Invariants of Lagrangian surfaces

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Abstract

We define a nonnegative integer \( \lambda(L, L_0; \phi) \) for a pair of diffeomorphic closed Lagrangian surfaces \( L_0, L \) embedded in a symplectic 4-manifold \( (M, \omega) \) and a diffeomorphism \( \phi \in \text{Diff}^+(M) \) satisfying \( \phi(L_0) = L \). We prove that if there exists \( \phi \in \text{Diff}^+_0(M) \) with \( \phi(L_0) = L \) and \( \lambda(L, L_0; \phi) = 0 \), then \( L_0, L \) are symplectomorphic. We also define a second invariant \( n(L_1, L_0; [L_t]) = n(L_1, L_0, [\phi_t]) \) for a smooth isotopy \( L_t = \phi_t(L_0) \) between two Lagrangian surfaces \( L_0 \) and \( L_1 \) with \( \lambda(L_1, L_0; \phi_1) = 0 \), which serves as an obstruction of deforming \( L_t \) to a Lagrangian isotopy with \( L_0, L_1 \) preserved.

1 Introduction

One subtle question in symplectic topology is to find the fine line between symplectic topology and differential topology. For example, what are the things that can be done diffeomorphically but not symplectically? The objects to be tested on are embedded compact Lagrangian surfaces. In their own worlds, individual Lagrangian surfaces do not know the existence of symplectic structures, until they try to communicate with each other via diffeomorphisms and/or homotopies. Diffeomorphic Lagrangian surfaces may be surprised to find that they live in quite different neighborhoods dictated by symplectic structures. Even if this is not the case, smoothly isotopic Lagrangian surfaces may still find that they are destined to meet the symplectic structure (by becoming non-Lagrangian) before they meet each other, no matter which path they choose. Such phenomena have been explored and studied by Fintushel-Stern [2] and Seidel [6] (see also [1]).

In this note we construct two invariants of embedded compact Lagrangian surfaces to address the two questions described above:

- When two diffeomorphic Lagrangian surfaces are symplectomorphic?
• When two smoothly isotopic Lagrangian surfaces are Lagrangian isotopic?

Here compact Lagrangian surfaces \( L_0, L_1 \) in a symplectic 4–manifold \((M, \omega)\) are said to be smoothly isotopic if there exists between \( L_0 \) and \( L_1 \) a smooth homotopy consists of embeddings. This is equivalent to the existence of a smooth family \( \phi_t \in \text{Diff}^+(M) \) with \( \phi_0 = \text{id} \) and \( \phi_1(L_0) = L_1 \). \( L_0, L_1 \) are Lagrangian isotopic if the smooth isotopy consists of Lagrangian surfaces.

Let \( L_0, L \) be two embedded Lagrangian surface such that \( L = \phi(L_0) \) for some \( \phi \in \text{Diff}^+(M) \). The first invariant \( \lambda(L, L_0; \phi) \) we construct is really a generalization of the \( \lambda(T) \) invariant (for Lagrangian torus \( T \)) constructed by Fintushel and Stern \[2\]. This invariant is related to the first question we mentioned above. In its most general form, it is really an invariant of a pair of diffeomorphic Lagrangian surfaces of positive genus (the invariant is trivial when the genus is 0) together with a diffeomorphism between them.

Define

\[
\text{Diff}_0^+(M; L_0 \to L) := \{ \phi \in \text{Diff}_0^+(M) \mid \phi(L_0) = L \}
\]

\[
\text{Symp}(M, \omega) := \{ \phi \in \text{Diff}^+(M) \mid \phi^* \omega = \omega \}
\]

Here \( \text{Diff}_0^+(M) \) denotes the connected component of \( \text{Diff}^+(M) \).

**Theorem 1.1.** Let \( L_0, L \) be two closed Lagrangian surfaces embedded in a symplectic 4-manifold \((M, \omega)\). Assume that \( \text{Diff}_0^+(M; L_0 \to L) \neq \emptyset \). Then \( \phi \in \text{Diff}_0^+(M; L_0 \to L) \) is homotopic in \( \text{Diff}_0^+(M; L_0 \to L) \) to some \( \psi \in \text{Symp}(M, \omega) \) iff \( \lambda(L, L_0; \phi) = 0 \).

**Remark 1.1.** The invariant \( \lambda(L, L_0; \cdot) \) actually assigned to each connected component of \( \text{Diff}_0^+(M; L_0 \to L) \) a nonnegative integer. Theorem 1.1 then says that this number is 0 iff the corresponding connected component of \( \text{Diff}_0^+(M; L_0 \to L) \) contains an element of \( \text{Symp}(M, \omega) \).

The construction of \( \lambda(L, L_0; \phi) \) implies that \( \lambda(L, L_0; \cdot) = 0 \) if \( L_0, L \) are of genus 0, i.e., if \( L_0, L \) are embedded Lagrangian spheres. Hence we have the following

**Corollary 1.1.** Let \( L_0, L \) be two embedded lagrangian spheres in a symplectic 4-manifold \((M, \omega)\). Suppose that \( L_0, L \) are smoothly isotopic, then \( L_0, L \) are symplectomorphic.

The second invariant \( n(L_1, L_0; [L_t]) \) (see Section 3) seems related to the generalized Dehn twist considered by Seidel \[6\]. It is an invariant of a smooth
homotopy between two Lagrangian surfaces, hence is related to the second question above.

The organization of this article is as follows: In Section 2 we construct a local version of \( \lambda \) on cotangent bundles \( T^*L \) and then introduce the definition of \( \lambda(L, L_0; \phi) \). The rest of Section 2 is devoted to the proof of Theorem 1.1. The invariant \( n(L_1, L_0; [L_t]) \) is constructed in Section 3 followed by some discussions and final remarks.

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2 The first invariant

2.1 Local definition of \( \lambda(L, \omega', \omega) \)

Let \( L \) be a closed orientable surface and \( T^*L \) denotes the cotangent bundles of \( L \). Let \( \lambda_{\text{can}} \) denote the canonical 1-form Let \( \omega_{\text{can}} = -d\lambda_{\text{can}} \) denote the canonical symplectic 2–form on \( T^*L \). Let \( x \) be a local coordinate on \( L \) and \( y \in \mathbb{R}^2 \) be a local coordinate on the fiber \( T^*_xL \), then \((x, y)\) are local coordinates of \( T^*L \), and \( \lambda_{\text{can}}, \omega_{\text{can}} \) are given by the formulae

\[
\lambda_{\text{can}} = ydx, \quad \omega_{\text{can}} = dx \wedge dy
\]

Let us fix an orientation for \( T^*L \) so that \( \omega_{\text{can}}^2 \) becomes a volume form on \( T^*L \). We will consider only symplectic 2–forms \( \omega \) on \( T^*L \) with \( \omega^2 > 0 \). View \( L \) as the zero section of \( T^*L \). Let \( \omega \) be be a symplectic 2–form on \( T^*L \) with \( L \) a Lagrangian surface. With local coordinates \((x, y)\) for \( T^*L \) we have the following isomorphisms

\[
T_{(x,0)}T^*L \cong T_xL \oplus T^*_xL \quad (1)
\]
\[
T_{(x,0)}^*T^*L \cong T^*_xL \oplus T_xL \quad (2)
\]

Being symplectic, \( \omega \) induces an isomorphism

\[
T_{(x,y)}T^*L \to T_{(x,y)}^*T^*L \\
v \to \omega(v, \cdot).
\]
Then by (1) and (2), \( \omega \) induces a bundle isomorphism

\[
\Omega_\omega : T_x L \to T^*_x L, \quad \Omega(v) := \omega(v, \cdot) \quad \text{for} \quad v \in T_x L.
\]

If we fix a Riemannian metric on \( L \) and let \( g \) denote the induced Riemannian metric on \( T^*L \) the cotangent bundle, then the splitting in (1) is orthogonal.

Let \( \omega' \) be another symplectic 2-form on \( T^*L \) with \( L \) Lagrangian, then

\[
\Omega_{\omega', \omega} := \Omega_{\omega'} \circ \Omega_\omega^{-1} : T^*_x L \to T^*_x L
\]

is an orientation-preserving linear automorphism for every \( x \in L \), hence is a section of the trivial \( GL(2, \mathbb{R}) \)-bundle over \( L \). By fixing a trivialization of the bundle we get

\[
\Omega_{\omega', \omega} : L \to GL(2, \mathbb{R}). \tag{3}
\]

Since \( GL(2, \mathbb{R}) \) is homotopic to \( S^1 \), the homotopy class of the map (3) is classified by

\[
[L, S^1] \cong H^1(L, \mathbb{Z}) \cong \mathbb{Z}^{2g}
\]

where \( g \) is the genus of \( L \), and is independent of the choice of the trivialization of the trivial \( GL(2, \mathbb{R}) \)-bundle. Note that the isomorphism \( H^1(L, \mathbb{Z}) \cong \mathbb{Z}^{2g} \) depends on a choice of a basis for \( H^1(L, \mathbb{Z}) \cong \mathbb{Z}^{2g} \). It is well-known that \( \text{Diff}^+(L) \) acts on \( H^1(L, \mathbb{Z}) \cong \mathbb{Z}^{2g} \) as the integral \( 2g \times 2g \) symplectic group \( Sp(2g, \mathbb{Z}) \). For each \( 0 \neq \sigma \in H^1(\Sigma, \mathbb{Z}) \) there is a unique positive integer \( m(\sigma) \) such that \( \sigma = m(\sigma)\sigma' \) where \( \sigma' \in H^1(\Sigma, \mathbb{Z}) \) is primitive. We call \( m(\sigma) \) the multiplicity of \( \sigma \). The multiplicity of \( 0 \in H^1(\Sigma, \mathbb{Z}) \) is defined to be 0.

Obviously, \( H^1(L, \mathbb{Z}) \) is trivial if \( L = S^2 \) is the 2–sphere. In this case \( \lambda(S^2, \omega', \omega) = 0 \) is trivial.

**Definition 2.1.** Assume that the genus of \( L \) is positive. Let \( \sigma \in H^1(L, \mathbb{Z}) \) be the class representing \( \Omega_{\omega', \omega} \). Define

\[
\lambda(L, \omega', \omega) := m(\sigma)
\]

**Lemma 2.1.** \( \lambda(L, \omega', \omega) = \lambda(L, \omega, \omega') \).

**Remark 2.1.** When \( L = T^2 \) is of genus 1, then \( TT^2 \) and \( T^*T^2 \) are trivial \( \mathbb{R}^2 \)-bundles over \( T^2 \). With trivializations of \( TT^2 \) and \( T^*T^2 \) fixed, the orientation-preserving bundle isomorphism \( \Omega_\omega := TT^2 \to T^*T^2 \) then induces a map \( T^2 \to GL^+(2, \mathbb{R}) \) and hence an absolute invariant \( \lambda(L, \omega) \in \mathbb{N} \cup \{0\} \) is also defined. In particular we have \( \lambda(T^2, \omega_{\text{can}}) = 0 \). This invariant \( \lambda(T^2, \omega) \) is actually the \( \lambda \)-invariant defined by Fintushel and Stern [2]. Our construction here can be thought as a relative extension of the Fintushel-Stern invariant to Lagrangian surfaces of any positive genus.
2.2 Realization of $\lambda(L, \omega', \omega)$

Consider the projection $\pi : T^*L \to L$, $(x, y) \to x$, and its differential

$$d\pi(x, y) : T_{(x,y)}T^*L \to T_xL.$$  \hfill (4)

For each $\rho : T^*L \to T^*L$ which is an orientation-preserving linear bundle automorphism over $L$, we define a 1-form $\lambda_{\rho} \in \Omega^1(T^*L)$ by

$$\lambda_{\rho}(x, y) = \rho(y) \circ d\pi(x, y) : T_{x,y}T^*L \to \mathbb{R}$$

and denote its negative differential $\omega_{\rho} := -d\lambda_{\rho}$. In particular, if $\rho = id$ then $\omega_{\rho} = \omega_{\text{can}}$ the canonical symplectic 2–form on $T^*L$. Note that for each $\rho$ as defined above, $\omega_{\rho}$ is symplectic and has the zero section $L$ as a Lagrangian surface.

Here is a way of constructing $\omega$ from $\omega_{\text{can}}$ and a smooth map $\rho : L \to S^1$. Fix a Riemannian metric $g_0$ on $L$. $g_0$ induces a Riemannian metric on $T^*L$. For any $\phi \in \text{Diff}^+(L)$ and any smooth map $\rho : L \to S^1$ consider the diffeomorphism

$$\Phi : T^*L \to T^*L$$

$$\Phi(x, y) := (\phi(x), e^{2\pi i \rho(\phi(x))}(\phi^{-1})^*y), \quad p \in T^*_yL$$

Here we use the polar coordinates for $T^*_xL \cong \mathbb{C}$. Clearly $\Phi(L) = L$. Let $\omega := \Phi^*\omega_{\text{can}}$. Then $L$ is also $\omega$–Lagrangian. Then $\lambda(L, \omega, \omega_{\text{can}})$ is the multiplicity of the element in $H^1(L, \mathbb{Z})$ corresponding to $\rho$. Note that $\omega = \omega_{\text{can}}$ if $\rho$ is a constant map.

2.3 The definition of $\lambda(L, L_0; \phi)$

The construction of $\lambda(L, \omega', \omega)$ in Section 2.1 can be applied straightforwardly to define an invariant $\lambda(L, L_0, \phi)$ for a pair of diffeomorphic Lagrangian surfaces $L_0, L = \phi(L_0)$ in a symplectic 4–manifold $(M, \omega)$ and $\phi \in \text{Diff}^+(M)$. We assume that $L_0$ is embedded and compact. Let $\omega_1 := (\phi^{-1})^*\omega$. Then we define

$$\lambda(L, L_0, \phi) := \lambda(L, \omega_1, \omega)$$

where $\lambda(L, \omega_1, \omega)$ is defined by using the fact that a tiny tubular neighborhood of $L$ is symplectomorphic to a tiny tubular neighborhood of the zero section of $T^*L$. This is really the Lagrangian Neighborhood Theorem due to Weinstein [7]. We state the theorem here for future reference (see [4]).
Theorem 2.1. Let \((M, \omega)\) be a symplectic manifold and \(L \subset M\) a compact Lagrangian submanifold. Then there exists a neighborhood \(U \subset T^*L\) of the zero section, a neighborhood \(V \subset M\) of \(L\), and a diffeomorphism \(\Phi : U \to V\) such that
\[
\Phi^*\omega = -d\lambda_{can}, \quad \Phi|_L = \text{id},
\]
where \(\lambda_{can}\) is the canonical 1–form on \(T^*L\).

2.4 Proof of Theorem 1.1

We start with the following lemma which is an easy consequence of the definition of \(\lambda(L, L_0; \phi)\).

Lemma 2.2. If \(\phi \in \text{Diff}^+_0(M; L_0 \to L)\) is homotopic in \(\text{Diff}^+_0(M; L_0 \to L)\) to some \(\psi \in \text{Symp}(M, \omega)\) then \(\lambda(L, L_0; \phi) = 0\).

The following Lemma will be frequently used in the proof of Theorem

Lemma 2.3 (Lemma 3.14 of [4]). Let \(M\) be a \(2n\)-dimensional smooth manifold and \(L \subset M\) be a compact submanifold. Suppose that \(\omega_0, \omega_1 \in \Omega^2(M)\) are closed 2–forms such that at each point \(q\) of \(L\) the forms \(\omega_0\) and \(\omega_1\) are equal and nondegenerate on \(T_qM\). Then there exist open neighborhoods \(N_0\) and \(N_1\) of \(L\) and a diffeomorphism \(\psi : N_0 \to N_1\) such that
\[
\psi|_L = \text{id}, \quad \psi^*\omega_1 = \omega_0.
\]

Let us also recall Moser’s argument on the isotopy of symplectic forms ([5], [4]). Here we follow the presentation in [4]. For every family of symplectic forms \(\omega_t \in \Omega^2(M)\) with an exact derivative \(\frac{d\omega_t}{dt} = d\sigma_t\), there exists \(\phi_t \in \text{Diff}(M)\) such that \(\phi_t^*\omega_t = \omega_0\). \(\phi_t\) can be chosen to be the time \(t\) map of the flow of a time dependent vector field \(X_t\), i.e.,
\[
\frac{d}{dt}\phi_t = X_t \circ \phi_t, \quad \phi_0 = \text{id},
\]
where \(X_t\) satisfies the equation
\[
d(\sigma_t + \iota(X_t)\omega_t) = 0
\]
In particular, since \(\omega_t\) is nondegenerate there exists a unique \(X_t\) satisfying
\[
-\sigma_t = \iota(X_t)\omega_t
\]
Lemma 2.4. Assume that $\lambda(L, L_0; \phi) = 0$ then $\phi$ can be smoothly isotoped to $\psi$ with $L$ fixed by the isotopy such that

$$(\psi^{-1})^* \omega = w \quad \text{near } L_0$$

Proof. Recall that

$$T_L M \cong T_L(T^* L) = TL \oplus T^* L$$

Since $\lambda(L, \omega_1, w) = \lambda(L, L_0; \phi) = 0$,

$$\rho := \Omega_{\omega_1, \omega} : T^* L \to T^* L$$

is homotopic to the identity map. Fix a trivialization of $TL \otimes T^* L$ then $\rho$ becomes a map $L \to GL^+(2, \mathbb{R})$ and is represented by $A := (a_{ij}) \in GL(2, \mathbb{R})$ which depends smoothly on $x \in L$. We may assume that $\omega = -d\lambda_{\text{can}}$ near $L$. Since $L$ is $\omega_1$–Lagrangian, $\omega_1$ is exact near $L$. Let $f \in C^\infty(M)$ be a smooth function supported in a small tubular neighborhood of $L \subset M$ and $f = 1$ near $L$. Define

$$\tau := \omega_1 + d(f \rho(y) \circ d\pi)$$

$\tau \in \Omega^2(M)$ is a closed 2–form which is exact near $L$ and $\tau|_{T} = 0$. Near $L$ we have $\tau = d\theta$ for some 1-form $\theta$ defined near $L$. Let $i : L \hookrightarrow M$ denote the inclusion of $L$. Then $i^* \theta$ is a closed 1-form on $L$. Then near $L$ we have

$$\omega = -d\lambda_{\text{can}}, \quad \omega_1 = -d(\rho(y) \circ d\pi) + d\theta$$

Let $\rho_t : T^* L \to T^* L$, $t \in [0, 1]$, be a smooth homotopy of linear automorphisms between $\rho_0 := \rho$ and $\rho_1 := id$. By using Theorem 2.1 we can consider a family of symplectic 2-forms defined near $L$:

$$\omega_t := -d(\rho_t(y) \circ d\pi) + td\theta.$$ 

Consider the vector field $Y_t$ defined by

$$\frac{d\rho_t}{dt} \circ d\pi + \theta = i(Y_t)\omega_t$$

Note that $Y_t = 0$ on $L$ for all $t$.

Now consider a smooth time dependent vector field $X_t$ supported in a tubular neighborhood of $L$ such that on $X_t = Y_t$ on $L$. Define $\phi'_t$ to be the time $t$ map of the flow of $X_t$. $\phi'_t$ fixes $L$ pointwise for all $t$, $\phi'_0 = id$. We obtain an isotopy $\phi'_t \circ \phi$ between $\phi$ and $\phi'_t \circ \phi$ such that $\psi := \phi'_1 \circ \phi$ satisfies

$$(\psi')^* \omega = \omega \quad \text{on } L_0$$

By Lemma 2.3 we can further isotope $\psi'$, with $L$ fixed, to a diffeomorphism $\psi$ such that $\psi(L_0) = L$ and $\psi^* \omega = \omega$ near $L_0$. 

Lemma 2.5. Let $\phi$ be as in Lemma 2.4. Assume that $\lambda(L, L_0; \phi) = 0$ and $\phi \in \text{Diff}_0^1(M)$, then there exists $\Phi_t \in \text{Diff}_0^1(M; L_0 \to L)$, $0 \leq t \leq 1$, such that

$$\Phi_0 = \phi, \quad (\Phi_t^{-1})^* \omega = \omega.$$  
That is, $L$ is symplectomorphic to $L_0$.

Proof. Let $\phi_t \in \text{Diff}_0^1(M)$, $0 \leq t \leq 1$, be a smooth homotopy between $\phi_0 := \text{id}$ and $\phi_1 := \phi$. Denote $\omega_t := (\phi_t^{-1})^* \omega$. $w_0 = w_1$ near $L$. Since $\lambda(L, L_0; \phi) = 0$, by Lemma 2.4 we may assume that $\omega_t = \omega_0$ and $\omega_{1-t} = w_1$ for all $0 \leq t \leq \delta_0$ for some constant $\delta_0 > 0$. $L$ need not be $\omega_t$-Lagrangian for $t \in (\delta_0, 1 - \delta_0)$.

Write $w_t = \omega + d\alpha_t$ with $\alpha_t \in \Omega^1(M)$, $\alpha_0 = 0$. Since $w_1 = w_0$ near $L$, $d\alpha_1 = 0$ near $L$. Let $i : L \hookrightarrow M$ denote the embedding of $L$, and $\sigma_t := i^* \alpha_t$. Then $\sigma_0 = 0$, $\sigma_1$ is a closed 1-form on $L$.

Claim: There exists a closed 1-form $\beta \in \Omega^1(M)$ such that $i^* \beta = \sigma_1$.

Proof of the Claim.
Consider the long exact sequence of cohomology groups

$$\cdots \to H^1(M) \xrightarrow{i^*} H^1(L) \xrightarrow{\delta^*} H^2(M, L) \to \cdots$$  
(5)

Since $\delta^* [\sigma_1] = [d\alpha_1] = 0$, $[\sigma_1] = c$ for some $c \in H^1(M)$. Hence there exists a closed 1-form $\beta' \in \Omega^1(M)$, $[\beta'] = c$, such that $i^* \beta' = \sigma_1 + df$ for some $f \in C^\infty(L)$. Extend $f$ to be a smooth function (also denoted by $f$) on $M$. Define $\beta := \beta' - df$. Then $\beta \in \Omega^1(M)$ is closed and $i^* \beta = \sigma_1$.

Subtract $t\beta$ from $\alpha_t$ and still call the resulting 1–form $\alpha_t$. Then

$$\omega_t = \omega_0 + d\alpha_t, \quad \alpha_0 = 0, \quad \sigma_0 = i^* \alpha_0 = 0, \quad \sigma_1 = i^* \alpha_1 = 0.$$  

Recall the projection $\pi : T^*L \to L$. Let $h$ be a smooth function on $T^*L$ that is compactly supported in a tiny tubular neighborhood of the zero section $L$ and satisfies $h = 1$ near $L$. By Lemma 2.3 the 1–form $\gamma_t := h \pi^* \sigma_t$ is a 1–form supported in a tubular neighborhood of $L \subset M$ such that $i^* \gamma_t = \sigma_t$.

For each $t$ consider the time independent vector field $X_t$ defined by

$$\gamma_t = \iota(X_t) \omega_t$$

Let $\psi_{t,s}$ denotes the time $s$ map of the flow of $X_t$. $\psi_{0,s} = id = \psi_{1,s}$ for all $s$. Denote $\omega_{t,s} := (\psi_{t,s}^{-1})^* \omega_t$. Then $i^* \omega_{t,1} = 0$ for all $t$, i.e., $L$ is $\omega_{t,1}$–Lagrangian.
Write $\omega'_t := \omega_{t,1}$. On $L$, as discussed in Lemma 2.4,
\[ \omega'_t = -d(\rho_t(y) \circ d\pi) + d\theta_t, \quad w'_t = w'_t \]
i* \theta_t \in \Omega^1(L) \text{ is closed,} \quad \rho_0(y) = \rho_1(y)

$\phi'_t := \psi_{t,1} \circ \phi_t$, $t \in [0, 1]$, is a homotopy between $id = \phi'_0$ and $\phi = \phi'_1$.

$(\phi'_t)^* \omega'_t = w_0$, $\omega'_0 = \omega$, $\omega'_1 = \omega_1$. $\phi'_1(L_0) = L$, $L$ is $\omega'_1$–Lagrangian $\forall t \in [0, 1]$. Note that $\lambda(L, L_0; \phi'_t) = 0$ for all $t$.

Again, by applying Lemma 2.4 to each of the triples $(L, L_0, \phi'_t)$, $t \in [0, 1]$, the homotopy $\phi'_t$ is perturbed, with $L$ fixed, to a new homotopy $\phi''_t$ between $\phi''_0 = id$ and $\phi''_1 = \phi$ such that

$\omega''_t := ((\phi''_t)^{-1})^* \omega$ equals $\omega$ near $L$

Hence $\omega''_t = \omega + d\eta_t$ with $\eta_0 = 0$ and $\eta_t \in \Omega^1(M)$ closed near $L$. Note that $\omega''_1 = \omega_1$. By applying the long exact sequence \[5\] we may assume that

$\eta_t = 0$ near $L$.

Define the vector field $Y_t$:
\[ \frac{d\eta_t}{dt} = \iota(Y_t) \omega''_t \]
and let $\psi_t$ be the time $t$ map of the flow of $Y_t$. $\psi_0 = id$, $\psi_t(L) = L$ for $t \in [0, 1]$, and $\psi_t^* \omega = \omega''_t = \omega_1$. Note that

$(\psi_1 \circ \phi)^* \omega = \phi^* \psi_1^* \omega = \phi^* \omega_1 = w$

Then $\Phi_t := \psi_t \circ \phi \in \text{Diff}^+_c(M; L_0 \to L)$ is an homotopy between $\Phi_0 = \phi$ and $\Phi_1 = \psi_1 \circ \phi \in \text{Symp}(M, \omega)$. This completes the proof of Lemma 2.5 and hence the proof of Theorem 1.1. \[\square\]

3 A smooth isotopy invariant

3.1 Local construction

Let $\tau$ be an area form on a closed oriented Riemann surface $L$. For $k_1, k_2 \in \mathbb{R}$ with $k_1 k_2 < 0$, $k_1 \tau \oplus k_2 \tau$ is a symplectic 2–form on $L \times L$. Let $\Delta := \{(x, x) \mid x \in L\}$ denote the diagonal of $L \times L$. $\Delta$ is $(k_1 \tau \oplus k_2 \tau)$–Lagrangian iff $k_1 + k_2 = 0$, $(k_1 \tau \oplus k_2 \tau)$–symplectic iff $k_1 + k_2 > 0$. By allpying Lemma 2.1 there exists a diffeomorphism $\phi : \mathcal{N}(L, T^*L) \to \mathcal{N}(\Delta, L \times L)$ from a tubular neighborhood $\mathcal{N}(L, T^*L)$ of the zero section of $T^*L$ to a tubular neighborhood $\mathcal{N}(\Delta, L \times L)$ of $\Delta \subset L \times L$ such that $\phi^*(k_1 \tau \oplus (-k_1) \tau) = \omega_{\text{can}}$. 9
We can get a second symplectic 2-form on $T^*L$ as follows: By scaling the fibers of $T^*L$ we get a fiber-preserving diffeomorphism $\psi : T^*L \to \mathcal{N}(L, T^*L)$. Fix constants $k_1, k_2$ with $k_1 k_2 < 0$ and $k_1 + k_2 > 0$, then the pullback $\omega^+ := (\phi \circ \psi)^* (k_1 \tau \oplus k_2 \tau)$ is a symplectic 2-form.

Fix a Riemannian metric $g$ on $T^*L$ for example, let $g$ be one that is induced by a Riemannian metric on $L$. Then we get an $S^2 = SO(4)/U(2)$–bundle $E$ over $T^*L$, whose sections are in one-one correspondence with $g$–skew-adjoint ($g$–skad) almost complex structures on $T^*L$. Recall from [4] (p.60–62) that for each symplectic 2–form $\omega$ on a Riemannian manifold $(M, h)$ there associates a unique $\omega$–compatible almost complex structure that is also $h$–skad. Now that we have two linearly independent symplectic 2–forms $\omega_0$ and $\omega^+$ on $(T^*L, g)$, each of $\omega_0$ and $\omega^+$ associates a unique $g$–skad almost complex structure denoted by $J_g$, $J_{g^+}$ respectively. $J_g$ and $J_{g^+}$, viewed as two sections of $E$, never intersect, hence the triple sections $(J_g, J_{g^+}, J_{g^+} \circ J_g)$ defines a trivialization

$$E \cong S^2 \times T^*L.$$  

Let $L' \subset T^*L$ be an embedded $\omega_{can}$–Lagrangian surface. Assume that there exists $\phi \in \text{Diff}_0^+(T^*L)$ such that $\phi(L) = L'$ and $\phi$ is $\omega_{can}$–symplectic near $L$, i.e., $\phi^* \omega_{can} = \omega_{can}$ in a tubular neighborhood of $L$. Take a smooth path $\phi_t \in \text{Diff}_0^+(T^*L)$ with $\phi_0 = id$ and $\phi_1 = \phi$. Define $\omega_t := (\phi^{-1})^* \omega_{can}$ and $L_t := \phi_t(L)$. We have that $L_t$ is $\omega_t$–Lagrangian, $L_0 = L$, $L_1 = L'$, and $\omega_0 = \omega_{can}$. Note in particular $w_1 = \omega_{can}$ near $L'$.

Let $J_t$ denote the $\omega_t$–compatible $g$–skad almost complex structure. Think of the family $L_t$, $0 \leq t \leq 1$, as the image of a map $\eta : L \times [0, 1] \to T^*L$. $\eta^* \mathcal{E}$ is trivial $S^2$–bundle over $L \times [0, 1]$ with the trivialization induced by $(J_g, J_{g^+}, J_{g^+} \circ J_g)$. Now the union $J(\eta) := \cup \eta^* (J_t|_L)$ is an element of

$$\Gamma(\eta^* \mathcal{E}, J_g) = \{ J \in \Gamma(\eta^* \mathcal{E}) \mid J = \eta^* J_g \text{ on } L \times \{0, 1\} \},$$

the set of all $C^\infty$ sections of $\eta^* \mathcal{E}$ which are equal to $\eta^* J_g$ over $L_0 \cup L_1$.

Clearly, a necessary condition for the path $L_t$ homotopic to a $\omega_{can}$–Lagrangian isotopy relative to $(L_0, L_1)$ is that $J(\eta)$ is homotopic to the ”constant” section $\eta^* J_g$ in $\Gamma(\eta^* \mathcal{E}, J_g)$, i.e., if $[J]$ is in the connected component of $\pi_0(\Gamma(\eta^* \mathcal{E}, J_g))$ containing $\eta^* J_g$.

We can think of $J_g$ as the north pole $*$ of the 2–sphere of almost complex structures of the given Riemannian metric $g$. The set $(S^2)_L$ of continuous maps from $L$ to $S^2$ has a special constant map denoted by $*_L$ which sends $L$ to the point $*$. Then

$$\Gamma(\eta^* \mathcal{E}, J_g) \hookrightarrow (S^2)^L_L.$$
where $\mathcal{M} := (S^2)^L$ is the connected component of $(S^2)^L$ containing the constant map $*_L$. Let $\Omega \mathcal{M}$ denote the loop space of the pointed space $(\mathcal{M}, *^L)$.

Given $\{f_t\} \in \Omega \mathcal{M}$, i.e., $f_t : L \to S^2$, $t \in [0,1]$, is a smooth path in $\mathcal{M}$ with $f_0 = *_L = f_1$. We can associate a map

$$F : X \to S^2$$

Since $f_0 = *_L = f_1$, $\hat{F} = F \circ u$ where

$$u := L \times [0,1] \to X = L \times [0,1]/L \times \{0,1\}$$

$X$ is the double cone of $L$ with the two vertices identified, and $u$ is the crushing map that sends $L \times \{0,1\}$ to the vertex of $X$, and $u|_{L \times (0,1)}$ is a homeomorphism. Then each $\{f_t\} \in \Omega \mathcal{M}$ induces a map

$$F : X \to S^2$$

It is easy to see that if $\{f_t\}, \{f'_t\} \in \Omega \mathcal{M}$ are homotopic, then the corresponding maps $F, F' : X \to S^2$ are homotopic as continuous maps from $X$ to $S^2$.

Think of $J = \{J_t\}$ as a loop $\{f_t\}$ in $\mathcal{M}$ and hence a map from $X \to S^2$, we arrive at the following

**Definition 3.1.** $n(J)$ is defined to be the corresponding homotopy class in $[X, S^2]$ that is represented by $F$.

Let $U := L \times (0,1)$ and $V := X \setminus L \times \{\frac{1}{2}\}$. Then $X = U \cup V$. By considering the Mayer-Vietoris sequence of $X = U \cup V$ we get a long exact sequence of homology groups (with $\mathbb{Z}$-coefficients)

$$\cdots \to H_i(U \cap V) \to H_i(U) \oplus H_i(V) \to H_i(X) \to H_{i-1}(U \cap V) \to \cdots$$

and get

$$H_1(X) \cong \mathbb{Z}, \quad H_2(X) \cong \mathbb{Z}^{2g}, \quad H_3(X) \cong \mathbb{Z}$$

hence the cohomology groups

$$H^1(X) \cong \mathbb{Z}, \quad H^2(X) \cong \mathbb{Z}^{2g}, \quad H^3(X) \cong \mathbb{Z}$$

The following lemma is due to Kuperberg [3]:

**Lemma 3.1.** The homotopy class of a map $F$ from $X$ to $S^2$ is described by $c := F^* \sigma$, where $\sigma \in H^2(S^2, \mathbb{Z})$ is the positive generator, and a Hopf degree $d \in \mathbb{Z}_n$ where $n$ is the maximal divisor of $c$.  

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Corollary 3.1. If \( L = S^2 \), then \( [X, S^2] \cong \mathbb{Z} \).

Example 3.1. Think of \( S^3 \) as the reduced suspension of \( S^2 \), then the Hopf map \( H: S^3 \to S^2 \) gives a loop \( f_t \) in \( (S^2)^S \) with \( f_0 = *_{S^2} = f_1 \) and \( f_t(p) = * \) for some \( p \in S^2 \), \( \forall t \in [0,1] \). The loop \( f_t \) represents a nontrivial element of \( \pi_1((S^2)^S) \). More generally, for any degree \( d \) map \( \Phi_d: S^3 \to S^3 \), the map \( H \circ \Phi_d \) induces an element of \( \pi_1((S^2)^S) \), and two such elements are distinct if \( d_1 \neq d_2 \).

Remark 3.1. When the genus of Lagrangian surfaces are positive, \( n(L_1, L_0; [\]t_1]) \) indicates the potential existence of symplectomorphisms which are smoothly but not symplectically isotopic to the identity map, and which are not generalized Dehn twists. Are there such symplectomorphisms? It will be very interesting if one can construct such an example or disprove the existence.

Note that a priori \( n(J) \neq 0 \) does not guarantee that \( L_1 \) is not \( \omega_{can} \)-Lagrangian isotopic to \( L_0 \), since in general \( n(J) \) depends on the choice of the path \( L_t \).

### 3.2 Definition of \( n(L_1, L_0; [\]t_1]) \)

Let \( L_t = \phi_t(L_0) \), \( \phi_t \in \text{Diff}_{\omega}(M) \), \( t \in [0,1] \), be a smooth isotopy of embedded surfaces in a symplectic 4–manifold \( (M, \omega) \) with \( L_0, L_1 \) being \( \omega \)-Lagrangian and \( \phi_t = id \), \( \phi_t^* \omega = \omega \) near \( L_0 \) (so \( L_0, L_1 \) are symplectomorphic). Fix an \( w \)-compatible almost complex structure \( J_\omega \) and let \( g_\omega = \omega \circ (Id \times J) \) be the induced Riemannian metric on \( M \). Let \( E \) denote the associated \( S^2 \)-bundle over \( M \). Fix a symplectic 2–form \( \omega_+ \) which is defined near \( L_0 \) such that \( L_0 \) is \( \omega^+ \)-symplectic. Define \( \omega_t := (\phi_t^{-1})^* \omega \), \( \omega_+^t := (\phi_t^{-1})^* \omega^+ \). \( L_t \) is \( \omega_t \)-Lagrangian and \( \omega_+^t \)-symplectic. Let \( J_t \) (resp. \( J_t^+ \)) denotes the \( \omega_t \)-compatible (resp. \( \omega_+^t \)-compatible) \( g_\omega \)-skad almost complex structure on \( T_L M \). Then the triple \( (J_t, J_t^+, J_t^+ \circ J_t) \) trivializes the \( S^2 \)-bundle \( E|_{L_t} \cong S^2 \times L_t \). Note that \( J_t|_{L_0} = J_\omega \) for \( t = 0,1 \). With respect to the \( (J_t, J_t^+, J_t^+ \circ J_t) \) trivialization, the section \( J_\omega|_{L_t} \) becomes a map \( L \times [0,1] \to S^2 \) with \( n(J_\omega, [\phi_t]) \) is 0 iff \( J_\omega \) is homotopic to \( J_t \) as a section of \( E|_{L_t} \) with boundary fixed.

Definition 3.2. \( n(L_1, L_0; [\]t_1]) = n(L_1, L_0; [\phi_t]) := n(J_\omega, [\phi_t]). \)

The following lemma is a easy consequence of the definition of \( n(L_1, L_0; [\phi_t]) \).

**Lemma 3.2.** If the path \( L_t := \phi_t(L_0) \) \( (\phi_0 = id) \) is homotopic to a Lagrangian isotopy with boundary \( L_0 \cup L_1 \) preserved, then \( n(L_1, L_0; [\phi_t]) = 0 \).
Question 3.1. Is it true that $n(L_1, L_0; [L_t]) = 0$ implies that the path $L_t := \phi_t(L_0)$ ($\phi_0 = id$) is homotopic to a Lagrangian isotopy with boundary $L_0 \cup L_1$ preserved?

Any answer to the above question will help us better understand the isotopy problem of Lagrangian surfaces. Also, with Theorem proved, it will be very important to construct or find examples Lagrangian surfaces (of positive genus) which are diffeomorphic but not symplectomorphic. Any such example will shed some new light on our understanding of symplectic topology. Moreover, by applying the philosophy behind the construction of $\lambda(L, L_0; \phi)$ and $n(L_1, L_0; [L_t])$ to the contact case, we can also define similar invariants for Legendrian knots in contact 3–manifolds, and use them to explore Legendrian isotopy problems. We will come back to these topics later.

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