RANTZER’S DENSITY FUNCTIONS FOR NONAUTONOMOUS DIFFERENTIAL SYSTEMS: A CONVERSE RESULT

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Abstract. We generalize a previous result about existence of density functions: if a linear nonautonomous system is globally asymptotically stable, it is well known that there exists a corresponding density function. In this paper, we prove that this result can be extend to a family of quasilinear systems, provided the existence of a preserving orientation \(C^2\) diffeomorphism between the solutions of both systems.

1. Introduction

Let us consider the system

\[
\dot{z} = g(t, z) \quad \text{with} \quad g(t, 0) = 0 \quad \text{for any} \quad t \in \mathbb{R},
\]

where the existence, uniqueness and unbounded continuation of the solutions is ensured.

Definition 1. A density function of (1.1) is a \(C^1\) function \(\rho: \mathbb{R} \times \mathbb{R}^n \setminus \{0\} \to [0, +\infty)\), integrable outside a ball centered at the origin that satisfies

\[
\frac{\partial \rho(t, z)}{\partial t} + \nabla \cdot [\rho(t, z)g(t, z)] > 0
\]

almost everywhere with respect to \(\mathbb{R}^n\) and for every \(t \in \mathbb{R}\), where

\[
\nabla \cdot [\rho g] = \nabla \rho \cdot g + \rho [\nabla \cdot g],
\]

and \(\nabla \rho, \nabla \cdot g\) denote respectively the gradient of \(\rho\) and divergence of \(g\).

The density functions were introduced by Rantzer in 2001 [12] in order to obtain sufficient conditions ensuring the almost global stability of autonomous systems, we refer the reader to [5] and [8] for a deeper discussion and applications. The extension to the nonautonomous case has been proved in [10] (see also Theorem 4 from [13]):

Proposition 1. Consider the system (1.1) such that \(z = 0\) is a locally stable equilibrium point. If there exists a density function associated to (1.1), then for every initial time to, the sets of points that are not asymptotically attracted by the origin has zero Lebesgue measure.
Converse results (i.e., global asymptotic stability implies the existence of a density function) were presented simultaneously by Rantzer [13] and Monzón [9] in the autonomous case. The extension to the nonautonomous framework was initiated by Monzón in 2006 [10], where a converse result for linear systems

\[ y' = A(t)y, \]

is obtained (see Proposition 3 in this article).

The purpose of this article is to extend the above converse result for the system

\[ x' = A(t)x + f(t, x), \]

where \( A(\cdot) \) is bounded and continuous and \( f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) is continuous with respect to \( t \) and \( x \). Moreover, \( f \) verifies:

1. \( f(t, 0) = 0 \) and \( ||f(t, x)|| \leq \mu \) for any \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^n \).
2. \( ||f(t, x_1) - f(t, x_2)|| \leq \gamma ||x_1 - x_2|| \) for any \( t \in \mathbb{R} \), where \( || \cdot || \) denotes a norm in \( \mathbb{R}^n \).

The fundamental ideas to prove the existence of a density function associated to (1.3) are to consider that, under some suitable conditions, there exists a \( C^2 \) diffeomorphism between the solutions of (1.2) and (1.3). Indeed, a remarkable result of Palmer [11] ensures that if (1.2) is exponentially stable and \( \gamma \) is small enough, an homeomorphism exists. On the other hand, the density function will be constructed by using the Monzón’s density function associated to (1.2).

The paper is organized as follows. In the section 2, we recall some previous results and basic notation. The section 3 states our main result and recall Palmer’s and Monzón’s results. The section 4 is devoted to the proof and is decomposed in two steps: firstly, we prove that Palmer’s homemorphism is a preserving orientation diffeomorphism, provided some additional conditions. Secondly, we construct a density function for (1.3) by using the Monzón’s function of (1.2) and Palmer’s diffeomorphism. The section 5 considers an extension of the result and an illustrative example is presented in section 6.

2. Basic notation

As usual, given a matrix \( M(t) \in M_n(\mathbb{R}) \), its trace will be denoted by \( \text{Tr}\ M(t) \) while its determinant by \( \det M(t) \), the identity matrix is denoted by \( I \). Given the function \( f(t, x) \) of system (1.3), its jacobian matrix is denoted by \( Df(t, x) \) and \( D^2f(t, x) \) is the corresponding bilinear form.

The solution of (1.3) passing through \( \xi \) at \( t_0 \) will be denoted by \( \phi(t, t_0, \xi) \). It will be interesting to consider the map \( \xi \mapsto \phi(t, t_0, \xi) \) and its properties. Indeed, if \( f \) is \( C^1 \), it is well known (see e.g. [3, Chap. 2]) that \( \partial \phi(t, t_0, \xi)/\partial \xi = \phi_{\xi}(t, t_0, \xi) \) satisfies the matrix differential equation

\[
\begin{cases}
\frac{d}{dt} \phi_{\xi}(t, t_0, \xi) = \{A(t) + Df(t, \phi(t, t_0, \xi))\} \phi_{\xi}(t, t_0, \xi), \\
\phi_{\xi}(t_0, t_0, \xi) = I.
\end{cases}
\]

Moreover, it is proved that (see e.g., Theorem 4.1 from [3 Ch.V]) if \( f \) is \( C^r \) with \( r > 1 \), then the map \( \xi \mapsto \phi(t, t_0, \xi) \) is also \( C^r \). In particular, if \( f \) is \( C^2 \), we can verify that the second derivatives \( \partial^2 \phi(s, t_0, \xi)/\partial \xi \xi_i \) are solutions of the system of
differential equations
\[
\frac{d}{dt} \frac{\partial^2 \phi}{\partial \xi_j \partial \xi_i} = \{A(t) + Df(t, \phi)\} \frac{\partial^2 \phi}{\partial \xi_j \partial \xi_i} + D^2 f(t, \phi) \frac{\partial \phi}{\partial \xi_j} \frac{\partial \phi}{\partial \xi_i}
\]
(2.2)
\[
\frac{\partial^2 \phi}{\partial \xi_j \partial \xi_i} = 0,
\]
with \( \phi = \phi(t, t_0, \xi) \), for any \( i, j = 1, \ldots, n \).

A fundamental matrix of (1.2) will be denoted by \( \Psi(t) \) and \( \Psi(t, s) = \Psi(t)\Psi^{-1}(s) \) is the corresponding transition matrix. Hence, the solution of (1.2) passing through \( y_0 \) at \( t_0 \) will be denoted by \( \Psi(t, t_0) y_0 \). Moreover, it is easy to verify that
\[
\frac{d}{dt} \Psi(t, s) = A(t)\Psi(t, s) \quad \text{and} \quad \frac{d}{ds} \Psi(t, s) = -\Psi(t, s)A(s).
\]

It will be useful to define the function \( F: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) and the \( n \times n \) matrix \( Z(s, r) \), where
(2.4)
\[
F(r, \xi) = \Psi(0, r)Df(r, \phi(r, 0, \xi))\Psi(r, 0)
\]
for any initial condition of (1.3), while \( Z(s, r) \) is the transition matrix of
(2.5)
\[
Z' = F(s, r)Z.
\]

3. MAIN RESULT

**Theorem 1.** If for any initial condition \( \xi \) of (1.3):

(A1) There exist some constants \( K \geq 1 \) and \( \alpha > 0 \) such that
(3.1)
\[
||\Psi(t, s)|| \leq Ke^{-\alpha(t-s)}, \quad \text{for any} \quad t \geq s.
\]

(A2) The Lipschitz constant of \( f \) satisfies:
(3.2)
\[
\gamma \leq \alpha/4K,
\]

(A3) \( f(\cdot, \cdot) \) is \( C^2 \) and its first derivative is such that
\[
\int_{-\infty}^{0} ||F(r, \xi)||_{\infty} \, dr < 1.
\]

(A4) \( A(t) \) and \( Df(t, \phi(t, 0, \xi)) \) are such that
\[
\lim_{s \to -\infty} \int_{s}^{0} \text{Tr} A(r) \, dr > -\infty \quad \text{and} \quad \liminf_{s \to -\infty} \int_{s}^{0} \text{Tr}\{A(r) + Df(r, \phi(r, 0, \xi))\} \, dr > -\infty.
\]

(A5) The second derivatives of \( \phi(s, 0, \xi) \) with respect to the initial condition are such that the following limit exists
(3.3)
\[
\lim_{s \to -\infty} \int_{s}^{0} Z(s, r)D^2 f(r, \phi(r, 0, \xi)) \frac{\partial \phi(s, 0, \xi)}{\partial \xi_j} \frac{\partial \phi(r, 0, \xi)}{\partial \xi_i} \, dr,
\]
then there exists a density function \( \bar{\rho} \in C(\mathbb{R} \times \mathbb{R}^n \setminus \{0\}, [0, +\infty)) \) associated to (1.3).

The proof is based in two results

**Proposition 2** (Palmer [11]). If the assumptions (A1)–(A2) are verified, there exists a unique function \( H: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) satisfying

i) \( H(t, x) - x \) is bounded in \( \mathbb{R} \times \mathbb{R}^n \),

ii) If \( t \mapsto x(t) \) is a solution of (1.3), then \( H[t, x(t)] \) is a solution of (1.2).
Moreover, $H$ is continuous in $\mathbb{R} \times \mathbb{R}^n$ and

$$|H(t, x) - x| \leq 4K\mu^{-1}$$

for any $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. For each fixed $t$, $H(t, x)$ is a homeomorphism of $\mathbb{R}^n$. $L(t, x) = H^{-1}(t, x)$ is continuous in $\mathbb{R} \times \mathbb{R}^n$ and if $y(t)$ is any solution of $(1.2)$, then $L[t, y(t)]$ is a solution of $(1.3)$.

**Proposition 3** (Monzón, [10]). If $(1.2)$ is globally asymptotically stable, then there exists a $C^1$ density function $\rho: \mathbb{R} \times \mathbb{R}^n \setminus \{0\} \to [0, +\infty)$.

**Remark 1.** In the next section, we will see that the construction of the homeomorphism $H$ considers the behavior of $\phi(t, 0, \xi)$ for any $t \in (-\infty, \infty)$. In particular, to prove that $H$ is a $C^2$ preserving orientation diffeomorphism will require to know the behavior on $(-\infty, 0]$. Indeed, notice that:

(A1) says that $(1.2)$ is exponentially stable at $+\infty$. In addition, let us recall that uniform asymptotical and exponential stability are equivalent in the linear case (see [11] or Theorem 4.11 from [7]).

(A2) is a necessary condition to prove the existence of the Palmer’s homeomorphism between the solutions of $(1.2)$ and $(1.3)$. Moreover, by Gronwall’s inequality, it is easy to deduce

$$||\phi(t, t_0, x_0)|| \leq Ke^{(-\alpha + K\gamma)(t-t_0)}||x_0||,$$

which implies the exponential stability of $(1.3)$ at $t = +\infty$.

(A3) is a technical assumption, introduced to ensure that the homeomorphism $H(t, x)$ stated in Proposition 2 is a $C^1$ diffeomorphism. Moreover, let us note that appears in some results about asymptotical equivalence (see e.g., [1, Sec.2]).

(A4) is introduced in order to ensure that $H$ is a preserving orientation diffeomorphism. We emphasize that this assumption is related to Liouville’s formula and is used in the asymptotic integration literature (see e.g., [15]).

(A5) is introduced to ensure that $H$ is a $C^2$ diffeomorphism. It is important to observe that the fundamental matrix of $(2.5)$ can be constructed by variation of parameters (see [1] and the next section for details) and is well defined by (A3).

### 4. Proof of main result

#### 4.1. Preparatory lemmatas.

**Lemma 1.** If (A1)–(A3) are satisfied, then, for any fixed $t$, the function $H(t, x)$ is a preserving orientation diffeomorphism.

**Proof.** By following the lines of the Palmer’s proof [11, Lemma 2] tailored for our purposes, $H$ is constructed as follows:

$$H[t, \phi(t, 0, \xi)] = \phi(t, 0, \xi) + \chi(t, (0, \xi)),$$

where $t \mapsto \chi(t, (0, \xi))$ is the unique bounded solution of the system

$$z' = A(t)z - f(t, \phi(t, 0, \xi)),$$

and $t \mapsto \phi(t, 0, \xi)$ is the unique solution of $(1.3)$ passing through $\xi$ at $t = 0$.

It is easy to prove that

$$\chi(t, (0, \xi)) = -\int_{-\infty}^{t} \Psi(t, s)f(s, \phi(s, 0, \xi)) \, ds$$
and the homeomorphism $H[t, \phi(t, 0, \xi)]$ can be written as follows

$$H[t, \phi(t, 0, \xi)] = \phi(t, 0, \xi) - \int_{-\infty}^{t} \Psi(t, s)f(s, \phi(s, 0, \xi)) \, ds.$$  

It is straightforward to verify that $H[t, \phi(t, 0, \xi)]$ is a solution of (1.2) passing through $H[0, \xi]$ at $t = 0$. In consequence, the uniqueness of the solution implies that

(4.1) $H[t, \phi(t, 0, \xi)] = \Psi(t, 0)H[0, \xi].$

The proof that $H[t, \phi(t, 0, \xi)]$ is an homeomorphism for any fixed $t$ is given by Palmer in [11, pp.756–757]. Turning now to the proof that is a preserving orientation diffeomorphism for any fixed $t$, we will decompose it in several steps.

**Step 1:** Some properties about the homeomorphism of initial conditions, $H[0, \xi].$

By using the fact that $f$ is $C^1$ combined with (2.1)–(2.3), we can deduce that:

$$\frac{\partial H[0, \xi]}{\partial \xi} = I - \int_{-\infty}^{0} \Psi(0, s)Df(s, \phi(s, 0, \xi)) \frac{\partial \phi(s, 0, \xi)}{\partial \xi} \, ds$$

(4.2)  

In consequence, the differentiability of $H[0, \xi]$ follows if and only if the limit above exists.

**Step 2:** $H[0, \xi]$ is a preserving orientation diffeomorphism.

By (2.1), we know that $\frac{\partial \phi(s, 0, \xi)}{\partial \xi}$ is solution of the equation:

$$\begin{aligned}
Y'(s) &= \{A(s) + Df(s, \phi(s, 0, \xi))\}Y(s) \\
Y(0) &= I.
\end{aligned}$$

(4.3)

By (2.3) and (4.3), the reader can verify that $\Psi(0, s)\frac{\partial \phi(s, 0, \xi)}{\partial \xi}$ is solution of the matrix differential equation

$$\begin{aligned}
Z'(s) &= \{\Psi(0, s)Df(s, \phi(s, 0, \xi))\Psi(s, 0)\}Z(s) \\
Z(0) &= I.
\end{aligned}$$

(4.4)

Moreover, we know that there exists a nonsingular matrix $C$ such that

$$\Psi(0, s)\frac{\partial \phi(s, 0, \xi)}{\partial \xi} = Z(s, 0)C,$$

(4.5)

where $Z(s, 0)$ a transition matrix related to (4.4).

A well known result of successive approximations (see e.g., Chap.1 of [1]) states that

$$Z(s, 0) = I - \int_{s}^{0} F(r, \xi) \, dr + \sum_{k=2}^{+\infty} (-1)^{k-1} \left( \int_{s}^{0} F(r_1, \xi) \, dr_1 \cdots \int_{r_{k-1}}^{0} F(r_k, \xi) \, dr_k \right),$$

where $F(r, \xi)$ is defined by (2.4). Moreover, we also know that

$$||Z(s, 0)|| \leq \exp \left( \int_{s}^{0} ||F(r, \xi)|| \, dr \right)$$
and (4.5) combined with (A3) implies that

\[ \frac{\partial H[0, \xi]}{\partial \xi} = \lim_{s \to -\infty} \Psi(0, s) \frac{\partial \phi(s, 0, \xi)}{\partial \xi}, \]

is well defined. Notice that the continuity of \( \frac{\partial \phi(s, 0, \xi)}{\partial \xi} \) (ensured by Theorem 4.1 from [6, Ch.V]) implies the continuity of \( \frac{\partial H[0, \xi]}{\partial \xi} \) and we conclude that \( H[0, \xi] \) is differentiable.

The Liouville’s formula (see e.g., Theorems 7.2 and 7.3 from [3]) says that \( \det \Psi(0, s) > 0 \) and \( \det \frac{\partial \phi(s, 0, \xi)}{\partial \xi} > 0 \) for any \( s \leq 0 \) and (A5) implies that these inequalities are preserved at \( s = -\infty \), and we conclude that \( H[0, \xi] \) is a preserving orientation diffeomorphism.

**Step 3:** End of proof.

By (4.1) combined with uniqueness of solutions, we can verify that

\[ H[t, \phi(t, 0, \xi) + \Delta] = \Psi(t) H[0, \xi + \Delta] \]

for any \( \Delta \in \mathbb{R}^n \).

Now, notice that

\[ H[t, \phi(t, 0, \xi) + \Delta] - H[t, \phi(t, 0, \xi)] = \Psi(t) \{ H[0, \xi + \Delta] - H[0, \xi] \}, \]

and, we can conclude that, as \( H[0, \xi] \) is a diffeomorphism, then \( H[t, \phi(t, 0, \xi)] \) is a diffeomorphism satisfying

\[ \frac{\partial H[t, \phi(t, 0, \xi)]}{\partial \phi(t, 0, \xi)} = \Psi(t) \frac{\partial H[0, \xi]}{\partial \xi}. \]

Finally, Liouville’s formula shows that

\[ \det \Psi(t, 0) = \det \Psi(t) \det \Psi^{-1}(0) \]

\[ = \exp \left( \int_{0}^{t} \text{Tr} A(s) ds \right) \det \Psi^{-1}(0) > 0 \]

and the preserving orientation of \( H[t, x(t)] \) for any fixed \( t \) follows from above. \( \square \)

**Remark 2.** The matrix differential equation (4.3) can be seen as a perturbation of the matrix equation

(4.6)

\[ \begin{cases} X'(s) = A(s)X(s) \\ X(0) = I \end{cases} \]

related to (1.2). In addition, (4.6) has a solution \( s \mapsto X(s) = \Psi(s, 0) \).

Notice that \( \Psi(0, s)X(s) = I \) while Lemma 1 says that \( s \mapsto \Psi(0, s)Y(s) \) exists at \( s = -\infty \). This fact prompt us that the behavior of (3.6) and (1.6) at \( s \to -\infty \) has some relation weaker than asymptotic equivalence. Indeed, in [2] it is proved that (A3) is a necessary condition for asymptotic equivalence between a linear system and a linear perturbation.

**Lemma 2.** If (A1)–(A5) are satisfied, then, for any fixed \( t \), the function \( H(t, x) \) is a \( C^2 \) function.

**Proof.** As in the previous result, the proof will be decomposed in several steps:
Step 1: About $\partial^2 H[0, \xi]/\partial \xi_j \partial \xi_i$.
For any $i, j \in \{1, \ldots, n\}$, we can verify that
\[
\frac{\partial^2 H}{\partial \xi_j \partial \xi_i}[0, \xi] = -\int_{-\infty}^{0} \Psi(0, s) D^2 f(s, \phi(s, 0, \xi)) \frac{\partial \phi(s, 0, \xi)}{\partial \xi_j} \frac{\partial \phi(s, 0, \xi)}{\partial \xi_i} \, ds
- \int_{-\infty}^{0} \Psi(0, s) D f(s, \phi(s, 0, \xi)) \frac{\partial^2 \phi(s, 0, \xi)}{\partial \xi_j \partial \xi_i} \, ds.
\]
Now, by using (2.2) and (2.3), the reader can verify that
\[
\frac{\partial^2 H}{\partial \xi_j \partial \xi_i}[0, \xi] = -\int_{-\infty}^{0} \frac{d}{ds} \left( \frac{\partial^2 \phi(s, 0, \xi)}{\partial \xi_j \partial \xi_i} \right) \, ds
= \lim_{s \to -\infty} \Psi(0, s) \frac{\partial^2 \phi(s, 0, \xi)}{\partial \xi_j \partial \xi_i}
\]
and the existence of $\partial^2 H[0, \xi]/\partial \xi_j \partial \xi_i$ follows if and only if the limit above exists.

Step 2: $\partial^2 H[0, \xi]/\partial \xi_j \partial \xi_i$ is well defined.

The reader can verify that:
\[
\begin{cases}
\frac{d}{ds} \Psi(0, s) \frac{\partial^2 \phi}{\partial \xi_j \partial \xi_i} = F(s, \xi) \frac{\partial \phi}{\partial \xi_i} + D^2 f(s, \phi) \frac{\partial \phi}{\partial \xi_j} \frac{\partial \phi}{\partial \xi_i}, \\
\Psi(0, s) \frac{\partial^2 \phi}{\partial \xi_j \partial \xi_i} = 0,
\end{cases}
\]
where $\phi = \phi(s, 0, \xi)$ and $F(s, \xi)$ is defined in (2.4). On the other hand, by (A5), we can see that
\[
\Psi(0, s) \frac{\partial^2 \phi(s, 0, \xi)}{\partial \xi_j \partial \xi_i} = -\int_{s}^{0} Z(s, r) D^2 f(r, \phi(r, 0, \xi)) \frac{\partial \phi(r, 0, \xi)}{\partial \xi_j} \frac{\partial \phi(r, 0, \xi)}{\partial \xi_i} \, dr,
\]
exists at $s \to -\infty$ and $\partial^2 H[0, \xi]/\partial \xi_j \partial \xi_i$ is well defined and continuous with respect to $\xi$.

Step 3: End of proof:
As before, a consequence of (4.1) combined with the uniqueness of the solution of (1.3) implies
\[
H[t, \phi + h_2 \bar{e}_i] - H[t, \phi] = \Psi(0, s) \left\{ H[0, \xi + h_2 \bar{e}_i] - H[0, \xi] \right\},
H[t, \phi + h_1 \bar{e}_j + h_2 \bar{e}_i] - H[t, \phi + h_1 \bar{e}_j] = \Psi(0, s) \left\{ H[0, \xi + h_1 \bar{e}_j + h_2 \bar{e}_i] - H[0, \xi + h_1 \bar{e}_j] \right\},
\]
where $\phi = \phi(t, 0, \xi) = (\phi_1(t, 0, \xi), \ldots, \phi_n(t, 0, \xi))$. Now, we can deduce
\[
\frac{\partial H[t, \phi]}{\partial \phi_i(t, 0, \xi)} = \Psi(0, s) \frac{\partial H[0, \xi]}{\partial \xi_i} \quad \text{and} \quad \frac{\partial H[t, \phi + h_1 \bar{e}_j]}{\partial \phi_i(t, 0, \xi)} = \Psi(0, s) \frac{\partial H[0, \xi + h_1 \bar{e}_j]}{\partial \xi_i}.
\]
The equality
\[
\frac{\partial H[t, \phi(t, 0, \xi) + h \bar{e}_j]}{\partial \phi_i(t, 0, \xi)} - \frac{\partial H[t, \phi(t, 0, \xi)]}{\partial \phi_i(t, 0, \xi)} = \Psi(t, 0) \left\{ \frac{\partial H[0, \xi + h \bar{e}_j]}{\partial \xi_i} - \frac{\partial H[0, \xi]}{\partial \xi_i} \right\},
\]
implies that
\[
\frac{\partial^2 H[t, \phi(t, 0, \xi)]}{\partial \phi_j(t, 0, \xi) \partial \phi_i(t, 0, \xi)} = \Psi(t, 0) \frac{\partial^2 H[0, \xi]}{\partial \xi_j \partial \xi_i}.
\]
is well defined for any fixed $t$. Finally, the continuity of the partial derivatives $\partial^2 H[0, \xi] / \partial \xi_j \partial \xi_k$ combined with the identity above implies that $H$ is $C^2$. □

4.2. Density function. As we pointed out in Remark 1, assumption (A1) implies that (1.2) is uniformly asymptotically stable, which is a particular case of global asymptotical stability. Now, by Proposition 3, there exists a density function $\rho \in C^1(\mathbb{R} \times \mathbb{R}^n \setminus \{0\}, [0, +\infty))$ associated to (1.2).

By following the ideas for the autonomous case studied by Monzón [9, Prop. III.1], we define

$$\bar{\rho}(t, x) = \rho(t, H(t, x)) \left| \frac{\partial H(t, x)}{\partial x} \right|,$$

where $H(\cdot, \cdot)$ is the $C^2$ preserving orientation diffeomorphism stated in Lemmas 1–2, $x$ is any initial condition of (1.3) and $| \cdot |$ denotes a determinant.

**Remark 3.** In spite that in the previous subsection, the initial condition and the determinant were respectively denoted by $\xi$ and $\det(\cdot)$, the notation of (4.7) is classical in the density function literature. The reader will not be disturbed by this fact.

We shall prove that (4.7) satisfies the properties of Definition 1 with $g(t, x) = A(t)x + f(t, x)$. Indeed, $\bar{\rho}$ is non–negative since $\rho$ is non–negative and $H$ is preserving orientation. In addition, $\bar{\rho}$ is $C^1$ since $H$ is $C^2$.

**Lemma 3.** The function (4.7) is integrable outside any ball centered at the origin.

**Proof.** Let $B$ be an open ball centered at the origin. By using a Liouville’s result (see e.g., [6, Corollary 3.1]), we know that

$$\frac{\partial \rho}{\partial \eta}(t, x) + \nabla \cdot (\bar{\rho}g)(t, x) > 0 \quad \text{a.e. in } \mathbb{R}^n.$$
where \( \eta = \tau + t \). Now, it is easy to verify that:

\[
\frac{\partial \bar{\rho}}{\partial t}(t, x) + \nabla \cdot (\bar{\rho} g)(t, x) = \frac{\partial}{\partial \eta} \left\{ \bar{\rho}(\tau + t, \phi(\tau + t, t, x)) \left| \frac{\partial \phi(\tau + t, t, x)}{\partial x} \right| \right\}_{\tau=0}
\]

\[
= \frac{\partial}{\partial \eta} \left\{ \rho(\tau + t, H[\tau + t, \phi(\tau + t, t, x)]) \right\}
\]

\[
\left| \frac{\partial H[\tau + t, \phi(\tau + t, t, x)]}{\partial \phi(\tau + t, t, x)} \right| \left| \frac{\partial \phi(\tau + t, t, x)}{\partial x} \right| \right\}_{\tau=0}
\]

\[
= \frac{\partial}{\partial \eta} \left\{ \rho(\tau + t, H[\tau + t, \phi(\tau + t, t, x)]) \right\}
\]

\[
\left| \frac{\partial H[\tau + t, \phi(\tau + t, t, x)]}{\partial x} \right| \right\}_{\tau=0}.
\]

Secondly, a consequence of (4.1) is

\[ H[\tau + t, \phi(\tau + t, t, x)] = \Psi(\tau + t, t) H(t, x), \]

which implies:

\[
\frac{\partial \bar{\rho}}{\partial t}(t, x) + \nabla \cdot (\bar{\rho} g)(t, x) = \frac{\partial}{\partial \eta} \left\{ \rho(\tau + t, H[\tau + t, \phi(\tau + t, t, x)]) \right\}
\]

\[
\left| \frac{\partial H[\tau + t, \phi(\tau + t, t, x)]}{\partial \phi(\tau + t, t, x)} \right| \left| \frac{\partial \phi(\tau + t, t, x)}{\partial x} \right| \right\}_{\tau=0}
\]

\[
= \mathcal{A}_1(\tau + t, x) + \mathcal{A}_2(\tau + t, x) \bigg|_{\tau=0},
\]

where \( \mathcal{A}_1(\cdot, \cdot) \) and \( \mathcal{A}_2(\cdot, \cdot) \) are respectively defined by

\[
\mathcal{A}_1(\tau + t, x) = \frac{\partial}{\partial \eta} \left\{ \rho(\tau + t, \Psi(\tau + t, t) H(t, x)) \right\} \left| \frac{\partial \Psi(\tau + t, t) H(t, x)}{\partial x} \right| \]

and

\[
\mathcal{A}_2(\tau + t, x) = \rho(\tau + t, \Psi(\tau + t, t) H(t, x)) \frac{\partial}{\partial \eta} \left\{ \left| \frac{\partial \Psi(\tau + t, t) H(t, x)}{\partial x} \right| \left| \frac{\partial H(t, x)}{\partial x} \right| \right\}
\]

As

\[
\mathcal{A}_1(t, x) = \left\{ \frac{\partial}{\partial \eta} \left( H(t, x)) + \nabla \rho(t, H(t, x)) A(t) H(t, x) \right) \right\} \bigg|_{\tau=0} \left| \frac{\partial H(t, x)}{\partial x} \right|
\]
and
\[ A_2(t, x) = \rho(t, H(t, x)) \operatorname{Tr} A(t) H(t, x) \left| \frac{\partial H(t, x)}{\partial x} \right|, \]
we can conclude that
\[ \frac{\partial \tilde{\rho}}{\partial t}(t, x) + \nabla \cdot (\tilde{\rho} g)(t, x) = A_1(t, x) + A_2(t, x) \]
\[ = \left\{ \frac{\partial \rho}{\partial \eta}(t, H(t, x)) + \nabla \cdot (\rho A)(t, H(t, x)) \right\} \left| \frac{\partial H(t, x)}{\partial x} \right|, \]
which is positive since it is the product of two positive terms. The positivity of the first one is ensured by Proposition 3, while the second follows by Lemma 1.

4.3. End of proof. As we commented before, the existence of density function associated to (1.3) is based on the homeomorphism \( H \) constructed by Palmer (Proposition 2) and the existence of the density function \( \rho(t, x) \) associated to (1.2) constructed by Monzón (Proposition 3). Lemmas 1 and 2 ensure that \( H \) is a \( C^2 \) preserving orientation diffeomorphism while Lemmas 3 and 4 state that the function described by (4.7)
\[ \tilde{\rho}(t, x) = \rho(t, H(t, x)) \left| \frac{\partial H(t, x)}{\partial x} \right| \]
is indeed a density function associated to (1.3) and the result follows.

5. An application to nonlinear systems

Let us consider the nonlinear system
\[ x' = g(t, x) \]
where \( g \) is a \( C^2 \) function satisfying
\[ (H1') g(t, 0) = 0 \text{ and } ||g(t, x)|| \leq \bar{\mu} \text{ for any } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n. \]
\[ (H2') ||g(t, x_1) - g(t, x_2)|| \leq L||x_1 - x_2|| \text{ for any } t \in \mathbb{R}, \text{ where } || \cdot || \text{ denotes a norm in } \mathbb{R}^n. \]

Theorem 2. If:

(G1) The linear system \( y' = Dg(t, 0)y \) is exponentially stable and its transition matrix satisfy
\[ ||\Phi(t, 0)|| \leq Ke^{-\alpha(t-s)} \text{ for some } K \geq 1 \text{ and } \alpha > 0. \]

(G2) The Lipschitz constant \( L \) satisfies
\[ L + ||Dg(t, 0)|| \leq \alpha/4K \text{ for any } t \in \mathbb{R}, \]

(G3) The first derivative of \( g \) is such that
\[ \int_{-\infty}^{0} ||\tilde{F}(r, \xi)||_\infty dr < \infty \]
with
\[ \tilde{F}(r, \xi) = \Phi(0, r)\{ Dg(r, \varphi(r, 0, \xi)) - Dg(r, 0)\} \Phi(r, 0), \]
where \( \varphi(r, 0, \xi) \) is the solution of (5.1) passing through \( \xi \) at \( r = 0. \)
(G4) \( Dg(t,0) \) and \( Dg(t, \varphi(t,0,\xi)) \) are such that

\[
\liminf_{s \to -\infty} - \int_s^0 \text{Tr} \, Dg(r,0) \, dr > -\infty \quad \text{and} \quad \liminf_{s \to -\infty} - \int_s^0 \text{Tr} \, Dg(r, \varphi(r,0,\xi)) \, dr > -\infty
\]

for any initial condition \( \xi \).

(G5) The second derivatives of \( \varphi(s,0,\xi) \) with respect to the initial condition are such that the following limit exists

\[
\lim_{s \to -\infty} \int_s^0 \tilde{Z}(s,r) D^2 g(r, \varphi(r,0,\xi)) \frac{\partial \varphi(s,0,\xi)}{\partial \xi_j} \frac{\partial \varphi(r,0,\xi)}{\partial \xi_i} \, dr,
\]

where \( \tilde{Z}(s,r) \) is the transition matrix of

\[
\tilde{Z}' = \tilde{F}(s,\xi) \tilde{Z}.
\]

then there exists a density function \( \bar{\rho} \in C(\mathbb{R} \times \mathbb{R}^n \setminus \{0\}, [0, +\infty)) \) associated to \((5.1)\).

6. Example

Let us consider the scalar equation

(6.1) \[
x' = -ax + h(t) \arctan(x),
\]

where \( a > 0 \) and \( h: \mathbb{R} \to \mathbb{R}_+ \) is a bounded and continuous. In addition, we will suppose that

(6.2) \[
t \mapsto h(r)e^{-ar} \quad \text{is integrable on} \quad (-\infty, 0].
\]

It is straightforward to verify that \((H1)-(H2)\) are satisfied with \( \mu = \gamma = ||h||_\infty \).

We can see that \((A1)\) is satisfied since \( \Psi(t,s) = e^{-a(t-s)} \) and \((A2)\) is satisfied if and only if \( 4||h||_\infty \leq a \).

Moreover, \((A3)\) is satisfied since for any solution \( \phi(t,0,\xi) \) of (6.1) it follows that

\[
\int_{-\infty}^0 \frac{h(r)}{1 + \phi^2(r,0,\xi)} \, dr < +\infty.
\]

It is interesting to point out \( \phi(t,0,\xi) \) is unbounded and have exponential growth at \( t = -\infty \), indeed. Now, it is easy to note that

\[
\lim_{s \to -\infty} -as = +\infty,
\]

which implies that

\[
\liminf_{s \to -\infty} - \int_s^0 \left\{ -a + \frac{h(r)}{1 + \phi^2(r,0,\xi)} \right\} \, dr > -\infty
\]

and \((A4)\) is satisfied.

Letting \( f(t,x) = h(t) \arctan(x) \), it is important to note that

\[
\frac{\partial \phi(s,0,\xi)}{\partial \xi} = \exp \left\{ -as - \int_s^0 Df(u, \phi(u,0,\xi)) \, du \right\}
\]

and a straightforward computation shows that \((A5)\) is satisfied if and only if

\[
e^{as} \frac{\partial^2 \phi}{\partial \xi^2} = -\exp \left\{ -\int_s^0 Df(u, \phi) \, du \right\} \left[ \int_s^0 \exp \left( -au - \int_u^0 Df(r, \phi) \, dr \right) D^2 f(u, \phi) \, du \right],
\]

has limit when \( s \to -\infty \). Above, \( \phi \) is either \( \phi(u,0,\xi) \) or \( \phi(r,0,\xi) \).
Finally, (6.2) implies that (A5) is satisfied since the function

$$u \mapsto -2h(u)e^{-au}\phi(u,0,\xi) = e^{-au}D^2 f(u,\phi(u,0,\xi))$$

is integrable on $(-\infty, 0]$.  

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