A SHARP COMPARISON THEOREM FOR COMPACT MANIFOLDS WITH MEAN CONVEX BOUNDARY

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Abstract. Let $M$ be a compact $n$-dimensional Riemannian manifold with nonnegative Ricci curvature and mean convex boundary $\partial M$. Assume that the mean curvature $H$ of the boundary $\partial M$ satisfies $H \geq (n-1)k > 0$ for some positive constant $k$. In this paper, we prove that the distance function $d$ to the boundary $\partial M$ is bounded from above by $\frac{1}{k}$ and the upper bound is achieved if and only if $M$ is isometric to an $n$-dimensional Euclidean ball of radius $\frac{1}{k}$.

1. Introduction

By a classical theorem of Bonnet and Myers, if a complete $n$-dimensional Riemannian manifold $M$ has Ricci curvature at least $(n-1)k$, where $k > 0$ is a constant, then the diameter of $M$ is at most $\frac{\pi}{\sqrt{k}}$. Applying this result to the universal cover $\tilde{M}$, we see that such manifolds must be compact and have finite fundamental group. In [2], Cheng proved the rigidity theorem that if the diameter is equal to $\frac{\pi}{\sqrt{k}}$, then $M$ is isometric to the $n$-sphere with constant sectional curvature $k$.

In this paper, we prove a similar result for compact manifolds with nonnegative Ricci curvature and mean convex boundary. Our main result is the following

Theorem 1.1. Let $M^n$ be a complete $n$-dimensional ($n \geq 2$) Riemannian manifold with nonnegative Ricci curvature and mean convex boundary $\partial M$. Assume the mean curvature $H$ of $\partial M$ with respect to the inner unit normal satisfies $H \geq (n-1)k > 0$ for some constant $k > 0$. Let $d$ denote the distance function on $M$. Then,

$$\sup_{x \in M} d(x, \partial M) \leq \frac{1}{k}. \quad (1.1)$$

Furthermore, if we assume that $\partial M$ is compact, then $M$ is also compact and equality holds in (1.1) if and only if $M^n$ is isometric to an $n$-dimensional Euclidean ball of radius $\frac{1}{k}$.

Remark 1.2. For any isometric embedding of a Riemannian $m$-manifold $N$ into a metric space $X$, Gromov [5] defined the filling radius, Fill Rad ($\partial M \subset M$), to be the infimum of those numbers $\epsilon > 0$ for which $N$ bounds in the $\epsilon$-neighborhood $U_\epsilon(N) \subset X$, that is the inclusion homomorphism of the $m$-th homology (over $\mathbb{Z}$ or $\mathbb{Z}_2$) $H_m(N) \to H_m(U_\epsilon(N))$ vanishes. Therefore, we can restate the conclusion of Theorem 1.1 as Fill Rad ($\partial M \subset M$) $\leq \frac{1}{k}$ and equality holds if and only if $M$ is the Euclidean ball of radius $\frac{1}{k}$.

Note that under the curvature assumptions in Theorem 1.1, the complete manifold $M$ may be non-compact. However, if we put a stronger convexity assumption on $\partial M$, then the boundary convexity could force $\partial M$ to be compact and hence
M would also be compact. In [6], Hamilton proved that any convex hypersurface in \( \mathbb{R}^n \) with pinched second fundamental form is compact. We conjecture that the result can be generalized to manifolds with nonnegative Ricci curvature.

**Conjecture 1.3.** Let \( M^n \) be a complete Riemannian \( n \)-manifold with nonempty boundary \( \partial M \). Assume \( M \) has nonnegative Ricci curvature and \( \partial M \) is uniformly convex with respect to the inner unit normal, i.e. the second fundamental form \( h \) of \( \partial M \) satisfies \( h \geq k > 0 \) for some constant \( k \). Then, \( M \) is compact and \( \pi_1(M) \) is finite.

Manifolds satisfying the assumptions in Conjecture 1.3 have been studied by several authors. Some rigidity results were obtained in [9] and [10]. In [4], J. Escobar gave upper and lower estimates for the first nonzero Steklov eigenvalue for these manifolds with boundary. However, all these results are proved under the assumption that \( M \) is compact. Conjecture 1.3 above would imply that this assumption is void and these manifolds have finite fundamental group.

2. Preliminaries

In this section, we collect some known facts which will be used in the proof of Theorem 1.1. Let \( M \) be a compact, connected \( n \)-dimensional Riemannian manifold with nonempty boundary \( \partial M \). We denote by \( \langle , \rangle \) the metric on \( M \) as well as that induced on \( \partial M \). Suppose \( \gamma : [0, \ell] \to M \) be a geodesic in \( M \) parametrized by arc length such that \( \gamma(0) \) and \( \gamma(\ell) \) lie on \( \partial M \) and \( \gamma(s) \) lies in the interior of \( M \) for all \( s \in (0, \ell) \). Assume that \( \gamma \) meets \( \partial M \) orthogonally, that is, \( \gamma'(0) \perp T_{\gamma(0)}\partial M \) and \( \gamma'(\ell) \perp T_{\gamma(\ell)}\partial M \). Hence, \( \gamma \) is a critical point of the length functional as a free boundary problem. We call such \( \gamma \) a free boundary geodesic. For any normal vector field \( V \) along \( \gamma \), the orthogonality condition implies that \( V \) is tangent to \( \partial M \) at \( \gamma(0) \) and \( \gamma(\ell) \), hence is an admissible variation to the free boundary problem. A direct calculation give the second variation formula of arc length

\[
\delta^s \gamma(V, V) = \int_0^\ell \left( |V'(s)|^2 - |V(s)|^2 K(\gamma'(s), V(s)) \right) ds
\]

\[
+ \langle \nabla_{V'(0)} V(0), \gamma'(0) \rangle - \langle \nabla_{V(0)} V(0), \gamma'(0) \rangle,
\]

where \( \nabla \) is the Riemannian connection on \( M \), and \( K(u, v) \) is the sectional curvature of the plane spanned by \( u \) and \( v \) in \( M \).

Let \( N \) be the inner unit normal of \( \partial M \) with respect to \( M \). The second fundamental form \( h \) of \( \partial M \) with respect to \( N \) is defined by \( h(u, v) = \langle \nabla_u v, N \rangle \) for \( u, v \) tangent to \( \partial M \). The mean curvature \( H \) of \( \partial M \) with respect to \( N \) is defined as the trace of \( h \), that is \( H = \sum_{i=1}^{n-1} h(e_i, e_i) \) for any orthonormal basis \( e_1, \ldots, e_{n-1} \) of the tangent bundle \( T\partial M \). The principal curvatures of \( \partial M \) are defined to be the eigenvalues of \( h \). Using a Frankel-type argument as in [7], we have the following Lemma.

**Lemma 2.1.** Let \( M \) be a compact, connected \( n \)-dimensional Riemannian manifold with nonempty boundary \( \partial M \). Suppose \( M \) has nonnegative Ricci curvature and the mean curvature \( H \) of \( \partial M \) with respect to the inner unit normal satisfies \( H \geq (n - 1)k > 0 \) for some positive constant \( k \). Then, \( \partial M \) is connected and the map

\[
\pi_1(\partial M) \xrightarrow{i} \pi_1(M)
\]

induced by inclusion is surjective, i.e. \( \pi_1(M, \partial M) = 0 \).
Proof. We follow the argument given in [7]. We show under the curvature assumptions, any free boundary geodesic must be unstable as a free boundary solution. To see this, let \( \gamma : [0, \ell] \to M \) be a free boundary geodesic. Fix an orthonormal basis \( e_1, \ldots, e_{n-1} \) of \( T_{\gamma(0)} \partial M \), let \( V_i(s) \) be the normal vector field along \( \gamma \) obtained from \( e_i \) by parallel translation, using the second variation formula (2.1), we have
\[
\sum_{i=1}^{n-1} \delta^2 \gamma(V_i, V_i) = - \int_0^\ell \text{Ric}(\gamma'(s), \gamma'(s)) \, ds - H_{\gamma}(s) - H_{\gamma(0)} < 0
\]
where Ric is the Ricci curvature of \( M \). Therefore, \( \delta^2 \gamma(V_i, V_i) < 0 \) for some \( i \) and therefore \( \gamma \) is unstable.

Suppose \( \partial M \) is not connected or \( \pi_1(M, \partial M) \neq 0 \). In either case, there exists a free boundary geodesic \( \gamma \) which minimize length in his homotopy class in \( \pi_1(M, \partial M) \), hence stable. This contradicts the fact that there is no stable free boundary geodesics in \( M \).

We will use the following Lemma which is a special case of Theorem 1 in [8].

**Lemma 2.2.** Let \( M \) be a compact \( n \)-dimensional Riemannian manifold with nonempty boundary \( \partial M \) and nonnegative Ricci curvature. If the mean curvature \( H \) of \( \partial M \) with respect to the unit inner normal satisfies
\[
H \geq \frac{n-1}{n} \frac{\partial M}{|M|},
\]
where \( |\partial M| \) and \( |M| \) denote the \( (n-1) \)- and \( n \)-dimensional volume of \( \partial M \) and \( M \) respectively, then \( M^n \) is isometric to a Euclidean ball.

3. **Proof of Theorem 1.1**

In this section, we give the proof of Theorem 1.1. We first prove the upper bound in (1.1). Fix any point \( x \) in the interior of \( M \), there exists a geodesic \( \gamma : [0, \ell] \to M \) parametrized by arc length such that \( \ell = d(x, \partial M) \) (the existence of such geodesic follows from the completeness of \( M \)). Note that \( \gamma \) lies in the interior of \( M \) except at \( \gamma(\ell) \). We want to prove that \( \ell \leq \frac{1}{n} \). The first variation formula tells us that \( \gamma'(\ell) \) is orthogonal to \( \partial M \) at \( \gamma(\ell) \). Moreover, the second variation of \( \gamma \) for any normal vector field \( V \) along \( \gamma \) where \( V(0) = 0 \) is nonnegative:
\[
\delta^2 \gamma(V, V) = \int_0^\ell \left( |V'(s)|^2 - |V(s)|^2 K(\gamma'(s), V(s)) \right) \, ds + \langle \nabla_{V(\ell)} V(\ell), \gamma'(\ell) \rangle \geq 0.
\]

Fix an orthonormal basis \( e_1, \ldots, e_{n-1} \) for \( T_{\gamma(\ell)} \partial M \), let \( E_i(s) \) be the parallel translate of \( e_i \) along \( \gamma \). Define \( V_i(s) = \frac{1}{\ell} E_i(s) \). Substitute into (3.1) and sum over \( i \) from 1 to \( n-1 \),
\[
\sum_{i=1}^{n-1} \delta^2 \gamma(V_i, V_i) = \int_0^\ell \left( \frac{n-1}{\ell^2} - \left( \frac{s}{\ell} \right)^2 \text{Ric}(\gamma'(s), \gamma'(s)) \right) \, ds - H_{\gamma}(s) \geq 0.
\]

Since \( \text{Ric} \geq 0 \) and \( H \geq \frac{(n-1)k}{\ell} > 0 \), (3.2) implies that \( \frac{n-1}{\ell^2} \geq \frac{(n-1)k}{\ell} \). Therefore, \( \ell \leq \frac{1}{n} \). Since the point \( x \) is arbitrary, we have proved inequality (1.1).

Assume now that \( \partial M \) is compact, then (1.1) implies that \( M \) is compact. Suppose equality holds in (1.1). By rescaling the metric of \( M \), we can assume that \( k = 1 \).
Then we want to prove that $M^n$ is isometric to the $n$-dimensional Euclidean unit ball. Since $M$ is compact, there exists some $x_0$ in the interior of $M$ such that
\begin{equation}
\label{eq3.3}
d(x_0, \partial M) = 1.
\end{equation}
The key step is to show that $M$ is equal to the geodesic ball of radius 1 centered at $x_0$, denoted by $B_1(x_0)$. From (3.3), it is clear that $B_1(x_0)$ is contained in $M$. Let $\rho = d(x_0, \cdot)$ denote the distance function from $x_0$. Since $M$ has nonnegative Ricci curvature, the Laplacian comparison theorem gives
\begin{equation}
\label{eq3.4}
\overline{\Delta} d \leq \frac{n-1}{d},
\end{equation}
where $\overline{\Delta}$ is the Laplacian operator on $M$, and $d = d(x, \cdot)$ is the distance function in $M$ from any point $x$.

Let $S = \{q \in \partial M : \rho(q) = 1\}$. We claim that $S = \partial M$. To prove the claim, it suffices to show that $S$ is an open and closed subset of $\partial M$, since $\partial M$ is connected by Lemma 2.1. Note that $S$ is closed by continuity of $\rho$. It remains to prove that $S$ is open in $\partial M$. Pick any point $q \in S$, we will show that $\rho \equiv 1$ in a neighborhood of $q$ in $\partial M$. If $q$ is not a conjugate point to $x_0$ in $M$, then the geodesic sphere $\partial B_1(x_0)$ is a smooth hypersurface near $q$ in $M$, whose mean curvature with respect to the inner unit normal is at most $n-1$ by the Laplacian comparison theorem (3.4). On the other hand, $\partial M$ has mean curvature at least $n-1$ with respect to the inner unit normal by assumption. The maximum principle for hypersurfaces in manifolds \cite{[H] implies that $\partial M$ and $\partial B_1(x_0)$ coincide in a neighborhood of $q$. Hence, $\rho \equiv 1$ in a neighborhood of $q$. Therefore, $S$ is open near any $q$ which is not a conjugate point to $x_0$ in $M$. If $q$ is a conjugate point of $x_0$, we want to show that $\Delta \rho \leq 0$ in the barrier sense $\overline{\Pi}$ in a neighborhood $q$, where $\Delta$ is the Laplacian operator on $\partial M$. Since $q$ is a minimum of $\rho$, we can then apply the strong maximum principle in $\overline{\Pi}$ for superharmonic function in the barrier sense to conclude that $\rho \equiv 1$ near $q$ in $\partial M$. To see why $\rho$ is superharmonic in $\partial M$. Let $\epsilon > 0$ be any small constant and $p$ be any point on $\partial M$ near $q$. We have to find an upper barrier $\rho_\delta$ which is $C^2$ in a neighborhood of $p$ in $\partial M$, i.e. $\rho_\delta(p) = \rho(p)$ and $\rho_\delta \geq \rho$ in a neighborhood of $p$ in $\partial M$. Let $\gamma : [0, 1] \to M$ be a minimizing geodesic from $x_0$ to $p$ parametrized by arc length. Let $\delta > 0$ be a small constant to be fixed later, and define
\begin{equation}
\rho_\delta(\cdot) = \delta + d(\gamma(\delta), \cdot),
\end{equation}
which is smooth in a neighborhood of $p$. Notice that $\rho_\delta(p) = \rho(p)$ and $\rho_\delta \geq \rho$ in a neighborhood of $p$ by the triangle inequality. By the Laplacian comparison theorem (3.4), we have
\begin{equation}
\label{eq3.5}
\overline{\Delta} \rho_\delta \leq \frac{n-1}{d(\gamma(\delta), \cdot)} = \frac{n-1}{\rho_\delta - \delta}.
\end{equation}
On a neighborhood of $p$ in $\partial M$, we have
\begin{equation}
\label{eq3.6}
\Delta \rho_\delta = \overline{\Delta} \rho_\delta + H \frac{\partial \rho_\delta}{\partial N} - \text{Hess} \rho_\delta(N, N),
\end{equation}
where $N$ is the inner unit normal of $\partial M$ with respect to $M$, $H$ is the mean curvature of $\partial M$ with respect to $N$ and $\text{Hess} \rho_\delta$ is the Hessian of $\rho_\delta$ in $M$. Observe that
\begin{equation*}
\rho_\delta(p) = \rho(p), \quad \frac{\partial \rho_\delta}{\partial N}(p) = -1 \quad \text{and} \quad \text{Hess} \rho_\delta(N, N)(p) = 0.
\end{equation*}
Choose a neighborhood $U \subset \partial M$ of $q$ such that for any $p \in U$ and $\delta > 0$ sufficiently small, we have

$$\rho_\delta \geq \rho \geq 1, \quad \frac{\partial \rho_\delta}{\partial N} \geq -1 + \delta \quad \text{and} \quad \text{Hess} \rho_\delta(N, N) \geq -\delta$$

on the neighborhood $U$. By assumption, $H \geq n - 1$, we see from (3.5), (3.6) and (3.7) that in the neighborhood $U$ around $p$, \[
\Delta \rho_\delta \leq \frac{n-1}{1-\delta} - (1-\delta)(n-1) + \delta \leq \epsilon
\]
if $\delta$ is sufficiently small. Since $\epsilon$ is arbitrary, this shows that $\rho$ is superharmonic near $q$ in the barrier sense and attains a local minimum at $q$. Therefore, $\rho$ is constant near $q$ by the maximum principle of [1]. This proves the claim that $S = \partial M$.

Now, we have shown that $M = B_1(x_0)$, the geodesic ball of radius 1 centered at $x_0$ in $M$. We first note that $\rho$ is smooth up to the boundary $\partial M$. This is true since any $q \in \partial M$ can be joined by a minimizing geodesic $\gamma$ of unit length from $x_0$ to $q$. As $\partial M = \partial B_1(x_0)$, $\gamma$ is orthogonal to $\partial M$ at $q$, hence is uniquely determined by $q$. Therefore, $q$ is not in the cut locus of $x_0$. Since $M$ has nonnegative Ricci curvature, the Laplacian comparison (3.4) for $\rho = d(x_0, \cdot)$ holds in the classical sense, that is,

$$\rho \Delta \rho \leq n - 1.$$ \hspace{1cm} (3.8)

Since $|\nabla \rho| = 1$ on $M$, $\rho \equiv 1$ and $\frac{\partial \rho}{\partial \nu} = 1$ on $\partial M$, where $\nu = -N$ is the outer unit normal of $\partial M$, integrating (3.8) over the whole manifold $M$ and applying Stokes theorem, we get

$$|\partial M| - |M| = \int_{\partial M} \rho \frac{\partial \rho}{\partial \nu} - \int_M |\nabla \rho|^2 = \int_M \rho \Delta \rho \leq \int_M (n-1) = (n-1)|M|.$$ \hspace{1cm} This implies that

$$\frac{1}{n} \frac{|\partial M|}{|M|} \leq 1.$$ \hspace{1cm} Since the mean curvature of $\partial M$ satisfies $H \geq n - 1$, by Lemma 2.2, $M$ is isometric to a Euclidean ball of radius $r$. It is clear that $r = 1$ as $M = B_1(x_0)$. This completes the proof of Theorem 1.1.

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