COEXISTENCE OF 1/2, 1/3–CAUSTICS FOR DEFORMATIVE NEARLY CIRCULAR BILLIARD MAPS

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Abstract. For symmetrically analytic deformation of the circle (with certain Fourier decaying rate), the necessary condition for the corresponding billiard map to keep the coexistence of 1/2, 1/3 caustics is that the deformation has to be an isometric transformation.

1. Introduction

Suppose Ω ⊂ R^2 is a strictly convex domain, with the boundary ∂Ω is C^r smooth, r ≥ 2. The billiard problem inside Ω can be described as the following:

A massless particle moves with unit speed and no friction following a rectilinear path inside the domain Ω. When the ball hits the boundary, it is reflected elastically according to the law of optical reflection: the angle of reflection equals the angle of incidence.

This problem was first investigated by Birkhoff (see [3]). Later we can see that such trajectories are called broken geodesics, as they correspond to local maximizers of the distance functional. The billiard map can be identified by the correspondence of the positions in one reflection φ: P_0 → P_1, see Fig. 1.

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Figure 1. The reflective angle keeps equal to the incident angle for every rebound.
Remark 1.2. Due to previous setting, we can see that for any \( \rho \) and the rotation number \( \phi \) the positive tangent to arc length \( s \) where \( X \) defined on the closed annulus \( \mathbb{A} = [0, 1] \times [0, \pi] \). Obviously \( \phi|_{\partial \mathbb{A}} = Id \), i.e.
\[
\phi(s, 0) = (s, 0), \quad \phi(s, \pi) = (s, \pi), \quad \forall s \in \mathbb{T}.
\]
Let’s denote by
\[
(1) \quad h(x, x') := -d(P_0, P_1), \quad (x, x') \in \mathbb{R}^2,
\]
where \( d(\cdot, \cdot) \) is the Euclid distance of \( \mathbb{R}^2 \), \( x \equiv s \mod 2\pi \) and \( x' \equiv s' \mod 2\pi \). It’s easily to find that
\[
(2) \quad h(x + 1, x' + 1) = h(x, x'), \quad (x, x') \in \mathbb{R}^2
\]
and
\[
(3) \quad \partial_1 h = \cos v, \quad \partial_2 h = -\cos v'.
\]
The twist property is implied by \(-\partial_2 h > 0 \) once the boundary is strictly convex [7].

Let’s make a rule for \( s \in \mathbb{T} = \mathbb{R}/[0, 2\pi] \): we always choose the counter clockwise direction to order the configurations, that means for any \( q \)-tuple \((s_0, s_1, \cdots, s_{q-1})\) with \( s_i \in \mathbb{T}, q \geq 2 \) and \( 0 \leq i \leq q - 1 \), we can fix a unique configuration \((x_0, x_1, \cdots, x_{q-1})\) in the universal covering space \( \mathbb{R} \), such that
\[
(4) \quad x_i \equiv s_i \mod 2\pi \) for \( 0 \leq i \leq q - 1, \quad 0 \leq x_i+x_{i+1} < 1 \) for \( 0 \leq i \leq q - 2, \quad x_0 \in [0, 1). \]
This unique configuration implies the following definition:

**Definition 1.1.** For a \( q \)-periodic tuple \( S = (s_0, s_1, \cdots, s_{q-1}, s_q) \) with \( s_0 = s_q \), we define the winding number of it by
\[
p := \frac{x_q - x_0}{2\pi} \in \mathbb{Z}^+
\]
and the rotation number by
\[
\rho(S) = \frac{p}{q},
\]
where \( X = (x_0, \cdots, x_q) \) is the configuration corresponding to \( S \) defined above.

**Remark 1.2.** Due to previous setting, we can see that for any \( q \)-periodic tuple \( S \), the rotation number \( \rho(S) \in [0, 1) \cap \mathbb{Q} \).

For \( q \geq 2 \) and a fixed \( s \in \mathbb{T} \), we can define a \( q \)-periodic configuration set \( C_{p/q}(s) \) by
\[
C_{p/q}(s) := \left\{ (x_0, \ldots, x_q) \in \mathbb{R}^{q+1} | 0 \leq x_{i+1} - x_i < 1 \text{ for } 0 \leq i \leq q-2, x_q - x_0 = p, x_0 \equiv s \mod 2\pi \right\}.
\]
Then the following action function is well defined
\[
(4) \quad F_{p/q}(s) := \inf_{\gamma \in C_{p/q}(s)} \sum_{i=0}^{q-1} h(\gamma_i, \gamma_{i+1})
\]
and the minimizer \( \gamma^* \) obeys the discrete Euler Lagrange equation
\[
(5) \quad \partial_1 h(\gamma^*_i, \gamma^*_{i+1}) + \partial_2 h(\gamma^*_{i-1}, \gamma^*_i) = 0, \quad \forall \ 1, 2, \ldots, q - 1.
\]
Moreover, due to [3], the corresponding \( \{(s_i, v_i)\}_{i=0}^q \) is an orbit of the billiard map \( \phi \), see [2].
Remark 1.3. Notice that \( \{(s_i, v_i)\}_{i=0}^q \) may not be a ‘real’ periodic orbit, because \( v_0 \neq v_q \) could happen. However, if we interpret \( F_{p/q}(s) \) as a function defined on \( \mathbb{T} \), then the minimizer \( s^* \) will definitely correspond to a ‘real’ periodic orbit \( \gamma^* \) of the billiard map.

Definition 1.4. We call a (possibly not connected) curve \( \Gamma \subset \Omega \) a \textbf{caustic} if any billiard orbit having one segment tangent to \( \Gamma \) has all its segments tangent to it. Precisely, \( \Gamma_{p/q} \) is an \( p/q \)-rational caustic if all the corresponding (noncontractible) tangential orbits are periodic with the rotation number \( p/q \).

Remark 1.5. In the remaining part of this paper, we agree that all caustics that we will consider will be smooth and convex; we will refer to such curves simply as caustics. By the Birkhoff’s Theorem of general exact monotone twist maps, such a rational caustic will correspond to a \( \phi \)-invariant curve in the phase space, i.e. formally we can express by

\[
\Gamma_{p/q} = \{(s, g_{p/q}(s)) \in \mathbb{A} | s \in \mathbb{T} \}
\]

which consists of periodic orbits of rotation number \( p/q \). Moreover, \( \Gamma_{p/q} \) is non-contractible and then \( g_{p/q}(s) \) is a Lipschitz graph, see [3].

Besides, there is reversibility of the billiard map \( \phi \), i.e. \( \phi \circ I(s, v) = I \circ \phi^{-1}(s, v) \) for all \( (s, v) \in \mathbb{A} \), where \( I(s, v) = (s, \pi - v) \) is a reflective transformation. Benefit from this, we can see that \( \Gamma_{p/q} = \Gamma_{1-p/q} \). So in the following we can impose that the rotation number of the caustic belongs to \([0, 1/2]\).

Theorem 1.6. [6] For an exact symplectic twist map, every orbit on a non contractible invariant curve is minimizing in the sense of variation.

By applying this Theorem to the billiard map, we get

Corollary 1.7. Suppose the billiard map \( \phi \) has a \( p/q \)-rational caustic, then \( F_{p/q}(s) \) equals a constant for all \( s \in \mathbb{T} \).

Remark 1.8. In [4], the authors defined the \textbf{width function} \( w(\alpha) \) of a convex billiard curve by the strip width formed by the tangent rays with the direction \( \alpha + \pi/2 \) and \( \alpha - \pi/2 \), for any unit vector \( \alpha \) in \( \mathbb{R}^2 \). We called the billiard boundary of constant width, if \( w(\alpha) \) is a constant. Observe that billiard boundaries of constant width has \( 1/2 \) caustic. The circle is a trivial example of constant width, of which the \( 1/2 \) caustic is the centre point. A popular non trivial example is the Releaux triangle, see [10].

We should remind the readers that elliptic boundary has no \( 1/2 \) caustic. Indeed, the elliptic billiard has a first integral, see [5], but the \( 1/2 \) periodic orbits just correspond to the rebound along the major (reps. minor) axis, which have separatriz arcs containing homoclinic orbits but not periodic ones.

Organization of this article. In Section 2 we state the rigid constraints on the boundaries to preserve certain rational caustics; we introduce the Projection Theorem and our main conclusion towards it; In Section 3 we get the 1st and 2nd order estimate of the action function, which leads to our harmonic equation; In Section 4 we analysis the harmonic equation and give the proof of our main conclusion. In Section 5 we make some heuristic comments and further generalizations of this direction.
2. Main conclusion and Scheme of the proof

Based on previous section’s setting, a natural question can be asked: How far can we constrain the boundary to preserve certain rational caustics?

Recently, Avila, de Simoi and Kaloshin proved the following result:

**Theorem 2.1.** [1] There exist $\varepsilon_0 > 0$ and $\eta_0 > 0$ such that for any $0 \leq \varepsilon \leq \varepsilon_0$, $0 \leq \varepsilon < \varepsilon_0$, any rationally integrable $C^{39}$-smooth domain $\Omega$ so that $\partial \Omega$ is $C^{39}$ close to the ellipse $E$ with the eccentricity $e$ is also an ellipse.

This implies that locally ellipse with small eccentricity is the only possible integrable convex billiard boundaries. The proof of this Theorem relies on an improved $1^{st}$ order estimate of the action function. To help the readers to get a clearer understanding of that, let us start by exploring the integrable infinitesimal deformations of a circle. Suppose $\Omega_0$ be the unit disk, and the polar coordinates on the plane be $(r, \theta)$. Let $\Omega_\varepsilon$ be a one-parameter family of deformations given by

$$\partial \Omega_\varepsilon = \{ r = 1 + \varepsilon n(\theta) + O(\varepsilon^2) \},$$

where the Fourier expansion of $n$:

$$n(\theta) = n_0 + \sum_{k>0} n_k' \sin k \theta + n_k'' \cos k \theta.$$

The following Theorem is available:

**Theorem 2.2** (Ramirez-Ros [9]). If $\Omega_\varepsilon$ has an integrable rational caustic $\Gamma_{1/q}$ of rotation number $1/q$ for all sufficiently small $\varepsilon$, then $n_k' = n_k'' = 0$ for all $k \in \mathbb{N}$.

If we impose that $\Omega_\varepsilon$ is rationally integrable for all $1/q$ with $q > 2$ and sufficiently small $\varepsilon$, then the above theorem implies that $n_k' = n_k'' = 0$ for all $k > 2$, i.e. we establish $n(\theta)$ by:

$$n(\theta) = n_0 + n_1' \cos \theta + n_1'' \sin \theta + n_2' \cos 2\theta + n_2'' \sin 2\theta$$

(6)

**Remark 2.3.** As in [1], we can find that in (6) $n_0$ corresponds to a homothety, $n_1'$ corresponds to a shift in the direction $\theta_1$ and $n_2'$ corresponds to a deformation into an ellipse of small eccentricity with the major axis coincides with the direction $\theta_2$.

It’s remarkable that in the above theorem, one have to face $\varepsilon \to 0$ as $q \to \infty$. So they need to replace the infinitesimal deformation by a fixed lower bound such that $\varepsilon > \varepsilon_0$; You can see [1] for more technical details.

A natural generalization of previous Theorem is reducing to ‘finitely many’ rational caustics, based on which we can still get the integrability of the billiard maps:

**The Projected Conjecture.** In a $C^r (r = 2, \cdots, \infty, w)$ neighborhood of the circle there is no other billiard domain of constant width and preserving $1/3$ caustics.

**Definition 2.4.** Denote by $BZ_q$ the manifold of codimension infinity containing all the strictly convex billiard boundaries preserving the $1/q$ caustic $\Gamma_q$, $q \geq 2$.

Here we propose a possible approach to prove it, still we start from the infinitesimal deformation of the circle:
Step 1. Find the tangent bundle of $BZ_q$ at the circle. Let $n(\theta)$ and $m(\theta)$ be functions with $\theta \in T^1 = \mathbb{R}/[0, 2\pi]$ given by the Fourier series

$$n(\theta) = \sum_{k \in \mathbb{Z}} n_k \exp(ik\theta), \quad m(\theta) = \sum_{k \in \mathbb{Z}} m_k \exp(ik\theta).$$

In the polar angle coordinate, for sufficiently small $\varepsilon$ the perturbed domain can be denoted by

$$\partial \Omega_\varepsilon = \{r = 1 + \varepsilon n(\theta) + \varepsilon^2 m(\theta) + O(\varepsilon^3)\} \quad (*)$$

Denote by $T_q$ the tangent space of $BZ_q$ at the circle, and $T_q^\perp$ be the orthogonal complement of $T_q$. Due to Theorem 2.2, $T_q$ consists of functions given by Fourier coefficients whose indices are not divisible by $q$.

Step 2. Describe $BZ_q$ as a graph over $T_q$. Namely, $BZ_q = \{(n, F_q(n)) \in T_q \times T_q^\perp\}$.

Step 3. Show that for all $n \in T_2 \cap T_3 \setminus 0$, there exists a upper bound $\varepsilon_0 = \varepsilon_0(n)$ such that for all $0 < \varepsilon \leq \varepsilon_0$, we have $F_2(\varepsilon n) \neq F_3(\varepsilon n)$. In particular, it implies the Projected Conjecture.

Remark 2.5. Notice that $\varepsilon_0(n)$ may be not uniform for different $n \in C^r(T, \mathbb{R})$, even though $\|n\|_{C^r} = 1$ is imposed. So we need a similar approach as [1] to avoid the collapse of the infinitesimal $\varepsilon_0(\cdot)$. To keep the consistency and readability, we don’t consider this part in this paper.

Following aforementioned strategy, we claim our main conclusion. Before we doing that, let’s formalize the symbol system first:

- $\gamma_0 \in \mathbb{R}^2$ be the unit circle and $\gamma_0^\varepsilon \in \mathbb{R}^2$ be a deformation, and in the polar coordinate $r_0, r_\varepsilon$ represent the corresponding axis length.

- $\{z_k^0 \in \Omega_0\}, \{z_k^\varepsilon \in \Omega_\varepsilon\}$ be the configuration of $q$-periodic, $k = 0, 1, \cdots, q - 1$.

- $s \in [0, 2\pi]$ be the arc length variable and $\theta \in [0, 2\pi]$ be the rotational angle;

- Suppose $\gamma_t(\theta)$ is a curve of strictly convex boundaries in $\mathbb{R}^2$ with the parameter $t \in [0, 1]$ and starting from the circle, i.e. $\gamma_0(\theta) = 1$. If $\gamma_t(\theta)$ is $C^3$ smooth of $t$, then we can expand the curve in the polar coordinate by

$$r_t(\theta) = 1 + tn(\theta) + t^2m(\theta) + O(t^4), \quad t \ll 1.$$  \hspace{1cm} (7)

We can assume $n, m \in C^\infty(T, \mathbb{R})$, $\rho > 0$. Moreover, by rescaling $t$, we can make

$$|n(\theta)|_{C^0} = 1, \quad |m(\theta)|_{C^0} \leq C.$$  \hspace{1cm} (8)

- Recall that two different deformation $\gamma_t$ and $\gamma'_t$ may be homogenous by a rigid transformation on $\mathbb{R}^2$ space (parallel shift, rotation), so we just need to choose a representation by imposing

$$n(0) = 0, n'(0) = 0.$$  \hspace{1cm} (9)

- Besides, we can fix the perimeter by $2\pi$ for all the deformations, i.e.

$$\int_T n(\theta)d\theta = 0.$$  \hspace{1cm} (10)
Theorem 2.6 (Main Conclusion). For non degenerate deformation of circle which can be formalized by (\ast) and satisfying (8), (9) and (10), if \(n(\theta)\) is an even analytic function with super exponential decaying rate, i.e.

\[
\lim_{i \to \infty} \frac{w(i)}{2^i} = \infty
\]

with the modular function \(w : \mathbb{Z} \to \mathbb{R}^+\) satisfying

\[
w(-i) = w(i),
\]

\[
n(\theta) = \sum_{k \in \mathbb{Z}} n_k e^{ik\theta}, \quad |n_k| \leq e^{-w(k)}
\]

and there exists a subsequence \(\{k_i\}_{i=1}^\infty\), such that

\[
\lim_{i \to \infty} \frac{|n_{k_i}|}{e^{-w(k_i)}} < \infty,
\]

then there exists \(\varepsilon_0(n)\) such that for all \(0 < \varepsilon \leq \varepsilon_0\), the deformed boundary couldn’t persist the coexistence of 1/2, 1/3-caustics.

Proof. This conclusion is proved in Section 4, Theorem 4.2. \(\Box\)

Corollary 2.7 (Trigonometric Polynomial). For any non degenerate deformation of the circle which can be formalized by (\ast) and satisfying (8), (9) and (10), if \(n(\theta)\) is an even trigonometric polynomial, then there exists \(\varepsilon_0(n)\) such that for all \(0 < \varepsilon \leq \varepsilon_0\), the deformed boundary couldn’t persist the coexistence of 1/2, 1/3-caustics.

Proof. This conclusion is proved in Section 4, Theorem 4.12. \(\Box\)

Let’s define the action function of \(q\)–periodic configuration by

\[
P_q(\theta, \Omega_\varepsilon) = \text{maximal perimeter of a } q \text{ gon inscribed into}
\]

the domain \(\Omega_\varepsilon\) starting and ending at \(\theta\),

then

\[
P_q(\theta, \Omega_\varepsilon) = -F_{1/q}(s(\theta))
\]

if we consider the arc length variable \(s\) as a function of the angle variable \(\theta\). Recall that \(s : [0, 2\pi] \to [0, 2\pi]\) is a diffeomorphism via the following:

\[
s = \int_0^\theta \sqrt{r_2^2(\tau) + r_1^2(\tau)} d\tau,
\]

so \(s(\theta + 2\pi) = s(\theta) + 2\pi\) can be easily achieved.

Lemma 2.8. Let \(n \in T_q\), then the domain \(\Omega_\varepsilon\) (\ast) has

\[
P_q(\theta, \Omega_\varepsilon) = P_q(\theta, \Omega_0) + \varepsilon c + \varepsilon^2 \left[2q \sin \frac{\pi}{q} m(q)(\theta) + D_q(\theta, \varepsilon)\right],
\]

for some \(c\), where

\[
m^{(q)}(\theta) = \sum_{k \in \mathbb{Z}} m_{kq} \exp ikq\theta
\]

is the averaging of \(1/q\)–frequency of \(m(\theta)\).

Proof. See section 3 for details. \(\Box\)
Consider the Fourier expansion of \( D_q \)

\[
D_q(\theta, \varepsilon) = \sum_{k \in \mathbb{Z}} D_{q,k}(\varepsilon) \exp(ik\theta),
\]

then the following property holds:

**Lemma 2.9.** In the above notations, if \( \gamma_\varepsilon \in BZ_q \),

\[
D_q(\theta, \varepsilon) = \sum_{k=mq, \, m \in \mathbb{Z}} D_{q,mq}(0) \exp(i(mq\theta)) + O(\varepsilon),
\]

i.e. \( D_q(\theta, \varepsilon) \) is \( 1/q \)-periodic of \( \theta \).

**Proof.** Since \( \gamma_\varepsilon \in BZ_q \), \( P_q(\theta, \Omega_\varepsilon) \) is a constant. Due to previous Lemma,

\[
2q \sin \frac{\pi}{q} m^{(q)}(\theta) + D_q(\theta, \varepsilon) = \text{const}.
\]

Recall that \( m^{(q)}(\theta) = \sum_l m_{ql} \exp(ql \cdot \theta) \) contains only harmonics divisible by \( q \), that means \( D_q(\theta, \varepsilon) \) has to contain only harmonics divisible by \( q \) as well, i.e.

\[
D_q(\theta) := D_q(\theta, 0) = \sum_{l \in \mathbb{Z}} D_{ql} \exp(iql \cdot \theta).
\]

So \( D_q(\theta) \) is \( 1/q \)-periodic of \( \theta \). \( \square \)

### 2.1. Obstruction to coexistence of two rational caustics.

In particular, for \( q = 2, 3 \) we want to show that for all 

\[
n \in (T_2 \cap T_3) \setminus 0
\]

the functions \( 3\sqrt{3}/4 D_2(\theta, \varepsilon) \) and \( D_3(\theta, \varepsilon) \) have at least one Fourier harmonic divisible by \( 6 \) whose coefficients are different. Once we did this, the contradiction with the action functions will lead to our main conclusion.

**Corollary 2.10.** Once \( 3\sqrt{3}/4 D_2(\theta) \) and \( D_3(\theta) \) have a distinct Fourier harmonic whose index is divisible by \( 6 \), then in a \( O(\varepsilon^2) \) neighborhood of \( n(\theta) \) there is no domain having \( 1/2 \) and \( 1/3 \) caustic.

**Proof.** Indeed,

\[
P_q(\theta, \Omega_\varepsilon) = P_q(\theta, \Omega_0) + c\varepsilon + \varepsilon^2 \left( D_q(\theta) + 2q \sin \frac{\pi}{q} m^{(q)}(\theta) \right) + O(\varepsilon^3)
\]

\[
P_p(\theta, \Omega_\varepsilon) = P_p(\theta, \Omega_0) + c\varepsilon + \varepsilon^2 \left( D_p(\theta) + 2p \sin \frac{\pi}{p} m^{(p)}(\theta) \right) + O(\varepsilon^3).
\]

If we take \( q = 2 \) and \( p = 3 \), then the necessary condition to preserve \( 1/2 \), \( 1/3 \) caustics is that \( P_2, P_3 \) should be both constant, which leads to

\[
D_2(\theta) + 4 m^{(2)}(\theta) = \text{const},
\]

\[
D_3(\theta) + 3\sqrt{3} m^{(3)}(\theta) = \text{const}.
\]

Once again we can take the averaging and get

\[
\frac{3\sqrt{3}}{4} D_2^{(3)}(\theta) = D_3^{(2)}(\theta).
\]

This is a necessary condition for boundary \( \Omega_\varepsilon \) to preserve both \( 1/2 \) and \( 1/3 \) caustics. \( \square \)

The Fourier coefficients of \( D_q(\theta) \) are obtained through a convolution of Fourier coefficients of \( n \). We will compute the corresponding formula in the next section.
3. Evaluation of $P_q$ action function

Let’s start with the circle $\Omega_0$ and the $q$-gon for some $q \geq 2$. Suppose the deformation in the polar coordinate has the form

$$\gamma_\varepsilon = \{r = 1 + \varepsilon n(\theta) + \varepsilon^2 m(\theta), \ \theta \in \mathbb{T}\}$$

for small $\varepsilon$ and

$$n(\theta) = \sum_{k \in \mathbb{Z}} n_k \exp 2\pi ik \cdot \theta \quad \& \quad m(\theta) = \sum_{k \in \mathbb{Z}} m_k \exp 2\pi ik \cdot \theta.$$ 

Consider a $q$-perimeter function $P_\varepsilon(\theta, \Omega_\varepsilon)$ as defined above. For $\varepsilon = 0$ we have

$$P_q(\theta, \Omega_0) \equiv 2q \sin \frac{\pi}{q}.$$ 

Let $(z_0, \ldots, z_q)$ be the right $q$-gon, i.e. $\theta_k^0 = \theta_k^0 + 2k\pi/q$. For small $\varepsilon$ we compute

$$z_k^\varepsilon = z_k^0 + \varepsilon \eta_k + \varepsilon^2 \xi_k + O(\varepsilon^3), \quad k = 0, \ldots, q - 1.$$ 

We postpone the computation of $\eta_k$ and $\xi_k$. Consider the $k$-th edge between $z_k$ and $z_{k+1}$. Taking a dot product of $z_k^\varepsilon - z_{k+1}^\varepsilon$ with itself we have

$$\left|z_k^\varepsilon - z_{k+1}^\varepsilon\right|^2 = \left|z_k^0 - z_{k+1}^0\right|^2 + \varepsilon^2 \left|\eta_k - \eta_{k+1}\right|^2$$

$$+ 2\varepsilon (z_k^0 - z_{k+1}^0) \cdot (\eta_k - \eta_{k+1}) + 2\varepsilon^2 (z_k^0 - z_{k+1}^0) \cdot (\xi_k - \xi_{k+1}) + O(\varepsilon^3).$$

Rewrite

$$|z_k^\varepsilon - z_{k+1}^\varepsilon| = |z_k^0 - z_{k+1}^0| + \varepsilon \frac{(z_k^0 - z_{k+1}^0, \eta_k - \eta_{k+1})}{|z_k^0 - z_{k+1}^0|}$$

$$- \frac{\varepsilon^2}{2} \frac{(z_k^0 - z_{k+1}^0, \eta_k - \eta_{k+1})^2}{|z_k^0 - z_{k+1}^0|^3} + \frac{\varepsilon^2}{2} \frac{|\eta_k - \eta_{k+1}|^2}{|z_k^0 - z_{k+1}^0|}$$

$$+ \varepsilon \frac{(z_k^0 - z_{k+1}^0, \xi_k - \xi_{k+1})}{|z_k^0 - z_{k+1}^0|} + O(\varepsilon^3).$$

Summing over $k$ we get

$$P_q(\theta, \Omega_\varepsilon) = P_q(\theta, \Omega_0) + \varepsilon \sum_{k=0}^{q-1} \left[ z_k^0 - z_{k+1}^0 \cdot (\eta_k - \eta_{k+1}) + O(\varepsilon^2) \right.$$ 

$$+ \varepsilon \frac{z_k^0 - z_{k+1}^0}{|z_k^0 - z_{k+1}^0|} \cdot (\xi_k - \xi_{k+1}) - \frac{\varepsilon}{2} \frac{(z_k^0 - z_{k+1}^0, \xi_k - \xi_{k+1})^2}{|z_k^0 - z_{k+1}^0|^3} + \frac{\varepsilon}{2} \frac{|\xi_k - \xi_{k+1}|^2}{|z_k^0 - z_{k+1}^0|} \left].
$$

where $z_k^0/|z_k^0| = e^{i\theta} = z_0^0/|z_0^0|$ (for convenience we can interpret the unit vector in $\mathbb{R}^2$ as a complex number in $\mathbb{C}$, this greatly simplifies our formulas without confusion). Also we can denote the unit vector $z_k^0 - z_{k+1}^0/|z_k^0 - z_{k+1}^0|$ by $e_k$ for short.

$$P_q(\theta, \Omega_\varepsilon) = P_q(\theta, \Omega_0) + \varepsilon \sum_{k=0}^{q-1} \left[ e_k \cdot (\eta_k - \eta_{k+1}) + O(\varepsilon^2) \right.$$ 

$$+ \varepsilon e_k \cdot (\xi_k - \xi_{k+1}) - \frac{\varepsilon}{2} \frac{(e_k, \eta_k - \eta_{k+1})^2}{|z_k^0 - z_{k+1}^0|} + \frac{\varepsilon}{2} \frac{|\eta_k - \eta_{k+1}|^2}{|z_k^0 - z_{k+1}^0|} \right].$$
Recall that $z_k^* = z_k^0$ and $z_k^0 = z_k^0 = z_k^0$, which leads to $\eta_k = \eta_k$ and $\xi_k = \xi_k$. Combining $e_k \cdot \eta_k$ and $-e_{k-1} \cdot \eta_k$ and observing that $(e_k - e_{k-1})$ is the outer normal vector to the boundary $\partial \Omega_0$ at $z_k^0$, then we have

$$P_q(\theta, \Omega_\varepsilon) = P_q(\theta, \Omega_0) + \varepsilon \sum_{k=0}^{q-1} \left[ (e_k - e_{k-1}) \cdot (\eta_k + \varepsilon \xi_k) + O(\varepsilon^2) \right]$$

$$- \frac{\varepsilon}{2} \left[ \left( e_k \cdot (\eta_k - \eta_k+1) \right)^2 + \frac{\varepsilon}{2} \left( \eta_k - \eta_k+1 \right)^2 \right] =$$

$$= P_q(\theta, \Omega_0) + \varepsilon \sum_{k=0}^{q-1} \left[ (e_k - e_{k-1}) \cdot (\eta_k + \varepsilon \xi_k) + O(\varepsilon^2) \right]$$

$$- \frac{\varepsilon}{2} \left[ \left( e_k \cdot (\eta_k - \eta_k+1) \right)^2 + \frac{\varepsilon}{2} \left( \eta_k - \eta_k+1 \right)^2 \right],$$

where $\eta_k^\perp$ (resp. $\xi_k^\perp$) is the component of $\eta_k$ (resp. $\xi_k$) perpendicular to $(z_{k-1}^0 - z_{k+1}^0)$ or, equivalently, the component of $\eta_k$ (resp. $\xi_k$) normal to the boundary at $z_k^0$.

### 3.1. The leading term in the case of the circle.

Consider the leading term of the expansion

$$P_q(q, \Omega_\varepsilon) = P_q(q, \Omega_0) + \varepsilon \sum_{k=0}^{q-1} \left[ (e_k - e_{k-1}) \cdot \eta_k^\perp \right] + O(\varepsilon^2).$$

Notice that

$$e_k - e_{k-1} = 2 \sin \frac{\pi}{q} n(\theta_k^0), \quad k = 1, \ldots, q$$

for $q \geq 2$. Here the bold $n(\cdot)$ is the normal vector, not the 1st jet of the deformation! Then we get

$$\eta_k^\perp = (\eta_k, n(\theta_k^0))n(\theta_k^0) = n(\theta_k^0)n(\theta_k^0) + O(\varepsilon).$$

This is because

$$z_k^\varepsilon = z_k^0 + \varepsilon \eta_k + \varepsilon^2 \xi_k + O(\varepsilon^3)$$

$$= r(\theta_k^0)e^{i\theta_k^0}$$

$$= \left[ 1 + \varepsilon n(\theta_k^0) + \varepsilon^2 m(\theta_k^0) \right] e^{i\theta_k^0},$$

and

$$\theta_k^\varepsilon = \theta_k^0 + \varepsilon \theta_k^1 + O(\varepsilon^2),$$

which leads to the 1st order equation:

**Lemma 3.1.** The 1st order estimate of $P_q(q, \Omega_\varepsilon)$ obeys

$$P_q(\theta, \Omega_\varepsilon) = P_q(\theta, \Omega_0) + 2\varepsilon q \sin \frac{\pi}{q} n(\theta) + O(\varepsilon^2), \quad q \geq 2.$$ 

### 3.2. The second order in the case of the circle.

In order to get a 2nd order estimate for $P_q(q, \Omega_\varepsilon)$, we need to get the exact expression of $\eta_k$ and $\xi_k$. Due to (14), we get

$$\eta_k(\theta_k^0) = \eta_k^\perp + \eta_k^\parallel = n(\theta_k^0)e^{i\theta_k^0} + \theta_k^1 e^{i(\pi/2 + \theta_k^0)} + O(\varepsilon)$$

and

$$\xi_k(\theta_k^0) = \left[ -\frac{1}{2} \theta_k^2 + n'(\theta_k^0)\theta_k^1 + m(\theta_k^0) \right] e^{i\theta_k^0} + n(\theta_k^0)\theta_k^1 e^{i(\pi/2 + \theta_k^0)} + O(\varepsilon).$$
Recall that $\theta_0^1 = \theta_q^1 = 0$, then

$$
\langle \mathbf{n}^0(\theta_k^0), \frac{z_k^0 - z_k^0}{\bar{z}_k^0 - \bar{z}_k^0} \rangle = 0, \quad k = 1, \ldots, q - 1.
$$

On the other side,

$$
\mathbf{n}^r(\theta_k^r) = \frac{e^{i\pi/2} \cdot \gamma_k(\theta)}{\theta_k^r} = e^{i(\pi/2 + \theta_k^r)} \left[ \varepsilon + \varepsilon^2 m'(\theta_k^r) + \varepsilon^2 m(\theta_k^r) \right] - e^{i\theta_k^r} \left[ 1 + \varepsilon n(\theta_k^r) + \varepsilon^2 m(\theta_k^r) \right]
$$

$$
= e^{i(\pi/2 + \theta_k^r)} \left[ \varepsilon + \varepsilon^2 n'(\theta_k^0) + \varepsilon^2 m'(\theta_k^0) \right] - \left( \varepsilon n(\theta_k^0) + \varepsilon^2 m(\theta_k^0) \right) + O(\varepsilon^3)
$$

$$
= -n(\theta_k^0) + \varepsilon n'(\theta_k^0) + e^0(\theta_k^0) - \varepsilon \eta_k(\theta_k^0) + O(\varepsilon^2)
$$

and

$$
z_k^0 - z_{k-1}^0 = \left[ z_k^{0+1} - z_k^{0+1} \right] + \varepsilon \left[ \eta_k(\theta_k^{0+1}) - \eta_k(\theta_k^{0-1}) \right] + O(\varepsilon^2),
$$

where we used the estimate

$$
e^{i(\phi + \psi)} - e^{i\phi} = \varepsilon \psi e^{i(\pi/2 + \phi)} - \frac{1}{2} \varepsilon^2 \psi^2 e^{i\phi} + O(\varepsilon^3), \quad \forall \phi, \psi \in \mathbb{T}.
$$

Then we turn back to (15) and get

$$
\langle \mathbf{n}^0(\theta_k^0), \eta_k+1(\theta_k^{0+1}) - \eta_k-1(\theta_k^{0-1}) \rangle + \langle \eta_k(\theta_k^0) - n'(\theta_k^0) \rangle = 0
$$

(17)

Simplify it:

$$
\left( n(\theta_k^0) - n(\theta_k^{0+1}) \right) \cos \frac{2\pi}{q} + \left( 2\theta_k^0 - \theta_k^{0+1} - \theta_k^{0-1} - 2n'(\theta_k^0) \right) \sin \frac{2\pi}{q} = 0
$$

(18)

we finally get a triple-diagonal linear equation group:

$$
4\theta_k^1 - \theta_k^{1+1} - 3\theta_k^{1-1} = 4n'(\theta_k^1) - \frac{n(\theta_k^{0+1}) - n(\theta_k^{0-1})}{\tan \frac{2\pi}{q}}, \quad k = 1, \ldots, q - 1,
$$

with $\theta_0^1 = \theta_q^1 = 0$. This is a general formula holding for all $q \geq 3$.

**Remark 3.2.** For $q = 2$, $\theta_1^1 = n'(\theta + \pi) - n'(\theta), \theta_0^1 = 0$. Mention that in this case (19) is invalid, but we can use that $\mathbf{n}^r(\theta_0^0) \parallel \mathbf{n}^r(\bar{\theta}_0^0)$.

For $q = 3$, we can solve (19) by

$$
\theta_1^1 = \frac{4}{13} \left[ n'(\theta + 4\pi/3) + n'(\theta + 2\pi/3) \right] + \frac{1}{13\sqrt{3}} \left[ 3n(\theta) + n(\theta + 2\pi/3) - 4n(\theta + 4\pi/3) \right],
$$

$$
\theta_2^1 = \frac{4}{13} \left[ 3n(\theta + 2\pi/3) + 4n'(\theta + 4\pi/3) \right] + \frac{1}{13\sqrt{3}} \left[ -n(\theta) - 3n(\theta + 4\pi/3) + 4n(\theta + 2\pi/3) \right].
$$
Recall that

\[ P_q(\theta, \Omega) = P_q(\theta, \Omega_0) + \varepsilon \sum_{k=0}^{q-1} \left[ (e_k - e_{k-1}) \cdot (\eta_k^+ + \varepsilon \xi_k^+) + O(\varepsilon^2) \right] \]

- \frac{\varepsilon}{2} \frac{[e_k \cdot (\eta_k - \eta_{k+1})]^2}{|z_k^0 - z_{k+1}^0|} + \frac{\varepsilon}{2} \frac{|\eta_k - \eta_{k+1}|^2}{|z_k^0 - z_{k+1}^0|},

Each \( \eta_k = \eta_k^+ + \eta_k^\| \). The above calculations show that

\[ \eta_k = n(\theta_k^0) \cdot n(\theta_k^0) + \theta_k^1 \cdot t(\theta_k^0) + O(\varepsilon) \]

with

\[ \theta_k^1 = \sum_{j=0}^{q-1} c_j^k n'(\theta_j^0) + d_j^k n(\theta_j^0), \quad k = 1, \ldots, q - 1 \]

which is solved from (19). Notice also that

\[ \xi_k(\theta_k^0) = \left[ -\frac{1}{2} \theta_k^1 + n'(\theta_k^0) \theta_k^0 + m(\theta_k^0) \right] \cdot n(\theta_k^0) + n(\theta_k^0) \theta_k^1 \cdot t(\theta_k^0) + O(\varepsilon). \]

So

\[ \xi_k^\| = \sum_{j=0}^{q-1} c_j^k n(\theta_k^0) n'(\theta_j^0) + d_j^k n(\theta_j^0) n(\theta_j^0) \]

and

\[ \xi_k^+ = -\frac{1}{2} \sum_{i,j=0}^{q-1} (c_j^k n'(\theta_j^0) + d_j^k n(\theta_j^0)) \cdot (c_i^k n'(\theta_i^0) + d_i^k n(\theta_i^0)) \]

\[ + \sum_{j=0}^{q-1} c_j^k n'(\theta_j^0) n'(\theta_j^0) + d_j^k n'(\theta_j^0) n(\theta_j^0) + m(\theta_k^0). \]

Substitute them in \( P_q \) and we get the \( \varepsilon^2 \)-term by

\[ \sum_{k=0}^{q-1} \left\{ 2 \sin \frac{\pi}{q} k - \frac{1}{16 \sin^3 \frac{\pi}{q}} \left[ (1 - \cos \frac{2\pi}{q}) (n(\theta_k^0) + n(\theta_{k+1}^0)) + \sin \frac{2\pi}{q} (\theta_k^1 - \theta_{k+1}^1) \right]^2 \right. \]

\[ + \frac{n^2(\theta_k^0) + \theta_k^1}{4 \sin \frac{\pi}{q}} \left. \left[ \cos \frac{2\pi}{q} + 2 \sin \frac{2\pi}{q} \left[ n(\theta_k^0) \theta_k^1 - n(\theta_{k+1}^0) \theta_{k+1}^1 \right] \right] \right\}. \]

(23)

Substituting for some computable matrix \( A_{q \times q}, B_{q \times q}, C_{q \times q} \) we have

\[ D_q(\theta, \varepsilon) = \sum_{i,j=0}^{q-1} \langle n'(\theta_i), B_{ij} \rangle \cdot n(\theta_j) + \langle n'(\theta_i), C_{ij} \rangle \cdot n'(\theta_j) \]

\[ + \langle n(\theta_i), A_{ij} \rangle \cdot n(\theta_j) + O(\varepsilon). \]

(24)

Recall that \( m^{(q)}(s) = \sum_{k \in \mathbb{Z}} m_{kq} \exp(ik\theta) \) and \( n(s) = \sum_{k \in \mathbb{Z}} n_k \exp ik\theta \), we have
\begin{align*}
n(\theta_i) n(\theta_j) &= \sum_{k,l \in \mathbb{Z}} n_k n_l \exp(i(k + l) \frac{2\pi}{q}) \exp(i[k + l] \theta). \\
n'(\theta_i) n(\theta_j) &= \sum_{k,l \in \mathbb{Z}} n_k n_l k \exp(i(k + l) \frac{2\pi}{q}) \exp(i[k + l] \theta). \\
n'(\theta_i) n'(\theta_j) &= \sum_{k,l \in \mathbb{Z}} n_k n_l k l \exp(i(k + l) \frac{2\pi}{q}) \exp(i[k + l] \theta).
\end{align*}

In the Appendix you can find a detailed calculation of $D^{(2)}$ and $D^{(3)}$.

**Theorem 3.3.** Suppose $\gamma_c$ can preserve 1/2 and 1/3 caustics, then for $q = 2, 3$, the second order estimate can be achieved by

\begin{align}
D_2(\theta) &= \frac{3n'^2(\theta + \pi) - n'^2(\theta) - n'(\theta) n'(\theta + \pi)}{2} \\
&= 2n'^2(\theta)
\end{align}

(25)

because $n^{(2)} = \text{const}$ leads to $n'(\theta + \pi) + n'(\theta) = 0$ for all $\theta \in \mathbb{T}$ and

\begin{align}
D_3(\theta) &= \sum_{k,l \in \mathbb{Z}} \frac{1}{13\sqrt{3}} \left[ 4 - 2e^{\frac{2\pi}{l} k} - 2e^{\frac{2\pi}{l} k} + (3\sqrt{3} - 2)e^{\frac{2\pi}{l} l} + (4 + 3\sqrt{3}l + 36kl)e^{\frac{2\pi}{l}(k+l)} \\
&\quad + (6kl - 2 - 6\sqrt{3}l)e^{\frac{2\pi}{l} k + \frac{2\pi}{l} l} - (2 + 3\sqrt{3}l)e^{\frac{2\pi}{l} l} + (2 + 6\sqrt{3}l + 30kl)e^{\frac{2\pi}{l} k + \frac{4\pi}{l} l} \\
&\quad + (4 - 3\sqrt{3}l + 36kl)e^{\frac{4\pi}{l}(k+l)} \right] n_k n_l \exp(i(k + l) \theta).
\end{align}

(26)

Denote by $q \mathbb{Z} := \{ qn | n \in \mathbb{Z} \}$ for $q \in \mathbb{Z}_+$ and $\mathcal{Z}_{2,3} = \mathbb{Z} \setminus (2\mathbb{Z} \cup 3\mathbb{Z})$, then due to a simple arithmetic deduction, we get

**Lemma 3.4.** $\mathcal{Z}_{2,3} = \{ 6l \pm 1 \mid l \in \mathbb{Z} \}$.

**Corollary 3.5.** For $\gamma_c \in B\mathbb{Z}_2 \cap B\mathbb{Z}_3$, we get conditional 2\textsuperscript{nd} order estimate by

(27)

\begin{align*}
D_2(\theta) &= \sum_{k \geq l} 2n_k n_l c^{(2)}(k, l) \exp(i(k + l) \theta),
\end{align*}

with

\begin{align*}
c^{(2)}(6p + 1, 6q + 1) = 0, \quad &\forall p, q \in \mathbb{Z}, \\
c^{(2)}(6p - 1, 6q - 1) = 0, \quad &\forall p, q \in \mathbb{Z}, \\
c^{(2)}(6p + 1, 6q - 1) = 2(6p + 1)(6q - 1), \quad &\forall p, q \in \mathbb{Z}, \\
c^{(2)}(6p - 1, 6q + 1) = 2(6p - 1)(6q + 1), \quad &\forall p, q \in \mathbb{Z},
\end{align*}

and

(28)

\begin{align*}
D_3(\theta) &= \sum_{k \geq l} 2n_k n_l c^{(3)}(k, l) \exp(i(k + l) \theta),
\end{align*}

with

\begin{align*}
c^{(3)}(6p + 1, 6q + 1) = 0, \quad &\forall p, q \in \mathbb{Z}, \\
c^{(3)}(6p - 1, 6q - 1) = 0, \quad &\forall p, q \in \mathbb{Z},
\end{align*}
$c^{(3)}(6p + 1, 6q - 1) = \frac{1}{13\sqrt{3}} \left[ (36 + 108kl) + i(27(l + k) + 24\sqrt{3}kl) \right], \quad \forall p, q \in \mathbb{Z},$

$c^{(3)}(6p - 1, 6q + 1) = \frac{1}{13\sqrt{3}} \left[ (36 + 108kl) - i(27(l + k) + 24\sqrt{3}kl) \right], \quad \forall p, q \in \mathbb{Z}.

Proof. Due to Lemma 2.9, we get $n^{(2)} = n^{(3)} = 0$, which implies $n_k = 0$ for all $k \in 2\mathbb{Z} \cup 3\mathbb{Z}$. Moreover, $D_2(\theta) = 2n^{(2)}(\theta) = -2n'(\theta)n'(\theta + \pi)$ is naturally $\pi$–periodic and $D_3(\theta) = D^{(3)}_3(\theta)$ due to a detailed computation in Appendix. This lead to the coefficient equalities. □

From previous corollary we also get the following property:

**Lemma 3.6.** Let’s simplify the notation by

$$D_q(\theta) = \sum_{k \geq l} n_k n_l c^{(q)}(k, l) \exp(i(k + l)\theta), \quad (29)$$

then the coefficient $c^{(q)}(k, l) \in \mathbb{C}$ satisfies

$$c^{(2)}(k, l) \# c^{(3)}(k, l).$$

Proof. This is a direct arithmetic observation. □

## 4. The harmonic analysis for $n \in T_2 \cap T_3$

In this section we analyze the harmonic behaviour of (24). Since $n \in T_2 \cap T_3$, then the Fourier coefficients satisfy

$$n_{2l} = n_{3l} = 0, \quad \forall l \in \mathbb{Z}$$

due to Lemma 3.1. Moreover, suppose the Fourier expansion of $D_q(\theta)$ satisfies

$$D_q(\theta) = \sum_{k \in \mathbb{Z}} D_{q, k} e^{ik\theta},$$

then from the second order estimate we get the following harmonic equalities: for all $l \in \mathbb{Z},$

$$D^{(2)}_{2l+1} = 0, \quad (\spadesuit)$$

$$D^{(3)}_{3l+1} = 0, \quad (\clubsuit)$$

$$D^{(3)}_{3l+2} = 0, \quad (\heartsuit)$$

$$4D^{(2)}_{6l} = 3\sqrt{3}D^{(3)}_{6l}. \quad (\diamondsuit) \quad (30)$$

As long as one of previous equalities is failed for non trivial $n(\theta)$, we would get $\gamma_\epsilon \notin BZ_2 \cap BZ_3$ and prove the Projected Theorem.

**Lemma 4.1.** $(\spadesuit), (\clubsuit)$ and $(\heartsuit)$ naturally hold for $n \in T_2 \cap T_3$.

Proof. This is a direct conclusion from Corollary 3.5. □
4.0.1. **the polynomial case: pyramid type harmonic equations.** Suppose \( n(\theta) \) is a trigonometric polynomial with the degree be \( N \), i.e.

\[
n(\theta) = \sum_{|k| \leq N} n_k e^{ik\theta},
\]

then (30) becomes a ‘pyramid’ type equation group with quadratic monomials. Benefit from this structure we can prove the following:

**Theorem 4.2** (Polynomial Avalanche). For even trigonometric polynomial \( n(\theta) \neq 0 \), the deformation boundary couldn’t survive the coexistence of 1/2, 1/3 caustics, as long as \( 0 < \varepsilon \leq \varepsilon_0(n) \).

**Proof.** We can prove this by contradiction. Suppose 1/2, 1/3 caustics coexist, then (30) should hold. Without loss of generality, we can assume \( 6P + 1 \) is the largest integer in \( \mathbb{Z}_{2,3} \) which doesn’t exceed \( N \), due to Lemma 3.6

(31) \quad n_{6p+1} \cdot n_{6q-1} = 0, \quad \forall q, p \in \mathbb{Z}, \quad -P \leq q, p \leq P.

Now if make an opposite pair by \((-a, a)\) of any integer \( a \in \mathbb{Z}_{2,3} \cap [-N, N] \), then we can make the following claim:

**Claim:** There exists only one opposite pair \((-a, a)\) for \( a \in \mathbb{Z}_{2,3} \cap [-N, N] \), such that \( n_{-a} \cdot n_a \neq 0 \).

The proof of this claim is straightforward. Without loss of generality, we assume \( a = 6\alpha + 1 \) \( \alpha \in \mathbb{Z} \). If there exists another pair \((-b, b)\) disapproves this claim, then \( b = 6\beta + 1 \) with \( \beta \in \mathbb{Z} \). For \( b = 6\beta + 1 \), we can get \( n_{6\alpha+1} \cdot n_{-6\beta-1} \neq 0 \); For \( b = 6\beta - 1 \), we can get \( n_{6\alpha+1} \cdot n_{6\beta-1} \neq 0 \); Anyway this disobeys (31) and leads to a contradiction.

Recall that \( n(\theta) \) is even, so \( n_k = n_{-k} \) for all \( k \in \mathbb{Z} \). Due to the claim, there will be only one couple \((-a, a)\), such that \( n_{-a} \cdot n_a \neq 0 \). But from [7] we know that \( n_a + n_{-a} = 0 \) should hold simultaneously. This implies that \( n_a = n_{-a} = 0 \) and \( n(\theta) = 0 \).

4.0.2. **the general analytic case: avalanche caused by a quantitative control of error terms.** In this section we generalize the idea in Theorem 4.2 and prove a similar result for general analytic \( n(\theta) \). Denote by \( C^w(\mathbb{T}, \mathbb{R}, \rho) \) the set of all analytic functions with radius \( \rho \), then it’s a Banach space under the analytic norm \( \|f\| \). The following estimate of the Fourier coefficients holds:

**Lemma 4.3.** If \( f(x) \in C^w(\mathbb{T}, \mathbb{R}, \rho) \), then \( f(x) = \sum_k f_k e^{ikx} \) with

\[
|f_k| \leq \|f\| e^{-|k|\rho}, \quad k \in \mathbb{Z}.
\]

Notice that previous Lemma is not always the optimal estimate of the Fourier coefficients for all functions in \( C^w(\mathbb{T}, \mathbb{R}, \rho) \), so we can use the following procedure to find the slowest decaying coefficient sequence, and define the corresponding modular function.

For \( n(\theta) = \sum n_k e^{ik\theta} \) consisting of infinitely many terms, since Lemma 4.3 is available, then we can pick \( k_1 \in \mathbb{Z} \) be the index satisfying

\[
|n_{k_1}| = \sup \left\{ |n_k| \ | k \in \mathbb{Z} \right\}.
\]

If there are several candidate index, we can choose the one with the greatest absolute value. Based on the same principle, we can choose \( k_2 \in \mathbb{Z} \), such that

\[
|n_{k_2}| = \sup \left\{ |n_k| \ | k \in \mathbb{Z}, |k| > |k_1| \right\}.
\]
Repeat this process we can get a sequence \( \{k_i\}_{i=1}^{\infty} \), and the corresponding coefficient sequence \( \{n_{k_i}\}_{i=1}^{\infty} \) is just the slowest decaying coefficient sequence of \( n(\theta) \).

**Definition 4.4.** We call a smooth decreasing, positive function \( w : [0, +\infty) \to \mathbb{R}^+ \) the modular function of \( n(\theta) \), if \( w(|k_i|) = |n_{k_i}| \) for all \( i = 1, 2, \ldots \).

**Remark 4.5.** Notice that the modular function \( w(x) \) is not uniquely defined, but any two modular functions \( w_1, w_2 \) corresponding to the same \( n(\theta) \) should satisfy:

\[
\lim_{i \to \infty} \frac{w_1(i)}{w_2(i)} = 1.
\]

Moreover, for \( n(\theta) \in C^u(\mathbb{T}, \mathbb{R}) \),

\[
\lim_{i \to \infty} \frac{w(i)}{i} \geq \rho.
\]

For any \( L \in \{k_i'\}_{i=1}^{\infty} \), which is a positive subsequence of \( \{k_i\}_{i=1}^{\infty} \), there exists a maximal \( P \in \mathbb{Z}_+ \), such that

\[
L = 6P + 1 \text{ or } 6P - 1.
\]

Without loss of generality, we just need to consider the first case. The following approximated equations can be derived from (32):

\[
\left| \sum_{k+l=6K, |k|, |l| \leq L} [c^{(2)}(k, l) - c^{(3)}(k, l)] n_k n_l \right| \leq \mathcal{E}, \quad -2P \leq K \leq 2P.
\]

Here we use \( \mathcal{E} \sim O(e^{-w(L)L^2}) \) is a Fourier reminder term. We can use the notation \( c^{(2)}(k, l) = c^{(2)}(k, l) - c^{(3)}(k, l) \) for short.

Denote by

\[
\mathcal{N}_\Diamond^k(K) := \{(k, l) \in \mathbb{Z}_{2,3} \times \mathbb{Z}_{2,3} | k + l = 6K, k > l, |k|, |l| \leq L \}
\]

for \(-2P \leq K \leq 2P\). This Lemma reveals the ‘pyramid’ structure of the main part of (32):

**Lemma 4.6.** For a fixed \( K \in \mathbb{Z} \) with \(|K| \leq 2P\), \( \sharp \mathcal{N}_\Diamond^k(K) = (1 + 2P - |K|) \).

**Proof.** To make \( k + l = 6K, k > l \) is always true. Moreover, we can assume \( k = 6\alpha \pm 1, l = 6\beta \mp 1 \), with \( \alpha, \beta \in \mathbb{Z} \). So we ascribe the problem to

\[
\alpha + \beta = K, \quad -P \leq \beta \leq \alpha \leq P.
\]

So for \( K > 0 \), the number of all possible \( \alpha \) is \( 2P - K + 1 \). For \( K < 0 \), the number of all possible \( \beta \) is \( 2P + K + 1 \). Then we deduce a unified estimate by \( 2P - |K| + 1 \), which is the number for all the possible couple \((\alpha, \beta)\). \( \square \)

**Remark 4.7.** From previous analysis, we can extra get \( \sharp \mathcal{N}_\Diamond^k(K) = \sharp \mathcal{N}_\Diamond^k(-K) \).

Now let’s explore the mechanism how the variables \( n_k, n_l \) relate with each other for \( \mathcal{N}_\Diamond^k(K) \):

**Definition 4.8.** We define the generation of \( \mathbb{Z}_{2,3} \cap [-L, L] \) by

\[
\mathcal{G}(k) := P + 1 - \max \left\{ \frac{|k|}{6}, \frac{|k|}{6} \right\}
\]
Lemma 4.9. For \( K-1 > 0 \), \( \pi_1N_0^L(K-1) \subset \pi_1N_0^L(K) \cup \pi_2N_0^L(K) \). For \( K < 0 \), \( \pi_1N_0^L(K) \subset \pi_1N_0^L(K-1) \cup \pi_2N_0^L(K-1) \). Here \( \pi_i \) is the coordinate projection to the corresponding component, \( i=1,2 \).

Lemma 4.10. \( \forall (k,l) \in N_0^L(K) \) with \( 0 < K \leq 2P \),
\[
G(k)+G(l) = 2P - K + 2, \quad \text{if } k > 0 \text{ and } l > 0
\]
and
\[
G(k) - G(l) = K - 2P, \quad \text{if } l < 0.
\]

We can easily prove Lemma 4.9 and Lemma 4.10 from observation. To prove the following Lemma, let’s assume \( \lim_{i \to \infty} \frac{w(i)}{i^{1/2}} = \infty \) and \( n(\theta) \) be even first:

Lemma 4.11. \( \forall (k,l) \in N_0^L(K) \) with \( -2P < K \leq 2P \) and \( K \neq 0 \),
\[
|e^{(2)-(3)}(k,l)| \cdot |n_k \cdot n_l| \lesssim 2^{2P-|K|} \sqrt{e^{-w(L)}} \cdot L^{2P-|K|} \frac{1}{2}.
\]

Proof. Because \( n(\theta) \) is even, so the modular function \( w(i) \) is even for \( i \in \mathbb{Z} \) and we just need to prove this Lemma for \( 0 < K \leq 2P \). Let’s start from the top level, i.e., \( K = 2P \), if we denote by
\[
\Delta_K = \max \left\{ |e^{(2)-(3)}(k,l)| \cdot |n_k n_l| \right\} (k,l) \in N_0^L(K), \quad 0 < K \leq 2P,
\]
then
\[
\Delta_K \leq L^3 \sqrt{\Delta_{K+1}}
\]
due to the pyramid structure of (32). Iterate this inequality we get (33) for all \( 0 < K \leq 2P \). Due to the symmetry of \( n(\theta) \) we can generalize to \( -2P \leq K < 0 \).

Theorem 4.12 (Analytic Avalanche). For even analytic \( n(\theta) \neq 0 \) with the modular function \( w(x) \) satisfying
\[
\lim_{i \to \infty} \frac{w(i)}{i^{1/2}} = \infty,
\]
there exists \( \varepsilon_0(n) \) depending on \( n(\theta) \), such that for all \( 0 < \varepsilon \leq \varepsilon_0(n) \), the deformative boundary couldn’t survive the coexistence of \( 1/2, 1/3 \) caustics.

Proof. Similar with Theorem 4.12, we make the following approximated claim:

**Claim:** There exists only one opposite pair \( (-a,a) \) for \( a \in \mathbb{Z}_{2,3} \cap [-N,N] \), such that
\[
\inf \{|n_a|, |n_a|\} \geq e^{-\frac{w(L)}{2^{|\beta|/3}}} L^7.
\]

We can prove this claim by contradiction. Without loss of generality, we assume \( a = 6 \alpha + 1 \) \( \alpha \in \mathbb{Z} \). If there exists another pair \( (-b,b) \) disapproves this claim, then \( b = 6 \beta \pm 1 \) with \( \beta \in \mathbb{Z} \). For \( b = 6 \beta + 1 \), we can get \( |n_{6 \alpha + 1} \cdot n_{-6 \beta - 1}| \geq e^{-\frac{w(L)}{2^{|\beta|/3}}} L^7 \); For \( b = 6 \beta - 1 \), we can get \( |n_{6 \alpha + 1} \cdot n_{6 \beta - 1}| \geq e^{-\frac{w(L)}{2^{|\beta|/3}}} L^7 \). Anyway this contradicts with Lemma 4.11 by taking \( L \to \infty \).

Recall that \( n(\theta) \) is even and obeys (9),
\[
2|n_a| = |n_{-a} + n_a| = \sum_{i \neq \pm a} |n_i| \leq \sum_{i \neq \pm a} |n_i| \leq 2 \sum_{i > L} |n_i| + \frac{2 \cdot L^8}{6} e^{-\frac{w(L)}{2^{|\beta|/3}}}
\]
\[
= 2 \sqrt{\varepsilon} + \frac{L^8}{3} e^{-\frac{w(L)}{2^{|\beta|/3}}},
\]
(34)
then
\[ |n(\theta)|_{C^0} \leq \sum_{i \in \mathbb{Z}} |n_i| \leq 4Le^{-w(L)} + \frac{2L^8}{3}e^{-w(L) + 4Le^{-w(L)}} + \frac{2L^8}{3}e^{-w(L) + \frac{2L^8}{3}e^{-w(L)}} \]
and let \( L \to 0, |n(\theta)|_{C^0} \to 0 \), which contradicts with the assumption \(|n|_{C^0} = 1\). \( \Box \)

5. Further comments and heuristic improvement

Here we list some facts that guides our further exploration towards the Projected Conjecture. Recall that \([19]\) and \([23]\) supply as a universal formula for all \( q \geq 2 \), so does Lemma 2.8 and Lemma 2.9. That gives us chance to propose a similar Conjecture:

**the Elliptic Projected Conjecture:** In a \( C^r \ (r = 2, \cdots, \infty, w) \) neighborhood of the circle there is no other billiard domain preserving both 1/3 and 1/5 caustics.

The strategy to prove this elliptic conjecture is more or less the same with the case of 1/2 and 1/3 caustics. But instead, some arithmetic properties related with the harmonics of \( n(\theta) \) will be changed, including the exact form of (32).

Another aspect we could do is to generalize our main conclusion to general analytic function space \( C^\infty(T, \mathbb{R}, \rho) \), or even finite smooth space \( C^r(T, \mathbb{R}) \). The crucial lies in (32), which is a homogeneous quadratic equation group of pyramid type. If we can make a better error control as solving it, we can reduce the decaying speed of the modular function \( w(i) \). Some evidence indeed impies so:

**Lemma 5.1.** If we impose that the coefficients of \( n(\theta) \) in the same generation are equivalent, i.e.
\[ \frac{1}{c}|n_{6P-1}| \leq |n_{6P+1}| \leq c|n_{6P-1}|, \quad \forall P \in \mathbb{Z} \text{ holds for some } c \sim O(1), \]
then we just need
\[ \lim_{L \to \infty} \frac{w(L)}{L^2 \ln L} = \infty. \]

Here we just give the essence for the proof: the condition we impose is actually for reducing the dimension of (32). The similar idea holds, if we impose that more caustics preserve:

Que: In a \( C^r \ (r = 2, \cdots, \infty, w) \) neighborhood of the circle there is no other billiard domain of constant width, and preserving 1/3, 1/5 caustics.

Que: For any decreasing rational sequence \( \{1/q_i\}_{i=1}^\infty \) with \( q_1 = 2 \), there exists a neighborhood of the circle, such that there is no other billiard domain preserving \( \{1/q_i\}_{i=1}^\infty \) caustics.

The former question can be generalized to the case with finitely many caustics preserved; As for the latter one, it can be considered as a generalization of the Theorem in \([1]\).
6. Appendix

6.1. the calculation of $D^{(3)}(\theta)$. This part can be embedded into (26):

$$D^{(3)}(\theta) = \sum_{k=1}^{2} -\frac{\sqrt{3}}{2} \theta_k^2 + \sqrt{3} n'(\theta_0^k) \theta_k + \frac{1}{8\sqrt{3}} \left[ n(\theta_0^k) - n(\theta_0^k) + \sqrt{3} \theta_{k+1} + \sqrt{3} \theta_k \right]^2$$

$$= -\frac{\sqrt{3}}{2} \left( \theta_1^2 + \theta_2^2 \right) + \sqrt{3} \left( n'(\theta_1^0) \theta_1^1 + n'(\theta_2^0) \theta_2^1 \right) + \frac{1}{8\sqrt{3}} \left[ (n(\theta_0^0) - n(\theta_1^0) + \sqrt{3} \theta_1^1)^2 + (n(\theta_0^0) - n(\theta_2^0) + \sqrt{3} \theta_2^1)^2 + (n(\theta_0^0) - n(\theta_0^0) + \sqrt{3} \theta_0^1)^2 \right]$$

$$= -\frac{\sqrt{3}}{2} \cdot 169 \left\{ (n_0, n_1, n_2) \cdot \left( \begin{array}{ccc} 3 & 1 & -4 \\ -4 & \frac{1}{3} & -\frac{4}{3} \\ \frac{1}{3} & \frac{1}{3} & 1 \end{array} \right) \cdot \left( \begin{array}{c} n_0 \\ n_1 \\ n_2 \end{array} \right) + 2(n_0, n_1, n_2) \right\}$$

$$+ (n_0, n_1, n_2) \cdot \left( \begin{array}{ccc} \frac{1}{3} & -\frac{4}{3} & -4 \\ -4 & 3 & 1 \\ 1 & -4 & \frac{1}{3} \end{array} \right) \cdot \left( \begin{array}{c} n_0 \\ n_1 \\ n_2 \end{array} \right) + 2(n_0, n_1, n_2) \right\}$$

$$+ \frac{\sqrt{3}}{13} \left\{ (n_0', n_1', n_2') \cdot \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 12 & 4 \end{array} \right) \cdot \left( \begin{array}{c} n_0' \\ n_1' \\ n_2' \end{array} \right) + (n_0', n_1', n_2') \cdot \left( \begin{array}{c} 0 \frac{1}{\sqrt{3}} 0 \\ 0 0 \frac{1}{\sqrt{3}} \end{array} \right) \right\}$$

$$+ \frac{1}{8\sqrt{3}} \left\{ \left[ (n_0, n_1, n_2) \left( \begin{array}{ccc} 16 & 13 & 4 \\ 13 & 13 & 1 \end{array} \right) \left( \begin{array}{c} n_0 \\ n_1 \\ n_2 \end{array} \right) \right]^2 + \left[ (n_0, n_1, n_2) \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 12 & 4 \end{array} \right) \left( \begin{array}{c} n_0' \\ n_1' \\ n_2' \end{array} \right) \right]^2 \right\}$$

$$+ \left[ (n_0, n_1, n_2) \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 12 & 4 \end{array} \right) \left( \begin{array}{c} n_0' \\ n_1' \\ n_2' \end{array} \right) \right]^2 \left\{ \right\}$$
\[ \begin{align*}
    & = -\frac{\sqrt{3}}{338} \left\{ N \left( \begin{array}{ccc}
    10 & -1 & -3 \\
    -1 & 12 & -16 \\
    \frac{3}{3} & -\frac{2}{3} & 4
    \end{array} \right) + 2N \left( \begin{array}{ccc}
    36 & 0 & -4 \\
    \frac{18}{3} & 0 & \frac{18}{3} \\
    \frac{3}{3} & \frac{3}{3} & \frac{3}{3}
    \end{array} \right) \right\} N' + \\
    & \quad + \frac{\sqrt{3}}{13} \left\{ N' \left( \begin{array}{ccc}
    0 & 0 & 0 \\
    0 & 400 & 256 \\
    0 & 256 & 272
    \end{array} \right) N'' + 2N \left( \begin{array}{ccc}
    0 & 0 & 0 \\
    0 & 16 & 12 \\
    0 & 16 & 16
    \end{array} \right) N' + \right\} N''
\end{align*} \]

\[ \begin{align*}
    & = \frac{1}{1352\sqrt{3}} \left\{ N \left( \begin{array}{ccc}
    456 & -212 & -244 \\
    -244 & 212 & -272 \\
    272 & 212 & 516
    \end{array} \right) + \right\} N' + 2N \left( \begin{array}{ccc}
    0 & 144\sqrt{3} & -120\sqrt{3} \\
    0 & 360\sqrt{3} & 376\sqrt{3} \\
    0 & -504\sqrt{3} & -256\sqrt{3}
    \end{array} \right) N' + \\
    & \quad + \frac{1}{1352\sqrt{3}} \left\{ N \left( \begin{array}{ccc}
    4 & -2 & -2 \\
    -2 & 4 & -2 \\
    -2 & -2 & 4
    \end{array} \right) + \right\} N' + 2N \left( \begin{array}{ccc}
    0 & 3\sqrt{3} & -3\sqrt{3} \\
    0 & 3\sqrt{3} & 6\sqrt{3} \\
    0 & -6\sqrt{3} & -3\sqrt{3}
    \end{array} \right) N' + \\
    & \quad + N' \left( \begin{array}{ccc}
    0 & 0 & 0 \\
    0 & 36 & 30 \\
    0 & 6 & 36
    \end{array} \right) N''
\end{align*} \]

\[ \begin{align*}
    = \sum_{k,l} \left[ n_k n_l V_{(3)}^t(k) A_{(3)} V_{(3)}(l) + i n_k n_l V_{(3)}^t(k) B_{(3)} V_{(3)}(l) \\
    + k n_k n_l V_{(3)}^t(k) C_{(3)} V_{(3)}(l) \right] \exp(i(k+l)\theta) \tag{\star}
    \end{align*} \]

\[ \begin{align*}
    = \sum_{k,l} n_k n_l c_{(3)}(k,l) \exp(i(k+l)\theta)
\end{align*} \]

with the coefficient \( c_{(3)}(k,l) \) denoted by

\[ c_{(3)}(k,l) = \frac{1}{13\sqrt{3}} \left[ 4 - 2e^{i\frac{2\pi}{3}k} - 2e^{i\frac{2\pi}{3}l} + (3\sqrt{3} - 2)e^{i\frac{2\pi}{3}k} + (4 + 3\sqrt{3}l + 36kl)e^{i\frac{2\pi}{3}(k+l)} \right. \\
+ (6kl - 2 - 6\sqrt{3}l)e^{i\frac{2\pi}{3}k + i\frac{2\pi}{3}l} - (2 + 3\sqrt{3}l)e^{i\frac{2\pi}{3}l} + \left. (-2 + 6\sqrt{3}l + 30kl)e^{i\frac{2\pi}{3}k + i\frac{2\pi}{3}l} \right] \\
+ (4 - 3\sqrt{3}l + 36kl)e^{i\frac{2\pi}{3}(k+l)}, \tag{35} \]

and

\[ A_{(3)} = \begin{pmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{pmatrix}, \quad B_{(3)} = \begin{pmatrix} 0 & 3\sqrt{3} & -3\sqrt{3} \\ 0 & 3\sqrt{3} & 6\sqrt{3} \\ 0 & -6\sqrt{3} & -3\sqrt{3} \end{pmatrix}, \quad C_{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 36 & 30 \\ 0 & 6 & 36 \end{pmatrix}, \]

\[ V_{(3)}(k) = \begin{pmatrix} \exp(i\frac{2\pi}{3}) \\ \exp(i\frac{2\pi}{3}) \end{pmatrix}, \]

\[ T = \begin{pmatrix} n(\theta), n(\theta + \frac{2\pi}{3}) \\ n(\theta), n(\theta + \frac{2\pi}{3}) \end{pmatrix} \] and \( N' = (n'(\theta), n'(\theta + \frac{2\pi}{3}), n'(\theta + \frac{2\pi}{3})) \). Recall that

\[ \begin{pmatrix} \theta_1' \\
\end{pmatrix} = \begin{pmatrix} n(\theta_0'), n(\theta_0'), n(\theta_0') \cdot \begin{pmatrix} \frac{1}{3\sqrt{3}} \\ \frac{1}{3\sqrt{3}} \end{pmatrix} + n'(\theta_0'), n'(\theta_0'), n'(\theta_0') \cdot \begin{pmatrix} \frac{1}{3\sqrt{3}} \\ \frac{1}{3\sqrt{3}} \end{pmatrix} \]
and
\[
\theta_2^1 = (n(\theta_0^1), n(\theta_1^0), n(\theta_2^0)) \cdot \left( \begin{pmatrix} 1 \\ -\frac{3\sqrt{3}}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix} + (n'(\theta_0^0), n'(\theta_1^0), n'(\theta_2^0)) \cdot \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \right)
\]

are used in the first few steps of previous computation, which are given by Remark 3.2. Recall that \( k, l \in \mathbb{Z}_{2,3} \) because \( n \in T_2 \cap T_3 \setminus \{0\} \), step (*) actually can be specified to the following properties: Notice that
\[
V_{(3)}(6k + 1) = \begin{pmatrix} 1 \\ \exp(i\frac{2\pi}{3}) \\ \exp(i\frac{4\pi}{3}) \end{pmatrix}, \quad \forall k \in \mathbb{Z}
\]

and
\[
V_{(3)}(6k - 1) = \begin{pmatrix} 1 \\ \exp(i\frac{4\pi}{3}) \\ \exp(i\frac{2\pi}{3}) \end{pmatrix}, \quad \forall k \in \mathbb{Z},
\]

then for an arbitrary \( 3 \times 3 \) matrix
\[
\Xi = \begin{pmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{pmatrix},
\]

we have
\[
V_{(3)}^r(6p + 1) \cdot \Xi \cdot V_{(3)}(6q + 1) = (\chi_{11} + \chi_{23} + \chi_{32}) + (\chi_{21} + \chi_{12} + \chi_{33})e^{i\frac{2\pi}{3}} + (\chi_{31} + \chi_{22} + \chi_{13})e^{i\frac{4\pi}{3}},
\]
\[
V_{(3)}^r(6p - 1) \cdot \Xi \cdot V_{(3)}(6q - 1) = (\chi_{11} + \chi_{23} + \chi_{32}) + (\chi_{21} + \chi_{12} + \chi_{33})e^{i\frac{4\pi}{3}} + (\chi_{31} + \chi_{22} + \chi_{13})e^{i\frac{2\pi}{3}},
\]
\[
V_{(3)}^r(6p + 1) \cdot \Xi \cdot V_{(3)}(6q - 1) = (\chi_{11} + \chi_{22} + \chi_{33}) + (\chi_{21} + \chi_{32} + \chi_{13})e^{i\frac{2\pi}{3}} + (\chi_{31} + \chi_{12} + \chi_{23})e^{i\frac{4\pi}{3}},
\]
\[
V_{(3)}^r(6p - 1) \cdot \Xi \cdot V_{(3)}(6q + 1) = (\chi_{11} + \chi_{22} + \chi_{33}) + (\chi_{21} + \chi_{32} + \chi_{13})e^{i\frac{4\pi}{3}} + (\chi_{31} + \chi_{12} + \chi_{23})e^{i\frac{2\pi}{3}}.
\]

Aforementioned equalities reveal the different summations of modulars of the matrix \( \Xi \). By taking \( \Xi = A_{(3)} + B_{(3)} + C_{(3)} \), we can show that:
\[
c^{(3)}(6p + 1, 6q + 1) = 0, \quad \forall p, q \in \mathbb{Z},
\]
\[
c^{(3)}(6p - 1, 6q - 1) = 0, \quad \forall p, q \in \mathbb{Z},
\]
\[
c^{(3)}(6p + 1, 6q - 1) = \frac{1}{13\sqrt{3}} \left[ 18 - 27i + kl(66 + 24 \exp(i\frac{4\pi}{3})) \right], \quad \forall p, q \in \mathbb{Z},
\]

and
\[
c^{(3)}(6p - 1, 6q + 1) = \frac{1}{13\sqrt{3}} \left[ 18 + 27i + kl(66 + 24 \exp(i\frac{2\pi}{3})) \right], \quad \forall p, q \in \mathbb{Z}.
\]

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