A note on the thermodynamics of ‘little string’ theory

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Abstract

We study the thermodynamics of the D1-D5 system on a five-torus, focussing on the roles of different scales. One can take a decoupling limit such that the tension of the ‘little string’ inside the fivebrane remains finite and the physics is 5+1 dimensional. The dual black geometry exhibits a boosted Hagedorn phase, as well as a phase describing a boosted fivebrane gas. The dependence on the boost yields information about the nature of the fivebrane modes and their interactions. In particular, the form of the equations of state suggests a description in terms of $k = Q_1Q_5$ degrees of freedom, which may lead to an explanation of the $Q_5^3$ growth in the fivebrane density of states below the Hagedorn transition.

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Introduction: Maldacena’s conjecture \[1\] has led to a systematic exploration of strongly coupled dynamics of non-gravitational theories. Zero temperature physics was studied in \[2, 3\]; some thermodynamic aspects were considered in \[4, 5\]; the phase diagrams of super Yang-Mills (SYM) and little string theories were charted in \[6, 7, 8\]. The reader is referred to \[9\] for a more complete list of references. In this note, we extend our previous analysis \[8\] of the D1-D5 system to demonstrate interesting features that are scaled away in the standard Maldacena limit of this system \[8\].

The black geometry of a configuration of $Q_1$ D1 branes and $Q_5$ D5 branes is dual to a non-gravitational theory with sixteen supercharges, a theory of little strings propagating in six dimensions. The theory is non-local on a scale set by the little string tension. We parameterize the phase structure of this theory as in \[8\] using some of the D1-D5 moduli: The six dimensional string coupling $g_6$; the cycle size $R$ on which the $Q_1$ D1 branes are wrapped; the string tension $\alpha'$ of the IIB theory; and the volume of the additional (square) four-torus on which the D5 branes are wrapped, denoted by $V_4 \equiv v\alpha'^2$. The little string tension is then given by the ’t Hooft coupling of the D5 branes

$$T_{ls} = \frac{1}{2\pi\alpha'_\text{eff}} = \frac{1}{2\pi Q_5 g_6 v^{1/2}\alpha'} = \frac{(2\pi)^2}{Q_5 (g_{YM}^{(D5)})^2},$$

which is the energy cost per unit length of instanton strings in the 5+1 gauge theory. It was proposed in \[10, 11\] that the limit of decoupling from the gravitational bulk corresponds to an energy regime set by this tension

$$\alpha' \to 0, \text{ with } g_6, R, \text{ and } v^{1/2}\alpha' (\text{hence } \alpha'_\text{eff}) \text{ held fixed.}$$

Thermodynamic phase diagrams of the theory in this energy regime and with $Q_1 = 0$ were studied in \[8\] in the context of D4 and D5 branes. An alternative limit, proposed in \[1\], was used in \[8\] for the D1-D5 system

$$\alpha' \to 0, \text{ with } g_6, R, \text{ and } v \text{ held fixed.}$$

In this limit, the little string tension is scaled away, $\alpha'_\text{eff} \to 0$, and the system exhibits 1+1d conformal symmetry. Yet a third energy regime can be identified, corresponding to that of Matrix theory \[12\] in the presence of a longitudinal fivebrane

$$\alpha' \to 0, \text{ with } g_6, R, \text{ and } v^{1/2}/\alpha' \text{ held fixed,}$$

\[3\]This line of thought was instigated by questions and comments from O. Aharony in regard to our previous paper \[8\].
which implies the ‘DLCQ’ limit

\[ l_{\text{pl}} \to 0, \quad \text{with} \quad \frac{l_{\text{pl}}^2}{R_{11}}, \quad \frac{l_{\text{pl}}}{R}, \quad \text{and} \quad \frac{l_{\text{pl}}^4}{V_4} \text{ held fixed}, \tag{5} \]

where \( l_{\text{pl}}, V_4, R \) and \( R_{11} \) are moduli in a description dual to the D1-D5 frame, via T-duality on the circle \( R \) and lifting to M-theory. This limit also sends the little string tension to infinity.

We encounter these three apparently different energy regimes because this system has two charges, and consequently two energy scales which can be taken as the appropriate powers of the SYM couplings \( g_{YM}^{(D1)} \) and \( g_{YM}^{(D5)} \) of the two sets of branes. The three regimes differ only in how the volume of the four-torus \( V_4 \) is arranged with respect to the string scale: in (2), \( V_4 \) is held fixed (i.e. \( V_4 \gg \alpha'^2 \)); \( V_4 \sim \alpha'^2 \) in the second case (3); and \( V_4 \sim 1/\alpha'^2 \) in the third scenario (4) (i.e. \( V_4 \ll \alpha'^2 \)). We discuss in detail the energy regime (2), where the little string tension is held fixed; we will then comment only briefly on the roles of the other two regimes, as no new physics arises in these cases (for a discussion of the regime (3), the reader may also consult [8]). It is useful to define the effective dual circle radius

\[ \hat{R} = \frac{\alpha'_{\text{eff}}}{R} \tag{6} \]

and the effective transverse box size (transverse volume per instanton string)

\[ \hat{V} = \left( \frac{Q_5 V_4}{Q_1} \right)^{1/4}, \tag{7} \]

since these are the dimensionful quantities that, together with \( \alpha'_{\text{eff}} \), parametrize all the strong coupling equations of state. We will treat the little string tension \( \alpha'_{\text{eff}} \) as the basic scale in the theory, referring all other dimensionful quantities (such as the cycle sizes \( \hat{R} \) and \( \hat{V} \)) to this scale. Phase diagrams are plotted for fixed \( \hat{R}/\alpha'_{\text{eff}}^{1/2} \) as a function of \( S \) and \( \hat{V}/\alpha'_{\text{eff}}^{1/2} \). The phase diagram differs qualitatively for \( \hat{R} \gg \alpha'_{\text{eff}}^{1/2} \) and \( \hat{R} \ll \alpha'_{\text{eff}}^{1/2} \). We use the same notation as in [8], in particular \( k \equiv Q_1 Q_5 \) and \( q = Q_1/Q_5 > 1 \).

**Phase diagram for \( \hat{R} \ll \alpha'_{\text{eff}}^{1/2} \):** Figure [4] shows the thermodynamic phase diagram of the theory; the reader is referred to [7], [8] for details on how to construct such a diagram. Consider the full form of the equation of state for the D1-D5 black geometry. The energy above extremality is given by

\[ E = \frac{R}{2g_6^2 \alpha'^2} \left( 1 + h_1 + h_5 \right) - M_{\text{BPS}}, \tag{8} \]
Figure 1: The thermodynamic phase diagram of the little string theory for $\hat{R} \ll \alpha'^{1/2}$ and $q = Q_1/Q_5 \gg 1$; we have defined $k \equiv Q_1 Q_5$, and consider the energy regime of Equation (2). The horizontal axis can be thought of either as the six-dimensional string coupling $g_6$, or as the effective transverse box size $\tilde{V}/\alpha'^{1/2} = (g_6 \sqrt{k})^{-1/2}$. The thermodynamic phases described by black geometries are labelled as follows: D1-D5 for the black D1-D5 system; and M5W for boosted five branes whose horizon is localized on a transverse circle. The Hagedorn transition of the little string theory is the horizon delocalization transition from the viewpoint of black geometry.
where $r_0$ is the location of the horizon, $M_{BPS}$ is the BPS mass of the system

$$M_{BPS} = \frac{R}{2g_6^2\alpha'^2} \left( \rho_1^2 + \rho_5^2 \right), \quad (9)$$

and

$$h_{1,5} \equiv \left( 1 + \frac{\rho_{1,5}^4}{r_0^4} \right)^{1/2},$$
$$\rho_1^2 \equiv \frac{2g_6\alpha'}{\nu^{1/2}} Q_1,$$
$$\rho_5^2 \equiv 2g_6\nu^{1/2}\alpha' Q_5. \quad (10)$$

The entropy is given by

$$S = \frac{2\pi R}{2g_6^2\alpha'^2} r_0^2 (1 + h_1)^{1/2} (1 + h_5)^{1/2}. \quad (11)$$

Given that the metric in the string frame has the standard relation between the time and radial components $g_{tt} = 1/g_{rr}$, the UV-IR relation is $E \propto r_0/\alpha'$. We then consider in (8) and (11) the energy regime (2) while holding $r_0/\alpha'$ fixed; this changes only the function $h_5$ associated to the fivebranes – the hierarchy of scales is $\rho_5 \gg r_0, \rho_1$, and in the scaling limit one drops the constant term in $h_5$. Equations (8) and (11) determine the energy above extremality as a function of the entropy. It takes the form of a relativistic dispersion relation

$$E = -p_\parallel + \sqrt{\left( S/2\pi \alpha'_{\text{eff}} \right)^{1/2} + p_\parallel^2}, \quad (12)$$

with the longitudinal momentum

$$p_\parallel = \frac{k}{\tilde{R}}. \quad (13)$$

in other words, the system can be interpreted as having $k$ units of longitudinal momentum on a (dual) circle of size $\tilde{R} = \alpha'_{\text{eff}}/R$. As evident from equation (12), the system behaves as a canonically boosted Hagedorn gas, with the invariant mass of thermal excitations given by

$$M = \frac{S}{2\pi \alpha'_{\text{eff}}^{1/2}}, \quad (14)$$

which is that of a string with tension $T_s$. The Hagedorn temperature is given by the effective string scale $T_{Hag} \sim 1/\alpha'_{\text{eff}}^{1/2}$. From the geometrical side, the issue of whether the boost or the
rest mass dominates correlates directly to the horizon radius being much smaller or much larger than the one-brane charge radius $\rho_1$.

The bulk phase labelled D1-D5 on Figure 1 and bounded by the solid lines is described by equation (12). Within this phase, we have two asymptotic regimes corresponding to a boosted Hagedorn phase at low entropies and a rest frame Hagedorn phase at high entropies. The wide dashed line running through this phase denotes the crossover at

$$S \sim \alpha'_{\text{eff}} p \parallel \sim k \frac{\alpha'_{\text{eff}}}{R}$$

between these two regimes. The standard decoupling limit (3) corresponds to sending this curve to infinity, scaling out the rest frame Hagedorn region. The transition curve (13) meets the vertical correspondence curve at $g_6 \sqrt{k} = \alpha'_{\text{eff}}/V^2 \sim 1$ provided $\hat{R} \ll \alpha'_{\text{eff}}$, as can be seen from Figure 1. This phase diagram is then valid for $\hat{R} \ll \alpha'_{\text{eff}}$. As we will show in the next section, for $\hat{R} \gg \alpha'_{\text{eff}}$, the scaling of the localization curve starts to change, and new phase structure emerges.

The upper right corner of the phase diagram is dominated by weakly coupled gases. For increasing $S > k$ and sufficiently large $V_4$, we first encounter a weakly coupled five-dimensional gas of $Q_5^2$ degrees of freedom with equation of state scaling as

$$E \sim \frac{S^{5/4}}{(Q_5^2 V_4)^{1/4}}.$$  

Further increasing $S$, the temperature eventually reaches $T \sim 1/R$, at which point the gas dynamics becomes six-dimensional

$$E \sim \frac{S^{6/5}}{(Q_5^2 RV_4)^{1/5}}.$$  

The lower left corner of the weakly-coupled gas regime consists of a two dimensional gas of $k$ degrees of freedom on a circle of size $R$ with the energy scaling as $E \sim S^2/kR$ (i.e. as in (12) with the boost dominating over the entropy). The boundary between the six- and two-dimensional gas phases can be found by minimizing the energies between (12) and (17) in the boost-dominated regime of (12). The result is

$$S \sim k \frac{\alpha'_{\text{eff}}}{RV}.$$  

4Recall that a $d+1$ dimensional weakly coupled gas has the equation of state

$$E \sim S^{d+1/d} (c V_d)^{-1/d},$$

where $c$ is the number of degrees of freedom and $V_d$ is the spatial volume.
Finally, the correspondence curve that marks the boundary between the Hagedorn regime and the six dimensional gas is found to scale as

$$S \sim k \frac{\tilde{V}^4}{R_{\alpha'}^{1/3/2}}.$$  \hfill (19)

This can also be determined by minimizing the energies between (12) and (17) in the entropy-dominated regime of (12).

At sufficiently low entropies ($S \ll k\tilde{V}^2/\alpha'_\text{eff}$), the appropriate duality frame for the near-horizon geometry is that of a boosted M5-brane; there is a circle of size $\alpha'/R$ transverse to the M5-branes, and the horizon eventually localizes along this circle to the phase we have labelled M5W (see [8]). This localization transition is the geometrical manifestation of the Hagedorn transition. Below, we will find an attractive interpretation of the equation of state (see equation (27) and subsequent discussion). This strong coupling phase transition curve meets the correspondence curve at $\tilde{V} \sim \alpha'_{\text{eff}}^{1/2}$, and $S \sim k$. The correspondence curve then continues horizontally at $S \sim k$ toward weak coupling, marking the boundary between the M5W phase and weakly coupled phases.

As mentioned above, the Maldacena limit (3) arises from the little string limit upon a further scaling that sends $\rho_1$ to infinity relative to $r_0$, which scales away the rest frame Hagedorn/6d gas/5d gas region in Figure 1 (compare to Figure 4 of [8]). The third (DLCQ) energy regime (4) of longitudinal five-branes in Matrix theory is related to the little string limit (2) as follows. The phase diagram of Figure 1 assumes that $q \gg 1$, i.e. $Q_1 \gg Q_5$. As we lower $q$, at $q \sim 1$, the four-torus, as measured at the horizon in the D1-D5 phase, becomes string scale (see [8] for the details); a T-duality on this torus is required to go beyond this point. The new thermodynamic vacuum is again that of a D1-D5 system with some of the parameters modified

$$v \rightarrow \frac{1}{v}, \quad q \rightarrow \frac{1}{q} \quad \text{with} \quad \alpha', \ g_6, \ R, \ \text{and} \ k \ \text{left unchanged.}$$  \hfill (20)

$Q_1$ and $Q_5$ are interchanged, while the four-torus in string units gets inverted. We then have

$$v^{1/2}\alpha' \rightarrow \frac{\alpha'}{v^{1/2}},$$  \hfill (21)

i.e. the limit (2) where one holds the little string tension fixed, is exchanged with the limit (4) of Matrix theory; $Q_5$ is now interpreted as the boost. In the limit, one still finds the little string theory; one is merely reinterpreting its parameters (as is usual in Matrix theory). In the new variables, the fivebrane function $h_5$ of (13) is not simplified as before, rather the hierarchy of distance scales is $\rho_1 \gg \rho_5, r_0$, and it is the onebrane function $h_1$ that loses
its constant term in the scaling limit. The interchange of one-brane and five-brane charges leads (after a further S-duality) to a decoupled little string theory whose tension is set by the NS5-brane before the map (20).

**Phase diagram for $\tilde{R} \gg \alpha'_{\text{eff}}^{1/2}$:** In this section, we chart the phase diagram for the regime $\tilde{R} \gg \alpha'_{\text{eff}}^{1/2}$, where the crossover infringes upon the localization transition. The diagram is shown in Figure (4). The structure of the smeared phase describing a boosted Hagedorn gas is exactly as before; new structure appears in the localized phase. The energy above extremality in the localized M5W region in the little string limit (2) is given by

$$e = y^3 \left( \frac{2}{3} - \frac{z^3}{y^3} + \left( 1 + \frac{z^6}{y^6} \right)^{1/2} \right), \quad (22)$$

with the definitions

$$e \equiv \frac{(2\pi)^6 g_6^6}{R^2} E, \quad (23)$$

$$z^3 \equiv (2\pi)^6 Q_1 \frac{g_6}{R v^{1/2} \alpha'}, \quad y \equiv \frac{r_0}{\alpha'}.$$

The entropy is given by

$$s = y^{5/2} \left( 1 + \left( 1 + \frac{z^6}{y^6} \right)^{1/2} \right)^{1/2}, \quad (24)$$

in terms of the quantity

$$s \equiv \frac{3}{2} (2\pi)^{9/2} \frac{g_6^2}{R^{3/2} \alpha'_{\text{eff}}^{1/2}} S. \quad (25)$$

The equation of state above the crossover at $s \sim 1$ takes the form

$$E \sim S^{6/5} \left( \frac{R}{g_6 Q_5^3 V_4^{3/2}} \right)^{1/5} = S^{6/5} \left( \frac{1}{k R V^4} \right)^{1/5}. \quad (26)$$

We will call this phase the $M5\text{ gas}$. In the first form of the equation of state, the scaling appears to be that of a six dimensional gas with $O(Q_5^3)$ degrees of freedom. However, the second way of parametrizing this energy suggests a somewhat more conventional number of degrees of freedom $k$, in a theory whose effective string tension/coupling (11) is rescaled by $Q_5$; the longitudinal circle has effective size $\tilde{R}$ (c.f. equation (6)), and the transverse torus...
Figure 2: The thermodynamic phase diagram of the little string theory for $\tilde{R} \gg \alpha_{\text{eff}}^{1/2}$ and $q \gg 1$. 
has effective size $\tilde{V}$ (c.f. equation (7)). The equation of state (22), (24) below the crossover at $s \sim 1$ reduces to

$$E \sim \frac{g_6}{k^{3/2}} \tilde{R}^3 \frac{S^3}{p_{||} \left[ \frac{S^{3/2}}{(k\tilde{V}^2)^{1/2}} \right]^2},$$

(27)

having the appearance of a weakly coupled 2+1d gas which has been boosted such that infinite momentum frame kinematics applies. This gas also has of order $k$ degrees of freedom and lives in a box of linear dimension $\tilde{V}$. Note that, unlike the smeared phase of equation (12), the M5W phase localized on the transverse cycle does not boost canonically.

The correspondence point for the M5 gas phase, i.e. the point where the curvature scale at the horizon becomes string scale, is given by

$$S \sim k \left( \frac{\tilde{V}}{\tilde{R}} \right)^4, \quad T \sim \frac{1}{\tilde{R}};$$

(28)

one again meets the scale of the dual circle $\tilde{R} = \alpha'_\text{eff}/R$. This curve meets the crossover transition curve at $S \sim k$; beyond that, the correspondence point becomes the line $S \sim k$ as before. The phase on the other side of this correspondence boundary is that of the five dimensional weakly coupled gas of $Q_5^2$ degrees of freedom, with equation of state scaling as (16). Minimizing the energy (24) with respect to (16) yields the transition curve (28). At temperatures less than $T \sim (q/V_4)^{1/4} = \tilde{V}^{-1}$, the five dimensional gas freezes its dynamics on the four-torus at $S \sim k$ as in the previous case $\tilde{R} \ll \alpha'_\text{eff}$. The effective coupling is dressed by the size of the torus and becomes of order one at $S \sim k$, while the effective volume of the four-torus appears again dressed by $q = Q_1/Q_5$. In some cases, such effective box sizes can be interpreted as resulting from the typical holonomies generated by the dynamics [13, 14]. It would be interesting to find a physical mechanism for the appearance of the factor of $q^{1/4}$ in $\tilde{V}$.

The crossover transition from (24) to (27) appears as an extension of the transition in which the weakly coupled gas freezes into its zero-modes; moreover, in the low-entropy M5W phase, there is no transition as we move toward strong coupling while staying below the crossover. This suggests that the strong-coupling crossover in the localized phase signals an analogous transition, in which the M5 gas freezes into zero-mode excitations of the fivebranes. Assuming infinite momentum frame kinematics, the invariant mass of these excitations is read off equation (27) and yields the equation of state of a 2+1d gas:

$$M \sim S^{3/2} \left( \frac{1}{k\tilde{V}^2} \right)^{1/2}.$$

(29)

It may be that the system can find the largest amount of available phase space by first creating a membrane embedded in the fivebrane along the transverse cycles; and then populating
that membrane with a gas of quasi-particle excitations. It is interesting that the LC matrix string which dominates the effective dynamics in the smeared phase, appears to become a LC matrix membrane in the localized phase. Of course, the LC string of the former phase is a membrane wrapped over the circle of the compactification transverse to the M5-brane (in the M-theory duality frame appropriate to this regime); one might imagine that the energetics requires this membrane to transfer its winding to cycles along the M5-brane as the entropy is lowered. What is seen as a transition between LC and rest frame kinematics of a Hagedorn string in the smeared phase, is seen in the localized phase as a transition between a six dimensional gas of $k$ degrees of freedom in a box of size $\tilde{R}\tilde{V}^4$ and a boosted three dimensional gas of $k$ degrees of freedom in a box of size $\tilde{V}^2$.

Motivated by the form of the equation of state having the appearance of a system with $k$ degrees of freedom, we can summarize the scaling of the various transition curves labelled on Figure 2 in terms of the entropy per degree of freedom $S/k$ in the system. Minimizing the free energy among the various equations of state we encounter on this diagram verifies the various phase transition curves determined from geometrical considerations (such as the Gregory-Laflamme localization transition and the correspondence principle):

- **Curve A:** $\frac{S}{k} \sim \left( \frac{\tilde{R}\tilde{V}^4}{\tilde{\alpha}'^{5/2}} \right)$.
- **Curve B:** $\frac{S}{k} \sim \left( \frac{\tilde{V}^2}{\tilde{\alpha}'_{\text{eff}}} \right)$.
- **Curve C:** $\frac{S}{k} \sim \left( \frac{\tilde{\alpha}'_{\text{eff}}^{1/2}}{\tilde{R}} \right)$.
- **Curve D:** $\frac{S}{k} \sim \left( \frac{\tilde{V}}{\tilde{R}} \right)^{2/3}$.
- **Curve E:** $\frac{S}{k} \sim \left( \frac{\tilde{V}}{\tilde{R}} \right)^4$.

The choice of parametrization of the thermodynamics at strong coupling in terms of the five independent variables $S$, $\tilde{R}$, $\tilde{V}$, $\tilde{\alpha}'_{\text{eff}}$, and $k$ is more than a mere reshuffling of the original parameters $S$, $R$, $V_4$, $g_6$, $k$, and $q$, which are six in number (the seventh parameter, $\alpha'$, is scaled out in the decoupling limit); the ratio $q = Q_1/Q_5$ has disappeared from the strong coupling equations of state (12), (26), and (27). This reduction of parameters indicates that we have found the proper interpretation of the thermodynamic data. Furthermore, the uniform linear scaling of the transition curves in $S$ with respect to $k$ (see also (13)) supports the interpretation of $k$ as the number of degrees of freedom in the system. The fact that
these curves are parametrized by dimensionless cycle sizes $\tilde{R}/\alpha_{\text{eff}}^{1/2}$ and $\tilde{V}/\alpha_{\text{eff}}^{1/2}$, with no other dependence on the combinations of charges $q$ or $k$, indicates a nontrivial scaling in the thermodynamics.

We hope this interpretation of the thermodynamics leads to further progress in understanding the dynamics of fivebranes. There seems to be a remarkable parallel with the Matrix model of M-theory. It is hard to understand what are the appropriate degrees of freedom to use in formulating M-theory in flat space; adding charge to the system in the form of $N$ units of momentum along a circle allows a truncation of the dynamics of the theory to $N^2$ degrees of freedom (the D0 brane sector). In the case of a system of $Q_5$ fivebranes, it is hard to understand how the apparently $O(Q_5^3)$ degrees of freedom originate. Again adding another charge to the system in the form of $Q_1$ units of momentum, one achieves an effective description of the system in terms of $O(k = Q_1Q_5)$ degrees of freedom.

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\footnote{In the duality frame appropriate to the M5 gas, $Q_1$ is a momentum rather than a winding quantum as in the original D1-D5 duality frame.}
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