The effective wavenumber of a pre-stressed nonlinear microvoided composite

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Abstract. By using nonlinear elasticity and a modified version of classical multiple scattering theory we derive an explicit form for the effective wavenumber for horizontally polarized shear (SH) elastic waves propagating through a pre-stressed inhomogeneous material consisting of well-separated cylindrical voids embedded in a neo-Hookean rubber host phase. The resulting effective (incremental) antiplane shear modulus is thus also derived.

1. Introduction
Multiple scattering in linear elastic inhomogeneous materials such as fibre reinforced or particulate composites (without pre-stress) in the quasi-static regime is relatively well understood. Much work has centred on the derivation of the effective wave number (and resulting effective elastic properties) given a random distribution of inhomogeneities (see e.g. [2], [22]). Additionally, the influence of nonlinear pre-stress on subsequent incremental linear wave propagation in elastic media has been studied extensively over the last few decades using the theory of small-on-large [9], [18]. In this theory, a linearization is performed about the nonlinear equilibrium state in order to determine the incremental wave propagation characteristics of the pre-stressed medium. However, to the authors’ knowledge, in the literature interest has centred almost exclusively on the influence of homogeneous stretch distributions (and hence induced anisotropy) on subsequent wave propagation, see e.g. [4], [10]. When the medium in question is inhomogeneous (for example a fibre-reinforced, or particulate composite material), where the host phase is nonlinear-elastic, pre-stress will almost always lead to non-homogeneous stretch distributions, except in very special cases (see e.g. [20]). Degtyar et al. [5] analysed the case of stressed composites where residual stresses occur in the vicinity of inhomogeneities, in the context of linear elasticity and a self consistent scattering formulation of the effective wavenumber was used. We emphasise that this analysis took place in the linear elastic (small displacement gradient) regime.

Of interest however, is how an initial nonlinear pre-stress affects subsequent multiple wave scattered fields in heterogeneous media. In the low frequency regime this information would allow us to determine the incremental behaviour of the material, the effective incremental homogenized properties and subsequently the overall effective strain energy function of the nonlinear inhomogeneous material. Recently, the so-called second-order homogenization method was applied to determine such homogenized behaviour in the static regime for fibre reinforced composites [3]. Nonlinear pre-stress can be extremely useful in practice, allowing us to tune...
materials in order to permit or restrict waves of specific frequency ranges to propagate through the structure. This property was described by Parnell [20] and discussed further in subsequent articles in different contexts [7], [1].

Although a significant amount of work exists regarding multiple scattering in stress-free configurations, there is no existing theory which can deal with the pre-stress problem where the host phase is nonlinear-elastic (e.g. rubber). A canonical scattering problem regarding pre-stress was analysed by Parnell and Abrahams [21]. In that article, the problem of elastic antiplane or horizontally-polarized shear (SH) wave scattering from a cylindrical void embedded in a host medium which is capable of finite deformation and is neo-Hookean in its constitutive behaviour, was studied. In particular it was shown that the initial pre-stress has no effect on the scattering coefficients of the scattered field due to an incident line source. Therefore in the far-field it is not possible to see the effect of the pre-stress: its effect is felt only within the local vicinity of the void. Since standard multiple scattering theory usually describes the effective wavenumber in terms of the far-field scattering pattern, this latter result suggests that the effective wavenumber is unaffected by the pre-stress. However, the modified theory that we develop in this paper suggests that this is not the case.

In section 2 we summarize the pre-stress problem regarding an isolated cylindrical void embedded in a neo-Hookean elastic material. We then derive an exact solution for incremental SH waves in this pre-stressed state in section 3 by using a mapping corresponding to the finite deformation associated with the pre-stress. The multiple scattering problem is then considered in section 4, assuming that each void is well separated. In this instance, each void sees the pre-stress as if it were isolated. By taking configurational averages and assuming a random distribution of voids in the pre-stressed state, we then derive an explicit expression for the effective wavenumber incorporating a parameter $P$ which characterizes the level of pre-stress within the material. We conclude in section 6 indicating where further work is required.

2. Pre-stress

Let us first consider the pre-stress problem associated with the finite deformation which ensues when a body of infinite extent, except for the presence of a cylindrical void of radius $A$, is subjected to hydrostatic pressure in the far-field, modifying its radius to $a$. We shall assume that the host phase is isotropic and nonlinear-elastic so that it is capable of finite deformation. Its constitutive behaviour is described by a strain energy function $W_{SE} = W_{SE}(I_1, I_2, I_3)$ where $I_j$ are the principal strain invariants of the deformation [9], [18]. We assume incompressibility of this phase, which is a very good approximation for materials of a rubber type. This means that the strain invariant $I_3 = 1$ and thus $W_{SE} = W_{SE}(I_1, I_2)$. Furthermore in this article we shall restrict attention to a material whose strain energy function is of the incompressible neo-Hookean type, whose strain energy function is $W_{SE} = \mu(I_1 - 3)/2$, where $\mu$ is the linear-elastic shear modulus of the material [18].

Let us define the cylindrical polar (Lagrangian) coordinate system $(R, \Theta, Z)$ with origin at the centre of the cylindrical void in the undeformed configuration. Additionally we define the cylindrical polar (Eulerian) coordinate system $(r, \theta, z)$ associated with the deformed configuration. In the far-field we impose a hydrostatic pressure $\sigma_{rr} = -p_{\infty}$ (per unit area in the deformed configuration, corresponding to a Cauchy stress) as $r \to \infty$ (uniform in the longitudinal direction), as shown in figure 1. The material is also held at a fixed stretch $L$ in the longitudinal $Z$ direction. The ensuing deformation is therefore described by the following relationships

$$R = rQ(r), \quad \Theta = \theta, \quad Z = \frac{z}{L},$$

where $Q(r)$ is a function which must be determined from the incompressibility condition and equilibrium equations. We shall use the convention that upper case variables correspond to the
undeformed configuration whilst lower case variables correspond to the deformed configuration.

\[ \text{Figure 1.} \ (a) \text{ Shows the undeformed cavity, with a line source indicated by crossed lines at the position } (R_0, \Theta_0). \ \text{In (b) we show how the cavity deforms due to the hydrostatic pressure } -p_\infty, \text{ successive dotted circles indicating decreasing strain as we move away from the cavity. Under this deformation the line source moves to the location } (r_0, \theta_0), \text{ noting that } \theta_0 = \Theta_0 \text{ due to symmetry.} \]

With reference to [21] we note that incompressibility gives
\[ Q(r) = \left( \sqrt{r^2 + M} \right) / r \]
where
\[ M = A^2 / L - a^2 \]
and where we remind the reader that \( A \) and \( a \) are the undeformed and deformed cylinder radii. Furthermore by consideration of the induced stress field, in [21] it was shown that when \( r \to \infty \), given that \( \sigma_{rr} \to -p_\infty \),
\[ p_\infty / \mu = 1 \left[ \frac{A^2}{L a^2} - 1 + \log \left( \frac{A^2}{L a^2} \right) \right], \tag{2} \]
and we see therefore that this is a (nonlinear) equation for the determination of the deformed to undeformed radius ratio \( a/A \) as a function of the ratio \( p_\infty / \mu \) and the stretch \( L \), both of which are assumed specified. We plot \( a/A \) as a function of \( p_\infty / \mu \) for three different prescribed values of \( L = 0.7, 1, 1.3 \) in figure 2. The exhibited behaviour is as one would expect.

3. Incremental deformation
In order to set up the multiple scattering problem in the pre-stressed medium, we consider the problem of incremental wave scattering from the finitely deformed medium described in section 2. We use the theory of small-on-large, i.e. linearization about a nonlinear deformation state [9]. The total displacement field may therefore be represented by
\[ \hat{u} = u + \eta u' \tag{3} \]
where
\[ u = u' g, \]
where \( u = u' g, \) is the displacement field derived from the finite deformation (1) and \( \eta \ll 1 \) is a small parameter associated with the magnitude of the incremental displacement.
Figure 2. Plot of the deformed radius $a/A$ as a function of $p_\infty/\mu$ for the three prescribed values of $L = 0.7, 1, 1.3$.

Note that $\mathbf{g}_i$ and $\mathbf{g}^i$ are covariant and contravariant basis vectors associated with the deformed configuration [21]. Let us assume that the incremental displacement is of an anti-plane nature, i.e. of the form

$$u' = u'_3 \mathbf{g}_3 = u^3 \mathbf{g}_3 = \Re \left[ w(r, \theta) \exp(-i\omega t) \right] \mathbf{g}_3$$

so that the time-harmonic wave is a horizontally-polarized shear (SH) wave, polarized in the $z$ direction and propagating in the $r\theta$ plane.

The initial finite deformation described in section 2 leads to the following modified wave equation for $w$ [21]

$$\mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r + \frac{M}{r} \frac{\partial w}{\partial r} \right) \right] + \frac{1}{(r^2 + M)} \frac{\partial^2 w}{\partial \theta^2} + \rho \omega^2 w = 0,$$

where we remind the reader that $M = A^2/L - a^2$.

In [21], it was shown, by mapping back to the undeformed configuration, that (5) can be solved analytically and its (outgoing) solution is

$$w(r) = \sum_{n = -\infty}^{\infty} i^n a_n H_n \left( k \sqrt{r^2 + M} \right) e^{in\theta},$$

where $H_n(x) = H_n^{(1)}(x) = J_n(x) + iY_n(x)$ is the Hankel function of the 1st kind of order $n$, where $J_n$ and $Y_n$ are Bessel functions of the first and second kind, of order $n$. Furthermore, we have introduced the notation $k^2 = L \rho \omega^2/\mu$, for the square of the modified wavenumber (modified due to pre-stress), noting that the original wavenumber $K$ is defined by $K^2 = \rho \omega^2/\mu$. The coefficients $a_n$ depend upon the type of forcing. In [21] a line source forcing was assumed and it was shown explicitly that the coefficients $a_n$ do not depend upon the pre-stress if the force of the source strength remains unchanged during the finite deformation. In the section to follow we shall describe the multiple scattering problem of interest. This problem is, in fact, an eigenvalue
problem in the sense that there is no assumed forcing since we take the limit where scatterers fill all space. The problem of an incident wave field is technically more difficult and leads to the same expression for the effective wavenumber and hence, for now at least, does not need to be considered.

4. Multiple scattering
Consider now the configuration consisting of \(N\) circular cylindrical voids of infinite extent in the \(Z\) direction distributed in an unbounded neo-Hookean host phase such that the voids are well-separated. In particular suppose that in the \(X,Y\) plane these voids occupy a finite domain \(D_0\), defined by \((X,Y) \in [-Q,Q] \times [-H,H]\) within unbounded space. This domain becomes \(D\) (defined by \((x,y) \in [-q,q] \times [-h,h]\)) after the pre-stress described in section 2 is imposed in the far-field. The area of \(D_0\) is denoted by \(|D_0|\) and analogously with \(D\). In the \(X,Y\) plane there are \(n_0\) inclusions per unit undeformed area so that \(N = n_0|D_0|\) and furthermore there are \(n\) inclusions per unit deformed area so that \(N = n|D|\). The volume fraction occupied by the voids is therefore denoted by \(\phi_0 = n_0 \pi a^2\) (per unit length) in the undeformed configuration which in the deformed configuration becomes \(\phi = n \pi a^2/L\), noting the factor of \(L\) in the latter.

Note that since the voids are well-separated we assume that under the pre-stress described in section 2, each void deforms as if it were isolated and embedded inside the homogeneous host phase. Thus its radius deforms from \(a\) to \(a\) and furthermore the local incremental field due to a superimposed horizontally polarized shear wave can be described by (5).

We wish to determine the wave field which arises due to multiple scattering from the \(N\) voids in the deformed configuration described above and in particular we wish to assess how the effective propagating wave is modified by the pre-stress. We will consider the limit when \(D\) expands to fill the all space, which is achieved by letting \(q,h \rightarrow \infty\) whilst taking the limit \(N \rightarrow \infty\) with \(n\) held fixed (Fig. 3). In the deformed configuration, the voids are located at \(p_j = (p_j,q_j)\), \(j = 1,2,\ldots,N\) with local polar coordinate system \((r_j,\theta_j)\) relating to the local Cartesian system \(x_j = r_j \cos \theta_j, y_j = r_j \sin \theta_j\), see Fig. 4.

Ensuring only an outgoing (scattered) field from each void, using (6), we may pose a solution of the form

\[
w = \sum_{j=1}^{N} \sum_{n=-\infty}^{\infty} a_n^j Z_n H_n \left( k \sqrt{r_j^2 + M} \right) e^{i \theta_j} \tag{7}
\]

in terms of the local coordinate system described above. We do not assume any incident forcing; we seek eigensolutions to the problem as we shall see shortly by taking the limit as \(N \rightarrow \infty\).

However, we note that if we had alternatively considered a half-space problem in spirit of, for example [13], the result for the resulting effective wavenumber in the pre-stressed medium would remain unchanged. The terms \(a_n^j\) are coefficients associated with the \(j\)th inclusion and \(n\)th mode and the coefficients \(Z_n\) are introduced for convenience as we shall show shortly. Locally to the \(n\)th inclusion we may write

\[
w(r_s, \theta_s) = \sum_{n=-\infty}^{\infty} a_n^s Z_n H_n (k \sqrt{r_s^2 + M}) e^{i \theta_s} + \sum_{j=1,j\neq s}^{N} \sum_{n=-\infty}^{\infty} a_n^j Z_n H_n (k \sqrt{r_j^2 + M}) e^{i \theta_j} \tag{8}
\]

and given that the inclusions are well-separated, we can approximate this as

\[
w(r_s, \theta_s) \approx \sum_{n=-\infty}^{\infty} a_n^s Z_n H_n (k \sqrt{r_s^2 + M}) e^{i \theta_s} + \sum_{j=1,j\neq s}^{N} \sum_{n=-\infty}^{\infty} a_n^j Z_n H_n (k r_j) e^{i \theta_j}. \tag{9}
\]
Figure 3. Inclusions reside inside a domain $\mathcal{D}$ in the deformed configuration, in an otherwise uniform and unbounded material. We consider the limit where $q, h \to \infty$, whilst letting the number of inclusions $N \to \infty$ with the volume fraction $\phi$ held fixed.

Referring to figure 4, we are now able to use Graf’s addition theorem directly on the second term in (9) to find that

$$w(r_s, \theta_s) \approx \sum_{n=-\infty}^{\infty} a_s^n Z_n H_n(k \sqrt{r_s^2 + M}) e^{in\theta_s}$$

$$+ \sum_{j=1, j \neq s}^{N} \sum_{n=-\infty}^{\infty} a_j^n Z_n \sum_{m=-\infty}^{\infty} H_{n-m}(k R_{js}) e^{i(n-m)\theta_s} J_m(k r_s) e^{im\theta_s}$$

where $R_{js}$ is the distance between the $j$th and $s$th void and $\theta_{js}$ is the angle between the $j$th and $s$th void.

On applying the zero traction condition on $r_s = a$ and exploiting the orthogonality of $e^{im\theta_s}$, we find that

$$a_m + \sum_{j=1, j \neq s}^{N} \sum_{n=-\infty}^{\infty} a_j^n Z_n H_{n-m}(k R_{js}) e^{i(n-m)\theta_s} = 0$$

(11)

where we have defined $Z_n$ as

$$Z_n = \frac{J_n'(ka)}{H_n'(Pka)}.$$

(12)
and where we have introduced the notation $P = A/(a\sqrt{L})$. We also used the relationship $k = K\sqrt{L}$ in the above.

In particular note that in the low frequency limit, $ka \ll 1$, (once again using $k = K\sqrt{L}$) we find that

$$Z_0 \sim -\frac{\pi}{4\delta}P^2(ka)^2 = \pi\hat{Z}_0(ka)^2, \quad Z_{-1} = Z_1 \sim \frac{\pi}{4\delta}P^3(ka)^2 = \pi\hat{Z}_1(ka)^2, \quad Z_n \sim o(ka)^2. \quad (13)$$

5. Effective wavenumber

Let us introduce a probability density function, defining the statistics of the random distribution of voids [15], [22] in the deformed configuration. This is written as

$$p(p_1, p_2, \ldots, p_N) = p(p_s)p(p_j|p_s)p(p_1, p_2, \ldots, p_j, \ldots, p_s, \ldots, p_N|p_j, p_s), \quad (14)$$

where $p_j$ indicates that $p_j$ is not present and $p(p_j|p_s)$ denotes the probability density of finding an inclusion at $p_j$ given that there is already one located at $p_s$. Thus, we can take the ensemble average of (11) as in Linton and Martin [13] and obtain

$$\langle a_{m}^1 \rangle_1 + (N - 1)\pi(ka)^2 \frac{1}{2} \sum_{n=-1}^{1} \int_{-\infty}^{\infty} \int_{-h}^{h} p(p_2|p_1)\langle a_{m}^1 \rangle_{12}\hat{Z}_2 H_{n-m}(kR_{21})e^{i(n-m)\theta_{21}} dp_2 dq_2 = 0 \quad (15)$$

where we were able to take $s = 1, j = 2$ without loss of generality and we have retained only terms of $O((ka)^2)$, employing (13). We recall that

$$\langle f \rangle_s = \int_D \int_D \ldots \int_D p(p_1, p_2, \ldots, p_s, \ldots, p_N|p_s) f dp_1 dp_2 dp_s \ldots dp_N, \quad (16)$$

$$\langle f \rangle_{js} = \int_D \int_D \ldots \int_D p(p_1, p_2, \ldots, p_j, \ldots, p_s, \ldots, p_N|p_j, p_s) f dp_1 dp_2 dp_j \ldots dp_s \ldots dp_N, \quad (17)$$

Figure 4. Geometry of the location of inclusions and their local coordinate systems.
are the ensemble averages of the field \( f \) with the \( s \)th inclusion fixed and the \( s \)th and \( j \)th inclusions fixed respectively [13], [22].

Let us use the hole correction correlation function [6], [22], defined by

\[
p(p_2|p_1) = \begin{cases} 
\frac{1}{|D|}, & R_{21} > 2a, \\
0, & R_{21} < 2a 
\end{cases} 
\]  

(18)

which implies that there is no interaction between voids other than asserting that they cannot overlap. Finally, as with most multiple scattering theories, in order to avoid using higher order correlation functions, a closure assumption is invoked; we use the Lax Quasi-Crystalline Approximation (QCA) [11]

\[
\langle a^2 \rangle_{12} = \langle a^2 \rangle_2. 
\]

(19)

In the limit as \( N, q, h \to \infty \), we find that (15) becomes

\[
\langle a^2 \rangle_1 + k^2 L \phi \sum_{n=-1}^{1} \int \langle a^2 \rangle_2 \hat{Z}_n H_{n-m}(kR_{21}) e^{i(n-m)\theta_{21}} dp_2 dq_2 = 0 
\]

(20)

where the notation on the integral sign indicates that the integral is over all space excluding the circular region of radius \( 2a \) centred at \( (p_2, q_2) = (p_1, q_1) \) and is known as the hole correction.

We now pose the effective wave solution [13]

\[
\langle a^2 \rangle_1 = i^m e^{i\beta p_1} C_m e^{i\gamma p_1}. 
\]

(21)

Under this assumption, (20) becomes (noting that \( \theta_{21} = \theta_{12} + \pi \))

\[
C_m e^{i\gamma p_1} + k^2 L \phi \sum_{n=-1}^{1} \hat{Z}_n (-i)^{n-m} e^{i\gamma p_1} C_n M_{n-m} = 0 
\]

(22)

where

\[
M_{n-m} = \oint_{-\infty}^{\infty} \oint_{-\infty}^{\infty} \psi_{n-m}(p_{21}, q_{21}) \Psi(p_{21}, q_{21}) dp_2 dq_2 
\]

(23)

and \( p_{21} = p_2 - p_1, q_{21} = q_2 - q_1 \). Furthermore, we have employed the same notation as Linton and Martin [13], i.e.

\[
\psi_n(x, y) = H_n(kr) e^{i n \theta}, \\
\Psi(x, y) = e^{i(\gamma x + \beta y)} = e^{i \Gamma r \cos(\theta - \zeta)},
\]

(24, 25)

where

\[
x = r \cos \theta, \quad y = r \sin \theta, \\
\gamma = \bar{\Gamma} \cos \zeta, \quad \beta = \bar{\Gamma} \sin \zeta.
\]

(26, 27)

Therefore since \( \alpha^2 + \beta^2 = k^2, \gamma^2 + \beta^2 = \bar{\Gamma}^2 \), we have that \( \gamma^2 - \alpha^2 = \bar{\Gamma}^2 - k^2 \).

We may follow the procedure of Linton and Martin [13] in order to evaluate the integrals in (23) using Green’s theorem. As such these integrals become line integrals along the faces at \( p_2 = 0 \) and around the boundary of the circle of radius \( 2a \) centred at \( (p_1, q_1) \) induced by the hole
correction. Omitting this detail and comparing coefficients of $e^{i\gamma_p}$ in the resulting equations, in the point scattering limit ($ka \to 0$) we find that

$$(1 - \Gamma^2)C_m - L\phi \sum_{n=-1}^{1} \bar{Z}_n N_{n-m}(\Gamma) C_n = 0, \quad m = 0, \pm 1 \quad (28)$$

where we have introduced the notation $\Gamma = \tilde{\Gamma}/k$ and where

$$N_n(\Gamma) = \lim_{\epsilon \to 0} 4\pi \epsilon [H_n'(2\epsilon) J_n(2\Gamma\epsilon) - (\Gamma \epsilon) H_n(2\epsilon) J_n'(2\Gamma\epsilon)] = 4i(\Gamma)^{|n|}. \quad (29)$$

Note the factor $L$ that arises in (28) since $\phi$ is the volume fraction of fibres whereas the probability density function describes the distribution of fibre locations in an area of the deformed configuration, see the discussion of the change in volume fraction at the start of section 4.

Given all of the above, the system (28) becomes

$$\begin{pmatrix}
1 - \Gamma^2 + L\phi \mathcal{M} & -L\phi D\Gamma & L\phi M \Gamma^2 \\
-L\phi M \Gamma & 1 - \Gamma^2 - L\phi \mathcal{D} & L\phi M \Gamma \\
L\phi M \Gamma^2 & -L\phi D\Gamma & 1 - \Gamma^2 + L\phi \mathcal{M}
\end{pmatrix}
\begin{pmatrix}
C_{-1} \\
C_0 \\
C_1
\end{pmatrix} = 0 \quad (30)$$

where

$$\mathcal{D} = P^2, \quad \mathcal{M} = P^3. \quad (31)$$

A nontrivial solution of (30) is only possible if the determinant of the matrix is zero which yields the following condition on $\Gamma$:

$$\Gamma^2 = (1 - L\phi \mathcal{D}) \left( \frac{1 + L\phi \mathcal{M}}{1 - L\phi \mathcal{M}} \right) + o(1), \quad (32)$$

so that in dimensional form

$$\tilde{\Gamma}^2 = L K^2 (1 - LP^2 \phi) \left( \frac{1 + LP^3 \phi}{1 - LP^3 \phi} \right) + o(1). \quad (33)$$

Furthermore, in the low frequency limit we may derive the effective incremental properties of the pre-stressed material,

$$\tilde{\Gamma}^2/K^2 = \frac{\rho_s \mu_0}{\rho_0 \mu_s} = L(1 - LP^2 \phi) \left( \frac{1 + LP^3 \phi}{1 - LP^3 \phi} \right) \quad (34)$$

and given that the effective density in the deformed configuration must have the form

$$\frac{\rho_s}{\rho_0} = (1 - \phi) \quad (35)$$

this leaves the following expression for the effective incremental shear modulus

$$\frac{\mu_s}{\mu_0} = \frac{(1 - \phi)(1 - LP^3 \phi)}{L(1 - LP^2 \phi)(1 + LP^3 \phi)}. \quad (36)$$
6. Conclusions

We have derived an explicit form for the effective wavenumber for SH elastic waves propagating through a pre-stressed inhomogeneous material consisting of well-separated cylindrical voids embedded in a neo-Hookean rubber host phase. The resulting effective (incremental) antiplane shear modulus has thus also been derived. Future work will include a study of the interaction of initial fields due to pre-stress and the error induced by the assumption which permitted the use of the standard Graf’s addition theorem to be applied to (9). The eventual goal is the study of vector problems in higher dimensions (i.e. for particulate composites), compressible phases and the consideration of nonlinear elastic inclusions in addition to voids.

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