Matrix Model for Yang-Mills Interactions

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Abstract
We introduce a $N_c \times N_c$ matrix model with $\mathcal{N} = 2$ supersymmetries and show its relation to the topological rigid string and the topological YM$_2$. This allows to connect the latter two theories directly. Moreover the construction leads to a new insight in the $N_c \to \infty$ limit. Finally a quantum mechanical matrix theory is proposed which may describe light-cone (light-front) dynamics of gauge fields.

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It is believed that the dynamics of Yang-Mills fields should be given by a string theory. Unfortunately no appropriate string theory has been built yet. There are several reasons to believe that the correct theory will incorporate rigidity \cite{1, 2}. But the rigid string cannot be quantized in standard manner although its topological sector is amazingly simple \cite{3}.

In this paper we shall consider quantum mechanical matrix models meant to describe dynamics of the Yang-Mills fields. Thus they are not supersymmetric in space-time. Quantum mechanics of free particles of any spin $s$ is well understood \cite{4}. It is given by $s = \mathcal{N}/2$ extended 1d supergravity coupled to matter multiplets. One would like to have a similar picture for many body interacting systems e.g. gauge particles. The hope is that matter multiplets in form of $N_c \times N_c$ matrices will provide a convenient representation of multi-particle states and will allow to introduce interaction between those states. The necessary gauge symmetry must be introduced on the world-line. Unfortunately this construction is not known if the world-line SUGRA is dynamical. There are two ways out. One can limit the discussion to theories which are world-line SUGRA independent i.e. topological. As we deal with supersymmetric quantum mechanics, the proposed matrix models possess natural topological symmetry if one assumes that the matrices do not depend on the world-line time variable.

An alternative way is to construct a non-relativistic theory with globally well defined world-line time $\tau$ for a multi-particle system. This theory maybe interpreted as a covariant theory in the light-cone gauge (or on the light-front) for which $\tau \propto x^+$. The advantages of these frames in investigations of many-particle systems are well known \cite{5, 6}. For the light-front frame one hopes to recover Lorentz invariance in the $N_c \to \infty$ limit. There is also a heuristic need for infinite matrices. Because we want to think about the matrix model as quantum mechanics of gauge particles we should not expect to get gauge invariant states built from a finite number of gluons.

This work was also inspired by the recent progress in (unifying) M/string theory. According to \cite{7} M-theory in the light-front frame is given by a certain space-time supersymmetric, quantum mechanical matrix model. Its basic ingredients are D0-branes. These are the partons.

The outline of the paper is the following. In the first section we shall recall certain facts about $\mathcal{N} = 2$ world-line supergravity coupled to matter and we shall show how it is related to abelian gauge particles. This section has a review character as all the results are well known. The next section is devoted to (time-independent) topological
matrix models and their relation to 2d topological strings and to the 2d topological YM$_2$ theory of Witten [9]. In the last section we shall shortly speculate on a dynamical matrix model.

1 Spin 1 ($\mathcal{N} = 2$) particle

It is well known that $\mathcal{N} = 2$ SUGRA on the world-line coupled to matter fields describes spin 1 particles in the target space-time [4]. The relevant action is

$$S = \frac{1}{2} \int d\tau \left( \frac{1}{e}(\dot{X} - i(\chi \bar{\psi} + \bar{\chi} \psi))^2 + i\psi D\bar{\psi} + i\bar{\psi} D\psi + eF^2 \right)$$

(1)

where $\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$ and $\psi_{1,2}$ are real. Also $D\psi \equiv (\partial_\tau - if)\psi$. The same convention holds for $\epsilon$ and $d\chi$. In (1) $(e, \chi, \bar{\chi}, f)$ is the gravity multiplet and $(X, \psi, \bar{\psi}, F)$ is the matter multiplet carrying a space-time index $i$.

The local supersymmetry transformations are

$$\delta X = i\epsilon \bar{\psi} + h.c.$$  
$$\delta \psi = \frac{\epsilon}{e}(-\dot{X} + i(\chi \bar{\psi} + \bar{\chi} \psi) + i\epsilon F)$$  
$$\delta F = \frac{\epsilon}{e}(D\bar{\psi} + i\chi F + \frac{1}{e}\bar{\chi} \dot{X} + i\chi \bar{\chi} \bar{\psi}) + h.c.$$  
$$\delta e = -2i\epsilon + h.c.$$  
$$\delta \chi = D\epsilon$$  
$$\delta f = 0$$

(2)

The theory possesses a local $U_f(1)$ symmetry of the 2 supersymmetry charges for which $f$ is the gauge field. The constraints on physical states are $Q_i|\text{phys} > = H|\text{phys} > = T_i|\text{phys} >= 0$ ($i = 1, 2$) where $Q$, $H$, $T^0$ are the generators of the supersymmetry, the local reparameterizations and the $U_f(1)$, respectively. The standard quantization procedure gives

$$\{\psi_i^m, \psi_j^n\} = \delta_{ij}\eta^{mn}.$$  

(3)

In this paper we shall mainly deal with $T^0$ thus we write it explicitly,

$$T^0 = i\psi_1^m \psi_2^m.$$  

(4)

$^1$Wherever it will be possible this index will be suppressed in order to simplify notation.
For 4d Minkowski space-time the constraint $T|\text{phys}\rangle = 0$ implies that physical states are in the symmetric part of the $(\frac{1}{2}, \frac{1}{2})$ representation of $\text{SL}(2, \mathbb{C})$. So we write $|\text{phys}\rangle = \Psi_{\alpha\beta}$. Then the equation $Q|\text{phys}\rangle = 0$ reads

$$\partial^\alpha \dot{\psi}_\alpha = 0 \quad (5)$$

We can write the state $\Psi_{\alpha\beta}$ as $\Psi_{\alpha\beta} = \sigma^{mn}_{\alpha\beta} F_{mn}$, where $F$ is an antisymmetric tensor. Because $\star \sigma^{mn} = i \sigma^{mn}$ we get $F_{mn} = F_{mn} - i F_{mn}$ for real $F_{mn}$. Then (5) yields the Maxwell equations,

$$\partial^m F_{mn} = 0, \quad \partial^m \star F_{mn} = 0. \quad (6)$$

### 2 Variables and interactions

The purpose of this section is to generalize the first quantized version of a single gauge particle to the multi-particle case. In the following we shall keep the $U_f(1)$ gauge field $f$ and global $N = 2$ SUSY as the only remnant of the local SUSY symmetry (3) of the theory. We shall in sec.4 that the $U_f(1)$ is necessary for having proper number of degrees of freedom. We introduce matter fields $(X, \psi, F)$ as hermitian $N_c \times N_c$ matrices and put all matter in a single $\mathcal{N} = 2$ supermultiplet

$$\hat{X} = X + i(\theta \bar{\psi} + \bar{\theta} \psi) + \theta \bar{\theta} F \quad (7)$$

We also impose a $\text{SU}(N_c)$ local gauge symmetry on the world-line under which the matter fields $(X, \psi, F)$ will transform in the adjoint representation of $\text{SU}(N_c)$. Thus we also need a gauge field $A$. The SUSY transformation rules are

$$\delta X = i e \bar{\psi} + \text{h.c.} \quad (8)$$

$$\delta \psi = \epsilon (-DX + ieF) \quad (9)$$

$$\delta F = \epsilon D \bar{\psi} + \text{h.c.} \quad (10)$$

All derivatives in the above are defined with the $\text{SU}(N_c)$ connection $A$ i.e. $DX \equiv (\partial_r X - i[A, X])$ and $D \bar{\psi} \equiv (\partial_r \bar{\psi} + if \bar{\psi} - i[A, \bar{\psi}])$. The kinetic term is given by

$$L_0 = \text{tr}(|D_\theta \hat{X}|^2)_{\theta \bar{\theta}}$$

$$= \text{tr} \left( (DX)^2 + i\psi D \bar{\psi} + i\bar{\psi} D \psi + F^2 \right) \quad (11)$$

where $D_\theta \hat{X} \equiv \partial_\theta \hat{X} - i \bar{\theta} D \hat{X}$. The covariant derivative is understood as above i.e. it has an extra contribution when acting on $\psi$ and $\bar{\psi}$. Now we build the interaction
of our system. We want the interaction to preserve shift symmetry $X \rightarrow X + a \mathbf{1}$. This corresponds to ordinary shift symmetry of space-time coordinates. If we bound considerations to terms which exist in space-times of any dimension then it appears that there is a unique lowest order non-derivative operator:

$$\lambda \text{tr}([\hat{X}^m, \hat{X}^n]^2) |_{\bar{\theta} \theta} = 4\lambda \text{tr}([X^m, X^n][X^m, F^n] - [X^m, X^n]\{\bar{\psi}^m, \psi^n\}
- [X^m, \psi^n][X^m, \bar{\psi}^n] - [X^m, \bar{\psi}^n][\bar{\psi}^m, X^n]) \quad (12)$$

In various specific dimensions we have more possibilities. For 3d space we could have

$$\epsilon_{mnr} \text{tr}([\hat{X}^m, \hat{X}^n] \hat{X}^r) |_{\bar{\theta} \theta}$$

while 4d spaces allow for:

$$\lambda \int d\tau \text{tr}([\hat{X}^m, \hat{X}^n] \ast [\hat{X}^m, \hat{X}^n]) |_{\bar{\theta} \theta}$$

where $\ast$ denotes the Hodge star in the target space. If one extends the gauge multiplet then there are more choices. We are going to discuss one of such operators in section 3.2. One can also add the unity operator $\text{tr}(1) = N$ to the action. It looks trivial but plays an important role in the limit of infinite matrices. Terms with derivatives are also allowed but they will not be discussed here.

## 3 Topological matrix models

In this section we shall construct several topological matrix models and show their equivalence with various 2d topological theories. We shall see that the BRST algebra is intimately related to $\mathcal{N} = 2$ quantum mechanics of the previous section. Relying on this relation we shall show that the topological rigid string [12] has a natural matrix counterpart. As a bonus we shall get immediately an extra BRST-like charge [12]. After taking the $N_c \rightarrow \infty$ limit we shall obtain the topological rigid string. A similar procedure will be applied to the matrix topological YM$_2$. After compactification of the target space on a torus we shall also get topological YM$_2$ [9]. Moreover we shall claim that both theories are equivalent. This will lead to a direct comparison of the topological rigid string and topological YM$_2$. Although all calculations are made for flat spaces we believe that the results should hold for arbitrary Riemann surfaces. The reason is that all the theories have the same topological symmetry and define the same
moduli problem. This is apparent in the matrix formulation. We also notice that the trace part of all matrices will not play any role in this section.

We take all quantities to be 1d time independent. Consequently we suppress also the 1d gauge fields form the algebra. Then (10) is

\[
\begin{align*}
\delta X &= i(\epsilon_1 \psi_1 + \epsilon_2 \psi_2) \\
\delta \psi_1 &= -\epsilon_2 F \\
\delta \psi_2 &= \epsilon_1 F \\
\delta F &= 0
\end{align*}
\]  

(13)

It is clear that we have two candidates for BRST charges \(Q_i\) (\(i = 1, 2\)). They also respect \(\{Q_1, Q_2\} = 0\). We choose \(Q_1\) to be our BRST charge. Then \(\psi_1\) is the ghost of the topological symmetry \(\delta X = \text{arbitrary matrix}\). Hence \((\psi_2, F)\) form an anti-ghost system.

### 3.1 Topological rigid string

In this subsection we shall consider the action

\[
S = \{Q_1, V\}, \quad V = \text{tr}(\psi_2^m ([X^n, [X^n, X^m]] + aF^m))
\]  

(14)

Let us also notice that analogously to [12] we have

\[
V = [Q_2, \text{tr}(-\frac{i}{4}[X^n, X^m]^2 + a \psi_2^m \psi_1^m)]
\]  

(15)

what is a simple consequence of the \(\mathcal{N} = 2\) SUSY. For finite matrices the first term of the action \(S\) is just (12) while the second is the leftover of the kinetic term (11). We can also perturb the theory by a unity operator.

The model (14) is localized on matrices respecting

\[
[X^n, [X^n, X^m]] = 0
\]  

(16)

For 2d space-times the moduli space of (16) can be given more explicitly. Simple calculations (e.g. with help of the Cartan-Weyl basis) show that (16) is equivalent to

\[
[X^1, X^2] = 0
\]  

(17)

We notice that if the target space is a torus, (17) is equivalent to \(F_{mn} = 0\). This will be crucial in establishing an equivalence of the topological matrix theory (14) and the topological gauge theory [9] for 2d target spaces.
In the following we shall show that a $N_c \to \infty$ limit of (14) leads naturally to the well known topological rigid string [12]. In this limit we substitute matrices by functions on a 2d compact parameter space (a Riemann surface $\Sigma_h$ of genus $h$). According to the prescription given in [10] the local $\text{SU}(N_c)$ symmetry goes to $\text{SDiff}(\Sigma_h)$ in this limit and we substitute

$$[A, B] \to \frac{\epsilon^{ab} \partial_a A \partial_b B}{\sqrt{g}}, \quad a = 1, 2$$

where $g$ is the determinant of a metric on the parameter space $\sigma^a$. In this paper we shall choose $g$ to be the induced metric $g_{ab} = \partial_a X \partial_b X \sqrt{g}$. In this way we force the r.h.s. of (18) to be explicitly $X$ dependent no matter what $A$ and $B$ are. The choice has several virtues which become apparent during the course of this article. After the substitution (18) the localization equations (16) go to

$$\Delta_g X^m = 0.$$  \hspace{1cm} (19)

The Laplacian $\Delta_g$ is defined with the 2d induced metric $g_{ab}$. In order to rewrite the action (14) on $\Sigma_h$ we substitute $\text{tr}(\ldots) \to \int_{\Sigma_h} \sqrt{g} \text{tr}(\ldots)$. With this prescription (14) defines the topological rigid string [12] in flat d-dimensional space-time. We also notice that $\text{tr}(1) \to \int_{\Sigma_h} \sqrt{g}$ i.e. the cosmological (Nambu-Goto) term. But what defines $h$? The clue to this point will be obtained in the next section where we shall discuss the relation of the matrix model with $\text{YM}_2$.

### 3.2 Topological string for 2d targets

In this section we shall show that a slight modification of the previous construction leads to other known topological theories in 2d target space-time.

First we notice that for 2d targets one has to be careful in concluding that $\Delta_g X^m = 0 \iff t^{mn} = 0$ where

$$t^{mn} = \frac{\epsilon^{ab} \partial_a X^m \partial_b X^n}{\sqrt{g}}$$

In fact $t^{mn} = 0$ has no nontrivial solutions in 2d targets, because $|t^{mn}| = 1$. It is clear that $\Delta_g X^m = 0 \Rightarrow t^{mn} = \pm \epsilon^{mn}$. Solutions to $t^{mn} = \pm \epsilon^{mn}$ are given by (both signs $\pm$) pseudo-holomorphic curves

$$\frac{\epsilon^b_a}{\sqrt{g}} \partial_b X^m \pm J^m_n \partial_a X^n = 0,$$  \hspace{1cm} (20)

One could use another metric on the l.h.s. of (18) e.g. a subsidiary elementary metric on $\Sigma_h$, but this possibility will not be discussed here.
where \( J_{nm} \) is 2d complex structure defined by a metric on \( M^2 \) with \( \epsilon^{mn} \). Appropriate gluing \([11, 12]\) of both spaces of maps (20) gives the space of minimal maps (19). In this sense the stronger statement \( \Delta g X^m = 0 \leftrightarrow t^{mn} = \pm \epsilon^{mn} \mid_{\text{comp}} \) holds. Thus we must conclude that for 2d targets the substitution \((18)\) is, in a sense, renormalized either to \([X^m, X^n] \to t^{mn} - \epsilon^{mn}\) or to \([X^m, X^n] \to t^{mn} + \epsilon^{mn}\).

In 2 dimensions one can build another simple topological theory if one extends the gauge multiplet to a full \( \mathcal{N} = 2 \) multiplet as in \((13)\). Previously under the BRST transformation we had \( \delta A = 0 \). Thus we take now

\[
\begin{align*}
\delta \lambda &= i(\epsilon_1 \eta_1 + \epsilon_2 \eta_2) \\
\delta \eta_1 &= -\epsilon_2 A \\
\delta \eta_2 &= \epsilon_1 A \\
\delta A &= 0
\end{align*}
\]

This is identical to the additional multiplets in section (3.1) of \([9]\). We take the following gauge fermion

\[
V = \text{tr}(\eta_2(\epsilon_{mn}[X^m, X^n] + bA) + [X^m, \lambda] \psi^m_1)
\]

(22)

The \( \mathcal{N} \rightarrow \infty \) limit of (22) is

\[
V_\pm = \int_{\Sigma_h} \eta_2[\epsilon_{mn}(\epsilon_{ab}\partial_a X^m \partial_b X^n \pm \epsilon^{mn} \sqrt{g}) + bA \sqrt{g}] + \epsilon_{ab}\partial_a X^m \partial_b \lambda \psi^m_1
\]

(23)

depending on the renormalization prescription discussed above. Thus (22) is localized on both pseudo-holomorphic curves (24) although it seems to have two \( \mathcal{N} \rightarrow \infty \) limits (23). Due to \( \Delta g X^m = 0 \leftrightarrow t^{mn} = \pm \epsilon^{mn} \mid_{\text{comp}} \) we can expect that theories (14) and (22) are equivalent.

### 3.3 Topological YM\(_2\)

Here we compactify the target space of the theory (22). In general, the problem is not easy as we know from (M)atrix. The compactification on \( M^2 = T^2 \) is the best known example \([4, 8]\). If one does it for the theory (22) then one gets the standard 2d topological YM\(_2\) theory \([9]\),

\[
V = \int_{T^2} (\phi(\epsilon^{mn} F_{mn} + bA) + (D_m \lambda) \psi^m_1)
\]

(24)

with the SU\((\mathcal{N})\) gauge group. In (24) \( D \) stands for the standard gauge covariant derivative. We recognize that (24) is topological YM\(_2\) localized on flat connections.
$F_{mn} = 0$. It would be interesting to show that a similar correspondence holds for other target spaces e.g. the Riemann surfaces $M^2 = \Sigma_G$ of genus $G$. We do not have the proof that this is true but note that the topological theories, under consideration, are insensitive to many changes e.g. they are insensitive to the metric chosen on the target manifold. Thus we take it for granted that (22) is the matrix representation of the topological YM$_2$.

3.4 Relation between 2d topological string models and topological YM$_2$

All matrix topological theories with 2d target space described in the previous subsections define the same moduli problem $[X^m, X^n] = 0$. After compactifying space-time on a compact Riemann surface of genus $G$ this appeared to be equivalent to $F_{mn} = 0$ i.e. the theory of flat SU($N_c$) connections [9]. On the other hand if we first take the limit $N_c \to \infty$ we get the topological rigid string, or equivalently, the theory of pseudo-holomorphic maps $t^{mn} = \pm 1$ from a world-sheet Riemann surface $\Sigma_h$ to the 2d target manifold. This shows that there should be a relation between both theories. In fact, the relation is well known [13, 11, 12]: both theories are equivalent in $1/N$ expansion:

$$Z_{YM_2}(SU(N_c), G) = \sum_h (\frac{1}{N_c})^{2h-2} Z_{\text{topological strings}}(X : \Sigma_h \to \Sigma_G)$$ (25)

The above gives the precise meaning to the mysterious genus $h$ which appeared when we have taken the limit $N_c \to \infty$.

4 Conclusions and outlook

In the previous section we have shown that the matrix model correctly reproduces known topological theories of Yang-Mills fields. The arguments shed light on the limit $N_c \to \infty$ and the relation $[A, B] \to \frac{\epsilon^{\mu\nu\rho\delta} A_{\mu} B_{\nu}}{\sqrt{g}}$. Although the discussed topological models left several problems unsolved we feel obliged to say something about the main initial motivation of this work i.e. the possible application to the gauge fields dynamics. In order to do this we must go beyond topological considerations.

It is conceivable that the supersymmetric rigid string [12] is the proper version of string theory. On the other hand this theory is hard to quantize so one can hope that

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$^3$This has been shown for $M^2 = \Sigma_G$ of genus at least 2 [11]. For $G = 1$ one needs the cosmological (Nambu-Goto) term in order the partition function does not vanish. For $G = 0$ the situation is more complicated [14].
the proposed matrix version will be easier to deal with. Unfortunately we do not have much to say about it now. Below we introduce a model which, in a sense, should be simpler because all variables have ordinary kinetic terms.

As discussed before we are bound to use non-covariant formulation in order to describe multi-particle quantum mechanics. Thus we should interpret the theory as the parton model of gauge particles in the light-front (-cone) gauge \[5\]. The hope is that after taking the \(N_c \to \infty\) limit we shall be able to recover the full Lorentz invariance of the theory. We stick to 4d space-time so the light-cone target space is 2-dimensional \((m, n = 1, 2)\). Thus the proposed action is

\[
L = (|D_\theta \hat{X}_m|^2 + \lambda \textrm{tr}([\hat{X}_m, \hat{X}_n]^2))|_{\theta \bar{\theta}}
\]

with the possible addition of a \(\textrm{tr}(1)\) term. We analyze first the constraints which follow from the gauge invariance \(U_f(1) \times SU(N_c)\). They are \(T^0|\text{phys} > = T^a|\text{phys} > = 0\) \((a = 1, ...N^2 - 1)\). We identify \(\psi^\dagger_i = \frac{1}{\sqrt{2}}(\psi_1^i + i\psi_2^i)\) and \(\psi_i = \frac{1}{\sqrt{2}}(\psi_1^i - i\psi_2^i)\) with annihilation and creation operators. Then the normal ordered operators \(T^0, T^a\) read

\[
T^0 = ie^{ij}\psi^\dagger_i \psi_j, \quad T^a = f^{abc}(i\psi^\dagger_b \psi^c + X^{bm} DX^{cm})
\]

\(T^0\) is the generator of the \(U_f(1)\) part of the gauge group with the gauge boson \(f\) as in (4), while \(T^a\) are generators of \(SU(N_c)\) group with gauge boson \(A\). In the limit \(N_c \to \infty\) the Lagrangian (26) with constraints (27) is a membrane theory \[15\].

We recall that in sec.4 we showed that the world line gauge symmetries leave only two physical states. The same procedure can be applied for the trace part of (26). Out of four degenerate states:

\[
|0 >, \psi^\dagger_1 |0 >, \psi^\dagger_2 |0 >, \psi^\dagger_1 \psi^\dagger_2 |0 >
\]

the two in the middle are projected out by the \(T^0|\text{phys} > = 0\) constraint. Hence we get only two physical states identified with the polarizations of photon in the light-front frame. We believe that similar mechanism holds for the non-abelian part of the model although we have no firm candidates for gluons. It is quite possible that one can not really see them in a simple way i.e. the weak coupling Yang-Mills theory might correspond to a non-perturbative region of this matrix theory \[1\]. The full analysis of (26) goes beyond the scope of this letter and will be the subject of a future publication.

\[4\] We refer the interested reader to the recent work \[16\] which makes some observations relevant at this point.
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References

[1] A.M.Polyakov, Nucl. Phys. B268 (1986) 406. L. Peliti and S. Leibler, Phys. Rev. Lett. 54 (1985) 1690; D. Forster, Phys. Lett. 114A (1986) 115; H. Kleinert, Phys. Lett. 114A (1986) 263, Phys. Lett. 174B (1987) 335.

[2] J.Pawelczyk, Phys. Lett. 387B (1996) 287, hep-th/9607198.

[3] J.Pawelczyk, Nucl. Phys. B491 (1997) 515, hep-th/9609196.

[4] P.Howe, S.Penati, M.Pernici and P.Townsend, Phys. Lett. 215B (1988) 555.

[5] J.Kogut and L.Susskind, Phys.Rept. 8 (1973) 75.

[6] Theory Of Hadrons And Light Front Qcd. Proceedings, 4th International Workshop On Light Front Quantization And Nonperturbative Dynamics, Polana Zgorzelisko, ed. S.D. Glazek, World Scientific, 1995.

[7] T. Banks, W. Fischler, S. H. Shenker, L. Susskind, Phys.Rev. D55 (1997) 5112, hep-th/9610043.

[8] A.Connes, M.R. Douglas and A.Schwarz, Noncommutative Geometry and Matrix Theory: Compactification on Tori, hep-th/9711162.

[9] E.Witten, J. Geom. Phys. 9 (1992) 303.

[10] J.Hoppe, PhD thesis, MIT (1982);
    E.G.Florates, J.Iliopoulos and G.Tiktopoulos, Phys. Lett. 217B (1989) 285;
    D.B.Fairlie and C.K.Zachos, Phys. Lett. 224B (1989) 101;
    C.N.Pope and K.S.Stelle, Phys. Lett. 226B (1989) 257.

[11] S.Corder, G.Moore and S.Ramgoolam, Comm. Math. Phys 185 (1997) 543, hep-th/9402107.
[12] P. Horava, Topological Strings and QCD in Two Dimensions, hep-th/9311156; Nucl.Phys. B463 (1996) 238, hep-th/9507060. A.M.Polyakov, Lectures given at the CRM-CAP Summer School Particle and Fields '94, August 16-24 1994, Banff, Alberta,Canada.

[13] D. J. Gross, Nucl. Phys. B400 (1993) 161, hep-th/9212149; D. J. Gross and W.Taylor, Nucl. Phys. B400 (1993) 181, hep-th/9301068; Nucl. Phys. B403 (1993) 395, hep-th/9303046.

[14] M.R. Douglas and V.A.Kazakov, Phys. Lett. B319 (1993) 219, hep-th/9305047; D.J. Gross and A. Matytsin, Nucl. Phys. B429 (1994) 50, hep-th/9404004.

[15] B. de Wit, J. Hoppe, H. Nicolai, Nucl. Phys. B305 (1988) 545.

[16] S.Hellerman and J.Polchinski, Compactification in the Lightlike Limit, hep-th/9711037.