Uniform Spanning Forests and the bi-Laplacian Gaussian field

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December 3, 2013

Abstract

We construct a natural discrete random field on \( \mathbb{Z}^d \), \( d \geq 5 \) that converges weakly to the bi-Laplacian Gaussian field in the scaling limit. The construction is based on assigning i.i.d. Bernoulli random variables on each component of the uniform spanning forest, thus defines an associated random function. To our knowledge, this is the first natural discrete model (besides the discrete bi-Laplacian Gaussian field) that converges to the bi-Laplacian Gaussian field.

Keywords Uniform spanning forest, bi-Laplacian Gaussian field, moment method

Acknowledgments. We are very grateful to Scott Sheffield for suggesting this problem, to Gregory Lawler for helping us complete the proof of Lemma 10 and to Richard Kenyon for discussions.

1 Introduction

Uniform spanning forest is an extensively studied combinatorial object [2], [17]. The uniform spanning forest measure on \( \mathbb{Z}^d \) can be defined in two equivalent ways: either as the weak limit of the uniform spanning tree measure on a sequence of finite subgraphs that exhaust \( \mathbb{Z}^d \), or as an output of the Wilson’s algorithm [21]. Detailed descriptions of these constructions are given in Section 2.2.

In this paper, we study the following random field associated with the USF on \( \mathbb{Z}^d \), \( d \geq 5 \). It is known that the USF on \( \mathbb{Z}^d \), \( d \geq 5 \) has infinitely many tree components a.s. Conditioned on the configuration of the whole forest \( \{T_i\}_{i \in \mathbb{N}} \), we assign i.i.d Bernoulli random variables on each tree \( T_i \), with probability \( 1/2 \) to be 1 and 1/2 to be \(-1\). We define a random function (which we call the spin of the spanning forest) \( h_1 \) from \( \mathbb{Z}^d \) to \( \{\pm 1\} \), such that for any \( x \in \mathbb{Z}^d \), \( h_1(x) \) equals the random variable associated with the tree component containing \( x \). This random function is constructed in a similar spirit as the Edward-Sokal coupling of the FK-Ising model [6].

We would like to study the scaling limit of \( h_1 \). For \( \varepsilon \geq 0 \), consider the lattice \( \varepsilon \mathbb{Z}^d \), let \( h_\varepsilon(x) = \varepsilon^{d/2-d/2} h_1(\varepsilon^{-1} x) \), \( \forall x \in \varepsilon \mathbb{Z}^d \). We extend \( h_\varepsilon \) to \( \mathbb{R}^d \) such that \( h_\varepsilon(y) = h_\varepsilon(x) \) for \( y \in B_{\varepsilon/2}(x) = (x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2})^d \).

It turns out that the limiting field of \( h_\varepsilon \) is a generalized Gaussian field (a random generalized distribution whose integral against any \( C_0^\infty \) test function is a Gaussian) closely related to bi-Laplacian operator \( \Delta^2 \), which we call the bi-Laplacian Gaussian field. We will give the precise definition of the bi-Laplacian Gaussian field in Section 2.1 here we offer an informal description. Intuitively, a bi-Laplacian Gaussian field is a generalized Gaussian field \( h \) whose covariance structure is given by \( \text{Cov}[h(x), h(y)] = |x - y|^{4-d} \). The rigorous formulation of this definition of given in Definition 4 of Section 2.1 where we also discuss its relation to bi-Laplacian equations. It is the
analogy of that of Gaussian free field (GFF) to Laplacian equation (for the definition and properties of Gaussian free field, see the survey [20]).

Here we point out that the bi-Laplacian Gaussian field fall into a bigger family of Gaussian fields called the fractional Gaussian fields (FGF) which is defined and studied in [16]. The relation of FGF and fractional Laplacian operator \((-\Delta)^s\) is analogous to both the Gaussian free field and the bi-Laplacian Gaussian field. Besides GFF and bi-Laplacian free field, the family of FGF also contains white noise, log-correlated Gaussian field and the fractional Brownian field [1] (a higher dimensional generalization of fractional Brownian motion).

The main result of this paper is that \(h, h_\varepsilon\) converge to \(h\) as random variables taking values in the space of generalized function. To be precise, we have the following theorem.

**Theorem 1.** For any \(\varphi \in C_0^\infty(\mathbb{R}^d)\), \((h, \varphi)\) converge to \(\sqrt{c_d}(h, \varphi)\) in distribution as \(\varepsilon \to 0\). The constant \(c_d\) can be computed by non-intersecting probability of a simple random walk and two loop erased random walks, see Lemma [17].

Gaussian fluctuations has been observed and studied for numerous physical systems. For systems in the critical regime, one expects the spatial or space-time fluctuation to be Gaussian free field. Typical examples come from domino tilings [9], random matrix theory [3][18] and random growth models [4]. In the subcritical regime, where the correlation decays faster, one expects Gaussian white noise fluctuations (see the example of edge process of spanning tree models in [8]). Our model can be viewed as a natural example in the supercritical regime.

[19][11][12] study the discrete bi-Laplacian Gaussian field (in physics literature, this is known as the membrane model) whose continuous counterpart is clearly the bi-Laplacian Gaussian field. Our model can be viewed as another natural discrete object that converges to the bi-Laplacian Gaussian field. In one dimensional case, Hammond and Sheffield constructed a reinforced random walk with long range memory [7], which can be associated with a spanning forest attached to \(\mathbb{Z}\). Our construction can also be viewed as a higher dimensional analogue of “forest random walks”.

Finally, we remark on universality features of our model. We can replace i.i.d. Bernoulli random variables by general i.i.d. random variables with mean 0 and variance 1, and obtain the same scaling limit. The same argument also goes through if we replace \(\mathbb{Z}^d\) by regular lattices, the constant \(c_d\) is lattice dependent. See Remark [11].

The strategy of the proof is moment method. Since \((h, \varphi)\) is a Gaussian random variable, to prove convergence in distribution, we only need to prove that all the moment of \((h, \varphi)\) converge to the corresponding moments of \((h_\varepsilon, \varphi)\). The paper is organized as follows. In Section 2 we give the necessary background on uniform spanning forest and the bi-Laplacian Gaussian field. In Section 3 we prove the convergence of second moment. It involves giving the precise asymptotics of the probability that two vertices are in the same tree of USF. In Section 4 we prove the convergence of higher moments. In Section 5 we discuss some further questions.

## 2 Preliminary

### 2.1 Bi-Laplacian Gaussian field

In this section, we will give a precise definition of the bi-Laplacian Gaussian field, which is a random variable taking value in the space of generalized functions (denoted by \((C_0^\infty(\mathbb{R}^d))^\prime\)). Or equivalently, a probability distribution on \((C_0^\infty(\mathbb{R}^d))^\prime\). For basic facts on generalized function, we refer to Appendix B, [15].
We first review some standard facts on white noise, [10, text]. White noise is the unique probability distribution on \((C_0^\infty(\mathbb{R}^d))'\) such that if \(W\) is a random generalized function with this distribution, then for any \(\varphi \in C_0^\infty(\mathbb{R}^d)\), \((W, \varphi)\) is a centred Gaussian variable with variance \((\varphi, \varphi)\). Here \((, )\) is the pair of a generalized function and a compact supported smooth function.

Formally speaking, we can say that \(W\) is a Gaussian process whose parameter is in \(\mathbb{R}^d\) and the covariance structure is given by

\[
\text{Cov}[(W(x), W(y))] = \delta(x - y).
\]

Another natural interpretation is that \(W\) is a standard normal distribution on the Hilbert space \(L^2(\mathbb{R}^d)\).

We give two equivalent definitions of the bi-Laplacian Gaussian field, that only differ by scalar multiplication. We can define the bi-Laplacian Gaussian field for all dimensions in a unified way, as in [16]. But to avoid technical details for \(d \leq 4\), we only define the field for \(d \geq 5\), which is sufficient for the purpose of this paper. From now on, we always assume \(d \geq 5\).

**Definition 2.** Bi-Laplacian Gaussian field is the unique probability distribution on \((C_0^\infty(\mathbb{R}^d))'\) such that if \(h\) is a random generalized function with this distribution, \(\Delta h\) is a white noise on \(\mathbb{R}^d\). Here \(\Delta\) is a well defined operator on \((C_0^\infty(\mathbb{R}^d))'\) by integration by part [15].

**Definition 3.** Bi-Laplacian Gaussian field is the unique probability distribution on \((C_0^\infty(\mathbb{R}^d))'\) such that for any \(\varphi \in C_0^\infty(\mathbb{R}^d)\), \((h, \varphi)\) is a centred Gaussian variable and

\[
\text{Var}[(h, \varphi)] = \int \int |x - y|^{4-d} \varphi(x)\varphi(y)dxdy. \tag{1}
\]

For this moment, we assume that there is a unique random generalized function satisfying the Definition 2 or 3, which we will explain later. Now we explain the equivalence of the two definitions.

We note that if \(\Delta h = W\). Then \((h, \Delta^2 f) = (\Delta h, \Delta f) = (W, \Delta f)\) is a centred Gaussian of variance \((\Delta f, \Delta f) = (f, \Delta^2 f)\). For \(\varphi \in C_0^\infty(\mathbb{R}^d)\), we can solve the bi-Laplacian equation

\[
\Delta^2 f = \varphi, \tag{2}
\]

for example, using Fourier transform. This is the place our assumption of \(d \geq 5\) plays a role since otherwise not all functions in \(C_0^\infty(\mathbb{R}^d)\) has bi-Laplacian inverse. There will be some extra assumption for \(\varphi\) when \(d \leq 4\) like in the case of 2 dimensional Gaussian free field in the whole plane [20]. Therefore the variance of \((h, \varphi)\) is given by

\[
(f, \Delta^2 f) = (f, \varphi) \tag{3}
\]

The presence of \(\Delta^2\) is the reason we call the field the bi-Laplacian Gaussian field. In the case of Gaussian free field, we want to solve a Laplacian equation. Here we want to solve a bi-Laplacian equation [2]. Bi-Laplacian equation is a standard object in potential theory and studied for many years. For information of this equation we refer to [5] and references therein. From [5], for \(d \geq 5\), the fundamental solution of bi-Laplacian equation is \(C_d|x - y|^{4-d}\) and we can use the fundamental solution to solve equation 2, which is

\[
f(x) = C_d \int_{\mathbb{R}^d} |x - y|^{4-d} \varphi(y)dy, \tag{4}
\]
where $C_d$ is a constant depending on $d$. From (3) and (4) we see that Definition 2 and 3 of a bi-Laplacian Gaussian field only differ by a constant $\sqrt{C_d}$. In Theorem 1 we use Definition 3 as our definition of a bi-Laplacian Gaussian field for convenience.

As mentioned before, the existence and uniqueness in the definition of bi-Laplacian field is not clear as a priori. The rigorous argument for existence and uniqueness is actually the same as in the definition of white noise [10]. Now we sketch the construction of white noise as a random distribution following [10]. The definition of a bi-Laplacian Gaussian field will follow by a similar argument.

**Definition 4 (Countably-Hilbert Space)**. Let $V$ be an infinite dimensional vector space over $\mathbb{C}$, and let $\{| \cdot |_n\}_{n \geq 1}$ be a collection of inner product norms on $V$. Define the metric $d$ on $V$ by

$$d(u, v) = \sum_{n=1}^{\infty} 2^{-n} \frac{|u - v|_n}{1 + |u - v|_n}, \quad u, v \in V,$$

If $V$ is complete with respect to $d$ then $(V, \{| \cdot |_n\}_{n \geq 1})$ is called a countably-Hilbert space.

**Definition 5 (Nuclear Spaces)**. Let $V$ be a countably-Hilbert space associated with an increasing sequence $\{| \cdot |_n\}_{n \geq 1}$ of norms, that is,

$|v|_1 \leq |v|_2 \leq \cdots \leq |v|_n \leq \cdots, \quad \forall v \in V.$

Let $V_n$ be the completion of $V$ with respect to the norm $| \cdot |_n$. We say that $V$ is a nuclear space if for any $m$, there exists $n \geq m$ such that the inclusion map $V_m$ into $V_n$ is a Hilbert-Schmidt operator, that is, there is an orthonormal basis $\{v_k\}$ for $V_m$ such that $\sum_{k=1}^{\infty} |v_k|_n^2 < \infty$. 

**Remark 6**. It is well known that $C_0^\infty(\mathbb{R}^d)$ is a nuclear space. For a proof, see [10].

If $V$ is a topological vector space, we denote by $V'$ the dual of $V$ (that is, the space of continuous linear functionals on $V$). We say that a complex-valued function $\varphi$ on $V$ is the characteristic function of a probability measure $\nu$ on $V'$ if

$$\varphi(v) = \int_{V'} e^{ix,v} \, d\nu(x), \quad \text{for all } v \in V.$$

For a proof of the following theorem, see ([10]).

**Theorem 7 (Bochner-Minlos theorem)**. Let $V$ be a real nuclear space. Then a complex-valued function $\Phi$ on $V$ is the characteristic function of a probability measure $\nu$ on $V'$ if and only if $\Phi(0) = 1$, $\Phi$ is continuous, and $\Phi$ is positive definite, that is,

$$\sum_{j,k=1}^{n} z_j \overline{z_k} \Phi(v_j - v_k) \geq 0,$$

for all $v_1, \ldots, v_n \in V$, and $z_1, \ldots, z_n \in \mathbb{C}$. Furthermore, $\Phi$ determines $\nu$ uniquely.

White noise will be defined as a Gaussian measure on the space of tempered distributions. To apply Theorem 7 we first note that $C_0^\infty(\mathbb{R}^d)$ is a nuclear space and that the function

$$C(\varphi) = e^{-\frac{1}{2} \langle \varphi, \varphi \rangle}, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d),$$

4
is continuous, positive definite, and satisfies $C(0) = 1$. Hence Theorem 7 implies that there is a unique probability measure $\mu$ on $(C_0^\infty(\mathbb{R}^d))'$ having $C$ as its characteristic function. which we define as white noise $W$. In particular we have the relation:

$$\int_{S'(\mathbb{R}^d)} e^{i(x, \varphi)} d\mu(x) = e^{-\frac{1}{2}(\varphi, \varphi)}, \quad \varphi \in C_0^\infty(\mathbb{R}^d),$$

which implies for every $\varphi \in C_0^\infty(\mathbb{R}^d)$ the random variable $(W, \varphi)$ is a mean zero Gaussian with variance $(\varphi, \varphi)$. Given $f, g \in C_0^\infty(\mathbb{R}^d)$ we may use polarization to see that

$$\text{Cov}[(W, f), (W, g)] = (f, g),$$

We may rewrite the above expression as

$$\text{Cov}[(W, f), (W, g)] = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \delta(x - y) f(x) g(y) \, dx \, dy,$$

and say that $W$ has covariance kernel $\delta(x - y)$.

To show the existence and uniqueness of the bi-Laplacian Gaussian field, we only need to find its characteristic function and apply Theorem 7. From the first definition of Definition 2, it is easy to see that the characteristic function for a bi-Laplacian Gaussian field is

$$C(\varphi) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2}(\Delta^{-1} \varphi, \Delta^{-1} \varphi) \right).$$

Here $(\Delta^{-1} \varphi, \Delta^{-1} \varphi)$ is understood as $(f, \phi)$ where $\Delta^2 f = \phi$.

**Lemma 8.** The functional $C(\varphi)$ defined by

$$C(\varphi) = \text{exp} \left( -\frac{1}{2}(\Delta^{-1} \varphi, \Delta^{-1} \varphi) \right),$$

is a continuous, positive definite functional on $C_0^\infty(\mathbb{R}^d)$ that satisfies $C(0) = 1$.

**Proof.** The continuity (continuity is taken with respect to the norm $(\Delta^{-1} \varphi, \Delta^{-1} \varphi)\frac{1}{2}$) of $C(\varphi)$ follows from Fourier transform and the fact that $|x|^4$ is integrable in $\mathbb{R}^d (d \geq 5)$. Further the statement $C(0) = 1$ is also clear. All that is left is to check that $C(\varphi)$ is positive definite.

Let $\varphi_1, \ldots, \varphi_n \in C_0^\infty(\mathbb{R}^d)$ be a set of functions, and define $V$ to be the subspace of $C_0^\infty(\mathbb{R}^d)$ spanned by $\{\varphi_i\}$. Define $\mu_V$ to be the Gaussian measure on $V$ with covariance matrix given by $\Xi_{i,j} = (\Delta^{-1} \varphi_i, \Delta^{-1} \varphi_j)$ so that its characteristic function is

$$\int_V e^{i(\Delta^{-1} \varphi, \Delta^{-1} y)} d\mu_V(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\Delta^{-1} \varphi, \Delta^{-1} \varphi)} = C(\varphi), \quad \varphi \in V.$$ 

Applying Bochner’s theorem for probability measures on $\mathbb{R}^n$ shows us that $C$ is positive definite. □

Now apply Milnos theorem we get the existence and uniqueness of the bi-Laplacian Gaussian field.

**Remark 9.** In [13], the authors define the so called fractional Gaussian field in the following way. Formally speaking, the $d$ dimensional fractional Gaussian field with index $s$ (denoted by $\text{FGF}_s^d$) is given by $(-\Delta)^{s} W$. Thus the bi-Laplacian Gaussian field is $\text{FGF}_2^d$. 


2.2 Uniform spanning forest

Here we review some facts about the uniform spanning forest model (USF) on \( \mathbb{Z}^d \). Most of the facts extend to general graphs as well. For more background, we refer the reader to the survey [2].

Given a finite graph \( G \subset \mathbb{Z}^d \), the (free) uniform spanning tree (UST) measure is the probability measure that assigns equal probability to the spanning trees of \( G \). When all the vertices on \( \mathbb{Z}^d \setminus G \) are contracted to a single vertex, the corresponding measure is called wired spanning tree.

Uniform spanning forest measure on \( \mathbb{Z}^d \) is the weak limit of uniform spanning trees on a sequence of exhausting subsets. Pemantle proved that the limit of free and wired spanning trees coincide [17], thus USF is uniquely defined and has trivial tail.

An alternative way to construct the USF is the Wilson’s algorithm [21], which we now describe. For any path \( \mathcal{P} \) in \( \mathbb{Z}^d \) that visits each vertex at most finitely many often, the loop erasure of \( \mathcal{P} \) is constructed as erasing cycles in \( \mathcal{P} \) in chronological order. Fix any ordering \((v_1, v_2, ...)\) of vertices, a growing sequence of forests \( \{F_i\}_{i \in \mathbb{N}} \) can be constructed inductively. Let \( F_0 = \emptyset \). Suppose the forest \( F_i \) has been generated. Start a simple random walk (SRW) at \( v_{i+1} \), and stop at the first time it hits \( F_i \), if it does, and otherwise let it run indefinitely. \( F_{i+1} \) is defined by adding the loop erasure of this SRW to \( F_i \) (for \( d \geq 3 \), SRWs are transient, so the loop erasure of SRW is well defined a.s.)

The algorithm yields \( \bigcup_{i \in \mathbb{N}} F_i \), it is shown in [2] that its distribution is independent of the ordering of vertices, and is USF.

Based on Wilson’s algorithm and properties of loop erased random walks (LERW), it is shown in [17] that on \( \mathbb{Z}^d \), the USF is a single tree a.s. if \( d \leq 4 \), and has infinitely many tree components a.s. when \( d \geq 5 \). The probability that two points are in the same tree is the intersection probability of a SRW and a LERW. This will be used in Lemma 10. Also, when \( 2 \leq d \leq 4 \), the USF has a single topological end a.s. (i.e. removing any vertex disconnect the tree into two components, one of them is infinite); when \( d \geq 5 \), each of the infinitely many trees a.s. has at most two topological ends.

3 Second moment

By definition of \( h_\varepsilon \),

\[
(h_\varepsilon, \varphi) = \sum_{x \in \mathbb{Z}^d} \varepsilon^{4-d} h_1(\varepsilon^{-1} x) \int \varphi(y) dy
\]

\[
= \sum_{x \in \mathbb{Z}^d} \varepsilon^{4-d} h_1(\varepsilon^{-1} x) \varphi(x) \varepsilon^d + R_\varepsilon(\varphi),
\]

where the remaining term \( \lim_{\varepsilon \to 0} R_\varepsilon = 0 \) almost surely.

So we only need to show that \( X_\varepsilon = \sum_{x \in \mathbb{Z}^d} \varepsilon^{4-d} h_1(\varepsilon^{-1} x) \varphi(x) \varepsilon^d \) converge to \( \sqrt{\alpha}(h, \varphi) \) in distribution. As explained in the introduction, we use the moment method. Since the first moment is just 0, we start from the second moment, which is the focus of this section.

Note that

\[
X_\varepsilon = \sum_{x \in \mathbb{Z}^d} \varepsilon^{4-d} h_1(x) \varphi(\varepsilon x).
\]

\[
\mathbb{E}(X_\varepsilon^2) = \sum_{x, y \in \mathbb{Z}^d} \varepsilon^{4+d} \varphi(\varepsilon x) \varphi(\varepsilon y) \mathbb{E}[h_1(x) h_1(y)].
\]
Let \( p(x, y) = P(x, y \text{ are in the same tree}) \), then

\[
E[h_1(x)h_1(y)] = p(x, y) \times 1 + (1 - p(x, y)) \times 0 = p(x, y).
\]

As explained in Section 2.2, uniform spanning forest can be generated using Wilson algorithm on \( \mathbb{Z}^d \). Therefore from Lemma 10 which we will prove in Section 3.1 we know that

\[
\lim_{|x-y| \to \infty} \frac{p(x, y)}{|x-y|^{d-\delta}} = c_d. \quad c_d \text{ is a constant which we could not explicitly evaluate explicitly because we cannot evaluate the number } q \text{ in (5)} \text{, Section 3.1.}
\]

Since \( \varphi \in C_0^\infty(\mathbb{R}^d) \), by dominate convergence theorem,

\[
\lim_{\varepsilon \to 0} E[X^2_\varepsilon] = c_d \int \int \varphi(x)\varphi(y)|x-y|^{4-d}dxdy.
\]

From Section 2.1 we recognize that the RHS of above formula is just \( c_d \) times the variance of \( (h, \varphi) \) as we defined in formula (3) in Section 2.1

### 3.1 Asymptotic correlation

In this section we explicitly determine the asymptotics of \( p(x, y) = p(0, y-x) \). This requires to evaluate the non-intersecting probability of a SRW starts at \( y-x \) and a LERW starts at 0. Using the bounds for intersection of SRWs, Penamle showed \( p(0, y-x) = O(|y-x|^{1-d}) \). Here we show this quantity actually converges in the scaling limit. This requires a more careful estimate of SRW hitting probabilities.

**Lemma 10.** Suppose \( S_1, S_2 \) are \( d \geq 5 \) dimensional SRWs starting from 0 and \( z \). Then \( P(\hat{S}_1[0, \infty] \cap S_2[0, \infty] \neq \emptyset) = c|z|^{4-d} + o(|z|^{4-d}) \text{ as } z \to \infty. \)

**Proof.** The proof is suggested by Lawler [13]. Let \( \rho \) be the first time \( S_2 \) hits \( \hat{S}_1[0, \infty] \), \( \tau \) be the largest time such that \( S_1(\tau) = S_2(\rho) \). For \( w \in \mathbb{Z}^d \), \( (j, k) \in \mathbb{N} \times \mathbb{N} \), let \( A_{w,j,k} \) be the event \( \{S_2(\rho) = w, \tau = j, \rho = k\} \). We can see that \( \{S_1[0, \infty] \cap S_2[0, \infty] \neq \emptyset\} = \sum_{w,j,k} A_{w,j,k} \) almost surely.

Let \( \hat{S}_1 \) be the time reversal of \( S_1 \) for \( \tau \) to 0, \( \hat{S}_2 \) be the time reversal of \( S_2 \) from \( \rho \) to 0, \( S_3 \) be \( S_1 \) from \( \tau \) to \( \infty \). Then

\[
A_{w,j,k} = \{\hat{S}_1(j) = 0, \hat{S}_2(k) = z, \hat{S}_1[0,j] \cap S_3[1, \infty] = \emptyset, \hat{S}_2[1,k] \cap \{\hat{S}_1[0,j] \cup \hat{S}_3[0, \infty]\} = \emptyset\}.
\]

Thus

\[
P(A_{w,j,k}) = P(\hat{S}_1(j) = 0, \hat{S}_2(k) = z)P(\hat{S}_1[0,j] \cap \hat{S}_3[1, \infty] = \emptyset, \hat{S}_2[1,k] \cap (\hat{S}_1[0,j] \cup \hat{S}_3[0, \infty]\} = \emptyset|\hat{S}_1(j) = 0, \hat{S}_2(k) = z).
\]

Now for simplicity of notation we assume that \( S_1, S_2, S_3 \) are three independent SRWs starting at \( w \). Then

\[
P(A_{w,j,k}) = P(S_1(j) = 0)P(S_2(k) = z)P(S_1[0,j] \cap S_3[1, \infty] = \emptyset, S_2[1,k] \cap (S_1[0,j] \cup S_3[0, \infty]\} = \emptyset|S_1(j) = 0, S_2(k) = z).
\]

Let

\[
q = P(\hat{S}_1[0, \infty] \cap S_3[1, \infty] = \emptyset, S_2[1, \infty] \cap \{\hat{S}_1[0, \infty] \cup S_3[0, \infty]\} = \emptyset)
\]

(5)
, \( G(\cdot, \cdot) \) be the Green function of SRW on \( \mathbb{Z}^d \). Now we show that the non-intersection probability

\[
\mathbb{P}(\hat{S}^1[0, \infty] \cap S^2[0, \infty] \neq \emptyset) = \sum_{w,j,k} \mathbb{P}(A_{w,j,k}) \sim q \sum_w G(0, w) G(w, z)
\]

, together with the fact that discrete Green’s function converges to the continuous whole space Green’s function [13], therefore \( G(z, w) = O\left(|z - w|^{2-d}\right) \) for \( z, w \) macroscopically apart, this implies Lemma [10].

To prove the upper bound, we fix small \( \varepsilon > 0 \) and large \( R > 0 \). Let \( w \) be in the range of \( |w| \geq \varepsilon |z|, |w - z| \geq \varepsilon |z| \) and \( j, k \) greater than \( |z|^2 \). Let \( \sigma_i \) be the last time when \( S^i \) hits the ball \( B_R \) centred at \( w \). For fixed \( R \) and \( w \), on the high probability event that \( \sigma_1 \ll j \) and \( \sigma_2 \ll k \), as \( |z| \to \infty \), the Radon-Nikodym derivative of the joint distribution \( \{S^1[0, \sigma_1], S^2[0, \sigma_2], S^3[0, \sigma_3]\} \) conditioned on that \( S^1(j) = 0, S^2(k) = z \) w.r.t the original unconditioned one tends to 1. For \( (w, i, j) \) satisfy the conditions prescribed,

\[
\mathbb{P}(A_{w,j,k}) \leq \mathbb{P}^w(S^1(j) = 0) \mathbb{P}^w(S^2(k) = z) \times \mathbb{P}(\hat{S}^1[0, \sigma_1] \cap S^3[1, \sigma_3] = \emptyset, S^2[1, \sigma_2] \cap \{\hat{S}^1[0, \sigma_1] \cup \hat{S}^3[0, \sigma_3]\} = \emptyset | S^1(j) = 0, S^2(k) = z).
\]

where \( \delta_{R,z} \to 0 \) as \( z \) tends to \( \infty \) and \( R \) fixed. On the other hand, the typical time for a SRW starting at \( w \) to hit 0 or \( z \) is \( O(|z|^2) \), thus \( \sum_{\delta_{R,z} < |z|^2} \mathbb{P}^w(S(j) = 0) \mathbb{P}^w(S^2(k) = z) \) tends to zero uniformly in \( w \) as \( z \to \infty \). Also, when summing over \( w \in \mathbb{Z}^d \), the contribution from \( |w| < \varepsilon |z| \) or \( |w - z| < \varepsilon |z| \) is negligible as \( \varepsilon \to 0 \). Since

\[
\lim_{R \to \infty} \mathbb{P}(\hat{S}^1[0, \sigma_1] \cap S^3[1, \sigma_3] = \emptyset, S^2[1, \sigma_2] \cap \{\hat{S}^1[0, \sigma_1] \cup S^3[0, \sigma_3]\} = \emptyset) = q.
\]

By summing over \( w, i, j \), first taking \( z \to \infty \), then \( R \to \infty \) and then \( \varepsilon \to 0 \), we know that

\[
\limsup_{z \to \infty} \frac{\sum_{w,j,k} \mathbb{P}(A_{w,j,k})}{\sum_w G(0, w) G(w, z)} \leq q.
\]

To show the lower bound, as before we first fix \( \varepsilon > 0 \) and \( w \) in the range \( |w| \geq \varepsilon |z|, |w - z| \geq \varepsilon |z| \) and \( j, k \geq |z|^2 \). When \( 1 \ll R \ll z \) is fixed but large enough, as \( z \) tends to \( \infty \), there is a high probability \( p_R \) such that the distance between \( S^1_{\sigma_1}, S^2_{\sigma_2}, S^3_{\sigma_3} \) is bigger than \( cR \), where \( c \) is a constant independent of \( R \) and \( p_R \) tends to 1 as \( R \) tends to infinity. This is because that as \( z \to \infty \), the \( S^1_{\sigma_1}, S^2_{\sigma_2}, S^3_{\sigma_3} \) are close to three uniform distribution on \( \partial B_R \) as we argued above. The probability that \( S^1[\sigma_1, \infty], S^2[\sigma_2, \infty], S^3[\sigma_3, \infty] \) have an intersection will tend to zero, as \( z \) first goes to \( \infty \) and then \( R \) goes to \( \infty \), which can be seen by bounding the intersection probabilities explicitly by Green’s functions. Using the asymptotic independence of \( S^1, S^2 \) in \( B_R \) and the event \( S^1(j) = 0, S^2(k) = z \), we obtain

\[
\mathbb{P}(\hat{S}^1[0, j] \cap S^3[1, \infty] = \emptyset, S^2[j, k] \cap \{\hat{S}^1[0, j] \cup \hat{S}^3[0, \infty]\} = \emptyset | S^1(j) = 0, S^2(k) = z) \geq (1 - \varepsilon_{R,z}) (1 - \delta_{R,z}) \mathbb{P}(\hat{S}^1[0, \sigma_1] \cap S^3[1, \sigma_3] = \emptyset, S^2[1, \sigma_2] \cap \{\hat{S}^1[0, \sigma_1] \cup S^3[0, \sigma_3]\} = \emptyset),
\]

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where $\delta_{R,z}$ tends to 0 as $z \to \infty$, and $\varepsilon_{R,z} \to 0$ as $z$ first goes to $\infty$ and then $R$ goes to $\infty$. Thus
\[
\liminf_{z \to \infty} \sum_w \frac{\mathbb{P}(A_{w,j,k})}{G(0, w)G(w, z)} \geq q.
\]

## 4 Higher moments

Recall the random field $\{h_\varepsilon\}$, defined for any $\varphi \in C_0^\infty(\mathbb{R}^d)$ as
\[
(h_\varepsilon, \varphi) = \varepsilon \sum_{x \in \varepsilon \mathbb{Z}^d} \varphi(x) h_1\left(\frac{x}{\varepsilon}\right) + O(\varepsilon).
\]

Therefore, for $k \geq 3$,
\[
\mathbb{E}\left((h_\varepsilon, \varphi)^k\right) = \varepsilon^{\frac{d+4}{2}} k \sum_{x_1, \ldots, x_k \in \varepsilon \mathbb{Z}^d} \varphi(x_1) \cdots \varphi(x_k) \mathbb{E}\left(h_1\left(\frac{x_1}{\varepsilon}\right) \cdots h_1\left(\frac{x_k}{\varepsilon}\right)\right) + O(\varepsilon)
\]
\[
= \varepsilon^{\frac{d+4}{2}} k \prod_{\Gamma=\{\gamma_l\}} \sum_{l} \mathbb{E}\left(\prod_{m \in \gamma_l} \varphi(x_m) h_1\left(\frac{x_m}{\varepsilon}\right)\right) + O(\varepsilon). \quad (6)
\]

Where in the last equality we group the vertices in terms of components of the uniform spanning forest: we sum over all the partitions $\Gamma$ of the index set $\{1, \ldots, k\}$, $h_1$ at vertices belong to different components of the forest are independent.

We claim the following Wick’s formula holds in the limit:
\[
\lim_{\varepsilon \to 0} \mathbb{E}\left((h_\varepsilon, \varphi)^k\right) = \begin{cases} (k-1)! \left(\lim_{\varepsilon \to 0} \mathbb{E}\left((h_\varepsilon, \varphi)^2\right)\right)^{k/2} & \text{k even} \\ 0 & \text{k odd} \end{cases}.
\]

It therefore uniquely identify the distribution of $\lim_{\varepsilon \to 0} (h_\varepsilon, \varphi)$ to be Gaussian. By the covariance structure given is Section 3, we complete the proof that $h_1$ converges weakly to $h$.

When $k$ is odd, at least one of the $\gamma_l$ contains odd number of elements, and therefore $\mathbb{E}\left(\prod_{m \in \gamma_l} h_1\left(\frac{x_m}{\varepsilon}\right)\right) = 0$. The independence of $h_1$ at different components implies $\mathbb{E}\left((h_\varepsilon, \varphi)^k\right) = 0$.

When $k$ is even, the non-vanishing contribution only comes from partitions such that each $\gamma_l$ contains even number of elements. By (6), it suffices to show that the contribution from those $\{\gamma_l\}$, with some $|\gamma_l| \geq 4$ is negligible in the limit. We claim:
\[
\mathbb{E}\left(\prod_{m=1}^{2l} h_1\left(\frac{x_m}{\varepsilon}\right)\right) = O\left(\varepsilon^{(d-4)(2l-1)}\right).
\]

And therefore, the contribution from the partition with a cycle of length $2l$ is
\[
\varepsilon^{\frac{d+4}{2}} \sum_{x_1, \ldots, x_{2l}} \mathbb{E}\left(\prod_{m=1}^{2l} \varphi(x_m) h_1\left(\frac{x_m}{\varepsilon}\right)\right) \leq O\left(\varepsilon^{\frac{d+4}{2}} \varepsilon^{-2dl} \varepsilon^{(d-4)(2l-1)}\right) = O\left(\varepsilon^{(d-4)(l-1)}\right),
\]

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which vanishes for $l \geq 2$.

Note that $E\left( \prod_{m=1}^{2l} h_1 \left( \frac{x_m}{\varepsilon} \right) \right)$ is the probability that $\frac{x_1}{\varepsilon}, ..., \frac{x_{2l}}{\varepsilon}$ belong to the same tree component.

This can be computed in terms of intersection probability of LERWs by Wilson’s algorithm (see Section 2.2). It is given by the probability of the following event: start a LERW from $x_1/\varepsilon$, and run indefinitely; then for $m = 2, ..., 2l$, start a SRW from $x_m/\varepsilon$ that eventually hit the union of the $m - 1$ walks, then stopped, and add its loop erasure to the union of the $m - 1$ walks. Since LERW is a subset of SRWs, the above quantity is bounded by the corresponding intersecting events of SRWs. The probability of each of such events can be bounded in a simple way. We prove it in detail for one example, the others are similar. For instance, let $A(x_1, ..., x_{2l})$ denote the event, that the SRW starting at $x_1/\varepsilon$ hits the SRW starting at $x_{2l}/\varepsilon$, the SRW starting at $x_2/\varepsilon$ hits the SRW starting at $x_1/\varepsilon$, and so on. Then

$$\mathbb{P} \left( A(x_1, ..., x_{2l}) \right) \leq \sum_{w_1, ..., w_{2l-1} \in \mathbb{Z}^d} \mathbb{P} \left( \text{SRW}_{x_1/\varepsilon} \text{hit } w_1; \text{SRW}_{x_2/\varepsilon} \text{hit } w_1, w_2; ...; \text{SRW}_{x_{2l-1}/\varepsilon} \text{hit } w_{2l-2}, w_{2l-1}; \text{SRW}_{x_{2l}/\varepsilon} \text{hit } w_{2l-1} \right)$$

$$\leq \sum_{w_1, ..., w_{2l-1} \in \mathbb{Z}^d} G(x_1/\varepsilon, w_1) G(x_2, w_1) G(x_2/\varepsilon, w_2) ... G(x_{2l}/\varepsilon, w_{2l-1})$$

$$= \left( \sum_{w_1 \in \mathbb{Z}^d} G(x_1/\varepsilon, w_1/\varepsilon) G(x_2/\varepsilon, w_1/\varepsilon) \right) ... \left( \sum_{w_{2l-1} \in \mathbb{Z}^d} G(x_{2l-1}/\varepsilon, w_{2l-1}/\varepsilon) G(x_{2l}/\varepsilon, w_{2l-1}/\varepsilon) \right)$$

$$= O \left( \varepsilon^{(d-4)(2l-1)} \right),$$

where the second inequality follows from the fact that the probability of a SRW hitting a point is bounded by the expected hitting time, which is given by the lattice Green’s function. The last inequality follows from the Green’s function asymptotics $G(x/\varepsilon, w/\varepsilon) = O \left( \varepsilon^{d-2} \right) \text{[14]}$. Since $E\left( \prod_{m=1}^{2l} h_1 \left( \frac{x_m}{\varepsilon} \right) \right)$ is a sum of finitely many such probabilities, it is at most $O \left( \varepsilon^{(d-4)(2l-1)} \right)$. And the proof is complete.

Remark 11. From the argument in Section 3 and 4 we can see that the proof does not require many special properties of Bernoulli random variables. What we need is that the sequence of i.i.d random variables have mean 0, variance 1, and all finite moments. Moreover, on other regular lattices, since the Green’s function has the same asymptotic decay rate (because the SRW still converges to Brownian motions), our result also holds for uniform spanning forest on other regular lattices. In this sense, Theorem [14] is universal.

5 Further questions

1. Bi-Laplacian Gaussian field is conformally invariant in four dimension. Are there any discrete random fields on $\mathbb{Z}^4$ that scale to some bi-Laplacian Gaussian field?

2. What geometric properties of uniform spanning forest can be inferred from the bi-Laplacian Gaussian field?
3. If one introduces short range interactions between the spins on different trees, do one obtain the same scaling limit?

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