STABLE RIGGED CONFIGURATIONS FOR QUANTUM AFFINE ALGEBRAS OF NONEXCEPTIONAL TYPES

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Abstract. For an affine algebra of nonexceptional type in the large rank we show the fermionic formula depends only on the attachment of the node 0 of the Dynkin diagram to the rest, and the fermionic formula of not type A can be expressed as a sum of that of type A with Littlewood-Richardson coefficients. Combining this result with [13] and [19] we settle the $X = M$ conjecture under the large rank hypothesis.

1. Introduction

Let $\mathfrak{g}$ be an affine Lie algebra and $I$ the index set of its Dynkin nodes. Let $\hat{\mathfrak{g}}$ be the classical subalgebra of $\mathfrak{g}$, namely, the finite-dimensional simple Lie algebra whose Dynkin nodes are given by $I_0 := I \setminus \{0\}$ where the node 0 is taken as in [10]. Let $U'_q(\mathfrak{g})$ be the quantized enveloping algebra associated to $\mathfrak{g}$ without the degree operator. Among finite-dimensional $U'_q(\mathfrak{g})$-modules there is a distinguished family called Kirillov-Reshetikhin (KR) modules, which have nice properties such as $T(Q, Y)$-systems, fermionic character formulas, and so on. See for instance [1, 9, 14, 20] and references therein. In [7, 6], assuming the existence of crystal basis $B_{r,s}(r \in I_0, s \in \mathbb{Z}_{>0})$ of a KR module we defined the one-dimensional (1-d) sum

$$X_{\lambda,B}(q) = \sum_{b \in B} q^{D(b)}$$

where the sum is over $I_0$-highest weight vectors in $B = B^{r_1,s_1} \otimes \cdots \otimes B^{r_m,s_m}$ with weight $\lambda$ and $D$ is a certain $\mathbb{Z}$-valued function on $B$ called the energy function (see e.g. (3.9) of [6]), and conjectured that $X$ has an explicit expression $M$ (see (2.12)) called the fermionic formula ($X = M$ conjecture). This conjecture is settled in full generality if $\mathfrak{g} = A_{n+1}^{(1)}$ [13], when $r_j = 1$ for all $j$ if $\mathfrak{g}$ is of nonexceptional affine types [20], and when $s_j = 1$ for all $j$ if $\mathfrak{g} = D_{n+1}^{(1)}$ [24]. It should also be noted that recently the existence of KR crystals for nonexceptional affine types were settled [21, 22] and their combinatorial structures were clarified [2].

Another interesting equality related to $X$ is the $X = K$ conjecture by Shimozono and Zabrocki [28, 27] that originated from the study of certain $q$-deformed operators on the ring of symmetric functions. Suppose $\mathfrak{g}$ is of nonexceptional type. If the rank of $\mathfrak{g}$ is sufficiently large, $X$ does not depend on $\mathfrak{g}$ itself, but only on the attachment of the affine Dynkin node 0 to the rest of the Dynkin diagram. See Table 2. Let $X^\diamondsuit_{\lambda,B}(q) (\diamondsuit = \emptyset, \emptyset, \bullet, \bullet, \bullet)$ denote the 1-d sum for $\mathfrak{g}$ of kind $\diamondsuit$. Then the $X = K$ conjecture, which has been settled in [27, 18, 19], states that if $\diamondsuit \neq \emptyset$, the following

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equality holds.

\[
(1.1) \quad X_{\lambda,B}^\diamond(q) = q^{-|\lambda|} \sum_{\mu \in \mathcal{P}_N^\diamond |\lambda|, \eta \in \mathcal{P}_N} c_{\mu \lambda}^{\eta} X_{\eta,B}^\diamond(q^{-1})
\]

Here \(|B| = \sum_{i=1}^{m} r_i s_i\), \(\mathcal{P}_N^\diamond\) is the set of partitions of \(N\) whose diagrams can be tiled from \(\diamond\), and \(c_{\mu \lambda}^{\eta}\) is the Littlewood-Richardson coefficient. Note also that \(X_{\lambda,B}^\diamond(q)\) in (1.1) is related to our \(X_{\lambda,B}^\diamond(q)\) by \(X_{\lambda,B}^\diamond(q) = X_{\lambda,B}^\diamond(q^{-1})\). Let us sketch out the proof in [19]. Since the 1-d sum depends only on \(\diamond\), for each \(\diamond\) we choose \(g = g^\diamond\) such that \(i \mapsto n - i\) for \(i \in I\) induces a Dynkin diagram automorphism. Namely, we choose \(g^\diamond = D_{n+1}(2), C_n(1), D_n(1)\) for \(\diamond = \emptyset\). Then on each KR crystal \(B^{r,s}\) for \(g^\diamond\) one can show there exists an automorphism satisfying \(\sigma \circ \hat{e}_i = \hat{e}_{n-i} \circ \sigma\) for any \(i \in I\), where \(\hat{e}_i\) is the Kashiwara operator. This automorphism \(\sigma\) can be extended to the tensor product \(B\) of KR crystals. Let \(\text{High}\) be the map sending an element \(b\) of \(B\) to the \(I_0\)-highest weight vector of the \(I_0\)-highest component containing \(b\).

Under the assumption that the rank is large, the composition \(\text{High} \circ \sigma\) has the following properties.

(i) The image of an \(I_0\)-highest weight vector can be regarded as an element of type \(A_n^{(1)}\).

(ii) For an \(I_0\)-highest weight vector \(b\) for type \(A_n^{(1)}\) of weight \(\eta\), the number of \(I_0\)-highest weight vectors of weight \(\lambda\) in the inverse image of \(b\) is given by

\[
\sum_{\mu \in \mathcal{P}_N |\lambda|} c_{\mu \lambda}^{\eta}.
\]

(iii) For an \(I_0\)-highest weight vector \(b\) for \(g^\diamond\), we have \(D(b) = D(\sigma(b)) = \frac{|B| - |\lambda|}{|\lambda|}\).

(1.1) is a direct consequence of these properties.

If we believe the \(X = M\) conjecture, we have the right to expect exactly the same relation in the \(M\) side under the same assumption of the rank. This is what we wish to clarify in this paper. Namely, if \(g\) is one of nonexceptional affine type and the rank is sufficiently large, we show the fermionic formula depends only on the symbol \(\diamond\), denoted by \(M^\diamond(\lambda, L; q)\), and if \(\diamond \neq \emptyset\) we have (Theorem 4.4)

\[
(1.2) \quad M^\diamond(\lambda, L; q) = q^{-|\lambda|} \sum_{\mu \in \mathcal{P}_N |\lambda|, \eta \in \mathcal{P}_L} c_{\mu \lambda}^{\eta} M^\diamond(\eta, L; q^{\frac{|\lambda|}{|\lambda|}}).
\]

Here \(L = (L_{i}^{(a)})_{a \in I_0, i \in \mathbb{Z}_{\geq 0}}\) is a datum such that \(L_{i}^{(a)}\) counts the number of \(B^{a,i}\) in \(B\) and \(|L| = \sum_{a \in I_0, i \in \mathbb{Z}_{\geq 0}} a L_{i}^{(a)}\).

The proof of (1.2) proceeds as follows. We first rewrite the fermionic formula as

\[
M^\diamond(\lambda, L; q) = \sum_{(\nu, J^*) \in \text{RC}^\diamond(\lambda, L)} q^{C(\nu, J^*)},
\]

by introducing the notion of stable rigged configuration. We then construct for \(\diamond \neq \emptyset\) a bijection

\[
\Psi : \text{RC}^\diamond(\lambda, L) \longrightarrow \bigcup_{\mu \in \mathcal{P}_N |\lambda|, \eta \in \mathcal{P}_L} \text{RC}^\diamond(\eta, L) \times LR_{\lambda \mu}^\diamond,
\]

where \(LR_{\lambda \mu}^\diamond\) is the set of Littlewood-Richardson skew tableaux of shape \(\eta/\lambda\) and weight \(\mu\). Roughly speaking, the bijection \(\Psi\) proceeds as follows. When the rank is sufficiently large, there exists \(k\) such that the \(a\)-th configuration \(\nu^{(a)}\) is the same for
\[a = k, k + 1, \ldots\] As opposed to the KKR algorithm that removes a box from \( \nu^{(a)}\) starting from \(a = 1\), we perform a similar algorithm starting from the largest \(a\). If we continue this procedure until all boxes are removed from \( \nu^{(a)}\) for sufficiently large \(a\), we can regard this as a rigged configuration of type \(A\). Reflecting this sequence of procedures we can also define a recording tableau, that is shown to be a Littlewood-Richardson skew tableau. This map can be reversed at each step, and therefore defines a bijection.

Finally we show
\[c(\nu^*, J^*) = c(\nu'^*, J'^*) - \frac{|L| - |\lambda|}{|\diamond|}\]
where \((\nu'^*, J'^*)\) is the first component of the image of \((\nu^*, J^*)\) by \(\Psi\). We note that two equalities \([1.1]\) and \([1.2]\) together with the result of \([13]\) implies
\[X_{\lambda, B}(q) = M_{\lambda, L; q}\]
for \(\diamond \neq \emptyset\) and therefore settle the \(X = M\) when \(g\) is of nonexceptional type and the rank is sufficiently large.

Let us summarize the combinatorial bijections that are relevant to our paper as the following schematic diagram:

\[
\begin{array}{c}
\{ \text{type } g \text{ path}\} \\
\downarrow (\text{b}) \\
\{ \text{type } g \text{ RC}\}
\end{array}
\quad
\begin{array}{c}
\{ \text{type } A_n^{(1)} \text{ path}\} \times \text{LR} \\
\downarrow (\text{a}) \\
\{ \text{type } A_n^{(1)} \text{ RC}\} \times \text{LR}
\end{array}
\quad
\begin{array}{c}
\downarrow \Psi \\
\end{array}
\]

Here “path” stands for the highest weight elements of \(\bigotimes_i B_{i,s_i}^r\) and “RC” stands for the rigged configurations. Our bijection \(\Psi\), that exists when the rank is large, corresponds to the bottom edge. Bijection (a), which we call type \(A_n^{(1)}\) RC-bijection, is established in full generality in the papers \([11, 12, 13]\). Algorithms for bijection (b) are known explicitly in the following cases:

- \((B^{1.1})^\otimes \) type paths for all nonexceptional algebras \(g\) \([23]\),
- \(\bigotimes B_{i,1}^r\) type paths for \(g = D_n^{(1)}\) \([25]\),
- \(\bigotimes B_{1,s_i}^1\) type paths for all nonexceptional algebras \(g\) \([26]\).

For the cases that the bijection (b) is established, our bijection \(\Psi\) thus gives the combinatorial bijection between the set of type \(g\) paths and the product set of the type \(A_n^{(1)}\) paths and the Littlewood-Richardson skew tableaux. We refer to \([27]\) for related combinatorial problems.

We expect that the bijection (b) exists in full generality even without the large rank hypothesis. It will give a combinatorial proof of the \(X = M\) conjecture. Furthermore, it also gives an essential tool for the study of a tropical integrable system known as the box-ball system (see e.g., \([4, 5, 8]\)) which is a soliton system defined on the paths and is supposed to give a physical background for the \(X = M\) identities. More precisely, the rigged configurations are identified with the complete set of the action and angle variables for the type \(A_n^{(1)}\) box-ball system \([15]\) (see \([17]\) for a generalization to type \(D_n^{(1)}\)). It is also interesting to note that by introducing a tropical analogue of the tau functions in terms of the charge \(c(\nu^*, J^*)\), the initial
value problem for the type $A_n^{(1)}$ box-ball systems is solved in \[10\]. Therefore the construction of the bijection (b) in full generality will be a very important future problem.

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## 2. Stable rigged configurations

### 2.1. Affine algebras

We recall necessary notations for affine Kac-Moody algebras. We adopt the notation of [6]. Let $\mathfrak{g}$ be a Kac-Moody Lie algebra of nonexceptional affine type $X_N^{(r)}$, that is, one of the types $A_n^{(1)}(n \geq 1)$, $B_n^{(1)}(n \geq 3)$, $C_n^{(1)}(n \geq 2)$, $D_n^{(1)}(n \geq 4)$, $A_{2n}^{(2)}(n \geq 1)$, $A_{2n-1}^{(2)}(n \geq 2)$, $D_{n+1}^{(2)}(n \geq 2)$. The nodes of the Dynkin diagram of $\mathfrak{g}$ are labeled by the set $I = \{0, 1, 2, \ldots, n\}$. See Table 2.1 of [6]. Let $\alpha_i, h_i, \Lambda_i (i \in I)$ be the simple roots, simple coroots, and fundamental weights of $\mathfrak{g}$. Let $\delta$ and $c$ denote the generator of imaginary roots and the canonical central element, respectively. Recall that $\delta = \sum_{i \in I} a_i \alpha_i$ and $c = \sum_{i \in I} a_i^\vee h_i$, where the Kac labels $a_i$ are the unique set of relatively prime positive integers giving the linear dependency of the columns of the Cartan matrix $A$, that is, $A^t(a_0, \ldots, a_n) = 0$. Explicitly,

$$
\delta = \begin{cases} 
\alpha_0 + \cdots + \alpha_n & \text{if } \mathfrak{g} = A_n^{(1)}, \\
\alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n & \text{if } \mathfrak{g} = B_n^{(1)}, \\
\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n & \text{if } \mathfrak{g} = C_n^{(1)}, \\
\alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n & \text{if } \mathfrak{g} = D_n^{(1)}, \\
2\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n & \text{if } \mathfrak{g} = A_{2n}^{(2)}, \\
\alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n & \text{if } \mathfrak{g} = A_{2n-1}^{(2)}, \\
\alpha_0 + \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n & \text{if } \mathfrak{g} = D_{n+1}^{(2)}. 
\end{cases}
$$

The dual Kac label $a_i^\vee$ is the label $a_i$ for the affine Dynkin diagram obtained by reversing the arrows of the Dynkin diagram of $\mathfrak{g}$. Note that $a_0^\vee = 1$.

Let $(\cdot | \cdot)$ be the normalized invariant form on the weight lattice $P$ [10]. It satisfies

$$
(\alpha_i | \alpha_j) = \frac{a_j^\vee}{a_i^\vee} A_{ij}
$$

for $i, j \in I$. In particular

$$
(\alpha_i | \alpha_i) = 2r
$$

if $\alpha_i$ is a long root. For $i \in I$ let

$$
t_i = \max\left(\frac{a_i}{a_i^\vee}, a_0^\vee\right), \quad t_i^\vee = \max\left(\frac{a_i^\vee}{a_i}, a_0\right).
$$
We shall only use $t_i^\vee$ and $t_i$ for $i \in I_0$, where we have set $I_0 = I \setminus \{0\}$. For $a \in I_0$ we have

$$t_a^\vee = 1 \text{ if } r = 1, \quad t_a = 1 \text{ if } r = 2.$$

We consider two finite-dimensional subalgebras of $\mathfrak{g}$: $\overset{\circ}{\mathfrak{g}}$, whose Dynkin diagram is obtained from that of $\mathfrak{g}$ by removing the 0 vertex, and $\mathfrak{g}_{\sigma}$ the subalgebra of $X_N$ fixed by the automorphism $\sigma$ given in [10, Section 8.3]. Let $\overset{\circ}{\mathfrak{g}}$ (resp. $\mathfrak{g}_{\sigma}$)

| $\mathfrak{g}$ | $X^{(1)}_N$ | $A^{(2)}_{2n}$ | $A^{(2)}_{2n-1}$ | $D^{(2)}_{n+1}$ |
|---------------|-------------|----------------|----------------|----------------|
| $\overset{\circ}{\mathfrak{g}}$ | $X_N$ | $C_n$ | $C_n$ | $B_n$ |
| $\mathfrak{g}_{\sigma}$ | $X_N$ | $B_n$ | $C_n$ | $B_n$ |

have weight lattice $\overset{\circ}{P}$ (resp. $\overset{\circ}{P}$), with simple roots and fundamental weights $\alpha_a, \Lambda a$ (resp. $\overset{\circ}{\alpha}_a, \overset{\circ}{\Lambda}_a$) for $a \in I_0$. Note that $\overset{\circ}{\mathfrak{g}} = \overset{\circ}{\mathfrak{g}}_{\sigma}$ for $\mathfrak{g} \neq A^{(2)}_{2n}$. For $\mathfrak{g} = A^{(2)}_{2n}$, $\overset{\circ}{\mathfrak{g}} = C_n$ and $\mathfrak{g}_{\sigma} = B_n$. $\overset{\circ}{P}$ is endowed with the bilinear form $(\cdot | \cdot)'$, normalized by

$$(\overset{\circ}{\alpha}_a | \overset{\circ}{\alpha}_a)' = 2r \quad \text{if } \overset{\circ}{\alpha}_a \text{ is a long root of } \mathfrak{g}_{\sigma}.$$

For $A^{(2)}_2$ the unique simple root $\overset{\circ}{\alpha}_1$ of $\mathfrak{g}_{\sigma} = B_1$ is considered to be short. Note that $\alpha_a, \Lambda a$ and $(\cdot | \cdot)$ may be identified with $\overset{\circ}{\alpha}_a, \overset{\circ}{\Lambda}_a$ and $(\cdot | \cdot)'$ if $\mathfrak{g} \neq A^{(2)}_{2n}$.

Define the $\mathbb{Z}$-linear map $\iota: \overset{\circ}{P} \to \overset{\circ}{P}$ by

$$(\iota(\alpha_a) = \epsilon_a \overset{\circ}{\Lambda}_a \quad \text{for } a \in I_0,$$

where $\epsilon_a$ is defined by

$$\epsilon_a = \begin{cases} 2 & \text{if } \mathfrak{g} = A^{(2)}_{2n} \text{ and } a = n \\ 1 & \text{otherwise.} \end{cases}$$

In particular $\iota(\alpha_a) = \epsilon_a \overset{\circ}{\alpha}_a$ for $a \in I_0$. If $\mathfrak{g} = A^{(2)}_{2n}$, we have $(\alpha_b | \alpha_b) = 4, (\overset{\circ}{\alpha}_b | \overset{\circ}{\alpha}_b)' = 2$ if $b = n$ and 2, 4 otherwise. In the rest of the paper we shall write $(\cdot | \cdot)$ in place of $(\cdot | \cdot)'$.

We now associate a kind $\diamondsuit$ to each nonexceptional affine algebra $\mathfrak{g}$ as follows. The kind depends precisely on the attachment of the affine Dynkin node 0 to the rest of the Dynkin diagram. We use the labeling of the Dynkin nodes in [10]. It is also related to the $I_0$-decomposition of the Kirillov-Reshetikhin crystal $B^{r,s}$ for $n$ large with respect to $r$ (see [19]).
2.2. Classical weights, Littlewood-Richardson skew tableaux. For classical simple Lie algebras $\mathfrak{g} = A_n, B_n, C_n, D_n$ we often identify the dominant integral weight without spin with a Young diagram. Namely, for $\lambda = \sum_{j=1}^{n'} m_j \Lambda_j$ ($n' = n$ for $A_n$, $n' = n - 1$ for $B_n, C_n$, $n' = n - 2$ for $D_n, m_j \in \mathbb{Z}_{\geq 0}$, $\Lambda_j$ is a fundamental weight of $\mathfrak{g}$), we associate the Young diagram such that there are exactly $m_j$ columns of height $j$. We utilize this identification throughout this paper. As usual, for a Young diagram (or partition) $\lambda$, $\ell(\lambda)$ denotes its depth and $|\lambda|$ the number of boxes.

Next we explain the Littlewood-Richardson skew tableau. A skew tableau $T$ is called a Littlewood-Richardson skew tableau, LR tableau for short, if it is semi-standard and its reverse row word is Yamanouchi, i.e., when the word $w = x_1 \cdots x_l$ is read from the first to any letter, the sequence $x_1 \cdots x_k$ contains at least as many $1$’s as it does $2$’s, at least as many $2$’s as $3$’s, and so on for all positive integers. A skew tableau is said to have weight $\mu = (\mu_1, \ldots, \mu_l)$ if it contains $\mu_1$ $1$’s, $\mu_2$ $2$’s, and so on up to $\mu_l$’s.

**Example 2.1.** The following skew tableau is an example of LR tableau of shape $(431^2)/ (2^21)$ and of weight $(32^22)$.

|   | 1 | 1 |   |
|---|---|---|---|
| 2 |   |   |   |
| 1 | 3 |   |   |
| 2 | 2 | 4 |   |
| 3 |   |   | 4 |

Its reverse row word is $1123142234$, which satisfies the Yamanouchi condition.

It is well known (see e.g. [4]) that the number of LR tableaux of shape $\eta/\lambda$ and of weight $\mu$ is equal to the Littlewood-Richardson coefficient $c^{\eta}_{\lambda \mu}$ that counts the multiplicity of $V_\eta$ in $V_\lambda \otimes V_\mu$, where $V_\lambda$ is the irreducible $\mathfrak{gl}_n$-module of highest weight $\lambda$.

2.3. Rigged configurations. We recall the fermionic formula given in [7, 6]. Let $P^+$ be the set of dominant integral weights of $\mathfrak{g}$. Fix $\lambda \in P^+$ and a matrix $L = (L_i^{(a)})_{a \in I_0, i \in \mathbb{Z}_{>0}}$ of nonnegative integers with finitely many positive ones. Let $\nu = (m_i^{(a)})_{a \in I_0, i \in \mathbb{Z}_{>0}}$ be another matrix of nonnegative integers. Say that $\nu$ is a $\lambda$-configuration if

$$\sum_{a \in I_0, i \in \mathbb{Z}_{>0}} i m_i^{(a)} \bar{\alpha}_a = \ell \left( \sum_{a \in I_0, i \in \mathbb{Z}_{>0}} i L_i^{(a)} \bar{\Lambda}_a - \lambda \right).$$

This is equivalent to assuming

$$\sum_{i \in \mathbb{Z}_{>0}} i m_i^{(a)} = \frac{2}{(\bar{\alpha}_a | \bar{\alpha}_a)} \sum_{b \in I_0} \epsilon_b \left( \sum_{i \in \mathbb{Z}_{>0}} i L_i^{(b)} - \langle \lambda, h_b \rangle \right) (\bar{\Lambda}_a | \bar{\Lambda}_b)$$

for $a \in I_0$, where $h_b$ is a simple coroot of $\mathfrak{g}$. Say that a configuration $\nu$ is $L$-admissible if

$$p_i^{(a)} \geq 0 \quad \text{for all } a \in I_0 \text{ and } i \in \mathbb{Z}_{>0},$$
\[ p_i^{(a)} = \sum_{k \in \mathbb{Z}_{>0}} \left( I_k^{(a)} \min(i, k) - \frac{1}{t_a^2} \sum_{b \in I_0} (\tilde{\alpha}_a | \tilde{\alpha}_b) \min(t_b i, t_a k) m_k^{(b)} \right). \]

Write \( C(\lambda, \mathbf{L}) \) for the set of \( \mathbf{L} \)-admissible \( \lambda \)-configurations. Define
\[
(2.11) \quad c(\nu) = \frac{1}{2} \left( \sum_{a,b \in I_0} \sum_{j, k \in \mathbb{Z}_{>0}} (\tilde{\alpha}_a | \tilde{\alpha}_b) \min(t_b j, t_a k) m_j^{(a)} m_k^{(b)} \right.
- \sum_{a \in I_0} t_a' \sum_{j, k \in \mathbb{Z}_{>0}} \min(j, k) L_j^{(a)} m_k^{(a)}.
\]

The fermionic formula is defined by
\[
(2.12) \quad M(\lambda, \mathbf{L}; q) = \sum_{\nu \in \mathbb{C}(\lambda, \mathbf{L})} q^{c(\nu)} \prod_{a \in I_0} \prod_{i \in \mathbb{Z}_{>0}} \left[ p_i^{(a)} + m_i^{(a)} \right] q^{t_a}. 
\]

The fermionic formula \( M(\lambda, \mathbf{L}) \) can be interpreted using combinatorial objects called rigged configurations. For \( a \in I_0 \), define
\[
(2.13) \quad v_a = \begin{cases} 
2 & \text{if } a = n \text{ and } \mathbf{g} = C_n^{(1)} \\
\frac{1}{2} & \text{if } a = n \text{ and } \mathbf{g} = B_n^{(1)} \\
1 & \text{otherwise}.
\end{cases}
\]

\( v_a \) is half the square length of \( \alpha_a \) for untwisted affine types and is equal to 1 for twisted types.

A quasipartition of type \( a \in I_0 \) is a finite multiset taken from the set \( \nu \mathbb{Z}_{>0} \). The diagram of a quasipartition has rows consisting of boxes with width \( v_a \). Denote by \( \{ \nu^*, J^* \} \) a pair where \( \nu^* = \{ \nu^{(a)} \}_{a \in I_0} \) is a sequence of quasipartitions with \( \nu^{(a)} \) of type \( a \) such that \( \nu^{(a)} = (v_a^{(n)}(\nu), 2(n) \nu, \ldots) \) and \( J = \{ J^{(a, i)} \}_{a \in I_0, i \in \mathbb{Z}_{>0}} \) is a double sequence of partitions. Then a rigged configuration is a pair \( (\nu^*, J^*) \) subject to the restriction \( \mathbb{L} \) and the requirement that \( J^{(a, i)} \) is a partition contained in a \( m_i^{(a)} \times p_i^{(a)} \) rectangle. \( \nu^* \) is called a configuration and \( J^* \) a rigging. The set of rigged configurations for fixed \( \lambda \) and \( \mathbf{L} \) is denoted by \( \text{RC}(\lambda, \mathbf{L}) \). Then \( (2.12) \) is equivalent to
\[
(2.12) \quad M(\lambda, \mathbf{L}; q) = \sum_{(\nu^*, J^*) \in \text{RC}(\lambda, \mathbf{L})} q^{c(\nu^*, J^*)} 
\]
where \( c(\nu^*, J^*) = c(\nu) + |J^*| \) and \( |J^*| = \sum_{a \in I_0, i \in \mathbb{Z}_{>0}} t_a' |J^{(a, i)}| \).

For a quasipartition \( \mu = (\nu^{(m_1)}, 2(\nu) \nu, \ldots) \) with boxes of width \( v \) and \( i \in \nu \mathbb{Z}_{>0} \), define
\[
(2.14) \quad Q_i(\mu) = \sum_k \min(i, vk) m_k,
\]
the area of \( \mu \) in the first \( i/v \) columns of width \( v \). We also set
\[
(2.15) \quad Q_{\max}(\mu) = v \sum_k km_k.
\]
For a configuration $\nu^* = \{\nu^{(a)}\}_{a \in I_\nu}$ we set $Q_i^{(a)} = Q_i(\nu^{(a)}), Q_i^{(a)} = Q_{\max}(\nu^{(a)}).$

Then the vacancy numbers $p_i^{(a)}$ are given by

\[
(2.16) \quad p_i^{(a)} = \sum_{k \in Z_{>0}} L_k^{(a)} \min(i, k) + Q_i^{(a-1)} - 2Q_i^{(a)} + Q_i^{(a+1)}
\]

except the ones below in each nonexceptional affine type. In the above formula $\nu^{(0)}$ should be considered to be empty.

$A_n^{(1)}$:
\[
p_i^{(n)} = Q_i^{(n-1)} - 2Q_i^{(n)}
\]

$B_n^{(1)}$:
\[
p_i^{(n-1)} = Q_i^{(n-2)} - 2Q_i^{(n-1)} + 2Q_i^{(n)}
\]
\[
p_i^{(n)} = 2Q_{i/2}^{(n-1)} - 4Q_{i/2}^{(n)}
\]

$C_n^{(1)}$:
\[
p_i^{(n)} = Q_i^{(n-1)} - Q_{2i}^{(n)}
\]

$D_n^{(1)}$:
\[
p_i^{(n-2)} = Q_i^{(n-3)} - 2Q_i^{(n-2)} + Q_i^{(n-1)} + Q_i^{(n)}
\]
\[
p_i^{(n-1)} = Q_i^{(n-2)} - 2Q_i^{(n-1)}
\]
\[
p_i^{(n)} = 2Q_i^{(n-2)} - 2Q_i^{(n)}
\]

$A_{2n}^{(2)}$:
\[
p_i^{(n)} = Q_i^{(n-1)} - Q_i^{(n)}
\]

$A_{2n-1}^{(2)}$:
\[
p_i^{(n-1)} = Q_i^{(n-2)} - 2Q_i^{(n-1)} + 2Q_i^{(n)}
\]
\[
p_i^{(n)} = Q_i^{(n-1)} - 2Q_i^{(n)}
\]

$D_{n+1}^{(2)}$:
\[
p_i^{(n)} = 2Q_i^{(n-1)} - 2Q_i^{(n)}
\]

We assumed $L_i^{(a)} = 0$ for any pair $(a, i)$ that $p_i^{(a)}$ appears in the above list, since it is enough for our calculations later.

We show $p_i^{(a)} \geq 0$ for all $i$ such that $m_i^{(a)} > 0$ implies $p_i^{(a)} \geq 0$ for all the other $i$.

**Lemma 2.1.** Suppose $m_i^{(a)} = 0$ for $k < i < l$. Then the vacancy numbers satisfy the following upper convex relation:

\[
p_k^{(a)} + p_i^{(a)} \leq 2p_i^{(a)}
\]

for $k < i < l$.

**Proof.** Suppose the vacancy number is given by (2.16). On the interval $[k, l]$, $Q_i^{(a)}$ is a linear function of $i$ since there is no length $i$ row of $\nu^{(a)}$ for $k < i < l$. On the other hand, $Q_i^{(a-1)}$ and $Q_i^{(a+1)}$ are upper convex functions of $i$. Similarly, $\sum_k L_k^{(a)} \min(i, k)$ is also an upper convex function of $i$. Hence, we obtain the result. Proof for other a’s are the same. \qed
Corollary 2.2. Suppose $p_i^{(a)} \geq 0$ for all $i$ such that $m_i^{(a)} > 0$. Then we have $p_i^{(a)} \geq 0$ for $i \in \mathbb{Z}_{>0}$.

Proof. If $i \leq \nu_1^{(a)}$ we obtain the result by Lemma 2.1. (Note that $k$ can be 0.) On the contrary, assume $\nu_1^{(a)} < i$. Then apply Lemma 2.1 with $k = \nu_1^{(a)}$, $l > \nu_1^{(a)}$. □

2.4. Stability. In this subsection we investigate the behavior of a rigged configuration when $n$ is large. We define an integer $\gamma$ by $\gamma = 2$ for $g = A_{2n}^{(2)}, D_{n+1}^{(2)}$, $\gamma = 1$ otherwise.

Lemma 2.3. Let $(\nu^*, J^*) \in RC(\lambda, L)$ and $k = \max\{a \mid \langle \lambda, h_a \rangle \neq 0 \text{ or } L_i^{(a)} \neq 0 \text{ for some } i\}$. Then we have

\[Q^{(k)}_{\max} = Q^{(k+1)}_{\max} = \cdots = Q^{(n)}_{\max} = 0 \text{ for } g = A_1^{(1)};\]
\[Q^{(k)}_{\max} = Q^{(k+1)}_{\max} = \cdots = Q^{(n-1)}_{\max} = 2Q^{(n)}_{\max} \text{ for } g = A_{2n-1}^{(2)}, B_n^{(1)};\]
\[Q^{(k)}_{\max} = Q^{(k+1)}_{\max} = \cdots = 2Q^{(n-1)}_{\max} = 2Q^{(n)}_{\max} \text{ for } g = D_n^{(1)};\]
\[Q^{(k)}_{\max} = Q^{(k+1)}_{\max} = \cdots = Q^{(n-1)}_{\max} = Q^{(n)}_{\max} \text{ otherwise.}\]

Proof. Look at the formula (2.13). The definition of $k$ restricts the summation over $b \in I_0$ to $1 \leq b \leq k$. Recalling (2.11) with (2.13) we obtain the desired result. □

Lemma 2.4. Let $k$ be as in Lemma 2.3. Then we have $\nu_1^{(k+1)} = \nu_1^{(k+2)} = \cdots = \nu_1^{(n)}$. In particular, the longest rows of $\nu^{(k+1)}, \nu^{(k+2)}, \ldots, \nu^{(n)}$ are singular.

Proof. We give the proof for $g = D_n^{(1)}$. Proofs for the other cases are similar. To begin with note that $Q^{(k)}_{\max} = Q^{(k+1)}_{\max} = \cdots = Q^{(n-2)}_{\max} = 2Q^{(n-1)}_{\max} = 2Q^{(n)}_{\max}$ from Lemma 2.3. Assume that $\nu_1^{(n-2)} > \nu_1^{(n)}$. Then we have $p_1^{(n)} = Q^{(n)}_{\nu_1^{(n)}} - 2Q^{(n-1)}_{\nu_1^{(n-2)}} < Q^{(n-2)}_{\nu_1^{(n-2)}} - 2Q^{(n-1)}_{\nu_1^{(n-2)}} = 0$, which is a contradiction. Therefore it has to be $\nu_1^{(n-2)} \leq \nu_1^{(n)}$.

Similarly we have $\nu_1^{(n-2)} \leq \nu_1^{(n-1)}$. Suppose that one of the inequalities is strict, say $\nu_1^{(n-2)} < \nu_1^{(n)}$. Then we have $p_1^{(n-2)} = Q^{(n-3)}_{\nu_1^{(n-2)}} - 2Q^{(n-2)}_{\nu_1^{(n-2)}} + Q^{(n-1)}_{\nu_1^{(n-2)}} + Q^{(n)}_{\nu_1^{(n-2)}} < Q^{(n-3)}_{\nu_1^{(n-2)}} - 2Q^{(n-2)}_{\nu_1^{(n-2)}} + Q^{(n-1)}_{\nu_1^{(n-2)}} + Q^{(n)}_{\nu_1^{(n-2)}} \leq 0$ (note that $Q^{(n-3)}_{\nu_1^{(n-2)}} \leq Q^{(n-3)}_{\nu_1^{(n-3)}}$), which is a contradiction. Hence, we have $\nu_1^{(n-2)} = \nu_1^{(n-1)} = \nu_1^{(n)}$. We can recursively do the same argument up to $\nu^{(k+1)}$ and finally we obtain $\nu_1^{(k)} \leq \nu_1^{(k+1)} = \nu_1^{(k+2)} = \cdots = \nu_1^{(n)}$. The statement about singularity also follows from this estimate. □

Example 2.2. The above lemma assures the singularity for the longest rows. If the row is not longest, there could be non-singular rows. For example, for type $D_6^{(1)}$, there is the following example $(\lambda = (3), L_4^{(1)} = 15, L_i^{(a)} = 0$ for other $(a, i)$, i.e., $k = 1$).
that \( \nu \) so on, arriving at the conclusion that \( \nu \) implies \( p \) in the second equality. From \( h \) we only consider the case such that \( m \geq 1 \).

The length 3 rows of \( \nu^{(a)} \) (2 \( \leq a \)) are singular. For the reader’s convenience, we record here the corresponding tensor product:

\[
1 \otimes 3 \otimes 2 \otimes 1 \otimes 2 \otimes 3 \otimes 2 \otimes 1 \otimes 2 \otimes 2 \otimes 2 \in (B^{11})^{\otimes 15}.
\]

**Proposition 2.5.** Let \( (\nu^*, J^*) \in \text{RC}(\lambda, L) \) and \( k \) as in Lemma 2.3. Then there exists a partition \( \nu^* = (1^{m_1}, 2^{m_2}, \ldots) \) tiled by \( \diamond \) such that by setting \( l^* = k + \ell(\nu^*) \)

\[
(1) \nu^{(a)} = \nu^* \text{ for } a \geq l^* \text{ except when } a = n \text{ for } g = A_{2n-1}^{(2)}, B_n^{(1)} \text{ and } a = n - 1, n \text{ for } g = D_n^{(1)}, \text{ in which case } \nu^{(a)} \text{ is given by halving each column of } \nu^*;
(2) p_i^{(a)} = 0 \text{ for any } i \text{ and } a > l^*;
(3) c(\nu) = \frac{\gamma}{2} \sum_{a, b \in \mathbb{Z}_{>0}} C_{ab} \min(j, k) m_j^{(a)} m_k^{(b)} - \gamma \sum_{a, b \in \mathbb{Z}_{>0}} \min(j, k) L_j^{(a)} m_k^{(a)} \text{ where } C_{ab} = (2 - \delta_{a+1}) \delta_{ab} - \delta_{a,b-1} - \delta_{a,b+1}.
\]

**Proof.** (1) For the case \( \diamond = \emptyset \), that is \( g = A_n^{(1)} \), \( Q_{\max}^{(a)} = 0 \) for \( a \geq k \) from Lemma 2.3. Hence \( \nu^{(a)} = \emptyset \) for \( a \geq l^* \). The proof for the other cases are more or less the same. We only consider the case \( g = C_n^{(1)} \).

Let \( w_a \) be the length of the longest row of \( \nu^{(a)} \). Note that \( w_a \) is even. Lemma 2.4 shows \( w_a = w_{a+1} \) for \( a \geq k + 1 \). Let us show the length \( h_a \) of the rightmost \( (w\text{-th}) \) column of \( \nu^{(a)} \) is equal for \( a \geq k + h_n \). Let \( m \) be the minimal integer such that \( m \geq k + 1, m \geq h_{m+1} = \cdots = h_m \). Then \( h_{m-1} < h_m \) since \( p_i^{(m-1)} = Q_{w-1}^{(m)} - 2Q_{w-1}^{(m+1)} + Q_{w-1}^{(m+1)} = Q_{w-1}^{(m)} - Q_{w-1}^{(m)} \geq 0 \), where we have used \( h_m = h_{m+1} \) in the second equality. From \( p_i^{(m-1)} \geq 0 \) we have \( h_{m-2} < h_{m-1} \leq h_m \). Hence we get \( h_{m-2} < h_{m-1} \). This argument continues until we get \( h_{k+1} < h_{k+2} < \cdots < h_m \). It implies \( m - k \leq h_n \). Therefore we have \( h_a = h_{a+1} \) for \( a \geq k + h_n \). Let \( h_a' \) be the length of the next rightmost \( ((w-1)\text{-th}) \) column of \( \nu^{(a)} \). A similar argument to the previous paragraph shows that \( h_a' = h_{a+1}' \) for \( a \geq k + h_n + h_n' \) and so on, arriving at the conclusion that \( \nu^{(a)} = \mu^{(a+1)} \) for \( a \geq k + \ell(\nu^{(a)}) \). The fact that \( \nu^* \) is tiled by \( \boxdot \) is a consequence from the fact that \( \nu^{(a)} \) is a quasipartition with boxes of width 2.

(2) is clear from (1) and the fact that \( p_i^{(a)} \) is calculated by \( \mu^{(a-1)}, \mu^{(a)}, \mu^{(a+1)} \). (3) can be proven using (1) by case-by-case checking. \( \square \)
From Proposition 2.5 we see if \( n \) is sufficiently large, the fermionic formula (2.12) can be rewritten as

\[
M(\lambda, L; q) = \sum_{\nu \in C(\lambda, L)} q^{c(\nu)} \prod_{a \leq l^*} \prod_{i \in \Bbb Z_{> 0}} \left[ \frac{p_i^{(a)} + m_i^{(a)}}{m_i^{(a)}} \right] q^{|\nu|} = \sum_{(\nu^*, J^*) \in RC(\lambda, L)} q^{c(\nu^*, J^*)},
\]

where \( c(\nu^*, J^*) = c(\nu) + |J^*| - |J^*| = \gamma \sum_{a \leq l^*, i \in \Bbb Z_{> 0}} |f^{(a,i)}| \) and \( l^* \) is as in Proposition 2.5. Apparently, it depends only on the symbol \( \diamond \), but not on the affine algebra belonging to the same group in Table 2. Hence, from now on we pick up one affine algebra \( g^\diamond = A_n^{(1)}, D_{n+1}^{(2)}, C_n^{(1)}, D_n^{(1)} \) for each symbol \( \diamond = \varnothing, \Box, \bigstar, \ast \), and define the set of stable rigged configurations \( RC^\diamond \) and stable fermionic formula \( M^\diamond(\lambda, L; q) \) by using rigged configurations of \( g^\diamond \) for \( n \) large. Therefore, we have

\[
M^\diamond(\lambda, L; q) = \sum_{(\nu^*, J^*) \in RC^\diamond(\lambda, L)} q^{c(\nu^*, J^*)}.
\]

**Remark 2.3.** The above choice of the affine algebra \( g^\diamond \) for each \( \diamond \) is not mandatory. Namely, the construction of our main bijection \( \Psi \) in the subsection section we use the symbol \( \diamond \) is defined by the following algorithm. Here \( l \) is the length of some row of \( \nu(1) \).

### 3. The bijection

#### 3.1. Definitions.

The goal of this subsection is to give definitions of our main algorithms \( \Psi \) and \( \tilde{\Psi} \). Roughly speaking, the algorithms consist of two parts: the one is box removing or adding procedure on the rigged configurations, and the other one is to create a kind of recording tableau \( T \) which eventually generates the \( L \) tableaux. We will divide the definition according to this distinction. During the algorithm, we will use \((\nu^*, J^*)\) in slightly generalized sense. More precisely, for the case \( \diamond = \bigstar \), we allow the vacancy number for the longest row of \( \nu(1) \) to be \(-1\) while its rigging is \( 0 \). Such a peculiar situation happens only if we consider odd powers of the operators \( \delta \) or \( \tilde{\delta} \) (see definitions below) for \( \diamond = \bigstar \), though the final algorithms \( \Psi \) and \( \tilde{\Psi} \) always contain even powers of such operators whenever \( \diamond = \bigstar \). In this section we use the symbol \( a^\diamond \) where \( a^\Box = n - 1 \) and \( a^\bigstar = n - 2 \).

**Definition 3.1.** The map \( \delta_l \)

\[
\delta_l : (\nu^*, J^*) \mapsto \{ (\nu'^*, J'^*), k \},
\]

is defined by the following algorithm. Here \( l \) is the length of some row of \( \nu(1) \).

(i) As the initial step, do one of the following:

(a) If \( \diamond = \Box \) or \( \bigstar \), choose one of the length \( l \) rows of \( \nu(1) \).
(b) The case $\diamond = \Box$. If $|\nu^{(n-1)}| = |\nu^{(n)}|$, choose one of the length $l$ rows of $\nu^{(n-1)}$. On the other hand, if $|\nu^{(n-1)}| < |\nu^{(n)}|$, choose one of length $l$ rows of $\nu^{(n)}$.

(ii) Choose one of the length $l$ rows of $\nu^{(a \Diamond)}$. Then choose rows of $\nu^{(a)} (a < a \Diamond)$ recursively as follows. Suppose that we have chosen a row of $\nu^{(a)}$. Choose a shortest singular row of $\nu^{(a-1)}$ whose length is equal to or longer than the chosen row of $\nu^{(a)}$, and continue, if such a row exists; otherwise (in particular when $a = 1$) set $k = a$ and stop.

(iii) $\nu^{\bullet}$ is obtained by removing one box from the right end of each chosen row at Step (i) and (ii).

(iv) The new riggings $J^{\bullet}$ are defined as follows. For the rows that are not changed in Step (iii), take the same riggings as before. Otherwise set the new riggings equal to the corresponding vacancy numbers computed by using $\nu^{\bullet}$.

**Definition 3.2.** The map $\Psi$

$$\Psi : (\nu^{\bullet}, J^{\bullet}) \mapsto \{(\nu^{\bullet}, J^{\bullet}), T\}$$

is defined as follows. As the initial condition, set $T$ to be the empty skew tableau with both inner and outer shape equal to the diagram representing the weight of $(\nu^{\bullet}, J^{\bullet})$. Let $h_i$ denote the height of the $i$-th column (counting from left) of the partition $\nu^{(a \Diamond)}$ and let $l = \nu^{(a \Diamond)}_l$.

(i) We will apply $\delta_l$ for $h_i$ times. Each time when we apply $\delta_l$, we recursively redefine $(\nu^{\bullet}, J^{\bullet})$ and $T$ as follows. Assume that we have done $\delta_l^{i-1}$ and obtained $\{(\nu^{\bullet}, J^{\bullet}), T\}$. Let us apply $\delta_l$ one more time:

$$\delta_l : (\nu^{\bullet}, J^{\bullet}) \mapsto \{(\nu^{\bullet}, J^{\bullet}), k\}.$$  

Using the output, do the following. Define the new $(\nu^{\bullet}, J^{\bullet})$ to be $(\nu^{\bullet}, J^{\bullet})$. Define the new $T$ by putting $i$ on the right of the $k$-th row of the previous $T$.

(ii) Recursively apply $\delta_l^{h_i-1}, \ldots, \delta_2^{h_2}, \delta_1^{h_1}$ by the same procedure as in Step (i). Then the final outputs $(\nu^{\bullet}, J^{\bullet})$ and $T$ give the image of $\Psi$.

**Example 3.1.** Let us consider the special case of the bijection $\Psi$ where the bijection between the rigged configurations and the tensor products of crystals is also available. Consider the following element of the tensor product $(B^{1,3})^{\otimes 3} \otimes (B^{1,2})^{\otimes 2} \otimes (B^{1,1})^{\otimes 2}$ of type $D_8^{(1)}$ ($n \geq 8$) crystals:

$$p = \begin{array}{c}
1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 2 & 2 & 2 & 2
\end{array}$$

Due to Theorem 8.6 of [26] all the isomorphic elements under the combinatorial $R$-matrices correspond to the same rigged configuration. Then the map $\Psi$ for the $D_8^{(1)}$ case proceeds as follows. We remark that one can slightly modify the definition of $\Psi$ so that the following computation can be done in $D_7^{(1)}$. In the following diagrams, the first rigged configuration corresponds to the above $p$. Here, we put the vacancy numbers (resp. riggings) on the left (resp. right) of the corresponding rows. The gray boxes represent the boxes to be removed by each $\delta$ indicated on the left of each arrow. The corresponding recording tableau $T$ is given on the right of each arrow.
The final rigged configuration and $T$ of the above diagrams give the image of $\Psi$. Under the bijection [12] the final rigged configuration corresponds to the following element:

$$p' = 1111 \otimes 2222 \otimes 1333 \otimes 1444 \otimes 3535 \otimes 4466$$

**Remark 3.2.** Let $p$ and $p'$ as in Example 3.1 and again consider them as elements of $B_{8}^{(1)}$. If we apply the involution $\sigma$ at Section 5.3 of [19], we have

$$\sigma(p) = 8888 \otimes 8877 \otimes 6868 \otimes 7676 \otimes 6666 \otimes 7777$$

Then $p'$ coincides with the $I_0$-highest element corresponding to $\sigma(p)$. We expect that the same relation holds for arbitrary image of $\Psi$.

Now we are going to give the description of the algorithm $\tilde{\Psi}$ which will be shown to be the inverse of $\Psi$.

**Definition 3.3.** The map $\tilde{\delta}_{k}$

$$\tilde{\delta}_{k} : (\nu^*, J^*) \mapsto (\nu'^*, J'^*)$$

is defined by the following algorithm. Here the integer $k$ should satisfy $k \leq a^\Diamond$.

(i) Starting with $\nu^{(k)}$, choose rows of $\nu^{(a)} \ (a \leq a^\Diamond)$ recursively as follows.

To initialize the process, let us tentatively assume that we have chosen an infinitely long row of $\nu^{(k-1)}$. Suppose that we have chosen a row of $\nu^{(a-1)}$. Find a longest singular row of $\nu^{(a)}$ whose length does not exceed the length of the chosen row of $\nu^{(a-1)}$. If there is no such row, suppose that we have chosen a length 0 row of $\nu^{(a)}$ and continue. Otherwise choose one such singular row and continue.
(ii) Suppose that we have chosen the length $l$ row of $\nu^{(a\D)}$. To finish the process, do one of the following:

(a) If $\varnothing = \square$ or $\square$, choose one of the length $l$ rows of $\nu^{(n)}$.
(b) The case $\varnothing = \Box$. If $|\nu^{(n-1)}| = |\nu^{(n)}|$, choose one of the length $l$ rows of $\nu^{(n)}$. On the other hand, if $|\nu^{(n-1)}| < |\nu^{(n)}|$, choose one of the length $l$ rows of $\nu^{(n-1)}$.

(iii) $\nu^\bullet$ is obtained by adding one box to each chosen row in Step (i) and (ii). If the length of the chosen row is 0, create a new row at the bottom of the corresponding partition $\nu^{(a)}$.

(iv) The new riggings $J'^\bullet$ are defined as follows. Take all entries of $J'^{(a)}$ to be 0 for $a > a\D$. The remaining parts are defined as follows. For the rows that are not changed in Step (iii), take the same riggings as before. Otherwise set the new riggings equal to the corresponding vacancy numbers computed by using $\nu^\bullet$.

Definition 3.4. The map $\tilde{\Psi}$

$$\tilde{\Psi} : \{(\nu^\bullet, J^\bullet), T\} \mapsto (\nu'^\bullet, J'^\bullet)$$

is defined as follows.

(i) Let $h_1$ be the largest integer contained in $T$. For $h_1$ do the following procedure. Fix the rightmost occurrence of $i$ in $T$ for each $1 \leq i \leq h_1$. Call these fixed $h_1$ integers of $T$ the first group. Remove all members of the first group from $T$ and do the same procedure for the new $T$. Call the integers that are fixed this time the second group. Repeat the same procedure recursively until all integers of $T$ are grouped. Let the total number of groups be $l$, the cardinality of the $i$-th group be $h_i$ and the position of the letter $j$ contained in the $i$-th group be the $k_{i,j}$-th row (counting from the top of $T$).

(ii) The output of $\tilde{\Psi}$ is defined as follows:

$$(\nu'^\bullet, J'^\bullet) = \tilde{\delta}_{k_{1,1}} \cdots \tilde{\delta}_{k_{2,2}} \cdots \tilde{\delta}_{k_{n,n}} \bar{\delta}_{k_{1,1}} \tilde{\delta}_{k_{n,n}} \bar{\delta}_{k_{1,1}} \cdots \bar{\delta}_{k_{1,h_1}} (\nu^\bullet, J^\bullet).$$

Example 3.3. As for an example of $\tilde{\Psi}$, one should read Example 3.1 in the reverse order. More precisely, reverse all arrows and apply $\tilde{\delta}_4$, $\tilde{\delta}_3$, $\tilde{\delta}_2$, $\tilde{\delta}_1$, $\tilde{\delta}_6$, $\tilde{\delta}_5$, $\tilde{\delta}_4$, $\tilde{\delta}_1$, $\tilde{\delta}_4$, $\tilde{\delta}_3$ in this order.

3.2. Main statements. The crux of the combinatorics is contained in the following two theorems on the well-definedness of both maps $\Psi$ and $\tilde{\Psi}$, which we will prove later.

Theorem 3.1. Assume that $(\nu^\bullet, J^\bullet) \in \text{RC}\D$. Suppose that the rank $n$ satisfies $a\D \geq \ell(\text{wt}(\nu^\bullet, J^\bullet)) + \ell(\nu^{(a\D)})$. Then the map $\Psi$

$$\Psi : (\nu^\bullet, J^\bullet) \mapsto \{(\nu'^\bullet, J'^\bullet), T\}$$

is well-defined. More precisely, $(\nu'^\bullet, J'^\bullet) \in \text{RC}\D$ and the LR tableau $T \in \text{LR}^\eta_{\lambda\mu}$ satisfy the following properties:

$$\lambda = \text{wt}(\nu^\bullet, J^\bullet), \quad \mu = \nu^{(a\D)}, \quad \eta = \text{wt}(\nu'^\bullet, J'^\bullet).$$

Theorem 3.2. Assume that $(\nu^\bullet, J^\bullet) \in \text{RC}\D$ and $T$ is the LR tableau that satisfy the following three properties: $T \in \text{LR}^\eta_{\lambda\mu}$ where $\lambda, \mu \in \mathcal{P}\D$ and $\eta = \text{wt}(\nu^\bullet, J^\bullet)$. 

Then the map $\tilde{\Psi}$;

$$\tilde{\Psi} : \{(\nu^*, J^*), T\} \mapsto (\nu^*, J^*),$$

is well-defined. More precisely, we have $(\nu^*, J^*) \in \text{RC}^\diamond$, $\text{wt}(\nu^*, J^*) = \lambda$ and $\nu^{(a)} = \mu$.

By construction, $\delta$ and $\tilde{\delta}$ are mutually inverse procedures. Therefore the above theorems imply the following main theorem.

**Theorem 3.3.** Assume that $(\nu^*, J^*) \in \text{RC}^\diamond$. Suppose that the rank $n$ satisfies $a^\diamond \geq \ell(\text{wt}(\nu^*, J^*)) + \ell(\nu^{(a)}))$. Then $\Psi$ gives a bijection between $\text{RC}^\diamond$ and the disjoint union of product sets of $\text{RC}^\diamond$ and the LR tableaux as follows:

$$\Psi : (\nu^*, J^*) \mapsto \{(\nu^*, J^*), T\},$$

$$(\nu^*, J^*) \in \text{RC}^\diamond(\lambda, L), \quad \{(\nu^*, J^*), T\} \in \text{RC}^\diamond(\eta, L) \times LR^\eta_{\lambda^\mu},$$

where $\lambda, \mu, \eta$ satisfy

$$\lambda = \text{wt}(\nu^*, J^*), \quad \mu = \nu^{(a)}, \quad \eta = \text{wt}(\nu^*, J^*).$$

That is, for fixed $\lambda$ and $L$, $\Psi$ defines a bijection

$$\text{RC}^\diamond(\lambda, L) \mapsto \bigsqcup_{\mu \in P_{L^\diamond - [\lambda], \eta \in T_{L^\diamond}}} \text{RC}^\diamond(\eta, L) \times LR^\eta_{\lambda^\mu}.$$

The inverse procedure is given by $\Psi^{-1} = \tilde{\Psi}$.

Let us examine the restrictions on the rank $n$ for $\Psi$ and $\tilde{\Psi}$. For simplicity, consider $\diamond = \emptyset$. Let $N_{\text{out}}$ (resp. $N_{\text{in}}$) be the length of the outer (resp. inner) shape of $T$ and $|T|$ be the number of (either filled or not filled) boxes in $T$. Then, by construction, $\tilde{\Psi}$ is defined if the rank $n$ satisfies $N_{\text{out}} + 2 \leq n$. Let us consider $\Psi$. Following Theorem 3.3, we see that the weight of the LR tableau $T$ coincides with the partition $\nu^{(a)}$ of $\text{RC}^\diamond$ and the inner shape of $T$ coincides with $\text{wt}(\nu^*, J^*)$. Since $T$ has column strict property, we have $N_{\text{out}} - N_{\text{in}} = \ell(\nu^{(a)})$. On the other hand, the condition in Theorem 3.1 can be rephrased as $N_{\text{in}} + \ell(\nu^{(a)}) + 2 \leq n$. To summarize the condition on the rank $n$ for $\Psi$ case is more restrictive and requires the rank $n$ to be larger than that for $\tilde{\Psi}$. We remark that in the present definition of $\Psi$, the partition $\nu^{(a)}$ is always removed entirely. For this point one can modify the definition of $\Psi$ slightly so that the required condition on the rank $n$ decreases by 1 compared with the original $\tilde{\Psi}$.

In order to get a feeling about the condition on the rank, let us recall the bijection between $\text{RC}^\diamond$ and tensor product of type $A_n^{(1)}$ crystals in [19]. Then if $\text{RC}^\diamond$ corresponds to an element of $\bigotimes_i B^{r_is_i}$, we have $|T| = \sum_i r_is_i$. Thus the quantity $\ell(\nu^{(a)})$ is always smaller than or equal to $\sum_i r_is_i$ (usually, much smaller than $\sum_i r_is_i$). Moreover, recall the bijection between $\text{RC}^\emptyset$ and tensor products of type $D_n^{(1)}$ crystals in [23]. Let the number of barred letters (i.e., $\bar{1}, \bar{2}, \ldots$) of the corresponding tensor product of crystals be $M$. Recall that under the present condition on the rank there is no $\bar{n}$. Then the total number of letters filled in $T$ is equal to $2M$. 

3.3. Proof of Theorem 3.1 Let us fix notations that will be used in this subsection. We will refer to Step (i), etc., of Definition 3.1 simply as Step (i), etc. We denote by $l^{(a)}$ the length of a row that is removed from $\nu^{(a)}$ by $\delta_t$. In other words, $l^{(a)}$ is the position of the box to be removed by $\delta_t$. Then we have $l^{(a-1)} > l^{(a)} > l^{(a+1)}$. We denote by $(\nu^*, J^*)$ the image of $\delta_t$ and $p_l^{(a)}$ the vacancy number with respect to $\nu^*$. If we further apply $\delta_t$ on $(\nu^*, J^*)$, we use $l^{(a)}$ to express the position of the box to be removed by the second $\delta_t$. We use the symbol $\Delta$ to express the differences of the quantities for $(\nu^*, J^*)$ minus $(\nu^*, J^*)$. For example, $\Delta p_l^{(a)} = p_l^{(a)} - p_l^{(a-1)}$.

Let the height of the $i$-th column (counting from left) of $\nu^{(a)}$ be $h_i$. Note that the procedure $\Psi$ does not change the quantum space. Therefore if the arguments concern only the differences of the vacancy numbers, we can neglect the effect of the quantum space on the vacancy numbers.

During this subsection, we assume that the rank $n$ satisfies the condition

\begin{equation}
\ell(wt(\nu^*, J^*)) + \ell(\nu^{(a)}) \leq a^\Diamond.
\end{equation}

We start with the following lemma, that makes the statement of Proposition 2.5 (1) more precise.

**Lemma 3.4.** Set $l = \nu_1^{(a)}$ and $a_{l+1} = \ell(wt(\nu^*, J^*)) + 1$. Then there exists an integer sequence $a_{l+1} \leq a_l \leq a_{l-1} \leq \cdots \leq a_1$ that satisfies

\begin{align*}
a_i - a_{i+1} &\leq m_i^{(a_i)}, \\
m_i^{(a_{i+1})} &< m_i^{(a_{i+1}+1)} < \cdots < m_i^{(a_i)} = \cdots = m_i^{(a)}
\end{align*}

for any $i$ such that $l \geq a \geq 1$.

**Proof.** We give the proof for $\Diamond = \Box$. Proofs for the other cases are similar. By Lemma 2.4 and $|\nu^{(n-2)}| = 2|\nu^{(n)}|$, we see that $p_l^{(n)} = 0$. On the other hand, from Corollary 2.2 we have $p_l^{(n)} - p_{l-1}^{(n)} = Q^{(n)}_l - Q^{(n)}_{l-1}$. Hence, $m_l^{(n-2)} \leq 2m_l^{(n)}$. Similarly, we have $m_l^{(n-2)} \leq 2m_l^{(n-1)}$. Next, from $p_l^{(n-2)} = 0$ and $p_{l-1}^{(n-2)} \geq 0$, we obtain $m_l^{(n-3)} = m_l^{(n-2)} > m_l^{(n-1)} + m_l^{(n)} \leq 0$. Combining this with the previously obtained inequalities $m_l^{(n-2)}/2 - m_l^{(n)} \leq 0$ and $m_l^{(n-2)}/2 - m_l^{(n-1)} \leq 0$, we deduce $m_l^{(n-3)} \leq m_l^{(n-2)}/2$. We can recursively do the same arguments and obtain $m_l^{(a-1)} \leq m_a^{(a)}$ for $a \leq n - 2$. Moreover, from the same arguments, we see that once $m_l^{(a-1)} < m_l^{(a)}$ happens then $m_l^{(a-1)} < m_l^{(a)}$ holds for all $k < a \leq a'$. Define $a_k$ to be the maximal such $a'$. If there is no such $a'$, set $a_k = k + 1$. Then if $m_l^{(n-2)} < a_l - k$ we have $m_l^{(k+1)} \leq 0$, which contradicts with Lemma 2.4. Therefore we conclude that $a_l - k \leq m_l^{(n-2)}$. In order to go further, we observe the following. By definition of $a_l$, we have $m_l^{(a_l)} = m_l^{(a_l+1)} = \cdots = m_l^{(n-2)} = m_l^{(n-1)}/2 = m_l^{(n)}/2$. From this we can deduce $p_l^{(a)} = p_l^{(a+1)} = \cdots = p_{l-1}^{(n-2)} = 0$. Then by the same arguments that are given in the previous case, we have $m_l^{(a-1)} \leq m_l^{(a)}$ for $a_l - a \leq n - 2$. Again, in decreasing $a$, once the inequality is strict so are the rest of inequalities. Suppose that $a_{l-1}$ is the maximal index where such a strict inequality appears. This time, in order to satisfy the condition $m_{l-1}^{(a_l)} \geq 0$, $a_{l-1}$ has to satisfy...
\[a_{l-1} - a_l \leq m_{l-1}^{(n-2)}\]. Starting from \(a_{l-1}\), we can recursively continue the similar arguments and finish the proof. \(\square\)

**Proposition 3.5.** \(\delta_l\) is well defined if \(\Diamond = \square\) or \(\Box\), and \(\delta_l^2\) is well defined if \(\Diamond = \Box\).

**Proof.** In order to check the well-definedness of \(\delta_l\) and \(\delta_l^2\), we need to check the positivity of the vacancy numbers and the inequality that the riggings are less than or equal to the corresponding vacancy numbers. To do this, we have to distinguish the following three cases.

Let us remark that, from the condition \(3.1\) on the rank \(n\), we can use Proposition 2.3 so that we can assume the specific nature of the rigged configurations stated in that proposition.

**Case 1.** Let us check the well-definedness for \(\nu^{(a)}\) \((a \geq a^\Diamond)\). First check the cases \(\Diamond = \square\) or \(\Box\). From Proposition 2.3, we see that \(\nu^{(n-1)} = \nu^{(n)}\) and \(J_i^{(n)} = 0\) for all \(i\). In Step (i), \(\delta_l\) removes one box from each \(l\)-th column of \(\nu^{(n-1)}\) and \(\nu^{(n)}\). Thus we have \(\nu^{(n-1)} = \nu^{(n)}\) so that the vacancy numbers for \(\nu^{(n)}\) are all 0. This shows the well-definedness for \(\nu^{(n)}\). Let us consider \(\nu^{(n-1)}\). In Step (iii), \(\delta_l\) may remove a row of \(\nu^{(n-2)}\) that is longer than or equal to \(l\). We see that it will contribute to \(\Delta Q_l^{(n-2)}\) as 0 or \(-1\). Then the vacancy numbers \(p_j^{(n-1)}\) for \(j \geq l\) will not decrease because \(\Delta p_j^{(n-1)} = \Delta Q_j^{(n-2)} - 2\Delta Q_j^{(n-1)} + \Delta Q_j^{(n)} = \Delta Q_j^{(n-2)} - 2(-1) + (-1) = \Delta Q_j^{(n-2)} + 1 \geq 0\). On the other hand, if \(j < l\) then \(\Delta p_j^{(n-1)} = 0\). This implies the well-definedness for \(\nu^{(n-2)}\).

Let us consider the case \(\Diamond = \Box\). Denote by \(\Delta'\) the differences of the quantities after \(\delta_l^2\) minus those before \(\delta_l^2\). Then we observe that \(\Delta' Q_l^{(n)} = \Delta' Q_l^{(n-3)} = -1, \Delta' Q_l^{(n-2)} = -2\) and \(-2 \leq \Delta' Q_l^{(n-3)} \leq 0\). Then we can use a similar discussion as in the previous case. For example, for \(j \geq l\), \(\Delta' p_j^{(n-2)} = \Delta' Q_j^{(n-3)} - 2\Delta' Q_j^{(n-2)} + \Delta' Q_j^{(n-1)} + \Delta' Q_j^{(n)} = \Delta' Q_j^{(n-3)} - 2(-2) + (-1) + (-1) = \Delta' Q_j^{(n-3)} + 2 \geq 0\) which implies the well-definedness for \(\nu^{(n-2)}\).

We remark that once the well-definedness for \(a \geq a^\Diamond\) is established, the rest of the proof does not depend on the choice of \(\Diamond\). Especially, it is enough to consider \(\delta_l\) and does not need to consider \(\delta_l^2\).

**Case 2.** Next let us check the well-definedness for rows of \(\nu^{(a)}\) \((a < a^\Diamond)\) that are not removed by \(\delta_l\). Recall that \(l^{(a-1)} \geq l^{(a)} \geq l^{(a+1)}\). Then by using the similar arguments of the previous case we conclude that \(\Delta p_j^{(a)} \geq 0\) if \(j \geq l^{(a)}\) and \(\Delta p_j^{(a)} = 0\) if \(l^{(a+1)} > j\), which imply the well-definedness in both cases. Now consider the case \(l^{(a)} > l \geq l^{(a+1)}\). Let the rigging corresponding to such a length \(j\) row of \(\nu^{(a)}\) be \(J(\geq 0)\). Since \(l^{(a)} > j\) the row will not be removed by \(\delta_l\) so that the rigging \(J\) will not change. In this case we have \(\Delta p_j^{(a)} = -1\). Remind that this row is not removed although it satisfies \(j \geq l^{(a+1)}\). Thus the row is not singular before the application of \(\delta_l\) so that it satisfies \(p_j^{(a)} - J \geq 1\). Combining this with \(\Delta p_j^{(a)} = -1\) we have \(p_j^{(a)} \geq J \geq 0\). This shows that the rigging does not exceeds the corresponding vacancy number and also that the positivity of the vacancy number.

**Case 3.** Consider the remaining case, i.e., rows of \(\nu^{(a)}\) \((a < a^\Diamond)\) that are removed by \(\delta_l\). The point is that \(\delta_l\) changes the length \(l^{(a)}\) row into \(l^{(a)} - 1\) row so that we
have to ensure the positivity \( p^{(a)}_{l(a)} \geq 0 \). We do not need to consider the rigging separately since the new rigging is taken equal to \( p^{(a)}_{l(a)} \). If \( p^{(a)}_{l(a)} > 0 \), we can use the similar arguments of the previous case to ensure the positivity. Therefore we assume \( p^{(a)}_{l(a)} = 0 \) in the rest of the proof. If \( l(a) > l^{(a+1)}(a) \) then \( p^{(a)}_{l(a)} = -1 \) which violates the positivity. Thus we ought to show \( l(a) = l^{(a+1)} \). In order to prove this, let us show that assuming \( l(a) > l^{(a+1)}(a) \) leads to a contradiction. To begin with, let us check that we can use Lemma 2.1 with \( l(a) \) instead of the length \( l^{(a+1)}(a) \). If there is no such row, set \( l(a) = l^{(a+1)}(a) \). In particular the length \( l(a) - 1 \) row of \( \nu(a) \), then it is singular (since the assumption \( p^{(a)}_{l(a)} = 0 \)) and it is longer than or equal to the length \( l^{(a+1)}(a) \) row of \( \nu(a+1) \). Thus \( \delta_l \) can remove the length \( l(a) - 1 \) row instead of the length \( l^{(a+1)}(a) \) row, which is a contradiction. Then Lemma 2.1 combined with \( p^{(a)}_{l(a)} = 0 \) gives \( p^{(a)}_{l(a)} = p^{(a)}_{l(a)-1} = 0 \) (recall that by definition we have \( p^{(a)}_{l(a)} = 0 \) and by Corollary 2.2 we have \( p^{(a)}_{l(a)-2} \geq 0 \)). From \( p^{(a)}_{l(a)-2} = 0 \) we can recursively do the same arguments and obtain the following result. Let \( j \) be the length of the longest rows among the rows of \( \nu(a) \) that are strictly shorter than \( l(a) \). If there is no such row, set \( j = 0 \). Then we have \( p^{(a)}_{l(a)} = p^{(a)}_{l(a)-1} = \cdots = p^{(a)}_j = 0 \). In particular the length \( j \) rows of \( \nu(a) \) are singular. Thus the assumption \( l(a) > j \) forces to \( l^{(a+1)} > j \) in order to avoid the removal of length \( j \) row by \( \delta_l \). Then on the interval \( j \leq i \leq l(a) \) the following properties hold as functions of \( i \):

\[
\begin{align*}
Q^{(a-1)}_i : & \text{ upper convex function (including linear function case)}, \\
Q^{(a)}_i : & \text{ linear function}, \\
Q^{(a+1)}_i : & \text{ strictly upper convex function due to the existence of the length } l^{(a+1)}(a) \text{ row that satisfy } l(a) > l^{(a+1)} > j.
\end{align*}
\]

Here we have used the terminology “strictly upper convex” in order to express the situation that the inequality in Lemma 2.1 is strict. Summing up all three contributions, we conclude that the function \( p^{(a)}_i \) is a strictly upper convex function with respect to \( i \) on this interval. However, as we have seen, \( p^{(a)}_i = 0 \) for \( j \leq \forall i \leq l(a) \). This is a contradiction. Hence we have \( l(a) = l^{(a+1)}(a) \) which ensures the well-definedness of \( \delta_l \) for Case 3.

\[\square\]

**Proposition 3.6.** Let us denote the image of the map \( \Psi \) as

\[\Psi : (\nu^*, J^*) \rightarrow \{(\nu^*, J^*), T\}.\]

Then \( \Psi \) is well-defined on \( (\nu^*, J^*) \) and the image \( \{(\nu^*, J^*), T\} \) does not depend on \( n \). Moreover, if the inequality of the condition \( (3.1) \) on the rank \( n \) is strict, then all the boxes of \( \nu(a^{(a)}) \) and \( \nu(a^{(a)}) \) are removed in exactly the same way during the whole procedure of \( \Psi \).

**Proof.** Suppose that the rank \( n \) satisfies the strict inequality \( a^{(a)} > k + \ell(\nu(a^{(a)})) \). We first show the latter statement. Then, combining \( \nu(k + \ell(\nu(a^{(a)}))) = \nu(k + \ell(\nu(a^{(a)})) + 1) = \cdots = \nu(a^{(a)}) \) from Proposition 2.5 \((1)\), the well-definedness of \( \Psi \) and the independence of the image on the rank \( n \) follow. During the proof, let \( m_j^{(a)}(a) \) be the multiplicity corresponding to the initial \( (\nu^*, J^*) \), let \( l = \nu_1^{(a)} \) and let \( h_i \) be the height of the \( i \)-th column (counting from left) of \( \nu(a^{(a)}) \).
The first step of $\Psi$ is the application of $\delta_{hi}$. Let us show that $\delta_{hi}$ remove the $l$-th columns of $\nu(a^\diamondsuit - 1)$ and $\nu(a^\diamondsuit)$. From this, we can deduce that the positions of the removed boxes are the same for these two partitions since the height of two columns are both $h_l$. From Lemma 2.4 we see that the longest rows of $\nu(k+1), \ldots, \nu(a^\diamondsuit)$ are length $l$ singular rows. Moreover, from Lemma 3.4 one of the followings is true: $m_{l_1}^{(k+1)} = m_{l_1}^{(k+2)} = \cdots = m_{l_1}^{(a^\diamondsuit)}$ or $m_{l_1}^{(k+1)} < m_{l_1}^{(k+2)} < \cdots < m_{l_1}^{(a_l)} = \cdots = m_{l_1}^{(a^\diamondsuit)}$ for some $k+1 < a_l < a^\diamondsuit$. Then we can analyze the consequences of $\delta_{hi}$ as follows. To begin with, $\delta_{hi}$ can remove only length $l$ rows of $\nu(k+1), \ldots, \nu(a^\diamondsuit)$ and, in particular, the first $\delta_l$ will remove one box from each partition $\nu(k+1), \ldots, \nu(a^\diamondsuit)$. To analyze the effects of $\delta_l$ after the first one, we have to clarify the changes of the vacancy numbers caused by $\delta_l$. Since the removed rows have the same length, we can do the analysis in the following way. Let $j > k + 1$.

(a) If length $l$ rows of $\nu(j-1), \nu(j), \nu(j+1)$ are removed, then the vacancy number $p_{l_1}^{(j)}$ will not change. Recall that initially we have $p_{l_1}^{(j)} = 0$ for $k < j \leq a^\diamondsuit$. Therefore we see that when we remove length $l$ rows of $\nu(j-1), \nu(j), \nu(j+1)$, the initial length $l$ singular rows of $\nu(j)$ remain as singular rows of the same length if they are not removed.

(b) If length $l$ rows of $\nu(j), \nu(j+1)$ are removed but $\nu(j-1)$ is not removed by the $i$-th $\delta_l$, then the vacancy number $p_{l_1}^{(j)}$ will increase by 1. Thus if we further apply $\delta_l$, $\nu(j)$ will not be removed since there is no singular rows of $\nu(j)$ whose length are greater than or equal to $l$. On the other hand, $\nu(j+1)$ will be removed. To show this, we have to distinguish two cases. If initially $m_{l_1}^{(j)} = m_{l_1}^{(a^\diamondsuit)} = h_l, \delta_l^i (i < h_l)$ will not exhaust the $l$-th column of $\nu(j+1)$. The remaining length $l$ rows are still singular as in the above case (a). On the contrary, if initially $m_{l_1}^{(j)} < m_{l_1}^{(a^\diamondsuit)} = h_l$, we have moreover $m_{l_1}^{(j)} < m_{l_1}^{(j+1)}$. Therefore, again we have remaining length $l$ singular rows of $\nu(j+1)$ which can be removed.

To summarize we obtain the following conclusion. When we apply $\delta_{hi}$, the $i$-th $\delta_l$ removes length $l$ rows of $\nu(a_1), \ldots, \nu(a_{hi})$ for some $a_1 \leq k + i$. Now consider the case $i = h_l$. Since $a^\diamondsuit - k > \ell(\nu(a^\diamondsuit)) > h_l$, we have $k + h_l < a^\diamondsuit$. In particular we see that $\delta_{hi}$ will remove both $l$-th columns of $\nu(a^\diamondsuit - 1)$ and $\nu(a^\diamondsuit)$, which is the desired fact.

Next let us consider $\delta_{h_l-1}^{hi}$. To begin with, we see the following facts concerning with the interaction with the effect of $\delta_{hi}$. Suppose that some length $l$ rows of $\nu(k+1), \ldots, \nu(a^\diamondsuit)$ are removed by $\delta_{hi}$ and set to be singular. Then all such rows remain as length $l - 1$ singular rows after applications of $\delta_{hi}$. This follows from the fact that $\delta_{hi}$ remove length $l$ rows of $\nu(k+1), \ldots, \nu(a^\diamondsuit)$ so that such removal does not affect the vacancy numbers $p_{l_1}^{(j)}$ for $k + 1 \leq j \leq a^\diamondsuit$. Therefore we see that the $i$-th $\delta_{h_l-1}$ will remove the same rows that have been removed by the $i$-th $\delta_l$. In other words, the $i$-th $\delta_{h_l-1}$ removes length $l - 1$ rows of $\nu(a_1), \ldots, \nu(a_{hi})$ for some $a_1 \leq k + i$.

Let us consider the remaining $\delta_{hi-1}^{h_l-1}$. Assume that $h_{l-1} > h_l$. Then we have $m_{l_1}^{(a_{hi-1})} > 0$. As before, let $a_l (> k + 1)$ be the maximal index such that $m_{l_1}^{(a_l-1)} < m_{l_1}^{(a_l)}$ happens. If there is no such $a_l$, set $a_l = k + 1$. Then we observe the
following two facts. (1) From Lemma 3.4, we have one of the followings: $m_{l-1}^{(a)} = m_{l-1}^{(a+1)} = \cdots = m_{l-1}^{(a^\diamond)}$ or $m_{l-1}^{(a)} < m_{l-1}^{(a+1)} < \cdots < m_{l-1}^{(a^\diamond)} = \cdots = m_{l-1}^{(a^\circ)}$ for some $a_i < a_{i-1} < a^{\circ}$. In particular, from $a_i \leq k + h_l$ and $m_{l-1}^{(a)} \geq 0$, we have $m_{l-1}^{(k+h_l+1)} > 0$. (2) Recall that the first $h_l$-th rows of $\delta_{l-1}$ remove totally $h_l$ length $l-1$ rows from each of $\nu^{(a)}$ $(k + h_l \leq a \leq a^{\diamond})$. Thus the length $l-1$ rows of $\nu^{(a)}$ $(k + h_l + 1 \leq a \leq a^{\circ})$ remain singular after application of $\delta_{l-1}^{h_l}$. From the observations (1) and (2), for the latter $h_l-1$ times $\delta_{l-1}$, we can use the same arguments that we have used in the case of $\delta_{l-1}^{h_l}$. To summarize, we see that the $i$-th $\delta_{l-1}$ removes length $l-1$ rows of $\nu^{(a)}, \ldots, \nu^{(a^{\circ})}$ for some $a \leq k + i$. Then by $a^{\diamond} - k \geq \ell(\nu^{(a^\circ)}) \geq h_l-1$ we have $h_l - 1 < a^{\diamond}$, which implies that $\delta_{l-1}^{h_l-1}$ will remove both $(l-1)$-th columns of $\nu^{(a^{\circ}-1)}$ and $\nu^{(a^{\circ})}$. Thus $\delta_{l-1}^{h_l-1}$ remove the same positions of $\nu^{(a^{\circ}-1)}$ and $\nu^{(a^{\circ})}$. Recursively, we conclude that every $\delta_i, \delta_{i-1}, \ldots, \delta_1$ removes from the same positions of $\nu^{(a^{\circ}-1)}$ and $\nu^{(a^{\circ})}$, which concludes the proof of the proposition.

Lemma 3.7. Consider the map $\Psi : (\nu^*, J^*) \mapsto \{(\nu^*, J^*), T\}$. Then $(\nu^*, J^*) \in \text{RC}^\diamond$ and the outer shape of $T$ coincides with the weight of $(\nu^*, J^*)$.

Proof. Under the assumption $\lambda \vdash n$ on the rank $n$, we see from Proposition 3.6 that we can apply all $\delta$’s of $\Psi$ in well-defined manner. Especially, from Proposition 3.6 we see that the image of each $\delta$ always belongs to $\text{RC}^\diamond$. Recall that $\Psi$ will entirely remove $\nu^{(a)}$ for $a^{\diamond} \leq a$. Therefore the image $(\nu^*, J^*)$ can be identified as the element of $\text{RC}^\diamond$.

Now we consider the statement about $T$. Let us consider a particular step of $\Psi$, say the application of $\delta_1$. Then $\delta_1$ removes one box from each $\nu^{(a)}, \nu^{(a+1)}, \ldots, \nu^{(a^{\circ})}$ for some $a$. On the level of the Young diagram that represents the weight of the rigged configuration, this procedure add a box to the $a$-th row of the Young diagram. On the other hand, according to the definition of $\Psi$, new $T$ is obtained by adding a box on the right of the $a$-th row of $T$. Since such coincidence occurs in every step of the $\Psi$, we obtain the statement.

Remark 3.4. From the above proof, we can also see that the outer shape of each intermediate $T$ coincides with the weight of the corresponding intermediate rigged configuration. In particular shapes of all intermediate $T$ are Young diagrams. Here we ignore the behavior of $\nu^{(a)}$ $(a > a^{\diamond})$ when $\diamond = \emptyset$.

We prove the following technical lemma which is useful to show that $T$ satisfies the definition of the LR tableaux.

Lemma 3.8. Consider the application of $\delta_1^2$ on $(\nu^*, J^*)$ where $l = \nu_1^{(a)}$. Then we have $l^{(a-1)} \leq l^{(a)}$ where $a \leq a^{\diamond}$. Here we assume that $l^{(a)} = \infty$ (resp. $l^{(a)} = \infty$) if $\nu^{(a)}$ is not removed by the first (resp. second) $\delta_1$ and allow $\infty \leq \infty$.

Proof. As the statement about $\nu^{(a)}$ for $a^{\diamond} < a$ is always true, we assume $a \leq a^{\diamond}$. Thus the proof does not depend on $\diamond$. Also it is enough to show the statement under the strict condition on the rank $n$ such as $a^{\diamond} < k + \ell(\nu^{(a^\circ)})$. We proceed by induction on $a$. By the condition on the rank $n$, we can assume the shape of $(\nu^*, J^*)$ as described in Proposition 2.5. From Proposition 3.6 we see that two $\delta_1$ remove the same positions of $\nu^{(a^\circ)-1}$ and $\nu^{(a^\circ)}$. In particular we have $l^{(a^\circ)-1} = l^{(a^\circ)} = l$, etc.
which proves the statement for $a = a^{\diamond}$. Assume that we have $l^{(a)} \leq l^{(a+1)}$ for some $a + 1 < a^{\diamond}$. Recall that by definition of $\delta_l$ we have $l^{(a)} \geq l^{(a+1)}$. Then the differences of the vacancy numbers by the first $\delta_l$ is as follows: $\Delta p^{(a)}_j = +1$ if $l^{(a-1)} > j \geq l^{(a)}$ and $\Delta p^{(a)}_j = 0$ if $j \geq l^{(a-1)}$. Note that if $\nu^{(a-1)}$ is not removed by the first $\delta_l$, we can formally set $l^{(a-1)} = \infty$ and obtain the same relation. In both cases, the length $j$ ($l^{(a-1)} > j \geq l^{(a)}$) rows of $\nu^{(a)}$ are non-singular and cannot be removed by the second $\delta_l$. On the other hand, combining the assumed inequality $l^{(a+1)} \geq l^{(a)}$ with $l^{(a)} \geq l^{(a+1)}$ that follows from the definition of $\delta_l$, we have $l^{(a)} \geq l^{(a)}$. Therefore $l^{(a)}$, i.e., the length of the row of $\nu^{(a)}$ that is removed by the second $\delta_l$ should satisfy $l^{(a)} \geq l^{(a-1)}$. Hence we complete the proof. 

\[\square\]

**Corollary 3.9.** Let us consider intermediate steps of $\Psi$. Let the recording tableau before (resp. after) the application of $\delta_l^{h_i}$ be $T$ (resp. $T'$). Then the difference between $T'$ and $T$, i.e., $T' \setminus T$, forms the vertical strip of cardinality $h_i$ and, moreover, the letters contained in the strip are $1, 2, \ldots, h_i$ from top to bottom.

**Proof.** Look at a pair of successive two $\delta_l$’s and apply the previous Lemma 3.8 for the corresponding intermediate rigged configuration. Let us use symbols $l^{(a)}$ and $l^{(a)}$ for positions of boxes that are removed by $\delta_l$’s. Suppose that $l^{(a)} < \infty$ and $l^{(a)} = \infty$. Then we add a letter, say $j$, to the $a$-th row of the recording tableau $T$. From Lemma 3.8 we have $l^{(a-1)} \leq l^{(a)}$, which forces $l^{(a)} = \infty$. In other words, the second $\delta_l$ cannot remove from $\nu^{(a)}$. Thus we have to put the letter $j+1$ to a row that is strictly lower than the $a$-th row of $T$. To summarize, $T' \setminus T$ contains at most one element for each row, and if we read $T' \setminus T$ from top to bottom it reads $1, 2, \ldots, h_i$.

Now recall from Lemma 3.9 that shapes of all the intermediate recording tableaux coincide with the corresponding intermediate rigged configurations. In particular their shapes are the Young diagrams. Therefore $T' \setminus T$ forms a vertical strip of cardinality $h_i$. 

\[\square\]

**Lemma 3.10.** Consider the applications of $\delta_l^{h_i}$ and $\delta_l^{h_i-1}$ of $\Psi$ for some $l$. Let the positions (row, column) of the letter $c$ of $T$ associated with $\delta_l^{h_i}$ (resp. $\delta_l^{h_i-1}$) be $(i, j)$ (resp. $(i', j')$). Then we have $i' \leq i$ and $j' > j$. In other words, we have the following two possibilities; $(i', j')$ is on the right of $(i, j)$, or $(i', j')$ is strictly above and strictly right of $(i, j)$.

**Proof.** We argue according to the computations of $\Psi$. Thus $\delta_l^{h_i}$ appears first and $\delta_l^{h_i-1}$ appears next. Let us look at a particular pair $\delta_l$. Denote the length of the row of $\nu^{(a)}$ that is removed by the first (resp. second) $\delta_l$ by $l^{(a)}$ (resp. $l^{(a)}$). Let us tentatively call the row that is removed by the first $\delta_l$ by $A$. Then we observe the following two facts.

1. If the row $A$ is removed by the first $\delta_l$, then it will not be removed by all the remaining $\delta_l$. In particular, the rigging of the row $A$ will not change after removed by the first $\delta_l$. To show this, let us look at the second $\delta_l$. Recall that from Lemma 3.8 we have $l^{(a)} \leq l^{(a+1)}$ and from definition of $\delta_l$ we have $l^{(a)} \geq l^{(a+1)}$, thus $l^{(a)} \geq l^{(a)}$. On the other hand the length of the row $A$ after the first $\delta_l$ is the same as that of $A$ after the second $\delta_l$. Hence it will not be removed by the second $\delta_l$. Recursively we can show the claim.
(2) After the first $\delta_l$, the vacancy number for the row $A$ will not be changed by the succeeding $\delta_l$’s. To show this, note that the length of the row $A$ after the first $\delta_l$ is $l^{(a)} - 1$. By Lemma 3.8 we have $l^{(a)} \leq l^{(a+1)}$ and from definition of $\delta_l$ we have $l^{(a-1)} \geq l^{(a)} \geq l^{(a+1)}$ so that the second $\delta_l$ will not change the vacancy number for the row $A$. Recursively we can show the claim.

To summarize, once a row is removed by some $\delta_l$, then the row is kept as singular after applications of all the remaining $\delta_l$’s. For the sake of the later arguments, we observe that from $l^{(a-1)} \leq l^{(a)}$ and $l^{(a-1)} \geq l^{(a)} \geq l^{(a+1)}$, we have $l^{(a-1)}$, $l^{(a)}$, $l^{(a+1)} \leq l^{(a)}$.

Let us consider $\delta_{l-1}$. To begin with, let us compare the locations of two letters 1 of $T$ corresponding to the first $\delta_l$ and $\delta_{l-1}$. According to the conclusion in the previous paragraph, we see that the first $\delta_{l-1}$ will remove a box from the same rows that are removed by the first $\delta_l$. Furthermore, the first $\delta_{l-1}$ may remove from leftward partitions that are not removed by the first $\delta_l$. Since the “left” of the rigged configurations corresponds to the “up” of the $T$, we obtain the statement in this case. Now we shall check that the box removing procedure of the first $\delta_{l-1}$ will not change the vacancy numbers for the rows that are removed by $\delta_l$’s except for the first $\delta_l$. For this, remind that the first $\delta_{l-1}$ removes a box from the same rows that are removed by the first $\delta_l$. Thus we can use the inequality at the end of the last paragraph to infer the property. In fact, we have $\Delta Q^{(a-1)} l^{(a)} = \Delta Q^{(a)} l^{(a)} = l^{(a+1)} - l^{(a)} = -1$, hence $\Delta p^{(a)} l^{(a)-1} = 0$. Thus we can use the same arguments to compare the second $\delta_{l-1}$ with the second $\delta_l$ to check the statement for this case. Recursively, we can check the statement for all $\delta_{l-1}$, and again recursively we can conclude the proof of the lemma.

$\square$

**Proposition 3.11.** Consider the map $\Psi : (\nu^*, J^*) \rightarrow ((\nu^*, J^*), T)$. Then the row word of $T$ satisfies the Yamanouchi condition.

**Proof.** Consider the subset of $T$ corresponding to the columns $h_1, h_{l-1}, \ldots, h_j$ and proceed by induction on $j$. For the initial case $j = l$, we see from Corollary 3.9 that the row word is 123····$h_l$ which is the Yamanouchi word. Assume that we have done for $h_1, h_{l-1}, \ldots, h_{j+1}$. Let the row word corresponding to the subset of $T$ corresponding to $h_1, h_{l-1}, \ldots, h_{j+1}$ be $w$. By the induction assumption $w$ is a Yamanouchi word. As we see in Corollary 3.9, the column $h_j$ will insert integers 1, 2, \ldots, $h_j$ to $w$ in this order. Let us rephrase Lemma 3.10 in the row word language. Then we see that each letter $i$ coming from the column $h_j$ will be inserted to the left of all $i$’s contained in $w$. Therefore insertion of the integers coming from the column $h_j$ does not violate the Yamanouchi condition.

$\square$

**Proof of Theorem 3.1**. By Lemma 3.7, $(\nu^*, J^*)$ is the type $\mathcal{Q} = \emptyset$ rigged configuration and the outer shape of $T$ coincides with the weight of $(\nu^*, J^*)$. Moreover, from definition of $\Psi$ we see that the inner shape of $T$ coincides with the weight of $(\nu^*, J^*)$ and from Corollary 3.9 we see that the weight of $T$ coincides with $\nu^{(\mathcal{Q})}$. Thus we have only to check that the resulting $T$ indeed satisfies the definition of the LR tableau. Recall that the heights of columns of $\nu^{(\mathcal{Q})}$ satisfy $h_j \geq h_{j+1}$. Recall also that the map $\Psi$ proceeds from $h_1$ to $h_l$.

Let us check that $T$ is a semi-standard tableau. Fix a particular entry of $T$, say $\alpha$. Assume that $\alpha$ corresponds to the column $h_j$. Denote the possible its neighbours
as in the following diagram.

\[
\begin{array}{c}
\alpha \\
\beta \\
\gamma
\end{array}
\]

If \( \beta \neq 0 \), let us check \( \alpha \leq \beta \). Consider the next column \( h_{j-1} \). According to Lemma 3.11 there are two possibilities. The first case is \( \alpha = \beta \), which immediately gives the statement. The other case is that when computing the column \( h_{j-1} \), \( \alpha \) appears strictly above and right of \( \beta \), \( \alpha \) appears strictly above and right of the previous \( \alpha \). Then \( \beta \) in the above diagram corresponds to the column \( h_i \) for some \( i < j \). By recursively using Lemma 3.11 we see that \( \alpha \) corresponding to the column \( h_i \) appears strictly above and right of the \( \alpha \) corresponding to the column \( h_j \). Recall from Corollary 3.9 that the entries corresponding to the column \( h_i \) forms a vertical strip and, moreover, its entry is strictly increasing integer sequence if we read from top. Within the vertical strip corresponding to the column \( h_i \), \( \beta \) in the above diagram appears strictly below \( \alpha \). Thus we have \( \alpha < \beta \), which gives the statement in this case. Next, let us check \( \alpha < \gamma \) if \( \gamma \neq \emptyset \). By Remark 3.3 shape of each subset of \( T \) corresponding to \( h_i, h_{i-1}, \ldots, h_j \) is a Young diagram. Thus \( \gamma \) in the above diagram corresponds to some column \( h_i \) for some \( i \leq j \). If \( \gamma \) corresponds to the same column \( h_j \), then we have \( \gamma = \alpha + 1 \geq \alpha \), which implies the statement in this case. On the contrary, suppose that \( \gamma \) corresponds to some column \( h_i \) \( (i < j) \). In this case, again we can compare \( \alpha \) and \( \gamma \) within the vertical strip corresponding to the column \( h_i \) to show \( \alpha < \gamma \). To summarize, we have checked that \( T \) is a semi-standard tableau.

Finally, from Proposition 3.11 we see that the row word of \( T \) satisfies the Yamanouchi condition. Thus \( T \) is the LR tableau. \( \square \)

3.4. Proof of Theorem 3.2. During this section, we denote by \( l(a) \) the length of the row of \( \nu(a) \) to which \( \delta_k \) has added a box. In other words, \( l(a) \) is the position of the added box. Then we have \( l(a-1) \geq l(a) \geq l(a+1) \). Again, during the computations of \( \Psi \) the quantum space does not change. Therefore we can neglect the effect of the quantum space when the arguments concern only with the differences of the vacancy numbers.

Proposition 3.12. For \( \diamondsuit = \square \), \( \diamondsuit b_k \) is well-defined. For \( \diamondsuit = \boxdot \) \( \delta^2_k \) is well-defined.

Proof. Again we have to check the positivity of the vacancy numbers and the inequality that the riggings are less than or equal to the corresponding vacancy numbers. During the proof, we use \( p(a) \) to express the vacancy numbers with respect to the image of \( \delta_k \) and the symbol \( \Delta \) to express the differences of the quantities for after \( \delta_k \) minus before \( \delta_k \).

Case 1. In order to check the well-definedness for \( \nu(a) (a \geq a^{\Diamond}) \), we can use case by case arguments similar to those appeared in Proposition 3.3. The rest of the proof does not depend on the specific choice of \( \Diamond \).

Case 2. Next we check the well-definedness for rows of \( \nu(a) (a < a^{\Diamond}) \) that are not added by \( \delta_k \). Assume \( l(a) = \infty \) if \( \nu(a) \) is not added by \( \delta_k \). Let \( j \) be the length of a row of \( \nu(a) \). Then \( \Delta p_j^{(a)} \) behaves as follows: \( \Delta p_j^{(a)} = 1 \) if \( l(a) > j \geq l(a+1) \), \( \Delta p_j^{(a)} = -1 \) if \( l(a-1) > j \geq l(a) \) and \( \Delta p_j^{(a)} = 0 \) otherwise. Thus the only case that may cause difficulty is \( l(a-1) > j \geq l(a) \). Let us call the length \( j \) row by \( A \) and let the corresponding rigging be \( J(\geq 0) \). By assumption the row \( A \) is not added by \( \delta_k \).
so it was not singular before application of \( \tilde{\delta}_k \), i.e., \( p_j^{(a)} - J > 0 \). Thus even if we have \( \Delta p_j^{(a)} = -1 \), we have \( p_j^{(a)} \geq J \geq 0 \), which assures the well-definedness.

**Case 3.** Consider the remaining case, i.e., the rows of \( \nu^{(a)} \) \((a < a')\) that are added by \( \tilde{\delta}_k \). In particular, we have to consider the case \( p_j^{(a)} = 0 \) before application of \( \tilde{\delta}_k \). If \( l^{(a-1)} = l^{(a)} \), there is no \( j \) such that \( \Delta p_j^{(a)} < 0 \) so that the well-definedness is assured. Thus we assume that \( l^{(a-1)} > l^{(a)} \) and lead to a contradiction. If there is a length \( l^{(a)} \) row of \( \nu^{(a)} \), then from \( p_j^{(a)} = 0 \) and \( l^{(a-1)} > l^{(a)} \), \( \tilde{\delta}_k \) would add a box to the length \( l^{(a)} \) row. This contradicts with the fact that \( \tilde{\delta}_k \) add a box to a length \( l^{(a)} - 1 \) row. Thus there is not a length \( l^{(a)} \) row of \( \nu^{(a)} \). Denote by \( S \) the shortest row among the rows of \( \nu^{(a)} \) that are strictly longer than \( l^{(a)} - 1 \). Let the length of the row \( S \) be \( s \). If there is no such a row, assume \( S = \emptyset \) and \( s = \infty \). Then from Lemma 2.1 we see that \( p_j^{(a)} = 0 \) (see the similar arguments in Proposition 5.6). If \( l^{(a-1)} \geq s \) then \( \tilde{\delta}_k \) would add a box to the row \( S \) which is the contradiction. Therefore we see that \( l^{(a-1)} < s \). Thus, for \( l^{(a)} < i \leq s \), we have the following properties as functions of \( i \):

\[
\begin{align*}
Q_i^{(a-1)} & : \text{strictly upper convex function due to the existence of the length } l^{(a-1)} \text{ row that satisfies } l^{(a)} < l^{(a-1)} < s. \\
Q_i^{(a)} & : \text{linear function,} \\
Q_i^{(a+1)} & : \text{upper convex function (including linear function case).}
\end{align*}
\]

Combining these three contributions, we conclude that \( p_i^{(a)} \) is a strictly upper convex function on the interval \( l^{(a)} \leq i \leq s \). However, as we have seen, \( p_i^{(a)} = 0 \) holds on this interval. This is the contradiction. Hence we conclude \( l^{(a-1)} = l^{(a)} \) which completes the proof of the proposition. 

**Lemma 3.13.** We follow the description at Definition 3.4. According to Step (i) we fix a group of letters contained in \( T \) and consider a pair of successive two integers within the group. Let the two letters of the pair be the \( k \)-th and the \( k' \)-th rows of \( T \) \((k > k')\), respectively. Let us consider \( \tilde{\delta}_k \circ \tilde{\delta}_{k'} \). Let \( \nu^{(a)} \) \((\text{resp. } l^{(a)})\) be the column of the added box by \( \tilde{\delta}_k \) \((\text{resp. } \tilde{\delta}_{k'})\). As usual, assume \( l^{(a)} = \infty \) \((\text{resp. } l^{(a)} = \infty)\) if \( \nu^{(a)} \) is not added by \( \tilde{\delta}_k \) \((\text{resp. } \tilde{\delta}_{k'})\). Then, as long as \( l^{(a)} < \infty \), we have \( l^{(a)} \leq l^{(a+1)} \).

**Proof.** We proceed by induction on \( a \). As the initial step, we show \( l^{(k-1)} \leq l^{(k)} \). Recall that by \( \tilde{\delta}_k \) the vacancy numbers for the rows of \( \nu^{(k-1)} \) that are longer than or equal to \( l^{(k)} \) are increased by 1, so that they are non singular and cannot be added by \( \tilde{\delta}_{k'} \). This forces to \( l^{(k-1)} - 1 < l^{(k)} \), i.e., \( l^{(k-1)} \leq l^{(k)} \). Note that \( \tilde{\delta}_{k'} \) will add to length \( l^{(k)} - 1 \) row of \( \nu^{(k)} \). As an induction hypothesis, for some \( k \leq a \), assume that we have \( l^{(a-1)} \leq l^{(a)} \). Since the vacancy numbers for the rows of \( \nu^{(a)} \) whose lengths \( j \) satisfy \( l^{(a)} > j \geq l^{(a+1)} \) are increased by 1 due to \( \tilde{\delta}_k \), they cannot be added by \( \tilde{\delta}_{k'} \). Then, combining this with the inequality \( l^{(a)} > l^{(a-1)} - 1 \geq l^{(a)} - 1 \), we have \( l^{(a+1)} > l^{(a)} - 1 \), i.e., \( l^{(a+1)} \geq l^{(a)} \). By induction, we finish the proof of the lemma. 

**Lemma 3.14.** We follow the description at Definition 3.4. According to Step (i) we fix a group of letters of \( T \). Once a row of \((\nu^*, J^*)\) becomes singular by the addition
of a box by \( \delta \), then the row remains singular during all the remaining procedure corresponding to the same group of \( T \).

**Proof.** As the proof is the same for all groups of letters of \( T \), we fix a group of the cardinality \( h \) and call it the group \( h \). Let the letter under consideration be integer \( j \) of the group \( h \) at the \( k \)-th row of \( T \). Denote by \( l^{(a)} \) (resp. \( l^{(a)} \)) the column coordinate of the box of \( \nu^{(a)} \) that is added by \( \delta_k \) (resp. \( \delta_{k'} \)), where \( k' \) is the row coordinate of the letter \( j + 1 \) of the group \( h \). Let us check that the rigging corresponding to the row that is added by \( \delta_k \) will not be changed by the rest of the procedures \( \delta_{k'}, \cdots \) corresponding to the rest of the letters \( j + 1, j + 2, \cdots, h \). From Lemma 3.13 we have \( l^{(a)} \leq l^{(a+1)} \) and from definition we have \( l^{(a)} \geq l^{(a+1)} \) so that we have \( l^{(a)} - 1 < l^{(a)} \). Thus \( \delta_k \) does not touch rows that are touched by \( \delta_k \), which implies invariance of the rigging. Recursively we see the invariance of the riggings during all the remaining procedure corresponding to the group \( h \).

Next we check the invariance of the vacancy numbers. Again, from Lemma 3.13 we have \( l^{(a-1)} \leq l^{(a)} \) and by definition we have \( l^{(a-1)} \geq l^{(a)} \geq l^{(a+1)} \). Thus the vacancy number for the length \( l^{(a)} \) rows of \( \nu^{(a)} \) will not change during the rest of the procedures corresponding to the group \( h \). Combining the invariance of the riggings and the vacancy numbers, the proof of the Lemma follows. \( \square \)

**Proof of Theorem 3.2** Consider the map \( \Psi : \{(\nu^*, J^*), T\} \mapsto (\nu^*, J^*) \). In Proposition 3.12 we checked the well-definedness of \( \Psi \). Thus the image \( (\nu^*, J^*) \) is a type \( \Box \) rigged configuration. Let us check that the weight of \( (\nu^*, J^*) \) coincides with the inner shape of \( T \). By assumption the outer shape of \( T \) coincides with the weight of \( (\nu^*, J^*) \). Then observe the fact that each \( \delta_k \) adds one box for each \( \nu^{(a)} (k \leq a) \).

On the level of the weight, this means that \( \delta_k \) removes one box from the \( k \)-th row of the Young diagram that represents the weight. By construction of \( \Psi \) we see that application of \( \Psi \) will remove from the Young diagram all boxes which correspond to filled boxes of \( T \). Thus we see that the weight of \( (\nu^*, J^*) \) is equal to the inner shape of \( T \).

Finally we have to check that the shape of weight of \( T \) is equal to \( \nu^{(a)} \). Let the cardinality of groups of \( T \) be \( h_1, h_2, \cdots, h_l \) where the labellings are according to the order in the procedure of \( \Psi \) (we will call the groups by their cardinality). In particular, we have \( h_i \geq h_{i+1} \). Then what we have to show is that the columns of \( \nu^{(a)} \) have height \( h_1, h_2, \cdots, h_l \) from left to right. The proof proceeds inductively on the number of groups of \( T \) that \( \Psi \) has processed. Consider the first group of the cardinality \( h_1 \). Let the length of the outer shape of \( T \) be \( N \). Then we have \( \nu^{(a)} = \emptyset \) for all \( N \leq a \). Thus corresponding to the letter \( h_1 \), \( \delta \) will create a new row for \( \nu^{(a)} = \emptyset (N \leq a) \). By applying Lemma 3.13 with \( l^{(a)} = 1 (N \leq a) \), we see that \( l^{(a)} = 1 (N \leq a) \), i.e., the second \( \delta \) corresponding to the letter \( h_1 - 1 \) will also create a new row for \( \nu^{(a)} (N \leq a) \). We can recursively continue the same argument and obtain the following result: after processing all letters of the group \( h_1 \) of \( T \), \( \nu^{(a)} (N \leq a) \) becomes the single column type partition whose height is \( h_1 \). Thus we have checked the assertion for the first group \( h_1 \).

Suppose that we have shown the property for the group \( h_i \). As the induction hypothesis, each \( \nu^{(a)} (N \leq a) \) has columns whose heights are \( h_1, h_2, \cdots, h_l \) from left to right. Let us apply Lemma 3.13 in this situation. Then all rows that are added by \( \delta \) corresponding to all letters of the group \( h_i \) remain as singular rows.
Recall that, by definition of $\hat{\Psi}$, the letter $j$ of the group $h_{i+1}$ is located on the same row or lower row of $T$ compared with the letter $j$ of the group $h_i$. To begin with let us consider the letter $h_{i+1}$ of the group $h_{i+1}$. Recall that $h_i \geq h_{i+1}$ so that there is also the letter $h_{i+1}$ within the group $h_i$. Then, on the level of $(\nu^*, J^*)$, the process corresponding with the letter $h_{i+1}$ of the group $h_{i+1}$ start from the same $\nu(a)$ or rightward $\nu(a)$ compared with the starting point of the process corresponding to the letter $h_{i+1}$ of the group $h_i$. Combining this with Lemma 3.13 we conclude that the operation $\delta$ corresponding to the letter $h_{i+1}$ of the group $h_{i+1}$ will add one more box to the same rows that are added by the process related with the letter $h_{i+1}$ of the group $h_i$. Especially we add the new column, i.e., the $(i+1)$-th column to each $\nu(a)$ ($N \leq a$). Observe that the procedure corresponding to the letter $h_{i+1}$ of the group $h_{i+1}$ will not violate the singular property of all rows that are added and set to be singular by the letters $h_{i+1} - 1, \ldots, 2, 1$ of the group $h_i$. This follows from the fact that the procedure for the letter $h_{i+1}$ of the group $h_{i+1}$ exactly follow that for the group $h_i$, hence we can apply Lemma 3.13 to claim the invariance of the vacancy numbers. Thus we can continue recursively the same arguments and obtain the following result: after processing all letters of the group $h_{i+1}$, each $\nu(a)$ ($N \leq a$) has $i + 1$ columns whose heights are $h_1, h_2, \ldots, h_{i+1}$ from left to right. This completes the induction step thereby finishing the whole proof of Theorem 3.12.

4. $M^\Diamond$ in terms of $M^\Box$

In this section we assume $\Diamond \neq \emptyset$. Note that $\gamma$ defined in section 2.4 is given by $\gamma = 2/|\Diamond|$ for $g^\Diamond$.

4.1. Change of statistic. Let $(\nu^*, J^*) \in RC^\Diamond$. Suppose the map $\delta$ sends $(\nu^*, J^*)$ to $(\tilde{\nu}^*, J^*)$, $k).$ Recall $l^*$ in Proposition 2.5. Let $\eta_a$ be the length of the row of $\nu(a)$ whose rightmost node is removed by $\delta$ for $k \leq a \leq l^*$.

**Lemma 4.1.**

$$(c(\nu) - c(\tilde{\nu})) = 2 \sum_{k \leq a < l^*} m_i(a) - \sum_{k \leq a < l^*} m_i(a) - \sum_{k \leq a < l^*} m_i(a+1) + \sum_{i \geq \eta_a} m_i(l) - \sum_{k \leq a < l^*} \left(1 - \delta_{\eta_a \eta_a+1}\right) - \frac{1}{2} \sum_{k \leq a < l^*} L_i(a).$$

**Proof.** Note that $m_i(a)$ changes to $m_i(a) - \delta_{\eta_a \eta_a+1}$ by $\delta$ and use Proposition 2.5 (3). A direct calculation shows the desired result. 

Let $\tilde{p}_i(a)$ be the vacancy number of $(\tilde{\nu}^*, J^*)$.

**Lemma 4.2.** For $k \leq a \leq l^*$,

$$\tilde{p}_{\eta_a} - \tilde{p}_{\eta_a-1} = \sum_{i \geq \eta_a} I_i(a) + \sum_{i \geq \eta_a} m_i(a-1) - 2 \sum_{i \geq \eta_a} m_i(a) + \sum_{i \geq \eta_a} m_i(a+1) + 1 - \delta_{\eta_a \eta_a+1}.$$

**Proof.** Direct calculation noting that $\eta_a - 1 \geq \eta_a \geq \eta_a + 1$.

**Proposition 4.3.** Suppose we get $(\tilde{\nu}^*, J^*)$ from $(\nu^*, J^*)$ by the map $\delta$. Then we have $c(\nu^*, J^*) - c(\tilde{\nu}^*, J^*) = -\gamma/2$. 

Proof. Let \( p_i^{(a)} \) be the vacancy number of \((\nu^*, J^*)\). By the definition of \( \delta \), we have
\[
\delta(\nu^*, J^*) - \delta(\tilde{\nu}^*, J^*) = (c(\nu) - c(\tilde{\nu})) + \gamma \sum_{k \leq a \leq l} (p_k^{(a)} - p_k^{(a-1)})
\]
where \( \eta_0 \) is the length of the row of \( \nu^{(a)} \) whose rightmost node is removed by \( \delta \) for \( k \leq a \leq l^* \). A direct calculation using Lemma 4.1 and 4.2 completes the proof. \( \square \)

Now we have the following theorem.

**Theorem 4.4.** For \( \Diamond = \Box \square \Box \) the stable fermionic formula \( M^{\Diamond} (\lambda, L; q) \) is expressed as a sum of \( M^{\Diamond} (\eta, L; q) \) as follows.
\[
M^{\Diamond} (\lambda, L; q) = q^{-\frac{1}{2}({|L| - |\lambda|})} \sum_{\mu \in P_{|L| - |\lambda|}^{\Diamond}, \eta \in P_L} c_{\lambda\mu}^{\Diamond} M^{\Diamond} (\eta, L; q^{\gamma})
\]
Here \( |L| = \sum_{a \in I_0, i \in \mathbb{Z}_{>0}} ai L_i^{(a)} \), \( P_{|L| - |\lambda|}^{\Diamond} \) is the set of partitions of \( N \) whose diagrams can be tiled by \( \Diamond \), and \( c_{\lambda\mu}^{\Diamond} \) is the Littlewood-Richardson coefficient.

**Proof.** In view of Theorem 3.3 Proposition 4.3 and the fact that the image of \( \Psi \) can be regarded as a rigged configuration of type \( \Diamond \) (although the statistic is multiplied by \( \gamma \)), it is sufficient to show

(i) \( \delta \) increases \( |\lambda| \) by 1,
(ii) in the image of \( \Psi \) we have \( |\lambda| = |L| \)

to prove the theorem.

Let \( \delta \) remove a box from \( \nu^{(a)} \) for \( a \geq k \). Then, from (2.7) it amounts to adding \( \alpha_k + \alpha_{k+1} + \cdots + \alpha_n = \epsilon_k \) (= standard orthonormal basis of the weight space) to \( \lambda \) when \( \Diamond = \Box \square \Box \). Hence, it increases \( |\lambda| \) by 1. If \( \Diamond = \Box \square \Box \), we see applying \( \delta \) two times increases \( |\lambda| \) by 2.

Next we show (ii). In the image of \( \Psi \), when \( k \geq \ell(\nu^*, J^*) \), from (2.8) we have
\[
0 = \frac{2}{(\alpha_k |\alpha_k|)} \sum_{b,i} (i L_i^{(b)} - \lambda_b)(a_k |a_k|).
\]
Since \( (\alpha_k |\alpha_k|) > 0 \) for any \( b \in I_0 \), we obtain \( \lambda_b = \sum_i i L_i^{(b)} \) for any \( b \in I_0 \), which concludes \( |\lambda| = \sum_b b \lambda_b = |L| \). \( \square \)

**Remark 4.1.** The minimum rank \( n \) that makes the theorem hold is determined by the condition that (3.1) is satisfied for any \( \nu = \nu^{(a)} \) such that \( c_{\lambda\mu}^{\Diamond} > 0 \). It is in fact given by
\[
\ell(\lambda) + \frac{|L| - |\lambda|}{\text{width}(\Diamond)} \leq a^{\Diamond}.
\]
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