Infinite Statistics and Black Holes in Matrix Theory

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Abstract

The concept of infinite statistics is applied to the analysis of black hole thermodynamics in Matrix Theory. It is argued that Matrix Theory partons, D0-branes, satisfy quantum infinite statistics, and that this observation justifies the recently proposed Boltzmann gas model of Schwarzschild black holes in Matrix Theory.
Various authors have recently shown that Matrix theory [1] can be used as a starting point for the analysis of basic qualitative features of Schwarzschild black holes [2], [3], [4], [5], [6]. (For other approaches to the same subject consult [7].) According to [2], [3], [4], [5], [6], neutral black holes can be understood as bound states of the zero modes of the induced super Yang-Mills theory which appears in the framework of a toroidally compactified Matrix Theory. For example, the Bekenstein-Hawking entropy formula can be deduced by analyzing the mean field theory of the induced super Yang-Mills zero modes. The fundamental relation between the black hole entropy and its mass then follows from the defining dispersion relation of the infinite momentum frame kinematics (one of the salient properties of Matrix theory). The original analysis of [2], [3] and [4] was done in a particular limit defined by the condition that the entropy $S$ of a neutral black hole configuration is proportional to the number of Matrix theory partons $N$. (The other important ingredient was the use of particular equations of state for $d + 1$ super Yang-Mills theories with 16 supercharges, derived in the study of the thermodynamics of near-extremal D-branes [15], [16].) It was further elucidated in [5] that the $N \sim S$ limit arises at a black hole/black string transition point, at which longitudinally stretched black strings become unstable to the formation of black holes. In this case the thermodynamics of neutral black holes, it was argued in [5], is completely determined by the mean field dynamics of the induced super Yang-Mills zero modes. Actually, the limit $N \gg S$ is in some sense much more appropriate for the infinite momentum frame black hole physics, for in this situation, as pointed out in [6], the black hole entropy is independent from the specifics of the boosting procedure. As it turns out, in both cases ($N \sim S$ and $N \gg S$) the fundamental properties of infinitely boosted neutral black holes can be understood in the framework of an interacting gas of Matrix theory partons, the nature of interactions being dictated by the fundamental
Matrix theory Hamiltonian [3], [5], [6].

It was emphasized in [3] that Matrix theory partons, i.e. D0-branes [23], [24], [25], [26], should be treated as distinguishable particles; otherwise the basic relations of black hole thermodynamics would not follow. According to [3], D0-branes are essentially distinguishable (Boltzmannian) because they can be understood as fluctuations around particular classical background configurations of Matrix theory (in the language of [3] D0-branes are tethered to particular classical background configurations). Different classical backgrounds which are responsible for the Boltzmannian character of D0-branes have to be identified in different dimensions [3].

More recently, the basic properties of a classical Boltzmann gas of D0-branes were analyzed in [14]. The classical partition function was computed for an interacting Boltzmann gas of D0-branes whose dynamics was modeled by a Born-Infeld type of lagrangian [17], [19], [23], [26]. It was pointed out in [14] that the absence of the Gibbs $1/N!$ factor ($N$ being the number of distinguishable particles) in the classical Boltzmann partition function is essential for computation of black hole thermodynamic functions, thus confirming the intuition of [3].

It is our aim in this article to further elaborate on this issue. We want to point out that the absence of the Gibbs factor and the Boltzmann nature of D0-branes follow from the fact that D0-branes obey quantum infinite statistics [9], [10]. Unlike the familiar Bose-Einstein or Fermi-Dirac statistics, somewhat exotic infinite statistics allow all representations of the symmetric group to be realized. Because of this fact, various quantum statistical properties of many-body systems described by infinite statistics are very similar to the corresponding properties of classical Boltzmann systems. Hence the intuition of [3] and the results of [14] are justified from a quantum statistical point of view. The resulting picture also turns out
to be in full agreement with the ideas of Strominger and Volovich [8] concerning black hole
statistics.

The note is organized as follows: First we review the basic set-up of the recently
proposed Matrix theory description of Schwarzschild black holes as presented in [2], [3],
[4], [5], [6]. Then we review the concept of infinite statistics following the original work of [9]
and apply it initially to a gas of free particles. Next we consider a general non-ideal gas of
particles described by infinite statistics. We establish constraints that have to be satisfied
by the effective Hamiltonian which describes the dynamics of such a gas in order for it to
qualify as a model of black hole thermodynamics in $D$ space-time dimensions ($D > 4$).
The resulting effective Hamiltonian is found to agree with Matrix theory predictions. Our
basic point is that quantum infinite statistics of D0-branes justifies the intuitive picture of
the Boltzmann gas model of Schwarzschild black holes in Matrix theory [3] and explains
the success of calculations performed in [14].

2. Let us review the basic physics of the Matrix theory approach to Schwarzschild
black holes [2], [3], [5], [6]. We consider Matrix theory compactified on a $d$-dimensional
torus $T^d$, the dimension of the uncompactified space time being $D = 11 - d$. Let the
characteristic size of the torus $T^d$ be $L$, and the extent of the compact M-theory dimension
be $R$ (where $R$, is eventually taken to infinity).

The entropy of a neutral black hole in $D$ space-time dimensions is proportional to the
transverse spatial volume and is given by the Bekenstein- Hawking formula

$$ S \sim R_s^{D-2}/G_D, \quad (1) $$

where $G_D = l_p^3/L^d$ is the $D$-dimensional Newton constant and the Schwarzschild radius
$R_s$ has the following dependence on the black hole mass $M$ (as determined by the $r^{3-D}$
form of the gravitational Newtonian potential in $D$ space-time dimensions)

$$R_s \sim (G_D M)^{\frac{1}{D-3}}. \quad (2)$$

The dynamics of black holes is studied in the infinite momentum frame (IMF) (one of the defining features of Matrix theory) which is characterized by the following fundamental kinematic dispersion relation between the light cone energy $E$ and the longitudinal momentum $P \sim N/R$

$$E \sim M^2 R/N. \quad (3)$$

Equation (3) together with the first law of thermodynamics $dE = TdS$ implies the following expression

$$S \sim (NT/R)^{\frac{D-2}{D-4}} G_D^{\frac{2}{D-4}}, \quad (4)$$

or equivalently,

$$R_s \sim (NT G_D/R)^{\frac{1}{D-4}}. \quad (5)$$

The above basic relations can be deduced from a mean field analysis of the dynamics of the zero modes of the induced super Yang-Mills theory [3],[5],[6]. The argument goes as follows: The energy and longitudinal momentum of a boosted neutral black hole, with the boosting parameter $e^\alpha$, are

$$E \sim M \cosh \alpha, \quad P \sim M \sinh \alpha. \quad (6)$$

These kinematic relations determine (at the black hole/black string transition point $N \sim S$, that is $P \sim S/R$) by how much the box of size $R$ has to expand in order to accommodate the black hole of size $R_s$, in the case of very large boosts, namely

$$\exp \alpha \sim R_s/R. \quad (7)$$
Assume that neutral black holes can be viewed as bound states of Matrix theory partons. It is well known that the mean field dynamics of $N$ zero modes of the induced super Yang-Mills theory is governed by the following effective Matrix theory Lagrangian [1], [24], [25] (derived in the limit of low parton velocities $v$ and large interparton separations $r$)

$$L_{\text{eff}} = \frac{N v^2}{R} + G_D \frac{N^2 v^4}{R^3 v^{D-4}}.$$  \hspace{1cm} (8)

Suppose one applies the virial theorem (i.e. equate the average kinetic and potential energies) to such a black hole-like bound state, by setting the characteristic size of the bound state to be essentially the Schwarzschild radius $R_s$ and the characteristic parton velocity to be proportional to the inverse of the boosting parameter $v \sim e^{-\alpha} \ll 1$ (or $vR_s/R \sim 1$, which is just the Heisenberg uncertainty bound). Then one can easily deduce the Bekenstein-Hawking relation (1) at the special point $N \sim S$, in other words, find that $N \sim R_s^{D-2}/G_D$! The use of the IMF on-shell relation (3) then implies $M \sim R_s^{D-3}$, which is identical to equation (2)! (Note that the first law of thermodynamics is not used in this part of the argument.)

On the other hand, if one examines the limit of $R \gg R_s$, one is lead to consider the case $N \gg S$, which in turn corresponds to the limit of low temperatures [6]. In this particular situation one may again apply the virial theorem in the following manner

$$N m < v^2 > \sim E \sim TS.$$ \hspace{1cm} (9)

Here the characteristic mean velocity is determined as $v \sim R_s T$ ($R_s$ being the typical distance and $T$- typical frequency of the system) and the parton mass $m \sim 1/R$. (Note that in this limit the first law of thermodynamics is used.) Then equation (9) implies $R_s \sim (NTG_D/R)^{\frac{1}{D-4}}$, which is identical to equation (5)! Moreover, as shown in [6], one can deduce the form of interactions needed to derive the equation of state (4) and find
that, indeed, the required interactions (which as it turns out involve spin [27], [28], [29]) can be deduced from the fundamental Matrix theory Hamiltonian.

3. In this section we want to show that all fundamental relations of section 2. follow from the quantum statistical mechanics of particles obeying infinite statistics (D0-branes being such particles).

We start with a review of infinite statistics following the original reference [9]. The construction of [9] represents a concrete realization of the results of [10] in which a particular example of statistics allowing for all possible representations of the symmetric group, was found (along with more familiar Bose-Einstein, Fermi-Dirac and parastatistics, the last being essentially equivalent to internal ”color” symmetry ). The explicit operator realization of infinite statistics in [9] is given in terms of the Cuntz algebra [11] (used, for example, in non-commutative probability theory [12] and its applications to the study of the large $N$ (’t Hooft) limit of matrix models [13])

$$a_i a_j^\dagger = \delta_{ij}. \tag{10}$$

As usual, a unique vacuum state is assumed. The vacuum state $|0\rangle$ is defined via

$$a_i |0\rangle = 0. \tag{11}$$

Notice that the infinite statistics commutation relations (10) can be thought of, rather formally, as a $q = 0$ limit of the $q$-deformed quantum commutation relations

$$a_i a_j^\dagger - qa_j^\dagger a_i = \delta_{ij} \quad \text{which describe } ”q\text{-ons” } [9].$$

Many unusual properties are implied by (10). For example, the number operator $N_i$ ($[N_i, a_j] \equiv -\delta_{ij} a_j$) is given by the following expression

$$N_i = a_i^\dagger a_i + \sum_k a_k^\dagger a_i a_k + \sum_{k_1,k_2} a_{k_1}^\dagger a_k^\dagger a_i a_{k_2} a_{k_1} + ... \tag{12}$$
Also, the inner product of two \( n \)-particle states is determined by

\[
<0|a_{i_n}...a_{i_1}a_{j_n}^\dagger...a_{j_1}^\dagger|0> = \delta_{i_1,j_1}...\delta_{i_n,j_n}.
\]  

(12a)

Similarly, the operators which create (annihilate) a particle in a given quantum state \( k \) read as follows

\[
A_k^\dagger(A_k) = \sum_{i_1,...,i_l} a_{i_1}^\dagger...a_{i_l}^\dagger a_k^\dagger(a_k)a_{i_l}...a_{i_1}.
\]  

(12b)

Because infinite statistics particles belong to many-dimensional representations of the symmetric group, the place in which these operators act is important.

It was argued in [9] that there exists no second quantized formulation of a local field theory in terms of such "free" operators (i.e. operators obeying (10)), yet there exist consistent non-relativistic theories. Moreover, the spin-statistics theorem implies no restriction on spin for particles satisfying infinite statistics. Furthermore, the principle of cluster decomposition and CPT theorem are found to hold [9].

We are interested in the quantum statistical properties of many-body systems described by infinite statistics. More specifically, we want to argue that a gas of \( N (N \rightarrow \infty) \) D0-branes behaves effectively as such a many-body system. The basic object of our study is the quantum partition function which is defined as usual

\[
Z \equiv \sum e^{-\beta H},
\]  

(13)

where the sum goes over all possible quantum states.

Let us first consider a free gas of particles satisfying infinite statistics. For such a free gas one can easily see that the quantum partition function (13) has the form of the classical Boltzmann partition function without the Gibbs \( 1/N! \) factor [9]. Put differently, the quantum partition function of a free gas of infinite statistics particles effectively describes the statistical properties of a system of identical particles with a very large (infinite)
number of internal degrees of freedom, so that particles can be distinguished by their internal quantum numbers. In the case of interest, that is, a free gas of $N$ ($N \to \infty$) D0-branes, the particular internal symmetry is $U(\infty)$ [1]. Our basic argument is simple: free D0-branes can be distinguished by their internal states and are thus effectively described by infinite statistics. Notice that this fact is the underlying reason why there exists no Gibbs paradox in this situation (which, we recall, is the generic feature of classical Boltzmann gases).

The explicit form of the quantum partition function for a free gas of particles obeying infinite statistics (in $D-2$ transverse dimensions) is

$$Z \sim V^N (T/R)^{N \frac{D-2}{2}}.$$  

(14)

Here $V$ denotes the volume of the transverse space and the parton mass $m \sim 1/R$. Notice how this differs from the usual expression for the classical partition function (which includes the Gibbs factor) by a factor $(1/N)^N$.

The energy of an ideal infinite statistics gas is therefore given by

$$E = -\frac{\partial \ln Z}{\partial \beta} \sim \frac{N}{\beta},$$

(15)

and its entropy

$$S = \ln Z + \beta E \sim \ln Z + N.$$  

(16)

Now, if we assume that $\ln Z \ll N$ i.e. $Z \sim 1$, we get that the entropy is proportional to the number of infinite statistics particles $S \sim N!$ The requirement $Z \sim 1$ (which is ”optimal” from the renormalization group point of view, like the condition $N \sim S$ [2], [5]) amounts to the condition

$$V(T/R)^{\frac{D-2}{2}} \sim 1.$$  

(16a)
What is the meaning of this relation? It is easy to see that, for $N \sim S$, equations (1), (2) and (3) (without using the first law of thermodynamics) lead to

$$S \sim (T/R)^{-\frac{D-2}{2}},$$

which is compatible with (16a) once the Bekenstein-Hawking formula (1) is recalled! In that sense the requirement $Z \sim 1$ is just another way of saying that the holographic principle [1], [21], [22] is at work here.

Let us now study a non-ideal, i.e. interacting gas of particles obeying infinite statistics. We are interested in the post-Newtonian limit (small velocities $v$, large separations $r$ between particles) which is quite natural from the point of view of the dynamics of D0-branes in Matrix theory [1], [23], [24], [25]. The most general form of such a mean field post-Newtonian lagrangian (if for the moment we imagine possible dependence on spin, so that time-reversal invariance is not violated) is

$$L \sim \sum_{l=0}^{l+k} a_{l,k} v^{l+2} r^{-k}, \quad (17)$$

where we have expanded around the quadratic kinetic term which describes the ideal gas, so that $a_{00} \sim N$. In general the partition function (13) cannot be evaluated in a closed form. But if the interaction potential is of the order of the kinetic term, the quantum partition function can be readily estimated. In fact, the quantum partition function reduces to equation (14) in this case. (Notice that our notation is quite schematic; the explicit sums over particles as well as indices determining particle positions have been suppressed for simplicity).

We would ultimately like to describe a Schwarzschild black hole as a bound state of interacting infinite statistics particles. Hence we apply the virial theorem to the lagrangian (17) and demand that the $i$-body interaction is of the same order as the $i+j$-body interaction and that the Bekenstein-Hawking relation (1) remains valid for $N \sim S$. The end
product is the following mean field lagrangian

\[ L \sim \sum_i b_i v^2 (Nv^m r^{-D+m+2})^i. \]  

(17a)

Note that here the general \( i \)-body interaction goes as \( N^i \), the factor which replaces explicit sums over interacting particles (also, \( b_0 \sim N \)). We naturally assume, as in section 2., that the characteristic size of the bound state is of the order of the Schwarzschild radius \( R_s \) and that the characteristic velocities of particles saturate the Heisenberg uncertainty bound. The virial theorem demands then that \( Nv^m \sim r^{D-m-2} \), which exactly corresponds to the Bekenstein-Hawking entropy formula (1), for \( N \sim S \). The quantum partition function is determined by (14), which is also consistent with the fact that \( N \sim S \).

Note that in case we wish to describe a gravity-like interaction in \( D \) space-time dimensions (and in the infinite momentum limit), the static potential is uniquely determined to be the static transverse-space Newtonian potential i.e. \( r^{-D+4} \) which immediately implies \( m = 2 \) in (17a).

The question arises: How does Matrix theory predictions fit into this particular result?

To start with, the holographic nature of the fundamental Hamiltonian of Matrix theory [1], [21], [22] determines the most general form of the effective post-Newtonian lagrangian (in case we neglect spin dependence) [17], [18], [19], [20]

\[ L_n \sim \sum_{m=0}^{\infty} c_{nm} \left( \frac{Nv^2}{R} \right)^m (\frac{Nv^2}{r^D})^n \left( \frac{v^2}{r^4} \right)^{m-n} \left( \frac{1}{R^2} \right)^m, \]  

(18)

\((n \) denotes the loop order). This lagrangian agrees with the form of (17). In the special case when \( n = m \)

\[ L_{\text{diag}} = \sum_n c_{nn} \left( \frac{Nv^2}{R} \right)^n (\frac{Nv^2}{R^2 r^{D-4}})^n. \]  

(18a)

For example, the effective lagrangian (8) represents the first two terms from (18a). It is again easily seen that (18a) agrees with the form of (17a).
Note that equation (18a) is also in full agreement with the post-Newtonian expansion of a $D$-dimensional Einstein-Hilbert action [17], [18], [19]. As a matter of fact (18a) represents the expansion of a Born-Infeld type of lagrangian considered in [14], [17] (with fixed coefficients $c_{nn}$ [18])

$$L_{B-I} \sim h^{-1}(\sqrt{1-hv^2} - 1), \quad (18b)$$

and with $h \sim r^{-(D-4)}$. In particular, the classical statistical mechanics of such an effective lagrangian without the Gibbs $1/N!$ term is studied in [14]. The classical partition function turns out to be, perhaps not surprisingly, equivalent to (14).

To recapitulate: The virial theorem (when applied to (18a) or equivalently (18b)) implies that $N\frac{v^2}{r^{D-4}} \sim 1$, so that the quantum partition function in this approximation becomes (14), as it should. In addition, the requirement that $v^{D-4} \sim Nv^2$ once again corresponds to the Bekenstein-Hawking formula (1), the moment we apply the same estimates for the velocity and distance as in section 2.

Note that all this indicates that a neutral black hole-like bound state of infinite statistics particles cannot be described by the full effective lagrangian (18), even though (18) is derived from a holographic theory. This particular fact is in accordance with the physical picture developed in [6] which states that a neutral black hole bound state is determined by a highly coherent set of interactions which are all of the same strength. The form of the selected interactions follows from the effective lagrangian (18a,b) or, in other words, from a post-Newtonian expansion of the leading term in the low energy action of Matrix theory, namely, the $D$ dimensional Einstein-Hilbert term.

The above analysis applies to the situation when $Z \sim 1$ or, as described by eq. (16) when $N \sim S$. What happens in the case when $N \gg S$? The result of [6] should apply
then. From this reference we read off the form of the necessary $l$-body effective interactions

$$U_{\text{eff}} \sim (N/R)^l \frac{v^{l+1}}{r^{(l-1)(D-3)}}, \quad (19)$$

where we have dropped the explicit spin dependence which is needed for time-reversal invariance.

Let us apply the virial theorem to this physical situation. The interaction terms are of the order of the kinetic term, and therefore, $N\frac{v}{r^{D-3}} \sim R$. This requirement leads (in the limit $N \gg S$) to the correct dependence of the Schwarzschild radius $R_s$ on temperature (when, as reviewed in section 2, $v \sim R_s T$), i.e. eq. (5). Note that if we consider the $N \sim S$ limit (when $v \sim 1/R_s$), the same interactions (19) would turn out to be compatible with the general expression (17a). The requirement $N\frac{v}{r^{D-3}} \sim R$ is then equivalent to the Bekenstein-Hawking relation (1), as expected.

To summarize: Schwarzschild black holes can be viewed as bound states of particles obeying quantum infinite statistics (such as D0-branes) if the effective interaction between these particles is given by the "diagonal" (Einstein-Hilbert or Born-Infeld) part of the Matrix theory effective lagrangian (18a). This interacting D0-brane gas is of Boltzmann nature [3], [14], because of the effective quantum infinite statistics of its constituents. The entropy of such a gas is proportional to the number of infinite statistics particles, when the value of the quantum partition function is of the order of unity, in accordance with the holographic principle of infinite momentum frame black hole dynamics [21], [22].

The resulting picture nicely meshes with the original ideas of Strominger and Volovich [8] concerning the issue of black hole statistics. Strominger suggested that the quantum statistics of extremal charged four-dimensional black holes is neither Bose-Einstein nor Fermi-Dirac, but rather infinite statistics. Strominger assumed that the black hole quantum state is a functional on the space of closed three-geometries, with each black hole
connected to an oppositely charged black hole via a spatial wormhole. By examining the process of black hole exchange, which is the exchange of two wormhole ends, and which generically results in a new three-geometry, Strominger concluded that the wave function does not have any specific symmetry properties under this operation; the wave function for many extremal black holes is a general function of black hole positions. Hence, black holes resemble distinguishable particles. (Volovich suggested an extension of this idea to D-branes, given the similarity of black holes and D-branes.)

4. In conclusion, in this note we have argued that quantum infinite statistics of D0-branes is what underlies the recently proposed Boltzmann gas model of Schwarzschild black holes in Matrix theory [3], [14]. It would be very interesting to apply the outlined formalism of infinite statistics [9] to the problem of Hawking radiation. For example, the form of creation/annihilation operators as given by (12b), is very suggestive when thinking about the analogy between D0-branes and Hawking particles. The obvious subtle point is that Hawking radiation can be thought of as a $1/N$ effect, whereas the quantum infinite statistics commutation relation (10) nicely fits the usual large $N$ framework [12], [13]. We can only speculate about the possibility that the general $q$-on algebra $a_ia_j^\dagger - qa_j^\dagger a_i = \delta_{ij}$ with $q \sim 1/N$ might turn out to be useful for the Hawking radiation problem.

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