Are Black Holes in Brans-Dicke Theory precisely the same as in General Relativity?

M. Campanelli* and C. O. Lousto**

Universität Konstanz, Fakultät für Physik,
Postfach 5560, D-7750 Konstanz, Germany.
E-mail: phlousto@dknkurz1

Abstract:

We study a three-parameters family of solutions of the Brans-Dicke field equations. They are static and spherically symmetric. We find the range of parameters for which this solution represents a black hole different from the Schwarzschild one. We find a subfamily of solutions which agrees with experiments and observations in the solar system. We discuss some astrophysical applications and the consequences on the "no hair" theorems for black holes.

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* On leave from the Dipartimento di Fisica, Università di Perugia, Italy. E-mail: phsirio@dknkurz1
** Permanent Address: Instituto de Astronomía y Física del Espacio, Casilla de Correo 67 - Sucursal 28, 1428 Buenos Aires, Argentina. E-mail: lousto@iafe.edu.ar
1. Introduction

Lately there have been some renewed interest in the Brans-Dicke theory of gravitation. On one hand, it has been applied to cosmological models of the universe during the inflationary era to make more natural bubble percolation \[1\]. Also, it was found that in the low-energy regime, the theory of fundamental strings can be reduced to an effective Brans-Dicke one\[2\]. The subject of gravitational collapse, however, has not yet been thoroughly studied. One of the outstanding results on this field is the Hawking theorem\[3\], that states that the Schwarzschild metric is the only spherically symmetric solution of vacuum Brans-Dicke field equations. The proof of this theorem goes through the fact that the Brans-Dicke scalar field \( \phi \) must be constant outside the black hole and the use of the weak energy condition. In this paper we study a three-parameters family of solutions of Brans-Dicke equations which is static and spherically symmetric. We study under which range of the parameters we can have non-singular (at the horizon) black hole solutions. We are able to obtain explicit examples where the metric represents a black hole solution different from the Schwarzschild one:

\[
ds^2 = -A(r)^{1-n} dt^2 + A(r)^{n-1} dr^2 + r^2 A(r)^n d\Omega^2 \quad A(r) = 1 - 2\frac{r_0}{r} \quad n \leq -1 \ . \quad (1)\]

where \( r_0 \) is an arbitrary constant and \( n \) represents a scalar hair.

Classical scalar hairs in General Relativity Black Hole solutions have already been found for several coupling. These include the case of an axion with an \( R \tilde{R} \) coupling \[4\], and a dilaton \[5\] and an axionlike scalar field \[6\] coupled to Einstein Maxwell theory. A conformally coupled scalar field can have a static \[7\], but unstable \[8\] solution.

The Brans-Dicke theory \[9\] incorporates the Mach principle, which states that the phenomenon of inertia must arise from accelerations with respect to the general mass distribution of the universe. This theory is self-consistent, complete and for \( |\omega| \geq 500 \) in accord with solar system observations and experiments \[10\]. It is, in some sense, the simplest extension of General Relativity. It introduces an additional long-range scalar field \( \phi \) besides the metric tensor of the spacetime \( g_{\mu\nu} \) from which are constructed the covariant derivative and the curvature tensors, in the usual manner. \( \omega \) is the Dicke dimensionless coupling constant.

The theory is metric, i.e. the weak equivalence principle is satisfied. The matter couples minimally to the metric and not directly to \( \phi \). The scalar field does not exert any direct influence on matter, its only role is that of participating in the field equations that determine the geometry of the spacetime.

The action for the Brans-Dicke theory is:

\[
S = \int dx^4 \sqrt{-g} [\phi R - \omega (\phi,\phi ; \alpha) / \phi + 16\pi L_{\text{matter}}] \ . \quad (2)
\]

The variational principle gives the field equations:

\[
G_{\alpha\beta} = \frac{8\pi}{\phi} T_{\alpha\beta} + \frac{\omega}{\phi^2} \left( \phi,\phi ; \alpha,\beta - \frac{1}{2} g_{\alpha\beta} \phi,\phi ; \mu,\mu \right) + \frac{1}{\phi} (\phi,\phi ; \alpha,\beta - g_{\alpha\beta} \Box \phi) \ . \quad (3)
\]
The matter stress-energy tensor and $\phi$ together generate the metric. The field equation for $\phi$ is:

$$\phi_{,\alpha}^\alpha = \square \phi = \frac{8\pi G}{3 + 2\omega} T .$$  \hspace{1cm} (4)$$

In the next section we study a solution\cite{9} of the vacuum field Eqs. (3)-(4). This is a three-parameters static spherically symmetric metric. We study the asymptotic behavior, the occurrence of singularities and event horizons. In the third section we study the special cases for which this metric can be of astrophysical relevance. We compute the geodesics equations, post Newtonian parameters, energy, period and redshift of the last stable circular orbit, dispersion cross sections, Kruskal transformations and Hawking temperature. We end the paper with the discussion of the obtained results, in particular, the relevance of the non-Schwarzschild-like black holes found in the Brans-Dicke theory.

2. Static Spherically Symmetric Vacuum Solutions

The Brans-Dicke vacuum field equations can be written as:

$$R_{\alpha\beta} = \frac{\omega}{\phi^2} \phi_{,\alpha} \phi_{,\beta} + \frac{\phi_{,\alpha\beta}}{\phi} ,$$  \hspace{1cm} (5)$$

$$\square \phi = 0 .$$  \hspace{1cm} (6)$$

It is easy to show that a power generalization of the Schwarzschild metric is a solution of this equations\cite{9,11}:

$$ds^2 = -A(r)^{m+1} dt^2 + A(r)^{n-1} dr^2 + r^2 A(r)^n d\Omega^2 ;$$  \hspace{1cm} (7)$$

$$d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta \varphi^2 ;$$

$$A(r) = 1 - 2 \frac{r_0}{r} ,$$

and the scalar field:

$$\phi(r) = \phi_0 A(r)^{-\frac{m+n}{2}} ;$$  \hspace{1cm} (8)$$

where $m$, $n$, $\phi_0$ and $r_0$ are arbitrary constants. The coupling constant is found from:

$$\omega = -2 \frac{(m^2 + n^2 + nm + m - n)}{(m + n)^2} .$$  \hspace{1cm} (9)$$

Either from (7) or (8) we can compute the components of the Ricci tensor:

$$R_{00} = (m + 1)(m + n) \frac{r_0^2}{r^4} A(r)^{m-n} ,$$  \hspace{1cm} (10)$$

$$R_{11} = (-m^2 + nm + 3n - m) \frac{r_0^2}{r^4} A(r)^{-2} + 2(m + n) \frac{r_0}{r^3} A(r)^{-1} ,$$  \hspace{1cm} (11)$$
\begin{align}
R_{22} &= -n(m+n) \left( \frac{r_0^2}{r^2} \right) A(r)^{-1} - (m+n) \frac{r_0}{r} , \label{eq:12} \\
R_{33} &= \sin^2 \vartheta R_{22} , \label{eq:13} \\
R_{\mu\nu} &= 0 \quad (\mu \neq \nu) . \label{eq:14}
\end{align}

And the curvature \( R \) is given by:
\begin{equation}
R = -2(m^2 + n^2 + mn + m - n) \frac{r_0^2}{r^4} A(r)^{-n-1} . \label{eq:14}
\end{equation}

We observe that it vanishes like \( r^{-4} \) as \( r \to \infty \).

We will study now the geometrical properties of the metric (7) for given values of the parameters \( m \) and \( n \).

To see that the metric (7) is asymptotically flat it is enough to show that the metric components behave in an appropriate way at large \( r \)-coordinate values, e.g., \( g_{\mu\nu} = \eta_{\mu\nu} + O(1/r) \) as \( r \to \infty \). By inspection of the coefficients, we verify that this is so. No matter which power of \( A(r) \), can be written as a binomial series:
\begin{equation}
A(r)^q = \left( 1 - 2 \frac{r_0}{r} \right)^q = 1 - q \frac{2r_0}{r} + q(q-1) \left( \frac{2r_0}{r} \right)^2 + .... \label{eq:15}
\end{equation}

Thus, asymptotically flatness is verified for every value of \( m \) and \( n \).

To study the occurrence of true singularities of the metric (7), (not coordinate system pathologies), it is enough for us to examine scalars formed out of the curvature. In particular, the scalar invariant:
\begin{align}
I &= R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \\
&= 4r_0^2 (r - r_0)^{-2(n+1)} r^{-4+2n} \left\{ \left( \frac{r_0}{r} \right)^2 I_1(m,n) + 4 \left( \frac{r_0}{r} \right) I_2(m,n) + 6I_3(m,n) \right\} , \label{eq:16}
\end{align}
where:
\begin{align}
I_1(m,n) &= 48 + 56m + 41m^2 + 10m^3 + m^4 - 56n - 34mn + \\
&\quad -20m^2n - 2m^3n + 29n^2 + 6mn^2 + 3m^2n^2 - 8n^3 + n^4 , \label{eq:17} \\
I_2(m,n) &= -12 - 13m - 8m^2 - m^3 + 13n + 4mn + 2m^2n - 6n^2 + n^3 , \label{eq:18} \\
I_3(m,n) &= (m+1)^2 + (n-1)^2 . \label{eq:19}
\end{align}

From this expression we observe that the invariant goes always to zero as \( r \to \infty \):
\begin{equation}
I \to O(r^{-6}) \quad , \label{eq:20}
\end{equation}
as \( r \to \infty \)

\begin{equation}
4
\end{equation}
unless \( m = -1 \) and \( n = 1 \). In this case, \( I_3(-1, 1) = 0 \) and also \( I_2(-1, 1) = 0 \). Thus,

\[
I(-1, 1) = 48r_0^4 r^{-8} .
\]  

(20)

The metric reads particularly simple in this case:

\[
ds^2 = -dt^2 + dr^2 + r^2 A(r) d\Omega^2 .
\]

We recall that for the Schwarzschild metric \( (m = n = 0) \):

\[
I_{\text{Schw.}} = 48r_0 .
\]  

(21)

We are also interested in studying the behavior of the scalar invariants as \( r \to 2r_0 \). From expression (16) we see that

\[
I \to O[(r - 2r_0)^{-2 - 2n}] ,
\]  

as \( r \to 2r_0 \)

thus, for not occurring a singularity at \( r = 2r_0 \) we must have:

\[
n \leq -1, \quad \text{no singularity at } r = 2r_0 .
\]  

(23)

This condition can also be obtained from asking a non-singular behavior of the scalar curvature \( R \), given by Eq. (14). Notice that \( n \leq -1 \) makes \( g_{\theta \theta} \) singular at the horizon. However, this is only a coordinate singularity since the scalar invariants, as we have seen, are all finite on the horizon.

One additional non-singular case is given when the term between curly brackets in Eq. (16) vanishes, i. e.,

\[
m = n = 0; \quad \text{the Schwarzschild metric} .
\]  

(24)

The other interesting value of the radial coordinate to study is \( r = 0 \). In this case, we see that:

\[
I \to O(r^{-6+2n}) ,
\]  

as \( r \to 0 \)

except when \( I_1 = 0 \), that is for \( n = -m = 2 \). In this case also \( I_2 = 0 \), then

\[
I(-2, 2) = \frac{48r_0^2}{r^6} ,
\]  

as \( r \to 0 \)
which is the same as in the Schwarzschild case. In fact, it is easy to show that for \( n = -m = 2 \) the metric (7) can be carried into Schwarzschild form. The transformation \( \chi = r - 2r_0 \) and the identification of \(-r_0\) with the mass \( M \) make the job.

We would like now to study the occurrence of an event horizon at \( r = 2r_0 \). Let us first observe that the Killing vector \( \xi(t) = \sqrt{-g_{00}}\partial_t = A(r)^{m+1} \partial_t \) becomes null at \( r = 2r_0 \) when \( m + 1 > 0 \). We can thus study the outgoing null geodesics from \( r \geq 2r_0 \) and see under which conditions \( r = 2r_0 \) is an outgoing null surface.

The first integral of the geodesics motion (related to the time and two angular variables) in our spherically symmetric gravitational field can be written as:

\[
\vartheta = \frac{\pi}{2} ,
\]

\[
r^2A(r)^n\frac{d\varphi}{d\lambda} = J ,
\]

\[
A(r)^{m+1}\frac{dt}{d\lambda} = E ,
\]

\[
\left(\frac{dr}{d\lambda}\right)^2 = A(r)^{-n+1}\left\{E^2A(r)^{-m-1} - J^2r^{-2}A(r)^{-n} + \epsilon\right\} ,
\]

where \( \epsilon = 0, \pm 1 \) for null, spacelike and timelike geodesics respectively.

We can describe the radial part of motion in terms of the effective potential. Then, for null geodesics, we define the impact parameter \( b \) as:

\[
b = \frac{J}{E} .
\]

The critical impact parameters \( b_c \) for which photons with \( b > b_c \) can escape to infinity and with \( b < b_c \) are absorbed by the black holes, can be found to be:

\[
\left.\frac{\partial V_{eff}}{\partial r}\right|_{r_c} = 0 , \quad \left.\frac{dr}{d\lambda}(b_c)\right|_{r_c} = 0 , \quad \left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r_c} < 0 .
\]

Thus, the radial coordinate of the critical periastrom \( r_c \) is:

\[
\frac{r_c}{r_0} = 3 + m - m ,
\]

and

\[
b_c = r_0 \frac{(3 + m - n)^{m-n+3}}{(1 + m - n)^{m-n+3}} .
\]

An observer at rest in our gravitational field measures the velocity of a photon relative to his orthonormal frame \([12]\):

\[
v_\phi = \frac{\sqrt{g_{\phi\phi}}d\phi/d\lambda}{\sqrt{-g_{00}}dt/d\lambda} = \frac{b}{r}A(r)^{\frac{m-n+1}{2}} .
\]
A photon at \( r < r_c \) will eventually escape to infinity instead of being trapped by the black hole at \( r = 2r_0 \) if \( \nu_r \) is positive and:

\[
\sin \delta < \frac{b_c}{r} A(r)^{-\frac{m(n+1)}{2}},
\]

where \( \delta \) is the angle between the propagation direction and radial direction.

Thus, we conclude that the surface \( r = 2r_0 \) will act as an event horizon whenever:

\[
m - n + 1 > 0.
\]

(36)

An alternative derivation of the horizon properties of the surface \( r = 2r_0 \) can be obtained by the study of the outgoing radial null geodesics. In fact, the time spent by a photon emitted at \( r_i \) to reach \( r_f \) as measured by an observer at infinity is given by

\[
\Delta t = \int_{t_i}^{t_f} dt = \int_{r_i}^{r_f} A(r) \frac{n-m+2}{2} dr = \left[ r A(r)^{\frac{n-m}{2}} \right]_{r_i}^{r_f} + O \left( A(r)^{\frac{n-m}{2}+1} \right).
\]

when \( m - n \geq 0 \) as \( r_i \rightarrow 2r_0 \) the photon will need a \( \Delta t \rightarrow \infty \) to leave the horizon neighborhood, thus indicating the presence of an event horizon at \( r = 2r_0 \). For \( n \leq -1 \) (eq(23)) we have not singularities on the surface \( r = 2r_0 \). However, \( g_{00} \) diverges there thus giving an infinite horizon area. This is only a purely geometrical divergence bringing no physical consequences. In fact, we have seen that the surface \( r = 2r_0 \) effectively acts as an event horizon with respect to null rays. For massive particles there neither any inconvenient to enter in to the black hole in a finite proper time since its effective potential, \( V_{eff} \):

\[
E^2 - V_{eff} = \left( \frac{dr}{d\tau} \right)^2 = A(r)^{1-n} \left[ E^2 A(r)^{-(m+1)} - \frac{J^2}{r^2} A(r)^{-n} - 1 \right],
\]

remains bounded at and outside the horizon. Besides, tidal effects on the horizon are finite since curvature tensors are well behaved there.

3. Astrophysical applications and discussion

For not only dealing with the mathematical aspects of the solution and to obtain further restrictions on \( m \) and \( n \), we will briefly study some astrophysical consequences. In particular, we will compute some physical quantities and show how much different they are from the General Relativistic results. Thus, confirming that metric(2) is indeed not the Schwarzschild one.

Nature has the final word to decide between mathematical models. Thus, to see if the family of solutions of Brans-Dicke equations could represent nature, we can start
by computing its post Newtonian parameters (PPN). For a static spherically symmetric metric we can write the PPN metric as \[ ds^2 = - \left[ 1 - 2 \left( \frac{M}{r} \right) + 2\beta \left( \frac{M}{r} \right)^2 \right] dt^2 + \left[ 1 + 2\gamma \left( \frac{M}{r} \right) \right] \left( dr^2 + r^2 d\Omega^2 \right) , \] (37)

where \( \beta \) and \( \gamma \) are two of the ten PPN parameters measuring, respectively, the amount of nonlinearity in the superposition law for \( g_{00} \) and the amount of space curvature produced by the unit rest mass.

By transforming our metric (7) to isotropic radial coordinates, \( \bar{r} \),
\[
\bar{r} = \bar{r}(1 + \bar{r}_0/\bar{r})^2 ; \quad \bar{r}_0 = r_0/2 ,
\]
we find \[ ds^2 = - \left( \frac{1 - \bar{r}_0/\bar{r}}{1 + \bar{r}_0/\bar{r}} \right)^{2(m+1)} (1 + \bar{r}_0/\bar{r})^4 \left( 1 - \bar{r}_0/\bar{r} \right)^{2n} \left( 1 + \bar{r}_0/\bar{r} \right)^{2n} \left( dr^2 + r^2 d\Omega^2 \right) . \] (39)

By expanding the coefficients of this metric and comparing them to those of Eq. (37) we obtain:
\[
\beta = 1 ; \quad \gamma = \frac{1 - n}{m + 1} ; \quad M = (m + 1)r_0 .
\]
(40)

Thus, when \( m \to -n \) we have agreement with the solar system experiments. In particular, results of time delay measurements gives \[ |\gamma - 1| < 10^{-3} . \]

To find observational differences between metric (7) with \( m \to -n \) (see (1)), and the Schwarzschild one, we must, then, look at strong gravitational field effects. We study some of such effects in accretion disks, scattering of photons and Hawking radiation.

The standard model of galactic hard X-ray sources is a binary stellar system formed by a normal star transferring matter onto its companion star, which is a compact object. This matter, falling inward in quasi-circular orbits, will form an accretion disk, which will emit the observed X-rays.

The friction due to viscosity will generate heat, which is radiated away through the disk surfaces. This energy is supplied by the loss of the total energy of the gas, while going through the disk, down to the last stable circular orbit. After this, the gas would fall almost without radiating \[ E \sim 1 . \]

Using Schwarzschild’s metric (see ref. \[ 14 \] also for the Kerr case), the last stable circular orbit has a radial coordinate \( r_c = 6M \), where \( M \) is the black hole mass. At this \( r_c \) the energy “at infinity” per rest energy is \( E_c = (8/9)^{1/2} \). If we take \( E \approx 1 \) at the external radius of the disk and a steady flux of matter (or its temporal average) the total luminosity of the accretion disk will be:
\[
L = (1 - E_c)\dot{M} ,
\]
(41)

with \( \dot{M} \) = mass per unit time entering the disk.
When we compute $E_c$ from metric (7) we obtain (for $-n$ large):

$$\frac{L^{BD}}{L^{GR}} = 0.958 \quad (42)$$

Another potentially observable quantities are the orbital frequency of the last stable circular orbit (as seen by an observer at infinity) and the redshift at infinity which are given by \cite{15}

$$\frac{\nu_c^{BD}}{\nu_c^{GR}} = 0.931 ; \quad \frac{Z_c^{BD}}{Z_c^{GR}} = 0.936 \quad , \quad n \to -\infty \quad (43)$$

As we have seen, the metric (7) is in agreement with the solar system observations and experiments when $m \to -n$. In addition, $n \leq -1$ for having a regular horizon. Thus, Eqs. (42)-(43) give results close to those of General Relativity.

Unfortunately, present uncertainties in the modeling and observation of accretion disk do not provide accurate enough data to discriminate between metric (1) and the Schwarzschild one.

From the study of null geodesics we made in the last section we can obtain the total scattering cross section for photons:

$$\sigma = \pi b_c^2 = \pi \frac{M^2}{(m+1)^2} \frac{(3+m-n)^{m-n+3}}{(1+m-n)^{m-n+1}} . \quad (44)$$

Let us observe that $m = n$ gives the same results as for a Schwarzschild black hole. This is so, because when $m = n$ the metric (7) can be written conformal to the Schwarzschild one, i.e. $ds^2 = A(r)^n ds^2_{Schw}$ and light rays do not "feel" conformal factors.

When we compare this cross section to the General Relativistic result, $\sigma^{GR} = 27\pi M^2$, in the case $m \to -n$ and $n \leq -1$, we find that, again, the results are close to those produced by the Schwarzschild metric. As $n$ goes to more negative values we have a quick convergence to the asymptotic value:

$$\frac{\sigma^{BD}}{\sigma^{GR}} = 1.095 \quad , \quad n \to -\infty \quad (45)$$

From the results above one sees that the Brans-Dicke gravitational field studied seems to be weaker than the Schwarzschild one. This conclusion will be reinforced when we compute the surface gravity on the horizon. Here we find the relatively strongest difference from the Schwarzschild’s results. The surface gravity plays an important role when one studies the thermodynamics of black holes because it is related to the temperature associated to quantum effects close to the horizon. For a static spherically symmetric system it is given by:

$$K = -\frac{1}{2} \frac{g_{00}'}{\sqrt{-g_{00}g_{rr}}} = (m + 1) \frac{r_0}{r^2} A(r)^{\frac{m-n}{2}} . \quad (46)$$
When we evaluate it at $r = 2r_0$ we find:

$$K_H = \begin{cases} 0, & \text{for } m > n \\ \infty, & \text{for } m < n \\ \frac{m+1}{4r_0} = \frac{1}{4M} = k_{Schw}, & \text{for } m = n \end{cases},$$

(47)

thus, we obtain the Schwarzschild value for the conformal case $m = n$.

We can write also our metric in terms of Kruskal-like variables. To this end, let us define first the null variables $\bar{u}$ and $\bar{v}$ by

$$d\bar{u} = dt - dr^*; \quad d\bar{v} = dt + dr^*,$$

(48)

where:

$$dr^* = rA(r)\frac{n-m}{2} - 1 dr,$$

(49)

and then to

$$u = -\exp(-K_H\bar{u}); \quad v = \exp(K_H\bar{v}),$$

(50)

where $K_H$ is the surface gravity evaluated at $r = 2r_0$.

Finally, we obtain \[16\]

$$ds^2 = -A(r)^{m+1}K_H^{-2}\exp(-2K_Hr^*)dudv + r^2d\Omega^2.$$

(51)

When $r \to 2r_0$ and $m = n$ we have

$$g_{uv}(2r_0) = e^{m+1}\left(\frac{4r_0}{m+1}\right)^2.$$

(52)

So, metric coefficient are finite on the horizon.

When we bring together all the conditions for having a regular black hole, we obtain: $n \leq -1$ for the horizon not being a singular surface (Eq. (23)) and $m - n + 1 > 0$ for $r = 2r_0$ acting as an event horizon (Eq. (36)). If in addition, we ask that the solution should be in agreement with the observations carried out in the solar system, the PPN parameters should coincide with those of General Relativity with great precision. As we have seen, this is achieved when $|\omega| \to \infty$, i.e. $m + n \cong 0$. This, in turn, gives a constant scalar field outside the horizon (see Eq. (8)). It is notably that in this case, the Ricci tensor has one of its components different from zero, i.e.

$$R_{11} = 2n(2-n)\frac{r_0^2}{r^4} \left(1 - 2\frac{r_0}{r}\right)^{-2}.$$

(53)

This is so, because in spite of the scalar field $\phi$ being constant and thus its derivatives going to zero, the coupling constant $|\omega|$ goes to infinity in such a way that the product appearing on the right hand side of the field equations (5) gives a finite value, i.e. (Eq.(53)). This fact has very important consequences for the Hawking theorem \[3\] establishing the identity
of Brans-Dicke and General Relativity Black Holes. Indeed, the limiting case $m + n \to 0^-$ is contained within our family of solutions and it is well defined (For example $m = -2$, $n = 2$ gives the Schwarzschild metric). In this black hole solution (1):

$$ds^2 = -A(r)^{1-n}dt^2 + A(r)^{n-1}dr^2 + r^2 A(r)^n d\Omega^2,$$

the parameter $n$ plays the role of a classical Brans-Dicke hair. It has its origin in the particular coupling of the Brans-Dicke scalar field. Their effects at large distances can be absorbed in a redefinition of the mass of the black hole and thus as we have seen, at Post-Newtonian level this metric coincides with the Schwarzschild one. However, as we study strong gravitational field effects, their results are dependent on the value of $n$. In some sense, thus, $n$ has an intermediate range of action.

When $\phi$ is not constant, black hole solutions are still possible due to the fact that the surface integral $\int (\varphi^2)_{,\alpha}d\Sigma_{\alpha}$ (where $\varphi = \phi - \phi_0$) assumed to vanish$^{[3,10]}$ (under the implicit supposition of $T_{\mu\nu}l^\mu l^\nu \geq 0$), here gives a non-zero contribution, i.e., $4\pi r_0\phi_0^2 (m + n) < 0$, thus compensating the positive value of the integral $\int (\varphi,_{\alpha})^2 \sqrt{-g} d^4x$. This is indeed so due to the particular form of the scalar field Eq. (8), which produces a stress tensor that violates the weak energy condition (with $(m + n) < 0$ ensuring regularity of the field on the horizon). The finiteness of the surface integral can be understood by the fact that $g_{\phi\phi}$ for $n \leq -1$ diverges on the horizon, thus producing a finite result when multiplied by the vanishing scalar field terms and integrated over the horizon surface.

It is worth to stress that as $m + n \to 0^-$ and $n \leq -1$, then $\omega \to -\infty$. This is perfectly acceptable because there is no theoretical reason to restrict $\omega$ to positive values$^{[17]}$ and experiments are consistent with $|\omega| \gtrsim 500$. For the allowed range of values of the parameters $m$ and $n$ (given by eqs. (23) and (36)), $-\infty < \omega < -4/3$. Let us remember that the string theory selects the value$^{[2]}$ $\omega_S = -1$; while the graceful exit problem is solved for $^{[1]} \omega_{EI} \leq 20$.

The no hair theorem can be overcome because the weak energy condition is violated by the energy momentum tensor of the Brans-Dicke field. In fact,

$$T_{00} = n(2 - n)\frac{r_0^2}{r^4}(1 - 2r_0/r)^{m-n},$$

that for $n \leq -1$, takes always negative values (independent of the limit $m \to -n$).

Another interesting result is that for the subfamily (1), in particular (in general see Eq. (36)), the surface gravity will be zero (see Eq. (47)). Hence, these black holes are truly "black", even at the semiclassical level, in the sense that not Hawking radiation is expected to take place here. It is worth to remark here that the divergence of the horizon surface not only does not affect the computation of relevant physical quantities, but can also be interpreted, with regards to the thermodynamics of black holes, as suggesting an infinite entropy for our black hole solutions. This, in turn, is consistent with its associated semiclassical zero temperature.
We remark that this kind of analysis can be also carried out for the generalization of the Kerr-Newmann metric in the Brans-Dicke theory \cite{11}.

The problem of stability of solution (1) is now under study by the present authors, but we can advance some comments: Matsuda \cite{18} has found, studying the spherically gravitational collapse of a star in Brans-Dicke theory, that it does not necessarily produce a Schwarzschild black hole, but can also produce the black hole solution given by metric Eq. (1).

The radiation of the scalar field will be damped by a factor $^{3,10} (2 + \omega)^{-1}$, which vanishes as $|\omega|$ goes to infinity. Thus, we think that metric (1) is a viable candidate to represents the black holes in nature.

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