WAVE MECHANICS AND GENERAL RELATIVITY: A RAPPROCHEMENT

Paul S. Wesson

Department of Physics, University of Waterloo,

Waterloo, Ontario N2L 3G1, Canada

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Abstract:

Using exact solutions, we show that it is in principle possible to regard waves and particles as representations of the same underlying geometry, thereby resolving the problem of wave-particle duality.

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Correspondence: Mail to address above, fax=(519)746-8115
1 Introduction

Wave-particle duality is commonly presented as a conceptual conflict between quantum and classical mechanics. The archetypal example is the double-slit experiment, where electrons as discrete particles pass through a pair of apertures and show wave-like interference patterns. However, particles and waves can both be given geometrical descriptions, which raises the possibility that these behaviours are merely different representations of the same underlying geometry. We will give a brief discussion involving exact solutions of extended geometry, to show that particles and waves may be the same thing viewed in different ways.

Certain technical results will be needed below. (Those readers more interested in results than method may like to proceed to section 2.) The basic idea is that waves and particles are different coordinate representations of the same geometry, or isometries [1-5]. Even in special relativity, which frequently uses as a basis four-dimensional Minkowski space \((M_{4})\), we can if we so wish change the form of the metric by a change of coordinates (or gauge). Thus, \(M_{4}\) is actually isometric to the Milne universe, which is often presented as a Friedmann-Robertson-Walker (FRW) model with negative 3D or spatial curvature in general relativity [4]. While the metrics may look
different, their equivalence is shown by the fact that in both cases the density and pressure of matter are zero as determined by the field equations. The latter in 4D read $R_{\alpha\beta} - R g_{\alpha\beta}/2 + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}$ (\(\alpha, \beta = 0, 1, 2, 3\) for time and space, where the speed of light and the constant of gravity have been set to unity). Here $R_{\alpha\beta}$ is the Ricci tensor, $R$ is the Ricci scalar, $g_{\alpha\beta}$ is the metric tensor, $\Lambda$ is the cosmological constant and $T_{\alpha\beta}$ is the energy-momentum tensor. While certain wave-like solutions of the latter equations are known [3], none has the properties of the deBroglie waves which are commonly used to describe the energy ($E$) and spatial momenta $p_{123}$ of particles in wave mechanics. Symbolically, these have wavelengths $\lambda^0 = h/E$, $\lambda^1 = h/p_1$ etc., where $h$ is Planck’s constant (which may also be set to unity). However, solutions of the field equations are known with deBroglie-like waves in dimensionally-extended gravity [5-7]. The latter is fundamentally Einstein’s theory of general relativity, extended to $N (> 4) D$, in order to unify gravity with the interactions of particle physics. The basic extension is to $N = 5$, where Campbell’s theorem ensures that any solution of the 5D field equations in vacuum is also a solution of the 4D field equations with matter [1, 8]. That is, we can always recover a solution of the 4D equations noted above from the 5D equations, which in terms of the extended Ricci tensor
are just $R_{AB} = 0 (A, B = 0, 123, 4)$. There are many exact solutions known of these equations, whereby the extended version is known to agree with observations, both in regard to the solar system [5, 9] and cosmology [5, 10]. Several are relevant to the present project [11-14]. For example, the Billyard solution [14] has a metric coefficient for the 3D or spatial part which naturally represents the 3D or momentum component of a deBroglie wave [15]. It is a remarkable solution, in that it is not only Ricci-flat ($R_{AB} = 0$) but also Riemann-flat ($R_{ABCD} = 0$). That is, it represents a flat 5D space, which by virtue of Campbell’s theorem satisfies Einstein’s 4D equations, and has a 3D deBroglie wave. However, it is deficient in some respects as regards the present project, notably in that it has a signature $(+----+)$ which is at variance with the one $(+---)$ indicated by particle physics. The latter subject is constrained by Lorentz invariance and experiments related to this.

There is a relation between the energy $E$, 3-momentum $p$ and rest mass $m$ of a particle, which is regarded as standard because it is closely obeyed in experiments (see ref. 16 for a review). Namely,

$$E^2 = p^2 + m^2$$

(1)

This is a strong constraint on any attempt to construct a geometric relation between the particle and wave descriptions of matter. From the viewpoint
of a theory like general relativity, (1) is perhaps not surprising, in that it
can be understood as a consequence of multiplying a constant $m$ onto the
conventional condition for normalizing the 4-velocities, viz $u^\alpha u_\alpha = 1$. (Here,
$u^\alpha \equiv dx^\alpha/ds$ where the 4D coordinates $x^\alpha$ are related to the proper time $s$ and the metric tensor $g_{\alpha\beta}$ via $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$.) From the viewpoint of a dimensionally-extended theory, (1) is also not so surprising, in that it follows for a wide range of metrics. The latter involve two aspects. First, the coordinates should be “canonical” in 5D, which means that the interval can be written as $dS^2 = (l/L)^2 ds^2 - dl^2$ where $x^4 \equiv l$, so that the extra coordinate plays the role of particle mass and the Weak Equivalence Principle is obeyed [17, 18]. Second, the paths of particles (or waves) should be null, so that a photon-like object in 5D appears as a massive object in 4D [19-21]. This latter condition enables us to cast our project into a new form: we are asking if there is a photon-like solution in 5D, which in 4D can be interpreted as being a massive particle or equivalently a deBroglie wave.

The technical results noted in the preceding paragraph may appear to be very restrictive as regards a possible resolution of the apparent dichotomy between wave mechanics and classical mechanics. However, wave-particle duality is a generic feature of matter, as shown by experiments more so-
phisticated than the old double-slit one for electrons, such as studies of the interference in a gravitational field of neutrons [22, 23]. It is therefore to be expected that if there is a geometric explanation in terms of isometries, that it also will be generic in some sense. This will turn out to be the case. Thus while there are numerous coordinate frames which are useful for our studies, it transpires that they share the property of describing a 5D manifold that is flat [24]. We will present two exact solutions which represent deBroglie waves but share this property in section 2. The inference is that particles and waves in 4D are isometries of flat 5D space.

2 Exact Wave-Particle Solutions

In this section, we will consider flat manifolds of various dimensionalities, with a view to showing that a 4D deBroglie wave which describes energy and momentum is isometric to a flat 5D space. The notation is the same as that introduced above, and standard.

2D manifolds, like that which describes the surface of the Earth, are locally flat. A brief but instructive account of their isometries is given by Rindler (1977, p.114; a manifold of any N is approximately flat in a small
enough region, and changes of coordinates that qualify as isometries should strictly speaking preserve the signature.) Consider, as an example, the line element $ds^2 = dt^2 - t^2 dx^2$. Then the coordinate transformation $t \rightarrow e^{i\omega t} / i\omega$, $x \rightarrow e^{i\kappa x}$ causes the metric to read $ds^2 = e^{2i\omega t} dt^2 - e^{2i(\omega t + \kappa x)} dx^2$, where $\omega$ is a frequency, $\kappa$ is a wave-number and the phase velocity $\omega / \kappa$ has been set to unity. It is clear from this toy example that a metric which describes a freely-moving particle (the proper distance is proportional to the time) is equivalent to one which describes a freely-propogating wave. For the particle, we can define its energy and momentum via $E \equiv m \left( dt / ds \right)$ and $p \equiv m \left( dx / ds \right)$. For the wave, $\tilde{E} \equiv m e^{i\omega t} \left( dt / ds \right)$ and $\tilde{p} \equiv m e^{i(\omega t + \kappa x)} \left( dx / ds \right)$. In both cases, the mass $m$ of a test particle has to be introduced ad hoc, a shortcoming which will be addressed below. The standard energy condition (1), in the form $m^2 = E^2 - p^2$, is recovered if the signature is $(+-)$. If on the other hand we have a Euclidean signature of the kind used in certain approaches to quantum gravity, it is instructive in the 2D case to consider the isometry $ds^2 = x^2 dt^2 + t^2 dx^2$. The transformation $t \rightarrow e^{i\omega t} / i\omega$, $x \rightarrow e^{i\kappa x} / i\kappa$ causes this to read $ds^2 = - \left( 1 / \kappa \right)^2 e^{2i(\omega t + \kappa x)} \left( dt^2 + dx^2 \right)$, after the absorption of a phase velocity as above. Thus a particle metric becomes one with a conformal factor which resembles a wave function.
3D manifolds add little to what has been discussed above. It is well known that in this case the Ricci and Riemann-Christoffel tensors can be written as functions of each other, so the field equations bring us automatically to a flat manifold as before.

4D manifolds which are isotropic and homogeneous, but non-static, lead us to consider the FRW metrics. These have line elements given by

$$ds^2 = dt^2 - \frac{R^2(t)}{(1 + k r^2/4)^2} (dx^2 + dy^2 + dz^2), \quad (2)$$

where $R(t)$ is the scale factor and $k = \pm 1$, 0 defines the 3D curvature. (This should not be confused with the wave number.) In the ideal case where the density and pressure of matter are zero, a test particle moves away from a local origin with a proper distance proportional to the time. (I.e., $R = t$ above where the spatial coordinates $xyz$ and $r \equiv \sqrt{x^2 + y^2 + z^2}$ are comoving and dimensionless.) This specifies the Milne model, which by the field equations requires $k = -1$. (One way think of this as a situation where the kinetic energy is balanced by the gravitational energy of a negatively-curved 3D space.) As noted in section 1, (2) with $R = t$ and $k = -1$ is isometric to $M_4$ [ref.4, p.205]. Indeed, the Milne model is merely a convenient non-static representation of flat 4D space. In the local limit where $|r^2/4| \ll 1$, the t-behaviour of the 3D sections of (2) allows us to specify a wave via the
same kind of coordinate transformation used in the 2D case. We eschew the
details of this, since the same physics is contained in more satisfactory form
if the dimensionality is extended.

5D manifolds which are canonical [5, 24] have remarkably simple dynamics. And since Campbell’s theorem [1, 8] ensures that any Ricci-flat 5D
solution has an Einstein 4D analog, it is natural to focus on the 5D version
of the Milne model discussed in the preceding paragraph. Consider therefore
the 5D line element

$$dS^2 = \left( \frac{l}{L} \right)^2 dt^2 - \left[ l \sinh \left( \frac{t}{L} \right) \right]^2 d\sigma^2 - dl^2 . \quad (3)$$

Here $l$ is the extra coordinate, and $L$ is a constant length which we will see
below is related inversely to the cosmological constant $\Lambda$. The 3-space is
the same as that above, namely $d\sigma^2 = (dx^2 + dy^2 + dz^2) \left( 1 + kr^2/4 \right)^{-2}$ with
$k = -1$. That the time-dependence of the 3-space in (3) is different from
that in (2) is attributable to the fact that we are using the 4D parameters
$(t, xyz$ or equivalently the 4D proper time $s$) to describe the motion in a
5D metric (whose proper time is $S \neq s$: see note 24). However, the local
situation for the 5D case (3) is close to that for the 4D case (2). To see this,
we note that for laboratory situations $t/L \ll 1$ in (3), so it reads

$$dS^2 \simeq \left( \frac{l}{L} \right)^2 dt^2 - \left( \frac{lt}{L} \right)^2 d\sigma^2 - dl^2. \quad (4)$$

This is of canonical form, namely $dS^2 = (l/L)^2 ds^2 - dl^2$ [5, 8, 14, 15, 18, 20]. For such metrics, the reduction of the 5D field equations to the 4D Einstein equations identifies the length $L$ via $\Lambda = 3/L^2$ (see e.g. ref. 5, p. 159). Such metrics effectively describe momentum manifolds rather than coordinate manifolds, since the identification of $l$ with $m$ defines the conventional action of particle physics ($\int mds$), and ensures agreement with the Weak Equivalence Principle [18]. More importantly for present purposes, the metric (3) from which (4) is derived satisfies not only $R_{AB} = 0$ but also $R_{ABCD} = 0$. This may be confirmed either by algebra or a fast computer package such as GRTensor. Since the 5D manifold is flat, the appropriate condition for the path of a particle in it is $dS = 0$ [19, 20]. With this condition, any canonical metric results in the constraint $L (dl/ds) = \pm l$. Let us use this constraint with (4), where we multiply it by $L^2$ and divide it by $ds^2$. The result is

$$0 \simeq l^2 \left( \frac{dt}{ds} \right)^2 - (l t)^2 \left[ \left( \frac{dx}{ds} \right)^2 + \left( \frac{dy}{ds} \right)^2 + \left( \frac{dz}{ds} \right)^2 \right] - l^2. \quad (5)$$

This with the identification $l = m$ (see above) and the recollection that
proper distances are defined by $\int t \, dx$ etc., simply reproduces the standard condition (1), in the form $0 = E^2 - p^2 - m^2$.

To convert the 5D metric (4) to a wave, we follow the lower-dimensional examples noted before. Specifically, we change $t \to e^{\omega t}/i\omega$, $x \to \exp(i\kappa_xx)$ etc., where $\omega$ is a frequency and $\kappa_x$ etc. are wave numbers for the $x, y, z$ directions. After setting the phase velocity to unity, (4) then reads

$$dS^2 \simeq \left(\frac{l}{L}\right)^2 e^{2\omega t} dt^2 - \left(\frac{l}{L}\right)^2 \left\{\exp \left[2i(\omega t + \kappa_xx)\right] dx^2 + \text{etc}\right\} - dl^2. \quad (6)$$

This with the null condition causes the analog of (5) to read

$$0 \simeq \left\{l e^{i\omega t} \frac{dt}{ds}\right\}^2 - \left\{l \exp \left[i(\omega t + \kappa_xx)\right] \frac{dx}{ds}\right\}^2 - \text{etc} - l^2. \quad (7)$$

We can again make the identification $l = m$ and define

$$\tilde{E} \equiv l e^{i\omega t} \frac{dt}{ds}, \quad \tilde{p} \equiv l \exp \left[i(\omega t + \kappa_xx)\right] \frac{dx}{ds} \text{ etc.} \quad (8)$$

Then (7) is equivalent to

$$0 \simeq \tilde{E}^2 - \tilde{p}^2 - m^2. \quad (9)$$

This is of course the wave analog of the standard relation (1) for a particle.

Another example of a 5D wave-like metric is the Billyard solution, which like (3) above satisfies $R_{AB} = 0$ and $R_{ABCD} = 0$ [14, 15, 24]. It may be
expressed in a form somewhat different from the original as

\[ dS^2 = \left( \frac{l}{L} \right)^2 dt^2 - \left( \frac{l}{L} \right)^2 \left\{ \exp \left[ 2i \left( \frac{t}{L} + \kappa x \right) \right] dx^2 + \text{etc} \right\} + dt^2. \quad (10) \]

This metric resembles (6), but now the frequency is constrained by the scale of the geometry and the extra dimension is timelike. (It may be verified that there is no solution for the opposite case.) The latter property means that for null 5D geodesics we have \( l = l_0 e^{\pm i \phi / L} \) where \( l_0 \) is a constant, so the mass parameter is itself a wave which oscillates around the hypersurface we call spacetime [21]. Such behaviour can also occur in string theory [6, 7], and may or may not be realistic. For present purposes, we note that while (10) can describe a deBroglie wave for the 3-momentum, it is not clear how to treat the energy, and the signature is at variance with that implied by the standard particle relation (1). This may seem strange, given the similarities between (6) and (10). However, it should be recalled that in trying to identify a 4D deBroglie wave from a 5D metric, we are dealing with a quantity \( Q = Q(x^\alpha, l) \) which is not necessarily preserved under the group of 5D coordinate transformations \( x^A \to \bar{x}^A(x^B) \) if the extra one \( x^4 = l \) is involved. This implies that the exact 5D solutions (3) and (10) are not equivalent in terms of their 4D physics. Indeed, the reduction of the field equations from 5D to 4D implies that the approximate form (4) of (3) has
Λ > 0, whereas (10) has Λ < 0, due to their different signatures [5]. This and other aspects of these solutions should be investigated in future work. At present, it appears that duality can best be described by (3), (4) for the particle and (6) for the wave.

3 Conclusion

Wave-particle duality may be approached through a consideration of flat manifolds of various dimensionalities. In the context of classical 4D general relativity, the standard energy condition (1) of particle physics in vacuum is consistent with the Milne model (2). This is an isometry of Minkowski space $M_4$, and a coordinate transformation can be used to make it wave-like. However, the concept of momentum is better handled by $M_5$, and we have examined an exact solution (3) in 5D which is not only Ricci-flat ($R_{AB} = 0$) but also Riemann-flat ($R_{ABCD} = 0$). The local limit of this solution is (4), which is basically the Milne model embedded in a 5D momentum (as opposed to coordinate) manifold. This describes a particle which obeys the standard energy condition, but a coordinate transformation puts a wave on it as in (6). Since the underlying manifold is flat, the natural condition on the interval
(action) is that it be zero, as in (7). Then obvious definitions for the energy and spatial momentum (8) result in both quantities being wave-like, and obeying a wave analog (9) of the particle energy condition. The solution (3), while it lends itself easily to both particle and wave interpretations, deserves further study to see what other physics it may imply. By contrast, the solution (10) which has been discussed in the literature is already in wave form, but does not lend itself so readily to an interpretation in terms of deBroglie waves. Our main conclusion, based on the solution (3), is that particles and waves are isometries of flat 5D space.

This technical result invites a philosophical discussion which would be inappropriate here. However, some comments are in order about coordinates. Physics should always be constructed in an $N$-dimensional space in a manner which is covariant; but if that space is embedded in ($N+1$), and the extra coordinate enters in a significant way, the physics in $ND$ will necessarily depend on the coordinates in ($N+1$) $D$. Traditional Kaluza-Klein theory is a good example of this, where the electromagnetic potentials (which are the cross-terms in the extended metric tensor) can be included or declined depending on how the 5 degrees of coordinate freedom for the line element are used. Even in manifolds of fixed $N$, the physical interpretation of a
solution can depend on the choice of coordinates or gauge. The Minkowski and Milne cases in 4D provide a good example of this, where the former describes a static spacetime and the latter describes an expanding cosmology. (Certain quantities are of course preserved, and in this case the density and pressure of matter are zero in both interpretations.) Likewise, the particle and wave descriptions for energy and momentum which we have discussed above depend on a choice of coordinates. The waves are not electromagnetic, and nor are they gravitational of the conventional type. For want of a better term, they can be called metric waves. They should not be regarded as merely technical accidents. Physics has over a long period given us large bodies of information which, because of the experimental approaches involved, we describe as pertaining to particles and waves. But it is really not surprising that these two physical phenomena have a common mathematical base. We have simply argued that this common base is geometrical, and that particles and waves are isometries.

Wave-particle duality has long been considered a paradox, but it may simply be that particles and waves are the same thing viewed in geometrically different ways.
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References

1. Campbell, J. 1926. A Course on Differential Geometry (Oxford: Clarendon Press).

2. Eisenhart, L.D. 1949. Riemannian Geometry (Princeton: Princeton University Press).

3. Kramer, D., Stephani, H., Herlt, E., MacCallum, M., Schmutzer, E. 1980. Exact Solutions of Einstein’s Field Equations (Cambridge: Cambridge University Press).

4. Rindler, W. 1977. Essential Relativity (New York: Springer).

5. Wesson, P.S. 1999. Space, Time, Matter (Singapore: World Scientific).

6. Gubser, S.S., Lykken, J.D. (editors), 2004. Strings, Branes and Extra Dimensions (Singapore: World Scientific).
7. Szabo, R.J., 2004. An Introduction to String Theory and D-Brane Dynamics (Singapore: World Scientific).

8. Seahra, S.S., Wesson, P.S. 2003. Class. Quant. Grav. 20, 1321.

9. Kalligas, D., Wesson, P.S., Everitt, C.W.F. 1995. Astrophys. J. 439, 548.

10. Ponce de Leon, J. 1988, Gen. Rel. Grav. 20, 539.

11. Davidson, A., Sonnenschein, J., Vozmediano, A.H. 1985. Phys. Rev. D 32, 1330.

12. McManus, D.J. 1994. J. Math. Phys. 35, 4889.

13. Abolghasem, G., Coley, A.A., McManus, D.J. 1996. J. Math. Phys. 37, 361.

14. Billyard, A., Wesson, P.S. 1996. Gen. Rel. Grav. 28, 129.

15. Wesson, P.S. 2003. Int. J. Mod. Phys. D 12, 1721.

16. Pospelov, M., Romalis, M. 2004. Phys. Today 57 (7), 40.

17. Will, C.M. 1993. Theory and Experiment in Gravitational Physics (Cambridge: Cambridge University Press).
24. There are several useful coordinate transformations which relate flat 5D manifolds and curved 4D ones. For example, the metrics $dS^2 = dT^2 - d\sigma^2 - dL^2$ and $ds^2 = l^2 dt^2 - d\sigma^2 - t^2 dl^2$ are related by the transformation $T = t^2 l^2 / 4 + \ln(t^{1/2} t^{-1/2})$, $L = t^2 l^2 / 4 - \ln(t^{1/2} t^{-1/2})$.

The “standard” 5D cosmologies of Ponce de Leon [10] have metrics of the second-noted form, and may by coordinate transformations be shown to be 5D flat. The full transformations, including those for the spatial part, are given elsewhere (ref. 5, p. 49). The Billyard wave [14] may similarly be shown to be a coordinate-transformed version of
de Sitter space, and flat in 5D. A generic discussion of cosmological models which are flat in 5D is due to McManus [12]. One of his solutions is a metric for a particle in a manifold whose 3D part is curved, which effectively generalizes the Billyard wave whose 3D part is flat. [See ref. 12, p. 4895, equation (30).] This can be seen by changing the 4D coordinates as discussed in the main text, which results in

$$dS^2 = \left(\frac{l}{L}\right)^2 dt^2 - \left(\frac{l}{L}\right)^2 \left(e^{it/L} + ke^{-it/L}\right)^2 \left[\exp(2i\kappa x) dx^2 + \text{etc}\right] + dl^2.$$  

When the curvature constant $k$ is zero, this gives back the Billyard wave. Another of the McManus solutions reproduces work by Davidson et al. [See ref. 11; and also ref. 12, p. 4893, equation (19).] This is effectively a 5D embedding of the 4D Milne model, and can be written as

$$dS^2 = dt^2 - t^2d\sigma^2 - dl^2,$$

where $d\sigma^2 \equiv (1 + kr^2/4)^{-2}$ with $k = -1$. The transformation $t \rightarrow l \sinh(t/L), \ l \rightarrow l \cosh(t/L)$ causes the metric to read

$$dS^2 = \left(\frac{l}{L}\right)^2 dt^2 - \left[l \sinh\left(t/L\right)\right]^2 d\sigma^2 - dl^2.$$  

This is quoted as (3) of the main text, and its local approximation is (4). The former has proper distances which vary as $\sinh t$, whereas the latter has proper distances which vary as $t$. The former is typical of motion in flat 5D space, when the 4D proper time $s$ (as opposed to the 5D proper time $S$) is used as parameter [5, p. 169]. The latter is typical of motion in flat
4D space, when the ordinary time $t$ is used as parameter [4, p. 205].

Both of the models used in the main text to illustrate the passage from particle to wave use metrics which are canonical in form, and there is a large literature on these. However, a more general class of metrics is given by

$$\text{d}S^2 = g_{\alpha\beta}(x^\gamma, l) \text{d}x^\alpha \text{d}x^\beta + \epsilon \Phi^2(x^\gamma, l) \text{d}l^2,$$

where $\epsilon = \pm 1$ and $\Phi$ is a scalar field. Einstein’s 4D equations are satisfied for this 5D metric if the effective or induced energy-momentum tensor is given by

$$8\pi T_{\alpha\beta} = \frac{\Phi_{,\alpha\beta}}{\Phi} - \frac{\epsilon}{2\Phi^2} \left\{ \frac{\Phi_{,\alpha4} g_{\beta,44}}{\Phi} - g_{\beta,44} + g^{\lambda\mu} g_{\alpha\lambda,4} g_{\beta\mu,4} \right\}$$

$$- \frac{g^{\mu\nu} g_{\mu\nu,4} g_{\alpha\beta,4}}{2} + g_{\alpha\beta} \left[ \frac{g^{\mu\nu} g_{\mu\nu,4}}{4} + (g_{\alpha\beta,4})^2 \right].$$

Here a comma denotes the partial derivative and a semicolon denotes the 4D covariant derivative. We have not discussed the matter which relates to the exact solutions (3), (10) of the main text because it is merely vacuum [5, 14]. But the matter properties of these and more complicated solutions may be evaluated for any choice of coordinates by using the noted expression.