Perturbatively Stable Resummed Small $x$ Evolution Kernels

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Abstract

We present a small $x$ resummation for the GLAP anomalous dimension and its corresponding dual BFKL kernel, which includes all the available perturbative information and nonperturbative constraints. Specifically, it includes all the information coming from next-to-leading order GLAP anomalous dimensions and BFKL kernels, from the constraints of momentum conservation, from renormalization-group improvement of the running coupling and from gluon interchange symmetry. The ensuing evolution kernel has a uniformly stable perturbative expansion. It is very close to the unresummed NLO GLAP kernel in most of the HERA kinematic region, the small $x$ BFKL behaviour being softened by momentum conservation and the running of the coupling. Next-to-leading corrections are small thanks to the constraint of gluon interchange symmetry. This result subsumes all previous resummations in that it combines optimally all the information contained in them.
1. Introduction

The theory of scaling violations of deep inelastic structure functions at small $x$ has progressed considerably over the last few years. This theory has been characterized by the long-standing puzzle posed by the fact that no clear deviation of the data from the next-to-leading order prediction has been observed (for instance at HERA [1]), even though higher order perturbative corrections grow rapidly in the small $x$ region. The recent determination [2] of the full set of three-loop (NNLO) splitting functions makes a resolution of this puzzle less of an academic exercise. Indeed, results obtained using these NNLO splitting functions already [3] show sign of the known small $x$ instability, for instance in the extraction of parton distributions. Hence, NNLO phenomenology becomes problematic unless this instability is treated in some way.

However, the reason why superficially large small $x$ corrections sum up to very moderate effects is now essentially understood. It is the consequence of two effects: first, momentum conservation which provides a constraint which limits the unbounded growth of small $x$ kernels, and second, renormalization group improvement which through the running of the coupling softens the asymptotic small $x$ growth. A systematic theory of small $x$ resummation which includes both these effects has been developed in refs. [4–8]. In particular, we have constructed a relatively simple analytical expression for the improved anomalous dimension which we argue is the best way to put together the information from LO BFKL and NLO GLAP. A shortcoming of the approach developed in these references is that even orders of the resummed perturbative expansion are subject to a large model dependence. This is due to the fact that the constraint of momentum conservation stabilizes the perturbative expansion of evolution kernels to all orders, but only in the collinear region. In the “anticollinear” region, where parton virtualities have reverse ordering (and thus only contribute due to $\ln \frac{1}{x}$ enhancement) the instability remains. As a consequence, the stabilization of the BFKL kernel is lost in the vicinity of its minimum, where the collinear and anticollinear regions weigh equally: the minimum is in fact only present at leading order of the perturbative expansion. Because the minimum is necessary for running coupling resummation, the inclusion of NLO BFKL resummation prevents it, unless one makes some assumption on the restoration of the minimum by higher order corrections; this, however, introduces a sizable model–dependence [8].

A similar approach to small $x$ resummation which also embodies the principles of momentum conservation and renormalization group improvement has been presented in refs. [9–11]. This approach differs [12] from ours essentially because it has the somewhat wider goal of determining the off-shell gluon Green function, and it is therefore rooted in the BFKL equation. Whereas this gives for example an exact treatment of small $x$ running coupling corrections, it only allows a numerical extraction of anomalous dimensions through deconvolution from a given boundary condition. Also, it does not naturally lead to a symmetric (double) leading log expansion of the anomalous dimension (for example, when $n_f \neq 0$ the inclusion of subleading GLAP terms in this approach has not yet been accomplished). However, in this approach it was pointed out that the underlying symmetry upon the interchange of gluon virtualities in the gluon-gluon scattering process which determines the BFKL kernel can be used for the treatment of the instability in the anticollinear region. This symmetry interchanges the collinear and anticollinear regions, and
therefore it enables the implementation of the momentum conservation constraint even in the anticollinear region.

In this paper, we exploit the symmetric collinear-anticollinear resummation [10] to stabilize the resummed and renormalization-group improved perturbative expansion of refs. [4–8]. This can be done by using the duality relations [4,5] which connect GLAP and BFKL kernels in order to translate the symmetry constraints on BFKL evolution into constraints on subleading contributions to the GLAP anomalous dimensions. Because the “double leading” perturbative expansion of refs. [5] systematically includes leading, next-to-leading, . . . logs of \( Q^2 \) and \( \frac{1}{x} \) simultaneously, its symmetrization must be done by inclusion of terms which are subleading order by order, and this requires a subtle treatment of double counting terms. Also, implementation of the symmetry is nontrivial when the coupling runs, because the choice of running coupling scale may break it. Finally, after symmetrization, even at LO, the BFKL intercept, which determines the asymptotic small \( x \) behaviour, depends nonlinearly on the strong coupling, and this makes the renormalization-group improvement of refs. [7,8] rather more elaborate: it is still possible to derive an asymptotically exact solution to the running-coupling BFKL equation in terms of special functions, but this now requires a double expansion about the BFKL minimum. Once this is done, a stable perturbative expansion is obtained, and in particular the NLO resummed result turns out [13,14] to be both close to the LO resummed one, and also quite close to the NLO fixed–order result, thereby explaining the lack of experimental evidence for large deviations from low fixed–order predictions at small \( x \): in fact, the resummed NLO splitting function is closer to the NLO than to the NNLO fixed order one.

In this paper we discuss the resummation of anomalous dimensions and splitting functions. To make direct connection to the structure functions we need to also implement coefficient functions and their resummation. This also entails a scheme choice: here we adopt the so-called \( Q_0 \) scheme [15], which coincides with the standard \( \overline{\text{MS}} \) scheme at fixed order but differs from it at the resummed level. Issues of scheme dependence were discussed in detail in ref. [6], where resummed coefficient functions and fits to the data were explicitly presented. The relation of \( Q_0 \) and \( \overline{\text{MS}} \) after small \( x \) running coupling resummation was discussed in ref. [7]. Whereas the use of the results presented in this paper for actual fits to data will require the formalism presented in [6],[7], all the important small \( x \) resummation effects are included in the \( Q_0 \)-scheme results presented here. In order to minimize issues of scheme dependence, in this paper we will determine all anomalous dimensions and splitting functions with \( n_f = 0 \).

The paper is organized as follows. After recalling in section 2 some background results from the resummation formalism of refs. [4–8], which will be needed in the sequel, in section 3 we explain how the underlying symmetry of the BFKL kernel can be coupled to the double–leading expansion of ref. [4] in order to produce a stable resummation at the fixed coupling level. The formalism which is required to generalize the running coupling resummation of ref. [7] to this symmetrized case is introduced in section 4. Resummed anomalous dimensions and splitting functions are explicitly constructed in section 5 at the leading order and in section 6 at the next-to-leading order. Results for anomalous dimensions and splitting functions are presented, discussed and compared in section 7.
2. Background

The behaviour of structure functions at small $x$ is dominated by the large eigenvalue of evolution in the singlet sector. Thus we consider the singlet parton density $G(\xi, t)$ with $\xi = \log 1/x$, $t = \log Q^2/\mu^2$, such that, for each moment

$$G(N, t) = \int_0^\infty d\xi e^{-N\xi} G(\xi, t),$$

(2.1)

the associated anomalous dimension $\gamma(\alpha_s(t), N)$ corresponds to the largest eigenvalue in the singlet sector. At large $t$ and fixed $\xi$ the evolution equation in $N$-moment space is then

$$\frac{d}{dt} G(N, t) = \gamma(\alpha_s(t), N) G(N, t),$$

(2.2)

where $\alpha_s(t)$ is the running coupling.

The perturbative anomalous dimension is known up to NNLO [2], given by

$$\gamma(\alpha_s, N) = \alpha_s \gamma_0(N) + \alpha_s^2 \gamma_1(N) + \alpha_s^3 \gamma_2(N) \ldots .$$

(2.3)

The corresponding splitting function is related by a Mellin transform to $\gamma(\alpha_s, N)$:

$$\gamma(\alpha_s, N) = \int_0^1 dx x^N P(\alpha_s, x).$$

(2.4)

Small $x$ for the splitting function corresponds to small $N$ for the anomalous dimension: more precisely $P \sim (1/x)(\log(1/x))^n$ corresponds to $\gamma \sim n!/N^{n+1}$. Even assuming that a leading twist description of scaling violations is still valid in some range of small $x$, as soon as $x$ is small enough that $\alpha_s \xi \sim 1$, with $\xi = \log 1/x$, all terms of order $(\alpha_s/x)(\alpha_s \xi)^n$ (LLx) and $\alpha_s(\alpha_s/x)(\alpha_s \xi)^n$ (NLLx) which are present in the splitting functions must be considered in order to achieve an accuracy up to terms of order $\alpha_s^2(\alpha_s/x)(\alpha_s \xi)^n$ (NNLLx).

These terms can in principle be derived from the knowledge of the kernel $\chi(\alpha_s, M)$ of the BFKL $\xi$-evolution equation

$$\frac{d}{d\xi} G(\xi, M) = \chi(\alpha_s, M) G(\xi, M),$$

(2.5)

which is satisfied at large $\xi$ by the Mellin transform of the parton distribution

$$G(\xi, M) = \int_{-\infty}^\infty dt e^{-M t} G(\xi, t).$$

(2.6)

The evolution kernel in eq. (2.5) is the Mellin transform of the BFKL kernel $K \left( \alpha_s, \frac{Q^2}{k^2} \right)$, which determines the high-$s$ limit of the cross-section for the scattering of two gluons with virtualities $k^2$ and $Q^2$ into $n$ gluons

$$\chi(\alpha_s, M) = \int_0^\infty \frac{dQ^2}{Q^2} \left( \frac{Q^2}{k^2} \right)^{-M} K \left( \alpha_s, \frac{Q^2}{k^2} \right).$$

(2.7)
This kernel has been computed to NLO accuracy [16–19]

\[ \chi(\alpha_s, M) = \alpha_s \chi_0(M) + \alpha_s^2 \chi_1(M) + \ldots \]  

(2.8)

Notice that in gluon–gluon scattering \( \xi = \ln(s/\sqrt{Q^2 k^2}) \), whereas in deep-inelastic scattering \( \xi \sim \ln(s/Q^2) \). The kernels that correspond to these two cases coincide at leading order, and beyond leading order are related in a simple way, as we shall discuss explicitly in sect. 3 below.

Quite generally, the anomalous dimension \( \gamma \) eq. (2.3) and the BFKL kernel are related due to the fact [20,5,6] that the solutions of the BFKL and GLAP equations coincide at leading twist if their respective evolution kernels are related by a “duality” relation. In the fixed coupling limit the duality relations are simply given by:

\[ \chi(\alpha_s, \gamma(\alpha_s, N)) = N, \]  

(2.9)

\[ \gamma(\alpha_s, \chi(\alpha_s, M)) = M. \]  

(2.10)

This implies that all relative corrections of order \((\alpha_s \log 1/x)^n\) to the splitting function can be derived from \( \chi_0(M) \), those of order \( \alpha_s(\alpha_s \log 1/x)^n \) from \( \chi_1(M) \) and so on: indeed, if we expand \( \gamma(\alpha_s, N) \) in powers of \( \alpha_s \) at fixed \( \alpha_s/N \)

\[ \gamma(\alpha_s, N) = \gamma_s(\alpha_s/N) + \alpha_s \gamma_{ss}(\alpha_s/N) + \ldots, \]  

(2.11)

eq (2.9) determines \( \gamma_s \) in terms of \( \chi_0 \), \( \gamma_{ss} \) in terms of \( \chi_0 \) and \( \chi_1 \) and so on.

Conversely, all relative corrections of order \((\alpha_s \log Q^2)^n\) to the BFKL kernel \( K(\alpha_s, Q^2) \) eq. (2.7) can be derived from \( \gamma_0(N) \), those of order \( \alpha_s(\alpha_s \log Q^2)^n \) from \( \gamma_1(N) \) and so on: expanding \( \chi(\alpha_s, M) \) in powers of \( \alpha_s \) at fixed \( \alpha_s/M \)

\[ \chi(\alpha_s, M) = \chi_s(\alpha_s/M) + \alpha_s \chi_{ss}(\alpha_s/M) + \ldots, \]  

(2.12)

eq (2.10) determines \( \chi_s \) in terms of \( \gamma_0 \), \( \chi_{ss} \) in terms of \( \gamma_0 \) and \( \gamma_1 \) and so on.

As is by now well known, the early wisdom on how to implement the information from \( \chi_0 \) was completely shaken by the much softer behaviour of the data at small \( x \) [21,22,23] and by the computation of \( \chi_1 \) [16–19], which showed that the naive expansion for the improved anomalous dimension had a hopelessly bad behaviour, in particular near \( M = 0 \) and \( M = 1 \). In refs. [6,5] we have shown that the physical requirement of momentum conservation for anomalous dimensions fixes by duality the value of \( \chi(\alpha_s, M) \) at \( M = 0 \):

\[ \chi(\alpha_s, 0) = 1. \]  

(2.13)

This implies that in order to cure the instability at \( M = 0 \) it is sufficient to ensure that the standard GLAP leading log resummation is performed in accordance to the perturbative expansion of the anomalous dimension eq. (2.3), in which momentum conservation is respected order by order.

The natural way to do this is by combining the small \( x \) resummation and the standard resummation of collinear singularities. As a result one obtains a pair of ‘double-leading’
perturbative expansions of the kernel $\chi$ and anomalous dimension $\gamma$, dual to each other order in perturbation theory (up to subleading terms). For example, in the DL LO expression for $\chi$ an infinite series of terms of order $(\alpha_s/M)^n$ is added to the fixed-order $\chi_0$ result, and similarly at DL NLO terms of order $\alpha_s(\alpha_s/M)^n$. The net result is to obtain a regular function at $M = 0$ which only violates momentum by small subleading terms, and interpolates between the BFKL result at small $x$ ($M \sim 1/2$) and the GLAP result at large $x$ ($M \sim 0$).

The DL expansion kernels are

$$
\chi_{DL} = \chi_{DL\ LO} + \alpha_s \chi_{DL\ NLO} + \ldots
$$

(2.14)

with

$$
\chi_{DL\ LO}(\alpha_s, M) = \alpha_s \chi_0(M) + \chi_s(\frac{\alpha_s}{M}) - \frac{n_c \alpha_s}{\pi M}
$$

(2.15)

$$
\chi_{DL\ NLO}(\alpha_s, M) = \alpha_s \chi_1(M) + \chi_{ss}(\frac{\alpha_s}{M}) - \alpha_s(\frac{f_2}{M^2} + \frac{f_1}{M}) - f_0
$$

(2.16)

The kernels $\chi_s$ and $\chi_{ss}$ are derived from $\gamma_0(N)$ and $\gamma_1(N)$ using the inverse duality relation

$$
\gamma(\alpha_s, \chi(\alpha_s, M)) = M
$$

(2.17)

and are then given by:

$$
\gamma_0 \left( \chi_s \left( \frac{\alpha_s}{M} \right) \right) = \frac{M}{\alpha_s},
$$

(2.18)

$$
\chi_{ss} \left( \frac{\alpha_s}{M} \right) = -\frac{\gamma_1 \left( \chi_s \left( \frac{\alpha_s}{M} \right) \right)}{\gamma_0 \left( \chi_s \left( \frac{\alpha_s}{M} \right) \right)}.
$$

(2.19)

while the numbers $f_0$, $f_1$, and $f_2$ are fixed so as to remove double counting between $\chi_1$ and $\chi_{ss}$.

The DL kernel eq. (2.14) has a stable perturbative expansion for small $M$, and indeed for all $M \lesssim 1/2$. However, it is still unstable near $M = 1$: while momentum conservation and duality fix $\chi = 1$ at $M = 0$, near $M = 1\chi_0$ behaves like $1/(1 - M)$ and $\chi_1$ behaves like $-1/(1 - M)^2$. Hence, the NLO approximation for $\chi_{DL}$ has no minimum — in fact, the minimum is absent at any even order of the perturbative expansion eq. (2.14). This prevents a precise treatment of non leading terms, for which a better control of the central region near $M = 1/2$ is needed, which in turn creates a problem for the evaluation of the small $x$ asymptotic behaviour, which depends on the central region. In ref. [6], however, it was found that phenomenology requires the all-order minimum of $\chi$ to significantly differ from its low-order one, and that a successful description of the data is obtained if the minimum value of $\chi$ is taken as a free parameter to be fitted to the HERA data.

So far we have assumed that the running of the coupling is treated perturbatively in the DL expansion. In particular, the running of the coupling is a leading $\ln Q^2$, but subleading $\ln \frac{1}{x}$ effect. As a consequence, when the coupling runs the duality relation eq. (2.9) is unaffected at the leading $\ln \frac{1}{x}$ level, while at the $N^k\text{LL}x$ it receives a perturbatively computable $O \left[ (\beta_0 \alpha_s \ln \frac{1}{x})^k \right]$ correction. Specifically, at NLO

$$
\gamma_{ss}(\alpha_s/N) = -\frac{\chi_1(\gamma_s(\alpha_s/N))}{\chi_0(\gamma_s(\alpha_s/N))} - \beta_0 \frac{\chi_0''(\gamma_s(\alpha_s/N))\chi_0(\gamma_s(\alpha_s/N))}{2[\chi_0(\gamma_s(\alpha_s/N))]^2}.
$$

(2.20)
The running coupling correction may be viewed as an effective contribution to $\chi$: defining
\[
\chi_{\text{pert.}}^\beta(M) = \beta_0 \frac{\chi''_0(M) \chi_0(M)}{2 \chi_0'(M)}
\] (2.21)

$\chi_{ss}$ eq. (2.20) is the naive (fixed coupling) dual of an effective running coupling $\chi$ whose NLO term is supplemented by $\chi_{\text{pert.}}^\beta_0$, eq. (2.20). However, it is apparent that this effective $\chi$ is singular as $M \to \frac{1}{2}$, which implies [4] that the corresponding splitting function is enhanced as $x \to 0$. In fact, the contribution $\Delta P_n^s(\alpha_s, x)$ to the $n^{\text{LLx}}$ splitting function induced by the running-coupling duality correction to the splitting function behaves as
\[
\frac{\Delta P_n^s(\alpha_s, x)}{P_s(\alpha_s, x)} \sim (\beta_0 \alpha_s \ln \frac{1}{x})^n.
\] (2.22)

The all-order resummation of these running coupling small $x$ corrections is therefore needed, and it turns out [7,8] to be, on top of momentum conservation, the second crucial ingredient of the resummation procedure. The running coupling resummation can be performed [7] by taking a quadratic approximation to the BFKL kernel in the vicinity of its minimum. Indeed, in $M$ space the usual running coupling $\alpha_s(t)$ becomes a differential operator: assuming the coupling to run in the usual way with $Q^2$, and taking only the one-loop beta function into account, one has
\[
\tilde{\alpha}_s = \frac{\alpha_s}{1 - \beta_0 \alpha_s \frac{d}{dM}},
\] (2.23)
where $\beta_0$ is the first coefficient of the $\beta$-function (so $\beta = -\beta_0 \alpha_s^2 + \cdots$), with the obvious generalization to higher orders. Eq. (2.23) implies that
\[
[\tilde{\alpha}_s, M] = \beta_0 \tilde{\alpha}_s^2.
\] (2.24)

Therefore, the explicit form of a general function of $\tilde{\alpha}_s$ and $M$ will depend on the choice of operator ordering. The $\xi$-evolution equation eq. (2.5) becomes [24]
\[
\frac{d}{d\xi} G(\xi, M) = \chi(\tilde{\alpha}_s, M) G(\xi, M),
\] (2.25)
where the derivative with respect to $M$ acts on everything to the right, and $\chi$ may be expanded as in eq. (2.8) keeping the powers of $\tilde{\alpha}_s$ on the left. It is easy to show that different choices of operator ordering in eq. (2.25) would correspond to different choices for the argument of the running coupling [24,11].

When $\chi$ is approximated by $\alpha_s \chi_0$, eq. (2.25) becomes, after taking a second Mellin transform,
\[
(1 - \beta_0 \alpha_s \frac{d}{dM}) NG(N, M) + F(M) = \alpha_s \chi_0(M) G(N, M),
\] (2.26)
where the function $F(M)$ is a boundary condition. By expressing the anomalous dimension in terms of the exact solution to eq. (2.26), it is possible to prove [7] that corrections
eq. (2.22) to naive duality eq. (2.9) which generalize the NLO term eq. (2.20) may be determined perturbatively, thereby ensuring that duality continues to hold at the running coupling level to all perturbative orders. Because the dual GLAP equation enjoys the standard factorization properties, this also implies perturbative factorization of the running coupling BFKL equation. Furthermore, the all-order sum of these running coupling corrections can be determined explicitly when $\chi_0(M)$ is replaced by its quadratic approximation near its minimum at $M = 1/2$: $\chi_0(M) \sim c + \frac{1}{2} k(M - \frac{1}{2})^2$, because in such case the inverse Mellin eq. (2.6) of the solution to eq. (2.26) can be determined exactly in terms of Airy function of an argument that depends on $\alpha_s(t)$ and $N$. Since the asymptotic small $x$ behaviour is determined by the central interval in $M$ of $\chi$, one can show that the quadratic expansion leads to the correct asymptotic behaviour implied by the kernel $\chi_0$. The all-order resummation of running coupling effects can then be obtained [7] by showing that this exact solution based on the quadratic approximation to $\chi$ determines the asymptotic small $x$ behaviour of the solution computed from the full $\chi$.

The running coupling resummation of the DL anomalous dimension is found to greatly soften the asymptotic small $x$ behaviour in comparison to the DL result with unresummed running coupling effects, by replacing the naive DL small $x$ intercept with an effective one [10,7,8]. As a consequence, it is no longer necessary to fit the small $x$ behaviour to the data in order to obtain reasonable phenomenology [7,8]. This, however, makes a precise description of the minimum of $\chi$ mandatory. Specifically, at NLO of the DL expansion where no minimum is present there is a loss of predictivity because it must be assumed that the minimum is present in the all order result [8]. Fortunately, the minimum is restored at each order by exploiting the underlying BFKL symmetry, as we explain in the next section.

3. Symmetrisation

We have seen that for a precise treatment of non leading terms a better control of the central region near $M = 1/2$ is needed and for this one has to stabilize the bad behaviour of $\chi$ near $M = 1$. This can be done using the symmetry of the BFKL kernel. The underlying Feynman diagrams which determine the kernel $K \left( \alpha_s, \frac{Q^2}{k^2} \right)$ in eq. (2.7) order by order in perturbation theory are symmetric upon the exchange of the incoming and outgoing gluon. But the interchange of the incoming and outgoing gluon virtualities $Q^2 \leftrightarrow k^2$ in the argument of the BFKL kernel $K \left( \alpha_s, \frac{Q^2}{k^2} \right)$ in eq. (2.7) corresponds to the transformation $M \rightarrow 1 - M$ in the argument of its Mellin transform $\chi(\alpha_s, M)$. Hence, $\chi(\alpha_s, M)$ must be symmetric upon this interchange.

The symmetry, however, is broken by two effects: the running of the coupling and the fact that in deep-inelastic scattering the $N$-Mellin transform eq. (2.1) is performed with respect to the variable $\xi = \ln(s/Q^2)$ which depends on the outgoing scale $Q^2$ only. We will start with the fixed $\alpha_s$ case, while running coupling effects and the associate symmetry breaking will be discussed in the next section. The kernel is then only asymmetric due to the choice of definition of $\xi$. As is well known [16], this asymmetry corresponds to a reshuffling of the arguments $M, N$ of $\chi$: namely, a symmetrization $\chi_\Sigma$ of the kernel $\chi$ of
eq. (2.25) can be obtained from a symmetric kernel $\chi_\sigma$ which corresponds to the symmetric choice $\xi = \ln(s/\sqrt{Q^2k^2})$ through the implicit equation \[16:\]

$$
\chi_\Sigma(\alpha_s, M + \frac{1}{2}\chi_\sigma(M)) = \chi_\sigma(\alpha_s, M). \tag{3.1}
$$

Hence, we can implement the symmetrization, thereby resumming the $M = 1$ poles, by performing the double-leading resummation of $M = 0$ poles of $\chi$, determining the associated $\chi_\sigma$ through eq. (3.1), then symmetrizing it (as $\chi_\sigma$ must be symmetric), and finally going back to DIS variables by using eq. (3.1) again in reverse.

The procedure can be understood most easily by first considering a toy case, where we pretend that the anomalous dimension exactly coincides with its leading–order form $\alpha_s \gamma_0(N)$, so that the BFKL kernel is just given by $\chi_s(\alpha_s/M)$ from eq. (2.18). This corresponds to a leading $\ln Q^2$ (or collinear) approximation to the BFKL kernel. Symmetrization is enforced by using eq. (3.1) to relate the DIS kernel to the symmetric one: we replace $M$ by $M + N/2$ and symmetrize for $M \to (1 - M)$. Note that this is allowed at the leading $\ln Q^2$ level, because the symmetrizing terms are by construction free of poles at $M = 0$ and are thus subleading in $\ln Q^2$.

We thus obtain the following pair of kernels, related by eq. (3.1):

$$
\bar{\chi}_{\sigma\text{ toy}}(\alpha_s, M, N) = \chi_s\left(\frac{\alpha_s}{M+N/2}\right) + \chi_s\left(\frac{\alpha_s}{1-M+N/2}\right), \tag{3.2}
$$

$$
\bar{\chi}_{\Sigma\text{ toy}}(\alpha_s, M, N) = \chi_s\left(\frac{\alpha_s}{M}\right) + \chi_s\left(\frac{\alpha_s}{1-M+N}\right), \tag{3.3}
$$

where $\bar{\chi}_{\sigma\text{ toy}}$ is symmetric, and $\bar{\chi}_{\Sigma\text{ toy}}(\alpha_s, M)$ is relevant for DIS, i.e. dual to the DIS anomalous dimension.

Equations (3.2), (3.3) should be viewed as implicit definitions of the corresponding kernels $\chi_\sigma(\alpha_s, M), \chi_\Sigma(\alpha_s, M)$ as solutions of equations of the form

$$
\chi(\alpha_s, M) = \bar{\chi}(\alpha_s, M, \chi(\alpha_s, M)) \tag{3.4}
$$

in respectively symmetric or asymmetric variables. So $\bar{\chi}(\alpha_s, M, N)$ can be viewed as “off-shell” continuations of the kernels $\chi(\alpha_s, M)$, which are obtained from them by putting $N$ “on-shell” (see eq. (2.9)). Alternatively we can use use the off-shell kernels as implicit definitions of the anomalous dimensions $\gamma(\alpha_s, N)$ through the duality condition (2.17):

$$
N = \bar{\chi}(\alpha_s, \gamma(\alpha_s, N), N). \tag{3.5}
$$

Summarising, we obtain $\chi(\alpha_s, M)$ or $\gamma(\alpha_s, N)$ by putting either $N$ or $M$ “on-shell” (i.e. set $N = \chi(\alpha_s, M)$ or $M = \gamma(\alpha_s, N)$ respectively) in the “off-shell” relation $N = \bar{\chi}(\alpha_s, M, N)$.

Note that as $N \to \infty$, the symmetrising contribution in eq. (3.3) $\chi_s\left(\frac{\alpha_s}{1-M+N}\right) \sim \chi_s\left(\frac{\alpha_s}{N}\right) \sim O(\frac{\alpha_s}{N^2})$. Thus this term does not spoil the identification of the resummed kernel $\chi$ with the GLAP result $\chi_s$ at large $N$, which in turn guarantees that the resulting anomalous dimension matches smoothly to the GLAP anomalous dimension.

However the symmetrizing contribution does violate the momentum conservation condition $\chi_{\Sigma\text{ toy}}(\alpha_s, 0) = 1$, albeit by subleading terms. Indeed, at the momentum
conservation point $M = 0$, $\chi_s(\alpha_s/M)$ by construction satisfies $\chi_s(\alpha_s/M)|_{M=0} = 1$. However, $\chi_s(\alpha_s/(1 - M))|_{M=0} = \chi_s(\alpha_s) = O(\alpha_s)$, and $\chi_s(\alpha_s/(1 - M + N))|_{M=0} = \chi_s(\alpha_s)(1 + O(N)) = O(\alpha_s)$, where the last step follows from duality eq. (2.9). Hence, the momentum violation is $O(\alpha_s) = O(\alpha_s^{n+1}M^{-n})$, or next-to-leading order in an expansion of $\chi$ in powers of $\alpha_s$ at fixed $\alpha_s/M$. A simple way of enforcing momentum conservation without spoiling the symmetrization properties is then to add to eqs. (3.2), (3.3) an extra term of the form

$$\chi_{\text{mom}}(\alpha_s, N) = c_m f_m(N),$$

(3.6)

where $f_m(1) = 1$ and $f_m(0) = f_m(\infty) = 0$, for example

$$f_m(N) = \frac{4N}{(N+1)^2},$$

(3.7)

and with the constant $c_m$ is chosen to fix up the momentum constraint. The condition $f_m(0) = 0$ ensures that $\chi_{\text{mom}}(\alpha_s, N)$ is subleading; because $c_m = O(\alpha_s)$, $\chi_{\text{mom}}(\alpha_s, N)$ eq. (3.6) is of order $O(\alpha_s N)[1 + O(\alpha_s) + O(N)]$. But by the duality eq. (2.9) $N = O(\alpha_s/M)$, hence $\chi_{\text{mom}}(\alpha_s, N)$ is $O(\alpha_s^{2}/M)$, i.e. next-to-leading order in $Q^2$. The condition $\lim_{N \to \infty} f_m(N) = 0$ again ensures that the extra term does not spoil the indentification of the resummed $\chi$ kernel with the GLAP result $\chi_s$ at large $N$.

The full symmetrized double-leading result can be obtained through a similar procedure, but starting with $\chi_{\text{DL LO}}(M)$, given in eq. (2.15). Clearly, the contribution from the leading-order BFKL kernel $\chi_0$ is already symmetric, but the symmetry should only hold in symmetric variables, while in DIS variables it must be asymmetric. Furthermore the collinear and anti-collinear singularities must match exactly with those derived from GLAP. To achieve this it is necessary to modify the BFKL kernel by subleading terms in the same way that we symmetrised the pure GLAP model above. Recall that the LO BFKL kernel

$$\chi_0(M) = \frac{n_c}{\pi} [2\psi(1) - \psi(M) - \psi(1 - M)],$$

(3.8)

where $\psi(M) \equiv \frac{d}{dx} \ln \Gamma(x)$ is the polygamma function. We can separate it into a sum of two pieces: a collinear piece $\chi_0^+$ with poles at $M = 0, -1, -2, \ldots$ coming from the collinear region $Q^2 > k^2$ and an anticollinear piece $\chi_0^-$ with poles at $M = 1, 2, 3, \ldots$ coming from the anti-collinear region $Q^2 < k^2$: $\chi_0 = \chi_0^+ + \chi_0^-$ with

$$\chi_0^+(M) = \frac{n_c}{\pi} (\psi(1) - \psi(M))$$

$$\chi_0^-(M) = \chi_0^+(1 - M).$$

(3.9)

It follows that in symmetric variables the off-shell extension of $\chi_0$ must be

$$\bar{\chi}_0(M, N) = \frac{n_c}{\pi} [\psi(1) + \psi(1 + N) - \psi(M + N/2) - \psi(1 - M + N/2)],$$

(3.10)

so that in DIS variables

$$\bar{\chi}_0(M - \frac{N}{2}, N) = \frac{n_c}{\pi} [\psi(1) + \psi(1 + N) - \psi(M) - \psi(1 - M + N)].$$

(3.11)
The leading collinear and anticollinear poles
\[ \tilde{\chi}^{dc}_0(M, N) = \frac{n_s}{\pi} \left[ \frac{1}{M+N/2} + \frac{1}{1-M+N/2} \right], \]  
(3.12)
then coincide with those from GLAP [eq. (3.2) expanded to \(O(\alpha_s)\)], while at the same time the only difference between (3.10) and (3.8) is in terms of \(O(N) = O(\alpha_s)\) [using (2.9) with \(\chi(M, \alpha_s) = \alpha_s \chi_0(M)\)]. Note that besides the shift in the argument of \(\psi(1-M)\) we also added a term to \(\chi_0\) proportional to \(\psi(1 + N) - \psi(1)\): this is to ensure that at large \(N\) in DIS variables the anticollinear terms \(\psi(1 + N) - \psi(1 - M + N) \sim O(1/N)\), and thus that the matching to GLAP in the collinear region (and in particular at large \(N\)) is not spoiled.

Putting together the GLAP and BFKL components, the symmetrized off-shell leading order DL \(\chi\) is then
\[ \tilde{\chi}_{\sigma \text{LO}}(\alpha_s, M, N) = \chi_s \left( \frac{\alpha_s}{M+N/2} \right) + \chi_s \left( \frac{\alpha_s}{1-M+N/2} \right) + \alpha_s \tilde{\chi}_0(M, N) + \chi_{\text{mom}}(\alpha_s, N), \]  
(3.13)
where if
\[ \tilde{\chi}_0(M, N) = \tilde{\chi}_0^+(M, N) + \tilde{\chi}_0^-(M, N), \]
\[ \tilde{\chi}_0^+(M, N) = \frac{n_s}{\pi} (\psi(1) - \psi(M + \frac{N}{2})), \]
\[ \tilde{\chi}_0^-(M, N) = \frac{n_s}{\pi} (\psi(1 + N) - \psi(1 - M + \frac{N}{2})), \]  
(3.14)
then
\[ \tilde{\chi}_0(M, N) = \tilde{\chi}_0(M, N) - \tilde{\chi}^{dc}_0(M, N) \]
\[ = \frac{n_s}{\pi} \left( \psi(1) + \psi(1 + N) - \psi(1 + M + \frac{N}{2}) - \psi(2 - M + \frac{N}{2}) \right), \]  
(3.15)
where in the last step we have used the identity \(\psi(x + 1) = \psi(x) + \frac{1}{x}\). In DIS variables we get
\[ \tilde{\chi}_{\Sigma \text{LO}}(\alpha_s, M, N) = \chi_s \left( \frac{\alpha_s}{M-N/2} \right) + \chi_s \left( \frac{\alpha_s}{1-M-N/2} \right) + \alpha_s \tilde{\chi}_0(M - \frac{N}{2}, N) + \chi_{\text{mom}}(\alpha_s, N), \]  
(3.16)
where
\[ \tilde{\chi}_0(M - \frac{N}{2}, N) = \frac{n_s}{\pi} (\psi(1) + \psi(1 + N) - \psi(1 + M) - \psi(2 - M + N)). \]  
(3.17)
This differs from the original unsymmetrized leading order DL result eq. (2.15) by subleading terms, as we now show. The symmetrizing contribution \(\chi_s[\alpha_s/(1-M)]\) is \(O(\alpha_s)\) and free of \(M = 0\) poles, hence next-to-leading \(\ln Q^2\). Because it is an \(O(\alpha_s)\) contribution to \(\chi\) it is leading \(\frac{1}{x}\). However, its \(O(\alpha_s)\) term is subtracted by the double counting term \(-\frac{\alpha_s n_s}{\pi (1-M+N)}\), and what remains is \(O(\alpha_s^2)\), namely next-to-leading \(\ln \frac{1}{x}\). Finally, the leading \(\ln \frac{1}{x}\) contribution to \(\chi_{\Sigma \text{LO}}\) eq. (3.16) differs from \(\chi_0\) because of the \(N\)-dependent shift of the argument \(M\): however, by duality eq. (2.9) \(N = O(\alpha_s)\) and thus the difference is \(O(\alpha_s^2)\), namely next-to-leading \(\ln \frac{1}{x}\). The violation of momentum conservation in the absence of \(\chi_{\text{mom}}\) is still subleading. This is proven by simply observing that \(\chi_{\text{mom}}(\alpha_s, N)\)
differs from the DL result by subleading terms. But the DL result respects momentum conservation up to subleading terms, and the remaining terms are subleading with respect to it everywhere.

In a similar way, starting from $\chi_{DL \ NLO}(M, \alpha_s)$ in eq. (2.16), one can write down the symmetrized off-shell NLO DL $\chi$:

$$\tilde{\chi}_{\sigma \ NLO}(\alpha_s, M, N) = \chi_{ss} \left( \frac{\alpha_s}{M+N} \right) + \chi_{ss} \left( \frac{\alpha_s}{1-M+N} \right) + \alpha_s \tilde{\chi}_1(M, N) + \chi_{\text{mom}}(\alpha_s, N). \quad (3.18)$$

In DIS variables this corresponds to:

$$\tilde{\chi}_{\Sigma \ NLO}(\alpha_s, M, N) = \chi_{ss} \left( \frac{\alpha_s}{M} \right) + \chi_{ss} \left( \frac{\alpha_s}{1-M+N} \right) + \alpha_s \tilde{\chi}_1(M - \frac{N}{2}, N) + \chi_{\text{mom}}(\alpha_s, N). \quad (3.19)$$

The function $\tilde{\chi}_1(M, N)$, which is related to the Fadin–Lipatov kernel $\chi_1$ (the next-to-leading contribution to the BFKL kernel eq. (2.8)) minus double counting, up to subleading corrections due to the symmetrization and to the running of the coupling, will be constructed explicitly in section 6 below.

Let us now turn to the generic properties of the symmetrized on-shell kernels, obtained by solving eq. (3.4) in either symmetric or DIS variables. First, it is clear that, because momentum conservation in asymmetric (DIS) variables fixes $\chi(\alpha_s, 0) = 1$, in symmetric variables the associate symmetric kernel will satisfy the constraint

$$\chi_{\sigma} \left( \alpha_s, -\frac{1}{2} \right) = \chi_{\sigma} \left( \alpha_s, \frac{3}{2} \right) = 1. \quad (3.20)$$

When transforming back to DIS variables, this will lead to

$$\chi_{\Sigma}(\alpha_s, 0) = \chi_{\Sigma}(\alpha_s, 2) = 1. \quad (3.21)$$

Since in the central region $M \sim \frac{1}{2}, \chi \sim O(\alpha_s)$ and thus rather less than one, at least for small enough values of $\alpha_s$, on the real axis of the $M$-plane the symmetrized kernel will have a minimum order by order in the (symmetrised) double leading expansion. Since the change of variables from symmetric to DIS variables is just a shift $M \rightarrow M - \frac{N}{2}$, there will be a minimum in DIS variables too. This is an important generic result, since it implies necessarily that at fixed coupling the splitting function must behave as a power of $x$, order by order in resummed perturbation theory.

Now consider the position and curvature around the minimum. While $\chi_{\sigma}$ has the minimum at $M = \frac{1}{2}$, the minimum of $\chi_{\Sigma}$ is displaced, but the value at the minimum and the curvature are the same. To see this, we rewrite the implicit equation eq. (3.1) as

$$\chi_{\sigma}(\alpha_s, M - \frac{1}{2} \chi_{\Sigma}(M)) = \chi_{\Sigma}(\alpha_s, M), \quad (3.22)$$

where $\chi_{\sigma}(\alpha_s, M) = \chi_{\sigma}(\alpha_s, 1 - M)$ is stationary at $M = \frac{1}{2}$. Eq. (3.22) implies that the derivatives of $\chi_{\Sigma}$ and $\chi_{\sigma}$ are related by

$$\chi'_{\Sigma}(\alpha_s, M) = \frac{\chi'_{\sigma}(\alpha_s, M - \frac{1}{2} \chi_{\Sigma}(M))}{1 + \frac{1}{2} \chi'_{\sigma}(\alpha_s, M - \frac{1}{2} \chi_{\Sigma}(M))}, \quad (3.23)$$

$$\chi''_{\Sigma}(\alpha_s, M) = \frac{\chi''_{\sigma}(\alpha_s, M - \frac{1}{2} \chi_{\Sigma}(M))}{(1 + \frac{1}{2} \chi'_{\sigma}(\alpha_s, M - \frac{1}{2} \chi_{\Sigma}(M)))^3}. \quad (3.24)$$
Figure 1: Plot of different symmetric small $x$ kernels $\chi$. From the top (in the central region): the LO BFKL kernel $\alpha_s \chi_0$, the NLO and LO resummed DL expansion kernels $\chi_\sigma$, both on-shell and expressed in symmetric variables, the NLO BFKL kernel $\alpha_s \chi_0 + \alpha_s^2 \chi_1$. All curves are determined with $\beta_0 = 0$ (fixed coupling), $\alpha_s = 0.2$ and $n_f = 0$. Note the unphysical branches of the LO and NLO BFKL curves, outside the interval $0 < M < 1$.

where the prime denotes differentiation with respect to $M$. It follows that the minimum of $\chi_\Sigma(\alpha_s, M)$ is at $M_s$, determined by

$$M_s = \frac{1}{2} + \frac{1}{2} \chi_\Sigma(\alpha_s, M_s),$$

(3.24)

which implies

$$\chi_\sigma(\alpha_s, \frac{1}{2}) = \chi_\Sigma(\alpha_s, M_s), \quad \chi_\sigma''(\alpha_s, \frac{1}{2}) = \chi_\Sigma''(\alpha_s, M_s),$$

(3.25)

i.e., the intercept and curvature of $\chi_\Sigma$ and $\chi_\sigma$ at the respective minima are the same (note that this is not true for higher derivatives of either function: after all, $\chi_\Sigma$ is asymmetric). Hence, whereas the location $M_s$ of the minimum is shifted away from $M_s = \frac{1}{2}$ by the asymmetric scale choice, the intercept and curvature at the minimum are unaffected.

Although $\chi_{DL}$ has poles at $M = \pm 1, \pm 2, \ldots$, in symmetric variables $\chi_\sigma$ is actually an entire function of $M$. To see this, note that the off-shell $\tilde{\chi}_\sigma(\alpha_s, M, N)$ is of the form $\varphi(M + \frac{N}{2}) + \varphi(1 - M + \frac{N}{2})$, where $\varphi(M)$ has poles on the negative real axis at $M = -n$, $n = 1, 2, \ldots$, so $\tilde{\chi}_\sigma(\alpha_s, M, N)$ will have poles at $M + \frac{N}{2} = -n$, near which it must go to infinity. Eq. (3.4) then shows that $N \to \infty$ there, i.e., as we go on-shell the corresponding singularity in $\chi_\sigma(\alpha_s, M)$ is shifted to the point at infinity. Similarly, in DIS variables $\tilde{\chi}_\Sigma(\alpha_s, M, N)$ is of the form $\varphi(M) + \varphi(1 - M + N)$, so by a similar argument the poles on
the positive real axis are shifted to infinity when we go on-shell, and \( \chi_\Sigma(\alpha_s, M) \) is free of singularities to the right of \( ReM = -1 \). The singularities at \( M = -1, -2, \ldots \) correspond of course to higher twists, which are outside the control of our approximations, but die off rapidly at high \( Q^2 \).

Note that whereas \( \chi_\sigma(M) \) is an entire function, its perturbative expansion still has order by order singularities at \( M = 0 \), just like \( \chi_{DL}(\alpha_s, M) \) and \( \chi_\Sigma(\alpha_s, M) \). In fact whereas the perturbative expansions \( \chi_{DL}(\alpha_s, M) \) and \( \chi_\Sigma(\alpha_s, M) \) have single collinear poles (of the form \( (\alpha_s/M)^n \)) the perturbative expansion of \( \chi_\sigma(\alpha_s, M) \) has double singularities of the form \( \alpha_s^n/M^{2n-1} \). To see this substitute the expansion eq. (2.8) in eq. (3.1) to get

\[
\chi_\Sigma^0(M) = \chi_\sigma^0(M), \quad \chi_\Sigma^1(M) = \chi_\sigma^1(M) - \frac{1}{2} \chi_0^1(M) \chi_0(M), \ldots
\]

whence the result. Similarly the anticollinear singularities in DIS variables, i.e in \( \chi_\Sigma(\alpha_s, M) \) at \( M = 1 \) are also double singularities, of the form \( \alpha_s^n/(1 - M)^{2n-1} \). The spurious cubic singularity in \( \chi_1 \) is well known [16,25] and will be discussed in section 5 when extending the symmetrization to next-to-leading order.

A comparison of the unresummed BFKL kernels \( \alpha_s \chi_0 \) and \( \alpha_s \chi_0 + \alpha_s^2 \chi_1 \) with the symmetrised DL expansions at LO and NLO is presented in fig. 1. The curves shown are, in symmetric variables, the on-shell versions obtained by using eq. (3.4), from eqs. (3.13), (3.18). First, we observe that the LO and NLO symmetrised DL curves are very close, indicating a good behaviour of the corresponding expansion, while the NLO BFKL kernel was in all respects completely different from its LO approximation. We also observe
that the DL expressions are very smooth and rather flat. This is due to the fact that the symmetrization procedure shifts the poles at infinity while momentum conservation imposes $\chi(-\frac{1}{2}) = \chi(\frac{3}{2}) = 1$. In the following section we will use a quadratic approximation of the kernels near their minimum. Clearly for the DL kernels the validity of this approximation is quite accurate in the whole physical region $0 < M < 1$. The resummation considerably reduces the value of $\chi$ at the minimum in comparison to the LO BFKL kernel, implying a softer small $x$ asymptotic behaviour. Amusingly, after resummation the NLO correction is actually small and positive, in complete contrast to the large negative correction suggested by a naive interpretation of NLO BFKL.

In fig. 2 we display a set of $\chi$ kernels expressed in asymmetric variables as relevant for DIS structure function evolution. The LO and NLO BFKL kernels, the LO and NLO DL curves and finally those derived at LO and NLO from the symmetrization procedure are compared with the dual of LO and NLO GLAP. The stabilization of the resummed expansion for $M \gtrsim \frac{1}{2}$ through the symmetrisation procedure is apparent: in fact it is guaranteed by the anticollinear momentum conservation constraint at $M = 2$. The position of the minimum is shifted slightly to the right or $M = \frac{1}{2}$ but its height and curvature are the same as in the resummed curves of fig. 1. However the resummed result is now very close to the GLAP result for all $M$ below the minimum of the resummed symmetrised curve. As we shall see in the next section, running coupling effects extend this agreement considerably.

Let us finally discuss the effect of the switch from symmetric to asymmetric variables on the expansion of the anomalous dimension. At the fixed-coupling level, this is simply determined using duality eq (2.9) in eq. (3.22): if $\gamma_\sigma$ and $\gamma_\Sigma$ are respectively dual to the symmetric and nonsymmetric $\chi_\sigma$ and $\chi_\Sigma$, then, to all perturbative orders

$$\gamma_\Sigma(\alpha_s, N) = \gamma_\sigma(\alpha_s, N) + \frac{1}{2} N.$$  \hspace{1cm} (3.27)

This shows that the effect of the scale choice is subleading, i.e. $O(\alpha_s^{k+1} N^{-k})$. If $\gamma_\sigma$ is expanded according to eq. (2.12), then the term induced by the scale choice (the last term in eq. (3.27)) is a contribution to $\gamma_{\sigma ss}$ of order $(\alpha_s/N)^{-1}$. Of course, if $\gamma$ is perturbative (i.e. if it vanishes in the $\alpha_s \rightarrow 0$ limit), this term will cancel against an equal and opposite $\frac{1}{2} N$ contribution to $\gamma_{\sigma ss}$. Furthermore, an important implication of eq. (3.27) is that the singularities of $\gamma(\alpha_s, N)$ in the complex $N$-plane are the same in both symmetric and asymmetric variables, and thus in particular the small-$N$ singularity which determines the small $x$ behaviour of the splitting function will also be the same.

4. Running coupling

So far, we have discussed the improvement of the $\chi$ kernel and its dual anomalous dimension at the fixed coupling level. As discussed in the introduction, the main motivation for the symmetrization of the previous section is that the symmetrized double-leading result has a stable minimum order by order in the resummed expansion, in contrast to the BFKl or unsymmetrized DL expansions which have an instability. The existence of a minimum enables a running coupling resummation, which is required [7,8] if we wish to get a stable
resummed anomalous dimension at small \( N \), and thus a uniformly stable splitting function at small \( x \). To this purpose, in line with the strategy devised in refs. [7,8], the running coupling evolution equation (2.25) is solved with a quadratic approximation to the kernel near the minimum, but with the running of the coupling treated exactly at the leading \( \ln Q^2 \) level. The solution is then used to improve the anomalous dimension computed within a perturbative treatment of the running of the coupling.

In refs. [7,8] the quadratic approximation near the minimum was applied to the LO BFKL kernel \( \chi_0 \), eq. (3.8). This kernel is symmetric and has a minimum at \( M = \frac{1}{2} \). However, we have seen that in order to preserve the symmetry and the existence of the minimum beyond the LO, we have to construct a different expansion for \( \chi \) which at LO is given by \( \chi_{\sigma \, LO}(\alpha_s, M) \) obtained from its off-shell version eq. (3.16) through eq. (3.4).

We are thus led to solve the equation

\[
NG(N, M) = \chi^q_\sigma(\hat{\alpha}_s, M) G(N, M) + F(M), \tag{4.1}
\]

where \( \chi_\sigma(\hat{\alpha}_s, M) \) has been replaced by its quadratic approximation \( \chi^q_\sigma(\hat{\alpha}_s, M) \) near the minimum

\[
\chi^q_\sigma(\alpha_s, M) = c(\alpha_s) + \frac{1}{2} \kappa(\alpha_s) (M - \frac{1}{2})^2 \tag{4.2}
\]

and \( F(M) \) is a boundary condition to perturbative evolution. Note that, in practice, because of eq. (3.25), the intercept \( c(\alpha_s) \) and curvature \( \kappa(\alpha_s) \) can be computed using the symmetric variable resummation \( \chi_\sigma \). The quality of the quadratic approximation, especially in the relevant region \( 0 < M < \frac{1}{2} \), can be appreciated from fig. 3, where the resummed curves obtained from the symmetrisation procedure, shown in fig. 1, are compared to their quadratic approximations.

There are two complicating features in the present case with respect to our previous work [7]. First, in earlier papers the curvature \( c \) and intercept \( \kappa \) of the kernel were linear in \( \alpha_s \); eq. (4.1) could then be solved to yield a solution expressed in terms of Airy functions. Here we need to expand \( c(\hat{\alpha}_s) \) and \( \kappa(\hat{\alpha}_s) \) in powers of \( \hat{\alpha}_s - \alpha_s \) and then, to the required accuracy, it turns out that the solution can be expressed in terms of Bateman functions.

The second complication is that, since \( \hat{\alpha}_s \) is an operator which does not commute with \( M \), any expression like \( \chi(\hat{\alpha}_s, M) \) is meaningful only after specifying the operator ordering. This amounts to specifying the argument of the running coupling. The first issue will be dealt with immediately, while the ordering issue will be postponed to the next two sections, where we will explicitly construct the resummed kernels, anomalous dimensions and splitting functions in the LO and NLO approximations.

When, as in the present case, \( c(\hat{\alpha}_s) \) and \( \kappa(\hat{\alpha}_s) \) are given as power series in \( \hat{\alpha}_s \), the differential equation (4.1) cannot be solved in general. However, we can determine the solution by expanding \( \hat{\alpha}_s \) about \( \alpha_s \) to first order in \( \hat{\alpha}_s \beta_0 \). It is easy to see that this procedure guarantees that the whole sequence of leading log contributions to \( \alpha_s \) is correctly included, up to subleading corrections. We have

\[
c(\hat{\alpha}_s) = c(\alpha_s) + (\hat{\alpha}_s - \alpha_s) c'(\alpha_s) + O \left[ (\beta_0 \frac{d}{dM})^2 \right], \tag{4.3}
\]

\[
\kappa(\hat{\alpha}_s) = \kappa(\alpha_s) + (\hat{\alpha}_s - \alpha_s) \kappa'(\alpha_s) + O \left[ (\beta_0 \frac{d}{dM})^2 \right], \tag{4.4}
\]

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Figure 3: Plot of the NLO and LO resummed DL expansion kernels $\chi_{\sigma}$, both on-shell and expressed in symmetric variables, compared with their quadratic approximations (dashed curves) near the minimum at $M = \frac{1}{2}$

and, from eq. (4.1),

$$
\left[ N - \bar{c}(\alpha_s) - \frac{1}{2} \bar{\kappa}(\alpha_s) (M - \frac{1}{2})^2 \right] G(N, M) = \hat{\alpha}_s \left( c'(\alpha_s) + \frac{1}{2} \kappa'(\alpha_s) (M - \frac{1}{2})^2 \right) G(N, M) + F(M),
$$

(4.5)

where we have defined

$$
\bar{c}(\alpha_s) \equiv c(\alpha_s) - \alpha_s c'(\alpha_s)
$$

(4.6)

and

$$
\bar{\kappa}(\alpha_s) \equiv \kappa(\alpha_s) - \alpha_s \kappa'(\alpha_s).
$$

(4.7)

Equation (4.5) can be simplified defining $\tilde{G}(N, M)$ through the implicit equation

$$
G(M, N) \equiv \frac{N}{N - \bar{c}(\alpha_s) - \frac{1}{2} \bar{\kappa}(\alpha_s) (M - \frac{1}{2})^2} \tilde{G}(M, N),
$$

(4.8)

so $\tilde{G}(N, M)$ satisfies

$$
N \tilde{G}(N, M) = \hat{\alpha}_s \phi(M, N) \tilde{G}(N, M) + F(M),
$$

(4.9)

with the kernel

$$
\phi(N, M) = \frac{N(c'(\alpha_s) + \frac{1}{2} \kappa'(\alpha_s) (M - \frac{1}{2})^2)}{N - \left( \bar{c}(\alpha_s) + \frac{1}{2} \bar{\kappa} (M - \frac{1}{2})^2 \right)}.
$$

(4.10)
Equation (4.10) has the form of a leading-order $M, N$ space running coupling BFKL equation, generalized to the case in which the kernel $\phi(M, N)$ depends on both $M$ and $N$, and not just on $M$ as the BFKL kernel $\chi_0$ does. As discussed in ref. [8], the anomalous dimension is entirely determined by the $Q^2$ dependence of the inverse $M$–Mellin transform of the inhomogeneous solution to this equation, viewed as a differential equation in $M$. Furthermore, this inhomogeneous solution to all perturbative orders satisfies a factorization property, whereby it can be written as the solution to the associate homogenous equation, times an $M$–independent, but $N$–dependent boundary condition. The anomalous dimension is therefore entirely determined by the solution to the associated homogeneous equation. The fact that the kernel now depends also on $N$ does not affect the form of the solution, since the inhomogeneous equation is a differential equation in $M$. Hence, using the general solution of ref. [7] (given as eq. (3.21) of that reference) we get

$$\tilde{G}(N, t) = \int_{\frac{1}{2} + i\infty}^{\frac{1}{2} - i\infty} dM \frac{M}{2\pi i} \exp \left[ \frac{M}{\beta_0 \alpha_s(t)} - \frac{1}{\beta_0} \int_{\frac{1}{2}}^{M} dM' \phi(M', N) \right]$$

$$= e^{1/2\beta_0 \alpha_s(t)} \int_{-i\infty}^{+i\infty} dm \frac{1 + A(\alpha_s, N)m}{1 - A(\alpha_s, N)m} B(\alpha_s, N) e^{m/\beta_0 \bar{\alpha}_s(t)},$$

where we have defined

$$A(\alpha_s, N) = \sqrt{\frac{\frac{1}{2} \bar{\kappa}(\alpha_s)}{(N - \bar{c}(\alpha_s))},} \quad (4.12)$$

$$B(\alpha_s, N) = \frac{1}{2\beta_0} \left( \frac{c'(\alpha_s)}{N - \bar{c}(\alpha_s)} + \frac{\kappa'(\alpha_s)}{\bar{\kappa}(\alpha_s)} \right) \sqrt{\frac{N - \bar{c}(\alpha_s)}{\frac{1}{2} \bar{\kappa}(\alpha_s)}}, \quad (4.13)$$

while

$$\frac{1}{\bar{\alpha}_s(t)} = \frac{1}{\alpha_s(t)} - \frac{\kappa'(\alpha_s)}{\kappa(\alpha_s)}. \quad (4.14)$$

The Mellin inversion integral can now be computed exactly, with the result

$$\tilde{G}(N, t) = \beta_0 \bar{\alpha}_s(t) B(\alpha_s, N) K_{2B(\alpha_s, N)} (1/\beta_0 \bar{\alpha}_s(t) A(\alpha_s, N)) \quad (4.15)$$

where $K_{\nu}(x)$ is the Bateman function, defined as the solution of the differential equation

$$-K''_{\nu}(x) + \left( 1 - \frac{\nu}{x} \right) K_{\nu} = 0, \quad (4.16)$$

with boundary condition $K_{\nu}(0) = 1$. Note that the argument of $A$ and $B$ eqs. (4.12)-(4.13) is the fixed coupling $\alpha_s$, not $\alpha_s(t)$. This is due to the fact that in the approximation eq. (4.3),(4.4), all functions of $\hat{\alpha}_s$ are linearized by expanding about the fixed coupling $\alpha_s$.

The anomalous dimension is determined by taking the logarithmic derivative of $\tilde{G}(N, t)$, and then setting $\alpha_s(t) = \alpha_s$. The extra factor which relates $\tilde{G}$ to $G$ eq. (4.8)
only affects the boundary condition which, as proven in ref. [7], does not contribute to the anomalous dimension. Therefore, we get

\[ \gamma_B(\alpha_s, N) = \frac{\partial}{\partial t} \ln \tilde{G}(N, t) \bigg|_{\alpha_s(t) = \alpha_s} = \frac{1}{2} - \beta_0 \bar{\alpha}_s + \frac{1}{A(\alpha_s, N)} \frac{K'_{2B(\alpha_s, N)}(1/\beta_0 \bar{\alpha}_s A(\alpha_s, N))}{K_{2B(\alpha_s, N)}(1/\beta_0 \bar{\alpha}_s A(\alpha_s, N))}. \] (4.17)

The Bateman anomalous dimension can be matched to the naive (i.e. fixed coupling) dual of the \( \chi \) with the same procedure which was used in ref. [7] for the Airy anomalous dimension. Namely, we determine \( \gamma_B(\alpha_s, N) \) in the \( \beta_0 \to 0 \) limit by saddle-point evaluation of the Mellin integral eq. (4.11). The leading order saddle condition is

\[ N = \alpha_s(t) \varphi(N, M_s), \] (4.18)

where \( \varphi(N, M) \) is given by eq. (4.10), which reduces to

\[ N = c_s(\alpha_s(t), \alpha_s) + \frac{1}{2} \kappa_s(\alpha_s(t), \alpha_s)(M_s - \frac{1}{2})^2, \] (4.19)

where

\[ c_s(\alpha_s(t), \alpha_s) = \alpha_s(t)c'(\alpha_s) + \bar{c}(\alpha_s) \] (4.20)
\[ \kappa_s(\alpha_s(t), \alpha_s) = \alpha_s(t)\kappa'(\alpha_s) + \bar{\kappa}(\alpha_s) \] (4.21)

are the linearized forms of \( c \) and \( \kappa \). Solving eq. (4.19) for \( M_s(N) \) gives

\[ M_s(N) \equiv \frac{1}{2} - \sqrt{\frac{(N - c(\alpha_s(t), \alpha_s))}{\frac{1}{2} \kappa(\alpha_s(t), \alpha_s)}}, \] (4.22)

which, if one lets \( \alpha_s(t) = \alpha_s \), reduces to the standard naive dual

\[ \gamma_s^B(\alpha_s, N) \equiv \frac{1}{2} - \sqrt{\frac{(N - c(\alpha_s))}{\frac{1}{2} \kappa(\alpha_s)}}, \] (4.23)

of \( \chi^q(\alpha_s, N) \) eq. (4.2) as it should.

Further terms in the expansion about the saddle generate an expansion of the anomalous dimension in powers of \( \frac{N \beta_0}{d \varphi(N, M_s)/dM} \). In particular, the fluctuations about the saddle lead to a contribution to the anomalous dimension which can be computed as in ref. [7]:

\[ \frac{1}{\sqrt{\phi'(N, M_s(\alpha_s(t), \alpha_s, N))}} = \exp \left[ \int_{t_0}^{t'} dt' \Delta \gamma_{ss}(\alpha_s(t'), \alpha_s, N) \right] \frac{1}{\sqrt{\phi'(N, M_s(\alpha_s(t), \alpha_s, N))}}, \] (4.24)
Figure 4: The Bateman anomalous dimension $\gamma_B(\alpha_s, N)$, computed with $\alpha_s = 0.2$ and the values of the parameters which correspond to the LO resummed curve of fig. 1, namely $c = 0.27, c' = 0.71, \bar{c} = 0.13, \kappa = 1.40, \kappa' = 0.99, \bar{\kappa} = 1.20$.

where the prime denotes differentiation with respect to $M$ (which is then set equal to $M_s$, eq. (4.23)), and

$$\Delta \gamma_{ss}(\alpha_s(t), \alpha_s, N) \equiv -\frac{1}{2} \beta_0 \alpha_s(t) \frac{\varphi''(N, M) \varphi(N, M)}{\varphi'(N, M)^2} \bigg|_{M=M_s}. \quad (4.25)$$

Use of the saddle condition eq. (4.22) with the $\varphi$ kernel eq. (4.10) in eq. (4.24) gives, after setting $\alpha_s(t) = \alpha_s$,

$$\gamma_{ss}^B(\alpha_s, N) = \gamma_{ss,0}^B(\alpha_s, N) + \frac{1}{4} \alpha_s^2 \beta_0 \frac{c'(\alpha_s)}{(c(\alpha_s) - N)}, \quad (4.26)$$

where

$$\gamma_{ss,0}^B(\alpha_s, N) = -\beta_0 \alpha_s + \frac{3}{4} \alpha_s^2 \beta_0 \frac{\kappa'(\alpha_s)}{\kappa(\alpha_s)}. \quad (4.27)$$

Note that $\gamma_{ss}^B$ depends on both $\alpha_s/N$ and $\alpha_s$: in fact, it differs by terms of order $O(\alpha_s)$ from a contribution of order $\gamma_{ss}$ in the expansion eq. (2.12) of $\gamma_B$ in powers of $\alpha_s/N$ at fixed $N$. The contribution $\gamma_{ss,0}^B(\alpha_s, N) = \lim_{N \to \infty} \gamma_{ss}^B(\alpha_s, N)$ is subleading in the expansion eq. (2.12) but leading–order in the expansion eq. (2.3) of the anomalous dimension in powers of $\alpha_s$ at fixed $N$. It follows that, at LO in the symmetrised DL expansion, as $N \to \infty$

$$\gamma_B(\alpha_s, N) \sim \gamma_{ss}^B(\alpha_s, N) + \gamma_{ss,0}^B(\alpha_s, N) + O(1/N). \quad (4.28)$$

The Bateman anomalous dimension eq. (4.17) is displayed in fig. 4, with the values of the parameters which correspond to the quadratic approximation to the LO resummed
kernel displayed in figs. 1, 3. It is apparent that its behaviour is similar to that of the Airy anomalous dimension discussed in ref. [7]: the anomalous dimension is regular and monotonically decreasing along the real axis, with a simple pole at \( N = N_B(\alpha_s) \), located at the rightmost zero of the Bateman function \( K_\nu(x) \). The small \( x \) behaviour is determined by the location of this pole, which in turn depends on the values of the intercept and curvature of the quadratic kernel \( c(\alpha_s) \) and \( \kappa(\alpha_s) \) eq. (4.2) and their first derivatives \( c'(\alpha_s) \), \( \kappa'(\alpha_s) \).

The qualitative behaviour of the Bateman running coupling resummation can be inferred from fig. 5, where we compare the quadratic approximation to the LO resummed kernel (same as in fig. 3) with the naive dual eq. (2.9) of the physical branch of the Bateman anomalous dimension (i.e. with \( N > N_B(\alpha_s) \)). For comparison, the corresponding Airy anomalous dimension is also shown. The latter is only defined when \( c \) and \( \kappa \) are linear functions of \( \alpha_s \), and it is thus determined by taking in eq. (4.2) \( c(\hat{\alpha}_s) \approx \hat{\alpha}_s c(\alpha_s) / \alpha_s \) and \( \kappa(\hat{\alpha}_s) \approx \hat{\alpha}_s \kappa(\alpha_s) / \alpha_s \). This is a worse approximation than that used to determined the Bateman anomalous dimension, since the leading log terms are not correctly reproduced. The resummation of running coupling effects effectively takes the minimum of \( \chi \) from one half out to infinity [7], thereby greatly extending the range of the physical region. For the anomalous dimension this means that the usual fixed coupling cut at \( N = c(\alpha_s) \) is replaced by a simple pole at \( N_B(\alpha_s) \) to the left, i.e. \( N_B(\alpha_s) < c(\alpha_s) \). This in turn means that the small \( x \) behaviour of the splitting function will be softer (more slowly rising) than that expected from naive duality.

In fig. 5 (right) we display the Bateman effective kernel determined after subtracting from the Bateman anomalous dimension its \( N \to \infty \) limit, namely, the dual of \( \gamma_B - \gamma_{ss,0}^B \), with \( \gamma_{ss,0}^B \) given by eq. (4.27), together with the corresponding Airy curve. This comparison shows that, after this constant subtraction, for \( M \leq 0 \) \( \gamma_B \) is very well approximated by the leading order asymptotic \( \beta \to 0 \) approximation eq. (4.23), i.e., by the fixed-coupling quadratic curve, as is \( \gamma_A \). In other words, the running coupling resummation for \( N > c(\alpha_s) \) (to the right of the fixed–coupling cut) is a very small correction.

The location of the rightmost singularity of the anomalous dimension, which determines the small \( x \) behaviour, is shown in fig. 6 as a function of \( \alpha_s \). This singularity is a cut at fixed coupling, and a simple pole after running coupling resummation. The nonlinear dependence of \( c \) on \( \alpha_s \) in the resummed fixed-coupling case (as opposed to the linear dependence in the unresummed BFKL LO case) is apparent, and it explains why a linear extrapolation from the origin would not be a good approximation for realistic values of \( \alpha_s \).

In all cases, running coupling resummation softens the leading singularity. The effect is most dramatic for the LO BFKL kernel \( \hat{\alpha}_s \chi_0 \) eq. (3.8), whose running coupling resummation is effected by the Airy method of ref. [7]. Even though the percentage softening is more moderate for the Bateman resummation discussed here (see also fig. 5), the symmetrization discussed in the previous section also reduces the fixed coupling intercept in comparison to the BFKL one. Hence, after running coupling resummation the LO linear (Airy) result and the NLO resummed (Bateman) result for the location of the pole end up being quite close. The stability of the resummed result when going from LO to NLO is further enhanced by running coupling resummation, so all three running coupling curves end up being close to each other. Interestingly, the contributions to the NLO kernel proportional to \( \beta_0 \) (to be discussed in detail in section 6) further reduce the difference between
the resummed LO and NLO minima, and in fact they pull the NLO resul below the LO one for $\alpha_s \approx 0.15$. For comparison, we also show the location of the NLO unresummed BFKL stationary point, though this is in fact unphysical for realistic $\alpha_s \gtrsim 0.1$ where the NLO BFKL kernel in fact has a minimum instead of a maximum, so the associated anomalous dimension becomes imaginary and the cross section oscillates [26,27].

The Bateman anomalous dimension will be used in the next sections to construct the improved anomalous dimensions in the symmetrised double leading expansion, first at LO and then at NLO.

5. Leading Order Resummation

We wish now to determine anomalous dimensions and splitting functions by combining the ingredients of the previous two sections: symmetrization of the double–leading expansion and its running coupling resummation. In previous work [7,6] we constructed resummed anomalous dimensions by exploiting the fact that the DL expansions of $\chi$ and $\gamma$ are dual of each other, up to subleading terms. We thus constructed the anomalous dimension $\gamma$ starting from its own DL expansion, namely

$$\gamma_{DL}(\alpha_s, N) = \gamma_{DL \, LO}(\alpha_s, N) + \alpha_s \gamma_{DL \, NLO}(\alpha_s, N) + \ldots$$

$$\gamma_{DL \, LO}(\alpha_s, N) = [\alpha_s \gamma_0(N) + \gamma_s(\frac{\alpha_s}{N^2}) - \frac{n_c \alpha_s}{\pi N}]$$

$$\gamma_{DL \, NLO}(\alpha_s, N) = [\alpha_s \gamma_1(N) + \gamma_{ss}(\frac{\alpha_s}{N}) - e_0 - \alpha_s \left(\frac{e_2}{N^2} + \frac{e_1}{N}\right)].$$
Figure 6: The leading (rightmost) singularity of various anomalous dimension as a function of $\alpha_s$, all determined with $n_f = 0$. BFKL LO , res fix LO and res fix NLO (dashed) denote respectively the location of the cut for $\alpha_s \chi_0$ eq. (3.8) and for the $\beta_0 = 0$ resummed kernels $\chi_{\sigma, LO}$ and $\chi_{\sigma, NLO}$ obtained putting on shell the kernels eq. (3.16) and (3.19). LO Airy, LO Bateman and NLO Bateman (solid curves) denote the location of the pole $\nu_B$ for the running coupling resummation of the corresponding three curves. BFKL NLO (dashed) denotes the location of the stationary point of the NLO kernel eq. (2.8).

Here, instead, in order to take advantage of the improved kernel $\chi_{\Sigma, LO}(\alpha_s, M)$, which is the on-shell version of eq. (3.16), we must define the anomalous dimension $\gamma_{\Sigma LO}(\alpha_s, N)$ using the duality equation (3.5):

$$\chi_{\Sigma, LO}(\alpha_s, \gamma_{\Sigma LO}(\alpha_s, N)) = N.$$

(5.2)

This coincides with the LO double–leading anomalous dimension $\gamma_{DL LO}$ up to subleading terms.

To go beyond the fixed coupling limit, in the kernel $\chi_{\Sigma}(\alpha_s, M)$ we must make the replacement $\alpha_s \rightarrow \check{\alpha}_s$. As proven in Ref. [7], factorized duality remains valid order by order in perturbation theory at the running coupling level, i.e. for any given $\chi$ there exists a $\gamma$ such that the solution to eq. (2.26) with the given $\chi$ and the solution to the usual evolution equation (2.2) coincide if, order by order, the boundary conditions are suitably matched and appropriate $\beta_0$-dependent corrections are added to the fixed coupling kernels. In particular, in the LO expression of $\chi_{\Sigma}(\alpha_s, M)$ eq. (3.16), the terms originating from $\chi_0$ are not modified and, given the basic commutator relation, eq. (2.24), they do not depend on operator ordering up to NLO corrections. For the $\chi_s$ terms it is easy to see that an operator ordering exists such that the $\chi_s$ which is dual to $\gamma_0$ coincides with the
fixed-coupling expression eq. (2.18). Indeed, the leading-order form of the duality relation eq. (2.17) in the running coupling case is

$$\gamma_0(\chi_s) = \hat{\alpha}_s^{-1} M, \quad (5.3)$$

which immediately implies that if $\chi_s$ satisfies eq. (2.18), i.e. it is the inverse function of $\gamma_0$, then

$$\chi_s \left[ \gamma_0 \left( (\hat{\alpha}_s^{-1} M)^{-1} \right) \right] = \chi_s \left( M^{-1} \hat{\alpha}_s \right) = N \quad (5.4)$$

by direct substitution of eq. (5.3). Hence, the running-coupling dual of $\gamma_0$ coincides with the fixed coupling dual $\chi_s$, provided $\chi_s$ is now evaluated as a function of the operator $M^{-1} \hat{\alpha}_s$.

A more detailed discussion of operator ordering issues in running coupling duality will be given elsewhere [28].

Let us now turn to the symmetrization of $\chi_s$. First, we should observe that the symmetry of the underlying Feynman diagrams which determines the symmetry of the BFKL kernel $K \left( \alpha_s, \frac{Q^2}{k^2} \right)$ eq. (2.7) holds for the unintegrated gluon distribution $G(\xi, Q^2)$, obtained by differentiation from the standard gluon distribution which enters the GLAP equation:

$$G(\xi, Q^2) \equiv \frac{d}{dt} G(\xi, Q^2), \quad (5.5)$$

i.e., upon Mellin transformation,

$$G(\xi, M) \equiv MG(\xi, M). \quad (5.6)$$

When the coupling runs, the $M$-space evolution kernel acquires a further contribution when switching from the integrated to unintegrated gluon distribution due to the commutator of $\hat{\alpha}_s$ and the $M$ prefactor in eq. (5.6). Indeed,

$$\chi_s \left( M^{-1} \hat{\alpha}_s \right) G(\xi, M) = \chi_s \left( M^{-1} \hat{\alpha}_s \right) M^{-1} G(\xi, M)
= M^{-1} \chi_s \left( \hat{\alpha}_s M^{-1} \right) G(\xi, M). \quad (5.7)$$

So, going from the integrated to the unintegrated distribution switches the ordering of $\hat{\alpha}_s$ and $M^{-1}$ in the argument of $\chi_s$. Note that this transformation is manifestly not symmetric upon $M \to 1 - M$, hence if the kernel at the unintegrated level is symmetric, the kernel at the integrated level won’t be.

At the running coupling level, the kernel is symmetric only if the argument of the running coupling is also treated symmetrically. In fact, upon Mellin transformation, a $\xi$ evolution equation of the form

$$\frac{\partial}{\partial \xi} G(\xi, Q^2) = \int_{-\infty}^{\infty} \frac{dk^2}{k^2} \sum_{p=1}^{\infty} \left[ \alpha_s^p(Q^2) K_L^{(p)}(Q^2/k^2) + \alpha_s^p(k^2) K_R^{(p)}(Q^2/k^2) \right] G(\xi, k^2) \quad (5.8)$$

becomes

$$\frac{\partial}{\partial \xi} G(\xi, M) = \sum_{p=1}^{\infty} \left[ \hat{\alpha}_s^p \chi_L^{(p)}(M) + \chi_R^{(p)}(M) \right] G(\xi, M), \quad (5.9)$$
where $\chi^{(p)}_L(M), \chi^{(p)}_R(M)$ are the Mellin transforms eq. (2.6) of the kernels $K^{(p)}_R(Q^2/k^2)$ and $K^{(p)}_L(Q^2/k^2)$, respectively. Because the symmetrization is based on the interchange $Q^2 \leftrightarrow k^2$, it follows that the symmetrized version of $\hat{\alpha}_s^k f(M)$ is $\frac{1}{2} (\hat{\alpha}_s^k f(M) + f(1-M)\hat{\alpha}_s^k)$. Furthermore, because

$$[\hat{\alpha}_s, M^{-1}] = -\beta_0 M^{-1} \hat{\alpha}_s^2 M^{-1} = -\beta_0 \hat{\alpha}_s M^{-2} \hat{\alpha}_s,$$

(5.10)

while

$$[\hat{\alpha}_s, (1-M)^{-1}] = \beta_0 (1-M)^{-1} \hat{\alpha}_s^2 (1-M)^{-1} = -\beta_0 \hat{\alpha}_s (1-M)^{-2} \hat{\alpha}_s,$$

(5.11)

it is easy to show that if

$$f(\hat{\alpha}_s M^{-1}) = \tilde{f}(M^{-1} \hat{\alpha}_s) = \sum_k \tilde{f}_k \hat{\alpha}_s^k M^{-k}$$

then

$$f((1-M)^{-1} \hat{\alpha}_s) = \tilde{f}(\hat{\alpha}_s (1-M)^{-1}) = \sum_k \tilde{f}_k (1-M)^{-k} \hat{\alpha}_s^k.$$

It follows that at the running-coupling level the symmetrized LO kernel is

$$\bar{\chi}_{\sigma \text{ LO}}(\hat{\alpha}_s, M, N) = \chi_s \left( \hat{\alpha}_s \left( M + \frac{N}{2} \right)^{-1} \right) + \chi_s \left( (1-M + \frac{N}{2})^{-1} \hat{\alpha}_s \right) + \bar{\chi}_0(\hat{\alpha}_s, M, N) + \chi_{\text{mom}},$$

(5.12)

where

$$\bar{\chi}_0(\hat{\alpha}_s, M, N) = \hat{\alpha}_s \left( \bar{\chi}_0^+(M, N) - \frac{1}{M + \frac{N}{2}} \right) + \left( \bar{\chi}_0^-(M, N) - \frac{1}{1-M + \frac{N}{2}} \right) \hat{\alpha}_s$$

$$= \frac{n_c}{\pi} \left[ \hat{\alpha}_s \left( \psi(1) - \psi(1 + M + \frac{N}{2}) \right) + \psi(1 + N) - \psi(2 - M + \frac{N}{2}) \right] \hat{\alpha}_s,$$

(5.13)

which are the running coupling analogues of eqs. (3.13), (3.14). Different orderings of the $\chi_s$ term can be determined using the commutators (5.10) and (5.11). Different orderings of the remaining terms are equivalent, up to subleading corrections which modify the form of the subleading $\chi_1$, and will be discussed in the next section. Here, we have chosen the operator ordering in the $\chi_0$ terms such that it matches the ordering of the corresponding $\chi_s$ terms, namely, so that the ordering is the same in the double counting contributions. Note that this corresponds to taking, after inversion of the Mellin transform, $\alpha_s(Q^2)$ in the collinear double-counting term, and $\alpha_s(k^2)$ in its anti-collinear counterpart.

Starting from eq. (5.12) one can expand $\bar{\chi}_{\sigma \text{ LO}}(\hat{\alpha}_s, M, N)$ in powers of $(M - \frac{1}{2})$ at fixed $N$ and $\hat{\alpha}_s$, around its minimum at $M = \frac{1}{2}$ with a result of the form:

$$\bar{\chi}_\sigma(\hat{\alpha}_s, M, N) = \bar{\chi}_0^q(\hat{\alpha}_s, M, N) + \bar{\chi}_s^q(\hat{\alpha}_s, M, N) + O(\beta_0 \hat{\alpha}_s) + O((M - \frac{1}{2})^4),$$

(5.14)

where

$$\bar{\chi}_0^q(\hat{\alpha}_s, M, N) = \hat{\alpha}_s \left( c_0(N) + \frac{1}{2} \kappa_0(N)(M - \frac{1}{2})^2 \right),$$

(5.15)

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while
\[ \bar{\chi}_s^q(\hat{\alpha}_s, M, N) = \bar{c}(\hat{\alpha}_s, N) + \frac{1}{2} \bar{\kappa}(\hat{\alpha}_s, N) \left( M - \frac{1}{2} \right)^2, \] (5.16)
and the expansion coefficients are given by
\[
\bar{c}(\hat{\alpha}_s, N) = 2c_s \left( \frac{\hat{\alpha}_s}{1 + N} \right),
\]
\[
\bar{\kappa}(\hat{\alpha}_s, N) = 2 \frac{1}{(1 + N)^2} \kappa_s \left( \frac{\hat{\alpha}_s}{1 + N} \right),
\] (5.17)
in terms of the coefficients of the Taylor expansions of \( \chi_s \):
\[
\chi_s \left( \frac{\alpha_s}{M} \right) = c_s(\alpha_s) + \delta_s(\alpha_s) \left( M - \frac{1}{2} \right) + \frac{1}{2} \kappa_s(\alpha_s) \left( M - \frac{1}{2} \right)^2 + \ldots
\] (5.18)
The on-shell quadratic expansion eq. (4.2) can now be obtained by putting \( N = \bar{\chi}_s(\hat{\alpha}_s, M, N) \) with the right-hand side given by eq. (5.14) for \( N \) consistently to order \( (M - \frac{1}{2})^2 \), and then identifying the result \( N = \chi_s(\hat{\alpha}_s, M) \). Alternatively one of course could first obtain the full on-shell \( \chi_s(\hat{\alpha}_s, M) \) by solving eq. (3.4) and then expanding the result about \( M = \frac{1}{2} \): the result will be the same.

The \( O(\beta_0 \hat{\alpha}_s) \) corrections in eq. (5.14) are due to operator ordering, and will be discussed in sect. 6. They are of relative order \( \hat{\alpha}_s \) with respect to the curvature and intercept, which are determined to all orders in \( \hat{\alpha}_s \) in order to obtain a good description of the \( \chi \) kernel in the vicinity of the minimum. Further \( O(\beta_0 \hat{\alpha}_s) \) corrections, also to be discussed in the next section, are generated when switching from the unintegrated distribution (which evolves with a symmetric kernel) to the standard integrated one which enters the GLAP equation.

The final expression for the improved anomalous dimension at LO is now obtained by implementing the running coupling resummation through the quadratic approximation. Namely, we write the evolution equation eq. (4.1) as
\[
NG(N, M) = \chi_s(\hat{\alpha}_s, M)G(N, M) + F(M),
\] (5.19)
and then exploit the symmetry of \( \chi_s \) to expand it about its minimum, up to a quartic term which leads to asymptotically subleading [7,8] corrections. The anomalous dimension determined from solution of eq. (5.19) in the quadratic approximation is then simply given by \( \gamma_B \) eq. (4.17), but with the replacement \( \frac{1}{2} \rightarrow \frac{1}{2} + \frac{1}{2}N \). The anomalous dimension in asymmetric (DIS) variables is obtained from it using eq. (3.27), which simply restores \( M_s \) to the value eq. (4.17).

Hence, the resummed anomalous dimension after running coupling resummation is given by
\[
\gamma_{\Sigma, LO}^C(N, \alpha_s(t)) = \gamma_{\Sigma, LO}(N, \alpha_s(t)) + \gamma_B(\alpha_s(t), N)
- \gamma_s(B(\alpha_s(t), N) - \gamma_0(B(\alpha_s(t), N) + \gamma_{\text{mom}}(\alpha_s(t), N),
\] (5.20)
where \( \gamma_{\Sigma, LO}(N, \alpha_s(t)) \) is determined from eq. (5.2), \( \gamma_B(\alpha_s(t), N) \) is the Bateman anomalous dimension eqs. (4.17), and \( \gamma_s(B(\alpha_s(t), N) \), \( \gamma_0(B(\alpha_s(t), N) \), obtained letting \( \alpha_s \rightarrow \alpha_s(t) \) in eqs. (4.23),(4.27), remove the double counting between the the first two terms. Notice
that \( \gamma^B \) removes the square–root branch cut of the fixed-coupling anomalous dimension, thanks to the fact that the curvature and intercept of the quadratic kernel are the same eq. (3.25) in symmetric variables (used to compute \( \gamma^B \)) and DIS variables (used to compute \( \gamma^\Sigma \)). The momentum subtraction term is needed to cancel small (and formally subleading) momentum violations incurred during the running coupling resummation: it can be chosen as

\[
\gamma_{\text{mom}}(\alpha_s(t), N) = d_m f_m(N),
\]

where \( f_m \) must satisfy \( f_m(1) = 1 \) and \( f_m(\infty) = 0 \) and can thus be implemented through the same function \( f_m \) introduced in eq. (3.7), and \( d_m \) is a constant of order \( \alpha_s^2 \).

6. Next-to-Leading Order Resummation

The extension of our results to next-to-leading order requires two steps: first, the symmetrization of the next-to-leading DL kernel eq. (2.16), and second, the next-to-leading treatment of the running coupling resummation. The general structure of the next-to-leading order symmetrized kernel was already given in eqs. (3.18),(3.19) in the fixed coupling case. At the running coupling level, operator ordering in the leading order contribution affects the form of the NLO term. We assume therefore that (in symmetric variables)

\[
\bar{\chi}_\sigma(\hat{\alpha}_s, M, N) = \bar{\chi}_{\sigma \text{LO}}(\hat{\alpha}_s, M, N) + \hat{\alpha}_s \bar{\chi}_{\sigma \text{NLO}}(\hat{\alpha}_s, M, N),
\]

where \( \bar{\chi}_{\sigma \text{LO}}(\hat{\alpha}_s, M, N) \) is given by eq. (5.12). The corresponding kernel in DIS variables is

\[
\bar{\chi}_\Sigma(\hat{\alpha}_s, M, N) = \bar{\chi}_\sigma(\hat{\alpha}_s, M - \frac{N}{2}, N).
\]

Note that at the running coupling level the kernel for evolution of the integrated and unintegrated parton distributions are not equal, as shown in eq. (5.7) for \( \chi_s \), while in fact, as discussed in the previous section, only the unintegrated distribution is symmetric. Hence, we will construct the symmetrized \( \chi \) kernel at the unintegrated level, and then switch back to the integrated level at the end when determining the resummed GLAP anomalous dimension from it.

At the leading order level, we have already shown in section 3 that the fixed coupling \( \bar{\chi}_{\Sigma \text{LO}}(\alpha_s, M, N) = \bar{\chi}_{\sigma \text{LO}}(\alpha_s, M - \frac{N}{2}, N) \) is equal to \( \chi_{DL \text{LO}}(M, N) \) up to subleading terms, and generalized this in section 4 to the running coupling \( \bar{\chi}_{\Sigma \text{LO}}(\hat{\alpha}_s, M, N) \). We must now check this equality up to NLO at the running coupling level, and compute the correction terms which are necessary to restore it when violated. This will lead to the determination of \( \bar{\chi}_{\Sigma \text{NLO}}(\hat{\alpha}_s, M, N) \) eq. (6.2), and in particular of the function \( \bar{\chi}_1(M) \) which was left implicit in eqs. (3.18),(3.19).

We start therefore with \( \bar{\chi}_{\Sigma \text{LO}}(\hat{\alpha}_s, M, N) \) given by eq. (5.12). This differs from the double leading \( \chi_{DL \text{LO}}(M, N) \) in two respects: because of the addition of the anticollinear term \( \chi_s((1 - M + N)^{-1}\hat{\alpha}_s) \), and because \( \chi_0 \) is put off shell. In order to compare to the DL result it is convenient to view the simple and double poles \((1 - M + N)^{-1}\hat{\alpha}_s\) and \(((1 - M + N)^{-1}\hat{\alpha}_s)^2\) of the anticollinear term as contributions to the off-shell \( \chi_0 \) and \( \chi_1 \), respectively.
Consider first the anticollinear term $\chi_s((1 - M + N)^{-1}\hat{\alpha}_s)$ of eq. (5.12). This term starts at $O(\alpha_s)$, but the first two terms are already included in $\chi_0$ and $\chi_1$, hence the remaining contribution starts at $O(\alpha_s^3)$, namely NNLO in an expansion in powers of $\alpha_s$ at fixed $M$, and $N^4$LO in an expansion in powers of $\alpha_s$ at fixed $\alpha_s/M$. Let us turn now to the rather more complicated case of the off-shell extension of $\chi_0(M)$, which is given by the $O(\hat{\alpha}_s)$ contribution to $\chi_{LO}(\hat{\alpha}_s, M, N)$ (in an expansion in powers of $\hat{\alpha}_s$ at fixed $M$). It is thus

$$\hat{\alpha}_s\chi^+_0(M - \frac{N}{2}, N) + \hat{\alpha}_s\chi^-_0(M - \frac{N}{2}, N) = \hat{\alpha}_s\left[\hat{\chi}^+_0(M - \frac{N}{2}, N) + \hat{\chi}^-_0(M - \frac{N}{2}, N)\right] - \hat{\alpha}_s^2\beta_0 \frac{n_s}{\pi} \psi'(1 - M) + O(\alpha_s^3),$$

(6.3)

where $\chi^\pm_0(M, N)$ are given by eq. (3.14), and in the last step we used the commutator eq. (2.24), which implies

$$[\hat{\alpha}_s, f(M)] = \hat{\alpha}_s^2\beta_0 f'(M) - \hat{\alpha}_s^2\beta_0 f''(M) + \ldots$$

(6.5)

The ordering of eq. (6.4) for the $O(\alpha_s)$ contribution in the expansion eq. (2.8) of the kernel has been adopted in refs. [5–6] for the construction of the double–leading expansion, though the ordering eq. (6.3), as discussed in sect. 4, is more convenient in a symmetrized approach.

Let us first consider the effect of the off-shell extension on terms which are next-to-leading in $\ln Q^2$, i.e. in the expansion of $\chi$ in powers of $\alpha_s$ at fixed $\alpha_s/M$. The effect of the off-shell extension can be determined by considering

$$[\chi_0^+(M) + \chi_0^-(M - N)] - \chi_0(M) = \frac{n_s}{\pi} \left[\psi(1 - M) - \psi(1 - M + N)\right]$$

$$= \frac{n_s}{\pi} \left[\psi(1) - \psi(1 + N)\right] - M \left[\psi'(1) - \psi'(1 + N)\right] + O(M^2),$$

(6.6)

where $\chi^\pm(M)$ are given by eq. (3.9). It follows that

$$\hat{\alpha}_s\chi^+_0(M - \frac{N}{2}, N) + \hat{\alpha}_s\chi^-_0(M - \frac{N}{2}, N)\hat{\alpha}_s - \hat{\alpha}_s\chi_0(M) = O(M) + O(\hat{\alpha}_s^2),$$

(6.7)

where now $\chi^\pm_0(M, N)$ is given by eq. (3.14), and the $O(\hat{\alpha}_s^2)$ correction eq. (6.4) due to operator ordering will be included in $\chi_1$ as we discuss below. Therefore, thanks to the replacement $\psi(1) \rightarrow \psi(1 + N)$ in $\chi^-(M, N)$ eq. (3.14) in comparison to $\chi^-(M)$ eq. (3.9), the combination eq. (6.4) which appears in the leading–order kernel $\chi_{LO}$, eq. (3.16), differs from the BFKL kernel $\alpha_s\chi_0$, eq. (3.8), by terms which are of order $O(\alpha_s M) = O(\alpha_s^2(\alpha_s/M)^{-1})$ i.e. next-to-next-to leading $\ln Q^2$ in an expansion in powers of $\alpha_s$ at fixed $\alpha_s/M$.

Next we consider the effect of the off-shell extension on terms which are next-to-leading $\ln x$, i.e. the expansion of the kernel in powers of $\alpha_s$ at fixed $N$. Because this expansion of the kernel (and in particular the NLO term $\chi_1(M)$) is usually computed in symmetric variables, it is useful to compare the combination which appears in the leading–order kernel $\chi_{LO}$, eq. (3.16) with the LO BFKL kernel $\alpha_s\chi_0$ in symmetric variables, too. It turns out that the off-shell extension of $\chi_0$ induces next-to-leading $\ln \frac{1}{x}$ contributions which must be subtracted. Indeed,

$$[\chi_0^+(M, N) + \chi_0^-(M, N)] - \chi_0(M) = \frac{1}{2} \frac{n_s}{\pi} \left[2\psi'(1) - \psi'(M) - \psi'(1 - M)\right] N + O(N^2).$$

(6.8)
In order to construct the appropriate subtraction, start with the NLO BFKL kernel eq. (2.8), written in symmetric variables. The NLO version of eq. (6.3) is (in symmetric variables)

\[
\chi_{\sigma \text{NLO}}(\hat{\alpha}_s, M, N) = \hat{\alpha}_s \chi_0^+(M, N) + \chi_0^-(M, N) \hat{\alpha}_s + 2 \hat{\alpha}_s ^2 \chi_1(M, N) + O(\hat{\alpha}_s^3), \tag{6.9}
\]

where the ordering in the NLO term \(\chi_1(M)\) is immaterial since it only affects NNLO terms.

With the symmetric ordering of \(\hat{\alpha}_s\) adopted in eq. (6.4), the NLO contribution \(\chi_1(M, N)\) is symmetric. The on-shell contribution \(\chi_1(M)\) has been determined in refs. [16–19], in the form

\[
\chi(\hat{\alpha}_s, M) = \hat{\alpha}_s \chi_0(M) + 2 \hat{\alpha}_s ^2 \chi_1^F(L)(M) + O(\hat{\alpha}_s^3) = \hat{\alpha}_s \chi_0^+(M) + \chi_0^-(M) \hat{\alpha}_s + 2 \hat{\alpha}_s ^2 \left[ \chi_1^F(L)(M) + \beta_0 \frac{n_c}{\sqrt{\pi}} \psi'(1 - M) \right] + O(\hat{\alpha}_s^3), \tag{6.10}
\]

where \(\chi_1^F(L)(M)\) is given by \(\delta(\gamma)\) in ref. [16], with the identification \(\gamma \to M\) and \(\chi_1^F = \frac{n_c^2}{4\pi^2} \delta\). In practice, the effect of the \(\beta_0\) term in eq. (6.11) is thus to reverse the sign of the \(\psi'(1 - M)\) contribution to \(\Delta(\gamma)\) (as defined in ref. [16]) thereby leading to a manifestly symmetric result for

\[
\chi_1(M) \equiv \chi_1^F(L)(M) + \beta_0 \frac{n_c}{\sqrt{\pi}} \psi'(1 - M). \tag{6.12}
\]

We can now finally remove the terms generated by the off-shell extension eq. (6.8) up to NLO by defining (in asymmetric variables)

\[
\chi_{\text{NLLx}}(\hat{\alpha}_s, M) = \hat{\alpha}_s \chi_0^+(M) + \chi_0^-(M - N) \hat{\alpha}_s + 2 \hat{\alpha}_s ^2 \tilde{\chi}_1(M) + O(\hat{\alpha}_s^3) \tag{6.13}
\]

\[
\tilde{\chi}_1(M) = \chi_1(M) - \frac{1}{2} \chi_0(M) \frac{n_c}{\sqrt{\pi}} \left[ 2 \psi'(1) - \psi'(M) - \psi'(1 - M) \right]. \tag{6.14}
\]

With this definition, the off-shell extension eq. (6.13) coincides with the on-shell \(O(\hat{\alpha}_s^2)\) kernel of refs. [16–19], up to sub-subleading corrections.

Note that although \(\chi_1(M)\) has triple poles at \(M = 0\) and \(M = 1\), these are cancelled in \(\tilde{\chi}_1(M)\), so that the leading small \(M\) singularity is \(O\left(\frac{\hat{\alpha}_s^2}{M^2}\right)\) as required by the matching to \(\chi_s\). This cancellation would be spoiled if one did not replace \(\chi_1\) with \(\tilde{\chi}_1\) on the r.h.s. of eq. (6.14). Of course this is all as it should be: we obtained \(\tilde{\chi}_1\) through eq. (6.8) as compensation for the off-shell extension of \(\chi_0\) eq. (3.14), which was in itself constructed precisely so that LO collinear evolution is not spoiled at large \(N\). The cancellation of the triple poles is then the NLLx manifestation of this fact.

Therefore, to match to the collinear contributions \(\chi_s\) and \(\chi_{ss}\), and their anticollinear counterparts, as described by eq. (3.13) and eq. (3.18), we define a function \(\tilde{\chi}(M)\) as

\[
\tilde{\chi}_1(M) = \tilde{\chi}_1(M) - \left[ g_1 \left( \frac{1}{M^2} + \frac{1}{1-M^2} \right) + g_2 \left( \frac{1}{M^2} + \frac{1}{(1-M)^2} \right) \right], \tag{6.15}
\]

with coefficients \(g_i\) are determined by demanding regularity as \(M \to 0\) and \(M \to 1\), and thus correct matching to \(\chi_s\) and \(\chi_{ss}\):

\[
g_1 = -\frac{n_f n_c}{\pi^2} \left( 10 + \frac{13}{n_c^2} \right), \tag{6.16}
\]

\[
g_2 = -\frac{11 n_f^2}{12 \pi^2} - \frac{3 n_f}{4 \pi^2 n_c}.
\]
Now, note that, of course, a shift by an amount proportional to \( N \) of the argument of \( \chi_1 \) is sub-subleading: 
\[
\tilde{\chi}_1(M + kN) = \tilde{\chi}_1(M) + O(\alpha_s).
\]
Hence, we can simply take in eqs. (3.18), (3.19)
\[
\tilde{\chi}_1(M, N) = \tilde{\chi}_1(M - \frac{N}{2}, N) + O(\alpha_s) = \tilde{\chi}_1(M),
\]
with \( \tilde{\chi}_1(M) \) given by eq. (6.15). With this choice, \( \chi_\Sigma(M) \) obtained putting on shell \( \tilde{\chi}_\Sigma(M, N) \) coincides with its double-lead counterpart up to subleading terms: eq. (6.7) shows that the off-shell extension of \( \chi_0 \) only adds terms which are subleading in an expansion in powers of \( \alpha_s \) at fixed \( \alpha_s/M \), and eq. (6.14) shows that it only adds terms which are subleading in an expansion in powers of \( \alpha_s \) at fixed \( M \), while the anticollinear contribution is also subleading.

The choice of \( \tilde{\chi}_1 \) eq. (6.17) has two shortcomings, however. First, eq. (6.17) implies that the pair of kernels in symmetric and asymmetric variables are only related by the shift eq. (3.22) up to subleading corrections as well, because the contribution \( \tilde{\chi}_1 \) does not satisfy eq. (3.22). This in particular means that also the equality of curvature and intercept eq. (3.25) of the pair of kernels is spoiled by subleading terms. As discussed in sect. 4, it then follows that the cancellation of the square-root cut in eq. (5.20) is spoiled by subleading corrections. Second, as discussed in sect. 3, \( \chi_\Sigma \) in symmetric variables is an entire function, provided that eq. (3.22) holds. This property is lost if eq. (6.17) is used, and \( \chi_\Sigma \) then has subleading poles on the real axis, so that in particular the quadratic approximation is much poorer.

It is therefore necessary to construct \( \chi_\Sigma^\pm(M, N) \) such that while it is symmetric in symmetric variables, in asymmetric variables \( \chi_\Sigma^\pm(M - \frac{N}{2}, N) \) reproduces the correct LO and NLO singularities, as in eq. (6.15), even at large \( N \), while at small \( N \) \( \tilde{\chi}_1(M, N) = \tilde{\chi}_1(M) + O(N) \). The construction follows the same procedure as at LO, in particular by first separating
\[
\chi_1(M) = \chi_1^+(M) + \chi_1^-(M),
\]
where \( \chi_1^+(M, N) \) \( (\chi_1^-(M, N)) \) is regular when \( \text{Re} \, M > \frac{1}{2} \) \( (\text{Re} \, M < \frac{1}{2}) \) just as in eq. (3.14). However the details are somewhat technical, and have been relegated to an appendix.

Because operator ordering issues are only relevant in the leading order term,\(^1\) this concludes the construction of \( \chi_\Sigma \) eq. (6.2): the NLO kernel \( \chi_\Sigma_{NLO} \) is found using the kernel \( \tilde{\chi}_1(M, N) \), which reduces to \( \chi_1(M) \) as \( N \to 0 \) in eqs. (3.18) or (3.19), while the collinear and anticollinear terms are given by \( \chi_{ss} \) eq. (2.19), with argument \( M^{-1} \hat{\alpha}_s \) in the collinear and \( \hat{\alpha}_s M^{-1} \) in the anticollinear terms, and off-shell extension as in eqs. (3.18), (3.19).

Let us now turn to the running-coupling resummation. First, we must determine the quadratic approximation to the NLO kernel obtained by putting on shell \( \tilde{\chi}_\Sigma \) eq. (6.1). Operator ordering issues are only relevant in the leading-order terms, as otherwise they

\(^1\) Strictly speaking, at the running coupling level the expression eq. (2.19) of \( \chi_{ss} \) is corrected by \( O(\beta_0) \) terms which are of the same order as \( \chi_{ss} \) (5.10). However, since our final aim is the determination of the anomalous dimension, we will neglect these terms, which do not affect the final double-leading anomalous dimension, but only the NLO symmetrization.
lead to sub-subleading corrections, whereas different choices of operator ordering in $\bar{\chi}_\Sigma LO$ lead to terms of order $\bar{\chi}_\Sigma NLO$.

The leading–order kernel $\bar{\chi}_\Sigma LO$ is determined with the canonical ordering eq. (5.12), whereas the quadratic kernel eq. (4.2) used in the running-coupling BFKL equation (4.5) is based on the ordering eq. (6.4), with all $\hat{\alpha}_s$ to the left. The quadratic approximation thus receives two contributions due to operator reordering. The first is due to reordering in $\bar{\chi}_0(M, N)$ eq. (5.13), analogous to eq. (6.4):

$$\bar{\chi}_0(\hat{\alpha}_s, M, N) = \hat{\alpha}_s \bar{\chi}_0(M, N) + \hat{\alpha}_s^2 \chi_1^{\beta_0}(M, N) + O(\beta_0^2 \hat{\alpha}_s^2)$$

$$\chi_1^{\beta_0}(M, N) = -\beta_0 \frac{\hat{s}}{\pi} \left( \psi(1 - M + \frac{N}{2}) - \frac{1}{(1 - M + \frac{N}{2})^2} \right) = -\beta_0 \frac{\hat{s}}{\pi} \psi'(2 - M + \frac{N}{2}),$$

(6.19)

where $\bar{\chi}_0(M, N)$ is given by eq. (3.14).

The second is due to ordering in the collinear and anticollinear contributions $\chi_s$ in eq. (5.12), which can be determined by expanding about $M = \frac{1}{2}$, just as in eq. (5.14), but now retaining commutator contributions up to order $\hat{\alpha}_s$. Note that, because the symmetry is broken by operator ordering, we now also get a cubic contribution. We obtain

$$\chi_s(\hat{\alpha}_s(M + \frac{N}{2})^{-1}) + \chi_s((1 - M + \frac{N}{2})^{-1}) = \bar{\chi}_s(\hat{\alpha}_s, M, N) + \chi_s^{\beta_0}(\hat{\alpha}_s, M, N) + O((\hat{\alpha}_s \beta_0)^2)$$

(6.20)

where $\bar{\chi}_s(\hat{\alpha}_s, M, N)$ is given by eq. (5.16) and the commutator correction is

$$\chi_s^{\beta_0}(\hat{\alpha}_s, M, N) = \bar{c}^{\beta_0}(\hat{\alpha}_s, N) + \bar{\delta}^{\beta_0}(\hat{\alpha}_s, N) \left( M - \frac{1}{2} \right) + \frac{1}{2} \bar{\kappa}^{\beta_0}(\hat{\alpha}_s, N) \left( M - \frac{1}{2} \right)^2 + O((M - \frac{1}{2})^3),$$

(6.21)

where

$$\bar{c}^{\beta_0}(\hat{\alpha}_s, N) = -2\beta_0 \frac{\hat{s}}{1 + N} c' \left( \frac{\hat{\alpha}_s}{1 + N} \right),$$

$$\bar{\delta}^{\beta_0}(\hat{\alpha}_s, N) = -\beta_0 \frac{\hat{s}}{(1 + N)^2} \delta' \left( \frac{\hat{\alpha}_s}{1 + N} \right),$$

$$\bar{\kappa}^{\beta_0}(\hat{\alpha}_s, N) = -2\beta_0 \frac{\hat{s}}{(1 + N)^3} \left[ 8c' \left( \frac{\hat{\alpha}_s}{1 + N} \right) - 4\delta' \left( \frac{\hat{\alpha}_s}{1 + N} \right) + \kappa' \left( \frac{\hat{\alpha}_s}{1 + N} \right) \right],$$

(6.22)

where the functions $c(\alpha_s)$, $\delta(\alpha_s)$ and $\kappa(\alpha_s)$ are defined in eq. (5.18), and the prime denotes differentiation with respect to their argument.

In summary, the quadratic approximation to the operator $\bar{\chi}_\Sigma LO(\hat{\alpha}_s, M, N)$ eq. (5.12) is equal to that which is found when $\alpha_s$ is an ordinary commuting function, plus the two commutator corrections $\chi_1^{\beta_0}$ eq. (6.19) and $\chi_s^{\beta_0}$ eq. (6.21), up to $O((\hat{\alpha}_s \beta_0)^2)$ corrections. Note that because of the cubic term, the minimum is shifted away from $M = \frac{1}{2}$. For instance, the minimum of the symmetrized collinear contributions is shifted to $M_0$ given by

$$M_0 = \frac{1}{2} - \frac{\bar{\delta}^{\beta_0}(\alpha_s, N)}{\bar{\kappa}(\alpha_s, N) + \bar{\kappa}^{\beta_0}(\alpha_s, N)},$$

(6.23)
Re-expanding the symmetrized collinear contribution about \( M_0 \) one gets

\[
\bar{\chi}_s^q(\hat{\alpha}_s, M, N) + \chi_s^{\beta_0}(M, N) = \bar{c}(\hat{\alpha}_s, N) + \bar{c}^{\beta_0}(\hat{\alpha}_s, N) - \frac{1}{2} \left( \bar{\alpha}_s^{\beta_0}(\alpha_s, N)^2 \right) \\
+ \frac{1}{2}(\bar{\kappa}(\hat{\alpha}_s, N) + \bar{\kappa}^{\beta_0}(\alpha_s, N))(M - M_0)^2 + O((M - M_0)^3)
\] (6.24)

Because operator ordering is immaterial in subleading contributions, the quadratic approximation at NLO can be obtained by simply adding the leading-order commutator corrections \( \chi_1^{\beta_0} \) eq. (6.19) and \( \chi_s^{\beta_0} \) eq. (6.21) to the NLO kernel \( \bar{\chi}_\sigma(\hat{\alpha}_s, M, N) \) eq. (6.1). The quadratic kernel can then be determined in two equivalent ways. One possibility is to expand \( \bar{\chi}_\sigma(\hat{\alpha}_s, M, N) \) about \( M = \frac{1}{2} \), and determine the shifted minimum due to \( \beta_0 \) corrections and the expansion about it using eq. (6.24), with \( \bar{c} \) and \( \bar{\kappa} \) replaced by the intercept and curvature of the quadratic expansion of \( \bar{\chi}_\sigma(\hat{\alpha}_s, M, N) \) about \( M = \frac{1}{2} \), and \( \bar{c}^{\beta_0} \), \( \bar{\kappa}^{\beta_0} \) and \( \bar{\kappa}^{\beta_0} \) replaced by the corresponding parameters of the quadratic expansion of \( \chi_1^{\beta_0} + \chi_s^{\beta_0} \). This then gives a result of the form of eq. (6.24), which can be put on shell by letting \( N = \chi_\sigma(\alpha_s, M) \), in turn obtained putting on shell the kernel \( \chi_\sigma(\alpha_s, M, N) \) eq. (6.2). A simpler option for the sake of numerical applications (which we will adopt for the computation of resummed anomalous dimensions and splitting functions) is to just put on shell the kernel

\[
\chi_\sigma(\alpha_s, M, N) + \hat{\alpha}_s^2 \chi_1^{\beta_0}(M, N) + \chi_s^{\beta_0}(M, N),
\] (6.25)

and then determine numerically the location of its minimum \( M_0 \) and the quadratic approximation about it. Of course, the two procedures are equivalent up to subleading corrections.

All the discussion so far applies to the unintegrated distribution, which evolves with a symmetric kernel (provided the symmetric operator ordering on the left-hand side of eq. (5.14) is adopted). It is easy to switch to the kernel for the unintegrated distribution by using eq. (5.7) in reverse. Specifically, given the quadratic approximation

\[
\bar{\chi}_\sigma^q(\hat{\alpha}_s, M) = \bar{c}_\sigma(\hat{\alpha}_s, N) + \frac{1}{2} \bar{\kappa}_\sigma(\hat{\alpha}_s, N) (M - \frac{1}{2})^2
\] (6.26)

to the kernel obtained putting on–shell the symmetric–variable kernel \( \bar{\chi}_\sigma(\hat{\alpha}_s, M, N) \) eq. (6.1), switching to the integrated distribution generates a further \( O(\hat{\alpha}_s^{2\beta_0}) \) correction to the kernel of the form

\[
\chi_1^{\beta_0} = c_i^{\beta_0}(\hat{\alpha}_s, N, M) + \frac{1}{2} \kappa_i^{\beta_0}(\hat{\alpha}_s, N, M) (M - \frac{1}{2})^2,
\] (6.27)

with

\[
c_i^{\beta_0}(\hat{\alpha}_s, N, M) = (M + \frac{N}{2})^{-1} [c_\sigma(\hat{\alpha}_s, N), M] \\
= \hat{\alpha}_s^2 \beta_0 (M + \frac{N}{2})^{-1} \frac{\partial c_\sigma(\hat{\alpha}_s, N)}{\partial \hat{\alpha}_s},
\]

\[
\kappa_i^{\beta_0}(\hat{\alpha}_s, N, M) = (M + \frac{N}{2})^{-1} [\kappa_\sigma(\hat{\alpha}_s, N), M] \\
= \hat{\alpha}_s^2 \beta_0 (M + \frac{N}{2})^{-1} \frac{\partial \kappa_\sigma(\hat{\alpha}_s, N)}{\partial \hat{\alpha}_s},
\] (6.28)
where we have taken into account the fact that eq. (5.7) is expressed in asymmetric variables so a further shift is necessary to bring it to symmetric variables as in eq. (5.12). Because the coefficients $c^\beta_0$ and $\kappa^\beta_0$ are $M$–dependent, they must be expanded about $M = \frac{1}{2}$. Using this expansion in eq. (6.27) and keeping terms up to second order, leads to a set of contributions to the intercept, linear term and curvature which must be added to those of eq. (6.22), and treated in the same way in order to obtain the quadratic approximation to the kernel for the integrated distribution.

Once the coefficients of the quadratic expansion about $M_0$ have been determined, they can be used to determine the Bateman anomalous dimension eq. (4.23), as well as its asymptotic expansion in the $\beta_0 \to 0$ limit eqs (4.26) and (4.27). We thus obtain the full NLO resummed anomalous dimension, as discussed in ref. [7,8] and summarized in sect. 2, by determining the perturbative running-coupling dual anomalous dimension $\gamma_{\sigma\,NLO}$, and then combining it with the Bateman resummation determined in the quadratic approximation, and subtracting the double counting.

The perturbative running coupling dual of $\chi_\Sigma(\alpha_s, M, N)$ eq. (6.1) is constructed as follows. We start from the observation that [7] the dual of the NLO double–leading $\gamma$ coincides with the NLO double–leading $\gamma$, up to subleading terms. However, recall that in order to determine the anomalous dimension we must first switch back to the integrated gluon distribution. Upon this transformation, $\chi_s$ changes according to eq. (5.7), while for $\chi_0$ we have

$$\hat{\alpha}_s \chi_0(M) G(\xi, M) = \hat{\alpha}_s \chi_0(M) M G(\xi, M) = M \left( \hat{\alpha}_s \chi_0(M) + \hat{\alpha}_s^2 \beta_0 \frac{\chi_0(M)}{M} + O(\hat{\alpha}_s^3) \right) G(\xi, M).$$  \hspace{1cm} (6.29)

Now, we note that the dual anomalous dimension to the kernel $\chi_s(\hat{\alpha}_s M^{-1}) + \hat{\alpha}_s \chi_{ss}(\hat{\alpha}_s M^{-1})$ which we get after switching back to the integrated level is just, by construction, the NLO anomalous dimension as given by eq. (2.3) up to order $\alpha_s^2$. On the other hand, the dual of the $O(\hat{\alpha}_s^2)$ kernel eq. (6.10) is given by eq. (2.20), supplemented by the $O(\hat{\alpha}_s^2)$ term of eq. (6.29). Note that the result holds with the operator ordering of eq. (6.10). The $O(\hat{\alpha}_s)$ contribution to $\chi_\sigma$ can be brought to this form by using eq. (6.11), and subtracting the double pole from the $\beta_0$ term of eq. (6.11), which is a double–counting contribution between $\chi_1$ and $\chi_s((1-M)^{-1}\hat{\alpha}_s)$. In practice, this means that the $\beta_0$ term of eq. (6.11) should be replaced by $\chi_1^\beta_0$ eq. (6.19).

Putting everything together, the perturbative running coupling dual to $\chi_\sigma\,NLO(M)$ differs from the fixed-coupling dual in three respects: first, the $O(\hat{\alpha}_s)$ contribution to $\chi_\Sigma$ must be reordered so that $\hat{\alpha}_s$ is to the left, and second it must be supplemented by the contribution eq. (6.29) which switches to the integrated distribution; finally, the fixed coupling duality must be corrected by the $O(\beta_0)$ term of eq. (2.20). We thus get

$$\gamma_{\Sigma\,NLO}^{rc, pert}(\alpha_s(t), N) = \chi_{\Sigma\,NLO}(\alpha_s(t), N) - \beta_0 \alpha_s(t) \left[ \chi_0''(\alpha_s(t)/N) \chi_0(\gamma_s(\alpha_s(t)/N)) \right] \frac{\chi_0''(\alpha_s(t)/N) \chi_0(\gamma_s(\alpha_s(t)/N))}{2 [\chi_0(\gamma_s(\alpha_s(t)/N))]^2} - 1,$$  \hspace{1cm} (6.30)

where the subtraction in the term in square brackets subtracts the double-counting between this term and $\gamma_0$, i.e. it is part of the standard subtraction $e_0$ in eq. (22) of ref. [5].
anomalous dimension $\gamma_{\Sigma NLO}(N,\alpha_s(t))$ in eq. (6.30) is the naive (fixed coupling) dual eq. (2.9) of $\chi_{\Sigma NLO}$ obtained by putting on shell the kernel in asymmetric variables,

$$\tilde{\chi}_{\Sigma NLO}(M - \frac{N}{2}, N) + \hat{\alpha}_s^2 \chi_1(M - \frac{N}{2}, N) + \hat{\alpha}_s^2 \beta_0 \left( \frac{\chi_0(M)}{M} - \frac{n_c}{\pi M^2} \right), \quad (6.31)$$

where $\tilde{\chi}_{\Sigma NLO}(M, N)$ is given by eq. (6.1) as discussed in this section, $\chi_1(M)$ is given by eq. (6.19), and the term in brackets is the transformation to the kernel for the integrated distribution, with the double counting between it and $\chi_s(M^{-1} \hat{\alpha}_s)$ subtracted.

We finally obtain the full resummed anomalous dimension by combining this result with the Bateman running coupling resummation:

$$\gamma_{\Sigma NLO}^{rc}(\alpha_s(t), N) = \gamma_{\Sigma NLO}^{rc, pert}(\alpha_s(t), N) + \gamma^B(\alpha_s(t), N) - \gamma^B_s(\alpha_s(t), N)$$

$$- \gamma^B_s(\alpha_s(t), N) + \gamma^B_{ss,0}(\alpha_s(t), N) + \gamma_{match}(\alpha_s(t), N) + \gamma_{mom}(\alpha_s(t), N). \quad (6.32)$$

In this equation, the Bateman anomalous dimension is given by eq. (4.17), computed with the curvature $c$ and intercept $\kappa$ determined from the quadratic expansion of the on-shell form of the kernel eq. (6.25) discussed above. However, in order to ensure complete cancellation of the singularities, the asymptotic double counting terms $\gamma^B_s(\alpha_s(t), N)$, $\gamma^B_{ss}(\alpha_s(t), N)$ and $\gamma^B_{ss,0}(\alpha_s(t), N)$ should be determined with values of the parameters $c$ and $\kappa$ which differ from those used in $\gamma^B$ itself by sub-subleading terms. Specifically, $\gamma^B_s(\alpha_s(t), N)$ should be computed with the values of $c$ and $\kappa$ which characterize $\gamma_{\Sigma NLO}$ eq. (6.30), namely, the intercept and curvature of the quadratic expansion of the kernel $\chi(\gamma_{\Sigma NLO})$ related to $\gamma_{\Sigma NLO}$ by fixed-coupling duality eq. (2.9). These differ from the values of $c^\beta_0$ of $\kappa^\beta_0$ of the quadratic expansion because of the contribution to the latter from $\chi_s^\beta_0$ eq. (6.21). In practice, we eliminate this sub-subleading singularity mismatch through the term

$$\gamma_{match}(\alpha_s(t), N)$$

$$= \sqrt{N - c} + c \sqrt{\frac{N}{2} - c^\beta_0} - \sqrt{\frac{N}{2} - \frac{c^\beta_0}{2}}$$

$$- \sqrt{\frac{N}{2}} + c \sqrt{\frac{N}{2} - \frac{c^\beta_0}{2}} + \frac{1 + c}{\sqrt{2\kappa(N + 1)}} - \frac{1 + c}{\sqrt{2\kappa(N + 1)}} : \quad (6.33)$$

the first two terms shift the square root cut, and the remaining ones ensure that at large $N$ the resummed anomalous dimension eq.(6.32) reproduces the NLO GLAP result. Furthermore, $\gamma^B_s(\alpha_s(t), N)$ is computed with the intercept and curvature of the leading order BFKL kernel $\chi_0$ eq. (3.8), which ensures cancellation of the pole at $M = \frac{1}{2}$ of the $O(\beta_0)$ term in eq. (6.30) which motivates the running coupling resummation, as discussed in refs. [7,8] and summarized in sect. 2. The values used in $\gamma^B$ differ from these due to the inclusion of all the subleading corrections discussed above, but this is immaterial because $\gamma^B_{ss}$ is already subleading. Finally, the term $\gamma_{mom}$ serves the purpose of improving the matching to large $N$, by removing terms which survive at the momentum-conservation point $N = 1$. These terms are of course sub-subleading, so they could be in principle omitted. In practice $\gamma_{mom}$ is constructed as in eq. (5.21), with $d_m$ readjusted to ensure that $\gamma_{SS NLO}(1) = 0$. 

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7. Results for the Anomalous Dimension and the Splitting Function

We are now finally in a position to collect and discuss our results. Our goal was to derive for the singlet anomalous dimension and for the associated splitting function expressions that at large $N$ or large $x$ reduce to the familiar perturbative GLAP results, and at small $N$ or small $x$ are improved by including in a suitably resummed form the information from the BFKL kernels with a procedure that takes into full account the constraints from momentum conservation and symmetrisation and includes running coupling effects. As mentioned in the introduction, all results presented in this section are computed with $n_f = 0$ in order to minimize the impact of issues of scheme dependence.

The results for the anomalous dimension are presented in figs. 7-8. In fig. 7 our most accurate prediction, denoted ‘NLO rc res’, is compared with the LO, NLO and NNLO GLAP perturbative results and with the DL LO approximation. The curve labelled ‘NLO rc res’ shows the anomalous dimension, which includes the information from the NLO BFKL kernel and perturbative anomalous dimension, to which it reduces at large $N$. It corresponds to the result given in eq. (6.32). The DL LO curve, obtained from $\gamma_{DLLO}$ given in eq. (5.1), corresponds to a naive application of the BFKL kernel $\chi_0$ to evaluate the small $N$ behaviour. The step corresponds to the cut starting at the value of $N$ which corresponds by duality to the minimum of $\chi_0$. It is quite evident that our final improved anomalous dimension is remarkably close to the GLAP NLO result down to very small values of $N$, where finally we can see a departure due to the position of the Bateman pole which is at a small positive value of $N$ and not at $N = 0$ as is the case for GLAP. The
instability of the NNLO GLAP result is due to an unresummed $\frac{\alpha_3^3}{N^2}$ pole, and it is removed by the resummation.

Figure 8 shows a magnification of the small $N$ region, and also compares the NLO resummed result to its LO counterpart eq. (5.20). The stability of the resummed perturbative expansion is manifest in the small difference between the LO and NLO results, which however are in perfect agreement with the respective GLAP curves for large $N \gtrsim 1$. At medium-small $N$ the NLO resummed curve is in fact rather closer to the unresummed one than the LO resummed. At very small $N$ the LO and NLO resummed curves are almost indistinguishable because of the closeness of the respective Bateman poles, displayed in Fig. 6, and rise somewhat more steeply than the unresummed result.

We now consider the splitting function obtained by inverse Mellin transformation from these anomalous dimension. In fig. 9 the resummed splitting function curves (labelled ‘LO rc res’ and ‘NLO rc res’) which are our main predictions at the leading and next-to-leading level, corresponding to the anomalous dimensions with the same labels in fig. 8, are compared with the LO, NLO and NNLO GLAP perturbative splitting functions (note that what is actually plotted is $xP$). The first important feature is the marked deviation of the curve NNLO GLAP from the approximate $1/x$ behaviour at small $x$ which is typical of the LO and NLO GLAP singlet splitting function. This pronounced deviation makes the need for resummation even more compelling, in the sense that it clearly shows that the ordinary perturbative expansion is unstable at small $x$. As the bulk of HERA data is concentrated at $x > 10^{-3} - 10^{-4}$ the departure from the NLO splitting functions used in most fits to
Figure 9: The LO and NLO resummed splitting function obtained by Mellin transformation of the anomalous dimensions eq. (5.20) and eq. (6.32) displayed in fig. 8 (labelled ‘rc res’) are compared to the LO, NLO and NNLO GLAP perturbative splitting functions (note that the plot shows $xP$).

the data is not too large, but, for the perturbative results, the difference NNLO-NLO is much larger than NLO-LO in the whole range of small $x$. The resummed NLO prediction, which constitutes our main result, closely follows the NLO GLAP curve down to $x$ values as small as $x \sim 10^{-5}$, thus explaining the success of the NLO perturbative QCD fits of the HERA data. In comparison the LO resummed prediction shows more pronounced deviations from the perturbative splitting function. It is only at very small $x$, $x < 10^{-6}$, that there starts to be a noticeable difference between the perturbative and the resummed splitting functions. This late take-over of the BFKL regime, softened by the resummation procedure, is due to the small residue of the Bateman pole.

In fig. 10 we compare our main results for the splitting function, i.e. the LO and NLO resummed curves of fig. 9, to the result obtained in our previous work, ref. [8], denoted by ‘LO lin rc res’ and with the result (labeled ‘NLO CCSS’) which was obtained by the authors of ref.[11] by a different technique based on the same physical ingredients. The prediction of ref. [8] is a simple analytical result that can be described as the best approximation that can be derived if only the 1-loop kernels $\gamma_0$ and $\chi_0$ are known. Note that, since $\chi_0$ is already symmetric for $M \to (1-M)$, there is no need of symmetrisation, while the result shown includes collinear resummation, momentum conservation and running coupling effects. The rise at small $x$ is steeper and takes off at larger $x$ than for the more refined results obtained in this paper and in ref. [11], because the symmetrised resummed DL kernels displayed in figs. 1 and 3 have a smaller value at the minimum and are much flatter than $\chi_0$. The main
Figure 10: The LO and NLO resummed splitting function of fig. 9 are compared to the result obtained in our previous work, ref. [8], denoted by ‘LO lin rc res’ and to the result of ref. [11], labeled ‘NLO CCSS’ advantage of the new result, however, is its perturbative stability, i.e. the closeness of LO and NLO results.

The approach of ref. [11], although based on similar physical principles, differs in many respects from ours, most notably in the fact that the whole approach is based on the BFKL equation, and attempts to derive the full gluon Green function, instead of focussing on the determination of anomalous dimensions, as in our approach, which treats the $\chi$ BFKL kernel and $\gamma$ anomalous dimension on the same footing. It is therefore to some extent unexpected that our NLO resummed result at small $x$ is very close to the NLO CCSS splitting function. In fact, the very small difference between our results and those of CCSS is surely accidental, given the uncertainty involved in various subleading resummation terms. Note that even though the approach of ref. [11] does not yet allow for the inclusion of NLO GLAP terms when $n_f \neq 0$, the CCSS curve of fig. 10, taken from refs. [12] and computed with $n_f = 0$, does include a NLO GLAP contribution, which when $n_f = 0$ vanishes as $x \to 0$.

A feeling for the uncertainty of our results can be obtained by determining their dependence on the choice scale $Q^2$, which in turn may be estimated from the dependence of the position of the rightmost singularity $N_B$ of the resummed anomalous dimensions eq. (5.20),(6.32) on the value of $\alpha_s$, which was displayed in fig. 6: the value of $N_B$ determines the asymptotic small $x$ behaviour of the splitting function $xP \sim x^{-N_B}$. We can also use this dependence to estimate the uncertainty on our results due to subleading higher order terms. To this purpose, we study the dependence of the value of $N_B(\alpha_s)$ on
the choice of renormalization scale. At leading order, we plot $N_B(\alpha_s(kQ^2))$, with $N_B$ the rightmost pole of the leading order $\gamma_{\sigma}^{rc}$ eq. (5.20), while at next-to-leading order we plot

$$N_B(k, Q^2) = N_B(\alpha_s(kQ^2)) + \beta_0 \ln k \alpha_s^2(Q^2) \frac{\partial}{\partial \alpha_s} N_B(\alpha_s(Q^2)), \quad (7.1)$$

with $N_B$ the rightmost pole of the next-to-leading order $\gamma_{\sigma}^{rc, NLO}$ eq. (6.32). In fig. 11 we compare the values of the leading order and next-to-leading order $N_B$ as a function of $\alpha_s(kQ^2)$ for $k = 1$, $k = 4$ and $k = \frac{1}{4}$, while in fig. 12 we show the dependence of these quantities on $k$ when $Q^2$ is chosen so that $\alpha_s(Q^2) = 0.2$. We see that the scale dependence of the leading–order result is about as large as the difference between leading and next-to-leading order results, whereas the scale dependence of the next-leading–order result is smaller by about a factor four. This is somewhat larger than the difference between our NLO result and that of ref. [11], as shown in fig. 10. Also, it appears from fig. 12 that whereas at leading order $N_B$ is monotonic as a function of the scale parameter $k$, at next-to-leading order the dependence of $N_B$ on $k$ is stationary around $k \sim 1$, confirming that indeed $Q^2$ is the appropriate scale for this process. We conclude from the significantly reduced dependence on renormalization scale at NLO that we have indeed succeeded in constructing a stable resummed perturbative expansion in the small $x$ region. If we were to go to NNLO in this expansion, we would expect the scale dependence to be reduced still further.
8. Conclusion

In this paper we have presented in detail a complete procedure to construct an improved singlet anomalous dimension or splitting function that reduces at large $x$ to the NLO perturbative approximation but also contains all resummed small $x$ improvements from NLO BFKL kernels. The problem of explaining the apparent smallness of the small $x$ corrections in the HERA data region in spite of the wild behaviour of the BFKL perturbative expansion is solved. The main physical reason for the moderate impact of the small $x$ corrections is the duality relation that implies that in the large domain where both $Q^2$ and $1/x$ are large the GLAP and the BFKL expansions are both valid and know about each other. Thus the corresponding leading twist terms must match and indeed a better convergence is obtained by the ‘double leading (DL)’ expansion where at each level the BFKL (GLAP) perturbative terms are used to resum the corrections to the GLAP (BFKL) kernel in a symmetric way. The constraints from momentum conservation and from the underlying symmetry of the gluon-gluon BFKL scattering kernel under interchange of the two gluons are used to construct a stable and physically motivated expansion at each level. An important effect is also obtained from the running coupling corrections which are large and considerably soften the small $x$ asymptotic behaviour. The technical implementation of the running coupling corrections in our approach is realized via a quadratic approximation of the BFKL kernel in the DL approximation near its central minimum followed by the analytic solution of the corresponding differential equation in terms of a Bateman function. The systematic use of duality and the Bateman method are the main qualitative
differences with respect to the approach of ref. [11] which uses a different path to reach similar conclusions from the same basic physical principles.

The main results of our work for the singlet anomalous dimension and the corresponding splitting function are displayed in figs. 7-11 where the comparison with the results of ref. [11] is also shown. In the region of the bulk of HERA data for \( x > 10^{-4} \) the corrected splitting function at NLO level shows a moderate depletion with respect to the perturbative NLO approximation. At still smaller values of \( x \) the asymptotic regime eventually takes over and a powerlike increase of the splitting function is observed, though at a much lower rate than that naively expected from the value of the Lipatov exponent.

We recall once again that to make direct connection to the structure functions we need to include the coefficient functions. This in principle is relatively easy, because the coefficient functions are available, even though properly accounting for scheme choices, diagonalization of the anomalous dimension matrix, and matching to large \( x \) coefficient functions is a laborious task, as discussed in ref. [6]. The effects of the coefficient functions are subdominant in the factorization scheme adopted here, but they deserve a full investigation.

In conclusion we think that by now the problem of the small \( x \) behaviour of the singlet splitting functions has been well understood in its physical principles and that the technical tools have been satisfactorily developed. The next step is to make a direct comparison of the resulting theoretical predictions with the data.

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**Appendix A**

To evaluate \( \chi_1(M) \) we use the result

\[
\varphi(M) \equiv -\int_0^1 \frac{dx}{1+x} (x^{M-1} + x^{-M}) \int_x^1 \frac{dt}{t} \ln(1-t) = \frac{\pi^3}{6 \sin \pi M} - \frac{\pi^2}{6M(1-M)} - \frac{1}{M^2} (\psi(1) - \psi(1+M)) - \frac{1}{(1-M)^2} (\psi(1) - \psi(2-M)) + \varphi_s(M),
\]

(A.1)

where

\[
\varphi_s(M) = \sum_{r=1}^{\infty} c_r (M - \frac{1}{2})^{2(r-1)},
\]

(A.2)

and

\[
c_r = -2 \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-)^m}{k^2} \frac{1}{(m+k+\frac{1}{2})^{2r-1}},
\]

(A.3)
so the sum over \( r \) has infinite radius of convergence. If we further note that

\[
\frac{2\pi}{\sin \pi M} = \psi(\frac{1}{2} + \frac{M}{2}) - \psi(\frac{M}{2}) + (M \leftrightarrow 1 - M),
\]

(A.4)

\[
\frac{4\pi^2 \cos \pi M}{\sin^2 \pi M} = \psi'(\frac{M}{2}) - \psi'(\frac{1}{2} + \frac{M}{2}) - (M \leftrightarrow 1 - M),
\]

(A.5)

then we may then use eq. (14) of ref. [16] to write \( \chi_1(M) = \frac{n^2}{4\pi^2}\delta(M) - \beta_0 \frac{n}{\pi} \psi'(M) \) as

\[
\chi_1(M) = -\frac{1}{2} \beta_0 \frac{n}{\pi} \left( \frac{n^2}{n_c^2} \chi_0(M)^2 - \psi'(M) - \psi'(1 - M) \right)
\]

\[\]

\[\]

+ \frac{n^2}{4\pi^2} \left[ \left( \frac{67}{9} - \frac{22}{9n_c} \right) (\psi(1) - \psi(M)) + 3 \zeta(3) + \psi''(M) \right]

+ 4 \left( \psi(\frac{1}{2} + \frac{M}{2}) - \psi(\frac{M}{2}) \right) + \frac{6\pi^2}{M^2} + \frac{1}{3M^2} (\psi(1) - \psi(1 + M) - \frac{1}{2} \varphi_s(M))

+ \frac{1}{4(1 - 2M)} \left( 3 + \frac{2n}{n_c^2} \frac{(2 + 3M(1 - M))}{(3 - 2M)(1 + 2M)} \right) \left( \psi'(\frac{1}{2} + \frac{M}{2}) - \psi'(\frac{M}{2}) \right)

+ (M \leftrightarrow 1 - M) \right].

(A.6)

The advantage of using this expression is that all terms in square brackets are regular for \( \text{Re} \, M > \frac{1}{2} \) (note that the poles at \( M = \frac{1}{2}, M = \frac{3}{2} \) and \( M = -\frac{1}{2} \) in the last term are all removable: they cancel against similar poles in the \( M \to 1 - M \) piece). Thus we may extend them off-shell following the same procedure as at LO. The first term may be treated by the simple observation that up to subleading terms we may substitute \( \alpha_s \chi_0(M) = N \), which is trivially extended off-shell as \( N = \alpha_s \bar{\chi}_0(M, N) \). The final result for \( \bar{\chi}_1(M, N) \) is then (in asymmetric variables for simplicity)

\[
\bar{\chi}_1(M - \frac{N}{2}, N) = -\frac{1}{2} \beta_0 \frac{n}{\pi} \left( \frac{n^2}{n_c^2} \bar{\chi}_0(M - \frac{N}{2}, N)^2 - \psi'(M) - \psi'(1 - M + N) \right)
\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

+ \frac{n^2}{4\pi^2} \left[ \left( \frac{67}{9} - \frac{22}{9n_c} \right) \left\{ \frac{1}{2} (\psi(1) + \psi(1 + N)) - \psi(M) \right\} + 3 \zeta(3) + \psi''(M) \right]

+ 4 \left\{ \pi^2 (\psi(\frac{1}{2} + \frac{M}{2}) - \psi(\frac{M}{2})) + \frac{6\pi^2}{M^2} + \frac{1}{3M^2} (\psi(1) - \psi(1 + M) - \frac{1}{2} \varphi_s(M)) \right\}

+ \frac{3}{4(1 - 2M)} \left( \psi'(\frac{1}{2} + \frac{M}{2}) - \psi'(\frac{M}{2}) + \psi'(\frac{3}{4}) - \psi'(\frac{5}{4}) \right)

+ \frac{1}{16} \left( 1 + \frac{2n}{n_c^2} \right) \left( 2 + 3M(1 - M) \right) \left\{ \frac{1}{(1 - 2M)} (\psi'(\frac{1}{2} + \frac{M}{2}) - \psi'(\frac{M}{2}) + \psi'(\frac{3}{4}) - \psi'(\frac{5}{4})) \right\}

+ \frac{1}{2(1 + 2M)} (\psi'(\frac{1}{2} + \frac{M}{2}) - \psi'(\frac{M}{2}) + \psi'(\frac{3}{4}) - \psi'(\frac{5}{4}))

+ \frac{1}{2(3 - 2M)} (\psi'(\frac{1}{2} + \frac{M}{2}) - \psi'(\frac{M}{2}) + \psi'(\frac{3}{4}) - \psi'(\frac{5}{4})) \right\}

+ \frac{3}{512} (1 + \frac{2n}{n_c^2}) (\psi'(\frac{1}{4}) - \psi'(\frac{5}{4}) + 3\psi'(\frac{3}{4}) - 3\psi'(\frac{1}{4})) \right\}

+ (M \leftrightarrow 1 - M + N) \].

(A.7)

where in the term in curly brackets we have taken care to ensure that we do not spoil the cancellation of the poles at \( M = \pm \frac{1}{2}, M = \frac{3}{2} \) by adding in subleading (ie \( O(N) \)) terms:
the growth of these extra terms at large $N$ must then be subtracted off again, and this is done in the last line. The result in symmetric variables may be found by the simple change of variables $M \to M + \frac{N}{2}$.

The off-shell extension of eq. (6.14) is then straightforward:

$$\tilde{\chi}_1(M, N) = \bar{\chi}_1(M, N) - \frac{1}{2} \bar{\chi}_0(M, N) \frac{n_c}{\pi} \left[ 2\psi'(1) - \psi'(M + \frac{N}{2}) - \psi'(1 - M + \frac{N}{2}) \right]. \quad (A.8)$$

The collinear subtraction eqn(6.15) is then

$$\bar{\chi}_1(M, N) = \tilde{\chi}_1(M, N) - g_1 \left( \frac{1}{M + \frac{N}{2}} + \frac{1}{1 - M + \frac{N}{2}} \right) - g_2 \left( \frac{1}{(M + \frac{N}{2})^2} + \frac{1}{(1 - M + \frac{N}{2})^2} \right), \quad (A.9)$$

with coefficients $g_i$ as before, eq. (6.16), which render in particular $\tilde{\chi}_1(M - \frac{N}{2}, N)$ finite in the collinear limit $M \to 0$. To improve the matching to GLAP at large $N$, we may further add to $\tilde{\chi}_1(M - \frac{N}{2}, N)$ a manifestly subleading contribution $-\tilde{\chi}_1(-\frac{N}{2}, N) + \tilde{\chi}_1(0, 0)$ to remove spurious constant terms at large $N$, and thus spurious $O(1/N)$ terms in the anomalous dimension.
References

[1] M. Dittmar et al., hep-ph/0511119.
[2] S. Moch, J. A. M. Vermaseren and A. Vogt, Nucl. Phys. B 691 (2004) 129;
A. Vogt, S. Moch and J. A. M. Vermaseren, Nucl. Phys. B 688 (2004) 101.
[3] See e.g. R. S. Thorne, A. D. Martin, R. G. Roberts and W. J. Stirling, hep-
ph/0507013.
[4] R. D. Ball and S. Forte, Phys. Lett. B465 (1999) 271.
[5] G. Altarelli, R. D. Ball and S. Forte, Nucl. Phys. B575, 313 (2000); see also hep-
ph/0001157.
[6] G. Altarelli, R. D. Ball and S. Forte, Nucl. Phys. B599 (2001) 383; see also hep-
ph/0104246.
[7] G. Altarelli, R. D. Ball and S. Forte, Nucl. Phys. B 621 (2002) 359.
[8] G. Altarelli, R. D. Ball and S. Forte, Nucl. Phys. B 674 (2003) 459; see also hep-
ph/0310016.
[9] G. Salam, Jour. High Energy Phys. 9807 (1998) 19.
[10] M. Ciafaloni, D. Colferai and G. P. Salam, Phys. Rev. D 60 (1999) 114036.
[11] M. Ciafaloni, D. Colferai, G. P. Salam and A. M. Stasto, Phys. Rev. D 66 (2002)
054014; Phys. Lett. B 576 (2003) 143; Phys. Rev. D 68 (2003) 114003.
[12] G. Altarelli et al., “Resummation”, in M. Dittmar et al., hep-ph/0511119.
[13] G. Altarelli, R. D. Ball and S. Forte, Nucl. Phys. Proc. Suppl. 135 (2004) 163
[14] M. Ciafaloni, D. Colferai, G. P. Salam and A. M. Stasto, Phys. Lett. B 587 (2004)
87; see also G. P. Salam, hep-ph/0501097.
[15] M. Ciafaloni, Phys. Lett. B 356, 74 (1995).
[16] V.S. Fadin and L.N. Lipatov, Phys. Lett. B429 (1998) 127.
[17] V.S. Fadin et al, Phys. Lett. B359 (1995) 181; Phys. Lett. B387 (1996) 593;
Nucl. Phys. B406 (1993) 259; Phys. Rev. D50 (1994) 5893; Phys. Lett. B389 (1996)
737; Nucl. Phys. B477 (1996) 767; Phys. Lett. B415 (1997) 97; Phys. Lett. B422
(1998) 287.
[18] V. del Duca, Phys. Rev. D54 (1996) 989;Phys. Rev. D54 (1996) 4474;
V. del Duca and C.R. Schmidt, Phys. Rev. D57 (1998) 4069;
Z. Bern, V. del Duca and C.R. Schmidt, Phys. Lett. B445 (1998) 168.
[19] G. Camici and M. Ciafaloni, Phys. Lett. B412 (1997) 396; Phys. Lett. B430 (1998)
349.
[20] R. D. Ball and S. Forte, Phys. Lett. B405 (1997) 317.
[21] R. D. Ball and S. Forte, Phys. Lett. B335 (1994) 77; B336 (1994) 77;
Acta Phys. Polon. B26 (1995) 2097.
[22] R. D. Ball and S. Forte, Phys. Lett. B351 (1995) 313;
    R.K. Ellis, F. Hautmann and B.R. Webber, Phys. Lett. B348 (1995) 582.
[23] R. D. Ball and S. Forte, hep-ph/9607291;
    I. Bojak and M. Ernst, Phys. Lett. B397 (1997) 296; Nucl. Phys. B508 (1997) 731;
    J. Blümlein and A. Vogt, Phys. Rev. D58 (1998) 014020.
[24] L.N. Lipatov, Sov. Phys. J.E.T.P. 63 (1986) 5.
[25] M. Ciafaloni and D. Colferai, Phys. Lett. B452 (1999) 372.
[26] R. D. Ball and S. Forte, hep-ph/9805313.
[27] D. A. Ross, Phys. Lett. B 431, 161 (1998).
[28] R. D. Ball and S. Forte, hep-ph/0601049.