VIRASORO TYPE ALGEBRAIC STRUCTURE HIDDEN IN THE CONSTRAINED DISCRETE KP HIERARCHY

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ABSTRACT. In this paper, we construct the additional symmetries of one-component constrained discrete KP (cdKP) hierarchy, and then prove that the algebraic structure of the symmetry flows is the positive half of Virasoro algebra.

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1. Introduction

In the past few years, lots of attention have been given to the study of Kadomtsev-Petviashvili (KP) hierarchy [1, 2] in the field of integrable systems. The Lax pairs, Hamiltonian structures, symmetries and conservation laws, the N-soliton, tau function, the gauge transformation, reductions etc. of the KP hierarchy and its sub-hierarchies have been discussed. A specific interesting aspect of the research of the KP hierarchy is additional symmetry [2, 3, 4, 5, 6, 7]. Additional symmetries are special symmetries which are not contained in the KP hierarchy and do not commute with each other. The additional symmetry flows of the KP hierarchy form an infinite dimensional algebra $W_{1+\infty}$ [4]. Recently, there are several new results about partition function in the matrix models and Seiberg-Witten theory associated with additional symmetries, string equation and Virasoro constraints of the KP hierarchy [3, 9, 10, 11, 12].

There are several sub-hierarchies of the KP by considering different reduction conditions on the Lax operator $L$. One of them is called constrained KP (cKP) hierarchy [13, 14, 15] by setting the Lax operator as $L = \partial + \sum_{i=1}^{m} \Phi_i \partial^{-i} \Psi_i$. Here $\Phi_i$ is an eigenfunction and $\Psi_i$ is an adjoint eigenfunction of the cKP hierarchy. The cKP hierarchy contains a large number of interesting soliton equations. The basic idea of this procedure is so-called symmetry constraint [13, 14, 15]. The negative part of the Lax operator of the constrained KP, i.e. $\sum_{i=1}^{m} \Phi_i \partial^{-i} \Psi_i$, is a generator [2] of the additional symmetry of the KP hierarchy. This observation inspires the study [16] of the additional symmetries of the c KP hierarchy. However, the additional symmetry flows of the KP hierarchy are not consistent with the special form of the Lax operator for the cKP hierarchy autonomously, and so it is highly non-trivial to find a suitable form of additional

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symmetry flows for this sub-hierarchy. The correct additional symmetry flows are given by means of a crucial modification associated with a complicated operator $X_i$ \[^{[16]}\]. Very recently, by a further modification of the additional flows, the additional symmetries of the constrained BKP and constrained CKP hierarchies are given in references \[^{[17, 18]}\]. In addition to the above-mentioned integrable systems, discrete system such as Toda hierarchy also has interesting algebraic structure of additional symmetry\[^{[19]}\].

At the same time the discovery of Hirota’s bilinear difference equation \[^{[20]}\] attracted much interest in looking for other integrable discrete equations and systems. This famous 3-dimensional difference equation is known to provide a canonical integrable discretization for most important types of soliton equations. There are several difference kinds of the discrete hierarchies including differential-difference KP hierarchy, semi-discrete systems, full discrete equations and so on. The differential-difference KP (dKP) hierarchy \[^{[21, 22, 23]}\] defined by the difference operator $\Delta$ is one interesting object of the integrable discrete systems. Note that, the additional symmetry of dKP hierarchy and it’s Sato Bäcklund transformations have been given in reference\[^{[24]}\]. So it is worthy to find the hidden algebraic structure in the constrained discrete KP(cdKP) hierarchy using the additional symmetry flows, which is the main purpose of us. This will be done by following four steps:

1) show the inconsistency between the additional symmetry flows eq.(2.19) of the dKP hierarchy and the Lax form eq.(3.1) of the cdKP hierarchy;
2) modify the additional flows to sort out this inconsistency;
3) prove that the modified additional flows are symmetry of the cdKP hierarchy;
4) identify the algebraic structure of the additional symmetry flows.

The crucial step of this process is to find a suitable modification of the additional flows of the dKP hierarchy such that these flows are consistent with the form of the Lax pair of the cdKP hierarchy.

The paper is organized as follows. Some basic results of of dKP hierarchy and the additional symmetry of dKP hierarchy are summarized in Section 2. After introducing of a definition of the Lax equation of cdKP hierarchy, the additional symmetry flows of the cdKP hierarchy are defined properly by means of a crucial modification from the additional symmetry flows of the dKP hierarchy in Section 3. Next the Virasoro type algebraic structure of these additional symmetry flows is also identified by a straightforward calculation in Section 4. Section 5 is devoted to conclusions and discussions.

2. The dKP hierarchy and it’s additional symmetry

Let us briefly recall some basic facts about the dKP hierarchy according to reference \[^{[22]}\]. Firstly a space $F$, namely

$$F = \{ f(n) = f(n, t_1, t_2, \cdots, t_j, \cdots); n \in \mathbb{Z}, t_i \in \mathbb{R} \}$$

(2.1)

is defined for the space of the discrete KP hierarchy. And $\Lambda, \Delta$ are denote for the shift operator and the difference operator, respectively. Their actions on function $f(n)$ are define for

$$\Lambda f(n) = f(n + 1)$$

(2.2)
and
\[
\triangle f(n) = f(n + 1) - f(n) = (\Lambda - I)f(n)
\]
respectively, where \( I \) is the identity operator.

For any \( j \in \mathbb{Z} \), the Leibniz rule of \( \triangle \) operation is,
\[
\triangle^j \circ f(n) = \sum_{i=0}^{\infty} \binom{j}{i} (\triangle f)(n + j - i) \triangle^{j-i}, \quad \binom{j}{i} = \frac{j(j-1) \ldots (j-i+1)}{i!}.
\]

So an associative ring \( F(\triangle) \) of formal pseudo difference operators is obtained, with the operation “ + ” and “ o ”, namely \( F(\triangle) = \{ R = \sum_{j=-\infty}^{d} f_j(n) \triangle^j, f_j(n) \in R, n \in \mathbb{Z} \} \), and denote \( R_+ := \sum_{j=0}^{d} f_j(n) \circ \triangle^j \) as the positive projection of \( R \) and by \( R_- := \sum_{j=-\infty}^{-1} f_j(n) \circ \triangle^j \), the negative projection of \( R \). The adjoint operator to the \( \triangle \) operator is given by \( \triangle^* \),
\[
\triangle^* \circ f(n) = (\Lambda^{-1} - I)f(n) = f(n - 1) - f(n),
\]
where \( \Lambda^{-1}f(n) = f(n - 1) \), and the corresponding “ o ” operation is
\[
\triangle^{*j} \circ f(n) = \sum_{i=0}^{\infty} \binom{j}{i} (\triangle^{*i} f)(n + i - j) \triangle^{*j-i}.
\]

Then the adjoint ring \( F(\triangle^*) \) to the \( F(\triangle) \) is obtained, and the formal adjoint to \( R \in F(\triangle) \) is defined by \( R^* \in F(\triangle^*) \) as \( R^* = \sum_{j=-\infty}^{d} \triangle^{*j} \circ f_j(n) \). The ” * ” operation satisfies rules as \( (F \circ G)^* = G^* \circ F^* \) for two operators and \( f(n)^* = f(n) \) for a function.

The discrete KP (dKP) hierarchy [22] is a family of evolution equations depending on infinitely many variables \( t = (t_1, t_2, \cdots) \)
\[
\frac{\partial L}{\partial t_k} = [B_k, L], \quad B_k := (L^k)_+, \quad (2.7)
\]
where \( L \) is a general first-order pseudo difference operator(PDO)
\[
L(n) = \triangle + \sum_{j=1}^{\infty} f_j(n) \triangle^{-j}.
\]

\( B_m = (L^m)_+ = \sum_{j=0}^{m} a_j(n) \triangle^j \), i.e. \( (L^m)_+ \) is the non-negative projection of \( L^m \), and \( (L^m)_- = L^m - (L^m)_+ \) is the negative projection of \( L^m \). The Lax operator in eq.(2.8) can be generated by a dressing operator
\[
W(n; t) = 1 + \sum_{j=1}^{\infty} w_j(n; t) \triangle^{-j}.
\]

through
\[
L = W \circ \triangle \circ W^{-1}.
\]
Further the flow equation (2.7) is equivalent to the so-called Sato equation,

$$\partial_t W = -(L^k)_- \circ W. \quad (2.11)$$

Now we introduce the additional symmetry flows\cite{24} of the dKP hierarchy as following. Set

$$\Gamma_\Delta = \sum_{i=1}^{\infty} (it_i \Delta^{i-1} + (-1)^i n \Delta^{i-1}), \quad (2.12)$$

and it is easy to find the following formula

$$[\partial_{t_k} - \Delta^k, \Gamma_\Delta] = 0. \quad (2.13)$$

Define another operator

$$M_\Delta = W \circ \Gamma_\Delta \circ W^{-1}. \quad (2.14)$$

There are the following commutation relations

$$[\Delta, \Gamma_\Delta] = 1, \quad [L, M_\Delta] = 1, \quad (2.15)$$

which can be verified by a straightforward calculation. By using the Sato equation, the isospectral flow of the $M_\Delta$ operator is given by

$$\partial_{t_k} M_\Delta = [L^k_+, M_\Delta]. \quad (2.16)$$

More generally,

$$\partial_{t_k} (M^n_\Delta L^l) = [L^k_+, M^n_\Delta L^l]. \quad (2.17)$$

Based on the above preparation, the additional symmetry flows\cite{24} of the dKP hierarchy are define by their actions on the dressing operator

$$\tilde{\partial}_{t, m} W = -(M^n_\Delta L^l)_- \circ W, \quad (2.18)$$

or equivalently on the Lax operator

$$\tilde{\partial}_{t, m} L = [-(M^n_\Delta L^l)_-, L], \quad (2.19)$$

where $\tilde{\partial}_{t, m}$ denotes the derivative with respect to an additional new variable $t^*_m$. The more general actions of the additional symmetry flows of the dKP are given by

$$\tilde{\partial}_{t, m} M^n_\Delta L^k = [-(M^n_\Delta L^l)_-, M^n_\Delta L^k]. \quad (2.20)$$

As the end of this section, we would like to point out two important technical identities as followings. For two pseudo-difference operators $X_i = f_i \Delta^{-1} g_i, i = 1, 2$, we have

$$X_1 \circ X_2 = X_1 (f_2) \Delta^{-1} g_2 + f_1 \Delta^{-1} X^*_2 (g_1). \quad (2.21)$$

For a pure-difference operator $K$ and two arbitrary smooth functions $(q, r)$, we have

$$[K, q \Delta^{-1} r]_- = K(q) \Delta^{-1} r - q \Delta^{-1} K^*(r). \quad (2.22)$$

The usual versions(non-discrete) of them are given by eq.(A.3) and eq.(A.4) in reference\cite{16}. 
3. ADDITIONAL SYMMETRY FLOWS OF THE CDKP HIERARCHY

The one-component cdKP hierarchy is defined by following Lax equation
\[
\frac{\partial L}{\partial t_l} = [(L^l_+, L)_l, l = 1, 2, \cdots, \tag{3.1}
\]
associated with a special Lax operator
\[
L = (L)_+ + q(t)\Delta^{-1}r(t), \tag{3.2}
\]
and \(q(t)\) is an eigenfunction, \(r(t)\) is an adjoint eigenfunction. The eigenfunction and adjoint eigenfunction \(q(t), r(t)\) are important dynamical variables in the cdKP hierarchy. Using identity eq.(2.22), one can check Lax equation (3.1) is consistent with the evolution equations of the eigenfunction(or adjoint eigenfunction)
\[
q_t = B_m q, \quad r_t = -B_m^* r, \quad B_m = (L^m)_+, \forall m \in \mathbb{N}. \tag{3.3}
\]
Therefore the cdKP hierarchy in eq.(3.1) is well defined. And Eq.(3.1) implies that the Sato equation of cdKP hierarchy is
\[
\partial_l W = -(L^l_+ \circ W), \tag{3.4}
\]
where \(\partial_l = \frac{\partial}{\partial t_l}\).

The central task of this section is to find the additional symmetry flows of the cdKP hierarchy, which can be realized by three steps as we have mentioned in the introduction. As usual calculation of the infinitesimal analysis, the desired action of the additional symmetry flows on the Lax operator \(L\) of the cdKP hierarchy should be
\[
(\tilde{\partial}_t L)_- = \tilde{\partial}_r (q(n, t))\Delta^{-1}r(n, t) + q(n, t)\Delta^{-1}\tilde{\partial}_r (r(n, t)). \tag{3.5}
\]
However, according to the definition eq.(2.19) of the dKP hierarchy, the action of original additional flows of the cdKP hierarchy is expressed by
\[
(\tilde{\partial}_{k,1} L)_- = [(M_\Delta L^k)_+, L]_+ + (L^k)_-. \tag{3.6}
\]
The following lemma and eq.2.22 show that it can not be rewritten as the desired form eq.(3.5) except \(k = 0, 1, 2\). Specifically, \(k = 3\), from \(L^3_\Delta\)
\[
(L^3)_- = L^2(q)\Delta^{-1}r + L(q)\Delta^{-1}L^*r + q\Delta^{-1}L^*_2, \tag{3.7}
\]
we can find the middle term can not be rewritten as the form of eq.(3.5). This demonstrates obviously the inconsistency between the additional symmetry flows eq.(2.19) of the dKP hierarchy and the Lax form eq.(3.1) of the cdKP hierarchy.

Lem 3.1. The Lax operator \(L\) of constrained dKP hierarchy given by (3.2) satisfied the relation of
\[
(L^k)_- = \sum_{j=0}^{k-1} L^{k-j-1}(q)\Delta^{-1}(L^*)^j(r). \tag{3.8}
\]
where \(L(q) = (L)_+(q) + q(t)\Delta^{-1}(r(t)q)\).
Proof. It can be reduced by induction with the help of technical identity in eq. (2.21). We omit it here. □

To overcome the inconsistency, we shall introduce an operator $Y_k$ to modify the additional symmetry eq. (2.19) of the dKP hierarchy. The following lemmas implied by identity (2.21,2.22) are necessary.

**Lemma 3.2.**

$$[X, L] - \sum_{k=1}^{l} [M_k \Delta^{-1} L^*(N_k) - L(M_k) \Delta^{-1} N_k] + [X(q) \Delta^{-1} r - q \Delta^{-1} X^*(r)], \quad (3.9)$$

with definitions (3.2) and

$$X = \sum_{k=1}^{l} M_k \Delta^{-1} N_k. \quad (3.10)$$

We now introduce a pseudo-difference operator $Y_k$,

$$Y_k = \sum_{j=0}^{k-1} \left[ j - \frac{1}{2} (k-1) \right] L^{k-1-j} (q) \Delta^{-1} (L^*)^j (r), k \geq 2, \quad (3.11)$$

$$Y_k = 0, k = -1, 0, 1. \quad (3.12)$$

and have the following property.

**Lemma 3.3.** The action of flows $\partial_l$ of the cdKP hierarchy on the $Y_k$ is

$$\partial_l Y_k = [(L^l)_+, Y_k]_. \quad (3.13)$$

Proof.

$$\partial_l Y_k = \partial_l \sum_{j=0}^{k-1} \left[ j - \frac{1}{2} (k-1) \right] L^{k-1-j} (q) \Delta^{-1} (L^*)^j (r) \quad (3.14)$$

$$= \sum_{j=0}^{k-1} \left[ j - \frac{1}{2} (k-1) \right] \{ \partial_l (L^{k-1-j} (q)) \Delta^{-1} (L^*)^j (r) + L^{k-1-j} (q) \Delta^{-1} \partial_l ((L^*)^j (r)) \} \quad (3.15)$$

by (2.22)

$$= (L^l)_+ \circ \sum_{j=0}^{k-1} \left[ j - \frac{1}{2} (k-1) \right] L^{k-1-j} (q) \Delta^{-1} (L^*)^j (r) \quad (3.16)$$

$$- \left[ \sum_{j=0}^{k-1} \left[ j - \frac{1}{2} (k-1) \right] L^{k-1-j} (q) \Delta^{-1} (L^*)^j (r) \right] \circ (L^l)_+ \quad (3.17)$$

$$= (L^l)_+ \circ \left[ (L^l)_+ \circ \left[ \sum_{j=0}^{k-1} \left[ j - \frac{1}{2} (k-1) \right] L^{k-1-j} (q) \Delta^{-1} (L^*)^j (r) \right] \right] \quad (3.18)$$

$$= (L^l)_+ [Y_k]_. \quad (3.19)$$
The first nontrivial example of (3.11) is given by
\[ Y_2 = [-\frac{1}{2}L(q)\Delta^{-1}r + \frac{1}{2}q\Delta^{-1}L^*(r)], \tag{3.14} \]
for \( k = 2 \). And
\[ [Y_2, L]_- = -(L^3)_- + \frac{3}{2}[L^2(q)\Delta^{-1}r + q\Delta^{-1}(L^*)^2(r)] + [Y_2(q)\Delta^{-1}r - q\Delta^{-1}Y_2^*(r)]. \tag{3.15} \]

Further, the following expression of \([Y_{k-1}, L]_-\) is also necessary to define the additional flows of the cdKP hierarchy.

**Lemma 3.4.** The Lax operator \( L \) of constrained dKP hierarchy and \( Y_{k-1} \) has the following relation,
\[ [Y_{k-1}, L]_- = -(L^k)_- + \frac{k}{2}[q\Delta^{-1}(L^*)^{k-1}(r) + L^{k-1}(q)\Delta^{-1}r] + Y_{k-1}(q)\Delta^{-1}r - q\Delta^{-1}Y_{k-1}^*(r) \tag{3.16} \]

**Proof.**
\[
[Y_{k-1}, L]_- = \sum_{j=0}^{k-2} [j - \frac{1}{2}(k - 2)]L^{k-2-j}(q)\Delta^{-1}L^{*j}(r), L]_- \\
\overset{\text{by (3.9)}}{=} \sum_{j=0}^{k-2} [j - \frac{1}{2}(k - 2)]L^{k-2-j}(q)\Delta^{-1}(L^*)^{j+1}(r) - \sum_{j=0}^{k-2} [j - \frac{1}{2}(k - 2)]L^{k-1-j}(q)\Delta^{-1}L^{*j}(r) \\
+ Y_{k-1}(q)\Delta^{-1}r - q\Delta^{-1}Y_{k-1}^*(r) \\
= -\sum_{j=1}^{k-2} L^{k-1-j}(q)\Delta^{-1}(L^*)^{j}(r) + (\frac{k}{2} - 1)[q\Delta^{-1}(L^*)^{k-1}(r) + L^{k-1}(q)\Delta^{-1}r] \\
+ Y_{k-1}(q)\Delta^{-1}r - q\Delta^{-1}Y_{k-1}^*(r) \\
= -(L^k)_- + \frac{k}{2}[q\Delta^{-1}(L^*)^{k-1}(r) + L^{k-1}(q)\Delta^{-1}r] + Y_{k-1}(q)\Delta^{-1}r - q\Delta^{-1}Y_{k-1}^*(r) \\
\]

Putting together (3.6) and (3.16), we define the additional flows of the cdKP hierarchy as
\[ \partial_k^* L = [-(M\Delta L^k)_- + Y_{k-1}, L], \tag{3.17} \]
where \( \partial_k^* = \overline{\partial_k} \) and \( Y_{l-1} = 0 \), for \( l = 0, 1, 2 \), such that the right-hand side of (3.17) is in the form of (3.5). It must be mentioned that the additional flows \( \partial_k^* \) of cdKP hierarchy is nothing but the additional flows \( \overline{\partial_k} \) of the dKP hierarchy for \( k = 0, 1, 2 \). Generally,
\[
\partial_k^*(M\Delta L^l) = [-(M\Delta L^k)_- + Y_{k-1}, M\Delta L^l]. \tag{3.18}
\]
Now we calculate the action of the additional flows (3.17) on the eigenfunction $q$ and $r$ of the cdKP hierarchy.

**Theorem 3.5.** The acting of additional flows of constrained dKP hierarchy on the eigenfunction $q$ and $r$ are

\[
\partial_k^* q = (M_\Delta L^k)_+(q) + Y_{k-1}(q) + \frac{k}{2} L^{k-1}(q), \tag{3.19}
\]
\[
\partial_k^* r = -(M_\Delta L^k)_+(r) - Y_{k-1}^*(r) + \frac{k}{2} (L^*)^{k-1}(r).
\]

**Proof.** Substitution of (3.16) to negative part of (3.17) shows

\[
\begin{align*}
(\partial_k^* L)_- &= (M_\Delta L^k)_+(q) \Delta^{-1}(r) - q \Delta^{-1}(M_\Delta L^k)_+(r) \\
&+ Y_{k-1}(q) \Delta^{-1}r - q \Delta^{-1}Y_{k-1}^*(r) + \frac{k}{2} q \Delta^{-1}(L^*)^{k-1}(r) + \frac{k}{2} L^{k-1}(q) \Delta^{-1}r.
\end{align*}
\tag{3.20}
\]

On the other side,

\[
(\partial_k^* L)_- = (\partial_k^* q) \Delta^{-1}r + q \Delta^{-1}(\partial_k^* r).
\tag{3.21}
\]

Comparing right hand sides of (3.20) and (3.21) implies the action of additional flows on the eigenfunction and the adjoint eigenfunction (3.19). □

And the case of (3.19) with $k = 3$ is

\[
\begin{align*}
\partial_k^* q &= (M_\Delta L^3)_+(q) + Y_2(q) + \frac{3}{2} L^2(q), \\
\partial_k^* r &= -(M_\Delta L^3)_+(r) - Y_2^*(r) + \frac{3}{2} L^2(r).
\end{align*}
\tag{3.22}
\]

Next we shall prove the commutation relation between the additional flows $\partial_k^*$ of cdKP hierarchy and the original flows $\partial_i$ of it.

**Theorem 3.6.** The additional flows of $\partial_k^*$ commute with all $\partial_i$ flows of the cdKP hierarchy.

**Proof.** According the action of $\partial_k^*$ and $\partial_i$ on the Sato operator $W$ (3.4, 3.17) and (3.13), then

\[
\begin{align*}
[\partial_k^*, \partial_i]W &= -\partial_k^* (L_i^- W) - \partial_i [-(M_\Delta L^k)^- + Y_{k-1}] W \\
&= (-\partial_k^* L_i^-)W - L_i^+ \partial_k^* W - [(M_\Delta L^k)^- - Y_{k-1}] L_i^- W \\
&+ [L_i^+, M_\Delta L^k]^- W - (\partial_i Y_{k-1}) W \\
&= [L_i^-, -Y_{k-1}]^- W + [Y_{k-1}, L_i^-]^- W - (\partial_i Y_{k-1}) W \quad \text{by Jacobi identity (3.13)} \\
&= 0.
\end{align*}
\]

Therefore, the additional flows $\partial_k^*$ commute with all flows $\partial_i$ of the cdKP hierarchy. □

**Remark:** This theorem implies that the additional flows $\partial_k^*$ (3.17) are symmetry flows of the cdKP hierarchy.
4. Virasoro type algebraic structure of the additional symmetry flows

In this section, we shall discuss the algebraic structure of the additional symmetry flows of the cdKP hierarchy. For this end, we need the actions of \( \partial_l^* \) on \( Y_k \) and \( L \).

Taking into account \( Y_{k-1} = 0 \) for \( k = 0, 1, 2 \), then Eqs. (3.19) becomes

\[
\begin{align*}
\partial_0^* q &= (M_\Delta)_+(q), \\
\partial_0^* r &= -(M_\Delta)_+^*(r), \\
\partial_1^* q &= (M_\Delta L)_+(q) + \alpha q, \\
\partial_1^* r &= -(M_\Delta L)_+^*(r) + \beta r, \quad \alpha + \beta = 1, \\
\partial_2^* q &= (M_\Delta L^2)_+(q) + L(q), \\
\partial_2^* r &= -(M_\Delta L^2)_+^*(r) + L^*(r).
\end{align*}
\]

(4.1)

We can rewrite the (4.1) for \( \partial_l^* \) for

\[
\begin{align*}
\partial_l^* q &= (M_\Delta L)_+(q) + \frac{1}{2} l L_l^{-1} q, \\
\partial_l^* r &= -(M_\Delta L)_+^*(r) + \frac{1}{2} l (L^*)_{l-1} r,
\end{align*}
\]

(4.2)

with \( l = 0, 1, 2 \).

Because of \( \partial_l^* = \overline{\partial}_{l,1} \), \( l = 0, 1, 2 \) as mentioned above, we have the following lemma.

**Lemma 4.1.** The additional flows \( \partial_l^* \) of cdKP hierarchy have the following relations for \( l = 0, 1, 2 \) and \( k \geq 0 \), namely,

\[
\begin{align*}
\partial_l^* L^k(q) &= (M_\Delta L)_+^l(L^k(q)) + (k + \frac{l}{2}) L^{k+l-1}(q), \\
\partial_l^* (L^*)_k^l(r) &= -(M_\Delta L)_+^l(L^*)^k(r) + (k + \frac{l}{2}) (L^*)^{k+l-1}(r).
\end{align*}
\]

(4.3)

**Proof.** It is easy to get this by using (2.20), (4.2) and the relation \( \overline{\partial}_{l,1}(L^k(q)) = \overline{(\partial_{l,1}(L^k))(q) + L^k \overline{\partial}_{l,1}(q)} \).

Moreover, the action of \( \partial_l^* \) on \( Y_k \) is given by the following lemma.

**Lemma 4.2.** The actions on \( Y_k \) of the additional symmetry flows \( \partial_l^* \) of the cdKP hierarchy are

\[
\partial_l^* Y_k = [(M_\Delta L)_+^l, Y_k]_+ + (k - l + 1) Y_{k+l-1},
\]

(4.4)

for \( l = 0, 1, 2 \) and \( k \geq 0 \).
Proof. A straightforward calculation implies

\[
\partial^*_t Y_k = \partial^*_t \sum_{j=0}^{k-1} [j - \frac{1}{2}(k-1)] L^{k-1-j}(q) \Delta^{-1}(L^*)^j(r)
\]

\[
= \sum_{j=0}^{k-1} [j - \frac{1}{2}(k-1)] (\partial^*_t (L^{k-1-j}(q)) \Delta^{-1}(L^*)^j(r) + L^{k-1-j}(q) \Delta^{-1}(\partial^*_t (L^*)^j(r)))
\]

\[
\text{(4.3)} \sum_{j=0}^{k-1} [j - \frac{1}{2}(k-1)] (M_\Delta L^j)_+ (L^{k-1-j}(q)) \Delta^{-1}(L^*)^j(r)
\]

\[
+ \sum_{j=0}^{k-1} [j - \frac{1}{2}(k-1)] (k-j-1 + \frac{l}{2}) L^{k+l-2-j}(q) \Delta^{-1}(L^*)^j(r)
\]

\[
- \sum_{j=0}^{k-1} [j - \frac{1}{2}(k-1)] L^{k-1-j}(q) \Delta^{-1}(M_\Delta L^j)_+ (L^*)^j(r)
\]

\[
+ \sum_{j=0}^{k-1} [j - \frac{1}{2}(k-1)] (j + \frac{l}{2}) L^{k-1-j}(q) \Delta^{-1}(L^*)^{j+l-1}(r).
\]

The first term and the third one of the right part of eq. (4.5) can be simplified to

\[
\sum_{j=0}^{k-1} [j - \frac{1}{2}(k-1)] (M_\Delta L^j)_+ (L^{k-1-j}(q)) \Delta^{-1}(L^*)^j(r)
\]

\[
- \sum_{j=0}^{k-1} [j - \frac{1}{2}(k-1)] L^{k-1-j}(q) \Delta^{-1}(M_\Delta L^j)_+ (L^*)^j(r)
\]

\[
= [(M_\Delta L^j)_+, Y_k]_-. \tag{4.6}
\]

Furthermore, on behaving of

\[
\sum_{j=0}^{k-1} [j - \frac{1}{2}(k-1)] (j + \frac{l}{2}) L^{k-1-j}(q) \Delta^{-1}(L^*)^{j+l-1}(r)
\]

\[
= \sum_{j=l-1}^{k-1+\ell-1} [j-l+1 - \frac{1}{2}(k-1)] (j-l+1 + \frac{l}{2}) L^{k+l-2-j}(q) \Delta^{-1}(L^*)^j(r),
\]
the second term and the fourth one of the right part of eq. (4.5) can also be simplified to

\[
\sum_{j=0}^{k-1} [j - \frac{1}{2}(k - 1)](k - j - 1 + \frac{l}{2})L^{k+l-2-j}(q)\Delta^{-1}(L^*)^j(r)
\]

\[
+ \sum_{j=0}^{k-1} [j - \frac{1}{2}(k - 1)](j + \frac{l}{2})L^{k-l-j}(q)\Delta^{-1}(L^*)^{j+l-1}(r)
\]

\[
= \sum_{j=0}^{k-1} [j - \frac{1}{2}(k - 1)](k - j - 1 + \frac{l}{2})L^{k+l-2-j}(q)\Delta^{-1}(L^*)^j(r)
\]

\[
+ \sum_{j=l-1}^{k-1} [j - l + 1 - \frac{1}{2}(k - 1)](j - l + 1 + \frac{l}{2})L^{k+l-2-j}(q)\Delta^{-1}(L^*)^j(r)
\]

\[
= \sum_{j=0}^{k-1} (k - l + 1)[j - \frac{1}{2}(k + l - 2)]L^{k+l-2-j}(q)\Delta^{-1}(L^*)^j(r)
\]

\[
= (k - l + 1)Y_{k+l-1}.
\]

Taking (4.6) and (4.7) back into eq. (4.5) we have

\[
\partial^*_k Y_k = [(M_\Delta L^l)_+, Y_k]_+ + (k - l + 1)Y_{k+l-1}.
\]

Now we are in a position to identity the algebraic structure of the additional symmetry flows of the cdKP hierarchy.

**Theorem 4.3.** The additional flows \(\partial^*_k\) of the cdKP hierarchy form the positive half of Virasoro algebra, i.e.,

\[
[\partial^*_k, \partial^*_l] = (k - l)\partial^*_k, 
\]

for \(l = 0, 1, 2,\) and \(k \geq 0.\)

**Proof.**

\[
[\partial^*_k, \partial^*_l]L = \partial^*_k[(-(M_\Delta L^k)_-, L) + [Y_{k-1}, L]] - \partial^*_l[(-(M_\Delta L^l)_-, L)]
\]

\[
= \partial^*_k[(-(M_\Delta L^k)_-, L) + [\partial^*_k Y_{k-1}, L] + [Y^{(1)}_{k-1}, \partial^*_k L] + [\partial^*_k (M_\Delta L^l)_-, L] + [(M_\Delta L^l)_-, \partial^*_k L]
\]

\[
= -(\partial^*_k (M_\Delta L^k)_-, L) + [-(M_\Delta L^k)_-, (\partial^*_k L)] + [(M_\Delta L^l)_+, Y_{k-1}]_+ + (k - l)Y_{k+l-2}, L
\]

\[
+ [Y_{k-1}, [-(M_\Delta L^k)_-, L]] + [[-(M_\Delta L^k)_-, Y_{k-1}, M_\Delta L^l]_-, L]
\]

\[
+ [(M_\Delta L^l)_-, [-(M_\Delta L^k)_-, Y_{k-1}, L]]
\]

\[
= (k - l)[-(M_\Delta L^{k+l-1})_-, (\partial^*_k L)] + (k - l)[Y_{k+l-2}, L]
\]

\[
- [[(M_\Delta L^l)_-, Y_{k-1}, L]] + (k - l)[Y_{k-1}, [-(M_\Delta L^k)_-, L]] + [(M_\Delta L^l)_-, [Y_{k-1}, L]]
\]

\[
= (k - l)\partial^*_k Y_{k+l-1}L.
\]
\[(M_\Delta L^i)_- Y_{k-1} = [(M_\Delta L^i)_-, Y_{k-1}]\] and the Jacobi identity have been used in the fourth identity.

5. Conclusions and Discussions

In this paper, the additional symmetry flows in eq. (3.17) for the cdKP hierarchy have been constructed by a modification of the corresponding one of the dKP hierarchy. In this process, the difference operator \(Y_k\) plays a very crucial role. The actions of the additional flows (3.17) on the eigenfunction \(q\) and \(r\) of one-component cdKP hierarchy were obtained in theorem 3.5. In theorem 4.3, these flows have been shown to provide a hidden algebraic structure, i.e., the Virasoro algebra (positive half), in the cdKP hierarchy. Thus we can say that the discretization from KP hierarchy to dKP hierarchy is good enough to retain several interesting mathematical structures including Lax pair [21], \(\tau\) function [22] and algebraic structure.

It is possible to extend our results to the \(n\)-component cdKP hierarchy associated with a Lax operator

\[L(n) = \Delta + \sum_{i=1}^{m} q_i(n,t)\Delta^{-1}r_i(n,t).\]

Solving the cdKP hierarchy is also an interesting topic. We shall do it a near future.

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