LACK OF ISOMORPHIC EMBEDDINGS OF SYMMETRIC FUNCTION SPACES INTO OPERATOR IDEALS

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Abstract. Let \( E(0, 1) \) be a symmetric space on \((0, 1)\) and \( C_F \) be a symmetric ideal of compact operators on the Hilbert space \( \ell_2 \) associated with a symmetric sequence space \( F \). We give several criteria for \( E(0, 1) \) and \( F \) so that \( E(0, 1) \) does not embed into the ideal \( C_F \), extending the result for the case when \( E(0, 1) = L_p(0, 1) \) and \( F = \ell_p, 1 \leq p < \infty \), due to Arazy and Lindenstrauss [5].

1. Introduction

This paper has been motivated by a beautiful result due to Arazy and Lindenstrauss [5, Theorem 6] (see also its antecedent [41, Theorem 6.1]) that

\[ L_p(0, 1) \not\hookrightarrow C_p, \ 2 < p < \infty, \]

where \( C_p \) is the Schatten \( p \)-class of compact operators on a separable Hilbert space \( \mathcal{H} \), and the notation \( A \hookrightarrow B \) (resp. \( A \not\hookrightarrow B \)) stands for indication that a Banach space \( A \) is (resp. not) isomorphic to a subspace of a Banach space \( B \).

The study of isomorphic classification of classical Banach spaces has a long history and it is one of the most essential topics in the theory of Banach spaces. It is well-known that \( \ell_p \hookrightarrow L_p(0, 1) \) if \( 1 \leq p < \infty \) [1, Lemma 5.1.1 and Proposition 6.4.1] (see also [28] and [48]). On the other hand, \( L_p(0, 1) \not\hookrightarrow \ell_q, p \in [1, \infty), q \in [1, \infty), \) if and only if \( p = q = 2 \) [9, Ch. XII, Theorem 9] (see also [27, 53]). The above-mentioned result by Arazy and Lindenstrauss [5] can be viewed as a noncommutative counterpart of the latter fact.

For the deep theory concerning symmetric structure of general symmetric function spaces \( E(0, 1)/E(0, \infty) \) and symmetric sequence spaces \( F \), we refer to outstanding monographs [26, 39, 40]. Let \( E(0, 1) \) be a symmetric space on \((0, 1)\) and \( F \) be a separable symmetric sequence space, and let \( C_F \) be a symmetric ideal of compact operators on the Hilbert space \( \ell_2 \), generated by \( F \). We are interested in the question:

\[ \text{does } E(0, 1) \text{ isomorphically embed into } C_F? \]

In this general setting, the situation becomes dramatically different. Consider, for instance, the sequence spaces \( \ell_{p,q} \) (resp. function spaces \( L_{p,q}(0, 1) \)), \( 1 < p < \infty, \)
1 \leq q < \infty$, which are the most natural generalizations of the \( \ell_p \)-spaces (resp. \( L_p(0,1) \)-spaces). It was shown recently in [37] and [49] that
\[
\ell_{p,q} \not\leftrightarrow L_{p,q}(0,1), \quad p \in (1,\infty), \quad q \in [1,\infty), \quad p \neq q,
\]
which is in strong contrast with the fact \( \ell_p \hookrightarrow L_p(0,1) \) mentioned above. This remark shows that the techniques used by Arazy and Lindentrau s may not be sufficient to treat the general case.

Below, we briefly introduce the structure of the present paper.

In Section 2, we provide all necessary preliminaries and technical results. Some of them, known for Schatten \( p \)-classes, we establish for general symmetric ideals.

It is well known that there are many fundamental differences in properties of the \( L_p \)-spaces in the cases when \( 1 \leq p < 2 \) and \( 2 < p < \infty \). The same observation holds also for symmetric spaces, which are located between the spaces \( L_1(0,1) \) and \( L_2(0,1) \), on the one hand, and between \( L_2(0,1) \) and \( L_\infty(0,1) \), on the other hand.

In particular, the subspace structure of symmetric spaces located between \( L_1(0,1) \) and \( L_2(0,1) \) is much richer. This fact stipulates the different approaches to these two cases.

Recall that \( L_p(0,1) \) has a subspace isomorphic to \( \ell_r \) for any \( p < r \leq 2 \) [1,16,26,40]. Also, every subspace of the Schatten class \( C_p \) has a subspace isomorphic to \( \ell_2 \) or to \( \ell_p \) [5, Proposition 4]. As a result, one can easily deduce the lack of isomorphic embeddings of \( L_p(0,1) \) into \( C_p \), when \( 1 \leq p < 2 \) [5, p. 197]. Arazy [3, Corollary 3.2] established the following deep result, which allows to use a similar reasoning for general ideals: for any \( p \in [1,2) \cup (2,\infty) \) and a separable symmetric sequence space \( F \), \( \ell_p \) is isomorphically embedded into the ideal \( C_F \) generated by \( F \) if and only if \( \ell_p \hookrightarrow F \).

In Section 3, we consider the case of symmetric spaces located between \( L_1(0,1) \) and \( L_2(0,1) \) and, by making use of this Arazy’s result, show that for any symmetric space \( E(0,1) \) such that \( E(0,1) \supset L_p(0,1) \), with some \( p < 2 \), we have
\[
E(0,1) \not\leftrightarrow C_F
\]
whenever a separable symmetric sequence space \( F \) satisfies the condition: for every \( \epsilon > 0 \) there exists \( r \in (2-\epsilon,2) \) with \( \ell_r \not\hookrightarrow F \). In particular, we show that for any \( 1 < p < 2 \) and \( 1 \leq q < \infty \) we have \( L_{p,q}(0,1) \not\leftrightarrow C_{p,q} \). Similarly, \( \Lambda_p^q(0,1) \not\leftrightarrow \Lambda_p^q \), where \( \Lambda_p^q \) is an arbitrary Lorentz function space such that \( \int_0^1 t^{-q/p}d\psi(t) < \infty \) for some \( 1 < p < 2 \) and \( \Lambda_p^q \) is an arbitrary Lorentz sequence space.

Section 4 contains the principal results of the paper (see Propositions 4.1 and 4.2). Here, we consider symmetric spaces \( E(0,1) \) located between the spaces \( L_2(0,1) \) and \( L_\infty(0,1) \) and operator ideals \( C_F \) generated by \( p \)-convex and \( q \)-concave symmetric sequence spaces \( F \), with some \( 2 < p < q < \infty \). In view of the classical Kadec–Pelczynski alternative for \( L_p \), \( p > 2 \) [28], in this case we cannot hope on the existence of symmetric sequence spaces \( G \) such that \( G \hookrightarrow E(0,1) \) and \( G \not\leftrightarrow C_F \).

In particular, a recent deep result in [22] shows that a subspace of \( L_p(0,1) \), \( p > 2 \), either isomorphically embeds into \( \ell_p \oplus \ell_2 \) or contains \( (\ell_2 \oplus \cdots \oplus \ell_2)_p \). However, both \( \ell_p \oplus \ell_2 \) and \( (\ell_2 \oplus \cdots \oplus \ell_2)_p \) are isomorphic to some subspaces of \( C_p \). This demonstrates why the case \( p > 2 \) is much harder than the case \( 1 \leq p < 2 \). Indeed, as one can see from [5, Theorem 6], the proof of the lack of isomorphic embeddings of \( L_p(0,1) \) into \( C_p \), \( p > 2 \), based on using the classical Haar basis, is rather complicated. Observe that the idea of argument in [5] can be traced back to the proof of Theorem 6.1 by Lindenstrauss and Pelczynski in [41], in which the authors stated that there are no isomorphisms from \( L_p(0,1) \) into the space \( (\ell_2 \oplus \ell_2 \oplus \cdots)_p \) but the proof there was
oversimplified and incomplete (see a related comment in [5]). In Section 4, we succeed in extending of [5, Theorem 6] to some classes of operator ideals generated by $p$-convex and $q$-concave symmetric sequence spaces, $2 < p \leq q < \infty$, in particular, to the class of distributionally concave spaces.

In Section 5, we collect applications of the results obtained in the previous section. In particular, in Corollary 5.3, we prove that $L_M(0, 1) \not\subseteq C_{\ell_M}$ for every submultiplicative Orlicz function $M$, which is equivalent to a $p$-convex Orlicz function for some $p > 2$. This result can be treated as a partial noncommutative extension of a well-known theorem by Lindenstrauss and Tzafriri that an Orlicz function space $L_M(0, 1)$, which is not isomorphic to a Hilbert space, is not isomorphically embedded into any separable sequence Orlicz space $\ell_N$ [38, Theorem 3]. Another application of the results obtained in the previous section relates to the Lorentz spaces $L_{p,q}$: we show that $L_{p,q}(0,1) \not\subseteq C_{p,q}$ if $2 < q \leq p < \infty$ (see Theorem 5.1).

In the final section of the paper, we focus on considering the spaces $L_{2,q}$, $1 \leq q < \infty$, which do not satisfy the assumptions on symmetric function spaces in the preceding sections. The space $L_{2,q}$, $1 \leq q < \infty$ is a typical example of a symmetric function space which is “very close” to the space $L_2$. We show that

$$L_{2,q}(0,1) \not\subseteq C_{2,q}, \quad q \in [1, \infty), \quad q \neq 2.$$ 

The main tools here are known properties of sequences of independent functions in $L_{2,q}$-spaces [6,13], combined with recent results on the lack of isomorphic embeddings from $\ell_{p,q}$ into $L_{p,q}(0,1)$ co-authored by the third named author [37,49] and with a result due to Arazy [3, Theorem 2.4] describing shell-block basic sequences in $C_F$.

2. Preliminaries and auxiliary results

We use [1,26,39,40] as main references to Banach space theory. General facts concerning operator ideals may be found in [19,20,31,36,42] and references therein. For convenience of the reader, some of the basic definitions are recalled.

2.1. Symmetric function and sequence spaces. Let $L_0 := L_0(I)$ be the space of finite almost everywhere Lebesgue measurable functions either on $I = [0,1]$ or $I = [0,\infty)$ (with identification $m$-a.e.) equipped with Lebesgue measure $m$ or the set $I = \mathbb{N}$ of all positive integers equipped with the counting measure (in the latter case, the space $L_0$ coincides with the space $\ell_\infty(\mathbb{N})$ of all bounded real-valued sequences). Denote by $\mathcal{S} := \mathcal{S}(I)$ the subset of $L_0$ which consists of all functions (or sequences) $f$ such that the distribution function

$$d_f(s) := m(\{t \in I : |f(t)| > s\})$$

is finite for some $s > 0$. Any two functions $f$ and $g$ from $\mathcal{S}$ are said to be equimeasurable if $d_f(s) = d_g(s)$ for every $s > 0$. We denote by $f^*$ the non-increasing right-continuous rearrangement of $|f|$ given by

$$f^*(t) := \inf\{s \geq 0 : \ d_f(s) \leq t\}, \quad t \in I.$$

Let $E = E(I)$ be a Banach space of real-valued Lebesgue measurable functions if $I = [0,1]$ or $[0,\infty)$ (resp. of real-valued sequences if $I = \mathbb{N}$). To specify the notation we shall also use $E(0,1)$ or $E(0,\infty)$ instead of $E$. The space $E$ is said to be an ideal lattice if the conditions $f \in E$ and $|g| \leq |f|$, $g \in \mathcal{S}$ imply that $g \in E$ and $\|g\|_E \leq \|f\|_E$. The ideal lattice $E \subseteq \mathcal{S}$ (respectively, $E \subseteq \ell_\infty$) is said to be a symmetric function space (respectively, symmetric sequence space) if the norm
The operator $\sigma$ defined by norms. Also, $I$ decreases on analogous properties.

For every symmetric function space $E$ its fundamental function $\phi_E$ is quasi-concave, that is, it is nonnegative, increases, $\phi_E(0) = 0$, and the function $\phi_E(t)/t$ decreases on $I$. The fundamental function of a symmetric sequence space has analogous properties.

Without loss of generality, for any symmetric function (resp. sequence) space $E$ we always assume that $\|\chi_{[0,1]}\|_E = 1$ (resp. $\|e_0\|_E = 1$).

If $\tau > 0$, the dilation operator $\sigma_\tau$ is defined by setting $\sigma_\tau f(s) = f(s/\tau)$, $s > 0$, in the case of the semi-axis. For a function on the interval $(0,1)$, the operator $\sigma_\tau$ is defined by

$$\sigma_\tau f(s) = \begin{cases} f(s/\tau), & s \leq \min\{1,\tau\}, \\ 0, & \text{otherwise.} \end{cases}$$

The operator $\sigma_\tau$ is bounded in every function symmetric space $E(I)$ and $\|\sigma_\tau\|_{E(I) \to E(I)} \leq \max(1,\tau)$ [36, Theorem II.4.5]. In particular, $\|\sigma_\tau\|_{L_p \to L_p} = \tau^{1/p}$, $1 \leq p \leq \infty$.

Similarly, in the case of sequence spaces, for each $m \in \mathbb{N}$ by $\sigma_m$ and $\sigma_{1/m}$ we define the dilation operators as follows: if $a = (a_n)_{n=0}^\infty$, then

$$\sigma_m a = ((\sigma_m a)_n)_{n=0}^\infty = (\overbrace{m, \ldots, m}^m, a_0, a_0, \ldots, a_1, a_1, \ldots, a_1, \ldots)$$

and

$$\sigma_{1/m} a = ((\sigma_{1/m} a)_n)_{n=0}^\infty = \left(\frac{1}{m} \sum_{k=m}^{(n+1)m-1} a_k\right)_{n=0}^\infty$$

(see, for example, [36, p. 223]). As in the case of function spaces, these operators are bounded in every symmetric sequence space $F$ with the same estimates for their norms. Also, $\|\sigma_{1/m}\|_{\ell_p \to \ell_p} = m^{-1/p}$ and $\|\sigma_m\|_{\ell_p \to \ell_p} = m^{1/p}$, $m \in \mathbb{N}$, $1 \leq p \leq \infty$.

The dilation function $M_f$ of a nonnegative function $f$ on the interval $(0,1)$ is defined by

$$M_f(t) := \sup_{0 < s \leq \min(1,1/t)} \frac{f(st)}{f(s)}, \quad t > 0.$$ 

Since the function $M_f$ is submultiplicative, there are the following dilation exponents:

$$\gamma_\varphi := \lim_{t \to +0} \frac{\ln M_\varphi(t)}{\ln t} \quad \text{and} \quad \delta_\varphi := \lim_{t \to \infty} \frac{\ln M_\varphi(t)}{\ln t}$$

For each quasi-concave function $\varphi$ we have $0 \leq \gamma_\varphi \leq \delta_\varphi \leq 1$ [36, § II.1].

We say that $x \in S(I)$, where $I = [0,1]$ or $I = (0,\infty)$ (resp. $x = (x_k)_{k=0}^\infty \in \ell_\infty$), is submajorized by $y \in S(I)$ (resp. $y = (y_k)_{k=0}^\infty \in \ell_\infty$) in the sense of Hardy–Littlewood–Pólya (briefly, $x \prec \prec y$) if

$$\int_0^t x^*(s) \, ds \leq \int_0^t y^*(s) \, ds, \quad t \in I$$
readily see that condition (\(\varepsilon\text{ is a constant } C\))
\[
\sum_{k=0}^{n} x_k^* \leq \sum_{k=0}^{n} y_k^*, \quad n = 0, 1, 2, \ldots).
\]

See [40, Definition 2.a.6 and Proposition 2.a.8] or [10, \S 2.3] for the main properties of this pre-order. Recall here only that the norm of every separable symmetric space \(E\) is monotone with respect to Hardy–Littlewood–Polya submajorization, i.e., if \(x \in S\) and \(y \in E\) such that \(x \prec\prec y\), then \(x \in E\) with \(\|x\|_E \leq \|y\|_E\) (see e.g. [40, Proposition 2.a.8]).

Let \(E = E(0,1)\) be a symmetric space, \(x_{k,j} := \phi_E(2^{-k})^{-1} \chi_{A_k}\), where \(A_k := [j2^{-k}, (j+1)2^{-k})\), \(k = 1, 2, \ldots\), \(j = 0, 1, \ldots, 2^k - 1\). Then, for each \(k = 1, 2, \ldots\), \(\{x_{k,j}\}_{j=0}^{2^k-1}\) is a normalized basis in the subspace \(E_k := [x_{k,j}, j = 0, \ldots, 2^k - 1]\) of \(E\).

**Lemma 2.1.** Suppose that a symmetric space \(E = E(0,1)\) satisfies the condition:

\[
(2.1) \quad \lim_{t \to \infty} \frac{M_{\phi_E}(t)}{t^{1/2}} = 0.
\]

Then, for arbitrary \(\varepsilon > 0\) there exists a positive integer \(k = k(\varepsilon)\) such that for every linear operator \(V : E_k \to \ell_2\) such that \(\|V\| = 1\) we have

\[
|\{j = 0, 1, \ldots, 2^k - 1 : \|V(x_{k,j})\|_{\ell_2} \geq \varepsilon\}| \leq \varepsilon 2^k.
\]

**Proof.** Denote for each \(k \in \mathbb{N}\)
\[
A_k(\varepsilon) := \{j = 0, 1, \ldots, 2^k - 1 : \|V(x_{k,j})\|_{\ell_2} \geq \varepsilon\}.
\]

Then, on the one hand, for any \(\theta_j = \pm 1\), \(j = 0, 1, \ldots, 2^k - 1\),
\[
\| \sum_{j \in A_k(\varepsilon)} \theta_j V(x_{k,j}) \|_{\ell_2} \leq \|V(\sum_{j \in A_k(\varepsilon)} \theta_j x_{k,j})\|_{\ell_2} \leq \sum_{j \in A_k(\varepsilon)} \|\theta_j x_{k,j}\|_{\ell_2} \leq 2^k \sum_{j \in A_k(\varepsilon)} \|V(x_{k,j})\|_{\ell_2}
\]
\[
= \frac{\phi_E(|A_k(\varepsilon)| \cdot 2^{-k})}{\phi_E(2^{-k})} \leq M_{\phi_E}(|A_k(\varepsilon)|).
\]

Hence, we have
\[
\left(\text{Ave}_{\theta_j = \pm 1} \left\| \sum_{j \in A_k(\varepsilon)} \theta_j V(x_{k,j}) \right\|_{\ell_2}^2 \right)^{1/2} \leq M_{\phi_E}(|A_k(\varepsilon)|).
\]

On the other hand, according to the parallelogram identity,
\[
\left(\text{Ave}_{\theta_j = \pm 1} \left\| \sum_{j \in A_k(\varepsilon)} \theta_j V(x_{k,j}) \right\|_{\ell_2}^2 \right)^{1/2} = \left( \sum_{j \in A_k(\varepsilon)} \|V(x_{k,j})\|_{\ell_2}^2 \right)^{1/2} \geq \varepsilon |A_k(\varepsilon)|^{1/2}.
\]

Consequently,
\[
\varepsilon |A_k(\varepsilon)|^{1/2} \leq M_{\phi_E}(|A_k(\varepsilon)|), \quad k = 1, 2, \ldots.
\]

Combining this inequality with the hypothesis of the lemma, we conclude that there is a constant \(C = C(\varepsilon)\) such that \(|A_k(\varepsilon)| \leq C\) for all \(k = 1, 2, \ldots\). Choosing \(k\) so that \(C 2^k > C\), we get the desired result.

**Remark 2.2.** In particular, by the definition of the dilation exponents, one can readily see that condition (2.1) follows from the inequality \(\delta_{\phi_E} < 1/2\).
2.2. Orlicz and Lorentz spaces. The most known and important symmetric spaces are the $L_p$-spaces, $1 \leq p \leq \infty$. Their natural generalization is the Orlicz spaces. Let $M$ be an Orlicz function, that is, an increasing convex function on $[0, \infty)$ such that $M(0) = 0$. Denote by $L_M := L_M(I)$, where $I = (0, 1)$ or $(0, \infty)$, the Orlicz space on $I$ (see e.g. [35]) endowed with the Luxemburg–Nakano norm

$$
\|f\|_{L_M} = \inf \left\{ v > 0 : \int_I M(|f(t)|/v) \, dt \leq 1 \right\}.
$$

In particular, if $M(u) = u^p$, $1 \leq p < \infty$, we obtain $L_p$. One can readily check that the fundamental function of $L_M$ is determined by the formula: $\phi_{L_M}(u) = 1/M^{-1}(1/u)$, $0 < u \leq 1$, where $M^{-1}$ is the inverse function for $M$.

Similarly, we can define an Orlicz sequence space. Specifically, the space $\ell_N$, where $N$ is an Orlicz function, consists of all sequences $(a_k)_{k=0}^\infty$ such that

$$
\|(a_k)_{k=0}^\infty\|_{\ell_N} := \inf \left\{ u > 0 : \sum_{k=0}^\infty N\left(\frac{|a_k|}{u}\right) \leq 1 \right\} < \infty.
$$

An Orlicz function $H$ satisfies the $\Delta^2_\infty$-condition ($H \in \Delta^2_\infty$) (resp. the $\Delta^1_\infty$-condition ($H \in \Delta^1_\infty$)) if

$$
\limsup_{t \to \infty} \frac{H(2t)}{H(t)} < \infty \quad (\text{resp.} \limsup_{t \to 0} \frac{H(2t)}{H(t)} < \infty).
$$

It is well known that an Orlicz function space $L_M$ on $[0, 1]$ (resp. an Orlicz sequence space $\ell_N$) is separable if and only if $M \in \Delta^\infty_\infty$ (resp. $N \in \Delta^1_\infty$).

Observe that the definition of an Orlicz sequence space $\ell_N$ depends (up to equivalence of norms) only on the behaviour of the function $N$ near zero. More precisely, in the separable case (i.e., when $N, N_1 \in \Delta^1_\infty$), the following conditions are equivalent: 1) $\ell_N = \ell_{N_1}$ (with equivalence of norms); 2) the unit vector bases of the spaces $\ell_N$ and $\ell_{N_1}$ are equivalent; 3) there are $C > 0$ and $t_0 > 0$ such that for all $0 \leq t \leq t_0$ it holds

$$
C^{-1}N_1(t) \leq N(t) \leq CN_1(t)
$$

(cf. [39, Proposition 4.a.5]). Quite similarly, the definition of an Orlicz function space $L_M$ on $[0, 1]$ depends only on the behaviour of the function $M$ for large values of the argument.

Another natural generalization of the $L_p$-spaces is the class of Lorentz spaces. Let $\psi$ be an increasing concave function on $I$ with $\psi(0) = \psi(\infty) = \psi(+0) = 0$, $\psi(\infty) = \infty$ and $1 \leq q < \infty$. The Lorentz space $\Lambda^q_\psi := \Lambda^q_\psi(I)$ consists of all measurable functions $f$ on $I$, for which

$$
\|f\|_{\Lambda^q_\psi} := \left( \int_I f^*(t)^q d\psi(t) \right)^{1/q} < \infty
$$

(see [32, 34, 39, 40]). It is well-known that $\Lambda^q_\psi(I)$ is separable for all $\psi$ and $1 \leq q < \infty$ [32].

Recall also the definition of Lorentz spaces $L_{p,q} := L_{p,q}(I)$ [10, 14, 16, 47]. If $1 < p < \infty$ and $1 \leq q \leq \infty$, then $L_{p,q}$ is the space of all measurable functions $f$ on $I$ such that

$$
\|f\|_{L_{p,q}} := \begin{cases} 
(\int_I f^*(t)^q d(t^{q/p}))^{1/q}, & q < \infty; \\
\sup_{t \in I} (t^{1/p} f^*(t)), & q = \infty
\end{cases}
$$

is finite. In particular, $L_{p,\infty}$, $1 < p < \infty$, are called often the weak $L_p$-spaces. It is clear that if $1 \leq q \leq p < \infty$ and $\psi(t) := t^{q/p}$, then $L_{p,q}(I) = \Lambda^q_\psi(I)$. In
this case \( \| \cdot \|_{p,q} \) defines a norm under which \( L_{p,q} \) is a separable symmetric space; for \( 1 < p < q \leq \infty \), \( \| \cdot \|_{p,q} \) is a quasi-norm which is known to be equivalent to a symmetric norm [10, Theorem 4.4.6].

Define also the Lorentz sequence space \( \ell^q_w \), where \( 1 \leq q < \infty \) and \( w = (w_n)_{n=0}^{\infty} \) is a decreasing sequence of positive numbers such that \( w_0 = 0 \), \( \lim_{n \to \infty} w_n = 0 \) and \( \sum_{n=0}^{\infty} w_n = \infty \), as the set of all sequences \( a = (a_n)_{n=0}^{\infty} \) such that

\[
\|a\|_{\ell^q_w} := \left( \sum_{n=0}^{\infty} (a_n^*)^q w_n \right)^{1/q} < \infty,
\]

where \( (a_n^*)_{n=0}^{\infty} \) is the decreasing permutation of the sequence \( (|a_n|)_{n=0}^{\infty} \).

Similarly, if \( 1 < p < \infty \) and \( 1 \leq q \leq \infty \), the space \( \ell_{p,q} \) consists of all sequences \( a = (a_n)_{n=0}^{\infty} \), for which

\[
\|a\|_{\ell_{p,q}} := \begin{cases} \left( \sum_{n=0}^{\infty} (a_n^*)^q (n^{q/p} - (n-1)^{q/p}) \right)^{1/q}, & q < \infty; \\ \sup_{n=0,1,2,...} (a_n^* n^{1/p}), & q = \infty \end{cases}
\]

is finite.

We note that the norm of all Orlicz and Lorentz spaces defined above is monotone with respect to Hardy–Littlewood–Pólya submajorization [10, 32, 36].

2.3. Operator ideals in \( B(\mathcal{H}) \). Define by \( B(\mathcal{H}) \) the *-algebra of all bounded linear operators acting on a separable infinite-dimensional Hilbert space \( \mathcal{H} \). For any \( x \in B(\mathcal{H}) \), we denote by \( \{\mu(n;x)\}_{n=0}^{\infty} \) the sequence of singular values of \( x \), i.e., the eigenvalues of \( (x^*x)^{1/2} \) arranged in a non-increasing ordering, counting multiplicity.

Let \( F \) be a symmetric sequence space. We work with the ideal \( C_F \) in the algebra \( B(\mathcal{H}) \) defined as follows

\[ C_F = \{ a \in B(\mathcal{H}) : \{\mu(k;a)\}_{k=0}^{\infty} \in F \}. \]

This ideal becomes Banach when equipped with the norm \( \|a\|_{C_F} = \|\mu(a)\|_F, \ a \in C_F \) [31, 42]. When \( F = \ell_p \), \( p \geq 1 \), we denote by \( C_p \) the corresponding operator ideal

\[ \{ a \in B(\mathcal{H}) : \mu(A) \in \ell_p \}, \quad \|a\|_p = \|\mu(a)\|_p, \]

which are the best known examples of Banach ideals in \( B(\mathcal{H}) \) (called Schatten-von Neumann p-class). When \( F = \ell_{p,q} \), \( 1 < p < \infty \), \( 1 \leq q \leq \infty \), we denote the corresponding operator ideal by \( C_{p,q} \).

Quite similarly, we can define also the ideals \( C_{\ell,M} \), for each Orlicz function, and \( C_{\ell,M}^q \), where \( w = (w_n)_{n=0}^{\infty} \) is a decreasing sequence of positive numbers such that \( w_0 = 1 \), \( \lim_{n \to \infty} w_n = 0 \), \( \sum_{n=0}^{\infty} w_n = \infty \) and \( 1 \leq q < \infty \).

We use the notion of the right support of the operator \( a \in B(\mathcal{H}) \) defined as follows

\[ r(a) = \bigwedge \{ p \text{ is a projection in } B(\mathcal{H}) : ap = a \}. \]

Operators \( a_k \in B(\mathcal{H}) \), \( k \geq 0 \), are called disjointly supported from the right if \( r(a_{k_1}) r(a_{k_2}) = 0 \) for \( k_1 \neq k_2 \). Equivalently, \( |a_{k_1}| \cdot |a_{k_2}| = 0 \) for \( k_1 \neq k_2 \). Whenever \( a_k \), \( k \geq 0 \), are disjointly supported from the right and the subsets \( A_l \in \mathbb{Z}, l \geq 0 \), are disjoint (i.e. \( A_{l_1} \cap A_{l_2} = \emptyset \) for \( l_1 \neq l_2 \)), the elements

\[ b_l = \sum_{k \in A_l} a_k \]
are also disjointly supported from the right.

Modifying the argument in [51, Proposition 2.3], we observe that if a basic sequence in \( C_F \) consists of elements which are pairwise disjointly supported from the left and from the right, then this sequence is (isometrically) equivalent to the corresponding basic sequence of pairwise disjointly supported elements in the symmetric sequence space \( F \). However, in general, this fact fails for elements in \( C_F \) which are pairwise disjointly supported only from the left (or, only from the right).

2.4. **Distributional concavity.** Fix a partition \( \mathbb{N} = \bigcup_{k=1}^{\infty} U_k \), where \( U_k, k = 1, 2, \ldots \), are infinite disjoint sets, and one-to-one mappings \( \kappa_k : U_k \to \mathbb{N}, k = 1, 2, \ldots \). Define now as a **disjoint sum** of a set \( a_k = (a_{k,i})_{i=1}^{\infty}, k = 1, 2, \ldots \), of sequences of real numbers the sequence

\[
\bigoplus_{k=1}^{\infty} a_k := \sum_{k=1}^{\infty} \sum_{i \in U_k} a_{k, \kappa_k(i)} e_i,
\]

where \( e_i \) are standard unit vectors. It is important to observe that the distribution function of a disjoint sum \( \bigoplus_{k=1}^{\infty} a_k \) does not depend on the particular choice of a partition \( \mathbb{N} = \bigcup_{k=1}^{\infty} U_k \) and mappings \( \kappa_k : U_k \to \mathbb{N}, k = 1, 2, \ldots \).

In particular, in the case when \( a_k = a \) if \( k = 1, \ldots, n \), and \( a_k = 0 \) if \( k > n \), we will denote \( \bigoplus_{k=1}^{\infty} a_k \) by \( a^{\oplus n} \).

Now we can adopt the well-known definition of distributionally concave symmetric function spaces on \( [0, 1] \) (see e.g. [8, Definition 2.2]) in the case of symmetric sequence spaces as follows.

**Definition 2.3.** A symmetric sequence space \( F \) is called **distributionally concave** if there is a constant \( c_F > 0 \) such that for every finite collection \( \{a_k\}_{k=1}^{n} \subset F \) we have

\[
\left\| \bigoplus_{k=1}^{n} a_k \right\|_F \geq c_F \min_{1 \leq k \leq n} \left\| a_k^{\oplus n} \right\|_F.
\]

As in the case of function spaces (see e.g. [46], [8, Proposition 2.5], [44, Proposition 19] and [52, Proposition 2.5]), it can be easily checked that all Orlicz sequence spaces \( \ell_M \) and Lorentz spaces \( \lambda^q \) (in particular, \( \ell_{p,q}, 1 \leq q \leq p < \infty \)) are distributionally concave.

**Remark 2.4.** For arbitrary Hilbert space \( \mathcal{H} \) and every positive integer \( n \) we clearly have \( \mathcal{H}^{\oplus n} \simeq \mathcal{H} \) a natural isomorphism. Then, if \( a_k \in B(\mathcal{H}), k = 1, 2, \ldots, n \), considering the image of the direct sum \( \bigoplus_{k=1}^{n} a_k \), under this isomorphism, as an element of \( \mathcal{H} \), by the definition of singular values of operators, we obtain

\[
\mu \left( \bigoplus_{k=1}^{n} a_k \right) = \left( \bigoplus_{k=1}^{n} \mu(a_k) \right)^*,
\]

where \( \bigoplus_{k=1}^{n} \mu(a_k) \) is the disjoint sum of the sequences \( \mu(a_k), k = 1, 2, \ldots, n \) and \( \left( \bigoplus_{k=1}^{n} \mu(a_k) \right)^* \) stands for the decreasing rearrangement of \( \bigoplus_{k=1}^{n} \mu(a_k) \) (see Subsection 2.1). Hence, for any distributionally concave symmetric sequence space \( F \) we

\[1\)This fact justifies that for the direct sum we use the same symbol as for a disjoint sum.
have
\[ \| {\bigoplus_{k=1}^{n} a_k } \|_{C_F} = \| \mu \left( \bigoplus_{k=1}^{n} a_k \right) \|_{F} \geq c_F \min_{1 \leq k \leq n} \| \mu(a_k)^{\oplus n} \|_{F} \]
\[ = c_F \min_{1 \leq k \leq n} \| a_k^{\oplus n} \|_{C_F}. \] (2.2)

2.5. The upper triangular part of $C_F$. Recall first that a pair $(X_0, X_1)$ of Banach spaces is called a Banach couple if $X_0$ and $X_1$ are both linearly and continuously embedded in some Hausdorff linear topological vector space. In particular, every two symmetric (function or sequence) spaces $E_0$ and $E_1$ form a Banach couple.

A Banach space $X$ is called interpolation with respect to a Banach couple $(X_0, X_1)$ (in brief, $X \in \text{Int}(X_0, X_1)$) whenever $X_0 \cap X_1 \subset X \subset X_0 + X_1$ and each linear operator $T : X_0 + X_1 \to X_0 + X_1$, which is bounded in $X_0$ and in $X_1$, is bounded in $X$. For a further information related to the theory of interpolation of operators we refer to the monographs [10, 36, 40].

The following lemma establishes an isomorphic embedding from an operator ideal onto its upper triangular part, which extends [5, Proposition 1].

Lemma 2.5. For every separable symmetric sequence space $F \in \text{Int}(\ell_p, \ell_q)$, $1 < p, q < \infty$, there exists an isomorphic embedding from $C_F$ onto its upper triangular part $U_F := \{ x \in C_F : x_{ij} = 0, i > j \}$.

Proof. Let $T$ be the upper triangular truncation operator (see e.g. [2, (1.1)]). Recall that $T$ is bounded on $C_F$ [2, Corollary 4.12]. Let $S$ be the transposition operator. In fact, it is an isometry on $C_F$ because it preserves the singular value function. Let $D$ be the diagonal cut. Note that the assumption that $F \in \text{Int}(\ell_p, \ell_q)$ implies that $F$ is monotone with respect to Hardy–Littlewood–Pólya submajorization (see e.g. [36] and [10], see also [12, Theorem 3.1]). Moreover, since $Dx \prec_\prec x$ for every $x$ (see e.g. [20, Lemma 6.1]), it follows that $D$ is bounded on $C_F$.

Define a bounded mapping $A : C_F \to U_F \oplus U_F := U_F^{\oplus 2}$ by the formula
\[ Ax = Tx \oplus TSx, \quad x \in C_F. \] (2.3)

The boundedness of $A$ follows immediately from the boundedness of $T : C_F \to C_F$. We now show that $A$ is an isomorphic embedding. Since $x = Tx + (STS)(x - Dx)$, it follows that
\[ \| x \|_{C_F} \leq \| Tx \|_{C_F} + \| (TS)(x - Dx) \|_{C_F} \leq \| Tx \|_{C_F} + \| TSx \|_{C_F} + \| T \| \| Dx \|_{C_F} \]
\[ \leq 2 \| Ax \|_{U_F^{\oplus 2}} + \| T \| \| Dx \|_{C_F}. \] (2.4)

Since $Dx = DTx \prec_\prec Tx$ and since the norm in $F$ is monotone with respect to the Hardy–Littlewood–Pólya submajorisation, it follows that
\[ \| Dx \|_{C_F} \leq \| Tx \|_{C_F} \leq \| Ax \|_{U_F^{\oplus 2}}. \]

Therefore, by (2.4), we have
\[ \| x \|_{C_F} \leq 3 \| Ax \|_{U_F^{\oplus 2}}, \]
which shows that the bounded operator $A : C_F \to U_F^{\oplus 2}$ is an isomorphic embedding.

We now claim that $U_F^{\oplus 2}$ admits an isomorphic embedding into $U_F$. Let $p_1, p_2$ be projections in $B(\mathcal{H})$ such that
1. $p_1$ and $p_2$ are diagonal.
2. $p_1p_2 = 0$.
3. each $p_k$ has infinite rank, i.e., $\text{Tr}(p_k) = \infty$, where $\text{Tr}$ is the standard trace on $B(H)$.
4. $p_1 + p_2 = 1$.

Since $p_k$ is diagonal, it follows that, for a given $l \geq 0$, either $p_ke_{ll} = e_{ll}$ or $p_ke_{ll} = 0$.

Set $B_k = \{l \geq 0 : p_k e_{ll} = e_{ll}\}, k = 1, 2$. Let $\theta_k : B_k \to \mathbb{Z}_+ \cup \{0\}$ be the monotone bijection, i.e.,

\[ \theta_k(l) = |\{j \in B_k : j \leq l\}| - 1, \quad l \in B_k. \]

Define the mapping $V_k : (p_k B(H)p_k, \text{Tr}) \to (B(H), \text{Tr})$ by the setting

\[ V_k : \sum_{l_1, l_2 \in B_k} a_{l_1 l_2} e_{l_1 l_2} = \sum_{l_1, l_2 \in B_k} a_{l_1 l_2} e_{\theta_k(l_1) \theta_k(l_2)}. \]

This is a trace-preserving $*$-homomorphism. Moreover, $V_k$ preserves the singular value function. Since $\theta_k$ is monotone, it follows that $V_k$ maps upper triangular matrices to the upper triangular ones. Thus, $V_k : p_k U_F p_k \to U_F$ is an isometry. Hence,

\[ U_F \approx p_1 U_F p_1, \quad U_F \approx p_2 U_F p_2, \]

\[ U_F^{\oplus 2} \approx p_1 U_F p_1 \oplus p_2 U_F p_2 \approx p_1 U_F p_1 + p_2 U_F p_2, \]

where the last equivalence can be seen as follows: for every $x_k \in p_k U_F p_k$, we have

\[ \mu(x_1 + x_2) = \mu(x_1) + \mu(x_2) \text{ and, therefore, } \|x_1 + x_2\|_{C_F} \approx \|x_1\|_{C_F} + \|x_2\|_{C_F}. \]

Since we have established that $C_F$ admits an isomorphic embedding into $U_F^{\oplus 2}$, it follows that $C_F$ admits an isomorphic embedding into $U_F$. This completes the proof.

2.6. Haar system and Rademacher functions. Recall that the Haar system [39, Definition 1.a.4.] can be defined for $l = 0, 1, \ldots, 2^k - 1$ and $k = 0, 1, \ldots$, by setting

\[ h_{2^k + l}(t) = \begin{cases} 1, & l \cdot 2^{-k} < t < (l + \frac{1}{2})2^{-k}, \\ -1, & (l + \frac{1}{2})2^{-k} < t < (l + 1)2^{-k}, \\ 0, & \text{otherwise}. \end{cases} \]

Lemma 2.6. Let $F$ be a separable symmetric sequence space and $F \in \text{Int}(\ell_p, \ell_q)$ for some $1 < p, q < \infty$. If a separable space $E(0, 1)$ isomorphically embeds into $U_F$, then there is another isomorphic embedding of $E(0, 1)$ into $U_F$ such that images of the Haar basis are disjointly supported from the right.

Proof. Let $e_{ll}$, $l \geq 0$, be the $l$-th matrix unit on the diagonal. It is known (see [2, Lemma 4.5]) that $\{U_F e_{ll}\}_{l \geq 0}$ is an unconditional finite dimensional decomposition (see e.g. [39, Chapter 1.g] for definition) in $U_F$. By [18, Theorem 5.2], there exists an increasing sequence $\{q_m\}_{m \geq 0} \subset \mathbb{Z}_+$ and a basic sequence $z_m \in U_F(\sum_{l=q_m}^{q_{m+1}-1} e_{ll})$ which is equivalent to the Haar basis. It is immediate that the support $r(z_m) \leq \sum_{l=q_m}^{q_{m+1}-1} e_{ll}$, $m \geq 0$. Therefore, $r(z_m) r(z_{m_2}) = 0$ for $m_1 \neq m_2$. Recall that the Haar system is a Schauder basis of any separable symmetric function space on $(0, 1)$ [40, Proposition 2.c.1]. Therefore, the isomorphism is given by the formula

\[ \sum_{m \geq 0} \alpha_m h_m \mapsto \sum_{m \geq 0} \alpha_m z_m. \]
satisfies all desired conditions.

Let \( r_k(t), k = 0, 1, 2, \ldots, t \in [0, 1] \), be the Rademacher functions. Then, for all positive integers \( n \geq k \) and \( j = 0, 1, \ldots, 2^k - 1 \), we set \( r_{n,k,j} := r_n \chi \left( \frac{k - j}{2^k} \right) \).

**Lemma 2.7.** Let \( E := E(0, 1) \) be a symmetric space with the fundamental function \( \phi_E \). Then, for every \( k \in \mathbb{N} \) and all \( j = 0, 1, \ldots, 2^k - 1 \), we have

\[
(2.5) \quad \left\| \sum_{n=k}^{\infty} a_n r_{n,k,j} \right\|_E \geq \frac{1}{32} \phi_E(2^{-k}) \|(a_n)\|_{\ell_2}.
\]

**Proof.** Denoting

\[
B := \left\{ t \in [0, 1] : \left| \sum_{n=k}^{\infty} a_n r_{n,k,j}(t) \right| \geq \frac{1}{2} \|(a_n)\|_{\ell_2} \right\},
\]

by the Paley–Zygmund inequality (see e.g. [29, p. 8]), we have

\[
m(B) = m\{ t \in [0, 1] : \left| \sum_{n=k}^{\infty} a_n r_{n,k,j}(t) \right| \geq \frac{1}{2} \|(a_n)\|_{\ell_2} \} \geq \frac{1}{16} 2^{-k}.
\]

Hence, from the quasi-concavity of \( \phi_E \) it follows

\[
\left\| \sum_{n=k}^{\infty} a_n r_{n,k,j} \right\|_E \geq \frac{1}{2} \|(a_n)\|_{\ell_2} \chi_B \left( \frac{2^{-k}}{16} \right) \geq \frac{1}{32} \|(a_n)\|_{\ell_2} \phi_E(2^{-k}).
\]

\[
(2.6) \geq \frac{1}{2} \|(a_n)\|_{\ell_2} \chi_B \left( \frac{2^{-k}}{16} \right) \geq \frac{1}{32} \|(a_n)\|_{\ell_2} \phi_E(2^{-k}).
\]

\[
(2.6) \geq \frac{1}{2} \|(a_n)\|_{\ell_2} \chi_B \left( \frac{2^{-k}}{16} \right) \geq \frac{1}{32} \|(a_n)\|_{\ell_2} \phi_E(2^{-k}).
\]

2.7. \( p \)-convexity, \( q \)-concavity and close notions. Recall that a symmetric (sequence/function/operator) space \( E \) is said to be \( p \)-convex (resp. \( q \)-concave) [4, 17, 39, 40] if there exists a constant \( K > 0 \) such that, for every \( x_1, \ldots, x_n \) in \( E \), we have

\[
\left\| \left( \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \right\|_E \leq K \left( \sum_{k=1}^{n} \|x_k\|^p \right)^{1/p}
\]

(resp.

\[
\left( \sum_{k=1}^{n} \|x_k\|^q \right)^{1/q} \leq K \left( \sum_{k=1}^{n} |x_k|^q \right)^{1/q}.
\]

Introduce also the following weaker notions. A symmetric (sequence/function/operator) space \( E \) is said to satisfy an upper \( p \)-estimate (resp. a lower \( q \)-estimate) [17, 34, 39, 40] if there exists a constant \( K > 0 \) such that, for every \( x_1, \ldots, x_n \) of pairwise left disjointly supported elements in \( E \),

\[
\left\| \sum_{k=1}^{n} x_k \right\|_E \leq K \left( \sum_{k=1}^{n} \|x_k\|^p \right)^{1/p}
\]
Clearly, if a symmetric (sequence/function/operator) space $E$ is $p$-convex (resp. $q$-concave), then it admits an upper $p$-estimate (resp. a lower $q$-estimate). Conversely, if a symmetric function/sequence space $E$ satisfies an upper $p$-estimate, $p > 1$ (resp. a lower $q$-estimate, $q < \infty$), then $E$ is $p_1$-convex for each $p_1 \in (1, p)$ (resp. $q_1$-concave for each $q_1 \in (q, \infty)$) \cite[Theorem 1.7]{40}.

According to \cite{46}, we will refer to an Orlicz function $M$ as $p$-convex (resp. $q$-concave) if the function $t \to M(t^{1/p})$ (resp. $t \to M(t^{1/q})$) is convex (resp. concave) on $(0, \infty)$. By \cite[Lemma 20]{46}, $M$ is equivalent to a $p$-convex (resp. $q$-concave) function on $(0, \infty)$ if and only if there exists a constant $C > 0$ such that for all $t > 0$ and $0 < s \leq 1$ we have

\begin{equation}
M(st) \leq Cs^p M(t) \tag{2.7}
\end{equation}

(resp.

\begin{equation}
s^q M(t) \leq CM(st) \tag{2.8}
\end{equation}

An Orlicz function $M$ is equivalent to a $p$-convex (resp. $q$-concave) function on the interval $[0, 1]$ if and only if (2.7) (resp. (2.8)) holds for all $0 < t \leq 1$ and $0 < s \leq 1$ (see also \cite[Lemma 6]{7} and \cite[Lemma 11]{24}). This is equivalent to the $p$-convexity (resp. $q$-concavity) of the sequence Orlicz space $\ell_M$ (see e.g. \cite[pages 121 and 124]{33}). Similarly, an Orlicz function $M$ is equivalent to a $p$-convex (resp. $q$-concave) function on the interval $[1, \infty)$ if and only if (2.7) (resp. (2.8)) holds for all $t \geq st \geq 1$, and this is equivalent to the $p$-convexity (resp. $q$-concavity) of the Orlicz space $L_M := L_M(0, 1)$.

It is well known that either of the spaces $L_{p,q}(I)$, where $I = (0, 1)$ or $I = (0, \infty)$, and $\ell_{p,q}$, $1 < p < \infty$, $1 \leq q < \infty$, is $q$-convex and admits a lower $p$-estimate in the case when $1 \leq q \leq p < \infty$, and it is $q$-concave and admits an upper $p$-estimate in the case when $1 \leq p < q < \infty$ (see e.g. \cite{14,16,34,39,40}).

It is shown in \cite[Corollary 5.3]{17} (see also \cite{4}) that if $F$ is a separable symmetric sequence space satisfying an upper $p$-estimate, $p \in (1, 2]$, then the corresponding operator space $C_F$ satisfies an upper $r$-estimate for each $1 \leq r < p$. If $F$ is a $q$-concave separable symmetric sequence space for some $q \geq 2$, then $C_F$ satisfies a lower $q$-estimate.

Under a slightly stronger assumption, for any $p > 0$, we have the following analogue.

**Lemma 2.8.** Let $F$ be a symmetric (or symmetrically quasi-normed) sequence space having an upper $p$-estimate for some $p > 0$ (or a lower $q$-estimate for some $q > 0$). Then, for every finite sequence $x_1, \ldots, x_n$ of pairwise left and right disjointly supported elements in $C_F$, we have

$$
\left\| \sum_{k=1}^{n} x_k \right\|_{C_F} \leq D_F \left( \sum_{k=1}^{n} \left\| x_k \right\|_{C_F}^p \right)^{1/p},
$$
respectively,
\[
\left( \sum_{k=1}^{n} \|x_k\|_{C_F}^q \right)^{1/q} \leq D'_F \left\| \sum_{k=1}^{n} x_k \right\|_{C_F}.
\]

**Proof.** Since \(x_1, \ldots, x_n\) are pairwise left and right disjointly supported, it follows that
\[
\mu \left( \sum_{k=1}^{n} x_k \right) = \left( \bigoplus_{k=1}^{n} \mu(x_k) \right)^*.
\]
By the assumption that \(F\) satisfies an upper \(p\)-estimate, we obtain that
\[
\left\| \sum_{k=1}^{n} x_k \right\|_{C_F} \leq \left\| \bigoplus_{k=1}^{n} \mu(x_k) \right\|_{F} \leq D_F \left( \sum_{k=1}^{n} \|\mu(x_k)\|_{F}^P \right)^{1/p} = D_F \left( \sum_{k=1}^{n} \|x_k\|_{C_F}^P \right)^{1/p}.
\]
The same argument yields the case for lower \(q\)-estimate. \(\square\)

The notion of \(p\)-convexity is closely connected with the important concept of Rademacher type. A Banach space \(X\) is said to have Rademacher \(q\)-type, where \(1 \leq q \leq 2\), if there exists a constant \(K > 0\) such that for all \(n \in \mathbb{N}\) and \(x_j \in X\), \(j = 1, 2, \ldots, n\), we have
\[
\left\| \sum_{j=1}^{n} x_j r_j(t) \right\|_{X} \leq K \left( \sum_{j=1}^{n} \|x_j\|_X^q \right)^{1/q}.
\]
Clearly, every Banach space has Rademacher 1-type. Moreover, if \(X\) has Rademacher \(q_1\)-type and \(1 \leq q_2 < q_1 \leq 2\), then \(X\) possesses also Rademacher \(q_2\)-type.

Assume that a symmetric sequence space \(F\) has Rademacher 2-type. Then, by \cite[Corollary from Theorem 4]{21}, the ideal \(C_F\) has Rademacher 2-type as well. This fact combined together with the Kahane-Khintchine inequality (see e.g. \cite[Theorem II.4]{29} or \cite[Theorem 1.e.13]{40}) implies that for every \(1 \leq p < \infty\) there exists \(K_0 \geq 1\) such that for all \(n \in \mathbb{N}\) and \(x_j \in C_F\), \(j = 1, 2, \ldots, n\), we have
\[
\left( \int_0^1 \left\| \sum_{j=1}^{n} x_j r_j(t) \right\|_{C_F}^p \ dt \right)^{1/p} \leq K_0 \left( \sum_{j=1}^{n} \|x_j\|_{C_F}^2 \right)^{1/2}.
\]
Let \(C_F\) be the ideal of the algebra \(B(H)\) generated by a symmetric sequence space \(F\). We define projections \(R_m\) and \(P_m\), \(m = 0, 1, 2, \cdots\), on \(C_F\) by setting: if \(x = (x_{i,j}) \in C_F\), then
\[
R_m x = \begin{cases} x_{i,j}, & \text{max}(i,j) \leq m, \\ 0, & \text{otherwise}, \end{cases}
\]
and
\[
P_m x = \begin{cases} x_{i,j}, & \text{min}(i,j) \leq m, \\ 0, & \text{otherwise}, \end{cases}
\]
Clearly, for any \(m = 0, 1, 2, \cdots\), \(\|R_m\|_{C_F \to C_F} \leq 1\) and \(\|P_m\|_{C_F \to C_F} \leq 2\) (see e.g. \cite[(2.7) and (2.8)]{5}). Moreover, \((P_m - P_n)x\) and \((P_n - P_l)x\) are disjointly supported from the left for any upper triangular operator \(x \in B(H)\) and any \(l \leq n \leq m\).

The following result is an extension of \cite[Lemma 4]{5}.
Proposition 2.9. Let $F$ be a separable symmetric sequence space admitting an upper $p$-estimate and having Rademacher $2$-type. Suppose that $\{u_n\}_{n=1}^{\infty} \subset F$ is a sequence of upper triangular operators such that $\|u_n\|_{C_F} \leq M$ and

$$\sum_{n=1}^{\infty} a_n u_n \|_{C_F} \geq M^{-1} \|a\|_{\ell_2}, \text{ for all } a = (a_n)_{n=1}^{\infty} \in \ell_2. \quad (2.11)$$

Then, if $0 < \gamma < (MK_0)^{-1}$ ($K_0$ is the constant from inequality (2.10)), we can find $m \in \mathbb{N}$ such that

$$\|P_m u_n\|_{C_F} \geq \gamma, \quad n = 1, 2, \ldots \quad (2.12)$$

Proof. Assume by contradiction that for any $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that

$$\|P_m u_n\|_{C_F} < \gamma. \quad (2.13)$$

Without loss of generality, we may assume that for every $m$ the last inequality holds for infinitely many $n$. Indeed, suppose that for a given $m$, there exist only finitely many $n$ (denoted by $m(1)$, $m(2)$, \ldots, $m(k)$) such that $\|P_m u_n\|_{C_F} < \gamma$. Since $P_m \uparrow 1$, where 1 is the identity, and $\|u_n\|_{C_F} \geq M^{-1} \geq (MK_0)^{-1} > \gamma$ for all $n \in \mathbb{N}$ (see (2.11)), it follows from the separability and [20, Theorem 3.1] (see also [15, 23]) that we can find $m'$ such that

$$\|P_m' u_n\|_{C_F} > \gamma \quad \text{if } n = m(1), m(2), \ldots, m(k).$$

Since

$$\|P_m' u_n\|_{C_F} \geq \|P_m u_n\|_{C_F} > \gamma, \quad n \neq m(1), m(2), \ldots, m(k),$$

it follows that (2.12) holds with $m'$ instead of $m$. Thus, next we may assume that for every given $m$ inequality (2.13) holds for infinitely many values of $n \in \mathbb{N}$. Consequently, there are two increasing sequences of positive integers $\{m_i\}_{i=0}^{\infty}$ ($m_0 = 0$) and $\{n_i\}_{i=1}^{\infty}$ such that

$$\|(1 - R_{m_i}) u_{n_i}\|_{C_F} < 2^{-i} \quad \text{and} \quad \|P_{m_{i-1}} u_{n_i}\|_{C_F} < \gamma, \quad i = 1, 2, \ldots \quad (2.14)$$

Next, by (2.11) and the triangle inequality for the $L_p$-norm, for each $k \in \mathbb{N}$, we have

$$M^{-1}k^{1/2} \overset{(2.11)}{\leq} \left( \int_0^1 \left\| \sum_{i=1}^{k} r_i(t) u_{n_i} \right\|_{C_F}^p \, dt \right)^{1/p} \overset{(2.15)}{\leq} \left( \int_0^1 \left\| \sum_{i=1}^{k} r_i(t) R_{m_i} (1 - P_{m_{i-1}}) u_{n_i} \right\|_{C_F}^p \, dt \right)^{1/p}
\quad + \left( \int_0^1 \left\| \sum_{i=1}^{k} r_i(t) \left[ u_{n_i} - R_{m_i} (1 - P_{m_{i-1}}) u_{n_i} \right] \right\|_{C_F}^p \, dt \right)^{1/p}.$$
where \( D_F \) is the \( p \)-upper estimate constant of \( F \). Moreover, applying (2.10), we see that the second term in (2.15) is not bigger than

\[
K_0 \left\{ \sum_{i=1}^{k} \left( \| (1 - R_{m_i}) u_{n_i} \|_{C_F} + \| R_{m_i} P_{m_{i-1}} u_{n_i} \|_{C_F} \right)^2 \right\}^{1/2} \leq K_0 (1 + \gamma k^{1/2}).
\]

Summing up the last estimates, we conclude that for all \( k \in \mathbb{N} \)

\[
M^{-1} k^{1/2} \leq D_F M^{1/p} + K_0 (1 + \gamma k^{1/2}),
\]

whence

\[
(M^{-1} - K_0 \gamma) k^{1/2} \leq D_F M^{1/p} + K_0.
\]

Since \( \gamma < (MK_0)^{-1} \), this contradicts the fact that \( p > 2 \).

---

Let \( F \) be a symmetric ideal of \( B(H) \). For \( 1 \leq p < \infty \), we define (see e.g. [17, 54]) the \( p \)-convexification \( F^{(p)} \) by setting

\[
F^{(p)} = \{ x \in B(H) : |x|^p \in F \}, \quad \| x \|_{F^{(p)}} = \| |x|^p \|_{F}^{1/p},
\]

and the \( p \)-concavification \( F_{(p)} \) by setting

\[
F_{(p)} = \{ x \in B(H) : |x|^{1/p} \in F \}, \quad \| x \|_{F_{(p)}} = \| |x|^{1/p} \|_{F}^{p}.
\]

The Lorentz–Shimogaki Theorem [44, Theorem 2 and Lemma 3] demonstrates that the notions of \( p \)-convexifications and \( p \)-concavifications are closely related to the interpolation theory of operators. We compare below the norms of \( \bigoplus_j x_j \) and \( \sum_j x_j \) in the ideal \( C_F \) generated by an interpolation space \( F \) between \( \ell_2 \) and \( \ell_\infty \) by applying the Lorentz–Shimogaki Theorem.

**Lemma 2.10.** Let \( F \in \text{Int}(\ell_2, \ell_\infty) \). There exists a constant \( C > 0 \) such that for any elements \( x_j \in C_F \), \( 1 \leq j \leq n \), which are disjointly supported from the right (or from the left), we have

\[
\left\| \sum_{j=1}^{n} x_j \right\|_{C_F} \geq C \left\| \bigoplus_{j=1}^{n} x_j \right\|_{C_F}.
\]

**Proof.** Let \( x_j = u_j |x_j| \), \( 1 \leq j \leq n \), be the polar decompositions. Due to the assumption that \( x_j \) have disjoint right supports, we have \( |x_j| |x_k| = 0 \) for \( j \neq k \).

Thus,

\[
\left( \sum_{j=1}^{n} x_j \right) = \sum_{1 \leq j, k \leq n} x_j x_k^* = \sum_{1 \leq j, k \leq n} u_j |x_j| |x_k| u_k^* = \sum_{j=1}^{n} u_j |x_j|^2 u_j^*.
\]

By [42, Lemma 3.3.7], we have that

\[
\bigoplus_{j=1}^{n} u_j |x_j|^2 u_j^* \preceq \sum_{j=1}^{n} u_j |x_j|^2 u_j^*.
\]

By the Lorentz–Shimogaki Theorem (see e.g. [12, Theorem 3.1], [44, Theorem 2 and Lemma 3] and [50]), the (quasi-)norm in \( (C_F)_{(2)} \) is monotone with respect to
submajorisation (up to a constant). It follows that
\[
\left\| \frac{1}{n} \sum_{j=1}^{n} x_j \right\|_{F} = \left\| \left( \sum_{j=1}^{n} x_j \right)^{2} \right\|_{(F) (2)} = \left\| \sum_{j=1}^{n} u_j |x_j|^2 u_j^* \right\|_{(F) (2)} \geq C \left\| \sum_{j=1}^{n} u_j |x_j|^2 u_j^* \right\|_{(F) (2)} = C \left\| \sum_{j=1}^{n} x_j \right\|_{F}.
\]

\[\square\]

3. Symmetric spaces located between \( L_1 \) and \( L_2 \)

We start with considering the simpler situation. It is well known that the \( L_p \)-spaces, for \( 1 \leq p < 2 \), have a much richer geometric structure than in the case when \( 2 < p < \infty \). The same observation is true also for symmetric spaces located between \( L_1(0,1) \) and \( L_2(0,1) \), comparing with spaces lying between \( L_2(0,1) \) and \( L_{\infty}(0,1) \). This observation combined with the above-mentioned Arazy’s result (see [3, Corollary 3.2]) allows rather simply to show that \( E \not\hookrightarrow F \) for wide classes of symmetric function spaces \( E \) and symmetric sequence spaces \( F \).

**Proposition 3.1.** Let \( E := E(0,1) \) be a symmetric function space and \( F \) be a separable symmetric sequence space. Then,

(i) If \( E \) contains a symmetric sequence space \( G \) such that \( G \not\hookrightarrow \ell_2 \oplus F \), then \( E \not\hookrightarrow C_F \). In particular, if \( E \) contains a subspace isomorphic to \( \ell_r \) for some \( 1 \leq r < \infty \), \( r \neq 2 \) and \( \ell_r \not\hookrightarrow F \), then \( E \not\hookrightarrow C_F \);

(ii) If \( t^{-1/r} \in E \) for some \( r \in (1,2) \), then \( E \not\hookrightarrow C_{p,q} := C_{\psi_{p,q}} \) for all \( 1 < p < \infty \), \( 1 \leq q < \infty \), and \( E \not\hookrightarrow C_{\Lambda_p} \) for any \( 1 \leq q < \infty \) and decreasing sequence of positive numbers \( w = (w_n)_{n=0}^{\infty} \) such that \( \lim_{n \to \infty} w_n = 0 \) and \( \sum_{n=0}^{\infty} w_n = \infty \). In particular, \( L_{p_1,q_1} \not\hookrightarrow C_{p_2,q_2} \) for all \( 1 < p_1 < 2 \), \( 1 < p_2 < \infty \), \( 1 \leq q_1, q_2 < \infty \), and \( \Lambda_{\psi_{p_1,q_1}}(0,1) \not\hookrightarrow C_{\psi_{p_2,q_2}} \) for arbitrary \( \Lambda_{\psi_{p_1,q_1}}(0,1) \not\hookrightarrow C_{\psi_{p_2,q_2}} \) for every increasing concave function \( \psi \), \( \psi(0) = 0 \), and any decreasing sequence of positive numbers \( w = (w_n)_{n=0}^{\infty} \) satisfying the same properties as in (ii);

(iii) If \( p_1, p_2 \in (1, \infty) \) and \( q_1, q_2 \in [1, \infty) \) such that \( q_1 \neq q_2, q_1 \neq 2 \), then \( L_{p_1,q_1}(0,1) \not\hookrightarrow C_{p_2,q_2} \) and \( \Lambda_{\psi_{p_1,q_1}}(0,1) \not\hookrightarrow C_{\psi_{p_2,q_2}} \) for every increasing concave function \( \psi \), \( \psi(0) = 0 \), and any decreasing sequence of positive numbers \( w = (w_n)_{n=0}^{\infty} \) satisfying the same properties as in (ii).

**Proof.** Assertion (i) is an immediate consequence of [3, Corollary 3.2].

(ii). Fix \( r \in (1,2) \), \( r \neq q \), such that \( t^{-1/r} \in E \). Then, by [40, Theorem 2.1.4], \( \ell_\psi \hookrightarrow E \). According to (i), it remains to show that \( \ell_r \not\hookrightarrow \ell_{p,q} \) and \( \ell_r \not\hookrightarrow \ell_{\Lambda_p} \). Since the proof of these relations is quite similar, we check only the first of them.

Assuming the contrary, we have a sequence \( \{x_k\} \subset \ell_{p,q} \), which is equivalent in \( \ell_{p,q} \) to the unit vector basis in \( \ell_r \) and hence is weakly null in \( \ell_{p,q} \). Therefore, applying the Bessaga–Pełczyński selection principle (see e.g. [39, Proposition 1.a.12] or [1, Proposition 1.3.10]), we can assume that \( \{x_k\} \) consists of pairwise disjoint elements. Then, by [16, Theorem 5], the span \( \{x_k\} \) contains a further subspace isomorphic to \( \ell_q \). Since the spaces \( \ell_q \) and \( \ell_r, q \neq r \), are totally incomparable (see e.g. [1, Corollary 2.1.6]), this is a contradiction. Hence, the first assertion of (ii) follows.
The second assertion is a direct consequence of the first one, because $t^{-1/r} \in L_{p,q}(0,1)$ if $p < r < 2$ (resp. $t^{-1/r} \in \Lambda_{q}^2(0,1)$ if $\int_0^1 t^{-q/r}d\psi(t) < \infty$).

(iii). The desired result follows from the fact that $\ell_{q_2} \hookrightarrow L_{p_2,q_2}(0,1)$ [16, Theorem 11] and the part (i) in the same way as (ii). □

4. Symmetric spaces located between $L_2$ and $L_{\infty}$

In this section we prove the main results of the paper. Key roles will be played by the following propositions. Extending Theorem 6 in [5], we provide conditions, under which a symmetric function space $E(0,1)$ fails to be isomorphically embedded into the ideal $C_F$ generated by a symmetric sequence space $F$. Examples of the spaces $E$ and $F$ which satisfy all the above conditions will be given in Section 5.

Proposition 4.1. Let a separable symmetric sequence space $F$ be $q$-concave and has an upper $p$-estimate for some $2 < p \leq q < \infty$. Assume also that a separable symmetric function space $E$ on $[0,1]$ is such that $\delta_{E} < 1/2$ and

\begin{equation}
\phi_E(1/n) \geq An^{-1/q}, \quad n \in \mathbb{N}.
\end{equation}

Then, we have

$E \not\hookrightarrow C_F$.

A disadvantage of Proposition 4.1 consists in the fact that a function space $E(0,1)$ must satisfy rather restrictive conditions. The next proposition provides an alternative criterion for $E(0,1)$ ensuring the lack of isomorphic embeddings into the ideal $C_F$. We relax the restriction imposed on $E(0,1)$ in Proposition 4.1, and ask for an additional condition of distributional concavity on the sequence space $F$. However, since all Orlicz spaces and weighted Lorentz spaces are distributionally concave [44], Proposition 4.2 has a much wider applicability.

Proposition 4.2. Let a separable symmetric function space $E := E(0,1)$ and a separable symmetric sequence space $F$ satisfy the conditions:

(a) $F$ admits an upper $p$-estimate for some $p > 2$;
(b) $F$ is distributionally concave;
(c) $F$ is $q$-concave for some $q < \infty$;
(d) there is $A > 0$ such that

$$
\|\sigma_{1/n}\|_{F \rightarrow F} \leq A\phi_E(1/n), \quad n \in \mathbb{N};
$$

(e)

$$
\lim_{t \to \infty} \frac{M_{\phi_E}(t)}{t^{1/2}} = 0.
$$

Then, $E \not\hookrightarrow C_F$.

Remark 4.3. The fact that $F$ is $q$-concave and has an upper $p$-estimate for $2 < p \leq q < \infty$ together the Boyd interpolation theorem (see e.g. [40, the discussion on p.132 and Theorem 2.b.11], [30] or [10, Theorem 3.5.16]) implies that $F$ is an interpolation space with respect to the couple $(\ell_2, \ell_q)$. Moreover, by the same reasons, $F$ has the Rademacher type 2 [40, Proposition 1.f.3 and Theorem 1.f.7]. These observations allow us to use Proposition 2.9 and Lemmas 2.5, 2.6 in the proof below.
4.1. Common part of two proofs. On the contrary, suppose that there exists an isomorphic embedding $T$ of $E$ into $C_F$. Applying Lemmas 2.5 and 2.6, we may assume that $T$ maps $E$ into the upper triangular part $U_F$ of $C_F$ and that, for each $n \geq 0$, the elements $T(h_{2^n+i})$, $0 \leq i < 2^n$, where $h_j$, $j = 1, 2, \ldots$, are the Haar functions, are disjointly supported from the right.

From the definitions of the Rademacher and Haar functions it follows
\[ r_k(t) = \sum_{i=0}^{2^k-1} h_{2^k+i}, \quad t \in [0,1], \quad k = 0, 1, 2, \ldots \]

Hence, recalling that, for any positive integers $n \geq k$ and $j = 0, 1, \ldots, 2^k - 1$, $r_{n,k,j} := r_n \chi_{\left(\frac{j}{2^k}, \frac{j+1}{2^k}\right)}$ (see Section 2.6), we have
\[ r_{n,k,j} = \sum_{i=0}^{2^n - k - 1} h_{2^n + j 2^{n-k} + i}. \]

Therefore, for any fixed positive integers $k$ and $n \geq k$, the elements $T(\hat{r}_{n,k,j})$, $j = 0, 1, \ldots, 2^k - 1$, where
\[ \hat{r}_{n,k,j} := \frac{r_{n,k,j}}{\phi_E(2^{-k})} \]

are also disjointly supported from the right.

Without loss of generality, we will suppose that $\|T^{-1}\| = 1$. Then, by Lemma 2.7,
\[
\left\| \sum_{n=k}^{\infty} a_n T(\hat{r}_{n,k,j}) \right\|_{C_F} \geq \left\| T \left( \sum_{n=k}^{\infty} a_n \hat{r}_{n,k,j} \right) \right\|_{C_F} \geq \left\| \sum_{n=k}^{\infty} a_n \hat{r}_{n,k,j} \right\|_E \geq \frac{1}{32} \|(a_n)\|_{\ell_2}.
\]

Applying now Proposition 2.9 to the sequence $\{u_n\}_{n=1}^{\infty}$, $u_n := T(r_n) = T(\hat{r}_{n,0,0})$, we find $m_1 \in \mathbb{N}$ such that
\[
\|P_{m_1} T(r_n)\|_{C_F} \geq \gamma, \quad n = 1, 2, \ldots,
\]
where $\gamma := (2K_0 \max(32, \|T\|))^{-1}$ ($K_0$ is the constant from inequality (2.10)).

Let $n, k \in \mathbb{N}$, $n \geq k$, be fixed. Denote by $Q_{n,k}$ the natural isometry of the span $E_k := [x_{k,j}, j = 0, 1, \ldots, 2^k - 1]$ in $E$, where $x_{k,j} := \phi_E(2^{-k})^{-1} \chi_{\Delta_k^j} (\Delta_k^j = [j2^{-k}, (j+1)2^{-k}), k = 1, 2, \ldots, j = 0, 1, \ldots, 2^k - 1)$, onto the span $Y_k := [\hat{r}_{n,k,j}, j = 0, 1, \ldots, 2^k - 1]$ in $E$. Observe that, for each $m \in \mathbb{N}$, the space $(P_m U_F, \|\|_{C_F})$ is isomorphic to $\ell_2$, and hence the operator $P_{m_1} T Q_{n,k}$ is bounded from $E_k$ in $\ell_2$. By the condition $\delta_{\phi_E} < 1/2$ (see also Remark 2.2), we can use Lemma 2.1. Consequently, for every $\epsilon > 0$ there is $k_1 \in \mathbb{N}$ such that for all $n \geq k_1$
\[
\left\{ j = 0, 1, \ldots, 2^k - 1 : \frac{\|P_{m_1} T(\hat{r}_{n,k,j})\|_{\ell_2}}{\|P_{m_1} T(\hat{r}_{n,k,j})\|_{C_F}} \geq \epsilon \right\} \leq \epsilon \cdot 2^k,
\]

where for a finite set $B$ we put $|B| := \text{card} \ B$. Noting that
\[
\frac{\|P_{m_1} T(\hat{r}_{n,k,j})\|_{C_F}}{\|P_{m_1} T(\hat{r}_{n,k,j})\|_{\ell_2}} \leq d(P_m U_F, \ell_2)
\]
and
\[
\|P_{m_1} T\|_{E_k \to \ell_2} \leq \|P_{m_1} T\|_{E \to \ell_2} \leq \|P_{m_1} T\|_{E \to C_F} d(P_m U_F, \ell_2),
\]

we conclude that
\[
\frac{\|P_{m_1} T(\hat{r}_{n,k_1,j})\|_{C_F}}{C(m_1, T)} \leq \frac{\|P_{m_1} T(\hat{r}_{n,k_1,j})\|_{\ell_2}}{\|P_{m_1} T\|_{E_k \to \ell_2}},
\]
where
\[
C(m_1, T) := \|P_{m_1} T\|_{E \to C_F} \cdot d(P_{m_1} U_{m_1}, \ell_2)^2
\]
\((d(X, Y)\) is the Banach–Mazur distance between Banach spaces \(X\) and \(Y\)), we obtain
\[
\left|\left\{ j = 0, 1, \ldots, 2^{k_1} - 1 : \|P_{m_1} T(\hat{r}_{n,k_1,j})\|_{C_F} \geq \epsilon C(m_1, T) \right\}\right| \leq \epsilon \cdot 2^{k_1}.
\]
Choose \(\epsilon \in (0, 1/2)\) so that \(\gamma > 2\epsilon C(m_1, T)\). Then, from the preceding inequality it follows that
\[
\left|\left\{ j = 0, 1, \ldots, 2^{k_1} - 1 : \|P_{m_1} T(\hat{r}_{n,k_1,j})\|_{C_F} \geq \gamma/2 \right\}\right| \leq 2^{k_1 - 1}.
\]
Therefore, letting
\[
A_n := \left\{ j = 0, 1, \ldots, 2^{k_1} - 1 : \|P_{m_1} T(\hat{r}_{n,k_1,j})\|_{C_F} \leq \gamma/2 \right\},
\]
for all \(n \geq k_1\) we get
\[(4.4) \quad |A_n| \geq 2^{k_1 - 1}.\]

Next, in view of inequality \((4.2)\) and Remark 4.3, we can apply Proposition 2.9 to each of the sequences \((T(\hat{r}_{n,k_1,j}))_{n \geq k_1}, j = 0, 1, \ldots, 2^{k_1} - 1,\) and find a positive integer \(m_2 > m_1\) such that
\[
\|P_{m_2} T(\hat{r}_{n,k_1,j})\|_{C_F} \geq \gamma, \quad n \geq k_1, \quad j = 0, 1, \ldots, 2^{k_1} - 1.
\]
Combining this inequality together with the definition of sets \(A_n\), we infer
\[(4.5) \quad \|P_{m_2} T(\hat{r}_{n,k_1,j}) - P_{m_1} T(\hat{r}_{n,k_1,j})\|_{C_F} \geq \gamma/2, \quad n \geq k_1, \quad j \in A_n.
\]
As was mentioned above, the elements \(T(\hat{r}_{n,k_1,j})\), \(0 \leq j < 2^k\), are disjointly supported from the right. Clearly, the elements
\[(P_{m_2} - P_{m_1}) T(\hat{r}_{n,k_1,j}), \quad 0 \leq j < 2^k,
\]
have the same property.

4.2. Proof of Proposition 4.1.

Proof of Proposition 4.1. Observe that
\[
T(r_n) = \phi_E(2^{-k_1}) \cdot \sum_{j=0}^{2^{k_1} - 1} T(\hat{r}_{n,k_1,j}).
\]
By the lower \(q\)-estimate of \(C_F\) [4] (see also [17, Proposition 5.1]), we have
\[
\|P_{m_2} - P_{m_1}\|_{C_F} \geq c_F \phi_E(2^{-k_1}) \left( \sum_{j=0}^{2^{k_1} - 1} \|P_{m_2} - P_{m_1}\|_{C_F} \right)^{1/q},
\]
where \(c_F\) is the lower \(q\)-estimate constant for \(C_F\). Moreover, thanks to condition \((4.1)\),
\[
\phi_E(2^{-k_1}) \cdot 2^{k_1/q} \geq A.
\]
Thus, from (4.4) and (4.5) it follows that for \( n \geq k_1 \)
\[
\|(P_{m_2} - P_{m_1})T(r_n)\|_{C_F} \geq c_F \cdot \phi_E(2^{-k_1}) \cdot 2^{k_1 - 1} \cdot \frac{\gamma}{2} \geq c_F \cdot A \cdot 2^{-1/q} \frac{\gamma}{2} \geq \frac{A}{4} c_F \gamma.
\]

Next, using Lemma 2.1 once more and repeating the same reasoning, we find a positive integer \( k_2 > k_1 \) such that
\[
\|\{j = 0, 1, \ldots, 2^{k_2} - 1 : \|P_{m_2} T(r_{n,k_2,j})\|_{C_F} \geq \gamma/2\}\| \leq 2^{k_2 - 1}.
\]

Applying then Proposition 2.9 to each of the sequences \( (T(r_{n,k_2,j}))_{n \geq k_2}, j = 0, 1, \ldots, 2^{k_2} - 1, \) and repeating the above argument, we find \( m_3 > m_2 \) satisfying the condition
\[
\|(P_{m_3} - P_{m_2})T(r_n)\|_{C_F} \geq \frac{A}{4} c_F \gamma \quad \text{for all } n \geq k_2.
\]

Proceeding in the same way, we obtain two increasing sequences of positive integers \( \{k_i\}^\infty_{i=1} \) and \( \{m_i\}^\infty_{i=1} \) such that for all \( n \geq k_i \)
\[
\|(P_{m_{i+1}} - P_{m_i})T(r_n)\|_{C_F} \geq \frac{A}{4} c_F \gamma, \quad i = 1, 2, \ldots
\]

Fix \( l \in \mathbb{N} \). For each \( n \geq k_1 \), the elements \( (P_{m_{i+1}} - P_{m_i})T(r_n), 1 \leq i \leq l - 1 \), are disjointly supported from the left with left supports \( \sum_{j=m_i+1}^{m_{i+1}} \varepsilon_{jj} \). Hence, by the lower \( q \)-estimate of \( F \) [4] (see also [17, Proposition 5.1]), we have
\[
\|T(r_n)\|_{C_F} \geq c_F \left( \sum_{i=1}^{l-1} \| (P_{m_{i+1}} - P_{m_i})T(r_n) \|_{C_F}^q \right)^{1/q} \geq c_F (l - 1)^{1/q} A c_F \gamma.
\]

Since \( l \in \mathbb{N} \) is arbitrary, this implies that the operator \( T \) is unbounded, which contradicts the hypothesis. \( \square \)

4.3. Proof of Proposition 4.2.

Proof of Proposition 4.2. Taking into account that
\[
T(r_n) = \phi_E(2^{-k_1}) \cdot \sum_{j=0}^{2^{k_1} - 1} T(r_{n,k_1,j})
\]
and \( (P_{m_2} - P_{m_1})T(r_{n,k_1,j}) \), \( 0 \leq j < 2^{k_1} \), are disjointly supported from the left, by Lemma 2.10, we have
\[
\|(P_{m_2} - P_{m_1})T(r_n)\|_{C_F} \geq \phi_E(2^{-k_1}) \left( \bigoplus_{j=0}^{2^{k_1} - 1} \| (P_{m_2} - P_{m_1})T(r_{n,k_1,j}) \|_{C_F} \right) \geq \phi_E(2^{-k_1}) \left( \bigoplus_{j \in A_n} \| (P_{m_2} - P_{m_1})T(r_{n,k_1,j}) \|_{C_F} \right).
\]

Here, \( \bigoplus_j (P_{m_2} - P_{m_1})T(r_{n,k_1,j}) \in \bigoplus_j \mathcal{H} \) stands for the direct sum of operators \( (P_{m_2} - P_{m_1})T(r_{n,k_1,j}) \) and, under the natural isomorphism from \( \bigoplus_j \mathcal{H} \) to \( \mathcal{H} \) (see Remark 2.4), summands in the direct sum can be viewed as elements in \( \mathcal{H} \), which are pairwise disjointly supported from the left and from the right. Since \( F \) is distributionally concave, say, with the constant \( c_F \), in view of the preceding relations, definition of the dilation operator and the quasi-concavity of the fundamental function \( \phi_E \), for all \( n \geq k_1 \) we obtain
\[ \|(P_{m_2} - P_{m_1})T(r_n)\|_{C_F} \geq \frac{c_F \phi(E(2^{k_1}))}{\| \sigma_{2^{l-k_1}} \|_{F \to F}} \cdot \min_{j \in A_n} \|(P_{m_2} - P_{m_1})T(r_{n,k_1,j})\|_{C_F} \geq \frac{c_F \phi(E(2^{k_1}))}{A \cdot \| \phi(E(2^{l-k_1})) \|_{F \to F}} \cdot \min_{j \in A_n} \|(P_{m_2} - P_{m_1})T(r_{n,k_1,j})\|_{C_F} \geq \frac{c_F \phi(E(2^{k_1}))}{2A} \cdot \min_{j \in A_n} \|(P_{m_2} - P_{m_1})T(r_{n,k_1,j})\|_{C_F}. \]

Thus, by (4.5), for all \( n \geq k_1 \) we have
\[ \|(P_{m_2} - P_{m_1})T(r_n)\|_{C_F} \geq c_0 \gamma, \]
where \( c_0 \) depends only on \( E \) and \( F \).

Next, using Lemma 2.1 once more and repeating the same reasoning, we find a positive integer \( k_2 > k_1 \) such that
\[ \| \{ j = 0, 1, \ldots, 2^{k_2} - 1 : \| P_m T(\hat{r}_{n,k_2,j})\|_{C_F} \geq \gamma/2 \} \| \leq 2^{k_2-1}. \]

Applying then Proposition 2.9 to each of the sequences \( (T(\hat{r}_{n,k_2,j}))_{n \geq k_2}, j = 0, 1, \ldots, 2^{k_2} - 1, \) we find \( m_3 > m_2 \) satisfying the condition
\[ \|(P_{m_3} - P_{m_2})T(r_n)\|_{C_F} \geq c_0 \gamma \text{ for all } n \geq k_2. \]

Proceeding in the same way, we obtain two increasing sequences of positive integers \( \{ k_i \}_{i=1}^\infty \) and \( \{ m_i \}_{i=1}^\infty \) such that for all \( n \geq k_i \)
\[ \|(P_{m_{i+1}} - P_{m_i})T(r_n)\|_{C_F} \geq c_0 \gamma, \quad i = 1, 2, \ldots. \]

Fix \( l \in \mathbb{N} \). For each \( n \geq k_l \), the elements \( (P_{m_{i+1}} - P_{m_i})T(r_n), 1 \leq i \leq l-1, \) are disjointly supported from the left. Hence, it follows from Lemma 2.10 that
\[ \|T(r_n)\|_{C_F} \geq \left\| \sum_{i=1}^{l-1} (P_{m_{i+1}} - P_{m_i})T(r_n) \right\|_{C_F} \geq \left\| C_F \right\|_{C_F}. \]

Moreover, since the space \( F \) is \( q \)-concave, it follows that \( F \) satisfies a lower \( q \)-estimate with a constant \( K_F \) [40, p.85]. Thus, applying Lemma 2.8, we obtain that for every positive integer \( l \) and all \( n \geq k_l \),
\[ \|T\| \geq \|T(r_n)\|_{C_F} \geq \left\| \sum_{i=1}^{l-1} (P_{m_{i+1}} - P_{m_i})T(r_n) \right\|_{C_F} \geq K_F^{q} \left( \|P_{m_1}T(r_n)\|_{C_F}^{q} + \|P_{m_2} - P_{m_1})T(r_n)\|_{C_F}^{q} + \cdots + \|P_{m_{l-1}} - P_{m_{l-2}})T(r_n)\|_{C_F}^{q} \right)^{1/q} \]
\[ \geq K_F^{q} c_0 \gamma (l-1)^{1/q}. \]

Since \( l \in \mathbb{N} \) is arbitrary, this implies that the operator \( T \) is unbounded, which contradicts the hypothesis. \( \square \)
5. Applications

The most natural examples satisfying all the conditions of Propositions 4.1 and 4.2 are $L_{p,q}$-spaces, with $2 < q \leq p < \infty$ and suitable Orlicz spaces.

Recall that the Lorentz sequence space $\ell_{r,s}$, $1 < r < \infty$, $1 \leq s < \infty$, admits an upper $\min(r,s)$-estimate and a lower $\max(r,s)$-estimate (see Section 2.7 and [14,16,34,39,40]). Therefore, $\ell_{r,s}$ has a lower $q$-estimate and an upper $p$-estimate for some $2 < p \leq q < \infty$ if and only if $2 < r, s < \infty$. Moreover, since $\phi_{L_{r_1,s_1}}(t) = t^{1/r_1}$, $1 < r_1 < \infty$, $1 \leq s_1 < \infty$, one can easily verify that the rest of conditions of Proposition 4.1 for the spaces $F = \ell_{r,s}$ and $E = L_{r_1,s_1}$ is satisfied if and only if $r_1 \geq \max(r,s)$. From these observations and Proposition 4.1 we get the following result that extends [5, Theorem 6] to the class of $L_{p,q}$-spaces. Namely, [5, Theorem 6] is the special case of the assertion below when $p_1 = p_2 = q_1 = q_2$.

**Theorem 5.1.** For arbitrary $1 \leq q_1 < \infty$, $q_2, p_2 > 2$, and $\max(p_2, q_2) \leq p_1 < \infty$, we have

$$L_{p_1,q_1}(0,1) \næ C_{p_2,q_2}.$$  

In particular, if $2 < q \leq p < \infty$, then

$$L_{p,q}(0,1) \næ C_{p,q}.$$  

We apply Proposition 4.2 below.

**Theorem 5.2.** Let $M$ and $N$ be Orlicz functions satisfying the following conditions:

(i) $N$ is equivalent to a $p$-convex and $q$-concave Orlicz function on $[0,1]$ for some $2 < p \leq q < \infty$;

(ii) there is $A > 0$ such that

$$N(uv) \leq AN(u)M(v) \text{ if } 0 < u \leq 1, \ v \geq 1;$$

(iii) $M$ is equivalent to a $r$-convex Orlicz function on $[1,\infty)$ for some $r > 2$.

Then, $L_M[0,1] \næ C_{\ell_N}$.

**Proof.** Let us check that conditions (a) — (e) of Proposition 4.2 are fulfilled for the spaces $E = L_M[0,1]$ and $F = \ell_N$. Without loss of generality, we assume that $N(1) = M(1) = 1$.

First of all, condition (i) implies that the space $F$ is $p$-convex and $q$-concave (and so is separable) [33]. Thus, since every Orlicz space is distributionally concave (see Section 2.4), the space $F$ satisfies conditions (a) — (c).

Let us check that (d) is a consequence of conditions (i) and (ii) of the theorem. Indeed, by [43, Theorem 6] (see also [45, §4, p. 28] and [11]), we have

$$\|\sigma_{1/n}\|_{F \rightarrow F} \leq 2 \cdot \sup_{m \in \mathbb{N}} \frac{\phi_F(m)}{\phi_F(nm)}.$$  

Therefore, since $\phi_F(m) = 1/N^{-1}(1/m)$, $m \in \mathbb{N}$, it follows

$$\|\sigma_{1/n}\|_{F \rightarrow F} \leq 2 \sup_{m \in \mathbb{N}} N^{-1}(1/(nm)) \leq 2 \sup_{0 < s \leq 1} N^{-1}(s/n).$$  

On the other hand,

$$\phi_E(t) = \frac{1}{M^{-1}(1/t)}, \ t > 0.$$  

(5.1)
and so condition (d) is satisfied if there is a constant $C > 0$ such that for all $0 < s, t \leq 1$

(5.2) \[ N^{-1}(st)M^{-1}(1/t) \leq CN^{-1}(s). \]

Passing to the inverse functions, we get that it is equivalent to the inequality

\[ N(C^{-1}N^{-1}(st)M^{-1}(1/t)) \leq s, \]

or, after the changes of variables $u = N^{-1}(st)$ and $v = M^{-1}(1/t)$, to the following:

\[ N(C^{-1}uv) \leq N(u)M(v), \quad 0 < u \leq 1, \quad v \geq 1. \]

Clearly, we can assume that the constant $A$ in (ii) is bigger than 1. Therefore, applying successively conditions (ii) and (i), we get

\[ N(A^{-1/p}uv) \leq AN(A^{-1/p}u)M(v) \leq N(u)M(v), \quad 0 < u \leq 1, \quad v \geq 1. \]

Thus, the preceding inequality and also (5.2) hold with $C = A^{1/p}$, which implies (d).

It remains to show that the space $E = L^M[0,1]$ satisfies condition (e). To this end, we observe that, by definition and (5.1), for every $u \geq 1$

\[ M(\phi_E(u)) = \sup_{0 < v \leq 1/u} \phi_E(vu) \sup_{w \geq u} M^{-1}(w/u). \]

Therefore, since $r > 2$, it suffices to prove that there is a constant $C > 0$ such that

(5.3) \[ M^{-1}(w) \leq Cu^{1/r}M^{-1}(w/u), \quad w \geq u \geq 1. \]

One can easily check that this is equivalent to the following:

\[ uM(z) \leq M(Cu^{1/r}z), \quad u, z \geq 1. \]

In turn, after the change $s = u^{-1/r}$ we come to the inequality

\[ M(z) \leq s^rM(Cz/s), \quad z \geq 1, \quad 0 < s \leq 1, \]

and then, setting $t = z/s$, to

\[ M(st) \leq s^rM(Ct), \quad t \geq st \geq 1. \]

On the other hand, it is easy to see that the latter inequality is an immediate consequence of condition (iii) of the theorem. Indeed, (iii) means (see Subsection 2.7) that for some constant $C' \geq 1$

\[ M(st) \leq C's^rM(t), \quad t \geq st \geq 1. \]

Combined this with the fact that $C'M(t) \leq M(C't), \quad t > 0$, because $M$ is an increasing convex function, we obtain the preceding inequality with $C = C'$. This completes the proof of the theorem. \qed

As a consequence we get the following result.

**Corollary 5.3.** Let $M$ be an Orlicz function satisfying the following conditions:

(a) $M$ is equivalent to an Orlicz function that is $p$-convex on $[0, \infty)$ and $q$-concave on $[0,1]$ for some $2 < p \leq q < \infty$;

(b) there is $A > 0$ such that

\[ M(uv) \leq AM(u)M(v) \quad \text{if} \quad 0 < u \leq 1, \quad v \geq 1. \]

Then, $L^M[0,1] \not\rightarrow C_{\ell^M}$.  

In particular, this result holds for every submultiplicative Orlicz function $M$, which is $p$-convex for some $p > 2$.

Proof. The first assertion of the corollary is a straightforward consequence of Theorem 5.2. To show the second one, it suffices to check that $M$ is equivalent to a $q$-concave Orlicz function on $[0, 1]$ for some finite $q$. To this end, let us observe that the submultiplicativity of $M$ clearly implies that $M \in \Delta_2^0$ (see Subsection 2.2). In turn, then there exist $q < \infty$ and $C > 0$ such that for all $0 < s, t \leq 1$ we have $s^q M(t) \leq CM(st)$ (see e.g. [33, Proposition on p. 121]). This observation finishes the proof (see Subsection 2.7). \hfill \square

6. Lack of isomorphic embeddings of $L_{2,q}$-spaces into ideals $C_{2,q}$.

In the previous sections, we considered symmetric spaces, which are located either between $L_2$ and $L_\infty$ or between $L_1$ and $L_2$. In this final part of the paper, we study the spaces $L_{2,q}$, $1 \leq q < \infty$, which are being the most typical examples of spaces “very close” to the space $L_2$ (in particular, they belong to neither the set $\text{Int}(L_1, L_2)$ nor $\text{Int}(L_2, L_\infty)$). We show that $L_{2,q}$ cannot be isomorphically embedded into the ideal $C_{2,q}$ for every $1 \leq q < \infty$. Here, we make use of the above-mentioned Arazy’s result [3, Corollary 3.2], some properties of sequences of independent functions in $L_{2,q}$-spaces obtained in [6,13] and recent results on the embeddings of $\ell_{p,q}$-spaces from [37,49].

Theorem 6.1. For every $1 \leq q < \infty$, $q \neq 2$, the space $L_{2,q} := L_{2,q}(0,1)$ fails to be isomorphically embedded into the ideal $C_{2,q}$.

Proof. Assume by contradiction that $L_{2,q}$ is isomorphically embedded into $C_{2,q}$. Further, we consider the cases when $1 \leq q < 2$ and $2 < q < \infty$ separately.

(a) $1 \leq q < 2$. By Corollary 3.6 from the paper [6] (see also [25]), for every $1 \leq q < 2$ the space $L_{2,q}$ contains a sequence of independent identically distributed mean zero functions $\{f_k\}_{k=1}^\infty$, which is not equivalent in $L_{2,q}$ to the unit vector basis in $\ell_2$. Moreover, according to [13, Corollary 3.14], for some $C > 0$ we have

\[(6.1) \quad \|\sum_{k=1}^\infty a_k f_k\|_{L_{2,q}} \leq C \|(a_k)\|_{\ell_{2,q}}.\]

Since $f_k$, $k = 1, 2, \ldots$, are independent and identically distributed, then $\{f_k\}_{k=1}^\infty$ is a symmetric basic sequence in $L_{2,q}$. Therefore, by [3, Corollary 3.2], the closed linear span $[f_k]$ in $L_{2,q}$ is isomorphically embedded into the space $\ell_2 \oplus \ell_{2,q}$. This means that there is a sequence $\{x_k\} \subset \ell_2 \oplus \ell_{2,q}$ such that for all $a_k \in \mathbb{R}$

\[(6.2) \quad \left\|\sum_{k=1}^\infty a_k x_k\right\|_{\ell_2 \oplus \ell_{2,q}} \asymp \left\|\sum_{k=1}^\infty a_k f_k\right\|_{L_{2,q}}.\]

Combining this with the fact that $\{f_k\}$ is not equivalent in $L_{2,q}$ to the unit vector basis in $\ell_2$ and with inequality (6.1), we see that the sequence $\{x_k\}$ is not equivalent in $\ell_2 \oplus \ell_{2,q}$ to the latter basis as well and

\[(6.3) \quad \left\|\sum_{k=1}^\infty a_k x_k\right\|_{\ell_2 \oplus \ell_{2,q}} \leq C_1 \|(a_k)\|_{\ell_{2,q}}.\]
Observe that \( f_k \) are independent, \( \int_0^1 f_k(s) \, ds = 0 \), \( k = 1, 2, \ldots \), and \( \|f_k\|_{L_2} = \|f_1\|_{L_2} \leq \|f_1\|_{L_{2,q}} \), \( k \geq 2 \) [10, Chapter 4, Proposition 4.2]. Hence, the functions \( f_k / \|f_1\|_{L_2} \), \( k = 1, 2, \ldots \), form an orthonormal system. Therefore, \( \int_0^1 f_k(s)g(s) \, ds \to 0 \) as \( k \to \infty \) for each \( g \in L_2 \). Since \( L_2 \) is dense in the dual space \( (L_{2,q})^* = L_{2,q'} \), \( 1/q + 1/q' = 1 \), we obtain that \( \{f_k\} \) is weakly null in \( L_{2,q} \). Thus, \( \{x_k\} \) is weakly null in \( \ell_2 \oplus \ell_{2,q} \) as well, and hence, applying the Bessaga–Pelczyński selection principle (see e.g. [39, Proposition 1.a.12]), we can assume that \( x_k, k = 1, 2, \ldots \), are pairwise disjointly supported.

Let \( x_k = y_k + z_k \), where \( y_k \in \ell_2 \) and \( z_k \in \ell_{2,q}, k = 1, 2, \ldots \). Then,

\[
\left\| \sum_{k=1}^{\infty} a_k x_k \right\|_{\ell_2} \geq \left\| \sum_{k=1}^{\infty} a_k y_k \right\|_{\ell_2} + \left\| \sum_{k=1}^{\infty} a_k z_k \right\|_{\ell_{2,q}}.
\]

If \( \liminf_{k \to \infty} \|z_k\|_{\ell_{2,q}} = 0 \), then passing to a subsequence if it is necessary, we get that \( \{x_k\} \) is equivalent in \( \ell_2 \oplus \ell_{2,q} \) to the unit vector basis in \( \ell_2 \), which contradicts our assumption. Therefore, \( \|z_k\|_{\ell_{2,q}} \geq c, k = 1, 2, \ldots, \) for some \( c > 0 \).

Let \( z_k = (z_k(i))_{i=1}^{\infty} \). We consider two cases: (i) \( \liminf_{k \to \infty} \sup_i |z_k(i)| = 0 \) and (ii) \( \liminf_{k \to \infty} \sup_i |z_k(i)| > 0 \).

In the case (i), by [16, Proposition 1], passing to a subsequence, we can assume that \( \{z_k\} \) is equivalent in \( \ell_{2,q} \) to the unit vector basis in \( \ell_q \), and then from the inequality \( q < 2 \) and (6.4) it follows that \( \{x_k\} \) is equivalent in \( \ell_2 \oplus \ell_{2,q} \) to the same basis, which contradicts inequality (6.3) because \( \ell_{q/2} \not\subset \ell_{2,q} \) for \( q < 2 \) [10, p.217].

In the case (ii), we can find \( \delta > 0 \) such that for each \( k = 1, 2, \ldots \) there is a positive integer \( i_k \) satisfying the inequality \( |z_k(i_k)| \geq \delta \). Then, since \( z_k, k = 1, 2, \ldots \), are pairwise disjointly supported, it follows that

\[
\left\| \sum_{k=1}^{\infty} a_k z_k \right\|_{\ell_{2,q}} \geq \delta \| (a_k) \|_{\ell_{2,q}}.
\]

Hence, we obtain

\[
\| (a_k) \|_{\ell_{2,q}} \geq C^{-1} \left( \sum_{k=1}^{\infty} a_k x_k \right)_{\ell_2} \geq C' \left( \sum_{k=1}^{\infty} a_k z_k \right)_{\ell_{2,q}} \geq c \| (a_k) \|_{\ell_{2,q}},
\]

for some \( c > 0 \). Therefore, taking into account equivalence (6.2), we conclude that the sequence \( \{f_k\} \) is equivalent in \( L_{2,q}(0, 1) \) to the unit vector basis in \( \ell_{2,q} \), which contradicts the fact that \( \ell_{2,q} \not\subset L_{2,q}(0, 1), q \in [1, 2) \cup (2, \infty) \) (see [37, 49]). As a result, in the case \( 1 \leq q < 2 \) the proof is completed.

(b) \( 2 < q < \infty \). Our argument will be based on using Proposition 3.11 from the paper [13]. According to this result, if \( \{g_k\}_{k=1}^{\infty} \) is a sequence of independent, symmetrically and identically distributed functions such that \( g_1 \in L_{2,q} \setminus L_2 \), then

\[
\frac{1}{\sqrt{n}} \left\| \sum_{k=1}^{n} g_k \right\|_{L_{2,q}} \to \infty \text{ as } n \to \infty.
\]
Since \( \{g_k\} \) is a symmetric basic sequence, as in the preceding case, by [3, Corollary 3.2], we can find a sequence \( \{x_k\} \subset \ell_2 \oplus \ell_{2,q} \) such that for all \( a_k \in \mathbb{R} \)
\[
\left\| \sum_{k=1}^{\infty} a_k x_k \right\|_{\ell_2 \oplus \ell_{2,q}} \geq C \left\| \sum_{k=1}^{\infty} a_k y_k \right\|_{L_{2,q}}.
\]
Then, clearly,
\[
(6.5) \quad \frac{1}{\sqrt{n}} \left\| \sum_{k=1}^{n} x_k \right\|_{\ell_2 \oplus \ell_{2,q}} \to \infty \text{ as } n \to \infty.
\]
Since \( g_k, k = 1, 2, \ldots, \) are independent, symmetrically and identically distributed, then the sequence \( \{g_k\} \) in \( L_{2,q} \) is equivalent to the sequence \( \{g_k(s) r_k(t)\} \) in \( L_{2,q}([0,1] \times [0,1]) \) (as above, \( r_k \) are the Rademacher functions). One can readily check that the latter sequence is weakly null in \( L_{2,q}([0,1] \times [0,1]) \) so is \( \{g_k\} \) in \( L_{2,q} \).
Thus, \( \{x_k\} \) is weakly null in \( \ell_2 \oplus \ell_{2,q} \), and, as above, we may assume that \( x_k, k = 1, 2, \ldots, \) are pairwise disjoint.
If \( x_k = y_k + z_k \), where \( y_k \in \ell_2 \) and \( z_k \in \ell_{2,q}, k = 1, 2, \ldots, \), then we have
\[
\left\| \sum_{k=1}^{\infty} a_k x_k \right\|_{\ell_2 \oplus \ell_{2,q}} \geq \left\| \sum_{k=1}^{\infty} a_k y_k \right\|_{\ell_2} + \left\| \sum_{k=1}^{\infty} a_k z_k \right\|_{\ell_{2,q}}.
\]
Thus, since the sequences \( \{y_k\} \) and \( \{z_k\} \) consist of pairwise disjoint elements and the space \( \ell_{2,q}, q > 2, \) admits an upper 2-estimate (see e.g. [16, Theorem 3]), this inequality yields
\[
\left\| \sum_{k=1}^{\infty} a_k x_k \right\|_{\ell_2 \oplus \ell_{2,q}} \leq C \left\{ \left( \sum_{k=1}^{\infty} |a_k|^2 \|y_k\|_{\ell_2}^2 \right)^{1/2} + \left( \sum_{k=1}^{\infty} |a_k|^2 \|z_k\|_{\ell_{2,q}}^2 \right)^{1/2} \right\} \leq C \|(a_k)\|_{\ell_2},
\]
which contradicts (6.5). \( \square \)

Summing up Theorems 5.1, 6.1 and Proposition 3.1, we get the following result.

**Theorem 6.2.** If a couple \((p, q)\) of positive numbers satisfies one of the following conditions:

1. \( 1 < p < 2 \) and \( 1 \leq q < \infty; \)
2. \( p = 2 \) and \( q \in [1, 2) \cup (2, \infty); \)
3. \( 2 < q \leq p < \infty, \)

then we have
\[
L_{p,q}(0,1) \not\cong C_{p,q}.
\]

**Question 6.3.** Does \( L_{p,q}(0,1) \) isomorphically embed into \( C_{p,q} \) when \( 2 < p < \infty \) and \( q \in [1, 2] \cup (p, \infty) \)?

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