ON THE EQUIFORM DIFFERENTIAL GEOMETRY OF AW(k)-TYPE CURVES
IN PSEUDO-GALILEAN 3-SPACE

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\textbf{Abstract.} The aim of this paper is to study AW(k)-type (1 ≤ k ≤ 3) curves according to the equiform differential geometry of the pseudo-Galilean space \(G^3_3\). We give some geometric properties of AW(k) and weak AW(k)-type curves. Moreover, we give some relations between the equiform curvatures of these curves. Finally, examples of some special curves are given and plotted to support our main results.

\section{1. Introduction}

The geometry of space is associated with mathematical group. The idea of invariance of geometry under transformation group may imply that, on some spacetimes of maximum symmetry there should be a principle of relativity which requires the invariance of physical laws without gravity under transformations among inertial systems [1]. The theory of curves and the curves of constant curvature in the equiform differential geometry of the isotropic spaces \(I^1_1\), \(I^2_2\) and the Galilean space \(G_3\) are described in [2] and [3], respectively. The pseudo-Galilean space is one of the real Cayley-Klein spaces. It has projective signature (0, 0, +, −) according to [2]. The absolute of the pseudo-Galilean space is an ordered triple \(\{w, f, I\}\) where \(w\) is the ideal plane, \(f\) a line in \(w\) and \(I\) is the fixed hyperbolic involution of the points of \(f\). In [4], from the differential...
geometric point of view, K. Arslan and A. West defined the notion of AW(k)-type submanifolds. Since then, many works have been done related to AW(k)-type submanifolds (see, for example, [5–10]). In [9], Özgür and Gezgin studied a Bertrand curve of AW(k)-type and furthermore, they showed that there is no such Bertrand curve of AW(1) and AW(3)-types if and only if it is a right circular helix. In addition, they studied weak AW(2)-type and AW(3)-type conical geodesic curves in Euclidean 3-space $E^3$. Besides, in 3-dimensional Galilean space and Lorentz space, the curves of AW(k)-type were investigated in [6, 8]. In [7], the authors gave curvature conditions and characterizations related to AW(k)-type curves in $E^n$ and in [10], the authors investigated curves of AW(k)-type in the 3-dimensional null cone.

This paper is organized as follows. In section 2, the basic notions and properties of a pseudo-Galilean geometry are reviewed. In section 3, properties of the equiform geometry of the pseudo-Galilean space $G^1_3$ are given. Section 4 contains a study of AW(k)-type equiform Frenet curves. Finally, some examples of special curves in $G^1_3$ are included in section 5.

2. Basic concepts

In this section, we recall some basic notions from pseudo-Galilean geometry [11,12]. In the inhomogeneous affine coordinates for points and vectors (point pairs) the similarity group $H_8$ of $G^1_3$ has the following form

\[
\begin{align*}
\bar{x} &= a + b.x, \\
\bar{y} &= c + d.x + r.\cosh \theta.y + r.\sinh \theta.z, \\
\bar{z} &= e + f.x + r.\sinh \theta.y + r.\cosh \theta.z,
\end{align*}
\]

(2.1)

where $a, b, c, d, e, f, r$ and $\theta$ are real numbers. Particularly, for $b = r = 1$, the group (2.1) becomes the group $B_6 \subset H_8$ of isometries (proper motions) of the pseudo-Galilean space $G^1_3$. The motion group leaves invariant the absolute figure and defines the other invariants of this geometry. It has the following form

\[
\begin{align*}
\bar{x} &= a + x, \\
\bar{y} &= c + d.x + \cosh \theta.y + \sinh \theta.z, \\
\bar{z} &= e + f.x + \sinh \theta.y + \cosh \theta.z.
\end{align*}
\]

(2.2)

According to the motion group in the pseudo-Galilean space, there are non-isotropic vectors $A(A_1, A_2, A_3)$ (for which holds $A_1 \neq 0$) and four types of isotropic vectors: spacelike ($A_1 = 0, A_2^2 - A_3^2 > 0$), timelike ($A_1 = 0, A_2^2 - A_3^2 < 0$) and two types of lightlike vectors ($A_1 = 0, A_2 = \pm A_3$). The scalar product of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $G^1_3$ is defined by

\[
\langle u, v \rangle = \begin{cases} 
 u_1v_1, & \text{if } u_1 \neq 0 \text{ or } v_1 \neq 0, \\
 u_2v_2 - u_3v_3, & \text{if } u_1 = 0 \text{ and } v_1 = 0.
\end{cases}
\]
We introduce a pseudo-Galilean cross product in the following way

\[ u \times_{G^3_1} v = \begin{vmatrix} 0 & -j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}, \]

where \( j = (0, 1, 0) \) and \( k = (0, 0, 1) \) are unit spacelike and timelike vectors, respectively. Let us recall basic facts about curves in \( G^1_3 \), that were introduced in [13–15].

A curve \( \gamma(s) = (x(s), y(s), z(s)) \) is called an admissible curve if it has no inflection points \( (\dot{\gamma} \times \ddot{\gamma} \neq 0) \) and no isotropic tangents \( (\dot{x} \neq 0) \) or normals whose projections on the absolute plane would be lightlike vectors \( (\dot{y} \neq \pm \dot{z}) \). An admissible curve in \( G^1_3 \) is an analogue of a regular curve in Euclidean space [12].

For an admissible curve \( \gamma : I \subseteq \mathbb{R} \to G^1_3 \), the curvature \( \kappa(s) \) and torsion \( \tau(s) \) are defined by

\[ \kappa(s) = \sqrt{\frac{\dot{y}(s)^2 - \ddot{z}(s)^2}{(\dot{x}(s))^2}}, \quad \tau(s) = \frac{\ddot{y}(s) \dddot{z}(s) - \ddot{z}(s) \dddot{y}(s)}{|\dot{x}(s)|^3 \cdot \kappa^2(s)}, \]

expressed in components. Hence, for an admissible curve \( \gamma : I \subseteq \mathbb{R} \to G^1_3 \) parameterized by the arc length \( s \) with differential form \( ds = dx \) is given by

\[ \gamma(x) = (x, y(x), z(x)). \]

The formulas (2.3) have the following form

\[ \kappa(x) = \sqrt{|y''(x)^2 - z''(x)^2|}, \quad \tau(x) = \frac{y''(x)z'''(x) - y'''(x)z''(x)}{\kappa^2(x)}. \]

The associated trihedron is given by

\[ \mathbf{e}_1 = \gamma'(x) = (1, y'(x), z'(x)), \]

\[ \mathbf{e}_2 = \frac{1}{\kappa(x)} \gamma''(x) = \frac{1}{\kappa(x)}(0, y''(x), z''(x)), \]

\[ \mathbf{e}_3 = \frac{1}{\kappa(x)}(0, \epsilon z''(x), \epsilon y''(x)), \]

where \( \epsilon = +1 \) or \( \epsilon = -1 \), chosen by criterion \( \det(e_1, e_2, e_3) = 1 \), that means

\[ |y''(x)^2 - z''(x)^2| = \epsilon(y''(x)^2 - z''(x)^2). \]

The curve \( \gamma \) given by (2.4) is timelike (resp. spacelike) if \( \mathbf{e}_2(s) \) is a spacelike (resp. timelike) vector. The principal normal vector or simply normal is spacelike if \( \epsilon = +1 \) and timelike if \( \epsilon = -1 \). For derivatives of the tangent \( \mathbf{e}_1 \), normal \( \mathbf{e}_2 \) and binormal \( \mathbf{e}_3 \) vector fields, the following Frenet formulas in \( G^1_3 \) hold:

\[ \mathbf{e}_1'(x) = \kappa(x)\mathbf{e}_2(x), \]

\[ \mathbf{e}_2'(x) = \tau(x)\mathbf{e}_3(x), \]

\[ \mathbf{e}_3'(x) = \tau(x)\mathbf{e}_2(x). \]
3. Frenet equations according to the equiform geometry of $G^1_3$

This section contains some important facts about equiform geometry. The equiform differential geometry of curves in the pseudo-Galilean space $G^1_3$ has been described in [11]. In the equiform geometry a few specific terms will be introduced. So, let $\gamma(s) : I \to G^1_3$ be an admissible curve in the pseudo-Galilean space $G^1_3$, the equiform parameter of $\gamma$ is defined by

$$\sigma := \int \frac{1}{\rho} ds = \int \kappa ds,$$

where $\rho = \frac{1}{\kappa}$ is the radius of curvature of the curve $\gamma$. Then, we have

$$\frac{d \sigma}{d \sigma} = \rho. \tag{3.1}$$

Let $h$ be a homothety with center at origin and the coefficient $\mu$. If we put $\bar{\gamma} = h(\gamma)$, then it follows

$$\bar{s} = \mu s \quad \text{and} \quad \bar{\rho} = \mu \rho,$$

where $\bar{s}$ is the arc-length parameter of $\bar{\gamma}$ and $\bar{\rho}$ is the radius of curvature of this curve. Therefore, $\sigma$ is an equiform invariant parameter of $\gamma$ (see [11]).

**Notation 3.1.** The functions $\kappa$ and $\tau$ are not invariants of the homothety group, then from (2.3) it follows that $\bar{\kappa} = \frac{1}{\mu} \kappa$ and $\bar{\tau} = \frac{1}{\mu} \tau$.

Now we define the Frenet formulas of the curve $\gamma$ with respect to its equiform invariant parameter $\sigma$ in $G^1_3$. The vector

$$T = \frac{d \gamma}{d \sigma},$$

is called a tangent vector of the curve $\gamma$. From (2.6) and (3.1), we get

$$T = \frac{d \gamma}{d s} \frac{d s}{d \sigma} = \rho \cdot \frac{d \gamma}{d s} = \rho \cdot \mathbf{e}_1. \tag{3.2}$$

Also, the principal normal and the binormal vectors are respectively, given by

$$N = \rho \cdot \mathbf{e}_2, \quad B = \rho \cdot \mathbf{e}_3. \tag{3.3}$$

It is easy to show that $\{T, N, B\}$ is an equiform invariant frame of $\gamma$. On the other hand, the derivatives of these vectors with respect to $\sigma$ are given by

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} \dot{\rho} & 1 & 0 \\ 0 & \dot{\rho} & \rho \tau \\ 0 & \rho \tau & \dot{\rho} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{3.4}$$

The functions $K : I \to \mathbb{R}$ defined by $K = \dot{\kappa}$ is called the equiform curvature of the curve $\gamma$ and $T : I \to \mathbb{R}$ defined by $T = \rho \tau = \frac{\xi}{\kappa}$ is called the equiform torsion of this curve. In the light of this, the formulas (3.4)
analogous to the Frenet formulas in the equiform geometry of the pseudo-Galilean space $G^1_3$ can be written as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} \mathcal{K} & 1 & 0 \\ 0 & \mathcal{K} & \mathcal{T} \\ 0 & \mathcal{T} & \mathcal{K} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \quad (3.5)$$

The equiform parameter $\sigma = \int \kappa(s) ds$ for closed curves is called the total curvature, and it plays an important role in global differential geometry of Euclidean space. Also, the function $\frac{\mathcal{Z}}{\mathcal{K}}$ has been already known as a conical curvature and it also has interesting geometric interpretation.

**Notation 3.2.** Let $\gamma : I \to G^1_3$ be a Frenet curve in the equiform geometry of $G^1_3$, the following statements are true (for more details, see [11, 13]):

1. If $\gamma(s)$ is an isotropic logarithmic spiral in $G^1_3$. Then, $\mathcal{K} = \text{const.} \neq 0$ and $\mathcal{T} = 0$.
2. If $\gamma(s)$ is a circular helix in $G^1_3$. Then, $\mathcal{K} = 0$ and $\mathcal{T} = \text{const.} \neq 0$.
3. If $\gamma(s)$ is an isotropic circle in $G^1_3$. Then, $\mathcal{K} = 0$ and $\mathcal{T} = 0$.

4. **AW($k$)-type curves in the equiform geometry of $G^1_3$**

Let $\gamma(s) : I \to G^1_3$ be a curve in the equiform geometry of the pseudo-Galilean space $G^1_3$. The curve $\gamma$ is called a Frenet curve of osculating order $l$ if its derivatives:

$$\gamma'(s), \gamma''(s), \gamma'''(s), ..., \gamma^{(l)}(s),$$

are linearly dependent and

$$\gamma'(s), \gamma''(s), \gamma'''(s), ..., \gamma^{(l+1)}(s),$$

are no longer linearly independent for all $s \in I$.

To each Frenet curve of order 3, one can associate an orthonormal 3-frame $\{T, N, B\}$ along $\gamma$, such that $\gamma'(s) = \frac{1}{\rho} T$, called the equiform Frenet frame (Eqs. (3.5)).

Now, we consider equiform Frenet curves of osculating order 3 in $G^1_3$ and discuss some important results.

Let $\gamma(s) : I \to G^1_3$ be a Frenet curve in the equiform geometry of the pseudo-Galilean space. By the use of Frenet formulas (3.5), we obtain the higher order derivatives of $\gamma$ as follows

$$\gamma'(s) = \frac{d\gamma}{d\sigma} \frac{d\sigma}{ds} = \frac{1}{\rho} T,$$

$$\gamma''(s) = \frac{1}{\rho^2} N,$$

$$\gamma'''(s) = \frac{1}{\rho^3} (-\mathcal{K}N + \mathcal{T}B),$$

$$\gamma''''(s) = \frac{1}{\rho^4} [(2\mathcal{K}^2 + \mathcal{T}^2 - \mathcal{K}'\mathcal{N} + (\mathcal{T}' - 3\mathcal{K}\mathcal{T})B].$$
Notation 4.1. Let us write

\[ Q_1 = \frac{1}{\rho^2} N, \]  
\[ Q_2 = \frac{1}{\rho^3} (-\kappa N + \tau B), \]  
\[ Q_3 = \frac{1}{\rho^4} [(2\kappa^2 + \tau^2 - \kappa')N + (\tau' - 3\kappa\tau)B]. \]  

Notation 4.2. \( \gamma'(s), \gamma''(s), \gamma'''(s) \) and \( \gamma''''(s) \) are linearly dependent if and only if \( Q_1, Q_2 \) and \( Q_3 \) are linearly dependent.

Definition 4.1. \[5\] Frenet curves (of osculating order 3) in the equiform geometry of the pseudo-Galilean space \( G_{3}^{1} \) are called curves of type:

1. equiform AW(1) if they satisfy \( Q_3 = 0 \),
2. equiform AW(2) if they satisfy \( ||Q_2||^2 Q_3 = (Q_3, Q_2)Q_2 \),
3. equiform AW(3) if they satisfy \( ||Q_1||^2 Q_3 = (Q_3, Q_1)Q_1 \),
4. weak equiform AW(2) if they satisfy

\[ Q_3 = (Q_3, Q_2^*) Q_2^*, \]  

(4.4)

5. weak equiform AW(3) if they satisfy

\[ Q_3 = (Q_3, Q_1^*) Q_1^*, \]  

(4.5)

where

\[ Q_2^* = \frac{Q_2 - (Q_2, Q_1^*) Q_1^*}{||Q_2 - (Q_2, Q_1^*) Q_1^*||}. \]  

(4.6)

Proposition 4.1. Let \( \gamma : I \rightarrow G_{3}^{1} \) be a Frenet curve (of osculating order 3) in the equiform geometry of the pseudo-Galilean space \( G_{3}^{1} \), therefore

(i) \( \gamma \) is of type weak equiform AW(2) if and only if

\[ 2\kappa^2 + \tau^2 - \kappa' = 0, \]  

(4.7)

(ii) \( \gamma \) is of type weak equiform AW(3) if and only if

\[ \tau' - 3\kappa\tau(s) = 0. \]  

(4.8)

Proof. Using Definition 4.1 and Notation 4.1, the proof will be obvious. \(\square\)
Theorem 4.1. Let $\gamma : I \rightarrow G^1_3$ be a Frenet curve (of osculating order 3) in the equiform geometry of the pseudo-Galilean space $G^1_3$. Then $\gamma$ is of type equiform AW(1) if and only if

$$-K' + 2K^2 + T^2 = 0,$$

$$3K' - T' = 0.$$  \hspace{1cm} (4.9)

Proof. Since $\gamma$ is of type equiform AW(1), then from (4.3), we obtain

$$\frac{1}{\rho^4}[(2K^2 + T^2(s) - K')N + (T' - 3K'T)B] = 0.$$  \hspace{1cm} (4.10)

As we know, the vectors $N$ and $B$ are linearly independent, so we can write

$$2K^2 + T^2 - K' = 0 \text{ and } T' - 3K'T = 0.$$

The converse statement is straightforward and therefore, the proof is completed.  \hfill \square

Theorem 4.2. Let $\gamma : I \rightarrow G^1_3$ be a Frenet curve (of osculating order 3) in the equiform geometry of the pseudo-Galilean space $G^1_3$. Then $\gamma$ is of type equiform AW(2) if

$$K^2T - KT' + TK' - T^3 = 0.$$  \hspace{1cm} (4.10)

Proof. Assuming that $\gamma$ is a Frenet curve in the equiform geometry of $G^1_3$, then from (4.2) and (4.3), one can write

$$Q_2 = a_{11}N + a_{12}B,$$

$$Q_3 = a_{21}N + a_{22}B,$$

where $a_{11}, a_{12}, a_{21}$ and $a_{22}$ are differentiable functions. Since $Q_2$ and $Q_3$ are linearly dependent, hence coefficients determinant equals zero, that is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0.$$  \hspace{1cm} (4.11)

where

$$a_{11} = \frac{-1}{\rho^4}K', \quad a_{12} = \frac{1}{\rho^4}T,$$

$$a_{21} = \frac{1}{\rho^4}[-K' + 2K^2 + T^2],$$

$$a_{22} = \frac{1}{\rho^4}[-3K'T + T'].$$  \hspace{1cm} (4.12)

From (4.11) and (4.12), we obtain (4.10).  \hfill \square
Theorem 4.3. Let $\gamma : I \rightarrow G^3_3$ be a Frenet curve (of osculating order 3) in the equiform geometry of $G^3_3$. Then $\gamma$ is of equiform AW(3)-type if
\[ T' - 3KT = 0. \] (4.13)

Proof. Using Definition 4.1 and Eqs. (4.1) and (4.3), we obtain (4.13). \qed

5. Computational examples

We consider some examples (timelike and spacelike curves [11, 12]) which characterize equiform general (circular) helices with respect to the Frenet frame \{T, N, B\} in the equiform geometry of $G^3_3$ which satisfy some conditions of equiform curvatures (i) $K = K(s)$, $T = T(s)$ (ii) $K = \text{const.} \neq 0$, $T = \text{const.} \neq 0$ (iii) $K = \text{const.} \neq 0$, $T = 0$.

Example 5.1. Consider the equiform timelike general helix $r : I \rightarrow G^3_3, I \subseteq \mathbb{R}$ which parameterized by the arc length $s$ with differential form $ds = dx$ is given by
\[ r(x) = (x, y(x), z(x)), \]
where
\[ x(s) = s, \]
\[ y(s) = \frac{e^{-as}}{(a^2 - b^2)^2} \left( (a^2 + b^2) \cosh (bs) + 2ab \sinh (bs) \right), \]
\[ z(s) = \frac{e^{-as}}{(a^2 - b^2)^2} \left( 2ab \cosh (bs) + (a^2 + b^2) \sinh (bs) \right); \]
\[ a, b \in \mathbb{R} - \{0\}. \]

The corresponding derivatives of $r$ are as follows
\[ r' = \left( 1, \frac{-e^{-as}}{(a^2 - b^2)} (a \cosh (bs) + b \sinh (bs)), \frac{e^{-as}}{(b^2 - a^2)} (b \cosh (bs) + a \sinh (bs)) \right), \]
\[ r'' = \left( 0, \frac{e^{-as}}{(a^2 - b^2)} \cosh (bs), \frac{e^{-as}}{(b^2 - a^2)} \sinh (bs) \right), \]
\[ r''' = \left( 0, \frac{e^{-as}}{(a^2 - b^2)} (-a \cosh (bs) + b \sinh (bs)), \frac{e^{-as}}{(b^2 - a^2)} (b \cosh (bs) - a \sinh (bs)) \right). \]

The tangent vector of $r$ has the form
\[ e_1 = (x', y', z') \]
\[ = \left( 1, \frac{-e^{-as}}{(a^2 - b^2)} (a \cosh (bs) + b \sinh (bs)), \frac{e^{-as}}{(b^2 - a^2)} (b \cosh (bs) + a \sinh (bs)) \right), \]
and the two normals (normal and binormal) of the curve are, respectively
\[ e_2 = (0, \cosh (bs), \sinh (bs)); \]
\[ e_3 = (0, \sinh (bs), \cosh (bs)); \]
\[ \text{det} [e_1, e_2, e_3] = 1. \]
Therefore, the curvature and torsion of \( \mathbf{r} \) are respectively, given by

\[
\kappa = e^{-as}, \quad \tau = b.
\]

From the equiform Frenet formulas, we can express the vector fields \( \mathbf{T}, \mathbf{N}, \mathbf{B} \) as follows

\[
\mathbf{T} = \left( e^{as}, \frac{-1}{(a^2 - b^2)} (a \cosh (bs) + b \sinh (bs)), \frac{1}{(b^2 - a^2)} (b \cosh (bs) + a \sinh (bs)) \right),
\]

\[
\mathbf{N} = (0, e^{as} \cosh (bs), e^{as} \sinh (bs)),
\]

\[
\mathbf{B} = (0, e^{as} \sinh (bs), e^{as} \cosh (bs)),
\]

respectively. In the light of this, the equiform curvatures are given by

\[
\kappa = ae^{as}, \quad \tau = -be^{as}.
\]

![Figure 1. Equiform timelike general helix with \( \kappa = 5e^{5s}, \tau = -2e^{5s} \).](image)

**Example 5.2.** Let \( \mathbf{r} : I \rightarrow G_3^1, I \subseteq \mathbb{R} \) be the equiform spacelike general helix, and it is given by

\[
\mathbf{r}(x) = (x, g(x), z(x)),
\]
where
\[
x(s) = s, \\
y(s) = \frac{e^{-as}}{(a^2 - b^2)^2} \left( 2ab \cosh(bs) + (a^2 + b^2) \sinh(bs) \right), \\
z(s) = \frac{e^{-as}}{(a^2 - b^2)^2} \left( (a^2 + b^2) \cosh(bs) + 2ab \sinh(bs) \right);
\]
\(a, b \in \mathbb{R} - \{0\}.
\]

For the coordinate functions of \(r\), we have
\[
\begin{align*}
r' &= \left(1, \frac{e^{-as}}{(b^2 - a^2)} (b \cosh(bs) + a \sinh(bs)), \frac{-e^{-as}}{(a^2 - b^2)} (a \cosh(bs) + b \sinh(bs)) \right), \\
r'' &= \left(0, e^{-as} \sinh(bs), e^{-as} \cosh(bs) \right), \\
r''' &= \left(0, e^{-as} (b \cosh(bs) - a \sinh(bs)), e^{-as} (b \sinh(bs) - a \cosh(bs)) \right).
\end{align*}
\]

Also, the associated trihedron is given by
\[
\begin{align*}
e_1 &= \left(1, \frac{e^{-as}}{(b^2 - a^2)} (b \cosh(bs) + a \sinh(bs)), \frac{-e^{-as}}{(a^2 - b^2)} (a \cosh(bs) + b \sinh(bs)) \right), \\
e_2 &= \left(0, \sinh(bs), \cosh(bs) \right), \\
e_3 &= \left(0, - \cosh(bs), - \sinh(bs) \right).
\end{align*}
\]

The curvature and torsion of this curve are
\[
\kappa = e^{-as}, \quad \tau = -b.
\]

Furthermore, the tangent, normal and binormal vector fields in the equiform geometry of \(G_3^1\) are obtained as follows
\[
\begin{align*}
T &= \left(e^{as}, \frac{1}{(b^2 - a^2)} (b \cosh(bs) + a \sinh(bs)), \frac{-1}{(a^2 - b^2)} (a \cosh(bs) + b \sinh(bs)) \right), \\
N &= \left(0, e^{as} \sinh(bs), e^{as} \cosh(bs) \right), \\
B &= \left(0, -e^{as} \cosh(bs), -e^{as} \sinh(bs) \right),
\end{align*}
\]
respectively.

The equiform curvatures of \(r\) are
\[
\kappa = ae^{as}, \quad \tau = -be^{as}.
\]

**Example 5.3.** Consider the equiform timelike circular helix \(r : I \rightarrow G_3^1, I \subseteq \mathbb{R}\) is given by
\[
r(x) = (x, y(x), z(x)),
\]
Figure 2. Equiform spacelike general helix with $K = 5e^{5s}, T = -2e^{5s}$.

where

\[
\begin{align*}
    x(s) &= s, \\
    y(s) &= \frac{a^3 s}{b(b^2 - a^2)} \left( b \sinh \left( \frac{b}{a} \ln(\alpha s) \right) - a \cosh \left( \frac{b}{a} \ln(\alpha s) \right) \right), \\
    z(s) &= \frac{a^3 s}{b(b^2 - a^2)} \left( b \cosh \left( \frac{b}{a} \ln(\alpha s) \right) - a \sinh \left( \frac{b}{a} \ln(\alpha s) \right) \right); \\
    a, b &\in \mathbb{R} - \{0\}.
\end{align*}
\]

For this curve, the equiform vector fields are obtained as follows

\[
\begin{align*}
    \mathbf{T} &= \left( \frac{s}{a}, \frac{as}{b} \cosh \left( \frac{b}{a} \ln(\alpha s) \right), \frac{as}{b} \sinh \left( \frac{b}{a} \ln(\alpha s) \right) \right), \\
    \mathbf{N} &= \left( 0, \frac{s}{a} \sinh \left( \frac{b}{a} \ln(\alpha s) \right), \frac{s}{a} \cosh \left( \frac{b}{a} \ln(\alpha s) \right) \right), \\
    \mathbf{B} &= \left( 0, \frac{s}{a} \cosh \left( \frac{b}{a} \ln(\alpha s) \right), \frac{s}{a} \sinh \left( \frac{b}{a} \ln(\alpha s) \right) \right),
\end{align*}
\]

respectively.

It follows that

\[
    K = \frac{1}{a}, T = -\frac{b}{a^2}.
\]
Example 5.4. Let the equiform spacelike circular helix \( r : I \rightarrow G^1_3, I \subseteq \mathbb{R} \) be

\[
\mathbf{r}(x) = (x, y(x), z(x)),
\]

where

\[
\begin{align*}
x(s) &= s, \\
y(s) &= \frac{a^3}{b(b^2 - a^2)} \left( b \cosh \left( \frac{b}{a} \ln(\alpha s) \right) - a \sinh \left( \frac{b}{a} \ln(\alpha s) \right) \right), \\
z(s) &= \frac{a^3}{b(b^2 - a^2)} \left( b \sinh \left( \frac{b}{a} \ln(\alpha s) \right) - a \cosh \left( \frac{b}{a} \ln(\alpha s) \right) \right); \\
a, b &\in \mathbb{R} - \{0\}.
\end{align*}
\]

Here, the equiform differential vectors respectively, are as follows

\[
\begin{align*}
\mathbf{T} &= \left( \frac{s a}{a}, \frac{s a}{b} \sinh \left( \frac{b}{a} \ln(\alpha s) \right), \frac{a s}{b} \cosh \left( \frac{b}{a} \ln(\alpha s) \right) \right), \\
\mathbf{N} &= \left( 0, \frac{s}{a} \cosh \left( \frac{b}{a} \ln(\alpha s) \right), \frac{s}{a} \sinh \left( \frac{b}{a} \ln(\alpha s) \right) \right), \\
\mathbf{B} &= \left( 0, -\frac{s}{a} \sinh \left( \frac{b}{a} \ln(\alpha s) \right), -\frac{s}{a} \cosh \left( \frac{b}{a} \ln(\alpha s) \right) \right).
\end{align*}
\]

Equiform curvature and equiform torsion are calculated as follows

\[
K = \frac{1}{a}, \quad T = \frac{b}{a^2}.
\]
Figure 4. Equiform spacelike circular helix with $\mathcal{K} = \frac{1}{3}, \mathcal{T} = \frac{4}{5}$.

Example 5.5. Let $\mathbf{r} : I \rightarrow C^1_3, I \subseteq \mathbb{R}$ be an equiform timelike isotropic logarithmic spiral which parameterized by the arc length $s$ with differential form $ds = dx$, and is given by

$$
\mathbf{r}(x) = (x, y(x), 0),
$$

where

$$
x(s) = s, \\
y(s) = \frac{as + b}{a^2} (\ln(as + b) - 1), \\
z(s) = 0; \\
a, b \in \mathbb{R} - \{0\}.
$$

For this curve, we get

$$
\mathbf{r}' = \left(1, \frac{\ln(as + b)}{a}, 0\right), \\
\mathbf{r}'' = \left(0, \frac{1}{as + b}, 0\right), \\
\mathbf{r}''' = \left(0, \frac{-a}{(as + b)^2}, 0\right).
$$
and

\[ e_1 = \left(1, \frac{\ln(as + b)}{a}, 0\right), \]

\[ e_2 = (0, 1, 0), \]

\[ e_3 = (0, 0, 1); \quad \kappa = \frac{1}{as + b}, \quad \tau = 0. \]

In this case, equiform Frenet vectors and equiform curvatures are as follows

\[ T = \left(as + b, \frac{(as + b) \ln(as + b)}{a}, 0\right), \]

\[ N = (0, as + b, 0), \]

\[ B = (0, 0, as + b), \quad K = a, \quad T = 0. \]

respectively.

![Equiform timelike isotropic logarithmic spiral](image)

**Figure 5.** Equiform timelike isotropic logarithmic spiral with \( K = 2, T = 0. \)

From aforementioned calculations, according to (Proposition 4.2 and Theorems 4.1 – 4.3), the first four examples are not characterize curves of equiform AW(1), weak equiform AW(2) or weak equiform AW(3)-types. On the other hand, the last example shows that the curve is of equiform AW(2) and AW(3)-types and it is not of equiform AW(1)-type. Also, this curve is of weak equiform AW(2) and not of weak equiform AW(3)-types.
6. Conclusion

In this paper, we have considered some special curves of equiform \( AW(k) \)-type of the pseudo-Galilean 3-space. Also, using the equiform curvature conditions of these curves, the necessary and sufficient conditions for them to be equiform \( AW(k) \) and weak equiform \( AW(k) \)-types are obtained. Furthermore, some examples to support our main results are given and plotted.

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