Existence theory for a Poisson-Nernst-Planck model of electrophoresis *

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Abstract

A system modeling the electrophoretic motion of a charged rigid macromolecule immersed in a incompressible ionized fluid is considered. The ionic concentration is governing by the Nernst-Planck equation coupled with the Poisson equation for the electrostatic potential, Navier-Stokes and Newtonian equations for the fluid and the macromolecule dynamics, respectively. A local in time existence result for suitable weak solutions is established, following the approach of [15].

1 Introduction

Electrophoresis is the motion of charged colloidal particles or molecules through a solution under the action of an applied electrical field. This phenomenon is one of the most powerful analytical tools in colloidal science, being often used in the characterization of colloidal systems [31]. Modern applications include drug delivery and screening, manipulation of particles in micro-/nanofluidic systems, sequencing of genome of the organisms, forensic analysis, micro-chip design and others [8], [32], [20]. Despite the fact that the analytical description of this type of electrokinetic phenomenon is a difficulty task (due to the complicated interplay between hydrodynamic and electric effects [11]), there is a vast literature on the subject. The majority dealing with special problems like electrophoresis of bio-molecules with some geometric symmetry property or small surface potentials (see, for example, [1], [19], [24], [35], [38]). Few of these papers deal with the rigorous mathematical aspects of the models involved.

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In this paper we present a model of electrophoresis of a single particle immersed in a viscous and incompressible fluid (a ionic water solution). Unlike the models where the concentrations of charged particles range from colloidal to nano size (see [36]), we deal with the case where the colloidal particle is a very large rigid molecule and obeys the laws of classical mechanics. Also we consider the situation where the particle and fluid are contained in a bounded domain $\mathcal{O} \subset \mathbb{R}^3$ which represents the enclosure of the system (a tube, in the capillary electrophoresis). These are fairly reasonable assumptions in a number of practical situations such as in the study of electrophoresis of proteins [12]. On the other hand, we assume that the particle is far from the boundary of $\mathcal{O}$. As a consequence we are able to analyze only the local motion of the particle, as we are considering the standard electrokinetics equations. In fact, in the standard models (see [11], [1], [24]), the analysis is restrict to a single particle that is suspended in a fluid which fills all the region in $\mathbb{R}^3$ exterior to the particle. This means that the boundaries are sufficiently far removed from the particle so that they have negligible effects on the electric and fluid velocity fields associated with the particle [23]. According to [37] when the geometry forces the mobile particle into close of proximity with a surface (which may itself be charged) complex phenomena arise and a more detailed analysis should be done.

Although we are interesting mostly in the local motion of the particle, we do not neglect the inertial terms in the Navier-Stokes equations for fluid velocity and pressure. The Reynolds number for fluid flows in a typical electrophoretic motion is small and many authors consider the Stokes equations instead the Navier-Stokes equations (see [11], [32], [3]). However, it is not clear to us under what circumstances the linear theory can be used. With respect to this issue Allison and Stigter [2] suggest that the use of linearization of the constitutive equations with respect to the perturbing electric/flow fields are allowed provided the perturbing fields are weak. Also we believe that the analysis of the non simplified model considered here can be given us a support from the treatment of more general problems where the inertial effects could not be negligible (see [19], [8]).

Another important aspect in the modeling of electrophoretic motion is the electrostatic potential. In more simplified models the total electrostatic potential of the system is given by the applied external electric field (see [3], [38]) and the local potential (due to particle charge distribution and ions) is neglect. On the other hand, the local electrostatic potential is usually modeled by the Poisson-Boltzman equation [5], [4], [7], [42]. This equation is derived from the assumption of thermodynamics equilibrium where the ionic distributions are not affected by fluid flows. According to [30], this is a reasonable assumption for the case of steady-state processes with stationary values of diffusive fluxes, but there are some important cases where convective transport of ions has non-trivial effects [27]. In this context, in the lines of [23] and [11], we consider a more convenient approach: the use of a convective-diffusion type equation, more precisely, the well known Nernst-Planck equation, for the ionic concentration and Poisson equation for the total electric potential. Moreover, we only
impose $C^{2+\alpha}$ regularity on the domain occupied by the particle (without other geometric symmetry restrictions).

Some of the mathematical aspects of an electrophoresis model based on the Poisson-Boltzmann theory have been discussed by the authors in the papers [4] and [5]. However, at least to the knowledge of the authors, there are no rigorous mathematical results on the Poisson-Nernst-Planck model of electrophoresis of the rigid macromolecules. From the mathematical point of view, the main feature relies in the highly coupled equations which must be solved in time dependent domains. We establish the proof of a existence theorem (see Theorem 3.1 in Section 3) for appropriate weak solutions for this system. The proof is based on the approximation technique introduced in the references [14], [15] for the study of the motion of rigid bodies in viscous fluid. A sequence of approximate smooth solutions is construct and the solution is obtained as the limit of this sequence. Our scheme is local in time in the sense that the macromolecule does not touch the boundary of the enclosure in the time interval of existence. Employing a linearization and regularization procedure, this permits us to obtain suitable energy bounds for the electric potential, fluid velocity and ionic concentration (see Lemmas 4.2, 4.3, 4.5). As a consequence, we establish special results on time compactness (see Lemmas 5.1 and 5.2) that give us strong convergence results for fluid velocity and ionic concentration. Then, the main result follows from the passage to the limit in the approximation equations (see Lemmas 5.3, 5.4, 5.5) and is announced in Theorem 1.

The outline of the paper is as follows. In Section 2 we introduce the coupled system of governing equations; in the Section 3 we obtain formal energy estimates that give us a weak formulation for the system; in the Section 4 we construct a sequence of approximate solutions that are uniformly bounded in energy spaces; in the Section 5 we obtain appropriate compactness results that permit us to pass the limit the approximate solutions.

In a subsequent work we will investigate the effect of proximal boundaries in the electrophoresis of many particles and consider the possible contact between two particles or with the boundary, in the lines of [18].

2 Governing Equations

We assume that the solvent and macromolecule occupy a bounded open connected set $\mathcal{O} \subset \mathbb{R}^3$ and the solvent contains $J$ ionic species. At the initial time, the macromolecule occupies a compact region $\overline{K}_0$, where $K_0 \subset \subset \mathcal{O}$ is a open connected set such that

$$\gamma_0 := \text{dist}(K_0, \partial \mathcal{O}) > 0. \quad (2.1)$$

The fluid domain at the initial time is denoted by $\Omega_0 := \mathcal{O} \setminus \overline{K}_0$. We also assume that $K_0$ and $\Omega_0$ are $C^{2+\alpha}$-domains, $0 < \alpha < 1$, and define $K_t = Q(t)K_0$ as the position of the particle at time $t$, $Q(t)$ is an affine isometry such that $Q(0) = I$.

Also, let us denote $\Omega_t = \mathcal{O} \setminus \overline{K}_t$, i.e., the fluid domain at time $t$ and assume
(formally) that there exists \( T > 0 \) such that
\[
\gamma(t) := \text{dist}(K_t, \partial O) > \gamma, \quad 0 < \gamma < \gamma_0
\] (2.2)
for all \( t \in [0, T] \).

It is important to remark that, although we have not imposed any symmetry hypothesis on \( K_0 \), the regularity imposed is not satisfactory in a large number of situations. In fact, the determination of the electrostatic interface in biological processes is not a trivial task. According to \[11\] (see also \[12\]), any model of molecular surface must follow, in some way, the boundary of the space from which other molecules are excluded. Usually, the molecular surface (the accessibility surface of the solvent) is taken to be the surface described by a point on the surface of an idealized spherical solvent probe as it is rolled around the solvated molecule. This lead us to a \( C^0 \) surface with the eventual presence of cusps and with multiple connected components. Accordingly, we are dealing with an idealized situation in order to obtain appropriate regularity results and \textit{a priori} estimates for the approximate solutions of the system. However, we believe that this condition can be relaxed using a smoothness domain technique. We will discuss this issue in a subsequent paper.

If the dielectric constant of the particle and of the solvent are \( \kappa_1 > 0 \) and \( \kappa_2 > 0 \), respectively, the electrostatic potential \( \psi \), for each \( t \in [0, T] \), is governed by the Poisson equation
\[
\nabla \cdot (k(x, t) \nabla \psi(x, t)) = -4\pi e \sum_{i=1}^{J} q_i(x, t) - 4\pi \rho(x, t), \ x \in O,
\] (2.3)
where
\[
q_i(x, t) = \begin{cases} 
Z_i N_i(x, t), & x \in \Omega_t, \\
0, & x \in K_t;
\end{cases}
\] (2.4)
\( k : O \times [0, T] \to L(\mathbb{R}^3, \mathbb{R}^3) \) is given by \( k_{ij}(x, t) = \delta_{ij} \kappa(x, t) \), with
\[
\kappa(x, t) := \delta_{ij} (\kappa_1 \mathcal{I}_{K_t} + \kappa_2 \mathcal{I}_{\Omega_t})(x, t);
\]
\( e \) is the electron charge, \( Z_i \) and \( N_i \) are the valence and concentration of the \( i \)th ionic species, respectively. Moreover,
\[
\rho(x, t) = \begin{cases} 
\rho_0(Q^{-1}(t)x), & x \in K_t, \\
0, & x \in \Omega_t,
\end{cases}
\] (2.5)
where \( \rho_0 \) is a fixed charge distribution on \( K_0 \) such that \( \text{supp} \rho_0 \subset \subset K_0 \). Note that (2.5) implies that the fixed charges of the particle are invariant under rigid motion. As a consequence, we have the conservation of total fixed charges
\[
\left( \int_{O} \rho dx \right)(t) = \int_{O} \rho_0(x) dx,
\] (2.6)
for all \( t \in [0, T] \). These assumptions on \( \rho \) corresponds, of course, to an idealized situation. In fact, the determination of the properties of fixed charges in colloidal molecules is a very difficult task (see \[12, 28\]).
We assume transmission boundary conditions for \( \psi \)

\[
\psi_2(\mathbf{x}, t) = \psi_1(\mathbf{x}, t), \quad \mathbf{x} \in \partial K_i, \quad (2.7)
\]

\[
(\kappa_1 \partial_n \psi_1 - \kappa_2 \partial_n \psi_2)(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial K_i, \quad (2.8)
\]

\[
\psi_2(\mathbf{x}, t) = \Psi(\mathbf{x}), \quad \mathbf{x} \in \partial \Omega, \quad (2.9)
\]

where \( \nu(\cdot, t) \) is the exterior normal to \( \partial K_i \) and \( \Psi \) represents a stationary electrostatic potential on \( \partial \Omega \), \( \psi_1 = \psi|_{K_1} \), \( \psi_2 = \psi|_{\Omega} \).

The evolution of \( \mathcal{N}_i \), for each \( i \in \{1, \ldots, J\} \), is described by the Nernst-Planck equation

\[
\partial_t \mathcal{N}_i(\mathbf{x}, t) = \nabla \cdot \left( -\mathcal{N}_i \mathbf{v}_f + d_i \nabla \mathcal{N}_i + \frac{d_i Z_i e}{\kappa_B \theta} \mathcal{N}_i \nabla \psi_2 \right)(\mathbf{x}, t), \quad (x, t) \in \Omega_t \times (0, T), \quad (2.10)
\]

where \( d_i \) is the ionic diffusion coefficient of the \( i \)th ionic species, \( \mathbf{v}_f \) is the solvent velocity field, \( \kappa_B \) is the Boltzmann constant and \( \theta \) is the temperature of the system (which we suppose constant). It is important to remark that, for obvious physical reasons, we seek non-negative functions \( \mathcal{N}_i \) in \( \Omega_t \times [0, T] \).

The boundary conditions correspond to no ion diffusion and no ion conduction through the boundaries (see \[23\]) are given by

\[
\left( \partial_n \mathcal{N}_i + \frac{Z_i e}{\kappa_B \theta} \mathcal{N}_i \partial_n \psi_2 \right)(\mathbf{x}, t) = 0, \quad (x, t) \in (\partial \Omega \cup \partial K_i) \times (0, T). \quad (2.11)
\]

The system is complemented with the initial condition

\[
\mathcal{N}_i(\mathbf{x}, 0) = \mathcal{N}_{i,0}(\mathbf{x}) \geq 0, \quad \text{a.e.} \quad \mathbf{x} \in \Omega_0. \quad (2.12)
\]

The solvent velocity field is governed by the Navier-Stokes equations for incompressible flows

\[
\mathbf{p}_f(\partial_t \mathbf{v}_f + \nabla \cdot (\mathbf{v}_f \otimes \mathbf{v}_f))(\mathbf{x}, t) + \eta \Delta \mathbf{v}_f(\mathbf{x}, t) + \nabla p(\mathbf{x}, t) = \mathbf{p}_f \mathbf{F}(\mathbf{x}, t), \quad (x, t) \in \Omega_t \times (0, T) \quad (2.13)
\]

\[
\nabla \cdot \mathbf{v}_f(\mathbf{x}, t) = 0, \quad (x, t) \in \Omega_t \times (0, T),
\]

where

\[
\mathbf{F}(\mathbf{x}, t) = -\sum_{i=1}^J \frac{c(q_i \nabla \psi_2)(\mathbf{x}, t)}{J} \quad (2.14)
\]

is the electrical forcing term, \( p \) is the pressure, \( \eta > 0 \) and \( \mathbf{p}_f > 0 \) are the viscosity and the mass density of the fluid, respectively.

Let denote by \( \mathbf{x}_c(t), \mathbf{v}_c(t) \) and \( \mathbf{w}(t) \) the center of mass, the translational velocity and the rotation vector of the particle in the time \( t \), respectively. If \( \mathcal{A}, \mathbf{v}_p \) and \( \mathbf{p}_p > 0 \) are the symmetric inertial matrix, velocity and the mass density of the particle, we have

\[
\mathbf{y}^T \mathcal{A} \mathbf{y} = \mathbf{p}_p \int_{K_0} |\mathbf{y} \times (\mathbf{x} - \mathbf{x}_c(0))|^2 d\mathbf{x}, \quad \forall \mathbf{y} \in \mathbb{R}^3, \quad (2.15)
\]

and

\[
\mathbf{v}_p(\mathbf{x}, t) = \mathbf{v}_c(t) + \mathbf{w}(t) \times (\mathbf{x} - \mathbf{x}_c(t)), \quad (x, t) \in K_i \times (0, T). \quad (2.16)
\]
Using (2.3) and (2.14), the electrical forcing term $F$ can be written in the divergence form

$$F = \nabla \cdot \sigma_E,$$

where $\sigma_E = ((\sigma_E)_{ij})$ is the well known electrostatic tensor,

$$(\sigma_E)_{ij} := \frac{\kappa^2}{4\pi} \left( \frac{\partial \psi_2}{\partial x_i} \frac{\partial \psi_2}{\partial x_j} - \frac{1}{2} \delta_{ij} (\nabla \psi_2)^2 \right),$$

(see [41]). If $M$ is the mass of the particle, the evolution law for its motion is given by the Newtonian dynamic equations

$$M \frac{dv_c}{dt}(t) = \int_{\partial K_t} \sigma_H(\mathbf{x},t) \cdot \nu(\mathbf{x},t) d\mathbf{s}(\mathbf{x}) + \int_{\partial K_t} \sigma_E(\mathbf{x},t) \cdot \nu(\mathbf{x},t) d\mathbf{s}(\mathbf{x}) \quad (2.17)$$

and

$$A \frac{dw}{dt}(t) = \int_{\partial K_t} (\mathbf{x} - \mathbf{x}_c(t)) \times (\sigma_H \cdot \nu)(\mathbf{x},t) d\mathbf{s}(\mathbf{x}) + \mathbf{w} \times (A \cdot \mathbf{w}) +$$

$$+ \frac{d}{dt} \int_{\partial K_t} (\mathbf{x} - \mathbf{x}_c(t)) \times (\sigma_E \cdot \nu)(\mathbf{x},t) d\mathbf{s}(\mathbf{x}),$$

where $\sigma_H$ is the stress tensor of the fluid. If we set $D(v_f) = \frac{1}{2}(\nabla v_f + (\nabla v_f)^T)$, then

$$\sigma_H(\mathbf{x},t) = 2\eta D(v_f(\mathbf{x},t)) - p(\mathbf{x},t) \mathbf{I}.$$

We assume the homogeneous Dirichlet condition for $v_f$ on $\partial \Omega$ and require the velocity and normal stress to be continuous at the interface between the solid body and fluid

$$v_f(\mathbf{x},t) = 0, \ (\mathbf{x},t) \in \partial \Omega \times (0,T), \quad (2.19)$$

$$v_f(\mathbf{x},t) = v_p(\mathbf{x},t) \in \partial K_t \times (0,T), \quad (2.20)$$

$$((\sigma_H + \sigma_E) \cdot \nu)(\mathbf{x},t) = (\Sigma \cdot \nu)(\mathbf{x},t), \ (\mathbf{x},t) \in \partial K_t \times (0,T), \quad (2.21)$$

where $\Sigma$ is the Cauchy stress tensor in the solid. Also, we have the following initial conditions for the velocities

$$v_c(0) = v_{c,0}, \quad w(0) = w_0,$$

$$v_p(\mathbf{x},0) = v_{p,0}(\mathbf{x}) := v_{c,0} + w_0 \times (\mathbf{x} - \mathbf{x}_c(0)), \ \mathbf{x} \in K_0,$$

$$v_f(\mathbf{x},0) = v_{f,0}(\mathbf{x}), \ \mathbf{x} \in \Omega_0,$$

$$v_{f,0}(\mathbf{x}) = v_{p,0}(\mathbf{x}), \ \mathbf{x} \in \partial K_0.$$

In the "zeta" potential approach for electrophoresis modeling, (2.20) is replaced by a "slip" boundary condition on $v_f$: a nonlinear Dirichlet condition that depends on $\psi$ and it is based on a Prandtl boundary layer approximation (see [3], [4]). This approximation is relatively accurate when the Debye screening length of the macromolecule is much smaller comparable with its radius of curvature [38]. Otherwise, in the more realistic cases, this approach is no longer correct [23].

### 3 Weak Formulation and Main Result

We can obtain a weak formulation for the problem (2.3)-(2.22) if we take into account the energy framework of the system. First, we need to introduce a global
formulation for \(2.13\), \(2.17\)–\(2.18\). Following \[14\], we define in \(O \times [0, T]\) the Eulerian densities
\[
\mu_p(x, t) = \mathcal{P}_p \mathcal{J}_{K_t}(x, t), \quad \mu_f(x, t) = \mathcal{P}_f \mathcal{J}_{\Omega_t}(x, t),
\]
the global density \(\mu = \mu_p + \mu_f\) and the global velocity
\[
u(x, t) = \begin{cases} v_p(x, t) & \text{if } x \in K_t, \\ v_f(x, t) & \text{if } x \in \Omega_t. \end{cases}
\]
In view of mass conservation, \(\mu\) must satisfies
\[
\partial_t \mu + \nabla \cdot (\mu \mathbf{u}) = 0, \tag{3.23}
\]
in \(D'(O \times (0, T))\). Also, in terms of the Eulerian quantities, \(2.13\), \(2.17\) and \(2.18\) can be expressed, in \(D'(O \times (0, T))^3\), as
\[
\partial_t (\mu_f \mathbf{u}) + \nabla \cdot (\mu_f \mathbf{u} \otimes \mathbf{u}) = \frac{1}{\mu_f} \nabla \cdot (\mu_f \mathbf{S} \mathbf{H}) + \frac{1}{\mu_p} \nabla \cdot \mathbf{S} \mathbf{p} + \nabla \cdot (\mu_f \mathbf{S} \mathbf{E}),
\]
\[
\partial_t (\mu_p \mathbf{u}) + \nabla \cdot (\mu_p \mathbf{u} \otimes \mathbf{u}) = \frac{1}{\mu_p} \nabla \cdot (\mu_p \mathbf{S}) - \frac{1}{\mu_f} \mathbf{S} \mathbf{H} \mathbf{f} + \frac{1}{\mu_f} \mathbf{S} \mathbf{E} \mathbf{f} \mathbf{f} - \frac{1}{\mu_f} \mathbf{S} \mathbf{E} \nabla \mathbf{f},
\]
where we have consider the balance the momentum in the fluid and particle (see \[14\]). Summing the above equations and using \(2.21\) we obtain the global formulation for \(2.13\), \(2.17\)–\(2.18\),
\[
\partial_t (\mu \mathbf{u}) + \nabla \cdot (\mu \mathbf{u} \otimes \mathbf{u}) = \nabla \cdot \left( \frac{1}{\mu_f} \mu_f \mathbf{S} \mathbf{H} + \frac{1}{\mu_p} \mu_p \mathbf{S} + \mu_f \mathbf{S} \mathbf{E} \right), \tag{3.24}
\]
in \(D'(O \times (0, T))^3\). Furthermore, from \(2.20\), \(2.16\), \(2.19\) and \(2.22\) it is clear that, in the sense of distributions in \(O \times (0, T)\),
\[
\nabla \cdot \mathbf{u} = 0, \quad \mu_D \mathbf{u} = 0, \tag{3.25}
\]
and that
\[
u(x, 0) = \mathbf{u}_0(\mathbf{x}) := (\mathcal{J}_{K_0} v_{p, 0})(\mathbf{x}) + (\mathcal{J}_{\Omega_0} v_{f, 0})(\mathbf{x}), \quad \mathbf{x} \in O \]
\[
\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) := \mathcal{P}_p \mathcal{J}_{K_0}(\mathbf{x}) + \mathcal{P}_f \mathcal{J}_{\Omega_0}(\mathbf{x}), \quad \mathbf{x} \in O, \tag{3.26}
\]
Formally, taking the inner product of \(3.24\) with \(\mathbf{u}\), integrating by parts, using \(3.25\) and \(3.23\) we obtain, for all \(t \in (0, T)\),
\[
\frac{d}{dt} E[\mu, \mathbf{u}, N_1, \ldots, N_J, \psi](t) + \frac{\eta}{\mu_f} \left( \int_O \mu_f D \mathbf{u} : D \mathbf{u} dx \right)(t) + \left( \int_O \mu_f F \cdot \mathbf{u} dx \right)(t) = 0. \tag{3.27}
\]
Note that \(E = E_k + E_d + E_p\), where
\[
E_k[\mu, \mathbf{u}, N_1, \ldots, N_J, \psi](t) = \left( \frac{1}{2} \int_O \mu_f |\mathbf{u}|^2 dx \right)(t)
\]
\[
= \left( \frac{1}{2} \int_O \mu_f |\mathbf{u}|^2 dx \right)(t) + \frac{1}{2} (M|\mathbf{u}_c(t)|^2 + \mathbf{w}(t)^T A \mathbf{w}(t))
\]
is the kinetic energy at time $t$, 

$$E_d[\mu, u, N_1, \ldots, N_J, \psi](t) = \frac{\eta}{\rho_f} \int_0^t \int_O \mu_f D_u : D ud\tau$$

is the viscous dissipation and

$$E_p[\mu, u, N_1, \ldots, N_J, \psi](t) = -\int_0^t \int_O \mu_f F \cdot u dx$$

is the total work of the electrical force. Then (3.27) corresponds to the energy conservation of the system. Moreover, integrating (3.27) with respect to $t$ we obtain the energy (mechanical) bound

$$\left(\frac{1}{2} \int_0^t \int_O |u|^2 dx\right)(t) + \eta \int_0^t \int_O |\nabla u|^2 dx \leq \int_0^t \int_O \mu_f F \cdot u dx d\tau + \frac{1}{2} \int_0^t \int_O \mu_0 |u_0|^2 dx.$$

As a consequence, recalling that there exists a positive constant $C_* = C_*(O)$ such that

$$\|u(., t)\|_{0,2,O} \leq C_* \|\nabla u(., t)\|_{0,2,O},$$

it is easy to obtain the estimate

$$\left(\int_0^t |\nabla u|^2 dx\right)(t) + \int_0^t \int_O |u|^2 dx \leq \frac{4\mathcal{P}_\text{max}^2 C_*^2}{\mathcal{P}_\text{min}^2} \int_0^t \|F(\cdot, \tau)\|_{0,2,\Omega}^2 d\tau + \frac{2\mathcal{P}_\text{max}^2}{\mathcal{P}_\text{min}^2} \int_0^t \int_O |u_0|^2 dx,$$

where $\mathcal{P}_\text{min} = \min\{2\eta, \mathcal{P}_f, \mathcal{P}_f\}$ and $\mathcal{P}_\text{max} = \max\{\mathcal{P}_p, \mathcal{P}_f\}$.

Now, supposing that $\Psi \in H^1(\partial O)$, we define

$$L_\Psi = \{\hat{\Psi} \in H^1(\Omega), \hat{\Psi}|_{\partial O} = \Psi\}. \quad (3.30)$$

If we take the product of (2.31) with $\psi - \hat{\Psi}$, where $\hat{\Psi} \in L_\Psi$, we obtain, using (2.7) and (2.8),

$$\left(\int_0^\kappa |\nabla \psi|^2 dx\right)(t) - 4\pi e \sum_{i=1}^J \left(\int_{\Omega_t} Z_i N_i \psi dx\right)(t) - 4\pi \left(\int_\Omega \rho \psi dx\right)(t)$$

$$= -4\pi \left(\int_\Omega \rho \hat{\Psi} dx\right)(t) - 4\pi e \sum_{i=1}^J \left(\int_{\Omega_t} Z_i N_i \hat{\Psi} dx\right)(t) + \left(\int_\Omega \kappa \nabla \psi \cdot \nabla \hat{\Psi} dx\right)(t),$$

for each $t \in [0, T]$. Using Cauchy and Young inequalities, we have

$$\frac{1}{2} \left(\int_\Omega \kappa |\nabla \psi|^2 dx\right)(t) - 4\pi e \sum_{i=1}^J \left(\int_{\Omega_t} Z_i N_i \psi dx\right)(t) +$$

$$- 4\pi \left(\int_\Omega \rho \psi dx\right)(t) \leq \frac{1}{2} \left(\int_\Omega \kappa |\nabla \psi|^2 dx\right)(t) +$$

$$- 4\pi e \sum_{i=1}^J \left(\int_{\Omega_t} Z_i N_i \hat{\Psi} dx\right)(t) - 4\pi \left(\int_\Omega \rho \hat{\Psi} dx\right)(t), \quad (3.31)$$
for each \( t \in [0, T] \). The left side of (3.31) corresponds to the electrostatic energy contribution (the negative of the free energy according to (33)) which we denote by \( E_{el} \); the first term in \( E_{el} \) corresponds to the self-energy of the electric field and the next two terms are the electrostatic energies of the ions and fixed charges (see [12], [28], [33]). As a consequence of (3.31), for known \( \mathcal{N}_i \) and \( u \) and each \( t \in [0, T] \),

\[
E_{el}[\mu, u, \mathcal{N}_1, \ldots, \mathcal{N}_J, \psi](t) \leq E_{el}[\mu, u, \mathcal{N}_1, \ldots, \mathcal{N}_J, \tilde{\psi}](t),
\]

(3.32)

for all \( \tilde{\psi} \in L_\psi \).

Note that (2.10) (as well as (3.23)) corresponds to a conservation type equation. In fact, from the transport theorem

\[
\frac{d}{dt} \left( \int_{\Omega_t} N_i dx \right)(t) = \left( \int_{\Omega_t} \partial_t N_i dx \right)(t) + \left( \int_{\partial \Omega_t} N_i \nu \cdot \nu ds \right)(t),
\]

then, integrating (2.10) in \( \Omega_t \times [0, t] \), using Gauss theorem, (2.11), (2.19) and (2.12), we obtain the molar conservation for the \( i \)th ionic species,

\[
\left( \int_{\Omega_t} N_i dx \right)(t) = \int_{\Omega_0} N_{i,0} dx, \forall t \in [0, T].
\]

(3.33)

In particular, using (2.8), we have the total charge conservation for the system.

From the above discussion, in particular, from (3.32) and (3.27), we have a natural weak formulation of the above system (see Definition 3.1 below), which is obtained from the following minimization problem with constraints: Find \( (\mu, u, \mathcal{N}_1, \ldots, \mathcal{N}_J, \psi) \) (in appropriate functional spaces) that minimizes the energy \( E \) in \( \Omega \times (0, T) \) and such that \( \psi(., t) \) minimizes \( E_{el}[\mu, u, \mathcal{N}_1, \ldots, \mathcal{N}_J, \ldots](t) \) in \( L_\psi \), for each \( t \in [0, T] \); (2.10), (2.11), (2.12), (3.23) - (3.26) are the constraints; furthermore, the pressure \( p \) and the Cauchy stress tensor \( \Sigma \) are the Lagrange multipliers of the problem.

The additional hypotheses on the data are

\[
\mathbf{H} \quad \rho_0 \in L^2(\Omega), \text{supp} \rho_0 \subset K_0, \quad \Psi \in H^1(\partial \Omega), \quad v_{f,0} \in L^2(\Omega_0)^3, \quad v_{f,0}|_{\partial \Omega} = 0
\]

\[
\mathcal{N}_{i,0} \in L^2(\Omega), \text{supp} \mathcal{N}_{i,0} \subset \subset \Omega_0, \quad \mathcal{N}_{i,0} \geq 0 \text{ a. e. in } \Omega_0.
\]

Despite the fact that the ions are concentrated very close to \( \partial K_i \), the hypotheses on the support of \( \rho_0 \) and \( \mathcal{N}_{i,0} \) are reasonable as there is an ion exclusion region close to the boundary of the molecule [12].

In order to obtain the suitable functional spaces of the solutions, we observe that (3.28) and (3.29) can give us \( H^1 \)-bounds for \( u \) and \( \psi \). However, we need uniform \( L^2 \)-estimates for \( F \) and for \( \mathcal{N}_i \). As we will see, for small \( T \) (depending on the initial data) uniform \( L^4(\Omega_t) \)-estimates for \( \nabla \psi \) and \( \mathcal{N}_i \) can be obtained; as a consequence, we can estimate \( F \) and the conduction term in the weak formulation for (2.10).

We need to define the following functional spaces

\[
L^2(0, T; H^1(\Omega_t)) = \{ v \in L^2(\bar{\Omega}_T), \nabla v \in L^2(\bar{\Omega}_T)^3 \},
\]

\[
L^\infty(0, T; L^p(\Omega_t)) = \{ v \in L^p(\bar{\Omega}_T), \text{ ess sup}_{t \in (0, T)} \| v(., t) \|_{0, p, \Omega_t} < \infty \}.
\]
where
\[ \hat{\Omega}_T = \bigcup_{t \in (0,T)} \{t\} \times \Omega_t. \]

**Definition 3.1.** Let us to assume that $K_0$ and $\Omega_0$ are $C^{2+\alpha}$-domains, $0 < \alpha < 1$, $\rho$ is invariant under rigid motion (see (2.7)), hypothesis $H$ and (2.11). Then $(\mu, \mathbf{u}, \psi, N_1, \ldots, N_J)$ is a weak solution of (3.2), (3.24), (3.25), (3.26), coupled with (2.3), (2.7)- (2.9), (2.10), (2.11)- (2.12) in $(0,T)$ if it satisfies

1. $\mu \in L^{\infty}(\mathcal{O} \times (0,T))$, $\mathbf{u} \in L^{\infty}(0,T; L^2(\mathcal{O}))^3 \cap L^2(0,T; H_0^1(\mathcal{O}))^3$,  
   $\psi \in L^{\infty}(0,T; H^1(\mathcal{O}))$, $\nabla \psi \in L^{\infty}(0,T; L^4(\mathcal{O}))^3$,  
   $N_i \in L^2(0,T; H^1(\Omega_i)) \cap L^\infty(0,T; L^2(\Omega_i))$, $\forall i \in \{1, \ldots, J\}$.

2. For $\mathbf{F}$ given in (2.14), $(\mathbf{u}, \mu)$ satisfies
   \begin{align}
   &\int_0^t \int_{\mathcal{O}} (\mu \mathbf{u} \cdot \partial_t \mathbf{w} + \mu \mathbf{u} \otimes \mathbf{u} : D(\mathbf{w}) - \eta D(\mathbf{u}) : D(\mathbf{w}) + \mu_f \mathbf{F} \cdot \mathbf{w}) \, dxd\tau \\
   &+ \int_0^t \int_{\mathcal{O}} \mu_0 \mathbf{u}_0 \cdot \mathbf{w}_0 \, dx = \left( \int_0^t \mu \mathbf{u} \cdot \mathbf{w} \, dx \right) (t),
   \end{align}
   (3.34)
   for all $\mathbf{w} \in \mathcal{S}$ and a.e. $t \in (0,T)$, where
   \[ \mathcal{S} = \{ \mathbf{w} \in H^1(\mathcal{O} \times (0,T))^3 / \mathbf{w}(.,t) \in \mathcal{S}(t), \forall t \in (0,T) \} \]
   and $\mathcal{S}(t) = \{ \mathbf{w} \in H_0^1(\mathcal{O})^3 / \nabla \cdot \mathbf{w} = 0, \mu_f D(\mathbf{w}) = 0 \}$. Furthermore,
   \begin{align}
   &\partial_t \mu + \nabla \cdot (\mu \mathbf{u}) = 0, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \mathcal{D}'(\mathcal{O} \times (0,T)), \quad (3.35) \\
   &\mu_f D(\mathbf{u}) = 0 \quad \text{in} \quad \mathcal{D}'(\mathcal{O} \times (0,T))^3. \quad (3.36)
   \end{align}

3. For a.e. $t \in (0,T)$, $\psi(.,t)$ is the solution of the
   \begin{align}
   &\left( \int_{\mathcal{O}} \kappa |\nabla \psi|^2 \, dx \right) (t) - 4\pi \epsilon \sum_{i=1}^J \left( \int_{\Omega_i} Z_i N_i \psi \, dx \right) (t) + \\
   &- 4\pi \left( \int_{\mathcal{O}} \rho \psi \, dx \right) (t) = -4\pi \left( \int_{\mathcal{O}} \rho \tilde{\psi} \, dx \right) (t) + \\
   &- 4\pi \epsilon \sum_{i=1}^J \left( \int_{\Omega_i} Z_i N_i \tilde{\psi} \, dx \right) (t) + \left( \int_{\mathcal{O}} \kappa \nabla \psi \cdot \nabla \tilde{\psi} \, dx \right) (t),
   \end{align}
   (3.37)
   for all $\tilde{\psi} \in L_\psi = \{ \tilde{\psi} \in H^1(\mathcal{O}), \tilde{\psi}|_{\partial \mathcal{O}} = \Psi \}.$

4. For each $i \in \{1, \ldots, J\}$, $N_i$ satisfies
   \begin{align}
   &\left( \int_{\Omega_i} N_i \xi \, dx \right) (t) + \int_0^t \int_{\Omega_i} \xi (\mathbf{u} \cdot \nabla N_i) \, dxd\tau + \\
   &+ d_i \int_0^t \int_{\Omega_i} \left( \nabla N_i + \frac{Z_i e}{\kappa BB} N_i \nabla \psi \right) \cdot \nabla \xi \, dxd\tau + \\
   &- \int_0^t \int_{\Omega_i} N_i \xi_t \, dx \, d\tau = \left( \int_{\Omega_0} N_i \xi_0 \, dx \right),
   \end{align}
   (3.38)
   for a.e. $t \in (0,T)$ and for all $\xi \in H^1(\hat{\Omega}_T)$. Furthermore, $N_i \geq 0$ a.e. in $\hat{\Omega}_T$ and $\left( \int_{\Omega_i} N_i \, dx \right) (t) = \int_{\Omega_0} N_i \xi_0 \, dx$, for a.e. $t \in (0,T)$. 

The main result of the paper is given below

**Theorem 3.1.** There exists \( T^* > 0 \) and a weak solution of (3.23)-(3.26), coupled with (2.3)-(2.5), (2.7)-(2.12) in \((0, T^*)\). Moreover,

\[ T^* = \inf \{ t > 0; \text{dist}(K_t, \partial \Omega) > 0 \} \]

As remarked in the introduction of the present work, the strategy of the proof follows the lines of the results in references [14] and [15] for the motion of rigid bodies in incompressible fluids. It is based on the solution of an approximated system, which gives us an approximate sequence of solutions \((u_n, \mu_n, \psi_n, \mathcal{N}_{1,n}, \ldots, \mathcal{N}_{J,n})\) for (3.34)-(3.38). The solution \((u, \mu, \psi, \mathcal{N}_1, \ldots, \mathcal{N}_J)\) is built up as a limit of these approximations.

The new feature here is the coupling between (3.34)-(3.38) which lead us to consider an appropriate fixed-point schema in order to construct the approximate sequence. The existence results for the Lagrangian version of the parabolic problem (3.38) depend on the suitable estimates on the term containing \(\nabla \psi_2\) (see Lemma 4.3). Then, the crucial point of the proof is the obtention of the uniform \(L^4\)-estimates for \(\nabla \psi_2\) (see Lemma 4.1 below), in the sense that the bound does not depend on the particle motion. Using results on singular integral operator and elliptic regularity, this is possible if we assume that (2.2) is valid.

### 4 Approximate solutions

Let us fix \(0 < \gamma < \gamma_0\). We begin with a technical result

**Lemma 4.1.** Suppose \( t > 0 \) and \( Q(t) \) an affine isometry such that \( Q(0) = I \) and \( K_t = Q(t)K_0, \Omega_t = \Omega \setminus K_t \) satisfy (2.2). Consider the situation in which

\[ -4\pi \varepsilon \sum_{i=1}^{J} q_i(x, t) - 4\pi \rho(x, t) \]

is replaced by \( f(\cdot, t) \in L^2(\Omega) \) and \( \Psi \) is replaced by \( g \in H^1(\partial \Omega) \). Then the problem (3.37) has a unique solution \( \psi(\cdot, t) \in H^1(\Omega) \) that satisfies

\[ \max \{ \| \nabla \psi_1(\cdot, t) \|_{0,4,K_t}, \| \nabla \psi_2(\cdot, t) \|_{0,4,\Omega_t} \} \leq C_1(\| f(\cdot, t) \|_{0,2,\partial \Omega} + \| g \|_{1,2,\partial \Omega}), \]

(4.39)

where \( C_1 = C_1(\Omega, K_0, \kappa_1, \kappa_2, \gamma) \). If, \( f(\cdot, t) \in C^\alpha(\overline{\Omega}) \) and \( g \in C^{2+\alpha}(\partial \Omega) \), we have

\[ \psi(\cdot, t) \in C^{2+\alpha}(K_t) \times C^{2+\alpha}(\partial K_t) \times C^{2+\alpha}(\Omega_t). \]

(4.40)

**Proof.** From a theorem due to Stampacchia (see [6]) it is a routine to check that there exists an unique solution in \( H^1(\Omega) \) for the problem (3.37). Let us to extend \( f \) to be zero outside of \( \Omega \). Then, if \( \omega(x, t) = -(G * f)(x, t) \) where \( G \) is the fundamental solution of the Laplace equation in \( \mathbb{R}^3 \), \( \omega(\cdot, t) \in H^2(\Omega), \)

\[ \Delta \omega(x, t) = f(x, t), \quad \text{a.e.} \quad x \in \mathbb{R}^3, \]

(4.41)
and there exists $C = C(\mathcal{O})$ such that
\[
\|\omega(\cdot, t)\|_{2,2,\mathcal{O}} \leq C\|f(\cdot, t)\|_{0,2,\mathcal{O}}. \tag{4.42}
\]

For the above results see Theorem 9.9 in [21].

We define $B_\gamma = \{x \in \mathcal{O}, 0 < \text{dist}(x, \partial\mathcal{O}) < \gamma/2\}$ and consider the following problem
\[
\begin{align*}
\Delta \Phi(x, t) &= 0, \quad x \in B_\gamma, \\
\Phi(x, t) &= 0, \quad x \in \partial B_\gamma \setminus \partial\mathcal{O}, \\
\Phi(x, t) &= \kappa_2 g(x) - \kappa_1^{-1} \omega(x, t), \quad x \in \partial\mathcal{O}.
\end{align*} \tag{4.43}
\]

From well known results (see Theorem 2.2 in Chapter 4 in [29]), this problem has a unique solution
\[
\Phi(\cdot, t) \in H^{3/2}(B_\gamma) \tag{4.44}
\]
that satisfies
\[
\|\Phi(\cdot, t)\|_{3/2,2,\mathcal{O}} \leq C(\|g\|_{1,2,\partial\mathcal{O}} + \|\omega(\cdot, t)\|_{1,2,\partial\mathcal{O}}), \tag{4.45}
\]
where $C = C(\mathcal{O}, \kappa_2, \gamma)$ and we have extended $\Phi$ to be zero in $\mathcal{O} \setminus B_\gamma$.

Now, let us consider $0 < \sigma < \gamma/4$ and define the set
\[
A_\sigma(t) = \{x \in \Omega_t, 0 < \text{dist}(x, \partial K_t) < \sigma\}.
\]

Note that, from [22], we have dist$(A_\sigma(t), B_\gamma) > \gamma/4$. We also define
\[
\tilde{\omega}(\xi, t) = \omega(Q(t)\xi, t), \quad \forall \xi \in K_0
\]
and consider the following auxiliary problem
\[
\begin{align*}
\Delta \tilde{\phi}_1(\xi, t) &= 0, \quad \xi \in K_0, \\
\Delta \tilde{\phi}_2(\xi, t) &= 0, \quad \xi \in A_\sigma(0), \\
(\kappa_2^{-1} \tilde{\phi}_2 - \kappa_1^{-1} \tilde{\phi}_1)(\xi, t) &= (\kappa_2^{-1} - \kappa_1^{-1}) \tilde{\omega}(\xi, t), \quad \xi \in \partial K_0, \\
(\partial_\nu \tilde{\phi}_2 - \partial_\nu \tilde{\phi}_1)(\xi, t) &= 0, \quad \xi \in \partial K_0, \\
\tilde{\phi}_2(\xi, t) &= 0, \quad \xi \in \partial A_\sigma(0) \setminus \partial K_0.
\end{align*} \tag{4.46}
\]

This problem has a unique solution
\[
(\tilde{\phi}_1(\cdot, t), \tilde{\phi}_2(\cdot, t)) \in H^{3/2}(K_0) \times H^{3/2}(\mathbb{R}^3 \setminus K_0), \tag{4.47}
\]
where we have extended $\tilde{\phi}$ to be zero outside $A_\sigma(0) \cup K_0$. In fact this follows from suitable modifications in the arguments of the Theorem 7.2 in the reference [40] (see also [22]). Furthermore, from the results on singular integral operators of [10] and [40] (for related results see [2]), there exists a constant $C > 0$, depending on $K_0$, $\kappa_1$, $\kappa_2$ and $\gamma$ such that
\[
\begin{align*}
\|\tilde{\phi}_1(\cdot, t)\|_{3/2,2,K_0} &\leq C\|\tilde{\omega}(\cdot, t)\|_{1,2,\partial K_0}, \\
\|\tilde{\phi}_2(\cdot, t)\|_{3/2,2,\mathbb{R}^3 \setminus K_0} &\leq C\|\tilde{\omega}(\cdot, t)\|_{1,2,\partial K_0}. \tag{4.48}
\end{align*}
\]
Now, if we define $\phi_i(x, t) = \tilde{\phi}_i(Q^{-1}(t)x, t), i = 1, 2$, we see that
\[
\Delta \phi_1(x, t) = 0, \quad x \in K_t, \\
\Delta \phi_2(x, t) = 0, \quad x \in A_\sigma(t), \\
(\kappa_2^{-1} \phi_2 - \kappa_1^{-1} \phi_1)(x, t) = (\kappa_1^{-1} - \kappa_2^{-1})\omega(x, t), \quad x \in \partial K_t, \\
(\partial_\nu \phi_2 - \partial_\nu \phi_1)(x, t) = 0, \quad x \in \partial K_t, \\
\phi_2(x, t) = 0, \quad x \in \partial A_\sigma(t) \setminus \partial K_t.
\] (4.49)

This follows from an elementary calculation using the properties of $Q(t)$. We claim that
\[
\psi(x, t) = \begin{cases} 
\psi_1(x, t) = \kappa_1^{-1}(\phi_1 + \omega + \Phi)(x, t), & x \in K_t, \\
\psi_2(x, t) = \kappa_2^{-1}(\phi_2 + \omega + \Phi)(x, t), & x \in \Omega_t.
\end{cases}
\] (4.50)

In fact, the transmission boundary conditions (2.7)-(2.9) are satisfied in the sense of traces, in particular, (2.7) implies that $\psi(t) \in H^1(\mathcal{O})$; also it it easy to check that (3.37) is valid. The result then follows from the uniqueness of the problem (3.37). Now, note that,
\[
\nabla_\zeta \tilde{\omega}(\xi, t) \cdot \nabla_\zeta \tilde{\omega}(\zeta, t) = \nabla_x \omega(x, t) \cdot \nabla_y \omega(y, t),
\] (4.51)

for $x = Q(t)\xi, y = Q(t)\zeta, \forall \xi, \zeta \in \mathbb{K}_0$. Then, if we consider the Slobodetskii norm
\[
\|\tilde{\omega}(., t)\|_{3/2, 2, K_0}^2 = \int_{K_0} \tilde{\omega}^2(\xi, t) d\xi + \int_{K_0} |\nabla_\zeta \tilde{\omega}(\xi, t)|^2 d\xi + \\
+ \int_{K_0} \int_{K_0} \frac{|\nabla_\zeta \tilde{\omega}(\xi, t) - \nabla_\zeta \tilde{\omega}(\zeta, t)|^2}{|\xi - \zeta|^4} d\xi d\zeta,
\]

it is easy to check that
\[
\|\tilde{\omega}(., t)\|_{3/2, 2, K_0}^2 = \|\omega(., t)\|_{3/2, 2, K_1}^2.
\] (4.52)

Henceforth, from (4.51), standard Sobolev embedding, (4.48), the trace theorem, (4.52) and (4.42), we have
\[
\left( \int_{\mathbb{R}^3 \setminus \mathbb{K}_1} |\nabla_x \phi_2(x, t)|^4 d\xi \right)^{1/4} = \left( \int_{\mathbb{R}^3 \setminus \mathbb{K}_0} |\nabla_\zeta \tilde{\phi}_2(\xi, t)|^4 d\xi \right)^{1/4} \\
\leq C\|\phi_2(., t)\|_{3/2, 2, \mathbb{R}^3 \setminus \mathbb{K}_0} \leq C\|\tilde{\omega}(., t)\|_{1, 2, \partial K_0} \\
\leq C\|\omega(., t)\|_{3/2, 2, K_0} = C\|\omega(., t)\|_{3/2, 2, K_1} \\
\leq C\|\omega(., t)\|_{3/2, 2, \mathcal{O}} \leq C\|f(., t)\|_{0, 2, \mathcal{O}},
\] (4.53)

where $C = C(K_0, \mathcal{O})$. Similarly,
\[
\|\nabla \phi_1(., t)\|_{3/4, K_1}^2 \leq C\|f(., t)\|_{0, 2, \mathcal{O}}.
\] (4.54)

Consequently, (4.59) follows from (4.50), (4.46), the trace theorem, Sobolev embedding, (4.42), (4.53) and (4.54).
Now, let us to suppose that \( f(.,t) \in C^n(\overline{\mathcal{O}}) \) and consider the extension of \( f(.,t) \) to \( \mathbb{R}^3 \) (until denoted by \( f \)) such that \( f(.,t) \in C_0^2(\mathbb{R}^3) \). Then, we have \( \omega(.,t) \in C^{2+\alpha}(\overline{\mathcal{O}}) \) (see Lemma 4.4 in [21]). If \( g \in C^{2+\alpha}(\partial\mathcal{O}) \), from classical results, \( \left(\phi_1(.,t),\phi_2(.,t)\right) \in C^{2+\alpha}(\overline{\mathcal{O}}_x) \times C^{2+\alpha}(\mathbb{R}^3\setminus K_I) \) (see Theorem 2.1, Chapter 14 in [26]) and 
\[
\Phi(.,t) \in C^{2+\alpha}(\overline{\mathcal{O}})
\]
(see Theorem 6.14 in [21]). Hence (4.40) follows from (4.50).

In what follows we describe how to construct the sequence of approximations. We adapt the proof in [15]. The idea is to introduce an approximation scheme which consists in solving a system of regularized equations: an “almost” linear problem related to the velocity field as well as appropriate linear problems for \( \mathcal{N}_i \) and \( \psi \). The existence of these regularized solutions is obtained from the use of a fixed point type theorem. As described below, this is done in small time intervals, chosen in such a way that the advecting vector field is close to the identity, (2.2) is valid in each time interval, and using a Lagrangian Galerkin method for the linear problems related to \( u \) and \( \mathcal{N}_i \) (a similar approach may be seen in the study of the free surface problems [39]).

Let us consider
\[
L_2 > \frac{1}{d_{\text{min}}^{1/2}} \max\{\|\mathcal{N}_{1,0}\|_{0,2,\Omega_0}, \ldots, \|\mathcal{N}_{J,0}\|_{0,2,\Omega_0}\},
\]
where \( d_{\text{min}} = \min\{1/2, d_{1}/8, \ldots, d_{J}/8\} \) and
\[
A := \frac{32\pi^2 d_{\text{max}} Z_{\text{max}}^2 C_2}{9 \kappa_B r^2 d_{\text{min}}} (L_2^2 + \|\rho_0\|_{0,2,\mathcal{K}_0}^2 + \|\Psi\|_{1,2,\mathcal{O}}^2),
\]
where \( C_2 = 2 C_2^2 \max\{32 \pi^2 c^2 \sum_{i=1}^J |Z_i|^2, 32 \pi^2\} \), \( d_{\text{max}} = \max\{d_1, \ldots, d_J\} \) and \( Z_{\text{max}} = \max\{|Z_1|, \ldots, |Z_J|\} \).

From the Sobolev embedding theorem, there exists a positive constant \( C_{**} = C_{**}(\Omega_0) \) such that
\[
\|f\|_{0,4,\Omega_0} \leq C_{**} \|f\|_{1,2,\Omega_0}.
\]
Let us define \( L_1 > 0 \) such that
\[
L_1^2 > \frac{8 \pi_{\text{max}}^2 C_2^2 C_2^2 c^2 L_0^2 \sum_{i=1}^J |Z_i|^2}{\pi_{\text{min}}^2} (L_2^2 + \|\rho_0\|_{0,2,\mathcal{K}_0}^2 + \|\Psi\|_{1,2,\mathcal{O}}^2) +
\]
\[
+ \frac{2 \pi_{\text{max}}}{\pi_{\text{min}}} \int_{\mathcal{O}} |u_0|^2 \, dx,
\]
and
\[
0 < T < \min\left\{\frac{\gamma - \gamma_0}{L_1}, \frac{1}{4 \pi^2}\right\}.
\]

Following [15] we observe that it is possible to show the existence of a homeomorphism \( \Theta \) from the space of incompressible vector fields \( u \in L^2(0,T; H^1_0(\mathcal{O}))^3 \cap L^\infty(0,T; L^2(\mathcal{O}))^3 \).
and which corresponds to a rigid motions in $K_\varepsilon$ into the representation space $Y_T = C^{0,1}([0,T];\mathbb{R}^3 \times \mathbb{R}^3) \times \lbrace \vec{v} \in L^\infty(0,T;L^2(\Omega_0))^3 \cap L^2(0,T;H^1_0(\Omega_0))^3, \nabla \cdot \vec{v} = 0 \rbrace$.

Let us also define $W_T = L^\infty(0,T;L^2(\Omega_0))$. Here we will consider the natural corresponding norms in $Y_T$ and $W_T$. Now, using standard regularization techniques, we consider

- $u_{0,\varepsilon} \in C^\infty_0(\Omega)^3$ such that $\nabla \cdot u_{0,\varepsilon} = 0$ in $\Omega$ and corresponds to a rigid motion in $K_0$; furthermore, $u_{0,\varepsilon}$ converges to $u_0$ in $L^2(\Omega)^3$ as $\varepsilon \to 0^+$.
- $N_{i,0,\varepsilon} \in C^{2+\alpha}(\Omega)$, $\text{supp}N_{i,0,\varepsilon} \subset \subset \Omega_0, N_{i,0,\varepsilon} \geq 0$ in $\Omega_0, N_{i,0,\varepsilon}$ converges to $N_{i,0}$ in $L^2(\Omega)$ as $\varepsilon \to 0^+$.
- $\Psi_{\varepsilon} \in C^{2+\alpha}(\partial \Omega)$, $\Psi_{\varepsilon}$ converges to $\Psi$ in $H^1(\partial \Omega)$ as $\varepsilon \to 0^+$.
- $\rho_{0,\varepsilon} \in C^\infty_0(K_0)$, $\rho_{0,\varepsilon}$ converges to $\rho_0$ in $L^2(K_0)$ as $\varepsilon \to 0^+$.

Also we consider, for each $f \in L^\infty(0,T;L^2(\Omega))$, $r_{\varepsilon}(f) \in L^\infty(0,T;C^\infty(\Omega))$ such that

$$\|r_{\varepsilon}(f)\|_{L^\infty(0,T;L^2(\Omega))} \leq \|f\|_{L^\infty(0,T;L^2(\Omega))},$$

(4.60)

for $\varepsilon > 0$ sufficiently small and $r_{\varepsilon}(f_{\varepsilon})$ converges to $f$ in $L^2(\Omega \times (0,T))$ with $f_{\varepsilon}$ converging to $f$ in $L^2(\Omega \times (0,T))$ as $\varepsilon \to 0^+$.

Let us also consider, for $f \in L^2(0,T)$,

$$[f]^{\varepsilon}(t) = \int_0^T g_{\varepsilon}(t-\tau)f(\tau)d\tau,$$

(4.61)

where $\{g_{\varepsilon}\}_{\varepsilon>0}$ is a family of regularizing kernels with respect to time such that $[f]^{\varepsilon}(\cdot) \in C^\infty[0,T]$. Furthermore, if $f \in L^4(0,T)$

$$\|f\|^4_{0,4,(0,T)} \leq \|f\|_{0,4,(0,T)},$$

(4.62)

for $\varepsilon > 0$ sufficiently small and $[f_{\varepsilon}]^{\varepsilon}$ converges to $[f]$ in $L^2(0,T)$ with $f_{\varepsilon}$ converging to $f$ in $L^2(0,T)$.

Now, as in [13], for any

$$\vec{v} = (x_{\varepsilon}(t), \theta(t), \vec{v}) \in Y_T,$$

we consider the incompressible field

$$\vec{v} = \Theta(\vec{v}) \in L^\infty(0,T;L^2(\Omega))^3 \cap L^2(0,T;H^1_0(\Omega))^3$$

and its regularized version $\vec{v}_{\varepsilon} = \mathcal{R}_{\varepsilon}(\vec{v})$, where $\mathcal{R}_{\varepsilon}$ is a regularization operator (see [14]), such that $\mathcal{R}_{\varepsilon}(\vec{v})$ is analytical in time and smooth in space (in particular, Lipschitz in space), $\mathcal{R}_{\varepsilon}(\vec{v}) \to \vec{v}$ in $L^2(\Omega \times (0,T))^3$ if $\vec{v}_{\varepsilon} \to \vec{v}$ in $L^2(\Omega \times (0,T))^3$, as $\varepsilon \to 0^+$. Furthermore, $\mathcal{R}_{\varepsilon}$ is divergence free in $\Omega$ and corresponds to a rigid motion in the particle domains $K_{\varepsilon,i} = \{X_{\varepsilon}(\xi,0,t), \xi \in K_0\}$, where $X_{\varepsilon}$ is the Lagrangian flow of $\vec{v}_{\varepsilon}$: for each $0 \leq s \leq T$ and $\xi \in \mathcal{O}$,

$$\frac{d}{dt}X_{\varepsilon}(\xi,s,t) = \vec{v}_{\varepsilon}(X_{\varepsilon}(\xi,s,t),t), 0 \leq t \leq T, t \neq s$$

$$X_{\varepsilon}(\xi,s,s) = \xi.$$
Also, we denote, for each \( t \in [0, T] \), \( \Omega_{t} = \mathcal{O} \setminus K_{t, \epsilon} \). From the construction of \( \Theta \) (see [15]), if \( v = \Theta(\nabla) \) for some \( \nabla \in Y_{T} \) and \( \| \nabla \|_{Y_{T}} \leq L_{1} \), we have

\[
\| v \|_{L^{\infty}(0, T; L^{2}(\mathcal{O}))} \leq L_{1}
\]  

(4.64)

for \( \epsilon > 0 \) sufficiently small. Then, let consider \( \epsilon' > 0 \) such that (4.60), (4.61), (4.62) and

\[
\| \nabla_{x} v \|_{0, 2, 0, \mathcal{O}}, \| \nabla v_{0, t} \|_{0, 2, 0, \mathcal{O}} \leq \| \nabla v_{0} \|_{0, 2, 0, \mathcal{O}} \quad \text{(4.65)}
\]

are valid for all \( 0 < \epsilon < \epsilon' \). As a consequence of (4.64) and (4.65), it is easy to check that

\[
\gamma_{\epsilon}(t) := \text{dist} \left( \partial K_{\epsilon, t}, \partial \mathcal{O} \right) > \gamma
\]

(4.66)

for all \( t \in [0, T] \) and \( 0 < \epsilon < \epsilon' \).

Now, for \( 0 < \epsilon < \epsilon' \) fixed, let us define \( \mathcal{F}_{\epsilon} = \nabla X_{\epsilon} \). From the smoothness of \( \nabla v_{\epsilon}, \) (4.64) and (4.65), it is possible to choose \( N \in \mathbb{N}^{+} \) (depending only on \( \epsilon \) and \( L_{1} \)) and \( t_{0} = T/N \) such that

\[
\sup_{0 \leq m \leq N} \sup_{t \in [mt_{0}, (m+1)t_{0}]} \| I - \mathcal{F}_{\epsilon}(\cdot, mt_{0}, t) \|_{L^{\infty}(\mathcal{O})} \leq 1/3.
\]  

(4.67)

The inequality (4.67) implies that, in each time interval \( (t_{k-1}, t_{k}) \), the advecting vector field is close to the identity. This is an important point in the proof and it is based on [15]. As we will see, from (4.67), the Lagrangian forms of certain operators that appear in the Lagrangian formulation for (3.37) and (5.37) are uniformly elliptic. In fact, from (4.67), it is easy to check that, for each \( m \in \{0, 1, \ldots, N\} \) and \( t \in [mt_{0}, (m+1)t_{0}] \),

\[
1/4|y|^{2} \leq |\mathcal{F}_{\epsilon}^{-1}(\xi, mt_{0}, t) \cdot y|^{2} < 4|y|^{2}, \quad \text{a.e.} \quad \xi \in \mathcal{O},
\]  

(4.68)

\( \forall \ y \in \mathbb{R}^{3} \). As a consequence, we have existence results for the related linearized problems, in each time interval \( (t_{k-1}, t_{k}) \).

Let us consider \( L_{2}, L_{1}, T, \epsilon' > 0 \), \( N \) and \( t_{0} \) as above. For each \( 0 < \epsilon < \epsilon' \) and in each time interval \( (t_{k-1}, t_{k}) \), we want to solve a set of linearized problems and to apply the Schauder’s fixed point theorem. Below we give the details for the first time interval \( (0, t_{0}) \).

We take \( (\nabla, \tilde{\Theta}_{1}, \ldots, \tilde{\Theta}_{j}) \in \mathcal{B} \times \mathcal{X} \times \ldots \times \mathcal{X} \), where

\[
\mathcal{B} = \{ \nabla \in Y_{0}/\| \nabla \|_{Y_{0}} \leq L_{1} \}, \quad \mathcal{X} = \{ \tilde{\Theta}_{i} \in W_{0}/\| \tilde{\Theta}_{i} \|_{W_{0}} \leq L_{2} \}.
\]

Extend \( \nabla(\cdot, t) \) to be zero for \( t \in (t_{0}, T) \), we want to solve (in the given order) the following linearized problems

**P1** Problem (5.37) in \( (0, t_{0}) \) with \( N_{i} \) replaced by

\[
\tilde{\theta}_{i, \epsilon}(\cdot, t) := r_{\epsilon}(\tilde{\theta}_{i})(\cdot, t),
\]  

(4.69)
where \( \tilde{\vartheta}_i(., t) = \tilde{\vartheta}_i(X_\epsilon(., t, 0), t) \) and \( \tilde{\vartheta}_i(., t) \) is extended to be zero outside of \( \Omega \); the domains \( (\Omega_t, K_t) \) replaced by \( (\Omega_{t, \epsilon}, K_{t, \epsilon}) \), \( \Psi \) replaced by \( \Psi_{t, \epsilon} \), \( \rho_0 \) replaced by \( \rho_{0, \epsilon} \), \( \kappa \) replaced by

\[
\kappa_{t, \epsilon}(x, t) = \kappa_{1, t, \epsilon}(x, t) + \kappa_{2, t, \epsilon}(x, t),
\]

(4.70)

and \( \rho, q \), \( i \), \( q_{i, \epsilon} \) give as in (2.33) and (2.34) (with the suitable modifications), respectively. Let us to denote this solution as \( \psi_{t, \epsilon} \).

**P2** Problem (3.33) in \( (0, t_0) \), with \( (\psi, u) \) replaced by \( (|\psi_{t, \epsilon}|, v_{t, \epsilon}) \), \( N_{t, 0} \) by \( N_{t, 0, \epsilon} \) and the domains \( (\Omega_t, K_t) \) replaced by \( (\Omega_{t, \epsilon}, K_{t, \epsilon}) \). Let us denote this solution as \( N_{t, \epsilon} \).

**P3** Problem (3.34) in \( (0, t_0) \) with \( \mu(x, t) \) replaced by

\[
\mu_{t, \epsilon}(x, t) = \mu_{1, t, \epsilon}(x, t) + \mu_{2, t, \epsilon}(x, t)
\]

(4.71)

(the solution of (3.23), considering \( v_{t, \epsilon} \) instead \( u \)), \( F \) by \( F_{t, \epsilon} \), \( u_0 \) by \( u_{0, \epsilon} \) and replacing the term \( \mu u \otimes u \) by \( \mu_{t, \epsilon} v_{t, \epsilon} \otimes u \). Here \( F_{t, \epsilon} \) corresponds to \( F \) given in (2.24) with \( (\psi, N_{t, \epsilon}) \) replaced by \( (\psi_{t, \epsilon}, N_{t, \epsilon}) \).

### 4.1 The Poisson System

**Lemma 4.2.** The problem **P1** has a unique solution \( \psi_{t, \epsilon} \in L^\infty(0, t_0; H^1(\Omega)) \) that satisfies

\[
(\psi_{t, \epsilon}(., t), \psi_{t, \epsilon}(., t)) \in C^{2+\alpha}(\Omega_{t, \epsilon}) \times C^{2+\alpha}(\Omega_{t, \epsilon}),
\]

(4.72)

for a. e. \( t \in (0, t_0) \). Furthermore, for a. e. \( t \in (0, t_0) \)

\[
\max\{|\nabla \psi_{t, \epsilon}(., t)|^2_{0, 4, K_{t, \epsilon}}, |\nabla \psi_{t, \epsilon}(., t)|^2_{0, 4, \Omega_{t, \epsilon}}\}
\]

\[
\leq C_2(L^2_2 + ||\rho_0||^2_{0, 2, K_0} + ||\Psi||^2_{1, 2, \partial \Omega}),
\]

(4.73)

where \( C_2 = 2C_1^2 \max\{32\pi^2 \alpha^2 \sum_{i=1}^J |Z_i|^2, 32\pi^2 \epsilon^2 \} \).

**Proof.** First we observe that, for all \( t \in [0, t_0] \), we have \( \nabla \cdot v_{t, \epsilon}(., t) = 0 \), so that \( \det F_{t, \epsilon}(., t, 0) = 1 \), as is easy to check. As a consequence, from (4.69) and (4.66), we have

\[
||\vartheta_{t, \epsilon}(., t)||_{0, 2, \Omega_{t, \epsilon}} \leq L_2
\]

(4.74)

for a. e. \( t \in (0, t_0) \). Furthermore, from the smoothness of \( X_\epsilon \) and the definition of \( r_\epsilon(\vartheta_{t, \epsilon}), \vartheta_{t, \epsilon}(., t) \in C^{\alpha}(\Omega_{t, \epsilon}), \) a. e. \( t \in (0, t_0) \). Then, recalling (4.66), the results follow from Lemma (3.1) with

\[
f = -4\pi \epsilon \sum_{i=1}^J q_{i, \epsilon}(., t) - 4\pi \rho_{i, \epsilon}(., t)
\]

and \( g = \Psi_{t, \epsilon} \). In (4.73) we have also used (4.65). □
Remark 4.1. We have also the estimates, for a.e. \( t \in (0, t_0) \),
\[
\| \omega_i(t) \|_{C^{2+a}(\overline{\Omega})} \leq C_\varepsilon (L_2 + \| \rho_0 \|_{0,2,K_0}),
\]
\[
\| \psi_i(t) \|_{C^{2+a}(\overline{\Omega})} \leq C(\| \psi_i \|_{C^{2+a}(\partial \Omega)} + \| \omega_i(t) \|_{C^{2+a}(\partial \Omega)})
\leq C_\varepsilon (L_2 + \| \rho_0 \|_{0,2,K_0} + \| \Psi \|_{1,2,\partial \Omega}),
\]
\[
\| \phi_{2,i}(t) \|_{C^{2+a}(\overline{\Omega})} \leq C(\| \omega_i(t) \|_{C^{2+a}(\partial K_{i,0}^a)}),
\]
where \( C_\varepsilon = C_\varepsilon (\Omega, K_0, e, \kappa_1, \kappa_2, \gamma, Z_1, \ldots, Z_J) \). As a consequence, for a.e. \( t \in (0, t_0) \),
\[
\| \psi_{2,i}(t) \|_{C^{2+a}(\overline{\Omega})} \leq C_\varepsilon (1 + \| \rho_0 \|_{0,2,K_0} + \| \Psi \|_{1,2,\partial \Omega}),
\] (4.75)
where \( C_\varepsilon = C_\varepsilon (\Omega, K_0, e, \kappa_1, \kappa_2, \gamma, L_2, Z_1, \ldots, Z_J) \). Moreover, for each \( \xi \in \overline{\Omega} \) the function \( \tilde{\psi}_i(\xi, \cdot) \) is measurable with respect to the variable \( t \in [0, t_0] \). The same is valid for the first and second derivatives of \( \tilde{\omega}_i(\xi, \cdot) \) with respect to \( \xi \). In fact, this is a consequence of (4.50) as \( \tilde{\omega}_i(\xi, \cdot) \) is clearly measurable and \( \tilde{\Phi}_i(\xi, \cdot) \), \( \tilde{\phi}_i(\xi, \cdot) \) can be represented in terms of simple and double layer potentials with measurable time dependent density functions (see the classical results on Laplace and transmission problems in [9]).

4.2 The Nernst Planck Equation

Let us define \( \nabla \nu^* = F_{\varepsilon}^{-1} \cdot \nabla \) and \( \tilde{\psi}_i(\cdot, t) = \psi_i(X_i(\cdot, 0, t), t) \). We consider the problem \( P_2 \) in its Lagrangian version: For each \( i \in \{1, \ldots, J\} \), find
\[
\tilde{N}_{i,e}(\cdot, t) := N_i(X_i(\cdot, 0, t), t)
\]
satisfying
\[
\partial_t \tilde{N}_{i,e}(\xi, t) = d_i \nabla \psi_i \cdot \left( \nabla \psi_i \tilde{N}_{i,e} + \frac{Z_i e}{\kappa B \theta} \tilde{N}_{i,e} \nabla \psi_i \tilde{\psi}_i \right) \quad (\xi, t) \in \Omega_0 \times (0, t_0]
\]
\[
\tilde{N}_{i,e}(\xi, 0) = N_{i,0,e}(\xi) \quad \xi \in \Omega_0,
\]
\[
\left( \partial_t \tilde{N}_{i,e} + \frac{Z_i e}{\kappa B \theta} \tilde{N}_{i,e} \partial_t \tilde{\psi}_i \right) (\xi, t) = 0 \quad (\xi, t) \in \partial \Omega \cup \partial K_0 \times (0, t_0],
\] (4.76)
where we have extended \( \tilde{\psi}_i(\cdot, t) \) to be zero in \( t \in (t_0, T) \). From the hypothesis \( H \) we have the compatibility condition
\[
\partial_t N_{i,0,e}(\xi) + \frac{Z_i e}{\kappa B \theta} N_{0,i,e}(\xi) \partial_t \psi_{2,i}(\xi, 0) = 0 \quad \xi \in \partial \Omega \cup \partial K_0.
\] (4.77)

Also, we consider the weak formulation for (4.76): Find
\[
\tilde{N}_{i,e} \in L^\infty (0, t_0; L^2(\Omega_0)) \cap L^2 (0, t_0; H^1(\Omega_0))
\]
satisfying
\[
\left( \int_{\Omega_0} \tilde{N}_{i,e} d\xi \right) (t) + d_i \int_{\Omega_0} \int_0^t \left( \nabla \psi_i \tilde{N}_{i,e} + \frac{Z_i e}{\kappa B \theta} \tilde{N}_{i,e} \nabla \psi_i \tilde{\psi}_i \right) \cdot \nabla \psi_i \tilde{\psi}_i d\xi d\tau +
\]
\[
- \int_0^t \int_{\Omega_0} \tilde{N}_{i,e} d\xi d\tau = \left( \int_{\Omega_0} N_{i,0,e} d\xi \right)
\] (4.78)
for all \( \varsigma \in H^1(\Omega_0 \times (0, t_0)) \) and a. e. \( t \in (0, t_0) \).

A direct calculation using (4.68) gives us that the problem (4.76) is uniformly parabolic. Using the regularity of the coefficients we can obtain the existence of a unique nonnegative solution \( \bar{N}_{i, \epsilon} \in C^{2+\alpha,1}(\Omega_0 \times [0, t_0]) \) of (4.76)- (4.77) (see Theorem 5.3, Chapter IV in [25]) that corresponds to the unique solution of (4.78). For sufficiently small \( t_0 \), we can obtain uniform estimates on \( \bar{N}_{i, \epsilon} \). In fact, we have the following Lemma.

**Lemma 4.3.** The solution \( \bar{N}_{i, \epsilon} \) of (4.76)- (4.77) satisfies

\[
\sup_{t \in [0,t_0]} \int_{\Omega_0} |\bar{N}_{i, \epsilon}(\xi, t)|^2 d\xi + \int_0^{t_0} \int_{\Omega_0} |
abla_\xi \bar{N}_{i, \epsilon}|^2 d\xi d\tau < L_2, \tag{4.79}
\]

where \( L_2 \) is given in (4.55).

**Proof.** First, we observe that a standard calculation gives us that, for all \( t \in [0, t_0] \)

\[
\frac{1}{2} \int_{\Omega_0} (|\bar{N}_{i, \epsilon}(\xi, t)|^2 - |\bar{N}_{i, \epsilon}(\xi, 0)|^2) d\xi + d_i \int_0^t \int_{\Omega_0} |\nabla_v \bar{N}_{i, \epsilon}|^2 d\xi d\tau =
\]

\[
= - \frac{d_i Z_\epsilon c}{k_B \theta} \int_0^t \int_{\Omega_0} \bar{N}_{i, \epsilon} \nabla_\xi \tilde{\psi}_\epsilon \cdot \nabla_v \bar{N}_{i, \epsilon} d\xi d\tau \tag{4.80}
\]

\[
\leq \frac{d_i}{2} \int_0^t \int_{\Omega_0} |\nabla_v \bar{N}_{i, \epsilon}|^2 d\xi d\tau + \frac{d_i Z_\epsilon^2 c^2}{2 k_B \theta^2} \int_0^t \int_{\Omega_0} |\bar{N}_{i, \epsilon}|^2 |\nabla_v \tilde{\psi}_\epsilon|^2 d\xi d\tau.
\]

Now, from (4.68), (4.69) and recalling the Remark 1, we have

\[
\int_{\Omega_0} |\nabla_v [\tilde{\psi}_\epsilon]^* (\xi, t)|^4 d\xi < 16 \int_{\Omega_0} |\nabla_v [\tilde{\psi}_\epsilon]^* (\xi, t)|^4 d\xi
\]

\[
= 16 \int_{\Omega_0} |\nabla \tilde{\psi}_\epsilon (\xi, t) \theta \xi (t - \tau) d\tau |^4 d\xi \tag{4.81}
\]

\[
= 16 \int_{\Omega_0} \int_0^{t_0} |\nabla \tilde{\psi}_\epsilon (\xi, \tau) \theta \xi (t - \tau) d\tau |^4 d\xi = 16 \int_{\Omega_0} |\nabla [\tilde{\psi}_\epsilon (\xi, t)]^*|^4 d\xi.
\]

From (4.67) and the definition of \( X_\epsilon \), we have \( \| \bar{F}_\epsilon (X_\epsilon (\xi, \tau), 0, \tau) \|_{0, \infty, C} \leq 4/3 \), for each \( \tau \in [0, t_0] \). Then, using (4.62) and (4.73), we have

\[
\int_0^{t_0} \int_{\Omega_0} |\nabla_v [\tilde{\psi}_\epsilon]^* (\xi, t)|^4 d\xi d\tau < 16 \int_{\Omega_0} \int_0^{t_0} |\nabla [\tilde{\psi}_\epsilon (\xi, \tau)]^*|^4 d\xi d\tau
\]

\[
\leq 16 \int_0^{t_0} \int_{\Omega_0} |\nabla [\tilde{\psi}_\epsilon (\xi, \tau)]|^4 d\xi d\tau
\]

\[
= 16 \int_0^{t_0} \int_{\Omega_0} |\bar{F}_\epsilon (X_\epsilon (\xi, \tau), 0, \tau) \nabla_x \psi_\epsilon (\xi, \tau)|^4 d\xi d\tau \tag{4.82}
\]

\[
\leq \frac{4096}{81} \int_0^{t_0} \int_{\Omega_0} |\nabla_x \psi_\epsilon (\xi, \tau)|^4 d\xi d\tau
\]

\[
\leq \frac{4096 C_\epsilon^2}{81} (L_2^2 + \| \rho_0 \|_{0, 2, \infty}^2 + \| \Psi \|_{1, 2, \infty}^2)^2.
\]
Using the Hörder’s inequality and (4.82) we have
\[
\int_0^t \int_{\Omega_0} |\tilde{N}_{i,\epsilon}(\xi,t)|^2 d\xi d\tau \\
\leq \left( \int_0^t \int_{\Omega_0} |\tilde{N}_{i,\epsilon}(\xi,t)|^4 d\xi d\tau \right)^{1/2} \left( \int_0^t \int_{\Omega_0} |\tilde{N}_{i,\epsilon}(\xi,t)|^4 d\xi d\tau \right)^{1/2} \\
\leq t_0^{1/2} 64C_2 9 (L_2 + \|\rho_0\|_{L_{2,\Omega_0}} + \|\Psi\|_{L_{2,\partial \Omega}}) \times \\
\times \left\{ \int_0^t \int_{\Omega_0} |\nabla \tilde{N}_{i,\epsilon}(\xi,t)|^2 d\xi d\tau + \sup_{t \in [0,t_0]} \|\tilde{N}_{i,\epsilon}(\cdot,t)\|_{L_{2,\Omega_0}}^2 \right\},
\]
where for \( \tilde{N}_{i,\epsilon} \) we have used a well known Sobolev inequality (see inequality (3.8) in Chapter II of [25]). From (4.68), (4.83) and (4.84), we can write (4.80) as
\[
\int_{\Omega_0} |\tilde{N}_{i,\epsilon}(\xi,t)|^2 d\xi + \int_0^t \int_{\Omega_0} |\nabla \tilde{N}_{i,\epsilon}(\xi,t)|^2 d\xi d\tau \\
\leq T^{1/2} A \left\{ \int_0^t \int_{\Omega_0} |\nabla \tilde{N}_{i,\epsilon}(\xi,t)|^2 d\xi d\tau + \sup_{t \in [0,t_0]} \|\tilde{N}_{i,\epsilon}(\cdot,t)\|_{L_{2,\Omega_0}}^2 \right\} + \frac{1}{2d_{\text{min}}} \int_{\Omega_0} |N_{i,0}|^2 d\xi,
\]
for all \( t \in [0,t_0] \), where \( A \) and \( d_{\text{min}} \) were defined in (4.66). Recalling (4.55) and (4.59), we obtain (4.79).

**Lemma 4.4.** The solution \( \tilde{N}_{i,\epsilon} \) of (4.76)-(4.77) satisfies
\[
\int_0^{t_0} \int_{\Omega_0} (\partial_t \tilde{N}_{i,\epsilon})^2 d\xi d\tau + \sup_{t \in (0,t_0)} \int_{\Omega_0} |\nabla \tilde{N}_{i,\epsilon}(\xi,t)|^2 d\xi \\
\leq C_{\epsilon} (1 + \|N_{i,0}\|_{L_{2,\Omega_0}}^2),
\]
where \( C_{\epsilon} = C_{\epsilon}(O, K_0, \kappa_1, \kappa_2, Z_1, \ldots, Z_J, d_i, \epsilon, \kappa_{B,\theta}, \gamma, t_0, L_1, L_2) \).

**Proof.** Taking the \( L^2 \)-internal product of (4.76) with \( \partial_t \tilde{N}_{i,\epsilon} \), we obtain, after integrating by parts and the use of the boundary conditions,
\[
\left( \int_{\Omega_0} (\partial_t \tilde{N}_{i,\epsilon})^2 d\xi + d_i \int_{\Omega_0} \nabla \tilde{N}_{i,\epsilon} \cdot \nabla \tilde{N}_{i,\epsilon} d\xi + \\
+ \frac{d_i e Z_i}{\kappa_{B\theta}} \int_{\Omega_0} \tilde{N}_{i,\epsilon} \nabla \tilde{v}_e \cdot \nabla \tilde{v}_e (\partial_t \tilde{N}_{i,\epsilon}) d\xi \right) (t) = 0,
\]
for each \( t \in (0,t_0) \). Now, integrating by parts again, we have
\[
\left( \int_{\Omega_0} \tilde{N}_{i,\epsilon} \nabla \tilde{v}_e \cdot \nabla (\partial_t \tilde{N}_{i,\epsilon}) d\xi \right) (t) = \left( \int_{\partial O \cup \partial K_0} \partial_t \tilde{N}_{i,\epsilon} \tilde{N}_{i,\epsilon} \nabla \tilde{v}_e \cdot \nu ds + \\
- \int_{\Omega_0} \partial_t \tilde{N}_{i,\epsilon} \nabla \tilde{v}_e \cdot (\tilde{N}_{i,\epsilon} \nabla \tilde{v}_e) d\xi \right) (t) = \left( \frac{1}{2} \int_{\partial O \cup \partial K_0} \partial_t (\tilde{N}_{i,\epsilon}) \nabla \tilde{v}_e \cdot \nu ds + \\
- \int_{\Omega_0} \partial_t \tilde{N}_{i,\epsilon} \nabla \tilde{v}_e \cdot (\tilde{N}_{i,\epsilon} \nabla \tilde{v}_e) d\xi \right) (t) = \left( \frac{1}{2} \int_{\partial O \cup \partial K_0} \partial_t ((\tilde{N}_{i,\epsilon})^2 \nabla \tilde{v}_e \cdot \nu ds + \\
- \frac{1}{2} \int_{\partial O \cup \partial K_0} (\tilde{N}_{i,\epsilon})^2 \partial_t \nabla \tilde{v}_e \cdot (\tilde{N}_{i,\epsilon} \nabla \tilde{v}_e) d\xi \right) (t).
\]
Also, it is easy to check that

\[
\nabla v_x \tilde{N}_{i,e} \cdot \nabla v_x \partial_t \tilde{N}_{i,e} = \frac{1}{2} |(\nabla^{-1}_v \nabla \tilde{N}_{i,e})|^2_t - \nabla v_x \tilde{N}_{i,e} \cdot ((\nabla^{-1}_v \nabla \tilde{N}_{i,e}))_t.
\]  

(4.88)

Then, from (4.87), (4.88) and integrating (4.80) in \((0, t)\), we obtain

\[
\int_0^t \int_{\Omega_0} (\partial_t \tilde{N}_{i,e})^2 d\xi d\tau +
\frac{d_i}{2} \left( \int_{\Omega} |\nabla^{-1}_v \nabla \tilde{N}_{i,e}|^2 d\xi \right)(t) - \frac{d_i}{2} \left( \int_{\Omega} |\nabla^{-1}_v \nabla \tilde{N}_{i,e}|^2 d\xi \right)(0)
\]

\[
- d_i \int_0^t \int_{\Omega_0} \nabla v_x \tilde{N}_{i,e} \cdot ((\nabla^{-1}_v \nabla \tilde{N}_{i,e})) d\xi d\tau +
\frac{d_i eZ_i}{2 \kappa_B \theta} \int_{\partial \Omega \cup \partial K_0} ((\tilde{N}_{i,e})^2 \nabla v_x [\tilde{e}]^\tau \cdot \nu ds)(t) +
\frac{d_i eZ_i}{2 \kappa_B \theta} \int_{\partial \Omega \cup \partial K_0} ((\tilde{N}_{i,e})^2 \nabla v_x [\tilde{e}]^\tau \cdot \nu ds)(0) +
\]

\[
- d_i eZ_i \int_0^t \int_{\partial \Omega \cup \partial K_0} \tilde{N}_{i,e}^2 \partial_t (\nabla v_x [\tilde{e}]^\tau \cdot \nu) ds d\tau +
\frac{d_i eZ_i}{\theta K_B} \int_0^t \int_{\partial \Omega \cup \partial K_0} \partial_t \tilde{N}_{i,e} \nabla \xi \cdot (\tilde{N}_{i,e} \nabla \xi [\tilde{e}]^\tau) d\xi d\tau = 0,
\]

for each \( t \in (0, t_0) \). Let us now obtain bounds on each term given above. First we observe that, from the incompressibility condition of \( v_x, \nabla^{-1}_v = (a_{ij}) \) corresponds to the adjoint matrix of \( v_x \), i.e., for each \( i, j \in \{1, 2, 3\} \),

\[
a_{ij} = (-1)^{i+j} \left( \frac{\partial X_{m,e}}{\partial \xi_r} \frac{\partial X_{n,e}}{\partial \xi_l} - \frac{\partial X_{n,e}}{\partial \xi_r} \frac{\partial X_{m,e}}{\partial \xi_l} \right),
\]

(4.89)

where \( X_e = (X_{1,e}, X_{2,e}, X_{3,e}) \). Furthermore, \( n, l \) are the largest elements of \( \{1, 2, 3\} \) satisfying \( n \neq i, l \neq j \), respectively; \( m, r \in \{1, 2, 3\}, m \neq i, n \) and \( r \neq j, l \). Now, from (4.89),

\[
X_{i,e}(\xi, 0, t) = \xi + \int_0^t v_{i,e}(X_e(\xi, 0, \tau), \tau) d\tau,
\]

for all \((\xi, t) \in \Omega \times [0, t_0]\) and \( i \in \{1, 2, 3\} \), where \( v_e = (v_{1,e}, v_{2,e}, v_{3,e}) \). As a consequence, if \( x_i = X_{i,e} \),

\[
\left\| \frac{\partial X_{i,e}}{\partial \xi_j}(\cdot, 0, t) \right\|_{0, \infty, \Omega} \leq \delta_{ij} + \int_0^t \sum_{k=1}^3 \left\| \frac{\partial v_{i,e}(\cdot, \tau)}{\partial x_k} \right\|_{0, \infty, \Omega} \left\| \frac{\partial X_{i,e}(\cdot, 0, \tau)}{\partial \xi_k} \right\|_{0, \infty, \Omega} d\tau
\]

\[
\leq 1 + \int_0^t \left\| \nabla v_e(\cdot, \tau) \right\|_{0, \infty, \Omega} \sum_{k=1}^3 \left\| \frac{\partial X_{i,e}(\cdot, 0, \tau)}{\partial \xi_k} \right\|_{0, \infty, \Omega} d\tau,
\]

for all \( i, j \in \{1, 2, 3\} \) and \( t \in [0, t_0] \). Then,

\[
\sum_{i,j} \left\| \frac{\partial X_{i,e}}{\partial \xi_j}(\cdot, 0, t) \right\|_{0, \infty, \Omega} \leq 1 + 3 \int_0^t \left\| \nabla v_e(\cdot, \tau) \right\|_{0, \infty, \Omega} \sum_{i,j} \left\| \frac{\partial X_{i,e}(\cdot, 0, \tau)}{\partial \xi_j} \right\|_{0, \infty, \Omega} d\tau
\]
which implies that
\[
\left\| \frac{\partial X_{i,t}}{\partial \xi_j} (., 0, .) \right\|_{0, \infty, \Omega \times (0, t_0)} \leq 1 + 3 \int_0^{t_0} \left\| \nabla_x \psi_{., \tau} \right\|_{0, \infty, \Omega} \exp \left( \int_0^{t_0} \left\| \nabla_x \psi_{., r} \right\|_{0, \infty, \Omega} dr \right) d\tau \quad (4.90)
\]
\[
\leq 1 + C \varepsilon \int_0^{t_0} \left\| \nabla_x \psi_{., \tau} \right\|_{0, \infty, \Omega} \exp \left( C \varepsilon \int_0^{t_0} \left\| \nabla_x \psi_{., r} \right\|_{0, \infty, \Omega} dr \right) d\tau
\leq C \varepsilon (t_0, L_1),
\]

using the Gronwall’s lemma. Analogously,
\[
\left\| \frac{\partial^2 X_{i,t}}{\partial \xi_j} (., 0, .) \right\|_{0, \infty, \Omega \times (0, t_0)} \leq C \varepsilon (t_0, L_1),
\]
for all \(i, j \in \{1, 2, 3\}\). Hence,
\[
\left\| (\partial_t \Psi^{-1}) (., 0, .) \right\|_{0, \infty, \Omega \times (0, t_0)} \leq C \varepsilon (t_0, L_1). \quad (4.91)
\]

Now we observe that, from (4.61) and (4.73),
\[
| \nabla_x \psi_{., \tau} (\xi, t) | \leq \int_0^{t_0} | g_{., t - \tau} | \left\| \Psi^{-1} (\xi, 0, t) \nabla_x \psi_{., \tau} (\xi, \tau) \right\| d\tau
\leq C \varepsilon \int_0^{t_0} \left\| \nabla_x \psi_{., \tau} (x, \tau) \right\| d\tau \leq C \varepsilon (t_0) \text{ess sup}_{t \in (0, t_0)} \left\| \psi_{., \tau} (\xi, \tau) \right\|_{C_t(\Omega, \tau)} \quad (4.92)
\]
\[
\leq C \varepsilon (\kappa_1, \kappa_2, \Omega, K_0, \gamma, Z_1, \ldots, Z_J, t_0, L_2) (1 + \left\| \rho_0 \right\|_{0, 2, K_0} + \left\| \Psi \right\|_{1, 2, \partial \Omega}),
\]
for all \(\xi \in \Omega\) and a.e. \(t \in (0, t_0)\). Similarly, we have
\[
| \nabla_x \cdot \left( \nabla_x \psi_{., \tau} \right) (\xi, t) | \leq \int_0^{t_0} | g_{., t - \tau} | \left\| \Delta_x \psi_{., \tau} (x, \tau) \right\| d\tau \leq C \varepsilon (\kappa_1, \kappa_2, \Omega, K_0, \gamma, Z_1, \ldots, Z_J, t_0, L_2) (1 + \left\| \rho_0 \right\|_{0, 2, K_0} + \left\| \Psi \right\|_{1, 2, \partial \Omega}), \quad (4.93)
\]
and
\[
| \partial_t (\nabla_x \psi_{., \tau}) (\xi, t) | \leq \int_0^{t_0} | \partial_t g_{., t - \tau} | \left\| \nabla_x \psi_{., \tau} (x, \tau) \right\| d\tau \leq C \varepsilon (t_0) \text{ess sup}_{t \in (0, t_0)} \left\| \psi_{., \tau} (x, \tau) \right\|_{C_t(\Omega, \tau)} \quad (4.94)
\]
\[
\leq C \varepsilon (\kappa_1, \kappa_2, \Omega, K_0, \gamma, Z_1, \ldots, Z_J, t_0, L_2) (1 + \left\| \rho_0 \right\|_{0, 2, K_0} + \left\| \Psi \right\|_{1, 2, \partial \Omega}),
\]
for all \(\xi \in \Omega\) and a.e. \(t \in (0, t_0)\).

Using (4.63), (4.91) and (4.79), we obtain the estimate
\[
\int_0^{t} \int_{\Omega_0} \nabla_x \cdot \left( (\Psi^{-1})_{t} \nabla_x \nabla_x \psi_{., \tau} \right) d\xi d\tau
\leq C \varepsilon (t_0, L_1) \int_0^{t} \left\| \nabla_x \nabla_x \psi_{., \tau} \right\|_{0, 2, \Omega_0} d\tau < C \varepsilon (t_0, L_1, L_2),
\]
for each \(t \in (0, t_0)\). If \(1/2 < s < 1\), the trace theorem and (4.92) give us,
\[
\left( \int_{\partial \Omega \cup \partial K_0} ((\nabla_x \psi_{., \tau})^2 \nabla_x \cdot \nu) \frac{1}{s} ds \right) (t)
\leq C \varepsilon \left( \int_{\partial \Omega} (\nabla_x \psi_{., \tau})^2 ds + \int_{\partial K_0} (\nabla_x \psi_{., \tau})^2 ds \right) \leq C \varepsilon \left\| \nabla_x \psi_{., \tau} \right\|_{s, 2, \Omega_0}^2,
where $C = C(e, \kappa, \kappa_2, \mathcal{O}, K_0, e, \gamma, Z_1, \ldots, Z_J, t_0, \rho_0, \Psi, L_2)$. Hence, from interpolation results (see Theorem 1.3.7 in [31]), Young’s inequality and (4.79), we have,

\[
\left(\int_{\partial \Omega \cup \partial K_0} ((\tilde{N}_{i,e})^2 \nabla_{\nu_e} [\tilde{\psi}_e]^\nu) \, ds \right)(t) \\
\leq C_\varphi \|\tilde{N}_{i,e}(, t)\|_{0,2,\Omega_0}^{(1-s)/2} \|\tilde{N}_{i,e}(, t)\|_{2,2,\Omega_0}^{s/2} \\
\leq C_\varphi^{1/(s-1)} (s-1)^{-1/(s-1)} \|\tilde{N}_{i,e}(, t)\|_{0,2,\Omega_0}^{2/2} + s\varphi^{1/s} \|\tilde{N}_{i,e}(, t)\|_{1,2,\Omega_0}^2 \\
\leq C_\varphi + s\varphi^{1/s} \|\nabla_{\xi} \tilde{N}_{i,e}(, t)\|_{0,2,\Omega_0}^2,
\]

where $\varphi > 0$ and $C = C(e, s, \varphi, \kappa, \kappa_2, \mathcal{O}, K_0, \gamma, Z_1, \ldots, Z_J, t_0, \rho_0, \Psi, L_2)$. Let us recall that supp $\tilde{N}_{i,0,e} \subset \subset \Omega_0$, so that

\[
\left(\int_{\partial \Omega \cup \partial K_0} ((\tilde{N}_{i,e})^2 \nabla_{\nu_e} [\tilde{\psi}_e]^\nu) \, ds \right)(0) = 0.
\]

Also, using (4.94), trace theorem and (4.79), for each $t \in (0, t_0]$,

\[
\int_0^t \int_{\partial \Omega \cup \partial K_0} (\tilde{N}_{i,e})^2 \partial_t (\nabla_{\nu_e} [\tilde{\psi}_e]^\nu) \, ds \, d\tau \leq C_\varphi \int_0^t \|\tilde{N}_{i,e}(, \tau)\|_{1,2,\Omega_0}^2 \, d\tau \leq C_\varphi,
\]

where $C = C(e, \kappa, \kappa_2, \mathcal{O}, K_0, \gamma, Z_1, \ldots, Z_J, t_0, \rho_0, \Psi, L_2)$.

Furthermore, using the Young’s inequality, (4.92), (4.93) and (4.68),

\[
\int_0^t \int_{\Omega_0} \partial_t (\tilde{N}_{i,e})^2 \nabla_{\nu_e} \cdot (\tilde{N}_{i,e} \nabla_{\nu_e} [\tilde{\psi}_e]^\nu) \, d\xi \, d\tau \\
\leq \frac{1}{2} \int_0^t \int_{\Omega_0} (\partial_t \tilde{N}_{i,e})^2 \, d\xi \, d\tau + \frac{1}{2} \int_{\Omega_0} |\nabla_{\nu_e} \cdot (\tilde{N}_{i,e} \nabla_{\nu_e} [\tilde{\psi}_e]^\nu)|^2 \, d\xi \, d\tau \\
\leq \frac{1}{2} \int_0^t \int_{\Omega_0} (\partial_t \tilde{N}_{i,e})^2 \, d\xi \, d\tau + \frac{1}{2} \int_{\Omega_0} |\nabla_{\nu_e} \tilde{N}_{i,e}|^2 |\nabla_{\nu_e} [\tilde{\psi}_e]^\nu|^2 \, d\xi \, d\tau + \\
+ \int_{\Omega_0} |\tilde{N}_{i,e}|^2 |\nabla_{\nu_e} \cdot (\nabla_{\nu_e} [\tilde{\psi}_e]^\nu)|^2 \, d\xi \, d\tau \\
\leq \frac{1}{2} \int_0^t \int_{\Omega_0} (\partial_t \tilde{N}_{i,e})^2 \, d\xi \, d\tau + C_\varphi \int_0^t \int_{\Omega_0} |\nabla_{\xi} \tilde{N}_{i,e}|^2 \, d\xi \, d\tau + C_\varphi \int_0^t \int_{\Omega_0} |\tilde{N}_{i,e}|^2 \, d\xi \, d\tau \\
\leq \frac{1}{2} \int_0^t \int_{\Omega_0} (\partial_t \tilde{N}_{i,e})^2 \, d\xi \, d\tau + C_\varphi,
\]

where $C = C(e, \kappa, \kappa_2, \mathcal{O}, K_0, \gamma, Z_1, \ldots, Z_J, t_0, \rho_0, \Psi, L_2)$. From the previous results, choosing $\varphi = \left(\frac{\alpha \rho \theta}{\sec Z_1}\right)^{\alpha}$ we obtain

\[
\frac{1}{2} \int_0^t \int_{\Omega_0} (\partial_t \tilde{N}_{i,e})^2 \, d\xi \, d\tau + \frac{d_i}{16} \int_{\Omega_0} |\nabla_{\xi} \tilde{N}_{i,e}(t, \xi)|^2 \, d\xi < C_\varphi \|\tilde{N}_{i,0,e}\|_{0,2,\Omega_0}^2 + 1
\]

for each $t \in (0, t_0)$, which implies in (4.85).

\[\square\]

### 4.3 The Navier-Stokes Equations

In order to deal with the problem P3, we define

\[
\tilde{F}_e(, t) = -e \sum_{i=1}^J (Z_i \tilde{N}_{i,e} \nabla_{\nu_e} \tilde{\psi}_e)(, t)
\]
and consider the linear problem related to (3.34) in Lagrangian coordinates: Find \( \tilde{u}_r(.,t) := u_r(X_\varepsilon(.,0,t),t) \) such that

\[
\int_0^t \int_\Omega (\mu_{p,0}\tilde{u}_r \cdot \partial_\varepsilon \varphi - 2\gamma D_{\varepsilon'}(\tilde{u}_r) : D_{\varepsilon'}(\varphi)) \, dx \, dt + \\
+ \int_0^t \sum_{j} \int_{\Omega_j} \pi_j F_{\varepsilon} \cdot \varphi \, dx \, dt = 0,
\]

for a.e. \( t \in (0, t_0) \) and \( \forall \varphi \in \mathcal{S} \), where

\[ \mathcal{S} = \{ \varphi \in H^1(\mathcal{O} \times (0, t_0))^3, \nabla_{\varepsilon'} \cdot \varphi = 0, \varphi|_{\partial \mathcal{O}} = 0, \nu_{p,0} D_{\varepsilon'} \varphi = 0 \} \times (0, t_0) \}.

**Lemma 4.5.** The problem (4.95) has a unique solution

\[ \tilde{u}_r \in L^\infty(0, t_0; L^2(\mathcal{O}))^3 \cap L^2(0, t_0; H^1_0(\mathcal{O}))^3, \]

such that \( \nabla_{\varepsilon'} \cdot \tilde{u}_r = 0, \mu_{p,0} D_{\varepsilon'}(\tilde{u}_r) = 0 \) a.e. in \( \mathcal{O} \times (0, t_0) \). Furthermore, \( u_r \), satisfies (3.34) and we have the estimates

\[
\text{ess sup}_{t \in (0, t_0)} \| u_r(.,t) \|^2_{1,2,\Omega_0} + \int_0^{t_0} \| \nabla u_r \|^2_{0,2,\Omega} \, d\tau < L_1^2,
\]

where \( L_1 \) is given in (4.58), and

\[
\text{ess sup}_{t \in (0, t_0)} \| u_r(.,t) \|^2_{1,2,\Omega_0} + \int_0^{t_0} \| \partial_{\varepsilon} u_r \|^2_{0,2,\Omega} \, d\tau \\
+ \int_0^{t_0} \| u_r \|^2_{2,2,\Omega} \, d\tau \leq C_s \left( \| u_0 \|^2_{0,2,\Omega} + L_2^2 + \| \rho_0 \|^2_{0,2,\Omega_0} + \| \Psi \|^2_{1,2,\Omega} \right),
\]

where \( C_s = C_s(\eta, \pi_p, \pi_f, K_0, \mathcal{O}, Z_1, \ldots, Z_J, \varepsilon, C_1, C_2) \).

**Proof.** The existence and uniqueness is based on a special Galerkin approximation technique and was established in [15] jointly with the estimate (3.29) for \( u_r \). Now, from Schwarz inequality, Sobolev embedding (see (4.57)), (4.73) and (4.79),

\[
\int_0^{t_0} \int_{\Omega_{\varepsilon,\tau}} |F_{\varepsilon}|^2 \, dx \, d\tau \leq 2\varepsilon^2 \int_0^{t_0} \sum_{i=1}^J |Z_i|^2 |N_{i,\varepsilon}|^2 |\nabla \psi_{\varepsilon}|^2 \, dx \, d\tau
\]

\[
\leq 2\varepsilon^2 \int_0^{t_0} \sum_{i=1}^J |Z_i|^2 \| \nabla \psi_{\varepsilon} \|^2_{0,4,\Omega_0} \| \nabla \psi_{\varepsilon} \|^2_{0,4,\Omega_{\varepsilon,\tau}} \, d\tau
\]

\[
\leq 2\varepsilon^2 \sum_{i=1}^J |Z_i|^2 C_{r,i}^2 \text{ess sup}_{t \in (0, t_0)} \| \nabla \psi_{\varepsilon}(.,t) \|^2_{0,4,\Omega_{\varepsilon,\tau}} \int_0^{t_0} \| \nabla \psi_{\varepsilon} \|^2_{1,2,\Omega_0} \, d\tau
\]

\[
\leq 2\varepsilon^2 \sum_{i=1}^J |Z_i|^2 C_{r,i}^2 L_2^2 C_2 (L_2^2 + \| \rho_0 \|^2_{0,2,\Omega_0} + \| \Psi \|^2_{1,2,\Omega}).
\]

Then, recalling (3.34), the estimates (4.98) and (4.58) give us (4.99).

From the regularity results established in [15], we have

\[
\text{ess sup}_{t \in (0, t_0)} \| u_r(.,t) \|^2_{1,2,\Omega_0} + \int_0^{t_0} \| \partial_{\varepsilon} u_r \|^2_{0,2,\Omega} \, d\tau \\
+ \int_0^{t_0} \| u_r \|^2_{2,2,\Omega_0} \, d\tau \leq C_s \left( \| u_0 \|^2_{0,2,\Omega} + \int_0^{t_0} \| F_{\varepsilon} \|^2_{0,2,\Omega_{\varepsilon,\tau}} \, d\tau \right),
\]
where $C_ε = C_ε(τ, π_p, π_f, K_0, O)$. Hence, (4.97) follows from (4.98).

4.4 Fixed Point Procedure

Let us recall that we have chosen $(ς, ϑ_1, \ldots, ϑ_J) \in B \times X \times \ldots \times X$. Now we observe that

$$u_ε(x, t) = v_{ε,t}(t) + w_ε(t) \times (x - \bar{x}_ε(t))$$

for $(x, t) \in K_{ε,t} \times (0, t_0)$, $\bar{x}_ε(t)$ is the center of mass of $K_{ε,t}$. Let us define the map

$$G_ε(ς, ϑ_1, \ldots, ϑ_J) = (ς, ϑ_1, \ldots, ϑ_J),$$

where $(ς, ϑ_1, \ldots, ϑ_J) \in Y_{t_0}$, $ς(x) = x_c(0) + \int_0^t v_{ε,t}(τ)dτ$, $ϑ_1(t) = \int_0^t w_ε(τ)dτ$ and $ϑ_2^{-1}(y_c)$ denotes the incompressible component in $Ω_0$ of $ϑ^{-1}(y_c)$; as described in [15], $v_ε$ is a divergence-free vector field constructed from $u_ε$, $v_ε$ and from suitable rigid current functions. In order to show that this map has a fixed point in $B \times X \times \ldots \times X$ we need to prove some technical results.

Lemma 4.6. $G_ε$ is a continuous map from $B \times X \times \ldots \times X$ into itself.

Proof. First we observe that $G_ε$ applies $B \times X \times \ldots \times X$ into itself. In fact, using (4.79) we have the estimate for $N_{t,ε}$. As in [15], the estimate for $(ς, \bar{ϑ}_1, \bar{ϑ}_J)$ follows from (4.96). Now, let us consider $(ς, ϑ_1, \ldots, ϑ_J) \in B \times X \times \ldots \times X$ and sequences $ς_n \to ς$ in $B$, $ϑ_{1,n} \to ϑ_1$, $ϑ_{J,n} \to ϑ_J$ in $X$, as $n \to \infty$. We need to show that

$$G_ε(ς_n, ϑ_{1,n}, \ldots, ϑ_{J,n}) \to G_ε(ς, ϑ_1, \ldots, ϑ_J) \in B \times X \times \ldots \times X,$$

as $n \to \infty$. As before, we denote $v_n = Θ(ς_n)$, $v = Θ(ς)$, $v_{n,ε} = R_ε(ς_n)$ and $v_ε = R_ε(ς)$. Also $X_{n,ε}$ and $X_ε$ represent the Lagrangian flows of $v_{n,ε}$ and $v_ε$, respectively. From the properties of the function $Θ$ (see [15]), for fixed $ε > 0$, $v_{n,ε} \to v_ε$ in $L^∞(0, t_0; L^2(O))^3 \cap L^2(0, t_0; H^1_0(O))^3$, as $n \to \infty$. Henceforth, denoting

$$F_{n,ε} = ∇X_{n,ε}$$

and $F_ε = ∇X_ε$,

we have

$$||F_{n,ε}^{-1} - F_ε^{-1}||_{0,∞,O \times (0, t_0)} \leq C(ε, t_0, O, L_1)||v_{n,ε} - v_ε||_{L^∞(0, t_0; L^2(O))^3 \cap L^2(0, t_0; H^1_0(O))^3} \to 0, \quad n \to \infty.$$  (4.99)

In fact, this follows from (4.89) combined with (4.89) and (4.90).

Now, we define

$$(ϑ_{1,n,ε}(., t) := ϑ_{1,n,ε}(X_{n,ε}(., t, 0), t), ϑ_{1,ε}(., t) := ϑ_{1,ε}(X_ε(., t, 0), t)),$$

(4.100)

and consider the Lagrangian versions of (2.3) – (2.9), for $t \in (0, t_0)$,

$$\nabla v_{n,ε} \cdot (k(., 0)∇v_{n,ε} ψ_{n,ε}) = -4πε \sum_{i=1}^J χ_{n,ε} - 4πρ_ε \quad \text{in} \quad (H^1(O))^′,$$

$$\tilde{ψ}_{n,2,ε}(x, t) = \tilde{ψ}_{n,1,ε}(x, t), \quad x \in ∂K_0,$n
to (4.101)

$$(κ_1 ϑ_ν ψ_{n,1,ε} - κ_2 ϑ_ν ψ_{n,2,ε})(x, t) = 0, \quad x \in ∂K_0,$n
to (4.101)

$$\tilde{ψ}_{n,2,ε}(x, t) = Ψ_ε(x), \quad x \in O,$n
\[
\n\nabla v_\epsilon \cdot (k(\xi, 0)\nabla v_\epsilon \tilde{\psi}_\epsilon) = -4\pi e \sum_{i=1}^{J} \bar{q}_{i,n,\epsilon} - 4\pi \bar{\rho}_\epsilon \quad \text{in} \quad (H^1(\Omega))',
\]

\[\bar{\psi}_{2,\epsilon}(\xi, t) = \bar{\psi}_{1,\epsilon}(\xi, t), \quad \text{a. e. } \xi \in \partial K_0, \quad (4.102)\]

\[
(\kappa_1 \partial_n \bar{\psi}_{1,\epsilon} - \kappa_2 \partial_n \bar{\psi}_{2,\epsilon})(\xi, t) = 0, \quad \text{a. e. } \xi \in \partial K_0, \quad \bar{\psi}_{2,\epsilon}(\xi, t) = \bar{\Psi}(\xi), \quad \text{a. e. } \xi \in \partial \Omega,
\]

where,

\[
\bar{q}_{i,n,\epsilon}(\xi, t) = \begin{cases} 
Z_i \tilde{\bar{\psi}}_{i,n,\epsilon}(\xi, t), & \xi \in \Omega_0, \\
0, & \xi \in K_0,
\end{cases}
\]

\[
\bar{\rho}_\epsilon(\xi, t) = \begin{cases} 
\rho_0, & \xi \in K_0, \\
0, & \xi \in \Omega.
\end{cases}
\]

Considering \(f_{n,\epsilon} := \bar{\psi}_{n,\epsilon} - \bar{\psi}_\epsilon\) as a test function in (4.101) and in (4.102), after some calculation, we obtain

\[
\left( \int_{\Omega} \kappa_0 |\nabla v_{n,\epsilon} f_{n,\epsilon}|^2 d\xi \right)(t) \leq \left( \int_{\Omega} \kappa_0 |(\nabla v_{n,\epsilon} \bar{\psi}_\epsilon - \nabla v_{\bar{\psi}} \bar{\psi}_\epsilon) \cdot \nabla v_{n,\epsilon} f_{n,\epsilon}| d\xi \right)(t) + \left( \int_{\Omega} \kappa_0 |(\nabla v_{n,\epsilon} f_{n,\epsilon} - \nabla v_{\psi} \bar{\psi}_\epsilon)| d\xi \right)(t) + 4\pi e \sum_{i=1}^{J} |Z_i| \left( \int_{\Omega} |\tilde{\bar{\psi}}_{i,n,\epsilon} - \tilde{\bar{\psi}}_{i,\epsilon}| |f_{n,\epsilon}| d\xi \right)(t)
\]

\[
\leq \|F_{n,\epsilon}^{-1}(0, 0)|_{0,0,\infty, \sigma} \max \{\kappa_1, \kappa_2\} \|\nabla_{\nabla_{\psi}} \bar{\psi}_\epsilon\|_{0,2,\sigma} \|\nabla_{\nabla_{v_{n,\epsilon}}} f_{n,\epsilon}(\cdot, t)\|_{0,2,\sigma} + \|\nabla_{\nabla_{f_{n,\epsilon}}} (\cdot, t)\|_{0,2,\sigma} \|\nabla_{\nabla_{\bar{\psi}}}(\cdot, t)\|_{0,2,\sigma} C \|\tilde{\bar{\psi}}_{i,n,\epsilon} - \tilde{\bar{\psi}}_{i,\epsilon}\|_{0,2,\sigma} f_{n,\epsilon}(\cdot, t)\|_{0,2,\sigma},
\]

where \(C = C(e, Z_1, \ldots, Z_J)\). Clearly, as \(\|\tilde{\bar{\psi}}_{i,n,\epsilon} - \tilde{\bar{\psi}}_{i,\epsilon}\|_{0,2,\sigma} \to 0\) with \(n \to \infty\), we have

\[
\left( \int_{\Omega} |\nabla_{\nabla_{f_{n,\epsilon}}} (\cdot, t)|^2 d\xi \right)(t) \to 0, \quad n \to \infty, \quad \text{a. e. } t \in (0, t_0) \quad (4.103)
\]

using (4.73), Poincaré’s inequality, (4.68) and (4.99).

We define \((N_{i,n,\epsilon}, N_{i,\epsilon})\) and \((\bar{u}_{n,\epsilon}, \bar{u}_\epsilon)\) as the solutions of (4.78) and (8.39) considering \((|\bar{\psi}_{n,\epsilon}|^2, v_{n,\epsilon})\) and \((|\bar{\psi}_\epsilon|^2, v_\epsilon)\), respectively. Taking \(U_{i,n,\epsilon} := N_{i,n,\epsilon} - \bar{u}_{n,\epsilon}\)
\( \tilde{N}_{i, \varepsilon} \) as a test function in (4.78) we obtain, after some calculation,

\[
\int_{\Omega_0} (\mathcal{U}_{n, \varepsilon}(\xi, t))^2 d\xi + d_i \int_{0}^{t} \int_{\Omega_0} |\nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon}|^2 d\xi d\tau =
\]

\[
= -d_i \int_{0}^{t} \int_{\Omega_0} (\nabla_{\varepsilon} \tilde{N}_{i, \varepsilon} - \nabla_{\varepsilon} \tilde{N}_{i, \varepsilon}) \cdot \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon} d\xi d\tau +
\]

\[
- d_i \int_{0}^{t} \int_{\Omega_0} (\nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon} - \nabla_{\varepsilon} \tilde{N}_{i, \varepsilon}) \cdot \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon} d\xi d\tau +
\]

\[
- \frac{d_i Z_{i, \varepsilon}}{\kappa_B^{1/2}} \int_{0}^{t} \int_{\Omega_0} \mathcal{U}_{n, \varepsilon} \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon} \cdot \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon} d\xi d\tau +
\]

\[
- \frac{d_i Z_{i, \varepsilon}}{\kappa_B^{1/2}} \int_{0}^{t} \int_{\Omega_0} \tilde{N}_{i, \varepsilon} \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon} - \nabla_{\varepsilon} \tilde{N}_{i, \varepsilon} \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon} d\xi d\tau +
\]

\[
- \frac{d_i Z_{i, \varepsilon}}{\kappa_B^{1/2}} \int_{0}^{t} \int_{\Omega_0} \tilde{N}_{i, \varepsilon} \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon} - \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon} d\xi d\tau +
\]

\[
- \frac{d_i Z_{i, \varepsilon}}{\kappa_B^{1/2}} \int_{0}^{t} \int_{\Omega_0} \tilde{N}_{i, \varepsilon} \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon} - \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon} d\xi d\tau,
\]

for each \( t \in (0, t_0) \). Then, using Hölder’s inequality, Sobolev embedding and an analogous estimate as in (1.83) and (1.84) we have

\[
\int_{\Omega_0} (\mathcal{U}_{n, \varepsilon}(\xi, t))^2 d\xi + \int_{0}^{t} \int_{\Omega_0} |\nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon}|^2 d\xi d\tau
\]

\[
\leq C \| (\mathcal{E}^{-1}_{n, \varepsilon} - \mathcal{E}^{-1}_{\varepsilon}) (\cdot, 0, \cdot) \|_{H_{\varepsilon}^{\infty}, 0} \times \int_{0}^{t} \int_{\Omega_0} \| \mathcal{U}_{n, \varepsilon}(\tau) \|_{L^2(\Omega)} \| \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon}(\tau) \|_{L^2(\Omega)} \| \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon}(\tau) \|_{L^2(\Omega)} d\tau +
\]

\[
+ \int_{0}^{t} \int_{\Omega_0} \| \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon}(\tau) \|_{L^2(\Omega)} \| \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon}(\tau) \|_{L^2(\Omega)} \int_{0}^{t} \int_{\Omega_0} \| \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon}(\tau) \|_{L^2(\Omega)} \| \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon}(\tau) \|_{L^2(\Omega)} d\tau +
\]

\[
+ C \| \mathcal{U}_{n, \varepsilon}(\cdot, 0, \cdot) \|_{H_{\varepsilon}^{\infty}, 0} \times \| \mathcal{U}_{n, \varepsilon}(\cdot, 0, \cdot) \|_{H_{\varepsilon}^{\infty}, 0} \int_{0}^{t} \int_{\Omega_0} \| \nabla_{\varepsilon} (\mathcal{U}_{n, \varepsilon}(\tau)) \|_{L^2(\Omega)} \| \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon}(\tau) \|_{L^2(\Omega)} \| \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon}(\tau) \|_{L^2(\Omega)} d\tau,
\]

where \( C = C(\Omega_0, \kappa_B, \theta, Z_{i, \varepsilon}, d_i) \). Hence, using (1.99), (1.62), (1.103) and (4.79) we obtain the convergence

\[
\text{ess sup}_{t \in (0, t_0)} \int_{\Omega_0} (\mathcal{U}_{n, \varepsilon}(\xi, t))^2 d\xi + \int_{0}^{t} \int_{\Omega_0} |\nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon}|^2 d\xi d\tau \rightarrow 0, \ n \rightarrow \infty. \quad (4.106)
\]

Now, we define

\[
\mathcal{F}_e(\cdot, t) = -e \sum_{i=1}^{J} (Z_{i} \tilde{N}_{i, \varepsilon} \nabla_{\varepsilon} \mathcal{U}_{n, \varepsilon}), \mathcal{F}_n(\cdot, t) = -e \sum_{i=1}^{J} (Z_{i} \tilde{N}_{i, \varepsilon} \nabla_{n, \varepsilon} \mathcal{U}_{n, \varepsilon}),
\]

respectively. Taking \( q_{n, \varepsilon} := \mathcal{U}_{n, \varepsilon} - \mathcal{U}_{\varepsilon} \) as a test function in (4.33), using (3.29) and (4.39) we have \( q_{n, \varepsilon} \rightarrow 0 \) in \( L^\infty(0, t_0; L^2(\Omega)) \cap L^2(0, t_0; H_{\varepsilon}^{1}(\Omega)) \) if we can show that

\[
\int_{0}^{t} \| (\mathcal{F}_n - \mathcal{F}_e)(\cdot, \tau) \|^2_{0, \varepsilon, \Omega} d\tau \rightarrow 0
\]

with \( n \rightarrow \infty \), for a. e. \( t \in (0, t_0) \). This result follows writing the above expression as in the right side of (4.104) and from the estimates obtained in
Then, following [13],

\[ (\mathcal{X}_{n, \epsilon, t}, \mathcal{Y}_{n, \epsilon, t}, \Theta^{-1}_2(y_{n, \epsilon})) \rightarrow (\mathcal{X}_t, \mathcal{Y}_t, \Theta^{-1}(y_t)) \] in \( B \),

as \( n \rightarrow \infty \). Hence, recalling (4.106) we obtain the continuity of \( G_\epsilon \).

**Proposition 4.1.** The operator \( G_\epsilon \) has a fixed point in \( B \times \mathcal{X} \times \ldots \times \mathcal{X} \).

**Proof.** First we observe that \( G_\epsilon \) is a compact operator. In fact, this follows from Lemma 4.6, the rigidity properties in \( K_0 \) and from the Sobolev embeddings in \( \Omega_0 \) for \( \tilde{N}_i, \epsilon \) (as a consequence of the estimates (4.79) and (4.80)) and \( u_\epsilon \) (from the estimates (4.96) and (4.97)). Then, the map \( G_\epsilon \) has a fixed point in \( B \), using Lemma 4.6 and Schauder’s theorem.

The first time step on \((0, t_0)\) is completed. From interpolation results (see Theorem 1.3.8 in [31]), \( u_\epsilon(\cdot, t_0) \in H^1(\Omega)^3 \). Therefore, we can proceed similarly on the interval \((t_0, t_1)\), considering \((u_\epsilon(\cdot, t_0), \tilde{N}_i(\cdot, t_0))\) as the initial conditions for the problems (4.95) and (4.78), respectively. We give below some details related to the obtention of the bound (4.79) in this second step of the procedure.

Let us consider \( \psi_\epsilon \) and \( N_i, \epsilon \) the solutions of the problems \( P1 \) and \( P2 \) in \((0, t_0)\), respectively. We extend these functions in \((t_0, t_1)\) to be the solutions of \( P1 \) and \( P2 \) in this interval. First, we observe that (4.73) is valid in (4.108), we obtain the validity of inequality (4.108) for all \( t \in [0, t_0] \),

\[
\frac{1}{2} \int_{\Omega_0} |\tilde{N}_i, \epsilon(\xi, t)|^2 d\xi + \frac{d_i}{8} \int_{t_0}^t \int_{\Omega_0} |\nabla \tilde{N}_i, \epsilon|^2 d\xi d\tau
\]

\[
\leq \frac{d_i \epsilon^2}{2\kappa_B^2 \theta^2} \int_{t_0}^t \int_{\Omega_0} |\tilde{N}_i, \epsilon|^2 |\nabla \psi_\epsilon| |\psi_\epsilon| d\xi d\tau +
\]

\[
\frac{1}{2} \int_{\Omega_0} |\tilde{N}_i(\xi, t_0)|^2 d\xi
\]

and, for all \( t \in [0, t_0] \),

\[
\frac{1}{2} \int_{\Omega_0} |\tilde{N}_i, \epsilon(\xi, t)|^2 d\xi + \frac{d_i}{8} \int_{t_0}^t \int_{\Omega_0} |\nabla \tilde{N}_i, \epsilon|^2 d\xi d\tau
\]

\[
\leq \frac{d_i \epsilon^2}{2\kappa_B^2 \theta^2} \int_{t_0}^t \int_{\Omega_0} |\tilde{N}_i, \epsilon|^2 |\nabla \psi_\epsilon| |\psi_\epsilon| d\xi d\tau +
\]

\[
\frac{1}{2} \int_{\Omega_0} |\tilde{N}_i(\xi, 0)|^2 d\xi.
\]

Considering \( t = t_0 \) in (4.108), we obtain the validity of inequality (4.108) for all \( t \in [0, t_1] \). As a consequence, we obtain a version of (4.79) in the interval \((0, t_1)\), following the lines of the proof of Lemma 4.3. In particular, (4.79) is valid if we consider the interval \((t_0, t_1)\). A similar argument can be used in order to obtain the estimate (4.96) in \((0, t_1)\).

Repeating all the process in each step \((t_{i-1}, t_i)\) (recall that the number of steps \( N = T/t_0 \) only depends on \( \epsilon \) and \( L_1 \)) we obtain, for each \( \epsilon > 0 \), \((\tilde{u}_\epsilon, \tilde{N}_i, \psi_\epsilon)\) as the solutions, respectively, of the problems (4.95), (4.78) and (3.37) in \((0, T)\).
and $\mu_\epsilon$ that satisfies (4.71). If we consider $\tilde{u}_\epsilon$ and $\tilde{N}_{i,\epsilon}$ in terms of the Eulerian coordinates, we see that, for each $0 < \epsilon' < \epsilon$, $(u_\epsilon, \mu_\epsilon, N_1, \ldots, N_{J,\epsilon}, \psi_\epsilon)$ is a solution of the problems $P1$, $P2$ and $P3$ and we can obtain an approximate sequence $(u_n, \mu_n, N_{1,n}, \ldots, N_{J,n}, \psi_n)_{n \in \mathbb{N}}$ of solutions for (3.34)-(3.38). Using (4.68), (4.79) and (4.96) in each time step $(t_{i-1}, t_i)$ we obtain, for all $n \in \mathbb{N}$, $i \in \{1, \ldots, J\}$,

\begin{align*}
\text{ess sup}_{t \in (0,T)} \| N_{i,n}(\cdot,t) \|_{0,2,\Omega_n,t}^2 + \int_0^T \| \nabla N_{i,n} \|_{0,2,\Omega_n,t}^2 \, dt < 4L_2^2, \\
\text{ess sup}_{t \in (0,T)} \| u_n(\cdot,t) \|_{0,2,\Omega,n}^2 + \int_0^T \| \nabla u_n \|_{0,2,\Omega,n}^2 \, dt < L_i^2.
\end{align*}

(4.109)

Also,

\begin{align*}
\text{ess sup}_{t \in (0,T)} \| \mu_n \|_{0,\infty,D} & \leq \max\{\bar{p}, \bar{p}_p\}, \\
\text{ess sup}_{t \in (0,T)} \max\{\| \nabla \psi_{1,n}(\cdot,t) \|_{0,4,K_m,t}, \| \nabla \psi_{2,n}(\cdot,t) \|_{0,4,\Omega,t} \} & \leq B_*, \quad (4.110)
\end{align*}

where $B_* = C_2^{1/2}(L_2^2 + \| \rho_0 \|_{0,2,K_0} + \| \Psi_1 \|_{1,2,\partial \Omega})^{1/2}$ and

\begin{equation}
\inf_{n \in \mathbb{N}} \inf_{t \in (0,T)} \text{dist}(K_{n,t}, \partial \Omega) \geq \gamma > 0. \quad (4.111)
\end{equation}

**Remark 4.2.** In order to establish convergence properties for $N_n$ it is convenient to consider its extension to fixed domains. So we consider the standard extension operator $E : H^1(\Omega_0) \to H^1(\mathbb{R}^3)$ such that $Ef = f \text{ a.e. in } \Omega_0$, $\| \nabla (Ef) \|_{0,2,\mathbb{R}^3} \leq C \| \nabla f \|_{0,2,\Omega_0}$, $\| Ef \|_{0,2,\mathbb{R}^3} \leq C \| f \|_{0,2,\Omega_0}$ and $Ef \geq 0 \text{ a.e. in } \mathbb{R}^3$ as $f \geq 0 \text{ a.e. in } \Omega_0$; the constant $C$ depends only on $\partial K_0$ and $\partial \Omega$ (see Chapter 5 in [17]). Then, from (4.109) and (4.68), we have

\begin{align*}
\text{ess sup}_{t \in (0,T)} \| N_{i,n}(\cdot,t) \|_{0,2,\mathbb{R}^3}^2 + \int_0^T \| \nabla N_{i,n} \|_{0,2,\mathbb{R}^3}^2 \, dt < 4CL_2^2, \quad (4.112)
\end{align*}

where $N_{i,n}(x,t) = \tilde{N}_{i,n}(X_n(x,t,0),t)$ and $\tilde{N}_{i,n} = \tilde{E}\tilde{N}_{i,n}$.

## 5 Convergence Results

As a consequence of the bounds (4.109), (4.110) and (4.112), there exist subsequences $(u_{n_k})_{k \in \mathbb{N}}, (\mu_{n_k})_{k \in \mathbb{N}}$ and $(N_{i,n_k})_{k \in \mathbb{N}}$ such that

\begin{align}
\text{weakly in } L^2(0,T; H^1_0(\Omega))^3, \quad & u_{n_k} \to u, \\
\text{weakly* in } L^\infty((0,T) \times D), \quad & \mu_{n_k} \to \mu, \\
\text{weakly in } L^2(0,T; H^1(\Omega)), \quad & N_{i,n_k} \to N^*_i
\end{align}

(5.13)

for some $(u, \mu, N^*_1, \ldots, N^*_J)$. From the compactness results for linear transport equations of Di Perna-Lions (see [16]), there exists a subsequence $(\mu_{n_{k_j}})_{j \in \mathbb{N}}$ such that

\begin{equation}
\mu_{n_{k_j}} \to \mu \text{ strongly in } C([0,T]; L^p(D)), \quad (5.14)
\end{equation}

where $p > 1$. Using the standard compactness results for $H^1_0(\Omega)$, we obtain a subsequence $(\nabla N^*_i)_{j \in \mathbb{N}}$ such that

\begin{equation}
\nabla N^*_i \to \nabla N_i^* \text{ strongly in } L^2(0,T; H^1(\partial \Omega)), \quad (5.15)
\end{equation}

and $\mu \in C([0,T]; L^p(D))$. Therefore, $u, \mu$ and $N^*_1, \ldots, N^*_J$ solve a problem in the sense of Di Perna-Lions.
for all $1 \leq p < +\infty$. Moreover, from (4.109) we can show that $u \in L^\infty(0, T; L^2(\mathcal{O}))^3$ and $N_{i}^* \in L^\infty(0, T; L^2(\mathcal{O}))$, using a uniqueness argument. For simplicity we denote $(u_{nk}, \mu_{nk}, N_{i,nk}^*)$ instead $(u_{nk}, \mu_{nk}, N_{i,nk}^*)$.

It is a routine to check that $(\mu, u)$ satisfies (3.35) and (3.36) and $u$ satisfies (4.109). In particular, from (4.111), this implies that (2.2) is valid for the domains $K_t$ and $\Omega_t$ that correspond to $u$. Below we establish some compactness results.

**Lemma 5.1.** For all $h > 0$ sufficiently small,

$$\sup_{n \in \mathbb{N}} \int_0^T \int_\mathcal{O} \mu_n(u_n(x, t + h) - u_n(x, t))^2 dx dt \leq C h^{2/5}, \quad (5.115)$$

where $C = C(T, L_2, L_1, B_s, K_0, e, \mathcal{O}, Z_1, \ldots, Z_n)$ and we have extended $u_n(\cdot, \tau, \mu_n(\cdot, \tau)$ to be zero if $\tau > T$.

**Proof.** Let us to denote $w_n(\cdot, t + h) = u_n(\cdot, t + h) - u_n(\cdot, t)$ for all $t \geq 0$. As a consequence of (4.111) and considering the construction and notation introduced in (15), it is possible to find $\Pi_\beta(w_n(\cdot, \tau)) \in S_n(\tau)$, $\forall \tau \in [t, t + h]$, such that

$$\int_0^T \|w_n(t) - \Pi_\beta(w_n(t))\|_{0, p, \mathcal{O}} dt \leq C_p \beta^{-s} \int_0^T \|\nabla w_n(t)\|_{0, 2, \mathcal{O}} dt,$$

$$\int_0^T \|\nabla \Pi_\beta(w_n(t))\|_{0, p, \mathcal{O}} dt \leq C \beta^{-s} \int_0^T \|\nabla w_n(t)\|_{0, 2, \mathcal{O}} dt,$$

for $s = 3(1/2 - 1/p)$ if $p \in [2, \infty)$, $s = 3/2$ if $p = \infty$.

If we extend $I_{\Omega_n}(\cdot, \tau)$, $N_{i,n}(\cdot, \tau)$, $\psi_n(\cdot, \tau)$ and $\Pi_\beta(w_n)(\cdot, \tau)$ to be zero if $\tau > T$, we have

$$\int_0^T \int_t^{t+h} I_{\Omega_n}(\cdot, \tau) \cdot \Pi_\beta(w_n(t)) dx d\tau dt$$

$$\leq C(e, \mathcal{O}) \sum_{i=1}^J |Z_i| \int_0^T \|\nabla \Pi_\beta(w_n(t))\|_{0, \infty, \mathcal{O}} \int_t^{t+h} \int_{\mathcal{O}} (|I_{\Omega_n}(\cdot, \tau) \cdot |\nabla \psi_n(t)|)(x, \tau) dx d\tau dt$$

$$\leq C(e, \mathcal{O}) \sum_{i=1}^J |Z_i| \left( \int_0^T \|\nabla \Pi_\beta(w_n(t))\|_{0, \infty, \mathcal{O}} dt \right)^{1/2} \times$$

$$\times \left( \int_0^T \left( \int_t^{t+h} \int_{\mathcal{O}} (|I_{\Omega_n}(\cdot, \tau) \cdot |\nabla \psi_n(t)|)(x, \tau) dx d\tau \right)^2 dt \right)^{1/2}$$

$$\leq C(e, \mathcal{O}, L_1) \beta^{-3/2} \sum_{i=1}^J |Z_i| \left( \int_0^T \left( \int_t^{t+h} \|N_{i,n}(t)\|_{0, 2, \Omega_n}(x, \tau) dx d\tau \right)^2 dt \right)^{1/2}$$

$$\leq C(e, \mathcal{O}, L_1, Z_1, \ldots, Z_J, L_2, B_s, T) \beta^{-3/2} h \leq Ch^{2/5}$$

choosing $\beta = h^{2/5}$; where we have used the Schwarz's and Poincaré's inequalities, (5.116) and (4.109). As a consequence, following the steps of the proof of Lemma 4.1 in the reference (15) (see also (13)) we obtain the inequality (5.115).
Lemma 5.2. For all \( h > 0 \) sufficiently small and each \( i \in \{1, \ldots, J\} \),
\[
\sup_{n \in \mathbb{N}} \int_0^T \int_{\mathbb{R}^3} I_{\Omega_{n,t}} |\mathcal{N}_{i,n}^*(x, t + h) - \mathcal{N}_{i,n}^*(x, t)|^2 \, dx \, dt \leq C h^{1/3}, \tag{5.117}
\]
where \( C = C(d_1, \ldots, d_n, L_2, L_1, C, T, \kappa_B, \theta, e, Z_1, \ldots, Z_J, \mathcal{O}, \mathcal{B}_n) \) and we have defined \( I_{\Omega_{n,t}}(\cdot, \tau) \), \( \mathcal{N}_{i,n}^*(\cdot, \tau) \) to be zero if \( \tau > T \).

Proof. Following [13], we have a scalar version of (5.116). More precisely, (5.116) is valid for functions in \( H^1(\mathbb{R}^3) \). In this case, \( \Pi_\beta(f) = f \ast \eta_\beta \), where \( \eta \in C_0^\infty(\mathbb{R}^3) \) satisfies \( \int_{\mathbb{R}^3} \eta dx = 1 \) and, for each \( \beta > 0 \), \( \eta_\beta(x) = \beta^{-2} \eta(\beta^{-1} x) \).

Let us consider \( i \in \{1, \ldots, J\} \) fixed and denote
\[
\mathcal{W}_{i,n}(\cdot, t, h) = \mathcal{N}_{i,n}^*(\cdot, t + h) - \mathcal{N}_{i,n}^*(\cdot, t).
\]
Then, for each \( n \in \mathbb{N} \), we bound by
\[
\int_0^T \int_{\mathbb{R}^3} I_{\Omega_{n,t}}(x) \mathcal{W}_{i,n}^2(x, t, h) dx \, dt \leq P_1 + P_2 + P_3,
\]
where,
\[
P_1 := \int_0^T \int_{\mathbb{R}^3} (I_{\Omega_{n,t+h}} - I_{\Omega_{n,t}})(x)(\mathcal{N}_{i,n}^*)^2(x, t) \, dx \, dt,
\]
\[
P_2 := \int_0^T \int_{\mathbb{R}^3} (I_{\Omega_{n,t}} - I_{\Omega_{n,t+h}})(x)(\mathcal{N}_{i,n}^*)^2(x, t + h) \, dx \, dt,
\]
\[
P_3 := \int_0^T \int_{\mathbb{R}^3} (I_{\Omega_{n,t+h}}(x)\mathcal{N}_{i,n}^*(x, t + h) - I_{\Omega_{n,t+h}}(x)\mathcal{N}_{i,n}^*(x, t)) \mathcal{W}_{i,n}(x, t, h) dx \, dt.
\]
Using (4.112) and Lemma 1 of the reference [13], we have
\[
P_1 \leq C \beta^{1/4} + C \int_0^T \int_{\mathbb{R}^3} (I_{\Omega_{n,t+h}} - I_{\Omega_{n,t}})(x)(\Pi_\beta(\mathcal{N}_{i,n}^*))^2(x, t) \, dx \, dt, \tag{5.118}
\]
where \( C = C(L_2, C, T) \). Now, extending \( \mathcal{R}_n(u_n) \) to be zero in \( \mathbb{R}^3 \setminus \overline{\mathcal{O}} \), we see that \( I_n(x, t) = I_{\Omega_{n,t}}(x) \) is the solution of the transport problem
\[
\partial_t I_n + \nabla \cdot (I_n \mathcal{R}_n(u_n)) = 0, \quad I_n(\cdot, 0) = I_{\Omega_0}.
\]
in \( \mathcal{D}'((0, T) \times \mathbb{R}^3) \). Then, using (5.118), we have
\[
P_1 \leq C \beta^{1/4} + C \int_0^T \int_{\mathbb{R}^3} \left| \int_t^{t+h} \nabla \cdot (I_n \mathcal{R}_n(u_n))(x, \tau)(\Pi_\beta(\mathcal{N}_{i,n}^*))^2(x, t) d\tau \, dx \right| d\tau \, dt,
\]
\[
\leq C \beta^{1/4} + C \int_0^T \int_{\mathbb{R}^3} \int_t^{t+h} \left| (I_n \mathcal{R}_n(u_n))(x, \tau)(\Pi_\beta(\mathcal{N}_{i,n}^*))^2(x, t) \right| d\tau \, dx \, dt.
\]
Now, from Hölder’s inequality, the Sobolev embedding theorem, (4.64), (4.109),
where
\[ \int_0^T \int_t^{t+h} \int_{\mathbb{R}^3} |(\mathcal{I}_n \mathcal{R}_n(\mathbf{u}_n))((\mathbf{x}, \tau))||| \Pi_\beta(N_{i,n}^\ast)(\mathbf{x}, t)||\nabla(\Pi_\beta(N_{i,n}^\ast))(\mathbf{x}, t)|dxd\tau dt \]
\[ \leq \int_0^T \int_t^{t+h} \int_{\mathbb{R}^3} |(\mathcal{I}_n \mathcal{R}_n(\mathbf{u}_n))((\mathbf{x}, \tau))||| \Pi_\beta(N_{i,n}^\ast) - N_{i,n}^\ast)(\mathbf{x}, t)||\nabla(\Pi_\beta(N_{i,n}^\ast))(\mathbf{x}, t)|dxd\tau dt + \]
\[ + \int_0^T \int_t^{t+h} \int_{\mathbb{R}^3} |(\mathcal{I}_n \mathcal{R}_n(\mathbf{u}_n))((\mathbf{x}, \tau))||N_{i,n}^\ast(\mathbf{x}, t)||\nabla(\Pi_\beta(N_{i,n}^\ast))(\mathbf{x}, t)|dxd\tau dt \]
\[ \leq \int_0^T \int_t^{t+h} \|\mathcal{R}_n(\mathbf{u}_n)(\cdot, \tau)\|_{0,6,\mathbb{R}^3} \|\nabla(\Pi_\beta(N_{i,n}^\ast))(\cdot, t)\|_{0,3,\mathbb{R}^3} \times \]
\[ \times \|\Pi_\beta(N_{i,n}^\ast) - N_{i,n}^\ast)(\cdot, t)\|_{0,2,\mathbb{R}^3} d\tau dt + \]
\[ + \int_0^T \int_t^{t+h} \|\mathcal{R}_n(\mathbf{u}_n)(\cdot, \tau)\|_{0,2,\mathbb{R}^3} \|\nabla(\Pi_\beta(N_{i,n}^\ast))(\cdot, t)\|_{0,3,\mathbb{R}^3} \|N_{i,n}^\ast(\cdot, t)\|_{0,6,\mathbb{R}^3} d\tau dt \]
\[ \leq C h^{1/2} \left( \int_0^T \|\mathbf{u}_n(\cdot, \tau)\|_{1,2,\mathcal{C}}^2 d\tau \right)^{1/2} \left( \int_0^T \|\nabla(\Pi_\beta(N_{i,n}^\ast))(\cdot, t)\|_{0,3,\mathbb{R}^3}^2 dt \right)^{1/2} \times \]
\[ \times \left( \int_0^T \|\Pi_\beta(N_{i,n}^\ast) - N_{i,n}^\ast)(\cdot, t)\|_{0,2,\mathbb{R}^3}^2 dt \right)^{1/2} + \]
\[ + Ch \left( \int_0^T \|\nabla(\Pi_\beta(N_{i,n}^\ast))(\cdot, t)\|_{0,3,\mathbb{R}^3}^2 dt \right)^{1/2} \left( \int_0^T \|N_{i,n}^\ast(\cdot, t)\|_{0,6,\mathbb{R}^3}^2 dt \right)^{1/2} \]
\[ \leq Ch^{1/2} \beta^{3/4} + C h^{1/2} \beta^{3/4} + h\beta^{-1/4}, \]
where \( C = C(L_2, L_1, T, \mathbf{C}) \). As a consequence,
\[ P_1 \leq C(\beta^{3/4} + h^{1/2} \beta^{3/4} + h\beta^{-1/4}), \quad (5.119) \]

A similar estimate can be obtained for \( P_2 \). Now, an elementary calculation show us that
\[ P_3 \leq \left| \int_0^T \int_{\mathbb{R}^3} ((\mathcal{I}_n N_{i,n}^\ast)(\mathbf{x}, t + h) - (\mathcal{I}_n N_{i,n}^\ast)(\mathbf{x}, t))(\mathcal{W}_{i,n} - \Pi_\beta(\mathcal{W}_{i,n}))(\mathbf{x}, t, h) d\mathbf{x} dt \right| + \]
\[ + \left| \int_0^T \int_{\mathbb{R}^3} ((\mathcal{I}_n N_{i,n}^\ast)(\mathbf{x}, t + h) - (\mathcal{I}_n N_{i,n}^\ast)(\mathbf{x}, t))\Pi_\beta(\mathcal{W}_{i,n})(\mathbf{x}, t, h) d\mathbf{x} dt \right|. \]

Denoting the first an the second terms in the right side above by \( P_3' \) and \( P_3'' \), respectively, we have
\[ P_3' \leq C \beta^{1/2} \left( \int_0^T \|\nabla \mathcal{W}_{i,n}\|_{0,2,\mathbb{R}^3}^2 dt \right)^{1/2} \times \]
\[ \times \left( \int_0^T \|(\mathcal{I}_n N_{i,n}^\ast)(\cdot, t + h) - (\mathcal{I}_n N_{i,n}^\ast)(\cdot, t)\|_{0,2,\mathbb{R}^3}^2 dt \right)^{1/2} \]
\[ \leq C(L_2, T, \mathbf{C}) \beta^{1/2}, \]

using Schwarz inequality and (4.112). In order to estimate \( P_3'' \), for \( t \in [0, T] \) fixed, we take \( \zeta(t, \tau) = \Pi_\beta(\mathcal{W}_{i,n})(\mathbf{x}, t), (\mathbf{x}, \tau) \in \mathbb{R}^3 \times [0, T] \) as a test function
expressions, we obtain, after some calculation,

\[
\begin{align*}
&\int_{\mathbb{R}^3} (\mathcal{I}_n N_{i,n}^\ast \Pi_\beta(W_{i,n}))(x, t)dx + \int_0^t \int_{\Omega_{n,\tau}} \Pi_\beta(W_{i,n})(x, t) (\mathcal{R}_n(u_n) \cdot \nabla N_{i,n}^\ast)(x, \tau)dx d\tau + \\
&+ d_1 \int_0^t \int_{\Omega_{n,\tau}} \left( \nabla N_{i,n}^\ast + \frac{\varepsilon Z_i}{\kappa B \theta} N_{i,n}^\ast \nabla [\psi_n]^n \right)(x, \tau) \cdot \nabla \Pi_\beta(W_{i,n})(x, t)dx d\tau \\
&= \int_{\Omega_0} N_{i,0,n}(x) \Pi_\beta(W_{i,n})(x, t)dx
\end{align*}
\]

and

\[
\begin{align*}
&\int_{\mathbb{R}^3} (\mathcal{I}_n N_{i,n}^\ast)(x, t + h)\Pi_\beta(W_{i,n})(x, t)dx + \\
&+ \int_0^{t+h} \int_{\Omega_{n,\tau}} \Pi_\beta(W_{i,n})(x, t) (\mathcal{R}_n(u_n) \cdot \nabla N_{i,n}^\ast)(x, \tau)dx d\tau + \\
&+ d_1 \int_0^{t+h} \int_{\Omega_{n,\tau}} \left( \nabla N_{i,n}^\ast + \frac{\varepsilon Z_i}{\kappa B \theta} N_{i,n}^\ast \nabla [\psi_n]^n \right)(x, \tau) \cdot \nabla \Pi_\beta(W_{i,n})(x, t)dx d\tau \\
&= \int_{\Omega_0} N_{i,0,n}(x) \Pi_\beta(W_{i,n})(x, t)dx,
\end{align*}
\]

where we have extended \([\psi_n]^n(\cdot, \tau)\) to be zero if \(\tau > T\). Subtracting the above expressions, we obtain, after some calculation,

\[
\begin{align*}
&\int_{\mathbb{R}^3} \int_t^{t+h} \partial_\tau (\mathcal{I}_n N_{i,n}^\ast)(x, \tau) \Pi_\beta(W_{i,n})(x, t)d\tau dx + \\
&+ \int_0^{t+h} \int_{\Omega_{n,\tau}} \Pi_\beta(W_{i,n})(x, t) (\mathcal{R}_n(u_n) \cdot \nabla N_{i,n}^\ast)(x, \tau)dx d\tau + \\
&+ d_1 \int_0^{t+h} \int_{\Omega_{n,\tau}} \left( \nabla N_{i,n}^\ast + N_{i,n}^\ast \nabla [\psi_n]^n \right)(x, \tau) \cdot \nabla \Pi_\beta(W_{i,n})(x, t)dx d\tau = 0.
\end{align*}
\]

Hence,

\[P_3'' = \left| \int_0^T \int_{\mathbb{R}^3} \int_t^{t+h} \partial_\tau (\mathcal{I}_n N_{i,n}^\ast)(x, \tau) \Pi_\beta(W_{i,n})(x, t)d\tau dx dt \right|
\]

\[
\leq \int_0^T \int_t^{t+h} \int_{\Omega_{n,\tau}} |\Pi_\beta(W_{i,n})(x, t)||\mathcal{R}_n(u_n)(x, \tau)||\nabla N_{i,n}^\ast(x, \tau)||dx d\tau dt + \\
+ d_1 \int_0^T \int_t^{t+h} \int_{\Omega_{n,\tau}} |\nabla N_{i,n}^\ast(x, \tau)||\nabla \Pi_\beta(W_{i,n})(x, t)||dx d\tau dt \\
+ d_1 \frac{|Z_i|}{\kappa B \theta} \int_0^T \int_t^{t+h} \int_{\Omega_{n,\tau}} |N_{i,n}^\ast(x, \tau)||\nabla [\psi_n]^n(x, \tau)||\nabla \Pi_\beta(W_{i,n})(x, t)||dx d\tau dt.
\]

From Hölder’s and Poincaré’s inequalities, (5.116), (4.112), (4.109) and (4.64).
we obtain the estimates
\[
\int_0^T \int_t^{t+h} \int_{\Omega_{n,\tau}} |\Pi_{\beta}(W_{i,n})(x,\tau)||\mathcal{N}_{i,n}^\ast(x,\tau)|dxd\tau dt \\
\leq \int_0^T \int_t^{t+h} \int_{\Omega_{n,\tau}} \|\Pi_{\beta}(W_{i,n})(..,\tau)||\mathcal{N}_{i,n}^\ast(x,\tau)|_0.2.\mathcal{O}d\tau dt \\
\leq C(L_2,\overline{C}) \int_0^T \left( \int_t^{t+h} \|\mathcal{N}_{i,n}^\ast(x,\tau)|_0.2.\mathcal{O}d\tau \right)^{1/2} \left( \int_t^{t+h} \|\nabla\mathcal{N}_{i,n}^\ast(x,\tau)|_0.2.\mathcal{O}d\tau \right)^{1/2} dt \\
\leq C(L_2,\overline{C},L_1,T)h^{1/2}\beta^{-3/4},
\]
(5.121)

\[
\int_0^T \int_t^{t+h} \int_{\Omega_{n,\tau}} |\nabla\mathcal{N}_{i,n}^\ast(x,\tau)||\nabla\Pi_{\beta}(W_{i,n})(x,\tau)|dxd\tau dt \\
\leq \int_0^T \left( \int_t^{t+h} \|\nabla\mathcal{N}_{i,n}^\ast(x,\tau)|_0.2.\mathcal{O}d\tau \right)^{1/2} \times \\
\times \left( \int_t^{t+h} \|\nabla\Pi_{\beta}(W_{i,n})(x,\tau)|_0.2.\mathcal{O}d\tau \right)^{1/2} dt \\
\leq C(L_2,\overline{C},T) \int_0^T \left( \int_t^{t+h} \|\nabla\Pi_{\beta}(W_{i,n})(x,\tau)|_0.2.\mathcal{O}d\tau \right)^{1/2} dt \\
\leq C(L_2,\overline{C},T)h^{1/2},
\]
(5.122)

\[
\int_0^T \int_t^{t+h} \int_{\Omega_{n,\tau}} |\mathcal{N}_{i,n}^\ast(x,\tau)||\nabla[\psi_n]^n(x,\tau)||\nabla\Pi_{\beta}(W_{i,n})(x,\tau)|dxd\tau dt \\
\leq \int_0^T \|\nabla\Pi_{\beta}(W_{i,n})(..,\tau)||_{0.\mathcal{O}} \int_t^{t+h} \|\mathcal{N}_{i,n}^\ast|_0.2.\mathcal{O}\nabla[\psi_n]^n|_0.2.\mathcal{O}d\tau dt \\
\leq C(L_2,\overline{B}_\omega,\mathcal{O})\beta^{-3/4}h.
\]
(5.123)

Choosing \(\beta = h^{1/2}\) and using (5.119)-(5.122) we obtain the result. \(\square\)

From Lemmas 5.1 and 5.2 and from Frechét-Kolmogorov Theorem (see Theorem IV.25 in [6]), up to a extraction of a subsequence,
\[
\begin{align*}
\mathbf{u}_n &\to \mathbf{u} \quad \text{strongly in} \quad L^2(\mathcal{O} \times (0,T))^3, \\
\mathcal{N}_{i,n}^\ast &\to \mathcal{N}_i^\ast \quad \text{strongly in} \quad L^2(\mathcal{O} \times (0,T)), \quad \forall i \in \{1,\ldots,J\}.
\end{align*}
\]
(5.124)

In particular this implies that \(\mathcal{N}_i^\ast \geq 0 \ a.e. \ \text{in} \ \mathbb{R}^3\), as it is easy to check. Now, let us consider the problem (3.37) with the data \((K_1,\Omega_1,\mathcal{N}_{i,1}^\ast,\ldots,\mathcal{N}_{i,J}^\ast,\rho,\mathbf{\Psi})\). From Lemma 4.2 this problem has a unique solution \(\psi(.,t) \in H^1(\mathcal{O})\) that satisfies the bound (1.73). As a consequence we can prove the following convergence result for \(\psi_n(.,t)\).

**Lemma 5.3.** Let consider \((\psi_n,\psi)\) the solutions of the problem (3.37) corresponding to \((K_{n,t},\Omega_{n,t},\mathcal{N}_{i,1,n}^\ast,\ldots,\mathcal{N}_{i,J,n}^\ast,\rho_n,\mathbf{\Psi}_n)\) and \((K_1,\Omega_1,\mathcal{N}_{i,1}^\ast,\ldots,\mathcal{N}_{i,J}^\ast,\rho,\mathbf{\Psi})\), respectively. Then,
\[
\int_0^T \int_\mathcal{O} |\nabla(\psi_n - \psi)|^2 dxd\tau \to 0, \quad n \to \infty.
\]
(5.125)
Proof. First we consider the standard extension operator

\[ E_O : H^1(\partial \mathcal{O}) \to H^1(\mathcal{O}), \]

i.e., \( E_O(f)|_{\partial \mathcal{O}} = f \) and \( \|E_O(f)\|_{1,2,\mathcal{O}} \leq C_\mathcal{O}\|f\|_{1,2,\partial \mathcal{O}}. \)

Then, defining \( \hat{\Psi} = E_O\Psi \) and \( \hat{\Psi}_n = E_O\Psi_n \) we have from (4.65),

\[ \|\hat{\Psi}_n\|_{1,2,\mathcal{O}} \leq C_\mathcal{O}\|\Psi\|_{1,2,\partial \mathcal{O}}. \tag{5.126} \]

Furthermore, it is clear that if \( \Upsilon_n := \hat{\Psi}_n - \hat{\Psi} \),

\[ \|\Upsilon_n\|_{1,2,\mathcal{O}} \to 0, \quad n \to \infty. \tag{5.127} \]

Note that \( \psi_n - \hat{\Psi}_n + \hat{\Psi} \in \mathbb{L}\mathcal{O} \); then, from (3.37), we have

\[
\left( \int_O \kappa_n \nabla \psi \cdot \nabla (\psi - \psi_n - \hat{\Psi} + \hat{\Psi}_n) dx \right) (t) +
- 4\pi \epsilon \sum_{i=1}^J \left( \int_{\Omega_i} Z_i N_i^* (\psi - \psi_n - \hat{\Psi} + \hat{\Psi}_n) dx \right) (t) +
- 4\pi \left( \int_O \rho (\psi - \psi_n - \hat{\Psi} + \hat{\Psi}_n) dx \right) (t) = 0.
\tag{5.128}
\]

Similarly, observing that \( \psi - \hat{\Psi} + \hat{\Psi}_n \in \mathbb{L}\mathcal{O}_n \), we have

\[
\left( \int_O \kappa_n \nabla \psi_n \cdot \nabla (\psi_n - \psi - \hat{\Psi}_n + \hat{\Psi}) dx \right) (t) +
- 4\pi \epsilon \sum_{i=1}^J \left( \int_{\Omega_{n,i}} Z_i r_i (N_i^* - r_i (N_i^*)) (\psi_n - \psi - \hat{\Psi}_n + \hat{\Psi}) dx \right) (t) +
- 4\pi \left( \int_O \rho_n (\psi_n - \psi - \hat{\Psi}_n + \hat{\Psi}) dx \right) (t) = 0.
\tag{5.129}
\]

Let us set \( A_{n,t} = K_{n,t} \cap K_t, B_{n,t} = \Omega_{n,t} \cap \Omega_t \) and \( f_n = \psi - \psi_n \). Then, summing (5.128) with (5.129), we obtain,

\[
\left( \int_O \kappa_n |\nabla f_n|^2 dx \right) (t) =
- \left( \int_O (\kappa - \kappa_n) \nabla \psi \cdot \nabla f_n dx \right) (t) - \left( \int_O (\kappa - \kappa_n) \nabla \psi \cdot \nabla Y_n dx \right) (t) +
- \left( \int_O \kappa_n \nabla f_n \cdot \nabla Y_n dx \right) (t) + 4\pi \epsilon \sum_{i=1}^J \left( \int_{B_{n,i}} Z_i (N_i^* - r_i (N_i^*)) (f_n + Y_n) dx \right) (t) +
+ 4\pi \epsilon \sum_{i=1}^J \left( \int_{K_{n,i} \setminus A_{n,t}} Z_i N_i^* (f_n + Y_n) dx \right) (t) +
- 4\pi \sum_{i=1}^J \left( \int_{K_i \setminus A_{n,t}} Z_i r_i (N_i^*) (f_n + Y_n) dx \right) (t) + 4\pi \left( \int_O (\rho - \rho_n) (f_n + Y_n) dx \right) (t).
\]
Hölder’s inequality gives us
\[
\left( \int_0^T |\nabla f_n|^2 \, dx \right) (t) \leq
\]
\[
\leq C((\kappa_1 - \kappa_2)||\nabla \psi(\cdot, t)||_{0,2,K_n,A_n,t} + ||\nabla f_n(\cdot, t)||_{0,2,\Theta} + ||\nabla Y_n||_{0,2,\Theta}) +
\]
\[
+ |\kappa_1 - \kappa_2||\nabla \psi(\cdot, t)||_{0,2,K_1,A_n,t} (||\nabla f_n(\cdot, t)||_{0,2,\Theta} + ||\nabla Y_n||_{0,2,\Theta}) +
\]
\[
+ ||\nabla Y_n||_{0,2,\Theta}^2 + \sum_{i=1}^J ||(\mathcal{N}_i^* - r_n(\mathcal{N}_i^*))(\cdot, t)||_{0,2,\Theta} (||f_n + Y_n(\cdot, t)||_{0,2,\Theta} +
\]
\[
+ \sum_{i=1}^J ||(\mathcal{N}_i^* - r_n(\mathcal{N}_i^*))(\cdot, t)||_{0,2,\Theta} + \sum_{i=1}^J |K_{n,t} \setminus A_{n,t}|^{1/4} ||N_i^* (\cdot, t)||_{1,2,\Theta} +
\]
\[
+ \sum_{i=1}^J |K_{n,t} \setminus A_{n,t}|^{1/4} ||r_n(\mathcal{N}_i^*)(\cdot, t)||_{1,2,\Theta} +
\]
\[
+ ||(\rho - \rho_n)(\cdot, t)||_{0,2,A_n} + |K_{n,t} \setminus A_{n,t}|^{1/4} ||r_n(\cdot, t)||_{0,2,\Theta} +
\]
\[
+ |K_{n,t} \setminus A_{n,t}|^{1/4} ||\rho_n(\cdot, t)||_{0,2,\Theta} + C ||\nabla Y_n||_{0,2,\Theta}^2,
\]
\[
(5.130)
\]
where \( C = C(O, Z_1, \ldots, Z_J, \epsilon, \kappa_1, \kappa_2) \). Then, integrating (5.130) in \((0, T)\), we obtain
\[
\int_0^T \int_0^T |\nabla f_n|^2 \, dx \, dt \leq
\]
\[
\leq C(B_*) + ||\Psi||_{1,2,\Theta} (B_*) + ||\rho_0||_{0,2,\Theta} \int_0^T (|K_{n,t} \setminus A_{n,t}|^{1/4} + |K_{n,t} \setminus A_{n,t}|^{1/4}) \, dt +
\]
\[
+ \sum_{i=1}^J \int_0^T (|N_i^* - r_n(\mathcal{N}_i^*))(\cdot, t)||_{0,2,\Theta} \, dt + \sum_{i=1}^J L_2^{1/2} \left( \int_0^T |K_{n,t} \setminus A_{n,t}|^{1/2} \, dt \right)^{1/2} +
\]
\[
+ \sum_{i=1}^J L_2^{1/2} \left( \int_0^T |K_{n,t} \setminus A_{n,t}|^{1/2} \, dt \right)^{1/2} + \int_0^T ||(\rho - \rho_n)(\cdot, t)||_{0,2,A_n} \, dt +
\]
\[
+ CT ||\nabla Y_n||_{0,2,\Theta}^2,
\]
\[
(5.131)
\]
using Schwarz’s inequality, (2.25), (4.112) and (4.60). Now, denoting by \( Q(t) \), \( Q_n(t) \) the affine isometries such that \( K_t = Q(t)K_0 \), \( K_{n,t} = Q_n(t)K_0 \), we have
\[
||Q_n(\cdot) - Q(\cdot)||_{0,\infty,(0,T)} \to 0, \ \ n \to \infty
\]
\[
(5.132)
\]
using (5.124) (see [14]). As a consequence,

\[\|\rho_n(t) - \rho_n(t)\|_{L^2}^2 \leq \int_{K_0} (\rho(Q_n(t)\xi) - \rho_n(Q_n(t)\xi))^2 d\xi\]

\[= \int_{K_0} (\rho_0(Q^{-1}(t)Q_n(t)\xi) - \rho_0(\xi))^2 d\xi\]

\[\leq 2 \int_{K_0} (\rho_0(Q^{-1}(t)Q_n(t)\xi) - \rho_0(Q^{-1}(t)Q(t)\xi))^2 d\xi +
+ 2 \int_{K_0} (\rho_0(\xi) - \rho_0(\xi))^2 d\xi \to 0, \ n \to \infty,\]

using \(L^2\)-continuity and the strong convergence \(\|\rho_0 - \rho_0, n\|_{L^2, K_0} \to 0, \ n \to \infty\).

Moreover, from (5.132), we have \(|K_{n,t} \setminus A_{n,t}| \to 0, \ |K_t \setminus A_{n,t}| \to 0\) as \(n \to \infty\). Then (5.124) follows from (5.131), (5.133), (5.124), (5.127) and the dominated convergence theorem.

\[\square\]

**Lemma 5.4.** For each \(i \in \{1, \ldots, J\}\), the function \(N^*_i\), satisfies (3.38), considering \(u\) and \(\psi\) given in (5.124) and Lemma 5.3 respectively.

**Proof.** For each \(\zeta \in H^1(\Omega)\), we consider its extension \(\tilde{\zeta} \in H^1(O \times (0,T))\) and a sequence \(\tilde{\zeta}_n \in C^{2+\alpha,1}(\Omega \times [0,T])\) such that

\[\|\tilde{\zeta}_n - \tilde{\zeta}\|_{L^2(O \times (0,T))} \to 0, \ n \to \infty. \quad (5.134)\]

Let us set \(\zeta_n := \tilde{\zeta}_n|\Omega_{n,T}\), where \(\Omega_{n,T} = \bigcup_{t \in (0,T)} \{t\} \times \Omega_{n,t}\) and, as before, \(\Omega_{n,t}\) denote the fluid domains related to \(\mathcal{R}_n(u_n)\). We want to pass to the limit in

\[\left(\int_{\Omega_{n,t}} N^*_{i,n} \zeta_n \ dx \right) + \int_0^t \int_{\Omega_{n,\tau}} \zeta_n (\mathcal{R}_n(u_n) \cdot \nabla N^*_{i,n}) \ dx \ d\tau +
+ d_i \int_0^t \int_{\Omega_{n,\tau}} \left(\nabla N^*_{i,n} \cdot \frac{Z_i e^{-\theta n B}}{\Theta_{n B}} N^*_{i,n} \nabla \psi_{n}\right) \cdot \nabla \zeta_n \ dx \ d\tau +
- \int_0^t \int_{\Omega_{n,\tau}} N^*_{i,n} \partial_{i,n} \ dx \ d\tau = \left(\int_{\Omega_0} N^*_{i,0,n} \zeta_{0,n} \ dx \right) (t)\]

for a. e. \(t \in (0,T)\).

First we observe that

\[\left|\int_{\Omega_0} N^*_{i,0,n} \zeta_{0,n} \ dx - \int_{\Omega_0} N^*_{i,0} \zeta_0 \ dx\right| \to 0, \ n \to \infty, \quad (5.135)\]

follows directly from (5.134) and from the strong convergence \(N^*_{i,0,n} \to N^*_{i,0}\) in \(L^2(\Omega_0)\).

Now, from (5.124), up to a extraction of a subsequence, we have

\(N^*_i(\cdot, t) \to \mathcal{N}_i^*(\cdot, t)\) in \(L^2(O)\),
a. e. \(t \in (0,T), \forall i \in \{1, \ldots, J\}\). Then, using the notation of the Lemma 5.3
and Hölder’s inequality, we have

\[
\left| \int_{\Omega_{n,t}} N_{i,n}^* \zeta dx - \int_{\Omega_t} N_i^* \zeta dx \right| \leq \int_{B_{n,t}} |N_{i,n}^* - N_i^*| |\zeta_n| dx + \int_{B_{n,t}} |\zeta_n - \zeta| |N_i^*| dx + \\
+ \int_{K_t \setminus A_{n,t}} |N_{i,n}^* \zeta_n| dx + \int_{K_n \setminus A_n} |N_i^* \zeta| dx
\]

\[
\leq \|N_{i,n}^*(\cdot,t) - N_i^*(\cdot,t)\|_{0,2,\mathcal{O}} \|\zeta_n(\cdot,t)\|_{0,2,\mathcal{O}} + \|N_i^*(\cdot,t)\|_{0,2,\mathcal{O}} \|\zeta_n - \zeta(\cdot,t)\|_{0,2,\mathcal{O}} + \\
+ |K_t \setminus A_{n,t}|^{1/6} \|N_{i,n}^*(\cdot,t)\|_{0,2,K_1} \|\zeta_n(\cdot,t)\|_{0,3,\mathcal{O}} + \\
+ |K_n \setminus A_n|^{1/6} \|N_i^*(\cdot,t)\|_{0,2,K_n} \|\zeta(\cdot,t)\|_{0,3,\mathcal{O}} \to 0, \ n \to \infty,
\]

(5.136)

for a. e. \( t \in (0, T) \). Indeed, from (5.133), \( \|\zeta_n\|_{1,2,\mathcal{O} \times (0,T)} \leq \|\zeta\|_{1,2,\mathcal{O} \times (0,T)} \) for \( n \) sufficiently large. As a consequence, using (5.124) and (5.134) the first two terms above tends to zero as \( n \to \infty \). As in Lemma 5.3, \( |K_t \setminus A_{n,t}|, |K_n \setminus A_n| \to 0, \ n \to \infty \); then from Sobolev embedding and (4.112), we obtain the convergence of the third and fourth terms above.

From (5.113), we have

\[
\nabla N_{i,n}^* \to \nabla N_i^* \text{ weakly in } L^2((0, T) \times \mathcal{O})^2,
\]

(5.137)

\( \forall i \in \{1, \ldots, J\} \). For a. e. \( t \in (0, T) \),

\[
\left| \int_0^t \int_{\Omega_{\tau,n}} \zeta_n (R_n(u_n) \cdot \nabla N_{i,n}^*) dx d\tau - \int_0^t \int_{\Omega_{\tau}} \zeta (u \cdot \nabla N_i^*) dx d\tau \right| \leq \\
\leq \int_0^t \int_{\Omega_{\tau,n}} |\nabla N_{i,n}^*| |\zeta_n - \zeta| |R_n(u_n)| dx d\tau + \\
+ \int_0^t \int_{\Omega_{\tau,n}} |\zeta u \cdot \nabla N_{i,n}^*| dx d\tau - \int_0^t \int_{\Omega_{\tau}} |\zeta u \cdot \nabla N_i^*| dx d\tau + \\
+ \int_0^t \int_{\Omega_{\tau,n}} |\zeta u \cdot \nabla N_{i,n}^*| dx d\tau - \int_0^t \int_{\Omega_{\tau,n}} |\zeta u \cdot \nabla N_i^*| dx d\tau \to 0, \ n \to \infty.
\]

(5.138)

In fact, Hölder and Poincaré inequality, Sobolev embedding, (4.109), (4.112), (5.134) and (4.64) furnish us with the estimate

\[
\int_0^T \int_{\Omega_{t,n}} |\nabla N_{i,n}^*| |\zeta_n - \zeta| |R_n(u_n)| dx d\tau \leq \\
\leq \int_0^T \|\nabla N_{i,n}^*\|_{0,2,\mathcal{O}} \|\zeta_n - \zeta\|_{0,3,\mathcal{O}} \|R_n(u_n)\|_{0,6,\mathcal{O}} \|dx d\tau \leq \\
\leq C(O,T) \|\zeta_n - \zeta\|_{1,2,\mathcal{O} \times (0,T)} \left( \int_0^T \|\nabla N_{i,n}^*\|_{0,2,\mathcal{O}}^2 \|dx d\tau \right)^{1/2} \left( \int_0^T \|R_n(u_n)\|_{0,2,\mathcal{O}}^2 \|dx d\tau \right)^{1/2} \leq \\
\leq C(O,T,L_1,L_2,\mathcal{O}) \|\zeta_n - \zeta\|_{1,2,\mathcal{O} \times (0,T)} \to 0, \ n \to \infty.
\]
Also, we have,

\[ \left| \int_{0}^{t} \int_{\Omega_{n,\tau}} \tilde{\mathbf{u}} \cdot \nabla N_{i,n}^* \, dx \, d\tau - \int_{0}^{t} \int_{\Omega_{\tau}} \tilde{\mathbf{u}} \cdot \nabla N_{i,n}^* \, dx \, d\tau \right| \]

\[ \leq \left| \int_{0}^{T} \int_{0}^{T} \mathcal{I}_{(0,t)} (\mathcal{I}_{\Omega_{n,\tau}} - \mathcal{I}_{\Omega_{\tau}}) \tilde{\mathbf{u}} \cdot \nabla N_{i,n}^* \, dx \, d\tau \right| + \]

\[ \left| \int_{0}^{T} \int_{0}^{T} \mathcal{I}_{(0,t)} \mathcal{I}_{\Omega_{\tau}} \tilde{\mathbf{u}} \cdot (N_{i,n}^* - \nabla N_{i}^*) \, dx \, d\tau \right| \]

The second term in the right-hand side above tends to zero as \( n \to \infty \), using (5.137) and the fact that \( \mathcal{I}_{(0,t)} \mathcal{I}_{\Omega_{\tau}} \tilde{\mathbf{u}} \in L^2(\mathcal{O} \times (0, T)) \). From (5.132), \( \mathcal{I}_{\Omega_{n,\tau}} - \mathcal{I}_{\Omega_{\tau}} \to 0 \) a.e. in \( \mathcal{O} \times (0, T) \), with \( n \to \infty \). Then,

\[ \left| \int_{0}^{T} \int_{0}^{T} \mathcal{I}_{(0,t)} (\mathcal{I}_{\Omega_{n,\tau}} - \mathcal{I}_{\Omega_{\tau}}) \tilde{\mathbf{u}} \cdot \nabla N_{i,n}^* \, dx \, d\tau \right| \]

\[ \leq \int_{0}^{T} \| \mathcal{I}_{\Omega_{n,\tau}} - \mathcal{I}_{\Omega_{\tau}} \|_{0,8,\mathcal{O}} \| \nabla N_{i,n}^* \|_{0,2,\mathcal{O}} \| \tilde{\mathbf{u}} \|_{0,16/3,\mathcal{O}} \, d\tau \]

\[ \leq C \| \tilde{c} \|_{1,2,\mathcal{O} \times (0,T)} \int_{0}^{T} \| \mathcal{I}_{\Omega_{n,\tau}} - \mathcal{I}_{\Omega_{\tau}} \|_{0,8,\mathcal{O}} \| \nabla N_{i,n}^* \|_{0,2,\mathcal{O}} \| \tilde{\mathbf{u}} \|_{0,16/3,\mathcal{O}} \, d\tau \]

\[ \leq C \| \tilde{c} \|_{1,2,\mathcal{O} \times (0,T)} \left( \int_{0}^{T} \| \tilde{\mathbf{u}} \|_{0,16/3,\mathcal{O}}^2 \, d\tau \right)^{1/2} \left( \int_{0}^{T} \| \nabla N_{i,n}^* \|_{0,2,\mathcal{O}}^2 \, d\tau \right)^{1/2} \]

\[ \leq C \| \tilde{c} \|_{1,2,\mathcal{O} \times (0,T)} \left( \int_{0}^{T} \| \tilde{\mathbf{u}} \|_{0,16/3,\mathcal{O}}^2 \| \mathcal{I}_{\Omega_{n,\tau}} - \mathcal{I}_{\Omega_{\tau}} \|_{0,8,\mathcal{O}}^2 \, d\tau \right)^{1/2} \to 0, \quad n \to \infty, \]

where \( C = C(\mathcal{O}, T) \) and we have used the dominated convergence theorem.
Also, from Hölder’s inequality, Sobolev embedding, (4.112), (4.109) and (5.124),

\[
\left| \int_0^t \int_{\Omega_{n,\tau}} \nabla N^*_{i,n} \cdot \nabla \nabla^*_{i,n} d\tau d\Omega \right| \leq \int_0^T \int_{\Omega} |\nabla \nabla^*_{i,n}| d\tau d\Omega \\
\leq \int_0^T \int_{\Omega} |\nabla \nabla^*_{i,n}| d\tau d\Omega \\
\leq \int_0^T \int_{\Omega} |\nabla \nabla^*_{i,n}| d\tau d\Omega \\
\leq \int_0^T \int_{\Omega} \left| \nabla \nabla^*_{i,n} \right| d\tau d\Omega \leq C \int_0^T \left( \int_{\Omega} \left| \nabla \nabla^*_{i,n} \right|^3 d\tau d\Omega \right)^{\frac{1}{3}} \times \\
\times \left( \int_0^T \left| \nabla \nabla^*_{i,n} \right|^3 d\tau d\Omega \right)^{\frac{1}{3}} \\
\leq C(O, T) \left( \int_0^T \left| \nabla \nabla^*_{i,n} \right|^3 d\tau d\Omega \right)^{\frac{1}{3}} \rightarrow 0, \quad n \rightarrow \infty.
\]

The convergences

\[
\left| \int_0^t \int_{\Omega_{n,\tau}} \nabla N^*_{i,n} \cdot \nabla \nabla^*_{i,n} d\tau d\Omega \right| \rightarrow 0,
\]

\[
\left| \int_0^t \int_{\Omega_{n,\tau}} N^*_{i,n} \partial_i \nabla^*_{i,n} d\tau d\Omega \right| \rightarrow 0, \quad n \rightarrow \infty
\]

for a. e. \( t \in (0, T) \), follows from analogous arguments.

Now, for a. e. \( t \in (0, T) \),

\[
\left| \int_0^t \int_{\Omega_{n,\tau}} N^*_{i,n} \nabla [\psi]^n \cdot \nabla \nabla^*_{i,n} d\tau d\Omega \right| \\
\leq \int_0^T \int_{B_{n,\tau}} |N^*_{i,n}||\nabla [\psi]^n| d\tau d\Omega + \int_0^T \int_{B_{n,\tau}} |N^*_{i,n}||\nabla [\psi]^n| d\tau d\Omega + \\
+ \int_0^T \int_{B_{n,\tau}} |N^*_{i,n}||\nabla [\psi]^n| d\tau d\Omega + \\
+ \int_0^T \int_{K_{n,\tau} \setminus A_{n,\tau}} |N^*_{i,n}||\nabla [\psi]^n| d\tau d\Omega + \\
+ \int_0^T \int_{K_{n,\tau} \setminus A_{n,\tau}} |N^*_{i,n}||\nabla [\psi]^n| d\tau d\Omega \rightarrow 0, \quad n \rightarrow \infty.
\]

(5.140)
In fact, from Hölder’s inequality, Sobolev embedding, \((4.62), (4.110), (4.112)\) and \((5.134)\) we have

\[
\int_0^T \int_{B_{n,\tau}} |N_{i,n}^*| \|\nabla [\psi_n]^n\| \|\nabla \xi_n - \nabla \xi\| dxd\tau \\
\leq \int_0^T \|\nabla \xi_n - \nabla \xi\|_{0,0,\mathcal{O}} \|N_{i,n}^*\|_{0,4,\mathcal{O}} \|\nabla [\psi_n]^n\|_{0,4,\Omega_{i,n}} d\tau \\
\leq C(\mathcal{O}, T) \|\xi_n - \xi\|_{1,2,\mathcal{O} \times (0,T)} \\
\times \left( \int_0^T \|N_{i,n}^*\|_{1,2,\mathcal{O}}^2 dt \right)^{1/2} \left( \int_0^T \|\nabla [\psi_n]^n\|_{0,4,\Omega_{i,n}}^2 d\tau \right)^{1/2} \\
\leq C(\mathcal{O}, B_*, L_2, C, T) \|\xi_n - \xi\|_{1,2,\mathcal{O} \times (0,T)} \rightarrow 0, \ n \rightarrow \infty
\]

and

\[
\int_0^T \int_{B_{n,\tau}} N_{i,n}^* |\nabla [\psi_n]^n - \nabla \psi|^2 dxd\tau \\
\leq \int_0^T \left( \int_{B_{n,\tau}} |N_{i,n}^*|^2 |\nabla [\psi_n]^n - \nabla \psi|^2 d\tau \right)^{1/2} \left( \int_{\Omega} |\nabla \xi|^2 dx \right)^{1/2} \\
\leq C(\mathcal{O}, T) \|\xi\|_{1,2,\mathcal{O} \times (0,T)}^3 \\
\times \int_0^T \|N_{i,n}^*\|_{0,16/3,\mathcal{O}} \|\nabla [\psi_n]^n - \nabla \psi\|_{0,4,\Omega_{i,n}}^{3/4} \|\nabla [\psi_n]^n - \nabla \psi\|_{0,2,\mathcal{O}}^{1/4} d\tau \\
\leq C(\mathcal{O}, T) \|\xi\|_{1,2,\mathcal{O} \times (0,T)} \left( \int_0^T \|N_{i,n}^*\|_{1,2,\mathcal{O}}^2 d\tau \right)^{1/2} \\
\times \left( \int_0^T \|\nabla [\psi_n]^n - \nabla \psi\|_{0,2,\mathcal{O}}^2 d\tau \right)^{3/8} \left( \int_0^T \|\nabla [\psi_n]^n - \nabla \psi\|_{0,4,\mathcal{O}}^2 d\tau \right)^{1/8} \\
\leq C(\mathcal{O}, T, B_*, L_2) \|\xi\|_{1,2,\mathcal{O} \times (0,T)} \left( \int_0^T \|\nabla [\psi_n]^n - \nabla \psi\|_{0,2,\mathcal{O}}^2 d\tau \right)^{3/8} \rightarrow 0, \ n \rightarrow \infty.
\]

(5.141)

Analogously,

\[
\int_0^T \int_{B_{n,\tau}} |N_{i,n}^* - N_{i,n}^*| \|\nabla \psi\| \|\nabla \xi\| dxd\tau \rightarrow 0, \ n \rightarrow \infty.
\]
Finally, we have,
\[
\int_0^T \int_{K_{n,t} \setminus A_{n,t}} |N^{*}_i| |\nabla \psi| |\nabla \varsigma| \, dx \, d\tau
\]
\[
\leq \int_0^T |K_{n,t} \setminus A_{n,t}|^{1/2} \|N^{*}_i\|_{0,6,\Omega} \|\nabla \psi\|_{0,4,K_{n,t}} \|\nabla \varsigma\|_{0,2,\Omega} \, d\tau
\]
\[
\leq C(\mathcal{O}, T, B_{*}) \|\varsigma\|_{1,2,\Omega \times (0,T)} \left( \int_0^T |K_{n,t} \setminus A_{n,t}|^{1/6} \, dt \right)^{1/2} \left( \int_0^T \|N^{*}_i\|_{1,2,\mathcal{O}} \, d\tau \right)^{1/2}
\]
\[
\leq C(\mathcal{O}, T, B_{*}, L^2) \|\varsigma\|_{1,2,\Omega \times (0,T)} \left( \int_0^T |K_{n,t} \setminus A_{n,t}|^{1/6} \, dt \right)^{1/2} \rightarrow 0, \ n \rightarrow \infty,
\]
using the dominated convergence theorem. Also,
\[
\int_0^T \int_{K_{n,t} \setminus A_{n,t}} |N^{*}_{i,n}| |\nabla [\psi_n]| |\nabla \varsigma_n| \, dx \, d\tau
\]
\[
\leq \int_0^T \int_{K_{n,t} \setminus A_{n,t}} |N^{*}_{i,n}| |\nabla \psi| - |\nabla \psi_n| |\nabla \varsigma_n| \, dx \, d\tau
\]
\[
+ \int_0^T \int_{K_{n,t} \setminus A_{n,t}} |N^{*}_{i,n}| |\nabla \psi| |\nabla \varsigma_n| \, dx \, d\tau \rightarrow 0, \ n \rightarrow \infty,
\]
using similar arguments as in (5.142) and (5.141). The result follows from (5.135), (5.136), (5.138), (5.139) and (5.140).

\[\square\]

**Lemma 5.5.** The function \( u \) satisfies (3.34), considering \( \psi, N_i \) and \( \mu \) given in (5.3), (5.124) and (5.114), respectively.

**Proof.** The proof follows from (5.124) and from the arguments in [15], observing that: given \( w \in S \) and \( w_n \in S_n \) such that \( \int_0^T \|w_n - w\|_{2,\Omega}^2 \, d\tau \rightarrow 0 \) with \( n \rightarrow \infty \), we obtain
\[
\int_0^T \mathbf{F}_n \cdot w_n \, d\tau - \int_0^T \mathbf{F} \cdot w \, d\tau \rightarrow 0, \ n \rightarrow \infty
\]
for \( \mathbf{F}_n = -e \sum_{i=1}^J \Omega_{i,n} Z_i N^{*}_{i,n} \nabla \psi_n \), \( \mathbf{F} = -e \sum_{i=1}^J Z_i \Omega_i N^{*}_i \nabla \psi \); this follows from analogous estimates as in Lemma 5.4.

\[\square\]

We have obtained \( T > 0 \) and a weak solution for the problem in \((0, T)\). From the bound (4.109), \( Q(t) \) is continuous in \([0, T] \) (see [14]), so that
\[
\lim_{t \to T^-} \text{dist}(K_t, \partial \mathcal{O}) \geq \gamma > 0
\]
and we can iterate the existence result in order to obtain Theorem 3.1.

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