NON INJECTIVITY OF THE “HAIR” MAP

BERTRAND PATUREAU-MIRAND

ABSTRACT. Kricker constructed a knot invariant $Z^{rat}$ valued in a space of Feynman diagrams with beads. When composed with the so called “hair” map $H$, it gives the Kontsevich integral of the knot. We introduce a new grading on diagrams with beads and use it to show that a non trivial element constructed from Vogel’s zero divisor in the algebra $\Lambda$ is in the kernel of $H$. This shows that $H$ is not injective.

INTRODUCTION

The Kontsevich integral $Z$ is a universal rational finite type invariant for knots (see the Bar-Natan survey [1]). For a knot $K$, $Z(K)$ lives in the space of Chinese diagrams isomorphic to $\hat{B}(\ast)$ (see Section 1.1). Rozansky conjectured ([3]) and Kricker proved ([3]) that $Z$ can be organized into a series of “lines” called $Z^{rat}$. They can be represented by finite $\mathbb{Q}$–linear combinations of diagrams whose edges are labelled, in an appropriate way, with rational functions. In [2], Garoufalidis and Kricker directly proved that the map $Z^{rat}$ with values in a space of diagrams with beads is an isotopy invariant and that $Z$ factors through $Z^{rat}$. For a knot $K$ with trivial Alexander polynomial, $Z(K) = H \circ Z^{rat}(K)$ where $H$ is the hair map (see Section 1.3). Rozansky, Garoufalidis and Kricker conjectured (see [4, Conjecture 3.18]) that $H$ could be injective. Theorem 4 gives a counterexample to this conjecture.

1. The hair map

1.1. Classical diagrams. Let $X$ be a finite set. A $X$–diagram is an isomorphism class of finite uni-tri-valent graphs $K$ with the following data:

- At each trivalent vertex $x$ of $K$, we have a cyclic ordering on the three oriented edges starting from $x$.
- A bijection between the set of univalent vertices of $K$ and the set $X$.

We define $A(X)$ to be the quotient of the $\mathbb{Q}$–vector space generated by $X$–diagrams by the relations:

1. The (AS) relations for “antisymmetry”:

$$\begin{array}{c}
\begin{array}{c}
\text{\quad}\text{\quad}\text{\quad}
\end{array}
\end{array}
\bigcirc
\\bigcirc
\begin{array}{c}
\begin{array}{c}
\text{\quad}\text{\quad}\text{\quad}
\end{array}
\end{array}
= 0$$

2. The (IHX) relations for three diagrams which differ only in a neighborhood of an edge:

$$\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{\quad}\text{\quad}\text{\quad}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{\quad}\text{\quad}\text{\quad}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{\quad}\text{\quad}\text{\quad}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{\quad}\text{\quad}\text{\quad}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{\quad}\text{\quad}\text{\quad}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{\quad}\text{\quad}\text{\quad}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{\quad}\text{\quad}\text{\quad}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{\quad}\text{\quad}\text{\quad}
\end{array}
\end{array}
\end{array}$$

These spaces are graded. The degree of an $X$–diagram is given by half the total number of vertices.

Let $[n] = \{1, 2, \ldots, n\}$ and define $F_n$ to be the subspace of $A([n])$ generated by...
connected diagrams with at least one trivalent vertex. The permutation group \( \mathcal{S}(X) \) acts on \( A(X) \). Let \( B(*) \) be the coinvariant space for this action:

\[
B(*) = \bigoplus_{n \in \mathbb{N}} A([n]) \otimes_{\mathcal{S}_n} \mathbb{Q}
\]

and let \( \hat{B}(*) \) be the completion of \( B(*) \) for the grading.

Finally let \( \Lambda \) be Vogel’s algebra generated by totally antisymmetric elements of \( F_3 \) (for the action of \( \mathcal{S}_3 \)).

We recall (see [6]) that \( \Lambda \) acts on the modules \( F_n \) and that for this action, \( F_0 \) and \( F_2 \) are free \( \Lambda \)–modules of rank one. Furthermore, the following elements are in \( \Lambda \):

\[
t = \frac{1}{2} \qquad x_n = \frac{1}{n+2}
\]

**Theorem 1.** (Vogel [6, Section 8 and Proposition 8.5]). The element \( t \) is a divisor of zero in \( \Lambda \).

**Corollary 2.** There exists an element \( r \in \Lambda \setminus \{0\} \) such that \( t \cdot r = 0 \). So one has

\[
\begin{align*}
\text{if } & \quad \begin{array}{c} \includegraphics[width=1cm]{t_diagram} \\ 3 \\ 1 \\ 2 \\ \end{array} \\
\text{then } & \quad \begin{array}{c} \includegraphics[width=1cm]{r_diagram} \\ 1 \end{array} = 0 \in F_3
\end{align*}
\]

**Proof:** \( F_0 \) is a free \( \Lambda \)–module of rank one generated by the diagram \( \Theta \) and the previous diagram of \( F_0 \) is \( r \cdot \Theta \neq 0 \). The diagram of \( F_3 \) of the corollary is the product \( r, \frac{1}{2}, \frac{1}{3} = 2t \cdot r = 0 \in \Lambda \).

**Remark:** Vogel shows that \( r \) can be chosen with degree fifteen in \( \Lambda \) (the degree in \( \Lambda \) is the degree in \( F_3 \) minus two), and in the algebra generated by the \( x_n \). This element is killed by all the weight systems coming from Lie algebras (but \( r \) is not killed by the Lie superalgebras \( D_{2,1,\alpha} \)).

1.2. **Diagrams with beads.** Diagrams with beads have been introduced by Kricker and Garoufalidis (see [3], [2]). A presentation of \( B \) which uses the first cohomology classes of diagrams is already present in [5]. Vogel explained me this point of view for diagrams with beads.

Let \( G \) be the multiplicative group \( \{b^n, n \in \mathbb{Z}\} \simeq (\mathbb{Z}, +) \) and consider its group algebra \( R = \mathbb{Q}G = \mathbb{Q}[b, b^{-1}] \). Let \( a \mapsto \sigma \) be the involution of the \( \mathbb{Q} \)–algebra \( R \) that maps \( b \) to \( b^{-1} \).

A diagram with beads in \( R \) is an \( \sigma \)–diagram with the following supplementary data: The beads form a map \( f: E \to R \) from the set of oriented edges of \( K \) such that if \( -e \) denotes the same edge than \( e \) with opposite orientation, one has \( f(-e) = \overline{f(e)} \). We will represent the beads by some arrows on the edges with label in \( R \). The value of the bead \( f \) on \( e \) is given by the product of these labels and we will not represent the beads with value 1. So with graphical notations, we have:

\[
\begin{align*}
\includegraphics[width=1cm]{f_bead} & = \includegraphics[width=1cm]{f_bead_bar} \\
\includegraphics[width=1cm]{f_bead_g_bead} & = \includegraphics[width=1cm]{f_bead_g_bead_bar}
\end{align*}
\]

The loop degree of a diagram with beads is the first Betti number of the underlying graph.

Let \( A^R(\emptyset) \) be the quotient of the \( \mathbb{Q} \)–vector space generated by diagrams with beads in \( R \) by the following relations:

1. (AS)
2. The (IHX) relations should only be considered near an edge with bead 1.
(3) PUSH:

(4) Multilinearity:

\[ \alpha f(b) + \beta g(b) = \alpha f(b) + \beta g(b) \]

\[ A^R(\emptyset) \] is graded by the loop degree:

\[ A^R(\emptyset) = \bigoplus_{n \in \mathbb{N}} A^R_n(\emptyset) \]

We will prefer another presentation of \( A^R(\emptyset) \):

- Remark that it is enough to consider diagrams with beads in \( G \) and the multilinear relation can be viewed as a notation.
- Next remark that for a diagram with beads in \( G \), the map \( f \) define a 1–cochain \( \tilde{f} \) with values in \( \mathbb{Z} \cong G \) on the underlying simplicial set of \( K \). The elements \( \tilde{f} \) are in fact 1–cocycles because of the condition \( f(-e) = f(e) \) which implies \( \tilde{f}(-e) = -\tilde{f}(e) \).
- The “PUSH” relation at a vertex \( v \) implies that \( \tilde{f} \) is only given up to the coboundary of the 0–cochain with value 1 on \( v \) and 0 on the other vertices. Hence \( A^R(\emptyset) \) is also the \( \mathbb{Q} \)–vector space generated by the pairs \( (3–\text{valent graph } D, x) \in H^1(D, \mathbb{Z}) \) quotiented by the relations (AS) and (IHX). With these notations one can describe the (IHX) relations in the following way:

Let \( K_I, K_H \) and \( K_X \) be three graphs which appear in a (IHX) relation on an edge \( e \). Let \( K_\bullet \) be the graph obtained by collapsing the edge \( e \). The maps \( p_? : K_? \rightarrow K_\bullet \) induce three cohomology isomorphisms. If \( x \in H^1(K_\bullet, \mathbb{Z}) \) then the (IHX) relation at \( e \) says that

\[ (K_I, p_i^* x) = (K_H, p_H^* x) - (K_X, p_X^* x) \]

holds in \( A^R(\emptyset) \).

1.3. The hair map. The hair map \( H : A^R(\emptyset) \rightarrow \hat{B}(\ast) \) replaces beads by legs (or hair): Just replace a bead \( b^n \) by the exponential of \( n \) times a leg.

\[ b^n \mapsto exp_\#(n) = +n + \frac{n^2}{2} + \cdots \]

\( H \) is well defined (see [2]).

2. Grading on diagrams with beads

Remark that for a 3–valent graph \( K \), \( H^1(K, \mathbb{Z}) \) is a free \( \mathbb{Z} \)–module. The beads \( x \in H^1(K, \mathbb{Z}) \) which occur in an (AS) or (IHX) relation are the same up to isomorphisms. We will call \( p \in \mathbb{N} \) the bead degree of \( (K, x) \) if \( x \) is \( p \) times an indivisible element of \( H^1(K, \mathbb{Z}) \).

Theorem 3. The bead degree is well defined in \( A^R_n(\emptyset) \). Thus we have a grading

\[ A^R_n(\emptyset) = \bigoplus_{p \in \mathbb{N}} A^R_{n,p}(\emptyset) \]

where \( A^R_{n,p}(\emptyset) \) is the subspace of \( A^R_n(\emptyset) \) generated by diagrams with bead degree \( p \). Furthermore, \( A^R_{n,0}(\emptyset) \cong A_n(\emptyset) \) and for \( p > 0 \), \( A^R_{n,p}(\emptyset) \cong A^R_{n+1}(\emptyset) \).
Proof: The second presentation we have given for $A^n(\emptyset)$ implies that this degree is well defined. Indeed, the elements in a IHX relation have the same degree because the set of indivisible elements of the cohomology is preserved by isomorphisms.

Now, the map $\psi : R \to \mathbb{Q}$ that sends $b$ to 1 induces the isomorphism $A^n(\emptyset) \simeq A_n(\emptyset)$ and the group morphism $\phi_p : G \to G$ that sends $b$ to $b^p$ (or the multiplication by $p$ in $H^1(\mathbb{Z})$) induces the isomorphism $A_{n,1}(\emptyset) \simeq A_{n,p}(\emptyset)$. These maps are isomorphisms because they have obvious inverses. □

3. A non trivial element in the kernel of $H$

Theorem 4. This non trivial element of $A^n(\emptyset)$ is in the kernel of $H$:

$$
\begin{array}{c}
\text{b-1} \\
\text{r}
\end{array}
$$

Thus $H$ is not injective.

Proof: This element is not zero because its bead degree zero part is the opposite of the element $r.\Theta$ of Corollary 2. Then, one has

$$
\begin{array}{c}
\text{b-1} \\
\text{r}
\end{array} \mapsto H \begin{array}{c}
\text{b-1} \\
\text{r}
\end{array} + \frac{1}{2!} \begin{array}{c}
\text{b-1} \\
\text{r}
\end{array} + \frac{1}{3!} \begin{array}{c}
\text{b-1} \\
\text{r}
\end{array} + \cdots
$$

but all these diagrams are zero in $B(\ast)$ because they contain, as a sub-diagram, the element of $F_3$ of Corollary 2 □

Remark: The element of theorem 4 has a loop degree seventeen.

The hair map is obviously injective on the space of diagrams with bead degree zero. I don’t know if the same is true in other degrees.

References

[1] D. Bar-Natan - On the Vassiliev knot invariants, Topologie 34 n° 2 1995, 423–472.
[2] S. Garoufalidis, A. Kricker - A rational noncommutative invariant of boundary links Geom. Topol. 8 (2004), 115–204.
[3] A. Kricker - The lines of the Kontsevich integral and Rozansky’s rationality conjecture, Tokyo Institute of Technology preprint, May 2000, [arXiv:math.GT/0005284]
[4] T. Ohtsuki - Problems on Invariants of Knots and 3-Manifold Geom. Topol., 4, Invariants of knots and 3-manifolds (Kyoto, 2001), iiv, 377–572, Geom. Topol. Publ., Coventry, 2002.
[5] L. Rozansky - Une structure rationnelle sur des fonctions génératrices d’invariants de Vassiliev Summer school of the University of Grenoble preprint, July 1999.
[6] P. Vogel - Algebraic structures on modules of diagrams, J. Pure Appl. Algebra 215 (2011), no. 6, 1292–1339.

LMAM, université de Bretagne-Sud, université européenne de Bretagne, BP 573, 56017 Vannes, France
E-mail address: bertrand.patureau@univ-ubs.fr