WALL-CROSSING STRUCTURES ON SURFACES

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ABSTRACT. Families of Bridgeland stability conditions induce families of stability data, wall-crossing structures and scattering diagrams on the motivic Hall algebra. These structures can be transferred to the quantum torus if the stability conditions of the family have global dimension at most 2. We show that geometric stability conditions on surfaces with nef anticanonical bundle have global dimension 2 and we study the resulting family of stability data. We formulate a conjecture relating this family to the family of stability data associated with a quiver with potential and we verify this conjecture for the projective plane.

CONTENTS

1. Introduction 2
Plan of the paper 6
Acknowledgments 7
2. Cones and semigroups 7
2.1. Cones 7
2.2. Strict semigroups 9
2.3. Families of strict semigroups 9
2.4. Cones and quadratic forms 11
2.5. Families of stable supports 12
3. Stability conditions 13
3.1. t-structures 14
3.2. Slicings 14
3.3. Bridgeland stability conditions 15
3.4. Stability functions on abelian categories 16
3.5. Families of stability conditions 17
3.6. Action on stability conditions 19
3.7. Phase behavior in families 20
3.8. Global dimension of a slicing 21
4. Stability data 21
4.1. Graded Lie algebras 21
4.2. Examples of graded Lie algebras 22
4.3. Factorizations 23
4.4. Stability data 24
4.5. Families of stability data 25
4.6. Wall-crossing structures 27
5. Wall-crossing formulas 32
5.1. Grothendieck ring of stacks 32
1. Introduction

Bridgeland stability conditions on triangulated categories were introduced in [13] as a mathematical incarnation of the notion of Π-stability in string theory [23, 3]. One can define the notion of a continuous family of stability conditions and consider the universal object in the category of all such families. This universal object is given by the set of all stability conditions equipped with a natural topology. It has a structure of a complex manifold [13].

On the other hand, one can also define the notion of a stability condition on an abelian category [32, 13]. Under appropriate assumptions, one can define moduli spaces and moduli stacks of semistable objects of the category [32, 33, 34, 36, 35, 37] and consider their invariants, known as Donaldson-Thomas (DT) invariants or BPS invariants. The behavior of these invariants is controlled by a formula, called the wall-crossing formula [33, 37], reflecting the fact that DT invariants change their values when stability condition crosses certain codimension one subspaces of the space of all stability conditions.
The above two approaches were combined in [41], where one constructed DT invariants associated to stability conditions on appropriate triangulated 3CY categories. Moreover, in [41] one conceptualized the structure formed by the collection of all DT invariants associated to a given stability condition by introducing the framework of stability data on graded Lie algebras (stability data \((Z, a)\) consists of a central charge \(Z: \Gamma \rightarrow \mathbb{C}\), where \(\Gamma\) is the grading lattice, and a collection \(a = (a_\gamma)_{\gamma \in \Gamma}\) of elements of the Lie algebra). In the case of unrefined DT invariants, the corresponding Lie algebra is the Lie algebra of a Poisson torus, and in the case of motivic DT invariants, the corresponding Lie algebra is the Lie algebra of a quantum torus. One can think about stability data on a graded Lie algebra as an analogue of the notion of a stability condition on a triangulated category. As in the case of stability conditions, one can introduce the notion of a continuous family of stability data on a graded Lie algebra [41] (also called a variation of BPS structures [17, 18]). Importantly, the wall-crossing formula is intrinsically embedded into the notion of continuity of a family of stability data (the Kontsevich-Soibelman wall-crossing formula). Given a continuous family of stability conditions on an appropriate triangulated 3CY category, one can define a continuous family of stability data on the corresponding quantum torus [41].

There exist alternative ways to encode a continuous family of stability data on a graded Lie algebra. One of them is a wall-crossing structure [43] which captures information of stability data along a fixed ray in \(\mathbb{C}\) (usually \(i\mathbb{R}_{>0}\) or \(-\mathbb{R}_{>0}\)). This new structure may carry less information than the original family of stability data unless the central charge rotates in the family. In the case of a wall-crossing structure on a vector space, it can be effectively and intuitively represented by a scattering diagram [43, 40, 29].

![Diagram](image)

While the above hierarchy seems to be rather natural, in some situations (for example, in the case of quivers with potential) there is no need to go through all of the above steps and one can go directly from a continuous family of stability conditions (parametrized by a vector space) to a scattering diagram [28, 16, 52]. The applications of this approach to the theory of cluster algebras are ubiquitous. The theory of generalized scattering diagrams, associated to wall-crossing structures on manifolds, has yet to be developed (cf. [12]).
In this paper we will give a detailed introduction to stability conditions, stability data and relationship between them. We will develop some tools to treat both of the above notions in a unified manner. For example, it was observed in [6, 4] that the support property with respect to a quadratic form can be extended from a stability condition to its neighborhood. The same property for stability data was made one of the axioms in the original definition of a continuous family of stability data [41, §2.3]. We will show that this axiom follows automatically from a weaker support property. In doing so, we introduce the notion of a continuous family of stable supports and prove that the support property for such family with respect to a fixed quadratic form is an open condition. The corresponding statements for stability conditions and stability data then follow immediately.

It was mentioned above that one can associate stability data with a stability condition on a triangulated 3CY category [41]. On the other hand, one can associate stability data with a stability condition on an abelian category of homological dimension one [33]. The reason for these restrictions on a category is that the precursor of the wall-crossing formula, the Harder-Narasimhan identity in the Hall algebra of the corresponding triangulated or abelian category, can not be transferred to the quantum torus in general. In this paper we will show that this still can be done if a stability condition $\sigma = (Z, P)$ has global dimension $\leq 2$. The global dimension is a real invariant of a slicing $\mathcal{P}$ [31, 58] which measures the distance between interacting slices §3.8. We will show that stability data associated with such a stability condition satisfies the wall-crossing formula (Corollary 5.7). We will also show that a continuous family of such stability conditions induces a continuous family of stability data (Theorem 5.9). Morally, these results are similar to [36, §6.4], where one proves a wall-crossing formula for Gieseker-semistable sheaves on a surface having a nef anticanonical bundle.

Our next goal is to find meaningful examples in which the above results can be applied. Given a smooth projective surface $X$, we say that a stability condition $\sigma = (Z, P)$ on $\mathcal{D} = D^b(\text{Coh} X)$ is special geometric if the sheaves $\mathcal{O}_x$, for $x \in X$, are $\sigma$-stable of the same phase and if the discriminant (157) is negative-definite on $\text{Ker} Z$. For any pair of real divisors $\beta, \omega$ with ample $\omega$, one can define some special geometric stability condition $\sigma_{\beta, \omega}$ [2, 14]. We will prove in Theorem 6.10 that every special geometric stability condition is equal to some $\sigma_{\beta, \omega}$ up to the action of $\widetilde{\text{GL}}_2^+ (\mathbb{R})$ on the space of stability conditions (cf. [14]). The space of special geometric stability conditions is an open subset of the space of all stability conditions.

We will prove that if $X$ is a surface having a nef anticanonical divisor, then the global dimensional of any special geometric stability condition is equal to 2 (Theorem 7.7). In the case of $X = \mathbb{P}^2$ this result was proved in [25] based on the method developed in [46]. This method, for a stability condition $\sigma_{\beta, \omega}$, relies on the fact that $K_X$ is proportional to $\omega$ and it doesn’t seem to generalize to other surfaces. In our approach we develop a method to control the behavior of mini/max phases of objects under deformations of stability conditions (Theorem 3.12) and we show that this method can be applied in our situation by proving a new Bogomolov-type inequality for $\sigma_{\beta, \omega}$-semistable objects (Theorem 6.16).
In Theorem 7.2 we prove that given two semistable objects \( E, F \) with \( \phi(E) < \phi(F) \) and a nef line bundle \( L \), we have

\[
(1) \quad \text{Hom}(F \otimes L, E) = 0.
\]

This statement is trivial for Gieseker-semistable sheaves, but it becomes more involved in the case of \( \sigma_{\beta,\omega} \)-semistable objects as the object \( F \otimes L \) is not necessarily \( \sigma_{\beta,\omega} \)-semistable.

The above results imply that for any smooth projective surface \( X \) with a nef anticanonical bundle we have a continuous family of stability data parametrized by the space \( \text{Stab}^+(\mathcal{D}) \) of special geometric stability conditions. Stability conditions contained in the same \( \widetilde{\text{GL}}_2^+(\mathbb{R}) \)-orbit produce equivalent stability data, hence it is enough to consider the family of stability data parametrized by pairs of divisors \((\beta, \omega)\) with ample \( \omega \). In the case of \( X = \mathbb{P}^2 \), one defined in [12] the corresponding scattering diagram by using the interpretation of DT invariants through intersection cohomology (see §7.3). This approach relies on the equation \( \text{Ext}^2(E, E) = 0 \) for semistable objects \( E \), hence one needs to restrict the possible phases (the equation is not true, for example, for \( E = \mathcal{O}_x \)). As we mentioned earlier, the wall-crossing structure (and the corresponding scattering diagram) associated to a continuous family of stability data does not capture in general the whole information of the family as it only preserves stability data along one ray, for every point of the family. On the other hand, this scattering diagram captures a significant part of the family of stability data and it is fully described in [12].

Another topic that we address in this paper is the relationship of the above stability data for surfaces and stability data for quivers with potentials. There exist many examples (see e.g. [8]) of smooth projective surfaces \( X \) with a nef anticanonical bundle such that \( D^b(\text{Coh} X) \) admits a tilting objects \( T' \) that can be lifted to a tilting object \( T \) on the canonical bundle \( Y = \omega_X \) considered as a non-compact 3CY variety (see §8.1). In these examples, the algebra \( \text{End}(T)^{\text{op}} \) can be interpreted as the Jacobian algebra \( J_W \) of some quiver with potential \((Q, W)\), and the algebra \( \text{End}(T')^{\text{op}} \) can be interpreted as the partial Jacobian algebra \( J_{W,I} \) for a certain cut \( I \subset Q_1 \) (see §8.4). By the tilting theory we have

\[
(2) \quad D^b(\text{Coh} X) \simeq D^b(\text{mod } J_{W,I}), \quad D^b_c(\text{Coh} Y) \simeq D^b(\text{mod } J_W),
\]

where \( D^b_c(\text{Coh} Y) \) denotes the derived category of sheaves with compact support. The first equivalence implies that stability conditions and stability data that we constructed earlier, can as well be associated with the algebra \( J_{W,I} \).

For a del Pezzo surface, under appropriate assumptions on stability conditions on \( D^b_c(\text{Coh} Y) \) and the phases of semistable objects, one can expect that these semistable objects are push-forwards of semistable objects in \( D^b(\text{Coh} X) \). Then one can expect that some components of stability data for \( D^b_c(\text{Coh} Y) \) coincide with components of stability data for \( D^b(\text{Coh} X) \). This is not what we are going to do.

Given a quiver with potential \((Q, W)\) and a stability function \( Z : \mathbb{Z}Q_0 \rightarrow \mathbb{C} \), one can define the corresponding stability data \( A_{2}^{Q,W} \) in the quantum torus with coefficients in the (localized) Grothendieck ring of varieties with exponentials §8.4. One could invoke
instead monodromic mixed Hodge structures \[42\], but this would be unnecessary. In the presence of a cut, the coefficients are contained in the usual (localized) Grothendieck ring of varieties. Morally, stability data \(A_{Z}^{Q,W}\) should correspond to some stability condition on the derived 3CY category of DG modules over the Ginzburg DG algebra associated to \((Q,W)\). The above stability data enjoys the wall-crossing formula (Theorem 8.5) and the family of such stability data (parametrized by stability functions \(Z: \mathbb{Z}Q_0 \rightarrow \mathbb{C}\)) is continuous.

Let us assume now that \((Q,W)\) and a cut \(I \subset Q_1\) arise from the tilting theory on a surface \(X\) with a nef anticanonical bundle, as we discussed earlier. Then we have a family of stability data for \(D^b(\text{Coh} X)\) (parametrized by special geometric stability conditions) and a family of stability data for \((Q,W)\) (parametrized by stability functions \(Z: \mathbb{Z}Q_0 \rightarrow \mathbb{C}\)). The quantum tori, where these families live, turn out to be the same. We conjecture that the above families are compatible, meaning that there exists a special geometric stability condition \(\sigma = (Z,P)\) on \(D^b(\text{Coh} X) \simeq D^b(\text{mod} J_{W,I})\) such that its heart is equal to \(\text{mod} J_{W,I}\) (so that we have a stability function \(Z: \mathbb{Z}Q_0 \rightarrow \mathbb{C}\)) and such that stability data \(A_\sigma\) corresponding to \(\sigma\) is equal to \(A_{Z}^{Q,W}\). This conjecture looks somewhat surprising as it gives a strong compatibility of DT invariants on a surface and on its canonical bundle. We verify this conjecture for \(X = \mathbb{P}^2\) in §8.6. In other examples from [8] we checked that, as long as there exists a special geometric stability condition with the heart equal to \(\text{mod} J_{W,I}\), the conjecture is true (see Remark 8.7). This is a manageable problem which will be addressed elsewhere. The above compatibility means that it is enough to know stability data \(A_{Z}^{Q,W}\) for some \(Z\) in order to determine stability data \(A_\sigma\) for all special geometric stability conditions \(\sigma\). A complete (conjectural) description of stability data \(A_{Z}^{Q,W}\) exists for \(\mathbb{P}^2\) and other surfaces [8, 55]. It is fully verified for small dimension vectors.

**Plan of the paper.** In §2.1 and §2.2 we introduce strict cones and strict semigroups and study their properties. This will be needed later in the definition of stability data and wall-crossing formulas. In §2.3 we introduce families of strict semigroups and study the behavior of their lower sets under deformations. In §2.4 we study the relationship between strict cones and quadratic forms. In §2.5 we introduce the notion of a continuous family of stable supports and prove a result about the behavior of such families with respect to quadratic forms. This result has an important applications in the case of stability conditions (Theorem 3.10) and in the case of stability data (Theorem 4.10).

In §3 we give a detailed introduction to Bridgeland stability conditions. In particular, in Theorem 3.10 we give a proof of the fact that the support property with respect to a fixed quadratic form is an open condition and in §3.7 we introduce a method to control the behavior of mini/max phases under deformations.

In §4 we give a detailed introduction to stability data. In particular, in §4.5 we introduce continuous families of stability data with a weakened support axiom and we prove in Theorem 4.10 that our definition is equivalent to the original definition of [41, §2.3]. In §4.6 we introduce wall-crossing structures and discuss their relation to continuous families of stability data.
In §5 we discuss various wall-crossing formulas. In §5.1 we introduce Grothendieck rings of algebraic varieties and stacks and in §5.2 we introduce motivic Hall algebras of exact categories. In §5.3 we prove the usual wall-crossing formula in the motivic Hall algebra for an appropriate stability condition on a triangulated category. In §5.4 we prove the wall-crossing formula in the quantum torus under the assumption that our stability condition has global dimension \( \leq 2 \). In §5.5 and §5.6 we construct continuous families of stability data associated with continuous families of stability conditions having global dimension \( \leq 2 \).

In §6 we study geometric stability conditions on surfaces. In particular, in §6.4 we show that every special geometric stability condition is equal to some \( \sigma_{\beta, \omega} \) up to the action of \( \widetilde{\text{GL}}_2^+ (\mathbb{R}) \) on the space of stability conditions. In §6.6 we discuss the relationship of the large volume limit with Gieseker and twisted stability. In §6.7 we prove a new Bogomolov-type inequality. In §6.8 we discuss alternative ways to parametrize geometric stability conditions.

In §7 we study stability data associated to special geometric stability conditions. In §7.1 we prove various vanishing results for semistable objects with respect to geometric stability conditions and we prove that special geometric stability conditions on surface with a nef anticanonical bundle have global dimension 2. In §7.2 we construct a continuous family of geometric stability data. In §7.3 we discuss a relation of geometric stability data to intersection cohomology of moduli spaces of semistable objects.

In §8 we discuss a relation of geometric stability data to stability data associated to quivers with potentials. In §8.1 we briefly discuss basic tilting theory for smooth projective algebraic varieties and their canonical bundles. In §8.3 we illustrate this approach on the example of \( \mathbb{P}^2 \). In §8.4 we introduce quivers with potentials, their Jacobian algebras, partial Jacobian algebras, and their motivic invariants (stability data). In §8.5 and §8.6 we prove that the family of geometric stability data on \( \mathbb{P}^2 \) and the family of stability data associated to the corresponding quiver with potential are compatible.

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2. Cones and semigroups

Given an abelian group \( M \) and a commutative ring \( R \), we define \( M_R = M \otimes_{\mathbb{Z}} R \). Given a module \( M \) over a commutative ring \( R \), we define \( M^\vee = \text{Hom}_R(M, R) \).

2.1. Cones. Let \( E \) be a real vector space of dimension \( n > 0 \). We equip it with a norm \( ||\cdot|| \). For any subset \( S \subset E \), let \( \text{conv}(S) \) denote its convex hull. By Carathéodory’s theorem, every element of \( \text{conv}(S) \) can be written as a convex combination of \( n+1 \) points in \( S \). This implies that if \( S \subset E \) is compact, then \( \text{conv}(S) \) is also compact.

A subset \( C \subset E \) is called a cone if \( \mathbb{R}_{\geq 0} C \subset C \) and it is called a convex cone if also \( C + C \subset C \). We will say that a cone \( C \) is blunt if \( 0 \notin C \). We define a ray to be a blunt cone of dimension 1. For any subset \( S \subset E \), we define the conical hull \( \text{cone}(S) \) of \( S \) (or the convex cone generated by \( S \)) to be the minimal convex cone that contains \( S \).
Explicitly,

\[
\text{cone}(S) = \text{conv}(\mathbb{R}_{>0}S) = \left\{ \sum_{i=1}^{k} a_i x_i \mid a_i \in \mathbb{R}_{>0}, x_i \in S, k \geq 1 \right\}.
\]

Note that traditionally one also includes 0 in \(\text{cone}(S)\). We will say that a convex cone \(C \subset E\) is strict if its closure \(\bar{C}\) does not contain a line or, equivalently, if \(\bar{C} \setminus \{0\}\) is convex.

**Lemma 2.1.** A convex cone \(C \subset E\) is strict if and only if there exists \(u \in E^\times\) such that

\[
C \subset C_u := \{ x \in E \mid u(x) \geq \|x\| \}.
\]

**Proof.** It is clear that \(C_u\) is a strictly closed cone. Conversely, let \(C\) be a strictly cone. We can assume that it is closed. Then the set \(C_1 = \{ x \in C \mid \|x\| = 1 \} \subset C \setminus \{0\}\) is compact, hence the convex hull \(\text{conv}(C_1) \subset C \setminus \{0\}\) is also compact. By the Hahn-Banach separation theorem, there exists \(u \in E^\times\) and \(\varepsilon > 0\) such that \(u(x) > \varepsilon\) for all \(x \in \text{conv}(C_1)\). This implies that \(\frac{1}{\varepsilon} u(x) \geq \|x\|\) for all \(x \in C\). \(\square\)

**Lemma 2.2.** Let \(Z : E \to F\) be a linear map between two finite-dimensional normed vector spaces. For any strict cone \(V \subset F\) and \(\varepsilon > 0\), consider the convex cone

\[
C(V, Z, \varepsilon) = \text{cone}\{ x \in E \mid Z(x) \in V, \|Z(x)\| \geq \varepsilon \|x\| \}.
\]

Then \(C(V, Z, \varepsilon)\) is strict and there exists \(\varepsilon' > 0\) (independent of \(Z\)) such that

\[
\|Z(x)\| \geq \varepsilon' \|x\| \quad \forall x \in C(V, Z, \varepsilon).
\]

**Proof.** There exists \(0 \neq u \in F^\times\) such that \(V \subset C_u = \{ y \in F \mid u(y) \geq \|y\| \}\). If \(Z(x) \in V\) and \(\|Z(x)\| \geq \varepsilon \|x\|\), then \(uZ(x) \geq \|Z(x)\| \geq \varepsilon \|x\|\). This implies that \(C(V, Z, \varepsilon)\) is contained in the strict cone \(C' = \{ x \in E \mid uZ(x) \geq \varepsilon \|x\| \}\). Let \(\|u\| > 0\) be the operator norm of the bounded operator \(u : F \to \mathbb{R}\). For any \(x \in C'\), we have \(\varepsilon \|x\| \leq |uZ(x)| \leq \|u\| \cdot \|Z(x)\|\) and we choose \(\varepsilon' = \varepsilon \|u\|^{-1}\). \(\square\)

Note that for a ray \(\ell \subset F\), we have

\[
C(\ell, Z, \varepsilon) = \{ x \in E \mid Z(x) \in \ell, \|Z(x)\| \geq \varepsilon \|x\| \}
\]
as the right hand side is automatically convex.

**Lemma 2.3.** Let \(C \subset E\) be a strict cone. Then there exists \(c > 0\) such that

\[
\sum \|x_i\| \leq c \left\| \sum x_i \right\| \quad \forall x_1, \ldots, x_k \in C.
\]

**Proof.** Let \(u \in E^\times\) be such that \(C \subset C_u\). For any \(x_1, \ldots, x_k \in C\), we have

\[
\sum \|x_i\| \leq \sum u(x_i) \leq \|u\| \cdot \left\| \sum x_i \right\|,
\]
where \(\|u\|\) is the operator norm of \(u : E \to \mathbb{R}\). \(\square\)

**Example 2.4.** Let \(V \subset \mathbb{C}\) be a strict cone of angle \(\theta < \frac{\pi}{2}\). Then

\[
\sum |x_i| \leq \frac{1}{\cos(\theta)} \left\| \sum x_i \right\| \quad \forall x_1, \ldots, x_k \in V.
\]

Note also that \(|x_j| \leq |\sum_i x_i|\) for any \(1 \leq j \leq k\).
2.2. **Strict semigroups.** Let \( S \) be a commutative semigroup, written additively. We define an ideal of \( S \) to be a subset \( I \subset S \) such that \( I + S \subset I \). For any subset \( X \subset S \), we define the semigroup generated by \( X \) to be

\[
\text{sgr}(X) = \{ a_1 + \cdots + a_k \mid a_i \in X, k \geq 1 \} \subset S.
\]

If \( S \) has cancellation property, meaning that \( a + b = a + c \) implies \( b = c \), then we can embed \( S \) into an abelian group generated by \( S \) (the Grothendieck group of \( S \)). In what follows we will always assume that \( S \) has cancellation property and satisfies \( S \cap (-S) \subset \{0\} \). In this case we define the partial order on \( S \)

\[
a \leq b \iff b - a \in S \cup \{0\}.
\]

A subset \( I \subset S \) is an ideal if and only if it is an upper set of the poset \( S \), meaning that \( a \leq b \) and \( a \in I \) imply \( b \in I \). Similarly, a subset \( I \subset S \) is the complement of an ideal if and only if it is a lower set of the poset \( S \), meaning that \( a \leq b \) and \( b \in I \) imply \( a \in I \).

Let \( \Gamma \) be a free abelian group of finite rank and \( S \subset \Gamma \) be a semigroup. We will say that \( S \) is strict if it generates a strict convex cone in \( \Gamma_\RR \). Then \( S \) has automatically the cancellation property and satisfies \( S \cap (-S) \subset \{0\} \). Recall that a map between topological spaces is called proper if the preimage of every compact set is compact. In particular, a map \( u: S \to \RR \) (where \( S \) has discrete topology) is proper if and only if the preimage of every bounded set in \( \RR \) is finite.

**Lemma 2.5.** Let \( S \subset \Gamma \) be a strict semigroup without zero. Then

1. There exists an additive proper map \( u: S \to \RR \) such that \( u(S) \subset \RR_{>0} \).
2. For any \( a \in S \), the lower set \( \downarrow a = \{ b \in S \mid b \leq a \} \) is finite.
3. For any \( a \in S \), the set \( \{ (a_1, \ldots, a_k) \in S^k \mid \sum a_i = a, k \geq 1 \} \) is finite.

**Proof.** Let \( \|\cdot\| \) be a norm on \( \Gamma_\RR \). By Lemma 2.1, there exists a linear map \( u: \Gamma \to \RR \) such that \( u(a) \geq \|a\| \) for all \( a \in S \). This map satisfies the required properties. The other statements of the lemma follow immediately. \( \Box \)

Note that if \( u: S \to \RR \) is an additive proper map such that \( u(S) \subset \RR_{>0} \), then the subset

\[
I_N = \{ a \in S \mid u(a) \leq N \}, \quad N \geq 1,
\]

is a finite lower set. Moreover, every finite lower subset of \( S \) is contained in \( I_N \) for some \( N \geq 1 \).

2.3. **Families of strict semigroups.** Let as before \( \Gamma \) be a free abelian group of finite rank and let \( \|\cdot\| \) be a norm on \( \Gamma_\RR \). For any strict cone \( V \subset \CC \), \( Z \in \Gamma_\CC^\vee = \text{Hom}(\Gamma, \CC) \) and \( \varepsilon > 0 \), we define the semigroup

\[
S(V, Z, \varepsilon) = \text{sgr} \{ a \in \Gamma \mid Z(a) \in V, |Z(a)| \geq \varepsilon \|a\| \} \subset \Gamma
\]

which is strict by Lemma 2.2. Note that for a ray \( \ell \subset \CC \), we have

\[
S(\ell, Z, \varepsilon) = \{ a \in \Gamma \mid Z(a) \in \ell, |Z(a)| \geq \varepsilon \|a\| \}.
\]
Lemma 2.6. Let $M$ be a topological space, $Z: M \to \Gamma_C$ be a continuous map and $x \in M$. Let $\varepsilon > 0$ and $V \subset \mathbb{C}$ be a strict cone such that

$$Z_x(\Gamma) \cap \partial V = \{0\}, \quad |Z_x(a)| \neq \varepsilon \|a\| \quad \forall 0 \neq a \in \Gamma.$$ 

Then, for any finite set $D \subset \Gamma$, there exists an open set $x \in U \subset M$ such that

$$D \cap S(V, Z_x, \varepsilon) = D \cap S(V, Z_y, \varepsilon) \quad \forall y \in U.$$

Proof. It is enough to assume that $D = \{\gamma\}$. Let us define

$$S^\varepsilon(V, Z, \varepsilon) = \{a \in \Gamma \mid Z(a) \in V, |Z(a)| \geq \varepsilon \|a\|\}$$

so that $S(V, Z, \varepsilon)$ is the semigroup generated by $S^\varepsilon(V, Z, \varepsilon)$. It follows from our assumptions that for any finite set $D' \subset \Gamma$, there exists an open set $x \in U \subset M$ such that

$$D' \cap S^\varepsilon(V, Z_x, \varepsilon) = D' \cap S^\varepsilon(V, Z_y, \varepsilon) \quad \forall y \in U.$$

If $\gamma \in S(V, Z_x, \varepsilon)$, then $\gamma = \sum a_i$ with $a_i \in S^\varepsilon(V, Z_x, \varepsilon)$. By the previous remark, there exists an open set $x \in U \subset M$ such that $\{a_i\} \subset S^\varepsilon(V, Z_y, \varepsilon)$ for all $y \in U$. This implies that $\gamma \in S(V, Z_y, \varepsilon)$ for all $y \in U$, proving one of the required inclusions.

Let us prove the other inclusion. There exists an open set $x \in U \subset M$ such that $\|Z_x - Z_y\| \leq \frac{1}{2} \varepsilon$ for all $y \in U$. If $|Z_y(a)| \geq \varepsilon \|a\|$, then $|Z_x(a)| \leq \frac{1}{2} \varepsilon \|a\|$. Therefore

$$S^\varepsilon(V, Z_y, \varepsilon) \subset S^\varepsilon(V, Z_x, \frac{1}{2} \varepsilon) \subset S(V, Z_x, \frac{1}{2} \varepsilon).$$

By Lemma 2.3, there exists $c > 0$ such that $\sum |a_i| \leq c \|\sum a_i\|$ for all $a_i \in S(V, Z_x, \frac{1}{2} \varepsilon)$. Let us consider the finite set $D' = \{a \in \Gamma \mid \|a\| \leq c \|\gamma\|\}$. Then we can shrink $U$ so that $D' \cap S^\varepsilon(V, Z_x, \varepsilon) = D' \cap S^\varepsilon(V, Z_y, \varepsilon)$ for all $y \in U$.

If $\gamma \in S(V, Z_y, \varepsilon)$ for some $y \in U$, then $\gamma = \sum a_i$ with $a_i \in S^\varepsilon(V, Z_y, \varepsilon) \subset S(V, Z_x, \frac{1}{2} \varepsilon)$, hence $\|a_i\| \leq c \|\sum a_i\| = c \|\gamma\|$ and $a_i \in D'$. This implies that $a_i \in S^\varepsilon(V, Z_x, \varepsilon)$, hence $\gamma \in S(V, Z_x, \varepsilon)$. \qed

Lemma 2.7. Let $M$ be a topological space, $Z: M \to \Gamma_C$ be a continuous map and $x \in M$. Let $\varepsilon > 0$ and $V \subset \mathbb{C}$ be a strict cone such that

$$Z_x(\Gamma) \cap \partial V = \{0\}, \quad |Z_x(a)| \neq \varepsilon \|a\| \quad \forall 0 \neq a \in \Gamma.$$ 

Then, for any finite lower set $I \subset S(V, Z_x, \varepsilon)$, there exists an open set $x \in U \subset M$ such that $I \subset S(V, Z_y, \varepsilon)$ is a lower set for all $y \in U$.

Proof. By the previous result, we can assume that $I \subset S(V, Z_y, \varepsilon)$ for all $y \in U$. Using the same argument as before, we can assume that

$$S(V, Z_y, \varepsilon) \subset S(V, Z_x, \frac{1}{2} \varepsilon) \quad \forall y \in U$$

and $c > 0$ is such that $\sum |a_i| \leq c \|\sum a_i\|$ for all $a_i \in S(V, Z_x, \frac{1}{2} \varepsilon)$. Let us consider the finite set

$$D = \bigcup_{\gamma \in I} \{a \in \Gamma \mid \|a\| \leq c \|\gamma\|\}.$$ 

By the previous result, we can assume that $D \cap S(V, Z_y, \varepsilon) = D \cap S(V, Z_x, \varepsilon)$ for all $y \in U$.

We need to show that for all $y \in U$ the subset $I \subset S(V, Z_y, \varepsilon)$ is a lower set, meaning that if $\gamma = a_1 + a_2 \in I$ and $a_i \in S(V, Z_y, \varepsilon)$, then $a_i \in I$. We have $a_i \in S(V, Z_y, \varepsilon) \subset$
$S(V, Z_x, \frac{1}{2}\varepsilon)$, hence $\|a_i\| \leq c\|\gamma\|$ and $a_i \in D \cap S(V, Z_y, \varepsilon) = D \cap S(V, Z_x, \varepsilon)$. As $I \subset S(V, Z_x, \varepsilon)$ is a lower set, we conclude that $a_i \in I$. □

**Remark 2.8.** We don’t know if a similar statement is true if one uses semigroups $C(V, Z_x, \varepsilon) \cap \Gamma$ instead of $S(V, Z_x, \varepsilon)$.

### 2.4. Cones and quadratic forms.

**Lemma 2.9.** [cf. [41, 6]] Let $Z: E \to F$ be a linear map between finite-dimensional real vector spaces, $Q: E \to \mathbb{R}$ be a quadratic form, negative semi-definite on $\text{Ker} Z$, and $\ell \subset F$ be a ray. Then the cone

$$C(\ell, Z, Q) = \{ x \in E \mid Z(x) \in \ell, Q(x) \geq 0 \}$$

is convex and $Q(x + y) \geq Q(x) + Q(y)$ for all $x, y \in C(\ell, Z, Q)$. This cone is strict if $Q$ is negative-definite on $\text{Ker} Z$.

**Proof.** It is enough to show that $Q(x + y) \geq Q(x) + Q(y)$ for all $x, y \in C(\ell, Z, Q)$. There exists $\lambda > 0$ such that $y - \lambda x \in \text{Ker} Z$. Let $q$ be the symmetric bilinear form corresponding to $Q$. Then $2\lambda q(x, y) = \lambda^2 Q(x) + Q(y) - Q(y - \lambda x) \geq 0$. Therefore $q(x, y) \geq 0$, hence $Q(x + y) \geq Q(x) + Q(y)$ as required. The last statement can be proved directly or by applying Lemma 2.11. □

**Corollary 2.10.** Let $Z: E \to F$ be a linear map between finite-dimensional real vector spaces, $Q: E \to \mathbb{R}$ be a quadratic form, negative semi-definite on $\text{Ker} Z$, and $\ell \subset F$ be a ray. Then the cone

$$C^+(\ell, Z, Q) = \{ x \in E \mid Z(x) \in \ell, Q(x) > 0 \}$$

is convex.

**Lemma 2.11.** Let $Z: E \to F$ be a linear map between finite-dimensional real vector spaces, $Q: E \to \mathbb{R}$ be a quadratic form, negative-definite on $\text{Ker} Z$, and $V \subset F$ be a strict cone. Then the convex cone

$$C(V, Z, Q) = \text{cone} \{ x \in E \mid Z(x) \in V, Q(x) \geq 0 \}$$

is strict.

**Proof.** Let $K = \text{Ker} Z$ and $K^\perp \subset E$ be its orthogonal complement with respect to $Q$. Then $E = K \oplus K^\perp$ and $Z: K^\perp \to F$ is injective. Let $F$ be equipped with an inner product and the induced norm. There exists $\varepsilon > 0$ such that $\varepsilon Q(y) < \|Z(y)\|^2$ for all $0 \neq y \in K^\perp$. We define the norm $\|\cdot\|$ on $E$ by

$$\|x\|^2 = \|Z(x)\|^2 - \varepsilon Q(x).$$

It is indeed a norm: for any $x \in K$ and $y \in K^\perp$ we have

$$\|x + y\|^2 = \|Z(y)\|^2 - \varepsilon Q(x) - \varepsilon Q(y) \geq -\varepsilon Q(x) \geq 0$$

and the equality is only possible for $x = y = 0$. For $x \in E$, we have $Q(x) \geq 0$ if and only if $\|x\| \leq \|Z(x)\|$. As $V$ is strict, there exists $u \in F^\vee$ such that $V \subset C_u = \{ y \in F \mid u(y) \geq \|y\| \}$. If $Z(x) \in V$ and $Q(x) \geq 0$, then $uZ(x) \geq \|Z(x)\| \geq \|x\|$. Therefore $C(V, Z, Q)$ is contained in the strict cone $\{ x \in E \mid uZ(x) \geq \|x\| \}$. □
Given a linear map $Z : E \to F$, we will say that a subset $S \subset E$ satisfies the support property (with respect to $Z$) if there exist norms on $E$ and $F$ such that

$$S \subset \{ x \in E \mid \|Z(x)\| \geq \|x\| \}. \quad (16)$$

**Remark 2.12.** The support property is equivalent to the condition that the following function on $\text{Hom}(E, F)$ defines a norm (cf. Lemma 3.4)

$$\|u\|_{Z, S} := \sup_{x \in S \setminus \{0\}} \frac{\|u(x)\|}{\|Z(x)\|}, \quad u \in \text{Hom}(E, F).$$

**Lemma 2.13.** A subset $S \subset E$ satisfies the support property if and only if there exists a quadratic form $Q$ on $E$ such that

1. $Q$ is negative definite on $\text{Ker} Z$.
2. For any $x \in S$, we have $Q(x) \geq 0$.

**Proof.** Let us assume that $S$ satisfies the support property. We can assume that the norms on $E$ and $F$ are induced by inner products. Then we define the quadratic form

$$Q(x) = \|Z(x)\|^2 - \|x\|^2$$

which satisfies the required properties.

Conversely, let us assume that $Q$ satisfies the required properties. Let $K = \text{Ker} Z$ and $K^\perp \subset E$ be its orthogonal complement with respect to $Q$. Then $E = K \oplus K^\perp$ and $Z : K^\perp \to F$ is injective. Let $F$ be equipped with an inner product and the induced norm. There exists $\varepsilon > 0$ such that $\varepsilon Q(y) < \|Z(y)\|^2$ for all $0 \neq y \in K^\perp$. We define the norm on $E$ by (cf. Lemma 2.11)

$$\|x\|^2 = \|Z(x)\|^2 - \varepsilon Q(x).$$

For $x \in S$, we have $Q(x) \geq 0$, hence $\|Z(x)\| \geq \|x\|$.

\[ \square \]

### 2.5. Families of stable supports

Let $\Gamma$ be a free abelian group of finite rank and $\| \cdot \|$ be a norm on $\Gamma \mathbb{R}$. We define a stable support to be a triple $(Z, r, s)$, where $Z : \Gamma \to \mathbb{C}$ is a linear map and $r \subset s \subset \Gamma \setminus \{0\}$ are subsets satisfying for some $\varepsilon > 0$ (cf. (16))

$$s \subset \{ \gamma \in \Gamma \mid \|Z(\gamma)\| \geq \varepsilon \|\gamma\| \} \quad (17)$$

and such that for every ray $\ell \subset \mathbb{C}$, we have

$$s \cap Z^{-1}(\ell) \subset \text{sgn}(r \cap Z^{-1}(\ell)). \quad (18)$$

Intuitively, one can think about $r$ as the set of classes of stable objects and about $s$ as the set of classes of semistable objects.

Let $M$ be a topological space. We define a continuous family of stable supports on $M$ to be a collection of stable supports $(Z_x, r_x, s_x)_{x \in M}$ such that $Z : M \to \Gamma^\vee \mathbb{C}$ is continuous and

1. For any $x \in M$, there exists $\varepsilon > 0$ and an open set $x \in U \subset M$ such that

   $$s_y \subset \{ \gamma \in \Gamma \mid \|Z_y(\gamma)\| \geq \varepsilon \|\gamma\| \} \quad \forall y \in U.$$

2. For any $\gamma \in \Gamma$, the set $\{ x \in M \mid \gamma \in r_x \}$ is open.
3. For any $\gamma \in \Gamma$, the set $\{ x \in M \mid \gamma \in s_x \}$ is closed.
Theorem 2.14. Let $M$ be path-connected and $(Z, r, s)$ be a continuous family of stable supports on $M$. Let $Q: \Gamma_{\mathbb{R}} \to \mathbb{R}$ be a quadratic form that is negative semi-definite on $\text{Ker} Z_x$ for all $x \in M$ and such that there exists $x \in M$ satisfying $Q(\gamma) \geq 0$ for all $\gamma \in s_x$. Then we have $Q(\gamma) \geq 0$ for all $\gamma \in s_y$ and $y \in M$.

Proof. By Lemma 2.9 it is enough to show that $Q(\gamma) \geq 0$ for all $\gamma \in r_y$ and $y \in M$. We can assume that $M = [0, 1]$ and that $Q(\gamma) \geq 0$ for all $\gamma \in s_1$. As $M$ is compact, there exists $\varepsilon > 0$ such that

$$s_t \subset \{ \gamma \in \Gamma \mid |Z_t(\gamma)| \geq \varepsilon \|\gamma\| \}$$

for all $t \in M$. Let $0 < \theta < \pi/4$. By dividing $[0, 1]$ into a finite union of intervals, we can assume that $|Z_t - Z_1| \leq \theta \sin(\theta)$ for all $t \in [0, 1]$. For any $\gamma \in s_t$, we have $|Z_t(\gamma) - Z_1(\gamma)| \leq \theta \|\gamma\| \leq \theta |Z_1(\gamma)|$, hence

$$|Z_1(\gamma)| \geq \frac{1}{2} |Z_t(\gamma)| \geq \frac{\varepsilon}{2} \|\gamma\|.$$ 

On the other hand, if $\gamma \in \Gamma \setminus \{0\}$ satisfies $|Z_1(\gamma)| \geq \frac{\varepsilon}{2} \|\gamma\|$, then

$$|Z_t(\gamma) - Z_1(\gamma)| \leq \theta \sin(\theta) \|\gamma\| \leq \sin(\theta) |Z_1(\gamma)|,$$

hence the angle between $Z_t(\gamma)$ and $Z_1(\gamma)$ is $\leq \theta$. If $\gamma = \gamma_1 + \gamma_2$ with $Z_t(\gamma_i)$ contained in the same ray and $|Z_1(\gamma_i)| \geq \frac{\varepsilon}{2} \|\gamma_i\|$, then the angle between $Z_t(\gamma_i)$ and $Z_1(\gamma_i)$ is $\leq \theta$, hence the angle between $Z_1(\gamma_1)$ and $Z_1(\gamma_2)$ is $\leq 2\theta < \pi/2$. This implies that

$$\frac{\varepsilon}{2} \|\gamma_i\| \leq |Z_1(\gamma_i)| < |Z_1(\gamma)|,$$

for a fixed $\gamma \in r_0$, we can assume by induction that the result is true for all elements of the finite set

$$D = \left\{ \gamma_1 \in \bigcup_t s_t \mid \frac{\varepsilon}{2} \|\gamma_1\| \leq |Z_1(\gamma_1)| < |Z_1(\gamma)| \right\}.$$ 

Let us choose maximal $t > 0$ such that $\gamma \in r_{t'}$ for all $t' \in [0, t)$. Then $\gamma \in s_t$. If $t = 1$, then we are done. Otherwise, we have $\gamma \in s_t \setminus r_t$, hence we can decompose $\gamma = \sum_{i=1}^n \gamma_i$, where $\gamma_i \in r_t \cap Z^{-1}(\ell)$, $\ell = \mathbb{R}_{>0} Z_t(\gamma)$ and $n \geq 2$. By the above discussion, we have $\gamma_i \in D$, hence $Q(\gamma_i) \geq 0$ by inductive assumption. By Lemma 2.9 we conclude that $Q(\gamma) \geq 0$. 

Corollary 2.15. Let $M$ be locally path-connected and $(Z, r, s)$ be a continuous family of stable supports on $M$. Let $Q: \Gamma_{\mathbb{R}} \to \mathbb{R}$ be a quadratic form and $x \in M$ be such that $Q$ is negative-definite on $\text{Ker} Z_x$ and $Q(\gamma) \geq 0$ for all $\gamma \in s_x$. Then we have $Q(\gamma) \geq 0$ for all $\gamma \in s_y$ and $y$ in some neighborhood of $x$.

Proof. There exists an open subset $x \in U \subset M$ such that $Q_x$ is negative-definite on $\text{Ker} Z_y$ for all $y \in U$. We can assume that $U$ is path-connected. Now we apply the previous result. 

3. Stability conditions

Given a triangulated category $\mathcal{D}$ and a family of subcategories (or objects) $(\mathcal{A}_i)_{i \in I}$ in $\mathcal{D}$, we denote by $(\mathcal{A}_i : i \in I)$ the minimal full subcategory of $\mathcal{D}$ that contains all $\mathcal{A}_i$ and is closed under extensions. For two subcategories $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{D}$, we will write $\text{Hom}(\mathcal{A}_1, \mathcal{A}_2) = 0$ if $\text{Hom}(X, Y) = 0$ for all $X \in \mathcal{A}_1, Y \in \mathcal{A}_2$. Given two objects
$X, Y \in \mathcal{D}$ and $n \in \mathbb{Z}$, we will sometimes denote $\text{Hom}(X, Y[n])$ by $\text{Hom}^n(X, Y)$. For any morphism $f : X \to Y$ in $\mathcal{D}$, we will write cone($f$) for the third object (unique up to an isomorphism) of the corresponding distinguished triangle. Sometimes we will use the word triangle for a distinguished triangle.

3.1. **t-structures.** A *t-structure* on a triangulated category $\mathcal{D}$ is a pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of full subcategories in $\mathcal{D}$ such that, using $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[−n]$ and $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[−n]$ for $n \in \mathbb{Z}$, we have

1. $\text{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$.
2. $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1}$.
3. For any $X \in \mathcal{D}$, there exists a distinguished triangle $X' \to X \to X'' \to$ with $X' \in \mathcal{D}^{\leq 0}$ and $X'' \in \mathcal{D}^{\geq 1}$.

We have [9]

\begin{align}
\mathcal{D}^{\geq 1} &= (\mathcal{D}^{\leq 0})^\perp := \{ Y \in \mathcal{D} \mid \text{Hom}(\mathcal{D}^{\leq 0}, Y) = 0 \}, \\
\mathcal{D}^{\leq 0} &= \perp (\mathcal{D}^{\geq 1}) := \{ X \in \mathcal{D} \mid \text{Hom}(X, \mathcal{D}^{\geq 1}) = 0 \},
\end{align}

hence a t-structure is uniquely determined by the subcategory $\mathcal{D}^{\leq 0}$. The category $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is abelian and is called the *heart* of the t-structure [9]. A t-structure is called *bounded* if every object in $\mathcal{D}$ is contained in $\mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq n}$ for some $n \geq 0$. In this case the t-structure is uniquely determined by its heart, namely,

\begin{equation}
\mathcal{D}^{\leq 0} = \langle \mathcal{A}[n] : n \geq 0 \rangle, \quad \mathcal{D}^{\geq 0} = \langle \mathcal{A}[n] : n \leq 0 \rangle.
\end{equation}

3.2. **Slicings.** We define a slicing $\mathcal{P}$ of a triangulated category $\mathcal{D}$ to be a collection $(\mathcal{P}_\phi)_{\phi \in \mathbb{R}}$ of full subcategories in $\mathcal{D}$ such that

1. $\mathcal{P}_{\phi+1} = \mathcal{P}_\phi[1]$ for all $\phi \in \mathbb{R}$.
2. $\text{Hom}(\mathcal{P}_\phi, \mathcal{P}_{\phi'}) = 0$ for all $\phi > \phi'$.
3. For any $0 \neq E \in \mathcal{D}$, there exists a sequence of maps

$$0 = E_0 \to E_1 \to \cdots \to E_n = E$$

such that cone($E_{k-1} \to E_k$) $\in \mathcal{P}_{\phi_k}$ for some $\phi_k \in \mathbb{R}$ satisfying $\phi_1 > \cdots > \phi_n$.

We will call the above sequence the Harder-Narasimhan (HN) filtration of $E$ and we will address axiom (3) as the HN property. The HN filtration of $E$ is uniquely determined up to isomorphism. We define $\phi^+_\mathcal{P}(E) = \phi_1$ and $\phi^-_\mathcal{P}(E) = \phi_n$. For any interval $I \subset \mathbb{R}$, let

\begin{equation}
\mathcal{P}_I = \langle \mathcal{P}_\phi : \phi \in I \rangle \subset \mathcal{D}.
\end{equation}

Note that traditionally one denotes $\mathcal{P}_\phi$ by $\mathcal{P}(\phi)$ and $\mathcal{P}_I$ by $\mathcal{P}(I)$. The categories

\begin{equation}
\mathcal{D}^{\leq 0} = \mathcal{P}_{>0} = \mathcal{P}_{(0, +\infty)}, \quad \mathcal{D}^{\geq 0} = \mathcal{P}_{\leq 1} = \mathcal{P}_{(-\infty, 1]}
\end{equation}

form a bounded t-structure with the heart $\mathcal{A} = \mathcal{P}_{(0, 1]}$, called the *heart* of $\mathcal{P}$ [13]. The categories $\mathcal{P}_\phi$ are also abelian [13, Lemma 5.2]. For any $a \in \mathbb{R}$, we define a new slicing $\mathcal{P}[a]$

\begin{equation}
\mathcal{P}[a]_\phi = \mathcal{P}_{\phi+a}, \quad \phi \in \mathbb{R}.
\end{equation}

Its heart is equal to $\mathcal{P}_{(a, a+1]}$. Note that $\mathcal{P}[n]_\phi = \mathcal{P}_{\phi+n} = \mathcal{P}_\phi[n]$ for $n \in \mathbb{Z}$.
Example 3.1. Given a full subcategory $\mathcal{A} \subset \mathbb{D}$, let us define $\mathcal{P}_n = \mathcal{A}[n - 1]$ for $n \in \mathbb{Z}$ and $\mathcal{P}_n = 0$ for $n \in \mathbb{R} \setminus \mathbb{Z}$. Then $\mathcal{A}$ is the heart of a bounded $t$-structure if and only if $\mathcal{P} = (\mathcal{P}_n)_{n \in \mathbb{R}}$ is a slicing (cf. [13, Lemma 3.2]). The heart of $\mathcal{P}$ is equal to $\mathcal{P}_1 = \mathcal{A}$.

3.3. Bridgeland stability conditions. Let $\mathbb{D}$ be a triangulated category, $\Gamma$ be a free abelian group of finite rank and $\text{cl} : K(\mathbb{D}) \to \Gamma$ be a linear map. We define a pre-stability condition on the triangulated category $\mathbb{D}$ (with respect to $\text{cl}$) to be a pair $\sigma = (Z, \mathcal{P})$, where

1. $Z : \Gamma \to \mathbb{C}$ is a linear map, called a central charge.
2. $\mathcal{P}$ is a slicing such that for any $0 \neq E \in \mathcal{P}_\phi$, we have $Z(E) := Z(\text{cl} E) \in \mathbb{R}_{>0} e^{i \pi \phi}$.

Nonzero objects of $\mathcal{P}_\phi$ are called $\sigma$-semistable objects of phase $\phi$. Simple objects of $\mathcal{P}_\phi$ are called $\sigma$-stable objects of phase $\phi$. The heart of $\mathcal{P}$ is also called the heart of the stability condition $\sigma$. For any $0 \neq E \in \mathbb{D}$, we will denote $\phi^+(\phi)(E)$ by $\phi^+(\phi)(E)$. Assuming that $\phi^+(\phi) - \phi^-(\phi) < 1$, we have $Z(E) \neq 0$ and we define the phase $\phi^+(\phi)$ to be the unique value such that

\[ \phi^+(\phi)(E) \in [\phi^-(\phi)(E), \phi^+(\phi)(E)], \quad Z(E) \in \mathbb{R}_{>0} e^{i \pi \phi^+(\phi)}. \]

We define the support of $\sigma$ to be

\[ \text{supp}(\sigma) = \{ \text{cl} E \mid E \text{ is } \sigma\text{-semistable} \} \subset \Gamma. \]

Let us equip $\Gamma_R = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ with a norm $\|\cdot\|$. A pre-stability condition $\sigma = (Z, \mathcal{P})$ on $\mathbb{D}$ is called a stability condition if there exists $\varepsilon > 0$ such that we have (the support property)

\[ |Z(E)| \geq \varepsilon \| \text{cl} E \| \]

for all $\sigma$-semistable objects $E$. It is enough to verify this property for $E \in \mathcal{P}_\phi$ with $\phi \in (0, 1]$.

Lemma 3.2. A pre-stability condition $\sigma = (Z, \mathcal{P})$ satisfies the support property if and only if there exists a quadratic form $Q$ on $\Gamma_R$ such that

1. $Q$ is negative definite on $\text{Ker}(Z : \Gamma_R \to \mathbb{C})$.
2. For any $\sigma$-semistable object $E$ we have $Q(\text{cl} E) \geq 0$.

Proof. See Lemma 2.13. \qed

Remark 3.3. Note that if $\text{rk} \Gamma = n$, then a quadratic form $Q$ as above has negative index $\geq n - 2$. If $Q$ has signature $(2, n - 2)$, then $\text{dim} \text{Ker}(Z) = n - 2$ and $Z : \Gamma_R \to \mathbb{C}$ is automatically surjective.

Lemma 3.4. A pre-stability condition $\sigma = (Z, \mathcal{P})$ satisfies the support property if and only if the following map defines a norm on $\Gamma_R^\vee = \text{Hom}(\Gamma, \mathbb{C})$

\[ \| \cdot \| : \Gamma_R^\vee \to [0, +\infty], \quad u \mapsto \sup \left\{ \frac{|u(E)|}{|Z(E)|} \left| E \text{ is } \sigma\text{-semistable} \right\} \]

Proof. The above map satisfies all properties of a norm, except that it may be not finite. Let $S = \{ \text{cl} E \mid 0 \neq E \in \mathcal{P}_\phi, \phi \in \mathbb{R} \} \subset \Gamma$. If $\sigma$ satisfies the support property, then there exists $\varepsilon > 0$ such that $|Z(x)| \geq \varepsilon \| x \|$ for all $x \in S$. A linear map $u : \Gamma_R \to \mathbb{C}$ is bounded,
meaning that there exists \( C > 0 \) such that \( |u(x)| \leq C \|x\| \) for all \( x \in \Gamma_R \). This implies that \( |u(x)| \leq \frac{C}{C} |Z(x)| \) for all \( x \in S \), hence \( \|u\|_\sigma \leq \frac{C}{C} \).

Conversely, we can assume that the norm on \( \Gamma_R \) is given by \( \|x\| = \sum_i |u_i(x)| \) for some basis \((u_1, \ldots, u_n)\) of \( \Gamma_R \). For any \( x \in S \), we have
\[
\|x\| = \sum_i |u_i(x)| \leq \sum_i \|u_i\|_\sigma \cdot |Z(x)| \leq \max_i \|u_i\|_\sigma \cdot |Z(x)| .
\]
This proves the support property. \( \square \)

**Remark 3.5.** In our applications the group \( \Gamma \) will be the numerical Grothendieck group of \( D \). More precisely, let us assume that \( \bigoplus_{n \in \mathbb{Z}} \text{Hom}^n(E, F) \) is finite-dimensional for all \( E, F \in D \) and let \( \chi(E, F) = \sum_{n \in \mathbb{Z}} (-1)^n \dim \text{Hom}^n(E, F) \) be the Euler form of \( D \) defined on the Grothendieck group \( K(D) \). Assuming that \( D \) has a Serre functor, meaning an automorphism \( S : D \to D \) such that there are natural isomorphisms \( \text{Hom}(E, F) \simeq \text{Hom}(F, SE)^\vee \) for all \( E, F \in D \), we conclude that the right and the left kernels of \( \chi \) are equal and we define the numerical Grothendieck group \( N(\mathcal{D}) \) to be the quotient of \( K(D) \) by this kernel. We define the linear map \( c_\ell : K(D) \to \Gamma = N(\mathcal{D}) \) to be the projection.

### 3.4. Stability functions on abelian categories.

Let \( \mathcal{A} \subset D \) be the heart of a bounded t-structure so that \( K(\mathcal{A}) \simeq K(D) \). We define a *stability function* on \( \mathcal{A} \) (with respect to \( c_\ell : K(\mathcal{A}) \to \Gamma \)) to be a linear map \( \phi : \Gamma \to \mathbb{C} \) such that for any \( 0 \neq E \in \mathcal{A} \), the number \( Z(E) := Z(c_\ell E) \) is contained in the (semi-closed) upper-half plane
\[
\mathbb{H} = \left\{ r e^{i \pi \phi} \mid r > 0, 0 < \phi \leq 1 \right\} \subset \mathbb{C} .
\]

Let us consider the argument map
\[
\text{Arg} : \mathbb{C}^* \to (-\pi, \pi], \quad re^{i \pi \phi} \mapsto \pi \phi, \quad r > 0, \phi \in (-1, 1],
\]
and define the *phase* of \( 0 \neq E \in \mathcal{A} \) to be
\[
\phi(E) = \frac{1}{\pi} \text{Arg} Z(E) \in (0, 1].
\]

An object \( 0 \neq E \in \mathcal{A} \) is called \( Z \)-semistable if for any subobject \( 0 \neq E' \subset E \) we have \( \phi(E') \leq \phi(E) \). Equivalently, this means that
\[
\text{Im}(Z(E) \cdot \overline{Z}(E')) \geq 0 .
\]

A stability function \( Z : \Gamma \to \mathbb{C} \) on \( \mathcal{A} \) is said to satisfy the *HN property* if, for any \( 0 \neq E \in \mathcal{A} \), there exists a (unique) filtration, called the *HN filtration* of \( E \),
\[
0 = E_0 \subset E_1 \subset \ldots \subset E_n = E
\]

such that \( E_k / E_{k-1} \) are \( Z \)-semistable and have strictly decreasing phases.

It is proved in [13, §5] that to give a pre-stability condition \( \sigma = (Z, \mathcal{P}) \) on the triangulated category \( D \) is equivalent to giving a pair \((Z, \mathcal{A})\), where \( \mathcal{A} \subset \mathcal{D} \) is the heart of a bounded t-structure and \( Z : \Gamma \to \mathbb{C} \) is a stability function on \( \mathcal{A} \) satisfying the HN property. The category \( \mathcal{A} \) is defined to be the heart \( \mathcal{P}_{[0,1]} \) of the slicing \( \mathcal{P} \). Conversely, one defines \( \mathcal{P}_\phi \) for \( \phi \in (0,1] \) to be the category of all semistable objects of \( \mathcal{A} \) having phase \( \phi \) (plus the zero object). Then one defines \( \mathcal{P}_{\phi+n} = \mathcal{P}_\phi [n] \) for \( n \in \mathbb{Z} \). In what follows we will sometimes write \((Z, \mathcal{A})\) for the pre-stability condition \( \sigma \).
3.5. Families of stability conditions. Let \( \text{Slice}(\mathcal{D}) \) be the set of all slicings of \( \mathcal{D} \) and \( \text{Stab}_{cl}(\mathcal{D}) \) be the set of all stability conditions on \( \mathcal{D} \) (satisfying the support property). Given two slicings \( \mathcal{P}, \mathcal{Q} \), we define the distance between them to be

\[
d(\mathcal{P}, \mathcal{Q}) = \sup_{\phi \in \mathcal{D}} \{ |\phi^+_\mathcal{P}(E) - \phi^+_{\mathcal{Q}}(E)|, |\phi^-_\mathcal{P}(E) - \phi^-_{\mathcal{Q}}(E)| \} \in [0, +\infty].
\]

Equivalently [13, Lemma 6.1],

\[
d(\mathcal{P}, \mathcal{Q}) = \inf \{ \varepsilon \geq 0 \mid \mathcal{Q}_\phi \subset \mathcal{P}_{[\phi - \varepsilon, \phi + \varepsilon]} \forall \phi \in \mathbb{R} \}.
\]

This distance function induces a topology on \( \text{Slice}(\mathcal{D}) \). We equip \( \text{Stab}_{cl}(\mathcal{D}) \) with the induced topology using the inclusion

\[
\text{Stab}_{cl}(\mathcal{D}) \subset \text{Hom}(\Gamma, \mathbb{C}) \times \text{Slice}(\mathcal{D}).
\]

It is proved in [13] that the forgetful map \( \text{Stab}_{cl}(\mathcal{D}) \to \text{Hom}(\Gamma, \mathbb{C}) \) is a local homeomorphism and one can equip \( \text{Stab}_{cl}(\mathcal{D}) \) with a structure of a complex manifold.

We define a continuous family of stability conditions to be a continuous map \( \sigma : M \to \text{Stab}_{cl}(\mathcal{D}) \), where \( M \) is a topological space. Equivalently, a family of stability conditions \( \{\sigma_x = (Z_x, \mathcal{P}_x)\}_{x \in M} \) is continuous if

1. The map \( Z : M \to \text{Hom}(\Gamma, \mathbb{C}), x \mapsto Z_x \), is continuous.
2. For any \( x \in M \) and \( \varepsilon > 0 \), there exists a neighborhood \( x \in U \subset M \) such that \( d(\mathcal{P}_x, \mathcal{P}_y) < \varepsilon \) for all \( y \in U \).

We can similarly define a continuous family of pre-stability conditions. In what follows, we will denote \( \phi_{\sigma_x}^+ \) by \( \phi_x^+ \). The support property is an open and closed condition by the following result.

**Lemma 3.6** ([13, Lemma 6.2]). Let \( \sigma = (Z, \mathcal{P}) \) and \( \sigma' = (Z', \mathcal{P}') \) be pre-stability conditions such that for some \( 0 < \eta < \frac{1}{4} \) we have

\[
\|Z - Z'\|_{\sigma} \leq \sin(\pi\eta), \quad d(\mathcal{P}, \mathcal{P}') < \eta.
\]

Then there exist \( c_1, c_2 > 0 \) such that

\[
c_1 \|u\|_{\sigma} \leq \|u\|_{\sigma'} \leq c_2 \|u\|_{\sigma} \quad \forall u \in \text{Hom}(\Gamma, \mathbb{C}).
\]

**Lemma 3.7.** Let \( \{\sigma_x\}_{x \in M} \) be a continuous family of pre-stability conditions. If \( x \in M \) is such that \( \sigma_x \) satisfies the support property, then there exists a neighborhood \( x \in U \subset M \) and \( \varepsilon > 0 \) such that

\[
|Z_y(E)| \geq \varepsilon \|\text{cl} E\|
\]

for all \( y \in U \) and \( \sigma_y \)-semistable objects \( E \).

**Proof.** Let \( \sigma = (Z, \mathcal{P}) = \sigma_x \) satisfy the support property. It is enough to show that for some neighborhood \( x \in U \subset M \) and \( C > 0 \), we have \( \|u\|_{\sigma_y} \leq C \|u\|_{\sigma} \) for all \( u \in \Gamma^\vee \) and \( y \in U \). Indeed, we can assume that the norm on \( \Gamma^\vee \) is given by \( \|\gamma\| = \sum_i |u_i(\gamma)| \), for a fixed basis \( (u_1, \ldots, u_n) \) of \( \Gamma^\vee \). Then, for any \( y \in U \), \( \sigma_y \)-semistable object \( E \) and \( \gamma = \text{cl} E \), we have

\[
\|\gamma\| = \sum_i |u_i(\gamma)| \leq \sum_i \|u_i\|_{\sigma_y} \cdot |Z_y(\gamma)| \leq C \sum_i \|u_i\|_{\sigma} \cdot |Z_y(\gamma)|
\]

which is equivalent to the statement of the lemma.
Let $0 < \eta < \frac{1}{2}$. For any $0 \neq E \in \mathcal{D}$ satisfying $\phi^+(E) - \phi^-(E) \leq \eta$ and any $u \in \Gamma^\vee_C$, we have

$$|u(\text{cl} E)| \leq \frac{\|u\|_\sigma}{\cos(\pi \eta)} |Z(E)|. \quad (37)$$

Indeed, let $F_1, \ldots, F_n$ be $\sigma$-semistable HN factors of $E$. Then (see Example 2.4)

$$|u(\text{cl} E)| \leq \sum_i |u(F_i)| \leq \|u\|_\sigma \sum_i |Z(F_i)| \leq \|u\|_\sigma \frac{|Z(E)|}{\cos(\pi \eta)}.$$

There exists a neighborhood $x \in U \subset M$ such that for all $y \in U$ we have

$$\|Z_y - Z\|_\sigma < \frac{1}{2} \cos(\pi \eta), \quad d(\mathcal{P}_y, \mathcal{P}) < \eta/2.$$ 

For any $0 \neq E \in \mathcal{P}_{y, \phi}$, we have $E \in \mathcal{P}(\phi - \eta/2, \phi + \eta/2)$, hence $\phi^+(E) - \phi^-(E) < \eta$ and (37)

$$|Z_y(E) - Z(E)| \leq \frac{\|Z_y - Z\|_\sigma}{\cos(\pi \eta)} |Z(E)| \leq \frac{1}{2} |Z(E)|.$$

This implies that $|Z_y(E)| \geq \frac{1}{2} |Z(E)|$ and for any $u \in \Gamma^\vee_C$ we have (37)

$$|u(\text{cl} E)| \leq \frac{\|u\|_\sigma}{\cos(\pi \eta)} |Z(E)| \leq \frac{\|u\|_\sigma}{\cos(\pi \eta)} 2 |Z_y(E)|.$$

Therefore $\|u\|_{\sigma_y} \leq \frac{2}{\cos(\pi \eta)} \|u\|_\sigma$ as required. \qed

The following result is well-known.

**Lemma 3.8.** Let $(\sigma_x)_{x \in M}$ be a continuous family of stability conditions and $0 \neq E \in \mathcal{D}$. Then

1. The set $\{x \in M \mid E \text{ is } \sigma_x\text{-stable}\}$ is open in $M$.
2. The set $\{x \in M \mid E \text{ is } \sigma_x\text{-semistable}\}$ is closed in $M$.

We will prove its variant.

**Lemma 3.9.** Let $(\sigma_x)_{x \in M}$ be a continuous family of stability conditions and let

$$(38) \quad \mathfrak{t}_x = \{\text{cl} E \mid E \text{ is } \sigma_x\text{-stable}\}, \quad \mathfrak{s}_x = \{\text{cl} E \mid E \text{ is } \sigma_x\text{-semistable}\}$$

for $x \in M$. Then, for every $\gamma \in \Gamma$,

1. The set $\{x \in M \mid \gamma \in \mathfrak{t}_x\}$ is open in $M$.
2. The set $\{x \in M \mid \gamma \in \mathfrak{s}_x\}$ is closed in $M$.

**Proof.** There exists $\varepsilon > 0$ and an open set $x \in U \subset M$ such that

$$\mathfrak{s}_y \subset \{ \gamma \in \Gamma \mid |Z_y(\gamma)| \geq \varepsilon \|\gamma\| \}$$

for all $y \in U$. We can assume that $|Z_y - Z_x| \leq \frac{\varepsilon}{2}$ for all $y \in U$. Then $|Z_y(\gamma) - Z_x(\gamma)| \leq \frac{\varepsilon}{2} \|\gamma\| \leq \frac{1}{2} |Z_y(\gamma)|$, hence $|Z_x(\gamma)| \geq \frac{1}{2} |Z_y(\gamma)| \geq \frac{\varepsilon}{2} \|\gamma\|$ for all $\gamma \in \mathfrak{s}_y$.

Let $\gamma = \text{cl} E$ for some $\sigma_x$-stable object $E \in \mathcal{P}_{x, \phi}$. For any $0 < \eta < 1/2$, we can assume that $d(\mathcal{P}_x, \mathcal{P}_y) < \eta/2$ for all $y \in U$. Then $\phi^+(E) \in (\phi - \eta/2, \phi + \eta/2)$. This implies that for every $\sigma_y$-HN-factor $F_i$ of $E$, we have $\phi^+_y(F_i) \in I = (\phi - \eta, \phi + \eta)$, hence $F_i \in \mathcal{P}_{x, I}$. All nonzero objects in $\mathcal{P}_{x, I}$ have classes in the semigroup $S(V, Z_x, \varepsilon)$, where $V = \mathbb{R}_{>0}e^{i\pi I}$ is a strict cone. There are finitely many such classes that are $\leq \gamma$. By decreasing $\eta$ (and shrinking $U$) we can assume that all such classes are contained in $S(\ell, Z_x, \varepsilon)$, where
\[ \ell = \mathbb{R}_{>0}e^{i\pi}\phi. \] This implies that \( \phi^\pm_x(F_i) = \phi. \) As \( E \) is \( \sigma_x \)-stable, the above HN-filtration is trivial and \( E \) is \( \sigma_y \)-semistable. Let \( E \in \mathcal{P}_{y,\phi'} \) and let \( 0 \neq F \subset E \) be its subobject in \( \mathcal{P}_{y,\phi'} \). Then \( \phi^\pm_x(F) \in (\phi' - \eta/2, \phi' + \eta/2) \subset I, \) hence \( F \in \mathcal{P}_{x,I}. \) As before, we obtain \( \phi^\pm_x(F) = \phi. \) As \( E \) is \( \sigma_x \)-stable, we conclude that \( F = E, \) hence \( E \) is \( \sigma_y \)-stable.

To show that \( \{ x \in M \mid \gamma \in \mathfrak{s}_x \} \) is closed, we need to prove that if \( x \in M \) is such that for every open \( x \in U \subset M, \) there exists \( y \in U \) with \( \gamma \in \mathfrak{s}_y, \) then \( \gamma \in \mathfrak{s}_x. \) Let us assume that \( \gamma = \text{cl } E \) for some \( 0 \neq E \in \mathcal{P}_{y,\phi'} \) and \( y \in U. \) By the above discussion, we can assume that \( |Z_x(\gamma)| \geq \frac{3}{2} ||\gamma||. \) Let \( \phi = \frac{1}{\pi} \text{Arg } Z_x(\gamma). \) As before, we assume that \( 0 < \eta < 1/2 \) and \( d(\mathcal{P}_x, \mathcal{P}_y) < \eta/2 \) for all \( y \in U. \) Then \( \phi^\pm_x(E) \in I' = (\phi' - \eta/2, \phi' + \eta/2). \) We can assume that \( E \) and \( \phi' \) are chosen in such way that \( \phi \in I'. \) Then \( \phi^\pm_x(E) \in I = (\phi - \eta, \phi + \eta), \) hence \( E \in \mathcal{P}_{x,I}. \) All nonzero objects in \( \mathcal{P}_{x,I} \) have classes in the semigroup \( S(V, Z_x, \epsilon), \) where \( V = \mathbb{R}_{>0}e^{i\pi\phi} \) is a strict cone. There are finitely many such classes that are \( \leq \gamma. \) By decreasing \( \eta \) (and shrinking \( U \)) we can assume that all such classes are contained in \( S(\ell, Z_x, \epsilon), \) where \( \ell = \mathbb{R}_{>0}e^{i\pi\phi}. \) This implies that if we can still find \( y \in U \) and a \( \sigma_y \)-semistable object \( E \) with \( \gamma = \text{cl } E, \) then \( E \) is automatically \( \sigma_x \)-semistable. \( \square \)

**Theorem 3.10 (Cf. [6, 4]).** Let \( M \) be path-connected, \( (\sigma_x)_{x \in M} \) be a continuous family of stability conditions on \( M, \) and \( Q: \Gamma_\mathbb{R} \to \mathbb{R} \) be a quadratic form negative semi-definite on \( \text{Ker } Z_x \) for all \( x \in M. \) If there exists \( x \in M \) such that \( Q(\text{cl } E) \geq 0 \) for all \( \sigma_x \)-semistable objects \( E, \) then the same is true for all points of \( M. \)

**Proof.** By Lemma 3.7 and Lemma 3.9, we can apply Theorem 2.14. \( \square \)

**Corollary 3.11.** Let \( M \) be locally path-connected and \( (\sigma_x)_{x \in M} \) be a continuous family of pre-stability conditions on \( M. \) Let \( Q: \Gamma_\mathbb{R} \to \mathbb{R} \) be a quadratic form and \( x \in M \) be such that \( Q \) is negative definite on \( \text{Ker } Z_x \) and \( Q(\text{cl } E) \geq 0 \) for all \( \sigma_x \)-semistable objects \( E. \) Then the same is true in some neighborhood of \( x. \)

### 3.6 Action on stability conditions.

Let \( \text{Aut}(\mathcal{D}) \) be the group of automorphisms of \( \mathcal{D}, \) \( \text{GL}^+_2(\mathbb{R}) \) be the group of elements in \( \text{GL}_2(\mathbb{R}) \) having positive determinant and let \( \text{GL}^+_2(\mathbb{R}) \) be its universal cover. There is a left action of the group \( \text{Aut}(\mathcal{D}) \) and the right action of the group \( \text{GL}^+_2(\mathbb{R}) \) on \( \text{Stab}_\mathcal{D}(\mathcal{D}) \) defined as follows [13, Lemma 8.2].

For any \( T \in \text{Aut}(\mathcal{D}) \) and stability condition \( \sigma = (Z, \mathcal{P}), \) we define

\[
T \sigma = (Z', \mathcal{P}'), \quad Z' = ZT^{-1}, \quad \mathcal{P}'_\phi = T\mathcal{P}_\phi.
\]

To describe \( \text{GL}^+_2(\mathbb{R}) \), let us consider the group homomorphism

\[
\text{GL}^+_2(\mathbb{R}) \to \text{Diff}(U(1)), \quad T \mapsto \bar{T}, \quad \bar{T}(z) := T(z)/|T(z)|,
\]

where \( \text{Diff}(U(1)) \) is the group of (orientation-preserving) diffeomorphisms of \( U(1) \subset \mathbb{C}^*. \) Let \( \text{Diff}^+(\mathbb{R}) \) be the group of (orientation-preserving) diffeomorphisms \( f: \mathbb{R} \to \mathbb{R} \) that commute with the translation \( \phi \mapsto \phi + 2. \) Any \( f \in \text{Diff}^+(\mathbb{R}) \) induces

\[
\tilde{f} \in \text{Diff}(U(1)), \quad e^{i\pi\phi} \mapsto e^{i\pi f(\phi)}.
\]

Then \( \text{GL}^+_2(\mathbb{R}) \) can be defined using the following diagram with Cartesian squares
Here $C^* \hookrightarrow \text{GL}_2^+(\mathbb{R})$ is given by $a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and
\begin{align}
\mathbb{C}^* &\hookrightarrow \text{Diff}(U(1)), \quad w \mapsto [z \mapsto zw/|w|] \\
\mathbb{C} &\hookrightarrow \text{Diff}^*(\mathbb{R}), \quad a + bi \mapsto [\phi \mapsto \phi + a].
\end{align}

For any stability condition $\sigma = (Z, \mathcal{P})$ and an element $g = (T, f) \in \tilde{\text{GL}}_2^+ (\mathbb{R})$ (where $T \in \text{GL}_2^+(\mathbb{R})$ and $f \in \text{Diff}^*(\mathbb{R})$ are mapped to the same element in $\text{Diff}(U(1))$), we define a new stability condition
\begin{equation}
\sigma[g] = (Z[g], \mathcal{P}[g]), \quad Z[g] = T^{-1}Z, \quad \mathcal{P}[g]_\phi = \mathcal{P}_{f(\phi)}.
\end{equation}

In particular, for $a + ib \in \mathbb{C}$, we define (cf. (24))
\begin{equation}
Z[a + ib] = e^{-i\pi a + \epsilon b}Z, \quad \mathcal{P}[a + ib]_\phi = \mathcal{P}_{\phi + a}.
\end{equation}

3.7. Phase behavior in families. Let $(\sigma_x)_{x \in M}$ be a continuous family of stability conditions. We denote $\phi^\pm_{a_x}$ by $\phi^\pm_x$. For any object $0 \neq E \in \mathcal{D}$ the maps
\begin{align}
M &\rightarrow \mathbb{R}, \quad x \mapsto \phi^-_x(E), \quad M \rightarrow \mathbb{R}, \quad x \mapsto \phi^+_x(E),
\end{align}
are continuous by (33).

**Theorem 3.12.** Let $(\sigma_t = (Z_t, \mathcal{P}_t))_{t \in [0, 1]}$ be a continuous family of stability conditions on $\mathcal{D}$ such that the map $Z : [0, 1] \rightarrow \Gamma_C^\vee$ is differentiable and
\begin{equation}
\text{Im}(Z'_t(E) \cdot \tilde{Z}_t(E)) \geq 0
\end{equation}
for all $t \in [0, 1]$ and $\sigma_t$-stable objects $E \in \mathcal{D}$. Then, for any object $0 \neq E \in \mathcal{D}$, the functions
\begin{equation}
t \mapsto \phi^-_t(E), \quad t \mapsto \phi^+_t(E)
\end{equation}
are weakly-increasing.

**Proof.** We will prove just that $\phi^-_t(E)$ is weakly-increasing. It is enough to show that if $E$ is $\sigma_{t_0}$-stable and $t \in (t_0, t_0 + \varepsilon)$ for $0 < \varepsilon \ll 1$, then $\phi^-_t(E) \geq \phi_{t_0}(E)$. We can assume that $E$ is $\sigma_t$-stable for all $t \in [t_0, t_0 + \varepsilon)$. The condition (47) means that
\begin{equation}
Z'_t(E) \in Z_t(E) \cdot \mathbb{H}, \quad \mathbb{H} = \{ x + iy \in \mathbb{C} \mid y \geq 0 \},
\end{equation}
as $\tilde{Z}_t(E)$ and $(Z_t(E))^{-1}$ are contained in the same ray. This condition implies that $Z_t(E) \in Z_{t_0}(E) \cdot \mathbb{H}$ for $t \in (t_0, t_0 + \varepsilon)$ and $0 < \varepsilon \ll 1$. Therefore $\phi_t(E) \geq \phi_{t_0}(E)$. \hfill \Box

**Remark 3.13.** Given a stability condition $\sigma = (Z, \mathcal{P})$ on a triangulated category $\mathcal{D}$, we define a stability condition $\tilde{\sigma} = (\tilde{Z}, \tilde{\mathcal{P}})$ on the opposite category $\mathcal{D}^{\text{op}}$ as follows. We define $\tilde{\mathcal{P}}_\phi = \mathcal{P}_{-\phi}$ and we define $\tilde{Z}(E)$ to be the conjugate of $Z(E)$. Note that if $0 \neq E \in \tilde{\mathcal{P}}_\phi$, then $Z(E) \in \mathbb{R}_{>0}e^{-i\pi \phi}$, hence $\tilde{Z}(E) \in \mathbb{R}_{>0}e^{i\pi \phi}$. Assuming that we have a family of stability conditions $(\tilde{\sigma}_t)_{t}$ on $\mathcal{D}$ as above, we obtain a family $(\tilde{\sigma}_t)_{t}$ of stability conditions on $\mathcal{D}^{\text{op}}$. Let $\tilde{\phi}^\pm_t$ be the phase functions corresponding to these stability conditions. We
have \( \text{Im}(\bar{Z}^t(E) \cdot Z_t(E)) \leq 0 \), hence we obtain from the first part of the above result that \( \tilde{\phi}^+_t(E) \) is weakly-decreasing, hence \( \phi_t^+(E) = -\tilde{\phi}^-_t(E) \) is weakly-increasing.

**Remark 3.14.** Let \( (\sigma_t = (Z_t, P_t))_{t \in [0,1]} \) be a continuous family of stability conditions such that the map \( Z: [0,1] \to \Gamma^\sigma_C \) is differentiable. For any \( t \in [0,1] \) we define the quadratic form

\[
(48) \quad Q_t: \Gamma_\mathbb{R} \to \mathbb{R}, \quad \gamma \mapsto \text{Im}(Z^t(\gamma) \cdot \bar{Z}_t(\gamma)),
\]

which is zero on \( \text{Ker} \ Z_t \). The requirement of Theorem 3.12 means that \( Q_t(\text{cl} E) \geq 0 \) for all \( \sigma_t \)-stable (or \( \sigma_t \)-semistable) objects \( E \in \mathcal{D} \).

### 3.8. Global dimension of a slicing

We define the **global dimension** of a slicing \( \mathcal{P} \) to be

\[
(49) \quad \text{gldim}(\mathcal{P}) = \sup \{ \phi' - \phi \mid \text{Hom}(\mathcal{P}_\phi, \mathcal{P}_{\phi'}) \neq 0 \} \in [0, +\infty].
\]

Note that if \( \phi' - \phi < 0 \), then \( \text{Hom}(\mathcal{P}_\phi, \mathcal{P}_{\phi'}) = 0 \) by assumption, hence the global dimension is indeed non-negative. If \( \sigma = (Z, \mathcal{P}) \) is a stability condition, then we define its global dimension to be \( \text{gldim}(\sigma) = \text{gldim}(\mathcal{P}) \).

**Example 3.15.** Let us assume that \( \mathcal{D} \) has the Serre functor, meaning an automorphism \( S: \mathcal{D} \to \mathcal{D} \) such that there are natural isomorphisms \( \text{Hom}(X, Y) \simeq \text{Hom}(Y, SX)^\vee \) for all \( X, Y \in \mathcal{D} \). Let us assume that there exists \( d \in \mathbb{R} \), such that \( S(\mathcal{P}_\phi) = \mathcal{P}_{\phi+d} \) for all \( \phi \in \mathbb{R} \).

If \( X \in \mathcal{P}_\phi, Y \in \mathcal{P}_{\phi'} \) and \( \phi' > \phi + d \), then \( \text{Hom}(X, Y) \simeq \text{Hom}(Y, SX)^\vee = 0 \). Therefore \( \text{gldim}(\mathcal{P}) \leq d \). Taking \( Y = SX \) with \( X \in \mathcal{P}_\phi \), we obtain that \( \text{gldim}(\mathcal{P}) = d \).

**Example 3.16.** Let us assume that the Serre functor has the form \( S = \Phi[d] \) for some \( d \in \mathbb{Z} \) and a functor \( \Phi \) that preserves the heart \( \mathcal{A} = \mathcal{P}_{(0,1]} \). If \( X \in \mathcal{P}_\phi, Y \in \mathcal{P}_{\phi'} \) for \( \phi \in (0,1] \) and \( \phi' > \phi + d + 1 \), then \( SX \in \mathcal{P}_{(d,d+1]} \), hence \( \text{Hom}(X, Y) \simeq \text{Hom}(Y, SX) = 0 \). This implies that \( \text{gldim}(\mathcal{P}) \leq d + 1 \). On the other hand, let \( X \in \mathcal{P}_\phi \) for \( \phi \in (0,1] \) and let \( Y \in \mathcal{P}_{\phi'} \) be the first term of the \( \text{HN} \)-filtration of \( SX \in \mathcal{P}_{(d,d+1]} \). Then \( \text{Hom}(X, Y) \neq 0 \) and \( \phi' > d \), hence \( \text{gldim}(\mathcal{P}) > d - 1 \).

### 4. Stability data

#### 4.1. Graded Lie algebras

Let \( \Gamma \) be a free abelian group of finite rank. Given a \( \Gamma \)-graded Lie algebra \( \mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma \), we define its support to be

\[
(50) \quad \text{supp}(\mathfrak{g}) = \{ \gamma \in \Gamma \mid \mathfrak{g}_\gamma \neq 0 \}.
\]

We will say that \( \mathfrak{g} \) has a strict support if \( \text{supp}(\mathfrak{g}) \) generates a strict semigroup without zero in \( \Gamma \). If \( \mathfrak{g} \) has a finite strict support, then \( \mathfrak{g} \) is nilpotent and we can define the corresponding nilpotent group \( G = \exp(\mathfrak{g}) \) equipped with the bijective exponential map \( \exp: \mathfrak{g} \to G \).

If \( \mathfrak{g} \) has a strict support, we define its pro-nilpotent completion \( \hat{\mathfrak{g}} \) as follows. Let \( S \subset \Gamma \) be the strict semigroup (without zero) generated by \( \text{supp}(\mathfrak{g}) \). We equip \( S \) with a partial order as in §2.2. Given a finite lower set \( I \subset S \), we define the Lie algebra

\[
(51) \quad \mathfrak{g}_I = \mathfrak{g}/m_I \simeq \bigoplus_{\gamma \in I} \mathfrak{g}_\gamma, \quad m_I = \bigoplus_{\gamma \in S \setminus I} \mathfrak{g}_\gamma.
\]
If \( I \subset J \) are two lower sets, then there is a canonical epimorphism of Lie algebras \( g_I \to g_J \). We define the completion
\[
\hat{g} = \lim_{\leftarrow I \in S} g_I,
\]
where the limit is taken over all finite lower sets \( I \subset S \). Note that we have an isomorphism of vector spaces \( \hat{g} \cong \prod_{\gamma \in \Gamma} g_{\gamma} \). The Lie algebra \( \hat{g} \) is pro-nilpotent as the Lie algebras \( g_I \), for finite lower sets \( I \subset S \), are nilpotent. We define the corresponding pro-nilpotent group \( \hat{G} = \exp(\hat{g}) = \lim_I \exp(g_I) \) and the bijective exponential map \( \exp: \hat{g} \to \hat{G} \) (we denote its inverse by log).

For an arbitrary \( \Gamma \)-graded Lie algebra \( g \), we define the vector space
\[
\hat{g} = \prod_{\gamma \in \Gamma \setminus \{0\}} g_{\gamma}.
\]
Given \( a \in \hat{g} \), we define
\[
\supp(a) = \{ \gamma \in \Gamma \mid a_{\gamma} \neq 0 \}.
\]

4.2. Examples of graded Lie algebras. The following two examples of graded Lie algebras usually appear in the study of stability data and wall-crossing structures.

Example 4.1. Let \( g \) be a semisimple Lie algebra over \( \mathbb{C} \), with a Cartan subalgebra \( \mathfrak{h} \), the set of roots \( \Delta \subset \mathfrak{h}^\vee \) and a set of positive roots \( \Delta_+ \subset \Delta \). Let \( \Gamma \subset \mathfrak{h}^\vee \) be the root lattice of \( g \), so that \( \Gamma_C \cong \mathfrak{h}^\vee \) and \( \text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C}) = \Gamma_C^\vee \cong \mathfrak{h} \). Using the root space decomposition \( g = \bigoplus_{\alpha \in \Delta \setminus \{0\}} g_\alpha \) with \( g_0 = \mathfrak{h} \), we can consider \( g \) as a \( \Gamma \)-graded Lie algebra. Let \( \Pi \subset \Delta_+ \) be the set of simple roots and \( e_i \in g_{\alpha_i} \), \( f_i \in g_{-\alpha_i} \), for \( \alpha_i \in \Pi \), be the standard generators. The algebra \( g \) is equipped with the Chevalley involution which is a Lie algebra automorphism \( \omega: g \to g \) given by
\[
\omega(e_i) = -f_i, \quad \omega(f_i) = -e_i, \quad \omega(h) = -h, \quad h \in \mathfrak{h}.
\]

One can also consider \( g = g_{\mathfrak{h}_{\mathbb{C}}}(\mathbb{C}) \) with the subalgebra \( \mathfrak{h} \subset g \) consisting of diagonal matrices and the root lattice \( \Gamma \subset \mathfrak{h}^\vee \) generated by \( \varepsilon_i \in \mathfrak{h}^\vee \), \( \sum_j a_j E_{jj} \mapsto a_i \). As before, we have a root space decomposition \( g = \bigoplus_{\alpha \in \Delta \setminus \{0\}} g_\alpha \) with \( g_0 = \mathfrak{h} \) and \( \Delta = \{ \varepsilon_i - \varepsilon_j \mid i \neq j \} \). Therefore \( g \) is \( \Gamma \)-graded.

Example 4.2. Let \( \Gamma \cong \mathbb{Z}^n \) be equipped with a skew-symmetric \( \mathbb{Z} \)-valued bilinear form \( \langle \cdot, \cdot \rangle \) and let \( R \) be a commutative ring with an invertible element \( s \in R \) (in our later considerations \( R \) is the ring of motivic classes and \( s = \mathbb{L}^{1/2} \), where \( \mathbb{L} \) is the class of the Lefschetz motive). We define the quantum torus to be a \( \Gamma \)-graded associative \( R \)-algebra
\[
\mathbb{T} = \bigoplus_{\gamma \in \Gamma} R x^\gamma, \quad x^\alpha \circ x^\beta = s^{\langle \alpha, \beta \rangle} x^{\alpha + \beta}.
\]
We can also consider it as a \( \Gamma \)-graded Lie algebra, where the Lie bracket is given by the commutator. The algebra \( \mathbb{T} \) is equipped with the involution \( x^\alpha \mapsto x^{-\alpha} \).

Assuming that \( s^2 - 1 \) is invertible, we define new generators \( \bar{x}^\alpha = x^\alpha / (s - s^{-1}) \), so that
\[
[\bar{x}^\alpha, \bar{x}^\beta] = \frac{s^{\langle \alpha, \beta \rangle} - s^{-\langle \alpha, \beta \rangle}}{s - s^{-1}} \bar{x}^{\alpha + \beta}.
\]
Taking the limit $s \to -1$, we define a new $\Gamma$-graded Lie algebra
\begin{equation}
\mathbb{T} = \bigoplus_{\gamma \in \Gamma} Q x^\gamma, \quad [\bar{x}^\alpha, \bar{x}^\beta] = (-1)^{(\alpha, \beta)} \langle \alpha, \beta \rangle \bar{x}^{\alpha + \beta},
\end{equation}
called the torus Lie algebra. It is equipped with the involution $\bar{x}^\alpha \mapsto \bar{x}^{-\alpha}$.

4.3. Factorizations. Let $\mathfrak{g}$ be a $\Gamma$-graded Lie algebra.

**Lemma 4.3.** Let $\mathfrak{g}$ be a graded Lie algebra with a finite strict support having a decomposition $\text{supp}(\mathfrak{g}) = S_1 \sqcup S_2$ such that $\mathfrak{g}_i = \bigoplus_{\gamma \in S_i} \mathfrak{g}_\gamma$ are subalgebras for $i = 1, 2$. Then the map
\[ \exp(\mathfrak{g}_1) \times \exp(\mathfrak{g}_2) \to \exp(\mathfrak{g}), \quad (g_1, g_2) \mapsto g_1 g_2, \]
is a bijection.

**Proof.** There exists $\gamma \in \text{supp}(\mathfrak{g})$ such that $[\mathfrak{g}_\gamma, \mathfrak{g}] = 0$. Let us assume that $\gamma \in S_1$ and let $G_\gamma = \exp(\mathfrak{g}_\gamma)$. We have the diagram
\[
\begin{CD}
\exp(\mathfrak{g}_1) \times \exp(\mathfrak{g}_2) @>>> \exp(\mathfrak{g}) \\
@VVV @VVV \\
\exp(\mathfrak{g}_1/\mathfrak{g}_\lambda) \times \exp(\mathfrak{g}_2) @>>> \exp(\mathfrak{g}/\mathfrak{g}_\lambda)
\end{CD}
\]
where both vertical arrows are $G_\gamma$-torsors, the top arrow is $G_\gamma$-equivariant, and the bottom arrow is a bijection by induction on the size of $\text{supp}(\mathfrak{g})$. We conclude that the top arrow is also a bijection. \hfill \Box

**Corollary 4.4.** Let $\mathfrak{g}$ be a graded Lie algebra with a strict support having a decomposition $\text{supp}(\mathfrak{g}) = S_1 \sqcup S_2$ such that $\mathfrak{g}_i = \bigoplus_{\gamma \in S_i} \mathfrak{g}_\gamma$ are subalgebras for $i = 1, 2$. Then the map
\[ \exp(\hat{\mathfrak{g}}_1) \times \exp(\hat{\mathfrak{g}}_2) \to \exp(\hat{\mathfrak{g}}), \quad (g_1, g_2) \mapsto g_1 g_2, \]
is a bijection.

Let $\mathfrak{g}$ be a $\Gamma$-graded Lie algebra with a strict support and $Z : \Gamma \to \mathbb{C}$ be a linear map. Let $\hat{\mathfrak{g}} \simeq \prod_{\gamma} \mathfrak{g}_\gamma$ be the pro-nilpotent completion of $\mathfrak{g}$ and $\hat{G}$ be the corresponding pro-nilpotent group. Given a strict cone $V \subset \mathbb{C}$, we define the Lie algebra
\begin{equation}
\mathfrak{g}_{V,Z} = \bigoplus_{Z(\gamma) \in V} \mathfrak{g}_\gamma \subset \mathfrak{g}
\end{equation}
and consider its pro-nilpotent completion $\hat{\mathfrak{g}}_{V,Z} \subset \hat{\mathfrak{g}}$ and the corresponding pro-nilpotent group $\hat{G}_{V,Z} \subset \hat{G}$. In particular, for any ray $\ell \subset \mathbb{C}$, we consider the pro-nilpotent Lie algebra $\hat{\mathfrak{g}}_{\ell,Z} \subset \hat{\mathfrak{g}}$ and the corresponding pro-nilpotent group $\hat{G}_{\ell,Z} \subset \hat{G}$.

**Lemma 4.5.** For any blunt strict cone $V \subset \mathbb{C}$, the map
\[ \prod_{\ell \subset V} \hat{G}_{\ell,Z} \to \hat{G}_{V,Z}, \quad (g_{\ell})_\ell \mapsto \prod_{\ell \subset V} g_{\ell}, \]
is a bijection, where the product is taken in the clockwise order over all rays $\ell \subset V$.

**Proof.** It is enough to prove the statement for $\mathfrak{g}$ having a finite strict support contained in $Z^{-1}(V)$. For any ray $\ell \subset V$, let $S_\ell = \{ \gamma \in \text{supp}(\mathfrak{g}) \mid Z(\gamma) \in \ell \}$. Then $\text{supp}(\mathfrak{g}) = \bigsqcup_{\ell} S_\ell$ and $\mathfrak{g}_{\ell,Z} = \bigoplus_{\gamma \in S_\ell} \mathfrak{g}_\gamma$. Now the statement follows from Lemma 4.3. \hfill \Box
4.4. **Stability data.** Let $\mathfrak{g}$ be a $\Gamma$-graded Lie algebra and let us equip $\Gamma_{\mathbb{R}} = \Gamma \otimes \mathbb{R}$ with a norm $\| \cdot \|$. We define *stability data* [41, §2.1] to be a pair $(Z, a)$, where $Z \in \text{Hom}(\Gamma, \mathbb{C})$ and $a = (a_\gamma)_\gamma \in \hat{\mathfrak{g}} = \prod_{\gamma \in \Gamma \setminus \{0\}} \mathfrak{g}_{\gamma}$ is such that there exists $\varepsilon > 0$ satisfying (the *support property*)

\begin{equation}
\text{supp}(a) \subset \{ \gamma \in \Gamma \mid |Z(\gamma)| \geq \varepsilon \|\gamma\| \}.
\end{equation}

Sometimes $Z \in \text{Hom}(\Gamma, \mathbb{C})$ will be clear from the context and we will call $a \in \hat{\mathfrak{g}}$ stability data.

For any blunt strict cone $V \subset \mathbb{C}$, let us consider the semigroup (11)

\begin{equation}
S(V, Z, \varepsilon) = \text{sgn} \{ \gamma \in \Gamma \mid Z(\gamma) \in V, \ |Z(\gamma)| \geq \varepsilon \|\gamma\| \} \subset \Gamma
\end{equation}

which is strict by Lemma 2.2. We define the corresponding Lie algebra and its pro-nilpotent completion

\begin{equation}
\mathfrak{g}_{V, Z, \varepsilon} = \bigoplus_{\gamma \in S(V, Z, \varepsilon)} \mathfrak{g}_{\gamma}, \quad \hat{\mathfrak{g}}_{V, Z, \varepsilon} \simeq \prod_{\gamma \in S(V, Z, \varepsilon)} \mathfrak{g}_{\gamma}.
\end{equation}

We consider the corresponding pro-nilpotent group $\hat{G}_{V, Z, \varepsilon}$ and the bijective exponential map $\exp : \hat{\mathfrak{g}}_{V, Z, \varepsilon} \to \hat{G}_{V, Z, \varepsilon}$. By Lemma 4.5, we also have a bijection

\begin{equation}
\exp_{Z} : \hat{\mathfrak{g}}_{V, Z, \varepsilon} \to \hat{G}_{V, Z, \varepsilon}, \quad a \mapsto \prod_{\ell \in V} \exp(a_\ell), \quad a_\ell = \sum_{Z(\gamma) \in \ell} a_\gamma \in \hat{\mathfrak{g}}_{\ell, Z, \varepsilon}.
\end{equation}

where the product is taken in the clockwise order over all rays in $V$.

If $\varepsilon > \varepsilon' > 0$, then $S(V, Z, \varepsilon) \subset S(V, Z, \varepsilon')$ and $\mathfrak{g}_{V, Z, \varepsilon} \subset \mathfrak{g}_{V, Z, \varepsilon'}$. We define the ind-pro-nilpotent Lie algebra

\begin{equation}
\hat{\mathfrak{g}}_{V, Z} = \lim_{\varepsilon \to 0} \hat{\mathfrak{g}}_{V, Z, \varepsilon} \subset \prod_{Z(\gamma) \in V} \mathfrak{g}_{\gamma} \subset \hat{\mathfrak{g}},
\end{equation}

the ind-pro-nilpotent group $\hat{G}_{V, Z} = \lim_{\varepsilon \to 0} \hat{G}_{V, Z, \varepsilon}$ and the bijective exponential map

\begin{equation}
\exp : \hat{\mathfrak{g}}_{V, Z} \to \hat{G}_{V, Z}.
\end{equation}

Note that $\hat{\mathfrak{g}}_{V, Z}$ is independent of a choice of a norm on $\Gamma_{\mathbb{R}}$. Note also that the vector space $\prod_{Z(\gamma) \in V} \mathfrak{g}_{\gamma}$ is not a Lie algebra in general. The maps in (63) induce a bijection

\begin{equation}
\exp_{Z} : \hat{\mathfrak{g}}_{V, Z} \to \hat{G}_{V, Z}.
\end{equation}

Let $a \in \hat{\mathfrak{g}}$ be stability data. For any blunt strict cone $V \subset \mathbb{C}$, we define

\begin{equation}
A_{V} = \prod_{\ell \in V} \exp(a_\ell) \in \hat{G}_{V, Z}, \quad a_\ell = \sum_{Z(\gamma) \in \ell} a_\gamma \in \hat{\mathfrak{g}}_{\ell, Z}.
\end{equation}

The above stability data $a \in \hat{\mathfrak{g}}$ is equivalent (see [41]) to the family $(A_{V} \in \hat{G}_{V, Z})_{V}$, where $V$ runs through all blunt strict cones in $\mathbb{C}$, such that for a decomposition $V = V_1 \cup V_2$ (in the clockwise order), we have the *factorization property* in $\hat{G}_{V, Z}$

\begin{equation}
A_{V} = A_{V_1}A_{V_2}.
\end{equation}
Note that rays $\ell \subset \mathbb{C}$ can be identified with elements of the quotient group $\mathbb{C}^*/\mathbb{R}_{>0} \simeq U(1)$, hence stability data is equivalent to a collection of elements
\begin{equation}
(a_\ell \in \hat{g}_{\ell,Z})_{\ell \in U(1)}.
\end{equation}

Let $\eta: \mathfrak{g} \to \mathfrak{g}$ be an involutive automorphism of Lie algebras that maps $\mathfrak{g}_\gamma$ to $\mathfrak{g}_{-\gamma}$. Then it induces an automorphism $\eta: \hat{g}_{\ell,Z} \to \hat{g}_{-\ell,Z}$. We say that stability data is $\eta$-symmetric if $\eta(a_\ell) = a_{-\ell}$ for all rays $\ell \subset \mathbb{C}$. Equivalently, this means that $\eta(a_\gamma) = a_{-\gamma}$ for all $\gamma \in \Gamma$.

4.5. **Families of stability data.** For more information on continuous families of stability data see [41, 44]. Let $M$ be a topological space. We define a **continuous family of stability data** on $M$ to be a pair $(Z, a)$, where $Z : M \to \Gamma_\mathbb{C}^\times$ is a continuous map and $a = (a_\ell \in \hat{g})_{\ell \in M}$ is a collection such that

1. For any $x \in M$, there exists an open neighborhood $x \in U \subset M$ and $\varepsilon > 0$ such that
   $$\text{supp}(a_y) \subset \{ \gamma \in \Gamma \mid |Z_y(\gamma)| \geq \varepsilon \|\gamma\| \} \quad \forall y \in U.$$
2. For any strict cone $V \subset \mathbb{C}$, we define (67)
   $$A_{x,V} = \prod_{\ell \in V} \exp(a_{x,\ell}) \subset \hat{G}_{V,z_x}, \quad a_{x,\ell} = \sum_{Z_\gamma \in \Gamma_\mathbb{C}^\times} a_{x,\gamma} \in \hat{g}_{\ell,Z_x}.$$

   We require that if $Z_x(\text{supp} a_x) \cap \partial V = \emptyset$ and $\gamma \in \Gamma$, then the map
   $$M \ni y \mapsto p_\gamma \log(A_{y,V}) \in \mathfrak{g}_\gamma$$
   is constant in a neighborhood of $x$, where $p_\gamma : \hat{g} \to \mathfrak{g}_\gamma$ is the projection.

The second axiom is called the **Kontsevich-Soibelman wall-crossing formula** [41, Def. 3]. Sometimes $Z : M \to \Gamma_\mathbb{C}^\times$ will be clear from the context and we will call $a = (a_\ell \in \hat{g})_{\ell \in M}$ a continuous family of stability data. Locally, we can perform calculations by applying Lemma 2.7.

**Remark 4.6.** The above definition is slightly different from [41, Def. 3]. Instead of the first axiom one requires in loc.cit. a seemingly stronger condition that if $a_x$ satisfies the support property with respect to a quadratic form $Q$, meaning that $Q$ is negative definite on $\text{Ker} Z_x$ (this is an open condition) and $Q(\gamma) \geq 0$ for all $\gamma \in \text{supp}(a_x)$, then stability data also satisfy the support property with respect to $Q$ in a neighborhood of $x$. We will prove in Theorem 4.10 that this condition is automatically satisfied if $M$ is locally path connected. For a similar result for families of stability conditions see Theorem 3.10.

**Remark 4.7.** Because of the continuity of $Z$, the first axiom is equivalent to the requirement that for any $x \in M$ there exists an open neighborhood $x \in U \subset M$ and $\varepsilon > 0$ such that
\begin{equation}
\text{supp}(a_y) \subset \{ \gamma \in \Gamma \mid |Z_x(\gamma)| \geq \varepsilon \|\gamma\| \} \quad \forall y \in U.
\end{equation}

**Lemma 4.8.** Let $Z : M \to \Gamma_\mathbb{C}^\times$ be continuous and $a = (a_\ell \in \hat{g})_{\ell \in M}$ be a collection satisfying the first axiom. Then the following conditions are equivalent

1. For any $x \in M$, $\gamma \in \Gamma$ and a strict cone $V \subset \mathbb{C}$ such that $Z_x(\text{supp} a_x) \cap \partial V = \emptyset$, the map $y \mapsto p_\gamma \log(A_{y,V})$ is constant in a neighborhood of $x$. 
(2) For any \( x \in M, \gamma \in \Gamma \) and a strict cone \( V \subset \mathbb{C} \) such that \( Z_x(\Gamma) \cap \partial V = \{0\} \), the map \( y \mapsto p_\gamma \log(A_{y,V}) \) is constant in a neighborhood of \( x \).

(3) For any \( x \in M, \gamma \in \Gamma \) and a strict cone \( V \subset \mathbb{C} \) such that \( Z_x(\gamma) \in V \), there exists a strict cone \( Z_x(\gamma) \in V' \subset V \) such that \( Z_x(\Gamma) \cap \partial V' = \{0\} \) and the map \( y \mapsto p_\gamma \log(A_{y,V'}) \) is constant in a neighborhood of \( x \).

Proof. It is clear that (1) implies (2) and (3). To see that (2) implies (1), let us choose \( \varepsilon > 0 \) such that the first axiom is satisfied for \( x \in M \) and \( |Z_x(\gamma)| \neq \varepsilon |\gamma| \) for all \( 0 \neq \gamma \in \Gamma \). We embed \( V \subset \mathbb{C} \) into a strict cone \( V' \subset \mathbb{C} \) such that \( Z_x(\Gamma) \cap \partial V' = \{0\} \). We consider the semigroup \( S = S(V', Z_x, \varepsilon) \) and the lower set \( I = \{ \gamma' \in S \mid \gamma' \leq \gamma \} \). By Lemma 2.7, we can assume that \( I \subset S(V', Z_y, \varepsilon) \) is a lower set for all \( y \). Therefore we can restrict our considerations from \( g \) to the nilpotent Lie algebra \( g_I = \bigoplus_{\gamma \in S} g_\gamma / \bigoplus_{\gamma \in S \setminus I} g_\gamma \simeq \bigoplus_{\gamma \in I} g_\gamma \). By assumption, the map \( y \mapsto A_{y,V'} \) (in the nilpotent group \( \exp(g_I) \)) is constant in a neighborhood of \( x \). This implies that the map \( y \mapsto A_{y,V} \) is also constant in a neighborhood of \( x \).

To see that (3) implies (2), let us choose \( \varepsilon > 0 \) as before. We consider the semigroup \( S = S(V, Z_x, \varepsilon) \) and the lower set \( I = \{ \gamma' \in S \mid \gamma' \leq \gamma \} \). By Lemma 2.7, we can assume that \( I \subset S(V, Z_y, \varepsilon) \) is a lower set for all \( y \). Therefore we can restrict our considerations from \( g \) to the nilpotent Lie algebra \( g_I \). By induction, we can assume that \( y \mapsto p_\gamma \log(A_{y,V}) \) is constant in a neighborhood of \( x \) for all \( \gamma' < \gamma \). By assumption, we can decompose \( V = V_1 \sqcup V_2 \sqcup V_3 \), where \( V_i \) are strict cones ordered clockwise, such that \( Z_x(\Gamma) \cap \partial V_i = \{0\} \), \( Z_x(\gamma) \in V_2 \) and the map \( y \mapsto p_\gamma \log(A_{y,V_2}) \) is constant in a neighborhood of \( x \). Shrinking the neighborhood of \( x \) if needed, we conclude that \( y \mapsto A_{y,V_i} \) (in the nilpotent Lie group \( \exp(g_I) \)) is constant for \( i = 1, 2, 3 \). Therefore also \( y \mapsto A_{y,V} \) is constant.

Note that the second axiom allows us to express \( a_{x,\gamma} \) locally as finite Lie expressions of elements \( a_{y,\gamma'} \). The following result shows that this can be done globally (cf. [41, §2.3]).

**Lemma 4.9.** Let \( M \) be path-connected and \((Z, a)\) be a continuous family of stability data on \( M \). For any \( x, y \in M \) and \( \gamma \in \Gamma \setminus \{0\} \), the element \( a_{x,\gamma} \) is a finite Lie expression of elements \( a_{y,\gamma'} \).

**Proof.** We can assume that \( M = [0, 1] \). We will show that for any \( t \in [0, 1] \) and \( \gamma \in \Gamma \setminus \{0\} \), the element \( a_{t,\gamma} \) is a finite Lie expression of elements \( a_{t,\gamma'} \). As \( M \) is compact, there exists \( \varepsilon > 0 \) such that

\[
\supp a_t \subset \{ \gamma \in \Gamma \setminus \{0\} \mid \|Z_t(\gamma)\| \geq \varepsilon \|\gamma\| \}
\]

for all \( t \in M \). Let \( 0 < \theta < \pi/4 \). By dividing \([0, 1]\) into a finite union of intervals, we can assume that \( \|Z_t - Z_1\| \leq \frac{\pi}{2} \sin(\theta) \) for all \( t \in [0, 1] \). For any \( \gamma \in \supp a_t \), we have \( |Z_t(\gamma) - Z_1(\gamma)| \leq \frac{\pi}{2} \|\gamma\| \leq \frac{\pi}{2} |Z_1(\gamma)| \), hence

\[
|Z_1(\gamma)| \geq \frac{1}{\theta} |Z_1(\gamma)| \leq \frac{\pi}{2} \|\gamma\|.
\]

On the other hand, if \( \gamma \in \Gamma \setminus \{0\} \) satisfies \( |Z_1(\gamma)| \geq \frac{\pi}{2} \|\gamma\| \), then

\[
|Z_t(\gamma) - Z_1(\gamma)| \leq \frac{\pi}{2} \sin(\theta) \|\gamma\| \leq \sin(\theta) |Z_1(\gamma)|,
\]

hence the angle between \( Z_t(\gamma) \) and \( Z_1(\gamma) \) is \( \leq \theta \). If \( \gamma = \gamma_1 + \gamma_2 \) with \( Z_1(\gamma_i) \) contained in the same ray as \( Z_t(\gamma) \) and \( |Z_1(\gamma_i)| \geq \frac{\pi}{2} \|\gamma_i\| \), then the angle between \( Z_t(\gamma_i) \) and \( Z_1(\gamma_i) \) is...
\[ \frac{\pi}{2} \| \gamma_i \| \leq |Z_1(\gamma_i)| < |Z_1(\gamma)|, \quad i = 1, 2. \]

For a fixed \( \gamma \in \Gamma \setminus \{0\} \), we can assume by induction that the result is true for all elements of the finite set
\[ D = \left\{ \gamma' \in \Gamma \setminus \{0\} \mid \frac{\pi}{2} \| \gamma' \| \leq |Z_1(\gamma')| < |Z_1(\gamma)| \right\}. \]

By the second axiom, for any \( t \in [0, 1] \), there exists an open set \( U \subset [0, 1] \) such that, for any \( t' \in U \), the element \( a_{t', \gamma} \) can be written as a finite Lie expression of \( a_{t', \gamma'} \) with \( Z_t(\gamma') \) contained in the same ray as \( Z_t(\gamma) \). Therefore, there exists a finite sequence \( 0 = t_0 < \cdots < t_n = 1 \) such that, for any \( t' \in [t_i, t_{i+1}] \), the element \( a_{t_i, \gamma} \) can be written as a finite Lie expression of \( a_{t', \gamma} \) with \( Z_{t_i}(\gamma') \) contained in the same ray as \( Z_{t_i}(\gamma) \). By the discussion above we conclude that \( \gamma' \in D \cup \{\gamma\} \). Therefore by the inductive assumption we only need to express \( a_{t_{i+1}, \gamma} \) in terms of \( a_{1, \gamma'} \). Now we repeat the previous step. \( \square \)

**Theorem 4.10.** Let \( M \) be a path-connected topological space and \((Z, a)\) be a continuous family of stability data on \( M \). Let \( Q : \Gamma \rightarrow \mathbb{R} \) be a quadratic form that is negative semidefinite on \( \text{Ker} Z_x \) for all \( x \in M \). If there exists \( x \in M \) such that \( \text{supp} a_x \subset \{Q \geq 0\} \), then the same is true for all points of \( M \).

**Proof.** We will use the same notation and assumptions as in the previous lemma and we assume that \( Q(\gamma) \geq 0 \) for all \( \gamma \in \text{supp} a_1 \). Let \( \gamma \in \text{supp} a_0 \) and let \( 0 \leq i \leq n \) be the maximal number such that \( \gamma \in \text{supp} a_i \). If \( i = n \), then we are done. Otherwise, we can express \( \gamma = \sum_{k=1}^m \gamma_k \), where \( \gamma_k \in \text{supp} a_{i+1} \) and \( Z_{t_i}(\gamma_k) \) are contained in the same ray as \( Z_{t_i}(\gamma) \). We conclude that \( \gamma_k \in D \), hence by induction \( Q(\gamma_k) \geq 0 \). By Lemma 2.9, we obtain \( Q(\gamma) \geq 0 \).

Alternatively, we can apply Theorem 2.14. For any \( x \in M \), let \( s_x = \text{supp}(a_x) \) and let \( t_x = \bigcup_{\ell \subset C} t_x, \ell \) where \( t_x, \ell \) is the set of minimal elements in the strict semigroup \( \text{sg}(s_x) \cap Z^{-1}(\ell) \). It is easy to see that \((Z_x, t_x, s_x)_{x \in M}\) is a continuous family of stable supports (cf. Lemma 3.9), hence we can apply Theorem 2.14. \( \square \)

The following results follows from [41, Theorem 3].

**Theorem 4.11.** Let \( M \) be a path-connected topological space, \((Z, a)\) be a continuous family of stability data on \( M \) and \( Q : \Gamma \rightarrow \mathbb{R} \) be a quadratic form that is negative-definite on \( \text{Ker} Z_x \) and satisfies \( \text{supp} a_x \subset \{Q \geq 0\} \) for all \( x \in M \). If \( x, y \in M \) are such that \( Z_x = Z_y \), then \( a_x = a_y \).

This result implies that it is enough to parametrize stability data by subsets of \( \text{Hom}(\Gamma, C) \) as long as stability data are supported on \( \{Q \geq 0\} \) for a fixed quadratic from \( Q \).

4.6. Wall-crossing structures. In this section we introduce wall-crossing structures following [43] and we discuss their relationship to continuous families of stability data. A simplified definition of a wall-crossing structure, without usage of WCS sheaves, is given in (91).
4.6.1. **Group slicing.** Let $\mathfrak{g}$ be a $\Gamma$-graded Lie algebra with a finite strict support. For any $x \in \Gamma_{\mathbb{R}}$, we define the subalgebras
\begin{equation}
\mathfrak{g}^+_x = \bigoplus_{x(\gamma) > 0} \mathfrak{g}_\gamma, \quad \mathfrak{g}^0_x = \bigoplus_{x(\gamma) = 0} \mathfrak{g}_\gamma, \quad \mathfrak{g}^-_x = \bigoplus_{x(\gamma) < 0} \mathfrak{g}_\gamma.
\end{equation}

The Lie algebra $\mathfrak{g}$ is nilpotent, hence we can define the corresponding nilpotent groups
\begin{equation}
G = \exp(\mathfrak{g}), \quad G^x = \exp(\mathfrak{g}^x), \quad \star \in \{+, 0, -\}.
\end{equation}

By Lemma 4.3, there is a bijection
\begin{equation}
G^+ = G_0 \times G^- \rightarrow G, \quad \ (g_+, g_0, g_-) \mapsto g = g_+ g_0 g_-.
\end{equation}

Taking the inverse and the projection to the middle term, we obtain a map (which is not a group homomorphism in general)
\begin{equation}
\pi_x : G \rightarrow G_0^x, \quad g \mapsto g_0.
\end{equation}

Similarly, let $\mathfrak{g}$ be a $\Gamma$-graded Lie algebra with a strict support, $\hat{\mathfrak{g}}$ be its pro-nilpotent completion and $\hat{G}$ be the corresponding pro-nilpotent group. For any $x \in \Gamma_{\mathbb{R}}$, we consider Lie algebras (71), the corresponding pro-nilpotent Lie algebras $\hat{\mathfrak{g}}^+_x$, $\hat{\mathfrak{g}}^0_x$ and the corresponding pro-nilpotent groups $\hat{G}^+_x$, $\hat{G}^0_x$. Then we have a bijection
\begin{equation}
\hat{G}^+ = \hat{G}_0 \times \hat{G}^- \rightarrow \hat{G}, \quad \ (\hat{g}_+, \hat{g}_0, \hat{g}_-) \mapsto \hat{g} = \hat{g}_+ \hat{g}_0 \hat{g}_-.
\end{equation}

Taking the inverse and the projection to the middle term, we obtain a map (which is not a group homomorphism in general)
\begin{equation}
\pi_x : \hat{G} \rightarrow \hat{G}_0^x, \quad \hat{g} \mapsto \hat{g}_0.
\end{equation}

These projections are natural with respect to morphisms of $\Gamma$-graded Lie algebras. More precisely, for any homomorphism $f : \mathfrak{g} \rightarrow \mathfrak{h}$ of $\Gamma$-graded Lie algebras with strict support and $\hat{G} = \exp(\hat{\mathfrak{g}}), H = \exp(\hat{\mathfrak{h}})$, we have a commutative diagram
\begin{equation}
\begin{array}{ccc}
\hat{G} & \xrightarrow{\pi_x} & \hat{G}_0^x \\
\exp(f) \downarrow & & \downarrow \exp(f) \\
\hat{H} & \xrightarrow{\pi_x} & \hat{H}_0^x
\end{array}
\end{equation}

4.6.2. **Special sheaf construction.** Let $M$ be a topological space and $(\pi_x : S \rightarrow S_x)_{x \in M}$ be a collection of maps between sets. Then we define the sheaf $S$ over $M$ to be the sheafification of the presheaf
\begin{equation}
M \ni U \mapsto \{(\pi_x(s))_{x \in U} \mid s \in S\} \subset \prod_{x \in U} S_x.
\end{equation}

In particular, let us assume that the maps $\pi_x : S \rightarrow S_x$ are surjective and $\forall s, s' \in S$ the set
\begin{equation}
\{x \in M \mid \pi_x(s) = \pi_x(s')\}
\end{equation}

is open in $M$ (cf. [41, §2.1.2]). Then the set of sections $\Gamma(U, S)$ over an open set $U \subset M$ consists of $a \in \prod_{x \in U} S_x$ such that, for any $x \in U$, there exists an open neighborhood $x \in U' \subset U$ and $s \in S$ satisfying
\begin{equation}
\pi_y(s) = a_y \quad \forall y \in U'.
\end{equation}
The stalk $S_x$ of the sheaf $S$ is equal to $S_x$ for all $x \in M$.

Let us assume additionally that, for all $x \in M$, we have $S_x \subset S$ such that $S_x \hookrightarrow S \xrightarrow{\pi_x} S_x$ is the identity. Then $\Gamma(U, S)$ consists of $a \in \prod_{x \in U} S_x$ (or $a \in S^U = \prod_{x \in U} S$) such that, for any $x \in U$, there exists an open neighborhood $x \in U' \subset U$ satisfying

\begin{equation}
\pi_y(a_x) = a_y \quad \forall y \in U'.
\end{equation}

4.6.3. **WCS sheaf: finite strict support.** Let $\mathfrak{g}$ be a $\Gamma$-graded Lie algebra with a finite strict support. As in §4.6.1, we have the maps $\pi_x: G \rightarrow G^{(x)}_0$ for $x \in \Gamma^\vee_{\mathbb{R}}$. For any $g, g' \in G$, the set

\begin{equation}
\{ x \in \Gamma^\vee_{\mathbb{R}} \mid \pi_x(g) = \pi_x(g') \}
\end{equation}

is open, see e.g. [52]. Applying construction from §4.6.2 to the topological space $\Gamma^\vee_{\mathbb{R}}$ and the collection of surjective maps $(\pi_x)_{x \in \Gamma^\vee_{\mathbb{R}}}$, we define the sheaf $\text{WCS}_{\mathfrak{g}}$ of wall-crossing structures over $\Gamma^\vee_{\mathbb{R}}$ as follows. For any open set $U \subset \Gamma^\vee_{\mathbb{R}}$, the set of sections $\Gamma(U, \text{WCS}_{\mathfrak{g}})$ consists of elements $a \in \prod_{x \in U} \mathfrak{g}^{(x)}_0$ (or $a \in \mathfrak{g}^U$) such that, for any $x \in U$, there exists an open neighborhood $x \in U' \subset U$ satisfying

\begin{equation}
\pi_y(e^{a_x}) = e^{a_y} \quad \forall y \in U'.
\end{equation}

4.6.4. **WCS sheaf: strict support.** Let $\mathfrak{g}$ be a $\Gamma$-graded Lie algebra having a strict support and let $S \subset \Gamma$ be the strict semigroup generated by this support. As in (52), we define the pro-nilpotent Lie algebra $\hat{\mathfrak{g}} = \lim_{I \subset S} \mathfrak{g}_I \simeq \prod_{\gamma \in \Gamma} \mathfrak{g}_\gamma$, where the limit runs over finite lower sets $I \subset S$ and $\mathfrak{g}_I$ denotes the corresponding nilpotent Lie algebra (51). We define the sheaf $\text{WCS}_{\mathfrak{g}}$ of wall-crossing structures over $\Gamma^\vee_{\mathbb{R}}$ to be

\begin{equation}
\text{WCS}_{\mathfrak{g}} = \lim_{I \subset S} \text{WCS}_{\mathfrak{g}_I},
\end{equation}

where the limit runs over finite lower sets $I \subset S$. For any open set $U \subset \Gamma^\vee_{\mathbb{R}}$, the set of sections is equal to

\begin{equation}
\Gamma(U, \text{WCS}_{\mathfrak{g}}) = \lim_{I \subset S} \Gamma(U, \text{WCS}_{\mathfrak{g}_I}).
\end{equation}

Explicitly, this means that $\Gamma(U, \text{WCS}_{\mathfrak{g}})$ consists of $a \in \prod_{x \in U} \mathfrak{g}^{(x)}_0$ (or $a \in \hat{\mathfrak{g}}^U$) such that

\begin{equation}
(p_I a_x)_{x \in U} \in \Gamma(U, \text{WCS}_{\mathfrak{g}_I}),
\end{equation}

where $p_I: \hat{\mathfrak{g}} \rightarrow \mathfrak{g}_I$ is the projection, for any finite lower set $I \subset S$. This means that $a \in \prod_{x \in U} \mathfrak{g}^{(x)}_0$ is contained in $\Gamma(U, \text{WCS}_{\mathfrak{g}})$ if and only if for any $x \in U$ and $\gamma \in S$, there exists an open set $x \in U' \subset U$ such that

\begin{equation}
a_{y, \gamma} = p_{\gamma} \log(\pi_y(e^{a_x})) \quad \forall y \in U',
\end{equation}

where $p_{\gamma}: \hat{\mathfrak{g}} \rightarrow \mathfrak{g}_{\gamma}$ is the projection and $\pi_y: \hat{G} \rightarrow \hat{G}^{(y)}_0$ is the map defined in §4.6.1.
4.6.5. WCS sheaf: arbitrary support. Let $\mathfrak{g}$ be an arbitrary $\Gamma$-graded Lie algebra. Let $\mathcal{P}$ be the set of all strict semigroups $S \subset \Gamma$, equipped with the partial order induced by inclusion. For every semigroup $S \in \mathcal{P}$, the Lie algebra $\mathfrak{g}_S = \bigoplus_{\gamma \in S} \mathfrak{g}_\gamma$ has a strict support. If $S \subset S'$, then $\mathfrak{g}_S \subset \mathfrak{g}_{S'}$ and we obtain the induced inclusion of sheaves $\text{WCS}_{\mathfrak{g}_S} \subset \text{WCS}_{\mathfrak{g}_{S'}}$. We define the sheaf $\text{WCS}_\mathfrak{g}$ of wall-crossing structures over $\Gamma_\mathfrak{g}$ to be the colimit of sheaves

\begin{equation}
\text{WCS}_\mathfrak{g} = \lim_{S \in \mathcal{P}} \text{WCS}_{\mathfrak{g}_S}.
\end{equation}

Note that for any open set $U \subset \Gamma_\mathfrak{g}$, we have $\Gamma(U, \text{WCS}_{\mathfrak{g}_S}) \subset \hat{\mathfrak{g}}_S^U \subset \hat{\mathfrak{g}}^U$, hence $\Gamma(U, \text{WCS}_\mathfrak{g}) \subset \hat{\mathfrak{g}}^U$. More explicitly, the set of sections $\Gamma(U, \text{WCS}_\mathfrak{g})$ consists of elements $a \in \prod_{x \in U} \hat{\mathfrak{g}}_x^{(e^a_x)} \subset \hat{\mathfrak{g}}^U$ such that, for any $x \in U$, there exists an open neighborhood $x \in U' \subset U$ such that the set $\bigcup_{y \in U'} \text{supp}(a_y) \subset \Gamma$ generates a strict semigroup $S \subset \Gamma$ and

\begin{equation}
(a_y)_{y \in U'} \in \Gamma(U', \text{WCS}_{\mathfrak{g}_S}).
\end{equation}

4.6.6. Wall-crossing structures. Given a topological space $M$ and a continuous map $\theta: M \to \Gamma_\mathfrak{g}$, we define the sheaf

\begin{equation}
\text{WCS}_{\mathfrak{g}, \theta} = \theta^* \text{WCS}_\mathfrak{g}
\end{equation}

of wall-crossing structures over $M$. We define a wall-crossing structure over $M$ to be a global section of $\text{WCS}_{\mathfrak{g}, \theta}$. We can actually describe the sheaf $\text{WCS}_{\mathfrak{g}, \theta}$ directly in the same way as the sheaf $\text{WCS}_\mathfrak{g}$:

1. If $\mathfrak{g}$ has a finite strict support, then the set of sections $\Gamma(U, \text{WCS}_{\mathfrak{g}, \theta})$ (for $U \subset M$) consists of $a \in \prod_{x \in U} \hat{\mathfrak{g}}_x^{(e^a_x)}$ such that, for any $x \in U$, there exists an open set $x \in U' \subset U$ such that $\pi_{\theta_x}(e^{a_x}) = e^{a_y}$ for all $y \in U'$, where $\pi_{\theta_x}: \hat{\mathfrak{g}} \to \mathfrak{g}_0^{(\theta_x)}$ is the projection.

2. If $\mathfrak{g}$ has a strict support generating the semigroup $S \subset \Gamma$, then $\Gamma(U, \text{WCS}_{\mathfrak{g}, \theta})$ (for $U \subset M$) consists of $a \in \prod_{x \in U} \hat{\mathfrak{g}}_x^{(e^a_x)}$ such that $(p_I a_x)_{x \in U} \in \Gamma(U, \text{WCS}_{\mathfrak{g}_I, \theta})$, where $p_I: \hat{\mathfrak{g}} \to \mathfrak{g}_I$ is the projection, for any lower set $I \subset S$.

3. If $\mathfrak{g}$ is arbitrary, then $\Gamma(U, \text{WCS}_{\mathfrak{g}, \theta})$ (for $U \subset M$) consists of $a \in \prod_{x \in U} \hat{\mathfrak{g}}_x^{(e^a_x)}$ such that, for any $x \in U$, there exists an open set $x \in U' \subset U$ such that $\bigcup_{y \in U'} \text{supp}(a_y)$ generates a strict semigroup $S \subset \Gamma$ and $(a_y)_{y \in U'} \in \Gamma(U', \text{WCS}_{\mathfrak{g}_I, \theta})$.

We conclude that a wall-crossing structure on $M$ (for a fixed continuous map $\theta: M \to \Gamma_\mathfrak{g}$) is a collection $a \in \prod_{x \in M} \hat{\mathfrak{g}}_x^{(e^a_x)}$ such that for any $x \in M$ and $\gamma \in \Gamma$, there exists an open set $x \in U \subset M$ such that $\bigcup_{y \in U} \text{supp}(a_y)$ generates a strict semigroup $S \subset \Gamma$ and

\begin{equation}
a_{y, \gamma} = p_{\gamma}(\log \pi_y(e^{a_y})) \quad \forall y \in U,
\end{equation}

where we consider projections $\pi_y: \hat{\mathfrak{g}}_S \to \mathfrak{g}_S^{(\theta_y)}$ with $\hat{\mathfrak{g}}_S = \exp(\hat{\mathfrak{g}}_S)$, $\mathfrak{g}_S = \bigoplus_{\gamma' \in S} \mathfrak{g}_{\gamma'}$.

Remark 4.12. It is proved in [43, Theorem 2.1.6] that if $\mathfrak{g}$ has a strict support, then

\begin{equation}
\Gamma(\Gamma_\mathfrak{g}, \text{WCS}_\mathfrak{g}) \simeq \hat{\mathfrak{g}}.
\end{equation}

Wall-crossing structures on $\Gamma_\mathfrak{g}$ can be identified with scattering diagrams [43].
4.6.7. WCS induced by families of stability data. Let $M$ be a topological space and $(Z_x, a_x)_{x \in M}$ be a continuous family of stability data on a $\Gamma$-graded Lie algebra $\mathfrak{g}$. Then every $Z_x \in \Gamma^\vee$ has the form $Z_x = -\theta_x + i\rho_x$ for some $\theta_x, \rho_x \in \Gamma^\vee_R$. We define the continuous map
\begin{equation}
\theta: M \to \Gamma^\vee_R, \quad \theta_x = -\Re(Z_x) = \Im(i^{-1} \cdot Z_x).
\end{equation}
Note that if $Z_x(\gamma) \in \mathbb{H}$ for some $\gamma \in \Gamma$, then $\theta_x(\gamma) = 0$ if and only if $Z_x(\gamma) \in \ell_0 := i\mathbb{R}_{>0}$.

**Lemma 4.13.** The collection
\begin{equation}
b = (b_x)_{x \in M}, \quad b_x = (a_{x, \gamma})_{Z_x(\gamma) \in \ell_0} \in \mathfrak{g}_0^{(\theta_x)},
\end{equation}
is a wall-crossing structure on $M$.

**Proof.** For any $x \in M$, there exists an open set $x \in U \subset M$ and $\varepsilon > 0$ such that
\begin{equation}
\supp(a_y) \subset \{ \gamma \in \Gamma \mid |Z_y(\gamma)| \geq \varepsilon \|\gamma\| \}.
\end{equation}
Let $0 < \eta < \pi/4$ and let us assume that $|Z_x - Z_y| \leq \frac{\varepsilon}{2} \sin(\eta)$ for all $y \in U$. Then, for any $\gamma \in \supp(a_y)$, we have $|Z_x(\gamma)| \geq \frac{\varepsilon}{2} \|\gamma\|$ and
\begin{equation}
|Z_x(\gamma) - Z_y(\gamma)| < \varepsilon \sin(\eta) \|\gamma\| \leq \sin(\eta) |Z_y(\gamma)|.
\end{equation}
This implies that the angle between $Z_x(\gamma)$ and $Z_y(\gamma)$ is $< \eta$. Let $V \subset \mathbb{C}$ be the cone around $\ell_0$ having angle $2\eta < \pi/2$. If $\gamma \in \supp(b_y)$, then $Z_y(\gamma) \in \ell_0$, hence $Z_x(\gamma) \in V$. We conclude that
\begin{equation}
\supp(b_y) \subset S = S(V, Z_x, \varepsilon/2) \quad \forall y \in U,
\end{equation}
hence $\bigcup_{y \in U} \supp(b_y)$ generates a strict semigroup. Let us consider $0 \neq \gamma \in \Gamma$ with $|Z_x(\gamma)| \geq \frac{\varepsilon}{2} \|\gamma\|$. If $Z_x(\gamma) \notin \ell_0$, then we can decrease $\eta$ and shrink $U$ so that $\gamma \notin S$ and the condition (91) is automatically satisfied. If $Z_x(\gamma) \in \ell_0$, then we can decrease $\eta$ and shrink $U$ so that the finite lower set $I = \{ \gamma' \in S \mid \gamma' \leq \gamma \}$ is contained in $Z_x^{-1}(\ell_0)$. Note the condition (91) follows from the second axiom of a continuous family of stability data.

Note that the above wall-crossing structure does not capture the full information of the family of stability data as it remembers only stability data along a particular ray. To solve this problem, we either need to assume that rays rotate in the family $M$ or we can introduce such rotation explicitly as follows [43]. Let us consider a new continuous family of stability data over $\tilde{M} = U(1) \times M$
\begin{equation}
U(1) \times M \ni (z, x) \mapsto (iz^{-1}Z_x, a_x).
\end{equation}
Then we define the corresponding continuous map
\begin{equation}
\tilde{\theta}: \tilde{M} \to \Gamma^\vee_R, \quad (z, x) \mapsto \Im(z^{-1}Z_x)
\end{equation}
By the previous lemma, we have a wall-crossing structure
\begin{equation}
b = (b_{z, x})_{(z, x) \in \tilde{M}}, \quad b_{z, x} = (a_{x, \gamma})_{Z_x(\gamma) \in \ell_{R>0}} \in \mathfrak{g}_0^{(\theta_{z, x})}.
\end{equation}
This wall-crossing structure can be used to completely recover the original continuous family of stability data on $M$. 
5. Wall-crossing formulas

5.1. Grothendieck ring of stacks. For more details see e.g. [15, 35, 66]. In this section we will consider only algebraic (or Artin) stacks locally of finite type over $\mathbb{C}$. Let $\text{St}^a$ denote the 2-category of algebraic stacks of finite type over $\mathbb{C}$ having affine stabilizers. Given an algebraic stack $S$ with affine stabilizers, we denote by $K(\text{St}^a/S)$ the corresponding Grothendieck group with rational coefficients generated by isomorphism classes $[X \to S]$ of objects in $\text{St}^a/S$ subject to usual relations [15, §3]. We can similarly consider the category $\text{Var}$ of finite type algebraic varieties over $\mathbb{C}$ and the Grothendieck group $K(\text{Var}/S)$ with rational coefficients.

In particular, for $S = \text{Spec} \mathbb{C}$, the corresponding Grothendieck groups $K(\text{St}^a) = K(\text{St}^a/\mathbb{C})$ and $K(\text{Var}) = K(\text{Var}/\mathbb{C})$ have a ring structure with the product defined by $[X] \cdot [Y] = [X \times Y]$. We define $\mathbb{L} = [\mathbb{A}^1]$ so that $[\text{GL}_n] = \prod_{i=0}^{n-1}(\mathbb{L}^n - \mathbb{L}^i)$. It is proved in [66] (see also [15]) that there is a natural isomorphism

$$K(\text{St}^a) \simeq K(\text{Var})[[\text{GL}_n]]^{-1} : n \geq 1 = K(\text{Var})[(\mathbb{L} - 1)^{-1}, \mathbb{L}^{-1}, [\mathbb{P}^n]^{-1} : n \geq 1].$$

For any algebraic stack $S$ with affine stabilizers, the Grothendieck group $K(\text{St}^a/S)$ is a module over $K(\text{St}^a)$ with the product defined by

$$[X] \cdot [Y \to S] = [X \times Y \to Y \to S].$$

Let us also define

$$K^\circ(\text{St}^a/S) = \text{Im} \left( K(\text{Var}/S)[\mathbb{L}^{-1}, [\mathbb{P}^n]^{-1} : n \geq 1] \to K(\text{St}^a/S) \right),$$

which is a module over $K^\circ(\text{St}^a) = K^\circ(\text{St}^a/\mathbb{C})$. Note that the above map is not necessarily injective as $\mathbb{L} - 1$ can be a zero divisor in $K(\text{Var}/\mathbb{C})$, cf. [11, 49].

Given a morphism of stacks $f : S \to T$, we define

$$f_* : K(\text{St}^a/S) \to K(\text{St}^a/T), \quad [X \to S] \mapsto [X \to S \to T].$$

If $f : S \to T$ is of finite type, then it induces

$$f^* : K(\text{St}^a/T) \to K(\text{St}^a/S), \quad [Y \to T] \mapsto [S \times_T Y \to S].$$

In what follows we will introduce the square root $\mathbb{L}^{\frac{1}{2}}$ of $\mathbb{L}$ and define

$$K(\text{St}^a/\mathbb{C}) = K(\text{St}^a/\mathbb{C})[\mathbb{L}^{\frac{1}{2}}], \quad K^\circ(\text{St}^a/\mathbb{C}) = K^\circ(\text{St}^a/\mathbb{C})[\mathbb{L}^{\frac{1}{2}}].$$

By the results of Deligne [22], there exists a ring homomorphism

$$E : K(\text{Var}/\mathbb{C}) \to \mathbb{Q}[u, v], \quad [X] \mapsto \sum_{p,q,n} (-1)^n \dim \left( \text{Gr}_p^W \text{Gr}_{p+q}^{W+} H_c^n(X, \mathbb{C}) \right) u^p v^q,$$

called the Hodge-Deligne polynomial map or the $E$-polynomial map. Note that $H_c^n(\mathbb{A}^1, \mathbb{Q}) = \mathbb{Q}(-1)[-2]$, hence $E(\mathbb{L}) = uv$. We can extend $E$ to

$$E : K(\text{St}^a/\mathbb{C}) \to \mathbb{Q}(u, \sqrt{uv}),$$

where $\mathbb{L}^{\frac{1}{2}} \mapsto -\sqrt{uv}$. Similarly, we define the Poincaré polynomial map

$$P : K(\text{Var}/\mathbb{C}) \to \mathbb{Q}[y], \quad [X] \mapsto E(X; y, y),$$

which extends to $P : K(\text{St}^a/\mathbb{C}) \to \mathbb{Q}(y)$, where $\mathbb{L}^{\frac{1}{2}} \mapsto -y$. 


5.2. **Motivic Hall algebra of an exact category.** For more details see e.g. [15, 33, 24]. Let $\mathcal{E}$ be an exact category, meaning an additive category equipped with a class of short exact sequences (to be called admissible exact sequences) satisfying Quillen’s axioms [59, 38]. We define admissible monomorphisms and epimorphisms to be respectively monomorphisms and epimorphisms of admissible short exact sequences. Let us recall the Waldhausen construction of the simplicial groupoid associated with an exact category $\mathcal{E}$. Let us define $M_n$ to be the groupoid of chains
\begin{equation}
0 = E_0 \hookrightarrow E_1 \hookrightarrow \ldots \hookrightarrow E_n,
\end{equation}
where $E_{i-1} \hookrightarrow E_i$ are admissible monomorphisms, together with choices of cokernels $E_{j/s}$ of $E_i \hookrightarrow E_j$. Note that $M_0$ is a point, $M = M_1$ parametrizes objects of $\mathcal{E}$, and $M_2$ parametrizes admissible exact sequences $0 \to E_1 \to E_2 \to E_{2/1} \to 0$ in $\mathcal{E}$.

Groupoids $M_n$ form a simplicial groupoid, where for any weakly increasing map $\alpha: [m] \to [n] = \{0, 1, \ldots, n\}$, we define the face map
\begin{equation}
\alpha^*: M_n \to M_m, \quad [E_0 \hookrightarrow \ldots \hookrightarrow E_n] \mapsto [E_{\alpha_0} \hookrightarrow \ldots \hookrightarrow E_{\alpha_m}] / E_{\alpha_0}.
\end{equation}
The fundamental property of this simplicial groupoid is the 2-Segal property [24]. We will identify increasing maps $\alpha: [m] \to [n]$ with subsets $\{\alpha_0, \ldots, \alpha_m\} \subset [n]$ and we will denote $\alpha^*$ by $p_{\alpha_0, \ldots, \alpha_m}$. In particular, we consider
\begin{equation}
q = (p_{01}, p_{12}): M_2 \to M \times M, \quad [E_1 \hookrightarrow E_2] \mapsto (E_1, E_{2/1}),
\end{equation}
\begin{equation}
p = p_{02}: M_2 \to M, \quad [E_1 \hookrightarrow E_2] \mapsto E_2.
\end{equation}

Let us assume that every $M_n$ is equipped with a structure of an algebraic stack locally of finite type over $\mathbb{C}$ with affine stabilizers, so that all face maps are morphisms of algebraic stacks and the map $q: M_2 \to M \times M$ is of finite type. We define the **motivic Hall algebra**
\begin{equation}
H(\mathcal{E}) = K(\text{St}^a / M)
\end{equation}
with the product given by
\begin{equation}
K(\text{St}^a / M) \otimes K(\text{St}^a / M) \to K(\text{St}^a / M^2) \xrightarrow{q^*} K(\text{St}^a / M_2) \xrightarrow{p_*} K(\text{St}^a / M),
\end{equation}
The associativity is a consequence of the 2-Segal property.

**Remark 5.1.** It is actually enough to equip groupoids $M_n$ with a slightly weaker structure than an algebraic stack structure. Given a groupoid $\mathcal{G}$, let us consider the set of pairs $(X, f)$, where $X \in \text{St}^a$ and $f: X(\mathbb{C}) \to \mathcal{G}$ is an equivalence of categories. We will say that two such pairs $(X_1, f_1)$ and $(X_2, f_2)$ are equivalent if there exist morphisms $g_i: Z \to X_i$ in $\text{St}^a$ such that the functors $Z(\mathbb{C}) \xrightarrow{g_i(\mathbb{C})} X_i(\mathbb{C}) \xrightarrow{f_i} \mathcal{G}$ for $i = 1, 2$ are naturally isomorphic. Note that morphisms $g_i: Z \to X_i$ are geometric bijections, meaning that $g_i: Z(\mathbb{C}) \to X_i(\mathbb{C})$ are equivalences of categories. Therefore there exist finite stratifications of $Z$ and $X_i$ such that $g_i$ induce isomorphisms between the strata [15, Lemma 3.2]. This implies that $g_i$ induce isomorphisms $g_*: K(\text{St}^a / Z) \to K(\text{St}^a / X_i)$, hence we obtain an isomorphism $K(\text{St}^a / X_1) \simeq K(\text{St}^a / X_2)$. We will call an equivalence class of above pairs an **atlas** of $\mathcal{G}$. In what follows, we will have a linear map $\text{cl}: K(\mathcal{E}) \to \Gamma$ and a decomposition $M = \bigsqcup_{\gamma \in \Gamma} M(\gamma)$, where $M(\gamma)$ is the groupoid parameterizing objects
$E \in \mathcal{E}$ with $\text{cl} E = \gamma$. For our considerations it will be enough to equip every groupoid $\mathcal{M}(\gamma)$ (as well as the fibers of $\mathcal{M}_n \xrightarrow{\rho_{n-1,n}} \mathcal{M}_n \to \Gamma^n$) with an atlas.

5.3. Wall-crossing in the Hall algebra. In this section we will restrict our considerations to the triangulated category $\mathcal{D} = D^b(\text{Coh} X)$, where $X$ is a smooth projective variety over $\mathbb{C}$, although most of the results can be formulated for other triangulated categories. By the results of [47], the stack $\mathcal{M}$ parameterizing objects $E \in \mathcal{D}$ with $\text{Hom}^{<0}(E, E) = 0$ is algebraic and locally of finite type over $\mathbb{C}$. More precisely, we consider the 2-functor

$$
\mathcal{M}: \text{Sch}/\mathbb{C} \to \text{Grpd}
$$

that sends a scheme $S$ to the groupoid of objects $E \in D^b(X \times S)$ that are relatively perfect [47] and satisfy $\text{Hom}^{<0}(E_s, E_s) = 0$ for $s \in S$. Then $\mathcal{M}$ is an algebraic stack locally of finite type over $\mathbb{C}$.

Remark 5.2. The stack $\mathcal{M}$ has affine stabilizers. Indeed, the stabilizer of any object $E$ can be identified with the group $\text{Aut}(E)$ which is the group of invertible elements of the finite-dimensional algebra $A = \text{Hom}_\mathbb{C}(E, E)$. Let us consider an injective morphism of algebras $A \hookrightarrow \text{End}_\mathbb{C}(A)$ given by the left multiplication. Then the group of invertible elements of $A$ can be identified with $A \cap \text{GL}(A)$, which is a closed subgroup of $\text{GL}(A)$.

We consider the numerical Grothendieck group $\Gamma = N(\mathcal{D})$ and the projection $\text{cl}: K(\mathcal{D}) \to \Gamma$. Let $\sigma = (Z, \mathcal{P})$ be a stability condition on $\mathcal{D}$ (with respect to $\text{cl}$). For any interval $I \subset \mathbb{R}$ of length $\leq 1$ (we assume that $I$ is non-closed if it has length 1), let $\mathcal{M}_{\sigma,I}$ be the substack of $\mathcal{M}$ parameterizing objects in $\mathcal{P}_I$. We will assume that

Assumption 1. For $I$ of length 1, the substack $\mathcal{M}_{\sigma,I}$ is open in $\mathcal{M}$.

Assumption 2. For any $\gamma \in \Gamma$ with $Z(\gamma) \neq 0$, the stack $\mathcal{M}_{\sigma}(\gamma)$ is of finite type, where $\mathcal{M}_{\sigma}(\gamma)$ is the stack of $\sigma$-semistable objects in $\mathcal{M}$ having class $\gamma$ and phase $\frac{1}{\tau} \text{Arg} Z(\gamma)$.

Note that the first assumption implies that for any interval $I$ of length $< 1$, the substack $\mathcal{M}_{\sigma,I}$ is open in $\mathcal{M}$. In particular, the stack $\mathcal{M}_{\sigma,\phi} = \mathcal{M}_{\sigma,[\phi,\phi]}$, $\phi \in \mathbb{R}$, is open in $\mathcal{M}$. The stack $\mathcal{M}_{\sigma}(\gamma)$ is the substack of $\mathcal{M}_{\sigma,\phi}$ for $\phi = \frac{1}{\tau} \text{Arg} Z(\gamma)$, hence it is also open in $\mathcal{M}$. The second assumption implies that, for any interval $I$ of length $< 1$, the stack $\mathcal{M}_{\sigma,I}(\gamma)$ of objects in $\mathcal{M}_{\sigma,I}$ having class $\gamma$ is of finite type. The above assumptions are proved for a large class of stability conditions on surfaces in [64] and on 3-folds in [57].

Let $I \subset \mathbb{R}$ be an interval of length $< 1$ and

$$
V = \mathbb{R}_{>0}e^{i\tau \phi} = \{ re^{i\tau \phi} \mid r > 0, \phi \in I \} \subset \mathbb{C}
$$

be the corresponding strict cone. The category $\mathcal{E} = \mathcal{P}_I$ is quasi-abelian [13]. In particular, it is automatically exact, with the class of admissible exact sequences consisting of all short exact sequences. By our assumptions, its objects are parametrized by the algebraic stack $\mathcal{M} = \mathcal{M}_{\sigma,I}$. The stacks $\mathcal{M}_n$ parameterizing chains (107) in $\mathcal{E}$ are also algebraic. For example, the fiber of $q: \mathcal{M}_2 \to \mathcal{M} \times \mathcal{M}$ over $(E_1, E_2)$ can be identified with the stack

$$
\text{Ext}^1(E_2, E_1)/\text{Hom}(E_2, E_1)
$$

and we can stratify $\mathcal{M} \times \mathcal{M}$ so that $q$ is a trivial fibration over the strata (cf. [15, Prop. 6.2]). This also implies that $q$ is of finite type.
We consider the decomposition $\mathcal{M} = \bigsqcup_{\gamma \in \Gamma} \mathcal{M}(\gamma)$, where $\mathcal{M}(\gamma) = \mathcal{M}_{\sigma, I}(\gamma)$ is the stack parameterizing objects $E \in \mathcal{E}$ with $\text{cl } E = \gamma$. By the support property of $\sigma$, there exists $\varepsilon > 0$ such that $|Z(E)| \geq \varepsilon \|\text{cl } E\|$ for any $\sigma$-semistable object $E$. Therefore, for any $0 \neq E \in \mathcal{E}$ we have
\begin{equation}
\text{cl } E \in S := S(V, Z, \varepsilon),
\end{equation}
where $S(V, Z, \varepsilon)$ is the strict semigroup defined in (11). Let $S_0 = S \cup \{0\}$. Then the Hall algebra $H(\mathcal{E}) = K(\text{St}^a / \mathcal{M})$ is $S_0$-graded
\begin{equation}
H(\mathcal{E}) = \bigoplus_{\gamma \in S_0} H_\gamma(\mathcal{E}), \quad H_\gamma(\mathcal{E}) = K(\text{St}^a / \mathcal{M}(\gamma)).
\end{equation}

We define the completion of the Hall algebra to be
\begin{equation}
\hat{H}(\mathcal{E}) = \lim_{J \subset S} H(\mathcal{E})_J, \quad H(\mathcal{E})_J = H(\mathcal{E}) / \bigoplus_{\gamma \in S \setminus J} H_\gamma(\mathcal{E}) \simeq \bigoplus_{\gamma \in J \cup \{0\}} H_\gamma(\mathcal{E}),
\end{equation}
where the limit is taken over all finite lower sets $J \subset S$. Note that $\hat{H}(\mathcal{E}) \simeq \prod_{\gamma} H_\gamma(\mathcal{E})$ as a vector space. Let us define
\begin{equation}
1_{\sigma, V} = 1 + \sum_{Z(\gamma) \in V} 1_\gamma \in \hat{H}(\mathcal{E}), \quad 1_\gamma = [\mathcal{M}(\gamma) \to \mathcal{M}(\gamma)] \in H_\gamma(\mathcal{E}).
\end{equation}
On the other hand, for any $\gamma \in \Gamma \cap Z^{-1}(V)$, we can consider $\mathcal{M}_\sigma(\gamma)$ as an open substack of $\mathcal{M}(\gamma)$. We define
\begin{equation}
1_{\sigma, \ell} = 1 + \sum_{Z(\gamma) \in \ell} 1_{\sigma, \gamma} \in \hat{H}(\mathcal{E}), \quad 1_{\sigma, \gamma} = [\mathcal{M}_\sigma(\gamma) \to \mathcal{M}(\gamma)] \in H_\gamma(\mathcal{E}),
\end{equation}
for any ray $\ell \subset V$. The following result is well-known \([33, 41]\).

**Theorem 5.3.** We have
\begin{equation}
1_{\sigma, V} = \prod_{\ell \subset V} 1_{\sigma, \ell}.
\end{equation}

**Proof.** Given $\gamma_1, \gamma_2 \in S \subset \Gamma$, we have $Z(\gamma_i) \in \mathbb{R}_{>0} e^{i\phi_i}$ for some $\phi_i \in I$, $i = 1, 2$. We will write $\gamma_1 \succ \gamma_2$ if $\phi_1 > \phi_2$. By the HN-property of $\sigma$, for any $E \in \mathcal{E}$, there exists a unique filtration
\begin{equation}
0 = E_0 \subset \ldots \subset E_n = E
\end{equation}
such that $E_i / E_{i-1} \in \mathcal{P}_{\phi_i}$ with $\phi_i \in I$ satisfying $\phi_1 > \cdots > \phi_n$. We define the $\sigma$-HN type of $E$ to be the tuple $(\gamma_1, \ldots, \gamma_n)$, where $\gamma_i = \text{cl}(E_i / E_{i-1}) \in S$ and $\gamma_1 \succ \gamma_2 \succ \cdots \succ \gamma_n$. By Lemma 2.5, for any fixed $\gamma \in S$, there exist finitely many tuples $(\gamma_1, \ldots, \gamma_n)$ in $S$ such that $\gamma = \sum_i \gamma_i$. This implies that $\mathcal{M}(\gamma)$ has a finite stratification with strata $\mathcal{M}_\sigma(\gamma_1, \ldots, \gamma_n)$ parameterizing objects in $\mathcal{M}(\gamma)$ having $\sigma$-HN type $(\gamma_1, \ldots, \gamma_n)$. By the uniqueness of the HN-filtration we have
\begin{equation}
[\mathcal{M}_\sigma(\gamma_1, \ldots, \gamma_n) \to \mathcal{M}] = [\mathcal{M}_\sigma(\gamma_1) \to \mathcal{M}] \ast \cdots \ast [\mathcal{M}_\sigma(\gamma_n) \to \mathcal{M}].
\end{equation}
This implies that
\begin{equation}
[\mathcal{M}(\gamma) \to \mathcal{M}] = \sum_{\gamma_1 \succ \cdots \succ \gamma_n = \gamma} [\mathcal{M}_\sigma(\gamma_1) \to \mathcal{M}] \ast \cdots \ast [\mathcal{M}_\sigma(\gamma_n) \to \mathcal{M}].
\end{equation}
which is a reformulation of the required formula.

\[ \square \]

**Remark 5.4.** Note that it is important in the above theorem that the interval \( I \) has length \(< 1 \). If \( I \) has length \( 1 \), for example \( I = (0, 1] \), then the corresponding semigroup \( S(V, Z, \varepsilon) \) is not necessarily strict and the ordered product in the theorem is not necessarily well-defined.

### 5.4. Wall-crossing in the quantum torus

We use the same notation and assumptions as in the previous section as well as Assumption 3.

**Assumption 3.** The stability condition \( \sigma = (Z, P) \) has global dimension \( \leq 2 \).

Let \( \chi \) be the Euler form on \( \Gamma = \mathbb{N}(D) \) and let \( \langle \cdot, \cdot \rangle \) be its anti-symmetrization

\[ \langle \gamma_1, \gamma_2 \rangle = \chi(\gamma_1, \gamma_2) - \chi(\gamma_2, \gamma_1). \]

We define the algebra over \( R = \mathbb{K}(\text{St}^a/\mathbb{C}) \)

\[ T = \bigoplus_{\gamma \in \Gamma} R x_\gamma, \quad x^{\gamma_1} \ast x^{\gamma_2} = L^{\frac{1}{2}\langle \gamma_1, \gamma_2 \rangle} \cdot x^{\gamma_1 + \gamma_2}, \]

called the *quantum torus*. For a strict semigroup \( S = S(V, Z, \varepsilon) \subset \Gamma \), we consider the subalgebra

\[ \mathbb{T}_S = \bigoplus_{\gamma \in S \cup \{0\}} R x_\gamma \]

and define its completion \( \mathbb{T}_S \) similarly to \((118)\). We define the *integration map*

\[ J: H(E) \to \mathbb{T}_S, \quad [X \to M_{\sigma, I}(\gamma)] \mapsto L^{\frac{1}{2}\langle \gamma_1, \gamma_2 \rangle}[X] \cdot x^\gamma \]

which extends to \( J: \hat{H}(E) \to \hat{T}_S \). Note that

\[ J(1_{\sigma, V}) = 1 + \sum_{Z(\gamma) \in V} L^{\frac{1}{2}\langle \gamma, \gamma \rangle}[M_{\sigma, I}(\gamma)]x^\gamma, \]

\[ J(1_{\sigma, \ell}) = 1 + \sum_{Z(\gamma) \in \ell} L^{\frac{1}{2}\langle \gamma, \gamma \rangle}[M_{\sigma}(\gamma)]x^\gamma. \]

**Lemma 5.5.** Let \( X_1, X_2 \in \text{St}^a \) and let \( X_i \to M, i = 1, 2, \) be morphisms such that for arbitrary \( x_i \in X_i \) and the corresponding objects \( E_i \in E \), we have \( \phi_{\sigma}^-(E_1) > \phi_{\sigma}^+(E_2). \) Then

\[ J([X_1 \to M] \ast [X_2 \to M]) = J([X_1 \to M]) \ast J([X_2 \to M]). \]

**Proof.** We can stratify \( X_i \) and assume that they are of the from \( Y_i / \text{GL}_{n_i} \), where \( Y_i \) are algebraic varieties. Furthermore, we can substitute \( X_i \to M \) by \( Y_i \to M \) and divide both sides of the equation by \( [\text{GL}_{n_1}] \cdot [\text{GL}_{n_2}] \). Therefore we can assume that \( X_i \) are algebraic varieties. We can also assume that \( X_i \) is mapped to \( M(\gamma_i) \) for some \( \gamma_i \in S \). Let \( \gamma = \gamma_1 + \gamma_2 \). Expression on the left is given by the motivic class of the Cartesian product

\[ \begin{array}{ccc}
Z & \longrightarrow & M_2 \\
\downarrow & & \downarrow \\
X_1 \times X_2 & \longrightarrow & M \times M
\end{array} \]
multiplied by $\mathbb{L}^\frac{1}{2}\chi(\gamma,\gamma)x^\gamma$. The fiber of $M_2 \to M \times M$ over $(E_1, E_2)$ is given by the stack $
abla^1(E_2, E_1)/\nabla(E_2, E_1)$.

For any $k \geq 2$, we have $\phi^-_\sigma(E_1[k]) - \phi^+_\sigma(E_2) > k \geq 2$, hence $\nabla^k(E_2, E_1) = 0$ by the assumption that $\sigma$ has global dimension $\leq 2$. We conclude that the motivic class of the fiber is equal to $\mathbb{L}^{-\chi(\gamma_2, \gamma_1)}$. We note that

$$\frac{1}{2}\chi(\gamma, \gamma) - \chi(\gamma_2, \gamma_1) = \frac{1}{2}\chi(\gamma_1, \gamma_1) + \frac{1}{2}\chi(\gamma_2, \gamma_2) + \frac{1}{2}\langle \gamma_1, \gamma_2 \rangle$$

which is exactly the power of $\mathbb{L}$ on the right hand side of the required equation. \hfill \square

As in Theorem 5.3 we will write $\gamma_1 \succ \gamma_2$ if $Z(\gamma_i) \in \mathbb{R}_{>0}e^{i\pi\phi_i}$ for $\phi_i \in I$ satisfying $\phi_1 > \phi_2$.

**Corollary 5.6.** Let $\gamma_1 \succ \cdots \succ \gamma_n$ be elements in $S(V, Z, \varepsilon)$. Then

$$J(1_{\sigma, \gamma_1} \ast \cdots \ast 1_{\sigma, \gamma_n}) = J(1_{\sigma, \gamma_1}) \ast \cdots \ast J(1_{\sigma, \gamma_n}).$$

**Corollary 5.7.** We have

$$J(1_{\sigma, V}) = \prod_{\ell \in V} J(1_{\sigma, \ell}).$$

**Proof.** By Theorem 5.3 we have $J(1_{\sigma, V}) = J\left(\prod_{\ell \in V} 1_{\sigma, \ell}\right)$ and by the previous result we have $J\left(\prod_{\ell \in V} 1_{\sigma, \ell}\right) = \prod_{\ell \in V} J(1_{\sigma, \ell})$. \hfill \square

5.5. **Stability data associated to a stability condition.** Let us consider the $\Gamma$-graded Lie algebra

$$g = \bigoplus_{\gamma \in \Gamma} Rx^\gamma = \mathbb{T}, \quad R = K(\text{St}^a/\mathbb{C}),$$

with the Lie bracket given by the commutator of $\mathbb{T}$ (123). For any strict semigroup $S \subset \Gamma$ without zero, we consider the Lie algebra $g_S = \bigoplus_{\gamma \in S} g_\gamma$ and the corresponding pro-nilpotent Lie algebra $\hat{g}_S$ and the pro-nilpotent group $\hat{G}_S$. Note that we can identify $\hat{g}_S$ with the ideal $\hat{T}_S^+ = \prod_{\gamma \in S} g_\gamma$ of $\hat{T}_S = \prod_{\gamma \in S \cup \{0\}} g_\gamma$ and identify $\hat{G}_S$ with the group $1 + \hat{T}_S^+ \subset \hat{T}_S$ so that we have a commutative diagram

$$\begin{array}{ccc}
\hat{g}_S & \xrightarrow{\exp} & \hat{G}_S \\
\downarrow & & \downarrow \\
\hat{T}_S^+ & \xrightarrow{\exp} & 1 + \hat{T}_S^+
\end{array}$$

Let $\varepsilon > 0$ be such that $|Z(E)| \geq \varepsilon \|\text{cl} E\|$ for any $\sigma$-semistable object $0 \neq E \in \mathcal{D}$. For any ray $\ell \subset \mathbb{C}$, we consider the strict semigroup $S(\ell, Z, \varepsilon)$ and define

$$A_{\sigma, \ell} = J(1_{\sigma, \ell}) = 1 + \sum_{Z(\gamma) \in \ell} \mathbb{L}^\frac{1}{2}\chi(\gamma, \gamma)[M_\sigma(\gamma)]x^\gamma \in 1 + \hat{T}_S^+ \simeq \hat{G}_S(\ell, Z, \varepsilon) \subset \hat{G}_\ell Z,$$

$$a_{\sigma, \gamma} = \log(A_{\sigma, \ell}) \in \hat{g}_S(\ell, Z, \varepsilon) \subset \hat{g}_\ell Z.$$
where \( \hat{g}_{\ell, Z} \) is the ind-pro-nilpotent Lie algebra defined in (64) and \( \hat{G}_{\ell, Z} \) is the corresponding ind-pro-nilpotent group. We define \textit{stability data associated to the stability condition} \( \sigma \) to be the family

\[
(135) \quad a_\sigma = (a_{\sigma, \gamma})_{\gamma \in \Gamma \setminus \{0\}} \in \hat{g}.
\]

\textbf{Remark 5.8.} In the above discussion we considered the quantum torus (123) over the algebra \( K(St^n/\mathbb{C}) \) and constructed stability data \( a_\sigma \) with values in \( K(St^n/\mathbb{C}) \). Alternatively, we can consider the quantum torus over the algebra \( \mathbb{Q}(u, \sqrt{uv}) \) and apply the Hodge-Deligne (polynomial) map (104) to construct stability data \( E(\sigma_\gamma) \) with values in \( \mathbb{Q}(u, \sqrt{uv}) \). Similarly, we can consider the quantum torus over the algebra \( \mathbb{Q}(y) \) and apply the Poincaré (polynomial) map (106) to construct stability data \( P(\sigma_\gamma) \) with values in \( \mathbb{Q}(y) \).

### 5.6. Families of stability data and stability conditions.

Let \( M \) be a topological space and \((\sigma_x = (Z_x, \mathcal{P}_x))_{x \in M}\) be a continuous family of stability conditions on \( \mathcal{D} \) satisfying assumptions 1, 2, and 3. For every \( x \in M \), we define stability data

\[
(136) \quad a_x = (a_{\sigma_x, \gamma})_{\gamma \in \Gamma \setminus \{0\}} \in \hat{g}
\]
as in the previous section.

\textbf{Theorem 5.9.} \textit{The family of stability data} \((Z_x, a_x)_{x \in M}\) \textit{is continuous.}

\textit{Proof.} Axiom (1) follows from Lemma 3.7. Let us verify axiom (2). Given a strict cone \( V = \mathbb{R}_{>0} e^{i\pi I} \) (where \( I \subset \mathbb{R} \) is an interval of length \( < 1 \)), we have by Corollary 5.7

\[
J(1_{\sigma_y, V}) = \prod_{\ell \in V} J(1_{\sigma_y, \ell}) = \prod_{\ell \in V} A_{y, \ell} = A_{y, V}.
\]

Therefore we need to show that components of \( J(1_{\sigma_y, V}) \) are constant in a neighborhood of a fixed \( x \in M \). The component of \( J(1_{\sigma_y, V}) \) at \( \gamma \in \Gamma \) is equal to \([M_{\sigma_y, I}(\gamma)] \) (up to some factor depending on \( \gamma \)). We will compare it to \([M_{\sigma_y}(\gamma)] \).

There exists \( \epsilon > 0 \) and an open set \( x \in U \subset M \) such that \( |Z_y(E)| \geq \epsilon ||cl E|| \) for all \( \sigma_y \)-semistable objects \( E \) and \( y \in U \). We can assume that \( ||Z_x - Z_y|| \leq \frac{\epsilon}{2} \) for all \( y \in U \). This implies that if \( |Z_y(\gamma)| \geq \epsilon \|\gamma\| \), then \( |Z_x(\gamma)| \geq \frac{\epsilon}{2} \|\gamma\| \). Therefore \( S(V, Z_y, \epsilon) \subset S(V, Z_x, \frac{\epsilon}{2}) \).

If \( M_{\sigma_y, I}(\gamma) \) is non-empty, then \( \gamma \in S(V, Z_y, \epsilon) \subset S(V, Z_x, \frac{\epsilon}{2}) \).

Let us fix \( \gamma \in S(V, Z_x, \frac{\epsilon}{2}) \) and let \( Z_x(\gamma) \in \ell_0 = \mathbb{R}_{>0} e^{i\pi \phi_0} \) for some \( \phi_0 \in I \). By Lemma 4.8 it is enough to consider open cones \( V \) around \( \ell_0 \) of arbitrarily small angle. For any \( 0 < \eta < \frac{1}{2} \), let \( I_{\eta} = (\phi_0 - \eta, \phi_0 + \eta) \) and \( V_{\eta} = \mathbb{R}_{>0} e^{i\pi I_{\eta}} \). We can assume that for any \( \gamma' \leq \gamma \) in \( S(V_{\eta}, Z_x, \epsilon/2) \), we have \( Z_x(\gamma') \in \ell_0 \). We can also assume that \( d(\mathcal{P}_x, \mathcal{P}_y) < \eta/2 \) for all \( y \in U \).

Let \( I = I_{\eta/2} \) and \( V = V_{\eta/2} \). If \( E \in \mathcal{P}_{y, I} \), then \( \phi_y^+(E) \in I \), hence \( \phi_x^+(E) \in I_{\eta} \). If \( \text{cl} E = \gamma \), then by our assumption, for any \( \sigma_x \)-HN-factor \( F \) of \( E \), we have \( Z_x(\text{cl} F) \in \ell_0 \), hence \( E \) is \( \sigma_x \)-semistable and \( E \in \mathcal{P}_{x, \phi_0} \). Conversely, if \( E \in \mathcal{P}_{x, \phi_0} \), then \( \phi_y^+(E) \in I \), hence \( E \in \mathcal{P}_{y, I} \). This proves that \( M_{\sigma_y, I}(\gamma) = M_{\sigma_y}(\gamma) \). The same can be done for all \( \gamma' \leq \gamma \) in \( S(V_{\eta}, Z_x, \epsilon/2) \). Taking the logarithm \( \log(A_{y, V}) \), we obtain that its \( \gamma \)-component is constant for \( y \in U \). \( \square \)
6. Stability conditions on surfaces

6.1. Some conventions.

6.1.1. Numerical intersection ring. Let $X$ be a smooth projective (connected) variety over $\mathbb{C}$ of dimension $d$. Let $A^*(X)$ be its intersection ring (also called the Chow ring) [26, §8.3]. It is a graded commutative ring with the involution

$$A^*(X) \to A^*(X), \quad a \mapsto a^* = \sum_k (-1)^k a_k,$$

where $a_k \in A^k(X)$ denotes the $k$-th component of $a \in A^*(X)$. We consider the degree map

$$\int: A^d(X) = A_0(X) \to \mathbb{Z}, \quad \sum_x n_x[x] \mapsto \sum_x n_x,$$

and extend it to $\int: A^*(X) \to \mathbb{Z}$. We define the intersection form on $A^*(X)$

$$(a, b) = -\int a^*b.$$

It satisfies

$$(a, b) = (-1)^d(b, a).$$

Let $A_{num}^*(X) \subset A^*(X)$ be the kernel of this form, having components

$$A_{num}^k(X) = \left\{ a \in A^k(X) \mid \int ab = 0 \quad \forall b \in A^{d-k}(X) \right\}.$$

It is an ideal of $A^*(X)$. We define the numerical intersection ring

$$N^*(X) = A^*(X)/A_{num}^*(X).$$

The above intersection form induces a non-degenerate form on $N^*(X)$. Every component $N^k(X)$ is a free abelian group of finite rank, as it can be embedded into the torsion free part of $H^{2k}(X, \mathbb{Z})$. We have $N^0(X) \simeq \mathbb{Z}$ and $N^d(X) \simeq \mathbb{Z}$. The group $N^1(X)$ is called the (torsion free) Néron-Severi group. The elements of $N^1(X)$ will be called divisors and the elements of $N^1_R(X) = N^1(X)_\mathbb{R}$ will be called $\mathbb{R}$-divisors. For $a \in N^k(X)$ and $b \in N^{d-k}(X)$, we will interpret $ab \in N^d(X)$ as an integer, using the isomorphism $N^d(X) \simeq \mathbb{Z}$. We also define $N_k(X) = N^{d-k}(X)$ for $k \in \mathbb{Z}$. The involution (137) produces adjoint operators, namely

$$(ab, c) = (b, a^*c), \quad a, b, c \in N^*(X).$$

Remark 6.1. The pairing $\sigma(a, b) = \int ab$ on $N^*(X)$ makes $N^*(X)$ a (graded) symmetric Frobenius algebra, meaning that $\sigma$ is symmetric, non-degenerate and $\sigma(ab, c) = \sigma(a, bc)$ for all $a, b, c \in N^*(X)$. However, we will not use this pairing.

If $X$ is a surface, then we can write the intersection form on $N^*_R(X)$ as

$$(a, b) = a_1b_1 - a_0b_2 - a_2b_0, \quad a, b \in N^*_R(X).$$

This intersection form has signature $(2, \rho)$, where $\rho = \text{rk } N^1(X)$ is the Picard number of $X$, because of the following result.
Theorem 6.2 (Hodge index theorem). Let $X$ be a smooth projective surface. Then the intersection form on $N^1_{\mathbb{R}}(X)$ has signature $(1, \rho - 1)$, where $\rho = \text{rk} N^1(X)$. In particular, for any ample divisor $H$, the intersection form restricted to $H^\perp \subset N^1_{\mathbb{R}}(X)$ is negative-definite.

6.1.2. Numerical Grothendieck group. Let $K(X) = K_0(\text{Coh} X)$ be the Grothendieck group of coherent sheaves on $X$. It is a commutative ring with an involution, where

$$[E] \cdot [F] = [E \otimes F], \quad [E]^* = [E^\vee],$$

for $E, F \in \text{Coh}(X)$ with $E$ locally-free. The Chern character map $\text{ch}: K(X) \to A^*(X)_\mathbb{Q}$ is a ring homomorphism preserving involutions. It induces an isomorphism [26, §18.3]

$$\text{ch}: K(X)_{\mathbb{Q}} \to A^*(X)_{\mathbb{Q}}.$$  

Let $\chi$ be the Euler form on $K(X)$, defined by

$$\chi([E], [F]) = \chi(E, F) := \sum_k (-1)^k \dim \text{Ext}^k(E, F).$$

By Serre duality, we have $\chi(E, F) = (-1)^d \chi(F, E \otimes \omega_X)$, where $\omega_X$ is the canonical bundle of $X$. This implies that the left and the right kernels of the Euler form coincide. The kernel $K_{\text{num}}(X) \subset K(X)$ is an ideal of the ring $K(X)$ and we define the numerical Grothendieck group (which is a ring)

$$N(X) = K(X)/K_{\text{num}}(X).$$

By the Grothendieck-Riemann-Roch theorem, we have

$$\chi([E], [F]) = -(\text{ch} E, \text{ch} F : \text{Td}(X)),$$

where the Todd class $\text{Td}(X) = \text{td}(T_X) \in A^*(X)_\mathbb{Q}$ is an invertible element. The Chern character induces the ring homomorphism

$$\text{ch}: N(X) \to N^*(X)_\mathbb{Q}$$

which becomes an isomorphism after tensoring with $\mathbb{Q}$.

6.1.3. Ample cone. Let $X$ be a smooth projective variety. We define the ample cone $\text{Amp}(X) \subset N^1_{\mathbb{R}}(X)$ to be the convex (blunt) cone generated by ample classes in $N^1(X)$. We define the cone of curves $\text{NE}(X) \subset N_1(X)_{\mathbb{R}}$

$$\text{NE}(X) = \left\{ \sum \mathbb{R}_{\geq 0}[C_i] \bigg | C_i \subset X \text{ is an integral curve} \right\}$$

and let $\overline{\text{NE}}(X)$ be its closure. Then

$$\text{Amp}(X) = \left\{ D \in N^1_{\mathbb{R}}(X) \big | DC > 0 \ \forall C \in \overline{\text{NE}}(X) \setminus \{0\} \right\}.$$

We define the nef cone to be the dual of $\overline{\text{NE}}(X)$

$$\text{Nef}(X) = \left\{ D \in N^1_{\mathbb{R}}(X) \big | DC \geq 0 \ \forall C \in \overline{\text{NE}}(X) \right\}.$$

It is equal to the closure of $\text{Amp}(X)$, while $\text{Amp}(X)$ is equal to the interior of $\text{Nef}(X)$ (see e.g. [45]). By the Nakai–Moishezon criterion, if $X$ is a surface, then $D \in N^1_{\mathbb{R}}(X)$ is ample if and only if $D^2 > 0$ and $DC > 0$ for all curves $C \subset X$. 
6.2. Construction of stability conditions. Let $X$ be a smooth projective surface and $\mathcal{D} = D^b(\text{Coh} X)$ be its bounded derived category. We consider the numerical Grothendieck group $\Gamma = N(X) = N(\mathcal{D})$ and the projection $\text{cl}: K(\mathcal{D}) \to \Gamma$. For any ample divisor $\omega \in N^1(\mathbb{R})$ and a torsion-free sheaf $E \in \text{Coh}(X)$, we define the slope
\begin{equation}
\mu_\omega(E) = \frac{c_1(E) \cdot \omega}{\text{rk}(E)}.
\end{equation}
We say that a torsion-free sheaf $E$ is $\mu_\omega$-semistable if, for any subsheaf $0 \neq E' \subset E$, we have $\mu_\omega(E') \leq \mu_\omega(E)$. Then every $E \in \text{Coh}(X)$ has a unique filtration (called the Harder-Narasimhan filtration)
\[ E_0 \subset E_1 \subset \ldots \subset E_n = E, \]
such that $E_0$ is the torsion part of $E$ and $E_i/E_{i-1}$ are $\mu_\omega$-semistable and have decreasing slopes. We define
\begin{equation}
\mu_\omega^+(E) = \begin{cases} +\infty & E_0 \neq 0, \\
\mu_\omega(E_1) & E_0 = 0,
\end{cases} \quad \mu_\omega^-(E) = \mu_\omega(E_n/E_{n-1}).
\end{equation}

**Remark 6.3.** The above discussion can be interpreted in the framework of §3.4 by considering the linear map
\begin{equation}
Z: N(X) \to \mathbb{C}, \quad E \mapsto -c_1(E) \cdot \omega + i \text{rk}(E).
\end{equation}
Note, however, that $Z(\mathcal{O}_x) = 0$ for $x \in X$, so that $Z$ is not a stability function on $\text{Coh}(X)$.

Let us define the discriminant $\Delta$ to be the quadratic form on $N(X)$ corresponding to the intersection form, meaning that
\begin{equation}
\Delta(E) := (\text{ch}(E), \text{ch}(E)) = a_2^2 - 2ra_2, \quad \text{ch}(E) = (r, a_1, a_2).
\end{equation}
Recall that it has signature $(2, \rho)$, where $\rho = \text{rk} N^1(X)$.

**Theorem 6.4** (Bogomolov inequality). Let $E \in \text{Coh}(X)$ be a $\mu_\omega$-semistable sheaf. Then $\Delta(E) \geq 0$.

For any $s \in \mathbb{R}$, let $\text{Coh}_{\omega, > s}$ be the subcategory of $\text{Coh}(X)$ consisting of $E \in \text{Coh}(X)$ with $\mu_\omega^-(E) > s$ and let $\text{Coh}_{\omega, \leq s}$ be the subcategory of $\text{Coh}(X)$ consisting of $E \in \text{Coh}(X)$ with $\mu_\omega^+(E) \leq s$. Then $(\text{Coh}_{\omega, > s}, \text{Coh}_{\omega, \leq s})$ is a torsion pair on $\text{Coh}(X)$ and we define the category
\begin{equation}
\text{Coh}_{\omega, \# s} = \langle \text{Coh}_{\omega, \leq s}[1], \text{Coh}_{\omega, > s} \rangle \subset \mathcal{D}
\end{equation}
which is the heart of a bounded $\text{t}$-structure on $\mathcal{D}$. In particular, for any $\beta \in N^1(\mathbb{R})$, we define
\begin{equation}
\mathcal{A}_{\beta, \omega} = \text{Coh}_{\omega, \# \beta \omega} = \langle \text{Coh}_{\omega, \leq \beta \omega}[1], \text{Coh}_{\omega, > \beta \omega} \rangle.
\end{equation}

The following result was formulated in [2, §2], where it was proved that the linear map $Z_{\beta, \omega}$ maps $\mathcal{A}_{\beta, \omega}$ to the upper half-plane. The support property and the HN property were proved in [65] (see also [6]). This result is a generalization of [14, Lemma 6.2].
Theorem 6.5. Let $\beta, \omega \in N^1_{\mathbb{R}}(X)$ and let $\omega$ be ample. Then the linear map
\begin{equation}
Z_{\beta,\omega} : N(X) \to \mathbb{C}, \quad Z_{\beta,\omega}(E) = (e^{\beta+\omega}, \text{ch}(E)) = - \int e^{-\beta-\omega} \text{ch}(E)
\end{equation}
is a stability function on the abelian category $A_{\beta,\omega}$.

We denote by $\sigma_{\beta,\omega}$ the stability condition corresponding to $(Z_{\beta,\omega}, A_{\beta,\omega})$. If $\text{ch}(E) = (r, a_1, a_2)$, then
\begin{equation}
Z_{\beta,\omega}(E) = \left( \frac{r}{2}(\omega^2 - \beta^2) + a_1 \beta - a_2 \right) + i(a_1 - r \beta) \omega.
\end{equation}
We can also write this expression in the form
\begin{equation}
Z_{\beta,\omega}(E) = \begin{cases}
\frac{1}{2r}(r^2 \omega^2 + \Delta(E) - (a_1 - r \beta)^2) + i(a_1 - r \beta) \omega & r \neq 0, \\
(a_1 \beta - a_2) + ia_1 \omega & r = 0.
\end{cases}
\end{equation}

It is convenient to define
\begin{equation}
\text{ch}^\beta(E) = e^{-\beta} \text{ch}(E) \in N^*(X).
\end{equation}
If $\text{ch}^\beta(E) = (r, b_1, b_2)$, then
\begin{equation}
Z_{\beta,\omega}(E) = (\frac{1}{2}r \omega^2 - b_2) + ib_1 \omega.
\end{equation}

Lemma 6.6. The discriminant $\Delta : N(X)_{\mathbb{R}} \to \mathbb{R}$ is negative-definite on $\text{Ker} \, Z_{\beta,\omega}$.

Proof. Let $E$ be such that $Z_{\beta,\omega}(E) = 0$ and let $\text{ch}^\beta(E) = (r, b_1, b_2)$. Then $b_1 \omega = 0$ and $b_2 = \frac{1}{2}r \omega^2$, hence $b^2_1 \leq 0$. If $(r, b_1, b_2) \neq 0$, then either $b^2_1 < 0$ or $r \neq 0$. Therefore
\begin{equation}
\Delta(E) = b^2_1 - 2rb_2 = b^2_1 - r^2 \omega^2 < 0.
\end{equation}

6.3. Cone of positive planes. Let $X$ be a smooth projective surface as before. We will say that a subspace $V \subset N^*_H(X)$ is a positive plane if $\dim V = 2$ and the intersection form restricted to $V$ is positive-definite. We define
\begin{equation}
\mathcal{P}(X) = \{ x + iy \in N^*_C(X) \mid \langle x, y \rangle \text{ is a positive plane} \},
\end{equation}
\begin{equation}
\bar{\mathcal{P}}(X) = \{ e^{\beta+\omega} \in N^*_C(X) \mid \beta, \omega \in N^1_{\mathbb{R}}(X), \omega^2 > 0 \},
\end{equation}
\begin{equation}
\bar{\mathcal{P}}^+(X) = \{ e^{\beta+\omega} \in N^*_C(X) \mid \beta \in N^1_{\mathbb{R}}(X), \omega \in \text{Amp}(X) \}.
\end{equation}

Lemma 6.7 (Cf. [14]). We have $\bar{\mathcal{P}}^+(X) \subset \bar{\mathcal{P}}(X) \subset \mathcal{P}(X)$ and
\begin{equation}
\bar{\mathcal{P}}(X) = \{ u \in N^*_C(X) \mid u_0 = 1, (u, u) = 0, (u, \bar{u}) > 0 \}.
\end{equation}
The action of $\text{GL}^+_2(\mathbb{R})$ on $\mathcal{P}(X)$ is free and $\bar{\mathcal{P}}(X) \to \mathcal{P}(X)/\text{GL}^+_2(\mathbb{R})$ is a bijection.

Proof. For $u \in N^*_C(X)$ to be of the form $u = e^{\beta+\omega}$, we require that $u_0 = 1$ and $u_2 = \frac{1}{2}u^2_1$. If $u_0 = 1$, then $(u, u) = u^2_1 - 2u_2$. Therefore we require $u_0 = 1$ and $(u, u) = 0$. If $u = e^{\beta+\omega}$, then
\begin{equation}
(u, u) = - \int u^*u = 0, \quad (u, \bar{u}) = - \int e^{-2\omega} = 2\omega^2.
\end{equation}
This implies that $\bar{\mathcal{P}}(X)$ is given by (168). If $u = x + iy = e^{\beta+\omega}$ and $\omega^2 > 0$, then
\begin{equation}
\langle x, x \rangle = \langle y, y \rangle = \omega^2 > 0, \quad \langle x, y \rangle = 0.
\end{equation}
hence the vectors $x, y$ span a positive plane and $\tilde{\mathcal{P}}(X) \subset \mathcal{P}(X)$.

If $u = x + iy \in \mathcal{P}(X)$ and $g = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in GL_2^+(\mathbb{R})$ are such that $gu \in \tilde{\mathcal{P}}(X)$, then $g \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) = \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)$, hence $a = 1, c = 0$. Therefore $gu = (x + by) + idy$. We conclude from (169) that $1 + b^2 = d^2$ and $bd = 0$, hence $b = 0$ and $d = \pm 1$. As $g \in GL_2^+(\mathbb{R})$, we conclude that $g$ is the identity. This proves that the map $\tilde{\mathcal{P}}(X) \to \mathcal{P}(X)/ GL_2^+(\mathbb{R})$ is injective.

If $u = x + iy \in \mathcal{P}(X)$, then by applying an element of $GL_2^+(\mathbb{R})$ we can assume that $(x, y) = 0$. If $x_0 = y_0 = 0$, then $(x + y) = x_1^2 + y_1^2 > 0$ and $(x, y) = x_1y_1 = 0$, which would imply that $N^1_{\text{ge}}(X)$ contains a positive plane. This is impossible, hence $(\frac{x_0}{y_0}) \neq 0$ and there exists a matrix in $\mathbb{R}_{>0} \cdot SO(2)$ that sends this vector to $(1)$. Therefore we can assume that $(x, y) = 0$, $x_0 = 1$, $y_0 = 0$. By applying a positive scalar to $y$, we can make $(x, x) = (y, y)$. Then $x + iy \in \tilde{\mathcal{P}}(X)$.

Let us define

$$\mathcal{P}^+(X) = GL_2^+(\mathbb{R}) \cdot \tilde{\mathcal{P}}(X) \subset \mathcal{P}(X) \subset N^1_{\text{ge}}(X).$$

Note that we have the quotient maps $\mathcal{P}(X) \to \tilde{\mathcal{P}}(X)$ and $\mathcal{P}^+(X) \to \tilde{\mathcal{P}}(X)$ and the subset $\tilde{\mathcal{P}}(X) \subset \mathcal{P}^+(X)$ is open, hence the subset $\mathcal{P}^+(X) \subset \mathcal{P}(X)$ is open.

6.4. Geometric stability conditions. In this section we will give a characterization of stability conditions that are equal to $\sigma_{\beta, \omega}$ up to the action of $GL_2^+(\mathbb{R})$. Let $X$ be a smooth projective surface, $\mathcal{D} = D^b(\text{Coh} X)$. We consider $\Gamma = \mathcal{N}(X) = \mathcal{N}(\mathcal{D})$ and the projection $\text{cl}: K(\mathcal{D}) \to \Gamma$. We will say that a stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ is geometric if the skyscraper sheaves $\mathcal{O}_x$ are stable and have the same phase for all $x \in X$. By rotating the stability condition (45), we can assume that all $\mathcal{O}_x$ have phase 1. We will say that a stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ is special if the discriminant $\Delta$ is negative-definite on $\text{Ker} Z$.

Lemma 6.8. Let $(\sigma_x)_{x \in M}$ be a continuous family of stability conditions on $\mathcal{D}$. Then the subsets

$$\{ x \in M \mid \sigma_x \text{ is geometric} \}, \quad \{ x \in M \mid \sigma_x \text{ is special} \}$$

are open in $M$.

Proof. The statement about special stability conditions is clear. The proof of the statement about geometric stability repeats the arguments of Lemma 3.9. \qed

Lemma 6.9 (See [14, Lemma 10.1]). Let $\sigma = (Z, \mathcal{P})$ be a stability condition on $X$ such that all sheaves $\mathcal{O}_x$ are stable of phase 1, and let $\mathcal{A} = \mathcal{P}_{[0, 1]}$ be its heart. Then

1. If $E \in \mathcal{A}$, then $H^i(E) = 0$ for $i \neq 0, -1$ and $H^{-1}(E)$ is torsion free.
2. If $E \in \mathcal{P}_{1}$ is stable, then either $E = \mathcal{O}_x$ for some $x \in X$ or $E[-1]$ is a locally-free sheaf.
3. If $E \in \text{Coh}(X)$, then $E \in \mathcal{P}_{(-1, 1]}$ and if $E$ is a torsion sheaf, then $E \in \mathcal{A}$.
4. The pair of categories

$$\mathcal{I} = \text{Coh}(X) \cap \mathcal{A}, \quad \mathcal{J} = \text{Coh}(X) \cap (\mathcal{A}[-1])$$

is a torsion pair in $\text{Coh}(X)$ and $\mathcal{A} = (\mathcal{I}[1], \mathcal{J})$. 
Theorem 6.10 (Cf. [14, Prop. 10.3]). Let $\sigma = (Z, \mathcal{P})$ be a special geometric stability condition. Then there exist unique $g \in \widetilde{\text{GL}}^+_2(\mathbb{R})$, $\beta \in N^1_X$ and $\omega \in \text{Amp}(X)$ such that $\sigma[g] = \sigma_{\beta, \omega}$.

Proof. There exists a unique $u = x + iy \in N^*_C(X)$ such that $Z(E) = (u, \text{ch} E)$ for all $E \in \mathcal{D}$. The condition that $Z$ is special means that $\langle x, y \rangle$ is a positive plane §6.3. By Lemma 6.7 there exist unique $T \in \text{GL}^+_2(\mathbb{R})$ and $\beta, \omega \in N^1_X$ with $\omega^2 > 0$ such that $T^{-1}u = e^{\beta+kw}$. We have $T^{-1}Z(\mathcal{O}_p) = (e^{\beta+kw}, \text{ch}(\mathcal{O}_p)) = -1$ for all $p \in X$. The kernel of $\widetilde{\text{GL}}^+_2(\mathbb{R}) \rightarrow \text{GL}^+_2(\mathbb{R})$ (see §3.6) acts on stability conditions by even shifts, hence there exists a unique $g \in \widetilde{\text{GL}}^+_2(\mathbb{R})$ that is mapped to $T \in \text{GL}^+_2(\mathbb{R})$ and such that $\mathcal{O}_p$ are $\sigma[g]$-stable objects of phase 1.

Therefore we can assume that $u = x + iy = e^{\beta+kw}$ and $\mathcal{O}_p$ are $\sigma$-stable objects of phase 1. Note that $y = (0, \omega, \beta \omega)$. For any $E$ with $\text{ch}(E) = (r, a_1, a_2)$, we have

$$\text{Im} Z(E) = a_1 \omega - r \beta \omega.$$  

If $C \subset X$ is a curve, then $\mathcal{O}_C$ is torsion, hence $\mathcal{O}_C \in \mathcal{A} = \mathcal{P}_{[0,1]}$ by Lemma 6.9(3). If $Z(\mathcal{O}_C)$ is real, then $\mathcal{O}_C \in \mathcal{P}_1$ which is impossible by Lemma 6.9(2). This implies that $\text{Im} Z(\mathcal{O}_C) = C \cdot \omega > 0$ for any curve $C \subset X$. As $\omega^2 > 0$, we conclude that $\omega$ is ample.

Now we only need to show that $\mathcal{A} = \mathcal{A}_{\beta, \omega}$. The proof of this statement goes through the lines of [14, Prop. 10.3 (Step 2)].

Let

$$\text{Stab}^+_\mathcal{A}(\mathcal{D}) \subset \text{Stab}^+_\mathcal{A}(\mathcal{D})$$

be the open subset of special geometric stability conditions. Let us identify $N^*_C(X)$ with $\text{Hom}(\Gamma, \mathbb{C})$ (where $\Gamma = N(X)$) by sending $u \in N^*_C(X)$ to

$$Z_u: \Gamma \rightarrow \mathbb{C}, \quad Z_u(E) = (u, \text{ch} E).$$

Then the open subsets $\mathcal{P}^+(X) \subset \mathcal{P}(X) \subset N^*_C(X)$ can be interpreted as open subsets of $\text{Hom}(\Gamma, \mathbb{C})$. By the previous theorem, the local homeomorphism $\text{Stab}^+_\mathcal{A}(\mathcal{D}) \rightarrow \text{Hom}(\Gamma, \mathbb{C})$, $(Z, \mathcal{P}) \mapsto Z$, restricts to a local homeomorphism

$$\text{Stab}^+_\mathcal{A}(\mathcal{D}) \rightarrow \mathcal{P}^+(X)$$

and we have bijections

$$\text{Stab}^+_\mathcal{A}(\mathcal{D})/\text{GL}^+_2(\mathbb{R}) \simeq \mathcal{P}^+(X)/\text{GL}^+_2(\mathbb{R}) \simeq \bar{\mathcal{P}}^+(X).$$

6.5. Gieseker and twisted stability. Given an ample divisor $\omega \in N^1_R(X)$, we define the Hilbert polynomial of $E \in \text{Coh}(X)$ to be

$$P_\omega(E, n) = \int e^{n \omega} \text{ch}(E) \text{Td}(X).$$

If $E$ is torsion-free, then we define its reduced Hilbert polynomial to be

$$p_\omega(E, n) = \frac{P_\omega(E, n)}{\text{rk} E}.$$
We say that a torsion-free sheaf $E$ is $\omega$-Gieseker semistable if for any $0 \neq E' \subset E$ we have

$$ p_\omega(E', n) \leq p_\omega(E, n), \quad n \gg 0. \quad (177) $$

Let us give a more explicit characterization of Gieseker stability. We have

$$ \text{Td}(X) = (1, -\frac{1}{2}K_X, \chi), \quad \chi = \chi(O_X). \quad (178) $$

If $\text{ch}(E) = (r, a_1, a_2)$, then

$$ P_\omega(E, n) = \frac{r}{2}n^2\omega^2 + n(a_1 - r\frac{K_X}{2})\omega + (a_2 - a_1\frac{K_X}{2} + r\chi). \quad (179) $$

Therefore we can substitute the reduced Hilbert polynomial by

$$ \bar{p}_\omega(E, n) = \frac{na_1\omega + (a_2 - a_1\frac{K_X}{2})}{r}, \quad \text{ch}(E) = (r, a_1, a_2), \quad (180) $$

and use it instead of $p_\omega(E, n)$ to test Gieseker stability. More generally, given another divisor $\beta \in N^1(X)$, we define for any torsion-free sheaf $E$ with $\text{ch}(E) = (r, a_1, a_2)$

$$ \nu_\beta(E) = \frac{a_2 - a_1\beta}{r}, \quad p_{\beta,\omega}(E, n) = n\mu_\omega(E) + \nu_\beta(E) = \frac{na_1\omega + (a_2 - a_1\beta)}{r}. \quad (181) $$

We say that a torsion-free sheaf $E$ is $(\beta, \omega)$-twisted semistable (cf. [14]) if, for any subsheaf $0 \neq E' \subset E$, we have

$$ p_{\beta,\omega}(E', n) \leq p_{\beta,\omega}(E, n), \quad n \gg 0. \quad (182) $$

Equivalently, this means that either $\mu_\omega(E') < \mu_\omega(E)$ or $\mu_\omega(E') = \mu_\omega(E)$ and $\nu_\beta(E') \leq \nu_\beta(E)$. In particular, $E$ is $\mu_\omega$-semistable.

6.6. **Large volume limit.** Let us consider $\beta, \omega \in N^1(X)$ with ample $\omega$ and let us define

$$ \sigma_t = (Z_t, \mathcal{P}_t) = \sigma_{\beta, t\omega}, \quad t > 0. \quad (183) $$

Note that the heart of $\sigma_t$ is equal to $\mathcal{A}_{\beta, \omega}$ for all $t > 0$. We will say that an object $0 \neq E \in \mathcal{D}$ is $\sigma_\infty$-semistable (or semistable in the large volume limit) if $E$ is $\sigma_t$-semistable for $t \gg 0$.

**Lemma 6.11** (See [14, Prop. 14.2]). Let $0 \neq E \in \mathcal{D}$ with $\text{ch}(E) = (r, a_1, a_2)$ be such that

$$ r > 0, \quad \text{Im} Z_{\beta, \omega}(E) = (a_1 - r\beta)\omega > 0. \quad (184) $$

Then $E$ is $\sigma_\infty$-semistable if and only if $E$ is a shift of a $(\beta, \omega)$-twisted semistable sheaf.

**Lemma 6.12** (Cf. [6, Lemma 2.7]). Let $E \in \mathcal{A}_{\beta, \omega}$ be $\sigma_\infty$-semistable. Then

- (1) If $\text{rk}(E) < 0$, then $H^{-1}(E)$ is a $\mu_\omega$-semistable sheaf and $H^0(E)$ has dimension 0.
- (2) If $\text{rk}(E) \geq 0$, then $E = H^0(E)$ is either a $\mu_\omega$-semistable or a torsion sheaf.

**Proof.** Let $\phi_t$ be the phase function on $\mathcal{A} = \mathcal{A}_{\beta, \omega}$ corresponding to $\sigma_t$. For any object $F \in \mathcal{D}$ with $\text{ch}(F) = (r, a_1, a_2)$, we obtain from (162)

$$ \phi_\infty(F) := \lim_{t \to \infty} \frac{1}{\pi} \text{Arg} Z_t(F) = \begin{cases} 0 & r > 0 \\ \frac{1}{2} & r = 0, \ a_1\omega > 0 \\ 1 & r = a_1 = 0, \ a_2 > 0 \end{cases} \quad (185) $$
Note that if $0 \neq F \in \mathcal{A}$, then $\phi_\infty(F) = \lim_{t \to \infty} \phi_t(F)$. Let
\[ \mathcal{F} = \text{Coh}_{\omega, > \beta \omega}, \quad \mathcal{F} = \text{Coh}_{\omega, \leq \beta \omega} \]
so that $\mathcal{A} = (\mathcal{F}[1], \mathcal{F})$. Then there is an exact sequence in $\mathcal{A}$
\[ 0 \to E_1[1] \to E \to E_0 \to 0 \]
where $E_1 \in \mathcal{F}$ and $E_0 \in \mathcal{F}$.

Let us assume that $E_1 \neq 0$. Then $E_1 \in \mathcal{F} = \text{Coh}_{\omega, \leq \beta \omega}$ is torsion-free, hence $\phi_\infty(E_1) = 0$ by (185). This implies that $\phi_\infty(E_1[1]) = 1$. If $E_0$ has dimension $2$ or $1$, then $\phi_\infty(E_0) = 0$ or $1$ by (185). But this implies that $\phi_t(E_1[1]) > \phi_t(E_0)$ for $t \gg 0$, which is a contradiction. We conclude that $E_0$ has dimension $0$, hence $Z_t(E) = -Z_t(E_1) - \chi_1(E_0)$. Let $\chi(E_1) = (r, a_1, a_2)$. If $(a_1 - r \beta)\omega = 0$, then $\mu_\omega(E_1) = \beta \omega$, hence $E_1 \in \text{Coh}_{\omega, \leq \beta \omega}$ is automatically $\mu_\omega$-semistable. Let us assume that $(a_1 - r \beta)\omega < 0$. If $E_1$ is not $\mu_\omega$-semistable, then there exists $0 \neq F \subset E_1$ with $\mu_\omega(F) > \mu_\omega(E)$ and $F, E_1/F \in \mathcal{F}$. This implies that we have an exact sequence in $\mathcal{A}$
\[ 0 \to F[1] \to E_1[1] \to (E_1/F)[1] \to 0 \]
Let $\chi(F) = (s, b_1, b_2)$. If $(b_1 - s \beta)\omega = 0$, then $\phi_t(F[1]) = 1 > \phi_t(E_1[1])$ for any $t$, which is a contradiction. Therefore $(b_1 - s \beta)\omega < 0$. We obtain from (161) and $\sigma_\infty$-semistability of $E$ that
\[ -r \geq -s \]
\[ \frac{(a_1 - r \beta)\omega}{(b_1 - s \beta)\omega} \]
which implies that $\mu_\omega(F) \leq \mu_\omega(E_1)$.

Let us assume that $E_1 = 0$, hence $E = E_0 \in \mathcal{F} = \text{Coh}_{\omega, > \beta \omega}$. Let $\chi(E) = (r, a_1, a_2)$. If $r > 0$, then $(a_1 - r \beta)\omega > 0$ and the statement follows from Lemma 6.11. If $r = 0$, then $E$ is a torsion sheaf.

\[ \Box \]

6.7. **Bogomolov-type inequalities.** In this section we will prove a new Bogomolov-type inequality for $\sigma_{\beta, \omega}$-semistable objects (Theorem 6.16). This inequality will be important in our analysis of the behavior of mini/max phases of objects under stability deformations. Let us first describe the known results.

Let $\beta, \omega \in N^1_\mathbb{R}(X)$ with ample $\omega$. We define
\[ (186) \quad \Delta_{\beta, \omega}(E) = (b_1 \omega)^2 - 2b_0b_2\omega^2, \quad b = \chi^2(E) = e^{-\beta} \chi(E). \]

Recall that for every $D \in \overline{\mathcal{NE}(X)} \setminus \{0\}$, we have $D\omega > 0$. Let us introduce a norm on $N_1(X)_\mathbb{R} \supset \overline{\mathcal{NE}(X)}$ and define
\[ (187) \quad C = \sup \left\{ 0, \frac{-D^2}{(D\omega)^2} \middle| D \in \overline{\mathcal{NE}(X)}, \|D\| = 1 \right\} \]
Then $C \geq 0$ and for every effective divisor $D$ we have
\[ (188) \quad C(D\omega)^2 + D^2 \geq 0. \]
We define
\[ (189) \quad \Delta^C_{\beta, \omega}(E) = \Delta(E) + C(b_1 \omega)^2, \quad b = \chi^2(E). \]

The following result is proved in [65, 7, 6].
Theorem 6.13. Let $E \in D^b(Coh X)$ be a $\sigma_{\beta,\omega}$-semistable object. Then
\[
\Delta_{\beta,\omega}(E) \geq 0, \quad \Delta^c_{\beta,\omega}(E) \geq 0.
\]

Before we will formulate a new Bogomolov-type inequality, let us derive a consequence of the classical Bogomolov inequality (Theorem 6.4).

Lemma 6.14. Let $\omega, \beta, H$ be divisors such that $\omega$ is ample and $H$ is nef. For any $\mu_{\omega}$-semistable sheaf $E$, we have
\[
\Delta_{\beta,\omega}(E) := (b_1 H)(b_1 \omega) - b_0 b_2(H \omega) \geq 0, \quad b = \text{ch}^\beta(E).
\]

Proof. We can assume that $\omega^2 = 1$. Let $t = H \omega \geq 0$ and $s = 0$ be such that $s^2 = H^2$. Then $H = t \omega + D_1$ for some $D_1 \in \omega^\perp$, hence $s^2 = H^2 = t^2 + D_1^2 \leq t^2$ and $s \leq t$. By the Bogomolov inequality, we have
\[
\Delta(E) = (\text{ch}(E), \text{ch}(E)) = (\text{ch}^\beta(E), \text{ch}^\beta(E)) = b_1^2 - 2b_0 b_2 \geq 0.
\]
Therefore $2b_0 b_2(H \omega) \leq b_1^2(H \omega)$ and we need to prove that
\[
b_1^2(H \omega) \leq 2(b_1 H)(b_1 \omega).
\]
We can represent $b_1 = x H + y \omega + D$, where $x, y \in \mathbb{R}$ and $D \in (H, \omega)^\perp$, hence $D^2 \leq 0$. It is enough to show that
\[
t(x^2 s^2 + y^2 + 2xyt) \leq 2(x s^2 + y t)(x t + y).
\]
This simplifies to $t s^2 x^2 + t y^2 + 2 s^2 x y \geq 0$. We have $t s^2 x^2 + t y^2 + 2 s^2 x y \geq s^2 x^2 + s y^2 + 2 s^2 x y = s(s x + y)^2 \geq 0$.

Lemma 6.15. The quadratic form $\Delta_{\beta,\omega}^H$ is negative semi-definite on $\text{Ker}(Z_{\beta,\omega} \colon \mathcal{N}(X)_\mathbb{R} \to \mathbb{C})$. Therefore, for any ray $\ell \subset \mathbb{C}$, the cone
\[
Z_{\beta,\omega}^{-1}(\ell) \cap \{\Delta_{\beta,\omega}^H \geq 0\}
\]
is convex.

Proof. Let $b = \text{ch}^\beta(E)$ and $Z_{\beta,\omega}(E) = \frac{1}{2} b_0 \omega^2 - b_2 + b_1 \omega = 0$. Then $b_1 \omega = 0$ and $b_2 = \frac{1}{2} b_0 \omega^2$. Therefore $\Delta_{\beta,\omega}^H(E) = -\frac{1}{2} b_0^2 \omega^2(H \omega) \leq 0$. The last assertion follows from Lemma 2.9.

Theorem 6.16. Let $\omega, \beta, H \in N^1_\mathbb{R}(X)$ be divisors such that $\omega$ is ample and $H$ is nef. For any $\sigma_{\beta,\omega}$-semistable object $E \in D^b(Coh X)$, we have
\[
\Delta_{\beta,\omega}^H(E) = (b_1 H)(b_1 \omega) - b_0 b_2(H \omega) \geq 0, \quad b = \text{ch}^\beta(E).
\]

Proof. Our proof is similar to the proof of [6, Theorem 3.5]. Let us assume first that $\beta, \omega \in N^1_\mathbb{Q}(X)$. Recall that if $a = \text{ch}(E)$, then
\[
Z_{\beta,\omega}(E) = \left(\frac{a_0}{2} (\omega^2 - \beta^2) + a_1 \beta - a_2 \right) + i (a_1 - a_0 \beta) \omega.
\]
Let $\rho(E) = \text{Im} Z_{\beta,\omega}(E)$. Then the image of the map $\rho : \mathcal{N}(X) \to \mathbb{R}$ is discrete.

Let us consider stability conditions $\sigma_t = \sigma_{\beta,\omega}$ for $t \geq 1$. If $E \in \mathcal{A}_{\beta,\omega}$ is $\sigma_t$-semistable for $t \gg 1$, then we can apply Lemma 6.12. If $E_1 = H^{-1}(E)$ is $\mu_{\omega}$-semistable and $E_0 = H^0(E)$ has dimension zero, let $\text{ch}^\beta(E_1) = b$ and $\text{ch}^\beta(E_0) = (0, 0, n)$. Then $b_0 > 0$, $n \geq 0$ and
\[
\Delta_{\beta,\omega}^H(E) = \Delta_{\beta,\omega}^H(E_1) + b_0 n(H \omega) \geq 0.
\]
by Lemma 6.14. If \( E = H^0(E) \) is \( \mu_\omega \)-semistable, then \( \Delta^H_{\beta,\omega}(E) \geq 0 \) by Lemma 6.14. If \( E = H^0(E) \) is torsion, let \( \chi^\beta(E) = (0, b_1, b_2) \). Then \( b_1 = c_1(E) \) and

\[
\Delta^H_{\beta,\omega}(E) = (b_1 H)(b_1 \omega) \geq 0.
\]

By Lemma 6.15, it is enough to prove the statement of the theorem for any stable object \( E \in \mathcal{A} = \mathcal{A}_{\beta,\omega} \). If \( \rho(E) = 0 \), then \( E \) is automatically \( \sigma_t \)-semistable of phase 1 for all \( t \), hence \( \Delta^H_{\beta,\omega}(E) \geq 0 \) by the previous discussion. Let \( E \in \mathcal{A} \) be a \( \sigma_{t_0} \)-stable object for some \( t_0 \geq 1 \) and let us assume by induction on \( \rho(E) \geq 0 \) (recall that \( \rho \) has discrete values) that if \( 0 \neq E' \in \mathcal{A} \) is \( \sigma_t \)-stable for some \( t \geq 1 \) and \( \rho(E') < \rho(E) \), then \( \Delta^H_{\beta,\omega}(E') \geq 0 \). If \( E \) is \( \sigma_t \)-stable for all \( t \geq t_0 \), then \( \Delta^H_{\beta,\omega}(E) \geq 0 \) by the previous discussion. Otherwise, there exists \( t_1 > t_0 \) such that \( E \) is \( \sigma_{t_1} \)-stable for all \( t \in [t_0, t_1) \) and is strictly \( \sigma_{t_1} \)-semistable (i.e. semistable, but not stable). Then all the \( \sigma_{t_1} \)-stable factors \( F_i \) of \( E \) will have the same \( \sigma_{t_1} \)-phase as \( E \), hence \( \rho(F_i) > 0 \). Therefore \( \rho(F_i) < \rho(E) \) and we conclude by induction that \( \Delta^H_{\beta,\omega}(F_i) \geq 0 \). As the \( \sigma_{t_1} \)-phases of \( E \) and \( F_i \) are the same, we conclude from Lemma 6.15 that \( \Delta^H_{\beta,\omega}(E) \geq 0 \).

Let us assume now that \( (\beta, \omega) \in N^1_\mathbb{R}(X) \times \text{Amp}(X) \) is arbitrary and let \( E \in D^b(\text{Coh} X) \) be \( \sigma_{\beta,\omega} \)-stable. Then there exists an open neighborhood \( U \ni (\beta, \omega) \) such that \( E \) is stable in this neighborhood. We can then find a sequence of rational points \( (\beta_n, \omega_n) \) in \( U \) that converges to \( (\beta, \omega) \). Then \( \Delta^H_{\beta_n,\omega_n}(E) \geq 0 \) by the previous discussion. Therefore \( \Delta^H_{\beta,\omega}(E) \geq 0 \).

We will also need later a minor modification of the above result.

**Theorem 6.17.** Let \( \omega, \beta, H \in N^1_\mathbb{R}(X) \) be divisors such that \( \omega \) and \( H \) are ample. For any \( \sigma_{\beta,\omega} \)-semistable object \( E \in D^b(\text{Coh} X) \) having non-integer phase, we have

\[
\Delta^H_{\beta,\omega}(E) = (b_1 H)(b_1 \omega) - b_0 b_2(H\omega) + \frac{1}{2} b_0^2(H\omega)^2 > 0, \quad b = \chi^\beta(E).
\]

**Proof.** Let us use the same notation as in the previous theorem. We can assume that \( E \in \mathcal{A}_{\beta,\omega} \) is \( \sigma_{t_0} \)-semistable of phase \( \neq 1 \). If \( E \) is \( \sigma_t \)-semistable for \( t \gg 0 \) and \( b = \chi^\beta(E) \), then either \( b_0 \neq 0 \) or \( b_0 = 0 \) and \( E = H^0(E) \) has a 1-dimensional support. In both cases we obtain \( \Delta^H_{\beta,\omega}(E) > 0 \). Now we repeat the argument of the previous theorem. More precisely, if we meet an object of zero rank in our inductive process, then \( \Delta^H_{\beta,\omega}(E) \geq \frac{1}{2}(H\omega)^2 \). And if \( E \) is \( \sigma_t \)-stable for \( t \in [t_0, t_1) \) and is strictly \( \sigma_{t_1} \)-semistable with all \( \sigma_{t_1} \)-stable factors having zero rank, then all these \( \sigma_{t_1} \)-semistable factors have the same \( \sigma_{t_1} \)-phase as \( E \), which is a contradiction. We conclude that \( E \) (having zero rank) is \( \sigma_t \)-stable for all \( t \gg 0 \). Therefore \( E = H^0(E) \) has a 1-dimensional support and \( \Delta^H_{\beta,\omega}(E) = (b_1 H)(b_1 \omega) > 0 \).

### 6.8. Alternative parametrization of stability conditions.

The following parametrization of stability conditions \( \sigma_{\beta,\omega} \) also appears in the literature. For any pair of divisors \((\beta, \omega) \in N^1_\mathbb{R}(X) \times \text{Amp}(X)\), there exists a unique tuple

\[
(D, H, s, t) \in N^1_\mathbb{R}(X) \times \mathbb{R} \times \mathbb{R}_{>0}
\]

such that

\[
H^2 = 1, \quad HD = 0,
\]

\[
\beta + i\omega = (sH + D) + itH = (s + it)H + D.
\]
Indeed, we require $\omega = tH$, hence $t = \sqrt{\omega^2}$ and $H = \frac{1}{t} \omega$. On the other hand, equation $\beta = sH + D$ determines $s$ and $D$ uniquely as we have a decomposition $N^1_{\mathbb{R}}(X) = \mathbb{R}H \oplus H^\perp$.

Note that $\beta \omega = st$ and we have $\mu_\omega(E) \leq \beta \omega$ if and only if $\mu_H(E) \leq s$. Therefore

$$A_{\beta,\omega} = \text{Coh}_{\omega,\# \beta \omega} = \text{Coh}_{H,\# s}.$$  

(194)

From now on, we fix divisors $D$ and $H$ as above and we consider $s$ and $t$ as parameters. Let us define linear maps

$$v_H : N^1_{\mathbb{R}}(X) \to \mathbb{R}^3, \quad a \mapsto (a_0, a_1H, a_2),$$  

(195)

and

$$v_{D,H} : K(X) \to \mathbb{R}^3, \quad E \mapsto v_H(\text{ch}^D(E)).$$  

(196)

We define a linear map $Z'_{s,t} : \mathbb{R}^3 \to \mathbb{C}$ such that

$$Z_{\beta,\omega}(E) = Z'_{s,t}(v_{D,H}(E)) \quad \forall E \in K(X).$$

 Explicitly, if $v = v_{D,H}(E) = (a_0, a_1H, a_2)$ with $a = \text{ch}^D(E)$, then

$$Z'_{s,t}(v) = - \int e^{-(s+it)H} \cdot a = \left(\frac{a_0}{2}(t^2 - s^2) + sH a_1 - a_2\right) + i(tHa_1 - sta_0)$$

$$= \left(\frac{v_0}{2}(t^2 - s^2) + sv_1 - v_2\right) + it(v_1 - sv_0) =: -\theta(v) + it\rho(v).$$

We can substitute $Z'_{s,t}$ by the equivalent stability function (semistability is preserved)

(197) $Z'_{s,t}(v) = -\theta(v) - s\rho(v) + i\rho(v) = (qv_0 - v_2) + i(v_1 - sv_0),$

(198) $q = \frac{1}{2} (s^2 + t^2) > \frac{1}{2}s^2.$

7. Stability data associated to geometric stability conditions

7.1. Vanishing results. Let $X$ be a smooth projective surface and $\beta, \omega \in N^1_{\mathbb{R}}(X)$ with ample $\omega$.

Lemma 7.1. For any line bundle $L = \mathcal{O}_X(D)$, the automorphism

$$D^b(\text{Coh} X) \to D^b(\text{Coh} X), \quad E \mapsto E(D) = E \otimes L,$$

maps $(Z_{\beta,\omega}, A_{\beta,\omega})$ to $(Z_{\beta+D,\omega}, A_{\beta+D,\omega})$.

Proof. We have $\mu_\omega(E) \leq \beta \omega$ if and only if $\mu_\omega(E(D)) = \mu_\omega(E) + D\omega \leq (\beta + D)\omega$. This implies that $A_{\beta,\omega}$ is mapped to $A_{\beta+D,\omega}$. If $a = \text{ch}(E)$, then

$$Z_{\beta+D,\omega}(E(D)) = (e^{\beta+D+i\omega}, e^D \text{ch}(E)) = (e^{\beta+i\omega}, \text{ch}(E)) = Z_{\beta,\omega}(E).$$

Therefore semistability in $A_{\beta,\omega}$ with respect to $Z_{\beta,\omega}$ corresponds to semistability in $A_{\beta+D,\omega}$ with respect to $Z_{\beta+D,\omega}$. \qed

Theorem 7.2. Let $E, F$ be $\sigma_{\beta,\omega}$-semistable objects with $\phi_{\beta,\omega}(E) < \phi_{\beta,\omega}(F)$. Then, for any nef line bundle $L$, we have

$$\text{Hom}(F \otimes L, E) = 0.$$
Proof. Let \( L = \mathcal{O}_X(H) \). We will prove that \( \phi_{\beta-H,\omega}(F) \geq \phi_{\beta,\omega}(F) \). Then
\[
\phi_{\beta-H,\omega}(F) \geq \phi_{\beta,\omega}(F) > \phi_{\beta,\omega}(E) = \phi_{\beta-H,\omega}(E(-H)).
\]
By Lemma 7.1, the object \( E(-H) \) is semistable with respect to \( \sigma_{\beta-H,\omega} \). Therefore
\[
\text{Hom}(F \otimes L, E) \simeq \text{Hom}(F, E(-H)) = 0.
\]
To prove the inequality \( \phi_{\beta-H,\omega}(F) \geq \phi_{\beta,\omega}(F) \), let us consider the family of stability conditions \( \sigma_t = (Z_t, \mathcal{P}_t) := \sigma_{\beta-tH,\omega} \) for \( t \in [0,1] \) and let \( \phi_t \) be the corresponding phase function. Then the required inequality \( \phi_{\bar{\sigma}}(F) \geq \phi_0(F) \) will follow from Theorem 3.12, after we prove that
\[
\text{Im}(Z'_t(E) \cdot \bar{Z}_t(E)) \geq 0
\]
for any \( \sigma_t \)-semistable object \( E \). It is enough to consider just \( t = 0 \). If \( b = \text{ch}^\beta(E) \), then
\[
Z_0(E) = \frac{d}{dt}Z_{\beta-tH,\omega}(E)|_{t=0} \quad \text{and} \quad Z'_0(E) = \frac{d}{dt}(e^{i\omega-tH}, b)|_{t=0} = (e^{i\omega}, Hb) = -b_1H + ib_0(H\omega).
\]
Therefore
\[
\text{Im}(Z'_0(E) \cdot \bar{Z}_0(E)) = (b_1H)(b_1\omega) - b_0b_2(H\omega) + \frac{1}{2}b_0^2(H\omega)\omega^2.
\]
By Theorem 6.16, for any \( \sigma_{\beta,\omega} \)-semistable object \( E \) with \( \text{ch}^\beta(E) = b \), we have
\[
(b_1H)(b_1\omega) - b_0b_2(H\omega) \geq 0.
\]
The summand \( \frac{1}{2}b_0^2(H\omega)\omega^2 \) is non-negative. Therefore \( \text{Im}(Z'_0(E) \cdot \bar{Z}_0(E)) \geq 0 \) as required.

Corollary 7.3. Let \( \sigma \) be a special geometric stability condition and \( E, F \) be \( \sigma \)-semistable objects with \( \phi_\sigma(E) < \phi_\sigma(F) \). Then, for any nef line bundle \( L \), we have
\[
\text{Hom}(F \otimes L, E) = 0.
\]
Proof. By Theorem 6.10 there exists \( g = (T,f) \in \tilde{\text{GL}}_2^+(\mathbb{R}) \) such that \( \sigma[g] = \sigma' = \sigma_{\beta,\omega} \) for some \( \beta \in N_2^+(X) \) and \( \omega \in \text{Amp}(X) \). Then \( E, F \) are \( \sigma' \)-semistable and \( \phi_{\sigma'}(E) < \phi_{\sigma'}(F) \). We conclude from Theorem 7.2 that \( \text{Hom}(F \otimes L, E) = 0 \) as required.

Theorem 7.4. Let \( E, F \) be \( \sigma_{\beta,\omega} \)-semistable objects with \( \phi_{\beta,\omega}(E) = \phi_{\beta,\omega}(F) \notin \mathbb{Z} \). Then, for any ample line bundle \( L \), we have
\[
\text{Hom}(F \otimes L, E) = 0.
\]
Proof. Let us use the same notation as in the previous theorem. We can assume that \( E, F \in A_{\beta,\omega} \) have phase \( \phi \in (0,1) \). In our analysis of the behavior of phases under deformations from the previous theorem, we note that by Theorem 6.17 (where \( L = \mathcal{O}_X(H) \) and \( b = \text{ch}^\beta(E) \))
\[
\text{Im}(Z'_0(E) \cdot \bar{Z}_0(E)) = (b_1H)(b_1\omega) - b_0b_2(H\omega) + \frac{1}{2}b_0^2(H\omega)\omega^2 = \Delta_{\beta,\omega}(E) > 0
\]
whenever \( E \) is a \( \sigma_{\beta,\omega} \)-semistable object having phase in \( (0,1) \). By the same argument as in Theorem 3.12, we conclude that \( \phi_{\bar{\sigma}}(F) > \phi_0(F) \). Therefore
\[
\phi_{\beta-H,\omega}(F) > \phi_{\beta,\omega}(F) = \phi_{\beta-H,\omega}(E(-H)).
\]
This implies that $\text{Hom}(F \otimes L, H) \simeq \text{Hom}(F, E(-H)) = 0$. 

**Remark 7.5.** Note that the sheaves $\mathcal{O}_x$ are $\sigma_{\beta,\omega}$-semistable objects of phase 1 while $\text{Hom}(\mathcal{O}_x \otimes L, \mathcal{O}_x) \neq 0$ for any line bundle $L$. On the other hand, the above theorem can be extended to semistable objects of phase 1 having non-zero rank (see Lemma 6.9).

**Corollary 7.6.** Let $\sigma$ be a special geometric stability condition and $E, F$ be $\sigma$-semistable objects with $\phi_\sigma(E) = \phi_\sigma(F) \not\in \phi_\sigma(\mathcal{O}_x) + \mathbb{Z}$ for $x \in X$. Then, for any ample line bundle $L$, we have

$$\text{Hom}(F \otimes L, E) = 0.$$ 

**Theorem 7.7.** Let $X$ be a surface with a nef anticanonical bundle. Then any special geometric stability condition $\sigma$ on $D^b(\text{Coh} X)$ satisfies

$$\text{gldim}(\sigma) = 2.$$ 

**Proof.** To prove that $\text{gldim}(\sigma) \leq 2$, we need to show that given two $\sigma$-semistable objects $E, F$ with $\phi_\sigma(F) > \phi_\sigma(E) + 2$, we have $\text{Hom}(E, F) = 0$. By Serre duality, this means that $\text{Hom}(F \otimes \omega_X^{-1}, E[2]) = 0$, where $\omega_X^{-1}$ is nef and $\phi_\sigma(E[2]) < \phi_\sigma(F)$. The last equation follows from Corollary 7.3. To see that $\text{gldim}(\sigma) = 2$, we note that for any $x \in X$ the object $\mathcal{O}_x$ is $\sigma$-stable and $\text{Ext}^2(\mathcal{O}_x, \mathcal{O}_x) \simeq \text{Hom}(\mathcal{O}_x, \mathcal{O}_x)^\vee \neq 0$. 

**Theorem 7.8.** Let $X$ be a del Pezzo surface (meaning that the anticanonical bundle is ample). Let $E, F$ be $\sigma_{\beta,\omega}$-semistable objects with $\phi_{\beta,\omega}(E) = \phi_{\beta,\omega}(F) \not\in \mathbb{Z}$. Then

$$\text{Hom}^k(E, F) = 0 \quad \forall k \geq 2.$$ 

**Proof.** We apply Serre duality and Theorem 7.4. 

**Remark 7.9.** It was proved in [46, Lemma 4] that $\text{Ext}^2(E, E) = 0$ for a $\sigma_{\beta,\omega}$-semistable object $E$ of nonzero rank under an implicit assumption that $-K_X$ is proportional to $\omega$. A similar method was used in [25] to show that the statement of Theorem 7.7 is true for $X = \mathbb{P}^2$. This method doesn’t seem to generalize to other surfaces with nef or ample anticanonical bundle.

### 7.2. Geometric stability data

In this section we will combine all previous results and obtain a continuous family of stability data associated to a surface.

Let $X$ be a smooth projective surface, $\mathcal{D} = D^b(\text{Coh} X)$ and $\Gamma = \mathcal{N}(X)$. Let $\beta \in N^1_\mathbb{R}(X)$, $\omega \in \text{Amp}(X)$ and $\sigma_{\beta,\omega}$ be the corresponding stability condition on $\mathcal{D}$. This stability condition satisfies assumptions (1) and (2) by the results of [64] (see also [57]). We conclude from Theorem 6.10 that the same is true for any special geometric stability condition $\sigma \in \text{Stab}^+(\mathcal{D})$ (171).

Let us assume now that $X$ has a nef anticanonical bundle. Then we conclude from Theorem 7.7 that assumption (3) is satisfied for any special geometric stability condition $\sigma \in \text{Stab}^+(\mathcal{D})$. Following §5.4 and §5.5, we consider the quantum torus $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} R x^\gamma$, $R = K(\text{St}^+ / \mathbb{C})$, and define, for any special geometric stability condition $\sigma = (Z_\sigma, \mathcal{P}_\sigma) \in \text{Stab}^+(\mathcal{D})$: 

$$\text{Hom}(F \otimes L, E) = 0.$$
Stab$^+_\cl(X)$, the corresponding stability data $a_\sigma = (a_{\sigma,\gamma})_{\gamma \in \Gamma \setminus \{0\}} \in \hat{\mathfrak{g}}$ by

\begin{equation}
    a_{\sigma,\ell} = \sum_{\mathcal{Z}_\sigma(\gamma) \in \ell} a_{\sigma,\gamma} = \log(A_{\sigma,\ell}) \in \hat{\mathfrak{g}}_\ell, \mathcal{Z}_\sigma,
\end{equation}

and

\begin{equation}
    A_{\sigma,\ell} = 1 + \sum_{\mathcal{Z}_\sigma(\gamma) \in \ell} \mathbb{L}^{\frac{1}{2} \chi(\gamma, \gamma)} [\mathcal{M}_\sigma(\gamma)] x^{\gamma} \in \hat{G}_\ell, \mathcal{Z}_\sigma,
\end{equation}

for any ray $\ell \subset \mathbb{C}$, where $\mathcal{M}_\sigma(\gamma)$ is the moduli stack of $\sigma$-semistable objects having class $\gamma$. $\hat{\mathfrak{g}}_\ell, \mathcal{Z}_\sigma$ is the ind-pro-nilpotent Lie algebra defined in (64) and $\hat{G}_\ell, \mathcal{Z}_\sigma$ is the corresponding ind-pro-nilpotent group. We obtain from Theorem 5.9

**Theorem 7.10.** For any smooth projective surface $X$ with the nef anticanonical bundle, the family of stability data $(\mathcal{Z}_\sigma, a_\sigma)_{\sigma \in \text{Stab}^+_\cl(D)}$ is continuous.

### 7.3. Relation to intersection cohomology

We briefly explain the relationship of the above invariants to the intersection cohomology of the moduli spaces of semistable objects.

Let $X$ be a del Pezzo surface, meaning that the anticanonical bundle is ample. By the results of [64, 57], for any $\sigma \in \text{Stab}^+_\cl(D)$ and $\gamma \in \Gamma$, we can construct the moduli space $\mathcal{M}_\sigma(\gamma)$ of $\sigma$-semistable objects having class $\gamma$ (and phase $\frac{1}{\ell} \text{Arg} Z_\sigma(\gamma)$) and the moduli space $\mathcal{M}_\sigma^\text{st}(\gamma)$ of $\sigma$-stable objects having class $\gamma$. The moduli space $\mathcal{M}_\sigma^\text{st}(\gamma)$ is smooth if $\text{Hom}^2(E, E) = 0$ for all $E \in \mathcal{M}_\sigma^\text{st}(\gamma)$ (or more generally, if $\text{Hom}^2(E, E)$ has constant dimension).

Given an algebraic variety $Y$ of dimension $d_Y$, let $\text{IC}_Y \mathcal{Q}^H \in \text{MHM}(Y)$ be the corresponding intersection complex (see e.g. [62]). It is a simple pure object of weight $d_Y$. If $Y$ is smooth, then $\text{IC}_Y \mathcal{Q}^H = \mathcal{Q}^H[d_Y]$. We define (cf. [48])

\begin{equation}
    \text{IH}^*(Y, \mathbb{Q}) = H^*(Y, \text{IC}_Y \mathcal{Q}^H)[-d_Y] \in D^b(\mathcal{M}H\mathcal{S}_\mathbb{Q}).
\end{equation}

Let us consider the Hodge-Deligne polynomial (cf. (104))

\begin{equation}
    E : K(D^b(\mathcal{M}H\mathcal{S}_\mathbb{Q})) \simeq K(\mathcal{M}H\mathcal{S}_\mathbb{Q}) \rightarrow \mathbb{Z}[u^{\pm 1}, v^{\pm 1}], \quad V \mapsto \sum_{p,q} \dim \left( \text{Gr}^p_F \text{Gr}^q_W V \right) u^p v^q.
\end{equation}

and define

\begin{equation}
    \text{IE}(Y; u, v) = E(\text{IH}^*(Y, \mathbb{Q})) \in \mathbb{Z}[u^{\pm 1}, v^{\pm 1}].
\end{equation}

Let us consider the closure $\overline{\mathcal{M}}_\sigma^\text{st}(\gamma)$ of $\mathcal{M}_\sigma^\text{st}(\gamma)$ $\subset \mathcal{M}_\sigma(\gamma)$ and define (recall that we defined earlier $E(\mathbb{L}^{\frac{1}{2}}) = -\sqrt{uv}$)

\begin{equation}
    \Omega_\sigma(\gamma) = E(\mathbb{L}^{\frac{1}{2}})^{\chi(\gamma, \gamma)} \cdot \text{IE}(\overline{\mathcal{M}}_\sigma^\text{st}(\gamma); u, v).
\end{equation}

On the other hand, given a ray $\ell \subset \mathbb{C}$, we consider (cf. (203))

\begin{equation}
    E(A_{\sigma,\ell}) = 1 + \sum_{\mathcal{Z}_\sigma(\gamma) \in \ell} E(\mathbb{L}^{\frac{1}{2}})^{\chi(\gamma, \gamma)} \cdot E(\mathcal{M}_\sigma(\gamma)) x^{\gamma}.
\end{equation}

Let us assume that a ray $\ell \subset \mathbb{C}$ is such that $\mathcal{Z}_\sigma(\mathcal{O}_x) \notin \pm \ell$ for $x \in X$ and

\begin{equation}
    \chi(\gamma, \gamma') = \chi(\gamma', \gamma) \quad \forall \gamma, \gamma' \in \Gamma \cap \mathcal{Z}_\sigma^{-1}(\ell).
\end{equation}

By Theorem 7.8, the first assumption implies that for any $\sigma$-semistable objects $E, F$ having the same phase and classes in $\mathcal{Z}_\sigma^{-1}(\ell)$, we have $\text{Hom}^2(E, F) = 0$. The second
We will say that an exceptional collection is strong where Exp is the plethystic exponential (see e.g. [E]).

The objects \( \pi \) variety and let \( \pi \) variety. We have

\[ 8.1. \]

\[ X \] in the case of \( \pi \) variety. By the results of [E], one can represent

\[ E(\sigma, \ell) = \exp \left( \frac{\sum_{\sigma(\gamma) \in \ell} \Omega_\sigma(\gamma) x^\gamma}{E(L^v) - E(L^{-v})} \right), \]

where Exp is the plethystic exponential (see e.g. [48]).

**Remark 7.11.** In [12] one used the last formula as the definition of the invariants \( E(\sigma, \ell) \) in the case of \( \pi = \mathbb{P}^2 \). Note that this formula cannot be used if \( Z_\sigma(\mathcal{O}_x) \in \ell \).

### 8. Relation to quivers with potentials

#### 8.1. Exceptional collections and tilting objects.

For more information on tilting theory see e.g. [1]. Let \( X \) be a smooth projective variety of dimension \( d \) and let

\[ (211) \quad E_1, \ldots, E_n \]

be an exceptional collection in \( D^b(\text{Coh} X) \), meaning that

1. \( \text{Hom}^k(E_i, E_i) = 0 \) for \( k \neq 0 \) and \( \text{Hom}(E_i, E_i) = \mathbb{C} \).
2. \( \text{Hom}^k(E_i, E_j) = 0 \) for \( i > j \) and \( k \in \mathbb{Z} \).

We will say that an exceptional collection is full if the objects \( E_i \) generate \( D^b(\text{Coh} X) \). We will say that an exceptional collection is strong if \( \text{Hom}^k(E_i, E_j) = 0 \) for all \( i, j \) and \( k \neq 0 \). Given a full and strong exceptional collection, let us define the algebra

\[ (212) \quad A = \text{End}(T')^{op}, \quad T' = \bigoplus_i E_i, \]

and let \( \text{mod} A \) denote the category of finite-dimensional left \( A \)-modules. Then there is an equivalence of categories [10, 61, 1]

\[ (213) \quad \Phi: D^b(\text{Coh} X) \to D^b(\text{mod} A), \quad E \mapsto \text{RHom}(T', E). \]

The objects \( E_i \) are mapped to the projective modules \( P_i = Ae_i \), where \( e_i \in A \) is the idempotent corresponding to \( \text{id} \in \text{End}(E_i) \). Let us introduce a \( \mathbb{Z} \)-grading \( A = \bigoplus_{k \geq 0} A_k \), \( A_k = \bigoplus_{j-i=k} \text{Hom}(E_i, E_j) \), and let \( \bar{A} = \text{Ker}(A \to A_0 = \bigoplus_{i=1}^n \mathbb{C}) \) be the augmentation ideal. By [20, Lemma A.1], one can represent \( A \) as a quotient \( \mathbb{C}Q'/I \) for a unique quiver \( Q' \) with the set of vertices \( Q'_0 = \{1, \ldots, n\} \) and the number of arrows from \( i \) to \( j \) equal to \( \dim e_j(\bar{A}/\bar{A}^2)e_i \).

On the other hand, let us consider the canonical bundle \( Y = \omega_X \) as an algebraic variety and let \( \pi: Y \to X \) be the projection. Note that \( Y \) is a non-compact Calabi-Yau variety. We have \( \pi_\ast \mathcal{O}_Y = \bigoplus_{i \geq 0} \omega_X^{-i} \) and \( Y = \text{Spec}(\mathcal{O}_Y) \). Let \( \text{ Coh}_c Y \subset \text{ Coh} Y \) be the subcategory of coherent sheaves with compact support and let \( \text{ Coh}_0 Y \subset \text{ Coh}_c Y \) be the subcategory of coherent sheaves supported on \( X \subset Y \) embedded as the zero section. A coherent sheaf \( \bar{E} \in \text{ Coh}_c Y \) can be identified with a pair \( (E, \phi) \), where \( E = \pi_\ast \bar{E} \in \text{ Coh} X \) and \( \phi: E \to E \otimes \omega_X \) is a morphism. We have \( \bar{E} \in \text{ Coh}_0 Y \) if and only if \( \phi: E \to E \otimes \omega_X \) is nilpotent, meaning that \( E \to E \otimes \omega_X \to E \otimes \omega_X^2 \to \ldots \) eventually vanishes.
The category $D^b(\text{Coh}_c X)$ can be identified with the subcategory $D^b_c(\text{Coh} Y) \subset D^b(\text{Coh} Y)$ consisting of complexes with cohomology having compact support. Similarly, the category $D^b(\text{Coh}_0 X)$ can be identified with the subcategory $D^b_0(\text{Coh} Y) \subset D^b(\text{Coh} Y)$ consisting of complexes with cohomology supported on $X$. Note that the full embedding $\text{Coh} X \hookrightarrow \text{Coh}_0 Y$, $E \mapsto (E, 0)$, induces a (non-full) functor $D^b(\text{Coh} X) \to D^b_0(\text{Coh} Y)$. The category $D^b(\text{Coh} X)$ generates $D^b_0(\text{Coh} Y)$ by extensions.

**Lemma 8.1** (See e.g. [54]). Let $\bar{E} = (E, \phi)$, $\bar{F} = (F, \psi)$ be objects in $\text{Coh}_c Y$. Then there is an exact sequence

$$0 \to \text{Hom}_Y(\bar{E}, \bar{F}) \to \text{Hom}_X(E, F) \to \text{Hom}_X(E, F \otimes \omega_X)$$

$$\to \text{Ext}_Y^1(\bar{E}, \bar{F}) \to \text{Ext}_X^1(E, F) \to \text{Ext}_X^1(E, F \otimes \omega_X) \to \ldots$$

**Corollary 8.2.** Let $\bar{E} = (E, \phi)$, $\bar{F} = (F, \psi)$ be objects in $\text{Coh}_c Y$. Then

$$\chi_Y(\bar{E}, \bar{F}) = \chi_X(E, F) - (-1)^d \chi_X(F, E) =: \langle E, F \rangle.$$

Let us consider the object $T = \pi^* T' = \bigoplus_i \pi^* E_i$ in $D^b(\text{Coh} Y)$. We have

$$\text{Hom}^k(T, T) = \text{Hom}^k(T', T' \otimes \pi_* \mathcal{O}_Y) = \bigoplus_{1 \leq i, j \leq n, m \geq 0} \text{Hom}^k(E_i, E_j \otimes \omega_X^{-m}).$$

Let us assume that

$$\text{Hom}^k(T, T) = 0 \quad \forall k \neq 0.$$

The object $T$ generates $D^b(\text{Coh} Y)$ in the sense that $\text{Hom}_Y^*(T, F) = 0$ for $F \in D^b(\text{Coh} Y)$ implies $F = 0$. Indeed, $\text{Hom}_Y^*(T, F) = \text{Hom}_X^*(T', \pi_* F) = 0$ implies $\pi_* F$, hence $F = 0$. This means that $T$ is a classical tilting object, hence there is an equivalence of categories [30, §7] (see also [20, Theorem 3.6])

$$\Phi: D^b_0(\text{Coh} Y) \to D^b(\text{mod} B), \quad E \mapsto \text{RHom}(T, E), \quad B = \text{End}(T)^{op}.$$

Let us consider the ideal $J = \bigoplus_{m \geq 1} \text{Hom}(T', T' \otimes \omega_X^{-m})$ of $B$ and let $\text{mod}_0 B \subset \text{mod} B$ consist of modules on which $J$ acts nilpotently. Then the above functor induces an equivalence

$$\Phi: D^b_0(\text{Coh} Y) \to D^b_0(\text{mod} B) = D^b(\text{mod}_0 B), \quad E \mapsto \text{RHom}(T, E).$$

Let us extend the above exceptional collection to a sequence $(E_i)_{i \in \mathbb{Z}}$ such that

$$E_{i+n} = E_i \otimes \omega_X^{-1}, \quad i \in \mathbb{Z}.$$

This is an example of a helix [20]. By our assumptions $\text{Hom}^k(E_i, E_j) = 0$ for $j > i - n$ and $k \neq 0$. On the other hand, $\text{Hom}(E_i, E_j) \simeq \text{Hom}^d(E_{j+n}, E_i)^{\vee} = 0$ for $i > j$. Therefore

$$B^{op} \simeq \bigoplus_{1 \leq i \leq n, j \geq i} \text{Hom}(E_i, E_j).$$
8.2. **Local surfaces.** Let $X$ be a smooth projective surface, $Y = \omega_X$ be its canonical bundle and $\pi: Y \to X$ be the projection as before. Then $Y$ is a non-compact 3-Calabi-Yau variety which is sometimes called a *local surface*. Recall that

\begin{align}
\chi_X(E,F) &= \int a^*b \cdot \text{Td}(X), \quad a = \text{ch}(E), \ b = \text{ch}(F), \\
\text{Td}(X) &= (1, -\frac{1}{2}K_X, \chi), \quad \chi = \chi(\mathcal{O}_X).
\end{align}

We have $a^*a = (a_0^2, 0, 2a_0a_2 - a_1^2)$, hence

\begin{equation}
\chi_X(E,E) = \chi \cdot a_0^2 - \Delta(E).
\end{equation}

We have $a^*b - b^*a = 2(0, a_0b_1 - a_1b_0, 0)$, hence

\begin{equation}
\langle E, F \rangle = (a_1b_0 - a_0b_1)K_X = (a, bK_X).
\end{equation}

For many surfaces $X$ (see e.g. [8]) one can find an exceptional collection (211) such that $T' = \bigoplus E_i$ and $T = \pi^*T'$ satisfy the assumptions of §8.1, the algebra $B = \text{End}(T)\text{op}$ can be identified with the Jacobian algebra $J_W$ of some quiver with potential $(Q,W)$ (see §8.4) and the algebra $A = \text{End}(T')\text{op}$ can be identified with the partial Jacobian algebra $J_{W,I}$ corresponding to a cut $I \subset Q_1$ (see §8.4). This implies that we can construct three, a priori unrelated, families of stability data:

1. For the category $D^b(\text{Coh} X) \simeq D^b(\text{mod } J_{W,I})$.
2. For the category $D^b_0(\text{Coh} Y) \simeq D^b_0(\text{mod } J_W)$.
3. For the category $D^b_0(\text{Coh} Y) \simeq D^b(\text{mod } J_W)$.

Because of Corollary 8.2, all these families are defined in the same quantum torus. We will show later that for $X = \mathbb{P}^2$ stability data on $D^b(\text{Coh} X)$ and on $D^b(\text{mod } J_W)$ can be glued together. This implies that knowing stability data on $D^b(\text{mod } J_W)$, we can reconstruct stability data on $D^b(\text{Coh} X)$. A complete (but still conjectural except for low dimension vectors) description of stability data on $D^b(\text{mod } J_W)$ was given in [8, 55].

A partial description of stability data on $D^b(\text{Coh} X)$ was given in [12] (the scattering diagram used there captures a significant, but incomplete information about the stability data). It should be not difficult to generalize our results to other del Pezzo surfaces. A complete (conjectural) description of stability data for $D^b(\text{mod } J_W)$ is also available in this case [8, 55].

Recall that for every $\beta \in N^1_\mathbb{R}(X)$ and $\omega \in \text{Amp}(X)$ we have a stability condition $\sigma_{\beta,\omega}$ on $D^b(\text{Coh} X)$. This stability condition can be extended to $D^b_0(\text{Coh} Y)$ (see e.g. [5]). Alternatively, one can use the approach of [31] to extend a stability condition from $D^b(\text{Coh} X)$ to $D^b_0(\text{Coh} Y)$. Motivic DT invariants can be defined for the 3CY categories $D^b_0(\text{Coh} Y)$ and $D^b(\text{Coh} Y)$ using the framework of [41], as long as one proves that various necessary assumptions are satisfied (and this is not a simple task). On the other hand, as was observed in [12] (see also [21, Prop. 3.1]), in many situations (assuming that $X$ is a del Pezzo surface) semistable objects in $D^b(\text{Coh} Y)$ are automatically semistable objects in $D^b(\text{Coh} X)$. Intuitively, the reason is that for any semistable object $(E, \phi)$, there is a morphism $\phi: (E, \phi) \to (E \otimes \omega_X, \phi \otimes \text{id}_{\omega_X})$ which has to vanish if the object on the right is semistable of phase less than the phase of $(E, \phi)$. This means that for del Pezzo surfaces
one can perform computations in $\mathcal{D}^b(\text{Coh}\ X)$ instead of the 3CY categories $\mathcal{D}^b_0(\text{Coh}\ Y)$ and $\mathcal{D}^b_c(\text{Coh}\ Y)$ (at least for some stability conditions and some Chern characters).

In the next sections we will consider the case of $X = \mathbb{P}^2$ and we will see how computation of stability data for $\mathcal{D}^b(\text{Coh}\ X)$ reduces to computations of stability data for a certain quiver with potential.

8.3. Relation to quiver representations. In this section we will identify the derived category $\mathcal{D} = \mathcal{D}^b(\text{Coh}\ X)$ for $X = \mathbb{P}^2$ with the derived category of some explicit finite-dimensional algebra $A$ of homological dimension two. We will see later that this algebra arises from a quiver with a potential and a cut.

8.3.1. Derived equivalence. Let us consider the Beilinson exceptional collection on $X = \mathbb{P}^2$

\begin{equation}
(224) \quad \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2).
\end{equation}

As in §8.1, we define the algebra

\begin{equation}
(225) \quad A = \text{End}(\mathcal{T}'), \quad T' = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2),
\end{equation}

and consider an equivalence of categories

\begin{equation}
(226) \quad \Phi: \mathcal{D}^b(\text{Coh}\ X) \to \mathcal{D}^b(\text{mod}\ A), \quad E \mapsto \text{RHom}(T', E).
\end{equation}

The algebra $A$ can be described as the quotient of the path algebra $\mathbb{C}Q'$ over the quiver $Q'$

\begin{equation}
(227) \quad 0 \leftrightarrow a_i \leftrightarrow 1 \leftrightarrow b_i \leftrightarrow 2
\end{equation}

by the relations

\begin{equation}
(228) \quad a_i b_j = a_j b_i, \quad i \neq j.
\end{equation}

Let $e_i \in A$ be the idempotent corresponding to the trivial path at the vertex $i \in Q'_0$.

Let $S_i \in \text{mod} A$ denote the corresponding simple module, $P_i = Ae_i \in \text{mod} A$ denote the projective indecomposable module, and $I_i = D(e_i A) \in \text{mod} A$ denote the injective indecomposable module, where $DV = \text{Hom}_\mathbb{C}(V, \mathbb{C})$ for a vector space $V$. Note that $S_0 = P_0$ and $S_2 = I_2$.

The objects of the exceptional sequence (224) are mapped to the projective modules $P_0$, $P_1$, $P_2$ respectively. Note that

$$\text{Hom}(\mathcal{O}, \mathcal{O}(1)) \simeq \mathbb{C}^3 \simeq \text{Hom}(P_0, P_1) = e_0 Ae_1,$$

$$\text{Hom}(\mathcal{O}(1), \mathcal{O}(2)) \simeq \mathbb{C}^3 \simeq \text{Hom}(P_1, P_2) = e_1 Ae_2,$$

$$\text{Hom}(\mathcal{O}, \mathcal{O}(2)) \simeq \mathbb{C}^6 \simeq \text{Hom}(P_0, P_2) = e_0 Ae_2.$$

8.3.2. Simple objects. From now on we will identify $\mathcal{D} = \mathcal{D}^b(\text{Coh}\ X)$ with $\mathcal{D}^b(\text{mod}\ A)$ using the equivalence $\Phi$ (226). We claim that the simple objects $S_0$, $S_1$, $S_2$ correspond to

\begin{equation}
(229) \quad \mathcal{O}, \Omega_X(1)[1], \mathcal{O}(-1)[2],
\end{equation}

where $\Omega_X$ is the cotangent bundle. It is clear that $S_0 = P_0 = \mathcal{O}$. Note that

\begin{equation}
(230) \quad \text{dim} \text{Hom}^k(P_i, S_j) = \delta_{ij} \delta_{k0}.
\end{equation}

It is proved in [20, Lemma 2.5] that for a full exceptional collection $(E_1, \ldots, E_n)$ there exists a unique collection $(F_n, \ldots, F_1)$ such that $\text{dim} \text{Hom}^k(E_i, F_j) = 0$ (the new
collection is also a full exceptional collection). Therefore it is enough to show that (229) has the required properties. Below we will give a direct proof instead.

The category $D^b(\text{Coh} X)$ has the Serre functor (note that $\omega_X = \mathcal{O}(-3)$)

$$S: D^b(\text{Coh} X) \to D^b(\text{Coh} X), \quad E \mapsto E \otimes \omega_X[2].$$

On the other hand, the category $D^b(\text{mod} A)$ also has the Serre functor (called the Nakayama functor in this context; the same construction works for any finite-dimensional algebra $A$ of finite homological dimension)

$$\nu: D^b(\text{mod} A) \to D^b(\text{mod} A), \quad M \mapsto DA \otimes_A M,$$

where we consider $DA = \text{Hom}_C(A, C)$ as an $A$-bimodule. Note that $\nu P_i = I_i$. We can identify the Serre functors $S$ and $\nu$, hence

$$I_0 = \mathcal{O}(-3)[2], \quad I_1 = \mathcal{O}(-2)[2], \quad S_2 = I_2 = \mathcal{O}(-1)[2].$$

Let us consider the universal exact sequence on $X = \mathbb{P}^2$ [26, B.5.7]

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^\oplus_3 \to \mathcal{Q} \to 0.$$ 

Then the tangent bundle is given by $T_X = \mathcal{Q}(1)$. Dualizing the above exact sequence, we obtain a distinguished triangle

$$\mathcal{O}^\oplus_3 \to \mathcal{O}(1) \to \mathcal{Q}^\vee[1] \to$$

which corresponds to the canonical short exact sequence in $\text{mod} A$

$$0 \to S_0^\oplus_3 \to P_1 \to S_1 \to 0.$$ 

We conclude that $S_1 = \mathcal{Q}^\vee[1] = \Omega_X(1)[1]$.

### 8.3.3. Chern classes and dimension vectors.

The above equivalence $\Phi: \mathcal{D} = D^b(\text{Coh} X) \to D^b(\text{mod} A)$ induces an isomorphism between the (numerical) Grothendieck groups. We have linear maps $\text{ch}: \mathcal{N}(\text{Coh} X) \to \mathbb{N}^*_0(X)$ and

$$\text{dim}: \mathcal{N}(\text{mod} A) \to \mathbb{Z}Q'_0, \quad M \mapsto (\text{dim } e_i M)_{i \in Q'_0}.$$ 

Given $E \in \mathcal{D}$, let us write $\text{ch}(E) \in \mathcal{N}^*_0(X)$ and $\text{dim } E := \text{dim } \Phi(E) \in \mathbb{Z}Q'_0$ as column vectors and let $M$ be the transition matrix such that $\text{ch } E = M \cdot \text{dim } E$. Let us also define matrices

$$A = (\chi(P_i, P_j))_{ij}, \quad B = (\chi(S_i, S_j))_{ij}, \quad C = (\text{ch } P_i)_{i}.$$ 

Then

$$A \cdot B^t = I, \quad M = C \cdot A^{-1}.$$ 

We can calculate $A$ and $C$ using (224) and we obtain

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -3 & -3 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 2 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

In particular, if $a = (a_0, a_1, a_2) = \text{ch}(E)$ and $d = (d_0, d_1, d_2) = \text{dim } E$, then

$$a_0 = d_0 - 2d_1 + d_2, \quad a_1 = d_1 - d_2, \quad 2a_2 = d_1 + d_2.$$
Remark 8.3. The above formula also follows from the fact that the Chern classes of simple objects (229) are

\[(242) \quad \text{ch } \mathcal{O} = (1, 0, 0), \quad \text{ch } \Omega_X(1)[1] = (-2, 1, \frac{1}{2}), \quad \text{ch } \Omega(-1)[2] = (1, -1, \frac{1}{2}).\]

Let us consider the quiver \(Q\) (cf. (227))

\[(243) \quad \begin{array}{ccc}
0 & \overset{1}{\rightarrow} & 1 \\
\overset{a_i}{\rightarrow} & & \overset{b_i}{\rightarrow} \\
& \overset{c_i}{\rightarrow} & 2 \\
\end{array}\]

and the corresponding Euler form

\[(244) \quad \chi_Q(d, d') = \sum_i d_i (d'_i - 3d'_{i-1}).\]

Then the above explicit formula for the matrix \(B\) implies

Lemma 8.4. Let \(E, F \in \mathcal{D}\) and \(d = \dim E, d' = \dim F\). Then

\[(245) \quad \chi_X(E, F) = \chi_Q(d, d') + 3d_0d'_0 + 3d_0d'_2.\]

In particular, we have

\[(246) \quad \chi_X(E, E) = \chi_Q(d, d) + 6d_0d_2,
(247) \quad \langle E, F \rangle = \chi_Q(d, d') - \chi_Q(d', d) =: \langle d, d' \rangle.\]

8.4. Quivers with potential and their motivic invariants. The quiver \(Q\) defined in (243) is the McKay quiver of the representation of \(\mathbb{Z}_3\) on \(\mathbb{C}^3\) given by \(1 \mapsto \text{diag}(\xi, \xi, \xi)\), \(\xi = e^{2\pi i/3}\). This quiver is automatically equipped with the potential [27, §4.4]

\[(248) \quad W = \sum_{\sigma \in \Sigma_3} \text{sgn}(\sigma) \cdot a_{\sigma_1}b_{\sigma_2}c_{\sigma_3},\]

where \(\Sigma_3\) is the permutation group.

Generally, let us assume that we have a quiver \(Q\) equipped with a potential \(W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]\), that is, a finite linear combination \(W = \sum u c_u u\) of cyclic paths considered up to cyclic shift. We define the Jacobian algebra of \((Q, W)\) to be

\[(249) \quad J_W = \mathbb{C}Q/(\partial W) = \mathbb{C}Q/(\partial W/\partial a : a \in Q_1),\]

where \(\partial u/\partial a = \sum_{i: a_i = a} a_{i-1}a_1a_2 \ldots a_{i+1}\) for any cycle \(u = a_n \ldots a_1\). Let \(I \subset Q_1\) be a cut, meaning a subset such that every non-zero term of \(W\) contains exactly one arrow from \(I\). We consider the quiver \(Q' = (Q_0, Q_1 \setminus I)\) and define the partial Jacobian algebra

\[(250) \quad J_{W,I} = \mathbb{C}Q'/(\partial W/\partial a : a \in I).\]

For any dimension vector \(d \in \mathbb{N}Q_0\), we define

\[(251) \quad R(Q, d) = \bigoplus_{a: i \rightarrow j} \text{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})\]

and we define \(R(J_W, d) \subset R(Q, d)\) and \(R(J_{W,I}, d) \subset R(Q', d)\) to be the closed subsets corresponding to the relations of the algebras \(J_W\) and \(J_{W,I}\) respectively. We define

\[(252) \quad \text{tr } W: R(Q, d) \rightarrow \mathbb{C}, \quad M \mapsto \text{tr } W|M = \sum_{u} c_u \cdot \text{tr } u|M,\]
where $\text{tr } u|M$ for a cycle $u = a_n \ldots a_1$ is defined to be $\text{tr}(M_{a_n} \ldots M_{a_1})$. Then $R(J_W, d)$ can be identified with the critical locus of $\text{tr } W$ (see e.g. [63]). Let us consider the skew-symmetric form on $\Gamma = \mathbb{Z}Q_0$

$$\langle d, d' \rangle = \chi_Q(d, d') - \chi_Q(d', d)$$

and the corresponding quantum torus $\mathbb{T}$ (123) (with coefficients in the Grothendieck ring of varieties with exponentials, see e.g. [53, §5]). We define the series

$$A^Q_W = \sum_{d \in \mathbb{Z}Q_0} \mathbb{L}^{\frac{1}{2} \chi_Q(d, d)} \frac{[\text{tr } W][R(Q, d), \text{tr } W]}{[\text{GL}_d]} x^d \in \hat{\mathbb{T}},$$

where $[\text{tr } W][R(Q, d), \text{tr } W]$ is the exponential motivic class (see e.g. [53, §5]) and $\text{GL}_d = \prod_i \text{GL}_{d_i}$. This expression can be simplified in the presence of a cut $I \subset Q_1$, namely [53, §5]

$$[\text{tr } W][R(Q, d), \text{tr } W] = \mathbb{L}^{\gamma_I(d, d)} \cdot [R(J_{W, I}, d)],$$

$$\gamma_I(d, d') = \sum_{(a: i \to j) \in I} d_i d'_j.$$

Let us assume now that we have a stability function $Z : \Gamma = \mathbb{Z}Q_0 \to \mathbb{C}$, meaning that $Z(e_i) \in \mathbb{H}$ for $i \in Q_0$. It defines the notion of stability on the abelian category $\text{Rep } Q$ §3.4. Let $R_Z(Q, d) \subset R(Q, d)$ denote the open subset of $Z$-semistable representations. For any ray $\ell \subset \mathbb{H}$, we define

$$A^Q_W = 1 + \sum_{Z(d) \in \ell} \mathbb{L}^{\frac{1}{2} \chi_Q(d, d)} \frac{[R_Z(Q, d), \text{tr } W]}{[\text{GL}_d]} x^d \in \hat{\mathbb{T}},$$

and $A^{QW}_Z = A^Q_W$. We define stability data of the quiver with potential $(Q, W)$ to be the collection $A^Q_W = (A^Q_W)_{\ell}$. It is parametrized by $Z \in \mathbb{H}^{Q_0} \subset \text{Hom}(\mathbb{Z}Q_0, \mathbb{C})$, although one can extend the set of parameters to $\text{GL}^+_{\ell}(\mathbb{R}) \cdot \mathbb{H}^{Q_0} \subset \text{Hom}(\mathbb{Z}Q_0, \mathbb{C})$.

The following result is well-known. It is important to stress that while we compute invariants $[\text{tr } W][R(Q, d), \text{tr } W]$ which morally correspond to counting representations of the Jacobian algebra (and these representations form the heart of a bounded t-structure on the 3CY derived category of DG modules over the Ginzburg DG algebra of $(Q, W)$ [39]), in reality all computations are performed on the level of $Q$-representations which form an abelian category of homological dimension one. Therefore the proof of the below wall-crossing formula is the same as in [60, 33] and we don’t need to invoke the framework of [41].

**Theorem 8.5.** For any stability function $Z : \Gamma \to \mathbb{C}$, we have

$$A^Q_W = \prod_{\ell \subset \mathbb{R}} A^{QW}_Z.$$

8.5. **Relating stability data.** Let us consider again the quiver $Q$ (243) and its potential $W$ (248)

$$W = \sum_{\sigma \in \Sigma_3} \text{sgn}(\sigma) \cdot a_{\sigma_1} b_{\sigma_2} c_{\sigma_3}.$$
Let us consider the cut \( I = \{ c_1, c_2, c_3 \} \subset Q_1 \). Then the quiver \( Q' = (Q_0, Q_1 \setminus I) \) has the form
\[
0 \xleftarrow{a_i} 1 \xrightarrow{b_i} 2
\]
which coincides with the quiver (227) and the relations of the algebra \( J_{W,I} \) coincide with (228). We conclude that
\[
A = \text{End}(T)_{\text{op}} = J_{W,I}, \quad T' = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2),
\]
and \( D = D^b(\text{Coh } X) \simeq D^b(\text{mod } J_{W,I}) \), where \( X = \mathbb{P}^2 \). Similarly, let \( \pi : Y = \omega_X \to X \) be the projection and \( T = \pi^*T' \). Then
\[
B = \text{End}(T)_{\text{op}} = J_W
\]
and \( D^b_c(\text{Coh } Y) \simeq D^b(\text{mod } J_W) \).

**Remark 8.6.** The canonical bundle \( Y = \omega_{\mathbb{P}^2} \) can be interpreted as a crepant resolution of \( \mathbb{C}^3/\mathbb{Z}_3 \). This implies that [19]
\[
D^b_c(\text{Coh } Y) \simeq D^b_{\mathbb{C}^3}(\text{Coh } \mathbb{C}^3) \simeq D^b(\text{mod } J_W).
\]
Note that the left hand side of (247) is the Euler form of \( D^b_c(\text{Coh } Y) \) and the right hand side of (247) is the Euler form of \( D^b(\text{mod } J_W) \).

We conclude from Lemma 8.4 that the Euler form \( \chi_X \) of the derived category \( D^b(\text{Coh } X) \simeq D^b(\text{mod } J_{W,I}) \) is given by
\[
\chi_X(E, F) = \chi_X(d, d'), \quad d = \dim E, \ d' = \dim F,
\]
\[
\bar{\chi}_X(d, d') := \chi_Q(d, d') + \gamma_I(d, d') + \gamma_I(d', d),
\]
where \( \gamma_I(d, d') \) was defined in (256).

**Remark 8.7.** This seems to be a rather general phenomenon. In [8] one considered a large family of examples of surfaces \( X \) with a nef anticanonical bundle (called weak Fano surfaces there) and quivers with potential \( (Q, W) \) such that \( D^b_c(\text{Coh } \omega_X) \simeq D^b(\text{mod } J_W) \). We verified in all of these examples that there exists a special cut \( I \subset Q_1 \) such that \( D^b(\text{Coh } X) \simeq D^b(\text{mod } J_{W,I}) \) and such that
\[
\chi_X(E, F) = \chi_Q(d, d') + \gamma_I(d, d') + \gamma_I(d', d), \quad d = \dim E, \ d' = \dim F.
\]
It would be interesting to give a conceptual explanation of this phenomenon.

Let \( Z : \Gamma = ZQ_0 \to \mathbb{C} \) be a stability function on the abelian category \( \text{mod } J_{W,I} \) and let \( \sigma = (Z, \mathcal{P}) \) be the corresponding stability condition on \( D = D^b(\text{Coh } X) \simeq D^b(J_{W,I}) \). For any ray \( \ell \subset \mathbb{H} \), we define stability data (133)
\[
A_{\sigma,\ell} = 1 + \sum_{Z(d) \in \ell} \mathbb{L}^\frac{1}{2} \bar{\chi}_X(d, d') \left[ R_Z(J_{W,I}, d) \right]_{\text{GL}_d} \chi^d
\]
and \( A_{\sigma,-\ell} = A_{\sigma,\ell} \). Note that the algebra \( J_{W,I} \) has homological dimension 2, hence we don’t have a wall-crossing formula for the above stability data. Nevertheless, we can formulate such a result if a stability condition on \( D^b(\text{mod } J_{W,I}) \) has global dimension \( \leq 2 \), similarly to Corollary 5.7. We will see later that there exist geometric stability conditions
σ on \(D^b(\text{Coh} \ X)\) with the heart \(A_\sigma = \text{mod} \ J_{W,I}\). These stability conditions have global dimension 2 by Theorem 7.7.

**Theorem 8.8.** Let \(Z: \Gamma = \mathbb{Z}Q_0 \to \mathbb{C}\) be a stability function such that the corresponding stability condition \(\sigma\) on \(D^b(\text{mod} \ J_{W,I})\) has global dimension \(\leq 2\). Then, for any ray \(\ell \subset \mathbb{H}\), we have

\[
A_{\sigma,\ell} = A_{Z,\ell}^{Q,W}.
\]

**Proof.** Let \(\ell_0 \subset V = \text{cone} \{Z(e_i) : i \in Q_0\} \subset \mathbb{H}\) be a ray and let \(Z_0: \Gamma \to \mathbb{C}\) be such that \(Z_0(e_i) \in \ell_0\) for all \(i \in Q_0\). Let \(\sigma_0\) be the corresponding stability condition on \(D^b(\text{mod} \ J_{W,I})\) (we will call it trivial as all objects in mod \(J_{W,I}\) have \(\sigma\)-stability condition). By Theorem 8.8, let \(\ell_0 \subset V = \text{cone} \{Z(e_i) : i \in Q_0\} \subset \mathbb{H}\) be a ray and let \(Z_0: \Gamma \to \mathbb{C}\) be such that \(Z_0(e_i) \in \ell_0\) for all \(i \in Q_0\). Let \(\sigma_0\) be the corresponding stability condition on \(D^b(\text{mod} \ J_{W,I})\) (we will call it trivial as all objects in mod \(J_{W,I}\) are automatically semistable). Then we have (267)

\[
A_{\sigma_0,\ell_0} = 1 + \sum_{Z_0(d) \in \ell_0} \mathbb{L}^\kappa_{X}(d,d)[R(J_{W,I},d)]_{\chi^d}.
\]

This series coincides with \(A_{\sigma,\ell_0}^{Q,W}\) (254) as

\[
\mathbb{L}^\kappa_{X}(d,d)[R(J_{W,I},d)] = \mathbb{L}^\kappa_{Q,W}(d,d)[R(J_{W,I},d)] = \mathbb{L}^\kappa_{Q,W}(d,d) \cdot [R(Q, d), \text{tr} \ W].
\]

By the assumption that \(\sigma\) has global dimension \(\leq 2\), we obtain (see Corollary 5.7)

\[
A_{\sigma_0,\ell_0} = A_{\sigma,V} = \prod_{\ell \subset \mathbb{H}} A_{\sigma,\ell}.
\]

By Theorem 8.5, we have \(A_{\sigma,V} = \prod_{\ell \subset \mathbb{H}} A_{Z,\ell}^{Q,W}\). We conclude that \(A_{\sigma,\ell} = A_{Z,\ell}^{Q,W}\).

**8.6. Gluing families of stability data.** Let \(X = \mathbb{P}^2\) and \(H\) be the ample divisor such that \(H^2 = 1\). We will identify \(\mathbb{N}^*_\mathbb{R}(X)\) with \(\mathbb{R}^3\) by sending \(a \mapsto (a_0, a_1 H, a_2)\). Let (cf. §6.8)

\[
\beta + i\omega = (s + it)H, \quad s \in \mathbb{R}, \quad t \in \mathbb{R}_{>0}.
\]

We consider the abelian category \(A_{\beta,\omega} = \text{Coh}_{\beta,\omega} = \text{Coh}_{H,\beta,\omega}\) and the stability function (see (179))

\[
Z_{\beta,\omega}(E) = Z_{s,t}(a) = (y a_0 + sa_1 - a_2) + i t (a_1 - sa_0), \quad a = \text{ch}(E),
\]

where \(y = \frac{1}{2}(t^2 - s^2) > -\frac{1}{2}s^2\). Let \(\sigma_{\beta,\omega}\) be the corresponding stability condition on \(D^b(\text{Coh} \ X) \simeq D^b(\text{mod} \ J_{W,I})\). The following result is similar to [12, §4], although one works with a different exceptional collection there.

**Lemma 8.9.** Assume that

\[
s > -\frac{1}{2}, \quad t > 0, \quad s + t < 0.
\]

Then the heart of \(\sigma_{\beta,\omega,1/2}\) is equal to \(\text{mod} \ J_{W,I}\).

**Proof.** Let \(\sigma = (Z, \mathcal{P}) := \sigma_{\beta,\omega}\) and let \(A_{\sigma,1/2}\) denote the heart of \(\sigma_{1/2}\). Recall from §8.3.2 that the category \(\mathcal{A}_0 = \text{mod} \ J_{W,I}\) has simple objects

\[
\mathcal{O}, \Omega_X(1)[1], \Omega(-1)[2].
\]

We will show that the objects

\[
(270) \quad \mathcal{O}, \Omega_X(1)[1], \Omega(-1)[1]
\]
are contained in $A_{\beta,\omega}$ and

$$\text{(271)} \quad \text{Re} \ Z(\mathcal{O}) < 0, \quad \text{Re} \ Z(\Omega_X(1)[1]) < 0, \quad \text{Re} \ Z(\mathcal{O}(-1)[1]) > 0. $$

Then $\text{Re} \ Z$ is negative on all simple objects of $\mathcal{A}_0$, hence all these simple objects are contained in $A_{\sigma, \frac{1}{2}}$. This implies that $\mathcal{A}_0 \subset A_{\sigma, \frac{1}{2}}$. As both categories are hearts of bounded t-structures, we conclude that $\mathcal{A}_0 = A_{\sigma, \frac{1}{2}}$ as required.

Let us show first that $\Omega_X(1)$ is $\mu_H$-semistable. As $\Omega_X(1) = \mathcal{Q}'$, where $\mathcal{Q}$ was defined in (234), it is enough to show that $\mathcal{Q}$ is $\mu_H$-semistable. We have $\text{ch} \mathcal{Q} = (2, 1, -1/2)$, hence $\mu_H(\mathcal{Q}) = 1/2$. There are no non-trivial morphisms $\mathcal{Q}(k) \to \mathcal{Q}$ for $k \geq 1$ because of the exact sequence (234). Therefore $\mathcal{Q}$ is indeed $\mu_H$-semistable.

We have

$$\mu_H(\mathcal{O}) = 0, \quad \mu_H(\Omega_X(1)) = -1/2, \quad \mu_H(\mathcal{O}(-1)) = -1. $$

As $-1/2 < s < 0$, we conclude that $\mathcal{O} \in \text{Coh}_{H, > s}$ and $\Omega_X(1), \mathcal{O}(-1) \in \text{Coh}_{H, \leq s}$. Therefore all objects in (270) are contained in $\text{Coh}_{H, \# s} = A_{\beta,\omega}$. Let us check that (271) is satisfied. Recall that (242)

$$\text{ch} \mathcal{O} = (1, 0, 0), \quad \text{ch} \Omega_X(1)[1] = (-2, 1, \frac{1}{2}), \quad \text{ch} \mathcal{O}(-1) = (1, -1, \frac{1}{2}).$$

Therefore we require (with $y = \frac{1}{2}(t^2 - s^2)$)

$$y < 0, \quad -2y + s - \frac{1}{2} < 0, \quad y - s - \frac{1}{2} < 0.$$ 

Under the assumptions $-\frac{1}{2} < s < 0$ and $-\frac{1}{2}s^2 < y < 0$, the other conditions follow. As $y = \frac{1}{2}(t^2 - s^2)$, these assumptions are equivalent to (269).

Choosing parameters $(s, t)$ as in the previous lemma, we obtain a special geometric stability condition $\sigma = \sigma_{\beta,\omega, \frac{1}{2}}$ and central charge $Z = Z_{\beta,\omega, \frac{1}{2}} = \iota^{-1}Z_{\beta,\omega}$ such that the heart of $\sigma$ is equal to $A_\sigma = \text{mod} \ J_{W,I}$. The corresponding linear map $Z: N(X) \simeq \mathbb{Z}Q_0 \to \mathbb{Z}$ satisfies the assumptions of Theorem 8.8, hence we obtain $A_{\sigma,\ell} = A_{\sigma,\ell}^{Q,W}$. On the other hand, the series $A_{\sigma,\ell}$ defined in (267) (for the category $D^b(\text{mod} \ J_{W,I})$) and $A_{\sigma,\ell}$ defined in (133) (for the category $D^b(\text{Coh} \ X)$) are the same.

Note that stability data $A_\sigma = (A_{\sigma,\ell})_\ell$ is defined for every special geometric stability condition $\sigma = (Z, \mathcal{P}) \in \text{Stab}_+^b(\mathcal{D})$ (171). The discriminant $\Delta$ is negative-definite on Ker $Z$, hence by Theorem 4.11 stability data $A_\sigma$ depends just on $Z \in \mathcal{P}^+(X) \subset \text{Hom}(\Gamma, \mathbb{C})$ and we will denote it by $A_Z$. On the other hand, stability data $A_{\sigma,\ell}^{Q,W}$ is defined for every $Z \in \mathcal{P}(Q) := \text{GL}_2^+(\mathbb{R}) \cdot \mathbb{H}^{Q_0} \subset \text{Hom}(ZQ_0, \mathbb{C}) \simeq \text{Hom}(\Gamma, \mathbb{C})$. We conclude from Theorem 8.8 and Lemma 8.9 that we can glue together stability data for the category.
$D^b(\text{Coh} X)$ and stability data for the quiver with potential $(Q, W)$ and define a family of stability data parametrized by $\mathcal{P}^+(X) \cup \mathcal{P}(Q)$.

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