A four-dimensional theory for quantum gravity with conformal and nonconformal explicit solutions

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Abstract

The most general version of a renormalizable $d = 4$ theory corresponding to a dimensionless higher-derivative scalar field model in curved spacetime is explored. The classical action of the theory contains 12 independent functions, which are the generalized coupling constants of the theory. We calculate the one-loop beta functions and then consider the conditions for finiteness. The set of exact solutions of power type is proven to consist of precisely three conformal and three nonconformal solutions, given by remarkably simple (albeit nontrivial) functions that we obtain explicitly. The finiteness of the conformal theory indicates the absence of a conformal anomaly in the finite sector. The stability of the finite solutions is investigated and the possibility of renormalization group flows is discussed as well as several physical applications.
1 Introduction

The considerable achievements that have been obtained in the field of two-dimensional quantum gravity have inspired different attempts to use it as a pattern for the construction of the more realistic theory of quantum gravity in four dimensions. Unfortunately the direct analogies of the two cases do not work here, for rather evident reasons. First of all, the quantum metric in $d = 4$ has more degrees of freedom, which include the physical degrees of freedom of spin two, what is quite different from the $d = 2$ case. Second, the Feynman integrals in $d = 4$ have worst convergence properties as compared with the $d = 2$ case, from what follows that higher-derivative terms have to be included in order to ensure renormalizability. An example of this sort is given by quantum $R^2$-gravity (for a review and a list of references see [1]), which is multiplicatively renormalizable [2] (not so is Einstein’s gravity) and also asymptotically free. However the presence of higher derivatives leads to the problem of massive spin-two ghosts, which violate the unitarity of the $S$-matrix. It has been conjectured, nevertheless, that the problem of non-unitarity in $R^2$-gravity might perhaps be solved in a non-perturbative approach.

The alternative approach is based on the assumption that gravity is the induced interaction and the equations for the gravitational field arise as effective ones in some more general theory, as the theory of (super)strings [3]. It is also interesting to notice that higher-derivative gravitational theories (like string-inspired models) often admit singularity-free solutions (for a recent discussion and a list of references, see [4, 5]). In string theory, higher-derivative actions also arise in quite a natural way. For instance, if one wants to study the massive higher-spin modes of the theory one has to modify the standard $\sigma$-model action by adding to it an infinite number of terms, which contain all possible derivatives. On the other hand, the effective action of gravity, which follows from string theory, contains higher-derivative terms, and the higher powers in derivatives correspond to the next order of string perturbation theory. One can expect that the unitarity of the theory will be restored when all the excitations are taken into account. Therefore, it is quite natural to consider fourth-order gravity as some kind of effective theory, which is valid as an approximation to a more fundamental theory, still unknown.

String-inspired models of gravity contain, at least, two independent fields, which are the metric and the scalar dilaton field. Hence, the aforementioned effective theory has to depend on the dilaton field as well. The more general action ([4]) for a renormalizable theory of this type has been recently formulated in [6]. Since this model is rather complicated, even the one-loop calculations are very tedious. At the same time it is possible to make quite a considerable simplification: since both the metric and the dilaton are dimensionless, higher-derivative fields, the structure of divergences is essentially the same even if the metric is taken as a purely classical background. Indeed, the renormalization constants are different, if compared with the complete theory, but their general structures have to be similar.

Let us recall that the theory of a quantum dilaton field has been recently proposed for the description of infrared quantum gravity [4] (see also [8] and [9]). Furthermore it has turned out that the quantum dilaton theory enables one to estimate the back reaction of the vacuum to the matter fields [10]. It is very remarkable that the effect of the quantum dilaton is qualitatively the same as the effect of the quantum metric, evaluated earlier in [11]. In a previous article [6] we have considered the one loop renormalization and asymptotic behaviour of the special constrained version of the dilaton theory. In fact the action of this special model
We start with an action of effects of the quantum metric. Section 7 contains the discussion of our results, including the possible role of the behaviour of the theory, together with a number of mathematical tools which are useful in nonconformal version is explored. In section 6 we present some analysis of the asymptotic construct three different examples of anomaly-free dilaton models. Then the more general of dilaton gravity (this model is an extension of the one formulated in \[15, 12\]) and thus construct three different examples of anomaly-free dilaton models. Then the more general nonconformal version is explored. In section 6 we present some analysis of the asymptotic behaviour of the theory, together with a number of mathematical tools which are useful in this field. Section 7 contains the discussion of our results, including the possible role of the effects of the quantum metric.

## 2 Description of the model

We start with an action of \(\sigma\)-model type which is renormalizable in a generalized sense. A basic assumption will be that the scalar \(\varphi\) be dimensionless in four-dimensional curved spacetime, namely that \([\varphi] = 0\). We will also admit that there is just one fundamental dimensional constant, which has dimension of mass squared. The only field, aside from the scalar, which will be present in the theory is the gravitational field \(g_{\mu\nu}\).

Then, dimensional considerations lead us to the following general action of sigma-model type

\[
S = \int d^4x \sqrt{-g} \{ b_1(\varphi)(\Box \varphi)^2 + b_2(\varphi)(\nabla_\mu \varphi)(\nabla^\mu \varphi) \Box \varphi + b_3(\varphi)[(\nabla_\mu \varphi)(\nabla^\mu \varphi)]^2 \\
+ b_4(\varphi)(\nabla_\mu \varphi)(\nabla^\mu \varphi) + b_5(\varphi) + c_1(\varphi) R(\nabla_\mu \varphi)(\nabla^\mu \varphi) + c_2(\varphi) R_{\mu\nu}(\nabla_\mu \varphi)(\nabla_\nu \varphi) \\
+ c_3(\varphi) R \Box \varphi + a_1(\varphi) R_{\mu\nu\alpha\beta} + a_2(\varphi) R_{\mu\nu}^2 + a_3(\varphi) R^2 + a_4(\varphi) R \} \quad \text{(s.t.)},
\]

where \(s.t.\) means ‘surface terms’. All generalized coupling constants are dimensionless, except for \(b_4, b_5\) and \(a_4\), for which we have: \([b_4(\varphi)] = 2, [b_5(\varphi)] = 4, [a_4(\varphi)] = 2\). All other possible terms that can appear in dimension 4 in the above model can be obtained from (II) by simple integration by parts, and thus differ from these structures of the above action by some surface terms (s.t.) only. One can easily verify the following reduction formulas

\[
c_4(\nabla_\mu R)(\nabla_\nu \varphi) = -c_4'(\nabla_\mu \varphi)^2 - c_4 R(\Box \varphi) + (s.t.)
\]

\[
c_5(\Box R) = c_5'' R(\nabla_\mu \varphi)^2 + c_5'/2 R(\Box \varphi) + (s.t.)
\]

\[
c_6 R_{\mu\nu}(\nabla_\mu \nabla_\nu \varphi) = c_6' R_{\mu\nu}(\nabla_\mu \varphi)(\nabla_\nu \varphi) + 1/2 c_6'' R(\nabla_\mu \varphi)^2 + 1/2 c_6 R(\Box \varphi) + (s.t.)
\]

\[
b_6(\nabla_\nu \varphi)(\Box \nabla_\nu \varphi) = -b_6'(\nabla_\nu \varphi)^2(\Box \varphi) - b_6(\Box \varphi)^2 + b_6 R_{\mu\nu}(\nabla_\mu \varphi)(\nabla_\nu \varphi) + (s.t.)
\]

\[
b_7(\nabla_\nu \nabla_\mu \varphi)^2 = 1/2 b_7'(\nabla_\mu \varphi)^2 + 3/2 b_7' R(\nabla_\mu \varphi)^2(\Box \varphi) + b_7(\Box \varphi)^2 - b_7 R_{\mu\nu}(\nabla_\mu \varphi)(\nabla_\nu \varphi) + (s.t.)
\]
\[ b_8(\nabla_\nu \varphi)(\nabla_\mu \varphi)(\nabla'^2 \varphi) = \left(-\frac{1}{2}\right)\left[b_6(\nabla_\mu \varphi)^4 + b_8(\nabla_\mu \varphi)^2(\Box \varphi)\right] + \text{(s.t.)} \]

\[ b_9(\nabla'^2 \Box \nabla_\nu \varphi) = b''_9(\nabla_\mu \varphi)^2(\Box \varphi) + b_9(\Box \varphi)^2 - b'_9 R_{\mu\nu}(\nabla_\mu \varphi)(\nabla_\nu \varphi) + \text{(s.t.)} \]

\[ b_{10}(\Box^2 \varphi) = b''_{10}(\nabla_\mu \varphi)^2(\Box \varphi) + b'_{10}(\Box \varphi)^2 + \text{(s.t.)} \]

\[ b_{11}(\nabla'^2 \varphi)(\nabla_\nu \Box \varphi) = -b'_{11}(\nabla_\mu \varphi)^2(\Box \varphi) - b_{11}(\Box \varphi)^2 + \text{(s.t.)} \]

Here \( c_{4,5,6} = c_{4,5,6}(\varphi), b_{6,...,11} = b_{6,...,11}(\varphi) \) are some (arbitrary) functions. We shall extensively use these formulas below. Notice that, for constant \( \varphi \), this theory represents at the classical level the standard R\(^2\) gravity.

Theory (1) is renormalizable in a generalized sense, i.e., assuming that the form of the scalar functions \( b_1(\varphi), \ldots, a_4(\varphi) \) is allowed to change under renormalization. As we see, also some terms corresponding to a new type of the non-minimal scalar-gravity interaction appear, with the generalized non-minimal couplings \( c_1(\varphi), c_2(\varphi) \) and \( c_3(\varphi) \).

It is interesting to notice that, at the classical level and for some particular choices of the generalized couplings, the action (1) may be viewed, in principle, as a superstring theory effective action — the only background fields being the gravitational field and the dilaton, see [3]. It has been known for some time that string-inspired effective theories with a massless dilaton lead to interesting physical consequences, as a cosmological variation of the fine structure constant and of the gauge couplings [3], a violation of the weak equivalence principle [16], etc. It could seem that all these effects are in conflict with existing experimental data. However, some indications have been given [17] that non-perturbative loop effects might open a window for the existence of the dilaton, being perfectly compatible with the known experimental data. This gives good reasons for the study of higher-derivative generalizations of theories of the Brans-Dicke type [18] and, in particular, of their quantum structure.

### 3 Calculation of the counterterms

In this section we shall present the details of the calculation of the one-loop counterterms of the theory for the dilaton in an external gravitational field. For the purpose of calculation of the divergences we will apply the background field method and the Schwinger-De Witt technique. The features of higher-derivative theories do not allow for the use of the last method in its original form. At the same time, a few examples of calculations in higher-derivative gravity theory are known [19]–[24] (see also [1] for a review and more complete list of references) which possess a more complicated structure than (1), because of the extra diffeomorphism symmetry. Let us start with the usual splitting of the field into background \( \varphi \) and quantum \( \sigma \) parts, according to

\[
\varphi \rightarrow \varphi' = \varphi + \sigma. \tag{2}
\]

The one-loop effective action is given by the standard general expression

\[
\Gamma = \frac{i}{2} \text{Tr} \ln H, \tag{3}
\]

where \( H \) is the bilinear form of the action (1). Substituting (2) into (1), and taking into account the bilinear part of the action only, after making the necessary integrations by parts.
(the surface terms give no contribution to $\Gamma$), we obtain the following self-adjoint bilinear form:

$$H = 2b_1(\Box^2 + L^{\alpha\beta\gamma} \nabla_\alpha \nabla_\beta \nabla_\gamma + V^{\alpha\beta} \nabla_\alpha \nabla_\beta + N^\alpha \nabla_\alpha + U),$$

where the $L^{\alpha\beta\gamma}$ have the specially simple structure

$$L^{\alpha\beta\gamma} \nabla_\alpha \nabla_\beta \nabla_\gamma = \frac{1}{2b_1} [4b'_1 (\nabla_\mu \varphi) \nabla^\mu \Box] = L^\lambda \nabla_\lambda \Box.$$

The quantities $V^{\alpha\beta}$, $N^\alpha$ and $U$ are defined according to

$$V^{\alpha\beta} \nabla_\alpha \nabla_\beta = \frac{1}{b_1} \{(c''_3 - c'_1) R + (3b'_1 - 2b_2)(\Box \varphi) + (b''_1 - 2b_3)(\nabla_\mu \varphi)^2 - b_4 m^2\} \Box$$

$$+ [-c_2 R_{\mu\nu} + (2b'_2 - 4b_3)(\nabla_\mu \varphi)(\nabla_\nu \varphi) + 2b_2(\nabla_\mu \nabla_\nu \varphi)] \nabla^\mu \nabla^\nu \}$$

$$N^\alpha \nabla_\alpha = \frac{1}{b_1} \{(c''_3 - c'_1) R(\nabla_\mu \varphi) \nabla^\mu + (c'_3 - \frac{1}{2} c_2 - c_1)(\nabla_\mu R)(\nabla^\mu \varphi) + (2b_2 - c_2) R_{\mu\nu}(\nabla^\mu \varphi) \nabla^\nu$$

$$+ 2(b'_1 - 2b_3)(\Box \varphi)(\nabla_\mu \varphi) \nabla^\mu + 2(b'_2 - 3b_3)(\nabla_\nu \varphi)^2(\nabla_\mu \varphi) \nabla^\mu + 4(b'_2 - 2b_3)(\nabla_\nu \varphi)(\nabla^\nu \nabla_\mu \varphi) \nabla_\mu$$

$$+ 2b'_1(\nabla^\mu \Box \varphi) \nabla_\mu - b'_4 m^2(\nabla_\mu \varphi) \nabla^\mu \}$$

$$U = \frac{1}{b_1} \{(c''_3 - c'_1) R(\Box \varphi) + (c'_3 - \frac{1}{2} c'_2 - c'_1)(\nabla_\mu R)(\nabla^\mu \varphi) + (\frac{1}{2} c''_3 - c''_1) R(\nabla_\mu \varphi)^2$$

$$+ (\frac{1}{2} b''_2 - \frac{3}{2} b''_1)(\nabla_\mu \varphi)^4 + (2b''_2 - 4b_3)(\nabla_\nu \varphi)(\nabla^\nu \nabla_\mu \varphi)$$

$$- c'_2 R_{\mu\nu}(\nabla^\mu \nabla^\nu \varphi) + b'_1 (\Box^2 \varphi)$$

$$+ 2b'_1(\nabla_\mu \varphi)(\nabla^\mu \Box \varphi) + b'_2(\nabla_\mu \nabla_\nu \varphi)^2 + \frac{1}{2} c'_2 (\Box R) - \frac{1}{2} b'_4 m^2(\nabla_\mu \varphi)^2$$

$$- b'_4 m^2(\Box \varphi) + \frac{1}{2} a''_1 R_{\alpha\beta\gamma\tau} + \frac{1}{2} a''_2 R_{\alpha\beta} + \frac{1}{2} a''_3 R + \frac{1}{2} a''_4 R + \frac{1}{2} b'_4 m^4 \}.$$  

The next problem is to separate the divergent part of the trace (4). First of all, let us note that (4) is just a particular case of the general fourth-order operator which has been considered in [25]. However, direct use of the general results in [25] leads to very cumbersome calculations and we use a different procedure, already employed in [20]. Let us rewrite the trace (4) under the form

$$\text{Tr} \ln H = \text{Tr} \ln(2b_1) + \text{Tr} \ln(\Box^2 + L^\mu \nabla_\mu \Box + V^{\mu\nu} \nabla_\mu \nabla_\nu + N^\mu \nabla_\mu + U),$$

and notice that the first term does not give contribution to the divergences. Let us explore the second term. From standard considerations based on power counting and covariance, it follows that the possible divergences have the form

$$\text{Tr} \ln(\Box^2 + L^\mu \nabla_\mu \Box + V^{\mu\nu} \nabla_\mu \nabla_\nu + N^\mu \nabla_\mu + U)|_{\text{div}}$$

$$= \text{Tr} \{k_1 U + k_2 L^\lambda N^\lambda + k_3 L^\lambda \nabla_\lambda V - k_4 L^\lambda \nabla^\tau V_\tau$$

$$- k_5 \nabla_\lambda L^\lambda - k_6 V_\lambda L^\lambda + k_7 \nabla_\lambda L_\tau \nabla^\lambda L^\tau + k_8 \nabla_\lambda L_\tau \nabla^\tau L^\lambda \}.$$
\[ +k_0 L^\tau L^\lambda \nabla_\tau L_\lambda + k_{10} L_\lambda L_\tau L^\lambda L^\tau + k_{11} R^2_{\mu\nu} + k_{12} R^2 \]
\[ + k_{13} RV + k_{14} R_{\mu\nu} V^{\mu\nu} + k_{15} V^2 + k_{16} V_{\mu\nu} V^{\mu\nu} \} + (\text{s.t.}), \]

where \( k_{1\ldots16} \) are some (unknown) divergent coefficients.

The questions is now to find their explicit values in the one-loop approximation. It is easy to classify the terms in (7) into several groups. The first group is formed by the structures with numerical factors \( k_{1,11\ldots,16} \)—those are the ones which do not depend on \( L^\mu \). The divergences of this type are just the same as for the operator

\[ \Box^2 + V^{\alpha\beta} \nabla_\alpha \nabla_\beta + N^\alpha \nabla_\alpha + U, \]

and we can use the well-known values from [20]. To the second group belong the structures with \( k_{7\ldots10} \). Here we will use the following method. Since these structures do not contain \( V, N \) and \( U \), it is clear that \( k_{7\ldots10} \) will be just the same as for (6) with \( V = N = U = 0 \). Hence we can simply put \( V = N = U = 0 \). Then, taking into account that \( L^{\alpha\beta} \nabla_\alpha \nabla_\beta \nabla_\gamma = L^\lambda \nabla_\lambda \Box \), we can write

\[ \text{Tr } \ln(\Box^2 + L^\alpha \nabla_\alpha \Box) = \text{Tr } \ln(\Box) + \text{Tr } \ln(\Box + L^\alpha \nabla_\alpha). \]

The first term gives contribution to the \( k_{11,12} \) only, which we have already taken into account. The second term has a standard structure, and its contribution has a well-known form (see, for example, [1]).

The third group is just the mixed sector with coefficients \( k_{2\ldots6} \). Here we use the following method [23]. Performing the transformation

\[ \text{Tr } \ln(\Box^2 + L^\alpha \nabla_\alpha \Box + V^{\alpha\beta} \nabla_\alpha \nabla_\beta + N^\alpha \nabla_\alpha + U) \]
\[ = \text{Tr } \ln(1 + L^\alpha \nabla_\alpha \Box^{-1} + V^{\alpha\beta} \nabla_\alpha \nabla_\beta \Box^{-2} + N^\alpha \nabla_\alpha \Box^{-2} + U \Box^{-2}) + \text{Tr } \ln(\Box^2), \]

we can easily find that the second term contributes only to \( k_{11,12} \). Then we can expand the logarithm in the first term into a power series (see [1] for details) and use the universal traces of [23]. After a little algebra, we obtain the final result in the form:

\[ \text{Tr } \ln H = \frac{2i}{\varepsilon} \text{Tr } \{-U + \frac{1}{4} L^\lambda N_\lambda + \frac{1}{6} L^\lambda \nabla_\lambda V - \frac{1}{6} L^\lambda \nabla_\lambda V_{\lambda\tau} \]
\[ - \frac{1}{24} V L_\lambda L^\lambda - \frac{1}{12} V_{\lambda\tau} L^\lambda L^\tau + \frac{1}{2} P^2 + \frac{1}{12} S_{\mu\nu} S^{\mu\nu} + \frac{1}{30} R^2_{\mu\nu} \]
\[ + \frac{1}{60} R^2 + \frac{1}{12} RV - \frac{1}{6} R_{\mu\nu} V^{\mu\nu} + \frac{1}{48} V^2 + \frac{1}{24} V_{\mu\nu} V^{\mu\nu} \}, \]

where

\[ P = \frac{1}{6} R - \frac{1}{2} \nabla_\lambda L^\lambda - \frac{1}{4} L_\lambda L^\lambda, \]
\[ V = V^{\mu}_{\mu}, \]
\[ S_{\mu\nu} = \frac{1}{2} (\nabla_\nu L_\mu - \nabla_\mu L_\nu) + \frac{1}{4} (L_\nu L_\mu - L_\mu L_\nu). \]

Finally, substituting (6) into (11) and after a very tedious algebra which uses the reduction formulas (2), we arrive at the following result:

\[ \Gamma^{(1-\text{loop})}_{\text{div}} = -\frac{2}{\varepsilon} \int d^4 x \sqrt{-g} \left[ A_1 R^2_{\alpha\beta\gamma\tau} + A_2 R^2_{\alpha\beta} + A_3 R^2 + A_4 R m^2 \right. \]
\[ + \left. P^2 + S_{\mu\nu} S^{\mu\nu} + \frac{1}{30} R^2_{\mu\nu} \right]. \]
\begin{align*}
+C_1 R(\nabla^\mu \varphi)^2 + C_2 R_{\mu\nu}(\nabla^\mu \varphi)(\nabla^\nu \varphi) + C_3 R(\square \varphi) \\
+ B_1(\square \varphi)^2 + B_2(\nabla^\mu \varphi)^2(\square \varphi) + B_3(\nabla^\mu \varphi)^4 + B_4 m^2(\nabla^\mu \varphi)^2 + B_5 m^4, \quad (12)
\end{align*}

where

\begin{align*}
A_1 &= \frac{1}{90} - \frac{1}{2b_1} a''_1 \\
A_2 &= -\frac{1}{90} - \frac{1}{2b_1} a''_2 + \frac{1}{24} \frac{(c_2^2)}{b_1^2} + \frac{c_2}{6b_1} \\
A_3 &= \frac{1}{36} - \frac{1}{2b_1} a''_3 + \frac{1}{48b_1^2} [(4c'_3 - 4c_1 - c_2)^2 + 4(c'_3 - c_1)(2c'_3 - 2c_1 - c_2)] - \frac{1}{6b_1} (c_1 - c'_3 + \frac{1}{2}c_2) \\
A_4 &= -\frac{1}{2b_1} a''_4 - \frac{1}{4b_1^2} b_4 (4c'_3 - 4c_1 - c_2) - \frac{1}{6b_1} b_4 \\
B_1 &= -\frac{b''_1}{2b_1} + \frac{1}{4b_1^2} (8(b'_1)^2 - 10b'_1 b_2 + 5b'_2) \\
B_2 &= -\frac{b''_2}{2b_1} + \frac{1}{2b_1^2} (b''_1 b_2 + 4b'_1 b'_2 - 2b'_2 + 10b'_1 b_3 + 10b_2 b_3) - \frac{1}{2b_1^2} (2(b'_1)^2 b_2 + b'_2 b'_1) \\
B_3 &= -\frac{b''_3}{2b_1} + \frac{1}{3b_1^2} (5b''_1 b_3 + \frac{3}{4} (b'_2)^2 - 5b'_2 b_3 + 15b'_2 b_3 + 5b'_1 b'_3 + b_3 b'_3) \\
&+ \frac{1}{6b_1^2} (20(b'_1)^2 b_3 + 4b'_1 b_2 b_3 + b''_3 b'_2 + 2b'_1 b'_2 b_2) + \frac{b''_3 b'_1}{2b_1^2} \\
B_4 &= -\frac{b''_4}{2b_1} + \frac{1}{2b_1^2} (4b''_1 b_4 - 4b'_1 b_4 + 6b_2 b_4 - 5b'_1 b'_4 - 3b'_4 b_2) + \frac{1}{b_1} (3b'_1 b_2 b_4 - 5(b'_1)^2 b_4) \\
B_5 &= \frac{b''_5}{2b_1} + \frac{1}{2} \left( \frac{b_4}{b_5} \right)^2 \\
C_1 &= -\frac{c''_1}{2b_1} - \frac{2b_3}{3b_1} - \frac{1}{b_1} (\frac{1}{2} c' b'_3 - 3c'_3 b_3 + \frac{1}{2} b''_1 c_1 - \frac{1}{2} c_1 b'_2 + 3c_1 b_3 + \frac{1}{6} b''_1 c_2 - \frac{1}{6} c_2 b'_2 + \frac{2}{3} c_2 b_3) \\
&+ c'_1 b'_1 + \frac{1}{6} b'_1 b'_2 - \frac{1}{12} c'_2 b'_2 + \frac{1}{12} c'_2 b'_1) + \frac{1}{6b_1^2} (b'_1 b_2 c_2 - 2(b'_1)^2 c_2 - 6(b'_1)^2 c_1) \\
C_2 &= -\frac{c''_2}{2b_1} + \frac{2b_3}{3b_1} + \frac{1}{6b_1^2} (5b'_1 c_2 + 2c_2 b_3 + c'_2 b'_2 - b'_2) - \frac{b'_2 b_2 c_2}{3b_1^2} \\
C_3 &= -\frac{c''_3}{2b_1} + \frac{1}{3b_1} (c'_1 - b_2) + \frac{1}{6b_1^2} (12b'_1 c'_3 - 9b_2 c'_3 - 9b'_1 c_1 + 9c_1 b_2 - 2b'_1 c_2 + 2c_2 b_2) \\&+ \frac{1}{6b_1^2} (12b'_1 c'_3 - 9b_2 c'_3 - 9b'_1 c_1 + 9c_1 b_2 - 2b'_1 c_2 + 2c_2 b_2). \quad (13)
\end{align*}

Let us now briefly analyze the above expression. First of all, notice that the divergences \((12), (13)\) have just the same general structure as the classical action \((11)\). This fact indicates that the theory under consideration is renormalizable, what is in full accord with the more direct analysis based on power counting. All the divergences can be removed by a renormalization transformation of the functions \(a(\varphi), b(\varphi), c(\varphi)\), in analogy with two-dimensional sigma models. We do not include the renormalization of the quantum field \(\varphi\), since in the case of arbitrary \(b_1\) it leads to unavoidable difficulties. Let us now say some words about the possible role of the matter fields. Suppose that the dilaton model under consideration
is coupled to a set of free massless matter fields of spin $0, \frac{1}{2}, 1$. Then the matter fields contributions to the divergences of vacuum type lead to the following change of the functions $A_{2,3}(\varphi)$ (see, for example, [20]).

$$A_2 \to A_2 = A_2 + \frac{1}{60} \left( N_0 + 6N_{1/2} + 12N_1 \right),$$

$$A_3 \to A_3 = A_3 - \frac{1}{180} \left( N_0 + 6N_{1/2} + 12N_1 \right) + \frac{1}{2} \left( \xi - \frac{1}{6} \right)^2 N_0,$$

where $N_0, N_{1/2}$ and $12N_1$ are the numbers of fields with the corresponding spin, and $\xi$ is the parameter of the non-minimal interaction in the scalar field sector. Here we have omitted the topological Gauss-Bonnet term for simplicity. Thus, we see that even in the presence of the matter fields all the divergences can be removed by the renormalization of the functions $a(\varphi), b(\varphi), c(\varphi)$. (In the case massive scalars and spinors a matter contribution to $B_5$ and $A_4$ will also appear). Below it will be shown that the above change of $A_{1,2,3}(\varphi)$ does not affect our results seriously.

It is important to notice that renormalization of the generalized couplings $a(\varphi), b(\varphi), c(\varphi)$ explicitly manifests the properties which are usual for any quantum field theory in an external gravitational field [1]. All these functions can be easily separated into three groups, with a different renormalization rule. The first group is constituted by the $b(\varphi)$ functions. The renormalization of these functions is independent of the other functions, $a(\varphi), c(\varphi)$, and is similar to the renormalization of matter fields couplings in usual models (like the $\phi^4$ coupling constant in the case of an ordinary scalar field). The second group are the $c(\varphi)$ functions, which renormalize in a manner similar to that for the nonminimal constant $\xi$ of the $\xi R \phi^2$ interaction [1]. This means that their renormalization transformations are independent on $a(\varphi)$, but strongly depend on $b(\varphi)$. The third group of couplings is composed by the $a(\varphi)$, and they are similar to the parameters of the action of the vacuum for ordinary matter fields. Furthermore, the renormalization of the dimensionless functions does not depend on that of the dimensional ones, $a_4, b_4, b_5$, what is in good accord with a well-known general theorem [20]. Thus, the theory under consideration possesses all the standard properties of the models on a curved classical background. The only distinctive feature of the present one is that the couplings in our theory are arbitrary functions of the field $\varphi$. This fact can be interpreted as pointing out to the presence of an infinite number of coupling constants.

Since the theory is renormalizable, one can formulate the renormalization group equations for the effective action and couplings and then explore its asymptotic behaviour. The renormalization group equations for the effective action have the standard form, since the number (finite or infinite) of coupling constants is not essential for the corresponding formalism [1]. The general solution of this equation has the form

$$\Gamma[e^{-2t}g_{\alpha\beta}, a_i, b_j, c_k, \mu] = \Gamma[g_{\alpha\beta}, a_i(t), b_j(t), c_k(t), \mu],$$

where $\mu$ is the renormalization parameter and the effective couplings satisfy renormalization group equations of the form

$$\frac{da_i(t)}{dt} = \beta_{a_i}, \quad a_i = a_i(0),$$

$$\frac{db_i(t)}{dt} = \beta_{b_i}, \quad b_i = b_i(0),$$

(15)
\[
\frac{dc_i(t)}{dt} = \beta_{c_i}, \quad c_i = c_i(0). \tag{16}
\]

Note that we do not take into account the dimensions of the functions \(a_4, b_4, b_5\). In fact we consider here these quantities as dimensionless and suppose that the dimension of the corresponding terms in the action is provided by some fundamental nonrenormalizable constant. The beta-functions are defined in the usual manner. For instance,

\[
\beta_{b_1} = \lim_{n \to 4} \mu \frac{db_1}{d\mu}. \tag{17}
\]

The derivation of the \(\beta\)-functions is pretty the same as in theories with finite number of couplings, and we easily get

\[
\beta_{a_i} = -(4\pi)^{-2} A_i, \quad \beta_{b_i} = -(4\pi)^{-2} B_i, \quad \beta_{c_i} = -(4\pi)^{-2} C_i. \tag{18}
\]

In the next sections we shall present the analysis of the renormalization group equations (16),(18). In accordance with the considerations above, one can first explore the equations for the effective couplings \(b_1,..,5\), then for \(c_1,2,3\) and finally for the “vacuum” ones \(a_1,..4\). All that analysis looks much more simple for the conformal version of the theory.

4 The conformally-invariant theory and some explicit solutions

Let us now consider the most general conformally-invariant version of the theory (1):

\[
S_c = \int d^4x \sqrt{-g} \left\{ f(\varphi) \nabla^4 \varphi + q(\varphi) C_{\mu\nu\alpha\beta}^2 + p(\varphi) [(\nabla_\mu \varphi)(\nabla_\mu \varphi)]^2 \right\}. \tag{19}
\]

Here \(f(\varphi), q(\varphi)\) and \(p(\varphi)\) are arbitrary functions, \(\nabla^4 = \Box^2 + 2R^{\mu\nu} \nabla_\mu \nabla_\nu - \frac{2}{3} R \Box + \frac{1}{3}(\nabla^\mu R) \nabla_\mu\) is a fourth-order conformally invariant operator, and we should recall that due to the fact that \([\varphi] = 0\) the conformal transformation of our dilaton is trivial:

\[
g_{\mu\nu} \rightarrow e^{-2\sigma} g_{\mu\nu}, \quad \varphi \rightarrow \varphi. \tag{20}
\]

Now, using expressions (2), one can integrate by parts the rhs in (19) and present the result as a particular case of the theory (1), with

\[
a_1(\varphi) = q(\varphi), \quad a_2(\varphi) = -2q(\varphi), \quad a_3(\varphi) = \frac{1}{3} q(\varphi),
\]

\[
b_1(\varphi) = f'(\varphi) \varphi + f(\varphi), \quad b_2(\varphi) = f''(\varphi) \varphi + 2f'(\varphi) = b'_1(\varphi), \quad b_3(\varphi) = p(\varphi),
\]

\[
c_1(\varphi) = \frac{2}{3} f'(\varphi) \varphi + \frac{2}{3} f(\varphi) = -\frac{1}{3} c_2, \quad c_2(\varphi) = -2f'(\varphi) \varphi - 2f(\varphi). \tag{21}
\]

The rest of the generalized couplings \(a_4, b_4, b_5, c_3\) are equal to zero. So, the general action (1) is invariant under the conformal transformation (20) when the functions \(a_i, b_j, c_k\) obey the constraints (21).
Substituting the relations \((27)\) into the general expression for the divergences of the effective action, we get the divergences of the conformal theory in the form \((12)\), where instead of \((13)\) we have
\[
B_1 = -\frac{b_1''}{2b_1} + \frac{3(b_1')^2}{4b_1^2}, \quad B_2 = B'_1, \quad C_1 = \frac{2}{3}B_1, \quad C_2 = -2B_1, \quad B_4 = B_5 = 0, \\
B_3 = -\frac{b_1''}{2b_1} + \frac{1}{3b_1} \left[ \frac{3}{4}(b_1')^2 + 15b_1^2 + 6b_1'b_3 \right] - \frac{1}{2b_1} \left[ 8(b_1')^2b_3 + b_1'(b_1')^2 \right] + \frac{(b_1')^4}{2b_1^4}, \\
A_1 = \frac{1}{90} - \frac{1}{2b_1}a_1''', \quad A_2 = -2A_1, \quad A_3 = \frac{1}{3}A_1, \quad A_4 = A_5 = 0. \tag{22}
\]

As one can see from the last expressions, the divergences of the conformally-invariant theory \((13)\) appear also in a conformally invariant form (up to the total divergence), as it should be. The functions \(A(\varphi), B(\varphi), C(\varphi)\) obey the same conformal constraints \((21)\) with some \(F(\varphi), Q(\varphi), P(\varphi)\) instead of \(f(\varphi), q(\varphi), p(\varphi)\). Below, we shall use the functions \(a_i, b_j, c_k\), taking into account the restrictions \((21)\), because in this way calculations become more compact. Hence, we have shown that the conformal invariant, higher-derivative scalar theory considered here is renormalizable at the one-loop level in a conformally invariant way, and therefore it is multiplicatively renormalizable at one-loop. One can suppose that the general proof of one-loop conformal renormalizability in an external gravitational field, given in \([27]\) (see also \([1]\)), is valid for the higher-derivative dimensionless scalar field as well. Note that taking into account the matter field contributions does not lead, according to \((14)\), to the violation of conformal invariance. Indeed the conformal value of \(\xi = \frac{1}{6}\) must be chosen.

From a technical point of view, the cancellation of non-conformal divergences gives a very effective tool for the verification of the calculations. It moreover enables us to hope that the general dilaton model \([1]\) might be asymptotically conformal invariant \([28,1]\), just as the special case considered in \([1]\).

As a byproduct, the above expression also gives us the conformal anomaly of the conformal invariant theory \((13)\): \(T^{\mu}_{\nu}\) is equal to the integrand of \((12), \,(22)\) (up to total derivatives, that we have dropped). Thus, if one finds the form of the functions \(f(\varphi), q(\varphi), p(\varphi)\) which provide the one-loop finiteness in the theory \((13)\), the last will be free from the conformal anomaly. Actually, the one-loop effective action obeys the equation
\[
- \frac{2}{\sqrt{-g}} g^{\mu\nu} \frac{\delta \Gamma^{(1)}}{\delta g_{\mu\nu}} = T^{(1)}, \tag{23}
\]
where \(T^{(1)}\) is the one-loop part of the anomaly trace of the energy-momentum tensor. Eq. \((23)\) allows one to define \(\Gamma^{(1)}\) with accuracy up to some conformally invariant functional. Hence if we find the solution of the equations
\[
A_i (f(\varphi), q(\varphi), p(\varphi)) = B_j (f(\varphi), q(\varphi), p(\varphi)) = C_k (f(\varphi), q(\varphi), p(\varphi)) = 0, \tag{24}
\]
taking into account the constraints \((21)\), the right-hand side of Eq. \((23)\) will be zero (up to surface terms), and \(\Gamma^{(1)}\) will be a conformally invariant (but probably nonlocal) functional. So the solution gives us the conformal invariant theory \((13)\) that is free from the anomaly (at least on the one-loop level). Moreover, according to the structure of the conformal Ward identities \([27,1]\) it is clear, that the two-loop divergences of the corresponding theory will be conformally invariant as well.
The conditions (24) are nothing but a set of nonlinear (and rather complicated) ordinary differential equations. Fortunately, one can use the results of the qualitative analysis of the previous section and divide the equations into three groups \( B_j = 0, C_k = 0 \) and \( A_i = 0 \), respectively. It turns out that the only nontrivial problem is to explore the equations of the first group. Note that, due to the conformal constraints (21), the equation for \( b_3(\varphi) \) can be factorized out and we just have to deal with the ones for \( b_1(\varphi) \) and \( b_2(\varphi) \) first. Since the variable \( b_2(\varphi) \) is not independent, we end up with only one equation for \( b_1(\varphi) \) that can be solved, in principle (actually just a very reduced number of explicit solutions could be obtained, see below). The only three solutions of power-like form are the following:

\[
\begin{align*}
    b_1 &= k, \quad k = \text{const.}, \quad b_2 = 0, \quad b_3 = 0, \\
    b_1 &= k, \quad b_2 = b_1' = 0, \quad b_3 = \frac{3k}{5(\varphi - \varphi_0)^2}, \quad \varphi_0 = \text{const.},
\end{align*}
\]

and

\[
\begin{align*}
    b_1 &= \frac{k^2}{(\varphi - \varphi_0)^2}, \quad b_2 = b_1' = -\frac{2k^2}{(\varphi - \varphi_0)^3}, \quad b_3 = \frac{k^2}{(\varphi - \varphi_0)^4}.
\end{align*}
\]

We should observe that the second solution (26) is a particular point of a whole surface of conformal fixed points (i.e., conformal solutions) which can be expressed as

\[
\begin{align*}
    b_1 &= k, \quad b_2 = 0, \quad b_3 = F^{-1}(\varphi - \varphi_0),
\end{align*}
\]

where the function \( F(p) = x \) is the solution of the differential equation \( p'' - \frac{10}{k} p^2 = 0 \), and is given by the quadrature:

\[
\pm \int \frac{dp}{\sqrt{\frac{20}{3k} p^3 + c_1}} = x,
\]

with \( c_1 \) an arbitrary constant.

Since within the conformal theory the functions \( c_{1,2,3} \) are not independent, the corresponding equations are satisfied automatically. The equations for \( a_{1,2,3} \) have the following corresponding solutions. For (25) and (26), the common one

\[
a_1(\varphi) = \frac{(\varphi)^2}{90} + a_{11} \varphi + a_{12},
\]

and for (27),

\[
a_1(\varphi) = -\frac{1}{45} \ln |\varphi - \varphi_0| + a_{11} \varphi + a_{12},
\]

where \( a_{11} \) and \( a_{12} \) are integration constants and \( a_2(\varphi) \) and \( a_3(\varphi) \) are both defined via the conformal constraints (21). Let us notice that the above finite solutions (with evident numerical modifications) is stable under the contributions of the matter fields, that directly follows from (14).

In this way we have constructed three explicit examples of one-loop finite, anomaly free, conformal theories. The model (25) is essentially the same theory which had been investigated in previous articles [6]. It is closely related with the theory of induced conformal factor [4, 12, 13, 9]. Since the only nontrivial interactions here are of “nonminimal” and “vacuum” type, it is renormalized in a manner similar to the one for the theory of a free (ordinary) scalar field in an external metric field. That is why the finiteness of this model is...
rather trivial. Not so are the solutions \((26), (27)\) and \((28)\). These models contain nontrivial interaction sectors and their finiteness does not look trivial at all. Moreover, the form of solution \((27)\) probably indicates that some extra symmetry is present. Notice also that both nontrivial solutions depend on the arbitrary value \(\varphi_0\) and are singular in the vicinity of this value. One could argue that this fact hints towards the existence of some different, nonsingular parametrization of the field variable. This conformally invariant finite model might be quite interesting in connection with some attempts to generalize the \(C\)-theorem \([29]\) to four dimensions \([30]\).

5 Explicit non-conformal solutions

We now turn to the search of finite solutions of the general model \((1), (13)\), free of the conformal constraints. Since we are looking for non-conformally invariant solutions, \(b_1'\) must be different from \(b_2\) (otherwise we get back to the conformally invariant case). From the mathematical point of view, to obtain solutions of the general system \((24), (13)\) is a rather difficult problem, because in this case the equation \(B_3 = 0\) is not factorized out. So, already at a first stage, we are faced up with a set of the nonlinear, higher-dimensional differential equations. Fortunately, these equations exhibit some homogeneity property, and hence it is natural to look for solutions of the exponential form

\[
 b_j(\varphi) = k_j \exp \left[ (\varphi - \varphi_0) \lambda_j \right],
\]

where \(k_j\) and \(\lambda_j\) are some constants.

Accurate analysis shows that all the \(\lambda_j\) are necessary equal to zero in the exponential case. Quite on the contrary, the search for solutions of power type yields the following three non-conformal fixed points:

\[
\begin{align*}
 b_1 &= \frac{k}{(\varphi - \varphi_0)^{5/3}}, & b_2 &= -\frac{2k}{(\varphi - \varphi_0)^{8/3}}, & b_3 &= \frac{16k}{15(\varphi - \varphi_0)^{11/3}}, \\
 b_1 &= \frac{k}{(\varphi - \varphi_0)^{5/3}}, & b_2 &= -\frac{4k}{3(\varphi - \varphi_0)^{8/3}}, & b_3 &= \frac{4k}{9(\varphi - \varphi_0)^{11/3}},
\end{align*}
\]

and

\[
 b_1 = \frac{k}{(\varphi - \varphi_0)^{1/3}}, & b_2 = 0, & b_3 = 0.
\]

These are in fact the only solutions of power type. The solution of the equations for \(a_i(\varphi)\) and \(c_k(\varphi)\) is then straightforward (but involved). We shall present only the results of this analysis. For all three solutions \((32)\)–\((34)\), the \(a_i(\varphi)\) are given by the integrals

\[
 a_i = \int_{\varphi_i1}^{\varphi} \int_{\varphi_i2}^{\varphi} 2b_1(\varphi) \left[ A_i(\varphi) + \frac{1}{2b_1(\varphi)} a''_i(\varphi) \right].
\]

Notice that in the last expression the integrands do not depend on \(a_i(\varphi)\) while \(\varphi_{i1}\) and \(\varphi_{i2}\) are arbitrary constants. In the case of the theory coupled to matter fields the values of \(A_{1,2,3}\) have to be substituted according to \((14)\). The solutions for \(b_{4,5}\) and \(c_{1,2,3}\) are written below. For the case \((34)\), these solutions have the form

\[
 c_2 = r_0 x^\frac{\tilde{\delta}}{2} + r_1
\]
\[ c_1 = -\frac{2}{7} r_0 x^{\frac{4}{3}} - \frac{1}{3} r_1 + r_2 x^{\frac{2}{3}} + r_3 x^{-\frac{1}{3}} \]

\[ c_3 = -\frac{45}{364} r_0 x^{\frac{13}{3}} - \frac{1}{12} r_1 x - \frac{9}{10} r_2 x^{\frac{2}{3}} + \frac{2k - 9r_3}{10} x^{\frac{2}{3}} + r_4 + r_5 x^{\frac{7}{3}} \]

\[ b_4 = r_6 x^{\frac{4 + \sqrt{22}}{3}} + r_7 x^{\frac{4 - \sqrt{22}}{3}} \]

\[ b_5 = r_8 + r_9 x + \frac{9}{44} r_6 r_7 x^{\frac{4}{3}} \]

\[ + \frac{r_6^2}{k \left( 4 + \frac{2}{3} \sqrt{22} \right) \left( 5 + \frac{3}{2} \sqrt{22} \right)} x^{5 + \frac{4}{3} \sqrt{22}} + \frac{r_7^2}{k \left( 4 - \frac{2}{3} \sqrt{22} \right) \left( 5 - \frac{3}{2} \sqrt{22} \right)} x^{5 - \frac{4}{3} \sqrt{22}}, \] (36)

where, for the sake of brevity, we have denoted \( x \equiv \varphi - \varphi_0 \) and introduced the set of integration constants \( r_0, ..., r_9 \). For the cases when the \( b_i \) are given by (32) and (33), the solutions for \( c_i, b_4 \) and \( b_5 \) are still easily found in a closed form, but we will not bother the reader with such lengthy expressions here. Thus we have constructed the finite nonconformal versions of the theory (31). The functions \( a_i(\varphi), b_j(\varphi) \) and \( c_k(\varphi) \) above correspond to the finite theory.

6 Renormalization group and stability analysis

Here we apply a method of analysis based on the renormalization group for the investigation of the general model (11). If we do not impose the conditions (24) on the interaction functions, then the theory is not finite (of course, it is possible that there exist some other nonconformal finite solutions) but renormalizable. As it was already pointed out above, the renormalization group \( \beta \)-functions are defined in a unique way (18), and we arrive at the following renormalization group equations for \( a_i(\varphi), b_j(\varphi) \) and \( c_k(\varphi) \):

\[ \frac{da_i}{dt'} = -A_i, \quad \frac{db_j}{dt'} = -B_j, \quad \frac{dc_k}{dt'} = -C_k, \] (37)

where \( t' = (4\pi)^{-2} t \), and \( t \) is the parameter of the rescaling of the background metric (13).

The renormalization group equations (37) have a complicated structure. In fact the effective couplings \( a, b, c \) depend not only on \( t \), but also on \( \varphi \) and, therefore, (37) is nothing but a set of nonlinear, higher-order differential equations in terms of partial derivatives. For this reason, to obtain the complete solution of these equations does not seem to be possible. At the same time, we already know the values of \( a, b, c \) which correspond to vanishing \( \beta \)-functions. From the renormalization group point of view these values are the fixed points of the theory. Thus, we can explore the stability of the fixed points (37) and then formulate some conjectures concerning the asymptotic behaviour of the theory.

We thus face the problem of the stability analysis of a system with an infinite number of variables. A possible way to attack it consists in combining the standard Lyapunov method and harmonic Fourier analysis. Let us first illustrate the method on the most simple example of the conformal fixed point (26). The advantage of this solution is that the equations for \( b_1 \) and \( b_3 \) do not depend on each other. One can start with the equation for \( b_1 \), and put \( k = 1 \) for the sake of simplicity. Moreover, we shall write \( t \) instead of \( t' \). According to the Lyapunov method we write \( b_1 = 1 + y(x) \), where \( x = \varphi - \varphi_0 \) and \( y \) is the infinitesimal variation of \( b_1 \).
Hence we preserve the conformal constraint $b_2 = 1 + y'(x)$, where the derivative is taken with respect to $x$. Substituting the above expressions into the renormalization group equation, we get

$$\frac{dy}{dt} = \frac{1}{4(1+y)} \left[ 2y''_{xx} (1 + y) - \frac{3}{4} (y'_x)^2 \right].$$

(38)

Since we are only interested in the behaviour at the vicinity of the fixed point, the nonlinear terms of the last equation can be safely omitted, and we obtain

$$\frac{dy}{dt} = \frac{1}{2} y''_{xx}.$$  

(39)

This equation looks very simple but it depends still on two variables. However (39) can be easily reduced to a set of ordinary differential equations. One can expand $y(x)$ in Fourier series with $t$ dependent coefficients:

$$y(x, t) = \frac{y_0(t)}{2} + \sum_{n=1}^{\infty} y_n(t) \cos nx + \tilde{y}_n(t) \sin nx.$$  

(40)

Substituting (40) into (39) we obtain

$$\frac{dy_0}{dt} = 0, \quad \frac{dy_n}{dt} = -\frac{n^2}{2} y_n, \quad \frac{d\tilde{y}_n}{dt} = -\frac{n^2}{2} \tilde{y}_n.$$  

(41)

From (41) it follows that all the coefficients except for $y_0$ vanish in the limit $t \to +\infty$. Since the infinitesimal variation cannot contain a zero mode, one can put $y_0 = 0$ and hence the fixed value $b_1 = 1$ is stable in the mentioned limit. Notice that if we do not input the conformal constraints, that is, if we take $\delta b_2 \neq (\delta b_1)'$, then the values $b_1 = 1$, $b_2 = 0$ give a saddle point of the theory.

Then one can start with $b_3$, what is a bit more complicated. If one introduces the infinitesimal variation $z$ as $b_3 = \frac{5}{3(\varphi - \varphi_0)} + z$ and omits all the nonlinear terms, the remaining equation is

$$\frac{dz}{dt} = \frac{1}{2} \frac{z''_{\varphi \varphi}}{\varphi' - \varphi_0} - \frac{50}{3} \frac{z^2}{\varphi - \varphi_0}.$$  

(42)

If we consider the behaviour of $z$ in a region far from the value of $\varphi_0$, then the factor $(\varphi - \varphi_0)^{-1}$ is slowly varying and one can regard it as a constant $x_0$. After expanding $z$ into a Fourier series, we get

$$\frac{d z_0}{dt} = -\frac{50}{3} \frac{z_0}{x_0^2}, \quad \frac{d z_n}{dt} = \left( -\frac{1}{2} n^2 - \frac{50}{3 x_0^2} \right) z_n, \quad \frac{d \tilde{z}_n}{dt} = \left( -\frac{1}{2} n^2 - \frac{50}{3 x_0^2} \right) \tilde{z}_n,$$  

(43)

what reveals the stable nature of this conformal fixed point. The exploration of the behaviour of the $c_{1,2,3}$ is not necessary, because they are related with $b_1$ by the conformal constraints, and thus their behaviour is completely determined.

If one takes the values of the infinitesimal corrections which violate the conformal constraints then this fixed point is a saddle one. The last claim is actually trivial, since we already knew this from the behaviour of $b_2$. Stability analysis performed on the last of the non-conformal solutions (34) shows that it is a saddle point of the non-conformally invariant theory. It is clear already from the behaviour of $b_1, b_2, b_3$ and hence further investigation is not necessary.
So we can see that among the fixed points of the theory there are some which are completely stable in UV limit and others which are partially stable, namely saddle points of the renormalization group dynamics. One can conjecture that the behaviour of the functions $a(\varphi), b(\varphi), c(\varphi)$ essentially depends on the choice of the initial data (with respect to the renormalization group parameter), which have to be postulated at some given energy. In particular, it is natural to expect that for some conformal models at high energies, asymptotic finiteness manifestly appears. Simultaneously, there is a cancellation of the conformal anomaly in this limit. In such way, the theory predicts the existence of renormalization group flows from arbitrary values of $a, b, c$ to the one which provides finiteness and conformal invariance of the theory.

7 Discussion

We have investigated the renormalization group behaviour of the general dilaton model on the background of a classical metric. The theory under consideration possesses interesting nontrivial features, as finite fixed points and plausible renormalization group flows between these points. This fact has important physical applications, if we make use of the hypothesis in [7] and regard the dilaton theory as an approximation to some more fundamental theory of quantum gravity (like the theory of strings) at low energies. The action of gravity, induced by string loop effects, has the form of a series in the string loop parameter $\alpha'$, and at second order it contains the terms with fourth derivatives of the target space metric and the dilaton [32]. Thus, within some accuracy, the effective action of the string is a particular case of our dilaton model (the well known arbitrariness in the second order effective action for the string does not affect our speculations here). This particular case is not a fixed point of our model (maybe only at one loop). One can suppose that our theory of the dilaton is valid at scales between the Planck energy $M_p$ and some energy $M_l$, where the effects of quantum gravity are weak and only matter fields can be regarded as quantum ones. It is rather remarkable that the action for the dilaton —generated by quantum effects of the matter fields— is an IR fixed point of our general dilaton model. Hence our dilaton model can describe the transition from string induced dilaton gravity at the $M_p$ scale to matter induced gravity at the $M_l$ scale.

We can also say some words about the expected effects of the quantum metric. In spite of the fact that the theory is rather involved, one can calculate the one-loop divergences with the use of the method proposed in [14]. Moreover, some conjectures concerning the renormalization of the theory of quantum gravity based on can be made even without carrying out calculations to the end explicitly. As has been already pointed out above, the general structure of the expressions for the counterterms will be similar to (13). This means that all the structures (but not necessarily the numerical coefficients, of course) will be actually the same. However, the structure of the renormalization might be much more complicated. In particular, for the theory of quantum gravity the hierarchy of the couplings is lacking and all the functions $b, c, a$ have to be renormalized simultaneously, what is rather more cumbersome as compared with the dilaton theory described above. However, the general structure of the counterterms in the case of the quantum metric must be the same as for our dilaton model. In particular, the functions $A_i, B_j, C_k$ are expected to be homogeneous just as in the case considered above. Hence one can hope to get similar finite solutions in the general theory.
The final point of our discussion is related with the conformal invariance properties at the quantum level. Some features of the theory of quantum gravity based on (19) (with \( p(\varphi) = 0 \)) have been recently discussed in [15]. It was shown there that, generally, the theory leads to a conformal anomaly. This anomaly appears already in the one loop counterterms and prevents the theory from being renormalizable. For this reason, we cannot expect from a theory of quantum gravity based on (1) to have a conformal fixed point. However, it should be possible to obtain conformal invariance at the quantum level within the general model (1), by introducing the loop expansion parameter in an explicit way.

It would be of interest to study the cosmological consequences that arise from the family of finite models (1), as the possible existence of solutions of black hole type and their influence on the evolution of the early universe. The dilaton in the starting theory is massive, owing to the nontrivial dimensions of the functions \( b_4(\phi), b_5(\phi) \). For the finite conformal versions of the theory it is not so more. However one can get massive parameters as a result of some symmetry breaking and for this purposes it is necessary, for instance, to derive the effective potential and to explore the possibility of a phase transition (see [31, 11] for the discussion of that approach). Indeed, the effective potential in dilatonic gravity under discussion has the form (in the linear curvature approximation)

\[
V = b_5(\varphi) + \frac{1}{2} B_5(\varphi) \ln \frac{\chi(\varphi)}{\mu^2} + R \left[ a_4(\varphi) + \frac{1}{2} A_4(\varphi) \ln \frac{\chi(\varphi)}{\mu^2} \right],
\]

where \( \chi(\varphi) \) is some combination of the dimensional functions \( a_4 \) and \( b_4 \). Its explicit form plays no role in this qualitative discussion. It is clearly seen that the one-loop level effective action of our theory at low energies represents the standard Einstein theory with \( \varphi \)-dependent cosmological and gravitational constants. Hence, our theory leads to the induction of general relativity at low energies, what serves as an additional physical motivation for its detailed study. Notice also that one can introduce massive terms even in the conformal case, what is something like soft breaking of the conformal invariance. It is possible to provide finiteness even in this case (as well as in nonconformal versions of the theory, of course). We expect to return to such questions elsewhere.

Summing up, we have explored some features of the general dilaton model (1) which can be regarded as a toy model for the same theory with a quantum metric. Some special versions of the model are finite at one loop and, moreover, some of them are conformally invariant both at the classical and at the quantum level. The lack of conformal anomaly holds even if the matter field contributions are taken into account. The last property is likely to survive for the more general model with a quantum metric. In this respect the theory discussed above is the first example of such kind. Furthermore we have investigated its stability of found several fixed points (developing by the way new mathematical tool for this purposes). This enables us to draw some conclusions on the possibility of renormalization group flows between the different versions of the theory. In particular, one can hope to apply our model to obtain the connection between the string induced gravity action at the Planck energy scale and the matter field induced action at some lower scale, what is certainly valuable for phenomenology purposes.

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