ZERO KINEMATIC VISCOSITY-MAGNETIC DIFFUSION LIMIT OF THE INCOMPRESSIBLE VISCOUS MAGNETOHYDRODYNAMIC EQUATIONS WITH NAVIER BOUNDARY CONDITIONS*

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Abstract We investigate the uniform regularity and zero kinematic viscosity-magnetic diffusion limit for the incompressible viscous magnetohydrodynamic equations with the Navier boundary conditions on the velocity and perfectly conducting conditions on the magnetic field in a smooth bounded domain \( \Omega \subset \mathbb{R}^3 \). It is shown that there exists a unique strong solution to the incompressible viscous magnetohydrodynamic equations in a finite time interval which is independent of the viscosity coefficient and the magnetic diffusivity coefficient. The solution is uniformly bounded in a conormal Sobolev space and \( W^{1,\infty}(\Omega) \) which allows us to take the zero kinematic viscosity-magnetic diffusion limit. Moreover, we also get the rates of convergence in \( L^\infty(0,T;L^2) \), \( L^\infty(0,T;W^{1,p}) \) (\( 2 \leq p < \infty \)), and \( L^\infty((0,T) \times \Omega) \) for some \( T > 0 \).

Key words incompressible viscous MHD equations; ideal incompressible MHD equations; Navier boundary conditions; zero kinematic viscosity-magnetic diffusion limit

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1 Introduction

We consider the incompressible viscous magnetohydrodynamic (MHD) equations (for example, see [6, 9])

\[
\begin{aligned}
\partial_t v^\epsilon - \epsilon \Delta v^\epsilon + v^\epsilon \cdot \nabla v^\epsilon - H^\epsilon \cdot \nabla H^\epsilon + \frac{1}{2} \nabla (p^\epsilon + |H^\epsilon|^2) &= 0, \\
\partial_t H^\epsilon - \epsilon \Delta H^\epsilon + v^\epsilon \cdot \nabla H^\epsilon - H^\epsilon \cdot \nabla v^\epsilon &= 0, \\
\nabla \cdot v^\epsilon &= \nabla \cdot H^\epsilon = 0
\end{aligned}
\]

(1.1)

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in \((0, T) \times \Omega\), where \(\Omega \subset \mathbb{R}^3\) is a smooth bounded domain. The unknowns \(v^\varepsilon, H^\varepsilon\) and \(p^\varepsilon\) represent the velocity, the magnetic field and the pressure of the fluid, respectively. The pressure \(p^\varepsilon\) can be recovered from \(v^\varepsilon\) and \(H^\varepsilon\) via an explicit Calderón-Zygmund singular integral operator ([7]). Here, we assume that the viscosity coefficient is equal to the magnetic diffusivity coefficient, denoted by \(\varepsilon > 0\). We add to system (1.1) the initial-boundary conditions

\[
\begin{align*}
&v^\varepsilon \cdot n = 0, \quad (Sv^\varepsilon \cdot n)_\tau = -\zeta v^\varepsilon \quad \text{on } \partial \Omega, \\
&H^\varepsilon \cdot n = 0, \quad n \times \omega^\varepsilon_H = 0 \quad \text{on } \partial \Omega, \\
&(v^\varepsilon, H^\varepsilon)|_{t=0} = (v_0, H_0) \quad \text{in } \Omega,
\end{align*}
\]

where \(\omega^\varepsilon_H = \nabla \times H^\varepsilon\), \(n\) stands for the outward unit normal vector to \(\Omega\), \(\zeta \in \mathbb{R}\) is a coefficient measuring the tendency of the fluid to slip on the boundary and \(S\) is the strain tensor defined by

\[
Su = \frac{1}{2} \left( \nabla u + (\nabla u)^T \right),
\]

where \((\nabla u)^T\) denotes the transpose of the matrix \(\nabla u\), and \(u_\tau\) stands for the tangential part of \(u\) on \(\partial \Omega\), i.e.,

\[
u_\tau = u - (u \cdot n)n.
\]

The boundary condition (1.2) was first introduced by Navier in [18] to show that the velocity is proportional to the tangential part of the stress; it allows the fluid slip along the boundary and is often used to model rough boundaries. The boundary condition (1.2) can be generalized to the following form ([11]):

\[
\begin{align*}
u \cdot n &= 0, \quad (Su \cdot n)_\tau + Au = 0. \tag{1.3}
\end{align*}
\]

Here \(A\) is a \((1, 1)\)-type tensor on the boundary \(\partial \Omega\). When \(A = \zeta \text{Id}\) (\(\text{Id}\) denotes the identity matrix), (1.3) is reduced to the standard Navier boundary conditions. In addition, for smooth functions, we can get the form of the vorticity

\[
u \cdot n = 0, \quad n \times \omega = [Bu]_\tau \quad \text{on } \partial \Omega, \tag{1.4}
\]

where \(\omega = \nabla \times u\) and \(B = 2(A - S(n))\) (see [27]).

In this paper, we are interested in the uniform regularity and the zero kinematic viscosity-magnetic diffusion limit of the problem (1.1)–(1.2), and taking the limit \(\varepsilon \to 0\) to obtain the ideal incompressible MHD equations (Suppose that the limits \(v^\varepsilon \to v\) and \(H^\varepsilon \to H\) as \(\varepsilon \to 0\) exist), i.e.

\[
\begin{align*}
&\partial_t v + v \cdot \nabla v - H \cdot \nabla H + \frac{1}{2} \nabla (|H|^2 + p) = 0, \\
&\partial_t H + v \cdot \nabla H - H \cdot \nabla v = 0, \\
&\nabla \cdot v = \nabla \cdot H = 0,
\end{align*}
\]

(1.5)

with the following initial-boundary conditions:

\[
\begin{align*}
&v \cdot n = H \cdot n = 0 \quad \text{on } \partial \Omega, \\
&(v, H)|_{t=0} = (v_0, H_0) \quad \text{in } \Omega. \tag{1.6}
\end{align*}
\]

When taking \(H^\varepsilon = 0\) in the system (1.1), it is reduced to the classical incompressible Navier-Stokes equations. There are a lot of works on the vanishing viscosity limit of the incompressible
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Navier-Stokes equations. The inviscid limit of the Cauchy problem has been studied by many authors, see [14, 16, 21]. When the boundary appears, the inviscid limit problem becomes challenging due to the possible appearance of the boundary layer [13, 19]. On one hand, for the incompressible Navier-Stokes equations with no-slip boundary condition, its vanishing viscosity limit is wildly open except when the initial datum is analytic [22, 23] or the initial vorticity is located away from the boundary in the two-dimensional half plane [15]. On the other hand, considering the incompressible Navier-Stokes system with Navier boundary conditions, more results are available, see [2–5, 11, 13, 17, 28] and the references cited therein. The uniform $H^3$ bound and a uniform existence time interval with respect to $\epsilon$ were obtained by Xiao and Xin [28] for the flat boundary. Subsequently, the conclusions in [28] were extended to $W^{k,p}(\Omega)$ with $k \geq 3$ and $p \geq 2$ in [2]. The main reason for this is that the boundary integrals vanish on the flat boundary, see also [3, 4]. However, such results can’t be expected for the general boundary since boundary layer may appear. In order to analyze the effect of the boundary layer in a general bounded domain, Iftimie and Sueur [13] constructed the boundary layer for the incompressible Navier-Stokes equations with the Navier boundary condition (1.2) in the form

$$u^\epsilon(t, x) = u(t, x) + \sqrt{\epsilon} V(t, x, \frac{\phi(x)}{\sqrt{\epsilon}}) + O(\epsilon),$$

(1.7)

where the function $V$ vanishes for $x$ outside a small neighborhood of $\partial\Omega$ and $\phi(x)$ is the distance between $x$ and $\partial\Omega$ for $x$ in a neighborhood of $\partial\Omega$. The boundary layers constructed in [13] are of width $O(\sqrt{\epsilon})$, like the Prandtl layer [19], but are of amplitude $O(\sqrt{\epsilon})$ (the Prandtl layer is of width $O(\sqrt{\epsilon})$ and of amplitude $O(1)$). Thus it is impossible to obtain the $H^3(\Omega)$ or $W^{2,p}(\Omega)$ ($p$ large enough) uniform estimates for the incompressible Navier-Stokes equations. Recently, Masmoudi and Rousset [17] considered the vanishing viscosity limit for the incompressible Navier-Stokes equation with the boundary condition (1.2) in the anisotropic conormal Sobolev spaces (defined below), and this can eliminate the effects of normal derivatives near the boundary. They obtained uniform regularity and the convergence of the viscous solutions to the inviscid one by compactness arguments. Subsequently, some results in [17] were extended to the compressible isentropic Navier-Stokes equations with Navier boundary conditions [20, 26]. Moreover, based on the results in [17], the rates of convergence in different spaces were obtained by Gie and Kelliher [11] and Xiao and Xin [27], respectively.

As for the 3D incompressible viscous MHD equations, the study of the inviscid limit is quite limited, see [29–31]. Specially, Xiao, Xin and Wu [29] investigated the inviscid limit for the system (1.1) with the boundary conditions

$$\begin{cases}
v^\epsilon \cdot n = 0, \quad n \times \omega_v^\epsilon = 0 & \text{on } \partial\Omega, \\
H^\epsilon \cdot n = 0, \quad n \times \omega_H^\epsilon = 0 & \text{on } \partial\Omega,$$

(1.8)

where $\omega_v = \nabla \times u^\epsilon$. They used the approaches in [28] and formulated the boundary value in a suitable functional setting so that the Stokes operator is well behaved and the nonlinear terms fall into the desired functional spaces. These facts allowed them to get the uniform regularity for the viscous incompressible MHD system through the Galerkin approximation and a priori energy estimates.

Motivated by [17], in this paper, we investigate the inviscid limit for the system (1.1) with
Navier boundary conditions for \((v^\varepsilon, H^\varepsilon)\) in a bounded domain \(\Omega \subset \mathbb{R}^3\) in the framework of anisotropic conormal Sobolev spaces. Due to the strong coupling between \(v^\varepsilon\) and \(H^\varepsilon\), a priori estimates become more complicated than that in [17]. We obtain the uniform regularity of the solution, which allow us to pursue the inviscid limit for the problem (1.1)–(1.2). Moreover, we obtain some rates of convergence for \((v^\varepsilon, H^\varepsilon)\) in different spaces.

Our uniform regularity result reads as follows:

**Theorem 1.1** Let \(m\) be an integer satisfying \(m > 6\) and \(\Omega\) be a \(C^{m+2}\) domain. Assume the initial data \((v_0, H_0) \in \mathcal{E}_{MHD}^m\). Then, there exist a time \(T_0 > 0\) and a constant \(\tilde{C}_0 > 0\), independent of \(\varepsilon \in (0,1)\) and \(|\zeta| \leq 1\), such that there exists a unique solution of the problem (1.1)–(1.2) satisfying \((v^\varepsilon, H^\varepsilon) \in C([0,T_0],\mathcal{E}_{MHD}^m)\) and

\[
\sup_{t \in [0,T_0]} \left\{ \| (v^\varepsilon, H^\varepsilon)(t) \|_m^2 + \| (v^\varepsilon, H^\varepsilon)(t) \|_{S^H_{m-1}}^2 + \| (\nabla v^\varepsilon, \nabla H^\varepsilon)(t) \|_{1,\infty}^2 \right\} \\
+ \varepsilon \int_0^{T_0} \left( \| \nabla^2 v^\varepsilon(t) \|_{m-1}^2 + \| \nabla^2 H^\varepsilon(t) \|_{m-1}^2 \right) dt \leq \tilde{C}_0, \tag{1.9}
\]

where \(\tilde{C}_0\) depends only on \((v_0, H_0)\) and the regularity of \(\partial \Omega\). The meanings of \(\mathcal{E}_{MHD}^m\), \(\| \cdot \|_m\) and \(\| \cdot \|_{m,\infty}\) will be explained in detail in next section.

**Remark 1.2** When the Navier boundary condition (1.2)_1 is replaced by the following conditions

\[
\begin{cases}
v^\varepsilon \cdot n = 0, \quad n \times \omega^\varepsilon = [Bv^\varepsilon]_\tau & \text{on } \partial \Omega, \\
H^\varepsilon \cdot n = 0, \quad n \times \omega^H = 0 & \text{on } \partial \Omega,
\end{cases} \tag{1.10}
\]

we can also obtain the same conclusions as those in Theorem 1.1, where \(B = 2(A - S(n))\) and \(A\) is a \((1,1)\)-type tensor on \(\partial \Omega\).

We now give some comments on the proof of Theorem 1.1. The main steps of the proof are, in some senses, similar to those in [17], however, due to the strong coupling between \(v^\varepsilon\) and \(H^\varepsilon\), we need to overcome some new difficulties and to face more complicated energy estimates. We explain them briefly below. First, we get a conormal energy estimate in \(H^m_{co}\) for \((v^\varepsilon, H^\varepsilon)\). Here, we define \(P^1_{\varepsilon} := |v^\varepsilon|^2 + |H^\varepsilon|^2\), where \(P^1_{\varepsilon}\) and \(P^2_{\varepsilon}\) satisfy corresponding boundary value problems (see (3.8) and (3.9) below). By doing this decomposition, we can avoid higher order terms which are out of control. In the second step, we estimate \(\|(\partial_n v^\varepsilon, \partial_n H^\varepsilon)\|_{m-1}\). Due to the incompressible conditions (1.1)_2, both \(\partial_n v^\varepsilon \cdot n\) and \(\partial_n H^\varepsilon \cdot n\) can be easily controlled by the \(H^m_{co}\) norm of \((v^\varepsilon, H^\varepsilon)\). In order to control the tangential part of \((\partial_n v^\varepsilon, \partial_n H^\varepsilon)\), we find that it is convenient to study \(\eta_{v}^\varepsilon = (Sv^\varepsilon n + \zeta v^\varepsilon)_\tau\) and \(\eta_{H}^\varepsilon = (SH^\varepsilon n + Sn H^\varepsilon)_\tau\). Here \(\eta_{v}^\varepsilon\) and \(\eta_{H}^\varepsilon\) satisfy equations with homogeneous Dirichlet boundary conditions. By performing energy estimates on the equations solved by \((\eta_{v}^\varepsilon, \eta_{H}^\varepsilon)\), we obtain the estimate of \(\|(\eta_{v}^\varepsilon, \eta_{H}^\varepsilon)\|_{m-1}\), which allows us to control \((\partial_n v^\varepsilon \cdot \tau, \partial_n H^\varepsilon \cdot \tau)\). The third step is to estimate \(P^1_{\varepsilon}\) and \(P^2_{\varepsilon}\). Note that they satisfy nonhomogeneous elliptic equations with Neumann boundary conditions. In view of the regularity theory of elliptic equations with Neumann boundary conditions, we get the estimates of the pressure terms. Finally, we need to estimate \(\| \nabla v^\varepsilon \|_{1,\infty}\) and \(\| \nabla H^\varepsilon \|_{1,\infty}\). Similarly to the second step, we introduce equivalent quantities \(\overline{\eta}_{v}\) and \(\overline{\eta}_{H}\). However, due to the strong coupling between \(\overline{\eta}_{v}\) and \(\overline{\eta}_{H}\), we cannot deal with the system solved by \((\overline{\eta}_{v}, \overline{\eta}_{H})\) directly as that in [17]. Instead, we need to further introduce another two quantities: \(\eta_1 = \overline{\eta}_{v} + \overline{\eta}_{H}\) and \(\eta_2 = \overline{\eta}_{v} - \overline{\eta}_{H}\). We then estimate \(\eta_1\) and \(\eta_2\) straightforwardly.
Based on the uniform estimates in Theorem 1.1, by strong compactness arguments similar to those of [17], we can justify the zero kinematic viscosity-magnetic diffusion limit, but without a convergence rate. In what follows, we are interested in the zero kinematic viscosity-magnetic diffusion limit with rates of convergence. First, thanks to the result in [24], we know that there exists a unique solution \((v, H) \in H^3\) of the ideal incompressible MHD equations (1.5) with the boundary condition (1.6) and the initial condition \((v_0, H_0) \in H^3\) which satisfies
\[
\sum_{k=0}^{3} \| (v, H) \|_{C^k([0,T_1]; H^{3-k})} \leq \tilde{C}_1,
\]
where \(\tilde{C}_1\) depends only on \((v_0, H_0)\) and \(\Omega\). Then we can state our convergence results on the zero kinematic viscosity-magnetic diffusion limit as follows:

**Theorem 1.3** Assume that \((v_0, H_0)\) belong to \(H^3(\Omega)\) and satisfy the same assumptions as in Theorem 1.1. Let \((v, H)\) be the smooth solution of (1.5)–(1.6) on \([0, T_1]\), and \((v^\varepsilon, H^\varepsilon)\) be the solution of (1.1)–(1.2) on \([0, T_0]\). Then there exists a time \(T_2 = \min\{T_0, T_1\} > 0\) and a constant \(\tilde{C}_2\) such that
\[
\|(v^\varepsilon - v)(t)\|_{L^2}^2 + \|(H^\varepsilon - H)(t)\|_{L^2}^2 \\
+ \varepsilon \int_0^t \left( \|(v^\varepsilon - v)(s)\|_{H^1}^2 + \|(H^\varepsilon - H)(s)\|_{H^1}^2 \right) ds \leq \tilde{C}_2 \varepsilon \frac{t}{2}, \quad t \in [0, T_2],
\]
\[
\|(v^\varepsilon - v)(t)\|_{H^1}^2 + \|(H^\varepsilon - H)(t)\|_{H^1}^2 \\
+ \varepsilon \int_0^t \left( \|(v^\varepsilon - v)(s)\|_{H^2}^2 + \|(H^\varepsilon - H)(s)\|_{H^2}^2 \right) ds \leq \tilde{C}_2 \varepsilon \frac{t}{2}, \quad t \in [0, T_2]
\]
for \(\varepsilon\) small enough. Consequently,
\[
\|(v^\varepsilon - v)(t)\|_{W^{1,p}}^p + \|(H^\varepsilon - H)(t)\|_{W^{1,p}}^p \leq \tilde{C}_2 \varepsilon \frac{t}{2}, \quad t \in [0, T_2]
\]
for \(2 < p < \infty\) and \(\varepsilon\) small enough, and
\[
\|v^\varepsilon - v\|_{L^\infty([0,T_2] \times \Omega)} + \|H^\varepsilon - H\|_{L^\infty([0,T_2] \times \Omega)} \leq \tilde{C}_2 \varepsilon \frac{t}{2},
\]
where \(\tilde{C}_2\) depends only on \((v_0, H_0)\).

We now outline the proof of Theorem 1.3. Our approach is similar to that of [27] in spirit. However, due to the strong coupling between the magnetic field and the velocity field, we meet some new difficulties. We first give the rates of the convergence in \(L^\infty(0, T_2; L^2(\Omega))\) and \(L^\infty([0, T_2] \times \Omega)\) by using an elementary energy estimate and Gagliardo-Nirenberg interpolation inequality. Next, we find that it is very difficult to estimate some boundary terms caused by multiplying (4.1)_1 by \(\Delta(v^\varepsilon - v)\) and (4.1)_2 by \(\Delta(H^\varepsilon - H)\) directly in the proof of the rate of the convergence in \(L^\infty(0, T_2; H^1(\Omega))\). Indeed, we get \(\|v^\varepsilon\|_{H^2} \leq \|P \Delta v^\varepsilon\| + \|v^\varepsilon\|\) and \(\|H^\varepsilon\|_{H^2} \leq \|P \Delta H^\varepsilon\| + \|H^\varepsilon\|\) from Section 2 in [27], where \(P\) is Leray projector. Therefore, we replace \(\Delta(v^\varepsilon - v)\) and \(\Delta(H^\varepsilon - H)\) by \(P \Delta(v^\varepsilon - v)\) and \(P \Delta(H^\varepsilon - H)\) to prove the rates of the convergence in \(L^\infty(0, T_2; H^1(\Omega))\) and \(L^\infty(0, T_2; W^{1,p}(\Omega))\).

The rest of this paper is organized as follows: in the next section, we give some assumptions on the domain and the definitions on conormal Sobolev spaces, and present some inequalities which will be used frequently. In Section 3, we prove a priori energy estimates and give the proof of Theorem 1.1. Theorem 1.3 will be verified in Section 4.
Throughout this paper, we shall denote by $\| \cdot \|_{H^m}$ and $\| \cdot \|_{W^{1,\infty}}$ the usual Sobolev norms in $\Omega$ and $\| \cdot \|$ for the standard $L^2$ norm. The letter $C$ is a positive constant which may change from line to line, but which is independent of $\epsilon \in (0, 1]$ and $|\zeta| \leq 1$.

2 Preliminaries

We first state the assumptions on the bounded domain $\Omega \subset \mathbb{R}^3$ and then introduce some norms. We assume that $\Omega$ has a covering such that

$$\Omega \subset \Omega_0 \cup_{k=1}^n \Omega_k,$$

where $\overline{\Omega_0} \subset \Omega$, and in each $\Omega_k$ there exists a function $\psi_k$ such that

$$\Omega \cap \Omega_k = \{ x = (x_1, x_2, x_3) \mid x_3 > \psi_k(x_1, x_2) \} \cap \Omega_k,$$

$$\partial \Omega \cap \Omega_k = \{ x = (x_1, x_2, x_3) \mid x_3 = \psi_k(x_1, x_2) \} \cap \Omega_k.$$

We say that $\Omega$ is $C^m$ if the functions $\psi_k$ are $C^m$ functions. Denote by $C_m$ a positive constant independent of $\epsilon \in (0, 1]$ and $|\zeta| \leq 1$ which depends only on the $C^k$-norm of the functions $\psi_k$, $k = 1, \ldots, n$.

To define the conormal Sobolev spaces, we consider $(Z_k)_{1 \leq k \leq N}$, a finite set of generators of vector fields that are tangential to $\partial \Omega$, and set

$$H^m_{cc}(\Omega) = \left\{ f \in L^2(\Omega) \mid Z^I f \in L^2(\Omega) \text{ for } \ |I| \leq m, \ m \in \mathbb{N} \right\},$$

where $I = (k_1, \ldots, k_m)$ and $Z^I = Z_{k_1} \cdots Z_{k_m}$. We define the norm of $H^m_{cc}(\Omega)$ as follows:

$$\| f \|_m^2 = \sum_{|I| \leq m} \| Z^I f \|_{L^2}^2.$$

We say that a vector field, $u$, is in $H^m_{cc}(\Omega)$ if each of its components is in $H^m_{cc}(\Omega)$ and

$$\| u \|_m^2 = \sum_{i=1}^3 \sum_{|I| \leq m} \| Z^I u_i \|_{L^2}^2$$

is finite. In the same way, we set

$$\| f \|_{m, \infty} = \sum_{|I| \leq m} \| Z^I f \|_{L^\infty}, \quad \| \nabla Z^m u \|^2 = \sum_{|I|=m} \| \nabla Z^I u \|_{L^2}^2,$$

and we say that $f \in W^{m, \infty}(\Omega)$ if $\| f \|_{m, \infty}$ is finite. By using the covering (2.1) of $\Omega$, we can assume that each vector field is supported in one of $\{ \Omega_k \}_{k=0}^n$. Also, we note that the $H^m_{cc}$ norm yields a control of the standard $H^m$ norm in $\Omega_0$, whereas if $\Omega_k \cap \partial \Omega \neq \emptyset$, there is no control of the normal derivatives.

Since $\partial \Omega$ is given locally by $x_3 = \psi(x_1, x_2)$ (we omit the subscript $k$ for notational convenience), it is convenient to use the coordinates

$$\Psi : (y, z) \mapsto (y, \psi(y) + z) = x.$$  (2.3)

A local basis is thus given by the vector fields $(\partial_{y_1}, \partial_{y_2}, \partial_z)$ where $\partial_{y_1}$ and $\partial_{y_2}$ are tangential to $\partial \Omega$ on the boundary and in general $\partial_z$ is not a normal vector field. We sometimes use the notation $\partial_{y^2}$ for $\partial_z$. By using this parametrization, we can take suitable vector fields compactly supported in $\Omega_k$ in the definition of the $H^m_{cc}$ norms

$$Z_i = \partial_{y_i} = \partial_i + \partial_i \psi \partial_z, \quad i = 1, 2, \quad Z_3 = \varphi(z) \partial_z,$$

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where \( \varphi(z) = \frac{1}{1 + rz} \) is a smooth, supported function in \((0, +\infty)\) and satisfies
\[
\varphi(0) = 0, \quad \varphi'(0) > 0, \quad \varphi(z) > 0 \quad \text{for} \quad z > 0.
\]

In this paper, we shall still denote by \( \partial_i (i = 1, 2, 3) \) or \( \nabla \) the derivatives with respect to the standard coordinates of \( \mathbb{R}^3 \). The coordinates of a vector field \( u \) in the basis \( (\partial_y, \partial_x, \partial_z) \) will be denoted by \( u^i \), thus,
\[
u = u^1 \partial_y + u^2 \partial_x + u^3 \partial_z.
\]

We denote by \( u_i \) the coordinates in the standard basis of \( \mathbb{R}^3 \), i.e.,
\[
u = u_1 \partial_1 + u_2 \partial_2 + u_3 \partial_3.
\]

Denote by \( n \) the unit outward normal vector which is given locally by
\[
n(x) = n(\Psi(y, z)) = \frac{1}{\sqrt{1 + |\nabla \psi(y)|^2}} \begin{pmatrix}
\partial_1 \psi(y) \\
\partial_2 \psi(y) \\
-1
\end{pmatrix},
\]
and by \( \Pi \) the orthogonal projection \( \Pi(x) = \Pi(\Psi(y, z))u = u - [u \cdot n(\Psi(y, z))]n(\Psi(y, z)) \), which gives the orthogonal projector onto the tangent space of the boundary. Note that \( n \) and \( \Pi \) are defined in the whole \( \Omega \) and do not depend on \( z \). By using these notations, (1.2) \(_1\) and (1.2) \(_2\) read as
\[
\nu^i \cdot n = 0, \quad \Pi \partial_i \nu^i = \theta(\nu^i) - 2\zeta \Pi \nu^i, \quad (2.4)
\]
\[
H^i \cdot n = 0, \quad \Pi \partial_i H^i = -\Pi((\nabla n)^T H^i), \quad (2.5)
\]
where \( \theta \) is the shape operator (second fundamental form) of the boundary, \( \theta(\nu^i) = \Pi((\nabla n)\nu^i) \) and \( \theta(H^i) = \Pi((\nabla n)H^i) \).

Since the boundary layers may appear in the presence of physical boundaries, in order to obtain the uniform estimates for the solutions of the incompressible MHD equations with the boundary conditions (1.2) \(_1\) and (1.2) \(_2\), we need to find a suitable functional space. Here, we define the functional space \( \mathcal{E}_m(T) \) for functions \((\nu, H)(t, x)\) as
\[
\mathcal{E}_m(T) = \left\{ (\nu, H) \in L^\infty([0, T], L^2) \mid \text{ess sup}_{0 \leq t \leq T} \| (\nu, H)(t) \|_{\mathcal{E}_m} < +\infty \right\}, \quad (2.6)
\]
where the norm \( \| \cdot \|_{\mathcal{E}_m} \) is given by
\[
\| (\nu, H)(t) \|_{\mathcal{E}_m} := \| (\nu, H)(t) \|_{L^2}^2 + \| (\nabla \nu, \nabla H)(t) \|_{L^1}^2 + \| (\nabla H, \nabla H)(t) \|_{L^1}^2. \quad (2.7)
\]
We note that the a priori estimates in Theorem 3.1 below are obtained in the situation in which the approximate solution is sufficiently smooth up to the boundary. Therefore, in order to obtain a self-contained result, we need to assume that the approximated initial data satisfies the boundary compatibility conditions (1.2) \(_1\) and (1.2) \(_2\). For the initial data \((\nu_0, H_0)\) satisfying (1.6) \(_2\), it is not clear if there exists an approximate sequence \((\nu_0^\delta, H_0^\delta)\) (\( \delta \) being a regularization parameter) which satisfies the boundary compatibility conditions and \( \| (\nu_0^\delta - \nu_0, H_0^\delta - H_0) \|_{\mathcal{E}_m} \to 0 \) as \( \delta \to 0 \). Thus, we set
\[
\mathcal{E}_{\text{MHD, ap}}^m := \{ (\nu, H) \in H^{3m}(\Omega) \times H^{3m}(\Omega) \mid (\nu, H) \text{ satisfy the incompressible condition and the boundary compatibility condition} \} \quad (2.8)
\]
and
\[ E_{\text{MHD}}^m := \text{The closure of } E_{\text{MHD,ap}}^m \text{ in the norm } \|(\cdot, \cdot)\|_{E^m}. \quad (2.9) \]

Now we introduce some inequalities. First, we give a well-known inequality.

**Lemma 2.1** ([25, 28]) For \( u \in H^s(\Omega) (s \geq 1) \), we have
\[
\|u\|_{H^s(\Omega)} \leq C\left(\|\nabla \times u\|_{H^{s-1}(\Omega)} + \|\nabla \cdot u\|_{H^{s-1}(\Omega)} + \|u\|_{H^{s-1}(\Omega)} + |u|_{H^{s-\frac{1}{2}}(\partial \Omega)}\right).
\]

Next, we introduce Korn’s inequality, which plays an important role in energy estimates.

**Lemma 2.2** (Korn’s inequality [10]) Let \( \Omega \) be a bounded Lipschitz domain of \( \mathbb{R}^3 \). There exists a constant \( C > 0 \) depending only on \( \Omega \) such that
\[
\|u\|_{H^1(\Omega)} \leq C\left(\|u\|_{L^2(\Omega)} + \|S(u)\|_{L^2(\Omega)}\right), \quad \forall u \in (H^1(\Omega))^3.
\]

Third, we also need the following anisotropic Sobolev embedding and trace estimates:

**Lemma 2.3** ([17, 26]) Let \( m_1 \geq 0 \) and \( m_2 \geq 0 \) be integers, and \( u \in H^{m_1}_{\mathrm{co}}(\Omega) \cap H^{m_2}_{\mathrm{co}}(\Omega) \) and \( \nabla u \in H^{m_2}_{\mathrm{co}}(\Omega) \). Then
\[
\|u\|_{L^\infty(\Omega)} \leq C\left(\|\nabla u\|_{m_2} + \|u\|_{m_2}\right)\|u\|_{m_1}, \quad m_1 + m_2 \geq 3,
\]
\[
|u|_{H^s(\partial \Omega)} \leq C\left(\|\nabla u\|_{m_2} + \|u\|_{m_2}\right)\|u\|_{m_1}, \quad m_1 + m_2 \geq 2s \geq 0.
\]

Finally, we introduce the following Gagliardo-Nirenberg-Moser inequality, which will be used frequently:

**Lemma 2.4** ([12]) Let \( u, v \in L^\infty(\Omega) \cap H^k(\Omega) \). Then
\[
\|Z^{\alpha_1}uZ^{\alpha_2}v\| \leq C\left(\|u\|_{L^\infty(\Omega)}\|v\|_k + \|v\|_{L^\infty(\Omega)}\|u\|_k\right), \quad |\alpha_1| + |\alpha_2| = k.
\]

3 A Priori Estimates and Proof of Theorem 1.1

The main aim of this section is to prove the following a priori estimates which is the crucial step in the proof of Theorem 1.1. To this end, let \((v^\epsilon, H^\epsilon)\) be the solution of the problem (1.1)–(1.2) with pressure \( p^\epsilon \). Moreover, the solution \((v^\epsilon, H^\epsilon, p^\epsilon)\) possesses proper regularity such that the procedure of formal calculations makes sense.

**Theorem 3.1** For \( m > 6 \) and a \( C^{m+2} \) bounded domain \( \Omega \), there exists a constant \( C_{m+2} \) independent of \( \epsilon \in (0, 1] \) and \( |\zeta| \leq 1 \), such that for any sufficiently smooth solution defined on \([0, T]\) of the problem (1.1)–(1.2) in \( \Omega \), we have
\[
N_m(t) \leq C_{m+2}\left(N_m(0) + (1 + t + \epsilon^3 t^2) \int_0^t (N_m^2(s) + N^m(s))ds\right), \quad \forall t \in [0, T], \quad (3.1)
\]
where
\[
N_m(t) = ||v^\epsilon||_m^2 + ||\nabla v^\epsilon||_{m-1}^2 + ||\nabla v^\epsilon||_{1, \infty}^2 + ||H^\epsilon||_m^2 + ||\nabla H^\epsilon||_{m-1}^2 + ||\nabla H^\epsilon||_{1, \infty}^2. \quad (3.2)
\]

Since the proof of Theorem 3.1 is quite complicated and lengthy, we divide it into subsections.

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3.1 Conormal Energy Estimates

In this subsection, we first give the basic a priori $L^2$ energy estimate.

**Lemma 3.2** For a smooth solution to the problem (1.1)–(1.2), it holds that, for any $\epsilon \in (0, 1]$ and $|\alpha| \leq 1$,

$$
\frac{1}{2} \frac{d}{dt} (\|v^\epsilon(t)\|_2^2 + \|H^\epsilon(t)\|_2^2) + 2\epsilon \|\nabla \cdot \nabla v^\epsilon\|_2^2 + \epsilon \|\nabla \times H^\epsilon\|_2^2 + 2\epsilon \zeta \int_{\partial \Omega} |v^\epsilon_{\tau}|^2 d\sigma = 0. \tag{3.3}
$$

**Proof** Multiplying (1.1)$_1$ and (1.1)$_2$ by $v^\epsilon$ and $H^\epsilon$, respectively, applying the boundary condition (1.2), and integrating by parts, we obtain

$$
\frac{1}{2} \frac{d}{dt} (\|v^\epsilon\|_2^2 + \|H^\epsilon\|_2^2) - \epsilon (\Delta v^\epsilon, v^\epsilon) - \epsilon (\Delta H^\epsilon, H^\epsilon) = 0, \tag{3.4}
$$

where $(\cdot, \cdot)$ stands for the $L^2$ scalar product. Now, let us treat the terms with the coefficient $\epsilon$ in (3.4). Integrating by parts and using the boundary condition (1.2)$_1$, we have

$$
(\epsilon \Delta v^\epsilon, v^\epsilon) = -2\epsilon \|S v^\epsilon\|_2^2 - 2\epsilon \zeta \int_{\partial \Omega} |v^\epsilon_{\tau}|^2 d\sigma, \tag{3.5}
$$

$$
(\epsilon \Delta H^\epsilon, H^\epsilon) = -\epsilon \|\nabla \times H^\epsilon\|_2^2. \tag{3.6}
$$

Putting (3.5) and (3.6) into (3.4), we then obtain (3.3).

We remark that the above basic energy estimation is insufficient to get the vanishing viscosity limit. Thus, higher order energy estimates are needed.

**Lemma 3.3** For every $m \in \mathbb{N}_+$, it holds that

$$
\frac{d}{dt} \left(\|v^\epsilon(t)\|_m^2 + \|H^\epsilon(t)\|_m^2\right) + \epsilon \left(\|\nabla v^\epsilon\|_m^2 + \|\nabla H^\epsilon\|_m^2\right)
\leq C_{m+2} \left\{ (1 + \|v^\epsilon\|_{W^{1, \infty}} + \|H^\epsilon\|_{W^{1, \infty}}) (\|v^\epsilon\|_m^2 + \|\nabla v^\epsilon\|_{m-1}^2 + \|H^\epsilon\|_m^2 + \|\nabla H^\epsilon\|_{m-1}^2)
+ \|\nabla^2 P^\epsilon\|_{m-1} \|v^\epsilon\|_m + \epsilon^{-1} \|\nabla P^\epsilon\|_{m-1}^2 \right\}. \tag{3.7}
$$

where the pressure $P^\epsilon := p^\epsilon + \frac{1}{2} |H^\epsilon|^2 := P^\epsilon_1 + P^\epsilon_2$. Here, $P^\epsilon_1$ is the “Euler” part of the pressure, which solves

$$
\begin{cases}
\Delta P^\epsilon_1 = -\nabla \cdot (v^\epsilon \cdot \nabla v^\epsilon - H^\epsilon \cdot \nabla H^\epsilon) & \text{in } \Omega, \\
\partial_n P^\epsilon_1 = -(v^\epsilon \cdot \nabla v^\epsilon - H^\epsilon \cdot \nabla H^\epsilon) \cdot n & \text{on } \partial \Omega,
\end{cases} \tag{3.8}
$$

and $P^\epsilon_2$ is the “Navier-Stokes” part of the pressure, which solves

$$
\begin{cases}
\Delta P^\epsilon_2 = 0 & \text{in } \Omega, \\
\partial_n P^\epsilon_2 = \epsilon \Delta v^\epsilon \cdot n & \text{on } \partial \Omega.
\end{cases} \tag{3.9}
$$

**Proof** In view of Korn’s inequality and Lemma 3.2, we can prove the case for $m = 0$. Now we assume that (3.7) has been proved for $|\alpha| \leq m - 1$. We shall prove that it also holds for $|\alpha| = m \geq 1$. Applying $Z^\alpha$ with $|\alpha| = m$ to (1.1)$_1$ and (1.1)$_2$, respectively, we obtain that

$$
\begin{cases}
\partial_t Z^\alpha v^\epsilon + v^\epsilon \cdot \nabla Z^\alpha v^\epsilon - H^\epsilon \cdot \nabla Z^\alpha H^\epsilon + Z^\alpha \nabla P^\epsilon = \epsilon Z^\alpha \Delta v^\epsilon + C_1, \\
\partial_t Z^\alpha H^\epsilon + v^\epsilon \cdot \nabla Z^\alpha H^\epsilon - H^\epsilon \cdot \nabla Z^\alpha v^\epsilon = \epsilon Z^\alpha \Delta H^\epsilon + C_2,
\end{cases} \tag{3.10}
$$

where

$$
C_1 = -[Z^\alpha, v^\epsilon \cdot \nabla]v^\epsilon + [Z^\alpha, H^\epsilon \cdot \nabla]H^\epsilon, \quad C_2 = -[Z^\alpha, v^\epsilon \cdot \nabla]H^\epsilon + [Z^\alpha, H^\epsilon \cdot \nabla]v^\epsilon.
$$

\(\Box\)
Multiplying (3.10)₁ and (3.10)₂ by $Z^α v^ε$ and $Z^α H^ε$, respectively, and integrating by parts, we have
\[
\frac{1}{2} \frac{d}{dt} \left( \|Z^α v^ε\|^2 + \|Z^α H^ε\|^2 \right) = \epsilon (Z^α \Delta v^ε, Z^α v^ε) + \epsilon (Z^α \Delta H^ε, Z^α H^ε)
\]
\[
+ (C_1, Z^α v^ε) + (C_2, Z^α H^ε) - (Z^α \nabla P^ε, Z^α v^ε). \tag{3.11}
\]

First, in view of Lemma 2.4, we obtain that
\[
\left| (C_1, Z^α v^ε) + (C_2, Z^α H^ε) \right| \leq C_{m+1} \left( \|v^ε\|_{W^{1,∞}} + \|H^ε\|_{W^{1,∞}} \right)
\times \left( \|v^ε\|_m^2 + \|\nabla v^ε\|^2_{m-1} + \|H^ε\|_m^2 + \|\nabla H^ε\|_{m-1}^2 \right). \tag{3.12}
\]

Next, we deal with the terms with the viscosity coefficient $\epsilon$. Notice that
\[
\epsilon \int_Ω Z^α \Delta v^ε \cdot Z^α v^ε dx = 2\epsilon \int_Ω (\nabla \cdot Z^α S v^ε) \cdot Z^α v^ε dx + 2\epsilon \int_Ω ([Z^α, \nabla]S v^ε) \cdot Z^α v^ε dx. \tag{3.13}
\]

For the first term of the right-hand side of (3.13), by integrating by parts, we get
\[
\epsilon \int_Ω (\nabla \cdot Z^α S v^ε) \cdot Z^α v^ε dx
\]
\[
= - \epsilon \int_Ω Z^α S v^ε \cdot \nabla Z^α v^ε dx + \epsilon \int_{\partial Ω} ((Z^α S v^ε) \cdot n) \cdot Z^α v^ε dσ
\]
\[
= - \epsilon \|S(Z^α v^ε)\|_2^2 - \epsilon \int_Ω [Z^α, S]v^ε \cdot \nabla Z^α v^ε dx + \epsilon \int_{\partial Ω} ((Z^α S v^ε) \cdot n) \cdot Z^α v^ε dσ. \tag{3.14}
\]

Thus, it follows from Lemma 2.2 that there exists a $c_0 > 0$ such that
\[
\epsilon \int_Ω (\nabla \cdot Z^α S v^ε) \cdot Z^α v^ε dx \leq - c_0 \epsilon \|\nabla (Z^α v^ε)\|^2 + C_1 \|v^ε\|_m^2 + C_{m+1} \epsilon \|\nabla Z^α v^ε\| \|\nabla v^ε\|_{m-1}
\]
\[
+ \epsilon \int_{\partial Ω} ((Z^α S v^ε) \cdot n) \cdot Z^α v^ε dσ. \tag{3.15}
\]

It remains to control the boundary term in (3.15). To this end, we have the following observations: due to the Navier boundary condition (2.4), we obtain
\[
\|\Pi \partial_n v^ε\|_{H^{m-1}(\partial Ω)} \leq |\theta v^ε|_{H^m(\partial Ω)} + 2\xi \|\Pi v^ε\|_{H^m(\partial Ω)} \leq C_{m+2} \|v^ε\|_{H^m(\partial Ω)}. \tag{3.16}
\]

In order to estimate the normal part of $\partial_n v^ε$, we can use the divergence free condition to write
\[
\nabla \cdot v^ε = \partial_n v^ε \cdot n + (\Pi \partial_{\mathbf{b}_1} v^ε)^1 + (\Pi \partial_{\mathbf{b}_2} v^ε)^2. \tag{3.17}
\]

Hence,
\[
|\partial_n v^ε \cdot n|_{H^{m-1}(\partial Ω)} \leq C_m \|v^ε\|_{H^m(\partial Ω)}. \tag{3.18}
\]

It follows from (3.16) and (3.18) that
\[
|\nabla v^ε|_{H^{m-1}(\partial Ω)} \leq C_{m+2} \|v^ε\|_{H^m(\partial Ω)}. \tag{3.19}
\]

Thanks to $v^ε \cdot n = 0$ on the boundary, we immediately obtain
\[
|(Z^α v^ε) \cdot n|_{H^1(\partial Ω)} \leq C_{m+2} \|v^ε\|_{H^m(\partial Ω)}, \quad |\alpha| = m. \tag{3.20}
\]

Based on the above observations, we return to estimate the boundary term in (3.15) as
\[
\int_{\partial Ω} ((Z^α S v^ε) \cdot n) \cdot Z^α v^ε dσ = \int_{\partial Ω} Z^α (\Pi (S v^ε \cdot n)) \cdot \Pi Z^α v^ε dσ
\]
\[
+ \int_{\partial Ω} Z^α (\partial_n v^ε \cdot n) Z^α v^ε \cdot ndσ + C_v^v, \tag{3.21}
\]

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where
\[ C^\alpha_0 = \int_{\partial \Omega} [\Pi, Z^\alpha](Sv^\alpha \cdot n) \cdot \Pi Z^\alpha v^\alpha \, d\sigma + \int_{\partial \Omega} [n, Z^\alpha](Sv^\alpha \cdot n)Z^\alpha v^\alpha \cdot n \, d\sigma \]
\[ - \sum_{|\alpha_1 + \alpha_2| = m, |\alpha_2| \geq 1} \int_{\partial \Omega} ((Z^{\alpha_1} Sv^\alpha) \cdot Z^{\alpha_2} n) \cdot Z^\alpha v^\alpha \, d\sigma. \]

By virtue of (3.19) and (1.2)1, we can easily obtain that
\[ |C^\alpha_0| \leq C_{m+1}|\nabla v^\alpha|_{H^{m-1}(\partial \Omega)}|v^\alpha|_{H^m(\partial \Omega)} \leq C_{m+1}|v^\alpha|^2_{H^m(\partial \Omega)}, \quad (3.21) \]
\[ \left| \int_{\partial \Omega} Z^\alpha (\Pi(Sv^\alpha \cdot n)) \cdot \Pi Z^\alpha v^\alpha \, d\sigma \right| \leq C_{m+1}|v^\alpha|^2_{H^m(\partial \Omega)}. \quad (3.22) \]

Integrating by parts along the boundary, we have
\[ \left| \int_{\partial \Omega} Z^\alpha (\partial_n v^\alpha \cdot n)Z^\alpha v^\alpha \cdot n \, d\sigma \right| \leq C_{m+2}|\partial_n v^\alpha \cdot n|_{H^{m-1}(\partial \Omega)}|Z^\alpha v^\alpha \cdot n|_{H^1(\partial \Omega)} \]
\[ \leq C_{m+2}|v^\alpha|^2_{H^m(\partial \Omega)}. \quad (3.23) \]

Hence, we get, from (3.14), (3.15), and (3.21)–(3.23), that
\[ \epsilon \int_{\Omega} (\nabla \cdot Z^\alpha Sv^\alpha) \cdot Z^\alpha v^\alpha \, dx \]
\[ \leq C_{m+2} \left( |v^\alpha|^2_m + \epsilon \left( \| \nabla Z^m v^\alpha \| \| \nabla v^\alpha \|_{m-1} + |v^\alpha|^2_{H^m(\partial \Omega)} \right) - C_0 \epsilon \| \nabla (Z^\alpha v^\alpha) \|^2. \quad (3.24) \]

As for the term \( \epsilon \int_{\Omega} (\{Z^\alpha, \nabla \} Sv^\alpha) \cdot Z^\alpha v^\alpha \, dx \) in (3.13), we can expand it as a sum of terms under the form
\[ \epsilon \int_{\Omega} \beta_k \partial_k (Z^\alpha Sv^\alpha) \cdot Z^\alpha v^\alpha \, dx, \quad |\beta| \leq m - 1, \]
where \( \beta_k \) \( (k = 1, 2, 3) \) are bounded smooth functions. Integrating by parts using (3.19), we have
\[ \epsilon \int_{\Omega} \beta_k \partial_k (Z^\alpha Sv^\alpha) \cdot Z^\alpha v^\alpha \, dx \leq C_{m+2} \epsilon \left( \| \nabla v^\alpha \|_{m-1} \| \nabla Z^m v^\alpha \| + \| \nabla v^\alpha \|_m + |v^\alpha|^2_{H^m(\partial \Omega)} \right). \]
\[ \int_{\Omega} (Z^\alpha \Delta v^\alpha \cdot Z^\alpha v^\alpha \, dx \leq C_{m+2} \left( |v^\alpha|^2_m + \epsilon \| \nabla Z^m v^\alpha \| \| \nabla v^\alpha \|_{m-1} + |v^\alpha|^2_{H^m(\partial \Omega)} \right. \]
\[ + \epsilon \| \nabla v^\alpha \|_m |v^\alpha|_m - C_0 \epsilon \| \nabla (Z^\alpha v^\alpha) \|^2. \quad (3.26) \]

For the term \( \epsilon(Z^\alpha \Delta H^\alpha \cdot Z^\alpha H^\alpha) \), by Lemma 2.1, we have
\[ \epsilon \int_{\Omega} Z^\alpha \Delta H^\alpha \cdot Z^\alpha H^\alpha \, dx \leq - \frac{3\epsilon}{4} \| \nabla (Z^\alpha H^\alpha) \|^2 + \delta^2 \| \nabla^2 H^\alpha \|^2_{m-1} + C_\delta \left( \| \nabla H^\alpha \|^2_{m-1} + |H^\alpha|^2_m \right), \quad (3.27) \]
where \( \delta \) is a small constant which will be chosen later. Finally, we estimate the term involving the pressure \( P^\alpha \) in (3.11). We have
\[ \left| \int_{\Omega} Z^\alpha \nabla P^\alpha \cdot Z^\alpha v^\alpha \, dx \right| \leq \| \nabla^2 P^\alpha \|_{m-1} |v^\alpha|_m + \int_{\Omega} Z^\alpha \nabla P^\alpha_2 \cdot Z^\alpha v^\alpha \, dx \]
\[ \leq \| \nabla^2 P^\alpha \|_{m-1} |v^\alpha|_m + C_{m+1} \| \nabla P^\alpha_2 \|_{m-1} |v^\alpha|_m \]
\[ + \int_{\Omega} \nabla Z^\alpha P^\alpha_2 \cdot Z^\alpha v^\alpha \, dx. \quad (3.28) \]
Now, we focus on the last term in (3.28). Integrating by parts yields that
\[ \int_{\Omega} \nabla Z^\alpha P_2 \cdot Z^\alpha v^\epsilon \, dx \leq \| \nabla P_2^\alpha \|_{m-1} \| \nabla Z^\alpha v^\epsilon \| + \int_{\partial \Omega} Z^\alpha P_2 Z^\alpha v^\epsilon \cdot n \, d\sigma. \]
To estimate the boundary term, we note that when \( m = 1 \), (3.7) can be obtained easily. Thus we assume that \( m \geq 2 \). Integrating by parts along the boundary, we get
\[ \int_{\partial \Omega} Z^\alpha P_2 Z^\alpha v^\epsilon \cdot n \, d\sigma \leq C_2 |Z^\alpha P_2|_{L^2(\partial \Omega)} \| Z^\alpha v^\epsilon \cdot n \|_{H^1(\partial \Omega)}, \]
where \( |\alpha| = m - 1 \). With the help of (3.20) and Lemma 2.3, we have
\[ \int_{\Omega} Z^\alpha \nabla P^\alpha \cdot Z^\alpha v^\epsilon \, dx \leq \| \nabla^2 P_1^\alpha \|_{m-1} \| v^\epsilon \|_m + C_{m+2} \| \nabla P_2^\alpha \|_{m-1} \| v^\epsilon \|_m \]
\[ + C_{m+2} \| \nabla P_2^\alpha \|_{m-1} \| \nabla Z^\alpha v^\epsilon \| + \epsilon^{-1} \| \nabla P_2^\alpha \|_{m-1}^2 \]
\[ + \epsilon \left( \| \nabla v^\epsilon \|_m \| v^\epsilon \|_m + \| v^\epsilon \|_{m-1}^2 \right). \] (3.29)
Substituting (3.12), (3.26), (3.27) and (3.29) into (3.11), we obtain that
\[ \frac{1}{2} \frac{d}{dt} \left( \| Z^\alpha v^\epsilon \|^2 + \| Z^\alpha H^\epsilon \|^2 \right) + c_0 \epsilon \left( \| \nabla (Z^\alpha v^\epsilon) \|^2 + \| \nabla (Z^\alpha H^\epsilon) \|^2 \right) \]
\[ \leq C_{m+2} \left( 1 + \| v^\epsilon \|_{W^{1,\infty}} + \| H^\epsilon \|_{W^{1,\infty}} \right) \left( \| v^\epsilon \|^2_m + \| \nabla v^\epsilon \|^2_{m-1} + \| H^\epsilon \|^2_m + \| \nabla H^\epsilon \|^2_{m-1} \right) \]
\[ + \delta \epsilon^2 \| \nabla^2 H^\epsilon \|^2_{m-1} + C_{m+2} \left( \epsilon \| Z^\alpha v^\epsilon \| \| \nabla v^\epsilon \|_{m-1} + \epsilon \| v^\epsilon \|_{H^1(\partial \Omega)} + \epsilon \| \nabla v^\epsilon \|_{m-1} \| v^\epsilon \|_m \right) \]
\[ + \epsilon \| \nabla Z^\alpha H^\epsilon \| \| \nabla H^\epsilon \|_{m-1} \| H^\epsilon \|_m \]
\[ + \| \nabla^2 P_1^\alpha \|_{m-1} \| v^\epsilon \|_m + \| \nabla P_2^\alpha \|_{m-1} \| v^\epsilon \|_m + \| \nabla P_2^\alpha \|_{m-1} \| \nabla Z^\alpha v^\epsilon \| \]
\[ + \epsilon^{-1} \| \nabla P_2^\alpha \|_{m-1}^2 + \epsilon \left( \| \nabla v^\epsilon \|_m \| v^\epsilon \|_m + \| v^\epsilon \|_{m-1}^2 \right). \] (3.30)
Consequently, using Lemma 2.3, Young’s inequality, and the assumptions with respect to \( |\alpha| \leq m - 1 \), we have
\[ \frac{1}{2} \frac{d}{dt} \left( \| v^\epsilon(t) \|^2_m + \| H^\epsilon(t) \|^2_m \right) + c_0 \epsilon \left( \| \nabla v^\epsilon \|^2_m + \| \nabla H^\epsilon \|^2_m \right) \]
\[ \leq C_{m+2} \left( 1 + \| v^\epsilon \|_{W^{1,\infty}} + \| H^\epsilon \|_{W^{1,\infty}} \right) \left( \| v^\epsilon \|^2_m + \| \nabla v^\epsilon \|^2_{m-1} + \| H^\epsilon \|^2_m + \| \nabla H^\epsilon \|^2_{m-1} \right) \]
\[ + \delta \epsilon^2 \| \nabla^2 H^\epsilon \|^2_{m-1} + \left( \| \nabla^2 P_1^\alpha \|_{m-1} \| v^\epsilon \|_m + \epsilon^{-1} \| \nabla P_2^\alpha \|_{m-1}^2 \right). \]
This ends the proof of Lemma 3.3. \( \square \)

3.2 Normal Derivative Estimates

In this subsection, we provide the estimates for \( \| \nabla v^\epsilon \|_{m-1} \) and \( \| \nabla H^\epsilon \|_{m-1} \). We note that
\[ \| \chi \partial_{\nu_i} v^\epsilon \|_{m-1} \leq C_m \| v^\epsilon \|_m, \quad \| \chi \partial_{\nu_i} H^\epsilon \|_{m-1} \leq C_m \| H^\epsilon \|_m, \quad i = 1, 2, \]
where \( \chi \) is compactly supported in one of \( \Omega_k \) and with value one in a vicinity of the boundary. Therefore, it suffices to estimate \( \| \chi \partial_{\nu_i} v^\epsilon \|_{m-1} \) and \( \| \chi \partial_{\nu_i} H^\epsilon \|_{m-1} \). We shall thus use the local coordinates (2.3).

Due to (3.17), we immediately obtain that
\[ \| \chi \partial_{\nu_i} v^\epsilon \cdot n \|_{m-1} \leq C_m \| v^\epsilon \|_m, \quad \| \chi \partial_{\nu_i} H^\epsilon \cdot n \|_{m-1} \leq C_m \| H^\epsilon \|_m. \] (3.31)
Thus, it remains to estimate \( \| \chi \Pi \partial_{\nu_i} v^\epsilon \|_{m-1} \) and \( \| \chi \Pi \partial_{\nu_i} H^\epsilon \|_{m-1} \). We define
\[ \eta^\epsilon_i = \chi \Pi (\nabla v^\epsilon + (\nabla v^\epsilon)^2) \eta + 2\zeta \chi \Pi v^\epsilon, \]
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\[ \eta_H = \chi \Pi ((\nabla H^e + (\nabla H^e)^T)n) + 2\chi \Pi (S(n)H^e). \]

In view of (1.2)1 and (1.2)2, we have
\[ \eta^e = 0, \quad \eta_H = 0 \quad \text{on} \quad \partial \Omega. \]

Moreover, since \( \eta_H^e \) and \( \eta_H^H \) can be rewritten as
\[ \eta_H^e = \chi \Pi \partial_n v^e + \chi \Pi (\nabla(v^e \cdot n) - \nabla n \cdot v^e - v^e \times (\nabla \times n) + 2\zeta v^e), \]
\[ \eta_H^H = \chi \Pi \partial_n H^e + \chi \Pi (\nabla(H^e \cdot n) - \nabla n \cdot H^e - H^e \times (\nabla \times n) + 2S(n)H^e), \]  
we have
\[ \|\chi \Pi \partial_n v^e\|_{m-1} \leq C_{m+1}(\|\eta_v\|_{m-1} + \|v^e\|_m), \]
\[ \|\chi \Pi \partial_n H^e\|_{m-1} \leq C_{m+1}(\|\eta_H\|_{m-1} + \|H^e\|_m). \]

Hence, we need only to estimate \( \|\eta_v^e\|_{m-1} \) and \( \|\eta_H^H\|_{m-1} \).

We have the following conormal estimates for \( \eta_v^e \) and \( \eta_H^H \):

Lemma 3.4 For every \( m \geq 1 \), it holds that
\[ \frac{1}{2} \frac{d}{dt}(\|\eta_v^e(t)\|^2_{m-1} + \|\eta_H^H(t)\|^2_{m-1}) + \epsilon (\|\nabla \eta_v^e\|^2_{m-1} + \|\nabla \eta_H^H\|^2_{m-1}) \leq C_{m+2} \left( (1 + \|v^e\|_{2,\infty} + \|\nabla v^e\|_{1,\infty} + \|H^e\|_{2,\infty} + \|\nabla H^e\|_{1,\infty}) \times (\|\eta_v^e\|^2_{m-1} + \|\eta_H^H\|^2_{m-1} + \|v^e\|^2_m + \|\nabla v^e\|^2_{m-1} + \|\nabla H^e\|^2_{m-1}) \right) \]
\[ \cdot \left( \|\nabla P^e_{1}\|^2_{m-1} + \|\nabla P^e_{2}\|^2_{m-1} + \epsilon^{-1}\|\nabla P^e_{2}\|^2_{m-1} \right). \]

Proof Setting \( M_v = \nabla v^e \) and \( M_H = \nabla H^e \) and applying the operator \( \nabla \) to (1.1)1 and (1.1)2, respectively, we get that
\[ \begin{cases} \partial_t M_v - \epsilon \Delta M_v + v^e \cdot \nabla M_v - H^e \cdot \nabla M_H = (M_H)^2 - (M_v)^2 - \nabla^2 P^e, \\ \partial_t M_H - \epsilon \Delta M_H + v^e \cdot \nabla M_H - H^e \cdot \nabla M_v = M_v M_H - M_H M_v. \end{cases} \]

Consequently, \( (\eta_v^e, \eta_H^H) \) solve the equations
\[ \begin{cases} \partial_t \eta_v^e - \epsilon \Delta \eta_v^e + v^e \cdot \nabla \eta_v^e - H^e \cdot \nabla \eta_H^H = F_v^b + F_v^\chi + F_v^e = 2\chi \Pi (\nabla^2 P^e \), \\ \partial_t \eta_H^H - \epsilon \Delta \eta_H^H + v^e \cdot \nabla \eta_H^H - H^e \cdot \nabla \eta_v^e = F_H^b + F_H^\chi + F_H^e, \end{cases} \]
where
\[ F_v^b = -\chi \Pi ((\nabla v^e)^2 + ((\nabla v^e)^T)^2 - (\nabla H^e)^2 - ((\nabla H^e)^T)^2)n - 2\chi \Pi \nabla P^e, \]
\[ F_v^\chi = -2\epsilon \chi \Pi (S v^e n + \zeta v^e) - 4\epsilon \nabla \chi \cdot \nabla (\Pi (S v^e n + \zeta v^e)) + 2(v^e \cdot \nabla \chi) \Pi (S v^e n + \zeta v^e) - 2(H^e \cdot \nabla \chi) \Pi (S H^e n + (S n - \zeta I_3) H^e), \]
\[ F_v^e = 2\chi (v^e \cdot \nabla \Pi) (S v^e n + \zeta v^e) + 2\chi \Pi (S v^e (v^e \cdot \nabla) n) - 2\epsilon \chi (\Delta \Pi) (S v^e n + \zeta v^e) - 4\epsilon \nabla \Pi \cdot \nabla (S v^e n + \zeta v^e) - 2\chi \Pi (S v^e \Delta n + 2\nabla S v^e \cdot n) - 2\chi (H^e \cdot \nabla \Pi) (S H^e n + (S n - \zeta I_3) H^e) - 2\chi \Pi (S H^e (H^e \cdot \nabla) n) - 2\chi (H^e \cdot \nabla ((S n - \zeta I_3) H^e)), \]
\[ F_H^b = -\chi \Pi ((M_H M_v + M_v M_H)^2 - M_v M_H - M_H^2 M_v) n), \]
\[ F_H^\chi = -2\epsilon \chi \Pi (S H^e n + S n H^e) - 4\epsilon \nabla \chi \cdot \nabla (\Pi (S H^e n + S n H^e)). \]
respectively, and integrating by parts, we obtain that

\[
F^c_H = 2\chi(v^e \cdot \nabla \chi)\Pi(SH^*n + SnH^* - 2\Pi(SH^*n + (\zeta I_3 - Sn)v^e),
\]

\[
- 2\chi\Delta(P\Pi(SH^*n + SnH^*) + v^e \cdot \nabla(Sn)H^*)
\]

\[
- 2\chi\Pi(SH^* \Delta n + 2\nabla \cdot SH^*\nabla n + \Delta S(n)H^* + 2\nabla \cdot Sn\nabla H^*)
\]

\[
- 2\chi(H \cdot \nabla)(Sv^e n + (\zeta I_3 - Sn)v^e) - 2\chi\Pi(Sv^e (H \cdot \nabla)n)
\]

\[
- 2\Pi(H^e \cdot \nabla ((\zeta I_3 - Sn)v^e)).
\]

Let us start with the case of \( m = 1 \). Multiplying (3.38)\(_1\) and (3.38)\(_2\) by \( \eta^c_v \) and \( \eta^c_H \), respectively, and integrating by parts, we obtain that

\[
\frac{1}{2} \frac{d}{dt}(\|\eta^c_v\|^2 + \|\eta^c_H\|^2) + \epsilon(\|\nabla \eta^c_v\|^2 + \|\nabla \eta^c_H\|^2)
\]

\[
= (F^b_v + F^c_v, \eta^c_v) + (F^b_H + F^c_H, \eta^c_H) - 2(\chi\Pi(\nabla^2 P^e n), \eta^c_v). \tag{3.39}
\]

To handle the right-hand side terms of (3.39), in view of Lemma 2.4, we arrive at

\[
\|F^c_v\|_{m-1} + \|F^c_v\|_{m-1} \leq C_m \left( \|v^e\|_{W^{1,\infty}} + \|H^e\|_{W^{1,\infty}} \right) \left( \|\nabla v^e\|_{m-1} + \|\nabla H^e\|_{m-1} \right)
\]

\[
+ \|\nabla P^e\|_{m-1}, \tag{3.40}
\]

\[
\|F^c_H\|_{m-1} + \|F^c_H\|_{m-1} \leq C_{m+2} \left( \epsilon(\|\nabla^2 v^e\|_{m-1} + \|\nabla^2 H^e\|_{m-1} + \|\nabla v^e\|_{m-1} \right)
\]

\[
+ \|\nabla H^e\|_{m-1} + \|\eta^e_v\|_{m} + \|H^e\|_{m} \right)
\]

\[
+ \left( \|v^e\|_{W^{1,\infty}} + \|H^e\|_{W^{1,\infty}} \right) \left( \|v^e\|_{m-1} + \|H^e\|_{m-1} \right)
\]

\[
+ \|\nabla v^e\|_{m-1} + \|\nabla H^e\|_{m-1}) \right). \tag{3.41}
\]

Since all terms in \( F^c_v \) and \( F^c_H \) are supported away from the boundary, we can control any derivatives by the \( \| \cdot \|_m \) norms. Therefore, it is easy to obtain that

\[
\|F^c_v\|_{m-1} + \|F^c_H\|_{m-1} \leq C_{m+1} \left( 1 + \|v^e\|_{W^{1,\infty}} + \|H^e\|_{W^{1,\infty}} \right) \left( \|v^e\|_{m} + \|H^e\|_{m} \right)
\]

\[
+ \epsilon(\|\nabla v^e\|_{m} + \|\nabla H^e\|_{m}) \right). \tag{3.42}
\]

Finally, we estimate \( \left( \chi\Pi(\nabla^2 P^e n), \eta^c_v \right) \). Recalling that \( P^e = P^e_1 + P^e_2 \), we get that

\[
\left| \left( \chi\Pi(\nabla^2 P^e n), \eta^c_v \right) \right| \leq \|\nabla^2 P^e_1\| \|\eta^c_v\| + \left| \int_{\Omega} \chi\Pi(\nabla^2 P^e_2 n) \cdot \eta^c_v dx \right| \tag{3.43}
\]

Since \( \eta^c_v = 0 \) on the boundary, we can integrate by the parts the last term in (3.43) to obtain that

\[
\left| \int_{\Omega} \chi\Pi(\nabla^2 P^e_2 n) \cdot \eta^c_v dx \right| \leq C_2 \|\nabla P^e_2\| \left( \|\nabla \eta^c_v\| + \|\eta^c_v\| \right). \tag{3.44}
\]

Consequently, substituting (3.40)–(3.42), (3.43) and (3.44) into (3.39), we have

\[
\frac{1}{2} \frac{d}{dt}(\|\eta^c_v\|^2 + \|\eta^c_H\|^2) + \epsilon(\|\nabla \eta^c_v\|^2 + \|\nabla \eta^c_H\|^2)
\]

\[
\leq C_3 \left( \epsilon(\|\nabla^2 v^e\| + \|\nabla^2 H^e\| + \|\nabla v^e\|_1 + \|\nabla H^e\|_1) \left( \|\eta^c_v\| + \|\eta^c_H\| \right)
\]

\[
+ (1 + \|v^e\|_{W^{1,\infty}} + \|H^e\|_{W^{1,\infty}}) \left( \|v^e\|_1^2 + \|H^e\|_1^2 + \|\nabla v^e\|^2 + \|\nabla H^e\|^2
\]

\[
+ \|\eta^c_v\|^2 + \|\eta^c_H\|^2) + \|\nabla P^e_1\| \left( \|\nabla \eta^c_v\| + \|\eta^c_v\| \right) + \|\nabla^2 P^e_1\| \|\eta^c_v\| + \|\nabla P^e_2\| \|\eta^c_v\| \right). \tag{3.45}
\]
In view of (3.31) and (3.32), we obtain
\[ \epsilon \| \nabla^2 v^\epsilon \|_{m-1} \leq C_2 \epsilon \left( \| \nabla \eta^\epsilon_c \|_{m-1} + \| \nabla v^\epsilon \|_m + \| v^\epsilon \|_m \right). \tag{3.46} \]

Similarly, we get
\[ \epsilon \| \nabla^2 H^\epsilon \|_{m-1} \leq C_2 \epsilon \left( \| \nabla \eta^\epsilon_h \|_{m-1} + \| \nabla H^\epsilon \|_m + \| H^\epsilon \|_m \right). \tag{3.47} \]

Furthermore, using (3.46), (3.47) and Young’s inequality, we have
\[
\frac{1}{2} \frac{d}{dt} (\| \eta^\epsilon_c \|^2 + \| \eta^\epsilon_h \|^2) + \epsilon \left( \| \nabla \eta^\epsilon_c \|^2 + \| \nabla \eta^\epsilon_h \|^2 \right) \\
\leq C_3 \epsilon \left( \| \nabla v^\epsilon \|_1 + \| \nabla H^\epsilon \|_1 \right) \\
+ (1 + \| v^\epsilon \|_{W^{1,\infty}} + \| H^\epsilon \|_{W^{1,\infty}}) \left( \| \nabla v^\epsilon \|^2 + \| \nabla H^\epsilon \|^2 + \| \nabla v^\epsilon \|^2 + \| \nabla H^\epsilon \|^2 \right) \\
+ \| \eta^\epsilon_c \|^2 + \| \eta^\epsilon_h \|^2) + \epsilon^{-1} \| \nabla P^\epsilon \|_2^2 + \| \eta^\epsilon_c \| \left( \| \nabla P^\epsilon \| + \| \nabla^2 P^\epsilon \| \right). \tag{3.48} \]

Since \( \epsilon (\| \nabla v^\epsilon \|_1 + \| \nabla H^\epsilon \|_1) \) has been estimated in Lemma 3.3, this yields (3.36) for the case of \( m = 1 \).

To prove the general case, we assume that (3.36) holds for \( |\alpha| \leq m - 2 \). Let us consider the situation where \( |\alpha| = m - 1 \). Applying \( Z^\alpha \) to (3.38)\(_1\) and (3.38)\(_2\), respectively, yields that
\[
\begin{align*}
\partial_t Z^\alpha \eta^\epsilon_c &- \epsilon Z^\alpha \Delta \eta^\epsilon_c + v^\epsilon \cdot \nabla Z^\alpha \eta^\epsilon_c - H^\epsilon \cdot \nabla Z^\alpha \eta^\epsilon_h \\
&= Z^\alpha F^\alpha_v + Z^\alpha F^\alpha_h + Z^\alpha F^\alpha_c - 2 Z^\alpha (\chi \Pi (\nabla^2 P^\epsilon \eta^\epsilon_c) + C_3, \\
\partial_t Z^\alpha \eta^\epsilon_h &- \epsilon Z^\alpha \Delta \eta^\epsilon_h + v^\epsilon \cdot \nabla Z^\alpha \eta^\epsilon_h - H^\epsilon \cdot \nabla Z^\alpha \eta^\epsilon_c \\
&= Z^\alpha F^\alpha_v + Z^\alpha F^\alpha_h + Z^\alpha F^\alpha_c + C_4,
\end{align*}
\tag{3.49} \]

where
\[
C_3 = -[Z^\alpha, v^\epsilon \cdot \nabla] \eta^\epsilon_c + [Z^\alpha, H^\epsilon \cdot \nabla] \eta^\epsilon_h, \\
C_4 = -[Z^\alpha, v^\epsilon \cdot \nabla] \eta^\epsilon_h + [Z^\alpha, H^\epsilon \cdot \nabla] \eta^\epsilon_c.
\]

Multiplying (3.49)\(_1\) and (3.49)\(_2\) by \( Z^\alpha \eta^\epsilon_c \) and \( Z^\alpha \eta^\epsilon_h \), respectively, and integrating by parts, we obtain that
\[
\frac{1}{2} \frac{d}{dt} \left( \| Z^\alpha \eta^\epsilon_c \|^2 + \| Z^\alpha \eta^\epsilon_h \|^2 \right) \\
\leq \epsilon (Z^\alpha \Delta \eta^\epsilon_c, Z^\alpha \eta^\epsilon_c) + \epsilon (Z^\alpha \Delta \eta^\epsilon_h, Z^\alpha \eta^\epsilon_h) \\
+ (C_3, Z^\alpha \eta^\epsilon_c) + (C_4, Z^\alpha \eta^\epsilon_h) - 2 (Z^\alpha (\chi \Pi (\nabla^2 P^\epsilon \eta^\epsilon_c), Z^\alpha \eta^\epsilon_h) \\
+ (Z^\alpha F^\alpha_v + Z^\alpha F^\alpha_h, Z^\alpha \eta^\epsilon_c) + (Z^\alpha F^\alpha_v + Z^\alpha F^\alpha_h, Z^\alpha \eta^\epsilon_h). \tag{3.50} \]

We first handle \( \epsilon (Z^\alpha \Delta \eta^\epsilon_c, Z^\alpha \eta^\epsilon_c) \) and \( \epsilon (Z^\alpha \Delta \eta^\epsilon_h, Z^\alpha \eta^\epsilon_h) \). We observe that
\[
\int_\Omega Z^\alpha \partial_i \eta^\epsilon_c \cdot Z^\alpha \eta^\epsilon_c dx = - \int_\Omega |\partial_i Z^\alpha \eta^\epsilon_c|^2 dx - \int_\Omega [Z^\alpha, \partial_i] \eta^\epsilon_c \cdot \partial_i Z^\alpha \eta^\epsilon_c dx \\
+ \int_\Omega [Z^\alpha, \partial_i] \partial_i \eta^\epsilon_c \cdot Z^\alpha \eta^\epsilon_c dx, \tag{3.51} \]

where \( i = 1, 2, 3 \). To estimate the last two terms on the right-hand side of (3.51), we use the structure of the commutator \([Z^\alpha, \partial_i]\) and the expansion \( \partial_i = \beta^1 \partial_{\eta^1} + \beta^2 \partial_{\eta^2} + \beta^3 \partial_{\eta^3} \) in the local basis. We have the following expansion:
\[
[Z^\alpha, \partial_i] \eta^\epsilon_c = \sum_{\gamma, |\gamma| = |\alpha|-1} c_{\gamma} \partial_\gamma Z^\alpha \eta^\epsilon_c + \sum_{\beta, |\beta| = |\alpha|} c_{\beta} Z^\beta \eta^\epsilon_c.
\]
This yields the estimates
\[
\left| \int \nabla Z^\alpha \eta_\alpha' \cdot \partial_i Z^\alpha \eta_\alpha' \, dx \right| \leq C_m \| \nabla Z^\alpha \eta_\alpha' \| \left( \| \nabla \eta_\alpha' \|_{m-2} + \| \eta_\alpha' \|_{m-1} \right), \tag{3.52}
\]
\[
\left| \int \partial_i \nabla Z^\alpha \eta_\alpha' \cdot Z^\alpha \eta_\alpha' \, dx \right| \leq C_{m+1} \| \nabla \eta_\alpha' \|_{m-1} \left( \| \nabla \eta_\alpha' \|_{m-2} + \| \eta_\alpha' \|_{m-1} \right). \tag{3.53}
\]
Therefore, substituting (3.52) and (3.53) into (3.51) gives
\[
|\varepsilon(Z^\alpha \Delta \eta_\alpha', Z^\alpha \eta_\alpha')| \leq C_{m+1} \varepsilon \left( \| \nabla \eta_\alpha' \|_{m-1} \left( \| \nabla \eta_\alpha' \|_{m-2} + \| \eta_\alpha' \|_{m-1} \right) \right.
+ \left. \| \nabla Z^\alpha \eta_\alpha' \| \left( \| \nabla \eta_\alpha' \|_{m-2} + \| \eta_\alpha' \|_{m-1} \right) \right) - \varepsilon \| \nabla Z^\alpha \eta_\alpha' \|^2. \tag{3.54}
\]
Arguing in a manner similar to (3.54), we can also obtain that
\[
|\varepsilon(Z^\alpha \Delta \eta_\alpha', Z^\alpha \eta_\alpha')| \leq C_{m+1} \varepsilon \left( \| \nabla \eta_\alpha' \|_{m-1} \left( \| \nabla \eta_\alpha' \|_{m-2} + \| \eta_\alpha' \|_{m-1} \right) \right.
+ \left. \| \nabla Z^\alpha \eta_\alpha' \| \left( \| \nabla \eta_\alpha' \|_{m-2} + \| \eta_\alpha' \|_{m-1} \right) \right) - \varepsilon \| \nabla Z^\alpha \eta_\alpha' \|^2. \tag{3.55}
\]
Secondly, we get, from (3.40)–(3.42), (3.46), and (3.47), that
\[
\| F^b_i \|_{m-1} + \| F^a_i \|_{m-1} + \| F^a_e \|_{m-1} \leq C_{m+2} \left( \| v^\alpha \|_{W^{1,\infty}} + \| H^\alpha \|_{W^{1,\infty}} \right) \left( \| v^\alpha \|_{m} \| \nabla v^\alpha \|_{m-1} + \| H^\alpha \|_{m} \right)
+ \| \nabla H^\alpha \|_{m-1} \right) + \varepsilon \| \nabla \eta_\alpha' \|_{m} + \varepsilon \| \nabla \eta_\alpha' \|_{m-1} + \| \nabla P^\alpha \|_{m-1}, \tag{3.56}
\]
\[
\| F^b_i \|_{m-1} + \| F^a_i \|_{m-1} + \| F^a_e \|_{m-1} \leq C_{m+2} \left( \| v^\alpha \|_{W^{1,\infty}} + \| H^\alpha \|_{W^{1,\infty}} \right) \left( \| v^\alpha \|_{m} \| \nabla v^\alpha \|_{m-1} + \| H^\alpha \|_{m} \right)
+ \| \nabla H^\alpha \|_{m-1} \right) + \varepsilon \| \nabla \eta_\alpha' \|_{m} + \varepsilon \| \nabla \eta_\alpha' \|_{m-1}, \tag{3.57}
\]
Next, we estimate \( \| C_3 \| \) and \( \| C_4 \| \). In the local coordinates, we observe that
\[
f \cdot \nabla g = f_1 \partial_\varphi g + f_2 \partial_\varphi g + f \cdot N \partial_2 g,
\]
which implies
\[
\left[ Z^\alpha, v^\alpha \cdot \nabla \right] \eta_\alpha' = \sum_{i=1,2} \sum_{|\beta|+|\gamma|+1=|\alpha|} Z^\beta v^\beta Z^\gamma \eta_\gamma' + \sum_{|\beta|+|\gamma|=|\alpha|} Z^\beta \left( v^\beta \cdot N \right) Z^\gamma \partial_\gamma \eta_\gamma',
\]
\[
= \sum_{i=1,2} \sum_{|\beta|+|\gamma|+1=|\alpha|} Z^\beta v^\beta Z^\gamma \eta_\gamma' + \sum_{|\beta|+|\gamma|+1=|\alpha|} Z^\beta \left( v^\beta \cdot N \right) Z^\gamma \partial_\gamma \eta_\gamma'. \tag{3.58}
\]
We can do similar calculations for the other terms in \( C_3 \) and \( C_4 \). Consequently, based on (1.2), (1.2) and Lemma 2.4, we get
\[
\| C_3 \| \leq C_m \left( \| v^\alpha \|_{2,\infty} + \| H^\alpha \|_{W^{1,\infty}} + \| Z \eta_\alpha' \|_{L^\infty} \right) \left( \| \eta_\alpha' \|_{m-1} + \| v^\alpha \|_{m} \right)
+ C_m \left( \| H^\alpha \|_{2,\infty} + \| H^\alpha \|_{W^{1,\infty}} + \| Z \eta_\alpha' \|_{L^\infty} \right) \left( \| \eta_\alpha' \|_{m-1} + \| H^\alpha \|_{m} \right), \tag{3.59}
\]
\[
\| C_4 \| \leq C_m \left( \| v^\alpha \|_{2,\infty} + \| H^\alpha \|_{W^{1,\infty}} + \| Z \eta_\alpha' \|_{L^\infty} \right) \left( \| \eta_\alpha' \|_{m-1} + \| v^\alpha \|_{m} \right)
+ C_m \left( \| H^\alpha \|_{2,\infty} + \| H^\alpha \|_{W^{1,\infty}} + \| Z \eta_\alpha' \|_{L^\infty} \right) \left( \| \eta_\alpha' \|_{m-1} + \| H^\alpha \|_{m} \right). \tag{3.60}
\]
Finally, it remains to deal with the terms involving the pressure \( P^\alpha \). As above, we use the split \( P^\alpha = F^b_i + P^e_i \). Integrating by parts the terms involving \( P^e_i \) leads to getting that
\[
\left| \left( Z^\alpha \left( \chi \Pi (\nabla^2 P^\alpha n) \right), Z^\alpha \eta_\alpha' \right) \right|
\]
\[ \leq C_{m+2}\left( \|\nabla^2 P\|_{m-1}\|\eta\|_{m-1} + \|\nabla P\|_{m-1}\left(\|\nabla Z^{-1}\eta\| + \|\eta\|_{m-1}\right) \right). \] (3.61)

By combining (3.50), (3.54)–(3.57) and (3.59)–(3.61), and using the induction assumption and Young’s inequality, we get (3.36).

\section{3.3 Pressure Estimates}

The aim of this subsection is to give the pressure estimates.

\textbf{Lemma 3.5} For every \( m \geq 2 \), we have the following estimates:

\[ \|\nabla P\|_{m-1} + \|\nabla^2 P\|_{m-1} \leq C_{m+2}\left( 1 + \|v\|_{W^{1,\infty}} \right) \left( \|v\|_{m} + \|\nabla v\|_{m-1} \right) + \left( 1 + \|H\|_{W^{1,\infty}} \right) \left( \|H\|_{m} + \|\nabla H\|_{m-1} \right). \quad (3.62) \]

\[ \|\nabla P\|_{m-1} \leq C_{m+2}\epsilon \left( \|v\|_{m} + \|\nabla v\|_{m-1} \right). \quad (3.63) \]

\textbf{Proof} Recall that \( P = P_1 + P_2 \), and that \( P_1 \) and \( P_2 \) are given in (3.8) and (3.9), respectively. In view of standard elliptic regularity theory with Neumann boundary condition, we obtain that

\[ \|\nabla P\|_{m-1} + \|\nabla^2 P\|_{m-1} \leq C_{m+1} \left( \|\nabla v \cdot \nabla v - \nabla H \cdot \nabla H\|_{m-1} + \|\nabla v \cdot \nabla v - H \cdot \nabla H\|_{m-1} \right) + \left( \|v\|_{m} + \|\nabla v\|_{m-1} \right) \left( \|H\|_{W^{1,\infty}} \right) \left( \|H\|_{m} + \|\nabla H\|_{m-1} \right). \quad (3.64) \]

Using Lemma 2.4, we have

\[ \|\nabla v \cdot \nabla v - \nabla H \cdot \nabla H\|_{m-1} + \|\nabla v \cdot \nabla v - H \cdot \nabla H\|_{m-1} \leq C_{m+2}\left( \|v\|_{W^{1,\infty}} \|\nabla v\|_{m-1} + \|H\|_{W^{1,\infty}} \|\nabla H\|_{m-1} \right). \quad (3.65) \]

As for the last term in (3.64), by virtue of \( v \cdot n = 0 \) and \( H \cdot n = 0 \) on \( \partial \Omega \), Lemma 2.3, and Lemma 2.4, we have

\[ \left| \left( \nabla v \cdot \nabla v - H \cdot \nabla H\right) \cdot n \right|_{H^{m-\frac{1}{2}}(\partial \Omega)} \leq C_{m+2}
\]

\[ \leq C_{m+2}\left( \|\nabla (v \otimes v)\|_{m-1} + \|v \otimes v\|_{m} + \|\nabla (H \otimes H)\|_{m-1} + \|H \otimes H\|_{m} \right) \]

\[ \leq C_{m+2}\left( \|v\|_{W^{1,\infty}} \|\nabla v\|_{m-1} + \|H\|_{W^{1,\infty}} \|\nabla H\|_{m-1} \right). \quad (3.66) \]

Putting (3.65) and (3.66) into (3.64) yields (3.62).

It remains to estimate \( P_2 \). By using the standard elliptic regularity theory with Neumann boundary conditions again, we obtain that

\[ \|\nabla P_2\|_{m-1} \leq C_{m}\epsilon \|\Delta v \cdot n\|_{H^{m-\frac{1}{2}}(\partial \Omega)}. \]

Since

\[ \Delta v \cdot n = 2 \left( \nabla \cdot (Sv \cdot n) - \sum_j (Sv \cdot \partial_j n)\right), \]

we infer that

\[ \|\Delta v \cdot n\|_{H^{m-\frac{1}{2}}(\partial \Omega)} \leq C_{m}\epsilon \|\nabla (Sv \cdot n)\|_{H^{m-\frac{1}{2}}(\partial \Omega)} + C_{m+1}\epsilon \|v\|_{H^{m-\frac{1}{2}}(\partial \Omega)}. \]

Based on (2.4) and (3.17), we can further arrive at

\[ \|\Delta v \cdot n\|_{H^{m-\frac{1}{2}}(\partial \Omega)} \leq C_{m}\epsilon \|\nabla (Sv \cdot n)\|_{H^{m-\frac{1}{2}}(\partial \Omega)} + C_{m+1}\epsilon \|v\|_{H^{m-\frac{1}{2}}(\partial \Omega)}. \quad (3.67) \]

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As for $|\nabla \cdot (Sv^n)|_{H^{m-\frac{2}{3}}(\partial \Omega)}$, we can use (3.17) to obtain that
\[
|\nabla \cdot (Sv^n)|_{H^{m-\frac{2}{3}}(\partial \Omega)} \leq C_{m+1}|\partial_n(Sv^n) \cdot n|_{H^{m-\frac{2}{3}}(\partial \Omega)} + C_{m+1}\left(|\Pi(Sv^n)|_{H^{m-\frac{2}{3}}(\partial \Omega)} + |\nabla v^\epsilon|_{H^{m-\frac{2}{3}}(\partial \Omega)}\right).
\]
Also, with the help of (1.2), we have
\[
|\nabla \cdot (Sv^n)|_{H^{m-\frac{2}{3}}(\partial \Omega)} \leq C_{m+1}\left(|\partial_n(Sv^n) \cdot n|_{H^{m-\frac{2}{3}}(\partial \Omega)} + |v^\epsilon|_{H^{m-\frac{2}{3}}(\partial \Omega)}\right).
\]
(3.68)
The first term on the right-hand side of (3.68) can be controlled as
\[
|\partial_n(Sv^n) \cdot n|_{H^{m-\frac{2}{3}}(\partial \Omega)} \leq C_{m+1}\left(|\partial_n(v^\epsilon \cdot n)|_{H^{m-\frac{2}{3}}(\partial \Omega)} + |\nabla v^\epsilon|_{H^{m-\frac{2}{3}}(\partial \Omega)}\right).
\]
By taking the normal derivative of (3.17) and using (2.4), we have
\[
|\partial_n(\partial_n v^\epsilon \cdot n)|_{H^{m-\frac{2}{3}}(\partial \Omega)} \leq C_{m+1}\left(|\Pi\partial_n v^\epsilon|_{H^{m-\frac{2}{3}}(\partial \Omega)} + C|\nabla v^\epsilon|_{H^{m-\frac{2}{3}}(\partial \Omega)}\right)
\]
\[
\leq C_{m+2}|v^\epsilon|_{H^{m-\frac{2}{3}}(\partial \Omega)}.
\]
(3.69) Consequently, it follows from (3.67)–(3.93) that
\[
|\Delta v^\epsilon \cdot n|_{H^{m-\frac{2}{3}}(\partial \Omega)} \leq C_{m+2}|v^\epsilon|_{H^{m-\frac{2}{3}}(\partial \Omega)}.
\]
By Lemma 3.3, we obtain (3.63). This complete the proof of Lemma 3.5.

Choosing $\delta$ small enough in Lemma 3.3, we get, from Lemma 2.3 and Lemmas 3.3–3.5, that, for $m \geq 4$,
\[
|v^\epsilon|^2_m + \|H^\epsilon\|^2_m + \|\nabla v^\epsilon\|^2_{m-1} + \|\nabla H^\epsilon\|^2_{m-1} + \epsilon \int^t_0 (\|\nabla^2 v^\epsilon\|^2_m + \|\nabla^2 H^\epsilon\|^2_m) \, ds
\]
\[
\leq C_{m+2}(|v^\epsilon(0)|^2_m + \|H^\epsilon(0)\|^2_m + \|\nabla v^\epsilon(0)\|^2_{m-1} + \|\nabla H^\epsilon(0)\|^2_{m-1})
\]
\[
+ C_{m+2} \int^t_0 (1 + \|\nabla v^\epsilon\|_{1,\infty} + \|\nabla H^\epsilon\|_{1,\infty} + \|v^\epsilon\|_m + \|\nabla v^\epsilon\|_{m-1} + \|H^\epsilon\|_m
\]
\[
+ \|\nabla H^\epsilon\|_{m-1}) (\|v^\epsilon\|^2_m + \|H^\epsilon\|^2_m + \|\nabla v^\epsilon\|^2_{m-1} + \|\nabla H^\epsilon\|^2_{m-1}) \, ds.
\]
(3.70)

### 3.4 $L^\infty$ estimates

In order to enclose the estimates in (3.70), we need to control $\|\nabla v^\epsilon\|_{1,\infty}$ and $\|\nabla H^\epsilon\|_{1,\infty}$.

We have

**Lemma 3.6** For $m > 6$, it holds that
\[
\|\nabla v^\epsilon\|^2_{1,\infty} + \|\nabla H^\epsilon\|^2_{1,\infty} \leq C_{m+2} \left( N_m(0) + (1 + t + \epsilon^3 t^2) \int^t_0 (N_m(s) + N_m(s)^2) \, ds \right).
\]

**Proof** We observe that, away from the boundary, the following estimates hold:
\[
\|\beta_i \nabla v^\epsilon\|_{1,\infty} + \|\beta_i \nabla H^\epsilon\|_{1,\infty} \leq C (\|v^\epsilon\|_m + \|H^\epsilon\|_m), \quad m \geq 4.
\]
Here $\{\beta_i\}$ is a partition of unity subordinated to the covering (2.1). In order to estimate the near boundary parts, we adopt the ideas of Proposition 21 of [17]. We use a local parametrization in the vicinity of the boundary given by a normal geodesic system
\[
\Psi^n(y, z) = \left( \begin{array}{c} y \\ \psi(y) \end{array} \right) - zn(y),
\]
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where
\[ n(y) = \frac{1}{\sqrt{1 + |\nabla \psi(y)|^2}} \begin{pmatrix} \partial_1 \psi(y) \\ \partial_2 \psi(y) \end{pmatrix}. \]

We can extend \( n \) and \( \Pi \) in the interior by setting
\[ n(\Psi(\eta, z)) = n(y), \quad \Pi(\Psi(\eta, z)) = \Pi(y). \]

We observe \( \partial_z = \partial_n \) and
\[ \left( \partial_{\nu^r} \right) \bigg|_{\Psi(\eta, z)} \cdot \left( \partial_z \right) \bigg|_{\Psi(\eta, z)} = 0. \]

Hence, the Riemann metric \( g \) has the following form:
\[ g(y, z) = \begin{pmatrix} \tilde{g}(y, z) & 0 \\ 0 & 1 \end{pmatrix}. \]

Consequently, the Laplacian in this coordinate system reads as
\[ \Delta f = \partial_{zz} f + \frac{1}{2} \partial_z \left( \ln |g| \right) \partial_z f + \Delta_{\tilde{g}} f, \]
where \( |g| \) is the determinant of the matrix \( g \) and \( \Delta_{\tilde{g}} \) is defined by
\[ \Delta_{\tilde{g}} f = \frac{1}{|\tilde{g}|^2} \sum_{1 \leq i, j \leq 2} \partial_{\nu^i} (\tilde{g}^{ij} |\tilde{g}| \partial_{\nu^j} f), \tag{3.71} \]
where \( \{\tilde{g}^{ij}\} \) is the inverse matrix to \( g \) and (3.71) only involves tangential derivatives.

With these preparations, we begin to estimate the near boundary parts. In view of Lemma 2.3 and the equality (3.17), we have
\[ \| \chi \nabla \nu^r \|_{1, \infty} \leq C_3 \left( \| \chi \Pi \partial_n \nu^r \|_{1, \infty} + \| \nu^r \|_m + \| \nabla \nu^r \|_{m-1} \right), \tag{3.72} \]
\[ \| \chi \nabla H^* \|_{1, \infty} \leq C_3 \left( \| \chi \Pi \partial_n H^* \|_{1, \infty} + \| H^r \|_m + \| \nabla H^r \|_{m-1} \right). \tag{3.73} \]

Hence, we need only to estimate \( \| \chi \Pi \partial_n \nu^r \|_{1, \infty} \) and \( \| \chi \Pi \partial_n H^* \|_{1, \infty} \). To this end, we introduce the vorticity
\[ \omega^r = \nabla \times \nu^r, \quad \omega_H^r = \nabla \times H^r. \]

We find that
\[ \Pi(\omega^r \times n) = \Pi \left( (\nabla \nu^r - (\nabla \nu^r)^T) n \right) = \Pi \left( \partial_n \nu^r - \nabla (\nu^r \cdot n) + \nu^r \cdot \nabla n + \nu^r \times (\nabla \times n) \right). \tag{3.74} \]

Consequently, we have
\[ \| \chi \Pi \partial_n \nu^r \|_{1, \infty} \leq C_3 \left( \| \chi \Pi (\omega^r \times n) \|_{1, \infty} + \| \nu^r \|_{2, \infty} \right). \tag{3.75} \]

Using Lemma 2.3, we get
\[ \| \chi \nabla \nu^r \|_{1, \infty} \leq C_3 \left( \| \chi \Pi (\omega^r \times n) \|_{1, \infty} + \| \nu^r \|_m + \| \nabla \nu^r \|_{m-1} \right). \tag{3.76} \]

By taking the same arguments as that on \( \nu^r \), we have the following estimate for \( H^r \):
\[ \| \chi \nabla H^r \|_{1, \infty} \leq C_3 \left( \| \chi \Pi (\omega_H^r \times n) \|_{1, \infty} + \| H^r \|_m + \| \nabla H^r \|_{m-1} \right). \tag{3.77} \]
Therefore, (3.76) and (3.77) yield that we only need to estimate \( \| \chi \Pi(\omega^e_v \times n) \|_{1, \infty} \) and \( \| \chi \Pi(\omega^e_H \times n) \|_{1, \infty} \). Here \( \omega^e_v \) and \( \omega^e_H \) satisfy
\[
\begin{cases}
\partial_t \omega^e_v - \epsilon \Delta \omega^e_v + v^e \cdot \nabla \omega^e_v - H^v \cdot \nabla \omega^e_H = F^v, \\
\partial_t \omega^e_H - \epsilon \Delta \omega^e_H + v^e \cdot \nabla \omega^e_H - H^v \cdot \nabla \omega^e_v = F^H,
\end{cases}
\]
where
\[
F^v = \omega^e_v \cdot \nabla v^e - \omega^e_H \cdot \nabla H^v, \quad F^H = [\nabla \times, H^v \cdot \nabla] v^e - [\nabla \times, v^e \cdot \nabla] H^v.
\]

We define the functions
\[
\begin{align*}
\tilde{\omega}_v^e(y, z) &= \omega^e_v(\Psi^e(y, z)), \\
\tilde{v}^e(y, z) &= v^e(\Psi^e(y, z)), \\
\tilde{\omega}_H^e(y, z) &= \omega^e_H(\Psi^e(y, z)), \\
\tilde{H}^e(y, z) &= H^e(\Psi^e(y, z))
\end{align*}
\]

in the support of \( \chi \), we then have
\[
\begin{align*}
\partial_t \tilde{\omega}_v^e + (\tilde{v}^e)^1 \partial_{y^1} \tilde{\omega}_v^e + (\tilde{v}^e)^2 \partial_{y^2} \tilde{\omega}_v^e + \tilde{v}^e \cdot n \partial_z \tilde{\omega}_v^e - (\tilde{H}^e)^1 \partial_{y^1} \tilde{\omega}_H^e - (\tilde{H}^e)^2 \partial_{y^2} \tilde{\omega}_H^e \\
- \tilde{H}^v \cdot n \partial_z \tilde{\omega}_H^e = \epsilon \left( \partial_{zz} \tilde{\omega}_v^e + \frac{1}{2} \partial_z \left( \ln |g| \right) \partial_z \tilde{\omega}_v^e + \Delta_{\tilde{g}} \tilde{\omega}_v^e \right) + \mathcal{T}^v, \\
\partial_t \tilde{\omega}_H^e + (\tilde{v}^e)^1 \partial_{y^1} \tilde{\omega}_H^e + (\tilde{v}^e)^2 \partial_{y^2} \tilde{\omega}_H^e + \tilde{v}^e \cdot n \partial_z \tilde{\omega}_H^e - (\tilde{H}^e)^1 \partial_{y^1} \tilde{\omega}_v^e - (\tilde{H}^e)^2 \partial_{y^2} \tilde{\omega}_v^e \\
- \tilde{H}^v \cdot n \partial_z \tilde{\omega}_v^e = \epsilon \left( \partial_{zz} \tilde{\omega}_H^e + \frac{1}{2} \partial_z \left( \ln |g| \right) \partial_z \tilde{\omega}_H^e + \Delta_{\tilde{g}} \tilde{\omega}_H^e \right) + \mathcal{T}^H,
\end{align*}
\]
where
\[
\begin{align*}
\mathcal{T}^v &= F^v(\Psi^e(y, z)), \\
\mathcal{T}^H &= F^H(\Psi^e(y, z)).
\end{align*}
\]
By virtue of (2.4) and (3.74), we get that, for \( z = 0 \),
\[
\Pi(\tilde{\omega}_v^e \times n) = 2 \Pi(\tilde{v}^e \cdot \nabla n - \zeta \tilde{v}^e), \quad \Pi(\tilde{\omega}_H^e \times n) = 0.
\]
Consequently, we introduce the following quantities:
\[
\begin{align*}
\tilde{n}_v^e(y, z) &= \chi \Pi(\tilde{\omega}_v^e \times n - 2 \tilde{v}^e \cdot \nabla n + 2 \zeta \tilde{v}^e), \\
\tilde{n}_H^e(y, z) &= \chi \Pi(\tilde{\omega}_H^e \times n - 2 \tilde{H}^v \cdot \nabla n + 2 \tilde{H}^v \cdot \nabla n).
\end{align*}
\]
Thus, we have \( \tilde{n}_v^e(y, 0) = 0 \) and \( \tilde{n}_H^e(y, 0) = 0 \). In view of (3.78), we easily get
\[
\begin{align*}
\partial_t \tilde{n}_v^e + ((\tilde{v}^e)^1 \partial_{y^1} + (\tilde{v}^e)^2 \partial_{y^2} + \tilde{v}^e \cdot n \partial_z) \tilde{n}_v^e - ((\tilde{H}^e)^1 \partial_{y^1} + (\tilde{H}^e)^2 \partial_{y^2} + \tilde{H}^e \cdot n \partial_z) \tilde{n}_H^e \\
- \tilde{H}^v \cdot n \partial_z \tilde{n}_H^e = \epsilon \left( \partial_{zz} \tilde{n}_v^e + \frac{1}{2} \partial_z \left( \ln |g| \right) \partial_z \tilde{n}_v^e \right) + \chi \Pi(\mathcal{T}^v \times n) + \mathcal{T}^v + \mathcal{T}^e + \mathcal{T}_v^e, \\
\partial_t \tilde{n}_H^e + ((\tilde{v}^e)^1 \partial_{y^1} + (\tilde{v}^e)^2 \partial_{y^2} + \tilde{v}^e \cdot n \partial_z) \tilde{n}_H^e - ((\tilde{H}^e)^1 \partial_{y^1} + (\tilde{H}^e)^2 \partial_{y^2} + \tilde{H}^e \cdot n \partial_z) \tilde{n}_v^e \\
- \tilde{H}^v \cdot n \partial_z \tilde{n}_v^e = \epsilon \left( \partial_{zz} \tilde{n}_H^e + \frac{1}{2} \partial_z \left( \ln |g| \right) \partial_z \tilde{n}_H^e \right) + \chi \Pi(\mathcal{T}^H \times n) + \mathcal{T}^H + \mathcal{T}^e, \tag{3.79}
\end{align*}
\]
where
\[
\begin{align*}
\mathcal{T}_v^e &= 2 \chi \Pi(\nabla P^v \cdot \nabla n - \zeta \nabla P^v) \circ \Psi^e, \\
\mathcal{T}_H^e &= 2 \chi \Pi(\nabla P^H \cdot \nabla n - \zeta \nabla P^H) \circ \Psi^e.
\end{align*}
\]
\[ T^\Lambda_v = \left( (\bar{\omega}^v) \partial_y + (\bar{\omega}^v)^2 \partial_y + \bar{\omega}^v \cdot n \partial_z \right) \Pi(\bar{\omega}^v \times n - 2\nabla n + 2\zeta \bar{\omega}^v) \]
\[ - \left( (\bar{H}^v) \partial_y + (\bar{H}^v)^2 \partial_y + \bar{H}^v \cdot n \partial_z \right) \Pi(\bar{\omega}^v_H \times n - 2\nabla n + 2\zeta \bar{\omega}^v) \]
\[ - \epsilon \left( \partial_{zz} \chi + 2\partial_z \chi \partial_z + \frac{1}{2} \partial_z (\ln |g|) \partial_z \chi \right) \Pi(\bar{\omega}^v \times n - 2\nabla n + 2\zeta \bar{\omega}^v), \]
\[ \mathcal{F}^\omega_v = \chi \left( (\bar{\omega}^v)^3 \partial_y^3 + (\bar{\omega}^v)^2 \partial_y^2 + \bar{\omega}^v \cdot n \partial_z \right) \Pi(\bar{\omega}^v \times n - 2\nabla n + 2\zeta \bar{\omega}^v) + \epsilon \chi \Pi(\Delta \bar{\omega}^v \times n) \]
\[ - \chi \left( (\bar{H}^v)^3 \partial_y^3 + (\bar{H}^v)^2 \partial_y^2 + \bar{H}^v \cdot n \partial_z \right) \Pi(\bar{\omega}^v_H \times n - 2\nabla n - 2\zeta \bar{\omega}^v) \]
\[ - 2\epsilon \chi \Pi(\Delta \bar{\omega}^v \cdot \nabla n) - 2\epsilon \chi \Pi \left( (\bar{\omega}^v)^3 \partial_y^3 + (\bar{\omega}^v)^2 \partial_y^2 \nabla n \right) \bar{\omega}^v \]
\[ + \chi \Pi(\bar{\omega}^v \times (\bar{\omega}^v)^3 \partial_y^3 + (\bar{\omega}^v)^2 \partial_y^2) n) + 2\zeta \epsilon \chi \Pi(\Delta \bar{\omega}^v), \]
\[ \mathcal{F}^H_H = \chi \left( (\bar{\omega}^v)^3 \partial_y^3 + (\bar{\omega}^v)^2 \partial_y^2 + \bar{\omega}^v \cdot n \partial_z \right) \Pi(\bar{\omega}^v_H \times n) \]
\[ - \chi \left( (\bar{H}^v)^3 \partial_y^3 + (\bar{H}^v)^2 \partial_y^2 + \bar{H}^v \cdot n \partial_z \right) \Pi(\bar{\omega}^v_H \times n + 2\nabla n - 2\zeta \bar{\omega}^v) \]
\[ - \chi \left( (\bar{H}^v)^3 \partial_y^3 + (\bar{H}^v)^2 \partial_y^2 ) \Pi(\bar{\omega}^v \times n + 2\nabla n - 2\zeta \bar{\omega}^v) + 2\epsilon \chi \Pi(\Delta \bar{\omega}^v \times n) \]
\[ + 2\epsilon \chi \Pi \left( (\bar{\omega}^v)^3 \partial_y^3 + (\bar{\omega}^v)^2 \partial_y^2 \nabla n \right) \bar{\omega}^v \]
\[ - \chi \Pi(\bar{\omega}^v \times (\bar{\omega}^v)^3 \partial_y^3 + (\bar{\omega}^v)^2 \partial_y^2) n) + 2\zeta \epsilon \chi \Pi(\Delta \bar{\omega}^v). \]

Note that in the derivation of the source terms above, we have used the fact that in the coordinate system just defined, \( \Pi \) and \( n \) do not depend on the normal variable. Since \( \Delta \bar{g} \) involves only the tangential derivatives, and the derivatives of \( \chi \) are compactly supported away from the boundary, the following estimates hold:

\[
\| \mathcal{F}^\omega_v \|_{1,\infty} \leq C_3 \| \Pi \nabla P^\omega \|_{1,\infty},
\]
\[
\| \mathcal{F}^\Lambda_v \|_{1,\infty} \leq C_3 \left( \| \omega^v \|_{1,\infty} \| \omega^v \|_{2,\infty} + \| K^v \|_{1,\infty} \| K^v \|_{2,\infty} + \epsilon \| v^\omega \|_{3,\infty} \right),
\]
\[
\| T^\Lambda_v \|_{1,\infty} \leq C_4 \left( \| \omega^v \|_{1,\infty} \| \nabla v^\omega \|_{1,\infty} + \| K^v \|_{1,\infty} \| \nabla v^\omega \|_{1,\infty} + \| v^\omega \|_{1,\infty}^2 + \| K^v \|_{1,\infty}^2 \right.
\]
\[ + \epsilon \| v^\omega \|_{1,\infty} + \| v^\omega \|_{3,\infty} + \epsilon \| \nabla v^\omega \|_{3,\infty}, \]
\[
\| \mathcal{F}^H_H \|_{1,\infty} \leq C_3 \left( \| \omega^v \|_{1,\infty} \| H^v \|_{2,\infty} + \| H^v \|_{1,\infty} \| v^\omega \|_{2,\infty} + \epsilon \| H^v \|_{3,\infty} \right),
\]
\[
\| T^\Lambda_H \|_{1,\infty} \leq C_4 \left( \| \omega^v \|_{1,\infty} \| \nabla v^\omega \|_{1,\infty} + \| H^v \|_{1,\infty} \| \nabla v^\omega \|_{1,\infty} + \| v^\omega \|_{1,\infty}^2 + \| H^v \|_{1,\infty}^2 \right.
\]
\[ + \epsilon \| H^v \|_{3,\infty} + \epsilon \| \nabla H^v \|_{3,\infty} \right). \]

A crucial estimate towards the proof of Lemma 3.6 is the following lemma:

**Lemma 3.7** (17) Let \( \rho \) be a smooth solution of
\[ \partial_t \rho + u \cdot \nabla \rho = \epsilon \partial_{zz} \rho + f, \quad z > 0, \quad \rho(t, y, 0) = 0, \]
where \( u \) satisfies the divergence free condition and \( u \cdot n \) vanishes on the boundary. Assume that \( \rho \) and \( f \) are compactly supported with respect to \( z \). Then
\[ \| \rho \|_{1,\infty} \leq C \| \rho(0) \|_{1,\infty} + C \int_0^t \left( \| u \|_{2,\infty} + \| \partial_z u \|_{1,\infty} \right) \times \left( \| \rho \|_{1,\infty} + \| \rho \|_{m_0 + 3} + \| f \|_{1,\infty} \right) \text{d}s \quad \text{for} \quad m_0 > 2. \]
In order to apply Lemma 3.7, we shall eliminate $\partial_z (\ln |g|) \partial_z \eta_v$ in (3.79) and $\partial_z (\ln |g|) \partial_z \eta_H$ in (3.79)$_2$, respectively. We define

$$\eta_v = \frac{1}{|g|^2} \bar{\eta}_v = \gamma \eta_v, \quad \eta_H = \frac{1}{|g|^2} \bar{\eta}_H = \gamma \eta_H.$$  

We note that

$$\|\bar{\eta}_v\|_{1,\infty} \sim \|\eta_v\|_{1,\infty}, \quad \|\bar{\eta}_H\|_{1,\infty} \sim \|\eta_H\|_{1,\infty}$$  

and $(\bar{\eta}_v, \eta_H)$ solve the system

$$\left\{ \begin{array}{l} \partial_t \eta_v + ((\bar{v})^1) \partial_{v^1} + ((\bar{v})^2) \partial_{v^2} + \bar{v} \cdot n \partial_z \eta_v - ((\bar{\eta})^1) \partial_{v^1} + (\bar{\eta})^2 \partial_{v^2} + \bar{\eta} \cdot n \partial_z \eta_H - \epsilon \partial_{zz} \eta_v \\
\quad - (\bar{v} \cdot \nabla \gamma) \eta_v + (\bar{\eta} \cdot \nabla \gamma) \eta_H = S_1, \\
\partial_t \eta_H + ((\bar{v})^1) \partial_{v^1} + ((\bar{v})^2) \partial_{v^2} + \bar{v} \cdot n \partial_z \eta_H - ((\bar{\eta})^1) \partial_{v^1} + (\bar{\eta})^2 \partial_{v^2} + \bar{\eta} \cdot n \partial_z \eta_H - \epsilon \partial_{zz} \eta_H \\
\quad - (\bar{v} \cdot \nabla \gamma) \eta_H + (\bar{\eta} \cdot \nabla \gamma) \eta_H = S_2. \end{array} \right.$$  

Furthermore, we set

$$\eta_1 := \eta_v, \quad \eta_2 := \eta_H,$$

and find that $\eta_1$ and $\eta_2$ satisfy, respectively, the following equations:

$$\begin{align*}
\partial_t \eta_1 + ((\bar{v})^1) \partial_{v^1} + ((\bar{v})^2) \partial_{v^2} + \bar{v} \cdot n \partial_z \eta_1 \\
- ((\bar{\eta})^1) \partial_{v^1} + (\bar{\eta})^2 \partial_{v^2} + \bar{\eta} \cdot n \partial_z \eta_1 - \epsilon \partial_{zz} \eta_1 &= S_1 + S_2, \\
\eta_2 + ((\bar{v})^1) \partial_{v^1} + ((\bar{v})^2) \partial_{v^2} + \bar{v} \cdot n \partial_z \eta_2 \\
+ ((\bar{\eta})^1) \partial_{v^1} + (\bar{\eta})^2 \partial_{v^2} + \bar{\eta} \cdot n \partial_z \eta_2 - \epsilon \partial_{zz} \eta_2 &= S_1 - S_2.
\end{align*}$$

By applying Lemma 3.7 to (3.86), we directly obtain that

$$\|\eta_1\|_{1,\infty} \leq C \|\eta_1(0)\|_{1,\infty} + C \int_0^t \left( \|v^*\|_{2,\infty} + \|H^*\|_{2,\infty} + \|\nabla v^*\|_{1,\infty} + \|\nabla H^*\|_{1,\infty} \right) \left( \|\eta_1\|_{1,\infty} + \|\eta_1\|_{m_0+3} + \|S_1\|_{1,\infty} + \|S_2\|_{1,\infty} \right) \|v^*\|_{1,\infty} ds$$  

for $m_0 > 2$. Thanks to Lemma 2.3, the definition of $N_m$ and (3.80)–(3.84), we get

$$\|\eta_1\|_{1,\infty} \leq C_m + 1 \left( \|\eta_1(0)\|_{1,\infty} + \int_0^t \left( N_m^2(s) + N_m(s) + \|\Pi \nabla P^*\|_{1,\infty} \right) \right)$$  

for $m > 6$.  

Based on Lemmas 2.3 and 3.5, we obtain

$$\|\Pi \nabla P^*\|_{1,\infty} \leq C_{m+2} \left( N_m^2(t) + N_m(t) \right)$$  

for $m \geq 4$.  

Finally, we deal with the terms with the coefficient $\epsilon$. Applying Lemma 2.3, we get

$$\left( \epsilon \int_0^t \left( \|\nabla v^*\|_{1,\infty} + \|\nabla H^*\|_{1,\infty} \right) ds \right)^2.$$  

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exists a positive $v$ of (3.93)–(3.96) can be obtained by using classical methods, for example, arguments similar to

Since in (0, $T$), one can prove that there

Therefore, we complete the proof of Lemma 3.6.

\[ \tag{3.92} \]

3.5 Proof of Theorem 3.1

Based on Lemma 3.6 and (3.70), we can easily prove Theorem 3.1. We omit the details here.

3.6 Proof of Theorem 1.1

In this subsection, we show how to combine our a priori estimates to obtain the uniform existence result. Let us fix $m \geq 6$ and consider the initial data $(v_0, H_0) \in \mathcal{E}^m_{\text{MHD}}$. For such initial data, we are not aware of local existence results for the problem (1.1)–(1.2), so we first need to prove the local existence results for (1.1)–(1.2) with the initial data $(v_0, H_0) \in \mathcal{E}^m_{\text{MHD}}$ by virtue of the definition of $\mathcal{E}^m_{\text{MHD}}$, there exists a sequence of smooth approximate initial data $(v_0^\delta, H_0^\delta) \in \mathcal{E}^m_{\text{MHD,app}}$ ($\delta$ being a regularization parameter). For fixed $\epsilon \in (0, 1]$, we construct the approximate solution as follows:

1. Define $(v^1_{\epsilon}, H^1_{\epsilon}) = (v_0^\delta, H_0^\delta)$.

2. Assume that $(v^k_{\epsilon}, H^k_{\epsilon})$ has been defined for $k \geq 1$. Let $(v^{k+1}_{\epsilon}, H^{k+1}_{\epsilon})$ be the unique solution to the linearized initial boundary value problem

$$
\begin{cases}
(v^{k+1}_{\epsilon})_t - \epsilon \Delta v^{k+1}_{\epsilon} + v^k_{\epsilon} \cdot \nabla v^{k+1}_{\epsilon} + \nabla P^{k+1}_{\epsilon} - H^k_{\epsilon} \cdot \nabla H^{k+1}_{\epsilon} = 0, \\
(H^{k+1}_{\epsilon})_t - \epsilon \Delta H^{k+1}_{\epsilon} + v^k_{\epsilon} \cdot \nabla H^{k+1}_{\epsilon} - H^{k+1}_{\epsilon} \cdot \nabla v^k_{\epsilon} = 0, \\
\nabla \cdot v^{k+1}_{\epsilon} = 0, \quad \nabla \cdot H^{k+1}_{\epsilon} = 0
\end{cases}
$$

(3.93)
in $(0, T) \times \Omega$ with the initial boundary conditions

$$
\begin{align*}
(v^{k+1}_{\epsilon}, H^{k+1}_{\epsilon})|_{t=0} &= (v_0^\delta, H_0^\delta) \quad \text{in} \quad \Omega, \\
v^{k+1}_{\epsilon} \cdot n &= 0, \quad (S v^{k+1}_{\epsilon} \cdot n)_\tau = -\zeta (v^{k+1}_{\epsilon})_\tau, \quad \text{on} \quad (0, T) \times \partial \Omega, \\
H^{k+1}_{\epsilon} \cdot n &= 0, \quad n \times (\nabla \times H^{k+1}_{\epsilon}) = 0, \quad \text{on} \quad (0, T) \times \partial \Omega.
\end{align*}
$$

(3.94)

Since $v^{k+1}_{\epsilon}$ and $H^{k+1}_{\epsilon}$ are decoupled, the existence of the global smooth solution $(v^{k+1}_{\epsilon}, H^{k+1}_{\epsilon})$ of (3.93)–(3.96) can be obtained by using classical methods, for example, arguments similar to those in [1, 8]. On the other hand, due to $(v_0^\delta, H_0^\delta) \in H^{3m} \times H^{3m}$, one can prove that there exists a positive $T_0 = T_0(\epsilon)$ which depends on $\epsilon$ and $\|(v_0^\delta, H_0^\delta)\|_{H^{3m}}$ such that

$$
\|(v^k_{\epsilon}, H^k_{\epsilon})\|_{H^{3m}}^2 \leq C_3, \quad t \in [0, T_0],
$$

(3.97)
where \( \tilde{C}_3 \) depends on \( \epsilon^{-1} \) and \( \| (v^0, H^0_\epsilon) \|_{H^{3m}} \). Based on the above uniform estimates for \((v^k_\epsilon, H^k_\epsilon)\), it is easy to show that there exists a uniform time \( \tilde{T}_1 \leq \tilde{T}_0 \) such that \( \{(v^k_\epsilon, H^k_\epsilon)\}_{k=1}^\infty \) is a Cauchy sequence in \( L^\infty(0, \tilde{T}_0; L^2) \) and that \((v^k_\epsilon, H^k_\epsilon)\) converges to a limit \((v^{\epsilon,\delta}, H^{\epsilon,\delta})\) as \( k \to +\infty \) in the following strong sense:

\[
(v^k_\epsilon, H^k_\epsilon) \to (v^{\epsilon,\delta}, H^{\epsilon,\delta}) \quad \text{in} \quad L^\infty(0, \tilde{T}_1; L^2)
\]

and

\[
(\nabla v^k_\epsilon, \nabla H^k_\epsilon) \to (\nabla v^{\epsilon,\delta}, \nabla H^{\epsilon,\delta}) \quad \text{in} \quad L^2(0, \tilde{T}_1; L^2).
\]

It is easy to check that \((v^{\epsilon,\delta}, H^{\epsilon,\delta})\) is a classical solution to the problem \((1.1)-(1.2)\) with initial data \((v^0_\epsilon, H^0_\epsilon)\). Then, by virtue of the lower semi-continuity of norms, we have

\[
\|(v^{\epsilon,\delta}, H^{\epsilon,\delta})\|^2_{H^{3m}} \leq \tilde{C}_3, \quad t \in [0, \tilde{T}_1].
\]

Applying the a priori estimates given in Theorem 3.1 to the solution \((v^{\epsilon,\delta}, H^{\epsilon,\delta})\), we obtain that there exist a uniform time \( \tilde{T}_2 \) and a constant \( \tilde{C}_4 \) (independent of \( \epsilon \) and \( \delta \)) such that it holds for \((v^{\epsilon,\delta}, H^{\epsilon,\delta})\) that

\[
\begin{aligned}
&\sup_{t \in [0, \tilde{T}_2]} \left\{ \| (v^{\epsilon,\delta}, H^{\epsilon,\delta}) (t) \|^2_m + \| (\nabla v^{\epsilon,\delta}, \nabla H^{\epsilon,\delta}) (t) \|^2_{m-1} + \| (\nabla v^{\epsilon,\delta}, \nabla H^{\epsilon,\delta}) (t) \|^2_{1,\infty} \right\} \\
&{} + \epsilon \int_0^{\tilde{T}_2} (\| \nabla^2 v^{\epsilon,\delta} (t) \|^2_{m-1} + \| \nabla^2 H^{\epsilon,\delta} (t) \|^2_{m-1}) dt \leq \tilde{C}_4,
\end{aligned}
\]

(3.99)

where \( \tilde{T}_2 = \min\{\tilde{T}_0, \tilde{T}_1\} \), and \( \tilde{T}_2 \) and \( \tilde{C}_4 \) depend only on \((v_0, H_0)\). Based on the uniform estimate (3.99) for \((v^{\epsilon,\delta}, H^{\epsilon,\delta})\), we can pass the limit \( \delta \to 0 \) to get a strong solution \((v^\epsilon, H^\epsilon)\) of the system \((1.1)-(1.2)\) with initial data \((v_0, H_0)\) by using a strong compactness argument. Indeed,

\[
(v^{\epsilon,\delta}, H^{\epsilon,\delta}) \quad \text{is bounded uniformly in} \quad L^\infty(0, \tilde{T}_2; H^{m-1}_\epsilon),
\]

\[
(\nabla v^{\epsilon,\delta}, \nabla H^{\epsilon,\delta}) \quad \text{is bounded uniformly in} \quad L^\infty(0, \tilde{T}_2; H^{m-1}_\epsilon).
\]

Then, we obtain that \((v^{\epsilon,\delta}, H^{\epsilon,\delta})\) is compact in \( C(0, \tilde{T}_2; H^{m-1}_\epsilon) \) and \((\nabla v^{\epsilon,\delta}, \nabla H^{\epsilon,\delta})\) is compact in \( C(0, \tilde{T}_2; H^{m-2}_\epsilon) \) by using the strong compactness argument. In particular, there exist a sequence \( \delta_k \to 0^+ \), and the functions \((v^\epsilon, H^\epsilon) \in C(0, \tilde{T}_2; H^{m-1}_\epsilon) \) and \((\nabla v^\epsilon, \nabla H^\epsilon) \in C(0, \tilde{T}_2; H^{m-2}_\epsilon) \), such that

\[
(v^{\epsilon,\delta_k}, H^{\epsilon,\delta_k}) \to (v^\epsilon, H^\epsilon) \quad \text{in} \quad C(0, \tilde{T}_2; H^{m-1}_\epsilon) \quad \text{as} \quad \delta_k \to 0^+.
\]

(3.100)

\[
(\nabla v^{\epsilon,\delta_k}, \nabla H^{\epsilon,\delta_k}) \to (\nabla v^\epsilon, \nabla H^\epsilon) \quad \text{in} \quad C(0, \tilde{T}_2; H^{m-2}_\epsilon) \quad \text{as} \quad \delta_k \to 0^+.
\]

(3.101)

Moreover, applying the lower semi-continuity of norms to the bounds (3.99), we obtain the bounds (3.99) for \((v^\epsilon, H^\epsilon)\). In view of (3.99), (3.100), (3.101) and Lemma 2.3, we have

\[
\sup_{0 \leq t \leq \tilde{T}_2} \|(v^{\epsilon,\delta_k}, H^{\epsilon,\delta_k}) - (v^\epsilon, H^\epsilon)\|_{L^\infty} \to 0.
\]

(3.102)

Hence, it is easy to deduce that \((v^\epsilon, H^\epsilon)\) is a weak solution of the incompressible MHD equations. The uniqueness of the solution \((v^\epsilon, H^\epsilon)\) comes directly from the Lipschitz regularity. Therefore, the whole family \((v^{\epsilon,\delta}, H^{\epsilon,\delta})\) converges to \((v^\epsilon, H^\epsilon)\). Thus, we have established the local solution of the system \((1.1)-(1.2)\) on \([0, \tilde{T}_2]\). Taking \( T_0 = \tilde{T}_2 \) and \( \tilde{C}_0 = \tilde{C}_4 \), we complete the proof of Theorem 1.1. \( \square \)

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4 Proof of Theorem 1.3

In this section, we study the zero kinematic viscosity-magnetic diffusion limit of viscous solutions to the inviscid one with rates of convergence in different spaces. For the convenience of calculations, we replace the boundary conditions in (1.2) by (1.10). Define

\[ \varphi^e = v^e - v, \quad \psi^e = H^e - H. \]

It then follows from (1.1) and (1.5) that \( \varphi^e \) and \( \psi^e \) satisfy

\[
\begin{aligned}
\partial_t \varphi^e - \epsilon \Delta \varphi^e + \Phi_1 + \nabla (p^e - p) &= \epsilon \Delta v, \\
\partial_t \psi^e - \epsilon \Delta \psi^e + \Phi_2 &= \epsilon \Delta H, \\
\nabla \cdot \varphi^e &= 0, \quad \nabla \cdot \psi^e = 0
\end{aligned}
\]

(4.1)

in \( \Omega \times (0, T_2) \) with the initial boundary conditions

\[
\begin{aligned}
(\varphi^e, \psi^e)|_{t=0} &= (0, 0) \quad \text{in} \ \Omega, \\
\varphi^e \cdot n &= 0, \quad n \times \omega^e = [B\varphi^e + Bv]_\tau - n \times \omega_v \quad \text{on} \ \partial \Omega \times (0, T_2), \\
\psi^e \cdot n &= 0, \quad n \times \omega^e = -n \times \omega_H \quad \text{on} \ \partial \Omega \times (0, T_2),
\end{aligned}
\]

(4.2)

where \( \omega^e = \nabla \times \varphi^e, \omega^v_\tau = \nabla \times \psi^e, \omega_v = \nabla \times v, \omega_H = \nabla \times H, \)

\[
\Phi_1 := v \cdot \nabla \varphi^e + \varphi^e \cdot \nabla v + \varphi^e \cdot \nabla \varphi^e - H \cdot \nabla \psi^e - \psi^e \cdot \nabla H - \psi^e \cdot \nabla \psi^e + \frac{1}{2} \nabla (|H^e|^2 - |H|^2),
\]

\[
\Phi_2 := \varphi^e \cdot \nabla H + \varphi^e \cdot \nabla \psi^e + v \cdot \nabla \psi^e - \psi^e \cdot \nabla v - \psi^e \cdot \nabla \varphi^e - H \cdot \nabla \varphi^e,
\]

and \( 0 < T_2 \leq \min\{T_0, T_1\} \) such that the incompressible viscous MHD equations and ideal incompressible MHD equations have solutions.

We start with the rates of convergence in \( L^\infty(0, T_2; L^2) \) and \( L^\infty([0, T_2] \times \Omega) \).

**Lemma 4.1** Under the assumptions in Theorem 1.3, it holds

\[
\|\varphi^e(t)\|^2 + \|\psi^e(t)\|^2 + \epsilon \int_0^t (\|\varphi^e\|^2_{H^1} + \|\psi^e\|^2_{H^1}) \, ds \leq C \epsilon^{\frac{3}{2}}, \quad t \in [0, T_2].
\]

Furthermore, we have

\[
\|\varphi^e\|_{L^\infty([0, T_2] \times \Omega)} + \|\psi^e\|_{L^\infty([0, T_2] \times \Omega)} \leq C \epsilon^{\frac{3}{2n}}.
\]

**Proof** Multiplying (4.1) and (4.1) by \( \varphi^e \) and \( \psi^e \), respectively, and integrating the results by parts, we have

\[
\frac{1}{2} \frac{d}{dt} (\|\varphi^e\|^2 + \|\psi^e\|^2) + \epsilon (\|\omega^e\|^2 + \|\omega^e_\tau\|^2) \geq \Phi_1(\varphi^e, \psi^e) + B_1 + B_2 = (\epsilon \Delta v, \varphi^e) + (\epsilon \Delta H, \psi^e),
\]

where

\[
B_1 = \epsilon \int_{\partial \Omega} (n \times \omega^e_\tau) \cdot \varphi^e \, d\sigma = \epsilon \int_{\partial \Omega} (B\varphi^e + Bv - n \times \omega_v) \cdot \varphi^e \, d\sigma,
\]

\[
B_2 = \epsilon \int_{\partial \Omega} (n \times \omega^e_\tau) \cdot \psi^e \, d\sigma = -\epsilon \int_{\partial \Omega} (n \times \omega_H) \cdot \psi^e \, d\sigma.
\]

It is easy to check that

\[
|\langle \epsilon \Delta v, \varphi^e \rangle| + |\langle \epsilon \Delta H, \psi^e \rangle| \leq C (\|\varphi^e\|^2 + \|\psi^e\|^2) + \epsilon^2.
\]

\[\Box\]
Next, we deal with the boundary integrals $B_1$ and $B_2$. For $B_1$, we have
\[
B_1 = \epsilon \int_{\partial \Omega} (B \varphi' + Bv - n \times \omega_v) \cdot \varphi' d\sigma \leq C \epsilon \int_{\partial \Omega} (|\varphi'|^2 + |\varphi'|) d\sigma.
\]
In view of Lemma 2.1, the trace theorem
\[
|u|_{L^1(\partial \Omega)} \leq C |u|_{L^2(\partial \Omega)} \leq C \|u\|_{H^{\frac{1}{2}}},
\]
and the interpolation inequality
\[
\|u\|_{H^{0,\nu}(\Omega)} \leq C \|u\|_{H^\frac{3}{4}} \|u\|_{H^1},
\]
we further obtain that
\[
B_1 \leq C \epsilon (\|\varphi\|^{\nu} \|\omega\| + |\varphi'|_{L^1(\partial \Omega)} \leq \delta \epsilon \|\omega\|^2 + C \delta \|\varphi\|^2 + \epsilon \frac{3}{2},
\]
where $\delta$ is small enough. Similarly, we also get that
\[
B_2 \leq \delta \epsilon \|\varphi\|^2 + C \delta \|\psi\|^2 + \epsilon \frac{3}{2}.
\]
Finally, we deal with $(\Phi_1, \varphi')$ and $(\Phi_2, \psi')$. We have
\[
|\langle \Phi_1, \varphi' \rangle + \langle \Phi_2, \psi' \rangle | = \left| \left( \varphi', v \cdot \nabla \varphi' + \varphi' \cdot \nabla \varphi' - H \cdot \nabla \varphi' - \psi' \cdot \nabla v - \varphi' \cdot \nabla \psi' \right.ight.
\[
+ \left. \frac{1}{2} \nabla (|H|^2 - |H|^2) \right) + (\varphi', \varphi' \cdot \nabla H + \varphi' \cdot \nabla \psi' + v \cdot \nabla \psi' \right.
\[
- \psi' \cdot \nabla v - \psi' \cdot \nabla \varphi' - H \cdot \nabla \varphi').
\]
Using (4.1), (4.2) and (4.3), we have
\[
\left( \frac{1}{2} \nabla (|H|^2 - |H|^2), \varphi' \right) = 0, \quad \left( \varphi', v \cdot \nabla \varphi' \right) = 0, \quad \left( \varphi', \varphi' \cdot \nabla \varphi' \right) = 0,
\]
\[
\left( \psi', v \cdot \nabla \psi' \right) = 0, \quad \left( \psi', \varphi' \cdot \nabla \psi' \right) = 0, \quad \left( \psi', \nabla \varphi', \psi' \right) + \left( \varphi', \nabla \varphi', \psi' \right) = 0,
\]
\[
\left( \psi', H \cdot \nabla \varphi' \right) + \left( \varphi', H \cdot \nabla \varphi' \right) = 0.
\]
Consequently, we obtain
\[
|\langle \Phi_1, \varphi' \rangle + \langle \Phi_2, \psi' \rangle | \leq C \left( \|\varphi'\|^2 + \|\psi'\|^2 \right).
\]
Based on Lemma 2.1, (4.3), (4.6), (4.7) and (4.8), we arrive at
\[
\frac{d}{dt} (\|\varphi'\|^2 + \|\psi'\|^2) + \epsilon (\|\omega\|^2 + \|\omega\|^2) \leq C (\|\varphi'\|^2 + \|\psi'\|^2) + \epsilon \frac{3}{2}.
\]
Then, by using Gronwall’s inequality, we get that, for $t \in (0, T_2]$,
\[
\|\varphi'(t)\|^2 + \|\psi'(t)\|^2 + \epsilon \int_0^t \left( \|\varphi'\|^2_{H^1} + \|\psi'\|^2_{H^1} \right) ds \leq C \epsilon \frac{3}{2}.
\]
Furthermore, using the Gagliardo-Nirenberg interpolation inequality, we have
\[
\|\varphi'\|_{L^\infty} + \|\psi'\|_{L^\infty} \leq C \left( \|\varphi'\|^\frac{3}{2} + \|\psi'\|^\frac{3}{2} \right) \leq C \epsilon \frac{3}{2}.
\]
This completes the proof of Lemma 4.1. \(\square\)

Next, we focus on proving the rates of convergence in $L^\infty(0, T; H^1)$ and $L^\infty(0, T; W^{1,p})$ ($2 \leq p < \infty$). \(\odot\) Springer
Lemma 4.2  Under the assumptions in Theorem 1.3, it holds
\[
\|\varphi'(t)\|_{H^1}^2 + \|\psi'(t)\|_{H^1}^2 + \epsilon \int_0^t (\|\varphi'\|_{H^2}^2 + \|\psi'\|_{H^2}^2) \, ds \leq C \epsilon^{\frac{1}{2}}, \quad t \in (0, T_2].
\] (4.11)

Also, we have
\[
\|\varphi'\|_{W^{1,p}}^p + \|\psi'\|_{W^{1,p}}^p \leq C \epsilon^{\frac{1}{2}}, \quad t \in (0, T_2]
\]
for \(2 < p < \infty\).

Proof \ Since \ \(\partial_t \varphi' \cdot n = 0, \ \partial_t \psi' \cdot n = 0\) on \ \(\partial \Omega,\)
multiplying (4.1) and (4.1), by \(\mathbb{P} \Delta \varphi'\) and \(\mathbb{P} \Delta \psi',\) respectively, and integrating by parts leads to
\[
\frac{1}{2} \frac{d}{dt}(\|\omega_{\varphi}'\|^2 + \|\omega_{\psi}'\|^2) + \epsilon (\|\mathbb{P} \Delta \varphi'\|^2 + \|\mathbb{P} \Delta \psi'\|^2)
= (\Phi_1, \mathbb{P} \Delta \varphi') + (\Phi_2, \mathbb{P} \Delta \psi') + B_3 + B_4 - (\epsilon \Delta \psi, \mathbb{P} \Delta \varphi') - (\epsilon \Delta H, \mathbb{P} \Delta \psi'),
\] (4.12)
where
\[
B_3 = \int_{\partial \Omega} \partial_t \varphi' \cdot (n \times \omega_{\psi}') \, ds, \quad B_4 = \int_{\partial \Omega} \partial_t \psi' \cdot (n \times \omega_{\varphi}') \, ds.
\]
It follows from (4.2) and (4.2), that
\[
B_3 + B_4 = \int_{\partial \Omega} \partial_t \varphi' \cdot (B \varphi' + B v - n \times \omega_v) \, ds - \int_{\partial \Omega} \partial_t \psi' \cdot (n \times \omega_H) \, ds
= \frac{1}{2} \frac{d}{dt} \left( \int_{\partial \Omega} \varphi' \cdot \varphi' \, ds + 2 \int_{\partial \Omega} \psi' \cdot (B v - n \times \omega_v) \, ds \right) - \tilde{B}_3
+ \frac{1}{2} \frac{d}{dt} \left( \int_{\partial \Omega} \psi' \cdot \psi' \, ds - 2 \int_{\partial \Omega} \psi' \cdot (n \times \omega_H) \, ds \right) - \tilde{B}_4,
\] (4.13)
where
\[
\tilde{B}_3 = \int_{\partial \Omega} \varphi' \cdot \partial_t (B v - n \times \omega_v) \, ds, \quad \tilde{B}_4 = - \int_{\partial \Omega} \psi' \cdot \partial_t (n \times \omega_H) \, ds.
\] (4.14)
It follows from Lemma 2.1, Lemma 4.1, (4.4) and (4.5) that
\[
|\tilde{B}_3 + \tilde{B}_4| \leq C \left( \int_{\partial \Omega} |\varphi'|^2 \, ds \right)^{\frac{1}{2}} + C \left( \int_{\partial \Omega} |\psi'|^2 \, ds \right)^{\frac{1}{2}} \leq \epsilon \left( \|\omega_{\varphi}'\|^2 + \|\omega_{\psi}'\|^2 \right) + C \epsilon^{\frac{1}{2}}.
\] (4.15)
As for the other terms in (4.12), using Hölder’s inequality, we obtain that
\[
\left| (\epsilon \Delta \psi, \mathbb{P} \Delta \varphi') + (\epsilon \Delta H, \mathbb{P} \Delta \psi') \right| \leq \epsilon \left( \|\mathbb{P} \Delta \varphi'\|^2 + \|\mathbb{P} \Delta \psi'\|^2 \right) + C \epsilon
\] (4.16)
and
\[
-(\Phi_1, \mathbb{P} \Delta \varphi') - (\Phi_2, \mathbb{P} \Delta \psi') = (\mathbb{P} \Phi_1, -\Delta \varphi') + (\mathbb{P} \Phi_2, -\Delta \psi')
= (\nabla \times \Phi_1, \omega_{\varphi}') + \int_{\partial \Omega} n \times \omega_{\varphi}' \cdot \mathbb{P} \Phi_1 \, ds + (\nabla \times \Phi_2, \omega_{\psi}')
+ \int_{\partial \Omega} n \times \omega_{\psi}' \cdot \mathbb{P} \Phi_2 \, ds
= (\nabla \times \Phi_1, \omega_{\varphi}') + \int_{\partial \Omega} (B \varphi' + B v - n \times \omega_v) \cdot \mathbb{P} \Phi_1 \, ds
+ (\nabla \times \Phi_2, \omega_{\psi}') - \int_{\partial \Omega} (n \times \omega_H) \cdot \mathbb{P} \Phi_2 \, ds.
\] (4.17)
From (4.13), (4.15), (4.16), and (4.17), we arrive at
\[
\frac{1}{2} \frac{d}{dt} E + \epsilon \frac{1}{2} \left( \| \nabla \phi \|^2 + \| \nabla \psi \|^2 \right) \leq I_1 + I_2 + I_3 + C \left( \| \phi \|_\infty^2 + \| \psi \|_\infty^2 + \epsilon^2 \right),
\] where
\[
E = \| \phi \|_\infty^2 + \| \psi \|_\infty^2 - \int_\Omega B \phi \cdot \psi \, ds - 2 \int_\Omega \phi \cdot (B v - n \times \omega_v) \, ds + 2 \int_\partial \Omega \psi \cdot (n \times \omega_H) \, ds,
\]
\[
I_1 = \left| (\nabla \times \Phi_1, \omega_\phi^\epsilon) + (\nabla \times \Phi_2, \omega_\psi^\epsilon) \right|, \quad I_2 = \left| \int_\partial \Omega B \phi \cdot \delta \Phi_1 \, ds \right|,
\]
\[
I_3 = \left| \int_\partial \Omega (B v - n \times \omega_v) \cdot \delta \Phi_1 \, ds - \int_\Omega (n \times \omega_H) \cdot \delta \Phi_2 \, ds \right|.
\]
Now we estimate the terms $I_1, I_2$ and $I_3$ in turn. First, we have
\[
I_1 = \left| (\nabla \times \Phi_1, \omega_\phi^\epsilon) + (\nabla \times \Phi_2, \omega_\psi^\epsilon) \right| \leq I_{11} + I_{12},
\]
where
\[
I_{11} = \left| (v \cdot \nabla \phi \cdot \omega_\phi^\epsilon + \phi \cdot \nabla \omega_\phi^\epsilon + \phi \cdot \nabla \omega_\psi^\epsilon - H \cdot \nabla \omega_\phi^\epsilon - \psi \cdot \nabla \omega_\psi^\epsilon - \psi \cdot \nabla \omega_\phi^\epsilon, \omega_\phi^\epsilon) \right|
\]
\[
+ \left| (\phi \cdot \nabla \omega_H + \psi \cdot \nabla \omega_\psi^\epsilon + v \cdot \nabla \omega_\psi^\epsilon - \psi \cdot \nabla \omega_\psi^\epsilon - \psi \cdot \nabla \omega_\phi^\epsilon - H \cdot \nabla \omega_\phi^\epsilon, \omega_\phi^\epsilon) \right|
\]
\[
I_{12} = \left| (\nabla \times, v \cdot \nabla) \left( \phi \cdot \omega_\phi^\epsilon + \psi \cdot \omega_\psi^\epsilon \right) + (\nabla \times, \psi \cdot \nabla) \left( \phi \cdot \omega_\phi^\epsilon - \psi \cdot \omega_\psi^\epsilon - H \cdot \nabla \omega_\phi^\epsilon, \omega_\phi^\epsilon \right) \right|
\]
\[
+ \left| (\nabla \times, \psi \cdot \nabla) \left( \phi \cdot \omega_\phi^\epsilon - \psi \cdot \omega_\psi^\epsilon - H \cdot \nabla \omega_\phi^\epsilon, \omega_\phi^\epsilon \right) \right|
\]
\[
- \left| (\nabla \times, \psi \cdot \nabla) \left| v \cdot \nabla \phi \cdot \omega_\phi^\epsilon + \psi \cdot \nabla \omega_\psi^\epsilon - \nabla \times H \cdot \nabla \phi \cdot \omega_\phi^\epsilon \right| \right|
\]
Notice that
\[
(v \cdot \nabla \omega_\phi^\epsilon, \omega_\phi^\epsilon) = 0, \quad (\phi \cdot \nabla \omega_\phi^\epsilon, \omega_\phi^\epsilon) = 0, \quad (v \cdot \nabla \omega_\psi^\epsilon, \omega_\psi^\epsilon) = 0, \quad (\phi \cdot \nabla \omega_\psi^\epsilon, \omega_\psi^\epsilon) = 0,
\]
\[
(H \cdot \nabla \omega_\phi^\epsilon + \psi \cdot \nabla \omega_\psi^\epsilon, \omega_\phi^\epsilon) + (\psi \cdot \nabla \omega_\psi^\epsilon + H \cdot \nabla \omega_\phi^\epsilon, \omega_\phi^\epsilon) = 0.
\]
Therefore, using Hölder’s inequality, we have
\[
I_1 \leq C \left( \| \phi \|_\infty^2 + \| \psi \|_\infty^2 \right).
\]
is well-defined in \( \Omega_\sigma = \{ x \in \Omega, r(x) \leq 2\sigma \} \) for some \( \sigma > 0 \) and \( \varphi(s) \in C_c^\infty [0, 2\sigma] \) satisfying
\[
\varphi(s) = 1 \quad \text{in} \quad [0, \sigma].
\]

Then, by integrating by parts, we can obtain
\[
I_2 = \left| \int_{\partial \Omega} B \phi^* \cdot P \Phi_1 \, d\sigma \right| = \left| \int_{\partial \Omega} \left( (n \times B \phi^*) \cdot (n \times P \Phi_1) \right) \, d\sigma \right|
= \left| \left( n \times B \phi^*, \nabla \times \Phi_1 \right) - \left( \nabla \times (n \times B \phi^*), P \Phi_1 \right) \right|.
\]
(4.20)

We easily get that
\[
\left| \left( \nabla \times (n \times B \phi^*), P \Phi_1 \right) \right| \leq C \left( \| \omega_{\phi}^* \|^2 + \| \omega_{\phi}^e \|^2 \right).
\]
(4.21)

As for the remaining terms in (4.20), by the same arguments as to those of \( I_1 \), we have
\[
\left| \left( n \times B \phi^*, \nabla \times \Phi_1 \right) \right| \leq C \left( \| \omega_{\phi}^* \|^2 + \| \omega_{\phi}^e \|^2 \right).
\]
(4.22)

It follows from (4.21) and (4.22) that
\[
I_2 \leq C \left( \| \omega_{\phi}^* \|^2 + \| \omega_{\phi}^e \|^2 \right).
\]
(4.23)

Finally, we deal with \( I_3 \), i.e.,
\[
\left| \int_{\partial \Omega} (B v - n \times \omega_v) \cdot P \Phi_1 \, d\sigma - \int_{\partial \Omega} (n \times \omega_H) \cdot P \Phi_2 \, d\sigma \right|.
\]

We observe that the estimate is trivial if the system (1.5) satisfies the same boundary conditions as the system (1.1) does. In general, \( [Bv]_\tau - n \times \omega_v \) and \( [BH]_\tau - n \times \omega_H \) may be not equal to zero. As a result, the boundary layer may occur, so we will experience more complicated estimates. Similarly to \( I_2 \), we obtain that
\[
I_3 = \left| \int_{\partial \Omega} (B v - n \times \omega_v) \cdot P \Phi_1 \, d\sigma - \int_{\partial \Omega} (n \times \omega_H) \cdot P \Phi_2 \, d\sigma \right|
= \left| \int_{\partial \Omega} \left( n \times (B v - n \times \omega_v) \right) \cdot (n \times P \Phi_1) \, d\sigma - \int_{\partial \Omega} \left( n \times (n \times \omega_H) \right) \cdot (n \times P \Phi_2) \, d\sigma \right|
= \left| \left( n \times (B v - n \times \omega_v), \nabla \times \Phi_1 \right) - \left( n \times (n \times \omega_H), \nabla \times \Phi_2 \right) \right|
- \left( \nabla \times \left( n \times (B v - n \times \omega_v) \right), P \Phi_1 \right) + \left( \nabla \times \left( n \times (n \times \omega_H) \right), P \Phi_2 \right) \right|
\leq I_{31} + I_{32},
\]
where
\[
I_{31} = \left| \left( n \times (B v - n \times \omega_v), \nabla \times \Phi_1 \right) - \left( n \times (n \times \omega_H), \nabla \times \Phi_2 \right) \right|,
I_{32} = \left| \left( \nabla \times \left( n \times (B v - n \times \omega_v) \right), P \Phi_1 \right) - \left( \nabla \times \left( n \times (n \times \omega_H) \right), P \Phi_2 \right) \right|.
\]

We first deal with the term \( I_{31} \). It follows from the definitions of \( \Phi_1 \) and \( \Phi_2 \) that
\[
I_{31} \leq L_1 + L_2 + L_3 + L_4 + L_5 + L_6,
\]
where
\[
L_1 = \left| \left( n \times (B v - n \times \omega_v), \nabla \times (\phi^* \cdot \nabla \phi^* - \nabla \times (H \cdot \nabla \psi^*)) \right) \right|,
L_2 = \left| \left( n \times (B v - n \times \omega_v), \nabla \times (\phi^* \cdot \nabla v - \nabla \times (\psi^* \cdot \nabla H)) \right) \right|,
\]
\( \Phi \) Springer
We first deal with the terms which contain higher derivatives. Integrating by parts leads to
\[
L_3 = \left| \left( n \times (Bv - n \times \omega_v), \nabla \times (\varphi^e \cdot \nabla \varphi^e) - \nabla \times (\psi^e \cdot \nabla \psi^e) \right) \right|,
\]
\[
L_4 = \left| \left( n \times (n \times \omega_H), \nabla \times (v \cdot \nabla \psi^e) - \nabla \times (H \cdot \nabla \varphi^e) \right) \right|,
\]
\[
L_5 = \left| \left( n \times (n \times \omega_H), \nabla \times (\varphi^e \cdot \nabla H) - \nabla \times (\psi^e \cdot \nabla v) \right) \right|,
\]
\[
L_6 = \left| \left( n \times (n \times \omega_H), \nabla \times (\psi^e \cdot \nabla \varphi^e) - \nabla \times (\varphi^e \cdot \nabla \psi^e) \right) \right|.
\]
We have
\[
L_1 = \left| \left( n \times (Bv - n \times \omega_v), \nabla \times (v \cdot \nabla \varphi^e) - \nabla \times (H \cdot \nabla \psi^e) \right) \right|
= \left| \left( n \times (Bv - n \times \omega_v), v \cdot \nabla \omega^e_v - H \cdot \nabla \omega^e_\psi + [\nabla \times, v \cdot \nabla] \varphi^e - [\nabla \times, H \cdot \nabla] \psi^e \right) \right|.
\]
We first deal with the terms which contain higher derivatives. Integrating by parts leads to
\[
\left| \left( n \times (Bv - n \times \omega_v), v \cdot \nabla \omega^e_v - H \cdot \nabla \omega^e_\psi \right) \right|
= \left| \left( v \cdot \nabla (n \times (Bv - n \times \omega_v)), \omega^e_\varphi \right) - \left( H \cdot \nabla (n \times (Bv - n \times \omega_v)), \omega^e_\psi \right) \right|
\leq \int_{\partial \Omega} \left( n \times \varphi^e \right) \cdot \left( v \cdot \nabla (n \times (Bv - n \times \omega_v)) \right) d\sigma
+ \int_{\Omega} \nabla \times \left( v \cdot \nabla (n \times (Bv - n \times \omega_v)) \right) \cdot \varphi^e d\mathbf{x}
+ \int_{\partial \Omega} \left( n \times \psi^e \right) \cdot \left( H \cdot \nabla (n \times (Bv - n \times \omega_v)) \right) d\sigma
+ \int_{\Omega} \nabla \times \left( H \cdot \nabla (n \times (Bv - n \times \omega_v)) \right) \cdot \psi^e d\mathbf{x}
\leq C(|\varphi^e|_{L^2(\partial \Omega)} + |\psi^e|_{L^2(\partial \Omega)} + \|\varphi^e\| + \|\psi^e\|)
\leq \sqrt{C(|\omega^e_\varphi|^2 + |\omega^e_\psi|^2 + \epsilon^2)}.
\]
We also note that each component of $[\nabla \times, v \cdot \nabla] \varphi^e - [\nabla \times, H \cdot \nabla] \psi^e$ is a combination of such terms as $\partial_k v \cdot \nabla \varphi^e_k$ and $\partial_k H \cdot \nabla \psi^e_k$. Therefore, we consider the term
\[
\left( n \times (Bv - n \times \omega_v) \right)_m, \partial_k v \cdot \nabla \varphi^e_k - \partial_k H \cdot \nabla \psi^e_k.
\]
Since $\nabla \cdot \partial_k v = 0$ and $\nabla \cdot \partial_k H = 0$, we have
\[
\left| \left( n \times (Bv - n \times \omega_v) \right)_m, \partial_k v \cdot \nabla \varphi^e_k - \partial_k H \cdot \nabla \psi^e_k \right|
= \left| \left( \partial_k v, \nabla \left( \nabla \left( n \times (Bv - n \times \omega_v) \right)_m \right), \nabla \left( n \times (Bv - n \times \omega_v) \right)_m, \varphi^e_k \right) \right|
= \left| \left( \partial_k v, \nabla \left( n \times (Bv - n \times \omega_v) \right)_m \right), \nabla \left( n \times (Bv - n \times \omega_v) \right)_m, \varphi^e_k \right|
+ \left| \left( \partial_k H, \nabla \left( n \times (Bv - n \times \omega_v) \right)_m \right), \nabla \left( n \times (Bv - n \times \omega_v) \right)_m, \varphi^e_k \right|
= \int_{\partial \Omega} \varphi^e_k \left( n \times (Bv - n \times \omega_v) \right)_m \partial_k v \cdot nd\sigma - \int_{\partial \Omega} \varphi^e_k \left( n \times (Bv - n \times \omega_v) \right)_m \partial_k H \cdot nd\sigma
- \int_{\Omega} \partial_k v \cdot \nabla \left( n \times (Bv - n \times \omega_v) \right)_m \varphi^e_k d\mathbf{x} + \int_{\Omega} \partial_k H \cdot \nabla \left( n \times (Bv - n \times \omega_v) \right)_m \varphi^e_k d\mathbf{x}
\leq C\left( |\varphi^e|_{L^1(\partial \Omega)} + |\psi^e|_{L^1(\partial \Omega)} + \|\varphi^e\| + \|\psi^e\| \right)
\leq C\left( |\omega^e_\varphi|^2 + |\omega^e_\psi|^2 + \epsilon^2 \right).
\]
Hence, we obtain that
\[
L_1 \leq C(|\omega^e_\varphi|^2 + |\omega^e_\psi|^2 + \epsilon^2).
\]
Compared to $L_1$, both $L_2$ and $L_3$ can be easily estimated. In fact, we have

\[
L_2 = \left| \int_{\partial \Omega} (n \times (Bv - n \times \omega_v), \nabla \times (\varphi^x \cdot \nabla v) - \nabla \times (\psi^x \cdot \nabla H)) d\sigma \right|
\]

\[
+ \int_{\Omega} \nabla \times (n \times (Bv - n \times \omega_v)) \cdot (\varphi^x \cdot \nabla v - \psi^x \cdot \nabla H) dx \leq C(\|\varphi^x \|_{L^1(\partial \Omega)} + \|\psi^x \|_{L^1(\partial \Omega)} + \|\varphi^x\| + \|\psi^x\|)
\]

\[
\leq C(\|\omega^x_\varphi\|^2 + \|\omega^x_\psi\|^2 + \varepsilon^2), \tag{4.25}
\]

\[
L_3 = \left| \int_{\partial \Omega} (n \times (Bv - n \times \omega_v), \nabla \times (\varphi^x \cdot \nabla \varphi^x) - \nabla \times (\psi^x \cdot \nabla \psi^x)) d\sigma \right|
\]

\[
= \left| (n \times (Bv - n \times \omega_v), [\nabla \times \varphi^x \cdot \nabla] \varphi^x - [\nabla \times \psi^x \cdot \nabla] \psi^x) \right|
\]

\[
+ (n \times (Bv - n \times \omega_v), \varphi^x \cdot \nabla \omega^x_\varphi - \psi^x \cdot \nabla \omega^x_\psi)
\]

\[
\leq C(\|\omega^x_\varphi\|^2 + \|\omega^x_\psi\|^2) \tag{4.26}
\]

We find that $L_4$, $L_5$ and $L_6$ have similar structures to those of $L_1$, $L_2$ and $L_3$, respectively, so we can get

\[
L_4 \leq C(\|\omega^x_\varphi\|^2 + \|\omega^x_\psi\|^2 + \varepsilon^2), \tag{4.27}
\]

\[
L_5 \leq C(\|\omega^x_\varphi\|^2 + \|\omega^x_\psi\|^2 + \varepsilon^2), \tag{4.28}
\]

\[
L_6 \leq C(\|\omega^x_\varphi\|^2 + \|\omega^x_\psi\|^2). \tag{4.29}
\]

It follows from (4.24)–(4.29) that

\[
I_{31} \leq C(\|\omega^x_\varphi\|^2 + \|\omega^x_\psi\|^2 + \varepsilon^2).
\]

Now, it remains to estimate the term $I_{32}$, i.e.,

\[
\left[ \left( \nabla \times (n \times (Bv - n \times \omega_v)), \nabla \times (n \times (n \times \Omega)) \right), \nabla \times (n \times (n \times \Omega)) \right].
\]

We consider the first term in $I_{32}$. Since it involves Leray projection, some terms which contain higher derivatives of $\varphi^x$ or $\psi^x$ cannot be estimated easily. We have the following observations:

\[
v \cdot \nabla \varphi^x - \varphi^x \cdot \nabla v = \nabla \times (\varphi^x \times v), \tag{4.30}
\]

\[
H \cdot \nabla \psi^x - \psi^x \cdot \nabla H = \nabla \times (\psi^x \times H). \tag{4.31}
\]

Furthermore, since $\varphi^x \cdot n = 0$, $v \cdot n = 0$, $\psi^x \cdot n = 0$ and $H \cdot n = 0$, we have

\[
\varphi^x \times v = \lambda_1 n, \quad \psi^x \times H = \lambda_2 n, \tag{4.32}
\]

where $\lambda_1$ and $\lambda_2$ are two scalar functions defined on $\partial \Omega$. Based on the above observations, we easily obtain that

\[
\nabla \times (\varphi^x \times v) \in \mathbb{H}, \quad \nabla \times (\psi^x \times H) \in \mathbb{H},
\]

where $\mathbb{H}$ is Leray projection space. Thus we have the equality

\[
\mathbb{P} \Phi_1 = v \cdot \nabla \varphi^x - \varphi^x \cdot \nabla v + \mathbb{P} \Phi_{1v} = (H \cdot \nabla \psi^x - \psi^x \cdot \nabla H) - \mathbb{P} \Phi_{1h},
\]

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where
\[ \mathbb{P} \Phi^1_v = \mathbb{P}[2 \varphi^\varepsilon \cdot \nabla v + \varphi^\varepsilon \cdot \nabla \varphi^\varepsilon], \quad \mathbb{P} \Phi^1_H = \mathbb{P}[2 \psi^\varepsilon \cdot \nabla H + \psi^\varepsilon \cdot \nabla \psi^\varepsilon]. \]

Hence, we have
\[
\begin{align*}
\left( \nabla \times (n \times (Bv - n \times \omega_v)), \mathbb{P} \Phi^1_1 \right) \\
= \left( \nabla \times (n \times (Bv - n \times \omega_v)), \mathbb{P} \Phi^1_v \right) - \left( \nabla \times (n \times (Bv - n \times \omega_v)), \mathbb{P} \Phi^1_H \right) \\
+ \left( \nabla \times (n \times (Bv - n \times \omega_v)), v \cdot \nabla \varphi^\varepsilon - \varphi^\varepsilon \cdot \nabla v - (H \cdot \nabla \psi^\varepsilon - \psi^\varepsilon \cdot \nabla H) \right).
\end{align*}
\]

First, it holds
\[
\begin{align*}
\| \mathbb{P} \Phi^1_v \| & \leq C(\|v\|_{W^{1, \infty}} + \|\varphi^\varepsilon\|_{W^{1, \infty}})\|\varphi^\varepsilon\|, \quad (4.33) \\
\| \mathbb{P} \Phi^1_H \| & \leq C(\|H\|_{W^{1, \infty}} + \|\psi^\varepsilon\|_{W^{1, \infty}})\|\psi^\varepsilon\|. \quad (4.34)
\end{align*}
\]

Then it follows from (4.33), (4.34) and Lemma 4.1 that
\[
\left| \left( \nabla \times (n \times (Bv - n \times \omega_v)), \mathbb{P} \Phi^1_v \right) - \left( \nabla \times (n \times (Bv - n \times \omega_v)), \mathbb{P} \Phi^1_H \right) \right| \leq C \varepsilon^\frac{3}{4}. \quad (4.35)
\]

Next, by integrating by parts, we have
\[
\begin{align*}
\left| \left( \nabla \times (n \times (Bv - n \times \omega_v)), v \cdot \nabla \varphi^\varepsilon \right) \right| \\
= \left| \left( v \cdot \nabla \left( \nabla \times (n \times (Bv - n \times \omega_0)), \varphi^\varepsilon \right) \right) \right| \leq C(\|v\|_{H^3} + \|v\|_{H^3}^2)\|\varphi^\varepsilon\| \leq C \varepsilon^\frac{3}{4}. \quad (4.36)
\end{align*}
\]

Similarly, we obtain that
\[
\begin{align*}
\left| \left( \nabla \times (n \times (Bv - n \times \omega_0)), H \cdot \nabla \psi^\varepsilon \right) \right| \\
= \left| \left( H \cdot \nabla \left( \nabla \times (n \times (Bv - n \times \omega_0)), \psi^\varepsilon \right) \right) \right| \leq C(\|H\|_{H^3} + \|v\|_{H^3} \|H\|_{H^3})\|\psi^\varepsilon\| \leq C \varepsilon^\frac{3}{4}. \quad (4.37)
\end{align*}
\]

In addition, we directly get that
\[
\left| \left( \nabla \times (n \times (Bv - n \times \omega_0)), \varphi^\varepsilon \cdot \nabla v + \psi^\varepsilon \cdot \nabla H \right) \right| \leq C(\|\varphi^\varepsilon\| + \|\psi^\varepsilon\|) \leq C \varepsilon^\frac{3}{4}. \quad (4.38)
\]

Therefore, in view of (4.35)–(4.38), we obtain that
\[
\left| \left( \nabla \times (n \times (Bv - n \times \omega_0)), \mathbb{P} \Phi_1 \right) \right| \leq C \varepsilon^\frac{3}{4}.
\]

By taking the same arguments as to those above, we observe that
\[
\mathbb{P} \Phi_2 = \Phi_2.
\]

Hence, we get
\[
\left| \left( \nabla \times (n \times (n \times \omega_H)), \mathbb{P} \Phi_2 \right) \right| = \left| \left( \nabla \times (n \times (n \times \omega_H)), \Phi_2 \right) \right| \leq C \varepsilon^\frac{3}{4}, \quad (4.39)
\]

which implies
\[
I_{32} \leq C \varepsilon^\frac{3}{4}. \quad (4.40)
\]

Thus, we conclude that
\[
I_3 \leq C(\|\omega_\varepsilon^\varepsilon\|^2 + \|\omega_\varepsilon^\varepsilon\|^2 + \varepsilon\varepsilon^\frac{3}{4}). \quad (4.41)
\]

In conclusion, it follows from (4.19), (4.23) and (4.41), that
\[
\frac{1}{2} \frac{d}{dt} E + \frac{\varepsilon}{2}(\|\mathbb{P} \Delta \varphi^\varepsilon\|^2 + \|\mathbb{P} \Delta \psi^\varepsilon\|^2) \leq C(\|\omega_\varepsilon^\varepsilon\|^2 + \|\omega_\varepsilon^\varepsilon\|^2 + \varepsilon^\frac{3}{4}).
\]
Now we need to deal with the left terms in the above inequality. Let us recall that
\[
E = \|\omega_\varepsilon\|^2 + \|\psi_\varepsilon\|^2 - \int_{\partial \Omega} B \varphi' \cdot \varphi' d\sigma - 2 \int_{\partial \Omega} \varphi' \cdot (Bv - n \times \omega_v) d\sigma + 2 \int_{\partial \Omega} \psi' \cdot (n \times \omega_H) d\sigma.
\]
In view of Lemma 2.1, we obtain
\[
\left| \int_{\partial \Omega} B \varphi' \cdot \varphi' d\sigma \right| \leq C\|\varphi'\|^2_{L^2(\partial \Omega)} \leq \delta \|\omega_\varepsilon\|^2 + C\delta \|\varphi'\|^2,
\]
\[
2 \int_{\partial \Omega} \varphi' \cdot (Bv - n \times \omega_v) d\sigma - 2 \int_{\partial \Omega} \psi' \cdot (n \times \omega_H) d\sigma \leq C(\|\varphi'\|_{L^1(\partial \Omega)} + \|\psi'\|_{L^1(\partial \Omega)}) \leq \delta (\|\omega_\varepsilon\|^2 + \|\psi_\varepsilon\|^2) + C\delta (\|\varphi'\|^2 + \|\psi'\|^2),
\]
where \(\delta\) is small enough. Consequently, we get
\[
\|\omega_\varepsilon(t)\|^2 + \|\psi_\varepsilon(t)\|^2 + \epsilon \int_0^t \left( \|P \Delta \varphi'\|^2 + \|P \Delta \psi'\|^2 \right) ds \leq C \int_0^t \left( \|\omega_\varepsilon\|^2 + \|\omega_\varepsilon\|^2 \right) ds + C\epsilon \delta. \tag{4.42}
\]
Applying Gronwall's inequality to (4.42) gives us
\[
\|\omega_\varepsilon(t)\|^2 + \|\psi_\varepsilon(t)\|^2 \leq C\epsilon \delta, \quad t \in [0, T_2]. \tag{4.43}
\]
Thus, thanks to Lemma 2.1, we have
\[
\|\varphi'\|^2_{H^1} + \|\psi'\|^2_{H^1} + \epsilon \int_0^t \left( \|P \Delta \varphi'\|^2 + \|P \Delta \psi'\|^2 \right) ds \leq C\epsilon \delta. \tag{4.44}
\]
Using the same arguments as those of Section 2 in [27], we get
\[
\|v'\|^2 \leq C\|\varphi'\| + \|P \Delta \varphi'\|, \quad \|H'\|^2 \leq C\|H'\| + \|P \Delta H'\|. \tag{4.45}
\]
Therefore, it follows from Lemma 4.1 and (4.44) that
\[
\epsilon \int_0^t \left( \|v' - v\|^2_{H^1} + \|H' - H\|^2_{H^1} \right) ds \leq C\epsilon \delta. \tag{4.46}
\]
In addition, it is well-known that the following inequality holds:
\[
\|\nabla f\|_{L^p} \leq C\|\nabla f\|_{L^{p-2}} \|\nabla f\|^2.
\]
Hence, we obtain that
\[
\|\nabla \varphi'\|_{L^p} + \|\nabla \psi'\|_{L^p} \leq C\epsilon \delta. \tag{4.47}
\]
This completes the proof of Lemma 2.1 with Lemma 4.2, we easily get Theorem 1.3.

\[\square\]

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