Optimal quadrature formulas in the sense of Sard in $W_2^{(m,m-1)}$ space

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Abstract This paper studies the problem of construction of optimal quadrature formulas in the sense of Sard in the $W_2^{(m,m-1)}(0, 1)$ space. Using the Sobolev’s method we obtain new optimal quadrature formulas of such type for $N + 1 \geq m$, where $N + 1$ is the number of the nodes. Moreover, explicit formulas of the optimal coefficients are obtained. We investigate the order of convergence of the optimal formula for $m = 1$ and prove an asymptotic optimality of such a formula in the Sobolev space $L_2^{(1)}(0, 1)$. It turns out that the error of the optimal quadrature formula in $W_2^{(1,0)}(0, 1)$ is less than the error of the optimal quadrature formula given in the $L_2^{(1)}(0, 1)$ space. The obtained optimal quadrature formula in the $W_2^{(m,m-1)}(0, 1)$ space is exact for $\exp(-x)$ and $P_{m-2}(x)$, where $P_{m-2}(x)$ is a polynomial of degree $m - 2$. Furthermore, some numerical results, which confirm the obtained theoretical results of this work, are given.

Keywords Optimal quadrature formulas · The error functional · Extremal function · Hilbert space · Optimal coefficients

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1 Introduction: statement of the problem

We consider the following quadrature formula

$$\int_0^1 \varphi(x) \, dx \approx \sum_{\beta=0}^{N} C_\beta \varphi(x_\beta), \quad (1)$$

with the error functional

$$\ell(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C_\beta \delta(x-x_\beta), \quad (2)$$

where $C_\beta$ are the coefficients and $x_\beta$ are the nodes of formula (1), $x_\beta \in [0, 1]$, $\varepsilon_{[0,1]}(x)$ is the indicator of the interval $[0, 1]$ and $\delta(x)$ is the Dirac’s delta-function.

We suppose that functions $\varphi(x)$ belong to the Hilbert space

$$W_2^{(m,m-1)}(0, 1) = \left\{ \varphi : [0, 1] \to \mathbb{R} \mid \varphi^{(m-1)} \text{ is absolutely continuous} \right\},$$

equipped with the norm

$$\| \varphi \|_{W_2^{(m,m-1)}(0, 1)} = \left\{ \int_0^1 (\varphi^{(m)}(x) + \varphi^{(m-1)}(x))^2 \, dx \right\}^{1/2}, \quad (3)$$

and $\int_0^1 (\varphi^{(m)}(x) + \varphi^{(m-1)}(x))^2 \, dx < \infty$.

The equality (3) is a semi-norm and $\| \varphi \| = 0$ if and only if $\varphi(x) = P_{m-2}(x) + d e^{-x}$, where $P_{m-2}(x)$ is a polynomial of degree $m - 2$ and $d$ is a constant.

It should be noted that for a linear differential operator of order $m$, $L \equiv P_m(\frac{d}{dx})$, Ahlberg et al. in the book [1, Chapter 6] investigated the Hilbert spaces $K_2(P_m)$ in the context of generalized splines. Namely, with the inner product

$$\langle \varphi, \psi \rangle = \int_0^1 L\varphi(x) \cdot L\psi(x) \, dx,$$

$K_2(P_m)$ is a Hilbert space if we identify functions that differ by a solution of $L\varphi = 0$. Also, such a type of spaces of periodic functions and optimal quadrature formulas were discussed in [6].
The difference

$$\langle \ell, \varphi \rangle = \int_0^1 \varphi(x) \, dx - \sum_{\beta=0}^N C_\beta \varphi(x_\beta) = \int_{-\infty}^{\infty} \ell(x) \varphi(x) \, dx$$

(4)

is called the error of the quadrature formula (1).

The error of the formula (1) is a linear functional in $W_2^{(m,m-1)}(0, 1)$, where $W_2^{(m,m-1)}(0, 1)$ is the conjugate space to the space $W_2^{(m,m-1)}(0, 1)$.

By the Cauchy–Schwarz inequality

$$|\langle \ell, \varphi \rangle| \leq \|\varphi\|_{W_2^{(m,m-1)}(0, 1)} \cdot \|\ell\|_{W_2^{(m,m-1)}(0, 1)}.$$ 

So, the error (4) of formula (1) is estimated with the help of the norm

$$\|\ell|_{W_2^{(m,m-1)}(0, 1)} = \sup_{\|\varphi\|_{W_2^{(m,m-1)}(0, 1)}=1} |\langle \ell, \varphi \rangle|$$

of the error functional (2). Consequently, the estimation of the error of the quadrature formula (1) over functions of the space $W_2^{(m,m-1)}(0, 1)$ is reduced to finding the norm of the error functional $\ell(x)$ in the conjugate space $W_2^{(m,m-1)*}(0, 1)$.

Obviously the norm of the error functional $\ell(x)$ depends on the coefficients $C_\beta$ and the nodes $x_\beta$. The problem of finding the minimum of the norm of the error functional $\ell(x)$ by coefficients $C_\beta$ and by nodes $x_\beta$, is called the Nikol’skii problem, and the obtained formula is called the optimal quadrature formula in the sense of Nikol’skii. This problem was first considered by Nikol’skii [17]. After this paper, this problem has been investigated by many authors for various cases (see e.g. [2–6,18,36] and references therein). Minimization of the norm of the error functional $\ell(x)$ by coefficients $C_\beta$ when the nodes are fixed is called Sard’s problem. And the obtained formula is called the optimal quadrature formula in the sense of Sard. This problem was first investigated by Sard [19].

There are several methods for constructing the optimal quadrature formulas in the sense of Sard such as the spline method, the $\varphi$-function method (see e.g. [3,27]) and the Sobolev’s method. Note that the Sobolev’s method is based on the construction of a discrete analog to a linear differential operator (see e.g. [30–32]). In different spaces based on these methods, the Sard problem was investigated by many authors (see, for example, [2,3,5,7,8,12–16,19–22,25–32,34,35] and references therein).

Furthermore, in the space $L_2^{(m)}$ the explicit expressions of the coefficients of the optimal quadrature formulas have been obtained in the works [13,20,21] for any $m \in \mathbb{N}$, with $N + 1$ nodes $x_\beta$ which are equally spaced. Here $L_2^{(m)}$ is the Sobolev space of functions which $(m - 1)$-st derivative is absolutely continuous and the $m$-th derivative is square integrable. In the work [25] the positiveness of the coefficients of optimal quadrature formulas in $L_2^{(m)}(0, 1)$ space is investigated.

It should be noted that a construction of optimal quadrature formulas in the sense of Sard, which are exact for solutions of linear differential equations, was given in
using the $\varphi$-function method, including several examples for some number of nodes.

The main aim of the present paper is to solve the Sard problem in
the space $W_2^{(m,m-1)}(0, 1)$ using the Sobolev’s method for any $N + 1$ nodes $x_\beta$, with $N + 1 \geq m$, i.e. to look for the coefficients $C_\beta$ that satisfy the following equality

$$
\|\ell| W_2^{(m,m-1)*}(0, 1)\| = \inf_{C_\beta} \|\ell| W_2^{(m,m-1)*}(0, 1)\|.
$$

(5)

Thus, to construct the optimal quadrature formula in the sense of Sard in the space $W_2^{(m,m-1)}(0, 1)$, we need to solve the following problems.

Problem 1 Find the norm of the error functional $\ell(x)$ of quadrature formulas (1) in the space $W_2^{(m,m-1)*}(0, 1)$.

Problem 2 Find the coefficients $C_\beta$ that satisfy equality (5) when the nodes $x_\beta$ are fixed.

The paper is organized as follows: in Sect. 2 the extremal function, which corresponds to the error functional $\ell(x)$, is found and, with its help, the representation of the norm of the error functional (2) is calculated, i.e. the Problem 1 is solved; in Sect. 3 we obtain the system of linear equations for the coefficients of the optimal quadrature formulas in the space $W_2^{(m,m-1)}(0, 1)$, moreover the existence and uniqueness of the solution of this system are proved; in Sect. 4 explicit formulas for the coefficients of optimal quadrature formula of the form (1) are found, i.e. the Problem 2 is solved; in Sect. 5 we calculate the norm of the error functional of the optimal quadrature formula in $W_2^{(1,0)}(0, 1)$ space, in Sect. 6 numerical results are showed.

2 The extremal function and the representation of the norm of the error functional

To solve the Problem 1, i.e., to calculate the norm of the error functional (2) in the space $W_2^{(m,m-1)*}(0, 1)$, we use the concept of the extremal function as follows. The function $\psi_\ell(x)$ is called the *extremal function* for the functional $\ell(x)$ (see [31]), if the following equality holds

$$
(\ell, \psi_\ell) = \|\ell| W_2^{(m,m-1)*}(0, 1)\| \cdot \|\psi_\ell| W_2^{(m,m-1)}(0, 1)\|.
$$

(6)

Since the space $W_2^{(m,m-1)}(0, 1)$ is a Hilbert space, then the extremal function $\psi_\ell(x)$ in this space, is found with the help of the general form of a linear continuous functional on Hilbert spaces given by the Theorem of Riesz. Then for the functional $\ell(x)$ and for any $\varphi(x) \in W_2^{(m,m-1)}(0, 1)$ there exists the function $\psi_\ell(x) \in W_2^{(m,m-1)}(0, 1)$ for which the following equation holds

$$
(\ell, \varphi) = (\psi_\ell, \varphi),
$$

(7)
where

\[ \langle \psi_\ell, \varphi \rangle = \int_0^1 \left( \psi^{(m)}_\ell(x) + \psi^{(m-1)}_\ell(x) \right) \left( \varphi^{(m)}(x) + \varphi^{(m-1)}(x) \right) \, dx \]  

is the inner product defined in the space \( W^{(m,m-1)}_2(0,1) \).

Further, we solve Eq. (7).

Suppose that \( \varphi(x) \) belongs to the space \( C^{(\infty)}(0,1) \), where \( C^{(\infty)}(0,1) \) is the space of the infinity differentiable, and finite, functions defined in the interval \( (0,1) \). Then from (8), integrating by parts, we obtain

\[ \langle \psi_\ell, \varphi \rangle = (-1)^m \int_0^1 \left( \psi^{(2m)}_\ell(x) - \psi^{(2m-2)}_\ell(x) \right) \varphi(x) \, dx. \]  

Keeping in mind (9), from (7) we get

\[ \psi^{(2m)}_\ell(x) - \psi^{(2m-2)}_\ell(x) = (-1)^m \ell(x). \]  

So, when \( \varphi(x) \in C^{(\infty)}(0,1) \) the extremal function \( \psi_\ell(x) \) is a solution of the Eq. (10).

But, we have to find the solution of Eq. (7) when \( \varphi(x) \in W^{(m,m-1)}_2(0,1) \). As is known from [31, Chapter 3], the space \( C^{(\infty)}(0,1) \) is dense in the space \( L^{(m)}_p(0,1) \). Then, from the definition of the space \( W^{(m,m-1)}_2(0,1) \), we get that the space \( C^{(\infty)}(0,1) \) is also dense in the space \( W^{(m,m-1)}_2(0,1) \). It should be noted that in [31, Theorem III.11] the density of the space \( C^{(\infty)}(0,1) \) in \( L_p, 1 \leq p < \infty \) is proved.

Since the space \( C^{(\infty)}(0,1) \) is dense in the space \( W^{(m,m-1)}_2(0,1) \), then we can approximate arbitrarily exact functions of the space \( W^{(m,m-1)}_2(0,1) \) by a sequence of functions of the space \( C^{(\infty)}(0,1) \).

Next for any \( \varphi(x) \in W^{(m,m-1)}_2(0,1) \) we consider the inner product \( \langle \psi_\ell, \varphi \rangle \) and, integrating by parts, we have

\[
\langle \psi_\ell, \varphi \rangle = \sum_{s=1}^{m-1} (-1)^s \varphi^{(m-s)}(x) \left( \psi^{(m+s)}_\ell(x) - \psi^{(m+s-2)}_\ell(x) \right) \bigg|_{x=1}^{x=0} \\
+ \varphi^{(m-1)}(x) \left( \psi^{(m)}_\ell(x) + \psi^{(m-1)}_\ell(x) \right) \bigg|_{x=1}^{x=0} \\
+ (-1)^m \int_0^1 \varphi(x) \left( \psi^{(2m)}_\ell(x) - \psi^{(2m-2)}_\ell(x) \right) \, dx.
\]
Hence from the arbitrariness of the choice of \( \varphi(x) \) and the uniqueness of the function \( \psi_\ell(x) \) (up to the function \( e^{-x} \) and polynomial of degree \( m - 2 \)), taking into account (10), the following equations have to be fulfilled

\[
\psi^{(2m)}_\ell(x) - \psi^{(2m-2)}_\ell(x) = (-1)^m \ell(x), \tag{11}
\]

\[
\left( \psi^{(m+s)}_\ell(x) - \psi^{(m+s-2)}_\ell(x) \right)|_{x=0} = 0, \quad s = 1, m - 1, \tag{12}
\]

\[
\left( \psi^{(m)}_\ell(x) + \psi^{(m-1)}_\ell(x) \right)|_{x=0} = 0. \tag{13}
\]

Thus, we conclude that the extremal function \( \psi_\ell(x) \) is the solution of the boundary value problem (11)–(13).

The following holds

**Theorem 2.1** The solution of the boundary value problem (11)–(13) is the extremal function \( \psi_\ell(x) \) of the error functional \( \ell(x) \) and has the following form

\[
\psi_\ell(x) = (-1)^m \ell(x) \ast G(x) + P_{m-2}(x) + de^{-x},
\]

where

\[
G(x) = \frac{\text{sign}x}{2} \left( \frac{e^x - e^{-x}}{2} - \sum_{k=1}^{m-1} \frac{x^{2k-1}}{(2k-1)!} \right) \tag{14}
\]

is a solution of the equation

\[
\psi^{(2m)}(x) - \psi^{(2m-2)}(x) = \delta(x), \tag{15}
\]

where \( d \) is any real number and \( P_{m-2}(x) \) is a polynomial of degree \( m - 2 \).

**Proof** As is very well known, the general solution of a nonhomogeneous differential equation consists of a sum of a particular solution of the nonhomogeneous differential equation and the general solution of the corresponding homogeneous differential equation.

The homogeneous equation for differential equation (11) has the form

\[
\psi^{(2m)}_\ell(x) - \psi^{(2m-2)}_\ell(x) = 0.
\]

It is easy to show that the general solution of this equation is given by the expression

\[
P_{2m-3}(x) + d_1 e^x + d_2 e^{-x}. \tag{16}
\]

It is not difficult to verify that a particular solution of the differential equation (11) is

\[
(-1)^m \ell(x) \ast G(x),
\]
where $G(x)$ is a fundamental solution of Eq. (11) and $G(x)$ is defined by (14), * is operation of convolution, where convolution of two functions is defined as

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y) \, dy = \int_{-\infty}^{\infty} f(y)g(x-y) \, dy.$$  

The rule of finding a fundamental solution of a linear differential operator

$$L \equiv \frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_n,$$

where $a_j$ are constants, is given in [33, p. 88]. Using this rule, it is found the function $G(x)$, which is a fundamental solution of the operator $d^{2m}_{2m} - d^{2m-2}_{2m}$ and has the form (14).

Thus, we have the following general solution of the Eq. (11)

$$\psi_\ell(x) = (-1)^m \ell(x) * G(x) + P_{2m-3}(x) + d_1 e^x + d_2 e^{-x},$$  

where $P_{2m-3}(x) = b_{2m-3}x^{2m-3} + \cdots + b_1 x + b_0$ is a polynomial of degree $2m - 3$ and $d_1, d_2$ are constants.

Since the function $\psi_\ell(x)$ is unique in the space $W^{(m,m-1)}_{2}(0, 1)$ (up to the function $e^{-x}$ and polynomial of degree $m - 2$), it has to satisfy the conditions (12) and (13). Here the derivative is in the generalized sense and

$$\psi^{(k)}_\ell(x) = (-1)^m \ell(x) * G^{(k)}(x) + P^{(k)}_{2m-3}(x) + d_1 e^x + (-1)^k d_2 e^{-x},$$  

$k = 1, 2, \ldots, 2m - 1$,

where

$$G^{(k)}(x) = \frac{\text{sign} x}{2} \left\{ \begin{array}{ll} \frac{e^x - e^{-x}}{2} - \frac{2m-3-k}{(2m-3-k)!} - \cdots - \frac{1}{2!} - x, & \text{when } k \text{ is even}, \\ \frac{e^x + e^{-x}}{2} - \frac{2m-3-k}{(2m-3-k)!} - \cdots - \frac{1}{2!} - 1, & \text{when } k \text{ is odd}. \end{array} \right.$$

From conditions (12), for $s = m - 1$ we get

$$\psi^{(2m-1)}_\ell(x) - \psi^{(2m-3)}_\ell(x) = (-1)^m \ell(x) * \frac{\text{sign} x}{2} - (2m - 3)!b_{2m-3}$$  

$$= (-1)^m \left( \ell(y), \frac{\text{sign}(x-y)}{2} \right) - (2m - 3)!b_{2m-3}.$$  

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Hence, for \( x = 0 \) and \( x = 1 \) we have

\[
\psi^{(2m-1)}_\ell(0) - \psi^{(2m-3)}_\ell(0) = (-1)^m \left( \ell(y), \frac{\text{sign}(-y)}{2} \right) - (2m - 3)!b_{2m-3} \\
= - \frac{(-1)^m}{2} (\ell(y), 1) - (2m - 3)!b_{2m-3} = 0,
\]

\[
\psi^{(2m-1)}_\ell(1) - \psi^{(2m-3)}_\ell(1) = (-1)^m \left( \ell(y), \frac{\text{sign}(1-y)}{2} \right) - (2m - 3)!b_{2m-3} \\
= \frac{(-1)^m}{2} (\ell(y), 1) - (2m - 3)!b_{2m-3} = 0.
\]

Then, from the last two equations, we get

\[
b_{2m-3} = 0, \text{ and } (\ell(y), 1) = 0. \tag{18}
\]

When \( s = m - 2 \), from conditions (12), we have

\[
\psi^{(2m-2)}_\ell(x) - \psi^{(2m-4)}_\ell(x) = (-1)^m \ell(x) \ast \frac{x \cdot \text{sign}x}{2} \\
- (2m - 3)!b_{2m-3}x - (2m - 4)!b_{2m-4}.
\]

Hence, taking into account (18), for \( x = 0 \) and \( x = 1 \) we obtain

\[
(-1)^m \left( \ell(y), \frac{0 - y}{2} \text{sign}(0 - y) \right) - (2m - 4)!b_{2m-4} \\
= \frac{(-1)^m}{2} (\ell(y), y) - (2m - 4)!b_{2m-4} = 0,
\]

\[
(-1)^m \left( \ell(y), \frac{1 - y}{2} \text{sign}(1 - y) \right) - (2m - 4)!b_{2m-4} \\
= - \frac{(-1)^m}{2} (\ell(y), y) - (2m - 4)!b_{2m-4} = 0.
\]

From here

\[
b_{2m-4} = 0 \text{ and } (\ell(y), y) = 0, \tag{19}
\]

and so on.

Continuing in this manner, for \( s = m - 3, m - 4, \ldots, 2, 1 \) we obtain

\[
b_{m-2+s} = 0, (\ell(y), y^{m-1-s}) = 0. \tag{20}
\]

Combining (18), (19) and (20) we get

\[
b_{2m-3-s} = 0, \quad s = 0, 1, \ldots, m - 2, \tag{21}
\]
and
\[(\ell(y), y^s) = 0, \ s = 0, 1, \ldots, m - 2. \tag{22}\]

From the condition (13), keeping in mind (21), we have
\[
\psi^m_{\ell}(x) + \psi^{m-1}_{\ell}(x) = (-1)^m \ell(x) \ast \left[ \frac{\text{sign} x}{2} \left( e^x - \frac{x^{m-2}}{(m-2)!} - \frac{x^{m-3}}{(m-3)!} - \ldots - x - 1 \right) \right] + d_1 e^x.
\]

Hence for \(x = 0\), taking into account (22), we obtain
\[
\psi^m_{\ell}(0) + \psi^{m-1}_{\ell}(0) = (-1)^m \left( \ell(y), \frac{\text{sign}(0 - y)}{2} \cdot \left( e^{-y} - \frac{(-y)^{m-2}}{(m-2)!} \right. \right.
\]
\[
\left. \left. - \frac{(-y)^{m-3}}{(m-3)!} - \ldots + y - 1 \right) \right) + d_1
\]
\[
= -\frac{(-1)^m}{2} \left( \ell(y), e^{-y} \right) + d_1 = 0
\]

and, for \(x = 1\) we get
\[
\psi^m_{\ell}(1) + \psi^{m-1}_{\ell}(1) = (-1)^m \left( \ell(y), \frac{\text{sign}(1 - y)}{2} \cdot \left( e^{1-y} - \frac{(1 - y)^{m-2}}{(m-2)!} \right. \right.
\]
\[
\left. \left. - \frac{(1 - y)^{m-3}}{(m-3)!} - \ldots - (1 - y) - 1 \right) \right) + d_1 e
\]
\[
= \frac{(-1)^m e}{2} \left( \ell(y), e^{-y} \right) + d_1 e = 0.
\]

Whence
\[d_1 = 0, \tag{23}\]

and
\[(\ell(y), e^{-y}) = 0. \tag{24}\]

Taking into account the equalities (21)–(24) and denoting \(d_2 = d\), we get the assertion of the theorem. Thus Theorem 2.1 is proved. \(\square\)

The equalities (22) and (24) mean that our quadrature formula is exact for the function \(e^{-x}\) and for any polynomial of degree \(\leq m - 2\).

It should be noted that to ensure the solvability of the system (22), (24) with respect to the coefficients \(C_{\beta}(\beta = 0, 1, \ldots, N)\), the condition \(N + 1 \geq m\) has to be imposed.
Now, using the result of Theorem 2.1, we immediately obtain the representation of the square of the norm of the error functional (2)

$$\|\ell |W_2^{(m,m-1)}*(0,1)\|^2 = (\ell, \psi_{\ell}) = (-1)^m \left[ \sum_{\beta=0}^{N} \sum_{\gamma=0}^{\beta} C_{\beta} C_{\gamma} G(x_\beta - x_\gamma) \right]$$

$$-2 \sum_{\beta=0}^{N} C_{\beta} \int_0^1 G(x - x_\beta) dx + \int_0^1 \int_0^1 G(x - y) dx dy \right] . \tag{25}$$

Thus the Problem 1 is solved.

Further in Sects. 3 and 4 we solve the Problem 2.

3 The system for the coefficients of optimal quadrature formulas in the space $W_2^{(m,m-1)}(0,1)$

Assume that the nodes $x_\beta$ of the quadrature formula (1) are fixed. The error functional (2) satisfies conditions (22) and (24). The norm of the error functional $\ell(x)$ is a multidimensional function with respect to the coefficients $C_\beta (\beta = 0, N)$. For finding the point of the conditional minimum of the expression (25), under the conditions (22) and (24), we apply the Lagrange method.

We denote $C = (C_0, C_1, \ldots, C_N)$ and $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{m-1})$.

Consider the function

$$\Psi(C, \lambda) = \|\ell(x)\|^2 - 2(-1)^m \sum_{\alpha=0}^{m-2} \lambda_\alpha \left( \ell(x), x^\alpha \right) - 2(-1)^m \lambda_{m-1} \left( \ell(x), e^{-x} \right).$$

Equating to 0 the partial derivatives of $\Psi(C, \lambda)$ by $C_\beta (\beta = 0, N)$ and $\lambda_0, \lambda_1, \ldots, \lambda_{m-1}$, we get the following system of linear equations

$$\sum_{\gamma=0}^{N} C_{\gamma} G(x_\beta - x_\gamma) + \sum_{\alpha=0}^{m-2} \lambda_\alpha x_\beta^\alpha + \lambda_{m-1} e^{-x_\beta} = f_m(x_\beta), \quad \beta = 0, N, \tag{26}$$

$$\sum_{\gamma=0}^{N} C_{\gamma} x_\gamma^\alpha = \frac{1}{\alpha + 1}, \quad \alpha = 0, 1, \ldots, m - 2, \tag{27}$$

$$\sum_{\gamma=0}^{N} C_{\gamma} e^{-x_\gamma} = 1 - e^{-1}, \tag{28}$$

where $G(x)$ is defined by the equality (14),

$$f_m(x_\beta) = \int_0^1 G(x - x_\beta) dx. \tag{29}$$
The system (26)–(28) has a unique solution and this solution gives the minimum to $\|\ell\|^2$ under the conditions (27), (28). The uniqueness of the solution of such type of systems is discussed in [31,32].

It should be noted that the existence and uniqueness of optimal quadrature formulas in the sense of Sard is also investigated in [14].

Now in (25) we make the change of variables $C_\beta = C_\beta + C_1\beta$. Then (25) and the system (26)–(28) have the following form

$$
\|\ell\|^2 = (-1)^m \left[ \sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} \overline{C}_\beta \overline{C}_\gamma G(x_\beta - x_\gamma) - 2 \sum_{\beta=0}^{N} (\overline{C}_\beta + C_1\beta) \int_0^1 G(x-x_\beta) \, dx \\
+ \sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} (2\overline{C}_\beta C_{1\gamma} + C_1\beta C_{1\gamma}) G(x_\beta - x_\gamma) + \int_0^1 \int_0^1 G(x-y) \, dx \, dy \right],
$$

(30)

$$
\sum_{\gamma=0}^{N} \overline{C}_\gamma G(x_\beta - x_\gamma) + \sum_{\alpha=0}^{m-2} \lambda_\alpha x_\beta^\alpha + \lambda_{m-1} e^{-x_\beta} = F_m(x_\beta), \quad \beta = 0, N,
$$

(31)

$$
\sum_{\gamma=0}^{N} \overline{C}_\gamma x_\gamma^\alpha = 0, \quad \alpha = 0, m-2,
$$

(32)

$$
\sum_{\gamma=0}^{N} \overline{C}_\gamma e^{-x_\gamma} = 0,
$$

(33)

where $F_m(x_\beta) = f_m(x_\beta) - \sum_{\gamma=0}^{N} C_{1\gamma} G(x_\beta - x_\gamma)$ and $C_{1\beta}$ is a partial solution of Eqs. (27), (28).

Hence, we directly deduce that the minimization of (25), under the conditions (27), (28), with respect to $C_\beta$, is equivalent to the minimization of the expression (30), with respect to $\overline{C}_\beta$, under the conditions (32), (33). Therefore it is sufficient to prove that the system (31)–(33) has a unique solution with respect to unknowns $\overline{C} = (\overline{C}_0, \overline{C}_1, \ldots, \overline{C}_N), \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{m-1})$ and this solution gives the minimum to the expression $\|\ell\|^2$.

From the theory of the conditional extremum, we know that the sufficient condition, in which the solution of the system (31)–(33) gives the conditional minimum to the expression $\|\ell\|^2$ on the manifold (32), (33), consists on the positiveness of the quadratic form

$$
\Phi(\overline{C}) = \sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} \frac{\partial^2 \Psi}{\partial \overline{C}_\beta \partial \overline{C}_\gamma} \overline{C}_\beta \overline{C}_\gamma,
$$

(34)

on the set of the vectors $\overline{C} = (\overline{C}_0, \overline{C}_1, \ldots, \overline{C}_N)$, under the condition
\[ S \overline{C} = 0, \quad (35) \]

where \( S \) is the following matrix of Eqs. (32), (33):

\[
S = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
x_0 & x_1 & \cdots & x_N \\
\vdots & \vdots & \ddots & \vdots \\
x_0^{m-2} & x_1^{m-2} & \cdots & x_N^{m-2} \\
e^{-x_0} & e^{-x_1} & \cdots & e^{-x_N}
\end{pmatrix}.
\]

We show that in our case this condition is fulfilled.

**Theorem 3.1** For any nonzero vector \( \overline{C} \in R^{N+1} \), lying in the subspace \( S \overline{C} = 0 \), the function \( \Phi(\overline{C}) \) is strictly positive.

**Proof** Using the definition of the function \( \Psi(C, \lambda) \) and Eqs. (30), (32), (33), from (34) we get

\[
\Phi(\overline{C}) = 2(-1)^m \sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} G(x_\beta - x_\gamma) \overline{C}_\beta \overline{C}_\gamma. \quad (36)
\]

Consider the linear combination of delta functions

\[
\delta_{\overline{C}}(x) = \sqrt{2} \sum_{\beta=0}^{N} \overline{C}_\beta \delta(x - x_\beta). \quad (37)
\]

By virtue of the condition (35) this functional belongs to the space \( W_{2}^{(m,m-1)}(0, 1) \). So, it has the extremal function \( u_{\overline{C}}(x) \in W_{2}^{(m,m-1)}(0, 1) \) which is a solution of the equation

\[
\left( \frac{d^2}{dx^{2m}} - \frac{d^2}{dx^{2m-2}} \right) u_{\overline{C}}(x) = (-1)^m \delta_{\overline{C}}(x). \quad (38)
\]

As \( u_{\overline{C}}(x) \) we can take the following linear combination of shifts of the fundamental solution \( G(x) \)

\[
u_{\overline{C}}(x) = \sqrt{2}(-1)^m \sum_{\beta=0}^{N} \overline{C}_\beta G(x - x_\beta).
\]

The square of its norm in the space \( W_{2}^{(m,m-1)}(0, 1) \) coincide with \( \Phi(\overline{C}) \), i.e.

\[
\|u_{\overline{C}}(x)\|_{W_{2}^{(m,m-1)}(0, 1)}^2 = (\delta_{\overline{C}}(x), u_{\overline{C}}(x)) = 2(-1)^m \sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} \overline{C}_\beta \overline{C}_\gamma G(x_\beta - x_\gamma).
\]
Hence, it is obvious that for any nonzero $\overline{C}$ the function $\Phi \left( \overline{C} \right)$ is strictly positive.

Theorem 3.1 is proved. $\square$

If the nodes $x_0, x_1, \ldots, x_N$ are selected such that the matrix $S$ has the right inverse matrix, then the system (31)–(33) has a unique solution. Then the system (26)–(28) has also a unique solution.

**Theorem 3.2** If the matrix $S$ has the right inverse matrix, then the main matrix $Q$ of the system (31)–(33) is nonsingular.

**Proof** We denote by $M$ the matrix of quadratic form $\left( \frac{-1}{2} \right)^m \Phi \left( \overline{C} \right)$, where $\Phi \left( \overline{C} \right)$ is defined by equality (36). As is very well known, if the homogenous system of linear equations has only the trivial solution then the corresponding nonhomogeneous system has a unique solution. Consider the homogeneous system corresponding to the system (31)–(33) in the following matrix form

$$Q \left( \overline{C}, \lambda \right) = \begin{pmatrix} M & S^* \\ S & 0 \end{pmatrix} \begin{pmatrix} \overline{C} \\ \lambda \end{pmatrix} = 0,$$

where $S^*$ is the transposed matrix to the matrix $S$.

We verify that the unique solution of (39) is identically zero. Suppose $(\overline{C}, \lambda)^*$ is the solution of (39). Consider the function $\delta \overline{C}(x)$, which is determined by the equality (37). As the extremal function for the function $\delta \overline{C}(x)$ we take the following function

$$u_{\overline{C}}(x) = \sqrt{2} \left( \sum_{\beta=0}^N \overline{C}_\beta G(x - x_\beta) + \sum_{\alpha=0}^{m-2} \lambda_\alpha x^\alpha + \lambda_{m-1} e^{-x} \right).$$

This is possible because the function $u_{\overline{C}}(x)$ belongs to the space $W_{2,2}(0, 1)$ and $u_{\overline{C}}(x)$ is a solution of Eq. (38). First $N + 1$ equations of the system (39) mean that $u_{\overline{C}}(x)$ takes 0 values at all nodes $x_\beta$. Then for the norm of the functional $\delta \overline{C}(x)$ in $W_{2,2}^{(m, m-1)}(0, 1)$ we have

$$\| \delta \overline{C}(x) \|^2 = (\delta \overline{C}(x), u_{\overline{C}}(x)) = 2(-1)^m \sum_{\beta=0}^N \overline{C}_\beta u_{\overline{C}}(x_\beta) = 0. \quad (40)$$

On the other hand, taking into account Eqs. (32), (33) we get

$$\| \delta \overline{C}(x) \|^2 = (\delta \overline{C}(x), u_{\overline{C}}(x)) = 2(-1)^m \sum_{\beta=0}^N \sum_{\gamma=0}^N \overline{C}_\beta \overline{C}_\gamma G(x_\beta - x_\gamma). \quad (41)$$

From (41) we conclude that (40) is possible if $\overline{C} = 0$. Then from first $N + 1$ equations of the system (39) we obtain

$$S^* \lambda = 0. \quad (42)$$
By assertion of the theorem, the matrix $S$ has the right inverse matrix, then $S^*$ has the left inverse matrix. Then from (42) we conclude that the solution $\lambda$ is also equal to zero.

Theorem 3.2 is proved. \hfill $\Box$

From (25) and Theorems 3.1, 3.2, it follows that in fixed values of the nodes $x_\beta$ the square of the norm of the error functional $\ell(x)$, being a quadratic functions of the coefficients $C_\beta$ has a unique minimum in some concrete value of $C_\beta = \hat{C}_\beta$.

As it was said in the first section, the quadrature formula with the coefficients $\hat{C}_\beta \ (\beta = \overline{0,N})$, corresponding to this minimum in fixed nodes $x_\beta$, is called the optimal quadrature formula in the sense of Sard, and $\hat{C}_\beta \ (\beta = \overline{0,N})$ are called the optimal coefficients.

Below, for the purposes of convenience, the optimal coefficients $\hat{C}_\beta$ will be denoted as $C_\beta$.

4 Coefficients of optimal quadrature formula in the sense of Sard

In the present section we solve the system (26)--(28) and we find the explicit formula for the coefficients $C_\beta$. Here we use a similar method to the one suggested by Sobolev [30] for finding the coefficients of optimal quadrature formulas in the space $L_2^{(m)}(0, 1)$. Here the main concept used is that of functions of discrete argument and operations on them. Theory of discrete argument functions is given in [31, 32]. For the purposes of completeness we give some definitions about functions of discrete argument.

Assume that the nodes $x_\beta$ are equal spaced, i.e. $x_\beta = h_\beta, h = \frac{1}{N}, N = 1, 2, \ldots$.

Suppose that $\varphi(x)$ and $\psi(x)$ are real-valued functions of real variable and are defined in real line $\mathbb{R}$.

**Definition 4.1** A function $\varphi(h_\beta)$ is called function of discrete argument if it is defined on some set of integer values of $\beta$.

**Definition 4.2** We define the inner product of two discrete functions $\varphi(h_\beta)$ and $\psi(h_\beta)$ as the following number

$$[\varphi(h_\beta), \psi(h_\beta)] = \sum_{\beta=-\infty}^{\infty} \varphi(h_\beta) \cdot \psi(h_\beta),$$

if the series on the right hand side of the last equality converges absolutely.

**Definition 4.3** We define convolution of two discrete functions $\varphi(h_\beta)$ and $\psi(h_\beta)$ the inner product

$$\varphi(h_\beta) * \psi(h_\beta) = [\varphi(h_\gamma), \psi(h_\beta - h_\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h_\gamma) \cdot \psi(h_\beta - h_\gamma).$$

Now we return to our problem.
Suppose that \( C_\beta = 0 \) when \( \beta < 0 \) and \( \beta > N \). Using the above definitions, we rewrite the system (26)–(28) in the following convolution form

\[
G(h\beta) * C_\beta + P_{m-2}(h\beta) + de^{-h\beta} = f_m(h\beta), \quad \beta = 0, 1, \ldots, N,
\]

\[
C_\beta = 0, \text{ when } \beta < 0 \text{ and } \beta > N,
\]

\[
\sum_{\beta=0}^{N} C_\beta \cdot (h\beta)^\alpha = \frac{1}{\alpha + 1}, \quad \alpha = 0, 1, \ldots, m - 2,
\]

\[
\sum_{\beta=0}^{N} C_\beta \cdot e^{-h\beta} = 1 - e^{-1},
\]

where

\[
f_m(h\beta) = \frac{e^{h\beta} + e^{-h\beta} + e^{1-h\beta} + e^{h\beta-1} - 4}{4} - \sum_{k=1}^{m-1} \frac{(h\beta)^{2k} + (1 - h\beta)^{2k}}{2 \cdot (2k)!},
\]

\( G(x) \) is defined by (14).

Consider the following problem

**Problem A** Find a discrete function \( C_\beta \), a polynomial \( P_{m-2}(h\beta) \) of degree \( m - 2 \) and a constant \( d \) which satisfy the system (43)–(46) for given \( f_m(h\beta) \).

Further we investigate the Problem 4 and instead of \( C_\beta \) we introduce the functions

\[ v(h\beta) = G(h\beta) * C_\beta \]

and

\[ u(h\beta) = v(h\beta) + P_{m-2}(h\beta) + d e^{-h\beta}. \]

In this statement it is necessary to express the coefficients \( C_\beta \) by the function \( u(h\beta) \).

For this, we have to construct such operator \( D_m(h\beta) \) which satisfies the equality

\[ D_m(h\beta) * G(h\beta) = \delta(h\beta), \]

where \( \delta(h\beta) \) is equal to 0 when \( \beta \neq 0 \) and is equal to 1 when \( \beta = 0 \), i.e. \( \delta(h\beta) \) is the discrete delta-function.

In [23,24] the discrete analogue \( D_m(h\beta) \) of the operator \( \frac{d^{2m}}{dx^{2m}} - \frac{d^{2m-2}}{dx^{2m-2}} \), which satisfies Eq. (50) is constructed and its some properties are investigated.

The following theorems are proved in [23,24].
**Theorem 4.1** The discrete analogues to the differential operator $\frac{d^{2m}}{dx^{2m}} - \frac{d^{2m-2}}{dx^{2m-2}}$ satisfying the Eq. (50) has the form

$$D_m(h\beta) = \frac{1}{p_{2m-2}} \begin{cases} \sum_{k=1}^{m-1} A_k \lambda_k^{\beta-1}, & |\beta| \geq 2, \\ -2e^h + \sum_{k=1}^{m-1} A_k, & |\beta| = 1, \\ 2C + \sum_{k=1}^{m-1} A_k, & \beta = 0, \end{cases}$$

where

$$C = 1 + (2m - 2)e^h + e^{2h} + \frac{e^h \cdot p_{2m-3}}{p_{2m-2}},$$

$$A_k = \frac{2(1 - \lambda_k)^{2m-2}[\lambda_k(e^{2h} + 1) - e^h(\lambda_k^2 + 1)]p_{2m-2}}{\lambda_k p_{2m-2}(\lambda_k)},$$

$$p_{2m-2}(\lambda) = \sum_{s=0}^{2m-2} p_s^{2m-2} \lambda^s = (1 - e^{2h})(1 - \lambda)^{2m-2} - 2(\lambda(e^{2h} + 1) - e^h(\lambda^2 + 1)) \times \left[ h(1 - \lambda)^{2m-4} + \frac{h^3(1 - \lambda)^{2m-6}}{3!} E_2(\lambda) + \cdots + \frac{h^{2m-3} E_{2m-4}(\lambda)}{(2m - 3)!} \right],$$

$p_{2m-2}, p_{2m-3}$ are the coefficients of the polynomial $p_{2m-2}(\lambda)$ defined by equality (54), $\lambda_k$ are the roots of the polynomial $p_{2m-2}(\lambda)$, $|\lambda_k| < 1$, $E_k(\lambda)$ is the Euler–Frobenius polynomial of degree $k$ (see [32]).

**Theorem 4.2** The discrete analogue $D_m(h\beta)$ of the differential operator $\frac{d^{2m}}{dx^{2m}} - \frac{d^{2m-2}}{dx^{2m-2}}$ satisfies the following equalities

1. $D_m(h\beta) * e^{h\beta} = 0$,
2. $D_m(h\beta) * e^{-h\beta} = 0$,
3. $D_m(h\beta) * (h\beta)^n = 0$, $n \leq 2m - 3$,
4. $D_m(h\beta) * G(h\beta) = \delta(h\beta),$

here $G(h\beta)$ is the function of discrete argument corresponding to the function $G(x)$, defined by equality (14) and $\delta(h\beta)$ is the discrete delta function.

Then taking into account (49), (50) and Theorems 4.1, 4.2, for the optimal coefficients we have

$$C_\beta = D_m(h\beta) * u(h\beta).$$

Thus, if we find the function $u(h\beta)$, then the optimal coefficients will be found from equality (55).

To calculate the convolution (55), it is required to find the representation of the function $u(h\beta)$ for all integer values of $\beta$. From equality (43), we get that $u(h\beta) =$
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When \( h \beta \in [0, 1] \). Now we need to find the representation of the function \( u(h \beta) \) when \( \beta < 0 \) and \( \beta > N \).

Since \( C_\beta = 0 \) when \( h \beta \notin [0, 1] \) then

\[
C_\beta = D_m(h \beta) * u(h \beta) = 0, \quad h \beta \notin [0, 1].
\]

Now we calculate the convolution \( v(h \beta) = G(h \beta) * C_\beta \) when \( h \beta \notin [0, 1] \).

Suppose \( \beta < 0 \) then, taking into account equalities (14), (44), (45), (46), we have

\[
v(h \beta) = G(h \beta) * C_\beta = \frac{1}{2} \sum_{\gamma=0}^{N-1} C_\gamma \left( \frac{e^{h \beta - h \gamma} - e^{-h \beta + h \gamma}}{2} - \sum_{k=1}^{m-1} \frac{(h \beta - h \gamma)^{2k-1}}{(2k-1)!} \right)
\]

\[
= -\frac{e^{h \beta}}{4} (1 - e^{-1}) + D e^{-h \beta} + Q^{(2m-3)}(h \beta) + Q_{m-2}(h \beta),
\]

(56)

where

\[
Q^{(2m-3)}(h \beta) = \frac{1}{2} \left[ \sum_{k=1}^{\left\lfloor \frac{m+1}{2} \right\rfloor} \sum_{\alpha=0}^{2k-1} \frac{(h \beta)^{2k-1-\alpha} (-1)^{\alpha}}{(2k-1-\alpha)! (\alpha+1)!} \right] + \sum_{k=\left\lfloor \frac{m+1}{2} \right\rfloor}^{m-1} \sum_{\alpha=m-1}^{m-2} \frac{(h \beta)^{2k-1-\alpha} (-1)^{\alpha}}{(2k-1-\alpha)! (\alpha+1)!}
\]

(57)

is the polynomial of degree \( 2m - 3 \) with respect to \( (h \beta) \),

\[
Q_{m-2}(h \beta) = \frac{1}{2} \sum_{k=\left\lfloor \frac{m+1}{2} \right\rfloor}^{m-1} \sum_{\alpha=m-1}^{2k-1} \frac{(h \beta)^{2k-1-\alpha} (-1)^{\alpha}}{(2k-1-\alpha)! \cdot \alpha!} \sum_{\gamma=0}^{N-1} C_\gamma (h \gamma)^{\alpha}
\]

(58)

is an unknown polynomial of degree \( m - 2 \) with respect to \( (h \beta) \), and

\[
D = \frac{1}{4} \sum_{\gamma=0}^{N-1} C_\gamma e^{h \gamma}.
\]

(59)

Similarly, in the case \( \beta > N \), for the convolution \( v(h \beta) = G(h \beta) * C_\beta \), we obtain

\[
v(h \beta) = \frac{e^{h \beta}}{4} (1 - e^{-1}) - D e^{-h \beta} - Q^{(2m-3)}(h \beta) - Q_{m-2}(h \beta).
\]

(60)
We denote
\[ Q_{m-2}^{(-)}(h\beta) = P_{m-2}(h\beta) + Q_{m-2}(h\beta), \quad a^- = d + D, \]  
\[ Q_{m-2}^{(+)}(h\beta) = P_{m-2}(h\beta) - Q_{m-2}(h\beta), \quad a^+ = d - D, \]  
and, taking into account (56), (60), (49), we get the following problem.

**Problem B**  Find the solution of the equation
\[ D_m(h\beta) \ast u(h\beta) = 0, \quad h\beta \notin [0, 1] \]  
having the form
\[ u(h\beta) = \begin{cases} \quad -\frac{h\beta}{4} (1 - \frac{1}{e}) + a^- e^{-h\beta} + Q^{(2m-3)}(h\beta) + Q_{m-2}^{(-)}(h\beta), & \beta < 0, \\ f_m(h\beta), & 0 \leq \beta \leq N, \\ \quad \frac{eh\beta}{4} (1 - \frac{1}{e}) + a^+ e^{-h\beta} - Q^{(2m-3)}(h\beta) + Q_{m-2}^{(+)}(h\beta), & \beta > N. \end{cases} \]  

Here \( Q_{m-2}^{(-)}(h\beta) \) and \( Q_{m-2}^{(+)}(h\beta) \) are unknown polynomials of degree \( m - 2 \) with respect to \( h\beta \), \( a^- \) and \( a^+ \) are unknown constants.

If we find \( Q_{m-2}^{(-)}(h\beta) \), \( Q_{m-2}^{(+)}(h\beta) \), \( a^- \) and \( a^+ \), then from (61), (62) we have
\[ P_{m-2}(h\beta) = \frac{1}{2} \left( Q_{m-2}^{(-)}(h\beta) + Q_{m-2}^{(+)}(h\beta) \right), \quad d = \frac{1}{2}(a^- + a^+), \]
\[ Q_{m-2}(h\beta) = \frac{1}{2} \left( Q_{m-2}^{(-)}(h\beta) - Q_{m-2}^{(+)}(h\beta) \right), \quad D = \frac{1}{2}(a^- - a^+). \]

Unknowns \( Q_{m-2}^{(-)}(h\beta) \), \( Q_{m-2}^{(+)}(h\beta) \), \( a^- \) and \( a^+ \) can be found from the Eq. (63), using the function \( D_m(h\beta) \). Then we can obtain the explicit form of the function \( u(h\beta) \) and find the optimal coefficients \( C_\beta \). Thus, the problem \( B \) and, respectively, the problem \( A \) can be solved.

But here we will not find \( Q_{m-2}^{(-)}(h\beta) \), \( Q_{m-2}^{(+)}(h\beta) \), \( a^- \) and \( a^+ \). Instead of them, using \( D_m(h\beta) \) and \( u(h\beta) \), taking into account (55), we will find the expressions for the optimal coefficients \( C_\beta \) when \( \beta = 1, \ldots, N - 1. \)

We denote
\[ a_k = \frac{A_k}{\lambda_k p} \sum_{\gamma=1}^{\infty} \lambda_k^\gamma \left( -\frac{e^{-h\gamma}}{4} (1 - e^{-1}) + Q^{(2m-3)}(-h\gamma) \\
+ Q_{m-2}^{(-)}(-h\gamma) + a^- e^{h\gamma} - f_m(-h\gamma) \right), \]  
\[ b_k = \frac{A_k}{\lambda_k p} \sum_{\gamma=1}^{\infty} \lambda_k^\gamma \left( \frac{e^{h\gamma+1}}{4} (1 - e^{-1}) - Q^{(2m-3)}(1 + h\gamma) \\
+ Q_{m-2}^{(+)}(1 + h\gamma) + a^+ e^{-1-h\gamma} - f_m(1 + h\gamma) \right). \]
where $\lambda_k$ are the roots, and $p$ is the leading coefficient of the polynomial $P_{2m-2}(\lambda)$ of degree $2m - 2$ defined by equality (54) and $|\lambda_k| < 1$. The series in the notations (65), (66) are convergent.

The following holds

**Theorem 4.3** (Theorem 3 of [22]). The coefficients of optimal quadrature formulas in the sense of Sard of the form (1) in the space $W_2^{(m,m-1)}(0,1)$ have the following form

$$C_\beta = D_m(h\beta) * f_m(h\beta) + \sum_{k=1}^{m-1} \left( a_k \lambda_k^\beta + b_k \lambda_k^{N-\beta} \right), \quad \beta = 1, 2, \ldots, N - 1,$$

(67)

where $a_k$ and $b_k$ are unknowns and have the form (65) and (66) respectively, $\lambda_k$ are the roots of the polynomial $P_{2m-2}(\lambda)$ which is defined by equality (54) and $|\lambda_k| < 1$.

From Theorem 4.3, it is clear that to obtain the explicit forms of the optimal coefficients $C_\beta$ in the space $W_2^{(m,m-1)}(0,1)$ it is sufficient to find $a_k$ and $b_k$ ($k = 1, m - 1$). But here we will not calculate series (65) and (66). Instead of that substituting the equality (67) into (43) we obtain the identity with respect to $(h\beta)$. Whence, equating the corresponding coefficients in the left and the right hand sides of Eq. (43), we will find $a_k$ and $b_k$. And the coefficient $C_0$ and $C_N$ will be found from (45) when $\alpha = 0$ and (46), respectively. Below we will do it.

It should be noted that the cases $m = 1$ and $m = 2$ are solved in the work [22] and the following theorems are proved.

**Theorem 4.4** (Theorem 4 of [22]). The coefficients of optimal quadrature formulas of the form (1) with equal spaced nodes in the space $W_2^{(1,0)}(0,1)$ are expressed by formulas

$$C_\beta = \begin{cases} \frac{e^h - 1}{e^{h+1}}, & \beta = 0, N, \\ \frac{2(e^h - 1)}{e^h + 1}, & \beta = 1, N - 1, \end{cases}$$

where $h = 1/N$, $N = 1, 2, \ldots$.

**Theorem 4.5** (Theorem 5 of [22]). The coefficients of optimal quadrature formulas of the form (1) with equal spaced nodes in the space $W_2^{(2,1)}(0,1)$ are expressed by formulas

$$C_\beta = \begin{cases} 1 - \frac{h}{e^{h-1}} - K(\lambda_1 - \lambda_1^N), & \beta = 0, \\ h + K \left( (e^h - \lambda_1)\lambda_1^\beta + (1 - \lambda_1 e^h)\lambda_1^{N-\beta} \right), & \beta = 1, N - 1, \\ -1 + \frac{e^h}{e^{h-1}} - K(\lambda_1 - \lambda_1^N)e^h, & \beta = N, \end{cases}$$
where

\[
K = \frac{(2e^h - 2 - he^h - h)(\lambda_1 - 1)}{2(e^h - 1)^2(\lambda_1 + \lambda_1^{N+1})},
\]

\[
\lambda_1 = \frac{h(e^{2h} + 1) - e^{2h} + 1 - (e^h - 1)\sqrt{h^2(e^h + 1)^2 + 2h(1 - e^h)} - e^{2h} + 2he^h}{1 - e^{2h} + 2he^h}, \quad |\lambda_1| < 1,
\]

\[h = 1/N, \quad N = 2, 3, \ldots\]

The main goal of the present section is to solve the system (43)–(46) for any \(m \geq 2\) and any natural \(N, N + 1 \geq m\). As it was mentioned above, it is sufficient to find \(a_k\) and \(b_k(k = 1, m - 1)\) in (67).

The main result of the present paper is the following theorem.

**Theorem 4.6** The coefficients of optimal quadrature formulas of the form (1) with the error functional (2) and with equal spaced nodes in the space \(W_2^{(m,m-1)}(0, 1)\) when \(m \geq 2\) and \(N + 1 \geq m\) are expressed by formulas

\[
C_0 = \frac{e^h - 1 - h}{e^h - 1} + \sum_{k=1}^{m-1} \left( a_k \frac{\lambda_k(e^h - e) + \lambda_k^2(e - 1) + \lambda_k^{N+1}(1 - e^h)}{(e - 1)(1 - \lambda_k)(e^h - \lambda_k)} \right.
\]

\[
+ b_k \frac{\lambda_k^{N+1}(e^h - e) + \lambda_k^N(e - 1) + \lambda_k(1 - e^h)}{(e - 1)(\lambda_k - 1)\lambda_k e^h - 1}) \right),
\]

\[
C_\beta = h + \sum_{k=1}^{m-1} \left( a_k \lambda_k^\beta + b_k \lambda_k^{N-\beta} \right), \quad \beta = 1, N - 1,
\]

\[
C_N = \frac{e^h h + 1 - e^h}{e^h - 1} + \sum_{k=1}^{m-1} \left( a_k \frac{\lambda_k(e - e^{h+1}) + \lambda_k^N(e^{h+1} - e^h) + \lambda_k^{N+1}(e^h - e)}{(e - 1)(1 - \lambda_k)(e^h - \lambda_k)} \right.
\]

\[
+ b_k \frac{\lambda_k^{N+1}(e - e^{h+1}) + \lambda_k^2(e^{h+1} - e^h) + \lambda_k(e^h - e)}{(e - 1)(1 - \lambda_k)(1 - \lambda_k e^h)} \right),
\]

where \(a_k\) and \(b_k(k = 1, m - 1)\) are defined by the following system of \(2m - 2\) linear equations

\[
\sum_{k=1}^{m-1} a_k \frac{\lambda_k}{(\lambda_k - 1)(\lambda_k - e^h)} + \sum_{k=1}^{m-1} b_k \frac{\lambda_k^{N+1}}{(\lambda_k - 1)(\lambda_k e^h - 1)} = \frac{h - 2}{2(e^h - 1)} + \frac{h}{(e^h - 1)^2};
\]

\[
\sum_{k=1}^{m-1} a_k \frac{\lambda_k^{N+1}}{(\lambda_k - 1)(\lambda_k - e^h)} + \sum_{k=1}^{m-1} b_k \frac{\lambda_k}{(\lambda_k - 1)(\lambda_k e^h - 1)} = \frac{h - 2}{2(e^h - 1)} + \frac{h}{(e^h - 1)^2};
\]
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For the purposes of convenience, we denote

\[
\sum_{k=1}^{m-1} a_k \left[ \sum_{l=2}^{j} \frac{h^{2l-2}}{(2l-2)!} \sum_{i=1}^{2l-2} \lambda_k \Delta^i 0^{2l-2} \right]
\]

\[
+ \sum_{k=1}^{m-1} b_k \left[ \sum_{l=2}^{j} \frac{h^{2l-2}}{(2l-2)!} \sum_{i=1}^{2l-2} \lambda_k^{N+i} \Delta^i 0^{2l-2} \right] = 0, \quad j = 2, \left\lfloor \frac{m}{2} \right\rfloor;
\]

\[
\sum_{k=1}^{m-1} a_k \left[ \sum_{l=1}^{j} \frac{h^{2l-1}}{(2l-1)!} \sum_{i=1}^{2l-1} \lambda_k \Delta^i 0^{2l-1} \right]
\]

\[
+ \sum_{k=1}^{m-1} b_k \left[ \sum_{l=1}^{j} \frac{h^{2l-1}}{(2l-1)!} \sum_{i=1}^{2l-1} \lambda_k^{N+i} \Delta^i 0^{2l-1} \right] = \sum_{l=1}^{j} \frac{h^{2l} B_{2l}}{(2l)!}, \quad j = 1, \left\lfloor \frac{m-1}{2} \right\rfloor;
\]

\[
\sum_{k=1}^{m-1} a_k \left[ \sum_{l=1}^{j} h^l C_{j}^{l} \sum_{i=1}^{l} \lambda_k^{N+i} \Delta^i 0^{l} \right]
\]

\[
+ \sum_{k=1}^{m-1} b_k \left[ \sum_{l=1}^{j} h^l C_{j}^{l} \sum_{i=1}^{l} \lambda_k \Delta^i 0^{l} \right] = \frac{1}{(j+1)!} B_{j+1-l} h^{j+1-l}, \quad j = 1, m-2.
\]

Here \( \lambda_k \) are the roots of the polynomial (54) and \( |\lambda_k| < 1 \), \( B_j \) are the Bernoulli numbers.

In the proof of Theorem 4.6 we use the following formulas from [11]

\[
\sum_{\gamma=0}^{n-1} q^\gamma \gamma^k = \frac{1}{1-q} \sum_{i=0}^{k} \left( \frac{q}{1-q} \right)^i \Delta^i 0^k - \frac{q^n}{1-q} \sum_{i=0}^{k} \left( \frac{q}{1-q} \right)^i \Delta^i \gamma^k |_{\gamma=n}, \quad (68)
\]

where \( \Delta^i 0^k = \sum_{l=1}^{i} (-1)^{i-l} C_{i}^{l} t^k \), \( \Delta^i \gamma^k \) is the finite difference of order \( i \) of \( \gamma^k \), and from [9]

\[
\sum_{\gamma=0}^{\beta-1} \gamma^k = \sum_{j=1}^{k+1} \frac{k! B_{k+1-j}}{j!(k+1-j)!} \beta^j, \quad (69)
\]

where \( B_{k+1-j} \) are the Bernoulli numbers,

\[
\Delta^\alpha x^v = \sum_{P=0}^{v} C_{v}^{P} \Delta^\alpha 0^P x^{v-P}. \quad (70)
\]

Proof of Theorem 4.6 For the purposes of convenience, we denote \( T = D_m(h\beta) * f_m(h\beta). \)
For the convolution $G(h\beta) * C_\beta$ of the equality (43) we have

$$g(h\beta) = C_0 \left( \frac{e^{h\beta} - e^{-h\beta}}{2} - \sum_{k=1}^{m-1} \frac{(h\beta)^{2k-1}}{(2k-1)!} \right) + g_1(h\beta) + g_2(h\beta), \quad (71)$$

where

$$g_1(h\beta) = \sum_{\gamma=1}^{\beta-1} C_\gamma \left( \frac{e^{h\beta - h\gamma} - e^{h\gamma - h\beta}}{2} - \sum_{k=1}^{m-1} \frac{(h\beta - h\gamma)^{2k-1}}{(2k-1)!} \right), \quad (72)$$

$$g_2(h\beta) = -\frac{1}{2} \sum_{\gamma=0}^{N} C_\gamma \left( \frac{e^{h\beta - h\gamma} - e^{h\gamma - h\beta}}{2} - \sum_{k=1}^{m-1} \frac{(h\beta - h\gamma)^{2k-1}}{(2k-1)!} \right). \quad (73)$$

From (72), using (67), (68), (69), (70) and taking into account that $\lambda_k$ are the roots of the polynomial (54), after some simplifications, we get

$$g_1(h\beta) = \frac{e^{h\beta}}{2} \left[ \frac{T}{e^h - 1} + \sum_{k=1}^{m-1} \left( a_k \frac{\lambda_k}{e^h - \lambda_k} + b_k \frac{\lambda_k^N}{\lambda_k e^h - 1} \right) \right]$$

$$- \frac{e^{-h\beta}}{2} \left[ \frac{T e^h}{1 - e^h} + \sum_{k=1}^{m-1} \left( a_k \frac{\lambda_k e^h}{1 - e^h \lambda_k} + b_k \frac{\lambda_k^N e^h}{\lambda_k - e^h} \right) \right]$$

$$+ \frac{T e^h}{2(1 - e^h)} + \frac{T}{2(1 - e^h)} - \sum_{\ell=1}^{m-1} \frac{T h^{-1}(h\beta)^{2\ell}}{(2\ell)!}$$

$$- \sum_{\ell=1}^{m-1} \frac{T h^{2\ell-1}}{(2\ell - 1)!} \sum_{j=0}^{2\ell-1} \frac{B_{2\ell-j}}{j!(2\ell - j)!} \beta^j$$

$$+ \sum_{\ell=1}^{m-1} \frac{h^{2\ell-1}}{(2\ell - 1)!} \sum_{j=0}^{2\ell-1} \frac{C_j^{2\ell-1} \beta_j}{(\lambda_k - 1)^j} \Delta_i \frac{0^{2\ell-1-j}}{(\lambda_k - 1)^i}$$

$$+ \sum_{\ell=1}^{m-1} \frac{h^{2\ell-1}}{(2\ell - 1)!} \sum_{j=0}^{2\ell-1} \frac{C_j^{2\ell-1} \beta_j}{(\lambda_k - 1)^j} \Delta_i \frac{0^{2\ell-1-j}}{(\lambda_k - 1)^i}.$$

Now, using the binomial formula and equalities (45), (46), from (73) we obtain

$$g_2(h\beta) = -\frac{1}{2} \left[ \frac{e^{h\beta}}{2} (1 - e^{-1}) - \frac{e^{-h\beta}}{2} \sum_{\gamma=0}^{N} C_\gamma e^{h\gamma} \right]$$

$$- \left( \sum_{k=1}^{m+1} \sum_{\alpha=0}^{2k-1} \frac{(h\beta)^{2k-1-\alpha}((-1)^{\alpha})}{(2k - 1 - \alpha)! (\alpha + 1)!} \right).$$


Using the binomial formula in equality (47), for \( f_m(h\beta) \) we deduce

\[
 f_m(h\beta) = \frac{e^{h\beta} + e^{-h\beta} + e^{1-h\beta} + e^{h\beta-1} - 4}{4} - \sum_{k=1}^{m-1} \frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{(2k-1-\alpha)!(\alpha+1)!}
\]

\[
 + \frac{1}{2} \sum_{k=\left[ \frac{m+1}{2} \right]}^{m-2} \sum_{\alpha=0}^{m-1} \frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{(2k-1-\alpha)!(\alpha+1)!}
\]

\[
 + \frac{1}{2} \sum_{k=\left[ \frac{m+1}{2} \right]}^{m-1} \sum_{\alpha=m-1}^{m-2} \frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{(2k-1-\alpha)!(\alpha+1)!}
\]

\[
 + \frac{1}{2} \sum_{k=\left[ \frac{m+1}{2} \right]}^{m-1} \sum_{\alpha=m-1}^{m-2} \frac{(h\beta)^{2k-1-\alpha}(-1)^\alpha}{(2k-1-\alpha)!(\alpha+1)!}.
\]

(76)

Taking into account (74), (75) and putting (71), (76) into (43), we get the following identity with respect to \((h\beta)\):

\[
 g(h\beta) + P_{m-2}(h\beta) + de^{-h\beta} = f_m(h\beta).
\]

(77)

From here, equating the coefficients of the term \((h\beta)^{2m-2}\), we obtain that for \( m \geq 2 \)

\[
 T = D_m(h\beta) \ast f_m(h\beta) = h,
\]

(78)

and correspondingly for the optimal coefficients (67) we have the following formula

\[
 C_\beta = h + \sum_{k=1}^{m-1} \left( a_k \lambda_k^\beta + b_k \lambda_k^{N-\beta} \right), \quad \beta = 1, N - 1.
\]

(79)

Note that the equality (78), for some \( m \), was proved by calculation of the convolution \( D_m(h\beta) \ast f_m(h\beta) \) in [24].

As it was said above, the equality (77) is the identity with respect to \((h\beta)\). Keeping in mind (74), (75), (76), equating the coefficients of \( e^{h\beta} \) and the terms which consist of \((h\beta)^\alpha, \alpha = m - 1, 2m - 3\) in both sides of (77), we get the following equations for \( a_k \) and \( b_k \)
\[ \sum_{k=1}^{m-1} \left( a_k \left( \frac{\lambda_k - \lambda_k^{N+1}}{e^h - \lambda_k} \right) + \frac{\lambda_k - \lambda_k^{N+1}}{(\lambda_k e^h - 1)(1 - \lambda_k)} b_k \right) = 0. \] (80)

\[ \sum_{\ell=0}^{m-1} \left( -C_0 \frac{(h\beta)^{2\ell-1}}{(2\ell - 1)!} - \sum_{j=m-1}^{2\ell-1} \frac{(h\beta)^j h^{2\ell-j} B_{2\ell-j}}{j!(2\ell - j)!} \right) + \sum_{j=m-1}^{2\ell-1} \frac{(h\beta)^j h^{2\ell-j-1}}{(2\ell - 1 - j)!j!} \sum_{k=1}^{m-1} a_k \sum_{i=0}^{m-1} a_k \frac{\lambda_k^{N+i} \Delta^i 0^{2\ell-1-j}}{(1 - \lambda_k)^{i+1}} \right] = 0. \] (81)

Now, from Eq. (45) when \( \alpha = 0 \) and (46), using identities (68), (69), (70), taking into account (79), after some simplifications for the coefficients \( C_0 \) and \( C_N \), we get the following expressions which are asserted in the theorem

\[ C_0 = \frac{e^h - 1 - h}{e^h - 1} + \sum_{k=1}^{m-1} \left( a_k \frac{\lambda_k (e^h - e) + \lambda_k^2 (e - 1) + \lambda_k^{N+1} (1 - e^h)}{(e - 1)(1 - \lambda_k)(e^h - \lambda_k)} + b_k \frac{\lambda_k^{N+1} (e - e^h) + \lambda_k^N (e - 1) + \lambda_k (1 - e^h)}{(e - 1)(\lambda_k - 1)(\lambda_k e^h - 1)} \right), \] (82)

\[ C_N = \frac{e^h h + 1 - e^h}{e^h - 1} + \sum_{k=1}^{m-1} \left( a_k \frac{\lambda_k (e - e^{h+1}) + \lambda_k^N (e^{h+1} - e^h) + \lambda_k^{N+1} (e^h - e)}{(e - 1)(1 - \lambda_k)(e^h - e)} + b_k \frac{\lambda_k^{N+1} (e - e^{h+1}) + \lambda_k^2 (e^{h+1} - e^h) + \lambda_k (e^h - e)}{(e - 1)(1 - \lambda_k)(1 - \lambda_k e^h)} \right). \] (83)

From (81), using (82), grouping the coefficients of same degrees of \( (h\beta) \) and equating to zero, for \( a_k \) and \( b_k \) we obtain the following m – 1 linear equations

\[ \sum_{k=1}^{m-1} a_k \left[ \sum_{i=1}^{m} \frac{h^{2i-2}}{(2i - 2)!} \sum_{i=0}^{2i - 2} \frac{\lambda_k \Delta^i 0^{2i-2}}{(\lambda_k - 1)^{i+1}} - \frac{\lambda_k (e^h - e) + \lambda_k^2 (e - 1) + \lambda_k^{N+1} (1 - e^h)}{(e - 1)(\lambda_k - 1)(\lambda_k e^h - 1)} \right] \]

\[ + \sum_{k=1}^{m-1} b_k \left[ \sum_{i=1}^{m} \frac{h^{2i-2}}{(2i - 2)!} \sum_{i=0}^{2i - 2} \frac{\lambda_k^{N+i} \Delta^i 0^{2i-2}}{(1 - \lambda_k)^{i+1}} - \frac{\lambda_k (1 - e^h) + \lambda_k^N (e - 1) + \lambda_k^{N+1} (e^h - e)}{(e - 1)(\lambda_k - 1) (\lambda_k e^h - 1)} \right] = \frac{e^h - 1 - h}{e^h - 1} - \frac{h}{2}, \quad j = 1, \left[ \frac{m}{2} \right]. \] (84)

\[ \sum_{k=1}^{m-1} a_k \left[ \sum_{i=1}^{m} \frac{h^{2i-1}}{(2i - 1)!} \sum_{i=0}^{2i - 1} \frac{\lambda_k \Delta^i 0^{2i-1}}{(\lambda_k - 1)^{i+1}} \right] + \sum_{k=1}^{m-1} b_k \left[ \sum_{i=1}^{m} \frac{h^{2i-1}}{(2i - 1)!} \sum_{i=0}^{2i - 1} \frac{\lambda_k^{N+i} \Delta^i 0^{2i-1}}{(1 - \lambda_k)^{i+1}} \right] = 0, \quad j = 1, \left[ \frac{m - 1}{2} \right]. \] (85)
Further, from (45) when \( \alpha = 1, \ldots, m - 2 \), using identities (68), (69), (70) and the expression (83), for \( a_k \) and \( b_k \) we have the following \( m - 2 \) linear equations

\[
\sum_{k=1}^{m-1} a_k \left[ h^j \sum_{i=0}^{j} \frac{\lambda_k - \lambda_k^{N+i}}{(1 - \lambda_k)^{i+1}} \Delta^i 0^j - \sum_{l=0}^{j-1} h^l C_j^l \sum_{i=0}^{l} \frac{\lambda_k^{N+i} \Delta^i 0^l}{(1 - \lambda_k)^{i+1}} \right] + \frac{\lambda_k(e - e^{h+1}) + \lambda_k^N(e^{h+1} - e^h) + \lambda_k^N(e^h - e)}{(e - 1)(\lambda_k - 1)(\lambda_k - e^h)} \\
+ \sum_{k=1}^{m-1} b_k \left[ h^j \sum_{i=0}^{j} \frac{\lambda_k^{N+1} - \lambda_k^i}{(\lambda_k - 1)^{i+1}} \Delta^i 0^j - \sum_{l=0}^{j-1} h^l C_j^l \sum_{i=0}^{l} \frac{\lambda_k \Delta^i 0^l}{(\lambda_k - 1)^{i+1}} \right] + \frac{\lambda_k^{N+1}(e - e^{h+1}) + \lambda_k(e^h - e) + \lambda_k^2(e^{h+1} - e^h)}{(e - 1)(\lambda_k - 1)(\lambda_k e^h - 1)} \\
= -\sum_{l=1}^{j} \frac{j! B_{j+1-l}}{l!(j+1-l)!} h^{j+1-l} - \frac{e^h h + 1 - e^h}{e^h - 1}, \quad j = 1, m - 2. \tag{86}
\]

After some simplifications in the system of Eqs. (80), (84), (85), (86), we get the system which is given in the assertion of the theorem.

Theorem 4.6 is proved. \( \square \)

From Theorem 4.6 when \( m = 2 \) we get Theorem 4.5.

For \( m = 3 \) and \( m = 4 \), from Theorem 4.6 we have the following results

**Corollary 4.1** The coefficients of optimal quadrature formulas of the form (1), with the error functional (2), and with equal spaced nodes in the space \( W_2^{(3,2)}(0, 1) \), are expressed by formulas

\[
C_0 = \frac{e^h - 1 - h}{e^h - 1} + \sum_{k=1}^{2} \left( a_k \frac{\lambda_k(e^h - e) + \lambda_k^2(e - 1)}{(e - 1)(1 - \lambda_k)(e^h - \lambda_k)} + b_k \frac{\lambda_k^{N+1}(e - e^{h+1}) + \lambda_k^N(e^{h+1} - e^h) + \lambda_k^{N+1}(e^h - e)}{(e - 1)(1 - \lambda_k)(e^h - \lambda_k)} \right),
\]

\[
C_\beta = h + \sum_{k=1}^{2} \left( a_k \lambda_k^\beta + b_k \lambda_k^{N-\beta} \right), \quad \beta = 1, N - 1,
\]

\[
C_N = \frac{e^h h + 1 - e^h}{e^h - 1} + \sum_{k=1}^{2} \left( a_k \frac{\lambda_k(e - e^{h+1}) + \lambda_k^N(e^{h+1} - e^h) + \lambda_k^{N+1}(e^h - e)}{(e - 1)(1 - \lambda_k)(e^h - \lambda_k)} + b_k \frac{\lambda_k^{N+1}(e - e^{h+1}) + \lambda_k^2(e^{h+1} - e^h) + \lambda_k(e^h - e)}{(e - 1)(1 - \lambda_k)(1 - \lambda_k e^h)} \right),
\]

where \( a_k \) and \( b_k (k = 1, 2) \) are defined by the following system of linear equations
The coefficients of optimal quadrature formulas of the form \( (1) \) are expressed by formulas

\[
\begin{align*}
\lambda_k & = \frac{\lambda_1^k}{(\lambda - 1)(\lambda - e^h)} + \frac{\lambda_2^k}{(\lambda - 1)(\lambda - e^{1-k}h)} = \frac{e^h - 1 - h}{e^h - 1} + \sum_{k=1}^{N+1} \left( a_k \lambda_1^k e^{h-k} + b_k \lambda_2^k (e - 1) + \lambda_1^{N+1} (1 - e^h) \right), \\
C_0 & = a_k \lambda_1^k + b_k \lambda_2^k, \quad C_N = a_k \lambda_1^k + b_k \lambda_2^k, \\
\end{align*}
\]

where \( a_k \) and \( b_k (k = 1, 3) \) are defined by the following system of linear equations
\[
\sum_{k=1}^{3} a_k \frac{\lambda_k}{(\lambda_k - 1)(\lambda_k - e^h)} + \sum_{k=1}^{3} b_k \frac{\lambda_k^{N+1}}{(\lambda_k - 1)(\lambda_k e^h - 1)} = \frac{h - 2}{2(e^h - 1)} + \frac{h}{(e^h - 1)^2};
\]

\[
\sum_{k=1}^{3} a_k \frac{\lambda_k^{N+1}}{(\lambda_k - 1)(\lambda_k - e^h)} + \sum_{k=1}^{3} b_k \frac{\lambda_k}{(\lambda_k - 1)(\lambda_k e^h - 1)} = \frac{h - 2}{2(e^h - 1)} + \frac{h}{(e^h - 1)^2};
\]

\[
\sum_{k=1}^{3} a_k \frac{\lambda_k}{(\lambda_k - 1)^2} + \sum_{k=1}^{3} b_k \frac{\lambda_k^{N+1}}{(\lambda_k - 1)^2} = \frac{h}{12};
\]

\[
\sum_{k=1}^{3} a_k \frac{\lambda_k^{N+1}}{(\lambda_k - 1)^2} + \sum_{k=1}^{3} b_k \frac{\lambda_k}{(\lambda_k - 1)^2} = \frac{h}{12};
\]

\[
\sum_{k=1}^{3} a_k \frac{\lambda_k}{(\lambda_k - 1)^3} + \sum_{k=1}^{3} b_k \frac{\lambda_k^{N+2}}{(1 - \lambda_k)^3} = -\frac{h}{24};
\]

\[
\sum_{k=1}^{3} a_k \frac{\lambda_k^2 - \lambda_k^{N+2}}{(1 - \lambda_k)^3} + \sum_{k=1}^{3} b_k \frac{\lambda_k^{N+1} - \lambda_k}{(\lambda_k - 1)^3} = 0;
\]

Here $\lambda_k$, $k = 1, 2, 3$ are the roots of the polynomial

\[
P_6(\lambda) = (1 - e^{2h})(1 - \lambda)^6 - 2(\lambda(e^{2h} + 1) - e^h(\lambda^2 + 1))
\times \left(h(1-\lambda)^4 + \frac{h^3}{6}(1-\lambda)^2(1+4\lambda^2+\lambda^4) + \frac{h^5}{120}(1+26\lambda+66\lambda^2+26\lambda^3+\lambda^4)\right)
\]

which $|\lambda_k| < 1$.

5 The norm of the error functional of the optimal quadrature formula in $W_2^{(1,0)}(0, 1)$ space

Here we calculate the square of the norm of the error functional (2), of the optimal quadrature formula (1), on the space $W_2^{(1,0)}(0, 1)$. Furthermore, we give an asymptotic analysis of this norm.

The following result holds

**Theorem 5.1** The square of the norm of the error functional (2), of the optimal quadrature formula (1), on the space $W_2^{(1,0)}(0, 1)$, has the form

\[
\left\| \ell \right\|^2 = 1 - \frac{2(e^h - 1)}{h(e^h + 1)},
\]

(87)
Proof For $m = 1$ the system (43)–(46) takes the form

$$
\sum_{\gamma=0}^{N} C_{\gamma} G(h\beta - h\gamma) + de^{-h\beta} = f_1(h\beta), \quad \beta = 0, 1, \ldots, N, \quad (88)
$$

$$
C_{\beta} = 0 \quad \text{when } \beta < 0 \text{ and } \beta > N, \quad (89)
$$

$$
\sum_{\beta=0}^{N} C_{\beta} e^{-h\beta} = 1 - e^{-1}, \quad (90)
$$

where

$$
G(x) = \frac{\text{sign}(x)}{4} \left( e^x - e^{-x} \right), \quad (91)
$$

$$
f_1(h\beta) = \frac{1}{4} \left( e^{h\beta} + e^{-h\beta} + e^{1-h\beta} + e^{h\beta-1} - 4 \right), \quad (92)
$$

and $C_{\beta}(\beta = 0, 1, \ldots, N), d$ are unknowns.

In this case the Problem B is expressed as follows.

Problem A Find the solution of the equation

$$
D_1(h\beta) \ast G(h\beta) = 0, \quad h\beta \notin [0, 1] \quad (93)
$$

having the form

$$
\begin{aligned}
\quad u(h\beta) &= \begin{cases} 
-\frac{e^{h\beta}}{4}(1 - e^{-1}) + a^- e^{-h\beta}, & \beta < 0, \\
\quad f_1(h\beta), & 0 \leq \beta \leq N, \\
\quad \frac{e^{h\beta}}{4}(1 - e^{-1}) + a^+ e^{-h\beta}, & \beta > N,
\end{cases}
\end{aligned} \quad (94)
$$

where $f_1(h\beta)$ is defined by (92), $a^-$ and $a^+$ are unknowns.

For $m = 1$, from Theorem 4.1 for $D_1(h\beta)$, we obtain

$$
D_1(h\beta) = \frac{1}{1 - e^{2h}} \begin{cases} 
0, & |\beta| \geq 2, \\
-2e^h, & |\beta| = 1, \\
2(1 + e^{2h}), & \beta = 0.
\end{cases} \quad (95)
$$

Now, taking into account (95), for the convolution $C_{\beta} = D_1(h\beta) \ast u(h\beta)$, we have

$$
D_1(h\beta) \ast u(h\beta) = D_1(h)(u(h\beta - h) + u(h\beta + h)) + D_1(0)u(h\beta).
$$

Hence, keeping in mind (93) for $\beta = -1$ and $\beta = N + 1$, we get the following system

$$
\begin{align*}
D_1(h)(u(-2h) + u(0)) + D_1(0)u(-h) &= 0, \\
D_1(h)(u(Nh) + u(Nh + 2h)) + D_1(0)u(Nh + h) &= 0.
\end{align*}
$$
Whence, taking into account (94), (95) for $a^-$ and $a^+$, we have

$$a^- = \frac{e - 1}{4}, \quad a^+ = -\frac{e - 1}{4}. \quad (96)$$

Then, using (96), from (61) and (62) we obtain

$$d = \frac{1}{2}(a^- + a^+) = 0, \quad (97)$$

$$D = \frac{1}{2}(a^- - a^+) = \frac{e - 1}{4}. \quad (98)$$

Substituting (96) into (94) for $u(h\beta)$ we have the following expression

$$u(h\beta) = \begin{cases} 
-\frac{e^{h\beta}}{4}(1 - e^{-1}) + \frac{e^{-h\beta}}{4}(e - 1), & \beta < 0, \\
\frac{1}{4}(e^{h\beta} + e^{-h\beta} + e^{1-h\beta} + e^{h\beta-1} - 4), & 0 \leq \beta \leq N, \\
\frac{e^{h\beta}}{4}(1 - e^{-1}) - \frac{e^{-h\beta}}{4}(e - 1), & \beta > N,
\end{cases} \quad (99)$$

Using (99) and (95), for optimal coefficients $C_{\beta} = D_1(h\beta) \ast u(h\beta)(\beta = 0, 1, \ldots, N)$ we obtain the assertion of Theorem 4.4.

Now for $m = 1$ we rewrite the equality (25) in the following form

$$\|\ell\|^2 = -\left[\sum_{\beta=0}^{N} C_{\beta} \left(\sum_{\gamma=0}^{N} C_{\gamma} G(h\beta - h\gamma) - f_1(h\beta)\right) - \sum_{\beta=0}^{N} C_{\beta} f_1(h\beta) + \int_{0}^{1} \int_{0}^{1} G(x - y) \, dx \, dy\right],$$

where $G(x)$ is defined by (91).

Taking into account (97), from (88) we get

$$\sum_{\gamma=0}^{N} C_{\gamma} G(h\beta - h\gamma) - f_1(h\beta) = 0.$$ 

Then taking into account (91), (92), for $\|\ell\|^2$ we have

$$\|\ell\|^2 = \frac{1}{4} \sum_{\beta=0}^{N} C_{\beta} (e^{h\beta} + e^{-h\beta} + e^{1-h\beta} + e^{h\beta-1} - 4) - \frac{e^2 - 2e - 1}{2e}.$$ 

Hence, using Theorem 4.4, we get (87).

Theorem 5.1 is proved.
Theorem 5.2 The square of the norm of the error functional (2), of the optimal quadrature formula (1), on the space $W_2^{(1,0)}(0, 1)$, has the form

$$
\| \circ \ell \|_2^2 = \frac{h^2}{12} - \frac{h^4}{5!} + \frac{h^6}{4 \cdot 7!} - \cdots .
$$  \tag{100}

Proof The expansion of the right hand side of (87) in a series of powers of $h$ gives the assertion of Theorem 5.2. $\square$

The next theorem gives an asymptotic optimality for our optimal quadrature formula.

Theorem 5.3 The optimal quadrature formula of the form (1), with the error functional (2), in the space $W_2^{(1,0)}(0, 1)$, is asymptotic optimal in the Sobolev space $L_2^{(1)}(0, 1)$, i.e.,

$$
\lim_{N \to \infty} \frac{\| \circ \ell | W_2^{(1,0)*}(0, 1) \|_2^2}{\| \circ \ell | L_2^{(1)*}(0, 1) \|_2^2} = 1.
$$  \tag{101}

Proof Using Corollary 5.2 from [25] (for $m = 1$ and $\eta_0 = 0$), for the square of the norm of the error functional (2), of the optimal quadrature formula of the form (1), on the Sobolev space $L_2^{(1)}(0, 1)$, we get the following expression

$$
\| \circ \ell | L_2^{(1)*}(0, 1) \|_2^2 = \frac{h^2}{12}.
$$  \tag{102}

Using (100) and (102) we obtain (101). Thus, Theorem 5.3 is proved. $\square$

Remark 5.1 Comparison of the formulas (100) and (102) shows that the error of the optimal quadrature formula in the space $W_2^{(1,0)}(0, 1)$ is less than the error of the optimal quadrature formula in the space $L_2^{(1)}(0, 1)$.

6 Numerical results

In this section we give the numerical results which confirm the theoretical results obtained in Sect. 4.

Taking into account (14), from (25) for the norm of the error functional of optimal quadrature formulas, in the space $W_2^{(m,m-1)}(0, 1)$, we obtain

$$
\| \ell(x) \|_2^2 = (-1)^m \left[ \sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} C_\beta C_\gamma \frac{\sin(h\beta - h\gamma)}{2} \left( \frac{e^{h\beta - h\gamma} - e^{h\gamma - h\beta}}{2} - \sum_{k=1}^{m-1} \frac{(h\beta - h\gamma)^{2k-1}}{(2k-1)!} \right) \right. \\
-2 \sum_{\beta=0}^{N} C_\beta \left( \frac{e^{h\beta} + e^{h\gamma} + e^{1-h\beta} + e^{1-h\gamma} - 4}{4} - \sum_{k=1}^{m-1} \frac{(h\beta)^{2k} + (1-h\beta)^{2k}}{2 \cdot (2k)!} \right) \\
+ \frac{e^2 - 2e - 1}{2e} - \sum_{k=1}^{m-1} \frac{1}{(2k+1)!} \right].
$$  \tag{103}
Table 1  The numerical results of $\|\ell\|$ for the cases $m = 1, 2, 3, 4$ and $N = 10, 50, 100$.

| $m$  | $N = 10$  | $N = 50$  | $N = 100$ |
|------|-----------|-----------|-----------|
| 1    | 0.02886   | 0.00577   | 0.00289   |
| 2    | 0.000424  | 0.00001534| 0.37802 $\times 10^{-5}$ |
| 3    | 0.0000108 | 0.5643 $\times 10^{-7}$ | 0.6435 $\times 10^{-8}$ |
| 4    | 0.5051 $\times 10^{-6}$ | 0.3854 $\times 10^{-9}$ | 0.1821 $\times 10^{-10}$ |

To compute the optimal coefficients, we need to calculate the roots $\lambda_k$, $k = 1, \ldots, m - 1$, $|\lambda_k| < 1$ of the polynomial defined by equality (54)

$$P_{2m-2}(\lambda) = \sum_{s=0}^{2m-2} \rho_s(2m-2)\lambda^s = (1 - e^{2h})(1 - \lambda)^{2m-2} - 2(\lambda(e^{2h} + 1) - e^h(\lambda^2 + 1))$$

$$\times \left[ h(1 - \lambda)^{2m-4} + \frac{h^3(1 - \lambda)^{2m-6}}{3!} E_2(\lambda) + \cdots + \frac{h^{2m-3} E_{2m-4}(\lambda)}{(2m-3)!} \right],$$

(104)

where $E_{2k}(\lambda) = \sum_{s=0}^{2k} e_s \lambda^s$ is the Euler–Frobenius polynomial of degree $2k$. To calculate the coefficients of the polynomial $E_{2k}(\lambda)$ we use the following formula given by Euler

$$e_s = \sum_{j=0}^{s} (-1)^j C_{2k+2}^j (s + 1 - j)^{2k+1}.$$

For the purposes of convenience we denote by $|R(\varphi)|$ the absolute value of the difference (4) of the quadrature formula (1). Then by the Cauchy–Schwarz inequality we have

$$|R(\varphi)| \leq \|\varphi\| W_{2}^{(m,m-1)}(0, 1) \cdot \|\ell\| W_{2}^{(m,m-1)^*}(0, 1).$$

The numerical results of $\|\ell(x)\| W_{2}^{(m,m-1)^*}(0, 1)$ for the cases $m = 1, 2, 3, 4$ and $N = 10, 50, 100$ are presented in Table 1.

These numerical results show that the error of the optimal quadrature formula decreases as $m$ and $N$ increase.

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