The Temperley–Lieb algebra and its generalizations in the Potts and $XXZ$ models

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Abstract. We discuss generalizations of the Temperley–Lieb algebra in the Potts and $XXZ$ models. These can be used to describe the addition of integrable boundary terms of different types.

We use the Temperley–Lieb algebra and its one-boundary, two-boundary, and periodic extensions to classify different integrable boundary terms in the two-, three-, and four-state Potts models. The representations always lie at critical points where the algebras becomes non-semisimple and possess indecomposable representations. In the one-boundary case we show how to use representation theory to extract the Potts spectrum from an $XXZ$ model with particular boundary terms and hence obtain the finite size scaling of the Potts models. In the two-boundary case we find that the Potts spectrum can be obtained by combining several $XXZ$ models with different boundary terms. As in the Temperley–Lieb case, there is a direct correspondence between representations of the lattice algebra and those in the continuum conformal field theory.

Keywords: algebraic structures of integrable models, integrable quantum field theory, symmetries of integrable models
Contents

1. Introduction ........................................... 3

2. Temperley–Lieb algebra .............................. 5
   2.1. Representations .................................. 6
      2.1.1. XXZ ........................................ 6
      2.1.2. Two-state Potts (Ising). .................. 6
      2.1.3. Three-state Potts. .......................... 6
      2.1.4. Four-state Potts. ........................... 7
   2.2. Potts spectra within XXZ spectra (review) ........... 8

3. One-boundary Temperley–Lieb algebra ................. 10
   3.1. Representations ................................ 11
      3.1.1. XXZ ........................................ 11
      3.1.2. Two-state Potts (Ising). .................. 11
      3.1.3. Three-state Potts. .......................... 12
      3.1.4. Four-state Potts. ........................... 12
   3.2. One-boundary TL representation theory ............... 13
      3.2.1. Generic: $\gamma$ and $\omega$ arbitrary. .... 14
      3.2.2. Critical: generic $\gamma$ but $\omega = \gamma \mathbb{Z}$. 15
      3.2.3. Doubly critical: $\gamma$ a rational multiple of $\pi$ and $\omega = \gamma \mathbb{Z}$. 16
   3.3. Potts spectra within XXZ spectra in the one-boundary cases .... 17
      3.3.1. Two-state Potts (Ising). .................. 17
      3.3.2. Three-state Potts. .......................... 17

4. Two-boundary Temperley–Lieb algebra ................. 19
   4.1. Representations ................................ 20
      4.1.1. XXZ ........................................ 20
      4.1.2. Two-state Potts (Ising). .................. 21
      4.1.3. Three-state Potts. .......................... 21
      4.1.4. Four-state Potts. ........................... 23
   4.2. Finding Potts spectra within the XXZ spectra ......... 26

5. Finite size scaling limits ........................... 29
   5.1. Temperley–Lieb and IBTL cases ..................... 29
   5.2. Truncation in the Temperley–Lieb quotient cases ....... 32
   5.3. Ising model .................................... 32
   5.4. Three-state Potts model .......................... 34

6. Periodic Temperley–Lieb algebra ...................... 36
   6.1. Representations ................................ 37
      6.1.1. XXZ ........................................ 37
      6.1.2. Two-state Potts (Ising). .................. 37
      6.1.3. Three-state Potts. .......................... 38
      6.1.4. Four-state Potts. ........................... 38
   6.2. Potts within XXZ ................................ 39

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1. Introduction

In this paper we shall use generalizations of the Temperley–Lieb algebra to discuss the XXZ and Potts models with integrable boundary terms of different types. This will allow us to find relations between the spectra of these models and hence derive finite size scaling results in the Potts models from known analytical results in the XXZ model.

The Ising model is probably the best studied statistical model in two dimensions. The $Q$-state Potts models are a generalization of this in which the symmetry group is the symmetric group, $S_Q$. We shall study the cases $Q = 2, 3, 4$ which are known to have a second-order phase transition [1]. The XXZ model is another extremely well-studied integrable model. In this paper we shall discuss representations of the Temperley–Lieb (TL) algebra [2, 3] and its one-boundary, two-boundary, and periodic extensions [4] which appear in the XXZ and Potts models. These are of interest as both the XXZ and Potts models can be written in terms of the same algebraic Hamiltonian. The XXZ model can be solved using Bethe ansatz techniques [5]–[7]. As we shall see, in many cases the XXZ spectra contain all the Potts spectra and using the underlying algebra the spectra and finite size scaling behaviour of the Potts models can now be extracted, in a completely controlled way, from those of the XXZ models.

In section 2 we review the generators of the TL algebra in the XXZ and Potts representations. The TL algebra depends on a single parameter $q$. In the two-, three-, and four-state Potts representations, $q$ takes the values $e^{\pi i/4}$, $e^{\pi i/6}$, and 1 respectively. In contrast the XXZ representation exists for any value of $q$. The Temperley–Lieb algebra is semisimple for generic values of $q$ and all representations are fully decomposable. However for exceptional points when $q$ is a root of unity (relevant for Potts models) the algebra becomes non-semisimple and possesses both ‘good’ irreducible and ‘bad’ indecomposable representations. For the XXZ representation this structure can also be understood by studying the centralizer, the quantum group $SU_q(2)$, which commutes with each generator [8].

The $L$-site two-, three-, and four-state Potts models with free boundary conditions and the $2L$-site XXZ model with $SU_q(2)$ boundary conditions at the points $q = e^{\pi i/4}$ and $q = e^{\pi i/6}$ and $q = 1$ can be written in terms of one Hamiltonian in two different representations of the TL algebra. The XXZ representation of TL is known to be faithful [9] and therefore evaluating a TL Hamiltonian in the XXZ representation gives all the eigenvalues allowed by the algebra and any other representation (e.g. Potts) must give, up to degeneracies, some subset of the XXZ eigenvalues. This explains the numerical
coincidences found in [5]–[7]. The fact that these spectral coincidences continue to be found for inhomogeneous chains emphasizes their algebraic, rather than integrable, nature. It is only the energy levels of the ‘good’ states of the \(SU_q(2)\) invariant \(XXZ\) models that are present in the Potts models with free boundary conditions.

An interesting generalization of the TL algebra known as the one-boundary TL (1BTL), or blob, algebra has been well studied in the mathematical literature [10]–[13]. The 1BTL algebra depends on two parameters, one for the bulk generators and one for the boundary generator. In section 3.1 we give the Potts and \(XXZ\) representations of the 1BTL algebra. The \(XXZ\) representation involves a non-diagonal boundary operator. It exists for general values of the parameters and its structure can also be understood from a detailed study [14] of its centralizer [15, 16]. In the Potts representations we shall see how the bulk symmetry can be broken in different ways by the presence of integrable boundary terms.

In section 3.2 we discuss the representation theory of the 1BTL algebra [10]–[13]. As in the quantum group case the values of the parameters realized in the Potts representations correspond to exceptional cases of the 1BTL algebra in which one has both ‘good’ irreducible and ‘bad’ indecomposable representations. In section 3.3 we show that the energy levels of the \(L\)-site two-and three-state Potts models with different types of integrable boundary term added to one end can be found within the \(2L\)-site \(XXZ\) models with a single non-diagonal boundary term. The explanation of this phenomenon is similar to that in the TL case—here they are simply two different representations of the 1BTL algebra. The energy levels from the \(XXZ\) chain which occur in the Potts models correspond to the ‘good’ representations of the 1BTL algebra. In contrast to the TL case the issue of faithfulness of the \(XXZ\) representation is less clear [17]. The results of this paper concern only the irreducible representations. They are compatible with the statement that the \(XXZ\) representation contains all possible irreducible structure allowed by the 1BTL algebra. In [14] we found a spectral equivalence between the \(XXZ\) model with an arbitrary boundary term added to one end and the same \(XXZ\) model with diagonal boundary terms. Therefore an understanding of the one-boundary chain immediately allows one to understand all of the numerical results for obtaining Potts spectra from diagonal chains [7, 18].

In section 4 we discuss a further generalization of the TL algebra: the two-boundary (2BTL) algebra. This algebra is different from the TL and 1BTL algebras in that the number of words is no longer finite and the representation theory is essentially unknown. All representations that occur in this paper lie in a finite dimensional quotient that involves an additional parameter \(b\). In [19] the 2BTL algebra with this quotient was studied and a conjecture was made for the values of the parameter \(b\) that correspond to exceptional points of the 2BTL algebra. The Potts representations correspond to very particular values of the parameters all of which are exceptional. The \(XXZ\) representation exists for arbitrary values of \(b\) and the parameters in the algebra. However, in contrast to the TL and 1BTL cases, it is certainly not faithful (as we shall see explicitly at two sites) and so we cannot hope to obtain all the Potts eigenvalues from a single \(XXZ\) chain. However we shall see that by combining sectors of \(XXZ\) chains with different exceptional values of \(b\) one can get all of the Potts eigenvalues. Here we shall give numerical results and sketch the mechanism that is responsible. A systematic procedure for extracting them requires a knowledge of 2BTL representation theory and we shall not describe this here. In certain
cases however the 2BTL representations lie in a quotient of the TL, or 1BTL, algebra and one can understand the representations and hence extract the Potts spectra from XXZ ones. This generalizes the results of [20, 21] for the Ising case and allows one to expand and cross-check many results.

In section 5 we shall discuss the finite size scaling (FSS) of the one-boundary chains. A central result of [14] concerning the spectral equivalence of the one-boundary and diagonal chains will be of great practical value as many results are known about the FSS limit of the diagonal chain [22]–[25]. All results can be understood algebraically in terms of the finite size scaling of an integrable 1BTL Hamiltonian acting on a particular (irreducible) representation. The structure of irreducible representations of the 1BTL is related, in the finite size scaling limit, to the embedding of Verma modules in the Virasoro minimal models [26]. One benefit of this algebraic point of view is that it does not depend on the use of a particular representation. We shall use the Potts representations of 1BTL to derive FSS results for the Ising and three-state Potts models with boundary terms. Our results are in complete agreement with previous studies [6, 7] and continuum results [28, 29]. In [27] generalizations of the Potts models were constructed which have extended multi-site interactions. The finite size scaling of these theories were also discussed from the point of view of the TL algebra.

In section 6 we discuss briefly the periodic Temperley–Lieb (PTL) algebra [10, 11, 30]. This algebra has also attracted attention in the mathematical literature [31, 32]. The algebra again has an infinite number of words; however, as with the 2BTL case, all representations occurring in this paper lie in a finite dimensional quotient with a single parameter $b$. The Potts representations are, as before, at the exceptional points of the algebra. The XXZ representation exists for generic values of the parameters where the parameter $b$ is related to the twist angle in the spin chain. As in the 2BTL case this is not a faithful representation. The Potts eigenvalues can again be obtained by combining sectors of several XXZ chains with different exceptional values of the parameter $b$ [18, 33]. We leave a detailed explanation of the truncation schemes and FSS results to future work.

In the appendix we have given numerical examples which confirm the arguments made in the text.

2. Temperley–Lieb algebra

The Temperley–Lieb algebra [2, 3], which we shall denote $\text{TL}_N(q)$, is given by generators $e_i$ with $i = 1, \ldots, N - 1$ obeying the relations
\[
\begin{align*}
    e_i^2 &= (q + q^{-1})e_i, \\
    e_i e_{i\pm 1} e_i &= e_i, \\
    e_i e_j &= e_j e_i \quad |i - j| > 1.
\end{align*}
\]

Using these relations one finds that there are only a finite number of distinct words for a given value of $N$. In the following subsections we shall give the XXZ and Potts representations of the TL algebra.

From the TL algebra a full, spectral parameter dependent, $R$-matrix can be found by a process of Baxterization [34]. From the expansion of this transfer matrix one can
obtain, algebraically, the integrable TL Hamiltonian

$$H_{TL}^{\text{int}} = -\sum_{i=1}^{N-1} e_i.$$  \hspace{1cm} (2.2)

2.1. Representations

2.1.1. XXZ. On an $L$-site $SU(2)$ spin chain we have a representation of $\text{TL}_L(q)$ with arbitrary parameter $q = e^{i\gamma}$:

$$e_i = -\frac{1}{2} \left\{ \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cos \gamma \sigma_i^z \sigma_{i+1}^z - \cos \gamma + i \sin \gamma \left( \sigma_i^z - \sigma_{i+1}^z \right) \right\}.$$ \hspace{1cm} (2.3)

2.1.2. Two-state Potts (Ising). On an $L$-site chain we have a representation of $\text{TL}_{2L}(e^{\pi i/4})$ realized by

$$e_{2i} = \frac{1}{\sqrt{2}} \left( 1 + \sigma_i^x \sigma_{i+1}^x \right) \hspace{1cm} i = 1, \ldots, L - 1$$

$$e_{2i-1} = \frac{1}{\sqrt{2}} \left( 1 + \sigma_i^y \right) \hspace{1cm} i = 1, \ldots, L.$$ \hspace{1cm} (2.4)

Each of these TL generators commutes with the element

$$\sigma = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \otimes \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \cdots \otimes \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right).$$ \hspace{1cm} (2.5)

This obeys $\sigma^2 = 1$ corresponding to the $Z_2$ (= $S_2$) symmetry of the model.

2.1.3. Three-state Potts. On an $L$-site chain we have a representation of $\text{TL}_{2L}(e^{\pi i/6})$ realized by

$$e_{2i} = \frac{1}{\sqrt{3}} \left( 1 + R_i R_{i+1}^2 + R_i^2 R_{i+1} \right) \hspace{1cm} i = 1, \ldots, L - 1$$

$$e_{2i-1} = \frac{1}{\sqrt{3}} \left( 1 + M_i + M_i^2 \right) \hspace{1cm} i = 1, \ldots, L$$ \hspace{1cm} (2.6)

where the matrices $R, M$ are given by

$$R = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{array} \right) \hspace{1cm} M = \left( \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right).$$ \hspace{1cm} (2.7)

\hspace{1cm}

1 As with any integrable model there are of course an infinite number of integrable Hamiltonians coming from the expansion of the transfer matrix. In this paper we shall only consider the simplest one which is linear in the generators.

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The Temperley–Lieb algebra and its generalizations in the Potts and XXZ models

Each of these TL generators commutes with the elements

\[
P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdots \otimes \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdots \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\]

(2.8)

These obey \( P_1^2 = 1 = P_2^2 \) and \((P_1P_2)^3 = 1\) corresponding to the \( D_3 (=S_3) \) symmetry of the model.

### 2.1.4. Four-state Potts

On an \( L \)-site chain we have a representation of TL\(_{2L}(1)\) realized by

\[
e_{2i} = \frac{1}{\sqrt{4}} (1 + R_iR_{i+1}^3 + R_i^2R_{i+1}^2 + R_i^3R_{i+1}) \quad i = 1, \ldots, L - 1
\]

\[
e_{2i-1} = \frac{1}{\sqrt{4}} (1 + M_i + M_i^2 + M_i^3) \quad i = 1, \ldots, L
\]

(2.9)

where the matrices \( R, M \) are given by

\[
R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \quad M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\]

(2.10)

These are invariant under a local \( S_4 \) symmetry generated by

\[
P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdots \otimes \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

\[
P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdots \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

\[
P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdots \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

(2.11)

On can form TL representations in general \( Q \)-state Potts models. However for \( Q > 4 \) these models do not have a second-order phase transition and we shall not discuss them here.
2.2. Potts spectra within \textit{XXZ} spectra (review)

The integrable TL Hamiltonian (2.2) formed from TL$_{2L}(q)$ is exactly the Hamiltonian for the $L$-site Potts models with free boundary conditions where the TL generators are given by (2.4), (2.6), and (2.9) for the Ising, three-state, and four-state Potts models respectively. It is also the Hamiltonian of the $2L$-site integrable $XXZ$ chain with $SU_q(2)$ invariant boundary terms where the TL generators are now given by (2.3) with $q = e^{\pi i/4}$, $e^{\pi i/6}$ and 1 for the Ising, three-state, and four-state Potts models respectively.

We shall use the representation theory of the TL algebra in order to determine which states in the $XXZ$ representation have energy levels present in the Potts model. For more details on TL representation theory see [3,35]. In this specific case it is also possible to use the quantum group $SU_q(2)$ which acts as the centralizer of the TL algebra in the $XXZ$ representation [8]. We shall not follow this approach as it cannot be generalized to the boundary and periodic cases.

Irreducible representations of the TL algebra TL$^N(q)$ are labelled by $V_j^{(N)}$ and are indexed by a ‘spin’ $0 \leq j \leq N/2$ which takes integer values for $N$ even and half-integer values for $N$ odd. These representations can be conveniently encoded in a Bratelli diagram (see figure 1). On the horizontal axis the value of $N$ is given. For each value of $N$ the different irreducible representations $V_j$ that are present are read off. The dimension of the representation $V_j^{(N)}$ is given by the number of paths that can be drawn between that point and the 0 at the far LHS of the diagram. It is given by

$$\dim V_j^{(N)} = \left( \frac{N}{2} - j \right) - \left( \frac{N}{2} + j + 1 \right).$$

In the $L$-site $XXZ$ representation of TL$^L(q)$ each of the irreducible representations $V_j$ occurs with multiplicity $2j + 1$. This results in the identity

$$\sum_{j=0}^{L/2} (2j + 1) \dim V_j^{(L)} = 2^L.$$

Figure 1.
In terms of the quantum group each term in this sum is simply the contribution from the spin \( j \) representation of \( SU_q(2) \).

When \( q \) is a root of unity we get not just irreducible but also indecomposable representations. We can now consider a truncated theory in which we discard the indecomposable representations. For the case \( q^p = \pm 1 \) the remaining irreducible representations \( V_j \) have \( 0 \leq j \leq (p - 2)/2 \).

- **Ising:**
  For the case of \( q = e^{i\pi/4} \) relevant to the Ising model the ‘good’ (irreducible) representations are \( V_0 \), \( V_{1/2} \), and \( V_1 \). These can be encoded in a truncated Bratelli diagram:

  ![Truncated Bratelli Diagram](image)

  This is obtained from the full Bratelli diagram (figure 1) by simply removing all representations with \( j > 1 \).

  The sizes of the irreducible representations can be read off from the truncated Bratelli diagram (again by counting the paths from the 0 on the left-hand side) and are given in the table below. In the XXZ chain of length \( 2L \) the energy levels which also occur in the Ising model with free boundary conditions come from \( V_0 \) and \( V_1 \). One can see that the total number of states contained in these levels is exactly the number of states (i.e. \( 2^L \)) present in an \( L \)-site Ising model.

| Length of XXZ chain | Representation | Ising model |
|---------------------|---------------|-------------|
|                     | \( V_0 \)     | \( V_{1/2} \) | \( V_1 \) | |
| 1                   | -             | 1           | -          |
| 2                   | 1             | -           | 1          |
| 3                   | -             | 2           | -          |
| 4                   | 2             | -           | 2          |
| 5                   | -             | 4           | -          |
| 6                   | 4             | -           | 4          |
| 7                   | -             | 8           | -          |
| 8                   | 8             | -           | 8          |

  For instance in the \( L = 2 \) Ising model we have the algebra \( TL_4(e^{i\pi/4}) \) and a single copy of the \( V_1 \) and \( V_0 \) irreducible representations.

- **Three-state Potts:**
  For the three-state Potts model we have \( q = e^{i\pi/6} \). The ‘good’ representations now have \( 0 \leq j \leq 2 \) and the truncated Bratelli diagram is given by
The Temperley–Lieb algebra and its generalizations in the Potts and $XXZ$ models

The three-state Potts model has symmetry $S_3$ with irreducible representations given by singlets and doublets. The entire spectrum of the Potts model is contained within the $XXZ$ spectrum and moreover the singlet states from the Potts come from $V_0 \oplus V_2$ and the doublets from $V_1$. The degeneracies from the Bratelli diagram and the Potts model are summarized in the following table:

| Length of $XXZ$ chain | Representation $V_0$ $V_{1/2}$ $V_1$ $V_{3/2}$ $V_2$ | Three-state Potts model Singlets Doublets Total size |
|-----------------------|-------------------|-------------------|-------------------|
| 1                     | – 1 – 1 – 1 – 1   | 1 1 3             |
| 2                     | 1 2 3 1 1        |                  |
| 3                     | – 2 – 3 – 4 – 4  | 3 3 9             |
| 4                     | 2 5 9 4 9 4      | 9 9 27            |
| 5                     | – 14 – 13 – 13  | 27 27 81          |

Now in the $L = 4$ case the three-state Potts representation of $\text{TL}_4(e^{\pi i/6})$ contains two copies of $V_1$ and one each of $V_0$ and $V_2$.

- Four-state Potts:
  In the $XXZ$ model this corresponds to $q = 1$ ($XXX$) model. The symmetry $SU_q(2)$ becomes simply $SU(2)$ and there is no truncation of the spectrum. The spectra of the Hamiltonian using the four-state Potts representation (2.9) and using the $XXZ$ one are found to be identical.

In section 5 we shall discuss the finite size scaling results of these integrable chains. In the continuum limit they become conformally invariant and each of the irreducible representations of the TL algebra becomes an irreducible representation of the Virasoro algebra.

3. One-boundary Temperley–Lieb algebra

The one-boundary Temperley–Lieb, or blob, algebra is an interesting generalization of the TL algebra that has been well studied in the mathematical literature [10]–[13]. In addition to the TL relations (2.1) we have in addition one extra boundary generator $f_0$. 

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The Temperley–Lieb algebra and its generalizations in the Potts and XXZ models

\[ f_0^2 = s_1 f_0 \]
\[ e_1 f_0 e_1 = e_1 \]
\[ e_i f_0 = f_0 e_i \quad i > 1. \]  

There is now one extra parameter \( s_1 \). The space of words is still finite dimensional. Again we can find XXZ and Potts representations of this algebra.

The integrable Hamiltonian for the 1BTL is given by

\[ H_{1BTL} = -a f_0 - \sum_{i=1}^{N-1} e_i \]  

where \( a \) is an arbitrary coefficient. Therefore by considering this extension of the TL algebra we are able to describe the addition of an integrable boundary operator.

### 3.1. Representations

#### 3.1.1. XXZ

The boundary generator is given by [10]

\[ f_0 = \frac{1}{2} \frac{1}{\sin(\omega + \gamma)} \left( -i \cos \omega - \sigma_1^z - \cos \phi \sigma_1^x + \sin \phi \sigma_1^y - \sin \omega \right). \]

This has

\[ s_1 = \frac{\sin \omega}{\sin(\omega + \gamma)}. \]  

The angle \( \phi \) is irrelevant as it can be changed by a rotation of \( \sigma_1^x \) and \( \sigma_1^y \) preserving the bulk generators (2.3). For convenience we shall set \( \phi = 0 \). Then we have

\[ f_0 = \frac{1}{2} \frac{1}{\sin(\omega + \gamma)} \left( -i \cos \omega \sigma_1^z - \sigma_1^x - \sin \omega \right). \]  

#### 3.1.2. Two-state Potts (Ising)

The most general left boundary generator, which must commute with \( e_2 \), is given by \( f_0 = a + b \sigma_1^z \). However the relation \( e_1 f_0 e_1 = e_1 \) fixes \( a = 1/\sqrt{2} \). Imposing now \( f_0^2 = s_1 f_0 \) we get, up to action by the bulk symmetry (2.5), a unique solution:

\[ f_0 = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \otimes 1 \otimes \cdots \otimes 1. \]  

This boundary term breaks the \( Z_2 \) symmetry (2.5) of the Ising model. In addition to \( f_0^2 = \sqrt{2} f_0 \) these two boundary generators satisfy the additional relation: \( f_0 e_1 f_0 = f_0 \) and so this 1BTL algebra is actually a quotient of the TL algebra \( T_{2L+1}(e^{\pi i/4}) \). This implies that the spectrum of the one-boundary Hamiltonian (3.2) can be extracted from a purely TL one. We shall return to this point in section 5.2.

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3.1.3. Three-state Potts. Writing the most general left boundary generator and requiring it to satisfy the 1BTL algebra gives, up to $S_3$ symmetry (2.8), the following possibilities:

- $f_0^2 = \sqrt{3}f_0$:
  \[ f_0 = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes 1 \otimes \cdots \otimes 1. \]  
  (3.6)

This boundary term breaks the $S_3$ symmetry down to $S_2 = Z_2$ as it only invariant under the generator $P_2$ (2.8). We also find, as with the Ising model, that we have $f_0 e_1 f_0 = f_0$ and therefore this is again a quotient of the TL algebra $T_{2L+1}(e^{\pi i/6})$.

- $f_0^2 = 3/2f_0$:
  \[ f_0 = \begin{pmatrix} \sqrt{3}/2 & 0 & 0 \\ 0 & \sqrt{3}/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes 1 \otimes \cdots \otimes 1. \]  
  (3.7)

These boundary conditions again break the $S_3$ symmetry down to $S_2 = Z_2$. This does not satisfy the TL quotient: $f_0 e_1 f_0 = f_0$. However it can be related by a linear transformation: $\sqrt{3} - 2f_0$ to the previous case (3.6). Although this is therefore also a quotient of $T_{2L+1}(e^{\pi i/6})$ we shall treat them separately, as in the Hamiltonian (3.2) this linear transformation reverses the sign of the boundary term and gives rise to different finite size scaling behaviour—see section 5.

3.1.4. Four-state Potts. In this case we have even more possibilities for the left boundary generator:

- $f_0^2 = \sqrt{4}f_0$:
  \[ f_0 = \begin{pmatrix} \sqrt{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes 1 \otimes \cdots \otimes 1. \]  
  (3.8)

This breaks the $S_4$ symmetry down to $S_3$. In this case we find, as with the Ising model, that we have $f_0 e_1 f_0 = f_0$ and therefore this is a quotient of the TL algebra $TL_{2L+1}(1)$.

- $f_0^2 = \sqrt{4}/2f_0$:
  \[ f_0 = \begin{pmatrix} \sqrt{4}/2 & 0 & 0 & 0 \\ 0 & \sqrt{4}/2 & 0 & 0 \\ 0 & 0 & \sqrt{4}/2 & 0 \\ 0 & 0 & 0 & \sqrt{4}/2 \end{pmatrix} \otimes 1 \otimes \cdots \otimes 1. \]  
  (3.9)

These sets of boundary conditions break the $S_4$ symmetry to $Z_2 \otimes Z_2$. These boundary terms are not in a TL quotient.
The Temperley–Lieb algebra and its generalizations in the Potts and XXZ models

- \( f_0^2 = \sqrt{4}/3 \):

\[
f_0 = \begin{pmatrix}
\frac{\sqrt{4}}{3} & 0 & 0 & 0 \\
0 & \frac{\sqrt{4}}{3} & 0 & 0 \\
0 & 0 & \frac{\sqrt{4}}{3} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \otimes 1 \otimes \cdots \otimes 1.
\]

(3.10)

These sets of boundary conditions break the \( S_4 \) symmetry to \( S_3 \). Again, as in the case of (3.7), although these are not in the TL quotient \( f_0 e_1 f_0 = f_0 \) they can be linearly related to the solutions (3.8).

In the two-, three-, and four-state Potts models we have given a complete list of possible one-site boundary generators of the 1BTL algebra that act only on the first site of the Potts chain. We shall see in the next subsection that these all occur at points in which the 1BTL algebra is non-semisimple and can possess indecomposable representations. These different boundary generators lead to different possible integrable boundary terms in the Potts Hamiltonian (3.2). By directly solving the reflection equation [36,37], in the two-, three-, and four-state Potts models, we did not find any further one-site integrable boundary terms. In [38] other integrable boundary terms in the Potts models were obtained by considering local boundary terms rather than simply single-site ones. In the continuum conformal field theory description these correspond to all the conformally invariant boundary conditions.

3.2. One-boundary TL representation theory

In order to understand the Potts representations we shall use 1BTL representation theory.

The representation theory of 1BTL was first worked out by Martin et al [10]–[13]. For the XXZ representation we have derived the same truncation schemes [14] based on the properties of the centralizer given in [15,16,39]. This approach has a close relation to the bulk quantum group case and lends support to the fact that the XXZ representation captures all of the irreducible structure of the algebra.

In the study of the 1BTL algebra (3.1) it is convenient to use the parametrization for \( s_1 \) that arises in the XXZ representation:

\[
s_1 = \frac{\sin \omega}{\sin(\omega + \gamma)}.
\]

(3.11)

In the representation theory a crucial role is played by the relation

\[
2\gamma Q + \omega = \pi \mathbb{Z}.
\]

(3.12)

There are three different cases depending on the number of solutions of (3.12) for \( 2Q \in \mathbb{Z} \):

- Generic case: no solutions.
- Critical case: only one solution.
- Doubly critical case: infinitely many solutions.
In the generic case the algebra is semisimple and only possesses irreducible representations. In the critical or doubly critical cases one also gets indecomposable representations. In a similar way to the TL case of section 2.2 there is a truncated sector in which there are only irreducible representations. In the doubly critical case, relevant for the Potts models, we always have a finite number of irreducible representations.

In the next three subsections we shall discuss the counting of the number of ‘good’ states in each of these cases.

3.2.1. Generic: $\gamma$ and $\omega$ arbitrary. The irreducible representations of the 1BTL algebra, $W_Q$, are indexed by $Q = -N/2, \ldots, (N/2) - 1, N/2$. They can be conveniently encoded in a Bratelli diagram as shown below:

The system size, $N$, is given on the horizontal axis. One of the paths to the irreducible representation $W_{1}$ is shown in bold.

The size of irreducible representations $W_Q$ is given by the number of paths from 0 to that point. In a chain of length $N$ it is given by

$$\left( \frac{N}{2} - Q \right).$$  

(3.13)
For example in a chain of length 6 there are \( \binom{6}{3} = 20 \) states with \( Q = 0 \). In the \( L \)-site 1BTL XXZ representation (3.4), in contrast to the TL case (2.13), each representation \( W_Q \) occurs only once:

\[
\sum_{Q=-L/2}^{L/2} \binom{L}{L/2 - Q} = 2^L.
\] (3.14)

3.2.2. Critical: generic \( \gamma \) but \( \omega = \gamma \mathbb{Z} \). If we have

\[\omega = k\gamma\] (3.15)

for some integer \( k \) then the 1BTL representation theory becomes ‘critical’. Then, as \( \gamma \) is generic, the relation (3.12) has only one solution: \( Q = -k/2 \). Let us concentrate on the case \( k > 0 \).

The space of ‘good’ states becomes truncated from below and the minimum ‘good’ value of \( Q \) is given by \( Q = (1 - k)/2 \). The Bratelli diagram for the case of \( \omega = 2\gamma \) is given by

At a given value of \( Q \geq (1 - k)/2 \) the number of ‘good’ states is given by

\[
\Gamma_Q^{(N)} = \left( \frac{N}{2} - Q \right) - \left( \frac{N}{2} + k + Q \right).
\] (3.16)

The first term is the number that would occur in the untruncated diagram and the second term is the number of paths which go ‘outside’ the truncated Bratelli diagram. For example in a chain of length 6, of the 20 states in the untruncated diagram with \( Q = 0 \) only \( 20 - \binom{6}{3+2} = 14 \) lie inside the truncated diagram shown above.

In the case of \( k < 0 \) the discussion is very similar but the Bratelli diagram now becomes truncated from above.
3.2.3. Doubly critical: $\gamma$ a rational multiple of $\pi$ and $\omega = \gamma \mathbb{Z}$. This is the most complicated case and also the one that is the most interesting. Now the relation (3.12) has an infinite number of solutions for $2Q \in \mathbb{Z}$. Here we shall consider the case

$$\gamma = \frac{\pi}{m+1}, \quad \omega = k\gamma$$

(3.17)

where both $m$ and $k$ are both positive integers. Other rational multiples of $\pi$ can be dealt with in a similar way.

The lowest ‘bad’ value of $Q$ is given by $-k/2$ and so we have $Q \geq (1-k)/2$. The highest ‘bad’ value of $Q$ is given by $(m+1-k)/2$ and so we have $Q \leq (m-k)/2$. Therefore we must have

$$\frac{1-k}{2} \leq Q \leq \frac{m-k}{2}.$$  

(3.18)

An example of a doubly truncated Bratelli diagram for the case of $\gamma = \pi/6$, $\omega = \pi/3$, i.e. $m = 5$, $k = 2$, is shown below:

The calculation of the number of ‘good’ representations, or paths in the Bratelli diagram, in the doubly critical cases is given by

$$\Omega_Q^{(N)} = \Gamma_Q^{(N)} - \Gamma_{m+1-k-Q}^{(N)} + \Gamma_{m+1+Q}^{(N)} - \Gamma_{2(m+1)-k-Q}^{(N)} + \cdots$$  

(3.19)

where the $\Gamma_Q^{(N)}$ were given before in (3.16). We shall use this result later when discussing the FSS results. It will give rise to the infinite number of subtractions which occur in characters of the minimal conformal field theories.

In the previous example with $\gamma = \pi/6$, $\omega = \pi/3$ the number of ‘good’ states in the six-site case is given by

$$\Omega_0^{(6)} = \Gamma_0^{(6)} - \Gamma_{-1}^{(6)} + \Gamma_{1}^{(6)} - \Gamma_{2}^{(6)} + \cdots = 14$$

(3.20)

$$\Omega_1^{(6)} = \Gamma_1^{(6)} - \Gamma_{-1}^{(6)} + \Gamma_{1}^{(6)} - \Gamma_{2}^{(6)} + \cdots = 13.$$  

(3.21)

In this case one can easily prove that

$$\Omega_0^{(2L)} + \Omega_1^{(2L)} = 3^L.$$  

(3.22)
As we shall see in the next subsection this is due to the fact that the ‘good’ states in this model can also be realized in a Potts representation of dimension $3^L$.

### 3.3. Potts spectra within XXZ spectra in the one-boundary cases

The integrable 1BTL Hamiltonian (3.2) in the Potts representations corresponds to Potts models with a boundary term added to the left side.

#### 3.3.1. Two-state Potts (Ising)

There is only one possible boundary term (3.5). It obeys $f_0^2 = \sqrt{2} f_0$ implying $\gamma = \pi/4$ and $\omega = \pi/2$. The truncated diagram in this case is given by

![Truncated diagram](image)

The dimensions of the irreducible representations are given by

| Length of XXZ chain | Representation $W_{-1/2}$ | $W_0$ | $W_{1/2}$ | Ising model total size |
|---------------------|---------------------------|-------|-----------|-----------------------|
| 1                   | 1                         | –     | –         | 1                     |
| 2                   | –                         | 2     | –         | 2                     |
| 3                   | 2                         | –     | 2         | 4                     |
| 4                   | –                         | 4     | –         | 8                     |
| 5                   | 4                         | –     | 4         |                       |
| 6                   | –                         | 8     | –         |                       |
| 7                   | 8                         | –     | 8         |                       |
| 8                   | –                         | 16    | –         | 16                    |

We see that the Ising model contains only the $W_0$ sector. A numerical example confirming this is given in appendix A.1.

Note that although the above table is superficially similar to that for the TL case of section 2.2 the symmetry group here is different and in the XXZ model each representation $W_Q$ occurs just once (3.14).

#### 3.3.2. Three-state Potts

A similar discussion can be given for each of the integrable boundary terms in the three-state Potts model.

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The Temperley–Lieb algebra and its generalizations in the Potts and $XXZ$ models

The first boundary term (3.6) obeys $f_0^2 = \sqrt{3} f_0$ implying $\gamma = \pi/6$ and $\omega = 2\pi/3$. The truncated diagram in this case is given by

![Diagram](image)

The dimensions of the irreducible representations are given by

| Length of $XXZ$ chain | $W_{-3/2}$ | $W_{-1}$ | $W_{-1/2}$ | $W_0$ | $W_{1/2}$ | Potts model total size |
|------------------------|------------|----------|------------|------|----------|----------------------|
| 0                      | -          | -        | -          | 1    | -        | 1                    |
| 1                      | -          | -        | 1          | -    | 1        | 3                    |
| 2                      | -          | 1        | -          | 2    | -        | 9                    |
| 3                      | 1          | -        | 3          | -    | 2        |                      |
| 4                      | -          | 4        | -          | 5    | -        |                      |
| 5                      | 4          | -        | 9          | -    | 5        |                      |
| 6                      | -          | 13       | -          | 14   | -        | 27                   |
| 7                      | 13         | -        | 27         | -    | 14       | 81                   |
| 8                      | -          | 40       | -          | 41   | -        |                      |

Note that in contrast to the bulk case of section 1 all states are singlets.

The second boundary term (3.7) obeys $f_0^2 = \sqrt{3}/2 f_0$ implying $\gamma = \pi/6, \omega = \pi/3$:  

![Diagram](image)
Again the dimensions of the irreducible representations can be easily calculated:

| Length of XXZ chain | Representation | Potts model |
|---------------------|----------------|-------------|
|                     | $W_{-1/2}$   | $W_0$  | $W_{1/2}$ | $W_1$ | $W_{3/2}$ | total size |
| 0                   | -             | 1      | -         | -      | -         | 1          |
| 1                   | 1             | -      | 1         | -      | -         | 3          |
| 2                   | -             | 2      | -         | 1      | -         | 9          |
| 3                   | 2             | -      | 3         | -      | 1         | 27         |
| 4                   | -             | 5      | -         | 4      | -         | 81         |
| 5                   | 5             | -      | 9         | -      | 4         |            |
| 6                   | -             | 14     | -         | 13     | -         |            |
| 7                   | 14            | -      | 27        | -      | 13        |            |
| 8                   | -             | 41     | -         | 40     | -         |            |

Numerical examples that confirm this picture are given in appendix A.2.

In [14] an exact spectral equivalence was given, between the one-boundary chain and the general XXZ chain with diagonal boundary terms. Therefore, for certain values of the parameters, we can find Potts spectra with a boundary term within the diagonal chain. In section 5 we shall use this fact to understand the finite size scaling of the Potts models with integrable boundary term added to one end. We shall find that in the finite size scaling limit each representation of the 1BTL algebra gives a particular representation of the Virasoro algebra. For the special cases $\omega = \pm \gamma$ it was shown in [14] that the boundary generator $f_0$ acts trivially on the ‘good’ truncated sector of 1BTL. This explains why Potts spectra with free boundary conditions could also be found in the spectrum of particular diagonal chains [7, 18].

4. Two-boundary Temperley–Lieb algebra

The addition of a second boundary term added at the right-hand end proceeds in a similar way to the 1BTL case. If we have the 1BTL algebra (3.1) with $f_0$ and $e_i$ ($i = 1, \ldots, N - 1$) then we introduce an extra element $f_N$ satisfying the relations

$$
\begin{align*}
    f_N^2 &= s_2 f_N \\
    e_{N-1} f_N e_{N-1} &= e_{N-1} \\
    e_i f_N &= f_N e_i, \quad 1 < i < N - 1 \\
    f_0 f_N &= f_N f_0.
\end{align*}
$$

(4.1)

The integrable Hamiltonian for the 2BTL is given by

$$
H^{2BTL} = -af_0 - a'f_N - \sum_{i=1}^{N-1} e_i
$$

(4.2)

where both $a$ and $a'$ are arbitrary coefficients.

In contrast to the TL and 1BTL cases the space of words of the 2BTL is no longer finite dimensional. However we shall find that the XXZ and Potts representations lie in particular finite dimensional quotients. Let us first define $I$ and $J$ by:
The Temperley–Lieb algebra and its generalizations in the Potts and XXZ models

- $N$ even:
  \begin{align}
  I &= e_1 e_3 \cdots e_{N-1} \\
  J &= f_0 e_2 e_4 \cdots e_{N-2} f_N. 
  \end{align}

- $N$ odd:
  \begin{align}
  I &= f_0 e_2 e_4 \cdots e_{N-1} \\
  J &= e_1 e_3 \cdots e_{N-2} f_N. 
  \end{align}

In the Potts models the value of $N$ is always even. In this case $I$ and $J$ are very different as $I$ has no boundary generators in it whereas $J$ contains both. In the odd $N$ cases there is one boundary generator in both $I$ and $J$.

We shall encounter representations which lie in three different types of finite dimensional quotient:

- Type I:
  \[ IJI = bI. \]

- Type II:
  \[ IJI = bI \quad \text{and} \quad JIJ = bJ. \]

  It is clear from considering $IJIJ$ that we could not have $JIJ = bJ$ and $IJI = b'I$ with $b' \neq b$.

- Type III:
  \[ J = 0. \]

It is clear that $II \subset I$ and that $III \subset II_{b=0} \subset I_{b=0}$.

4.1. Representations

4.1.1. XXZ. The XXZ representation (on $L$ sites) is given by

\begin{align}
  f_0 &= -\frac{1}{2\sin(\omega_1 + \gamma)} (-i \cos \omega_1 \sigma_1^x - \sigma_1^z - \sin \omega_1) \\
  f_L &= -\frac{1}{2\sin(\omega_2 + \gamma)} (i \cos \omega_2 \sigma_L^x + \cos \phi \sigma_L^z - \sin \phi \sigma_L^y - \sin \omega_2). 
\end{align}

This has

\begin{align}
  s_1 &= \frac{\sin \omega_1}{\sin(\omega_1 + \gamma)} \\
  s_2 &= \frac{\sin \omega_2}{\sin(\omega_2 + \gamma)}. 
\end{align}

Note that we have an additional arbitrary angle $\phi$. In the one-boundary case, discussed in section 3.1.1, in which we have only the generator $f_0$, this angle could be removed by
a global $U(1)$ rotation. However in the two-boundary case we cannot remove them from both ends simultaneously. We find that the $XXZ$ representation always lives in a type II quotient and the angle $\phi$ affects the value of $b$ differently for odd and even values of $L$:

- **$L$ odd:**
  \[
  b = \frac{\sin ((\omega_1 - \omega_2 - \phi)/2) \sin ((-\omega_1 + \omega_2 + \phi)/2)}{\sin(\gamma + \omega_1) \sin(\gamma + \omega_2)}.
  \]

- **$L$ even:**
  \[
  b = \frac{\sin ((\gamma + \omega_1 + \omega_2 - \phi)/2) \sin ((\gamma + \omega_1 + \omega_2 + \phi)/2)}{\sin(\gamma + \omega_1) \sin(\gamma + \omega_2)}.
  \]

The conventions we use here differ slightly from those of [19].

4.1.2. Two-state Potts (Ising). In the $L$-site model if we have two boundary generators $f_0$ and $f_{2L}$ then we have only two possible cases. These will be labelled $(+, +)$ and $(+, -)$:

- $(+, +)$: same boundary terms:
  \[
  f_0 = \frac{1}{\sqrt{2}} (1 + \sigma^z_1) = \left( \begin{array}{cc}
  \sqrt{2} & 0 \\
  0 & 0
  \end{array} \right) \otimes 1 \cdots \otimes 1 \quad (4.15)
  \]
  \[
  f_{2L} = \frac{1}{\sqrt{2}} (1 + \sigma^z_L) = 1 \otimes \cdots \otimes 1 \otimes \left( \begin{array}{cc}
  \sqrt{2} & 0 \\
  0 & 0
  \end{array} \right). \quad (4.16)
  \]
  In this case we find this lies in a type II quotient: $II = \sqrt{2}I$, $JIJ = \sqrt{2}J$.

- $(+, -)$: different boundary terms:
  \[
  f_0 = \frac{1}{\sqrt{2}} (1 + \sigma^z_1) = \left( \begin{array}{cc}
  \sqrt{2} & 0 \\
  0 & 0
  \end{array} \right) \otimes 1 \cdots \otimes 1 \quad (4.17)
  \]
  \[
  f_{2L} = \frac{1}{\sqrt{2}} (1 - \sigma^z_L) = 1 \otimes \cdots \otimes 1 \otimes \left( \begin{array}{cc}
  0 & 0 \\
  0 & \sqrt{2}
  \end{array} \right). \quad (4.18)
  \]
  In this case we find this lies in a type III quotient: $J = 0$.

Other possibilities are equivalent and can be related to these by the global symmetry. In all these cases the bulk $Z_2$ symmetry is completely broken.

4.1.3. Three-state Potts. We label the three different states of the Potts model by $A$, $B$, and $C$. The different possibilities for the boundary terms in the $L$-site model are then labelled according to the non-zero entries in the matrices:

(1) $(A, A)$:

\[
  f_0 = \left( \begin{array}{ccc}
  \sqrt{3} & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
  \end{array} \right) \otimes 1 \cdots \otimes 1 \quad (4.19)
  \]
\[
  f_{2L} = 1 \otimes \cdots \otimes 1 \otimes \left( \begin{array}{ccc}
  \sqrt{3} & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
  \end{array} \right). \quad (4.20)
  \]

This breaks $S_3$ to $S_2$. This lies in a type II quotient: $II = \sqrt{3}I$, $JIJ = \sqrt{3}J$. 

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The Temperley–Lieb algebra and its generalizations in the Potts and XXZ models

(2) \((A, B)\):

\[
f_0 = \left( \begin{array}{ccc} \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \otimes 1 \otimes \cdots \otimes 1 \tag{4.21}
\]

\[
f_{2L} = 1 \otimes \cdots \otimes 1 \otimes \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 \end{array} \right).
\tag{4.22}
\]

This breaks \(S_3\) completely. This lies in a type III quotient: \(J = 0\).

(3) \((A, AB)\):

\[
f_0 = \left( \begin{array}{ccc} \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \otimes 1 \otimes \cdots \otimes 1 \tag{4.23}
\]

\[
f_{2L} = 1 \otimes \cdots \otimes 1 \otimes \left( \begin{array}{ccc} \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right).
\tag{4.24}
\]

This breaks \(S_3\) completely. This lies in a type II quotient: \(IJJ = \frac{\sqrt{3}}{2} I\), \(JIJ = \frac{\sqrt{3}}{2} J\).

(4) \((A, BC)\):

\[
f_0 = \left( \begin{array}{ccc} \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \otimes 1 \otimes \cdots \otimes 1 \tag{4.25}
\]

\[
f_{2L} = 1 \otimes \cdots \otimes 1 \otimes \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} \end{array} \right).
\tag{4.26}
\]

This breaks \(S_3\) to \(S_2\). This lies in a type III quotient: \(J = 0\).

(5) \((AB, AB)\):

\[
f_0 = \left( \begin{array}{ccc} \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 \end{array} \right) \otimes 1 \otimes \cdots \otimes 1 \tag{4.27}
\]

\[
f_{2L} = 1 \otimes \cdots \otimes 1 \otimes \left( \begin{array}{ccc} \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right).
\tag{4.28}
\]

This breaks \(S_3\) to \(S_2\). This lies in a type I quotient: \(IJJ = \frac{3}{2} I\).
(6) $(AB, BC)$:
\[
\begin{align*}
    f_0 &= \left( \frac{\sqrt{3}}{2} \ 0 \ 0 \ 0 \\
    & \ 0 \ \frac{\sqrt{3}}{2} \ 0 \ 0 \end{align*} \right) \otimes 1 \otimes \cdots \otimes 1 \quad (4.29)
\]
\[
\begin{align*}
    f_{2L} &= 1 \otimes \cdots \otimes 1 \otimes \left( 0 \ \frac{\sqrt{3}}{2} \ 0 \ 0 \\
    & \ 0 \ 0 \ \frac{\sqrt{3}}{2} \ 0 \end{align*} \right). \quad (4.30)
\]

This breaks $S_3$ completely. This lies in a type II quotient: $IIJ = \frac{\sqrt{3}}{2} I, JIJ = \frac{\sqrt{3}}{2} J$.

4.1.4. Four-state Potts. Now we have four states $A, B, C,$ and $D$. As in the three-state Potts model the different possibilities for the boundary terms in the $L$-site model are labelled according to the non-zero entries in the matrices:

(1) $(A, A)$:
\[
\begin{align*}
    f_0 &= \left( \sqrt{4} \ 0 \ 0 \ 0 \\
    & \ 0 \ 0 \ 0 \ 0 \\
    & \ 0 \ 0 \ 0 \ 0 \\
    & \ 0 \ 0 \ 0 \ 0 \end{align*} \right) \otimes 1 \otimes \cdots \otimes 1 \quad (4.31)
\]
\[
\begin{align*}
    f_{2L} &= 1 \otimes \cdots \otimes 1 \otimes \left( \sqrt{4} \ 0 \ 0 \ 0 \\
    & \ 0 \ 0 \ 0 \ 0 \\
    & \ 0 \ 0 \ 0 \ 0 \\
    & \ 0 \ 0 \ 0 \ 0 \end{align*} \right). \quad (4.32)
\]

This breaks $S_4$ to $S_3$. This lies in a type II quotient: $IIJ = 2I, JIJ = 2J$.

(2) $(A, B)$:
\[
\begin{align*}
    f_0 &= \left( \sqrt{4} \ 0 \ 0 \ 0 \\
    & \ 0 \ 0 \ 0 \ 0 \\
    & \ 0 \ 0 \ 0 \ 0 \\
    & \ 0 \ 0 \ 0 \ 0 \end{align*} \right) \otimes 1 \otimes \cdots \otimes 1 \quad (4.33)
\]
\[
\begin{align*}
    f_{2L} &= 1 \otimes \cdots \otimes 1 \otimes \left( 0 \ 0 \ 0 \ 0 \\
    & \ 0 \ \sqrt{4} \ 0 \ 0 \\
    & \ 0 \ 0 \ 0 \ 0 \\
    & \ 0 \ 0 \ 0 \ 0 \end{align*} \right). \quad (4.34)
\]

This breaks $S_4$ to $S_2$. This lies in a type III quotient: $J = 0$.

(3) $(A, AB)$:
\[
\begin{align*}
    f_0 &= \left( \sqrt{4} \ 0 \ 0 \ 0 \\
    & \ 0 \ 0 \ 0 \ 0 \\
    & \ 0 \ 0 \ 0 \ 0 \\
    & \ 0 \ 0 \ 0 \ 0 \end{align*} \right) \otimes 1 \otimes \cdots \otimes 1 \quad (4.35)
\]
\[
\begin{align*}
    f_{2L} &= 1 \otimes \cdots \otimes 1 \otimes \left( \sqrt{4} \ 0 \ 0 \ 0 \\
    & \ 0 \ 0 \ \sqrt{4} \ 0 \\
    & \ 0 \ 0 \ 0 \ 0 \\
    & \ 0 \ 0 \ 0 \ 0 \end{align*} \right). \quad (4.36)
\]
The Temperley–Lieb algebra and its generalizations in the Potts and XXZ models

This breaks $S_4$ to $S_2$. This lies in a type II quotient: $IJI = I$, $JIJ = J$.

(4) $(A, BC)$:

$$f_0 = \begin{pmatrix} \sqrt{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes 1 \otimes \cdots \otimes 1$$  \hspace{1cm} (4.37)

$$f_{2L} = 1 \otimes \cdots \otimes 1 \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (4.38)

This breaks $S_4$ to $S_2$. This lies in a type III quotient: $J = 0$.

(5) $(A, ABC)$:

$$f_0 = \begin{pmatrix} \sqrt{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes 1 \otimes \cdots \otimes 1$$  \hspace{1cm} (4.39)

$$f_{2L} = 1 \otimes \cdots \otimes 1 \otimes \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (4.40)

This breaks $S_4$ to $S_2$. This lies in a type II quotient: $IJI = \frac{2}{3}I$, $JIJ = \frac{2}{3}J$.

(6) $(A, BCD)$:

$$f_0 = \begin{pmatrix} \sqrt{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes 1 \otimes \cdots \otimes 1$$  \hspace{1cm} (4.41)

$$f_{2L} = 1 \otimes \cdots \otimes 1 \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (4.42)

This breaks $S_4$ to $S_3$. This lies in a type III quotient: $J = 0$.

(7) $(AB, AB)$:

$$f_0 = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes 1 \otimes \cdots \otimes 1$$  \hspace{1cm} (4.43)

$$f_{2L} = 1 \otimes \cdots \otimes 1 \otimes \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (4.44)

This breaks $S_4$ to $S_2 \otimes S_2$. This lies in a type I quotient: $IJI = I$. 

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The Temperley–Lieb algebra and its generalizations in the Potts and XXZ models

(8) \((AB, BC)\):

\[
f_0 = \begin{pmatrix} 
\sqrt{4} & 0 & 0 & 0 \\
0 & \sqrt{4} & 0 & 0 \\
0 & 0 & \sqrt{4} & 0 \\
0 & 0 & 0 & 0 
\end{pmatrix} \otimes 1 \otimes \cdots \otimes 1 
\]

\[(4.45)\]

\[
f_{2L} = 1 \otimes \cdots \otimes 1 \otimes \begin{pmatrix} 
0 & 0 & 0 & 0 \\
0 & \sqrt{4} & 0 & 0 \\
0 & 0 & \sqrt{4} & 0 \\
0 & 0 & 0 & 0 
\end{pmatrix} .
\]

\[(4.46)\]

This breaks \(S_4\) completely. This lies in a type II quotient: \(IJJ = \frac{1}{2}J, IJI = \frac{1}{2}I\).

(9) \((AB, CD)\):

\[
f_0 = \begin{pmatrix} 
\sqrt{4} & 0 & 0 & 0 \\
0 & \sqrt{4} & 0 & 0 \\
0 & 0 & \sqrt{4} & 0 \\
0 & 0 & 0 & 0 
\end{pmatrix} \otimes 1 \otimes \cdots \otimes 1 
\]

\[(4.47)\]

\[
f_{2L} = 1 \otimes \cdots \otimes 1 \otimes \begin{pmatrix} 
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{4} & 0 \\
0 & 0 & 0 & 0 
\end{pmatrix} .
\]

\[(4.48)\]

This breaks \(S_4\) to \(S_2 \otimes S_2\). This lies in a type III quotient: \(J = 0\).

(10) \((AB, ABC)\):

\[
f_0 = \begin{pmatrix} 
\sqrt{4} & 0 & 0 & 0 \\
0 & \sqrt{4} & 0 & 0 \\
0 & 0 & \sqrt{4} & 0 \\
0 & 0 & 0 & 0 
\end{pmatrix} \otimes 1 \otimes \cdots \otimes 1 
\]

\[(4.49)\]

\[
f_{2L} = 1 \otimes \cdots \otimes 1 \otimes \begin{pmatrix} 
\sqrt{3} & 0 & 0 & 0 \\
0 & \sqrt{3} & 0 & 0 \\
0 & 0 & \sqrt{3} & 0 \\
0 & 0 & 0 & 0 
\end{pmatrix} .
\]

\[(4.50)\]

This breaks \(S_4\) to \(S_2\). This lies in a type I quotient: \(IJJ = \frac{2}{3}I\).

(11) \((AB, BCD)\):

\[
f_0 = \begin{pmatrix} 
\sqrt{4} & 0 & 0 & 0 \\
0 & \sqrt{4} & 0 & 0 \\
0 & 0 & \sqrt{4} & 0 \\
0 & 0 & 0 & 0 
\end{pmatrix} \otimes 1 \otimes \cdots \otimes 1 
\]

\[(4.51)\]

\[
f_{2L} = 1 \otimes \cdots \otimes 1 \otimes \begin{pmatrix} 
0 & 0 & 0 & 0 \\
0 & \sqrt{3} & 0 & 0 \\
0 & 0 & \sqrt{3} & 0 \\
0 & 0 & 0 & \sqrt{3} 
\end{pmatrix} .
\]

\[(4.52)\]

This breaks \(S_4\) to \(S_2\). This lies in a type II quotient: \(IJJ = \frac{1}{3}J, IJI = \frac{1}{3}I\).
(12) \((ABC, ABC)\):

\[
f_0 = \begin{pmatrix}
\frac{\sqrt{3}}{3} & 0 & 0 & 0 \\
0 & \frac{\sqrt{3}}{3} & 0 & 0 \\
0 & 0 & \frac{\sqrt{3}}{3} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \otimes 1 \otimes \cdots \otimes 1
\]

\[
f_{2L} = 1 \otimes \cdots \otimes 1 \otimes \begin{pmatrix}
\frac{\sqrt{3}}{3} & 0 & 0 & 0 \\
0 & \frac{\sqrt{3}}{3} & 0 & 0 \\
0 & 0 & \frac{\sqrt{3}}{3} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

This breaks \(S_4\) to \(S_3\). This lies in a type I quotient: \(IJI = \frac{2}{3}I\).

(13) \((ABC, BCD)\):

\[
f_0 = \begin{pmatrix}
\frac{\sqrt{3}}{3} & 0 & 0 & 0 \\
0 & \frac{\sqrt{3}}{3} & 0 & 0 \\
0 & 0 & \frac{\sqrt{3}}{3} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \otimes 1 \otimes \cdots \otimes 1
\]

\[
f_{2L} = 1 \otimes \cdots \otimes 1 \otimes \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{\sqrt{3}}{3} & 0 & 0 \\
0 & 0 & \frac{\sqrt{3}}{3} & 0 \\
0 & 0 & 0 & \frac{\sqrt{3}}{3}
\end{pmatrix}.
\]

This breaks \(S_4\) to \(S_2\). This lies in a type I quotient: \(IJI = \frac{1}{3}I\).

4.2. Finding Potts spectra within the XXZ spectra

In the TL and 1BTL algebras the XXZ representation was faithful and therefore evaluating a Hamiltonian in this representation gives all the possible eigenvalues allowed by the algebra. Moreover in the Potts models we understood how to extract the spectra from the XXZ one. In the next section we shall use this fact, together with known analytical results from the XXZ model, to deduce finite size scaling properties of the Potts models with boundaries.

In the case of the 2BTL the XXZ representation is not a faithful representation even of the finite dimensional quotients \(IJI = bI\) that we considered\(^2\). Therefore a naive extension of the TL and 1BTL results is impossible. However, as we shall see, by combining the XXZ results with several different values of the parameter \(b\) one can obtain all the eigenvalues of the Potts models. This phenomenon is a consequence of the existence of indecomposable representations that can occur at the exceptional points of the 2BTL algebra.

The first case in which this phenomenon occurs is the three-state Potts model \((q = e^{\pi i/6})\). Let us consider the two different types of boundary term given by \(s_1 = s_2 = \sqrt{3}\).

\(^2\) This can be seen already at two sites where the quotient \(IJI = bI\) is \(c_1 f_0 f_2 e_1 = b e_1\). We have twenty possible words: \(1, f_0, e_1, f_2, f_0 e_1, f_0 f_2, e_1 f_0, e_1 f_2, f_2 e_1, f_0 f_2 e_1, e_1 f_0 f_2, f_2 e_1 f_0, f_0 e_1 f_2, f_2 e_1 f_2, e_1 f_0 f_2, f_2 e_1 f_0 f_2, f_0 f_2 e_1 f_2, f_0 f_2 e_1 f_0 f_2\). If we also have the relation \(J I J = b J\) then we must remove the final word from this set. The two-site XXZ representation is only of dimension \(4 \times 4 = 16\) and so cannot possibly be faithful.
In the first (4.19) denoted as \((A, A)\) an unbroken \(Z_2\) symmetry remains. The states can therefore be labelled by their parity under this symmetry. All the states lie in the quotient \(IJ = \sqrt{3}I\). In the second case \((A, B)\) the \(S_3\) symmetry is fully broken and all the states lie in the quotient \(J = 0\).

For each of these cases one might hope to be able to obtain these eigenvalues from those of a single \(2L\)-site \(XXZ\) representation with a particular value of \(b\). In appendix A.3. we have given the eigenvalues of the Potts chains. We took an inhomogeneous chain to emphasize that integrability plays no role in our arguments. For the \((A, B)\) case, which has \(J = 0\), all the eigenvalues are indeed found within an \(XXZ\) model with \(b = \sqrt{3}\). We find numerically that the odd parity states can be obtained by instead taking the \(XXZ\) representation with \(b = -\sqrt{3}\). This is not a value of \(b\) that occurs in the Potts model. What is special about these values of \(b\) and why should there be mixing between them?

Recently in [19] a conjecture was made, based on studies at a low number of sites, for the critical values of \(b\). These are points at which the 2BTL algebra in the quotient \(IJ = bI\) has indecomposable representations. In terms of the parametrization arising from the \(XXZ\) model, (4.13) and (4.14), they are given by:

- \(N\) odd:
  \[\pm \phi = 2k\gamma + \epsilon_1 w_1 + \epsilon_2 w_2 + 2\pi Z.\] (4.57)

- \(N\) even:
  \[\pm \phi = (2k + 1)\gamma + \epsilon_1 w_1 + \epsilon_2 w_2 + 2\pi Z\] (4.58)

where \(k\) is a non-negative integer \(k < L/2\) and \(\epsilon_1, \epsilon_2 = \pm 1\). Therefore each choice of \(k\) gives rise to four critical points except for \(N\) odd and \(k = 0\) in which case there are only two. There are precisely \(2N\) critical points in total.

The \(L\)-site Potts model has \(N = 2L\) even. In the case of the three-state Potts models in which we have boundary parameters \(s_1 = s_2 = \sqrt{3}\) (i.e. \(\omega_1 = \omega_2 = 2\pi/3\)) we find the only exceptional values are \(b = -\sqrt{3}, 0, \sqrt{3}\). Firstly we see that, as with the TL and 1BTL algebras, the Potts representations of 2BTL are at (a subset of) the critical points.

We do not yet have a full understanding of the 2BTL algebra [40]. Here we shall present arguments based on the structure which we have deduced from studying a low number of sites. At generic points the full space of 2BTL words in the quotient \(IJ = bI\) can be fully decomposed into irreducible representations:

\[\mathcal{V}^{(b)} = V^{(b)}_{XXZ} \oplus V_1 \oplus V_2 \oplus \cdots \oplus V_{2N}.\] (4.59)

The \(XXZ\) representation, of dimension \(2^N\), is generically found to be irreducible. The quotient also contains \(2N\) additional irreducible representations. We have indicated the dependence of the representations on the parameter \(b\). A crucial point is that all the representations in \(\mathcal{V}^{(b)}\) except \(V_{XXZ}\) are annihilated by \(I\) and are therefore insensitive to the value of \(b\). Therefore on the full space \(\mathcal{V}^{(b)}\) many of the eigenvalues of the 2BTL Hamiltonian (4.2) do not depend on \(b\).

At each of the \(2N\) exceptional points, given by (4.57) or (4.58), we find that a different representation \(V_i\) mixes with \(V_{XXZ}\) and becomes part of a larger indecomposable
The Temperley–Lieb algebra and its generalizations in the Potts and $XXZ$ models

This implies that, at an exceptional point, some of the eigenvalues of the 2BTL Hamiltonian on the $XXZ$ space are the same as some of those from another space $V_i$. By allowing $b$ to take different exceptional values we can ‘read’ the eigenvalues from different spaces $V_i$ by just studying the $XXZ$ one. The fact that we had to combine several $XXZ$ representations with different values of $b$ to find all the Potts eigenvalues is simply due to the fact that the Potts representation is composed of several irreducible representations of the 2BTL algebra.

In the tables below we give the sectors of the $XXZ$ model which contribute to the Potts models with boundary terms of different types.

- Two-state Potts: Ising:
  In the Ising case the boundary terms are named as in section 4.1.2. In the table below we have indicated with a $\ast$ the values of $b$ in the $XXZ$ models in which the Potts eigenvalues appear. It is important to stress that it is only some of the $XXZ$ eigenvalues that are used.

| Boundary term $XXZ$ term | $b = 0$ | $b = \sqrt{2}$ |
|-------------------------|---------|----------------|
| $(+, +)$                |         | $\ast$         |
| $(+, -)$                |         | $\ast$         |

- Three-state Potts:
  In the three-state Potts case the boundary terms are named as in section 4.1.3. In some of the cases the boundary terms are still invariant under a $Z_2$ symmetry. In these cases the states can be labelled by their parity under this symmetry. We have indicated this in table 1 below with $\pm$ signs. The ground state is always found to be in the $+$ parity sector.

| Boundary term $XXZ$ term | Residual symmetry $b = -\sqrt{3}$ | $b = 0$ | $b = \sqrt{2}$ | $b = \sqrt{3}$ |
|-------------------------|-----------------------------------|---------|----------------|----------------|
| $(A, A)$                | $Z_2$                             | $-$     | $\ast$         | $+$            |
| $(A, B)$                |                                   |         | $\ast$         |                |
| $(A, AB)$               |                                   |         | $\ast$         |                |
| $(A, BC)$               | $Z_2$                             |         | $+$            | $-$            |
| $(AB, AB)$              | $Z_2$                             |         | $-$            | $\ast$         |
| $(AB, BC)$              |                                   |         |                | $+$            |

One can immediately see that in the cases in which there is a residual symmetry the different parity components are in different irreducible representations. We shall see later that the number of irreducible representations of 2BTL algebra present with each boundary term exactly matches the number of primary fields in the continuum CFT. This fact was previously found in the study of particular cases with $\gamma = \pi/2$ and $\pi/3$ [19].

The Potts representations lie at multi-critical points, generalizing the notation of doubly critical from section 3.2.3, in which the TL and 1BTL subalgebras are also non-semisimple. It remains a challenge to understand the 2BTL representation theory and the structure of the Potts representations we have given here. The Bethe ansatz for the
The Temperley–Lieb algebra and its generalizations in the Potts and XXZ models

XXZ model with general boundary terms has (only) been constructed \([41]–[43]\) in the cases in which they satisfy an additional constraint. This constraint is exactly the same one as defines the critical points of the 2BTL algebra \([19]\). An understanding of the Potts representations will allow one to systematically extract the Potts spectra, and hence FSS results, from the XXZ one.

5. Finite size scaling limits

5.1. Temperley–Lieb and 1BTL cases

In this section we shall discuss the finite size scaling (FSS) limits of the integrable TL and 1BTL Hamiltonians. We shall be able to use these to discuss FSS limits for the Potts models with different boundary terms. The FSS result for the integrable TL Hamiltonian is well known \([8]\). The 1BTL case is new and, as in the TL case, there will be a clear connection between the representation theory of the finite lattice algebra (i.e. 1BTL) and the continuum conformal field theory.

The results for the finite size scaling of the TL chain, as given in \([8]\), can be understood in the following way. We are concerned with the integrable Hamiltonian

\[
H = -\gamma \frac{\pi \sin \gamma}{\pi} \sum_{i=1}^{N-1} e_i. \tag{5.1}
\]

The pre-factor is necessary so that the resulting theory is conformally invariant \([23]\).

Let us consider the following quantities:

\[
\bar{F}_{ji}(N; \gamma) = \frac{N}{\pi} \{ E_{ji}(N; \gamma) - E_{00}(N; \gamma) \} \tag{5.2}
\]

\[
\mathcal{F}_j(N; \gamma) = \sum_i z^{F_{ji}(N; \gamma)} \tag{5.3}
\]

where \(E_{ji}(N; \gamma)\) denotes the energy levels, indexed by \(i\), of the Hamiltonian \((5.1)\) occurring in the irreducible representation \(V_j\). In the case of the XXZ representation we have \(2j+1\) copies of each \(V_j\) \((2.13)\). The FSS limit is defined as

\[
\mathcal{F}_j(\gamma) = \lim_{N \to \infty} \mathcal{F}_j(N; \gamma). \tag{5.4}
\]

Now setting

\[
\gamma = \frac{\pi}{m+1} \tag{5.5}
\]

we find the central charge of the theory is given by

\[
c = 1 - \frac{6}{m(m+1)} \tag{5.6}
\]

For \(m = 3, 4, \ldots\) this is the central charge of the \(c_{m+1,m}\) minimal model. In this case there are only a finite number of irreducible representations with \(j \leq (m-1)/2\). These minimal models have Virasoro degenerate fields \([44]\) given by

\[
h(r, s) = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}. \tag{5.7}
\]
The Temperley–Lieb algebra and its generalizations in the Potts and XXZ models

Then the FSS limit (5.4) of the energy levels of the integrable Hamiltonian (5.1) occurring in the irreducible representation $V_j$ are given by $h_{1,2j+1}$ [8]. The main point of abstracting away from the XXZ chain is that it allows one to use any representation (e.g. Potts) of the TL algebra.

Now we shall discuss the FSS limit for the integrable 1BTL Hamiltonian:

$$H^{nd} = \frac{\gamma}{\pi \sin \gamma} \left( -af_0 - \sum_{i=1}^{N-1} e_i \right).$$

(5.8)

Let us begin by using the XXZ representation given in (2.3) and (3.4). Then parametrizing $a$ by

$$a = \frac{2 \sin \gamma \sin(\omega + \gamma)}{\cos \omega + \cos \delta},$$

(5.9)

we get the integrable XXZ chain with arbitrary left boundary term:

$$H^{nd} = \frac{\gamma}{\pi \sin \gamma} \left\{ -\frac{\sin \gamma}{\cos \omega + \cos \delta} (i \cos \omega \sigma^x_1 + \sigma^x_1 + \sin \omega) \\
+ \frac{1}{2} \sum_{i=1}^{N-1} \left( \sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1} + \cos \gamma \sigma^z_i \sigma^z_{i+1} - \cos \gamma \right) + i \sin \gamma (\sigma^z_1 - \sigma^z_N) \right\}.$$

(5.10)

In order to understand the FSS limit of the one-boundary XXZ chain we shall use the spectral equivalence found in [14] between this system and the XXZ chain with purely diagonal boundaries:

$$H^d = -\frac{1}{2} \frac{\gamma}{\pi \sin \gamma} \left\{ \sum_{i=1}^{N-1} (\sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1} - \cos \gamma \sigma^z_i \sigma^z_{i+1} + \cos \gamma) \\
+ \sin \gamma \left[ \tan \left( \frac{\omega + \delta}{2} \right) \sigma^z_i + \tan \left( \frac{\omega - \delta}{2} \right) \sigma^z_N + \frac{2 \sin \omega}{\cos \omega + \cos \delta} \right] \right\}. \quad (5.11)$$

The diagonal Hamiltonian $H^d$ has the obvious local charge:

$$S^z = \frac{1}{2} \sum_{i=1}^{N} \sigma^z_i.$$ 

(5.12)

In [14] it was found that for generic values of the parameters the energy levels of $H^{nd}$ in the sector $W_Q$ (for the definition of 1BTL irreducible sectors $W_Q$ see section 3.2) are exactly the same as the energy levels of $H^d$ in the sector with total $S^z$ component equal to $Q$.

The Bethe ansatz for the diagonal chain has been well studied [7]. The finite size scaling limit is well understood only if certain numerical truncation schemes are employed. Here, by using the one-boundary Hamiltonian, we will be able to understand all of this properly using the 1BTL representation theory of section 3.2.

The finite size scaling is defined like in the TL case (5.4). We start by considering the following quantities:

$$\bar{F}_{Q;i}(N; \gamma, \omega, \delta) = \frac{N}{\pi} \left\{ E_{Q;i}(N; \gamma, \omega, \delta) - E_{0;0}(N; \gamma, \omega = \gamma, \delta) \right\}$$

(5.13)

$$\mathcal{F}_Q(N; \gamma, \omega, \delta) = \sum_i \bar{F}_{Q;i}(N; \gamma, \omega, \delta).$$

(5.14)
where again we use $i$ to index the energy levels now within an irreducible representation $W_Q$. The reason for subtracting the ground state energy of the chain with $\omega = \gamma$ is that this is where the $h_{1,1} = 0$ state lies and we want to keep this energy as our reference state\(^3\).

The limit of large $N$ is independent of $\delta$ as long as $\alpha > 0$ (see (5.9)) and we get

$$
F_Q(\gamma, \omega) = \lim_{N \to \infty} F_Q(N; \gamma, \omega, \delta)
$$

where

$$
h = \frac{1}{4} \left( 1 - \frac{\gamma}{\pi} \right)^{-1}
$$

$$
\phi = \frac{2h\omega}{\pi}
$$

$$
\alpha = \frac{2h\gamma}{\pi}.
$$

The value of $\gamma$ is the same as for the bulk theory (5.5). Let us parametrize

$$
\omega = r\gamma = \frac{r\pi}{m+1}
$$

$$
Q = \frac{s-r}{2}.
$$

Then we see that

$$
F_Q(\gamma, \omega) = z^{h(r,s)} \prod_{n=1}^{\infty} (1 - z^n)^{-1}.
$$

The leading form of this expression gives the dimension of an operator degenerate at level $rs$ in the Kac table of the $c_{m+1,m}$ model (5.7). However at this stage this is purely formal as we are at a generic point and so $r, s \notin \mathbb{N}$. In the exceptional cases we must subtract the ‘bad’ part to obtain the irreducible characters.

For the ‘critical’ case, where $r, s \in \mathbb{N}$ but $m \notin \mathbb{N}$, we have a single subtraction (3.16):

$$
K_{r,s} \equiv F_Q(\gamma, \omega) = (z^{h(r,s)} - z^{h(r,-s)}) \prod_{n=1}^{\infty} (1 - z^n)^{-1}
$$

whereas for the ‘doubly critical’ case there are an infinite number of subtractions (3.19):

$$
\chi_{r,s} = K_{r,s} - K_{r,2(m+1)-s} + K_{r,2(m+1)+s} - K_{r,4(m+1)-s} + \cdots
$$

where $K_{r,s}$ was defined in (5.23). This is precisely the character of the $h_{r,s}$ field in the $c_{m+1,m}$ minimal model [26].

We can now rephrase these results in a more algebraic way. We are concerned with the integrable Hamiltonian (5.8) with $a > 0$. Then the finite size scaling limit,

\(^3\) It was shown in [14] that for $\omega = \gamma$ the energy levels of the ‘good’ states are the same, up to a constant, as the ‘good’ states appearing in the Temperley–Lieb Hamiltonian. Therefore we know that the $h_{1,1} = 0$ state must lie in this sector.

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defined in (5.15), of the energy levels occurring in the irreducible representation $W_Q$ is given by (5.22), (5.23) and (5.24) for the generic, critical, and doubly critical cases (see section 3.2) respectively. As in the TL case the advantage of this algebraic viewpoint is that it makes no reference to the actual realization of the 1BTL algebra.

If one is presented with a new 1BTL representation, for example from a loop or RSOS model, once one makes contact with the 1BTL representation theory one can simply read off the finite size scaling results.

One should note that the results for the TL case can also be obtained from the 1BTL algebra in the case $\omega = \pm \gamma$. This is due to the fact, as shown in [14], that in these cases the boundary operator $f_0$ acts like a constant on the irreducible 1BTL space and therefore has a trivial effect on the spectrum. In both cases in the finite size scaling limit we get $h_{1,s}$ fields. This explains why energy levels from the $SU_q(2)$ spectra were also observed in the diagonal chain [45].

Before we discuss the FSS in the Potts models we shall discuss briefly in the next subsection why the spectra of certain 1BTL and 2BTL problems can be found within the spectra of a TL Hamiltonian. This will be useful in order to cross-check, and considerably extend, the TL and 1BTL finite size scaling results.

5.2. Truncation in the Temperley–Lieb quotient cases

We have already seen several examples in the Potts models in which the one-boundary Temperley–Lieb algebra on $N$ sites is actually a quotient of a TL algebra with one extra generator $T_{N+1}(q)$. In these cases the spectrum of

$$H^{\text{quotient}} = -f_0 - \sum_{i=1}^{N-1} e_i$$

must be contained in the spectrum of the integrable Hamiltonian for $T_{N+1}(q)$:

$$H^{\text{TL}} = -\sum_{i=1}^{N} e_i$$

when evaluated in a faithful representation like the $XXZ$ one. If the coefficient of the boundary term is different from that of (5.25) then we have coincidence between a boundary chain and an inhomogeneous TL chain. In these cases one cannot compare to known finite size scaling results for integrable TL chains.

A similar situation occurs for certain choices of the right boundary term, $f_N$, when the 2BTL algebra also lies in a quotient of TL with two extra generators. A numerical example illustrating this is given in section A.4. The existence of these quotients allows one to considerably extend the FSS results.

In the next two sections we shall use the FSS results and representation theory of TL and 1BTL to derive the finite size scaling of the Potts models with boundary terms.

5.3. Ising model

In this section we shall show how the knowledge of TL and 1BTL representation theory together with the finite size scaling results for the $XXZ$ model allows us to derive results
in the Ising model. Our conclusions fully agree with the continuum results of [28] if we identify the fixed boundary conditions in the continuum with the appropriate boundary terms added to the Hamiltonian.

The Kac table for the \( c_{4,3} = \frac{1}{2} \) minimal model is given below:

| \( r \) | \( \frac{1}{2} \) | 0 | \( \frac{1}{16} \) | \( \frac{1}{16} \) | 1 | 0 | \( \frac{1}{2} \) | 1 | 2 |
|---|---|---|---|---|---|---|---|---|---|

The entries correspond to the primary fields \( h_{r,s} \) as given in (5.7). The values of \( r \) and \( s \) are given on the horizontal and vertical axis respectively.

As we reviewed in section 1 the spectra of the \( L \)-site Ising model with free boundary conditions are contained in the spectra of the \( 2L \)-site \( SU_q(2) \) invariant \( XXZ \) model. The ‘good’ states come from \( V_0 \oplus V_1 \). We know that the FSS limit of \( V_j \) is \( h_{1,2j+1} \) which therefore implies that the finite size scaling limit of the Ising model with free boundary conditions gives \( h_{1,1} = 0 \) and \( h_{1,3} = \frac{1}{2} \) fields. This is in agreement with [28].

For the \( L \)-site Ising model with an integrable boundary term (3.5) added to one end (there is only one possibility) we saw in section 3.3.1 that we have only the \( W_0 \) representation. Using the FSS scaling results for 1BTL we find this gives \( h_{2,2} = \frac{1}{16} \). This also agrees with the results of [28]. These states also lie in a quotient of TL and come from the \( V_{1/2} \) representation of the \((2L + 1)\)-site \( XXZ \) model. We have denoted this as \((+, \text{free})^q\) in the table below where the \( q \) superscript indicates the use of a quotient. Using the TL finite size scaling results we find this field becomes \( h_{1,2} = \frac{1}{16} \). Therefore we have perfect agreement between the two approaches. In the case of \((+,+)\) boundary terms, defined in section 4.2, these can be understood in terms of TL quotients, 1BTL quotients, or the 2BTL algebra. In the two quotient cases we obtain \( h = 0 \). The case of \((+,-)\) boundary terms can be treated in a similar manner and we obtain \( h = \frac{1}{2} \). Although we do not yet know the FSS limit of the 2BTL Hamiltonian (4.2) directly we can see that in the Ising model with \((+,+)\) or \((+,-)\) boundary terms we had just one irreducible representation of 2BTL (as seen in table 1) and one continuum field.

These results are summarized in the following table. We use a \( q \) superscript to denote the use of a quotient of the algebra:

| \( XXZ \) chain | Sector | \( L \)-site Ising model with boundary term | FSS limit |
|---|---|---|---|
| \( 2L; 2L + 2 \) | \( V_0 \) | \((+, \text{free}); (+, +)^q\) | \( h_{1,1} = 0 \) |
| \( 2L + 1 \) | \( V_{1/2} \) | \((+, \text{free})^q\) | \( h_{1,2} = \frac{1}{16} \) |
| \( 2L; 2L + 2 \) | \( V_1 \) | \((+, \text{free}); (+, -)^q\) | \( h_{1,3} = \frac{1}{2} \) |
| \( 1BTL (f_0^2 = \sqrt{2}f_0) \) | \( 2L + 1 \) | \( W_{-1/2} \) | \((+, -)^q\) | \( h_{2,1} = \frac{1}{2} \) |
| \( 2L \) | \( W_0 \) | \((+, \text{free})^q\) | \( h_{2,2} = \frac{1}{16} \) |
| \( 2L + 1 \) | \( W_{1/2} \) | \((+, +)^q\) | \( h_{2,3} = 0 \) |
5.4. Three-state Potts model

In this section we shall again use TL and 1BTL representation theory to derive results in the three-state Potts model. Our conclusions once again fully agree with those of Cardy [28].

The Kac table for the \( c_{6,5} = \frac{4}{5} \) minimal model is given below:

\[
\begin{array}{cccccc}
5 & 3 & 7 & 2 & 5 & 0 \\
4 & 13 & 21 & 1 & 1 & 2 \\
 & 8 & 40 & 40 & 8 & 5 \\
3 & 2 & 1 & 1 & 2 & 3 \\
 & 3 & 15 & 15 & 3 & 3 \\
2 & 1 & 1 & 21 & 13 & 4 \\
 & 8 & 40 & 40 & 8 & 5 \\
1 & 0 & 2 & 7 & 5 & 3 \\
 & 1 & 2 & 3 & 4 & 4 \\
\end{array}
\]

As for the previous Ising example we have labelled \( r \) on the horizontal and \( s \) on the vertical axis.

For the case of free boundary conditions we saw that the singlets came from \( V_0 \) and \( V_2 \) and the doublets from \( V_1 \). In the FSS limit this means singlets come from \( h_{1,1} = 0 \) and \( h_{1,5} = 3 \) and doublets from \( h_{1,3} = \frac{2}{3} \).

With the boundary term corresponding to \( f_0^2 = \sqrt{3} f_0 \) (i.e. \( \omega = 2\pi/3 \)) we find only the irreducible representations \( W_0 \) and \( W_{-1} \). In the FSS limit these become \( h_{4,4} = \frac{1}{8} \) and \( h_{4,2} = \frac{13}{8} \) respectively. These fields also lie in the \( V_{1/2} \) and \( V_{3/2} \) irreducible representations of TL. In the FSS limit these become \( h_{1,2} = \frac{1}{8} \) and \( h_{1,4} = \frac{13}{8} \). Therefore we see total consistency between the two approaches.

For the case of a boundary term corresponding to \( f_0^2 = \sqrt{3}/2 f_0 \) (i.e. \( \omega = \pi/3 \)) we find the irreducible representations \( W_0 \) and \( W_1 \). In the FSS limit these become \( h_{2,2} = \frac{1}{40} \) and \( h_{2,4} = \frac{21}{40} \). Although these boundary conditions are linearly related to the previous ones at \( \omega = \frac{2\pi}{3} \) this is of no help as the FSS results are only for the Hamiltonian (5.1) and one cannot expect the same result if some of the terms are reversed.

The finite size scaling results are shown below where again we use a \( q \) superscript to denote the use of a quotient of the algebra:
The Temperley–Lieb algebra and its generalizations in the Potts and \( XXZ \) models

| Length of \( XXZ \) chain | Sector | \( L \)-site three-state Potts model with boundary term | FSS limit |
|---------------------------|--------|-------------------------------------------------|----------|
| Temperley–Lieb:          |        |                                                 |          |
| \( 2L; 2L + 2 \)         | \( V_0 \) | \( \text{(free, free); } (A, A)^g \)          | \( h_{1,1} = 0 \) |
| \( 2L + 1 \)             | \( V_{1/2} \) | \( (A, \text{free})^g \)                       | \( h_{1,2} = \frac{1}{3} \) |
| \( 2L \)                 | \( V_1 \) | \( \text{(free, free); } (A, B)^g \)          | \( h_{1,3} = \frac{2}{3} \) |
| \( 2L + 1 \)             | \( V_{3/2} \) | \( (A, \text{free})^g \)                       | \( h_{1,4} = \frac{13}{8} \) |
| \( 2L; 2L + 2 \)         | \( V_2 \) | \( \text{(free, free); } (A, A)^g \)          | \( h_{1,5} = 3 \) |

| 1BTL \( (f_0^2 = \sqrt{3} f_0) \): |        |                                                 |          |
| \( 2L + 1 \) | \( W_{-3/2} \) | \( (A, A)^g \) | \( h_{4,1} = 3 \) |
| \( 2L \)     | \( W_{-1} \) | \( (A, \text{free}) \) | \( h_{4,2} = \frac{13}{8} \) |
| \( 2L + 1 \) | \( W_{-1/2} \) | \( (A, B)^g \) | \( h_{4,3} = \frac{4}{3} \) |
| \( 2L \)     | \( W_0 \) | \( (A, \text{free}) \) | \( h_{4,4} = \frac{1}{4} \) |
| \( 2L + 1 \) | \( W_{1/2} \) | \( (A, A)^g \) | \( h_{4,5} = 0 \) |

| 1BTL \( (f_0^2 = \frac{\sqrt{3}}{2} f_0) \): |        |                                                 |          |
| \( 2L + 1 \) | \( W_{-1/2} \) | \( (AB, C)^g \) | \( h_{2,1} = \frac{2}{5} \) |
| \( 2L \)     | \( W_0 \) | \( (AB, \text{free}) \) | \( h_{2,2} = \frac{1}{40} \) |
| \( 2L + 1 \) | \( W_{1/2} \) | \( (AB, A)^g \) | \( h_{2,3} = \frac{1}{15} \) |
| \( 2L \)     | \( W_1 \) | \( (AB, \text{free}) \) | \( h_{2,4} = \frac{21}{40} \) |
| \( 2L + 1 \) | \( W_{3/2} \) | \( (AB, C)^g \) | \( h_{2,5} = \frac{3}{4} \) |

These FSS results, some of which can be obtained in several ways, are consistent with the continuum boundary conformal field results of Cardy [28]. We would like however to stress one important point. We have always referred to the addition of boundary terms rather than boundary conditions. We have at no point attempted to construct transfer matrices corresponding to fixed boundary conditions. However consider the Hamiltonian

\[
H = -a \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \sum_{i=1}^{N-1} e_i
\]

(5.27)

with \( a > 0 \). If, under the FSS scaling, \( a \) becomes very large, then we will flow towards the situation in which the first spin is fixed into one state.

The other boundary term \( AB \) will correspond to a fixed boundary condition which allows oscillation between \( A \) and \( B \) [28, 46].

We have discussed in this section the finite size scaling of the TL and 1BTL integrable Hamiltonians. As we have emphasized throughout this paper the representation theoretic arguments, and subtractions to obtain a unitary theory, are completely algebraic in nature and rely only on the structure of the TL algebra or its appropriate extension. Consider,
for example, the Hamiltonian
\[ H^{\text{non-critical}} = -\sum_{i=1}^{L-1} e_{2i} - \lambda \sum_{i=1}^{L} e_{2i-1}. \]  

(5.28)

For \( \lambda \neq 1 \) this is the Hamiltonian of the off-critical Potts models. The spectrum of this (non-integrable) theory is now contained within the spectrum of an inhomogeneous \( XXZ \) chain and, although the finite size scaling results would be completely different, one uses exactly the same truncation of the lattice TL algebra to obtain the unitary Potts spectrum.

6. Periodic Temperley–Lieb algebra

This is another extension of the TL algebra, which we shall call periodic Temperley–Lieb (PTL), in which a new element \( e_N \) is added to the TL algebra (2.1) satisfying [10,11,30]
\[
\begin{align*}
e_N^2 &= (q + q^{-1}) e_N, \\
e_i e_N e_i &= e_i, \quad i = 1, N-1, \\
e_N e_i e_N &= e_N, \quad i = 1, N-1, \\
e_i e_N &= e_N e_i, \quad i \neq 1, N-1.
\end{align*}
\]

(6.1)

This algebra, and slight variations of it, have been studied in the mathematical literature [31,32].

The integrable Hamiltonian for the PTL is given by
\[ H^{\text{PTL}} = -\sum_{i=1}^{N} e_i. \]  

(6.2)

The space of words is now infinite dimensional. Let us define:

- \( N \) even:
  \[
  I = e_1 e_3 \cdots e_{N-1} \\
  J = e_2 e_4 \cdots e_N.
  \]  

(6.3, 6.4)

- \( N \) odd:
  \[
  I = e_2 e_4 \cdots e_{N-1} \\
  J = e_1 e_3 \cdots e_{N-2} e_N.
  \]  

(6.5, 6.6)

In the \( L \)-site Potts models the value of \( N = 2L \) is always even. In this paper we shall encounter representations which lie in three different types of quotient. In terms of the quantities \( I \) and \( J \) defined here they take exactly the same form (types I, II and II) as those in the 2BTL case (4.7)–(4.9). For \( N \) even these quotients are all finite dimensional; however for \( N \) odd one must take a further quotient [11].

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6.1. Representations

6.1.1. XXZ. The global symmetry in this case is \( U(1) \) and therefore we can consider the toroidal boundary conditions

\[
\sigma_{L+1}^\pm = e^{\pm i \phi} \sigma_1^\pm
\]

(6.7)

\[
\sigma_{L+1}^z = \sigma_1^z
\]

(6.8)

where \( \sigma^\pm = \sigma^x \pm i\sigma^y \). The extra Temperley–Lieb generator is given by

\[
e_L = -\frac{1}{2} \left\{ 2e^{-i \theta} \sigma_L^+ \sigma_1^- + 2e^{i \theta} \sigma_L^- \sigma_1^+ + \cos \gamma \sigma_L^z \sigma_1^z - \cos \gamma + i \sin \gamma (\sigma_L^z - \sigma_1^z) \right\}.
\]

(6.9)

Although here we have put all the twist angle \( \phi \) dependence into \( e_L \) it is a global property of the system.

For an even number of sites \( L \) we find that this representation lies in a type II quotient with

\[
b = 2 + e^{i \phi} + e^{-i \phi}.
\]

(6.10)

For odd number of sites the algebra lies in a type II quotient only if \( e^{i \phi} = 1 \) with \( b = -1 \) or \( e^{i \phi} = -1 \) with \( b = 1 \).

In the XXZ representation all the generators are equivalent and therefore one can introduce an additional ‘translation’ operator \( \tau \) with

\[
\tau e_i \tau^{-1} = e_{i+1}.
\]

(6.11)

It is clear that this operator commutes with the integrable Hamiltonian (6.2) and therefore they can be simultaneously diagonalized. In terms of the XXZ model the operator \( \tau \) is simply the momentum and it has been used when discussing the spectrum of the model and its finite size scaling [18].

6.1.2. Two-state Potts (Ising). The global symmetry of the bulk model is \( S_2 = \mathbb{Z}_2 \). There are two conjugacy classes corresponding to untwisted and twisted periodic boundary conditions:

- **Untwisted**: \( \sigma_{L+1}^z = \sigma_1^z \):

  \[
e_{2L} = \frac{1}{\sqrt{2}} (1 + \sigma_1^z \sigma_L^z).
\]

(6.12)

In this case we find this lies in a type I quotient: \( JJI = 2I \).

- **Twisted**: \( \sigma_{L+1}^z = -\sigma_1^z \):

  \[
e_{2L} = \frac{1}{\sqrt{2}} (1 - \sigma_1^z \sigma_L^z).
\]

(6.13)

In this case we find this lies in a type III quotient: \( J = 0 \).

These conclusions do not depend on the number of sites \( L \).
6.1.3. Three-state Potts. The different possible periodic boundary conditions are in correspondence with the conjugacy classes of $S_3$.

\[ e_{2L} = \frac{1}{\sqrt{3}} (1 + R_L R_{L+1}^2 + R_L^2 R_{L+1}) . \]  

- 123:

\[ R_{L+1} = R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix} . \]  

This lies in a type I quotient: $IJI = 3I$.

- (12)3:

\[ R_{L+1} = \begin{pmatrix} e^{2\pi i/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix} . \]  

This lies in a type II quotient: $IJI = I$ and $JIJ = J$.

- (123):

\[ R_{L+1} = \begin{pmatrix} e^{4\pi i/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{2\pi i/3} \end{pmatrix} . \]  

This lies in a type III quotient: $J = 0$.

6.1.4. Four-state Potts. The different possible periodic boundary conditions are in correspondence with the conjugacy classes of $S_4$.

\[ e_{2L} = \frac{1}{\sqrt{4}} (1 + R_L R_{L+1}^3 + R_L^2 R_{L+1}^2 + R_L^3 R_{L+1}) . \]  

- 1234:

\[ R_{L+1} = R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} . \]  

This lies in a type I quotient: $IJI = 4I$.

- (12)34:

\[ R_{L+1} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} . \]  

This lies in a type I quotient: $IJI = 2I$.  

\[ \text{doi:10.1088/1742-5468/2006/01/P01003} \]
6.2. Potts within $XXZ$

The critical values of $b$ are given in terms of the parametrization arising in the $XXZ$ representation (6.9) by [10,11]

$$\phi = 2\gamma Z$$

(6.24)

where $q = e^{i\gamma}$. For the Ising model with $q = e^{i\pi/4}$ this gives $b = 0, 2, 4$ and for the three-state Potts with $q = e^{i\pi/6}$ it gives $b = 0, 1, 3, 4$. We see that all the values realized in the Potts representation do indeed lie at critical points of the PTL algebra.

It has been discussed in [18,33] how to obtain the spectra of integrable Potts models numerically from the $XXZ$ ones. For each choice of conjugacy class in the Potts model one must combine several different twist sectors from the $XXZ$ model. We have used the relation between the twist angle and parameter $b$ (6.10) in the tables below. The labelling of states $(h, \bar{h})$ is from the continuum CFT. However the combinations of $XXZ$ sectors, and position of the ground state sector, remain the same in finite size systems.

- Two-state Potts: Ising:

| Potts conjugacy class | $XXZ$ |
|-----------------------|-------|
| $b = 2 \phi = \pi/2$  | $b = 0 \phi = \pi$ |
| $12 \ IJI = 2I$       | $(0, 0) + (\frac{1}{2}, \frac{1}{2})$ | $(\frac{1}{16}, \frac{1}{16})$ |

(12) $J = 0$

$(\frac{1}{16}, \frac{1}{16}) + (0, \frac{1}{2}) + (\frac{1}{2}, 0)$
The Temperley–Lieb algebra and its generalizations in the Potts and XXZ models

- Three-state Potts:

| Potts conjugacy class | \( X X Z \) | \( X X Z \) |
|-----------------------|-----------|-----------|
| \( I J I = 3I \)      | \( b = 3 \ (\phi = \pi/3) \) | \( b = 1 \ (\phi = 2\pi/3) \) | \( b = 0 \ (\phi = \pi) \) |
| \( (123) J = 0 \)    | \( (0, 0) + (\frac{2}{5}, \frac{2}{5}) \) | \( (0, 3) + (3, 0) \) | \( (0, 2) + (2, 0) \) |
|                       | \( + (\frac{7}{5}, \frac{7}{5}) + (3, 3) \) | \( + (\frac{2}{5}, \frac{2}{5}) + (\frac{7}{5}, \frac{2}{5}) \) | \( + (\frac{1}{15}, \frac{1}{15}) + (\frac{7}{5}, \frac{3}{5}) \) |

One can see, like in the 2BTL case of section 4.2, that we have a mixing between states with different critical values \((6.24)\) of the parameter \(b\). It seems very plausible that all the mixing, truncations, and FSS results of \([18,33]\) can be deduced purely from the representation theory of the periodic Temperley–Lieb algebra but we shall not attempt this here.

7. Conclusion

In this paper we have studied the Temperley–Lieb algebra and its one-boundary, two-boundary, and periodic extensions in the Potts and \( X X Z \) models. These were used to understand the structure and relations between the spectra, as well as the finite size scaling behaviour. Throughout this paper, with the exception of the FSS results, we have used algebraic techniques rather than relying on integrability. The same truncation schemes therefore continue to hold in off-critical and inhomogeneous Potts models.

The first, and from our point of view the most basic, algebra that appears is the Temperley–Lieb algebra (TL) discussed in section 2. The Potts models with free boundary conditions and the \( X X Z \) model with \( SU_q(2) \) invariant boundary terms can be written in terms of the same algebraic Temperley–Lieb Hamiltonian. The existence of this underlying algebra allows one to understand relations between the spectra of these different physical models. Moreover the special structure of the algebra at exceptional points gives one information on additional degeneracies as well as providing the possibility of truncation to unitary (e.g. Potts) representations.

We discussed one-boundary, two-boundary, and periodic generalizations of the TL algebra. In each case, as for Temperley–Lieb, for generic values of the parameters the...
algebra was semisimple and had only irreducible representations. However at exceptional points the algebra becomes non-semisimple and possesses additional indecomposable representations.

For the two-, three-, and four-state Potts models all the representations of these algebras were explicitly given. This gives one a simple classification of possible integrable single-site boundary terms in these models. In every case one finds that the resulting algebra is at an exceptional point. In the case of the one-boundary Temperley–Lieb algebra (1BTL) we were able to use representation theory to understand the Potts representations and extract the spectra from an $XXZ$ model. For the 2BTL case we found that the spectra could be obtained from combining the spectra of several different $XXZ$ spin chains. This fact was again completely algebraic in origin.

In section 5 we discussed the finite size scaling of the Potts models with different boundary terms. One can separate the parameters into those occurring in the algebras and those in the Hamiltonian. This is important as it is just the parameters of the algebra which control the structure of the lattice theory as well as its finite size scaling limit. In the case of the TL, 1BTL, and every case of the two-boundary and periodic extensions that we understood, each representation of the algebra becomes a single representation of the continuum Virasoro algebra. This connection has been well known for the TL algebra but it is rather amazing that it continues to hold for all these generalizations.

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Appendix A. Numerical results

A.1. One-boundary Temperley–Lieb: Ising

We take $L = 2$, $q = e^{\pi i/4}$ and $w = \pi/2$. The Hamiltonian is

\[ H = -\alpha_0 f_0 - \sum_{i=1}^{3} \alpha_i e_i \]  

(A.1)

with

\[ (\alpha_0, \ldots, \alpha_3) = (0.68792, 0.72537, 0.33053, 0.95574). \]  

(A.2)
The Temperley–Lieb algebra and its generalizations in the Potts and $XXZ$ models

The splitting into $Q$ sectors for all the one-boundary examples was performed as follows. On each energy eigenstate the action of the 1BTL centralizer \cite{16} was computed. The connection between this centralizer, $X$, and the 1BTL representation theory has been studied in \cite{14}. However, as the eigenvalues of $X$ with $Q = -2, 0, 2$ are degenerate for $\gamma = \pi/4$, it is difficult to see from which of these sectors a particular eigenstate of the Hamiltonian came. To solve this problem we kept $w = \pi/2$ but perturbed $\gamma$ slightly away from $\pi/4$. As shown in the bottom row of the table this splits the degeneracy between levels and one can again clearly see which sectors they came from.

As explained in section 3.3.1 the Ising model contains only the $Q = 0$ sector. We must however discard the doublets which come from indecomposable representations marked with $\ast$.

A.2. One-boundary Temperley–Lieb: three-state Potts

We take $L = 2$, $q = e^{\pi i/6}$ and $w = 2\pi/3$. In this case we have $f_0^2 = \sqrt{3} f_0$. The Hamiltonian is

$$H = -\alpha_0 f_0 - \sum_{i=1}^{3} \alpha_i e_i$$

where

$$(\alpha_0, \ldots, \alpha_3) = (0.68792, 0.72537, 0.33053, 0.95574).$$

"\input{I2N-13/14441/14441.sty}"

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$Q$ sector in $XXZ$ & -2 & -1 & 0 & 1 & 2 \\
\hline
$X(\gamma = \pi/4)$ & -3.32069 & -3.32069 & -2.66105 & -2.07891 & -1.92688 \\
$X(\gamma = \pi/4 + \epsilon)$ & -1.87591 & -1.37886 & -0.97287 & -0.81231 & -0.36325 & -0.49708 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{tabular}
\end{table}
The Temperley–Lieb algebra and its generalizations in the Potts and XXZ models

We can see that after ignoring any doublets coming from indecomposable representations we keep only the $Q = -1, 0$ sectors to get the Potts spectrum. This is in exact agreement with the truncated Bratelli diagram in section 3.3.2.

For the second boundary term $\omega = 4\pi/3$ we take $L = 4$, $q = e^{\pi i/6}$, and $w = 4\pi/3$. In this case we have $f_0^2 = \sqrt{3}/2f_0$. The Hamiltonian is

$$
H = -\alpha_0 f_0 - \sum_{i=1}^{3} \alpha_i e_i
$$

(A.5)

where

$$(\alpha_0, \ldots, \alpha_3) = (0.68792, 0.72537, 0.33053, 0.95574).$$

(A.6)

| $Q$ sector in XXZ | Potts model |
|------------------|-------------|
| $-2$             | $-1$        | $0$     | $1$     | $2$     |
| $-3.81188$       | $-3.12797$  | $-2.49890$ | $-2.24237$ |
| $-2.09965$       | $-1.80385$  | $-1.19151^*$ | $-1.91511^*$ |
| $-0.71399$       | $-0.70109$  | $-0.63655$  |
| $-0.21686$       | $-0.05829$  |

0

| $Q$ sector in XXZ | Potts model |
|------------------|-------------|
| $-2$             | $-1$        | $0$     | $1$     | $2$     |
| $-3.57991$       | $-2.54529$  | $-2.13969$ | $-1.87428$ |
| $-2.01982$       | $-1.75615$  | $-0.85611$  |
| $-1.68686$       | $-0.67184$  | $-0.59576$  |
| $-0.42747$       | $-0.42396$  | $-0.18014$  |

0* 0*

We can see that after ignoring any doublets coming from indecomposable representations we keep only the $Q = -1, 0$ sectors to get the Potts spectrum. This is in exact agreement with the truncated Bratelli diagram in section 3.3.2.

For the second boundary term $\omega = 4\pi/3$ we take $L = 4$, $q = e^{\pi i/6}$, and $w = 4\pi/3$. In this case we have $f_0^2 = \sqrt{3}/2f_0$. The Hamiltonian is

$$
H = -\alpha_0 f_0 - \sum_{i=1}^{3} \alpha_i e_i
$$

(A.5)

where

$$(\alpha_0, \ldots, \alpha_3) = (0.68792, 0.72537, 0.33053, 0.95574).$$

(A.6)
We can see that after ignoring any doublets coming from indecomposable representations we keep only the \( Q = 0, 1 \) sectors to get the Potts spectrum. This is again in exact agreement with the truncated Bratelli diagram in section 3.3.2.

**A.3. Two-boundary Temperley–Lieb: three-state Potts from \( XXZ \)**

We take again \( L = 2 \) and the boundary term of the three-state Potts model with \( f_0^2 = \sqrt{3} f_0 \) and \( f_4^2 = \sqrt{3} f_4 \). Either the two are the same (4.19) or they are different (4.21):

\[
H = -\alpha_0 f_0 - \alpha_4 f_4 - \sum_{i=1}^{3} \alpha_i e_i \tag{A.7}
\]

with as before

\[
(\alpha_0, \ldots, \alpha_4) = (0.68792, 0.72537, 0.33053, 0.95574, 0.99239). \tag{A.8}
\]

| \( XXZ \) representation | \( b = -\sqrt{3} \) | \( b = 0 \) | \( b = \sqrt{3} \) | Potts model \((A, A)\) | \((A, B)\) |
|---------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| -4.68898 + 0.40346i       | -4.71928        | -4.99459        | -4.99459 +      | -4.71928        |
| -4.68898 - 0.40346i       | -4.4806         | -3.93145        | -3.93145        | -3.93145        |
| -3.64363 + 0.18862i       | -3.47444        | -3.34783        | -3.34783 +      | -3.47444        |
| -3.64363 - 0.18862i       | -2.96446        | -3.10834        | -3.10834        | -2.93234        |
| -2.74265                  | -2.93234        | -2.91038        | -2.74265 +      | -2.84228        |
| -2.21333 - 0.34342i       | -2.46344 - 0.16479i | -2.74922        | -2.74922 +      | -2.74922 +      |
| -2.21333 + 0.34342i       | -2.46344 + 0.16479i | -2.48295        |                |                |
| -2.12550                  | -2.16901        | -2.31532        | -2.16901        |                |
| -2.01588                  | -1.9856         | -1.91307        | -2.01588 +      |                |
| -1.7012 + 0.07186i        | -1.71887        | -1.54222        |                |                |
| -1.7012 - 0.07186i        | -1.38747        | -1.46238        | -1.46238 +      | -1.38747        |
| -1.24121                  | -1.19151        | -0.99767        |                |                |
| -1.04463                  | -0.93583        | -0.81619        | -1.04463 -      | -0.93583        |
| -0.591482                 | -0.631387       | -0.64404        | -0.59148 -      | -0.63139        |
| 0                        | -0.09190        | -0.23527        | -0.23527 +      | -0.09190        |

We have underlined the states from the different \( XXZ \) representations which are also to be found in the Potts model. We have labelled the states from \( Z_{A,A} \) with their parity under the residual \( Z_2 \) symmetry. Note that all the + parity states are found in the \( XXZ \) model with \( b = \sqrt{3} \) and the − parity ones in \( b = -\sqrt{3} \). All the states from \( Z_{A,B} \) come from the \( XXZ \) model with \( b = 0 \).

**A.4. Two-boundary Temperley–Lieb: quotient case**

We take again \( L = 2 \) and the boundary term of the three-state Potts model with \( f_0^2 = \sqrt{3} f_0 \) and \( f_4^2 = \sqrt{3} f_4 \). Either the two are the same (4.19) or they are different (4.21):

\[
H = -\alpha_0 f_0 - \alpha_4 f_4 - \sum_{i=1}^{3} \alpha_i e_i \tag{A.9}
\]

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The Temperley–Lieb algebra and its generalizations in the Potts and \( XXZ \) models

with as before

\[
(\alpha_0, \ldots, \alpha_4) = (0.68792, 0.72537, 0.33053, 0.95574, 0.99239).
\]

\[(A.10)\]

| \( S^2 \) sector in \( XXZ \) | Potts model         | Same | Different |
|---------------------------------|-------------------|------|-----------|
| \( 0 \)                        |                  |      |           |
| \( \pm 1 \)                     |                  |      |           |
| \( \pm 2 \)                     |                  |      |           |
| \( \pm 3 \)                     |                  |      |           |
| \( -4.99459 \)                  |                  |      |           |
| \( -4.71928 \)                  | \( -4.71928 \)    |      | \( 4.71928 \) |
| \( -3.47444 \)                  | \( -3.47444 \)    |      | \( -3.47444 \) |
| \( -3.34783 \)                  |                  |      |           |
| \( -2.93234 \)                  | \( -2.93234 \)    |      | \( -2.93234 \) |
| \( -2.84228 \)                  | \( -2.84228 \)    |      | \( -2.84228 \) |
| \( -2.74922 \)                  |                  |      |           |
| \( -2.74265 \)                  | \( -2.74265 \)    |      | \( -2.74265 \) |
| \( -2.16901 \)                  | \( -2.16901 \)    |      | \( -2.16901 \) |
| \( -2.01588 \)                  | \( -2.01588 \)    |      | \( -2.01588 \) |
| \( -1.46238 \)                  |                  |      |           |
| \( -1.38747 \)                  | \( -1.38747 \)    |      | \( -1.38747 \) |
| \( -1.04463 \)                  | \( -1.04463 \)    |      | \( -1.04463 \) |
| \( -0.93583 \)                  | \( -0.93583 \)    |      | \( -0.93583 \) |
| \( -0.63139 \)                  | \( -0.63139 \)    |      | \( -0.63139 \) |
| \( -0.59148 \)                  | \( -0.59148 \)    |      | \( -0.59148 \) |
| \( -0.23527 \)                  |                  |      |           |
| \( -0.09190 \)                  | \( -0.09190 \)    |      | \( -0.09190 \) |
| \( 0, 0^* \)                    | \( 0, 0^* \)      |      | \( 0, 0^* \) |
| \( 0 \)                        |                  |      | \( 0 \) |

One can clearly see that the different boundary terms come from \( V_1 \) and the same ones from \( V_0 \oplus V_2 \).

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