Optimal quantum tomography of permutationally invariant qubits

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Mutually unbiased bases determine an optimal set of measurements to extract complete information about the quantum state of a system. However, quite often a priori information about the state exist, making some of the measurement bases superfluous. This is, for example, the case when multiqubit states belong to the permutationally invariant subspace. In this paper we derive the minimal sets of mutually unbiased bases needed to tomographically reconstruct such states.

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I. INTRODUCTION

The quantum state is a mathematical entity that encodes complete information about a system: once it is known, the probabilities of the outcomes of any measurement can be predicted. It seems thus indisputable that ascertaining an unknown state accurately turns out to be of uttermost importance for modern quantum technologies. Broadly speaking, this is the scope of quantum tomography [1] which, over the past years, has evolved from the initial theoretical [2] and experimental concepts [3] to a widely acknowledged and fairly standard method used extensively for both discrete [4] and continuous [5] variables.

However, the tomographic task becomes harder as we explore more intricate systems. For example, for the simple case of $n$ qubits, $2^n - 1$ real numbers are required for its complete characterization, while any von Neumann measurement gives only $2^n - 1$ independent data. Consequently, one will have to make at least $2^n + 1$ different such measurements before one can claim to know everything about an a priori unknown system. With such a scaling, it is clear that the methods rapidly become intractable for present state-of-the-art experiments.

As a result, more sophisticated tomographical techniques are called for. New protocols try to exploit the idea that the scheme is explicitly optimized only for a particular kind of states [6]. In that perspective, we look here at the specific but not unimportant example of $n$ qubits prepared in an arbitrary state that is, however, known to be invariant with respect to any qubit permutation. This may be due to, e.g., a permutationally invariant preparation Hamiltonian. In this instance, the associated Hilbert space has dimension $n + 1$, and therefore it should be possible to reconstruct such a state with only $n + 2$ von Neumann measurements.

Permutationally invariant qubit states are employed in diverse quantum information strategies [7]. They are also optimal for quantum metrology [8] and play an important role in the characterization of locally non-interconvertible entanglement classes [9]. Through all this paper we take permutational invariance for granted; theoretical tests of this property (others than full tomography) have been put forward in [10].

Recently, a number of suggestions have appeared for an efficient generation of different entangled permutationally symmetric qubit states [11]. The tomography of such states has already been discussed in Ref. [12], and a four-qubit experiment has been performed [13]. However, in these proposals the measurements have been chosen as a set of informationally complete projectors. This may provide a certain experimental simplicity (e.g., for spin states it may be possible to simply use the orientation of a Stern-Gerlach apparatus to chose the projector), but is by no means an optimal strategy.

The number of separate von Neumann measurements needed for a complete state determination is optimal when the bases in which those measurements are performed are mutually unbiased [14] (in the standard $n$ qubit Pauli tomography, $3^n$ different settings are needed, while in this optimal approach, $2^n + 1$ are enough). In fact, the notion of mutually unbiased bases (MUBs) emerged in the seminal work of Schwinger [15] and has turned into a cornerstone of quantum information, mainly due to the spotlight placed on them by the elegant work of Wootters and coworkers [16]. MUBs are endowed with the property of being maximally incompatible, in the sense that a state giving precise results in one set (i.e., one of the basis states) produces maximally random results when measured in another basis in the MUB set.

Another remarkable advantage of the MUB-based tomography is that each measured probability determines a single element of the density matrix so, in principle, there should be no need for a numerical data inversion to reconstruct the corresponding state. In practice, however, experimental noise and measurement imperfections may yield an unphysical density matrix, so a fitting procedure might still be needed.

For all these compelling reasons, we think it is worthwhile to first prove that minimal complete sets of MUBs exist for the tomography of a permutationally invariant $n$-qubit state, and subsequently show how to construct them. This is precisely the goal of this paper. Of course, these advantages come with a price: those minimal MUB sets are, in general, comprised of entangled projectors, which renders their experimental implementation more challenging than the product Pauli projectors.
II. MUTUALLY UNBIASED MEASUREMENTS FOR QUBITS

A compact way of labeling $n$-qubit states consists in using the finite field $\mathbb{F}_{2^n}$ (the reader interested in mathematical details is referred e.g., to the excellent monograph by Lidl and Niederreiter [17]). For our purposes, this can be considered as a linear space spanned by an abstract basis \( \{ \theta_1, \ldots, \theta_n \} \), so that given a field element \( v \) (henceforth, they will be denoted by Greek letters) the expansion \( v = \sum_{i=1}^{n} n_i \theta_i \) (with \( n_i \in \mathbb{Z}_2 \)) allows us the identification \( v \leftrightarrow (n_1, n_2, \ldots, n_n) \).

Moreover, the basis can be chosen to be orthonormal with respect to the trace operation (the self-dual basis); that is, \( \text{tr}(\theta \theta') = \delta_{jj'} \), where \( \text{tr}(\cdot) = v + v^2 + \ldots + v^{2^n-1} \) and maps $\mathbb{F}_{2^n}$ onto the base field $\mathbb{Z}_2$. In this way, to each qubit we associate a particular element of the self-dual basis: \( \alpha \equiv \theta_i \).

Let \( \{|\nu\rangle\} \) be an orthonormal basis in the Hilbert space of the system, which is isomorphic to $\mathbb{C}^{2^n}$. Operationally, the elements of the basis are labeled by powers of a primitive element. These vectors are eigenvectors of the operators $\mathcal{Z}_\alpha$ belonging to the generalized Pauli group $\{18\}$, whose basic generators are

\[
\mathcal{Z}_\alpha = \sum_v (-1)^{v \cdot \alpha} |v\rangle \langle v|, \quad \mathcal{X}_\beta = \sum_v |v + \beta\rangle \langle v|, \tag{2.1}
\]

with $\alpha, \beta \in \mathbb{F}_{2^n}$. Notice that in the self-dual basis these operators factorize as

\[
\mathcal{Z}_\alpha = \sigma_{z_1}^{a_1} \cdots \sigma_{z_n}^{a_n}, \quad \mathcal{X}_\beta = \sigma_{x_1}^{b_1} \cdots \sigma_{x_n}^{b_n}, \tag{2.2}
\]

where $a_i = \text{tr}(\alpha \theta_i)$ and $b_i = \text{tr}(\beta \theta_i)$ are the corresponding expansion coefficients for $\alpha$ and $\beta$ in that basis. The single-qubit Pauli operators $\sigma_z$ and $\sigma_x$ can be expressed in the standard basis of the two-dimensional Hilbert space $\mathbb{C}^2$ as

\[
\sigma_z = |1\rangle \langle 1| - |0\rangle \langle 0|, \quad \sigma_x = |0\rangle \langle 1| + |1\rangle \langle 0|. \tag{2.3}
\]

In addition, we have the commutation relation

\[
\mathcal{Z}_\alpha \mathcal{X}_\beta = (-1)^{\text{tr}(\alpha \beta)} \mathcal{X}_\beta \mathcal{Z}_\alpha. \tag{2.4}
\]

This is the discrete counterpart of the Heisenberg-Weyl group for continuous variables and the hallmark of noncommutativity. Moreover, $\mathcal{X}_\alpha$ and $\mathcal{Z}_\alpha$ are related through the finite Fourier transform [19]

\[
\mathcal{F} = \frac{1}{\sqrt{2^n}} \sum_{v,v'} (-1)^{v \cdot v'} |v\rangle \langle v'|, \tag{2.5}
\]

so that $\mathcal{F} \mathcal{Z}_\alpha \mathcal{F} = \mathcal{Z}_{\alpha^*}$.

We next recall [13] that the grid specifying the phase space for $n$ qubits can be appropriately labeled by the discrete points $(\alpha, \beta)$, which are precisely the indices of the operators $\mathcal{Z}_\alpha$ and $\mathcal{X}_\beta$: $\alpha$ is the “horizontal” axis and $\beta$ the “vertical” one. On this grid one can introduce a variety of geometrical structures with much the same properties as in the continuous case [21]: the simplest are the straight lines passing through the origin (also called rays). These rays have a quite remarkable property: the monomials $\{\mathcal{Z}_\alpha \mathcal{X}_\beta\}$ labeled by points of the same ray commute with each other, and thus have a common system of eigenvectors, which we shall label as $|v, \mu\rangle$. Without going into details, they can be constructed as

\[
|v, \mu\rangle = \mathcal{X}_v \mathcal{Z}_\mu |0\rangle \otimes \cdots \otimes |0\rangle, \tag{2.6}
\]

where $\mathcal{Z}_\mu$ is a finite rotation that changes the slope $\mu$ of the rays. Therefore, states with the same slope index $\mu$ span an orthogonal basis. The explicit construction of $\mathcal{Z}_\mu$ can be found in Ref. [21]. In this way, both the state index $v$ and the slope (i.e., basis) $\mu$ run over the $2^n$ elements of $\mathbb{F}_{2^n}$. Of course, there is an extra basis of “infinite” slope (corresponding to the “vertical” axis) that cannot be obtained through a rotation:

\[
|\bar{v}\rangle = \mathcal{X}_v |0\rangle \otimes \cdots \otimes |0\rangle. \tag{2.7}
\]

Let us enumerate for the time being these vectors by $|v, k\rangle$, where $k$ runs the $2^n$ values of $\mu$ in (2.6) plus the extra basis in (2.7). One can easily check that

\[
|\langle v, k|v', k'\rangle|^2 = \delta_{kk'} \delta_{vv'} + \frac{1}{2^n} (1 - \delta_{kk'}), \tag{2.8}
\]

so they constitute a set of MUBs. In other words, the complete set of $(2^n + 1)$ mutually unbiased projectors

\[
\mathcal{P}_{v,k} = |v, k\rangle \langle v, k|, \tag{2.9}
\]

defines a complete scheme, in the sense that the measured probabilities

\[
p_{v,k} = \text{Tr} \left( \rho \mathcal{P}_{v,k} \right), \quad \sum_v p_{v,k} = 1, \tag{2.10}
\]

determine completely the density matrix through

\[
\rho + \mathbb{I} = \sum_{k=1}^{2^n} \sum_v p_{v,k} \mathcal{P}_{v,k}. \tag{2.11}
\]

Here, we have used $\text{Tr}$ (with capital $T$) for the standard trace in Hilbert space. Note that the structure of this MUB set is preserved under any local unitary transformation, so any factorizable, complete basis can be chosen as a computational basis.

III. OPTIMAL MEASUREMENT SCHEME UNDER PERMUTATIONAL SYMMETRY

As heralded in the Introduction, we wish to design a minimal MUB tomographical scheme for systems that remain invariant under all possible interchanges of its different particles. This invariance could be simply stated as

\[
\Pi_{pq} \rho \Pi_{pq} = \rho, \tag{3.1}
\]

where $p \neq q$, $p, q \in \{1, \ldots, n\}$. The elements $\Pi_{pq}$ of the permutation group are known as swap operators, as they exchange the states of the $p$-th and the $q$-th qubits; i.e.,

\[
\Pi_{pq} |a_1, a_2, \ldots, a_n\rangle = |a_1, a_2, \ldots, a_n\rangle, \tag{3.2}
\]
It has recently been shown \[22\] that any permutationally invariant \(n\)-qubit state defined via Eq. (3.1) can be written as
\[
\rho_{pq} = \frac{n}{2} \sum_{j=J_{\text{min}}}^{J_{\text{max}}} p_j \rho_j \otimes R_j. \tag{3.3}
\]
The summation runs over different total spin numbers starting from \(J_{\text{min}} \in \{0,1/2\}\), depending on whether \(n\) is even or odd. \(\rho_j\) is the density matrix of a \(j\)-spin state and \(p_j\) are the associated probabilities. In addition, \(R_j = \mathbb{I}/\text{dim} \mathcal{H}_j\) and the factor \(\text{dim} \mathcal{H}_j\) comes from the degeneracy of the subspaces appearing in the decomposition of the total Hilbert space \(\mathcal{H} = \mathbb{C}^2\) in the form
\[
\mathcal{H}_j = \frac{n}{2} \bigoplus_{j=J_{\text{min}}}^{J_{\text{max}}} \mathcal{H}_j \otimes \mathcal{H}_j. \tag{3.4}
\]
Here, \(\mathcal{H}_j\) is a spin Hilbert space of dimension \(\text{dim} \mathcal{H}_j = 2j + 1\), while \(\mathcal{H}_j\) are referred as multiplicative spaces that account for the different possibilities to obtain a spin \(j\). One can show that \[23\]
\[
\text{dim} \mathcal{H}_j = \left(\frac{n}{2} - j\right) - \left(\frac{n}{2} - j - 1\right). \tag{3.5}
\]
This means that a permutationally invariant density operator only contains nontrivial parts of the spin Hilbert spaces, and there are no coherences between different spin states. In other words, any of these states can be parsed into a block-diagonal form that has been exploited in several contexts \[24\].

The crucial observation for what follows is that, at the level of field elements, the action of the permutation operator \(\Pi_{pq}\) on a state is tantamount of
\[
\mu \mapsto \mu + \varepsilon \text{tr}(\mu \varepsilon), \tag{3.6}
\]
where \(\varepsilon = \theta_p + \theta_q\), with \(\theta_p\ (\theta_q)\) being the \(p\)-th \((q\)-th\) element of the self-dual basis. Since the field element addition is commutative, the operator is symmetric in \(p\) and \(q\), as it should. In algebraic terms, we have then
\[
\Pi_{pq} = \sum_\kappa |\kappa + \varepsilon \text{tr}(\varepsilon \kappa)\rangle \langle \kappa |. \tag{3.7}
\]
Using this field representation, we can check that the mutually unbiased projectors \[2.9\] transform as
\[
\Pi_{pq} \mathcal{P}_{\nu,\mu} \Pi_{pq} = \mathcal{P}_{\nu+\varepsilon \text{tr}(\nu \varepsilon),\mu+\varepsilon \text{tr}(\mu \varepsilon)}; \tag{3.8}
\]
whence the density matrix in the tomographic representation \[2.11\] is transformed as
\[
\Pi_{pq} (\rho + \mathbb{I}) \Pi_{pq} = \sum_{\mu,\nu} p_{\nu,\mu} \mathcal{P}_{\nu+\varepsilon \text{tr}(\nu \varepsilon),\mu+\varepsilon \text{tr}(\mu \varepsilon)} + \sum_{\nu} \hat{\rho}_{\nu} \mathcal{P}_{\nu+\varepsilon \text{tr}(\nu \varepsilon)}. \tag{3.9}
\]
The last term is just the contribution from the basis \[2.7\], which we split for notational simplicity.

If we perform the change of variables
\[
\nu' = \nu + \varepsilon \text{tr}(\nu \varepsilon), \quad \mu' = \mu + \varepsilon \text{tr}(\mu \varepsilon), \tag{3.10}
\]
we can recast \[3.9\] in the form
\[
\Pi_{pq} (\rho + \mathbb{I}) \Pi_{pq} = \sum_{\nu',\mu'} p_{\nu',\mu'} \mathcal{P}_{\nu',\mu'} + \sum_{\nu} \hat{\rho}_{\nu} \mathcal{P}_{\nu}. \tag{3.11}
\]
Consequently, the invariance condition \[3.1\] leads to the following restrictions on the measured probabilities:
\[
p_{\nu+\varepsilon \text{tr}(\nu \varepsilon),\mu+\varepsilon \text{tr}(\mu \varepsilon)} = p_{\nu,\mu}, \quad \forall \varepsilon. \tag{3.12}
\]
Obviously, the probabilities \(p_{\nu,\mu}\) should be also invariant under all consecutive index permutations.

### IV. PHYSICAL DISCUSSION

The above basic expression can be given a transparent physical meaning. Indeed, let us expand \(\nu\) and \(\mu\) in the self-dual basis
\[
\mu = \sum_{i=0}^{n} m_i \theta_i, \quad \nu = \sum_{i=0}^{n} n_i \theta_i, \tag{4.1}
\]
with \(m_i, n_i \in \mathbb{Z}_2\) and analogous expansions for the transformed indices in \[3.10\]. For a given \(\varepsilon = \theta_p + \theta_q\), one can check that \(m'_i = m_i\), except for \(m'_q = m_q\) and \(m'_p = m_p\), and similarly for \(n'_i\). That is, a change of the index \(\nu\) in the states of a basis simply results in a reshuffling of its states. Therefore, such transformations do not give any new tomographic projectors for a permutationally invariant state.

The transformation of \(\mu\) implies that measurements by MUBs corresponding to \(\mu\) indices with the same number of non-zero components in the self-dual basis [the length of the word \(|\mu|\) corresponding to the binary string \((a_0, a_1, \ldots, a_n)\)] give the same information. In short, the projectors labeled by, e.g., \(\mu = (1, 1, 0, \ldots, 0)\) and \(\mu' = (0, \ldots, 0, 1, 1)\) are equivalent. The computational basis, associated with \(\mu = 0\), automatically satisfies \[3.12\] for all \(\nu\). Similarly, the X basis also satisfies \[3.12\] because it has no \(\mu\) dependence \([\Pi_{pq}, \mathcal{F}] = 0\). Therefore, these two bases remain invariant under any qubit permutation. This allows us to count the total number of measurements needed for a complete reconstruction of the density matrix, which is just \(n + 2\). This result could be expected, for the Hilbert space dimension of the permutation invariant system is \(n + 1\).

Since the permutation group acts simultaneously on both indices \(\mu\) and \(\nu\), there are different orbits of equivalent probabilities that are defined not only by the length \(|\mu|\) but also by the mutual symmetry properties of the indices representing the number of the basis and the element in each basis. In particular, for \(\mu \neq 0\) each orbit representative is labeled by three lengths \(m = |\mu|, l = |\nu|, s = |\mu + \nu|\), i.e. \(p_{\nu,\mu} = p_{(m,l,s)}\). For the computational and the Fourier bases the orbits are characterized only by \(|\nu|\); for instance, \(p_{\nu,0} = p_{0|(l)}\). Accordingly,
in each basis not all the probabilities should be measured, but only those that belong to different orbits, which leads to a reduction of the experimental errors. Since for given \(m\) and \(l\), \(s\) runs from \([m-\lfloor m \rfloor] \) to \([m+\lfloor m \rfloor, 2n-m-\lfloor m \rfloor] \) in steps of two, the number of orbits turns out to be \(1+n(n^2+6n+17)/6\). Bearing in mind the normalization condition (2.10), we find that there are \(n(n^2+6n+17)/6\) independent probabilities \(p(m,l,s)\), which completely define the density matrices appearing in the decomposition (3.3). Projectors corresponding to the same probabilities are given by the condition (3.12).

The final reconstruction takes the form

\[
\rho + \mathbb{1} = \sum_{m=0}^{n} \sum_{l=1}^{\text{max}(m+n, l)} \sum_{s=m-l}^{\text{min}(m+l, 2n-m-l, n)} p(m,l,s) \sum_{\mu} \mathcal{P}_{\mu}(l,s) \mu(m,s) + \sum_{l=1}^{n} p_0(l) \sum_{\mu} \mathcal{P}_{\mu}(l,0) + \sum_{\mu} \sum_{\nu} p_{0,\nu} \mathcal{P}_{0,\mu}
\]

where the sum on \(\mu\) and \(\nu\) run over all the field elements such that \(|\mu(m,s)| = m, l = |\nu(l,s)|, s = |\mu + \nu|, \) and \(p_{0,\mu} = 1 - \sum_{\nu} p_{\nu,\mu}, \) and \(p_0 = 1 - \sum_{\nu} \tilde{p}_\nu.\)

For instance, for two qubits, the field \(\mathbb{F}_{2^2}\) has the primitive element defined by the irreducible polynomial \(\theta^2 + \theta + 1 = 0\). Therefore, \(\theta_1 = \theta\) and \(\theta_2 = \theta^2\), so that \(\theta^3 = \theta\theta^2 = \theta(1 + \theta) = \theta_1 + \theta_2.\) In this case, only measurements in the bases with \(\mu = \theta_1\) (or \(\mu = \theta_2\)) and \(\mu = \theta_1 + \theta_2\) (apart from measurements in the computational and \(X\) bases) are required. The 9 independent measured probabilities \(p_{\theta_1,0}, p_{\theta_1,\theta_2}, \) and \(p_{\theta_1,\theta_1}\) from the \(Z\) basis, \(p_{\theta_2,0}, p_{\theta_2,\theta_1}, \) and \(p_{\theta_2,\theta_2}\) from basis 1, \(p_{\theta_1,\theta_1}\) and \(p_{\theta_1,\theta_2}\) from basis 2, and \(p_{\theta_1,\theta_2}\) from the \(X\) basis are representatives of the equivalent probabilities orbits. This selection gives an explicit reconstruction form that reads as

\[
\rho + \mathbb{1} = p_{\theta_1,0} \left( \mathcal{P}_{\theta_1,0} + \mathcal{P}_{\theta_2,0} \right) + p_{\theta_1,\theta_2} \left( \mathcal{P}_{\theta_1,\theta_2} + \mathcal{P}_{\theta_2,\theta_1} \right) + p_{\theta_1,\theta_1} \left( \mathcal{P}_{\theta_1,\theta_1} + \mathcal{P}_{\theta_1,\theta_2} \right) + p_{\theta_1,\theta_2} \left( \mathcal{P}_{\theta_1,\theta_1} + \mathcal{P}_{\theta_1,\theta_2} \right)
\]

while the one corresponding to (2.7) turns out to be

\[
\{|\tilde{\nu}\rangle\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}. \tag{4.6}
\]

Of these five MUBs, only e.g., \(\{\tilde{\nu}\}\), \(\{X_\theta|\theta_1\}\), \(\{X_\theta|\theta_1 + \theta_2\}\), and the \(\{|\tilde{\nu}\rangle\}\) are needed to tomographically reconstruct a permutationally invariant two-qubit state. If we permute the second and the third qubit (and the state to be tomographed would not change due to such permutation) it is readily seen that the permuted basis \(\{X_\theta|\theta_2\}\) becomes the non-permuted basis \(\{X_\theta|\theta_1\}\) (but with the middle two vectors interchanged). Hence, the two bases extract identical information from the state, and hence one of them can be disregarded. They both have one nonzero component in the self dual basis

| TABLE I. Allowed values of \(m\), \(l\) and \(s\) for the 24 independent orbits in the three-qubit case. The tilde indicates that the corresponding probabilities are measured in the \(X\) basis. The last row (denoted \#) gives the number of (equivalent) probabilities in each orbit. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \(m\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) |
| \(l\) | \(0\) | \(1\) | \(2\) | \(3\) | \(0\) | \(1\) | \(2\) | \(3\) | \(0\) | \(1\) | \(2\) | \(3\) |
| \(s\) | \(0\) | \(1\) | \(2\) | \(3\) | \(0\) | \(1\) | \(2\) | \(3\) | \(0\) | \(1\) | \(2\) | \(3\) |
| \# | \(1\) | \(3\) | \(1\) | \(3\) | \(1\) | \(3\) | \(1\) | \(3\) | \(1\) | \(3\) | \(1\) | \(3\) |
and are therefore directly related by a permutation as shown by eq. (3.12).

Before we conclude, let us briefly address the case of three qubits. In Table II we give the values of \( l, m, s \) for the 24 independent orbits (all in all, we get 72 probabilities). Taking into account that 5 probabilities (each one defining an orbit) can be determined from the normalization condition (2.10) [for example, we can fix \( p(0,0,0) \), \( p(0,0,0) \), \( p(1,0,1) \), \( p(2,0,2) \), and \( p(3,0,3) \)], we arrive at 19 orbits that determine any symmetric density matrix.

V. CONCLUSIONS

We have developed a method to generate a minimal set of MUBs needed to tomographically reconstruct a state consisting of \( n \) qubits, where the state is invariant under the permutation of the qubits. Such a state spans an \( n+1 \) dimensional Hilbert space. Consequently the smallest set of bases one can hope to use is \( n+2 \), and indeed our method provides a minimal set.

MUBs are not strictly necessary to reconstruct such a state, but they have the advantage of capturing maximally different aspects of the state. Moreover, as the bases constitute complete sets of states in the \( 2^n \)-dimensional space of \( n \) qubits, they can in principle be implemented as von Neumann measurements and not as individual projectors or positive operator valued measures. The price is that the MUB projectors are for the most part highly entangled, so their experimental implementation can be difficult.

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