Scattering of obliquely incident guided waves by a stiffener bonded to a plate

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Abstract. Nondestructive testing of aerospace structures often requires their immobilization or disassembly. Structural health monitoring (SHM) can overcome these problems; the use of guided elastic waves (GW) in SHM is of great interest, because they propagate long distance in the structure thickness. Structures being stiffened, optimally positioning sensors implies to determine the number of stiffeners a wave can go through while remaining detectable. Here, the diffraction of GW by a stiffener bonded to a plate is considered. Elastic and geometric invariances along stiffener axis lead to 2D computations involving the three components of wave particle displacement, whatever its incidence on stiffener. A hybrid model is developed combining the semi-analytical finite element method for GW propagation and a finite element method (FE) for the stiffener diffraction. Optimal hybridization is obtained thanks to the development of transparent boundaries of the FE domain. Such boundaries have been obtained for GW normally incident onto a scattering feature, thanks to Fraser’s bi-orthogonality relation, which unfortunately does not hold for oblique incidence. A numerical approach is developed to numerically approximate it, then, to derive boundary conditions in the wanted form. Their use minimizes the size of the FE domain and avoids any artificial reflection. They provide a mean for projecting diffracted fields on modes reflected on or transmitted through the stiffener; corresponding coefficients are obtained as functions of the direction of incidence.

1. Introduction
Structural Health Monitoring (SHM) by guided waves (GW) is studied as an alternative or complementary mean for the non-destructive examination (NDE) of aeronautical structures, since classical NDE methods often require partial immobilization and disassembly [1]. Generally speaking, SHM allows both permanent monitoring and scheduled tests to be achieved. Many aircrafts parts are thin plate-like structures made of composite materials or Aluminium alloy, which are stiffened for reinforcement. SHM by GW is particularly promising since GW propagate long distance in plates while probing their whole thickness. A stiffener or a set of stiffeners of course scatter GW.

Present work aims at quantitatively predicting scattering phenomena arising when a GW is incident on a stiffener. This question arises when designing a SHM method: optimally positioning sensors on the structure requires the knowledge of reflection upon and transmission through a stiffener of GW. Once corresponding coefficients are known for the various propagating modes, it becomes easier to optimally position sensors needed to achieve full coverage in the examination of the structure with sufficient signal amplitude for further processing and interpretation.
This work is part of a more general attempt to develop simulation tools capable of addressing typical SHM configurations for optimizing them at a lower cost and more efficiently than with the sole experimentation. In this work, plate-like structures can be made either of homogeneous and isotropic material (Aluminium alloy) or of multiple layers of viscoelastic anisotropic material (composite). A study of GW propagation in these latter structures with application to SHM was described in [2].

It is also part of a general development at CEA of a tool for simulating NDT methods based on GW [3] to be included in the CIVA-platform. Inclusion in this industrial software implies that tools must be generic and compatible with intensive use. The overall approach consists in dealing with the whole testing configuration as a series of connected sub-problems: GW radiation and reception by transducers, propagation in uniform portions where waves propagate as GW in an unbounded guide, scattering by whatever scatters (a defect, a junction, a stringer, etc.). It fundamentally relies on a description of GW as modal series in uniform portions; then, complex phenomena involving local non-uniformity of the configuration as far as GW are concerned are interpreted as factors affecting modal amplitudes and phases. Scattering phenomena are treated by various approaches [3], a most promising one consisting in developing a hybrid model linking modal descriptions in uniform guides with a local numerical method (e.g., FE) to deal with non-uniformities. Note that complex scattering behaviour of GW does not generally admit accurate and fast asymptotic solutions since typical scatterer dimensions are of the same order as guide dimension and as the wavelength. A crucial ingredient in the development of the hybrid model was the obtaining of exact transparent boundary conditions adapted to the modal description of waves in uniform guides [3, 4]. This allows restricting the FE computation zone to a minimum.

Typical stiffener geometry and elastic properties are however invariant along one direction; as it will be shown, one can benefit of this symmetry for deriving an appropriate hybrid modelling approach minimizing computation effort. Therefore, the problem can be solved in a 2D computation. The existing hybrid method is developed for GW propagating along a single guiding direction. In the case considered herein, an arbitrary direction of propagation must be taken into account to predict wave behaviour between several sensors or actuators arbitrarily positioned on the stiffened plate.

The paper is therefore organized as follows. At first, the theoretical derivation of a hybrid modelling approach to solve the problem in hands is described; this section includes a brief reminder of the originally proposed hybrid approach restricted to normal incidence. In the course of the derivation, a modified Semi-Analytical Finite Element method is proposed to compute in one run all possible reflected and transmitted modes than can be generated by the scattering process. Then, simple numerical examples are given to check and prove the accuracy of the numerical scheme. Lastly, a study using it is carried out to evaluate how scattering by a realistic stiffener varies with incident direction depending on the incident mode considered. Implications to practical design of SHM configurations are briefly tackled.

The subject of our paper was recently considered independently in [5]; the aims of both studies are identical and the methods themselves are quite similar. The difference is the way transparent conditions are obtained for hybridizing modal series and FE scheme: in [5] this relies on the use of absorbing boundary conditions. Advantage and drawbacks of both methods will be briefly discussed.

2. Hybrid SAFE/FE method for a GW at oblique incidence

2.1. Governing equations for the symmetry of the configuration in hand

The structure considered is a plate with a straight infinitely long scattering feature (called “the scatterer” in what follows) along the y-axis (see figure 1). The whole 3D structure is entirely described by its 2D trace in the (x-z) plane. It is assumed that the same symmetry holds for elastic properties.
The field radiated by an arbitrary time-dependent source in a plate can always be decomposed as a sum of monochromatic guided waves propagating in all the directions in the plate, by means of adapted space and time Fourier transforms. Let us consider a source radiating in the stiffened plate under study. Thanks to the same decomposition, predicting the field radiated in the presence of the stiffener can be made by solving the problem of a monochromatic GW making an arbitrary angle of incidence relatively to the straight scatterer, followed by the required inverse Fourier transforms to get the space and time-dependent solution. Our aim is therefore to find the elementary solution for each possible incident mode existing at a given frequency parameterized by the angle of incidence $\beta$ its wavevector makes with the normal to the scatterer. Thanks to the symmetry of the problem in hand, modes reflected on or transmitted through the scatterer resulting from the interaction of a given incident mode, have all in common that the $y$-component of their wavevector $k_y$ is equal to that of the incident mode considered. The conservation law at the basis of this result is nothing but an expression of the Snell-Descartes’ law. Let us call this component $\gamma$ defined as

$$\gamma = -k \sin \beta.$$  

For the symmetry considered and by taking into account this conservation law for wavevectors, the dependence of any elastodynamic quantity upon the $y$-coordinate can be factorized as

$$u(x, y, z) = u^i(x, z)e^{-i\gamma y}. \quad (2)$$

Operators can be defined by accounting for the symmetry: for instance, the divergence of a vector writes

$$\nabla \cdot u = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \left( \frac{\partial u'_x}{\partial x} - i\gamma u'_y + \frac{\partial u'_z}{\partial z} \right) e^{-i\gamma y} = \nabla \cdot (u^i)e^{-i\gamma y}. \quad (3)$$

Eventually, the general 3D problem to be solved can be mathematically dealt with a sum over $\gamma$ of solutions of simpler 2D problems in the $(x, z)$ plane (vector variables having three components still) at fixed $\gamma$. The $\gamma$-dependent equation of wave motion in the volume $V$ of the part considered, of free boundary $\partial V$ is given by:

$$\begin{cases}
\nabla \cdot (\sigma^\gamma (u^i)) = -\omega^2 \rho u^i & \text{in } V \\
\sigma^\gamma (u^i).n = 0 & \text{on } \partial V
\end{cases} \quad (4)$$

where $\omega$ denotes the pulsation and $\rho$ the mass density.

2.2. SAFE method at fixed $\gamma$

The Semi-Analytical Finite Element (SAFE) method is becoming a standard to efficiently compute guided modes in a variety of guide geometries made of arbitrary material. Classically, the SAFE method allows computing, at a given frequency, a series of guided modes (propagative, inhomogeneous and evanescent) assuming one given guiding direction. The FE computation is limited
to a solution of the wave equation in a plane perpendicular to the guiding direction considered, the dependency upon this direction of the solution being handled analytically. The SAFE formulation leads to the prediction of wavevectors and associated particle displacements as being the eigenvalues and eigenvectors of the SAFE system of equations.

Here, we can derive a specific SAFE formulation to compute all the modes having the same $\gamma$. The unknown eigenvalues are no longer the wave numbers, $k$, but their projection $k_z$ on the $z$-axis. Classical matrices of SAFE formulation (recalled in [2]) are slightly modified; the system now becomes

$$ (A - k_z D)Q = 0 $$

(5)

with,

$$ Q = \begin{bmatrix} d \\ k_z d \end{bmatrix}, $$

(6)

$$ A = \begin{bmatrix} 0 & \gamma^2 K_{22} - j\gamma K_{12} + K_{11} - \omega^2 M \\ \gamma^2 K_{22} - j\gamma K_{12} + K_{11} - \omega^2 M & (-jK_{13} + \gamma K_{23}) \end{bmatrix}, $$

(7)

$$ D = \begin{bmatrix} \gamma^2 K_{22} - j\gamma K_{12} + K_{11} - \omega^2 M & 0 \\ 0 & -K_{33} \end{bmatrix}. $$

(8)

$\omega$ denotes the angular frequency and $d$ the nodal displacement, $K_{mn}$ are the stiffness matrices (in the FE sense of the word) and $M$ the mass matrix. Stiffness and mass matrices are defined in [2].

An example of solution is shown on figure 2 where an incident mode gives rise to two reflected modes. The figure actually displays the slowness curves of wave modes existing below the first cut-off frequency in a plate made of unidirectional (transversely isotropic) composite material (see [2] for the stiffness constants used). The symmetry axis of the composite makes an angle of $\pi/4$ with the $y$-direction. The value of $\gamma$ of the incident wave mode is such that

1) there is no mode conversion in the reflection into the fastest mode (in red) ii) but the two other modes contribute to the reflected field.

![Figure 2. Slowness curves of guided modes in a unidirectional composite plate illustrating Snell’s law as it applies to reflected modes computed for a given $\gamma$ in the modified SAFE-$\gamma$ method.](image-url)
2.3. Transparent boundaries: short review and brief recall on previous work for normal incidence

In a hybrid model such as the one we want to derive, a boundary is defined between two physical portions of the volume, where two different methods apply. At this boundary, the two methods exchange data that represent physical quantities of interest for the overall computation. Suppose now this boundary is at a position in the volume where the physical quantities studied are continuous; in the present case, consider an artificial boundary arbitrarily positioned in the propagation medium where particle displacement and stress associated with the propagating waves are continuous. The way the two methods exchange data must ensure that no artificial reflection takes place at it.

In the literature, transparent boundaries have been developed for elastodynamics. Methods based on Perfectly Matched Layers (PML) originally developed for electromagnetic waves [6] constitute a class of computationally efficient methods available for various equations. However, in the case of elastic guided waves, the method may be unstable in the presence of inverse modes, but problems occur since the sign of the normal component of group velocity and phase velocity are different [7, 8].

A classical way of obtaining locally transparent boundaries for wave equations consists in defining absorbing zones [5, 9] to artificially decrease unwanted contributions. The size of the absorbing zone that must be meshed by finite elements is proportional to the wavelength in the direction of absorption, this leading to costly computation at typical frequencies involved in GW testing configurations. In acoustic problems, the Dirichlet-to-Neumann (DtN) method consists in a boundary condition where the normal derivative of the pressure is expressed as a (non-local) function of the pressure on the same boundary. The natural extension to elasticity would be a boundary condition with a normal stress expressed as a function of the displacement. This is unfortunately not tractable. Instead, hybrid variables \(X\) and \(Y\) (mixing components of both the displacement and the normal stress) have been used by Baronian et al. [4] to develop the YtX method adapted to guided modes. This last method will be extended in this study to the case of an obliquely incident GW on a scatterer invariant in one direction. We briefly recall the underlying principle of this recently developed method before deriving what is needed for its extension.

Two mixed variables \(X\) and \(Y\) are first introduced, defined as

\[
X = \begin{pmatrix} t_s \\ u_x \\ \sigma_{xz} \\ u_z \end{pmatrix}, \quad Y = \begin{pmatrix} u_s \\ t_x \\ u_y \\ \sigma_{yz} \end{pmatrix}. \tag{9}
\]

As GW displacement and stress can be expressed as modal series, both variables can similarly be decomposed as:

\[
\begin{pmatrix} X \\ Y \end{pmatrix} = \sum_n \mathbf{A}_n(z)(\mathbf{X}_n(x) + \mathbf{Y}_n(x)). \tag{10}
\]

Let \(\langle \cdot \rangle_{S}\) denotes the following product

\[
\langle Y \left| X \right. \rangle_{S} = \int_S (t_s \cdot u_s + t_x \cdot u_x) \, dS, \tag{11}
\]

where \(S\) is the section of the guide. Fraser’s relation [10] expresses the bi-orthogonality of two modes \(m\) and \(n\) through the bi-linear relation given by equation (11) as

\[
\langle \mathbf{Y}_n \left| \mathbf{X}_m \right. \rangle_{S} = \delta_{mn}. \tag{12}
\]

Let us now consider the following problem depicted on figure 3. We want to define a transparent boundary between a homogeneous guide for which a modal decomposition is known, called the exterior domain, and a scattering feature, called the interior domain. The transparent boundary condition to be built must relate the two variables \(X\) and \(Y\) in both the two domains while ensuring continuity of these fields across it.
The so-called YtX operator must be obtained to relate $X^{\text{int}}$ to $Y^{\text{int}}$ of the internal field at the boundary $\Sigma$ between the scatterer and the guide. It is built in three steps, as shown on figure 3.

In steps 1 and 3, continuity relations for $X$ and $Y$ fields across the boundary separating the interior domain and the exterior domain are used. These relations are:

$$
\begin{align*}
Y^{\text{ext}} &= Y^{\text{int}} \quad \text{on } \Sigma, \\
X^{\text{int}} &= X^{\text{ext}} \quad \text{on } \Sigma.
\end{align*}
$$

(14)

In step 2, using Fraser’s relation, it is possible to decompose the exterior field onto the modal solution. The amplitude of the field for the various modes is simply obtained as

$$
A_n^+ = \left( Y^{\text{ext}} | X_n' \right).
$$

(15)

Noting that in equation (10), the same coefficient $A_n^+$ appears in the modal decomposition for the two mixed variables $X$ and $Y$ and using equation (15) for the projection of the external field on modes, a relation between $X^{\text{ext}}$ and $Y^{\text{ext}}$ is readily obtained, given by

$$
X^{\Sigma} = \sum_{n \in \mathbb{N}} \left( Y^{\text{ext}} | X_n' \right) \mathcal{X}^n.
$$

(16)

By combining equation (16) with the continuity relation in equations (14), the YtX operator denoted by $T$ coupling $X^{\text{int}}$ and $Y^{\text{int}}$ can be expressed as

$$
X^{\Sigma} = \sum_{n \in \mathbb{N}} \left( Y^{\text{int}} | X_n' \right) \mathcal{X}^n = T(Y^{\text{int}}),
$$

(17)

with

$$
T = \left( \sum_{n \in \mathbb{N}} \left( * | \mathcal{T}_S^n \right) \mathcal{T}_S^{\text{in}} \mathcal{U}_c^n \mathcal{T}_S^{\text{in}} \right),
$$

(18)

where $\mathcal{U}_c^n$ denotes the axial displacement of mode $n$, and $\mathcal{T}_S^n$ denotes the following vector:

$$
\mathcal{T}_S^n = \begin{pmatrix} \sigma_x^n \\ \sigma_y^n \end{pmatrix}.
$$

(19)

2.4. **Transparent boundaries for an obliquely incident wave**

This sub-section describes how to derive transparent boundary conditions by generalizing the YtX method to deal with obliquely incident guided waves. The formulae given hereafter concern the two...
boundaries $\Sigma^\pm$ between the scatterer and the portions of the uniform guide, as shown on figure 4. The appropriate field quantities are again the mixed variables which now depend on $\gamma$ to account for the symmetry described in the first section. These variables are defined as follows:

$$X^\gamma = \begin{pmatrix} t^\gamma_s \\ u^\gamma_x \\ u^\gamma_y \end{pmatrix} \quad \text{and} \quad Y^\gamma = \begin{pmatrix} u^\gamma_x \\ t^\gamma_x \\ -\sigma^\gamma_{xz} \end{pmatrix}. \quad (20)$$

where the stress and the displacement are calculated by the modified SAFE method at fixed $\gamma$, and notations with $\gamma$ are defined in relations (2) et (3).

Figure 4. FE domain for diffraction problem.

The originally developed YtX method relied on Fraser’s relation. For obliquely incidence (as for anisotropic materials), the Fraser’s relation is no longer verified so that modes cannot be projected straightforwardly on each others. To overcome this difficulty, a new way of projecting modes must be developed. At first, a Gramm matrix $\mathcal{O}$ is defined, whose components are given by:

$$\mathcal{O}_{nm} = \left( \mathcal{Y}_n^\gamma | \mathcal{X}_m^\gamma \right)_S \quad (21)$$

where $(\cdot | \cdot)_S$ denotes the same bilinear relation as that defined by equation (11) for $\gamma$ dependent variables. For normal incidence, Fraser’s relation implies that $\mathcal{O}$ is a diagonal matrix, defined in equation (12). But for oblique incidence, the matrix is no longer diagonal.

The YtX operator, to be defined at fixed $\gamma$, is derived similarly in three steps. Since the overall procedure to get the new transparent conditions is basically the same as that already shown in the case of normal incidence, we restrict the description of the procedure for oblique incidence to what differs in the two cases. Moreover, as this was done in the previous subsection, the formulæ are detailed for the case of the right boundary $\Sigma^+$ where waves leave the interior domain, assuming that the incident waves arise from the left. Boundary conditions on $\Sigma^-$ will be deduced from their expression on $\Sigma^+$ at the end of this subsection.

The external field $Y^{\text{ext}}_m$ is projected on a mode $m$ of the modal series $\mathcal{X}_m^\gamma$ using the matrix $\mathcal{O}$, defined by equation (21). One gets

$$\left( Y^{\text{ext}}_m | \mathcal{X}_m^\gamma \right)_S = \sum_{n \in \mathbb{N}} A^+_n \left( \mathcal{Y}_n^\gamma | \mathcal{X}_m^\gamma \right)_S = \sum_{n \in \mathbb{N}} \left( \mathcal{O}^+ \right)_{nm} A^+_n \quad (22)$$

The inversion of the system given by equation (22) leads to a new expression of the modal amplitude of the field on $\Sigma^+$, written as

$$A^+_n = \sum_{m \in \mathbb{N}} \left( \mathcal{O}^+ \right)^{-1}_{nm} \left( Y^{\text{ext}}_m | \mathcal{X}_m^\gamma \right)_S \quad (23)$$

The Gramm matrix being neither unitary nor diagonal, its numerical inversion requires a truncation of the modal series.

Now, the modal decomposition of the external field $X^{\text{ext}}_m$ can be rewritten as follows:
\[ X_{\Sigma}^{\text{in}} = \sum_{n \in \mathbb{N}} A_n^* \mathcal{X}_n^* \]
\[ = \sum_{n \in \mathbb{N}} \mathcal{X}_n^* \sum_{m \in \mathbb{N}} \left( (\mathcal{O}^T)^{-1} \right)_{nm} \left( \mathcal{Y}_{\Sigma}^{\text{in}} | \mathcal{X}_m^* \right)_S \]
\[ = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \mathcal{X}_n^* \left( (\mathcal{O}^T)^{-1} \right)_{nm} \left( \mathcal{Y}_{\Sigma}^{\text{in}} | \mathcal{X}_m^* \right)_S. \]  

(24)

Finally, combining equation (24) with continuity relations for the interior and exterior fields at the boundary, the \( \text{YtX} \) operator \( T_{\gamma}^+ \) for obliquely incident guided waves on \( \Sigma^+ \) is expressed as:

\[ X_{\Sigma}^{\text{in}} = T_{\gamma}^+ (Y_{\Sigma}^{\text{in}}), \]  

(25)

\[ T_{\gamma}^+ = \left( \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \mathcal{T}_{S,n}^+ \left( (\mathcal{O}^T)^{-1} \right)_{nm} \left( \mathcal{T}_{S,m}^+ \right)_S \right) + \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \mathcal{U}_{c,n}^+ \left( (\mathcal{O}^T)^{-1} \right)_{nm} \left( \mathcal{U}_{c,m}^+ \right)_S, \]  

(26)

where \( \mathcal{U}_{c,m}^+ \) denotes the axial displacement for the mode \( m \), and \( \mathcal{T}_{S,m}^+ \) denotes the following vector:

\[ \mathcal{T}_{S,m}^+ = \left( \sigma_{c,m}^{\gamma} \sigma_{c,m}^{\gamma} \right). \]  

(27)

For the diffraction problem, YtX operator is used to model the outgoing conditions on the boundaries of the FE domain. The construction of the YtX operator \( T_{\gamma}^- \) on the left boundary \( \Sigma^- \) is similar and involves the modal decomposition of the field on the left-going modes. For example, if the plate is isotropic, one has \( X_{-n} = -X_n \) and \( Y_{-n} = Y_n \) so that \( T_{\gamma}^- = -T_{\gamma}^+ \), right-going modes being again used in its expression.

We can now write the variational formulation for the FE computation at fixed \( \gamma \) of the diffraction by a straight scatterer of volume \( \Omega \), with \( \Sigma^{\text{i+}} \) transparent boundary conditions and with \( \partial \Omega - \Sigma^{\text{i+}} \) modelled as free boundaries. This formulation is given by the following system of equations

\[ \text{div}_\gamma (\mathbf{u}) = -\alpha^2 \rho \mathbf{u} \text{ in } \Omega, \]
\[ \sigma^T (\mathbf{u}) \cdot \mathbf{n} = 0 \text{ on } \partial \Omega - \Sigma^{\text{i+}}, \]
\[ X_{\Sigma^{\text{i+}}}^{\text{m}} = T_{\gamma}^{\text{i+}} (Y_{\Sigma^{\text{i+}}}^{\text{m}}) \text{ on } \Sigma^{\text{i+}}. \]  

(28)

3. Validations of the hybrid method

The hybrid method has been implemented in the Melina code (FE code developed at IRMAR and POEMS [11]). In this section, elementary tests are computed to check the accuracy of the proposed hybrid method. A first validation test consists in considering a scattering zone to be computed by FE which has the same geometric and elastic characteristics as the guiding plate; expected results must show that the propagation computed by FE in this zone remains identical to that in the plate as computed by the SAFE-\( \gamma \) (no mode-conversion, no reflection, no energy lost). The second test consists in checking energy conservation in the presence of a non-trivial scattering feature.

The various examples of simulated results given in this section and in the next one have been computed by considering 50 modes (propagative, evanescent and inhomogeneous) in the series taken into account for obtaining transparent boundaries as it was explained en section 2. It must be noted that, presently, the influence of this number on accuracy has not been fully studied. The number
considered, as it will be shown, leads to accurate transparent conditions, but is probably higher than really needed. Optimizing this number could be studied for minimizing computation times.

3.1. “Scattering” by a zone geometrically and elastically identical to the plate
This first numerical experiment is useful for proving the efficiency and quality of transparent boundaries: the FE computation must leave unperturbed the incident wave so that one expects both null-reflection and total transmission without mode-conversion. A 5-mm-thick plate made of Aluminium alloy is considered and the FE zone is 10-mm-long, as shown by Figure 5.

For an excitation frequency of 500 kHz, there are five propagative modes in the unperturbed plate: three Lamb modes A0, S0 and A1 and two shear horizontal modes SH0 and SH1. Each mode is successively taken to be an incident wave for various values of $\gamma$. For each mode, a quadratic error for the total displacement field at the input boundary $\Sigma^-$ and the output boundary $\Sigma^+$ is computed between the FE solution that includes transparent boundaries and the modified-SAFE solution (identical at the two boundaries). All the values of quadratic errors are less that 0.03% at both boundaries.

Further tests not discussed herein demonstrated that transparency is accurately obtained even for incident angles of 89.5°. As it may be anticipated, the 90° case is singular but it has no actual physical meaning as far as the SHM applications are concerned.

3.2. Energy conservation in the presence of a scatterer
Let us consider now the case of a stiffened isotropic plate. The geometry considered is shown on Figure 6. The plate in Aluminium alloy is 1.6-mm-thick.

In the present case, reflection and transmission phenomena are not trivial. The test consists in checking that the numerical method ensures energy conservation. For this, the sum of squared modulus of reflection and transmission coefficients (each coefficient being normalized by the square root of the ratio between its power flow and that of the incident mode, measured in the direction normal to the straight feature) shall equal 1, according to

$$E = E_{\text{reflected}} + E_{\text{transmitted}} = \sum_i |R_i|^2 + |T_i|^2 = 1,$$

where $R_i$ (respectively $T_i$) denotes the reflection (respectively, transmission) coefficient of the $i$-th mode for the $j$-th incident mode.
In this example, we also check the influence of the distance of the transparent boundaries to the stiffener. Boundaries can be considered as being transparent if transmission and reflection coefficients do not depend on that distance. Distances $D$ considered are 7.5 (I), 10 (II) and 20 (III) mm. Previous studies in the case of normal incidence shown that the boundary position can be all the closer to the scatterer while ensuring transparency since the number of modes accounted for in the series is high.

Whatever the distance $D$ considered, energy conservation is obtained with a total energy equal to the incident energy within an error less than 0.01 %. This was checked for all the possible incident modes and for various incident angles.

**4. Example of parametric study for a realistic stiffened plate**

The same stiffened plate with $D = 10$ mm is now used in several parametric studies given to exemplify the interest of the method for practical applications.

At first, transmission and reflection coefficients for a S0 incident mode are shown on figure 7 at a frequency of 300 kHz. The behaviour of these coefficients is rather complex to describe; globally, transmission tends to decrease (and reflection to increase) for increasing incident angle. At about 40°, both the reflection and transmission coefficients of the S0 incident mode are of low value whereas other coefficients corresponding to mode conversion are higher.

![Figure 7](image)

**Figure 7.** Transmission (left) and reflection (right) coefficients for a S0 incident mode at 300 kHz vs. incident angle (°).

Now, at 500 kHz, still below the first cut-off frequency so that there are only three propagative modes A0, S0 and SH0, we compare transmission coefficients for two different incident modes, S0 and A0, as shown by figure 8.
The behaviour of the various transmission coefficients very much differs depending on the incident mode. As it may have been anticipated, in the case of an A0 incident mode, two critical angles are observed, since the A0 mode is the slowest of the three modes. Interestingly, it is also observed that for incident angles higher than 59°, even A0 is not transmitted anymore through the stiffener.

We can also compare figures 7 and 8 to study the influence of the frequency on the transmission coefficients for a S0 incident mode. The global behaviour tends to become more complex as the frequency increases.

At last, to illustrate total transmission and total reflection phenomena observed in the case of a A0 incident mode depending on the incident angle considered, the following figure display the various components of the total field inside the FE zone for two different incident angles. For a 50° of incidence, total transmission is observed while for a 70° incident angle, total reflection occurs.

![Figure 9. Components of the displacement for A0-incident mode at 500 kHz. Left: total transmission at 50° of incidence; right: total reflection at 70° of incidence.](image-url)
These first results of scattering coefficients for obliquely incident guided waves demonstrate how complex are the scattering phenomena that take place in a stiffened plate and how these phenomena are highly dependent on the angle of incidence, on the working frequency and on the type of incident mode considered. To quantitatively address such a complexity, adapted simulation tools are obviously required.

5. Conclusion
A hybrid method has been proposed to simulate the scattering of elastic guided waves in a plate by an obliquely incident wave onto a straight feature. The model hybridizes a semi analytical finite element method developed for computing all the guided modes in the unperturbed plate that have the same wavevector component along the axis of the scattering feature and a finite element method for computing the scattering phenomena. For the latter method, a classical finite element method is used and complemented by specific transparent boundary conditions. These conditions are derived in details in the paper; they are expressed through the decomposition of the fields on modes existing in the unperturbed plate. They have been shown to be very accurately transparent and can be very close to the scatterer, all the closer since more inhomogeneous and evanescent modes are taken into account in the modal series used.

The method has been validated by checking the quality of transparency of the specifically developed boundaries and its ability to satisfy global energy conservation. The method has also been used in a dedicated parametric study of the scattering of guided waves by a realistic stiffener, expressed in terms of variation of reflection and transmission coefficients with the incident angle, the type of incident mode etc. These latter results are typical of what is needed for optimizing the number and position of transducers in SHM applications.

The overall computations being modal, the interpretation of simulated results in terms of mode reflection, transmission, conversion etc. is straightforward. Moreover, the FE zone being reduced to a minimum thanks to the specific boundaries developed, results in typical parametric studies needed for demonstrating the performance of a method can be efficiently computed in an industrial context.

The hybrid method has been implemented at first to deal with isotropic materials as the examples given show. Its implementation to deal with multilayered anisotropic case was under test at the time of this conference; it may be noticed that its application to composite plate and stiffener has been considered since [12].

Our purpose was the development of a simulation tool to predict these phenomena. In practice, the full understanding of the complex wave behavior can be obtained using this tool to study in details how the elastic energy propagates in the stiffener depending on the given set of configuration parameters; results obtained suggest that an optimal SHM configuration could be designed but would be specific to a given material and a given geometry of the stiffener.

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