Representation of superoperators in double phase space

Marcos Saraceno¹ and Alfredo M Ozorio de Almeida²

¹Departamento de Física Teórica, GlyA, Comisión Nacional de Energía Atómica, Av. Libertador 8250, C1429BNP Buenos Aires, Argentina
²Centro Brasileiro de Pesquisas Físicas, Rua Xavier Sigaud 150, 22290–180, Rio de Janeiro, R.J., Brazil

E-mail: saraceno@tandar.cnea.gov.ar

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Abstract
Operators in quantum mechanics—either observables, density or evolution operators, unitary or not—can be represented by c-numbers in operator bases. The position and momentum bases are in one-to-one correspondence with lagrangian planes in double phase space, but this is also true for the well known Wigner–Weyl correspondence based on translation and reflection operators. These phase space methods are here extended to the representation of superoperators. We show that the Choi–Jamiolkowski isomorphism between the dynamical matrix and the linear action of the superoperator constitutes a ‘double’ Wigner or chord transform when represented in double phase space. As a byproduct several previously unknown integral relationships between products of Wigner and chord distributions for pure states are derived.

Keywords: superoperators, phase space methods, Wigner–Weyl transform

(Some figures may appear in colour only in the online journal)

1. Introduction

The action of a unitary operator on states in quantum Hilbert space corresponds semi-classically to a classical transformation, x → x+, in the corresponding classical phase space R²ⁿ, with points x = (q₁, ..., qₙ, p₁, ..., pₙ). Alternatively, one may place the points x⁺ and x⁻ each in its own phase space so that the graph of a canonical transformation becomes a 2N dimensional surface within the 4N dimensional double phase space, built from the product space R²ᴺ × R²ᴺ, with points X = (x⁺, x⁻) [1]. Transformations in this double phase space allow the introduction of different representations of this surface and constitute the basis of the general method of generating functions in classical mechanics.
While the evolution of a pure state can be seen semiclassically as the evolution of an \( N \)-dimensional lagrangian surface in \( 2N \)-dimensional phase space \([2, 3]\), the evolution of an operator, mediated in quantum mechanics by a superoperator, corresponds to the motion of a \( 2N \)-dimensional lagrangian surface in \( 4N \)-dimensional double phase space. Different systems of canonical coordinates can be utilized to describe this evolution, which in quantum mechanics correspond to different operator bases.

One should keep in mind some essential differences between the classical and the quantum scenarios. Above all there is the uncertainty principle: while vectors in Hilbert space correspond semiclassically to (complex) functions on the lagrangian manifolds on single phase space—say the position or the momentum basis wave functions—operators are represented by their matrix elements from a pair of such manifolds. When viewed in double phase space these pairs are themselves \( 2N \)-dimensional complementary lagrangian surfaces.

However, some special coordinate transformations are allowed that take these obvious lagrangian coordinate planes of double phase space, that is, initial and final positions or momenta, to new coordinate planes that cannot be so decomposed. This is just the case of phase space labels underlying the Weyl representation (i.e. the Wigner function in the case of the density operator) and its Fourier transform (FT). These labels correspond to the affine operations of translations and reflections in phase space. These operations form a group, the affine group, which has a unitary representation in quantum mechanics consisting of the Weyl translation and the Wigner reflection operators. In the quantum optics literature, they are well known as displacement operators and their Fourier transform, the Wigner operator. Besides this active role we stress here the fact that they constitute two orthogonal bases of operators, and these bases are complementary, in the sense of the uncertainty principle. Thus, in double phase space, an operator can be represented in the translation or the reflection bases, just as a vector in Hilbert space can be represented in the coordinate or the momentum bases.

But the analogy of an operator represented by a lagrangian surface in double phase space to the state represented by a lagrangian surface in single phase space can now be pushed a step further: what about the ‘double Weyl transform’ of a superoperator, represented as a function in double phase space? The answer invokes the adaptation of the Choi–Jamiolkowski isomorphism \([5]\), well known in the theory of quantum information, to the operator basis underlying the Weyl representation and its Fourier transform. When the superoperator represents a quantum channel, i.e. a transformation mapping density operators, the Choi–Jamiolkowski isomorphism implements the channel-state duality \([6]\), which relates completely positive, trace preserving linear maps to bipartite density operators. Here we take a broader view and consider the isomorphism in its more general form, as it applies to arbitrary superoperators, and not only to quantum channels. We derive the general form it takes in an arbitrary orthogonal basis, but focus on the complementary phase space representations based on translation and reflection operators.

The purpose of this paper is to undertake this adaptation, showing that the Choi conjugate of a superoperator is the ‘double Weyl transform’ of its matrix elements in the Wigner basis. We achieve this by defining reflection and translation superoperators, labeled by points in double phase space, in strict analogy to the corresponding definitions in single phase space. Moreover we show that they take a simple monomial form in the Choi-conjugate basis.

Another surprise is that the Choi-conjugate pairs of operators representing pure states are identical. This leads to unexpected Fourier identities involving integrals of products of ordinary Wigner functions. In some cases this in turn leads to new identities involving the special functions of analysis.
The paper is organized as follows: in section 2 we review the concept of double phase space [1, 7–9] and how it is related to the labeling of complementary bases of operators in terms of lagrangian surfaces in double phase space.

In section 3 we extend the methods to the representation of superoperators. The concept of Choi-conjugate bases is defined and applied to the phase space bases that are determined by reflections and translations. In this context Choi-conjugation takes on the character of a ‘double’ Wigner–Weyl transformation in double phase space. The identification of such double Wigner functions with the Wigner–Weyl transformation from alternative matrix representations of a superoperator is examined in section 4.

Superoperators for various kinds of quantum evolution are then discussed in section 5. Finally, in section 6 the pullback of the double phase space results for single phase space Wigner functions generates several new identities for pure states.

2. Double phase space and operator representations

The concept of double phase space underlies in classical mechanics the general theory of generating functions of canonical transformations. It provides an elegant way to visualize a canonical transformation in a doubled phase space as the gradient of a generating function [1, 7, 8]. In quantum mechanics it provides a flexible mechanism to represent unitary propagators in the semiclassical limit [11] as functions on these manifolds and underlies the presentation of quantum mechanics in terms of phase space path integrals [12, 13].

A canonical transformation $x_+ \mapsto x_-$ can be specified implicitly by a generating function $S(q_+, q_-)$ whose differential is

$$dS(q_+, q_-) = p_+ dq_+ - p_- dq_-.$$  \hspace{1cm} (1)

With a simple redefinition of coordinates

$$Q_1 = (q_+, q_-), \quad P_1 = (p_+, -p_-)$$  \hspace{1cm} (2)

we rewrite $dS = P_1 \cdot dQ_1$ and the transformation is defined implicitly as $P_1(Q_1) = \partial S(Q_1)/\partial Q_1$. The new coordinates $Q_1, P_1$ can now be interpreted as canonical coordinates in a phase space with doubled dimensions. In the elementary theory other canonical pairs are well known and are obtained by various Legendre transforms. In the standard notation of [14–16] they are:

$$dS = dF_1 = p_+ dq_+ - p_- dq_- \equiv P_1 \cdot dQ_1 \quad Q_1 = (q_+, q_-), \quad P_1 = (p_+, -p_-)$$

$$dF_2 = q_+ dp_+ + p_+ dq_- \equiv P_2 \cdot dQ_2 \quad Q_2 = (p_+, q_-), \quad P_2 = (q_+, p_-)$$

$$dF_3 = -q_- dp_- - p_- dq_+ \equiv Q_2 \cdot dP_2$$

$$dF_4 = -q_- dp_+ - p_+ dq_+ \equiv Q_1 \cdot dP_1$$  \hspace{1cm} (3)

The general theory developed by [1, 7, 8] interprets these alternative pairs $Q_j, P_j$ as the coordinates for hyperplanes lying in a phase space with doubled dimension. We give here a synthetic overview with the purpose of identifying the structures that will reappear in quantum mechanics in the representation of operators and superoperators. Starting from the elementary statement of area preservation in a canonical transformation

3 For simplicity from now on we restrict to the 1-D case $x = (q, p)$. 

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\[ \oint p_i \, dq_i = \oint p_j \, dq_j = \frac{1}{2} \oint p_i \, dq_i - q_i \, dp_i = \frac{1}{2} \oint p_j \, dq_j - q_j \, dp_j. \]  

Rewritten as

\[ \frac{1}{2} \oint (x_+ \cdot J \, dx_+ - x_- \cdot J \, dx_-) = 0, \]

where as usual \( x_{\pm} \equiv (q, p)_{\pm} \) are points in single phase space and \( J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) is the standard symplectic matrix in 2D which defines the symplectic product as \( \langle x, y \rangle = x \cdot J y \). Placing ourselves in the 4D direct product space spanned by \( (x_+, x_-) \) we rewrite equation (5) as

\[ \frac{1}{2} \oint (x_+ \cdot x_-) \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \begin{pmatrix} dx_+ \\ dx_- \end{pmatrix} = 0. \]

Thus the canonical transformation \( x_- \mapsto x_+ \) is defined by a 2D lagrangian surface in the 4D direct product phase space. On this surface

\[ dA = \frac{1}{2} (x_+ \cdot x_-) \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \begin{pmatrix} dx_+ \\ dx_- \end{pmatrix} \]

is an exact differential. However this surface is lagrangian with respect to the non-standard symplectic matrix \( \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \). To introduce canonical coordinates with respect to the standard canonical matrix \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), we consider a linear transformation

\[ \begin{pmatrix} x_+ \\ x_- \end{pmatrix} = U \begin{pmatrix} Q \\ P \end{pmatrix} \equiv \begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix} \]

such that

\[ dA = \frac{1}{2} (x_+ \cdot J \, dx_+ - x_- \cdot J \, dx_-) = \frac{1}{2} (P \cdot dQ - Q \cdot dP). \]

The matrix \( U \) should satisfy

\[ U^t \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

Various choices of the matrix \( U \) lead to different pairs \( Q, P, \) of which those defined in equation (3) are the most common examples. All such pairs are related by canonical transformations in 4D phase space. Notice that the first form of \( dA \) in equation (9) is invariant under canonical transformations \( x_{\pm} \mapsto y_{\pm} \) in single phase space while the second form becomes invariant under fully two degrees of freedom canonical transformations \( Q, P \mapsto Q', P' \). To proceed to the definition of the canonical transformation in these coordinates a Legendre transformation on \( dA \) leads to

\[ d\tilde{A} = d \left( A + \frac{1}{2} P \cdot Q \right) = P \cdot dQ \]

and therefore

\[ P(Q) = \frac{\partial \tilde{A}(Q)}{\partial Q}. \]
The canonical transformation is then specified via the parametric equations

\[
\begin{pmatrix}
  x_+(Q) \\
  x_-(Q)
\end{pmatrix} = \begin{pmatrix}
  U_{00} & U_{01} \\
  U_{10} & U_{11}
\end{pmatrix} \begin{pmatrix}
  Q \\
  P
\end{pmatrix}.
\]

(13)

Alternatively a different Legendre transform leads to a conjugate representation exchanging coordinates and momenta

\[
\begin{pmatrix}
  x_+(P) \\
  x_-(P)
\end{pmatrix} = \begin{pmatrix}
  U_{00} & U_{01} \\
  U_{10} & U_{11}
\end{pmatrix} \begin{pmatrix}
  Q \\
  P
\end{pmatrix}.
\]

(14)

In this general approach a canonical transformation is characterized by (a) a linear transformation \( \mathcal{U} \) satisfying equation (10) and the canonical coordinates \( Q, P \) and (b) by generating functions \( \hat{A}(Q) (\hat{A}(P)) \) on the lagrangian surfaces \( P = \text{const.} \) (\( Q = \text{const.} \)). The two generating functions are themselves related by the Legendre transform \( \hat{A} = \hat{A} + P \cdot Q \).

Avoidance of singularities in the process of inverting the functions \( x_+(Q) (x_+(P)) \) in equations (13) and (14) will lead to transversality conditions guaranteeing the existence of the transformation. This general scheme can be easily identified in equation (3) where \( Q_1, P_1 \) and \( Q_2, P_2 \) are canonically conjugate pairs and \( \mathcal{U} \) is a simple permutation.

A different choice of canonical variables in double phase space is related to the well-known Wigner–Weyl representation of quantum mechanics in phase space. It is characterized by the matrix

\[
\mathcal{U} = \begin{pmatrix}
  1 & \frac{1}{2}J \\
  -\frac{1}{2}J & 1
\end{pmatrix},
\]

leading to the transformation

\[
\begin{pmatrix}
  x_+ \\
  x_-
\end{pmatrix} = Q + \frac{1}{2}JP \quad \begin{pmatrix}
  x_+ \\
  x_-
\end{pmatrix} = Q - \frac{1}{2}JP
\]

(16)

and its inverse

\[
\begin{pmatrix}
  Q \\
  P
\end{pmatrix} = \frac{x_+ + x_-}{2} \quad \begin{pmatrix}
  Q \\
  P
\end{pmatrix} = -J(x_+ - x_-).
\]

(17)

However it is customary to use, instead of \((Q, P)\), the single phase space coordinates

\[
\begin{pmatrix}
  x \\
  \xi
\end{pmatrix} \equiv Q = (x_+ + x_-)/2 \quad \xi \equiv JP = x_+ - x_-
\]

(18)

Notice that the single phase space coordinates \( x = (q, p) \) are now treated as a pair of position coordinates \((Q_1, Q_2)\) in double phase space, while \( \xi = (\xi_q, \xi_p) \) are their canonical conjugates \((P_1, P_2)\). The lagrangian planes of constant \( x \) represent the transformation \( x_+ = 2x - x_- \) which is a phase space reflection about the point \( x \), while those of constant \( \xi \) lead to the translation \( x_+ = x_- + \xi \). Thus \( x, \xi \) are natural labels in quantum mechanics for the operators that represent translations and reflections. They have a simple geometrical interpretation in single phase space as centers and chords of a pair of phase space points [17]. In figure 1 we display this relationship. The Poincaré generating function, \( S(x) \) and its Legendre transform \( \hat{S}(\xi) \) provide ways of representing canonical transformations alternative to the usual ones in equation (3) \((F_1, F_2, F_3, F_4)\) and have many desirable properties [10, 17].

Given this firm double phase space structure for classical transformations we now turn to quantum operators \( \hat{A} \) and their representation. We adopt here a dual notation for operators:
when they are considered as vectors in the Hilbert–Schmidt sense we use the double Dirac notation $A \langle \rangle$, but we keep the standard notation $\hat{A}$ when they are two sided arrays with the usual matrix multiplication rules. The latter notation emphasizes the active aspect of $\hat{A}$ as operators acting on vectors as $\hat{A} |\psi\rangle$ while the former is meant to bring out their passive role as basis elements in which to expand operators and superoperators. Thus we adopt the double bracket notation $AB \langle \rangle$ for the Hilbert–Schmidt scalar product and then understand $BA \langle \rangle$ as the projection of the operator $\hat{A}$ in a generic operator basis $B$, just as the single bracket $i$ is the projection of $\psi$ in a generic Hilbert space basis $|i\rangle$.

Lagrangian coordinates in double phase space provide the natural labels for orthogonal operator bases in quantum mechanics. To treat the generic case we consider a basis of operators $B_\alpha \hat{a}$ labeled by a double index $2 \alpha \in \mathbb{R}^2$ which can be any of the lagrangian planes considered. The main properties that we assume for these bases are

$$\langle B_\alpha | B_\beta \rangle = \Lambda_\delta (\alpha - \beta) \quad \text{Orthogonality}$$

$$\frac{1}{\Lambda} \int d^2\alpha \langle B_\alpha | B_\alpha \rangle = 1 \quad \text{Completeness.}$$

The factor $\Lambda$ is conventional and could be removed by a simple rescaling.

In the context of next section notice that equation (20) is in fact a representation of the unit superoperator in the operator basis $B_\alpha$. Equation (20) implies that any operator $\hat{O}$ can be expanded in the basis $\langle B_\alpha |$, i.e.

$$\hat{O} = \frac{1}{\Lambda} \int d^2\alpha \hat{B}_\alpha \langle B_\alpha | O \rangle,$$

with coefficients $\langle B_\alpha | O \rangle$ that are the c-number representatives of $\hat{O}$.

We can now identify the bases corresponding to the lagrangian planes in equation (3). $\langle Q_1 | = | q_+ \rangle \langle q_+ |$, $\langle P_1 | = | p_+ \rangle \langle p_+ |$ are of course the usual position and momentum bases. $\langle Q_2 | = | p_+ \rangle \langle q_+ |$, $\langle P_2 | = | q_+ \rangle \langle p_+ |$ are mixed position and momentum. The complementarity of these bases is reflected in the property $\langle P_1 | Q_1 \rangle = \exp - i Q_1 \cdot P_1 / \hbar$. The c-number representatives of an operator in these bases are simply its position, momentum or mixed position–momentum matrix elements.

Another pair of complementary operator bases is obtained when considering the unitary representation of reflections and translations in phase space. They are defined in terms of symmetrized Fourier transforms of the previous bases as follows

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{Geometrical relationship between $x_-$, $x_+$ and centers and chords $x$, $\xi$.}
\end{figure}
\[ R_\xi \equiv \hat{R}_{q,p} = \int dq' \left| q + \frac{q'}{2} \right\rangle \left\langle q - \frac{q'}{2} \right| \omega^{pq'} \]  
(22)

\[ \hat{T}_\xi \equiv \hat{T}_{\xi',\xi} = \int dp' \left| p' + \frac{\xi_p}{2} \right\rangle \left\langle p' - \frac{\xi_p}{2} \right| \omega^{-\xi p'.} \]  
(23)

Here the first is in the position basis and the latter in the momentum basis, but either can be used\(^4\). To simplify notation we have defined \( \omega \equiv \exp i/\hbar \). As a complementary set of orthogonal bases they have the properties

\[ \langle R_x | R_x' \rangle = 2\pi\hbar (x - x'), \quad \langle T_\xi | T_\xi' \rangle = 2\pi\hbar (\xi - \xi'), \quad \langle T_\xi | R_x \rangle = \omega^{-\xi x} \]  
(24)

so that for these bases \( \Lambda = 2\pi\hbar \), whereas \( \Lambda = 1 \) for the position or momentum bases. The action of these operators and further properties are reviewed in the appendix. It should be noted that in the quantum optics literature these are the well-known Wigner (or phase space point) operator and its Fourier transform, the displacement operator. The c-number representatives of operators in these bases are phase space functions that constitute the starting point for the formulation of quantum mechanics in phase space [18–21]. The reflection basis,

\[ O(x) \equiv \langle R_x | \hat{O} \rangle = \int dq' \left| q + \frac{q'}{2} \right\rangle \hat{O} \left| q - \frac{q'}{2} \right\rangle \omega^{-q'p'}, \]  
(25)

leads to the Wigner–Weyl symbol of the operator while the translation basis

\[ \hat{O}(\xi) \equiv \langle T_\xi | \hat{O} \rangle = \int dp' \left| p' + \frac{\xi_p}{2} \right\rangle \hat{O} \left| p' - \frac{\xi_p}{2} \right\rangle \omega^{p\xi}, \]  
(26)

leads to its characteristic function. In line with [17] we will refer to them respectively as center and chord representations. They are related by a symplectic Fourier transform

\[ O(x) = \frac{1}{2\pi\hbar} \int d\xi^2 \omega^{-\xi x} \hat{O}(\xi). \]  
(27)

In the special case of the density operator \( \hat{\rho} \), \( \rho(x) \) and \( \rho(\xi) \) are the Wigner function and the characteristic function, or chord function:

\[ \chi(\xi) = \frac{1}{2\pi\hbar} \langle T_\xi | \rho \rangle \quad \text{and} \quad W(x) = \frac{1}{2\pi\hbar} \langle R_x | \rho \rangle, \]  
(28)

with the special normalization properties

\[ 2\pi\hbar \chi(0) = \text{tr} \hat{\rho} = 1 \quad \text{and} \quad \int dx^2 W(x) = \text{tr} \hat{\rho} = 1, \]  
(29)

corresponding to classical distributions.

### 3. Superoperator representations

#### 3.1. General representations and the Choi–Jamiolkowsky isomorphism

A superoperator \( \mathbf{S} \) is a general linear transformation in the space of operators. Simple examples are the unitary evolution of a density matrix \( \hat{\rho} \to \hat{U} \hat{\rho} \hat{U}^\dagger \), symmetry operations, time reversal, non-unitary evolution, etc. Once the action of \( \mathbf{S} \) on general operators has been

\(^4\) It should be noted that the unitary reflection operator is \( \hat{R}_x/2 \), but our definition is more convenient as a basis.
specified as $\mathbf{S}(\hat{O}) = \hat{O}^\dagger$, its matrix elements in the $B_\alpha$ basis are

$$\langle B_\alpha \parallel \mathbf{S} \parallel B_\beta \rangle = \text{tr} \left[ \hat{B}_\alpha \mathbf{S} \hat{B}_\beta \right].$$

Thus a general superoperator can be expanded as

$$\mathbf{S} = \frac{1}{N^2} \int d^2\alpha d^2\beta \langle B_\alpha \parallel \mathbf{S} \parallel B_\beta \rangle \langle B_\beta \parallel \hat{O} \parallel B_\alpha \rangle.$$ \hfill (31)

The superoperators $\langle B_\alpha \parallel \mathbf{S} \parallel B_\beta \rangle$ constitute then a complete orthonormal basis labeled by a double index $\alpha, \beta \in \mathbb{R}^2$. Their action on an operator $\hat{O}$ follows the standard bra and ket rules so that $\langle B_\alpha \parallel \langle B_\beta \parallel \hat{O} \parallel B_\alpha \rangle \parallel B_\beta \rangle = \hat{B}_\alpha \text{tr} \hat{B}_\beta \hat{O}$. The unitary relations between different choices of bases also allow us to define the trace of a superoperator uniquely as

$$\text{Tr} \mathbf{S} = \frac{1}{N^2} \int d^2\alpha \langle B_\alpha \parallel \mathbf{S} \parallel B_\alpha \rangle.$$ \hfill (32)

Nonetheless, it is convenient to consider a different action related to the pair of operators $\hat{B}_\alpha, \hat{B}_\beta^\dagger$

$$\hat{B}_\alpha \cdot \hat{B}_\beta^\dagger \left( \hat{O} \right) \equiv \hat{B}_\alpha \hat{O} \hat{B}_\beta^\dagger.$$ \hfill (33)

Indeed, $\hat{B}_\alpha \cdot \hat{B}_\beta^\dagger$ is a different orthonormal superoperator basis related to the former by a unitary transformation in $\mathbb{R}^2$. The transformation is given by

$$\hat{B}_x \cdot \hat{B}_y^\dagger = \frac{1}{N^2} \int d^2\alpha d^2\beta \text{tr} \left( \hat{B}_x \hat{B}_y^\dagger \hat{B}_\alpha \hat{B}_\beta^\dagger \right) \langle B_\alpha \parallel B_\beta \rangle,$$ \hfill (34)

with the inverse:

$$\langle B_\alpha \parallel B_\beta \rangle = \frac{1}{N^2} \int d^2x d^2y \text{tr} \left( \hat{B}_x^\dagger \hat{B}_y^\dagger \hat{B}_\alpha \hat{B}_\beta \right) \hat{B}_x \cdot \hat{B}_y^\dagger.$$ \hfill (35)

We prove for example (34) by computing with equation (30) the matrix element $\langle B_\alpha \parallel \hat{B}_x \cdot \hat{B}_y^\dagger \parallel B_\beta \rangle$. In the following we will refer to the pair of superoperators $\langle A \parallel B \rangle$ and $\hat{A} \cdot \hat{B}^\dagger$ as Choi conjugates and extend this name to the two bases.

$\mathbf{S}$ can also be expanded in the new basis as

$$\mathbf{S} = \frac{1}{N^2} \int d^2x d^2y \mathbf{C}_S(x, y) \hat{B}_x \cdot \hat{B}_y^\dagger,$$ \hfill (36)

so that the coefficients $\mathbf{C}_S(x, y)$ constitute an alternative representation of $\mathbf{S}$, related to its matrix elements by equation (35)

$$\langle B_\alpha \parallel \mathbf{S} \parallel B_\beta \rangle = \frac{1}{N^2} \int d^2x d^2y \mathbf{C}_S(x, y) \text{tr} \left( \hat{B}_x^\dagger \hat{B}_y^\dagger \hat{B}_\alpha \hat{B}_\beta \right).$$ \hfill (37)

The nature of the transformation between the two representations of $\mathbf{S}$ is specific to a given basis and depends on the trace of four basis elements. The simplest transformation occurs when a transition basis $\{ i \} \langle j \}$ of orthonormal states is used. In that case (setting $\Lambda = 1$) the quadruple trace results in a product of four delta functions which reduces the relationship to a partial transposition of indices

$$\langle E_i \parallel \mathbf{S} \parallel E_i \rangle = \mathbf{C}_S(ik, jl).$$ \hfill (38)

In this case one obtains two different linear transformations related to $\mathbf{S}$. They are well known in the quantum open systems literature. $\mathbf{C}_S(ik, jl)$ is known as the Choi matrix [22], dynamical matrix [23, 24], chi matrix [25] and the relationship between them as the
Choi–Jamiolkowski isomorphism. It is also of importance in the literature on matrix product states for spin chains [26]. When $S$ represents a quantum operation on density matrices the physical requirement of complete positivity requires that $C_S(ik, jl)$ be hermitian and positive [22, 27]. In that case $C_S$ maps the space of positive density matrices into itself. Although related, the spectral properties of the two transformations are very different. Thus if $C_S(ik, jl)$ is hermitian it can be brought to diagonal form leading to the Kraus representation [27]. On the other hand $\langle E_{ij} | S | E_{ij} \rangle$ as a linear transformation may not even be normal, leading in general to a complex spectral decomposition.

3.2. The Choi–Jamiolkowski relation in phase space

Consider first the reflection basis $R_x$. A general superoperator $S$ in this basis has the matrix elements $\langle R_{x+\frac{1}{2}l} | S | R_{x-\frac{1}{2}k} \rangle$. The labels, $x, l$, span a full set of reflection centers, each covering a full phase space. On the other hand a rotation in double phase space allows us to consider $\langle R_{x+\frac{1}{2}l} |S | R_{x-\frac{1}{2}k} \rangle$ as an alternative position basis for superoperators, just as valid as $\langle q_{+} | q_{-} \rangle$ in the operator case. Hence, one defines a double Weyl transform in analogy to equation (25) as a symmetrized Fourier transform:

$$S(x, \xi) \equiv \frac{1}{2\pi\hbar} \int d^2x \langle R_{x+\frac{1}{2}l} | S | R_{x-\frac{1}{2}k} \rangle \omega^{<x,\xi>},$$

where we use the phase space coordinates $(x, \xi)$ which change the Fourier phase from $-Q \cdot P$ to $x_1 \cdot J\xi \equiv <x_1, \xi>$. This leads to a very natural definition of a reflection superoperator in this basis as

$$R_{x,\xi} \equiv \frac{1}{2\pi\hbar} \int d^2x \langle R_{x+\frac{1}{2}l} | R_{x-\frac{1}{2}k} \rangle \langle R_{x-\frac{1}{2}k} |\rangle \omega^{-<x,\xi>}.$$

The Weyl transform of $S$ can now be expressed as a superoperator trace,

$$S(x, \xi) = \text{Tr} \; SR_{x,\xi},$$

in complete analogy to equation (25). The ‘super’ reflection properties of $R_{x,\xi}$ are contained in the easily derived actions on translations and reflections,

$$R_{x,\xi} |R_{x_0}\rangle = 4 |R_{x-x_0}\rangle \omega^{<x-x_0,\xi>},$$
$$R_{x,\xi} |T_{\xi_{\xi}}\rangle = 4 |T_{\xi_{-\xi}}\rangle \omega^{-<\xi,\xi_{\xi}>},$$

in obvious similarity to equation (A.6).

Just as any other superoperator, one can now expand $R_{x,\xi}$ in the Choi-conjugate basis, $\hat{R}_{x} \cdot \hat{R}_{x'}$, using the general relationship in equation (35):

$$\langle R_{x+\frac{1}{2}l} | R_{x-\frac{1}{2}k} \rangle = \frac{1}{(2\pi\hbar)^2} \int d^2\alpha d^2\beta \hat{R}_\alpha \cdot \hat{R}_\beta \delta \left( x - \frac{\alpha + \beta}{2} \right) \omega^{<\alpha,\alpha-\beta>},$$

The quadruple trace computed in equation (A.13) reduces this to

$$\langle R_{x+\frac{1}{2}l} | R_{x-\frac{1}{2}k} \rangle = \frac{1}{(2\pi\hbar)^2} \int d^2\alpha d^2\beta \hat{R}_\alpha \cdot \hat{R}_\beta \delta \left( x - \frac{\alpha + \beta}{2} \right) \omega^{<\alpha,\alpha-\beta>},$$

$$= \frac{1}{2\pi\hbar} \int d^2\xi \; \hat{R}_{x+\frac{1}{2}l} \cdot \hat{R}_{x-\frac{1}{2}k} \; \omega^{<x,\xi>} \tag{44}$$

5 The denominator $2\pi\hbar = \Lambda$ is usually absent because $\Lambda = 1$ for the position representation.
and its inverse
\[
\hat{R}_{x+\xi/2} \cdot \hat{R}_{x-\xi/2} = \frac{1}{2\pi\hbar} \int d^2\xi \|T_{\xi+\frac{\xi}{2}}\rangle\langle T_{\xi-\frac{\xi}{2}}\| \omega^{-\langle \xi, x \rangle}. \tag{45}
\]
Comparison with equation (40) then shows that \(\mathbf{R}_{x,\xi}\) has a very simple (monomial) form in the Choi-conjugate basis:
\[
\mathbf{R}_{x,\xi} = \hat{R}_{x+\xi/2} \cdot \hat{R}_{x-\xi/2} = \hat{R}_{x,}\cdot\hat{R}_{x}. \tag{46}
\]
Very similar results can be obtained in the translation basis. A translation superoperator is defined as
\[
\mathbf{T}_{x,\xi} \equiv \frac{1}{2\pi\hbar} \int d^2\xi \|T_{\xi+\frac{\xi}{2}}\rangle\langle T_{\xi-\frac{\xi}{2}}\| \omega^{-\langle \xi, x \rangle}, \tag{47}
\]
with the translation properties
\[
\mathbf{T}_{x,\xi}\|T_{\xi}\rangle = \|T_{\xi+\frac{\xi}{2}}\rangle \omega^{\langle x, \xi+\frac{\xi}{2}\rangle},
\]
\[
\mathbf{T}_{x,\xi}\|\mathbf{R}_{x}\rangle = \|\mathbf{R}_{x+\xi}\rangle \omega^{\langle \xi, x+\xi \rangle}. \tag{48}
\]
It defines the double chord representation of \(\mathbf{S}\) as
\[
\mathbf{S}(x,\xi) = \frac{1}{2\pi\hbar} \int d^2\xi \|T_{\xi+\frac{\xi}{2}}\rangle\langle T_{\xi-\frac{\xi}{2}}\| \omega^{-\langle \xi, x \rangle} = \text{Tr} \mathbf{ST}_{x,\xi}. \tag{49}
\]
An almost identical derivation as in equation (45) yields the relationship between the Choi-conjugate translation bases:
\[
\|T_{\xi+\frac{\xi}{2}}\rangle\langle T_{\xi-\frac{\xi}{2}}\| = \frac{1}{2\pi\hbar} \int d^2\xi \|T_{\xi+\frac{\xi}{2}}\rangle\langle T_{\xi-\frac{\xi}{2}}\| \omega^{-\langle \xi, x \rangle}, \tag{50}
\]
which again leads to a monomial representation of the translation superoperator:
\[
\mathbf{T}_{x,\xi} = \hat{T}_{x+\xi/2} \cdot \hat{T}_{x-\xi/2} = \hat{T}_{x,}\cdot\hat{T}_{x}. \tag{51}
\]
Equations (46)–(51) are our main general results: in double phase space the reflection and translation superoperators can be defined as usual in terms of symmetrized Fourier transforms of matrix elements, but they also acquire a simple monomial form in their respective Choi-conjugate basis. The interplay between their action in double phase space with the single phase space factor translations and reflections in \(\hat{R}_{x,}\cdot\hat{R}_{x}\) and \(\hat{T}_{x,}\cdot\hat{T}_{x}\) will be further discussed in section 5.

We now use these identities to derive the Choi–Jamiolkowsky relationship between this pair of conjugate bases that represent \(\mathbf{S}\). Expanding a general superoperator as in equation (36)
\[
\mathbf{S} = \int \frac{d^2x_+d^2x_-}{(2\pi\hbar)^2} C_{\mathbf{S}}(x_+, x_-) \hat{R}_{x_+} \cdot \hat{R}_{x_-} = \int \frac{d^2x_+d^2x_-}{(2\pi\hbar)^2} \tilde{C}_{\mathbf{S}}(x_+, x_-) \hat{T}_{x_+} \cdot \hat{T}_{x_-}, \tag{52}
\]
defines \(C_{\mathbf{S}}(x_+, x_-)\) and \(\tilde{C}_{\mathbf{S}}(x_+, x_-)\) as the Choi or dynamical matrices of \(\mathbf{S}\) in the reflection and translation basis. Changing the double phase space variables from \((x_+, x_-) \rightarrow (x, \xi)\) as in equation (18) (with unit jacobian) we rewrite it as
\[ S = \int \frac{d^2x d^2\xi}{(2\pi\hbar)^2} C_S(x + \frac{\xi}{2}, x - \frac{\xi}{2}) R_{x, \xi} = \int \frac{d^2x d^2\xi}{(2\pi\hbar)^2} \tilde{C}_S(x + \frac{\xi}{2}, x - \frac{\xi}{2}) T_{x, \xi}. \]

That is, the superoperator is expanded as a superposition of reflections (translations), such that the coefficients are identified with the center (chord) representations in double phase space. Thus, we have obtained

\[
S(x, \xi) = \text{Tr}(SR_{x, \xi}) = C_S(x_+, x_-) \quad \text{(54)}
\]

\[
\tilde{S}(x, \xi) = \text{Tr}(ST_{x, \xi}) = \tilde{C}_S(x_+, x_-), \quad \text{(55)}
\]

which is the desired Choi–Jamiołkowski relation in these bases: the Choi matrix appears as a rotated double Weyl or chord transform of the superoperator. The property can be inverted to retrieve the matrix elements from an integral over the Choi matrix

\[
\langle R_{x, \frac{1}{2}x} | S \rangle \langle R_{x, \frac{1}{2}x} \rangle = \frac{1}{2\pi\hbar} \int d^2\xi \ C_S(x + \frac{1}{2}\xi, x - \frac{1}{2}\xi) \omega^{-<\xi, \xi>}
\]

\[
\langle T_{\xi, \frac{1}{2}x} | S \rangle \langle T_{\xi, \frac{1}{2}x} \rangle = \frac{1}{2\pi\hbar} \int d^2\xi \ \tilde{C}_S(x + \frac{1}{2}\xi, x - \frac{1}{2}\xi) \omega^{-<\xi, x>}. \quad \text{(56)}
\]

It is important to note that these relationships follow directly from equation (37), but the direct derivation would then miss the interpretation of the Choi matrix as a phase space distribution in double phase space. Furthermore, the complete analogy between the Weyl transform in double phase space to its familiar version in ordinary phase space allows us to directly import formulae that are familiar in the latter context into the superoperator scenario. For instance, the trace of a superoperator, \( S \), takes on the alternative forms:

\[
\text{Tr} S = \frac{1}{2\pi\hbar} \int d^2x \langle S \rangle \langle R_x \rangle = \frac{1}{(2\pi\hbar)^2} \int d^2x d^2\xi \ S(x, \xi)
\]

\[
= \frac{1}{2\pi\hbar} \int d^2\xi \ \langle T_{\xi} \rangle \langle S \rangle \langle T_{\xi} \rangle = \tilde{S}(x = 0, \xi = 0). \quad \text{(57)}
\]

Again, the trace of a product of superoperators may be expressed as

\[
\text{Tr} S_2 S_1 = \int \frac{d^2x_1 d^2x_2}{(2\pi\hbar)^2} \langle R_{x, 1} \rangle \langle S_2 \rangle \langle R_{x, 2} \rangle \langle S_1 \rangle \langle R_{x, 1} \rangle
\]

\[
= \int \frac{d^2x d^2\xi}{(2\pi\hbar)^2} S_2(x, \xi) S_1(x, \xi)
\]

\[
= \int \frac{d^2\xi d^2\xi}{(2\pi\hbar)^2} \langle T_{\xi, 1} \rangle \langle S_2 \rangle \langle T_{\xi, 2} \rangle \langle S_1 \rangle \langle T_{\xi, 1} \rangle
\]

\[
= \int d^2x d^2\xi \ \tilde{S}_1(x, \xi) S_1^*(x, \xi). \quad \text{(58)}
\]

The Wigner–Weyl expressions of partial traces over subsystems can also be immediately generalized from their versions in single phase space, which are discussed in [9].

### 4. Generality of the double Weyl transform

In the previous section the double Weyl transform was implemented by choosing the conjugate lagrangian planes \((x, \xi)\) in equation (16). We now show that many other equivalent definitions can be defined using alternative coordinates. Take for example the coordinates \(Q_1, P_1\) in equation (2) and the operator basis
\[ \hat{Q}_a \equiv \| Q_a \| = |q_a|, \quad \hat{P}_\alpha \equiv \| P_\alpha \| = |p_\alpha|, \]  
\[ (59) \]

\( \mathbf{a} = (q, \mathbf{p}) \), \( \mathbf{a} = (p, -\mathbf{p}) \) are conjugate position and momentum variables in double phase space. Starting again from the double position basis \( \langle Q_a \| S \| Q_a \rangle \) an alternative double Weyl transform in this basis is obtained in exact analogy to equation (39) as

\[ S'(\mathbf{a}, \alpha) = \int d^2 \mathbf{a}' \langle Q_{a'}^{\alpha} \| S \| Q_{a'}^{\alpha} \| \omega^{\mathbf{a}'\mathbf{a}}. \]  
\[ (60) \]

Again a reflection superoperator results as

\[ R'(\mathbf{a}, \alpha) = \int d^2 \mathbf{a}' \langle Q_{a'}^{\alpha} \| S \| Q_{a'}^{\alpha} \| \omega^{-\mathbf{a}'\mathbf{a}}. \]  
\[ (61) \]

where now the reflection is in the position basis \( \langle Q_a \rangle \). In this case the Choi conjugation (35) between the two position bases consists of a simple transposition of indices

\[ \| Q_{a,b} \| \| Q_{c,d} \| = \hat{Q}_{a,c} \cdot \hat{Q}_{b,d}. \]  
\[ (62) \]

Applying this transformation to equation (61)

\[ R'(\mathbf{a}, \alpha) = \int dq_+ dq_- \left\langle q_+ + \frac{q_+}{2} \| q_+ - \frac{q_+}{2} \| q_- - \frac{q_-}{2} \| q_- + \frac{q_-}{2} \| \omega^{p_+ q_- - p_- q_+}. \right\]  
\[ (63) \]

we recognize again the appearance of the reflection superoperator \( \hat{R}_x \cdot \hat{R}_x \). Thus

\[ R'(\mathbf{a}, \alpha) = \hat{R}_x \cdot \hat{R}_x. \]  
\[ (64) \]

where \( (\mathbf{a}, \alpha) = (q, p, p, -p) \) is related to \( (x, x) = (q, p, q, p) \) by a signed permutation matrix \( \mathcal{U}_1 \) that satisfies equation (10). Just as in equation (54) we then obtain again

\[ S'(\mathbf{a}, \alpha) = C_S(x_+, x_-). \]  
\[ (65) \]

Thus a rotation of the arguments of the Choi matrix (by a linear transformation \( \mathcal{U}_1 \) satisfying equation (10)) results in the double Wigner transform of \( S \) in a different basis. Of course, the reversal of the Weyl transform (56) can also be used to retrieve the double position or the double momentum representation:

\[ \langle Q_{a'}^{\alpha} \| S \| Q_{a'}^{\alpha} \| \omega^{-\mathbf{a}'\mathbf{a}}. \]  
\[ (66) \]

Moreover, comparing equation (65) with equation (54) we also obtain

\[ S'(\mathbf{a}, \alpha) = S(x, \xi) = C_S(x_+, x_-). \]  
\[ (67) \]

In short, we obtain that the single function \( C_S(x_+, x_-) \) supports the Weyl transform of \( S \) in two different bases just as would be expected from symplectic invariance for the ordinary Weyl transform in single phase space. We note here that the classical transformation \( (\mathbf{a}, \alpha) \mapsto (x, -J\xi) \) preserves the symplectic form in 4D and therefore belongs to the symplectic group \( \text{Sp}(4) \). An almost identical derivation involving the bases \( \| Q_2 \rangle = |p_1 \rangle |q_1 \rangle \| P_2 \rangle = |q_1 \rangle |p_1 \rangle \) yields \( S'(Q_2, P_2) = C_S(x_+, x_-) \). Clearly an analogous discussion relates the various double chord transforms to the Choi matrix \( \hat{C}_S(x_+, x_-) \) in the translation basis.

Indeed, the immediate generalization of the above discussion would show that the transit between the various operator representations corresponding to different choices of double phase space coordinates (3) among themselves (as well as with the Weyl and the chord
representations) is universally obtained from a single double Weyl representation by a symplectic change of coordinates. To complete the generalization one would need to attempt the explicit construction of metaplectic superoperators in double phase space that would implement the change of basis. We intend to proceed along this path in future publications.

5. Quantum evolutions

Consider the superoperator

\[
\mathbf{U} = \hat{U} \cdot \hat{U}^\dagger,
\]

which propagates unitarily the density matrix as \( \mathbf{U} \rho \mathbf{U}^\dagger \). The matrix elements in the reflection basis are \( R_{\ddot{r} \ddot{r}} \).

This is the kernel that propagates unitarily the Weyl representation of the density matrix \( R_{\ddot{x} \ddot{x}} \). As a particular case of the operators treated in section 3, the Choi matrix of \( \mathbf{U} \) is easily computed in a separable form as

\[
\langle \langle R_{\ddot{x} \ddot{x}} \rangle \mathbf{U} \rangle | \langle \langle R_{\ddot{x} \ddot{x}} \rangle \mathbf{U} \rangle \rangle = \frac{1}{2\pi} \int d^2\xi \mathbf{U} \left( x + \frac{1}{2} \xi \right) \mathbf{U}^* \left( x - \frac{1}{2} \xi \right) \omega^{x_1, \xi^2}.
\]

At first sight, this centre–centre propagator seems to result from a double Weyl transform from the single Weyl propagator, \( \mathbf{U} \ddot{x} \ddot{x} \), but the derivation of equation (56) clarifies its true role as the inverse transform from the double Wigner function, that is the Choi representation of the super evolution operator. In the semiclassical regime the ordinary Weyl propagators have explicit formulae in terms of generating functions [17,32]. Further evaluation by stationary phase leads to semiclassical approximations for this propagator. Very similar formulae for the chord–chord propagator can be derived in the translation basis: \( \langle \langle T_{\ddot{x} \ddot{x}} \rangle \mathbf{U} \rangle | \langle \langle T_{\ddot{x} \ddot{x}} \rangle \mathbf{U} \rangle \rangle \).

A more general evolution is generated by the Kraus superoperator [27],

\[
\mathbf{K} = \sum_j \hat{K}_j \cdot \hat{K}_j^\dagger
\]

which is again easily accommodated in the present framework. Indeed, linearity of the Fourier transforms then specifies the integral kernel for evolving Wigner functions as

\[
\langle \langle R_{\ddot{x} \ddot{x}} \rangle \mathbf{K} \rangle | \langle \langle R_{\ddot{x} \ddot{x}} \rangle \mathbf{K} \rangle \rangle = \frac{1}{2\pi} \sum_j \int d^2\xi \hat{K}_j \left( x + \frac{1}{2} \xi \right) \hat{K}_j^* \left( x - \frac{1}{2} \xi \right) \omega^{x_1, \xi^2}.
\]

An alternative generalization from ordinary unitary evolution, which is included in the general formulæ of the previous section, is the superoperator

\[
\mathbf{U} = \hat{U}_1 \cdot \hat{U}_2^\dagger.
\]

\(^6\) One should note that this product of simple Weyl propagators in the integrand was mistakenly identified in [31] with the mixed center-chord propagator, defined in [30].
This determines the evolving kernel for the quantum fidelity, or the quantum Loschmidt echo [33–35], that is, the overlap of two different evolutions for the same initial state. Unlike the previous examples, the trace of the evolving operator is not preserved and the identity operator is not invariant.

The super-reflection, \( R_{\mathbf{x}, \xi} \), in equation (46) is precisely of this form (within factors of two), choosing \( \hat{U}_1 = \hat{R}_{\mathbf{x}, \frac{1}{2} \xi} \) and \( \hat{U}_2 = \hat{R}_{\mathbf{x}, \frac{1}{2} \xi} \). The same goes for the super-translation, \( T_{\mathbf{x}, \xi} \), defined in (51). As revealed by equations (42) and equation (48), these superoperators can be viewed as active agents, rather than mere passive Choi bases. For instance, the general rules for products of reflections and translations, which are reviewed in the appendix, determine the action of a super-translation, \( \hat{T}_{\xi} \cdot \hat{T}_{\xi} \), acting on the translation operator \( \hat{T}_{\xi} \), as just \( \omega^{\xi_1} \hat{T}_{-\xi} \), with \( \xi_1 = \xi_2 + \xi_3 + \xi_4 \) as depicted in figure 2. Likewise, the action of the superoperator \( \hat{R}_{\mathbf{x}_1} \cdot \hat{R}_{\mathbf{x}_2} \) on the reflection \( \hat{R}_{\mathbf{x}_1} \) is simply \( \omega^{\xi_1} \hat{R}_{\mathbf{x}_2} \), with \( \mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_3 - \mathbf{x}_1 \). Note that the four reflection centers \( \mathbf{x}_1 \) form a parallelogram with half the symplectic area of the circumscribed quadrilateral, \( \Delta_4 \), formed by the translation chords, \( \xi_n \), so one obtains the same phase for the composition of three translations or three reflections. Indeed, this is also the same phase as determines the traces of four translations or four reflections that are discussed in the appendix.

The extent to which equations (46) and (51) are natural definitions for the reflection and translation superoperators is illuminated by considering the graphs of classical translations and reflections as planes in double phase space. First one notes that each operator, \( \hat{T}_{\xi} \), corresponds to the translation, \( \mathbf{x}_+ \rightarrow \mathbf{x}_+ = \mathbf{x}_+ + \xi \); the locus of all pairs \( (\mathbf{x}_+, \mathbf{x}_-) \) in a given initial plane in double phase space. Then the pair of translations, \( \mathbf{x}_+ \rightarrow \mathbf{x}_+ = \mathbf{x}_+ + \xi \) and \( \mathbf{x}_- \rightarrow \mathbf{x}_- = \mathbf{x}_- - \xi \), that is, the double phase space translation \( X \rightarrow X' \), takes all double phase space points in the original plane to the new plane \( \mathbf{x}_+ = \mathbf{x}_- - \xi \). Thus, the new and the old planes representing translations are themselves related by just the double phase space translation corresponding to \( \hat{T}_{\xi} \cdot \hat{T}_{\xi} \), that is, \( \hat{T}_{\xi} \cdot \hat{T}_{\xi}(T_{\xi}) = \omega^{\xi_1} \hat{T}_{-\xi} \). In the same way, one also finds that the combination of phase space reflections within each of the phase spaces \( \mathbf{x}_\pm \), corresponding to \( \hat{R}_{\mathbf{x}_1} \cdot \hat{R}_{\mathbf{x}_2} \), transports the double phase space points \( X = (\mathbf{x}_+ = \mathbf{x}_- - \frac{1}{2} \xi, \mathbf{x}_+ = \mathbf{x}_- + \frac{1}{2} \xi) \) in an initial reflection, to a new reflection, \( \mathbf{x}_+ + \frac{1}{2} \xi \rightarrow \mathbf{x}_+ = \frac{1}{2} \xi \).

### 6. Phase space correlations and general pure state conditions

Consider a general quantum state represented by its density matrix \( \hat{\rho} \). Its phase space translation correspondence to the operator

\[
\hat{\rho}_x = \hat{T}_x \hat{\rho} \hat{T}_x^\dagger,
\]

and the (non-normalized) phase space correlation function [9, 37] is naturally defined as

\[
C_\mathbf{x} \equiv \text{tr} \, \hat{\rho} \hat{\rho}_x = \text{tr} \left( \hat{\rho} \hat{T}_x \hat{T}_x^\dagger \right).
\]

Likewise, from the reflected state \( \hat{\rho}^\mathbf{x} = \hat{R}_x \hat{\rho} \hat{R}_x \), an analogous anticorrelation function can be defined as

\[
C^\mathbf{x} \equiv \text{tr} \, \hat{\rho} \hat{\rho}^\mathbf{x} = \text{tr} \left( \hat{\rho} \hat{R}_x \hat{R}_x \right).
\]

To provide a link with our previous results it is then convenient to define the density superoperator,
in terms of which one can identify
\[ C_{T} = \langle T_{x} \parallel P \parallel T_{x} \rangle \quad \text{and} \quad C^{x} = \langle R_{x} \parallel P \parallel R_{x} \rangle \]
as the diagonal matrix elements of \( P \) in the translation and reflection bases. Utilizing the cyclic invariance of the trace in equations (75) and (76), we can alternatively rewrite
\[ C_{x} = \langle \rho \parallel T_{x} \cdot T_{x}^{\dagger} \parallel \rho \rangle \quad \text{and} \quad C^{x} = \langle \rho \parallel R_{x} \cdot R_{x} \parallel \rho \rangle. \]The Choi-conjugate relations (50) and (45) lead to
\[ C_{x} = \frac{1}{2\pi\hbar} \int d^{2}\xi \langle \rho \parallel T_{\xi} \parallel T_{\xi} \rangle \quad \text{(80)} \]
\[ C^{x} = \frac{1}{2\pi\hbar} \int d^{2}\xi \langle \rho \parallel R_{\xi+\frac{1}{2}\nu} \parallel R_{\xi-\frac{1}{2}\nu} \rangle \quad \text{(81)} \]
that is, both correlations can then be computed directly from the chord and the (ordinary) Wigner function and chord function (28).
In the case of pure states, \( \hat{\rho} = |\psi \rangle \langle \psi | \), the correlations can be computed directly:
\[ C_{x} = |\langle T_{x} \parallel \rho \rangle|^{2} = (2\pi\hbar)^{2}|\chi(x)|^{2} \quad \text{and} \quad C^{x} = |\langle R_{x} \parallel \rho \rangle|^{2} = (2\pi\hbar)^{2}W(x)^{2} \]and then comparison with equations (80) and (81) leads to the identities [36, 37],
\[ |\chi(x)|^{2} = \frac{1}{2\pi\hbar} \int d^{2}y \omega_{<x,y>}|\chi(y)|^{2}, \]and
\[ W(x)^{2} = \int \frac{d^{2}y}{2\pi\hbar} W\left(x + \frac{y}{2}\right) W\left(x - \frac{y}{2}\right). \]
Further relationships between products of center and chord functions for pure states are obtained from the off-diagonal matrix elements of $P$, such as the matrix element

$$
\langle T_{x+\frac{i}{2}\xi} \mid P \mid T_{x+\frac{i}{2}\xi} \rangle = \langle \rho \mid T_{x+\frac{i}{2}\xi} \star T_{x+\frac{i}{2}\xi}^{-1} \mid \rho \rangle
$$

$$
= \frac{1}{2\pi\hbar} \int d^2\xi \langle \rho \mid T_{\xi+\frac{i}{2}\xi} \rangle \langle T_{\xi+\frac{i}{2}\xi} \mid \rho \rangle \omega^{\xi,\star \xi}. \tag{85}
$$

Now we introduce the remarkable property of pure states that the superoperator $P$ is self-Choi-conjugate, i.e. $\hat{\rho} \star \hat{\rho} = \| \rho \| \langle \rho \|$, as can be easily checked by acting on any operator. It is then possible to pull back the Choi-conjugation relations as unsuspected properties of ordinary pure state Wigner or chord functions in single phase space. Indeed the l.h.s. of equation (85) is then reduced to $\langle \langle T_{x+\frac{i}{2}\xi} \| \rho \| \rangle \langle \rho \| T_{x+\frac{i}{2}\xi} \rangle \rangle$ so that

$$
\chi \left( x + \frac{1}{2} \xi \right) \chi \left( x - \frac{1}{2} \xi \right)^* = \frac{1}{2\pi\hbar} \int d^2\xi_i \chi \left( \xi_1 + \frac{1}{2} \xi \right)^* \left( \xi_1 - \frac{1}{2} \xi \right) \omega^{\xi,\star \xi}. \tag{86}
$$

is identified as a general requirement for chord functions of pure states. The previous case of equation (83) is included as $\xi = 0$. The off-diagonal elements in the reflection basis yield a similar property for Wigner functions,

$$
W \left( x + \frac{1}{2} \xi \right) W \left( x - \frac{1}{2} \xi \right) = \frac{1}{2\pi\hbar} \int d^2\xi \chi \left( \xi + \frac{1}{2} \xi \right) \chi^* \left( \xi - \frac{1}{2} \xi \right) \omega^{\xi,\star \xi}. \tag{87}
$$

and a mixed case relating Wigner and chord functions is

$$
W \left( x + \frac{1}{2} \xi \right) W \left( x - \frac{1}{2} \xi \right)^* = \frac{1}{2\pi\hbar} \int d^2\xi \chi \left( \xi + \frac{1}{2} \xi \right)^* \chi \left( \xi - \frac{1}{2} \xi \right) \omega^{\xi,\star \xi}. \tag{88}
$$

These formulae establish families of Fourier identities parametrized continuously by $x$ or $\xi$. They show e.g. that the product of symmetrically displaced Wigner functions, $W \left( x + \frac{1}{2} \xi \right) W \left( x - \frac{1}{2} \xi \right)$, is its own Fourier transform for all $x$ and $\chi \left( x - \frac{1}{2} \xi \right)^* \chi \left( x + \frac{1}{2} \xi \right)$ is likewise invariant for all values of $\xi$. Many particular cases give rise to interesting properties. Besides the ones already noted for the correlations, we list the following special cases

$$
\chi \left( \frac{1}{2} \xi \right)^2 = \frac{1}{2\pi\hbar} \int d^2\xi \chi \left( \xi_1 + \frac{1}{2} \xi \right) \chi^* \left( \xi_1 - \frac{1}{2} \xi \right) \tag{89}
$$

$$
W \left( \frac{1}{2} \xi \right) W \left( -\frac{1}{2} \xi \right) = \frac{1}{2\pi\hbar} \int d^2\xi \chi \left( \frac{1}{2} \xi \right) W \left( -\frac{1}{2} \xi \right) \omega^{\xi,\star \xi}. \tag{90}
$$

$$
W \left( \frac{1}{2} \xi \right) W \left( -\frac{1}{2} \xi \right) = \frac{1}{2\pi\hbar} \int d^2\xi \chi \left( \xi + \frac{1}{2} \xi \right) \chi \left( \xi - \frac{1}{2} \xi \right) \omega^{\xi,\star \xi}. \tag{91}
$$

$$
W \left( x \right)^2 = \frac{1}{2\pi\hbar} \int d^2\xi \chi \left( \frac{1}{2} \xi \right) \chi^* \left( -\frac{1}{2} \xi \right) \omega^{\xi,\star \xi}. \tag{92}
$$

The last one, taking into account that $\chi \left( x \right) = \chi^* \left( -x \right)$ and Parseval’s relation, yields the further integral
Furthermore, for $x = \xi = 0$, one obtains:

$$W(0)^2 = \frac{1}{2\pi\hbar} \int d^2x W\left(\frac{1}{2}x\right) W\left(-\frac{1}{2}x\right) = \frac{1}{2\pi\hbar} \int d^2\xi_1 \chi\left(\frac{1}{2}\xi_1\right) \chi^*\left(-\frac{1}{2}\xi_1\right).$$  

The way the general formulae (86)–(88) were derived implies that they are necessary conditions for pure state distributions. By setting $x = \frac{1}{2}\xi$, it is easily shown that equation (86) is equivalent to the pure state condition $\hat{\rho} = \hat{\rho}^2$ in the chord representation, and so it is also sufficient. The case of the Wigner function is not so transparent, but the integral product rule for the Weyl representation [17] for $\hat{\rho}^2$,

$$\rho^2(x) = 4 \int d^2x_1 d^2x_2 \ W(x_1) \ W(x_2) \ \omega^{2<(x_1-x_2)(x_1-x_2)>}$$

$$= 4 \int d^2\xi \int d^2\xi' \ W\left(x + \frac{1}{2}\xi\right) W\left(x - \frac{1}{2}\xi\right) \omega^{2<(x-x')(\xi-\xi')},$$  

is immediately simplified by equation (87) so that

$$\rho^2(x) = 8\pi\hbar \ W(x) \int d^2\xi W\left(2\xi - x\right) = 2\pi\hbar \ W(x) = \rho(x).$$

6.1. Airy functions: an example

There are some notable cases where the Wigner function of pure states can be described in terms of standard special functions found in e.g. [38]. Then the identity,

$$W\left(x + \frac{Y}{2}\right) W\left(x - \frac{Y}{2}\right) = \int \frac{dx'}{(2\pi\hbar)} W\left(x + \frac{x'}{2}\right) W\left(x - \frac{x'}{2}\right) \omega^{<x'Y>},$$  

implies a possibly unsuspected Fourier identity for a symmetrized product of analytic functions. Such is the case for the eigenfunctions of the harmonic oscillator, given in terms of Laguerre polynomials [19], or the unnormalized eigenfunctions of the hyperbolic hamiltonian $H(x) = p \cdot q$, calculated in [39] in terms of Laguerre functions of complex index. We develop here the important example of the linear potential $V(q) = q$. If $m = 1/2$ and $\hbar = 1$, so that the hamiltonian is simply $H(x) = p^2 + q$, the zero energy eigenfunction is proportional to the Airy function,

$$Ai(q) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \exp\left[i\left(\frac{p^3}{3} + pq\right)\right],$$  

which is not normalizable and hence has no Fourier transform. Evidently, the corresponding momentum representation of this state is just

$$\langle p|\psi\rangle = \frac{1}{\sqrt{2\pi}} \exp\left[i\left(\frac{p^3}{3}\right)\right].$$  

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so that the corresponding Wigner function is just \[ W(x) = \frac{1}{2\pi} \int dp' \left\langle p + \frac{p'}{2} \right| \left[ \psi(p - \frac{p'}{2}) \right] \exp\left[ ip'q \right] \]

\[
= \frac{1}{(2\pi)^2} \int dp' \exp\left[ i \left( \frac{p'^3}{12} + (p'^2 + q)p' \right) \right] 
= \sqrt{\frac{2^{1/3}}{\pi}} \text{Ai}\left(\frac{2^{1/3}}{\pi}H(x)\right),
\] (100)

which is also not square integrable.

Nonetheless, the product \( W\left(x + \frac{x'}{2}\right) W\left(x - \frac{x'}{2}\right) \) decays exponentially in the \( x' \)-phase plane outside the region limited by the pair of reflected parabolae, \( H\left(x \pm \frac{x'}{2}\right) = 0 \), so that it is square integrable. Thus the direct verification of the Fourier identity (87) in the case of the Airy function proceeds from the integral representation:

\[
W\left(x_1 + \frac{x_2}{2}\right)W\left(x_1 - \frac{x_2}{2}\right) = \frac{1}{(2\pi)^2} \int dp' dp'' \exp\left[ i \left( \frac{p'^3}{12} + \left( \left( p_1 + \frac{p_2}{2} \right)^2 + q_1 + \frac{q_2}{2} \right)p' \right) \right] 
\times \exp\left[ -i \left( \frac{p''^3}{12} - \left( \left( p_1 - \frac{p_2}{2} \right)^2 + q_1 - \frac{q_2}{2} \right)p'' \right) \right] 
\] (101)

Then the transformation \( p' = a + \frac{b}{2}, p'' = a - \frac{b}{2} \) simplifies this into

\[
W\left(x_1 + \frac{x_2}{2}\right)W\left(x_1 - \frac{x_2}{2}\right) = \frac{1}{(2\pi)^4} \int da \int db \exp\left[ i \left( \frac{b^3}{48} + \frac{a^2b}{4} + (q_2 + 2p_1p_2)a + \left( p_1^2 + \frac{p_2^2}{4} + q_1 \right)b \right) \right].
\] (102)

Thus, the Fourier transform with respect to \( x_2 \) becomes

\[
\frac{1}{2\pi} \int dx_2 \text{e}^{i<x_1,x_2>} W\left(x_1 + \frac{x_2}{2}\right)W\left(x_1 - \frac{x_2}{2}\right) = \frac{1}{(2\pi)^3} \int da \int db dq_2 dp_2 \exp\left[ i \left( \frac{b^3}{48} + \frac{a^2b}{4} + (q_2 + 2p_1p_2)a + \left( p_1^2 + \frac{p_2^2}{4} + q_1 \right)b + 2p_1q_2 - q_2p_1 \right) \right] 
= \frac{1}{(2\pi)^2} \int db \int dp_2 \exp\left[ i \left( \frac{b^3}{48} + \frac{a^2b}{4} + (q_2 + 2p_1p_2)a + \left( p_1^2 + \frac{p_2^2}{4} + q_1 \right)b + p_2q_3 \right) \right] 
= W\left(x_1 + \frac{x_2}{2}\right)W\left(x_1 - \frac{x_2}{2}\right).
\] (103)
which implies the Fourier invariance for a product of symmetrized Airy functions,

$$\frac{1}{2\pi} \int dx_2 \, e^{i < x_2, x_1 >} \, \text{Ai}\left(2^{2/3}H\left(x_1 + \frac{x_2}{2}\right)\right) \, \text{Ai}\left(2^{2/3}H\left(x_1 - \frac{x_2}{2}\right)\right)$$

which is not encountered even in [40], a book dedicated specifically to Airy functions, or [41]. In figure 3 we show this product in the plane $x_3$ for $x_1 = (-3, 0)$. Notice the central symmetry implied by the Fourier invariance.

An important feature of this example is that the semiclassical transitional approximation of pure state Wigner functions for general WKB-quantized states near the closed energy eigencurve was shown by Berry [42] to be just the Airy function over the approximating parabola. Thus one finds that both the exact Wigner function and its transitional approximation satisfy the new Fourier invariance, even though this is not the case for other semiclassical approximations that are more refined in other respects.

7. Conclusions and outlook

Two possible representations of a superoperator can be naturally derived from the same operator basis. They are unitarily related, and we have referred to them as Choi conjugates. We have developed here the general relationship between them—when the basis is orthogonal—and we have studied in particular the form of this relationship when the unitary bases of translations and reflections are used. It turns out that the representation in terms of the Choi or dynamical matrix $C_S$ can be interpreted as a double Weyl or Wigner transform of the matrix elements of the superoperator. This is because the representation in terms of the Choi matrix is actually an expansion in terms of translation and reflection superoperators, in strict analogy to the expansion of an ordinary operator in terms of translations and reflections, yielding its Weyl or Wigner transform. The definition of $T_{\kappa, \xi}$ and $R_{\kappa, \xi}$ opens up the possibility for a full

![Figure 3. The product of two Airy functions as in equation (104) as a function of $x_3$ for fixed $x_1 = (-3, 0)$. In the color density plot blue is positive and yellow negative.](image-url)
A study of the affine geometry of these superoperators in double phase space, including the definition of unitary superoperators that implement symplectic transformations belonging to \( \text{Sp}(4) \). We intend to pursue this analysis in the future.

Our treatment here has been for the simplest case of a phase space with no boundaries and with one degree of freedom. The extension to \( D \) degrees is immediate and needs no further comment. The adaptation of our techniques to a phase space with boundaries needs more care. The case of torus topology, i.e. periodic boundary conditions both in position and momentum, is the closest to the present approach and leads to a finite dimensional Hilbert space of integer dimension \( d = \text{area}/2\pi\hbar \) [43-46] and is of great current interest in quantum information theory. Translation and reflection operators can still be defined [45] and provide a basis for a similar treatment as the one developed here. In this context, the display in double phase space of the properties of superoperators can provide new insights into their actions, just as the celebrated Wigner and Weyl representation displayed properties of quantum states in single phase space. The action of Gaussian noise channels in the chord representation [47] is a first step in that direction. Another avenue of generalization would be to incorporate frames instead of orthogonal bases. The theory of frames is well developed in the context of quantum information [49, 50] and in the present context would allow consideration of coherent state bases, which are non-orthogonal and overcomplete, and would require the consideration of complex double phase space. This remains to be done.

In the application of these methods we have found some previously unknown identities relating products of Wigner and Weyl distributions for pure states. These identities generalize the pure state conditions and in some cases produce new relationships for the special functions of analysis.

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Appendix

We review the well-known definitions and properties of reflection and translation operators. They constitute the foundation for the Weyl representation of quantum mechanical operators as phase space c-number functions (the Wigner quasiprobability distribution in the case density matrices). We start with the usual \( \hat{q}, \hat{p} \) operators that we subsume in a phase space operator \( \hat{x} = (\hat{q}, \hat{p}) \) and a phase space label \( \mathbf{x} = (q, p) \in \mathbb{R}^2 \). The corresponding position and momentum bases are denoted \( |q\rangle, |p\rangle \). The symplectic product is defined as

\[
\langle \mathbf{x}, \mathbf{x}' \rangle = (q, p) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix} = pq'-qp'.
\]  

(A.1)

For notational simplicity we also introduce the quantities

\[
\tau = e^{i/(2\hbar)}, \quad \omega = e^{i/\hbar}.
\]  

(A.2)
Reflections and translation operators are defined as
\[
\hat{R}_q \equiv \hat{R}_{q,p} = \int dq' \left| q + \frac{q'}{2} \right\rangle \left\langle q' - \frac{q}{2} \right| \omega^{pq'}
\]
(A.3)
\[
\hat{T}_\xi \equiv \hat{T}_{\xi,p} = \int dp' \left| p + \frac{\xi_p}{2} \right\rangle \left\langle p' - \frac{\xi_{p'}}{2} \right| \omega^{-\xi_{p'}}
\]
(A.4)
where the first is in the position and the latter in the momentum basis. They have the properties
\[
\hat{T}_\xi^\dagger = \hat{T}_{-\xi}, \quad \hat{R}_q^\dagger = \hat{R}_{-q}, \quad \left( \frac{1}{2} \hat{R}_x \right)^2 = 1.
\]
(A.5)
Their action on the position and momentum basis justifies their names
\[
\hat{R}_x \left| q_0 \right\rangle = 2 \left| 2q - q_0 \right\rangle \omega^{2(q-q_0)p} \quad R_x \left| p_0 \right\rangle = 2 \left| 2p - p_0 \right\rangle \omega^{-2q(p-p_0)}
\]
(A.6)
and
\[
\hat{T}_\xi \left| p_0 \right\rangle = \left| p_0 + \xi_p \right\rangle \omega^{-\xi_p(p_0+\xi_p)} \quad \hat{T}_\xi \left| q_0 \right\rangle = \left| q_0 + \xi_q \right\rangle \omega^{\xi_q(q_0+\xi_q)}.
\]
(A.7)
Moreover they form a group which is the representation of the affine group of reflections and translations, with the following composition laws
\[
\hat{T}_\xi \hat{T}_\zeta = \tau^{<\xi,\zeta>} \hat{T}_{\xi+\zeta} \quad \hat{R}_x \hat{R}_y = 4\omega^{2<\xi_x,\zeta_x>} \hat{T}_{\xi_x+\zeta_x} \quad \hat{R}_x \hat{T}_\xi = \omega^{-<\xi_x,\epsilon_x>} \hat{R}_{x-\xi/2} \quad \hat{T}_\xi \hat{R}_y = \omega^{-.<\xi_y,\epsilon_y>} \hat{R}_{x+\xi/2}.
\]
(A.8)
(A.9)
They constitute a pair of complementary orthonormal bases with the properties
\[
\text{tr} \hat{T}_\xi^\dagger \hat{T}_\beta = 2\pi \hbar \delta (\xi - \beta), \quad \text{tr} \hat{R}_x \hat{R}_y = 2\pi \hbar \delta (x - y), \quad \text{tr} \hat{R}_x \hat{T}_\xi = \omega^{-<\xi_x,\epsilon_x>},
\]
(A.10)
Switching to the double Dirac notation we rewrite the above as
\[
\langle \hat{T}_\xi | \hat{T}_\beta \rangle = 2\pi \hbar \delta (\xi - \beta),
\]
\[
\langle \hat{R}_x | \hat{R}_y \rangle = 2\pi \hbar \delta (x - y),
\]
\[
\langle \hat{R}_x | \hat{T}_\xi \rangle = \omega^{-<\xi_x,\epsilon_x>},
\]
(A.11)
The labels \( x \) and \( \xi \) are related to the conjugate variables \( Q, P \) of equation (17) as \( x = Q \) and \( \xi = J P \). Thus we can think of reflection and translation operators as alternative position and momentum bases in double phase space. Using these properties we compute the quadruple traces needed in the main text:
\[
\text{tr} \left( \hat{T}_\xi, \hat{T}_{\xi'}, \hat{T}_{\xi''}, \hat{T}_{\xi'''} \right) = 2\pi \hbar \delta (\xi + \xi' + \xi'' + \xi''') \tau^{<\xi,\xi'>+<\xi',\xi''>}
\]
(A.12)
\[
\text{tr} \left( \hat{R}_x, \hat{R}_y, \hat{R}_z, \hat{R}_y \right) = 2\pi \hbar \delta \left( \frac{x_1 + x_3 + x_4}{2} - \frac{x_2 + x_4}{2} \right) \omega^{2<\xi_1,\xi_3>+2<\xi_0,\xi_3>}
\]
(A.13)
We should remark at this point that traces of unitary operators are related semiclassically to classical periodic orbits and their actions \([2, 3]\), even when the classical and quantum evolutions are broken up into several steps \([48]\). In the case of four operators \( \hat{T}_\xi \) such a trajectory is composed of four segments, each giving a phase space translation and so as to form a closed quadrilateral. The action of this trajectory is the symplectic area of the quadrilateral \( \Delta_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{1}{2}(\langle \xi_1, \xi_2 \rangle + \langle \xi_3, \xi_4 \rangle) \) (in units of \( \hbar \)). In the case of
reflections the trajectory connects the centers of the segments of this quadrilateral, which is a parallelogram $x_1 - x_2 + x_3 - x_4 = 0$. Both cases are illustrated in figure 2.

References

[1] Abraham R and Marsden J 1978 Foundations of Mechanics (Reading, MA: Benjamin)
[2] Gutzwiller M 1990 Chaos in Classical and Quantum Mechanics (New York: Springer)
[3] Ozorio de Almeida A M 1988 Hamiltonian Systems: Chaos and Quantization (Cambridge: Cambridge University Press)
[4] Arnold V I 1978 Mathematical Methods of Classical Mechanics (Berlin: Springer)
[5] Jamiolkowsky A 1972 Rep. Math. Phys. 3 275
[6] Min J, Shunlong L and Shuanghuang F 2013 Phys. Rev. A 87 022310
[7] Amiet J P and Huguenin P 1981 Mécaniques classique et quantique dans l’espace de phase (Neuchâtel: Universite de Neuchâtel)
[8] Littlejohn R G 1986 Phys. Rep. 138 193–291
[9] Littlejohn R G 1990 J. Math. Phys. 31 7
[10] Weinstein A 1972 Invent. Math. 16 202
[11] Miller W H 1974 Classical limit quantum mechanics and the theory of molecular collisions Adv. Chem. Phys. 25 69
[12] Feynman R P 1948 Rev. Mod. Phys. 20 367
[13] Schulman L S 1981 Techniques and Applications of Path Integration (New York: Wiley)
[14] Goldstein H 1980 Classical Mechanics 2nd edn (Reading, MA: Addison-Wesley)
[15] Synge J L 1960 in Encyclopedia of Physics vol 3 (Berlin: Springer) p157
[16] Jose J V and Saletan E J 1998 Classical Dynamics: A Contemporary Approach (Cambridge: Cambridge University Press)
[17] Ozorio de Almeida A M 1998 Phys. Rep. 295 265
[18] Wigner E P 1932 Phys. Rev. 40 749
[19] Groenewold H J 1946 Physica 12 405
[20] Moyal J E 1949 Proc. Camb. Phil. Soc. 45 99124
[21] Balazs N L and Jennings B K 1984 Phys. Rep. 104 347
[22] Choi M D 1984 Phys. Rep. 104 285
[23] Bengtsson I and Zyczkowski K 2006 Geometry of Quantum States (Cambridge: Cambridge University Press)
[24] Sudarshan E C G, Mathews P M and Rau J 1961 Phys. Rev. 121 920
[25] Nielsen M A and Chuang I L 2003 Quantum Computation and Quantum Information (Cambridge University Press)
[26] Rommer S and Östlund S 1997 Phys. Rev. B 55 2164
[27] Kraus K 1983 States, Effects and Operations (Lecture Notes in Physics vol 190) (Berlin: Springer)
[28] de P, Rios M and Ozorio de Almeida A M 2002 J. Phys. A 35 2609
[29] Dittrich T, Viviescas C and Sandoval L 2006 Phys. Rev. Lett. 96 070403
[30] Ozorio de Almeida A M and Brodier O 2006 Ann. Phys. 321 1790
[31] Ozorio de Almeida A M and Brodier O 2011 Phil. Trans. R. Soc. A 369 260
[32] Berry M V 1989 Proc. R. Soc. Lond. A 423 219–31
[33] Gorin T, Prosen T, Seligman T H and Znidaric M 2006 Phys. Rep. 435 33
[34] Zambrano E and Ozorio de Almeida A M 2011 Phys. Rev. E 84 045201 R
[35] Ozorio de Almeida A M, Vallejos R O and Zambrano E 2013 J. Phys. A 46 135304
[36] Chountasis S and Vorudas A 1998 Phys. Rev. A 58 848–55
[37] Ozorio de Almeida A M, Vallejos R O and Saraceno M 2005 J. Phys. A: Math. Gen. 38 1473–90
[38] Abramowitz M and Stegun I A 1964 Handbook of Mathematical Functions (Washington, D.C.: National Bureau of Standards)
[39] Balazs N L and Voros A 1990 Ann. Phys. 199 123–40
[40] Vallée O and Soares M Airy Functions and Applications to Physics (London: Imperial College Press)

22
[41] Abramochkin E and Razueva E 2011 Opt. Lett. 36 3732
see also Razueva E 2015 thesis University of Samara
[42] Berry M V 1977 Phil. Trans. R. Soc. Lond 287 237
[43] Leonhardt U 1996 Phys. Rev. A 53 2998
[44] Wootters W K 1987 Ann. Phys. 176 1
[45] Rivas F and Ozorio de Almeida A M 1999 Ann. Phys. 276 223
[46] Miquel C, Paz J P and Saraceno M Phys. Rev. A 65 62309
[47] Aolita M L, Garcia-Mata I and Saraceno M 2004 Phys. Rev. A 70 62301
[48] Ozorio de Almeida A M and Brodier O 2015 arXiv:1507.04707v1
[49] Ferrie C and Emerson J 2008 J. Phys. A: Math. Theor. 41 352001
[50] Ferrie C and Emerson J 2009 New J. Phys. 11 063040