ADM approach to 2+1 dimensional gravity
coupled to particles $^1$. 

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Abstract

We develop the canonical ADM approach to 2+1 dimensional gravity in presence of point particles. The instantaneous York gauge can be applied for open universes or universes with the topology of the sphere. The sequence of canonical ADM equations is solved in terms of the conformal factor. A simple derivation is given for the solution of the two body problem. A geometrical characterization is given for the apparent singularities occurring in the N-body problem and it is shown how the Garnier Hamiltonian system arises in the ADM treatment by considering the time development of the conformal factor at the locations where the extrinsic curvature tensor vanishes.

1 Introduction

After the seminal paper by Deser Jackiw and ’t Hooft [1] the problem of supplying a solution to 2+1 dimensional gravity has been approached from different viewpoints and with different techniques. Moncrief [2] and Hosoya and Nakao [3] gave the hamiltonian treatment in absence of particles. In [2, 3] the complete reduction of the hamiltonian to the physical parameters i.e. the moduli of the time slices was obtained in the York (minimal surface) gauge; the problem can be dealt with explicitly for the torus while for higher genus, even if well defined, is far from trivial. The solution of [2] and [3] were exploited in [4] and [5] for the quantization of the theory.

In presence of particles progress was made in the papers by Bellini, Ciafaloni and Valtancoli [6] and by Welling [7] in the first order formalism by going over to the instantaneous gauge; in these papers it was shown that the problem is equivalent to the solution of a Riemann- Hilbert problem. It can be solved explicitly for the case of two particles, in terms of hypergeometric functions. For three or more particles one encounters a feature known in the mathematical literature as apparent singularities. Such apparent singularities evolve according to a hamiltonian system of equations known as the Garnier system [8] which is derived in the mathematical literature from the isomonodromicity condition.

A different approach was put forward by ’t Hooft [9] by describing the evolving Cauchy surfaces in terms of polygonal tiles which join along segments where the extrinsic curvature
is singular. The dynamics of the system is codified in transition rules which intervene when e.g. the length of a side of a polygon goes to zero or when a particle collides with the side of a polygon. A similar but different approach was given by Waelbroek [10]. Quantization schemes were given in [11, 12, 13].

In this paper we shall consider $2 + 1$ dimensional gravity in presence of massive particles by exploiting the hamiltonian formulation. Most techniques apply to massless particles as well, but here we shall not be concerned with this case. Thus our setting is a second order formalism and the basic equations are those derived systematically from the variation of the ADM action. The reason for such a development is to give a treatment which resides completely within the canonical framework. We shall see that all results obtained in [6] come out in simple fashion from such an approach; moreover we shall prove that the Garnier hamiltonian system is a direct outcome of the canonical ADM formalism. Again the exploitation of the instantaneous York gauge plays a major role; the technical advantage of such a gauge is to reduce an equation of sinh-Gordon to one of Liouville type to which powerful methods of complex analysis apply. Unfortunately the applicability of such a gauge is restricted to universes of spherical topology or to open ones.

We shall see that such an approach provides an elementary way to solve the two body problem: the exact solution of the motion can be derived even before the explicit computation of the metric, which combined with the qualitative knowledge to the asymptotic metric gives a complete description of the scattering. The exact metric is obtained by solving a Liouville equation, leading of course to the same results obtained in [6] and [7].

One of the aims of presenting a canonical solution of the problem is to provide a framework for the quantization of the theory but here we shall be concerned solely with the classical theory.

The plan of the paper is the following: After recalling in sect.2 the ADM canonical formulation of gravity coupled with point particles we proceed in sect.3 to the solution of the canonical equations. These will be solved in the following order: First one computes the momenta canonically conjugate to the space metric; they turn out to be rational analytic or antianalytic functions of the complex coordinates of the plane. The knowledge of such canon-
ical momenta allows one to write down the partial differential equation for the conformal factor or better for a more fundamental quantity which we shall call the reduced conformal factor. This is a Liouville equation whose sources are located at the particle position; in addition when the number of particles $N$ is greater than 2 new sources appear. These are located at the points where the extrinsic curvature tensor vanishes and are identified as the position of the apparent singularities of the fuchsian differential equation which underlies the expression of the conformal factor. In the ADM equation for the conformal factor $\sigma$ the sources appear as function of the conformal factor itself. On the other hand it is shown in the text that due to the vanishing of the inverse $g^{ij}$ of the spatial metric on the particles, the sources simplify and are just given by the particle masses. The Liouville equation admits a whole family of solutions parametrized by a number $M$ whose values runs from zero to $4\pi$ and it is related to the asymptotic behavior of the conformal factor; it is nothing else than the total energy of the system in adimensional units. A byproduct of the ADM equation for the time derivative of $\sigma$ is that such parameter $M$ is constant i.e. the energy is conserved.

The Lagrange multiplier $N$ which appears in the ADM metric in the combination $-N^2 dt^2$ satisfies a differential equation, which again due to the vanishing of $g^{ij}$ on the particles reduces to a homogeneous linear differential equation. This is shown to be simply solved by the derivative of the conformal factor with respect to the parameter $M$. An elementary theorem proves that such a solution is unique. The final ingredient of the ADM metric i.e. the shift functions $N^i$ are expressed in terms of the spatial derivative of $N$.

In sect.4 we write the Hamilton equation for the time variation of the particle positions and momenta. These are explicited for the general $N$-body problem in terms of the coefficient of the fuchsian equation which underly the conformal factor. In the two body case, which is dealt with in sect.5, they give rise to an elementary system of two differential equations for the particle positions and momenta, which solved provides the particle trajectories. The exact metric can be given in terms of the classical solution of the Liouville equation. In sect.6 we give the general form of the asymptotic metric in the York instantaneous gauge.

In sect.7 we give the treatment of the problem for $N > 2$. It is shown how the fuchsian nature of the differential equation is related to the conservation of momenta and to a gen-
eralized conservation law which appears related to the transformation of the action under space dilatations.

As mentioned above, with more than two particles there are apparent singularities and one has to provide evolution equations for their positions and residues. We show that all this information is contained in the ADM equation for the time derivative of the conformal factor. In fact by exploiting Schwarz’s relation between the coefficient of the fuchsian differential equation and the reduced conformal factor one obtains the Garnier equation by equating the residues at the polar singularities at the position of the apparent singularities.

2 The action

The action for the gravitational field in any dimension, including the boundary terms is given by [14]

\[ S_{\text{Grav}} = \frac{c^3}{16\pi G_N} \int_M d^{(D+1)}x \sqrt{g} \, \mathcal{R} + \frac{c^3}{8\pi G_N} \int_{\Sigma_0}^{\Sigma_1} d^Dx \sqrt{g} \, K + \]

\[ + \frac{c^3}{8\pi G_N} \int_B d^Dx \sqrt{\gamma} \Theta + \frac{c^3}{8\pi G_N} \int_{B_0}^{B_1} d^{D-1}x \sqrt{\sigma} \eta \]  

where \( g_{\mu\nu}, \mathcal{R} \) are the \( D + 1 \) dimensional metric and curvature, \( g_{ij} \) the \( D \)-dimensional metric of the constant time slices \( \Sigma_t; \Sigma_0 \) and \( \Sigma_1 \) the initial and final time slices; \( K_{ij} \) the second fundamental form of the time slices \( \Sigma_t \) and \( \Theta_{ij} \) the second fundamental form of the lateral boundary \( B \) whose volume form is \( \sqrt{-\gamma} \); \( n^\mu \) is the future pointing unit normal to the time slices and \( w^\mu \) the outward pointing unit normal to \( B \); \( B_t = \Sigma_t \cap B \), \( \sqrt{\sigma} \) is the volume form induced on \( B_t \) and \( \sinh \eta = n_\mu w^\mu \). In presence of particles we must add to the gravitational action the matter term

\[ S_m = -\int dt \sum_n m_n c \sqrt{-g_{\mu\nu}(q_n)} \dot{q}_n^\mu \dot{q}_n^\nu \]  

(2)
where we have chosen the gauge \( q_n^0 = t \). At the classical level it is convenient to multiply the action by \( \frac{16\pi G_N}{c^3} \); the matter action e.g. becomes

\[
S_m = -\int dt \sum_n m_n \sqrt{-g_{\mu\nu}(q_n)\dot{q}_n^\mu\dot{q}_n^\nu}
\]

(3)

with \( m_n = \frac{16\pi G_N m_n}{c^2} \); it is a real number \( 0 < m_n < 4\pi \). In the ADM notation where the metric is written as \[15\]

\[
ds^2 = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)
\]

(4)

the volume term can be rewritten as \[14\]

\[
S_H = \int_M d^Dx dt \sqrt{\sigma} \left[ R + K_{ij}K^{ij} - K^2 + 2(n^\alpha K - a^\alpha)\gamma_{,\mu} \right],
\]

(5)

where \( R \) is the \( D \)-dimensional curvature related to \( g_{ij} \) and \( a^\mu \) is the acceleration vector \( a^\mu = n^\nu n^\mu_{,\nu} \). Going over to the canonical variables

\[
P_{ni} = \frac{\partial L}{\partial \dot{q}_n^i} = m_n \frac{g_{\mu\sigma}(q_n)\dot{q}_n^\mu\dot{q}_n^\sigma}{\sqrt{-g_{\rho\sigma}(q_n)\dot{q}_n^\rho\dot{q}_n^\sigma}} = m_n \frac{g_{ij}(q_n)(\dot{q}_n^i + N^i)}{\sqrt{N^2 - g_{ij}(\dot{q}_n^i + N^i)(\dot{q}_n^j + N^j)}}
\]

(6)

the matter action can be rewritten as

\[
S_m = \int dt \sum_n \left( P_{ni} \dot{q}_n^i + N^i(q_n)P_{ni} - N(q_n)\sqrt{P_{ni}P_{nj}g^{ij}(q_n) + m_n^2} \right).
\]

(7)

On the other hand the gravitational action expressed in terms of the canonical variables becomes \[14, 16, 17\]

\[
S_{Grav} = \int_M dt d^Dx \left[ \pi^{ij} \dot{g}_{ij} - N^i H_{,i} - NH \right] +
+ 2 \int dt \int_{B_t} d^{(D-1)}x \sqrt{\sigma} N \left( K_{B_t} + \frac{\eta}{\cosh\eta} \nabla_\alpha v^\alpha \right) - 2 \int dt \int_{B_t} d^{(D-1)}x r_\alpha \pi^{\alpha\beta}_{(\sigma)} N_\beta.
\]

(8)

The symbol \( K_{B_t} \) stands for the extrinsic curvature of \( B_t \) as a surface embedded in \( \Sigma_t \), \( v^\alpha \equiv \frac{1}{\cosh\eta} (n^\alpha - \sinh\eta u^\alpha) \) and \( r_\alpha \) is the versor normal to \( B_t \) in \( \Sigma_t \). The subscript \( \sigma \) in \( \pi^{\alpha\beta}_{(\sigma)} \) is a reminder that it has to be considered a tensor density with respect to the measure \( \sqrt{\sigma} \).
3 The field equations

3.1 The conjugate momenta

Variation of the action with respect to $N^i$ gives

$$H_i = -2\sqrt{g} \nabla_j \frac{\pi^j}{\sqrt{g}} - \sum_n \delta^2(x - q_n) P_{ni} = 0 \quad (9)$$

where $\nabla$ is the covariant derivative with respect to the induced metric $g_{ij}$. In a York $K = \text{const.}$ gauge we have

$$2\sqrt{g} \nabla_j \frac{\pi^T_j}{\sqrt{g}} = -\sum_n \delta^2(x - q_n) P_{ni} \quad (10)$$

where $\pi^T$ is the traceless part of $\pi^{ij}$. When $K$ vanishes, $\pi^{ij}$ becomes traceless since in 2 + 1 dimensions

$$\pi^{ij} \equiv \sqrt{g}(K^{ij} - g^{ij}K) \quad (11)$$

and thus $\pi^i_j$ has only two independent components. Using the complex coordinates $z = x + iy$ and $\bar{z} = x - iy$, eq. (10) reduces to

$$\partial_z \pi^z = -\frac{1}{2} \sum_n P_{nz} \delta^2(z - z_n) \quad (12)$$

$$\partial_{\bar{z}} \pi^z = -\frac{1}{2} \sum_n P_{n\bar{z}} \delta^2(z - z_n) \quad (13)$$

with $\pi^z = \pi^x - i \pi^y$ and $\int \delta^2(z) dx dy = 1$. They are solved by

$$\pi^z = -\frac{1}{2\pi} \sum_n \frac{P_{nz}}{z - z_n} \quad (14)$$

$$\pi^z = -\frac{1}{2\pi} \sum_n \frac{P_{n\bar{z}}}{\bar{z} - \bar{z}_n} \quad (15)$$
We shall work in the c.m. frame i.e. with \( \sum_n P_n = 0 \). With such a restriction \( \pi^z \) and \( \pi^\bar{z} \) decrease at infinity at least like \( 1/|z|^2 \). In principle one could add to the solution (14) (and (13)) and arbitrary analytic (antianalytic) function. Our choice to maintain the behavior \( \pi^a = O(1/|z|^2) \) will give rise, as discussed in sect.3.3 to a well defined asymptotic behavior of the conformal factor \( e^{2\sigma} \sim (z\bar{z})^{-\mu} \) and \( N \sim \ln(z\bar{z}) \). Thus the choice of solutions (14,15) is a way to impose the form of the asymptotic metric.

As we already mentioned in the introduction and will be shown below, one can apply the instantaneous York gauge \( K = 0 \) only to the topology of the plane (open universes) or the topology of the sphere. The sphere can be described by the metric

\[
g_{ij} = \frac{8e^{2\sigma}}{(1 + z\bar{z})^2} \tag{16}
\]

where \( \sigma \) is a conformal factor regular at infinity. The equations for the \( \pi^a \) are still provided by eq.(10) and their solution by eq.(14,15). On the other hand for the sphere the point \( z = \infty \) is a regular point. Imposition of the regularity of the tensor \( g^{-1/2}\pi^a \) at infinity, obtained through the transformation \( z' = 1/z \) imposes for \( \pi^z \) the behavior \( \pi^z \approx z^{-n} \) with \( n \geq 4 \). This is reflected on the following sum rules on the momenta \( \sum_n P_{nz} = 0, \sum_n P_nz_n = 0, \sum_n P_nz^2_n = 0 \). Thus except for the static case in which all momenta vanish we need at least four particles on the sphere. In the following we shall be mainly interested in the plane topology.

The tracelessness of the momenta \( \pi^{ij} \) in the \( K = 0 \) gauge entails some conservation laws which relate combinations of particle positions and momenta with no reference to the field variables. We shall see in sect.7 that such relations can be derived directly from the equations of motions.

We want to examine here the expression of the conservation of angular momentum in the \( K = 0 \) gauge in the context of the hamiltonian formalism. Under a rotation

\[
q_n^i \to q_n^i + \varepsilon q_n^j \epsilon^{-j}_i; \quad P_{ni} \to P_{ni} - \varepsilon \epsilon^{-j}_i P_{nj};
\]

\[
g_{ij}(x^k) \to g_{ij}(x^k - \varepsilon x'^k_i) - \varepsilon \epsilon^j_i g_{ij}(x) - \varepsilon \epsilon^k_i g_{ij}(x) \tag{17}
\]
and similar transformations on the remaining field variables, the volume hamiltonian extended to a disk of radius \( r \) and the boundary hamiltonian computed on a circle of radius \( r \) are left invariant. The term \( \sum_n P_n \dot{q}_n \) is also left invariant while the term \( \pi^{ij} \dot{g}_{ij} \) is left identically zero. As a consequence the quantity

\[
\sum_n x^i_n \epsilon^j_n P_{nj}
\]  

(18)

is a constant of motion. This is the expression of the conservation of angular momentum in the \( K = 0 \) gauge.

### 3.2 The conformal factor

The variation of the action under \( N \) gives the hamiltonian constraint \([13, 17]\)

\[
H = \frac{1}{\sqrt{g}} \left[ \pi^a_b \pi^b_a - (\pi^c_c)^2 \right] - \sqrt{g} R + \sum_n \delta^2(z - z_n) \sqrt{m_n^2 + P_{n_i} g^{ij} P_{nj}} = 0.
\]  

(19)

which in the instantaneous York gauge, i.e. \( K = 0 \) becomes the non linear equation

\[
2 \Delta \sigma = -\pi^a_b \pi^b_a e^{-2\sigma} - \sum_n \delta^2(z - z_n) \sqrt{m_n^2 + 4 P_{nz} P_{n\bar{z}} e^{-2\sigma}}.
\]  

(20)

Integrating eq. (19) over all space and using the Gauss-Bonnet theorem we see that, due to the positivity of \( \pi^a_b \pi^b_a \) the \( K = 0 \) gauge is applicable only to closed surfaces of genus 0 or to open universes \([7]\). We shall be concerned in the following with open universes.

The term \( \pi^a_b \pi^b_a = 2 \pi^z_{\bar{z}} \pi^{\bar{z}}_{z} \) is the product of an analytic \( \pi^{z}_{\bar{z}}(z) \), and an antianalytic function and as such the laplacian of its logarithm is zero except on the singularities. The structure of \( \pi^{z}_{\bar{z}} \), in the c.m. frame where \( \sum_n P_n = 0 \), is

\[
\pi^{z}_{\bar{z}} = \frac{p_{N-2}(z)}{\prod_n (z - z_n)}
\]  

(21)

with \( p_{N-2}(z) \) a polynomial of degree \( N - 2 \), being \( N \) the number of particles. It will possess \( N - 2 \) zeros in the complex plane and this is the origin in the present approach of the so called apparent singularities. Thus in the ADM approach the apparent singularities are given by
the zeros of the extrinsic curvature tensor $K_{ij}$ because we have the general relation eq.(11) and in our gauge $K = 0$. Due to the Gauss–Codazzi relations they are also the points where the intrinsic curvature of the two dimensional time slice vanish. We shall denote by $z_A$ the location of the apparent singularities. We shall now go over to the function

$$2\tilde{\sigma} = 2\sigma - \ln(2\pi^2 z^2 \bar{z}^2)$$

and thus we have

$$2\Delta \tilde{\sigma} = -e^{-2\tilde{\sigma}} - \sum_n \delta^2(z - z_n)(\sqrt{m_n^2 + 4P_{nz}P_{n\bar{z}}e^{-2\sigma} - 4\pi}) - \sum_A 4\pi\delta^2(z - z_A)$$

which we want now to discuss near the particle singularities.

Near the $n$ particle, which to simplify the notation we suppose to be placed at $z = 0$, eq.(23) reduces to

$$2\Delta \tilde{\sigma} = -e^{-2\tilde{\sigma}} - 4\pi(a_n - 1)\delta^2(z)$$

with

$$4\pi a_n = \sqrt{m_n^2 + 4C_nP_{nz}P_{n\bar{z}}}$$

and $C_n = e^{-2\sigma(0)}$. $C_n$ cannot be infinity otherwise $\tilde{\sigma}$ is not a solution of eq.(24). Eq.(24) can be solved exactly [19] by performing the transformation $w = (z/\Lambda)^{a_n}$ on the solution of the Liouville equation $2\Delta \tilde{\sigma} = -e^{-2\tilde{\sigma}}$ representing the Poincaré pseudosphere, to obtain

$$e^{-2\tilde{\sigma}} = \frac{8a_n^2}{\Lambda^2} \frac{(z\bar{z}/\Lambda^2)^{a_n-1}}{1 - (z\bar{z}/\Lambda^2)\mu_n^2}$$

with arbitrary $\Lambda$. In particular, for $|z| \to 0$ we have

$$e^{-2\tilde{\sigma}} \sim \frac{8a_n^2}{\Lambda^2} \left(\frac{z\bar{z}}{\Lambda^2}\right)^{a_n-1}.$$  

Combining such a behavior with the factor $\pi^a \pi^b \pi^c$ and keeping in mind that $m_n/4\pi > 0$ we have that the factor $e^{-2\sigma}$ vanishes on the sources, which implies $4\pi a_n = m_n$. This simplifies the source term in eq.(23). We have

$$e^{-2\tilde{\sigma}} \sim \frac{8a_n^2}{\Lambda^2} \left(\frac{z\bar{z}}{\Lambda^2}\right)^{\mu_n-1}$$

with $\mu_n = m_n/4\pi$. 

9
The analogous calculation on the apparent singularities gives a \( e^{-2\sigma} \) regular at those points as expected from eq. (20) and the equation for \( 2\tilde{\sigma} \) takes the simpler form

\[
2\Delta \tilde{\sigma} = -e^{-2\tilde{\sigma}} - 4\pi \sum \delta^2(z - z_n)(\mu_n - 1) - 4\pi \sum_{A} \delta^2(z - z_A).
\] (29)

### 3.3 The lapse function

The variation of the action with respect to \( g_{ij} \) gives

\[
\frac{\partial \pi^{ij}}{\partial t} = \frac{Ng^{ij}}{2\sqrt{g}}(\pi^a_b \pi^b_a - (\pi^a_a)^2) - \frac{2N}{\sqrt{g}}(\pi^{im} \pi^m_j - \pi^{ij} \pi^a_a) + \sqrt{g}(\nabla^i \nabla^j N - g^{ij} \nabla^m \nabla_m N) +
\]

\[
+ \nabla_m (\pi^{ij} N^m - \pi^{jm} \nabla_m N^i + \pi^{im} \nabla_m N^j + N \sum_n \delta^2(x - q_n) \frac{P_{na} P_{nb} g^{ai} g^{bj}}{2\sqrt{m_n^2 + P_{na} P_{nb} g^{ab}}}. \] (30)

where we took into account that in two dimensions \( R^{ij} - g^{ij} R/2 = 0 \), and the variation of the action with respect to \( \pi^{ij} \) gives

\[
\dot{\pi}^{ij} = \frac{2N}{\sqrt{g}}(\pi^{ij} - g^{ij} \pi^a_a) + \nabla_j N_i + \nabla_i N_j. \] (31)

Using the time evolution given by the above two equations, the relation \( K = \text{const} = 0 \) becomes

\[
\Delta N = \pi^a_b \pi^b_a e^{-2\sigma} N + Ne^{-2\sigma} \sum_n \frac{4P_{nz} P_{\bar{n}z}}{2m_n} \delta^2(z - z_n) \] (32)

where we took into account that on the particles \( e^{-2\sigma} \) vanishes. This is a linear partial differential equation for \( N \). We are interested in the behavior of \( N \) near the particle singularity. Near the particle \( n \) the equation takes the form

\[
\Delta N = \frac{P_{nz} P_{\bar{n}z}}{2\pi^2 z \bar{z}} e^{-2\sigma} N + \frac{4P_{nz} P_{\bar{n}z}}{2m_n} e^{-2\sigma} N \delta^2(z).
\] (33)

A solution to eq. (33) must be such \( c = \lim_{r \to 0} Ne^{-2\sigma} \) is finite. Moreover it is not difficult to prove that \( c = 0 \). In fact by integrating twice eq. (33) in \( |z| \) we find

\[
N \approx \frac{c P_{nz} P_{\bar{n}z}}{4\pi^2} \ln^2 |z| + c_1 \ln |z| + c_0,
\] (34)
which substituted in the definition of $c$ gives $c = 0$. Moreover as for $c = 0$ the $\delta$ contribution in eq. (33) is absent we have also $c_1 = 0$. The conclusion is that $N$ is finite at the particles and the $\delta$ term in eq. (32) is absent due to the vanishing of $Ne^{-2\sigma}$ on the particles and $N$ satisfies the linear homogeneous equation

$$\Delta N = e^{-2\sigma}N.$$  \hfill (35)

Now we analyze the equation

$$\Delta N = \frac{P_n z \bar{z}}{2\pi^2} e^{-2\sigma}N$$  \hfill (36)

for small $z$. Substituting the behavior of $e^{-2\sigma}$ we have

$$\Delta N = \frac{8\mu_n^2}{\Lambda^2} (\frac{z\bar{z}}{\Lambda^2})^{\mu_n-1}N.$$  \hfill (37)

Going over to the variable $r = |z|/\Lambda$ and integrating twice we have for the $s$-wave around the singularity

$$\frac{1}{r} \frac{d}{dr} \frac{d}{dr} N = 8\mu_n^2 (r^2)^{\mu_n-1}N$$  \hfill (38)

and we find

$$N = \text{const.} \left[ 1 + 2(z\bar{z}/\Lambda^2)^{\mu_n} + O(z) + O(\bar{z}) \right].$$  \hfill (39)

We examine now the large distance behavior of the conformal factor. Integrating

$$\Delta(2\sigma) = -\pi^a_b \pi^b_a e^{-2\sigma} - \sum_n \delta^2(z - z_n)m_n,$$  \hfill (40)

we obtain

$$\lim_{R \to \infty} 2 \oint_R \nabla \sigma \cdot \bar{n} \ dl = -M \equiv -4\pi \mu,$$  \hfill (41)

where

$$M = \sum_n m_n + \int \pi^a_b \pi^b_a e^{-2\sigma} d^2z.$$  \hfill (42)
which gives for $\sigma$ the asymptotic behavior

$$2\sigma \sim -\frac{M}{4\pi} \ln(z\bar{z}) + \ln s. \quad (43)$$

This result holds if the integral appearing in eq.(42) converges. As $\sum_n P_n = 0$ we have

$$\pi^a b^b \approx \frac{1}{\pi^4}$$

and the integral converges provided $-1 + \mu < 0$, which means that the opening of the cone at infinity cannot be negative. (We notice that with a choice $\sum_n P_nz \neq 0$ the integral in eq.(42) would be divergent). Actually one has infinite solutions to eq.(40) depending on the total energy $M$ of the system which has to be specified in solving it. We shall see at the end of this section that $M$ cannot depend on time.

Similarly applying Gauss theorem to

$$\Delta N = e^{-2\tilde{\sigma}} N = \pi^a b^b_a e^{-2\sigma} N \quad (44)$$

we obtain

$$N \sim \frac{n}{4\pi} \ln(z\bar{z}) \quad (45)$$

with

$$n = \int d^2 z \pi^a b^b_a e^{-2\sigma} N. \quad (46)$$

The behavior eq.(43) is consistent with the convergence of the integral eq.(46) provided $M/4\pi \equiv \mu < 1$ which is the condition we already met. On the other hand it is easily seen that a behavior $N \sim |z|^\beta$ which makes the integral (46) divergent at infinity is inconsistent with the differential equation (44).

We want now to connect the solution $2\tilde{\sigma}$ of eq.(29) with the solutions $N$ of eq.(35). We already noticed that $2\tilde{\sigma}$ depends on the free parameter $M$, which is the total energy of the system, and such a parameter does not appear in eq.(29). Then it is immediately seen that

$$N = \frac{\partial(-2\tilde{\sigma})}{\partial M} \quad (47)$$

is a solution to eq.(35). The equation for $N$ being homogeneous, its normalization is conventional and could also be taken time dependent. With our choice eq.(47), the behavior of
$N$ at infinity is, from eq.(13) $N \approx \frac{1}{4\pi} \ln(z\bar{z})$. On the other hand it is very simple to prove the following uniqueness theorem: If $N$ behaves at infinity like $N \approx c_0 + c_1 r^{-a}$, $a > 0$ and on the particle singularities as in eq.(39), then $N \equiv 0$.

In fact multiplying by $N$ eq.(35)

$$\int_A d^2z \nabla(N \nabla N) = \int_A d^2z \nabla N \cdot \nabla N + \int_A d^2z e^{-2\sigma} N^2. \quad (48)$$

But for $r \to \infty$

$$\int_{C_r} N \nabla N \cdot \vec{n} \, dl \approx 2\pi (-a) r^{-a} \to 0 \quad (49)$$

and on the particle singularity for $r \to 0$

$$\int_{C_r} N \nabla N \cdot \vec{n} \, dl \approx 2\pi r [c_0 + c_1 (r^2) \mu_n] [2c_1 r \mu_n (r^2) \mu_n - 1] \to 0 \quad (50)$$

which proves our assertion.

### 3.4 The shift functions

The traceless part of eq.(31) gives

$$\partial_z N^z = -\pi_2^z e^{-2\sigma} N \quad (51)$$

$$\partial_{\bar{z}} N^\bar{z} = -\pi_2^\bar{z} e^{-2\sigma} N. \quad (52)$$

Use eq.(44) on the r.h.s. of the first. Then we have

$$2\pi_2^z(z) \partial_z N^z = -4 \partial_{\bar{z}} \partial_z N \quad (53)$$

whose general solution is

$$N^z = -\frac{2}{\pi_2^z(z)} \partial_z N + g(z) \quad (54)$$
where \( g(z) \) is an analytic function of \( z \). A similar solution holds for \( \bar{N}^z \). The analytic function \( g(z) \) must be chosen as to kill the poles generated by the zeros of \( \pi^z_{z}(z) \), which occur only in presence of three or more particles, and if we are interested in describing a reference system which does not rotate at infinity \( g(z) \) has to be chosen as to give \( N^z < |z| \) at infinity. We shall see in sect.7 that, due to \( \sum P_{nz} = 0 \), such a function can diverge at infinity at most linearly. As we shall see \( g(z) \) encodes the time evolution of all the quantities of the problem.

The trace part of eq.(31) gives

\[
2 \dot{\sigma} = N^z \partial_z (2\sigma) + \partial_z N^z + \text{c.c.} \tag{55}
\]

By substituting the asymptotic behavior for \( 2\sigma \) eq.(43) into eqs.(54,55) we see that \( \dot{M} = 0 \) as the logarithmic term \( \dot{M} \ln(z\bar{z})/4\pi \) on the l.h.s. is not matched on the r.h.s.

### 4 The particle equations of motion

In the present section we shall derive the general expression for the time derivative of the particle position \( z_n \) and of the particle conjugate momenta \( P_n \).

Variation of the action with respect to \( P_{nz} \) gives

\[
\dot{z}_n = -N^i(z_n, t) = -g(z_n). \tag{56}
\]

The equation for \( \dot{P}_n \)

\[
\dot{P}_{ni} = \frac{\partial \left( P_{na} N^a - N \sqrt{P_{na} P_{nb} g^{ab} + m_n^2} \right)}{\partial x^i}(z_n) \tag{57}
\]

requires a little attention because in general the metric is divergent at the particle position. Let us consider the expression of \( P_n \)

\[
P_{ni} = m_n \frac{g_{ij}(z_n)(\dot{z}_n^i + N^i)}{\sqrt{N^2 - g_{ij}(\dot{z}_n^i + N^i)(\dot{z}_n^j + N^j)}} \tag{58}
\]

At the point \( z = z_n \) we have \( \dot{z}_n^i + N^j = 0 \) but it does not mean that \( P_{ni} = 0 \) as \( g_{ij} = \hat{g}_{ij} e^{2\sigma} \) is divergent at that point. We have to compute the behavior of \( N^z(z) - N^z(z_n) \) for \( z \to z_n \).
From eq.(54) we have
\[ N^z(z) - N^z(z_n) \approx \frac{2m_n N(z_n)}{P_{nz}} \left( \frac{(z - z_n)(\bar{z} - \bar{z}_n)}{\Lambda^2} \right)^{\mu_n} + O(z - z_n) + O(\bar{z} - \bar{z}_n) \]  (59)
and
\[ \epsilon^{2 \sigma} \approx \frac{P_{nz} P_{nz}}{m_n^2} \left( \frac{\Lambda^2}{(z - z_n)(\bar{z} - \bar{z}_n)} \right)^{-\mu_n} + O(z - z_n) + O(\bar{z} - \bar{z}_n). \]  (60)
The square root in the denominator goes over to \( N(z_n) \) and thus in the limit \( z \to z_n \) we have
\[ m_n \frac{g_{ij}(z_n)(\dot{z}_n^j + N^j)}{\sqrt{N^2 - g_{ij}(\dot{z}_n^i + N^i)(\dot{z}_n^j + N^j)}} \to P_{nz}. \]  (61)
With regard to the convergence of the numerator we notice that it is of the type
\[ ((z - z_n)(\bar{z} - \bar{z}_n))^{-\mu_n} |((z - z_n)(\bar{z} - \bar{z}_n))^{\mu_n} + O(z - z_n) + O(\bar{z} - \bar{z}_n)| \]  (62)
which converges absolutely to 1 only for \( \mu_n < 1/2 \) i.e. for not too heavy particle. On the other hand if we average the l.h.s. of eq.(61) on a circle around \( z_n \) and then take the limit for zero radius it always converges to \( P_{nz} \). We find a similar problem in writing the equation for \( \dot{P}_{nz} \). Taking into account the behavior eqs.(38,59) of the functions \( N \) and \( N^z \) for \( z \to z_n \), eq.(57) becomes
\[ \dot{P}_{nz} = P_{na} \frac{\partial N^a}{\partial z} - m_n \frac{\partial N}{\partial z} \]  (63)
and a similar equation for \( \dot{P}_{nz} \). Eq.(63) has to be understood as taking the average over a circle around \( z_n \) and then taking the limit for zero radius. The procedure is similar to the one adopted in [18]. The situation is analogous to the one which occurs in electrostatics in presence of point charges. The electric field is infinite or better not defined on the particle, but the average of the electric field on a small sphere surrounding the particle gives the effective field which acts on the particle as in such a process the field generated by the particle itself averages to zero and what survives is the external field. We shall see several application of such equations in the following.
5 The two body problem

5.1 The conformal factor

In the two particle case the equation for $\tilde{\sigma}$ has only two sources.

$$\Delta(2\tilde{\sigma}) = -e^{-2\tilde{\sigma}} + 4\pi(1 - \mu_1)\delta^2(z - z_1) + 4\pi(1 - \mu_2)\delta^2(z - z_2)$$

and we have no apparent singularities as $\pi^z_\tilde{z}(z)$ has no zeros in the complex plane. The procedure for solving such equation is well known [19]

$$e^{-2\tilde{\sigma}} = \frac{8f'(z)f'(\bar{z})}{(1 - f(z)f(\bar{z}))^2}$$

(65)

where the function $f(z)$ is given by the ratio of two independent solutions of the second order fuchsian differential equation with two regular singularities in $z_1$ and $z_2$ and one at infinity [20]

$$f(z) = \frac{ky_1(z)}{y_2(z)}.$$  

(66)

In fact neglecting for the moment the sources one has simply to check that, being $\ln(f'(z)f'(\bar{z}))$ an harmonic function,

$$4\partial_f\partial_{\bar{f}}\ln(1 - ff) = -\frac{8}{(1 - ff)^2}.$$  

(67)

On the singularities the difference of the indices given by [20]

$$y_1 \approx (z - z_n)^{\alpha_n}, \quad y_2 \approx (z - z_n)^{\beta_n}$$

(68)

has to be such that on the singularity the term $\ln(f'(z)f'(\bar{z}))$ gives

$$4\partial_z\partial_{\bar{z}}(\alpha_n - \beta_n - 1)\ln((z - z_n)(\bar{z} - \bar{z}_n)) = -4\pi(1 - \mu_n)\delta^2(z - z_n)$$

(69)

i.e. $\alpha_n - \beta_n = \mu_n$. The differential equation for $y$ is provided by

$$y''(\zeta) + Q(\zeta)y(\zeta) = 0$$

(70)
with \( \zeta = (z - z_1)/(z_2 - z_1) \) and

\[
Q(\zeta) = \frac{1}{4} \left( \frac{1 - \mu_1^2}{\zeta^2} + \frac{1 - \mu_2^2}{(\zeta - 1)^2} + \frac{1 - \mu_1^2 - \mu_2^2 + \mu_\infty^2}{\zeta(1 - \zeta)} \right). \tag{71}
\]

In fact with two singularities at the finite and one at infinity the linear residues are fixed by the Fuchs relations in terms of the \( \mu_1, \mu_2 \) and \( \mu_\infty \) (see also eq.(127)). To understand the behavior at infinity let us rewrite

\[
e^{-2\tilde{\sigma}} = \frac{8w_{12} \bar{w}_{12}}{(y_2 \bar{y}_2 - k k \bar{k} y_1 \bar{y}_1)^2} \tag{72}
\]

where \( w_{12} = k(y'_1 y_2 - y_1 y'_2) \) is the constant wronskian. From eq.(72) we see that at infinity

\[
-2\tilde{\sigma} \approx -(1 + \mu_\infty) \ln(\zeta \bar{\zeta}) \tag{73}
\]

or

\[
-2\tilde{\sigma} = -2\sigma + \ln(\pi^a \pi^b) \approx \frac{M}{4\pi} \ln(\zeta \bar{\zeta}) - 2 \ln(\zeta \bar{\zeta}) \tag{74}
\]

i.e. \( \mu_\infty = 1 - M/4\pi \equiv 1 - \mu \).

A further requirement on the mapping function \( f \) is to give a well defined, i.e. monodromic \( \tilde{\sigma} \) when one goes around the singularities 0, 1 and consequently \( \infty \) in the variable \( \zeta \). From the general theory of fuchsian differential equations we know that the vector \( (ky_1, y_2) \) goes over to a \( SL(2, C) \) transformed vector \( (ay_1 + by_2, cy_1 + dy_2) \) due to the constancy of the wronskian. From eq.(72) we see that in order that \( e^{-2\tilde{\sigma}} \) be well defined we need

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1, 1). \tag{75}
\]

The simplest way to realize this constraint is to choose \( f = ky_1/y_2 \) with \( y_1 \) and \( y_2 \) behaving as in eq.(68); then \( f \) satisfies the monodromic condition around the point \( \zeta = 0 \) for any \( k \). The point now is to determine \( k \) such as the \( \tilde{\sigma} \) is monodromic also around the point 1 and as a consequence at \( \infty \). For completeness we give in the Appendix the expression of \( y_1, y_2 \) and the derivation of \( k \).
The so determined value of $k$ gives through eq.(72) and eq.(22) the conformal factor $e^{2\sigma}$. Then we obtain $N$ from

$$N = \frac{\partial(-2\tilde{\sigma})}{\partial M}$$

(76)

whose behavior for large $|z|$ we already know to be $N \approx \frac{1}{4\pi} \ln(z\bar{z})$.

5.2 Determination of the trajectory

It is interesting that in order to write down and solve the equations of motion of the particles we do not need to know the explicit form of $2\sigma$ and $N$. In fact from eq.(56) we see that what we need is $N^z$ in $z_1$ and $z_2$. For two particles let

$$2P = P_1 - P_2; \quad P = P_1$$

(77)

from which

$$\pi^z_\bar{z}(z) = \frac{P_2 z - z_1}{2\pi (z - z_1)(\bar{z} - \bar{z_2})}$$

(78)

and thus

$$\pi^a_b \pi^b_a = \frac{2P_2 P_\bar{z}}{4\pi^2} \frac{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)}{(z - z_1)(\bar{z} - \bar{z_1})(z - z_2)(\bar{z} - \bar{z_2})}$$

(79)

and

$$N^z = \frac{4\pi (z - z_1)(z - z_2)}{P_\bar{z}} \frac{z_2 - z_1}{z_2 - z_1} \bar{z}_2 N + g(z).$$

(80)

The first term at infinity behaves like

$$-\frac{z}{P_\bar{z}(z_2 - z_1)}$$

(81)

and in order to have a non rotating reference frame at infinity we must choose $g(z)$ such as to remove this divergent behavior at infinity i.e.

$$g(z) = \frac{z}{P_\bar{z}(z_2 - z_1)}.\quad (82)$$
As $\pi_z$ has poles in $z_1, z_2$ we have
\[
\dot{z}_1 = -N^z(z_1) = -g(z_1) = -\frac{z_1}{P_z(z_2 - z_1)}; \quad \dot{z}_2 = -\frac{z_2}{P_z(z_2 - z_1)} \tag{83}
\]
and then defining $l = z_1 - z_2$ we have
\[
\dot{l} = \frac{1}{P_z}. \tag{84}
\]
We now turn to the equation for $\dot{P}_z$. We show in general i.e. for any number of particles that $\dot{P}_z$ is given in terms of the residue of the simple poles at the point $z_n$ of the function $Q(z)$ which appears in the fuchsian differential equation.

Let the singularity at $z_n$ be
\[
Q(z) = \frac{1 - \mu_n^2}{4z^2} + \frac{\beta_n}{2z} + Q_n(z) \tag{85}
\]
where for the time being $z$ denotes the difference $z - z_n$. The expansion of the singular solution $y_2$ around the singularity is
\[
y_2 = z^{1-\mu_n} (1 + b_n z + O(z^2)) \tag{86}
\]
with
\[
b_n = -\frac{\beta_n}{2(1 - \mu_n)}. \tag{87}
\]
Then substituting into eq.(83) we have in the limit $z \to z_n$
\[
\dot{P}_{nz} = P_{na} \frac{\partial N^a}{\partial z} - m_n \frac{\partial N}{\partial z} = 4\pi \frac{\partial \beta_n}{\partial M} + P_{nz}g'(z_n). \tag{88}
\]
As we already mentioned, such formulas hold in the general $\mathcal{N}$-particle system. In our specific two particle case we have from eq.(71)
\[
\beta_1 = -\beta_2 = \frac{1 + \mu_\infty^2 - \mu_1^2 - \mu_2^2}{2(z_2 - z_1)}. \tag{89}
\]
where working in the $z$ variable the $\beta$’s get divided by $z_2 - z_1$ with respect to the $\beta$’s appearing in eq.(71). Substituting into eq.(88) we have

$$\dot{P}_z = \frac{M}{4\pi(z_2 - z_1)} = \frac{\mu}{z_2 - z_1}.$$  

(90)

Thus we have to solve the elementary system

$$\dot{l} = \frac{1}{P_z}; \quad \dot{P}_z = -\frac{\mu}{l}.$$  

(91)

Taking the time derivative of the first equation and introducing the variable $\lambda = \frac{\ln(l)}{dt}$ we reach

$$\frac{d\lambda}{\lambda^2} = (\mu - 1)dt.$$  

(92)

which solved gives

$$l = \text{const } [(1 - \mu)(t - t_0) - iL/2]^{\frac{1}{1-\mu}}.$$  

(93)

which agrees with the solution given in [3].

It is interesting that the same solution can be obtained using simply the first of eq.(91) i.e. $\dot{l} = 1/P_z$ and the general conservation laws we shall derive in sect.7. For two particles the conserved angular momentum is given by

$$L = \sum_{n=1,2} x_n^i \epsilon_i^j P_{nj} = i(lP_z - \bar{lP}_z).$$  

(94)

On the other hand eq.(131) gives for two particles

$$lP_z + \bar{lP}_z = 2(1 - \mu)t + \text{const}.$$  

(95)

which gives

$$lP_z = (1 - \mu)t - \frac{L}{2} + \text{const}.$$  

(96)

Using now $\dot{l} = 1/P_z$ we reach immediately the solution eq.(93).
6 The asymptotic metric

In the previous section we gave the exact metric for the two particle case. In case of \( N \) particles again the solution of the Liouville equation is given by eq.(72) where \( y_1, y_2 \) are solution of a fuchsian differential equation with singularity at infinity of index \( \mu_\infty = 1 - M/4\pi \equiv 1 - \mu \). Then the asymptotic behavior of \( \tilde{\sigma} \) for large \( z \) is given by

\[
2\tilde{\sigma} = 2\ln(y_2\bar{y}_2 - k^2y_1\bar{y}_1) - \ln|w_{12}|^2 \approx (2 - \mu)\ln(z\bar{z}) - k^2(z\bar{z})^{\mu-1} + ... \tag{97}
\]
Thus

\[
e^{2\sigma} \approx \text{const} (z\bar{z})^{-\mu}. \tag{98}
\]

It follows that the asymptotic behavior of \( N \) is

\[
N = \frac{\partial(-2\tilde{\sigma})}{\partial M} \approx \frac{1}{4\pi}\ln(z\bar{z}) + \frac{k^2}{4\pi}\ln(z\bar{z})(z\bar{z})^{\mu-1} + ... \tag{99}
\]
and then

\[
N^z = \text{const} z (1 + k^2(\mu - 1)\ln(z\bar{z})(z\bar{z})^{\mu-1} + ...) + g(z). \tag{100}
\]
If \( g(z) \) is chosen as to remove the behavior proportional to \( z \) (non-rotating frame at infinity) the behavior of \( N^z \) becomes

\[
N^z \approx \text{const} z \ln(z\bar{z})(z\bar{z})^{\mu-1}. \tag{101}
\]
Formulas (98,99,101) provide the general asymptotic behavior of the metric.

In this formalism the appearance of a Gott pair is signalled by an oscillating behavior of \( f(z) \) as \( |z| \to \infty \) (the total mass is now an imaginary number)[3]. This pathological behavior is not surprising because both the formalism and the gauge used here assume the existence of a global time.

7 The \( \mathcal{N} \)-particle case

We shall consider in this section the \( \mathcal{N} \)-particle case with particular attention to \( \mathcal{N} = 3 \). Again the key role is played by the conformal factor \( e^{2\sigma} \) or equivalently \( e^{2\tilde{\sigma}} \) which is given
by eq. (65) where $y_1, y_2$ are solution of a fuchsian differential equation. It is interesting that $N$ can be rewritten in the following form

$$N = \frac{\partial G(f)}{\partial f} + \frac{2\bar{f}G(f)}{1 - ff} + \text{c.c.} \quad (102)$$

where

$$G(f) = \frac{\partial f}{\partial M}. \quad (103)$$

In fact

$$\frac{\partial (-2\tilde{\sigma})}{\partial M} = \frac{1}{f'\partial M} \frac{\partial f'}{\partial M} + \frac{2\bar{f}G(f)}{1 - ff} + \text{c.c.} \quad (104)$$

and it is easily checked that

$$\frac{1}{f'\partial M} \frac{\partial f'}{\partial M} = \frac{\partial}{\partial f} \left( \frac{\partial f}{\partial M} \right). \quad (105)$$

With regard to the conjugate momenta we have

$$\pi^z_\pm(z) = \frac{p_{N-2}(z)}{\prod_n (z - z_n)} \quad (106)$$

where as $\sum_n P_n = 0$, $p_{N-2}(z)$ is a polynomial of degree $N - 2$. In flat space the number of Lorentz invariants is easily seen to be $3N - 3$ as for $n > 3$ the $2 + 1$-vector $P_n$ is determined by the scalar product of $P_n$ with $P_1, P_2, P_3$. In curved space we have the same number of invariants. As it is well known here the invariants are replaced by the traces of the holonomies around the world lines of an arbitrary collection of particles [1, 21, 22]. The parallel transport monodromy matrices which belong to $SO(2,1)$ or in the fundamental representation to $SU(1,1)$ are defined up to a conjugation. Keeping in mind that every $SU(1,1)$ holonomy has three real degrees of freedom we have $3N$ degrees of freedom to which we have to subtract the three degrees of freedom of the $SU(1,1)$ conjugation thus reaching as expected the same number of invariants as in the flat case.

Given at the time $t = 0$ the position $z_n$ and the momenta $P_n$ of the particles we know from eq. (14) also the position of the apparent singularities $z_A$. Knowledge of the masses $m_n$
fix also the second order residues of the poles in \( Q(z) \) at \( z_n \) to \((1 - \mu_n^2)/4\) while the second order residue at the apparent singularities in fixed to \(-3/4\) \[8\].

In order to understand the problem better let us consider the case \( \mathcal{N} = 3 \). It is useful to go over from the equation in projectively canonical form

\[
y'' + Qy = 0
\]

where the general form of \( Q(z) \) is \[8, 20\]

\[
Q(z) = \sum_n \left[ \frac{1 - \mu_n}{4(z - z_n)^2} + \frac{\beta_n}{2(z - z_n)} \right] + \sum_B \left[ -\frac{3}{4(z - z_B)^2} + \frac{\beta_B}{2(z - z_B)} \right],
\]

(108)

to the equivalent form \[8\]

\[
y'' + py' + qy = 0
\]

(109)

with

\[
Q = q - \frac{p'}{2} - \frac{p^2}{4}.
\]

(110)

The transformation can be so chosen as to have the following Riemann scheme

\[
\begin{pmatrix}
0 & 1 & z_3 & z_A & \infty \\
0 & 0 & 0 & 0 & \rho_\infty \\
\mu_1 & \mu_2 & \mu_3 & 2 & \rho_\infty + \mu_\infty
\end{pmatrix}
\]

(111)

where the Fuchs relation

\[
\mu_1 + \mu_2 + \mu_3 + \mu_\infty + 2\rho_\infty = 1
\]

(112)

fixes \( \rho_\infty \) in terms of the other parameters and

\[
p(z) = \frac{1 - \mu_1}{z} + \frac{1 - \mu_2}{z - 1} + \frac{1 - \mu_3}{z - z_3} - \frac{1}{z - z_A}
\]

(113)

\[
q(z) = \kappa \frac{z(z_3 - 1)}{z(z - 1)} - \frac{z_3(z_3 - 1)H}{z(z - 1)(z - z_3)} + \frac{z_A(z_A - 1)b_A}{z(z - 1)(z - z_A)}
\]

(114)
with \( \kappa = \rho_\infty (\rho_\infty + \mu_\infty) \). Here we use the unconstrained parameterization of ref.\[8\]; the connection with the parameters \( \mu_n \) and \( \beta_n \) can be easily obtained by comparing the residues in eq.\( (110) \).

The parameters \( \mu_i \) and \( \mu_\infty \) are free real parameters which give the trace of the holonomies around the particles and infinity (total energy). \( z_A \) is given in terms of the \( z_n \) and the \( P_n \) \( (n = 1, 2, 3) \) while \( H \) is given by the no-logarithm condition i.e. the requirement that the point \( z_A \) is a regular point for \( 2 \sigma \) \[8\]. Such \( H \) is easily found

\[
H = \frac{z_A (z_A - 1) (z_A - z_3)}{z_3 (z_3 - 1)} \left\{ b_A^2 - \left( \frac{\mu_1}{z_A} + \frac{\mu_2}{z_A - 1} + \frac{\mu_3 - 1}{z_A - z_3} \right) b_A + \frac{\kappa}{z_A (z_A - 1)} \right\}. \tag{115}
\]

Thus we have the following free real parameters \( \mu_1, \mu_2, \mu_3, \mu_\infty, \text{Re}(z_A), \text{Im}(z_A), \text{Re}(b_A), \text{Im}(b_A) \) which are eight real parameters, to be contrasted with the six real invariants we have enumerated at the beginning of this section. The reason is that the constraint that the holonomies described by eq.\( (109) \) belong to \( SU(1, 1) \), fixes the value of \( \text{Re}(b_A), \text{Im}(b_A) \) in terms of all other parameters. The explicit form of this relation has been investigated in the mathematical literature \[24\] but we are not aware of any explicit form of it. Adding a new particle introduces a new particle singularity with exponent \( \mu_4 \) i.e. \( p(z) \) acquires a new term of the type

\[
\frac{1 - \mu_4}{z - z_4} \tag{116}
\]

but in addition a new apparent singularity is generated say at \( z_B \). The indices of this new singularity have to be integers, 0 and 2 while the new simple residue \( b_B \) has to be a function of all other parameters, and in particular of \( z_B \), because the \( SU(1, 1) \) character of the holonomies allow only an increase of 3 in the number of free real parameters i.e. \( \mu_4, \text{Re}z_B, \text{Im}z_B \), but as we said above we do not know an explicit solution of this problem. Adding other particles does not alter the procedure.

We come now to the problem of determining the change in time of the position of the particles \( z_n \) and of the position of the apparent singularities \( z_B \). If \( f(z) = \frac{ky_1(z)}{y_2(z)} \) provides a monodromic \( 2\tilde{\sigma}, N \) is still given by eq.\( (17) \) as the sources are independent of \( M \); the equation
for \( N^z, N^\bar{z} \) are solved as before by

\[
N^z = -\frac{2}{\pi^z(z)} \partial_z N + g(z). \tag{117}
\]

The new fact with respect to the two particle case is that now \( \pi^z(z) \) has \( \mathcal{N} - 2 \) zeros in the complex plane which are the position of the apparent singularities. We write

\[
-\frac{\pi^z(z)}{2} = \frac{1}{4\pi} \sum_n \frac{P_{nz}}{z - z_n} = \frac{\prod_B (z - z_B)}{\mathcal{P}(z)}. \tag{118}
\]

Taking the derivative of the previous equation with respect to \( z \) and then setting \( z = z_A \) we obtain a relation which will be useful in the following

\[
\sum_n \frac{P_{nz}}{(z_A - z_n)^2} = -4\pi \frac{\prod_{B \neq A}(z_A - z_B)}{\mathcal{P}(z_A)}. \tag{119}
\]

which for a single apparent singularity reduces to

\[
\sum_n \frac{P_{nz}}{(z_A - z_n)^2} = -\frac{4\pi}{\mathcal{P}(z_A)}. \tag{120}
\]

The general expression of \( N^z \) is

\[
N^z = \frac{\mathcal{P}(z)}{\prod_B (z - z_B)} \partial_z N + g(z) \tag{121}
\]

and thus to remove the poles from \( N^z \) we must have

\[
g(z) = \sum_B \frac{\partial \beta_B}{\partial M} \frac{1}{z - z_B} \frac{\mathcal{P}(z_B)}{\prod_{C \neq B}(z_B - z_C)} + p_1(z), \tag{122}
\]

where \( p_1(z) \) is a polynomial, as

\[
\partial_z N(z_B) = -\frac{\partial \beta_B}{\partial M}. \tag{123}
\]

In fact the expansion of the singular solution around the apparent singularity \( z_A \) is given by

\[
y_2 = (z - z_A)^{-\frac{1}{2}} \left( 1 + \frac{\beta_A}{2}(z - z_A) + O((z - z_A)^2) \right) \tag{124}
\]
which substituted in $2\sigma$ gives the above result. Recalling the equation of motion of $z_n$ we have now

$$
\dot{z}_n = -N^z(z_n) = -g(z_n) = -\sum_B \frac{\partial \beta_B}{\partial M} \frac{1}{z_n - z_B} \frac{\mathcal{P}(z_B)}{\prod_{C \neq B} (z_B - z_C)} - p_1(z_n). \quad (125)
$$

Thus the time variation of $z_n$ is given in terms of the momenta $P_n$ which determine completely the polynomial $\mathcal{P}(z)$ and the derivative with respect to $M$ of the residues $\beta_A$.

We come now to the equations for $\dot{P}_n$. We already saw on general grounds that

$$
\dot{P}_{nz} = 4\pi \frac{\partial \beta_n}{\partial M} + P_{nz} g'(z_n) = 4\pi \frac{\partial \beta_n}{\partial M} - P_{nz} \sum_B \frac{\partial \beta_B}{\partial M} \frac{\mathcal{P}(z_B)}{(z_n - z_B)^2 \prod_{C \neq B} (z_B - z_C)} + P_{nz} p_1'(z_n). \quad (126)
$$

From the fuchsianity of the differential equation in projectively canonical form we know that

$$
\sum_n \beta_n + \sum_b \beta_B = 0; \quad 1 - \mu^2_{\infty} = \sum_n (1 - \mu^2_n + 2\beta_n z_n) + \sum_B (-3 + 2\beta_B z_B) \quad (127)
$$

so that due to $\mu_{\infty} = 1 - M/4\pi$,

$$
\sum_n \frac{\partial \beta_n}{\partial M} + \sum_B \frac{\partial \beta_B}{\partial M} = 0; \quad \frac{1}{4\pi} (1 - \frac{M}{4\pi}) = \sum_n \frac{\partial \beta_n}{\partial M} z_n + \sum_B \frac{\partial \beta_B}{\partial M} z_B. \quad (128)
$$

We see that provided $p_1(z)$ is a first degree polynomial, $p(z) = c_0 + c_1 z$, the consistency of the equation $\sum_n P_n = 0$ is assured by the the first equation in (128). Similarly one easily checks that the second relation in eq.(128) provides the law

$$
\frac{d}{dt} \sum_n z_n P_{nz} = \frac{d}{dt} \sum_n \dot{z}_n \bar{P}_{nz} = (1 - \frac{M}{4\pi}) \quad (129)
$$

where we have exploited eq.(113) and $\sum_n P_{nz}/(z_B - z_n) = 0$. Such equation gives both the law of conservation of angular momentum

$$
\dot{L} = \frac{d}{dt} \sum_n i(z_n P_{nz} - \bar{z}_n \bar{P}_{nz}) = 0 \quad (130)
$$
which we derived in sect. 3 and the “generalized conservation law”

\[
\frac{d}{dt} \sum_{n} (z_n P_{nz} + \bar{z}_n \bar{P}_{nz}) = 2(1 - \frac{M}{4\pi}) \tag{131}
\]

which appears to be related to a dilatation transformation \[23\]. To complete the scheme we must provide the expression of the time derivative of the position of the apparent singularities \(z_A\) and the time derivatives of the simple residues at such singularities \(\beta_A\). The equation for \(\dot{z}_A\) is easy to obtain. One takes the time derivative of

\[
0 = \sum_{n} \frac{P_{nz}}{z_A - z_n} \tag{132}
\]

which allows to express \(\dot{z}_A\) in terms of \(\dot{z}_n\) and \(\dot{P}_{nz}\) which are already known. Using eq.(132) and eq.(119) we obtain

\[
0 = \prod_{C \neq A} (z_A - z_C) \left[ \dot{z}_A + c_1 z_A + c_0 \right] + \sum_{n} \frac{\partial \beta_n}{\partial M} \frac{1}{z_A - z_n} + \sum_{B \neq A} \frac{\partial \beta_B}{\partial M} \frac{1}{z_A - z_B} \left[ \prod_{C \neq A} (z_A - z_C) \frac{\mathcal{P}(z_A)}{\mathcal{P}(z)} + \prod_{C \neq B} (z_B - z_C) \frac{\mathcal{P}(z_B)}{\mathcal{P}(z)} \right]. \tag{133}
\]

Using the no-log condition the second term in the above equation can be rewritten as

\[
-\beta_A \frac{\partial \beta_A}{\partial M} - \sum_{B \neq A} \frac{\partial \beta_B}{\partial M} \frac{1}{z_A - z_B} \tag{134}
\]

and thus we reach

\[
\dot{z}_A = -c_0 - c_1 z_A + \beta_A \frac{\partial \beta_A}{\partial M} \frac{\mathcal{P}(z_A)}{\prod_{C \neq A} (z_A - z_C)} - \sum_{B \neq A} \frac{\partial \beta_B}{\partial M} \frac{1}{z_A - z_B} \frac{\mathcal{P}(z_B)}{\prod_{C \neq B} (z_B - z_C)}. \tag{135}
\]

In the case of three particles i.e. one apparent singularity, the above equation reduces to

\[
\dot{z}_A = -c_0 - c_1 z_A + \mathcal{P}(z_A) \beta_A \frac{\partial \beta_A}{\partial M}. \tag{136}
\]

In order to connect with the mathematical literature we shall choose the meromorphic function \(g(z)\) such as to keep the position of particle 1 at 0 and of particle 2 at 1. For clarity we
shall refer first to the case of three particles i.e. one apparent singularity, and then give the
generalization to $N$ particles. Thus
\[ g(z) = \mathcal{P}(z_A) \frac{\partial \beta_A}{\partial M} \frac{z(z-1)}{(z-z_A)z_A(z_A-1)} = \mathcal{P}(z_A) \frac{\partial \beta_A}{\partial M} \left[ \frac{1}{z-z_A} + \frac{1}{z_A} + \frac{z}{z_A(z_A-1)} \right]. \]
(137)

Then substituting in eq.(136) we reach
\[ \dot{z}_A = \mathcal{P}(z_A) \frac{\partial \beta_A}{\partial M} \left( \beta_A - \frac{1}{z_A-1} - \frac{1}{z_A} \right). \]
(138)

If we divide the previous expression by $\dot{z}_3$ which is simply given by $-g(z_3)$ we obtain
\[ \frac{\partial z_A}{\partial z_3} = \frac{\partial H}{\partial b_A} = \frac{z_A(z_A-1)(z_A - z_3)}{z_3(z_3-1)} \left[ 2b_A - \frac{\mu_1}{z_A} - \frac{\mu_2}{z_A-1} - \frac{\mu_3 - 1}{z_A-3} \right] \]
(139)
i.e. the first Garnier equation, with hamiltonian eq.(115) where we took into account that
\[ \frac{\beta_A}{2} = b_A + \frac{1}{2} \left( \frac{1 - \mu_1}{z_A} + \frac{1 - \mu_2}{z_A-1} + \frac{1 - \mu_3}{z_A-3} \right). \]
(140)

We come now to the more difficult task of finding the time variation of the simple residues $\beta_B$. To accomplish this we shall exploit the equation
\[ \frac{d(2\sigma)}{dt} = N^z \partial_z (2\sigma) + \partial_z N^z + N^z \partial_z (2\sigma) + \partial_z N^z \]
evaluated in the neighborhood of an apparent singularity. The above equation embodies
part of the information carried by the dynamical equation for $\dot{g}_{ij}$.

It is more useful to rewrite equation (141) in terms of the reduced conformal factor
\[ 2\tilde{\sigma} = 2\sigma - \ln(2\pi_z^z \pi_z^z). \] In order to accomplish this we need the evolution equation for $\pi_z^z$ and $\pi_z^z$ i.e. for the traceless part of $\pi^a_b$. These are given, outside the particle singularities by
\[ \pi_z^z = 2e^{2\sigma} \partial_z (e^{-2\sigma} \partial_z N) + 2\pi_z^z \partial_z N^z + N^z \partial_z \pi_z^z. \]
(142)

Using eqs.(135,142) we reach
\[ 2\tilde{\sigma} = g(z) \partial_z (2\tilde{\sigma}) - g'(z) + c.c. \]
(143)
We now recall the fundamental relation of the theory of fuchsian differential equation \[8, 19\], which can be easily proved starting from eq.(65) which holds in the general \(N\)-particle case
\[
Q(z) = -\frac{1}{2}[\partial_z^2(2\tilde{\sigma}) + \frac{1}{2}(\partial_z(2\tilde{\sigma}))^2]
\] (144)
where the r.h.s. plays the role of the analytic component of the energy momentum tensor in two dimensional conformal Liouville theories. Taking the time derivative of eq.(144) we have
\[
\dot{Q}(z) = \frac{1}{2}g'''(z) + 2g'(z)Q(z) + g(z)Q'(z).
\] (145)
It is interesting that this equation contains the whole system of the Garnier equations for the time evolution of the apparent singularities. If we Laurent-expand \(g(z)\) about the apparent singularity \(z_A\)
\[
g(z) = \frac{g_{-1}}{z-z_A} + g_0 + g_1(z-z_A) + \frac{g_2}{2}(z-z_A)^2 + O((z-z_A)^2)
\] (146)
and
\[
Q(z) = -\frac{3}{4(z-z_A)^2} + \frac{\beta A}{2(z-z_A)} + Q_A(z_A) + Q'_A(z_A)(z-z_A) + O((z-z_A)^2) =
\]
\[
= -\frac{3}{4(z-z_A)^2} + \frac{\beta A}{2(z-z_A)} - \frac{\beta A^2}{4} + Q'_A(z_A)(z-z_A) + O((z-z_A)^2)
\] (147)
The fourth order poles in eq.(143) cancel identically without producing any information; the matching of the residue of the third order pole gives
\[
\dot{z}_A = g_{-1}\beta A - g_0
\] (148)
while the matching of the first order pole gives
\[
\dot{\beta}_A = -2g_{-1}Q'_A(z_A) + g_1\beta A - \frac{3}{2}g_2.
\] (149)
Eq. (148) reproduces the results obtained in eq. (135), as can be seen by computing \( g_{-1} \) and \( g_0 \) from eq. (122). As an example we can apply the above equations to the three particle case. If we keep particle 1 at 0 and particle 2 at 1, the \( g(z) \) function takes the form

\[
g(z) = \mathcal{P}(z_A) \frac{\partial \beta_A}{\partial M} \left[ \frac{1}{z - z_A} + \frac{z - z_A}{z_A(z_A - 1)} + \frac{1}{z_A - 1} + \frac{1}{z_A} \right].
\]

Then substituting into eq. (148) we have

\[
\dot{z}_A = \mathcal{P}(z_A) \frac{\partial \beta_A}{\partial M} \left[ \beta_A - \frac{1}{z_A - 1} - \frac{1}{z_A} \right] = \dot{z}_3 \frac{\partial H}{\partial b_A},
\]

and

\[
\dot{\beta}_A = \mathcal{P}(z_A) \frac{\partial \beta_A}{\partial M} \left[ \beta_A - \frac{1}{z_A - 1} - 2Q'_A(z_A) \right] = -\dot{z}_3 \frac{\partial H}{\partial z_A}.
\]

These are the Garnier equations with Hamiltonian eq. (115) once we keep in mind that according to the general equation eq. (125)

\[
\dot{z}_3 = \mathcal{P}(z_A) \frac{\beta}{\partial M} \frac{z_3(z_3 - 1)}{z_A(z_A - 1)(z_A - z_3)}. \tag{153}
\]

It is important to remark that the matching of the residues of eq. (145) on the particle singularities reproduce the equation of motion \( \dot{z} = -g(z_n) \).

### 8 Conclusions

It is useful to summarize here the basic features of our derivations which are rather straightforward. In the instantaneous York gauge \( K = 0 \) the momenta \( \pi^a{}_b \) conjugate to the space metric have only two independent components \( \pi^z{}_z \) which are meromorphic (antimeromorphic) functions of \( z \) whose residues are the particle momenta \( P_n \). Knowledge of such momenta \( \pi^a{}_b \) allows us to write a Liouville equation for the reduced conformal factor \( \tilde{\sigma} \), in which the sources are the particle singularities i.e. the poles of \( \pi^z{}_z \) and the apparent singularities, i.e. the zeros of \( \pi^z{}_z \). The function \( N \) is given by the derivative of \( \tilde{\sigma} \) with respect to the total energy while \( N^z \) is given in terms of \( N \) by

\[
N^z = -\frac{2}{\pi^z} \frac{\partial z}{\partial \tilde{z}} N + g(z), \tag{154}
\]
where \( g(z) \) is a meromorphic function which cancels the polar singularities of the first term and grows at infinity not faster than \( z \). The linear term in \( g(z) \) is fixed if we want to deal with a reference frame which does not rotate at infinity.

The equation of motion for the particle position are
\[
\dot{z}_n = -N\dot{z}(z_n) = -g(z_n)
\]
while those for the particle momenta
\[
\dot{P}_n = P_{na}\partial_z N^a - m_n\partial_z N. \tag{155}
\]

On the other hand the change in time of the position of the apparent singularities and their residues are given by the ADM equation for \( \dot{\tilde{\sigma}} \) computed on the apparent singularities, thus providing the Garnier hamiltonian system.

In the solution for \( N \) and consequently in \( g(z) \) and \( \dot{P}_n \) there appears the derivative with respect to the total energy \( M \) of the coefficients of the fuchsian differential equation which underlies the determination of \( \dot{\tilde{\sigma}} \). I.e. the change in the \( \beta \)'s has to be such as to remain in the \( SU(1,1) \) class of differential equation. It has a very simple solution in the case of two particles while in general it is obviously related to the Riemann-Hilbert problem even if it is less specific.

### 9 Appendix

The two solutions of eq.\((70)\) which near the origin are a power multiplied by an analytic function are given by \([25]\)
\[
y_i(\zeta) = \zeta^{1+\mu_i} (\zeta - 1)^{1-\mu_i} u_i(\zeta) \tag{156}
\]
where
\[
u_1 = F(a, b; c; \zeta); \quad u_2 = \zeta^{-c} F(a - c + 1, b - c + 1; 2 - c; \zeta) \tag{157}
\]
and
\[
a = \frac{1}{2} (1 + \mu_1 - \mu_2 + \mu_\infty); \quad b = \frac{1}{2} (1 + \mu_1 - \mu_2 - \mu_\infty); \quad c = 1 + \mu_1. \tag{158}
\]
Under a circuit around point 1 they mix as \( u_i \to B_{i1} u_1 + B_{i2} u_2 \) and thus the demand that \((ky_1, y_2)\) transforms under a representation of \(SU(1, 1)\) imposes

\[
|k^2| = \left| \frac{B_{21}}{B_{12}} \right|
\]

We have \(^{25}\)

\[
B_{12} = -2\pi i \left( e^{i\pi(c-a-b)} \frac{\Gamma(c)\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(b)\Gamma(a)} \right)
\]

\[
B_{21} = 2\pi i \left( e^{i\pi(c-a-b)} \frac{\Gamma(2-c)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1+a-c)\Gamma(1+b-c)} \right)
\]

which gives

\[
|k^2| = |\Delta(a)\Delta(b)\Delta(1-c)\Delta(2-c)\Delta(c-a)\Delta(c-b)|
\]

where as usual \(\Delta(x) \equiv \Gamma(x)/\Gamma(1-x)\).

\[
|k^2| = |\Delta(a)\Delta(b)\Delta(1-c)\Delta(2-c)\Delta(c-a)\Delta(c-b)|
\]

\[
B_{12} = -2\pi i \left( e^{i\pi(c-a-b)} \frac{\Gamma(c)\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(b)\Gamma(a)} \right)
\]

\[
B_{21} = 2\pi i \left( e^{i\pi(c-a-b)} \frac{\Gamma(2-c)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1+a-c)\Gamma(1+b-c)} \right)
\]

\[
|k^2| = |\Delta(a)\Delta(b)\Delta(1-c)\Delta(2-c)\Delta(c-a)\Delta(c-b)|
\]

where as usual \(\Delta(x) \equiv \Gamma(x)/\Gamma(1-x)\).

\[
|k^2| = |\Delta(a)\Delta(b)\Delta(1-c)\Delta(2-c)\Delta(c-a)\Delta(c-b)|
\]

\[
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\]

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