Photon-pair blockade in a Josephson-photonics circuit with two nondegenerate microwave resonators

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Abstract

We propose to generate photon-pair blockade in a Josephson-photonics circuit that consists of a dc voltage-biased Josephson junction in series with a superconducting charge qubit and two nondegenerate microwave resonators. The two-level charge qubit is utilized to create anticorrelations of the charge transport; that is, the simultaneous Cooper pair tunneling events are inhibited. When the Josephson frequency matches the sum of resonance frequencies of the charge qubit and the two resonators, we demonstrate that the two resonators can release their energies in the form of antibunched pairs of two strongly correlated photons. The present work provides a practical way for producing a bright microwave source of antibunched photon pairs, with potential applications ranging from spectroscopy and metrology to quantum information processing.

1. Introduction

Solid-state superconducting circuit studies the interaction between artificial atoms and quantized electromagnetic fields in the microwave frequency domain [1]. This architecture has emerged as one of the leading platforms for realizing quantum computation and simulation [2, 3]. Commonly, superconducting qubits are used for encoding quantum information, with superconducting resonators acting as date buses. However, the efficient generation, manipulation, and transmission of nontrivial quantum states in a linear resonator are also crucial to different kinds of quantum information tasks [4]. The quantum states of a harmonic oscillator are extraordinarily rich, but are hard to access due to the infinitely equally spaced energy levels. This difficulty can be overcome by interposing a nonlinear artificial atom, and various quantum states in a resonator can be synthesized via the deliberate use of classical control signals, such as Fock state [5] and Schrödinger cat state [6].

In recent years, the Josephson-photonics circuit of a dc voltage-biased Josephson junction in series with a microwave resonator has emerged as an alternative tool for efficient on-chip generation of coherent microwave photons [7–16]. It relies on the exceptionally strong nonlinearity of light–charge interaction, and eliminates the need for any microwave drives [17–21]. The Josephson junction acts as a highly nonlinear driving element; that is, the inelastic Cooper pair tunneling across a junction can convert different numbers of photons from an easily controlled bias voltage source into a microwave cavity [22, 23]. By modulating the charge tunneling effect via a specifically tailored electromagnetic environment, the nonclassical microwave light can be generated [24–28]. The resulting quantum electrodynamics of this simple circuit has been demonstrated to realize Josephson junction lasers [29, 30, 31], single-photon sources [32, 33–35], multi-photon sources [36, 37, 38], and near quantum-limited amplifiers [39].

In parallel, a significant development in this field is to connect Josephson junctions with multiple resonators for various quantum technological applications, such as the implementations of entangled quantum microwaves [40] and microwave single-photon detectors [41]. While the simplest case is one
voltage-biased junction in series with two cavities of incommensurate frequencies. When the bias voltage is tuned to match the energy required to simultaneously produce one photon in each cavity for a single Cooper pair traversing the circuit, the system can be effectively reduced as a nondegenerate parametric amplifier [42–44]. It allows the continuous emission of correlated photon pairs. Recent experiments have observed the amplitude squeezing [45] and entanglement [46] of these output microwave beams. Especially, when the cavities possess the impedance of 4.1 kΩ, there is no matrix element for a transition between the one and two photon states [22, 34]. So, the two cavities can be regarded as two-level systems, leading to an antibunched photon-pair source [42]. However, the fabrication of coplanar waveguide resonators with such high impedances is highly challenging as the standard cavity designs only yields characteristic impedances of the order of 100 Ω.

To go beyond this limitation, we propose a more practical way to realize photon-pair blockade by regulating the anticorrelated behavior of the charge transport via a two-level charge qubit. In our scheme, we study the Josephson-photonics device of a voltage-biased Josephson junction in series with a charge qubit and two nondegenerate microwave resonators. The nonlinear qubit-resonator coupling can be sculpted via the phase difference across the junction. For each tunneling Cooper pair, the suitably set bias voltage enables the excitation of charge qubit and the creation of one photon in each resonator concurrently. Since the charge qubit is an ideal anharmonic element with two quantum energy levels, the anticorrelations of the tunneling Cooper pairs can be created, preventing the simultaneous tunnel events. Combined with the dissipation, we show that the two resonators can release their energies in the form of antibunched photon pairs in a controllable manner. Compared with the previous work [42], the present one constitutes a significant step forward; that is, the photon-pair source can be achieved with the standard coplanar waveguide resonator designs, eliminating the need for ultrahigh cavity impedance that is not accessible in current experiment. Our work offers an appealing method for generating a bright nonclassical antibunched photon pairs in a controllable manner. Compared with the previous work [42], the present work [42] allows the continuous emission of correlated photon pairs. Recent experiments have shown that the two cavities can be regarded as two-level systems, leading to an antibunched photon-pair source [42].

2. Model

As shown in figure 1, we investigate the Josephson-photonics circuit of a voltage-biased dc superconducting quantum interference device (dc-SQUID) in series with a charge qubit and two nondegenerate LC resonators. We focus on the situation where the bias voltage \( V \) is smaller than the gap voltage, and no quasi-particle excitation can be produced in the superconducting electrodes. So, the quantum transport of Cooper pairs through the circuit will supply energies to both the charge qubit and the two LC resonators.

The model Hamiltonian describing the entire setup takes the form (see appendix A)

\[
\begin{align*}
H_T &= E_c(n_q - n_g)^2 - E_{iq} \cos \phi_q + \sum_{j=1,2} \left[ \frac{\tilde{q}_j^2}{2C_j} + \left( \frac{\hbar}{2e} \right)^2 \frac{\tilde{\phi}_j^2}{2L_j} \right] \\
&\quad - E_I \cos \phi_I - 2em(V - V_q - V_{R1} - V_{R2}). \quad (1)
\end{align*}
\]

The first two terms denote the part of charge qubit, the third term describes the part of two LC resonators, and the last two terms represent the part of dc-SQUID, where \( V_q = -\hbar \phi_q / 2e \) is the voltage drop at the qubit, and \( V_{Rj} = -\hbar \phi_j / 2e \) is the voltage drop at the \( j \)th resonator.

To exclude the Cooper pair number \( n_j \), we perform a unitary transformation \( U(t) = \exp[i(\omega_j t + \phi_q + \phi_1 + \phi_2)n_j] \) on the full Hamiltonian, where \( \omega_j = 2eV / \hbar \) is the Josephson frequency. Then, we can obtain

\[
\begin{align*}
\tilde{H}_T &= U^\dagger(t)H_T U(t) + i\hbar \frac{dU^\dagger(t)}{dt} U(t) \\
&= E_c(\tilde{n}_q - \tilde{n}_g)^2 - E_{iq} \cos \phi_q + \sum_{j=1,2} \left[ \frac{\tilde{q}_j^2}{2C_j} + \left( \frac{\hbar}{2e} \right)^2 \frac{\tilde{\phi}_j^2}{2L_j} \right] \\
&\quad - E_I \cos(\omega_j t + \phi_q + \phi_1 + \phi_2), \quad (2)
\end{align*}
\]

where \( \tilde{n}_q = n_q + n_j, \tilde{q}_j = q_j + 2em \) are the transformed number and charge operators, arising from the charge fluctuations regard to the flow of Cooper pairs through the dc-SQUID. As described by the last term in \( \tilde{H}_T \), the nonlinear coupling between the charge qubit and the two cavities is established via the phase difference across the junctions of dc-SQUID.
3.1. Anticorrelations of the charge transport

In this section, we will illustrate the procedure for the realization of photon-pair blockade in the aforementioned two-mode superconducting circuit. The central idea is to create anticorrelations of the charge transport, which gives rise to the desired antibunching of the photon pairs leaking out of the two microwave resonators.

To this end, we start to derive the effective Hamiltonian of the system, which helps to uncover the mechanism of our scheme. In the interaction picture with respect to the frame rotating exp(−iH₀t), where H₀ = 1/2δσₓ + ωjaj†aj − E₁ cos[ω₁t + φ₁ + 2δl(a†jaj + ajaj)l][120x94][120x103][120x107][120x110][120x113][120x116][120x120][120x123][120x126][120x129][120x132][120x135][120x138][120x141][120x144][120x147][120x150][120x153][120x156][120x159][120x162][120x165][120x168][120x171][120x174][120x177][120x180][120x183][120x186][120x189][120x192][120x195][120x198][120x201][120x204][120x207][120x210][120x213][120x216][120x219][120x222][120x225][120x228][120x231][120x234][120x237][120x240][120x243][120x246][120x249][120x252][120x255][120x258][120x261][120x264][120x267][120x270][120x273][120x276][120x279][120x282][120x285][120x288][120x291][120x294][120x297][120x300][120x303][120x306][120x309][120x312][120x315][120x318][120x321][120x324][120x327][120x330][120x333][120x336][120x339][120x342][120x345][120x348][120x351][120x354][120x357][120x360][120x363][120x366][120x369][120x372][120x375][120x378][120x381][120x384][120x387][120x390][120x393][120x396][120x399][120x402][120x405][120x408][120x411][120x414][120x417][120x420][120x423][120x426][120x429][120x432][120x435][120x438][120x441][120x444][120x447][120x450][120x453][120x456][120x459][120x462][120x465][120x468][120x471][120x474][120x477][120x480][120x483][120x486][120x489][120x492][120x495][120x498][120x501][120x504][120x507][120x510][120x513][120x516][120x519][120x522][120x525][120x528][120x531][120x534][120x537][120x540][120x543][120x546][120x549][120x552][120x555][120x558][120x561][120x564][120x567][120x570][120x573][120x576][120x579][120x582][120x585][120x588][120x591][120x594][120x597][120x600][120x603][120x606][120x609][120x612][120x615][120x618][120x621][120x624][120x627][120x630][120x633][120x636][120x639][120x642][120x645][120x648][120x651][120x654][120x657][120x660][120x663][120x666][120x669][120x672][120x675][120x678][120x681][120x684][120x687][120x690][120x693][120x696][120x699][120x702][120x705][120x708][120x711][120x714][120x717][120x720][120x723][120x726][120x729][120x732][120x735][120x738][120x741][120x744][120x747][120x750][120x753][120x756][120x759][120x762][120x765][120x768][120x771][120x774][120x777][120x780][120x783][120x786][120x789][120x792][120x795][120x798][120x801][120x804][120x807][120x810][120x813][120x816][120x819]}
Thus, the anticorrelations of the charge transport are created: a former Cooper-pair tunneling event acts as a blockade. However, the charge qubit is operated in the regime in which the charging energy is much larger than the Josephson coupling energy $E_J$. Consequently, the photon-pair blockade can be realized in this two-cavity system, which leads to antibunching in the photon emission.

With

$$\beta_{n+1}^m(\lambda) = \sqrt{\frac{n!}{(n+1)!}} (2i\lambda)^{n+1} L_{n}^{(0)}(4\lambda^2).$$

Here $\beta_{n+1}^m(\lambda)$ is a generalized Frank–Condon factor that describes an $l$-photon transition rate, and $L_{n}^{(0)}(4\lambda^2)$ is a Laguerre polynomial.

To go a further step, we should tune the bias voltage $V$ to meet the resonance condition $\omega_1 = \delta + \omega_1 + \omega_2$, i.e., the energy provided by the voltage source upon the transfer of a Cooper pair matches the sum of the excitation energy of the qubit and the photon energies of the two oscillators. In this case, we can retain the resonant terms, but discard those fast oscillating terms under the rotating-wave approximation provided that the condition $\delta, \omega_1, |\omega_1 - \omega_2| \gg \frac{\lambda}{e} (\beta_{n+1}^m(\lambda_1) \beta_{m+1}^n(\lambda_2))$ is satisfied. Thus, with all the other possibilities strongly suppressed, the effective Hamiltonian is derived as

$$H_{\text{eff}} = \sum_{n,m=0}^{\infty} H_{nm}$$

with

$$H_{nm} = g_{\text{eff}}^{nm} |n, m, g\rangle \langle n+1, m+1, e| + \text{h.c.},$$

where $g_{\text{eff}}^{nm} = \frac{\lambda}{e} (\beta_{n+1}^m(\lambda_1) \beta_{m+1}^n(\lambda_2))$ is the effective coupling strength, and $|n, m, g\rangle$ is the tensor product of $|n\rangle \otimes |m\rangle \otimes |g\rangle$.

The effective Hamiltonian $H_{\text{eff}}$ elucidates the process of energy conversion of Josephson frequency to qubit excitation and photon production in the two resonators. Note that each subunit $H_{nm}$ can induce a coherent quantum transition from the state $|n, m, g\rangle$ to $|n+1, m+1, e\rangle$; that is, each Cooper pair can tunnel to simultaneously populate the qubit and add one photon in each cavity. Since the charge qubit is a two-level system, its excitation will greatly inhibit the whole system’s transitions to higher occupations.

Thus, the anticoherences of the charge transport are created: a former Cooper-pair tunneling event acts back onto the next one; that is, a second Cooper pair can not pass through the circuit until the flip of the qubit state. This is the key point to induce antibunching in the photon emission.

If the system is initially prepared in the ground state $|0, 0, g\rangle$, the Hamiltonian $H_0$ will dominate the dynamics, enabling a Rabi oscillation $|0, 0, g\rangle \leftrightarrow |1, 1, e\rangle$. Obviously, the absorption of one photon in each cavity is accompanied with a excitation in the charge qubit. Since the charge qubit is treated as a two-level system, its excitation will inhibit further photon absorption, which is the mechanism involved in photon blockade [55, 56]. Consequently, the photon-pair blockade can be realized in this two-cavity system, which offers a source of antibunched pairs of two strongly correlated photons. It is also pointed out here that the charge qubit has finite nonlinearity in practice, and many other higher excited states exist. So, the excitation of these states is accompanied with the higher photon number excitations, which will degrade the photon blockade. However, the charge qubit is operated in the regime in which the charging energy $E_c$ is much larger than the Josephson coupling energy $E_J$ [57]. The third energy level has an eigenfrequency...
Figure 3. Steady-state photon correlation functions versus $\kappa \tau$: (a) and (b) for the stand second-order correlation functions $g^{(2)}_{11}(\tau)$, $g^{(2)}_{22}(\tau)$; (c) for the cross-correlation function $g^{(2)}_{12}(\tau)$; (d) for the generalized second-order correlation functions $g^{(2)}_{12,12}(\tau)$. The cavity damping rate is chosen as $\kappa/2\pi = 0.1$ GHz, and the other parameters are chosen to be the same as those in figure 2.

about $6E_c$, which is far greater than the transition frequency $E_{Jq}$ of the lowest two energy levels. So, the higher-order qubit excitations are greatly inhibited, which has negligible effect on the photon blockade. Compared with the previous proposal [42], our scheme can be implemented with the standard coplanar waveguide resonator designs, greatly lowering the requirement for ultrahigh cavity impedance that is not accessible in current experiment.

To check the validity of the approximation, we now investigate the dynamics of the system by numerically solving the Schrödinger equation with both the full Hamiltonian $H_I$ and the effective Hamiltonian $H_{\text{eff}}$. With the system initialized in the ground state $|0, 0, g\rangle$, the time evolution of populations $P_{00g}$ in the state $|0, 0, g\rangle$ and $P_{11e}$ in the state $|1, 1, e\rangle$ is shown in figure 2. The perfect Rabi oscillations $|0, 0, g\rangle \leftrightarrow |1, 1, e\rangle$ are observed with both of these two Hamiltonians $H_I$ and $H_{\text{eff}}$, implying that our approximation is valid.

3.2. Antibunched photon pair emission

As a trigger of quantum emission of antibunched photon pairs, dissipation has to be taken into account. When the system–environment coupling is considered in the Born–Markov approximation, the time evolution of density matric $\rho$ of the whole system is now governed by the master equation

$$\frac{d\rho}{dt} = -i[H_{\text{eff}}, \rho] + \sum_{j=1,2} \frac{\kappa_j}{2} D[a_j]\rho + \frac{\gamma}{2} D[\sigma_-]\rho, \quad (10)$$

where $D[\sigma]\rho = 2\sigma\rho\sigma - \sigma\rho - \rho\sigma$ is the standard Lindblad operator for a given operator $\sigma$, and $\kappa (\gamma)$ denotes the energy damping rate of cavities (qubit). In the presence of dissipation, a pair of two strongly correlated photons will be transferred outside of the two cavities for each tunneling Cooper pair. We now detail the underlying principle of the fundamental dynamics of this photon-pair emission below.

Specifically, we prepare the system initially in the ground state, and the tunneling of a Cooper pair will draw energy quanta from the bias voltage, inducing the coherent transition $|0, 0, g\rangle \rightarrow |1, 1, e\rangle$. Due to the energy upper limit of the two-level charge qubit, the system populated in the state $|1, 1, e\rangle$ can not be excited to higher energy levels. This indicates that a second Cooper pair cannot pass through the circuit only after the spontaneous emission of the charge qubit. So, to guarantee the desired antibunching, it is crucial to meet the condition $\kappa \gg \gamma$, i.e., the coherence time of charge qubit is much longer than that of cavities. In this situation, the two cavities will take the lead to emit two correlated photons within the lifetime $1/\kappa$, stemming from the spontaneous emission of $|1, 1, e\rangle$ state via the photonic dissipation. Then, the wavefunction of the system is collapsed into the state $|0, 0, e\rangle$ but without Rabi flopping. Only after a quantum jump $|0, 0, e\rangle \rightarrow |0, 0, g\rangle$ of the qubit state within a relative longer lifetime $1/\gamma$, a second Cooper
Figure 4. The photon-pair emission rate $S$ versus $\gamma/\kappa$. The relevant parameters are chosen to be the same as those in figure 3.

pair can tunnel to restart the coherent transition $|0, 0, g\rangle \rightarrow |1, 1, e\rangle$ for the next emission of a photon pair. This is the mechanism for generating antibunched photon pairs.

On the other hand, it is worth noting here that we also can not make an arbitrary small $\gamma$. For $\gamma \rightarrow 0$, the system seems to behave as a completely antibunched photon-pair source. However, it will take a very long time to flip the state of charge qubit $|0, 0, e\rangle \rightarrow |0, 0, g\rangle$ and reconstruct the state $|1, 1, e\rangle$ for the two cavities to emit a next correlated photon pair. This will result in an extremely low emission rate. Therefore, there is a tradeoff between the emission rate and the nonclassical property of the radiation field, which can be balanced by tuning the ratio $\gamma/\kappa$.

To describe the quantum statistics of the photon emission, we further study the following time-delay correlation functions

$$g^{(2)}_{pq}(\tau) = \frac{\langle a_{p}^\dagger(0)a_{p}(\tau)a_{q}(\tau)a_{q}(0)\rangle}{\langle a_{p}^\dagger a_{p}(0)\rangle\langle a_{q}^\dagger a_{q}(\tau)\rangle},$$

with $p, q = 1, 2$ and $p \neq q$. $g^{(2)}_{pq}(\tau)$ is just the standard second-order correlation function that can quantify the photon correlation emitted by a single cavity. While for $p \neq q$, it represents the cross-correlation between the photons emitted by different cavities. Besides, we should also introduce the generalized second-order correlation function

$$g^{(2)}_{12,12}(\tau) = \frac{\langle a_{1}^\dagger(0)a_{2}^\dagger(0)a_{1}(\tau)a_{2}(\tau)a_{1}(0)a_{2}(0)\rangle}{\langle a_{1}^\dagger a_{1}(0)\rangle\langle a_{2}^\dagger a_{2}(\tau)\rangle},$$

where the joint two-photon emission event by the two cavities is considered as a single entity [58, 59]. Here, $g^{(2)}_{12,12}(\tau)$ can capture the fundamental dynamics of photon-pair emission, and characterize the quantum statistics of photon pairs.

In figure 3, we plot the different steady-state correlation functions by numerically solving the master equation (10). For $\kappa \gg \gamma$, the zero-delay second-order correlation functions $g^{(2)}_{11}(0) \rightarrow 0$ and $g^{(2)}_{22}(0) \rightarrow 0$ are observed in figures 3(a) and (b), exhibiting distinct antibunching effects. So, each cavity behaves as an excellent single-photon emitter. As seen in figure 3(c), the zero-delay cross-correlation function yields $g^{(2)}_{12}(0) \approx 1$. This indicates that a pair of strongly correlated photons are emitted simultaneously by the two cavities. Moreover, the generalized zero-delay second-order correlation function $g^{(2)}_{12,12}(0) \ll 1$ in figure 3(d) manifests clearly that the two cavities release their energies in the form of antibunched photon pairs. In addition, as expected before, the changes of the different zero-delay correlation functions indicate that our device approaches a perfect photon-pair emitter with the decrease of $\gamma$.

Finally, we investigate the emission rate of our photon-pair source. It is defined as

$$S = \kappa \overline{n},$$

where $\overline{n}$ is the average photon numbers of the two cavities. The emission rate $S$ versus $\gamma/\kappa$ is displayed in figure 4. Under the premise of $\kappa \gg \gamma$, we can observe that a tunable emission rate can be achieved, i.e., $S$ gradually increases with the increase of $\gamma$. Hence, the emission rate can be experimentally controlled by changing the distance between the cavity and the transmission line to adjust the value of $\kappa$ [34]. With the
currently available parameters $\omega_1/2\pi = 9$ GHz, $\omega_2/2\pi = 7$ GHz, $\delta/2\pi = 5$ GHz, $E_f/2\pi = 0.5$ GHz, $\lambda_1 = \lambda_2 = 0.2$, we can obtain an emission rate of the order of MHz.

4. Conclusion

In conclusion, we have proposed a practical approach to generate antibunched photon pairs in a Josephson-photonics circuit of a dc voltage-biased Josephson junction coupled to both a superconducting charge qubit and two nondegenerate microwave cavities. Under an appropriate bias voltage, each Cooper pair can tunnel inelastically to cause the excitation of charge qubit and the creation of one photon in each cavity. We demonstrate that the charge transport can be controlled via the two-level charge qubit, preventing the simultaneous Cooper pair tunneling events. As a result, the photon-pair blockade can be realized, i.e., the presence of a qubit excitation will impede further photon absorption. Together with the photonic dissipation, the two cavities can emit antibunched pairs of two strongly correlated photons with a tunable emission rate. Moreover, the generation of such a nonclassical microwave source is compatible with current experimental architectures, and may stimulate a variety of applications in the field of quantum information science.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix A. Detailed derivation of equation (1) in the main text

As illustrated in figure 1, the charge qubit is a mesoscopic superconducting island connected by a tunnel junction with capacitance $C_q$ and Josephson coupling energy $E_{iJ}$ [51–53]. A gate voltage $V_g$ is coupled to the island to control the tunneling of Cooper pairs. Its Hamiltonian is given by

$$H_q = E_c(n_q - n_g)^2 - E_{iJ} \cos \phi_{iJ}, \quad (A.1)$$

where $E_c = 2e^2/(C_0 + C_q)$ denotes the charging energy of a Cooper pair, and $n_q = C_q V_g/2e$ is the dimensionless gate charge as a control parameter. $n_g$ is the number of Cooper pairs on the island, and $\phi_{iJ}$ is the phase difference across the junction, which are conjugate variables that obey $[\phi_{iJ}, n_q] = i$.

The LC resonator consists of a capacitor $C_j$ connected to an inductor $L_j$, and its Hamiltonian is written as

$$H_e = \sum_{j=1,2} \frac{q_j^2}{2C_j} + \left( \frac{\hbar}{2e} \right)^2 \frac{\phi_j^2}{2L_j}, \quad (A.2)$$

where $q_j$ and $\phi_j$ represent the charge and phase operators acting on the capacitor $C_j$ and the inductor $L_j$, respectively. They form a canonical conjugate pair, and satisfy the commutation relation $[\phi_j, q_j] = 2i\hbar$.

The dc-SQUID is made up of two identical Josephson junctions and can be treated as a tunable Josephson junction with effective Josephson energy $E_J = 2E_{J0} \cos(\pi \Phi_{ext}/\Phi_0)$, which can be tuned by controlling the magnetic flux $\Phi_{ext}$ penetrating its loop area [45]. Here $E_{J0}$ is the Josephson junction energy of a single junction, and $\Phi_0 = h/2e$ is the magnetic flux quantum. The Hamiltonian of the SQUID takes the form

$$H_J = -E_J \cos \phi_J - 2e\eta_1 V_j, \quad (A.3)$$

where $\phi_J$ is the phase difference across the SQUID, and $\eta_1$ counts the number of Cooper pairs that have transferred the junctions. These two variables obey the commutation relation $[\phi_J, \eta_1] = i$. Additionally, $V_j = -h\dot{\phi}_J/2e$ represents the voltage drop across the SQUID. In the presence of the bias voltage $V$, we have $V_1 = V - V_q - V_{R1} - V_{R2}$ according to Kirchoff’s rules, where $V_q = -h\dot{\phi}_J/2e$ is the voltage drop at the qubit, and $V_{Rj} = -h\dot{\phi}_J/2e$ is the voltage drop at the $j$th resonator. By replacing $V_j$ with $V - V_q - V_{R1} - V_{R2}$ in equation (A.3), we have

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\[ H_1 = -E_l \cos \phi_l - 2\text{en}_l(V - V_q - V_{R1} - V_{R2}). \]  
\[ \text{(A.4)} \]

Together with the equations \( (A.1) \), \( (A.2) \) and \( (A.4) \), we can give the total Hamiltonian of the whole system
\[ H_T = H_q + H_e + H_1, \]
\[ \text{(A.5)} \]

which is just the equation (1) in the main text.

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