A CURIOUS GEOMETRICAL FACT ABOUT ENTANGLEMENT

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Abstract

I sketch how the set of pure quantum states forms a phase space, and then
point out a curiosity concerning maximally entangled pure states: they form
a minimal Lagrangian submanifold of the set of all pure states. I suggest that
this curiosity should have an interesting physical interpretation.

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1. From classical to quantum states

I will state a curious fact concerning maximally entangled pure states of a bipartite quantum system [1]. I do not know any physical interpretation of it, but I do expect that there is one. Before I can state my fact, I will have to give a bird’s eye view of some aspects of quantum mechanics—aspects that may be worth recalling anyway, in a reconsideration of the foundations of quantum theory.

We begin at the beginning. A classical state—in classical probability theory—is a probability distribution, that is a set of $N$ numbers $p_i$ such that

$$p_i \geq 0, \quad \sum_{i=1}^{N} p_i = 1.$$  \hspace{1cm} (1)

The set of classical states forms a simplex; a triangle if $N = 3$, which is a good case to choose as an illustrative example. If you want to introduce dynamics on a probability simplex, the options are rather limited. You can use stochastic maps, tending to shrink the simplex towards some fixed point, and then combine such maps into Markov chains. If we also require the stochastic maps to take pure states into pure states, we only have permutations of the corners to play with.

The set of pure states is isomorphic to the sample space. To define a symplectic structure on sample space, it needs to be a continuous space—the discrete space forming the corners of a finite dimensional simplex simply will not do. To get classical mechanics, with Hamiltonians and symplectic forms, it is necessary to go to the infinite dimensional case. (I am aware that this statement must be qualified [2].)

To generalize the classical case, we first rewrite the definitions in terms of diagonal matrices. Thus a state $P$ is a matrix

$$P = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix}, \quad P \geq 0, \quad \text{Tr} P = 1.$$  \hspace{1cm} (2)

A random variable $A$ is an otherwise unrestricted diagonal matrix,

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}. $$  \hspace{1cm} (3)
The expectation value of a random variable, given a state, is then

$$\langle A \rangle = \text{Tr}PA .$$  \hspace{1cm} (4)

There is a fairly obvious generalization of all this. We replace the diagonal matrices with “diagonalizable”, that is Hermitian, matrices. This still leaves it open whether they should be real or complex matrices, but I will soon argue that complex numbers are the preferred choice, so let us make it right away. A state is now a positive (and therefore Hermitian) matrix,

$$\rho = \rho^\dagger \geq 0 , \quad \text{Tr}\rho = 1 .$$  \hspace{1cm} (5)

Such a state is known as a density matrix. A random variable is an otherwise unrestricted Hermitian matrix,

$$A = A^\dagger .$$  \hspace{1cm} (6)

For expectation values, we keep the classical formula (4).

The quantum generalization is a significant one. The random variables, and the states, now belong to a non-commutative algebra. The $N - 1$ dimensional classical state space is turned into an $N^2 - 1 = (N + 1)(N - 1)$ dimensional one. This set forms an interesting convex body, which can be understood by observing that a general density matrix can be obtained from a diagonal one by means of a unitary transformation

$$\rho \rightarrow U \rho U^\dagger .$$  \hspace{1cm} (7)

It is only the $SU(N)$ subgroup that acts effectively on the states. This is a subgroup of $SO(N^2 - 1)$, so the convex body of density matrix is obtained by performing quite special rotations of the classical simplex. This gives the convex body a subtle structure, but our concern here is with its pure states.

Pure quantum states are density matrices of rank one,

$$\rho = \frac{\psi^\alpha \bar{\psi}^\beta}{\psi \cdot \bar{\psi}} , \quad \psi \cdot \bar{\psi} \equiv \bar{\psi}^\alpha \bar{\psi}_\alpha = \langle \psi | \psi \rangle , \quad 1 \leq \alpha, \beta \leq N .$$  \hspace{1cm} (8)

Equivalently, they are vectors in an $N$ dimensional Hilbert space up to normalization and phase, that is to say they can be regarded as points in complex projective space $\mathbb{CP}^{N-1}$. The real dimension of this space is $2(N - 1)$. Hence
the pure states in quantum mechanics form a continuous manifold already in the finite dimensional case.

2. Quantum mechanics

Because we decided to work with complex numbers, we can now justify the name “quantum mechanics”. This happens because, as noted by Strocchi [3] (and later by others [4, 5, 6, 7]), the Hilbert space scalar product conceals a symplectic form.

To see this, split the complex vectors into real and imaginary parts, and reassemble them into real $2N$ dimensional vectors:

$$\psi^\alpha = x^\alpha + iy^\alpha , \quad X^I = \begin{pmatrix} x^\alpha \\ y^\alpha \end{pmatrix} , \quad 1 \leq I, J \leq 2N .$$

(9)

Similarly, Hermitian operators may be split into real symmetric and imaginary anti-symmetric parts. The scalar product becomes

$$X \cdot Y = X^I g_{IJ} Y^J + iX^I \Omega_{IJ} Y^J ,$$

(10)

where

$$g_{IJ} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Omega_{IJ} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

(11)

Therefore $g_{IJ}$ will serve as a metric on state space, while $\Omega_{IJ}$ is an anti-symmetric symplectic form.

Recall that a symplectic form on a vector space is simply a non-degenerate anti-symmetric matrix. If $\Omega^{IJ}$ denotes its inverse, one can use it to define a Poisson bracket between any two functions on phase space, as

$$\{ f(X), g(X) \} = \Omega^{IJ} \partial_I f(X) \partial_J g(X) .$$

(12)

If the symplectic form itself depends on $X^I$, an extra condition—that the form be closed—must be imposed in order to ensure that the Jacobi identities hold.

The point we are driving at is that, once these definitions are in place, the Schrödinger equation takes precisely the form that time evolution always takes in classical Hamiltonian mechanics.
\[ i \partial_t \psi^\alpha = H^\alpha_\beta \psi^\beta \quad \Leftrightarrow \quad \dot{X}^I = \Omega^{IJ} \partial_J \langle H \rangle = \{ X^I, \langle H \rangle \} . \]  

(13)

It is just that the Hamiltonian function takes a quite special form:

\[ \langle H \rangle = \langle \psi | H | \psi \rangle . \]  

(14)

Hence the name quantum mechanics is indeed justified: the space of its pure states is rich enough to support time evolution of Hamiltonian form. We have not identified classical and quantum mechanics, since we admit Hamiltonians of a very special form only. On inspection, one finds that quantum mechanics admits precisely those Hamiltonian flows that preserve the metric which is also present on phase space.

In classical theory, one needs a continuous sample space to support Hamiltonian mechanics. In quantum theory Hamiltonian mechanics is always there, because one uses a continuous set of discrete (or continuous) sample spaces.

We note in passing that nothing like this would work over the real numbers. Real projective space is not a symplectic manifold.

For the record, we will need the symplectic form also on complex projective space. On \( \mathbb{CP}^{N-1} \) we have the Fubini-Study metric

\[ ds^2 = \psi \cdot \bar{\psi} d\psi \cdot d\bar{\psi} - d\psi \cdot \bar{\psi} \psi \cdot d\bar{\psi} \]  

(15)

and its relative, the symplectic form

\[ \Omega = i \psi \cdot \bar{\psi} d\psi \cdot \wedge d\bar{\psi} - d\psi \cdot \bar{\psi} \wedge \psi \cdot d\bar{\psi} \]  

(16)

Basically this is what we had before, rewritten to be invariant under the transformation \( \psi^\alpha \rightarrow z \psi^\alpha \), where \( z \) is an arbitrary non-zero complex number.

3. Composite systems

We are now halfway to our fact. Before we get there, we must consider composite systems, described by means of the Hilbert space \( \mathcal{H}^{N^2} = \mathcal{H}^N \otimes \mathcal{H}^N \). For this purpose, it is helpful to introduce a notation that regards vectors as square arrays of numbers. Thus we write
\[ |\Psi\rangle = \psi^\alpha |e_\alpha\rangle = \frac{1}{\sqrt{N}} \Gamma^{ij} |e_i\rangle |e_j\rangle , \quad 1 \leq \alpha \leq N^2 , \quad 1 \leq i,j \leq N . \]  

(17)

Although \( \Gamma^{ij} \) is primarily a square array, not a matrix, the terminology of matrix theory applies to it. If the matrix is of rank one, the state is a product state:

\[ |\Psi\rangle = \phi^i \lambda^j |e_i\rangle |e_j\rangle = |\phi\rangle |\lambda\rangle . \]  

(18)

Such states are also known as separable. Geometrically, the set of separable states are embedded in \( \mathbb{CP}^{N^2-1} \) through the Segre embedding

\[ \mathbb{CP}^N \times \mathbb{CP}^N \rightarrow \mathbb{CP}^{N^2-1} . \]  

(19)

They form an orbit under the group of local unitaries, \( SU(N) \times SU(N) \), which is a subgroup of \( SU(N^2) \).

Once vectors get two indices, density matrices get four. The rank one density matrix describing a pure state is

\[ \rho^{ij}_{kl} = \psi^\alpha \bar{\psi}^\beta = \frac{1}{N} \Gamma^{ij} \Gamma^*_{kl} . \]  

(20)

A basic point is this: if we control operators of the form \( A \otimes 1 \) only, then the relevant state is given by the partial trace

\[ \text{Tr}_2 \rho = \frac{1}{N} \sum_{j=1}^{N} \Gamma^{ij} \Gamma^*_{kj} = \frac{1}{N} (\Gamma \Gamma^\dagger)^i_k . \]  

(21)

If the original state is separable, this is still a pure state on \( \mathcal{H}^N \). On the other hand, if

\[ \Gamma \Gamma^\dagger = 1 , \]  

(22)

then \( \Gamma \in U(N) \), the reduced state is maximally mixed, and the original state is said to be maximally entangled—all the information in the original state is in (non-classical) correlations.

Projectively the phase is irrelevant, and the set of maximally entangled states in \( \mathcal{H}^N \otimes \mathcal{H}^N \) is isomorphic to
\[ SU(N)/Z_N \in \mathbb{CP}^{N^2 - 1}. \]  

This is a group manifold, and as such it is again an orbit under \( SU(N) \times SU(N) \). But precisely what kind of submanifold is it?

4. The curious fact

Observe that
\[ \dim [SU(N)/Z_N] = N^2 - 1 = \frac{1}{2} \dim \left[ \mathbb{CP}^{N^2 - 1} \right]. \]  

This should make you suspicious. If we insert a state vector given by a unitary matrix in the \( N^2 \) dimensional version of eq. (16), we find [1] that
\[ \Omega|_{SU(N)/Z_N} = 0. \]  

The dimension of the set of maximally entangled state is one half of the dimension of the set of all pure states, and the symplectic form vanishes when restricted to this submanifold. A submanifold having these two properties is called a Lagrangian submanifold. To a physicist, Lagrangian submanifolds are known as configuration spaces. In a phase space spanned by \( qs \) and \( ps \), the subspace spanned by the \( qs \) is Lagrangian.

Considered as a fact about differential geometry, our example is well known to mathematicians [8]. They also know that this particular example is a minimal submanifold—if you wiggle it, its volume grows. In a technical sense, its two properties make our submanifold into a special submanifold of a Kähler space.

We have arrived at our curious fact: The set of maximally entangled pure states is a Lagrangian and minimal submanifold of the set of all pure states. From a physical point of view, why? I do not know, but I imagine that there should be a reason, perhaps one that can be phrased in control theoretic terms. I did offer a free dinner to any participant who would tell me, but there were no takers. The offer still stands.
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