Generalized Seniority Description of Cold Fermi Gases

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We suggest that extension of the Racah seniority description of strongly interacting fermions in the nuclear shell model is directly generalizable to describe pairing of atoms in cold Fermi systems. We illustrate this by the fermionic pairing in the much studied cold two-component gas of $^6$Li atoms. Our pairing interaction is two orders of magnitude stronger than that used in the usual BCS approach. We also explain why the Racah scheme is less applicable to nuclei, and discuss the similarities of the strongly-coupled matter in cold fermion systems and the new form of matter found in RHIC close to $T_c$.

I. INTRODUCTION

Ultracold atomic gases have become a medium to realize novel phenomena in condensed matter physics and test many-body theories in new regimes [1]. Recently Regal et al. [2] observed a preexisting condensation of fermionic atom pairs in the region of magnetic field $B$ above the Feshbach resonance, generally referred to as the BCS side. Whereas Regal et al. used a trapped gas of fermionic $^{40}$K atoms, Zwierlein et al. [1] found with $^6$Li atoms a much larger condensation of fermionic atoms in the same BCS region. Such a large condensation of essentially zero momentum molecules on the BCS side is difficult to understand in the weak coupling BCS theory, following from the Fermi-Thomas approximation usually employed in theoretical discussions.

By staying in the harmonic oscillator representation (rather than approximating by plane waves), we utilize the large degeneracy of atoms within a major shell $N$. This gives a factor of about 60 increase in pairing strength. Pairing is between time reversed states $m$ and $-m$, giving a simple explanation for the zero total momentum of each molecule in the condensate.

II. GENERALIZED SENIORITY

Racah [3] showed that for a $\delta$-function potential the ground state solution for the $J^n$ shell was given by coupling all pairs to $J = 0$. This is reasonable, because with fermions, antisymmetry requires that

$$\delta(r_{12}) = \frac{1}{3} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \delta(r_{12}),$$

so that fermions in one pair will have zero average interaction with those in another pair. Thus, the energy of a $J^n$ configuration is the sum of pairing energies (the seniority quantum number being the number of unpaired particles).

Even though the seniority representation gave the exact answer only for an attractive zero range interaction in the $J^n$ configuration, it could easily be extended to other degenerate configurations [4]. With a pure pairing interaction

$$H = -G \sum_{m,m'} a^+_m a^+_m a^-_{m'} a^-_{m'}$$

with matrix elements assumed equal for all angular momenta $l$ in a major shell of a principal quantum number $N$, the collective wave function

$$\Psi = \frac{1}{\sqrt{\Omega}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

where the states are the paired states of azimuthal quantum numbers $m$ and $-m$ and $\Omega$ is the number of pairs $(m,-m)$, is moved downward in energy with full trace of the secular matrix. This is just the bosonization of the fermion pairs. We have chosen an example of the $N = 15$ shell, with a trap energy $15 \hbar \omega_0$, of a cold Fermi gas with $^6$Li atoms in mind, with all levels through $N = 15$ being filled. The number of atoms is 1632.

We first make an estimate of the degeneracy. Since in the harmonic trap

$$\langle r^2 \rangle_{N1} = \frac{\hbar}{m \omega_0} \left( N + \frac{3}{2} \right),$$

then in dimensional analysis, the Slater integral

$$\int R^2_{N1}(r_1) R^2_{N1}(r_2) \delta(r_{12}) d^3r_1 d^3r_2 \sim \left( \frac{\hbar (N + \frac{3}{2})}{m \omega_0} \right)^{-3/2}$$

and the total number of states [5] in shell $N$ is $(N+3/2)^2$ in our assumption where all Slater integrals, both for $l = l'$ and $l \neq l'$, contribute equally, we find a pairing energy*

$$\Delta = \left( N + \frac{3}{2} \right)^{1/2} \left( \frac{m \omega_0}{\hbar} \right)^{3/2} \frac{4 \pi \hbar^2 |a|}{m},$$

*The coupling constant in eq. (2) is given in terms of the negative scattering length $a$ by $G = 4 \pi \hbar^2 |a|/m$. 

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so that
\[
\frac{\Delta}{\hbar \omega_0} = 4\pi \left( N + \frac{3}{2} \right)^{1/2} \frac{|a|}{a_{osc}} = 51 \frac{|a|}{a_{osc}},
\]
(7)
where \( a_{osc} = (\hbar/m\omega_0)^{1/2} \).

We compare this with Heiselberg’s eq. (16) [6],
\[
\frac{\Delta}{\hbar \omega_0} = \frac{32 \sqrt{2N_f}}{15\pi^2} \frac{|a|}{a_{osc}} = 1.18 \frac{|a|}{a_{osc}},
\]
(8)
with the last shell \( N_f = 15 \).

A shell-model calculation, without use of the Thomas-Fermi approximation, in the \( N_f = 12 \) shell gave 1.14\(|a|/a_{osc} \) to compare with Heiselberg’s 1.06, showing the Thomas-Fermi approximation to be good.

Because of the large factor between our rough estimate, eq. (7) and Heiselberg’s, we undertook a numerical diagonalization of the pairing interaction (eq. (2)) in the \( N = 15 \) shell without assumption of equal matrix elements using harmonic oscillator wave functions. The result was that the factor 51 in eq. (7) was replaced by 58.6.\(^1\) We note that this latter number is 4\(\pi \) times 58% of the trace of the matrix we diagonalized. Using the Brown-Bolsterli model [7], in which off-diagonal matrix elements are approximated by
\[
M_{ij} = \sqrt{M_{ii} M_{jj}},
\]
(9)
the factor would be 4\(\pi \) times the trace of the matrix, so that, within factor of about 2, the Brown-Bolsterli factorisable model, which involves calculation of only the diagonal matrix elements, can be used. This should facilitate calculations involving many shells.

From the similarity of the mathematics with the Brown-Bolsterli model, we see that it is the degeneracy of the configurations within the shell of a given principal quantum number which gives us our factor of about 60 as compared with weak coupling BCS. In fact, as can be seen, the usual BCS treatment gives a rather weak pairing, with a gap
\[
\Delta \approx E_f \exp \left( - \frac{\pi}{2|a| k_f} \right)
\]
(10)
aside from a factor of about unity. In general, \( k_f \) is taken as \( \sqrt{2N_f}/a_{osc} \), where \( N_f = 15 \) in our case. This pairing is a surface effect, holes being made over a region of momenta about the Fermi surface so that particles of equal magnitude, but opposite momenta, can scatter with them.

In our case of a volume effect, the gap begins linearly with \( a \), increasing rapidly with increasing degeneracy.

\[\text{With the large binding energy of eq. (7) we can easily increase } \Delta \text{ so that it cancels the trap energy for the } N_f = 15 \text{ shell. In fact, we can make the pair of fermions into a boson, following a scenario somewhat similar to that suggested by Falco and Stoof [8]. The molecular binding energy of the boson is}
\]
\[
\frac{\hbar^2}{ma^2} = \hbar \omega_0 \left( \frac{a_{osc}}{a} \right)^2.
\]
\(\text{(11)}\)

We can provide the total binding energy from the mean field energy on the right hand side of eq. (7):
\[
30 + \left( \frac{a_{osc}}{a} \right)^2 = 4\pi \left( N_f + \frac{3}{2} \right)^{1/2} \frac{|a|}{a_{osc}},
\]
(12)
which gives
\[
\frac{a_{osc}}{|a|} = 1.75.
\]
(13)

Using the \( k_f = \sqrt{2N_f}/a_{osc} \) this gives
\[
\frac{1}{k_f |a|} = 0.32.
\]
(14)
For \( 8 \times 1632 \) atoms \((k_f |a|)^{-1} \) would be about half of this.

Our factor 60 increase is achieved chiefly by correlation in angle. By pairing the \( m \) and \(-m \) projections of angular momentum and summing over \( m \) one gets using the addition theorem for angular momenta
\[
\Psi_{L=0}(\theta, \phi) = \sqrt{2L+1} P_l(\cos \theta_{12})
\]
for the angular dependence of the paired state [4]. The factors \( \sqrt{2L+1} \) were used in the Brown-Bolsterli model [7] which was formulated in the representation of good angular momentum; in the \( m \) representation the matrix in eq. (2) with all matrix elements equal is equivalent.

Of course, the \( \delta \)-function interaction sets \( P_l(\cos \theta_{12}) = 1 \) so that the factor \((2L+1)\) from squaring \( \Psi_{L=0} \) resulted in Racah’s seniority scheme. We gain additionally by including the off-diagonal Slater integrals.

In fact, our interaction is increased by the two-body correlations, so that \( k_f \), which is determined by the total number of particles divided by the volume of the trap; \( \text{i.e., by the average density, should still be given by } \sqrt{2N_f}/a_{osc}. \) We can, therefore, rewrite our eq. (7) for \( \Delta \) as
\[
\frac{\Delta}{\hbar \omega_0} = 11 k_f |a|,
\]
(15)
neglecting the \( 3/2 \) as compared with \( N \), so that the \( N \) dependence is taken up in \( k_f \). In this way we do not have to know \( a_{osc}. \) The \( \sqrt{2N_f} \) times 11 gives the factor of about 60 in our enhancement in the \( N = 15 \) shell.

It should be noted that for the spherical trap we have assumed that 34 out of this 60 comes from the 31 \( l = 15 \) pairs and that our seniority 0 solution is exact.\(^1\)This is an underestimate because backward-going ladders (ground-state correlations) which we have not included give additional attraction.
for a $\delta$-function interaction [3]. In the experiments the traps are not spherical, typically being cigar shaped with quite large ratios of frequencies. In nuclear physics, although with lesser ratios, one uses the Nilsson model, with $\tilde{\omega} = (\omega_x \omega_y \omega_z)^{1/3}$. Since, $\langle r^2 \rangle^{1/2}_{N_1}$, in eq. (4), goes as $\omega_0^{-1/2}$ this means that the dependence of $\langle r^2 \rangle^{1/2}_{N_1}$ on any particular $\omega_i$, say $\omega_z$, goes as $\omega_z^{-1/6}$. Now in a cigar-shaped trap there will still be the correlation in $\theta_{12}$ enhancing the densities of time-reversed orbits $m$ and $-m$, but, especially at the equator, the distance $r$ from the center of the trap to the edge of the cigar will be smaller than that of a spherical trap with the same number of $m$ values. Thus, the distance $r \theta_{12}$ will be shorter than in the spherical trap, so the enhancement in probability will be greater. However, this should not be a large effect because of the weak dependence of $\langle r^2 \rangle^{1/2}_{N_1}$ on any particular $\omega_i$.

For small values of $|a|$, the particles in the levels below $N = 15$ will interact with those in $N = 15$ so as to push the level up (although in the middle levels they will push the levels above up and the those below down). However, the attraction goes as $\sqrt{N}$, as can be seen from eq. (7), whereas the trap energy goes linearly with $N$, which decreases faster than the attraction with decreasing $N$, so that the lower levels will have already gone into molecules, the atoms from shell $N_f$ being the last to make the transition.

We have not done dynamics, but we believe that for very small $|a|$ there will be a collisionless regime, in which the RPA vibrations of Bruun and Mottelson [9] occur. The binding energy of the system, the sum of bubbles which make up the RPA vibrations or zero-point energy of the vibrations [10] can be incorporated, as noted by Bruun and Mottelson, as a mean field which slightly increases the oscillator energy spacings. But then as $a$ becomes larger in magnitude the inner shells start losing their support as the attractive field from the interaction reaches the trap energy in magnitude and the shells start collapsing in what becomes a hydrodynamical (strongly interacting) regime. The end result, already early on for negative values of $(k_f a)^{-1}$ is that the strongly bound molecules are formed. The higher shells of filled atoms may remain, but their $\hbar \omega_0$ will be lowered by the attractive mean field.

III. SENIORITY IN NUCLEI

We explain why nuclei, for which Racah introduced seniority, do not look like the cold Fermi gases we have been discussing. Although seniority was very useful in discussion of the energies of nuclei, it has turned out that the short range attraction in nuclei is totally obliterated by the short-range repulsion, so that the effective interaction in $s$-wave is only the long-range part of the potential, from about $1 fm$ outwards. This is most clearly understood in the Moszkowski-Scott separation method in the calculations of Holt and Brown [11]. In the calculation of nuclear interactions, as done by Kuo and Brown [12] which used this separation method, the inner part of the $s$-wave two-body interaction up to separation distance $d \sim 1 fm$ is simply removed, leaving the long-range part of the two-body interaction $V_i$ to be used in shell-model calculations. Holt and Brown showed that this $V_i$ is the configuration space form of $V_{low-k}$, which is the more effective effective field theory (MEEFT) interaction and contains all experimental information in the nucleon-nucleon scattering phase shifts [13]. This long range $V_i$ is clearly more suitable for shell-model calculations than for pairing. The latter in nuclei is of the weak coupling BCS type and takes place only in the last shells, near the Fermi energy.

IV. RELATION TO RHIC PHYSICS

Edward Shuryak [14] has often emphasized that the elliptic flow seen by O’Hara et al. [15] of the cold Fermi liquid, when released from the trap, is similar to that formed at RHIC, although there are many orders of magnitude difference in them (e.g. microseconds versus fermi/c). Our description of the pairing involves the same wave function $\Psi$ (eq. (3) as that used by Brown, Lee, Rho and Shuryak [16], (Appendix A. in [17]), although at RHIC the bosonization of quarks and antiquarks into chirally restored mesons above $T_c$ is involved, rather than fermion pairs. There the quark and antiquark masses from lattice gauge calculations are found to be $\sim 1 GeV$, and the $\pi$ and $\sigma$ meson masses must be brought to zero at $T = T_c$, in the chiral limit, which requires extremely strong interactions. The attractive interactions are built up enormously through the degeneracies.

Thus, although the Brown-Bolsterli model for giant resonances in nuclei suggested the use of the degeneracies both here and in RHIC, they are immensely strong, because the numbers are larger and the degeneracy is not partially broken by spin-orbit coupling, as it is in nuclear physics. Inclusion of the degeneracy brings about the strong coupling.

V. DISCUSSION

Our result is consistent with the results of Carlson et al. [18] who found $\Delta$ to be about half of the Fermi energy in quantum Monte Carlo studies of superfluid Fermi gases. In [17] Brown et al. discuss fluid formed in RHIC in terms of Nambu Jona-Lasinio theory. They find that the interactions above $T_c$ are much stronger, because of the large degeneracy, than below $T_c$, where the unperturbed spectrum of quarks and quark-holes is continuous.
as in the Fermi gas.\textsuperscript{4}

We believe the experiments of Regal et al. \cite{Regal2004} and Zwierlein et al. \cite{Zwierlein2003} found pairs of atoms with properties such as our scenario would produce. First of all, our molecules are formed on the BCS side so one would expect large condensate fractions there when the temperature was lowered. They find a large fraction of molecules to be in a zero-momentum state after fast ramping in magnetic field over to the BEC side, “this means that the nearest neighbors had opposite momenta” \cite{Regal2003}. In our seniority scenario, atoms of time-reversed states \( m \) and \(-m\) are paired, states of equal in magnitude and opposite in direction angular momenta. In the long-range Cooper pairs discussed in other works one would expect the transfer into a tightly bound molecular state would randomly pick out one of the nearest neighbors, resulting in a thermal molecule. Zwierlein et al. \cite{Zwierlein2003} regard their high condensate fraction as evidence of condensed atomic pairs in the BCS sector which are smaller in size than the inter atomic distance and, therefore, molecular. The size of our pairing modes \( \Psi \), eq. (3), decreases roughly inversely with increasing \( \Omega \), the number coupled together, forming a highly compact molecular state.

The condensate drains particles from the Fermi sea on the BCS side as we have outlined, but only after they are tightly paired. They go into zero-momentum molecules forming a highly compact molecular state. Such as our scenario would produce. First of all, our pairing modes \( \Psi \), eq. (3), decreases roughly inversely with increasing \( \Omega \), the number coupled together, forming a highly compact molecular state.

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\begin{equation}
\Psi (r_1 \sigma_1 ... r_N \sigma_N) = A \phi (r_1 \sigma_1, r_2 \sigma_2)
\phi (r_3 \sigma_3, r_4 \sigma_4) \phi (r_{N-1} \sigma_{N-1}, r_N \sigma_N)
\end{equation}

with the quasi molecular wave function \( \phi \) as a variational parameter. The \( \phi \), common to all pairs, is

\[ \phi (r_1 \sigma_1, r_2 \sigma_2) = \frac{1}{\sqrt{2}} (\uparrow \downarrow - \downarrow \uparrow) \phi (|r_1 - r_2|) . \]

It reduces to a Bose condensation of tightly bound diatomic molecules in his limit \( \xi = (k_{\text{F}} a)^{-1} \to \infty \) and to the standard BCS result in the limit \( \xi \to -\infty \). This \( \phi \) is

\textsuperscript{4}In a longer paper we shall show schematizing further the schematic model of a Cooper pair \cite{Leggett1980} that it is possible to go smoothly from the weak coupling BCS to our strong coupling one, as the degeneracy is introduced.

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