Stripping the planar Quantum Compass Model to its basics

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(Dated: 5th January 2021)

We introduce a novel mean field theory (MFT) around the exactly soluble two leg ladder limit for the
planar quantum compass model (QCM). In contrast to usual MFT, our construction respects the
stringent constraints imposed by emergent, lower (here \(d = 1\)) dimensional gauge like symmetries of
the QCM. Specializing our construction to the QCM on a periodic 4-leg ladder, we find that a first
order transition separates two mutually dual Ising nematic phases, in good accord with state-of-the-
art numerics for the planar QCM. One pseudo spin-flip excitations in the ordered phase turn out to be
two (Jordan-Wigner) fermion bound states, reminiscent of spin waves in spin-1/2 Heisenberg chains.
We discuss the novel implications of our work on (1) emergence of coupled orbital and magnetic
ordered and liquid like disordered phases, and (2) a rare instance of orbital-spin separation in \(d > 1\),
in the context of a Kugel-Khomskii view of multi-orbital Mott insulators.

Introduction. Frustration in many body systems continues to evince interest for over four decades
now. The wide range of exotic physical responses exhibited by transition metal oxides (TMO) are believed to originate from a complex interplay between strongly coupled charge, orbital, spin, and lattice degrees of freedom [1]. Adding frustration results in exponential degeneracies of classical ground states. Quantum fluctuations are believed to select novel ordered states out of this manifold by “order by disorder” mechanism [2]. The upshot is the possible emergence of truly novel, unconventional ordered phases of matter, characterized by fractionalized excitations in spatial dimension, \(d > 1\). These issues have been addressed extensively using a variety of analytic and numerical techniques each having their own strengths and limitations [3]. In absence of exact solutions in \(d > 1\), controlled approximations that capture the main essence of a given problem remain an attractive option.

The quantum compass model (QCM) on a square lattice (\( H_{QC} = \frac{1}{2} \sum_{i,j} J_{ij} \sigma^x_{i} \sigma^x_{j} + J_{z} \sigma^z_{i} \sigma^z_{j} \)) is a particularly illustrative case in point. The “orbital only” QCM has received extensive attention as a model for orbital order in TMO based Mott insulators [4], and in the context of qubit implementation in Josephson junction arrays [5]. Moreover, QCM on a square lattice is dual to the Xu-Moore model for \( p + ip \) superconductor arrays [6] and Kitaev’s toric code (TC) model in transverse field [7]. These dualities are useful to unveil the hidden connections between classical and topological orders [8] in higher dimensions. On the analytic front, QCM has been extensively studied in 1d chains [9] and on 2-leg ladders [10, 11]: the latter one has been very recently studied [12] in the context of many-body localization.

Inspired by the “chain-mean-field” approach [13], one may wonder whether the exactly soluble 2-leg ladder QCM may serve as an appropriate template to investigate the QCM on a square lattice as a collection of coupled 2-leg ladders. Ref. [13] finds a first-order quantum phase transition (QPT) between two states with \([|\sigma^x_i\rangle = m^x > 0, |J_z| > \alpha |J_z|\] and \([|\sigma^z_i\rangle = m^z > 0, |J_z| < \alpha |J_z|\] with \( \alpha \neq 1 \); thus, the chain-mean-field theory (MFT) violates the rigorous self-duality of square lattice QCM, which requires \( \alpha = 1 \). The (expected) inability of chain-MFT to respect self-duality can be traced back to its inherent inability to treat the lower \((d = 1)\) dimensional gauge like symmetries (GLS) [6] rigorously. In the QCM, various 1d GLS constrain the finite temperature responses. These GLS are: (i) \( P_j = \prod_i \sigma^x_{j,i}, \) acting on the row \( j \) and (ii) \( Q_l = \prod_j \sigma^z_{l,j} \) acting along the column \( l \) of the square lattice. Both \( P_j \) and \( Q_l \) commute with \( H_{QC} \), but \([P_j, Q_l] \neq 0 \) \forall \( i, j \) while \([P_j, P_l] = [Q_l, Q_j] = 0 \) \forall \( i, j \). Also, \([P_j, P_k, Q_l] = [Q_j, Q_l, P_k] = 0 \). Hence, all the eigenstates are two fold degenerate [5] and there are exactly \( 2^L \) low energy states of \( H_{QC} \) (here \( L \) denotes linear size of the system). At \( J_z = J_x \), \( H_{QC} \) is also invariant under a global \((d = 2)\) reflection symmetry \( \sigma^x \leftrightarrow \sigma^z \) (implemented by an operator \( R = \prod_i \exp \left[ \frac{i}{4 \sqrt{2}} (\sigma^x_i + \sigma^z_i) \right] \)). In the thermodynamic limit, all the \( 2^L \) low lying states collapse into each other for \( J_x \neq J_z \) (and \( 2^{L+1} \) states for \( J_x = J_z \) [14], leading to infinite but sub-extensive degeneracy (scales with linear size) of the compass ground state.

Proliferation of non-local defects (like domain walls in 1d Ising model), generated by these 1d GLS, completely obliterate any conventional magnetic order at any temperature, \( T > 0 \) due to Elitzur’s theorem [15]. Remarkably, the directional spin nematic order [6], described by an Ising like variable, \( \langle D \rangle = (\sigma^x_{\hat{x}} - \sigma^z_{\hat{x}}) \) survives the strong fluctuations implied by these 1d GLS, even at finite \( T \). Thus, any approach must obey (i) these GLS and (ii) self-duality. These are stringent constraints for any analytical approximation.
Figure 1. QCM on a periodic square lattice is represented as collection of coupled 2-leg QC ladders; intra-(inter) 2-leg ladder exchange interactions are shown in thick blue (dashed red) arrows.

**Formulation.** In this communication, we construct a novel “mean-field like” approach as a first step toward the full 2d QCM. Specifically, we exploit Mattis’s exact solution for the $H_{QC}$ on a 2-leg ladder [11] by coupling such 2-leg ladders as shown in Fig.1. Now, the Hamiltonian is $H_{QC} = \sum_{j=1}^{N/2} H_{i,j}^{(0)} + H_{int}$, where $H_{i,j}^{(0)}, H_{int}$ denote intra-ladder and inter-ladder interaction terms [16].

$$H_{i,j}^{(0)} = \sum_{j=1}^{N} \left[ J_{x} (S_{j}^{x} S_{j+1}^{x} + T_{j}^{x} T_{j+1}^{y}) + J_{y} S_{j}^{y} T_{j}^{y} \right]$$  \hspace{1cm} (1)

$$H_{int} = J_{z} \sum_{j=1}^{N} \sum_{l=1}^{N/2} S_{j}^{z} T_{j+l}^{z}$$  \hspace{1cm} (2)

We now use the following canonical transformations [11],

$$(S_{j}^{x}, S_{j}^{y}, S_{j}^{z}) = (P_{j}^{x}, 2P_{j}^{y} Q_{j}^{z}, 2P_{j}^{y} Q_{j}^{z})$$

$$(T_{j}^{x}, T_{j}^{y}, T_{j}^{z}) = (-2Q_{j}^{z} P_{j}^{z}, 2Q_{j}^{z} P_{j}^{z}, Q_{j}^{z})$$  \hspace{1cm} (3)

This transformation is nothing but the two site version of well known Kramer-Wannier (KW) duality with an additional rotation of spin basis. To see this, we rotate only the $P^{x}$ basis about $y$-axis by an angle $\pi/2$, $(P_{j}^{z} \rightarrow P_{j}^{z}, P_{j}^{y} \rightarrow -P_{j}^{z})$, then the Eq. (3) implies (i): $2S_{j}^{x} T_{j}^{z} = P_{j}^{z}, S_{j}^{y} = -P_{j}^{z}, T_{j}^{x} = 2(-P_{j}^{y})(-Q_{j}^{z})$ and (ii): $2S_{j}^{y} T_{j}^{z} = -Q_{j}^{z}, T_{j}^{x} = Q_{j}^{z}, S_{j}^{z} = 2Q_{j}^{z} P_{j}^{z}$. Both of these are conventional expressions of the KW duality (apart from the minus signs which could be absorbed by a further rotation, $R_{z}(\pi)$ of $P$, $Q$ spins).

Now using Eq. (3), $H_{i,j}^{(0)}$ and $H_{int}$ read following,

$$H_{i,j}^{(0)} = J_{x} \sum_{j=1}^{N} P_{j}^{x} P_{j+1}^{x} (1 + 4Q_{j}^{z} Q_{j+1}^{z}) + \frac{J_{y}}{2} \sum_{j=1}^{N} P_{j}^{y}$$  \hspace{1cm} (4a)

$$H_{int} = 2J_{z} \sum_{j=1}^{N} \sum_{l=1}^{N/2} P_{j}^{y} Q_{j}^{z} Q_{j+l}^{z}$$  \hspace{1cm} (4b)

When $H_{int}$ is absent, we get a collection of transverse field Ising chains with spins ($P_{j}^{z}$) coupled to static $Z_{2}$ variables ($Q_{j}^{z}$). With $H_{int}$, the $Q^{z}$’s become fully dynamical, pre-empting exact solubility. At this stage, MF decoupling of $P^{\mu}$ and $Q^{\nu}$’s in both Eq. (4a), (4b) gives,

$$H_{1} = \sum_{j=1}^{N/2} \left[ J_{x} \sum_{j=1}^{N} (1 + 4Q_{j}^{z} Q_{j+1}^{z}) P_{j}^{x} P_{j+1}^{x} + \frac{J_{y}}{2} \sum_{j=1}^{N} (1 + 4Q_{j}^{z} Q_{j+1}^{z}) P_{j}^{y} \right]$$  \hspace{1cm} (5a)

$$H_{2} = J_{x} \sum_{j=1}^{N} \sum_{l=1}^{N/2} (4P_{j}^{y} P_{j+1}^{y}) Q_{j}^{z} Q_{j+l}^{z} + \frac{J_{y}}{2} \sum_{j=1}^{N} \sum_{l=1}^{N/2} (2P_{j}^{y}) Q_{j}^{z} Q_{j+l}^{z}$$  \hspace{1cm} (5b)

Now $H_{1}$ is a collection of $N/2$ 1d TFIMs with couplings determined by two-spin correlations of the $Q^{z}$, ($\nu = x, z$), while $H_{2}$ is another 2d QCM (but on a $N \times N/2$ rectangular lattice) whose coefficients are the correlators of the 1d TFIM. Thus, it may look as if we have complicated the problem. However, this is not so, as we explain now.

First we notice that (i) $\Delta_{z,j}^{\ell} = 4(\langle P_{j}^{x} P_{j+1}^{x} \rangle)$ corresponds to $\langle \sigma^{x}_{j} \sigma^{x}_{j+\ell} \rangle$ in the original spin language; similarly, (ii) $\Delta_{z,j}^{\ell} = 2(\langle P_{j}^{z} \rangle \sim \langle \sigma^{z}_{j} \sigma^{z}_{j+\ell} \rangle)$, (iii) $\Theta_{x,j}^{\ell} = 4(\langle Q_{j}^{z} Q_{j+1}^{z} \rangle \sim \langle \sigma^{z}_{j} \sigma^{z}_{j+\ell+1} \rangle)$, (iv) $\Omega_{j}^{\ell} = 4(\langle Q_{j}^{z} Q_{j+1}^{z} \rangle \sim \langle \sigma^{z}_{j} \sigma^{z}_{j+\ell+2} \rangle)$. Remarkably, all these MF averages thus respect the rigorous 1d GLS of the QCM. This feature, counter-intuitive for any MFT, will play a central role in our analysis, as we show below.

We now notice that if we restrict ourselves to just two coupled QC ladders, Eq. (5b) then reads

$$H_{2} = \sum_{j=1}^{N} \sum_{l=1}^{N/2} \left[ J_{x} \Delta_{z,j}^{\ell} Q_{j}^{z} Q_{j+1}^{z} + J_{y} \Delta_{z,j}^{\ell} Q_{j}^{z} Q_{j+l}^{z} \right]$$  \hspace{1cm} (6)

which is precisely another 2-leg QC ladder for $Q^{z}$ ! Here we assume periodic boundary conditions both along leg and rung directions of the ladder.

Thus we can use the Mattis’s transformation for the $Q^{\nu}$’s in Eq. (6). Writing $Q_{j}^{z} = -2V_{j}^{x} V_{j+1}^{x}$, $Q_{j}^{y} = W_{j}$, and $Q_{j}^{z} = -V_{j}^{z}$, $Q_{j}^{x} = -2V_{j}^{z}$. Eq. (6) reads

$$H_{2} = J_{x} \sum_{j=1}^{N} \left[ \Delta_{z,j}^{\ell} + 4(\langle V_{j}^{x} V_{j+1}^{x} \rangle) \Delta_{z,j}^{\ell} \right] W_{j}^{x} W_{j+1}^{x}$$

$$+ \frac{J_{y}}{2} \sum_{j=1}^{N} \left[ (\Delta_{z,j}^{\ell} + \Delta_{z,j+1}^{\ell}) W_{j}^{x} \right]$$  \hspace{1cm} (7)

So $H_{2}$ is another 1d TFIM of $W^{\mu}$ spins coupled to static $Z_{2}$ fields $V_{j}^{z}$. These local $Z_{2}$ variables are $V_{j}^{z} = 8S_{j}^{x} T_{j+1}^{x} S_{j+1}^{x} T_{j}^{x}$. Remarkably, these are just one of the 1d GLS of $H_{QC}$ for the two coupled QC ladders. We assume $V_{j}^{z} = \pm 1/2, \forall j$, which is strictly valid only for
This restores translation invariance, giving 
\[\Delta_{\mu}^{x} \equiv \Delta_{\mu} \ (\mu = x, z), \ \Theta_{\mu}^{x} \equiv \Theta_{x} \text{ and } \Omega_{\mu}^{x} \equiv \Omega_{z}. \]
So finally,
\begin{align*}
H_{1}^{(l)} &= J_{x}(1 + \Theta_{x}) \sum_{j=1}^{N} P_{j}^{x} P_{j+1}^{x} + \frac{J_{z}}{2} (1 + \Omega_{z}) \sum_{j=1}^{N} P_{j}^{z} \\
H_{2} &= 2J_{x}\Delta_{x} \sum_{j=1}^{N} W_{j}^{x} W_{j+1}^{x} + J_{z}\Delta_{z} \sum_{j=1}^{N} W_{j}^{z}
\end{align*}

We solve the coupled Eqs. (8a), (8b) using the exact solution of 1d TFIM [17]. The four self-consistency equations are written in the following compact manner. Define two vectors \(M^{x}, M^{z}\), with two components \(M^{x}_{a} = \Delta_{x}, \ M^{x}_{b} = \Theta_{x}, \ M^{z}_{a} = \Delta_{z}\), and \(M^{z}_{b} = \Omega_{z}\).

\begin{align*}
M^{x}_{a} &= \int_{0}^{\pi} \frac{dk}{\pi} \left( h_{\sigma} \cos k - 1 \right) \tanh \left( \frac{\beta E_{k}^{x}}{2} \right) \\
M^{z}_{a} &= \int_{0}^{\pi} \frac{dk}{\pi} \left( h_{\sigma}^{-1} \cos k - 1 \right) \tanh \left( \frac{\beta E_{k}^{z}}{2} \right) \quad (9)
\end{align*}

where \(E_{k}^{x} = |J_{x}|(1 + \Theta_{x})\sqrt{1 + h_{\sigma}^{2} - 2h_{\sigma} \cos k} \), \(E_{k}^{z} = |J_{x}|(1 + \Omega_{z})\sqrt{1 + h_{\sigma}^{2} - 2h_{\sigma} \cos k} \), and \(\beta = 1/T\).

We could equivalently have applied the Mattis’s relations (for \(Q^{x}\)) before Eq. (7) to Eqs. (4a), (4b) and then done a MF decoupling, again leading to Eqs. (8a), (8b). Notice that the MF self-consistency Eqs. (9) and (10) faithfully reflect the exact self-duality of the square lattice QCM at \(J_{x} = J_{z} \ (M^{x}_{a} \leftrightarrow M^{z}_{a} \text{ when } J_{x}, \ \Delta_{x}, \ \Omega_{x} \leftrightarrow J_{z}, \ \Delta_{z}, \ \Omega_{z}). \) This is a very positive feature of the present approach, in contrast to that of Ref. [13], which violates this constraint. This important difference can be traced back to the fact that our MF decoupling is performed at a specific two-(particle) spin channel that preserves the GLS.

**Results.** We now present our results. The 2-leg QC ladder exhibits a quantum critical point (see Ref. [11]) separating a “magnetically ordered” and “quantum disordered” phase at \(T = 0\). In original spin variables, this is a continuous transition between \(xx\)-ordered phase to \(zz\)-ordered phase, with Ising nematic order parameter \(\langle D \rangle = \langle +(-)D_{0} \rangle \text{ for } |J_{x}| > \langle < \rangle |J_{z}|\), clearly shown in Fig. 2. This QPT belongs to the well known 2d classical Ising universality class. Remarkably, the 4-leg QC ladder reveals a clear first order transition between the two Ising nematic phases above, (see Fig. 2) precisely at the self-dual point \((J_{x} = J_{z})\). This agrees fully with both exact arguments and numerical results [18, 19]. Interestingly, the MF ground state energy per lattice site, \(e_{gs}(\theta)\), as a function of \(\theta = \tan^{-1}(|J_{x}|/|J_{z}|)\) exhibits a clear cusp (see Fig. 2) at \(\theta = \pi/4\) \((J_{x} = J_{z})\), and agrees closely, both in its functional form and magnitude, with the PCUT results of Vidal et al [7], iPEPS results of Orús et al [19], and Green function Monte-Carlo results of Dorier et al [14]. In fact, at \(\theta = \pi/4\), the point of maximum frustration, our \(e_{gs} = -0.19\) is very close to the \(e_{gs} \approx -0.2\) found from the above numerical techniques. Considering the approximations made here, this is remarkable accord. Thus, most of the ground state correlation energy for the full 2d QCM already seems to be captured by a 4-leg ladder! Equivalences between the QCM and Xu-Moore as well as the transverse field-TC model also imply first-order QPTs at self-dual points in these models [7]. Even more interestingly, we also uncover a hitherto unnoticed (to our best knowledge) duality between a plaquette order, \(\Theta_{x}\) and a next-neighbour \(zz\)-correlation (this “hidden” duality could also be proven analytically, see [20]) in Fig. 3. Both of these also exhibit a clear jump at \(J_{x} = J_{z}\). While possible “hidden” dimer order has been studied earlier [21] in the 2d QCM, the duality between \(\Theta_{x}\) and \(\Omega_{z}\) is a new finding.

What about the excitations? (1) In the \(xx\)-ordered state, the pseudo-spin fluctuation spectrum, given by \(\text{Im} \chi_{(\text{pcut})}(\bar{q}, \omega)\), is the two Jordan-Wigner (JW) fermion continuum. Thus remarkably, a pseudospin-flip in the QCM is a two fermion bound state, reminiscent of the des Cloizeaux-Pearson spin wave spectrum of the \(S = 1/2\) AF Heisenberg chain; (2) The \(\langle \sigma_{x}^{+} \sigma_{x} \rangle\) correlations along the legs (x-direction) rigorously vanish, both at zero and finite T (see [22]), and (3) \(\langle \sigma_{z}^{+} \sigma_{z} \rangle\) correlations along x, z-direction are consistent with a \(T = 0\) symmetry broken ordered state (see [19]) when \(|J_{x}| > |J_{z}|\), as well as with short-range correlation at any finite \(T\). It is indeed remarkable that nematic and plaquette correlators nevertheless dominate; this is a consequence of our novel MF like decoupling, which is carried out by decoupling the “four-spin interaction” at a “two-spin” level, in a way that preserves the GLS. At \(J_{x} = J_{z}, \ T = T_{c}\), the second order endpoint has a Landau mean-field character (see [23]). This is an
expected artifact of any MFT, and proper inclusion of beyond-MF effects to “restore” the classical 2d Ising critical exponents remains a hard, open issue.

Finally, what about fluctuations beyond MFT? There are two types of fluctuations: (1) interactions between the JW fermions, and (2) sudden local quenches due to flipping of one or more of $V_j^\alpha$. In [24], we argue that self-consistent perturbative processes beyond MFT converge almost everywhere in parameter space, except possibly at $J_x = J_z$, $T = T_c$. Thus, we expect our main results to survive beyond-MF fluctuation effects.

**Magnetism in TMO with $t_{2g}$ orbital degeneracy**

Buoyed by very good accord we find with both, iPEPS and PCUT results, as well as by comparison with a 8-leg ladder MFT (see [25]), we assume that our MFT is a good approximation to the planar QCM. Consider a two dimensional TMO with active degenerate orbital degrees of freedom per lattice site. In cases, where the MO orbital octahedron is squashed, the $xy$ orbital is “pushed above” the 2-fold degenerate $xz$, $yz$ orbitals, whence orbital degeneracy is relevant for the $d^1$ and $d^4$ configurations of the TM ion. When the local Hubbard interaction is large compared to the $d$ electron kinetic energy, the Mott insulator is effectively described by Kugel-Khomskii Hamiltonian [26],

$$H_{KK} = J_1 \sum_{i,\alpha=x,z} \left( S_i \cdot S_{i+\hat{\alpha}} + \frac{1}{4} \right) T_i^\alpha T_{i+\hat{\alpha}}^\alpha + J_2 \sum_{\langle (i,j) \rangle} S_i \cdot S_j - J_f \sum_{\langle (i,j) \rangle} S_i \cdot S_j$$

(11)

For a $d^1$ TM ion $S = \frac{1}{2}$, while for the $d^4$ TM ion, $S = 1$ (we assume that the crystal field splitting between $t_{2g}$ and $e_g$ states is larger than the Hund coupling, so can neglect $e_g$ states). Here $J_2$ is the diagonal antiferromagnetic (AF) interaction while $J_f$ is the direct ferromagnetic (FM) term [27].

Suppose orbital order occurs before magnetic order ($T_c > T_N$). Starting from high temperature, the orbital ordering (OO) is captured by the “orbital only” part of Eq. (11); $H_{oo} = \frac{J_2}{4} \sum (\tau_i^\alpha \tau_{i+\hat{\alpha}}^\alpha + \lambda \tau_i^x \tau_{i+\hat{\alpha}}^x)$, where $\lambda$ can differ from unity, either via coupling to the spin fluctuations in Eq. (11) or a coupling to Jahn-Teller modes, or both [4]. The effective Heisenberg super-exchange now explicitly depends on the orbital correlations, and the effective spin couplings read $J_{1x} = J_1 (\tau_i^x \tau_{i+\hat{\alpha}}^x) - J_f$, $J_{1z} = J_1 (\tau_i^z \tau_{i+\hat{\alpha}}^z) - J_f$. Remarkably, the resulting spin model,

$$H_s = \sum_i \sum_{\alpha=x,z} J_{1\alpha} S_i \cdot S_{i+\hat{\alpha}} + J_2 \sum_{\langle (i,j) \rangle} S_i \cdot S_j$$

(12)

is precisely the $J_{1x}$-$J_{1z}$-$J_2$ Heisenberg model successfully used in the Fe-arsenide [28] context, but should obviously be more generally valid. Furthermore, the sign of the coupling strengths $J_{1\alpha}$ could be AF or FM type depending on the values of orbital correlations. If we deal with $S > 1/2$, spin excitations below $T_N$ are qualitatively described by renormalized spin wave theory (RSWT) [29] and are dressed propagating magnons of a stripe magnetic order with $q = (\pi, 0)$.

Remarkably, a most interesting feature reveals itself. At finite $T$, we unearth a rare instance of higher dimensional orbital-spin separation. While the generally gapped orbitons are now two JW fermion bound states, the magnons are over-damped ($T > T_N$) or under-damped ($T < T_N$) bosons. This OSS is especially manifest above $T_b$, when orbitons “break up” into two JW fermions, while spin excitations are over-damped bosons. Thanks to emergent GLS, this OSS should happen in $d = 2, 3$ as well. Following [30], resonant inelastic X-ray scattering (RIXS) experiments could unveil the OSS proposed here. Below $T_N < T_o$, a (magnon) Goldstone mode, along with an amplitude (Higgs) mode, co-existing with a gapped orbiton mode should obtain. For $T_N < T < T_o$, a broad continuum of spin excitations will co-exist with damped (but still well defined) orbitons. Thanks to the emergent 1d GLS, a carrier doped in such a Mott insulator will propagate predominantly along the

\[ \text{Figure 3. Various MF averages for the 4-leg QC ladder at finite T. (a) $\Delta_x$ and dual $\Delta_\tau$ (shown in the inset). (b) the other dual pair $\Theta_x$ and $\Omega_x$ (shown in the inset) are plotted as function of $\theta$, for different T values. All these averages continue showing a sudden jump at $\theta = \pi/4$ ($J_x = J_z$) up to $T = T_c \approx 0.125$, where the discontinuity just vanishes.} \]
legs [31] (physically to maximize kinetic energy gain in the $xx$-orbital ordered phase) when $\langle D \rangle > 0$, leading to complete fractionalization of a doped carrier: we dub this orbital-spin-charge separation! Such an anisotropic conductive state can spawn a multitude of complex orders at lower $T$, but this is beyond the scope of this work.

Conclusion. We have constructed a novel MFT for the planar QCM and implemented it for the 4-leg case. A distinguishing feature of our approach is that the MF decoupling is done at the level of two-spin averages, in contrast to usual MFT. For a 4-leg ladder, our MFT preserves the lower-$d$ ($= 1$) GLS crucial to proper analysis of the QCM. In very good accord with exact arguments and numerics for the full 2$d$ QCM, we find that a first order transition separates two, mutually dual Ising nematic phases at $J_z = J_x$. Our results reveal an exciting (pseudo-) spin fractionalization that may survive in the full planer QCM, and point a way to realization of (i) strongly coupled orbital and magnetic orders, and (ii) orbital-spin-charge separation in 2$d$ multi-orbital Mott insulators. Considering fluctuations beyond ladder-MFT, as well as extension to $d = 2$, are complex avenues for future work and subject of future study.

Acknowledgement. We thank the Department of Atomic Energy, Govt. of India, for funding.

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[24] See section (VI) of the supplementary material for discussions about fluctuations beyond MFT.
[25] See section (V) of the supplementary material for the results of 8-leg QCM ladder MFT.
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Supplemental Material: Stripping the planar Quantum Compass Model to its basics

In this supplemental material, we begin with the derivation of self-consistency equations (Eqs. (9), (10)) and the expression for ground state energy, starting from the 1d transverse field Ising model (TFIM) Hamiltonians (Eqs. (8a), (8b)). Then the exact duality between $\Theta_z$ and $\Omega_z$ is established and different two-spin correlation functions are computed at the mean-field level. Next we discuss the mean-field theory (MFT) results of 8-leg compass ladder. Finally, the effects of possible fluctuations around the mean field theory are explained briefly.

I. DERIVATION OF THE SELF-CONSISTENCY EQUATIONS AND THE EXPRESSION FOR MEAN-FIELD GROUND STATE ENERGY

Exact solution of 1d TFIM using Jordan-Wigner (JW) fermionization is well known (See [1] and references therein), we provide here only few steps to derive the mean-field self-consistency conditions. The mapping between spins $(S = 1/2)$ and spinless JW fermions are following,

$$P_{jl}^\tau = a_{jl}^\dagger \prod_{m<j} (2a_{ml}^\dagger a_{ml} - 1), \quad (l = 1, 2), \quad W_j^\tau = b_j^\dagger \prod_{m<j} (2b_{jm}^\dagger b_{jm} - 1)$$

Using this, Eqs. (8a) and (8b) become

$$H = \frac{t_\alpha}{2} \sum_{j=1}^N (\psi_{j,\alpha} - \psi_{j,\alpha}^\dagger)(\psi_{j+1,\alpha} + \psi_{j+1,\alpha}^\dagger) + \frac{\mu_\alpha}{2} \sum_{j=1}^N (\psi_{j,\alpha}^\dagger \psi_{j,\alpha} - \frac{1}{2}) \quad (\alpha = a, b)$$

here $t_\alpha = \frac{J_\alpha}{2}(1 + \Theta_x)$, $\mu_\alpha = J_z(1 + \Omega_z)$, $t_\alpha = J_x \Delta_x$, $\mu_\beta = 2J_z \Delta_z$ are hopping, pairing, and chemical potentials of the JW fermions. To obtain the bulk excitation spectrum, we simply go to momentum space, and perform Bogoliubov transformation to diagonalize the momentum space $(-\pi \leq k < \pi)$ Hamiltonian. The transformation is given by

$$\psi_{k,\alpha} = u_k^\alpha \gamma_{k,\alpha} - i v_k^\alpha \gamma_{-k,\alpha}, \quad u_k^\alpha = \frac{1}{\sqrt{2}} \left( 1 + \frac{E_k}{E_k^\alpha} \right)^{1/2}, \quad v_k^\alpha = \frac{\Delta_k}{\sqrt{2|\Delta_k^1|}} \left( 1 - \frac{E_k^\alpha}{E_k} \right)^{1/2}$$

with $E_k^\alpha = -t_\alpha \cos k + (\mu_\alpha/2), \quad \Delta_k^\alpha = t_\alpha \sin k, \quad E_k = \sqrt{(\xi_k^\alpha)^2 + (\Delta_k^\alpha)^2}$.

The diagonalized Hamiltonian is following,

$$H = \sum_{k=-\pi}^\pi E_k^\alpha (\gamma_{k,\alpha}^\dagger \gamma_{k,\alpha} - \frac{1}{2})$$

Now we compute various expectation values using the above exact solution. We have defined $\Delta_x = 4\langle P_{j,l}^x P_{j+1,l}^x \rangle \equiv 4\langle P_{j,l}^x P_{j+1,l}^+ \rangle = (\langle a_j - a_j^\dagger \rangle^2 + \langle a_{j+1} + a_{j+1}^\dagger \rangle^2)$. Similarly, we have $\Theta_x = 4\langle Q_{j,1}^x Q_{j+1,1}^x \rangle$. Applying Mattis's duality,

$$4\langle Q_{j,1}^{x} Q_{j+1,1}^{x} \rangle = 4\langle W_{j}^{x} W_{j+1}^{x} \rangle, \quad \langle Q_{j,1}^{x} Q_{j+1,1}^{+} \rangle = 16 \langle V_{j}^{x} V_{j+1}^{z} W_{j}^{x} W_{j+1}^{+} \rangle = 4\langle W_{j}^{x} W_{j+1}^{+} \rangle, \quad \Theta_x = 4\langle W_{j}^{x} W_{j+1}^{x} \rangle = \langle (b_j - b_j^\dagger)(b_{j+1} + b_{j+1}^\dagger) \rangle. \quad \text{So,}

$$\langle \psi_j \psi_{j+1} \rangle + h.c. = -\frac{1}{N} \sum_k (2u_k^\alpha v_k^\alpha \sin k \tanh \left( \frac{\beta E_k^\alpha}{2} \right) = -\frac{1}{N} \sum_k t_\alpha \sin^2 k \frac{E_k^\alpha}{E_k} \tanh \left( \frac{\beta E_k^\alpha}{2} \right)$$

$$\langle \psi_j^\dagger \psi_{j+1} \rangle + h.c. = \frac{1}{N} \sum_k (2 \cos k) |(u_k^\alpha)^2 n(E_k^\alpha) + (v_k^\alpha)^2 (1 - n(E_k^\alpha))| = \frac{1}{N} \sum_k \cos k \left[ 1 - \frac{E_k^\alpha}{E_k} \right] \tanh \left( \frac{\beta E_k^\alpha}{2} \right)$$

Subtracting (1.6) from (1.5), we arrive at Eq. (9). In a similar way, Eq. (10) could be found from (1): $\Delta_z = 2\langle P_{j,z}^z \rangle = 2(a_j^\dagger a_j) - 1$ and (2): $\Omega_z = 4\langle Q_{j,1}^{z} Q_{j+1,1}^{+} \rangle = 2\langle W_{j}^{z} \rangle = 2(b_{j}^\dagger b_{j}) - 1$.

The mean field Ground state (GS) energy of the 4-leg compass ladder is following,

$$E_{gs} = -4J_z \sum_{j=1}^{N} \sum_{i=1}^{2} \langle P_{j,l}^x P_{j+1,l}^+ \rangle \langle Q_{j,1}^{x} Q_{j+1,1}^{+} \rangle - 2J_z \sum_{j=1}^{N} \langle P_{j,1}^{z} + P_{j,2}^{+} \rangle \langle Q_{j,1}^{z} Q_{j,2}^{+} \rangle - \frac{1}{2} \sum_{k=-\pi}^{\pi} (2E_k^\alpha + E_k^\alpha)$$

(1.7)
The first two position space summations come from the mean field decoupling of (4a) and (4b). The last one involving momentum space sum, is the condensation energies of \( p \)-wave superconductor chains, there are two identical fermionic chains of \( 'a' \) (for \( l = 1, 2 \)) and one of \( 'b' \) JW fermions. Simplifying (I.7), the GS energy per lattice site \( (e_{gs}) \) could be written as

\[
e_{gs} = E_{gs}/4N = -\frac{1}{8}(J_x \Delta_x \Theta_x + J_z \Delta_z \Omega_z) - \int_0^\pi \frac{dk}{8\pi} \left[ \sqrt{J_x^2 \Delta_x^2 + J_z^2 \Delta_z^2 - 2J_x J_z \Delta_x \Delta_z \cos k} + \sqrt{J_x^2(1 + \Theta_x)^2 + J_z^2(1 + \Omega_z)^2 - 2J_x J_z(1 + \Theta_x)(1 + \Omega_z) \cos k} \right] \tag{I.8}
\]

We have chosen \( J_x = J \cos \theta \), \( J_z = J \sin \theta \), and \( J = 1 \) for doing calculations.

II. DERIVATION OF DUALITY BETWEEN \( \Theta_x \) AND \( \Omega_z \)

First we establish the exact duality between \( \Theta_x \) and \( \Omega_z \) (in addition to the duality between \( \Delta_x \) , \( \Delta_z \)) for 4-leg compass ladder in the restricted subspace where all local \( \mathbb{Z}_2 \) symmetry operators \( (V_j^z) \) are frozen to \( \pm 1/2 \). To do that, we start from the Hamiltonian,

\[
H_{QC}(S, T) = J_x \sum_{j=1}^N \sum_{l=1}^2 (S_{jl}^x S_{j+l, 1,l}^x + T_{jl}^x T_{j+l, 1,l}^x) + J_z \sum_{j=1}^N \sum_{l=1}^2 S_{jl}^z T_{j+l, 1,l}^z + J_s \sum_{j=1}^N \sum_{l=1}^2 S_{jl}^z T_{j+l, 1,l}^z \tag{II.1}
\]

We successively apply two Mattis’s transformations,

\[
(S_{jl}^x, S_{jl}^y, S_{jl}^z) = (P_{jl}^x, 2P_{jl}^y Q_{jl}^y, 2P_{jl}^y Q_{jl}^z), \quad (T_{jl}^x, T_{jl}^y, T_{jl}^z) = (-2Q_{jl}^y P_{jl}^x, 2Q_{jl}^y P_{jl}^z, Q_{jl}^z) \tag{II.2a}
\]

\[
(Q_{jl,1}^x, Q_{jl,1}^y, Q_{jl,1}^z) = (-2W_{jl,1}^x V_{jl,1}^z, 2W_{jl,1}^y V_{jl,1}^z, W_{jl,1}^z), \quad (Q_{jl,2}^x, Q_{jl,2}^y, Q_{jl,2}^z) = (-V_{jl,2}^x, 2V_{jl,2}^y W_{jl,2}^z, -2V_{jl,2}^z W_{jl,2}^z) \tag{II.2b}
\]

As explained in the main text, both (II.2a), (II.2b) are just two-site Kramers-Wannier (KW) dualities with additional rotations of spin basis (the two sites are positioned along rungs of the ladder). Then, Eq. (II.1) reads

\[
H_{QC}(P, W, V^z) = J_x \sum_{j=1}^N \left[ P_{jl,1}^x P_{j+1,l,1}^x (1 + 4W_{jl,1}^z W_{j+1,l,1}^z) + P_{jl,2}^x P_{j+1,l,2}^x (1 + 4(W_{jl,2}^z V_{j+1,l,2}^z)W_{jl,2}^z W_{j+1,l,2}^z) \right] + \frac{J_s}{2} \sum_{j=1}^N \sum_{l=1}^2 (1 + 2W_{jl}^z) P_{jl}^z \tag{II.3}
\]

Here \( V_{jl}^z \)s are conserved operators. We define a projector \( (P) \) onto the subspace where all \( V_{jl}^z = \pm 1/2 \), i.e. \( P = \sum_k \langle \phi_k | V_{jl}^z = \pm 1/2 \rangle \langle \phi_k | \). The projected Hamiltonian looks following,

\[
H_{QC}^P(P, W) = PH_{QC}P = J_s \sum_{j=1}^N \sum_{l=1}^2 P_{jl}^y P_{jl+1,l,1}^y (1 + 4W_{jl,1}^z W_{jl+1,l,1}^z) + \frac{J_s}{2} \sum_{j=1}^N \sum_{l=1}^2 (1 + 2W_{jl}^z) P_{jl}^z \tag{II.4}
\]

We have the following correspondences between the averages of transformed (II.4) and original (II.1) model, it could be easily shown by inverting (II.2a) and (II.2b). We see, \( \Delta_x = 4\langle P_{jl}^x P_{jl+1,l}^x \rangle \equiv 4\langle S_{jl}^x S_{j+1,l,1}^x \rangle \sim \langle \sigma_{jl,1}^x \sigma_{jl,2}^x \rangle, \quad \Delta_z = 2\langle P_{jl}^z \rangle \equiv 4\langle S_{jl}^z T_{jl}^z \rangle \sim \langle \sigma_{jl,1}^z \sigma_{jl,2}^z \rangle, \quad \Theta_x = 4\langle W_{jl}^z W_{jl+1,l}^z \rangle \equiv 16\langle S_{jl}^x T_{jl}^z S_{jl}^y T_{jl+1,l}^z T_{jl,1}^z T_{jl+1,l,1}^z \rangle \sim \langle \sigma_{jl,1}^z \sigma_{jl,2}^z \rangle, \quad \Omega_z = 2\langle W_{jl}^z \rangle \equiv 4\langle T_{jl,1}^z T_{jl,2}^z \rangle \sim \langle \sigma_{jl,1}^z \sigma_{jl,2}^z \rangle \).

We consider another 4-leg ladder compass model, where spins \( (\tilde{S}^x, \tilde{T}^x) \) reside on the ‘dual’ lattice sites \( (\tilde{l} = l, \tilde{j} = j + 1/2) \) with coupling strengths being swapped with the original model.

\[
H_{QC}^{\text{dual}}(\tilde{S}, \tilde{T}) = J_s \sum_{j=1}^N \sum_{l=1}^2 (\tilde{S}_{jl}^x \tilde{S}_{jl+1,l,1}^x + \tilde{T}_{jl}^x \tilde{T}_{jl+1,l,1}^x) + J_x \sum_{j=1}^N \sum_{l=1}^2 \tilde{S}_{jl}^z \tilde{T}_{jl}^z + J_x \sum_{j=1}^N \sum_{l=1}^2 \tilde{S}_{jl}^z \tilde{T}_{jl+1,l}^z \tag{II.5}
\]
The dual spins and co-ordinates are denoted by tilde symbols. Applying transformations like (II.2a), (II.2b) on these ‘dual’ spins, we get the following,

\[
H_{\text{QC}}^{\text{dual}}(\tilde{P}, \tilde{W}, \tilde{V}) = J_z \sum_{j=1}^{N} \left[ \tilde{P}_{j,1}^{x} \tilde{P}_{j+1,1}^{x} (1 + 4\tilde{W}_{j}^{z}\tilde{W}_{j+1}^{z}) + \tilde{P}_{j,2}^{x} \tilde{P}_{j+1,2}^{x} (1 + 4(4\tilde{V}_{j}^{z}\tilde{V}_{j+1}^{z})\tilde{W}_{j}^{z}\tilde{W}_{j+1}^{z}) \right] + \frac{J_z}{2} \sum_{j=1}^{N} \sum_{l=1,2} (1 + 2\tilde{W}_{j}^{z}) \tilde{P}_{j,l}^{z}
\]

(II.6)

We project onto the subspace where all \( \tilde{V}_{j}^{z} = \pm \frac{1}{2} \),

\[
H_{\text{QC}}^{\text{dual}, P}(\tilde{P}, \tilde{W}) = \tilde{P} H_{\text{QC}}^{\text{dual}} \tilde{P} = J_z \sum_{j=1}^{N} \sum_{l=1,2} \tilde{P}_{j,l}^{x} \tilde{P}_{j+1,l}^{x} (1 + 4\tilde{W}_{j}^{z}\tilde{W}_{j+1}^{z}) + \frac{J_z}{2} \sum_{j=1}^{N} \sum_{l=1,2} (1 + 2\tilde{W}_{j}^{z}) \tilde{P}_{j,l}^{z}
\]

(II.7)

Here we have \( \Delta_{\tilde{x}}^{\text{dual}} = 4 \langle \tilde{P}_{j,l}^{x}\tilde{P}_{j+1,l}^{x} \rangle = 4 \langle \tilde{S}_{j,l}^{z}\tilde{S}_{j+1,l}^{z} \rangle \sim \langle \tilde{\sigma}_{\tilde{x}}^{-}\tilde{\sigma}_{\tilde{x}}^{+} \rangle \), \( \Delta_{\tilde{x}}^{\text{dual}} = 2 \langle \tilde{P}_{j,l}^{z}\tilde{P}_{j+1,l}^{z} \rangle = 4 \langle \tilde{S}_{j,l}^{z}\tilde{S}_{j+1,l}^{z} \rangle \sim \langle \tilde{\sigma}_{\tilde{x}}\tilde{\sigma}_{\tilde{x}}^{-} \rangle \), \( \Omega_{\tilde{x}}^{\text{dual}} = 2 \langle \tilde{W}_{j}^{z}\tilde{W}_{j+1}^{z} \rangle \sim \langle \tilde{\sigma}_{\tilde{x}}\tilde{\sigma}_{\tilde{x}}^{+} \rangle \), and \( \Theta_{\tilde{x}}^{\text{dual}} = 4 \langle \tilde{W}_{j}^{z}\tilde{W}_{j+1}^{z} \rangle \sim \langle \tilde{\sigma}_{\tilde{x}}\tilde{\sigma}_{\tilde{x}}^{-} \rangle \).

We see that Eqs. (II.4) and (II.7) could be connected by an “infinite chain type” KW duality (like the one used in the context of 1d TFIM),

\[
W_{j}^{z} \rightarrow 2\tilde{W}_{j}^{z}\tilde{W}_{j+1}^{z} , \ W_{j}^{z}\tilde{W}_{j+1}^{z} \rightarrow \frac{1}{2} \tilde{W}_{j}^{z}
\]

\[
P_{j,l}^{z} \rightarrow 2\tilde{P}_{j,l}^{x}\tilde{P}_{j+1,l}^{x} , \ P_{j,l}^{z}\tilde{P}_{j+1,l}^{z} \rightarrow \frac{1}{2} \tilde{P}_{j,l}^{z}
\]

(II.8)

Using Eq. (II.8), we see \( \Delta_{\tilde{x}} \) maps to \( \Delta_{\tilde{x}}^{\text{dual}} \). Similarly, we have \( \Delta_{\tilde{x}} \rightarrow \Delta_{\tilde{x}}^{\text{dual}}, \Theta_{\tilde{x}} \rightarrow \Omega_{\tilde{x}}^{\text{dual}}, \Omega_{\tilde{x}} \rightarrow \Theta_{\tilde{x}}^{\text{dual}} \). So we have proved the duality in a projected subspace of all \( \tilde{V}_{j}^{z} = \pm \frac{1}{2} \) (and \( \tilde{V}_{j}^{z} = \pm \frac{1}{2} \)).

We now extend the above proof to arbitrary subspaces (of \( \tilde{V}_{j}^{z}, \tilde{V}_{j}^{z} \)) for planar compass model by taking a different route. We start from the Hamiltonian (II.1) (change the upper limit of \( l \) to \( N/2 \)) and apply (II.2a), we get

\[
H_{\text{QC}}(p, q) = J_z \sum_{j=1}^{N/2} \sum_{l=1}^{N/2} p_{j,l}^{x} p_{j+l,1,l}^{x} (1 + q_{j,l}^{x} q_{j+l,1,l}^{x}) + \frac{J_z}{4} \sum_{j=1}^{N/2} \sum_{l=1}^{N/2} p_{j,l}^{x} (1 + q_{j,l}^{x} q_{j+l,1,l}^{x})
\]

(II.9)

Here \( p_{j,l}^{x} = 2p_{j,l}^{x} \), \( q_{j,l}^{x} = 2q_{j,l}^{x} \). The self-duality is already visible from this expression as we have 1d TFIM of \( p^{x} \) coupled to a compass model of \( q^{x} \), both of them show self-dual property. To be rigorous mathematically, we prove it by applying the following set of duality relations successively on (II.9), (we define \( l = j = j + 1/2 \))

(1): We use the “Infinite chain like” KW duality for \( p^{x} \) and \( q^{x} \) spins,

\[
p_{j,l}^{x} p_{j+l,1,l}^{x} \rightarrow \tilde{p}_{j,l}^{z} , \ p_{j,l}^{x} \rightarrow \tilde{p}_{j,l}^{z} p_{j+l,1,l}^{z} , \ q_{j,l}^{x} q_{j+l,1,l}^{x} \rightarrow \tilde{q}_{j,l}^{z} , \ q_{j,l}^{x} \rightarrow \tilde{q}_{j,l}^{z} p_{j+l,1,l}^{z}
\]

(II.10a)

(2): Next we implement Xu-Moore duality relation [2] for \( \tau^{z} \) spins,

\[
\tilde{\tau}_{j,l}^{z} = \tau_{j+1/2,l}^{z} \tau_{j-1/2,l+1/2}^{z} , \ \tilde{\tau}_{j,l}^{z} = \Pi_{\tilde{r} < (j,l)} \tau_{\tilde{r}}^{z}
\]

(II.10b)

here \( \tilde{r} < (j,l) \) means the \( x \) and \( z \) co-ordinates of \( \tilde{r} \) is less than \( j \) and \( l \) respectively.

(3): Finally, we use again “Infinite chain like” KW duality, for \( \tilde{p}^{z} \) spins,

\[
\tilde{q}_{j,l}^{z} \rightarrow \tilde{q}_{j,l}^{z} , \ \tilde{q}_{j,l}^{z} \rightarrow \tilde{q}_{j,l}^{z} p_{j+l,1,l}^{z}
\]

(II.10c)

Then (II.9) reads following,

\[
H_{\text{QC}}^{\text{dual}}(\tilde{P}, \tilde{Q}) = J_z \sum_{j=1}^{N/2} \sum_{l=1}^{N/2} \tilde{P}_{j,l}^{x} \tilde{P}_{j+l,1,l}^{x} (1 + 4\tilde{Q}_{j,l}^{z} \tilde{Q}_{j+l,1,l}^{z}) + \frac{J_z}{2} \sum_{j=1}^{N/2} \sum_{l=1}^{N/2} \tilde{P}_{j,l}^{z} (1 + 4\tilde{Q}_{j,l}^{z} \tilde{Q}_{j+l,1,l}^{z})
\]

(II.11)

Here \( \tilde{P}_{j,l}^{x} = \tilde{p}_{j,l}^{x}/2 \), \( \tilde{Q}_{j,l}^{z} = \tilde{q}_{j,l}^{z}/2 \). Using the inverse of (II.2a) for \( \tilde{P}_{j,l}^{x}, \tilde{Q}_{j,l}^{z} \), the Hamiltonian (II.11) reduces to the ‘dual’ compass model (see (II.5)). We see \( \Theta_{\tilde{x}} \sim \langle \tilde{q}_{j,l}^{x} q_{j+l,1,l}^{x} \rangle \) maps to \( \Theta_{\tilde{x}}^{\text{dual}} \sim \langle \tilde{q}_{j,l}^{z} q_{j+l,1,l}^{z} \rangle \) under the web of dualities (II.10a)-(II.10c), similar mapping also holds true between \( \Omega_{\tilde{x}} \) and \( \Theta_{\tilde{x}}^{\text{dual}} \). So the duality between \( \Theta_{\tilde{x}} \) and \( \Omega_{\tilde{x}} \) holds generally true for compass like interactions, not just restricted to a particular subspace or 4-leg ladder geometries.
III. MFT RESULTS FOR FINITE TEMPERATURE CRITICALITY

In the Fig. 3 of main text, we see that the jump discontinuity (at \( J_x = J_z \)) of various mean-field averages vanishes near \( T \approx 0.125 \) and the graphs for higher \( T \) values show continuous behaviour. Here we provide some additional plots which show that the point \( T \approx 0.125, J_x = J_z \) actually corresponds to a second order critical endpoint (\( T_c \)) to the first order transitions below \( T_c \).

We find that the Ising like nematic order parameter, \( |\langle D \rangle| = |\langle \sigma^z_x \sigma^z_{x+z} - \sigma^z_x \sigma^z_{x+z} \rangle| \) continuously decreases with increasing \( T \) and goes to zero at \( T = T_c = 0.1225 \) (\( J = 1 \)) when \( J_x \) and \( J_z \) are equal (see Fig. 4(a)). In our calculations, we have used \( \theta = \tan^{-1}(J_z/J_x) = \pi/4 \pm 10^{-6} \). We also compute the “nematic susceptibility”, defined as \( \chi_D = (\partial |\langle D \rangle|/\partial T)|_{h=0} \), where \( h = 1 - (J_x/J_z) \), is the anisotropy in coupling strengths which acts as a fictitious external field. We find that the susceptibility diverges near \( T = T_c \) (see Fig. 4(b)). The critical exponents, \( \beta \) (defined as \( |\langle D \rangle| \sim (T_c - T)^{\beta} \)) and \( \gamma \) (defined as \( \chi_D \sim A_{\pm} |T - T_c|^{-\gamma} \)) are almost equal to the classical Landau theory exponents, we find \( \beta = 0.4776 \) and \( \gamma = 1.0 \). For the calculation of \( \gamma \), we have fitted \( \chi_D^{-1} \) with \( a_{\pm} |T - T_c| + b_{\pm} \); we find \( a_- = 10.7, b_- = 3.6 \times 10^{-3} < T < T_c \), and \( a_+ = 4.648, b_+ = 5.02 \times 10^{-4} > T > T_c \). So even the ratio of \( A_{\pm} \) (<1/a_\pm) is almost like Landau theory. The specific heat \( (C_V) \) calculated from MFT shows a finite jump at \( T_c \), meaning the corresponding exponent, \( \alpha = 0.0 \) (See Fig. 4(c)). This is also same as classical Landau theory.

![Figure 4.](image)

We find the critical exponents are same as classical Landau mean-field theory, whereas the numerical results [3] show signatures of 2\( d \) classical Ising universality class. Fluctuations around MFT play a significant role near this critical region (as we will explain below).

IV. TWO-SPIN CORRELATION FUNCTIONS

One novel feature of our MFT is that the spatial correlations are partially retained, which are unusual in conventional MF descriptions. We evaluate here some important time independent two-spin correlation functions both at zero and finite temperatures. Explicit computations are not performed here as the MF Hamiltonians are just 1\( d \) TFIMs. We will use Mattis’s dualities and some well-known results of 1\( d \) TFIM [1, 4] to derive the following correlation functions.

\[(A): \langle \sigma^x_j \sigma^x_{j+n} \rangle \text{ for } n > > 1, T = 0 : \]

\[
4|\langle S^x_{j,1} S^x_{j+n,1} \rangle| = 4|\langle P^x_{j,1} P^x_{j+n,1} \rangle| \quad \text{for } |J_x| > |J_z| = (1 - \lambda^2_0)^{1/4} \left[ 1 + \frac{1}{2\pi n^2} \frac{\lambda^{2n+2}_0}{1 - \lambda^2_0} + \cdots \right], \quad \lambda_a = \frac{|J_z| (1 + \Omega_x)}{|J_x| (1 + \Omega_x)} < 1 \quad \text{(IV.1a)}
\]

\[
4|\langle S^x_{j,1} S^x_{j+n,1} \rangle| = \frac{\lambda^2_0}{\sqrt{\pi n}} \frac{\lambda^{1/2}_a}{(\lambda^2_a - 1)^{1/4}} e^{-|n|/\xi_a}, \quad \xi_a = \frac{1}{\ln (\lambda_a)}, \quad \lambda_a > 1 \quad \text{(IV.1b)}
\]
By applying Mattis’s transformations (II.2a) and (II.2b), we get
\[ 4T_{j,1}^x T_{j+n,1}^x = 16Q_{j,1}^x P_{j,1}^x Q_{j+n+1,1}^x P_{j+n+1,1}^x W_{j,1}^x W_{j+n,1}^x P_{j+n+1,1}^x. \]
So,
\[ 4|(T_{j,1}^x W_{j+n,1}^x)| \approx 4|\langle P_{j,1}^x P_{j+n,1}^x \rangle| \times 4|\langle W_{j,1}^x W_{j+n,1}^x \rangle| \quad \text{(MFT decouples P and W spins)} \]
for \(|J_x| > |J_z| = \{(1 - \lambda_a^2)(1 - \lambda_b^2)\}^{1/4} \left[ 1 + \frac{1}{2\pi n^2} \left( \frac{\lambda_a^{2n+2}}{1 - \lambda_a^2} + \frac{\lambda_b^{2n+2}}{1 - \lambda_b^2} + \cdots \right) \right], \lambda_a < 1, \lambda_b = \frac{|J_x|}{|J_z|} \Delta_x < 1 \]  
(IV.2a)
\[ \text{for } |J_x| > |J_z| = \frac{1}{\pi n} (\lambda_a \lambda_b)^{1/2} e^{-\frac{|n|}{\xi_z}}, \quad \frac{1}{\xi_z} = \frac{1}{\xi_a} + \frac{1}{\xi_b}, \xi_a, b = \frac{1}{\ln(\lambda_a, b)}, \lambda_a, \lambda_b > 1 \]  
(IV.2b)
We have \(S_{j,1}^x S_{j+n,1}^x = P_{j,1}^x P_{j+n,1}^x\) and \(T_{j,1}^x T_{j+n,1}^x = 64V_j^z V_{j+n}^z W_{j,1}^x P_{j,1}^x W_{j+n,1}^x P_{j+n,1}^x \). We take all \(V_j^z = \pm 1/2\), this holds rigorously at \(T = 0\). So we see \(|\langle S_{j,1}^x S_{j+n,1}^x \rangle| = |\langle T_{j,1}^x T_{j+n,1}^x \rangle| = |\langle T_{j,1}^x T_{j+n,1}^x \rangle| . \]

(B): \(\langle \sigma_{j,1}^z \sigma_{n+2}^z \rangle\) for \(m = 1, 2, T = 0:\)

These operators do not commute with the 1d symmetries, \(Z_l = \lim_{N \to \infty} \prod_{l=1}^N \sigma_{j,l}^z\); so according to Elitzur’s theorem [5], their expectation values should vanish rigorously above \(T = 0\). Although at \(T = 0\), this theorem permits spontaneous breaking of these 1d \(Z_2\) symmetries in the thermodynamic limit, resulting finite average values of these symmetry non-invariant operators. We now show that these non-zero averages result from the long range ordering of \(P\) and \(W\) Ising chains \((\langle P_{j}^z \rangle, \langle W_{j}^z \rangle \neq 0)\) in the \(|J_x| > |J_z|\) region. At first, we consider the nearest neighbour (along ladder rungs) or \(m = 1\) case,
\[ |\langle \sigma_{j,1}^z \sigma_{j,2}^z \rangle| \sim |\langle S_{j,1}^x T_{j,1}^x \rangle| \sim |\langle Q_{j,1}^z \rangle| \sim |\langle W_{j}^x \rangle| = (1 - \lambda_b^2)^{1/8} \]
(IV.3a)
Similarly, it is easy to show \( |\langle \sigma_{j,1}^z \sigma_{j,4}^z \rangle| \sim |\langle S_{j,1}^x T_{j,1}^x \rangle| \sim |\langle Q_{j,1}^z \rangle| \sim |\langle W_{j}^x \rangle| = (1 - \lambda_a^2)^{1/8}\)
(IV.3b)
Here the spins \(P_{j,1}^x P_{j,2}^x\) are governed by identical Hamiltonians and they don’t interact with each other, so we can write \( |\langle P_{j,1}^x P_{j,2}^x \rangle| = |\langle P_{j,1}^x \rangle|^2\). Similarly, \( |\langle \sigma_{j,1}^z \sigma_{j,4}^z \rangle| = |\langle \sigma_{j,1}^z \sigma_{j,3}^z \rangle| = (1 - \lambda_b^2)^{1/4}(1 - \lambda_a^2)^{1/8} \). Due to periodic boundary condition, \((j, 1)\) and \((j, 4)\) are nearest neighbour sites. Next we calculate the second neighbour or \(m = 2\) case,
\[ |\langle \sigma_{j,1}^z \sigma_{j,3}^z \rangle| \sim |\langle T_{j,1}^x T_{j,2}^x \rangle| \sim 16|\langle Q_{j,1}^x P_{j,1}^x Q_{j,2}^x P_{j,2}^x \rangle| \sim 4|\langle P_{j,1}^x \rangle|^4 \sim |\langle P_{j,1}^x \rangle|^2 = (1 - \lambda_b^2)^{1/4}(1 - \lambda_a^2)^{1/8} \]
(IV.3c)
Finally, we see \( |\langle \sigma_{j,1}^z \sigma_{j,4}^z \rangle| \sim |\langle S_{j,1}^x S_{j,2}^x \rangle| \sim |\langle P_{j,1}^x P_{j,2}^x \rangle| \) is identical to (IV.3c). As expected here, the second neighbour \((m = 2)\) correlations are weaker than the nearest neighbour \((m = 1)\) ones.

(C): \(\langle \sigma_{j,1}^z \sigma_{j,n+2}^z \rangle\) for any \(n, \mu = y, z\), and \(T = 0:\)

These operators violate local \((d = 0)\) symmetries \((V^z_{j,1} \sim \prod_{l=1}^4 \sigma_{j,l}^z)\) of the Hamiltonian. Elitzur’s theorem tells that spontaneous breaking of local gauge symmetries is impossible even at \(T = 0\). So, these spatial correlations are ultra-local in nature, \(\langle \sigma_{j,1}^z \sigma_{j,n+2}^z \rangle = \delta_{n,0}\). We now prove this statement using our MF construction. We show here two such examples,
\[ |\langle \sigma_{j,1}^z \sigma_{j,2}^z \rangle| \sim |\langle S_{j,1}^x S_{j,2}^x \rangle| \sim 16|\langle P_{j,1}^x Q_{j,1}^x P_{j,2}^x \rangle|^2 \sim 64|\langle P_{j,1}^x W_{j,1}^z V_{j,2}^z P_{j,2}^z \rangle|^2 = \delta_{n,0} \]
(IV.4a)

Similarly,
\[ |\langle \sigma_{j,1}^y \sigma_{j,2}^y \rangle| \sim |\langle T_{j,1}^y T_{j,2}^y \rangle| \sim 16|\langle P_{j,1}^y Q_{j,1}^y P_{j,2}^y \rangle|^2 \sim 64|\langle P_{j,1}^y W_{j,1}^y V_{j,2}^y P_{j,2}^y \rangle|^2 = \delta_{n,0} \]
(IV.4b)
The MF eigenstates are common eigenstates of \(H_{OC} \) and \(V^z_j\), so flipping of \(V^z_j\) will map to different symmetry sector and thus makes the overlap zero. This is independent of any \(|J_x|/|J_z|\).

**Conclusion (1):** When \(T = 0\) and \(|J_x| > |J_z|\), there is a long range magnetic order \((\langle \sigma^z_j \rangle \neq 0)\) in the system where spins are mostly “aligned” in the \(x\)-direction. This ordering occurs due to spontaneous breaking of \(d = 1\) Ising symmetries. The magnetic order suddenly drops to zero at the self-dual point, \(J_x = J_z\) and continues to be zero in \(|J_x| > |J_z|\) region with a finite two-spin correlation length.
Now we will show that long-range magnetic order completely disappears as we go above \( T = 0 \) and the two-spin correlation functions become short ranged. Elitzur’s theorem forbids spontaneous breaking of \( d=1 \) symmetries, \( Z_l \) at any \( T > 0 \) (like in 1d TFIM), thus the long-range order vanishes. We define following parameters, \( \Delta_a = |J_x| (1 + \Theta_x) (1 - \lambda_a) \), \( \Delta_b = |J_x| \Delta_x (1 - \lambda_b) \), \( a = 2 |J_x| (1 + \Theta_x) \), \( b = 2 |J_x| \Delta_x \). Using Mattis’s relations and the rigorous finite \( T \) results of 1d TFIM [1], we find the following,

When \( \Delta_a, \Delta_b >> T > 0 \) (or \( |J_x| > |J_z|, T << T_c \)) :

\[
4 \langle S^x_{jl}, S^x_{j+l+n, l} \rangle \sim \Delta_a^{-1/4} e^{-|n|/\xi_a(T)} , \quad \xi_a^{-1} = \left( \frac{2 |\Delta_a| T}{\pi v_a^2} \right)^{1/2} e^{-|\Delta_a|/T} \quad (IV.5a)
\]

\[
4 \langle T^x_{jl}, T^x_{j+l+n, l} \rangle \sim \Delta_b^{-1/4} e^{-|n|/\xi_b(T)} , \quad \xi_b^{-1} = \xi_b^{-1} + \xi_b^{-1} , \quad \xi_b^{-1} = \left( \frac{2 |\Delta_b| T}{\pi v_b^2} \right)^{1/2} e^{-|\Delta_b|/T} \quad (IV.5b)
\]

So the temperature induces a finite two-spin correlation length, destroying the \( T = 0 \) magnetic order.

When \( \Delta_a, \Delta_b < 0, |\Delta_a|, |\Delta_b| >> T > 0 \) (or \( |J_z| > |J_x|, T << T_c \)) :

\[
4 \langle S^x_{jl}, S^x_{j+l+n, l} \rangle \sim \frac{T}{|\Delta_a|^{3/4}} e^{-|n|/\xi_a(T)} , \quad \xi_a^{-1} = \frac{|\Delta_a| T}{v_a} + \left( \frac{2 |\Delta_a| T}{\pi v_a^2} \right)^{1/2} e^{-|\Delta_a|/T} \quad (IV.6a)
\]

\[
4 \langle T^x_{jl}, T^x_{j+l+n, l} \rangle \sim \frac{T^2}{|\Delta_b|^{3/4}} e^{-|n|/\xi_b(T)} , \quad \xi_b^{-1} = \xi_b^{-1} + \xi_b^{-1} , \quad \xi_b^{-1} = \frac{|\Delta_b| T}{v_b} + \left( \frac{2 |\Delta_b| T}{\pi v_b^2} \right)^{1/2} e^{-|\Delta_b|/T} \quad (IV.6b)
\]

When \( |\Delta_a|, |\Delta_b| << T \) (or \( J_x = J_z, T \rightarrow T^- \)) :

This is the region near second order critical point where the nematic order parameter smoothly goes to zero (Fig. 4(a)) and the elementary JW fermionic excitations become gapless (Fig. 7). Although fluctuations become important (as we show below) in this region, we continue using the MFT to see what minimal features could be extracted from it. We find

\[
4 \langle S^x_{jl}, S^x_{j+l+n, l} \rangle \sim T^{1/4} e^{-|n|/\xi_a^c} , \quad (\xi_a^c)^{-1} \approx \frac{\pi T_c}{4 v_a} \quad (IV.7a)
\]

\[
4 \langle T^x_{jl}, T^x_{j+l+n, l} \rangle \sim T^{1/2} e^{-|n|/\xi_b^c} , \quad (\xi_b^c)^{-1} = (\xi_b^c)^{-1} + (\xi_b^c)^{-1} , \quad (\xi_b^c)^{-1} \approx \frac{\pi T_c}{4 v_b} \quad (IV.7b)
\]

So the two-spin correlation length remains finite even in this critical region, reflecting that finite \( T \) phase transition is non-magnetic in nature. In the finite \( T \) calculations, we assumed that all the classical Ising variables, \( V^z \) are frozen to \( \pm 1/2 \), this is rigorously valid only at \( T = 0 \). We should expect that at any finite \( T \), \( \langle V^z_{j} V^z_{j+n} \rangle \sim e^{-|n|/\xi_z} \), where \( \xi_z^{-1} = \ln \coth (J_z/T) \). Here \( J_z \) represents some unknown energy scale depending on \( J_x, J_z \). We already have exponentially decaying finite \( T \) correlations at the MF level. An additional decay of \( \langle V^z_{j} V^z_{j+n} \rangle \) will just bring quantitative changes in correlation lengths, the functional forms of these correlations remain same as MFT.

V. MEAN-FIELD RESULTS OF 8-LEG COMPASS LADDER

Construction of lower dimensional symmetry preserving MFT for 8-leg compass ladder is straightforward. We skip the lengthy algebra and provide only some crucial steps to derive the MF self-consistency relations.

We start form the 8-leg compass ladder Hamiltonian (\( H_{QC} \)) and apply Mattis’s transformations (labelling it M1). Like in the main text (see Eqs. (4a),(4b)), we decouple the resulting four-spin interacting model in a two-spin channel which preserves all lower dimensional symmetries. This decoupling (calling it D1) results four identical 1d TFIMs (\( H^{(l)}_1 \), \( l = 1, 2, 3, 4 \)) and a 4-leg compass ladder (\( H'^{C} \)). Coupling constants in both of these Hamiltonians depend on bare strengths (\( J_x, J_z \)) plus various mean-field averages which have to be determined later using self-consistency conditions. Now a similar scheme would be repeated for \( H'^{C} \); we use M2 and D2 [6] which gives two more 1d TFIMs (\( H^{(m)}_2 \), \( m = 1, 2 \)) and a 2-leg compass ladder (\( H''^{C} \)). Applying M3 on \( H'^{C} \), we get the final 1d TFIM (\( H_3 \)) coupled to static \( \mathbb{Z}_2 \) fields which are one of the gauge-like symmetries of \( H_{QC} \).
We arrive at the following Hamiltonians,

\[ H_1^{(l)} = J_x \sum_{j=1}^{N} (1 + \Theta_{jl}^x) P_{j,l}^x P_{j+1,l}^x + \frac{J_z}{2} \sum_{j=1}^{N} (1 + \Omega_{jl}^x) P_{j,l}^z \quad (l = 1, 2, 3, 4) \]  

\[ H_2^{(m)} = J_x \sum_{j=1}^{N} \alpha_{jl}^x Q_{jm}^x Q_{j+1,m}^x + \frac{J_z}{2} \sum_{j=1}^{N} \beta_{jl}^x Q_{jm}^z \quad (m = 1, 2) \]  

\[ H_3 = J_x \sum_{j=1}^{N} \left[ j_{jl}^x + 4V_j^z j_{jl+1}^z \right] W_{j,l}^x + \frac{J_z}{2} \sum_{j=1}^{N} \gamma_j W_{j}^z \]  

Like in the 4-leg case, here also we find the following MF order parameters, \( \Delta_{jl}^x = 4(S_{jl}^x S_{jl+1}^x + T_{jl}^x T_{jl+1}^x), \) \( \Theta_{jl}^x = 16(S_{jl}^x S_{jl+1}^x T_{jl}^x T_{jl+1}^x), \) and \( \Omega_{jl}^x = 4(T_{jl}^x T_{jl+1}^x) \) (with \( j = 1 \) to \( N \) and \( l = 1, 2, 3, 4 \)). In addition, we find a rectangular loop-like object, \( \Phi_{jl}^x = 2^5 (S^x_{jl} S^x_{jl+1} T^x_{jl} T^x_{jl+1} S^x_{jl+1} T^x_{jl+1} S^x_{jl+1} T^x_{jl+1}) \) and a fourth neighbour \( zz \)-correlation, \( \Pi_j^x = 4(T^x_{j,3} T^x_{j,1}) \) (\( \sim (\sigma^x_{j} \sigma^x_{j+4 \pm}) \)), these two are also dual to each other as we will show below.

The conserved \( \mathbb{Z}_2 \) fields are \( V_j^x = \frac{1}{2} \prod_{l=1}^{4} (4S^x_{jl} T^x_{jl}) \), which are one of the gauge-like symmetries of 8-leg compass ladder.

Next we assume translationally invariant ansatz for the MF order parameters, i.e. (a) take all \( V_j^x = \pm 1/2 \), this helps us to get rid of \( j \) dependence, and (b) \( \Delta_{jl}^x \equiv \Delta_x \) and \( \Delta_{jl}^z \equiv \Delta_z \) (independent of \( j, l \)). In the 4-leg case, the condition (b) comes as a consequence of condition (a), but here we have to impose it separately. It is straightforward to verify that these conditions (a) and (b) will make all the order parameters (hence the coupling constants) to be completely independent of spatial co-ordinates (\( j, l \)).

Finally, we arrive at the following self-consistently coupled, spatially uniform 1d TFIMs,

\[ H_1^{(l)} = J_x (1 + \Theta_x) \sum_{j=1}^{N} P_{j,l}^x P_{j+1,l}^x + \frac{J_z}{2} (1 + \Omega_x) \sum_{j=1}^{N} P_{j,l}^z \quad (l = 1, 2, 3, 4) \]  

\[ H_2^{(m)} = J_x \Delta_x (1 + \Phi_x) \sum_{j=1}^{N} Q_{jm}^x Q_{j+1,m}^x + \frac{J_z}{2} \Delta_x (1 + \Pi_x) \sum_{j=1}^{N} Q_{jm}^z \quad (m = 1, 2) \]  

\[ H_3 = 2J_x \Delta_x \Theta_x \sum_{j=1}^{N} W_{j}^x W_{j+1}^x + J_x \Delta_x \Omega_x \sum_{j=1}^{N} W_{j}^z \]  

Which could be solved easily as shown in the section I, we just write the final equations. Like in the 4-leg case, we define \( M_x^z = (\Delta_x, \Theta_x, \Phi_x) \) and \( M_z^z = (\Delta_z, \Omega_x, \Pi_x) \).

\[ M_x^z = \int_0^\pi \frac{dk}{\pi} \frac{\left( h_x \cos k - 1 \right) \tanh (\beta E_x^z/2)}{\sqrt{1 + h_x^2 - 2h_x \cos k}}, \quad M_z^z = \int_0^\pi \frac{dk}{\pi} \frac{\left( h_z \cos k - 1 \right) \tanh (\beta E_z^z/2)}{\sqrt{1 + h_z^2 - 2h_z \cos k}} \]  

Here \( E_x^a = \frac{|J_x|}{2} (1 + \Theta_x) \sqrt{1 + h_x^2 - 2h_x \cos k} \), \( E_k^b = \frac{|J_x|}{2} \sqrt{1 + h_x^2 - 2h_x \cos k} \), and \( E_k^c = |J_x| \Delta_x \Theta_x \times \sqrt{1 + h_x^2 - 2h_x \cos k} \) are elementary excitation spectrum of the above Ising chains (V.4)-(V.6) with \( h_a = \frac{J_x (1 + \Omega_x)}{\sqrt{1 + h_x^2}}, \) \( h_b = \frac{J_x \Delta_x (1 + \Pi_x)}{\sqrt{1 + h_x^2}}, \) \( h_c = \frac{J_x \Delta_x \Omega_x}{\sqrt{1 + h_x^2}} \), and \( \beta = 1/T \).

The ground state energy per site is given by following,

\[ e_{gs} = -\frac{1}{16} \left[ J_x \Delta_x \Theta_x (1 + 2\Phi_x) + J_x \Delta_x \Omega_x (1 + 2\Pi_x) \right] - \frac{1}{2} \sum_{k=-\pi}^{\pi} \left[ 4E_k^a + 2E_k^b + E_k^c \right] \]  

We now proceed to the results. We find that the ground state energy density is almost same as what is found in 4-leg ladder case, except close to isotropic or self-dual point \( J_x = J_z \) where very small changes are observed (see Fig. 5). At the self-dual point, we find \( e_{gs} = -0.189623 \) for 8-leg case and \( e_{gs} = -0.190205 \) for 4-leg case (\( J = 1 \)). This negligible difference strongly supports our previous argument (in the main text) that most of the ground state correlation energy of 2d compass model is captured in a 4-leg ladder and increasing number of legs has only negligible
effect. Although we have performed the mean-field decoupling twice (in 8-leg case), various loop/plaquette correlators \((\Theta_x, \Phi_x)\) and beyond nearest neighbour \(zz\)-correlations \((\Omega_z, \Pi_z)\) (which don’t violate gauge-like symmetries) actually capture the correlations along rungs in a self-consistent way. This should be a reason behind such nice convergence of \(e_{gs}\).

Figure 5. Comparison between ground state energy density \(e_{gs}\) of 4-leg (green triangles) and 8-leg (red circles) compass ladder as function of \(\theta = \tan^{-1}(|J_z|/|J_x|)\), there is a negligibly small difference between two energies, only near \(\theta = \pi/4\) \((J_z = J_x)\).

\[
\begin{align*}
\Theta_x & \rightarrow (a) \\
\Phi_x & \rightarrow (b) \\
\Omega_z & \rightarrow (c)
\end{align*}
\]

Figure 6. Various MF order parameters of 8-leg compass ladder as function of anisotropy, \(\theta = \tan^{-1}(|J_z|/|J_x|)\). The dual observables are shown in the inset figures.

Next we plot various MF order parameters as function of anisotropy \(\theta\) for different values of \(T\) (see Fig. 6). We observe that \(\Delta_x, \Delta_z\) show properties similar to 4-leg case. The main thing to notice is that the discontinuity of these
VI. FLUCTUATIONS AROUND MEAN FIELD THEORY

Finally we discuss qualitatively about various fluctuation effects around the 4-leg ladder MFT. There are mainly two different sources of fluctuations which we have neglected in MFT: (1) the interaction between $P$ and $W$ spins (or between ‘a’ and ‘b’ JW fermions), this effect is present at both $T = 0$, and $T > 0$. (2) thermal ($T > 0$) fluctuations of static $\mathbb{Z}_2$ fields, $V^z_j$. These fields are present in (a) the interaction terms (between $P$ and $W$) and (b) the quadratic MF Hamiltonian of $W$ spins ($H_2$). The timescale associated with collective fluctuations of JW fermions is of the order of $1/J_x$ or $1/J_w$, so that these are switched on gradually. On the other hand, the local quenches (flipping of $V^z_j$ at $T > 0$) are suddenly switched on. Thus, there is a clear separation of timescales associated with different fluctuation processes. Starting with the “fast” process (flipping of $V^z_j$), these are non-singular except around $J_x = J_z$ and $T = T_c$, as we will argue below in detail. Over the timescale on which the “slow” processes (fluctuations of JW fermions) operate, there will be many such incidents of “sudden” flipping processes. Thus, the “disorder” generated by local flips of the $V^z_j$ is annealed out at times of order of $1/J_x$, restoring the translational invariance. So it is reasonable to “decouple” the two effects (1) and (2) in the spirit of an adiabatic approximation: treat the fast process first, and then consider the slow process in a medium already renormalized by the fast fluctuations.

(1): Fluctuations arising from the interaction between JW fermions:

We have neglected the following interaction Hamiltonian in MFT description.

$$V_f = \frac{J_x}{4} \sum_{j=1}^{N} \left[ (p^x_{j,1} p^x_{j+1,1} - \Delta^x_{j,1})(w^z_{j} w^z_{j+1} - \Theta^x_{j,1}) + (p^x_{j,2} p^x_{j+1,2} - \Delta^x_{j,2})(v^z_{j} v^z_{j+1} w^x_{j} w^x_{j+1} - \Theta^x_{j,2}) \right]$$

$$+ \frac{J_z}{4} \sum_{j=1}^{N} \left[ (p^z_{j,1} - \Delta^z_{j,1})(w^z_{j} - \Omega^z_{j}) + (p^z_{j,2} - \Delta^z_{j,2})(w^z_{j} - \Omega^z_{j}) \right]$$

(VI.1)

here $p^\mu_{jl} = 2P^\mu_{jl}$, $w^\mu_{j} = 2W^\mu_{j}$, and $v^\mu_{j} = 2V^\mu_{j}$. We have argued in the main text that if we take $v^z_{j} = \pm1 \forall j$ (which is rigorously true at $T = 0$), then for a periodic 4-leg ladder, we have $\Delta^x_{jl} \equiv \Delta_x$, $\Delta^z_{jl} \equiv \Delta_z$, $\Theta^x_{jl} \equiv \Theta_x$, and $\Omega^z_{j} \equiv \Omega_z$. We continue to assume this spatially uniform configuration of $v^z_{j}$ for $T > 0$, according to the reasons explained above. The MF Hamiltonians corresponding to $p^\mu_{jl}$ are same for both $l = 1, 2$ and the interaction is only between $p$ and $w$. So we consider only one of them for further discussions and use common notation $p^\mu_{jl}$ for both $l = 1, 2$. So the above Hamiltonian now reads the following,

$$V_f = \frac{J_x}{4} \sum_{j=1}^{N} \left[ p^x_{j} p^x_{j+1} w^z_{j} w^z_{j+1} - 2\Delta_x w^x_{j} w^x_{j+1} - \Theta_x p^x_{j} p^x_{j+1} \right] + \frac{J_z}{4} \sum_{j=1}^{N} \left[ p^z_{j} w^z_{j} - 2\Delta_z w^z_{j} - \Omega_z p^z_{j} \right]$$

(VI.2)

Apart from the four-spin Ising type interaction (proportional to $J_z$), the rest is precisely a 2-leg QCM ladder, but now with “chain-dependent” exchanges and magnetic fields.

$$- \sum_{j=1}^{N} \left[ J_p p^x_{j} p^x_{j+1} + J_w w^z_{j} w^z_{j+1} + J_z^{zz} w^z_{j} - \sum_{j=1}^{N} \mu_p p^z_{j} + \mu_w w^z_{j} \right]$$

(VI.3)

Here $J_p = J_x \Theta_x/4$, $J_w = J_x \Delta_x/2$, $\mu_p = J_x \Omega_x/4$, and $\mu_w = J_z \Delta_z/2$. Now it’s easy to see that the fluctuations in the $zz$-sector are suppressed for all $\mu_{xx} \neq 0$, $\sigma = p, w$. While the remaining $xx$-part can order at $T = 0$ (i.e. $\chi_{xx}(q, \omega)$ can diverge at $\omega = 0$, $T = 0$ for $q = 0$ or $\pi$), it cannot order at $T > 0$. Moreover, even with inclusion of the four-spin term, having $J_p \neq J_w$ and $\mu_p \neq \mu_w$ ensures the non-closure of the spin excitation gap. Thus neither the $xx$ nor the $zz$ fluctuations get singular. This qualitatively implies that MF results are stable against fluctuations.

While the argument above is a symmetry-based one, actual computation of the renormalization caused by $V_f$ is rather involved. We now present a perturbative argument that can be made self-consistent, and which reinforces the above conclusions.
We apply JW fermionization (I.1) on Eq. (VI.2),

\[
V_f = \left[ \frac{J_x}{4} \sum_{j=1}^{N} (a_j - a_j^\dagger)(a_{j+1} + a_{j+1}^\dagger)(b_j - b_j^\dagger)(b_{j+1} + b_{j+1}^\dagger) + \frac{J_z}{4} \sum_{j=1}^{N} (2a_j^\dagger a_j - 1)(2b_j b_j - 1) \right] \\
- \left[ \frac{J_x}{4} \sum_{j=1}^{N} \left( 2\Delta_x (b_j - b_j^\dagger)(b_{j+1} + b_{j+1}^\dagger) + \Theta_x (a_j - a_j^\dagger)(a_{j+1} + a_{j+1}^\dagger) \right) + \frac{J_z}{4} \sum_{j=1}^{N} \left( 2\Delta_z (2b_j b_j - 1) + \Omega_z (a_j^\dagger a_j - 1) \right) \right] 
\]

(VI.4)

The terms inside first square bracket represents the quartic interactions between \(a\) and \(b\) fermions. It contains several terms, some of the interaction vertices don’t even preserve the fermion numbers (like \(a^\dagger a b^\dagger b\)). The terms inside second square bracket denotes quadratic “external fields”. We write the above interaction in momentum space with spinor notations for fermionic fields. we define \(\psi_k^\dagger \equiv (a_k^\dagger, a_{-k}^\dagger), \phi_k^\dagger \equiv (b_k^\dagger, b_{-k}^\dagger),\)

\[
V_f = \frac{1}{2} \sum_{k, q, \alpha, \beta} \sum_{\sigma} \gamma^{\alpha \beta \gamma \delta} \psi_{k+q, \alpha, \sigma}^\dagger \phi_{k, \gamma, \delta}^\dagger \psi_k \phi_k - \sum_{k, \alpha, \beta} \left[ h^\alpha_{a,k} \psi_k^\dagger \phi_k + h^\beta_{b,k} \phi_k^\dagger \psi_k \right] 
\]

(VI.5)

Our target is to find how the single particle Green’s functions (SPGF) of ‘\(a\)’ and ‘\(b\)’ get modified by (VI.5), the modified SPGF will cause corrections in different MF averages. We define the matrix Green’s functions for ‘\(a\)’ fermions as \(G^{a}_{\alpha \sigma}(k, \tau) = -(T_{\tau} \psi_{k, \alpha}(\tau) \psi_{k, \alpha}^\dagger(0))\) and similar for ‘\(b\)’ fermions, \(G^{b}_{\alpha \sigma}(k, \tau) = -(T_{\tau} \psi_{k, \alpha}(\tau) \psi_{k, \alpha}^\dagger(0))\). The bare/ non-interacting SPGF are following,

\[
G_{11}^{0,\sigma}(k, i\omega) = \frac{(w_k^\sigma)^2}{i\omega - E_k} + \frac{(v_k^\sigma)^2}{i\omega + E_k^\sigma}, \quad G_{22}^{0,\sigma}(k, i\omega) = -[G_{11}^{0,\sigma}(k, i\omega)]^* \\
G_{21}^{0,\sigma}(k, i\omega) = \frac{\Delta_k^\sigma}{2E_k} \left[ \frac{1}{i\omega - E_k} - \frac{1}{i\omega + E_k^\sigma} \right], \quad G_{12}^{0,\sigma}(k, i\omega) = [G_{21}^{0,\sigma}(k, i\omega)]^* 
\]

(VI.6)

We find that the Feynman diagrams arising from “external fields” in \(V_f\) will always cancel with the zero-momentum transfer Hartree diagrams arising from the quartic interactions in the perturbative expansion of SPGF. In this way, the leading order diagrams are those of second order in \(V_f\). It is straightforward to check that second order perturbative corrections (without Hartree diagrams) have the following general structure,

\[
\sim \frac{1}{\beta N} \sum_{q, \alpha, \beta, \alpha', \beta'} \chi^{(b),p}_{\alpha' \beta' \alpha \beta'}(q, i\omega) G^{a}_{\alpha \alpha'}(p, ip) G^{a}_{\beta' \beta}(p - q, ip - iq) G^{a}_{\beta' \beta}(q, i\omega) 
\]

(VI.7)

the generalized susceptibility or so called bubble diagram is expressed following,

\[
\chi^{(b),p}_{\alpha' \beta' \alpha \beta'}(q, i\omega) = \frac{1}{\beta N} \int_0^\beta d\tau e^{i\omega \tau} \sum_{k, k'} \sum_{\gamma, \delta} \gamma^{\alpha' \beta' \gamma \delta} \gamma^{\alpha \beta \gamma' \delta'} \int_{-\infty}^{\infty} \left[ T_{\tau} \{ \phi_{k', \gamma, \delta}^\dagger \phi_{k', \gamma, \delta} \} \right] 
\]

(VI.8)

Explicit computation of these diagrams are straightforward but lengthy. We use here a simple physical argument. The excitation spectrum of ‘\(a\)’ and ‘\(b\)’ JW fermions are always gapped for all \(J_x/J_z\) and 0 \(\leq T < T_c\), which means that single particle density of states (DOS), \(D(\omega)\) is zero at low energies (for \(\omega \leq \Delta\), the energy gap). This will put a infrared cut-off scale for all energy/ frequency integrations in the above Feynman diagrams. So as a result, the diagrams will not diverge and the pole structure of the SPGF remains preserved with some quantitative renormalization of the energy gap [7]. As a whole, the MFT phase diagram qualitatively remains same in the presence of these interactions.

(2): Thermal fluctuations of \(V_f^z\) :

A glance at the Hamiltonian (VI.1) shows that while spatial uniformity of the bond-product of \(Z_2\) variables \(v^z\) enables solubility of the MFT equations that lead to our results, fluctuations in these variables at finite \(T\) are akin to sudden local quenches. Indeed, we can see from Eq. (VI.1) that a flip of a single \(v^z\), say from +1 to −1 (or vice-versa), acts as a “suddenly switched on” potential for the JW fermions. However, since these are described by a fully gapped \(p\)-wave superconductor, the orthogonality catastrophe (OC) associated with sudden switching of such a local potential in a Fermi sea is suppressed by the \(p\)-wave gap. This is true everywhere, except at a single point \(J_x = J_z = T = T_c\), where the gapless \(1d\) like fermionic spectrum (see Fig. 7) causes an OC to occur. Had this
occurred at $T = 0$, it would have generated an infrared singularity and invalidated any picture based on well-defined JW fermions, or of $p$-wave pairs formed from them [8]. However, at finite $T$, it is well known that this singularity is smeared by a scale $k_B T$, $k_B = $ Boltzmann constant. The resulting Doniach-Sunjic lineshape [9] carries the memory of the infrared singularity, and would correspond to a branch cut, instead of a renormalized pole structure, in the spin-fluctuation propagators (remember that, in JW fermionic language, the spin-fluctuation propagators are density fluctuation propagators).

The above discussion shows that the sudden local quenches associated with fluctuations of the $\mathbb{Z}_2$ variables ($V_{jz}$) do not invalidate MFT results qualitatively either, except in a narrow window around $J_x = J_z$ and $T = T_c$.

Taken together, the above arguments show that our MFT results should remain qualitatively valid even when fluctuations beyond MFT are taken into account, except for a narrow interval around $J_x = J_z$ and $T = T_c$. Additionally, the surprisingly good accord found with $i$PEPS (for the 2d QCM) [11] and PCUT (for the transverse field Toric code) results [10] also hint at fluctuation effects being finite.

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