SPECIAL REPRESENTATIONS OF WEYL GROUPS: A POSITIVITY PROPERTY

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INTRODUCTION

Let $W$ be an irreducible Weyl group with length function $l : W \to \mathbb{N}$ and let $S = \{ s \in W ; l(s) = 1 \}$. Let $\text{Irr} W$ be a set of representatives for the isomorphism classes of irreducible representations of $W$ (over $\mathbb{C}$). In [L1] a certain subset of $\text{Irr} W$ was defined. The representations in this subset were later called special representations; they play a key role in the classification of unipotent representations of a reductive group over a finite field $\mathbb{F}_q$ for which $W$ is the Weyl group. (The definition of special representations is reviewed in 3.1.)

It will be convenient to replace irreducible representations of $W$ with the corresponding simple modules of the asymptotic Hecke algebra $J$ (see [L6, 18.3]) associated to $W$ via the canonical isomorphism $\psi : \mathbb{C}[W] \xrightarrow{\sim} J$ (see 3.1); let $E_\infty$ be the simple $J$-module corresponding to $E \in \text{Irr} W$ under $\psi$.

In this paper we show that a special representation $E$ of $W$ is characterized by the following positivity property of $E_\infty$: there exists a $\mathbb{C}$-basis of $E_\infty$ such that any element $t_u$ in the standard basis of $J$ acts in this basis through a matrix with all entries in $\mathbb{R}_{\geq 0}$.

The fact that for a special representation $E$, $E_\infty$ has the positivity property above was pointed out (in the case where $W$ is of classical type) in [L9]. In this paper I will recall the argument of [L9] (see 3.3) and I give two other proofs which apply for any $W$. One of these proofs (see 4.4) is based on the interpretation [L3], [BFO], of $J$ (or its part attached to a fixed two-sided cell) in terms of $G$-equivariant vector bundles on $X \times X$ where $X$ is a finite set with an action of a finite group $G$. Another proof (see Section 2) is based on the use of Perron’s theorem for matrices with all entries in $\mathbb{R}_{\geq 0}$. (Previously, Perron’s theorem has been in used in the context of canonical bases in quantum groups in the study [L5] of total positivity and, very recently, in the context of the canonical basis [KL1] of $\mathbb{C}[W]$, in [KM]; in both cases the positivity properties of the appropriate canonical bases were used). We also show that the Hecke algebra representation corresponding to a special representation $E$ can be realized essentially by a $W$-graph (in the sense supported in part by National Science Foundation grant 1303060.
of [KL1]) in which all labels are natural numbers. Some of our results admit also an extension to the case of affine Weyl groups (see Section 5).

1. Statement of the main theorem

1.1. Let \( v \) be an indeterminate and let \( \mathcal{A} = \mathbb{Z}[v, v^{-1}] \). Let \( \mathcal{H} \) be the Hecke algebra of \( W \) that is, the associative \( \mathcal{A} \)-algebra with 1 with an \( \mathcal{A} \)-basis \( \{ T_w; w \in W \} \) (where \( T_1 = 1 \)) and with multiplication such that \( T_wT_w' = T_{ww'} \) if \( l(ww') = l(w) + l(w') \) and \( (T_s+1)(T_s-v^2) = 0 \) if \( s \in S \). Let \( \{ c_w; w \in W \} \) be the \( \mathcal{A} \)-basis of \( \mathcal{H} \) denoted by \( \{ C'_w; w \in W \} \) in [KL1] (with \( q = v^2 \)); see also [L6, 5.2]. For example, if \( s \in S \), we have \( c_s = v^{-1}T_s + v^{-1} \). The left cells and two-sided cells of \( W \) are the equivalence classes for the relations \( \sim_L \) and \( \sim_{LR} \) on \( W \) defined in [KL1], see also [L6, 8.1]; we shall write \( \sim \) instead of \( \sim_L \). For \( x,y \) in \( W \) we have \( c_xc_y = \sum_{z \in W} h_{x,y,z}c_z \) where \( h_{x,y,z} \in \mathbb{N}[v, v^{-1}] \). As in [L6, 13.6], for \( z \in W \) we define \( a(z) \in \mathbb{N} \) by \( h_{x,y,z} \in v^{a(z)}\mathbb{Z}[v^{-1}] \) for all \( x,y \in W \) and \( h_{x,y,z} \notin v^{a(z)-1}\mathbb{Z}[v^{-1}] \) for some \( x,y \) in \( W \). (For example, \( a(1) = 0 \) and \( a(s) = 1 \) if \( s \in S \).) For \( x,y,z \) in \( W \) we have \( h_{x,y,z} = \gamma_{x,y,z-1}v^{a(z)} \mod v^{a(z)-1}\mathbb{Z}[v^{-1}] \) where \( \gamma_{x,y,z-1} \in \mathbb{N} \) is well defined. Let \( \mathbf{J} \) be the \( \mathbb{C} \)-vector space with basis \( \{ t_w; w \in W \} \). For \( x,y \) in \( W \) we set \( t_xt_y = \sum_{z \in W} \gamma_{x,y,z-1}t_z \in \mathbf{J} \). This defines a structure of associative \( \mathbb{C} \)-algebra on \( \mathbf{J} \) with unit element of the form \( \sum_{d \in D} t_d \) where \( D \) is a certain subset of the set of involutions in \( W \), see [L6, 18.3]. For any subset \( X \) of \( W \) let \( \mathbf{J}_X \) be the subspace of \( \mathbf{J} \) with basis \( \{ t_w; w \in X \} \); let \( \mathbf{J}_X^+ \) be the set of elements of the form \( \sum_{w \in X} f_w t_w \in \mathbf{J}_X \) with \( f_w \in \mathbb{R}_{>0} \) for all \( w \in X \). We have \( \mathbf{J} = \bigoplus c \mathbf{J}_c \) where \( c \) runs over the two-sided cells of \( W \). Each \( \mathbf{J}_c \) is a subalgebra of \( \mathbf{J} \) with unit element \( \sum_{d \in D_c} t_d \) where \( D_c = c \cap D \); moreover, \( \mathbf{J}_c \mathbf{J}_c = 0 \) if \( c \neq c' \).

Until the end of Section 4 we fix a two-sided cell \( c \).

Let \( L \) be the set of left cells that are contained in \( c \). We have

\[
\mathbf{c} = \bigcup_{\Gamma \in L} \Gamma = \bigcup_{\Gamma, \Gamma'} \text{in } L(\Gamma \cap \Gamma'^{-1});
\]

moreover, \( \Gamma \cap \Gamma'^{-1} \neq \emptyset \) for any \( \Gamma, \Gamma' \) in \( L \). It follows that

\[
\mathbf{J}_c = \bigoplus_{\Gamma \in L} \mathbf{J}_\Gamma = \bigoplus_{\Gamma, \Gamma'} \text{in } L\mathbf{J}_{\Gamma \cap \Gamma'^{-1}};
\]

moreover, \( \mathbf{J}_{\Gamma \cap \Gamma'^{-1}} \neq 0 \). Note that for \( \Gamma \in L \), \( \mathbf{J}_\Gamma \) is a left ideal of \( \mathbf{J}_c \).

A line \( \mathcal{L} \) in \( \mathbf{J}_{\Gamma \cap \Gamma'^{-1}} \) is said to be positive if \( \mathcal{L}^+ := \mathcal{L} \cap \mathbf{J}_{\Gamma \cap \Gamma'^{-1}} \neq \emptyset \); in this case, \( \mathcal{L}^+ \) consists of all \( \mathbb{R}_{>0} \)-multiples of a single nonzero vector. We now state our main result.

**Theorem 1.2.** (a) Let \( \Gamma \in L \). There is a unique left ideal \( \mathbf{M}_\Gamma \) of \( \mathbf{J}_c \) such that property (\( \forall \)) below holds:

\[
(\forall) \quad \mathbf{M}_\Gamma = \bigoplus_{\Gamma' \in L} \mathbf{M}_{\Gamma, \Gamma'} \text{ where for any } \Gamma' \in L, \quad \mathbf{M}_{\Gamma, \Gamma'} := \mathbf{M}_{\Gamma} \cap \mathbf{J}_{\Gamma \cap \Gamma'^{-1}} \text{ is a positive line.}
\]

(b) Let \( \Gamma, \Gamma' \in L, u \in \mathbf{c} \). We have \( u \in \hat{\Gamma} \cap \hat{\Gamma'}^{-1} \) for well-defined \( \hat{\Gamma}, \hat{\Gamma}' \) in \( L \). If \( \hat{\Gamma} \neq \hat{\Gamma}' \), then \( t_u \mathbf{M}_{\Gamma \cap \Gamma'} = 0 \). If \( \hat{\Gamma} = \hat{\Gamma}' \), then \( t_u \mathbf{M}_{\Gamma \cap \Gamma'} = \mathbf{M}_{\Gamma \cap \Gamma'} \), and \( t_u \mathbf{M}_{\Gamma \cap \Gamma'}^+ = \mathbf{M}_{\Gamma \cap \Gamma'}^+ \).
(c) The $J_c$-module $M_\Gamma$ in (a) is simple. Its isomorphism class is independent of $\Gamma \in L$.

(d) The subspace $I = \oplus_{\Gamma \in L} M_\Gamma$ of $J_c$ is a simple two-sided ideal of $J_c$.

(e) Let $\Gamma, \Gamma', \bar{\Gamma}, \bar{\Gamma}'$ be in $L$. If $\Gamma \neq \bar{\Gamma}'$ then $M_{\Gamma, \Gamma'} M_{\bar{\Gamma}, \bar{\Gamma}'} = 0$. If $\Gamma = \bar{\Gamma}'$, then multiplication in $J_c$ defines an isomorphism $M_{\Gamma, \Gamma'} \otimes M_{\bar{\Gamma}, \bar{\Gamma}'} \cong M_{\bar{\Gamma}, \bar{\Gamma}'}$, and a surjective map $M_{\Gamma, \Gamma'}^+ \times M_{\bar{\Gamma}, \bar{\Gamma}'}^+ \rightarrow M_{\bar{\Gamma}, \bar{\Gamma}'}^+$.

(f) Let $\Gamma, \Gamma'$ be in $L$. The antiautomorphism $\theta : J_c \rightarrow J_c$ given by $t_x \mapsto t_{x^{-1}}$ for all $x \in c$ maps $M_{\Gamma, \Gamma'}$ onto $M_{\Gamma', \Gamma}$ and $M_{\Gamma, \Gamma'}^+$ onto $M_{\Gamma', \Gamma}^+$.

The proof is given in Section 2.

1.3. As a consequence of Theorem 1.2, the simple $J_c$-module $M_\Gamma$ admits a $C$-basis \{1\} with the following property:

(i) If $u \in c$ and $\Gamma' \in L$, then $t_u \tilde{e}_{\Gamma'}$ is an $R_{\geq 0}$-linear combination of elements $\tilde{e}_{\Gamma''}$ with $\Gamma'' \in L$; more precisely, if $u \in \bar{\Gamma} \cap \bar{\Gamma}'^{-1}$ with $\bar{\Gamma}, \bar{\Gamma}'$ in $L$, then $t_u \tilde{e}_{\Gamma'} = \lambda_{u, \bar{\Gamma}', \bar{\Gamma}'} \tilde{e}_{\Gamma'}$

with $\lambda_{u, \bar{\Gamma}', \bar{\Gamma}'} \in R_{> 0}$ if $\bar{\Gamma} = \bar{\Gamma}'$ and $\lambda_{u, \bar{\Gamma}', \bar{\Gamma}'} = 0$ if $\bar{\Gamma} \neq \bar{\Gamma}'$.

Indeed, we can take for $\tilde{e}_{\Gamma'}$ any element of $M_{\Gamma, \Gamma'}^+$ and we use 1.2(b).

1.4. Let $\leq$ be the standard partial order on $W$. By [KL1], to any $y \neq w$ in $W$ one can attach a number $\mu(y, w) \in Z$ such that for any $s \in S$ and any $w \in W$ with $sw > w$ we have $c_s c_w = \sum_{y \in W, s y < y} \mu(y, w) c_y$. By [KL2] we have $\mu(y, w) \in N$.

1.5. Let $H = C(v) \otimes_A H$ where we use the obvious imbedding $A \rightarrow C(v)$; we denote $1 \otimes c_w$ again by $c_w$. Let $J = C(v) \otimes_C J$ where we use the obvious imbedding $C \rightarrow C(v)$. We have a homomorphism of $C(v)$-algebras (with 1) $\Psi : H \rightarrow J$ given by $\Psi(c_x) = \sum_{d \in D, z \in W, d \sim z} h_{x, d, z} t_z$

for all $x \in W$, see [L6, 18.9]. (Note that $\Psi$ is in fact the composition of a homomorphism in loc.cit. with an automorphism of $H$.)

1.6. For any $\Gamma \in L$ let $S_\Gamma$ be the set of all $t \in S$ such that $rt < r$ for some (or equivalently any) $r \in \Gamma$.

We fix $\Gamma \in L$. Let $\{\tilde{e}_{\Gamma'}; \Gamma' \in L\}$ be a $C$-basis of $M_{\Gamma}$ as in 1.3; we use the notation of 1.3. We shall view $C(v) \otimes M_{\Gamma}$ as an $H$-module via $\Psi$. Let $s \in S$ and let $\Gamma' \in L$; let $\delta$ be the unique element in $\Gamma' \cap D$. We show:

(a) If $s \in S_{\Gamma'}$, then $\Psi(T_s) \tilde{e}_{\Gamma'} = v^2 \tilde{e}_{\Gamma'}$.

(b) If $s \notin S_{\Gamma'}$, then $\Psi(T_s) \tilde{e}_{\Gamma'} = -\tilde{e}_{\Gamma'} + \sum_{\Gamma \in L, s \in S_\Gamma, v^{-1} \Gamma} f_{\Gamma, \Gamma'} v^{-1} \tilde{e}_{\Gamma}$

where $f_{\Gamma, \Gamma'} = \sum_{u \in \Gamma' \cap \bar{\Gamma}^{-1}} \mu(u, \delta) \lambda_{u, \Gamma', \bar{\Gamma}} \in R_{\geq 0}$. 

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By definition, we have

\[ \Psi(c_s) \tilde{e}_{\Gamma'} = \sum_{d \in \mathcal{D}, u \in c, d \sim u} h_{s, d, u} t_u \tilde{e}_{\Gamma'} = \sum_{\tilde{\Gamma} \in L} \sum_{d \in \mathcal{D}, u \in \Gamma' \cap \tilde{\Gamma}^{-1}, d \in \Gamma'} h_{s, d, u} \lambda_{u, \Gamma', \tilde{\Gamma}} \tilde{e}_{\tilde{\Gamma}}. \]

Since in the last sum we have \( d \in \Gamma' \) we see that we can assume that \( d = \delta \). Thus we have

\[ \Psi(c_s) \tilde{e}_{\Gamma'} = \sum_{\tilde{\Gamma} \in L} \sum_{\Gamma' \cap \tilde{\Gamma}^{-1}, d \in \Gamma'} h_{s, \delta, u} \lambda_{u, \Gamma', \tilde{\Gamma}} \tilde{e}_{\tilde{\Gamma}}. \]

If \( s \delta < \delta \) (that is, \( s \in S_{\Gamma'} \)) we have \( c_s \delta = (v + v^{-1}) \delta \) hence \( h_{s, \delta, u} \) is \((v + v^{-1})\) for \( u = \delta \) and is 0 for \( u \neq \delta \); hence in this case

\[ \Psi(c_s) \tilde{e}_{\Gamma'} = (v + v^{-1}) \tilde{e}_{\Gamma'}; \]

(we use that \( \lambda_{\delta, \Gamma', \Gamma'} = 1 \).)

We now assume that \( s \delta > \delta \) (that is, \( s \notin S_{\Gamma'} \)). In this case, \( h_{s, \delta, u} \) is \( \mu_{u, \delta} \) if \( su < u \) and is 0 if \( su > u \) (see 1.4); hence

\[ \Psi(c_s) \tilde{e}_{\Gamma'} = \sum_{\tilde{\Gamma} \in L} \sum_{\Gamma' \cap \tilde{\Gamma}^{-1}, su < u} \mu(u, \delta) \lambda_{u, \Gamma', \tilde{\Gamma}} \tilde{e}_{\tilde{\Gamma}} \]

\[ = \sum_{\tilde{\Gamma} \in L; s \in S \Gamma' \cap \tilde{\Gamma}^{-1}} \sum_{u \in \Gamma' \cap \tilde{\Gamma}^{-1}} \mu(u, \delta) \lambda_{u, \Gamma', \tilde{\Gamma}} \tilde{e}_{\tilde{\Gamma}}. \]

Now (a),(b) follow.

Note that (a),(b) show that in the \( H \)-module \( C(v) \otimes M_{\Gamma} \) the generators \( T_s \) act with respect to the basis \( \{ \tilde{e}_{\tilde{\Gamma}'}; \tilde{\Gamma}' \in L \} \) essentially by formulas which are those in a \( W \)-graph (in the sense of [KL1]) in which all labels are in \( \mathbb{R}_{\geq 0} \).

1.7. In Section 4 we will give another proof of the existence part of 1.2(a) which also shows that \( \tilde{e}_{\Gamma'} \) in 1.3 can be chosen so that

(i) each \( \tilde{e}_{\Gamma'} \) is a \( \mathbb{Z}_{\geq 0} \)-linear combination of elements in \( \{ t_x; x \in \Gamma \cap \Gamma'^{-1} \} \),

(ii) \( \lambda_{u, \Gamma', \Gamma'} \in \mathbb{Z}_{\geq 0} \) (notation of 1.3).

In particular, with this choice of \( \tilde{e}_{\Gamma'} \), the constants \( f_{\tilde{\Gamma}, \Gamma'} \) in the "\( W \)-graph formulas" in 1.6 are in \( \mathbb{Z}_{\geq 0} \).

2. Proof of Theorem 1.2

2.1. From [L6,§15] we see that, for \( x, y, u \) in \( W \) we have:

(a) \( \gamma_{x, y, u} = \gamma_{y, u, x} = \gamma_{u, x, y} \),

(b) \( \gamma_{x, y, u} \neq 0 \iff x \sim y^{-1}, y \sim u^{-1}, u \sim x^{-1} \).

By [L6, 18.4(a)]:

(c) for \( y, z \) in \( W \) we have \( y \sim z \) if and only if \( t_y t_{z^{-1}} \neq 0 \).
2.2. Let \( \Gamma, \Gamma', \hat{\Gamma}, \hat{\Gamma}' \) be in \( L \). From 2.1(b) we deduce:

(a) If \( \Gamma \neq \hat{\Gamma}' \), then \( J_{\Gamma \cap \Gamma' - 1} J_{\Gamma \cap \Gamma' - 1} = 0 \),

(b) \[ J_{\Gamma \cap \Gamma' - 1} J_{\Gamma \cap \Gamma' - 1} \subset J_{\hat{\Gamma} \cap \hat{\Gamma}' - 1} \]

We show:

(c) If \( u \in \Gamma \cap \Gamma' - 1 \), then \( t_u J^+_{\Gamma \cap \Gamma' - 1} \subset J^+_{\hat{\Gamma} \cap \hat{\Gamma}' - 1} \).

Let \( \xi = \sum_{y \in \Gamma \cap \Gamma' - 1} f_y t_y \in J_{\Gamma \cap \Gamma' - 1} \) with \( f_y \in \mathbb{R}_{>0} \) for all \( y \). We must show that \( t_u \xi \in J^+_{\Gamma \cap \Gamma' - 1} \); it is enough to show that for any \( z \in \hat{\Gamma} \cap \Gamma' - 1 \) there exists \( y \in \hat{\Gamma} \cap \Gamma' - 1 \) such that \( \gamma_{u,y,z}^{-1} \neq 0 \) or that there exists \( y \in W \) such that \( \gamma_z^{-1,u,y} \neq 0 \) (see 2.1(a)); such \( y \) is automatically in \( \hat{\Gamma} \cap \Gamma' - 1 \). Hence it is enough to show that for any \( z \in \hat{\Gamma} \cap \Gamma' - 1 \) we have \( t_{z-1} t_u \neq 0 \). This holds since \( z^{-1} \sim u^{-1} \) (see 2.1(c)).

From (c) we deduce

(d) \[ J^+_{\Gamma \cap \Gamma' - 1} J^+_{\Gamma \cap \Gamma' - 1} \subset J^+_{\hat{\Gamma} \cap \hat{\Gamma}' - 1} \]

2.3. Let \( \Gamma \in L \). For any \( \Gamma' \in L \) we define a \( \mathbb{C} \)-linear map \( T_{\Gamma'} : J_{\Gamma \cap \Gamma' - 1} \to J_{\Gamma \cap \Gamma' - 1} \) by

\[ T_{\Gamma'}(t_x) = \sum_{y \in \Gamma \cap \Gamma' - 1} t_x t_y = \sum_{y \in \Gamma \cap \Gamma' - 1, z \in \Gamma} \gamma_{x,y,z}^{-1} t_z. \]

We show:

(a) the matrix representing \( T_{\Gamma'} \) with respect to the basis \( \{ t_w; w \in \Gamma \cap \Gamma' - 1 \} \) has all entries in \( \mathbb{R}_{>0} \).

An equivalent statement is: for any \( x, z \) in \( \Gamma \cap \Gamma' - 1 \), the sum \( \sum_{y \in \Gamma \cap \Gamma' - 1} \gamma_{x,y,z}^{-1} \) is \( > 0 \). Since \( \gamma_{x,y,z}^{-1} \in \mathbb{N} \) for all \( y \), it is enough to show that for some \( y \in \Gamma \cap \Gamma' - 1 \) we have \( \gamma_{x,y,z}^{-1} \neq 0 \) or equivalently (see 2.1(a)) that for some \( y \in W \) we have \( \gamma_{x,y,z}^{-1} \neq 0 \) (we then have automatically \( y \in \Gamma \cap \Gamma' - 1 \)). Thus, it is enough to show that \( t_{z-1} t_x \neq 0 \). This follows from 2.1(c) since \( z^{-1} \sim x^{-1} \).

Applying Perron's theorem [Pe] to the matrix in (a) we see that there is a unique \( T_{\Gamma'} \)-stable positive line \( L_{\Gamma',\Gamma} \) in \( J_{\Gamma \cap \Gamma' - 1} \) (the "Perron line").

Now let \( u \in \mathfrak{c} \); we have, \( u \in \hat{\Gamma} \cap \hat{\Gamma}' - 1 \) with \( \hat{\Gamma}, \hat{\Gamma}' \) in \( L \). From 2.2(a), 2.2(d), we deduce

(b) If \( \hat{\Gamma} \neq \Gamma' \), then \( t_u J_{\Gamma \cap \Gamma' - 1} = 0 \) hence \( t_u L_{\Gamma,\Gamma'} = 0 \);

(c) if \( \hat{\Gamma} = \Gamma' \), then \( t_u L^+_{\Gamma,\Gamma'} \subset J^+_{\Gamma \cap \Gamma' - 1} \), hence \( t_u L_{\Gamma,\Gamma'} \) is a positive line in \( J_{\Gamma \cap \Gamma' - 1} \).

In the setup of (c), we have \( t_u(T_{\Gamma'}(\xi)) = T_{\Gamma'}(t_u \xi) \) for any \( \xi \in J_{\Gamma \cap \Gamma' - 1} \). It follows that \( t_u L_{\Gamma,\Gamma'} \) is a \( T_{\Gamma'} \)-stable line in \( J_{\Gamma \cap \Gamma' - 1} \). Thus,

(d) if \( \hat{\Gamma} = \Gamma' \), then \( t_u L_{\Gamma,\Gamma'} = L_{\Gamma,\Gamma'} \).

We set \( \mathcal{M}_\Gamma = \oplus_{\Gamma' \in L} L_{\Gamma,\Gamma'} \). From (b),(d) we see that \( \mathcal{M}_\Gamma \) is a \( \mathfrak{c} \)-submodule of \( J_{\Gamma} \).

We now see that the existence part of 1.2(a) is proved: we can take \( M_\Gamma = \mathcal{M}_\Gamma \).
2.4. Let $\Gamma \in L$ and let $M_\Gamma$ be any $J_\mathfrak{c}$-submodule of $J_\Gamma$ for which property (\(\heartsuit\)) in 1.2(a) holds. We show that 1.2(b) holds for $M_\Gamma$. Let $u \in \tilde{\Gamma} \cap \tilde{\Gamma'}^{-1}$ be as in 1.2(b) and let $\Gamma' \in L$. If $\tilde{\Gamma} \neq \Gamma'$, then $t_u J_{\Gamma \cap \Gamma'}^{-1} = 0$ hence $t_u M_{\Gamma,\Gamma'} = 0$. Now assume that $\tilde{\Gamma} = \Gamma'$. By 2.2(c), left multiplication by $t_u$ maps $J_{\Gamma \cap \Gamma'}^+ \rightarrow J_{\Gamma \cap \Gamma'}^+$; it maps any positive line in $J_{\Gamma \cap \Gamma'}^+$ onto a positive line in $J_{\Gamma \cap \Gamma'}$. In particular, maps $M_{\Gamma,\Gamma'}$ onto a line in $J_{\Gamma \cap \Gamma'}$, which, being also contained in $M_\Gamma$, must be equal to $M_{\Gamma,\Gamma'}$; moreover, it maps $M_{\Gamma,\Gamma'}^+$ into $J_{\Gamma \cap \Gamma'}^+$ hence onto $M_{\Gamma,\Gamma'}^+$. Thus, 1.2(b) holds for $M_\Gamma$.

We now choose a basis $\{\tilde{e}_{\Gamma'}; \Gamma' \in L\}$ of $M_\Gamma$ such that $\tilde{e}_{\Gamma'} \in M_{\Gamma,\Gamma'}^+$ for any $\Gamma' \in L$; then for any $u \in \mathfrak{c}$, the matrix of the $t_u$-action on $M_\Gamma$ in this basis has entries in $\mathbb{R}_{\geq 0}$. Thus,

(a) $\text{tr}(t_u, M_\Gamma) \in \mathbb{R}_{\geq 0}$ for all $u \in \mathfrak{c}$.

We show:

(b) the $\mathbb{C}$-linear map $\nu : J_\mathfrak{c} \rightarrow \text{End}_\mathbb{C}(M_\Gamma)$ given by the $J_\mathfrak{c}$-module structure on $M_\Gamma$ is surjective.

It is enough to show that for any $\Gamma' \in L$ there exists $u \in \mathfrak{c}$ such that $\nu(t_u)$ carries the line $M_{\Gamma,\Gamma'}$ onto the line $M_{\Gamma,\Gamma'}$ and carries the line $M_{\Gamma,\Gamma''}$ (where $\Gamma'' \in L$, $\Gamma'' \neq \Gamma'$) to zero. Note that any $u \in \Gamma' \cap \Gamma'^{-1}$ has the required properties. This proves (b).

It follows that the $J_\mathfrak{c}$-module $M_\Gamma$ is simple. We show:

(c) Assume that $M'$ is any simple $J_\mathfrak{c}$-module such that $\text{tr}(t_u, M') \in \mathbb{R}_{\geq 0}$ for all $u \in \mathfrak{c}$. Then $M'$ is isomorphic to $M_\Gamma$.

Assume that this is not so. We use the orthogonality formula

$$\sum_{u \in \mathfrak{c}} \text{tr}(t_u, M_\Gamma)\text{tr}(t_{u^{-1}}, M') = 0,$$

which is a special case of [L6, 19.2(e)] (taking into account [L6, 20.1(b)] and using that $u \mapsto u^{-1}$ maps $\mathfrak{c}$ into $\mathfrak{c}$). Since each term in the last sum is in $\mathbb{R}_{\geq 0}$, it follows that each term in the last sum is 0. In particular, we have $\text{tr}(t_d, M_\Gamma)\text{tr}(t_{d^{-1}}, M') = 0$ for any $d \in \mathcal{D}_\mathfrak{c}$. We show that for any $d \in \mathcal{D}_\mathfrak{c}$ we have $\text{tr}(t_d, M_\Gamma) \in \mathbb{R}_{>0}$. Using the basis of $M_\Gamma$ employed in the proof of (a), it is enough to show that some diagonal entry of the $t_d$-action in this basis is $\neq 1$ (all entries are in $\mathbb{R}_{\geq 0}$). We have $d \in \Gamma'$ for a unique $\Gamma' \in L$; then $t_d M_{\Gamma,\Gamma'}^+ = M_{\Gamma,\Gamma'}^+$, and the desired property holds.

From $\text{tr}(t_d, M_\Gamma)\text{tr}(t_{d^{-1}}, M') = 0$ and $\text{tr}(t_d, M_\Gamma) \in \mathbb{R}_{>0}$ we deduce that $\text{tr}(t_d, M') = 0$ for any $d \in \mathcal{D}_\mathfrak{c}$. Since $\sum_{d \in \mathcal{D}_\mathfrak{c}} t_d$ is the unit element $1_\mathfrak{c}$ of $J_\mathfrak{c}$, it follows that $\text{tr}(1_\mathfrak{c}, M') = 0$. This is a contradiction. This proves (c).

Let $I$ be the simple ideal of $J_\mathfrak{c}$ such that $IM_\Gamma \neq 0$. It is a $\mathbb{C}$-vector space of dimension $N^2$ where $N$ is the number of elements in $L$. If $\tilde{\Gamma} \in L$, then $M_{\tilde{\Gamma}}$ is a simple $J_\mathfrak{c}$-module such that $\text{tr}(t_u, M_{\tilde{\Gamma}}) \in \mathbb{R}_{\geq 0}$ for all $u \in \mathfrak{c}$ (we use (a) with $M_\Gamma$ replaced by $M_{\tilde{\Gamma}}$); hence, by (c), we have $M_{\tilde{\Gamma}} \cong M_\Gamma$ as $J_\mathfrak{c}$-modules. In particular,
the isomorphism class of $\mathcal{M}_\Gamma$ is independent of $\tilde{\Gamma}$. We see that the (necessarily direct) sum $\sum_{\Gamma \in \mathcal{L}} \mathcal{M}_\Gamma$ is contained in $I$ and has dimension $N^2$ hence it is equal to $I$; we also see that $M_\Gamma \subset I$ and, taking intersections with $\mathbf{J}_\Gamma$, we see that $M_\Gamma \subset \mathcal{M}_\Gamma$, hence $M_\Gamma = \mathcal{M}_\Gamma$ (since $\dim M_\Gamma = \dim \mathcal{M}_\Gamma = N$). We now see that the uniqueness part of 1.2(a) is proved. Note that 1.2(b), 1.2(c), 1.2(d) are also proved and we have $I = I$.

2.5. We prove 1.2(e). In the setup of (e), if $\Gamma \neq \tilde{\Gamma}$ then, using 2.2(a), we have $M_{\Gamma,\Gamma'} M_{\tilde{\Gamma},\Gamma'} = 0$. Assume now that $\Gamma = \tilde{\Gamma}$. Using 2.2(d), we see that $M_{\Gamma,\Gamma'}, M_{\tilde{\Gamma},\Gamma'}$ is contained in $\mathbf{J}_{\Gamma \cap \Gamma'}^{+}$; it is also contained in $I$ (since $I$ is closed under multiplication), hence it is contained in $I \cap \mathbf{J}_{\Gamma \cap \Gamma'}^{+} = M_{\Gamma,\Gamma'}^{+}$. Thus, multiplication restricts to a map $M_{\Gamma,\Gamma'}^{+} \times M_{\tilde{\Gamma},\Gamma'}^{+} \rightarrow M_{\Gamma,\Gamma'}^{+}$. This map is necessarily surjective since $M_{\Gamma,\Gamma'}^{+}$ is a single orbit of $\mathbf{R}_{>0}$ under scalar multiplication. This implies that the linear map between lines $M_{\Gamma,\Gamma'} \otimes M_{\tilde{\Gamma},\Gamma'} \rightarrow M_{\Gamma,\Gamma'}$ is an isomorphism. This proves 1.2(e).

2.6. We prove 1.2(f). Any element $\xi \in \mathbf{J}_c$ defines a linear map $t(\theta(\xi)) : M_{\Gamma}^{*} \rightarrow M_{\Gamma}^{*}$ where $M_{\Gamma}^{*}$ denotes the dual space and $t$ denotes the transpose. This defines a $\mathbf{J}_c$-module structure on $M_{\Gamma}^{*}$ such that for any $x \in c$ we have $\text{tr}(t_x, M_{\Gamma}^{*}) = \text{tr}(t_{-x}, M_{\Gamma})$. By the argument in [L6, 20.13(a)], $\text{tr}(t_{x}, M_{\Gamma})$ is the complex conjugate of $\text{tr}(t_{x}, M_{\Gamma})$. But the last trace is a real number, so that $\text{tr}(t_x, M_{\Gamma}^{*}) = \text{tr}(t_{x}, M_{\Gamma})$. It follows that $M_{\Gamma}^{*} \cong M_{\Gamma}$ as $\mathbf{J}_c$-modules. From the definitions, the simple two-sided ideal $\mathbf{I}'$ of $\mathbf{J}_c$ such that $\mathbf{I}' M_{\Gamma}^{*} \neq 0$ satisfies $\mathbf{I}' = \theta(\mathbf{I})$. It follows that $\theta(\mathbf{I}) = \mathbf{I}$. Since $\mathbf{I} = \bigoplus_{\Gamma \cap \Gamma'} I \otimes M_{\Gamma,\Gamma'}^{+} \cap \bigoplus_{\Gamma \cap \Gamma'} I \otimes M_{\Gamma,\Gamma'}^{+}$, and $\theta(\mathbf{J}_{\Gamma,\Gamma'}) = \mathbf{J}_{\Gamma',\Gamma}$, it follows that $\theta(M_{\Gamma,\Gamma'}) \subset \mathbf{J}_{\Gamma',\Gamma} \cap \bigoplus_{\Gamma \cap \Gamma'} I \otimes M_{\Gamma,\Gamma'}^{+} = M_{\Gamma',\Gamma}$.

Since $\theta$ is a vector space isomorphism, it follows that $\theta(M_{\Gamma,\Gamma'}) = M_{\Gamma',\Gamma}$. Note that $\theta(\mathbf{J}_{\Gamma,\Gamma'}) = \mathbf{J}_{\Gamma',\Gamma}$; hence $\theta(M_{\Gamma,\Gamma'}) \subset \mathbf{J}_{\Gamma',\Gamma} \cap M_{\Gamma',\Gamma} = M_{\Gamma',\Gamma}$. This forces the equality $\theta(M_{\Gamma,\Gamma'}) = M_{\Gamma,\Gamma'}$ (since $M_{\Gamma,\Gamma'}^{+}$ is a single orbit of $\mathbf{R}_{>0}$ under scalar multiplication). This proves 1.2(f). Theorem 1.2 is proved.

2.7. After an earlier version of this paper was posted, P. Etingof told me that the line $M_{\Gamma,\Gamma'}$ in 1.2 is the same as the line associated in [EGNO, 3.4.4] to the right $\mathbf{J}_{\Gamma \cap \Gamma' - 1}$-module $\mathbf{J}_{\Gamma \cap \Gamma' - 1}$ (viewed as a based module over a based ring) that is, the unique positive line $\mathcal{L}$ in $\mathbf{J}_{\Gamma \cap \Gamma' - 1}$ such that $\mathcal{L}$ is a right $\mathbf{J}_{\Gamma \cap \Gamma' - 1}$-submodule. (The discussion in loc.cit. concerns left (instead of right) indecomposable based modules over a fusion ring.) Indeed, from the definitions we see that $\mathcal{L}$ must be the same as $\mathcal{L}_{\Gamma,\Gamma'}$ in 2.3, hence the same as $M_{\Gamma,\Gamma'}$.

3. Special representations

3.1. When $\mathcal{H}$ is tensored with $\mathbf{C}$ (using the ring homomorphism $\mathcal{A} \rightarrow \mathbf{C}, v \mapsto 1$), then it becomes $\mathbf{C}[W]$, the group algebra of $W$. (For $w \in W$ we have $1 \otimes T_w = w \in$
\[ C[W]; \text{we denote } 1 \otimes c_w \text{ again by } c_{w^*}. \] We have a homomorphism of \( C \)-algebras (with 1) \( \psi : C[W] \rightarrow J \) given by

\[
\psi(c_x) = \sum_{d \in D, z \in W, d \sim z} h_{x, d, z} |_{v=1} t_z
\]

for all \( x \in W \), see [L6, 18.9]; this is an isomorphism, see [L6, 20.1]. For example, if \( W = \{ 1, s \} \) is of type \( A_1 \) we have \( \psi(c_1) = t_1 + t_s, \psi(c_s) = 2t_s \); hence \( \psi(1) = t_1 + t_s, \psi(s) = -t_1 + t_s \).

For each \( E \in \text{Irr} W \) let \( E_\infty \) be the simple \( J \)-module corresponding to \( E \) under \( \psi \) and let \( c_E \) be the unique two-sided cell of \( W \) such that \( J_{c_E} E \neq 0 \). (Note that \( E = E_\infty \) as \( C \)-vector spaces.) Let \( \text{Irr}^c W = \{ E \in \text{Irr} W ; c_E = c \} \) and let \( a' = a(xw_0) \) for any \( x \in c \), where \( w_0 \) is the longest element of \( W \).

For any \( k \in \mathbb{N} \) let \( G^k \) be the \( k \)-th symmetric power of the reflection representation of \( W \), viewed as a representation of \( W \) in an obvious way. For \( E \in \text{Irr} W \) let \( b_E \) be the smallest integer \( k \geq 0 \) such that \( E \) is a constituent of \( G^k \). Now for any \( E \in \text{Irr}^c W \) we have \( b_E \geq a' \) and there is a unique \( E \in \text{Irr}^c W \) such that \( b_E = a' \); this \( E \) is denoted by \( E^c \) and is called the special representation associated to \( c \). (This is a reformulation of the definition of special representations given in [L1].)

**Theorem 3.2.** In the setup of Theorem 1.2, for any \( \Gamma \in L \), we have \( M_\Gamma \cong E^c_\infty \) as \( J_c \)-modules.

We give two proofs; one is contained in 3.3.3.4,3.5. The other is given in 3.4,3.6.

3.3. In this subsection we assume that \( W \) is of type \( A, B \) or \( D \).

Let \( \Gamma \in L \). For any \( \Gamma' \in L \) we set

\[
\epsilon_{\Gamma'} = \sum_{z \in \Gamma \cap \Gamma'-1} t_z \in J^+_{\Gamma \cap \Gamma'-1}.
\]

By [L9, 4.8(b)], \( \{ \epsilon_{\Gamma'; \Gamma' \in L} \} \) is a \( C \)-basis of the unique \( J \)-submodule of \( J_\Gamma \) isomorphic to \( E^c_\infty \). By the uniqueness part of 1.2(a) this \( J \)-submodule of \( J_\Gamma \) (viewed as a \( J_c \)-module) must be the same as \( M_\Gamma \) in 1.2(a). We see that in this case, \( M_\Gamma \) is isomorphic to \( E^c_\infty \) and \( M_{\Gamma, \Gamma'} \) is the line spanned by \( \epsilon_{\Gamma'} \). In particular, 3.2 holds in our case.

3.4. In this subsection we assume that \( c \) is such that \( \text{Irr}^c W \) consists of exactly 2 irreducible representations. In this case, \( W \) is of type \( E_7 \) (resp. \( E_8 \)) and the 2 irreducible representations in \( \text{Irr}^c W \) have degree 512 (resp. 4096). Let \( \Gamma \in L \) and let \( d \in D \cap \Gamma \). The \( C \)-linear map \( r : J_\Gamma \rightarrow J_\Gamma \) given by left multiplication by \( (-1)^{l(d)} \psi(w_0) \) is in fact \( J \)-linear (since \( w_0 \) is central in \( W \)) and \( r(t_x) = t_{x^*} \) for any \( x \in \Gamma \), where \( x \rightarrow x^* \) is a certain fixed point free involution of \( \Gamma \), see [L7]. Then \( J^0_\Gamma = \{ \xi \in J_\Gamma ; r(\xi) = \xi \} \) is a simple \( J_c \)-submodule of \( J_\Gamma \) with \( C \)-basis \( \{ t_x + t_{x^*} ; x \in \Gamma_1 \} \) where \( \Gamma_1 \) is a set of representatives for the orbits of \( x \rightarrow x^* \) on \( \Gamma \). Note that, if \( x \in \Gamma \), then \( \{ x, x^* \} \) is the intersection of \( \Gamma \) with the inverse
of a left cell $\Gamma' \in L$; hence $t_x + t_{x'} \in J^+_T \cap \Gamma_{\Gamma' \cap -1}$. By the uniqueness part of 1.2(a), we must have $M_{\Gamma} = J^+_T$ and that for any $\Gamma' \in L$, $M_{\Gamma, \Gamma'}$ is the line spanned by $\sum_{x \in \Gamma_{\Gamma' \cap -1}} t_x$.

Now let $E \in \text{Irr}^c W$ be such that $E_{\infty} = J^+_T$. From the definitions we have $\text{tr}((-1)^{l(d)}w_0, E) = |\Gamma_1| = \dim E$. Hence, if $\epsilon = \pm 1$ is the scalar by which $w_0$ acts on $E$, then $(-1)^{l(d)}\epsilon = 1$. We have $l(d) = a(d) \mod 2$; hence $\epsilon = (-1)^{a(d)}$. But this equality characterizes the special representation in $\text{Irr}^c W$ (the special representation satisfies it, the nonspecial representation doesn’t satisfy it). We see that $M_{\Gamma} = J^+_T \cong E_{\infty}$. In particular, 3.2 holds in our case.

3.5. In this subsection we assume that $W$ is of exceptional type, but that $c$ is not as in 3.4. In this case, $E^c$ is the only representation in $\text{Irr}^c W$ of dimension equal to $|L|$; since $M_{\Gamma}$ (in 1.2(a)) has dimension equal to $|L|$, it follows that $M_{\Gamma} = J^+_T \cong E_{\infty}$. In particular, 3.2 holds in our case. This completes the proof of Theorem 3.2.

3.6. In this subsection we give a second proof of Theorem 3.2 assuming that $c$ is not as in 3.4. Let $a = a(w)$ for any $w \in c$. Let $\Gamma \in L$. Let $X = \sum_{w \in W} v^{-l(w)} T_w \in \mathcal{H}_x$. We can view $\mathcal{C}(v) \otimes J^+_T$ and $\mathcal{C}(v) \otimes M_{\Gamma}$ as $\mathcal{H}$-modules via $\Psi$ in 1.5. By [L9, 4.6], for any $x \in \Gamma$ we have

$$Xt_x = v^a \sum_{z \in c} t_z t_x \mod \sum_{i < a} v^i J^+_T.$$ 

By an argument as in the proof of 2.2(c), we see that $\sum_{z \in c} t_z t_x \in J^+_T$. It follows that if $\Gamma' \in L$ and $\xi \in M_{\Gamma, \Gamma'}$ then

$$X\xi = v^a \xi' \mod \sum_{i < a} v^i J^+_T$$

where $\xi' \in J^+_T$. In particular, we have $X\xi \neq 0$. Thus, $X(\mathcal{C}(v) \otimes M_{\Gamma}) \neq 0$. Using this and Theorem 4.2 in [L9] we deduce that the simple $\mathcal{H}$-module $\mathcal{C}(v) \otimes M_{\Gamma}$ is a constituent of the ”involution module” $M$ in [L9, 0.1] (with $Q(u)$ replaced by $\mathcal{C}(v)$). According to [L8] if a simple $\mathcal{H}$-module appears in $\mathcal{C}(v) \otimes J^+_T$ for every $\Gamma' \in L$ and it appears in $M$, then that $\mathcal{H}$-module corresponds to $E^c$. We deduce that $M_{\Gamma} \cong E_{\infty}$. This completes the second proof of Theorem 3.2, assuming that $c$ is not as in 3.4.

4. Equivariant vector bundles

4.1. In this section we fix a reductive, not necessarily connected algebraic group $G$ over $\mathbb{C}$ acting on a finite set $X$. Let $G \times X$ be the set of $G$-orbits on $X$. Representations of reductive groups over $\mathbb{C}$ are always assumed to be of finite dimension over $\mathbb{C}$ and algebraic. For $x \in X$ let $G_x = \{g \in G; gx = x\}$. Now $G$ acts diagonally on $X \times X$ and we can consider the Grothendieck group $K_G(X \times X)$ of $G$-equivariant complex vector bundles ($G$-eq.v.b.) on $X \times X$. This is an (associative) ring with
1 under convolution, denoted by \( * \) (see [L3, 2.2], [L4, 10.2]). For a \( G \)-eq.v.b. \( V \) on \( X \times X \) we denote by \( V_{x,y} \) the fibre of \( V \) at \((x, y) \in X \times X \). Let \( B \) be the set of pairs \((\Omega, \rho)\) where \( \Omega \) is a \( G \)-orbit \( \Omega \) in \( X \times X \) and \( \rho \) is an irreducible representation of \( G_{\Omega} \) (the isotropy group of a point \((x, y) \in \Omega\)). For any \((\Omega, \rho) \in B\) we denote by \( V_{\Omega, \rho} \) the \( G \)-eq.v.b. on \( X \times X \) such that \( V_{\Omega, \rho}|_{X \times X - \Omega} = 0 \) and the action of \( G_{\Omega} \) on \( V_{x,y}^{\Omega, \rho} \) is equivalent to \( \rho \). Now \( \{V_{\Omega, \rho}; (\Omega, \rho) \in B\} \) is a \( \mathbb{Z} \)-basis of \( K_{G}(X \times X) \). Let 

\[ K_{G}(X \times X) = C \otimes K_{G}(X \times X), \text{ viewed as a } C\text{-algebra.} \]

4.2. In this subsection we assume that \( G \) is finite. Let \( \omega, \omega' \) be in \( G \setminus X \). Let \( V_{\omega, \omega'} \) be the \( G \)-eq.v.b. on \( X \times X \) such that \( V_{a,b}^{\omega, \omega'} = C[G] \) if \((a, b) \in \omega \times \omega'\), \( V_{a,b}^{\omega, \omega'} = 0 \) if \((a, b) \notin \omega \times \omega'\). (Here \( C[G] \) is the left regular representation of \( G \).) The \( G \)-action

\[ g : V_{a,b}^{\omega, \omega'} \rightarrow V_{ga,gb}^{\omega, \omega'} \]

is left translation by \( g \) on \( C[G] \) (if \((a, b) \in \omega \times \omega'\) and is 0 if \((a, b) \notin \omega \times \omega'\).

We show:

(a) Let \((Om, \rho) \in B\); we have \( \Omega \subset \omega_1 \times \omega'_1 \) where \( \omega_1, \omega'_1 \) are in \( G \setminus X \). Then

\[ U' := V_{\Omega, \rho}^{\omega, \omega'} \]

is isomorphic to a direct sum of copies of the single \( G \)-eq.v.b.

\[ V_{\omega_1, \omega'}^{\omega, \omega'} \]. More precisely, if \( \omega_1 \neq \omega \), we have \( U' = 0 \); if \( \omega_1 = \omega \), we have \( U' = V_{\omega_1, \omega'}^{\omega, \omega'} \). For \((a, b) \in X \times X \) we have

\[ U_{a,b}' = \oplus_{z \in \omega_1 \times \omega_1} V_{a,z}^{\Omega_1, \rho} \otimes C[G] \text{ if } b \in \omega', \]

\[ U_{a,b}' = 0 \text{ if } b \notin \omega'. \]

Thus the support of \( U' \) is contained in \( \omega_1 \times \omega' \) and \( U' = 0 \) unless \( \omega_1 = \omega \). We now assume that \( \omega_1 = \omega \) and \((a, b) \in \omega_1 \times \omega'\). Then \( U_{a,b}' = \oplus_{z \in \omega_1 \times \omega_1} V_{a,z}^{\Omega_1, \rho} \otimes C[G] \). We have \( \dim U_{a,b}' = d[G]|\Omega_1|/|\omega_1| \). We show:

(b) as a \( G_a \cap G_b \)-module, \( U_{a,b}' \) is a multiple of the regular representation.

Let \( \sigma_1, \ldots, \sigma_k \) be the various \( G_a \cap G_b \)-orbits contained in \( \omega' \). We have \( U_{a,b}' = \sum_{i=1}^{k} R_i \), where \( R_i = \oplus_{z \in \sigma_i} V_{a,z}^{\Omega_1, \rho} \otimes C[G] \). We pick \( z_i \in \sigma_i \). Now

\[ R_i = \text{ind}_{G_a \cap G_b \cap G_{z_i}}^{G_a \cap G_b} (A \otimes B) \]

where

\[ A = \text{res}_{G_a \cap G_b \cap G_{z_i}}^{G_a \cap G_b \cap G_{z_i}} (V_{a,z_i}^{\Omega_1, \rho}), \quad B = \text{res}_{G_a \cap G_b \cap G_{z_i}}^{G_a \cap G_b \cap G_{z_i}} (C[G]). \]

It is enough to show that \( R_i \) is a multiple of the regular representation of \( G_a \cap G_b \).

Since \( R_i \) is induced, it is enough to show that \( A \otimes B \) is a multiple of the regular representation of \( G_a \cap G_b \cap G_{z_i} \). It is also enough to show that \( B \) is a multiple of the regular representation of \( G_a \cap G_b \cap G_{z_i} \). This follows from the fact that \( C[G] \) is a multiple of the regular representation of \( G_{z_i} \cap G_b \). This proves (b). Now (a) follows.

Note that in \( K_{G}(X \times X) \) we have

\[ V_{\omega, \omega'}^{\omega, \omega'} = \sum_{(\Omega, \rho) \in B; \Omega \subset \omega \times \omega'} \dim \rho|\Omega| V_{\Omega, \rho}. \]
4.3. We now drop the assumption (in 4.2) that \( G \) is finite. Let \( \hat{K}_G(X \times X) \) be the \( C \)-vector space consisting of formal (possibly infinite) linear combinations \( \sum_{(\Omega, \rho) \in B} f_{\Omega, \rho} V^{\Omega, \rho} \) where \( f_{\Omega, \rho} \in C \). The left \( K_G(X \times X) \)-module structure on \( K_G(X \times X) \) given by left multiplication extends naturally to a left \( K_G(X \times X) \)-module structure on \( \hat{K}_G(X \times X) \). If \( \omega, \omega' \) are as in \( G \setminus X \) then we can define \( V^{\omega, \omega'} \in K_G(X \times X) \) by the sum 4.2(c) (which is now a possibly infinite sum); we set \( \hat{V}^{\omega, \omega'} = |\omega|^{-1} V^{\omega, \omega'} \) so that

\[
V^{\omega, \omega'} = \sum_{(\Omega, \rho) \in B; \Omega \subseteq \omega \times \omega'} \dim \rho |\Omega| |\omega|^{-1} V^{\Omega, \rho}.
\]

Now formula 4.2(a) extends to the present case as follows. Let \( (\Omega, \rho) \in B; \) we have \( \Omega \subseteq \omega_1 \times \omega_1' \) where \( \omega_1, \omega_1' \) are \( G \)-orbits in \( X \). Then

\[
V^{\Omega, \rho} V^{\omega, \omega'} = NV^{\omega_1, \omega'}
\]

where \( N \in Z \) is 0 if \( \omega_1' \neq \omega \) and \( N = \dim \rho |\Omega||\omega_1|^{-1} \) if \( \omega_1' = \omega \). Hence

\[
V^{\Omega, \rho} \hat{V}^{\omega, \omega'} = N' \hat{V}^{\omega_1, \omega'}
\]

where \( N' \in Z \) is 0 if \( \omega_1' \neq \omega \) and \( N' = \dim \rho |\Omega||\omega|^{-1} \) if \( \omega_1' = \omega \).

Let \( \omega' \in G \setminus X \). Let \( \hat{R}_{\omega'} \) be the subspace of \( \hat{K}_H(X \times X) \) consisting of formal (possibly infinite) linear combinations \( \sum_{(\Omega, \rho) \in B; \text{pr}_2 \Omega = \omega'} f_{\Omega, \rho} V^{\Omega, \rho} \) with \( f_{\Omega, \rho} \in C \). Here \( \text{pr}_2 : X \times X \to X \) is the second projection. Note that \( \hat{R}_{\omega'} \) is a \( K_G(X \times X) \)-submodule of \( \hat{K}_H(X \times X) \).

Let \( R_{\omega'} \) be the subspace of \( \hat{K}_G(X \times X) \) with basis formed by the elements \( \hat{V}^{\omega, \omega'} \) for various \( \omega \in G \setminus X \). Using (c) we see that \( R_{\omega'} \) is a (simple) \( K_G(X \times X) \)-submodule of \( \hat{K}_G(X \times X) \); we have \( R_{\omega'} \subset \hat{R}_{\omega'} \). Using (c) we see also that if \( \omega'' \in G \setminus X \) then \( \hat{V}^{\omega, \omega'} \mapsto \hat{V}^{\omega, \omega''} \) defines an isomorphism of \( K_G(X \times X) \)-modules \( R_{\omega'} \iso R_{\omega''} \). Hence the isomorphism class of the \( K_G(X \times X) \)-module \( R_{\omega'} \) is independent of the choice of \( \omega' \).

4.4. We now assume that \( G \) is the finite group associated to \( c \) in \([L2]\) and that \( X \) is the finite \( G \)-set \( \bigoplus_{\Gamma \in L} G/H_{\Gamma} \) where \( H_{\Gamma} \) is the subgroup of \( G \) defined in \([L3, 3.8]\). In this case we have \( \hat{K}_G(X \times X) = K_G(X \times X) \). By a conjecture in \([L3, 3.15]\), proved in \([BFO]\), there exists an isomorphism of \( C \)-algebras \( \chi : K_G(X \times X) \iso J_c \) carrying the basis \( (V^{\Omega, \rho}) \) of \( K_G(X \times X) \) onto the basis \( \{t_x; x \in c\} \) of \( J_c \). Under \( \chi \), the left ideal \( J_{\Gamma} \) of \( J_c \) (for \( \Gamma \in L \)) corresponds to the left ideal \( \hat{R}_{\omega'} \) of \( K_G(X \times X) \) where \( \omega' \in G \setminus X \) corresponds to \( \Gamma \), and the basis \( \{t_x; x \in \Gamma\} \) of \( J_{\Gamma} \) corresponds to the intersection of the basis \( (V^{\Omega, \rho}) \) of \( K_G(X \times X) \) with \( \hat{R}_{\omega'} \). The basis of \( R_{\omega'} \) formed by the elements \( \hat{V}^{\omega, \omega'} \) corresponds to a family of elements \( \{e_{\Gamma'}; \Gamma' \in L\} \) in \( J_{\Gamma} \).
From 4.3(a) we see that \( e_{\Gamma'} \in J_{\Gamma \cap \Gamma' - 1}^+ \) for any \( \Gamma' \in L \) (in fact the coefficients of the various \( t_x, x \in \Gamma \cap \Gamma' - 1 \) are in \( \mathbb{Z}_{>0} \)) and from 4.3(c) we see that for \( u \in c \) the product \( t_u e_{\Gamma'} \) is a \( \mathbb{Z}_{\geq 0} \) multiple of an element \( e_{\Gamma''} \). We see that the \( C \)-subspace of \( J_{\Gamma} \) spanned by \( \{ e_{\Gamma'}; \Gamma' \in L \} \) satisfies property (\( \forall \)) in 1.2(a) hence it is equal to \( M_{\Gamma} \). This provides another proof in our case for the existence part of 1.2(a), with the additional integrality properties in 1.7.

5. Final remarks

5.1. Theorem 1.2 and its proof remain valid if \( W \) is replaced by an affine Weyl group (with \( c \) assumed to be finite) or by a finite Coxeter group; in the last case we use the positivity property of \( h_{x,y,z} \) established in [EW]. In these cases, the simple \( J_c \)-module given by Theorem 1.2 will be called the special \( J_c \)-module.

5.2. Assume now that \( W \) is an (irreducible) affine Weyl group and that \( c \) is a not necessarily finite two-sided cell of \( W \). We denote again by \( L \) the set of left cells of \( W \) that are contained in \( c \); this is a finite set. Then the \( C \)-algebra \( J_c \) with its basis \( \{ t_x; x \in c \} \) is defined. Let \( J J_c \) be the set of formal (possibly infinite) linear combinations \( \sum_{u \in c} f_u t_u \) where \( f_u \in C \). This is naturally a left \( J_c \)-module. For any subset \( X \) of \( c \) let \( \hat{J}_X \) be the set of all \( \sum_{u \in c} f_u t_u \in \hat{J}_c \) such that \( f_u = 0 \) for \( u \in c - X \). If \( \Gamma \in L \), then \( \hat{J}_\Gamma \) is a \( J_c \)-submodule of \( J_c \). We have \( \hat{J}_\Gamma = \oplus_{\Gamma' \in L} \hat{J}_{\Gamma \cap \Gamma' - 1} \).

According to a conjecture in [L4, 10.5], proved in [BFO], we can find \( G, X \) as in 4.1 and an isomorphism of \( C \)-algebras \( \chi : K_G(X \times X) \xrightarrow{\sim} J_c \) carrying the basis \( \{ V^{\Omega, \rho}; (\Omega, \rho) \in B \} \) of \( K_G(X \times X) \) onto the basis \( \{ t_x; x \in c \} \) of \( J_c \). This extends in an obvious way to an isomorphism \( \hat{\chi} : K_G(X \times X) \xrightarrow{\sim} \hat{J}_c \) under which the left \( K_G(X \times X) \)-module structure on \( \hat{J}_c \) extends to \( J_c \). If \( \Gamma \in L \), there is a unique \( \omega' \in G \setminus X \) (the set of \( G \)-orbits in \( X \)) such that \( \hat{\chi} \) carries \( R_{\omega'} \) (see 4.3) onto a (simple) \( J_c \)-submodule \( M_{\Gamma} \) of \( \hat{J}_\Gamma \) whose isomorphism class is independent of \( \Gamma \); we say that this is the special \( J_c \)-module.

The \( J_c \)-module \( M_{\Gamma} \) admits a basis \( \{ e_{\Gamma'}; \Gamma' \in L \} \) in which any \( t_u \) (with \( u \in c \)) acts by a matrix with all entries in \( \mathbb{Z}_{\geq 0} \), namely the basis corresponding to the basis \( \{ \tilde{V}^{\omega, \omega'}; \omega \in G \setminus X \} \) of \( R_{\omega'} \).

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