**Conformal window from conformal expansion**

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**Abstract:** We study the conformal window of asymptotically free gauge theories containing $N_f$ flavors of fermion matter transforming to the vector and two-index representations of $SO(N)$, $SU(N)$ and $Sp(2N)$ gauge groups. For $SO(N)$ we also consider the spinorial representation. We determine the critical number of flavors $N_f^{cr}$, corresponding to the lower end of the conformal window, by using the conjectured critical condition on the anomalous dimension of the fermion bilinear at an infra-red fixed point, $\gamma_{\bar{\psi}\psi,IR} = 1$ or equivalently $\gamma_{\bar{\psi}\psi,IR}(2 - \gamma_{\bar{\psi}\psi,IR}) = 1$. To compute $\gamma_{\bar{\psi}\psi,IR}$ we employ the scheme-independent (Banks-Zaks) conformal expansion at the 4th order in $\Delta N_f$, where the expansion parameter is the distance from the onset of the loss of asymptotic freedom, $\Delta N_f = N_f^{AF} - N_f$.

To quantify the uncertainties in our analysis, which potentially originate from nonperturbative effects, we propose two distinct approaches by assuming the large order behavior of the conformal expansion separately, either convergent or divergent asymptotic. In the former case, we take the difference in the Padé approximants to the two definitions of the critical condition, whereas in the latter case the truncation error associated with the singularity in the Borel plane is taken into account. Our results are further compared to other analytical methods as well as lattice results available in the literature. In particular, we find that $SU(2)$ with six and $SU(3)$ with ten fundamental flavors are likely on the lower edge of the conformal window, which are consistent with the recent lattice results. We also predict that $Sp(4)$ theories with fundamental and antisymmetric fermions have the critical numbers of flavors, approximately ten and five, respectively.
1 Introduction

Since it was discovered that $SU(N)$ gauge theories with $N_f$ flavors of fundamental fermions could exhibit an interacting conformal phase at an infra-red (IR) fixed point with a nonzero coupling constant [1, 2], a substantial amount of work has been devoted to investigate its properties as well as near-conformal behavior in the vicinity of the phase boundary. Besides its own theoretical interests, there has also been considerable interest in its applications to phenomenological model building for physics beyond the standard model. (For instance, see the recent review paper in Ref. [3].) In order to access the whole range of the IR conformal phase (conformal window), we typically assume that the number of flavors $N_f$ varies continuously. One end of the conformal window is identical to the critical point at which the theory loses asymptotic freedom. The corresponding number of flavors $N_f^{AF}$ can exactly be determined by investigating the renormalization group (RG) beta function in the usual perturbative expansion with respect to the coupling constant. We also know that there should be the other end of the conformal window at $N_f^{cr}$ with $0 < N_f^{cr} < N_f^{AF}$, because the theories with a sufficiently small number of flavors, including $N_f = 0$ pure Yang-Mills, are in the confining phase, and the dynamically generated confinement scale breaks the conformal symmetry. In contrast to the upper bound of the conformal window, however, it is a highly nontrivial task to identify its lower end since we may be in the large coupling regime in general and thus have to deal with nonperturbative effects.

Recently, our understanding of the phase structure of nonabelian gauge theories with fermionic matter has been further extended to asymptotically unfree theories for $N_f > N_f^{AF}$.
Figure 1. Conjectured phase structure of large $N$ QCD in the Veneziano limit at zero temperature and chemical potential. The continuous variable $x_f$ is defined as $x_f = N_f/N$ with both $N_f$ and $N$ taken to be infinite.

[4, 5]. Just above $N_f^{AF}$ the perturbative beta function yields that the theory possess a Landau pole, and thus it is not well defined in the ultraviolet (UV) while it is trivial in the IR. If $N_f$ further increases and becomes larger than $N_f^{safe}$, however, the theory develops an ultraviolet fixed point with a non-zero value of the coupling, which has been discussed in the context of asymptotic safety [4].

The conjectured phase diagram of non-abelian gauge theories with fermionic matter fields at zero temperature and chemical potential can then be drawn as in Fig. 1. For illustration purposes we consider that fermions are in the fundamental representation and take the large $N$ limit while keeping the ratio $x_f = N_f/N$ is fixed, i.e. the Veneziano limit. However, we note that without losing generosity the discussion below can be applied to all the theories with a gauge group $G$ and $N_f$ fermions in the representation $R$ considered in this work. There are two different phases in which the theory is asymptotically free, chirally broken and IR conformal. In the asymptotically unfree regime, two other phases are expected to exist, QED-like and UV safe. Analytical understanding of the chirally broken phase at small $x_f$ is highly limited because the standard perturbation technique is not applicable due to the absence of a small expansion parameter. One should instead rely on fully nonperturbative methods such as the lattice Monte-Carlo calculations.

In the vicinity of $x_f^{AF} = 11/2$, onset of the loss of asymptotic freedom, the coupling expansion of the beta function finds an IR fixed point in the weak coupling regime for $x_f < x_f^{AF}$, i.e. IR conformal, but it does not for $x_f > x_f^{AF}$ except the Gaussian fixed point at the origin, i.e. non-abelian QED in the IR. In this perturbative regime one may also consider an alternative series expansion by taking the difference, $\delta x_f \equiv x_f^{AF} - x_f$, as a small parameter. Such an expansion, so-called the Banks-Zaks conformal expansion, has been shown to be a useful tool for the investigation of the IR conformal phase [2]. In particular, the scheme-independent conformal expansions of physical quantities, such as the anomalous dimension of a fermion bilinear operator $\gamma_{\bar{\psi}\psi,IR}$ and the derivative of the beta function $\beta'_{IR}$, have been extensively studied in a series of papers [6–12].

For $x_f \gg x_f^{AF}$ the coupling or conformal expansion is no longer useful, but one can still analytically explore the phase diagram by means of the large $N_f$ expansion [19, 20] which in turns has proven its worth
by discovering the aforementioned UV safe phases \[4, 5\].

The purpose of this work is to estimate the critical number of flavors \( N_f^{cr} \), corresponding to the phase boundary between the chirally broken and the IR conformal, in 
\[ G = SU(N), Sp(2N) \text{ and } SO(N) \] gauge theories with fermion matter content in a single representation \( R \). In particular, we consider the fundamental (F), adjoint (Adj), two-index symmetric (S2) and antisymmetric (AS), and spinorial (S) representations. To do this we follow the approach discussed in Ref. [21] (see also Ref. [22] for an earlier work along this direction): \( N_f^{cr} \) is determined by using the conjectured critical condition to the anomalous dimension of a fermion bilinear operator at an IR fixed point, \( \gamma_{\bar{\psi}\psi, IR} = 1 \) or equivalently \( \gamma_{\bar{\psi}\psi, IR}(2 - \gamma_{\bar{\psi}\psi, IR}) = 1 \), which characterizes the chiral phase transition through the annihilation of infra-red and ultraviolet fixed points \[23\] and Schwinger-Dyson analysis in the ladder approximation \[22, 24, 25\]. To make our analysis scheme-independent, we employ the conformal expansion for the computation of \( \gamma_{\bar{\psi}\psi, IR} \) \[6\]. At finite order in the conformal expansion the two critical conditions lead to different results in general, and the latter definition is often used because it does not only show better convergence to the known orders but also reproduces the value of the critical coupling \( \alpha^{cr} \) obtained from the Schwinger-Dyson analysis in the ladder approximation. For the rest of this paper we reluctantly use the simplified notation \( \gamma_{IR} \) for \( \gamma_{\bar{\psi}\psi, IR} \).

In this work we put one step forward by taking account of the uncertainties associated with the truncation of the conformal expansions at finite order. It is largely unknown whether the conformal expansion is convergent or divergent asymptotic, and how far the expansion can be reliably applicable.\(^2\) There has been much evidence that the conformal expansion may reach the lower end of the conformal window (e.g. see Refs. \[7, 8\]), and thus we first assume that this is the case.\(^3\) While accounting for the full nonperturbative effects is beyond the scope of this work, our analysis will capture part of them implied by the inconsistency in the perturbative expansion of the critical condition.\(^4\) We then treat

\(^2\)The conformal expansion is expected to be free of factorially increasing coefficients due to renormalons which dominate the large-order behavior of the coupling expansion in QCD-like theories \[26, 27\]. Such a fact results in better-behaved series expansions which have been explicitly shown in the higher-order calculations of \( \gamma_{\bar{\psi}\psi, IR} \) and \( \beta_{IR} \) \[9, 10\]. Of course, the absence of renormalons is not sufficient to conclude that the conformal series is convergent, since other types of factorial growth such as the one related to the multiplicity of diagrams could be involved. Furthermore, in Ref. \[28\] it has been argued that the conformal series would be divergent asymptotic if the coupling expansion turns out to be divergent.

\(^3\)The relatively well-behaved conformal series expansion, compared to the coupling expansion in the \( \overline{\text{MS}} \) scheme which is broken for small \( N_f \) (but supposed to be within the conformal window) at the 5th order \[29\], should be taken with care. Namely, the conformal expansion is blind to the existence of the IR fixed point and thus requires an external input in order to determine the valid region in the parameter space. In this work, the critical condition on \( \gamma_{\bar{\psi}\psi, IR} \) plays the role of this input.

\(^4\) In Ref. \[28\], the authors have accounted for nonperturbative effects by introducing new terms involving \( e^{-\frac{\beta_0 \alpha}{m}} \) with \( 2 \leq m \) through the notion of trans-series to the coupling expansion of the RG beta function \( \beta(a) \) with \( \beta_0 \) the coefficient of the lowest-order term, and similarly to the conformal expansions. Here, such corrections may not be related to the renormalons or instantons, but could rather be associated with higher dimensional operators which become marginal in the onset of the conformality loss. We believe that this approach is natural and plausible, but have a concern about the way how they arrived at the final form of \( e^{-\frac{\beta_0 \alpha}{m}} \). In particular, near the boundary of the chiral phase transition the emergence of a dynamical scale could be largely different to the one in QCD. For example, the Berezinskii-Kosterlitz-Thouless (BKT)-type
the two possibilities for the asymptotic behavior of conformal expansion, separately. In the case the expansion is convergent, we employ the Padé approximation to approximate the closed forms for the two definitions of the critical condition. We then take the difference in the resulting values of $N_{f}^{cr}$ as the uncertainty of our analysis. The best Padé approximants are determined by comparing their asymptotic behaviors at sufficiently large values of $N_{f}$ to the large $N_{f}$ expansion. If the conformal series is assumed to be divergent asymptotic, on the other hand, we roughly estimate the uncertainty, associated with the truncation at the largest order available up to date, by approximating the size of an ambiguity in the perturbative expansion which is closely related to the singularity in the Borel plane.

The paper is organized as follows. In Sec. 2.1 we discuss the generic infra-red properties of non-abelian gauge theory coupled to fermionic matter at zero temperature. In particular, we recall that both the truncated Schwinger-Dyson analysis and the mechanism of fixed-point annihilation imply the same critical condition, $\gamma_{\bar{\psi}\psi, \text{IR}} = 1$ or equivalently $\gamma_{\bar{\psi}\psi, \text{IR}}(2 - \gamma_{\bar{\psi}\psi, \text{IR}}) = 1$, characterizing the loss of IR conformality. In Sec. 2.2, we briefly review the conformal expansion for $\gamma_{\bar{\psi}\psi, \text{IR}}$ defined at an IR fixed point. We then describe our strategy to determine the lower edge of the conformal window in a scheme independent way in Sec. 2.3: we apply the critical condition, which is responsible for the chiral phase transition, to $\gamma_{\bar{\psi}\psi, \text{IR}}$ computed from the conformal expansion at finite order. Sec. 2.4 is devoted to estimate the size of systematic effects associated with the finite-order perturbative calculations by assuming different large-order behaviors, convergent or divergent asymptotic. We present our main results on the conformal window of $SO(N)$, $SU(N)$ and $Sp(2N)$ gauge theories with $N_{f}$ Dirac (Weyl) fermions in various representations, in Sec. 3.1 for the large $N$ limit and in Secs. 3.2, 3.3 and 3.4 for finite values of $N$, respectively. We critically assess our results by comparing to other analytical methods and the most recent nonperturbative lattice results available in the literature. Finally, we conclude by summarizing our findings in Sec. 4.

2 Background and methods

2.1 Infra-red conformal phase in asymptotically free gauge theories

We consider a generic non-abelian gauge theory containing $N_{f}$ flavors of massless fermionic matter in distinct representations $R$ of the gauge group $G = SO(N)$, $SU(N)$, and $Sp(2N)$ \textsuperscript{5}. The evolution of the gauge coupling constant $g$ is described by the renormalization group beta function

$$\beta(g) = \frac{dg}{dt},$$

(2.1)

where $t = \ln \mu$ with $\mu$ the renormalization scale. For a small value of $g$ the RG evolution can be studied by perturbation technique in a reliable way, which is equivalent to the Feynman scenario \cite{23} may render the non-perturbative corrections occurring only in the broken phase with the form of $e^{-\frac{\sigma}{\sqrt{\alpha}}}$, where $\alpha$ is a tunable parameter (e.g. slowly varying gauge coupling at an approximate IR fixed point) which triggers the phase transition at $\alpha = \alpha^{cr}$.

\textsuperscript{5}Throughout this section $N_{f}$ denotes the dummy variable for either the Dirac or Weyl flavors unless explicitly specified.
loop expansion. After rewriting the coupling constant as \( \alpha = g^2/4\pi \) to be positive definite, one can write the perturbative beta function as

\[
\beta(\alpha) = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \left( \frac{\alpha}{4\pi} \right)^\ell,
\]

(2.2)

where the \( \ell \)-loop coefficient \( b_\ell \) depends on the details of the theory, such as the number of flavors \( N_f \), the number of colors \( N \), the representation \( R \), and the gauge group \( G \).

The essential features of the perturbative theory are encoded in the lowest two terms which are independent of the renormalization scheme. Note that in general the series expansions at finite order in \( \alpha \) for \( \ell \geq 3 \) are scheme-dependent. With \( N_f \) Dirac fermions in the representation \( R \) of the gauge group \( G \) the explicit expressions of \( b_1 \) [30, 31] and \( b_2 \) [1] are

\[
b_1 = \frac{11}{3} C_2(G) - \frac{4}{3} N_f T(R), 
\]

(3.3)

\[
b_2 = \frac{34}{3} C_2(G)^2 - \frac{4}{3} (5C_2(G) + 3C_2(R)) N_f T(R),
\]

(3.4)

where \( T(R) \) is the trace normalization factor and \( C_2(R) \) is the quadratic Casimir invariant with \( C_2(G) = C_2(\text{Adj}) \) \(^6\). The beta function in Eq. 2.2 has a trivial fixed point at \( \alpha = 0 \), a Gaussian fixed point, for which the theory is free. In the vicinity of this fixed point the coupling constant can be arbitrarily small and the behavior of the RG flow is governed by the slope of the beta function, i.e. the sign of \( b_1 \). Consider that we fix the gauge group, the fermion representation, and the number of colors, but continuously vary the number of flavors \( N_f \) for which only non-negative integer values are physically meaningful. For sufficiently small numbers of flavors the coefficient \( b_1 \) has a positive value and the coupling constant approaches zero as the momentum scale flows from the IR to the UV, indicating that the theory is asymptotically free at high energy. If the number of flavors is larger than \( N_f^{\text{AF}} = 11C_2(G)/4T(R) \) or equivalently \( b_1 < 0 \), on the other hand, the theory loses the asymptotic freedom and the IR theory is trivial. The focus of our interest is in the asymptotically free theory and thus we restrict our attention to \( N_f < N_f^{\text{AF}} \).

The 2-loop results further divide the asymptotically free region into two nontrivial phases whose IR behaviors are distinct from each other. If the number of flavors is sufficiently small such that \( b_2 > 0 \), including the extreme case of the pure Yang-Mills \( (N_f = 0) \), from the UV to the IR the coupling runs to infinity and the theory is expected to confine by developing a dynamical scale. In the presence of fermionic matter the global (flavor) symmetry is also expected to be broken due to the non-zero fermion condensate. From the fact that a negative value of \( b_2 \) and an arbitrarily small positive value of \( b_1 \) are realized if \( N_f \) is just below \( N_f^{\text{AF}} \), on the other hand, one finds a coupling constant satisfying \( \beta(\alpha_{\text{BZ}}) = 0 \) at \( \alpha_{\text{BZ}} = -4\pi b_1/b_2 \ll 1 \) in a reliable manner within the perturbation theory [1, 2]. The corresponding BZ fixed point, named after Banks-Zaks, suggests the existence of interacting IR

\(^6\)The group theoretical invariants are defined as \( \text{Tr}[T^a_R T^b_R] = T(R) \delta^{ab} \) and \( T^a_R T^b_R = C_2(R) I \), where the summation runs over \( a = 1, \cdots, d_G \) with \( d_G \) the dimension of the gauge group \( G \). Here, \( T_R^a \) are the generators in the representation \( R \) of \( G \) and the group invariants are related by \( C_2(R) d_R = T(R) d_G \) with \( d_R \) the dimension of the representation \( R \).
conformal theories (even beyond the weak coupling regime) with certain numbers of flavors ranged over $N_f^L < N_f < N_f^A$. Such an interval in $N_f$ is commonly called the **conformal window** (CW). The conformal phase near the upper bound can systematically be studied by the perturbative analysis as discussed above. However, it is difficult to investigate the phase near the lower bound, because in general $\alpha_{\text{BZ}}$ grows to a large value as $N_f$ decreases and thus the perturbative analysis would fail. In this region, nonperturbative effects are also expected to be sizable in the IR. It is even a nontrivial task to determine the value of $N_f^L$.

In Ref. [23], it has been argued that the underlying mechanisms responsible for the loss of conformality could generally be classified by the following three criteria from the RG point of view: (a) the coupling at an IR fixed point, $\alpha_{\text{IR}}$, goes to zero, (b) $\alpha_{\text{IR}}$ runs off to infinity, or (c) an IR fixed point merges with a counterpart UV fixed point. The transition between asymptotically free and unfree phases at $N_f^A$, the upper bound of CW, belongs to scenario (a), where the BZ fixed point annihilates with the Gaussian fixed point at zero coupling. Similarly, the lower end of the conformal window might be determined from scenario (b) using the 2-loop results, i.e. $b_2 = 0$ such that $\alpha_{\text{BZ}} \to \infty$. However, such a naive estimation is limited by the reliability of the perturbative expansion and, more severely, by the unphysical values of physical quantities, e.g. the anomalous dimension of a fermion bilinear operator which may violate the unitarity bound, $\gamma_{\bar{\psi}\psi, \text{IR}} \leq 2$ [32]. Nevertheless, we note that mechanism (b) successfully describes the conformal transition at the lower end of the conformal window in $\mathcal{N} = 1$ supersymmetric QCD (SQCD) through the electro-magnetic (Seiberg) duality [33, 34], where the loss of conformality in the dual magnetic theory is described by scenario (a) in a weak coupling regime.

The last scenario was realized in the exemplified cases of certain nonrelativistic and relativistic quantum theories, and conjectured to explain the loss of conformality at the lower end of the conformal window in nonsupersymmetric theories in the large $N$ limit [23].

The UV and IR fixed points correspond to the solutions, $g_{\text{UV}}$ and $g_{\text{IR}}$, of the suggested RG equation, $\beta(g; \alpha) \equiv (\alpha^{\text{ct}} - \alpha) - (g - g^{\text{ct}})^2 = 0$ with $(\alpha^{\text{ct}} - \alpha) > 0$, respectively. If we consider the chiral phase transition in non-abelian gauge theories coupled to fermionic matter in $d = 4$ space-time dimension, the interpretation of the couplings is as follows: $g$ is the dimensionless running coupling associated with a certain higher-dimensional gauge-singlet operator $O_g$ which becomes marginal at the chiral phase transition, $\alpha$ is the gauge coupling at an approximate IR fixed point which is (almost) constant and thus treated as an external parameter, and $g^{\text{ct}}$ and $\alpha^{\text{ct}}$ are the critical couplings in the onset of the transition. The coupling $\alpha$ can be tuned by varying the number of flavors $N_f$ (or $x_f = N_f/N_c$ in the Veneziano limit). Note that this RG prescription is expected to be only valid near the lower end of the conformal window at which $\alpha \sim \alpha^{\text{ct}}$ is strong enough to cause substantially large dimensional transmutation of $O_g$ such that the operator becomes relevant to the RG evolution in the IR, i.e. $d_{O_g} - \gamma_{O_g} = 4$ with $d_{O_g}$ the mass dimension and $\gamma_{O_g}$ the anomalous dimension of $O_g$. The operator $O_g$ is weakly relevant, $g \sim g^{\text{ct}} \ll 1$, such that

\footnote{Although mechanism (c) still remains a conjecture because no rigorous proof exist, it is encouraging that UV complete 4D gauge models explicitly realizing the scenario of merging two fixed points were found in the weakly-coupled regime [35].}
the perturbative analysis is a valid description for the RG flow connecting the UV and IR
fixed points. The most natural candidate for $O_g$ in the large $N$ limit would be the chirally
symmetric four-fermion operators of double-trace form \[36\], e.g., $\overline{\psi}\gamma^\mu\psi^2$, which leads to
the critical condition $\gamma_{\overline{\psi}\psi,\text{IR}} = (dO_g - 4)/2 = 1 \ [23, 37]$. At finite $N$ the marginal operator
at $N_f^\text{ct}$ could take the form of other than double-trace or a mixture.\(^8\) If this is the case, the
critical condition on $\gamma_{\overline{\psi}\psi,\text{IR}}$ may not be directly connected with the chiral phase transition
and should be taken as an approximation at best.

Surprisingly, the critical condition mentioned above is equivalent to the one suggested
by Schwinger-Dyson (SD) analysis in the ladder approximation, but with a different form,
$\gamma_{\overline{\psi}\psi,\text{IR}}(2 - \gamma_{\overline{\psi}\psi,\text{IR}}) = 1 \ [24]$. As argued in Ref. \[25\], this critical condition is also believed
to persist beyond the ladder approximation. In terms of the gauge coupling the truncated
SD equation yields the critical condition, $\alpha_{\text{IR}} = \alpha_{\text{cr}}$ with $\alpha_{\text{cr}} = \pi/3C_2(R)$. The naive and
traditional way to determine $N_f^\text{ct}$ using the SD analysis is to calculate $\alpha_{\text{IR}}$ from the 2-loop beta function and match it to $\alpha_{\text{cr}}$. However, if we want to proceed the analysis beyond
the 2-loop, we cannot avoid renormalization-scheme dependence and the critical condition
becomes ambiguous. We therefore exploit the anomalous dimension $\gamma_{\overline{\psi}\psi,\text{IR}}$, which is phys-
ical, to estimate the critical number of flavors $N_f^\text{ct}$ by using the aforementioned critical
condition. This condition satisfies the unitarity condition by construction. We note that
in the case of SQCD the unitarity condition is the same with the onset of the conformality
loss and is often used to determine the conformal window even for nonsupersymmetric
theories. However, these two conditions could largely be different in general, because the
underlying mechanism of the loss of conformality is expected to depend on the details of
the theory as discussed above.

2.2 Conformal expansion for the anomalous dimension $\gamma_{\overline{\psi}\psi,\text{IR}}$

One of the consequences of the perturbative BZ fixed point is that the IR coupling can be
expanded in terms of the distance from $N_f^\text{AF}$, $\Delta_{N_f} \equiv N_f^\text{AF} - N_f$, i.e. the Banks-Zaks
conformal expansion \[2\],

$$\frac{\alpha_{\text{IR}}}{4\pi} = \sum_{j=1}^{\infty} a_j (\Delta_{N_f})^j,$$  \hspace{1cm} (2.5)

where the coefficients $a_j$ are independent of $N_f$. The leading order term is solely determined
from the two-loop results as

$$\alpha_{\text{IR}} = 4\pi a_1 \Delta_{N_f} + \mathcal{O}(\Delta_{N_f}^2),$$  \hspace{1cm} (2.6)

with

$$a_1 = \frac{1}{b_2} \left. \frac{\partial b_1}{\partial N_f} \right|_{N_f = N_f^\text{AF}} = \frac{4T(R)}{3C_2(G)(7C_2(G) + 11C_2(R))}. \hspace{1cm} (2.7)$$

Similarly, the $j$th order coefficient $a_j$ can be determined from a power series solution to
$\beta(\alpha) = 0$ with $\beta(\alpha)$ in Eq. 2.2 truncated at the $(j+1)$th order.

\(^8\) For instance, see Ref. \[35\] for the discussion about the mixture of single- and double-trace operators
at finite $N$ in the context of weakly coupled and UV complete gauge models realizing the scenario of fixed
point merger.
The most notable feature of the conformal expansion is that the series coefficients of the expansion for a physical observable are universal, in the sense that they are independent on the renormalization scheme order by order. Such a fact can be understood on general grounds, because the expansion parameter $\Delta_{N_f}$, defined through the scheme-independent 2-loop beta function, is a well-defined physical quantity. The conformal expansion relevant to us is the one for the anomalous dimension of a fermion bilinear operator

$$\gamma_{\bar{\psi}\psi,\text{IR}}(\Delta_{N_f}) = \sum_{j=1}^{\infty} c_j(\Delta_{N_f})^j.$$  \hspace{1cm} (2.8)

The coefficients $c_j$ are determined by combining the results of the coupling expansion of $\gamma_{\bar{\psi}\psi,\text{IR}}(\alpha)$ at the $j$th order and $\beta(\alpha)$ at the $(j + 1)$th order, respectively. As mentioned above, the conformal expansion is scheme-independent at finite order and does not require any information from higher-order terms. This is a somewhat distinctive feature compared to other expansions, alternative to the coupling expansion, such as the large-$N_f$ expansion for which all orders in $\alpha$ are necessary to compute the coefficient at each order in $1/N_f$.

The recent computations of the perturbative beta function at the 5th order in the gauge coupling [38–41], along with the results for the anomalous dimension at the 4th order [42, 43], within the modified minimal subtraction (MS) scheme enable to determine the coefficients $c_j$ to the 4th order in $\Delta_{N_f}$, where the explicit results in terms of group invariants for fermions transforming according to the representation $R$ of a generic gauge group $G$ are presented in Ref. [10].

In general, besides the scheme-independence, the conformal expansion of $\gamma_{\bar{\psi}\psi,\text{IR}}$ better behaves compared to the coupling expansion. For instance, we refer the reader to the results in Tables. 1-5 of Ref. [10], where the resulting values of $\gamma_{\bar{\psi}\psi,\text{IR}}$, computed using both the conformal and coupling expansion up to 4th order for $SU(N)$ gauge theories coupled to $N_f$ Dirac fermions in the fundamental, adjoint, two-index symmetric and antisymmetric representations, are present. First of all, at fixed $N_f$, $\gamma_{\bar{\psi}\psi,\text{IR}}$ monotonically increases with the order of $\Delta_{N_f}$ for all the theories considered, indicating that the coefficients are all positive, while those in the coupling expansion are not. Furthermore, the difference of $\gamma_{\bar{\psi}\psi,\text{IR}}$ between the adjacent orders of $\Delta_{N_f}^j$ and $\Delta_{N_f}^{j+1}$ typically decreases if $j$ increases, except for $j = 2$ and 3 in a few theories with small values of $N_f$ ($x_f$). Interestingly, these exceptional cases consistently reside in the broken phase just outside the conformal window estimated in this work. If the conformal expansion turns out to be a convergent series, such an agreement seems to indicate that the radius of convergence is closely related with the phase boundary at which the loss of conformality occurs. As we have no good understanding of this interesting observation at the moment, however, it should not be generalized as a generic feature of the chiral phase transition in nonsupersymmetric theories without further investigation. Last but not the least, $\gamma_{\bar{\psi}\psi,\text{IR}}$ at each order in the conformal expansion also monotonically increases with $\Delta_{N_f}$ due to the positive coefficients. Accordingly, with a few lowest coefficients the perturbative calculations of $\gamma_{\bar{\psi}\psi,\text{IR}}$ seem to be stretched to the very small $N_f$. However, we note that the results at small $N_f$ should be taken with some care, because the conformal expansion only makes sense when an IR fixed point exists, where its existence is not known a priori.
2.3 Determination of the lower edge of the conformal window

As discussed in Sec. 2.1, we assume that the chiral phase transition in nonsupersymmetric gauge theories occurs through mechanism (c) rather than (b), i.e. the coupling at an IR fixed point disappears by annihilating the UV fixed point instead of running to infinity. Consequently, we determine the critical number of flavors $N_{c}\text{f}$, corresponding to the lower edge of conformal window, by adopting the critical condition on the anomalous dimension of a fermion bilinear,

$$\gamma_{\bar{\psi}\psi, \text{IR}} = 1 \text{ or } \gamma_{\bar{\psi}\psi, \text{IR}}(2 - \gamma_{\bar{\psi}\psi, \text{IR}}) = 1.$$  \hspace{1cm} (2.9)

In an earlier work along this direction [44], the coupling expansion including higher order terms in the $\overline{\text{MS}}$ scheme was used to compute $\gamma_{\bar{\psi}\psi, \text{IR}}(\alpha_{\text{IR}})$ and $\alpha_{\text{IR}}$. Furthermore, the authors employed the critical condition $\gamma_{\bar{\psi}\psi, \text{IR}}(2 - \gamma_{\bar{\psi}\psi, \text{IR}}) = 1$, not $\gamma_{\bar{\psi}\psi, \text{IR}} = 1$, because the 1-loop result turned out to be identical to the critical condition on $\alpha$ in the traditional Schwinger-Dyson analysis. In this work we instead use the Banks-Zaks conformal expansion for the computation of $\gamma_{\bar{\psi}\psi, \text{IR}}$, since it shows better behavior as discussed in the previous section. More importantly both forms of the critical condition can be expanded order by order in a scheme independent way, so be the conformal window. Comparisons between the conformal and coupling expansions for the determination of $N_{c}\text{f}$ in $SU(3)$ gauge theories coupled to $N_{f}$ fundamental Dirac fermions are found in Ref. [21].

Two equivalent critical conditions in Eq. 2.9 should be identical to each other if all orders of the conformal expansion are considered. If the left-hand sides of those equations are truncated at the finite order $n$, however, it leads to two different critical conditions. Accordingly, the resulting values of $N_{c}\text{f}$ are different in general. To be explicit, we first define the finite-order critical condition of the former

$$\gamma_{\text{IR}}(\Delta_{N_{f}}^{n}) = \sum_{j=1}^{n} k_{j} (\Delta_{N_{f}})^{j} \equiv 1,$$  \hspace{1cm} (2.10)

where the coefficients $k_{j}$ are known to the 4th order [10]. Similarly, the latter critical condition at each order $n$ can be written as

$$\gamma_{\text{IR}}(2 - \gamma_{\text{IR}})(\Delta_{N_{f}}^{n}) = \sum_{j=1}^{n} \kappa_{j} (\Delta_{N_{f}})^{j} \equiv 1.$$  \hspace{1cm} (2.11)

The coefficients $\kappa_{j}$ are related to $k_{j}$ as

$$\kappa_{1} = 2k_{1}, \quad \kappa_{2} = 2k_{2} - k_{1}^{2}, \quad \kappa_{3} = 2k_{3} - k_{1}k_{2}, \quad \kappa_{4} = 2k_{4} - 2k_{1}k_{3} - k_{2}^{2}, \quad \cdots.$$  \hspace{1cm} (2.12)

To illustrate the typical behavior of the left-hand sides of Eq. 2.10 and Eq. 2.11, we consider $SU(N)$ gauge theories coupled to $N_{f}$ fundamental Dirac fermions in the Veneziano limit, i.e. $N \to \infty$ and $N_{f} \to \infty$ with the ratio $x_{f} = N_{f}/N$ fixed. We define $\Delta_{x_{f}} = x_{f}^{AF} - x_{f}$ with $x_{f}^{AF} = 11/2$. In Fig. 2, we show the results for $x_{f} \leq x_{f}^{AF}$. As discussed in the previous section, $\gamma_{\text{IR}}(\Delta_{x_{f}}^{n})$ monotonically increases as we go to the higher order in $\Delta_{x_{f}}$ over the whole range of $x_{f}$ considered, so does $\gamma_{\text{IR}}(2 - \gamma_{\text{IR}})(\Delta_{x_{f}}^{n})$. We also observe that the latter receives
smaller corrections from higher order terms along the black dotted line, corresponding to the critical condition, which can be understood as follows. First of all, the monotonic increment of $\gamma_{\text{IR}}$ with $n$ implies the positiveness of the coefficients $c_i$, which in turn results in $\frac{\kappa_2}{\kappa_1} < \frac{k_2}{k_1}$ as seen in Eq. 2.12, i.e. for a given value of $\Delta x_f$ the ratio between the second and first terms of $\gamma_{\text{IR}}(2 - \gamma_{\text{IR}})$ is smaller than that of $\gamma_{\text{IR}}$. We note that such an inequality cannot always be true for higher order coefficients. Secondly, the leading-order result of $\gamma_{\text{IR}}(2 - \gamma_{\text{IR}})$ starts by twice larger than that of $\gamma_{\text{IR}}$. Combined with the positive coefficients, it leads us to find higher-order results at smaller $\Delta x_f$ along the black solid line, which allows us to be in the better controlled regime of the perturbative series expansion. As we approach the unity from below, therefore, we find that the conformal expansion of $\gamma_{\text{IR}}(2 - \gamma_{\text{IR}}) = 1$ provides better performance to estimate $x_f^c$. As shown in the left panel of Fig. 3, in particular, the differences in the resulting values of $\Delta x_f$ (or equivalently $x_f^c$) among $n = 2, 3, \text{ and 4th orders (red circle)}$ are within $10\%$ level, while those obtained from the conformal expansion of $\gamma_{\text{IR}} = 1$ (blue circle) are about $20 \sim 40\%$. Nevertheless, we find that red and blue circles approach to each other as we go to the higher order in $\Delta x_f$, which is consistent with our expectation. Similar conclusions are drawn for the other theories considered in this work. Therefore, we use the critical number of flavors $N_f^c$ determined from the finite-order critical condition defined in Eq. 2.11 with $n = 4$ as our best estimate for the lower end of the conformal window.

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**Figure 2.** Conformal expansions of $\gamma_{\bar{\psi}\psi, \text{IR}}$ (dashed lines) and $\gamma_{\bar{\psi}\psi, \text{IR}}(2 - \gamma_{\bar{\psi}\psi, \text{IR}})$ (solid lines) at finite order in $\Delta x_f = 11/2 - x_f$ for $SU(N)$ gauge theories coupled to the $N_f$ Dirac flavors of fundamental fermions in the Veneziano large $N$ limit with fixed $x_f = N_f/N$. Purple, green, blue and red colors denote the results obtained by truncating the series expansions at $n = 1, 2, 3 \text{ and } 4$, respectively. The black dotted line corresponds to the critical condition for the loss of conformality.
2.4 Error estimates

Aside from higher order corrections to the critical condition, our approach discussed in the previous section could suffer from various unknown systematic effects like as in other analytical methods. Let us first discuss the nonperturbative effects implied by the inconsistency of the perturbative expansion. The best known example might be the IR renormalon in QCD-like theories which are asymptotically free in the UV but confining in the IR. (See Ref. [45] for a classical review.) The conformal expansion, defined at an IR fixed point, is expected to have no IR renormalons by construction [27]. Other types of divergence, for instance, sourced by instanton anti-instanton configurations, may appear in the expansion, but we expect that these are not as severe as that of renormalons. Below, we account for such nonperturbative effects, if exist, by investigating the coefficients of the conformal expansion and estimate the errors from there.

We start by assuming that the conformal expansion is a divergent asymptotic series. If this is the case, the perturbative series expansion only provides an approximate description at best, and we expect to have the best accuracy to the approximate when it is truncated at a certain optimal order \( k_{\text{opt}} \). To see this, we consider a divergent series which is asymptotic to \( f(z) \) in a domain \( C \) in a complex \( z \) plane

\[
\sum_{j=0}^{J} c_j z^j. \tag{2.13}
\]

Then, for all \( z \) in \( C \) there exist numbers \( A_J \) such that the error is bounded as

\[
\left| f(z) - \sum_{j=0}^{J-1} c_j z^j \right| < A_J z^J. \tag{2.14}
\]

If it assumed that the coefficients factorially increase for \( j \gg 1 \), \( c_j \sim j! t^{-J}_0 \), one often find that \( A_J \sim J! t^{-J}_0 \). It is also found that the best accuracy can be achieved if the series is truncated at the order at which the subsequent higher order becomes larger, i.e. \( j_{\text{opt}}(z) \sim |t_0|/z \). For this optimal truncation one finds that the uncertainty is given by

\[
\delta(z) \sim e^{-|t_0|/z}. \tag{2.15}
\]

It might be also instructive to understand the ambiguity of the asymptotically divergent series using the standard Borel summation technique. The Borel transformation of \( f(z) \) is defined as

\[
Bf(t) = \sum_{j=0}^{\infty} \frac{c_j}{j!} t^j, \tag{2.16}
\]

and its inverse is

\[
f_B(z) = \int_0^{\infty} dt e^{-t} Bf(t z). \tag{2.17}
\]
With the factorally growing coefficients \( c_j \) as before one finds that \( Bf(t) \) has a pole at \( t = |t_0|/z \) on the positive real axis. According to the inverse Borel transformation, this singularity in the Borel plane yields the same result in Eq. 2.15.

Currently, the coefficients of the conformal expansion for \( \gamma_{\text{IR}} \) are known to the 4th order in \( \Delta N_f \). With this limited number of coefficients the search for the optimal truncation discussed above should only be understood in a practical sense. To demonstrate this, again, consider the Veneziano limit of \( SU(N) \) gauge theories with \( N_f \) fundamental fermions. Since we determine the best estimate of \( N_f^{\text{cr}} \) using the critical condition in Eq. 2.11, we focus on the conformal series of \( \gamma_{\text{IR}}(2 - \gamma_{\text{IR}})(\Delta x_f) \). When the series is truncated at \( n = 2, 3 \) and 4, we obtain \( t_0 \approx 13, 32 \) and 6 by examining the coefficients \( \kappa_1, \ldots, 4 \). As expected, \( t_0 \) varies significantly at different orders. Assuming that the highest order result well approximates the large order behavior, we read off that \( j_{\text{opt}} \sim 3 \) near the lower end of the conformal window. This result indicates that the 4th order truncation used for the determination of \( x_f^{\text{cr}} \) is roughly the optimal one.

However, we find that \( j_{\text{opt}} \) is substantially dependent on \( G, N \) and \( R \). In particular, in the cases of \( SU(N) \) and \( Sp(2N) \) for all the values of \( N \) with fundamental fermions, as well as some other theories at small \( N \), the typical values of \( t_0/\Delta x_f^{\text{cr}} \) estimated from the highest-order coefficients are smaller than the truncation order, \( n = 4 \), where \( \Delta x_f^{\text{cr}} = N_f^{\text{AF}} - N_f^{\text{cr}} \). For all the other cases, on the other hand, we find that they are much larger than 4, indicating that the optimal truncation is expected to be at far beyond the currently available highest order of the conformal expansion. By accounting for such large variances in the optimal order we therefore estimate the truncation error of \( N_f^{\text{cr}} \), associated with the ambiguity in the conformal expansion, in a conservative and practical way as

\[
\delta_i^{\pm} = N_f^{\text{cr}, \pm} - N_f^{\text{cr}},
\]  

(2.18)

where \( N_f^{\text{cr}, \pm} \) are solutions to

\[
\gamma_{\text{IR}}(2 - \gamma_{\text{IR}})(\Delta x_f^{n=4}) = 1 \mp \exp \left[ -j_{\text{min}} \right],
\]  

(2.19)

with \( j_{\text{min}} = \text{Min} \left[ 3, t_0/\Delta x_f^{\text{cr}} \right] \) and \( t_0 = 4\kappa_3/\kappa_4 \).

There could be totally different types of nonperturbative effects which may not be captured in the perturbative series expansion. One possibility might be the contribution of certain higher dimensional operators which are expected to be marginal in the onset of chiral phase transition. Then, it may not be fair to compute \( \gamma_{\text{IR}} \) by solely using the conformal expansion, as the IR evolution can be affected by the new relevant operators. However, we believe that the corresponding effects to our approach are not significant since those operators are expected to be weakly relevant within the conformal window as discussed in Sec. 2.1. Nevertheless, if these effects are present, the critical conditions in Eqs. 2.10 and 2.11 will result in two different values of \( N_f^{\text{cr}} \) even when the series expansions are exactly known to \( n \to \infty \). We therefore consider this difference for the uncertainties associated with the augmented critical conditions away from the ones defined with the conformal series.
To compute the difference, we first approximate the closed form of $\gamma_{IR}(\Delta_{N_f})$ by employing Padé approximants (for a review see Ref. [46]). The underlying assumption is that $\gamma_{IR}$ is an analytic function of $\Delta_{N_f}$ and its Taylor series expansion at $\Delta_{N_f} = 0$ is convergent with a finite radius in the complex plane. Provided that the series coefficients are calculated to the maximum order $n$ as in Eq. 2.10, the $[p/q]$ Padé approximant can be given as

$$
\gamma_{IR,[p/q]} = c_1\Delta_{N_f} \left[ \frac{1 + \sum_{j=1}^{p} b_j \Delta_{N_f}^j}{1 + \sum_{k=1}^{q} d_k \Delta_{N_f}^k} \right],
$$

with $p + q + 1 = n$. The coefficients are uniquely determined by matching the terms with those in Eq. 2.10 order-by-order after expanding $\gamma_{IR,[p/q]}$ around $\Delta_{N_f} = 0$. The differences among $[p/q]$ Padé approximants are at $O(\Delta_{N_f}^{(n+1)})$. With $n = 4$, there are four possibilities: $[3/0]$, $[2/1]$, $[1/2]$, and $[0/3]$ Padé approximants, where $[3/0]$ is nothing but the original conformal expansion $\gamma_{IR}(\Delta_{N_f}^{(n=4)})$.

The $[p/q]$ Padé approximant in Eq. 2.20 is a meromorphic function by construction, which is defined in a certain domain of the complex plane including the origin, with $q$ poles. To make it useful to our analysis, the Padé approximant should not have a pole within the bound estimated by which 2-loop beta function loses an IR fixed point. There, several exemplified $SU(N)$ theories coupled to $N_f$ fermions in the representation $R$ for some small integer values of $N$ and the large $N$ limit were extensively investigated, where the suggested condition has been used to find the valid Padé approximants.

In this work, we determine the best Padé approximant by exploiting the large-$N_f$ technique [19, 20] as follows. For illustration purposes consider the Veneziano limit of $SU(N)$ gauge theories coupled to $N_f$ fundamental fermion matter as in Sec. 2.3. We first find all the four possible Padé approximants defined in Eq. 2.20. Also, we compute the anomalous dimension $\gamma_{IR}(\lambda_{f,IR})$ using the leading-order result of the large-$N_f$ expansion in the $\overline{MS}$ scheme [47, 48], where the IR coupling $\lambda_{f,IR}$ is determined from the two lowest terms of the large $N_f$ beta function [19, 20], i.e. $\beta(\lambda_f) \simeq \frac{4T_F}{\pi} \lambda_f^2 + \frac{\beta^{(1)}(\lambda_f)}{N_f} = 0$ with $\lambda_f = N_f \alpha/4\pi$.\footnote{We refer the reader to the Appendix B of Ref. [28] for comprehensive results of $\gamma_{\psi\psi}$ and $\beta$ at $O(x_f^{-1})$ in the large $x_f$ expansion.} For $x_f \leq x_f^{AF}$ both calculations of $\gamma_{IR}$ are physically well defined, but the large $x_f$ results are not reliable as they receive sizable corrections from higher order terms at $O(x_f^{-2})$. If $x_f \gg x_f^{AF}$, on the other hand, the Padé approximants, which are supposed to have analytic continuations of the IR fixed point with the perturbative origin by construction, do not correctly describe the nonanalytical behavior at $|\lambda_{f,IR}| \gg 1$ shown by the large $x_f$ expansion. Note that for $x_f > x_f^{AF}$ the IR fixed points are found in the nonunitary regime with $\alpha_{IR} < 0$.

Fortunately, there exists a limited window in $x_f$, $10 \lesssim x_f \lesssim 14$, where the large $N_f$ expansion is under control and analytically continued [20]. We therefore compare the two
Figure 3. In the left panel, we present the resulting values of $\Delta x_f = x_f^{AF} - x_f^{cr}$ obtained from the finite-order critical conditions in Eqs. 2.10 and 2.11 truncated at nth order, denoted by blue and red dots, respectively. In the same figure, we also present the results obtained by using the [0/3] Padé approximants for the critical conditions in Eq. 2.9. In the right panel, we show the results of the anomalous dimension of a fermion bilinear at an IR fixed point in the vicinity of which asymptotic freedom is lost. Red dashed line represents for the conformal expansion truncated at the 4th order, black solid line for the leading-order large $N_f$ expansion, and purple, blue and green dashed lines for [2/1], [1/2] and [0/3] Padé approximants.

resulting values of $\gamma_{IR}$ over this region and find the best Padé approximant from their qualitative agreement. In the right panel of Fig. 3, we show the results of $\gamma_{IR}$: the black solid line is for the large $N_f$ expansion, while the red, blue and green dashed lines are for [3/0], [1/2] and [0/3] Padé approximants, respectively. As seen in the figure, [0/3] Padé approximant is in good agreement with the large $N_f$ result over the aforementioned range. Although we do not present the results here, this qualitative picture holds for all the other finite and infinite $N$ theories considered in the next section. Therefore, we use [0/3] Padé approximant for our best estimate of $\gamma_{IR}$, [p/q] throughout this work.\(^\text{10}\)

We carry out the same analysis for $\gamma_{IR}(2 - \gamma_{IR})$, where we also find that [0/3] Padé approximants provide the best results as they qualitatively agree with the large $N_f$ results over the range in $x_f$ mentioned before. Furthermore, as shown in the left panel in Fig. 3, it turns out that [0/3] Padé approximants obtained from both critical conditions result in better agreement than those at finite order. We therefore determine the critical number of flavors $N_f^{cr1,[0/3]}$ and $N_f^{cr2,[0/3]}$ from both critical conditions, $\gamma_{IR,[0/3]}(\Delta N_f) = 1$ and $[\gamma_{IR}(2 - \gamma_{IR})][0/3] (\Delta N_f) = 1$, respectively, and take the difference to $N_f^{cr}$ as our estimate for the uncertainty related to the inconsistency of the critical condition:

\[
\delta_2^+ = N_f^{cr2,[0/3]} - N_f^{cr} \quad \text{and} \quad \delta_2^- = N_f^{cr1,[0/3]} - N_f^{cr},
\]

\(^{10}\) In most theories considered in this work [0/3] Padé approximant also satisfied the condition suggested in Ref. [12]. There are a few cases, however, that it has a pole slightly within the conformal boundary estimated by the 2-loop beta function analysis. Nevertheless, we believe that our conclusion does not conflict with the restriction of which the Padé approximant must not have a pole in the conformal window, because in general the would-be conformal window is expected to be narrower than the one determined from that the 2-loop IR coupling runs to infinity.
where we recall that $N_f^{cr}$ is obtained from the critical condition defined in Eq. 2.11 with $n = 4$.

### 3 Conformal window in nonsupersymmetric gauge theories

In this section, we present our main results for the conformal window in nonsupersymmetric gauge theories coupled to fermionic matter using the strategies discussed in the previous section. In particular, we consider non-abelian gauge theories with $SU(N)$, $Sp(2N)$ and $SO(N)$ gauge groups and fermionic matter fields in the fundamental (F), adjoint (Adj), two-index symmetric (S2) and antisymmetric (AS) representations. In the case of $SO(N)$, we also consider the spinorial (S) representation. The group invariants necessary for the computation of $k_1, ..., 4$ in the conformal expansion of $\gamma_{\text{IR}}$ in Eq. 2.10 are basically the same with the ones used for calculations of $\beta(\alpha)$ and $\gamma(\alpha)$ at the 4th order in the coupling expansion [42, 43, 49]. The results have further been generalized to two-index and spinorial representations using the general fourth-order Casimir invariants for simple Lie groups [50, 51], where the detailed discussions and notations are found in the Appendices of Refs. [11, 52] and [53], respectively. (See also Tables in the Appendix A of Ref. [21] for the summary of the resulting expressions relevant to this work.)

The lower end of the conformal window has been estimated by a number of different analytical methods. Among those known in the literature, we consider the following three scheme-independent analytical approaches for a comparison:

- The 2-loop beta function loses the BZ fixed point when $\alpha_{\text{BZ}} = -\frac{4\pi b_1}{b_2}$ runs to infinity, which yields that
  \[ N_f^{\text{cr,2-loop}} = \frac{17C_2(G)^2}{T(R)[10C_2(G) + 6C_2(R)]}, \]  \[ (3.1) \]

- The SD analysis in the ladder approximation suggests that the loss of the conformality happens if the IR coupling satisfies the condition, $\alpha_{\text{IR}} = \alpha_{\text{cr}}$ with $\alpha_{\text{cr}} = \frac{\pi}{3C_2(R)}$. Conventionally the IR coupling is approximated by $\alpha_{\text{BZ}}$ and one finds
  \[ N_f^{\text{cr,SD}} = \frac{C_2(G)(17C_2(G) + 66C_2(R))}{T(R)(10C_2(G) + 30C_2(R))}. \]  \[ (3.2) \]

- A closed form of the beta function was proposed in Ref. [54], which is scheme-independent in the sense that it involves the first two universal coefficients of $\beta(\alpha)$, the first universal coefficient of $\gamma_{\bar{\psi}\psi}(\alpha)$, and a physical input for $\gamma_{\bar{\psi}\psi,\text{IR}}$.\footnote{In Ref. [55] a modified version of the all-order beta function was also proposed. By setting $\gamma_{\bar{\psi}\psi,\text{IR}} = 1$, even though we do not present the results in this paper, we find that the resulting values of $N_f^{\text{cr,2-loop}}$ for the fundamental and spinorial representations, while they are roughly lying in the middle between $N_f^{\text{cr,2-loop}}$ and our results.}

To be consistent with the critical condition used for this work, we set $\gamma_{\bar{\psi}\psi,\text{IR}} = 1$. It leads to a simple expression for the critical number of flavors
  \[ N_f^{\text{cr,BF}} = \frac{11C_2(G)}{6T(R)}. \]  \[ (3.3) \]

For some theories at finite $N$ we also compare our results with recent lattice results.
3.1 Large $N$ limit

Before we present the results at finite $N$, let us first consider an appropriate large $N$ limit in which the fermionic flavors are still relevant to the dynamics. For the fundamental flavors we take the Veneziano limit, i.e. both $N$ and $N_f$ are infinite while keeping $x_f = N_f/N$ finite. For $SU(N)$ and $Sp(2N)$ we find

$$\left[ x_f^\text{cr}, x_f^\text{AF} \right] = \left[ 3.39^{+0.10}_{-0.09} +0.08, 5.5 \right].$$  \hspace{1cm} (3.4)

Throughout this work we denote the errors to $x_f^\text{cr}$ and $N_f^\text{cr}$ by $\delta_1$ and $\delta_2$, respectively. For $SO(N)$ the conformal window is basically the same as that of $SU(N)$ except that now the Veneziano limit is defined with Weyl fermions, $x_{Wf} = N_{Wf}/N$,

$$\left[ x_{Wf}^\text{cr}, x_{Wf}^\text{AF} \right] = \left[ 3.39^{+0.10}_{-0.09} +0.08, 5.5 \right].$$  \hspace{1cm} (3.5)

For a comparison we also present the resulting values of $x_f^\text{cr}$ ($x_{Wf}^\text{cr}$) defined in Eqs. 3.1, 3.2, and 3.3

$$x_f^{\text{cr,2-loop}} = \frac{34}{13}, \quad x_f^{\text{cr,SD}} = 4, \quad \text{and} \quad x_f^{\text{cr,BF}} = \frac{11}{3}.$$

(3.6)

In the cases of the adjoint and two-index symmetric and antisymmetric representations, we take the ’t Hooft large $N$ limit, i.e. $N \to \infty$ with fixed $N_f$. For the adjoint representation the results are same for all the three gauge groups, which is also true for the other analytical methods mentioned above, and we find

$$\left[ N_f^\text{cr}, N_f^\text{AF} \right] = \left[ 1.90^{+0.03}_{-0.03} <0.01, 2.75 \right],$$

(3.7)

or equivalently,

$$\left[ N_{Wf}^\text{cr}, N_{Wf}^\text{AF} \right] = \left[ 3.79^{+0.07}_{-0.07} +0.01, 5.5 \right].$$

(3.8)

Using Eqs. 3.1, 3.2, and 3.3, we also find that

$$N_f^{\text{cr,2-loop}} = \frac{17}{16}, \quad N_f^{\text{cr,SD}} = \frac{83}{40}, \quad \text{and} \quad N_f^{\text{cr,BF}} = \frac{11}{6}.$$

(3.9)

For both two-index symmetric and antisymmetric representations of $SU(N)$ the conformal windows are exactly twice larger than that for the adjoint representation

$$\left[ N_f^\text{cr}, N_f^\text{AF} \right] = \left[ 3.79^{+0.07}_{-0.07} +0.01, 5.5 \right].$$

(3.10)

In the case of $Sp(2N)$ with fermions in the antisymmetric representation, the conformal window is equivalent to that for the adjoint representation. Similarly, for $SO(N)$ the conformal window for the symmetric representation is same with that for the adjoint representation. Note that in these theories the other two-index representations are identical to the adjoint representation by construction. Analogously, the other analytical calculations for the two-index representations yield the same results of Eq. 3.9 up to a factor of two.
Figure 4. The boundary between conformal and chirally broken phases in $SU(N)$ gauge theories with $N_f$ Dirac flavors of fermion in the fundamental (top-left), adjoint (top-right), two-index antisymmetric (bottom-left) and symmetric (bottom-right) representations. The black solid line is estimated from the finite-order critical condition in Eq. 2.11 with $n = 4$, where red and blue bands denote the systematic errors computed according to Eqs. 2.18 and 2.21, respectively. Dotted, dashed, and dot-dashed lines are for the analytical results estimated by the 2-loop beta function, the truncated Schwinger-Dyson, and the all-order beta function analyses in Eqs. 3.1, 3.2, and 3.3.

| $N$ | $F$ | $\text{Adj}$ | $\text{AS}$ | $S2$ |
|-----|-----|--------------|-------------|------|
| 2   | $6.22^{+1.32}_{-1.01} +0.31_{-0.36}$ | $2.75$ | $N/A$ | $1.92^{+0.35}_{-0.39} +0.03_{-0.07}$ |
| 3   | $9.79^{+0.94}_{-0.82} +0.31_{-0.36}$ | $9.79^{+1.31}_{-0.82} +0.31_{-0.36}$ | $7.18^{+1.31}_{-0.36}$ | $2.31^{+0.04}_{-0.01} +0.04_{-0.01}$ |
| 4   | $13.29^{+1.77}_{-1.01} +0.44_{-0.36}$ | $1.90^{+0.03}_{-0.10}$ | $7.18^{+1.31}_{-0.36}$ | $2.57^{+0.04}_{-0.10}$ |
| 5   | $16.74^{+0.70}_{-0.69} +0.51_{-0.36}$ | $1.90^{+0.03}_{-0.10}$ | $6.06^{+0.11}_{-0.38}$ | $2.74^{+0.05}_{-0.15}$ |
| 6   | $20.18^{+0.79}_{-0.83} +0.51_{-0.36}$ | $1.90^{+0.03}_{-0.11}$ | $5.50^{+0.10}_{-0.36}$ | $2.89^{+0.05}_{-0.16}$ |
| 7   | $23.60^{+0.84}_{-0.97} +0.59_{-0.36}$ | $1.90^{+0.03}_{-0.11}$ | $5.16^{+0.09}_{-0.34}$ | $2.99^{+0.05}_{-0.16}$ |
| 8   | $27.01^{+0.91}_{-0.83} +0.66_{-0.36}$ | $1.90^{+0.03}_{-0.11}$ | $4.94^{+0.09}_{-0.32}$ | $3.07^{+0.05}_{-0.17}$ |
| 9   | $30.42^{+0.99}_{-0.97} +0.71_{-0.36}$ | $1.90^{+0.03}_{-0.11}$ | $4.78^{+0.09}_{-0.31}$ | $3.14^{+0.05}_{-0.17}$ |
| 10  | $33.83^{+1.07}_{-1.02} +0.82_{-0.36}$ | $1.90^{+0.03}_{-0.11}$ | $4.65^{+0.09}_{-0.30}$ | $3.17^{+0.05}_{-0.18}$ |

Table 1. Conformal window of $SU(N)$ gauge theories coupled to fermion matter in the fundamental (F), adjoint (Adj), anti-symmetric (AS), and symmetric (S2) representations. The lower and upper bounds, denoted by $[N_f^L, N_f^U]$, correspond to which conformality and asymptotic freedom are lost. The first and second errors to $N_f^L$ are computed according to Eqs. 2.18 and 2.21, respectively.
3.2 $SU(N)$ gauge theory with $N_f$ Dirac fermions in various representations

In Fig. 4 we present our results for the lower edge of the conformal window in $SU(N)$ gauge theories coupled to $N_f$ Dirac fermions in the fundamental, adjoint, antisymmetric and symmetric representations, where the resulting values of $N_{\text{cr}}^f$ are denoted by black solid lines. Note that we take both $N$ and $N_f$ as continuous variables. In each figure, red and blue bands denote the errors, $\delta_1$ and $\delta_2$, defined in Eqs. 2.18 and 2.21, respectively. We recall that these errors should not be taken simultaneously since the underlying assumptions for the error estimates are incompatible to each other as discussed in Sec. 2.4. For the integer values of $N$ ranged over $2 \leq N \leq 10$ we present the explicit values of $N_{\text{cr}}^f$ with errors in Table 1, where the values of $N_{\text{AF}}^f$ are also presented.

As shown in the figures, $\delta_2$ persists to be sizable for all the values of $N$ at $\sim 6\%$ level at most. On the other hand, $\delta_1$ is relatively large at small $N$, but comparable or smaller than $\delta_2$ at large $N$. If we concern the truncation error $\delta_1$, such a result implies that $SU(N)$ theories at small $N$ receive significant nonperturbative corrections near the lower end of the conformal window, which result in the large ambiguity of the perturbative conformal series expansion.

We compare our results to other analytical methods in the figures: dotted lines are for the 2-loop beta function analysis, dashed lines for the traditional SD method, and dot-dashed lines for the all-order beta function with $\gamma_{\text{IR}} = 1$. We find that our results are in between $N_{\text{cr},2\text{-loop}}^f$ and $N_{\text{cr,SD}}^f$, and more or less comparable to $N_{\text{cr,BF}}^f$, except the fundamental representation at large $N$, if the errors $\delta_2$ are concerned. Note that the adjoint and symmetric representations are identical for $N = 2$, while the antisymmetric and fundamental representations are identical for $N = 3$

Studying the infra-red dynamics of an interacting (near) conformal theory has also been a rich subject of lattice gauge theories, because the corresponding IR coupling is typically in the strong coupling regime and thus it requires reliable nonperturbative techniques. However, it is a highly nontrivial task to investigate the deep IR regime of the would-be conformal theories using lattice simulations due to the large finite-size effects as expected from the fact that the correlation length diverges at an IR fixed point. Nevertheless, in recent years various numerical techniques have been developed to tackle such a problem, and turned out to be successful as they revealed various aspects of the (near) conformal dynamics from first principles. (See Sec. V in [56] for a brief summary of the recent developments and challenges along this direction.)

In particular, $SU(3)$ theories with many fundamental flavors have been extensively studied, because they are not only easily utilizing the state-of-the-art lattice techniques justified by successfully simulating QCD, but also provide useful benchmark studies for more interesting UV models in the context of physics beyond the standard model. Although it is not yet conclusive, the most recent lattice results suggest that $N_f = 12$ is conformal [57], 8 is chirally broken but nearly conformal [59], and 10 is likely conformal but controversial [60–63]. In the case of $SU(2)$, much evidence has been found that $N_f = 4$

\footnote{See also e.g. [58] for a different point of view.}
is chirally broken while 6 is likely conformal, e.g. see [64] and references therein. As shown in Table 1, our results are in excellent agreement with these lattice results for both cases.

Besides the fundamental representation, adjoint and two-index representations have also been studied by the means of nonperturbative lattice calculations. Lattice studies of \( SU(2) \) with adjoint fermion matter find evidence for that \( N_f = 2, 3/2 \) are conformal, \( 1/2 \) is chirally broken, and 1 is likely conformal (e.g. see [65] and references therein), which is somewhat different to what we have found. However, we note that the adjoint \( SU(2) \) turned out to receive significant nonperturbative corrections, as indicated by the large truncation errors shown in the right-top panel of Fig. 4 and Table 1. For this specific case, it would be interesting to further investigate the nonperturbative effects in details by improving the error analyses. \( SU(3) \) theory with 2 symmetric (sextet) Dirac fermions has also been studied extensively by lattice techniques, as it has been shown to exhibit near conformal behaviors like as in the 8 fundamental-flavor \( SU(3) \) theories (for the most recent progress see Ref. [66] and references therein). Our results strongly support such lattice results.

We summarize our findings on the conformal window of \( SU(N) \) gauge theories with \( N_f \) flavors of fermion in the fundamental, adjoint, two-index symmetric and antisymmetric representations in Fig. 5.

### 3.3 \( Sp(2N) \) gauge theory with \( N_f \) Dirac fermions in various representations

In the case of \( Sp(2N) \) we consider the fundamental, adjoint (= two-index symmetric) and two-index antisymmetric representations. We present our results with black solid lines in Fig. 6, where the red and blue bands denote the errors, \( \delta_1 \) and \( \delta_2 \), defined in Eqs. 2.18 and 2.21, respectively. Again, we take \( N \) and \( N_f \) as continuous variables to obtain the results.
Figure 6. The boundary between conformal and chirally broken phases in $Sp(2N)$ gauge theories with $N_f$ Dirac flavors of fermion in the fundamental (top-left), adjoint (top-right) and two-index antisymmetric (bottom) representations. The black solid line is estimated from the finite-order critical condition in Eq. 2.11 with $n = 4$, where red and blue bands denote the systematic errors computed according to Eqs. 2.18 and 2.21, respectively. Dotted, dashed, and dot-dashed lines are for the analytical results estimated by the 2-loop beta function, the truncated Schwinger-Dyson, and the all-order beta function analyses in Eqs. 3.1, 3.2, and 3.3. in the figure. For several small integer values of $N = 2, \cdots, 6$, we present the resulting values of $N_f^c$ with errors and $N_f^{AF}$ in Table 2. Note that $Sp(2) = SU(2)$. For a comparison, we also present the results of other analytical approaches defined in Eqs. 3.1, 3.2 and 3.3. The generic trend is similar to what we found for $SU(N)$, except that the truncation errors $\delta_1$ are much larger than $\delta_2$ and those of $SU(N)$ in the fundamental representation at the same values of $N$.

Nonperturbative lattice studies of $Sp(2N)$ gauge groups are barely found in the literature. Only recently has a research program for $Sp(4)$ lattice theories with fermions in the fundamental and antisymmetric representations begun by aiming to explore the composite dynamics of the electroweak symmetry breaking and composite dark matter [67–69]. Our results suggest that $Sp(4)$ theory would exhibit near conformal behaviour if it couples to $8 \sim 9$ fundamental or $4 \sim 5$ antisymmetric flavors of fermion.\footnote{A phenomenologically interesting minimal $Sp(4)$ composite Higgs model requires to contain 2 fundamental and 3 antisymmetric fermions [70]. In this case, analytical studies using the same strategy used for this work, but truncated at the 3rd order in the conformal expansion of $\gamma_{IR}$ for multiple representations,}
Table 2. Conformal window of $Sp(2N)$ gauge theories with fermion matter in the fundamental (F), adjoint (Adj) and anti-symmetric (AS) representations. The lower and upper bounds, denoted by $[N_{cr}^F, N_{AF}^F]$, correspond to which conformality and asymptotic freedom are lost. The first and second errors to $N_{cr}^F$ are computed according to Eqs. 2.18 and 2.21, respectively.

| $N$ | F     | Adj    | AS     |
|-----|-------|--------|--------|
| 2   | $[9.68 \pm 1.93, -0.42, 16.5]$ | $[1.91 \pm 0.08, -0.09, 2.75]$ | $[5.46 \pm 0.09, -0.09, 8.25]$ |
| 3   | $[13.09 \pm 1.56, -0.34, 22.0]$ | $[1.90 \pm 0.03, 0.01, 2.75]$ | $[3.66 \pm 0.07, -0.23, 5.5]$ |
| 4   | $[16.50 \pm 1.61, -0.49, 27.5]$ | $[1.90 \pm 0.03, 0.01, 2.75]$ | $[3.07 \pm 0.06, -0.15, 4.58]$ |
| 5   | $[19.90 \pm 1.65, -0.03, 33.0]$ | $[1.90 \pm 0.03, -0.10, 2.75]$ | $[2.77 \pm 0.05, -0.18, 4.13]$ |
| 6   | $[23.30 \pm 1.70, -0.77, 38.5]$ | $[1.90 \pm 0.03, -0.11, 2.75]$ | $[2.60 \pm 0.05, -0.16, 3.85]$ |

Figure 7. Conformal window of $Sp(2N)$ gauge theory with $N_f$ Dirac flavors of fermion in the fundamental (blue), adjoint (red), and antisymmetric (green) representations. The upper bound is determined by the perturbative beta function in the onset of the loss of an asymptotic freedom, while the lower bound is estimated from the finite-order critical condition truncated at the 4th order of the conformal expansion defined in Eq. 2.11.

$Sp(2N)$ at large $N$ are also being pursued by the same research group [71], we should note that $Sp(6)$ and $Sp(8)$ with 3 antisymmetric fermions would be good candidates for near conformal theories.

We summarize our findings on the conformal window of $Sp(2N)$ gauge theories coupled to $N_f$ flavors of fermion in the fundamental, adjoint and two-index antisymmetric representations in Fig. 7.
Figure 8. The boundary between conformal and chirally broken phases in \( SO(N) \) gauge theories with \( N_{Wf} \) Weyl flavors of fermion in the fundamental (top-left), adjoint (top-right), two-index symmetric (middle-left) representations. In the case of the spinorial representation, we show the results for odd (middle-right) and even (bottom) integer values of \( N \), separately. The black solid line is estimated from the finite-order critical condition in Eq. 2.11 with \( n = 4 \), where red and blue bands denote the systematic errors computed according to Eqs. 2.18 and 2.21, respectively. Dotted, dashed, and dot-dashed lines are for the analytical results estimated by the 2-loop beta function, the truncated Schwinger-Dyson, and the all-order beta function analyses in Eqs. 3.1, 3.2, and 3.3.

3.4 \( SO(N) \) gauge theory with \( N_{Wf} \) Weyl fermions in various representations

For \( SO(N) \) gauge theories we consider \( N_{Wf} \) Weyl fermions in the spinorial representation in addition to the fundamental, adjoint (= antisymmetric) and symmetric representations. suggest that the model resides slightly outside the conformal window [21].
Table 3. Conformal window of $SO(N)$ gauge theories with fermion matter in the fundamental (F), adjoint (Adj), two-index symmetric (S2), and spinorial (S) representations. The lower and upper bounds, denoted by $[N^c_{\text{F}}, N^w_{\text{F}}]$, correspond to which conformality and asymptotic freedom are lost. The first and second errors to $N^c_{\text{F}}$ are computed according to Eqs. 2.18 and 2.21, respectively.

We also restrict our attention to $N \geq 6$ since the results for the smaller values of $N$ could be deduced from $SU(2)$ and $Sp(4)$ gauge theories using the fact that $SO(3) \sim SU(2)$, $SO(4) \sim SU(2) \times SU(2)$ and $SO(5) \sim Sp(4)$ for which only even numbers of $N_W$ are allowed to avoid a Witten anomaly [72].

We present our results for the lower edge of the conformal window by black solid lines in Fig. 8. The blue and red bands denote the errors, $\delta_1$ and $\delta_2$, defined in Eqs. 2.18 and 2.21, respectively. Again, the results in the figures are obtained by treating $N$ and $N_f$ as continuous variables, but the physical system should take the integer values of $N$. In particular, the results for the spinorial representation displayed in the right-middle and the bottom panels of Fig. 8 are physical only at odd and even integer values of $N$, respectively.

Our results are further compared to other analytical approaches, where the dotted lines are the results obtained from the loss of IR fixed point in 2-loop beta function, the dashed lines from the traditional Schwinger-Dyson analysis, and the dot-dashed lines from the all-order beta function with $\gamma_R = 1$. Similar to $SU(N)$ and $Sp(2N)$, our results are comparable to $N^c_{\text{F},BF}$, but smaller than $N^c_{\text{F},SD}$ and larger than $N^c_{\text{F},2\text{-loop}}$, respectively.

For several integer values of $N$ over the region, 6 $\leq N \leq$ 12, we report our results in Table 3. Except in the case of the spinorial representation for $SO(6)$, $\delta_2$ is larger than $\delta_1$ for all the considered theories. For the adjoint representation the conformal windows are identical to those of $SU(N)$ gauge theories. In the case of $SO(6)$, the result of the spinorial representation is identical to that of the fundamental representation of $SU(4)$ up to a factor of 2, as expected from $SO(6) \sim SU(4)$.

We summarize our findings on the conformal window of $SO(N)$ gauge theories coupled to $N_W$ flavors of fermion in the fundamental, adjoint, two-index symmetric and spinorial representations in Fig. 9.

4 Conclusion

We have investigated the conformal window of asymptotically free $SU(N)$, $Sp(2N)$ and $SO(N)$ gauge theories coupled to $N_f$ flavors of fermion in various representations, where
Figure 9. Conformal window of $SO(N)$ gauge theory with $N_{Wf}$ Weyl flavors of fermion in the fundamental (blue), adjoint (red), symmetric (brown), spinorial (purple and green are for odd and even integer values of $N$, respectively) representations. The upper bound is determined by the perturbative beta function in the onset of the loss of an asymptotic freedom, while the lower bound is estimated from the finite-order critical condition truncated at the 4th order of the conformal expansion defined in Eq. 2.11.

their appropriate large $N$ limits are also considered. The upper end of the conformal window is identical to which asymptotic freedom is lost, and the corresponding number of flavors $N^\text{AF}_f$ is determined by the perturbative RG beta function as usual. To find the lower end of the conformal window, in this work we have adopted the conjectured critical condition on the anomalous dimension of a fermion bilinear at an IR fixed point, $\gamma_{\bar{\psi}\psi,\text{IR}}$, in which the theory is expected to lose the conformality due to the dynamically generated scale and fall into a chirally broken phase. In addition, we calculate $\gamma_{\bar{\psi}\psi,\text{IR}}$ by employing the Banks-Zaks conformal expansion whose expansion parameter $\Delta_{N_f} = N^\text{AF}_f - N_f$, as well as the coefficient at each order, is free from the renormalization scheme. Following the strategy proposed in Ref. [21], we particularly use the finite-order definition of the critical condition in Eq. 2.11 up to the 4th order, the highest order available up to date, to determine the critical number of flavors, $N^\text{cr}_f$, which corresponds to the onset of the conformality lost.

The highlight of this work lies in the estimation of the uncertainties in the resulting values of $N^\text{cr}_f$ by treating the following two scenarios separately: the conformal series expansion is either convergent or divergent asymptotic. For the latter we have searched for the optimal truncation by assuming the factorial growth of the coefficients at larger order. Such divergent asymptotic behavior is related to the singularity in the Borel plane. We found that the truncation order used for the determination of $N^\text{cr}_f$ was roughly the optimal one for the fundamental $SU(N)$ theories in the Veneziano limit. However, it turned out
that the optimal truncation order estimated from the coefficients of the two largest orders was highly dependent on the details of the theories considered in this work. We therefore estimate the truncation error $\delta_1$ only in a practical sense, as defined in Eqs. 2.18 and 2.19, but to be conservative enough to account for potentially severe nonperturbative effects implied by the inconsistency of the series expansion. In the former case, on the other hand, we first approximate the closed forms of $\gamma_{\bar{\psi} \psi, \text{IR}}$ and $\gamma_{\bar{\psi} \psi, \text{IR}}(2 - \gamma_{\bar{\psi} \psi, \text{IR}})$ using the Padé approximation. The best approximant has been determined by comparing its qualitative behavior to the large $N_f$ expansion in a certain range of $N_f$ for which both calculations are supposed to be under control. The two definitions of the critical condition, $\gamma_{\bar{\psi} \psi, \text{IR}} = 1$ and $\gamma_{\bar{\psi} \psi, \text{IR}}(2 - \gamma_{\bar{\psi} \psi, \text{IR}}) = 1$, approximated by the best Padé approximants, generally lead to different values of $N_f^{\text{cr}}$. And we take the difference as the uncertainty of our analysis denoted by $\delta_2$. This uncertainty may simply reflect the inexactness of the Padé approximation constructed from limited number of coefficients in the conformal expansion, but it is also plausible that it is due to the incorrectness of the critical condition potentially affected by newly emerging marginal operators which become relevant to the RG flow in the IR in the vicinity of the chiral phase transition.

In the large $N$ limit we basically found two different results for the conformal window, up to an overall factor of two, thanks to the large $N$ universality: one for the fundamental representation in the Veneziano limit, and the other for the adjoint and two-index representations in the ’t Hooft limit. Our results show that for the former the two distinct errors, $\delta_1$ and $\delta_2$, are comparable to each other, but for the latter $\delta_2$ is about three times larger than $\delta_1$. At finite, but not too small, $N$, we find that $\delta_2$ is much smaller than $\delta_1$ except for the fundamental $SU(N)$ and $Sp(2N)$ theories. For the very small integer values of $N$ the truncation error $\delta_2$ turned out to be much larger than $\delta_1$ and prevented us from narrowing down the location of the lower edge of the conformal window. In these small $N$ theories, more dedicated studies with improved error analysis techniques and a better understanding of the critical condition are highly desired to further discriminate their IR nature near the phase boundary.

Among various analytical methods found in the literature, we have considered three widely used ones to be compared with our results. For all the theories considered in Sec. 3, we find that our values of $N_f^{\text{cr}}$ are typically larger than those estimated by the 2-loop beta function, but smaller than those by the conventional Schwinger-Dyson analysis. Within the uncertainties estimated in this work, our results are roughly consistent with those determined by the all-order beta function analysis with $\gamma_{\bar{\psi} \psi, \text{IR}} = 1$. Compared with recent lattice results, we find that our results of $N_f^{\text{cr}}$ are in excellent agreement for $SU(3)$ with fermions in the fundamental and symmetric representations, and for $SU(2)$ with fermions in the fundamental representation. In the case of the adjoint $SU(2)$ our estimation for $N_f^{\text{cr}}$ is somewhat larger than what has been found in lattice studies. Again, such a discrepancy may be understood from the fact that for this specific case with very small $N = 2$ our result could receive significant nonperturbative corrections as indicated by the large ambiguity of the perturbative conformal expansion, e.g. see the top-right panel of Fig. 4. For $Sp(4)$ theories coupled to fundamental and antisymmetric fermions, which have received considerable attention in recent years, we predict that $N_f^{\text{cr}} = 9 \sim 11$ and
5 ∼ 6, respectively. Besides the lattice methods, it would be also interesting to compare our results to Bootstrap techniques, as the recent calculation of $\gamma_{\bar{\psi}\psi,\text{IR}}$ for the $N_f = 12$ fundamental $SU(3)$ theory showed a promising result comparable to various lattice and analytical methods [73].

In this work we have continued our journey to the end of conformal window in non-supersymmetric and asymptotically free gauge theories. To reach there we assume that the loss of conformality in the onset of chiral phase transition is featured by the critical condition to $\gamma_{\bar{\psi}\psi}$ in Eq. 2.9. We also assume that $\gamma_{\bar{\psi}\psi,\text{IR}}$ can perturbatively be calculated by the scheme-independent conformal expansion over the whole range of the conformal window. The determination of $N_f^{\text{CT}}$ based on these assumptions, however, is challenged by our limited understanding on the nature of chiral phase transition and the potentially sizable nonperturbative effects. To address these issues, we have tried to estimate the errors in $N_f^{\text{CT}}$ in two folds, as discussed in details in Sec. 2.4, and argue that part of such systematic effects are well captured by the errors. In several finite $N$ theories, we find that our values of $N_f^{\text{CT}}$ are consistent with the recent lattice results. In this respect, we believe that our approach to the determination of the conformal window, along with the error analyses, would provide a useful analytical supplement and a guidance to more rigorous and better controlled nonperturbative calculations of (near)-conformal theories.

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