Gravity-induced four-fermion contact interaction implies gravitational intermediate W and Z type gauge bosons

Jens Boos\textsuperscript{1,2} and Friedrich W. Hehl\textsuperscript{1,3,4}

\textsuperscript{1} Perimeter Inst. Theor. Physics, Waterloo, ON N2L 2Y5, Canada
\textsuperscript{2} Dept. Physics & Astron., Univ. of Waterloo, Waterloo, ON N2L 3G1, Canada
\textsuperscript{3} Inst. Theor. Physics, Univ. of Cologne, 50923 Köln, Germany
\textsuperscript{4} Dept. Physics & Astron., Univ. of Missouri, Columbia, MO 65211, USA

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Abstract

Coupling fermions to gravity necessarily leads to a non-renormalizable, gravitational four-fermion contact interaction. In this essay, we argue that augmenting the Einstein–Cartan Lagrangian with suitable kinetic terms quadratic in the gravitational gauge field strengths (torsion and curvature) gives rise to new, massive propagating gravitational degrees of freedom. This is to be seen in close analogy to Fermi’s effective four-fermion interaction and its emergent W and Z bosons.

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The idea that spin gives rise to torsion should not be regarded as an ad hoc modification of General Relativity. On the contrary, it has a deep group theoretical and geometric basis. If history had been reversed and the spin of the electron discovered before 1915, I have little doubt that Einstein would have wanted to include torsion in his original formulation of General Relativity. On the other hand, the numerical differences which arise are normally very small, so that the advantages of including torsion are entirely theoretical.

— Dennis W. Sciama, priv. comm. (1979)

1 Introduction

Physics thrives, whenever concepts of different disciplines are combined to find something new. One may think of electromagnetism, the archetype of classical field theory, which in many ways served as a precursor for the more elaborate theory of General Relativity. Up to today, physicists use and employ similarities between these two theories to learn something new about the other, and in the light (or sound) of the gravitational wave detection GW150914 this becomes even more apparent. Other examples are renormalization and massive gauge theories, enriched by physical understanding via the condensed matter phenomena of block spin coarse-graining and spontaneous symmetry breaking, respectively.

Undoubtedly, the quantum nature of gravity has been a very puzzling question of both the twentieth and twenty-first century. While we do not attempt to settle this issue here, we would like to employ an analogy from particle theory, that could perhaps open a fruitful and interesting direction for future research.

In 1933, Fermi proposed a phenomenological model of a four-fermion interaction that could potentially describe the $\beta$ decay. Due to the negative mass dimension coupling constant, $G_F \simeq 1.17 \times 10^{-5} \text{GeV}^{-2} (hc)^3$, which is equivalent to a typical interaction range of $l_{\text{Fermi}} \approx 10^{-20} \text{m}$, this theory is non-renormalizable and should be viewed as an effective field theory instead, as it turned out later. However, a massive gauge boson propagator can be approximated as

$$-\frac{-i}{p^2 - M^2 + i\epsilon} \left( g_{ij} - \frac{p_i p_j}{M^2} \right) \approx \frac{ig_{ij}}{M^2} \propto i G_F g_{ij},$$

provided that $p \ll M$. Here $p$ is the momentum of the particle, $M$ its mass, $g_{ij}$ the metric of spacetime (here $i, j, ...$ are coordinate indices running from 0 to 3). Hence, an effective field theory, even though non-renormalizable, can give important hints to the more accurate physics of nature (in this case, the underlying massive SU(2) gauge structure of the weak interaction).

Let us now turn back to gravity—at this point, we would like to quote John L. Synge, who once said “Newton successfully wrote apple = moon, but you cannot write apple = neutron.” This is true due to the intrinsically fermionic character of matter. As a historical remark, at times
of the discovery of the field equation of General Relativity, spin was still not known. It is hence
not surprising that General Relativity fails taking intrinsic angular momentum (the classical
analogue of spin) into account: it only couples to the symmetric Hilbert energy-momentum
tensor, which is blind to the spin current.

Hence Riemannian geometry, the geometrical arena of General Relativity, is not large enough
to accommodate spin. However, it has long been shown that the Einstein–Cartan theory,
naturally endowed with a Riemann–Cartan geometry allowing for non-vanishing torsion $T_{\alpha\beta i}$,
is a consistent candidate theory for coupling spin to gravity; here $\alpha, \beta, \ldots$ are frame indices taking
the values $\hat{0}, \hat{1}, \hat{2}, \hat{3}$. Therein, energy-momentum sources curvature, and spin sources torsion.
As it turns out, the tetrad 1-form $e_i^\alpha dx^i$ (coframe) and the connection 1-form $\Gamma_{\alpha\beta i} dx^i$
are now independent translational and rotational gauge fields, respectively. Together they incorporate
local Poincaré invariance $[1]$. Their field equations read

$$\text{Ric}_{\alpha i} - \frac{1}{2} R e_{\alpha} + \Lambda e_{\alpha} = \kappa \mathcal{S}_{\alpha i}, \quad T_{\alpha\beta i} + \delta_{\alpha}^{\gamma} T_{\beta\gamma i} - \delta_{\beta}^{\gamma} T_{\alpha\gamma i} = \kappa \mathcal{S}_{\alpha\beta i}, \quad (2)$$

where $e_\alpha^i := \delta \mathcal{L}/\delta e_{\alpha}^i$ is the canonical energy-momentum of matter, $e \mathcal{S}_{\alpha\beta i} := \delta \mathcal{L}/\delta \Gamma_{\alpha\beta i}$ is
the canonical spin current of matter, and $e := \det (e_{\alpha})$. On the other hand, the symmetric
Hilbert energy-momentum tensor of General Relativity is given by $\sqrt{-g} t_{ij} := 2\delta \mathcal{L}/\delta g_{ij}$, with $g := \det (g_{ij})$. In order to establish the difference of Einstein–Cartan theory as compared
to General Relativity, one can write (see $[1]$ and also Poplawski $[2]$, Magueijo et al. $[3]$, and
Khriplovich and Rudenko $[4]$)

$$EC_{\alpha i} := t_{ij} + \kappa \left[ -4 \mathcal{S}_{ikl} [j \mathcal{S}_{kl}^j] - 2 \mathcal{S}_{ik}^j \mathcal{S}_{kl}^j \mathcal{S}_{kl}^j + \frac{1}{2} g^{ij} \left( 4 \mathcal{S}_{m}^k [i \mathcal{S}_{ml}^k] + \mathcal{S}_{mkl}^j \mathcal{S}_{mlk}^j \right) \right], \quad (3)$$

where $\kappa = 8\pi G/c^4$ is Einstein’s gravitational constant. From the perspective of General Relativity,
we integrated out torsion and thereby created an additional spin–spin contact interaction,
or a gravitational four-fermion interaction beyond General Relativity. The “coupling constant”
of this four-fermion interaction is indeed of mass dimension $-2$, that is, $\kappa = \hbar/(c^3 M^2)$, where
$M = \sqrt{\hbar c/(8\pi G)} = 2.4 \times 10^{-18}$ GeV is the reduced Planck mass. Moreover, this four-fermion
interaction in general contains parity-even $\overline{\Psi} \gamma^\alpha \Psi$ and parity-odd $\overline{\Psi} \gamma^5 \gamma^\alpha \Psi$ pieces (where $\Psi$
is the spinor field, $\overline{\Psi}$ its Dirac adjoint, and $\gamma^\alpha$ are the Dirac matrices), in close analogy to the
electroweak scenario where the analogy began in the first place.

Of the two field equations in (2), the first one corresponds to translational invariance (generator
$P_{\alpha}$) and energy-momentum, the second one to Lorentz invariance (generator $J_{\alpha\beta}$) and spin
angular momentum. These facts are also manifest in the study of the gravitational phase shift
integrated along an infinitesimal closed loop $\gamma$ bordering the area element $d\sigma^{ij}$. This yields, as
shown by Anandan $[5]$,

$$\Phi_{\gamma} = 1 - \frac{i}{2} \left( T_{ij}^\alpha P_{\alpha} + R_{ij}^\alpha \alpha_{\alpha\beta} J_{\alpha\beta} \right) d\sigma^{ij}. \quad (4)$$

Torsion is related to translation and curvature to Lorentz transformations, as a geometrical
interpretation of the Riemann–Cartan geometry would suggest, and hence the holonomy of the closed loop \( \gamma \) is a Poincaré transformation.

# 2 Gravitational four-fermion interaction

Let us now estimate at which energy scales this interaction, should it be realized in nature, becomes relevant. Since it is related to particles that carry spin, we demand that the quantities on the right-hand side of Eq. (3) are of the same order of magnitude, that is,

\[ t^{ij} \approx \kappa (\mathcal{E}^2)^{ij}, \]

see the phenomenological oriented discussion of Ni [6]. Given a certain number density \( n \) of fermions with mass \( m \), the mass density is \( \rho = mn \). Next, we need to estimate the spin density \( s \). In standard equilibrium configurations and in the absence of external magnetic fields, spins tend to average out, and hence \( s \in \mathcal{O}(h) \). In order for the gravitational four-fermion interaction to become relevant, we either need to assume large magnetic fields or special phases of matter, like a ferromagnetic phase, for instance. Both criteria can be met in extreme situations: neutron stars are known to have extremely strong magnetic fields of the order of \( 10^4 \ldots 10^{11} \) T, and anomalous phases like the low temperature A phase of \(^3\text{He}\) indeed have a macroscopic net spin, see the books by Vollhardt and Wölfle [7] and Volovik [8]. It is therefore conceivable that there are, indeed, both astrophysical and cosmological scenarios where the spin density can be approximated by \( s \approx h n \).

Substitution into Eq. (5) yields the critical or Einstein–Cartan number density \( n_{EC} \approx m/(\kappa h)^2 \).

Defining the reduced Compton wavelength of the fermion under consideration, \( \lambda_{\text{Compton}} := h/(mc) \), one finally has for the critical Einstein–Cartan density

\[ \rho_{EC} = m n_{EC} = \frac{m}{\lambda_{\text{Compton}}^2 \ell_{\text{Planck}}^2}. \]

For a typical nucleon, \( m \approx 1 \text{GeV}/(c^2) \), and hence \( \rho_{EC} \approx 10^{50} \) kg/m\(^3\), which is much smaller than the reduced Planck density of \( \rho_{\text{Planck}} = 10^{96} \) kg/m\(^3\) at the big bang. For comparison, a typical nuclear density is \( \rho_{\text{nucl}} = 10^{18} \) kg/m\(^3\). Analogously, the Einstein–Cartan length scale \( \ell_{EC} = (\lambda_{\text{Compton}}^2 \ell_{\text{Planck}}^2)^{1/3} \approx 10^{-29} \) m is seven orders of magnitude larger than the reduced Planck scale \( \ell_{\text{Planck}} \approx 10^{-36} \) m.

PLANCK data indicate [9] that General Relativity can be verified to scales of \( \approx 10^{-28} \) m. Hence, noticeable effects due to Einstein–Cartan corrections can be expected to emerge soon, if present.
3 Liberating the gravitational W and Z bosons

Similar as in Fermi’s original contact interaction, this gravitational four-fermion contact interaction is probably not the end of the story. Contact interactions are unphysical, since they are mediated by some Heaviside potential, differentiations of which produce infinite, delta function-like forces.

Like in the electroweak case, we may now introduce additional, massive short-range degrees of freedom by modifying the Lagrangian of the Einstein–Cartan theory. However, there are two key ingredients that allow us to make an almost unique choice: (i) the coupling constant is an inverse mass squared, hence we know that the additional degree of freedom has to correspond to massive bosons. (ii) The Einstein–Cartan theory results from a gauge approach to gravity, hence there is a natural way to include kinetic terms of the gauge potentials $e_i^{\alpha}$ and $\Gamma_i^{\alpha\beta}$: the squares of their respective curvatures, that is, torsion $T_{\alpha\beta}^i$ (translational curvature) and curvature $R_{\alpha\beta}^{ij}$ (rotational curvature), correspond to the only gauge-invariant kinetic terms allowed.

Demanding that the field equations for coframe and connection be linear in second derivatives, one can construct the following type of extended Einstein–Cartan Lagrangian:

$$L_{\text{EC}}^{\text{ext.}} \sim e \left[ \frac{1}{(\mu \ell_{\text{Planck}})^4} + \frac{1}{L^2} \left( T_{\alpha\beta}^i T^{\alpha\beta}^i + \frac{1}{\chi} R_{\alpha\beta}^{ij} e_i^{\alpha} e_j^{\beta} + \frac{1}{k} R_{\alpha\beta}^{ij} R^{\alpha\beta}^{ij} \right) \right]$$

(7)

Here, $\mu$, $\chi$, and $k$ are new, dimensionless parameters, and $L$ is a length scale such that $\chi L^2 = \ell_{\text{Planck}}^2$. For an explicit form of the Lagrangian (7)—including parity-even and parity-odd terms—see Eq. (5.13) in Blagojević and Hehl [1]. Lagrangians like the above have been studied, see Sezgin and van Nieuwenhuizen [10] or Kuhfuss and Nitsch [11], and found to be ghost-free under certain conditions; see also the recent work of Shie et al. [12] or Bjorken [13] in the context of cosmology. As a finger exercise, we recently studied quadratic curvature terms in the context of exact solutions of Einstein’s equations [14].

The coupling constant of the curvature-squared piece is dimensionless, just like one would expect for Yang–Mills theory, and we call the propagating degrees of freedom liberated by this kinetic term “tordions.” As it turns out, they are also massive, $m = \sqrt{k}/\ell_{\text{Planck}}$, and for the interaction range to be of order $\ell_{\text{EC}}$, as argued above, one has $k \approx (\ell_{\text{Planck}}/\ell_{\text{EC}})^2 \approx 10^{-14}$. Hence, we arrived at a theory with a weak, Einstein gravity sector mediated by $e_i^{\alpha}$ (gravitons), and a strong, massive Yang–Mills sector mediated by $\Gamma_i^{\alpha\beta}$ (“gravitational W and Z type bosons”).
4 Conclusion

In this essay, we have argued that the gravitational four-fermion interaction can possibly be described as the effective field theory limit of a curvature-squared Lagrangian in the framework of Poincaré gauge theory of gravity. We leave a more detailed analysis for the future.

The new gravitational degrees of freedom are of a geometric origin rooted in the Poincaré group. Their similarities to the electroweak W and Z bosons are purely based on effective field theory considerations, and we cannot see any geometric relation between torsion and electromagnetism, see, however, Poplawski [15], e.g., and references therein.

*Note added in proof:* The recent paper of Donoghue [16] seems to have a similar aim as ours. Donoghue first reiterates the standard point of view of Poincaré gauge theory, compare [1]. What he calls ‘the constraint of metricity for the vierbeins,’ translates into our language as the vanishing of Cartan’s torsion tensor. The torsion itself is introduced by Donoghue in his Eq. (29). A link variable for torsion is suggested by comparing Donoghue’s Eq. (32) with our Eq. (4), namely as proportional to $P_a e_{\mu}^a$.

Donoghue has shown that the beta function of an SO(1,3) gauge theory is negative. Interpreting this as the Lorentz sector of Poincaré gauge theory, it implies that possibly new propagating degrees of freedom (tordions, or “gravitational W and Z bosons”) are confined, similar as the fermionic quarks in quantum chromodynamics. Due to the positivity of mass, however, this confinement might be of different origin than the screening mechanism encountered in particle theory. Nevertheless, this is a highly interesting observation and should be compared with the approach advocated here, since both mechanisms render the new degrees of freedom irrelevant for large scale physics.

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