ABC-type estimates via Garsia-type norms

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Abstract. We are concerned with extensions of the Mason–Stothers abc theorem from polynomials to analytic functions on the unit disk $D$. The new feature is that the number of zeros of a function $f$ in $D$ gets replaced by the norm of the associated Blaschke product $B_f$ in a suitable smoothness space $X$. Such extensions are shown to exist, and the appropriate abc-type estimates are exhibited, provided that $X$ admits a “Garsia-type norm”, i.e., a norm sharing certain properties with the classical Garsia-type norm on BMO. Special emphasis is placed on analytic Lipschitz spaces.

1. Introduction

One of the famous (and notoriously difficult) open problems in number theory is the so-called abc conjecture of Masser and Oesterlé. It states that to every $\varepsilon > 0$ there is a constant $K(\varepsilon)$ with the following property: whenever $a$, $b$ and $c$ are relatively prime positive integers satisfying $a + b = c$, one has

$$c \leq K(\varepsilon) \cdot \{\text{rad}(abc)\}^{1+\varepsilon}.$$ 

Here, $\text{rad}(\cdot)$ stands for the radical of the integer in question, defined as the product of the distinct primes that divide it. (In other words, $\text{rad}(m)$ is the greatest square-free divisor of $m$.) We refer to [7, 10] for a discussion of the abc conjecture and its potential applications.

The conjecture was inspired by the following abc theorem for polynomials. When stating it, we write $\text{deg } p$ for the degree of a polynomial $p$ (in one complex variable) and $\widetilde{N}(p)$ for the number of its distinct zeros in $\mathbb{C}$.

**Theorem A.** Suppose $a$, $b$ and $c$ are polynomials, not all constants, having no common zeros and satisfying $a + b = c$. Then

$$\max\{\text{deg } a, \text{deg } b, \text{deg } c\} \leq \widetilde{N}(abc) - 1.$$ 

This result is also known as Mason’s theorem. It is indeed contained – in a more general form – in Mason’s book [11], but the current version is essentially due to Stothers [13]. The situation seems to be in full accordance with V. I. Arnold’s principle: a personal name, when attached to a mathematical notion or statement,
is never the name of the true discoverer. (Needless to say, Arnold’s principle applies to itself as well.)

Various approaches to Theorem A can be found in [7, 8, 10, 12]. Let us also mention the following generalization involving any finite number of polynomials; see [1, 5, 8] for this and other related results.

**Theorem B.** Let \( p_0, \ldots, p_n \) be linearly independent polynomials and put \( p_{n+1} = p_0 + \cdots + p_n \). Assume further that the zero-sets of \( p_0, \ldots, p_{n+1} \) are pairwise disjoint. Then

\[
\max\{\deg p_0, \ldots, \deg p_{n+1}\} \leq n \tilde{N}(p_0 p_1 \cdots p_{n+1}) - \frac{n(n + 1)}{2}.
\]

Quite recently, in [4, 5], we came up with some abc-type estimates that make sense in a much more general setting. Namely, we were concerned with analytic functions on a (reasonably decent) planar domain, rather than just polynomials on \( \mathbb{C} \). In fact, [4] dealt with the case of the disk only, while the functions were assumed to be analytic in a neighborhood of its closure. Retaining these hypotheses, we now go on to describe part of what we did in [4].

Write \( D \) for the unit disk \( \{ z \in \mathbb{C} : |z| < 1 \} \) and \( \partial D \) for its boundary. Suppose \( f_0, \ldots, f_n \) are functions that are analytic on the (closed) disk \( D \cup \partial D \), and set

\[
f_{n+1} = f_0 + \cdots + f_n.
\]

For each \( j = 0, \ldots, n + 1 \), we associate with \( f_j \) the (finite) Blaschke product \( B_j \) built from the function’s zeros. This means that

\[
z \mapsto \prod_{k=1}^{s_j} \left( \frac{z - a_k}{1 - \overline{a}_k z} \right)^{m_k},
\]

where \( a_k = a_k^{(j)} (1 \leq k \leq s = s_j) \) are the distinct zeros of \( f_j \) in \( D \), and \( m_k = m_k^{(j)} \) are their respective multiplicities. Further, we let \( B \) denote the least common multiple (defined in the natural way) of the Blaschke products \( B_0, \ldots, B_{n+1} \), to be written as

\[
B := \text{LCM}(B_0, \ldots, B_{n+1}),
\]

and we put

\[
B := \text{rad}(B_0 B_1 \cdots B_{n+1}).
\]

In the latter formula, we use the notation \( \text{rad}(B) \) for the radical of a Blaschke product \( B \). This is, by definition, the Blaschke product that arises when the zeros of \( B \) are all converted into simple ones. In other words, given a Blaschke product of the form (1.3), its radical is obtained by replacing each \( m_k \) with 1.

Finally, let \( W = W(f_0, \ldots, f_n) \) be the Wronskian of the (analytic) functions \( f_0, \ldots, f_n \), so that

\[
W := \begin{vmatrix}
f_0 & f_1 & \cdots & f_n \\
f'_0 & f'_1 & \cdots & f'_n \\
\cdots & \cdots & \cdots & \cdots \\
f^{(n)}_0 & f^{(n)}_1 & \cdots & f^{(n)}_n \\
\end{vmatrix}.
\]

We then introduce the quantities

\[
\kappa = \kappa(W) := \|W'\|_1 \|1/W\|_\infty
\]

and

\[
\mu = \mu(W) := \|W\|_\infty \|1/W\|_\infty.
\]
where \( N \) stands for \( N_{\mathbb{D}} \), counting multiplicities.

It was explained in [4] how to derive the original abc inequality (1.1) from (1.6). Basically, the idea is to apply Theorem C, with \( n = 1 \), to the three polynomials, rescaling everything for the disk \( \mathbb{D} = \{ z : |z| < 1 \} \), and then pass to the limit as \( R \to \infty \). In the case of an arbitrary \( n \), we similarly deduce Theorem B from Theorem C; see [5] for details.

Also, in [4], inequality (1.6) was supplemented with a certain alternative estimate, which we do not cite here. Further developments, as contained in [5], included the situation where the functions \( f_j \) are merely analytic on \( \mathbb{D} \) and suitably smooth up to \( \mathbb{T} \) (but not necessarily analytic on \( \mathbb{D} \cup \mathbb{T} \)); in particular, the case of infinitely many zeros was dealt with. In addition, other – fairly general – domains were considered in place of the disk.

In this note, we extend (1.6) in yet another direction. Let us observe that, for a Blaschke product \( B \), the number of its zeros \( N_\mathbb{D}(B) \) coincides with the quantity \( \| B' \|_1 \); moreover, a similar quantity \( \| W' \|_1 \) appears in the definition of the coefficient \( \kappa \) above. Therefore, (1.6) reflects a certain fact about the Hardy–Sobolev space \( H^1_1 : = \{ f \in H^1 : f' \in H^1 \} \), equipped with the norm \( \| f \|_{H^1_1} : = \| f' \|_1 \). We may ask, then, what other “smooth” analytic spaces \( X \) admit (under the hypotheses of Theorem C) the abc-type estimate

\[
(1.7) \quad c \| B \|_X \leq \kappa_X + n \mu \| B \|_X,
\]

with \( \kappa_X : = \| W \|_X \| 1/W \|_\infty \) and with some constant \( c = c_X > 0 \). While for \( X = H^1_1 \) we have \( c = 1 \), it seems reasonable to allow for an unspecified factor \( c \) in the general case; this \( c \) should depend neither on \( n \) nor on the functions involved.

Our main result, to be stated in Section 3, will provide us with a collection of spaces \( X \) that enjoy the required property. Roughly speaking, the “smoothest” of these is the space \( H^\infty_1 : = \{ f \in H^\infty : f' \in H^\infty \} \) (i.e., the class of analytic functions satisfying the Lipschitz 1-condition) with the natural norm \( \| f \|_{H^\infty_1} : = \| f' \|_\infty \). Indeed, while \( H^\infty_1 \) does admit the abc-type estimate (1.7), no further increase of smoothness (e.g., in the sense of passing to higher order Lipschitz spaces) is possible. We first state the positive part of that endpoint result as follows.

**Proposition 1.1.** Under the hypotheses of Theorem C, one has

\[
(1.8) \quad c \| B' \|_\infty \leq \| 1/W \|_\infty \cdot (\| W' \|_\infty + n \| W \|_\infty \| B' \|_\infty),
\]

where \( c > 0 \) is an absolute constant.

This means that the abc-type inequality

\[
(1.9) \quad c \| B \|_X \leq \| 1/W \|_\infty \cdot (\| W \|_X + n \| W \|_\infty \| B \|_X),
\]
or equivalently (1.7), holds with $X = H_1^\infty$. Moreover, we shall see that there is a certain “privileged” norm on $H_1^\infty$ which, when used in place of $\|f'\|_\infty$ above, makes the corresponding statement true with $c = 1$.

It is precisely the implementation of a special norm that is crucial to our approach. Our method also applies to the analytic Lipschitz spaces $A_\omega$ associated with certain slower majorants $\omega$, not just to $H_1^\infty$ (in which case the majorant is $\omega_1(t) := t$). Here, by saying that $\omega$ is a majorant we mean that $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing continuous function on $\mathbb{R}_+ := (0, \infty)$ with $\lim_{t \to 0^+} \omega(t) = 0$ such that $\omega(t)/t$ is nonincreasing for $t > 0$. The space $A_\omega(E)$ on a set $E \subset \mathbb{C}$ is then formed by the functions $f : E \to \mathbb{C}$ satisfying

$$\|f\|_{A_\omega(E)} := \sup \left\{ \frac{|f(z_1) - f(z_2)|}{\omega(|z_1 - z_2|)} : z_1, z_2 \in E, z_1 \neq z_2 \right\} < \infty.$$  

When $\omega$ is of the form $\omega_\alpha(t) := t^\alpha$, with $0 < \alpha \leq 1$, we write $A^\alpha(E)$ rather than $A_{\omega_\alpha}(E)$. Further, we define the analytic Lipschitz space $A_\omega$ to be $H_1^\infty \cap A_\omega(\mathbb{D})$ and endow it with the norm $\| \cdot \|_{A_\omega} := \| \cdot \|_{A_\omega(\mathbb{D})}$. When $\omega = \omega_\alpha$, the corresponding $A_\omega$-space is denoted by $A^\alpha$; thus, in particular, $H_1^\infty = A^1$ (with equality of norms).

Finally, we recall that a majorant $\omega$ is said to be regular if

$$\int_0^\delta \frac{\omega(t)}{t} \, dt + \delta \int_\delta^\infty \frac{\omega(t)}{t^2} \, dt \leq C \omega(\delta), \quad 0 < \delta < 2,$$

for some fixed $C = C_\omega > 0$. The basic examples of regular majorants are the $\omega_\alpha$’s with $0 < \alpha < 1$.

**Proposition 1.2.** For every regular majorant $\omega$ one has, under the hypotheses of Theorem $C$,

$$(1.10) \quad c_\omega \|B\|_{A_\omega} \leq \|1/W\|_\infty \cdot (\|W\|_{A_\omega} + n \|W\|_\infty \|B\|_{A_\omega}),$$

where $c_\omega > 0$ is a constant depending only on $\omega$.

Both propositions will follow as special cases from our main result, Theorem 3.1 in Section 3 below. Indeed, the hypothesis of that theorem (concerning the existence of a “Garsia-type norm” on the space in question) is fulfilled for our Lipschitz spaces $A_\omega$ and $H_1^\infty$; this is explained in Section 2.

We conclude this introduction with an example showing that the $H_1^\infty$ result (Proposition 1.1) is both sharp and best possible, at least within the Lipschitz scale, as we said before. Consider the functions $f_0(z) = 1$ and $f_j(z) = \varepsilon z^j/j!$ for $j = 1, \ldots, n$, with a suitable $\varepsilon > 0$; then define $f_{n+1}$ by (1.3). If $\varepsilon$ is small enough, then $f_{n+1}$ is zero-free on $\mathbb{D}$. The Blaschke products that arise are $B_0(z) = B_{n+1}(z) = 1$ and $B_j(z) = z^j$ for $1 \leq j \leq n$. We have, therefore, $B(z) = z^n$ and $B(\zeta) = z$, whence $\|B'\|_\infty = n$ and $\|B'\|_\infty = 1$. Also, the Wronskian matrix being upper triangular, one easily finds that $W = \varepsilon^n (= \text{const})$; this yields $\|W\|_\infty = 0$ and $\|W\|_\infty/\|W\|_\infty = 1$. Consequently, equality holds in (1.8) with $c = 1$.

Now let $1 < \alpha < 2$ and consider the (higher order) Lipschitz space $A^\alpha := \{ f \in H^\infty : f' \in A^{\alpha-1} \}$, normed in a natural way. This time, our current $f_j$’s provide a counterexample to the abc-type estimate (1.9) with $X = A^\alpha$. Indeed, the left-hand side in (1.9) is now a constant times $\|z^n\|_{A^\alpha}$, which is comparable to $n^\alpha$. As to the right-hand side, it is still $O(n)$, so the inequality breaks down.
2. Preliminaries on Garsia-type norms

The classical Garsia norm on the space BMOA := BMO ∩ H^2 is given by

\[ \|f\|_G := \sup_{z \in \mathbb{D}} \{ \mathcal{P}(|f|^2)(z) - |f(z)|^2 \}^{1/2}. \]

Here, the notation \( \mathcal{P} \) stands for the Poisson integral of a function \( \varphi \in L^1(\mathbb{T}) \), so that

\[ \mathcal{P}(\varphi)(z) := \frac{1}{2\pi} \int_{\mathbb{T}} \varphi(\zeta) \frac{1 - |z|^2}{|z - \zeta|^2} d\zeta, \quad z \in \mathbb{D}. \]

The definition (2.1) actually makes sense for all \( f \in H^2 \), and it is well known that BMOA = \{ \( f \in H^2 : \|f\|_G < \infty \) \}. Moreover, the Garsia norm \( \| \cdot \|_G \) is equivalent to the original BMO-norm \( \| \cdot \|_* \) defined in terms of mean oscillation; see [6] Chapter VI or [9] Chapter X.

Recently, we introduced in [3] a more general concept of Garsia-type norm (GTN) as follows. Suppose \( X \) is a Banach space of analytic functions on \( \mathbb{D} \) such that \( X \subset H^p \) for some \( p > 0 \). Write \( \|H^p\| \) for the set of all nonnegative functions \( g \in L^p(\mathbb{T}) \) satisfying either \( \log g \in L^1(\mathbb{T}) \) or \( g = 0 \) a.e.; these are precisely (the boundary values of) the moduli of \( H^p \)-functions. Further, assume that there exist a mapping \( \Psi : |H^p| \times \mathbb{D} \to [0, +\infty] \) and a function \( k : \mathbb{D} \to \mathbb{R}_+ \) with the following properties:

- \( \Psi(\lambda g, z) = \lambda^p \Psi(g, z) \) whenever \( \lambda \in \mathbb{R}_+ \), \( g \in |H^p| \) and \( z \in \mathbb{D} \),
- \( \Psi(|f|, z) \geq |f(z)|^p \) for all \( f \in H^p \) and \( z \in \mathbb{D} \),
- the quantity

\[ \mathcal{N}(f) = \mathcal{N}_{p, \Psi, k}(f) := \sup_{z \in \mathbb{D}} \left\{ \frac{\Psi(|f|, z) - |f(z)|^p}{k(z)} \right\}^{1/p}, \quad f \in H^p, \]

is comparable to \( \|f\|_X \) with constants not depending on \( f \) (it is understood that \( \|f\|_X = \infty \) for \( f \in H^p \setminus X \)).

Then we say that \( \mathcal{N}(\cdot) \) is a GTN on \( X \), and accordingly, that \( X \) admits a GTN. In fact, the \( X \)'s we have in mind will always be analytic subspaces of certain smoothness classes on \( T \), and the constants will have zero \( X \)-norm. Therefore, we should also have \( \mathcal{N}(1) = 0 \) (where \( 1 \) is the constant function 1), and this reduces to saying that

\[ \Psi(1, z) = 1, \quad z \in \mathbb{D}, \]

an assumption to be imposed hereafter.

The reason why GTN's are useful is that, once available, such a norm makes it easy to separate the contributions of the two factors in the canonical (inner-outer) factorization of functions in \( X \). Indeed, let \( f = h\theta \), where \( h \in H^p \) and \( \theta \) is an inner function. Since \( |f| = |h| \) a.e. on \( T \), we have

\[ \Psi(|f|, z) - |f(z)|^p = \{ \Psi(|h|, z) - |h(z)|^p \} + \{ |h(z)|^p (1 - |\theta(z)|^p) \} \]

for \( z \in \mathbb{D} \). Both terms in curly brackets on the right are nonnegative, so

\[ \{ \Psi(|f|, z) - |f(z)|^p \}^{1/p} \leq \{ \Psi(|h|, z) - |h(z)|^p \}^{1/p} + |h(z)|(1 - |\theta(z)|^p)^{1/p} \]

(meaning that the ratio of the two sides lies between two positive constants that depend only on \( p \)), and hence

\[ \mathcal{N}_{p, \Psi, k}(f) \leq \mathcal{N}_{p, \Psi, k}(h) + \mathcal{S}_{p, k}(h, \theta), \]
where
\[ S_{p,k}(h, \theta) := \sup_{z \in \mathbb{D}} \frac{|h(z)|(1 - |\theta(z)|^p)^{1/p}}{k(z)}. \]

More precisely,
\[ (2.3) \quad \max \{ N_{p,\Psi,k}(h), S_{p,k}(h, \theta) \} \leq N_{p,\Psi,k}(f) \leq c_p \{ N_{p,\Psi,k}(h) + S_{p,k}(h, \theta) \} \]

for a suitable constant \( c_p > 0 \); we can take \( c_p = 1 \) if \( p \geq 1 \).

We see, in particular, that
\[ (2.4) \quad N_{p,\Psi,k}(h\theta) \geq N_{p,\Psi,k}(h) \]

for \( h \in H^p \) and \( \theta \) inner, which means that division by inner factors preserves membership in \( X \). (Equivalently, \( X \) enjoys the so-called \( f \)-property.) On the other hand, given \( h \in X \) and an inner function \( \theta \), we have \( h\theta \in X \) if and only if \( S_{p,k}(h, \theta) < \infty \). When \( h = 1 \), this gives a criterion for an inner function \( \theta \) to be in \( X \); moreover, \( (2.2) \) implies that
\[ N_{p,\Psi,k}(\theta) = S_{p,k}(1, \theta). \]

The parameters corresponding to the Garsia norm \( \| \cdot \|_G \) are obviously \( p = 2 \), \( \Psi(g, z) = \mathcal{P}(g^2)(z) \) and \( k(z) \equiv 1 \). Keeping the same \( p \) and \( \Psi \) while putting \( k(z) = \omega(1 - |z|) \), with a majorant \( \omega \), gives rise to the GTN
\[ \| f \|_{G, \omega} := \sup_{z \in \mathbb{D}} \frac{\{ \mathcal{P}(|f|^2)(z) - |f(z)|^2 \}^{1/2}}{\omega(1 - |z|)} \]

on the space \( \text{BMOA}_\omega := \{ f \in H^2 : \| f \|_{G, \omega} < \infty \} \). If \( \omega(t) \) tends to 0 slowly enough as \( t \to 0^+ \) (e.g., if \( \omega(t) = (\log \frac{1}{t})^{-\epsilon} \) with a suitably small \( \epsilon > 0 \)), then \( \text{BMOA}_\omega \) will retain many features of \( \text{BMOA} \); in particular, it will contain unbounded and discontinuous functions. For faster \( \omega \)'s (such as \( \omega(t) = t^\alpha \) with \( 0 < \alpha < \frac{1}{2} \)), it becomes a Lipschitz space. In fact, it was proved in [2] that if \( \omega \) and \( \omega^2 \) are both regular majorants, then \( \text{BMOA}_\omega \) coincides with \( \Lambda^2 \), the norm \( \| \cdot \|_{\Lambda^2} \) being equivalent to \( \| \cdot \|_{\Lambda_\omega} \). The assumption on \( \omega^2 \) cannot be dropped here: just note that the identity function \( f_0(z) := z \) has \( \| f_0 \|_{G, \omega} = \infty \) whenever \( \omega(t)/\sqrt{t} \to 0 \) as \( t \to 0^+ \).

Furthermore, assuming that \( \omega \) alone is a regular majorant, we proved in [2] that the functional
\[ \mathcal{M}_\omega(f) := \| f \|_{\Lambda^2} + \sup_{z \in \mathbb{D}} \frac{(\mathcal{P}|f|)(z) - |f(z)|}{\omega(1 - |z|)} \]

provides an equivalent norm on \( \Lambda^2 \). In particular, this is the case for \( \omega(t) = t^\alpha \) with \( 0 < \alpha < 1 \) (but not with \( \alpha = 1 \)). Clearly, we have \( \mathcal{M}_\omega(f) = N_{1,\Psi,k}(f) \) with
\[ \Psi(g, z) = \| g \|_{\Lambda^2} \cdot \omega(1 - |z|) + (\mathcal{P}g)(z) \]

and \( k(z) = \omega(1 - |z|) \), so \( \mathcal{M}_\omega \) is a GTN on \( \Lambda^2 \).

Finally, the extreme case \( \omega(t) = t \) was studied in [3]. There we showed that a GTN can be defined on \( H^1_{\infty} \) by taking \( p = 1, k(z) = 1 - |z| \) and
\[ \Psi(g, z) = \| g \|_{\Lambda^2} \cdot (1 - |z|) + \left| g \left( \frac{z}{|z|} \right) - \exp\{ (\mathcal{P} \log g)(z) \} \right| + \exp\{ (\mathcal{P} \log g)(z) \} \]
(with the understanding that $\Psi(g, z) = \infty$ if $g \notin \Lambda^1(\mathbb{T})$, and $z/|z| = 1$ if $z = 0$).

The norm that arises is thus $\tilde{N}_1(\cdot) := N_{1, \Psi, k}(\cdot)$, or equivalently,

$$\tilde{N}_1(f) = \|f\|_{\Lambda^1(\mathbb{T})} + \sup_{z \in \mathbb{D}} \frac{|f(z/|z|)| - |O_{f1}|(z)| + |O_{f1}|(z) - |f(z)|}{1 - |z|},$$

where $O_{f1}$ is the outer function with modulus $|f|$ on $\mathbb{T}$.

At the same time, it turns out [3] that the higher order $A^\omega$-spaces (i.e., those with $\alpha > 1$) or, more generally, the classes $A^\omega_n := \{f : f^{(n)} \in A_\omega\}$ with $n \geq 1$ admit no GTN at all. This accounts for the special role of the endpoint space $H^\infty$ in our story.

3. Main result

**Theorem 3.1.** Let $X$ be a space that admits a Garsia-type norm $N = N_{p, \Psi, k}$, where $p \geq 1$ and $\Psi$ satisfies (2.2). Assume that the functions $f_j$ $(j = 0, 1, \ldots, n+1)$, related by (1.3), are analytic on $\mathbb{D} \cup \mathbb{T}$ and that the Wronskian (1.4) vanishes nowhere on $\mathbb{T}$. Write

$$B := \text{LCM}(B_0, \ldots, B_{n+1}) \quad \text{and} \quad B := \text{rad}(B_0 B_1 \ldots B_{n+1}),$$

where $B_j$ is the Blaschke product associated with $f_j$. Then

$$N(B) \leq \gamma + \mu n^{1/p} N(B),$$

where

$$\gamma = \gamma(W) := N(W)\|1/W\|_\infty \quad \text{and} \quad \mu = \mu(W) := \|W\|_\infty\|1/W\|_\infty.$$  

Also, for some constant $c = c(X, p, \Psi, k) > 0$, we have

$$c\|B\|_X \leq \kappa_X + \mu n^{1/p} \|B\|_X,$$

with

$$\kappa_X = \kappa_X(W) := \|W\|_X\|1/W\|_\infty$$

and $\mu$ as above.

**Proof.** Of course, it suffices to prove (3.2). This done, (3.3) will follow readily, the norms $N(\cdot)$ and $\|\cdot\|_X$ being equivalent.

As in [5], we begin by verifying that the ratio $WB^n/B$ is analytic on $\mathbb{D}$ (and in fact on $\mathbb{D} \cup \mathbb{T}$). We need not worry about the zeros of $B$ whose multiplicity is at most $n$, since these are obviously killed by the numerator, $WB^n$. So let $z_0 \in \mathbb{D}$ be a zero of multiplicity $k$, $k > n$, for $B$. Then there is a $j \in \{0, \ldots, n+1\}$ such that $B_j$ vanishes to order $k$ at $z_0$, and so does $f_j$. Expanding the determinant (1.5) along the column which contains $f_j, \ldots, f_j^{(n)}$, while noting that $f_j^{(l)}$ vanishes to order $k - l$ at $z_0$, we see that $W$ has a zero of multiplicity $\geq k - n$ at $z_0$. (In case $j = n + 1$, one should observe that, by (1.3), the determinant remains unchanged upon replacing any one of its columns by $(f_{n+1}, \ldots, f_{n+1}^{(n)})^T$.) And since $B$ has a zero at $z_0$, it follows that $WB^n$ vanishes at least to order $k$ at that point.

We conclude that $WB^n$ is indeed divisible by $B$. In other words, we have

$$WB^n = FB,$$

where $F$ is analytic on $\mathbb{D}$. In addition, this $F$ is analytic across $\mathbb{T}$, because the other factors in (3.3) have this property and because

$$|B| = |B| = 1 \quad \text{on} \quad \mathbb{T}.$$
Factoring $F$ canonically (see [6, Chapter II]), we write $F = IO$, where $I$ is inner and $O$ is outer. Furthermore, a glance at (3.4) and (3.5) reveals that $|F| = |W|$ on $T$, so the outer factor $O = O_F$ coincides with $O_{|W|}$. An application of (2.4) with $h = O_{|W|}B$ and $\theta = I$ now shows that
\[ N(W^n) = N(FB) = N(O_{|W|}IB) \geq N(O_{|W|}B). \]
This and (2.3) together imply
\[ N(W^n) \geq N(O_{|W|}B) \]
\[ \geq \sup_{z \in D} |O_{|W|}(z)| \cdot \{1 - |B(z)|^p\}^{1/p} k(z) \]
\[ \geq \left( \inf_{z \in D} |O_{|W|}(z)| \right) \cdot \sup_{z \in D} \{1 - |B(z)|^p\}^{1/p} k(z) \]
\[ = \left( \inf_{z \in D} |O_{|W|}(z)| \right) \cdot N(B). \]  
We further observe that
\[ 1/O_{|W|} = O_{1/|W|} \in H^\infty \]
(because $1/W \in L^\infty(T)$) and
\[ \sup_{z \in D} |O_{|W|}(z)|^{-1} = \|1/O_{|W|}\|_\infty = \|1/W\|_\infty, \]
whence
\[ \inf_{z \in D} |O_{|W|}(z)| = \|1/W\|_\infty^{-1}. \]
Substituting this into (3.6), we obtain
\[ N(W^n) \geq \|1/W\|_\infty^{-1} \cdot N(B). \]  
Another application of (2.3) (with $c_p = 1$), coupled with the elementary inequality
\[ 1 - t^n \leq n(1 - t) \quad \text{for} \quad t \in [0, 1], \]
yields
\[ N(W^n) \leq N(W) + \sup_{z \in D} |W(z)| \cdot \{1 - |B(z)|^p\}^{1/p} k(z) \]
\[ \leq N(W) + \|W\|_\infty \cdot n^{1/p} \cdot \sup_{z \in D} \{1 - |B(z)|^p\}^{1/p} k(z) \]
\[ = N(W) + \|W\|_\infty \cdot n^{1/p} \cdot N(B). \]  
Thus,
\[ N(W^n) \leq N(W) + n^{1/p} \|W\|_\infty N(B). \]  
Finally, we combine (3.7) and (3.8) to get
\[ N(B) \leq \|1/W\|_\infty \cdot \left\{ N(W) + n^{1/p} \|W\|_\infty N(B) \right\}, \]
which is the required estimate (3.2). □
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