ON SUBLEVEL SET ESTIMATES AND THE LAPLACIAN

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Abstract. Carbery proved that if $u : \mathbb{R}^n \to \mathbb{R}$ is a positive, strictly convex function satisfying $\det D^2 u \geq 1$, then we have the estimate

$$|\{x \in \mathbb{R}^n : u(x) \leq s\}| \lesssim n^{\alpha / 2}$$

and this is optimal. We give a short proof that also implies other results. Our main result is an estimate for the sublevel set of functions $u : [0, 1]^2 \to \mathbb{R}$ satisfying $\Delta u \geq 1$: for any $\alpha > 0$, we have

$$|\{x \in [0, 1]^2 : |u(x)| \leq \varepsilon\}| \leq \sqrt{\varepsilon + \varepsilon^{\alpha - 1} - \frac{1}{\varepsilon} \int_{[0, 1]^2} |\nabla u|^2 |u|^\alpha dx.$$ 

For 'typical' functions, we expect the integral to be finite for $\alpha < 1$. While Carbery-Christ-Wright have shown that no sublevel set estimates independent of $u$ exist, this result shows that for 'typical' functions satisfying $\Delta u \geq 1$, we expect the sublevel set to be $\lesssim \varepsilon^{1/2}$. It is an interesting problem whether and to which extent similar inequalities are possible in higher dimensions.

1. INTRODUCTION

1.1. Introduction. Sublevel set estimates encapsulate the notion that 'if a real-valued function $u$ has a large derivative, then it cannot spend too much time near any fixed value' [4]. If $u : \mathbb{R} \to \mathbb{R}$ satisfies $u^{(k)} \geq 1$ for some integer $k \geq 2$, then it cannot be close to any constant for a long time (since, ultimately, it has to 'curve upward'). This is formalized in the van der Corput Lemma

$$|\{x \in \mathbb{R} : |u(x)| \leq t\}| \lesssim k \frac{t^2}{2},$$

where the implicit constant is independent of $u$: the extremal case behaves, up to constants, essentially like the monomial $u(x) = x^k/k!$ (see [1, 5, 21] for explicit constants). These questions are classical [1, 21, 24] and well understood in one dimension. The problem becomes a lot harder in higher dimensions [5, 6, 9, 18]. The seminal paper of Carbery, Christ & Wright [5] shows that if $u : [0, 1]^n \to \mathbb{R}$ satisfies $D^\beta u \geq 1$ for some multi-index $\beta$, then there is a constant $\varepsilon > 0$, depending only $n$ and $\beta$, such that

$$|\{x \in [0, 1]^n : |u(x)| \leq t\}| \lesssim \varepsilon^{1/2} t.$$ 

One could perhaps assume that $\varepsilon = 1/|\beta|$ but this is far from known, the problem seems very difficult and intimately connected to problems in combinatorics, see [5]. The problem is even open in $n = 2$ dimensions for the differential inequality

$$\frac{\partial^2 u}{\partial x \partial y} \geq 1$$

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for which it is known \[5\] that
\[\left| \{ x \in [0, 1]^2 : |u(x)| \leq t \} \right| \lesssim \sqrt{t} \sqrt{\log(1/t)}\]
but where it is not known whether the logarithm is necessary. The related combinatorial problems have been studied in their own right \[10, 11\].

1.2. Carbery’s Sublevel Set Estimate. Carbery asked whether it is possible to replace the condition \(D^\beta u \geq 1\) by a nonlinear condition. Motivated by the corresponding theory for oscillatory integral operators, the condition \(\det D^2 u \geq 1\) seems like a natural first step, however, the example \(u(x, y) = xy\) shows that some further conditions are required.

**Theorem 1** (Carbery \[4\]). Let \(K\) be a convex domain, let \(u : K \to \mathbb{R}\) be strictly convex and satisfy \(u \geq 0\) as well as
\[\det D^2 u \geq 1,\]
Then, for any \(s > 0\),
\[\left| \{ x \in K : u(x) \leq s \} \right| \lesssim_n s^{n/2},\]
where the implicit constant depends only on the dimension.

We refer to the original paper \[4\] for other related statements of a similar type. This is the optimal scaling: consider the function \(u : \mathbb{R}^n \to \mathbb{R}\)
\[u(x) = a_1 x_1^2 + \cdots + a_n x_n^2\]
for some positive real numbers \(a_1, \ldots, a_n > 0\) that satisfy \(\det D^2 u = 2^na_1 \cdots a_n = 1\). The function is strictly convex, the sublevel sets are ellipsoids and
\[\left| \{ x \in K : u(x) \leq s \} \right| \sim_n s^{n/2} .\]
The quantity \(\det D^2 u\) is also invariant under affine transformations, something that is required for these types of statements to hold. Carbery’s proof, albeit short, is fairly nontrivial. One of our contributions is a simpler proof.

1.3. The Laplacian. One could also wonder whether similar results are possible for other differential operators. The Laplacian \(\Delta\) is a natural starting point. This case has been analyzed by Carbery, Christ & Wright \[5\] who proved that such statements must necessarily fail. More precisely, they show

**Proposition 1** (Proposition 5.2., Carbery-Christ-Wright \[5\]). For \(0 < \delta < 1/2\), there exists \(u \in C^\infty((0, 1)^2)\) such that \(\Delta u \equiv 1\) on \((0, 1)^2\) and
\[\left| \{ x \in [0, 1]^2 : |u(x)| \geq \delta \} \right| \leq \delta.\]
This shows a striking failure of the condition \(\Delta u \geq 1\) to prevent the function from being close to a constant on a set of large measure. The construction uses the Mergelyan theorem and is thus intimately connected to two dimensions. The main contribution of our paper is to show that such constructions are ‘rare’, in a certain sense, since \(|\nabla u|\) has to rather large in regions where \(|u|\) is small. We also prove the existence of a constant \(c_n\) such that \(\left| \{ x \in [0, 1]^n : |u(x)| \geq c_n \} \right| \parallel u\parallel_{L^\infty} \geq c_n.\)
2. Main Results

2.1. Revisiting Carbery’s estimate. A natural question is whether it is possible to weaken the assumptions in Carbery’s estimate (this was also discussed by Carbery-Maz’ya-Mitrea-Rule [7] by very different means). We note that, with the inequality of arithmetic and geometric mean, we obtain

\[
1 \leq \det D^2 u = \prod_{i=1}^{n} \lambda_i(D^2 u) \leq \frac{1}{n^n} \left( \sum_{i=1}^{n} \lambda_i(D^2 u) \right)^n = \frac{(\Delta u)^n}{n^n},
\]

where we used strict convexity (positivity of the eigenvalues of the Hessian) to invoke the inequality. One could thus wonder whether the weaker condition \(\Delta u \geq 1\) by itself is sufficient and it is fairly easy to see that this is not the case: consider \(u(x) = x_1^2 + \varepsilon x_2^2\), then the set \(\{x : u(x) \leq 1\}\) can be arbitrarily large if we make \(\varepsilon\) sufficiently small. We also see that the shape of this sublevel set is rather eccentric and, as it turns out, this is necessary. For spherical sublevel sets, we can indeed establish the desired result under weaker conditions.

**Proposition 2.** Suppose \(u : \{x \in \mathbb{R}^n : \|x\| \leq r\} \to \mathbb{R}\) satisfies \(\Delta u \geq 1\), then

\[
\max_{\|x\| \leq r} u - \min_{\|x\| \leq r} u \geq \frac{r^2}{2n}.
\]

This result is a consequence of the maximum principle (and, as such, has presumably been used many times in the literature). It implies a short proof of Theorem 1: consider the domain \(\Omega = \{x \in K : |u(x)| \leq s\}\). Since \(\Omega\) is convex, by John’s ellipsoid theorem it contains an ellipsoid \(E \subset \Omega\) such that \(|E| \sim_n |\Omega|\). Let us apply the diagonal volume-preserving affine transformation that maps \(E\) to a ball. Since the condition \(\det D^2 u \geq 1\) is affinely invariant, it is preserved. Then the arithmetic-geometric inequality implies \(\Delta u \gtrsim_n 1\) on the ball and we can apply Proposition 2: since \(u \geq 0\), we obtain

\[
\|u\|_{L^\infty(K)} \gtrsim |\Omega|^{\frac{2}{n}}
\]

which is the desired result. We will give two proofs of Proposition 2: one simple and using only the maximum principle and one that is slightly more complicated that will set the stage for the arguments in §2.3.

2.2. Another sublevel set estimate. We return to the problem of understanding functions \(u : [0, 1]^2 \to \mathbb{R}\) that satisfy \(\Delta u \geq 1\). As was shown by Carbery-Christ-Wright (see §1.3.), no classical sublevel set estimates (i.e. depending only on \(\varepsilon\) but not on the function \(u\)) are possible. One could then wonder about sublevel set estimates that somehow depend on \(u\). The Markov inequality can be written as

\[
|\{x \in [0, 1]^2 : |u(x)| \leq \varepsilon\}| \leq \varepsilon^n \int_{[0,1]^2} \frac{1}{|u(x)|^n} dx.
\]

Needless to say, this estimate is not exactly of great interest. It is obviously true for all functions and not just those that satisfy \(\Delta u \geq 1\). We discovered a second inequality, one that requires the assumption \(\Delta u \geq 1\), which is of a very different flavor and which we consider to be the main contribution of this paper.
Theorem 2. Assume \( u : [0,1]^2 \to \mathbb{R} \) satisfies \( \Delta u \geq 1 \). Then, for all \( \alpha > 0 \),
\[
|\{ x \in [0,1]^2 : |u(x)| \leq \varepsilon \}| \lesssim \sqrt{\varepsilon} + (2\varepsilon)^{\alpha - \frac{1}{2}} \int_{[0,1]^2} \frac{|\nabla u|}{|u|^\alpha} dx,
\]
where the implicit constant is universal.

We can prove a slightly sharper result: our proof does not apply in a \( \sim \sqrt{\varepsilon} \) neighborhood of the boundary. If we set \( Q = [100\sqrt{\varepsilon}, 1 - 100\sqrt{\varepsilon}] \), then, for all \( \alpha > 0 \),
\[
|\{ x \in Q : |u(x)| \leq \varepsilon \}| \lesssim (2\varepsilon)^{\alpha - \frac{1}{2}} \int_{[0,1]^2} \frac{|\nabla u|}{|u|^\alpha} dx
\]
and the implicit constant does not depend on \( \alpha \) or \( u \). It is interesting that this curious estimate gives the sharp (up to the endpoint in \( \alpha \)) results for several different types of functions. We consider the simple example \( u(x_1, x_2) = x_1^2 + x_2^2 \) where
\[
|\{ x \in [0,1]^2 : |u(x)| \leq \varepsilon \}| \lesssim \varepsilon.
\]

We have \( |\nabla u| \lesssim \max(|x_1|, |x_2|) \) and obtain
\[
\int_{[0,1]^2} \frac{|\nabla u|}{|u|^\alpha} dx \lesssim \int_{[0,1]^2} \frac{\max(|x_1|, |x_2|)}{\max(|x_1|^{2\alpha}, |x_2|^{2\alpha})} dx \lesssim \int_{[0,1]^2} \frac{1}{|x_1|^{2\alpha - 1}} dx_1
\]
which is finite up to \( \alpha < 3/2 \). We also emphasize that Theorem 2, when considering analytic functions, seems to connect to a type of inverse Lojasiewicz inequality \[\text{[3 8 12 14 15 16]}\]. We recall the inequality: if \( f : \mathbb{R}^n \to \mathbb{R} \) is analytic in a neighborhood of the origin and \( f(0) = 0 \) and \( \nabla f(0) = 0 \), then there is an open neighborhood around the origin as well as two constants \( c > 0 \) and \( \rho < 1 \) such that
\[
|\nabla f(x)| \geq c |f(x)|^\rho.
\]

Explicit estimates on \( \rho \) are available when \( f \) is a polynomial (in terms of \( n \) and the degree, see \[\text{[8]}\] and references therein). This interesting connection suggests the possibility of applications of Theorem 2 to polynomials or analytic functions.

Our proof is strictly two-dimensional (exploiting a geometric argument that fails in higher dimensions). It is not clear to us whether and to which extent similar results could hold in higher dimensions, i.e. for \( u : [0,1]^n \to \mathbb{R} \) satisfying \( \Delta u \geq 1 \). We believe that this could be quite interesting.

2.3. A flatness estimate. Let us again assume \( \Delta u \geq 1 \) on \([0,1]^n\). As discussed in §1.3. above, it is possible that
\[
|\{ x \in [0,1]^n : |u(x)| \geq \varepsilon \}| \leq \varepsilon
\]
for arbitrarily small values of \( \varepsilon \). How does such a function look like? Applying Proposition 2 immediately shows that the set \( \{ x \in [0,1]^n : |u(x)| \leq \varepsilon \} \) cannot contain a ball of radius \( 2\sqrt{n}\varepsilon \). At the same time, the condition \( \Delta u \geq 1 \) implies that there are no local maxima inside, therefore every connected component of
\[
\{ x \in [0,1]^n : u(x) \geq \varepsilon \}
\]
must necessarily touch the boundary.

Figure 1 shows how such a function could possibly look like.
The Carbery-Christ-Wright construction shows that the set \( \{ x : u(x) \geq \varepsilon \} \) can indeed be arbitrarily small: we were interested in whether this required the function to be large in some places and this motivated our result: if solutions \( \Delta u \geq 1 \) are flat on a very large subset, then \( u \) must be very large on the complement.

**Theorem 3.** There exists a constant \( c_n > 0 \) depending only on the dimension such that if \( u : [0,1]^n \to \mathbb{R} \) satisfies \( \Delta u \geq 1 \), then
\[
\left| \{ x \in [0,1]^n : |u(x)| \geq c_n \} \right| \cdot \| u \|_{L^\infty([0,1]^n)} \geq c_n.
\]

One way of interpreting the Theorem is to say that solutions of \( \Delta u \geq 1 \) may indeed be very flat on a set of large measure but that requires the function to be rather large in other places. It is not at all clear how the extremal examples behave, whether Theorem 3 has the optimal scaling. Is there an estimate like
\[
\left| \{ x \in [0,1]^n : |u(x)| \geq c_n \} \right| \cdot \| u \|_{L^p([0,1]^n)} \geq c_n
\]
for some \( \alpha < 1 \) or \( p < \infty \) or both?

3. Proofs

3.1. First Proof of Proposition 2.

**Proof.** The estimate is invariant under addition of constants. We can thus, by adding \( -\min_{\|x\| \leq r} u \), assume that \( u \geq 0 \) and it suffices to show that
\[
\max_{\|x\| \leq r} u \geq \frac{r^2}{2n}.
\]
We define the function \( w \) as the solution of
\[
\Delta w = 1 \quad \text{for } \|x\| < r
\]
\[
w = \max_{\|x\|=r} u \quad \text{for } \|x\| = r.
\]
We see that \( w - u \) is positive on the boundary and that
\[
\Delta (w - u) = \Delta w - \Delta u \leq 0.
\]
This implies that the minimum of $w - u$ is assumed on the boundary, where the function is nonnegative. Therefore $w \geq u \geq 0$. However, we can actually compute $w$ in closed form: the solution is radial and the radial Laplacian can be written as

$$\Delta s = \frac{1}{s^{n-1}} \frac{\partial}{\partial s} \left( s^{n-1} \frac{\partial f}{\partial s} \right)$$

which shows that the solution is given by

$$w(s) = \frac{s^2}{2n} + c,$$

where $c$ is a constant chosen such that the boundary conditions are satisfied. However, $w \geq 0$ and thus $c \geq 0$. This implies that

$$\max_{\|x\|=r} u = w(r) \geq \frac{r^2}{2n}.$$

It is clear that the estimate is sharp since

$$\Delta \left( \frac{\|x\|^2}{2n} \right) = \Delta \left( \frac{x_1^2 + \cdots + x_n^2}{2n} \right) = 1.$$

### 3.2. Second Proof of Proposition 2.

**Proof.** Our second proof of Proposition 2 is based on representing the function $u$ as the stationary solution of the heat equation. By itself, this argument is more difficult than the one based on the maximum principle but it introduces a line of thought that will be useful for a later proof (indeed, this type of argument has proven useful for several different problems \cite{2, 13, 19, 22, 23}). We study

$$v_t + \Delta v = \Delta u \quad \text{in } \Omega$$
$$v(0, x) = u(x) \quad \text{in } \Omega$$
$$v(t, x) = u(x) \quad \text{on } \partial \Omega.$$ The Feynman-Kac formula then implies a representation of the function $u(x) = v(t, x)$ as a weighted average of its values in a neighborhood to which standard estimates can be applied. We denote a Brownian motion started in $x \in \Omega$ at time $t$ by $\omega_x(t)$. The Dirichlet boundary conditions require us to demand that the boundary is ‘sticky’ and that a particle remains at the boundary once it touches it. The Feynman-Kac formula implies that for all $t > 0$

$$u(x) = \mathbb{E}u(\omega_x(t)) + \mathbb{E} \int_0^{t \wedge \tau} (\Delta u)(\omega_x(t))dt,$$

where $\tau$ is the stopping time for impact on the boundary. A simple way to derive the scaling for Proposition 2 is now as follows: suppose we are on a ball of radius $r$. We have, for all $\|x\| < r$ and all $t > 0$,

$$|u(x) - \mathbb{E}u(\omega_x(t))| \leq \max_{\|x\| \leq r} u - \min_{\|x\| \leq r} u.$$

The expected lifetime of Brownian motion in a ball of radius $r$ when started near the center until hitting the boundary is $\sim r^2$. Since $\Delta u \geq 1$ this implies

$$\mathbb{E} \int_0^{t \wedge \tau} (\Delta u)(\omega_x(t))dt \gtrsim r^2.$$
and this establishes the desired result. □

One obvious advantage of this kind of approach is that \( \Delta u \geq 1 \) is clearly not strictly required as long as it is true ‘in the aggregate’. Indeed, if we have \( \Delta u \geq \phi(x) \), then approaches of this flavor could be used to deduce analogous bounds as long as ‘\( \phi \geq 1 \)’ in a suitable averaged sense.

3.3. Proof of Theorem 2. We start with a quick variation on Proposition 2 that will prove useful in the proof of Theorem 2 where we require a pointwise statement.

Proposition 3. Suppose \( u : \{ x \in \mathbb{R}^n : \| x \| \leq r \} \to \mathbb{R} \) satisfies \( \Delta u \geq 1 \), then, for all \( \| y \| < r \)

\[
\max_{\| x \| = r} u(x) \geq \frac{r^2 - \| y \|^2}{2n} + u(y).
\]

Proof. The proof is almost completely analogous. We define the function \( w \) as the solution of

\[
\Delta w = 1 \quad \text{for} \quad \| x \| < r \\
w = \max u \quad \text{for} \quad \| x \| = r
\]

and observe that, as before, \( w \geq u \). Moreover, we know that

\[ w(r) = \frac{r^2}{2n} + c \]

for some constant \( c \geq 0 \) chosen so that the boundary conditions are satisfied. The argument is finished by observing that

\[
\frac{r^2}{2n} - \frac{\| y \|^2}{2n} = \left( \frac{r^2}{2n} + c \right) - \left( \frac{\| y \|^2}{2n} + c \right) = \max_{\| x \| = r} (u(x) - w(y)) \leq \max_{\| x \| = r} (u(x) - u(y)).
\]

□

We will use this statement to conclude, for all \( y \) and all \( r > 0 \),

\[
\max_{\| x - y \| = r} u(x) - u(y) \gtrsim_n r^2.
\]

The proof of Theorem 2 uses the coarea formula which we recall for the convenience of the reader. If \( \Omega \subset \mathbb{R}^n \) is an open set, \( u : \Omega \to \mathbb{R} \) is Lipschitz and \( g \in L^1(\Omega) \), then

\[
\int_{\Omega} g(x) |\nabla u(x)| dx = \int_{\mathbb{R}} \left( \int_{u^{-1}(t)} g(x) d\mathcal{H}^{n-1} \right) dt.
\]

We will use this identity for the function

\[ g(x) = \chi_{\| u(x) \| \leq 2 \varepsilon} \frac{\varepsilon^{n/2}}{|u|^{n/2}} \]

which is bounded and thus in \( L^1 \).
Proof of Theorem 2. Let us fix $\varepsilon > 0$. We use the coarea formula to estimate
\[
\int_{[0,1]^2} \frac{\nabla u}{|u|^\alpha} dx \geq \int_{[0,1]^2} \frac{\nabla u}{|u|^\alpha} \chi_{\varepsilon \leq u(x) \leq 2\varepsilon} dx
\]
and thus
\[
\frac{1}{\varepsilon} \int_{\varepsilon < t < 2\varepsilon} \frac{\mathcal{H}^1 (\{x : u(x) = t\})}{|t|^\alpha} dt \leq \varepsilon^{-1} \int_{[0,1]^2} \frac{\nabla u}{|u|^\alpha} dx
\]
and therefore there exists $\varepsilon \leq t \leq 2\varepsilon$ such that
\[
\mathcal{H}^1 (\{x : u(x) = t\}) \leq (2\varepsilon)^{\alpha-1} \int_{[0,1]^2} \frac{\nabla u}{|u|^\alpha} dx.
\]
We fix this value of $t$ and observe two basic facts: if $|u(x)| \leq t$, then there a point $|y - x| \leq 10\sqrt{t}$ where $u(y) \geq t$ (this follows from Proposition 3). We will also use a basic Bonnensen-style isoperimetric inequality (this one in particular is [17, Eq. 14]): for any simply connected domain $D \subset \mathbb{R}^2$, we have
\[
|\partial D| \cdot \text{inrad}(D) \geq |D| + \pi \cdot \text{inrad}(D)^2 \geq |D|.
\]
We observe that every connected component of $\{x \in [0,1]^2 : u(x) \leq t\}$ is simply connected because of the maximum principle. Consider the connected components of $\{x \in [0,1]^2 : u(x) \leq t\}$ (all of which are simply connected). Some of them may have a boundary that is strictly contained in $[0,1]^2$. Let us denote their union by $\Omega_1$. Since their inradius is less than $10\sqrt{t}$, we have with the Bonnensen-style inequality that
\[
|\Omega_1| \lesssim t^{1/2} \cdot \mathcal{H}^1 (\{x : u(x) = t\}) \leq (2\varepsilon)^{\alpha-1/2} \int_{[0,1]^2} \frac{\nabla u}{|u|^\alpha} dx
\]
which would be the desired result.

Figure 2. A sketch of the second part of the argument: if $|u(x)| \leq t$, then a $10\sqrt{t}$ ball contains an element from the level set and since that level set has to either move to the boundary, a $20\sqrt{t}$ ball contains level set of length at least $\gtrsim \sqrt{t}$.

However, it is certainly conceivable that several connected components of the set $\{x \in [0,1]^2 : u(x) \leq t\}$ do not close up and instead touch the boundary (in which
case the Bonnensen-style inequality would not be valid since, geometrically interpreted, it would also count part of the boundary which is not counted in our level set estimate). We now deal with this remaining case. We now decompose \([0, 1]^2\) into \(\sim t^{-1}\) boxes of size \(\sqrt{t} \times \sqrt{t}\) and will ignore the boxes that are distance \(\leq 10\sqrt{t}\) from the boundary of the unit square (this accounts for \(\sqrt{\varepsilon}\) error term). We will prove an upper bound on the number of boxes \(B\) which contain a point \(x\) for which \(|u(x)| \leq t\) and have the property that the boundary of the connected component touches the boundary. We proceed as follows: for any such box \(B\), we consider the \(10\sqrt{t}\) neighborhood of the box \(B\). There exists a point \(y\) in that neighborhood for which \(u(y) \geq t\). As a consequence, the same neighborhood also contains an element of the level set \(\{x: u(x) = t\}\). Since that level set has to connect to the boundary, we see that the \(20\sqrt{t}\) neighborhood has to contain a level set of length at least \(20\sqrt{t}\). Simple double-counting shows that every line segment of the level set is at most associated to \(\sim t^{-1/2}\) boxes and thus we can bound the number of boxes by the length of the level set. Altogether, we see

\[
t^{-1/2} \cdot \# \{B: \exists x \in B: |u(x)| \leq t\} \lesssim \mathcal{H}^1(\{x: u(x) = t\}).
\]

Each box has area \(t\) and thus, recalling the \(\sim t^{-1/2}\) boxes close to the boundary that we did not consider,

\[
\left| \{x \in [0, 1]^2: |u(x)| \leq t\} \right| \lesssim t \cdot \left( t^{-1/2} + \# \{B: \exists x \in B: |u(x)| \leq t\} \right) \lesssim \sqrt{t} + t^{1/2} \mathcal{H}^1(\{x: u(x) = t\}).
\]

Since \(\varepsilon \leq t \leq 2\varepsilon\), we get

\[
\left| \{x \in [0, 1]^2: |u(x)| \leq \varepsilon\} \right| \lesssim \sqrt{\varepsilon} + (2\varepsilon)^{\alpha - \frac{1}{2}} \int_{[0, 1]^2} \frac{\nabla u}{u|\alpha|} dx
\]

which is the desired result.

\[\square\]

Remarks.

1. The reason why this proof is restricted to two dimensions is the way we count boxes: both the coarea formula and Proposition 2 can be used in any dimension. However, in higher dimensions, the mere existence of a piece of level set in a box does not guarantee that the \(\mathcal{H}^{n-1}\) measure of that level set is large even if the level sets are known to connect to the boundary.

2. There is a natural analogue to Proposition 2 for more general elliptic operators; the proof is stable and can be modified to account for functions where

\[-\text{div}(a(x)\nabla u) \geq 1.\]

The arising results will then depend on the ellipticity constant of \(a(x)\).

3.4. Proof of Theorem 3.

Proof. We want to prove the existence of a constant \(c_n\), depending only on \(n\), such that for all \(u: [0, 1]^n \to \mathbb{R}\) satisfying \(\Delta u \geq 1\), we have

\[
\left| \{x \in [0, 1]^n: |u(x)| \geq c_n\} \right| \cdot \|u\|_{L^\infty} \geq c_n.
\]

We will prove it for the unit ball instead of the unit cube

\[
\left| \{|x| \leq 1: |u(x)| \geq c_n\} \right| \cdot \|u\|_{L^\infty((x: |x| \leq 1))} \geq c_n
\]
which implies the original result since the unit cube contains a ball of radius $1/2$ and we only work up to constants depending only on the dimension. Now suppose the desired statement is false. Then, for any $\varepsilon > 0$, there exists a function satisfying $\Delta u \geq 1$ for which

$$|\{\|x\| \leq 1 : |u(x)| \geq \varepsilon\}| \cdot \|u\|_{L^\infty} \leq \varepsilon.$$  

We will now argue for a fixed (but unspecified) value of $\varepsilon$ and will then see that $\varepsilon$ cannot be chosen arbitrarily small. Fubini’s theorem combined with the pigeonhole principle implies the existence of a $0.99 < t < 1$ for which

$$\mathcal{H}^{n-1}\left(\{\|x\| = t : |u(x)| \geq \varepsilon\}\right) \cdot \|u\|_{L^\infty(\|x\| \leq 1)} \lesssim n \varepsilon.$$  

We fix this value of $t$.

![Figure 3](image_url)

**Figure 3.** If a set $\{x : u(x) \geq c_n\}$ inside a ball is small, then there also exists a slight shrinking of the ball, such that the $(n-1)$–dimensional size of the set $\cap \{x : \|x\| = t\}$ is small.

The next step is to argue as in the (second) proof of Proposition 2. We rewrite the function $u$ as the stationary solution of a heat equation and obtain the equation

$$u(x) = \mathbb{E}u(\omega_x(t)) + \mathbb{E} \int_0^{t \land \tau} (\Delta u)(\omega_x(t)) dt.$$  

We now let $t \to \infty$. In that regime, all the Brownian motion particles are impacted on the boundary and we can reinterpret $\mathbb{E}u(\omega_x(t))$ as an integral over the boundary with respect to harmonic measure. We integrate this identity in the ball $\{\|x\| \leq 1/100\}$. The symmetry of the ball (and the inherited symmetry of the harmonic measure) implies, for some positive constants $c_{n,1}, c_{n,2} > 0$ that only depend on the dimension (and, very mildly and in a way that can be controlled, on $t$)

$$\int_{\|x\| \leq 1/100} u(x) dx = c_{n,1} \int_{\|x\| = t} u(x) d\mathcal{H}^{n-1} + c_{n,2} \int_{\|x\| \leq 1/100} \mathbb{E} \int_0^\tau (\Delta u)(\omega_x(t)) dt dx.$$  

The argument can now be concluded as follows: the first integral is certainly small since

$$\int_{\|x\| \leq 1/100} u(x) dx \leq \varepsilon + |\{|u(x)| \geq \varepsilon\}| \cdot \|u\|_{L^\infty} \lesssim \varepsilon.$$  

The second integral is also small. Ignoring the constant in front, which only depends on the dimension, we can estimate
\[
\int_{\|x\|=t} u(x) d\mathcal{H}^{n-1} \lesssim_n \epsilon + \mathcal{H}^{n-1}(\{\|x\| = t : |u(x)| \geq \epsilon\}) \cdot \|u\|_{L^\infty(\|x\| \leq 1)} \lesssim_n \epsilon.
\]
However, the third time is an integral over the expected exit time: starting in the center of the ball, that expected exit time is \(\gtrsim_n 1\) and thus
\[
\int_{\|x\| \leq 1/100} E \int_0^\tau (\Delta u)(\omega_x(t)) dt \, dx \gtrsim \int_{\|x\| \leq 1/100} E \int_0^\tau 1 \, dt \, dx \gtrsim \int_{\|x\| \leq 1/100} E r \, dx \gtrsim_n 1.
\]
This leads to a contradiction for \(\epsilon\) sufficiently small. \qed

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