Symmetry group classification for general Burgers’ equation

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Abstract

The present paper solves the problem of the group classification of the general Burgers’ equation
\[ u_t = fa(x, u)u_x^2 + gb(x, u)u_{xx}, \]
where \( f \) and \( g \) are arbitrary smooth functions of the variable \( x \) and \( u \), by using Lie method. The paper is one of the few applications of an algebraic approach to the problem of group classification: the method of preliminary group classification. A number of new interesting nonlinear invariant models which have nontrivial invariance algebras are obtained. The result of the work is a wide class of equations summarized in table form.

Key words: Infinitesimal generator, General Burgers’ equation, Optimal system, Preliminarily group classification.

1 Introduction

It is well known that the symmetry group method plays an important role in the analysis of differential equations. The history of group classification methods goes back to Sophus Lie. The first paper on this subject is [1], where Lie proves that a linear two-dimensional second-order PDE may admit at most a three-parameter invariance group (apart from the trivial infinite parameter symmetry group, which is due to linearity). He computed the maximal invariance group of the one-dimensional heat conductivity equation and utilized this symmetry to construct its explicit solutions. Saying it the modern way, he performed symmetry reduction of the heat equation. Nowadays symmetry reduction is one of the most powerful tools for solving nonlinear partial differential equations (PDEs). Recently, there have been several generalizations of the classical Lie group method for symmetry reductions. Ovsiannikov [2] developed the method of partially invariant solutions. His approach is based on the concept of an equivalence group, which is a Lie transformation group acting in the extended space of independent variables, functions and their derivatives, and preserving the class of partial differential equations under study [3,4,5].

The investigation of the exact solutions plays an important role in the study of nonlinear physical systems. A wealth of methods have been developed to find these exact physically significant solutions of a PDE though it is rather difficult. Some of the most important methods are the inverse scattering method [6], Darboux and Bäcklund transformations [7], Hirota bilinear method [7,8], Lie symmetry analysis [9,10,11], etc. The paper [12], based on the Lie group method, is investigated a very famous and important equation, which is the general Burgers’ equation as the form

\[ u_t = au_x^2 + bu_{xx}, \]  

(1.1)

where \( u = u(x, t) \) is the unknown real function, \( a, b \in \mathbb{R} \) and \( ab \neq 0 \). In the present paper, we consider the general Burgers’ equation as the form

\[ u_t = f(x, u)u_x^2 + g(x, u)u_{xx}, \]  

(1.2)

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where \( u = u(x,t) \) is the unknown real function, \( f \) and \( g \) are arbitrary smooth functions of the variable \( x \) and \( u \). Eq. (1.2) represents the Burgers’ equation combining both dissipative and nonlinear effects, therefore appears in a wide variety of physical applications. So it is important to lucubrate the exact explicit solutions and similarity reductions for this equation [13,14]. Here, we get the preliminary group classification of Eq. (1.2) by means of Lie point symmetry, and the constructed optimal systems of subalgebras. The knowledge of the optimal system of reductions for this equation [13,14]. Here, we get the preliminary group classification of Eq. (1.2) by means of Lie point symmetry, and the constructed optimal systems of subalgebras. The knowledge of the optimal system of subalgebras gives the possibility of constructing the optimal system of solutions [2,15,16] and permits the generation of new solutions starting from invariant or non-invariant solutions.

2 Symmetry Methods

Let a partial differential equation contains \( p \) dependent variables and \( q \) independent variables. The one-parameter Lie group of transformations

\[
\begin{align*}
\mathbf{\xi}_i &= x_i + \epsilon \xi_i(x, u) + O(\epsilon^2); \\
\mathbf{\eta}_\alpha &= u_\alpha + \epsilon \varphi^\alpha(x, u) + O(\epsilon^2),
\end{align*}
\]  

(2.3)

where \( \xi_i = \frac{\partial \mathbf{\xi}_i}{\partial \epsilon} \bigg|_{\epsilon=0}, \; i = 1, \ldots, p \), and \( \varphi^\alpha = \frac{\partial \mathbf{\eta}_\alpha}{\partial \epsilon} \bigg|_{\epsilon=0}, \; \alpha = 1, \ldots, q \), are given. The action of the Lie group can be recovered from that of its associated infinitesimal generators. We consider general vector field

\[
X = \sum_{i=1}^{p} \xi_i(x, u) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u) \frac{\partial}{\partial u^\alpha},
\]  

(2.4)

on the space of independent and dependent variables. Therefore, the characteristic of the vector field \( X \) given by (2.4) is the function

\[
Q^\alpha(x, u^{(1)}) = \varphi^\alpha(x, u) - \sum_{i=1}^{p} \xi_i(x, u) \frac{\partial \varphi^\alpha}{\partial x_i}, \quad \alpha = 1, \ldots, q.
\]  

(2.5)

The second prolongation of the infinitesimal operator

\[
X = \xi^1(x, t, u) \frac{\partial}{\partial x} + \xi^2(x, t, u) \frac{\partial}{\partial t} + \varphi(x, t, u) \frac{\partial}{\partial u},
\]  

(2.6)

obtained via the following prolongation formulas:

\[
X^{(2)} = X + \varphi_x \frac{\partial}{\partial u_x} + \varphi_t \frac{\partial}{\partial u_t} + \varphi_{xt} \frac{\partial}{\partial u_{xt}} + \varphi_{xx} \frac{\partial}{\partial u_{xx}},
\]

the coefficients are obtained by

\[
\begin{align*}
\varphi_x &= D_x \varphi - u_x D_x \xi^1 - u_t D_x \xi^2, \\
\varphi_t &= D_t \varphi - u_x D_t \xi^1 - u_t D_t \xi^2, \\
\varphi_{xx} &= D_x \varphi_x - u_{xx} D_x \xi^1 - u_{xt} D_x \xi^2, \\
\varphi_{xt} &= D_t \varphi_x - u_{xx} D_t \xi^1 - u_{xt} D_t \xi^2,
\end{align*}
\]  

(2.8-2.11)

where the operators \( D_x \) and \( D_t \) denote the total derivatives with respect to \( x \) and \( t \): 

\[
D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \ldots, \quad D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \ldots,
\]
By the theorem 6.5. in [17], \( X(2)[u_t - f(x,u)u_x^2 - g(x,u)u_{xx}](1.2) = 0 \). Since
\[
X(2)[u_t - f(x,u)u_x^2 - g(x,u)u_{xx}] = \varphi' - (f_x \xi^1 + f_u \varphi)u_x^2 - (g_x \xi^1 + g_u \varphi)u_{xx} - 2f \varphi^x u_x - g \varphi^{xx},
\]
therefore we obtain the following determining function:
\[
[\varphi' - (f_x \xi^1 + f_u \varphi)u_x^2 - (g_x \xi^1 + g_u \varphi)u_{xx} - 2f \varphi^x u_x - g \varphi^{xx}](1.2) = 0.
\]
In the case of arbitrary \( f(x,u) \) and \( g(x,u) \) it follows that
\[
\xi^1 = \varphi = \varphi^x = \varphi' = \varphi^{xx} = 0,
\]
or
\[
\xi^1 = \varphi = 0, \quad \xi^2 = C.
\]
Therefore, for arbitrary \( f(x,u) \) and \( g(x,u) \) Eq. (1.1) admits the one-dimensional Lie algebra \( g_1 \), with the basis
\[
X_2 = \frac{\partial}{\partial t}.
\]

\( g_1 \) is called the principle Lie algebra for Eq. (1.1). So, it is remained to specify the coefficient \( f \) and \( g \) such that Eq. (1.1) admits an extension of the principal algebra \( g_1 \). Usually, the group classification is obtained by inspecting the determining equation. But in our case the complete solution of the determining equation (2.12) is a wasteful venture. Therefore, we don’t solve the determining equation but, instead we obtain a partial group classification of Eq. (1.1) via the so-called method of preliminary group classification. This method was applied when an equivalence group is generated by a finite-dimensional Lie algebra \( \mathfrak{g}_E \). The essential part of the method is the classification of all nonsimilar subalgebras of \( \mathfrak{g}_E \). Actually, the application of the method is simple and effective when the classification is based on finite-dimensional equivalence algebra \( \mathfrak{g}_E \).

3 Equivalent transformations

An equivalence transformation is a nondegenerate change of the variables \( t, x, u \) taking any equation of the form (1.1) into an equation of the same form, generally speaking, with different \( f(x,u) \) and \( g(x,u) \). The set of all equivalence transformations forms an equivalence group \( \mathcal{E} \). We shall find a continuous subgroup \( \mathcal{E}_c \) of it making use of the infinitesimal method.

We consider an operator of the group \( \mathcal{E}_c \) in the form
\[
Y = \xi^1(x,t,u)\frac{\partial}{\partial x} + \xi^2(x,t,u)\frac{\partial}{\partial t} + \varphi(x,t,u)\frac{\partial}{\partial u} + \mu(x,t,u,f,g)\frac{\partial}{\partial f} + \nu(x,t,u,f,g)\frac{\partial}{\partial g},
\]
from the invariance conditions of Eq. (1.1) written as the system:
\[
u_t - f(x,u)u_x^2 - g(x,u)u_{xx} = 0, \quad f_t = g_t = 0,
\]
where \( u \) and \( f, g \) are considered as differential variables: \( u \) on the space \( (x,t) \) and \( f, g \) on the extended space \( (x,t,u) \).

The invariance conditions of the system (3.17) are
\[
Y^{(2)}[u_t - f(x,u)u_x^2 - g(x,u)u_{xx}] = 0,
Y^{(2)}(f_t) = Y^{(2)}(g_t) = 0,
\]
where $Y^{(2)}$ is the prolongation of the operator (3.16):

$$Y^{(2)} = Y + \varphi^x \frac{\partial}{\partial u_x} + \varphi^t \frac{\partial}{\partial u_t} + \varphi^{xt} \frac{\partial}{\partial u_{xt}} + \varphi^{xx} \frac{\partial}{\partial u_{xx}} + \mu^t \frac{\partial}{\partial f_t} + \nu^t \frac{\partial}{\partial g_t}. \quad (3.20)$$

The coefficients $\varphi^x, \varphi^t, \varphi^{xt}, \varphi^{xx}$ are given in (2.7) and the other coefficients of (3.20) are obtained by applying the prolongation procedure to differential variables $f$ and $g$ with independent variables $(x, u)$. In view of (3.18), we have

$$\mu^t = \tilde{D}_t(\mu) - f_x \tilde{D}_t(\xi^1) - f_u \tilde{D}_t(\varphi), \quad \nu^t = \tilde{D}_t(\nu) - g_x \tilde{D}_t(\xi^1) - g_u \tilde{D}_t(\varphi), \quad (3.21)$$

where $\tilde{D}_t = \frac{\partial}{\partial t}$. So, we have the following prolongation formulas:

$$\mu^t = \mu_t - f_x \xi^1_t - f_u \varphi_t, \quad \nu^t = \nu_t - g_x \xi^1_t - g_u \varphi_t. \quad (3.22)$$

The invariance conditions (3.19) give rise to

$$\mu^t = \nu^t = 0, \quad (3.23)$$

that is held for every $f$ and $g$. Substituting (4.49) into (4.48), we obtain

$$\mu_t = \nu_t = 0, \quad \xi^1 = \varphi_t = 0. \quad (3.24)$$

Moreover with substituting (3.20) into (3.19) we obtain

$$\nu^t - 2f(x, u)u_x \varphi^x - g(x, u)\varphi^{xx} - \mu u_x^2 - \nu u_{xx} - \nu = 0. \quad (3.25)$$

Substituting (3.27), (3.28) and (3.29) into invariance condition (3.25), and introducing the relation $u_t = f(x, u)u_x^2 + g(x, u)u_{xx}$ to eliminate $u_2$, we use a particular equation involving the various derivatives of $u(x, t)$ whose coefficients are certain derivatives of $\xi^1, \xi^2, \varphi, \mu,$ and $\nu$. We can equate the individual coefficients to zero, leading to the complete set of determining equations:

$$\xi^1 = \xi^1(x) \quad (3.26)$$

$$\xi^2 = \xi^2(t) = 0 \quad (3.27)$$

$$\varphi_u = \xi^2_t \quad (3.28)$$

$$\nu = -g \xi^2_t + 2 \xi^1_x \quad (3.29)$$

$$\mu = -f \xi^2_t - f(\varphi_u - 2 \xi^1_x) - g \varphi_{uu} \quad (3.30)$$

so, we find that

$$\xi^1(x) = a(x), \quad \xi^2(t) = c_1 t + c_2, \quad \varphi(x, u) = c_1 u + b(x), \quad \mu = -2f(c_1 - a(x)), \quad \nu = -g(c_1 - a'(x)) \quad (3.31)$$

with constants $c_1, c_2$ and two arbitrary functions $a(x)$ and $b(x)$ such that $b''(x) = c_1 - a'(x)$.

We summarize: The class of Eq. (1.2) has an infinite continuous group of equivalence transformations generated by the following infinitesimal operators:

$$Y = a(x) \frac{\partial}{\partial x} + (c_1 t + c_2) \frac{\partial}{\partial t} + (c_1 u + b(x)) \frac{\partial}{\partial u} - 2f(c_1 - a(x)) \frac{\partial}{\partial f} - g(c_1 - a'(x)) \frac{\partial}{\partial g}. \quad (3.32)$$

Therefore the symmetry algebra of the Burgers’ equation (1.2) is spanned by the vector fields

$$Y_1 = \frac{t}{u} \frac{\partial}{\partial t} + u \frac{\partial}{\partial u} - 2f \frac{\partial}{\partial f} - g \frac{\partial}{\partial g}, \quad Y_2 = \frac{\partial}{\partial t} \quad (3.33)$$

$$Y_3 = a(x) \frac{\partial}{\partial x} + 2fa(x) \frac{\partial}{\partial f} + ga'(x) \frac{\partial}{\partial g}, \quad Y_4 = b(x) \frac{\partial}{\partial u}.$$
Table 1
Commutation relations satisfied by infinitesimal generators in (4.35)

|   | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ | $Y_5$ |
|---|---|---|---|---|---|
| $Y_1$ | 0 | 0 | 0 | 0 | 0 |
| $Y_2$ | 0 | 0 | 0 | $Y_2$ | 0 |
| $Y_3$ | 0 | 0 | 0 | $Y_3$ | 0 |
| $Y_4$ | 0 | $-Y_2$ | $-Y_3$ | 0 | 0 |
| $Y_5$ | 0 | 0 | 0 | 0 | 0 |

Table 2
Adjoint relations satisfied by infinitesimal generators in (4.35)

|   | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ | $Y_5$ |
|---|---|---|---|---|---|
| $Y_1$ | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ | $Y_5$ |
| $Y_2$ | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4 - s Y_2$ | $Y_5$ |
| $Y_3$ | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4 - s Y_3$ | $Y_5$ |
| $Y_4$ | $Y_1$ | $e^s Y_2$ | $e^s Y_3$ | $Y_4$ | $Y_5$ |
| $Y_5$ | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ | $Y_5$ |

Moreover, in the group of equivalence transformations there are included also discrete transformations, i.e., reflections

\[ t \rightarrow -t, \quad x \rightarrow -x, \quad u \rightarrow -u, \quad f \rightarrow -f, \quad g \rightarrow -g. \]  

(3.34)

4 Preliminary group classification

One can observe in many applications of group analysis that most of extensions of the principal Lie algebra admitted by the equation under consideration are taken from the equivalence algebra $g_E$. We call these extensions $\mathcal{E}$-extensions of the principal Lie algebra. The classification of all nonequivalent equations (with respect to a given equivalence group $G_E$) admitting $\mathcal{E}$-extensions of the principal Lie algebra is called a preliminary group classification. Here, $G_E$ is not necessarily the largest equivalence group but, it can be any subgroup of the group of all equivalence transformations.

So, we can take any finite-dimensional subalgebra (desirable as large as possible) of an infinite-dimensional algebra with basis (3.33) and use it for a preliminary group classification. We select the subalgebra $g_5$ spanned on the following operators:

\[
Y_1 = \frac{\partial}{\partial x}, \quad Y_2 = \frac{\partial}{\partial t}, \quad Y_3 = \frac{\partial}{\partial u},
\]

\[
Y_4 = t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u} - 2f \frac{\partial}{\partial f} - g \frac{\partial}{\partial g}, \quad Y_5 = \frac{\partial}{\partial x} + 2f \frac{\partial}{\partial f} + g \frac{\partial}{\partial g}.
\]

(4.35)

The communication relations between these vector fields is given in Table 1. To each $s$-parameter subgroup there corresponds a family of group invariant solutions. So, in general, it is quite impossible to determine all possible group-invariant solutions of a PDE. In order to minimize this search, it is useful to construct the optimal system of solutions. It is well known that the problem of the construction of the optimal system of solutions is equivalent to that of the construction of the optimal system of subalgebras [2,15]. Here, we will deal with the construction of the optimal system of subalgebras of $g_5$.

Let $G$ be a Lie group, with $\mathfrak{g}$ its Lie algebra. Each element $T \in G$ yields inner automorphism $T_a \rightarrow TT_aT^{-1}$ of the group $G$. Every automorphism of the group $G$ induces an automorphism of $\mathfrak{g}$. The set of all these automorphisms is a Lie group called the adjoint group $G^A$. The Lie algebra of $G^A$ is the adjoint algebra of $\mathfrak{g}$, defined as follows. Let us have two infinitesimal generators $X, Y \in L$. The linear mapping $\text{Ad}_X(Y) : Y \rightarrow [X, Y]$ is an automorphism of
The set of all inner derivations \( \text{ad}X(Y)(X,Y \in \mathfrak{g}) \) together with the Lie bracket \([\text{ad}X,\text{ad}Y] = \text{ad}[X,Y]\) is a Lie algebra \(\mathfrak{g}^s\) called the adjoint algebra of \(\mathfrak{g}\). Clearly \(\mathfrak{g}^s\) is the Lie algebra of \(G^s\). Two subalgebras in \(\mathfrak{g}\) are conjugate (or similar) if there is a transformation of \(G^s\) which takes one subalgebra into the other. The collection of pairwise non-conjugate \(s\)-dimensional subalgebras is the optimal system of subalgebras of order \(s\). The construction of the one-dimensional optimal system of subalgebras can be carried out by using a global matrix of the adjoint transformations as suggested by Ovsiiannikov [2]. The latter problem, tends to determine a list (that is called an optimal system) of conjugacy inequivalent subalgebras with the property that any other subalgebra is equivalent to a unique member of the list under some element of the adjoint representation i.e. \(\mathfrak{g}/\text{Ad}(\mathfrak{g})\) for some \(\mathfrak{g}\) of a considered Lie group. Thus we will deal with the construction of the optimal system of subalgebras of \(\mathfrak{g}_5\).

The adjoint action is given by the Lie series

\[
\text{Ad}(s Y_i))Y_j = Y_j - s [Y_i, Y_j] + \frac{s^2}{2} [Y_i, [Y_i, Y_j]] - \cdots,
\]

where \(s\) is a parameter and \(i, j = 1, \ldots, 5\). The adjoint representations of \(\mathfrak{g}_5\) is listed in Table 2, it consists the separate adjoint actions of each element of \(\mathfrak{g}_5\) on all other elements.

**Theorem 4.1.** An optimal system of one-dimensional Lie subalgebras of general Burgers’ equation (1.2) is provided by those generated by

1) \(Y^1 = Y_1 = \partial_t\),
2) \(Y^2 = Y_2 = \partial_x\),
3) \(Y^3 = Y_3 = \partial_u\),
4) \(Y^4 = Y_4 = t\partial_t + u\partial_u - 2f\partial_f - g\partial_g\),
5) \(Y^5 = Y_5 = \partial_x + 2f\partial_f + g\partial_g\),
6) \(Y^6 = Y_1 + Y_2 = \partial_t + \partial_x\),
7) \(Y^7 = -Y_1 + Y_2 = -\partial_t + \partial_x\),
8) \(Y^8 = Y_1 + Y_4 = (t + 1)\partial_t + u\partial_u - 2f\partial_f - g\partial_g\),
9) \(Y^9 = -Y_1 + Y_4 = (t - 1)\partial_t + u\partial_u - 2f\partial_f - g\partial_g\),
10) \(Y^{10} = Y_1 + Y_5 = \partial_t + \partial_x + 2f\partial_f + g\partial_g\),
11) \(Y^{11} = -Y_1 + Y_5 = -\partial_t + \partial_x + 2f\partial_f + g\partial_g\),
12) \(Y^{12} = Y_4 + Y_5 = t\partial_t + \partial_x + u\partial_u\),
13) \(Y^{13} = -Y_4 + Y_5 = -t\partial_t + \partial_x - u\partial_u + 4f\partial_f + 2g\partial_g\),
14) \(Y^{14} = Y_1 + Y_4 + Y_5 = (t + 1)\partial_t + \partial_x + u\partial_u\),
15) \(Y^{15} = -Y_1 + Y_4 + Y_5 = (t - 1)\partial_t + \partial_x + u\partial_u\),
16) \(Y^{16} = Y_1 - Y_4 + Y_5 = (1 - t)\partial_t - u\partial_u + 4f\partial_f + 2g\partial_g\),
17) \(Y^{17} = -Y_1 - Y_4 + Y_5 = -(1 + t)\partial_t - \partial_x - u\partial_u + 4f\partial_f + 2g\partial_g\).

**Proof.** Let \(\mathfrak{g}_4\) is the symmetry algebra of Eq. (1.2) with adjoint representation determined in Table 2 and

\[
Y = a_1 Y_1 + a_2 Y_2 + a_3 Y_3 + a_4 Y_4 + a_5 Y_5,
\]

is a nonzero vector field of \(\mathfrak{g}\). We will simplify as many of the coefficients \(a_i; i = 1, \ldots, 5\), as possible through proper adjoint applications on \(Y\). We follow our aim in the below easy cases:

**Case 1:**
At first, assume that \(a_5 \neq 0\). Scaling \(Y\) if necessary, we can assume that \(a_5 = 1\) and so we get

\[
Y = a_1 Y_1 + a_2 Y_2 + a_3 Y_3 + a_4 Y_4 + Y_5.
\]

Using the table of adjoint (Table 2), if we act on \(Y\) with \(\text{Ad}(\exp(a_2 Y_2))\), the coefficient of \(Y_2\) can be vanished:

\[
Y' = a_1 Y_1 + a_3 Y_3 + a_4 Y_4 + Y_5.
\]

Then we apply \(\text{Ad}(\exp(a_3 Y_3))\) on \(Y'\) to cancel the coefficient of \(Y_3\):

\[
Y'' = a_1 Y_1 + a_4 Y_4 + Y_5.
\]

**Case 1a:**
If \(a_1, a_4 \neq 0\) then we can make the coefficient of \(Y_1\) and \(Y_4\) either +1 or -1. Thus any one-dimensional subalgebra
generated by $Y$ with $a_3, a_4 \neq 0$ is equivalent to one generated by $\pm Y_1 \pm Y_4 + Y_5$ which introduce parts 14), 15), 16) and 17) of the theorem.

**Case 1b:**
For $a_1 = 0, a_4 \neq 0$ we can see that each one-dimensional subalgebra generated by $Y$ is equivalent to one generated by $\pm Y_4 + Y_5$ which introduce parts 12) and 13) of the theorem.

**Case 1c:**
For $a_1 \neq 0, a_4 = 0$ we can see that each one-dimensional subalgebra generated by $Y$ is equivalent to one generated by $\pm Y_1 + Y_5$ which introduce parts 10) and 11) of the theorem.

**Case 2:**
The remaining one-dimensional subalgebras are spanned by vector fields of the form $Y$ with $a_5 = 0$.

**Case 2a:**
If $a_4 \neq 0$ then by scaling $Y$, we can assume that $a_4 = 1$. Now by the action of $\text{Ad}(\exp(a_2 Y_2))$ on $Y$, we can cancel the coefficient of $Y_2$:

$$\nabla = a_1 Y_1 + a_3 Y_3 + Y_4.$$  \hspace{1cm} (4.42)

Then by applying $\text{Ad}(\exp(a_3 Y_3))$ on $\nabla$ the coefficient of $Y_3$ can be vanished and we have

$$\nabla = a_1 Y_1 + Y_4.$$  \hspace{1cm} (4.43)

The one-dimensional subalgebra generated by $Y$ is equivalent to one generated by $\pm Y_1 + Y_4$ which introduce parts 8) and 9) of the theorem.

**Case 2b:**
Let $a_4 = 0$ then $Y$ is in the form

$$\hat{Y} = a_1 Y_1 + a_2 Y_2 + a_3 Y_3.$$  \hspace{1cm} (4.44)

Suppose that $a_2 \neq 0$ then if necessary we can let it equal to 1 and we obtain

$$\hat{Y}' = a_1 Y_1 + Y_2 + a_3 Y_3.$$  \hspace{1cm} (4.45)

By acting $\text{Ad}(\exp(a_3 Y_3))$ on $\hat{Y}'$, it changed to $a_1 Y_1 + Y_2$:

**Case 2b-1:**
Let $a_1$ be nonzero. In this case we can make the coefficient of $Y_1$ in $\hat{Y}$ either +1 or −1 and find 6), 7) sections of the theorem.

**Case 2b-2:**
If $a_1$ is zero then $Y_2$ is remained. Hence this case suggests part 2).

**Case 2c:**
Finally if in the latter case $a_2$ be zero, then no further simplification is possible and then $Y$ is one of cases of (4.37). There is not any more possible case for studying and the proof is complete.

The coefficients $f, g$ of Eq. (1.2) depend on the variables $x, u$. Therefore, we take their optimal system’s projections on the space $(x, u, f, g)$. The nonzero in $x$-axis or $u$-axis projections of (4.37) are

1) $Z^1 = Y^2 = Y^6 = Y^7 = \partial_x,$
2) $Z^2 = Y^3 = \partial_u,$
3) $Z^3 = Y^4 = Y^8 = Y^9 = -Y^{16} = u \partial_u - 2 f \partial_f - g \partial_g,$
4) $Z^4 = Y^5 = Y^{10} = Y^{11} = \partial_x + 2 f \partial_f + g \partial_g,$
5) $Z^5 = Y^{12} = Y^{14} = Y^{15} = \partial_x + u \partial_u,$
6) $Z^6 = Y^{13} = Y^{17} = \partial_x - u \partial_u + 4 f \partial_f + 2 g \partial_g.$  \hspace{1cm} (4.46)

**Proposition 4.2.** Let $\mathfrak{g}_m := \langle Y_1, \ldots, Y_m \rangle$, be an $m$-dimensional algebra. Denote by $Y^i(i = 1, \ldots, r, 0 < r \leq m, r \in \mathbb{N})$ an optimal system of one-dimensional subalgebras of $\mathfrak{g}_m$ and by $Z^i (i = 1, \ldots, t, 0 < t \leq r, t \in \mathbb{N})$ the projections of $Y^i$, i.e., $Z^i = \text{pr}(Y^i)$. If equations

$$f = \Phi(x, u), \quad g = \Psi(x, u),$$  \hspace{1cm} (4.47)
are invariant with respect to the optimal system \( Z^i \) then the equation

\[ u_t = \Phi(x, u)u_x^2 + \Psi(x, u)u_{xx}, \]

admits the operators \( X^i = \text{projection of } Y^i \) on \((t,x,u)\).

**Proposition 4.3.** Let Eq. (4.48) and the equation

\[ u_t = \Phi'(x, u)u_x^2 + \Psi'(x, u)u_{xx}, \]

be constructed according to Proposition 4.2. via optimal systems \( Z^i \) and \( Z'^i \), respectively. If the subalgebras spanned on the optimal systems \( Z^i \) and \( Z'^i \), respectively, are similar in \( g \), then Eqs. (4.48) and (4.49) are equivalent with respect to the equivalence group \( G_m \), generated by \( g \).

Now we apply Proposition 4.2. and Proposition 4.3. to the optimal system (4.46) and obtain all nonequivalent Eq. (1.2) admitting \( \mathcal{E} \)-extensions of the principal Lie algebra \( g \), by one dimension, i.e., equations of the form (1.2) such that they admit, together with the one basic operators (4.50) of \( g \), also a second operator \( X^{(2)} \). For every case, when this extension occurs, we indicate the corresponding coefficients \( f, g \) and the additional operator \( X^{(2)} \).

We perform the algorithm passing from operators \( Z^i (i = 1, \ldots, 6) \) to \( f, g \) and \( X^{(2)} \) via the following example. Let consider the vector field

\[ Z^6 = \partial_x - u\partial_u + 4f\partial_f + 2g\partial_g, \]

then the characteristic equation corresponding to \( Z^6 \) is

\[ dx = \frac{du}{-u} = \frac{df}{4f} = \frac{dg}{2g}, \]

and can be taken in the form

\[ I_1 = ue^x, \quad I_2 = \frac{f}{e^x}, \quad I_3 = \frac{g}{e^{2x}}. \]

From the invariance equations we can write

\[ I_2 = \Phi(I_1), \quad I_3 = \Psi(I_1), \]

it follows that

\[ f = e^{x^4}\Phi(\lambda), \quad g = e^{x^2}\Phi(\lambda), \]

where \( \lambda = I_1 \).
From Proposition 4.2. applied to the operator $Z^6$ we obtain the additional operator $X^{(2)}$

\[ -t\partial_t + \partial_x - u\partial_u, \quad -(1+t)\partial_t + \partial_x - u\partial_u. \]  

(4.55)

After similar calculations applied to all operators (4.46) we obtain the following result (Table 3) for the preliminary group classification of Eq. (1.2) admitting an extension $g_3$ of the principal Lie algebra $g_1$.

5 Conclusion

In this paper, following the classical Lie method, the preliminary group classification for the class of general Burgers’ equation (1.2) and investigated the algebraic structure of the symmetry groups for this equation, is obtained. The classification is obtained by constructing an optimal system with the aid of Propositions 4.2. and 4.3.. The result of the work is summarized in Table 3. Of course it is also possible to obtain the corresponding reduced equations for all the cases in the classification reported in Table 3. We omitted these for brevity.

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