Quantum knots

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Abstract

We construct an exactly solvable example of Sturmian bound states which exist in the absence of any confining potential. Their origin is topological – these states are found to live on certain “knotted” contours $C(N)$ of complexified coordinates.

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1 Introduction

A physical framework and motivation of our forthcoming considerations lies partially (though not only) in the standard radial Schrödinger equation

\[- \frac{d^2}{d\xi^2} \psi(\xi) + \frac{\ell(\ell + 1)}{\xi^2} \psi(\xi) + \lambda V(\xi^2) \psi(\xi) = E \psi(\xi)\]  (1)

where the radial coordinate \( \xi \) runs over the half-axis \( \mathbb{R}^+ \) and where the standard Dirichlet boundary conditions are usually imposed at \( \xi = \infty \) and in the origin (at \( \xi = 0 \), with the well known exceptions for strongly singular \( V(\xi^2) \) [1]). In phenomenological setting, such an ordinary differential equation is usually obtained from a “realistic”, spherically symmetric \( D \)-dimensional single-particle Hamiltonian \( \tilde{H} = -\Delta + \lambda V(|\vec{x}|^2) \) acting in the most common representation \( L^2(\mathbb{R}^D) \) of the Hilbert space of bound states. With \( \ell = (D - 3)/2 + m \) in the \( m \)-th partial wave, one has to distinguish between \( D = 1 \) and \( D > 1 \) [2]. At \( D = 1 \) the situation is exceptional and a due care is needed when one tries to work, purely formally, with \( m = 0 \) (for the even-parity states) and \( m = 1 \) (for the odd-parity states). At all the higher dimensions \( D \geq 2 \), the correspondence between \( \tilde{H} \) and eq. (1) is more standard and the sequence of the angular-momentum indices becomes infinite, \( m = 0, 1, \ldots \). In practice, one usually works with a fixed strength \( \lambda = 1 \) of the interaction and studies the spectrum of the bound-state energies \( E_n, n = 0, 1, \ldots \). Alternatively, one can choose and fix the energy (say, \( E = 1 \)) and compute the related eigencouplings \( \lambda_n, n = 0, 1, \ldots \) which correspond to the normalizable solutions \( \psi(\xi) \) called Sturmians (this will also be our choice in what follows).

An immediate mathematical inspiration of our present note can be traced back to the 1993 paper by Buslaev and Grecchi [2] who complexified, purely formally, the variable \( \xi \) in eq. (1) (for more details cf. section 2 below). From the historical point of view it proved a bit unfortunate that the Buslaev’s and Grecchi’s considerations did not attract too much attention. It took further five years before Bender, Milton and Boettcher [3] returned to the subject, revealed and emphasized its formal appeal and persuaded many physicists about many phenomenological potentialities hidden in similar models. In this setting, our present brief note can be read as a part and
a continuation of the deeper analysis of the redefined models (1) using complex $\xi$ which, strictly speaking, shouldn’t be called a “particle coordinate” anymore [4].

We shall study the maximally simplified, analytically solvable version of the Schrödinger differential equation without any interaction,

$$-\frac{d^2}{d\xi^2}\psi(\xi) + \frac{\ell(\ell + 1)}{\xi^2}\psi(\xi) = E\psi(\xi).$$

(2)

For compensation, the complexification of coordinates will be assumed more sophisticated than usual. In the formal definition of $\xi \in \mathcal{C}^{(N)}$, the complex contours $\mathcal{C}^{(N)}$ will be specified as highly unusual and topologically nontrivial (cf. section 3 below). As a consequence, we shall be able to obtain bound states by imposing the corresponding more or less standard complexified asymptotic boundary conditions (cf. a broader context outlined in refs. [3] and/or [5, 6, 7]). The simplicity of the dynamics encoded in eq. (2) will enable us to construct our bound-state solutions in closed analytic form (cf. section 4). A more detailed discussion concerning the interpretation and perspectives of applicability of our $\psi(\xi) \in L^2(\mathcal{C}^{(N)})$ will be added in section 5 and in a brief summary.

Marginally, let us note that sometimes, one could need a slight extension of the scope of our model (2) beyond its purely kinematical version. This can be easily achieved by an addition of a trivial potential $V(r) = \gamma/r^2$ and by the subsequent redefinition of the effective $\ell$ in (2),

$$\ell(\ell + 1) = \gamma + \left(m + \frac{D-3}{2}\right) \left(m + \frac{D-1}{2}\right), \quad m = 0, 1, \ldots.$$  

(3)

In this way one can treat $\ell = \ell(\gamma)$ as a continuous, not necessarily just a (half)integer real parameter.

2 The Buslaev’s and Grecchi’s model as a guide

2.1 Isospectral Hamiltonians

Let us briefly return to the Buslaev’s and Grecchi’s paper [2] where a constant shift $\epsilon > 0$ has been used to define the following straight line of “unmeasurable”
complexified coordinates,

\[ \mathcal{C}^{(BG)} = \{ \xi = x - i \epsilon \mid \epsilon > 0, x \in \mathbb{R} \} \tag{4} \]

A very specific anharmonic-oscillator potential has further been chosen as acting along \( \mathcal{C}^{(BG)} \). Under the most common Dirichlet asymptotic boundary conditions one reveals that with \( \psi[\xi(\pm \infty)] = 0 \) we have \( \phi^{(BG)}(x) \equiv \psi[\xi(x)] \in L^2(\mathcal{C}^{(BG)}) \) obtainable from the differential equation

\[
-\frac{d^2}{dx^2} + \frac{\ell(\ell + 1)}{[\xi(x)]^2} + \lambda V^{(BG)}\{[\xi(x)]^2\} - E^{(BG)} \phi^{(BG)}(x) = 0 . \tag{5}
\]

This is a non-Schrödinger, non-selfadjoint eigenvalue problem with \( \mathcal{PT} \)–symmetry defined in terms of the spatial reflection \( \mathcal{P} \) and temporal reflection \( \mathcal{T} \) and exhibited by the Hamiltonian \( H^{(BG)} \) \[2\].

After a “naive” choice of the Hilbert space \( \mathcal{H}^{(original)} \equiv L^2(\mathcal{C}^{(BG)}) \) the Buslaev’s and Grecchi’s Hamiltonian \( H^{(BG)} \) proves manifestly non-Hermitian (and, hence, apparently “unphysical”). Fortunately, one of the main results of ref. \[2\] tells us that \( H^{(BG)} \) proves isospectral to another operator

\[ h^{(BG)} = \Omega H^{(BG)} \Omega^{-1} \tag{6} \]

which happens to be self-adjoint and, hence, physical. This observation settled the questions of physics beyond BG model and re-established the correct probabilistic interpretation of all the observables in the system in question.

Several papers (cf., e.g., \[8\] or \[9\]) re-analysed the Buslaev’s and Grecchi’s conclusions recently. This partially motivated also our forthcoming considerations. One of our reasons was that for the model \( H^{(BG)} \) it was trivial to guarantee, by construction, that the Hamiltonian \( h^{(BG)} \) becomes self-adjoint in its own Hilbert space \( \mathcal{H}^{(physical)} \).

The challenge of a search for some other simple models was imminent.

### 2.2 General formalism and an amended Dirac’s notation

On a formal level needed in our forthcoming considerations one should refer to the review paper \[10\] where the authors emphasized that \( h^{(BG)} \) in (6) can be self-adjoint...
(in the Dirac’s transposition-plus-complex-conjugation sense, i.e., \( h^{(BG)} = \left( h^{(BG)} \right)^\dagger \)) only if \( H^{(BG)} \) is quasi-Hermitian (i.e., only if \( \left( H^{(BG)} \right)^\dagger = \Theta H^{(BG)} \Theta^{-1} \)) where, in our present notation, \( \Theta = \Omega^\dagger \Omega \)). From such a point of view, the Buslaev’s and Grecchi’s original choice of their very specific anharmonic-oscillator model can be interpreted as a “mixed blessing”. On the negative side, the narrow-minded results of ref. \[2\] did not prove too inspiring. In fact, they looked so exceptional that the physics community accepted them as a mere mathematical curiosity. On the positive side, the tractability of the model seems to have opened new perspectives.

The point is that in principle, the lower-case “correct” Hamiltonian operator can be interpreted as acting in another, different Hilbert space \( \mathcal{H}^{(\text{physical})} \). Thus, the Hamiltonian \( h^{(BG)} \) can be, in general, very different from its original upper-case representation introduced as acting in a “tentative”, unitarily non-equivalent Hilbert space \( \mathcal{H}^{(\text{original})} \). Exceedingly complicated versions of the “physical” \( h^{(BG)} \) may be encountered in some realistic models, e.g., in nuclear physics \[10\].

In the standard Dirac’s notation all the elements \(|\Psi\rangle\) of the original vector space and of its dual \( \left( \mathcal{H}^{(\text{original})} \right)^\dagger \) may be treated and denoted as the usual kets \(|\Psi\rangle\) and bras \( \langle \Psi | \), respectively. After the change of the spaces \( \mathcal{H}^{(\text{original})} \rightarrow \mathcal{H}^{(\text{physical})} \) it is necessary to keep the trace of the changes in order to avoid the possible ambiguity of the notation. Thus, the elements of the physical Hilbert space will be denoted here by the specific, curly ket symbols \(|\Psi\rangle \in \mathcal{H}^{(\text{physical})}\) while in the dual space of linear functionals \[11\] we shall write \( \langle \Psi | \in \left( \mathcal{H}^{(\text{physical})} \right)^\dagger \).

Puzzling as it may seem at the first sight, our emphasis on the difference between the spaces of kets \(|\Psi\rangle \in \mathcal{H}^{(\text{physical})}\) and \(|\Psi\rangle \in \mathcal{H}^{(\text{original})}\) did in fact play a key role in some misunderstandings which appeared in the current literature \[12\]. Paradoxically, the most natural and transparent resolution of the whole puzzle is virtually trivial. According to our recent proposal \[13\] one can simply add an auxiliary, third Hilbert space \( \mathcal{H}^{(\text{third})} \) exhibiting the following properties:

- as a vector space without inner product, the set \( \mathcal{H}^{(\text{third})} \) coincides with \( \mathcal{H}^{(\text{original})} \), i.e., we may write \(|\Psi\rangle \in \mathcal{H}^{(\text{physical})}\) as well;
the spaces of duals (or, if you wish, linear functionals) are different, i.e.,

\( (\mathcal{H}^{(original)})^\dagger := \mathcal{T}^{(original)}\mathcal{H}^{(original)} \neq (\mathcal{H}^{(third)})^\dagger := \mathcal{T}^{(third)}\mathcal{H}^{(third)} \);

for the auxiliary, innovated conjugation \(^\dagger\) and functionals \( \langle\langle \Psi | \in (\mathcal{H}^{(third)})^\dagger \) we have to postulate the defining relation

\[ \langle\langle \Psi | := \langle \Psi | \Theta \equiv (|\Psi\rangle)^\dagger \neq (|\Psi\rangle)^\dagger, \quad \Theta = \Omega^\dagger \Omega. \] (7)

As long as the simultaneous use of both the conjugations would almost certainly lead to dangerous confusions, we shall always employ just the Dirac’s transposition-plus-complex-conjugation one here. This means that \( \mathcal{T}^{(original)} \) or \( \mathcal{T}^{(physical)} \) will be both characterized by the same single-cross superscripts \(^\dagger\) and by the usual bra-ket correspondence. In contrast, the double cross \(^\ddagger\) will not be used at all. Thus, we shall always treat the space \( (\mathcal{H}^{(third)})^\dagger \) and its double-bra elements \( \langle\langle \Psi | \neq \langle \Psi | \) as mere abbreviations.

All these conventions are summarized in Table 1. They immediately imply that

\( \langle \psi | \psi' \rangle = \langle\langle \psi | \psi' \rangle \) so that the spaces \( \mathcal{H}^{(physical)} \) and \( \mathcal{H}^{(third)} \) are, by construction, unitarily equivalent. Any one of them may be employed as physical, therefore. Of course, the same language and physical interpretation is applicable not only to the BG model but also to all the models with real spectra (including the quantum knots to be described below) which may only look non-Hermitian due to the naive initial choice of the “wrong” inner product in \( \mathcal{H}^{(original)} \).

| Hilbert space   | element | dual       | inner product | Hamiltonian          |
|-----------------|---------|------------|---------------|----------------------|
| \( \mathcal{H}^{(original)} \) | \( |\psi\rangle \) | \( \langle\psi| = (|\psi\rangle)^\dagger \) | \( \langle\psi|\psi'\rangle \) | \( H^{(BG)} \neq (H^{(BG)})^\dagger \) |
| \( \mathcal{H}^{(physical)} \) | \( |\psi\rangle \equiv \Omega |\psi\rangle \) | \( \langle\psi| = (\langle\psi|\Omega)^\dagger \) | \( \langle\psi|\psi'\rangle \) | \( h^{(BG)} = (h^{(BG)})^\dagger \) |
| \( \mathcal{H}^{(third)} \) | \( |\psi\rangle \) | \( \langle\langle \psi| \equiv \langle \psi| \Omega \) | \( \langle\langle \psi|\psi'\rangle \) | \( H^{(BG)} = (H^{(BG)})^\dagger \) |
3 Integration paths \( C^{(N)} \)

Let us now return to our ordinary linear differential eq. (2) of second order, the general solution of which can always be expressed as a superposition of some of its two linearly independent components. In particular, in a small complex vicinity of the origin we may write

\[
\psi(\xi) = c_+ \psi^+(\xi) + c_- \psi^-(\xi)
\]

where

\[
\psi^+(\xi) = \xi^{\ell+1} + \text{corrections}, \quad \psi^-(\xi) = \xi^{-\ell} + \text{corrections}, \quad |\xi| \ll 1.
\]  

In the asymptotic domain we shall prefer another option with \( \kappa = \sqrt{E} \) in

\[
\psi(\xi) = c_1 \psi^{(1)}(\xi) + c_2 \psi^{(2)}(\xi)
\]

where

\[
\psi^{(1,2)}(\xi) = \exp(\pm i \kappa \xi + \text{corrections}) + \text{corrections}, \quad |\xi| \gg 1.
\]
In between these two extremes, our differential eq. \( \text{(2)} \) is smooth and analytic so that we may expect that all its solutions are locally analytic.

In the vicinity of the origin \( \xi = 0 \) our centrifugal pole with its real parameter \( \ell \) dominates our eq. \( \text{(2)} \). Once \( \ell \) is assumed irrational, both the components of our wave functions (as well as their arbitrary superpositions) would behave, *globally*, as multivalued analytic functions defined on a certain multisheeted Riemann surface \( \mathcal{R} \). In the other words, our wave functions would possess a logarithmic branch point in the origin, i.e. a branch point with an infinite number of Riemann sheets connected at this point \[14\].

Separately, one should study the simplified models with the rational \( \ell \)s which correspond to the presence of an algebraic branch point at \( \xi = 0 \). A finite number of sheets \[14\] would be connected there. In the simplest possible scenario of such a type we may take \( \ell(\ell + 1) = 0 \) with either \( \ell = -1 \) or \( \ell = 0 \). In such a setting, eq. \( \text{(8)} \) just separates \( \psi(\xi) \) into its even and odd parts so that the Riemann surface itself remains trivial, \( \mathcal{R} \equiv \mathcal{C} \).

In the generic case of a multisheeted \( \mathcal{R} \) we intend to show that the asymptotically free form of our differential eq. \( \text{(2)} \) with the independent solutions \( \text{(11)} \) *can generate*
bound states. One must exclude, of course, the contours running, asymptotically, along the real line of $\xi$ since, in such a case, both our independent solutions $\psi^{(1,2)}(\xi)$ remain oscillatory and non-localizable. The same exclusion applies to the parallel, horizontal lines $C^{(BG)}$ in the complex plane of $\xi$. In the search for bound states, both the “initial” and “final” asymptotic branches of our integration paths $C^{(N)}$ must have the specific straight-line form $\xi = \pm |s| e^{i\varphi}$ with a non-integer ratio $\varphi/\pi$. Thus, we may divide the asymptotic part of the complex Riemann surface of $\xi \in \mathcal{R}$ into the sequence of asymptotic sectors

$$S_0 = \{ \xi = -i \varrho e^{i\varphi} \mid \varrho \gg 1, \ \varphi \in (-\pi/2, \pi/2) \}. \quad (12)$$

$$S_{\pm k} = \{ \xi = -i e^{\pm ik\pi} \varrho e^{i\varphi} \mid \varrho \gg 1, \ \varphi \in (-\pi/2, \pi/2) \}, \quad k = 1, 2, \ldots . \quad (13)$$

We are now prepared to define the integration contours $C^{(N)}$. For the sake of convenience we shall set all their “left” asymptotic branches $C^{(left)}$ in the same sector $S_0$ and specify $\xi = (s + s_0) (1 + i\varepsilon)$ where $s \in (-\infty, -s_0)$, $\varepsilon > 0$ and $s_0 > 0$. The subsequent middle part of $C^{(N)}$ must make $N$ counterclockwise rotations around the origin inside $\mathcal{R}$ while $s \in (-s_0, s_0)$. Finally, the “outcoming” or “right” asymptotic branch of our integration contour $C^{(N)}$ with $s \in (s_0, \infty)$ must lie in another sector $S_{2N}$ of $\mathcal{R}$, i.e., in the Riemann sheet where the requirement of $\mathcal{PT}$-symmetry [6] forces us to set $\xi = (s - s_0) (1 - i\varepsilon)$.

4 Bound states along nontrivial paths

From now on we shall assume that the integration contour $C^{(N)}$ is fixed and that the variability of $\xi$ is confined to it. In this spirit we also adapt our notation writing $\xi = \xi(s) \equiv r \in C^{(N)}$. Our illustrative Figures [1]–[3] sample the choice of $N = 1$, $N = 2$ and $N = 3$, respectively.

It remains for us to impose the asymptotic boundary conditions requiring that our wave functions vanish at $s \rightarrow \pm \infty$. As long as our integration path $C^{(N)}$ performs $N$ counterclockwise rotations around the origin, this form of the asymptotic boundary conditions will already guarantee the normalizability of our bound-state
wave functions $\psi(r) \in L^2(C^{(N)})$ (cf. the similar situation encountered in the models with confining potentials [2, 5, 15]).

In our bound-state problem (2) with the (by assumption, real) $E = \kappa^2$ we may set $z = \kappa r$ and $\psi(r) = \sqrt{z} \varphi(z)$. This reduces eq. (2) to the Bessel differential equation with the pair of the two well known independent special-function (say, Hankel-function [16]) solutions which may be inserted in our ansatz

$$\psi(r) = c_1 \sqrt{r} H_{\nu}^{(1)}(\kappa r) + c_2 \sqrt{r} H_{\nu}^{(2)}(\kappa r), \quad \nu = \ell + 1/2. \quad (14)$$

At $|\arg z| < \pi$ and $\operatorname{Re} \nu > -1/2$, the asymptotics of its components are given by the respective formulae 8.451.3 and 8.451.4 of ref. [16],

$$\sqrt{\frac{\pi z}{2}} H_{\nu}^{(1)}(z) = \exp \left[ i \left( z - \frac{\pi(2\nu + 1)}{4} \right) \right] \left( 1 - \frac{\nu^2 - 1/4}{2iz} + \ldots \right),$$

$$\sqrt{\frac{\pi z}{2}} H_{\nu}^{(2)}(z) = \exp \left[ -i \left( z - \frac{\pi(2\nu + 1)}{4} \right) \right] \left( 1 + \frac{\nu^2 - 1/4}{2iz} + \ldots \right).$$

This implies that inside the even-subscripted sectors $S_{2k}$ our ansatz (14) combines the asymptotically growing (and, hence, unphysical) component $H_{\nu}^{(1)}(z)$ with the asymptotically vanishing and normalizable, physical component $H_{\nu}^{(2)}(z)$. Vice versa,
in all the odd-subscripted sectors \( S_{2k+1} \) we would have to eliminate, in principle, the asymptotically growing \( H^{(2)}_\nu(z) \) and to keep the asymptotically vanishing \( H^{(1)}_\nu(z) \).

We may start our discussion of the existence of the localized bound states from the straight-line contour \( C = C^{(BG)} = C^{(0)} \) which is all contained in the zeroth sector \( S_0 \). This immediately implies that with \( \text{Im} \, r \ll -1 \), the asymptotically vanishing solution \( \psi^{(1)}(r) = \sqrt{r} H^{(2)}_\nu(\kappa r) \) remains unconstrained at all the real \( \kappa \). Obviously, the spectrum remains non-empty and bounded from below. This means that the low-lying states remain stable with respect to a random perturbation. A less usual feature of such a model is that its energies densely cover all the real half-line \( \mathbb{R}^+ \). This feature is fairly interesting \textit{per se}, although a more detailed analysis of its possible physical consequences lies already beyond the scope of our present brief note.

Our eigenvalue problem becomes not too much more complicated when we turn attention to the spiral- or knot-shaped integration contours \( C^{(N)} \) with \( N > 0 \). In such a case, fortunately, the exact solvability of our differential equation enables us to rewrite ansatz (10) in its fully explicit form which remains analytic on all our Riemann surface \( \mathcal{R} \). Once we choose our “left” asymptotic sector as \( S_0 \), the “left” physical boundary condition fixes and determines the acceptable solution on the initial sheet,

\[
\psi(r) = c \sqrt{r} H^{(2)}_\nu(\kappa r), \quad r \in S_0. \tag{15}
\]

After the \( N \) counterclockwise turns of our integration path \( C^{(N)} \) around the origin this solution gets transformed in accordance with formula 8.476.7 of ref. [16] which plays a key role also in some other solvable models [17],

\[
H^{(2)}_\nu \left( z e^{im\pi} \right) = \frac{\sin(1 + m)\pi \nu}{\sin \pi \nu} H^{(2)}_\nu(z) + e^{im\pi} \frac{\sin m\pi \nu}{\sin \pi \nu} H^{(1)}_\nu(z). \tag{16}
\]

Here we have to set \( m = 2N \). This means that the existence of a bound state will be guaranteed whenever we satisfy the “right” physical boundary condition, i.e., whenever we satisfy the elementary requirement of the absence of the unphysical component \( H^{(1)}_\nu(z) \) in the right-hand side of eq. (16).

The latter requirement is equivalent to the doublet of conditions

\[
2N\nu = \text{integer}, \quad \nu \neq \text{integer}. \tag{17}
\]
This means that at any fixed and positive value of the energy $E = \kappa^2$ and at any fixed winding number $N = 1, 2, \ldots$, our present quantum-knot model generates the series of the bound states at certain irregular sequence of angular momenta avoiding some “forbidden” values,

$$\ell = \frac{M - N}{2N}, \quad M = 1, 2, 3, \ldots, \quad M \neq 2N, 4N, 6N, \ldots \quad (18)$$

These bound states exist and have the analytically continued Hankel-function form \((15)\) if and only if the kinematical input represented by the angular momenta $\ell$ is restricted to the subset represented by formula \((18)\).

Our construction is completed. Once we restrict our attention to the purely kinematic model with $\gamma = 0$, we can summarize that at the odd dimensions $D = 2p+1$ giving $\ell = n + p - 3/2$ we may choose any index $n$ and verify that formula \((18)\) can be read as a definition of the integer quantity $M = (2n+2p-1)N$ which is not forbidden. At the even dimensions $D = 2p$ we equally easily verify that the resulting $M$ is always forbidden so that our quantum-knot bound states do not exist at $V(r) = 0$ at all.

The latter dichotomy appears reminiscent of its well-known non-quantum real-space analogue, but the parallel is misleading because in quantum case the freedom of employing an additional coupling constant $\gamma$ enables us to circumvent the restrictions. Indeed, once we select any dimension $D$, angular-momentum index $m$, winding number $N$ and any “allowed” integer $M$, our spectral recipe \((18)\) may simply be re-read as an explicit definition of the knot-supporting value of the coupling constant

$$\gamma = \left( \frac{M}{2N} \right)^2 - \left( m + \frac{D-2}{2} \right)^2.$$ 

This implies that at non-vanishing $\gamma$s, the quantum knots do exist in any dimension.

5 Discussion

In the language of physics, our present construction and solution of a new and fairly unusual exactly solvable quantum model of bound states is based on the freedom of choosing the knot-shaped, complex contours of integration $\mathcal{C}$. This trick is not
new and may be perceived as just a consequence of the admitted loss of the observability of the coordinates in PT-symmetric Quantum Mechanics.

From an experimentalist’s point of view, the omission of the standard assumption that the coordinate “should be” an observable quantity is not entirely unacceptable since the current use of the concept of quasi-particles paved the way for similar constructions. Related Hamiltonians could be called, in certain sense, manifestly non-Hermitian. Still, they are currently finding applications in nuclear physics (where they are called quasi-Hermitian). The loss of the reality of the coordinates is also quite common in field theory where the similar unusual Hamiltonians are being rather called CPT-symmetric or crypto-Hermitian.

In a pragmatic phenomenological setting, the fairly unusual nature of the new structures of spectra seems promising. At the same time, the formalism itself is now considered fully consistent with the standard postulates of quantum theory. In the language of mathematics, the emergence of its innovative features may be understood as related to non-locality, i.e., to the replacement of the standard scalar product

\[ \langle \psi | \phi \rangle = \int \psi^*(x) \phi(x) dx \]

by its generalized, nonlocal modifications

\[ \langle \psi | \phi \rangle = \int \psi^*(x) \Theta(x, y) \phi(y) dx dy. \]

Although this leaves an overall mathematical consistency and physical theoretical framework of Quantum Theory virtually unchanged, a new space is being open, inter alia, to the topology-based innovations. In principle, they might inspire new developments of some of the older successful applications of the formalism ranging from innovative supersymmetric constructions to cosmology, occasionally even leaving the domain of quantum physics.

In section we summarized briefly the key ingredients of quantum theory where the “correct” metric is assumed nontrivial, \( \Theta \neq I \). Let us now add a few comments which may have emerged during our subsequent transition to the quantum-knot models of section. Of course, our eq. (at \( N > 1 \) can still be treated as compatible
with the standard postulates of quantum theory in principle. We only have to repeat that the necessary proof of the latter compatibility statement is nontrivial. For each individual Hamiltonian (with the property $H \neq H^\dagger$ with respect to the specific Dirac’s definition of the $\dagger$–conjugation) the rigorous demonstration is indispensable that the spectra are real and that they are discrete and bounded from below. This demonstration represents, in fact, the main part of our present contribution.

It is precisely the difficulty of the latter step which motivated our present start from the dynamically trivial version (2) of the BG model with vanishing $\lambda$. In the nearest future we shall have to pay attention to the related operator $\Theta(x, y)$, feeling inspired by the Mostafazadeh’s explicit formula

$$\Theta(x, y) \approx \sum_{n=0}^{M} \Psi_n(x) s_n \Psi_n(y), \quad M \gg 1 \quad (19)$$

where the normalized eigenstates $\Psi_n(x)$ of $H^\dagger$ have to be constructed in $\mathcal{H}^{\text{original}}$ and where the real and positive constants $s_n > 1$ are, in principle, arbitrary [26]. Of course, the letter-format of our present message does not allow us to get too far beyond the citation of the encouraging observation that this type of formula exhibited a quick convergence to the exact $\Theta$ in the square-well model where a fairly good approximation has already been obtained at $M \approx 10$ [4].

6 Summary

In spite of the absence of any confining force, our Schrödinger eq. (2) defined along topologically nontrivial integration paths has been shown to generate certain bound states $\psi(r) \in L^2(C^{(N)})$ at a discrete set of the centrifugal coupling $\gamma$. We may emphasize that in such an exemplification of the more or less standard quantum theory

- the complexified coordinates are loosing their immediate observability,

- arbitrary complex potentials $V(r) \in \mathcal{C}$ are allowed, provided only that the spectrum remains real,
• a redefinition of the inner product in the Hilbert space is required in order to return to the standard probabilistic framework of quantum theory,

• a challenging general open problem arises concerning the role and tractability of the complex coordinate paths with a nontrivial topological structure.

Of course, our present, exactly solvable $N > 1$ quantum-knot bound-state problem would probably become purely numerical after its immersion in virtually any external confining potential. In this setting, even the question of survival of the reality of the new bound-state spectra at $\lambda \neq 0$ remains open.

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Figure captions

Figure 1. Sample of the curve $C^{(N)}$ with $N = 1$.

Figure 2. Sample of the curve $C^{(N)}$ with $N = 2$.

Figure 3. Sample of the curve $C^{(N)}$ with $N = 3$. 
References

[1] M. Znojil, Phys. Rev. A 61 (2000) 066101 [quant-ph/9811088].

[2] V. Buslaev and V. Grecchi, J. Phys. A: Math. Gen. 26 (1993) 5541.

[3] C. M. Bender and K. Milton, Phys. Rev. D 55 (1997) R3255;
    C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243.

[4] A. Mostafazadeh and A. Batal, J. Phys. A: Math. Gen. 37 (2004) 11643.

[5] M. Znojil, Phys. Lett. A 342 (2005) 36.

[6] M. Znojil, J. Phys. A: Math. Gen. 39 (2006) 13325.

[7] M. Znojil, Phys. Lett. A, to appear [arXiv:0708.0087v1 [quant-ph] 1 Aug 2007,
    doi:10.1016/j.physleta.2007.07.072).

[8] M. Znojil, Phys. Lett. A 259 (1999) 220.

[9] H. F. Jones and J. Mateo, Phys. Rev. D 73 (2006) 085002;
    H. F. Jones, J. Mateo and R. J. Rivers, Phys. Rev. D 74 (2006) 125022.

[10] F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, Ann. Phys. (NY) 213 (1992)
    74.

[11] A. Messiah, Quantum Mechanics (North Holland, Amsterdam, 1961).

[12] A. Mostafazadeh, Phys. Lett. B 650 (2007) 208;
    M. Znojil, Time-dependent quasi-Hermitian Hamiltonians and the unitarity of
    quantum evolution, arXiv: 0710.5653 [quant-ph];
    A. Mostafazadeh, Comment on “Time-dependent quasi-Hermitian Hamiltonians
    and the unitary quantum evolution”, arXiv: 0711.0137 [quant-ph];
    M. Znojil, Reply to Comment on “Time-dependent quasi-Hermitian Hamiltoni-
    ans and the unitary quantum evolution”, arXiv: 0711.0514 [quant-ph];
A. Mostafazadeh, Comment on "Reply to Comment on Time-dependent Quasi-Hermitian Hamiltonians and the Unitary Quantum Evolution", arXiv: 0711.1078 [quant-ph].

[13] M. Znojil, Which operator generates time evolution in Quantum Mechanics? arXiv: 0710.0535 [quant-ph].

[14] see e.g. ”branch point” in http://eom.springer.de

[15] Y. Sibuya, Global Theory of Second Order Linear Differential Equation with Polynomial Coefficient, North Holland, Amsterdam, 1975;
G. Alvarez, J. Phys. A: Math. Gen. 27 (1995) 4589.

[16] I. S. Gradshteyn and I. M. Ryzhik, Tablicy integralov, summ, ryadov i proizvedenii, Nauka, Moscow, 1971.

[17] F. Cannata, G. Junker and J. Trost, Phys. Lett. A 246 (1998) 219.

[18] C. M. Bender, Reports on Progress in Physics 70 (2007) 947.

[19] C. M. Bender, D. C. Brody and H. F. Jones, Phys. Rev. Lett. 89 (2002) 0270401 (quant-ph/0208076).

[20] A. V. Smilga, Cryptogauge symmetry and cryptoghosts for crypto-Hermitian Hamiltonians, arXiv:0706.4064.

[21] H. Geyer, D. Heiss and M. Znojil, editors, J. Phys. A: Math. Gen. 39 (2006), Nr. 32 (dedicated special issue: pp. 9965 - 10261).

[22] C. M. Bender and K. A. Milton, Phys. Rev. D 57 (1998) 3595;
M. Znojil, F. Cannata, B. Bagchi and R. Roychoudhury, Phys. Lett. B 483 (2000) 284;
A. Mostafazadeh, Nucl. Phys. B 640 (2002) 419;
M. Znojil, J. Phys. A: Math. Gen. 35 (2002) 2341.
[23] A. Mostafazadeh, Class. Quantum Grav. 20 (2003) 155.

[24] U. Guenther, F. Stefani and M. Znojil, J. Math. Phys. 46 (2005) 063504.

[25] A. Mostafazadeh, J. Math. Phys. 43 (2002) 205 [math-ph/0107001] and 2814 [math-ph/0110016].

[26] M. Znojil, SIGMA 4 (2008), 001, 9 pages, arXiv: 0710.4432v3 [math-ph].