Quantum and classical probability as Bayes-optimal observation.

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Abstract

We propose a simple abstract formalisation of the act of observation, in which the system and the observer are assumed to be in a pure state and their interaction deterministically changes the states such that the outcome can be read from the state of the observer after the interaction. If the observer consistently realizes the outcome which maximizes the likelihood ratio that the outcome pertains to the system under study (and not to his own state), he will be called Bayes-optimal. We calculate the probability if for each trial of the experiment the observer is in a new state picked randomly from his set of states, and the system under investigation is taken from an ensemble of identical pure states. For classical statistical mixtures, the relative frequency resulting from the maximum likelihood principle is an unbiased estimator of the components of the mixture. For repeated Bayes-optimal observation in case the state space is complex Hilbert space, the relative frequency converges to the Born rule. Hence, the principle of Bayes-optimal observation can be regarded as an underlying mechanism for the Born rule. We show the outcome assignment of the Bayes-optimal observer is invariant under unitary transformations and contextual, but the probability that results from repeated application is non-contextual. The proposal gives a concise interpretation for the meaning of the occurrence of a single outcome in a quantum experiment as the unique outcome that, relative to the state of the system, is least dependent on the state of the observer at the instant of measurement.

1 Introduction

As early as 1935, Schrödinger wrote: “The rejection of realism has logical consequences. In general, a variable has no definite value before I measure it; then measuring it does not mean ascertaining the value that it has. But then what
As the advent of quantum mechanics solved the long standing problem of providing an adequate description for several important and unexplained experiments, the problem of realism in quantum mechanics was initially perceived mainly as a challenge to the construction of a new philosophy of natural science. In support of this perception, is the fact that almost all later theoretical advances with experimental consequences came about without any serious progress with this very basic problem. Yet at the same time, a growing number of people recognized that progress in this problem would likely have deep consequences for the quantum-classical transition, the attempt to produce a successful unification of quantum mechanics and relativity theory, and the related problem of quantum cosmology. Halfway the sixties two important advances were made. In 1964, John Bell showed that any local hidden variable theory will yield predictions that are at odds with quantum mechanics. A few years later, Kochen and Specker [23] presented an explicit set of measurements, for which the simultaneous attribution of values for each of these measurements, leads to a logical contradiction. These two results have been of such importance, that the notion of realism in quantum physics is usually considered automatically as having either the meaning of ‘locally realistic’ (Bell), or that of ‘the impossibility of attributing predetermined outcome values to the set of observables’ (Kochen and Specker). The apparent lack of realism in quantum mechanics has been illustrated again and again by clever theoretical constructions ranging from Bell-type arguments to impossible coloring games, and the countless attempts to produce an as loophole free as possible experimental verification of these arguments.

However, the commonly accepted notion that “measuring a variable does not mean ascertaining the value that it has”, does not mean that the answer to Schrödinger’s question is that the occurrence of a particular outcome has no meaning. Every proper quantum experiment is a testimony to the contrary, for if a single outcome has no informational content about the system at all, then how are we to derive anything at all from the sum of a great number of informationally empty statements? Whether we perform a tomographic state reconstruction, or experimentally estimate the value of a physical quantity of a system, we accept that in a well constructed experiment every outcome presents a piece of information, a piece of evidence, that brings us closer to the true state of affairs, whatever that may be. To give a more detailed answer to the question, more information is needed.

1Because local theories, by Bell’s theorem, cannot give rise to some of the experimentally verifiable predictions of quantum mechanics, the requirement of locality, or so-called “local-realism” takes a prominent role. However, realism seems more fundamental than locality, in the sense that the latter is only well-defined if we can attribute some form of reality with respect to the whereabouts of the system. Moreover, the derivation of the quantum correlation for most Bell-type experiments do not, at any point, invoke spatial coordinates. As far as concerns the actual application of quantum theory, it is quite immaterial whether we calculate the correlations between various outcomes that are obtained in a single location or at space-like separated locations. Of course, for a locally realistic theory, the difference is huge.
we are in need of a model that shows how a single outcome is obtained. We will provide such a model in an attempt to understand the meaning of the occurrence of a single outcome in a quantum mechanical experiment. More specifically, we will show that an observer actively seeking to minimize his own influence on the produced outcome, will, with the aid of Bayesian decision theory, give outcomes whose relative frequency converges to the Born rule in a natural way. This in turn will give us a possible interpretation for the occurrence of a particular outcome.

2 Probabilities of outcomes for a single observable quantity

Let us assume we have a system $S$ for which we write $\Sigma_S$ to denote its set of states, and $A$ for an observable that can take any single outcome out of $n$ distinct values in the outcome set $X = \{x_1, \ldots, x_n\}$. At the most trivial level, there is a counting measure on the set of outcomes. If $\mathcal{P}(X)$ denotes the set of all subsets of $X$, then the probability that a measurement of observable $A$ on the system in a state $\psi \in \Sigma_S$ yields an outcome in a given subset $X_1 \in \mathcal{P}(X)$, is a mapping

$$p(\cdot | \cdot) : \mathcal{P}(X) \times \Sigma_S \to [0, 1]$$

such that for disjoint $X_i \in \mathcal{P}(X)$, we have:

$$p(\cup X_i | \psi) = \sum_i p(X_i | \psi)$$

The additive property described by (2) is generally accepted both in quantum and classical probability and provides the rationale for the use of normalized states, that is, states $\psi$ that satisfy:

$$p(X | \psi) = 1$$

In this way, (3) reduces the number of free parameters in state space by one. We have written $p(x | \psi)$ to emphasize that it represents the probability that the outcome $x$ obtains when (we know that) the system is prepared in the state $\psi$. The classical interpretation for the arisal of probabilities, is one of a lack-of-knowledge about the precise state being measured. From a naive epistemic perspective, the outcome $x$ is then an objective attribute of each measured state, and the probability related to each outcome is simply the fraction of states having the “$x$-attribute” in the ensemble of systems that we measure. As indicated in the introduction, such an interpretation for the probabilities in quantum mechanics is problematic. Even for a single spin 1/2 particle, one can show [1] three measurements suffice to exclude such an interpretation, even without taking recourse to locality issues.
2.1 Quantum probability for a single observable quantity

In orthodox quantum mechanics, the state space $\Sigma_S$ is the complex Hilbert space $\mathcal{H}$. The set of states of the observed system that we will consider, is the set of unit vectors in an $n$-dimensional Hilbert space $\mathcal{H}_n$,

$$\Sigma_S = \{ \psi \in \mathcal{H}_n : |\psi| = 1 \}$$  (4)

As usual, the norm $|\cdot|$ is defined through the (sesquilinear) inner product that we will denote $\langle \cdot | \cdot \rangle$. Alternatively, one can take rays or even density operators for the states. Since both lead to essentially the same results, we will stick to unit norm vectors. Let $\mathcal{L}(\mathcal{H}_n)$ be the set of linear operators that act on the elements of $\mathcal{H}_n$, then an observable $A$ is represented by a self-adjoint element of $\mathcal{L}(\mathcal{H}_n)$:

$$A \in \mathcal{L}(\mathcal{H}_n) : A^\dagger = A$$  (5)

Throughout this presentation, we assume $A$ has a discrete, finite, non-degenerate spectrum, which implies that eigenvectors belonging to different eigenvalues are orthogonal. Let $F_A$ be the set of the eigenvectors of $A$:

$$F_A = \{ \psi_i \in \mathcal{H}_n : A|\psi_i\rangle = c_i|\psi_i\rangle, \; c_i \in \mathbb{R} \}$$  (6)

We now have $\langle \psi_i | \psi_j \rangle = \delta_{i,j}$ and $\sum_i |\psi_i\rangle\langle\psi_i| = I$, and, because the spectrum is assumed non-degenerate, we have that $F_A$ is a basis or a complete orthonormal frame. From linear algebra we know that an arbitrary element $\psi^s$ of $\mathcal{H}_n$ can be written in this frame $F_A$ as:

$$|\psi^s\rangle = \sum_{i=1}^n \alpha_i |\psi_i\rangle$$  (7)

If $\psi^s$ satisfies (3), then it lies in $\Sigma_S \subset \mathcal{H}_n$, and the $\alpha$’s obey:

$$\sum_i \alpha_i \alpha_i^* = 1$$  (8)

Moreover, one can easily verify that the observable $A$ can be written as

$$A = \sum_i a_i |\psi_i\rangle\langle\psi_i|$$  (9)

Hence the observable $A$ is in a one-to-one correspondence with an orthonormal frame $F_A$ of eigenvectors of $A$ and we will represent the observable by its associated frame. Throughout this paper, we reserve superscripts of states as a mnemotechnical aid for system recognition (i.e. $\psi^s$ is a system state and $\psi^m$ the state of the measurement apparatus) and subscripts of states to denote eigenstates. If a system is in an eigenstate corresponding to outcome $x_i$, we will...
denote the corresponding eigenstate as \( \psi_i \). For an arbitrary eigenstate \( \psi_j \) we have

\[
p(x_i|\psi_j) = \delta_{i,j}
\]  

(10)

Thus for an eigenstate, and also for a statistical mixture of eigenstates, the classical interpretation of probability as “proportion of system having the \( x \)-attribute” is tenable. The more interesting case, however, is the probability for the occurrence of an outcome \( x_i \) when the system is in a general state \( \psi_s \), which is given by the Born rule:

\[
p(x_i|\psi_s) = |\langle \psi_i, \psi_s \rangle|^2 = |\alpha_i|^2
\]  

(11)

The analog with the classical situation would be that \( \psi_s \) represents a mixture of states that have attribute \( x_k \) in the right proportion such that the Born rule holds. However the Born rule holds even when the system is in a pure state, i.e. a state which cannot be obtained as a statistical mixture of states. We will show that it is possible to regard the probabilities as arising from a lack of knowledge about the detailed state of the observer if the observer actively attempts to choose the outcome that maximizes a specific likelihood ratio that we will present shortly.

3 The process of observation

3.1 The deterministic observer

Let us first define what we mean by an observer. An observer is a physical system that takes a question as input, and yields in reply an outcome which is a member of a discrete set. This outcome can be freely copied, and hence communicated to many other observers. In general, this definition of observer will include the experimental setup, apparatus, sensors, and the human operator. It is however quite irrelevant to our purposes whether we consider an apparatus or a detector, an animal or a human being as observer, as long as we agree that it is this system that has produced the outcome. We will furthermore assume the observer comes to this outcome through a physical, deterministic interaction. That is, if we have perfect knowledge of the initial state of the system and of the potentials that act on the system, we can in principle predict the future state of the system perfectly. Besides the fact that all fundamental theories of physics (even classical chaotic systems and quantum dynamics) postulate deterministic evolution laws, the requirement of determinism allows to derive probability as a secondary concept. So let us assume that the outcome of an observation is the result of a deterministic interaction:

\[
\tau : \Sigma_S \times \Sigma_M \rightarrow X
\]  

(12)

Here \( \tau \) is the interaction rule, \( \Sigma_S \) is the set of states of the observed system, \( \Sigma_M \) the set of states of the observing system and \( X \) the set of outcomes that
observable \( A \) can have. We will deal only with a single observable, so no further notational reference is made to the particular observable. The mapping \( \tau \) encodes how an observer in a state \( \psi^m \in \Sigma_M \), observing a system in the state \( \psi^s \in \Sigma_S \), comes to the outcome \( x \in X \). Because our observer is deterministic, we assume \( \tau \) is single-valued. Probability will only arise as a lack of knowledge on deterministic events. The observer faces the task of selecting an outcome from the set \( X \) that tells something about the system under observation. But the outcome is always formulated by the observer, it has to be encoded somehow in the state of the observer after the observation. Hence the outcome itself is also an observable quantity of the post-measurement state of the observer. The outcome will then have to share its story among the two participating systems that gave rise to its existence: it will always have something to say about both the observer and the system under study. In \cite{6} it was shown by a diagonal argument, that even in the most simple case of a perfect observer, observing only classical properties\(^3\), there exist classical properties pertaining to himself that he cannot perfectly observe. More specifically, even if the observer can observe a given (classical) property perfectly, he cannot perfectly observe that he observes this classical property perfectly. There is no logical certainty with respect to faithfulness of a single shot, deterministic observation. On the other hand, observation is an absolutely indispensable part of doing science, hence it is only natural that every scientist believes that faithful observation can and does indeed occur. Living in the real world, somewhere between the extremes of the ideal and the impossible, we wonder whether there is a strategy for the observer so that he is guaranteed that each outcome he picks uses his observational powers to the best of his ability.

### 3.2 Repeated measurement and the randomization of probe states of the observer

Rather than attempting to measure observables in a single trial of an experiment, our observer turns to a new strategy. First he prepares an ensemble of a large number of identical system states. Next he will interact with each of the members of this ensemble in turn. For each and every single interaction, he will pick the outcome that somehow ‘has the largest likelihood’ of pertaining to the system. By randomizing his probe state and picking the outcomes in this way, the observer hopes to restore objectivity, so that he will eventually obtain information that pertains solely to the system under observation. To calculate \( p(x|\psi^s) \) within the deterministic setting of the previous section \cite{12} is in principle straightforward. The experiment our observer will perform is a

\(^3\)We say the property \( a \) of a system \( S \) in the state \( s \) is actual, iff the testing of property \( a \) for \( S \) in the state \( s \), would yield an affirmation with certainty. A property is called classical when the outcome of the observation to test that property, was predetermined by the state of the system (whatever that state was) prior to the test. For a classical property we can define a negation in the lattice of properties that is simply the Boolean NOT. A property \( a \) is then classical for \( S \) iff for each state of \( S \) the property, or its negation, is actual. For details, see \cite{9}.
repeated one, in which the set of states of the system under study is reduced to a singleton, and the set of states for the observer is the whole of \( \Sigma_M \). The set of states for the observer that leads to a given outcome \( x \in X \) when the observer observes a system in the state \( \psi^s \) will be denoted as \( \text{eig}(x, \psi^s) \):

\[
\text{eig}(x, \psi^s) = \{ \psi \in \Sigma_M : \tau(\psi^s, \psi) = x \}
\]  

(13)

From the single-valuedness of \( \tau \) in (12), we have for \( x_i \neq x_j \):

\[
\text{eig}(x_i, \psi^s) \cap \text{eig}(x_j, \psi^s) = \emptyset \quad (14)
\]

If we assume that the act of observation of an observable leads to an outcome for every state of the system investigated, we have

\[
\bigcup_{i=1}^n \text{eig}(x_i, \psi^s) = \Sigma_M \quad (15)
\]

In this way \( \tau \) defines in a trivial way a partition of the state space of the observer with each member \( \text{eig}(x_i, \psi^s) \) in the partition belonging to exactly one outcome.

We are now ready to introduce probability. With \( B(\Sigma_M) \) a \( \sigma \)-algebra of Borel subsets of \( \Sigma_M \), (which we tacitly assume includes \( \text{eig}(x_i, \psi^s) \) for every \( i \)), we define a probability measure \( \mu \) that acts on the measure space \((\Sigma_M, B(\Sigma_M))\).

For any two disjoint \( \sigma_i, \sigma_j \) in \( B(\Sigma_M) \), we have

\[
\mu : B(\Sigma_M) \to [0, 1] \\
\mu(\sigma_i \cup \sigma_j) = \mu(\sigma_i) + \mu(\sigma_j) \\
\mu(\Sigma_M) = 1
\]  

(16)

In order to calculate \( p(x|\psi^s) \), we need to evaluate the probability measure over the set of states for the observer giving rise to the outcome \( x \) when they interact with a state \( \psi^s \):

\[
p(x|\psi^s) = \frac{\mu(\text{eig}(x, \psi^s))}{\mu(\bigcup_{i=1}^n \text{eig}(x_i, \psi^s))} = \frac{\mu(\text{eig}(x, \psi^s))}{\mu(\Sigma_M)}
\]  

(17)

(18)

This last formula is fundamental to this paper. It says that for a repeated experiment on a set of identical pure system states, the probability \( p(x|\psi^s) \) is given as the ratio of observer states that, given \( \psi^s \), tell the outcome is \( x \), to the total number of observer states.

Note that the sets \( \text{eig}(x_i, \psi^s) \) are not sets of eigenvectors in the algebraic sense of the word\(^5\). However, if it happens to be the case that, for a given \( \psi^s \) and for almost every \( \psi \in \Sigma_M \), we have \( \tau(\psi^s, \psi) = x_k \) in the sense that

\[
\mu(\text{eig}(x_k, \psi^s)) = \mu(\Sigma_M)
\]  

(19)

\(^4\)In accordance with the literature on the subject, we used Dirac’s bra-ket notation for our brief introduction to quantum probability. In what follows we will not make use of the duality between a Hilbert space and the space of linear functionals on this Hilbert space, so all vectors are written without brackets.

\(^5\)The sets \( \text{eig} \) are called in eigensets in accordance with \[4\].
then, for that particular \( \psi^s \), we have \( p(x_k|\psi^s) = 1 \). The vector \( \psi^s \) thus defined, will coincide with a regular eigenvector if the state space is a Hilbert space.

The relation between (16) and (1) is through the mapping \( \tau \) and the measure \( \mu \). It is obvious that (16) is additive in \( X \) too

\[
\mu(\text{eig}(x_i, \psi^s) \cup \text{eig}(x_j, \psi^s)) = \mu(\text{eig}(x_i, \psi^s)) + \mu(\text{eig}(x_j, \psi^s))
\]

(20)

because of (14). Hence, if the probabilities of (16) and (1) coincide for every single outcome (the singletons in \( P(X) \)), they will coincide for all of \( P(X) \). In what follows we will therefore restrict our discussion to the probability related to the occurrence of a single outcome. In conclusion, the success of the program to model the probabilities in quantum mechanics as coming from a lack of knowledge about the precise state of the observer stands or falls with the question of defining a natural mapping \( \tau \) (which determines the outcome and hence \( \text{eig}(x, \psi^s) \) ) such that the measure \( \mu \) of the eigenset \( \text{eig}(x_i, \psi^s) \) pertaining to outcome \( x_i \) is identical with the probability obtained by the Born rule (11).

### 3.3 The Bayes-optimal observer

We can see from (17) that the system state \( \psi^s \) can be associated with a probability in a fairly trivial way: the probability of an outcome \( x \) when the system is in a pure state \( \psi^s \), is the proportion of observer states that attribute outcome \( x \) to that state. Even for a repeated measurement on a set of identical pure states, fluctuations in the outcomes can arise if there is a lack of knowledge concerning the precise state of the observer. Suppose now the observer, considered as a system in its own right, is in a state \( \psi^m \). Then in exactly the same way we can associate a probability with that state too. The operational meaning of this association is given either by a secondary observer observing an ensemble of observers in the state \( \psi^m \), or by the observer consistently (mis)identifying his own state \( \psi^m \) for a state of the system \( \psi^s \). We have argued that every outcome will say something about the observer, (that is, about \( \psi^m \)), and something about the system (that is, about \( \psi^s \)). The problem is that this information is mixed up in a single outcome. Some outcomes will contain more information about the state of the system, and some more about the state of the apparatus. Eventually, we, as operators of our detection apparatus, will have to decide whether we will retain a given outcome, or reject it. Such decisions are a vital part of experimental science. For example, an outcome that is deemed too far off the limit (so-called outliers), is rejected and hence excluded in the subsequent analysis. The rationale for this exclusion is that an outlier does not contain information about the system we seek to investigate, but rather that it represents a peculiarity of the measurement. In practice, rejection or acceptance of an outcome does not depend on a rational analysis, but on the common sense and expectations of the experimenter. Suppose however, that the observer does have absolute knowledge about the state of the system \( \psi^s \) and his own state \( \psi^m \), and recognizes the fact that the outcome he delivers may eventually be rejected. The observer considers this rejection to be based on the following binary
hypotheses:

\[ H_0 : \text{the outcome } x_i \text{ was inferred from } \psi^s \quad (21) \]

\[ H_1 : \text{the outcome } x_i \text{ was inferred from } \psi^m \]

In full, the hypotheses should actually read: “The outcome \( x_i \) yields as a consequence of the observer attributing the state \( \psi^s \) (or \( \psi^m \)) to the system”. To combat rejection, the observer chooses the outcome that maximizes the likelihood that \( H_0 \) prevails, as if the outcome he delivers will eventually be judged for acceptance or rejection by one with absolute knowledge about \( \psi^s \) and \( \psi^m \).

If, in an experiment, it is possible with (non-vanishing probability) to get an outcome \( x_i \) under either hypothesis, then a factual occurrence of this outcome in an experiment supports both hypotheses simultaneously. What really matters in deciding between \( H_0 \) and \( H_1 \) on the basis of a single outcome, is not the probability of the correctness of each hypothesis itself, but rather whether one hypothesis has become more likely than the other as a result of getting outcome \( x_i \). From Bayesian decision theory \[21\], we have that all the information in the data that is relevant for deciding between \( H_0 \) and \( H_1 \), is contained in the so-called likelihood ratios or, in the binary case, the odds \( \Lambda_i \):

\[ \Lambda_i = \frac{p(x_i|\psi^s)}{p(x_i|\psi^m)}, i = 1, \ldots, n \quad (22) \]

In this last formula, the numerator and denominator are given by (17). We are now in position to state our proposed strategy for the Bayes-optimal observer.

**Definition 1 (Bayes-optimal observer)** We call a system \( M \) in a state \( \psi^m \) a Bayes-optimal observer iff, after an interaction with a system in a state \( \psi^s \), the state of \( M \) will transform to a state that expresses the outcome \( x_i \) that corresponds to the maximal likelihood ratio \( \Lambda_i \) \[24\].

Picking the outcome \( x_i \) from \( X \) that maximizes the corresponding likelihood ratio \( \Lambda_i \), is simply optimizing the odds for \( H_0 \), given his information. This concludes our description of the observer. To see what probability arises for a repeated experiment when an observer is Bayes-optimal, we need a state space. We are especially interested in complex Hilbert space, but we will first have a look at statistical mixtures.

### 3.4 The Bayes-optimal observer for statistical mixtures

If the conditional probabilities \( p(x_i|\psi^s) \) are well-defined (which we will just accept for now), we can make a summary of them in a single vector \( \mathbf{x}(\psi^s) \):

\[ \mathbf{x}(\psi^s) = \sum_{i=1}^{n} p(x_i|\psi^s)x_i \quad (23) \]

First we define the convex closure of a number of elements \( a_1, \ldots, a_n \in A_i \):

\[ [a_1, \ldots, a_n] = \{ a \in \mathbb{R}^n : a = \sum \lambda_i a_i, 0 \leq \lambda_i \in \mathbb{R}, \sum \lambda_i = 1 \} \quad (24) \]
If we write \([C]\), as we shortly will, we mean the convex closure of the elements in \(C\). The standard \((n-1)\) simplex \(\Delta_{n-1}\) generated by the outcome set \(X\) is:

\[
\Delta_{n-1}(X) = [x_1, \ldots, x_n]
\] (25)

We see from (23), (19) that \(x(\psi^s)\) belongs to \(\Delta_{n-1}(X)\). By identification of the axes of \(\mathbb{R}^n\) with the members of \(X\), we have \(\Delta_{n-1}(X) \subset \mathbb{R}^n(X)\), the free vector space generated by the outcome set \(X\). Vectors like \(x(\psi^s)\) are often called ‘statistical states’ or ‘mixtures’ in the literature. Suppose now that all we can or care to know about the system \(S\) and the observer \(M\), are the statistical states, i.e. the probabilities related to the outcomes of a single experiment. Within this constraint, the vector \(x(\psi^s)\) represents all there is to know about \(S\) and the state spaces \(\Sigma_S\) and \(\Sigma_M\) reduce to \(\Delta_{n-1}(X)\):

\[
\Sigma_S = \Sigma_M = \Delta_{n-1}(X)
\] (26)

Having identified \(\psi^s\) with \(x(\psi^s)\) in this particular case, the conditional probability \(p(x_1|x(\psi^s))\) denotes the probability that outcome \(x_1\) occurs when our knowledge about the system is encoded in the statistical state \(x(\psi^s)\):

\[
x(\psi^s) = p(x_1|x(\psi^s))x_1 + \ldots + p(x_n|x(\psi^s))x_n
\] (27)

In this section \(\langle , \rangle\) denotes the standard inner product in Euclidean space, and with \(\langle x_i, x_j \rangle = \delta_{ij}\), we have from this last equation

\[
p(x_i|x(\psi^s)) = \langle x(\psi^s), x_i \rangle
\] (28)

For a statistical state, the magnitude of the \(i^{th}\) coordinate equals the probability of outcome \(x_i\). We have a state space (25), and we have a rule to extract a probability from a state (28), so we can characterize the sets \(\text{eig}(x_k, x(\psi^s))\). Let \(x(\psi^s)\) and \(x(\psi^m)\) be arbitrary states in \(\Delta_{n-1}(X)\), written as:

\[
x(\psi^s) = \sum_{i=1}^{n} t_i x_i
\]

\[
x(\psi^m) = \sum_{i=1}^{n} r_i x_i
\] (29)

By the definition of Bayes-optimal observation, we have that the outcome \(x_k\) is chosen, if for all \(j \neq k\), the corresponding likelihood ratio’s satisfy \(\Lambda_k > \Lambda_j\). By (22) and (23), \(x_k\) is chosen, if for all \(j = 1, \ldots, n\) \((j \neq k)\), we have:

\[
\frac{p(x_k|x(\psi^s))}{p(x_k|x(\psi^m))} > \frac{p(x_j|x(\psi^s))}{p(x_j|x(\psi^m))}
\] (30)

The regions \(\text{eig}(x_k, x(\psi^s))\), are found by substitution of (29) in (28) and then into (31). With \(j = 1, \ldots, n; \quad j \neq k\), we obtain:

\[
\text{eig}(x_k, x(\psi^s)) = \{ x(\psi^m) \in \Delta_{n-1} : \frac{t_k}{r_k} > \frac{t_j}{r_j} \}
\] (31)
According to (17), the probability of the outcome $x$ for the repeated experiment on a set of identical system states, is the ratio of observer states that tell the outcome is $x$, to the total number of observer states. Because the state space is Euclidean, it is natural to take for $\mu$ the $(n-1)$-Lebesgue measure in $\Delta_{n-1}$, assumed to be normalized: $\mu(\Delta_{n-1}(X)) = 1$. The probability $p_{BO}(x_k|x(\psi^s))$ that the Bayes-optimal observer obtains the outcome $x_k$ is then given by

$$p_{BO}(x_k|x(\psi^s)) = \mu(eig(x_k,x(\psi^s)))$$

(32)

However, because of the way we defined the statistical state, the probability is also given directly by components of the state. So the question is whether the Bayes-optimal observer (32) can recover that probability, i.e. is it true that (32) equals (28):

$$\mu(eig(x_k,x(\psi^s))) = \langle x(\psi^s), x_i \rangle$$

(33)

To see if this is the case, we first define the open convex closure of a number of elements $x_1,\ldots,x_n \in \mathbb{R}^n$ as

$$[x_1,\ldots,x_n] = \{ x \in \mathbb{R}^n : x = \sum \lambda_i x_i, 0 < \lambda_i \in \mathbb{R}, \sum \lambda_i = 1 \}$$

(34)

We can now characterize $eig(x_k,x(\psi^s))$ for the statistical state as being ‘almost equal’ to

$$C^s_k = [x_1,\ldots,x_{k-1},x(\psi^s),x_{k+1},\ldots,x_n]$$

(35)

A graphical representation of the eigensets in the simplex state space can be found in Figure (1).

**Lemma 2** Let $C^s_k$ be defined as in (35), $[C^s_k]$ be the convex closure of $C^s_k$, and $eig(x_k,x(\psi^s))$ by (31), then:

$$C^s_k \subset eig(x_k,x(\psi^s)) \subset [C^s_k]$$

The proof of this lemma can be found in appendix A. To obtain the probability (32), we calculate the $\mu$–measure of $[C^s_k]$, which is simply the $(n-1)$-dimensional volume of the simplex $[C^s_k]$.

**Lemma 3** If $\mu$ is a (probability) measure such that $\mu(\Delta_{n-1}(X)) = 1$, and $C^s_k$ is defined by the convex closure of (35), then we have $\mu([C^s_k]) = t_k$

One can calculate of the volume of a simplex straightforwardly by determinant calculus, as was done in [2]. For completeness, we have included an alternative in the form of a simple geometric argument in appendix B. We then easily obtain:

**Theorem 4** $\mu(eig(x_k,\psi^s)) = t_k$

**Proof.** By the first lemma, we have $C^s_k \subset eig(x_k,x(\psi^s)) \subset [C^s_k]$. Because $A \subset B \implies \mu(A) \leq \mu(B)$ we have

$$\mu(C^s_k) \leq \mu(eig(x_k,x(\psi^s))) \leq \mu([C^s_k])$$
Figure 1: Illustration of the scheme in the simplex state space. We start with the discrete outcome set, depicted in figure (a). The state space for an outcome set with three outcomes, is the standard 2-simplex in the free vector space generated by the outcome set over the field of real numbers, as depicted in picture (b). In figure (c), we see the eigensets $C_s^k$, so we can see what outcome will be obtained from a Bayes-optimal measurement. An apparatus state picked from the darkest region, $C_s^2$, will lead to the outcome $x_2$, in the lightest region, $C_s^1$, to $x_1$, and the intermediately shaded region, $C_s^3$, leads to the outcome $x_3$. The probability is the Lebesgue measure over the depicted eigensets. I.e., the probability of obtaining the outcome $x_2$, is the normalized area of the darkest triangle.
Figure 2: A graphical exposition of the contextuality of the outcome assignment.

(a) If we pick an observer state from the open white triangle $|\psi^s, x_2, x_3\rangle$, then a measurement of the state $\psi_s$ will yield outcome $x_1$. (b) If we interchange the second and third component of $\psi^s$, we obtain $\psi_1^s$. The probability of obtaining outcome $x_1$ is the same as in picture (a), because the two triangles have the same area. However, an observer state chosen from the black shaded region would yield outcome $x_1$ in picture (a), whereas it would yield $x_2$ in picture (b). Note that we did not change the $x_1$ component in the state $\psi_s$ to obtain $\psi_1^s$.

By the second lemma we have $\mu([C_k^s]) = t_k$. To calculate $\mu(C_k^s)$, we note that $\mu(C_k^s) = \mu([C_k^s]) - \mu([C_k^s] \cap C_k^s)$. Because $[C_k^s] \cap C_k^s$ is the collection of faces of $C_k^s$, a set of finite cardinality whose members have an affine dimension maximally equal to $n - 2$, it is $\mu$-negligible, hence we also have $\mu(C_k^s) = t_k$, establishing the result.

We see that indeed the Bayes-optimal observer recovers the probability that was encoded in the statistical state:

$$p_{BO}(x_k|x(\psi^s)) = t_k = p(x_k|x(\psi^s))$$

In this way the observer succeeds in obtaining a quantity that, in the limit of infinite measurements, depends only on the state of the system under investigation, and not on his own state. The results we have obtained for the simplex state space are identical to those in [2], where the scheme was proposed under the name “hidden measurements” to indicate the origin of the lack of knowledge. In [2] the eigensets are postulated ad hoc, whereas we have derived their simplicial shape from the principle of Bayes-optimal observation. We will use this principle in the next section to extend the results of [2] to systems with a complex state space.

Before we do so, two remarks are in order. First, we did not specify whether the state $x(\psi^s)$ is the result of mixing ‘pure’ components with appropriate weights, as indicated by the components of the state, or whether it represents a statistical tendency, somewhat like a propensity, of an ensemble of identical
‘pure’ states to reveal itself in the different outcomes. That is, if all we are allowed to do is perform a single experiment on each member of the ensemble, then from the resulting statistics of a single observable, we cannot distinguish between these two situations. In other words, if we have an urn filled with coins and we are allowed to inspect the coin only after a single throw of the coin, for every coin in the urn, then we cannot know whether it is a tendency of the coin to show heads with probability 1/2, or whether half of the coins have both sides heads and half of them have both sides tails (or indeed a mixture of these two situations). Secondly, it is interesting that, even for the conceptually simple statistical mixtures, the outcome assignment given by the Bayes-optimal observer is contextual in the following sense: given a state for the observer and system that lead to the outcome \(x_l\), then the mere interchanging of the coefficients \(t_j\) and \(t_k\) (equal to the probability for the outcomes \(x_j\) and \(x_k\)) can easily result in a different outcome than \(x_l\), even if neither \(x_j\) nor \(x_k\) is equal to \(x_l\)!

This can readily be verified in Figure (2). However, the probability \(p(x_i|x(\psi^s))\) of the outcome is a function of \(t_i\) only, hence the probability itself is non-contextual. Conversely, given a state of an observer \(\psi^m\) and a system state \(\psi^s\) that interact to yield the outcome \(x_k\), it is often possible to change the outcome of the Bayes-optimal observer to a different outcome by interchanging suitable coefficients of the observer, leaving \(r_k\) untouched. This means that changing only the observer’s preferences over the outcomes \(x_j\) and \(x_l\), may let the Bayes-optimal observer decide another outcome than \(x_k\) is more optimal, even if \(j, l\) and \(k\) are all different! This contextual aspect of the outcome assignment can here be understood as a result of the inescapable bias introduced by the state of the observer in producing a single outcome, for the coefficients of his state represent his tendencies for each outcome\(^6\). Perhaps somewhat paradoxically, it is precisely through the averaging procedure over all the different possibilities for this bias, that a non-contextual probability emerges. From Figure (2) we see that the contextuality of the outcome assignment depends on the classical entropy of the state. According to a well-known theorem due to Shannon, the higher the entropy of the state \(\psi^s\), the closer the coefficients of \(\psi^s\) in (23) are to \(1/n\) and the closer this state will reside near the centre of the simplex, effectively limiting the possibilities for producing a contextual outcome change by interchanging coefficients.

### 3.5 Bayes-optimal observation in complex Hilbert space

Complex Hilbert spaces are of considerable interest as they arise naturally in many prominent scientific areas including quantum theory, signal analysis (both in time-frequency and in wavelet analysis), electromagnetism and electronic networks\(^7\), and the more recently founded shape theory \[22\]. The natural setting

---

\(^6\)This tendency could be revealed if we fix the state of the observer and observe (by means of a second observer) the relative frequencies for the outcomes he produces when he measures members of an ensemble of randomly choosen states.

\(^7\)Interestingly, the name probability amplitude, and indeed the Born interpretation of the wave vector in quantum mechanics, were conceived by Born in analogy with electromagnetic
for the discrete state space in these examples, is the space of square summable functions on a Hilbert space \( \mathcal{H}_n(\mathbb{C}) \) over the field of complex numbers. A general state of the system \( \psi^s \in \Sigma_S = \mathcal{H}_n(\mathbb{C}) \) can then be written as:

\[
\psi^s = \sum_{i=1}^{n} q_i x_i \tag{36}
\]

where \( q_i \in \mathbb{C} \) and \(|\psi^s| = 1\). In this case the outcome set \( X \) consists of an orthonormal frame of complex vectors \( \{x_i\} \). An observer (or a detector, which is quite the same for our purposes) usually has a very large number of internal degrees of freedom. Accordingly it lives in a Hilbert space of appropriately high dimensionality. However, by the Schmidt bi-orthogonal decomposition theorem, we know we can model every possible interaction between two systems, one living in a Hilbert space of dimension \( n \) and one in a Hilbert space of dimension \( m \) with \( m > n \), by an interaction of two systems, each one living in a Hilbert space of dimension \( n \). With this in mind, we model the set of states of the observer as unit vectors in \( \mathcal{H}_n \):

\[
\Sigma_M = \{ \psi \in \mathcal{H}_n(\mathbb{C}) : |\psi| = 1 \}
\]

The reader should take note of the fact that, every time we speak about “the state of the observer”, we mean the state in the subspace indicated by the Schmidt bi-orthogonal decomposition theorem. The state of the observer, to us, always means only that part of the state that is of relevance to the production of the outcome. This is especially relevant for the interpretation of sentences such as “uniform distribution of initial observer states”, which taken too literally, would indicate the observer is perhaps doing something completely different than observing. The state of an observer with respect to an experiment with outcome set \( X \) can be written as \((r_i \in \mathbb{C})\)

\[
\psi^m = \sum_{i=1}^{n} r_i x_i \tag{37}
\]

Because the coefficients now assume complex values, they cannot be interpreted as probabilities because we do not have a total order relation in the field of complex numbers. This difference also affects the deeper, deterministic level of the description in a profound way. Let us explain why this is the case. For the statistical states of the former section, each eigenset is a subsimplex of the state space. A simplex is a (very) special case of a convex set. Because the eigensets share at most a lower dimensional face, any two different eigensets (for a fixed system state) can be separated\(^8\) by a single hyperplane. But in a complex waves. Here the norm is not unity, but equal to the energy in the wave, and probability conservation is replaced by conservation of energy.

\(^8\)If \( C_1 \) and \( C_2 \) are two sets in \( \mathbb{R}^n \), then a hyperplane \( H \) is said to separate \( C_1 \) and \( C_2 \) iff \( C_1 \) is contained in one of the closed halfspaces associated with \( H \) and \( C_2 \) lies in the opposite closed half-space. Two convex sets in \( \mathbb{R}^n \) that share at most an affine set of dimension \( n - 1 \), can be separated by a hyperplane.
space a hyperplane does not separate that space in two half-spaces. To apply
the criterion of Bayes-optimality, one needs to decomplexify the space to restore
the order relation, but this can be done in a variety of ways. On the other hand,
this plurality of decomplexifications need not bother us too much. Just as in
the case of the statistical states of the former section, the observer can check the
statistical validity of his outcome assignment by verifying that the probability
(in the sense of a relative frequency) that results from repeated application of his
outcome assignment, equals the assumed probability. In the same way, we can
simply postulate, or even guess, a specific form of the probability assign-ment
and justify it a posteriori: If the relative frequency of an outcome (as a result
of the observers’ outcome assignment, based on the Bayes-optimal condition),
converges to a limit that yields (a monotone function of) the very probability
assignment he used to obtain those outcomes, the Bayes-optimal observer knows
he was Bayes-optimal. Let us attempt a minimal generalization of the real case
(31), with \( \psi^* \) and \( \psi^m \) defined as in (36), (37) and \( j = 1, \ldots, n; \ j \neq k \):

\[
eig^C(x_k, \psi^s) = \{ \psi^m \in \Sigma_M : \frac{|q_k|}{|r_k|} > \frac{|q_j|}{|r_j|} \} \quad (38)
\]

The only difference with (31), is that we take the modulus of the coefficients and
that the set contains complex vectors, which is why we have given the eigenset
the superscript \( \mathbb{C} \). To check the consistency of our Bayes-optimal observer in
the complex state space, we evaluate the Lebesgue measure \( \nu(eig^C(x_k, \psi^s)) \).
Therefore we regard the measure \( \nu \) in \( \mathbb{C}^n \) as the Lebesgue measure \( \mu \) over \( \mathbb{R}^{2n} \).
The calculation of the measure by direct integration can be avoided by use of a
mapping \( \omega \) that preserves measures. A measurable mapping \( \omega \) between measure
spaces \( (\Sigma, A, \mu) \) and \( (\Sigma, B, \nu) \) is called a measure-preserving mapping if, for every
\( B \in B \), we have \( \mu(\omega^{-1}(B)) = \nu(B) \). In appendix \( C \) we demonstrate that the
component-wise (or Haddamard) product of a complex vector with its complex
conjugate, that sends elements of the complex unit-sphere \( S_n = \{ z \in \mathbb{C}^n : \sum_{i=1}^n z_i z_i^* = 1 \} \) onto the \( (n-1) \)-simplex \( \Delta_{n-1} = \{ x \in \mathbb{R}_+^n : \sum_{i=1}^n x_i = 1 \} \) is
indeed measure preserving in this sense. We have given a graphic representation
of the action of \( \omega \) in Figure (3). We are now in a position to prove our main
result.

**Theorem 5**

\[
p(x_k|\psi^s) = |\langle x_k, \psi^s \rangle|^2
\]

**Proof.** With \( eig^C(x_k, \psi^s) \) defined by (38), and

\[
C^s_k = [\omega(x_1), \ldots, \omega(x_{k-1}), \omega(\psi^s), \omega(x_{k+1}), \ldots, \omega(x_n)],
\]

it is straightforward to show that (for more details, see [4]) we have:

\[
C^s_k \supset \omega(eig^C(x_k, \psi^s)) \subset [C^s_k]
\]

Let \( \tilde{\mu} \) and \( \tilde{\nu} \) stand for the normalized versions of the measures \( \mu \) and \( \nu \) in
the proof in appendix \( C \), so that their constant of proportionality equals one:
Figure 3: The action of the mapping $\omega$ sends elements of the unit sphere to the standard simplex (upper figure). The probability for the occurrence of outcome $x_k$ is the measure of the eigenset corresponding to outcome $x_k$ and is calculated in the simplex using the measure preserving mapping $\omega$. The eigensets are depicted in the lower figure for the simplex; it is not possible to show graphically what these sets look like in the complex unit sphere.
\( \tilde{\nu}(\omega^{-1}(A)) = \tilde{\mu}(A) \). By definition \( p(x_k|\psi^s) = \tilde{\nu}(eig(x_k, \psi^s)) \), and by the previous lemma, we have

\[
\tilde{\nu}(eig(x_k, \psi^s)) = \tilde{\nu}(\omega^{-1}(C_k^s)) = \tilde{\mu}(C_k^s)
\]

The normalized measure \( \tilde{\mu}(C_k^s) \) of the real simplex \( C_k^s \) was calculated in the real state space. A completely equivalent calculation gives us

\[
\tilde{\mu}(C_k^s) = \langle \omega(x_k), \omega(\psi^s) \rangle = |q_k|^2
\]

We see that indeed the Bayes-optimal observer recovers the Born rule as a result of his attempt to maximize the odds with respect to the outcome that pertains to the system. To be precise, we did not maximize the odds, because substitution of the Born rule for the probability in (22) gives:

\[
\Lambda_k = \frac{|\langle x_k, \psi^s \rangle|^2}{|\langle x_k, \psi^m \rangle|^2} = \frac{|q_k|^2}{|r_k|^2}
\]

Whereas our observer, by (38), calculated the ratio’s:

\[
\tilde{\Lambda}_k = \frac{|q_k|}{|r_k|}
\]

where the tilde denotes the fact that, strictly speaking, this is not a likelihood, because \(|q_k|\) and \(|r_k|\) aren’t probabilities (they are square roots of probabilities). Yet, it is obvious that the value of \( k \) for which (39) and (40) are maximal, is the same because one is the square of the other, which is clearly a monotone function. As a consequence, it does not matter if the Bayes-optimal observer works with (39) or with (40): repeated application of either strategy on the same pure state will make the relative frequency converge to the Born rule in exactly the same way in both cases.

### 4 Consequences of Bayes-optimal observation

#### 4.1 Decision invariance and unitarity

The outcome chosen by a Bayes-optimal observers, is the one that maximizes the corresponding likelihood ratio \( \Lambda_i \). Any monotonously increasing function of the likelihood ratio’s preserves their relative order, and hence their maximum. By (31) and (38), this carries over to the coefficients of the state vectors in both the real and the complex state space. The same is true for multiplication by a phase factor, which is cancelled by taking the moduli in (38). As a result, the state space is not only a vector space, it is a projective vector space: if the vectors in the state space are multiplied \( z \in \mathbb{C} \), \( 0 < |z| < \infty \), this does
not change the result of the decision procedure adopted by the Bayes-optimal observer. There is another interesting class of transformations that leaves the Bayes-optimal decision unaltered. For any $\psi^s$, the probability of $x_k$ is defined as:

$$p(x_k|\psi^s) = \mu(eig^C(x_k, \psi^s))$$

Because $\omega(eig^C(x_k, \psi^s)) \subset [C^s_k]$, $\omega$ continuous, and because the elements of $[C^s_k]$ have finite norm, the norm of the vectors in $eig^C(x_k, \psi^s)$ is finite too. We can then apply a linear transformation to the base vectors of the state space:

$$T : \Sigma_S \rightarrow \Sigma_S$$

$$T(x_j) = \sum_i \sigma_{ij} x_j$$

The eigenset $eig^C(x_k, \psi^s)$ will accordingly be transformed by applying $T$ to $x_k$ and $\psi^s$. By Lebesgue measure theory, the volume of the transformed set is proportional to the volume of the original set, the constant of proportionality being the determinant of the transformation:

$$\mu(T(eig^C(x_k, \psi^s))) = |\det(T)|\mu(eig^C(x_k, \psi^s))$$

for all $eig^C(x_k, \psi^s) \in B(\Sigma)$. This is a classic result\(^9\), and we refer the interested reader to \([26], p54\) for a proof. Note that this would typically be untrue for a nonlinear transformation. As a result, all transformations with $|\det(T)| = 1$ leave the probabilities invariant, which means we have invariance under unitary transformations. Intuitively this is obvious: if the probabilities have their origin in a measure on state space, then scaling, phase shifting, forming the mirror image, or ‘rotating’ the entire state space, does not alter the relative proportions of the eigensets, hence the invariance. Of course, it is easy to derive from the Born rule that the probabilities are invariant under unitary transformations, because the Born rule is the square modulus of an inner product and a unitary transformation can be defined as a linear operator that leaves the inner product invariant. Our invariance principle tells us the same story at a deeper level, for not only the probabilities are invariant under unitary transformation, but also each obtained outcome will be the same whether or not we unitarily transform the eigensets.

### 4.2 The elusive quantum to classical transition

Suppose we have a particular statistical mixture

$$\varphi = \xi\psi_1 + (1 - \xi)\psi_2$$

of two (pure) states $\psi_1$ and $\psi_2$ with $\xi \in [0, 1]$. Suppose furthermore that

$$p(x_i|\psi_1) = q_1$$
$$p(x_i|\psi_2) = q_2$$

\(^9\)As before, we regard the complex $n$–space as a real $2n$–space, for which the theorem is applicable.
Then an observing system is said to satisfy the linear mixture property iff

\[ p(x_i|\varphi) = \xi q_1 + (1 - \xi)q_2 \quad (43) \]

In words: the probability of a mixture equals the mixture of the probabilities.

Does the Bayes-optimal observer satisfy the linear mixture property? Well, \( \varphi \) is a statistical mixture, as defined in the section on Bayes-optimal observation of statistical mixtures, and each of the constituents in the mixture is a pure state, as defined in the section Bayes-optimal observation in Hilbert space. So clearly, our Bayes-optimal observer satisfies the linear mixture property. In essence, this stems from his initial states being uniformly random (almost everywhere). Indeed, suppose the distribution of the initial states of the observer is not uniform a.e.. Then one can always find a convex region \( S \) in state space with surface measure \( A \), for which the density of observer states is not equal to \( 1/A \). Without giving a formal proof, one can see that, it is always possible to find two states \( \psi_1, \psi_2 \notin S \) and a real number \( \xi \in ]0, 1[ \), such that \( \xi \psi_1 + (1 - \xi)\psi_2 \in S \) and for which the linear mixture property will be violated.

The linear mixture property is essential to experimental observation: no experimenter would put his faith in the hands of a detection apparatus that manifestly fails this most basic requirement. From this perspective, the difficulty of finding an intermediate region between the classical and the quantum, originates from the lack of a principle that determines how the observer should behave in order to objectively observe the intermediate region in absence of the linear mixture property. As an example, suppose we want to determine the length of a linearly extend system. In a classical setting, we are in principle free to choose the number of outcomes, and we are allowed to make many observations before we settle on the result of a single measurement. For example, we can align the zero of the measuring rod with one end point of the system and read the outcome at the other end point as many times as we want to. If we are not satisfied with the precision that the measuring rod affords, we can pick a better one, or improve it by adding a nonius (or vernier) system to it. As long as we are able to do this, we are still in a classical regime of observation. In the classical regime of observation, the distribution of observer states will be highly non-uniform. Ideally, of all possible measurements, the only uncertainty we have about the state of the observer that is assumed to be of relevance to the measurement outcome, is an uncertainty of the order of the smallest number the measuring rod can represent. To decrease the uncertainty about the result, even beyond the precision offered by the smallest number the rod can represent, it is common scientific practice to perform the measurement many times. Assuming identical, independent observations, one can apply standard error theory. In the beginning of the eighties, Wootters has shown (\[22, \ 23\]), using standard error theory, that the distance (angle) between two states on the unit sphere in (real) Hilbert space, is proportional to the number of maximally discriminating observations along the geodesic between those two points. This beautiful result gains in richness when considered from the point of view that the probabilities arise in a Bayes-optimal way. In our search for ever more precise measurements
or measurements on ever smaller constituents of nature, we eventually reach a region where we cannot repeat measurements without absorbing the system or altering its state. We may not even be able to choose freely the set of outcomes for a particular measurement, as is the case in the quantum regime. It is then no longer possible to directly obtain the “true” value of a physical quantity, because the eigenstate of the observing system may not (and in general will not) coincide with the state of the system under investigation. We cannot attempt the same measurement (or one with altered eigenstates) on the same system, because the state of the system has been altered, or even destroyed. In view of this impossibility, we are led to statistical observation on ensembles. We have shown it is possible to recover an objective probability if the distribution of observer states is uniform. We see that the best possible observation scheme in the classical regime entails a minimal uncertainty (i.e. about the interpretation of the last digit only) in the state of the observer, and in the quantum regime a maximal uncertainty (any outcome is in principle possible) about the state of the observer. The consequence of such an interpretation is, that we will only be able to identify intermediate regions when we allow for a more complete description of the observing system. In essence, we need to describe how to go from this minimal to this maximal uncertainty state. There are good reasons for cautiously entering this intermediate region. Some of the beautiful properties of the classical and the quantum regime will not hold. For example, the linear mixture property cannot be universally satisfied. Moreover, we will obtain probability distributions that depend not only on the system, but at least partially on the dynamics of the observing system. It is possible to construct explicit models that show one can identify an intermediate region where the probabilities satisfy neither the classical statistical Bonferroni inequalities\footnote{The Bonferroni inequalities indicate when a set of (joint) probabilities can be derived from a Kolmogorovian probability model. The best known example of a Bonferroni inequality in the foundations of quantum mechanics, is the Bell inequality.} indicating the absence of a straightforward Kolmogorovian model, nor the Accardi-Fedullo inequalities\footnote{The Bonferroni inequalities indicate when a set of (joint) probabilities can be derived from a Kolmogorovian probability model. The best known example of a Bonferroni inequality in the foundations of quantum mechanics, is the Bell inequality.} that constrain the set of probabilities that are derivable from a Hilbert space model. This opens up a whole new area of investigation, but only if we are willing to take the bold step of abandoning the full generality of the linear mixture property.

4.3 Is the Bayes-optimal observer objective?

The purpose of objective observation is to obtain a probability for the outcome that depends only on the system under study. How fast the sequence of outcomes converges to this probability, depends on how well the observer manages to distinguish his state from the state of the system under study. This aspect was neglected in the previous discussion. If we apply the Born rule to calculate the quantities \( p(x_i|H_0) \) and \( p(x_i|H_1) \), we imply that \( \sum_j p(x_i|H_0) = \sum_j p(x_i|H_1) = 1 \). However, if the choice between \( H_0 \) and \( H_1 \) is indeed a binary decision problem,
we should have:

\[ \sum_i p(x_i | H_0) = \alpha \quad \text{(44)} \]
\[ \sum_i p(x_i | H_1) = 1 - \alpha \]

The reason why this is not contradictory, is because the observer chooses his outcome, as if the outcome will be judged afterwards as a binary decision problem. The observer himself has a priori no clue what the value of \( \alpha \) might be. But even if he would estimate the value of \( \alpha \) after repeated measurements, then still this knowledge cannot not help him to give a more optimal outcome. To the Bayes-optimal observer, knowledge of \( \alpha \) would merely have the effect of scaling the odds in \( \frac{\sum_j p(x_j | H_0)}{\sum_j p(x_j | H_1)} \) by \( \frac{1}{\alpha} \). The choice of the outcome for the Bayes-optimal observer is based on the maximal likelihood and a monotone function of the likelihoods will not change the maximum. Thus we see that the specific value of \( \alpha \) has no influence on the actual choice. It turns out we can always pick an outcome that supports \( H_0 \) more than it supports \( H_1 \) if \( \alpha > 1/2 \). To see this, we proceed ad absurdum. If no outcome supports \( H_0 \) more than it supports \( H_1 \), then for all \( x_j \),

\[ \frac{p(x_j | H_0)}{p(x_j | H_1)} \leq 1 \]

But then\(^{11}\) we have:

\[ \sum_j^n p(x_j | H_0) \geq \sum_j^n p(x_j | H_1) \leq 1, \quad \text{(45)} \]

which implies \( \alpha \leq 1 - \alpha \). We obtain the contradiction iff \( \alpha > 1/2 \). In words: if we can do only slightly better than completely arbitrary in letting the outcome probability depend on the system, we can guarantee the existence of an outcome that maximizes the odds and is greater than unity. In fact, for any value of \( \alpha \) we can find an (almost always unique) outcome that maximizes the odds, but

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\(^{11}\)This specific condition is known in the literature as majorization. It plays an important role in the investigation of bipartite state conversions by local operations and classical communications (LOCC). This may seem relevant in connection to our problem, as the basic scheme we present can be described as a bipartite state conversion problem. However, we cannot use the many interesting results in the literature on bipartite state conversion because LOCC’s in this particular problem are operationally defined by means of local unitary transformations and a local measurement, and it is the local measurement that we seek to understand!
when $\alpha > 1/2$, the maximal likelihood ratio enjoys the property of being greater than one.

4.4 The Bayes-optimal observer as a paradigm for observation

The proposed principle of observation is based on a Bayesian treatment of a binary decision problem, but is not used in its usual decision-theoretic form. In decision theory we seek to establish which of the hypotheses enjoys the strongest support in evidence of the data. In our case, there is no data to feed the likelihood with, because we produce the data by means of the odds. The way we employ the principle is like an inverse decision problem, as if anticipating that the result will be judged afterwards by a decision procedure performed by one with absolute knowledge of the system and observer states prior to the measurement. The possibility of applying Bayesian decision theory in quantum mechanics came to me through the realization that the criterion established by Aerts D. at the end of [2] to characterize the so-called hidden measurements, is a monotone function of the Bayesian odds and hence leads to the same choice for the outcome. In this sense, this paper can be seen as providing a Bayesian foundation for the structure of the hidden measurements as given in, for example [2] and [3], and extending the results to the complex Hilbert space.

More recently it has come to my attention that a somewhat similar paradigm (without reference to quantum mechanics) is proposed in several papers that deal with visual perception by humans. The idea that the visual system is rooted in inference, can be traced back to the work of Helmholtz [20], who proposed the notion of unconscious inference. It was only in the last decade that it was accepted and translated into a mathematical framework, not in the least because computer scientists who want to model the human vision system are faced with the apparent complexity that underlies human perception. The Bayesian framework provides the tools necessary to understand and explain a wide variety of sometimes baffling visual illusions that occur in human perception [19]. In retrospect, we have borrowed the term ‘Bayes-optimal’ from this literature, because the term so neatly describes the principle and it did not seem appropriate to introduce a new term. There are however some differences in the application of the principle with respect to our proposal. In the literature on visual perception, the prior distributions are derived from real world statistics. Of course, this begs the question how these prior distributions were obtained in the first place. There are two basic possibilities to obtain a prior: either a prior distribution is based on some theoretical assumption, or it is established by looking at the relative frequency of actual recordings. The first option is the one we pursued in this article, where we assumed a uniform distribution of observer states\textsuperscript{12}. In the second case, which is the one adopted in the literature on perception, one has the advantage of being able to explain a wide variety of

\textsuperscript{12}The absence of a more informative prior distribution effectively reduces the criterion of Bayes-optimality to a Neyman-Pearson maximum-likelihood criterion.
visual effects in human perception, and how the priors can be adapted through
the use of Bayesian updating, but we cannot explain observation itself. The
relative frequency needed to obtain the prior, is rooted in the observation of
data, which requires another prior and so on ad infinitum. One can break from
this loop by reconsideration of what a state is. In the literature on perception
states are considered only as (real) statistical mixtures, severely limiting both
the applicability and the philosophical scope of the paradigm. The state, as we
have defined it here, can be a complex vector, not obtainable as a mixture in
principle, and yet give rise to probabilities if we attempt to observe it as good
as possible. So the state is simultaneously a description of the ‘mode of being’
(the pure state that physically interacts), and a ‘catalogue of information’ (the
probabilities the Bayes-optimal observer obtains).

The possibility that the same principle governs human perception and quan-
tum mechanical observation, strengthens the Bayes-optimal paradigm. Mea-
surement apparata and human perception can be rooted in the same principle:
the attempt to relate the outcome to the object under investigation as unam-
biguously as possible by choosing the outcome that has the largest odds (22).
By repeating the observation many times, each time randomizing the internal
state of the sensor, we obtain an invariant of the observation that pertains solely
to the system.

Another interesting link with the existing literature was pointed out to me
by Thomas Durt [15]. The regions of the Bohm-Bub model [11] coincide with
our definition of the eigensets in the complex case [38]. Moreover, Bohm and
Bub propose a uniform measure of states that they interpret as apparatus states.
They perform the integration directly for the two dimensional case, and indicate
the integration scheme can be extended to the more dimensional case. Their
result, like ours, is the reproduction of the Born rule. From the perspective of
this paper, Bayes-optimal observation yields an interpretation for the regions
employed by Bohm and Bub.

5 Concluding remarks

The search for a Bayesian or decision-theoretic framework for quantum probabil-
ity has recently been subject of a number of interesting publications ([15], [16],
[18], [24], [25], and [31]). One important motivation for seeking such
an interpretation, is that it allows for a subjective interpretation of quantum
probability by regarding the state vector as a mathematical representation of
the knowledge an agent has about a system. An often heard critique of Bayesian
interpretations of quantum probability is that, from a strictly Bayesian point of
view, the state vector represents the knowledge available to the agent that deals
with it. A majority of physicists rejects this notion, mainly because they feel
the relative frequencies obtained in actual experiments are objective features of
the system, and not of the knowledge of the agent. The Bayesian pragmatic
response to this, is that what can be inferred about a system always depends on
one’s prior knowledge of the system. However, in a theory that takes observation
as a primitive concept, one cannot assume to have *a priori* knowledge. This is what we have modelled here as the uniform distribution of initial observer states and Bayes-optimal observation of an ensemble of identical states will then result in an unbiased probability. If it is physically possible to obtain unbiased estimates for a sufficient number of observables so that we can reconstruct the state vector, then, at least in an operational sense, the state can be truly assigned in an objective way to a system. Besides the objective informational content of the state, the state may also represent an objective reality. This is in agreement with the fact that we started from assumption (12); that the state is a realistic description of the system, and it is the state of the system and the observer that physically and deterministically interact to produce the measurement outcome. Systems are in a state, and that state uniquely determines every possible interaction. The state vector truly represents complete information about a system, but not merely as a collection of objective attributes, but as a representation of the possible deterministic interactions with any other system, in particular observing systems. A classically objective attribute, from this perspective, is then the limiting case where the same result follows for the vast majority of states of Bayes-optimal observing systems that the system can interact with.

The proposed interpretation is falsifiable in principle but there are obstacles along the way. If we succeed in preparing the relevant degrees of freedom of the states of the apparatus, we could produce a non-uniform distribution for the initial states. Such a prepared apparatus would be able to distinguish some pairs of states better, and some pairs of states worse than the usual Born rule allows, which means it can only be used to our advantage if we possess some information about the state prior to the measurement. It also means that the probability for the occurrence of an outcome when we measure a mixture of states, depends nonlinearly on the probabilities for each component of the mixture; a failure of what we have called the “linear mixture property”. This would most likely lead to a rejection of the validity of the apparatus by the majority of experimentalists. And, we hope to have shown, in complete absence of prior information, it is not evidently desirable to deviate from a complete lack of knowledge of the apparatus state.

Perhaps there is another, still deeper, reason why it is not possible to completely control the state of the observer at the quantum level. The source of probability in observation, the randomness in the state of the observer, may very well at some point become *fundamentally* incontrollable. Logical arguments seem to defend at least the possibility of such a thesis. In [12], it is shown by an elegant construction, that for every observer there will be different states of himself that he cannot distinguish. In [6] it is shown that, on purely logical grounds, no observer can determine whether his observations are entirely faithful\(^{13}\). It seems that, for every single measurement outcome, there is a trade-off between the information an observer can choose to extract about himself, and about the system he is observing. This trade-off can be quantified.

\[^{13}\text{To the best of our knowledge, the relation to such logical arguments and the quantum measurement problem was pointed out for the first time in 1977 in a remarkable pioneering paper by Dalla Chiara in [14].}\]
It is argued in [30] and [8] on thermodynamical grounds, that any gain in information about a system is accompanied by an equal increase of entropy about the state of the observing system. If this is indeed the underlying structure for the occurrence of the quantum probabilistic structure, then the probabilities in quantum mechanics are indeed ontic and epistemic at the same time. From an absolute perspective, probability always arises because there is a lack of knowledge situation; it is a measure over deterministic events. But to the one who observes, this lack of knowledge may be fundamentally irreducible. It might turn out that, after all, Einstein and Bohr were both right about the origin of probabilities in quantum mechanics.

Acknowledgements: I want to thank Freddy De Ceuninck and Michiel Seevinck for reading and discussing the content of this paper. This work was supported by the Flemish Fund for Scientific Research (FWO) project G.0362.03.

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6 Appendices

6.1 Appendix A

Lemma 6 If \( \mu \) is a (probability) measure such that \( \mu(\Delta_{n-1}(X)) = 1 \), and \( C^s_k \) is defined by the convex closure of \([23]\), then we have \( \mu([C^s_k]) = t_k \)

Proof.

Let \( \rho_{n-1} \) be the (not necessarily normalized) \((n - 1)\) -Lebesgue measure in \( \Delta_{n-1}(X) \). Then we have

\[
\mu([C^s_k]) = \frac{\rho_{n-1}([C^s_k])}{\rho_{n-1}(\Delta_{n-1})} = \frac{\rho_{n-1}([x_1, \ldots, x_{k-1}, X(\psi^s), x_{k+1}, \ldots, x_n])}{\rho_{n-1}([x_1, \ldots, x_n])} = \frac{\rho_{n-2}(B) d(B, X(\psi^s))}{\rho_{n-2}(B) d(B, x_k)}
\]

In this last equation, \( B = [x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n] \) is the face shared by the two simplices, and \( d(B, a) \) the smallest Euclidean distance between point \( a \) and each point of face \( B \), which is proportional to the norm of the orthogonal projection of \( a \) onto a unit vector \( b \) perpendicular to \( B \). In \( \mathbb{R}^n \) no unique vector is perpendicular to \( B \) (which only has affine dimension \( n - 2 \)), but as long as we...
stick to the same vector $b$ for both simplices, the same constant of proportionality will apply, and the ratio will eliminate that constant. Pick the $x_k$ base vector as $b$, which is obviously unit-norm and perpendicular to $B$. The orthogonal projection of the top of $C^*_k$ to $b$ is: $x(\psi^s) \downarrow b = \langle x(\psi^s), x_k \rangle x_k = t_k x_k$. For $\Delta_{n-1}$, the top is the vector $x_k$ itself and its projection $x_k \downarrow b = \langle x_k, x_k \rangle x_k = x_k$. Hence we have

$$\frac{d(B, x(\psi^s)) }{d(B, x_k) } = \frac{ ||(x(\psi^s) \downarrow b)|| }{ ||(x_k \downarrow b)|| } = t_k x_k / x_k = t_k$$

6.2 Appendix B

Lemma 7 Let $C^*_k$ be defined as in (35) and $eig(x_k, x(\psi^s))$ by (37), then:

$$C^*_k \subset eig(x_k, x(\psi^s)) \subset [C^*_k]$$

Proof. We start with the first inclusion. Suppose $x(\psi^m)$ is in one of the open $(n-1)$-simplices $C^*_k$, then, by definition, there exist $\lambda_i$ such that, with $0 < \lambda_i < 1$, $\sum \lambda_i = 1$,

$$x(\psi^m) = \sum_{i \neq k}^{n} \lambda_i x_i + \lambda_k x(\psi^s) \quad (46)$$

On the other hand we have that $x(\psi^s) \in \Delta_{n-1}$ and hence there exist $t_l \geq 0$, $\sum t_l = 1$ such that (23) holds:

$$x(\psi^s) = \sum_{l=1}^{n} t_l x_l \quad (47)$$

Substitution of (47) into (46) yields

$$x(\psi^m) = \lambda_k t_k x_k + \sum_{i \neq k}^{n} (\lambda_i + \lambda_k t_i) x_i$$

Calculating the likelihood ratios (22), we obtain $\Lambda_k = \frac{1}{\lambda_k}$, and for $i \neq k$ we have:

$$\Lambda_i = \frac{t_i}{\lambda_i + \lambda_k t_i}$$

We easily see that $\Lambda_k > \Lambda_i$ iff $\lambda_i > 0$, which is satisfied by assumption. Hence, by (31) every $x(\psi^m) \in C^*_k$ gives an outcome $x_k$, establishing the result. For the second inclusion, suppose there exists some $x(\psi^m) \in \Delta_{n-1}$ with $x(\psi^m) \notin [C^*_k]$. The sets $C^*_k$ in our theorem, as can be seen from the definition (35), are disjoint open $(n-1)$-simplices. If we had defined them by means of the closed
convex closure, they would maximally share the \((n - 2)\) simplex \(\Delta_{n-2}^{s}(j, k) = [x(\psi^s), x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n] : [C_k^s] \cap [C_k^s] = \Delta_{n-2}^{s}(j, k)\)

Assume first \(a\) is not in the boundary of \([C_k^s]\), i.e. not in one of the lower dimensional sub-simplices \(\Delta_{n-2}^{s}(j, k)\). Then \(x(\psi^m) \in C_i^s\) with \(i \neq k\). Because of the above demonstrated first inclusion we have \(x(\psi^m) \in eig(x(\psi^m))\) and hence \(x(\psi^m) \notin eig(x_k, x(\psi^s))\). If on the other hand \(x(\psi^m) \in \Delta_{n-2}^{s}(j, k)\), our outcome assignment on the basis of the maximum likelihood principle is ambiguous, as there will be two equal maxima, and even more when \(x(\psi^m)\) is chosen in a still lower dimensional sub-simplex. However, we are free to choose whatever outcome we like as long as it is one of the maxima. Because the maxima coincide, these points lie in the boundary and hence the conclusion remains \(eig(x_k, x(\psi^s)) \subset [C_k^s]\). ■

6.3 Appendix C

Lemma 8 The mapping \(\omega\)

\[
\omega : S_n \to \Delta_{n-1} \\
\omega(z) = (z_1 z_1^s, z_2 z_2^s, \ldots, z_n z_n^s)
\]

is measure-preserving, i.e. for two measure spaces \((\Delta_{n-1}, B(\Delta_{n-1}), \mu)\) and \((S_n, B(S_n), \nu)\) and \(A \in B(\Delta_{n-1})\) and \(\omega^{-1}(A) \in B(S_n)\), we have:

\[
\nu(\omega^{-1}(A)) = \frac{2\pi^n}{\sqrt{n}} \mu(A)
\]

Proof. \(^{14}\)Let \(A\) be an arbitrary open convex set in \(\Delta_1 : A = \{(x_1, x_2) : a < x_1 < b, \ x_2 = 1 - x_1\}\). Evidently, \(\mu(A) = \sqrt{2}(b - a)\). Let \(B\) be the pull-back of \(A\) under \(\omega\):

\[
B = \{(z_1, z_2) \in Z_1 \times Z_2 \subset C^2 : Z_1 = \{z_1 : a < |z_1|^2 < b\}, Z_2 = \{z_2 : z_2 = \sqrt{1 - |z_1|^2} e^{i\theta}, \theta \in [0, 2\pi]\}\}
\]

Clearly,

\[
\nu(B) = \nu(Z_1)\nu(Z_2) = \pi(b - a) \frac{2\pi^2}{\sqrt{2}} \mu(A)
\]

Hence the theorem holds for convex sets in \(\Delta_2\). This conclusion can readily be extended to an arbitrary \((n - 1)\)-dimensional rectangle set \(A\) in \(\Delta_{n-1}\):

\[
A = \{(x_1, \ldots, x_n) : \forall i = 1, \ldots, n - 1 : a_i < x_i < b_i; a_i, b_i \in [0, 1]\}
\]

\(^{14}\)This proof was first presented in \(\text{[5]}\), but we include it for completeness. The author is grateful for a valuable hint from Wade Ramey that was helpful in proving the theorem.
Its measure factorizes into:

$$\mu(A) = \sqrt{n} \prod_{i=1}^{n-1} (b_i - a_i)$$

Next consider \(n\)-tuples of complex numbers:

- \(B = \{(z_1, z_2, \ldots, z_n) \in Z_1 \times \ldots \times Z_n\}\)
- \(Z_i = \{z_i \in \mathbb{C} : a_i < |z_i|^2 < b_i, i \neq n\}\),
- \(z_n = \sqrt{1 - |z_1|^2 - \ldots - |z_{n-1}|^2} e^{i\theta_n}, \theta_n \in [0, 2\pi]\)\}

Clearly \(\omega(B) = A\). The measure of \(B\) can be factorized as:

$$\nu(B) = \nu(Z_1)\nu(Z_2)\ldots\nu(Z_n)$$

$$= 2\pi \prod_{i=1}^{n-1} \pi(b_i - a_i) = \frac{2\pi^n}{\sqrt{n}} \mu(A)$$

Hence the theorem holds for an arbitrary rectangle set \(A \subset \Delta_{n-1}\). But every open set in \(\Delta_{n-1}\) can be written as a pair-wise disjoint countable union of rectangular sets. It follows that \(\nu(\omega^{-1}(\cdot)) = \frac{2\pi^n}{\sqrt{n}} \mu(\cdot)\) for all open sets in \(\Delta_{n-1}\).

Both \(\nu\) and \(\mu\) are finite Borel measures because \(\Delta_{n-1}\) and \(S_n\) are both compact subsets of a vector space of countable dimension. Therefore they must be regular measures (26, p47), which are completely defined by their behavior on open sets. Hence \(\omega\) is measure preserving for Borel sets. ■