An improvement on the Rado bound for the
centerline depth

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March 1, 2018

Abstract

Let $\mu$ be a Borel probability measure in $\mathbb{R}^d$. For a $k$-flat $\alpha$ consider the value
$\inf \mu(H)$, where $H$ runs through all half-spaces containing $\alpha$. This infimum is
called the half-space depth of $\alpha$.

Bukh, Matoušek and Nivasch conjectured that for every $\mu$ and every $0 \leq k < d$
there exists a $k$-flat with the depth at least $\frac{k+1}{k+d+1}$. The Rado Centerpoint The-
orem implies a lower bound of $\frac{1}{d+1}$ (the Rado bound), which is, in general,
much weaker. Whenever the Rado bound coincides with the bound conjectured
by Bukh, Matoušek and Nivasch, i.e., for $k = 0$ and $k = d - 1$, it is known to be
optimal.

In this paper we show that for all other pairs $(d, k)$ one can improve on the
Rado bound. If $k = 1$ and $d \geq 3$ we show that there is a 1-dimensional line with
the depth at least $\frac{1}{d} + \frac{1}{3d}$. As a corollary, for all $(d, k)$ satisfying $0 < k < d - 1$
there exists a $k$-flat with depth at least $\frac{1}{d+1-k} + \frac{1}{3(d+1-k)}$.

Keywords: Half-space depth, centerflat, centerline, Rado theorem.

MSC classification: 52C35, 52A30, 68U05.

1 Introduction

Let $\alpha$ be a $k$-flat and $\mu$ a Borel probability measure in $\mathbb{R}^d$ ($0 \leq k < d$). Define the
depth of $\alpha$ as follows:

$$\text{depth}_\mu(\alpha) = \inf \{\mu(H) : H \text{ is a closed half-space, } \alpha \subset H\}. $$

Sometimes the depth defined above is called half-space depth or Tukey depth in or-
der order to distinguish it from other commonly used notions of depth. We will write sim-
ply $\text{depth}(\alpha)$ if the measure is clear from the context.

One of the most important results concerning the notion of half-space depth is
the Rado Centerpoint Theorem.

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Theorem (Rado Centerpoint Theorem, [8]). For every Borel probability measure $\mu$ in $\mathbb{R}^d$ there exists a point $x$ such that $\text{depth}(x) \geq \frac{1}{d+1}$.

Bukh, Matoušek and Nivasch proposed the following conjecture, which, if proved, would be a generalization of the Rado Theorem.

Conjecture (Bukh, Matoušek, Nivasch, [2]). Let $(d, k)$ be a pair of integers with $0 \leq k < d$. Then for every Borel probability measure $\mu$ in $\mathbb{R}^d$ there exists a $k$-flat $\alpha$ in $\mathbb{R}^d$ (a centerflat) such that

$$\text{depth}(\alpha) \geq \frac{k + 1}{k + d + 1}. \quad (1)$$

The conjecture is true for $k = 0$ (in this case the conjecture turns exactly into the Rado Theorem), $k = d - 1$ (a trivial case), and $k = d - 2$ (a case settled by Bukh, Matoušek and Nivasch [2] themselves).

A result by Klartag [6] implies that, if $d - k$ is fixed, then for every $\varepsilon > 0$, with $d$ sufficiently large depending on $\varepsilon$, and for every Borel probability measure $\mu$ in $\mathbb{R}^d$ there exists a $k$-flat $\alpha$ in $\mathbb{R}^d$ such that

$$\text{depth}(\alpha) > \frac{1}{2} - \varepsilon.$$

One can see that for $k = 0$ and $k = d - 1$ the constant $\frac{k + 1}{k + d + 1}$ in (1) cannot be increased. This is also the case for $k = 1$, as shown by Bukh and Nivasch [3].

For the purposes of our paper it will be convenient to think about a depth of a flat in terms of projections. If $\mu$ is a Borel probability measure in $\mathbb{R}^d$, and $\alpha$ is a $k$-flat, we will write $\pi_\alpha$ for the orthogonal projection from $\mathbb{R}^d$ onto the $(d - k)$-space $\beta = \alpha^\perp$ (i.e., $\pi_\alpha(\alpha)$ is a single point). Let $\mu_\alpha$ be a projection of $\mu$ along $\alpha$, i.e., a measure in $\beta$ such that for every Borel set $X \subseteq \beta$ one has

$$\mu_\alpha(X) = \mu(\pi_\alpha^{-1}(X)).$$

(Of course, $\mu_\alpha$ is a Borel probability measure in $\beta$.) Then for the point $o = \pi_\alpha(\alpha)$ one has the identity

$$\text{depth}_{\mu_\alpha}(o) = \text{depth}_\mu(\alpha).$$

We also note that a projection of a measure is sometimes called a marginal, see [6].

The Rado Centerpoint Theorem implies that for every $d$, $k$ and $\mu$ as above one can find a $k$-flat $\alpha$ such that

$$\text{depth}(\alpha) \geq \frac{1}{d - k + 1}. \quad (2)$$

(In fact, such a $k$-flat exists in any $k$-dimensional direction.) The bound of (2) will be called the Rado bound.

In this paper we prove that for $k = 1$ the Rado bound (2) is not optimal, except for the case $d = 2$. Namely, we have the following result:

Theorem 1. For every $d \geq 3$ and for every Borel probability measure $\mu$ in $\mathbb{R}^d$ there exists a $(1$-dimensional) line $\ell$ with

$$\text{depth}(\ell) \geq \frac{1}{d} + \frac{1}{3d^3}.$$
Theorem \[1\] also implies that the Rado bound \[2\] is optimal only for the cases \(k = 0\) and \(k = d - 1\), as stated in the following Corollary \[2\]. We emphasize that there is still a huge gap between the inequality \(3\) we were able to prove, and the conjectured inequality \(\[\]\).

**Corollary 2.** For every \(d \geq 3\), every \(k\) such that \(1 \leq k \leq d - 2\) and every Borel probability measure \(\mu\) in \(\mathbb{R}^d\) there exists a \(k\)-dimensional flat \(\alpha\) with

\[
\text{depth}(\alpha) \geq \frac{1}{d - k + 1} + \frac{1}{3(d - k + 1)^3}. \tag{3}
\]

**Reduction to Theorem \[1\].** Choose an arbitrary \((k - 1)\)-dimensional flat \(\beta\). After projecting along \(\beta\) onto \(\mathbb{R}^{d-k+1}\) we can apply Theorem \[1\]. Namely, we conclude that there is a line \(\ell \subset \mathbb{R}^{d-k+1}\) such that

\[
\text{depth}_{\mu_{\beta}}(\ell) \geq \frac{1}{d - k + 1} + \frac{1}{3(d - k + 1)^3}.
\]

To finish the proof it is enough to put

\[\alpha = \pi_{\beta}^{-1}(\ell)\.

In the rest of the paper we prove Theorem \[1\]. The body of the argument is contained in Sections 2–4. Sections 5–9 incorporate the proofs of the technical statements declared in Section 2.

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## 2 Geometric part: statements

First, it will be convenient for us to prove Theorem \[1\] for the \((d + 1)\)-dimensional space rather than for the \(d\)-dimensional. Next, we aim for a proof by contradiction. Therefore we assume that for every one-dimensional direction \(\ell\) no point of the \(d\)-dimensional plane \(\ell^\perp\) has depth (with respect to the projected measure \(\mu_{\ell}\)) \(\frac{1}{d+1} + \frac{1}{3(d+1)^3}\) or greater. For brevity, we will write

\[a_0 = a_0(d + 1) = \frac{1}{d + 1} + \frac{1}{3(d + 1)^3}.

### 2.1 Nice measures

A Borel probability measure \(\mu\) in a Euclidean \(d\)-space \(V\) will be called a nice measure if it has a density function \(f_\mu : V \rightarrow \mathbb{R}\) satisfying the following properties:

1. \(f_\mu\) is continuous.
2. \(f_\mu(x) > 0\) for every \(x \in V\).
3. There exist \(C_1, C_2 > 0\) such that \(f_\mu(x) < C_1 e^{-C_2|x|}\) for every \(x \in V\).

We supply the space \(\mathcal{M}(V)\) of nice measures in \(V\) with a metric — the \(L^1\) distance between density functions:

\[
\|\mu - \mu'\| = \|f_\mu - f_{\mu'}\|_{L^1} = \int_V |f_\mu(x) - f_{\mu'}(x)| \, dx.
\]
Let $Iso(V)$ be the group of all isometries of $V$. Then every element $F \in Iso(V)$ admits a natural push-forward $F_*: \mathcal{M}(V) \to \mathcal{M}(V)$. Namely, we define the measure $F_* (\mu)$ via

$$F_* (\mu) (X) = \mu (F^{-1} (X)) \quad \text{for every Borel set } X \subseteq V.$$ 

Recall that $Iso(V)$ has a natural topology. Every $F \in Iso(V)$ can be represented as $F(x) = Ax + v$, where $A \in O(V)$, $v \in V$, and the convergence $F \to Id$ is equivalent to the simultaneous convergence $A \to Id$ and $v \to 0$.

In the next proposition (Proposition 3) we collect the most important facts about nice measures that we will use in the paper. We omit the proof, as it is plainly standard.

**Proposition 3.** The following assertions hold:

1. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$. Then
   $$\lim_{F \to Id} \| \mu - F_* (\mu) \| = 0,$$
   where $F$ runs through $Iso(\mathbb{R}^d)$.

2. Let $\alpha \subset \mathbb{R}^d$ be a $k$-flat, where $k < d$, $\mu \in \mathcal{M}(\mathbb{R}^d)$. Then $\mu | \alpha$ is a nice measure.

3. Let $\alpha \subset \mathbb{R}^d$ be a fixed $k$-flat. Consider $\mu : \mathcal{M}(\mathbb{R}^d) \to \mathcal{M}(\alpha^\perp)$ as a function of $\mu$. Then $\mu | \alpha$ is continuous in $\mathcal{M}(\alpha^\perp)$.

4. Consider $\operatorname{depth}_\mu (x) : \mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ as a function of $\mu$ and $x$. Then $\operatorname{depth}_\mu (x)$ is continuous in $\mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d$ (with the product topology).

In the proof of Theorem 1 we will assume that $\mu$ is nice. Then the case of an arbitrary $\mu$ will follow from a standard approximation argument.

### 2.2 Properties of Tukey medians

Write

$$\mathcal{M}_a(V) = \{ \nu \in \mathcal{M}(V) : \sup_{x \in V} \operatorname{depth}_\nu (x) < a \}. $$

We will consider $\mathcal{M}_a(V)$ as a subspace of $\mathcal{M}(V)$ with the induced topology. Note that the Rado Centerpoint Theorem implies $\mathcal{M}_a(V) = \emptyset$ for all $a \leq \frac{1}{d+1}$.

Recall the notation $a_0 = \frac{1}{d+1} + \frac{1}{3(d+1)^3}$. In order to prove Theorem 1 in $\mathbb{R}^{d+1}$ by contradiction we have to assume that

$$\mu | \ell \in \mathcal{M}_{a_0}(\ell^\perp)$$

for every line $\ell \in \mathbb{R}^d$.

For $\nu \in \mathcal{M}(V)$ we will call a point $o \in V$ a Tukey median of $\nu$ if $\operatorname{depth}_\nu (o) = \sup_{x \in V} \operatorname{depth}_\nu (x)$. The following Lemma 4 concerns the properties of Tukey medians. The idea of such statement is certainly not new, see, for instance, [1].

**Lemma 4.** Let $V$ be a Euclidean $d$-space, $a \in \left( \frac{1}{d+1}, \frac{1}{d} \right)$. Then the following assertions hold.

1. Let $\nu \in \mathcal{M}_a(V)$, $o$ be a Tukey median of $\nu$. Then there exists $d+1$ half-spaces $H_1, H_2, \ldots, H_{d+1} \subset V$ such that $o \in \partial H_1$, $\bigcap_{i=1}^{d+1} H_i = o$, and $\nu(H_i) = \operatorname{depth}_\nu (o)$. 


2. For every $v \in \mathcal{M}_a(V)$ the Tukey median of $v$ is unique.

3. Let $o(v)$ denote the Tukey median of $v$ for every $v \in \mathcal{M}_a(V)$. Then depth$_1(o(v))$ depends continuously on $v$.

4. $o(v)$ is a continuous function of $v$ if $v$ runs through $\mathcal{M}_a(V)$.

The proof will be given in Section 5.

Define

$$\mathcal{M}^o_a(V) = \{v \in \mathcal{M}_a(V) : o(v) = 0\}.$$ 

I.e., $\mathcal{M}^o_a(V)$ contains all those measures in $\mathcal{M}_a(V)$ whose Tukey median is the origin.

By Lemma 4, assertion 2, for every $v \in \mathcal{M}_a(V)$ there exists a unique translation $F$ such that the translated measure $v^o = F(v)$ belongs to $\mathcal{M}^o_a(V)$ (namely, $F$ is the translation by $-o(v)$).

Lemma 4, assertion 4, and Proposition 3, assertions 1 and 4, imply that $v^o$ is a continuous function of $v$.

2.3 Geometry of measures in $\mathcal{M}^o_a(V)$

Let $V$ be a Euclidean $d$-space. Denote by $T(V)$ the set of all unordered $(d+1)$-tuples $(e_1, e_2, \ldots, e_{d+1}), e_i \in V$ such that

$$\dim \text{conv}\{e_1, e_2, \ldots, e_{d+1}\} = d; \quad 0 \in \text{int} \text{conv}\{e_1, e_2, \ldots, e_{d+1}\}.$$ 

Of course, $T(V)$ can be considered as a topological space with the topology induced from $V^{d+1}/\mathcal{S}_{d+1}$, where $\mathcal{S}_{d+1}$ is the symmetric group with the usual action on the $(d+1)$-th power of $V$.

The main geometric statement of the paper is provided below in Lemma 5. We say that an isometry $F : V_1 \to V_2$ between two Euclidean $d$-spaces is linear if it maps the origin of $V_1$ to the origin of $V_2$. Every such isometry naturally defines a push-forward map $F^* : \mathcal{M}^o_a(V_1) \to \mathcal{M}^o_a(V_2)$ and the map $F : T(V_1) \to T(V_2)$ resembling the usual notation:

$$F([e_1, e_2, \ldots, e_{d+1}]) = [F(e_1), F(e_2), \ldots, F(e_{d+1})].$$

**Lemma 5 (Structural Lemma).** Let $a \in \left(\frac{1}{d+1}, a_0\right)$. Then for every Euclidean $d$-space $V$ one can define a continuous map

$$T^V_a : \mathcal{M}^o_a(V) \to T(V)$$

such that

1. $T^V_a$ is continuous.

2. For any two Euclidean $d$-spaces $V_1$ and $V_2$ and any linear isometry $F : V_1 \to V_2$ the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{M}^o_a(V_1) & \xrightarrow{F^*} & \mathcal{M}^o_a(V_2) \\
\downarrow{T^V_1} & & \downarrow{T^V_2} \\
T(V_1) & \xrightarrow{F} & T(V_2)
\end{array}$$
The intuition behind the Structural Lemma can be roughly explained considering, in some sense, a “typical” representative of \( \mathcal{M}_d^2(V) \) for some \( a \in \left( \frac{1}{d+1}, a_0 \right) \). Let \((e_1, e_2, \ldots, e_{d+1}) \in \mathcal{T}(V)\), and let

\[
\nu = \frac{1}{d+1}(v_1 + v_2 + \ldots + v_{d+1}),
\]

where \( v_i \) is a nice measure sharply concentrated around \( e_i \). (We also require \( a(v) = 0 \), but this can also be settled by the particular choice of \( v_i \).) It is not hard to check that depth

\[
\nu(0) \text{ is close to } \frac{1}{d+1},
\]

so, in particular, \( \nu \in \mathcal{M}_d^2(V) \). If we were restricted only to this type of measures, then it would have been natural to put

\[
T^V_\nu(e_1, e_2, \ldots, e_{d+1}).
\]

Our goal will be to formalize this intuition showing that every \( \nu \in \mathcal{M}_d^2(V) \) behaves, in a certain sense, similarly to the described measures.

\section{Topological part}

We write \( \mathbb{R}P^d \) for the space of all one-dimensional directions in \( \mathbb{R}^{d+1} \) as this is indeed the real projective space.

Let \( \xi = (E, \mathbb{R}P^d, p) \) be the tautological quotient bundle \cite[§2.2.3]{4} over \( \mathbb{R}P^d \). I.e., the total space \( E \) can be written as a quotient space

\[
E = \{(u, v) : u \in S^d, v \in \mathbb{R}^{d+1}, \langle u, v \rangle = 0 \}/\sim,
\]

where the equivalence relation \( \sim \) is defined by \( (u, v) \sim (-u, v) \), and the projection \( p : E \to \mathbb{R}P^d \) is as follows:

\[
p(u, v) = \ell \iff \ell \parallel u.
\]

There is a natural way to identify the fiber \( p^{-1}(\ell) \) and the hyperplane \( \ell^\perp \): a point \( (u, v) \sim (-u, v) \), where \( u \parallel \ell \) is identified with the point \( v \in \ell^\perp \). (The last inclusion is due to the property \( \langle u, v \rangle = 0 \).)

Let us state and prove the key lemma of the topological part. After presenting the formal proof we will also give its less formal (but also non-rigorous) interpretation.

The term \textit{k-fold covering} of \( \mathbb{R}P^d \) will refer to a projection \( p : X \to \mathbb{R}P^d \) appearing in the common definition of a covering space (see, for example, \cite[§1.3]{5}), and \( k \) denotes the cardinality of each set \( p^{-1}(\ell) \), where \( \ell \) runs through \( \mathbb{R}P^d \).

\textbf{Lemma 6.} Let \( d \geq 2 \), \( \xi = (E, \mathbb{R}P^d, p) \) be the tautological quotient bundle as above. Then there is no space \( E' \subset E \) such that

\begin{enumerate}
  \item The projection \( p|_{E'} \) is a \((d+1)\)-fold covering of \( \mathbb{R}P^d \) by \( E' \).
  \item For every \( \ell \in \mathbb{R}P^d \) one has \( E' \cap p^{-1}(\ell) \in \mathcal{T}(p^{-1}(\ell)) \).
\end{enumerate}

\textbf{Proof.} Suppose that such \( E' \) exists. Since \( \pi_1(\mathbb{R}P^d) = \mathbb{Z}_2 \), then \( E' \) splits into 1-fold and 2-fold subcovers.

We will show that in each case \( \xi \) admits a non-vanishing section. That is, there exists \( F \subset E \) such that \( p|_F \) is a homeomorphism from \( F \) to \( \mathbb{R}P^d \), and for every \( \ell \in \mathbb{R}P^d \) the point \( F \cap p^{-1}(\ell) \) is not the origin of the fiber \( p^{-1}(\ell) \).
If there is a 1-fold subcover $F \subset E$, then it is a non-vanishing section itself. If there is a 2-fold subcover $G \subset E'$, then for each $\ell \in \mathbb{R}P^d$ define

$$f(\ell) = g_1 + g_2,$$

where $\{g_1, g_2\} = G \cap p^{-1}(\ell)$. 

(The sum of vectors is well defined in the fiber $p^{-1}(\ell)$.) Put

$$F = \{f(\ell) : \ell \in \mathbb{R}P^d\}.$$  

Since $d \geq 2$, and $g_1$ and $g_2$ are two vertices of the simplex $S(\ell)$, which contains the origin inside, the sum $g_1 + g_2$ cannot vanish. Hence $F$ is a non-vanishing section.

Consider the cohomology ring $H^*(\mathbb{R}P^d, \mathbb{Z}_2)$. We have

$$H^*(\mathbb{R}P^d, \mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^{d+1}),$$

where $x \in H^1(\mathbb{R}P^d, \mathbb{Z}_2)$ (see [7, Lemma 4.3]). If $sw(\xi) \in H^*(\mathbb{R}P^d, \mathbb{Z}_2)$ is the Stiefel-Whitney class of the bundle $\xi$, then we have $sw(\xi) = 1 + x + \ldots + x^d$ (see [7] §4, Example 3).

Hence the top ($d$-th) Stiefel-Whitney class of $\xi$ is non-zero, and, consequently, $\xi$ cannot have a non-vanishing section (see [7] §12 and references therein). A contradiction finishes the proof.

\[\square\]

**Remark (non-rigorous “proof” of Lemma 4)**. We reduce the lemma to non-existence of a non-vanishing section of $\xi$ just as above. Assume, for a contradiction, that $\xi$ has a non-vanishing section. Equivalently, there exists an even non-vanishing tangent vector field $v$ for the sphere $\mathbb{S}^d$ (i.e., $v(-x) = v(x)$).

Let us also allow $v$ to have isolated finite-index singularities. An example is the field $v_0$ of unit vectors pointing northwards. The singularities of $v_0$ are the north and the south poles. One can see that the index of $v_0$ at the poles is $\pm 1$. Hence the points with odd index split into an odd number of pairs (each pair consists of two mutually antipodal points). We claim that this property holds not only for $v_0$, but also for all possible vector fields.

Let $\mathcal{C}$ be a centrally symmetric simplicial subdivision of $\mathbb{S}^d$ isomorphic to the $(d+1)$-crosspolytope such that the poles of $\mathbb{S}^d$ do not belong to the $(d-1)$-skeleton of $\mathcal{C}$.

Let $\Sigma$ be a simplex of $\mathcal{C}$, and $u_\Sigma$ be a vector field on $\partial \Sigma$. We assume only that $u_\Sigma$ is tangent to $\mathbb{S}^d$ and has no singularities. Then all continuations of $u_\Sigma$ onto $\Sigma$ have the same sum of indices over all singularities. Denote this sum by $s(\Sigma, u_\Sigma)$.

If a vector field $u$ is defined on $sk_{d-1}(\mathcal{C})$, we write

$$s(u) = \sum_{\Sigma, u \mid \Sigma} s(\Sigma, u \mid \Sigma).$$

(In the summation above we account for each pair $(\Sigma, -\Sigma)$ of antipodal simplices exactly once; the order in which the pair is accounted for does not affect the parity of $s(u)$, because $s(\Sigma, u \mid \Sigma) \equiv s(-\Sigma, u \mid -\Sigma)$ (mod 2) for every even field $u$.) We want to prove that $s(u)$ has to be odd.

We can assume that $u$ coincides with $v_0$ on $sk_{d-2}(\mathcal{C})$; this can be done by a continuous perturbation of $u$.

Let us notice that, whenever a change of $u$ affects only a pair of antipodal $(d-1)$-faces of $\mathcal{C}$, the parity of $s(u)$ remains unchanged. Indeed, if the simplices $\Sigma$ and $\Sigma'$ share one of the chosen faces, then the two terms,

$$s(\Sigma, u \mid \Sigma) \quad \text{and} \quad s(\Sigma', u \mid \Sigma)$$
may either change or not change their parity simultaneously, and all the other terms remain unaffected. Hence, identifying \( u \) with \( v_0 \) face-by-face, we indeed get

\[
s(u) \equiv s(v_0 |_{sk_{d-1}(\xi)}) \equiv 1 \pmod{2}.
\]

**Remark.** The present version of Lemma 6 was suggested by R. Karasev and works for every dimension. For all dimensions, except \( d = 3 \) and \( d = 7 \), one can consider the tangent bundle of \( S^d \) with an embedded \((d + 1)\)-fold cover. This gives \( d + 1 \) affinely independent vector fields in \( S^d \) (and therefore \( d \) linearly independent vector fields), which is also impossible.

## 4 Deduction of Theorem 1 from Lemmas 4, 5 and 6

Let \( \mu \) be a nice measure in \( \mathbb{R}^{d+1} \). If \( \xi = (E, RP^d, p) \) is the tautological quotient bundle over \( RP^d \) as in previous section, we can think of a projected measure \( \mu_\ell \) as a measure in \( p^{-1}(\ell) \).

We will argue by contradiction. If Theorem 1 is false, then there exists a measure \( \mu \in \mathcal{M}(\mathbb{R}^{d+1}) \) such that for every \( \ell \in RP^d \) one has \( \mu_\ell \in \mathcal{M}_0(p^{-1}(\ell)) \).

Let us introduce the notation that we will refer to as the local picture at \( \ell_0 \).

Let \( \ell_0 \) be an arbitrary point of \( \mathbb{R}^p \). Then we can choose a neighborhood \( U \subset \mathbb{R}^p \) of \( \ell_0 \) and a homeomorphism

\[
\phi: p^{-1}(U) \rightarrow U \times V,
\]

where \( V \) is a Euclidean \( d \)-space, such that the restriction \( \phi \mid_{p^{-1}(\ell)} \) is a linear isometry of the spaces \( p^{-1}(\ell) \) and \( \ell \times V \). (We set the origin of \( \ell \times V \) to be the point \( (\ell, 0) \), where \( 0 \) is the origin of \( V \).) Let \( \pi: U \times V \rightarrow V \) be a projection preserving the \( V \)-component.

In the local picture at \( \ell_0 \) define

\[
v_\ell = (\pi \circ \phi \mid_{p^{-1}(\ell)}) \ast (\mu_\ell).
\]

It is clear that \( v(\ell) \in \mathcal{M}_0(V) \). Also, assertions 1–3 of Proposition 3 imply that \( v_\ell \) depends continuously on \( \ell \). By Lemma 4, the Tukey median of \( v_\ell \) is unique and continuous, so the measure \( (v_\ell)^{\ast} \) depends continuously on \( \ell \) as well. Finally, it is evident that

\[
(v_\ell)^{\ast} = (\pi \circ \phi \mid_{p^{-1}(\ell)}) \ast (\mu_\ell^{\ast}).
\]

Write \( a(\ell) = \text{depth}_{\mu_\ell}((0)_{p^{-1}(\ell)}) \). In the local picture at \( \ell_0 \) we have

\[
a(\ell) = \text{depth}_{v_\ell}((0),
\]

where \( 0 \) is the origin of \( V \). Thus, by Lemma 4 and the continuous dependence of \( v_\ell \) on \( \ell \), we conclude that \( a(\ell) \) depends continuously on \( \ell \) in \( U \). In particular, \( a(\ell) \) is continuous at the point \( \ell = \ell_0 \). But \( \ell_0 \) is arbitrary, hence the continuity of \( a(\ell) \) in the entire \( \mathbb{R}^p \) follows.

By compactness of \( \mathbb{R}^p \), the function \( a(\ell) \) attains its maximum, so \( \sup_{\ell \in \mathbb{R}^p} a(\ell) < a_0 \). Therefore one can choose \( a_1 < a_0 \) such that \( (\mu_\ell)^{\ast} \in \mathcal{M}_{a_1}(p^{-1}(\ell)) \) for every \( \ell \in \mathbb{R}^p \).
Define a space $E' \subset E$ using Lemma 5 as follows:

$$E' = \bigcup_{\ell \in \mathbb{R}^{d+1}} T_{\ell}^{P^{-1}(\ell)}((\mu_\ell)^\circ).$$

(Recall that we use the notation $\xi = (E, \mathbb{R}^d, p)$ for the tautological quotient bundle over $\mathbb{R}^d$.)

We claim that the $(d + 1)$-tuple of points $E' \cap p^{-1}(\ell) (= T_{\ell}^{P^{-1}(\ell)}((\mu_\ell)^\circ))$ depends continuously on $\ell$.

Consider the local picture at $\ell_0$. Since $\phi$ is a homeomorphism and $\ell_0$ is arbitrary, it will be sufficient to prove that

$$\phi \left( T_{\ell_0}^{P^{-1}(\ell)}((\mu_\ell)^\circ) \right)$$

depends continuously on $\ell$ at $\ell = \ell_0$. Since the $U$-coordinate of all points in the above expression is $\ell$ (and so depends continuously on $\ell$), we can ignore it. Thus, effectively, we need to prove the continuity of

$$(\pi \circ \phi) \left( T_{\ell_0}^{P^{-1}(\ell)}((\mu_\ell)^\circ) \right).$$

By condition 2 of Lemma 5 we have

$$(\pi \circ \phi) \left( T_{\ell_0}^{P^{-1}(\ell)}((\mu_\ell)^\circ) \right) = (\pi \circ \phi) |_{P^{-1}(\ell)} \left( T_{\ell_0}^{P^{-1}(\ell)}((\mu_\ell)^\circ) \right) = T_{\ell_0}^{V}(\pi \circ \phi |_{P^{-1}(\ell)}) (\mu_\ell)^\circ) = T_{\ell_0}^{V}(V_{\ell}).$$

But $V_{\ell}$ depends continuously on $\ell$, and, by condition 1 of Lemma 5, $T_{\ell_0}^{V}$ is continuous. Therefore $T_{\ell_0}^{V}(V_{\ell})$ indeed depends continuously on $\ell$. The claim is proved.

Consequently, the projection $p |_{E'}$ is a $(d + 1)$-fold covering of $\mathbb{R}^d$ by $E'$. Moreover, by construction, $E' \cap p^{-1}(\ell) \in \mathcal{F}(p^{-1}(\ell))$. Hence $E'$ satisfies the conditions of Lemma 5 which is impossible. The contradiction finishes the proof of Theorem 1.

5 The Tukey median of a measure

This entire section is devoted to the proof of Lemma 5.

If $n \in \mathbb{R}^d$ is a unit vector, denote by $H(n)$ the half-space such that the origin $\mathbf{0}$ belongs to $\partial H(n)$ and $n$ is the outer normal to $\partial H(n)$, i.e., $n$ is orthogonal to $\partial H(n)$ and is directed outwards the half-space $H(n)$.

**Assertion 1.** Without loss of generality assume that $o$ coincides with the origin $\mathbf{0}$.

Define $\mathcal{N}$ to be the set of all unit vectors $n \in \mathbb{R}^d$ such that $\nu(H(n)) = \text{depth}_n(\mathbf{0})$. Clearly, the set $\mathcal{N}$ is compact.

Assume that $\mathbf{0} \notin \text{conv.} \mathcal{N}$. Then conv. $\mathcal{N}$ can be separated from $\mathbf{0}$ by a hyperplane. Or, equivalently, there exists a unit vector $\nu$ such that

$$\inf_{n \in N} \langle n, \nu \rangle > 0.$$ 

This is impossible, since for small enough $\delta > 0$ we have depth$(\delta \cdot \nu) > a$. Indeed, if we translate $\nu$ by a vector $-\delta \cdot \nu$, then depth$(\mathbf{0})$ increases. The reason is that the translation of $\nu$ by $-\delta \cdot \nu$ increases the measure of each $H(n)$ for all $n$ in some open
neighborhood of $\mathcal{N}$ and does not sufficiently decrease the measure of all other half-spaces with $0$ in the boundary. The contradiction shows that $0 \in \text{conv} \mathcal{N}$.

The Carathéodory Theorem implies that $0 \in \text{conv}\{n_1, n_2, \ldots, n_k\}$ ($n_i \in \mathcal{N}, 2 \leq k \leq d+1$).

By the choice of $n_i$ we have

$$\nu\left(\bigcap_{i=1}^{k} H(-n_i)\right) = 0,$$

because the intersection is a $(d+1-k)$-dimensional affine plane through $0$. Equivalently,

$$\nu\left(\bigcup_{i=1}^{k} H(n_i)\right) = 1.$$

But $\nu(H(n_i)) = 1 - \nu(H(-n_i)) = \text{depth}_\nu(0) < a$, so

$$ka > \sum_{i=1}^{k} \nu(H(n_i)) \geq \nu\left(\bigcup_{i=1}^{k} H(n_i)\right) = 1.$$

Thus $k > \frac{a}{d} > d$. This leaves the only option $k = d+1$.

Therefore we have obtained a $(d+1)$-tuple of half-spaces

$$(H(-n_1), H(-n_2), \ldots, H(-n_{d+1}))$$

satisfying the requirements of Assertion 1.

**Assertion 2.** Without loss of generality assume that $0$ and $o \neq 0$ are two different Tukey medians of $\nu$. Since $0$ is a Tukey median of $\nu$, we can choose a $(d+1)$-tuple of half-spaces

$$(H(-n_1), H(-n_2), \ldots, H(-n_{d+1}))$$

as in the proof of Assertion 1. Then for some $i$ we have $o \in \text{int} H(n_i)$.

Consider the half-space $H$ such that

$$o \in \partial H, \quad \partial H \perp n_i, \quad \text{and} \quad H \subset H(n_i).$$

By construction, $\mu(H) < \mu(H(n_i)) = \text{depth}_\nu(0)$. Hence $\text{depth}_\nu(o) < \text{depth}_\nu(0)$. But we assumed that $o$ is a Tukey median, i.e., $\text{depth}_\nu(o) = \sup_{x \in \nu} \text{depth}_\nu(x) = \text{depth}_\nu(0)$, a contradiction.

**Assertion 3.** For every $\nu, \nu' \in \mathcal{M}_a(V)$ and their respective Tukey medians $o, o'$ we have

$$\text{depth}_\nu(o) - \|\nu - \nu'\| \leq \text{depth}_\nu(o) \leq \text{depth}_\nu(o') \leq \text{depth}_\nu(o') + \|\nu - \nu'\| \leq \text{depth}_\nu(o) + \|\nu - \nu'\|. \quad (4)$$

Hence Assertion 3 follows.

**Assertion 4.** Let $\nu \in \mathcal{M}_a(V)$. Denote $o = o(\nu)$.

To prove the assertion, it is enough to prove the following claim: given an arbitrary neighborhood $U$ of $o$ there exists $\varepsilon > 0$ such that $o(\nu') \in U$ whenever $\nu' \in \mathcal{M}_a(V)$ and $\|\nu - \nu'\| < \varepsilon$.

Choose a $(d+1)$-tuple of half-spaces

$$(H_1, H_2, \ldots, H_{d+1})$$
satisfying the requirements of Assertion 1.

Next, choose a \((d + 1)\)-tuple of half-spaces
\[
(H'_1, H'_2, \ldots, H'_{d+1})
\]
such that
\[
\partial H_i \parallel \partial H'_i, \quad H_i \subset \text{int } H'_i, \quad \text{and } S = \bigcap_{i=1}^{d+1} H'_i \subset U.
\]

By construction, \(S\) is a \(d\)-simplex, and \(o \in \text{int } S\).

Denote
\[
\delta = \min_{1 \leq i \leq d+1} \nu(H'_i) - \nu(H_i).
\]

Since \(\nu\) is nice, \(\delta > 0\). We will show that \(\epsilon = \delta / 2\) is sufficient.

Indeed, let \(\nu'\) satisfy the assumptions of our claim. Assume, for a contradiction, that \(o(\nu') \not\in S\). Then for some \(i\) we have \(o(\nu') \in \mathbb{R}^d \setminus H'_i\). Therefore
\[
\text{depth}_{\nu'}(o(\nu')) \leq \nu'(\mathbb{R}^d \setminus H'_i) \leq \nu(\mathbb{R}^d \setminus H'_i) + \|\nu - \nu'\| \leq \nu(\mathbb{R}^d \setminus H_i) - \delta + \|\nu - \nu'\|
\]
\[
< \text{depth}_\nu(0) - \delta + \frac{\delta}{2} = \text{depth}_\nu(0) - \frac{\delta}{2}.
\]

On the other hand, according to (4), \(\text{depth}_{\nu'}(o(\nu')) > \text{depth}_\nu(0) - \delta / 2\). A contradiction shows that \(o(\nu') \in S \subset U\). Thus our claim and Assertion 4 are proved.

6 Generating \((d + 1)\)-tuples of half-spaces

Let \(V\) be a Euclidean \(d\)-space. A \((d + 1)\)-tuple of closed half-spaces
\[
(H_1, H_2, \ldots, H_{d+1}), \quad H_i \subset V
\]
will be called \textit{generating} if it satisfies
\[
\bigcap_{i=1}^{d+1} H_i = \{0\}. \quad \text{(See also the two-dimensional illustration, Figure[1])}
\]

This definition implies, in particular, \(0 \in \partial H_i\) for \(i = 1, 2, \ldots, d + 1\). Another equivalent definition would be as follows: a \((d + 1)\)-tuple of closed half-spaces
\[
(H_1, H_2, \ldots, H_{d+1}), \quad H_i \subset V
\]
is generating if for every \(i\) one has \(0 \in \partial H_i\) and
\[
V \setminus 0 = \bigcup_{i=1}^{d+1} \text{int } H_i.
\]

Note that Lemma[4] assertion 1 in the case \(o = 0\) claims exactly the existence of a generating \((d + 1)\)-tuple with a certain property.

In this section we prove several auxiliary facts concerning generating \((d + 1)\)-tuples of cones.

For a generating \((d + 1)\)-tuple \((H_1, H_2, \ldots, H_{d+1})\) define the \textit{corresponding} \((d + 1)\)-tuple of simplicial cones \((B_1, B_2, \ldots, B_{d+1})\) by
\[
B_i = \bigcap_{1 \leq j \leq d+1 \atop j \neq i} H_j.
\]

Clearly, different \((d + 1)\)-tuples of half-spaces generate different \((d + 1)\)-tuples of simplicial cones.
Lemma 7. Let \((H_1, H_2, \ldots, H_{d+1})\) be a generating \((d+1)\)-tuple of half-spaces in the Euclidean \(d\)-space \(V\), \((B_1, B_2, \ldots, B_{d+1})\) — the corresponding \((d+1)\)-tuple of simplicial cones. Assume that a point \(b_i \in \text{int} \, B_i\) be chosen for each \(i = 1, 2, \ldots, d+1\). Then \(\text{conv}\{b_1, b_2, \ldots, b_{d+1}\}\) is a non-degenerate \(d\)-simplex, and \(0 \in \text{int} \, \text{conv}\{b_1, b_2, \ldots, b_{d+1}\}\).

Proof. For each plane \(\partial H_i\) denote by \(n_i\) the unit normal vector directed outwards \(H_i\). We have
\[0 \in \text{int} \, \text{conv}\{n_1, n_2, \ldots, n_{d+1}\},\]
otherwise
\[
\bigcap_{i=1}^{d+1} H_i \neq \{0\}.
\]

By construction, for every \(i \neq j\) we have
\[
\langle b_i, n_i \rangle > 0, \quad \langle b_i, n_j \rangle < 0.
\]

Now we argue by contradiction. Assuming that the statement of lemma is false, there is a plane \(\alpha\) that separates \(0\) from every \(b_i\). (The separation need not be strict.) If \(n\) is the normal vector to \(\alpha\) pointing towards the open half-space with all \(b_i\), then
\[
\langle b_i, n \rangle \geq 0 \quad \text{for all } i.
\]

Due to (6), we may assume without loss of generality that
\[
n = \lambda_1 n_1 + \lambda_2 n_2 + \ldots + \lambda_d n_d,
\]
where \(\lambda_i \geq 0\) and not all \(\lambda_i\) are zero. Then
\[
\langle b_{d+1}, n \rangle = \lambda_1 \langle b_{d+1}, n_1 \rangle + \lambda_2 \langle b_{d+1}, n_2 \rangle + \ldots + \lambda_d \langle b_{d+1}, n_d \rangle < 0,
\]
a contradiction.

A generating \((d+1)\)-tuple \((H_1, H_2, \ldots, H_{d+1})\) of half-spaces is said to have weight \(a\) with respect to a measure \(v\), if
\[
\min_i v(H_i) = 1 - a.
\]
Then we will write
\[
\text{weight}_\nu(H_1, H_2, \ldots, H_{d+1}) = a \quad \text{or} \quad \text{weight}_\nu(n_1, n_2, \ldots, n_{d+1}) = a,
\]
where \( n_i \) is an outer normal for \( H_i \).

**Lemma 8.** Let \( V \) be a Euclidean \( d \)-space, \( \nu \in \mathcal{M}(V), \varepsilon \in \left(0, \frac{1}{(d+1)(2d+1)}\right) \). Assume that a generating \((d+1)\)-tuple \((H_1, H_2, \ldots, H_{d+1})\) of half-spaces in \( V \) satisfies
\[
\text{weight}_\nu(H_1, H_2, \ldots, H_{d+1}) < \frac{1}{d+1} + \varepsilon.
\]
Then the corresponding \((d+1)\)-tuple of simplicial cones \((B_1, B_2, \ldots, B_{d+1})\) satisfies
\[
1. \quad \sum_{i=1}^{d+1} \nu(B_i) > 1 - (d + 1)\varepsilon.
\]
\[
2. \quad \frac{1}{d+1} - (2d + 1)\varepsilon < \nu(B_i) < \frac{1}{d+1} + \varepsilon.
\]

**Proof.** One can see that the set \( \bigcup_{i=1}^{d+1} B_i \) is exactly the region that is covered exactly \( d \) times by the covering family of half-spaces \( H_i \). In turn, the set \( V \setminus \bigcup_{i=1}^{d+1} B_i \) is covered at most \( d - 1 \) times. Hence
\[
d \sum_{i=1}^{d+1} \nu(B_i) + (d - 1) \left(1 - \sum_{i=1}^{d+1} \nu(B_i)\right) > (d + 1) \left(1 - \frac{1}{d+1} - \varepsilon\right),
\]
or
\[
(d - 1) + \sum_{i=1}^{d+1} \nu(B_i) > d - (d + 1)\varepsilon.
\]
This proves Assertion 1.

Without loss of generality let
\[
\nu(B_1) \leq \nu(B_2) \leq \ldots \leq \nu(B_{d+1}).
\]

Assume that the second inequality in Assertion 2 is false, and \( \nu(B_{d+1}) \geq \frac{1}{d+1} + \varepsilon \). Then
\[
\nu(H_{d+1}) < 1 - \nu(B_{d+1}) \leq 1 - \frac{1}{d+1} - \varepsilon.
\]
Consequently, weight\(_\nu(H_1, H_2, \ldots, H_{d+1}) > \frac{1}{d+1} + \varepsilon \), a contradiction.

Assume that the first inequality in Assertion 2 is false, and \( \nu(B_1) \leq \frac{1}{d+1} - (2d + 1)\varepsilon \). Then, by Assertion 1,
\[
\sum_{i=2}^{d+1} \nu(B_i) \geq \frac{d}{d+1} + d\varepsilon.
\]
Thus \( \nu(B_{d+1}) \geq \frac{1}{d+1} + \varepsilon \), contradicting the second inequality of Assertion 2, which has already been proved. Therefore Assertion 2 is proved completely.

**Lemma 9** (Bijection Lemma). Let \( V \) be a Euclidean \( d \)-space, \( \nu \in \mathcal{M}(V), \varepsilon \in \left(0, \frac{1}{(d+1)(2d+2)}\right) \). Assume that there are two generating \((d+1)\)-tuples of half-spaces in \( V \),
\[
(H_1, H_2, \ldots, H_{d+1}), \quad (H'_1, H'_2, \ldots, H'_{d+1}),
\]

satisfying
\[
\text{weight}_v(H_1, H_2, \ldots, H_{d+1}), \text{weight}_v(H'_1, H'_2, \ldots, H'_{d+1}) < \frac{1}{d+1} + \epsilon.
\]

Then the corresponding \((d + 1)\)-tuples of simplicial cones 
\[
(B_1, B_2, \ldots, B_{d+1}), \quad (B_1, B_2, \ldots, B_{d+1})
\]
satisfy
\[
\nu \left( B_i \cap B'_\sigma(i) \right) > \frac{1}{d+1} - (3d + 2)\epsilon \quad \nu \left( B_i \cap B'_\sigma(j) \right) = 0
\]
for every \(i, j \in \{1, 2, \ldots, d + 1\}, i \neq j\) and some permutation \(\sigma\) of the set \(\{1, 2, \ldots, d + 1\}\).

**Proof.** Consider the bipartite graph \(G\) whose vertex set is \(\{B_1, B_2, \ldots, B_{d+1}\} \cup \{B'_1, B'_2, \ldots, B'_{d+1}\}\) (the parts are \(\{B_1, B_2, \ldots, B_{d+1}\}\) and \(\{B'_1, B'_2, \ldots, B'_{d+1}\}\)), and an edge \((B_i, B'_j)\) is present in \(G\) if and only if \(\nu(B_i \cap B'_j) > 0\).

We claim that \(G\) is a perfect matching. First we prove that \(G\) contains a perfect matching as a subgraph.

Assume that there is no perfect matching in \(G\). Then, up to a permutation of indices, there exist \(k, m \in \mathbb{N}, k < m\), such that \(B_1, B_2, \ldots, B_m\) are connected only with \(B'_1, B'_2, \ldots, B'_k\). Indeed, this is a direct consequence of the Hall's Marriage Theorem [10 Theorem 25.1].

Thus the set \(\bigcup_{i=1}^{m} B_i\) is covered by the set 
\[
\bigcup_{i=1}^{k} B'_i \cup \left( V \setminus \bigcup_{i=1}^{d+1} B'_i \right).
\]
Hence 
\[
\bigcup_{i=m+1}^{d+1} B_i \cup \left( V \setminus \bigcup_{i=1}^{d+1} B_i \right) \cup \bigcup_{i=1}^{k} B'_i \cup \left( V \setminus \bigcup_{i=1}^{d+1} B'_i \right) = V,
\]
and, consequently,
\[
\sum_{i=m+1}^{d+1} \nu(B_i) + \nu \left( V \setminus \bigcup_{i=1}^{d+1} B_i \right) + \sum_{i=1}^{k} \nu(B'_i) + \nu \left( V \setminus \bigcup_{i=1}^{d+1} B'_i \right) \geq 1.
\]
As \(\epsilon < \frac{1}{(d+1)(3d+2)} < \frac{1}{(d+1)^2}\), it is possible to apply Lemma [8] replacing \(\nu(B_i)\), \(\nu(B'_i)\), \(\nu \left( V \setminus \bigcup_{i=1}^{d+1} B_i \right)\) and \(\nu \left( V \setminus \bigcup_{i=1}^{d+1} B'_i \right)\) with their respective upper bounds. This yields
\[
(d + 1 + k - m) \left( \frac{1}{d+1} + \epsilon \right) + 2(d+1)\epsilon > 1,
\]
or
\[
\epsilon > \frac{m - k}{(d+1)(3d+3+k-m)} \geq \frac{1}{(d+1)(3d+2)},
\]
a contradiction. Therefore \(G\) contains a perfect matching.

Up to a permutation of indices we can assume that for each \(i\) an edge \((B_i, B'_i)\) is present in \(G\). Now we aim to show that no other edge of \(G\) exists.

Without loss of generality, assume that \((B_1, B'_1)\) is also an edge of \(G\). Choose a point \(b_1 \in \text{int}B_1 \cap B'_1\). For each \(i > 1\) choose a point \(b_i \in B_i \cap B'_i\).
We have $b_i \in \text{int} B_i$ for each $i$. Hence, by Lemma 7
\[ 0 \in \text{int conv}\{b_1, b_2, \ldots, b_{d+1}\}. \]
On the other hand,
\[ 0 \notin \text{int} H_i' \supset \text{conv}\{b_1, b_2, \ldots, b_{d+1}\}, \]
a contradiction. Therefore $G$ is exactly a perfect matching.

Finally, according to Lemma 8,
\[ \nu(B_i) > \frac{1}{d+1} - (2d + 1)\epsilon, \]
and
\[ \nu(B_i \setminus B_i') \leq \nu(V \setminus \bigcup_{i=1}^{d+1} B_i') \leq (d+1)\epsilon. \]
Hence
\[ \nu(B_i \cap B_i') > \frac{1}{d+1} - (3d + 2)\epsilon. \]

\[ \Box \]

7 The central ray of a simplicial cone

Let us assume for this entire section that $V$ is a Euclidean $d$-space, and a measure $\nu \in \mathcal{M}(V)$ is fixed.

Let $B$ be a $d$-dimensional simplicial cone in $V$ with vertex $0$. For $t \in (0, 1]$ define
\[ \mathcal{H}(B, t) = \{ H : H \text{ is a half-space, } 0 \in \partial H, \text{ and } \nu(H \cap B) \geq t \nu(B) \}. \]

The cone $B$ does not contain any straight line entirely. Therefore we can choose a unit vector $n$ such that $\langle n, b \rangle > 0$ for every $b \in B \setminus 0$.

Consider the central projection $\pi_c$ of $V$ with the center at $0$ onto the plane
\[ \Pi = \{ y : \langle n, y \rangle = 1 \}. \]
Define the probability measure $\nu^*$ in the plane $\Pi$ as follows:
\[ \nu^*(X) = \frac{\nu(\pi_c^{-1}(X) \cap B)}{\nu(B)} \]
for every measurable $X \subseteq \Pi$. This means, for instance, that the support of $\nu^*$ is $\pi_c(B)$, so $\nu^* \in \mathcal{M}(\Pi)$, but we will not need the inclusion.

Let $H$ be a half-space in $V$ such that $\partial H$ is not orthogonal to $n$. Then $H \cap \Pi$ is a half-space in $\Pi$ and
\[ \nu^*(H \cap \Pi) = \frac{\nu(H \cap B)}{\nu(B)}. \]
If $t \geq \frac{d+1}{d}$, then, by the Rado theorem,
\[ \bigcap_{H \in \mathcal{H}(B, t)} (H \cap \Pi) \neq \emptyset. \]
The intersection above is contained in $B$, hence there exists a non-zero vector $e \in B$ such that
\[ \bigcap_{H \in \mathcal{H}(B, t)} (H \cap B) \ni \{ \lambda e : \lambda \geq 0 \}. \]
The above argument for \( t = \frac{d}{d+1} > \frac{d-1}{d} \) implies that the set
\[
C(B) = \bigcap_{H \in \mathcal{H}(B, d)} (H \cap B)
\]
is a convex cone of positive measure. We will call \( C(B) \) the \textit{central cone} of \( B \).

Let \( \mathbb{S}^{d-1} \) be the unit sphere in \( V \) centered at the origin. Define
\[
e(B) = \int_{\mathbb{S}^{d-1} \cap C(B)} x \, dx,
\]
where \( dx \) is the element of the \((d-1)\)-dimensional Lebesgue measure in \( \mathbb{S}^{d-1} \). We will call \( e(B) \) the \textit{central vector} of \( B \), and the ray along \( e(B) \) the \textit{central ray} of \( B \).

Let us state some properties of central cones and central vectors.

**Proposition 10.** \( C(B; \nu) \) and \( e(B; \nu) \) change continuously with a continuous change of \( \nu \) and \( B \).

**Remark.** In order to make Proposition 10 explicit, we need to define a continuous change of a cone. For instance, it is enough to define a basic neighborhood of a convex \( d \)-dimensional cone \( C \subset V \) containing no entire straight line. Namely, choose an arbitrary pair of closed convex cones \( C_{\text{int}} \) and \( C_{\text{ext}} \) such that
\[
C_{\text{int}} \subset \text{int} C_{\text{int}}, \quad C \subset \text{int} C_{\text{ext}}.
\]
Then \( C_{\text{int}} \) and \( C_{\text{ext}} \) span the following basic neighborhood of \( C \): the set of all closed cones \( C' \) satisfying
\[
C_{\text{int}} \subset \text{int} C', \quad C \subset \text{int} C_{\text{ext}}.
\]

The proof of Proposition 10 is routine, and we therefore skip it.

**Lemma 11.** Let \( B \) and \( B' \) be simplicial cones, both with vertex \( 0 \). Suppose that
\[
\max(\nu(B), \nu(B')) \leq \frac{1}{d+1} + \frac{1}{3(d+1)^3},
\]
\[
\nu(B \cap B') \geq \frac{1}{d+1} - \frac{3d+2}{3(d+1)^3}.
\]
Then
\[
B \supseteq C(B') \quad \text{and} \quad B' \supseteq C(B).
\]

**Proof.** The conditions are symmetric for \( B \) and \( B' \), so it is enough to prove that \( B \supseteq C(B') \). Let \( C(B') \setminus B \neq \emptyset \). Then there exists a half-space \( H \) such that
\[
0 \in \partial H, \quad B \subset H, \quad \text{and} \quad C(B') \setminus H \neq \emptyset.
\]

But we have
\[
\nu(H \cap B') \geq \nu(B \cap B') \geq \frac{1}{d+1} - \frac{3d+2}{3(d+1)^3} \geq \frac{d}{d+1} \left( \frac{1}{d+1} + \frac{1}{3(d+1)^3} \right) = \frac{d}{d+1} \nu(B').
\]
Hence \( C(B') \subset H \) by definition of \( C(B') \). A contradiction. \( \square \)
Remark. This is the point of the paper where the strongest assumptions are made. Indeed, the lower bound for \(v(B \cap B')\) will come from Lemma 9. To make this lower bound work in Lemma \(\mathbf{(11)}\) we need \(\varepsilon \leq \frac{1}{3(d+1)^3}\), which is much stronger than the assumption of Lemma 9.

Remark. Notice that the value \(t = \frac{d-1}{d}\) for \(\mathcal{H}(B, t)\) is not optimal. We could use any \(t > \frac{d-1}{d}\), but decreasing \(t\) gives only a minor improvement to our results.

\textbf{Corollary 12.} If \(B, B'\) are as in Lemma \(\mathbf{(11)}\), then
\[e(B) \in B' \quad \text{and} \quad e(B') \in B.\]

\section{Ordered \((d+1)\)-tuples of small weight}

In this section we continue to write \(V\) for a Euclidean \(d\)-space. For brevity, we write
\[a_0 = \frac{1}{d+1} + \frac{1}{3(d+1)^3}.\]

From now on, we start to distinguish ordered and unordered \((d+1)\)-tuples of half-spaces (cones). To emphasize the distinction, we write unordered \((d+1)\)-tuples in circle brackets, and the ordered \((d+1)\)-tuples in square brackets.

Given a measure \(v \in \mathcal{A}^\ast(d)\) and \(a \in (0, 1)\), let \(\mathcal{R}_v(a)\) denote the family of all (unordered) generating \((d+1)\)-tuples of half-spaces \((H_1, H_2, \ldots, H_{d+1})\) satisfying
\[\text{weight}_v(H_1, H_2, \ldots, H_{d+1}) \leq a.\]

Assume, in addition, that \(v \in \mathcal{A}^\ast(V)\) for some \(a \in (0, a_0)\). Starting from the family \(\mathcal{R}_v(a)\), we want to obtain a family of ordered \((d+1)\)-tuples with the following natural properties.

\textbf{(R1)} There is a bijection between \(\mathcal{R}_v(a)\) and \(\mathcal{R}_v^\ast(a)\): every unordered \((d+1)\)-tuple \((H_1, H_2, \ldots, H_{d+1})\) in \(\mathcal{R}_v(a)\) corresponds to a unique ordered \((d+1)\)-tuple
\[[H_{\sigma(1)}, H_{\sigma(2)}, \ldots, H_{\sigma(d+1)}] \in \mathcal{R}_v^\ast(a),\]
where \(\sigma\) is some permutation of \(1, 2, \ldots, d+1\).

\textbf{(R2)} If \([H_1, H_2, \ldots, H_{d+1}], [H'_1, H'_2, \ldots, H'_{d+1}] \in \mathcal{R}_v^\ast(a)\),
\[[B_1, B_2, \ldots, B_{d+1}]\text{ and } [B'_1, B'_2, \ldots, B'_{d+1}]\text{ are the corresponding } (d+1)\text{-tuples of cones},\]
then \(v(B_i \cap B'_i) > \frac{1}{d+1} - \frac{3d+1}{2d(d+1)}\).

\textbf{Lemma 13.} For every \(a \in (\frac{1}{d+1}, a_0)\) and every \(v \in \mathcal{A}^\ast(V)\) there exists a family of ordered \((d+1)\)-tuples of half-spaces \(\mathcal{R}_v^\ast(a)\) satisfying the conditions (R1) and (R2).

\textbf{Proof.} Choose an arbitrary unordered \((d+1)\)-tuple from \(\mathcal{R}_v(a)\) and select an arbitrary order for it, say,
\[[H_1, H_2, \ldots, H_{d+1}].\]

According to Lemma \(\mathbf{9}\) for any unordered \((d+1)\)-tuple
\[(H'_1, H'_2, \ldots, H'_{d+1}) \in \mathcal{R}_v(a)\]
Proof. We start with the observation that\((R1), (R2)\) and the additional condition
\[\|B_i\| \leq 1\]
Then, combining (7) and Lemma 8 for the cones\(a\) because
\[\|\|\|B_i\|\| \leq 1\]
Therefore
\[\|\|\|B_i\|\| > \frac{1}{2}\]
Lemma 9 implies
\[\|\|\|B_i\|\| > \frac{1}{2}\]
Hence \(R_i^*(a)\) meets the required conditions.

Let us also notice that the proof of Lemma 13 implies the following proposition.

**Proposition 14.** If a family \(R_i^*(a)\) satisfies the conditions \((R1)\) and \((R2)\), then any other family satisfying these conditions is obtained by choosing an arbitrary permutation \(\sigma\) and applying it to each element of \(R_i^*(a)\).

**Lemma 15.** Let \(a \in \left(0, \frac{1}{d+1}\right], V \in \mathcal{M}_d^*(V), a_1 \in (a, a_0)\). Assume that the family of ordered \((d+1)\)-tuples \(R_i^*(a_1)\) is chosen. Then for every measure \(v \in \mathcal{M}_d^*(V)\) satisfying \(\|v\| < a_1 - a\) one can choose the family of ordered \((d+1)\)-tuples \(R_i^*(a)\) satisfying \((R1), (R2)\) and the additional condition
\[R_i^*(a) \subseteq R_i^*(a_1).\]

**Proof.** We start with the observation that \(R_i^*(a) \neq \emptyset\). Indeed, this follows from the assumption \(v \in \mathcal{M}_d^*(V)\) and assertion 1 of Lemma 3 Therefore we can choose a \((d+1)\)-tuple \((H_1, H_2, \ldots, H_{d+1}) \in R_i^*(a)\).

Since \(\|v\| < a_1 - a\), we conclude that
\[(H_1, H_2, \ldots, H_{d+1}) \in R_i^*(a_1).\]
Without loss of generality assume that \([H_1, H_2, \ldots, H_{d+1}] \in R_i^*(a_1)\).

Choose \(R_i^*(a)\) so that \([H_1, H_2, \ldots, H_{d+1}] \in R_i^*(a)\).

Assume that \([H_1', H_2', \ldots, H_{d+1}'] \in R_i^*(a)\). Then
\[(H_1', H_2', \ldots, H_{d+1}') \in R_i^*(a_1).\]
But we have \(v'(B_i \cap B_i') \geq \frac{1}{d+1} - \frac{3d+2}{3(d+1)^3}\), and therefore
\[v(B_i \cap B_i') \geq \frac{1}{d+1} - \frac{3d+2}{3(d+1)^3} - (a_1 - a) > 0,\]
because \(a_1 - a < a_0 - a < \frac{1}{3(d+1)^3}\). Hence indeed \([H_1', H_2', \ldots, H_{d+1}'] \in R_i^*(a_1)\).
To state the next lemma note that \( R^*_\nu(a) \) can be treated as a subset of the compact space \( (\mathbb{S}^{d-1})^{d+1} \) with the natural topology. Indeed, an ordered \((d + 1)\)-tuple of half-spaces can be identified with the ordered \((d + 1)\)-tuple of their outer unit normals.

**Lemma 16.** For every \( a \in \left( \frac{1}{\pi}, a_0 \right) \), \( \nu \in \mathcal{M}_\nu(V) \) the set \( R^*_\nu(a) \) is compact.

**Proof.** Consider an arbitrary converging sequence

\[
[H_1^{(j)}, H_2^{(j)}, \ldots, H_{d+1}^{(j)}] \in R^*_\nu(a) \quad (j = 1, 2, \ldots).
\]

Let \([B_1^{(j)}, B_2^{(j)}, \ldots, B_{d+1}^{(j)}] \) be the respective \((d + 1)\)-tuples of simplicial cones. Write

\[
H_i = \lim_{j \to \infty} H_i^{(j)}.
\]

By property (R2) of \( R^*_\nu(a) \), we have

\[
\nu(B_i^{(1)} \cap B_i^{(j)}) \geq \frac{1}{d+1} - \frac{3d+2}{3(d+1)^3}.
\]

Lemma 11 implies

\[
C(B_i^{(1)}) \subseteq B_i^{(j)}.
\]

Therefore, for each \( 1 \leq i, i' \leq d + 1, i \neq i' \) one has

\[
C(B_i^{(1)}) \subset V \setminus H_i^{(j)}, \quad C(B_i^{(1)}) \subset H_{i'}^{(j)}.
\]

For each \( i = 1, 2, \ldots, d + 1 \) take a unit vector \( b_i \) pointing to the interior of \( C(B_i^{(1)}) \). If \( H \) is a half-space with \( 0 \in \partial H \), let \( n(H) \) denote the outer unit normal to \( \partial H \).

As \( b_i \) is separated from the boundary of \( C(B_i) \), there exists \( \delta > 0 \), independent of \( j \), such that for any \( 1 \leq i, i' \leq d + 1, i \neq i' \) one has

\[
\langle n(H_i^{(j)}), b_i \rangle \geq \delta, \quad \langle n(H_{i'}^{(j)}), b_i \rangle \leq -\delta.
\]

Taking the limit, we obtain

\[
\langle n(H_i), b_i \rangle \geq \delta, \quad \langle n(H_{i'}), b_i \rangle \leq -\delta.
\]

By Lemma 7

\[
0 \in \text{int conv}\{b_1, b_2, \ldots, b_{d+1}\}.
\]

Hence, if \( H_i^* \) is a half-space such that \( b_i = n(H_i^*) \), then the \((d + 1)\)-tuple

\[
(H_1^*, H_2^*, \ldots, H_{d+1}^*)
\]

is generating. Let

\[
(B_1^*, B_2^*, \ldots, B_{d+1}^*)
\]

be the corresponding \((d + 1)\)-tuple of simplicial cones. Then one has

\[
n(H_i) \in \text{int } B_i^*.
\]

Lemma 7 immediately yields

\[
0 \in \text{int conv}\{n(H_1), n(H_2), \ldots, n(H_{d+1})\}.
\]

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Hence the \((d + 1)\)-tuple \([H_1, H_2, \ldots, H_{d+1}]\) is generating. Denote the corresponding simplicial cones by \(B_i\) \((i = 1, 2, \ldots, d + 1)\).

By continuity of the weight function,

\[
\text{weight}_{\nu}(H_1, H_2, \ldots, H_{d+1}) \leq a.
\]

Finally, suppose that

\[
[H_1, H_2, \ldots, H_{d+1}] \notin \mathcal{R}^*_{\nu}(a).
\]

Then there is a non-trivial permutation \(\sigma\) such that

\[
[H_{\sigma(1)}, H_{\sigma(2)}, \ldots, H_{\sigma(d+1)}] \in \mathcal{R}^*_{\nu}(a).
\]

Due to the property (R2), for each \(j = 1, 2, \ldots\) we have

\[
\nu(B_{\sigma(i)} \cap B_i(j)) \geq \frac{1}{d + 1} - \frac{3d + 2}{3(d + 1)^3}.
\]

Taking the limit for \(j \to \infty\) yields

\[
\nu(B_{\sigma(i)} \cap B_i) \geq \frac{1}{d + 1} - \frac{3d + 2}{3(d + 1)^3},
\]

which is impossible. A contradiction shows that

\[
[H_1, H_2, \ldots, H_{d+1}] \in \mathcal{R}^*_{\nu}(a).
\]

Thus \(\mathcal{R}^*_{\nu}(a)\) is closed and hence compact.

\[\square\]

9 Proof of Lemma 5

We continue using the notation of the previous section. Also, throughout this section we will assume that \(a \in (\frac{1}{d + 1}, a_0)\) is fixed. We prove Lemma [17] to enable the definition of \(T^V_a\). Lemma [18] shows that the image of \(T^V_a\) is indeed a subset of \(\mathcal{T}(V)\). Property 1 of Lemma [5] follows from Lemma [19] and Corollary [20]. Property 2 follows immediately from the definition, because all the auxiliary objects we use change naturally under isometries.

Let us define a vector function

\[
e_i(v; n_1, n_2, \ldots, n_{d+1}) : \mathcal{M}_a^*(V) \times (\mathbb{S}^{d-1})^{d+1} \rightarrow V.
\]

I.e., the arguments are a measure \(v \in \mathcal{M}_a^*(V)\) and \(d + 1\) unit vectors in \(V\).

In order to do that, choose, according to Lemma [13] the family \(\mathcal{R}^*_\nu(a)\) of ordered \((d + 1)\)-tuples of half-spaces satisfying the properties (R1) and (R2) from the previous section.

Given a unit vector \(n \in V\) denote by \(H(n)\) the half-space in \(V\) such that \(0 \in \partial H(n)\) and \(n\) is the outer unit normal for \(H(n)\). Let us also write \(H_i = H(n_i)\). The definition of the function \(e_i\) will consist of two mutually disjoint cases.

Case 1. \([H_1, H_2, \ldots, H_{d+1}] \notin \mathcal{R}^*_\nu(a)\). Put

\[
e_i(v; n_1, n_2, \ldots, n_{d+1}) = 0.
\]
Case 2. \( [H_1, H_2, \ldots, H_{d+1}] \in \mathcal{R}^\circ_\nu(a) \). Then, writing \( B_i \) for the \( i \)th simplicial cone corresponding to the generating \((d+1)\)-tuple \((H_1, H_2, \ldots, H_{d+1})\), put
\[
e_i(v; n_1, n_2, \ldots, n_{d+1}) = \{a - \text{weight}_{\nu}(H_1, H_2, \ldots, H_{d+1})\}e(B_i; v).
\] (8)

Here \( e(B_i; v) \) denotes, as in Section [7], the central vector of the cone \( B_i \) with respect to the measure \( \nu \).

The definition of \( e_i(v; n_1, n_2, \ldots, n_{d+1}) \) is complete. Let us prove the following continuity property.

**Lemma 17.** Let \( v \in \mathcal{M}^\circ_\nu(V) \) be fixed. Then \( e_i(v; n_1, n_2, \ldots, n_{d+1}) \) is continuous as a function from \((\mathbb{S}^{d-1})^{d+1}\) to \( V \).

**Proof.** Consider the two cases.

**Case 1.** \( e_i(v; n_1, n_2, \ldots, n_{d+1}) = 0 \) holds. Then there are two subcases.

**Subcase 1.1.** \( [H_1, H_2, \ldots, H_{d+1}] \notin \mathcal{R}^\circ_\nu(a) \). By Lemma [16] the set \( \mathcal{R}^\circ_\nu(a) \) is compact. Therefore
\[
[H(n'_1), H(n'_2), \ldots, H(n'_{d+1})] \notin \mathcal{R}^\circ_\nu(a)
\]
for any \((d+1)\)-tuple \([n'_1, n'_2, \ldots, n'_{d+1}]\) of unit vectors close enough to \([n_1, n_2, \ldots, n_{d+1}]\).

Hence
\[
e_i(v; n'_1, n'_2, \ldots, n'_{d+1}) = 0
\]
in some neighborhood of \([n_1, n_2, \ldots, n_{d+1}]\).

**Subcase 1.2.** \( [H_1, H_2, \ldots, H_{d+1}] \in \mathcal{R}^\circ_\nu(a) \) and \( \text{weight}_{\nu}(H_1, H_2, \ldots, H_{d+1}) = a \). Let arbitrary \( \varepsilon > 0 \) be given. Then for any \((d+1)\)-tuple \([n'_1, n'_2, \ldots, n'_{d+1}]\) of unit vectors close enough to \([n_1, n_2, \ldots, n_{d+1}]\) one has
\[
\text{weight}_{\nu}(H(n'_1), H(n'_2), \ldots, H(n'_{d+1})) \geq a - \varepsilon.
\]
Consequently, \( \|e_i(v; n'_1, n'_2, \ldots, n'_{d+1})\| < \varepsilon \). This ends the proof of the subcase, since \( \varepsilon \) is arbitrary.

**Case 2.** \( e_i(v; n_1, n_2, \ldots, n_{d+1}) \neq 0 \). Then
\[
[H_1, H_2, \ldots, H_{d+1}] \in \mathcal{R}^\circ_\nu(a) \quad \text{and} \quad \text{weight}_{\nu}(H_1, H_2, \ldots, H_{d+1}) < a.
\]
Let arbitrary \( \varepsilon > 0 \) be given. Then, if a \((d+1)\)-tuple \([n'_1, n'_2, \ldots, n'_{d+1}]\) of unit vectors is close enough to \([n_1, n_2, \ldots, n_{d+1}]\), one has
\[
\text{weight}_{\nu}(H(n'_1), H(n'_2), \ldots, H(n'_{d+1})) < a.
\]
The property (R2) and the uniqueness part of the property (R1) imply
\[
[H(n'_1), H(n'_2), \ldots, H(n'_{d+1})] \in \mathcal{R}^\circ_\nu(a).
\]
Thus in some neighborhood of \([n_1, n_2, \ldots, n_{d+1}]\) the function \( e_i \) is defined according to (9). But both multipliers in the right-hand side of (9) are continuous (the second one due to Proposition [16]), hence Case 2 follows.

We continue by defining the function \( e_i(v) : \mathcal{M}^\circ_\nu(V) \to V \) as follows:
\[
e_i(v) = \int_{(\mathbb{S}^{d-1})^{d+1}} e_i(v; n_1, n_2, \ldots, n_{d+1}) \, d\nu_1 \, d\nu_2 \ldots d\nu_{d+1}.
\] (9)
Due to Lemma \[17\] the integration is indeed possible. We emphasize that the ordered \((d + 1)\)-tuple
\[
[e_1(v), e_2(v), \ldots, e_{d+1}(v)]
\]
depends on the choice of \(\mathcal{R}^*_a(a)\) from \((d + 1)!\) possible variants, but the unordered \((d + 1)\)-tuple
\[
(e_1(v), e_2(v), \ldots, e_{d+1}(v))
\]
does not. Therefore we have a map \(T^V_a: \mathcal{H}_a^*(V) \to V^{d+1}/\mathfrak{S}_{d+1}\) defined by
\[
T^V_a(v) = (e_1(v), e_2(v), \ldots, e_{d+1}(v)).
\]

Our aim will be to show that \(T^V_a\) satisfies the requirements of Lemma \[5\]. We do it in the next lemmas, leaving aside property 2, which is straightforward from the definition of \(T^V_a\).

**Lemma 18.** For every \(v \in \mathcal{H}_a^*(V)\) one has \(T^V_a(v) \in \mathcal{F}(V)\).

*Proof.* Equivalently, we have to prove
\[
0 \in \text{int} \text{ conv}(e_1(v), e_2(v), \ldots, e_{d+1}(v)). \tag{10}
\]

Assertion 1 of Lemma \[4\] implies that
\[
\text{weight}_v(H'_1, H'_2, \ldots, H'_{d+1}) = \text{depth}_a(0) < a
\]
for some generating \((d + 1)\)-tuple \((H_1, H_2, \ldots, H_{d+1})\). Consequently,
\[
e_i(v; n_1, n_2, \ldots, n_{d+1}) \neq 0.
\]

Let \(B_i\) denote the simplicial cones corresponding to the \((d + 1)\)-tuple \((H_1, H_2, \ldots, H_{d+1})\). Choose an arbitrary \((d + 1)\)-tuple
\[
[H'_1, H'_2, \ldots, H'_{d+1}] \in \mathcal{R}^*_a(a)
\]
satisfying
\[
\text{weight}_v(H'_1, H'_2, \ldots, H'_{d+1}) < a.
\]
If \([B'_1, B'_2, \ldots, B'_{d+1}]\) is the corresponding \((d + 1)\)-tuple of simplicial cones, and \(n'_j\) is the outer unit normal to \(H'_j\), then
\[
e_i(v, n'_1, n'_2, \ldots, n'_{d+1}) \in \text{int} C(B'_i) \subset \text{int} B_i. \tag{11}
\]

Note that \[11\] holds in a set of positive measure, for instance, in some neighborhood of \([n_1, n_2, \ldots, n_{d+1}]\).

If a \((d + 1)\)-tuple of unit vectors \([n'_1, n'_2, \ldots, n'_{d+1}]\) cannot be obtained in such a way
\[
e_i(v; n'_1, n'_2, \ldots, n'_{d+1}) = 0. \tag{12}
\]

Integrating according to \[9\], one obtains that
\[
e_i(v) \in \text{int} B_i,
\]
and, in particular, \(e_i(v) \neq 0\).

Applying Lemma \[7\] we immediately get \[10\].

\[\square\]
Lemma 19. Let $v \in \mathcal{M}_d^*(V)$. Writing $a_1 = \frac{a_0 + a}{2}$, assume that there is an infinite sequence of measures $v_j \in \mathcal{M}_d^*(V) (j = 1, 2, \ldots)$ such that $\|v_j - v\| < \frac{a_0 - a}{2}$ for each $j$, and $\lim_{j \to \infty} v_j = v$. Let $\mathcal{R}_{v_j}^*(a) \subseteq \mathcal{R}_{v}^*(a_1)$ for each $j$, and also $\mathcal{R}_{v}^*(a) \subseteq \mathcal{R}_{v}^*(a_1)$. Then for every sequence $n_1, n_2, \ldots, n_{d+1}$ ($n_k \in \mathbb{R}^{d-1}$) and every $i = 1, 2, \ldots, d + 1$ one has

$$\lim_{j \to \infty} e_i(v_j; n_1, n_2, \ldots, n_{d+1}) = e_i(v; n_1, n_2, \ldots, n_{d+1}).$$

Proof. Consider the two possible cases.

Case 1. $e_i(v; n_1, n_2, \ldots, n_{d+1}) = 0$. If, as before, $H_j$ is a half-space with outer normal $n_j$ and $0 \in \partial H_j$, then there are three subcases.

Subcase 1.1. $[H_1, H_2, \ldots, H_{d+1}] \not\in \mathcal{R}_{v}^*(a_1)$. Then for every $j$

$$[H_1, H_2, \ldots, H_{d+1}] \not\in \mathcal{R}_{v_j}^*(a_1),$$

hence $e_i(v_j; n_1, n_2, \ldots, n_{d+1}) = 0$.

Subcase 1.2. $[H_1, H_2, \ldots, H_{d+1}] \in \mathcal{R}_{v}^*(a_1)$ and $\text{weight}_{v_j}(H_1, H_2, \ldots, H_{d+1}) \geq a$. Let arbitrary $\varepsilon > 0$ be given. Then for every $j \geq j_0$ one has

$$\text{weight}_{v_j}(H_1, H_2, \ldots, H_{d+1}) > a - \varepsilon,$$

hence $\|e_i(v_j; n_1, n_2, \ldots, n_{d+1})\| < \varepsilon$. Since $\varepsilon$ is arbitrary, Subcase 1.2 is proved.

Subcase 1.3. $[H_1, H_2, \ldots, H_{d+1}] \in \mathcal{R}_{v}^*(a_1)$, weight$_{v_j}(H_1, H_2, \ldots, H_{d+1}) < a$, but

$$[H_1, H_2, \ldots, H_{d+1}] \not\in \mathcal{R}_{v}^*(a).$$

We claim that this subcase is impossible. Indeed, (R1) implies that there exists a non-trivial permutation $\sigma$ of $\{1, 2, \ldots, d + 1\}$ such that

$$[H_{\sigma(1)}, H_{\sigma(2)}, \ldots, H_{\sigma(d+1)}] \not\in \mathcal{R}_{v}^*(a) \subseteq \mathcal{R}_{v}^*(a_1).$$

A contradiction to the uniqueness part of (R1) applied to $\mathcal{R}_{v}^*(a_1)$.

Case 2. $e_i(v; n_1, n_2, \ldots, n_{d+1}) \neq 0$. Then

$$\text{weight}_{v_j}(H_1, H_2, \ldots, H_{d+1}) < a.$$

Consequently, for some $j_0$ and every $j > j_0$ one has

$$\text{weight}_{v_j}(H_1, H_2, \ldots, H_{d+1}) < a.$$

Then $[H_1, H_2, \ldots, H_{d+1}] \in \mathcal{R}_{v}^*(a)$. Indeed, otherwise there exists a non-trivial permutation $\sigma$ of $\{1, 2, \ldots, d + 1\}$ such that

$$[H_{\sigma(1)}, H_{\sigma(2)}, \ldots, H_{\sigma(d+1)}] \in \mathcal{R}_{v}^*(a) \subseteq \mathcal{R}_{v}^*(a_1),$$

which leads to a contradiction similarly to Subcase 1.3.

Hence for $j > j_0$ the vector $e_i(v_j; n_1, n_2, \ldots, n_{d+1})$ is defined according to (9). We have

$$\lim_{j \to \infty} \text{weight}_{v_j}(H_1, H_2, \ldots, H_{d+1}) = \text{weight}_{v}(H_1, H_2, \ldots, H_{d+1}),$$

$$\lim_{j \to \infty} e(B_j; v_j) = e(B_j; v)$$

(the last identity is due to Proposition[10]. Hence Case 2 follows. □
Corollary 20. The map $T^V_a(\nu)$ is continuous.

Proof. Choose an arbitrary measure $\nu_0 \in \mathcal{M}_a^\circ(V)$. Let us prove the continuity of $T^V_a(\nu)$ at $\nu = \nu_0$.

Since Cauchy and Heine definitions of continuity are equivalent in our case, we will use the latter. I.e., for an arbitrary sequence $\nu_j \to \nu_0$ ($j = 1, 2, \ldots, \nu_j \in \mathcal{M}_a^\circ(V)$) we will prove

$$\lim_{j \to \infty} T^V_a(\nu_j) = T^V_a(\nu_0).$$

Without loss of generality we can assume that $\|\nu_j - \nu_0\| < a_1 - a$, where $a_1 = \frac{a + a_0}{2}$.

Choose $R^*_\nu_0(a_1)$ and the families

$$R^*_\nu_j(a) \subseteq R^*_\nu_0(a_1) \quad (j = 0, 1, 2, \ldots)$$

satisfying the requirements of Lemma 15.

By Lemma 19 the sub-integral function for $\nu_j$ in (9) converges pointwise to that of $\nu_0$. Also, by definition,

$$\|e_i(\nu_j; n_1, n_2, \ldots, n_{d+1})\| \leq a.$$

Hence, by the Bounded Convergence Theorem (see, for example, [9, Section 4.2]), $e_i(\nu_j)$ converges to $e_i(\nu)$.

\[\square\]

10 Acknowledgements

The authors acknowledge the hospitality of the Alfréd Rényi Mathematical Institute, Budapest, where the research was done. We are thankful to Imre Bárány for making the joint research possible, and for participating in inspiring discussions. We appreciate the extremely useful comments and suggestions by Roman Karasev who changed our understanding of the topological part. The first author also thanks Boris Bukh, Bo’az Klartag, Micha Sharir, and Gabriel Nivasch for useful discussions of the result.

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