Stability of the solutions of the Gross-Pitaevskii equation

A. D. Jackson\textsuperscript{1}, G. M. Kavoulakis\textsuperscript{2}, and E. Lundh\textsuperscript{3}

\textsuperscript{1}Niels Bohr Institute, Blegdamsvej 17, DK-2100, Copenhagen \textit{Ø}, Denmark
\textsuperscript{2}Mathematical Physics, Lund Institute of Technology, P.O. Box 118, SE-22100 Lund, Sweden
\textsuperscript{3}Department of Physics, KTH, SE-10691, Stockholm, Sweden

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We examine the static and dynamic stability of the solutions of the Gross-Pitaevskii equation and demonstrate the intimate connection between them. All salient features related to dynamic stability are reflected systematically in static properties. We find, for example, the obvious result that static stability always implies dynamic stability and present a simple explanation of the fact that dynamic stability can exist even in the presence of static instability.

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INTRODUCTION

The question of excitations is central to the study of cold atoms. Numerous experimental and theoretical investigations have been devoted to the study of a variety of collective and elementary excitations in these gases including vortex states, solitary waves, and normal modes. When determining the properties of an excited state, it is natural to consider two kinds of possible instabilities—static and dynamic. In the former case, one wishes to determine whether a state which extremizes the energy is a genuine local minimum of the energy subject to certain physically motivated constraints. Since the wave function extremizes the energy, infinitesimal perturbations will make no first-order change of the energy. The state will be stable provided that an arbitrary infinitesimal variation of the wave function necessarily increases the energy to second order.

For the problem of dynamic stability, one conventionally considers the temporal evolution of arbitrary infinitesimal perturbations to the wave function using approximate linearized equations (e.g., the Bogoliubov equations). While the full time-dependent equations conserve probability, the linearized equations do not, and the resulting eigenvalue problem is non-Hermitian. If the corresponding eigenvalues are real, the system is dynamically stable. In this case small-amplitude time-dependent perturbations of the solution lead to bounded motion about the original equilibrium position. Instability is indicated by complex eigenvalues, and the associated exponential growth of small-amplitude perturbations drives the system to a new state beyond the scope of the linearized equations.

Here, we will explore the intimate connection between these two seemingly-different criteria for stability. Two main points will emerge. First, static stability implies dynamic stability, but dynamic stability can exist even in the absence of static stability. Second, the transition from dynamic stability to instability is reflected in the features of the corresponding problem of static stability (subject to appropriate constraints). Here, inspired by the experiment of Ref. \textsuperscript{[1]} and by the numerical studies of Refs. \textsuperscript{[2,3,4]}, we will focus on the specific problem of the stability (static and dynamic) of a doubly-quantized vortex state. Our study is, however, more general, and our results can be applied to the study of the stability of any problem described by the non-linear Gross-Pitaevskii equation. One remarkable observation obtained from numerical simulations of this problem \textsuperscript{[2,3,4]} is that the system alternates between regions of dynamical stability and instability as the strength of the interatomic interaction increases. Since some (and probably all) regions of dynamical stability coincide with regions of static instability, it is useful to seek a simple description of this surprising phenomenon.

In the following we first examine the questions of static and dynamic stability separately. We will then demonstrate the connection between them.

MODEL

We consider $N$ atoms subject to a spherically-symmetric single-particle Hamiltonian, $h_i$, interacting through a short-range effective interaction,

$$H = \sum_{i=1}^{N} h_i + U_0 \sum_{i \neq j=1}^{N} \delta(r_i - r_j)/2. \quad (1)$$

Here $U_0 = 4\pi\hbar^2a/M$ is the strength of the effective two-body interaction, $a$ is the scattering length for elastic atom-atom collisions, and $M$ is the atomic mass.

For simplicity we assume strong confinement along the $z$ axis, which is also chosen as the axis of rotation. This assumption implies that the cloud is in its lowest state of motion in the $z$ direction, and the problem thus becomes two-dimensional with higher degrees of freedom along the $z$ axis frozen out. We also assume that $h_i$ is rotationally symmetric about the $z$ axis (e.g., a harmonic oscillator Hamiltonian) with the result that angular momentum is conserved.
ENERGETIC STABILITY

To be concrete, we will consider the problem of a doubly-quantized vortex state. We start by examining the static stability of this state within the subspace of states of the lowest Landau level. We will generalize our arguments below. The corresponding nodeless eigenfunctions of the (two-dimensional) single-particle Hamiltonian, $H_k$, will be denoted as $\phi_m$, with $m$ the angular momentum; their energy eigenvalues are $\epsilon_m$.

We now consider the state

$$\phi = c_0\phi_0 + c_2\phi_2 + c_4\phi_4,$$

which describes a doubly-quantized vortex state when $c_2 = 1$ and $c_0 = c_4 = 0$. Since we wish to consider the effect of small admixtures of states with $m = 0$ and $m = 4$, we expand the energy to second order in $c_0$ and $c_4$ to obtain the expectation value of the energy per particle

$$E = E_2 + c_0^2c_0M_{00} + c_4^2c_4M_{44} + M_{04}(c_0^*c_4^* + c_0c_4),$$

where

$$E_2 = \epsilon_2 + \frac{1}{2}gI_{2222},$$

$$M_{00} = \epsilon_0 - \epsilon_2 + g(2I_{0202} - I_{2222}),$$

$$M_{44} = \epsilon_4 - \epsilon_2 + g(2I_{4244} - I_{2222}),$$

$$M_{04} = gI_{0422} = gI_{2204},$$

with $I_{nmkl} \equiv \int \phi_n^*\phi_m^*\phi_k\phi_l d\mathbf{r}$ and $g = NU_0$. The elements, $M_{mn\nu}$, are real due to the Hermiticity of $H$.

The energy $E$ is then equal to

$$E = E_2 + |c_0|^2M_{00} + |c_4|^2M_{44} + 2|c_0||c_4|M_{04}\cos(\theta_0 + \theta_4),$$

where $c_m = |c_m|e^{i\theta_m}$. The absence of terms linear in $c_0$ and $c_4$ is a consequence of the spherical symmetry of $H$ and implies that $c_2 = 1$ is a solution to the variational problem in this restricted space. For fixed $|c_0|$ and $|c_4|$, it is elementary that $E$ has extrema for $\cos(\theta_0 + \theta_4) = \pm 1$.

In general, the linearly independent curvatures of this quadratic energy surface are obtained as the eigenvalues of the real symmetric matrix

$$M_s = \begin{pmatrix} M_{00} & M_{04} \\ M_{40} & M_{44} \end{pmatrix}$$

with $M_{40} = M_{04}^\dagger$.

This problem, it is evidently of physical interest to consider the stability of the doubly quantized vortex subject to the constraint that angular momentum is conserved. This constraint will be satisfied if $|c_0| = |c_4| = |c|$, and the energy $E$ then becomes

$$E = E_2 + 2g|c|^2(I_{0202} + I_{2424} - I_{2222} \pm I_{0442}).$$

The doubly-quantized vortex will be an energy minimum if the coefficient of $|c|^2$ is positive. In other words, the doubly-quantized vortex will be energetically stable if

$$I_{0202} + I_{2424} - I_{2222} \geq I_{0442}. \quad (8)$$

This condition is not satisfied when only lowest Landau levels are retained, and a doubly-quantized vortex state is energetically unstable within this approximation.

Several comments are in order before turning to the dynamical behavior of the same (truncated) problem. We have tested the stability of the state with respect to the admixture of states $\phi_0$ and $\phi_4$. A complete test of stability requires the investigation of the admixture of $\phi_m$ for all $m$. Fortunately, the spherical symmetry of $H$ ensures that the generalized quadratic form of Eq. (4) will only mix states with $m = 2 \pm k$. With the inclusion of additional values of $k$, the matrix $M_s$ becomes block diagonal and leads to a sequence of equations for each $k$ completely analogous to those considered above. (The $k = 2$ choice considered here is known to be the most unstable case.) The problem of finding the eigenvalues of the Hermitian matrix, $M_s$, is subject to familiar variational arguments. Expansion of the dimension of this matrix, either by including more values of $k$ or by relaxing the restriction to the lowest Landau level, can only decrease the smallest eigenvalue of $M_s$. Thus, the fact that a given state is energetically unstable for a given choice of the finite space used to construct $M_s$ is conclusive proof of instability. Finally, we note that changing the state under investigation, e.g., through the inclusion of higher Landau levels, leads to fundamental changes in $M_s$. Under such circumstances, simple variational arguments cannot tell us whether the improved state will be more or less stable.

DYNAMIC STABILITY

As mentioned above, dynamic stability probes the temporal behavior of the system. We again start with the wave function of Eq. (4) and construct the Lagrangian to second order in the small parameters $c_0$ and $c_4$. Variations with respect to the coefficients $c_0^\dagger$ and $c_4$ then lead to the equations

$$i\dot{c}_0 = \epsilon_0c_0 + 2g_c0c_2c_4I_{0202} + gc_4^*c_2^*I_{0442},$$

$$-i\dot{c}_4 = \epsilon_4c_4^* + 2g_c^*c_2^*c_4I_{4244} + gc_0c_2^*I_{0442},$$

$$\dot{c}_2 = \epsilon_2c_2 + gc_0c_4^*I_{2222},$$

where the dot denotes a time derivative. We can solve these equations with the ansatz $c_0(t) = c_0(0)e^{-i(\mu + \omega)t}$ and $c_4^*(t) = c_4^*(0)e^{i(\mu - \omega)t}$. (The time dependence of the solution, $\phi_2$, is given by the factor $e^{-\mu t}$ with $\mu = \epsilon_2 + gI_{2222}$.) The resulting Bogoliubov equations can be
written in the form
\[
\begin{pmatrix}
\epsilon_0 - \mu + 2g I_{0202} & g I_{0422} \\
-g I_{0422} & -\epsilon_4 + \mu - 2g I_{2424}
\end{pmatrix}
\begin{pmatrix}
\epsilon_0 \\
\epsilon_4
\end{pmatrix} =
\omega_d
\begin{pmatrix}
\epsilon_0 \\
\epsilon_4
\end{pmatrix},
\]
(10)

This equation can also be written as
\[
M_s \begin{pmatrix}
\epsilon_0 \\
\epsilon_4
\end{pmatrix} = \omega_d \sigma \begin{pmatrix}
\epsilon_0 \\
\epsilon_4
\end{pmatrix}
\]
with \( \sigma \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \),
(11)

where \( M_s \) is the matrix appearing in the static stability problem. The dynamic problem is thus governed by the eigenvalues of the non-Hermitian Eq. (10), and these eigenvalues can be complex. Since \( M_s \) is real Hermitian, the dynamic eigenvalues are either real or come in conjugate pairs. For each eigenfunction, \( \psi_d \), with a complex eigenvalue, \( \omega_d \), \( \psi_d^* \) will also be an eigenfunction with eigenvalue \( \omega_d^* \). As a result, the roots move along the real axis, touch (i.e., become degenerate) and then move off in the complex plane. Evidently, the existence of complex eigenvalues indicates exponential divergence and dynamic instability.

From Eqs. (11) or (12) we see that the Bogoliubov eigenvalues are real and the state under investigation is dynamically stable provided that \( I_{0202} + I_{2424} - I_{2222} \geq I_{0422} \). This condition is identical to the condition for static stability found in Eq. (2). As we show below in greater generality, static stability always implies dynamic stability, but dynamic stability can occur even in the presence of static instability.

**GENERAL RESULTS**

At this point it is useful to generalize our static and dynamic formalisms to allow for the inclusion of an arbitrary number of Landau levels in the description of both the doubly quantized vortex and the potentially unstable states with \( m = 0 \) and \( m = 4 \). The matrix appropriate for the static problem now has the Hermitean block form
\[
M_s = \begin{pmatrix}
M_{00} & M_{04} \\
M_{40} & M_{44}
\end{pmatrix},
\]
(12)

with \( M_s^\dagger = M_s^T = M_s \). Aside from its dimension, the only change in the construction of \( M_s \) lies in the replacement of the lowest Landau state, \( \phi_2 \), by a superposition of Landau states, \( \psi_2 \), which satisfies the obvious Euler equation
\[
h_0 \psi_2 + g|\psi_2|^2 \psi_2 = \mu \psi_2.
\]
(13)

Here, \( \mu \) is a Lagrangian multiplier introduced to ensure that \( \psi_2 \) is normalized to unity. It is understood that the replacement \( \phi_2 \rightarrow \psi_2 \) is to be made as appropriate in the integrals, \( I_{nml} \), contributing to \( M_s \). As before, positive eigenvalues of \( M_s \) imply energetic stability.

The dynamic Bogoliubov equations again assume the form of Eq. (11) and can be written as
\[
M_s \begin{pmatrix}
\epsilon_0 \\
\epsilon_4
\end{pmatrix} = \omega_d \sigma \begin{pmatrix}
\epsilon_0 \\
\epsilon_4
\end{pmatrix},
\]
(14)

where \( M_s \) is again the real Hermitean matrix governing energetic stability. The quantities \( \epsilon_0 \) and \( \epsilon_4 \) now represent column vectors; their dimensions need not be equal. The reality of all eigenvalues, \( \omega_d \), is again an indication of dynamical stability.

As we have seen, \( H \) is Hermitean and spherically symmetric. The time evolution of an arbitrary wave function, \( \psi \), is governed by the full time-dependent Gross-Pitaevskii equation
\[
h_0 \psi + |\psi|^2 \psi = i\hbar \frac{\partial \psi}{\partial t}.
\]
(15)

It thus comes as no surprise that energy and angular momentum are, in general, constants of the motion. On the other hand, given that the (linearized) Bogoliubov equations do not even conserve probability, it is not obvious that energy and momentum are conserved when these equations are used to describe the temporal evolution of the system. This point merits some attention. Consider a dynamic eigenvector of the Bogoliubov equations, \( \psi_d \), and its (possibly complex) eigenvalue, \( \omega_d \). Given the Hermiticity of \( M_s \), standard arguments reveal that
\[
(\omega_d - \omega_d^*) \langle \psi_d | \sigma | \psi_d \rangle = 0.
\]
(16)

This equation is evidently trivial when \( \omega_d \) is real, but it shows that \( \langle \psi_d | \sigma | \psi_d \rangle = 0 \) when \( \omega_d \) is complex. This tells us that the probabilities of finding \( m = 0 \) is equal to that for \( m = 4 \) for all times. In this sense, angular momentum is rigorously conserved in spite of the approximations leading to the Bogoliubov equations.

The extension of this familiar conservation law presumes that the trial state \( \psi_2 \) is deformed by the inclusion (with arbitrary amplitude) of a single Bogoliubov eigenvector with complex eigenvalue. Angular momentum is not in general conserved with arbitrary deformations of \( \psi_2 \) involving either single Bogoliubov eigenvectors with real eigenvalues or superpositions of Bogoliubov eigenvectors. Since the primary rationale for studying the Bogoliubov equations at all is to determine the existence or non-existence of complex eigenvalues, this point is of interest in spite of such caveats. Further, it suggests that we are most likely to reveal relations between the problems of static and dynamic stability if we consider the question of static stability subject to the constraint of constant angular momentum. Indeed, this constraint was imposed trivially in Eq. (11) above, where it was crucial in establishing the identity of static and dynamic stability criteria in the simple case of single Landau levels.
For the more general problem considered here, it is easiest to proceed by introducing real Lagrange multipliers, $\lambda$ and $\eta$, in order to impose the constraints of overall normalization and constant angular momentum, respectively, in the static problem. The constrained static eigenvalue problem then reads

$$M_s \tilde{\psi}_s = \lambda \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix} \tilde{\psi}_s.$$  \hfill (17)

In practice, the real parameter $\eta$ is adjusted so that the resulting static eigenfunctions $\tilde{\psi}_s$ satisfy $\langle \tilde{\psi}_s | \sigma | \tilde{\psi}_s \rangle = 0$. The desired extrema of $\langle M_s \rangle$ then follow as $\tilde{\omega}_s \equiv \langle \tilde{\psi}_s | M_s | \tilde{\psi}_s \rangle$. The similarity between the constrained static problem of Eq. (17) and the dynamic eigenvalue problem of Eq. (14) is now obvious. We shall exploit this connection below.

**CONNECTION BETWEEN STATIC AND DYNAMIC STABILITY**

Given the fact that $\langle \psi_d | \sigma | \psi_d \rangle = 0$, we see that the inner product of $\langle \psi_d | M_s | \psi_d \rangle = 0$. If $\psi_2$ is deformed by the inclusion (with arbitrary amplitude) of a single Bogoliubov eigenvector with complex eigenvalue, the energy is rigorously conserved and identical to that obtained for the pure state $\psi_2$. We can use any $\psi_d$ with complex eigenvalue as a trial function to provide a variational upper bound of zero for the lowest constrained static eigenvalue, $\tilde{\omega}_s$. Said somewhat more simply, it is elementary that the smallest $\tilde{\omega}_s$ cannot be positive if there exists a state $\psi_d$ such that $\langle \psi_d | M_s | \psi_d \rangle = 0$. (Henceforth, we will consider only the constrained static problem and will refer to it simply as the “static problem”.) Thus, if the Bogoliubov problem has complex eigenvalues, the lowest eigenvalue of the static problem must be negative (or zero). Conversely, if all $\tilde{\omega}_s > 0$, none of the Bogoliubov eigenvalues can be complex. In other words, dynamic instability guarantees static instability, and static instability guarantees dynamic stability. This establishes the first rigorous connection between the two stability problems. While this result may appear obvious, a similar argument shows that it is equally impossible to find a state $\psi_d$ such that $\langle \psi_d | M_s | \psi_d \rangle = 0$ if all $\tilde{\omega}_s < 0$. This yields the more surprising result that complete static instability also leads to dynamic stability.

In order to find additional ties between the static and dynamic stability problems, it is useful to follow the trajectory of two Bogoliubov eigenvalues (as a function of the coupling constant) as they evolve from distinct real values to a conjugate pair. To see this, it is sufficient to consider the two-dimensional space of $(\psi_0, 0)$ and $(0, \psi_4)$. (Here, $\psi_0$ and $\psi_4$ are of arbitrary dimension and normalized.) We thus write

$$M_s \tilde{\psi}_s = \lambda \begin{pmatrix} E_0 & gV \\ gV & E_4 \end{pmatrix},$$  \hfill (18)

with all quantities real. The Bogoliubov problem of Eq. (14) is readily solved with the following results. For $|g| < g_c = |(E_0 + E_4)/2V|$, the $\omega_d$ are real and distinct. The eigenvectors are real and do not satisfy the constraint of constant angular momentum. For $|g| = g_c$, the eigenvalues are degenerate with $\omega_d = (E_0 - E_4)/2$. The eigenvectors are identical and given as $(\psi_0, -\psi_4)/\sqrt{2}$. (Recall that the problem is non-Hermitian.) For $g > g_c$, the eigenvalues have a conjugate pair, the eigenvectors also form a conjugate pair with a non-trivial phase, and the magnitudes of their $\psi_0$ and $\psi_4$ components are equal (i.e., they conserve angular momentum).

We can also solve the static problem using Eq. (17) in the same truncated basis. The values of $\eta$ required to impose the constraint of conserved angular momentum are found to be $\eta = (E_4 \pm gV)/(E_0 + gV)$. The corresponding static eigenvalues are $\tilde{\omega}_s = (E_0 + E_4)/2 \pm gV$ with $\psi_s = (\psi_0, \psi_4)/\sqrt{2}$, as expected. For $|g| < g_c$, the static eigenvalues are positive. For $|g| > g_c$, one of the static eigenvalues is negative. For $|g| = g_c$, one static eigenvalue is precisely zero with $\psi_s = \psi_d = (\psi_0, -\psi_4)/\sqrt{2}$. The condition for a static eigenvalue of zero is the same as the condition for degenerate dynamic eigenvalues.

We also see that the static and dynamic eigenvectors are identical at this point. This should come as no surprise. At $|g| = g_c$, the Lagrange multiplier, $\eta$, is $-1$. Equation (17), which determines $\tilde{\omega}_s$, becomes identical to the Bogoliubov equation, Eq. (14), and identical solutions must result. This result is general. While the form of these two equations is similar, the fact that $M_s$ is a real symmetric matrix forces $\tilde{\omega}_s$ and $\tilde{\omega}_s$ to be real (up to a trivial phase). When $\omega_d \neq \omega_d^*$, it is clear that the associated dynamic eigenvectors, $\psi_d$ and $\tilde{\psi}_d^*$ cannot be chosen equal. Thus, the phase of $\psi_d$ is non-trivial, and $\psi_d$ cannot be the desired solution to Eq. (17). (The case of $\omega_d$ real can be dismissed summarily, since the amplitudes of $m = 0$ and 4 states are not equal.) When two Bogoliubov roots are degenerate, however, $\psi_d$ satisfies both the constraints of angular momentum conservation and reality. Under such conditions, $\tilde{\omega}_s = \psi_d$, and $\tilde{\omega}_s = 0$. In other words, the number of conjugate pairs of complex Bogoliubov eigenvalues changes by one when a static eigenvalue passes through zero.

It might be thought that the number of conjugate pairs of dynamic eigenvalues was equal to the number of negative $\tilde{\omega}_s$ and that a stronger statement was thus possible. As we will show below, this is not the case. The present statement can nevertheless give us a useful corollary. Start from a manifestly stable choice of $\psi_2$ for which all $\tilde{\omega}_s > 0$ and dynamical stability is ensured. Vary some external parameter (e.g., the coupling constant, $g$). As demonstrated above, a single conjugate
pair of Bogoliubov eigenvalues will appear when the first \( \bar{\omega}_s \) passes through zero, and the system becomes dynamically unstable. With every subsequent passage of an \( \bar{\omega}_s \) through zero, the number of conjugate pairs changes (i.e., increases or decreases) by one. It is then clear that an odd number of \( \bar{\omega}_s \) must correspond to an odd number of conjugate Bogoliubov pairs and, hence, dynamic instability.

**DYNAMIC STABILITY IN THE ABSENCE OF STATIC STABILITY**

As mentioned earlier, it is possible for the system to be dynamically stable even in the presence of static instability. This effect is more subtle but is also useful in tightening the connection between the problems of static and dynamic stability. A simple calculation will be helpful. Start with two solutions to the constrained static problem of Eq. (17), \(|1\rangle\) and \(|2\rangle\). The corresponding static eigenvalues are \( \bar{\omega}_{s1} \), which is assumed to be negative, and \( \bar{\omega}_{s2} \), which can pass through zero. Approximate the Bogoliubov equations by truncating the basis to include these two states and the related states \(|3\rangle = \sigma|1\rangle\) and \(|4\rangle = \sigma|2\rangle\). (This calculation is exact when the dimension of \( M_s \) is equal to four.) Since these states are not necessarily orthogonal, the Bogoliubov equations assume the form

\[
M_s \psi_d = \omega_2 \Sigma \psi_d, \tag{19}
\]

with \((M_s)_{ij} = \langle i|M_s|j\rangle\) and \((\Sigma)_{ij} = \langle i|\sigma|j\rangle\). Both matrices are real symmetric. The first two diagonal elements of \( M_s \) are the constrained static eigenvalues, \( \bar{\omega}_{s1} \) and \( \bar{\omega}_{s2} \). The remaining two diagonal elements, \( \bar{\omega}_{s3} \) and \( \bar{\omega}_{s4} \), will be assumed to be positive. The matrix, \( \Sigma \), is intimately related to the overlap matrix with elements \((\langle i|j\rangle)\). Since the basis states conserve angular momentum, the diagonal elements of \( \Sigma \) vanish. Given the origin of the \(|1\rangle\) and \(|2\rangle\) as constrained extrema of \((M_s)\), it is clear that the elements of \( M_s \) and \( \Sigma \) are not independent. We shall ignore this fact in the following qualitative arguments.

First, consider the case where these four states are maximally linearly dependent. State \(|1\rangle\) has been assumed to be distinct from \(|2\rangle\) and is explicitly orthogonal to \(|3\rangle\). Let us assume it is identical to \(|4\rangle\). This immediately implies that \(|3\rangle\) is identical to \(|2\rangle\). The approximate Bogoliubov equation is thus reduced to a \(2 \times 2\) matrix equation in the space of \(|1\rangle\) and \(|2\rangle\) and assumes the form

\[
\frac{1}{\langle 1|\sigma|2\rangle} \left( \begin{array}{c} 1 \langle 1|M_s|2\rangle \\ \bar{\omega}_{s1} \end{array} \right) \langle 1|\sigma|2\rangle \psi_d = \omega_d \psi_d. \tag{20}\]

The eigenvalues of this problem are

\[
\omega_d = \frac{1}{\langle 1|\sigma|2\rangle} \left( \langle 1|M_s|2\rangle \pm \sqrt{\bar{\omega}_{s1} \bar{\omega}_{s2}} \right), \tag{21}\]

and the corresponding (unnormalized) eigenvectors are

\[
\psi_d \sim \sqrt{\bar{\omega}_{s2}} |1\rangle + \sqrt{\bar{\omega}_{s1}} |2\rangle. \tag{22}\]

When \( \bar{\omega}_{s2} > 0 \), the Bogoliubov eigenvalues form a conjugate pair. Further, Eq. (22) shows that \( \psi_d \) acquires a non-trivial phase and that \( \langle \psi_d|\sigma|\psi_d\rangle \sim \text{Re}[\langle \bar{\omega}_{s1} \bar{\omega}_{s2} \rangle^{1/2}] \) is zero. The system is dynamically unstable. When \( \bar{\omega}_{s2} = 0 \), the Bogoliubov eigenvalues are degenerate and real. In this case, \( \psi_d \) is equal to \(|2\rangle\), which is real and conserves angular momentum. These results are all consistent with those found above. When \( \bar{\omega}_{s2} < 0 \), however, the Bogoliubov eigenvalues are real and distinct. The corresponding \( \psi_d \) can thus be chosen real, and angular momentum is no longer conserved. The presence of two negative static minima of \( \langle M_s \rangle \) is thus capable of creating dynamic stability in spite of manifest static instability. This argument is actually more detailed than necessary. We have already seen that the number of conjugate pairs of Bogoliubov eigenvalues necessarily changes by one every time a static eigenvalue passes through zero. Linear dependence has reduced the present problem to one spanned by a two-dimensional space. The fact that \( \bar{\omega}_{s1} < 0 \) ensures that there are initially two complex eigenvalues in the space. We have seen in general that the number of complex eigenvalues must change by two when \( \bar{\omega}_{s2} \) crosses zero. Since all eigenvalues of this \(2 \times 2\) problem are already complex, the only possibility is thus that the number of complex eigenvalues is reduced to zero with dynamic stability as a consequence, as we have seen. States \(|1\rangle\) and \(|2\rangle\) are both essential to this process.

Now consider the case where \(|i\rangle\) are all mutually orthogonal and maximally linearly independent. Partition the matrix into \(2 \times 2\) blocks spanned by the states \(|1\rangle\) and \(|3\rangle\) and the states \(|2\rangle\) and \(|4\rangle\), respectively. Each of the diagonal sub-blocks of \(M_s\) now has a form familiar from Eq. (20). The diagonal elements are equal in each case. The products of off-diagonal elements are \( \bar{\omega}_{s1}\bar{\omega}_{s3} \) and \( \bar{\omega}_{s2}\bar{\omega}_{s4} \), respectively. If the off-diagonal blocks of this matrix are sufficiently small, the Bogoliubov eigenvalues will be given by the eigenvalues of these two \(2 \times 2\) matrices. Since \( \bar{\omega}_{s1}\bar{\omega}_{s3} < 0 \) by assumption, this block produces the original conjugate pair of dynamical eigenvalues independent of the properties of \(|2\rangle\). As \( \bar{\omega}_{s2} \) passes through zero, \( \bar{\omega}_{s2}\bar{\omega}_{s4} \) becomes negative, and an additional conjugate pair of eigenvalues appears independent of the properties of \(|1\rangle\). This behavior will persist whenever the off-diagonal blocks of \(M_s\) are sufficiently small.

We thus see that there are two possible outcomes when a second static extremum becomes negative. If the corresponding states are “strongly” coupled with \( \langle 1|\sigma|2\rangle = \langle 1|4\rangle \approx 1 \), the initial dynamic instability will be eliminated. If these states are “weakly” coupled with \( \langle 1|\sigma|2\rangle \approx 0 \), a second unstable dynamic mode will appear. There is another and potentially fruitful way to distinguish these alternatives. Consider the evolution of the closed surfaces with \( \langle M_s \rangle = 0 \) which bound domains
doubly quantized vortex. This is equivalent to finding a potential this is known not to be the case \[6, 7\].

For this reason, we now turn to the question of the stability. Although the dynamic instability.) The question of what constitutes “strong” or “weak” coupling would thus seem to require more details regarding the physical system in question.

As we have shown, the reality and Hermiticity of \(M_s\) and the non-Hermiticity introduced by the specific form of \(\sigma\) are not sufficient to provide a unique answer to the question of dynamic stability in the presence of an even number of unstable static modes. (As noted above, an odd number of negative static modes always implies dynamic instability.) The question of what constitutes “strong” or “weak” coupling would thus seem to require more details regarding the physical system in question. For this reason, we now turn to the question of the stability of vortex solutions to the physically relevant Gross-Pitaevskii equation.

**NUMERICAL RESULTS**

We now study the static and dynamic stability of a doubly quantized vortex numerically. The single-particle Hamiltonian, \(h\phi = -\hbar^2\nabla^2/(2M) + M\omega^2r^2/2\), represents a harmonic oscillator potential of strength \(\omega\). In an anharmonic potential, a doubly quantized vortex is energetically stable for weak couplings, but in a purely harmonic potential this is known not to be the case \[2, 5\].

In our simulations Eq. (14) was first solved to find the doubly quantized vortex. This is equivalent to finding a stationary solution to the Gross-Pitaevskii equation subject to the constraint that the system has a \(4\pi\) phase singularity at the origin. This minimization was carried out within a truncated basis composed of the \(N_c\) lowest radially excited states and is valid for small values of \(g\) such that \(g/(2\hbar\omega) < N_c\). The static and dynamic eigenvalues were then calculated according to Eqs. (14) and (17) in a basis consisting of \(N_c\) radially excited states for both the \(m = 0\) and \(m = 4\) components. The computation of the eigenvalues was carried out in Matlab. The results of this calculation are shown in Fig. 1. Results are shown for \(N_c = N_x = 8\), and we have checked convergence with respect to both \(N_c\) and \(N_x\) for the range of couplings displayed in the figure. The dynamic eigenvalues \(\omega_d\) coincide with those previously found numerically in Refs. 2, 3, 4.

The figure clearly shows the correlation between the “window” structure of the complex frequencies and the eigenvalues, \(\bar{\omega}_s\), of the static problem in agreement with our general arguments above. For small \(g\) such that \(g \ll \hbar\omega\), the properties of the system follow from the perturbative analysis of the lowest Landau level as described above: The system possesses one negative static eigenvalue and therefore one pair of complex dynamic eigenvalues. As the coupling increases, a second static eigenvalue crosses zero at \(g/\hbar\omega \approx 6\), and the system becomes dynamically stable despite its manifest static instability. As a third static eigenvalue crosses zero, one pair of dynamic eigenvalues again become complex but
become real again when a fourth static eigenvalue becomes negative. The numerical findings indicate that as $g$ is further increased, a succession of such alternations occurs with the result that the system is found to be dynamically stable (unstable) when there are an even (odd) number of negative static eigenvalues.

For the range of couplings, $g$, studied here, there is never more than one complex pair of dynamic eigenvalues. It is also seen that the system is always statically unstable. In the limit of large $g$, well-known results for vortices in an infinite system apply. In that limit the energy of two vortices increases monotonically with decreasing separation, thereby implying that at least one eigenvalue of the static stability matrix is negative. Hence it is reasonable to conclude that the system is statically unstable for all values of $g$. We thus conclude that the “window” structure of alternating dynamically stable and unstable regions arises from the non-trivial interplay between negative modes of the static stability matrix.

CONCLUSIONS

The purpose of this paper has been to point out the intimate connections between the static and dynamic stability of solutions to the Gross-Pitaevskii equations. We have shown that (suitably constrained) static stability necessarily implies dynamic stability, that the number of complex conjugate pairs of dynamic eigenvalues changes by one every time a constrained static eigenvalue passes through zero, and that an odd number of negative static eigenvalues thus implies dynamic instability. Numerical investigations revealed that the doubly-quantized vortex solution to the Gross-Pitaevskii equation is statically unstable over the full range of coupling constants explored but displays windows of dynamic stability. The general nature of the arguments presented here suggests that similar connections between the problems of static and dynamic stability are likely to be wide-spread. Further, we believe that additional insight obtained by studying both stability problems is likely to be well worth the minimal additional effort required.

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