Towards a unified approach to information-disturbance tradeoffs in quantum measurements

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Abstract. We show that the global balance of information dynamics for general quantum measurements given in [F. Buscemi, M. Hayashi, and M. Horodecki, Phys. Rev. Lett. 100, 210504 (2008)] makes it possible to unify various and generally inequivalent approaches adopted in order to derive information-disturbance tradeoffs in quantum theory. We focus in particular on those tradeoffs, constituting the vast majority of the literature on the subject, where disturbance is defined either in terms of average output fidelity or of entanglement fidelity.
1. Introduction

The general idea for which information extraction always causes disturbance is nowadays widely accepted as one of the most fundamental and distinctive principles of quantum theory as opposed to classical theory. However, without precisely defining what we mean with the vague words “information” and “disturbance”, such a statement is nothing but an empty sentence. In order to obtain some mathematically sound results, one first needs to provide a natural and meaningful way to measure both information and disturbance. Given these definitions, an information-disturbance tradeoff relation is then a lower bound on (some monotonic function of) disturbance given in terms of (some other monotonic function of) information gain, in such a way that the conclusion “disturbance is null only if information gain is null” can be drawn from it. With this at hand, a second order of problems is to identify least-disturbing measurements, namely, those measurements causing the minimum disturbance compatible with a given amount of information extracted. For these kind of “optimal” measurements then one would like to see that also the converse is true, that is, if information gain is null, then also minimum disturbance is null, thus establishing an equivalence relation between information and minimum disturbance. This is however beyond the scope of the present paper: our main concern will be to unify different and usually inequivalent ways to quantify information and disturbance, hoping that such a clarification could simplify the derivation of optimal measurements, which in general is an awkward, yet important, task.

The setting of the problem is usually as follows: a letter \( x \), drawn from an input alphabet \( \mathcal{X} \) according to the probability distribution \( p(x) \), is encoded into some quantum state \( \rho_x \). Subsequently, a quantum measurement \( \mathcal{M} \) (for the moment, let us think of the measurement as a kind of black box performing some fixed operation) is performed on the given state \( \rho_x \), whose label \( x \) is unknown, producing a measurement readout letter \( m \), belonging to the set \( \mathcal{X} \) of possible outcomes, together with the corresponding reduced state \( \rho_{m|x} \). Given the input letter \( x \), the probability of getting the result \( m \) is written as a conditional probability \( p(m|x) \). Then, information is usually understood as “information in \( m \) about \( x \)” and disturbance as “how well one can undo the state change \( \rho_x \mapsto \rho_{m|x} \) for all possible couples \( (x,m) \in \mathcal{X} \times \mathcal{X} \)\(^\dagger\).\(^\dagger\)

While information can be defined to be one of the many known and basically equivalent measures provided by classical information theory to quantify the input-output correlations given the joint probability distribution \( p(x,m) \)^\§, we have (at least) two in principle inequivalent—still both meaningful—ways to measure disturbance in a quantum scenario: the first one, adopted e. g. in Ref. \[2\], measures disturbance depending on “how close” the states \( \rho_x \) and \( \rho_{m|x} \) are, for all \( x \) and \( m \), possibly after a correcting operation performed onto the output states. We tend to call this way the average output fidelity approach. The second way, that we call entanglement fidelity approach, adopted e. g. in Ref. \[3, 4\], measures disturbance depending on

\[^\dagger\] We notice that, since a quantum measurement process is defined by several multidimensional parameters, i. e. the source and the measurement apparatus, it is extremely demanding to require these latter to be meaningfully summarized by means of two positive parameters only, i. e. information and disturbance: the very definition of information and disturbance hence represents a major problem in itself.

\[^\dagger\dagger\] Different definitions of information gain could however lead, in principle, to different least-disturbing measurements. For example, it is known that, for a given ensemble of states, the measurement optimizing the mutual information is in general different from the measurement optimizing the minimum discrimination error: see the analysis of the two-states case in Ref. \[1\].

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“how reversible”, or, equivalently, “how coherent” is the transformation causing the state change, and it is related to the possibility of reliably transmitting quantum entanglement through the measurement apparatus. The second approach is generally much more stringent than the first one, as the following example shows. Let us consider the simple situation in which the input quantum states $\rho_x$ are actually pure orthogonal states. Then, the von Neumann measurement projecting onto such states is able to fully extract the information about $x$, without causing any disturbance, when we understand it in the first sense (indeed, in this case, $\rho_{m|x} = \rho_x$ and $p(x,m) = p(x)\delta_{m,x}$). On the other hand, a complete projection like a von Neumann measurement causes a sudden and complete decoherence of any input state onto the measurement basis: it is hence amongst the most disturbing state changes possible, if disturbance is understood in the second sense, since the associated state change completely destroys entanglement. (We will be more quantitative in the following.)

In fact, there is a third remarkable way to define disturbance, and it is the one adopted by Ozawa in Ref. [5]. Ozawa analyzed the case of a subsequent measurement of two generally non-commuting observables and obtained a universally valid uncertainty relation, in the spirit of Heisenberg [6] and Robertson [7], as a formula explicitly involving noise, disturbance, and pre-measurement uncertainties (i.e., standard deviations in the preparation stage). However, we prefer to keep Ozawa’s analysis in a separate class, since he describes the evolution in the Heisenberg picture, considering observables evolving in time, while quantum states are fixed. Therefore, the notion of disturbance Ozawa uses is about observables and cannot be straightforwardly translated into a notion of disturbance about states, which is the one usually adopted when dealing with quantum communication scenarios, like e.g., quantum cryptography.

The question we address in this paper is: how can we reconcile the first two mentioned approaches to quantify disturbance, one using average-output-fidelity–like criteria, the other using entanglement-fidelity–like criteria? The motivation comes from a previous work of ours [4] where we derived the closed balance of information in a quantum measurement process by introducing a quantum information gain, which constitutes an upper bound to the information that the apparatus is able to extract, independently of how this information is encoded, and a quantum disturbance, which is related to the possibility of deterministically and coherently undoing the corresponding state change. Such a balance, by construction, incorporates a tight information-disturbance tradeoff where disturbance is understood in the second sense. In a way, quantum information gain and quantum disturbance defined in [4] are intrinsic features of the measuring instrument, much like the “technical specs” of the apparatus, independent of the source that is to be measured. Moreover, being a closed balance, the formula of Ref. [4] is able to explain the physical causes for the existence of the tradeoff, the latter being a direct consequence of the appearance of some hidden correlations between the input system (undergoing the measurement) and some inaccessible degrees of freedom (like e.g., the environment). The approach we proposed in Ref. [4] hence seems to possess some advantages with respect to other approaches, at least in terms of compactness and generality. It has been unclear for a while, however, how to derive from it a general tradeoff relation also when disturbance is understood in the first sense, namely, when it is measured in terms of average output fidelity. The previous example involving orthogonal states and von Neumann measurements seems to stand as an insurmountable obstacle to this program. Nevertheless, we will prove that it is indeed possible to circumvent it by showing that, whenever the
input alphabet is encoded onto non-orthogonal pure states (we will be more precise in the following), then the two notions of disturbance are equivalent, in the sense that when one is sufficiently small, the other is correspondingly small. Such a conclusion basically comes as a corollary of Lemma 3 in Ref. [8], whose proof we adapted here for our needs. It also provides the fact that, the more orthogonal the input states get, the looser the equivalence relation between the two notions of disturbance becomes, until when, for orthogonal ensembles, the two definitions become inequivalent, as in the mentioned example. In this sense, we will have a formula which describes the continuous transition from the quantum theory, where information extraction implies disturbance (in whichever way we understand it), to classical theory, where information can be freely read and copied.

As a drawback of the generality of our analysis (in particular, no symmetry is assumed for the input ensemble), the mathematical statement of the main result (Theorem 1) will turn out to be not very efficient, mainly because of two reasons: the first reason is that, as we said, we will not assume any sort of symmetry which could simplify the analysis; the second reason is that we will relate extensive quantities (like entropy and entropic measures of information) with non-extensive ones (like fidelity and fidelity-based measure of disturbance). On the contrary, when considering very special cases of highly symmetric input ensembles, we will see that, as one would expect, it is indeed possible to obtain much more compact formulas. Such a simplification is achieved by considering alternative entropy-based measures of disturbance which can be more elegantly related with informational quantities than fidelities. However, a generalization of this alternative approach seems to be difficult, and the results we have in this direction are still largely unsatisfactory. Nonetheless, we present them here in Section 5 as a preliminary step towards further investigations.

2. Notation and basic concepts

In this section we introduce notations and recall basic concepts and facts that will be useful in the following. Let us consider the following common situation: some source \( s \) draws the letter \( x \) from an alphabet \( X \), according to the probability distribution \( p(x) \), and correspondingly emits the signal state \( \rho_Q^x \) \((\rho_Q^x \geq 0 \text{ and } \text{Tr}[\rho_Q^x] = 1) \) belonging to the \( d \)-dimensional Hilbert space \( Q \). (Throughout this paper, all Hilbert spaces will be finite dimensional.) Such a source is modeled as a quantum ensemble

\[
\mathcal{S} := \{(p(x), \rho_Q^x) \}_{x \in X},
\]

whose average state is denoted by \( \rho_S^Q := \sum_x p(x) \rho_Q^x \). A measurement \( \mathcal{M}^Q \) is then performed onto the system \( Q \) (the input of the measurement apparatus) to get information about the encoded letter \( x \in X \). As proved by Ozawa in Ref. [12], the most general description of any experimentally realizable quantum measurement is given in terms of a completely positive (CP) quantum instrument, that is, a CP-map-valued measure \( \{E_Q^m\} \in X \), defined on a set of classical outcomes \( m \in X \), and normalized such that \( \sum_{m \in X} E_Q^m \) is trace-preserving (i. e. a channel). When the input state is described by the density matrix \( \rho_Q^e \), the probability of obtaining the outcome \( m \) is given by \( p(m|\rho) := \text{Tr}[E_Q^m(\rho_Q^e)] \); correspondingly, the state change \( \rho_Q^e \mapsto \sigma_{m|\rho}^Q := E_Q^m(\rho_Q^e)/p(m|\rho) \) occurs, \( Q' \) denoting the output Hilbert space. With a little abuse of notation, it is also helpful to represent the average state change caused by the measurement \( \mathcal{M}^Q \) as a black box performing some fixed operation on the input
state, namely, as a channel with hybrid quantum-classical output system

\[ \mathcal{M}_Q(q) := \sum_{m \in \mathcal{X}} p(m|q) \sigma^Q_{m|q} \otimes |m\rangle \langle m|_X, \]

(2)

where \(|m\rangle \langle m|_X\) are a set of orthonormal (hence perfectly distinguishable) vectors in the classical register space \(\mathcal{X}\) of outcomes.

In addition, we will find it useful to introduce an auxiliary reference \(\mathcal{R}\) (that we choose isomorphic with \(Q\)) purifying the input state \(q\) into \(|\Psi_{\mathcal{R}Q}\rangle\) and going untouched through the whole measurement process, in such a way that

\[ |\Psi_{\mathcal{R}Q}\rangle \mapsto (id_{\mathcal{R}} \otimes \mathcal{M}_Q)(|\Psi_{\mathcal{R}Q}\rangle) = \sum_{m \in \mathcal{X}} p(m|q) \Sigma^Q_{m|q} \otimes |m\rangle \langle m|_X, \]

(3)

where

\[ \Sigma^Q_{m|q} := \left( id_{\mathcal{R}} \otimes E^Q_m \right)(|\Psi_{\mathcal{R}Q}\rangle) \]

(4)

It is clear that \(\text{Tr}_\mathcal{R}[\Sigma^Q_{m|q}] = \sigma^Q_{m|q}\), for all \(m \in \mathcal{X}\). Moreover, by denoting \(\tau^Q_{m|q} := \text{Tr}_Q[\Sigma^Q_{m|q}]\), we have

\[ \sum_{m \in \mathcal{X}} p(m|q) \tau^Q_{m|q} = \text{Tr}_Q[|\Psi_{\mathcal{R}Q}\rangle \langle \Psi_{\mathcal{R}Q}|] =: q, \]

(5)

where \(\Psi_{\mathcal{R}Q}\) stands for the projector \(|\Psi_{\mathcal{R}Q}\rangle \langle \Psi_{\mathcal{R}Q}|\). It means that the action of the instrument \(\mathcal{M}_Q\) on \(Q\) induces the ensemble decomposition \(\{p(m|q), \tau^Q_{m|q}\}_{m \in \mathcal{X}}\) on \(\mathcal{R}\).

Even being a formally defined—hence non directly accessible—system, the reference \(\mathcal{R}\) will play a major role in our analysis. In order to better understand its meaning, it is suitable to think about \(\mathcal{R}\) as the remote system about which we actually want to extract information by measuring the system \(Q\), the latter representing in fact just the carrier undergoing the measurement. Indeed, consider for example the case in which the state \(|\Psi_{\mathcal{R}Q}\rangle\) is a purification of some state \(\rho^Q\). Then, all possible sources \(s = \{p(x), \rho^Q_x\}_x\) having average state \(\rho^Q\) equal to \(\rho^Q\), namely, \(\rho^Q = \sum_x p(x) \rho^Q_x\), are in one-to-one correspondence with positive-operator–valued measures (POVM) on \(\mathcal{R}\) through the duality relation

\[ p(x) \rho^Q_x =: \text{Tr}_\mathcal{R} \left([R^\mathcal{R}_x \otimes 1_Q] \Psi_{\mathcal{R}Q}\right). \]

(6)

In other words, when thinking of a classical-to-quantum communication scenario, the system \(Q\) represents the receiver, the source \(s\) models the quantum channel \(x \mapsto \rho^Q_x\) along which the classical information is sent, while the reference system \(\mathcal{R}\) can be understood as the sender’s system which retains the information about the encoded index \(x\).

3. Information gains and disturbances

3.1. Information gains: mutual information, accessible information, and quantum information gain

Let us suppose that we know exactly the source \(s\), in the sense that we know the probability distribution \(p(x)\) and the states \(\rho^Q_x\) which are associated with the letters \(x \in \mathcal{X}\). In this case, given the instrument \(\mathcal{M}_Q\), it is straightforward to compute
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the joint probability distribution $p(m, x) = \text{Tr}[\mathcal{E}_m(\rho_x)]$, and, from it, the correlations between the input letter and the measurement readout. We choose here to measure them as classical mutual information

$$I(X : X') := \sum_{x \in X} \sum_{m \in X} p(x, m) \log_2 \frac{p(x, m)}{p(x)p(m)},$$

where we implicitly used the Bayes’ rule $p(x, m) = p(x)p(m|x)$.

Imagine now that we ignore the exact structure of the source $s$, but its average state $\rho_s$. Even in this case we can define a mutual information–like quantity that describes how much information the instrument is able to extract: this quantity is the accessible information, when the measurement is fixed, and we maximize over all possible ensemble $s' = \{p'(x'), \rho'_x\}_{x' \in X'}$ with average state $\rho_{s'} = \rho_s$, that is

$$I_{\text{acc}}(\rho_s, M) := \max_{s' : \rho_{s'} = \rho_s} I(X' : X).$$

This quantity is well-known to be extremely hard to compute explicitly. It is then useful to introduce another quantity, which is much easier to calculate, yet equivalent to $I_{\text{acc}}$ for finite dimensional systems. In Ref. [4], we introduced the quantum information gain defined as the entropic defect [14] of the ensemble induced on the reference $R$ by the action of the instrument on $Q$

$$\iota(\rho_s, M) := S(\rho^R) - \sum_{m \in X} p(m|\rho_s) S(\tau^R_m|\rho_s),$$

where $S(\rho) := -\text{Tr}[\rho \log_2 \rho]$ is the von Neumann entropy, and other notations follows Eqs. (3)-(5). This quantity is what we are searching for, in that it is easy to compute and in Appendix A we prove that

$$I_{\text{acc}}(\rho_s, M) \leq \iota(\rho_s, M) \leq t \left( (2d - 1) \sqrt{2I_{\text{acc}}(\rho_s, M)} \right),$$

where $d := \dim Q$ and $t(x)$ is a positive, continuous, monotonic increasing function such that $t(0) = 0$. (The first inequality is the Holevo bound on accessible information [15].) In other words, for finite dimensional systems, $I_{\text{acc}}(\rho_s, M) \to 0$ if and only if $\iota(\rho_s, M) \to 0$. About this point, as we mentioned before, an important caveat is in order: the appearance of the dimension $d$ in Eq. (10) makes the equivalence relation quite weak as dimension increases. Such a factor is reminiscent of the phenomenon known as locking of classical correlations [27], and it is in general unavoidable [28].

As a concluding remark, notice that, even if the reference system $R$ explicitly appears in the definition (9), quantum information gain only depends on the average input state $\rho_s^Q$ and on the instrument $M^Q$, regardless of the particular purification $|\Psi_{RQ}\rangle$ chosen.

3.2. Disturbances: average output fidelity, entanglement fidelity, and quantum disturbance

Let us suppose that we know exactly the source $s$, in the sense that we know the probability distribution $p(x)$ and the states $\rho_x$ which are associated with the letters $x \in X$. Moreover, let us suppose that we know exactly the quantum instrument $M^Q$, in the sense that we know the outcome set $X$ and the set of associated CP-maps $\{\mathcal{E}_m\}_{m \in X}$. Having access to the measurement readout $m$, but not to the input letter

\| Via bounds that, being dimension-dependent, must be handled with care: see the following.
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A suitable definition for disturbance. The quantity to consider in this case is its average state $\bar{F}_{\text{av}}(s, \mathcal{M}) := \max_{\{\mathcal{R}_m\}} \sum_{x \in \mathcal{X}} p(x) F\left(\frac{\mathcal{R}_m^Q(\mathcal{E}_m^Q(\varrho_x^Q))}{p(m|x)}, \varrho_x^Q\right)$

for all $m \in \mathcal{X}$, where $F(\varrho, \sigma) := (\text{Tr} \sqrt{\sqrt{\varrho} \sqrt{\sigma}})^2$ is the fidelity. We however relax this condition by considering a situation where we are content to see that just the average value

$$F_{\text{av}}(\varrho, \mathcal{M}) := \sum_{x \in \mathcal{X}} p(x) F\left(\sum_{m \in \mathcal{X}} \mathcal{R}_m^Q(\mathcal{E}_m^Q(\varrho_x^Q)), \varrho_x^Q\right)$$

is close to one. This means that, whenever $F_{\text{av}} < 1$, there exists at least one state $\varrho_x$ which gets irrecoverably disturbed by the corrected channel $\sum_m \mathcal{R}_m \circ \mathcal{E}_m$. In turns, due to concavity of fidelity, this means that, for such state $\varrho_x$, there exists at least one outcome $m$ for which the corrected fidelity (11) is strictly less than one.

Suppose now that we instead ignore the exact structure of the source $s$, but know its average state $\bar{\varrho}_s$ only. In this case, the entanglement fidelity approach provides a suitable definition for disturbance. The quantity to consider in this case is

$$F_{\text{av}}(\bar{\varrho}_s, \mathcal{M}) := \max_{\{\mathcal{R}_m\}} F\left(\frac{(\text{id} \mathcal{R} \otimes \mathcal{R}_m^Q \circ \mathcal{E}_m^Q)(\varphi^Q)}{p(m|\bar{\varrho}_s)}, \varphi^Q\right)$$

for all possible outcomes $m \in \mathcal{X}$, where, as usual, $\varphi^Q$ is a purification of the average state $\bar{\varrho}_s^Q$. As before, we relax a bit the request, also to simplify the notation, and focus on the average quantity

$$F_{\text{av}}(\bar{\varrho}_s, \mathcal{M}) := \min_{\mathcal{R}_m} F\left(\sum_{m \in \mathcal{X}} (\text{id} \mathcal{R} \otimes \mathcal{R}_m^Q \circ \mathcal{E}_m^Q)(\varphi^Q), \varphi^Q\right)$$

Incidentally, notice that, in this setting, the restoring channels $\mathcal{R}_m^Q$ cannot depend on the ensemble $s$, but on its average state $\bar{\varrho}_s$ only. A value close to one for $F_{\text{av}}(\bar{\varrho}_s, \mathcal{M})$ means that, on the support of $\bar{\varrho}_s$, the channel $\sum_{m \in \mathcal{X}} \mathcal{R}_m^Q \circ \mathcal{E}_m^Q$ is close to the identity channel $\text{id}^Q$. This in turn implies that, not only states are transmitted with high output fidelity, but also entanglement can be reliably sent. Entanglement fidelity is known to provide a lower bound on the worst possible average output fidelity, in the sense that

$$F_{\text{av}}(\bar{\varrho}_s, \mathcal{M}) \leq \min_{\varrho' : \psi' | \varrho' = \bar{\varrho}_s} F_{\text{av}}(\varrho', \mathcal{M}).$$

This is the reason why the entanglement fidelity criterion is generally much more severe than the average output fidelity criterion. The example provided in the introduction exhibits the extreme case in which the two criteria are strictly inequivalent. Notice that a value $F_{\text{av}} < 1$ means that there is at least one outcome $m$ for which the corresponding state change map is irreversible on the support of $\varrho_x$.

The quantum disturbance, defined in Ref. [4] as

$$\delta(\varrho_s, \mathcal{M}) := S(\varrho_s^Q) - F_{\text{av}}(\varphi^Q, \mathcal{M}) \left(\text{id} \mathcal{R} \otimes \mathcal{M}(\varphi^Q)\right),$$

where

$$x$$ being unknown to the receiver, the best (deterministic) correction we could try in order to reduce state disturbance, is in principle to engineer, depending on the source and on the instrument, a set of correcting channels $\{\mathcal{R}_m^Q\}_{m \in \mathcal{X}}$, mapping $\mathcal{Q}'$ back to $\mathcal{Q}$, and to apply them onto the output state according to the measurement readout $m$. The maximum output fidelity approach aims at finding channels $\mathcal{R}_m^Q$ achieving the quantity

$$\max_{\{\mathcal{R}_m\}} \sum_x p(x) F\left(\frac{\mathcal{R}_m^Q(\mathcal{E}_m^Q(\varrho_x^Q))}{p(m|x)}, \varrho_x^Q\right),$$

for all $m \in \mathcal{X}$.
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where $I^\mathcal{A}_\mathcal{B}^{\mathcal{B}}(\varrho_{\mathcal{A}\mathcal{B}}) := S(\varrho_{\mathcal{B}}^\mathcal{B}) - S(\varrho_{\mathcal{A}\mathcal{B}}^{\mathcal{A}\mathcal{B}})$ is the coherent information from subsystem $\mathcal{A}$ to subsystem $\mathcal{B}$ for the bipartite state $\varrho_{\mathcal{A}\mathcal{B}}$, is a quantity bounded between zero and $S(\varrho_{\mathcal{A}}^\mathcal{A})$ and it is equivalent to the entanglement fidelity \cite{11}, in the sense that

$$\frac{1}{4} \left( 1 - F_e(\varrho_s, \mathcal{M}) \right)^2 \leq \delta(\varrho_s, \mathcal{M}) \leq g(1 - F_e(\varrho_s, \mathcal{M})), \quad (17)$$

where $g(x)$ is some positive, continuous, monotonic increasing function such that $g(0) = 0$. Moreover, the tradeoff relation is always obeyed \cite{11}

$$\nu(\varrho_s, \mathcal{M}) \leq \delta(\varrho_s, \mathcal{M}), \quad (18)$$

where $\nu(\varrho_s, \mathcal{M})$ is defined in \cite{18}. The equality in (18) is achieved by “single-Kraus” or “multiplicity free” instruments, that are those for which every map $\mathcal{E}_m$ defining the instrument $\mathcal{M}$ is represented by a single contraction, that is, $\mathcal{E}_m(\varrho) = E_m \varrho E_m^\dagger$. In this sense, single-Kraus instruments can be considered as least-disturbing measurements.

At this point, the question remains open whether or not quantum disturbance $\delta(\varrho_s, \mathcal{M})$ is equivalent to average output fidelity $F_{av}(s, \mathcal{M})$ as well, and, if it is, to which extent—in fact, they cannot be always equivalent, as the von Neumann measurement example, for which $F_{av}(s, \mathcal{M}) = 1$ while $\delta(\varrho_s, \mathcal{M}) = S(\varrho_s)$, proves. To answer this question will be the aim of the next section.

4. Unifying disturbances for irreducible ensembles of pure states

Let us suppose now that the source $s$ produces pure states, that is, $\varrho_x^s = \psi_x^s$, for all $x \in \mathcal{X}$. Now, let us consider all possible $N$-complete paths, namely, all possible $N$-sequences $x^N = (x_1, x_2, \ldots, x_N) \in \mathcal{X}^N$ such that the corresponding set of states

$$\{ |\psi_{x_1}^s \rangle, |\psi_{x_2}^s \rangle, \ldots, |\psi_{x_N}^s \rangle \} \quad (19)$$

contains, at least once, every state $|\psi_x^s \rangle$ emitted by the source. Let us denote the set of all possible $N$-complete path for a given source $s$ as $\mathcal{P}_s^N$, and let $\mathcal{P}_s = \bigcup_N \mathcal{P}_s^N$ be the set of all possible complete paths. Let us moreover define the following function of a given ensemble $s$

$$\eta(s) := \frac{\max_{x^N \in \mathcal{P}_s} \min_{1 \leq i \leq N-1} \langle \psi_{x_i}^s | \psi_{x_{i+1}}^s \rangle}{N}, \quad (20)$$

where the maximum is taken over all possible complete paths. Following Ref. [3], we say that the source $s$ is irreducible if and only if $\eta(s) > 0$, that is equivalent to say that the states $|\psi_x^s \rangle$ cannot be divided into two or more orthogonal subsets. The notion of irreducibility makes rigorous the notion of “non-orthogonal ensembles” used in the introduction: in fact, within a given ensemble of pure states, some of them could well be orthogonal, yet the ensemble being irreducible. Notice moreover that, implicit in Eq. (20) there is another tradeoff: we can make the numerator larger by considering “more complicated” paths, but, eventually, also the denominator gets larger. Hence, to find the optimum value of $\eta(s)$ is a tricky procedure. However, for sources emitting a finite number of states, the solution (maybe corresponding to non-unique choices of optimal complete paths) always exists.

The quantity $\eta(s)$ is a good indicator of the degree of irreducibility, or quantumness, of the ensemble $s$. There is however a hidden subtlety: till now we did

\footnote{We could relax this condition by asking valid complete paths to form spanning sets for $\text{Supp}(\varrho_s^s)$. This technicality, however, even being useful in some cases, would force our analysis to be notationally heavy. We hence prefer to loose a bit in the efficiency of our bounds while gaining in clarity.}
not consider the role of the probability distribution \( p(x) \) of the source yet. Imagine, for example, that a source is irreducible, in the sense that \( \eta(s) > 0 \), but only because of one state \(|\psi_0\rangle\) being non-orthogonal with any other, whose probability of occurrence \( p(0) \) is however very small. Such a source is then irreducible, but only “weakly” irreducible: we should take this possibility into account. We then define the function

\[
\zeta(s) := \eta(s) \cdot \min_x p(x).
\]

Again, since by definition \( \min_x p(x) > 0 \), the source of pure states \( s \) is irreducible if and only if \( \zeta(s) > 0 \), but now, contrarily to what happens for \( \eta(s) \), we are automatically weighing the role of the probability distribution \( p(x) \). In the following theorem we see how \( \zeta(s) \) enters in our discussion:

**Theorem 1** Let \( s = \{p(x), |\psi_x\rangle\}_{x \in X} \) be an ensemble of pure states, with average state \( \varrho_s \), undergoing the quantum measurement described by the instrument \( \mathcal{M}^{Q} \). Then

\[
\frac{1}{4} \left(1 - F_{av}(s, \mathcal{M})\right)^2 \leq \frac{1}{4} \left(1 - F_c(\varrho_s, \mathcal{M})\right)^2 \leq \delta(\varrho_s, \mathcal{M}) \leq f\left(\frac{\sqrt{1 - F_{av}(s, \mathcal{M})}}{\zeta(s)}\right),
\]

where \( f(x) \) is some positive, continuous, monotonic increasing function such that \( f(0) = 0 \).

In other words, whenever the ensemble \( s \) is irreducible, namely, \( \zeta(s) > 0 \), the notions of disturbance originating from average output fidelity, from entanglement fidelity, and from quantum disturbance are all equivalent, in the sense that \( F_{av}(s, \mathcal{M}) \rightarrow 1 \Leftrightarrow F_c(\varrho_s, \mathcal{M}) \rightarrow 1 \Leftrightarrow \delta(\varrho_s, \mathcal{M}) \rightarrow 0 \), and, equivalently, \( F_{av}(s, \mathcal{M}) < 1 \Leftrightarrow F_c(\varrho_s, \mathcal{M}) < 1 \Leftrightarrow \delta(\varrho_s, \mathcal{M}) > 0 \). On the other hand, as \( \zeta(s) \rightarrow 0 \) (a situation happening when \( \eta(s) \rightarrow 0 \) or \( \min_x p(x) \rightarrow 0 \)), while a high entanglement fidelity always implies a high average output fidelity, the converse direction becomes weaker and weaker. The example we mentioned in the introduction (involving orthogonal states and von Neumann measurement) where the two disturbances are irreconcilably inequivalent, even looking somehow over-simplified, actually captures all the essential features of those situations where average output fidelity and entanglement fidelity are really inequivalent quantities.

**Proof of Theorem 1** Inequalities \( \text{(22)} \) and \( \text{(23)} \) follow directly from Eqs. \( \text{(15)} \) and \( \text{(17)} \). The last inequality \( \text{(24)} \) is a consequence of the result proved in the Appendix B, which in turn follows the proof of Lemma 3 in Ref. \[8\]. In fact, the quantum disturbance \( \delta(\varrho_s, \mathcal{M}) \) is defined as the coherent information loss of the channel \( \mathcal{M}^{Q} \), as it is defined in Eq. \( \text{(2)} \), mapping the input system \( Q \) in the output quantum-classical system \( Q', X \). Moreover, whatever channel \( \mathcal{R}^{Q', X} \), applied after \( \mathcal{M}^{Q} \) in order to map the hybrid output system \( Q', X \) back to the input system \( Q \), returns in fact a set of channels \( \{\mathcal{R}_m^{Q}\}_{m \in X} \), each of them applied conditionally on the readout

\[+\] As noted in a previous footnote, by allowing a generalized notion of complete paths, we could also get rid here of signals—states with their corresponding probabilities—which are not necessary to span the subspace \( \text{Supp}(\varrho_s) \), and possibly obtain an effective parameter \( \zeta(s) \) which is larger than the one defined in Eq. \( \text{(21)} \). For the sake of clarity, however, we will leave this improvement as an exercise for the interested reader.
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m, all of them mapping \( Q' \) back to \( Q \). This is indeed the way in which we defined \( F_{av}(s, M) \) in Eq. (14). Therefore, we can straightforwardly apply the statement of Proposition 1 in Appendix B and get Eq. (24).

5. Entropy-based notion of average disturbance

The main content of Theorem 1 is to make explicit the relation existing between disturbances defined in terms of average output fidelity and entanglement fidelity. Quantum disturbance appears there because of Eq. (17), which states that quantum disturbance is essentially equivalent to entanglement-fidelity–based disturbance. It is now reasonable to ask whether one can devise an entropy-based analog also for average-output-fidelity–based disturbance, in the hope that the relation existing between such a new measure of disturbance and quantum disturbance can be expressed in a form simpler than Eqs. (22)-(24).

Let us consider an input ensemble \( s := \{p(x), \rho_x \}_{x \in X} \) undergoing a channel \( E \). It is known that its entropy defect \( \chi(s) := S(\rho_s) - \sum_x p(x)S(\rho_x) \) behaves monotonically under the action of a channel, that is, the entropy defect loss \( \Delta \chi(s, E) := \chi(s) - \chi(E(s)) \) (25) is always nonnegative. Moreover (see Ref. [31] and Lemma 2 in Appendix B) it is known that the condition \( F_{av}(s, E) \to 1 \) implies that \( \Delta \chi(s, E) \to 0 \) correspondingly. On the other hand, the opposite direction is known to hold only in the exact case, that is, \( \Delta \chi(s, E) = 0 \) implies that \( F_{av}(s, E) = 1 \) (see for example [32]). In other words, at the moment we cannot state that entropy defect loss and corrected average output fidelity are truly equivalent measures of disturbance, since we do not know whether a small entropy defect loss implies a correspondingly small average fidelity loss.

Anyway, let us postulate for the moment that entropy defect loss \( \Delta \chi(s, E) = 0 \) is the sought informational analog of average output fidelity, much like quantum disturbance represents the informational analog of entanglement fidelity. In the case of a quantum measurement process, when we consider the “channelized” action of the instrument \( M \) as given in Eq. (2), it is straightforward to compute

\[
\chi(M(s)) = I(X : X) + \sum_m p(m)\chi(\bar{\sigma}_m),
\]

where \( I(X : X) \) represents the (classical) mutual information between the input letter and the measurement outcome, as defined in Eq. (7), and

\[
\bar{\sigma}_m := \left\{ p(x|m), \frac{E_m(\rho_x)}{p(m|x)} \right\}_{x \in X}
\]

is the ensemble output by the measuring apparatus, given the outcome readout \( m \). Notice that, for every conditional output ensemble \( \bar{\sigma}_m \), the average state \( \sigma_{\bar{\sigma}_m} \) is

\[
\sigma_{\bar{\sigma}_m} := \sum_{x \in X} p(x|m)\frac{E_m(\rho_x)}{p(m|x)} = \frac{E_m(\rho_x)}{p(m)},
\]

according to Bayes’ rule. Then, the entropy defect loss is equal to

\[
\Delta \chi(s, M) := \chi(s) - \chi(M(s)) = \sum_m p(m) [\chi(s) - \chi(\bar{\sigma}_m)] - I(X : X).
\]
Let us now focus on a very special case of input ensemble (introduced in Ref. [33] and generalized in Ref. [34]). It is possible to define such an ensemble for any given ensemble average state $\rho_s$ as follows: given the average state $\rho_s$ on system $Q$, after having purified it as $|\Psi_{RQ}\rangle$ by means of a reference system $R$ isomorphic to $Q$, one defines the pure states $|\psi_x^Q\rangle$ along with their occurrence probabilities $p(x)$ as

$$p(x)\psi_x^Q := \text{Tr}_R[\phi_x^R \otimes 1^Q \Psi_{RQ}].$$

In the above equation, the vectors $\{\phi_x^R\}_{x \in X}$ constitute a rank-one POVM on $R$ with $2d$ elements, such that $\text{Tr}[\phi_x] = 1/2d$ for all $x$, and defined as follows: $d$ of them form a basis diagonalizing $\text{Tr}_Q[\Psi_{RQ}]$, while the other $d$ vectors form another basis, mutually unbiased with respect to the first one.

For this particular input ensemble of pure states, that we call of Christandl-Winter type, it is known that $\Delta\chi(s, M) \leq \delta(\rho_s, M) \leq 2\Delta\chi(s, M)$, that is, entropy defect loss and quantum disturbance are equivalent quantities, and their relation is strikingly more stringent than the relation existing between average output fidelity and entanglement fidelity as given in Theorem 1. However, as we stated in the introduction, the extension of this approach to the more general case where the input ensemble does not come from such a highly symmetric construction seems to be a hard path to pursue, yet worth further investigations.

6. Conclusion and discussion

With Theorem 1 we showed that, whenever the input source is sufficiently rich in its structure—i.e. it is irreducible—the vast majority of definitions of disturbance, introduced in the literature so far, actually turn out to be equivalent. In particular, we focused on the relations existing between average output fidelity and entanglement fidelity, without imposing any symmetry on the source distribution. Our analysis could hence unify different approaches to the quantification of information-disturbance tradeoffs in quantum theory. The general bounds obtained here exploit Fannes-like bounds on entropic quantities, and, for this reason, relate extensive quantities with non-extensive ones. In the very particular case of input ensembles of the Christandl-Winter type, we showed a way to circumvent this difficulty and obtain much more stringent bounds. A future research direction is to extend this latter approach to more general situations.

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Appendix A. Accessible information and quantum information gain are equivalent criteria to measure information gain

We exploit here a method introduced in Ref. [25]. Let us consider an ensemble $s = \{p(x), \rho_x\}_{x \in X}$. The accessible information is defined as the maximum over all
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Possible POVM’s of the mutual information\(^\text{[9]}\)

\[ I_{\text{acc}}(s) := \max_{\{P_m\}_{m \in X}} I(X : X). \]  

(A.1)

Now, let \( \chi(s) \) be the entropy defect of the ensemble \( s \) \(^{[14]}\)

\[ \chi(s) := S(\rho_s) - \sum_x p(x) S(\rho_x), \]  

(A.2)

where \( S(\rho) := -\text{Tr}[\rho \log_2 \rho] \) is the von Neumann entropy. Holevo bound states that \(^{[15]}\)

\[ I_{\text{acc}}(s) \leq \chi(s). \]  

(A.3)

By rearranging terms in Eq. (A.2), we can write

\[ \chi(s) = \sum_x p(x)(S(\rho_s) - S(\rho_x)). \]  

(A.4)

By Fannes inequality \(^{[23]}\), we know that

\[ \chi(s) = \sum_x p(x) (S(\rho_s) - S(\rho_x)) \]
\[ \leq \sum_x p(x)(S(\rho_s) - S(\rho_x)) \]
\[ \leq \sum_x p(x) t(|\rho_x - \rho_s|_1) \]
\[ \leq t\left( \sum_x p(x) |\rho_x - \rho_s|_1 \right). \]  

(A.5)

where \( t(x) \) is a positive, continuous, monotonic increasing, concave function such that \( t(0) = 0 \) and \( |X|_1 := \text{Tr}|X| \) is the trace-norm. In particular, for \( x \leq 1 \), we can take

\[ t(x) = x \log_2 \frac{2\sqrt{d} - 1}{x}, \]  

(A.6)

where \( d \) is the dimension of the underlying space \(^{[30]}\).

Let us now make a small detour, and introduce so-called informationally complete POVM’s, namely, those POVM’s whose elements also form an operator basis. This means that, being \( \{P_m\}_m \) an info-complete POVM, there exists a dual frame \( \{K_m\}_m \), with Hermitian operators \( K_m \), such that

\[ X = \sum_m \text{Tr}[XP_m]K_m, \]  

(A.7)

for all operators \( X \). We consider here a particular info-complete POVM, that is, \( \{P_g\}_{g \in G} \) defined as

\[ P_g := \frac{1}{d} U_g \phi U_g^\dagger, \]  

(A.8)

where \( U_g \) is a unitary representation of the group \( \text{SU}(d) \), and \( \phi \) is a \( d \)-dimensional pure state. In Ref. \(^{[29]}\) the canonical dual frame has been explicitly calculated, and it holds that

\[ |K_g|_1 = 2d - 1, \quad \forall g. \]  

(A.9)

* In fact, in the text we came to accessible information from the dual point of view, where the measurement (i.e., the POVM) is fixed and the maximization is done with respect to the input ensemble; thanks to the POVM/ensembles duality relation \(^{[4]}\) the two approaches are equivalent.
Let us go back to our original aim, i.e., to give an upper bound to $\chi(s)$, and continue from Eq. (A.5)

$$\chi(s) \leq t \left( \sum_x p(x) \left\| \varrho_x - \varrho_s \right\|_1 \right)$$

$$= t \left( \sum_x p(x) \left\| \sum_g p(g|x) K_g - p(g) K_g \right\|_1 \right)$$

$$\leq t \left( (2d-1) \sum_x p(x) \sum_g \left\| p(g|x) - p(g) \right\|_1 \right)$$

$$\leq t \left( (2d-1) \sqrt{2I(X : G)} \right)$$

$$\leq t \left( (2d-1) \sqrt{2I_{\text{acc}}(s)} \right).$$

(A.10)

(A.11)

(A.12)

Eq. (A.10) uses the expansion formula (A.7) specialized to the info-complete POVM (A.8), while to obtain Eq. (A.11) we used Pinsker inequality

$$\left\| \varrho - \sigma \right\|_1^2 \leq 2D(\varrho \| \sigma),$$

(A.13)

where $D(\varrho \| \sigma) := \text{Tr} \left[ \varrho \log_2 \varrho - \varrho \log_2 \sigma \right]$ is the quantum relative entropy. Eq. (A.12) comes from the very definition of accessible information, which is defined as the maximum over all possible POVM’s.

In conclusion, we have that

$$I_{\text{acc}}(s) \leq \chi(s) \leq t \left( (2d-1) \sqrt{2I_{\text{acc}}(s)} \right),$$

(A.14)

that means that, for finite dimensional systems, $I_{\text{acc}}(s) \rightarrow 0$ if and only if $\chi(s) \rightarrow 0$. In other words, accessible information and entropy defect are equivalent quantities.

Notice that in passing by we actually proved that

$$\sum_x p(x) \left\| \varrho_x - \varrho_s \right\|_1 \leq (2d-1) \sqrt{2I_{\text{acc}}(s)}.$$

(A.15)

Such a bound is a little better than the one proved, with different techniques, in Ref. [27].

Appendix B. Average output fidelity and entanglement fidelity are equivalent criteria to measure disturbance for irreducible pure state ensembles

In the following, let us consider a channel, that is, a completely positive trace-preserving map $\mathcal{E} : \mathcal{Q} \rightarrow \mathcal{Q}'$, and a state $\varrho$ which undergoes $\mathcal{E}$. As it is often done, we proceed here with the tripartite purification of the whole setting. The first step is to purify the input state $\varrho^{\mathcal{Q}}$ into a bipartite pure state $\ket{\Psi^{\mathcal{R}\mathcal{Q}}}$, shared with a reference system $\mathcal{R}$, as we did in Section 2. The second step consists in constructing the isometry $V : \mathcal{Q} \rightarrow \mathcal{Q}' \otimes \mathcal{A}$ (the isometric condition reads $V^\dagger V = 1^{\mathcal{Q}}$) such that

$$\mathcal{E}(\varrho) = \text{Tr}_{\mathcal{A}} [V \varrho V^\dagger],$$

(B.1)

for all input state $\varrho$. The auxiliary system $\mathcal{A}$ is sometimes called ancilla (or environment) and the construction in Eq. (B.1) is usually referred to as the
Stinespring’s dilation of $E$. Moreover, Stinespring’s dilation induces an ancillary channel $\tilde{E} : Q \rightarrow A$ defined as
\[
\tilde{E}(\varrho) := \text{Tr}_Q[V\varrho V^\dagger],
\]for all $\varrho$. The channel $\tilde{E}$ is defined up to local isometric transformations on $A$. By combining these two “purifications”, we can write the tripartite pure state
\[
|\Upsilon_{RQ}^{'A}\rangle := (I_R \otimes V_Q)|\Psi_{RQ}\rangle,
\]
such that $\text{Tr}_{R,A}[|\Upsilon_{RQ}^{'A}\rangle] = E(\varrho)$.

For our later convenience, we now introduce two useful notation. Given an ensemble $s := \{p(x), \varrho_x\}_x$, we denote with $\chi(s)$ its entropy defect, in formula
\[
\chi(s) := S(\varrho_s) - \sum_x p(x)S(\varrho_x),
\]
where $S(\varrho) := -\text{Tr}[\varrho \log_2 \varrho]$ is the von Neumann entropy. By the monotonicity property of quantum relative entropy, after the action of a channel, the entropy defect drops, and we denote the difference between input $\chi$ and output $\chi$,
\[
\Delta \chi(s,E) := \chi(s) - \chi(\tilde{E}(s)).
\]
We also define the coherent information loss $\delta(\varrho, E)$, due to the action of a channel $E : Q \rightarrow Q'$ onto a state $\varrho^Q$ which is a subsystem of a larger entangled pure state $|\Psi_{RQ}\rangle$ as
\[
\delta(\varrho, E) := S(\varrho^Q) - I_{C}^{R \rightarrow Q'}(\text{id}_R \otimes E^{Q})(|\Psi_{RQ}\rangle),
\]
where $I_{C}^{A \rightarrow B}(\varrho^{AB}) := S(\varrho^B) - S(\varrho^{AB})$ is the coherent information from subsystem $A$ to subsystem $B$ for the bipartite state $\varrho^{AB}$. We can now state the following

**Lemma 1** Let $E : Q \rightarrow Q'$ be a channel. Let $s := \{p(x), |\psi_x\rangle\}_x$ be an input ensemble of pure states. Then,
\[
\delta(\varrho_s, E) = \Delta \chi(s, E) + \chi(\tilde{E}(s)),
\]
that is, coherent information loss equals the sum of input ensemble entropy defect loss plus the entropy defect of the induced ancillary ensemble. 

**Proof.** The proof simply follows by direct inspection, as done in Ref. [29]. Notice that the condition of having pure input states is crucial. ■

Now, analogously to Eq. (12), let us define the corrected average output fidelity for a channel $E : Q \rightarrow Q'$ (there are no classical outcome here, only quantum, as the former is a special case of the latter) when the input ensemble is $s := \{p(x), \varrho_x\}_{x \in X}$ as
\[
\overline{F}_{av}(s, E) := \max_{R} \sum_{x \in X} p(x)F(R(E(\varrho_x)), \varrho_x),
\]
where the maximum is taken over all possible channels $R$ mapping $Q'$ back to $Q$.

**Lemma 2** We have that
\[
\Delta \chi(s, E) \leq f_1(1 - \overline{F}_{av}(s, E)),
\]
where $f_1(x)$ is a positive, continuous, monotonic increasing, concave function such that $f_1(0) = 0$. ■
Proof. Following Ref. [31], given two sources of $K$ states $s := \{p(x), \rho_x\}_x$ and $s' := \{p(x), \rho_x'\}_x$, we can apply the refined Fannes’ continuity relation for von Neumann entropy [30] and get
\begin{equation}
|\chi(s) - \chi(s')| \leq 2K\sqrt{\epsilon} \log_2 \frac{d\rho}{\epsilon},
\end{equation}
where $\epsilon := 1 - \sum_x p(x) F(\rho_x, \rho_x')$, for the fidelity defined as $F(\rho, \sigma) := (\text{Tr} \sqrt{\rho \sigma})^2$.

The statement is simply recovered by specializing such a relation to the ensembles $s$ and $R(E(s))$, with $f_1(x) := 2K\sqrt{x} \log_2 (d\rho / x)$, for sufficiently small $x$ (typically $x \leq 2/e^2$).

Lemma 3 We have that
\begin{equation}
\chi(\tilde{E}(s)) \leq f_2 \left( \frac{\sqrt{1-F_{av}(s, E)}}{\zeta(s)} \right),
\end{equation}
where $\zeta(s)$, for a given ensemble $s$, is defined in Eq. (21), and $f_2(x)$ is a positive, continuous, monotonic increasing, concave function such that $f_2(0) = 0$.

Proof. We carefully follow here the proof of Lemma 3 given in Ref. [8]. Noticing that we can always take $dA \leq d^2$, and denoting $\epsilon := 1 - F_{av}(s, E)$ and $\epsilon' := \sqrt{\epsilon / \zeta(s)} \geq \epsilon$, provided that $\epsilon' \leq 1$ we have that
\begin{equation}
\chi(\tilde{E}(s)) \leq 4N\sqrt{\epsilon} \log_2 \frac{d\rho}{\epsilon'},
\end{equation}
where $N \geq K$ is the number of states in the optimal path in Eq. (21), which is always greater of equal than the number of states $K$ emitted by the source.

Finally we obtained the following

Proposition 1 Let $E : \mathcal{Q} \rightarrow \mathcal{Q}'$ be a channel. Let $s := \{p(x), |\psi_x\rangle\}_x$ be an input ensemble of pure states. Then, provided that $F_{av}(s, E) \geq 1 - (\zeta(s))^2$ and that $F_{av}(s, E)$ is sufficiently close to one, we have that
\begin{equation}
\delta(\rho_s, E) \leq f \left( \frac{\sqrt{1-F_{av}(s, E)}}{\zeta(s)} \right),
\end{equation}
where $f(x)$ is a positive, continuous, monotonic increasing, concave function such that $f(0) = 0$. In other words, whenever the ensemble $s$ is irreducible, that means $\zeta(s) > 0$, the condition $F_{av}(s, E) \rightarrow 1$ implies that $\delta(\rho_s, E) \rightarrow 0$.

Proof. Simply by combining Eqs. (B.7), (B.10), and (B.12), and noticing that $\epsilon' \geq \epsilon$ and $N \geq K$ we see that
\begin{equation}
\delta(\rho_s, E) \leq 6N\sqrt{\epsilon} \log_2 \frac{d\rho}{\epsilon'},
\end{equation}
for
\begin{equation}
\epsilon' := \frac{\sqrt{1-F_{av}(s, E)}}{\zeta(s)},
\end{equation}
which is the statement.
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