Determination of accessory parameters in a system of the Okubo normal form

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Abstract

A system of differential equations of the Okubo normal form containing accessory parameters is considered. A condition for determining special values of the accessory parameters is given. It is shown that the special values give the differential equation satisfied by a product of the Gauss hypergeometric functions.

Keywords
Okubo normal form; accessory parameters; local solutions; system of difference equations; Gauss hypergeometric function

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1 Introduction

We are concerned with the system of differential equations satisfied by a product of the Gauss hypergeometric functions. Set

\[ w = \begin{pmatrix} f_1 f_2 \\ x f_1 f_2' \\ x f_1' f_2 \\ x^2 f_1 f_2' \end{pmatrix}, \quad f_j = \frac{d}{dx} f_j(x), \quad f_j' = 2F_1(\alpha_j, \beta_j; \gamma_j; x), \quad f_j'' = \frac{d^2 f_j}{dx^2} \quad (j = 1, 2). \]

The function \( f_j \) satisfies the differential equation

\[ f_j'' - \frac{p_j(x)}{x} f_j' - q_j(x) f_j = 0, \]

where

\[ p_j(x) = -\frac{\gamma_j}{x} + \frac{\gamma_j - 1 - \alpha_j - \beta_j}{x - 1}, \quad q_j(x) = -\frac{\alpha_j \beta_j}{x(x - 1)}, \]

and hence \( w \) satisfies the system of differential equations

\[ \frac{dw}{dx} = \begin{pmatrix} 0 & \frac{1}{x} & \frac{1}{x} & 0 \\ x q_1(x) & \frac{1}{x} + p_1(x) & 0 & \frac{1}{x} \\ x q_2(x) & 0 & \frac{1}{x} + p_2(x) & \frac{1}{x} \\ 0 & x q_2(x) & x q_1(x) & \frac{1}{x} + p_1(x) + p_2(x) \end{pmatrix} w, \]

which is a Fuchsian system of normal form, namely,

\[ \frac{dw}{dx} = \left( \frac{1}{x} H_0 + \frac{1}{x - 1} H_1 \right) w \quad (1) \]

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We denote by $I_m$ the $m \times m$ identity matrix. Since the characteristic polynomial 

$$
\varphi(t) = \det(tI_4 + H_0 + H_1) \text{ of the residue matrix } -(H_0 + H_1) \text{ at } x = \infty \text{ is factored into }
$$

$$
\varphi(t) = (t - \alpha_1 - \alpha_2)(t - \beta_1 - \beta_2)(t - \alpha_1 - \beta_2)(t - \alpha_2 - \beta_1),
$$

the Riemann scheme of (1) is

$$
\begin{aligned}
&x = 0 & x = 1 & x = \infty \\
&0 & 0 & \alpha_1 + \alpha_2 \\
1 - \gamma_1 & \gamma_1 - 1 - \alpha_1 - \beta_1 & \beta_1 + \beta_2 \\
1 - \gamma_2 & \gamma_2 - 1 - \alpha_2 - \beta_2 & \beta_1 + \beta_2 \\
2 - \gamma_1 - \gamma_2 & \gamma_1 + \gamma_2 - 2 - \alpha_1 - \alpha_2 - \beta_1 - \beta_2 & \beta_1 + \beta_2
\end{aligned}
$$

and the spectral type of (1) is $((1111), (1111), (1111))$ if the parameters $\alpha_j$, $\beta_j$, $\gamma_j$ ($j = 1, 2$) are generic.

In this paper we consider a case of repeated local exponents. We suppose that

$$1 - \gamma_2 = 1 - \gamma_1 \quad \text{and} \quad \gamma_1 + \gamma_2 - 2 - \alpha_1 - \alpha_2 - \beta_1 - \beta_2 = 0,$$

or equivalently

$$
\gamma_j = \frac{1}{2}(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) + 1 \quad (j = 1, 2).
$$

In this case the system (1), the spectral type of which is $((211), (211), (1111))$, can be transformed into what is called a system of the Okubo normal form. We denote by $\tilde{H}_0$, $\tilde{H}_1$ the residue matrices $H_0$, $H_1$ with $\gamma_j$ ($j = 1, 2$) replaced by the right member of (2). Introducing the notation

$$
\lambda_{i_1 i_2 i_3 i_4} = \frac{1}{2}\{i_1 \alpha_1 + (i_2 \alpha_2) + (i_3 \beta_1) + (i_4 \beta_2)\}, \quad \iota_j = +, -, 0,
$$

we can write the matrices $\tilde{H}_0$, $\tilde{H}_1$ in the form

$$
\tilde{H}_0 = \lambda_{- - - -} I_4 + \begin{pmatrix}
\lambda_{+++} & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & \lambda_{--- -}
\end{pmatrix},
$$

$$
\tilde{H}_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
-4\lambda_{+++} \lambda_{00+0} & \lambda_{- - - -} & 0 & 0 \\
-4\lambda_{0+0+} \lambda_{000+} & 0 & \lambda_{+++ -} & 0 \\
0 & -4\lambda_{0+0+} \lambda_{000+} & -4\lambda_{+++} \lambda_{00+0} & 0
\end{pmatrix}.
$$
Here note that \( \lambda_{\vec{i} \vec{j} \vec{k} \vec{l}} = -\lambda_{\vec{j} \vec{i} \vec{k} \vec{l}} \) holds for \( \vec{i} = -\vec{j} \). We change the variable \( w \) to \( u \) by

\[
w = x^{\lambda_{+-+}} Pu, \quad P = \begin{pmatrix}1 & 1 & 0 & 0 \\ 0 & \lambda_{---} & \lambda_{++-} & 0 \\ 0 & \lambda_{---} & 0 & \lambda_{++-} \\ 0 & \lambda_{---} & 4\lambda_{0++0000+} & 4\lambda_{++-0000+} \end{pmatrix}
\]

under the condition \( \det P = \lambda_{---} \lambda_{++-} \lambda_{++-} \lambda_{++-} \neq 0 \). The inverse matrix of \( P \) is

\[
P^{-1} = \frac{1}{\lambda_{++-} \lambda_{++-} \lambda_{++-}} \begin{pmatrix} \lambda_{++-} - \lambda_{++-} & \frac{4\lambda_{0++0000+}}{\lambda_{++-}} & \frac{4\lambda_{0++0000+}}{\lambda_{++-}} & \frac{\lambda_{++-} \lambda_{++-} \lambda_{++-}}{\lambda_{++-} \lambda_{++-} \lambda_{++-}} \\ 0 & \lambda_{---} & \lambda_{++-} & 0 \\ 0 & \lambda_{---} & 0 & \lambda_{++-} \\ 0 & \lambda_{---} & \frac{4\lambda_{0++0000+}}{\lambda_{++-}} & \frac{\lambda_{++-} \lambda_{++-} \lambda_{++-}}{\lambda_{++-} \lambda_{++-} \lambda_{++-}} \\ \frac{4\lambda_{0++0000+}}{\lambda_{++-}} & \frac{4\lambda_{0++0000+}}{\lambda_{++-}} & \lambda_{++-} & 0 \\ \frac{4\lambda_{0++0000+}}{\lambda_{++-}} & \frac{4\lambda_{0++0000+}}{\lambda_{++-}} & \lambda_{++-} & 0 \\ \frac{4\lambda_{0++0000+}}{\lambda_{++-}} & \frac{4\lambda_{0++0000+}}{\lambda_{++-}} & \lambda_{++-} & 0 \\ \frac{4\lambda_{0++0000+}}{\lambda_{++-}} & \frac{4\lambda_{0++0000+}}{\lambda_{++-}} & \lambda_{++-} & 0 \end{pmatrix}
\]

that consists of left eigenvectors of \( \tilde{H}_1 \) and \( \tilde{H}_0 - \lambda_{---} I_4 \) with respect to the eigenvalue \( 0 \). Then we have

\[
P^{-1}(\tilde{H}_0 - \lambda_{---} I_4) P = \begin{pmatrix} \lambda_{++-} & 0 & \lambda_{++-} & \lambda_{++-} \\ 0 & \lambda_{---} & \lambda_{++-} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
P^{-1} \tilde{H}_1 P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_{++-} \lambda_{++-} & 0 & 0 \\ \lambda_{++-} & 0 & 0 & 0 \\ \lambda_{++-} & \lambda_{++-} & 0 & 0 \end{pmatrix},
\]

Thus the resulting system

\[
\frac{du}{dx} = \left( \frac{1}{x} P^{-1}(\tilde{H}_0 - \lambda_{---} I_4) P + \frac{1}{x - 1} P^{-1} \tilde{H}_1 P \right) u
\]

by the transformation \( (3) \) is written in the Okubo normal form

\[
(xI_4 - \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}) \frac{du}{dx} = A_0 u,
\]

where the coefficient matrix \( A_0 \) is given by

\[
A_0 = P^{-1}(\tilde{H}_0 - \lambda_{---} I_4) P + P^{-1} \tilde{H}_1 P
\]

\[
= \begin{pmatrix} \lambda_{++-} & \lambda_{++-} & \lambda_{++-} & \lambda_{++-} \\ \lambda_{++-} & \lambda_{++-} & \lambda_{++-} & \lambda_{++-} \\ \lambda_{++-} & \lambda_{++-} & \lambda_{++-} & \lambda_{++-} \\ \lambda_{++-} & \lambda_{++-} & \lambda_{++-} & \lambda_{++-} \end{pmatrix}
\]

with

\[
J = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

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Note that $A_0$ is similar to a diagonal matrix of the form
\[
\begin{pmatrix}
\lambda_{++-} J \\
\lambda_{+-+} J 
\end{pmatrix}.
\]

Consider a size four system of the Okubo normal form
\[
(xI_4 - T) \frac{du}{dx} = Au,
\] (5)
where $T$ and $A$ are $4 \times 4$ matrices of the form
\[
T = \begin{pmatrix}
0I_2 \\
1I_2
\end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix}
aJ & A_{12} \\
A_{21} & bJ
\end{pmatrix},
\]
and $A$ is assumed to be similar to a diagonal matrix of the form
\[
\begin{pmatrix}
cJ \\
dJ
\end{pmatrix}.
\]
Throughout this paper, we assume the condition
\[
a, b, c, d, 2a, 2b, 2c, 2d, a \pm b, a \pm c, a \pm d, b \pm c, b \pm d, c \pm d \notin \mathbb{Z}.
\] (6)
The system (5) has the same local exponents as (4) if $a = \lambda_{++}, b = \lambda_{--}, c = \lambda_{+-}, d = \lambda_{-+}$, while contains two accessory parameters as a Fuchsian system of normal form since its spectral type is $((211),(211),(1111))$ and then its index of rigidity is
\[
\iota = (1 - 2) \cdot 4^2 + ((2^2 + 1^2 + 1^2) + (2^2 + 2^2 + 1^2) + (1^2 + 1^2 + 1^2 + 1^2)) = 0
\]
(see Haraoka [5, 7.4.2] for the relation of the index of rigidity and the number of accessory parameters, see also Proposition 2.1 and Remark 2.2). The system (4) is nothing but (5) with special values of the accessory parameters.

The system (5) has convergent power series solutions near each singularity. Coefficient vectors of the series solution satisfy a system of linear difference equations. In this paper we propose a condition about the systems of linear difference equations satisfied by the coefficient vectors for determining values of accessory parameters.

Recently Ebisu [9] (see also Ebisu et al. [3]) developed the theory of invariants of scalar linear difference equations of higher order, and defined the notion termed essentially the same for the difference equations. Moreover, he expands the notion to systems of linear difference equations of the first order.

**Definition 1.1** (Ebisu [2]). Let $B(z)$ and $C(z)$ be $n \times n$ matrices that consist of rational functions of $z$. Two systems of difference equations
\[
f(z + 1) = B(z)f(z) \quad \text{and} \quad h(z + 1) = C(z)h(z),
\]
where $f(z)$ and $h(z)$ are unknown $n$-vectors, are said to be essentially the same if there exists a diagonal transformation of the form
\[
f(z) = \begin{pmatrix}
\Gamma_1(z) \\
\Gamma_2(z) \\
\vdots \\
\Gamma_n(z)
\end{pmatrix} h(z),
\] (7)
where $\Gamma_j(z)$ satisfies a linear difference equation of the first order

$$\Gamma_j(z + 1) = g_j(z)\Gamma_j(z),$$

$g_j(z)$ being a rational function of $z \ (j = 1, 2, \ldots, n)$.

However, we make the following restrictive definition.

**Definition 1.2.** Two systems $f(z + 1) = B(z)f(z)$ and $h(z + 1) = C(z)h(z)$ of essentially the same are said to be substantially the same if we can take

$$\Gamma_1(z) = \Gamma_2(z) = \cdots = \Gamma_n(z)$$

in the transformation (7).

We shall show in Theorem 4.1 that it becomes a condition for determining values of accessory parameters that the system of difference equations for the series solution near $x = 1$ and that for the series solution near $x = \infty$ are substantially the same. Moreover, we shall show in Theorem 4.2 that the system of the Okubo normal form determined in Theorem 4.1 coincides with the system (4) up to a diagonal transformation.

## 2 Preliminary

In this section we give a parametrization of the coefficient matrix $A$ in (5).

**Proposition 2.1.** Assume the condition (6). For $\xi = \pm c, \pm d$ let $v_\xi$ be a left eigenvector of the matrix $A$ in (5) with respect to the eigenvalue $\xi$. Provided that $v_{\pm c} = (v_{\pm c}^1, v_{\pm c}^2, v_{\pm c}^3, v_{\pm c}^4)$ satisfy

$$v_{\pm c}^k \neq 0 \ (k = 1, 2, 3, 4) \quad \text{and} \quad \det \begin{pmatrix} v_{\pm c}^1 & v_{\pm c}^1 l+1 \\ v_{\pm c}^2 & v_{\pm c}^2 l+1 \end{pmatrix} \neq 0 \ (l = 1, 3),$$

then there exists a diagonal matrix $D$ such that $DAD^{-1}$ has a parametrization of the form

$$DAD^{-1} = \begin{pmatrix} aJ & \frac{(b-c)r_2-(b+c)r_3}{r_1-r_2} & \frac{(b-c)r_2-(b+c)r_4}{r_2-r_1} \\ \frac{(a-c)r_2-(a+c)r_3}{r_1-r_2} & bJ & \frac{(b+c)r_1-(b-c)r_4}{r_1-r_2} \\ \frac{(a-c)r_2-(a+c)r_4}{r_2-r_1} & \frac{(b+c)r_1-(b-c)r_4}{r_1-r_2} & \frac{(b+c)r_2-(b-c)r_3}{r_2-r_1} \end{pmatrix}, \quad (9)$$

where the $r_k$’s satisfy

$$r_1 r_2 r_3 r_4 \neq 0, \quad r_1 - r_2 \neq 0, \quad r_3 - r_4 \neq 0 \quad (10)$$

and

$$\frac{(a + b + c)^2 r_1 r_3 - (a - b + c)^2 r_1 r_4 - (a - b - c)^2 r_2 r_3 + (a + b - c)^2 r_2 r_4 - 4abr_1 r_2 - 4abr_3 r_4}{(r_1 - r_2)(r_3 - r_4)} = d^2. \quad (11)$$
Proof. Put \( L = \begin{pmatrix} v-c \\ v_c \\ v-d \end{pmatrix} \) and \( D = \begin{pmatrix} v_1^2 \\ v_2^3 \\ v_4^4 \end{pmatrix} \), and write

\[
L = \begin{pmatrix} L_{11} \\ L_{21} \\ L_{22} \end{pmatrix}, \quad L^{-1} = \begin{pmatrix} L'_{11} \\ L'_{21} \\ L'_{22} \end{pmatrix}, \quad D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix},
\]
where all the submatrices are \( 2 \times 2 \) matrices. Note that \( D_1, D_2, L_{11}, L_{12} \) are invertible because of (3). Set

\[
\tilde{A} = \begin{pmatrix} D & L \\ A & I \end{pmatrix} \begin{pmatrix} D & L \\ d^2I_4 - A^2 & -A \end{pmatrix} \begin{pmatrix} D & L \\ L & L \end{pmatrix}^{-1},
\]

Then \( \tilde{A} \) satisfies

\[
\tilde{A}^2 = d^2I_8.
\]

Since \( \tilde{A} \) has an expression of the form

\[
\tilde{A} = \begin{pmatrix} aJ & D_1A_12D_2^{-1} & D_1L'_{11} \\ bJ & D_2L'_{21} \\ (d^2-c^2)L_{11}D_1^{-1} & (d^2-c^2)L_{12}D_2^{-1} \end{pmatrix},
\]

we obtain

\[
L_{11}D_1^{-1}D_1A_12D_2^{-1} + bL_{12}D_2^{-1}J + cJL_{12}D_2^{-1} = O, \quad (12a)
\]
\[
L_{12}D_2^{-1}D_2A_21D_1^{-1} + aL_{11}D_1^{-1}J + cJL_{11}D_1^{-1} = O \quad (12b)
\]

from the \((3,2)\)-block, the \((3,1)\)-block of \( \tilde{A}^2 = d^2I_8 \). Setting

\[
r_k = \frac{v_k}{v_c} \quad (k = 1, 2, 3, 4),
\]

we have

\[
L_{11}D_1^{-1} = \begin{pmatrix} r_1 & r_2 \\ 1 & 1 \end{pmatrix}, \quad L_{12}D_2^{-1} = \begin{pmatrix} r_3 & r_4 \\ 1 & 1 \end{pmatrix}. \quad (13)
\]

Note that \( r_1r_2r_3r_4 \neq 0 \) and

\[
r_1 - r_2 = \det(L_{11}D_1^{-1}) \neq 0, \quad r_3 - r_4 = \det(L_{12}D_2^{-1}) \neq 0
\]

by the condition (10). Substituting (13) into (12a)–(12b), we obtain

\[
D_1A_12D_2^{-1} = -\begin{pmatrix} r_1 & r_2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} r_3 & -r_4 \\ 1 & -1 \end{pmatrix} + c \begin{pmatrix} r_3 & r_4 \\ -1 & -1 \end{pmatrix},
\]
\[
D_2A_21D_1^{-1} = -\begin{pmatrix} r_3 & r_4 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} r_1 & -r_2 \\ 1 & -1 \end{pmatrix} + c \begin{pmatrix} r_1 & r_2 \\ -1 & -1 \end{pmatrix},
\]

which give (2).
Moreover, as the $(1, 1)$-block, the $(1, 2)$-block of $\tilde{A}^2 = d^2 I_8$ we have

$$a^2 I_2 + D_1 A_{12} D_2^{-1} D_2 A_{21} D_1^{-1} + (d^2 - c^2)D_1 L_{11}' L_{11}^{-1} D_1^{-1} = d^2 I_2,$$

$$a J D_1 A_{12} D_2^{-1} + b D_1 A_{12} D_2^{-1} J + (d^2 - c^2)D_1 L_{11}' L_{12}^{-1} D_2^{-1} = O.$$ 

Combining these, we obtain

$$a^2 I_2 + D_1 A_{12} D_2^{-1} D_2 A_{21} D_1^{-1} - (a J D_1 A_{12} D_2^{-1} + b D_1 A_{12} D_2^{-1} J) D_2 L_{12}^{-1} L_{11}^{-1} D_1^{-1} = d^2 I_2.$$ 

A diagonal element of this relation gives (11). This completes the proof. $\square$

We put

$$A_1 = \begin{pmatrix}
\frac{aJ}{(a+c)r_1-(a-c)r_4} & \frac{(b-c)r_2-(b+c)r_3}{r_1-r_2} & \frac{(b+c)r_2-(b-c)r_4}{r_2-r_1} \\
\frac{(a-c)r_2-(a+c)r_4}{r_4-r_3} & \frac{(b-c)r_1-(b+c)r_3}{r_2-r_1} & \frac{(b+c)r_1-(b-c)r_4}{r_1-r_2} \\
\frac{(a+c)r_1-(a-c)r_3}{r_3-r_4} & \frac{(a-c)r_2-(a+c)r_3}{r_4-r_3} & bJ
\end{pmatrix}, \quad (14)$$

where we regard $r_k$ ($k = 1, 2, 3, 4$) as arbitrary parameters not relating to $v_{k,l}^k (k = 1, 2, 3, 4)$ but satisfying (10) and (11). In what follows, we investigate the system

$$(xI_4 - T)\frac{dy}{dx} = A_1 y, \quad (15)$$

where $T = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$.

Remark 2.2. All the elements of the non-diagonal blocks of $A_1$ are expressed by a ratio of homogeneous polynomials in $r_k$ ($k = 1, 2, 3, 4$) of degree one. Taking the relation (11) into account, we find that they can be expressed by two parameters, for instance, by

$$t_k = \frac{r_k}{r_4} \quad (k = 1, 2)$$

in addition to the local exponents $a, b, c, d$. The two parameters are accessory parameters.

Remark 2.3. The local exponents $(0, 0, a, -a), (0, 0, b, -b), (c, -c, d, -d)$ of (5) are essential for the parametrization (9). A generalization of the local exponents to

$$(0, 0, a, a'), \quad (0, 0, b, b'), \quad (c, c', d, d')$$

is impossible.

3 Local solutions

In this section we study local solutions of (15) near its singular points. The system (11) has regular singularity at $x = 0, 1$ and $\infty$. Assume the condition (10). Near $x = 1$ it has solutions of the form

$$y(x) = \sum_{r=0}^{\infty} g(r)(x - 1)^{r+\rho}, \quad \rho = 0, \pm b,$$
where the coefficient vectors \( g(r) \) \((r = 0, 1, 2, \ldots)\) are determined by the system of difference equations

\[
\begin{cases}
(r + \rho + 1)(T - I_4)g(r + 1) = (r + \rho)I_4 - A_1)g(r), \\
\rho(T - I_4)g(0) = 0.
\end{cases}
\] (16)

Near \( x = \infty \) the system (15) has solutions of the form

\[
y(x) = \sum_{s=0}^{\infty} h(s)(x - 1)^{-s-\sigma}, \quad \sigma = \pm c, \pm d,
\]

where the coefficient vectors \( h(s) \) \((s = 0, 1, 2, \ldots)\) are determined by the system of difference equations

\[
\begin{cases}
((s + \sigma + 1)I_4 + A_1)h(s + 1) = (s + \sigma)(T - I_4)h(s), \\
(sI_4 + A_1)h(0) = 0.
\end{cases}
\] (17)

Note that we express the local solutions near \( x = \infty \) by means of powers not in \( x^{-1} \) but in \((x - 1)^{-1}\).

In the system (16) we set \( z = r + \rho \) and \( \tilde{g}(z) = g(z - \rho) \). Then we obtain the system of difference equations

\[(z + 1)(T - I_4)\tilde{g}(z + 1) = (zI_4 - A_1)\tilde{g}(z).\] (18)

Besides, in the system (17) we set \( z = s + \sigma + 1 \) and \( \tilde{h}(z) = h(z - 1 - \sigma) \). Then we obtain the system of difference equations

\[(zI_4 + A_1)\tilde{h}(z + 1) = (z - 1)(T - I_4)\tilde{h}(z).\] (19)

Since \( T - I_4 = \begin{pmatrix} -1 & I_2 \\ 0 & I_2 \end{pmatrix} \), both (18) and (19) are reducible to a size two system of difference equations. Indeed, we can write (18), (19) in the form

\[
(z + 1)(zI_4 - A_1)^{-1}(T - I_4)\tilde{g}(z + 1) = \tilde{g}(z),
\] (20)

\[
\tilde{h}(z + 1) = (z - 1)(zI_4 + A_1)^{-1}(T - I_4)\tilde{h}(z).
\] (21)

Writing

\[
\tilde{g}(z) = \begin{pmatrix} \tilde{g}_1(z) \\ \tilde{g}_2(z) \end{pmatrix}, \quad \tilde{h}(z) = \begin{pmatrix} \tilde{h}_1(z) \\ \tilde{h}_2(z) \end{pmatrix}
\]

and

\[
(zI_4 - A_1)^{-1} = \frac{1}{(z^2 - c^2)(z^2 - d^2)} \begin{pmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{pmatrix},
\] (22)

where \( \tilde{g}_j(z) \) and \( \tilde{h}_j(z) \) \((j = 1, 2)\) are 2-dimensional, and \( A_{jk}(z) \) \((j, k = 1, 2)\) are 2 × 2 matrices, we obtain

\[
- \frac{z + 1}{(z^2 - c^2)(z^2 - d^2)} A_{11}(z)\tilde{g}_1(z + 1) = \tilde{g}_1(z),
\] (23a)

\[
- \frac{z + 1}{(z^2 - c^2)(z^2 - d^2)} A_{21}(z)\tilde{g}_1(z + 1) = \tilde{g}_2(z)
\] (23b)
from (20), and
\[\tilde{h}_1(z + 1) = \frac{z - 1}{(z^2 - c^2)(z^2 - d^2)} A_{11}(-z)\tilde{h}_1(z),\]  \hspace{1cm} (24a)
\[\tilde{h}_2(z + 1) = \frac{z - 1}{(z^2 - c^2)(z^2 - d^2)} A_{21}(-z)\tilde{h}_1(z)\]  \hspace{1cm} (24b)
from (21).

4 Main results

Recall Definition 1.2.

Theorem 4.1. Suppose that the system (15) satisfies (6), (10) and (11). The systems (23a) and (24a) for (15) are substantially the same if and only if the coefficient matrix \(A_1\) in (15) is equal to

\[
\begin{pmatrix}
  aJ & \frac{(a-b+c)^2-d^2}{4a} & \frac{(a+b+c)^2-d^2}{4a} \\
  \frac{(a-b-c)^2-d^2}{4b} & \frac{(a+b+c)^2-d^2}{4a} & \frac{d^2-(a-b-c)^2}{4a} \\
  \frac{d^2-(a+b+c)^2}{4b} & \frac{d^2-(a-b-c)^2}{4b} & bJ
\end{pmatrix}
\]  \hspace{1cm} (25)

that is obtained by substituting

\[r_1 = \frac{(a + b - c)^2 - d^2}{(a + b + c)^2 - d^2} r_4,\]  \hspace{1cm} (26a)
\[r_2 = \frac{(a - b + c)^2 - d^2}{(a - b - c)^2 - d^2} r_4,\]  \hspace{1cm} (26b)
or

\[r_1 = \frac{(a - b - c)^2 - d^2}{(a + b + c)^2 - d^2} r_3,\]  \hspace{1cm} (27a)
\[r_2 = \frac{(a + b + c)^2 - d^2}{(a + b - c)^2 - d^2} r_3,\]  \hspace{1cm} (27b)

into (14), where the denominators \((a + b + c)^2 - d^2\)((a - b - c)^2 - d^2)\) and \((a + b - c)^2 - d^2\)((a - b + c)^2 - d^2)\) never vanish simultaneously.

Theorem 4.2. If

\[a = \lambda_{++--}, \quad b = \lambda_{-++-}, \quad c = \lambda_{++-}, \quad d = \lambda_{+-++}\]  \hspace{1cm} (28)

and these values satisfy both of the conditions

\[(a + b + c)^2 - d^2 \neq (a - b - c)^2 - d^2\]  \hspace{1cm} (29a)

and

\[(a + b - c)^2 - d^2 \neq (a - b + c)^2 - d^2\]  \hspace{1cm} (29b)
in addition to (10), then the system (15) with the coefficient matrix (25) coincides with the system (11) up to a diagonal transformation.
5 Proofs

Recall $A_1 = \begin{pmatrix} aJ & A'_{12} \\ A'_{21} & bJ \end{pmatrix}$, where $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and

$$A'_{12} = \begin{pmatrix} \frac{(b-c)r_2-(b+c)r_3}{r_1-r_2} & \frac{(b+c)r_2-(b-c)r_3}{r_1-r_2} \\ \frac{(b-c)r_1-(b+c)r_3}{r_2-r_1} & \frac{(b+c)r_1-(b-c)r_3}{r_2-r_1} \end{pmatrix},$$

$$A'_{21} = \begin{pmatrix} \frac{(a+c)r_3-(a-c)r_1}{r_4-r_3} & \frac{(a-c)r_3-(a+c)r_1}{r_4-r_3} \\ \frac{(a+c)r_1-(a-c)r_3}{r_4-r_3} & \frac{(a-c)r_1-(a+c)r_3}{r_4-r_3} \end{pmatrix}.$$

Set

$$\varepsilon = b(a+c)r_1 + b(a-c)r_2 - a(b+c)r_3 - a(b-c)r_4,$$

$$\delta = r_1r_2 - r_3r_4,$$

and

$$\varepsilon' = \frac{2\varepsilon}{(r_1 - r_2)(r_3 - r_4)}, \quad \delta' = \frac{2\delta}{(r_1 - r_2)(r_3 - r_4)}.$$

Lemma 5.1. Assume the conditions (6), (10) and (11). When we write the submatrix $A_{11}(z)$ in (22) in the form

$$A_{11}(z) = z^3I_2 + z^2Q_{11} + zR_{11} + S_{11},$$

we have

$$Q_{11} = aJ, \quad \text{(30a)}$$

$$R_{11} = \frac{1}{2}(a^2 - b^2 - c^2 - d^2)I_2 + \varepsilon' \begin{pmatrix} r_2 \\ r_1 \end{pmatrix} - b\delta' \begin{pmatrix} c \\ a-c \end{pmatrix}, \quad \text{(30b)}$$

$$S_{11} = bA'_{12}JA'_{21} - ab^2J - 2abc\delta' I_2. \quad \text{(30c)}$$

Moreover, we have

$$A_{11}(z)^{-1} = \frac{1}{(z^2 - b^2)(z^2 - c^2)(z^2 - d^2)} \left(z^3I_2 + z^2\tilde{Q}_{11} + z\tilde{R}_{11} + \tilde{S}_{11}\right),$$

where

$$\tilde{Q}_{11} = -aJ, \quad \text{(31a)}$$

$$\tilde{R}_{11} = \frac{1}{2}(a^2 - b^2 - c^2 - d^2)I_2 - \varepsilon' \begin{pmatrix} r_2 \\ r_1 \end{pmatrix} + b\delta' \begin{pmatrix} c \\ a-c \end{pmatrix}, \quad \text{(31b)}$$

$$\tilde{S}_{11} = ab^2J - bA'_{12}JA'_{21}. \quad \text{(31c)}$$

Proof. Since $(A_1^2 - c^2I_4)(A_1^2 - d^2I_4) = O$, we have

$$(zI_4 - A_1^2)^{-1} = \frac{1}{(z - c^2)(z - d^2)} \left((z - c^2 - d^2)I_4 + A_1^2\right),$$
and hence
\[(zI_4 - A_1)^{-1} = (zI_4 + A_1)(z^2I_4 - A_1^2)^{-1}\]
\[= \frac{1}{(z^2 - c^2)(z^2 - d^2)}(zI_4 + A_1)\left\{(z^2 - c^2 - d^2)I_4 + A_1^2\right\}\]
\[= \frac{1}{(z^2 - c^2)(z^2 - d^2)}\{z^3I_4 + z^2A_1 + z(A_1^2 - (c^2 + d^2)I_4) + A_1^3 - (c^2 + d^2)A_1\}.
\]

From the (1,1)-block of this formula we have
\[Q_{11} = aJ,\]
\[R_{11} = A_{12}'A_{21}' + (a^2 - c^2 - d^2)I_2,\]
\[S_{11} = bA_{12}J A_{21}' + aJA_{12}'A_{21}' + aA_{12}'A_{21}J + a(a^2 - c^2 - d^2)J.\]

Besides, with the aid of a computer algebra system, we can easily check that
\[A_{12}'A_{21}' = \frac{1}{2}(c^2 + d^2 - a^2 - b^2)I_2 + \varepsilon'(\begin{pmatrix} r_1 & r_2 \\
1 & 2 \end{pmatrix}) - b\delta'(\begin{pmatrix} c & a + c \\
a - c & -c \end{pmatrix})\]
holds, where we have used \((11)\) for \(d^2\). Combining these expressions, we can obtain \((30a) - (30c)\).

As for \(A_{11}(z)^{-1}\), we have
\[A_{11}(z)^{-1} = \frac{1}{\det A_{11}(z)} \adj A_{11}(z),\]
\[\adj A_{11}(z) = z^3I_2 + z^2((\tr Q_{11})I_2 - Q_{11}) + z((\tr R_{11})I_2 - R_{11}) + (\tr S_{11})I_2 - S_{11}.\]

Here the determinant of \(A_{11}(z)\) follows from the equality
\[\begin{pmatrix} A_{11}(z) & A_{12}(z) \\
I_2 & I_2 \end{pmatrix}(zI_4 - A_1) = \begin{pmatrix} (z^2 - c^2)(z^2 - d^2)I_2 \\
-A_{21}' & zI_2 - bJ \end{pmatrix}.
\]

Taking the determinant of both sides, we have
\[\det A_{11}(z) \cdot \det (zI_4 - A_1) = (z^2 - c^2)(z^2 - d^2)^2 \cdot \det (zI_2 - bJ)\]
and hence
\[\det A_{11}(z) = (z^2 - b^2)(z^2 - c^2)(z^2 - d^2).
\]

From \((30a) - (30c)\) it is trivial that
\[\tr Q_{11} = 0, \quad \tr R_{11} = a^2 - b^2 - c^2 - d^2, \quad \tr S_{11} = b \cdot \tr (A_{12}'J A_{21}') - 4abc\delta'.\]

hold. By direct calculation we can obtain \(\tr (A_{12}'J A_{21}') = 2ac\delta'\) and hence
\[\tr S_{11} = -2abc\delta'.\]

Substituting these, we obtain \((31a) - (31c)\). This completes the proof. □

**Lemma 5.2.** Assume the conditions \((6), (10)\) and \((11)\). The systems \((23a)\) and \((24a)\) are substantially the same if and only if the condition
\[\varepsilon = 0 \quad \text{and} \quad \delta = 0\]
holds.
**Proof.** By Lemma 5.1 we can write the systems (23a) and (24a) in the form

\[
\tilde{g}_1(z + 1) = -\frac{1}{(z + 1)(z^2 - b^2)}(z^3I_2 - z^2aJ + z\tilde{R}_{11} + \tilde{S}_{11})\tilde{g}_1(z),
\]

\[
\tilde{h}_1(z + 1) = -\frac{z - 1}{(z^2 - c^2)(z^2 - d^2)}(z^3I_2 - z^2aJ + zR_{11} - S_{11})\tilde{h}_1(z).
\]

From these expressions it is obvious that the systems (23a) and (24a) are substantially the same if (32) holds. So, we only have to show that the condition (32) holds if (23a) and (24a) are substantially the same. Throughout the proof, we write

\[
z^3I_2 - z^2aJ + z\tilde{R}_{11} + \tilde{S}_{11} = \begin{pmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{pmatrix},
\]

\[
z^3I_2 - z^2aJ + zR_{11} - S_{11} = \begin{pmatrix} c_{11}(z) & c_{12}(z) \\ c_{21}(z) & c_{22}(z) \end{pmatrix}.
\]

Assume that the systems (23a) and (24a) are substantially the same. Then we have

\[
c_{jk}(z)b_{lm}(z) - b_{jk}(z)c_{lm}(z) = 0 \quad (j, k, l, m = 1, 2).
\]

By direct calculation we obtain

\[
c_{11}(z)b_{22}(z) - b_{11}(z)c_{22}(z) = -4bc\delta'z^4 + \cdots,
\]

\[
c_{12}(z)b_{22}(z) - b_{12}(z)c_{22}(z) = 2\{r_2\varepsilon' - (a + c)b\delta'\}z^4 + \cdots,
\]

\[
c_{21}(z)b_{22}(z) - b_{21}(z)c_{22}(z) = 2\{r_1\varepsilon' - (a - c)b\delta'\}z^4 + \cdots,
\]

which gives \(\delta' = 0\) and \(\varepsilon' = 0\). This completes the proof. \(\square\)

**Proof of Theorem 4.1.** It is easy to check that by substituting (26a)–(26b) or (27a)–(27b) into (11) we obtain (25). So, by Lemma 5.2 we only have to show that either or both of (26a)–(26b) and (27a)–(27b) holds if and only if (32) holds.

First, we assume (32). Using \(\varepsilon\) and \(\delta\), we can write the numerator of the left hand side of (11) as

\[
(r_1 - r_2)(r_3 - r_4)d^2
\]

\[
= (a + b + c)^2r_1r_3 - (a - b + c)^2r_1r_4 - (a - b - c)^2r_2r_3 + (a + b - c)^2r_2r_4
\]

\[
- 4abr_1r_2 - 4abrr_3r_4
\]

\[
= \{4b(a + c)r_1 + (a - b - c)^2r_3 - (a - b + c)^2r_4\}(r_1 - r_2) - 4r_1\varepsilon + 4ab\delta
\]

\[
= \{-4b(a - c)r_2 + (a + b + c)^2r_3 - (a + b - c)^2r_4\}(r_1 - r_2) - 4r_2\varepsilon + 4ab\delta.
\]

When \(\varepsilon = 0\), \(\delta = 0\) and \(r_1 - r_2 \neq 0\), we have

\[
r_1 = -\frac{((a - b - c)^2 - d^2)r_3 - ((a + b - c)^2 - d^2)r_4}{4b(a + c)}, \quad (33a)
\]

\[
r_2 = \frac{((a + b + c)^2 - d^2)r_3 - ((a - b + c)^2 - d^2)r_4}{4b(a - c)}. \quad (33b)
\]

Substituting these into \(\delta = r_1r_2 - r_3r_4\), we have

\[
\delta = -\frac{\{(a + b + c)^2 - d^2\}((a - b - c)^2 - d^2)r_3
\]

\[
- ((a + b - c)^2 - d^2)((a - b + c)^2 - d^2)r_4\}(r_3 - r_4)}{16b^2(a^2 - c^2)}.
\]

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When $\delta = 0$ and $r_3 - r_4 \neq 0$, we obtain (26b) and/or (27b). Here the coefficient of $r_3$ and that of $r_4$ never vanish simultaneously, since

$$((a + \iota_1 b - \iota_2 c)^2 - d^2)((a - \iota_1 b + \iota_2 c)^2 - d^2) = \iota_1 \iota_2 16bc(a + \iota_1 b)(a + \iota_2 c) \neq 0$$

if $(a + \iota_1 b + \iota_2 c)^2 - d^2 = 0$, where $\iota_j = +$ or $- (j = 1, 2)$. Substituting (26b) or (27b) into (33a)–(33b), we obtain (26a) or (27a), respectively.

Inversely, we assume (26a)–(26b) or (27a)–(27b). By direct calculation we can easily check that $\varepsilon = 0$ and $\delta = 0$ hold. This completes the proof.

**Proof of Theorem 4.3** First, substituting (28) into (29a), (29b), we have

$$((a + b + c)^2 - d^2)((a - b - c)^2 - d^2) = 64\lambda_{0+00}\lambda_{00+0}\lambda_{++--}\lambda_{----} \neq 0,$$

$$((a + b - c)^2 - d^2)((a - b + c)^2 - d^2) = 64\lambda_{+000}\lambda_{000+}\lambda_{++--}\lambda_{----} \neq 0.$$  

Substituting (28) into (25), we have

$$A_1 = \begin{pmatrix} \lambda_{++--} J & 2\lambda_{0+00}\lambda_{++--} & 2\lambda_{0+00}\lambda_{++--} & 2\lambda_{0+00}\lambda_{++--} \\ 2\lambda_{++--} & \lambda_{++--} & \lambda_{++--} & \lambda_{++--} \\ 2\lambda_{++--} & \lambda_{++--} & \lambda_{++--} & \lambda_{++--} \\ \lambda_{++--} & \lambda_{++--} & \lambda_{++--} & \lambda_{++--} \end{pmatrix}.$$  

For this $A_1$, taking

$$D_1 = \begin{pmatrix} 1 \\ \frac{\lambda_{++--}}{4\lambda_{0+00}\lambda_{00+0}} \\ \frac{\lambda_{++--}}{2\lambda_{0+00}} \\ \frac{\lambda_{++--}}{2\lambda_{0+00}} \end{pmatrix},$$

we see that $D_1^{-1}A_1D_1$ agrees with the coefficient matrix $A_0$ of (4). Namely, the transformation $y = D_1 u$ changes (15) to (4). This completes the proof.

6 Some remarks

6.1 Local solutions near another singular point

We can obtain the same result as Theorem 4.1 from local solutions

$$y(x) = \sum_{r=0}^{\infty} g(r)x^{r+\rho}, \quad \rho = 0, \pm a, \quad \text{near } x = 0$$

and

$$y(x) = \sum_{s=0}^{\infty} h(s)x^{-s-\sigma}, \quad \sigma = \pm c, \pm d, \quad \text{near } x = \infty.$$  

For these solutions, similarly to (18) and (19), we have

$$(z + 1)T\hat{g}(z + 1) = (zI_4 - A_1)\hat{g}(z) \quad \text{for } \hat{g}(z) = g(z - \rho)$$

on
and

\[(zI_4 + A_1)\hat{h}(z + 1) = (z - 1)T\hat{h}(z) \quad \text{for} \quad \hat{h}(z) = h(z - 1 - \sigma).\]

Moreover, since \(T = \begin{pmatrix} 0I_2 & \frac{z+1}{(z^2-c^2)(z^2-d^2)} \end{pmatrix},\) writing \(\hat{g}(z) = \begin{pmatrix} \hat{g}_1(z) \\ \hat{g}_2(z) \end{pmatrix}\) and \(\hat{h}(z) = \begin{pmatrix} \hat{h}_1(z) \\ \hat{h}_2(z) \end{pmatrix},\) we have

\[
\frac{z+1}{(z^2-c^2)(z^2-d^2)}A_{12}(z)\hat{g}_2(z+1) = \hat{g}_1(z),
\]

(34a)

\[
\frac{z+1}{(z^2-c^2)(z^2-d^2)}A_{22}(z)\hat{g}_2(z+1) = \hat{g}_2(z)
\]

(34b)

and

\[
\hat{h}_1(z+1) = -\frac{z-1}{(z^2-c^2)(z^2-d^2)}A_{12}(-z)\hat{h}_2(z),
\]

(35a)

\[
\hat{h}_2(z+1) = -\frac{z-1}{(z^2-c^2)(z^2-d^2)}A_{22}(-z)\hat{h}_2(z),
\]

(35b)

where \(A_{12}(z)\) and \(A_{22}(z)\) are the submatrices in (22). In the same way as the proof of Theorem 4.1 we can prove the following theorem.

**Theorem 6.1.** Suppose that the system (15) satisfies (9), (10) and (11). The systems (34b) and (35b) for (15) are substantially the same if and only if the coefficient matrix \(A_1\) in (15) is equal to (25).

### 6.2 Relation with the equation treated by Ebisu et al.

The present work is inspired by Ebisu et al. [3]. They investigated the scalar differential equation of the fourth order satisfied by

\[y = x^{-A_0}F_1(A_{+++}, A_{+++}, 1 - A_0, x)F_1(A_{++-}, A_{++-}, 1 - A_0; x)\]

(36)

under the parametrization

\[A_{\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3} = \frac{1}{2}\{(\varepsilon_0 A_0) + (\varepsilon_1 A_1) + (\varepsilon_2 A_2) + (\varepsilon_3 A_3) + 1\}, \quad \varepsilon_j = \pm\]

in their notation. Note that the product (36) is equivalent to

\[y = x^{\gamma-1}F_1(\alpha_1, \beta_1, \gamma'; x)F_1(\alpha_2, \beta_2, \gamma'; x)\]

(37)

with the condition

\[\gamma' = \frac{1}{2}(\alpha_1 + \alpha_2 + \beta_1 + \beta_2),\]

(38)

which is different from the condition (2).

We here explain the relation between our system and their equation. For the system (41) we change the variable \(u\) to \(v\) by

\[u = R^{-1}v, \quad R = \begin{pmatrix} 1 & \frac{\lambda_{++} + \lambda_{+-} + \lambda_{-+} + \lambda_{--}}{4\lambda_{+++} + \lambda_{++0} + \lambda_{+0+} + \lambda_{0++}} & \frac{\lambda_{++} + \lambda_{+-} + \lambda_{-+} + \lambda_{--}}{2\lambda_{+++} + \lambda_{++0} + \lambda_{+0+} + \lambda_{0++}} \\ 1 & \frac{\lambda_{++} + \lambda_{+-} + \lambda_{-+} + \lambda_{--}}{4\lambda_{++0} + \lambda_{++00} + \lambda_{+0+} + \lambda_{0++0}} & \frac{\lambda_{++} + \lambda_{+-} + \lambda_{-+} + \lambda_{--}}{2\lambda_{++0} + \lambda_{++00} + \lambda_{+0+} + \lambda_{0++0}} \\ 1 & \frac{\lambda_{++} + \lambda_{+-} + \lambda_{-+} + \lambda_{--}}{4\lambda_{+0+} + \lambda_{+0+0} + \lambda_{+000} + \lambda_{0++0}} & \frac{\lambda_{++} + \lambda_{+-} + \lambda_{-+} + \lambda_{--}}{2\lambda_{+0+} + \lambda_{+0+0} + \lambda_{+000} + \lambda_{0++0}} \\ 1 & \frac{\lambda_{++} + \lambda_{+-} + \lambda_{-+} + \lambda_{--}}{4\lambda_{0++} + \lambda_{0++0} + \lambda_{0++00} + \lambda_{0++000}} & \frac{\lambda_{++} + \lambda_{+-} + \lambda_{-+} + \lambda_{--}}{2\lambda_{0++} + \lambda_{0++0} + \lambda_{0++00} + \lambda_{0++000}} \end{pmatrix}\]

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under the conditions \( \lambda_{0000}\lambda_{0001}\lambda_{0010}\lambda_{0011} \neq 0 \) and

\[
\det R = \frac{\lambda_{++++} \lambda_{+-+-} \lambda_{+--+} \lambda_{+--+} \lambda_{+000}^2 \lambda_{000+}^2 \lambda_{00+0}^2 \lambda_{0+00}^2 \lambda_{+00+}^2 \lambda_{+0+0}^2 \lambda_{0+0+}^2 \lambda_{00+0}^2 \lambda_{000+}^2}{16\lambda_{+++0}^2 \lambda_{++-0}^2 \lambda_{+-+0}^2 \lambda_{+-0+}^2 \lambda_{-++0}^2 \lambda_{-+0+}^2 \lambda_{--++}^2 \lambda_{--+-}^2 \lambda_{--0+}^2 \lambda_{--0+}^2} \neq 0.
\]

The matrix \( R \) consists of left eigenvectors of the coefficient matrix \( A_0 \) of (36) to satisfy

\[
RA_0R^{-1} = \begin{pmatrix} \lambda_{++++} & \lambda_{+-+-} \\ \lambda_{+--+} & \lambda_{+--+} \end{pmatrix}.
\]

Then \( v \) satisfies the system of differential equations

\[
\frac{dv}{dx} = R(xI_4 - T)^{-1}R^{-1} \begin{pmatrix} \lambda_{++++} & \lambda_{+-+-} \\ \lambda_{+--+} & \lambda_{+--+} \end{pmatrix} v.
\]  

(39)

Besides, by direct calculation we have

\[
RP^{-1} = \begin{pmatrix} 1 & \frac{1}{\alpha_1} & \frac{1}{\alpha_2} & \frac{1}{\alpha_1\alpha_2} \\ 1 & \frac{1}{\beta_1} & \frac{1}{\beta_2} & \frac{1}{\beta_1\beta_2} \\ 1 & \frac{1}{\alpha_1} & \frac{1}{\beta_2} & \frac{1}{\alpha_1\beta_2} \\ 1 & \frac{1}{\beta_1} & \frac{1}{\alpha_2} & \frac{1}{\beta_1\alpha_2} \end{pmatrix}.
\]

For \( w = \begin{pmatrix} f_1 f_2 \\ x f'_1 f_2 \\ x f_1 f'_2 \\ x^2 f'_1 f'_2 \end{pmatrix} \), \( f_j = 2F_1(\alpha_j, \beta_j, \gamma_j; x) \) \( (j = 1, 2) \), using the relation

\[
f_j + \frac{x}{\alpha_j} f'_j = 2F_1(\alpha_j + 1, \beta_j, \gamma_j; x), \quad f_j + \frac{x}{\beta_j} f'_j = 2F_1(\alpha_j, \beta_j + 1, \gamma_j; x),
\]

we have

\[
v = x^\gamma_1RP^{-1}w = x^\gamma_1 \begin{pmatrix} 2F_1(\alpha_1 + 1, \beta_1, \gamma_1; x)2F_1(\alpha_2 + 1, \beta_2, \gamma_2; x) \\ 2F_1(\alpha_1, \beta_1 + 1, \gamma_1; x)2F_1(\alpha_2, \beta_2 + 1, \gamma_2; x) \\ 2F_1(\alpha_1 + 1, \beta_1, \gamma_1; x)2F_1(\alpha_2, \beta_2 + 1, \gamma_2; x) \\ 2F_1(\alpha_1, \beta_1 + 1, \gamma_1; x)2F_1(\alpha_2 + 1, \beta_2, \gamma_2; x) \end{pmatrix}, \tag{40}
\]

where the parameters \( \alpha_j, \beta_j, \gamma_j \) \( (j = 1, 2) \) satisfy the condition (2), which is written in the form

\[
\gamma_1 = \gamma_2 = \frac{1}{2}(\alpha_1 + 1 + \alpha_2 + 1 + \beta_1 + \beta_2), \text{ etc.}
\]

This means that the scalar differential equation of the fourth order satisfied by a component of \( v \) is equivalent to the equation satisfied by (37) with (38) by suitable change of parameters. For example, the equation satisfied by the first component of \( v \) agrees with the equation satisfied by (37) with \( \alpha_1, \alpha_2 \) and \( \gamma' \) replaced by \( \alpha_1 + 1, \alpha_2 + 1 \) and \( \gamma' + 1 \), respectively.
A The Dotsenko-Fateev system

Ebisu et al. [3] have shown that the Dotsenko-Fateev equation is obtained from the equation for (36) by the middle convolution and the addition.

In this appendix we shall show that the Euler transformation of a certain order of the system (39) gives the Dotsenko-Fateev system. It is a size three system of the form

\[ \frac{dz}{dx} = \left( \frac{1}{x} C_0 + \frac{1}{x-1} C_1 \right) z, \]  

(41)

where

\begin{align*}
C_0 &= \begin{pmatrix} 2a + 2c + g & 0 & b \\ 0 & -a - c & 0 \\ 0 & 2b + g & a + c \end{pmatrix}, \\
C_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2b + 2c + g & a \\ 2a + g & 0 & b + c \end{pmatrix}
\end{align*}

(see Haraoka [4]). For later use, we remark that

\begin{align*}
C_0 - (a + c)I_3 &= \begin{pmatrix} a + c + g & 0 & b \\ 0 & -a - c & 0 \\ 0 & 2b + g & 0 \end{pmatrix} \sim \begin{pmatrix} a + c + g \\ -a - c \\ 0 \end{pmatrix}, \quad (42a) \\
C_1 - (b + c)I_3 &= \begin{pmatrix} -b - c & 0 & 0 \\ 0 & b + c + g & a \\ 2a + g & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} b + c + g \\ -b - c \\ 0 \end{pmatrix} \quad (42b)
\end{align*}

and

\begin{align*}
C_0 + C_1 - (a + b + 2c)I_3 &= \begin{pmatrix} a - b + g & 0 & b \\ 0 & b - a + g & a \\ 2a + g & 2b + g & 0 \end{pmatrix} \sim \begin{pmatrix} g \\ a + b + g \\ -a - b \end{pmatrix}. \quad (42c)
\end{align*}

Note that we can write the system (39) in the form

\[ (xI_4 - RT R^{-1}) \frac{dv}{dx} = \Lambda v, \]

(43)

where \( \Lambda = \begin{pmatrix} \lambda_{++-J} & -J \\ \lambda_{+---} & +J \end{pmatrix} \). The Euler transformation

\[ v_\mu(x) = \int_C (t-x)^{\mu-1} v(t) \, dt, \]

(44)

where \( C \) is an appropriate path of integration, transforms the system (43) into

\[ (xI_4 - RT R^{-1}) \frac{dv_\mu}{dx} = (A + \mu I_4) v_\mu, \]

which is written in the form

\[ \frac{dv_\mu}{dx} = R(xI_4 - T)^{-1} R^{-1} (A + \mu I_4) v_\mu. \]

(45)
If \( \mu = \lambda_{-+++} \), then

\[
\Lambda + \mu I_4 = A + \lambda_{-+++} I_4 = \begin{pmatrix} 0 & 2\lambda_{-+++} & \lambda_{0-0+} \\ 0 & 2\lambda_{0-0+} & 0 \end{pmatrix},
\]

and hence the system (45) is reducible. We write

\[
R(\lambda_{-+++} I_4 - T)^{-1} R^{-1} (A + \lambda_{-+++} I_4) = \frac{1}{x} \begin{pmatrix} 0 & \ast \\ 0 & K_0 \end{pmatrix} + \frac{1}{x-1} \begin{pmatrix} 0 & \ast \\ 0 & K_1 \end{pmatrix},
\]

where \( K_0, K_1 \) are 3 \times 3 matrices, and

\[
v_{\lambda_{-+++}} = \begin{pmatrix} \ast \\ \tilde{v} \end{pmatrix},
\]

where \( \tilde{v} \) is a 3-dimensional vector. Then \( \tilde{v} \) satisfies the system

\[
\frac{d\tilde{v}}{dx} = \left( \frac{1}{x} K_0 + \frac{1}{x-1} K_1 \right) \tilde{v}.
\]

(47)

Since the matrices in the right hand side of (46) satisfy

\[
\begin{pmatrix} 0 & \ast \\ 0 & K_0 \end{pmatrix} = R \begin{pmatrix} I_2 & O \\ O & I_2 \end{pmatrix} (A + \lambda_{-+++} I_4) R^{-1}
\]

and

\[
\begin{pmatrix} 0 & \ast \\ 0 & K_1 \end{pmatrix} = R \begin{pmatrix} O & I_2 \\ I_2 & O \end{pmatrix} (A + \lambda_{-+++} I_4) R^{-1}
\]

we have

\[
K_0 \sim \begin{pmatrix} \lambda_{++++} J + \lambda_{-+++} I_2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \beta_1 + \beta_2 & -\alpha_1 - \alpha_2 \\ -\alpha_1 - \alpha_2 & 0 \end{pmatrix},
\]

\[
K_1 \sim \begin{pmatrix} \lambda_{-+++} J + \lambda_{-+++} I_2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\alpha_1 + \beta_2 & -\alpha_2 + \beta_1 \\ -\alpha_2 + \beta_1 & 0 \end{pmatrix}
\]

and

\[
K_0 + K_1 = \begin{pmatrix} 2\lambda_{-+++} & 0 \\ 0 & 2\lambda_{-+++} \end{pmatrix} = \begin{pmatrix} -\alpha_1 - \alpha_2 + \beta_1 + \beta_2 & -\alpha_2 + \beta_2 \\ -\alpha_2 + \beta_2 & -\alpha_1 + \beta_1 \end{pmatrix}.
\]
Comparing these with (42a)–(42c), we set
\[ \alpha_1 = a, \quad \alpha_2 = c, \quad \beta_1 = -b, \quad \beta_2 = a + b + c + g. \]  

Theorem A.1. **Under the situation (48), the change of variable**
\[ z = x^{2\lambda_{++00}}(x - 1)^{2\lambda_{0+00} - 0} \tilde{Q} \tilde{v}, \]
where
\[ \tilde{Q} = \begin{pmatrix} 2\lambda_{00+0} & 2\lambda_{00000\lambda_{0}+0} - \lambda_{++-+} & -2\lambda_{0000\lambda_{0}+0} - \lambda_{++--} \\ 2\lambda_{+000} & 2\lambda_{+0000\lambda_{0}+0} - \lambda_{++--} & -2\lambda_{+0000\lambda_{0}+0} - \lambda_{++--} \\ 2\lambda_{+0000} & 4\lambda_{+00000\lambda_{0}+0} - \lambda_{++--} & -2\lambda_{+0000\lambda_{0}+0} - \lambda_{++--} \end{pmatrix}, \]
transforms (47) into (41).

**Proof.** For \( C_0 + C_1 - (a + b + 2c)I_3 \), we take
\[ Q = \begin{pmatrix} -bp & q & -b(2b + g)r \\ ap & q & -a(2a + g)r \\ (a - b)p & 2q & (2a + g)(2b + g)r \end{pmatrix}, \]
where \( p, q, r \) are non-zero constants, that consists of right eigenvectors of the matrix to satisfy
\[ Q^{-1}(C_0 + C_1 - (a + b + 2c)I_3)Q = \begin{pmatrix} g & a + b + g \\ a + b \\ -(a + b) \end{pmatrix}. \]
This means that
\[ Q^{-1}(C_0 + C_1 - (a + b + 2c)I_3)Q = K_0 + K_1 \]
in the case (48). Moreover, by direct calculation we can verify that
\[ Q^{-1}(C_0 - (a + c)I_3)Q = K_0, \]
\[ Q^{-1}(C_1 - (b + c)I_3)Q = K_1 \]
hold separately if
\[ q = -\frac{a(a + b + g)}{2a + 2b + g}p, \quad r = \frac{a + b}{(2a + g)(2a + 2b + g)}p. \]

The matrix \( \tilde{Q} \) is nothing but (50) with (51) and \( p = 1 \) represented by \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) instead of \( a, b, c, g \). By the transformation (49), we have
\[ \frac{dz}{dx} = x^{2\lambda_{++00}}(x - 1)^{2\lambda_{0+00} - 0} \left( \frac{2\lambda_{++00} + 0}{x} + \frac{2\lambda_{00+0} + 0}{x - 1} \right) \tilde{Q} \tilde{v} \\
+ x^{2\lambda_{++00}}(x - 1)^{2\lambda_{0+00} - 0} \tilde{Q} \left( \frac{1}{x}K_0 + \frac{1}{x - 1}K_1 \right) \tilde{v} \\
= \left\{ \left( \frac{a + c}{x} + \frac{b + c}{x - 1} \right) I_3 + \tilde{Q} \left( \frac{1}{x}K_0 + \frac{1}{x - 1}K_1 \right) \tilde{Q}^{-1} \right\} z \\
= \left( \frac{1}{x}C_0 + \frac{1}{x - 1}C_1 \right) z. \]
This completes the proof. ☐
Following the process stated above, we can construct a solution of (11).

**Theorem A.2.** The system (11) has a solution of the form

\[
z = x^{a+c}(x-1)^{b+c} \left( \begin{array}{c}
-a-b - \frac{a(a+b+g)}{2a+2b+g} - \frac{b(a+b)(2a+2b+g)}{(2a+g)(2a+2b+g)} \\
a - \frac{a(a+b+g)}{2a+2b+g} - \frac{2a+b+g}{a+b} \\
a-b - \frac{2a(a+b+g)}{2a+2b+g} 
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\int_{0}^{x} (t-x)^{\frac{g}{2}-1}t^{a+c+\frac{g}{2}} _{2}F_{1}(a,-b+1,a+c+\frac{g}{2}+1;1) \\
\times _{2}F_{1}(c,a+b+c+g+1,a+c+\frac{g}{2}+1;1) dt \\
\int_{0}^{x} (t-x)^{\frac{g}{2}-1}t^{a+c+\frac{g}{2}} _{2}F_{1}(a+1,-b,a+c+\frac{g}{2}+1;1) \\
\times _{2}F_{1}(c,a+b+c+g+1,a+c+\frac{g}{2}+1;1) dt \\
\int_{0}^{x} (t-x)^{\frac{g}{2}-1}t^{a+c+\frac{g}{2}} _{2}F_{1}(a,-b+1,a+c+\frac{g}{2}+1;1) \\
\times _{2}F_{1}(c+1,a+b+c+g,a+c+\frac{g}{2}+1;1) dt 
\end{array} \right)
\]

where we assume that \(\Re\left(\frac{g}{2}\right) > 0\) and \(\Re(a+c+\frac{g}{2}+1) > 0\).

**Proof.** Under the situation (18), we have

\[
\gamma = \frac{1}{2}(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) + 1 = a + c + \frac{g}{2} + 1,
\]

\[
\mu = -\lambda_{++-} = \frac{g}{2},
\]

\[
\tilde{Q} = \left( \begin{array}{ccc}
-a-b & -\frac{a(a+b+g)}{2a+2b+g} & -\frac{b(a+b)(2a+2b+g)}{(2a+g)(2a+2b+g)} \\
a & -\frac{a(a+b+g)}{2a+2b+g} & -\frac{2a+b+g}{a+b} \\
a-b & -\frac{2a(a+b+g)}{2a+2b+g} & -\frac{2a+b+g}{a+b}
\end{array} \right).
\]

Combining these with (19), (21) and (23), we obtain the solution above. \(\square\)

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