Pre-Plactic Algebra and Snakes

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Abstract

We study a factor Hopf algebra $\mathcal{PP}$ of the Malvenuto-Reutenauer convolution algebra of functions on symmetric groups $\mathbb{S} = \bigoplus_{n \geq 0} \mathbb{C}[S_n]$ that we coined pre-plactic algebra. The pre-plactic algebra admits the Poirier-Reutenauer algebra based on Standard Young Tableaux as a factor and it is closely related to the quantum pseudo-plactic algebra introduced by Krob and Thibon in the non-commutative character theory of quantum group comodules. The connection between the quantum pseudo-plactic algebra and the pre-plactic algebra is similar to the connection between the Lascoux-Schützenberger plactic algebra and the Poirier-Reutenauer algebra. We show that the dimensions of the pre-plactic algebra are given by the numbers of alternating permutations (coined snakes after V.I. Arnold). Pre-plactic algebra is instrumental in calculating the Hilbert-Poincaré series of the quantum pseudo-plactic algebra.

1 Quantum Pseudo-Plactic Algebra

The diagonal matrix elements of the coordinate ring $\mathbb{C}[GL_q(V)]$ of a quantum group close a subalgebra which is the noncommutative reincarnation of the algebra of the functions on the torus. Krob and Thibon conjectured that the diagonal subalgebra in $\mathbb{C}[GL_q(V)]$ is a cubic algebra that will be referred to as the quantum pseudo-plactic algebra.

DEFINITION 1 The quantum pseudo-plactic algebra $\mathcal{PP}_q(W)$ of the vector space $V$ with ordered basis, $W = \bigoplus_{i=1}^n \mathbb{C}(q)a_i$, is the quotient of the free associative algebra $T(W)$ of $W$ by the two-sided ideal $\mathcal{I}$ generated by the cubic relations

\begin{align*}
[[a, c], b] &= 0 \quad a < b < c \\
[[a, b], a]_{q^2} &= 0 \quad a < b \\
[b, [a, b]]_{q^2} &= 0 \quad a < b
\end{align*}

(1)

which we are referring to as quantum pseudo-Knuth relations. Here $[x, y]_q$ stands for the deformed commutator $[x, y]_q := xy - qyx$ and $[x, y] = [x, y]_1$. Let $\mathcal{I}_q$ be the two-sided ideal generated by the quantum pseudo-Knuth relations then one has

$$\mathcal{PP}_q(W) = T(W)/\mathcal{I}_q(W).$$
Remark. The algebra $\mathcal{P}(W)$ has as a factor the coordinate ring of the quantum torus, $xy = q^2 xy$ for $x < y$.

The specialization at $q = 1$ of the quantum pseudo-plactic algebra relations

\begin{align}
[[a, c], b] &= 0 \quad a < b < c \\
[[a, b], a] &= 0 \quad a < b \\
[b, [a, b]] &= 0 \quad a < b
\end{align}

involve only commutators thus $\mathcal{P}(W) = T(W)/\mathcal{J}(W)$ is an universal enveloping algebra of a Lie algebra. The specialization $\mathcal{P}(W)$ will be referred to as pseudo-plactic algebra and denoted by $\mathcal{P}(W) = T(W)/\mathcal{J}(W)$.

Remark. The specialization at $q = 1$ of the diagonal subalgebra $\mathbb{C}[GL_q(V)]$ is a commutative algebra which is different from the cubic pseudo-plactic algebra $\mathcal{P}(W)$.

The quantum Schur-Weyl duality is the double centralizing property on $\text{End}(T(V))$ of the action of the Hecke algebra $\mathcal{H}(q) = \bigoplus_{r \geq 0} \mathcal{H}_r(q)$ and the (co)action of the quantum group $\mathbb{C}[GL_q(V)]$. A Schur functor is mapping a $\mathcal{H}(q)$-module to a $\mathbb{C}[GL_q(V)]$-comodule (or equivalently to a $U_qB$($V$)-module). The quantum pseudo-Knuth ideal $\mathcal{J}_q(W)$ inherits $\mathbb{C}[GL_q(V)]$-comodule structure upon restriction from the diagonal subalgebra of $\mathbb{C}[GL_q(V)]$. Thus the quantum pseudo-plactic algebra $\mathcal{P}(q)(W)$ allow for a description \cite{14} based on Schur functors

$$\mathcal{P}(q)(W) = \bigoplus_{n \geq 0} \mathcal{P}(q)(W)_n = \bigoplus_{n \geq 0} \mathcal{P}(q)(n) \otimes \mathcal{H}_n(q) W^\otimes n$$

where $W$ is the diagonal in $V \otimes V^*$, i.e., the elements invariant under transposition. Let $\mathcal{H}(q)$-module $\mathcal{J}_q = \bigoplus_{r \geq 0} \mathcal{J}_q(r)$ be the Schur functor pre-image of the quantum pseudo-Knuth ideal $\mathcal{J}_q(W)$. Then the the collection $\{\mathcal{P}(q)(r)\}_{r \geq 0}$ of $\mathcal{H}_r(q)$-modules determines a $\mathcal{H}(q)$-module

$$\mathcal{P}(q) = \bigoplus_{r \geq 0} \mathcal{P}(q)(r) = \bigoplus_{r \geq 0} \mathcal{H}_r(q)/\mathcal{J}_q(r)$$

which will be referred to as quantum pre-plactic algebra.

Similarly we define the pre-plactic algebra $\mathcal{P}$ to be a collection $\{\mathcal{P}(r)\}_{r \geq 0}$ of $\mathfrak{S}_r$-modules (see Definition \cite{14}). We are going to endow the pre-plactic algebra $\mathcal{P}$ with a structure of a Hopf algebra induced from the Malvenuto-Reutenauer Hopf structure on $\mathfrak{S} = \bigoplus_{n \geq 0} \mathbb{Q}\mathfrak{S}_n$ which we now briefly revise.

## 2 Malvenuto-Reutenauer Hopf Algebra

Malvenuto and Reutenauer \cite{11} have considered a Hopf algebra structure on $\mathfrak{S}$ related to the Solomon descent algebra $\mathfrak{S}_n$. There is a dual Hopf structure on $\mathfrak{S}$ known also as the algebra of free quasi-symmetric functions $\mathfrak{S}$.

Given two permutations $\alpha \in \mathfrak{S}_r$ and $\beta \in \mathfrak{S}_p$, the product $\star$ on $\mathfrak{S} = \bigcup_{n \geq 0} \mathbb{Q}[\mathfrak{S}_n]$ is induced (by linearity) from

$$\star : \mathbb{Q}[\mathfrak{S}_r] \times \mathbb{Q}[\mathfrak{S}_p] \rightarrow \mathbb{Q}[\mathfrak{S}_{r+p}]_{s_{\mathfrak{S}_r} \times s_{\mathfrak{S}_p}},$$

$$\alpha \times \beta \mapsto \alpha \star \beta = \sum_{st(v) = \beta} \alpha \beta.$$

Here $st(w)$ is the standard word in $\mathfrak{S}_n$ of the word $w = w_1 \ldots w_n$ of length $n$, i.e., the image of the unique increasing injective function from $\{w_1, \ldots, w_n\}$ to $\{1, \ldots, n\}$.
Similarly the coproduct $\Delta$ in $\mathfrak{S}$ is induced from

$$
\Delta : \mathbb{Q}[\mathfrak{S}] \rightarrow \bigoplus_{i=0}^{r} \mathbb{Q}[\mathfrak{S}] \otimes \mathbb{Q}[\mathfrak{S}]^{r-i}, \quad \alpha \mapsto \Delta(\alpha) = \sum_{i=0}^{r} \alpha_{(i+1,...,r)} \otimes \text{st}(\alpha_{(1,...,i)})
$$

Here $\alpha_I$ is the word obtained from $\alpha$ by erasing the letters outside $I$.

Let $\langle | \rangle$ stands for the scalar product such that the basis of permutation words in $\mathfrak{S}$ is orthonormal $\langle \alpha | \beta \rangle = \delta_{\alpha, \beta}$, $\alpha, \beta \in \mathfrak{S}$. The dual Hopf structure $\star', \Delta'$ on $\mathfrak{S}$ is in linear duality with the structure $\star, \Delta$, i.e., it is a convolution algebra defined through

$$
\langle \alpha \star' \beta | \gamma \rangle = \langle \alpha \otimes \beta | \Delta' \gamma \rangle \quad \langle \alpha' \star \beta | \gamma \rangle = \langle \alpha \otimes \beta | \Delta \gamma \rangle \quad \alpha, \beta, \gamma \in \mathfrak{S}.
$$

giving rise to the product and the coproduct

$$
\alpha \star' \beta = sh(\alpha, \tilde{\beta}), \quad \Delta'(\alpha) = \sum_{u,v} st(u) \otimes st(v).
$$

The dual Hopf algebra $(\mathfrak{S}, \star', \Delta')$ is isomorphic to the algebra of the free quasi-symmetric functions $\mathbb{F} \text{QSym}$ $[5]$. Throughout this paper we will be dealing only with the bialgebra structures of $\mathfrak{S}$, for the explicit formulas for the antipode see $[1]$.

### 3 Poirier-Reutenauer Hopf algebra

The celebrated Robinson-Schensted algorithm $[7]$ establishes a one-to-one correspondence between permutations $\alpha$ and pairs $(P, Q)$ of Standard Young Tableaux (SYT) of the same form

$$
RS : \mathfrak{S} \rightarrow \text{SYT} \times \text{SYT} \quad RS : \alpha \mapsto (P(\alpha), Q(\alpha)).
$$

Two permutations $\alpha$ and $\beta$ are $P$-equivalent, i.e., $P(\alpha) = P(\beta)$, if and only if their words are congruent $\alpha \sim \beta$ with respect to the plactic congruence generated by the Knuth equivalence relations $[7]$ (without repeating letters)

$$
bc_a \sim bea \quad acb \sim cab \quad a < b < c.
$$

**DEFINITION 2** Let $\text{Tab}$ be the quotient $\text{Tab} := \mathfrak{S}/\mathfrak{R}$ where $\mathfrak{R}$ is the “Knuth” submodule of $\mathfrak{S}$ generated by the elements $\alpha - \beta$, $\alpha, \beta \in \mathfrak{S}$, such that $\alpha \sim \beta$.

To any $t \in \text{SYT}$ we can associate its plactic class $c(t)$, which is the sum of the elements in $\alpha \in \mathfrak{S}$, such that $P(\alpha) = t$

$$
c(t) = \sum_{\alpha : t = P(\alpha)} \alpha \subset \mathfrak{S}.
$$

**DEFINITION 3** Let us denote by $\text{Tab'} = \bigoplus_{t \in \text{SYT}} \mathbb{Q} c(t)$ the $\mathbb{Q}$-span of all plactic classes.

Poirier and Reutenauer proved that the Malvenuto-Reutenauer Hopf structures on $\mathfrak{S}$ induce Hopf structures on the $\mathfrak{S}$-modules $\text{Tab}$ and $\text{Tab'}$.

**THEOREM 1** $[12]$. The submodule $\mathfrak{R}$ of $\mathfrak{S}$ is an ideal and a coideal of $(\mathfrak{S}, \star, \Delta)$. The quotient $\text{Tab}$ is a Hopf factor-algebra $(\text{Tab}, \star, \Delta)$. 

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THEOREM 2 \footnote{[12]. The submodule \( \text{Tab}^i \subset \mathcal{S} \) is a Hopf subalgebra \((\text{Tab}^i, *, \Delta')\) of the Hopf algebra \((\mathcal{S}, *, \Delta')\). The Hopf algebras \((\text{Tab}, *, \Delta)\) and \((\text{Tab}^i, *, \Delta')\) are canonically dual.

The Hopf subalgebra \((\text{Tab}^i, *, \Delta')\) of the Malvenuto-Reutenauer algebra \((\mathcal{S}, *, \Delta')\) is isomorphic to the algebra of the free symmetric functions \(\mathbb{FSym}^3\).

4 Pre-Plactic Hopf Algebra

Let \( u \in \mathcal{S}_p \) and \( v \in \mathcal{S}_r \) be (the words of) two permutations. We denote by \( \mathcal{I} \) the two-sided ideal in \( \mathcal{S} \) generated by elements

\[
u [ab]v := u(bac - bca - acb)v \in \mathbb{Z}[\mathcal{S}_{p+3+r}] \quad a < b < c \quad \tag{7}
\]

The relations in the ideal are among the relations of the quantum pseudo-plactic algebra \(\mathbb{PP}_q(W)\), namely the multilinear part of the ideal \(\mathcal{I}_q(W)\) and \(\mathcal{I}(W)\). The idea is to have a kind of “operadic approach” to the algebra \(\mathbb{PP}_q(W)\).

LEMMA 1 The submodule \( \mathcal{I} \subset \mathcal{S} \) is a Hopf ideal of the Hopf algebra \((\mathcal{S}, *, \Delta)\).

Proof. The ideal \( \mathcal{I} \) is generated in \( \mathbb{Z}[\mathcal{S}_3] \subset \mathcal{S} \) by the element \([2[13]]\). By definition it is a graded ideal \( \mathcal{I} = \oplus_{n \geq 0} \mathcal{I}(n) \) with degrees

\[
\mathcal{I}(n) = \sum_{i,j,i+j+3=n} \mathbb{Q}[\mathcal{S}_i] \ast [2[13]] \ast \mathbb{Q}[\mathcal{S}_j] \subset \mathbb{Q}[\mathcal{S}_n] \quad \Rightarrow \quad \mathcal{I} = ([2[13]]) \quad \tag{8}
\]

The module \( \mathcal{I} \) is a submodule of the module \( \mathcal{R} \), having as generators differences of the “Knuth” generators in \( \mathcal{R} \) hence the inclusions of submodules \( \mathcal{I} \subset \mathcal{R} \subset \mathcal{S} \). It is easy to see that \( \mathcal{I} \) is also coideal with respect to \( \Delta \). \( \square \)

DEFINITION 4 The pre-plactic algebra \( \mathbb{PP} \) is the quotient Hopf algebra \( \mathcal{S}/\mathcal{I} \)

\[
\mathbb{PP} := \mathcal{S}/([2[13]])
\]

The factor algebra \( \mathbb{PP} \) inherits the grading of \( \mathcal{S} \),

\[
\mathbb{PP} = \bigoplus_{n \geq 0} \mathbb{PP}(n) = \bigoplus_{n \geq 0} \mathbb{Q}[\mathcal{S}_n]/\mathcal{I}(n)
\]

We are going to calculate the dimensions \( \dim \mathbb{PP}(n) \). To this end we take up with the dimensions of the graded ideal \( \mathcal{I} \). We first introduce some notations about partitions.

Let \( \lambda = (\lambda_1, \ldots, \lambda_k) \) be a partition \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0 \) with \( r \) nonzero parts. The graphic representation of \( \lambda \) is the Young diagram whose \( i \)-th row is of length \( \lambda_i \), \( i = 1, \ldots, k \). By \( \lambda^t \) we denote the diagram transposed to \( \lambda \). A dual description \( \lambda = [1^{m_1}2^{m_2} \ldots k^{m_k}] \) is provided by the multiplicities \( m_i(\lambda) = \lambda'_i - \lambda'_{i+1} \) yielding the number of rows of length \( i \). One has \( |\lambda| = \sum_{i=1}^k i m_i(\lambda) = \sum_{i=1}^k \lambda_i \) with \( k = \sum m_i(\lambda) \).

PROPOSITION 1 The dimensions of the graded ideal \( \mathcal{I} = \bigoplus_{n \geq 0} \mathcal{I}(n) \) are counted by

\[
\dim \mathcal{I}(n) = \sum_{k > 0} \sum_{\Lambda \vdash n} \frac{(-1)^{1 + (|\lambda| - k) / 2} k! \prod_{i=1}^k m_i(\Lambda)! n!}{\prod_{i=1}^k \Lambda_i!} \quad \tag{9}
\]

where the sum runs over all partitions \( \Lambda = (\Lambda_1, \ldots, \Lambda_k) \) of \( n \) with \( k \) odd parts \( (\Lambda_i = 2\lambda_i + 1) \) except the partition \( \Lambda = 1^n \).
Proof. Our strategy in counting the dimension $\dim I(n)$ is a recurrent use of the formula for the dimension of an intersection
\[
\dim \sum_{\alpha} A_{\alpha} = \sum_{\alpha} \dim A_{\alpha} - \sum_{\alpha_1 < \alpha_2} \dim A_{\alpha_1} \cap A_{\alpha_2} + \sum_{\alpha_1 < \alpha_2 < \alpha_3} \dim A_{\alpha_1} \cap A_{\alpha_2} \cap A_{\alpha_3} - \ldots + (-1)^{l+1} \dim A_{\alpha_1} \cap A_{\alpha_2} \cap \ldots \cap A_{\alpha_l}
\]
where the family of spaces $\{A_{\alpha}\}$ such that $I(n) = \sum_{\alpha} A_{\alpha}$ is indexed by a finite ordered set of cardinality $l$.

We put into correspondence the composition $c = 1^{i}31^{j}$ of $n = i + j + 3$ with the subspace $A_c = \mathbb{Q}[S_1] * [2[13]] * \mathbb{Q}[S_3] \subset I(n)$. It is convenient to set $A_1 := \mathbb{Q}[S_1]$ then one has
\[
A_{1j} = A_1^i = \mathbb{Q}[S_j] \quad A_3 = \mathbb{Q}[2[13]]
\]
where $A_3$ is identified with the generator of the ideal $I = ([2[13]])$. With the new conventions the expression (8) for the graded ideal takes the succinct form
\[
I(n) = \sum_{i=0}^{n-3} A_{1j} * A_3 * A_{1n-3-i} = \sum_{i=0}^{n-3} A_1 \cap [13] \cap 31 \cap 13. \quad (11)
\]

**Lemma 2** The neighboring subspaces of $I(n)$ in the sum eq. (11) have zero intersection
\[
A_{1k31} \cap A_{1k+33} = 0. \quad (12)
\]

Proof of the lemma. The $S_3$-module $I(4)$ is a sum of two subspaces $I(4) = A_{31} + A_{13}$ and its dimension is given by the intersection formula eq. (10)
\[
\dim I(4) = \dim A_{31} + \dim A_{13} - \dim A_{13} \cap A_{31}.
\]
One can write explicitly the basis of $A_{31} = A_3 * A_1$, it consists of the four elements in $\mathbb{Q}[S_3]$
\[
\begin{align*}
 r_1 &= [13]2 \quad r_2 = [14]2 \quad r_3 = [14]3 \quad r_4 = [24]3
\end{align*}
\]
wheras the basis of $A_{13} = A_1 * A_3$ is spanned by
\[
\begin{align*}
 l_1 &= 4[13]2 \quad l_2 = 3[14]2 \quad l_3 = 2[14]3 \quad l_4 = 1[24]3.
\end{align*}
\]
An element $w$ belongs to the intersection $A_{13} \cap A_{31}$ when it can be represented $w = \sum c_i l_i = \sum d_i r_i$ with coefficients $c_i$ and $d_i$. However the only solution turns out to be $c_i = 0 = d_i$. We conclude that the intersection is empty,
\[
A_{13} \cap A_{31} = 0.
\]

The $S_5$-module $I(5)$ is in the linear envelope of three subspaces,
\[
I(5) = A_{123} + A_{131} + A_{312}
\]
and the vanishing $A_{13} \cap A_{31} = 0$ readily implies the zero intersections $A_{123} \cap A_{131} = 0$ and $A_{311} \cap A_{312} = 0$.

The statement eq. (12) follows by induction of the degree $I(n)$ of the ideal $I$. \(\square\)
However the intersection $A_{123} \cap A_{312}$ is not trivial (see Lemma 3 below).

Let us introduce some concise notations for the intersections
\[
A_{123} \cap A_{312} = : A_3, \quad A_{313} \cap A_{123} = : A_5, \quad A_{313} \cap A_{131} = : A_{51}, \quad A_{313} \cap A_{132} = : A_{53}.
\]

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Then the latter lemma implies the succinct expressions of the dimensions

\[
\begin{align*}
\dim \mathcal{I}(i) & = 0 \quad i < 3 \\
\dim \mathcal{I}(3) & = \dim A_3 \\
\dim \mathcal{I}(4) & = \dim A_{31} + \dim A_{13} \\
\dim \mathcal{I}(5) & = \dim A_{312} + \dim A_{131} + \dim A_{123} - \dim A_3 \\
\dim \mathcal{I}(6) & = \dim A_{313} + \dim A_{132} + \dim A_{1231} + \dim A_{133} \\
& \quad - \dim A_{31} - \dim A_{15} - \dim A_{33} \\
\end{align*}
\] (13)

A non-trivial (maximal) intersection \( A_7 := A_{314} \cap A_{1231} \cap A_{143} \) will appear in \( \mathcal{I}(7) \). In general on any odd degree \( 2k + 1 \) an intersection of \( k \) subspaces takes place.

**LEMMA 3** Let the space \( A_{2n+1} \) be the intersection of \( n \) subspaces

\[
A_{2n+1} := A_{312n-2} \cap A_{12312n-4} \cap A_{14312n-6} \cap \ldots \cap A_{12n-3} \in \mathcal{I}(2n+1) . \quad (14)
\]

The intersection \( A_{2n+1} \) is maximal in \( \mathcal{I}(2n+1) \) and it is one dimensional

\[
\dim A_{2n+1} = 1 .
\]

**Proof of the lemma.** For \( n = 1 \) the statement of the lemma is trivial

\[
\dim \mathcal{I}(3) = \dim A_3 = 1 .
\]

The first non-trivial statement is for the space \( A_5 \). For a commutative algebra with three generators \( x, y, z \) one has the antisymmetric combination

\[
x \wedge y \wedge z := x[yz] + y[zx] + z[xy] = [yz]x + [zx]y + [xy]z \quad (15)
\]

which spans a one dimensional space. Note that the generators \( x = [15], y = [24] \) and \( z = 3 \) in \( \mathbb{P}\mathbb{P}(5) \) commute and generate a maximal commutative subalgebra. Hence we have a one dimensional subspace

\[
[15] \wedge [24] \wedge 3 \in \mathcal{I}(5) .
\]

The following element belongs to the intersection \( A_5 = A_{123} \cap A_{312} \)

\[
[15][24][3] + [24][3][15] = [[24][3][15] + [3][15]][24] = [15] \wedge [24] \wedge 3 \mod [[15][24]] .
\]

The relation \([15], [24] \) can be represented also as \([15] \wedge [24] \) thus for the intersection space we get

\[
A_5 = A_{123} \cap A_{312} = Q[15] \wedge [24] \wedge 3 \mod [15] \wedge [24] .
\]

Therefore \( A_5 \) belongs to the ideal \( \mathcal{I}(5) \) and it is one dimensional

\[
\dim A_5 = 1 .
\]

So we have proven the lemma when \( n = 2 \).

We proceed by induction. In \( \mathcal{I}(6) \) the maximal commutative subalgebra is generated by \([16], [25] \) and \([34] \). Hence according to eq. (15) we get the antisymmetrized combination \([16] \wedge [25] \wedge [34] \in \mathcal{I}(6) \) which generates between others the following relations in \( \mathbb{P}\mathbb{P}(7) \),

\[
4[17] \wedge [26] \wedge [35] \in \mathcal{I}(7) \\
[17] \wedge [26] \wedge [35][4] \in \mathcal{I}(7) .
\]

---

1 Note that only next to neighboring terms from the sum \( \mathcal{I}(2n+1) = \sum A_{k+1} \wedge A_{2n-k} \wedge A_k \) appear in the intersection, as the intersections of neighboring terms is empty due to lemma 2.

2 The relation \([15], [24] = ([15][24] + [2][15][4]) \) is implied by the Jacoby identity.
The maximal commutative subalgebra is $\mathfrak{P}(7)$ is generated by the “long” generators \cite{17, 26, 35} and the “short” generator $4$. The commutative subalgebra gives rise to the space

$$[17] \land [26] \land [35] \land 4 \mod [17] \land [26] \land [35]$$

which belongs to the one-dimensional intersection $A_2 = A_{31^4} \cap A_{1312} \cap A_{1^4}$. By induction on $n$ we obtain an unique element in the $\mathfrak{S}_2n+1$-module $\mathcal{J}(2n+1)$

$$[1 2n + 1] \land [2 2n] \land \ldots \land [n n + 2] \land n + 1,$$

which after factoring gives rise to the intersection space \cite{4}

$$A_{2n+1} = \mathbb{Q}[1 2n+1] \land [2 2n] \land \ldots \land [n n+2] \land n + 1 \mod [1 2n+1] \land [2 2n] \land \ldots \land [n n+2].$$

We are done with the lemma. □

From the above examples the general pattern for $\dim \mathcal{J}(n)$ is clear, we have summation with alternating signs over the dimensions of subspaces $A_c = A_{c_1} \ast \ldots \ast A_{c_k} \subset \mathbb{Q}[\mathfrak{S}_n]$ labelled by compositions $c$ of $n$ having only odd parts

$$\dim \mathcal{J}(n) = \sum_{c:|c|=n} \pm \dim A_c$$

where the sum is over all compositions of $n$ with odd parts (except $c = 1^n$) and the sign $\pm$ is $(-1)^{k+1}$ if $A_c$ stems from intersection of $k$ elementary subspaces $A_{1^k1^k1\ldots}$. Subspaces $A_c$ and $A_{c'}$ whose compositions after ordering are mapped to the same partition $\Lambda$ will have same dimension $\dim A_c = \dim A_{c'} = \dim A_\Lambda$. Therefore the sum over compositions $c$ can be replaced by the sum over partitions $\Lambda$ at the expense of the multiplicities

$$\dim \mathcal{J}(n) = \sum_{k>0} \sum_{\Lambda \vdash n \atop \Lambda \neq 1^k} \pm \frac{k!}{\prod_{i=1}^k m_i(\Lambda)!} \dim A_\Lambda$$

Since $\dim A_{2n+1} = 1$, the dimension of a subspace $A_\Lambda = A_{\lambda_1} \ast \ldots \ast A_{\lambda_k} \subset \mathbb{Q}[\mathfrak{S}_n]$ is

$$\dim A_\Lambda = \frac{n!}{\prod_{i=1}^k \lambda_i!} \ldots |\Lambda| = n.$$ Any subspace $A_{\lambda_1} = A_{2^\lambda_1+1}$ is in the intersection of $\lambda_1$ elementary subspaces $A_{1^k1\ldots}$ hence the subspace $A_\Lambda$ lays in the intersection of $\sum_{i=1}^k \lambda_i$ subspaces. The summation over partitions $\Lambda$ having only odd parts is equivalent to the summation over partitions $\lambda$ (skipping the empty diagram $\lambda = 0$ corresponding to $\Lambda \neq 1^n$). The alternating sign factor is fixed by the number of intersections $\sum_{i=1}^k \lambda_i = (|\Lambda| - k)/2$. These observations together yields the final expression eq. \cite{9}.

5 Counting snakes and pre-plactic algebra

The term “snake” for an alternating permutation was coined by Vladimir Arnold \cite{3}.

**DEFINITION 5** An alternating permutation or snake is a permutation

$$\sigma = \left( \begin{array}{cccc} 1 & 2 & \ldots & n \\ \sigma_1 & \sigma_2 & \ldots & \sigma_n \end{array} \right) \in \mathfrak{S}_n \quad \text{such that} \quad \sigma_1 > \sigma_2 < \sigma_3 > \ldots$$

The word of the permutation $\sigma$ is the word $\sigma_1 \sigma_2 \ldots \sigma_n$. We denote by $E_n$ the number of snakes, $E_n = \# \{ \sigma \in \mathfrak{S}_n | \sigma_1 > \sigma_2 < \sigma_3 > \ldots \}$. 

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The alternating permutations are counted by Désiré André [2]. Their number is generated by the series
\[ y(x) := \sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x \]
satisfying the equation
\[ 2y' = y^2 + 1 \quad y(0) = 1 \quad (16) \]
Arnold also referred to the numbers \( E_n \) as to Bernoulli-Euler numbers for the following reason: the Euler numbers \( E_{2n} \) appear from the expansion of
\[ \sec x = \sum_{n \geq 0} E_{2n} \frac{x^{2n}}{(2n)!} \]
whereas the tangent numbers \( E_{2n+1} \)
\[ \tan x = \sum_{n \geq 0} E_{2n+1} \frac{x^{2n+1}}{(2n+1)!} \]
are related to the Bernoulli numbers \( E_{2n+1} = B_{2n+1} \frac{4^n (4^n-1)}{2^{2n+1}} \). Arnold introduced “snakes” for other Coxeter groups than \( A_n \), their numbers are topological invariants of bifurcation diagrams [3].

We found an expression of the snakes numbers \( E_n \) through summation over Young partitions which to the best of our knowledge is new.

**PROPOSITION 2** The number of snakes is given by the formula
\[ \frac{E_{n+1}}{n!} = \sum_{k>0} \sum_{\Lambda \vdash n} \frac{(-1)^{|\Lambda|-k}/2k!}{\prod_{i=1}^k \Lambda_i! \prod_{i=1}^k m_i(\Lambda)!} \quad \text{with} \quad \Lambda_i = 2\lambda_i + 1 \quad (17) \]
where the sum is over all partitions \( \Lambda \) of \( n \) with only odd parts.

**Proof.** Differentiating the generating series \( y(x) \) one gets
\[ y'(x) = \sum_{n \geq 0} E_{n+1} \frac{x^n}{n!} = \frac{1}{1 - \sin x} = \sum_{k \geq 0} \sin^k x = \sum_{k \geq 0} \left( \sum_{\lambda = 0}^\infty (-1)^\lambda \frac{x^{2\lambda+1}}{(2\lambda+1)!} \right)^k \]
The result follows by the binomial expansion and equating the coefficients of \( x^n \). \( \square \)

We are now ready to prove the main result of this paper namely the snakes numbers are the coefficients in the generating series for the \( \mathfrak{S} \)-module \( \mathfrak{P} \).

**THEOREM 3** The generating series of the \( \mathfrak{S} \)-module \( \mathfrak{P} = \bigoplus_{n \geq 0} \mathfrak{P}(n) \) is given by
\[ \sum_{n \geq 0} \dim \mathfrak{P}(n) \frac{x^n}{n!} = \sum_{n \geq 0} E_{n+1} \frac{x^n}{n!} . \]

**Proof.** The homogeneous degree \( \mathfrak{P}(n) \) of the pre-plactic algebra is \( \mathfrak{P}(n) = \mathbb{Q}[\mathfrak{S}_n]/\mathfrak{I}(n) \) hence
\[ \dim \mathfrak{P}(n) = n! - \dim \mathfrak{I}(n) . \quad (18) \]
The dimension of the ideal \( \dim \mathfrak{I}(n) \) was found in eq. (9) from where we get
\[ \dim \mathfrak{P}(n) = \sum_{k>0} \sum_{\Lambda \vdash n} \frac{(-1)^{|\Lambda|-k}/2k!}{\prod_{i=1}^k m_i(\Lambda)! \prod_{i=1}^k \Lambda_i!} \frac{n!}{\prod_{i=1}^k \Lambda_i!} . \]

---

3For a survey on alternating permutations see [15].
Note that by removing the restriction $\Lambda \neq 1^n$ in eq. (9) we incorporated the extra term $n!$ in $\dim \mathfrak{P}(n)$. We now realize that the expression for $\dim \mathfrak{P}(n)$ coincides with the formula eq. (17) for the snakes number $E_{n+1}$ which readily implies the result $\dim \mathfrak{P}(n) = E_{n+1}$.

6 Hopf structure and Hilbert-Poincaré Series

**Tensor algebra.** The free associative algebra $T(V)$ generated by $D$-dimensional vector space $V$ is naturally a $GL(V)$-module. Its character reads

$$ch_{T(V)}(x) = \frac{1}{1 - (x_1 + \ldots + x_D)} = \sum_{n \geq 0} (x_1 + \ldots + x_D)^n = \sum_{\lambda} s^\lambda(x) f^\lambda .$$

By Schur-Weyl duality the image of $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ is the $\mathfrak{S}$-module $\mathfrak{S} = \bigoplus_{n \geq 0} \mathfrak{S}_n$ with generating series

$$g_{\mathfrak{S}}(x) = \frac{1}{1 - x} = \sum_{n \geq 0} n^x n! = \exp \ln \frac{1}{1 - x} = \exp \sum_{k \geq 1} \frac{x^k}{k} = \exp \sum_{k \geq 1} (k - 1)! \frac{x^k}{k!}$$

where $(k - 1)!$ is the dimension of the space $\text{Lie}(k) \subset \mathbb{Q}[\mathfrak{S}_k]$ of primitive Lie elements.

**Parastatistics algebra** $PS(V)$. The universal enveloping algebra (UEA) of the two step nilpotent Lie algebra $PS(V) := U(V \oplus \lambda^2 V)$ arises in the parastatistics Fock spaces [4] and it will be referred to as the **parastatistics algebra** $PS(V)$. The algebra

$$PS(V) = T(V)/([V,V],[V])$$

is a factor algebra of the tensor algebra $T(V)$ by the two-sided ideal generated in $\text{Lie}(V)_3 := [[V,V],[V]]$ which is also a Hopf ideal for the standard Hopf structure on $T(V)$. The character of the $GL(V)$-module $PS(V)$ is also representable as the sum of all Schur polynomials

$$ch_{PS(V)}(x) = \prod_{i=1}^{D} \frac{1}{1 - x_i} \prod_{1 \leq i < j \leq D} \frac{1}{1 - x_i x_j} = \sum_{\lambda} s^\lambda(x_1, \ldots, x_D) . \quad (20)$$

The latter identity implies that in the decomposition of $PS(V)$ into irreducible representation every Schur module appears once and exactly once,

$$PS(V) = \bigoplus_{n \geq 0} PS(n) \otimes_{\mathfrak{S}_n} V^{\otimes n} \cong \bigoplus_{\lambda} S^\lambda(V) .$$

The $\mathfrak{S}$-module $PS = \bigoplus_{n \geq 0} PS(n) \cong \bigoplus_{\lambda} S^\lambda$ is the Schur-Weyl dual of the $GL(V)$-module $PS(V)$. The dimension of the irreducible $\mathfrak{S}_n$-module $S^\lambda$ is the number $f_\lambda$ of the Standard Young tableaux of shape $\lambda$, $f_\lambda := \dim S^\lambda$. Hence the generating series of the Standard Young Tableaux yields the generating series of the $\mathfrak{S}$-module $PS$

$$g_{PS}(x) = \exp \left( x + \frac{x^2}{2} \right) = \sum_{n \geq 0} x^n \sum_{\lambda \vdash n} f_\lambda . \quad (21)$$

**Plactic algebra** $\mathfrak{P}(V)$. The plactic algebra $\mathfrak{P}(V)$ turns out to be a “crystal” limit of a quantum deformation $PS_q(V)$ on the parastatistics algebra $PS(V)$ for the singular value $q = 0$ of the deformation parameter, $\mathfrak{P}(V) = PS_0(V) \oplus$. The deformed algebra
\( PS_q(V) \) is a module of the quantum UEA \( U_q gl.D \). The irreducible \( U_q gl.D \)-modules are the quantum Schur modules \( S_q^\lambda(V) \) having the same dimensions as the Schur modules \( S^\lambda(V) \). Hence the character formula \( (20) \) gives also the multi-graded dimensions of the deformed algebra \( PS_q(V) \) and its specialization: the plactic algebra \( \mathfrak{P}(V) \)

\[
ch_{PS(V)}(x) = ch_{PS_q(V)}(x) = ch_{\mathfrak{P}(V)}(x)
\]

By quantum Schur-Weyl duality to the \( U_q gl.D \)-module \( PS_q(V) = \bigoplus_{r \geq 0} PS_q(r) \otimes_{\mathcal{H}_q(q)} V^\otimes r \cong \bigoplus_{\lambda} S^\lambda_q(V) \)

one attaches the Hecke algebra modules \( PS_q \cong \bigoplus_{\lambda} S^\lambda_q \).

The Poirier-Reutenauer algebra can be seen as the \( q = 0 \) limit of direct sum of Hecke algebra modules \( [10] \). The plactic classes \( [6] \) close a commutative Hopf subalgebra (with respect to the shuffle) of the Malvenuto-Reutenauer algebra. Thus the generating series \( (21) \) is also counting the plactic classes which span the Poirier-Reutenauer algebra \( Tab' \)

\[
g_{PS}(x) = g_{PS_q}(x) = g_{Tab}(x).
\]

The series eq.\( (21) \) is a truncation of the generating series \( \exp \sum_{k \geq 1} \frac{x^k}{k!} \) reflecting the factorization of the free Lie algebra to the two-step nilpotent Lie algebra. As a digression we now prove an identity which is one more truncation of the generating series of \( \mathfrak{E} \)

**Lemma 4** The snake numbers satisfy the identity

\[
\sum_{n \geq 0} E_{n+1} \frac{x^n}{n!} = \exp \sum_{n \geq 1} E_{n-1} \frac{x^n}{n!}.
\]

**Proof.** The number of alternating permutations \( E_n \) is generated by the series \( y(x) = \sec x + \tan x \) which is a solution of the equation \( 2y' = y^2 + 1 \). The identity is equivalent to the following integro-differential equation

\[
y'(x) = \sum_{n \geq 0} E_{n+1} \frac{x^n}{n!} = \exp \int_0^x y(t) dt
\]

for the generating series \( y(x) = \sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x \) of the snakes numbers \( E_n \).

Indeed the differentiation of the eq. \( (23) \) yields

\[
y''(x) = y(x) \exp \int_0^x y(t) dt \quad \Rightarrow \quad y'' = yy' \quad \Leftarrow \quad (2y')' = (y^2 + 1)'\]

which is the same as the differentiation of the equation \( (16) \) for the generating function \( y(x) \) of the snakes numbers. □

### Pre-plactic algebra \( \mathfrak{P} \) and pseudo-plactic algebra \( \mathfrak{PP}(W) \)

The left hand side of the identity \( (22) \) is the generating series of the pre-plactic algebra \( \mathfrak{P} \). Then the natural interpretation of the numbers in the exponent of the identity \( (22) \) is as dimensions of the space of primitive Lie elements \( \mathfrak{P}^{Lie}(n) \) of the Hopf algebra \( \mathfrak{P} \)

\[
g_{\mathfrak{PP}}(x) = \sum_{n \geq 0} \dim \mathfrak{P}^{Lie}(n) \frac{x^n}{n!} = \exp \left( \sum_{n \geq 1} \dim \mathfrak{P}^{Lie}(n) \frac{x^n}{n!} \right).
\]
The Malvenuto-Reutenauer algebra $\mathcal{S}$ is a Hopf algebra which is not cocommutative and it is not an UEA of its Lie algebra. The primitive elements of $\mathcal{S}$ with respect to $\Delta$ are indexed by the connected permutations $\mathcal{S}_n$. However among the primitives of $\mathcal{S}_n$ we single out the Lie primitives which are Lie elements $\text{Lie}(n), \dim \text{Lie}(n) = (n-1)!$. Some of these primitive elements are in the ideal of $\mathcal{P} \mathcal{P}$ dimension of the non-vanishing Lie elements indexed by the connected permutations $[5, 12]$. However among the primitives of the Malvenuto-Reutenauer algebra $\mathcal{S}$ it is not an UEA of its Lie algebra. The primitive elements of $\mathcal{E}$ algebra of (pre-plactic) classes in $\mathcal{P} \mathcal{P}$ is a categorification of the factorial numbers $n!$. An interesting open problem is to construct bijection between $\mathcal{S}$ and the primitive elements of $\mathcal{P} \mathcal{P}$.

**THEOREM 4** The character of the $GL(V)$-module $\mathcal{P} \mathcal{P}(W)$ is given by

$$
ch_{\mathcal{P} \mathcal{P}(W)}(x_1, \ldots, x_D) = \prod_{n=1}^{\infty} \prod_{1 \leq i_1 < \cdots < i_n \leq D} (1 - x_1 x_2 \cdots x_n)^{-E_{n-1}}. \quad (24)
$$

It coincides with the character of the $U_q \mathfrak{gl}_D$-module $\mathcal{P} \mathcal{P}_q(W)$

$$
ch_{\mathcal{P} \mathcal{P}_q(W)}(x_1, \ldots, x_D) = ch_{\mathcal{P} \mathcal{P}_q(W)}(x_1, \ldots, x_D).
$$

**Proof.** The primitive Lie elements $[[a, b], a]$ and $[b, [a, b]], 1 \leq a < b \leq D$ with repeating letters are generators of the ideal $\mathcal{I}(W)$ of $\mathcal{P} \mathcal{P}(W)$. Hence any bracketing of Lyndon words with repeating letter is in the ideal $\mathcal{I}(W)$. Therefore the non-vanishing primitive Lie elements in $\mathcal{P} \mathcal{P}^{\text{Lie}}(V)_n$ stem from Lyndon words of length $n$ without repeating letter (from the alphabet $\{x_1, \ldots, x_D\}$). Then the Poincaré-Birkhoff-Witt theorem implies the expression for the character $ch_{\mathcal{P} \mathcal{P}(W)}(x_1, \ldots, x_D)$. The character $ch_{\mathcal{P} \mathcal{P}_q(W)}(x_1, \ldots, x_D)$ is the same since $\mathcal{P} \mathcal{P}_q(W)$ is a deformation of $\mathcal{P} \mathcal{P}(W)$.

**COROLLARY 1** The Hilbert series of the pseudo-plactic algebra $\mathcal{P} \mathcal{P}(W)$ and $\mathcal{P} \mathcal{P}_q(W)$ reads

$$
H_{\mathcal{P} \mathcal{P}(W)}(t) = \prod_{n=1}^{\infty} (1 - t^n)^{-E_{n-1}} \left( \frac{\dim V}{n} \right). \quad (25)
$$

**Outlook and perspectives**

We designed the quantum pre-plactic algebra $\mathcal{P} \mathcal{P}_q$ as a tool for proving the conjecture of Krob and Thibon about the isomorphism between the quantum pseudo-plactic algebra and the diagonal subalgebra of $\mathbb{C}[GL_q(V)]$. The sketch of this approach is given in the proceedings [14].

The pre-plactic algebra $\mathcal{P} \mathcal{P}$ turn out to be an object with a rich combinatorial structure. An interesting open problem is to construct bijection between snakes and the primitive Lie elements in $\mathcal{P} \mathcal{P}$. Another problem is to clarify the connection between the pre-plactic algebra $\mathcal{P} \mathcal{P}$ and the non-commutative trigonometric functions in $\text{FQSym}$ realization of snakes of type $A$. More distant goal will be to study the diagonal subalgebra of functions on quantum groups $\mathbb{C}[SO_q(n)]$ and $\mathbb{C}[Sp_q(2n)]$ and how they are related to the Arnold’s snakes of type $B$ and $D$. 
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