Testing for the mean shifts for long memory time series in presence of breaks in variance

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Abstract. This paper considers the detection problem of mean shifts in presence of variance changes for long memory time series. In performing the test, we employ the Ratio test introduced by Horváth L. et al. (2008). Given for the case of breaks in variance, the asymptotic properties of the tests are established. It is shown that the test statistics are not robust due to neglected of variance changes. And the numerical simulation results are consistent with our theoretical analysis.

1. Introduction

The presence of structural breaks in key macroeconomic and financial variables appears to be relatively common because a myriad of political and economic factors can cause these relationships among economic variables to change over time. So the change point studies is an important part of statistics, economic and financial analysis. Since the paper of Page [1], a vast amount of relevant articles have appeared, especially about the mean and variance shifts. Shao [2] showed a testing procedure on breaks in mean in long-range dependent series. Jin [3] extend the CUSUM test for mean shifts with heavy-tailed innovations. Related contributions include Hwang [4], Kokoszka [5], Teverosky [6], and the articles cited therein.

Most of the work has focused on the case in which these variables maintain mean constancy and involve variance breaks in statistic and econometric literatures. And the problem of testing for variance breaks has become an important issue since variance is often interpreted as a risk in econometrics. Therefore, it is significant work to study the effect of the variance changes. Jin [7] investigated the problem of testing for variance changes in the linear autoregressive processes including AR (p) processes meanwhile auto-parameters shifts occur. Horváth [8] considered testing for change points in unconditional variance in the case of a conditionally heteroskedastic time series. For more details about research on variance changes detection, see Steinebach [9], Wang [10], Zhao [11], and the articles cited therein.

However, all of the works above are concentrated on the case where the series only exists shifts in mean or variance. But it is more that the economic and financial time series exist change points both in variance and mean and there has been less work done to procedures that are specially designed to detect breaks in mean in presence of variance changes. Bai [12] established the consistency of the estimated common shifts in mean and variance for panel data and derived the limiting distribution for the estimated breaks. However, these common breaks are imposed to occur at the same position.
Therefore, it is urgent to investigate the effects of variance changes on testing for non-constant mean, while the data of shifts in mean and variance do not essentially require to simultaneously happen.

It is worth mentioning that most of the papers above studied short memory or other processes. But long memory time series appears frequently in various areas such as hydrology, economics and telecommunications. It is apt to be applied to time series that extend over a long period of time, in which circumstances the possibility of structural breaks is likely to be entertained. Horváth [13] examined the effect of long memory on the change-point estimators for the mean shift Gaussian long memory model. Wang [14] investigated the local linear regression estimation for the nonparametric regression models with locally stationary long memory errors. For more details about research on long memory processes, see, Beran [15], Doukhan [16]. It is no doubt that there is a growing need for a thorough analysis of the statistical methodologies based on long memory observations.

In this paper we check the effect of variance shifts on testing for mean changes under long memory observations. The limiting distribution of the Ratio tests are derived under the null and alternative hypothesis. Then, the results show that variances changes can be easily confused with detection for breaks in mean, the test no longer robust. Finally, simulation results are consistent with our theoretical analysis.

The paper is organized at follow. Section 2 introduces the detail of data generating process, the hypothesis and the ratio of the CUSUM test statistic. Assuming existence of variance changes, the asymptotic theory for the test statistic is developed in Section 3. The finite sample properties of the tests are explored through Monte Carlo simulation in Section 4. Section 5 concludes the paper. All proofs are given in the Appendix.

2. Model and Test
In order to analyze the effect of variance shifts on testing for mean changes, we start by considering the following model

$$X_t = \begin{cases} 
\mu_1 + \sigma_t e_t^d, & 1 \leq t \leq [\lambda T] \\
\mu_2 + \sigma_t e_t^d, & [\lambda T] < t \leq T 
\end{cases} \tag{1}$$

Where

$$e_t^d = (1 - L)^d \varepsilon_t, \quad \sigma_t = \sigma_1 \mathbb{1}_{[t \leq [\lambda T]]} + \sigma_2 \mathbb{1}_{[t > [\lambda T]]}. \tag{2}$$

Since \(\{\varepsilon_t\}\) is a sequence of independent and identically distributed (i.i.d.) random variables with zero mean and variance one. And the sequence \(\{e_t^d\}\) defined in (2) is a linear (moving average) stationary sequence with the long memory index \(d\). The notation \(\lfloor \cdot \rfloor\) denotes the largest integer less than equal to its argument and \(1_{[\cdot]}\) is an indicator function. In this time series \(\{X_t\}_{t=1}^T\) has a mean shift at \(\lambda \in [0,1]\) and a variance change at \(\tau \in [0,1]\).

The aim of the paper is check whether the mean of the observation has changed at an unknown time in presence of a variance shift for long memory time series, so there is a variance change in the whole series, i.e. \(\sigma_1 \neq \sigma_2\). And the null hypothesis \(H_0\) is that there is no break in mean throughout the sample period

$$H_0: \{X_t\} \text{ is a sample with } \mu_1 = \mu_2$$

against the alternative hypothesis that the time series exist a mean change

$$H_1: \{X_t\} \text{ is a sample with } \mu_1 \neq \mu_2.\tag{3}$$

About the test statistics, Horváth [17] develop tests which compute the ratio of the CUSUM functions. Then our test statistics is defined by

$$V_T = \max_{T\delta \leq k \leq T-\delta} \max_{0 \leq j \leq \delta} \frac{\sum_{k} |\sum_{j=1}^{k} (X_j - \tilde{X}_k)|}{\sum_{k} |\sum_{j=1}^{k} (X_j - \tilde{X}_k)|}, \tag{4}$$

where \(0 < \delta < 1/2\) and \(\tilde{X}_k = \frac{1}{k} \sum_{i=1}^{k} X_i, \ \bar{X}_k = \frac{1}{T-k} \sum_{i=k}^{T} X_i.\)
3. Main Results

In this section we discuss the impact of changing innovation variances on asymptotic distributions of $V_T$ statistic. We derive representations for the asymptotic (null) distributions of the mean change test under $H_0$ and analyze the large sample behavior under alternative $H_1$. The following theorem collects their limiting behavior under the null hypothesis of no mean shifts.

**Theorem 1.** Assume that $H_0$ hold, if $0 < d < 1/2$ and $\epsilon_1 B_0 < \infty$, then as $T \to \infty$,
\[
V_T \overset{D^2[0,1]}{\to} \max_{0 < |\tau| < 1} \phi_{\psi}(t),
\]
where $\phi_{\psi}(t)$ is a function of Brownian motion, i.e. if $k_0 < k$, then $V_T \overset{D^2[0,1]}{\to} \phi_{\psi}(t)$, if $k_0 > k$, then
\[
V_T \overset{D^2[0,1]}{\to} \psi_2(t)
\]
Where
\[
\phi_1(t) = \max_{0 < t < s} \{ |B_d(s) + \left(1 - \frac{\sigma_2}{\sigma_1} \right) B_d(t) - \frac{\sigma_2}{\sigma_1} B_d(t)|, |B_d(s) + \left(1 - \frac{\sigma_2}{\sigma_1} \right) \left(1 - \frac{s}{t}\right) B_d(t) - \frac{\sigma_2}{\sigma_1} B_d(t)| \},
\]
\[
\psi_1(t) = \frac{\sigma_2}{\sigma_1} \max_{0 < t < s} |B_d(s) - \frac{s}{t} B_d(t)|,
\]
\[
\psi_2(t) = \max_{t < s} \left( |B_d(s) + \frac{s}{1-t} B_d(t) + \frac{\sigma_2^2}{\sigma_1^2} B_d^*(t)| + \frac{1-s}{1-t} B_d(t) \right),
\]
\[
\psi_3(t) = \max_{\tau \leq s} \left( |B_d(s) - \frac{s}{t} B_d(t)| + \frac{1-s}{1-t} B_d(t) \right),
\]
and $B_d^*(t) = B_d(1 - B_d(t))$. $B_d(t)$ is the a fractional Brownian motion, which is defined by the stochastic integral $B_d(t) \equiv \frac{1}{\Gamma(1-d)} \int_{0}^{t} \left( f^r(t-s) dB_0(s) + \frac{1}{\Gamma(1-d)} \int_{0}^{\infty} (t-s)^{d-1} (-t)^{d} d\Gamma(\cdot) \right)$ is Gamma function, and $B_0(t)$ is the standard Brownian motion, see Mandelbrot [18].

**Remark 1.** The most significant finding from Theorem 1 is that in presence of breaks in variance. The single variance shift occurring at time $k_0 = \lfloor \tau T \rfloor$, where $\tau$ represents the date of the variance break, i.e. the variance changes from $\sigma_1^2$ to $\sigma_2^2$ at $k_0$. We assume without loss of generality that $\mu_1 = \mu_2 = \mu$. It is easy to find that the asymptotic distributions of the test statistic are also the function of the alternative $H_1$, but it is more complicated.

**Remark 2.** As is discussed above, when the errors are homoskedastic, i.e. the GDP (1)-(2) with $\mu_1 = \mu_2 = \mu$ and $\sigma_2/\sigma_1 = 1, t = 1, \cdots, T$. As $T \to \infty$, then
\[
V_T \overset{D}{\to} \max_{0 < t < s} \left( \frac{1}{\Gamma(1-d)} \int_{0}^{t} \left( f^r(t-s) dB_0(s) + \frac{1}{\Gamma(1-d)} \int_{0}^{\infty} (t-s)^{d-1} (-t)^{d} d\Gamma(\cdot) \right) \right)
\]
It is no doubt that the limiting distribution is a simple fractional Brownian motion. And the result also has been given in Zhao [11]. Then, we analyze the large sample behavior under the alternative $H_1$ and give the asymptotic distributions of the test statistic as follows.

**Theorem 2.** Assume that $H_1$ hold, if $0 < d < 1/2$, and $\epsilon_1 B_0 < \infty$, then as $T \to \infty$, $V_T \overset{p}{\to} \infty$.

4. Monte Carlo simulation

In this section we use the Monte Carlo simulation methods to investigate the finite sample behavior of the tests for mean shift if series involves a break in variance. Representations for critical values from the limiting null distributions of the foregoing statistics in the constant unconditional volatility case, $\sigma_1 = \sigma_2$, for all $t$. For each scenario, we simulate the replications 2000 times and report empirical rejection frequencies of the tests with sample size $T = 100,250,500$ for tests run at 5% critical value in various combinations, where all the long memory index is 0.3.

Then, we present the percentage of rejections of the null hypothesis $H_0$, under which no break mean is assumed to happen and evaluate the size performance of the test statistics through a simulation.

In order to analyze the effects caused by variance change, we set $\sigma_1 = 1$ in all case without loss of
It is obvious that the existence of variance changes, which may be positive ($\delta > 1$) or negative ($\delta < 1$), are allowed and the reverse conclusions are obtain when positive volatility.

Table 1. Empirical size of the tests with/without change points in variance.

| $\delta$ | $T = 100$ | $T = 250$ | $T = 500$ |
|----------|-----------|-----------|-----------|
|          | $\tau=0.3$ | $\tau=0.5$ | $\tau=0.7$ | $\tau=0.3$ | $\tau=0.5$ | $\tau=0.7$ | $\tau=0.3$ | $\tau=0.5$ | $\tau=0.7$ |
| 4        | 0.0155     | 0.0130    | 0.0005    | 0.0235     | 0.0120    | 0.0010    | 0.0205     | 0.0105    | 0.0012    |
| 2        | 0.0230     | 0.0115    | 0.0020    | 0.0265     | 0.0105    | 0.0020    | 0.0280     | 0.0145    | 0.0020    |
| 1        | 0.0495     | —         | —         | 0.0520     | —         | —         | 0.0545     | —         | —         |
| 1/2      | 0.2380     | 0.3125    | 0.3920    | 0.2755     | 0.3540    | 0.4300    | 0.3025     | 0.3940    | 0.4910    |
| 1/4      | 0.7205     | 0.8350    | 0.8740    | 0.7680     | 0.8765    | 0.9040    | 0.7960     | 0.9093    | 0.9445    |

We now discuss the size conclusions in table 1 that can be drawn from our simulation. If $\delta = 1$, the test is much simple and our null specifies the time series with no variance change and mean shift, so the empirical size tends to significant level 5% as if the observations have constant variance. However, it is clear from the results that the size is very seriously affected by breaks in variance. The $V_T$ is grossly over-(under-) sized where $\delta < 1$ ($\delta > 1$). Other things equal, the size distortions from the nominal level are larger the further $\delta$ is from 1 (either above or below). This phenomenon is consistent with the conclusion of Theorem 1.

Then, we present the rejection power of the test under the alternative hypothesis (table 2). The ratio of standard deviation $\delta = \sigma_2/\sigma_1$ is same with previous description and the location among $\tau \in \{0.3, 0.5, 0.7\}$. Without loss of generality, about broken mean we set the abrupt magnitude $\Delta = 0.5$. The date $\lambda$ belongs to $\{0.3, 0.5, 0.7\}$, allowing for mean breaks which take place either towards the middle or the start and end of the sample. For save space, the qualitatively similar conclusions for other cases do not collect here, but they are available on request.

Table 2. Empirical power of the tests with/without variance changes ($\Delta = 0.5$).

| $\delta$ | $T = 100$ | $T = 250$ | $T = 500$ |
|----------|-----------|-----------|-----------|
|          | $\tau=0.3$ | $\tau=0.5$ | $\tau=0.7$ | $\tau=0.3$ | $\tau=0.5$ | $\tau=0.7$ | $\tau=0.3$ | $\tau=0.5$ | $\tau=0.7$ |
| 0.3      | 0.0265     | 0.0290    | 0.0295    | 0.0375     | 0.0410    | 0.0210    | 0.0495     | 0.0745    | 0.0240    |
| 4        | 0.0120     | 0.0135    | 0.0095    | 0.0150     | 0.0245    | 0.0135    | 0.0175     | 0.0305    | 0.0140    |
| 0.7      | 0.0020     | 0        | 0         | 0.0020     | 0.0020    | 0.0020    | 0.0025     | 0.0045    | 0.0035    |
| 0.3      | 0.0525     | 0.0580    | 0.0290    | 0.1240     | 0.1375    | 0.0465    | 0.2355     | 0.2535    | 0.0560    |
| 2        | 0.0370     | 0.0375    | 0.0190    | 0.0620     | 0.0935    | 0.0265    | 0.1610     | 0.2005    | 0.0415    |
| 0.7      | 0.0130     | 0.0170    | 0.0030    | 0.0435     | 0.0550    | 0.0085    | 0.1495     | 0.1505    | 0.0170    |
| 1        | 0.2465     | 0.2110    | 0.0855    | 0.5155     | 0.4720    | 0.1230    | 0.7930     | 0.7615    | 0.2370    |
| 0.3      | 0.7360     | 0.6660    | 0.3495    | 0.9355     | 0.9315    | 0.5350    | 0.9960     | 0.9965    | 0.7290    |
| 1/2      | 0.7520     | 0.7180    | 0.4245    | 0.9465     | 0.9465    | 0.6035    | 0.9960     | 0.9965    | 0.7780    |
| 0.7      | 0.7650     | 0.7380    | 0.5020    | 0.9345     | 0.9340    | 0.6545    | 0.9975     | 0.9945    | 0.8420    |
| 0.3      | 0.9670     | 0.9720    | 0.8175    | 0.9990     | 0.9990    | 0.9490    | 1          | 1         | 0.9930    |
| 1/4      | 0.5        | 0.9865    | 0.9840    | 0.9010     | 0.9990    | 0.9730    | 1          | 1         | 0.9950    |
| 0.7      | 0.9755     | 0.9770    | 0.9355    | 1          | 1         | 0.9840    | 1          | 1         | 0.9995    |

We will show the different story about the rejection power, allowing for mean breaks and variance changes occur in one series. As might be expected, the increased sample size can help our test obtain better rejection power. For example, when $\delta = 1/4$ and $\tau = \lambda = 0.5$, the rejection frequency are 98.4% and 100% for $T = 100$ and $T = 500$. However, it is obvious that the existence of variance changes has a great influence in the tests of mean shifts. The negative volatility lead to the obvious power gain, and the reverse conclusions are obtain when positive volatility. For mean magnitude $\Delta$, we can get the
result that the power must increase obviously as magnitude increased in many past works. The result is consistent with the conclusion of Theorem 2.

5. Conclusions
In this paper, we show that the Ratio test on a mean shift becomes bias when the data generating process has a break in variance. The size of the tests are grossly over-(under-) sized where \( \delta < 1 \) (\( \delta > 1 \)). And the size distortions from the nominal level are larger the further \( \delta \) is from 1 (either above or below). The power depends heavily on the location of the breaks in variance. If the standard deviation decreases, the case will gain in power, and the reverse conclusions are obtained if the deviation increase. At the same time, the increase of the mean magnitude can help us get better power. But we know our Ratio test for detecting mean shift is valid. Therefore, the null (alternative) without (with) mean shift is falsely rejected by the test due to the variance changes.

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Appendix
Proof of theorem 1
According to Theorem 2 of Avram and Taqqu [19],
\[
c_0^{-1} T^{-\frac{d}{2}(t+1)} \sum_{j=1}^{[Tt]} e_j \rightarrow B_d(t), \quad c_0^{-1} T^{-\frac{d}{2}(t+1)} \sum_{j=[Tt]+1}^{T} e_j \rightarrow B_d'(t), \quad t \in [0,1]
\]
where \( \rightarrow \) denotes weak convergence and \( B_d'(t) = B_d(1) - B_d(t) \). \( B_d \) is a fractional Brownian motion. Then, we assume without loss of generality that \( \mu_1 = \mu_2 = \mu \). And there is a single variance shift occurring at time \( k_0 = [Tt] \), i.e. the variance changes from \( \sigma_1^2 \) to \( \sigma_2^2 \) at \( k_0 \).

Let \( k_0 < k \), \( k_0 = [Tt] \), \( k = [Tt] \), and \( i = [Ts] \), then the limit distribution of numerator of \( V_T \) is
\[
c_0^{-1} T^{-\frac{d}{2}(t+1)} \cdot \max_{i \in [Ts]} \left| \sum_{j=1}^{i} X_j - \frac{i}{k} \sum_{j=1}^{k} X_j \right|
\]
\[= c_0^{-1}T^{-(\epsilon^d+1)} \max_{0 < \epsilon \leq k} \left\{ \sum_{j=1}^{k} \sigma_j e_j - \frac{i}{k} \sum_{j=1}^{k} \sigma_j e_j - \frac{i}{k} \sum_{j=k+1}^{k+1} \sigma_j e_j \right\}, \]

\[\geq c_0^{-1}T^{-(\epsilon^d+1)} \max_{0 < \epsilon \leq k} \left\{ \sum_{j=1}^{k} \sigma_j e_j + \sum_{j=k+1}^{k+1} \sigma_j e_j \right\}, \]

\[\to \max_{0 < \epsilon \leq 1} \left\{ |B_d(s) + (\sigma_1 - \sigma_2) \frac{s}{\epsilon} B_d(\tau) - \frac{s}{\epsilon} B_d(\tau)|, |B_d(s) + (\sigma_1 - \sigma_2) \left(1 - \frac{s}{\epsilon}\right) B_d(\tau) - \frac{s}{\epsilon} B_d(\tau)| \right\}. \]

\[= \sigma_1 * \phi_1(t). \]

Above the case, a variance shift appears in the first half, so there is no change in \(X_k, X_{k+1}, \ldots, X_T\), so the limit distribution of denominator of \(V_T\) is

\[c_0^{-1}T^{-(\epsilon^d+1)} \max_{k \leq T} \left| \sum_{j=1}^{k} X_j - \frac{n-k}{n-k} \sum_{j=k+1}^{T} X_j \right| \to \max_{t < \epsilon \leq 1} \left| \sigma_2 B_d(s) - \frac{1-s}{1-t} B_d(t) \right| = \sigma_1 * \psi_1(t). \]

So, the limit distribution of \(V_T\) is \(V_T^{b^{[0,1]} \phi_1(t)} \psi_1(t)\).

If the variance shift occurs in the second half, let \(k < k_0\), i.e. there is a variance change point in second half, but not in the first half. Similarly, the limit distribution of \(V_T\) is \(V_T^{b^{[0,1]} \phi_2(t)} \psi_2(t)\), where \(\phi_2(t) = \max_{t < \epsilon \leq 1} \left| B_d(s) - \frac{s}{\epsilon} B_d(\tau) \right|\),

\[\psi_2(t) = \max_{t < \epsilon \leq 1} \left\{ \left( B_d(s) + \frac{s-t}{1-t} \left( B_d(\tau) + \frac{\sigma_2}{\sigma_1} B_d^*(\tau) \right) \right) - \frac{1-s}{1-t} B_d(t) \right\}. \]

Hence, the limit distribution of \(V_T\) is \(V_T^{b^{[0,1]} \phi_2(t)} \psi_2(t)\), The proof is complete.

**Proof of theorem 2**

Let \(k_1 = \lfloor T \lambda \rfloor\), \(k_0 = \lfloor T \tau \rfloor\), and \(k = \lfloor T t \rfloor\), then the limit distribution of numerator of \(V_T\) is

\[c_0^{-1}T^{-(\epsilon^d+1)} \max_{k \leq T} \left| \sum_{j=1}^{k} X_j - \frac{k_1}{k} \sum_{j=k+1}^{T} X_j \right| \geq c_0^{-1}T^{-(\epsilon^d+1)} \max_{k \leq T} \left| \sum_{j=1}^{k} X_j - \frac{k_1}{k} \sum_{j=1}^{k} X_j \right|, \]

\[\geq c_0^{-1}T^{-(\epsilon^d+1)} \max_{k \leq T} \left| \sum_{j=1}^{k} (\mu_j + \sigma_j e_j) - \frac{k_1}{k} \sum_{j=1}^{k} X_j \right|. \]

If \(0 < k_1 < k_0 < k\), then

\[c_0^{-1}T^{-(\epsilon^d+1)} \max_{k \leq T} \left| \sum_{j=1}^{k} (\mu_j + \sigma_j e_j) - \frac{k_1}{k} \sum_{j=1}^{k} X_j \right|. \]

\[= O_p(1) + \frac{k_1(k-k_1)}{k} (\mu_1 - \mu_2) c_0^{-1}T^{(\epsilon^d+1)}. \]

Similarly, if \(0 < k_0 < k_1 < k\), then

\[c_0^{-1}T^{-(\epsilon^d+1)} \max_{k \leq T} \left| \sum_{j=1}^{k} (\mu_j + \sigma_j e_j) - \frac{k_1}{k} \sum_{j=1}^{k} X_j \right|. \]

In the limit distribution of numerator of \(V_T\) is \(V_T \to 0\).