The Integrals of Motion for
the Deformed Virasoro Algebra

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Dedicated to Professor Masaki Kashiwara on the occasion on the 60th birthday

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Abstract

We explicitly construct two classes of infinitely many commutative operators in terms of the deformed Virasoro algebra. We call one of them local integrals and the other nonlocal one, since they can be regarded as elliptic deformations of the local and nonlocal integrals of motion obtained by V. Bazhanov, S. Lukyanov and Al. Zamolodchikov [1].
1 Introduction

The purpose of this paper is to construct two classes of infinitely many commutative operators, in terms of the deformed Virasoro algebra. Let us recall some facts about soliton equation. The classical sine-Gordon equation

\[ \partial_t \partial \phi(t, \tau) = e^{\phi(t, \tau)} - e^{-\phi(t, \tau)}, \]  

(1.1)
is one of the simplest example of the Toda field theory. One of the authors and E. Frenkel [4] considered the so-called local integrals of motion \( I^{(cl)} \) for the classical sine-Gordon theory

\[ \{ I^{(cl)}, H^{(cl)} \}_{P.B.} = 0, \]  

(1.2)
where \( H^{(cl)} = \frac{1}{2} \int (e^{\phi(t)} + e^{-\phi(t)}) dt \) is the Hamiltonian. They showed the existence of infinitely many commutative integrals of motion by a cohomological argument, and showed that they can be regarded as the conservation laws for the KdV equation

\[ \partial_t W(\tau, t) = \partial^3_t W(\tau, t) + 3W(\tau, t) \partial_t W(\tau, t). \]  

(1.3)
They gave similar construction for the Toda field theory associated with the root system of finite and affine type.

In [4] they constructed the quantum deformation of the local integrals of motion. In other words they showed the existence of quantum deformation of the conservation laws of the KdV equation. After quantization Gel’fand-Dickij bracket \( \{ , \}_{P.B.} \) for the second Hamiltonian structure of the KdV, gives rise to the Virasoro algebra

\[ [L_m, L_n] = (m - n)L_{m+n} + \frac{c_{\text{CFT}}}{12} \delta_{m+n,0}. \]  

(1.4)
V. Bazhanov, S. Lukyanov, Al. Zamolodchikov [1] constructed field theoretical analogue of the commuting transfer matrix \( T(z) \), acting on the irreducible highest weight module of the Virasoro algebra. They constructed this commuting transfer matrix \( T(z) \) as the trace of the monodromy matrix associated with the quantum affine symmetry \( U_q(\widehat{sl}_2) \), and commutatin relation \( [T(z), T(w)] = 0 \) is a direct consequence of the Yang-Baxter relation. The coefficients of the asymptotic expansion of the operator log \( T(z) \) at \( z \to \infty \), produce the local integrals of motion for the Virasoro algebra, which recover the conservation laws.
of the KdV equation in the classical limit $c_{CFT} \to \infty$. They call the coefficients of the 
Taylor expansion of the operator $T(z)$ at $z = 0$, the nonlocal integrals of motion for the 
Virasoro algebra. They have explicit integral representation in terms of the screening 
currents.

The purpose of this paper is to construct the elliptic version of the integrals of motion 
given by V. Bazhanov, S. Lukyanov, Al. Zamolodchikov [1]. Their construction is based on 
the free field realization of the Borel subalgebra $B_\pm$ of $U_q(\hat{sl}_2)$ in terms of the screening 
currents, which cannot be extended to the full symmetry $U_q(\hat{sl}_2)$. By using this realization 
they construct the monodromy matrix as the image of the universal $R$-matrix $\tilde{R} \in B_+ \otimes B_-$, and derive the transfer matrix $T(z)$ as the trace of this monodromy matrix. Because 
the universal $R$-matrix $\tilde{R}$ of the elliptic quantum group does not exist in the tensor of 
the Borel subalgebras $B_+ \otimes B_-$, it is impossible to construct the elliptic deformation of 
the transfer matris $T(z)$ as the same manner as [1]. Hence our method of construction 
should be completely different from those of [1]. Instead of considering the transfer matrix 
$T(z)$, we directly give the integral representations of the integrals of motion $I_n, G_n$ for 
the deformed Virasoro algebra. The commutativity of the integrals of motion are not 
understood as a direct consequence of the Yang-Baxter equation. They are understood 
as a consequence of the commutative subalgebra of the Feigin-Odesskii algebra [5].

We would like to mention about two important degenerating limits of the deformed 
Virasoro algebra. One is the CFT-limit [1] and the other is the classical limit [6]. In the 
CFT-limit V. Bazhanov, S. Lukyanov and Al. Zamolodchikov [1] constructed infinitely many 
integrals of motion for the Virasoro algebra, as we mentioned above. In the classical limit, 
the deformed Virasoro algebra degenerates to the Poisson-Virasoro algebra introduced by 
E. Frenkel and N. Reshetikhin [6]. In the classical limit, E. Frenkel [7] constructed infinitely 
many integrals of motion for the Poisson-Virasoro algebra.

The organization of this paper is as follows. In Section 2, we review the deformed 
Virasoro algebra, including free field realization, screening currents [3, 4]. In Section 3, 
we give explicit formulae for the local integrals of motion $I_n$, and show the commutation 
relation $[I_m, I_n] = 0$ and Dynkin-automorphism invariance $\eta(I_n) = I_n$. In Section 4, we 
give explicit formulae for the nonlocal integrals of motion $G_n$, and show the commutation 
relation $[G_m, G_n] = 0$, $[I_m, G_n] = 0$ and the Dynkin-automorphism invariance $\eta(G_n) = G_n$. 
In Section 5, we study specialization to $s = 2$. We discuss about relation to the Poisson-
Virasoro algebra \cite{6, 7} and compare our results with those of CFT by \cite{1}. In Appendix we summarize the normal ordering of the basic operators.

2 The Deformed Virasoro Algebra

In this section we review the deformed Virasoro algebra and its screening currents \cite{3, 4}. We prepare the notations to be used in this paper. Throughout this paper, we fix generic three parameters $0 < x < 1$, $r \in \mathbb{C}$ and $s \in \mathbb{C}$. Let us set $z = x^{2u}$. Let us set $r^* = r - 1$.

The symbol $[u]_r$ for $\text{Re}(r) > 0$ stands for the Jacobi theta function

$$[u]_r = x_{r^*}^{x^2-r^*} \frac{\Theta_{x^{2u}}(x^{2u})}{(x^{2r}; x^{2r^*})^3}, \quad \Theta_q(z) = (z; q)_\infty(qz^{-1}; q)_\infty(q; q)_\infty,$$

where we have used the standard notation

$$ (z; q)_\infty = \prod_{j=0}^{\infty} (1 - q^j z).$$

We set the parametrizations $\tau, \tau^*$

$$ x = e^{-\pi \sqrt{-1}/r\tau} = e^{-\pi \sqrt{-1}/r^*\tau^*}. $$

The theta function $[u]_r$ enjoys the quasi-periodicity property

$$ [u + r]_r = -[u]_r, \quad [u + r\tau]_r = -e^{-\pi \sqrt{-1}/r^*\tau^*} [u]_r. $$

The symbol $[a]$ stands for

$$ [a] = \frac{x^a - x^{-a}}{x - x^{-1}}. $$

2.1 Free Field Realization

Let $\beta_m^1, \beta_m^2$ be the oscillators ($m \in \mathbb{Z}_{\neq 0}$) with the commutation relations

$$ [\beta_n^i, \beta_m^j] = n\frac{[r^* n]}{[r m]} \frac{[s - 1 n]}{[s n]} \delta_{n+m,0} \quad (i = 1, 2), $$

$$ [\beta_n^i, \beta_m^j] = -n\frac{[r^* n]}{[r m]} \frac{[n]}{[s n]} x^{sn} \text{sgn}(i-j) \delta_{n+m,0} \quad (1 \leq i \neq j \leq 2). $$

Let $P, Q$ be the zero mode operators with the commutation relations

$$ [P, iQ] = 2. $$
Let us map $\eta$ on this algebra by

$$\eta(\beta_m^1) = x^{-sm}\beta_m^2, \quad \eta(\beta_m^2) = x^{sm}\beta_m^1, \quad \eta(P) = -P, \quad \eta(Q) = -Q.$$ (2.9)

It preserves the commutation relation, $[\beta_i^1, \beta_j^2] = [\eta(\beta_i^1), \eta(\beta_j^2)], \quad [P, iQ] = [\eta(P), i\eta(Q)], \quad \eta^2 = id$. In what follows we call the map $\eta$ Dynkin-automorphism.

We deal with the bosonic Fock space $F_{l,k}(l, k \in \mathbb{Z})$ generated by $\beta_{-m}(m > 0, i = 1, 2)$ over the vacuum vectors $|l, k\rangle$:

$$\beta_m^i|l, k\rangle = 0, (m > 0, j = 1, 2),$$ (2.10)

$$P|l, k\rangle = (l\sqrt{r}/r - k\sqrt{r^*}/r)|l, k\rangle,$$ (2.11)

$$|l, k\rangle = e^{l\sqrt{r^*}-k\sqrt{r^*}}P|0, 0\rangle.$$ (2.12)

We use the abbreviation $\hat{\pi} = \sqrt{rr^*}P$. In what follows we work on the Fock space $F_{l,k}$ with fixed values $l, k \in \mathbb{Z}$.

### 2.2 The Deformed Virasoro Algebra

Let us set the fundamental operators $\Lambda_j(z)$ ($j = 1, 2$) associated with vertex of Dynkin-diagram of the affine symmetry $A_1^{(l)}$

$$\Lambda_1(z) = x^{-\hat{\pi}}: \exp \left( \sum_{m \neq 0} \frac{1}{m} (x^{rm} - x^{-rm}) \beta_m^1 z^{-m} \right) :,$$ (2.13)

$$\Lambda_2(z) = x^{\hat{\pi}}: \exp \left( \sum_{m \neq 0} \frac{1}{m} (x^{rm} - x^{-rm}) \beta_m^2 z^{-m} \right) :.$$ (2.14)

Here the symbol $:\ast :$ stands for usual normal ordering of bosons, i.e., $\beta_m^i$ with $m > 0$ should be moved to the right. Let us set the operators $T_1(z), T_2(z)$ by

$$T_1(z) = \Lambda_1(z) + \Lambda_2(z), \quad T_2(z) = : \Lambda_1(x^{-1}z)\Lambda_2(xz) :.$$ (2.15)

**Proposition 2.1** *The operators $T_j(z)$ ($j = 1, 2$) satisfy*

$$f_{11}(z_2/z_1)T_1(z_1)T_1(z_2) - f_{11}(z_1/z_2)T_1(z_2)T_1(z_1) = c(\delta(x^2 z_2/z_1)T_2(x^{-1}z_1) - \delta(x^2 z_1/z_2)T_2(x^{-1}z_2)),$$ (2.16)

$$f_{12}(z_2/z_1)T_1(z_1)T_2(z_2) = f_{21}(z_1/z_2)T_2(z_2)T_1(z_1),$$ (2.17)

$$f_{22}(z_2/z_1)T_2(z_1)T_2(z_2) = f_{22}(z_1/z_2)T_2(z_2)T_2(z_1).$$ (2.18)
Here the structure functions $f_{ij}(z)$ and the constant $c$ are given by

$$f_{11}(z) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m}(1 - x^{2m})(1 - x^{-2r^*m}) \frac{(1 - x^{2m(s-1)})}{(1 - x^{2sm})} z^m\right),$$

$$f_{12}(z) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m}(1 - x^{2m})(1 - x^{-2r^*m}) \frac{(1 - x^{2m(s-2)})}{(1 - x^{2sm})} (xz)^m\right) = f_{21}(z),$$

$$f_{22}(z) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m}(1 - x^{2m})(1 - x^{-2r^*m}) \frac{(1 + x^{2m})(1 - x^{2m(s-2)})}{(1 - x^{2sm})} z^m\right),$$

$$c = -\frac{(1 - x^{-2r^*})(1 - x^{2r})}{(1 - x^2)}.$$

Note that the structure functions $f_{11}(z), f_{12}(z) = f_{21}(z)$ and $f_{22}(z)$ enjoy the fusion properties

$$f_{11}(z) = \frac{1}{1 - z} \frac{(x^{2r^*} z - x^{2s})}{(x^{2s} - x^{2r^*} z)} \frac{(x^r z, x^s)}{(x^r, x^s)},$$

$$f_{12}(z) = \frac{f_{11}(xz)}{\Delta(z)} = f_{21}(z), \quad f_{22}(z) = f_{12}(x^{-1}z)f_{21}(xz),$$

where we have set

$$\Delta(z) = \frac{(1 - x^{r+r^*} z)(1 - x^{-r-r^*} z)}{(1 - xz)(1 - x^{-1}z)}.$$

The Dynkin automorphism $\eta$ acts on $\Lambda_j(z)$ as

$$\eta(\Lambda_1(z)) = \Lambda_2(x^s z), \quad \eta(\Lambda_2(z)) = \Lambda_1(x^{-s} z).$$

**Definition 2.1** The deformed Virasoro algebra is defined by the generators $\hat{T}_m^{(1)}, \hat{T}_m^{(2)}, \ (m \in \mathbb{Z})$ with the defining relations (2.16), (2.17) and (2.18). Here we should understand $\hat{T}_m^{(1)}, \hat{T}_m^{(2)}$ as the Fourier coefficients of the operators $\hat{T}_1(z) = \sum_{m \in \mathbb{Z}} \hat{T}_m^{(1)} z^{-m}$, $\hat{T}_2(z) = \sum_{m \in \mathbb{Z}} \hat{T}_m^{(2)} z^{-m}$.

**2.3 Screening Current**

We introduce the screening currents $F_j(z), E_j(z) \ (j = 0, 1)$ as

$$F_1(z) = e^{i\sqrt{-1}Q z \frac{1}{1 + z}} : \exp\left(\sum_{m \neq 0} \frac{1}{m}(\beta_m^1 - \beta_m^2) z^{-m}\right) :,$$
The adjoint actions of the operator $F_0(z) = e^{-i\sqrt{\pi}Qz^{-\frac{1}{2}+i\frac{r}{r}}} : \exp \left( \sum_{m \neq 0} \frac{1}{m} \left( -x^{sm} \beta_m^1 + x^{-sm} \beta_m^2 \right) z^{-m} \right) :$  
(2.28)

$E_1(z) = e^{-i\sqrt{\pi}Qz^{-\frac{1}{2}+i\frac{r}{r}}} : \exp \left( - \sum_{m \neq 0} \frac{1}{m} \frac{[rm]}{r^{*m}} (\beta_m^1 - \beta_m^2) z^{-m} \right) :$  
(2.29)

$E_0(z) = e^{i\sqrt{\pi}Qz^{-\frac{1}{2}+i\frac{r}{r}}} : \exp \left( - \sum_{m \neq 0} \frac{1}{m} \frac{[rm]}{r^{*m}} (-x^{sm} \beta_m^1 + x^{-sm} \beta_m^2) z^{-m} \right) :$  
(2.30)

We summarize convenient commutation relations

$[\beta_m^1 - \beta_m^2, \beta_n^1 - \beta_n^2] = m \frac{[rm]}{[rm]} (x^m + x^{-m}) \delta_{m+n,0},$  
(2.31)

$[\beta_m^1 - \beta_m^2, -x^{sm} \beta_n^1 + x^{-sn} \beta_n^2] = m \frac{[rm]}{r^{*m}} (x^{(s-1)m} + x^{(1-s)m}) \delta_{m+n,0},$  
(2.32)

$[-x^{sm} \beta_m^1 + x^{-sm} \beta_m^2, -x^{sn} \beta_n^1 + x^{-sn} \beta_n^2] = m \frac{[rm]}{r^{*m}} (x^m + x^{-m}) \delta_{m+n,0}.$  
(2.33)

The Dynkin-automorphism $\eta$ maps the screening currents as

$\eta(F_1(z)) = F_0(z), \quad \eta(F_0(z)) = F_1(z), \quad \eta(E_1(z)) = E_0(z), \quad \eta(E_0(z)) = E_1(z).$  
(2.34)

The adjoint actions of the operator $T_1(z) = \Lambda_1(z) + \Lambda_2(z)$ on the screening currents $F_j(z), E_j(z)$ can be regarded as the differences of currents $A(z), B(z).$

**Proposition 2.2**  
The adjoint actions of $\Lambda_j(z)$ on the screenings $F_j(z), E_j(z)$ are

$[\Lambda_1(z_1), F_1(z_2)] = (x^{-r^*} - x^{-r}) \delta(x^r z_1/z_2) A(x^{-r} z_2),$  
(2.35)

$[\Lambda_2(z_1), F_1(z_2)] = (x^{r^*} - x^{-r^*}) \delta(x^{-r} z_1/z_2) A(x^r z_2),$  
(2.36)

$[\Lambda_1(z_1), F_0(z_2)] = (x^{r^*} - x^{-r^*}) \delta(x^{-r+s} z_1/z_2) \eta(A(x^r z_2)),$  
(2.37)

$[\Lambda_2(z_1), F_0(z_2)] = (x^{-r^*} - x^{-r}) \delta(x^{-r-s} z_1/z_2) \eta(A(x^{-r} z_2)),$  
(2.38)

$[\Lambda_1(z_1), E_1(z_2)] = (x^{-r^*} - x^{-r}) \delta(x^{-r^*} z_1/z_2) B(x^{-r} z_2),$  
(2.39)

$[\Lambda_2(z_1), E_1(z_2)] = (x^{-r^*} - x^{-r}) \delta(x^{-r^*} z_1/z_2) B(x^{-r} z_2),$  
(2.40)

$[\Lambda_1(z_1), E_0(z_2)] = (x^{-r} - x^{-r}) \delta(x^{-r+s} z_1/z_2) \eta(B(x^{-r} z_2)),$  
(2.41)

$[\Lambda_2(z_1), E_0(z_2)] = (x^{-r} - x^{-r}) \delta(x^{-r-s} z_1/z_2) \eta(B(x^{-r} z_2)).$  
(2.42)

Here we have set

$A(z) = e^{i\sqrt{\pi}Qz^{\frac{1}{2}+i\frac{r}{r}}} : \exp \left( \sum_{m \neq 0} \frac{1}{m} \frac{[rm]}{r^{*m}} (x^{r^*} \beta_m^1 - x^{-r^*} \beta_m^2) z^{-m} \right) :$  
(2.43)

$B(z) = e^{-i\sqrt{\pi}Qz^{-\frac{1}{2}+i\frac{r}{r}}} : \exp \left( \sum_{m \neq 0} \frac{1}{m} \frac{[rm]}{r^{*m}} (-x^{-r^*} \beta_m^1 + x^{-r} \beta_m^2) z^{-m} \right) :$  
(2.44)
Proposition 2.3  

For regime \( \Re(r) > 0 \), the commutation relations of the currents \( F_j(z) \) are given as

\[
\frac{[u_1 - u_2]}{[u_1 - u_2 - 1]} F_j(z_1) F_j(z_2) = \frac{[u_2 - u_1]}{[u_2 - u_1 - 1]} F_j(z_2) F_j(z_1), \quad (j = 1, 2) \tag{2.45}
\]

\[
\frac{[u_1 - u_2 + \frac{s}{2} - 1]}{[u_1 - u_2 + \frac{s}{2}]} F_0(z_1) F_1(z_2) = \frac{[u_2 - u_1 + \frac{s}{2} - 1]}{[u_2 - u_1 + \frac{s}{2}]} F_1(z_2) F_0(z_1). \tag{2.46}
\]

For regime \( \Re(r) < 0 \), we have

\[
\frac{[u_1 - u_2]}{[u_1 - u_2 + 1]} F_j(z_1) F_j(z_2) = \frac{[u_2 - u_1 - 1]}{[u_2 - u_1 + 1]} F_j(z_2) F_j(z_1), \quad (j = 1, 2) \tag{2.47}
\]

\[
\frac{[u_1 - u_2 - \frac{s}{2} + 1]}{[u_1 - u_2 - \frac{s}{2}]} F_0(z_1) F_1(z_2) = \frac{[u_2 - u_1 - \frac{s}{2} + 1]}{[u_2 - u_1 - \frac{s}{2}]} F_1(z_2) F_0(z_1). \tag{2.48}
\]

For regime \( \Re(r^*) > 0 \), we have

\[
\frac{[u_1 - u_2]}{[u_1 - u_2 + 1]} E_j(z_1) E_j(z_2) = \frac{[u_2 - u_1 - 1]}{[u_2 - u_1 + 1]} E_j(z_2) E_j(z_1), \quad (j = 1, 2) \tag{2.49}
\]

\[
\frac{[u_1 - u_2 - \frac{s}{2} + 1]}{[u_1 - u_2 - \frac{s}{2}]} E_0(z_1) E_1(z_2) = \frac{[u_2 - u_1 - \frac{s}{2} + 1]}{[u_2 - u_1 - \frac{s}{2}]} E_1(z_2) E_0(z_1). \tag{2.50}
\]

For regime \( \Re(r^*) < 0 \), we have

\[
\frac{[u_1 - u_2]}{[u_1 - u_2 - 1]} E_j(z_1) E_j(z_2) = \frac{[u_2 - u_1 - 1]}{[u_2 - u_1 + 1]} E_j(z_2) E_j(z_1), \quad (j = 1, 2) \tag{2.51}
\]

\[
\frac{[u_1 - u_2 + \frac{s}{2} - 1]}{[u_1 - u_2 + \frac{s}{2}]} E_0(z_1) E_1(z_2) = \frac{[u_2 - u_1 + \frac{s}{2} - 1]}{[u_2 - u_1 + \frac{s}{2}]} E_1(z_2) E_0(z_1). \tag{2.52}
\]

We have

\[
\frac{[E_1(z_1), F_1(z_2)]}{[E_0(z_1), F_0(z_2)]} = \frac{1}{x - x^{-1}} (\delta(xz_2/z_1)H(x^r z_2) - \delta(xz_1/z_2)H(x^{-r} z_2)), \tag{2.53}
\]

\[
\frac{[E_0(z_1), F_1(z_2)]}{[E_0(z_1), F_0(z_2)]} = \frac{1}{x - x^{-1}} (\delta(xz_2/z_1)\eta(H(x^r z_2)) - \delta(xz_1/z_2)\eta(H(x^{-r} z_2))). \tag{2.54}
\]

Here we have set

\[
H(z) = e^{-\frac{1}{\sqrt{4r^*}}\frac{Q}{2} z - \frac{1}{\sqrt{4r^*}} p + \frac{1}{r^*}} \exp \left( - \sum_{m \neq 0} \frac{1}{m} \frac{[m]}{[r^* m]} z^{-m} (\beta_1^m - \beta_2^m) \right). \tag{2.55}
\]

We have

\[
E_1(z_1) F_0(z_2) = F_0(z_2) E_1(z_1), \quad E_0(z_1) F_1(z_2) = F_1(z_2) E_0(z_1). \tag{2.56}
\]
2.4 Comparision with another definition

At first glance, our definition of the deformed Virasoro is different from those in [3]. In this subsection we compare two definitions, and show they are essentially the same thing. We show that operators $T_1(z), T_2(z)$ are realized as the tensor product of the deformed Virasoro algebra $\text{Vir}_{q,t}$, $(q = x^{2r}, t = x^{2r-2})$ and a proper operator $Z(z)$. Let us set the auxiliary bosons $B^1_m, B^2_m$ by

$$B^1_m = \beta^1_m - \beta^2_m, \quad (2.57)$$
$$B^2_m = x^m \beta^1_m + x^{-m} \beta^2_m, \quad (2.58)$$

We have

$$[B^1_n, B^1_m] = n \frac{(r^n + x^{-n})}{[rn]} \delta_{m+n,0}, \quad [B^1_n, B^2_m] = 0, \quad (2.59)$$
$$[B^2_n, B^2_m] = n \frac{(s^n + x^{-n})}{[sn]} (x^n + x^{-n}) \delta_{m+n,0} \quad (m, n \in \mathbb{Z} \neq 0). \quad (2.60)$$

Then we have the following decomposition

$$\Lambda_1(z) = \Lambda_1^{DV}(z) Z(z), \quad \Lambda_2(z) = \Lambda_2^{DV}(z) Z(z), \quad (2.61)$$

where we have set

$$\Lambda_1^{DV}(z) = x^{-\hat{\pi}} : \exp \left( \sum_{m \neq 0} \frac{1}{x^m} \left( \frac{x^m}{x^m + x^{-m}} B^1_m (xz)^{-m} \right) \right)^:, \quad (2.62)$$
$$\Lambda_2^{DV}(z) = x^{-\hat{\pi}} : \exp \left( -\sum_{m \neq 0} \frac{1}{x^m} \left( \frac{x^m}{x^m + x^{-m}} B^1_m (x^{-1}z)^{-m} \right) \right)^:, \quad (2.63)$$
$$Z(z) = : \exp \left( \sum_{m \neq 0} \frac{1}{x^m} \left( \frac{x^m}{x^m + x^{-m}} B^2_m z^{-m} \right) \right)^:. \quad (2.64)$$

Here we have

$$T_1(z) = T^{DV}(z) Z(z), \quad T_2(z) = : Z(x^{-1}z) Z(xz) :, \quad (2.65)$$

where

$$T^{DV}(z) = \Lambda_1^{DV}(z) + \Lambda_2^{DV}(z). \quad (2.66)$$
Proposition 2.4  The operator $T^{DV}(z)$ is the generating function of the deformed Virasoro algebra $Vir_{q,t}$ defined in [3] with $q = x^{2r}, t = x^{2r-2}$

$$f^{DV}(z_2/z_1)T^{DV}(z_1)T^{DV}(z_2) - f^{DV}(z_1/z_2)T^{DV}(z_2)T^{DV}(z_1) = c(\delta(x^2z_2/z_1) - \delta(x^2z_1/z_2)),$$  

(2.67)

where

$$f^{DV}(z) = \exp \left( \sum_{m=1}^{\infty} \frac{1}{m} \left( 1 - x^{2rm} \right) \left( 1 - x^{-2r^*m} \right) z^{-m} \right),$$

(2.68)

$$c = -\frac{(1 - x^{-2r^*})(1 - x^{2r})}{(1 - x^2)}. $$

(2.69)

We have

$$T^{DV}(z_1)Z(z_2) = Z(z_2)T^{DV}(z_1),$$

(2.70)

$$f^Z(z_2/z_1)Z(z_1)Z(z_2) = f^Z(z_1/z_2)Z(z_2)Z(z_1),$$

(2.71)

where

$$f^Z(z) = \exp \left( - \sum_{m=1}^{\infty} \frac{1}{m} \left( 1 - x^{2rm} \right) \left( 1 - x^{-2r^*m} \right) \left( x^{2m} - x^{2(s-1)m} \right) \left( 1 + x^{2m} \right) z^{-m} \right).$$

(2.72)

Therefore three parameter deformed Virasoro algebra $T_1(z), T_2(z)$ is realized as an extension of two parameter deformed Virasoro algebra $T^{DV}(z)$. Note that upon the specialization $s = 2$ we have

$$[B_n^2, B_m^2] = 0, \quad [B_n^1, B_m^2] = 0, \quad f^Z(z) = 1.$$

(2.73)

Hence we can regard $B_m^2 = 0$ and $T_1(z) = T^{DV}(z), T_2(z) = 1.$

## 3 Local Integrals of Motion $\mathcal{I}_n$

In this section we construct the local integrals of motion $\mathcal{I}_n$ and $\mathcal{I}^*_n$. We study the generic case : $0 < x < 1$, $r \in \mathbb{C}$ and Re($s$) > 0.

### 3.1 Local Integral of Motion

Let us set the function $h(u)$ and $h^*(u)$ by

$$h(u) = \frac{[u]_s[u + r]_s}{[u + 1]_s[u + r^*]_s}, \quad h^*(u) = \frac{[u]_s[u - r^*]_s}{[u + 1]_s[u - r]_s},$$

(3.1)

where we have set $z = x^{2u}$. We have $h^*(u) = h(u)|_{r \rightarrow -r^*}$.
Definition 3.1  We define $I_n$ for regime $\text{Re}(s) > 2$ and $\text{Re}(r^*) < 0$ by

$$I_n = \int \cdots \int_C \prod_{j=1}^n \frac{dz_j}{2\pi \sqrt{-1}z_j} \prod_{1 \leq j < k \leq n} h(u_k - u_j)T_1(z_1) \cdots T_1(z_n) \quad (n = 1, 2, \cdots). \quad (3.2)$$

Here, the contour $C$ encircles $z_j = 0$ in such a way that $z_j = x^{-2+2sl}z_k, x^{-2r+2sl}z_k$ ($l = 0, 1, 2, \cdots$) is inside and $z_j = x^{2-2sl}z_k, x^{2r-2sl}z_k$ ($l = 0, 1, 2, \cdots$) is outside for $1 \leq j < k \leq n$. We define $I_n^*$ for regime $\text{Re}(s) > 2$ and $\text{Re}(r) > 0$ by

$$I_n^* = \oint \cdots \oint_{C^*} \prod_{j=1}^n \frac{dz_j}{2\pi \sqrt{-1}z_j} \prod_{1 \leq j < k \leq n} h^*(u_k - u_j)T_1(z_1) \cdots T_1(z_n) \quad (n = 1, 2, \cdots). \quad (3.3)$$

The contour $C^*$ encircles $z_j = 0$ in such a way that $z_j = x^{-2+2sl}z_k, x^{2r+2sl}z_k$ ($l = 1, 2, \cdots$) is inside and $z_j = x^{2-2sl}z_k, x^{-2r-2sl}z_k$ ($l = 1, 2, \cdots$) is outside for $1 \leq j < k \leq n$.

We call $I_n$ and $I_n^*$ the local integrals of motion for the deformed Virasoro algebra. The definitions of $I_n, I_n^*$ for generic $\text{Re}(s) > 0$ and $r \in \mathbb{C}$ should be understood as analytic continuation.

We have the involution $I_n^* = I_n|_{r \to -r^*}$ and $I_n = I_n^*|_{r^* \to -r}$. The followings are some of Main Results of this paper.

Theorem 3.1  The local integrals of motion $I_n$ and $I_n^*$ commute with each other

$$[I_n, I_m] = [I_n^*, I_m^*] = 0 \quad (m, n = 1, 2, \cdots). \quad (3.4)$$

Theorem 3.2  The local integrals of motion $I_n$ and $I_n^*$ are invariant under the action of the Dynkin-automorphism

$$\eta(I_n) = I_n, \quad \eta(I_n^*) = I_n^* \quad (n = 1, 2, \cdots). \quad (3.5)$$

Conjecture 3.3

$$[I_m, I_n^*] = 0 \quad (m, n = 1, 2, \cdots). \quad (3.6)$$

3.2 Another Formulae

In this subsection we prepare another formulae of the local integrals of motion $I_n$. Because the integral contour of the definition of the local integrals of motion $I_n$ is not
annulus. i.e. \(|x^{-2}z_k| < |z_j| < |x^2z_k|\), the defining relations of the deformed Virasoro (2.16), (2.17), (2.18) should be used carefully. Hence, in order to show the commutation relations \([\mathcal{I}_m, \mathcal{I}_n] = 0\), it is better for us to deform the integral representations of the local integrals of motion \(\mathcal{I}_n\) to another formulae, in which the defining relations of the deformed Virasoro (2.16), (2.17), (2.18) can be used safely.

Let us set the auxiliary function \(s(z)\) and \(s^*(z)\) by \(h(u) = s(z)f_{11}(z)\), \(h^*(u) = s^*(z)f_{11}(z)\), where \(h(u)\) and \(f_{11}(z)\) are given in the previous section. We have explicitly

\[
s(z) = x^{-2r^*} \frac{(z; x^{2s})_\infty (x^{2s-2r^*}z; x^{2s})_\infty}{(x^{2s-2}z; x^{2s})_\infty (x^{-2r^*}z; x^{2s})_\infty} \times \frac{(1/z; x^{2s})_\infty (x^{2s-2r^*}/z; x^{2s})_\infty}{(x^{2s-2}/z; x^{2s})_\infty (x^{-2r^*}/z; x^{2s})_\infty},
\]

(3.7)

and

\[
s^*(z) = x^{2r} \frac{(z; x^{2s})_\infty (x^{2s+2r^*}z; x^{2s})_\infty}{(x^{2s-2}z; x^{2s})_\infty (x^{2r}z; x^{2s})_\infty} \times \frac{(1/z; x^{2s})_\infty (x^{2s+2r^*}/z; x^{2s})_\infty}{(x^{2s-2}/z; x^{2s})_\infty (x^{2r}/z; x^{2s})_\infty}.
\]

(3.8)

Let us introduce a “weak sense” equality

**Definition 3.2** We say the operators \(\mathcal{P}(z_1, z_2, \ldots, z_n)\) and \(\mathcal{Q}(z_1, z_2, \ldots, z_n)\) are equal in the “weak sense” if

\[
\prod_{1 \leq i < j \leq n} s(z_j/z_i)\mathcal{P}(z_1, z_2, \ldots, z_n) = \prod_{1 \leq i < j \leq n} s(z_j/z_i)\mathcal{Q}(z_1, z_2, \ldots, z_n).
\]

(3.9)

We write \(\mathcal{P}(z_1, z_2, \ldots, z_n) \sim \mathcal{Q}(z_1, z_2, \ldots, z_n)\), showing the weak equality.

For example \(\delta(z_1/z_2) \sim 0\) and \(\frac{1}{z_1-z_2} \delta(z_1/z_2) \sim 0\).

Let us set the auxiliary functions \(g_{11}(z), g_{12}(z)\) and \(g_{22}(z)\) by fusion procedure

\[
g_{12}(z) = g_{11}(xz)g_{11}(x^{-1}z) = g_{21}(z), \quad g_{22}(z) = g_{21}(x^{-1}z)g_{12}(xz),
\]

(3.10)

where \(g_{11}(z) = f_{11}(z)\) is defined in Proposition 2.1.

**Proposition 3.4** The following equalities hold in the weak sense

\[
\delta(x^2z_3/z_2)g_{12}(x^{-1}z_2/z_1)T_1(z_1)T_2(x^{-1}z_2) \sim \delta(x^2z_3/z_2)g_{21}(xz_1/z_2)T_2(x^{-1}z_2)T_1(z_1),
\]

(3.11)

\[
\prod_{j=3,4} \delta(x^2z_j/z_1)g_{22}(z_2/z_1)T_2(z_2)T_2(z_2) \sim \prod_{j=3,4} \delta(x^2z_j/z_1)g_{22}(z_1/z_2)T_2(z_2)T_2(z_1).\)

(3.12)
Proof. By defining relation of the deformation of the Virasoro algebra, we have

\[ g_{12}(x^{-1} z_2/z_1)T_1(z_1)T_2(x^{-1} z_2) - g_{21}(xz_1/z_2)T_2(x^{-1} z_2)T_1(z_1) \]
\[ = f_{12}(x^{-1} z_2/z_1)(\delta(z_2/z_1) - \delta(x^{-2} z_2/z_1))T_1(z_1)T_2(x^{-1} z_2), \quad (3.13) \]

where we have used delta-function relation

\[ \Delta(z) - \Delta(z^{-1}) = c(\delta(xz) - \delta(x^{-1}z)), \quad \Delta(z) = \frac{(1 - x^{2r-1}z)(1 - x^{-2r+1}z)}{(1 - x)(1 - x^{-1}z)}. \quad (3.14) \]

Because \( \prod_{1 \leq i < j \leq 3} s(z_j/z_i) \delta(x^2 z_3/z_2) f_{12}(x^{-1} z_2/z_1)(\delta(z_2/z_1) - \delta(x^{-2} z_2/z_1))T_1(z_1)T_2(x^{-1} z_2) = 0 \), we conclude the first equation of proposition. The second equation is obtained in the same manner. Q.E.D.

Let us introduce \( S_n \)-invariant in the "weak sense".

**Definition 3.3** We call the operator \( \mathcal{P}(z_1, z_2, \cdots, z_n) \) is \( S_n \)-invariant in the "weak sense" if

\[ \mathcal{P}(z_1, z_2, \cdots, z_n) \sim \mathcal{P}(z_{\sigma(1)}, z_{\sigma(2)}, \cdots, z_{\sigma(n)}), \quad (\sigma \in S_n). \quad (3.15) \]

**Example** The operator \( \mathcal{O}_2(z_1, z_2) = g_{11}(z_2/z_1)T_1(z_1)T_1(z_2) - c\delta(x^2 z_2/z_1)T_2(x^{-1} z_1) \) is \( S_2 \)-invariant.

In what follows we use the notation of the ordered product

\[ \prod_{i \in L} T_1(z_i) = T_1(z_{l_1})T_1(z_{l_2}) \cdots T_1(z_{l_n}), \quad (L = \{l_1, \cdots, l_n|l_1 < l_2 < \cdots < l_n\}). \quad (3.16) \]

Let us set the auxiliary operator \( \mathcal{O}_n(z_1, z_2, \cdots, z_n) \) by

\[
\mathcal{O}_n(z_1, z_2, \cdots, z_n) = \sum_{\alpha = 0}^{[n]} (-1)^{\alpha} c^{\alpha} \sum_{A_1, A_2, \cdots, A_n} \prod_{1 \leq j \leq n} \delta\left(\frac{x^2 z_{\text{Max}(A_j)}}{z_{\text{Min}(A_j)}}\right) \\
\times \prod_{1 \leq j \leq n} T_1(z_j) \prod_{1 \leq j \leq n} T_2(x^{-1} z_j) \\
\times \prod_{1 \leq j < k \leq n} g_{11}(z_j/z_k) \prod_{1 \leq j < k \leq n} g_{22}(z_j/z_k) \prod_{1 \leq j, k \leq n} g_{12}(x^{-1} z_k/z_j), \quad (3.17)
\]
where the summation $\sum_{A_1,\ldots,A_n}$ is taken over the set $A_1, A_2, \ldots, A_n \subset \{1, 2, \ldots, n\}$ such that $A_j \cap A_k = \emptyset$ ($1 \leq j \neq k \leq \alpha$), $|A_j| = 2$ ($1 \leq j \leq \alpha$), and $\text{Min}(A_j) < \text{Min}(A_k)$ ($1 \leq j < k \leq \alpha$). We have set $A_{\text{Min}} = \{\text{Min}(A_1), \ldots, \text{Min}(A_n)\}$, $A_{\text{Max}} = \{\text{Max}(A_1), \ldots, \text{Max}(A_n)\}$, and $A = A_1 \cup A_2 \cup \cdots \cup A_n$.

**Example** We summarize the current $O_n$ very explicitly for $n = 1, 2, 3, 4$.

\[
O_1(z_1) = T_1(z_1),
\]
\[
O_2(z_1, z_2) = g_{11}(z_2/z_1)T_1(z_1)T_1(z_2) - c\delta(x^2z_2/z_1)T_2(x^{-1}z_1),
\]
\[
O_3(z_1, z_2, z_3) = g_{11}(z_2/z_1)g_{11}(z_3/z_1)g_{11}(z_3/z_2)T_1(z_1)T_1(z_2)T_1(z_3)
- cg_{12}(x^{-1}z_2/z_1)T_1(z_1)T_2(x^{-1}z_2)\delta(x^2z_3/z_2)
- cg_{12}(x^{-1}z_1/z_2)T_1(z_2)T_2(x^{-1}z_1)\delta(x^2z_3/z_1)
- cg_{12}(x^{-1}z_1/z_3)T_1(z_3)T_2(x^{-1}z_1)\delta(x^2z_2/z_1),
\]
\[
O_4(z_1, z_2, z_3, z_4) = \prod_{1 \leq j < k \leq 4} g_{11}(z_k/z_j)T_1(z_1)T_1(z_2)T_1(z_3)T_1(z_4)
- cg_{11}(z_2/z_1)g_{12}(x^{-1}z_3/z_1)g_{12}(x^{-1}z_3/z_2)T_1(z_1)T_1(z_2)T_2(x^{-1}z_3)\delta(x^2z_4/z_3)
- cg_{11}(z_3/z_1)g_{12}(x^{-1}z_2/z_1)g_{12}(x^{-1}z_2/z_3)T_1(z_1)T_1(z_3)T_2(x^{-1}z_2)\delta(x^2z_4/z_2)
- cg_{11}(z_4/z_1)g_{12}(x^{-1}z_2/z_1)g_{12}(x^{-1}z_2/z_4)T_1(z_1)T_1(z_4)T_2(x^{-1}z_2)\delta(x^2z_3/z_2)
- cg_{11}(z_3/z_2)g_{12}(x^{-1}z_1/z_2)g_{12}(x^{-1}z_1/z_3)T_1(z_2)T_1(z_3)T_2(x^{-1}z_1)\delta(x^2z_4/z_1)
- cg_{11}(z_4/z_2)g_{12}(x^{-1}z_1/z_2)g_{12}(x^{-1}z_1/z_4)T_1(z_2)T_1(z_4)T_2(x^{-1}z_1)\delta(x^2z_3/z_1)
- cg_{11}(z_4/z_3)g_{12}(x^{-1}z_1/z_3)g_{12}(x^{-1}z_1/z_4)T_1(z_3)T_1(z_4)T_2(x^{-1}z_1)\delta(x^2z_2/z_1)
+ c^2g_{22}(z_3/z_1)\delta(x^2z_2/z_1)\delta(x^2z_4/z_3)T_2(x^{-1}z_1)T_2(x^{-1}z_3)
+ c^2g_{22}(z_2/z_1)\delta(x^2z_3/z_1)\delta(x^2z_4/z_2)T_2(x^{-1}z_1)T_2(x^{-1}z_2)
+ c^2g_{22}(z_2/z_1)\delta(x^2z_4/z_1)\delta(x^2z_3/z_2)T_2(x^{-1}z_1)T_2(x^{-1}z_2).
\]

**Proposition 3.5** The operator $O_n$ defined in (3.17) is $S_n$-invariant in the weak sense.

\[
O_n(z_1, z_2, \ldots, z_n) \sim O_n(z_{\sigma(1)}, z_{\sigma(2)}, \ldots, z_{\sigma(n)}) \quad (\sigma \in S_n).
\]

Before showing the complete proof, we consider some simple examples as warming-up exercise. When $n = 2$ case, $O_2(z_1, z_2) = O_2(z_2, z_1)$ is exactly the same as the defining relation of the deformed Virasoro algebra Proposition 2.1. When $n = 3$ case, it is enough
to show $S_3$-invariance for two generators $\sigma = (1, 2), (2, 3)$. Let us study the case $\sigma = (1, 2)$ case. We have

$$O_3(z_1, z_2, z_3) - O_3(z_2, z_1, z_3)$$

$$= g_{11}(z_3/z_1)g_{11}(z_3/z_2)(g_{11}(z_2/z_1)T_1(z_1)T_1(z_2) - g_{11}(z_1/z_2)T_1(z_2)T_1(z_1))T_1(z_3) \quad (3.23)$$

$$- cT_1(z_3)(g_{12}(x^{-1}z_1/z_3)T_2(x^{-1}z_1)\delta(x^2z_2/z_1) - g_{12}(x^{-1}z_2/z_3)T_2(x^{-1}z_2)\delta(x^2z_1/z_2)).$$

Changing the ordering of $T_1(z_1)T_1(z_2), T_1(z_2)T_1(z_1)$ and $T_1(z_3)$ in the first term, we have the following in the "weak sense"

$$T_1(z_3)g_{11}(z_1/z_3)g_{11}(z_2/z_3)(g_{11}(z_2/z_1)T_1(z_1)T_1(z_2) - g_{11}(z_1/z_2)T_1(z_2)T_1(z_1)) \quad (3.24)$$

$$- cT_1(z_3)(g_{12}(x^{-1}z_1/z_3)T_2(x^{-1}z_1)\delta(x^2z_2/z_1) - g_{12}(x^{-1}z_2/z_3)T_2(x^{-1}z_2)\delta(x^2z_1/z_2)).$$

Substituting the defining relation of the deformed Virasoro algebra (2.16), (2.17), (2.18) in Proposition 2.1, we have

$$cT_1(z_3)g_{11}(z_1/z_3)g_{11}(z_2/z_3)(T_2(x^{-1}z_1)\delta(x^2z_2/z_1) - T_2(x^{-1}z_2)\delta(x^2z_1/z_2)) \quad (3.25)$$

$$- cT_1(z_3)(g_{12}(x^{-1}z_1/z_3)T_2(x^{-1}z_1)\delta(x^2z_2/z_1) - g_{12}(x^{-1}z_2/z_3)T_2(x^{-1}z_2)\delta(x^2z_1/z_2)) = 0.$$

Here we have used the fusion relation $g_{12}(z) = g_{11}(x^{-1}z)g_{11}(xz)$. The case $\sigma = (2, 3)$ is similar.

**Proof.** We show $S_n$-invariance in the weak sense for general $n$ case. It is enough to prove theorem for generators $\sigma = (i, i + 1) \quad (1 \leq i \leq n - 1)$. At first we prepare convenient formula. Moving $T_1(z_i)$ and $T_1(z_{i+1})$ to the right, by using Proposition 2.1, we have

$$(T_1(z_i)T_1(z_{i+1})g_{11}(z_{i+1}/z_i) - T_1(z_{i+1})T_1(z_i)g_{11}(z_i/z_{i+1}))$$

$$\times T_1(z_{i+2})\cdots T_1(z_n) \prod_{j=i+2}^{n} g_{11}(z_j/z_i)g_{11}(z_j/z_{i+1}) \sim cT_1(z_{i+2})\cdots T_1(z_n) \quad (3.26)$$

$$\times (\delta(x^2z_{i+1}/z_i)T_2(x^{-1}z_i) \prod_{j=i+2}^{n} g_{21}(x^{-1}z_i/z_j) - \delta(x^2z_i/z_{i+1})T_2(x^{-1}z_{i+1}) \prod_{j=i+2}^{n} g_{21}(x^{-1}z_{i+1}/z_j)).$$

Let us study $O_n(z_1, \cdots, z_i, z_{i+1}, \cdots, z_n) - O_n(z_1, \cdots, z_i, z_i, \cdots, z_n)$. Collecting $T_1(z_i), T_1(z_{i+1})$ and $T_2(x^{-1}z_i), T_2(x^{-1}z_{i+1})$ in the center, by using Proposition 2.1 and Proposition 3.4, we have

$$O_n(z_1, \cdots, z_i, z_{i+1}, \cdots, z_n) - O_n(z_1, \cdots, z_{i+1}, z_i, \cdots, z_n)$$
\[
\sim \sum_{\alpha = 0} \sum_{A_1, A_2, \ldots, A_{n-1} \subset \{1, 2, \ldots, n\} - \{i, i+1\}} \prod_{1 \leq j \leq \alpha} \delta(x^2 z_{\text{Max}(A_j)}/z_{\text{Min}(A_j)}) \\
\times \prod_{1 \leq j, k \leq n, \ j \neq i, i+1, j \notin A} T_1(z_j)(g_{11}(z_{i+1}/z_i)T_1(z_i)T_1(z_{i+1}) - g_{11}(z_i/z_{i+1})T_1(z_{i+1})T_1(z_i)) \prod_{1 \leq k \leq n, \ k \in A_{\text{Min}}} T_2(x^{-1}z_k) \\
\times \prod_{1 \leq j \leq n, \ j < k \leq n, j \notin A} g_{11}(z_k/z_j) \prod_{1 \leq j \leq n, j \neq i, i+1, j \notin A} g_{11}(z_i/z_j)g_{11}(z_{i+1}/z_j) \prod_{1 \leq j \leq n, j \neq i, i+1, j \notin A} g_{12}(x^{-1}z_k/z_j) \prod_{1 \leq k \leq n, k \in A_{\text{Min}}} g_{22}(z_k/z_j) \\
+ \sum_{\beta = 1} \sum_{B_1, B_2, \ldots, B_{\gamma} \subset \{1, 2, \ldots, n\} \text{ for some } \gamma} \prod_{1 \leq j, k \leq n, \ j \notin B} \delta(x^2 z_{\text{Max}(A_j)}/z_{\text{Min}(A_j)}) \prod_{1 \leq j \leq n, \ j \notin B} T_1(z_j) \\
\times (\prod_{1 \leq j \leq n, \ j \notin B} g_{12}(x^{-1}z_i/z_j) \prod_{1 \leq k \leq n, k \notin B, k \in A_{\text{Min}}} g_{22}(x^{-1}z_k/z_i)T_2(x^{-1}z_i)\delta(x^2 z_{i+1}/z_i) \\
- \prod_{1 \leq j \leq n, \ j \notin B} g_{12}(x^{-1}z_{i+1}/z_j) \prod_{1 \leq k \leq n, k \notin B, k \in A_{\text{Min}}} g_{22}(x^{-1}z_k/z_{i+1})T_2(x^{-1}z_{i+1})\delta(x^2 z_i/z_{i+1})) \\
\times \prod_{1 \leq j \leq n, \ j \notin B} T_2(x^{-1}z_k) \prod_{1 \leq j \leq n, \ j \notin B} g_{11}(z_k/z_j) \prod_{1 \leq j \leq n, j \neq i, i+1, j \notin B} g_{22}(z_k/z_j) \prod_{1 \leq k \leq n, k \in A_{\text{Min}}} g_{12}(x^{-1}z_k/z_j).
\]

(3.27)

In the second sum, the operator part is deformed by the fusion relation of coefficients functions.

\[
\prod_{1 \leq j \leq n, \ j \notin B} g_{12}(x^{-1}z_i/z_j) \prod_{1 \leq k \leq n, k \notin B, k \in A_{\text{Min}}} g_{22}(x^{-1}z_k/z_i)T_2(x^{-1}z_i)\delta(x^2 z_{i+1}/z_i) \\
- \prod_{1 \leq j \leq n, \ j \notin B} g_{12}(x^{-1}z_i/z_j) \prod_{1 \leq k \leq n, k \notin B, k \in A_{\text{Min}}} g_{22}(x^{-1}z_k/z_{i+1})T_2(x^{-1}z_{i+1})\delta(x^2 z_i/z_{i+1}) \\
= \prod_{1 \leq j \leq n, \ j \notin B} g_{11}(z_i/z_j)g_{11}(z_{i+1}/z_j) \prod_{1 \leq k \leq n, k \notin B, k \in A_{\text{Min}}} g_{12}(x^{-1}z_k/z_i)g_{12}(x^{-1}z_k/z_{i+1}) \\
\times (T_2(x^{-1}z_i)\delta(x^2 z_{i+1}/z_i) - T_2(x^{-1}z_{i+1})\delta(x^2 z_i/z_{i+1})).
\]

(3.28)

Substituting this into the second sum in (3.27) and changing the summation variable \(\beta = \alpha + 1\), we conclude \(O_n(z_1, \ldots, z_i, z_{i+1}, \ldots, z_n) \sim O_n(z_1, \ldots, z_{i+1}, z_i, \ldots, z_n)\) is direct consequence of the commutation relation of \(T_1(z)\) in Proposition 2.1. Q.E.D.
Let us set the formal power series \( A(z_1, z_2, \cdots, z_n) \) by
\[
A(z_1, z_2, \cdots, z_n) = \sum_{k_1, \cdots, k_n \in \mathbb{Z}} a_{k_1, \cdots, k_n} z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}.
\] (3.29)

We define the symbol \([\cdot \cdot \cdot]_{1, z_1 \cdots z_n}\) by
\[
[A(z_1, z_2, \cdots, z_n)]_{1, z_1 \cdots z_n} = a_{0,0,\cdots,0}.
\] (3.30)

Let us set \( D = \{ (z_1, \cdots, z_n) \in \mathbb{C}^n | \sum_{k_1, \cdots, k_n \in \mathbb{Z}} |a_{k_1, \cdots, k_n} z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}| < +\infty \} \). When we assume closed curve \( J \) is contained in \( D \), we have
\[
[A(z_1, z_2, \cdots, z_n)]_{1, z_1 \cdots z_n} = \int \cdots \int \prod_{j=1}^n \frac{dz_j}{2\pi i z_j} A(z_1, z_2, \cdots, z_n).
\] (3.31)

Let us set the auxiliary functions, \( s_{11}(z) = s(z) \), \( h_{11}(z) = h(u) \), and
\[
s_{12}(z) = s_{21}(z) = s_{11}(x^{-2}z)s_{11}(x^2z), \ s_{22}(z) = s_{11}(x^{-2}z)s_{11}(z)^2s_{11}(x^2z),
\] \( h_{12}(z) = h_{21}(z) = h_{11}(x^{-2}z)h_{11}(x^2z), \ h_{22}(z) = h_{11}(x^{-2}z)h_{11}(z)^2h_{11}(x^2z) \). (3.32)

We will use the following formulae of the local integrals of motion to show the commutation relations, \([\mathcal{I}_m, \mathcal{I}_n] = [\mathcal{I}_m^*, \mathcal{I}_n^*] = 0\).

**Theorem 3.6** For \( \text{Re}(s) > 2 \) and \( \text{Re}(r) < 0 \), the local integrals of motion \( \mathcal{I}_n \) are written as
\[
\mathcal{I}_n = \left[ \prod_{1 \leq j < k \leq n} s(z_j/z_k) \mathcal{O}_n(z_1, z_2, \cdots, z_n) \right]_{1, z_1 \cdots z_n}.
\] (3.34)

For \( \text{Re}(s) > 2 \) and \( \text{Re}(r^*) > 0 \), the local integrals of motion \( \mathcal{I}_n^* \) are written as
\[
\mathcal{I}_n^* = \left[ \prod_{1 \leq j < k \leq n} s^*(z_j/z_k) \mathcal{O}_n(z_1, z_2, \cdots, z_n) \right]_{1, z_1 \cdots z_n}.
\] (3.35)

**Proof.** We start from
\[
\mathcal{I}_n = \int \cdots \int_C \prod_{j=1}^n \frac{dz_j}{2\pi i z_j} \prod_{1 \leq j < k \leq n} h(u_k - u_j) \prod_{1 \leq j < k \leq n} T_1(z_j),
\] (3.36)

where \( C \) is given by
\[
C : |x^{-2}z_k|, |x^{2+2s}z_k|, |x^{-2r^*}z_k| < |z_j| < |x^{-2-2s}z_k|, |x^2z_k|, |x^{2r^*}z_k| (1 \leq j < k \leq n).
\] (3.37)
Let us take the residue at \( z_n = x^{-2}z_J \) (\( 1 \leq J \leq n-1 \)). We have

\[
\mathcal{I}_n = \int \cdots \int_{\mathcal{C}(1)} \prod_{j=1}^{n} \frac{dz_j}{2\pi \sqrt{-1}z_j} \prod_{1 \leq j < k \leq n} h(u_k - u_j) \prod_{1 \leq j \leq n} T_1(z_j) \\
- \sum_{J=1}^{n-1} \int \cdots \int_{\mathcal{C}(J)} \prod_{j=1}^{n-1} \frac{dz_j}{2\pi \sqrt{-1}z_j} \text{Res}_{z_n = x^{-2}z_J} \frac{dz_n}{z_n} \prod_{1 \leq j < k \leq n} h(u_k - u_j) \prod_{1 \leq j \leq n} T_1(z_j) \bigg|_{z_n = x^{-2}z_J} \tag{3.38}
\]

Here the contours \( \mathcal{C}(J) \), \( 1 \leq J \leq n-1 \), are annulus. Hence defining relations of the deformed Virasoro can be used. Hence we can use the following relation, which is derived by defining relations of the deformed Virasoro algebra.

\[
\text{Res}_{w_2 = x^{-2}w_1} \frac{dw_2}{w_2} \left( \prod_{j=1}^{M} h(u_j - v_1)h(v_2 - u_j) \right) h(v_2 - v_1) \cdot T_1(w_1) \left( \prod_{1 \leq j \leq M} T_1(z_j) \right) T_1(w_2) \\
= c \prod_{j=1}^{M} h_{21}(xz_j/w_1) \cdot T_2(x^{-1}w_1) \left( \prod_{1 \leq j \leq M} T_1(z_j) \right). \tag{3.40}
\]

By the same arguments as above, we can use the weak sense relation

\[
\delta(x^2z_3/z_2)g_{12}(x^{-1}z_2/z_1)T_1(z_1)T_2(x^{-1}z_2) \sim \delta(x^2z_3/z_2)g_{21}(xz_1/z_2)T_2(x^{-1}z_2)T_1(z_1).
\]

Hence we have

\[
\mathcal{I}_n = \int \cdots \int_{\mathcal{C}(1)} \prod_{j=1}^{n} \frac{dz_j}{2\pi \sqrt{-1}z_j} \prod_{1 \leq j < k \leq n} h(u_k - u_j) \prod_{1 \leq j \leq n} T_1(z_j) \\
- c \sum_{J=1}^{n-1} \int \cdots \int_{\mathcal{C}(J)} \prod_{j=1}^{n-1} \frac{dz_j}{2\pi \sqrt{-1}z_j} \prod_{1 \leq j < k \leq n} h_{11}(z_j/z_k) \prod_{1 \leq k \leq n-1 \atop k \neq J} h_{12}(x^{-1}z_J/z_k) \\
\times \prod_{1 \leq j \leq n-1 \atop j \neq J} T_1(z_j) \cdot T_2(x^{-1}z_J) \tag{3.41}
\]
Here the contours \( C(J) \) for \( 1 \leq J \leq n - 1 \) are given by
\[
|z_{2s-2}^k| < |z_j| < |z_{-2s+1}^k| \quad (1 \leq k \neq J \leq n - 1),
\]
\[
|z_{-2}^k| < |z_j| < |z_2^k| \quad (1 \leq j < k \leq n - 1; j, k \neq J).
\] (3.42)

Repeating the same arguments for \( z_{n-1}, z_{n-2}, \ldots \), we have this theorem. Q.E.D.

### 3.3 Proof of \([\mathcal{I}_m, \mathcal{I}_n] = [\mathcal{I}_m^*, \mathcal{I}_n^*] = 0\)

In this section we show the commutation relation \([\mathcal{I}_m, \mathcal{I}_n] = [\mathcal{I}_m^*, \mathcal{I}_n^*] = 0\).

**Proposition 3.7**

\[
\mathcal{O}_n(z_1, \ldots, z_n) \mathcal{O}_m(z_{n+1}, \ldots, z_{n+m}) \sim \prod_{1 \leq j \leq n} \frac{1}{f_{11}(z_k/z_j)} \mathcal{O}_{n+m}(z_1, \ldots, z_{n+m}). \tag{3.43}
\]

**Proof** This is direct consequence of the following explicit formulae
\[
\mathcal{O}_{n+m}(z_1, \ldots, z_{n+m}) \sim \prod_{1 \leq j \leq n} g_{11}(z_k/z_j) \mathcal{O}_n(z_1, \ldots, z_n) \mathcal{O}_m(z_{n+1}, \ldots, z_{n+m})
\]
\[
+ \sum_{\alpha=1}^{\frac{n+m}{2}} (-1)^\alpha c_\alpha (\alpha!)^2 \sum_{L \subseteq \{1, \ldots, n\}} \sum_{R \subseteq \{1, \ldots, n+m\}} \prod_{1 \leq j \leq n+m} T_1(z_j) \prod_{1 \leq t \leq \alpha} \delta \left(x^2 \bar{z}_{\bar{R}(t)} \bar{z}_{\bar{L}(t)} \right) \\
\times \prod_{1 \leq j, k \leq n+m} g_{12}(z_k/z_j) \prod_{j, k \in L} g_{22}(z_k/z_j) \prod_{j, k \in R} g_{21}(x^{-1}z_k/z_j). \tag{3.44}
\]

Here we have set the index \( L(t) \) and \( R(t) \) by \( L = \{L(1) < L(2) < \cdots < L(\alpha)\} \) and \( R = \{R(1) < R(2) < \cdots < R(\alpha)\} \). Q.E.D.

**Proof of Theorem 3.1** At first we restrict ourself to the regime, \( \text{Re}(s) > 2 \) and \( \text{Re}(r) < 0 \), in order to use the power series formulae of the local integrals of motion, \( \mathcal{I}_n \). In Proposition 3.5 we showed
\[
\prod_{1 \leq j < k \leq n} s(z_k/z_j) \mathcal{O}_n(z_1, \ldots, z_n) = \prod_{1 \leq j < k \leq n} s(z_{\sigma(k)/z_{\sigma(j)}}) \mathcal{O}_n(z_{\sigma(1)}, \ldots, z_{\sigma(n)}) \quad (\sigma \in S_n). \tag{3.45}
\]

Hence we have
\[
\mathcal{I}_n \cdot \mathcal{I}_m
\]
\[
\begin{align*}
&= \left[ \prod_{1 \leq j < k \leq n} s(z_k/z_j) \mathcal{O}_n(z_1, \ldots, z_n) \prod_{n+1 \leq j < k \leq n+m} s(z_k/z_j) \mathcal{O}_m(z_{n+1}, \ldots, z_{n+m}) \right]_{1, z_1 \cdots z_{n+m}} \\
&= \left[ \frac{1}{(n+m)!} \sum_{\sigma \in S_{n+m}} \prod_{j=1}^{n} \prod_{k=n+1}^{n+m} \frac{1}{h(u_{\sigma(k)} - u_{\sigma(j)})} \prod_{1 \leq j < k \leq n+m} s(z_k/z_j) \mathcal{O}_{n+m}(z_1, \ldots, z_{n+m}) \right]_{1, z_1 \cdots z_{n+m}}.
\end{align*}
\]

Hence the commutation relation \( \mathcal{I}_n \cdot \mathcal{I}_m = \mathcal{I}_m \cdot \mathcal{I}_n \) is reduced to the following theta identity.

\[
\text{LHS}(n, m) = \text{RHS}(n, m) \quad (3.47)
\]

where we have set

\[
\text{LHS}(n, m) = \sum_{J \subset \{1, 2, \ldots, n+m\}} \prod_{j \in J \setminus J} \prod_{k \in J} \frac{[u_k - u_j + 1]_s [u_k - u_j + r^*]_s}{[u_k - u_j]_s [u_k - u_j + r]_s}, 
\]

\[
\text{RHS}(n, m) = \sum_{J' \subset \{1, 2, \ldots, n+m\}} \prod_{j \in J' \setminus J'} \prod_{k \in J'} \frac{[u_k - u_j + 1]_s [u_k - u_j + r^*]_s}{[u_k - u_j]_s [u_k - u_j + r]_s}. 
\]

where we have used \( h(u) = \frac{[u]}{[u + r]_s [u + r^*]_s} \). LHS \((n, m)\) and RHS \((n, m)\) are an elliptic functions. Therefore, from Liouville theorem, it is enough to check whether all the residues of LHS \((n, m)\) and RHS \((n, m)\) coincide or not. Candidates of poles are \( u_\alpha = u_\beta (\alpha \neq \beta) \) and \( u_\alpha = u_\beta - r (\alpha \neq \beta) \). Let us consider \( u_\alpha = u_\beta (\alpha \neq \beta) \)

\[
\text{LHS}(n, m) = \left( \frac{[u_\alpha - u_\beta + 1]_s [u_\alpha - u_\beta + r^*]_s}{[u_\alpha - u_\beta]_s [u_\alpha - u_\beta - r]_s} + \frac{[u_\beta - u_\alpha + 1]_s [u_\beta - u_\alpha + r^*]_s}{[u_\beta - u_\alpha]_s [u_\beta - u_\alpha - r]_s} \right) \times \sum_{J \cup J' \subset \{1, 2, \ldots, n+m\} - \{\alpha, \beta\}} \prod_{j \in J \setminus J'} \prod_{k \in J'} \frac{[u_k - u_j + 1]_s [u_k - u_j + r^*]_s}{[u_k - u_j]_s [u_k - u_j + r]_s} 
\]

\[
(3.50)
\]

Hence we have \( \text{Res}_{u_\alpha = u_\beta} \text{LHS}(n, m) = 0 \). As the same manner we have \( \text{Res}_{u_\alpha = u_\beta} \text{RHS}(n, m) = 0 \). Therefore \( u_\alpha = u_\beta \) is not pole. We only have to consider poles \( u_\alpha = u_\beta (\alpha \neq \beta) \). We show the LHS \((n, m)\) = RHS \((n, m)\) by the induction of the number \( n + m \). We assume \( n > m \geq 1 \) without losing generality. At first we show the starting point \( n > m = 1 \).

\[
\sum_{k=1}^{n+1} \prod_{j=1 \atop j \neq k}^{n+1} \frac{[u_j - u_k + 1]_s [u_j - u_k]_s}{[u_j - u_k]_s [u_j - u_k + r]_s} = \sum_{k=1}^{n+1} \prod_{j=1 \atop j \neq k}^{n+1} \frac{[u_k - u_j + 1]_s [u_k - u_j]_s}{[u_k - u_j]_s [u_k - u_j + r]_s} 
\]

\[
(3.51)
\]

Both LHS \((n, 1)\) and RHS \((n, 1)\) have simple poles at \( u_\alpha = u_\beta - r (\alpha \neq \beta) \) modulo \( \mathbb{Z} + \mathbb{Z} \tau \). Because both LHS \((n, 1)\) and RHS \((n, 1)\) are symmetric with respect with \( u_1, u_2, \ldots, u_{n+1} \).
it is enough to check the pole at \( u_2 = u_1 - r \). We have

\[
\text{Res}_{u_2 = u_1 - r} \text{LHS}(n, 1) = \text{Res}_{u_2 = u_1 - r} \text{RHS}(n, 1)
\]

\[
= \text{Res}_{u=0} \frac{[-r^*]_s[-1]_s \prod_{j=3}^{n+1} [u_j - u_1 + 1]_s [u_j - u_1 + r^*]_s}{[u_j - u_1]_s [u_j - u_1 + r]_s}. \tag{3.52}
\]

We have shown \( n > m = 1 \) case. We show general \( n > m \geq 1 \) case. We assume the equation \( \text{LHS}(n, 1) = \text{RHS}(n, 1) \) for some \((m, n)\). Because both \( \text{LHS}(n + 1, m + 1) \) and \( \text{RHS}(n + 1, m + 1) \) are symmetric with respect with \( u_1, u_2, \ldots, u_{n+m+2} \), it is enough to check the pole at \( u_2 = u_1 - r \). Let us take the residue at \( u_2 = u_1 - r \) for \((m + 1, n + 1)\).

\[
\text{Res}_{u_2 = u_1 - r} \left( \sum_{J \subset \{1,2,\ldots,n+m+2\}} \prod_{|J| = n+1} \prod_{j \in J} \prod_{k \notin J} \frac{[u_k - u_j + 1]_s [u_k - u_j + r^*]_s}{[u_k - u_j]_s [u_k - u_j + r]_s} \right)
\]

\[
= \left( \sum_{J \subset \{3,4,\ldots,n+m+2\}} \prod_{|J| = n} \prod_{j \in J} \prod_{k \in J^c} \frac{[u_k - u_j + 1]_s [u_k - u_j + r^*]_s}{[u_k - u_j]_s [u_k - u_j + r]_s} \right) \times \left( \sum_{J \subset \{1,2,\ldots,n+m+2\}} \prod_{|J| = m+1} \prod_{j \in J^c} \prod_{k \notin J} \frac{[u_k - u_j + 1]_s [u_k - u_j + r^*]_s}{[u_k - u_j]_s [u_k - u_j + r]_s} \right)
\]

\[
= 0. \tag{3.53}
\]

We have used the assumption of induction \( \text{LHS}(n, m) = \text{RHS}(n, m) \). We have proved the equation \([\mathcal{I}_n, \mathcal{I}_m] = 0\) for general \( n > m \geq 1 \). Generalization to generic parameter case : \( \text{Re}(s) > 0 \) and \( r \in \mathbb{C} \), should be understood as analytic continuation. The commutation relation \([\mathcal{I}_n^*, \mathcal{I}_m^*] = 0\) is shown by the same manner as above. Q.E.D.

### 3.4 Proof of Dynkin Automorphism Invariance \( \eta(\mathcal{I}_n) = \mathcal{I}_n \)

In this section we show the Dynkin automorphism invariance \( \eta(\mathcal{I}_n) = \mathcal{I}_n \). Before showing the complete proof, we consider some simple examples as warming-up exercise. We study the case : \( \text{Re}(s) > 2 \) and \( 1 - \text{Re}(s) < \text{Re}(r) < 0 \). Let us study \( n = 2 \) case. The following
Using the above relation, we have
\[ h_{11}(z) - h_{11}(x^2 z) = c_h(\delta(x^2 z) - \delta(x^{2r-2+2s} z)), \] (3.54)
\[ c_h = -\frac{(x^2 ; x^{2s})_\infty (x^{2r-2} ; x^{2s})_\infty (x^{2s-2} ; x^{2s})_\infty (x^{2s-2r+2} ; x^{2s})_\infty}{(x^{2r-4} ; x^{2s})_\infty (x^{2s} ; x^{2s})_\infty (x^{2s} ; x^{2s})_\infty (x^{2r-2+2} ; x^{2s})_\infty}. \]

Using the above relation, we have
\[ \eta(h_{11}(z_2/z_1)T_1(z_1)T_1(z_2)) = h_{11}(z_2/z_1)(A_1(x^{-s} z_1)A_1(x^{-s} z_2) + A_2(x^s z_1)A_2(x^s z_2)) + h_{11}(x^{2s} z_2/z_1)A_1(x^s z_1)A_2(x^s z_2) + c_h(\delta(x^{2s} z_2/z_1) - \delta(x^{2r-2+2s} z_2/z_1))A_1(x^{-s} z_1)A_2(x^s z_2) - c_h(\delta(x^{2r-2} z_2/z_1) - \delta(x^{2r-2} z_2/z_1))A_2(x^s z_1)A_1(x^{-s} z_2). \] (3.55)

The delta-function yields
\[ \delta(x^{2r-2+2s} z_2/z_1)A_1(x^{-s} z_1)A_2(x^s z_2) = \delta(x^{2r-2} z_2/z_1)A_2(x^s z_1)A_1(x^{-s} z_2) = 0, \] (3.56)
\[ c_h\delta(x^{2s} z_2/z_1)A_2(x^s z_1)A_1(x^{-s} z_2) = cs(x^{-2})\delta(x^{2s} z_2/z_1)T_2(x^{-1} z_1), \] (3.57)
\[ c_h\delta(x^{2s} z_2/z_1)A_1(x^{-s} z_1)A_2(x^s z_2) = cs(x^{-2})\delta(x^{2s} z_2/z_1)\eta(T_2(x^{-1} z_1)). \] (3.58)

Hence we have \( \eta(I_2) = I_2 \). Let us study \( n = 3 \) case. In what follows we use the following abbreviation.
\[ h_{12}(z) = h_{11}(x^{-1} z)h_{11}(x z), \quad h_{12}^\eta(z) = h_{11}(x^{-1} z)h_{11}(x^{-s+1} z). \] (3.59)

We have
\[ \eta \left( \prod_{1 \leq j < k \leq 3} h_{11}(z_k/z_j)A_2(z_j)A_2(z_k)A_1(z_3) \right)_{1,z_1 z_2 z_3} \]
\[ = \prod_{1 \leq j < k \leq 3} h_{11}(z_k/z_j)A_1(z_1)A_1(z_2)A_2(z_3)_{1,z_1 z_2 z_3} \]
\[ + \left[ cs(x^{-2})h_{12}^\eta(x^{-1} z_1/z_2)\delta(x^2 z_3/z_1)A_1(z_2)\eta(T_2(x^{-1} z_1)) \right]_{1,z_1 z_2 z_3} \]
\[ + \left[ cs(x^{-2})h_{12}^\eta(x^{-1} z_2/z_1)\delta(x^2 z_3/z_2)A_1(z_1)\eta(T_2(x^{-1} z_2)) \right]_{1,z_1 z_2 z_3}. \] (3.60)

Moreover we have
\[ \eta \left( \prod_{1 \leq j < k \leq 3} h_{11}(z_k/z_j)T_1(z_1)T_1(z_2)T_1(z_3) \right)_{1,z_1 z_2 z_3} \]
As the same manner, we have

\[
\eta(\{h_{12}(x^{-1}z_2/z_1)\delta(x^2z_3/z_2)T_1(z_1)T_2(x^{-1}z_2)\})_{z_1, z_2, z_3}
\]

Summing up all the above, we have \( \eta(\mathcal{I}_3) = \mathcal{I}_3 \).

**Proof.** Let us start general \( n \) case. For a while we study: \( \text{Re}(s) > 2 \) and \( 1 - \text{Re}(s) < \text{Re}(r) < 0 \). In what follows we use the following abbreviations.

\[
h_{22}^\eta(z) = h_{11}(x^{-s}z)h_{11}(x^{-s+2}z)h_{11}(x^s)h_{11}(x^{-2s}z),
\]

\[
h_{22}^{\eta\eta}(z) = h_{11}(x^{-2s+2}z)h_{11}(z)^2h_{11}(x^{2s-2}z).
\]

We have

\[
\eta \left( \prod_{1 \leq j < k \leq n} h_{11}(z_k/z_j)T_1(z_1)T_1(z_2) \cdots T_1(z_n) \right)_{1, z_1 \ldots z_n}
\]

\[
= \sum_{0 \leq \alpha, \beta \leq 0} (-1)^\alpha (cs(x^{-2}))^{\alpha + \beta} \sum_{A_1, \ldots, A_\alpha, B_1, \ldots, B_\beta} \prod_{1 \leq j \leq \alpha} \delta \left( x^{2z_{\text{Max}}(A_j)} \right) \prod_{1 \leq j \leq \beta} \delta \left( x^{2z_{\text{Min}}(A_j)} \right) \times \prod_{j \in A_{\text{Min}}} T_1(z_j) \prod_{j \in B_{\text{Min}}} T_2(x^{-1}z_j) \prod_{j, k \in A_{\text{Min}}} h_{11}(z_k/z_j) \prod_{j \in A_{\text{Min}}, k \in B_{\text{Min}}} h_{12}(z_k/z_j) \prod_{j \in A_{\text{Min}}, k \in B_{\text{Min}}} h_{12}^\eta(z_k/z_j) \prod_{j \in A_{\text{Min}}, k \in B_{\text{Min}}} h_{22}(z_k/z_j) \prod_{j \in A_{\text{Min}}, k \in B_{\text{Min}}} h_{22}^{\eta\eta}(z_k/z_j) \right)_{1, z_1 \ldots z_n}.
\]
We explain the notation of the above formulae (3.65). The summation $\sum_{A_1, \ldots, A_\alpha, B_1, \ldots, B_\beta}$ is taken over the set $A_1, A_2, \ldots, A_\alpha, B_1, B_2, \ldots, B_\beta \subset \{1, 2, \ldots, n\}$ such that $A_j \cap A_k = \phi$ ($1 \leq j \neq k \leq \alpha$), $B_j \cap B_k = \phi$ ($1 \leq j \neq k \leq \beta$), $|A_j| = |B_k| = 2$ ($1 \leq j \leq \alpha, 1 \leq k \leq \beta$), and $\text{Min}(A_j) < \text{Min}(A_\beta)$ ($1 \leq j < k \leq \alpha$), $\text{Min}(B_j) < \text{Min}(B_\beta)$ ($1 \leq j < k \leq \beta$). We have set $A_{\text{Min}} = \{\text{Min}(A_1), \ldots, \text{Min}(A_\alpha)\}$, $A_{\text{Max}} = \{\text{Max}(A_1), \ldots, \text{Max}(A_\alpha)\}$, $B_{\text{Min}} = \{\text{Min}(B_1), \ldots, \text{Min}(B_\beta)\}$, $B_{\text{Max}} = \{\text{Min}(B_1), \ldots, \text{Min}(B_\beta)\}$, and $A = A_1 \cup \cdots \cup A_\alpha$, $B = B_1 \cup \cdots \cup B_\beta$. We have

$$\eta\left(\prod_{1 \leq j < k \leq n} h_{11}(z_k/z_j) \prod_{n+1 \leq j < k \leq n+m} h_{12}(x^{-1}z_k/z_j) \prod_{n+1 \leq j < k \leq n+m} h_{22}(z_k/z_j) \times T_1(z_1) \cdots T_1(z_n) T_2(x^{-1}z_{n+1}) \cdots T_2(x^{-1}z_{n+m}) \prod_{1 \leq j \leq m} \delta(x^{2}z_{n+m+j}/z_{n+j})\right)$$

$$= \sum_{a, \beta \geq 0} (-1)^a (\text{cs}(x^{-2}))^{a+\beta} \sum_{A_1, \ldots, A_\alpha, B_1, \ldots, B_\beta} \prod_{1 \leq j \leq \alpha} \delta(x^{2}z_{\text{Max}(A_j)}/z_{\text{Min}(A_j)}) \prod_{1 \leq j \leq \beta} \delta(x^{2}z_{\text{Max}(A_\beta)}/z_{\text{Min}(A_\beta)})$$

$$\times \prod_{1 \leq j \leq m} \delta\left(x^{2}z_{n+m+j}/z_{n+j}\right) \times \prod_{1 \leq j \leq n, j \notin A \cup B} T_1(z_j) \prod_{j \in A_{\text{Min}}} T_2(x^{-1}z_j) \prod_{j \in B_{\text{Min}} \cup \{n+1, \ldots, n+m\}} \eta(T_2(x^{-1}z_j))$$

$$\times \prod_{1 \leq j \leq n, j \notin A \cup B} h_{11}(z_j/z_j) \prod_{j \in A_{\text{Min}}} h_{12}(x^{-1}z_j/z_j) \prod_{j \in B_{\text{Min}} \cup \{n+1, \ldots, n+m\}} h_{12}^{\eta}(x^{-1}z_j/z_j)$$

$$\times \prod_{j < k} h_{22}(z_k/z_j) \prod_{j \in A_{\text{Min}}} h_{22}^{\eta}(z_k/z_j) \prod_{j, k \in B_{\text{Min}} \cup \{n+1, \ldots, n+m\}} h_{22}^{\eta}(z_k/z_j)$$

Using the formulae (3.17), (3.65) and (3.66), we have $\eta(I_n) = I_n$ for general $n$. Cancellations of coefficients of $\prod T_1 \prod T_2 \prod \eta(T_2)$ come from relation $0 = (1-1)^k = \sum_{l=0}^{k} (-1)^k kC_l$. Generalization to generic parameter Re($s$) > 0 and $r \in \mathbb{C}$ should be understood as analytic continuation. Q.E.D.
4 Nonlocal Integrals of Motion $G_n$

In this section we explicitly construct the nonlocal integrals of motion $G_n$ and $G_n^*$. In this section we study generic case: $0 < x < 1$, $\text{Re}(r) \neq 0$ and $s \in \mathbb{C}$ (resp. $0 < x < 1$, $\text{Re}(r^*) \neq 0$ and $s \in \mathbb{C}$). Let us use the parametrization: $z_j = x^{2w_j}, w_j = x^{2v_j}$.

4.1 Nonlocal Integrals of Motion $G_n$

We explicitly construct the nonlocal integrals of motion $G_n$ and $G_n^*$, and state the main results.

Let us set the theta function $\vartheta_{\alpha,r}(u)$ by

$$
\vartheta_{\alpha,r}(u) = [u - \pi + \alpha]_r [u - \alpha]_r + [u - \pi - \alpha]_r [u + \alpha]_r.
$$

(4.1)

This function $\vartheta_{\alpha,r}(u)$ satisfies

$$
\vartheta_{\alpha,r}(u + r\tau) = e^{-2\pi \sqrt{1-r} + 2\pi \sqrt{r}(\pi + 2u)} \vartheta_{\alpha,r}(u),
\vartheta_{\alpha,r}(u) = \vartheta_{-\alpha,r}(u).
$$

(4.2)

**Definition 4.1**

- **We define $G_n$ ($n = 1, 2, \cdots$) for the regime $\text{Re}(r) > 0$ and $0 < \text{Re}(s) < 2$ by**

$$
G_n = \int \cdots \int \prod_{j=1}^{n} \frac{dz_j}{2\pi \sqrt{1-z_j}} \prod_{j=1}^{n} \frac{dw_j}{2\pi \sqrt{1-w_j}} \prod_{1 \leq j \leq n} F_1(z_j) \prod_{1 \leq j \leq n} F_0(w_j)
\times \frac{\prod_{1 \leq i < j \leq n} [u_i - u_j]_r [u_j - u_i - 1]_r [v_i - v_j]_r [v_j - v_i - 1]_r}{\prod_{i,j=1}^{n} [u_i - v_j + \frac{s}{2}]_r [v_j - u_i + \frac{s}{2} - 1]_r} \vartheta_{\alpha,r} \left( \sum_{j=1}^{n} (u_j - v_j) \right).
$$

(4.3)

Here the contour $I$ encircles $z_i, w_i = 0$ in such a way that

1. $z_j = x^{s+2lr} w_i, x^{2-s+2lr} w_i$ ($l = 0, 1, 2, \cdots$) is inside and $z_j = x^{-s-2lr} w_i, x^{-2-s-2lr} w_i$ ($l = 0, 1, 2, \cdots$) is outside for $i, j = 1, 2, \cdots, n$,
2. $z_p = x^{2r^*+2lr} z_q$ ($l = 0, 1, 2, \cdots$) is inside $z_p = x^{-2r^*-2lr} z_q$ ($l = 0, 1, 2, \cdots$) is outside for $1 \leq p < q \leq n$,
3. $w_p = x^{2r^*+2lr} w_q$ ($l = 0, 1, 2, \cdots$) is inside $w_p = x^{-2r^*-2lr} w_q$ ($l = 0, 1, 2, \cdots$) is outside for $1 \leq p < q \leq n$.
We define $G_n$ $(n = 1, 2, \cdots)$ for the regime $\text{Re}(r) < 0$ and $0 < \text{Re}(s) < 2$ by

$$G_n = \int \cdots \int \prod_{j=1}^{n} \frac{dz_j}{2\pi \sqrt{-1}z_j} \prod_{j=1}^{n} \frac{dw_j}{2\pi \sqrt{-1}w_j} \prod_{1 \leq j \leq n} F_1(z_j) \prod_{1 \leq j \leq n} F_0(w_j)$$

$$\times \frac{\prod_{1 \leq i < j \leq n} [u_i - u_j]^{-1} [v_j - v_i + 1]^{-1} [v_i - v_j]^{-1} [v_j - v_i + 1]^{-1} \theta_{\alpha \cdot r} \left( \sum_{j=1}^{n} (v_j - u_j) \right)}{\prod_{i,j=1}^{n} [u_i - v_j - \frac{s}{2}]^{-1} [v_j - u_i - \frac{s}{2} + 1]^{-1}}.$$

Here the contour $I$ encircles $z_i, w_i = 0$ in such a way that

1. $z_j = x^{-2s-2r} w_i, x^2-2-2r w_i$ $(l = 0, 1, 2, \cdots)$ is inside and $z_j = x^{-2s-2r} w_i, x^2-2-2r w_i$ $(l = 0, 1, 2, \cdots)$ is outside for $i, j = 1, 2, \cdots, n$,

2. $z_p = x^{-2r} x^{-2r} z_q$ $(l = 0, 1, 2, \cdots)$ is inside $z_p = x^{-2r} x^{-2r} z_q$ $(l = 0, 1, 2, \cdots)$ is outside for $1 \leq p < q \leq n$,

3. $w_p = x^{-2r} x^{-2r} w_q$ $(l = 0, 1, 2, \cdots)$ is inside $w_p = x^{-2r} x^{-2r} w_q$ $(l = 0, 1, 2, \cdots)$ is outside for $1 \leq p < q \leq n$.

We define $G_n^*$ $(n = 1, 2, \cdots)$ for the regime $\text{Re}(r^*) > 0$ and $0 < \text{Re}(s) < 2$ by

$$G_n^* = \int \cdots \int \prod_{j=1}^{n} \frac{dz_j}{2\pi \sqrt{-1}z_j} \prod_{j=1}^{n} \frac{dw_j}{2\pi \sqrt{-1}w_j} \prod_{1 \leq j \leq n} E_1(z_j) \prod_{1 \leq j \leq n} E_0(w_j)$$

$$\times \frac{\prod_{1 \leq i < j \leq n} [u_i - u_j]^{-1} [v_j - v_i + 1]^{-1} [v_i - v_j]^{-1} [v_j - v_i + 1]^{-1} \theta_{\alpha \cdot r^*} \left( \sum_{j=1}^{n} (v_j - u_j) \right)}{\prod_{i,j=1}^{n} [u_i - v_j - \frac{s}{2}]^{-1} [v_j - u_i - \frac{s}{2} + 1]^{-1}}.$$

Here the contour $I^*$ encircles $z_i, w_i = 0$ in such a way that

1. $z_j = x^{s+2r} w_i, x^{2-2+2r} w_i$ $(l = 0, 1, 2, \cdots)$ is inside and $z_j = x^{s-2r} w_i, x^{2-2-2r} w_i$ $(l = 0, 1, 2, \cdots)$ is outside for $i, j = 1, 2, \cdots, n$,

2. $z_p = x^{2r+2r} z_q$ $(l = 0, 1, 2, \cdots)$ is inside $z_p = x^{2r-2r} z_q$ $(l = 0, 1, 2, \cdots)$ is outside for $1 \leq p < q \leq n$,

3. $w_p = x^{2r+2r} w_q$ $(l = 0, 1, 2, \cdots)$ is inside $w_p = x^{2r-2r} w_q$ $(l = 0, 1, 2, \cdots)$ is outside for $1 \leq p < q \leq n$.

We define $G_n^*$ $(n = 1, 2, \cdots)$ for the regime $\text{Re}(r^*) < 0$ and $0 < \text{Re}(s) < 2$ by

$$G_n^* = \int \cdots \int \prod_{j=1}^{n} \frac{dz_j}{2\pi \sqrt{-1}z_j} \prod_{j=1}^{n} \frac{dw_j}{2\pi \sqrt{-1}w_j} \prod_{1 \leq j \leq n} E_1(z_j) \prod_{1 \leq j \leq n} E_0(w_j)$$
\[ \prod_{1 \leq i < j \leq n} [u_i - u_j]_{-r^*} [u_j - u_i - 1]_{-r^*} [v_j - v_i - 1]_{-r^*} \prod_{i,j=1}^{n} [v_j - u_i + \frac{s}{2} - 1]_{-r^*}^\theta_{\alpha,-r^*} \left( \sum_{j=1}^{n} (u_j - v_j) \right). \] (4.6)

Here the contour \( I^* \) encircles \( z_i, w_i = 0 \) in such a way that

1. \( z_j = x^{s-2r^*} w_i, x^{2-s-2r^*} w_i (l = 0, 1, 2, \cdots) \) is inside and \( z_j = x^{-s+2r^*} w_i, x^{s-2+2r^*} w_i (l = 0, 1, 2, \cdots) \) is outside for \( i, j = 1, 2, \cdots, n \).
2. \( z_p = x^{-2r-2r^*} z_q (l = 0, 1, 2, \cdots) \) is inside \( z_p = x^{2r+2r^*} z_q (l = 0, 1, 2, \cdots) \) is outside for \( 1 \leq p < q \leq n \).
3. \( w_p = x^{-2r-2r^*} w_q (l = 0, 1, 2, \cdots) \) is inside \( w_p = x^{2r+2r^*} w_q (l = 0, 1, 2, \cdots) \) is outside for \( 1 \leq p < q \leq n \).

We call \( G_n \) and \( G_n^* \) the nonlocal integrals of motion for the deformed Virasoro algebra. The definitions of \( G_n \) and \( G_n^* \) for generic \( s \in \mathbb{C} \), should be understood as analytic continuation.

**Example** For \( \text{Re}(r) > 0 \) and \( 0 < \text{Re}(s) < 2 \) we have

\[ G_1 = \int \int \frac{dz_1}{2\pi \sqrt{-1} z_1} \frac{dz_2}{2\pi \sqrt{-1} z_2} F_1(z_1) F_0(z_2) \frac{\theta_{\alpha,r}(u_1 - u_2)}{[u_1 - u_2 + \frac{s}{2}]_{-r}[u_1 - u_2 - \frac{s}{2} + 1]_{-r}}. \] (4.7)

For \( \text{Re}(r) < 0 \) and \( 0 < \text{Re}(s) < 2 \) we have

\[ G_1^* = \int \int \frac{dz_1}{2\pi \sqrt{-1} z_1} \frac{dz_2}{2\pi \sqrt{-1} z_2} F_1(z_1) F_0(z_2) \frac{\theta_{\alpha,r^*}(u_2 - u_1)}{[u_1 - u_2 + \frac{s}{2}]_{-r}[u_1 - u_2 + \frac{s}{2} - 1]_{-r}}. \] (4.8)

For \( \text{Re}(r^*) > 0 \) and \( 0 < \text{Re}(s) < 2 \) we have

\[ G_1^* = \int \int \frac{dz_1}{2\pi \sqrt{-1} z_1} \frac{dz_2}{2\pi \sqrt{-1} z_2} E_1(z_1) E_0(z_2) \frac{\theta_{\alpha,r^*}(u_2 - u_1)}{[u_1 - u_2 + \frac{s}{2}]_{-r^*}[u_1 - u_2 - \frac{s}{2} + 1]_{-r^*}}. \] (4.9)

For \( \text{Re}(r^*) < 0 \) and \( 0 < \text{Re}(s) < 2 \) we have

\[ G_1^* = \int \int \frac{dz_1}{2\pi \sqrt{-1} z_1} \frac{dz_2}{2\pi \sqrt{-1} z_2} E_1(z_1) E_0(z_2) \frac{\theta_{\alpha,-r^*}(u_1 - u_2)}{[u_1 - u_2 + \frac{s}{2}]_{-r^*}[u_1 - u_2 - \frac{s}{2} + 1]_{-r^*}}. \] (4.10)

The contour \( I \) and \( I^* \) encircles \( z_1 = 0 \) in such a way that \( z_1 = x^{s+2r} z_2, x^{2-s+2r} z_2 (l = 0, 1, 2, \cdots) \) is inside and \( z_1 = x^{-s-2r} z_2, x^{s-2-2r} z_2 (l = 0, 1, 2, \cdots) \) is outside.

The followings are some of **Main Results** of our paper.
Theorem 4.1 The nonlocal integrals of motion $\mathcal{G}_n$ (resp. $\mathcal{G}_n^*$) commute with each other for generic $\text{Re}(r) \neq 0$ and $s \neq 2$ (resp. $\text{Re}(r^*) \neq 0$ and $s \neq 2$)

$$[\mathcal{G}_n, \mathcal{G}_m] = 0, \quad [\mathcal{G}_n^*, \mathcal{G}_m^*] = 0 \quad (n, m = 1, 2, \cdots). \quad (4.11)$$

Theorem 4.2 The nonlocal integrals of motion $\mathcal{G}_n$ and $\mathcal{G}_n^*$ commute with each other for generic $0 < \text{Re}(r) < 1$ and $s \neq 2$

$$[\mathcal{G}_n, \mathcal{G}_m^*] = 0 \quad (n, m = 1, 2, \cdots). \quad (4.12)$$

Theorem 4.3 The nonlocal integrals of motion $\mathcal{G}_n$ (resp. $\mathcal{G}_n^*$) are invariant under the action of Dynkin automorphism $\eta$, for generic $\text{Re}(r) \neq 0$ and $s \neq 2$ (resp. $\text{Re}(r^*) \neq 0$ and $s \neq 2$).

$$\eta(\mathcal{G}_n) = \mathcal{G}_n, \quad \eta(\mathcal{G}_n^*) = \mathcal{G}_n^* \quad (n = 1, 2, \cdots). \quad (4.13)$$

Theorem 4.4 For generic parameter $\text{Re}(r) \neq 0$, $\text{Re}(s) > 0$, we have

$$[\mathcal{I}_n, \mathcal{G}_m] = 0, \quad [\mathcal{I}_n^*, \mathcal{G}_m] = 0 \quad (n, m = 1, 2, \cdots). \quad (4.14)$$

For generic parameter $\text{Re}(r^*) \neq 0$, $\text{Re}(s) > 0$, we have

$$[\mathcal{I}_n, \mathcal{G}_m^*] = 0, \quad [\mathcal{I}_n^*, \mathcal{G}_m^*] = 0 \quad (n, m = 1, 2, \cdots). \quad (4.15)$$

4.2 Proof of $[\mathcal{G}_n, \mathcal{G}_m] = [\mathcal{G}_n^*, \mathcal{G}_m^*] = 0$

In this section we show the commutation relations $[\mathcal{G}_n, \mathcal{G}_m] = [\mathcal{G}_n^*, \mathcal{G}_m^*] = 0.$ At first we show the theta function identity, which gives a generalization of the one in [5].

Theorem 4.5 The following theta function identity holds

$$\sum_{\substack{|K| = n, \ K \cap \mathbb{C}^* = \emptyset \atop |K| = n, \ K \cap \mathbb{C}^* = \emptyset}} \mathcal{I}_n \left( \sum_{j \in K} u_j - \sum_{j \in L} v_j \right) \mathcal{I}_n^* \left( \sum_{j \in K} u_j - \sum_{j \in L} v_j \right)$$

$$\times \prod_{i \in K} \prod_{j \in L} \frac{[v_j - u_k + \frac{s}{2} r][u_i - v_j + \frac{s}{2} r][u_k - v_j + \frac{s}{2} - 1][v_i - u_j + \frac{s}{2} - 1]}{[u_i - u_k][v_j - v_i][u_k - u_i - 1][v_j - v_i - 1]}$$

$$= \sum_{\substack{|K| = n, \ K \cap \mathbb{C}^* = \emptyset \atop |K| = n, \ K \cap \mathbb{C}^* = \emptyset}} \mathcal{I}_n \left( \sum_{j \in K} u_j - \sum_{j \in L} v_j \right) \mathcal{I}_n^* \left( \sum_{j \in K} u_j - \sum_{j \in L} v_j \right)$$

$$\times \prod_{i \in K} \prod_{j \in L} \frac{[v_i - u_i + \frac{s}{2}][u_k - v_j + \frac{s}{2}][u_i - v_i + \frac{s}{2} - 1][v_k - u_k + \frac{s}{2} - 1]}{[u_k - u_i][v_i - v_j][u_i - u_k - 1][v_j - v_i - 1]}.$$  \quad (4.16)
Here the theta functions $\hat{\vartheta}_{\alpha,r}(u)$ and $\hat{\vartheta}_{\beta,r}(u)$ are characterized by

$$\begin{align*}
\hat{\vartheta}_{\alpha,r}(u + r) &= \hat{\vartheta}_{\alpha,r}(u), \\
\hat{\vartheta}_{\alpha,r}(u + r \tau) &= e^{-2\pi \sqrt{-1} r + \frac{2\pi i}{r} (\pi + 2a) + \nu \hat{\vartheta}_{\alpha,r}(u)}, \quad (\nu \in \mathbb{C}), \\
\hat{\vartheta}_{\beta,r}(u + r) &= \hat{\vartheta}_{\beta,r}(u), \\
\hat{\vartheta}_{\beta,r}(u + r \tau) &= e^{-2\pi \sqrt{-1} r + \frac{2\pi i}{r} (\pi + 2b) + \nu \hat{\vartheta}_{\beta,r}(u)}, \quad (\nu \in \mathbb{C}).
\end{align*}$$

**Proof** In order to consider elliptic function, we divide the above theta identity by $\hat{\vartheta}_{r,\gamma}(\sum_{j=1}^{n+m} (u_j - v_j))$ with $\nu \gamma \in \mathbb{C}$. Let us set

$$\begin{align*}
\text{LHS}(n, m) &= \sum_{K \cup K^c = \{1, \ldots, n+m\}} \sum_{|K| = n, K \cap K^c = \emptyset} \sum_{L \cap L^c = \emptyset}
\times \frac{\hat{\vartheta}_{\alpha,r}(\sum_{j \in K} u_j - \sum_{j \in L} v_j)\hat{\vartheta}_{\beta,r}(\sum_{j \in K^c} u_j - \sum_{j \in L^c} v_j)}{\hat{\vartheta}_{\gamma,r}(\sum_{j=1}^{n+m} (u_j - v_j))}
\times \prod_{j \in K \cap L} \prod_{j \in K^c \cap L^c} [u_j - u_k + \frac{s}{2}r] [u_i - v_j + \frac{s}{2}r] [u_k - v_j + \frac{s}{2}r - 1] [v_i - u_j + \frac{s}{2}r - 1].
\end{align*}$$

$$\begin{align*}
\text{RHS}(n, m) &= \sum_{K \cup K^c = \{1, \ldots, n+m\}} \sum_{|K| = n, K \cap K^c = \emptyset} \sum_{L \cap L^c = \emptyset}
\times \frac{\hat{\vartheta}_{\beta,r}(\sum_{j \in K} u_j - \sum_{j \in L} v_j)\hat{\vartheta}_{\alpha,r}(\sum_{j \in K^c} u_j - \sum_{j \in L^c} v_j)}{\hat{\vartheta}_{\gamma,r}(\sum_{j=1}^{n+m} (u_j - v_j))}
\times \prod_{j \in K \cap L} \prod_{j \in K^c \cap L^c} [v_i - u_i + \frac{s}{2}r] [u_k - v_j + \frac{s}{2}r] [u_i - v_i + \frac{s}{2}r - 1] [v_j - u_k + \frac{s}{2}r - 1].
\end{align*}$$

We will show $\text{LHS}(n, m) = \text{RHS}(n, m)$ by induction. Both LHS$(n, m)$ and RHS$(n, m)$ are elliptic functions. Therefore, from Liouville theorem, it is enough to check whether all residues of LHS$(n, m)$ and RHS$(n, m)$ coincide or not. Candidates of poles are $u_\alpha = u_\beta$, $v_\alpha = v_\beta$, $u_\alpha = u_\beta + 1$, $v_\alpha = v_\beta + 1$ and $\hat{\vartheta}_{\gamma,r}(\{u_\alpha\} \{u_\beta\}) = 0$. (Some of them are real pole and some of them are fake.) Let us consider $u_\alpha = u_\beta$ ($\alpha \neq \beta$). Take the residue of the LHS$(n, m)$ at $u_\alpha = u_\beta$. We have

$$\begin{align*}
\text{Res}_{u_\alpha = u_\beta} \left( \frac{1}{[u_\alpha - u_\beta][u_\beta - u_\alpha - 1]} + \frac{1}{[u_\beta - u_\alpha][u_\alpha - u_\beta - 1]} \right)
\times \sum_{L \cap L^c = \emptyset} \prod_{j \in L^c} [u_i - u_j + \frac{s}{2}r] [u_i - v_j + \frac{s}{2}r] [v_i - u_j + \frac{s}{2}r - 1] [v_j - u_i + \frac{s}{2}r - 1][v_i - v_j + \frac{s}{2}r - 1].
\end{align*}$$
\[
\times \sum_{K\cup K^c = \{1, \ldots, n+m\} - \{\alpha, \beta\}} \prod_{\substack{\alpha \in K^c \cup \{k\} \in L \cup \{\alpha\} \in L \cup k \in K^c \cup \{k\} \in L \cup k \in L \cup \{\alpha\} \in L}} \prod_{\substack{[v_i - u_i + \frac{s}{2}]_r [u_k - v_j + \frac{s}{2}]_r [u_i - u_k + \frac{s}{2} - 1]_r [v_j - u_k + \frac{s}{2} - 1]_r}} [u_k - u_i]_r [v_i - v_j]_r [u_i - u_k - 1]_r [v_j - v_i - 1]_r
\]

\[
\times \hat{\gamma}_{\alpha, r} (\sum_{j=1}^{n+1} u_j) \sum_{j=1}^{n+1} v_j \hat{\gamma}_{\beta, r} (\sum_{j=1}^{n+1} u_j - \sum_{j=1}^{n+1} v_j) = 0. \quad (4.23)
\]

Hence \( u_{\alpha} = u_{\beta}, (\alpha \neq \beta) \) are not pole. They are regular points. By the same manner, we have

\[
\text{Res}_{u_{\alpha} = u_{\beta}} \text{RHS}(n, m) = 0, \quad \text{Res}_{v_{\alpha} = v_{\beta}} \text{LHS}(n, m) = 0, \quad \text{Res}_{u_{\alpha} = u_{\beta}} \text{RHS}(n, m) = 0. \quad (4.24)
\]

Therefore points \( u_{\alpha} = u_{\beta} \) and \( v_{\alpha} = v_{\beta} \) are not poles. Therefore candidates of poles are restricted to only \( u_{\alpha} = u_{\beta} + 1, \ v_{\alpha} = v_{\beta} + 1 \) and \( \hat{\gamma}_{\gamma, r} (\{u_{\alpha}\}\{u_{\beta}\}) = 0 \). We show the equation \( \text{LHS}(n, m) = \text{RHS}(n, m) \) by the induction of the number \( n + m \). We assume \( n > m \geq 1 \) without losing generality. (The case \( n = m \) is trivial.) At first we show the starting point \( n > m = 1 \). We would like to show \( \text{LHS}(n, 1) = \text{RHS}(n, 1) \). Let us take the residue at \( u_1 = u_2 + 1 \) and \( v_1 = v_2 + 1 \). We have

\[
\text{Res}_{u_{1} = u_{2} + 1} \text{Res}_{v_{1} = v_{2} + 1} \text{LHS}(n, 1) = \text{Res}_{u_{1} = u_{2} + 1} \text{Res}_{v_{1} = v_{2} + 1} \text{RHS}(n, 1).
\]

Because both \( \text{LHS}(n, 1) \) and \( \text{RHS}(n, 1) \) are symmetric with respect with \( u_1, u_2, \ldots, u_{n+1} \) and \( v_1, v_2, \ldots, v_{n+1} \), we have

\[
\text{Res}_{u_{\alpha} = u_{\beta} + 1} \text{Res}_{v_{\gamma} = v_{\delta} + 1} \text{LHS}(n, 1) = \text{Res}_{u_{\alpha} = u_{\beta} + 1} \text{Res}_{v_{\gamma} = v_{\delta} + 1} \text{RHS}(n, 1). \quad (4.26)
\]

After taking the residues finitely many times, every residue relation which comes from \( \text{LHS}(n, 1) = \text{RHS}(n, 1) \), is reduced to the above residue relation \( (4.26) \) at \( u_{\alpha} = u_{\beta} + 1, v_{\gamma} = v_{\delta} + 1 \). Hence we have proved the starting point \( n > m = 1 \). For the second, we show the general \( n > m \geq 1 \) case. We assume the equation \( \text{LHS}(n-1, m-1) = \text{RHS}(n-1, m-1) \)
for some \((n, m)\). Let us take the residue at \(u_1 = u_2 + 1\) and \(v_1 = v_2 + 1\) of \(L(n, m) - R(n, m)\). We have

\[
\text{Res}_{u_1=u_2+1, v_1=v_2+1} (LHS(n, m) - RHS(n, m)) = \text{Res}_{u=0, v=0} \left[ \frac{[u_1 - v_1 + \frac{s}{2}]_r [v_1 - u_1 + \frac{s}{2}]_r [u_1 - v_1 + \frac{s}{2}]_r [v_1 - u_1 + \frac{s}{2} - 1]_r}{[u]_r [v]_r [1]_r [1]_r} \times \prod_{j=3}^{n+m} \frac{[v_1 - u_j + \frac{s}{2} - 1]_r [u_j - v_1 + \frac{s}{2}]_r [v_j - u_1 + \frac{s}{2}]_r [u_1 - v_1 + \frac{s}{2} - 1]_r}{[u_1 - u_j - 1]_r [v_1 - u_1]_r [v_j - v_1]_r [v_j - v_1]_r} \times \sum_{K \cup K' = \{\ldots, n+m\}} \sum_{\{K|m-1, K'|=n-1\}} \prod_{j \in K^c} \prod_{l \in L^c} \prod_{i \in L} \prod_{i \in K} \frac{[v_j - u_k + \frac{s}{2}]_r [u_k - v_j + \frac{s}{2}]_r [v_j - u_i + \frac{s}{2} - 1]_r \times \prod_{j \in L^c} \prod_{l \in L} \prod_{i \in K^c} \prod_{j \in L^c} \frac{[u_k - u_i]_r [u_k - u_i - 1]_r \times \prod_{j \in L^c} \prod_{l \in L} \prod_{i \in K} \prod_{j \in L^c} \frac{[u_k - v_j + \frac{s}{2}]_r [v_j - u_k + \frac{s}{2} - 1]_r}{[u_k - u_i]_r [u_k - u_i - 1]_r \times \prod_{j \in L^c} \prod_{l \in L}} = 0. \right)
\]

(4.27)

We have used the hypothesis of the induction: \(LHS(n - 1, m - 1) = RHS(n - 1, m - 1)\).

Because both \(LHS(n, m)\) and \(RHS(n, m)\) are symmetric with respect with \(u_1, u_2, \ldots, u_{m+n}\) and \(v_1, v_2, \ldots, v_{m+n}\), we have

\[
\text{Res}_{u_\alpha = u_{\beta+1}, \alpha} \text{Res}_{v_{\gamma} = v_{\delta+1}, \gamma} LHS(n, m) = \text{Res}_{u_\alpha = u_{\beta+1}, \alpha} \text{Res}_{v_{\gamma} = v_{\delta+1}, \gamma} RHS(n, m), \tag{4.28}
\]

for arbitrary \(1 \leq \alpha \neq \beta \leq n+m\) and \(1 \leq \gamma \neq \delta \leq n+m\). After taking the residues finitely many times, every residue relation which comes from \(LHS(n, m) = RHS(n, m)\), is reduced to the above residue relation (4.28). Hence we have shown \(LHS(n, m) = RHS(n, m)\) for general \(n, m = 1, 2, \ldots\) Q.E.D.

Now let us show the commutation relation, \([G_n, G_m] = 0\).

**Proof of Theorem 4.1** We study the case: \(0 < \text{Re}(s) < 2\) and \(\text{Re}(r) > 0\). Other cases are

\[31\]
similar. Hence the integrand of the nonlocal integrals of motion satisfies the $S_n$-invariance

$$
F_1(z_1) \cdots F_1(z_n) \cdots F_0(w_1) \cdots F_0(w_n)
\times \prod_{1 \leq i < j \leq n} [u_{\sigma(i)} - u_{\sigma(j)}] [u_{\sigma(j)} - u_{\sigma(i)} - 1] [v_{\rho(j)} - v_{\rho(i)}] [v_{\rho(i)} - v_{\rho(j)}]
\times \prod_{i,j=1}^n [u_{\sigma(i)} - v_{\rho(j)} + \frac{s}{2} r] [v_{\rho(j)} - u_{\sigma(i)} + \frac{s}{2} - 1] r

= \prod_{1 \leq j \leq n} F_1(z_j) \prod_{1 \leq j \leq n} F_0(w_j)
\times \prod_{1 \leq i < j \leq n} [u_i - u_j] [u_j - u_i - 1] [v_i - v_j] [v_j - v_i - 1] r
\times \prod_{i,j=1}^n [u_i - v_j + \frac{s}{2} r] [v_j - u_i + \frac{s}{2} - 1] r,

for $\sigma, \rho \in S_n$. (4.29)

Hence we have

$$
\mathcal{G}_n \cdot \mathcal{G}_m = \left[ \prod_{1 \leq j \leq n+m} F_1(z_j) \prod_{1 \leq j \leq n+m} F_0(w_j) \times \prod_{1 \leq i < j \leq n+m} [u_i - u_j] [u_j - u_i - 1] [v_i - v_j] [v_j - v_i - 1] r \times \frac{1}{(n+m)!^2} \sum_{\sigma, \rho \in S_{n+m}} \vartheta_{\alpha, r} \left( \sum_{j=1}^n (u_{\sigma(j)} - v_{\rho(j)}) \right) \vartheta_{\beta, r} \left( \sum_{j=n+1}^{n+m} (u_{\sigma(j)} - v_{\rho(j)}) \right) \times \prod_{i=1}^n \prod_{j=n+1}^{n+m} \frac{[u_{\sigma(i)} - u_{\sigma(j)}] [v_{\rho(i)} - v_{\rho(j)}]}{[u_{\sigma(i)} - u_{\sigma(j)}] [v_{\rho(i)} - v_{\rho(j)}] r} \times \prod_{i=1}^n \prod_{j=n+1}^{n+m} \frac{[u_{\sigma(i)} - v_{\rho(i)} + \frac{s}{2} - 1] r [v_{\rho(i)} - u_{\sigma(i)} + \frac{s}{2} - 1] r}{[u_{\sigma(i)} - u_{\sigma(i)} - 1] r [v_{\rho(i)} - v_{\rho(i)} - 1] r} \right]_{1, z_1 \cdots z_{n+m} = w_1 \cdots w_{n+m}}. (4.30)
$$

Therefore we have the following theta function identity as a sufficient condition of the commutation relation $\mathcal{G}_n \cdot \mathcal{G}_m = \mathcal{G}_m \cdot \mathcal{G}_n$.

$$
\sum_{K \cup K^c = \{1, \ldots, n+m\}, \|K\| = n, \|K^c\| = m} \sum_{L \cup L^c = \{1, \ldots, n+m\}, \|L\| = m, \|L^c\| = n} \vartheta_{\alpha, r} \left( \sum_{j \in K} u_j - \sum_{j \in L} v_j \right) \vartheta_{\beta, r} \left( \sum_{j \in K^c} u_j - \sum_{j \in L^c} v_j \right)
\times \prod_{k \in K^c} \prod_{l \in L^c} \frac{[v_j - u_k + \frac{s}{2} r] [u_i - u_l + \frac{s}{2} r] [u_k - v_j + \frac{s}{2} - 1] r [v_l - u_i + \frac{s}{2} - 1] r}{[u_i - u_k] r [v_j - v_l] r [u_k - u_i - 1] r [v_l - v_j - 1] r}

= \sum_{K \cup K^c = \{1, \ldots, n+m\}, \|K\| = n, \|K^c\| = m} \sum_{L \cup L^c = \{1, \ldots, n+m\}, \|L\| = m, \|L^c\| = n} \vartheta_{\alpha, r} \left( \sum_{j \in K} u_j - \sum_{j \in L} v_j \right) \vartheta_{\beta, r} \left( \sum_{j \in K^c} u_j - \sum_{j \in L^c} v_j \right)
\times \prod_{k \in K^c} \prod_{l \in L^c} \frac{[v_j - u_k + \frac{s}{2} r] [u_i - u_l + \frac{s}{2} r] [u_k - v_j + \frac{s}{2} - 1] r [v_l - u_i + \frac{s}{2} - 1] r}{[u_k - u_i] r [v_l - v_j] r [u_i - u_k - 1] r [v_j - v_l - 1] r}.

(4.31)
This is special case $\nu_\alpha = \nu_\beta = 0$ of the equation (4.16). (In order to use induction, we have introduced additional parameters $\nu_\alpha, \nu_\beta$ to (4.16).) We have shown the commutation relation, $[G_m, G_n] = 0$. Q.E.D.

### 4.3 Proof of $[G_m, G_n^*] = 0$

In this section we show the commutation relation $[G_m, G_n^*] = 0$. The screening currents $E_j(z)$ and $F_j(z)$ almost commute

$$[E_1(z_1), F_1(z_2)] = \frac{1}{x - x^{-1}}(\delta(xz_2/z_1)H(x^r z_2) - \delta(xz_1/z_2)H(x^{-r} z_2)).$$

Hence, in order to show the commutation relation, remaining task for us is to check whether delta-function factors cancell out or not.

**Proof of Theorem 4.2** For a while we study the following parameter case: $0 < \text{Re}(r) < 1$ and $0 < \text{Re}(s) < 2$. For reader’s convenience, we show the simple case $[G_1^*, G_1] = 0$ at first. Using the commutation relations of the screening currents $E_j(z), F_j(z)$, we have

$$[G_1^*, G_1] = \int \int \int_{C_1} \frac{dz_1}{2\pi \sqrt{-1z_1}} \frac{dz_2}{2\pi \sqrt{-1z_2}} \frac{dw_2}{2\pi \sqrt{-1w_2}} B_1(x^r z_1 | z_1, z_2, w_2)$$

$$- \int \int \int_{C_1} \frac{dz_1}{2\pi \sqrt{-1z_1}} \frac{dz_2}{2\pi \sqrt{-1z_2}} \frac{dw_2}{2\pi \sqrt{-1w_2}} B_1(x^{-r} z_1 | z_1, z_2, w_2)$$

$$+ \int \int \int_{C_2} \frac{dz_1}{2\pi \sqrt{-1z_1}} \frac{dz_2}{2\pi \sqrt{-1z_2}} \frac{dw_1}{2\pi \sqrt{-1w_1}} B_2(x^r z_2 | z_1, z_2, w_1)$$

$$- \int \int \int_{C_2} \frac{dz_1}{2\pi \sqrt{-1z_1}} \frac{dz_2}{2\pi \sqrt{-1z_2}} \frac{dw_1}{2\pi \sqrt{-1w_1}} B_2(x^{-r} z_2 | z_1, z_2, w_1). \tag{4.32}$$

Here we have set

$$B_1(z|z_1, z_2, w_2) = \frac{1}{x - x^{-1}}H(z)E_0(w_2)F_0(z_2)$$

$$\times \frac{\partial_{\alpha,r}(u_1 - u_2)\partial_{\beta,-r^*}(u - v_2 - \frac{r^*}{2})}{[u - \frac{r^*}{2} - v_2 + \frac{s}{2}]_{-r}[u - \frac{r^*}{2} - v_2 - \frac{s}{2} - 1]_{-r}[u_1 - u_2 + \frac{s}{2}]_r[u_1 - u_2 - \frac{s}{2} + 1]_r}, \tag{4.33}$$

$$B_2(z|z_1, z_2, w_1) = \frac{1}{x - x^{-1}}F_1(z_1)E_1(w_1)\tau(H(z))$$

$$\times \frac{\partial_{\alpha,r}(u_1 - u_2)\partial_{\beta,-r^*}(v_1 - u - \frac{r^*}{2})}{[v_1 - u + \frac{r^*}{2} + \frac{s}{2}]_{-r}[v_1 - u + \frac{r^*}{2} - \frac{s}{2} + 1]_{-r}[u_1 - u_2 + \frac{s}{2}]_r[u_1 - u_2 - \frac{s}{2} + 1]_r}. \tag{4.34}$$
where the integral contours $C_1, C_2, \tilde{C}_1, \tilde{C}_2$ are given by

\begin{align*}
C_1 & : |x^s z_2|, |x^{2-s} z_2| < |z_1| < |x^{-2r+s-2} z_2|, |x^{-2r-s} z_2|, \\
| x^{s-1-2r} w_2 |, | x^{-s+1-2r} w_2 | < |z_1| < | x^{-s-1} w_2 |, | x^{s-3} w_2 |, | z_2 | < |x w_2 |,
\end{align*}

\begin{align*}
\tilde{C}_1 & : | x^{s+2r} z_2 |, | x^{2-s+2r} z_2 | < |z_1| < | x^{s-2} z_2 |, | x^{-s} z_2 |,
| x^{s+1} w_2 |, | x^{-s+3} w_2 | < |z_1| < | x^{-s-1+2r} w_2 |, | x^{s+3+2r} w_2 |, | z_2 | < |x w_2 |,
\end{align*}

and

\begin{align*}
C_2 & : | x^{s-1-2r} w_1 |, | x^{-s+1+2r} w_1 | < |z_2| < |x^{s-3} w_1|, |x^{-s-1} w_1|,
| x^{s} z_1 |, | x^{2-s} z_1 | < |z_2| < |x^{s-2-2r} z_1|, |x^{-s-2r} z_1|, |z_1| < |x w_1|,
\end{align*}

\begin{align*}
\tilde{C}_2 & : |x^{s+1} w_1|, |x^{-s+3} w_1| < |z_2| < |x^{s+2r-1} w_1|, |x^{-s+2r} w_1|,
| x^{s+2r} z_1 |, | x^{2-s+2r} z_1 | < |z_2| < |x^{s-2} z_1|, |x^{-s} z_1|, |z_1| < |x w_1|.
\end{align*}

When we change the variable $z_1 \to x^{-2r} z_1$ in the first term in RHS of the equation (4.32), the integrand $B_1(x^r z_1 | z_1, z_2, w_2)$, is deformed to $B_1(x^{-r} z_1 | z_1, z_2, w_2) = B_1(x^{-r} z_1 | x^{-2r} z_1, z_2, w_2)$, and the contour $C_1$ is deformed to exactly the same as $\tilde{C}_1$. Therefore we have

\begin{align*}
\int \int \int_{C_1} \frac{dz_1}{2\pi \sqrt{-1} z_1} \frac{dz_2}{2\pi \sqrt{-1} z_2} \frac{dw_2}{2\pi \sqrt{-1} w_2} B_1(x^r z_1 | z_1, z_2, w_2) \\
= \int \int \int_{\tilde{C}_1} \frac{dz_1}{2\pi \sqrt{-1} z_1} \frac{dz_2}{2\pi \sqrt{-1} z_2} \frac{dw_2}{2\pi \sqrt{-1} w_2} B_1(x^{-r} z_1 | z_1, z_2, w_2).
\end{align*}

As the same manner we have

\begin{align*}
\int \int \int_{C_2} \frac{dz_1}{2\pi \sqrt{-1} z_1} \frac{dz_2}{2\pi \sqrt{-1} z_2} \frac{dw_1}{2\pi \sqrt{-1} w_1} B_2(x^r z_1 | z_1, z_2, w_1) \\
= \int \int \int_{\tilde{C}_2} \frac{dz_1}{2\pi \sqrt{-1} z_1} \frac{dz_2}{2\pi \sqrt{-1} z_2} \frac{dw_1}{2\pi \sqrt{-1} w_1} B_2(x^{-r} z_1 | z_1, z_2, w_1).
\end{align*}

Therefore we have the commutation relation $[G_1, G_1^*] = 0$. Generalization to generic parameter $s \in \mathbb{C}, 0 < \text{Re}(r) < 1$ should be understood as analytic continuation.

For the second we show the commutation relation $[G_n, G_m^*] = 0$. For a while we study the following parameter case: $0 < \text{Re}(r) < 1$ and $0 < \text{Re}(s) < 2$. Using the commutation relations of the screening currents, we have

\begin{align*}
[G_n^*, G_m] = \sum_{i=1}^{n} \sum_{j=1}^{m} \int \int \int_{C_{ij}} \prod_{k=1}^{n} \frac{dz_k}{2\pi \sqrt{-1} z_k} \prod_{k \neq i}^{m} \frac{dw_k}{2\pi \sqrt{-1} w_k} B_{i,j}^{(n,m)}(x^r z_i, \{ z_k \}_{k=1}^{2n}, \{ w_k \}_{k=1}^{2m})
\end{align*}
\[ - \sum_{i=1}^{n} \sum_{j=1}^{m} \int \cdots \int_{\mathcal{C}_{ij}} \prod_{k=1}^{m} \frac{dz_{k}}{2\pi \sqrt{-1}z_{k}} \prod_{k \neq i}^{m} \frac{dw_{k}}{2\pi \sqrt{-1}w_{k}} B_{i,j}^{(n,m)}(x^{-r}z_{j}, \{z_{k}\}_{k=1}^{2n}, \{w_{k}\}_{k \neq i}^{2m}) \]
\[ + \sum_{i=n+1}^{2n} \sum_{j=m+1}^{2m} \int \cdots \int_{\mathcal{C}_{ij}} \prod_{k=1}^{m} \frac{dz_{k}}{2\pi \sqrt{-1}z_{k}} \prod_{k \neq i}^{m} \frac{dw_{k}}{2\pi \sqrt{-1}w_{k}} B_{i,j}^{(n,m)}(x^{-r}z_{j}, \{z_{k}\}_{k=1}^{2n}, \{w_{k}\}_{k \neq i}^{2m}) \]
\[ - \sum_{i=n+1}^{2n} \sum_{j=m+1}^{2m} \int \cdots \int_{\mathcal{C}_{ij}} \prod_{k=1}^{m} \frac{dz_{k}}{2\pi \sqrt{-1}z_{k}} \prod_{k \neq i}^{m} \frac{dw_{k}}{2\pi \sqrt{-1}w_{k}} B_{i,j}^{(n,m)}(x^{-r}z_{j}, \{z_{k}\}_{k=1}^{2n}, \{w_{k}\}_{k \neq i}^{2m}) \]

(4.41)

Here we have set for \(1 \leq i \leq n\) and \(1 \leq j \leq m\),

\[
B_{i,j}^{(n,m)}(z, \{z_{k}\}_{k=1}^{2n}, \{w_{k}\}_{k \neq i}^{2m}) = \frac{1}{x^{-r}x^{-1}} F_{1}(z_{1}) \cdots F_{1}(z_{j-1}) E_{1}(w_{1}) \cdots E_{1}(w_{i-1}) H(z) \times F_{1}(z_{j+1}) \cdots F_{1}(z_{m}) E_{1}(w_{i+1}) \cdots E_{1}(w_{n}) E_{0}(w_{n+1}) \cdots E_{0}(w_{2n}) F_{0}(z_{m+1}) \cdots F_{0}(z_{2m})
\]
\[ \times \vartheta_{\alpha,-r} \left( \sum_{k=1}^{n} v_{k} - \sum_{l=n+1}^{2n} u_{l} \right) \vartheta_{\beta,r} \left( \sum_{k=1}^{m} u_{k} - \sum_{l=m+1}^{2m} u_{l} \right) \prod_{1 \leq k < l \leq n} \left[ v_{k} - v_{l} \right]_{-r} \prod_{n+1 \leq k < l \leq 2n} \left[ v_{k} - v_{l} - 1 \right]_{-r} \prod_{1 \leq k \leq l \leq m} \left[ u_{k} - u_{l} \right]_{-r} \prod_{m+1 \leq k < l \leq 2m} \left[ u_{k} - u_{l} - 1 \right]_{r} \right], \] (4.42)

and for \(n+1 \leq i \leq 2n\) and \(m+1 \leq j \leq 2m\),

\[
B_{i,j}^{(n,m)}(z, \{z_{k}\}_{k=1}^{2m}, \{w_{k}\}_{k \neq i}^{2n}) = \frac{1}{x^{-r}x^{-1}} F_{1}(z_{1}) \cdots F_{1}(z_{j-1}) E_{1}(w_{1}) \cdots E_{1}(w_{n}) \times F_{0}(z_{m+1}) \cdots F_{0}(z_{j-1}) E_{0}(w_{n+1}) \cdots E_{0}(w_{i-1}) \tau(H(z)) \times F_{0}(z_{j+1}) \cdots F_{0}(z_{2m}) E_{0}(w_{i+1}) \cdots E_{0}(w_{2n})
\]
\[ \times \vartheta_{\alpha,-r} \left( \sum_{k=1}^{n} v_{k} - \sum_{l=n+1}^{2n} v_{l} \right) \vartheta_{\beta,r} \left( \sum_{k=1}^{m} u_{k} - \sum_{l=m+1}^{2m} u_{l} \right) \right), \]
where the integral contours $C_{i,j}, \tilde{C}_{i,j}$ are given by as follows. For $1 \leq i \leq n, 1 \leq j \leq m$ we set

$$C_{i,j} : |x^{s-1-2r^*} w_k|, |x^{-s+1-2r^*} w_k| < |z_j| < |x^{-s-1} w_k|, |x^{s-3} w_k|, \quad (n + 1 \leq k \neq i \leq 2n),$$

$$|x^s z_k|, |x^{s-2} z_k| < |z_j| < |x^{-2r+s} z_k|, |x^{-2r-s} z_k|, \quad (m + 1 \leq k \neq j \leq 2m), \quad (4.44)$$

$$\tilde{C}_{i,j} : |x^{s+1} w_k|, |x^{-s+3} w_k| < |z_j| < |x^{s+1+2r^*} w_k|, |x^{s-1+2r^*} w_k|, \quad (n + 1 \leq k \neq i \leq 2n),$$

$$|x^s z_k|, |x^{s+2} z_k| < |z_j| < |x^{s+2r} z_k|, |x^{-s+2} z_k|, \quad (m + 1 \leq k \neq j \leq 2m). \quad (4.45)$$

and for $n + 1 \leq i \leq 2n, m + 1 \leq j \leq 2m$ we set

$$C_{i,j} : |x^{s-1-2r^*} w_k|, |x^{-s+1+2r^*} w_k| < |z_j| < |x^{s-3} w_k|, |x^{-s-1} w_k|, \quad (n + 1 \leq k \neq i \leq 2n),$$

$$|x^s z_k|, |x^{s-2} z_k| < |z_j| < |x^{s-2-2r^*} z_k|, |x^{-s-2r^*} z_k|, \quad (m + 1 \leq k \neq j \leq 2m), \quad (4.46)$$

$$\tilde{C}_{i,j} : |x^{s+1} w_k|, |x^{-s+3} w_k| < |z_j| < |x^{s+1+2r^*} w_k|, |x^{s-1+2r} w_k|, \quad (n + 1 \leq k \neq i \leq 2n),$$

$$|x^s z_k|, |x^{s+2} z_k| < |z_j| < |x^{s+2r} z_k|, |x^{-s} z_k|, \quad (m + 1 \leq k \neq j \leq 2m). \quad (4.47)$$

When we change the variable $z_j \rightarrow x^{-2r} z_j$ in the first term in LHS of (4.41), both integrand and integral contour are deformed to the same as the second term of (4.41). Hence we have

$$\int \cdots \int \prod_{C_{i,j}} \prod_{k=1}^{n} \frac{dz_k}{2\pi \sqrt{-1} z_k} \prod_{k=1}^{m} \frac{dw_k}{2\pi \sqrt{-1} w_k} D_{i,j}^{(n,m)} \left( x^r z_i, \{ z_k \}_{k=1}^{2n}, \{ w_k \}_{k=1}^{2m} \right)$$

$$= \int \cdots \int \prod_{\tilde{C}_{i,j}} \prod_{k=1}^{n} \frac{dz_k}{2\pi \sqrt{-1} z_k} \prod_{k=1}^{m} \frac{dw_k}{2\pi \sqrt{-1} w_k} D_{i,j}^{(n,m)} \left( x^{-r} z_j, \{ z_k \}_{k=1}^{2n}, \{ w_k \}_{k=1}^{2m} \right) \quad (4.48)$$

By the same arguments as above we have $[\mathcal{G}^*_n, \mathcal{G}_m] = 0$ for $0 < \text{Re}(r) < 1$ and $0 < \text{Re}(s) < 2$. Generalization to generic parameter $0 < \text{Re}(r) < 1$ and $s \in \mathbb{C}$ should be understood as analytic continuation. Q.E.D.
4.4 Proof of Dynkin Automorphism Invariance \( \eta(\mathcal{G}_n) = \mathcal{G}_n \)

In this section we show the invariance condition \( \eta(\mathcal{G}_n) = \mathcal{G}_n \) and \( \eta(\mathcal{G}_n^*) = \mathcal{G}_n^* \). For reader’s convenience, we explain \( \eta(\mathcal{G}_1) = \mathcal{G}_1 \) at first. We study the case \( 0 < \text{Re}(r) \) and \( 0 < \text{Re}(s) < 2 \). From the definition of the Dynkin automorphism \( \eta \), we have

\[
\eta(\mathcal{G}_1) = \int \int \prod_{j=1}^{n} \frac{dz_j}{2\pi \sqrt{-1} z_j} F_0(z_1) F_1(z_2) \frac{\vartheta_{\alpha,r}(u_1 - u_2)}{[u_1 - u_2 + \frac{s}{2}]_r [u_2 - u_1 + \frac{s}{2} - 1]_r}.
\]

(4.49)

Exchanging the ordering of \( F_1(z_1) \) and \( F_0(z_2) \), and changing the variables \( u_1 \to u_2 \) and \( u_2 \to u_1 \), we have

\[
\eta(\mathcal{G}_1) = \int \int \prod_{j=1}^{n} \frac{dz_j}{2\pi \sqrt{-1} z_j} F_1(z_1) F_0(z_2) \frac{\vartheta_{-\alpha,r}(u_1 - u_2)}{[u_1 - u_2 + \frac{s}{2}]_r [u_2 - u_1 + \frac{s}{2} - 1]_r}.
\]

(4.50)

The relation \( \vartheta_{\alpha,r}(u) = \vartheta_{-\alpha,r}(-u) \) implies \( \eta(\mathcal{G}_1) = \mathcal{G}_1 \). Generalization to generic parameter \( 0 < \text{Re}(r) \) and \( s \in \mathbb{C} \) should be understood as analytic continuation.

Proof. Let us show \( \tau(\mathcal{G}_n) = \mathcal{G}_n \). We study the case \( 0 < \text{Re}(r) \) and \( 0 < \text{Re}(s) < 2 \). From the definition of the Dynkin automorphism \( \tau \), we have

\[
\eta(\mathcal{G}_n) = \int \cdots \int \prod_{j=1}^{n} \frac{dz_j}{2\pi \sqrt{-1} z_j} \prod_{j=1}^{n} \frac{dw_j}{2\pi \sqrt{-1} w_j} \prod_{1 \leq j \leq n} F_0(z_j) \prod_{1 \leq j \leq n} F_1(w_j)
\]

\[
\times \frac{\prod_{1 \leq i < j \leq n} [u_i - u_j]_r [u_j - u_i - 1]_r [v_i - v_j]_r [v_j - v_i - 1]_r}{\prod_{i,j=1}^{n} [u_i - v_j + \frac{s}{2}]_r [v_j - u_i + \frac{s}{2} - 1]_r} \vartheta_{\alpha,r} \left( \sum_{j=1}^{n} (u_j - v_j) \right) \mid_{\bar{\pi} \to -\bar{\pi}}.
\]

(4.51)

Exchanging the ordering of \( F_1(z_j) \) and \( F_0(w_k) \), and changing the variables \( u_j \to v_j \) and \( v_j \to u_j \), we have

\[
\eta(\mathcal{G}_n) = \int \cdots \int \prod_{j=1}^{n} \frac{dz_j}{2\pi \sqrt{-1} z_j} \prod_{j=1}^{n} \frac{dw_j}{2\pi \sqrt{-1} w_j} \prod_{1 \leq j \leq n} F_1(w_j) \prod_{1 \leq j \leq n} F_0(z_j)
\]

\[
\times \frac{\prod_{1 \leq i < j \leq n} [u_i - u_j]_r [u_j - u_i - 1]_r [v_i - v_j]_r [v_j - v_i - 1]_r}{\prod_{i,j=1}^{n} [u_i - v_j + \frac{s}{2}]_r [v_j - u_i + \frac{s}{2} - 1]_r} \vartheta_{\alpha,r} \left( \sum_{j=1}^{n} (v_j - u_j) \right) \mid_{\bar{\pi} \to -\bar{\pi}}.
\]

(4.52)

Using the relation \( \vartheta_{\alpha,r}(u) = \vartheta_{\alpha,r}(-u) \mid_{\bar{\pi} \to -\bar{\pi}} \), we have \( \eta(\mathcal{G}_n) = \mathcal{G}_n \). Generalization to generic parameter \( 0 < \text{Re}(r) \) and \( s \in \mathbb{C} \) should be understood as analytic continuation. The proof of the invariance \( \eta(\mathcal{G}_n^*) = \mathcal{G}_n^* \) is given by the same manner. Q.E.D.
4.5 Proof of $[I_m, G_n] = 0$

In this section we show the commutation relations $[I_m, G_n] = 0$. The operators $\Lambda_j(z)$ and $F_j(z)$ commute almost everywhere.

$$[\Lambda_1(z_1), F_1(z_2)] = (x^{r^*} - x^r)\delta(x^r z_1/z_2)A(x^{-r} z_2).$$

Hence, in order to show the commutation relation, remaining task for us is to check whether delta-function factors cancell out or not. At first we summarize simple case $[I_1, G_n] = 0$, for reader’s convenience.

**Proof of $[I_1, G_n] = 0$**

For a while we restrict our interest to the case $0 < \text{Re}(s) < 2$, $0 < \text{Re}(r) < 1$, and $\text{Re}(2r) < \text{Re}(s)$. We have

$$[I_1, G_n] = (x^{r^*} - x^r) \sum_{j=1}^n \int \cdots \int_{I(j)} \prod_{k=1}^{n} \frac{dz_k}{2\pi\sqrt{-1}z_k} \frac{dw_k}{2\pi\sqrt{-1}w_k} F_1(z_1) \cdots F_1(z_{j-1})$$
$$\times A(x^r z_j) F_1(z_{j+1}) \cdots F_1(z_n) F_0(w_1) \cdots F_0(w_n)$$
$$\times \prod_{1 \leq k < l \leq n} [u_k - u_l]_r [u_k - u_l + 1]_r [v_k - v_l]_r [v_k - v_l + 1]_r \frac{\partial_{\alpha, r}}{\prod_{k, l=1}^{n} [u_k - v_l + \frac{\alpha}{2}]_r [v_l - u_k + \frac{\alpha}{2} - 1]_r} (\sum_{j=1}^{n} (u_j - v_j))$$
$$+ \eta \left( \sum_{j=1}^{n} \int_{C} \frac{d\zeta}{2\pi\sqrt{-1}\zeta} \int \cdots \int_{I(\zeta)} \prod_{k=1}^{n} \frac{dz_k}{2\pi\sqrt{-1}z_k} \frac{dw_k}{2\pi\sqrt{-1}w_k} F_0(z_1) \cdots F_0(z_n) \right.$$
$$\times F_1(w_1) \cdots F_1(w_{j-1}) [T_1(\zeta), F_1(w_j)] F_1(w_{j+1}) \cdots F_1(w_n)$$
$$\times \prod_{1 \leq k < l \leq n} [u_k - u_l]_r [u_k - u_l + 1]_r [v_k - v_l]_r [v_k - v_l + 1]_r \frac{\partial_{\alpha, r}}{\prod_{k, l=1}^{n} [u_k - v_l + \frac{\alpha}{2}]_r [v_l - u_k + \frac{\alpha}{2} - 1]_r} (\sum_{j=1}^{n} (u_j - v_j)) \left). \right)$$

(4.53)

Here the contours $\tilde{I}(j)$ and $\tilde{I}'(j)$ are given by

$$\tilde{I}(j) : |x^{4r-2} z_{j+1}|, \cdots, |x^{4r-2} z_n| < |z_j| < |x^{-2r+2} z_1|, \cdots, |x^{-2r+2} z_{j-1}|,$$
$$|x^s w_k|, |x^{2r-s+2} w_k| < |z_j| < |x^{s-2} w_k|, |x^{-s+2r} w_k|, \quad (1 \leq k \leq n),$$

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Let us change the variable $z_j \rightarrow x^{2r}z_j$ of the first term of (4.53), upon condition $0 < \Re(s) < 2$, $0 < \Re(r) < 1$, $\Re(2r) < \Re(s)$. Using periodicity of the integrand, we have

\[
\int \cdots \int_{\mathcal{P}(j)} \prod_{k=1}^{n} \frac{dz_k}{2\pi \sqrt{-1} z_k} \frac{dw_k}{2\pi \sqrt{-1} w_k} F_1(z_1) \cdots F_1(z_{j-1}) \\
\times A(x^{-r}z_j) F_1(z_{j+1}) \cdots F_1(z_n) F_0(w_1) \cdots F_0(w_n) \\
\times \prod_{1 \leq k < l \leq n} [u_k - u_l] [u_k - u_l + 1] [v_k - v_l] [v_k - v_l + 1] r \frac{\prod_{k,l=1}^{n} [u_k - v_l + \frac{a}{2}] [v_l - u_k + \frac{a}{2} - 1] r}{\prod_{k,l=1}^{n} [u_k - v_l + \frac{a}{2}] [v_l - u_k + \frac{a}{2} - 1] r} \theta_{\alpha,r} \left( \sum_{j=1}^{n} (u_j - v_j) \right) 
\]

By the same manner as above, we have

\[
\int_{C} \frac{d\zeta}{2\pi \sqrt{-1} \zeta} \int \cdots \int \prod_{k=1}^{n} \frac{dz_k}{2\pi \sqrt{-1} z_k} \frac{dw_k}{2\pi \sqrt{-1} w_k} F_0(z_1) \cdots F_0(z_n) \\
\times F_1(w_1) \cdots F_1(w_{j-1}) |T_1(\zeta), F_1(w_j)| F_1(w_{j+1}) \cdots F_1(w_n) \\
\times \prod_{1 \leq k < l \leq n} [u_k - u_l] [u_k - u_l + 1] [v_k - v_l] [v_k - v_l + 1] r \frac{\prod_{k,l=1}^{n} [u_k - v_l + \frac{a}{2}] [v_l - u_k + \frac{a}{2} - 1] r}{\prod_{k,l=1}^{n} [u_k - v_l + \frac{a}{2}] [v_l - u_k + \frac{a}{2} - 1] r} \eta \left( \theta_{\alpha,r} \left( \sum_{j=1}^{n} (u_j - v_j) \right) \right) = 0. 
\]

Therefore we have $[\mathcal{I}_1, \mathcal{G}_n] = 0$. Generalization to generic $\Re(s) > 0$ and $\Re(r) > 0$ case, should be understood as analytic continuation. Q.E.D.

Proof of general case $[\mathcal{I}_m, \mathcal{G}_n] = 0$ is given essentially the same manner as above. Because the integral contour of the local integral motion $\mathcal{I}_m$ is not cylinder, we are not allowed to use the symbol $\delta(z)$ directly. Before starting general proof of $[\mathcal{I}_m, \mathcal{G}_n] = 0$, we review elementary fact about $\delta(z)$. Let us set the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ by

\[
\mathcal{F}_1 = \int \cdots \int_{C_1} \prod_{j=1}^{m} \frac{dw_j}{2\pi \sqrt{-1} w_j} f_1(w_1, \ldots, w_m) \Lambda_1(w_1), \tag{4.56}
\]

\[
\mathcal{F}_2 = \int \cdots \int_{C_2} \prod_{j=1}^{n} \frac{dz_j}{2\pi \sqrt{-1} z_j} f_2(z_1, \ldots, z_n) F_1(z_1), \tag{4.57}
\]

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where \( f_1(w_1, \ldots, w_m) \) and \( f_2(z_1, \ldots, z_n) \) are meromorphic functions. Let us set the region \( D_1 \) in which Laurent-series of \( f_1(w_1, \ldots, w_m) \) exists, and the region \( D_2 \) in which Laurent-series of \( f_2(z_1, \ldots, z_n) \) has Laurent-expansion exists. Let us set \( \tilde{C}_1(z_1) = \{(x^{-r}z_1, w_2, \ldots, w_m) \in \mathbb{C}^m | (w_1, \ldots, w_m) \in C_1 \} \). If the contours \( C_1 \subset D_1, C_2 \subset D_2, \) and \( \tilde{C}_1(z_1) \subset D_1 \) for any \((z_1, z_2, \ldots, z_n) \in C_2, \) we have the following formulae, which we have already used above.

\[
[F_1, F_2] = (x^{-r} - x^r) \left[ \delta(x^{-r}z_1/w_1)A(x^{-r}z_1)f_2(z_1, \ldots, z_n)f_1(x^{-r}z_1, w_2, \ldots, w_m) \right]_{1,z_1 \ldots z_n w_1 \ldots w_m}.
\]

Even if \( \tilde{C}_1(z_1) \) is not included in \( D_1 \) for some \((z_1, z_2, \ldots, z_n) \in C_2, \) we have a similar formulae.

**Proposition 4.6**

\[
[F_1, F_2] = (x^{-r} - x^r) \int \cdots \int_{C_2} \prod_{j=1}^{\infty} \frac{dz_j}{2\pi \sqrt{-1} z_j} f_2(z_1, \ldots, z_n)A(x^{-r}z_1) \quad (4.58)
\]

\[
\times \left( \int \cdots \int_{C_1 \cap \{|x^{-r}z_1| < 1\}} \prod_{j=1}^{m} \frac{dw_j}{2\pi \sqrt{-1} w_j} \frac{f_1(w_1, w_2, \ldots, w_m)}{(1 - x^{-r}z_1/w_1)} \right) + \int \cdots \int_{C_1 \cap \{|x^{-r}z_1| > 1\}} \prod_{j=1}^{m} \frac{dw_j}{2\pi \sqrt{-1} w_j} \frac{x^r w_1 f_1(w_1, w_2, \ldots, w_m)}{(1 - x^r w_1/z_1)} .
\]

Now let us start to show theorem for general case.

**Proof of \([I_m, \mathcal{G}_n] = 0\)** For a while we restrict our interest to the regime \(0 < \text{Re}(s) < 2, \)
\( 0 < \text{Re}(r) < 1 \) and \( \text{Re}(2r) < \text{Re}(s) \). We have

\[
[I_m, \mathcal{G}_n] = (x^{-r} - x^r) \sum_{i=1}^{n} \sum_{j=1}^{n} \int \cdots \int_{I_{(j)}} \prod_{k=1}^{n} \frac{dz_k}{2\pi \sqrt{-1} z_k} \prod_{k=1}^{n} \frac{dw_k}{2\pi \sqrt{-1} w_k}
\]

\[
\times \prod_{1 \leq k < l \leq n} [u_k - u_l][u_k - u_l + 1/2][v_k - v_l][v_k - v_l + 1/2] \theta_{\alpha, r} \left( \sum_{j=1}^{n} (u_j - v_j) \right)
\]

\[
\times \left( \int_{1 \cap \{|x^{-r}z_1| < 1\}} \prod_{k=1}^{m} \frac{dz'_k}{2\pi \sqrt{-1} z'_k} \frac{1}{(1 - x^{-r}z_j/z'_j)} + \int_{1 \cap \{|x^{-r}z_1| > 1\}} \prod_{k=1}^{m} \frac{dz'_k}{2\pi \sqrt{-1} z'_k} \frac{x^r z'_j/z_j}{(1 - x^r z'_j/z_j)} \right)
\]

\[
\times \prod_{1 \leq k < l \leq m} h(u'_k - u'_l)T_1(z'_1) \cdots T_1(z'_{i-1})F_1(z_1) \cdots F_1(z_{i-1})
\]

\[
\times A(x^{-r}z_j)F_1(z_{j+1}) \cdots F_1(z_n)F_0(w_1) \cdots F_0(w_n)T_1(z'_{i+1}) \cdots T_1(z'_m)
\]
Here \( \bar{T}(j) \) and \( \bar{\bar{T}}(j) \) are the same as given in proof of \([T_1, G_n] = 0\). Let us change the variable \( z_j \to x^{2r}z_j \) of the first term, upon condition \( 0 < \text{Re}(s) < 2, 0 < \text{Re}(r) < 1 \) and \( \text{Re}(2r) < \text{Re}(s) \). Using periodicity of integrand, we have

\[
\begin{align*}
& \left( x^{-r^*} - x^{r^*} \right) \sum_{i,j=1}^{n} \int \cdots \int_{\bar{T}(j)} \prod_{k=1}^{n} \frac{dz_k}{2\pi \sqrt{1 - z_k^2}} \prod_{k=1}^{n} \frac{dw_k}{2\pi \sqrt{-1 w_k}} \\
& \times \left( \prod_{1 \leq k < l \leq n} [u_k - u_l] [u_k - u_l + 1] [v_k - v_l] [v_k - v_l + 1] \right) \phi_{\alpha, r} \left( \sum_{j=1}^{n} (u_j - v_j) \right) \\
& \times \left( \prod_{1 \leq k < l \leq m} h(u'_k - u'_l) \right) \prod_{1 \leq l < m} \left( x^{r^*} \right) \\
& \times \left( T_1(z'_1) \cdots T_1(z'_{j-1}) F_1(z_1) \cdots F_1(z_{j-1}) A(x^{-r} z_j) F_1(z_{j+1}) \cdots F_1(z_n) \right) \\
& \times \left( F_0(w_1) \cdots F_0(w_n) T_1(z'_{j+1}) \cdots T_1(z'_m) \right) \\
& \left( \prod_{1 \leq k < l \leq m} [u_k - u_l]' [u_k - u_l + 1]' [v_k - v_l]' [v_k - v_l + 1]' \right) \phi_{\alpha, r} \left( \sum_{j=1}^{n} (u_j - v_j) \right) \\
& \times \left( \prod_{1 \leq k < l \leq m} h(u'_k - u'_l) \right) \prod_{1 \leq l < m} \left( x^{r^*} \right) \\
& \times \left( T_1(z'_1) \cdots T_1(z'_{j-1}) F_1(z_1) \cdots F_1(z_{j-1}) A(x^{-r^*} z_j) F_1(z_{j+1}) \cdots F_1(z_n) \right) \\
& \times \left( F_0(w_1) \cdots F_0(w_n) T_1(z'_{j+1}) \cdots T_1(z'_m) \right).
\end{align*}
\]

(4.59)
In what follows we restrict our interest to the case \( 1 < \text{Re}(s) \leq 2 \).

5.1 Local Integrals of Motion

By the similar way, we have

\[
\eta \left( \sum_{j=1}^{m} \sum_{j=1}^{n} \int \prod_{k=1}^{n} \frac{dz_k}{2\sqrt{-1}z_k} \prod_{k=1}^{n} \frac{dw_k}{2\sqrt{-1}w_k} \right) = 0. \tag{5.1}
\]

Therefore we have \( [I_m, G_n] = 0 \). Generalization to generic \( \text{Re}(s) > 0 \) and \( \text{Re}(r) > 0 \) case, should be understood as analytic continuation. Q.E.D.

5 Specialization upon \( s = 2 \)

In this section we study the specialization to \( s = 2 \), and discuss relation to the Poisson-Virasoro algebra [6].

5.1 Local Integrals of Motion

In what follows we restrict our interest to the case \( 1 < \text{Re}(s) \leq 2 \).

Definition 5.1 We set the local integrals of motion \( I_n^{DV} \) for the deformed Virasoro algebra \( s = 2 \) by

\[
I_n^{DV} = \prod_{j=1}^{n} \int_{|z_j|=1} \frac{dz_j}{2\sqrt{-1}z_j} \prod_{1 \leq j < k \leq n} h(u_k - u_j)_{|s=2}^{DV}(z_j) \cdots T^{DV}(z_n). \tag{5.1}
\]

Conjecture 5.1 Upon specialization \( s = 2 \) we have

\[
[I_n^{DV}, T_m^{DV}] = 0, \quad \eta(T_n^{DV}) = T_n^{DV}, \quad (m, n = 1, 2, \cdots). \tag{5.2}
\]
In what follows we give a supporting argument of this conjecture.

**Definition 5.2** Let us set the auxiliary operators $I_{m,l}$ by

$$I_{m,l} = \prod_{j=1}^{m-l} \int_{|z_j|=1} \frac{dz_j}{2\pi i z_j} T_1(z_1) \cdots T_1(z_m) T_2(x^{-1}z_{m+1}) \cdots T_2(x^{-1}z_{m+l})$$

$$\times \prod_{1 \leq i < j \leq m} h(u_j - u_i) \prod_{i=1}^{m} \prod_{j=m+1}^{m+l} h_{12}(u_j - u_i) \prod_{m+1 \leq i < j \leq m+l} h_{22}(u_j - u_i). \quad (5.3)$$

Here we have set

$$h_{12}(u) = h(u-1)h(u+1), \quad h_{22}(u) = h_{12}(u-1)h_{12}(u+1). \quad (5.4)$$

**Proposition 5.2** The local integrals of motion $I_n$ are written by linear combination of the auxiliary operators $I_{m,l}$

$$I_n = I_{n,0} + \sum_{\alpha=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (-c)^\alpha s(x^{-2})^\alpha \frac{n!}{\alpha!(n-2\alpha)!} I_{n-2\alpha,\alpha}. \quad (5.5)$$

Example

$$I_2 = I_{2,0} - 2cs(x^{-2})I_{0,1}, \quad I_3 = I_{3,0} - 6cs(x^{-2})I_{1,1}, \quad (5.6)$$

$$I_4 = I_{4,0} - 12cs(x^{-2})I_{2,1} + 12c^2s(x^{-2})^2 I_{0,2}, \quad (5.7)$$

$$I_5 = I_{5,0} - 20cs(x^{-2})I_{3,1} + 60c^2s(x^{-2})^2 I_{1,2}, \quad (5.8)$$

$$I_6 = I_{6,0} - 30cs(x^{-2})I_{4,1} + 180c^2s(x^{-2})^2 I_{2,2} - 120c^3s(x^{-2})^3 I_{0,3}. \quad (5.9)$$

**Definition 5.3** Let us set “renormalized” local integrals of motion $I_n(s)$ by

$$I_n(s) = I_n + \sum_{\alpha=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{\beta=\alpha}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^{\alpha+\beta} c^{\beta} \cdot s(x^{-2})^{\beta} \frac{n!}{\beta!(n-2\beta)!} \left( \sum_{k_1,\cdots,k_\alpha \geq 1} \frac{\beta!}{k_1!k_2!\cdots k_\alpha!} \right) I_{n-2\beta}. \quad (5.10)$$

**Proposition 5.3** We have the commutation relation

$$[I_n(s), I_m(s)] = 0 \quad (m, n = 1, 2, \cdots). \quad (5.11)$$
Example

\[ I_2(s) = 2c(s^{-2}) - I_0, \quad I_3(s) = 6c(s^{-2}) - I_1, \quad I_4(s) = 12c(s^{-2}) - 2I_2, \quad I_5(s) = 20c(s^{-2}) - 60c^2(s^{-2}) - I_3, \quad I_6(s) = 30c(s^{-2}) - 180c^2(s^{-2}) - 120c^3(s^{-2}) - I_4. \]

Proposition 5.4  

“Renormalized” local integrals of motion \( I_n(s) \) is written by linear combination of difference \( (I_m,l - I_{m,0}) \).

\[
I_n(s) = I_{n,0} + \sum_{\delta = 0}^{[\frac{n}{2}]} (-c)^\delta \frac{n!}{\delta!(n-2\delta)!} (I_{n-2\delta,0} - I_{n-2\delta,0}) s(x^{-2})^\delta
\]

\[
- \sum_{\delta=0}^{[\frac{n}{2}]-1} \sum_{\alpha=0}^\delta \sum_{\beta=0}^\delta (-1)^\alpha (-c)^\delta \frac{n!}{(n-2\delta)!\beta!}
\times \left( \sum_{k_1,k_2,\ldots,k_\alpha \geq 1\atop k_1+k_2+\cdots+k_\alpha = \delta - \beta} \frac{\delta!}{k_1!k_2!\cdots k_\alpha!} \right) s(x^{-2})^\delta (I_{n-2\delta,\beta} - I_{n-2\delta,0}).
\]

Example

\[
I_2(s) = I_{2,0} - 2c(s^{-2})(I_{0,1} - I_{0,0}),
\]

\[
I_3(s) = I_{3,0} - 6c(s^{-2})(I_{1,1} - I_{1,0}),
\]

\[
I_4(s) = I_{4,0} - 12c(s^{-2})(I_{2,1} - I_{2,0}) + 12c^2(s^{-2})^2(2I_{0,2} - I_{0,0}) - 24c^2(s^{-2})^2(I_{2,1} - I_{2,0}),
\]

\[
I_5(s) = I_{5,0} - 20c(s^{-2})(I_{3,1} - I_{3,0}) + 60c^2(s^{-2})^2(I_{1,2} - I_{1,0}) - 120c^2(s^{-2})^2(I_{1,1} - I_{1,0}),
\]

\[
I_6(s) = I_{6,0} - 30c(s^{-2})(I_{4,1} - I_{4,0}) + 180c^2(s^{-2})^2(I_{2,2} - I_{2,0}) - 360c^2(s^{-2})^2(I_{2,1} - I_{2,0}) - 120c^3(s^{-2})^3(I_{0,2} - I_{0,0}) + 360c^3(s^{-2})^3(I_{0,1} - I_{0,0}).
\]

When we take the limit \( s \to 2 \), we have

\[
\frac{1}{(s - 2)^{\delta}} (I_{n-2\delta,0} - I_{n-2\delta,0})
\]
Conjecture 5.5  When we take the limit $s \to 2$, we have
\[ \mathcal{I}_n(s) \to \mathcal{I}_n^{DV}, \quad (s \to 2). \] (5.24)

We have checked relation (5.24) for small $n$, i.e., $1 \leq n \leq 9$. As a consequence of (5.24), we have the commutation relation $[\mathcal{I}_m^{DV}, \mathcal{I}_n^{DV}] = 0$ and invariance $\eta(\mathcal{I}_n^{DV}) = \mathcal{I}_n^{DV}$.

Example  (The Virasoro-Poisson algebra)

Let us set two parameters $(q, \beta)$ by $q = x^{2r}$, $\beta = \frac{r-1}{r}$. When we take the limit $\beta \to 0$ with $q$ fixed, we have the Virasoro-Poisson algebra [6].
\[ \{T_m, T_n\}_{P.B.} = \sum_{l \in \mathbb{Z}} \frac{1 - q^l}{1 + q^l} T_{n-l} T_{m+l} + (q^n - q^{-n}) \delta_{n+m,0}. \] (5.25)

In this limit the integral of motion $\mathcal{I}_1^{DV}$ degenerates to the conservation law $H_1$ in [7].

Example  (CFT-limit)

Let us set two parameters $(h, \beta)$ by $e^h = x^{2r}$, $\beta = \frac{r-1}{r}$. We take the limit $h \to 0$ with $\beta$ fixed under the following $h$-expansion
\[ T_n = 2\delta_{n,0} + \beta \left( L_n + \frac{(1 - \beta)^2}{4\beta} \right) h^2 + O(h^4), \] (5.26)
we have the defining relation of the Virasoro algebra
\[ [L_n, L_m] = (n-m)L_{m+n} + \frac{c_{CFT}}{12} n(n^2-1) \delta_{n+m,0}, \quad c_{CFT} = 1 - \frac{6(1 - \beta)^2}{\beta}. \] (5.27)

We call this limit the CFT-limit. In this limit the local integrals of motion $\mathcal{I}_1^{DV}, \mathcal{I}_2^{DV}$ reproduce those of CFT $I_1, I_3$ by [1].
5.2 Nonlocal Integrals of Motion

In what follows we restrict our interest to the case $1 < \text{Re}(s) \leq 2$.

**Definition 5.4** We set the nonlocal integrals of motion $G_n^{\text{DV}}$ for the deformed Virasoro algebra $(s = 2)$ by

$$G_n^{\text{DV}} = \int \cdots \int_{I_{\text{Arg}}} \prod_{j=1}^{n} \frac{dz_j}{2\pi \sqrt{-1}z_j} \prod_{j=1}^{n} \frac{dw_j}{2\pi \sqrt{-1}w_j} \times \frac{F_{\text{DV}}^1(z_1)F_{\text{DV}}^0(w_1)}{[u_1 - v_1 + 1]_r[v_1 - u_1]_r} \cdots \frac{F_{\text{DV}}^1(z_n)F_{\text{DV}}^0(w_n)}{[u_n - v_n + 1]_r[v_n - u_n]_r}$$

\begin{equation}
\prod_{1 \leq i < j \leq n} \frac{[u_i - u_j]_r[v_i - u_j + 1]_r[v_i - v_j + 1]_r}{[u_i - v_j + 1]_r[v_i - u_j + 1]_r[v_i - u_j]_r} \vartheta_{\alpha,r} \left( \sum_{j=1}^{n} (u_j - v_j) \right) \bigg|_{s=2},
\end{equation}

where the contour $I_{\text{Arg}}$ are given by

$$|x^2w_n|, |x^2w_n| < |z_1| < |w_1| < |z_2| < |w_2| < \cdots < |z_n| < |w_n|.$$ \hspace{1cm} (5.29)

**Conjecture 5.6** Upon specialization $s = 2$ we have

$$[G_n^{\text{DV}}, G_m^{\text{DV}}] = 0, \quad \eta(G_n^{\text{DV}}) = G_n^{\text{DV}}, \quad (m, n = 1, 2, \cdots).$$ \hspace{1cm} (5.30)

**Conjecture 5.7** Upon specialization $s = 2$ we have

$$[I_n^{\text{DV}}, G_m^{\text{DV}}] = 0, \quad (m, n = 1, 2, \cdots),$$ \hspace{1cm} (5.31)

In what follows we give a supporting argument of the above conjecture.

**Definition 5.5** Let us set the the auxiliary operators $G_{m,l}$ by

$$G_{m,l} = \int \cdots \int_{I_{\text{(m,l)}}} \prod_{i=1}^{m} \prod_{j=m+1}^{m+l} \frac{dz_j}{2\pi \sqrt{-1}z_j} \prod_{i=1}^{m} \prod_{j=m+1}^{m+l} \frac{dw_j}{2\pi \sqrt{-1}w_j} \prod_{1 \leq i < j \leq m+l} j_{1,1}(u_i, v_i|u_j, v_j)$$

\begin{equation}
\times \prod_{i=1}^{m} \prod_{j=m+1}^{m+l} j_{1,2}(u_i, v_i|v_j) \prod_{m+1 \leq i < j \leq m+l} j_{2,2}(v_i|v_j) \times \vartheta_{\alpha,r} \left( \sum_{j=1}^{m} (u_j - v_j) + l\left(\frac{s}{2} - 1\right) \right)
\end{equation}

\begin{equation}
\times \frac{F_{\text{DV}}^1(z_1)F_{\text{DV}}^0(w_1)}{[u_1 - v_1 + \frac{s}{2}]_r[v_1 - u_1 + \frac{s}{2} - 1]_r} \cdots \frac{F_{\text{DV}}^1(z_m)F_{\text{DV}}^0(w_m)}{[u_m - v_m + \frac{s}{2}]_r[v_m - u_m + \frac{s}{2} - 1]_r}
\end{equation}

\begin{equation}
\times F_1(x^{2-s}w_{m+1})F_0(w_{m+1}) : \cdots : F_1(x^{2-s}w_{m+l})F_0(w_{m+l}) : \hspace{1cm} (5.32)
\end{equation}
Here we set

\[ j_{1,1}(u_1, v_1|u_2, v_2) = \frac{[u_1 - u_2]_r[u_1 - u_2 + 1]_r[v_1 - v_2]_r[v_1 - v_2 + 1]_r\cdot [u_1 - v_2 + \frac{s}{2}]_r[v_2 - u_1 + \frac{s}{2} - 1]_r[v_1 - u_2 + \frac{s}{2}]_r[u_2 - v_1 + \frac{s}{2} - 1]_r}{[u_1 - v_2 + \frac{s}{2}]_r[v_2 - u_1 + \frac{s}{2} - 1]_r[v_1 - u_2 + \frac{s}{2}]_r[u_2 - v_1 + \frac{s}{2} - 1]_r}, \]

\[ j_{1,2}(u_1, v_1|v_2) = j_{1,1}\left(\frac{u_1}{v_1}, v_1|v_2 + 1 - \frac{s}{2}, v_2\right), \]

\[ j_{2,2}(v_1|v_2) = j_{1,1}\left(v_1 + 1 - \frac{s}{2}, v_1|v_2 + 1 - \frac{s}{2}, v_2\right). \]  

(5.33)

Here the contour \( \bar{I}(m, l) \) is given by

\[ \bar{I}(m, l) : |x^m w_m|, |x^{2r} w_m| < |z_1| < |w_1| < |z_2| < |w_2| < \cdots < |z_m| < |w_m|, \]

\[ |x^{2r-2} w_j|, |x^{4-2s} w_j| < |w_i| < |x^{2-2s} w_j|, |x^{2s-4} w_j|, \quad (m + 1 \leq i < j \leq m + l), \]  

\[ (5.34) \]

\[ |x^s w_j| < |z_i| < |x^{s-2} w_j|, |x^{4-2s} w_j| < |w_i| < |x^{2-2s} w_j|, \quad (1 \leq i \leq m, \ m + 1 \leq j \leq m + l). \]

**Proposition 5.8** The local integrals of motion \( \mathcal{G}_n \) are written by linear combination of \( \mathcal{G}_{m,l} \).

\[ \mathcal{G}_n = \mathcal{G}_{n,0} + \sum_{\alpha=1}^{n} \frac{n(n-1)\cdots(n-\alpha+1)}{\alpha!} \left(\frac{t(s)}{[s-2]_r}\right)^\alpha \mathcal{G}_{n-\alpha,\alpha}, \]  

(5.35)

where we have set

\[ t(s) = \frac{x^{2r-s}}{[1]_r} \frac{(x^{2r+2s-4}; x^{2r})_\infty (x^{2r-2}; x^{2r})_\infty}{(x^{2s-2}; x^{2r})_\infty (x^{2r}; x^{2r})_\infty}. \]  

(5.36)

**Example**

\[ \mathcal{G}_1 = \mathcal{G}_{1,0} + \frac{t(s)}{[s-2]_r} \mathcal{G}_{0,1}, \]  

(5.37)

\[ \mathcal{G}_2 = \mathcal{G}_{2,0} + 4 \frac{t(s)}{[s-2]_r} \mathcal{G}_{1,1} + 2 \left(\frac{t(s)}{[s-2]_r}\right)^2 \mathcal{G}_{0,2}, \]  

(5.38)

\[ \mathcal{G}_3 = \mathcal{G}_{3,0} + 9 \frac{t(s)}{[s-2]_r} \mathcal{G}_{2,1} + 18 \left(\frac{t(s)}{[s-2]_r}\right)^2 \mathcal{G}_{1,2} + 6 \left(\frac{t(s)}{[s-2]_r}\right)^3 \mathcal{G}_{0,3}. \]  

(5.39)

**Definition 5.6** Let us set “renormalized” nonlocal integrals of motion, \( \mathcal{G}_n(s) \) by

\[ \mathcal{G}_n(s) = \mathcal{G}_n + \sum_{\alpha=1}^{n} \sum_{\beta=\alpha}^{n} (-1)^\beta \left(\frac{t(s)}{[s-2]_r}\right)^\beta \left(\frac{n!}{(n-\beta)!}\right)^2 \left(\sum_{k_1+k_2+\cdots+k_\alpha=\beta}^{1} \frac{1}{k_1! k_2! \cdots k_\alpha!}\right) \mathcal{G}_{n-\beta}. \]  

(5.40)

**Proposition 5.9** We have the commutation relation.

\[ [\mathcal{G}_n(s), \mathcal{G}_m(s)] = 0 \quad (m, n = 1, 2, \cdots). \]  

(5.41)
Example

\[ G_1(s) = G_1 - \frac{t(s)}{s-2} G_0, \quad (5.42) \]

\[ G_2(s) = G_2 - 4 \frac{t(s)}{s-2} G_1 + 2 \left( \frac{t(s)}{s-2} \right)^2 G_0, \quad (5.43) \]

\[ G_3(s) = G_3 - 9 \frac{t(s)}{s-2} G_2 + 18 \left( \frac{t(s)}{s-2} \right)^2 G_1 - 6 \left( \frac{t(s)}{s-2} \right)^3 G_0. \quad (5.44) \]

Proposition 5.10  
“Renormalized” nonlocal integrals of motion \( G_n(s) \) is written by linear combination of difference \( (G_{m,1} - G_{m,0}) \).

\[ G_n(s) = G_{n,0} + \sum_{\delta=1}^{n} \left( \frac{t(s)}{s-2} \right)^{\delta} \left( \frac{n!}{(n-\delta)!} \right)^2 \frac{1}{\delta!} (G_{n-\delta,\delta} - G_{n-\delta,0}) \]

\[ + \sum_{\alpha=1}^{n-1} \sum_{\delta=\alpha+1}^{n} \sum_{\beta=1}^{\delta-\alpha} (-1)^\alpha \left( \frac{t(s)}{s-2} \right)^{\delta} \left( \frac{n!}{(n-\delta)!} \right)^2 \frac{1}{\beta!} \]

\[ \times \left( \sum_{k_1, k_2, \ldots, k_\alpha \geq 1} \frac{1}{k_1! k_2! \cdots k_\alpha!} \right) (G_{n-\delta,\beta} - G_{n-\delta,0}). \quad (5.45) \]

Example

\[ G_1(s) = G_{1,0} + \frac{t(s)}{s-2} (G_{0,1} - G_{0,0}), \quad (5.46) \]

\[ G_2(s) = G_{2,0} + 4 \frac{t(s)}{s-2} (G_{1,1} - G_{1,0}) \]

\[ - 4 \left( \frac{t(s)}{s-2} \right)^2 (G_{0,1} - G_{0,0}) + 2 \left( \frac{t(s)}{s-2} \right)^2 (G_{0,2} - G_{0,0}), \quad (5.47) \]

\[ G_3(s) = G_{3,0} + 9 \frac{t(s)}{s-2} (G_{2,1} - G_{2,0}) \]

\[ + 18 \left( \frac{t(s)}{s-2} \right)^2 (G_{1,2} - G_{1,0}) - 36 \left( \frac{t(s)}{s-2} \right)^2 (G_{1,1} - G_{1,0}) \quad (5.48) \]

\[ + 6 \left( \frac{t(s)}{s-2} \right)^3 (G_{0,3} - G_{0,0}) - 18 \left( \frac{t(s)}{s-2} \right)^3 (G_{0,2} - G_{0,0}) \]

\[ + 18 \left( \frac{t(s)}{s-2} \right)^3 (G_{0,1} - G_{0,0}). \]

When we take the limit \( s \to 2 \), we have

\[ \frac{1}{(s-2)^{\delta}} (G_{n-\delta,\beta} - G_{n-\delta,0}) \]
\[ \rightarrow \int \cdots \int \frac{dz_j}{2\pi \sqrt{-1}z_j} \prod_{j=1}^{n-\delta} \frac{dw_j}{2\pi \sqrt{-1}w_j} \prod_{1 \leq i < j \leq n-\delta} j_{1,1}(u_i, v_i | u_j, v_j) \]

\[
\times \frac{F_1(z_1) F_0(w_1)}{[u_1 - v_1 + \frac{s}{2}] [v_1 - u_1 + \frac{s}{2} - 1] \cdots [u_{n-\delta} - v_{n-\delta} + \frac{s}{2}] [v_{n-\delta} - u_{n-\delta} + \frac{s}{2} - 1]} \times (\frac{\partial}{\partial s})^{\delta - n-\delta} \prod_{j=1}^{n-\delta} \prod_{i<j}^{n-\delta+\beta} j_{1,2}(u_i, v_i | v_j) \prod_{n-\delta+1 \leq i < j \leq n-\delta+\beta} j_{2,2}(v_i | v_j) \]

\[ \times : F_1(x^{2-s}w_{n-\delta+1}) F_0(w_{n-\delta+1}) : \cdots : F_1(x^{2-s}w_{m-\delta+\beta}) F_0(w_{m-\delta+\beta}) : . \] (5.49)

Simplifications occur for \( s = 2 \).

\[ j_{1,2}(u_1, v_1 | v_2) |_{s=2} = 1, \quad j_{2,2}(u_2 | v) |_{s=2} = 1, \quad : F_1(x^{2-s}w) F_0(w) : |_{s=2} = \text{id}, \] (5.50)

We conjecture the following.

**Conjecture 5.11** When we take the limit \( s \to 2 \), we have

\[ \mathcal{G}_n(s) \to \mathcal{G}^DV_n, \quad (s \to 2). \] (5.51)

As a consequence of (5.51), we have the commutation relation \([\mathcal{G}^DV_m, \mathcal{G}^DV_n] = 0, [\mathcal{T}^DV_m, \mathcal{G}^DV_n] = 0\) and the invariance \( \eta(\mathcal{G}^DV_n) = \mathcal{G}^DV_n \).

**Example (CFT-limit)**

Let us set two parameters \((h, \beta)\) by \( e^h = x^{2r}, \beta = \frac{r-1}{r} \). We call the limit \( h \to 0 \) with \( \beta \) fixed, the CFT-limit. In this limit the nonlocal integrals of motion, \( \mathcal{G}^DV_k(k = 1, 2, \cdots) \) reproduce those of CFT, \( Q_k(k = 1, 2, \cdots) \) by [1].

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A Normal Ordering

We summarize the normal orderings of the currents.

\[ \Lambda_i(z_1)\Lambda_i(z_2) = (1 - z_2/z_1)\frac{\left(x^{2r+2s-2}z_2/z_1; x^{2s}\right)_\infty\left(x^{2s-2r}z_2/z_1; x^{2s}\right)_\infty}{\left(x^{2s-2}z_2/z_1; x^{2s}\right)_\infty} (i = 1, 2), \quad (A.1) \]

\[ \Lambda_i(z_1)\Lambda_2(z_2) = (1 - z_2/z_1)\frac{\left(x^{2r}z_2/z_1; x^{2s}\right)_\infty\left(x^{2s}z_2/z_1; x^{2s}\right)_\infty}{\left(x^{2s}z_2/z_1; x^{2s}\right)_\infty} (A.2) \]

\[ \Lambda_2(z_1)\Lambda_1(z_2) = (1 - z_2/z_1)\frac{\left(x^{2s-2}z_2/z_1; x^{2s}\right)_\infty\left(x^{2s+2r}z_2/z_1; x^{2s}\right)_\infty}{\left(x^{2s+2}z_2/z_1; x^{2s}\right)_\infty} (A.3) \]

\[ \Lambda_1(z_1)F_1(z_2) = \Lambda_1(z_1)F_1(z_2) : x^{2(r-1)}\frac{1 - x^{r-2}z_2/z_1}{1 - x^{r-2}z_2/z_1}, \quad (A.4) \]

\[ F_1(z_1)\Lambda_1(z_2) = F_1(z_1)\Lambda_1(z_2) : (1 - x^{2-r}z_2/z_1)\frac{1}{1 - x^{r}z_2/z_1}, \quad (A.5) \]

\[ \Lambda_2(z_1)F_1(z_2) = \Lambda_2(z_1)F_1(z_2) : x^{2(r-1)}\frac{1 - x^{2-r}z_2/z_1}{1 - x^{r}z_2/z_1}, \quad (A.6) \]

\[ F_1(z_1)\Lambda_2(z_2) = F_1(z_1)\Lambda_2(z_2) : \frac{1 - x^{r-2}z_2/z_1}{1 - x^{r-2}z_2/z_1}, \quad (A.7) \]

\[ \Lambda_1(z_1)F_0(z_2) = \Lambda_1(z_1)F_0(z_2) : x^{2(r-1)}\frac{1 - x^{2-r-s}z_2/z_1}{1 - x^{r-s}z_2/z_1}, \quad (A.8) \]

\[ F_0(z_1)\Lambda_1(z_2) = F_0(z_1)\Lambda_1(z_2) : \frac{1 - x^{r+s-2}z_2/z_1}{1 - x^{r+s}z_2/z_1}, \quad (A.9) \]

\[ \Lambda_2(z_1)F_0(z_2) = \Lambda_2(z_1)F_0(z_2) : x^{2(r-1)}\frac{1 - x^{r+s-2}z_2/z_1}{1 - x^{r+s}z_2/z_1}, \quad (A.10) \]

\[ F_0(z_1)\Lambda_2(z_2) = F_0(z_1)\Lambda_2(z_2) : \frac{1 - x^{2-r-s}z_2/z_1}{1 - x^{r-s}z_2/z_1}, \quad (A.11) \]

\[ \Lambda_1(z_1)E_1(z_2) = \Lambda_1(z_1)E_1(z_2) : x^{2r}\frac{1 - x^{-1}z_2/z_1}{1 - x^{-1}z_2/z_1}, \quad (A.12) \]

\[ E_1(z_1)\Lambda_1(z_2) = E_1(z_1)\Lambda_1(z_2) : \frac{1 - x^{r+1}z_2/z_1}{1 - x^{r+1}z_2/z_1}, \quad (A.13) \]

\[ \Lambda_2(z_1)E_1(z_2) = \Lambda_2(z_1)E_1(z_2) : x^{-2r}\frac{1 - x^{r+1}z_2/z_1}{1 - x^{r+1}z_2/z_1}, \quad (A.14) \]
\[ E_1(z_1)A_2(z_2) = E_1(z_1)A_2(z_2) : \frac{(1 - x^{-r-1}z_2/z_1)}{(1 - x^{-r}z_2/z_1)}, \quad (A.15) \]

\[ \Lambda_1(z_1)E_0(z_2) = \Lambda_1(z_1)E_0(z_2) : x^{-2r} \frac{(1 - x^{r+1-s}z_2/z_1)}{(1 - x^{-r+1-s}z_2/z_1)}, \quad (A.16) \]

\[ E_0(z_1)A_1(z_2) = E_0(z_1)A_1(z_2) : \frac{(1 - x^{-r-1+s}z_2/z_1)}{(1 - x^{-r+1+s}z_2/z_1)}, \quad (A.17) \]

\[ \Lambda_2(z_1)E_0(z_2) = \Lambda_2(z_1)E_0(z_2) : x^{2r} \frac{(1 - x^{-r-1+s}z_2/z_1)}{(1 - x^{-r+1+s}z_2/z_1)}, \quad (A.18) \]

\[ E_0(z_1)A_2(z_2) = E_0(z_1)A_2(z_2) : \frac{(1 - x^{r+1-s}z_2/z_1)}{(1 - x^{-r+1-s}z_2/z_1)}, \quad (A.19) \]

\[ E_j(z_1)E_j(z_2) = E_j(z_1)E_j(z_2) : \frac{x^{2r} (1 - z_2/z_1)}{(x^{2r+2} z_2/z_1; x^{2r})_\infty}, \quad (A.20) \]

\[ E_1(z_1)E_0(z_2) = E_1(z_1)E_0(z_2) : \frac{(z^{2r-s} z_2/z_1; x^{2r})_\infty (x^{2r+2-s} z_2/z_1; x^{2r})_\infty}{(x^{-s} z_2/z_1; x^{2r})_\infty (x^{s-2} z_2/z_1; x^{2r})_\infty}, \quad (A.21) \]

\[ E_0(z_1)E_1(z_2) = E_0(z_1)E_1(z_2) : \frac{(z^{2r-s} z_2/z_1; x^{2r})_\infty (x^{2r+2-s} z_2/z_1; x^{2r})_\infty}{(x^{-s} z_2/z_1; x^{2r})_\infty (x^{s-2} z_2/z_1; x^{2r})_\infty}, \quad (A.22) \]

\[ F_j(z_1)F_j(z_2) = F_j(z_1)F_j(z_2) : \frac{x^{2r} (1 - z_2/z_1)}{(x^{2r-2} z_2/z_1; x^{2r})_\infty}, \quad (A.23) \]

\[ F_1(z_1)F_0(z_2) = F_1(z_1)F_0(z_2) : \frac{x^{-(2r-s)} z_2/z_1; x^{2r})_\infty (x^{2r-s} z_2/z_1; x^{2r})_\infty}{(x^{-s} z_2/z_1; x^{2r})_\infty (x^{s-2} z_2/z_1; x^{2r})_\infty}, \quad (A.24) \]

\[ F_0(z_1)F_1(z_2) = F_0(z_1)F_1(z_2) : \frac{x^{-(2r-s)} z_2/z_1; x^{2r})_\infty (x^{2r-s} z_2/z_1; x^{2r})_\infty}{(x^{s} z_2/z_1; x^{2r})_\infty (x^{s-2} z_2/z_1; x^{2r})_\infty}, \quad (A.25) \]

\[ F_1(z_1)E_1(z_2) = F_1(z_1)E_1(z_2) : \frac{1}{z_1^2 (1 - x z_2/z_1)(1 - x^{-1} z_2/z_1)}, \quad (A.26) \]

\[ E_1(z_1)F_1(z_2) = E_1(z_1)F_1(z_2) : \frac{1}{z_1^2 (1 - x z_2/z_1)(1 - x^{-1} z_2/z_1)}, \quad (A.27) \]

\[ F_0(z_1)E_0(z_2) = F_0(z_1)E_0(z_2) : \frac{1}{z_1^2 (1 - x z_2/z_1)(1 - x^{-1} z_2/z_1)}, \quad (A.28) \]

\[ E_0(z_1)F_0(z_2) = E_0(z_1)F_0(z_2) : \frac{1}{z_1^2 (1 - x z_2/z_1)(1 - x^{-1} z_2/z_1)}, \quad (A.29) \]

\[ F_1(z_1)E_0(z_2) = F_1(z_1)E_0(z_2) : \frac{1}{z_1^2 (1 - x z_2/z_1)(1 - x^{-1} z_2/z_1)}, \quad (A.30) \]

\[ E_0(z_1)F_1(z_2) = E_0(z_1)F_1(z_2) : \frac{1}{z_1^2 (1 - x z_2/z_1)(1 - x^{-1} z_2/z_1)}, \quad (A.31) \]

\[ F_0(z_1)E_1(z_2) = F_0(z_1)E_1(z_2) : \frac{1}{z_1^2 (1 - x z_2/z_1)(1 - x^{-1} z_2/z_1)}, \quad (A.32) \]

\[ E_1(z_1)F_0(z_2) = E_1(z_1)F_0(z_2) : \frac{1}{z_1^2 (1 - x z_2/z_1)(1 - x^{-1} z_2/z_1)}, \quad (A.33) \]

\[ \mathcal{A}(z_1) F_0(z_2) = \mathcal{A}(z_1) F_0(z_2) : \frac{z_1^2 (x^{3r-3} z_2/z_1; x^{2r})_\infty (x^{r+s-2} z_2/z_1; x^{2r})_\infty}{(x^{r-s+2} z_2/z_1; x^{2r})_\infty (x^{-r+s+2} z_2/z_1; x^{2r})_\infty}, \quad (A.34) \]
\begin{align}
F_0(z_1)A(z_2) &= z_1^2 \frac{(x^{\text{r}+s-z_2}/z_1; x^{2\text{r}})_{\infty}}{(x^r+2 z_2/z_1; x^{2\text{r}})_{\infty}} \frac{(x^{r+s-2} z_2/z_1; x^{2\text{r}})_{\infty}}{(x^{-r+s} z_2/z_1; x^{2\text{r}})_{\infty}}, \quad (A.35) \\
A(z_1)F_1(z_2) &= z_1^2 \frac{(x^{r+s-2} z_2/z_1; x^{2\text{r}})_{\infty}}{(x^{r+s} z_2/z_1; x^{2\text{r}})_{\infty}}, \quad (A.36) \\
F_1(z_1)A(z_2) &= z_1^2 \frac{(x^{r+s} z_2/z_1; x^{2\text{r}})_{\infty}}{(x^{r+s} z_2/z_1; x^{2\text{r}})_{\infty}}, \quad (A.37) \\
A(z_1)E_0(z_2) &= z_1^2 (1 - x^{r+1-s} z_2/z_1)(1 - x^{-r+s} z_2/z_1), \quad (A.38) \\
E_0(z_1)A(z_2) &= z_1^2 (1 - x^{r+1-s} z_2/z_1)(1 - x^{-r+s} z_2/z_1), \quad (A.39) \\
A(z_1)E_1(z_2) &= \frac{1}{z_1^2 (1 - x^{-1} z_2/z_1)(1 - x^{-1} z_2/z_1)}, \quad (A.40) \\
E_1(z_1)A(z_2) &= \frac{1}{z_1^2 (1 - x^{-1} z_2/z_1)(1 - x^{-1} z_2/z_1)}, \quad (A.41) \\
B(z_1)F_0(z_2) &= z_1^2 (1 - x^{s+r} z_2/z_1)(1 - x^{s^{-1}} z_2/z_1), \quad (A.42) \\
F_0(z_1)B(z_2) &= z_1^2 (1 - x^{s+r} z_2/z_1)(1 - x^{s^{-1}} z_2/z_1), \quad (A.43) \\
B(z_1)F_1(z_2) &= \frac{1}{z_1^2 (1 - x^{r+1} z_2/z_1)(1 - x^{-1} z_2/z_1)}, \quad (A.44) \\
F_1(z_1)B(z_2) &= \frac{1}{z_1^2 (1 - x^{r+1} z_2/z_1)(1 - x^{-1} z_2/z_1)}, \quad (A.45) \\
B(z_1)E_0(z_2) &= z_1^{-2} \frac{(x^{3r}-z_2/z_1; x^{2r})_{\infty}}{(x^{r-s} z_2/z_1; x^{2r})_{\infty}} \frac{(x^{r+s} z_2/z_1; x^{2r})_{\infty}}{(x^{-r+s} z_2/z_1; x^{2r})_{\infty}}, \quad (A.46) \\
E_0(z_1)B(z_2) &= z_1^{-2} \frac{(x^{3r}-z_2/z_1; x^{2r})_{\infty}}{(x^{r-s} z_2/z_1; x^{2r})_{\infty}} \frac{(x^{r+s} z_2/z_1; x^{2r})_{\infty}}{(x^{-r+s} z_2/z_1; x^{2r})_{\infty}}, \quad (A.47) \\
B(z_1)E_1(z_2) &= z_1^{-2} \frac{(x^{r-s} z_2/z_1; x^{2r})_{\infty}}{(x^{r-s} z_2/z_1; x^{2r})_{\infty}}, \quad (A.48) \\
E_1(z_1)B(z_2) &= z_1^{-2} \frac{(x^{r-s} z_2/z_1; x^{2r})_{\infty}}{(x^{r-s} z_2/z_1; x^{2r})_{\infty}}. \quad (A.49) \\

\text{For } \text{Re}(r) > 0 \text{ we have} \\
F_0(z_1)H(z_2) &= z_1^2 \frac{(x^{r+s} z_2/z_1; x^{2r})_{\infty}}{(x^{r+s} z_2/z_1; x^{2r})_{\infty}} \frac{(x^{r+s} z_2/z_1; x^{2r})_{\infty}}{(x^{r+s} z_2/z_1; x^{2r})_{\infty}}, \quad (A.50) \\
F_1(z_1)H(z_2) &= z_1^{-2} \frac{(x^{r+s} z_2/z_1; x^{2r})_{\infty}}{(x^{r+s} z_2/z_1; x^{2r})_{\infty}}, \quad (A.51) \\
H(z_1)F_0(z_2) &= z_1^2 \frac{(x^{r+s} z_2/z_1; x^{2r})_{\infty}}{(x^{r+s} z_2/z_1; x^{2r})_{\infty}} \frac{(x^{r+s} z_2/z_1; x^{2r})_{\infty}}{(x^{r+s} z_2/z_1; x^{2r})_{\infty}}, \quad (A.52) \\
H(z_1)F_1(z_2) &= z_1^{-2} \frac{(x^{r+s} z_2/z_1; x^{2r})_{\infty}}{(x^{r+s} z_2/z_1; x^{2r})_{\infty}}, \quad (A.53)
\end{align}
For \( \text{Re}(r^*) < 0 \) we have

\[
E_0(z_1)H(z_2) = z_1^s \frac{(x^{r^*+s}z_2/z_1; x^{-2r^*})_\infty (x^{-r^*+2s}z_2/z_1; x^{-2r^*})_\infty}{(x^{-r^*-s}z_2/z_1; x^{-2r^*})_\infty}, \quad (A.54)
\]

\[
E_1(z_1)H(z_2) = z_1^s \frac{(x^{r^*+2}z_2/z_1; x^{-2r^*})_\infty}{(x^{-r^*-2}z_2/z_1; x^{-2r^*})_\infty}, \quad (A.55)
\]

\[
H(z_1)E_0(z_2) = z_1^s \frac{(x^{r^*+s}z_2/z_1; x^{-2r^*})_\infty (x^{-r^*+2s}z_2/z_1; x^{-2r^*})_\infty}{(x^{-r^*-s}z_2/z_1; x^{-2r^*})_\infty}, \quad (A.56)
\]

\[
H(z_1)E_1(z_2) = z_1^s \frac{(x^{r^*+2}z_2/z_1; x^{-2r^*})_\infty}{(x^{-r^*-2}z_2/z_1; x^{-2r^*})_\infty}. \quad (A.57)
\]

References

[1] V.Bazhanov, S.Lukyanov, Al.Zamolodchikov : Integral Structure of Conformal Field Theory, Quantum KdV Theory and Thermodynamic Bethe Ansatz Commun.Math.Phys.177, 381-398, (1996). V.Bazhanov, S.Lukyanov, Al.Zamolodchikov : Integrable Structure of Conformal Field Theory III, The Yang Baxter Relation Commun.Math.Phys.200, 297-324, (1999).

[2] B.Feigin and E.Frenkel : Integrals of Motion and Quantum Groups, Lecture Notes in Mathematics1620 Integral Systems and Quantum Groups, 1995.

[3] J.Shiraishi, H.Kubo, H.Awata, S.Odake: A Quantum Deformation of the Virasoro Algebra and the Macdonald Symmetric Functions, Lett.Math.Phys.38, 647-666, (1996).

[4] B.Feigin and E.Frenkel : Quantum W-Algebra and Elliptic Algebras, Commun.Math.Phys.178, 653-678 (1996).

[5] B.Feigin and A.Odesskii : A Family of Elliptic Algebras, Internat.Math.Res.Notices no.11, 531-539, (1997).

[6] E.Frenkel and N.Reshetikhin : Quantum affine algebras and deformations of the Virasoro and W-algebras, Commun. Math. Phys.178, 237-264, (1996).

[7] E.Frenkel : Deformations of the KdV hierarchy and related soliton equations, Internat.Math.Res.Notices, no.2, 55-76, (1996).
[8] B. Feigin, T. Kojima, J. Shiraishi, H. Watanabe: The Integrals of Motion for the Deformed $W$-Algebra $W_{q,t}(\widehat{sl}_N)$, *Proceedings for Representation Theory 2006*, Atami, Japan, 102-114, (2006) [ISBN-9902328-2-8]