Transient quantum beats, Rabi-oscillations and delay-time of modulated matter-waves

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Transient phenomena of phase modulated cut-off wavepackets are explored by deriving an exact general solution to Schrödinger’s equation for finite range potentials involving arbitrary initial quantum states. We show that the dynamical features of the probability density are governed by a virtual self-induced two-level system with energies \(E_+\) and \(E_-\), due to the phase modulation of the initial state. The asymptotic probability density exhibits Rabi-oscillations characterized by a frequency \(\Omega = (E_+ - E_-)/\hbar\), which are independent of the potential profile. It is also found that for a system with a bound state, the interplay between the virtual levels with the latter causes a quantum beat effect with a beating frequency, \(\Omega\). We also find a regime characterized by a time-diffraction phenomenon that allows to measure unambiguously the delay-time, which can be described by an exact analytical formula. It is found that the delay-time agrees with the phase-time only for the case of strictly monochromatic waves.

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I. INTRODUCTION

Transient phenomena have lead to the development of powerful analytical and experimental techniques to explore the dynamics of matter-waves \([1,2]\). Transient effects emerge as a response of a quantum system to various types of interactions, state preparation or initial boundary conditions. The most representative example of transient phenomena due to a sudden interaction is the Moshinsky quantum-shutter setup \([3]\). The latter involves the time-dependent solution of free Schrödinger’s equation for a cut-off plane wave of momentum \(k\), initially confined to the left of a perfect absorbing shutter located at \(x = 0\). By opening the shutter at a time \(t = 0\), the quantum state evolves freely in space, and the probability density in the time-domain exhibits a distinctive oscillatory pattern known as diffraction in time \([3,4]\), analogous to the intensity profile of a light beam diffracted by a semi-infinite plane. The time-diffraction effect was verified decades later in experiments using ultracold atoms \([5]\), cold-neutrons \([6]\), and atomic Bose-Einstein condensates \([7]\). The quantum-shutter problem has allowed to translate spatial features of light optics to the time-domain of matter-waves in experiments with atoms that exhibit diffraction and interference \([5,8]\), as well as phase modulation \([9,10]\) and quantum beat phenomena \([10]\).

After Moshinsky’s proposal, several theoretical works have addressed the study of transients of tunneling matter-waves within a quantum shutter setup for systems involving delta potentials \([1,11-14]\), potential barriers \([15,17,20]\), and multibarrier resonant structures \([21,23]\).

Some of these transients involve electronic transport and buildup in quantum structures \([21,23]\), and time-scales \([1,11,14,18,19,21]\). Although most of these works have relied on cut-off plane wave initial conditions \([11,14,16,20]\) and wave packets with Gaussian distributions \([13,14,23]\), to our knowledge, the study and characterization of transient effects of phase modulated cut-off waves has not been addressed before.

In this work we explore transient phenomena of phase modulated cut-off wavepackets by deriving an exact analytical solution to Schrödinger’s equation based on a resonant state expansion. The modulation involves a superposition of quantum states with slightly different momenta. The aim of this work is to show that the phase modulation of initial cut-off quantum waves gives rise to interesting transient phenomena, such as Rabi-oscillations, quantum beats, delay-time, and time-diffraction effects. We discuss how these effects are the result of a virtual self-induced two-level system that governs the dynamics of probability density in the time-domain.

Our work is organized as follows. In Sec. \([I]\) we present the main equations involving the solution of the quantum shutter approach for a general potential to describe the dynamics of arbitrary cut-off initial quantum states. In Sec. \([II]\) we study the time-dependent features of phase modulated wavepackets along the transmission region of a potential, which exhibits different regimes governed by a virtual self-induced two-level system. The effects of the modulation on the delay-time of the system are also presented. Finally, in Sec. \([V]\) we present the conclusions.

II. MODEL FOR A GENERAL INITIAL STATE

Our approach deals with an exact analytical solution to Schrödinger’s equation for a finite range potential involv-
ing an arbitrary cut-off initial state, based on a resonant state expansion. We stress that our approach is non-Hermitian since the resonant states are eigenfunctions of the Hamiltonian with outgoing boundary conditions, which leads to complex energy eigenvalues. The Hermiticity of the system Hamiltonian is not only related to the operator itself but also to the functions on which it acts on [24]. Non-Hermitian approaches provide alternative exact analytical descriptions of physical processes. In fact, the equivalence of non-Hermitian dynamical descriptions and those based on continuum wave expansions of standard quantum mechanics has been demonstrated in Ref. [19]. Within this framework, the time-dependent features of quantum waves with different initial conditions can be explored. Our model is a generalization of the work originally developed for a cut-off plane wave in Ref. [21]. The general approach to the problem involves the time-evolution of particles of energy \( E = \hbar^2 k^2/2m \) incident from the left onto a one-dimensional finite range potential \( V(x) \) of arbitrary shape that extends along the region \( 0 \leq x \leq L \), and vanishes thereafter. The wavefunction \( \Psi(x,t) \) of the problem is a solution of Schrödinger’s equation \( \hat{H} \Psi(x,t) = i\hbar \partial_t \Psi(x,t) \), with \( \hat{H} = (-\hbar^2/2m)\partial_x^2 + V(x) \), for an initial condition at \( t = 0 \) given by,

\[
\Psi(x,0) = \begin{cases} 
\phi(x,k), & \text{for } x \leq 0; \\
0, & \text{for } x > 0.
\end{cases}
\]  

(1)

The function \( \phi(x,k) \) represents a general initial state that extends from \(-\infty < x \leq 0 \). The solution is obtained by Laplace-transforming Schrödinger’s equation and the corresponding initial condition, following the procedure of Ref. [21] that is,

\[
\left[ \partial_x^2 + p^2 \right] \tilde{\Psi}(x,p) = \begin{cases} 
i\alpha \phi(x,k), & \text{for } x \leq 0; \\
0, & \text{for } x > L.
\end{cases}
\]  

(2)

The solutions to Eq. (2) are,

\[
\tilde{\Psi}(x,p) = \begin{cases} 
\bar{c}e^{ipx} + e^{ipx} \bar{F}(x,k,p) e^{-ipx}, & \text{for } x > L; \\
e^{-ipx} + \alpha \bar{F}(x,k,p) e^{-ipx}, & \text{for } x \leq 0,
\end{cases}
\]  

(3)

where we have defined \( p = (i\alpha s)^{1/2} \) with \( \alpha = 2m/\hbar \), and \( \bar{c}^\pm \) are constants to be determined by the matching conditions of the wavefunctions. The function \( \bar{F}(x,k,p) \) is the particular solution to the inhomogeneous equation in [2], given by

\[
\bar{F}(x,k,p) = \int e^{2ipx} \left[ \int i \phi(x,k) e^{-ipx} dx \right] dx,
\]  

(4)

obtained following the procedure of Ref. [25].

The differential equation for the internal region reads,

\[
\left[ \partial_x^2 + p^2 - \bar{V}(x) \right] \tilde{\Psi}(x,p) = 0, \quad 0 \leq x \leq L,
\]  

(5)

with \( \bar{V}(x) = 2mV(x)/\hbar^2 \). We obtain the solution for the internal region by using the corresponding equation of the outgoing Green’s function of the system, \( G^+(x,x';p) \), that is,

\[
\left[ \partial_x^2 + p^2 - \bar{V}(x) \right] G^+(x,x';p) = \delta(x-x'), \quad 0 \leq x \leq L
\]  

(6)

with the outgoing boundary conditions,

\[
\partial_x G^+(x,x';p)|_{x=0} = -ip G^+(0,x';p), \quad \partial_x G^+(x,x';p)|_{x=L} = ip G^+(L,x';p).
\]  

(7)

We write \( \bar{\Psi}(x,p) \) in terms of \( G^+ \) by multiplying Eq. (5) by \( G^+(x,x';p) \) and Eq. (6) by \( \bar{\Psi}(x,p) \), and subtracting the equations. Thus, by integrating the resulting expression along \( x \in [0,L] \), with the help of Eq. (3) and the boundary conditions (7) and (8), we get

\[
\bar{\Psi}(x,p) = \alpha \tilde{\Phi}(0,k,p) G^+(0,x';p)
\]  

(9)

where \( \tilde{\Phi}(0,k,p) = \partial_x \bar{F}(x,k,p)|_{x=0} \). By evaluating Eq. (9) in \( x' = L \), and Eq. (3) in \( x = L \), we obtain the constant \( \alpha \) which allows to write the solution along the transmission region \( (x \geq L) \) as

\[
\bar{\Psi}(x,p) = \alpha C(p,k) e^{ipx}, \quad x \geq L,
\]  

(10)

with \( C(p,k) = \tilde{\Phi}(0,k,z) G^+(0,x';z) (1-iL) \), by considering the integration contour \( C = \Gamma + C_p + \sum_n C_n \), with \( C_n \) the contours of \( \bar{\Phi}(0,k,z) \), which is also a meromorphic function. The complex integration leads us to,

\[
\frac{1}{2\pi i} \left[ \int_{C_p} \frac{z C(z,k)}{z-p} dz \right. \\
+ \left. \sum_n \int_{C_n} \tilde{\Phi}(0,k,z) G^+(0,x';z) e^{-izL} dz \right. \\
+ \left. \sum_m \int_{C_m} \tilde{\Phi}(0,k,z) G^+(0,x';z) e^{-izL} dz \right] = 0.
\]  

(12)

We perform the integration in Eq. (12) by applying the Cauchy residue formula, and using the following expansions for \( \tilde{\Phi}(0,k,z) \), and \( G^+(0,x';z) \). For the case \( \tilde{\Phi}(0,k,z) \) we assume a Mittag-Leffler expansion

\[
\tilde{\Phi}(0,k,z) = \tilde{\Phi}(0,k,0) + \sum_{m'=1}^{\infty} \alpha_{m'} \left( \frac{1}{z - \kappa_{m'}} + \frac{1}{\kappa_{m'}} \right)
\]  

(13)
where $\alpha_m^*$ are the residues at the simple complex poles $\kappa_m^*$. For the case of $G^+(0,x';z)$, we use the expansion of the Green function in terms of the complex poles $k_n$ and the resonance states $\tilde{\Psi}(x,p)$, namely

$$G^+(x,x';z) = \sum_{n=-\infty}^{\infty} \frac{u_n(x)u_n(x')}{2k_n(z-k_n)},$$

(14)

where the $u_n$’s are one-dimensional resonant states (quasirmormal states), which are eigensolutions of the Schrödinger equation,

$$\frac{d^2u_n(x)}{dx^2} + \left[k_n^2 - V(x)\right] u_n(x) = 0,$$

(15)

with outgoing boundary conditions:

$$\frac{du_n(x)}{dx}|_{x=L} = +ik_n u_n(L);$$

$$\frac{du_n(x)}{dx}|_{x=0} = -ik_n u_n(0).$$

(16)

The index $n$ ($n = 1, 2, 3, ...$) spans both the third and fourth quadrant poles of the complex $k$-plane, given respectively by $k_{-n}$ and $k_n$. These poles are related through $k_{-n} = -k_n^*$, which follows from time-reversal invariance. The complex energies associated to $k_n$ are given by $E_n = \hbar^2 k_n^2/2m = \varepsilon_n - i\Gamma_n/2$, where $\varepsilon_n$ is the resonance energy, and $\Gamma_n$ the resonance width. The resonant states also satisfy the time-reversal property $u_{-n}(x) = u_n^*(x)$. Since our approach is non-Hermitian, the resonant states do not obey the usual rules of normalization and orthonormality [26]. Instead, the $u_n$’s fulfill the following condition,

$$\int_0^L u_n(x)u_m(x) \, dx + i\frac{u_n(0)u_m(0) + u_n(L)u_m(L)}{k_n + k_m} = \delta_{nm}.\tag{17}$$

For the particular case $n = m$, Eq. (17) yields the corresponding normalization for the resonant states,

$$\int_0^L u_n^2(x) \, dx + i\frac{u_n^2(0) + u_n^2(L)}{2k_n} = 1,\tag{18}$$

which also satisfy the closure relation,

$$\frac{1}{2} \sum_{-\infty}^{\infty} u_n(x)u_n(x') = \delta(x-x'), \quad 0 \leq (x,x')^t \leq L,\tag{19}$$

where the notation $(x,x')^t$ means that the expansion does not hold at $x = x' = 0 = L$. The resonant states form a complete set [26] along the internal region of the potential, and any arbitrary function $\psi(x,k)$ can be expressed as a linear combination of these states using the fundamental relation $\psi(x,k) = 2ikG^+(0,x;k)$.

By using the expansions (14) and (15) to evaluate the integrals in Eq. (12), by a direct computation of the corresponding residues, we obtain and expression for $C(p,k)$ that can be substituted in Eq. (10), leading us to,

$$\tilde{\Psi}(x,p) = \alpha \sum_{n=1}^{\infty} \frac{2\alpha_m^* k_m}{p(p-k_m)} G^+(0,L,\kappa_m)e^{-ik_m L e^{ipx}} + \sum_{n=-\infty}^{\infty} \Phi(0,k,k_n) e^{-ik_n L u_n(0) u_n(x)/p(p-k_n)} e^{px}.\tag{20}$$

We substitute in Eq. (20) the fundamental relationship between the transmission amplitude $t(k)$ and the Green function of the system [26], namely

$$t(k) = 2ke^{-ikL}G^+(0,L;k),\tag{21}$$

and from the resulting expression we obtain $\Psi(x,t)$ by performing the inverse Laplace-transform by means of the Bromwich integral,

$$\Psi(x,t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \tilde{\Psi}(x;s) e^{st} \, ds.\tag{22}$$

Equation (22) is evaluated by using the Moshinsky functions:

$$M(x,q,t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \left(\frac{ia}{\sqrt{a^2}}\right) \left(\frac{e^{\sqrt{a^2}x}}{\sqrt{a^2} - q}\right) e^{st} \, ds,\tag{23}$$

which are also related to the complex error function [27]

$$\text{w}(z) = e^{-z^2} \text{erfc}(-iz),\quad M(x,q,t) \equiv M(y_q) = \frac{1}{2} e^{im^2/2ht} w(iy_q),\tag{24}$$
with,
\[ y_q = e^{-i\pi/4} \frac{(x - \hbar q t/m)}{\sqrt{2\hbar t/m}}, \] (25)
where \( q \) stands for \( k \) or \( k_n \). We finally write the general time-dependent solution \( \Psi(x, t) \) along the transmission region as:
\[ \Psi(x, t) = -\left[ \sum_{m=1}^{\infty} \alpha_m t(k_m) M(y_{k_m}) \right. + i \sum_{n=-\infty}^{\infty} t_n \Phi(0, k, k_n) M(y_{k_n}) \bigg|_{x=0}, \]
\[ x \geq L. \] (26)
In the above formula we have defined \( t_n = e^{-ik_n L}u_n(0)\varphi_n(L) \), and the \( Q \)-transform function \( \Phi(0, k, Q) \),
\[ \Phi(0, k, Q) = \left[ \int \varphi(x, k) e^{-iQx} dx \right]_{x=0}. \] (27)
The coefficients \( \alpha_m \) are the residues at the complex poles \( Q = \kappa_m \) of the \( Q \)-transform, \( \Phi(0, k, Q) \). The solution given by Eq. (26) holds for any finite range potential, and for initial states \( \varphi(x, k) \), as long as the \( Q \)-transform \( \Phi(0, k, Q) \) yields simple \( Q \)-poles. To verify our result Eq. (26), let us consider as an example the case of an absorbing plane wave shutter \( \varphi(x, k) = e^{ikx} \). The \( Q \)-transform is \( \Phi(0, k, Q) = (k - Q)^{-1} \) with one single pole at \( Q = \kappa_1 = k \), with a residue \( \alpha_1 = -1 \), that yields by direct substitution in Eq. (26) the transmitted time-dependent wavefunction of Ref. 21.

A. Phase modulated wavepackets

We use as an initial condition a cut-off phase modulated wavepacket given by,
\[ \Psi(x, 0) = \begin{cases} 2 \sin(k x) \cos(\Delta k x), & \text{for } x \leq 0; \\ 0, & \text{for } x > 0, \end{cases} \] (28)
constructed by a superposition of quantum states with different momenta i.e. \( \Psi(x, 0) = \psi_+(x, k_+, 0) + \psi_-(x, k_-, 0) \), with \( \psi_\pm(x, k_\pm, 0) = (e^{ik\pm x} - e^{-ik\pm x})/2i \), where \( k_\pm = k \pm \Delta k \), corresponding to different kinetic energies, \( E_\pm = \hbar^2 k_\pm^2/2m \). The detailed analysis of the transient dynamics of these type of phase-modulated cut-off waves, to our knowledge, has not been performed before, although a similar condition has been applied in the context of time-dependent potentials 28.

We apply our general result Eq. (26), for a modulated wavepacket \( \varphi(x, k) = 2 \sin(k x) \cos(\Delta k x) \), and obtain,
\[ \Phi(0, k, Q) = i \left[ \frac{k_-}{(Q + k_-)(Q - k_-)} + \frac{k_+}{(Q + k_+)(Q - k_+)} \right], \] (29)
which has complex poles at \( Q = \pm k_+ \) (residues \( \pm i/2 \)), and \( Q = \pm k_- \) (residues \( \pm i/2 \)). Thus, Eq. (26) allows us to write,
\[ \Psi(x, t) = \psi_+(x, k_+, t) + \psi_-(x, k_-, t); \quad x \geq 0, \] (30)
with
\[ \psi_\pm(x, k_\pm, t) = (1/2i) \left[ t(k_\pm) M(y_{k_\pm}) - t(-k_\pm) M(y_{-k_\pm}) \right] + \sum_{n=-\infty}^{\infty} t_n \Phi(0, k, k_n) M(y_{k_n}). \] (31)
In the next section we shall apply our general result Eq. (30) to explore the features of modulated wavepackets for a particular potential that involves a bound state.

III. DYNAMICS OF PHASE MODULATED WAVEPACKETS

Based on our result Eq. (30), we shall explore the dynamics of the probability density of modulated cut-off wavepackets in quantum systems. As we shall show, the modulation of the initial state governs the time-dependent features of the probability density in different regimes, and some of these features are independent of potential. To begin our study, we consider the case of a simple potential that allows to explore the dynamics in systems with a single bound state, as is the case of the Dirac delta potential well, \( V(x) = -\lambda \delta(x) \), with \( \lambda > 0 \). We obtain the corresponding solution from Eqs. (30), and (31), noting that only a single term in the resonance sum is required since the spectrum of the system is composed by a single bound state at \( k_n = i\lambda \), with energy \( E_k = -\hbar^2 \lambda^2/2m \), where \( \lambda = m\lambda/\hbar^2 \). In the limit \( L \to 0 \), one can show with the help of Eq. (18) that \( u_n^2(0) \to \lambda \), and thus \( t_n \to \lambda \), leading us to the solution:
\[ \Psi^d(x, t) = \psi^d_+(x, k_+, t) + \psi^d_-(x, k_-, t); \quad x \geq 0, \] (32)
with
\[ \psi^d_\pm(x, k_\pm, t) = (1/2i) \left[ t(k_\pm) M(y_{k_\pm}) - t(-k_\pm) M(y_{-k_\pm}) \right] + \lambda \bar{\Phi}(0, k, i\lambda) M(y_{i\lambda}); \] (33)
where \( t(k) = k/(k - i\lambda) \) is the stationary transmission amplitude of the delta potential well. Also, the free modulated wavepacket solution is obtained directly from Eqs. (32), and (33), by letting \( t(\pm k_\pm) = 1 \), and \( \lambda = 0 \), leading us to
\[ \Psi_f(x, t) = (1/2i)[M(y_{k_+}) + M(y_{k_-}) - M(y_{-k_+}) - M(y_{-k_-})], \quad x \geq 0. \] (34)
In what follows, we use Eq. (32) for exploring different regimes of probability density \( \rho(x, t) \) defined as,
\[ \rho(x, t) = |\psi^d_+(x, k_+, t) + \psi^d_-(x, k_-, t)|^2; \quad x \geq 0, \] (35)
to investigate the effects of the phase modulation on the transient dynamics of the system.
A. Quantum beats

Let us now explore the time-dependent features of \( \rho(x,t) \) [Eq. \( \text{(35)} \)] at a fixed position \( x = 0 \). In Fig. 2(a) we plot \( \rho(0,t) \) as function of time, for an unmodulated wavepacket \( \Delta k = 0 \). The probability density exhibits an oscillatory effect known as persistent oscillations, reported in Ref. \footnote{16} for the case of an absorbing plane wave shutter. The effect is a result of the interaction of the quasi-stationary state induced by the incident wave, and the bound state of system. We argue that the persistent oscillations also manifest themselves in a reflecting shutter setup, since the particular case with \( \Delta k = 0 \) obtained from Eq. \( \text{(32)} \), corresponds in fact (up to a phase factor \(-i\)) to a reflecting cut-off plane wave initial condition. However, when we consider the case of a modulated wavepacket \( \Delta k \neq 0 \), we show in Fig. 2(b) that \( \rho(0,t) \) exhibits high-frequency oscillations in the time-domain, similar to a superposition effect of quantum waves with different frequencies. To identify the underlying superposition observed in Fig. 2(b), we plot the probability density for different values of \( \Delta k \) over a larger time interval.

This is shown in Fig. 3 where \( \rho(0,t) \) exhibits quantum beats with a well defined frequency. To determine the beating frequency of the oscillatory phenomenon, we derive an approximate solution for \( \Psi^a(0,t) \) [Eq. \( \text{(32)} \)], by using the properties of the Moshinsky functions. We simplify our notation for the arguments of the Moshinsky function at \( x = 0 \), by

\[
y(0,q,t) = y^a_q = -e^{-i\pi/4} (\hbar/2m)^{1/2} q t^{1/2}.
\]

By using the identity \( M(y^o_q) = e^{-i\hbar q^2 t/2m} - M(y^o_{-q}) \), and keeping only the exponential contributions, we can approximate \( \Psi^a(0,t) \) by,

\[
\Psi^a_\alpha(0,t) \approx \frac{1}{\sqrt{2}} \left[ t(k_+)e^{-iE_+t/\hbar} + t(k_-)e^{-iE_-t/\hbar} + 2\lambda \Phi(0,k,i\lambda) e^{-iE_0t/\hbar} \right],
\]

In Eq. \( \text{(37)} \) we have neglected the contributions of Moshinsky functions of the form \( M(y_{-q}) \) since these are fast decreasing functions of time. The expression for the probability density \( \rho_a(0,t) \) corresponding to Eq. \( \text{(37)} \) is:

\[
\rho_a(0,t) \approx \frac{1}{2} \left\{ \left[ |t(k_+)|^2 + |t(k_-)|^2 + 4\lambda^2|\Phi(0,k,i\lambda)|^2 \right] + a_{1,r} \cos \Omega t + a_{1,i} \sin \Omega t + (a_{2,r} + a_{3,r}) \cos (\Omega t/2) + (a_{2,i} - a_{3,i}) \sin (\Omega t/2) + (a_{2,i} + a_{3,i}) \cos (\Omega t/2) + (a_{3,r} - a_{2,r}) \sin (\Omega t/2) \right\}.
\]

where the coefficients in Eq. \( \text{(38)} \) are calculated as \( a_{n,r} = \text{Re}[a_n] \), and \( a_{n,i} = \text{Im}[a_n] \), with \( a_1 = t(k_+) + t(k_-) + 4\lambda\Phi(0,k,i\lambda) \), and \( a_2 = 2t(k_+)\Phi(0,k,i\lambda) \), and \( a_3 = 2t(k_-)\Phi(0,k,i\lambda) \).

We show from Eq. \( \text{(38)} \) that the dynamics of \( \rho(0,t) \) in Figs. 3(a)-(c) is governed by two types of frequencies, \( \Omega \) and \( \bar{\Omega} \), where \( \bar{\Omega} = (\omega_+ + \omega_-)/2 \), is an average frequency associated to persistent oscillations, with
frequencies associated to the virtual states, \( \omega_\pm \), induced by the source, and that of the bound state of the system, \( E_b \). The amplitude of the quantum wave is modulated by an envelope with a beating frequency, \( \Omega = \omega_+ - \omega_- = (E_+ - E_-)/\hbar \). Therefore, the quantum beats in Figs. 3(a)-(c) arise due to the interaction between the virtual two-level system with energies \( E_+ \) and \( E_- \), induced by the modulation of the initial state. In Fig. 4(d) we compare the exact \( \rho (t) \) [Eq. (35)] with \( \rho_{\text{int}}(t) \) [Eq. (38)], and a perfect agreement is observed. Note in Fig. 3(a) that the period \( T = 2\pi \Omega^{-1} \) indicated in the figure is of the order of picoseconds, and the corresponding frequency of the quantum beats lies in the terahertz range. Interestingly, the terahertz frequency regime is currently accessible to experiments in the field of femtosecond pulse technology.

B. Dynamics of a virtual two-level system

We investigate the effect of the self-induced virtual levels, \( E_+ \) and \( E_- \), on the dynamics of the probability density at distances far away from the potential. These virtual levels arise due to the quantum superposition in our initial state, due to the phase modulation. This is illustrated in Fig. 4 where we show the time-evolution of \( \rho (x,t) \) for large values of position, \( x \). The probability density features a sharp rise after the time of flight, \( t = t_+ \equiv x/v_+ \), where \( v_+ = \hbar k_+/m \), giving rise to a peculiar oscillatory-phenomenon known as diffraction in time [3]. However, we note that from \( t \approx t_- \equiv x/v_- \) onward, where \( v_- = \hbar k_-/m \), the time-diffraction profile becomes a highly-oscillatory pattern. The behavior observed in Fig. 4 can be explained by analyzing the contributions to the exact solution \( \Psi(x,t) \) [Eq. (32)] given by \( \psi_{\pm}(x,t) \) [Eq. (33)], and their corresponding interference. This is shown in Fig. 5 where we present the contributions to the exact probability density by rewriting Eq. (35) as \( \rho (x,t) = \rho_+ + \rho_- + \rho_{\text{int}} \), with \( \rho_\pm = |\psi_\pm(x,k_\pm,t)|^2 \), and the interference term \( \rho_{\text{int}} = 2\text{Re}[\psi_+(x,k_+,t)\psi_-^*(x,k_-,t)] \). The time-dependence of \( \rho (x,t) \) is characterized by the propagation of two wavefronts traveling with different speeds, \( v_+ \) and \( v_- \). Since \( v_+ > v_- \), it is expected that these traveling structures arrive at different times (\( t_+ < t_- \)) at a fixed position, \( x \). Therefore, in Fig. 5(a) we argue that the dynamics of \( \rho (x,t) \) for times \( t \approx t_+ \) is characterized by a time-diffraction effect originated by the fast momentum components \( k_+ \), which is a manifestation of the virtual state with energy \( E_+ \). At a later time, when \( t \approx t_- \), the slow momentum components \( k_- \) begin to build the virtual level of energy \( E_- \). From \( t \gtrsim t_- \) onward, the interplay between the two self-induced virtual levels originates the highly-oscillatory profile observed in Fig. 4 due to the \( \rho_{\text{int}} \) contribution shown in Fig. 5(b).

Next, we shall explore the effect of the virtual two-level system in the long-time behavior of the probability density. This is illustrated in Fig. 6(a), where we show the exact probability density [Eq. (35)] exhibits periodic oscillations with a frequency \( \Omega \). We shall show below that this transient feature is independent of the potential pro-
file. We proceed by deriving an asymptotic solution from Eq. (30) with the help of the asymptotic properties of the $M$-functions. In the long-time limit ($t \to \infty$), and large values of the position, the argument of the $M$-functions is such that $|y_q| \gg 1$, leading to the following asymptotic behavior [26]:

$$M(y_q) \simeq e^{imx^2/2ht} \frac{2}{\pi^{1/2}y_q}, \quad -\pi/2 < \arg(y_q) < \pi/2;$$

$$M(y_q) \simeq e^{imx^2/2ht}e^{i\phi_q}, \quad \pi/2 < \arg(y_q) < 3\pi/2.$$  

By using the above expressions in Eq. (30), we can see by inspection that the contributions involving the $M$-functions with $q = k_n, -k_\pm$, are vanishingly small since $M(y_q) \sim y_q^{-1}$ due to Eq. (39). Thus, only the $M$-functions with $q = k_\pm$, which are governed by Eq. (10), contribute to the dynamics, leading us to the result:

$$\Psi_{2l}(x, t) \simeq \frac{1}{2t} \left[ t(k_+)e^{i(k_+x-E_{+}t)/\hbar} + t(k_-)e^{i(k_-x-E_{-}t)/\hbar} \right].$$

The corresponding probability density $\rho_{2l}(x, t)$ is given by,

$$\rho_{2l}(x, t) \simeq \frac{1}{4} \left[ (|t(k_+)| + |t(k_-)|)^2 - 4|t(k_+)| |t(k_-)| \sin^2(\Delta k x - \Omega t/2 - \phi/2) \right].$$  

with $\phi = \arctan(a_{1,1}/a_{1,1})$. Note that $\rho_{2l}(x, t)$ exhibits an oscillatory profile typical of a two-level system (Rabi-model), with a frequency $\Omega$ (Rabi-frequency). In Fig. 6(a) we compare $\rho_{2l}(x, t)$ [Eq. (12)], with the exact $\rho(x, t)$ for the delta well [Eq. (35)], an a perfect agreement is observed. Furthermore, in Fig. 6(b) we show that the oscillatory phenomenon also manifests itself in the free probability density, computed from Eq. (44), and given by

$$\rho_{2l,f}(x, t) \simeq 1 - \sin^2(\Delta k x - \Omega t/2).$$

Our results in Fig. 6(b) also show that $\rho_{2l}(x, t)$ and $\rho_{2l,f}(x, t)$ oscillate with the same frequency $\Omega$. From a physical point of view, the Rabi-dynamics observed in the transient regime, is the result of the virtual self-induced two-level system, with energies $E_+$ and $E_-$, where the frequency of oscillation $\Omega$ is proportional to the semi-classical propagation speed, $v_k$, and $\Delta k$ of the initial state, a feature that is independent of the potential. Only in the case where $\Delta k = k$, the oscillatory phenomenon disappears, as illustrated in Fig. 6(b). In this case, the effective cancellation of one of the channels occurs i.e. $E_+ = 0$, which hinders the Rabi-mechanism. Alternatively, we can also argue that in the asymptotic regime our system behaves like a two-level system, by starting from Eq. (41) with the following change in notation,

$$|\Psi(t)\rangle = c_+e^{-iE_+t/\hbar}|+\rangle + c_-e^{-iE_-t/\hbar}|-\rangle,$$

where $\{|+\rangle, |-\rangle\}$ is an orthogonal base with eigenstates $\{E_+, E_-, \}$, and the coefficients $c_\pm$ take into account the $t(k_\pm)$, and other factors. By eliminating the phase factor $e^{-iE_-t/\hbar}$ from Eq. (44), we can obtain the probability of finding the state $|\Psi(t)\rangle$ at an earlier time $t = t'$ which is given by $P_{t,t'} = |\langle \Psi(t')|\Psi(t)\rangle|^2$, where the system was initially at one of the virtual states ($E_+$ or $E_-$), leading us to

$$P_{t,t'} = (|c_+|^2 + |c_-|^2)^2 - 4|c_+|^2|c_-|^2 \sin^2[\Omega(t - t')/2],$$

which is the well known Rabi-formula. These persistent asymptotic oscillations are a feature of the initial modulated state, and are present for any finite range potential.

C. Delay-time for modulated wavepackets

The delay-time [30] as a transient effect has been observed in quantum structures, such as resonant double barrier systems [21], and delta potentials [11 12 14 16 [33], and depending of the system, this time scale may exhibit advance or delay-time. For example, in the case of repulsive delta potentials [12], a delay-time is observed, while in the case of attractive potentials [16], a time-advance is obtained. It is the purpose of this section to study the effect of the virtual states, $E_+$, and $E_-$, on such a transient time-scale. We calculate the dynamical delay-time $\Delta t$ following the prescription of previous work on
By substituting in Eq. (48) the transmission amplitude, phase-time \( t \), and the free-case \( \rho_f(x,t) \) (red dashed line), at fixed position \( x = 1000.0 \text{ Å} \), and \( \Delta k = 0.02 \text{ Å}^{-1} \). The corresponding delay-time is \( \Delta t = -1.055 \times 10^{-3} \text{ ps} \).

We also compute the energies \( E \). The time-scale Eq. (47) is explicitly governed by the energies \( E_+ \) and \( E_0 \), and is also included in Fig. 8, where an excellent agreement is observed. The red solid circle corresponds to delay-time for the case of \( \Delta k = k \). The latter corresponds to a plane wave. The red dashed line corresponds to the phase-time Eq. (49), which yields \( t_\phi = -2.05 \times 10^{-3} \text{ ps} \).

\[
\Delta t = t_\delta - t_f, \quad (46)
\]

as illustrated in Fig. 7. Note also in this figure, that at a fixed position, \( x \), the maximum of the probability density for the delta well case appears before the free-propagation peak, leading to a time-advance. In both cases, the dynamics is governed by the virtual level, \( E_+ \).

In Fig. 8 we present the measurements of \( \Delta t \) as function of \( \Delta k \), for two different values of the position, \( x \), far from the potential. We show in Fig. 8 that for small values of \( \Delta k \), the delay-time is very close to phase time. In contrast, the delay-time tends to zero for large values of \( \Delta k \) because the initial state is very energetic, and it is not affected by the potential.

We derive an exact analytical expression for \( \Delta t \) by following the procedure of Ref. 33, which yields the result:

\[
\Delta t \simeq \frac{(-\lambda/2)}{v_+ (E_+ - E_0)}. \quad (47)
\]

The time-scale Eq. (47) is explicitly governed by the energies \( E_+ \) and \( E_0 \), and is also included in Fig. 8, where an excellent agreement is observed. We also compute the phase-time \( t_\phi \) of the system by using the definition:

\[
t_\phi = \hbar \frac{d\phi}{dE} = \hbar \text{Im} \left[ \frac{1}{t(k)} \frac{dt(k)}{dE} \right]. \quad (48)
\]

By substituting in Eq. (48) the transmission amplitude, \( t(k) \), for the delta potential, we obtain the explicit phase-time,

\[
t_\phi = \frac{(-\lambda/2)}{v_+ (E_+ - E_0)}. \quad (49)
\]

Note that in the limit \( \Delta k \rightarrow 0, E_+ \rightarrow E \), and \( v_+ \rightarrow v_k \), thus \( \Delta t \rightarrow t_\phi \), that is, the phase-time is recovered for a quasi-monochromatic initial state.

**IV. CONCLUSIONS**

We explore the transient dynamics of phase modulated cut-off wavepackets by deriving an exact analytical solution to Schrödinger’s equation for finite range potentials, within a quantum shutter approach involving a general initial quantum state. The modulated cut-off quantum wave allows us to explore matter-wave transient phenomena such as time-diffraction, wave superposition, quantum beats, Rabi-oscillations, and delay-time. In particular, it is demonstrated that the time-dependent features of the probability density are governed by a virtual self-induced two-level system, with energies \( E_+ \) and \( E_- \), due to the modulation of the initial state. We found that in the long-time regime the probability density exhibits Rabi-dynamics which is independent of the potential profile, with a frequency \( \Omega = (E_+ - E_-)/\hbar \). Interestingly, since \( \Omega = 2v_k \Delta k \), where \( v_k \) is the velocity of a plane wave.
with momentum $k$, the Rabi-frequency can be tuned by controlling the incidence energy $E$ and $\Delta k$ of the modulated wave. The Rabi-mechanism can be effectively turned off by setting $\Delta k = k$, which corresponds to the closing of the channel $E_-$, or by letting $\Delta k = 0$ (unmodulated case). For the particular case of a system with a bound state, $E_b$, we found that the virtual two-level system gives rise (at $x = 0$) to a series of quantum beats in the probability density, with a beating frequency $\Omega$. These quantum beats in the probability density modulate a highly-oscillatory pattern of average frequency $\Omega$ associated to persistent oscillations, resulting from an interplay between the virtual states, and the bound state of the system. We also demonstrate the existence of a transient regime governed by $E_+$, where the probability density exhibits a diffraction in time effect, which let us identify a well defined traveling wavefront. This feature allows to measure the delay-time $\Delta t$ of the system at far distances from the potential. We also derive a simple analytical formula for describing this time scale, which depends on the energy difference between $E_+$ and $E_b$. It is also shown that only for the case of plane waves, $\Delta t$ agrees with the phase-time. The latter is in agreement with the results obtained for the delay time in Refs. [12] and [16] involving the time-evolution of cut-off plane waves.

We stress that both the quantum beat effect, as well as the Rabi oscillations, occur in a frequency range of the order of terahertz, which is currently accessible to experiments involving femtosecond spectroscopy [29]. These matter-wave phenomena associated to electrons can be explored as discussed in Ref. [10] by using interferometric techniques based on electron holographic microscopy [34].

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