Simulation of Temperature Distribution
In a Rectangular Cavity Using Finite Element Method

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Abstract
This paper presents the study and implementation of finite element method to find the temperature distribution in a rectangular cavity which contains a fluid substance. The fluid motion is driven by a sudden temperature difference applied to two opposite side walls of the cavity. The remaining walls were considered adiabatic. Fluid properties were assumed incompressible. The problem has been approached by two-dimensional transient conduction which applied on the heated sidewall and one-dimensional steady state convection-diffusion equation which applied inside the cavity. The parameters which investigated are time and velocity. These parameters were computed together with boundary conditions which result in temperature distribution in the cavity. The implementation of finite element method was resulted in algebraic equation which is in vector and matrix form. Therefore, MATLAB programs used to solve this algebraic equation. The final temperature distribution results were presented in contour map within the region.

Keywords: conduction, convection-diffusion, finite element method

I. INTRODUCTION

In order to analyze an engineering system, a mathematical model is developed to describe the behavior of the system. The mathematical expression usually consists of differential equations and given conditions. These differential equations are usually very difficult to obtain solution if handled analytically. The alternative way to solve the differential equation is using numerical method.

The numerical method has some advantage, not only it can solved complicated equations which are hardly to solve analytically but also can reduce cost which needed to make experiments. Moreover, numerical method is able to predict the physical phenomena so it can be studied and implemented to make some device.

Considering the advantage of the numerical method, this paper uses it to analyze physics phenomena in heat transfer problem. Heat transfer problem that will be discussed is a temperature distribution over a rectangular cavity which is exists because of the sudden temperature difference applied to two opposite side walls of the cavity.

The numerical method which is used is finite element method. The reason why in this paper chooses the finite element method is because the finite element method is one of the numerical method that has received popularity due to its capability for solving complex structural problem. Moreover, the finite element method provides high computational flexibility and on the other hand facilitates a rigorous mathematical error analysis.

After the mathematical analyses take place, the finite element programming requires software to solve algebraic equations which is vector and matrix manipulation. Therefore, for this paper the programming uses MATLAB interactive software.

II. PROBLEM DEFINITION

The objective of this paper is finding the temperature distribution in a rectangular cavity as shown in Figure 1.
The cavity considered built by thin metal sheet homogenous material properties as the boundaries, inside the cavity consist a homogenous fluid substance. The initial temperature inside the cavity is considered to be $0^\circ C$. Suddenly, the cavity heated on the left sidewall while the right one is keeps at zero temperature, meanwhile the other side of the cavity is remain insulated.

The heated sidewall (on $x$ and $z$ axis) is shown in Fig. 2.

\[ T = 100 \quad \text{for} \quad 0.75 < x < 1.25 \text{ and } 0.75 < z < 1.25 \]

Next, the fluid inside the cavity is considered homogenous and incompressible (constant density and pressure) which flow by the present of constant velocity $v$ in $y$ axis direction (one-dimensional direction). This yields to a laminar flow inside the cavity. This means the flow inside the cavity is concerned to be a natural convective flow. In this paper, the analysis of the flow is considered in steady state condition or time independent. The equation which governs this problem is convection-diffusion equation which is similar with diffusion equation except the present of the convective terms.

\[ \frac{1}{\alpha} \frac{d^2T}{dy^2} = \nu \frac{dT}{dy} \]

Similar with thermal diffusivity $\alpha$, the velocity $\nu$ explained how quickly fluid can carry heat away from hot source.

### III. FINITE ELEMENT METHOD

The basic idea in the finite element method is to find the solution of a complicated problem by replacing it by a simpler one. Since the actual problem is replaced by a simpler one; the solution is only an approximation rather than the exact one. In the finite element method the solution domain is divided into small domain called elements; these elements connected each other by nodes, therefore this method called finite element. This subdivision process is called discretization.

In each element, a convenient approximate solution is assumed and the conditions of overall equilibrium of the structure are derived. The derivation is using several mathematical analyses, likewise integration by parts, weighted residual and Galerkin’s method. Once this analyses done, each element will form matrix equation, which resulting in global matrix equations. These global matrix equations solved by MATLAB to get the solution for each node.

All this processes are the basic procedures for the finite element method. Systematically, each of the basic procedures will be discussed separately for conduction and convection problem.

**Finite Element Method for Conduction Problem**

Figure 3 shows the discretization process for the conduction on left side of the cavity. The region is considered as 128 triangular elements (red numbers) and 81 interconnected nodes (blue numbers).

After the discretization process, the mathematical analyses take place; it does begin with method of weighted residual. Weighted Residual method is used for finding the approximation for the governing equation. The method is multiply the trial function and a weight function, then we take the integral value in all domains, the value must be 0 (this value means, we make the error for the assumption function equal to zero or no error at all).
The triangular element becomes linear in variable interpolation within the element is linear and it shown in Fig. 4 of Galerkin’s Method.

The approximation and it can be easily differentiate because this function can be easily differentiate and integrated. It also has advantages; if necessary the approximation quality can be improved by increase the polynomial degree. The most useful function is polynomial function.

Here, insulated there is no flux across the boundary term. When the boundary is insulated so there is no flux in the boundary term. When the boundary term vanish.

The $T$ function or trial function must be simple and easily to differentiate and integrated, the most useful function is polynomial function because this function can be easily differentiate and integrated. It also has advantages; if necessary the approximation quality can be improved by increase the polynomial degree. The $w$ function is take as same as $T$ function, this technique called Galerkin’s Method.

From Fig. 3, the element is triangular element and it shown in Fig. 4. The triangular element has three nodes at the vertices of the triangle and the variable interpolation within the element is linear in $x, z$, and $t$. The interpolation for linear triangular element becomes

\[ T(x, z, t) = H_1(x, z)T_1(t) + H_2(x, z)T_2(t) + H_3(x, z)T_3(t) \]

\[ I = \int_{\Omega} (wT) d\Omega = 0, \quad (3) \]

\[ \int_{\Gamma} w \frac{\partial T}{\partial n} d\Gamma = 0 \quad (4) \]

Here, $\Omega$ is all domain and $\Gamma$ is boundary condition. The integration result in another term, the boundary term. When the boundary is insulated there is no flux \( \left( \frac{\partial T}{\partial n} = 0 \right) \), the boundary term vanish.

The $T$ function or trial function must be simple and easily to differentiate and integrated, the most useful function is polynomial function because this function can be easily differentiate and integrated. It also has advantages; if necessary the approximation quality can be improved by increase the polynomial degree. The $w$ function is take as same as $T$ function, this technique called Galerkin’s Method.\(^1\)

Here, $H_i(x, y)$ are the shape functions,

\[ H_1(x, z) = \frac{1}{2A} \left[ (x_2z_3 - x_3z_2 + (z_2 - z_3)x + (x_3 - x_2)z) \right] \]

\[ H_2(x, z) = \frac{1}{2A} \left[ (x_3z_1 - x_1z_3 + (z_3 - z_1)x + (x_1 - x_3)z) \right] \]

\[ H_3(x, z) = \frac{1}{2A} \left[ (x_1z_2 - x_2z_1 + (z_1 - z_2)x + (x_2 - x_1)z) \right] \]

Here, $A$ is the area of the triangle.

Equation (4) consists of two terms, the spacial terms and transient terms. The spacial terms with substitution of the shape function together with Galerkin’s Method becomes

\[ [K^e] \{T^e\} = \int_{\Omega^e} \left( \frac{\partial w}{\partial x} \frac{\partial T}{\partial x} + \frac{\partial w}{\partial z} \frac{\partial T}{\partial z} \right) d\Omega = \]

\[ \int_{\Omega^e} \left( \begin{array}{ccc} \frac{\partial H_1}{\partial x} & \frac{\partial H_1}{\partial z} & \frac{\partial H_1}{\partial x} \\ \frac{\partial H_2}{\partial x} & \frac{\partial H_2}{\partial z} & \frac{\partial H_2}{\partial x} \\ \frac{\partial H_3}{\partial x} & \frac{\partial H_3}{\partial z} & \frac{\partial H_3}{\partial x} \end{array} \right) \{T^e\} d\Omega \]

Here, $\Omega^e$ \( \subset \Omega \) the element domain. Performing integration after substituting the shape functions will gives symmetric matrix.
Meanwhile, the element matrix for transient term becomes

\[
[M^e]\{\tilde{T}^e\} = \int_{\Omega} w \frac{dT}{dt} d\Omega = \\
\int_{\Omega} \begin{bmatrix} H_1 & H_2 & H_3 \\ H_2 & H_3 & H_1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} d\Omega
\]  
(8)

Here, \( \tilde{T} \) is temperature derivative with respect to time, computation Eq. (8) results in

\[
[M^e] = \frac{\Delta t}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}
\]  
(9)

Once again, this matrix form result in symmetrical matrix. Therefore, the final matrix equation for Eq. (4) becomes

\[
[M]\{\tilde{T}\}^t + [K]{\{\tilde{T}\}}^t = \{F\}^t
\]  
(10)

The column vector \( \{F\}^t \) is the boundary conditions to satisfy the matrix equation. Because this equation should be true for any time, the superscript \( t \) in Eq. (10) placed to denote the time when the equation is satisfied. Furthermore, matrices \([M]\) and \([K]\) are independent of time.

To solve Eq. (10), the finite difference method takes place. Equation (10) can be written at time \( t + \Delta t \)

\[
[M]\{\tilde{T}\}^{t+\Delta t} + [K]{\{\tilde{T}\}}^{t+\Delta t} = \{F\}^{t+\Delta t}
\]  
(11)

The time derivatives in the backward difference is

\[
\{\tilde{T}\}^{t+\Delta t} = \frac{\{\tilde{T}\}^{t+\Delta t} - \{\tilde{T}\}^t}{\Delta t}
\]  
(12)

Use Eq. (11) with Eq. (10) results in

\[
([M] + \Delta t[K]){\{\tilde{T}\}}^{t+\Delta t} = \Delta t\{F\}^{t+\Delta t} + [M]{\{\tilde{T}\}}^t
\]  
(13)

By choosing the appropriate step time and start with \( t = 0 \), the final solution will be found. All these calculation implemented with MATLAB structured program.

**Finite Element Method for Convection Problem**

Meanwhile, the discretization process for convection inside the cavity is shown in Fig. 5. The domain considers build up by 9 equal size small elements and interconnected by 10 nodes.

**Figure 5. Discretization Process for Heat Convection-Diffusion Problem**

Figure 5 shows representation from the actual problem. The actual problem consist 5 unit lengths on y axis, this means the actual problem consist 45 elements with 50 nodes.

After the discretization, similar with conduction problem, the method of weighted residual and Galerkin’s takes place, Eq. (2) becomes

\[
l = \int_{\Omega} \left( w \frac{d^2 T}{dy^2} - \nu w \frac{dT}{dy} \right) d\Omega = 0
\]  
(14)

Here, \( \Omega \) is the element domain. Thus, by Integration by parts Eq. (14) becomes

\[
l = \int_{\Omega} \left( - \frac{d\omega}{dy} \frac{dT}{dy} - \nu w \frac{dT}{dy} \right) d\Omega + \left[ w \frac{dT}{dy} \right]_{\Omega} = 0
\]  
(15)

Equation (15) contain extra boundary terms which neglected due to weight function in boundary is zero. The interpolation for one-dimensional element is

\[
T = H_1(y)T_1 + H_2(y)T_{i+1}
\]  
(16)

Where

\[
H_1(y) = \frac{y - y_i}{h_i}
\]  
(17a)

\[
H_2(y) = \frac{y_i - y}{h_i}
\]  
(17b)

\[
h_i = y_{i+1} - y_i
\]  
(17c)

Subscript \( i \) denotes node number and \( h_i \) is element length. By using the interpolation formula in Eq. (16) and the Galerkin’s method, Eq. (15) becomes

\[
\int_{y_i}^{y_{i+1}} \left( - \frac{d\omega}{dy} \frac{dT}{dy} - \nu w \frac{dT}{dy} \right) dy = \\
\left( - \int_{y_i}^{y_{i+1}} \left\{ H_1' \right\} \left\{ H_2' \right\} d\Omega \right) \{ T_1 \} + \{ T_{i+1} \}
\]  
(18)

where \( ()' \) denotes the derivatives with respect to x. Evaluating Eq. (18) gives

\[
[K^e]{\{T^e\}} = \left( - \frac{1}{h_i} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\nu}{2} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right) \begin{bmatrix} T_1 \\ T_{i+1} \end{bmatrix}
\]  
(19)
The matrix final form becomes
\[
[K][T] = [F]
\]  
(20)

Similar with conduction analysis, the column vector \([F]\) is the boundary conditions to satisfy the matrix equation. All these calculation implemented with MATLAB structured program.

**IV. THE APPROACHING MODEL**

To solve this problem, the inside of the cavity was divided into seven layers, each layer consist of seven lines, but not include the boundaries. That means the three dimension problem was approached by two dimension model and then by one dimension model. This approaching model is shown in Fig. 6.

![Figure 6. The Approaching Model: The Cavity Divided Into Seven Layers, Each Layer Consist of Seven Lines](image)

To get the temperature distribution inside the cavity several steps took place. First, temperature distribution on the left side of the cavity solved using the conduction finite element formulation which yields to the temperature value in each node on the left side of the cavity. Second, these values became the boundary conditions for convection-diffusion finite element formulation which yields to the temperature distribution on each line. Next, these line temperature distributions which in the same layer were combined to get the contour map for each layer. These contour maps were the final solution which will be given in the next sections.

**V. RESULT AND DISCUSSION**

The simulations of the temperature distribution inside the cavity involve two parameters, the time \(t\) and the velocity \(v\). The time parameter governs the left side heat conduction. Meanwhile, the velocity parameter governs the heat convection-diffusion. Therefore, both parameters were discussed separately.
The velocity based analysis took the comparison between velocity 0.5 m/s and 5 m/s at the same time (10 s). The results are shown in Fig. 9 and Fig. 10. This velocity differences made the heat flow faster as the velocity increase. Therefore, when the velocity has low value, the heat distribution inside the cavity is quite even, means that the high temperature and low temperature was spread evenly. As the velocity higher, the heat distribution inside the cavity was not spread evenly anymore. Therefore, the high temperature dominated the low one.

VI. CONCLUSION

This paper presents the study and implementation of finite element method to find the temperature distribution inside a rectangular cavity. The time and velocity parameters that influence the temperature distribution were analyzed by the help of the contour map.

It was shown that the time parameter governs the conduction on the heated sidewall, this implication is due to the conduction is time dependent problem (Eq. 1). As the time increase, the overall temperature in the sidewall becomes increase too and finally it reached steady state condition after 10 second (Fig. 9 and Fig. 10). The results of the heated sidewall conduction yield to
the boundary condition for the convection-diffusion inside the cavity. It was shown in contour map for each layer inside of the cavity.

Meanwhile, the velocity parameter governs the convection-diffusion inside the cavity. This implication is due to the convective terms in convection-diffusion steady state equation (Eq. 2). As the velocity increase, the temperature distribution inside the cavity was dominated by the temperature conditions on the heated sidewall. These findings were shown in contour map layers (Fig. 9 and Fig.10).

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