A web of confocal parabolas in a grid of hexagons

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Abstract
If one erects regular hexagons upon the sides of a triangle \( T \), several surprising properties emerge, including: (i) the triangles which flank said hexagons have an isodynamic point common with \( T \), (ii) the construction can be extended iteratively, forming an infinite grid of regular hexagons and flank triangles, (iii) a web of confocal parabolas with only three distinct foci interweaves the vertices of hexagons in the grid. Finally, (iv) said foci are the vertices of an equilateral triangle.

Keywords Hexagon · Flank · Map · Isodynamic · Parabola · Confocal

Mathematics Subject Classification 51M04 · 51N20 · 51N35 · 68T20

1 Introduction

Napoleon’s theorem, studied in Martini (1996), is illustrated in Fig. 1 (left): the centroids of the 3 equilaterals erected upon the sides of a reference triangle \( T = ABC \) are vertices of an equilateral known as the “outer” Napoleon triangle (Weisstein 2002). A related construction due to Lamoen appears in Fig. 1 (right), whereby squares are erected upon the sides of \( T \) and triangles which “flank” said squares are defined (Lamoen 2001).

We borrow from those ideas and study a construction whereby regular hexagons \( H_a, H_b, H_c \) are erected upon the sides of \( T \), with 3 “flank” triangles \( F_a, F_b, F_b \) defined between them, see Fig. 2. As it turns out, many surprising properties emerge, listed below.

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Fig. 1  Left: Napoleon’s theorem states that a triangle $T' = A'B'C'$ (dashed magenta) whose vertices are the centroids of 3 equilaterals (green) erected upon the sides of a triangle $T = ABC$ (blue) is itself equilateral. Furthermore, its centroid coincides with the barycenter $X_2$ of $T$ (Weisstein 2002, Napoleon’s Thm). Right: Lamoen (2001) has studied properties of “flank” triangles $F$ defined between neighboring squares $Q_a, Q_b, Q_c$ erected upon the sides of a $T = ABC$, as well as those of additional squares $Q'_a, Q'_b, Q'_c$ erected upon the flanks (color figure online)

Fig. 2 Given a reference triangle $T = ABC$, erect three regular hexagons $H_a, H_b, H_c$ on its sides; these define three flank triangles with a vertex at $A$, $B$, or $C$ (respectively), and two others which are proximal vertices of erected hexagons sharing $A$, $B$, or $C$, respectively

Summary of the results

- The second isodynamic points (Weisstein 2002) of $T$ and the flanks are common;
- A contiguous, infinite grid of regular hexagons can be constructed, see Fig. 8; all flanks in the grid have a common second isodynamic point and conserve a special quantity;
• Sequences of hexagon vertices are crisscrossed by a web of confocal parabolas with only 3 distinct foci;
• Their foci and directrix intersections are vertices of 2 new equilateral triangles;
• Phenomena are described when the reference triangle in Fig. 2 are triangles in two special Poncelet triangle families (for a survey on Poncelet’s porism, see Dragović and Radnović 2014).

Related work

In Čerin (2002), Lamoen (2001) properties of “flank” triangles located between squares and/or rectangles erected on a triangle’s sides are studied. Works (Dosa 2007; Hoehn 2001) study triangles centers (taken as triples or not) analogues of the intouch and/or extouch triangles erected upon each side of a reference triangle. In Fukuta (1996a), Fukuta (1996b), Stachel (2002), Čerin (1998) a construction related to Napoleon’s theorem is described which associates to a generic triangle a regular hexagon. A study of properties and invariants of regular polygons erected upon the homothetic Poncelet family appears in Morozov and Nilov (2022).

Article structure

In Sect. 2 we describe properties of the trio of flank triangles. In Sect. 3 we fix a central regular hexagon and consider properties of 6 “satellite” triangles built around it, and this implies a contiguous grid can be iteratively built. In Sect. 4 we show said grid is interwoven by three groups of confocal parabolas. In Sect. 5 we show two examples of grids controlled by Poncelet poristic triangles. Section 6 provides a table of videos illustrating some results as well as a list of open questions. In Appendix A we provide computer-usable expressions for some objects described in Sect. 4.

2 Hexagonal flanks

Adhering to Kimberling’s $X_k$ notation for triangle centers (Kimberling 2019), recall a triangle’s isodynamic points $X_{15}$ and $X_{16}$ are the limiting points of a pencil$^1$ of circles containing a triangle’s circumcircle and the Brocard circle (Weisstein 2002, Isodynamic Points). For foundations and properties of Brocard geometry and/or the isodynamic points see (Casey 1888; Johnson 1960).

Referring to Fig. 3:

**Proposition 1** The triangle with vertices on the $X_{15}$-of-flanks is perspective with $\mathcal{T}$ at the latter’s symmedian point $X_6$.

**Proof** A straightforward derivation for the barycentrics of $X_{15}$ of the $A$-flank yields $(a^2 - 3b^2 - 3c^2, b^2, c^2)$, and those of the $B$- and $C$-flanks can be obtained cyclically. Recall from the barycentrics of $X_6$ are $(a^2, b^2, c^2)$ (see Kimberling 2019), so the result follows. □

$^1$ This pencil is known as the Schoutte pencil (Johnson 1917).
The following proof sketch was kindly contributed by D. Grinberg (2021):

**Proof** Let two triangles $\mathcal{T} = ABC$ and $\mathcal{T}' = A'B'C'$ be called “30-twins around B” iff $B = B'$ and $\angle BAA' = \angle AA'B = 30^o$ and $\angle B'C'C = \angle C'C'B = 30^o$ (angles are in degrees and are directed angles mod 180°). In other words, this is the relation between $\mathcal{T}$ its flank triangle at B. It is a symmetric relation, up to relabeling $ABC$ as $CBA$ to get the correct orientation.

Let $ABC$ and $A'B'C'$ be two triangles that are 30-twins around $B$. Fix a nontrivial circle $K$ around $B$, and let $X, Z, X'$ and $Z'$ be, respectively, the images of $A, C, A'$ and $C'$ under the inversion with respect to $K$. It can be shown that it follows that triangles $XBZ$ and $X'BZ'$ are again 30-twins around $B$.

The same inversion takes $X_{16}$ of $ABC$ to the third vertex of an equilateral triangle erected inwardly on the side $XZ$ of triangle $XBZ$. The proof is an angle chase, using the fact that $X_{16}$ is the unique point of a triangle which sees the three vertices at $120^o$ angles (Weisstein 2002, Isodynamic point). The same applies for $X_{15}$ and an equilateral triangle erected outwardly. Likewise for triangle $A'B'C'$ and $X'Z'$.

Thus, it remains to be shown that the third vertex of an equilateral triangle erected inwardly on the side $XZ$ of triangle $XBZ$ is identical to the third vertex of an equilateral triangle erected inwardly on side $X'Z'$ of triangle $X'BZ'$. This is easy to prove using complex numbers: $w^2 + w + 1 = 0$ with $w = e^{2\pi i/3}$.

\(\blacksquare\)
2.1 Zero-area flanks

Consider a family of triangles $ABC$ where $A$, $B$ are fixed and $C$ is free. Let $F_c$ denote the flank triangle between the regular hexagons erected upon $AC$ and $CB$. As shown in Fig. 4:

**Observation 1** $F_c$ will be zero-area if $C$ subtends a $120^\circ$ angle, i.e., it lies on a circular arc centered on the centroid $O$ of an equilateral erected upon $AB$ and with radius $|OA|$.

Consider the construction for the two flank triangles $F_1$ and $F_2$ shown in Fig. 5, namely, departing from $T = ABC$, erect regular hexagons $H_1$ and $H_2$ on sides $AC$ and $BA$, respectively. Consider the first flank triangle $F_1$ between $H_1$ and $H_2$. As shown in the figure, erect a third regular hexagon $H_3$ on the unused side of $F_1$, and let a second flank triangle $F_2$ sit between $H_2$ and $H_3$. While holding $B$ and $C$ fixed, there are positions for $A$ such that $F_2$ has positive, zero, or negative signed area. Referring to Fig. 6:

**Proposition 3** $F_2$ will have positive (resp. negative) area if $A$ is exterior (resp. interior) to the circumcircle $C$ of an equilateral triangle whose base is $B$ and the midpoint of $BC$. In particular, the vertices of $F_2$ become collinear if $A$ is on $C$.

![Fig. 4](image-url) The locus of vertex $C$ such that it subtends a $120^\circ$ angle, i.e., the $C$-flank is degenerate, is a circular arc (dashed blue) such centered on $O$, the centroid of an equilateral (dashed brown) erected upon $AB$, and with radius $|OA|$ (a mirror arc corresponding to the top equilateral centered on $O'$ is also shown). The left (resp. right) picture shows $C$ in two distinct positions (color figure online).
For certain positions of $A$ (while keeping $B$ and $C$ stationary), flank triangle $F_2$ will be non-versed (left), degenerate (middle), or versed (right).

With $B$ and $C$ fixed, the locus of $A$ such that the area of $F_2$ in Fig. 5 is zero is the circumcircle (dashed red) of an equilateral (brown) with base on $B$ and the midpoint of $BC$. Two positions of $A$ are shown (left, right) (color figure online).

### 3 Satellite triangles

Consider the construction shown in Fig. 7: given a triangle $\mathcal{T} = ABC$, erect regular hexagons $H_0$ and $H_1$ on sides $BC$ and $AB$. These define a first flank triangle $F_1$. Continue adding hexagons $H_2, \ldots, H_5$, defining with $H_0$ new flank triangles $F_2, \ldots, F_5$. Let $D$ (resp. $E$) be a vertex of $F_5$ which is common to $H_0$ and $H_5$ (resp. located on $H_5$ and adjacent to $D$).

Let $A'$ be the reflection of $A$ about $BC$. The following lemma was kindly contributed by A. Akopyan (2021):

**Lemma 1** The reflection of the apexes of $F_1, \ldots, F_5$ all coincide with $A'$.

Still referring to Fig. 7, let $CE$ be the side of $F_5$ not on $H_0$ nor on $H_5$. We can use Lemma 1 to show that:

**Proposition 4** $|CE| = |AC|$ and $\angle ECA = (2\pi)/3$.

This implies that a sixth, regular hexagon $H_6$ can be erected on $AC$ and one of its vertices will snap perfectly against vertex $E$ of $H_5$. Referring to Fig. 8:

**Corollary 1** This construction can be extended ad infinitum, creating, modulo self-intersections, a locally-consistent contiguous grid.
Fig. 7 The grid “closure” property: departing from $T = ABC$ add regular hexagons $H_0, \ldots, H_5$ and flanks $F_1, \ldots, F_5$. The reflection of the apexes of $F_i$ about their based (sides of $H_i$) is a common point $A'$. This implies the regular hexagon $H_6$ erected on $AC$ fits perfectly between $H_1$ and $H_5$. The $X_{16}$ common to $T$ and the five flanks is shown in the upper left of the picture.

Fig. 8 Departing from $T = ABC$ a contiguous grid can be built by adding more triangles and regular hexagons as needed. The second isodynamic point $X_{16}$ of all interstitial triangles is common with that of $T$.

Since in this infinite grid one can isolate a central triangle and the three flanks around it (see Fig. 2), you get the following propagation:

**Corollary 2** The second isodynamic points $X_{16}$ of all flank triangles in any contiguous grid coincide in a single point.

Consider the related construction in Fig. 9: let $H$ denote a fixed regular hexagon with vertices $Q_i$, $i = 1, \ldots, 6$. Given a point $P$, let $Q_1Q_2P$ be a first “satellite” triangles $F_1$. Create 5 new satellite triangles $F_i$ as follows: Let $F_i$ be the flank triangle obtained by erecting a regular hexagon on a side of $F_{i-1}$, $i = 2, \ldots, 6$. 
Given a fixed central hexagon $H$ with vertices $Q_i, i = 1, \ldots, 6$, let 6 “satellite” flank triangles (green) be constructed around $H$ departing from a first triangle $PQ_1Q_2$. The sum of areas of said 6 satellites is independent of $P$ and equal to the area of $H$. In the left (resp. right) $P$ is positioned so the construction is regular (resp. irregular).

From Lemma 1 and Proposition 4, the reflected images of the flanks about their bases fill the interior of $H$, therefore:

**Corollary 3** The sum of the areas of the 6 satellite triangles is independent of $P$ and equal to the area of $H$.

Referring to Fig. 10:

The apexes of the 6 satellite triangles (green) around a fixed hexagon (blue) lie on a conic iff $P$ is on either line $Q_3Q_2$ or line $Q_6Q_1$. When $P$ is at the intersection of said lines, the 6 apexes are concyclic, as in Fig. 9 (left) (color figure online).
Proposition 5 The apexes of the 6 satellite triangles lie on a conic iff $P$ lies on either $Q_3Q_2$ or line $Q_6Q_1$.

Referring to Fig. 11, consider the 6 auxiliary regular hexagons erected successfully on the 6 satellite triangles parametrized by $P$. Let $O$ denote the intersection of $Q_3Q_2$ and $Q_6Q_1$.

Proposition 6 The iso-curves of $P$ such that the sum of the areas of the 6 satellite hexagons is constant are circles centered on $O$.

3.1 Second-level satellites

Referring to Fig. 12, consider the 6 satellite flanks $F_i$ surrounding a central hexagon which are obtained as above by sequentially erecting 6 regular hexagons $\mathcal{H}_i$. Consider 6 “2nd-level” flanks $F'_i$ nestled between the $\mathcal{H}_i$. Let $\mathcal{H}'_k$ denote a hexagon whose vertices are the $X_k$ of the $F'_i$.

Proposition 7 For all $X_k$ on the Euler line, $\mathcal{H}'_k$ has invariant internal angles.

We thank A. Akopyan for the following argument (Akopyan 2021):
Fig. 12 Shown are 5 hexagons $H'_i$ (red) whose vertices are the $X_k$ of second-level satellites (yellow). If $X_k$ is on the Euler line, the $H'_i$ have identical internal angles (color figure online).

**Proof** This follows from the fact that their area is the sum of squares of distances from the reflected point $A'$ to vertices $Q_i$ of the central hexagon. □

### 3.2 Properties of the second Fermat point

The second Fermat point $X_{14}$ of a triangle is the isogonal conjugate of the second isodynamic point $X_{16}$ (Weisstein 2002, Fermat Points). Let $H = ABCDEF$ be a regular hexagon with centroid $O$, and let $P$ be a point anywhere. Define six “inner” triangles $\mathcal{T}_1 = ABP$, $\mathcal{T}_2 = BCP$, ..., $\mathcal{T}_6 = FAP$. Referring to Fig. 13 (left):

**Proposition 8** The $X_{14}$ of the $\mathcal{T}_i$ will lie on $PO$.

Let $\mathcal{T}'_i$ be the six triangles with (i) the base a side of $H$, and (ii) the apex $P'_i$ the reflection of $P$ about said side. Assume in this case $P$ is interior to $H$. Referring to Fig. 13 (right):

[Diagram showing relationships between points and triangles as described in the text]
Fig. 13  Left: Let $H = ABCDEF$ be a regular hexagon, and $P$ a point. The second Fermat points $X_{14}$ of the “inner” triangles $T_i \in \{ABP, BCP, \ldots, FAP\}$ are collinear with $P$ and the centroid $O$ of $H$. Right: Let $T_i'$ be triangles with base a side of $H$, and apex $P_i'$ the reflection of $P$ about said side. The $X_{14}$ of the $T_i'$ lie on a rectangular hyperbola (green) concentric with $H$. Also shown is the (dotted magenta) line of the $X_{14}$ of the inner triangles

**Proposition 9** The $X_{14}$ of the $T_i'$ lie on a rectangular hyperbola (green) concentric with $H$.

### 4 A web of confocal parabolas

Referring to Fig. 14, let, let $H_i, H_{i+1},$ etc., be adjacent hexagons in the grid sharing antipodal vertices $U_i$, such that one of the $U_i$ is $A$. Let $H_i', H_{i+1}',$ etc., be a second sequence of adjacent hexagons running along the same “grain” in the grid. The following was discovered by A. Akopyan (2021):

**Proposition 10** The sequence ..., $U_{i-1}, A, U_{i+1}, \ldots$ lies on a parabola which we call the A-parabola. Furthermore, similarly-constructed sequences of vertices along hexagons in the same diagonal direction lie on parabolas confocal with the A-parabola at $f_a$.

Referring to Fig. 15, a total of three groups of confocal parabolas can be constructed, along three “grains” in the grid. Let $a, b, c$ denote the sides of $T = ABC$, and $S$ is Conway’s notation for twice the area of $T$. Referring to Fig. 16:

**Theorem 1** The foci of the three groups of confocal parabolas are vertices of an equilateral triangle with centroid at the $X_{16}$ common to all flanks and $T$. The barycentric coordinates of $f_a$ are given by:

$$f_a = \left[ \frac{\sqrt{3}(7a^2b^2 + 7a^2c^2 + 2b^2c^2 - 4a^4 - b^4 - c^4) - 2S(8a^2 + b^2 + c^2)}{\sqrt{3}(2a^2b^2 - a^2c^2 + 3b^2c^2 - 4b^4 + c^4) + 2S(2a^2 - 2b^2 - c^2)} \right]$$
From hexagons $\mathcal{H}_{i-1}$, $\mathcal{H}_i$, etc., take antipodal vertices $U_{i-2}$, $U_{i-1}$, $A$, $U_{i+1}$. These lie on the $A$-parabola (blue), whose focus is $f_a$. Confocal parabolas pass through antipodal vertices along any other “row” of adjacent hexagons in the grid, e.g., $\mathcal{H}_{i-1}$, $\mathcal{H}_i$, $\mathcal{H}_{i+1}$, \ldots (color figure online)

Three groups of confocal parabolas (red, green, blue) run along antipodal vertices of hexagons along three 3 major directions. The foci $f_a$, $f_b$, $f_c$ of each confocal group are the vertices of an equilateral triangle whose centroid is the $X_{16}$ common to all flank triangles in the grid (color figure online)
Fig. 16  Zooming into the focal equilateral $f_a, f_b, f_c$. Also shown (dashed gray) are the common axes of the 3 confocal parabola groups, all of which pass through the common $X_{16}$ at 120° angles.

Furthermore, side $s$ of the focal equilateral is given by:

$$s^2 = \frac{3}{32} (a^2 + b^2 + c^2 - 2S\sqrt{3}) = \frac{3}{16} (\cot \omega - \sqrt{3})S$$

where $\omega$ is the Brocard angle of a triangle, given by $\cot(\omega) = (a^2 + b^2 + c^2)/(2S)$.

Let $a_i, b_i, c_i$ (resp $S_i$) be the sidelengths (resp. twice the area) of a given triangle $\mathcal{T}_i$ in the grid. Since any $\mathcal{T}_i$ can be used to start the grid:

**Corollary 4** The quantity $a_i^2 + b_i^2 + c_i^2 - 2S_i\sqrt{3}$ is invariant over all $\mathcal{T}_i$.

As shown in Fig. 16:

**Proposition 11** The axes of the three confocal groups concur at the common $X_{16}$ at 120° angles.

**A directrix equilateral**

The anticomplement$^2$ of the second Fermat point $X_{14}$ is labeled $X_{617}$ on Kimberling (2019).

Referring to Fig. 17:

$^2$ This is the double-length reflection about the barycenter $X_2$. Springer
Fig. 17 The triangle $A'B'C'$ bounded by the directrices (dashed blue, red, and green) of the $A$-, $B$-, and $C$-parabolas (blue, red, red, and green) is also an equilateral, whose centroid is $X_{617}$. Interestingly, the triangle (brown) connecting the vertices $V_a$, $V_b$, $V_c$ of said parabolas is in general a scalene (color figure online)

**Proposition 12** The triangle bounded by the directrices of the $A$-, $B$-, and $C$-parabolas is an equilateral whose centroid is $X_{617}$, and whose sidelengths $s'$ is given by:

$$(s')^2 = \left( \frac{S}{4} \right) \left( \frac{5\sqrt{3} + 11 \cot \omega + 16(\sqrt{3} + \cot \omega)}{2 \cos(2\omega) - 1} \right)$$

Note: an expression of the $A$-vertex of the above appears in Appendix A. Note also that the sidelength is of the directrix equilateral is not conserved across all flank triangles in the grid since each of these will be associated with a different directrix equilateral.

Interestingly, the triangle formed by the vertices of said parabolas is not an equilateral.

**Skip-1 confocal parabolas**

Let $Q_1, A, Q_3, Q_5, \ldots$ (resp. $Q_2, B, Q_4, Q_6, \ldots$) be a sequence of odd (resp. even) side vertices of adjacent hexagons, as shown in Fig. 18.

**Proposition 13** The sequence of odd (resp. even) vertices lies on a parabola. The former (resp. latter) is confocal with the $A$-parabola (resp $B$-parabola). Furthermore, their axes are parallel to the axis of the $C$-parabola, and pass through $f_b$ and $f_c$, respectively.
Fig. 18 Two additional groups of confocal parabolas exist which pass through the odd (resp. even) side vertices along a given grain of the grid. A member of the first (resp. second) confocal group is shown in dashed blue (resp. red), passing through odd vertices $[\ldots, Q_1, A, Q_3, Q_5, Q_7, \ldots]$ (resp. even vertices) $[\ldots, Q_2, B, Q_4, Q_6, \ldots]$. The focus of the odd (resp. even) group is $f_a$ (resp. $f_b$). The major axes of the odd, even, and original $C$-parabola group are parallel (dashed red, blue, green) (color figure online).

The above statements are valid cyclically, i.e., there are families of confocal odd and even parabolas along each of the 3 major directions in the grid, e.g., corresponding to the diagonals of a hexagon erected upon a side of $\mathcal{T}$.

Computer-usable explicit equations for some of the objects in this section appear in Appendix A.

5 Controlled by poncelet

Recall Poncelet’s theorem: if a polygon inscribed in one conic simultaneously circumscribes a second one, a 1d family of such polygons exists inscribed/circumscribed about the same conics (Dragović and Radnović 2014). In Reznik and Garcia (2021) two related families of Poncelet triangles are studied: (i) the homothetic family, and (ii) the Brocard porism, see Fig. 19. Geometric details about this family are provided in Table 1.

Homothetic phenomena

Referring to Fig. 20, consider our basic construction such that the reference triangle $\mathcal{T}$ is one in the homothetic family.

Referring to the last column of Table 1, notice that the homothetic family conserves both the sum of squared sidelengths and area. Another fact proved in Reznik and Garcia (2021) is that the locus of $X_k$, $k = 13, 14, 15, 16$ over this family are 4 distinct circles.

Since the the side of said equilateral only depends on the sum of squared sidelengths and area Theorem 1:
Fig. 19  Left: The “homothetic” family is interscribed between two concentric, homothetic families centered on the fixed barycenter $X_2$. Right: The Brocard porism are circle-inscribed triangles circumscribing a fixed conic known as the Brocard inellipse, whose foci are the stationary Brocard points $\Omega_1, \Omega_2$ of the family.

Table 1  Geometric details about the homothetic and Brocard porism triangle families

| family          | Outer Conic     | Inner Conic     | Stationary | Conserves                  |
|-----------------|-----------------|-----------------|------------|----------------------------|
| Homothetic      | Steiner Ellipse | Steiner Inellipse | $X_2$      | $\sum s_i^2, A, \omega$   |
| Brocard Porism  | Circumcircle    | Brocard Inellipse | $X_3, X_6, X_{39}, \Omega_1, \Omega_2, \ldots$ | $\sum s_i^2 / A, \omega$ |

**Corollary 5**  Over homothetic triangle family controlling our grid, the focal equilateral has invariant sidelength and circumradius. Furthermore, the locus of its centroid is a circle concentric with the homothetic pair of ellipses.

Still referring to Fig. 20, experimentally, we observe:

**Observation 2**  Over the homothetic family, (i) the locus of the barycenters of the three flanks is an ellipse concentric and axis-aligned with the homothetic pair, though of distinct aspect ratio, and (ii) the locus of the centroids of the three regular hexagons erected on $\mathcal{T}$ is an ellipse which is a $90^\circ$-rotated copy of (i).

**Brocard porism phenomena**

Referring to Fig. 21, consider our basic construction such that the reference triangle $\mathcal{T}$ is one in the Brocard porism.

Let $\mathcal{A} = S/2$ denote the area of a reference triangle. Rewrite the expression for $s^2$ in Theorem 1 as $s^2 = (3/8) \left( \cot \omega - \sqrt{3} \right) \mathcal{A}$. Referring to the last column of Table 1, note the Brocard porism conserves $\omega$ (though not area), therefore:

**Corollary 6**  In a (dynamic) grid controlled by triangles $\mathcal{T}$ in the Brocard porism, the focal equilateral rotates about a fixed centroid $X_{16}$. Its area is variable and proportional to the area of $\mathcal{T}$. 
Fig. 20  Phenomena manifested by objects in our basic construction over homothetic triangles (blue): (i) the focal equilateral (orange) has fixed sidelength and (ii) its centroid moves along a circle (black); (iii) the centroids $X_2$ of the 3 flank triangles move along a first ellipse (green) concentric with the homothetic pair; (iv) the centroids $O_a$, $O_b$, $O_c$ of the three regular hexagons (purple) move along a second ellipse (dotted purple) which is a $90^\circ$-rotated copy of (iii) (color figure online)

Fig. 21  The isodynamic points $X_{15}$ and $X_{16}$ of Brocard porism triangles (blue) are stationary isodynamic. This entails that over the porism, flank triangle (green) $X_{16}$’s will also be stationary. Also shown are corresponding focal equilaterals, whose area is variable and proportional to the area of porism triangles (color figure online)
Table 2  Videos of some constructions. The last column is clickable and provides the YouTube code

| id | Title                                                                 | youtube.be/                |
|----|----------------------------------------------------------------------|---------------------------|
| 01 | Basic construction of hexagonal flanks                              | e3MkijszDEA               |
| 02 | Circular Loci of $X_k$, $k = 13, 14, 15, 16$ over the homothetic family | ZwTfwaJJitE               |

Since $X_6$ is stationary over the Brocard porism, recalling Proposition 1:

**Corollary 7** Over the Brocard porism, the triangle whose vertices are the $X_{15}$ is perspective with $T$ at a fixed point ($X_6$).

### 6 Videos and questions

Animations illustrating some constructions herein are listed on Table 2.

**Open questions**

- What dynamic properties underlie the fact that certain sequence of hexagonal vertices are spanned by parabolas?
- Does the sequence of hexagons and flank triangles tend to regular shapes away from the foci of the parabolas?
- What are new or different properties of both the basic and grid constructions if hexagons are erected inwardly upon each side of $T = ABC$?
- Depending on the amount of self-intersection (Fig. 5) for one or more triangles in the grid, a certain condition is crossed such that the second isodynamic point wanders away from its fixed common locations. What is that condition? Would using $X_{15}$ be correct?
- Is there an $N$ other than 3, 4, 6 such that interesting properties of similar constructions can be found?
- What happens if self-intersected hexagons are erected, e.g., with vertices common with a simple regular hexagon?

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**Appendix A: Explicit formulas**

To facilitate computational reproduction of our results, we provide code-friendly expressions for a few objects. In the expressions below $a, b, c$ refer to sidelengths and $x, y, z$ are barycentric coordinates.
A.1 A-parabola

It is given implicitly by:

\[2(a^8-4a^6b^2+6a^4b^4-4a^2b^6+b^8-4a^6c^2+13a^4b^2c^2-14a^2b^4c^2+5b^6c^2+6a^4c^4-23a^2b^2c^4+14b^6c^2+6a^4c^4-4a^2b^2c^4+15b^6c^2+6c^8)x\cdot yz-3c^2(2a^6-8a^4b^2+10a^2b^4-4b^6-5a^4c^2+18a^2b^2c^2-11b^4c^2+4a^2c^4-8b^2c^4+4c^6)y^2z+2*(a^8-4a^6b^2+6a^4b^4-4a^2b^6+b^8-4a^6c^2+13a^4b^2c^2-23a^2b^4c^2+14b^6c^2+6a^4c^4-35a^2b^2c^4+36b^2c^4-7a^2c^2+6+17b^6c^6)+215a^4c^4-15a^4b^2c^2+24a^2b^4c^2+14b^6c^2+6c^8)yz-3b^2(2a^6-8a^4b^2+10a^2b^4-4b^6-5a^4c^2+18a^2b^2c^2-11b^4c^2+4a^2c^4-8b^2c^4+c^6)yz+2*sqrt(3)*S*(2a^6-3a^4b^2+3a^2b^4-b^6-5a^4c^2+9a^2b^2c^2-7a^2b^4c^2-4a^2c^4-b^2c^4+b^6c^2+6c^8)x\cdot yz=0\]

The points in the order of \(A, B, C\) and \(C, A, B\) are cyclically permuted:

A.2 The two “skip-1” parabolas

The skip-1 parabola through \(B\) is given by:

\[(\sqrt{3}(2a^6-5a^4b^2+4a^2b^4-b^6-7a^2b^2c^2+b^4c^2-a^2c^4+b^2c^4)-2*(a^2-4a^4b^2+2a^2b^4-4a^2b^2c^2-28b^2c^2+5c^4))x^2+(\sqrt{3}(3a^6-7a^4b^2+5a^2b^4-b^6+6a^4c^2+10a^2b^2c^2-2b^4c^2+3a^2c^4-2b^2c^4)-2*(13a^4-8a^2b^2-2b^4c^4+17a^2b^2c^2+4a^2c^4-12b^2c^4+4c^6)x^2+2*sqrt(3)(a^2-c^2)*x(yz+2\cdot a^2-10a^2b^2+5b^4-4a^4c^2-28b^2c^2+22c^4)*z^2=0\]

The skip-1 parabola through \(C\) is obtained with a bicentric substitution, i.e.,

\((a, b, c, x, y, z) \to (a, c, b, x, y, z)\).

A.3 Center (at infinity) of the A-parabola group

The \(A\)-parabola and \(BC\) skip-1 pair of parabolas have parallel axes through \(f_a, f_b, f_c\), therefore their axes will cross the line at infinity at the same point given by the following barycentrics:

\[x=8a^6-8a^4b^2+2a^2b^4-4a^2b^6+8a^4c^2+2a^2b^2c^2+b^4c^2+2a^2c^4+b^2c^4-2c^6+2sqrt(3)\cdot (b^2-c^2)^2\cdot S\]

\[y=-4a^6+6a^4b^2+2a^2b^4+4a^2b^6+6a^4c^2+2a^2b^2c^2-5b^4c^2+2a^2c^4+b^2c^4-4c^6-2sqrt(3)\cdot (a^2-c^2)^2\cdot S\]

\[z=-4a^6+7a^4b^2+2a^2b^4+4a^2b^6+6a^4c^2-3a^2b^2c^2+2a^2c^4+b^2c^4+2c^6+2sqrt(3)\cdot (a^2-b^2)^2\cdot (b^2-c^2)^2\cdot S\]
A.4 Directrix of the A-parabola

Is is the line given by:

\[
\begin{align*}
(a^4-8a^2b^2+12b^4-8a^2c^2+27b^2c^2+12c^4-2\sqrt{3}(2a^2-5b^2-5c^2)S)x + \\
(6a^4-23a^2b^2+22b^4-30a^2c^2+58b^2c^2+39c^4+2\sqrt{3}(a^2-3b^2-2c^2)S)y + \\
(6a^4-30a^2b^2+39b^4-23a^2c^2+58b^2c^2+22c^4+2\sqrt{3}(a^2-2b^2-3c^2)S)z &= 0
\end{align*}
\]

The other two directrices can be obtained via cyclic substitution.

A.5 Directrix equilateral

The barycentrics \(x, y, z\) of the \(A\)-vertex of the directrix equilateral (Proposition 12) are given by:

\[
\begin{align*}
x &= -(\sqrt{3})(12a^6-13a^4b^2-a^2b^4+2b^6-13a^4c^2-8a^2b^2c^2-2b^4c^2-a^2c^4-2b^2c^4+2c^6) - 6(3a^2b^2+3a^2c^2+2b^2c^2)S; \\
y &= \sqrt{3}(5a^6+2a^4b^2-10a^2b^4+3b^6-7a^4c^2-10a^2b^2c^2-a^2c^4-6b^2c^4+3c^6) - 6(a^4-5a^2b^2+6b^4-3b^2c^2-2-c^4)S; \\
z &= \sqrt{3}(5a^6-7a^4b^2-a^2b^4+3b^6+2a^4c^2-10a^2b^2c^2-2-6b^2c^4+3c^6) - 6(a^4-b^4+5a^2c^2+2-3b^2c^2+2c^4)S.
\end{align*}
\]

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