Mean-field dynamics of sequence processing neural networks with finite connectivity

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Abstract

A recent dynamic mean-field theory for sequence processing in fully connected neural networks of Hopfield-type (Düring, Coolen and Sherrington, 1998) is extended and analyzed here for a symmetrically diluted network with finite connectivity near saturation. Equations for the dynamics and the stationary states are obtained for the macroscopic observables and the precise equivalence is established with the single-pattern retrieval problem in a layered feed-forward network with finite connectivity.

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1 Introduction

A path-integral approach [1, 2] has been successfully applied already some time ago [3] to study the dynamics near saturation of the Hopfield model [4] with a symmetric Hebbian learning rule that favors the retrieval of single patterns. The results for the stationary states were found to be those obtained by means of the replica method in equilibrium statistical mechanics [5]. The performance of networks trained with a sequence of stored patterns has been of interest over some time [6 − 13], for both biological and artificial neural networks [14], and the path-integral approach has been applied to study the single-pattern stationary states in a mean-field dynamics near saturation for a fully connected recurrent network trained with a sequence of stored patterns [15].

More recently, the transients in that network have been investigated in a statistical neurodynamics approach, which was shown to be equivalent to the path-integral method [16]. These methods are particularly suitable to handle dynamical systems with asymmetric interactions, as in the case of sequential patterns, due to the absence of detailed balance where equilibrium statistical mechanics cannot be used.

Explicit equations were obtained for the retrieval overlap with single patterns in a sequence, for the response function to an external field and for the two-state correlation function [15, 16]. By means of exact analytic work, supported by numerical simulations, it has been found that the critical storage capacity in the fully connected recurrent network for single-pattern retrieval in a sequence is $\alpha_c = 0.269$ [15]. This is the same as the critical storage capacity for a different problem in another network architecture, namely, the usual single-pattern retrieval problem in the layered feed-forward network [17]. It has been pointed out that this is due to the equivalence between the two models in the absence of synaptic noise ($T = 0$), and that this equivalence does not extend beyond the $T = 0$ limit. The performance of the network turns out to be limited by a non-local (in time) Gaussian noise in the effective single-spin problem [15].

The equilibrium behavior of Hopfield-type networks with finite random decay (dilution) of synapses is of interest for biological [18] and artificial neural networks and has already been considered for the single-pattern retrieval problem in systems with or without microscopic detailed balance [17, 19−21]. Dense networks with connectivity $c = O(1)$ share most of the features of fully
connected networks and have a non-trivial dynamics, in contrast to the extremely diluted network [22].

We discuss here the extension of the sequence processing dynamics of Düiring and al. to finite synaptic dilution and study the precise relationship with the single-pattern dynamics and the asymptotic stationary state of the layered feed-forward network. This is a network that consists of identical layers of $N$ non-interacting units each, with synaptic interaction $J_{ij}^l$ between units $j$ and $i$ in layers $l$ and $l + 1$, respectively.

The outline of the paper is the following. In Sec. 2 we present the model and derive the connected generating functional. In Sec. 3 we discuss the effective single-site problem in dynamic mean-field theory and present the specific equations for the macroscopic observables, including the stationary state. We discuss there the explicit relationship with the results for the layered feed-forward network and summarize our conclusions in Sec. 4.

2 The model

2.1 Defining relations

We consider a network of $N$ Ising neurons in a microscopic state $\sigma(t) = \{\sigma_1(t), \ldots, \sigma_N(t)\}$, at the time step $t$ in which each $\sigma_i(t) = \pm 1$, that is updated simultaneously according to the alignment of each unit to its local field

$$h_i(t) = \sum_j J_{ij}^d \sigma_j(t) + \theta_i(t),$$

following a stochastic dynamics with transition probability

$$w_i[\sigma_i(t + 1)|\sigma_i(t)] = \frac{1}{2}[1 + \sigma_i(t + 1) \tanh(\beta \sigma_i(t) h_i(t))],$$

ruled by the noise-control parameter $\beta = T^{-1}$. The dynamics is a deterministic one when $T = 0$ and it is fully random when $T = \infty$. In the former case, the state of a spin at a given time is completely determined by the sign of its local field following the relation $\sigma_i(t + 1) = \text{sgn}[h_i(t)]$. Here, $J_{ij}^d$ is the synaptic coupling to other neurons and $\theta_i(t)$ is an external field introduced in order to study the response of the network to an external stimulus. Thus, given the probability $p[\sigma(0)]$ of an initial state, Eq.(2) determines the path
probability \( p(\sigma(0), \ldots, \sigma(t)) \) that takes the system to the state \( \sigma(t) \) through a Markov chain.

A macroscopic sequential set of \( p = \alpha N \) independent and identically distributed random cyclic patterns \( \xi^\mu = (\xi^\mu_1, \ldots, \xi^\mu_N) \), \( \mu = 1, \ldots, p \), such that \( \xi^{\mu+1} = \xi^1 \) and each \( \xi^\mu_i = \pm 1 \) with probability \( \frac{1}{2} \), is assumed to be stored in the network according to the learning rule

\[
J^d_{ij} = \frac{c_{ij}}{cN} \sum_{\mu=1}^{p} \xi_{i}^{\mu+1} \xi_{j}^{\mu},
\]

and we are interested in the behavior of the network near saturation, that is, for finite \( \alpha \). The synaptic interactions are assumed to be symmetrically diluted by means of the set of couplings \( \{c_{ij}\} \), with \( c_{ij} = c_{ji} \) taken from a set of independent identically distributed random variables, which are the same for every time step, such that \( c_{ij} = 1 \) with probability \( c \) and zero with probability \( 1-c \), while \( c_{ii} = 0 \). Thus, the average \( \langle c_{ij} \rangle = c \) becomes the connectivity of the network and when \( c = 1 \) one has the usual synaptic coupling for the fully connected case. We consider a dense network, with \( c = O(1) \), and take \( cN >> 1 \) allowing eventually for a vanishingly small \( c \) after taking the thermodynamic limit in the final results of the dynamics. Once the network has a finite connectivity it is expected to have a non-trivial dynamics which is different from that of a purely random network as long as \( \beta \neq 0 \). Thus, one has to ensure first a finite connectivity and we do this next.

### 2.2 The connected generating functional

Since the matrix \( J^d = \{J^d_{ij}\} \) is non-symmetric, one has to resort to a dynamical procedure to study the time evolution and the stationary states of the network and we follow previous works using the generating functional or path-integral method \([15, 16]\). We introduce a connected generating functional

\[
Z_c(\psi) = \sum_{\sigma(0), \ldots, \sigma(t)} \langle p(\sigma(0), \ldots, \sigma(t)) \rangle_{c_{ij}} e^{-\sum_{s<t} \sigma(s).\psi(s)} ,
\]

where \( \langle \ldots \rangle_{c_{ij}} \) denotes the path probability for the connected network obtained by averaging \( p(\sigma(0), \ldots, \sigma(t)) \) over the set \( \{c_{ij}\} \) and, as usual, \( x.y = x_1y_1 + \ldots + x_ny_n \).
\[ \sum_i x_i y_i. \] Here, \( \psi_i(s) = \{ \psi(0), \ldots, \psi(t-1) \} \) is introduced and afterwards set to zero in order to generate state averages over the path probability. Using an integral representation of unity to account for the defining relation for the set of local fields \( \{ h_i(s) \} \) given above, by means of an auxiliary set \( \{ \hat{h}_i(s) \} \) [15], we can separate the configurational average over the set \{c_{ij}\} and obtain

\[ Z_c(\psi) = \sum_{\sigma(0), \ldots, \sigma(t)} p[\sigma(0)] \int \{ d\hat{h} \} \prod_{s<t} \exp\{ \beta \sigma(s+1) h(s) - \sum_i \ln 2 \cosh(\beta h_i(s)) \}
+ i\hat{h}(s) [h(s) - \theta(s)] - i\psi(s) . \sigma(s) \} \langle e^{-i\hat{h}(s).J^i\sigma(s)} \rangle_{\text{c}_{ij}}, \quad (5) \]

where \( \{ d\hat{h} \} = \prod_i \prod_{s<t} [dh_i(s) d\hat{h}_i(s)/2\pi] \).

We proceed next to calculate the configurational average along similar lines to those used in statistical mechanics of diluted spin-glass and neural network models [23, 21] and write

\[ \langle e^{-i\hat{h}(s).J^i\sigma(s)} \rangle_{\text{c}_{ij}} = \prod_{i,j} \{ 1 + c(\exp[-i\hat{h}_i(s) \tilde{J}_{ij} \sigma_j(s)] - 1) \}
= \exp[\sum_{i,j} \ln(1 + c\eta_{i,j}(s))] , \quad (6) \]

where the coupling \( \tilde{J}_{ij} = J^d_{ij} \) for \( c_{ij} = 1 \) and \( \eta_{i,j}(s) \equiv \exp[-i\hat{h}_i(s) \tilde{J}_{ij} \sigma_j(s)] - 1. \) Keeping in mind that we want results for finite \( c \) and noting that \( \tilde{J}_{ij} \equiv O(\sqrt{\alpha/cN}) \) is small, we use first \( \eta_{i,j} \) as an expansion parameter up to second order and then expand in \( \tilde{h}_i \), to that order, and obtain

\[ \langle e^{-i\hat{h}(s).J^i\sigma(s)} \rangle_{\text{c}_{ij}} = \exp\{ -i\hat{h}(s) . J \sigma(s) - \frac{1}{2} c(1-c) \sum_{i,j} [\hat{h}_i(s) \tilde{J}_{ij} \sigma_j(s)]^2 \} . \quad (7) \]

The couplings on the right \( J = \{ J_{ij} \} \), with \( J_{ij} = N^{-1} \sum_{\mu} \xi^{\mu+1}_{i} \xi^{\mu}_{j} \), are now the synapses for the fully connected network with sequential patterns and the second term in the argument of the exponential contains the full effect of the dilution.

Before doing the average of the exponential over the patterns following the standard procedure below, one may replace here the pattern dependence of the last term, up to terms of higher order which are irrelevant in mean-field theory, by its average over the pattern distribution becoming thereby
pattern independent. In this way, the state $\sigma(s)$ becomes decoupled from the auxiliary field $\hat{h}(s)$ and, noting that $\sum_i \sigma_i^2(s) = N$, we obtain

$$\langle e^{-i \hat{h}(s) . J \sigma(s)} \rangle_{c_{ij}} = e^{-i \hat{h}(s) . J \sigma(s)} e^{-\Delta^2 \hat{h}^2(s)/2} ,$$

(8)

where $\Delta^2 = \alpha (1 - c)/c$ and $\hat{h}^2(s) = \sum_i \hat{h}_i^2(s)$. Note that $J_{ij}$ is of order $1/\sqrt{N}$, the local field $h_i$ and $\hat{h}_i$ are of order one and, consequently, the argument of both exponents is, as one would expect, of order $N$. Incidentally, the result in Eq.(8) is precisely the average $\langle \exp( -i \hat{h}(s) . J^{eff} \sigma(s)) \rangle_{\{\delta_{ij}\}}$ over the pattern-independent set $\{\delta_{ij}\}$ of independent Gaussian random variables with mean zero and variance $\Delta^2$, in which the effective coupling $J^{eff}_{ij} = J_{ij} + \delta_{ij}$. Thus, as in the case of the equilibrium behavior of networks, the finite symmetric dilution in the dynamics amounts to the addition of a static Gaussian noise to the synaptic couplings of the fully connected network. [19].

This completes the derivation of the formal expression for $Z_c(\psi)$, in which the connected path probability takes the form

$$\langle p[\sigma(0), \ldots, \sigma(t)] \rangle_{c_{ij}} = p[\sigma(0)] \int \{dh \hat{h}\} \prod_{s<t} \exp \{ \beta \sigma(s+1) . h(s) - \sum_i \ln 2 \cosh(\beta h_i(s)) + i \hat{h}(s) . [h(s) - \theta(s)] - i \hat{h}(s) . J \sigma(s) - \frac{1}{2} \Delta^2 \hat{h}^2(s) \} .$$

(9)

Clearly, the auxiliary variable $\hat{h}$ has the role of relating microscopic physical quantities with each other and this will show up next in constructing the macroscopic observables.

2.3 The macroscopic observables

The macroscopic dynamic observables depend on the problem one wants to study. If one is interested in the retrieval of a single pattern, say $\xi^s$ at each time step $s$, the case to which we restrict ourselves in this work, one assumes that that pattern is ‘condensed’ which means that the overlap with the state of the network, $\sigma(s)$, at that time step is of $O(1)$ and that all other overlaps are of $O(N^{-\frac{1}{2}})$. The non-condensed overlaps introduce a stochastic noise into the dynamics which may be thought as a quenched disorder. One then
expects that the relevant macroscopic variables for single-pattern retrieval
are the overlap \( m_N(s) \), the single-site response and correlation functions,
\( G_N(s,s') \) and \( C_N(s,s') \), respectively, defined and given, for large but finite
\( N \), by

\[
m_N^*(s) \equiv \frac{1}{N} \sum_i \xi_i^s \langle \sigma_i(s) \rangle = i \lim_{\psi \to 0} \frac{1}{N} \sum_i \xi_i^s \frac{\partial Z_c(\psi)}{\partial \psi_i(s)} ,
\]

\[
G_N(s,s') \equiv \frac{1}{N} \sum_i \frac{\partial \langle \sigma_i(s) \rangle}{\partial \theta_i(s')} = i \lim_{\psi \to 0} \frac{1}{N} \sum_i \frac{\partial^2 Z_c(\psi)}{\partial \psi_i(s) \partial \theta_i(s')},
\]

\[
C_N(s,s') \equiv \frac{1}{N} \sum_i \langle \sigma_i(s) \sigma_i(s') \rangle = -i \lim_{\psi \to 0} \frac{1}{N} \sum_i \frac{\partial^2 Z_c(\psi)}{\partial \psi_i(s) \partial \psi_i(s')},
\]

where the brackets now denote the thermal averages over the connected path
probability given by Eq.(9). Note that, due to the form of \( Z_c(\psi) \), Eq.(5), the
response function is given by the \( \psi(s) \to 0 \) limit of \(-i N^{-1} \sum_i \langle \sigma_i(s) \hat{h}_i(s) \rangle \). In
principle, there could be other observables, as \( k_N(s) = N^{-1} \sum_i \xi_i^{s+1} \hat{h}_i(s) \)
and \( Q_N(s,s') = N^{-1} \sum_i \hat{h}_i(s) \hat{h}_i(s') \), but they do not represent physical quantities
so that one would expect that they do not show up in mean-field theory.
Indeed, it has been found that the saddle-point solutions for these observables
turn out to vanish in mean-field theory [15].

### 3 Dynamic mean-field theory

As usual in neural network studies, one can only deal with a finite number of
condensed patterns. Having assumed that at each time-step a single pattern
is condensed, a finite sequence of patterns with \( \mu \leq t \) will be condensed in
fixed time \( t \). Thus, the dynamical mean-field theory will be restricted to
finite time scales although we keep the limit \( N \to \infty \). In that limit, the
theory should be self-averaging and we are then left with an explicit average
of the generating functional over the non-condensed patterns, \( \overline{Z_c(\psi)} \). Noting
that only the part of the exponential depending on \( \mathbf{J} \) on the right in Eq.(9)
is pattern dependent, one can follow the steps in ref. 15 to do this average
and the relevant macroscopic observables in the dynamic mean-field theory,
in the limit \( N \to \infty \), become

\[
m_N^*(s) = \lim_{N \to \infty} \frac{1}{N} \sum_i \xi_i^s \langle \sigma_i(s) \rangle = i \lim_{\psi \to 0} \lim_{N \to \infty} \frac{1}{N} \sum_i \xi_i^s \frac{\partial Z_c(\psi)}{\partial \psi_i(s)} ,
\]
\[ G(s, st) = -i \lim_{N \to \infty} \frac{1}{N} \sum_i \langle \sigma_i(s) \hat{h}_i(st) \rangle = i \lim_{\psi \to 0} \lim_{N \to \infty} \frac{1}{N} \sum_i \frac{\partial^2 Z_c(\psi)}{\partial \psi_i(s) \partial \theta_i(st)} , (14) \]

\[ C(s, st) = \lim_{N \to \infty} \frac{1}{N} \sum_i \langle \sigma_i(s) \sigma_i(st) \rangle = - \lim_{\psi \to 0} \lim_{N \to \infty} \frac{1}{N} \sum_i \frac{\partial^2 Z_c(\psi)}{\partial \psi_i(s) \partial \psi_i(st)} . (15) \]

### 3.1 Single-site relations

An explicit expression is then obtained in the form of an effective single-site normalized average for any function \( f[\{\sigma\}] \) of the states of the diluted network with finite connectivity, denoted as \( \langle f[\{\sigma\}] \rangle^* \), for \( p = \alpha N \) in the limit \( N \to \infty \). The normalization is such that the average is unity when \( f[\{\sigma\}] = 1 \), and the result becomes independent of the specific site after the gauge transformation \( \sigma(s) \to \sigma(s) \xi_s \) and \( h(s) \to h(s) \xi_t^s \) [15]. This expression reads,

\[
\langle f[\{\sigma\}] \rangle^* = \sum_{\sigma(0), \ldots, \sigma(t)} \int \{d\hat{h}\} p(\sigma(0)) f[\{\sigma\}] \exp\{ \sum_{s<t} [\beta \sigma(s+1) h(s) - \ln 2 \cosh(\beta h(s))] + i \sum_{s<t} \hat{h}(s)[h(s) - \theta(s) - m^*(s)] - \frac{1}{2} \alpha \sum_{s,s',<t} \hat{h}(s) D(s, st) \hat{h}(st) \}, \tag{16}
\]

in which \( \{d\hat{h}\} = \prod_{s<t} [d\hat{h}(s) d\hat{h}(s)/2\pi] \) and all quantities are single-site quantities. Here,

\[ D(s, st) = R(s, st) + \frac{1-c}{c} \delta_{s, st} , \tag{17} \]

is the covariance \( D(s, st) = \langle v(s)v(st) \rangle \) of the zero-average Gaussian random noise \( v(s) \) in the effective local field

\[ h(s) = m^*(s) + \theta(s) + \sqrt{\alpha} v(s) , \tag{18} \]

as can be seen readily by integration over \( \hat{h} \). Thus, the effect of the dilution appears as a local term (in time) contributing to the variance \( \langle v^2(s) \rangle \) of the noise. It will be seen below that, precisely because of its local character, the dilution term has a crucial role in determining the stationary state of the network. The non-local contribution is that in the covariance of the noise,
\( R(s, st) \), for the fully connected network and this is given by the recurrence relation [16]

\[
R(s, st) = C(s, st) + G(s, s - 1)G(st, st - 1)R(s - 1, st - 1) .
\]

(19)

Combining Eqs. (17) and (19) with the appropriate average over the states given by Eq. (16), one obtains the evolution equations for the macroscopic observables.

### 3.2 Equations for the dynamics

The single-site equations for the time evolution of the macroscopic observables are then given by

\[
m(s) = \int \{dvdw\} e^{iv\cdot w - \frac{1}{2}w\cdot Dw} \\
\quad \times \tanh \beta[m(s - 1) + \theta(s - 1) + \sqrt{\alpha}v(s - 1)] ,
\]

(20)

\[
G(s, st) = \beta \delta_{s, st + 1} \{1 - \int \{dvdw\} e^{iv\cdot w - \frac{1}{2}w\cdot Dw} \\
\quad \times \tanh^2 \beta[m(s - 1) + \theta(s - 1) + \sqrt{\alpha}v(s - 1)]\} ,
\]

(21)

\[
C(s, st) = \delta_{s, st} + (1 - \delta_{s, st}) \int \{dvdw\} e^{iv\cdot w - \frac{1}{2}w\cdot Dw} \\
\quad \times \tanh \beta[m(s - 1) + \theta(s - 1) + \sqrt{\alpha}v(s - 1)] \\
\quad \times \tanh \beta[m(st - 1) + \theta(st - 1) + \sqrt{\alpha}v(st - 1)] ,
\]

(22)

where \( D \) is the covariance matrix of elements \( D(s, st) \). Note that the arguments of the tanh are \( \beta \) times the local field at time \( s \) or \( st \). From here on we drop the pattern index in the overlap \( m^s(s) \) with the understanding that \( m(s) \) means the overlap with the pattern in the sequence that corresponds to that time step.

It has been pointed out before, for the fully connected network [15], that the response function differs from zero only for changes of state one time step after the perturbation in the external field and that, consequently, stationary macroscopic states should be reached in finite timescales. Clearly, that is still the case for the diluted network.
The equations can be reduced by explicit integration over $w$, noting that only the matrix element $D(s - 1, s - 1)$ is needed for the first two of them and we obtain

$$m(s) = \int Dz \tanh \beta [m(s - 1) + \theta(s - 1) + z\sqrt{\alpha D(s - 1, s - 1)}] , \quad (23)$$

$$G(s, s - 1) = \beta \{1 - \int Dz \times \tanh^2 \beta [m(s - 1) + \theta(s - 1) + z\sqrt{\alpha D(s - 1, s - 1)}]\} , \quad (24)$$

where $z$ is a Gaussian random variable with mean zero and unit variance and, as usual, $Dz = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$. The third equation becomes

$$C(s, st) = (2\pi |D_{11}|)^{-1/2} \int dz \ e^{-\frac{1}{2}z \ D_{11}^{-1} \ z} \times \tanh \beta [m(st - 1) + \theta(st - 1) + \sqrt{\alpha z(st - 1)}] \times \tanh \beta [m(s - 1) + \theta(s - 1) + \sqrt{\alpha z(s - 1)}] , \quad (25)$$

when $s \neq st$ and $C(s, s) = 1$. Here, $D_{11}$ is the $2 \times 2$ submatrix of $D$ involving the elements at times $s - 1$ and $st - 1$, while $z$ is the vector with components $z(s - 1)$ and $z(st - 1)$. The diagonal elements of $D_{11}$ are given by

$$D(s, s) = R(s, s) + (1 - c)/c$$
$$R(s, s) = 1 + G^2(s, s - 1)R(s - 1, s - 1) \quad (26)$$

and only the off-diagonal elements depend on the correlation function, which enters the dynamics through Eq.(25).

In the case where $c = 1$, our equations, coincide with those for the transients in the fully connected network [16]. In the extremely dilute limit $c \to 0$, the storage ratio $\alpha$ has to be replaced by the ratio $p/cN$, that is, the number of patterns per connected units, which means $\alpha/c$. The equations for the overlap and the response function become then those for the extremely diluted network, and the correlation function vanishes in that limit. Given an initial value for $R(s, s)$, Eqs.(23),(24) and (26) form a closed set of equations for the time evolution of the macroscopic observables of primary interest. It is important to keep in mind that these quantities do not depend on the correlation function.
It may be noted at this point that, for general $c$, Eqs.(23),(24) and (26) are formally the same as the recurrence relations for the overlap and the stochastic noise in the layered feed-forward network with finite connectivity for the single-pattern retrieval problem [17]. The task in this problem is to recognize a given single pattern as the states evolve from one layer to the next one. The network model consists of $N$ binary units placed on layers in which all units of a layer are updated simultaneously according to the alignment with their local field produced by the units in the previous layer and there is no feedback from any layer to previous ones. A set of independent random patterns, $\{\xi_1^\mu(l), \ldots, \xi_N^\mu(l)\}$, $\mu = 1, \ldots, p$, each $\xi_i^\mu(l) = \pm 1$ with probability $\frac{1}{2}$, is generated on every layer $l$ independently of other layers, and the patterns are assumed to be stored in the network according to the learning rule

$$J_{ij}^d(l) = \frac{c_{ij}}{cN} \sum_{\mu=1}^{p} \xi_i^\mu(l+1)\xi_j^\mu(l) \ .$$

(27)

The local field at unit $i$ on layer $l$ may then be written as

$$h_i(l) = \xi_i^1(l+1)m^1(l) + \theta_i(l) + z_i(l) \ ,$$

(28)

where the first term is the signal from the finite overlap with pattern $\xi_1^1$, say, $m^1(l)$ and

$$z_i(l) = \sum_{\mu \neq 1} \xi_i^\mu(l+1)m^\mu(l)$$

is the noise due to the remaining patterns with overlap $m^\mu(l)$. The recurrence relations for the overlap $m^1(l)$ and for the variance of the noise, $D(l,l)$, are then given precisely by Eqs.(23), (24) and (26) if the time step is replaced by the layer index and writing $G(l,l-1) = \beta[1 - \tilde{q}(l-1)]$, where [17]

$$\tilde{q}(l-1) = \int Dz \tanh^2 \beta[m(l-1) + \theta(l-1) + z\sqrt{\alpha D(l-1,l-1)}]$$

(30)

with

$$R(l,l) = 1 + \beta^2[1 - \tilde{q}(l-1)]^2R(l-1,l-1)$$

(31)

and the relationship with $D(l,l)$ given above. The variable $\tilde{q}(l)$ is, clearly, the single-site average $\langle \sigma_i(l) \rangle$ in the layered network, where the brackets denote here the average with the distribution given by Eq.(2).

Consider now the correlation function in the layered network. To obtain $C(l,l)$ one needs to establish a recurrence relation for the covariance
\[ \langle z_i(l) z_i(l') \rangle \] of the noise. Following a standard procedure [16, 17], one finds that, for \( \alpha \neq 0 \), the covariance is zero due to the independence of the patterns in different layers and, consequently, \( C(l, l) = \delta_{l,l'} \).

### 3.3 The stationary state

The stationary state of the network is obtained from the time-translation invariant limit cycle solutions

\[ m(s) = m, \ C(s,s') = C(s-st), \ G(s,s') = G(s-st) \quad (32) \]

for the macroscopic observables. Then, \( R(s,s') = R(s-st) \) and \( D(s,s') = D(s-st) \) with \( R(s,s') \) given by Eq.(19). The equations for the first two macroscopic observables in the stationary state become then

\[ m = \int Dz \tanh \beta [m + \theta + z \sqrt{\alpha (\rho + (1-c)/c)}] , \quad (33) \]

\[ \tilde{q} = \int Dz \tanh^2 \beta [m + \theta + z \sqrt{\alpha (\rho + (1-c)/c)}] , \quad (34) \]

where \( \tilde{q} \) is given by \( G(1) = \beta (1 - \tilde{q}) \). Here,

\[ \rho = \frac{1}{1 - \beta^2 (1 - \tilde{q})^2} , \quad (35) \]

is the stationary value of the variance in the internal field for the fully connected network. These are, again, the equations for the stationary state for the single-pattern retrieval problem in the layered feed-forward network with finite connectivity and with an effective Gaussian noise of variance \( \tilde{\rho} = \alpha (\rho + (1-c)/c) \). The results at \( T = 0 \) have already been discussed, and it has been found that both, the critical storage capacity \( \alpha_c \) and the limiting overlap of the recalled pattern, are monotonic decreasing functions of the dilution [17].

To obtain the correlation function in the stationary state it is convenient to separate in Eq.(22) the persistent and non-persistent parts in \( R(\tau) \) [15],

\[ R(\tau) = r + \tilde{R}(\tau) , \quad \lim_{\tau \to \infty} \tilde{R}(\tau) = 0 , \quad (36) \]

which yields \( D(\tau) = r + \tilde{D}(\tau) \), where

\[ \tilde{D}(\tau) = \tilde{R}(\tau) + [(1-c)/c] \delta_{\tau,0} \quad (37) \]
also vanishes in that limit. The separation of the persistent part in \( D(\tau) \) and a linearization introduces a new Gaussian integration over a variable \( z \) of mean zero and variance \( \alpha r \), where \( r = q \rho \) and \( q = \lim_{\tau \to \infty} C(\tau) \) is the persistent part of \( C(\tau) \). This may be interpreted as the spin-glass order parameter. The remaining Gaussian random variable has covariance \( \alpha \tilde{D}(\tau) \) and, in the limit \( \tau \to \infty \), \( v(\tau) \) and \( v(0) \) become uncorrelated. We are left with the variance \( \alpha \tilde{D}(0) \), where

\[
\tilde{D}(0) = (1 - q)\rho + \frac{1 - c}{c}, \tag{38}
\]

and we obtain for the persistent part of the correlation function

\[
q = \int Dz \left\{ \int Dx \tanh \beta [m + \theta + z\sqrt{\alpha q \rho} + x\sqrt{\alpha((1 - q) \rho + (1 - c)/c)}] \right\}^2. \tag{39}
\]

We consider next the solutions of this equation combined with the response function in the absence of recall for \( \theta = 0 \). This requires \( \tilde{q} \), which is always non-zero and vanishes at high \( T \) as

\[
\tilde{q} = \alpha \beta^2 (1 + \frac{1 - c}{c}) + O(\beta^4). \tag{40}
\]

In the limit \( T \to 0 \), \( \beta(1 - \tilde{q}) \) remains finite and this leads to \( \rho = 1 + 2/\pi \alpha \) when \( m = 0 \). Consider next the equation for \( q \) and note that it always has the paramagnetic solution \( q = 0 \) in the absence of recall. In the limit \( T \to 0 \), we find

\[
q = \int Dz \text{erf} \left( \frac{z\sqrt{q \rho}}{\sqrt{2[(1 - q) \rho + (1 - c)/c]}} \right), \tag{41}
\]

for non-zero \( \alpha \), and analysis of this equation shows that, for any connectivity \( c < 1 \), the only solution is \( q = 0 \). In contrast, there is also the solution \( q = 1 \) in the case of the fully connected network. We also conclude that there is no solution other than \( q = 0 \) for finite \( T \), regardless of the connectivity, in extension of earlier work [15]. Thus, the persistent part of the correlation function vanishes for all \( T \) in the sequence processing network when \( c < 1 \). In contrast, it is always zero in the layered network, as we saw above.
4 Summary and conclusions

We extended the path-integral method to study the retrieval dynamics of consecutive single patterns in a sequence for a symmetrically diluted network near saturation in a dynamic mean-field theory. This is an exact procedure for an asymptotically large network and the close agreement with numerical simulations obtained in recent works [15, 16] seem to indicate that the theory accounts for the essentials of single-sequence processing. The main feature is the presence of an, in general, non-local Gaussian stochastic noise in time in the effective single-site problem. Only the local part of that noise enters in the equations for the dynamics and the stationary states for the overlap with a pattern and for the response function to an external field.

The symmetric dilution of the synapses, which is equivalent to the addition of a static noise to the coupling of the fully connected network, introduces a further local Gaussian noise which is relevant for the retrieval performance of the network. We have shown that the resulting equations for the dynamics and for the stationary states of the sequence processing network are precisely the same as those for the single-pattern retrieval problem in the layered feed-forward network, for all $T$. We also showed that the particular form of the resulting Gaussian stochastic noise is exactly the same in both systems for any finite connectivity.

Nevertheless, the two systems are not fully equivalent. Indeed, although the two-time correlation function is zero in the layered network, there is a finite, persistent correlation $q = 1$ for the fully connected sequence processing network at $T = 0$ [15]. However, we also showed that this finite value disappears as soon as there is a dilution of the couplings, restoring in that case the formal equivalence between the performance of both systems. We insist here that the task of the network and its architecture are different for both. In the case of finite $T$ there is no persistent part of the two-time correlation function in both systems, regardless of the connectivity.

Finally, all the results discussed here with the path-integral method also follow from the statistical neurodynamics approach. It would be interesting to study the effects of synaptic dilution on other tasks in a sequence processing network, like mixture-state retrieval, in which the overlap with more than one pattern in a sequence remains finite. It may also be interesting to introduce a dynamical dilution process that could account for deteriorating synapses. The random symmetric dilution used here is a static one that is
not altered in the evolution of the network. These are issues that will be studied in future work.

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