equivalent to an $n$-IG model, an $O(n^2)$ time algorithm to solve REP is obtained.

The unweighted version $S$ of $S(\ell)$ is a succinct representation of $M$ and, for this reason, Pirlot refers to $S$ as the synthetic graph of $M$. [19] continued the series by arguing that the minimal representation problem can be solved in $O(n) \log n$ time. Although her algorithm has a flaw and the correct version runs in $O(n^2)$ time [27], it correctly solves REPUIG in $O(n)$ time. Her algorithm follows by observing that $S$ admits a peculiar plane drawing in which the vertices occupy the entries of an imaginary matrix.

[15] rediscovered and extended Pirlot’s system $S_\ell$ to solve the bounded representation problem for UIG models in nearly quadratic time. Later, [26, 27] generalized $S_\ell$ to a new system $S_{c,\ell}$ to solve the problem of deciding if $M$ is equivalent to a $(c, \ell)$-CA model when $M$, $c$, and $\ell$ are given as input. The algorithm runs in $O(n^2)$ time and it can be adapted to solve the bounded representation problem for UCA models in $O(n^2)$ time as well. Furthermore, Soulignac adapted Mitas’ drawings to UCA models to design a certifying algorithm for REP that runs in $O(n)$ time or logspace. Moreover, he proved that every UCA model is equivalent to some minimal UCA model, though he left open the problem of computing such a minimal model in polynomial time.

Besides the previous five works, other articles apply systems of difference constraints to solve numerical representation problems related to intervals and circular-arc graphs (e.g. [2, 3, 12]).
1.4 Our contributions

In this article we follow the path described above. In Section 3, we define a system $S^k_{c,\ell}$ with $O(n)$ difference constraints to solve $k$-MULT for the particular case in which the output $U$ is required to be a $(c, \ell+1)$-CA model. The algorithm obtained runs in $O(n^2)$ time. In Section 4, we study the structure of the unweighted graph $S^k$ that represents $S^k_{c,\ell}$. As part of this section we provide an analogous of Mitas’ drawings for $S^k$. In Section 5 we exploit these drawings to devise a simple $O(n)$ time algorithm to solve $k$-MULT. The algorithm in this section outputs a negative certificate when the answer is no. Theorems 8 and 9 are the main theoretical contributions in Section 5, as they provide characterizations of those PCA models that have equivalent $k$-multiplicative UCA models. As far as our knowledge extends, these theorems are new even for $k = 1$, and they yield the simplest algorithm currently known to solve REP. Finally, in Section 6 we show how to build a $k$-multiplicative UCA model equivalent to $M$ when the answer to $k$-MULT is yes. Theorem 10 is the main theoretical contribution in this section, as it gives us an alternative characterization of those PCA models that are equivalent to $k$-multiplicative UCA models. For $k = 1$, Theorem 10 is a restatement of a theorem by [26] that, in turn, is a generalization Tucker’s characterization.

The algorithm to transform a PCA model $M$ into an equivalent $k$-multiplicative UCA model in Section 6 is rather similar to the one given by [27]. However, the theoretical framework developed to prove that the algorithm is correct is new. For instance, Theorems 8 and 9 in Section 5 are new and can be applied to solve other open problems, such as the minimal representation problem, in polynomial time. These applications, preliminarily described in [28], will be discussed in forthcoming articles. The major difference with respect to previous contributions is that we exploit a powerful geometric framework arising from the combination of Mitas’ drawings and the loop unrolling technique. When $M$ is a PIG model, the Mitas’ drawing of its synthetic graph $S^k$ is a plane drawing for every $k \geq 0$; this property is lost when $M$ is a PCA model. The problem is that the edges of $S^k$ corresponding to external arcs of $M^k$ cross other edges. This is hard to deal with when $k$ is large, as many arcs in $M^k$ are external. To apply the loop unrolling technique (Fig. 4), the idea is to replicate $\lambda$ times the arcs of a PCA model $M$, for a sufficiently large $\lambda$. This yields a new model $\lambda \cdot M$ in which every arc of $M^k$ has several internal copies. Interestingly, the PIG model $M'$ obtained after removing all the arcs of $\lambda \cdot M$ that are external in $(\lambda \cdot M)^k$ has enough information to solve $k$-MULT. The idea, then, is to study the Mitas’ drawing of the synthetic graph of $M'$ as if it were a planar representation of $S^k$.

It is important to remark that none of the previous tools is required to formally state our characterizations; all of our results can be easily translated to the idiom of PCA models. Clearly, synthetic graphs and Mitas’ drawings are not new concepts, while loop unrolling is a natural and old technique that, unsurprisingly, has already been applied to circular-arc models (e.g. [30]). Yet, we are not aware of any work that combines them together to obtain structural results about PCA and UCA models.
Figure 4: From left to right: a PCA model $M$, its loop unrolling $\lambda \cdot M$ with $\lambda$ copies, and the model $\tilde{M}$ obtained by removing the external arcs of $\lambda \cdot M$. Every external arc of $\tilde{M}$ has many internal copies in $\lambda \cdot M$.

2 Preliminaries

This section recalls how to solve a system of difference constraints and it introduces the remaining non-standard definitions that we use throughout the article. For $k \in \mathbb{N}$, we write $(k) = (0, k) \cap \mathbb{Z}$, $[k] = [k] \cup \{0\}$ and $[k] = [k] \cup \{k\}$. For a logical predicate $b$, we write $\beta(b) \in \{0, 1\}$ to denote 1 if and only if $b$ is true. For partial functions $f, g : \mathbb{R} \to \mathbb{R}$, we say that $f$ is bounded below by $g$, and that $g$ is bounded above by $f$, when $f(x) \geq g(x)$ for every $x \in \mathbb{R}$ that belongs to the domain of both $f$ and $g$. We sometimes write $vw$ or $v \to w$ to denote an ordered pair $(v, w)$. As usual, we reserve $n$ to denote the number of arcs of an input PCA model $M$.

A walk $W$ in a digraph $D$ is a sequence of vertices $v_0, \ldots, v_k$ such that $v_iv_{i+1}$ is an edge of $D$ for $i \in [k]$. Walk $W$ goes from (or begins at) $v_0$ to (or ends at) $v_k$. If $v_k = v_0$, then $W$ is a circuit, if $v_i \neq v_j$ for every $0 \leq i < j \leq k$, then $W$ is a path. And if $W$ is a circuit and $v_0, \ldots, v_{k-1}$ is a path, then $W$ is a cycle. If $W' = v_k, \ldots, v_j$ is a walk, then $W + W' = v_1, \ldots, v_j$ is also a walk, if $W$ is a circuit, then $j : W = \sum_{i=1}^{j} W$ is also a circuit for every $j \geq 1$, and if $D$ has no cycles, then $D$ is acyclic. For the sake of notation, we say that $W$ is a circuit when $v_0 \neq v_k$ to mean that $W, v_0$ is a circuit.

An edge weighting, or simply a weighting, of a digraph $D$ is a function $w : E(D) \to \mathbb{G}$ where $\mathbb{G}$ is a totally ordered additive group. The value $w(e)$ is referred to as the weight of $e$ (with respect to $w$). For any multiset of edges $E$, the weight of $E$ (with respect to an edge weighting $w$) is $w(E) = \sum_{e \in E} w(e)$. We use two distance measures on a digraph $D$ with a weighting $w$. For vertices $u, v$, we denote by $d^w(D, u, v)$ the maximum $w(W)$ among the walks $W$ from $u$ to $v$, while $d^w(D, u, v)$ denotes the maximum $w(W)$ among the paths $W$ starting at $u$ and ending at $v$. Note that $d^w(D, u, v) < \infty$ for every $u, v$, while $d^w(D, u, v) = d^w(D, u, v)$ when $D$ contains no cycle of positive weight [4, Section 24.1]. For the sake of notation, we omit the parameter $D$ when no ambiguities are possible.

A system of difference constraints is a system $S$ with $m$ linear inequalities and one equation over a set $x_0, \ldots, x_{n-1}$ of indeterminates. The unique equation of $S$ is $x_0 = 0$ while each of the difference constraints is an inequality of the form $x_j \geq x_i + c_{ij}$ for $i, j \in [n]$, where $c_{ij}$ is a constant. For each $i \in [n]$, $i \neq 0$, one of the inequalities is the non-negativity constraint $x_i \geq x_0 + 0$. The system $S$ defines a constraint digraph $D$ with $n$ vertices and $m$ edges that has a weighting sep. The digraph $D$ has a vertex $v_i$ corresponding to $x_i$, $i \in [n]$, and an edge $v_i v_j$ with weight $\text{sep}(v_i v_j) = c_{ij}$ corresponding to each inequality $x_j \geq x_i + c_{ij}$ of $S$. Vertex $v_0$ is the initial vertex of $D$. Clearly, $S$ is
fully determined by \(D\), \(sep\), and \(v_0\). The following well-known theorem gives a method to solve \(S\).

**Theorem 3** (e.g. [4, Theorem 24.9]). Let \(D\) be the constraint digraph of a system of difference constraints \(S\) with indeterminates \(x_0, \ldots, x_{n-1}\). Then, \(S\) has a feasible solution if and only if \(\text{sep}(W) \leq 0\) for every cycle \(W\) of \(D\). Moreover, if \(S\) has a feasible solution, then \(x_i = \text{dsep}(v_0, v_i)\) is a feasible solution to \(S\).

If \(S\) has \(m\) constraints, then the Bellman-Ford algorithm applied to \(D\) outputs in \(O(nm)\) time a set of values for \(x_0, \ldots, x_{n-1}\) or a cycle \(W\) of \(D\) with \(\text{sep}(W) > 0\). In the former case, we refer to \(x_i = \text{dsep}(v_0, v_i)\) as the canonical solution to \(S\). Say that \(S\) and a system \(S'\) are equivalent when they have the same canonical solution. An edge \(v_i v_j\) of \(D\) is implied by a path \(W\) from \(v_i\) to \(v_j\) when \(\text{sep}(W) \geq \text{sep}(v_i v_j)\); if the inequality is strict, then \(v_i v_j\) is strongly implied by \(W\). By definition, the digraph \(D'\) obtained by removing all the strongly implied edges of \(D\) defines a system equivalent to \(S\). Moreover, if every edge of \(D\) is implied by a path of a spanning subgraph \(D'\) of \(D\), then \(D'\) defines a system equivalent to \(S\).

In the above description, there is at most one inequality \(x_j \geq x_i + c_{ij}\) in \(S\) for each ordered pair \(x_i x_j\), while \(D\) is a digraph. Of course, there is no need for another inequality on \(x_i x_j\) as one of these would be strongly implied. Yet, for the sake of simplicity, it is sometimes convenient to describe a system with more than one constraint for each ordered pair \(x_i x_j\). In these situations, the corresponding constraint digraph \(D\) is a multidigraph. For the sake of notation we ignore this fact and regard the edge \(v_i v_j\) as representing both inequalities.

### 3 The synthetic graph of a model

The goal of \(k\)-**MULT** is to decide if a PCA model \(M\) is equivalent to a \(k\)-multiplicative model. In this section we define a compact system of difference constraints to solve the simpler problem \((k, c, \ell)\)-**MULT**: given \(M\), \(k \in \omega\), and even values \(c\) and \(\ell\), determine if \(M\) is equivalent to a \(k\)-multiplicative \((c, \ell + 1)\)-CA model.

By definition, if \(M\) is equivalent to an even \(k\)-multiplicative \((c, \ell + 1)\)-CA model \(U\), then \(i \times U\) is equivalent to \(M^i\) for every \(i \in [k]\). So, if \(A_0 < \ldots < A_{n-1}\) are the arcs of \(M\) and \(U_0 < \ldots < U_{n-1}\) are the arcs of \(U\), then \(\beta\{s(i \times U_i) \in i \times U_i\} = \beta\{s(A_i) \in A_i\}\) for every \(i \in [n]\). Moreover, \(U_x\) and \(U_y\) satisfy the following inequalities because \(s(i \times U_i) = s(U_i)\) is even whereas \(i\ell + 1\) is odd (e.g. Figs. 1 and 2):

\[
\begin{cases}
    s(U_x) \leq s(U_i) + i\ell - c\beta\{x \geq y\} & \text{for } i \in [k] \text{ if } s(A_i) \in A_i \\
    s(U_y) \geq s(U_i) + i\ell + 2 - c\beta\{x \geq y\} & \text{for } i \in [k] \text{ if } s(A_i) \not\in A_i
\end{cases}
\]

It is not hard to see that the converse of the previous reasoning is also true, regardless of whether \(c\) and \(\ell\) are even or odd. That is, if \(U\) is a \((c, \ell + 1)\)-CA model with arcs \(U_0 < \ldots < U_{n-1}\) that satisfy the above system, then \(U\) is \(k\)-multiplicative and equivalent to \(M\). Moreover, as the position of 0 in \(C(U)\) is irrelevant in the above inequalities, we can take \(s(U_0) = 0\). Therefore, a PCA model \(M\) is equivalent to a \(k\)-multiplicative UCA model if and only if the full system \(F_{k,c,\ell}(M)\) below has a solution; see Fig. 5. Moreover,
Lemma 1. If $M$ is a PCA model, $k \in \{0\}$, and $i \in [k]$, then every $i$-attract of $F^k$ that is not a hollow is strongly implied.

Proof. Suppose that an $i$-attract $B \rightarrow A$ is not a hollow, thus $i > 0$ and either $B \neq F_i^k(A)$ or $A \neq F_i^k(B)$. We prove only the former case, as the latter case is analogous. Hence,
\( s(B), s(R(B)), \) and \( t(F_i^r(A)) \) appear in this order in \( C(M) \) (Fig. 6(a)). By definition, \( R(B) \rightarrow A \) is an \( i \)-attract and \( B \rightarrow R(B) \) is a 0-repel. Moreover, either \( A \leq B \leq R(B) \) or \( R(B) \leq A \leq B \) or \( B \leq R(B) \leq A \), thus \( \beta \{ A \geq R(B) \} = \beta \{ B \geq R(B) \} = \beta \{ A \geq B \} \). Altogether, it follows that \( B \rightarrow A \) is strongly implied by the path \( W = B, R(B), A \) of \( F^k \) because

\[
\text{sep}_{c,i}(W) = 2 - c\beta \{ B \geq R(B) \} - i\ell + c\beta \{ A \geq B \}.
\]

\( \square \)

The fact that most \( i \)-repel edges are strongly implied can be proven with similar arguments. Say that an \( i \)-repel \( A \rightarrow B \) is an \( i \)-nose when \( A = H_i^r \circ L(B) \). The next lemma provides a symmetric definition for \( i \)-noses: \( A \rightarrow B \) is an \( i \)-nose when \( B = H_i^r \circ R(A) \). Colloquially, \( A \rightarrow B \) is an \( i \)-nose when \( A' \) is the rightmost arc not reaching \( B' \) and \( B' \) is the leftmost arc not reached by \( A' \).

**Lemma 2.** Let \( M \) be a PCA model, \( k \in (\omega), \) and \( i \in [k] \). The following statements are equivalent for \( A, B \in A(M) \), and each of them implies that \( A \rightarrow B \) is an \( i \)-repel of \( F^k \).

\[ \text{S1: } A = H_i^r \circ L(B). \]
\[ \text{S2: } F_i^w(A) = H_i^{r-w} \circ L(B) \text{ for every } w \in [i]. \]
\[ \text{S3: } \text{S2 and } R \circ F_i^x(A) = H_i^r \circ R(A) \text{ for every } x \in [i]. \]
\[ \text{S4: } F_i^r(B) = H_i^{i-r} \circ R(A) \text{ for every } z \in [i]. \]
\[ \text{S5: } \text{S4 and } L \circ F_i^r(B) = H_i^r \circ L(B) \text{ for every } y \in [i]. \]
\[ \text{S6: } B = H_i^r \circ R(A). \]

**Proof.** In this proof we use the following facts, and we write IH to reference the active inductive hypothesis, if any.

\[ \text{F1: } \text{If } X \in A(M) \text{ and } H_i(X) \neq \bot, \text{ then } F_i \circ H_i(X) = X. \]
\[ \text{F2: } \text{If } X \in A(M) \text{ and } H_i(X) \neq \bot, \text{ then } F_i \circ H_i(X) = X. \]
\[ \text{F3: } \text{If } X, Y \in A(M) \text{ and } X = H_i(Y), \text{ then } R(Y) = H_i \circ R(X). \]
\[ \text{F4: } \text{If } X, Y \in A(M) \text{ and } X = H_i(Y), \text{ then } L(Y) = H_i \circ L(X). \]

**Proof of Facts 1.** **F1** and **F2** follow directly from the definition of \( H_i \) and \( H_r \), respectively. For **F3**, observe that \( s(R(Y)) \) is the extreme immediately after \( t(X) \) in a clockwise traversal of \( C(M) \) (Fig. 6(b)). Then, \( F_i \circ R(Y) = R(X) \), thus \( R(Y) = H_i \circ R(X) \) because \( F_i(Y) \neq R(X) \). The proof of **F4** is omitted as it is analogous to that of **F3**. \( \triangle \)

\[ \text{S1 } \Rightarrow \text{S2 is proven by induction. The base case } w = 0 \text{ is trivial. For } w + 1 \leq i, \text{ note that } H_i^{r-w} \circ L(B) \neq \bot \text{ because } H_i^r \circ L(B) = A \neq \bot. \text{ Then, } H_i^{r-(w+1)} \circ L(B) \overset{\text{S2}}{=} F_r \circ H_i^{r-w} \circ L(B) = F_i \circ F_r^w(A) = F_i^{w+1}(A). \]

\[ \text{S2 } \Rightarrow \text{S3 is proven by induction. The base case } x = 0 \text{ is trivial. For } x + 1 \leq i, \text{ let } X = H_i^{r-x} \circ L(B) \text{ and } Y = H_i^{r-(x+1)} \circ L(B). \text{ By S2, } X \neq \bot \text{ and } Y \neq \bot, \text{ thus } X, Y \in A(M). \text{ Then, } H_i^{x+1} \circ R(A) = H_r \circ H_i^r \circ R(A) = H_r \circ R \circ F_i^x(A) \overset{\text{S2}}{=} H_r \circ R \circ H_i^{r-x} \circ L(B) = H_r \circ R(X) \overset{\text{F3}}{=} R(Y) = R \circ H_i^{r-(x+1)} \circ L(B) \overset{\text{S2}}{=} R \circ F_r^{x+1}(A). \]
\[ F^j(A) \quad B \quad R(B) \quad F^{j+1}(A) \]

(a)

\[ F^j(Y) \quad R(X) \quad Y \quad R(Y) \]

(b)

\[ F^j(A) \quad A' \quad X \quad F^j(X) \]

(c)

Figure 6: Removal of strongly implied constraints: (a) Lemma 1; (b) Lemma 2 (F3); (c) Lemma 3 for \( j < i \). In (c), \( F^j_{i-r}(A) = F_i(X) = H^{i-j-1}_i \circ L(B) \).

\[ S3 \Rightarrow S4 \] is proven by induction. If \( z = 0 \), then \( B = R \circ H^1_0 \circ L(B) \Rightarrow R \circ F^1(A) = H^1_0 \circ R(A) \), whereas if \( z + 1 \leq i \), then \( F^i_{i-r}(B) = F_i \circ F^i_r(B) \Rightarrow F_i \circ H^{z-i}_r \circ R(A) = H^{i-(z+1)}_i \circ R(A) \).

Implication \( S4 \Rightarrow S5 \) is analogous to \( S2 \Rightarrow S3; S5 \Rightarrow S6 \) is trivial; and the chain \( S6 \Rightarrow S4 \Rightarrow S5 \Rightarrow S2 \Rightarrow S3 \Rightarrow S1 \) is analogous to the chain \( S1 \Rightarrow S2 \Rightarrow S3 \Rightarrow S4 \Rightarrow S5 \Rightarrow S6 \). Finally, note that if any of the statements is true, then \( F^i_r(A) = F_i \circ H^1_0 \circ L(B) = L(B) \), thus \( A \to B \) is an \( i \)-repel of \( F^k \) by \( S2 \).

**Lemma 3.** If \( M \) is a PCA model, \( k \in \{\omega\} \), and \( i \in [\|k\|] \), then every \( i \)-repel of \( F^k \) that is not a nose is strongly implied.

**Proof.** Suppose that an \( i \)-repel \( A \to B \) is not a nose. By Lemma 2, \( F^j(A) \neq H^{i-j}_i \circ L(B) \) for some \( j \in [\|k\|] \). Among all the possible choices, take the one maximizing \( j \). Note that either \( j = i \) or \( F^{j+1}_r(A) = H^{i-j}_i \circ L(B) \). In the latter case \( H^{i-j}_i \circ L(B) \neq \perp \), while in the former case \( H^{i-j}_i \circ L(B) = L(B) \neq \perp \). So, regardless of whether \( i = j \), \( H^{i-j}_i \circ L(B) = X \neq \perp \) for some \( X \in A(M) \) (Fig. 6(c)). Moreover, \( s(R(X)) \notin A' \) because otherwise either \( i = j \) and \( A \to B \) is an \( i \)-attract or \( i < j \) and \( F^{j+1}(A) \neq F_i(X) = H^{i-j}_i \circ L(B) \) (Fig. 6(c)). Then, \( A \to X \) is a \( j \)-repel because \( F^j(A) \neq X \), i.e., \( s(X) \notin A' \) (Fig. 6(c)). By Lemma 2, \( X \to B \) is an \( (i-j) \)-repel. Altogether, \( A \to B \) is implied by the path \( W = A, X, B \) of \( F^k \) because either \( A \leq X \leq B \) or \( X \leq B \leq A \) or \( B \leq A \leq X \) and

\[
\text{sep}_{c,t}(W) = j\ell + 2 - c\beta \{ A \geq X \} + (i-j)\ell + 2 - c\beta \{ X \geq B \}
\]

\[ = i\ell + 4 - c\beta \{ A \geq B \} = 4 + \text{sep}_{c,t}(A \to B). \]

\[ \square \]

**Some (weakly) implied edges.** By Lemmas 1 and 3, the spanning subgraph \( F' \) of \( F^k \) formed by the hollows and noses, together with \( \text{sep}_{c,t} \), describes a system equivalent to \( F^k_{c,t} \). The digraph \( F' \) has \( O(kn) \) edges and it can be further simplified. For \( k \in \{\omega\} \), let \( S^k_{c,t}(M) \) be the spanning subgraph of \( F^k \) whose edges are the 1-hollows and \( i \)-noses of \( M \), for \( i \in [\|k\|] \).

**Lemma 4.** If \( M \) is a PCA model and \( k \in \{\omega\} \), then \( S^k_{c,t}(M) \), together with \( \text{sep}_{c,t} \), describes a system equivalent to \( F^k_{c,t} \).
Proof. For \( i > 1 \), consider an \( i \)-hollow \( A_i \rightarrow A_0 \) of \( \mathbf{F}^k \) and let \( A_j = F^i_j(A_0) \) for \( j \in \mathbb{Z} \). By definition \( A_{j+1} \rightarrow A_j \) is a 1-attract for \( j \in \mathbb{Z} \). Since \( k < \omega \), it follows that \( A_j \leq \ldots \leq A_2 \leq A_1 \leq \ldots \leq A_{j-1} \) for the unique \( j \in \mathbb{Z} \) such that 0 belongs to \( (s(A_{j-1}), s(A_j)) \) (with indices modulo \( i \)). Thus, \( A_0 \rightarrow A_i \) is implied by the path \( \mathbf{W} = A_i, \ldots, A_0 \) of \( \mathbf{F}^k \) because

\[
\text{sep}_{c, \ell}(\mathbf{W}) = \sum_{j=1}^{i} (c\beta \{ A_{j-1} \geq A_j \} - \ell) = c\beta \{ A_0 \geq A_i \} - \ell = \text{sep}_{c, \ell}(A_i \rightarrow A_0).
\]

If \( A_{j+1} \rightarrow A_j \) is not a 1-hollow for some \( j \in \mathbb{Z} \), then \( A_{j+1} \rightarrow A_j \) is strongly implied (Lemma 1) and, consequently, \( A_i \rightarrow A_0 \) is strongly implied. Otherwise, \( \mathbf{W} \) is a path of \( \mathbf{S}^k \) and the result follows by Lemma 1.

An argument similar to that of Lemma 4 can be used to remove most of the noses. Roughly speaking, the \( i \)-noses that must be kept have the largest possible \( i \) (note that \( i \) need not be equal to \( k \) because one cannot assure that a \( k \)-nose from \( A \) exists for every \( A \in A(M) \)). Say that an \( i \)-nose \( A \rightarrow B \) is short when \( i < k \) and either \( H_i(A) \neq \perp \) or \( H_i(B) \neq \perp \). Those \( i \)-noses that are not short are said to be long. For \( k \in \{0\} \), define \( S^k(M) \) as the spanning subgraph of \( S^i(M) \) obtained by removing all the short noses, and \( S^0(M) \) as the spanning subgraph of \( S^0(M) \) having the 0-noses and 1-hollows. Note that \( S^k = \bigcup \{ S^i | i \in [k] \} \); as usual, we omit the parameter \( M \). The digraph \( \mathbf{S}^k \) is called the \( k \)-order synthetic graph of \( M \).

Lemma 5. If \( M \) is a PCA model and \( k \in \{0\} \), then \( S^k(M) \), together with \( \text{sep}_{c, \ell} \), describes a system equivalent to \( \mathbf{F}^k_{c, \ell} \).

Proof. By Lemma 4, it suffices to prove that every \( i \)-nose \( A \rightarrow B \) is implied by a path of \( \mathbf{S}^k \). The proof is by induction on \( k - i \). The base case \( i = k \) is trivial because \( A \rightarrow B \) is long. For the inductive step, in which \( i < k \) and \( A \rightarrow B \) is short, we have that either \( H_i(A) \neq \perp \) or \( H_i(B) \neq \perp \). We prove only the former case, as the latter case is analogous. Thus, \( A \rightarrow H_i(A) \) is a 1-attract, whereas \( H_i(A) \rightarrow B \) is an \( (i+1) \)-repel by Lemma 2. Then, since \( i \in \{0\} \), we obtain that either \( H_i(A) \leq A \leq B \) or \( A \leq B \leq H_i(A) \) or \( B \leq H_i(A) \leq A \). Consequently, \( \mathbf{W} = A, H_i(A), B \) is a path of \( \mathbf{S}^k \) that implies \( A \rightarrow B \) because

\[
\text{sep}_{c, \ell}(\mathbf{W}) = -\ell + c\beta \{ H_i(A) \geq A \} + (i + 1)\ell + 2 - c\beta \{ H_i(A) \geq B \} = i\ell + 2 - c\beta \{ A \geq B \} = \text{sep}_{c, \ell}(A \rightarrow B).
\]

If \( A \rightarrow H_i(A) \) is not a hollow or \( H_i(A) \rightarrow B \) is not a nose, then \( A \rightarrow B \) is strongly implied by Lemmas 1 and 3. Otherwise, by induction, either \( A, H_i(A), B \) is a path of \( \mathbf{S}^k \) or \( H_i(A) \rightarrow B \) is a short nose and \( A \rightarrow B \) is implied by a path of \( \mathbf{S}^k \).}

Finally, the following corollary of Theorem 3 sums up this section.

Theorem 4. Let \( A_0 \) be the initial arc of a PCA model \( M \), \( k \in \{0\} \), and \( c, \ell \in \mathbb{N} \). If \( M \) is equivalent to an even \( k \)-multiplicative \( (c, \ell+1) \)-CA model, then \( \text{sep}_{c, \ell}(\mathbf{W}) \leq 0 \) for every cycle \( \mathbf{W} \) of \( \mathbf{F}^k \). Conversely, if \( \text{sep}_{c, \ell}(\mathbf{W}) \leq 0 \) for every cycle \( \mathbf{W} \) of \( \mathbf{S}^k \), then \( M \) is equivalent to the \( k \)-multiplicative \( (c, \ell+1) \)-CA model \( U \) that has an arc with beginning point \( d\text{sep}_{c, \ell}(A_0, A) \) for every \( A \in A(M) \).
The weighting of a walk. By Theorem 4, the weighting of each cycle of \( S^k \) plays a fundamental role in deciding if a PCA model \( M \) is equivalent to a \( k \)-multiplicative \((c, \ell + 1)\)-model; we find it useful to define \( \text{sep}_{c,\ell} \) as a linear function on \( c \) and \( \ell \). For a walk \( W \) of \( S^k \), let:

- \( \eta(W) \) and \( \mu(W) \) be the number of hollows and noses of \( W \), respectively,
- \( \eta_{\text{ext}}(W) \) and \( \mu_{\text{ext}}(W) \) be the number of external hollows and noses of \( W \), respectively,
- \( \mu(i, W) \) be the number of \( i \)-noses of \( W \),
- \( \text{bal}(W) = \sum_{i=0}^{k} i \mu(i, W) - \eta(W) \), and \( \text{ext}(W) = \eta_{\text{ext}}(W) - \mu_{\text{ext}}(W) \).

By definition, \( \text{sep}_{c,\ell}(B \rightarrow A) = -\ell + c \beta \{ A \geq B \} \) for every 1-hollow \( B \rightarrow A \), and also \( \text{sep}_{c,\ell}(A \rightarrow B) = i\ell + 2 - c \beta \{ A \geq B \} \) for every \( i \)-nose \( A \rightarrow B \). With the above terminology,

\[
\text{sep}_{c,\ell}(W) = \ell \text{bal}(W) + \text{ext}(W) + 2\mu(W). \tag{1}
\]

Intuitively, a walk \( W = A_1, \ldots, A_k \) can be seen as a traversal of the beginning points \( s(A_1) \), \ldots, \( s(A_k) \) of \( M \) in this order. The weight \( \text{sep}_{c,\ell}(A_i \rightarrow A_{i+1}) \) is a lower bound on how far \( s(A_i) \) and \( s(A_{i+1}) \) must be in \( C(M) \), thus \( \text{sep}_{c,\ell}(W) \) is a lower bound for the separation of \( s(A_1) \) and \( s(A_k) \). In this sense, \( \text{bal}(W) \) accumulates the separation according to \( \ell \), while \( \text{ext}(W) \) denotes the number of times that 0 is crossed in \( C(M) \), taking into account if the cross is in a clockwise (nose) or anticlockwise (hollow) sense.

### 3.1 Building the synthetic graph of a model

Lemma 2 implies that noses have a well defined structure, that is depicted by the gray arrows of Figure 7. To obtain this picture, suppose \( A_0 \rightarrow B_i \) is an \( i \)-nose of \( S^k \). By definition (i.e., \( S_1 \)), \( B_1 = H^j_i \circ R(A_0) \), thus \( B_j = H^j_i \circ R(A_0) \neq \perp \) for every \( j \in [k] \).

Similarly, \( A_0 = H^j_i \circ L(B_i) \) by \( S_6 \), thus \( A_j = H^{j-i}_{i-j} \circ L(B_i) \neq \perp \). That \( A_j = H_1(A_{j+1}) \) and \( B_{j+1} = H_1(B_j) \) when \( j < i \) follow by definition, while \( A_{j+1} = F_i(A_j) \) and \( B_j = F_i(B_{j+1}) \) follow by \( S_2 \) and \( S_4 \), respectively. Also, \( S_3 \) (or, symmetrically, \( S_5 \)) implies that \( B_j = R(A_j) \). Finally, \( S_2 \) and \( S_4 \) imply that \( F^k_i(A_0) \rightarrow R \circ F^i_1(A_0) \) is a \((j-h)\)-nose for every \( h \in [j] \). Summing up, the existence of an \( i \)-nose \( A \rightarrow R \circ F^i_1(A) \) implies the existence of many other noses in \( S^k \). Conversely, the 0-nose \( A \rightarrow R(A) \) generates the \( i \)-nose \( A \rightarrow R \circ F^i_1(A) \) when \( H^j_i \circ R(A) \neq \perp \).

By definition, any long \( i \)-nose \( A \rightarrow B \) of \( S^k \) with \( i < k \) is an \( i \)-nose of \( S^{k+1} \) as well. Therefore, to transform \( S^k \) into \( S^{k+1} \) it suffices to insert the \((k+1)\)-noses of \( S^k \) and to remove the \( k \)-noses of \( S^k \) that are short in \( S^{k+1} \). In contrast to Theorem 4, the next result holds for \( k = 0 \). This highlights the importance of \( S^0 \): it is the base case for building \( S^k \).

**Theorem 5.** Let \( M \) be a PCA model and \( k \in \{0, 1\} \). Then, \( S^{k+1} \) can be computed from \( S^k \) in two phases: first, each \( k \)-nose \( A \rightarrow B \) such that \( H_1(B) \neq \perp \) is iteratively replaced by the \((k+1)\)-nose \( A \rightarrow H_1(B) \); then, each remaining \( k \)-nose \( A \rightarrow B \) with \( H_1(A) \neq \perp \) is removed.
The problem of computing $S^k$ and bal, when a PCA model $M$ and a value $k \in \{0\}$ are given, can be solved in $O(n)$ time. With this information, $\text{sep}_{c,l}(A \rightarrow B)$ can be obtained in $O(1)$ time when $c, l \in \mathbb{N}$ are given, for $k \in \{0\}$.

**Proof.** We assume that $L(A)$, $R(A)$, $F_1(A)$, $F_r(A)$, $H_1(A)$, and $H_r(A)$ can be obtained in $O(1)$ time for a given $A \in \mathbf{A}(M)$. There is no loss of generality, as they can be computed in $O(n)$ time from any reasonable representation of $M$ (e.g. [25, Algorithm 5.2]). To find the 1-hollows of $S^k$ with bal = −1, it suffices to traverse each arc $A$ of $M$ to determine if $F_r \circ F_j(A) = A$. This step requires $O(n)$ time; we now discuss how to find all the $i$-noses.

Let $\mu$ be the function such that $\mu(A) = \max \{i \mid i \leq k \text{ and } H_j(A) \neq \perp\}$ and $H$ be the function such that $H(A) = H_j(A)$. $j = \mu(A)$, for every $A \in \mathbf{A}(M)$. By Lemma 2, there is a long nose $L(A) \rightarrow H(A)$ if and only if either $\mu(A) = k$ or $H_j \circ L(A) = \perp$. Moreover, if existing, $L(A) \rightarrow H(A)$ is the unique $\mu(A)$-nose of $S^k$ starting at $L(A)$. Thus, once $\mu$ and $H$ are known, the problem of finding each nose $A \rightarrow B$ of $S^k$, together with bal($A \rightarrow B$), can be accomplished in $O(n)$ time.

Let $D$ be the digraph that has one vertex $v(A)$ for each $A \in \mathbf{A}(M)$ and one edge $v(A) \rightarrow v(B)$ when $H_j(A) = B$. Clearly, every vertex of $D$ has at most one out neighbor. Moreover, since there is at most one arc $A$ such that $B = H_j(A)$ for every $B \in \mathbf{A}(M)$, it follows that all the vertices in $D$ have at most one in neighbor as well. Therefore, every component $D'$ of $D$ is either a path or a cycle. If $D'$ is a path $v(A_1), \ldots, v(A_f)$, then

$$
H_0^i \circ L(B_i) = L(B_0) = A_0
$$

$$
H_1^i \circ L(B_i) = L(B_{i-1}) = A_{i-1}
$$

$$
H_2^i \circ L(B_i) = L(B_1) = A_1
$$

$$
H_3^i \circ L(B_i) = L(B_i) = A_i
$$

$$
B_0 = R(A_0) ...
$$

$$
B_i = R(A_i) = H_j \circ R(A_0)
$$

Figure 7: Noses of $S^k$ from $A_0$ and to $R(A_i)$ for an $i$-nose $A_0 \rightarrow R(A_i)$ as implied by Lemma 2. A label $x$ is attached to each $x$-nose.

**Proof.** By Lemma 2, the graph obtained after the first phase has all the $(k+1)$-noses and every $k$-nose $A \rightarrow B$ satisfies $H_r(B) = \perp$. The second step removes the remaining short $k$-noses. 

Theorem 5 yields a simple method to compute $S^k$ in $O(nk)$ time when $M$ is given (Fig. 8). Starting with $S^0$, execute $k$ steps to transform $S^i$ into $S^{i+1}$ for each $0 \leq i < k$. However, Theorem 5 can be reinterpreted to design a faster algorithm. Just note that $A \rightarrow B$ is an $i$-nose of $S^k$ if and only if $B = H_j(R(A))$ and either $i = k$ or both $H_j(R(A)) = \perp$ and $H_r(A) = \perp$. For the next theorem, extend the weighting bal of $S^k$ to work with edges: bal$(A \rightarrow B) = -1$ if $A \rightarrow B$ is a 1-hollow and bal$(A \rightarrow B) = i$ if $A \rightarrow B$ is an $i$-nose, for $i \in \{1\}$. 

**Theorem 6.** The problem of computing $S^k$ and bal, when a PCA model $M$ and a value $k \in \{0\}$ are given, can be solved in $O(n)$ time. With this information, $\text{sep}_{c,l}(A \rightarrow B)$ can be obtained in $O(1)$ time when $c, l \in \mathbb{N}$ are given, for $k \in \{0\}$.
\( \mu(A_i) = \min\{k, j - i\} \) and \( H(A_i) = A_{i+\mu(A_i)} \). Similarly, if \( D' \) is a cycle \( v(A_1), \ldots, v(A_j) \), then \( \mu(A_i) = k \) and \( H(A_i) = A_{i+k \mod j} \). Then, by keeping two pointers, \( \mu \) and \( H \) can be computed for all the arcs corresponding to vertices in \( D' \) in \( O(j) \) time. Hence, the total time required to compute \( H \) and \( \mu \) is \( O(n) \) and, therefore, \( O(n) \) time suffices to compute \( S^k \) and bal. \( \square \)

**Corollary 1.** The problem \((k, c, \ell)\)-MULT can be solved in \( O(n^2) \) time for any PCA model \( M \), \( k \in \{\emptyset\} \), and \( c, \ell \in \mathbb{N} \). The algorithm outputs either a \( k \)-multiplicative \((c, \ell + 1)\)-CA model equivalent to \( M \) or a minimal family of difference constraints of \( S^k \) with no feasible solution.

**Proof.** By Theorem 4, \((k, c, \ell)\)-MULT is solved with an execution of Bellman-Ford’s algorithm on \( S^k \) weighted by \( \text{sep}_{c, \ell} \) from the initial arc of \( M \). The algorithm requires \( O(n^3) \) time because \( S^k \) has \( O(n) \) edges and can be computed in \( O(n) \) time. If a cycle of positive weight \( W \) is found, the corresponding family \( F \) of constraints is given as output. By Theorem 3, \( F \) has no solution because its constraint digraph is isomorphic to \( W \) (with weight \( \text{sep}_{c, \ell} \)), while \( F \) is minimal because the constraint digraph of \( F - F' \) has no cycles for every nonempty subsystem \( F' \). \( \square \)

If \( S^k \) has no feasible solution, then the family of difference constraints \( F \) given by Corollary 1 defines a submodel \( M' \) of \( M \) that contains an arc \( A \) of \( M \) if and only if \( A \) is referred by a constraint of \( F \). Note that \( M' \) is equivalent to no \( k \)-multiplicative \((c, \ell + 1)\)-CA model because \( S^k(M') \) contains \( F \).

### 4 Mitas’ drawing of a synthetic graph

[19] observed that \( S^1 \) admits a peculiar drawing in the plane when \( M \) is PIG. These drawings were later adapted to PCA models by [26], and provide a powerful tool for solving numerical representation problems [19, 21, 26, 27]. In this section we define an analogous of Mitas’ drawings for \( S^k \). Although drawings are defined for general PCA models, the results are restricted to connected models for simplicity.

In Mitas’ drawings, each arc \( A \) of a PCA model \( M \) occupies an entry of an imaginary matrix. The row of \( A \in A(M) \) is (Fig. 8):

\[
\text{row}(A) = \begin{cases} 
0 & \text{if } A \text{ is the initial arc} \\
\text{row}(B) + 1 & \text{if } A = H_i(B) \text{ for some } B < A \\
\text{row}(L(A)) & \text{otherwise}
\end{cases}
\]

The number of rows of \( M \) is defined as \( \text{rows}(M) = 1 + \max\{\text{row}(A) \mid A \in A(M)\} = 1 + \text{row}(L(A_0)) \) for the initial arc \( A_0 \). The family \( L \) of arcs with row = \( r \) is referred to as the row \( r \) of \( M \). It is not hard to see that \( L \) forms a contiguous sequence. We refer to those arcs that are the leftmost and rightmost of its row as being leftmost and rightmost, respectively. A nose \( A \rightarrow B \) is said to be backward when it is internal and \( A \) is rightmost. All the hollows and non-backward noses that are internal are called forward. A walk of \( S^k \) is internal if all its edges are internal and it is forward if all its
Figure 8: A PCA model $M$, $S^0$, and $S^1$ are shown from left to right. The vertices of $S^0$ and $S^1$ are displayed black in rows according to (row), while white vertices correspond to the black vertex with the same label. Noses are black, hollows are gray, double arrows are for external edges, and dashed arrows are for backward noses. Note that $S^1(M)$ can be obtained from $S^0(M)$ as in Theorem 5.

edges are forward. The interior and backbone of $S^k_i$ are the spanning subgraphs of $S^k_i$ formed by the interior and forward edges, respectively.

Besides helping with the drawing of $S^k$, the assignment of rows to arcs allows us to classify the hollows and noses according to how many rows are skipped when a hollow or nose is traversed. Specifically, the **jump** of an internal\(^1\) edge $A \to B$ of $S^k_i$ is the number $\text{jmp}(A \to B) = \text{row}(B) - \text{row}(A)$ of rows crossed by $A \to B$.

**Lemma 6.** If $M$ is a connected PCA model, $k \in [\omega)$, and $i \in [k]$, then internal hollows have $\text{jmp} = -1$, forward i-noses have $\text{jmp} = i$, and backward i-noses have $\text{jmp} = i + 1$.

**Proof.** If $A \to B$ is an internal 1-hollow, then $B = F_1(A)$ and $A = F_i(B)$, thus $H_i(B) \neq \bot$ is such that $H_i(B) \leq A$. Moreover, since $F_j(X) = F_i \circ H_i(B) = F_i(A) = B$ for every arc $X$ with $H_i(B) < X \leq A$, it follows that no arc $Z$ with $H_i(Z) = X$ exists. Consequently, $\text{row}(A) = \text{row}(H_i(B)) = \text{row}(B) + 1$ by (row) because $A \to B$ is internal.

If $A \to B$ is an internal $i$-nose, then $B = H_i \circ R(A)$ by Lemma 2. Observe that $H_i^{j+1} \circ R(A) > H_i^j \circ R(A)$ for every $j \in [\omega)$ because $M$ is connected, $R(A)$ is not the initial arc, and $A \to B$ is internal. Hence, by (row), $\text{row}(B) = \text{row}(H_i \circ R(A)) = \text{row}(H_i^{j+1} \circ R(A)) + 1 = \ldots = \text{row}(R(A)) + i$. Recall that $\text{jmp}(A \to R(A)) \in \{0, 1\}$ and it equals 0 if and only if $A \to R(A)$ is forward. Therefore, $\text{jmp}(A \to B) \in \{i, i + 1\}$ and it equals $i$ if and only if $A \to B$ is forward. \hfill $\square$

Lemma 6 can be extended to general walks. For this purpose, define the jump of an internal walk $W = A_1, \ldots, A_k$ as $\text{jmp}(W) = \sum_{i=1}^{k-1} \text{jmp}(A_i \to A_{i+1})$.

**Corollary 2.** Let $M$ be a connected PCA model and $k \in [\omega)$. If $W$ is an internal walk of $S^k_i$, then $\text{jmp}(W) = \text{bal}(W) + \mu_b$ where $\mu_b$ is the number of backward noses of $W$.

\(^1\)Although this definition can be properly applied to any edge of $S^k_i$, as it is on [26], in this work we will be concerned just with internal edges.
Proof. If \( f_i \) and \( b_i \) are the number of forward and backward \( i \)-noses of \( W \), respectively, then

\[
\text{jmp}(W) = \sum_{i=1}^{k} (i f_i + (i + 1) b_i) - \eta(W) = \sum_{i=1}^{k} i \mu(i, W) - \eta(W) + \sum_{i=1}^{k} b_i = \text{bal}(W) + \mu_b
\]

by Lemma 6.

The column \( \text{col}(A) \) of the arc \( A \) is defined according to a “topological ordering” of the backbone of \( S^k_{\phi} \). The fact that such an ordering exists follows from the next lemma.

Lemma 7. If \( M \) is a connected PCA model and \( k \in [\omega] \), then the backbone of \( S^k_{\phi} \) is acyclic.

Proof. It suffices to show that \( A_0 < A_j \) for any forward walk \( W = A_0, \ldots, A_j \) of \( S^k_{\phi} \) with \( j > 0 \) such that \( \text{row}(A_0) = \text{row}(A_j) \). The proof is by induction on \( \eta(W) \) and \( |W| \).

By Lemma 6, every edge of \( W \) is a 0-nose when \( \eta(W) = 0 \), thus \( A_i < A_{i+1} = R(A_i) \) for every \( i \in [j] \) and, hence, \( A_0 < A_j \). For the inductive step, consider the following alternatives.

Case 1: \( \text{row}(A_i) = \text{row}(A_0) \) for some \( i \in [j] \). Let \( W_0 \) and \( W_1 \) be the subpaths of \( W \) from \( A_0 \) to \( A_i \) and from \( A_i \) to \( A_j \), respectively. Clearly, \( \max\{\eta(W_0), \eta(W_1)\} \leq \eta(W) \) and \( \max\{|W_0|, |W_1|\} < j \), thus \( A_0 < A_i \) and \( A_i < A_j \) by induction.

Case 2: \( \text{row}(A_i) > \text{row}(A_0) \) for some \( i \in [j] \). By Lemma 6, there exists \( x, y \in [j] \), \( x \leq y \), such that \( A_{x-1} \rightarrow A_x \) is a \( p \)-nose, \( p > 0 \), and \( A_y \rightarrow A_{y+1} \) is a 1-hollow. Among all the possible combinations for \( x \) and \( y \), take one minimizing \( y - x \). In this configuration, \( \text{row}(A_i) = \text{row}(A_y) \) as every edge in the subpath \( A_x, \ldots, A_y \) of \( W \) is a 0-nose. By Lemma 2 (Fig. 7), \( F_i(A_x) = H^p_{x-1} \circ R(A_{x-1}) \) and \( A_{x-1} \rightarrow F_i(A_x) \) is a \((p - 1)\)-nose. The former condition implies that \( s(A_{x-1}), s(F_i(A_x)), s(A_x) \) appear in this order in a clockwise traversal of \( C(M) \). Then, taking into account that \( A_{x-1} < A_x \) because \( W \) is internal, we conclude that \( A_{x-1} < F_i(A_x) < A_x \) and, thus, \( A_{x-1} \rightarrow F_i(A_x) \) is internal as well. Moreover, \( A_{x-1} \rightarrow F_i(A_x) \) is forward because it starts at the same vertex as the forward edge \( A_{x-1} \rightarrow A_x \). Then, by Lemma 6 and recalling that \( A_y \rightarrow A_{y+1} \) is a 1-hollow, we obtain that \( \text{row}(F_i(A_x)) = \text{row}(A_y) - 1 = \text{row}(A_y) - 1 = \text{row}(A_{y+1}) \). And, since \( F_i(A_x) \leq F_i(A_y) = A_{y+1} \) because \( A_y \leq A_x \), the walk \( W_0 \) of 0-noses of \( S^k_{\phi} \) from \( F_i(A_x) \) to \( A_{y+1} \) is forward as well. Summing up, \( W' = A_0, \ldots, A_{x-1}, F_i(A_x) + W_0 + A_{y+1}, \ldots, A_j \) is a forward walk of \( S^k_{\phi} \). Then \( A_0 < A_j \) follows by induction because \( \eta(W') = \eta(W) - 1 \).

Case 3: \( \text{row}(A_i) < \text{row}(A_0) \) for every \( i \in [j] \). By Lemma 6, \( A_0 \rightarrow A_1 \) is a 1-hollow and \( A_{j-1} \rightarrow A_j \) is a \( p \)-nose, \( p > 0 \). By Lemma 2 (Fig. 7), \( A_j = H_r \circ F_i(A_j) \) and \( A_{j-1} \rightarrow F_i(A_j) \) is a \((p - 1)\)-nose that is forward because it starts at the same vertex as the forward edge \( A_{j-1} \rightarrow A_j \). Then, \( W' = A_1, \ldots, A_{j-1}, F_i(A_j) \) is a forward walk of \( S^k_{\phi} \). Clearly, \( |W'| > 1 \), \( \eta(W') = \eta(W) - 1 \) and, by Lemma 6, \( \text{row}(A_1) = \text{row}(A_0) - 1 = \text{row}(A_{j-1}) - 1 = \text{row}(F_i(A_j)) \). Then, \( A_1 \leq L \circ F_i(A_j) < F_i(A_j) \) follows by induction and, consequently, \( A_0 = F_r(A_1) \leq F_r \circ L \circ F_i(A_j) < H_r \circ F_i(A_j) = A_j \).

\qed
The column of an arc \(A\) of \(M\) is defined as:

\[
\text{col}_i(A) = \max(\{0\} \cup \{1 + \text{col}_i(B) \mid B \to A \text{ is a forward edge of } S_{\omega-1}^\phi\}).
\]

Note that \(\text{col}_i(A) \geq 0\) is well defined since the backbone of \(S_{\omega-1}^\phi\) is acyclic (Lemma 7).

The number of columns of \(M\) is \(\text{cols}(M) = 1 + \max\{\text{col}_i(A) \mid A \in A(M)\}\).

Let \(\text{col}_i(A) = i\text{cols}(M) + \text{col}_0(A)\) and \(\text{pos}_i(A) = (\text{col}_i(A), \text{row}(A))\) for every \(i > 0\) and \(A \in A(M)\). The (Mitas') drawing of each subdigraph \(D\) of \(S_{\omega-1}^\phi\) is obtained by placing, in \(\mathbb{R}^2\) and for every \(i \geq 0\), a straight \(i\)-arrow from \(\text{pos}_i(A)\) to \(\text{pos}_i(B)\) for each forward edge \(A \to B\) of \(D\) and a straight \(i\)-arrow from \(\text{pos}_i(A)\) to \(\text{pos}_{i+1}(B)\) for each backward nose \(A \to B\) of \(D\) (Fig. 9). We write \(v \to w\) to denote the arrow from \(v\) to \(w\) for \(v, w \in \mathbb{R}^2\) and, for simplicity, we say that \(v \to w\) is an arrow of \(D\) to mean that \(v \to w\) is an arrow corresponding to an edge of \(D\). For every \(p \geq 0\), every internal walk \(W = A_0, \ldots, A_j\) of \(D\) defines a curve \(\text{Gr}_p(W)\) in \(\mathbb{R}^2\) that starts at \(q_0 = \text{pos}_p(A_0)\) and, for \(x \in [\overline{0,j}]\), it takes the \(i\)-arrow of \(A_x \to A_{x+1}\) to move from \(q_x\) to \(q_{x+1}\) for the unique \(i\) such that \(q_x = \text{pos}_i(A_x)\).

**Corollary 3.** Let \(M\) be a connected PCA model. If \(W\) is an internal walk of \(S_{\omega-1}^\phi\) from \(A\) to \(B\) with \(b\) backward noses, then \(\text{Gr}_p(W), p \geq 0\), is the graph of a continuous function with domain \([\text{col}_p(A_0), \text{col}_{p+b}(A_j)]\).

The drawing of \(S^k\) is so attractive because it is “plane”, thus it provides a geometric framework to reason about PCA models.

**Theorem 7.** Let \(M\) be a connected PCA model. For every \(k < \omega\), two internal walks \(W\) and \(W'\) of \(S^k\) have a common vertex if and only if \(\text{Gr}_p(W)\) and \(\text{Gr}_q(W')\) share a point for some \(p, q \geq 0\). Furthermore, a vertex \(A\) belongs to \(W\) and \(W'\) if and only if \(\text{pos}_i(A)\) belongs to both \(\text{Gr}_p(W)\) and \(\text{Gr}_q(W')\) for some \(i, p, q \geq 0\).

**Proof.** Recall that for \(k < \omega\), \(S^k\) is the subgraph of \(S_k^\phi\) obtained by removing every short nose. That is, an \(i\)-nose \(A \to B\) is removed from \(S^k\) when \(i < k\) and either \(H_i(A) \neq \bot\) or \(H_i(B) \neq \bot\). Let \(S^k_\bot\) be the subgraph of \(S^k_k\) obtained by removing each \(i\)-nose \(A \to B\) such that \(i < k\), \(H_i(A) \neq \bot\), and \(H_i(B) \neq \bot\). Clearly, \(S^k_\bot\) is a supergraph of \(S^k\).

For technical reasons, we prove that the theorem holds even when \(S^k\) is replaced by \(S^k_\bot\), where \(S^0_\bot = S^0\). This stronger version of the theorem can be of interest in other.
applications. In this article we hide the definition of $S^k_r$ here, to avoid distractions in the main text.

Bending $\mathbb{R}^2$, we can identify the vertical lines passing through $\text{cols}(M)$ at the $x$ axis to obtain a cylinder $\mathbb{Y}$ in which $\text{pos}_j(A)$ is mapped to $\text{pos}_{q_0}(A)$ for every $A \in \mathcal{A}(M)$ and $i \geq 0$. This mapping transforms the drawing of $S^k_r$ from $\mathbb{R}^2$ to $\mathbb{Y}$, where each $i$-arrow for $A \rightarrow B$ is mapped to the 0-arrow of $A \rightarrow B$. Clearly, all the curves $\text{Gr}_p(W)$, $p \geq 0$, of an internal walk $W$ are mapped to the same curve of $\mathbb{Y}$. Thus, it suffices to show that $S^k_r$ has no crossing arrows when drawn in $\mathbb{Y}$ or, equivalently, that every edge has a corresponding $b$-arrow ($b \geq 0$) that is crossed by no other $b$-arrow. We prove the latter by induction on $k$.

For the base case $k = 0$ we observe four facts that together imply the no 0-arrow of $S^0_r = S^0$ is crossed by another 0-arrow. For $r \in \|\text{rows}(M)\|$, let $A_0^r < \ldots < A_q^r$ be the vertices of $S^0_r$ in row $r$ ($q$ depends on $r$). Recall that $A_i^r = R(A_i')$ for $i \in \|q\|$ by (row). Hence, $A_i' \rightarrow A_i'^{r+1}$ is a forward 0-nose and $W_r = A_0^r, \ldots, A_q^r$ is a path of the backbone of $S^0_r$. Fact 1 then follows by Corollary 3: $\text{Gr}_0(W_r)$ is the graph of the constant function $x \mapsto r$ in the domain $[\text{col}_0(A_1'), \text{col}_0(A_q')]$. Fact 2 follows by definition: if $r < \|\text{rows}(M)\| - 1$, then the 0-arrow corresponding to the backward 0-nose $A_i'^{-1} \rightarrow A_i'^{r+1}$ goes from $(\text{col}_0(A_i'), r)$ to $(\text{col}_1(A_i'^{-1}), r + 1)$. Fact 3 follows by Lemma 6: if $r > 0$, then the 0-arrow corresponding to a 1-hollow $A_i' \rightarrow B$ starts at $(\text{col}_0(A_i'), r)$ and ends at $(\text{col}_0(B), r - 1)$. Finally, since $B \leq B'$ when $A \rightarrow B$ and $A' \rightarrow B'$ are 1-hollows with $A \leq A'$, Facts 1 and 3 imply Fact 4: if $A_i' \rightarrow B$ and $A_i'^{-1} \rightarrow B'$ are hollows for $i < j$ (i.e., $\text{col}_0(A_i') < \text{col}_0(A_j')$), then $\text{row}(B) = \text{row}(B') = r - 1$ and $\text{col}_0(B) < \text{col}_0(B')$. Clearly, Facts 1–4 imply that no 0-arrow of $S^0_r$ is crossed by another 0-arrow.

For the inductive step we apply an algorithm to transform $S^k_r$ into $S^{k+1}_r$. In a first phase, the algorithm inserts a nose $A \rightarrow H_j(B)$ for every $k$-nose $A \rightarrow B$ of $S^k_r$ with $H_j(B) \neq \bot$. In a second phase, the algorithm removes those $k$-noses that do not belong to $S^{k+1}_r$. This algorithm is correct by Lemma 2. Clearly, the removal of edges and the insertion of external noses create no crossing arrows in the drawing of $S^{k+1}_r$. Then, to prove the inductive step, it suffices to consider only the first phase of the algorithm for the case in which the inserted nose is internal. Let $S_r$ be the graph obtained immediately after the $i$-th $(k + 1)$-nose was inserted. We show by induction on $i$ that some $b$-arrow corresponding to the inserted edge crosses no other $b$-arrows ($b \geq 0$). The case $i = 0$ follows by induction on $k$ because $S_0 = S^k_r$. For $i > 0$, suppose the $(k + 1)$-nose $A \rightarrow H_j(B)$ is inserted to obtain $S_i$ from $S_{i-1}$ (Fig. 10).

For $j \in \lfloor k + 1 \rfloor$, let $A_j = F_j^{r'}(A_0)$, $B_j = R(A_j)$, $X_j = F_j(A_{j+1})$, and $Y_j = F_j(B_j)$. If $j \leq k$, then $A_{j+1} \rightarrow X_j$ and $Y_{j+1} \rightarrow B_j$ are a 1-hollows because $F_j(X_j) = F_j(A_j) = A_{j+1}$ and $F_j(Y_{j+1}) = F_j(B_{j+1}) = B_j$. Thus, $A_{j+1} \rightarrow X_j$ and $Y_{j+1} \rightarrow B_j$ belong to $S_r$ (Fig. 10). Moreover, if $Z$ is a vertex with $X_j < Z < A_j$, then $F_j(Z) = F_j \circ R(Z) = F_j(A_j)$, thus $H_j(Z) = \bot$ and, consequently, $Z \rightarrow R(Z)$ is a 0-nose of $S^k_r$ and of $S_{i-1}$. That is, $S_{i-1}$ has a path of 0-noses from $X_j$ to $A_j$. Similarly, $S_{i-1}$ has a path of 0-noses from $B_j$ to $Y_{j+1}$ (Fig. 10). By Lemma 2, $H_j(B) = B_{k+1}$, thus $S_i$ has a path $W$ from $A_{k+1}$ to $B_0$ that contains the $(k + 1)$-nose $A_0 \rightarrow B_{k+1}$ together with every hollow $A_{j+1} \rightarrow X_j$, every hollow $Y_{j+1} \rightarrow B_j$, every path of 0-noses from $X_j$ to $A_j$, and every path of 0-noses from $B_{j+1}$ to $Y_{j+1}$, for $j \in \lfloor k \rfloor$. By Corollary 3, $\text{Gr}_0(W)$ is the graph of a partial function
from pos\(_0(A_{k+1})\) to pos\(_i(B_0)\), where \(b\) is the number of backward noses of \(W\) (Fig. 10 depicts the case \(b = 0\)).

Let \(b(j)\) be the number of backward noses of the subpath of \(W\) from \(A_{k+1}\) to \(A_j\) for \(j \in [k]\). Also, let \(\pi = 1\) if \(A_0 \rightarrow B_{k+1}\) is backward and \(\pi = 0\) otherwise. By definition, the arrow corresponding to \(A_0 \rightarrow B_{k+1}\) in \(Gr_0(W)\) is a \(b(0)\)-arrow that goes from pos\(_{b(0)}(A_0)\) to pos\(_{b(0)+\pi}(B_{k+1})\). By Lemma 2, \(B_{j+1} = H_r(B_j)\), thus row\((B_{j+1})\) = row\((B_j)\) + 1 by (row). Then, the subpath of \(W\) from \(B_{j+1}\) to \(B_j\) has exactly one 1-hollow and jmp = \(-1\) and, therefore, it has no backward noses by Corollary 2. Then, by induction, it follows that the arrow corresponding to \(Y_1 \rightarrow B_0\) in \(Gr_0(W)\) is a \((b(0) + \pi)\)-arrow, thus \(b = b(0) + \pi\).

By definition, \(B_j = R(A_j)\) for every \(j \in [k+1]\). Then, row\((A_j)\) is equal to either row\((B_j)\) or row\((B_j) - 1\), the latter being true if and only if \(B_j\) is leftmost. In other words, the subpath of \(W\) from \(A_{j+1}\) to \(A_j\) has jmp \(\in \{-1, 0\}\) and exactly one hollow. Thus, by Corollary 2, this subpath has either 0 or 1 backward noses and it has 1 backward nose only if row\((A_{j+1})\) = row\((A_j)\) = row\((B_j)\). Consequently, \(b(j) + \pi \leq 1\) follows by induction for every \(j \in [k]\). Altogether, this means that \(S^0\) has a \(b(j)\)-arrow from pos\(_{b(j)}(A_j)\) to pos\(_{b(j)}(B_j)\) that corresponds to the 0-nose \(A_j \rightarrow B_j\). As discussed in the base case, this implies that no vertex \(A\) in \(S_{i-1}\) has pos\(_p(A)\), \(p \geq 0\), in the interior of the polygon \(P\) whose borders are determined by the curve of \(Gr_0(W)\) from pos\(_0(A_{k+1})\) to pos\(_0(A_0)\), the curve of \(Gr_0(W)\) from pos\(_b(B_{k+1})\) to pos\(_b(B_0)\), the line from pos\(_b(A_0)\) to pos\(_b(B_0)\), and the line from pos\(_0(A_{k+1})\) to pos\(_b(B_{k+1})\) (Fig. 10).

By Lemma 2, \(A_1 \rightarrow B_{k+1}\) is a \(k\)-nose of \(S^0_k\) that also belongs to \(S_{i-1}\) because the second phase of the algorithm was not executed yet. Thus, there is a \(b(1)\)-arrow \(A_1 \rightarrow B_{k+1}\) in the drawing of \(S_{i-1}\) that, by construction, is completely inside \(P\). Similarly, the \(b(0)\)-arrow corresponding to the \(k\)-nose \(A_0 \rightarrow B_k\) is also inside in \(P\). Therefore, the sides of \(Q\) are arrows of the drawing of \(S_{i-1}\) for the polygon \(Q\) whose boundary is determined by the \(b(0)\)-arrow \(A_0 \rightarrow B_k\), the \(b(1)\)-arrow \(A_1 \rightarrow B_{k+1}\), the curve of \(Gr_0(W)\)
between \( \text{pos}_{0}^{(1)}(A_{1}) \) and \( \text{pos}_{0}^{(0)}(A_{0}) \) and the curve of \( \text{Gr}_{0}(W) \) between \( \text{pos}_{0}(B_{k+1}) \) and \( \text{pos}_{0}(B_{k}) \). By construction, \( Q \subseteq P \), thus no point in the interior of \( Q \) corresponds to \( \text{pos}_{p}^{(1)}(A) \) for every arc \( A \) of \( M \) and every \( p \geq 0 \) (Fig. 10). Then, by the inductive hypothesis, no arrow of \( S_{s-1} \) crosses the borders of \( Q \). Moreover, by Lemma 2, \( A_{1} \rightarrow B_{k} \) is the only edge of \( S_{k+1}^{k} \) with an arrow inside \( Q \), but it does not belong to \( S_{k}^{k} \) because \( H_{i}(A_{1}) = A_{0} \neq \bot \) and \( H_{i}(B_{k}) = B_{k+1} \neq \bot \). Consequently, no arrow of \( S_{s-1} \) intersects the interior of \( Q \) and, therefore, the \( b(0) \)-arrow \( A_{0} \rightarrow B_{k+1} \), that belongs to the interior of \( Q \) as it goes from \( \text{pos}_{0}^{(0)}(A_{0}) \) to \( \text{pos}_{0}^{(1)}(B_{k+1}) \) (Fig. 10), is crossed by no arrows of \( S_{s-1} \). Summing up, \( S_{i} \) has no pair of crossing arrows.

\[ \square \]

5 A simple and efficient algorithm for the multiplicative problem

In this section we devise a simple and efficient algorithm to solve \( k \)-MULT for an input PCA model \( M \). By Theorem 4, it suffices to determine if there exist \( c \) and \( \ell \) such that \( \text{sep}_{c,\ell}(W) \leq 0 \) for every cycle \( W \) of \( S_{i}^{i} \). One of the advantages of arranging the arcs of \( M \) into rows is that we can immediately conclude that \( \text{sep}_{c,\ell}(W) \leq 0 \) when \( W \) is internal and \( \ell \geq 2n \). Just note that: \( \text{ext}(W) = 0 \) because \( W \) is internal, \( \text{ JMP}(W) = 0 \) because \( W \) starts and ends at the same row, and \( W \) has at least one backward edge by Lemma 7. Then, \( \text{sep}_{c,\ell}(W) \leq -\ell + 2\mu(W) \leq 0 \) by Corollary 2 and (1). To deal with the cycles of \( S_{i}^{i} \) that have external edges, we create internal copies of these cycles via the loop unrolling technique.

Let \( c \) be the circumference of the circle of a PCA model \( M \) and \( \lambda \in \mathbb{N} \). The \( \lambda \)-unrolling of \( M \) (Fig. 11) is the PCA model \( \lambda \cdot M \) whose circle has circumference \( \lambda c \) that has \( \lambda \) arcs \( A_{0}, A_{1}, \ldots, A_{\lambda-1} \) for every \( A \in A \) such that, for \( i \in [\lambda] \):

\[
s(A_{i}) = s(A) + ic \quad \text{and} \quad t(A_{i}) = t(A) + c(i + \beta \{ s(A) > t(A) \}) \mod \lambda c.
\]

We refer to \( A_{i} \) as being a copy of both \( A \) and \( A_{j} \) (for \( j \in [\lambda] - \{ i \} \)); see Fig. 11. Also, we say that a row \( L \) of \( \lambda \cdot M \) is a copy of another row \( L' \) of \( \lambda \cdot M \) when \( |L| = |L'| \) and the \( i \)-th arc of \( L \) is a copy of the \( i \)-th arc of \( L' \) for every \( 0 \leq i \leq |L| \). An important feature of \( \lambda \cdot M \) is that it is highly repetitive when \( \lambda \) is large enough.

**Lemma 8.** Let \( M \) be a PCA model and \( r = \text{rows}(\lambda \cdot M) \) for \( \lambda > 0 \). If \( r \geq n \), then there exist \( x, z \in [n] - \{ 0 \} \) such that the row \( i + jz \) of \( M \) is a copy of the row \( i \) of \( M \) for every \( i \geq x, j \geq 0, \) and \( i + jz < r \).

**Proof.** Let \( L_{i} \) be the leftmost arc in the \( i \)-th row \( \lambda \cdot M \) for \( i \in [n] \). Since \( r \geq n \), there exist \( x, z \in [n] - \{ 0 \} \) such that \( L_{x+z} \) is a copy of \( L_{x} \). By (row), if \( L_{x} \) is a copy of \( L_{x+j} \), then \( L_{x} \) is a copy of \( L_{x+j} \) and, by induction, \( L_{x+j} \) is a copy of \( L_{x} \) for \( i \geq x, j \geq 0, \) and \( i + jz < r \). \[ \square \]

For \( k \in [\omega] \), we write \( \lambda \cdot S_{k}(M) \) as a shortcut for \( S_{k}^{k}(\lambda \cdot M) \) and we drop the parameter \( M \) when no confusions are possible. Following our naming convention, \( \lambda \cdot S_{k}^{k} \) has a vertex called \( A \) for each arc \( A \in \lambda \cdot M \). Thus, each vertex of \( \lambda \cdot S_{k}^{k} \) is a copy of an arc of \( M \). We say that a hollow (resp. nose) \( A' \rightarrow B' \) of \( \lambda \cdot S_{k}^{k} \) is a copy of the hollow (resp.
Thus, jmp a portion of the drawing. Inside this box, a dot labeled $\lambda$ to depict the drawing of when PCA (SPCA) models, i.e. PCA models that are not PIG, because $\lambda$ no interest in solving this system for $\lambda$. Thus, even though were internal walks of $A \to A$ for every $\lambda$, $\lambda$ is internal when $\lambda$ is PIG. For these models we can take advantage of Theorem 7 and Corollary 3.

Considering a large enough $\lambda$, we can manipulate all the circuits of $S^k$ as if they were internal walks of $\lambda \cdot S^k$. In this work this is the sole purpose of unrolled models. Thus, even though $\lambda \cdot S^k$ arises from its own system of difference constraints, we have no interest in solving this system for $\lambda \cdot M$. Moreover, we restrict ourselves to strict PCA (SPCA) models, i.e. PCA models that are not PIG, because $\lambda \cdot M$ is disconnected when $M$ is PIG. For these models we can take advantage of Theorem 7 and Corollary 3 to depict the drawing of $\lambda \cdot S^k$ (Figs. 12 to 16). A box labeled $\lambda \cdot S^k$ is used to frame a portion of the drawing. Inside this box, a dot labeled $A$ is used to represent $\text{pos}_p(A)$ for every $A \in M$ and $p \geq 0$. As $\text{pos}_p(A)$ is defined for every $p \geq 0$, different dots share a same label. Similarly, for a walk $W$ of $\lambda \cdot S^k$, $\text{Gr}_p(W)$ is depicted with an unlabeled curve; the identity of $\text{Gr}(W)$ can be decoded from the traversed dots. Dashed horizontal and vertical lines are used to represent rows and columns of $\lambda \cdot S^k$. The labels outside the box indicate the number of the corresponding row or column. Note

![Diagram of models](image-url)
that figures are out of scale because their purpose is to explain a behavior described in the text. Thus, these pictures as referred to as schemes.

Twister values are the key concept to determine if $\lambda$ is large enough. Say that $\gamma \in \mathbb{N}$ is a twister of $S^k$, if for every $\lambda \geq \gamma$, no walk $W$ of $S^k$ has a forward copy $T$ in $\lambda \cdot S^k$ with jmp$(T) \geq \gamma$. Although we do not prove it explicitly, the techniques in this section can be applied to show that, geometrically, if $\gamma$ is a twister and $T$ is a large path of $\lambda \cdot S^k$ moving upward (jmp$(T) \geq \gamma$), then Gr$_p(T)$ resembles a helix when drawn in the cylinder $\mathcal{Y}$ obtained by identifying the vertical lines passing through $i$cols$(M)$, $i \geq 0$. Theorem 8 applies a restricted notion of twisters as a tool to characterize those SPCA models that are equivalent to some $k$-multiplicative model. Specifically, $\gamma \in \mathbb{N}$ is a hollow (resp. nose) cycle twister of $S^k$ if for every cycle $W$ of $S^k$ with ext$(W) > 0$ (resp. ext$(W) < 0$) and every $\lambda \geq \gamma$ it happens that no internal copy of $\gamma \cdot W$ in $\lambda \cdot S^k$ is forward. For simplicity, $\gamma$ is a cycle twister of $S^k$ when $\gamma$ is either a nose or hollow cycle twister. Clearly, every twist is a nose cycle twister, and thus a cycle twister, because jmp$(T) \geq -\gamma ext(W)$ when $T$ is a copy of $\gamma \cdot W$, for every cycle $W$ with ext$(W) < 0$.

Before Theorem 8, Lemma 9 records the fact that the point 0 of $C(M)$ is crossed in the clockwise direction when enough noses are traversed. The analogous fact that 0 can be crossed in a counterclockwise direction follows as Corollary 4.

Lemma 9. If $M$ is a connected PCA model and $k \in \langle \omega \rangle$, then $S^k$ has a cycle with ext < 0.

Proof. Suppose $W$ is a circuit of $S^k$ with ext$(W) \geq 0$. As stated above, if $\lambda > |W|$, then $W$ has an internal copy $T$ in $\lambda \cdot S$ with jmp$(T) \leq 0$. By Corollary 2, bal$(T) = jmp(T) - b$ where $b$ is the number of backward noses in $T$. Clearly, $T$ is a circuit when jmp$(T) = 0$ because $T$ joins two copies of a same vertex of $W$. Then, bal$(T) < 0$ follows by Lemma 7 because $b > 0$ if $T$ is a circuit, while jmp$(T) < 0$ otherwise. By definition, $W$ and $T$ have the same number of noses and hollows; equivalently, bal$(W) = bal(T) < 0$. Consequently, by (1), sep$_{0.3h}(W) = 3n \cdot bal(W) + 2 \mu(W) < 0$.

Then, by Theorem 4, either $S^k$ has a cycle with ext < 0 or $M$ is equivalent to a $k$-multiplicative $(0, 3f)$-CA model. The latter is clearly impossible.

Theorem 8. The following statements are equivalent for an SPCA model $M$ and $k \in \langle \omega \rangle$.

1. $M$ is equivalent to a $k$-multiplicative model.
2. Some $\gamma \in \mathbb{N}$ is a cycle twister of $S^k$.
3. Any two cycles $W$ and $W'$ of $S^k$ with ext$(W) \cdot$ ext$(W') < 0$ have a vertex in common.

Proof. 1 $\Rightarrow$ 2. Let $h = \max\{10, n(k + 1)\}$ and suppose $\gamma = h^4$ is not a nose cycle twister of $S^k$. Then, there exists a cycle $W_N$ of $S^k$ with ext$(W_N) < 0$ and a value $\lambda \geq \gamma$ such that $\lambda \cdot S^k$ has a forward copy $T$ of $\gamma \cdot W_N$. For convenience, we first locate a forward copy of a portion of $T$ in a controlled location of $\lambda \cdot S^k$ (Fig. 12(a)). For this purpose, let $A_0$ be the first vertex of $T$ and $A_1, \ldots, A_{\gamma}$ be the other copies of $A_0$ in the order they are traversed by $T$. Note that $row(A_{i+1}) > row(A_i)$ for $i \in [\gamma]$ because ext$(W_N) < 0$ and, hence, row$(A_{4h}) \geq 4h$ (Fig. 12(a)). Then, by Lemma 8, $A_{4h}$ has a copy $X_0$ such that $row(X_0) \in [3h, 4h)$ and the row containing $X_0$ is a copy of the row containing $A_{4h}$ (Fig. 12(a)). For $i \in [h^3]$, let $T_i$ be the copy of $i \cdot W_N$ that begins at $X_0$ and $X_1, \ldots, X_i$ be
the other copies of $X_0$ in the order they are traversed by $T_i$. By Corollary 2, every internal edge of $\lambda \cdot S^k$ has $\text{jmp} \geq -1$. Thus, if $\text{row}(X_{i-1}) \geq 3h$, then all the edges traversed by $T_i$ between $X_{i-1}$ and $X_i$ are internal, thus all the vertices of $T_i$ have $\text{row} \geq 2h$ because $|W_N| \leq h$. Hence, $\text{row}(X_i) > \text{row}(X_{i-1})$ because $\text{ext}(W_N) < 0$. By induction, this means that $T_i$ is internal and traverses vertices with $\text{row} \geq 2h$. Altogether, Lemma 8 implies that the row containing the $j$-th vertex traversed by $T_i, j \in [\|T_i\|]$, is a copy of the row containing the $j$-th vertex visited by $T$ after $A_{4h}$. Then, every edge of $T_i$ is a copy of an edge of $T$ and, therefore, $T_i$ is forward as well (Fig. 12(a)).

Summing up, the previous paragraph proves that if $\gamma$ is not a cycle twister of $S^k$, then $\lambda \cdot S^k$ has a forward copy $T_N = T_h$ of $h \cdot W_N$ whose first vertex $X_0$ has $\text{row}(X_0) \in [3h, 4h)$. Similar arguments to those above imply that $S^k$ has a cycle $W_H$ with $\text{ext}(W_H) > 0$ such that $\lambda \cdot S^k$ has a forward copy $T_H$ of $(h^3) \cdot W_H$ whose last vertex $Y_0$ has $\text{row}(Y_0) \in [2h, 3h)$. Let $Y_0, \ldots, Y_{h^3}$ be the copies of $Y_0$ traversed by $T_H$ in the reverse order. (For the sake of notation, we write $h^i$ as a replacement of $h_i$ to avoid double subscripting.) Note that $\text{row}(Y_{i+1}) > \text{row}(Y_i)$ for $i \in [h^3]$ because $\text{ext}(W_H) > 0$.

By Lemma 6, every edge of $T_H$ has $\text{jmp} \in [-1, h]$. Thus, $T_H$ has an vertex $Z_i$ at $\text{row}(X_i)$ for every $i \in [h]$ because $\text{row}(Y_0) < 3h \leq \text{row}(X_0)$ and $\text{row}(X_i) \leq \text{row}(X_0) + h|W_N| < h^3 \leq \text{row}(Y_{h^3})$. Then, as $S^k$ has $n \leq h$ vertices, it follows that $Z_0$ and $Z_1$ are copies of the same vertex of $W_H$ for some $x, y \in [h]$, $x < y$. Summing up, $T_N$ has a forward subpath $K_N$ from $X_0$ to $X_{h^3}$, whereas $T_H$ has a forward subpath $K_H$ from $Z_0$ to $Z_h$ (Fig. 12(b)). By definition, $\text{row}(X_0) = \text{row}(Z_0)$ and $\text{row}(X_{h^3}) = \text{row}(Z_h)$. Then, by Corollary 2,

$$\text{bal}(K_N) = \text{jmp}(K_N) = \text{row}(X_{h^3}) - \text{row}(X_0) = -\text{jmp}(K_H) = -\text{bal}(K_H). \quad (*)$$

By construction, $K_N$ is a copy of $W_N = (y - x) \cdot W_N$. Thus, $K_N$ and $W_N'$ have the same number of noses and hollows and, therefore, $\text{bal}(K_N) = \text{bal}(W_N')$. Similarly, $K_H$ is a copy of $W_H' = z \cdot W_H$ for some $z \geq 0$, thus $\text{bal}(K_H) = \text{bal}(W_H')$. Moreover, $\text{ext}(W_N')$ equals one plus the number of copies of $X_0$ between $X_i$ and $X_{i+1}$ that, in turn, equals one plus the number $-\text{ext}(W_H')$ of copies of $Z_0$ between $Z_i$ and $Z_0$. Then, by (1), (\text{\textit{(*)}}), and
the fact that $W_N$ has at least one nose (because $y > x$), we obtain that:

$$\text{sep}_{c,\ell}(W_N) = \ell \text{bal}(W_N) + c \text{ext}(W_N) + 2\mu(W_N) > \ell \text{bal}(K_N) + c \text{ext}(W_N)$$

$$\text{sep}_{c,\ell}(W'_H) = \ell \text{bal}(W'_H) + c \text{ext}(W'_H) + 2\mu(W'_H) \geq -\ell \text{bal}(K_N) - c \text{ext}(W'_N),$$

for every $c, \ell > 0$. Then, $\text{sep}_{c,\ell}(W_N) + \text{sep}_{c,\ell}(W'_H) > 0$ and, by Theorem 4, $M$ is equivalent to no $k$-multiplicative $(c, \ell + 1)$-CA model regardless of the values of $c$ and $\ell$.

2 \Rightarrow 3. Suppose $\gamma \in \mathbb{N}$ is a nose cycle twister of $S^k$ and let $W_N$ be a cycle of $S^k$ with $\text{ext}(W_N) < 0$. If $\lambda > h^6$ for $h = \max\{10 + n(k + 1)\}$, then $h^3 \cdot W_N$ has a copy $T$ whose first vertex has row $\in (2h^3, 2h^3 + h)$. By Lemma 6, every edge of $\lambda \cdot S^k$ has jmp $\in [-1, h]$, thus $T$ is internal and all its vertices have row $\in (h^3, h^4)$. Moreover, as $\gamma$ is a nose cycle twister and $h \geq \gamma$, it follows that $T$ has at least $h^2$ backward edges, thus $T$ traverses $h$ leftmost copies $A_0, \ldots, A_h$ of some vertex of $W_N$ (Fig. 13(a)). If $A = A_0$ and $B \rightarrow A_h$ is the edge of $T$ to $A_h$, then $\text{row}(B) > \text{row}(A)$ because $\text{row}(A_h) \geq \text{row}(A) + h$ as $\text{ext}(W_N) < 0$. Hence, the walk $T_N$ from the leftmost vertex $A$ to the rightmost vertex $B$ traverses vertices with row $\in (h^3, h^4)$, starting from row $A$ and ending at row $B$ (Fig. 13(a)).

Let $b - 1$ be the number of backward edges in $T_N$ and $j \geq 0$. By Corollary 3, the curve $Gr_{b,j}(T_N)$ depicts a continuous function $f_j$ with domain $[\text{col}_{b,j}(A), \text{col}_{b,j+1}(B)]$ that is bounded below and above by the constant functions $x \mapsto h^3$ and $x \mapsto h^4$, respectively (Fig. 13(b)). Moreover, if the curve of $f_j$ is extended with the line $L_j$ from $\text{pos}_{b,j+1}(B)$ to $\text{pos}_{b,j}(A)$, then the curve of a continuous function $g_j$ with domain $[\text{col}_{b,j}(A), \text{col}_{b,j+1}(A)]$ is obtained (Fig. 13(b)). Therefore, $g = \bigcup_{j \geq 0} g_j$ is a continuous function with domain $[\text{col}_0(\lambda), \infty)$ that is bounded below and above by $x \mapsto h^3$ and $x \mapsto h^4$ (Fig. 13(b)).

Let $W_H$ be any cycle with $\text{ext}(W_H) > 0$. For $i \geq 0$, let $T_i$ be the copy of $i \cdot W_H$ that starts at a vertex $X$ with row $(X) \in [h^3, h^4 + h]$. Similarly as above, Lemma 6 implies that $T_i$ is internal when its last vertex has row $> h^2$. Then, there exists $i$ such that $T_H = T_i$ is internal and ends at a vertex $Y$ with row $(Y) < h^2$. By Corollary 3, $Gr_1(T_H)$ is the
curve of a continuous function that starts at \( \text{pos}_1(X) \) and ends at a point with ordinate \( \text{row}(Y) \). Since \( \text{row}(X) \geq h^2 \) and \( \text{row}(Y) < h^2 \), it follows that \( \text{Gr}_j(T_H) \) crosses the graph of \( g \) at some point (Fig. 13(b)). Among such crossing points, let \( p = (x, y) \) be the one minimizing \( x \). We claim that \( p \) belongs to the graph of \( f_j \) for some \( j \geq 0 \). Otherwise, \( p \) would be a point of \( L_j \) for some \( j \geq 0 \). As \( L_j \) starts at \( \text{pos}_{b(j+1)-1}(B) \) and ends at \( \text{pos}_{b(j)-1}(A) \) and \( \text{row}(B) > \text{row}(A) \), Lemma 6 implies that the edge \( X' \rightarrow Y' \) of \( T_H \) whose arrow contains \( p \) satisfies \( \text{row}(B) < \text{row}(X') \) and \( \text{row}(Y') > \text{row}(A) \). As this is impossible, because the first cross between \( g \) and \( \text{Gr}_j(T_H) \) happens at a point in which the slope of \( \text{Gr}_j(T_H) \) is smaller (Fig. 13(b)), it follows that \( p \in \text{Gr}_j(T_N) \cap \text{Gr}_j(T_H) \). Then, \( T_N \) and \( T_H \) have a vertex in common by Theorem 7, and so do \( W_N \) and \( W_H \).

The proof for the case in which \( \gamma \) is a hollow cycle twister is similar, thus we omit it for the sake of succinctness. We remark, however, that every cycle twister of \( S^K \) is a nose cycle twister (this is proven later in Theorem 9).

3 \( \Rightarrow \) 1. By Lemma 9, \( S^K \) has a cycle \( W \) with \( \text{ext}(W) < 0 \). Let \( \ell = 4n^2 \) and \( c \) be the minimum such that \( \text{sep}_{c,\ell}(W) \leq 0 \) for every cycle \( W \) of \( S^K \) with \( \text{ext}(W) < 0 \). Note that \( c \) exists because \( c \) can be as large as to bound all the other terms of (1). Moreover, as \( c \) is minimum, there exists a cycle \( W_N \) with \( \text{ext}(W_N) < 0 = \text{sep}_{c,\ell}(W_N) \). We prove that \( \text{sep}_{c,\ell}(W) \leq 0 \) for every cycle \( W \) of \( S^K \) and, therefore, \( M \) is equivalent to a \( k \)-multiplicative \((c, \ell + 1)\)-CA model by Theorem 4. Consider the following possibilities for \( \text{ext}(W) \).

**Case 1:** \( \text{ext}(W) < 0 \), thus \( \text{sep}_{c,\ell}(W) \leq 0 \) by the definition of \( c \).

**Case 2:** \( \text{ext}(W) = 0 \). If \( \lambda > |W| \), then \( W \) has an internal copy \( T \) in \( \lambda \cdot S^K \). Clearly, \( \text{jmp}(T) = 0 \) because \( \text{ext}(W) = 0 \). Hence, \( T \) is a circuit that has \( b > 0 \) backward edges by Lemma 7. Moreover, \( \text{bal}(T) = \text{bal}(W) \) because they traverse the same number of noses and hollows. Then, by (1) and Corollary 2,

\[
\text{sep}_{c,\ell}(W) = \ell \text{bal}(W) + c \text{ext}(W) + 2\mu(W) = \ell \text{bal}(T) + 2\mu(W) = -\ell b + 2\mu(W) \leq 0.
\]

**Case 3:** \( \text{ext}(W) > 0 \). By hypothesis, \( W \) and \( W_N \) have a vertex \( A \) in common. Starting at \( A \), let \( W_0 = \text{ext}(W) \cdot W_N + |\text{ext}(W_N)| \cdot W \), i.e., \( W_0 \) is the circuit of \( S^K \) that begins at \( A \), repeatedly traverses \( \text{ext}(W) \) times \( W_N \) and then it repeatedly traverses \( |\text{ext}(W_N)| \) times \( W \). Observe that \( \text{ext}(W_0) = 0 \) by construction, whereas \( \text{ext}(W) \leq |W| \leq n \) and \( |\text{ext}(W_N)| \leq |W_N| \leq n \) by definition. Thus, \( 2|W_0| \leq 4n^2 = \ell \). Then, if \( \lambda > \ell^2 \), we obtain that \( W_0 \) has an internal copy \( T_0 \) in \( \lambda \cdot S^K \). Note that \( T_0 \) is a circuit because \( W_0 \) is a circuit with \( \text{ext}(W_0) = 0 \) and, thus, \( \text{jmp}(T_0) = 0 \). Hence, \( T_0 \) has \( b > 0 \) backward noses by Lemma 7 and, consequently, \( \text{bal}(T_0) = -b < 0 \) by Corollary 2. Then, by (1) and the fact that \( \text{bal}(W_0) = \text{bal}(T_0) \) because \( T_0 \) is a copy of \( W_0 \), we obtain that:

\[
|\text{ext}(W_N)| \cdot \text{sep}_{c,\ell}(W) = |\text{ext}(W_N)| \cdot \text{sep}_{c,\ell}(W) + \text{ext}(W) \cdot \text{sep}_{c,\ell}(W_N) = \\
= \text{sep}_{c,\ell}(W_0) = \ell \text{bal}(W_0) + 2\mu(W_0) \leq -\ell + 2|W_0| \leq 0.
\]

**Corollary 4.** If \( M \) is an SPCA model and \( k \in \langle 0 \rangle \), then \( S^K \) has a cycle with \( \text{ext} > 0 \).
Proof. We refer to the proof of Theorem 8 (3 ⇒ 1), where \( \ell = 4n^2 \) and \( c \) is defined as the minimum such that \( \text{sep}_{d,c}(W) \leq 0 \) for every cycle \( W \) of \( S^k \) with \( \text{ext}(W) < 0 \). Suppose that every cycle of \( S^k \) has \( \text{ext} \leq 0 \) and let \( W \) be any cycle of \( S^k \) and \( d = c\ell n^2 \). If \( \text{ext}(W) < 0 \), then \( \text{sep}_{d,c}(W) \leq 0 \) because \( d \geq c \), while if \( \text{ext}(W) = 0 \), then \( \text{sep}_{d,c}(W) \leq 0 \) as in Case 2 of Theorem 8 (3 ⇒ 1). Then, by Theorem 4, \( M \) is equivalent to a \((d, \ell)\)-CA model \( U \). Clearly, some point of \( C(U) \) is crossed by no arc of \( U \) because \( d > n \ell \).

But this is impossible, as it implies that \( U \) is a UIG model and so is \( M \) because it is equivalent to \( U \). \( \square \)

By Theorem 8, \( k \)-MULT can be solved by checking that every pair of cycles \( W_N \) and \( W_H \) of \( S^k \) with \( \text{ext}(W_N) < 0 < \text{ext}(W_H) \) have a vertex in common. Theorem 9 yields an efficient method in which only two cycles are traversed (Corollary 5). Moreover, Theorem 9 generalizes Theorem 8 by replacing the restricted notion of cycle twisters with the general notion of twisters. Greedy walks play a central role in this theorem, as they do in the characterization by [29] (see [26]). A walk \( W = B_0, \ldots, B_j \) of \( S^k \) is greedy hollow (resp. greedy nose) when \( S^k \) has no hollows (resp. noses) from \( B_i \) when \( B_i \to B_{i+1} \) is a nose (resp. hollow), for \( i \in \[ j \] \). By Theorem 5, at most two edges of \( S^k \) begin at \( B_i \), one nose and one hollow. Thus, in other words, \( W \) is greedy hollow (resp. nose) in \( S^k \) if hollows (resp. noses) are preferred over noses (resp. hollows) when choices are possible. As hollows (resp. noses) have \( \text{jmp} < 0 \) (resp. \( \text{jmp} \geq 0 \)), greedy hollow (resp. nose) cycles usually have \( \text{ext} > 0 \) (resp. \( \text{ext} \leq 0 \)). This could be false, as a greedy hollow cycle is also greedy nose when no choices are possible (e.g. Fig. 17).

Lemma 10 proves that at least one greedy hollow cycle with \( \text{ext} > 0 \) and one greedy nose cycle with \( \text{ext} < 0 \) exist.

Lemma 10. If \( M \) is a connected PCA model and \( k \in (\omega) \), then \( S^k \) has a greedy nose cycle with \( \text{ext} < 0 \). Furthermore, if \( M \) is SPCA, then \( S^k \) has a greedy hollow cycle with \( \text{ext} > 0 \).

Proof. By Lemma 9, \( S^k \) has a cycle \( W_N \) with \( \text{ext}(W_N) < 0 \). Let \( h = \max \{10, n(k+1)\} \). If \( \lambda \geq h^4 \), then \( \lambda \cdot S^k \) has an internal copy \( T_N \) of \( h^2 \cdot W_N \) that starts at some copy \( A_0 \) of a vertex \( A \) of \( W_N \). Note that \( T_N \) has a vertex with \( \text{row} \geq \text{row}(A_0) + h^2 \) because \( \text{ext}(W_N) < 0 \). Then, by Corollary 3, the co-domain of the function whose graph is \( \text{Gr}(T_N) \) contains \( \{\text{row}(A_0), \text{row}(A_0) + h^2\} \) (Fig. 14(a)).

For \( i \geq 0 \), let \( G_i \) be the unique greedy nose walk of \( S^k \) that starts at \( A \) and has \( i \) vertices, and \( T_i \) be the copy of \( G_i \) in \( \lambda \cdot S^k \) that starts at \( A_0 \). Note that if \( \text{pos}_{p}(X) \in \text{Gr}(T_i) \cap \text{Gr}(T_N) \) for some vertex \( X \) and \( p \geq 0 \), then the slope of the \( p \)-arrow leaving \( \text{pos}_{p}(X) \) in \( \text{Gr}(T_i) \) is not smaller than the slope of the \( p \)-arrow leaving \( \text{pos}_{p}(X) \) in \( \text{Gr}(T_N) \). Otherwise, by Lemma 6, \( T_i \) would take a hollow from \( X \), whereas \( T_N \) would take a nose from \( X \), contradicting the fact that \( G_i \) is greedy. Then, as \( \text{pos}_{0}(A_0) \in \text{Gr}(T_i) \cap \text{Gr}(T_N) \), induction and Theorem 7 imply that \( \text{Gr}(T_i) \) is bounded below by \( \text{Gr}(T_N) \) for every \( i \) such that \( T_i \) is internal (Fig. 14(a)). This means that \( \text{Gr}(T_i) \) reaches \( \text{row}(A_0) + h^2 \) for some \( i \) sufficiently large, thus \( T_i \) has a vertex \( B \) with \( \text{row}(B) \geq \text{row}(A_0) + h^2 \) (Fig. 14(a)). Then, as \( S^h \) has \( n \leq h \) vertices, it follows that \( T_i \) contains a subwalk with \( \text{jmp} > 0 \) joining two copies of a same vertex of \( S^k \). The corresponding subpath of \( G_i \) is a greedy nose cycle \( G_N \) that has a copy in \( \lambda \cdot S^k \) with \( \text{jmp} > 0 \), i.e., \( \text{ext}(G_N) < 0 \).
Figure 14: Schemes for Lemma 10. In (a) the black walk has a copy of a greedy nose cycle, in (b) the black walk has a copy of a greedy hollow cycle.

Regarding the case in which $M$ is SPCA, recall that $S^k$ has a cycle $W_H$ with $\text{ext}(W_H) > 0$ by Corollary 4. Arguments similar to those above, where the role of $W_N$ is played by $W_H$, allows us to conclude that $S^k$ has a greedy hollow cycle $G_H$ with $\text{ext}(G_H) > 0$. We omit the details for the sake of succinctness; see Fig. 14(b). \qed

**Theorem 9.** The following statements are equivalent for an SPCA model $M$ and $k \in \omega$.

1. $M$ is equivalent to a $k$-multiplicative model.
2. Some greedy nose cycle of $S^k$ having $\text{ext} < 0$ shares a vertex with a greedy hollow cycle of $S^k$ having $\text{ext} > 0$.
3. Some $\gamma \in \mathbb{N}$ is a twister of $S^k$.

**Proof.**

1 $\Rightarrow$ 2. By Lemma 10, $M$ has a greedy nose cycle $G_N$ with $\text{ext} < 0$ and a greedy hollow cycle $G_H$ with $\text{ext} > 0$. By Theorem 8, $G_N$ and $G_H$ have a vertex in common.

2 $\Rightarrow$ 3. Let $\lambda \geq \gamma = h^k$ for $h = \max\{10, (k+1)n\}$, and $A_0$ be a copy of $A$ in $\lambda \cdot S^k$ with $\text{row}(A_0) \in (2h^2, 2h^2 + h]$. By hypothesis, a greedy hollow cycle $G_H$ of $S^k$ having $\text{ext}(G_H) > 0$ shares a vertex $A$ with a greedy nose cycle $G_N$ of $S^k$ having $\text{ext}(G_N) < 0$. Define $T_N$ as the copy of $2\text{ext}(G_H) \cdot G_N$ in $\lambda \cdot S^k$ that starts at $A_0$ and ends at a copy $A_2$ of $A$ (Fig. 15). As usual, Observation 6 implies that $T_N$ is an internal walk whose vertices have row $\in (h^2, h^2]$. Similarly, the copy $T_H$ of $2|\text{ext}(G_N)| \cdot G_H$ in $\lambda \cdot S^k$ that starts at $A_2$ is also internal and ends at $A_0$ (Fig. 15). Let $a$ and $b$ be the number of backward noses of $T_N$ and $T_H$, respectively. Clearly, both $T_N$ and $T_H$ pass through another copy $A_1$ of $A$ with $\text{row}(A_0) < \text{row}(A_1) < \text{row}(A_2)$ (Fig. 15). Thus, the walk obtained by traversing $T_N$ from $A_i$ to $A_{i+1}$, $i \in [2]$, and then traversing $T_H$ from $A_{i+1}$ to $A_i$ is an internal circuit. By Lemma 7, this circuit has at least one backward edge, hence $a + b \geq 2$.

We claim that $b = 0$, i.e., $T_H$ has no backward edges. Contrary to our claim, suppose $X \rightarrow Y$ is a backward nose in $T_H$. By definition, $X$ is rightmost, thus $R(X)$ is leftmost and, by (row), $R(X) = H_r(Z) = H_r \circ R \circ L(Z)$ for some vertex $Z$. Then, by Lemma 2, $H_l(X) = L(Z) \neq \perp$, thus $F_l \circ F_l(X) = X$. In other words, $X \rightarrow F_l(X)$ is a
1-hollow. But this is impossible if $X \rightarrow Y$ is a backward nose, because $T_H$ is greedy hollow. Hence, $b = 0$ and, therefore, $T_N$ has $a \geq 2$ backward edges.

To prove that $\gamma$ is a twister of $S^k$, we have to show that any internal copy $T$ of a walk $W$ of $S^k$ with jmp($T$) $\geq \gamma$ has a backward edges. To prove this, we use a copy $K$ of $T$ whose drawing is below $T_N$. Let $X_0$ be the last vertex of $T$ with row($X_0$) $\leq 3h$. If $T$ has no such vertices, then let $X_0$ be a vertex with minimum row in $T$. By definition, the subwalk $T'$ of $T$ from $X_0$ has jmp($T'$) $\geq \gamma - \lambda h \geq \lambda h^3$ and visits vertices with row $\geq 2h$. Then, by Lemma 8, $X_0$ has a copy $Y_0$ with row($Y_0$) $\in (h, 2h]$ such that the row containing $Y_0$ is a copy of the row containing $X_0$. Moreover, if $K$ is the copy of $T'$ from $Y_0$, then the $i$-th vertex of $K$ belongs to a row that is a copy of the row containing the $i$-th vertex of $T'$ that is traversed after $X_0$. Then, $T$ has at least as many backward edges as $K$.

By Corollary 3, $Gr_0(T_N)$ is a curve that joins pos$_0(A_0)$ and pos$_a(A_2)$. As in the proof of Lemma 10, Theorem 7 together with the facts that $T_N$ is greedy and row($Y_0$) $< row(Z)$ for every $Z \in T_N$, implies that $Gr_1(K)$ is bounded above by $Gr_0(T_N)$. Hence, since $K$ ends at a row greater than $h^3$ (because jmp($K$) $\geq h^3$), it follows that $Gr_1(K)$ crosses 2cols($M$), thus $K$ has a backward edge, and so does $T$ as desired (Fig. 15(b)).

3 $\Rightarrow$ 1 follows by Theorem 8 because every twister is a cycle twister. □

**Corollary 5.** The problem $k$-MULT can be solved in $O(n)$ time for every PCA model $M$ and every $k \in \{\omega\}$. If the output is no, then a negative certificate that can be authenticated in $O(n)$ time is obtained as a by-product.

**Proof.** If $M$ is PIG, then the algorithm outputs yes because $M$ is $\infty$-multiplicative [5, 10] (see Theorem 10 for an alternative proof). Otherwise, $S^k$ is built in $O(n)$ time with Theorem 6. Then, a subgraph $S_H$ (resp. $S_N$) of $S^k$ is obtained in $O(n)$ time by removing each nose (resp. hollow) $A \rightarrow B$ when a hollow (resp. nose) $A \rightarrow X$ exists. By construction, the walks of $S_H$ (resp. $S_N$) are precisely the greedy hollow (resp. nose) walks of $S^k$. By Lemma 2 and Theorem 5, at most one hollow (resp. nose) begins at each vertex $A$, thus the family of greedy hollow (resp. nose) cycles is obtained from $S_H$ (resp. $S_N$) in $O(n)$ time. By Lemma 10, at least one of these greedy cycles has ext $> 0$ (resp. ext $< 0$). Let $G_N$ and $G_H$ be greedy cycles with ext($G_N$) $< 0 <$ ext($G_H$) that are
computed in $O(n)$ time. The algorithm outputs yes if and only if $G_N$ and $G_H$ have a vertex in common, a fact that can be checked in $O(n)$ time. The algorithm is correct by Theorems 8 and 9. When the output is no, the pair of cycles $(G_H, G_N)$ is returned. To authenticate this certificate, $S^k$ is generated in $O(n)$ time to verify that $G_H$ and $G_N$ are cycles that have no vertices in common.

If $M$ is PIG, then $M$ is $\omega$-multiplicative. If $G(M)$ is co-bipartite, then $\omega = 2$ and $M$ is 1-multiplicative [29]. Finally, if $M$ is SPCA and $G(M)$ is not co-bipartite, then $M$ is the unique PCA model representing $G(M)$, up to equivalence and full reversal [13]. In this last case, the pair of cycles $(G_H, G_N)$ with $\text{ext}(G_H) \cdot \text{ext}(G_N) < 0$ and no vertices in common defines a submodel $F = (C(M), \{A \in A(M) | A \in G_H \cup G_N\})$ of $M$ that is equivalent to no $k$-multiplicative UCA model. Therefore, Theorem 8 implies a characterization by forbidden induced subgraphs for the class of PCA graphs that admit $k$-multiplicative UCA models. For $k = 1$, this is the characterization by [29].

6 A certifying algorithm for the multiplicative problem

Suppose $k$-MULT answers yes for an SPCA model $M$, and let $U$ be a $k$-multiplicative $(c, 2n^2 + 1)$-CA model equivalent to $M$, where $c$ is the minimum such that $\text{sep}_{c\ell}(W) \leq 0$ for every cycle $W$ with $\text{ext}(W) < 0$. The existence of $U$ follows by Theorem 8. Since $|\text{ext}(W)| \leq n$ for every cycle $W$ of $S^k$, Theorem 4 and (1) imply that $c$ is polynomial in $n$, thus we can compute $U$ in polynomial time. In this section we design a certifying algorithm for $k$-MULT that runs in $O(n)$ time. Although the algorithm is a simple generalization of one by [26, 27] for REP = 1-MULT, its correctness follows by simpler and shorter arguments that exploit the loop unrolling technique. We remark that the algorithm works for every PCA model, thus we do not assume that $M$ is SPCA beyond this point.

For a cycle $W$ of $S^k$, let $\text{ratio}(W) = -\text{bal}(W)(\text{ext}(W))^{-1}$. By Theorem 4 and (1), if $M$ is equivalent to a $k$-multiplicative $(c, \ell + 1)$-CA model, then either $\text{ext}(W) < 0$ and $c \geq \ell \text{ratio}(W) + 2\mu(W)$ or $\text{ext}(W) > 0$ and $c \leq \ell \text{ratio}(W)$. Then, as $W$ has at least one nose when $\text{ext}(W) < 0$, we obtain that $c = \ell \text{ratio}^k(M) + e \leq \ell \text{RATIO}^k(M)$ for some $e > 0$, where

$$\text{ratio}^k(M) = \max \{\text{ratio}(W) | W \text{ is a cycle of } S^k \text{ with } \text{ext}(W) < 0\},$$

and

$$\text{RATIO}^k(M) = \min \{\text{ratio}(W) | W \text{ is a cycle of } S^k \text{ with } \text{ext}(W) > 0\}.$$ 

We omit $M$ as usual; note that $\text{RATIO}^k = \infty$ when every cycle of $S^k$ has ext $\leq 0$.

The fact that $\text{ratio}^1 < \text{RATIO}^1$ is a restatement of a well-known result by [29]. Specifically, [29] proved that a PCA model $M$ is equivalent to a 1-multiplicative model if and only if $a/b > x/y$ every $(a, b)$-independent and $(x, y)$-circuit. We shall not define what an $(a, b)$-independent is or what an $(x, y)$-circuit is. Instead, we remark that, as discussed by [26, Theorem 4], each $(a, b)$-independent corresponds to a circuit $W_N$ of $S^1$ with $\text{ext}(W_N) < 0$ and, similarly, each $(x, y)$-circuit corresponds to a circuit $W_H$ of $S^1$ with $\text{ext}(W_H) > 0$. Moreover, $a/b = \text{ratio}(W_N) + h$ and $x/y = \text{ratio}(W_H) + h$ for some constant $h$ [26, Theorem 4]. Therefore, Tucker’s characterization not only
implies that $\text{ratio}^k < \text{RATIO}^k$ when $M$ is equivalent to a 1-multiplicative model, it also implies the converse. [26, Theorem 2] describes alternative characterizations of 1-multiplicative models that are described in terms of $\text{ratio}^1$ and the parameters $\text{len}^1$ and $\text{lex}^1$ that we define next for every $k \geq 1$. In few words, $\text{len}^k$ and $\text{lex}^k$ are special weightings of $S^k$ that can be used to discard some edges of $S^k$ that are implied when some specific values of $c$ and $\ell$ are used. As an acyclic digraph is obtained after discarding these “redundant” edges, the canonical solution to the full system $F_{c,\ell}$ can be computed more efficiently. Theorem 10 below is the generalization of Soulignac’s characterization for $k \geq 1$ and is the theoretical foundation for the algorithm that we develop in this section.

By definition, $\text{bal}$ is a weighting of $S^k$ where, for every edge $A \rightarrow B$, $\text{bal}(A \rightarrow B) = -1$ if $A \rightarrow B$ is a 1-hollow and $\text{bal}(A \rightarrow B) = i$ if $A \rightarrow B$ is an $i$-nose. Similarly, $\text{ext}$ is a weighting of $S^k$ where $\text{ext}(A \rightarrow B) = \beta \{ B \geq A \}$ if $A \rightarrow B$ is a hollow and $\text{ext}(A \rightarrow B) = -\beta \{ A > B \}$ if $A \rightarrow B$ is a nose. Let $\text{len}^k$ and $\text{lex}^k$ be the weightings of $S^k$ such that

$$\text{len}^k(A \rightarrow B) = \text{bal}(A \rightarrow B) + \text{ratio}^k \text{ext}(A \rightarrow B), \quad \text{and} \quad \text{lex}^k(A \rightarrow B) = (\text{len}^k(A \rightarrow B), \text{ext}(A \rightarrow B)).$$

By (1), if $W$ is a walk of $S^k$ and $c = \ell \text{ratio}^k + e$, then

$$\text{se}_{c,\ell}(W) = \ell \text{bal}(W) + c \text{ext}(W) + 2\mu(W)$$
$$= \ell \text{bal}(W) + \ell \text{ratio}^k \text{ext}(W) + e \text{ext}(W) + 2\mu(W)$$
$$= \ell \text{len}^k(W) + e \text{ext}(W) + 2\mu(W) = (\ell, e) \times \text{lex}^k(W) + 2\mu(W). \quad (2)$$

Let $A_0$ be the initial arc of $M$. Say that an edge $A \rightarrow B$ of $S^k$ is redundant when

$$\text{dlex}^k(A_0, B) > \text{dlex}^k(A_0, A) + \text{lex}^k(A \rightarrow B),$$

where $>$ denotes the lexicographically greater relation. Let $R^k(M)$ be the spanning subgraph of $S^k$ obtained by removing all the redundant edges; as usual, we omit $M$ from $R^k$. Our final characterization yields an alternative algorithm that provides a $k$-multiplicative model equivalent to $M$ at the cost of having longer arcs. 

**Theorem 10.** Let $A_0$ be the initial arc of a connected PCA model $M$, $k \in (\mathbb{N})$, $c = \ell \text{ratio}^k + e$, $\ell = e^3$, and $e = 4n$. The following statements are equivalent:

1. $M$ is equivalent to a $k$-multiplicative UCA model.
2. $\text{ratio}^k < \text{RATIO}^k$.
3. $\text{lex}^k(W) < (0, 0)$ for every cycle $W$ of $S^k$.
4. $R^k$ is acyclic.
5. $\text{dsep}_{c,\ell}(S^k, A_0, A) = \text{dsep}_{c,\ell}(R^k, A_0, A)$ for every $A \in A(M)$.

**Proof.** 1 $\Rightarrow$ 2 follows by Theorem 4 and (1); see above.

2 $\Rightarrow$ 3. If $\text{ext}(W) = 0$ and $M$ is PIG, then $W$ is internal because $S^k$ has no external hollows. Hence, by Lemma 7 and Corollary 2, $\text{len}^k(W) = \text{bal}(W) < 0$. Similarly, if $\text{ext}(W) = 0$ and $M$ is not PIG, then $W$ has an internal copy $T$ in $|W| \cdot S^k$ that is a circuit
and has \( \text{bal}(T) = \text{bal}(W) \), thus \( \text{len}^k(W) = \text{bal}(W) = \text{bal}(T) < 0 \) by Lemma \( \text{7} \) and Corollary \( \text{2} \). If \( \text{ext}(W) > 0 \), then \( -\text{bal}(W)(\text{ext}(W))^{-1} = \text{ratio}(W) \geq \text{ratio}^k > \text{ratio}^k \), thus \( \text{len}^k(W) < 0 \). Finally, if \( \text{ext}(W) < 0 \), then \( -\text{bal}(W)(\text{ext}(W))^{-1} = \text{ratio}(W) \leq \text{ratio}^k \), thus \( \text{len}^k(W) \leq 0 \).

3 \( \Rightarrow \) 4. If \( R^k \) has some circuit \( W = B_0, \ldots, B_j (B_0 = B_j) \), then \( B_i \to B_{i+1} \) is not redundant for \( i \in [j] \) and, consequently,

\[
\text{dlex}^k(A_0, B_j) \leq \text{dlex}^k(A_0, B_{j-1}) + \text{lex}^k(B_{j-1} \to B_j) \leq \ldots \leq \text{dlex}^k(A_0, B_0) + \text{lex}^k(W) = \text{dlex}^k(A_0, B_j) + \text{lex}^k(W).
\]

4 \( \Rightarrow \) 5. Note that \( d\text{sep}_{c,l}(S^k, A_0, A) \geq d^*\text{sep}_{c,l}(R^k, A_0, A) \) for \( A \in A(M) \) because every path of \( R^k \) is a walk of \( S^k \). For the other inequality, it suffices to prove that \( \text{sep}(W) \leq d^*\text{sep}_{c,l}(R^k, A_0, A) \) for every walk \( W = A_0, \ldots, A_j \) with \( A = A_j \) and \( j \leq n \).

We prove this fact by induction on \( j \). The base case \( j = 0 \) is trivial. In the inductive step \( j > 0 \), let

- for \( i \in [j] \), \( W_i \) be a walk of \( R^k \) from \( A_0 \) to \( A_i \) with \( \text{lex}^k(W_i) = d^*\text{lex}^k(R^k, A_0, A_i) \), and

- \( W_S \) be the walk obtained by traversing \( A_{j-1} \to A_j \) after \( W_{j-1} \).

By the inductive hypothesis, \( \text{sep}(W) \leq \text{sep}(W_{j-1}) + \text{sep}(A_{j-1} \to A_j) = \text{sep}(W_S) \), thus \( \text{sep}(W) \leq \text{sep}(W_j) \) when \( A_{j-1} \to A_j \) is an edge of \( R^k \). Suppose, then, that \( A_{j-1} \to A_j \) is redundant in \( S^k \). In this case, taking into account that no edge of \( W_j \) is redundant, it follows by induction that \( \text{lex}^k(W_j) = \text{dlex}^k(S^k, A_0, A_j) \). Consequently, there are only two possibilities for \( \text{lex}^k(W_j) \) and \( \text{lex}^k(W_S) \) because

\[
\text{lex}^k(W_j) = \text{dlex}^k(S^k, A_0, A_j) > \text{dlex}^k(S^k, A_0, A_{j-1}) + \text{lex}^k(A_{j-1} \to A_j) = \text{lex}^k(W_S).
\]

**Case 1:** \( \text{len}^k(W_j) > \text{len}^k(W_S) \). If \( W \) is a cycle with \( \text{ext}(W) < 0 \) and \( \text{ratio}(W) = \text{ratio}^k \), then

\[
|\text{ext}(W)|\text{ratio}^k = -|\text{ext}(W)|\text{bal}(W)(\text{ext}(W))^{-1} = \text{bal}(W)
\]

is integer. Therefore,

\[
|\text{ext}(W)|(\text{len}^k(W_j) - \text{len}^k(W_S)) = |\text{ext}(W)|(\text{bal}(W_j) - \text{bal}(W_S)) + \text{bal}(W)(\text{ext}(W_j) - \text{ext}(W_S))
\]

is also integer. Then, as \( |\text{ext}(W)| \leq |W| \leq n \), we get that \( \text{len}^k(W_j) - \text{len}^k(W_S) \geq n^{-1} \). On the other hand, by definition, \( \mu(W_S) \leq n \), \( \text{ext}(W_j) \geq -|W_j| \geq -n \) and \( \text{ext}(W_S) \leq |W_S| \leq n \). Then, by (2),

\[
\text{sep}_{c,l}(W_j) - \text{sep}_{c,l}(W_S) = \ell(\text{len}^k(W_j) - \text{len}^k(W_S)) + e(\text{ext}(W_j) - \text{ext}(W_S)) + 2(\mu(W_j) - \mu(W_S)) \geq 4n^3n^{-1} + 4n(-2n) - 2n > 0.
\]

**Case 2:** \( \text{len}^k(W_j) = \text{len}^k(W_S) \) and \( \text{ext}(W_j) > \text{ext}(W_S) \). In this case, (2) implies that

\[
\text{sep}_{c,l}(W_j) - \text{sep}_{c,l}(W_S) = \ell(\text{len}^k(W_j) - \text{len}^k(W_S)) + e(\text{ext}(W_j) - \text{ext}(W_S)) + 2(\mu(W_j) - \mu(W_S)) \geq 4n - 2n > 0.
\]
Summing up, \( d^* \text{sep}_{c,i}(R^k, A_0, A_j) = \text{sep}_{c,i}(W_j) > \text{sep}_{c,i}(W_S) \geq \text{sep}_{c,i}(W) \) in the case when \( A_{j-1} \rightarrow A_j \) is redundant.

5 \Rightarrow 1. If \( M \) is equivalent to no \( k \)-multiplicative \((c, \ell + 1)\)-CA model, then \( S^k \) has a cycle \( W \) with \( \text{sep}_{c,i}(W) > 0 \) by Theorem 4. Then, \( \infty = d^* \text{sep}_{c,i}(S^k, A_0, A) > d^* \text{sep}_{c,i}(R^k, A_0, A) \) for every \( A \in W \).

Theorem 10 yields the following algorithm to compute a \( k \)-multiplicative model equivalent to an input PCA model \( M \) when \( k \)-MULT answers yes; \( A_0 \) is the initial arc of \( M \):

1. Insert an arc intersecting \( L(A) \) and \( A \) for every \( A \in A(M) \setminus \{A_0\} \) such that \( L(A) \cap M = \emptyset \). (After this step, \( M \) is a connected model.)
2. Compute \( \text{ratio}^k \) to obtain the weighting \( \text{lex}^k \) of \( S^k \).
3. Determine \( d^\text{lex}^k(S^k, A_0, A) \) for every \( A \in A(M) \).
4. Obtain \( R^k \) by removing each redundant edge \( A \rightarrow B \) of \( S^k \).
5. Set \( s(A) = d^\text{sep}_{c,i}(R^k, A_0, A) \) for every \( A \in A(M) \), where \( \ell = (4n)^3 \) and \( c = \text{\( k \)-ratio}^k + 4n \).
6. Remove all the arcs inserted at Step 1.
7. Output \( (C, \{s(A), s(A) + \ell \mod c \mid A \in A(M)\}) \) for a circle \( C \) with \( |C| = c \).

Steps 1 and 4–7 can be easily implemented in \( O(n) \) time; just recall that \( R^k \) is acyclic by Theorem 10. In the following sections we discuss how to implement Steps 2 and 3.

### 6.1 Step 2: computation of the ratios

To efficiently compute \( \text{ratio}^k \), the key is to observe that every greedy nose cycle \( W \) of \( S^k \) with \( \text{ext}(W) < 0 \) has \( \text{ratio}(W) = \text{ratio}^k \). A weaker form of this result, stating that at least one greedy nose cycle \( W \) has \( \text{ratio}(W) = \text{ratio}^k \), is already known for \( k = 1 \) [26, Lemma 2].

**Lemma 11.** If \( M \) is a connected PCA model and \( k \in \{0\}, \) then \( \text{ratio}^k = \text{ratio}(G_N) \) for every greedy nose cycle \( G_N \) of \( S^k \) with \( \text{ext}(G_N) < 0 \).

**Proof.** Let \( G_N \) be a greedy nose cycle of \( S^k \) with \( \text{ext}(G_N) < 0 \) and \( W \) be a cycle of \( S^k \) with \( \text{ext}(W) < 0 \). The existence of \( G_N \) follows by Lemma 10. We shall prove that \( \text{ratio}(G_N) \geq \text{ratio}(W) \) to obtain that \( \text{ratio}^k = \text{ratio}(G_N) \).

Suppose first that \( M \) is PIG and let \( A_0 \) be the initial arc of \( M \). By hypothesis, \( G_N \) and \( \text{ext}(G_N) \) both contain the unique external nose \( L(A_0) \rightarrow A_0 \) of \( S^k \), thus \( \text{ext}(G_N) = \text{ext}(W) = -1 \) and \( \text{jmp}(G') = \text{jmp}(W) = \text{rows}(M) - 1 \). Moreover, the subpaths \( G' \) of \( G_N \) and \( W' \) of \( W \) from \( A_0 \) to \( L(A) \) are internal. Then, as \( G' \) is greedy, Theorem 7 implies that \( \text{Gr}_0(G') \) is bounded below by \( \text{Gr}_0(W') \), thus the number of backward edges of \( G' \) is not greater than the number of backward edges of \( W' \). Consequently, \( \text{bal}(G') \geq \text{bal}(W') \) by Corollary 2, thus \( \text{bal}(G_N) \geq \text{bal}(W) \) and, therefore, \( \text{ratio}(G_N) \geq \text{ratio}(W) \).

Suppose now that \( M \) is not PIG. Fix a vertex \( A \) of \( G_N \) and let \( A_0 < \ldots < A_N < B_A \) be the copies of \( A \) in \( \lambda \cdot S^k \) for \( \lambda \gg n^6 \). Similarly, let \( B_0 < \ldots < B_Z \) be the copies in \( \lambda \cdot S^k \) of a vertex \( B \) of \( W \). Let \( w = n^2, i = \text{ext}(G_N) \text{ext}(W), \) and \( z \in [n] - \{0\} \) be such that row \( w + zj \) is a copy of row \( w \) for every \( j \geq 0 \) with \( w + zj < \text{rows}(\lambda \cdot S^k) \). The existence
of \( z \) follows by Lemma 8. Note that \( i \leq n^2 \) because every edge has \( \text{ext} \in [-1,1] \). Moreover, for \( j \in [n] \), the copy \( T_N(j) \) of \( -z \text{ext}(W) \cdot G_N \) that starts at \( A_{w+z_i+j} \) in \( \lambda \cdot S^k \) is internal and ends at \( A_{w+z(i+1)} \). Similarly, the copy \( T(j) \) of \( -z \text{ext}(G_N) \cdot W \) that starts at \( B_{w+z(j+1))} \) in \( \lambda \cdot S^k \) is internal and ends at \( B_{w+2(z(j+1))} \). By definition, the rows of \( \lambda \cdot S^k \) between \( A_{w+z_i+j} \) and \( B_{w+z(j+1))} \) are copies of the rows between \( A_w \) and \( B_w \) and, consequently, \( \text{jmp}(T_N(j)) = \text{jmp}(T(j)) \).

For \( j \in [n] \), let \( a(j) \) be the number of backward edges in \( T_N(j) \) and \( b(j) \) be the number of backward noses in \( T(j) \). Clearly, \( K_N = T_N(1) + \ldots + T_N(n-1) \) is greedy and internal because \( G_N \) is greedy and \( T_N(j) \) is internal for \( j \in [n] \). Moreover, \( K_N \) has \( x = \sum_{j=1}^{n} a(j) \) backward edges. Similarly, \( K = T(0) + \ldots + T(n) \) is internal and has \( y = \sum_{j=0}^{n} b(j) \) backward edges. By Corollary 3, \( Gr_0(K_N) \) is a continuous curve from \( \text{pos}_0(A_{w+z_i}) \) to \( \text{pos}_0(A_{w+z(j)}) \), whereas \( Gr_1(K) \) is a continuous curve from \( \text{pos}_1(B_w) \) to \( \text{pos}_1(B_{w+z(i+1)}) \). Since \( K_N \) is greedy, Theorem 7 implies that \( Gr_0(K_N) \) is bounded below by \( Gr_1(K) \). Consequently, \( x \leq y + 2 \). Then, since \( a(j) \) and \( b(j) \) are integer, there exists \( j \in [n] \) such that \( a(j) \geq b(j) \). By definition, \( T_N(j) \) has exactly \( -z \text{ext}(W) \cdot G_N \) copies of each edge in \( G_N \), thus \( \text{bal}(T_N(j)) = -z \text{ext}(W) \cdot \text{bal}(G_N) \). Similarly, \( \text{bal}(T(j)) = -z \text{ext}(G_N) \cdot \text{bal}(W) \). Then, by Corollary 2:

\[
z(i)(\text{ratio}(G_N) - \text{ratio}(W)) = -z \text{ext}(G_N)^{-1} \cdot \text{bal}(G_N) + z \text{ext}(W)^{-1} \cdot \text{bal}(W)
= -z \text{ext}(W) \cdot \text{bal}(G_N) + z \text{ext}(G_N) \cdot \text{bal}(W)
= \text{bal}(T_N(j)) - \text{bal}(T(j))
= \text{jmp}(T_N(j)) - a(j) - \text{jmp}(T(j)) + b(j) \geq 0.
\]

As \( z(i) > 0 \), we conclude that \( \text{ratio}(G_N) \geq \text{ratio}(W) \) and, therefore, \( \text{ratio}(G_N) = \text{ratio}(W) \).

The analogous of Lemma 11 for \( \text{RATIO}^k \) is stated below without proof. When \( M \) is SPCA, \( \text{ratio}^k(M) \) and \( \text{RATIO}^k(M) \) can be obtained in \( O(n) \) time by considering a greedy nose cycle \( G_N \) with \( \text{ext}(G_N) < 0 \) and a greedy hollow cycle \( G_H \) with \( \text{ext}(G_H) > 0 \). The cycles \( G_N \) and \( G_H \) exist by Lemma 10 and can be computed in \( O(n) \) time as in Corollary 5. This yields another algorithm for \( k \)-MULT that is just a restatement of the one discussed in Corollary 5: instead of looking for the intersection of \( G_N \) and \( G_H \), compare their ratios. This algorithm is a simplification of the one designed by [14] in which all the greedy cycles are traversed.

**Lemma 12.** If \( M \) is an SPCA model and \( k \in (\omega) \), then \( \text{RATIO}^k = \text{ratio}(G_H) \) for every greedy hollow cycle \( G_H \) of \( S^k \) with \( \text{ext}(G_H) > 0 \).

**Corollary 6.** Given a connected PCA model \( M \) and \( k \in (\omega) \), it takes \( O(n) \) time to compute \( \text{ratio}^k \) and \( \text{RATIO}^k \).

### 6.2 Step 3: determining the distances according to \( \text{lex} \)

Let \( A_0 \) be the initial arc of \( M \). The key to efficiently determine \( d_{\text{lex}}^k(A_0,A) \) is to observe that some path \( W \) of \( S^k \) from \( A_0 \) to \( A \) with \( \text{lex}^k(W) = d_{\text{lex}}^k(A_0,A) \) is “dually greedy”; we remark that a restricted version of this fact is already known for \( k = 1 \) [26,
Lemma 4. A walk $W = B_0, \ldots, B_j$ of $S^k$ is greedy anti-hollow (resp. anti-nose) when $S^k$ has no hollows (resp. noses) reaching $B_{i+1}$ when $B_i \rightarrow B_{i+1}$ is a nose (resp. hollow), for $i \in \mathbb{Z}$. In other words, $W$ is greedy anti-hollow (resp. anti-nose) when hollows (resp. noses) are preferred to noses (resp. hollows) in a backward traversal of $W$. The walk $W$ is a dually greedy hollow (resp. nose) when there exists $i \in \mathbb{Z}$ such that:

- $B_0, \ldots, B_i$ is a greedy hollow (resp. nose) walk of $S^k$, and
- $B_i, \ldots, B_j$ is a greedy anti-nose (resp. anti-hollow) walk of $S^k$.

**Lemma 13.** Let $M$ be a connected PCA model that is equivalent to a $k$-multiplicative model for $k \in (\omega)$, and $A_0$ be the initial arc of $M$. If $W$ is a path of $S^k$ from $A_0$ to a vertex $A$, then $S^k$ has a dually greedy nose (resp. hollow) walk $W_D$ from $A_0$ to $A$ with $\text{lex}^k(W_D) = \text{lex}^k(W)$.

**Proof.** We only prove the existence of the dually greedy nose walk as the existence of the dually greedy hollow walk is analogous. Suppose first that $M$ is not PIG. By Theorem 9, $S^k$ has a twister $\gamma$. Let $\lambda \gg h^9$ for $h = \max\{10 + \gamma, n(k+1)\}$, and consider a copy $B$ of $A_0$ in $\lambda \cdot S^k$ with row$(B) \in (h^4, h^4 + h]$. Let $T$ be the copy of $W$ in $\lambda \cdot S^k$ that starts at $B$ and ends at a copy $X$ of $A$, $T_N$ be the greedy nose path of $\lambda \cdot S^k$ with $|T_N| = 3h^2$ that starts at $B$ and ends at a vertex $Y$, and $T_H$ be the greedy anti-hollow walk of $\lambda \cdot S^k$ with $|T_H| = 7h^4$ that starts at a vertex $Z$ and ends at $X$. Moreover, let $x, y,$ and $z$ be the number of backward edges in $T$, $T_N$, and $T_H$, respectively.

By Lemma 10, $S^k$ has a greedy nose cycle $G$ with $\text{ext}(G) < 0$. If some vertex of $G$ has a copy in $T_N$, then a copy $T_G$ of $G$ is included in $T_N$ because $T_N$ is greedy. Otherwise, $T_N$ contains the copy $T_G$ of a greedy nose cycle disjoint from $G$ that also has $\text{ext} < 0$ by Theorem 8. Thus, whichever the case, $\text{jmp}(T_N) > 0$. Moreover, $T_N$ visits at least $2h$ copies of $T_G$ because $|T_G| \leq h$ and $|T_N| = 3h^2$. Hence, $y > h$ because $h \geq y$ and $\gamma$ is a twister. As usual, Lemma 6 implies that $T$ and $T_N$ are internal, $T$ visits vertices with row $\in [h^4 - h, h^4 + 2h^2]$, and $T_N$ visits vertices with row $\in [h^4 - 2h^2, h^4 + 4h^2]$. Moreover, by Theorem 7, $Gr_z(T)$ is bounded above by $Gr_z(T_N)$ because both start at $\text{pos}_z(B)$ and $T_N$ is greedy.
Similarly, $T_H$ is internal and visits vertices with row $\in [h^2, h^5]$ by Lemma 6, thus $Gr_z(T)$ is bounded above by $Gr_z(T_H)$ because both end at $\text{pos}_{z+1}(X)$ and $T_H$ is greedy anti-hollow. Consider the following alternatives to prove that $Gr_z(T_N)$ and $Gr_z(T_H)$ share a point $p$.

**Case 1:** $\text{jmp}(T_H) < 0$. By Corollary 2, $|\text{jmp}(T_H)| \geq 7h^3$ because $T_H$ traverses at least $7h^3$ copies of $Z$, thus $\text{row}(X) \leq \text{row}(Y) \leq \text{row}(Z)$. Then, as $x \leq |T| \leq h < y$ it follows that $Gr_z(T_N)$ and $Gr_z(T_H)$ have a point in common (Fig. 16(a)).

**Case 2:** $\text{jmp}(T_H) \geq 0$. Let $q \in [z, x + z]$. Recall that $Gr_z(T)$ is bounded above by $Gr_z(T_H)$. Then, Lemma 6 implies that $\text{row}(Z') \in [h^4 - h, h^3 + 4h^2]$ for every $Z'$ such that $Gr_z(T_H)$ traverses $\text{pos}_q(Z')$. As there are less than $6h^2$ such possible $Z'$ for each $q$ and $x \leq h$, it follows that $Gr_z(T_H)$ passes through $\text{pos}_{z+1}(Z')$ for some vertex $Z'$, i.e., $x < z$. Then, $Gr_z(T_N)$ and $Gr_z(T_H)$ share some point (Fig. 16(b)).

Summing up, $\lambda \cdot S^k$ has a walk $T_D$ such that $Gr_z(T_D)$ starts at $\text{pos}_{(B_0)}$, takes the arrows of $Gr_z(T_N)$ until reaching $p$, and then it takes the arrows of $Gr_z(T_H)$ until reaching $\text{pos}_{x+z}(B)$. The path $T_D$ is dually greedy by construction, and so is the walk $W_D$ of $S^k$ whose copy is $T_D$. Moreover, $T_D$ has $x$ backward edges and $\text{jmp}(T_D) = \text{jmp}(T)$. Then, by Corollary 2, $\text{bal}(W_D) = \text{bal}(T_D) = \text{bal}(T_D) - x = \text{bal}(T) - x = \text{bal}(W)$, whereas $\text{ext}(W_D) = \text{ext}(W)$ is the number of copies of $A_0$ between $B$ and $X$ in $\lambda \cdot S$. Therefore, $\text{lex}^k(W_D) = \text{lex}^k(W)$ as desired.

The proof for the case in which $M$ is PIG is analogous, although loop unrolling is avoided. We succinctly describe it here for the sake of completeness. By Lemma 9, $S^k$ has a greedy nose path $G$ that is internal, starts at $A_0$, and ends at $L(A_0)$. The path $W$ is also internal because it starts at $A_0$ and, thus, it cannot take the unique external nose $L(A_0) \rightarrow A_0$. By Theorem 7, $Gr_1(G)$ is bounded below by $Gr_1(W)$. Let $W_H$ be the maximal greedy anti-hollow walk that is internal and has a drawing $Gr_p(W_H)$, $p \geq 0$, that ends at the same point as $Gr_1(W)$. By Theorem 7, $Gr_1(W_H)$ is bounded below by $Gr_1(W)$. Moreover, $Gr_p(W_H)$ is bounded above by $x \rightarrow \text{rows}(M) - 1$ because $\text{ext}(W_H) \leq 0$ as $M$ is PIG. Then, $Gr_1(G)$ and $Gr_p(W_H)$ share a point $q$ because there is a finite number of vertex positions that $Gr(W_H)$ can traverse without either reaching the column 0 or taking the external nose. Moreover, as above, the path $W_D$ whose drawing takes $Gr_1(G)$ until $q$ and then takes $Gr_1(W_H)$ is dually greedy and has $\text{lex}^k(W_D) = \text{lex}^k(W)$. }

**Lemma 14.** If $M$ is a connected PCA model that is equivalent to a $k$-multiplicative model for $k \in (0, 1)$, then $\text{dlex}^k$ can be computed in $O(n)$ time.

**Proof.** We prove that the next algorithm is linear and computes a function $\psi = \text{dlex}^k$.

1. Let $G = A_0, \ldots, A_p$ be the maximal greedy nose path from $A_0$. For $i \in [p]$, let $\phi(A_i) = i$ and $\alpha(A_i) = \text{lex}^k(A_0, \ldots, A_i)$. For $A \in G$, let $\phi(A) = n + 1$ and $\alpha(A) = (-\infty, -\infty)$.

2. For $A \in A(M)$, let $x(A)$ be the unique vertex that precedes $A$ in every greedy anti-hollow path of $S^k$ that traverses $A$. Let $D$ be the digraph with a vertex $v(A)$ and an edge $v(x(A)) \rightarrow v(A)$ for every $A \in A(M)$.

3. For each cycle $W$ of $D$, let $A_W$ be arc of $A(M)$ with minimum $\phi$ among those arcs $A$ such that $v(A) \in W$. The digraph $F$ that is obtained after removing $v(x(A_W)) \rightarrow A_W$
for every cycle $W$ of $D$ is a forest: each root has in-degree 0, whereas $v(x(A))$ is the parent of $A$ for each edge $v(x(A)) \rightarrow v(A)$.

4. The algorithm outputs the function $\psi$ below, that is well defined because $F$ is a forest:

$$\psi(A) = \begin{cases} 
\alpha(A) & \text{if } A \text{ is a root of } F \\
\max\{\psi(x(A)) + \text{lex}^k(x(A) \rightarrow A), \alpha(A)\} & \text{otherwise}
\end{cases}$$

To see that the algorithm is correct, we prove that $\psi(A) = \text{dlex}^k(A_0, A)$ for every $A \in A(M)$. By Theorem 10, $\text{lex}^k(W) < (0,0)$ for every cycle $W$ of $S^k$, so $\text{dlex}^k(A_0, A) = \text{d}^*\text{lex}^k(A_0, A)$ is well defined. By Lemma 13,

$$\text{dlex}^k(A_0, A) = \max\{\text{lex}^k(W) \mid W \text{ is a dually greedy hollow path from } A_0 \text{ to } A\}.$$ 

In other words, there exists a dually greedy nose path $G_D$ from $A_0$ to $A$ with $\text{lex}^k(G_D) = \text{dlex}^k(A_0, A)$. By definition, $G_D = G' + G_A'$, where $G' = A_0, \ldots, A_q, q \in [p]$, is a subpath of the greedy nose path $G$ computed at Step 1 and $G_A'$ corresponds to the path from $x(A_q)$ to $x(A)$ in the digraph $D$ computed at Step 2. The proof that $\psi(A) = \text{lex}^k(G_D)$ is by induction on the length of the unique path of $F$ from a root to $A$.

In the base case, $\psi(A)$ is a root of $F$. Clearly, $\text{lex}^k(G_D) \geq \alpha(A)$ by Step 1, whereas $\psi(A) = \alpha(A)$ by Step 4. Suppose, to obtain a contradiction, that $\text{lex}^k(G_D) > \alpha(A)$. Then, $A \neq A_q$ and, moreover, $B \neq A_i$ for every $B \in G_A'$ and every $i \in \{q\}$ because $G_D$ is a path. By Step 1, it follows that $\phi(B) > \phi(A_q)$, thus $G_A'$ corresponds also to a path of the digraph $F$ computed at Step 3. But this is impossible because $\psi(A)$ is a root of $F$. Hence, $\text{lex}^k(G_D) = \alpha(A) = \psi(A)$ when $\psi(A)$ is a root of $F$.

In the inductive step, $v(x(A))$ is the parent of $v(A)$ in $F$. Clearly, $\text{lex}^k(G_D) \geq \alpha(A)$ by Step 1, whereas $\text{lex}^k(G_D) \geq \text{lex}^k(W) + \text{lex}^k(x(A) \rightarrow A)$ for every dually greedy nose path $W$ from $A_0$ to $x(A)$. Hence, $\text{lex}^k(G_D) \geq \psi(A)$ follows by Step 4 and the inductive hypothesis. Conversely, if $A = A_q$, then $\text{lex}^k(G_D) = \alpha(A)$, whereas if $A \neq A_q$, then $G_D = W.A$ for a dually greedy path $W$ from $A_0$ to $x(A)$. Then, $\text{lex}^k(G_D) \leq \psi(A)$ also follows by Step 4 and the inductive hypothesis.

Regarding the time complexity, $S^k$ is computed in $O(n)$ time via Theorem 6 before the algorithm is invoked. Then, the greedy nose path $G$ of Step 1 can be obtained in $O(n)$ time, while $\phi$ and $\alpha$ are calculated in $O(n)$ time with a single traversal of $G$. Similarly, Step 2 is implemented in $O(n)$ time by traversing the edges of $S^k$ in a backward direction. Step 3 consumes $O(n)$ time as well as each vertex of $D$ has at most one in-neighbor. Finally, $\psi$ is calculated in $O(n)$ time at Step 4 with a traversal of $F$ from the roots to its leaves.

**Theorem 11.** The problem $k$-MULT can be solved in $O(n)$ time for every PCA model $M$ and every $k \in \{1, \ldots, \omega\}$. The algorithm outputs either a $k$-multiplicative model equivalent to $M$ or a negative certificate that can be authenticated in $O(n)$ time.

**Proof.** If $k = 0$, the algorithm returns $M$ in $O(1)$ time. Otherwise, $O(n)$ time spent by Corollary 5 to decide if $M$ is equivalent to some $k$-multiplicative model. If the answer is no, then a negative certificate is obtained as a by-product. If the answer is yes, then the algorithm implied by Theorem 10 (Section 6) is executed to build the $k$-multiplicative
model $U$ equivalent to $M$. By Corollary 6 and Lemma 14, this last step requires $O(n)$ time as well.

7 Concluding remarks

In this article we designed a certifying and linear time algorithm to solve $k$-$MULT$. As a by-product, we obtained a certifying and $O(n^2)$ time algorithm for $(k,c,\ell)$-$MULT$. From a theoretical point of view, we provided a new characterization of those PCA models that are equivalent to a $k$-multiplicative UCA model, for every $k < \omega$. The proof of this characterization exploits a powerful geometric framework given by Mitas’ drawings and the loop unrolling technique. Mitas’ drawings allow us to treat the internal cycles of the synthetic graphs as if they were curves in $\mathbb{R}^2$. The intersection of two curves corresponds to the intersection of the paths. The loop unrolling technique, on the other hand, provides an internal copy of every cycle of the synthetic graph. In a forthcoming article [28, see the preprint] we combine Mitas’ drawings of the synthetic graphs with the loop unrolling technique to solve the minimal representation problem.

In some sense, the algorithm that we provide is a generalization of the one given by [26, 27] that, in turn, generalizes the algorithm by [19] for UIG graphs. Even though many properties of UIG models hold naturally in UCA models, this is not always the case, as UCA models have a much richer structure than UIG models. This is the case for many of algorithms that solve REPUIG. The fact that the algorithm for REPUIG based on synthetic graphs generalizes to REP is a plus for this tool.

As discussed in Section 1, every PIG model $M$ is equivalent to an $\infty$-multiplicative UIG model. This fact can be easily proven by looking at the algorithms by [5] and [17], and it also follows by Theorem 8. We note that there exist PCA models that are not PIG and are equivalent to $\infty$-multiplicative models as well. For instance, \{$(2i, 2(i + k) + 1 \mod 2n) \mid 1 \leq i \leq n$\} is an $\infty$-multiplicative model representing $C_n^k$, $n \gg k$, where $C_n$ is the cycle with $n$ vertices.

Say that a PCA (resp. PIG) graph $G$ is $k$-UCA (resp. $k$-UIG), for $k \geq 0$, when $G^i$ is UCA (resp. UIG) for every $1 \leq i \leq k$. By definition, if a PCA model $M$ is equivalent to a $k$-multiplicative UCA model, then $G(M)$ is $k$-UCA. Theorem 2 implies that the converse is also true for PIG graphs: if $G$ is $k$-PIG, then $G$ is PIG and, thus, it admits an $\infty$-multiplicative UIG model. One is tempted to think that the converse is also true for UIG graphs: if $G$ is $k$-UCA and $k < \omega$, then $G$ is $k$-multiplicative. Unfortunately, this is false (Fig. 17). Note that $\infty$-UCA graphs is the subclass of UCA graphs closed under taking powers; its graphs can be recognized in $O(n^2)$ time. Several classes of graphs that are closed under taking powers were studied, including PIG, interval, PCA, and circular-arc graphs [22, 8]. Computing models representing powers of circular-arc graphs is an important problem with different applications [1]. We leave open the problem of recognizing these graphs in $o(n^2)$ time.

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Figure 17: The graph $G$ in the left is 2-UCA because the $(50,21)$-CA model in the center represents $G^2$. However, the greedy cycles $A_0, A_2, A_4, A_7, A_0$ and $A_1, A_5, A_6, A_5, A_3, A_1$ of $S^2(M)$ do not intersect, for the PCA model $M$ representing $G$ (that is unique up to full reversal and movement of 0).

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