Shape optimization in thermal convection field considering a slight compressibility

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Abstract

In this study, we investigate the shape optimization problem for maximizing heat dissipation in a thermal convection field considering a slight compressibility. This study aims to obtain an optimal shape that will maximize the heat dissipation on arbitrary boundaries considering a slight compressibility. The adjoint variable and traction methods are applied to obtain the gradient for the optimal shape. Consequently, it is found that the trend of increase in heat dissipation is different at certain shape update steps in comparison with the results obtained when considering slight compressibility and using the Boussinesq approximation.

Keywords shape optimization, adjoint variable method, finite element method, thermal convection field, traction method

Research Activity Group Mathematical Design

1. Introduction

In this study, we focus on the heat dissipation maximization, as a shape optimization problem in thermal convection fields, to increase heat dissipation. To improve the cooling performance of electronic devices, it is important to increase the heat dissipation. Therefore, the shape optimization problem, wherein the heat source of the device was the design domain, was formulated. The effect of a slight compressibility on the optimal shape results is specifically discussed in this study. The Boussinesq approximation is generally introduced in the numerical analysis of thermal fluids, and the density in the pressure and viscosity terms of the Navier–Stokes equation, which is the equation of motion of the thermal convection field, becomes constant, and compressibility is neglected. However, the Boussinesq approximation applies in cases where the temperature difference between the fields in the analysis region is insignificant [1, 2]. For example, when the fluid is water, the temperature difference of the field in the analysis domain should be within 2 (K). In the case of larger temperature changes, it is necessary to consider the change in density with temperature because the flow velocity or temperature distribution may vary. Therefore, in this study, we consider a shape optimization analysis that takes into account the effects of a slight compressibility by considering the density, which is a coefficient in the pressure and viscosity terms, to be a function of temperature. An approximation similar to slight compressibility is a method called the low Mach number approximation. This applies in air and is used in the analysis of phenomena involving combustion, such as thermal environments and fires. Because oil is considered as the fluid in the thermal convection field in this study, the analysis is performed considering the slight compressibility instead of the low Mach number approximation.

Considering the above, for the heat dissipation maximization problem, the squared error of the heat dissipation obtained by the calculation is defined as the objective function, and the shape optimization problem is formulated based on the adjoint variable and finite element methods. The state and adjoint variables are calculated, and the gradient of the Lagrange function is derived with respect to the nodal coordinates on the design boundary. The shape of the region is updated while suppressing the numerical oscillations of the gradient by applying the traction method [3] to the derived gradient. In the analysis considering the slight compressibility, shape optimization analysis is performed when the density of the fluid is considered as a function of the appropriate temperature, and the results are compared with those of shape optimization when the Boussinesq approximation is used. The effect of this treatment on the results of shape optimization is discussed. FreeFEM++ [4] is used for the numerical analysis.

2. Formulation for the heat dissipation maximization problem

2.1 Definition of the objective function

The objective function is first defined to obtain the shape that optimally satisfies the objectives. In this study, it is defined as shown in (1), considering the objective of the heat dissipation being maximized at a certain boundary.
\[ J = -\frac{1}{2} \int_{t_0}^{t_f} \int_{\Gamma} Q \rho^2 \, d\Gamma \, dt, \]  

where \( Q \) indicates the weight constant, and the computed heat dissipation \( q \) is expressed by \( q = \hat{h}(\theta - \hat{\theta}) \). \( \hat{h}, \theta, \) and \( \hat{\theta} \) represent the heat transfer coefficient, temperature at each computation step, and outside air temperature, respectively. \( t_0, t_f \), and \( \Gamma \) denote the initial and terminal times, and computational boundaries, respectively. The weight constant \( Q \) is given as 1 on the boundary where the heat dissipation is calculated, and as 0 on other boundaries.

### 2.2 Constraint conditions for the objective function

The Navier–Stokes equation, continuity equation, and heat transfer equation are employed as the governing equations. Let the density at the Navier–Stokes equation be a function of temperature to consider the density change \([5]\). The governing equations can be expressed as (2)–(4), and are written according to the Einstein summation convention.

\[ \dot{u}_i + u_j u_{i,j} + \frac{1}{\rho(\theta)} p_{,i} - \frac{\mu}{\rho(\theta)} u_{i,j} - \rho(\theta) - \rho_0 = 0 \quad \text{in} \quad t \in [t_0, t_f] \quad \text{in} \quad \Omega, \]  

\[ u_{i,j} = 0 \quad \text{in} \quad t \in [t_0, t_f] \quad \text{in} \quad \Omega, \]  

\[ \dot{\theta} + u_i \theta_{,i} - \alpha \theta_{,ii} = 0 \quad \text{in} \quad t \in [t_0, t_f] \quad \text{in} \quad \Omega, \]  

where \( u_i, p, \) and \( \theta \) indicate the flow velocity, pressure, and temperature, respectively. \( \mu, \alpha, \rho, \) and \( \Omega \) indicate the viscosity coefficient, temperature diffusivity, gravity acceleration, and computational domain, respectively. \( e_1 = e_3 = 0 \) and \( e_2 = -1 \). Moreover, \( \rho(\theta) \) is a density that is a function of temperature. The density is set as a function of temperature, as shown in (5).

\[ \rho(\theta) = \frac{\rho_0}{1 + \beta(\theta - \theta_0)}. \]  

The initial and boundary conditions shown in (6)–(12) are introduced as the constraint conditions for the objective function. The area constraint condition shown in (13) is also considered in this study.

\[ u_i = \hat{u}_i \quad \text{at} \quad t = t_0 \quad \text{in} \quad \Omega, \]  

\[ u_i = \hat{u}_i \quad \text{in} \quad t \in [t_0, t_f] \quad \text{on} \quad \Gamma_{1u} \quad \text{and} \quad \Gamma_{design}, \]  

\[ t_i = \hat{t}_i \quad \text{in} \quad t \in [t_0, t_f] \quad \text{on} \quad \Gamma_{2a}, \]  

\[ \theta = \hat{\theta} \quad \text{at} \quad t = t_0 \quad \text{in} \quad \Omega, \]  

\[ \theta = \hat{\theta} \quad \text{in} \quad t \in [t_0, t_f] \quad \text{on} \quad \Gamma_{1 \theta}, \]  

\[ b = \hat{b} - \alpha \theta_{,n_i} \quad \text{in} \quad t \in [t_0, t_f] \quad \text{on} \quad \Gamma_{3 \theta}, \]  

\[ q = \hat{q}(\theta - \hat{\theta}) = -\alpha \theta_{,n_i} \quad \text{in} \quad t \in [t_0, t_f] \quad \text{on} \quad \Gamma_{2a}, \]  

\[ \sum_{i=1}^{m_2} A_i = A_{initial} \quad \text{in} \quad \Omega, \]  

where \( \beta, \rho_0, \delta_{ij}, n_{ij}, \) and \( \hat{\cdot} \) indicate the thermal expansion coefficient, reference density, Dirac’s delta function, direction cosine of the unit outward normal of the boundary, and known function, respectively. \( \Gamma_{1u}, \Gamma_{2a}, \Gamma_{1 \theta}, \Gamma_{2 \theta}, \) and \( \Gamma_{3 \theta} \) represent the Dirichlet boundaries in the flow field, the Neumann boundaries in the flow field, the Dirichlet boundaries in the temperature field, the Neumann boundaries in the temperature field, and Robin boundaries, respectively. \( \Gamma_{design} \) indicates the design boundary. In addition, in (13), \( m_2 \) and \( A_{initial} \) represent the total number of elements and the area of the entire domain at the initial iteration.

### 2.3 Formulation by the adjoint variable method

On introducing the adjoint variables, the objective function is extended to (14), which is referred to as the Lagrange function.

\[ J^* = J + \Lambda \left\{ \sum_{i=1}^{m_2} A_i - A_{initial} \right\} \]  

\[ + \int_{t_0}^{t_f} \int_{\Omega} u_i^* \{ \dot{u}_i + u_j u_{i,j} + \frac{1}{\rho(\theta)[1 + \beta(\theta - \theta_0)]} p_{,i} - \frac{\mu}{\rho(\theta)} u_{i,j} - \rho(\theta) - \rho_0 \} \, d\Omega \, dt \]  

\[ - \int_{t_0}^{t_f} \int_{\Omega} p^* u_{i,j} \, d\Omega \, dt \]  

\[ + \int_{t_0}^{t_f} \int_{\Omega} \theta^* \left( \dot{\theta} + u_i \theta_{,i} - \alpha \theta_{,ii} \right) \, d\Omega \, dt = 0, \]  

where \( u_i^*, p^*, \) and \( \theta^* \), and \( \Lambda \) indicate the adjoint variables for the flow velocity, pressure, temperature, and area constraint, respectively.

The first variation of the Lagrange function must be zero to satisfy the stationary condition. Considering the stationary condition of the Lagrange function, the adjoint equation and the conditions for adjoint variables shown in (15)–(24) are obtained.

\[ - \dot{u}_i^* + u_j u_{i,j}^* - (u_j u_{i,j})^* - p_{,i}^* \]  

\[ - \frac{\mu}{\rho_0} (u_j^* [1 + \beta(\theta - \theta_0)])_{,jj} + \theta^* \theta_{,ii} = 0 \]  

\[ \text{in} \quad t \in [t_0, t_f] \quad \text{in} \quad \Omega, \]  

\[ - \frac{1}{\rho_0} (u_i^* [1 + \beta(\theta - \theta_0)])_{,i} = 0 \]  

\[ \text{in} \quad t \in [t_0, t_f] \quad \text{in} \quad \Omega, \]  

\[ - \dot{\theta}^* - (\theta^* u_{i,j})_{,i} - \alpha \theta_{,ii}^* + \frac{\beta}{\rho_0} p_{,i}^* u_i^* - \frac{\mu \beta}{\rho_0} u_{i,j}^* + u_i^* \beta e_i = 0 \]  

\[ \text{in} \quad t \in [t_0, t_f] \quad \text{in} \quad \Omega, \]  

\[ u_i^* = 0 \quad \text{at} \quad t = t_f \quad \text{in} \quad \Omega, \]  

\[ \theta^* = 0 \quad \text{at} \quad t = t_f \quad \text{in} \quad \Omega, \]  

\[ u_i^* = 0 \quad \text{in} \quad t \in [t_0, t_f] \quad \text{on} \quad \Gamma_{1u} \quad \text{and} \quad \Gamma_{design}, \]  

\[ \theta^* = 0 \quad \text{in} \quad t \in [t_0, t_f] \quad \text{on} \quad \Gamma_{1 \theta}, \]  

\[ -45 - \]
\[ t_i^* = \{ u_j u_i^* + p^* \delta_{ij} + \frac{\mu}{\rho_0} (u_i^* [1 + \beta(\theta - \theta_0)])_j \} n_j \]
\[ \quad \text{in} \quad t \in [t_0, t_f] \quad \text{on} \quad \Gamma_{2u}, \quad (22) \]
\[ b^* = b^* = (\theta^* u_i + \alpha \theta^*) n_i \]
\[ \quad \text{in} \quad t \in [t_0, t_f] \quad \text{on} \quad \Gamma_{2\theta}, \quad (23) \]
\[ s^* = -Q h^2 (\theta - \hat{\theta}_f) \]
\[ \quad \text{in} \quad t \in [t_0, t_f] \quad \text{on} \quad \Gamma_{3\theta}. \quad (24) \]

The gradient of the Lagrange function with respect to the coordinates on the design boundary, \( G_i \), is derived by (25).

\[ \int_{t_0}^{t_f} \left( \int_{\Gamma} t_i^* \delta u_i d\Gamma + \int_{\Gamma} b^* \delta \theta d\Gamma \right) dt \]
\[ = \int_{t_0}^{t_f} \left( \int_{\Gamma_{1u} + \Gamma_{2u}} t_i^* \delta u_i d\Gamma + \int_{\Gamma_{\text{design}}} t_i^* u_{i,j} \delta x_j d\Gamma \right. \]
\[ + \left. \int_{\Gamma_{1\theta} + \Gamma_{2\theta}} b^* \delta \theta d\Gamma + \int_{\Gamma_{\text{design}}} b^* \theta_{i} \delta x_i d\Gamma \right) dt \]
\[ = \int_{t_0}^{t_f} \left\{ \int_{\Gamma_{1u} + \Gamma_{2u}} t_i^* \delta u_i d\Gamma + \int_{\Gamma_{\text{design}}} \left[ u_j u_i^* + p^* \delta_{ij} \right] n_j u_{i,j} \delta x_j d\Gamma \right. \]
\[ \quad + \left. \int_{\Gamma_{1\theta} + \Gamma_{2\theta}} b^* \delta \theta d\Gamma \right. \]
\[ \quad \quad + \left. \int_{\Gamma_{\text{design}}} (\theta^* u_i + \alpha \theta^*) n_i \theta_{i} \delta x_i d\Gamma \right) \]  
\[ = \int_{t_0}^{t_f} \left\{ \int_{\Gamma_{1u} + \Gamma_{2u}} t_i^* \delta u_i d\Gamma + \int_{\Gamma_{1\theta} + \Gamma_{2\theta}} b^* \delta \theta d\Gamma \right\} dt \]
\[ + \int_{\Gamma_{\text{design}}} G_i \delta x_i d\Gamma. \quad (25) \]

The gradient \( G_i \) is modified by the traction method, wherein the gradient \( G_i \) is employed as the external force for the linear elastic body, and the displacement value is used as the modified gradient \( G_i^* \). The treatment of the area constraint is considered in the traction method [4]. The coordinates on the design boundary are updated using the modified gradient \( G_i^* \), as shown in (26).

\[ x_i^{l+1} = x_i^l - \eta G_i^{l+1} \quad \text{on} \quad \Gamma_{\text{design}}, \quad (26) \]

where \( \eta \) and \( l \) indicate the step length and number of iterations for the shape update.

3. Numerical experiments

3.1 Numerical conditions

Subsequently, shape optimization analysis considering the slight compressibility is conducted by using the adjoint variable and the finite element methods. The boundary conditions and computational model are shown in Table 1 and Fig. 1, respectively. In Fig. 1, the objective function is calculated on the Robin boundary \( \Gamma_{3\theta} \), and the Dirichlet boundary \( \Gamma_{1\theta} \) is given as the design boundary \( \Gamma_{\text{design}} \). The Neumann boundary \( \Gamma_{2\theta} \) is given as \( q = 0 \) (W/m²) as an adiabatic boundary, and the non-slip condition \( u_i = 0 \) (m/s) is given for all boundaries. The initial conditions are given as \( u_i = 0 \) (m/s) and \( \theta = 0 \) (K) in the entire domain. The dimensions of the computational model are \( L = 1 \) (m) and \( R = 0.2 \) (m), as shown in Fig. 1. When updating the shape, remeshing was performed using the function “adaptmesh” on FreeFEM++. As for the discretization in space, in the governing and the adjoint equations, the P2/P1 elements are employed for the velocity and the pressure, and the P1 element is used for the temperature. The backward difference method is applied to discretize the governing and the adjoint equations in time.

3.2 Numerical results

The finite element mesh for the initial and improved shapes obtained from the optimization analysis are shown in Figs. 2 and 3. In Fig. 3, the improved shape was spread out in the \( x_2 \) direction, resulting in faster temperature transfer and increased heat dissipation. In addition to the analysis considering the slight compressibility, the analysis with the Boussinesq approximation and the analysis with Boussinesq approximation with the step length of the shape update set to \( \eta = 0.0005 \) were also performed to compare the analysis results. The variation of each objective function is shown in Fig. 4. In case of \( \eta = 0.001 \), the heat dissipation was approximately 35% higher than the initial shape when considering the slight compressibility. It can be confirmed that the objective function around the 14th iteration when considering the slight compressibility increases slightly slowly than when using the Boussinesq approximation. It can be confirmed that when a slight compressibility
is considered, the same heat dissipation could be obtained with fewer shape updates than when using the Boussinesq approximation. On the other hand, the optimal shape could not be obtained due to mesh distortion when considering the slight compressibility in case of $\eta = 0.0005$. From this result, it is found that the effect of the slight compressibility can be obtained, when the step length $\eta$ is large. In addition, (27) is obtained by comparing the gradients when considering the slight compressibility and when using the Boussinesq approximation.

\[
\Delta G_i = G_i - G_i^{(\text{Boussinesq})}
\]

\[
= \int_0^{t_i} \left[ \frac{\mu}{\rho_0} u_{j,i}^* \frac{\partial (\theta - \theta_0)}{\partial x_j} n_{j,i} \right] \, dt \right] n_{j,i},
\]

(27)

where $G_i^{(\text{Boussinesq})}$ indicates the gradient when using the Boussinesq approximation. The case of $\eta = 0.001$, the distributions of the modified gradient $G_i^*$ and the difference of modified gradient $\Delta G_i^*$ at the initial iteration are depicted in Figs. 5 and 6, respectively; the distributions of the modified gradient $G_i^*$ and the difference of modified gradient $\Delta G_i^*$ at the 14th iteration are illustrated in Figs. 7 and 8, respectively. From Figs. 5 and 6, it can be confirmed that the distribution of the difference in the modified gradient $\Delta G_i^*$ on the design boundary at the initial iteration is in the opposite direction to that of the modified gradient $G_i^*$. Conversely, from Figs. 7 and 8, it can be seen that the distribution of the difference in the modified gradient $\Delta G_i^*$ is distributed in the opposite direction to the modified gradient $\Delta G_i^*$ only near the left and right sides of the design boundary. Consequently, the change in shape near the left and right sides of the design boundary affects the maximization of heat dissipation, and it is expected that the heat dissipation can be slowly increased by considering the slight compressibility.

4. Conclusions

In this study, we presented the shape optimization problem for maximizing heat dissipation considering slight compressibility. Comparing the results obtained when considering slight compressibility and using the Boussinesq approximation, it was confirmed that the trend of increase in heat dissipation was different at certain shape update steps.

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