Agut Masses

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Abstract. From considerations of the number of matrix elements of different orders of magnitude in the quark and charged lepton mass matrices, we suggest that the underlying gauge group responsible for the spectrum should have several—actually of the order of 7 to 10—cross product factors. This is taken to support our AGUT gauge group $SMG^3 \times U(1)_f$ which, under certain simple conditions, is the maximal group transforming the known 45 Weyl quark-lepton fields into each other. We describe the AGUT fit to the charged fermion mass spectrum, and briefly discuss baryogenesis and the neutrino mass problem.

1. Introduction

What is the origin of the well-known pattern of large ratios between the quark and lepton masses and of the small quark mixing angles? This is the problem of the hierarchy of Yukawa couplings in the Standard Model (SM). We suggest [1] that the natural resolution to this problem is the existence of some approximately conserved chiral charges beyond the SM. These charges, which we assume to be gauged, provide selection rules forbidding the transitions between the various left-handed and right-handed fermion states (except for the top quark).

For example, we suppose that there exists some charge (or charges) $Q$ for which the quantum number difference between left- and right-handed Weyl states is larger for the electron than for muon:

$$|Q_{eL} - Q_{eR}| > |Q_{\mu L} - Q_{\mu R}|$$

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It then follows that the SM Yukawa coupling for the electron $g_e$ is suppressed more than that for the muon $g_\mu$, when $Q$ is taken to be approximately conserved. This is what is required if we want to explain the electron-muon mass ratio.

In section 2 we give arguments motivating our identification of the above chiral gauge charges with those of the anti-grand unification theory (AGUT) based on the non-simple gauge group $SMG^3 \times U(1)_f$, where $SMG \equiv SU(3) \times SU(2) \times U(1)$. We also give a crude statistical argument for the number of cross-product factors in the gauge group beyond the SM, suggested by the observed fermion spectrum. In section 3 we discuss the structure of the AGUT gauge group and how it can be rather simply characterized, as the maximal gauge group satisfying a few simple principles. The Higgs fields responsible for breaking the AGUT gauge group $SMG^3 \times U(1)_f$ to the diagonal $SMG$ subgroup, identified as the SM gauge group, are considered in section 4. The structure of the resulting fermion mass matrices are presented in section 5, together with details of a fit to the charged fermion spectrum. In sections 6 and 7, we briefly discuss the problems of baryogenesis and neutrino oscillations respectively. Finally we present our conclusions in section 8.

2. Motivation for Anti-GUT

As pointed out in the introduction, the quark-lepton mass matrices—written in a basis of what we can call proto-flavours—have matrix elements typically suppressed relative to the electroweak scale ($<\phi_{WS}> = 246$ GeV) by rather large factors. We shall take the point of view that, in the fundamental theory beyond the SM, the Yukawa couplings allowed by gauge invariance are all of order unity and, similarly, all the mass terms allowed by gauge invariance are of order the fundamental mass scale of the theory—say the Planck scale. Then, apart from the matrix element responsible for the top quark mass, the quark-lepton mass matrix elements are only non-zero due to the presence of other Higgs fields having vacuum expectation values (VEVs) smaller (typically by one order of magnitude) than the fundamental scale. These Higgs fields will, of course, be responsible for breaking the fundamental gauge group $G$ down to the SM group. In order to generate a particular effective SM Yukawa coupling matrix element, it is necessary to break the symmetry group $G$ by a combination of Higgs fields with the appropriate quantum number combination $\Delta \vec{Q}$. When this $\Delta \vec{Q}$ is different for two matrix elements, they will typically deviate by a large factor. If we want to explain the observed spectrum of quarks and leptons in this way, it is clear that we need charges which—possibly in a complicated way—separate the generations and, at least for $t-b$ and $c-s$, also quarks in the same generation. Just using the usual simple $SU(5)$
GUT charges does not help, because both ($\mu_R$ and $e_R$) and ($\mu_L$ and $e_L$) have the same $SU(5)$ quantum numbers. So we prefer to keep each SM irreducible representation in a separate irreducible representation of $G$ and introduce extra gauge quantum numbers distinguishing the generations, by adding extra cross-product factors to the SM gauge group.

In order to be guided into a specific model and to convince ourselves that something like the model we propose should represent nature, we should like to estimate how complicated the gauge group $G$ should be. In principle we could estimate over how many different orders of magnitude the various matrix elements in the mass matrices should distribute themselves, because these matrix elements are to a large extent accessible to phenomenological—almost experimental—measurement. We could then adjust the degree of complication of the group—say the number of cross product factors in the gauge group $G$ that should be used.

We consider the three charged particle mass matrices—since we have only rather uncertain information on the neutrinos anyway. With 3 generations they clearly contain together $3 \times 9 = 27$ different matrix elements. The number of matrix elements that are essentially measurable in practice is estimated as ca 12, corresponding to the measurement of $3 \times 3$ masses and three mixing angles. How many different order of magnitude classes of matrix elements then needed will, of course, depend somewhat on the attitude as to when a couple of estimated matrix elements deviate by more than of “order of magnitude unity”, but we take roughly the number of classes to be around 7. We get this number 7 by saying that, of the “observable” 12 matrix elements in our model to be presented below, we have used 8 different orders of magnitude, but that there is one clear case in which there are used two different orders of magnitude in principle, although the experimental numbers give that the two matrix elements have the same order of magnitude: The matrix element giving the 2nd to 3rd generation mixing has, see eq. (33), the order $T^3$, while the one dominating the b-quark mass is of order $W T^2$, whereas they are experimentally numerically almost equal.

If now we could guess that the non-observed matrix elements would distribute themselves into classes with the same order of magnitude in much the same way—i.e. with much the same number of matrix elements in the same class—then we would come to around $\frac{7}{12} \times 27 \approx 16$ classes for the elements in all three matrices. This number, in succession, must now depend on how many combination possibilities there are for the breaking of the symmetries, in various ways, by means of the Higgs fields. Obviously there should be more classes the more pieces the gauge group consists of.

The connection between the number of cross product factors $n$, say in the gauge group $G = A_1 \times A_2 \times \cdots \times A_n$, and the number of order of magnitude classes of mass matrix elements may be seen crudely estimated in fig. 1.

This figure was constructed using three examples and a hand drawn
curve through the three points, based on a vague expectation of how it should be curved. The examples we chose were the SM itself, the maximal AGUT group model and the reduced AGUT model (in which one of the Higgs fields $S$ has a VEV of order unity in fundamental units) which we actually use to fit the experimental data. In the pure SM there are no suppression factors, and all 27 mass matrix elements are expected to belong to the same class and be of order the electroweak scale. Thus the SM corresponds to the point with the number of order of magnitude classes equal to one and number of cross product factors $n = 3$ on fig. 1. The other end of the curve corresponds to the maximal AGUT group $SMG^3 \times U(1)_f$, having $n = 10$ cross product factors. In this case the matrix elements are much more separated into different order of magnitude classes. As we shall see below the corresponding diagonal elements in each of the 3 Yukawa coupling matrices $Y_U$, $Y_D$ and $Y_E$ have the same order of magnitude. Furthermore we have $(Y_U)_{12} \simeq (Y_D)_{12}$ and $(Y_D)_{32} \simeq (Y_E)_{32}$, giving a total of 8 constraints and hence $27 - 6 - 2 = 19$ in principal different order of magnitude classes. If we take into account a couple of extra numerical coincidences in order of magnitude, the number of classes is reduced to 17. The third point on the curve corresponds to the AGUT model, in which the maximal AGUT group is replaced by the group to which it is broken by the non-suppressing Higgs field $S$, namely $SMG^2 \times U(1)$. The latter group
has 7 factors in the cross product, and the reduced Yukawa matrices satisfy 2 extra constraints: \((Y_U)_{12} = (Y_U)_{21}\) and \((Y_D)_{12} = (Y_E)_{21}\). Thus the number of different order of magnitude classes in this case is \(17 - 2 = 15\).

The idea now is to estimate the number of cross product factors in the “true” gauge group underlying the fermion mass spectrum, by looking up in figure 1 what number of factors would give the number of order of magnitude classes found phenomenologically. We get about 7 to 8 factors being suggested, not surprisingly since we used our model with seven cross product factors and the fact that it fits rather well to determine one point on the curve. In spite of this way of constructing the curve and what we consider the phenomenological number, we do not consider it an empty exercise to use our curve to estimate with what accuracy we must expect that a true model would have to have around the 7 cross product factors. Our point, of course, is to argue that only by a rather strange accident should it be possible for models, with a very different number of cross product factors, to fit the experimental data. In particular if the model had significantly less than 7 factors, there should be too many mass matrix elements not separated by the orders of magnitude found experimentally. It is anyway not possible to have more cross product factors than in the maximal AGUT model \((n = 10)\) and still have an anomaly free group.

At the end of the day, of course, the real motivation for considering the rather specific AGUT gauge group close to the Planck scale, with order unity fundamental Yukawa couplings, is its success in fitting the masses of the quarks and leptons and the quark mixing angles. This fit is discussed in section 5.

3. The “maximal” AGUT gauge group

The AGUT model is based on extending the SM gauge group \(SMG = S(U(2) \times U(3))\) in a similar way to grand unified \(SU(5)\), but rather to the non-simple \(SMG^3 \times U(1)_f\) group. Here we follow Michel and O’Raifeartaigh [2], in using not only the Lie algebra but even the Lie group for a Yang-Mills theory. The global properties of the \(SMG\) group imply that the SM particle representations of the \(SU(3) \times SU(2) \times U(1)\) Lie algebra should obey the charge quantisation rule

\[
\frac{y}{2} + \frac{1}{2} \text{“duality”} + \frac{1}{3} \text{“triality”} \equiv 0 \pmod{1} \quad (2)
\]

which expresses the somewhat complicated way in which the electric charge is quantised in the SM. Here \(y\) is the conventional weak hypercharge, and “duality” is 1 for the fundamental representation of \(SU(2)\) (the doublet) and 0 for the singlet. Similarly “triality” is 1 for the \(SU(3)\) triplet, −1 for the anti-triplet and 0 for the singlet.
The existence of the $SMG^3 \times U(1)_f$ group means that, near the Planck scale, there are three sets of SM-like gauge particles. Each set only couples to its own proto-generation [e.g. the proto- $u$, $d$, $e$ and $\nu_e$ particles], but not to the other two proto-generations [e.g. the proto- $c$, $s$, $\mu$, $\nu_\mu$, $t$, $b$, $\tau$ and $\nu_\tau$ particles]. There is also an extra abelian $U(1)_f$ gauge boson, giving altogether $3 \times 8 = 24$ gluons, $3 \times 3 = 9$ $W$'s and $3 \times 1 + 1 = 4$ abelian gauge bosons. The couplings of the $SMG_i = S(U(2) \times U(3))_i \approx SU(3)_i \times SU(2)_i \times U(1)_i$ group to the $i$'th proto-generation are identical to those of the SM group. Consequently we have a charge quantisation rule, analogous to eq. (2), for each of the three proto-generation weak hypercharge quantum numbers $y_i$.

To first approximation—namely in the approximation that the quark mixing angles $V_{us}$, $V_{cb}$, $V_{ub}$ are small—we may ignore the prefix “proto-”. However we really introduce in our model some “proto-fields” characterized by their couplings to the 37 gauge bosons of the $SMG^3 \times U(1)_f$ group. The physically observed $u$-quark, $d$-quark etc. are then superpositions of the proto-quarks (or proto-leptons), with the same named proto-particle dominating. Actually there is one deviation from this first approximation rule that proto-particles correspond to the same named physical particles. In the AGUT fit to the quark-lepton mass spectrum, discussed below, we find that to first approximation the right-handed components of the top and the charm quarks must be permuted:

$$c_R \text{ proto} \approx t_R \text{ physical} \quad t_R \text{ proto} \approx c_R \text{ physical}$$

But for all the other components we have:

$$t_L \text{ proto} \approx t_L \text{ physical} \quad b_R \text{ proto} \approx b_R \text{ physical}$$

and so on.

The AGUT group breaks down an order of magnitude or so below the Planck scale to the SM group, as the diagonal subgroup of its $SMG^3$ subgroup. The gauge coupling constants do not, of course, unify, but their values have been successfully calculated using the so-called multiple point principle.

At first sight, this $SMG^3 \times U(1)_f$ group with its 37 generators seems to be just one among many possible SM gauge group extensions. However, we shall now argue it is not such an arbitrary choice, as it can be uniquely specified by postulating 4 reasonable requirements on the gauge group $G$ beyond the SM. As a zeroth postulate, of course, we require that the gauge group extension must contain the Standard Model group as a subgroup $G \supseteq SMG$. In addition it should obey the following 4 postulates:

1. $G$ should transform the presently known (left-handed, say) Weyl particles into each other, so that $G \subseteq U(45)$. Here $U(45)$ is the group of
all unitary transformations of the 45 species of Weyl fields (3 generations with 15 in each) in the SM.

2. No anomalies, neither gauge nor mixed. We assume that only straightforward anomaly cancellation takes place and, as in the SM itself, do not allow for a Green-Schwarz type anomaly cancellation [4].

3. The fifteen irreducible representations of Weyl fields for the SM group remain irreducible under $G$. This is the most arbitrary of our assumptions about $G$. It is motivated by the observation that combining SM irreducible representations into larger unified representations introduces symmetry relations between Yukawa coupling constants, whereas the particle spectrum does not exhibit any exact degeneracies (except possibly for the case $m_b = m_{\tau}$). In fact AGUT only gets the naive $SU(5)$ mass predictions as order of magnitude relations: $m_b \approx m_{\tau}, m_s \approx m_\mu, m_d \approx m_e$.

4. $G$ is the maximal group satisfying the other 3 postulates. We already argued, in the previous section, that the large number of order of magnitude classes of fermion mass matrix elements indicates the need for a large number of cross product factors in $G$.

With these four postulates a somewhat cumbersome calculation shows that, modulo permutations of the various irreducible representations in the Standard Model fermion system, we are led to our gauge group $SMG^3 \times U(1)_f$. Furthermore it shows that the SM group is embedded as the diagonal subgroup of $SMG^3$, as in our AGUT model. The most difficult part of the calculation is to decide which identifications of the abelian groups have to be made in order to avoid anomalies, i.e. how big a subgroup of $U(1)^{15}$ can avoid having anomalies and be allowed in $G$. In searching for the generators of an allowed subgroup, one may expand them in terms of the generators for these 15 $U(1)$'s and they have to obey some first order (linear) relations for the coefficients, in order to avoid anomalies involving also the non-abelian or gravitational fields. Also there are third order relations that have to be satisfied, in order that there be no anomalies involving only the subspace of abelian generators. It turns out that, with the three proto-generations of fermions, there are too many constraints to be solved with an abelian subgroup of dimension higher than 4. It is found that three of the allowed abelian generators in $G$ can be taken to be the 3 weak hypercharges, each defined to act on only one proto-generation. After that choice the scheme becomes so tight that, apart from various rewritings and permutations of the particle names, there is a unique fourth $U(1)$ allowed and that is what we call $U(1)_f$. Several of the anomalies involving this $U(1)_f$ are cancelled by assigning equal and opposite values
of the $U(1)_f$ charge to the analogous particles belonging to second and third proto-generations, while the first proto-generation particles have just zero charge $[4]$. In fact the $U(1)_f$ group does not couple to the left-handed particles and the $U(1)_f$ quantum numbers can be chosen as follows for the proto-states:

$$Q_f(t_R) = Q_f(b_R) = Q_f(c_R) = 1$$  \hspace{1cm} (5)

$$Q_f(\mu_R) = Q_f(s_R) = Q_f(t_R) = -1$$  \hspace{1cm} (6)

Thus the quantum numbers of the quarks and leptons are uniquely determined in the AGUT model. However we do have the freedom of choosing the gauge quantum numbers of the Higgs fields responsible for the breaking the $SMG^3 \times U(1)_f$ group down to the SM gauge group. These quantum numbers are chosen with a view to fitting the fermion mass and mixing angle data $[3]$, as discussed in the next section.

4. Symmetry breaking by Higgs fields

There are obviously many different ways to break down the large group $G$ to the much smaller SMG. However, we can first greatly simplify the situation by assuming that, like the quark and lepton fields, the Higgs fields belong to singlet or fundamental representations of all non-abelian groups. The non-abelian representations are then determined from the $U(1)_i$ weak hypercharge quantum numbers, by imposing the charge quantisation rule eq. (3) for each of the $SMG_i$ groups. So now the four abelian charges, which we express in the form of a charge vector

$$\vec{Q} = \left( \frac{y_1}{2}, \frac{y_2}{2}, \frac{y_3}{2}, Q_f \right)$$

can be used to specify the complete representation of $G$. The constraint, that we must eventually recover the SM group as the diagonal subgroup of the $SMG_i$ groups, is equivalent to the constraint that all the Higgs fields (except for the Weinberg-Salam Higgs field which of course finally breaks the SMG) should have charges $y_i$ satisfying:

$$y = y_1 + y_2 + y_3 = 0$$  \hspace{1cm} (7)

in order that their SM weak hypercharge $y$ be zero.

We wish to choose the charges of the Weinberg-Salam (WS) Higgs field, so that it matches the difference in charges between the left-handed and right-handed physical top quarks. This will ensure that the top quark mass in the SM is not suppressed relative to the WS Higgs field VEV. However, as we remarked in the previous section, it is necessary to associate the physical right-handed top quark field not with the corresponding third
proto-generation field $t_R$, but rather with the right-handed field $c_R$ of the second proto-generation. Otherwise we cannot suppress the bottom quark and tau lepton masses. This is because, for the proto-fields, the charge differences between $t_L$ and $t_R$ are the same as between $b_L$ and $b_R$ and also between $\tau_L$ and $\tau_R$. So now it is simple to calculate the quantum numbers of the WS Higgs field $\phi_{WS}$:

$$\vec{Q}_{\phi_{WS}} = \vec{Q}_{c_R} - \vec{Q}_{t_L} = \left(0, \frac{2}{3}, 0, 1\right) - \left(0, 0, \frac{1}{6}, 0\right) = \left(0, \frac{2}{3}, -\frac{1}{6}, 1\right)$$  \hfill (8)

This means that the WS Higgs field will in fact be coloured under both $SU(3)_2$ and $SU(3)_3$. After breaking the symmetry down to the SM group, we will be left with the usual WS Higgs field of the SM and another scalar which will be an octet of $SU(3)$ and a doublet of $SU(2)$. This should not present any phenomenological problems, provided this scalar doesn’t cause symmetry breaking and doesn’t have a mass less than the few TeV scale. In particular an octet of $SU(3)$ cannot lead to baryon decay.

We can now choose the charges of the other Higgs fields in our model, by considering the charge differences between left-handed and right-handed fermions with the inclusion of the WS Higgs. Since we have the constraint of eq. (7), the charges of these Higgs fields must be chosen to span a 3 dimensional vector space of charges represented, for example, by $\frac{3y_2}{2}$, $\frac{y_3}{2}$ and $Q_f$ with $\frac{y_2}{2}$ being determined by eq. (7). This means that we will need at least 3 Higgs fields to break the gauge group down to the SMG. This gives us a lot of freedom, so we will choose the charges on these Higgs fields by considering phenomenological relations between fermion masses.

Since we are assuming that the fundamental Yukawa couplings are of order 1 but not exactly 1, we can only produce order of magnitude results. So we wish to choose, for example, 2 fermions with similar masses but not order of magnitude equal masses. We can then assume that the lighter fermion is suppressed relative to the heavier fermion by 1 Higgs with a VEV given approximately by the ratio of the 2 fermion masses. For example we would say that the bottom quark and tau lepton masses were of the same order of magnitude (remembering that we take all relations at the Planck scale). However we can take the following 2 ratios of effective Yukawa couplings to be significantly different from 1:

$$\frac{g_c}{g_b} \equiv \frac{<W>}{M_F} \approx \frac{1}{5}$$  \hfill (9)

$$\frac{g_\mu}{g_b} \equiv \frac{<T>}{M_F} \approx \frac{1}{13}$$  \hfill (10)

where we have defined 2 Higgs fields, $W$ and $T$, to have appropriate VEVs, relative to the fundamental mass scale $M_F$ of the theory, to cause $m_c$ and $m_\mu$ to be suppressed relative to $m_b$. 

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First we define $\tilde{b}$ to be the difference in charges between $b_L$ and $b_R$ proto-fields with the inclusion of the WS Higgs field. So we have:

$$\tilde{b} = \tilde{Q}_{b_L} - \tilde{Q}_{b_R} - \tilde{Q}_{WS} \quad (11)$$

Similarly we define $\tilde{c}$ and $\tilde{\mu}$ to be:

$$\tilde{c} = \tilde{Q}_{c_L} - \tilde{Q}_{t_R} + \tilde{Q}_{WS} \quad (12)$$
$$\tilde{\mu} = \tilde{Q}_{\mu_L} - \tilde{Q}_{\mu_R} - \tilde{Q}_{WS} \quad (13)$$

Note that $\tilde{c}$ has been defined using the $t_R$ proto-field, since we have essentially swapped the right-handed charm and top quarks. Also the charges of the WS Higgs field are added rather than subtracted for up-type quarks.

We observe that:

$$\tilde{b} + \tilde{c} + \tilde{\mu} = \tilde{0} \quad (14)$$

Now we can express these charges in terms of those of the Higgs fields. We can define:

$$\tilde{b} = \alpha \tilde{Q}_W + \beta \tilde{Q}_T + \tilde{Q}_X \quad (15)$$

where we have chosen the overall sign of the charges on the Higgs fields $W$ and $T$ so that $\alpha$ and $\beta$ are not negative. $\tilde{Q}_X$ is the total charges of all other Higgs fields used to suppress $m_b$ relative to $m_t$. We will assume that $\tilde{Q}_X$ cannot be expressed as a linear combination of $\tilde{Q}_W$ and $\tilde{Q}_T$. Now eqs. (11) and (15) require that:

$$\tilde{c} = \pm (\alpha + 1) \tilde{Q}_W \pm \beta \tilde{Q}_T \pm \tilde{Q}_X \quad (16)$$
$$\tilde{\mu} = \pm \alpha \tilde{Q}_W \pm (\beta + 1) \tilde{Q}_T \pm \tilde{Q}_X \quad (17)$$

The presence of the $\pm$ signs is due to the fact that we can use the fields $W^\dagger$ and $T^\dagger$ as well as $W$ and $T$.

So we can rewrite eq. (14) as:

$$
\begin{pmatrix}
3\alpha + 1 \\
\alpha + 1 \\
\alpha - 1 \\
-\alpha - 1
\end{pmatrix}
\tilde{Q}_W +
\begin{pmatrix}
3\beta + 1 \\
\beta - 1 \\
\beta + 1 \\
-\beta - 1
\end{pmatrix}
\tilde{Q}_T +
\begin{pmatrix}
3 \\
1 \\
1 \\
-1
\end{pmatrix}
\tilde{Q}_X = \tilde{0} \quad (18)
$$

where the 4 coefficients for each term correspond to the 4 combinations of signs in front of the terms in eqs. (16) and (17), giving 64 cases altogether.

All possible choices of coefficient of $\tilde{Q}_X$ are non-zero and, by assumption, $\tilde{Q}_X$ is linearly independent of $\tilde{Q}_W$ and $\tilde{Q}_T$; so eq. (18) cannot hold. We must therefore conclude that there are no Higgs fields, other than $W$ and $T$, used to suppress $m_b$ relative to $m_t$. So we must set $\tilde{Q}_X = \tilde{0}$. We can now use the fact that $\alpha$ and $\beta$ are not negative, along with the assumption that $\tilde{Q}_T$ is not directly proportional to $\tilde{Q}_W$, to conclude that eq. (18) requires that:

$$\alpha = \beta = 1 \quad (19)$$
Figure 2. Feynman diagram for bottom quark mass in the AGUT model. The crosses indicate the couplings of the Higgs fields to the vacuum.

and that the combination of signs is chosen so that:

\[ \vec{b} = \vec{Q}_W + \vec{Q}_T \]  \hspace{1cm} (20)

\[ \vec{c} = -2\vec{Q}_W + \vec{Q}_T \]  \hspace{1cm} (21)

\[ \vec{\mu} = \vec{Q}_W - 2\vec{Q}_T \]  \hspace{1cm} (22)

We note that this immediately implies the reasonably good Planck scale relation:

\[ g_b = \frac{\langle W \rangle < T >}{M_F M_F} \approx \frac{1}{65} \]  \hspace{1cm} (23)

This expression for the effective SM bottom quark Yukawa coupling constant arises from the Feynman diagram in figure 2. Here we have assumed the existence of a rich spectrum of vector-like Dirac fermions, with unsuppressed masses of the order of the fundamental mass scale \( M_F = M_{Planck} \), which provides the required intermediate states. Also the fundamental Yukawa couplings \( \lambda_i \) are taken of order unity.

It is now a simple matter to calculate the charges of the Higgs fields \( W \) and \( T \). We have:

\[ \vec{Q}_W = \frac{1}{3}(2\vec{b} + \vec{\mu}) = \left( 0, -\frac{1}{2}, \frac{1}{2}, -\frac{4}{3} \right) \]  \hspace{1cm} (24)

From this we can then calculate:

\[ \vec{Q}_T = \vec{b} - \vec{Q}_W = \left( 0, -\frac{1}{6}, \frac{1}{6}, -\frac{2}{3} \right) \]  \hspace{1cm} (25)

We notice that the charges of \( W \) and \( T \) do not cover the 2 dimensional space of charges \( \frac{y_1}{2} \) and \( Q_f \), since only even \( Q_f \) charges can be constructed with integer numbers of these Higgs fields. Therefore, since both \( W \) and \( T \) have \( \frac{y_1}{2} = 0 \), we will need at least 2 more Higgs fields to fully cover the 3-dimensional space.
dimensional charge space required to break $G$ down to the SM group. We will now choose 2 more Higgs fields which, together with $W$ and $T$, will fully cover this space.

Another parameter in the SM, which is within one order of magnitude from unity, is the mixing matrix element between the 1st and 2nd generations, which we associate with another Higgs field VEV:

$$V_{us} \equiv < \xi > \approx 0.2$$  \hspace{1cm} (26)

With the mass matrix texture in our model, $V_{us}$ is approximately given by the ratio of the mass matrix transition element from $d_L$ to $s_R$ to the transition from $s_L$ to $s_R$. This means that we must have:

$$\vec{Q}_\xi = \vec{Q}_{d_L} - \vec{Q}_{s_L} = \left( \frac{1}{6}, 0, 0, 0 \right) - \left( 0, \frac{1}{6}, 0, 0 \right) = \left( \frac{1}{6}, -\frac{1}{6}, 0, 0 \right)$$  \hspace{1cm} (27)

From the well-known Fritzsch relation\footnote{\textcite{Fritzsch1976}} $V_{us} \approx \sqrt{m_d/m_s}$, it is suggested that the two off-diagonal mass matrix elements connecting the d-quark and the s-quark be equally big. We achieve this approximately in our model by introducing a special Higgs field $S$, with quantum numbers equal to the difference between the quantum number differences for these 2 matrix elements in the down quark matrix. Then we postulate that this Higgs field has a VEV of order unity in fundamental units, so that it does not cause any suppression but rather ensures that the two matrix elements get equally suppressed. Henceforth we will consider the VEVs of the new Higgs fields as measured in units of $M_F$, and so we have:

$$< S > = 1$$  \hspace{1cm} (28)

and the charges of $S$ are given by:

$$\vec{Q}_S = [\vec{Q}_{s_L} - \vec{Q}_{d_R}] - [\vec{Q}_{d_L} - \vec{Q}_{s_R}]$$

$$= \left[ \left( 0, \frac{1}{6}, 0, 0 \right) - \left( \frac{1}{3}, 0, 0, 0 \right) \right] - \left[ \left( \frac{1}{6}, 0, 0, 0 \right) - \left( 0, -\frac{1}{3}, 0, -1 \right) \right]$$

$$= \left( \frac{1}{6}, -\frac{1}{6}, 0, -1 \right)$$  \hspace{1cm} (29)

The existence of a non-suppressing field $S$ means that we cannot control phenomenologically when this $S$-field is used. Thus the quantum numbers of the other Higgs fields $W$, $T$, $\xi$ and $\phi_{WS}$ given above have only been determined modulo those of the field $S$. 
5. Mass matrices, predictions

We define the mass matrices by considering the mass terms in the SM to be given by:

\[ \mathcal{L} = Q_L M_u U_R + Q_L M_d D_R + L_L M_l E_R + \text{h.c.} \]  

(30)

The mass matrices can be expressed in terms of the effective SM Yukawa matrices and the WS Higgs VEV by:

\[ M_f = Y_f \frac{\phi_{WS}}{\sqrt{2}} \]  

(31)

We can now calculate the suppression factors for all elements in the Yukawa matrices, by expressing the charge differences between the left-handed and right-handed fermions in terms of the charges of the Higgs fields. They are given by products of the small numbers denoting the VEVs, in fundamental units, of the fields \( W, T, \xi \) and the of order unity VEV of \( S \). In the following matrices, we simply write \( W \) instead of \( < W > \) etc. for the VEVs. With the quantum number choice given above, the resulting matrix elements are—but remember that “random” order unity factors are supposed to multiply all the matrix elements—for the uct-quarks:

\[
Y_U \approx \left( \begin{array}{ccc}
S^\dagger W^\dagger T^2 (\xi^\dagger)^2 & W^\dagger T^2 \xi & (W^\dagger)^2 T \xi \\
S W^\dagger T^2 (\xi^\dagger)^3 & W^\dagger T^2 & (W^\dagger)^2 T \\
S^\dagger (\xi^\dagger)^3 & 1 & W^\dagger T^\dagger
\end{array} \right)
\]  

(32)

the dsb-quarks:

\[
Y_D \approx \left( \begin{array}{ccc}
S W (T^\dagger)^2 \xi^2 & W (T^\dagger)^2 \xi & T^3 \xi \\
S W (T^\dagger)^2 \xi & W (T^\dagger)^2 & T^3 \\
S W^2 (T^\dagger)^4 \xi & W^2 (T^\dagger)^4 & WT
\end{array} \right)
\]  

(33)

and the charged leptons:

\[
Y_E \approx \left( \begin{array}{ccc}
S W (T^\dagger)^2 \xi^2 & W (T^\dagger)^2 (\xi^\dagger)^3 & (S^\dagger)^2 W T^4 \xi^4 \\
S W (T^\dagger)^2 \xi^5 & W (T^\dagger)^2 & (S^\dagger)^2 W T^4 \xi^2 \\
S^3 W (T^\dagger)^5 \xi^3 & (W^\dagger)^2 T^4 & WT
\end{array} \right)
\]  

(34)

We can now set \( S = 1 \) and fit the nine quark and lepton masses and three mixing angles, using 3 parameters: \( W, T \) and \( \xi \). That really means we have effectively omitted the Higgs field \( S \), and replaced the maximal AGUT gauge group \( SMG^3 \times U(1) \) by the reduced AGUT group \( SMG_{12} \times SMG_3 \times U(1) \), which survives the spontaneous breakdown due to \( S \). In order to find the best possible fit, we must use some function which measures how good a fit is. Since we are expecting an order of magnitude fit, this function
Table 1. Best fit to conventional experimental data. All masses are running masses at 1 GeV except the top quark mass which is the pole mass.

|      | Fitted      | Experimental |
|------|-------------|--------------|
| \(m_u\) | 3.6 MeV    | 4 MeV       |
| \(m_d\) | 7.0 MeV    | 9 MeV       |
| \(m_c\) | 0.87 MeV   | 0.5 MeV     |
| \(m_t\) | 1.02 GeV   | 1.4 GeV     |
| \(m_s\) | 400 MeV    | 200 MeV     |
| \(m_{\mu}\) | 88 MeV    | 105 MeV     |
| \(M_t\) | 192 GeV    | 180 GeV     |
| \(m_{\tau}\) | 8.3 GeV   | 6.3 GeV     |
| \(V_{us}\) | 1.27 GeV   | 1.78 GeV    |
| \(V_{cb}\) | 0.18       | 0.22        |
| \(V_{ub}\) | 0.018      | 0.041       |
| \(V_{ub}\) | 0.0039     | 0.0035      |

should depend only on the ratios of the fitted masses to the experimentally determined masses. The obvious choice for such a function is:

\[ \chi^2 = \sum \left[ \ln \left( \frac{m}{m_{\text{exp}}} \right) \right]^2 \]  

(35)

where \(m\) are the fitted masses and mixing angles and \(m_{\text{exp}}\) are the corresponding experimental values. The Yukawa matrices are calculated at the fundamental scale, which we take to be the Planck scale. We use the first order renormalisation group equations (RGEs) for the SM to calculate the matrices at lower scales.

We cannot simply use the 3 matrices given by eqs. (32)–(34) to calculate the masses and mixing angles, since only the order of magnitude of the elements is defined. Therefore we calculate statistically, by giving each element a random complex phase and then finding the masses and mixing angles. We repeat this several times and calculate the geometrical mean for each mass and mixing angle. In fact we also vary the magnitude of each element randomly, by multiplying by a factor chosen to be the exponential of a number picked from a Gaussian distribution with mean value 0 and standard deviation 1.

We then vary the 3 free parameters to find the best fit given by the \(\chi^2\) function. We get the lowest value of \(\chi^2\) for the VEVs:

\[ \langle W \rangle = 0.179 \]  

(36)

\[ \langle T \rangle = 0.071 \]  

(37)
The fitted value of \( \langle \xi \rangle \) is approximately a factor of two smaller than the estimate given in eq. (26). This is mainly because there are contributions to \( V_{us} \) of the same order of magnitude from both \( Y_U \) and \( Y_D \). The result of the fit is shown in table 1. This fit has a value of:

\[
\chi^2 = 1.87
\]  

This is equivalent to fitting 9 degrees of freedom (9 masses + 3 mixing angles - 3 Higgs VEVs) to within a factor of \( \exp(\sqrt{1.87/9}) \approx 1.58 \) of the experimental value. It is better than what one might have expected from an order of magnitude fit.

We can also fit to different experimental values of the 3 light quark masses by using recent results from lattice QCD, which seem to be consistently lower than the conventional phenomenological values. The best fit in this case is shown in table 2. The values of the Higgs VEVs are:

\[
\langle W \rangle = 0.123
\]
\[
\langle T \rangle = 0.079
\]
\[
\langle \xi \rangle = 0.077
\]

and this fit has a larger value of:

\[
\chi^2 = 3.81
\]

But even this is good for an order of magnitude fit.

6. Baryogenesis

A very important check of our model is whether or not it can be consistent with baryogenesis. In our model we have just the SM interactions up to about one or two orders of magnitude under the Planck scale. So we have no way, at the electroweak scale, to produce the baryon number in the universe. There is insufficient CP violation in the SM. Furthermore, even if created, the baryon number would immediately be washed out by sphaleron transitions after the electroweak phase transition. Our only chance to avoid the baryon number being washed out at the electroweak scale is to have a non-zero \( B - L \) (i.e. baryon number minus lepton number) produced from the high, i.e. Planck, scale action of the theory. That could then in turn give rise to the baryon number at the electroweak scale. Now, in our model, the \( B - L \) quantum number is broken by an anomaly involving the \( U(1)_f \) gauge group. This part of the gauge group in turn is broken by the Higgs field \( \xi \) which, in Planck units, is fitted to have an expectation value
around 1/10. The anomaly keeps washing out any net $B - L$ that might appear, due to CP-violating forces from the Planck scale physics, until the temperature $T$ of the universe has fallen to $\xi = 1/10$. The $U(1)_f$ gauge particle then disappears from the thermal soup and thus the conservation of $B - L$ sets in. The amount of $B - L$ produced at that time should then be fixed and would essentially make itself felt, at the electroweak scale, by giving rise to an amount of baryon number of the same order of magnitude.

The question now is whether we should expect, in our model, to have a sufficient amount of time reversal symmetry breaking, at the epoch when $B - L$ settles down to be conserved, that the amount of $B - L$ relative to the entropy (essentially the amount of 3 degree Kelvin background radiation) becomes large enough to agree with the well-known phenomenological value of the order of $10^{-9}$ or $10^{-10}$. We shall use purely dimensional arguments, assuming all couplings are generically of order unity, to estimate the effects of Planck scale physics. At the time of the order of the Planck scale, when the temperature was also of the order of the Planck temperature, we expect even the CP or time reversal violations were of order unity (in Planck units). So at that time there existed particles, say, with order of unity CP-violating decays. However, they had also, in our purely dimensional approximation, lifetimes of the order of the Planck scale too. Thus the $B - L$ biased decay products would mainly be dumped at time 1 in Planck units, rather than at the time of $B - L$ conservation setting in. In a radiation dominated universe, as we shall assume, the temperature is proportional to $1/a$ where

Table 2. Best fit using alternative light quark masses extracted from lattice QCD. All masses are running masses at 1 GeV except the top quark mass which is the pole mass.

|        | Fitted    | Experimental |
|--------|-----------|--------------|
| $m_u$  | 1.9 MeV   | 1.3 MeV      |
| $m_d$  | 3.7 MeV   | 4.2 MeV      |
| $m_s$  | 0.45 MeV  | 0.5 MeV      |
| $m_c$  | 0.53 GeV  | 1.4 GeV      |
| $m_t$  | 327 MeV   | 85 MeV       |
| $m_b$  | 75 MeV    | 105 MeV      |
| $M_t$  | 192 GeV   | 180 GeV      |
| $m_{\tau}$ | 6.4 GeV   | 6.3 GeV      |
| $m_{\tau}$ | 0.98 GeV  | 1.78 GeV     |
| $V_{ub}$ | 0.0054    | 0.0035       |
| $V_{cb}$ | 0.033     | 0.041        |
| $V_{us}$ | 0.15      | 0.22         |
a is the radius parameter—the size or scale parameter of the universe. Now the time goes as the square of this size parameter $a$. Thus the time in Planck units is given as the temperature to the negative second power:

$$t = \frac{0.3}{\sqrt{g_*} \times T^2}$$

(44)

where $g_*$ is the number of degrees of freedom—counted as 1 for bosons but as 7/8 per fermion degree of freedom—entering into the radiation density. In our model $g_*$ gets a contribution of $\frac{7}{8} \times 45 \times 2$ from the fermions and $2 \times 37$ from the gauge bosons, and in addition there is some contribution from the Higgs particles. So we take $g_*$ to be of order 100, in our crude estimate of the time $t$ corresponding to the temperature $T = \xi = \frac{1}{10}$ in Planck units, when $B - L$ conservation sets in:

$$t \simeq \frac{0.3}{\sqrt{100}} \times \left( \frac{1}{10} \right)^{-2} = 3$$

(45)

By that time we expect a fraction of the order of $\exp(-3)$ of the particles from the Planck era is still present and able to dump its CP-violating decay products. Of course here the uncertainty of, say, an order of magnitude in $t$ would appear in the exponent, meaning a suppression factor anywhere between $\exp(0)$ and $\exp(-30)$, which could thus easily be in agreement with the value wanted for baryogenesis of order $5 \times 10^{-10}$. This result is encouraging, but clearly a more careful analysis is required.

7. Neutrino oscillations, a problem?

We expect the neutrinos to get a mass in the AGUT model, by the exchange of the WS Higgs field $\phi_{WS}$ twice—the see-saw mechanism [9, 10]—or the exchange of a weak isosinglet Higgs field $\Delta$ [11]. In general it will also be necessary to exchange other AGUT Higgs scalars, in order to balance the AGUT gauge quantum numbers beyond the SM. This mechanism naturally generates an effective three generation light neutrino mass matrix $M_\nu$:

$$L_m = (M_\nu)_{ij} \nu_L^i C \nu_L^j + h.c.$$  

(46)

The neutrino mass matrix elements arise from Feynman diagrams similar to those of figure 2, but they involve two WS Higgs fields and the transitions are between $\nu_L$ and $\nu_R$. It follows that

$$M_\nu = H_\nu \frac{(\phi_{WS})^2}{2M_F}$$

(47)

where the dimensionless coupling matrix $H_\nu$ is analogous to the quark-lepton Yukawa matrices $Y_U$, $Y_D$ and $Y_E$, with elements expressed as products of the AGUT Higgs field VEVs like in eqs. (32)-(34).
Since the fundamental scale of the AGUT model is $M_F = M_{Planck}$, the overall neutrino mass scale in the model, as given by eq. (47), is:

$$\frac{\langle \phi WS \rangle^2}{2M_{Planck}} \sim 3 \times 10^{-6} \text{ eV}$$

(48)

This overall mass scale essentially provides an upper limit to the neutrino masses. So there is a basic problem for the AGUT model, since this mass scale is too small to explain neutrino oscillation phenomenology, except possibly for “just-so” vacuum oscillations of solar neutrinos.

The neutrino mass matrix $M_\nu$ is, by its very definition eq. (46), symmetric. Also, in models like AGUT with approximately conserved chiral $U(1)$ charges, the matrix elements are generally of different orders of magnitude, due to the presence of various suppression factors. Thus the generic structure for $M_\nu$ is a matrix in which the various elements typically each have their own order of magnitude, except in as far as they are forced to be equal by the symmetry $M_\nu = M_\nu^T$. The largest neutrino mass eigenvalue is then given by the largest matrix element of $M_\nu$. If it happens to be one of a pair of equal off-diagonal elements, we get two very closely degenerate states as the heaviest neutrinos with essentially maximal mixing; the third neutrino will be much lighter and, in first approximation, will not mix with the other two. In fact this is what happens in the AGUT model, if we assume that no new Higgs fields are introduced.

We can calculate the contributions to the neutrino mass matrix from the Higgs fields already introduced in our model and find:

$$H_\nu \sim \begin{pmatrix} (S^\dagger)^2(W^\dagger)^2T^4(\xi^\dagger)^4 & (S^\dagger)^2(W^\dagger)^2T^4\xi^\dagger & (W^\dagger)^2T(\xi^\dagger)^3 \\ (S^\dagger)^2(W^\dagger)^2T^4\xi^\dagger & W(T^\dagger)^5 & (W^\dagger)^2T \\ (W^\dagger)^2T(\xi^\dagger)^3 & (W^\dagger)^2T & S^2(W^\dagger)^2(T^\dagger)^2(\xi^\dagger)^2 \end{pmatrix}$$

(49)

where, as usual, we assume that all fundamental Yukawa couplings are of order 1. Clearly all the elements of $H_\nu$ are suppressed. The largest element is off-diagonal and of order $\langle W \rangle^2(T)$. We find the following eigenvalues for $H_\nu$:

$$h_{\nu_\mu} \simeq h_{\nu_\tau} \simeq \langle W \rangle^2(T) \simeq 2.3 \times 10^{-3}$$

(50)

$$h_{\nu_e} \simeq \langle W \rangle^2(T)^4\langle \xi \rangle^4 \simeq 7.8 \times 10^{-11}$$

(51)

There is almost maximal mixing ($\sin^2 2\theta = 1$) between $\nu_\mu$ and $\nu_\tau$ with very small mixing ($\sin^2 2\theta \simeq \langle \xi \rangle^6 \simeq 10^{-6}$) of $\nu_e$. This is not suitable for observable solar neutrino oscillations and, although we do have the correct mixing structure for atmospheric neutrino oscillations, the masses

$$m_{\nu_\mu} \simeq m_{\nu_\tau} \simeq h_{\nu_\tau} \frac{\langle \phi WS \rangle^2}{2M_F} \simeq 7 \times 10^{-9} \text{ eV}$$

(52)
are far too small.

So we predict no observable neutrino oscillations, unless we modify our AGUT model and introduce a new mass scale into the theory. Either some intermediate mass see-saw fermions or a weak isotriplet Higgs field $\Delta$ is required. The latter could acquire a vacuum expectation value of say $\langle \Delta^0 \rangle \sim 1$ eV, via its interaction with two WS Higgs fields $\phi_{WS}$ and the other Higgs fields $W, T, \xi$ and $S$; but only if, for some as yet unknown reason, it has a very small coefficient of $\Delta^2$ in the Higgs potential compared to $M_{Planck}^2$. Furthermore, it is difficult to explain both the solar and atmospheric neutrino problems in the above scenario where a single pair of equal off-diagonal elements dominates $M_\nu$. Although it can provide the large $\nu_\mu - \nu_\tau$ mixing needed for the atmospheric neutrino problem, their quasi-degeneracy in mass implies that their mass differences with $\nu_e$ are too large to explain the solar neutrino problem \[12\]. In fact it appears necessary for $M_\nu$ to have at least two independent large elements being of the same order of magnitude. Such an “accidental” order of magnitude degeneracy is perhaps not so unlikely, in light of our discussion, in section 2, of the number of order of magnitude degeneracies among the AGUT charged fermion mass matrix elements.

8. Conclusions

We have tried to motivate the AGUT model from the characteristic features of the quark-lepton masses and mixing angles, which point rather strongly to proto-flavour mass matrices having elements typically of different orders of magnitude. We considered roughly how many different orders of magnitude would be represented by the proto-flavour mass matrices, using experimental data and interpolating by theoretical considerations. In this way, we estimated that the number of these order of magnitude classes should be around 16. In turn this estimate suggests around 7 to 8 cross product factors in the gauge group responsible for the generation splittings etc.

The largest anomaly-free gauge group acting on just the 45 SM Weyl fermions, without any unification of the SM irreducible representations, is the AGUT group $SMG^3 \times U(1)_f$, which has 10 cross-product factors giving 37 generators in all. This is broken down to the reduced AGUT group $SMG^2 \times U(1)$ with 7 cross product factors (which we actually use to fit the mass spectrum), by the Higgs field $S$ with its VEV of order unity in Planck units.

Now, inspired by the experimental data, we introduced three more Higgs fields $W, T$ and $\xi$, with appropriate quantum numbers made to break the AGUT group down to the SM group. We then presented a rather good fit,
that in principle should only work to order of magnitude accuracy, to the charged quark-lepton masses and mixing angles, using three parameters corresponding to the VEVs of $W$, $T$ and $\xi$. The most characteristic feature is that, apart from the $t$ and $c$ quarks, the masses of the particles in the same generation are predicted to be degenerate, but only order of magnitude-wise, at the Planck scale. The worst feature of the fit is the deviation by a factor of about 2 between the fitted and experimental values for $m_s$ and $V_{cb}$. However this is what can be expected in an order of magnitude fit.

With this promising fit to the charged fermion masses, we considered the predictions of the model for the baryon number of the universe and neutrino masses. Using a crude dimensional argument, together with the expected number of degrees of freedom being of order 100, we obtain results for baryogenesis consistent with the observed baryon number, but with the uncertainty occurring inside an exponent. If anything we tend to get too high a prediction for the baryon number. However our predictions for the neutrino masses are set by a Planck mass see-saw mechanism, and are therefore too low to give observable neutrino oscillations. This suggests we may have to modify our model and introduce a new mass scale into the theory.

Apart from the neutrino puzzle, the AGUT model is successful in explaining the quark-lepton masses and mixing angles.

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