SZABÓ OSSERMAN IP PSEUDO-RIEMANNIAN MANIFOLDS

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ABSTRACT. We construct a family of pseudo-Riemannian manifolds so that
the skew-symmetric curvature operator, the Jacobi operator, and the Szabó
operator have constant eigenvalues on their domains of definition. This pro-
vides new and non-trivial examples of Osserman, Szabó, and IP manifolds.
We also study when the associated Jordan normal form of these operators is
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1. INTRODUCTION

Let \((M,g)\) be a pseudo-Riemannian manifold of signature \((p,q)\). Let \(R\) be the
Riemann curvature:

\[
R(Z_1, Z_2) := \nabla_{Z_1} \nabla_{Z_2} - \nabla_{Z_2} \nabla_{Z_1} - \nabla_{[Z_1, Z_2]},
\]

\[
R(Z_1, Z_2, Z_3, Z_4) := (R(Z_1, Z_2)Z_3, Z_4).
\]

We can use \(R\) and \(\nabla R\) to define several natural operators:

1. The Jacobi operator \(J(X) : Y \to R(Y, X)X\) is a symmetric operator with
   \(J(X)X = 0\). It plays an important role in the study of geodesic sprays. Since
   \(J(cX) = c^2 J(X)\), the natural domains of definition for \(J\) are the pseudo-
sphere bundles \(S^\pm(M, g) := \{X \in TM : (X, X) = \pm 1\}\).

2. The Szabó operator \(\mathcal{S}(X) : Y \to \nabla_X R(Y, X)X\) is a symmetric operator with
   \(\mathcal{S}(X)X = 0\). It plays an important role in the study of totally isotropic
   manifolds. Since \(\mathcal{S}(cX) = c^3 \mathcal{S}(X)\), the natural domains of definition for \(\mathcal{S}\)
   are \(S^\pm(M, g)\).

3. Let \(\{X_1, X_2\}\) be an oriented orthonormal basis for a non-degenerate 2 plane
   \(\pi\). The skew-symmetric curvature operator \(\mathcal{R}(\pi) : Y \to R(X_1, X_2)Y\) depends
   on the orientation of \(\pi\) but not on the particular orthonormal basis chosen.
   The natural domains of definition for \(\mathcal{R}(\cdot)\) are the oriented Grassmannians of
timelike, mixed (signature \((1, 1)\)), and spacelike 2 planes.

The spectrum \(\text{Spec}(T) \subset \mathbb{C}\) of a linear map \(T\) is the set of complex eigenvalues
of \(T\). It is natural to ask what are the geometric constraints that are imposed by
assuming that the spectrum (or more generally the Jordan normal form) of one of
these 3 natural operators is constant on the appropriate domains of definition.

Let \((M, g)\) be a pseudo-Riemannian manifold. \((M, g)\) is said to be spacelike
Osserman if \(\text{Spec}(J(\cdot))\) is constant on \(S^+(M, g)\), \((M, g)\) is said to be spacelike
Szabó if \(\text{Spec}(\mathcal{S}(\cdot))\) is constant on \(S^+(M, g)\), and \((M, g)\) is said to be spacelike IP
if \(\text{Spec}(\mathcal{R}(\pi))\) is constant on the Grassmannian of oriented spacelike 2 planes in
the tangent bundle \(TM\). One defines timelike Osserman, timelike Szabó, timelike
IP, and mixed IP similarly. The eigenvalue \(\{0\}\) plays a distinguished role. We say

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\( (M, g) \) is nilpotent Osserman if \( \text{Spec} (J(X)) = \{0\} \) for all \( X \), nilpotent Szabó and nilpotent IP are defined similarly.

The names Osserman, Szabó, and IP are used because the seminal papers for this subject in the Riemannian setting are due to Osserman \( [9] \) for the operator \( J(\cdot) \), to Szabó \( [11] \) for the operator \( \mathcal{O}(\cdot) \), to Ivanov and Petrova \( [7] \) and Stanilov and Ivanova \( [8] \) for the operator \( R(\cdot) \). The spectral properties of the operators \( J(\cdot) \) and \( R(\cdot) \) have been studied extensively; we refer to \( [4, 5] \) for a more complete historical discussion and bibliography. By contrast, the operator \( \mathcal{O}(\cdot) \) has received considerably less attention.

Suppose \( p \geq 1 \) and \( q \geq 1 \). Then the notions spacelike Osserman (resp. spacelike Szabó) and timelike Osserman (resp. timelike Szabó) are equivalent, so one simply says that \( (M, g) \) is Osserman (resp. Szabó). Similarly, if \( p \geq 2 \) and \( q \geq 2 \), then spacelike, mixed, and timelike IP are equivalent notions so \( (M, g) \) is said to be IP. See \( [3] \) for details. We shall use the words ‘nilpotent’, ‘Osserman’, ‘Szabó’, and ‘IP’ as adjectives. Thus, for example, to say that a manifold is nilpotent Osserman Szabó IP means that it is simultaneously nilpotent Szabó, nilpotent Osserman, and nilpotent IP. We say that \( (M, g) \) is locally symmetric if \( \nabla R = 0 \) and locally homogeneous if the local isometries of \( (M, g) \) act transitively on \( M \). We say that \( (M, g) \) is Ricci flat if the Ricci tensor vanishes identically.

Let \( (x, y) = (x_1, \ldots, x_p, y_1, \ldots, y_p) \) be coordinates on the manifold \( M := \mathbb{R}^{2p} \). Let \( \psi_{ij}(x) = \psi_{ji}(x) \) be a symmetric 2 tensor \( \psi \) on \( \mathbb{R}^p \). We define a non-degenerate pseudo-Riemannian metric \( g_\psi \) of balanced signature \( (p, p) \) on \( M \) by setting:

\[
\begin{align*}
  ds_\psi^2 := \sum_i x^i \circ dy^i + \sum_{i,j} \psi_{ij}(x) dx^i \circ dx^j.
\end{align*}
\]

In Section 2 we will prove the following result:

**Theorem 1.1.** Let \( p \geq 2 \) and let \( \psi(x) \) be a symmetric 2 tensor. Then \( (M, g_\psi) \) is:

1. a pseudo-Riemannian manifold of signature \((p, p)\);
2. nilpotent Szabó Osserman IP;
3. Ricci flat and Einstein;
4. neither locally homogeneous nor locally symmetric for generic \( \psi \).

Nilpotent Osserman manifolds have been constructed previously \( [1, 2] \). We can describe one family which arises from affine geometry as follows. Let \( \Gamma_{ij}^k(x) \) be the Christoffel symbols of an arbitrary torsion free connection \( \nabla \) on \( \mathbb{R}^p \). Let \( R_\nabla \) be the associated curvature operator and let \( J_\nabla (X_1) : X_2 \to R_\nabla (X_2, X_1)X_1 \) be the associated Jacobi operator on \( \mathbb{R}^p \). We say that the connection \( \nabla \) is nilpotent affine Osserman if for all \( X \in T \mathbb{R}^p \), \( \text{Spec} (J_\nabla (X)) = \{0\} \), i.e. \( J_\nabla (X)^p = 0 \).

Following Garcíá-Rió, Kupeli, and Vázquez-Lorenzo \( [2] \) (see page 147), define an associated metric on \( M = \mathbb{R}^{2p} \) by setting:

\[
\begin{align*}
  ds_\nabla^2 = \sum_i x^i \circ dy^i - 2 \sum_{i,j} y_k \Gamma_{ij}^k(x) dx^i \circ dx^j.
\end{align*}
\]

Then \( (M, ds_\nabla^2) \) is nilpotent Osserman if and only if \( \nabla \) is nilpotent affine Osserman. This metric is quite different in flavor from ours as the coefficients depend on the \( y \) variables as well as on the \( x \) variables. There does not seem to be any direct connection between the metrics defined in equations (1.1) and (1.2).

Nilpotent IP manifolds have also been constructed previously \( [4, 5] \). However, comparatively little is known about Szabó manifolds - see \( [4, 5] \) for some preliminary results in the algebraic setting. In particular, the manifolds \( (M, g_\psi) \) are the only known irreducible Szabó manifolds which are not locally symmetric.
The eigenvalue structure does not determine the conjugacy class (i.e. the real Jordan normal form) of a symmetric or skew-symmetric linear operator in the higher signature setting. We will use the words ‘timelike’, ‘spacelike’, and ‘Jordan’ as adjectives. Thus, for example, to say \((M, g)\) is timelike Jordan IP means that the Jordan normal form of the skew-symmetric curvature operator is constant on the Grassmannian of timelike oriented 2-planes. We shall omit the accompanying adjectives ‘timelike and spacelike’ if both apply. Thus \((M, g)\) is Jordan Osserman means \((M, g)\) is both timelike Jordan Osserman and spacelike Jordan Osserman, i.e. the Jordan normal form of \(J\) is constant on \(S^+(M, g)\) and on \(S^-(M, g)\).

There are no known timelike or spacelike Jordan Szabó manifolds which are not locally symmetric. Section 3 is devoted to the proof of:

**Theorem 1.2.** If \((M, g_\psi)\) is not locally symmetric, then \((M, g_\psi)\) is neither spacelike Jordan Szabó nor timelike Jordan Szabó.

It is useful to consider a subfamily of the metrics defined in equation (1.1). Let \(f\) be a real-valued function on \(\mathbb{R}^p\). In equation (1.1) we set \(\psi = df \circ df\) and define

\[
d_{f}^2 := \sum_i dx_i \otimes dy_i + \sum_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} dx_i \otimes dx_j.
\]

We can realize \((M, g_f)\) as a hypersurface in a flat space. Let \(\{\alpha_1, ..., \alpha_p, \beta_1, ..., \beta_p, \gamma\}\) and \(\{\alpha_1^*, ..., \alpha_p^*, \beta_1^*, ..., \beta_p^*, \gamma^*\}\) be a basis and associated dual basis for vector spaces \(W\) and \(W^*\), respectively. We give \(W\) the inner product of signature \((p, p+1)\):

\[
ds_{W}^2 := \alpha_i \circ \beta_i + ... + \alpha_p \circ \beta_p + \gamma \circ \gamma^*.
\]

Let \(F : M \to W\) be the isometric embedding:

\[
F(x, y) := \sum_i (x_i \alpha_i + y_i \beta_i) + f(x)\gamma.
\]

The normal \(\nu\) to the hypersurface is given by \(\nu = -\frac{\partial f}{\partial x_i} \beta_i + \gamma\), so the second fundamental form of the embedding is

\[
L(Z_1, Z_2) = Z_1Z_2(f).
\]

We define distributions

\[
\mathcal{X} := \text{span} \{\partial_{x_1}^1, ..., \partial_{x_p}^p\} \quad \text{and} \quad \mathcal{Y} := \text{span} \{\partial_{y_1}^1, ..., \partial_{y_p}^p\}.
\]

We then have \(L(Z_1, Z_2) = 0\) if \(Z_1 \in \mathcal{Y}\) or \(Z_2 \in \mathcal{Y}\) so the restriction \(L_X\) of the second fundamental form \(L\) to the distribution \(\mathcal{X}\) carries the essential information. If, for example, we set \(f(x) = \sum_i \varepsilon_i x_i^2\), then:

\[
L_X(\partial_{x_i}^e, \partial_{x_j}^f) = \varepsilon_i \delta_{ij}
\]

so there are non-trivial examples where \(L_X\) is non-degenerate on \(M\). In Sections 4 and 5, we prove:

**Theorem 1.3.** Assume the quadratic form \(L_X\) is non-degenerate.

1. If \(p = 2\) or if \(p \geq 3\) and if \(L_X\) is definite, then \((M, g_f)\) is:
   (a) nilpotent Jordan Osserman;
   (b) nilpotent spacelike and timelike Jordan IP;
   (c) not mixed IP.

2. If \(p \geq 3\) and if \(L_X\) is indefinite, then \((M, g_f)\) is:
   (a) neither spacelike Jordan Osserman nor timelike Jordan Osserman;
   (b) nilpotent spacelike and timelike Jordan IP;
   (c) not mixed IP.
So far, we have discussed the balanced (or neutral) signature \( p = q \). There are also results available when \( p \neq q \). Give \( \mathbb{R}^{(a,b)} \) the canonical flat metric of signature \((a,b)\). We conclude the paper in Section 3 by proving:

**Theorem 1.4.** Assume the quadratic form \( L_X \) of \((M,g_f)\) is positive definite. Let \( N := M \times \mathbb{R}^{(a,b)} \) and let \( g_N \) be the product metric on \( N \).

1. \((N,g_N)\) is a nilpotent Osserman Szabó IP manifold of signature \((p+a,q+b)\).
2. For generic \( f \), \((N,g_N)\) is
   - (a) neither spacelike Jordan Szabó,
   - (b) nor timelike Jordan Szabó,
   - (c) nor locally homogeneous,
   - (d) nor locally symmetric.
3. \((N,g_N)\) is not mixed Jordan IP.
4. Suppose that \( b = 0 \). Then \((N,g_N)\) is:
   - (a) neither timelike Jordan Osserman nor timelike Jordan IP;
   - (b) spacelike Jordan Osserman and spacelike Jordan IP.
5. Suppose that \( a = 0 \). Then \((N,g_N)\) is:
   - (a) timelike Jordan Osserman and timelike Jordan IP;
   - (b) neither spacelike Jordan Osserman nor spacelike Jordan IP.
6. Suppose that \( a > 0 \) and \( b > 0 \). Then \((N,g_N)\) is:
   - (a) neither timelike Jordan Osserman nor timelike Jordan IP;
   - (b) neither spacelike Jordan Osserman nor spacelike Jordan IP.

**Note:** By Theorem 1.3, Jordan Osserman and Jordan IP are different notions. By Theorem 1.4, timelike Jordan Osserman (resp. timelike Jordan IP) and spacelike Jordan Osserman (resp. spacelike Jordan IP) are different notions as well.

## 2. Nilpotent Jordan Szabó Osserman Manifolds

We begin the proof of Theorem 1.1 by determining the curvature tensor of \((M,g_\psi)\). Let \( \psi_{ij/k} := \partial^k \psi_{ij} \) and let \( \psi_{ij/kl} := \partial^l \psi_{ij/k} \).

**Lemma 2.1.** Let \( Z_\nu \) be vector fields on \((M,g_\psi)\). We have:

1. \( \nabla \partial^\nu = 0 \);
2. \( R(Z_1, Z_2, Z_3, Z_4) = 0 \) if one of the \( Z_\nu \in \mathcal{Y} \) for \( 1 \leq \nu \leq 4 \);
3. \( \nabla R(Z_1, Z_2, Z_3, Z_4; Z_5) = 0 \) if one of the \( Z_\nu \in \mathcal{Y} \) for \( 1 \leq \nu \leq 5 \);
4. \( R(\partial^k, \partial^\ell, \partial^l, \partial^k) = -\frac{1}{4} \left( \psi_{ijkl} + \psi_{jikl} - \psi_{ikjl} - \psi_{ijk/l} \right) \).

**Proof.** Let \( Z_1 \) and \( Z_2 \) be coordinate vector fields. We then have:

\[
(\nabla_{Z_1} \partial^\nu, Z_2) = \frac{1}{2} \{ \partial^\nu (Z_1, Z_2) + Z_1 (\partial^\nu, Z_2) - Z_2 (Z_1, \partial^\nu) \} = 0.
\]

Assertion (1) now follows. We use it to see

(2.1) \( R(Z_1, Z_2, \partial^\nu, Z_3) = (\nabla_{Z_1} \nabla_{Z_2} - \nabla_{Z_2} \nabla_{Z_1} - \nabla_{[Z_1, Z_2]} \partial^\nu, Z_3) = 0 \).

This proves assertion (2) if \( Z_3 \in \mathcal{Y} \); the curvature symmetries then show that \( R(Z_1, Z_2, Z_3, Z_4) = 0 \) if any of the remaining vectors belong to \( \mathcal{Y} \). Since \( \nabla \partial^\nu = 0 \), we can covariantly differentiate equation (2.1) and get

(2.2) \( \nabla_{Z_1} R(Z_2, Z_3, \partial^\nu, Z_4) = 0 \);

assertion (3) now follows from equation (2.2) and from the curvature symmetries.

Since \( (\nabla_{\partial^i} \partial^j, \partial^k) = 0 \) and \( (\nabla_{\partial^i} \partial^j, \partial^k) = \frac{1}{4} (\psi_{ik/j} + \psi_{jk/i} - \psi_{ij/k}) \), we have

\[
\nabla_{\partial^i} \partial^j = \frac{1}{4} \sum_k (\psi_{ik/j} + \psi_{jk/i} - \psi_{ij/k}) \partial^k.
\]
We complete the proof of the Lemma by computing:
\[
R(\partial^r_i, \partial^r_j, \partial^r_k, \partial^r_l) = ((\nabla_{\partial^r_i} \nabla_{\partial^r_j} - \nabla_{\partial^r_j} \nabla_{\partial^r_i})\partial^r_k, \partial^r_l)
\]
\[
= \frac{1}{2}(\{\partial_i(\psi_{jk/l} + \psi_{kl/j} - \psi_{jl/ki}) - \partial_j(\psi_{l/kj} + \psi_{ij/lk} - \psi_{lk/ji})\} \partial^r_k, \partial^r_l)
\]
\[
= \frac{1}{2}(\psi_{jk/ki} + \psi_{kl/ji} - \psi_{jk/ul} - \psi_{ul/jk} - \psi_{kl/ij} + \psi_{ik/jl}).
\]

\[\square\]

Proof of Theorem 1.1. 1) It is clear from equation (1.1) that \((M, g_\psi)\) is a pseudo-Riemannian manifold of signature \((p, p)\).

2) We use Lemma 2.1 to see that
\[
\mathcal{V} \subset \ker(J(Z_1)), \quad \mathcal{V} \subset \ker(R(Z_1, Z_2)), \quad \mathcal{V} \subset \ker(\mathfrak{S}(Z_1)).
\]

Since \(\mathcal{V}\) is a totally isotropic subspace of \(T_PM\) of dimension \(p\), \(\mathcal{V}^\perp = \mathcal{V}\). Since \((R(Z_1, Z_2)Z_3, \partial^0_i) = 0\) and \((\nabla_Z, R(Z_2, Z_3)Z_4, \partial^0_i) = 0\), we have:

\[
\text{range}(J(Z_1)) \subset \mathcal{V}, \quad \text{range}(R(Z_1, Z_2)) \subset \mathcal{V}, \quad \text{range}(\mathfrak{S}(Z_1)) \subset \mathcal{V}.
\]

We use equations (2.3) and (2.4) to show
\[
J(Z_1)^2 = 0, \quad \mathfrak{S}(Z_1)^2 = 0, \quad \text{and} \quad R(Z_1, Z_2)^2 = 0.
\]

This shows that
\[
\text{Spec}(J(Z_1)) = \{0\}, \quad \text{Spec}(\mathfrak{S}(Z_1)) = \{0\}, \quad \text{and} \quad \text{Spec}(R(Z_1, Z_2)) = \{0\}
\]
for any vector fields \(Z_i\). Consequently \((M, g_\psi)\) is nilpotent Jordan Szabó IP.

3) Let \(g\) be the Ricci tensor. Since \(g(Z, Z) = \text{trace}(J(Z))\) and since \(J(Z)^2 = 0\), \(g(Z, Z) = 0\). We polarize to see that \(g \equiv 0\). Thus \((M, g_\psi)\) is Ricci flat and Einstein.

4) Clearly, \((M, g_\psi)\) is generically neither locally homogeneous nor locally symmetric. \(\square\)

3. Jordan Szabó manifolds

Theorem 1.2 will follow from the following Lemma:

Lemma 3.1. Let \(p \geq 2\) and let \(P \in M\). If \(\nabla R_\psi(P)\) does not vanish identically, then \(\text{rank}(\mathfrak{S}(\cdot))\) is constant neither on \(S^+(T_PM)\) nor on \(S^-(T_PM)\). Thus \((M, g_f)\) is neither spacelike Jordan Szabó nor timelike Jordan Szabó.

Proof. Suppose \(\text{rank}(\mathfrak{S}(\cdot)) = r > 0\) is constant on \(S^+(T_PM)\); the timelike case is similar. Let \(\mathcal{V}^+\) be a maximal spacelike subspace of \(T_PM\) and let \(\mathcal{V}^- := (\mathcal{V}^+)^\perp\) be the complementary timelike subspace. Let \(\rho^\pm\) be orthogonal projection on \(\mathcal{V}^\pm\). If \(Z \in S^+(\mathcal{V}^+)\), then we define:

\[
\mathfrak{S}(Z) := \rho^+ \mathfrak{S}(Z) \rho^+.
\]

We wish to show that \(\text{rank}\mathfrak{S}(Z) = r\). Let \(\{Z_1, ..., Z_r\}\) be tangent vectors at \(P\) so \(\{\mathfrak{S}(Z)Z_1, ..., \mathfrak{S}(Z)Z_r\}\) is a basis for \(\text{range}(\mathfrak{S}(Z))\). As \(T_PM = \mathcal{V}^+ + \mathcal{V}^-\), we may decompose \(Z_i = V_i^+ + Y_i\), where \(V_i^+ \in \mathcal{V}^+\) and \(Y_i \in \mathcal{V}^-\). Since \(\mathcal{V} \subset \ker(\mathfrak{S}(Z))\), \(\mathfrak{S}(Z)Z_i = \mathfrak{S}(Z)V_i^+\) and thus \(\{\mathfrak{S}(Z)V_1^+, ..., \mathfrak{S}(Z)V_r^+\}\) is a basis for \(\text{range}(\mathfrak{S}(Z))\). As \(\ker(\rho^+) = \mathcal{V}^-\) is timelike, as \(\mathcal{V}^+\) is totally isotropic, and as \(\text{range}(\mathfrak{S}(Z)) \subset \mathcal{V}^\perp\), the vectors

\[
\{\rho^+ \mathfrak{S}(Z) \rho^+ V_1^+ + ..., \rho^+ \mathfrak{S}(Z) \rho^+ V_r^+\}
\]
are linearly independent. Consequently, rank $\mathfrak{S}(Z)) = r$. Since the reverse inequality is immediate, we have as desired that

$$\text{rank}(\mathfrak{S}(Z)) = r \quad \text{for} \quad Z \in S^{p-1} := S^{+}(V^{+}).$$

Since $\mathfrak{S}$ is self-adjoint and $\rho^{+}$ is self-adjoint, $\mathfrak{S}(Z)$ is a self-adjoint map of $V^{+}$. Let $E_{+}$ and $E_{-}$ be the span of the eigenvectors with positive and negative eigenvalues respectively; these are non-trivial as $r > 0$. Since $\mathfrak{S}$ is a self-adjoint map with constant rank, $E_{+}$ and $E_{-}$ are vector bundles over $S^{p-1}$. Since $\mathfrak{S}$ is self-adjoint, and since $\mathfrak{S}(Z)Z = 0$,

$$E_{+}(Z) \perp E_{-}(Z) \quad \text{and} \quad E_{\pm}(Z) \subset Z^\perp = T_{Z}S^{p-1}.$$

Let $N$ be the north pole of $S^{p-1}$. Since $S^{p-1} - \{N\}$ is contractable, there exists a section $s_{+}$ to $E_{+}$ vanishing only at $N$. Since $\mathfrak{S}(-Z) = -\mathfrak{S}(Z)$, $E_{+}(Z) = E_{-}(-Z)$. Thus $s_{-}(Z) := s_{+}(-Z)$ is a section to $E_{-}$ which only vanishes at $-N$. Since $E_{+} \perp E_{-}$, we have $s_{+}(Z) \perp s_{-}(Z)$. Consequently, the vector field

$$s(Z) := s_{+}(Z) + s_{-}(Z)$$

is nowhere vanishing on $S^{p-1}$. Furthermore, we have that $s(Z) = s(-Z)$. This contradicts a result of Szabó and shows that $r = 0$. Hence, $\mathfrak{S}(\cdot)$ vanishes identically on $S^{+}(T_{P}M)$. Consequently, $\nabla R = 0$ on $T_{P}M$, see for example [3].

4. JORDAN OSSERMAN MANIFOLDS

Let $L$ be the second fundamental form of the hypersurface $(M, g_{f})$ and let $R$ be the curvature tensor. We use Lemma [1] and equation (1.4) to see:

$$R(\partial^{e}_{i}, \partial^{e}_{j}, \partial^{e}_{k}, \partial^{e}_{l}) = -\frac{1}{2}\{\partial^{e}_{i} \partial^{e}_{j} (\partial^{e}_{k} f \cdot \partial^{e}_{l} f) + \partial^{e}_{j} \partial^{e}_{k} (\partial^{e}_{i} f \cdot \partial^{e}_{l} f)

\quad - \partial^{e}_{i} \partial^{e}_{j} (\partial^{e}_{k} f \cdot \partial^{e}_{l} f) - \partial^{e}_{j} \partial^{e}_{k} (\partial^{e}_{i} f \cdot \partial^{e}_{l} f)\}

\quad = \partial^{e}_{i} \partial^{e}_{j} f \cdot \partial^{e}_{k} \partial^{e}_{l} f - \partial^{e}_{i} \partial^{e}_{k} f \cdot \partial^{e}_{j} \partial^{e}_{l} f

\quad = L(\partial^{e}_{i}, \partial^{e}_{j}, \partial^{e}_{k}, \partial^{e}_{l}) - L(\partial^{e}_{i}, \partial^{e}_{k}) L(\partial^{e}_{j}, \partial^{e}_{l}).$$

This agrees with the well known formula for the curvature of a hypersurface [3]:

(4.1) \hspace{1cm} R(Z_{1}, Z_{2}, Z_{3}, Z_{4}) = L(Z_{1}, Z_{4}) L(Z_{2}, Z_{3}) - L(Z_{1}, Z_{3}) L(Z_{2}, Z_{4}).

Assertions (1a) and (2a) of Theorem [1.3] will follow from the following Lemma.

**Lemma 4.1.** Assume that $L_{X}$ is non-degenerate on a non-empty connected open subset $O$ of $M$.

1. If $p = 2$, then $(O, g_{f})$ is Jordan Osserman.
2. If $p \geq 3$ and if $L_{X}$ is definite on $O$, then $(O, g_{f})$ is Jordan Osserman.
3. If $p \geq 3$ and if $L_{X}$ is indefinite on $O$, then $(O, g_{f})$ is neither spacelike Jordan Osserman nor timelike Jordan Osserman.

**Proof.** We use an argument motivated by results of Stavrov [10]. Let $P \in M$ and suppose $L_{X}$ is non-degenerate on $T_{P}M$. Let $Z \in S^{\pm}(T_{P}M)$. We decompose $Z = X + Y$ for $X \in \mathfrak{X}(P)$ and $Y \in \mathfrak{Y}(P)$. Since $(Z, Z) \neq 0$ and since $\mathfrak{Y}$ is totally isotropic, $X \neq 0$. By Lemma [3], $J(Z) = J(X)$. As $J(X)^{2} = 0$, rank $(J(X))$ determines the Jordan normal form of $J(X)$. Let $0 \neq X \in \mathfrak{X}(P)$. By equation (1.1), we have:

(4.2) \hspace{1cm} (J(X_{1})X_{2}, X_{3}) = L(X_{1}, X_{1}) L(X_{2}, X_{3}) - L(X_{1}, X_{2}) L(X_{1}, X_{3}).
We have \( J(X)\partial_l^p = 0 \) and \( J(X)X = 0 \). Thus
\[
\text{rank } (J(X)) \leq p - 1.
\]

Suppose first that \( L(X, X) \neq 0 \). We can then choose a basis \( \{X_1,...,X_p\} \) for \( \mathcal{X}(P) \) so \( X_1 = X \) and so \( L(X_i, X_j) = \varepsilon_i\delta_{ij} \), where \( \varepsilon_i \neq 0 \) for \( 1 \leq i \leq p \). We use equation (4.2) to show rank \( (J(X)) = p - 1 \) by computing:
\[
(J(X)X_i, X_j) = \varepsilon_i^2\delta_{ij} \text{ for } i, j \geq 2.
\]

Suppose next that \( L(X, X) = 0 \). We can then choose a basis so \( L(X_1, X_2) = 1 \) and so \( L(X_1, X_i) = 0 \) for \( i \neq 2 \). We show that rank \( (J(X)) = 1 \) by computing
\[
(J(X)X_1, X_i) = -\delta_{1,i} \quad \text{and} \quad (J(X)X_1, X_j) = 0 \text{ for } i \neq 1.
\]

Consequently, if \( 0 \neq X \in \mathcal{A} \), then:
\[
\text{rank } (J(X)) = \begin{cases} p - 1 & \text{if } (X, X) \neq 0, \\ 1 & \text{if } (X, X) = 1. \end{cases}
\]

Suppose \( L_X \) is definite. Let \( X \neq 0 \). Then \( L(X, X) \neq 0 \), so rank \( (J(X)) = p - 1 \) by equation (4.3). This shows that \((\mathcal{O},g_f)\) is timelike and spacelike Jordan Osserman. If \( p = 2 \), then \( p - 1 = 1 \). Equation (4.3) implies rank \( (J(X)) = 1 \) and again \((\mathcal{O},g_f)\) is timelike and spacelike Jordan Osserman. Finally, if \( L_X \) is indefinite and if \( p > 2 \), then rank \( (J(X)) = 1 \) if \( L(X, X) = 0 \) and rank \( (J(X)) = p - 1 \neq 1 \) if \( L(X, X) \neq 0 \). Consequently, \((\mathcal{O},g_f)\) is neither spacelike Jordan Osserman nor timelike Jordan Osserman.

\[5. \text{ JORDAN IP MANIFOLDS}\]

We complete the proof of Theorem 1.3 by proving:

**Lemma 5.1.** Let \( p \geq 2 \). Assume that \( L_X \) is non-degenerate on a non-empty connected open subset \( \mathcal{O} \) of \( M \). Then \((\mathcal{O}, g_f)\) is:

1. spacelike Jordan IP and timelike Jordan IP;
2. not mixed Jordan IP.

**Proof.** We adopt arguments of [6] (see Section 5). Let \( \{Z_1, Z_2\} \) be an orthonormal basis for a non-degenerate 2 plane in \( T_PM \) for \( P \in \mathcal{O} \). We expand \( Z_\nu = X_\nu + Y_\nu \) and use Lemma 2.1 to see \( \mathcal{R}(\pi) = R(X_1, X_2) \). As \( \mathcal{R}(\pi)^2 = 0 \), rank \( (\mathcal{R}(\pi)) \) determines the Jordan normal form. Equation (4.4) implies:
\[
(\mathcal{R}(\pi)X_1, X_4) = L(X_1, X_4)L(X_2, X_3) - L(X_1, X_3)L(X_2, X_4).
\]

If \( \pi \) is spacelike or timelike, then \( \pi \) contains no null vectors and thus \( \{X_1, X_2\} \) are linearly independent vectors. We extend this set to a basis \( \{X_1,...,X_p\} \) for \( \mathcal{X}(P) \). Since \( L_X(P) \) is non-degenerate, we can choose a basis \( \{\hat{X}_1,...,\hat{X}_p\} \) for \( \mathcal{X}(P) \) which is dual (with respect to \( L \)) to the original basis, i.e. \( L(X_i, \hat{X}_j) = \delta_{ij} \). By equation (5.3),
\[
(\mathcal{R}(X_1, X_2)\hat{X}_i, \hat{X}_j) = \begin{cases} 1 & \text{if } i = 2, j = 1, \\ -1 & \text{if } i = 1, j = 2, \\ 0 & \text{otherwise}. \end{cases}
\]

It now follows that rank \( (\mathcal{R}(\pi)) \) = rank \( (\mathcal{R}(X_1, X_2)) \) = 2. Thus \((\mathcal{O},g_f)\) is spacelike Jordan IP and timelike Jordan IP.

To see that \((\mathcal{O},g_f)\) is not mixed IP, we consider the following 2 planes:
\[
\pi_1 := \text{span } \{\partial_1^p, \partial_1^r\} \quad \text{and} \quad \pi_2(\varepsilon) := \text{span } \{\varepsilon^{-1}\partial_1^p + \varepsilon\partial_1^r, -\varepsilon^{-1}\partial_2^p + \varepsilon\partial_2^r\},
\]
respectively, where $\varepsilon$ is a real parameter. The matrices giving the induced inner products on $\pi_1$ and $\pi_2(\varepsilon)$ are given by:

$$A_1 := \begin{pmatrix} 0 & 1 \\ 1 & \varrho_{11} \end{pmatrix} \quad \text{and} \quad A_2 := \begin{pmatrix} 2 + \varepsilon^2 \varrho_{11} & \varepsilon^2 \varrho_{12} \\ \varepsilon^2 \varrho_{12} & -2 - 2\varepsilon^2 \varrho_{22} \end{pmatrix},$$

where $\varrho_{ij} := (\partial_i^x, \partial_j^y)$. Since $\det(A_1) = -1$ and $\det(A_2) = -4 + O(\varepsilon^2)$, $\pi_1$ and $\pi_2(\varepsilon)$ are mixed 2 planes for $\varepsilon$ small. Since $\mathcal{R}(\pi_1) = 0$ and $\mathcal{R}(\pi_2(\varepsilon)) = c(\varepsilon)\mathcal{R}(\partial_1, \partial_2) \neq 0$, $\mathcal{R}(\pi_1)$ and $\mathcal{R}(\pi_2(\varepsilon))$ are not Jordan equivalent and hence $(\mathcal{O}, g_f)$ is not mixed Jordan IP.

6. Manifolds of signature $p \neq q$

Proof of Theorem 1.4. Let $N$ be the isometric product of $\mathbb{R}^{(a,b)}$ with $(M, g_f)$ this has signature $(p + a, q + b)$. Let $R^N$ and $R^M$ be the curvature tensors on $N$ and $M$ respectively. Let $U_\nu$ be tangent vectors on $N$. We decompose $U_\nu = W_\nu + Z_\nu$, where $W_\nu$ is tangent to $\mathbb{R}^{(a,v)}$ and $Z_\nu$ is tangent to $M$. Since

$$R^N(U_1, U_2)U_3 = R^M(Z_1, Z_2)Z_3, \quad \text{and} \quad \nabla_{U_1}R^N(U_2, U_3) = \nabla_{Z_1}R^M(Z_2, BZ_3),$$

we use (2.3) to see that $(N, g_N)$ is nilpotent Osserman Szabó IP; this proves assertion (1); assertions (2) and (3) follow from the corresponding assertions for $(M, g_f)$.

Suppose that $b = 0$. Let $0 \neq U \in T^N$ be spacelike. Expand $U = W_Z$ and $Z = X + Y$. If $X = 0$, then $Z \in \mathcal{Y}$ so $(Z, Z) = (U, U) \leq 0$, which is false. Thus $X \neq 0$ and by equation (4.3)

$$\text{rank}(J(U)) = \text{rank}(J(X)) = p - 1,$$

Thus $(N, g_N)$ is spacelike Jordan Osserman. One shows similarly that $(N, g_N)$ is spacelike Jordan IP. This proves assertion (4b); assertion (5a) follows similarly.

Suppose $b > 0$. We can choose $0 \neq W \in T(\mathbb{R}^{(a,b)})$ spacelike. Then we have that rank $(J(W)) = 0$. We can choose $0 \neq Z \in TM$ spacelike so rank $(J(Z)) = p - 1$. Thus $(N, g_N)$ is not timelike Jordan Osserman. Similarly, we may show that $(N, g_N)$ is not timelike IP. This proves assertions (5b) and (6b); the proof of assertion (6a) is similar.

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