Steiner-Minkowski Polynomials of Convex Sets in High Dimension, and Limit Entire Functions.

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Abstract

For a convex set $K$ of the $n$-dimensional Euclidean space, the Steiner-Minkowski polynomial $M_K(t)$ is defined as the $n$-dimensional Euclidean volume of the neighborhood of the radius $t$. Being defined for positive $t$, the Steiner-Minkowski polynomials are considered for all complex $t$. The renormalization procedure for Steiner polynomial is proposed. The renormalized Steiner-Minkowski polynomials corresponding to all possible solid convex sets in all dimensions form a normal family in the whole complex plane. For each of the four families of convex sets: the Euclidean balls, the cubes, the regular cross-polytopes and the regular symplexes of dimensions $n$, the limiting entire functions are calculated explicitly.

1 Renormalized Steiner-Minkowski Polynomials.

NOTATION. $\kappa_l$ is the $n$-dimensional volume of the unite ball in $\mathbb{R}^n$:

$$\kappa_l = \frac{\pi^{l/2}}{\Gamma(l/2 + 1)}$$  \hfill (1.1)

Let $K, K \subset \mathbb{R}^n$, be a compact convex set. For $t > 0$, there defined the function

$$M_K(t) \overset{\text{def}}{=} \text{Vol}_n(K + tB^n), \quad t > 0.$$  \hfill (1.2)
It is known (H.Minkowski) that \( M_K(t) \), considered for \( t > 0 \), is a polynomial in \( t \) of degree \( n \):

\[
M_K(t) = \sum_{0 \leq l \leq n} m_j(K)t^l.
\]  

(1.3)

This polynomial, which was defined originally by (1.2) for \( t > 0 \) only, will be considered for all \textit{complex} \( t \).

\textbf{DEFINITION 1.1.} The polynomial \( M_K(t) \) is said to be the \textit{Steiner-Minkowski polynomial} of the set \( K \).

The following normalizations for the coefficients of the polynomial \( M_K(t) \) are common:

\[
M_K(t) = \sum_{0 \leq l \leq n} \binom{n}{l} W_l(K)t^l,
\]  

(1.4)

and

\[
M_K(t) = \sum_{0 \leq l \leq n} \kappa_l V_{n-l}(K)t^l.
\]  

(1.5)

So,

\[
m_l(K) = \binom{n}{l} W_l(K) = \kappa_l V_{n-l}(K), \quad 0 \leq l \leq n.
\]  

(1.6)

The value \( W_j(K) \) is said to be \textit{l-th cross-sectional measure} (or \textit{l-th quer-massintegral} - in the German manner) of the set \( K \).

The value \( V_l(K) \) is said to be \textit{l-th intrinsic volume} of the set \( K \).

The constant term \( m_0(K) \) of the polynomial \( M_K(t) \) is the \( n \)-dimensional volume of \( K \), the coefficient \( m_1(K) \) of its linear term is the \((n-1)\)-dimensional ‘area’ of the ‘boundary surface’ \( \partial K \):

\[
m_0(K) = \text{Vol}_n(K) = W_0(K) = V_n(K),
\]  

(1.7a)

\[
m_1(K) = \text{Vol}_{n-1}(\partial K) = nW_1(K) = \kappa_1 V_{n-1}(K).
\]  

(1.7b)

We introduce the following normalization of the Steiner-Minkowski polynomial of a convex set \( K \), under the \textbf{extra assumption}:

The set \( K \) is \textbf{solid}, that is the interior \( \bar{K} \) is non-empty. This extra assumption can be reformulated as:

\[
\text{Vol}_n(K) > 0.
\]  

(1.8)

Under the extra assumption (1.8), the area of the surface is automatically positive: \( \text{Vol}_{n-1}(\partial K) > 0 \).
The ratio 
\[
\sigma_K = \frac{\text{Vol}_{n-1}(\partial K)}{\text{Vol}_n(K)}
\] 
(1.9)
is said to be the \textit{shape factor} of the set \( K \). In terms of cross-sectional measures the shape-factor is expressed as 
\[
\sigma_K = \frac{n W_1(K)}{W_0(K)}. 
\] 
(1.10)
The shape factor has dimension (length)\(^{-1}\).

\textbf{DEFINITION 1.2.} \textit{Given a solid compact convex set} \( K \), \textit{we present the Steiner-Minkovski polynomial} \( M_K(t) \) \textit{in the form:} 
\[
M_K(t) = \text{Vol}_n(K) \cdot M_K(\tau), 
\] 
(1.11)
where 
\[
\tau = \sigma_K t 
\] 
(1.12)
is a dimensionless parameter, and \( M_K(\tau) \) is a polynomial in \( \tau \) of degree \( n \).

The polynomial \( M_K(\tau) \) in the variable \( \tau \) is said to be the \textit{renormalized Steiner-Minkowski polynomial for the set} \( K \).

Let us present the sequence of the coefficients of the normalizes Steiner-Minkowski polynomial in the form: 
\[
M_K(\tau) = \sum_{0 \leq l \leq n} j_{n,l} \frac{\mu_l(K)}{l!} \tau^l, 
\] 
(1.13)
where the factors \( j_{n,l} \), so called Jensen multipliers, are \[3\]
\[
j_{n,0} = 1, \quad j_{n,l} = \prod_{0 \leq r \leq l-1} \left(1 - \frac{r}{n}\right), \quad \text{for} \quad l = 1, 2, \ldots, n, \quad j_{n,l} = 0 \quad \text{for} \quad l > n. 
\] 
(1.14)
The Jensen multipliers possesses properties
\[
0 \leq j_{n,l} \leq 1, \quad \text{for all} \; n, \; l, \quad j_{n,l} \to 1, \quad \text{as} \; l \; \text{is fixed}, \quad n \to \infty. 
\] 
(1.15)

\textbf{DEFINITION 1.3.} \textit{The coefficients} \( \mu_l(K) \) \textit{which appear in (1.13) are said to be the renormalized Steiner-Minkowski coefficients for the convex set} \( K \).

\(^3\)In other words, \[ \frac{j_{n,l} \cdot n^l}{l!} = \binom{n}{l} \] for \( 0 \leq l \leq n \).
In view of (1.3), (1.7), (1.10), (1.11), (1.12),
\[
\mu_l(K) = \frac{(W_0(K))^{l-1}W_l(K)}{(W_1(K))^l}, \quad 0 \leq l \leq n. \tag{1.16a}
\]
In particular,
\[
\mu_0(K) = 1, \quad \mu_1(K) = 1. \tag{1.16b}
\]

**LEMMA 1.1.** For any solid compact convex set \( K \subset \mathbb{R}^n \), the sequence \( \mu_l(K), \ l = 0, 1, 2, \ldots, n \) of its renormalized Steiner-Minkowski coefficients possesses the properties:

1. \( 0 < \mu_l(K) \leq 1, \quad 0 \leq l \leq n \). \tag{1.17}

2. It is logarithmically concave, that is the inequalities
   \[
   (\mu_l(K))^2 \geq \mu_{l-1}(K)\mu_{l+1}(K), \quad 1 \leq l \leq n - 1 \tag{1.18}
   \]
   hold.

**PROOF.** For any convex set \( K, \ K \subset \mathbb{R}^n \), its cross-sectional measures \( W_l(K), \ 0 \leq l \leq n \), are non-negative, and if the interior of \( K \) is non-empty, they are strictly positive:
\[
W_l(K) > 0, \quad 0 \leq l \leq n. \tag{1.19}
\]
The positivity property (1.19) is a special case of the positivity property of mixed volumes. (See, for example, [.].) The inequalities \( 0 < \mu_l(K), \ 0 \leq l \leq n \), are consequence of (1.16a) and (1.19).

The cross-sectional measures of any convex set satisfy the Alexandrov-Fenchel inequalities
\[
W_j^2(K) \geq W_{j-1}(K)W_{j+1}(K), \quad 1 \leq j \leq n - 1. \tag{1.20}
\]
The inequalities (1.18) are the Alexandrov-Fenchel inequalities (1.20), recalculated, according to (1.16a).

The inequalities \( \mu_l(K) \leq 1, \ 0 \leq l \leq n \), are consequence of the equalities (1.16b) and the logarithmic concavity inequalities (1.18).

The following statement is an immediate consequence of (1.13) and (1):

**LEMMA 1.2.** The renormalized Steiner-Minkowski polynomial \( M_K(\tau) \) of any solid compact convex set \( K \) satisfies the inequality
\[
|M_K(\tau)| \leq \exp \{|\tau|\} \quad \text{for every} \quad \tau \in \mathbb{C}. \tag{1.21}
\]
COROLLARY 1.1. The family \( \{ M_K(\tau) \} \) of all renormalized Steiner Minkowski polynomials corresponding to all solid compact convex sets \( K \subset \mathbb{R}^n \) in any dimension \( n : 1 \leq n < \infty \), is a normal family of analytic functions in the complex plane \( \mathbb{C} \).

This property of normality prompts us the following statement of the problem:

Given a sequence \( \{ K_p \} \) of solid compact sets, \( K_p \subset \mathbb{R}^{n_p} \), of increasing dimension: \( n_1 < n_2 < n_3 \ldots \). It is required to study the cluster set of functions for the family \( \{ M_{K_p}(\tau) \} \) of the corresponding Steiner-Minkowski polynomials. In particular, one need to clarify under which conditions this cluster set consists of one function, that is there exists the limit

\[
M(\tau) = \lim_{p \to \infty} M_{K_p}(\tau). \tag{1.22}
\]

Every function \( M(\tau) \) from this cluster set is an entire function satisfying the condition

\[
|M(\tau)| \leq \exp\{|\tau|\} \quad \text{for all } \tau \in \mathbb{C}. \tag{1.23}
\]

It is natural to restrict our consideration to the case

\[
K_1 \subset K_2 \subset \cdots \subset K_p \cdots, \tag{1.24}
\]

where we may consider the ambient Euclidean spaces as naturally embedded:

\[
\mathbb{R}^{n_1} \subset \mathbb{R}^{n_2} \subset \cdots \subset \mathbb{R}^{n_p} \subset \cdots. \tag{1.25}
\]

2 Some Examples.

To calculate explicitly the Steiner-Minkowski polynomials for concrete families of convex sets is difficult. Here we take several examples where such calculations can be done.

1. The family of Euclidean balls \( B^n \).

\[
B^n = \{ x = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n : \sum_{1 \leq i \leq n} |\xi_i|^2 \leq \rho^2 \}, \quad \rho > 0. \tag{2.1}
\]

Since \( B^n + tB^n = (\rho + t)B_n \) for \( t > 0 \),

\[
\text{Vol}_n(B^n + tB^n) = \text{Vol}_n(B^n)(\rho + t)^n \quad \text{for } t > 0.
\]
Thus the Steiner-Minkowski polynomial $M_{B^n}(t)$ is

$$M_{B^n}(t) = \kappa_n(\rho + t)^n.$$  

The shape constant $\sigma_{B^n}$ for the $n$-dimensional ball of radius $\rho$ is:

$$\sigma_{B^n} = n/\rho. \quad (2.2)$$

Thus, the renormalized Steiner-Minkowski polynomial $M_{B^n}(\tau)$ is equal to:

$$M_{B^n}(\tau) = (1 + \tau/n)^n. \quad (2.3)$$

The limit $M_{B^n}(\tau) \overset{\text{def}}{=} \lim_{n \to \infty} M_{B^n}(\tau)$ is:

$$M_{B^n}(\tau) = \exp\{\tau\}, \quad (2.4)$$

or, in the term of Taylor series,

$$M_{B^n}(\tau) = \sum_{0 \leq l < \infty} \frac{1}{\Gamma(l + 1)} \tau^l. \quad (2.5)$$

The next three families of convex sets which we consider are the families of cubes, regular crosspolytopes and regular symplexes. All these sets are regular polytopes. The Minkowski polynomial for the polytope $K, K \subset \mathbb{R}^n$, can be expressed in terms of the intrinsic volumes $V_r(K), r = 0, 1, \ldots, n$, by (1.5). The $r$-th intrinsic volume $V_r(K)$ of the polytope $K$ can be calculated by the formula

$$V_r(K) = \sum_{F_r} \gamma(F_r) \text{Vol}_r(F_r), \quad (2.6)$$

where the sum is taken over all $r$-faces $F_r$ of the polytope $K$, and $\gamma(F_r)$ is the external angle at the face $F_r$, normalized so that the total angle is 1. (See [Gr], Chapter 14, or [Schn]). For regular polytope $K$, the $r$-volumes $\text{Vol}_r(F_r)$ of all its $r$-faces $F_r$ are equal, and all external angles $\gamma_{F_r}$ are equal. Their common value is denoted by $v_r$ and $\gamma_r$ respectively:

$$\text{Vol}_r(F_r) = v_r \quad \text{for any} \quad r - \text{face } F_r, \quad (2.7)$$

$$\gamma_r(F_r) = \gamma_r \quad \text{for any} \quad r - \text{face } F_r. \quad (2.8)$$
The cardinality of the set $\mathcal{F}_r$ of all $r$-faces is denoted by $\nu_r$: 
\[ \nu_r = \#(\mathcal{F}_r). \] (2.9)

For the regular polytope $K$, the formula (2.6) takes the form 
\[ V_r(K) = \nu_r \gamma_r v_r. \] (2.10)

So, the formula (1.5) takes the form 
\[ M_K(t) = \sum_{0 \leq l \leq n} \kappa_l \nu_{n-l} \gamma_{n-l} v_{n-l} t^l. \] (2.11)

In the dimension $n$, $n \geq 5$, there are only three regular polytopes: cube, crosspolytope and symplex. We calculate the Minkowski polynomials for every of these three families. It is easy to calculate the volumes and the total numbers of $r$-faces, however to calculate the external angles for regular crosspolytopes and symplexes is more difficult. These values can not be expressed ‘in elementary functions’. In theyr expressions the Gauss’ integral oshibok appears.

\[ \diamond \hspace{1em} \diamond \hspace{1em} \diamond \]

2. The family of cubes $Q^n$.

$Q^n = \{ x = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n : |\xi_l| \leq \rho, \ 1 \leq l \leq n \}, \ \rho > 0$. (2.12)

The total number of $l$-faces $\nu_l(Q^n)$ is:
\[ \nu_l(Q^n) = 2^{n-l} \binom{n}{l}, \ \ l = 0, 1, \ldots, n. \] (2.13)

The $l$ dimensional volume of the $l$-face $v_l(Q^n)$ is:
\[ v_l(Q^n) = (2\rho)^l, \ \ l = 0, 1, \ldots, n. \] (2.14)

The external angle at $l$-face $\gamma_l(Q^n)$ is:
\[ \gamma_l(Q^n) = 2^{-(n-l)}, \ \ l = 0, 1, \ldots, n. \] (2.15)

According to (2.10), $V_r(Q^n) = 2^{(n-r)} \binom{n}{r} 2^{-(n-r)} (2\rho)^r$, or 
\[ V_r(Q^n) = \binom{n}{r} (2\rho)^r. \] (2.16)
Thus, for the cube $Q^n$, (2.12),

$$
M_{Q^n}(t) = \sum_{0 \leq l \leq n} \kappa_l \left( \frac{n}{n-l} \right)^{n-l} t^l,
$$

or

$$
M_{Q^n}(t) = (2\rho)^n \sum_{0 \leq l \leq n} \kappa_l j_{n,l} \frac{1}{l!} (2\rho)^{-l} (nt)^l.
$$

(2.17)

The shape factor (1.9) for the cube $Q^n$, (2.1), is

$$
\sigma_{Q^n} = n/\rho.
$$

(2.18)

Thus, the renormalized Steiner-Minkowski polynomial for the cube $Q^n$, (2.12), is

$$
M_{Q^n}(\tau) = \sum_{0 \leq l \leq n} j_{n,l} \left( \frac{\sqrt{\pi}}{2} \right)^l \frac{1}{\Gamma \left( \frac{l}{2} + 1 \right) \Gamma(l+1)} \tau^l.
$$

(2.19)

where $j_{n,l}$ is the Jensen multiplier defined in (1.14). Taking into account (1.15), we pass to the limit as $n \to \infty$ in the expression (2.19). The limiting entire function $M_{Q^n}(\tau)$ is:

$$
M_{Q^n}(\tau) = \sum_{0 \leq l \leq \infty} \left( \frac{\sqrt{\pi}}{2} \right)^l \frac{1}{\Gamma \left( \frac{l}{2} + 1 \right) \Gamma(l+1)} \tau^l.
$$

(2.20)

3. The family of Regular Cross-Polytopes $C^n$.

$$
C^n = \{ x = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n : \sum_{1 \leq l \leq n} |\xi_l| \leq \rho \}, \quad \rho > 0.
$$

(2.21)

The total number of $l$-faces $\nu_l(C^n)$ is:

$$
\nu_l(C^n) = 2^{l+1} \binom{n}{l+1}, \quad l = 0, 1, \ldots, n - 1.
$$

(2.22)

The $l$-dimensional volume of the $l$-face $v_l(C^n)$ is:

$$
v_l(C^n) = \rho^l \frac{\sqrt{l+1}}{l!}
$$

(2.23)
The external angle at $l$-face $\gamma_l(C^n)$ is calculated in [BeHe] (See Lemma 2.1 there):

$$
\gamma_l(C^n) = 2^{n-l-1} \frac{1}{\pi^{(n-l)/2}} \int_0^\infty e^{-x^2} \left( \frac{x}{\sqrt{l+1}} \right)^{n-l-1} dx.
$$

So, the $r$-th intrinsic volume $V_r(C^n)$ is:

$$
V_r(C^n) = 2^{r+1} \left( \frac{n}{r+1} \right)^{\sqrt{r+1}} \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-x^2} \left( \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{l+1}} e^{-y^2} dy \right)^{n-r-1} dx \cdot \rho^r,
$$

for $r = 0, 1, \ldots, n-1$;

$$
V_n(C_n) = \frac{2^n}{n!} \rho^n.
$$

The renormalized Steiner-Minkowski polynomial $M_{C^n}$ for the family of the regular cross-polytopes $\{C^n\}$, (2.21), is:

$$
M_{C^n}(t) = \frac{2^n}{n!} \rho^n + \sum_{1 \leq l \leq n} \kappa_l V_{n-l}(C^n) =
\frac{2^n}{n!} \rho^n + \sum_{1 \leq l \leq n} \kappa_l 2^{n-l} \left( \frac{n}{n-l+1} \right)^{\sqrt{n-l+1}} \cdot I_{n,l} \cdot \rho^{n-l} \cdot \frac{l!}{l!},
$$

where

$$
I_{n,l} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} \left( \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{n-l+1}} e^{-y^2} dy \right)^{l-1} dx,
$$

for $1 \leq l \leq n$.

The shape factor (1.9) for the regular cross-polytope $C^n$, (2.21), is

$$
\sigma_{C^n} = \frac{n^{3/2}}{\rho}.
$$

The renormalized Steiner-Minkowski polynomial $M_{C^n}$ for the family of the regular cross-polytopes $\{C^n\}$, (2.21), is:

$$
M_{C^n}(\tau) = 1 + \sum_{1 \leq l \leq n} \left( \frac{\sqrt{\pi}}{2} \right)^l (j_{n,l})^2 \frac{\sqrt{n}}{\sqrt{n-l+1}} \frac{2n^{l-1/2}}{\Gamma(l/2)} I_{n,l} \frac{\tau^l}{l!}.
$$
To pass to the limit as $n \to \infty$ in (2.19), we need some information about the values $I_{n,l}$, (2.27). Since

$$\int_0^\infty e^{-\lambda^2} d\lambda = \sqrt{\pi}/2,$$

$$|I_{n,l}| < 1, \quad l = 1, 2, \ldots, n.$$ 

Moreover, for every fixed $l$,

$$I_{n,l} = \left(\frac{2}{\sqrt{\pi}}\right)^l \int_0^\infty e^{-x^2} \left(\frac{x}{\sqrt{n-l+1}}\right)^{l-1} dx \left(1 + o(1)\right), \quad \text{as } n \to \infty,$$

or

$$I_{n,l} = \left(\frac{2}{\sqrt{\pi}}\right)^l \frac{1}{n^{l/2}} \int_0^\infty x^{l-1} e^{-x^2} dx \left(1 + o(1)\right), \quad \text{as } n \to \infty,$$

and finally

$$I_{n,l} = \frac{1}{2} \left(\frac{2}{\sqrt{\pi}}\right)^l \frac{1}{n^{l/2}} \Gamma\left(\frac{l}{2}\right) \left(1 + o(1)\right), \quad \text{as } n \to \infty, \quad (2.30)$$

Taking into account (1.15) and (2.30), we pass to the limit as $n \to \infty$ in (2.29). The limiting entire function is:

$$\mathcal{M}_{C^\infty}(\tau) = \sum_{0 \leq l < \infty} \frac{1}{\Gamma(l+1)} \tau^l. \quad (2.31)$$

or

$$\mathcal{M}_{C^\infty}(\tau) = \exp\{\tau\}. \quad (2.32)$$

4. The family of Regular Symplexes $S^n$.

$$S^n = \{x = (\xi_1, \xi_2, \ldots, \xi_{n+1}) \in \mathbb{R}^{n+1} : \sum_{1 \leq l \leq n+1} \xi_l = \rho, \quad \text{and } \xi_l \geq 0 \text{ for every } l\}, \quad (2.33)$$

The symplex $S^n$ is the $n$-dimensional polytop, which is considered as a subset of the $n$-dimensional space $\mathbb{R}^n$, :

$$R^n = \{x = (\xi_1, \xi_2, \ldots, \xi_{n+1}) \in \mathbb{R}^{n+1} : \sum_{1 \leq l \leq n+1} \xi_l = \rho\}. \quad (2.34)$$
The total number of $l$-faces $\nu_l(S^n)$ is:

$$\nu_l(S^n) = \binom{n+1}{n-l}, \quad l = 0, 1, \ldots, n.$$  \hfill (2.35)

The $l$-dimensional volume of the $l$-face $v_l(S^n)$ is:

$$v_l(S^n) = \rho \sqrt{l+1} \frac{1}{l!} \cdot$$  \hfill (2.36)

The external angle at $l$-face $\gamma_l(C^n)$ is calculated in [BöHe] (See Chapter 6, Section 6.5, Theorem 3, the formula 8 on the page 283 in [BöHe]):

$$\gamma_l(S^n) = 1 \sqrt{\pi} \int_{-\infty}^{\infty} e^{-x^2} \left( \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy \right)^{n-l} dx. \quad (2.37)$$

Remark that

$$\gamma_0(S^n) = \frac{1}{n+1}, \quad \gamma_{n-1}(S^n) = \frac{1}{2}, \quad \gamma_n(S^n) = 1.$$  \hfill (2.38)

So, the $r$-th intrinsic volume $V_r(S^n)$ is:

$$V_r(S^n) = \binom{n+1}{r+1} \frac{\sqrt{r+1}}{r!} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} \left( \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy \right)^{n-r} dx \cdot \rho^r, \quad r = 0, 1, \ldots, n.$$  \hfill (2.39)

$$M_{Sn}(t) = \sum_{0 \leq l \leq n} \kappa_l V_{n-l}(S^n) =$$

$$= \sum_{0 \leq l \leq n} \kappa_l \left( \frac{n+1}{n-l+1} \right) \frac{\sqrt{n-l+1}}{(n-l)!} \cdot I_{n,l} \cdot \rho^{n-l} t^l, \quad (2.40)$$

where

$$I_{n,l} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} \left( \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy \right)^l dx, \quad 0 \leq l \leq n.$$  \hfill (2.41)
Remark that
\[ I_{n,0} = 1, \quad I_{n,1} = \frac{1}{2}, \quad I_{n,n} = \frac{1}{n+1}. \] (2.42)
(These are the relations (2.38) in other notation). The shape factor (1.9) for the regular symplex \( S^n \), (2.33), is
\[ \sigma_{S^n} = \frac{n^{3/2}(n+1)^{1/2}}{\rho}. \] (2.43)

The renormalized Steiner-Minkowski polynomial \( M_{S^n} \) for the family of the regular symplexes \( \{S^n\}_n \), (2.33), is:
\[ M_{S^n}(\tau) = \sum_{0\leq l\leq n} \kappa_l \sqrt{\frac{n-l+1}{n+1}} j_{n+1,l} j_{n,l} \frac{(n+1)^{1/2}}{n^{l/2}} I_{n,l} \frac{\tau^l}{l!}, \] (2.44)
or
\[ M_{S^n}(\tau) = \sum_{0\leq l\leq n} \pi^{l/2} I_{n,l} \sqrt{\frac{n-l+1}{n+1}} \frac{(n+1)^{1/2}}{n^{l/2}} j_{n+1,l} j_{n,l} \frac{1}{\Gamma(\frac{l}{2}+1)\Gamma(l+1)} \tau^l, \] (2.45)

To pass to the limit as \( n \to \infty \) in (2.45), we need some information about the values \( I_{n,l} \), (2.41). Since \( \int_{-\infty}^{\infty} e^{-\lambda^2} d\lambda = \sqrt{\pi} \),
\[ 0 < I_{n,l} < 1, \quad l = 0, 1, \ldots, n. \]
Moreover, for every fixed \( l \),
\[ I_{n,l} = 2^{-l}(1 + o(1)), \quad n \to \infty. \] (2.46)

Taking into account (1.15) and (2.46), we pass to the limit as \( n \to \infty \) in (2.44). The limiting entire function is:
\[ M_{S^\infty}(\tau) = \sum_{0\leq l<\infty} \left( \frac{\sqrt{\pi}}{2} \right)^l \frac{\tau^l}{\Gamma(\frac{l}{2}+1)\Gamma(l+1)}. \] (2.47)

3 Discussion.
We have considered four families \( \{K^n\}_n \) of convex sets:
- balls \( \{B^n\}_n \), (2.1);
- cubes \( \{Q^n\}_n \), (2.12);
- regular cross-polytopes \( \{C^n\}_n \), (2.21);
- regular symplexes \( \{S^n\}_n \), (2.33).

For every of these families and for every convex set \( K_n \) of these family, we calculated the Steiner-Minkowski polynomial \( M_{K_n}(t) \) explicitly. The expression of \( M_{K_n}(t) \) contains the parameter \( \rho \), which can be considered as an inherent linear size of \( K_n \). For example, for the ball \( B_n \), (2.1), \( \rho \) is the radius of the ball; for the cube \( Q_n \), (2.12), \( 2\rho \) is the length of its edge, for the regular cross-polytope \( C^n \), \( \sqrt{2}\rho \) is the length of the edge. The choice of \( \rho \) is rather uncertain. If \( \rho \) is replaced with \( a\rho \), the value \( \rho^n \), which appears in the expression for \( \text{Vol}_n(K^n) \), will be replaced with \( a^n\rho^n \), and the shape factor \( \sigma_{K^n} \) will be replaced with \( a^{-1}\sigma_{K^n} \). We are interested in how the values \( \text{Vol}_n(K^n) \) and \( \sigma_{K^n} \) depend on \( n \). In view of aforesaid, the factors of the form \( a^n \) in the expression for \( \text{Vol}_n(K^n) \) and \( a \) in the expression for \( \sigma_{K^n} \) may be considered as non-essential. This uncertainty in the choice of \( \rho \) disappear after the renormalization of Minkowski polynomials. The renormalized Minkowski polynomial \( M_{K^n}(\tau) \) describe somehow the shape of \( K^n \), and do not depend on the size of \( K^n \). The limiting entire function \( M_{K^n}(\tau) \) describes somehow the shape of the convex set \( K^n \) of the family \( \{K_n\}_n \) in the very high dimension.

It looks very unexpectedly that the limiting entire functions for the families of balls and cross-polytopes coincide:

\[
M_{B^n}(\tau) = M_{C^n}(\tau) = \sum_{0 \leq l < \infty} \frac{1}{\Gamma(l+1)} \tau^l,
\]

and that the limiting entire functions for the families of cubes and symplexes coincide:

\[
M_{Q^n}(\tau) = M_{S^n}(\tau) = \sum_{0 \leq l < \infty} \left( \frac{\sqrt{n}}{2} \right)^l \frac{1}{\Gamma(l+1)\Gamma(l+1)} \tau^l.
\]

Recall that the appropriate shape factors are:

\[
\sigma_{B^n} = n, \quad \sigma_{Q^n} = n, \quad \sigma_{C^n} = n^{3/2}, \quad \sigma_{S^n} = n^{3/2}(n+1)^{1/2}.
\]

(We omitted multiplicative constants, and retain the dependence on \( n \) only in the expressions (3.3).)
Now we focus our attention on the location of zeros of Minkowski polynomials for the considered families of regular polytopes. Both entire functions which appear in the right hand sides of (3.1) and (3.2) belongs to the Laguerre-Polya class $L$-$P$-I of entire functions. (See [RaSc] for the definition. See also numerous papers by Th. Craven and G. Csordas.) For the function
\[
E_1(\tau) = \sum_{0 \leq l < \infty} \frac{1}{\Gamma(l + 1)} \tau^l,
\]
this is evident, for the function
\[
E_2(\tau) = \sum_{0 \leq l < \infty} \frac{1}{\Gamma(l + 1)} \tau^l,
\]
where
\[
\Psi(\tau) = \left(\frac{\sqrt{\pi}}{2}\right)^\tau \frac{1}{\Gamma\left(\frac{\tau}{2}\right) + 1},
\]
this is a consequence of the Laguerre Theorem, since the function $\Psi(\tau)$ belongs to the Laguerre-Polya class $L$-$P$-I. Since for the families of balls and cubes, the renormalized Minkowski polynomials $M_{K^n}(\tau)$ can be obtained from the entire function
\[
M_{K^n}(\tau) = \sum_{0 \leq l < \infty} j_{n,l} \frac{\mu_l}{l!} \tau^l,
\]
of the class $L$-$P$-I according to the rule
\[
M_{K^n}(\tau) = \sum_{0 \leq l < \infty} j_{n,l} \frac{\mu_l}{l!} \tau^l,
\]
where $j_{n,l}$ are Jensen multipliers, (1.14), the zeros of the polynomials $M_{B^n}(\tau)$ and $M_{Q^n}(\tau)$ are simple and negative. (For the polynomials $M_{B^n}(\tau)$, (2.3), this is evident.) However, the polynomials $M_{C^n}(\tau)$ and $M_{S^n}(\tau)$ are obtained from the appropriate entire functions $M_{C^n}(\tau)$ and $M_{S^n}(\tau)$ (of the class of the class $L$-$P$-I) by the rule more complicated than (3.4) - (3.5). So, we can not conclude in the above described way that the zeros of the polynomials $M_{C^n}(\tau)$ and $M_{S^n}(\tau)$ are negative.

OPEN PROBLEMS.
1. Are all zeros of polynomials $M_{C^n}(\tau)$ and $M_{S^n}(\tau)$ negative?
2. Are all zeros of these polynomials located in the left half plane?
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