PHASE TRANSITION OF PARABOLIC GINZBURG–LANDAU EQUATION WITH POTENTIALS OF HIGH-DIMENSIONAL WELLS

YUNING LIU

Abstract. In this work, we study the co-dimensional one interface limit and geometric motions of parabolic Ginzburg–Landau systems with potentials of high-dimensional wells. The main result generalizes the one by Lin et al. (Comm. Pure Appl. Math., 65(6):833-888, 2012) to a dynamical case. In particular combining modulated energy methods and weak convergence methods, we derive the limiting harmonic heat flows in the inner and outer bulk regions segregated by the sharp interface, and a non-standard boundary condition for them. These results are valid provided that the initial datum of the system is well-prepared under natural energy assumptions.

1. Introduction

In the work of Keller–Rubinstein–Sternberg [32, 33], they created a general gradient flow theory in the descriptions of fast reaction and slow diffusion, and established its relations to the mean curvature flow (MCF) and the harmonic heat flows into manifolds. These works involve some formal statements associated with the multiple components phase transitions with higher dimensional wells. Such statements are also referred to as the Keller–Rubinstein–Sternberg problem. More precisely they investigated the vectorial Allen–Cahn equation (also called Ginzburg–Landau equation)

\[
\partial_t u_\varepsilon = \Delta u_\varepsilon - \varepsilon^2 \partial F(u_\varepsilon),
\]

(1.1)

where \( u_\varepsilon(x,t) : \Omega \times (0,T) \mapsto \mathbb{R}^n \) is a mapping depending on a small parameter \( \varepsilon > 0 \) and \( \Omega \subset \mathbb{R}^d \) is a bounded domain with \( C^1 \) boundary. Here \( F(u) \) is a double equal-well potential with ground state being the disjoint union of two smooth closed submanifolds \( m_\pm \subset \mathbb{R}^n \), and \( \partial F(u) \) is the differential of \( F \) at \( u \). The Keller–Rubinstein–Sternberg problem is concerned with the limiting behavior of \( u_\varepsilon \) as \( \varepsilon \) tends to zero. In the work of Lin–Pan–Wang [25], they set up an analytic program to rigorously justify the formally asymptotic analysis given in the aforementioned works of Keller–Rubinstein–Sternberg. To be more precise they considered the minimizers of the Ginzburg–Landau functional

\[
A_\varepsilon(u_\varepsilon) := \int_\Omega \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) \right) \, dx
\]

(1.2)

that satisfy well-prepared boundary conditions on \( \partial \Omega \). They established the co-dimensional one interface limit of (1.2), which essentially generalizes the \( \Gamma \)-convergence of Modica–Mortola [29] to vectorial cases (though they did not state their main theorems in such an abstract manner). More importantly, they showed that the limits of \( u_\varepsilon \) in the bulk regions correspond to minimizing harmonic maps into \( m_\pm \), and they derived a non-standard boundary condition (also called the minimal pair condition, cf. (1.19) below) which services as a constraint on the limiting harmonic maps when restricted on the interface. Such a boundary condition is a new feature that arises due to minimization of surface tensions in vectorial cases. Note that such a condition holds trivially in the case of scalar Allen–Cahn equation.

In this work we shall try to generalize Lin–Pan–Wang [25] to the parabolic system (1.1) by proving the following statements: Firstly, for well-prepared initial datum, as \( \varepsilon \) tends to 0, the solution gradients of (1.1) will undergo phase transitions across a moving interface \( \Sigma_t \) that propagates...
according to (two-phase) MCF. Secondly, in the two bulk regions $\Omega^\pm_t$ segregated by the interface $\Sigma_t$, the solutions will converge to harmonic heat flows mapping into $m_\pm$ respectively. Finally, the one-sided traces of the limiting harmonic heat flows on $\Sigma_t$ must satisfy the minimal pair condition firstly derived in [25].

Our first result is a vectorial analogy of the co-dimensional one scaling limit of scalar parabolic Allen–Cahn equation to the MCF, i.e. the special case of (1.1) when $m_\pm$ are two distinct points $a_\pm \in \mathbb{R}^1$. There have been major progresses in the scalar case over the last thirty years, made under different frameworks. Here we mention two classes of results and leave the discussions of other classes in the sequel. One is the convergence to a Brakke’s flow by Ilmanen [19] using a version of Huisken's monotonicity formula together with tools from geometric measure theory. See also [6, 18, 37, 31, 30, 34] and the references therein for further renovations. Despite of its energetic nature, a major difficulty of such an approach is the control of the so called discrepancy measure, and in every existing literature in this direction the method relies crucially on a version of Modica’s maximum principle [28]. There have been attempts to generalize such a method to vectorial cases. However, it is not clear whether Modica’s maximum principle holds for elliptic system. Another approach, which relies more on the parabolic comparison principle, is the global in time convergences to the viscosity solution of MCF. These are weak solutions to the MCF built independently by Chen–Giga-Goto [7] and Evans–Spruck [12]. Concerning the convergence of scalar Allen–Cahn equation to such solutions, we refer the readers to the work of Evans–Soner–Souganidis [11], the work of Soner [35] and the references therein. These two approaches both give global in time (weak) convergences to weakly defined solutions of MCFs up to their life spans. However, as their technics involve parabolic maximum principle and comparison principle in one way or another, it is not clear how to use them to attack vectorial cases in general. It is worth mentioning that for radially symmetric initial datum and when $m_\pm$ are two concentric circles, Bronsard–Stoth [4] obtain global in time convergences to MCF of planar circles.

To the best of our knowledge, there are mainly two approaches to rigorously justify the convergences of the vectorial Allen–Cahn equations, both assuming that the limiting interface propagation problem has a (local in time) classical solution. Compared with the aforementioned methods which lead to global in time (weak) convergences, they have quite different natures. One of these methods is the asymptotical expansion technics developed by De Mottoni–Schatzman [9] and later by Alikakos–Bates–Chen [1], which has been used recently in [13, 14] for matrix-valued cases of (1.1). In particular, Fei–Lin–Wang–Zhang [13] studied the case when $m_\pm = \mathcal{O}^\pm(n)$, the $n$-dimensional orthogonal group. By inner-outer expansions together with a gluing procedure, such an approach reduces the convergence problem to a linear stability problem given that the limiting system (not merely the limiting interface motion) is strongly well-posed. The major challenge of this approach is the analysis of the spectrum of the linearized operator at the minimal orbits (also called optimal profile) or their variants. Indeed, one of the novelties of [13] is to devise the so called quasi-minimal orbits, overcoming the lack of minimal orbits in the bulk regions, and to derive the spectrum stability of the linearized operator at such orbits making use of the minimal pair condition. Finally we refer Lin–Wang [26] for a general theory for the strong well-posedness of the limiting system.

Another approach, which also assumes a regular solution of the limiting interface motion but not the limiting harmonic heat flows, is the relative entropy method developed by Fischer–Laux–Simon [16], motivated by Jerrard–Smets [20] and Fischer–Hensel [15]. A generalization to matrix-valued case has been done by Laux–Liu [22] to study the isotropic–nematic transition in Landau–De Gennes model of liquid crystals. More recently, in [27] the author used these methods, together with those developed by Lin–Wang [24], to investigate the convergence problem of an anisotropic 2D Ginzburg–Landau model.
Now we introduce a minimum amount of terminologies necessary for stating the main result of this work. Let
\[ m_\pm \text{ be two disjoint smooth, closed, connected submanifold in } \mathbb{R}^n. \] (1.3)
For technical purposes we assume \( 0 \in m_- \). Let \( F : \mathbb{R}^n \to [0, \infty) \) be a smooth function with \( m_\pm \) being its double equal-wells:
\[ \text{Arg min } F = m := m_+ \sqcup m_. \] (1.4)
We assume that \( F(u) \) only depends on the distance from \( u \) to \( m \). That is,
\[ F(u) = f(d_m^2(u)), \] (1.5)
where \( d_m(u) \) is the distance (see (2.14) below for the full definition), and \( f(s) \in C^2(\mathbb{R}^+, \mathbb{R}^+) \) satisfies
\[ \begin{cases} 
  f(s) = s & \text{if } 0 \leq s \leq \delta_0^2, \\
  f(s) = 2\delta_0^2 & \text{if } s \geq 2\delta_0^2. 
\end{cases} \] (1.6)
In (1.6) \( \delta_0 > 0 \) is a small number so that the nearest-point projection \( P_m \) from \( B_{2\delta_0}(m) \), the \( 2\delta_0 \)-tubular neighborhood of \( m \), to \( m \) is smooth.

We consider the following initial boundary value problems on a bounded domain \( \Omega \subset \mathbb{R}^d \) with \( C^1 \) boundary:
\[ \begin{align*}
  \partial_t u_\varepsilon &= \Delta u_\varepsilon - \varepsilon^{-2} \partial F(u_\varepsilon) & \text{in } \Omega \times (0,T), \\
  u_\varepsilon &= u_\varepsilon^m & \text{in } \Omega \times \{0\}, \\
  u_\varepsilon &= g & \text{on } \partial\Omega \times (0,T). 
\end{align*} \] (1.7)
Here \( \partial F(u) \) is the gradient of \( F(u) \), and \( g : \Omega \to m_- \) is a given smooth mapping. Our main result is concerned with the asymptotical behaviors of solutions to (1.7) for well-prepared initial datum.

To give an analytic characterization of such initial datum, we need to set up the geometry of the interface motion. To this end, we assume that \( \Sigma = \bigcup_{t \in [0,T]} \Sigma_t \times \{t\} \) is a smoothly evolving closed hypersurface in \( \Omega \),
\[ \Sigma = \bigcup_{t \in [0,T]} \Sigma_t \times \{t\} \text{ is a smoothly evolving closed hypersurface in } \Omega, \] (1.8)
starting from a closed smooth surface \( \Sigma_0 \subset \Omega \). We denote by \( \Omega^\pm_t \) the domain segregated by \( \Sigma_t \), and by
\[ d_\Sigma(x,t) \text{ the signed-distance from } x \text{ to the set } \Sigma_t \text{ taking positive values in } \Omega^+_t, \] (1.9)
and taking negative values in \( \Omega^-_t = \Omega \setminus \Omega^+_t \). In other words,
\[ \Omega^\pm_t := \{ x \in \Omega \mid d_\Sigma(x,t) \geq 0 \}. \] (1.10)
To avoid contact angle problems, we assume that \( \Sigma \) stays at least \( 4\delta_0 \) distant away from \( \partial\Omega \).

Following [20, 15, 16], we define the modulated energy (also called the relative entropy energy) by
\[ E_\varepsilon[u_\varepsilon | \Sigma](t) := \int_{\Omega} \left( \frac{\varepsilon}{2} \left| \nabla u_\varepsilon(\cdot, t) \right|^2 + \frac{1}{\varepsilon} F(u_\varepsilon(\cdot, t)) - \xi \cdot \nabla \psi_\varepsilon(\cdot, t) \right) dx. \] (1.11)
Here \( \xi \) is an appropriate extension of the unit normal vector field of \( \Sigma \) (see [2,26] below), and \( \psi_\varepsilon \) is the scalar function
\[ \psi_\varepsilon(x,t) := d_F \circ u_\varepsilon(x,t) \] (1.12)
with $d_F$ defined by (2.14) below. As we shall see later on, the integrand of (1.11) is non-negative, and enjoys several coercivity estimates including controls of discrepancy and calibration of the Ginzburg–Landau energy (1.2). We also need the surface tension coefficient

$$c_F := 2 \int_0^1 \frac{\text{dist}_m}{\sqrt{2f(\lambda^2)}}d\lambda,$$

(1.13)

where $\text{dist}_m$ is the Euclidean distance between $m_+$ and $m_-$, and another modulated energy controlling the bulk errors:

$$B[u_\varepsilon|\Sigma](t) := \int_\Omega \left( c_F \chi - c_F + 2(\psi_\varepsilon - c_F)^- \right) \eta \circ d_\Sigma dx + \int_\Omega (\psi_\varepsilon - c_F)^+ \eta \circ d_\Sigma dx.$$  (1.14)

In (1.14) $\chi(\cdot, t) = 1_{\Omega_+^t} - 1_{\Omega_-^t}$ and $g^\pm$ denotes the positive/negative parts of a function $g$ respectively, and $\eta$ is an appropriate truncation of the identity function (eq. (3.8)). In particular, $(\eta \circ d_\Sigma) \chi \geq 0$ holds in $\Omega$ due to our convention on the signed-distance function, and thus the two integrands in (1.14) are both non-negative. We refer the readers to the proof of Theorem 3.2 below for more details on the positivity of (1.14).

The main result of this work is the following:

**Theorem 1.1.** Assume that the family of hypersurfaces $\Sigma$ (1.8) evolves by mean curvature flow during $[0, T]$. If the initial datum of (1.7) is well-prepared in the sense that

$$\|u_\varepsilon(\cdot, 0)\|_{L^\infty} + B[u_\varepsilon|\Sigma](0) + E_\varepsilon[u_\varepsilon|\Sigma](0) \leq C_1 \varepsilon$$

(1.15)

for some constant $C_1$ that is independent of $\varepsilon$, then there exists $C_2$ independent of $\varepsilon$ so that

$$\sup_{t \in [0, T]} E_\varepsilon[u_\varepsilon|\Sigma](t) \leq C_2 \varepsilon,$$

(1.16a)

$$\sup_{t \in [0, T]} B[u_\varepsilon|\Sigma](t) \leq C_2 \varepsilon,$$

(1.16b)

$$\sup_{t \in [0, T]} \int_\Omega |\psi_\varepsilon - c_F 1_{\Omega_+^t}| dx \leq C_2 \varepsilon^{1/2}.$$  (1.16c)

Moreover, for some subsequence $\varepsilon_k \downarrow 0$ there holds

$$u_{\varepsilon_k} \overset{k \to \infty}{\rightharpoonup} u^\pm \text{ weakly in } L^2(0, T; H^1_{\text{loc}}(\Omega^\pm_t)),$$

(1.17)

where $u^\pm$ are weak solutions to the harmonic heat flows into $m_\pm$ respectively and

$$u^\pm \in L^\infty(0, T; H^1(\Omega^\pm_t; m_\pm)), \quad \partial_t u^\pm \in L^2(0, T; L^2_{\text{loc}}(\Omega^\pm_t)).$$  (1.18)

Furthermore, for every $t \in (0, T)$,

$$|u^+ - u^-|_{\mathbb{R}^n}(x, t) = \text{dist}_m \text{ for } \mathcal{H}^{d-1} \text{- a.e. } x \in \Sigma_t.$$  (1.19)

A few comments are in order. Firstly, in (1.15) the $L^\infty$ bound of the initial datum is used (together with (1.6)) to obtain an uniform in space-time $L^\infty$-bound of $u_\varepsilon$, i.e.

$$\|u_\varepsilon\|_{L^\infty(\Omega \times (0, T))} \leq c_0$$

(1.20)

for some $\varepsilon$-independent constant $c_0$. Such an estimate, derived by applying the maximum principle to (1.7a), enables us to avoid several technical complications in the passage of the limit $\varepsilon \downarrow 0$. Indeed, even in the case when $d = 2$, severe difficulties arise in the anisotropic model considered in [27] where an estimate like (1.20) is not available. Secondly, if we denote the second fundamental forms of $m_\pm$ at points $p^\pm$ by $A^\pm(p^\pm)(\cdot, \cdot)$, respectively, then the theorem above claims that the pair of mappings

$$u^\pm(\cdot, t) : \Omega^\pm_t \mapsto m_\pm \subset \mathbb{R}^n$$

(1.21)
satisfy the following system in the weak sense:

\[
\begin{align*}
\partial_t u^\pm - \Delta u^\pm &= A^\pm(u^\pm)(\nabla u^\pm, \nabla u^\pm) \quad \text{in } \bigcup_{t \in [0,T]} \Omega_t^\pm \times \{t\}, \\
|u^+ - u^-|_{\mathbb{R}^n}(x, t) &= \text{dist}_m \
\text{for } \mathcal{H}^{d-1}-\text{a.e } x \in \Sigma_t, \\
\text{on } \partial \Omega.
\end{align*}
\]

The first equation in (1.22) says that \( u^\pm \) are (weak) harmonic map heat flows from the moving domains \( \Omega_t^\pm \) to the target manifolds \( m_\pm \) respectively. The second equation in (1.22) is referred to as the minimal pair boundary condition (cf. [25, 26]), and in the last equation \( g : \Omega \mapsto m_- \) is a prescribed smooth mapping.

To make the main theorem applicable, we shall show that the class of initial datum fulfilling the condition (1.13) is geometrically rich. This is stated in the following result.

**Theorem 1.2.** For any \( \delta \in (0, \delta_0) \) and any pair of mappings \( u^{in}_\pm \in H^1(\Omega_0^\pm; m_\pm) \) with

\[
|u^{in}_+ - u^{in}_-|_{\mathbb{R}^n}(x) = \text{dist}_m, \quad \text{for } \mathcal{H}^{d-1}-\text{a.e } x \in \Sigma_0,
\]

there exist \( u^{in}_\varepsilon \in H^1(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega) \) and a constant \( C = C(\delta, u^{in}_\pm) \) so that

\[
\begin{align*}
u^{in}_\varepsilon &= u^{in}_\pm \quad \text{in } \Omega_0^\varepsilon \setminus B_{25}(\Sigma_0), \\
E_\varepsilon[u^{in}_\varepsilon|\Sigma_0] &\leq C\varepsilon, \\
B[u^{in}_\varepsilon|\Sigma_0] &\leq C\varepsilon.
\end{align*}
\]

The rest of the work will be organized as follows: in Section 2, we shall recall fundamental results that will be employed throughout the work. These include the compactness and closure of special function with bounded variation (cf. [3, Chapter 4]), the theory of minimal connection developed by Sternberg [36] and Lin–Pan–Wang [25], the elements of differential geometry used in the description of interface motion, and finally the relative entropy method by Fischer–Lau–Simon [16]. In particular, in Subsection 2.5 we shall adapt this later method to system (1.7), and then derive a differential inequality, i.e. Proposition 2.9. This proposition, when combined with Chen–Struwe [8] along with results in Section 3 leads to the convergences to harmonic heat flows locally away from the moving interface \( \Sigma_t \). Another important consequence of Proposition 2.9 is an \( L^1 \)-convergence rate estimate of \( \psi_\varepsilon \), obtained in Theorem 3.2. This theorem will be used to derive fine estimates of the level sets of \( \psi_\varepsilon \) in Lemma 3.4 as well as convergences of some corrections of \( u_\varepsilon \) up to the moving interface \( \Sigma_t \). All of these will be done in Section 3 and we shall use them to derive the minimal pair boundary condition (1.19) in Section 4 and thus finish the proof of Theorem 1.1. Finally we prove Theorem 1.2 in Section 5.

We end the introductory part by introducing some notations and conventions that will be employed throughout this work. Unless specified otherwise \( C > 0 \) is a generic constant depending only on the geometry of the interface \( \Sigma \) (cf. (1.8)) and that of the wells \( m \) (cf. (1.3)), but not on \( \varepsilon \) or \( t \in [0,T] \). The value of such a constant might change from line to line. In order to simplify the presentation, we shall sometimes abbreviate the estimates like \( X \lesssim CY \) by \( X \lesssim Y \) for two non-negative quantities \( X,Y \).

We provide a list of symbols for the convenience of the readers:

- \( A:B \) is the Frobenius inner product of two square matrices \( A, B \), defined by \( \text{tr } A^TB \).
- \( \partial_i = \partial_{x_i} \) (\( 0 \leq i \leq d \)) with the convention that \( \partial_d = \partial_0 \).
- \( \nabla f = (\partial_1 f, \cdots, \partial_d f) \) is the (distributional) gradient of a function \( f \) with variables \( x = (x_1, \cdots, x_d) \).
- \( \partial W = (\partial_{u_1} W, \cdots, \partial_{u_n} W) \) is the gradient of a function \( W = W(u) \).
• $\partial d_F(u)$: the generalized gradient of $d_F$ (cf. (2.35) below).
• $\partial U$: measure-theoretic boundary of a set $U$ of finite perimeter with measure-theoretic outer normal vector $\nu$.
• $\text{dist}(u, A)$: Euclidean distance from $u$ to a set $A \subset \mathbb{R}^n$.
• For two vectors $u, v \in \mathbb{R}^n$, $|u - v|$ is their Euclidean distance $|u - v|_{\mathbb{R}^n}$.
• $\text{dist}_m$: the distance between $m_\pm$ in $\mathbb{R}^n$, namely $\text{dist}_m := \inf_{p \in m_\pm} |p^+ - p^-|_{\mathbb{R}^n}$.
• $d_m(u)$: the distance from $u \in \mathbb{R}^n$ to $m = m_+ \cup m_-$ (cf. (2.14) below).
• $B_\delta(U)$: the $\delta$-(tubular) neighborhood of a set $U$ in the corresponding Euclidean space. In particular, $B_\delta(x)$ is the open ball centered at $x$.
• $d_{\Sigma}(x, t)$: signed distance from $x$ to $\Sigma_t$ (cf. (1.9)).

2. Preliminaries

2.1. Special function of bounded variation.

Definition 2.1. We say that $u \in BV(\Omega; \mathbb{R}^n)$ is a special function with bounded variation and we write $u \in SBV(\Omega; \mathbb{R}^n)$, if the Cantor part of its distributional derivative $\nabla^c u$ vanishes, i.e.

$$\nabla u = \nabla^a u \mathcal{L}^d + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{d-1} \mathcal{L} J_u$$

(2.1)

where $\nabla^a$ denotes the absolutely continuous part of the distributional derivative (with respect to Lebesgue measure $\mathcal{L}^d$) and $J_u$ is the jump set of $u$ with measure theoretical outer normal vector $\nu_u$.

The following two results will be used to obtain convergences up to the free boundary. We refer the readers to the monograph of Ambrosio–Fusco–Pallara [3] for the proofs.

Proposition 2.2. (Closure of $SBV$) Let $\varphi : [0, \infty) \to [0, \infty]$ be lower semicontinuous increasing functions and assume that $\lim_{t \to \infty} \frac{\varphi(t)}{t} = \infty$. Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $\{u_k\} \subset SBV(\Omega)$ such that

$$\sup_k \int_\Omega \varphi(|\nabla^a u_k|) \, dx + \sup_k \int_{J_{u_k}} |u_k^+ - u_k^-| \, d\mathcal{H}^{d-1} < \infty.$$  

(2.2)

where $\nabla^a u_k$ is the absolute continuous part of the distributional gradient $\nabla u_k$, and $(u_k^+, u_k^-)$ are the approximate one-sided limits on the jump set $J_{u_k}$. If $\{u_k\}$ weakly-star converges in $BV(\Omega)$ to $u$, then the following statements hold

• $u \in SBV(\Omega)$.
• $\nabla^a u_k$ weakly converge to $\nabla^a u$ in $L^1(\Omega)$.
• The jump part of the gradient $\nabla^j u_k$ weakly-star converge to $\nabla^j u$ in $\Omega$.
• For any convex function $\varphi$, there holds

$$\int_\Omega \varphi(|\nabla^a u|) \, dx \leq \liminf_{k \to \infty} \int_\Omega \varphi(|\nabla^a u_k|) \, dx.$$  

(2.3)

Proposition 2.3. (Compactness of $SBV$) Let $\varphi, \Omega$ be as in Proposition 2.2. Let $\{u_k\} \subset SBV(\Omega)$ be satisfying (2.2) and assume, in addition, that $\|u_k\|_{L^\infty(\Omega)}$ is uniformly bounded in $k$. Then, there exists a subsequence

$$u_k \xrightarrow{k \to \infty} u \in SBV(\Omega) \text{ weakly star in } BV(\Omega).$$  

(2.4)
2.2. Minimal connections. We shall briefly describe some basic properties of minimal orbits. For any \( p^\pm \in m_\pm \), we define their minimal connection

\[
C_F(p^+, p^-) := \inf \left\{ \int_\mathbb{R} \frac{1}{2} |\gamma'(t)|^2 + F(\gamma(t)) \, dt \mid \gamma \in H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n), \gamma(\pm \infty) = p^\pm \in m_\pm \right\}. \tag{2.5}
\]

We also introduce the centralized potential, which is the even function

\[
F(\lambda) := \begin{cases} f \left( \left( \frac{\text{dist}_m}{2} + \lambda \right)^2 \right) & \text{if } \lambda \leq 0, \\ f \left( \left( \frac{\text{dist}_m}{2} - \lambda \right)^2 \right) & \text{if } \lambda \geq 0,
\end{cases}
\]

and the associated scalar-valued minimal connection problem

\[
c_F := \min \left\{ \int_\mathbb{R} \left( \frac{1}{2} |\gamma'(s)|^2 + F(\gamma(s)) \right) \, ds \mid \gamma \in H^1_{\text{loc}}(\mathbb{R}), \gamma(\pm \infty) = \pm \frac{\text{dist}_m}{2} \right\}. \tag{2.7}
\]

The following result was obtained in \([17, 36]\).

**Lemma 2.4.** It holds that

\[
c_F := 2 \int_0^{\text{dist}_m} \sqrt{2\bar{F}(\lambda)} \, d\lambda = c_F \left( 2 \int_0^{\text{dist}_m} \sqrt{2f(\lambda^2)} \, d\lambda \right). \tag{2.8}
\]

Moreover, there exists a minimizer \( \alpha(s) \) of \((2.7)\) that satisfies

\[
\alpha(s) \in C^\infty \left( \mathbb{R}; \left(-\frac{\text{dist}_m}{2}, \frac{\text{dist}_m}{2}\right) \right) \text{ is odd and strictly increasing in } \mathbb{R}. \tag{2.9a}
\]

\[- \alpha''(s) + \bar{F}'(\alpha(s)) = 0, \quad s \in \mathbb{R}; \quad \alpha(\pm \infty) = \pm \frac{\text{dist}_m}{2}. \tag{2.9b}
\]

\[\alpha'(s) = \sqrt{2\bar{F}(\alpha(s))}, \quad \forall \ s \in \mathbb{R}. \tag{2.9c}\]

\[|\alpha'(s)| + |\alpha(s) + \frac{\text{dist}_m}{2}| \leq C e^{-C|s|} \quad \text{as } s \to \pm \infty. \tag{2.9d}\]

We need an equivalent condition to the minimal pair one stated at \((1.19)\). To this end, we introduce

\[
M^+ := \left\{ p^+ \in m_+ \mid \exists p^- \in m_- \text{ s.t. } |p^+ - p^-| = \text{dist}_m \right\}, \tag{2.10a}
\]

\[
M^- := \left\{ p^- \in m_- \mid \exists p^+ \in m_+ \text{ s.t. } |p^+ - p^-| = \text{dist}_m \right\}. \tag{2.10b}
\]

**Lemma 2.5.** \([25, \text{Theorem 2.1}]\) The function \( C_F(\cdot, \cdot) \) defined by \((2.5)\) satisfies

\[
C_F(p^+, p^-) = c_F \quad \forall p^\pm \in m_\pm. \tag{2.11}
\]

Moreover, we have the following equivalence for a pair \( p^\pm \in m_\pm \):

\[
\bullet \quad C_F(p^+, p^-) \text{ is attained by a minimal orbit } \gamma. \tag{2.12a}
\]

\[
\bullet \quad p^\pm \in M^\pm, \quad |p^+ - p^-| = \text{dist}_m. \tag{2.12b}
\]

Furthermore, assuming one of the above two conditions hold, the corresponding minimal orbit \( \gamma \) attaining the minimum of \( C_F(p^+, p^-) \) is

\[
\gamma(t) = \frac{p^+ + p^-}{2} + \alpha(t) \frac{p^+ - p^-}{\text{dist}_m}, \quad t \in \mathbb{R}, \tag{2.13}
\]

where \( \alpha \in H^1_{\text{loc}}(\mathbb{R}) \) is a solution to \((2.7)\) (and equivalently to \((2.9b)\)).

Seemly more complicated, the condition \((2.12a)\) is more compatible with the variational structure of the functional \((1.2)\) than \((2.12b)\).
2.3. Quasi-distance function. We define the distance from \( u \) to \( m \) according to its relative distance to its two components:

\[
d_m(u) = \begin{cases} 
  d_{m+}(u) & \text{when } d_{m+}(u) \leq \frac{\text{dist}_m}{2}, \\
  d_{m-}(u) & \text{when } d_{m-}(u) \leq \frac{\text{dist}_m}{2}.
\end{cases}
\]

(2.14)

In general such a function is not \( C^1 \) unless \( u \) is close to but not in \( m \) (cf. [21, Section 4.4]). We assume that the nearest point projection \( P_m : B_{\delta_0}(m) \mapsto m \) is smooth. Using this, we have

\[
d_m(u) := |u - P_m(u)|, \quad \forall \, u \in B_{\delta_0}(m).
\]

(2.15)

Such a function is \( C^1 \) in \( B_{\delta_0}(m) \setminus m \), and this motivates the following unit vector field:

\[
\nu_m(u) := \begin{cases} 
  \partial d_m(u) = \frac{u - P_m(u)}{|u - P_m(u)|} & \forall \, u \in B_{\delta_0}(m) \setminus m, \\
  0 & \forall \, u \in m.
\end{cases}
\]

(2.16)

Note that such a normal vector field is in general not continuous up to \( m \) unless \( m_\pm \) are hypersurfaces of \( \mathbb{R}^n \).

With these preparations, we introduce the quasi-distance function

\[
d_F(u) := \begin{cases} 
  \int_0^{d_m(u)} \sqrt{2f(\lambda^2)} \, d\lambda & \text{if } d_{m-}(u) \leq \frac{\text{dist}_m}{2}, \\
  \frac{1}{2}c_F & \text{if } d_m(u) > \frac{\text{dist}_m}{2}, \\
  c_F - \int_0^{d_m(u)} \sqrt{2f(\lambda^2)} \, d\lambda & \text{if } d_{m+}(u) \leq \frac{\text{dist}_m}{2},
\end{cases}
\]

(2.17)

where \( c_F = 2 \int_0^{\frac{\text{dist}_m}{2}} \sqrt{2f(\lambda^2)} \, d\lambda \) is the surface tension coefficient (cf. (1.13)). Such a function is a modification of the one used in [36, 17]. We list some of its important properties here.

**Lemma 2.6.** The function \( d_F : \mathbb{R}^n \mapsto [0, c_F] \) is Lipschitz continuous in \( \mathbb{R}^n \), and satisfies

\[
|\partial d_F(u)| \leq 2F(u) \quad \text{a.e. } u \in \mathbb{R}^n,
\]

(2.18)

\[
d_F(u) = \begin{cases} 
  0 & \text{if and only if } u \in m_-, \\
  c_F & \text{if and only if } u \in m_+.
\end{cases}
\]

(2.19)

\[
d_F \in C^1(B_{\delta_0}(m)), \quad \text{and } \partial d_F|_{B_{\delta_0}(m)}(u) = 0 \text{ if and only if } u \in m.
\]

(2.20)

Moreover, \( \frac{\partial d_F(u)}{\partial d_F(u)} \) is a continuous unit vector field in \( B_{\delta_0}(m) \setminus m \) so that

\[
\frac{\partial d_F(u)}{\partial d_F(u)} = \nu_m(u) \quad \forall u \in B_{\delta_0}(m) \setminus m.
\]

(2.21)

**Proof.** It is obvious that \( d_F \) is continuous in \( \mathbb{R}^n \), and is Lipschitz in each subdomain where it is defined. It suffices to check the Lipschitz condition across adjacent regions. If \( d_m(u_1) > \frac{\text{dist}_m}{2} \) and \( d_{m-}(u_2) \leq \frac{\text{dist}_m}{2} \), then

\[
0 \leq d_F(u_1) - d_F(u_2) = \int_{d_{m-}(u_2)}^{d_{m-}(u_1)} \sqrt{2f(\lambda^2)} \, d\lambda
\]

\[
\leq \int_{d_{m-}(u_2)}^{d_{m-}(u_1)} \sqrt{2f(\lambda^2)} \, d\lambda \leq C|d_{m-}(u_1) - d_{m-}(u_2)| \leq C|u_2 - u_1|.
\]

Other cases can be treated in a similar way. The inequality (2.18) and the formula (2.19) follows directly from the definition (2.17).
To show $d_F \in C^1(B_{\delta_0}(m))$, it suffices to write it as a function of $d_m^2$ which is smooth up to $m$. Indeed, by (1.6) one can verify that

$$h(s) = \int_0^{s^{1/2}} \sqrt{2f(\lambda^2)} \, d\lambda \in C^1[0, \infty).$$

As a result, $d_F(u) = h(d_m^2(m))$ in $B_{\delta_0}(m-1)$, and a similar argument applies to $B_{\delta_0}(m+1)$. The rest assertions are due to $d_m \in C^1(B_{\delta_0}(m))$. □

2.4. Geometry of interfaces. Under a local parametrization $\varphi_t(s) : U \to \Sigma_t$ on an open set $U \subset \mathbb{R}^{d-1}$, the MCF equation writes

$$\partial_t \varphi_t(s) = \kappa(\varphi_t(s), t) n(s, t),$$

where $\kappa$ is the mean curvature and $n$ is the inward normal. For $\delta > 0$, the $\delta$-neighborhood of $\Sigma_t$ is the open set

$$B_\delta(\Sigma_t) := \{ x \in \Omega : |d_{\Sigma}(x, t)| < \delta \}.$$  

We shall choose the $\delta_0$ (first appeared in (1.6)) small enough so that the nearest point projection $P_{\Sigma}(\cdot, t) : B_{4\delta_0}(\Sigma_t) \to \Sigma_t$ is smooth for any $t \in [0, T]$, and the interface keeps at least $4\delta_0$ distance away from the boundary of the domain $\partial \Omega$. Analytically we have

$$P_{\Sigma}(x, t) = x - \nabla d_{\Sigma}(x, t)d_{\Sigma}(x, t).$$

So for each fixed $t \in [0, T]$, any point $x \in B_{4\delta_0}(\Sigma_t)$ corresponds to a unique pair $(r, s)$ with $r = d_{\Sigma}(x, t)$ and $s \in U$, and thus the identity

$$d_{\Sigma}(\varphi_t(s) + r n(s, t), t) \equiv r$$

holds with independent variables $(r, s, t)$. Differentiating this identity with respect to $r$ and $t$ leads to the following identities:

$$\nabla d_{\Sigma}(x, t) = n(s, t), \quad -\partial_t d_{\Sigma}(x, t) = \partial_t \varphi_t(s) \cdot n(s, t) =: V(s, t).$$

These formulas extend the inward normal vector $n$ and the normal velocity $V$ of $\Sigma_t$ to $B_{4\delta_0}(\Sigma_t)$.

Now we come to the definition of $\xi$ in the modulated energy $E_{\varepsilon}(u_{\cdot, |\Sigma|})(t)$ (1.11). This is done by extending the inward normal vector field $n$ through

$$\xi(x, t) = \phi \left( \frac{d_{\Sigma}(x, t)}{\delta_0} \right) \nabla d_{\Sigma}(x, t).$$

Here $\phi(x) \geq 0$ is an even, smooth function on $\mathbb{R}$ that decreases for $x \in [0, 1]$, and satisfies

$$\begin{cases} 
\phi(x) > 0 & \text{for } |x| < 1, \\
\phi(x) = 0 & \text{for } |x| \geq 1, \\
1 - 4x^2 \leq \phi(x) \leq 1 - \frac{1}{2}x^2 & \text{for } |x| \leq 1/2.
\end{cases}$$
To fulfill these requirements, we can simply choose
\[ \phi(x) = e^{-\frac{1}{4} x^2} + 1 \quad \text{for } |x| < 1 \quad \text{and} \quad \phi(x) = 0 \quad \text{for } |x| \geq 1. \] (2.28)

We also need to extend the curvature of the domain \( B_{4\delta_0}(\Sigma_t) \). To this end, choose a cut-off function
\[ \eta_0 \in C_c^\infty(\mathbb{R}^d) \quad \text{with} \quad \eta_0 = 1 \quad \text{in} \quad B_{\delta_0}(\Sigma_t), \] (2.29)
and we define
\[ H(x,t) = \kappa \nabla d_{\Sigma}(x,t) \quad \text{with} \quad \kappa(x,t) = -\Delta d_{\Sigma}(P_{\Sigma}(x,t),t) \eta_0(x,t). \] (2.30)

By (2.29), \( H \) is extended constantly in the normal direction. So we have
\[ \eta \cdot \nabla H = 0 \quad \text{and} \quad \xi \cdot \nabla H = 0 \quad \forall t \in [0,T], \quad x \in B_{2\delta_0}(\Sigma_t). \] (2.31)
\[ \xi = 0 \quad \text{and} \quad H = 0 \quad \forall t \in [0,T], \quad x \in \partial \Omega. \] (2.32)

We end this part by the following identities which will be employed to prove the modulated energy inequalities:
\[ \nabla \cdot \xi + H \cdot \xi = O(d_{\Sigma}), \] (2.33a)
\[ \partial_t d_{\Sigma}(x,t) + (H(x,t) \cdot \nabla) d_{\Sigma}(x,t) = 0 \quad \text{in} \quad B_{\delta_0}(\Sigma_t), \] (2.33b)
\[ \partial_t \xi + (H \cdot \nabla) \xi + (\nabla H)^T \xi = 0 \quad \text{in} \quad B_{\delta_0}(\Sigma_t), \] (2.33c)
\[ \partial_t |\xi|^2 + (H \cdot \nabla) |\xi|^2 = 0 \quad \text{in} \quad B_{\delta_0}(\Sigma_t), \] (2.33d)

where \( \nabla H := \{ \partial_i H_i \}_{1 \leq i,j \leq d} \) is a matrix with \( i \) being the row index.

**Proof of (2.33).** Recalling (2.26), \( \phi_0(t) := \phi(t) \) is an even function. So it follows from \( \phi_0(0) = 0 \) and Taylor’s expansion in \( d_{\Sigma} \) that
\[ \nabla \cdot \xi = |\nabla d_{\Sigma}|^2 \phi_0'(d_{\Sigma}) + \phi_0(d_{\Sigma}) \Delta d_{\Sigma}(x,t) \]
\[ = O(d_{\Sigma}) + \phi_0(d_{\Sigma}) \Delta d_{\Sigma}(P_{\xi}(x,t),t), \]
and this together with (2.30) leads to (2.33a). Using (2.25) and (2.30), we can write (2.23) as the transport equation (2.33b):
\[ -\partial_t d_{\Sigma} = \partial_t \phi_t(s) \cdot n(s,t) = \kappa(\phi_t(s),t) = H \cdot \nabla d_{\Sigma} \quad \text{in} \quad B_{\delta_0}(\Sigma_t). \]

This equation implies that the following two identities hold in \( B_{\delta_0}(\Sigma_t) \):
\[ \partial_t |\nabla d_{\Sigma}|^2 + (H \cdot \nabla) \nabla d_{\Sigma} + (\nabla H)^T \nabla d_{\Sigma} = 0, \]
\[ \partial_t \phi_0(d_{\Sigma}) + (H \cdot \nabla) \phi_0(d_{\Sigma}) = 0. \]
These two equations together imply (2.33c). Finally (2.33d) is a consequence of (2.33b). □

2.5. Modulated energy method. As the gradient flow of the Ginzburg–Landau energy (1.2), the system (1.7a) has the following energy dissipation law

\[ A_\varepsilon(u_\varepsilon(\cdot, T)) + \int_0^T \int_{\Omega} \varepsilon |\partial_t u_\varepsilon|^2 \, dx \, dt = A_\varepsilon(u_\varepsilon(\cdot, 0)), \quad \text{for all } T > 0. \]  

(2.34)

For initial datum undergoing a phase transition across the initial interface \( \Sigma_0 \), due to concentrations of \( \nabla u_\varepsilon \) on \( \Sigma_0 \), the dissipation law (2.34) is not sufficient to derive quantitative convergences of \( u_\varepsilon \), not even away from \( \Sigma_0 \). Following a recent work of Fisher–Laux–Simon [16], we shall derive an inequality which modulates the concentrations and leads to compactness of \( \{u_\varepsilon\} \) in Sobolev spaces.

We shall start by discussing the differentiability of \( \psi_\varepsilon(x, t) = d_F \circ u_\varepsilon(x, t) \) (cf. (1.12)). It follows from Lemma 2.6 that \( d_F(\cdot) \) is a Lipschitz function in \( \mathbb{R}^n \) with \( d_F(0) = 0 \) under the assumption that \( 0 \in \mathcal{m}^- \). Following Laux–Simon [23], for every \( (x, t) \in \Omega \times [0, T] \), we consider the restriction of \( d_F(u_\varepsilon(x, t)) \) to the affine space

\[ T_{x,t}^{u_\varepsilon} := u_\varepsilon(x, t) + \text{span}\{\partial_1 u_\varepsilon(x, t), \ldots, \partial_d u_\varepsilon(x, t)\}, \]

denoted by \( d_F|_{T_{x,t}^{u_\varepsilon}} \). By the generalized chain rule of Ambrosio–Dal Maso [2], \( d_F|_{T_{x,t}^{u_\varepsilon}} \) is differentiable at \( u_\varepsilon(x, t) \). Now we denote the orthogonal projection from \( \mathbb{R}^n \) to the subspace \( T_{x,t}^{u_\varepsilon} - u_\varepsilon(x, t) \) by \( \Pi_{x,t}^{u_\varepsilon} \), and define the generalized differential by

\[ \partial d_F(u_\varepsilon) := \partial \left( d_F|_{T_{x,t}^{u_\varepsilon}} \right) \bigg|_{u_\varepsilon(x, t)} \circ \Pi_{x,t}^{u_\varepsilon}. \]  

(2.35)

Then we have for \( 0 \leq i \leq d \) that

\[ \partial d_F(u_\varepsilon) \cdot \partial_i u_\varepsilon = \partial \left( d_F|_{T_{x,t}^{u_\varepsilon}} \right) \bigg|_{u_\varepsilon(x, t)} \cdot \partial_i u_\varepsilon = \partial_i (d_F \circ u_\varepsilon) \]  

(2.36)

where the second equality is due to the directional derivative at \( u_\varepsilon(x, t) \) pointing to \( \partial_i u_\varepsilon(x, t) \). This proves the generalized chain rule

\[ \partial_i \psi_\varepsilon(x, t) = \partial_i u_\varepsilon(x, t) \cdot \partial d_F(u_\varepsilon(x, t)) \quad \text{for } 0 \leq i \leq d \text{ and a.e. } (x, t). \]  

(2.37)

Moreover, the point-wise differential inequality (2.18) is valid for the generalized differential (2.35):

\[ |\partial d_F(u_\varepsilon)| \leq \sqrt{2F(u_\varepsilon)}. \]  

(2.38)

Note that the above inequality, rather than its classical version (2.18), will be used to prove the modulated energy inequalities.

To proceed, we define the phase-field analogues of the normal vector and the mean curvature vector respectively by

\[ \mathbf{n}_\varepsilon(x, t) := \begin{cases} \frac{\nabla \psi_\varepsilon}{|\nabla \psi_\varepsilon|}(x, t) & \text{if } \nabla \psi_\varepsilon(x, t) \neq 0, \\ 0 & \text{otherwise}. \end{cases} \]  

(2.39a)

\[ \mathbf{H}_\varepsilon(x, t) := \begin{cases} -\left( \varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \partial F(u_\varepsilon) \right) \cdot \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} & \text{if } \nabla u_\varepsilon(x, t) \neq 0, \\ 0 & \text{otherwise}. \end{cases} \]  

(2.39b)

Note that in (2.39b), the inner product is made with the column vectors of \( \nabla u_\varepsilon = (\partial_1 u_\varepsilon, \ldots, \partial_d u_\varepsilon) \).
Using (2.16) and (2.21) we define the linear projection

$$
\Pi_{\mathbf{u}_\varepsilon} \partial_t \mathbf{u}_\varepsilon := \begin{cases}
\partial_t \mathbf{u}_\varepsilon \cdot \nu_m(\mathbf{u}_\varepsilon) \nu_m(\mathbf{u}_\varepsilon) & \text{if } \mathbf{u}_\varepsilon \in B_{\delta_0}(\mathbf{m}), \\
\left( \partial_t \mathbf{u}_\varepsilon \cdot \frac{\partial d_F(\mathbf{u}_\varepsilon)}{|\partial d_F(\mathbf{u}_\varepsilon)|} \right) \frac{\partial d_F(\mathbf{u}_\varepsilon)}{|\partial d_F(\mathbf{u}_\varepsilon)|} & \text{if } \mathbf{u}_\varepsilon \notin B_{\delta_0}(\mathbf{m}) \text{ and } \partial d_F(\mathbf{u}_\varepsilon) \neq 0,
\end{cases}
$$

(2.40)

where $\partial d_F$ is interpreted as the generalized differential (2.35) in case $d_F$ is not classically differentiable at $\mathbf{u}_\varepsilon$.

**Lemma 2.7.** The following two identities hold:

$$
|\nabla \psi_\varepsilon| = |\Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon||\partial d_F(\mathbf{u}_\varepsilon)|
$$

(a.e. $(x, t)$),

(2.41)

$$
\Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon = \frac{|\nabla \psi_\varepsilon|}{|\partial d_F(\mathbf{u}_\varepsilon)|^2} \partial d_F(\mathbf{u}_\varepsilon) \otimes \mathbf{n}_\varepsilon
$$

if $\partial d_F(\mathbf{u}_\varepsilon) \neq 0$,

(2.42)

where in (2.42) the projection applies to each column vector of $\nabla \mathbf{u}_\varepsilon$.

**Proof.** Concerning (2.41), we distinguish two cases:

(a) When $\mathbf{u}_\varepsilon \in B_{\delta_0}(\mathbf{m})$, then according to (2.20), $d_F$ is $C^1$ and the generalized differential (2.35) coincide with the classical one. Thus on the set $\{x \mid \mathbf{u}_\varepsilon(x, t) \in B_{\delta_0}(\mathbf{m}) \cap \mathbf{m}\}$, we have

$$
\partial_t \psi_\varepsilon = \partial_t \mathbf{u}_\varepsilon \cdot \frac{\partial d_F(\mathbf{u}_\varepsilon)}{|\partial d_F(\mathbf{u}_\varepsilon)|} |\partial d_F(\mathbf{u}_\varepsilon)|.
$$

(2.43)

This together with the first case of (2.40) leads to (2.41). On the set $\{x \mid \mathbf{u}_\varepsilon(x, t) \in \mathbf{m}\}$, we have from (2.20) that $\partial d_F(\mathbf{u}_\varepsilon) = 0$. By (2.37), both sides of (2.41) vanishes.

(b) When $\mathbf{u}_\varepsilon \notin B_{\delta_0}(\mathbf{m})$: if $\partial d_F(\mathbf{u}_\varepsilon) \neq 0$, then owning to (2.37) and the second case in (2.40), we obtain (2.41): if $\partial d_F(\mathbf{u}_\varepsilon) = 0$, then by (2.37), we have $\nabla \psi_\varepsilon = 0$ too. This finishes the proof of (2.41).

Now we turn to the proof of the formula (2.42) under the assumption that $\partial d_F(\mathbf{u}_\varepsilon) \neq 0$. It holds when $\nabla \psi_\varepsilon = 0$ because (2.41) then implies $\Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon = 0$. When $\nabla \psi_\varepsilon \neq 0$, then

$$
\frac{|\nabla \psi_\varepsilon|}{|\partial d_F(\mathbf{u}_\varepsilon)|^2} \partial d_F(\mathbf{u}_\varepsilon) \otimes \mathbf{n}_\varepsilon \overset{(2.39a)}{=} \partial_t \mathbf{u}_\varepsilon \cdot \nu_m(\mathbf{u}_\varepsilon) \frac{\partial d_F(\mathbf{u}_\varepsilon)}{|\partial d_F(\mathbf{u}_\varepsilon)|^2} \otimes \nabla \psi_\varepsilon
$$

$$
\overset{(2.37)}{=} \partial_t \mathbf{u}_\varepsilon \cdot \nu_m(\mathbf{u}_\varepsilon) \frac{\partial d_F(\mathbf{u}_\varepsilon)}{|\partial d_F(\mathbf{u}_\varepsilon)|^2} \otimes \nabla \mathbf{u}_\varepsilon \cdot \partial d_F(\mathbf{u}_\varepsilon).
$$

(2.44)

We distinguish two cases:

1. When $\mathbf{u}_\varepsilon \notin B_{\delta_0}(\mathbf{m})$, the last term of (2.44) is exactly the second case defining $\Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon$ (2.40).
2. When $\mathbf{u}_\varepsilon \in B_{\delta_0}(\mathbf{m})$, by (2.20) we have $\mathbf{u}_\varepsilon \notin \mathbf{m}$ (under the assumption $\partial d_F(\mathbf{u}_\varepsilon) \neq 0$). So by (2.21), the last term of (2.44) simplifies to $\nabla \mathbf{u}_\varepsilon \cdot \nu_m(\mathbf{u}_\varepsilon) \otimes \nu_m(\mathbf{u}_\varepsilon)$. This coincides with the first case of (2.40) defining $\Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon$. This finishes the proof of (2.42).  

As we shall not integrate the time variable $t$ throughout this section, we shall abbreviate the spatial integration $\int_{\Omega}$ by $\int$ and sometimes we omit the $dx$. The following lemma gives various coercivity estimates of $E_c[\mathbf{u}_\varepsilon \Sigma]$ (1.11). It was due to [22], generalizing the one by [16] to vectorial cases. We present the proof for the convenience of the readers.
Lemma 2.8. There exists a universal constant $C > 0$ which is independent of $t \in [0, T]$ and $\varepsilon$ such that the following estimates hold for every $t \in [0, T]$: 

\[
\int \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) - |\nabla \psi_\varepsilon| \right) \, dx \leq E_\varepsilon[u_\varepsilon][\Sigma], \tag{2.45a}
\]

\[
\varepsilon \int |\nabla u_\varepsilon - \Pi u_\varepsilon \nabla u_\varepsilon|^2 \, dx \leq 2E_\varepsilon[u_\varepsilon][\Sigma], \tag{2.45b}
\]

\[
\int \left( \sqrt{\varepsilon} |\Pi u_\varepsilon \nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}} |\partial d_F(u_\varepsilon)| \right)^2 \, dx \leq 2E_\varepsilon[u_\varepsilon][\Sigma], \tag{2.45c}
\]

\[
\int \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) + |\nabla \psi_\varepsilon| \right) (1 - \xi \cdot n_\varepsilon) \, dx \leq CE_\varepsilon[u_\varepsilon][\Sigma], \tag{2.45d}
\]

\[
\int \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) + |\nabla \psi_\varepsilon| \right) \min \left( d_\varepsilon^2, 1 \right) \, dx \leq CE_\varepsilon[u_\varepsilon][\Sigma]. \tag{2.45e}
\]

Proof. Using (2.39a), we obtain $|\nabla \psi_\varepsilon| = |\nabla \psi_\varepsilon| n_\varepsilon$. Note also that (2.40) implies

\[
|\nabla u_\varepsilon - \Pi u_\varepsilon \nabla u_\varepsilon|^2 + |\Pi u_\varepsilon \nabla u_\varepsilon|^2 = |\nabla u_\varepsilon|^2. \tag{2.46}
\]

Altogether, we can write

\[
\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) - \xi \cdot \nabla \psi_\varepsilon
\]

\[
= \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) - |\nabla \psi| + |\nabla \psi| (1 - \xi \cdot n_\varepsilon)
\]

\[
= \frac{\varepsilon}{2} |\nabla u_\varepsilon - \Pi u_\varepsilon \nabla u_\varepsilon|^2 + |\nabla \psi| (1 - \xi \cdot n_\varepsilon)
\]

\[
+ \frac{\varepsilon}{2} |\Pi u_\varepsilon \nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) - |\nabla \psi|.
\]

This combined with (2.38) and (2.41) yields

\[
\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) - \xi \cdot \nabla \psi_\varepsilon
\]

\[
\geq \frac{\varepsilon}{2} |\nabla u_\varepsilon - \Pi u_\varepsilon \nabla u_\varepsilon|^2 + |\nabla \psi| (1 - \xi \cdot n_\varepsilon)
\]

\[
+ \frac{1}{2} \left( \sqrt{\varepsilon} |\Pi u_\varepsilon \nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}} |\partial d_F(u_\varepsilon)| \right)^2.
\]  

(2.47)

This inequality implies (2.45a), (2.45b) and (2.45c).

Combining (2.45a) with $E_\varepsilon[u_\varepsilon][\Sigma] \geq \int (1 - \xi \cdot n_\varepsilon) |\nabla \psi| \, dx$ and $1 - \xi \cdot n_\varepsilon \leq 2$ yields (2.45d). Finally, by (2.27) and $\delta_0 \in (0, 1)$ we have

\[
1 - \xi \cdot n_\varepsilon \geq 1 - \phi \left( \frac{d_\varepsilon}{\delta_0} \right) \geq \min \left( \frac{d_\varepsilon^2}{2\delta_0^2}, 1 - \phi \left( \frac{1}{2} \right) \right) \geq C_{\phi, \delta_0} \min(d_\Sigma^2, 1).
\]

(2.48)

This together with (2.45d) implies (2.45e). \qed

The following result was first proved in [16] in the scalar case, and was generalized to a matrix-valued model in [22]. We present the proof in Appendix A for the convenience of the readers.
Proposition 2.9. There exists a constant $C = C(\Sigma)$ independent of $\varepsilon$ such that
\[
\frac{d}{dt} E_\varepsilon[u_\varepsilon] + \frac{1}{2\varepsilon} \int (\varepsilon^2 \partial_t u_\varepsilon^2 - |H_\varepsilon|^2) \, dx + \frac{1}{2\varepsilon} \int |H_\varepsilon - \varepsilon \nabla u_\varepsilon| H_\varepsilon \, dx \\
+ \frac{1}{2\varepsilon} \int \varepsilon \partial_t (u_\varepsilon - (\nabla \cdot \xi) \partial F(u_\varepsilon))^2 \, dx \leq C E_\varepsilon[u_\varepsilon|\Sigma].
\] (2.49)

3. Estimates of level sets

The main task of this section is to derive the convergence rate estimate (1.16c) and use it to obtain fine estimates of the level sets of $\psi$. We start with a corollary of Proposition 2.9.

Lemma 3.1. There exists a universal constant $C = C(\Sigma)$ such that
\[
\sup_{t \in [0,T]} \int_{\Omega} \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) - \xi \cdot \nabla \psi \right) \, dx \leq C \varepsilon,
\] (3.1a)
\[
\sup_{t \in [0,T]} \int_{\Omega} \left( |\nabla u_\varepsilon - \Pi u_\varepsilon \nabla u_\varepsilon|^2 \right) \, dx + \int_0^T \int_{\Omega} \left( |\partial_t u_\varepsilon - \Pi u_\varepsilon \partial_t u_\varepsilon|^2 \right) \, dx dt \leq C,
\] (3.1b)
\[
\sup_{t \in [0,T]} A_\varepsilon(u_\varepsilon(\cdot, t)) + \sup_{t \in [0,T]} \|\nabla \psi(\cdot, t)\|_{L^1(\Omega)} \leq C,
\] (3.1c)
\[
\sup_{t \in [0,T]} \int_{\Omega} \left( |\nabla \psi| - \xi \cdot \nabla \psi \right) \, dx \leq C \varepsilon.
\] (3.1d)

Moreover, for any fixed $\delta \in (0, \delta_0)$, there holds
\[
\sup_{t \in [0,T]} \int_{\Omega^c \setminus B_3(\Sigma_\varepsilon)} \left( \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2} F(u_\varepsilon) \right) \, dx \leq \delta^{-2} C,
\] (3.2a)
\[
\int_0^T \int_{\Omega^c \setminus B_3(\Sigma_\varepsilon)} |\partial_t u_\varepsilon|^2 \, dx dt \leq \delta^{-2} C.
\] (3.2b)

Proof. To prove (3.1a), we need to show that the first integral on the left-hand side of (2.49) is non-negative so that the Grönwall’s lemma can be applied. It follows from (2.49) and the assumption (1.16b) that
\[
\sup_{t \in [0,T]} \frac{1}{\varepsilon} E_\varepsilon[u_\varepsilon|\Sigma](t) + \frac{1}{\varepsilon^2} \int_0^T \int_{\Omega} \left( \varepsilon \partial_t u_\varepsilon - \partial F(u_\varepsilon)(\nabla \cdot \xi) \right)^2 \, dx dt \\
+ \frac{1}{\varepsilon^2} \int_0^T \int_{\Omega} \left( \varepsilon^2 |\partial_t u_\varepsilon|^2 - |H_\varepsilon|^2 + |H_\varepsilon - \varepsilon H \nabla u_\varepsilon|^2 \right) \, dx dt \\
\leq \frac{1}{\varepsilon} e^{(1+T)C(\Sigma)} E_\varepsilon[u_\varepsilon|\Sigma](0) \leq C_1 e^{(1+T)C(\Sigma)}.
\] (3.3)

Now we show the third term on the left-hand side of (3.3) has a non-negative integrand. By (1.7a) and (2.39b) we have $H_\varepsilon = -\varepsilon \partial_t u_\varepsilon \cdot \nabla u_\varepsilon|\Sigma_\varepsilon$. Using this formula, we can expand the integrand of the third term on the LHS of (3.3) and then apply the Cauchy-Schwarz inequality to obtain
\[
\varepsilon^2 |\partial_t u_\varepsilon|^2 - |H_\varepsilon|^2 + |H_\varepsilon - \varepsilon H \nabla u_\varepsilon|^2 \\
= \varepsilon^2 |\partial_t u_\varepsilon|^2 + \varepsilon^2 |H_\varepsilon|^2 |\nabla u_\varepsilon|^2 + 2 \varepsilon (H \cdot \nabla) u_\varepsilon \cdot \partial_t u_\varepsilon \\
\geq \varepsilon^2 |\partial_t u_\varepsilon| + (H \cdot \nabla) u_\varepsilon|^2.
\]
This together with (3.3) implies
\[ \int_0^T \int_\Omega |\partial_t u_\varepsilon + (H \cdot \nabla) u_\varepsilon|^2 \, dx \, dt \leq e^{(1+T)C(\Sigma)}. \] (3.4)

On the other hand, it follows from (2.21) and (2.40) that \( \Pi_{u_\varepsilon} \partial_t u_\varepsilon \parallel \partial d_F(u_\varepsilon) \). So we can decompose
\[ \varepsilon \partial_t u_\varepsilon - \partial d_F(u_\varepsilon)(\nabla \cdot \xi)^2 = \varepsilon \partial_t u_\varepsilon - \varepsilon \Pi_{u_\varepsilon} \partial_t u_\varepsilon + \varepsilon \Pi_{u_\varepsilon} \partial_t u_\varepsilon - \partial d_F(u_\varepsilon)(\nabla \cdot \xi)^2. \]

This together with (3.3) yields
\[ \frac{1}{\varepsilon^2} \int_0^T \int_\Omega \left| \varepsilon \partial_t u_\varepsilon - \varepsilon \Pi_{u_\varepsilon} \partial_t u_\varepsilon \right|^2 \leq e^{(1+T)C(\Sigma)}. \] (3.5)

The above estimate and (2.45b) imply (3.1b). Concerning (3.1c),
\[ \int_\Omega |\nabla \psi_\varepsilon| \, dx \leq \int_\Omega \left( \frac{\varepsilon}{2} |\Pi_{u_\varepsilon} \nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon} |\partial d_F(u_\varepsilon)|^2 \right) \, dx \]
\[ \leq \frac{2\varepsilon}{\varepsilon} A_\varepsilon(u_\varepsilon) \leq A_\varepsilon(u_\varepsilon(\cdot,0)). \] (3.6c)

The estimate (3.1d) follows from (2.45d). Finally, (3.2a) is a consequence of (2.45c) and when it is combined with (3.4) leads us to (3.2b). \( \square \)

We shall use (3.2) together with the method of Chen–Struwe [8] to show that the weak limits of \( u_\varepsilon \) are harmonic heat flows from the bulk regions \( \Omega^+_t \) to \( m_\pm \) respectively. However, the bulk potential \( F(u) \) (cf. (1.5)) depends on the relative distances to these two manifolds, and we must find a quantitative way to distinguish them. This is done in the following:

**Theorem 3.2.** Under the assumptions of Theorem 1.1, there exists \( C_2 > 0 \) independent of \( \varepsilon \) so that

\[ \sup_{t \in [0,T]} B[u_\varepsilon|\Sigma](t) \leq C_2 \varepsilon, \] (3.6a)
\[ \sup_{t \in [0,T]} \int_\Omega |\psi_\varepsilon - c_F 1_{\Omega^+_t}| \, dx \leq C_2 \varepsilon^{1/2}. \] (3.6b)

**Proof.** Within the proof, \( h^\pm \) will denote the positive/negative part of a scalar function \( h \). And we shall use the decomposition \( h = h^+ - h^- \). For \( h \in W^{1,1}(\Omega) \), we have the following formula (cf. [10], pp. 153):
\[ \partial_x_i(h(x))^+ = (\partial_x_i h(x)) 1_{\{x;h(x)>0\}}(x) \text{ for a.e. } x. \] (3.7)

The proof will be done in two steps.

**Step 1: Derivation of differential inequalities.** Let \( \chi(\cdot,t) = 1_{\Omega^+_t} - 1_{\Omega^-_t} \) and let \( \eta(\cdot) \) be the truncation of the identity map
\[ \eta(x) := \begin{cases} x & \text{when } x \in [-\delta_0, \delta_0], \\ \delta_0 & \text{when } x \geq \delta_0, \\ -\delta_0 & \text{when } x \leq -\delta_0, \end{cases} \] (3.8)
and \( \zeta := |\eta| \). It follows from (2.40) and the generalized chain rule (2.37) that
\[ \partial_t \psi_\varepsilon = (\partial_t u_\varepsilon + (H \cdot \nabla) u_\varepsilon) \cdot \partial d_F(u_\varepsilon) \cdot (\nabla \cdot \psi_\varepsilon) \] (3.9)

Motivated by the decomposition
\[ 2\psi_\varepsilon - c_F = 2(\psi_\varepsilon - c_F)^+ + c_F - 2(\psi_\varepsilon - c_F)^-, \] (3.10)
we shall establish differential inequalities of the following two energies which sum up to $B[u_\varepsilon|\Sigma](t)$ (cf. (1.14)):

\begin{align}
  g_\varepsilon(t) &:= \int (\psi_\varepsilon - c_F)^+ \zeta \circ d_\Sigma \, dx, \\
  h_\varepsilon(t) &:= \int (c_F \chi - c_F + 2(\psi_\varepsilon - c_F)^-) \eta \circ d_\Sigma \, dx. 
\end{align}

(3.11a, 3.11b)

Since $\psi_\varepsilon \geq 0$, we have $(\psi_\varepsilon - c_F)^- \in [0, c_F]$ and thus $c_F - 2(\psi_\varepsilon - c_F)^- \in [-c_F, c_F]$. In view of (3.8), we have $c_F \chi \eta \circ d_\Sigma \geq 0$, so the integrands of these two energies are both non-negative. Moreover, (1.15) implies that

\begin{equation}
  g_\varepsilon(0) + h_\varepsilon(0) \lesssim \varepsilon. 
\end{equation}

(3.12)

Now we proceed in the derivation of Grönwall’s inequalities of $g_\varepsilon$ and $h_\varepsilon$. Using (3.9)

\begin{align}
  g_\varepsilon'(t) &\leq \int_{\{\psi_\varepsilon > c_F\}} \left| \partial_t u_\varepsilon + (H \cdot \nabla) u_\varepsilon \right| \partial d_F(u_\varepsilon) \zeta(d_\Sigma) \\
  &\quad - \int_{\{\psi_\varepsilon > c_F\}} H \cdot \nabla \psi_\varepsilon \zeta(d_\Sigma) + \int (\psi_\varepsilon - c_F)^+ \partial_t \zeta(d_\Sigma) \\
  &\quad - \int_{\{\psi_\varepsilon > c_F\}} (\partial_t u_\varepsilon + (H \cdot \nabla) u_\varepsilon) \cdot \partial d_F(u_\varepsilon) \zeta(d_\Sigma) \\
  &\quad - \int H \cdot \nabla (\psi_\varepsilon - c_F)^+ \zeta(d_\Sigma) - \int (\psi_\varepsilon - c_F)^+ H \cdot \nabla \zeta(d_\Sigma) \\
  &\quad + \int (\partial_t \zeta(d_\Sigma) + H \cdot \nabla \zeta(d_\Sigma))(\psi_\varepsilon - c_F)^+ 
\end{align}

Finally by an integration by part, we can merge the second and the third integral in the last display:

\begin{align}
  g_\varepsilon'(t) &\leq \int_{\{\psi_\varepsilon > c_F\}} \left| \partial_t u_\varepsilon + (H \cdot \nabla) u_\varepsilon \right| \frac{\partial d_F(u_\varepsilon)}{\partial d_F(u_\varepsilon)} \sqrt{2F(u_\varepsilon)} \zeta(d_\Sigma) \\
  &\quad + \int (\text{div } H) (\psi_\varepsilon - c_F)^+ \zeta(d_\Sigma) + \int (\partial_t \zeta(d_\Sigma) + H \cdot \nabla \zeta(d_\Sigma))(\psi_\varepsilon - c_F)^+ \\
  &\leq \int \varepsilon \left| \partial_t u_\varepsilon + (H \cdot \nabla) u_\varepsilon \right|^2 + \int \frac{1}{\varepsilon} F(u_\varepsilon) \zeta^2(d_\Sigma) + C g_\varepsilon(t) \\
  &\leq \int \varepsilon \left| \partial_t u_\varepsilon + (H \cdot \nabla) u_\varepsilon \right|^2 + C E\varepsilon[u_\varepsilon|\Sigma] + C g_\varepsilon(t).
\end{align}

In view of (3.4), we can apply Grönwall’s lemma and obtain $g_\varepsilon(t) \lesssim C \varepsilon$. Similar calculation shows $h_\varepsilon'(t) \lesssim C h_\varepsilon(t)$. For simplicity we denote

\begin{equation}
  c_F \chi - c_F + 2(\psi_\varepsilon - c_F)^- =: w_\varepsilon.
\end{equation}

(3.13)

Using $\partial_t \chi \eta(d_\Sigma) \equiv 0$ (in the sense of distributions), we find

\begin{equation}
  \partial_t w_\varepsilon \eta(d_\Sigma) = 2 \partial_t \psi_\varepsilon 1_{\{\psi_\varepsilon < c_F\}} \eta(d_\Sigma) \quad \text{for a.e. } x.
\end{equation}

(3.14)
So by the same calculation for $g_\varepsilon$ we obtain
\[
\begin{align*}
\hat{h}'(t) & \leq \int_{\{\psi_\varepsilon < c_F\}} 2 \left| (\partial_t u_\varepsilon + (H \cdot \nabla) u_\varepsilon) \cdot \frac{\partial F(u_\varepsilon)}{\partial F(u_\varepsilon)} \sqrt{2F(u_\varepsilon)} \right| \zeta(d\Sigma) \\
& \quad + \int (\operatorname{div} H) w_\varepsilon \eta(d\Sigma) + \int \left( \partial_t \eta(d\Sigma) + H \cdot \nabla \eta(d\Sigma) \right) w_\varepsilon \\
& \leq \int \varepsilon \left| \partial_t u_\varepsilon + (H \cdot \nabla) u_\varepsilon \right|^2 + CE_\varepsilon |u_\varepsilon| \Sigma + C\hat{h}(t),
\end{align*}
\]
and $h_\varepsilon(t) \leq C\varepsilon$ follows from the Grönwall lemma, and we thus prove (3.6a). Finally,
\[
\int |2\psi_\varepsilon - c_F - c_F \chi| \zeta(d\Sigma) \leq \int 2(\psi_\varepsilon - c_F)^+ \zeta(d\Sigma) + \int |c_F - 2(\psi_\varepsilon - c_F)^- - c_F \chi| \zeta(d\Sigma) = 2g_\varepsilon + h_\varepsilon \leq C\varepsilon. \tag{3.14}
\]

**Step 2:** Pass to the unweighted inequality. We first note that (3.14) implies (3.6b) with $\Omega$ replaced by $\Omega \setminus B_{\delta_0}(\Sigma_t)$. So we shall focus on the estimate in $B_{\delta_0}(\Sigma_t)$. We shall use the following elementary estimate
\[
\left( \int_0^{\delta_0} |f(y)| \, dy \right)^2 \leq 2 \| f \|_\infty \int_0^{\delta_0} |f(y)| \, dy \quad \forall f \in L^\infty(0, \delta_0). \tag{3.15}
\]
Let $\chi_\varepsilon = \frac{2\psi_\varepsilon - c_F}{c_F}$. It follows from (1.20) that $\| \chi_\varepsilon \|_\infty$ is uniformly bounded. For each fixed $p \in \Sigma_t$ and $y \in [-\delta_0, \delta_0]$, we have $|d\Sigma|(p + y n(p), t) = |y|$. So we can apply the above inequality to estimate
\[
\begin{align*}
\left( \int_0^{\delta_0} |\chi(p + y n, t) - \chi_\varepsilon(p + y n, t)| \, dy \right)^2 & \leq 2 \left( 1 + \| \chi_\varepsilon \|_\infty \right) \int_0^{\delta_0} |\chi(p + y n, t) - \chi_\varepsilon(p + y n, t)| |d\Sigma|(p + y n, t) \, dy.
\end{align*}
\]
So by area formula,
\[
\begin{align*}
\left( \int_{B_{\delta_0}(\Sigma_t)} |\chi(x, t) - \chi_\varepsilon(x, t)| \, dx \right)^2 & = \left( \sum_{\pm} \int_{\Sigma_t} \int_0^{\delta_0} |\chi(p \pm y n, t) - \chi_\varepsilon(p \pm y n, t)| \, dy \, dS(p) \right)^2 \\
& \leq C \int_{\Sigma_t} \int_{-\delta_0}^{\delta_0} |\chi(p + y n, t) - \chi_\varepsilon(p + y n, t)| |d\Sigma|(p + y n, t) \, dy \, dS(p) \\
& = \int_{B_{\delta_0}(\Sigma_t)} |\chi(x, t) - \chi_\varepsilon(x, t)| \, |d\Sigma(x, t)| \, dx.
\end{align*}
\]
This implies the estimate in $B_{\delta_0}(\Sigma_t)$.

**Corollary 3.3.** There exists a sequence of $\varepsilon_k \downarrow 0$ and $u^\pm(x, t)$ so that
\[
u^\pm \in L^\infty(0, T; L^\infty(\Omega) \cap H^1_{\text{loc}}(\Omega_T^+; m_\pm)), \partial_t u^\pm \in L^2(0, T; L^2_{\text{loc}}(\Omega_T^+)), \tag{3.16}
\]
This together with (3.6b) and (2.19) yields that

\[ u = \mu \]

We deduce that

Moreover, by a diagonal argument we obtain (3.17a) and (3.17b), and also (3.17c) by the Aubin–Lions lemma.

Proof. It follows from (1.20) and (3.2) that, for any \[ \delta \in (0, \delta_0) \], there exists a subsequence \( \varepsilon_k = \varepsilon_k(\delta) > 0 \) such that

\[ u_{\varepsilon_k} \overset{k \to \infty}{\rightharpoonup} u^\pm \text{ weakly in } L^2(0, T; L^2(\Omega_0^{\pm})), \quad \text{(3.17a)} \]

\[ \nabla u_{\varepsilon_k} \overset{k \to \infty}{\rightharpoonup} \nabla u^\pm \text{ weakly in } L^\infty(0, T; L^\infty(\Omega_0^{\pm})), \quad \text{(3.17b)} \]

\[ u_{\varepsilon_k} \overset{k \to \infty}{\to} u^\pm \text{ strongly in } C([0, T]; L^2_{\text{loc}}(\Omega_0^{\pm})). \quad \text{(3.17c)} \]

and \( u^\pm = \bar{u}^\pm \) a.e. in \( U_\pm(\delta) := \cup_{t \in [0, T]} (\Omega_0^{\pm} \setminus B_\delta(\Sigma_t)) \times \{t\} \). By the arbitrariness of \( \delta \) we deduce

\[ u^\pm \in L^\infty(0, T; H^1_{\text{loc}}(\Omega_0^{\pm})) \text{ with } \partial_t u \in L^2(0, T; L^2_{\text{loc}}(\Omega_0^{\pm})). \quad \text{(3.19)} \]

Moreover, by a diagonal argument we obtain (3.17a) and (3.17b), and also (3.17c) by the Aubin–Lions lemma.

It remains to show that \( u^\pm \) are mappings into \( m_{\pm} \). Using (3.17c), (3.2a), and Fatou’s lemma, we deduce that \( F(u^\pm) = 0 \) a.e. in \( \Omega \). In view of (1.4) we deduce that the images of \( u^\pm \) lie in \( m = m_+ \cup m_- \). Owning to (3.17c) and (1.20),

\[ \psi_{\varepsilon_k} \overset{(1.12)}{=} d_F \circ u_{\varepsilon_k} \overset{k \to \infty}{\rightharpoonup} d_F \circ u^\pm \text{ strongly in } C([0, T]; L^2_{\text{loc}}(\Omega_0^{\pm})). \quad \text{(3.20)} \]

This together with (3.6b) and (2.19) yields that \( u^\pm \) maps into \( m_{\pm} \) respectively. Combining this with (3.19) yields (3.16).

Lemma 3.4. For any \( \delta \in (0, \delta_0) \), there exist \( b^\pm_\delta \in [\delta, 2\delta] \) s.t. the sets

\[ \{x : \psi_\varepsilon > c_F - b^+_\delta\} \text{ and } \{x : \psi_\varepsilon < b^-_\delta\} \quad \text{(3.21)} \]

have finite perimeters and

\[ \left| H^{d-1} \left( \{x : \psi_\varepsilon = c_F - b^+_\delta\} \right) - H^{d-1}(\Sigma_t) \right| \leq C \varepsilon^{1/2} \delta^{-1}, \quad \text{(3.22a)} \]

\[ \left| H^{d-1} \left( \{x : \psi_\varepsilon = b^-_\delta\} \right) - H^{d-1}(\Sigma_t) \right| \leq C \varepsilon^{1/2} \delta^{-1}. \quad \text{(3.22b)} \]

Proof. For any \( \delta < \delta_0 \ll c_F \), we denote (within the proof of the lemma)

\[ \Omega_t^{\varepsilon, \delta} = \{x \in \Omega : c_F - 2\delta < \psi_\varepsilon(x, t) < c_F - \delta\}. \quad \text{(3.23)} \]

We shall also denote the \( d \)-Lebesgue’s measure of a set \( A \) by \( |A| \).

Using (3.1d) and the co-area formula (cf. [10] section 5.5), we deduce that for almost every \( \delta \in (0, \delta_0) \),

\[ C \varepsilon \geq \int_{\Omega_t^{\varepsilon, \delta}} |\nabla \psi_\varepsilon| - \xi \cdot \nabla \psi_\varepsilon \, dx \geq 0 \]

\[ = \int_{c_F - 2\delta}^{c_F} H^{d-1}(\{x : \psi_\varepsilon = s\}) \, ds - \int_{\partial \Omega_t^{\varepsilon, \delta}} \xi \cdot \nu \psi_\varepsilon \, dH^{d-1} + \int_{\Omega_t^{\varepsilon, \delta}} (\text{div } \xi) \psi_\varepsilon \, dx. \]
where $\nu$ is the outward unit normal of the set under integration, defined on its (measure-theoretic) boundaries. Note that $\|\psi_\varepsilon\|_\infty$ is uniformly bounded due to (1.20). So we can estimate

$$\left| \int_{c_F-\delta}^{c_F-\delta} \mathcal{H}^{d-1}(\{x : \psi_\varepsilon = s\}) \, ds - \int_{\partial\Omega_t^{c_F-\delta}} \xi : \nu \psi_\varepsilon \, d\mathcal{H}^{d-1} \right|$$

$$\leq C(\varepsilon + \|\psi_\varepsilon\|_\infty|\Omega_t^{c_F-\delta}|) \leq C\varepsilon^{1/2}.$$

(3.24)

where we use the Chebyshev inequality and (3.6b) in the last step. On the other hand, applying the divergence theorem and adding zero,

$$\int_{\partial\Omega_t^{c_F-\delta}} \xi : \nu \psi_\varepsilon \, d\mathcal{H}^{d-1} = (c_F - 2\delta) \int_{\{\psi_\varepsilon > c_F - 2\delta\}} \text{div} \, \xi \, dx + (c_F - \delta) \int_{\{\psi_\varepsilon < c_F - \delta\}} \text{div} \, \xi \, dx,$$

$$= -\delta \mathcal{H}^{d-1}(\Sigma_t) = -(c_F - 2\delta) \int_{\Omega_t^+} \text{div} \, \xi \, dx - (c_F - \delta) \int_{\Omega_t^-} \text{div} \, \xi \, dx.$$

Adding the above two equations and substituting into (3.24), we obtain

$$\left| \int_{c_F-\delta}^{c_F-\delta} \mathcal{H}^{d-1}(\{x : \psi_\varepsilon = s\}) \, ds - \delta \mathcal{H}^{d-1}(\Sigma_t) \right|$$

$$\leq C \left( \varepsilon^{1/2} + \left| \Omega_t^+ \Delta \{x : \psi_\varepsilon > c_F - 2\delta\} \right| + \left| \Omega_t^- \Delta \{x : \psi_\varepsilon < c_F - \delta\} \right| \right),$$

(3.25)

where $A \Delta B = (A - B) \cup (B - A)$ is the symmetric difference of two sets $A, B$. We rewrite the last two terms by

$$r_\varepsilon^+ := \left| \Omega_t^+ \Delta \{x : \psi_\varepsilon > c_F - 2\delta\} \right|$$

$$= \left| \{x \in \Omega_t^+ : \psi_\varepsilon < c_F - 2\delta\} \right| + \left| \{x \in \Omega_t^- : \psi_\varepsilon > c_F - 2\delta\} \right|,$$

$$r_\varepsilon^- := \left| \Omega_t^- \Delta \{x : \psi_\varepsilon < c_F - \delta\} \right|$$

$$= \left| \{x \in \Omega_t^- : \psi_\varepsilon \geq c_F - \delta\} \right| + \left| \{x \in \Omega_t^+ : \psi_\varepsilon < c_F - \delta\} \right|.$$

Now using the Chebyshev inequality and (3.6b) we get $r_\varepsilon^- + r_\varepsilon^+ \leq C\varepsilon^{1/2}$. Substituting this estimate into (3.25) leads to

$$\left| \frac{1}{\delta} \int_{c_F-\delta}^{c_F-\delta} \mathcal{H}^{d-1}(\{x : \psi_\varepsilon = s\}) \, ds - \delta \mathcal{H}^{d-1}(\Sigma_t) \right| \leq C\varepsilon^{1/2} \delta^{-1}.$$

(3.26)

So the existence of $b_\varepsilon^+ \in [\delta, 2\delta]$ satisfying (3.22a) follows from Fubini’s theorem. The inequality (3.22b) can be done in the same way and we omit the proof. \qed

**Proposition 3.5.** Let $\varepsilon_k \downarrow 0$ be the sequence in Corollary 3.3. Then there exists $b_k^+ \in [\varepsilon_k^{1/6}, 2\varepsilon_k^{1/6}]$ so that the sets

$$\Omega_{t_k}^{k+} := \{x \in \Omega : \psi_{\varepsilon_k}(x, t) > c_F - b_k^+\},$$

(3.27a)

$$\Omega_{t_k}^{k-} := \{x \in \Omega : \psi_{\varepsilon_k}(x, t) < b_k^-\}$$

(3.27b)
have uniformly bounded perimeter. Moreover, up to the extraction of a subsequence, we have
\[ \left| \mathcal{H}^{d-1}(\partial\Omega_{t}^{k,+}) - \mathcal{H}^{d-1}(\Sigma_{t}) \right| \leq C \varepsilon_{k}^{1/3}, \quad (3.28a) \]
\[ 1_{\Omega_{t}^{k,+}} \xrightarrow{k \to \infty} 1_{\Omega_{t}^{+}} \text{ weakly-star in } BV(\Omega), \quad (3.28b) \]
\[ \partial\Omega_{t}^{k,+} \xrightarrow{k \to \infty} \Sigma_{t} \quad \text{under Hausdorff metric}. \quad (3.28c) \]

Finally there exists \( K_{1} \in \mathbb{N}^{+} \) so that for any \( k > K_{1} \), the solution \( u_{\varepsilon_{k}} \) satisfies
\[ u_{\varepsilon_{k}}(\Omega_{t}^{k,+}) \subset B_{\delta_{0}}(m_{\pm}), \quad (3.29a) \]
\[ \sup_{t \in [0,T]} \int_{\Omega_{t}^{k,+}} \left| \nabla P_{m}(u_{\varepsilon_{k}}) \right|^{2} dx + \int_{0}^{T} \int_{\Omega_{t}^{k,+}} \left| \partial_{t} P_{m}(u_{\varepsilon_{k}}) \right|^{2} dx dt \leq C. \quad (3.29b) \]

Proof. Choosing \( \delta = \delta_{k} := \varepsilon_{k}^{1/6} \) in Lemma 3.3 yields \( b_{k}^{+} \in [\varepsilon_{k}^{1/6}, 2\varepsilon_{k}^{1/6}] \) so that
\[ \left| \mathcal{H}^{d-1} \left( \{ x : \psi_{\varepsilon} = c_{F} - b_{k}^{+} \} \right) - \mathcal{H}^{d-1}(\Sigma_{t}) \right| \leq \varepsilon_{k}^{1/3}. \quad (3.30) \]

This leads to the ‘plus’ case of (3.28a) and the ‘minus’ case can be done in the same way. By (3.6b),
\[ 1_{\{ x : \psi_{\varepsilon} > c_{F} - b_{k}^{+} \}} \xrightarrow{\varepsilon \to 0} 1_{\Omega_{t}^{+}} \text{ strongly in } L^{1}(\Omega), \quad \text{for each fixed } k. \quad (3.31) \]

By a diagonal argument, we find a subsequence of \( \varepsilon_{k} \) (without relabeling) so that
\[ 1_{\Omega_{t}^{k,+}} \xrightarrow{k \to \infty} 1_{\Omega_{t}^{+}} \text{ strongly in } L^{1}(\Omega). \quad (3.32) \]

This combined with (3.28a) implies the ‘plus’ case of (3.28b). The ‘minus’ cases can be done in a similar way. The convergence (3.28c) is a consequence of the Blaschke's Theorem (cf. [5 Chapter 7]). By (2.17), (2.19) and \( b_{k}^{+} \to 0 \), there exists \( K_{1} > 0 \) so that for any \( k \geq K_{1} \) there holds
\[ d_{F} \circ u_{\varepsilon_{k}} > c_{F} - b_{k}^{+} \text{ implies } u_{\varepsilon_{k}} \in B_{\delta_{0}}(m_{+}), \]
\[ d_{F} \circ u_{\varepsilon_{k}} < b_{k}^{+} \text{ implies } u_{\varepsilon_{k}} \in B_{\delta_{0}}(m_{-}), \]

and thus (3.29a) holds.

It remains to use (3.29a) to derive (3.29b). So we shall always assume \( k \geq K_{1} \). If we denote the co-dimension of \( m \) to be \( \ell \in \mathbb{N}^{+} \), then as the nearest point projection \( P_{m} \) is smooth in \( B_{\delta_{0}}(m) \), any vector \( u \in B_{\delta_{0}}(m) \) can be written as
\[ u = P_{m}u + d_{m}(u)\nu_{m}(u) = P_{m}u + \sum_{j=1}^{\ell} d_{j}(u)\nu_{j}(P_{m}u) \quad (3.33) \]

where \( \{\nu_{j}\}_{j=1}^{\ell} \) is an orthonormal frame of the normal space at \( P_{m}u \) and \( \{d_{j}(u)\}_{j=1}^{\ell} \) is the coordinate of \( u_{\varepsilon_{k}} - P_{m}u_{\varepsilon_{k}} \) in such a frame. So we have
\[ d_{m}^{2}(u) = \sum_{j=1}^{\ell} d_{j}^{2}(u) \quad \text{and} \quad \partial d_{j}(u) = \nu_{j}(P_{m}u), \quad \forall u \in B_{\delta_{0}}(m). \quad (3.34) \]

Note that \( \{d_{j}\}_{j=1}^{\ell} \subset C^{1}(B_{\delta_{0}}(m)) \) and \( d_{m}(u) \in C^{1}(B_{\delta_{0}}(m) \setminus m) \), and in general \( d_{m} \) is not differentiable on \( m \). On the (open) set \( \{ x | u_{\varepsilon_{k}}(x,t) \in B_{\delta_{0}}(m) \} \), we can differentiate (3.33) and get
\[ \partial_{x_{i}} u_{\varepsilon_{k}} = \partial_{x_{i}}(P_{m}u_{\varepsilon_{k}}) + \sum_{j=1}^{\ell} d_{j}(u_{\varepsilon_{k}})\partial_{x_{j}}\nu_{j}(P_{m}u_{\varepsilon_{k}}) + \sum_{j=1}^{\ell} \partial_{x_{i}}d_{j}(u_{\varepsilon_{k}})\nu_{j}(P_{m}u_{\varepsilon_{k}}) \quad (3.35) \]
Theorem 3.6. The mappings

\[ u^\pm : \bigcup_{t \in [0, T]} \Omega_t^\pm \times \{t\} \mapsto m_\pm \]  

(3.38)

obtained in Corollary 3.3 are weak solutions of harmonic heat flows that satisfy

\[ u^\pm \in L^2(0,T;H^1(\Omega_T^\pm; m_\pm)) \]  

(3.39)

Additionally, moreover, with the notations in Proposition 3.5, the functions

\[ v_k(\cdot, t) := \sum_{\pm} P_{m_\pm} \circ u_{\epsilon_k}(\cdot, t) \mathbf{1}_{\Omega_t^\pm} \]  

(3.40)
satisfy the following properties for a.e. \( t \in [0,T] \):

\[
v_k(\cdot,t) \xrightarrow{k \to \infty} u = \sum_{\pm} u_{\pm}^k(\cdot,t) \ 1_{\Omega_{\pm}} \text{ weakly-star in } BV(\Omega),
\]

\[
\nabla^a v_k \xrightarrow{k \to \infty} 1_{\Omega_{\pm}}^a \nabla u_{\pm} \text{ weakly in } L^1(\Omega),
\]

\[
\sum_{\pm} \int_{\Omega_{\pm}} |\nabla u_{\pm}(\cdot,t)|^2 \, dx \leq \liminf_{k \to \infty} \sum_{\pm} \int_{\Omega_{\pm}^k} |\nabla^a v_k(\cdot,t)|^2 \, dx.
\]

Here in (3.41c) \( \nabla^a v_k \) is the absolute continuous part of the distributional gradient \( \nabla \).

**Proof.** The sequence \( v_k(\cdot,t) \) is bounded in \( L^\infty(\Omega) \), and by (3.29b) we deduce that their distributional derivatives have no Cantor parts. Moreover, the absolute continuous parts and the jump sets enjoy the estimates (3.29b) and (3.28a) respectively. So it follows from Proposition 2.3 that \( \{v_k(\cdot,t)\} \) is compact in \( SBV(\Omega) \): there exists \( v \in SBV(\Omega) \) so that \( v_k \to v \) weakly-star in \( BV(\Omega) \) as \( k \to \infty \), and the absolute continuous part of the gradient

\[
\nabla^a v_k = \sum_{\pm} \nabla P_{m_{\pm}}(u_{\pm}^k) 1_{\Omega_{\pm}^k} \xrightarrow{k \to \infty} \nabla^a v \text{ weakly in } L^1(\Omega).
\]

To identify \( v \), we combine (3.28b) with (3.17c) and deduce that \( v = \sum_{\pm} 1_{\Omega_{\pm}^\infty} u_{\pm} \) a.e. in \( \Omega \), and thus (3.41a) is proved. The lower semicontinuity of \( SBV \) functions (cf. (2.22)) implies (3.41c), and thus we can improve the spatial regularity in (3.16) to (3.39). Finally combining (3.2), (3.17) with Chen–Struwe [8] imply that, \( u_{\pm} \) are weak solutions to harmonic heat flows respectively.\( \square \)

## 4. Proof of Theorem 1.1

We first recall that the estimate (1.16a) is proved in Lemma 3.1 (cf. (3.1a)). The estimates (1.16b) and (1.16c) are obtained in Theorem 3.2. The convergence (1.17) is obtained in Corollary 3.3. The limit \( u_{\pm} \) being harmonic heat flow with regularity (1.18) has been done in Theorem 3.6. It remains to prove the minimal pair boundary conditions (1.19). To this end, we introduce the semi-distance function

\[
d_F^*(v_+,v_-) = \inf_{\xi \in C^1((a_-,a_+),\mathbb{R}^n)} \int_{a_-}^{a_+} |\xi'(t)| \sqrt{2F(\xi(t))} \, dt.
\]

for any \(-\infty \leq a_- < a_+ \leq \infty \). Note that the above definition is independent of the choice of \( a_\pm \) (cf. [25]). Such a function can be used to define a semi-distance between closed sets \( S_{\pm} \subset \mathbb{R}^n \):

\[
d_F^*(S_+,S_-) = \inf_{v_+ \in S_+} d_F^*(v_+,v_-).
\]

Let \( P_{m_{\pm}} \) be the nearest point projection to \( m_{\pm} \). For any subset \( S_{\pm} \subset m_{\pm} \), we define

\[
\mathcal{N}(S_{\pm},\rho) := \bigcup_{u_{\pm} \in S_{\pm}} \mathcal{N}(u_{\pm},\rho),
\]

where \( \mathcal{N}(u_{\pm},\rho) \) is the ‘normal sphere’ of radius \( \rho \) centered at \( u_{\pm} \in m_{\pm} \) under the metric \( d_F^* \):

\[
\mathcal{N}(u_{\pm},\rho) := \{ u \in \mathbb{R}^n : d_F^*(u,u_{\pm}) = \rho, P_{m_{\pm}} u = u_{\pm} \}.
\]

Inspired by [38], we introduce a function \( \kappa : m_+ \times m_- \times [0,1] \to \mathbb{R} \) by

\[
\kappa(u_+,u_-,\rho) := d_F^* \left( \mathcal{N}(\bar{B}_{\rho}(u_+) \cap m_+,\delta_0), \mathcal{N}(\bar{B}_{\rho}(u_-) \cap m_-,\delta_0) \right) + 2\delta_0 - c_F,
\]
where \( c_F = 2 \int_0^{\frac{\text{dist}_m}{2}} \sqrt{2f(X^2)}d\lambda \) (cf. (1.13)). It is obvious that \( \kappa \) is continuous w.r.t all its variables and non-increasing w.r.t \( \rho \). Note that \( \mathcal{N}(\bar{B}_\rho(u_+)) \cap m_+ \) can be visualized as a tube of thickness \( \delta_0 \) centered at \( B_\rho(u_+) \cap m_+ \).

**Lemma 4.1.** \( \kappa(u_+, u_-, \rho) \) is non-negative. Moreover, \( \kappa(u_+, u_-, 0) = 0 \) if and only if \( (u_+, u_-) \in m_+ \times m_- \) is a minimal pair.

**Proof.** It is easy to check that \( d_F^*(v_+, v_-) \) is continuous, and it vanishes if and only if either \( v_+ = v_- \) or \( v_\pm \) both lie in one of \( m_\pm \). So if we identify the sets \( m_\pm \) as two points, the resulting quotient space \( \mathbb{R}^n/m_\pm, d_F^* \) is a metric space.

For any \( v_\pm \in \mathcal{N}(\bar{B}_\rho(u_\pm)) \cap m_\pm, \delta_0 \), by triangle inequality

\[
d_F^*(v_+, v_-) + 2\delta_0 = d_F^*(v_+, v_-) + \sum_\pm d_F^*(v_\pm, P_m v_\pm) \geq d_F^*(P_m v_+, P_m v_-) = c_F.
\]

Minimizing \( v_\pm \) implies that \( \kappa(u_+, u_-, \rho) \) is non-negative.

If \( (u_+, u_-) \) is a minimal pair, then the line segment \( \overline{u_- u_+} \) meets \( m_\pm \) perpendicularly. Let \( v_\pm = \overline{u_- u_+} \cap \mathcal{N}(u_\pm, \delta_0) \). Then

\[
\kappa(u_+, u_-, 0) = d_F^*(v_+, v_-) + \sum_\pm d_F^*(v_\pm, u_\pm) - c_F = 0.
\]

Now we assume \( (u_+, u_-) \) is NOT a minimal pair, i.e. \( |u_+ - u_-| > \text{dist}_m \). We claim that

\[
\mathcal{N}(u_+, \frac{1}{2}c_F) \cap \mathcal{N}(u_-, \frac{1}{2}c_F) = \emptyset.
\] (4.6)

If \( \mathbf{4.6} \) were wrong, there would exist \( u \in \cap_\pm \mathcal{N}(u_\pm, \frac{1}{2}c_F) \). It follows from \( \mathbf{4.3} \) that \( \frac{1}{2}c_F = d_F^*(u, u_\pm) \) and \( P_m u = u_\pm \). For any curve \( \xi_\pm : [0, 1] \to \mathbb{R}^n \) with \( \xi_\pm(0) = u \) and \( \xi_\pm(1) = u_\pm \), by the co-area formula, we have

\[
\int_0^1 \sqrt{2F'(\xi(t))|\xi'(t)|} \, dt \geq \int_0^1 \frac{1}{2}f(d_m^2(\xi(t))) \frac{d}{dt} d_m(\xi(t)) \, dt = \int d_m(u) \sqrt{2f(\lambda^2)}d\lambda = \int_0^{\frac{|u-u_\pm|}{2}} \sqrt{2f(\lambda^2)}d\lambda.
\]

Note that the last step is due to \( P_m u = u_\pm \). Taking the infimum among all such curves we find

\[
\int_0^{\frac{|u-u_\pm|}{2}} \sqrt{2f(\lambda^2)}d\lambda \leq d_F^*(u, u_\pm) = \frac{1}{2}c_F = \int_0^{\text{dist}_m} \sqrt{2f(\lambda^2)}d\lambda.
\]

This implies that \( |u_\pm - u| \leq \frac{1}{2}\text{dist}_m \), and thus \( |u_+ - u_-| \leq \text{dist}_m \) by triangle inequality. This leads to a contradiction, and the claim \( \mathbf{4.6} \) is proved.

Using \( \mathbf{4.6} \) and the continuity of \( d_F^* \), we deduce

\[
d_F^*(\mathcal{N}(u_+, \frac{1}{2}c_F), \mathcal{N}(u_-, \frac{1}{2}c_F)) =: \beta > 0.
\]
This combined with the triangle inequality yields
\[
\begin{align*}
\text{d}_F^* (N(u_+, \delta_0), N(u_-, \delta_0)) \\
\leq \text{d}_F^* (N(u_+, \delta_0), N(u_+, \frac{1}{2}c_F)) \\
+ \text{d}_F^* (N(u_+, \frac{1}{2}c_F), N(u_-, \frac{1}{2}c_F)) \\
+ \text{d}_F^* (N(u_-, \frac{1}{2}c_F), N(u_-, \delta_0)) \\
= c_F - 2\delta_0 + \beta.
\end{align*}
\]
This implies \(\kappa(u_+, u_-, 0) = \beta > 0\).

**Lemma 4.2.** There exist constants \(C_0 = C_0(m)\) and \(\rho_0 = \rho_0(m)\) which only depend on the geometry of \(m\) so that the following holds:

for any curve \(\gamma \in H^1([-\delta, \delta], \mathbb{R}^n)\) with \(\gamma(\pm \delta) \in B_{\delta_0}(m_{\pm})\),

and any \(\rho \in (0, \rho_0)\), there holds
\[
\int_{-\delta}^{\delta} \frac{1}{2} |\gamma'|^2 + \frac{1}{\epsilon^2} F(\gamma) - \frac{1}{\epsilon} (\text{d}_F \circ \gamma)' \geq \min \left\{ \frac{C_0 \rho^2, \kappa(P_m \gamma(\delta)), P_m \gamma(-\delta)}{\max\{\epsilon, \delta\}} \right\}. \tag{4.8}
\]

**Proof.** For any curve \(\gamma\) satisfying (4.7), we define its first exit time of \(B_{\delta_0}(m_-)\) and last entrance time of \(B_{\delta_0}(m_+)\) respectively by
\[
t_- = \inf \{ t \in (-\delta, \delta) : \gamma(s) \in B_{\delta_0}(m_-) \text{ for } s \in (-\delta, t) \},
\]
\[
t_+ = \sup \{ t \in (-\delta, \delta) : \gamma(s) \in B_{\delta_0}(m_+) \text{ for } s \in (t, \delta) \}.
\]
We shall estimate three integrals
\[
I_- + I_0 + I_+ := \left( \int_{-\delta}^{t_-} + \int_{t_-}^{t_+} + \int_{t_+}^{\delta} \right) \left( \frac{1}{2} |\gamma'|^2 + \frac{1}{\epsilon^2} F(\gamma) - \frac{1}{\epsilon} (\text{d}_F \circ \gamma)' \right). \tag{4.9}
\]
For \(s \in (-\delta, t_-) \cup (t_+, \delta)\), we have \(\gamma(s) \in B_{\delta_0}(m)\). So we can define the normal projection of \(\gamma'\) by \(\Pi_{\gamma} \gamma' = \gamma' \cdot \nu_m(\gamma) \nu_m(\gamma)\). Recall that \(\nu_m(\gamma) \parallel \partial d_F(\gamma)\) (cf. (2.16) and (2.21)). By a similar calculation as (3.37), we obtain
\[
|\gamma' - \Pi_{\gamma} \gamma'|^2 \geq (1 - C\delta_0)(P_m \gamma)'|^2 \quad \text{on } (-\delta, t_-) \cup (t_+, \delta). \tag{4.10}
\]
Here \(C\) is a constant depending on the geometry of \(m\). Choosing \(\delta_0 \ll 1\) so that \(1 - C\delta_0 : = 2c_0 > 0\), we find
\[
I_+ \geq \frac{1}{2} \int_{t_+}^{\delta} \left( |\gamma'|^2 + \frac{1}{\epsilon^2} |\partial d_F(\gamma)|^2 - \frac{2}{\epsilon} \partial d_F(\gamma) \cdot \gamma' \right)
\]
\[
\geq \frac{1}{2} \int_{t_+}^{\delta} |\gamma' - \Pi_{\gamma} \gamma'|^2 + |\Pi_{\gamma} \gamma' - \frac{4}{\epsilon} \partial d_F(\gamma)|^2 \tag{4.10}
\]
\[
\geq c_0 \int_{t_+}^{\delta} |(P_m \gamma)'|^2.
\]
The same calculation of \(I_-\) leads to \(I_- \geq \frac{c_0}{\delta - t_+} |P_m \gamma(t_-) - P_m \gamma(-\delta)|^2\).

For \(\rho \in (0, \rho_0)\) with \(\rho_0 = \rho_0(m)\) being sufficiently small, we have at least one of the following three cases:
(1) If \( \gamma(t_+) \notin N\left(B_\rho(P_m \gamma(\delta)) \cap m_+\right) \), then \( |P_m \gamma(t_+) - P_m \gamma(\delta)| \geq C_1 \rho \) for some \( C_1 \) that only depends on the geometry of \( m \). Thus \( I_+ \geq \frac{C_\rho^2}{\delta - t_+} \).

(2) If \( \gamma(t_-) \notin N\left(B_\rho(P_m \gamma(\delta)) \cap m_-\right) \), then \( I_- \geq \frac{C_\rho^2}{\delta + t_-} \).

(3) If neither of the above cases happen, i.e. \( \gamma(t_\pm) \in N\left(B_\rho(P_m \gamma(\pm \delta)) \cap m_\pm\right) \), then it follows from Cauchy-Schwarz's inequality, (4.1) and (2.17) that
\[
I_0 = \frac{1}{\epsilon} \int_{t_-}^{t_+} \left| \gamma' \right| \sqrt{2F(\gamma)} \, d\tau - \frac{d_F \circ \gamma(t_+) - d_F \circ \gamma(t_-)}{\epsilon} \\
\geq \frac{1}{\epsilon} \left( d_F^*(\gamma(t_+), \gamma(t_-)) - c_F - 2\delta_0 \right) \\
\geq \frac{1}{\epsilon} \left( N\left( B_\rho(P_m \gamma(\delta)) \cap m_+, \delta_0 \right), N\left( B_\rho(P_m \gamma(\delta)) \cap m_-, \delta_0 \right) \right) - \frac{c_F - 2\delta_0}{\epsilon} \\
\geq \frac{1}{\epsilon} \kappa\left( P_m \gamma(\delta), P_m \gamma(-\delta), \rho \right).
\]

Proof of (1.19). We shall argue for every \( t \in [0, T] \) without mentioning each time in the sequel. We first use (3.17c) to deduce strong convergence of \( u_{\delta_k} \) on almost every slices. More precisely, there exists a null set \( N \subset [0, \delta_0] \), namely \( L^1(N) = 0 \), so that
\[
u_{\delta_k}(p \pm \delta \mathbf{n}(p)) \overset{k \to \infty}{\to} \nu^{\pm}(p \pm \delta \mathbf{n}(p)) \text{ strongly in } L^2(\Sigma_t) \quad \forall \delta \notin N. \tag{4.11}
\]
Here \( \mathbf{n}(p) \) is the normal vector of \( p \in \Sigma_t \). Note that the limit in (4.11) makes sense due to (3.39) and Sobolev’s trace theorem. Moreover,
\[
u^{\pm}(p \pm \delta \mathbf{n}(p)) \to \nu^{\pm}(p) \text{ strongly in } L^2(\Sigma_t) \quad \forall \delta \downarrow 0, \delta \notin N. \tag{4.12}
\]
Combining (4.11) with (4.12) and a diagonal argument, we find a sequence \( \delta_k \downarrow 0 \) so that
\[
u_{\delta_k}(p \pm \delta_k \mathbf{n}(p)) \overset{k \to \infty}{\to} \nu^{\pm}(p) \text{ strongly in } L^2(\Sigma_t). \tag{4.13}
\]
It follows from (5.1a), (2.26) and the orthogonal decomposition \( \nabla = \nabla_\Sigma + \mathbf{n} \partial_\mathbf{n} \) that
\[
\sup_{t \in [0, T]} \int_{B_{\delta_0}(\Sigma_t)} \left( \frac{\epsilon}{2} \left| \nabla_{\Sigma} u_{\varepsilon_k} \right|^2 + \frac{\epsilon}{2} |\partial_\mathbf{n} u_{\varepsilon_k}|^2 + \frac{1}{\epsilon} F(u_{\varepsilon_k}) - \xi : \mathbf{n} \partial_\mathbf{n} \psi_{\varepsilon_k} \right) \, dx \leq C\varepsilon.
\]
Owing to (2.27) and (2.45c), we have
\[
\sup_{t \in [0, T]} \int_{B_{\delta_0}(\Sigma_t)} \left( \frac{1}{2} \left| \partial_\mathbf{n} u_{\varepsilon_k} \right|^2 + \frac{1}{\epsilon} F(u_{\varepsilon_k}) - \frac{1}{\epsilon} \partial_\mathbf{n} \psi_{\varepsilon_k} \right) \, dx \\
\leq C \varepsilon^{-1} \sup_{t \in [0, T]} \int_{B_{\delta_0}(\Sigma_t)} \left( 1 - \phi \left( \frac{d_{\Sigma}}{\delta_0} \right) \right) |\nabla \psi_{\varepsilon_k}| \, dx \\
\leq C + \varepsilon^{-1} \sup_{t \in [0, T]} \int_{B_{\delta_0}(\Sigma_t)} \min(d_{\Sigma}^2, 1) |\nabla \psi_{\varepsilon_k}| \, dx \\
\leq C \left( 1 + \varepsilon^{-1} \sup_{t \in [0, T]} E_{\varepsilon}[u_{\varepsilon_k} | \Sigma] \right).
If we write \( u_{\varepsilon_k}(x,t) = u_{\varepsilon_k}(p + \tau n(p),t) \), then by the area formula, we deduce that

\[
\sup_{t \in [0,T]} \int_{\Sigma_t} \int_{-\delta_k}^{\delta_k} \left( \frac{1}{2} |\partial_T u_{\varepsilon_k}|^2 + \frac{1}{\varepsilon^2} F(u_{\varepsilon_k}) - \frac{1}{\varepsilon} \partial_T (d_F \circ u_{\varepsilon_k}) \right) \, dt \, dS(p) \leq C. \tag{4.14}
\]

For each fixed \( p \in \Sigma_t \) and \( k \), we consider the curves

\[
\gamma_k(\tau; p) := u_{\varepsilon_k}(p + \tau n(p)) : [-\delta_k, \delta_k] \rightarrow \mathbb{R}^n. \tag{4.15}
\]

Assume that there exists \( t \in [0,T] \) and a compact subset \( E_t^* \subset \Sigma_t \) with \( \mathcal{H}^{d-1}(E_t^*) \geq \alpha \) for some \( \alpha \in (0,1/2) \) s.t. the pairs \((u^+(p), u^-(p))_{p \in E_t^*} \subset \mathbb{R}^+ \times \mathbb{R}^-\), obtained by taking the one-sided trace of (3.39), are not minimal pairs. Then it follows from Lemma 4.1 that

\[
\kappa(u^+(p), u^-(p), 0) > 0, \quad \forall p \in E_t^*. \tag{4.16}
\]

By (4.13) and Egorov’s theorem, there exists a compact subset \( E_t \subset E_t^* \) with \( \mathcal{H}^{d-1}(E_t) \geq 0.9 \mathcal{H}^{d-1}(E_t^*) \geq \alpha \) s.t.

\[
\gamma_k(\pm \delta_k; p) = u_{\varepsilon_k}(p \pm \delta_k n(p)) \xrightarrow{k \to \infty} u^\pm(p) \text{ uniformly on } E_t. \tag{4.17}
\]

As a result, \( u^\pm(\cdot) \) are continuous on \( E_t \) because \( u_{\varepsilon_k}(p \pm \delta_k n(p)) \) are continuous on \( \Sigma_t \). As a result, there exists \( K > 0 \) so that

\[
\gamma_k(\pm \delta_k; p) = u_{\varepsilon_k}(p \pm \delta_k n(p)) \subset B_{\delta_0}(m_{\pm}) \quad \forall k \geq K, p \in E_t. \tag{4.18}
\]

The continuity of \( u^\pm(\cdot) \) and (4.15) imply that

\[
\kappa(u^+(\cdot), u^-(\cdot), \cdot) : E_t \times [0,1] \rightarrow [0,\infty) \text{ is continuous.}
\]

So we deduce from (4.16) that there exists \( \rho \in (0,\rho_0) \) s.t.

\[
\inf_{p \in E_t} \kappa(u^+(p), u^-(p), \rho) =: 2\beta > 0. \tag{4.19}
\]

Note that \( \beta \) is independent of \( k \). Owning to (4.17), the continuities of \( \kappa \) and that of the projection \( P_m \), there exists \( K > 0 \) s.t.

\[
\inf_{p \in E_t} \kappa(P_m \gamma_k(\delta_k; p), P_m \gamma_k(-\delta_k; p), \rho) \geq \beta, \quad \forall k \geq K. \tag{4.20}
\]

By (4.18), the curves \( \gamma_k(\cdot; p) \) satisfy the condition (4.17) so that Lemma 4.2 applies to the 1D integral \( \Theta_k \) defined in (4.14):

\[
\Theta_k(p, t) \geq \frac{1}{\max\{\varepsilon_k, \delta_k\}} \min\{C_0 \rho^2, \kappa(P_m \gamma_k(\delta_k; p), P_m \gamma_k(-\delta_k; p), \rho)\} \\
\geq \frac{\min\{C_0 \rho^2, \beta\}}{\max\{\varepsilon_k, \delta_k\}}, \quad \forall p \in E_t, k \geq K.
\]

However, this would contradict (4.14) and we thus finish the proof of (4.19). \qed

5. PROOF OF THEOREM 1.2: CONSTRUCTION OF INITIAL DATA

We shall first modify and extend \( u^0 \) in the transitional region \( B_{2\delta}(\Sigma_0) \) so that we can glue them into a new mapping that fulfills the desired properties in Theorem 1.2. To this aim, let \( \Psi^\pm_\delta : \Omega^0_\delta \cup \tilde{B}_\delta(\Sigma_0) \rightarrow \Omega^\pm_\delta \) be global \( C^1 \) diffeomorphisms up to the boundaries which result from gluing the identity mapping in \( \Omega^0_\delta \setminus B_{2\delta}(\Sigma_0) \) and the projection mapping \( P_{\Sigma_0} \) in \( \tilde{B}_\delta(\Sigma_0) \):

\[
\Psi^\pm_\delta(x) = \begin{cases} 
  P_{\Sigma_0}(x) & \text{in } \tilde{B}_\delta(\Sigma_0), \\
  x & \text{in } \Omega^\pm_\delta \setminus B_{2\delta}(\Sigma_0). 
\end{cases} \tag{5.1}
\]
We extend $u^{in}_{\pm}$ by defining
$$u^{\pm}_{0} := u^{in}_{\pm} \circ \Psi_{\delta}^{\pm} \in H^{1}(\Omega_{0}^{\pm} \cup \bar{B}_{\delta}(\Sigma_{0}), m_{\pm}).$$
(5.2)

This combined with (1.23) implies that
$$(u^{+}_{0}(x), u^{-}_{0}(x))|_{x \in \bar{B}_{\delta}(\Sigma_{0})}$$ are minimal pairs.
(5.3)

We shall construct $u^{in}_{\pm}$ by gluing $u^{\pm}_{0}$. To this end, we define a cut-off function
$$\eta_{\delta} \in C_{c}^{\infty}(B_{\delta}(\Sigma_{0}), [0, 1])$$ with $\eta_{\delta} = 1$ in $B_{\delta/2}(\Sigma_{0})$.
(5.4)

Recall the optimal profile $\alpha$ (2.9). We define
$$S_{\delta}(x) = \eta_{\delta}(x)\alpha \left( \frac{d(\delta, x)}{\delta} \right) + (1 - \eta_{\delta}(x))\frac{\text{dist}_{\Omega}}{2} \left( 1_{\Omega_{0}^{+}}(x) - 1_{\Omega_{0}^{-}}(x) \right).$$
(5.5)

It is easy to verify that $S_{\delta}$ is a smooth function (as the discontinuity caused by $1_{\Omega_{0}^{\pm}}$ is cut off by $\eta_{\delta}$). We write
$$S_{\delta}(x) = \alpha \left( \frac{d(x, 0)}{\delta} \right) - \hat{S}_{\delta}(x),$$
(5.6)

where $\hat{S}_{\delta}$ is the tail term
$$\hat{S}_{\delta}(x) = (1 - \eta_{\delta}(x)) \left[ \alpha \left( \frac{d(x, 0)}{\delta} \right) - \frac{\text{dist}_{\Omega}}{2} \left( 1_{\Omega_{0}^{+}}(x) - 1_{\Omega_{0}^{-}}(x) \right) \right].$$
(5.7)

By Rademacher’s theorem, $d_{\Sigma}(x, 0)$ is Lipschitz continuous in $\Omega$, and $|\nabla d_{\Sigma}(x, 0)| \leq 1$ a.e. in $\Omega$. This combined with (2.9a) yields
$$||\hat{S}_{\delta}||_{L^{\infty}(\Omega)} + ||\nabla \hat{S}_{\delta}||_{L^{\infty}(\Omega)} \leq C e^{-C/\delta}.$$ (5.8)

and thus
$$S_{\delta}(x) = \alpha \left( \frac{d(x, 0)}{\delta} \right) + O(e^{-C/\delta}) \text{ in } \Omega,$$
(5.9a)
$$\nabla S_{\delta}(x) = \nabla d_{\Sigma}(x, 0)\alpha' + O(e^{-C/\delta}) \quad \text{a.e. in } \Omega,$$
(5.9b)
$$S_{\delta}(x) \xrightarrow{\varepsilon \to 0} \frac{\text{dist}_{\Omega}}{2} \left( 1_{\Omega_{0}^{+}}(x) - 1_{\Omega_{0}^{-}}(x) \right) \quad \text{a.e. in } \Omega.$$ 
(5.9c)

Using (5.2) and (5.5), we define $u^{in}_{\pm}$ by
$$u^{in}_{\pm}(x) = \frac{u^{+}_{0}(x) + u^{-}_{0}(x)}{2} + S_{\varepsilon}(x)\frac{u^{+}_{0}(x) - u^{-}_{0}(x)}{\text{dist}_{\Omega_{0}}}. $$
(5.10)

We claim (1.24a) holds. Indeed in the domains $\Omega_{0}^{\pm} \backslash B_{2\delta}(\Sigma_{0})$, we have $\eta_{\delta} = 0$ and thus
$$u^{in}_{\pm} \equiv \frac{u^{+}_{0} + u^{-}_{0}}{2} + \frac{\text{dist}_{\Omega}}{2}(1_{\Omega_{0}^{+}} - 1_{\Omega_{0}^{-}})\frac{u^{+}_{0} - u^{-}_{0}}{\text{dist}_{\Omega_{0}}} = \sum_{\pm} u^{\pm}_{0} 1_{\Omega_{0}^{\pm}} \text{ in } \Omega_{0}^{\pm} \backslash B_{2\delta}(\Sigma_{0}).$$
(5.11)

This combined with (5.11) yields
$$u^{in}_{\pm}(x) = u^{in}_{\pm} \circ \Psi_{\delta}^{\pm}(x) = u^{in}_{\pm}(x), \quad \forall x \in \Omega_{0}^{\pm} \backslash B_{2\delta}(\Sigma_{0}),$$
(5.12)

and (1.24a) is proved.

Now we turn to the proof of (1.24b). Substituting (5.9a) and (5.9b) into (5.10), we obtain
$$\nabla u^{in}_{\varepsilon} = \frac{\nabla u^{+}_{0} + \nabla u^{-}_{0}}{2} + \frac{\nabla d_{\Sigma}(x, 0)\alpha'(\frac{\|u^{\pm}_{0}-u^{-}_{0}\|}{\text{dist}_{\Omega_{0}}})}{\varepsilon} + \alpha(\frac{\|u^{\pm}_{0}-u^{-}_{0}\|}{\text{dist}_{\Omega_{0}}}) + O(e^{-C/\delta}) \left( |\nabla u^{+}_{0}| + |\nabla u^{-}_{0}| + 1 \right) \quad \text{a.e. in } \Omega.$$ 
(5.13)
5.1. Proof of (1.24b): Estimate near $\Sigma_0$. By (5.3)

\[(u_0^+(x) - u_0^-(x)) \perp \mathbb{R}^n, \quad \forall x \in B_\delta(\Sigma_0).\]  

(5.14)

As $u_0^\pm$ are mappings into $\mathbb{m}_\pm$ respectively, we have $\nabla u_0^\pm(x) \in T_{u_0^\pm(x)}\mathbb{m}_\pm$. So

\[(u_0^+(x) - u_0^-(x)) \cdot \partial x_i u_0^\pm(x), \quad \text{a.e. } x \in B_\delta(\Sigma_0), \quad 1 \leq i \leq d.\]  

(5.15)

The square of the second term on the right-hand side of (5.13) is

\[
\left|\nabla_{\Sigma_0} \alpha' \left(\frac{d\Sigma_0}{\varepsilon}\right) \frac{u_0^+ - u_0^-}{\text{dist}_m}\right|^2 \leq \frac{3}{\varepsilon} - \varepsilon^{-2} \left(\alpha' \left(\frac{d\Sigma_0}{\varepsilon}\right)\right)^2 \quad \text{in } B_\delta(\Sigma_0).
\]  

(5.16)

This together with (5.15) enables us to compute the square of (5.13) by

\[
\left|\nabla u_\varepsilon^{in}\right|^2 = \varepsilon^{-2} \left(\alpha' \left(\frac{d\Sigma_0}{\varepsilon}\right)\right)^2 + \left|\nabla u_0^+ + \nabla u_0^- + \alpha \left(\frac{d\Sigma_0}{\varepsilon}\right) \nabla u_0^+ - \nabla u_0^-\right|^2 + O(e^{-C/\varepsilon}) \left(\left|\nabla u_0^+\right|^2 + \left|\nabla u_0^-\right|^2 + 1\right) \quad \text{in } B_\delta(\Sigma_0).
\]  

(5.17)

By (5.9a) and (5.10), we have

\[F(u_\varepsilon^{in}) = F\left(\frac{u_0^+ + u_0^-}{2} + \alpha \left(\frac{d\Sigma_0}{\varepsilon}\right) \frac{u_0^+ - u_0^-}{\text{dist}_m}\right) + O(e^{-C/\varepsilon}) \quad \text{in } \Omega.
\]  

(5.18)

To compute the RHS of (5.18), we first deduce from (2.9a) that

\[\gamma(s) : \mathbb{R} \mapsto \frac{u_0^+ + u_0^-}{2} + \alpha(s) \frac{u_0^+ - u_0^-}{\text{dist}_m}\]  

(5.19)

parametrizes the line segment $\overline{u_0^+ u_0^-}$ with $\gamma(0) = \frac{u_0^+ + u_0^-}{2}$ the middle point. For $x \in B_\delta(\Sigma_0) \cap \Omega_0^+$, we have $\alpha \left(\frac{d\Sigma_0(x)}{\varepsilon}\right) > 0$ and by (2.9a), the distance from $\gamma \left(\frac{d\Sigma_0(x)}{\varepsilon}\right)$ to $\mathbb{m}$ equals to the distance between $\gamma \left(\frac{d\Sigma_0(x)}{\varepsilon}\right)$ and $u_0^+ \in \mathbb{m}_+$. So

\[
\left|\text{dist}_m \left(\frac{u_0^+ + u_0^-}{2} + \alpha \left(\frac{d\Sigma_0}{\varepsilon}\right) \frac{u_0^+ - u_0^-}{\text{dist}_m}\right)\right| = \left|\text{dist}_m \frac{2}{2} - \alpha \left(\frac{d\Sigma_0}{\varepsilon}\right)\right| \left|\frac{u_0^+ - u_0^-}{\text{dist}_m}\right| = \left(\text{dist}_m \frac{2}{2} - \alpha \left(\frac{d\Sigma_0}{\varepsilon}\right)\right)\]  

(5.20)

on $B_\delta(\Sigma_0) \cap \Omega_0^+$, and thus

\[F\left(\frac{u_0^+ + u_0^-}{2} + \alpha \left(\frac{d\Sigma_0}{\varepsilon}\right) \frac{u_0^+ - u_0^-}{\text{dist}_m}\right) \overset{\text{def}}{=} \int \left(\left|\text{dist}_m \frac{2}{2} - \alpha \left(\frac{d\Sigma_0}{\varepsilon}\right)\right|^2\right) = \int \left(\text{dist}_m \frac{2}{2} - \alpha \left(\frac{d\Sigma_0}{\varepsilon}\right)\right) \overset{\text{def}}{=} F\left(\alpha \left(\frac{d\Sigma_0}{\varepsilon}\right)\right).
\]  

(5.21)

Similar calculation leads to the case when $x \in B_\delta(\Sigma_0) \cap \Omega_0^-$, and altogether we have (from (5.18) and (5.21)) that

\[F(u_\varepsilon^{in}) = F\left(\alpha \left(\frac{d\Sigma_0}{\varepsilon}\right)\right) + O(e^{-C/\varepsilon}) \quad \text{in } B_\delta(\Sigma_t).
\]  

(5.22)

Now we compute $\xi \cdot \nabla (d_F \circ u_\varepsilon^{in})$:

\[-\int_\Omega \eta_0 \xi \cdot \nabla (d_F \circ u_\varepsilon^{in}) \overset{\text{def}}{=} \int_\Omega \text{div}(\eta_0 \xi) d_F \circ u_\varepsilon^{in}\]  

\[\overset{\text{(5.5)}}{=} \int_\Omega \text{div}(\eta_0 \xi) d_F \left(\frac{u_0^+ + u_0^-}{2} + S_\varepsilon \frac{u_0^+ - u_0^-}{\text{dist}_m}\right)\]  

(5.23)

Using (5.9a) and the Lipschitz property of $d_F$, we can write

\[-\int_\Omega \eta_0 \xi \cdot \nabla (d_F \circ u_\varepsilon^{in}) = \int_\Omega \text{div}(\eta_0 \xi) d_F \left(\frac{u_0^+ + u_0^-}{2} + \alpha \left(\frac{d\Sigma_0}{\varepsilon}\right) \frac{u_0^+ - u_0^-}{\text{dist}_m}\right) + O(e^{-C/\varepsilon}).
\]  

(5.24)
To evaluate $d_F$ on $B_\delta(\Sigma_\ell)$, we note that a line segment inside a minimal connection will not involve the 2nd case defining $d_F$ (cf. (2.17)). As a result, at any $x \in B_\delta(\Sigma_0) \cap \Omega_0^+$, since $\alpha(\frac{d\Sigma}{\varepsilon}) > 0$ and $(u^+_0, u^-_0)$ is a minimal pair (cf. (5.3)), we have

\[
d_F \left( \frac{u^+_0 + u^-_0}{2} + \alpha(\frac{d\Sigma}{\varepsilon})(u^+_0 - u^-_0) \right) = c_F - \int_0^{\text{dist}_m} \left( \frac{u^+_0 + u^-_0}{2} + \alpha(\frac{d\Sigma}{\varepsilon})(u^+_0 - u^-_0) \right) \sqrt{2f(\lambda^2)} \, d\lambda \tag{1.13}
\]

\[
c_F - \int_0^{\text{dist}_m} \alpha(\frac{d\Sigma}{\varepsilon}) \sqrt{2f(\lambda^2)} \, d\lambda \tag{2.6}
\]

\[
c_F - \int_0^{\text{dist}_m} \sqrt{2\tilde{F}(\lambda)} \, d\lambda \quad \forall x \in B_\delta(\Sigma_0) \cap \Omega_0^+.
\]

Similar calculation applies to $B_\delta(\Sigma_0) \cap \Omega_0^-$ and $\alpha(\frac{d\Sigma}{\varepsilon}) < 0$:

\[
d_F \left( \frac{u^+_0 + u^-_0}{2} + \alpha(\frac{d\Sigma}{\varepsilon})(u^+_0 - u^-_0) \right) = c_F - \int_0^{\text{dist}_m} \left( \frac{u^+_0 + u^-_0}{2} + \alpha(\frac{d\Sigma}{\varepsilon})(u^+_0 - u^-_0) \right) \sqrt{2f(\lambda^2)} \, d\lambda \tag{1.13}
\]

\[
c_F - \int_0^{\text{dist}_m} \alpha(\frac{d\Sigma}{\varepsilon}) \sqrt{2f(\lambda^2)} \, d\lambda \tag{2.6}
\]

\[
c_F - \int_0^{\text{dist}_m} \sqrt{2\tilde{F}(\lambda)} \, d\lambda \quad \forall x \in B_\delta(\Sigma_0) \cap \Omega_0^-.
\]

To summarize, we have

\[
d_F \left( \frac{u^+_0 + u^-_0}{2} + \alpha(\frac{d\Sigma}{\varepsilon})(u^+_0 - u^-_0) \right) = \begin{cases} 
  c_F - \int_0^{\text{dist}_m} \sqrt{2\tilde{F}(\lambda)} \, d\lambda & \forall x \in B_\delta(\Sigma_0) \cap \Omega_0^+, \\
  c_F - \int_0^{\text{dist}_m} \sqrt{2\tilde{F}(\lambda)} \, d\lambda & \forall x \in B_\delta(\Sigma_0) \cap \Omega_0^-.
\end{cases} \tag{5.25}
\]

Recall from (5.4) that $\eta$ vanishes outside $B_\delta(\Sigma_0)$. So substituting (5.25) into (5.24) and integrating by parts yield

\[
\int_\Omega \eta \xi \cdot \nabla (d_F \circ u^{m}_\varepsilon) = \int_\Omega \varepsilon^{-1} \eta \xi \cdot \nabla d_\Sigma \alpha'(\frac{d\Sigma}{\varepsilon}) \sqrt{2\tilde{F}(\alpha(\frac{d\Sigma}{\varepsilon}))} \, dx + O(e^{-C/\varepsilon}). \tag{5.26}
\]
This combined with (5.22) and (5.17) yields
\[
\int_{\Omega} \left( \frac{1}{2} \nabla u_{\varepsilon}^{in} \right)^2 + \frac{F(u_{\varepsilon}^{in})}{\varepsilon^2} - \frac{1}{\varepsilon} \xi \cdot \nabla (d_F \circ u_{\varepsilon}^{in}) \right) \eta_{\delta} \\
= \int_{\Omega} \left( - \frac{1}{2} \left( \alpha' \left( \frac{d_F}{\varepsilon} \right) \right)^2 + \varepsilon^{-2} \tilde{F}(\alpha'(\frac{d_F}{\varepsilon})) - \varepsilon^{-2} \xi \cdot \nabla d_\Sigma \alpha' \left( \frac{d_F}{\varepsilon} \right) \right) \eta_{\delta} \\
+ \int_{\Omega} \frac{1}{2} \eta_{\delta} \left| \nabla u_{\varepsilon}^{in} + \nabla u_{\varepsilon}^{in} \right|^2 + \alpha \left( \frac{d_F}{\varepsilon} \right) \left| \nabla u_{\varepsilon}^{in} - \nabla u_{\varepsilon}^{in} \right|^2 \\
+ O(\varepsilon^{-C/\varepsilon}) \int_{\Omega} (1 + |\nabla u_0^+|^2 + |\nabla u_0^-|^2) \eta_{\delta}.
\]
By (2.9c) the first term on the right-hand side above simplifies to
\[
\int_{\Omega} \eta_{\delta}^{-2} (1 - \xi \cdot \nabla d_\Sigma) \left( \alpha'(\frac{d_F}{\varepsilon}) \right)^2 = \int_{\Omega} O(1) \eta_{\delta}^{-\frac{d_\delta}{\varepsilon}} \left( \alpha'(\frac{d_F}{\varepsilon}) \right)^2 = O(\varepsilon).
\]
The above two equations together implies
\[
\int_{\Omega} \frac{1}{2} \left( \nabla u_{\varepsilon}^{in} \right)^2 + \frac{F(u_{\varepsilon}^{in})}{\varepsilon^2} - \frac{1}{\varepsilon} \xi \cdot \nabla (d_F \circ u_{\varepsilon}^{in}) \right) \eta_{\delta} \\
= \int_{\Omega} \frac{1}{2} \eta_{\delta} \left| \nabla u_{\varepsilon}^{in} + \nabla u_{\varepsilon}^{in} \right|^2 + \alpha \left( \frac{d_F}{\varepsilon} \right) \left| \nabla u_{\varepsilon}^{in} - \nabla u_{\varepsilon}^{in} \right|^2 + O(\varepsilon) \int_{\Omega} (1 + |\nabla u_0^+|^2 + |\nabla u_0^-|^2) \eta_{\delta}.
\]
\[\text{(5.29)}\]

5.2. Proof of (1.24b): Estimates away from $\Sigma_0$. Using (2.9d), we have
\[
|\alpha'(\frac{d_F}{\varepsilon})| + \left| \alpha \left( \frac{d_F}{\varepsilon} \right) - \frac{\text{dist}(1_{\Omega_0^+} - 1_{\Omega_0^-})}{\varepsilon} \right| \leq C e^{-C/\varepsilon} \text{ in } \Omega \setminus B_{\delta/2}(\Sigma_0).
\]
Applying the above estimates to (5.13) yields
\[
\nabla u_{\varepsilon}^{in} = \nabla u_{\varepsilon}^{in} + \frac{\text{dist}(1_{\Omega_0^+} - 1_{\Omega_0^-})}{\varepsilon} \nabla u_{\varepsilon}^{in} - \nabla u_{\varepsilon}^{in} \\
+ O(e^{-C/\varepsilon}) \left( 1 + |\nabla u_0^+|1_{\Omega_0^+} + |\nabla u_0^-|1_{\Omega_0^-} \right) \\
= \nabla u_{\varepsilon}^{in} 1_{\Omega_0^+} + \nabla u_{\varepsilon}^{in} 1_{\Omega_0^-} \\
+ O(e^{-C/\varepsilon}) \left( 1 + |\nabla u_0^+|1_{\Omega_0^+} + |\nabla u_0^-|1_{\Omega_0^-} \right) \text{ a.e. in } \Omega \setminus B_{\delta/2}(\Sigma_0).
\]
By (5.4) the function $(1 - \eta_{\delta})$ vanishes on $B_{\delta/2}(\Sigma_0)$. So multiplying this function to (5.31) yields
\[
(1 - \eta_{\delta})||\nabla u_{\varepsilon}^{in}||^2 = (1 - \eta_{\delta}) \sum_\pm |\nabla u_0^\pm|^2 1_{\Omega_0^\pm} \\
+ O(e^{-C/\varepsilon}) \left( 1 + \sum_\pm |\nabla u_0^\pm|^2 1_{\Omega_0^\pm} \right) \text{ a.e. in } \Omega.
\]
Similar but easier calculation of (5.18) yields
\[
(1 - \eta_{\delta})F(u_{\varepsilon}^{in}) = O(e^{-C/\varepsilon}) \text{ in } \Omega.
\]
By (5.10), the Lipschitz continuity of $d_F$ and (5.9a),
\[
d_F \circ u_{\varepsilon}^{in} = d_F \left( \frac{u_0^++u_0^-}{2} + S_\varepsilon \frac{u_0^+-u_0^-}{\text{dist}(1_{\Omega_0^+} - 1_{\Omega_0^-})} \right) \\
= d_F \left( \frac{u_0^++u_0^-}{2} + \alpha \frac{u_0^+-u_0^-}{\text{dist}(1_{\Omega_0^+} - 1_{\Omega_0^-})} \right) + O(e^{-C/\varepsilon}) \text{ in } \Omega.
\]
This combined with (5.30) implies that
\[ d_F \circ u_{\varepsilon}^{in} = d_F \left( \frac{u_0^+ + u_0^-}{2} + \text{dist}_m (1_{\Omega_0^+} - 1_{\Omega_0^-}) \frac{u_0^+ - u_0^-}{\text{dist}_m} \right) + O(e^{-C/\varepsilon}) \]
\[ = d_F \left( u_0^+ 1_{\Omega_0^+} + u_0^- 1_{\Omega_0^-} \right) + O(e^{-C/\varepsilon}) \quad \text{in } \Omega \setminus B_{\delta/2}(\Sigma_0). \]

As \( u_0^\pm \) are mappings into \( \mathbb{m}_\pm \) (cf. (5.22)), we have from (2.17) that
\[ d_F \circ u_{\varepsilon}^{in} = c_F 1_{\Omega_0^+} + O(e^{-C/\varepsilon}) \quad \text{in } \Omega \setminus B_{\delta/2}(\Sigma_0). \] (5.35)

Since \((1 - \eta_\delta)\) vanishes on \( B_{\delta/2}(\Sigma_0) \),
\[ \text{div} \left( (1 - \eta_\delta) \xi \right) d_F \circ u_{\varepsilon}^{in} \overset{\text{(2.17)}}{=} \text{div} \left( (1 - \eta_\delta) \xi \right) c_F 1_{\Omega_0^+} + O(e^{-C/\varepsilon}). \] (5.36)

So we have
\[ - \int_\Omega (1 - \eta_\delta) \xi \cdot \nabla \left( d_F \circ u_{\varepsilon}^{in} \right) = \int_\Omega \text{div} \left( (1 - \eta_\delta) \xi \right) d_F \circ u_{\varepsilon}^{in} = \int_{\Omega_0^+} \text{div} \left( (1 - \eta_\delta) \xi \right) c_F + O(e^{-C/\varepsilon}). \] (5.37)

Putting (5.32), (5.33) and (5.37) together, we obtain
\[ \int_\Omega \left( \frac{1}{2} |\nabla u_{\varepsilon}^{in}|^2 + \frac{F(u_{\varepsilon}^{in})}{\varepsilon^2} - \frac{1}{\varepsilon} \xi \cdot \nabla \left( d_F \circ u_{\varepsilon}^{in} \right) \right) (1 - \eta_\delta) \]
\[ = \sum_\pm \int_{\Omega_0^\pm} \frac{1 - \eta_\delta}{2} |\nabla u_{\delta_0}^{\pm}|^2 + O(e^{-C/\varepsilon}) + O(e^{-C/\varepsilon}) \sum_\pm \int_{\Omega_0^\pm} |\nabla u_{\delta_0}^{\pm}|^2. \] (5.38)

Combining this with (5.29) leads to
\[ \int_\Omega \left( \frac{1}{2} |\nabla u_{\varepsilon}^{in}|^2 + \frac{F(u_{\varepsilon}^{in})}{\varepsilon^2} - \frac{1}{\varepsilon} \xi \cdot \nabla \left( d_F \circ u_{\varepsilon}^{in} \right) \right) \]
\[ = \int_\Omega \frac{1}{2} \eta_\delta \left| \nabla u_{\varepsilon}^{+} + \nabla u_{\varepsilon}^{-} \right| + \alpha \left( \frac{d_F}{\varepsilon} \right) \frac{|\nabla u_{\varepsilon}^{+} - \nabla u_{\varepsilon}^{-}|}{\text{dist}_m} \right| \right|^2 + \sum_\pm \int_{\Omega_0^\pm} \frac{1 - \eta_\delta}{2} |\nabla u_{\delta_0}^{\pm}|^2 + O(\varepsilon). \] (5.39)

This leads to (1.24b). Using \( \alpha \left( \frac{d_F}{\varepsilon} \right) \overset{\varepsilon \to 0}{\longrightarrow} \frac{\text{dist}_m (1_{\Omega_0^+} - 1_{\Omega_0^-})}{2} \) a.e. in \( \Omega \), we can apply the dominated convergence to the first integral on the RHS of (5.39) and get
\[ \lim_{\varepsilon \to 0} \int_\Omega \left( \frac{1}{2} |\nabla u_{\varepsilon}^{in}|^2 + \frac{F(u_{\varepsilon}^{in})}{\varepsilon^2} - \frac{1}{\varepsilon} \xi \cdot \nabla \left( d_F \circ u_{\varepsilon}^{in} \right) \right) = \sum_\pm \int_{\Omega_0^\pm} |\nabla u_{\delta_0}^{\pm}|^2. \] (5.40)

5.3. **Proof of (1.24e).** Recall from (1.14) that
\[ B[u_{\varepsilon}^{in} | \Sigma_0] := \int_\Omega \left( c_F \chi - c_F + 2 \left( d_F \circ u_{\varepsilon}^{in} - c_F \right) \right) \eta \circ d\Sigma \, dx \]
\[ + \int_\Omega \left( d_F \circ u_{\varepsilon}^{in} - c_F \right)^+ |\eta \circ d\Sigma| \, dx, \] (5.41)
where $\chi = 1_{\Omega_0^+} - 1_{\Omega_0^-}$, and $\eta$ is defined by (5.8). We also recall from (5.8) that $c_F = 2 \int_{\Omega_0^+} \frac{dist}{\varepsilon} \sqrt{2F(\lambda)} d\lambda$.

Concerning the first integral of (5.41), we first deduce from (5.35) that its integrand is of order $O(\varepsilon^{-C/\varepsilon})$ on $\Omega \setminus B_{\delta/2}(\Sigma_0)$. So it suffices to estimate the integral in the transitional region $B_\delta(\Sigma_0)$: by (5.34), (5.29) and a change of variable

$$
\int_{\Omega_0^+ \cap B_\delta(\Sigma_0)} \left( c_F \chi - c_F + 2 \left( d_F \circ u^{in}_\varepsilon - c_F \right)^- \right) \eta \circ d\Sigma dx
$$

In a similar way

$$
\int_{\Omega_0^+ \cap B_\delta(\Sigma_0)} \left( c_F \chi - c_F + 2 \left( d_F \circ u^{in}_\varepsilon - c_F \right)^- \right) \eta \circ d\Sigma dx
$$

Similar calculation shows that the second integral of (5.41) is of order $O(\varepsilon)$. All together we finish the proof of (1.24c).

Acknowledgements. Y. Liu is partially supported by NSF of China under Grant 119 71314. We would like to thank professor Wei Wang for sharing with us the notes [38] and stimulating discussions.

Appendix A. Proof of Proposition 2.9

As we shall not integrate the time variable $t$ throughout this section, we shall abbreviate the spatial integration $\int_{\Omega}$ by $\int$ and sometimes we omit the $dx$.

Lemma A.1. The following identity holds

$$
\int \nabla H : (\xi \otimes n_x) |\nabla \psi_\varepsilon| dx - \int (\nabla \cdot H) \xi \cdot \nabla \psi_\varepsilon dx
$$

$$
= \int \nabla H : (\xi - n_x) \otimes n_x |\nabla \psi_\varepsilon| dx + \int H_\varepsilon \cdot H |\nabla u_\varepsilon| dx
$$

$$
+ \int \nabla \cdot H \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) - |\nabla \psi_\varepsilon| \right) dx + \int \nabla \cdot H (|\nabla \psi_\varepsilon| - \xi \cdot \nabla \psi_\varepsilon) dx
$$

$$
- \sum_{i,j=1}^d \int (\nabla H)^{ij} \varepsilon (\partial_i u_\varepsilon \cdot \partial_j u_\varepsilon) dx + \int \nabla H : (n_x \otimes n_x) |\nabla \psi_\varepsilon| dx. \quad (A.1)
$$

Proof. We introduce the energy stress tensor $(T_\varepsilon)^{ij} = \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) \right) \delta_{ij} - \varepsilon \partial_i u_\varepsilon \cdot \partial_j u_\varepsilon$. By (2.39b), we have the identity $\nabla \cdot T_\varepsilon = H_\varepsilon |\nabla u_\varepsilon|$. Testing this identity by $H$, integrating by parts and using (2.32), we obtain

$$
\int H_\varepsilon \cdot H |\nabla u_\varepsilon| dx = - \int \nabla H : T_\varepsilon dx
$$

$$
= - \int \nabla \cdot H \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) \right) dx + \sum_{i,j} \int (\nabla H)^{ij} \varepsilon (\partial_i u_\varepsilon \cdot \partial_j u_\varepsilon) dx.
$$
So adding zero leads to
\[
\int \nabla H : \mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon |\nabla \psi_\varepsilon| \, dx
= \int \mathbf{H}_\varepsilon \cdot \mathbf{H} |\nabla \mathbf{u}_\varepsilon| \, dx + \int \nabla \cdot \mathbf{H} \left( \frac{\varepsilon}{2} |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon} F(\mathbf{u}_\varepsilon) - |\nabla \psi_\varepsilon| \right) \, dx + \int \nabla \cdot \mathbf{H} |\nabla \psi_\varepsilon| \, dx
- \sum_{i,j} \int (\nabla \mathbf{H})_{ij} \varepsilon \left( \partial_i \mathbf{u}_\varepsilon \cdot \partial_j \mathbf{u}_\varepsilon \right) \, dx + \int (\nabla \mathbf{H}) : (\mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon) |\nabla \psi_\varepsilon| \, dx.
\]

This easily implies (A.1). \qed

The second lemma gives the expansion of the time derivative of (1.11).

**Lemma A.2.** Under the assumptions of Theorem 1.1, the following identity holds

\[
\frac{d}{dt} E[u_\varepsilon|\Sigma|] + \frac{1}{2\varepsilon} \int \left( \varepsilon^2 |\partial t \mathbf{u}_\varepsilon|^2 - |\mathbf{H}_\varepsilon|^2 \right) \, dx
+ \frac{1}{2\varepsilon} \int |\varepsilon \partial_t \mathbf{u}_\varepsilon - (\nabla \cdot \mathbf{H}) \partial d_F(\mathbf{u}_\varepsilon)|^2 \, dx + \frac{1}{2\varepsilon} \int |\mathbf{H}_\varepsilon - \varepsilon |\nabla \mathbf{u}_\varepsilon| \mathbf{H}|^2 \, dx
= \frac{1}{2\varepsilon} \int |(\nabla \cdot \mathbf{H}) \partial d_F(\mathbf{u}_\varepsilon)| |\nabla \mathbf{u}_\varepsilon| \mathbf{n}_\varepsilon + \varepsilon |\Pi_{u_\varepsilon} \nabla \mathbf{u}_\varepsilon| \mathbf{H}|^2 \, dx
+ \frac{\varepsilon}{2} \int |\nabla \mathbf{H}|^2 \left( |\nabla \mathbf{u}_\varepsilon|^2 - |\Pi_{u_\varepsilon} \nabla \mathbf{u}_\varepsilon|^2 \right) \, dx - \int \nabla \mathbf{H} \cdot (\mathbf{H} - \mathbf{n}_\varepsilon)^{\otimes 2} |\nabla \psi_\varepsilon| \, dx
+ \int (\nabla \cdot \mathbf{H}) (1 - \mathbf{n}_\varepsilon \cdot \mathbf{n}_\varepsilon) |\nabla \psi_\varepsilon| \, dx + \int (J^1_\varepsilon + J^2_\varepsilon) \, dx,
\]

where $J^1_\varepsilon, J^2_\varepsilon$ are given by

\[
J^1_\varepsilon := \nabla H : \mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon \left( |\nabla \psi_\varepsilon| - \varepsilon |\nabla \mathbf{u}_\varepsilon|^2 \right)
+ \varepsilon \nabla H : (\mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon) \left( |\nabla \mathbf{u}_\varepsilon|^2 - |\Pi_{u_\varepsilon} \nabla \mathbf{u}_\varepsilon|^2 \right)
- \sum_{i,j} \varepsilon (\nabla H)_{ij} \left( \partial_i \mathbf{u}_\varepsilon - \Pi_{u_\varepsilon} \partial_i \mathbf{u}_\varepsilon \right) \cdot \left( \partial_j \mathbf{u}_\varepsilon - \Pi_{u_\varepsilon} \partial_j \mathbf{u}_\varepsilon \right),
\]

\[
J^2_\varepsilon := - \left( \partial_t \mathbf{\xi} + (\mathbf{H} \cdot \nabla) \mathbf{\xi} + (\nabla \mathbf{H})^T \mathbf{\xi} \right) \cdot \nabla \psi_\varepsilon.
\]

**Proof of Lemma A.2.** The proof here is exactly the same as in [22, Lemma 4.4]. Note that in the statement of this lemma, the term $\frac{1}{2\varepsilon} \int \left( \varepsilon^2 |\partial_t \mathbf{u}_\varepsilon|^2 - |\mathbf{H}_\varepsilon|^2 \right) \, dx$ is missing, but the proof of the identity there is correct (cf. see [22, equation (4.33)]).

We shall employ the Einstein summation convention by summing over repeated Latin indices. Using the energy dissipation law (2.34) and adding zero, we compute the time derivative of the energy (1.11) by

\[
\frac{d}{dt} E_{\Sigma}[u_\varepsilon] + \varepsilon \int |\partial_t \mathbf{u}_\varepsilon|^2 \, dx - \int (\nabla \cdot \mathbf{H}) \partial d_F(\mathbf{u}_\varepsilon) \cdot \partial_t \mathbf{u}_\varepsilon \, dx
= \int (\mathbf{H} \cdot \nabla) \mathbf{\xi} \cdot \nabla \psi_\varepsilon \, dx + \int (\nabla \mathbf{H})^T \mathbf{\xi} \cdot \nabla \psi_\varepsilon \, dx + \int J^2_\varepsilon \, dx.
\]
Due to the symmetry of the Hessian of $\psi_\varepsilon$ and the boundary conditions (2.32), we have
\[
\int \nabla \cdot (\xi \otimes H) \cdot \nabla \psi_\varepsilon \, dx = \int \nabla \cdot (H \otimes \xi) \cdot \nabla \psi_\varepsilon \, dx.
\]
Hence, the first integral on the right-hand side of (A.5) can be rewritten as
\[
\int (H \cdot \nabla) \xi \cdot \nabla \psi_\varepsilon \, dx
= \int \nabla \cdot (\xi \otimes H) \cdot \nabla \psi_\varepsilon \, dx - \int (\nabla \cdot H) \xi \cdot \nabla \psi_\varepsilon \, dx
= \int (\nabla \cdot \xi) H \cdot \nabla \psi_\varepsilon \, dx + \int (\xi \cdot \nabla) H \cdot \nabla \psi_\varepsilon \, dx - \int (\nabla \cdot H) \xi \cdot \nabla \psi_\varepsilon \, dx.
\]
Therefore
\[
\frac{d}{dt} E_\varepsilon[u_\varepsilon; \Sigma] + \varepsilon \int |\partial_t u_\varepsilon|^2 \, dx - \int (\nabla \cdot \xi) \partial_t d_F(u_\varepsilon) \cdot \partial_t u_\varepsilon \, dx
= \int (\nabla \cdot \xi) H \cdot \nabla \psi_\varepsilon \, dx + \int (\xi \cdot \nabla) H \cdot \nabla \psi_\varepsilon \, dx + \int \nabla H : (\xi - n_\varepsilon) \otimes n_\varepsilon |\nabla \psi_\varepsilon| \, dx
+ \int \nabla H : n_\varepsilon \otimes n_\varepsilon |\nabla \psi_\varepsilon| \, dx + \int \nabla^2 \psi_\varepsilon \, dx.
\]
Using (A.1) to replace the 3rd and 4th integrals on the right-hand side above yields
\[
\frac{d}{dt} E_\varepsilon[u_\varepsilon; \Sigma] + \varepsilon \int |\partial_t u_\varepsilon|^2 \, dx - \int (\nabla \cdot \xi) \partial_t d_F(u_\varepsilon) \cdot \partial_t u_\varepsilon \, dx
= \int (\nabla \cdot \xi) H \cdot \nabla \psi_\varepsilon \, dx + \int (\xi \cdot \nabla) H \cdot \nabla \psi_\varepsilon \, dx + \int \nabla H : (\xi - n_\varepsilon) \otimes n_\varepsilon |\nabla \psi_\varepsilon| \, dx
+ \int \nabla^2 \psi_\varepsilon \, dx.
\]
We claim that $J^1_\varepsilon$ arises from the 2nd and 3rd to last integral. Indeed, when $\partial d_F(u_\varepsilon) \neq 0$ and $u_\varepsilon \notin B_{\delta_0}(m)$, then it follows from (2.40) and (2.37) that
\[
\Pi_{u_\varepsilon}\partial_i u_\varepsilon \cdot \Pi_{u_\varepsilon} \partial_j u_\varepsilon |\partial d_F(u_\varepsilon)|^2 = \partial_i \psi_\varepsilon \partial_j \psi_\varepsilon \overset{(2.41)}{=} n^i_\varepsilon n^j_\varepsilon |\Pi_{u_\varepsilon} \nabla u_\varepsilon|^2 |\partial d_F(u_\varepsilon)|^2,
\]
where $(n^i_\varepsilon)_{1 \leq i \leq d} = n_\varepsilon$. This implies
\[
\Pi_{u_\varepsilon}\partial_i u_\varepsilon \cdot \Pi_{u_\varepsilon} \partial_j u_\varepsilon = n^i_\varepsilon n^j_\varepsilon |\Pi_{u_\varepsilon} \nabla u_\varepsilon|^2.
\]
In other cases defining (2.40), the equation (A.7) also holds. Using the orthogonality of (2.40), adding zero and using (A.7), we find
\[
\nabla H : n_\varepsilon \otimes n_\varepsilon |\nabla \psi_\varepsilon| - (\nabla H)_{ij} \varepsilon (\partial_i u_\varepsilon \cdot \partial_j u_\varepsilon)
= \nabla H : n_\varepsilon \otimes n_\varepsilon |\nabla \psi_\varepsilon| - \varepsilon (\nabla H)_{ij}(\Pi_{u_\varepsilon} \partial_i u_\varepsilon \cdot \Pi_{u_\varepsilon} \partial_j u_\varepsilon)
= (\nabla H)_{ij} \varepsilon (\partial_i u_\varepsilon - \Pi_{u_\varepsilon} \partial_i u_\varepsilon) \cdot (\partial_j u_\varepsilon - \Pi_{u_\varepsilon} \partial_j u_\varepsilon) = J^1_\varepsilon,
\]
and this finish the proof of the claim.
Now we write the sum of the integrands of 2nd and 3rd integrals on the right-hand side of (A.6) into a quadratic term of the difference: using the definition (2.39a) of $n_\varepsilon$ and $\nabla H : (\xi \otimes \xi) = 0$ (due to (2.26) and (2.31)), we have

\[
(\xi \cdot \nabla) H \cdot \nabla \psi_\varepsilon + \nabla H : (\xi - n_\varepsilon) \otimes n_\varepsilon |\nabla \psi_\varepsilon| \\
= (\xi \cdot \nabla) H \cdot n_\varepsilon |\nabla \psi_\varepsilon| + \nabla H : (\xi - n_\varepsilon) \otimes n_\varepsilon |\nabla \psi_\varepsilon| - \nabla H : (\xi \otimes \xi) |\nabla \psi_\varepsilon| \\
= \nabla H : (n_\varepsilon \otimes \xi) |\nabla \psi_\varepsilon| + \nabla H : (\xi - n_\varepsilon) \otimes n_\varepsilon |\nabla \psi_\varepsilon| - \nabla H : (\xi \otimes \xi) |\nabla \psi_\varepsilon| \\
= - \nabla H : (\xi - n_\varepsilon) \otimes |\nabla \psi_\varepsilon|.
\]

Using this identity, we can merge the 2nd and 3rd integrals on the right-hand side of (A.6):

\[
\frac{d}{dt} E_\varepsilon[u_\varepsilon; \Sigma] = -\varepsilon \int |\partial_t u_\varepsilon|^2 \, dx + \int (\nabla \cdot \xi) \partial_T (u_\varepsilon) \cdot \partial_t u_\varepsilon \, dx \\
+ \int (\nabla \cdot H) \nabla \psi_\varepsilon \, dx + \int H_\varepsilon \cdot H \nabla u_\varepsilon \, dx - \int \nabla H : (\xi - n_\varepsilon) \otimes |\nabla \psi_\varepsilon| \, dx \\
+ \int (\nabla \cdot H) \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) - |\nabla \psi_\varepsilon| \right) \, dx \\
+ \int (\nabla \cdot H) (1 - \xi \cdot n_\varepsilon) |\nabla \psi_\varepsilon| \, dx + \int (J_1^1 + J_2^1) \, dx. \quad (A.8)
\]

Now we complete squares for the first four terms on the right-hand side of (A.8). Reordering terms, we have

\[
-\varepsilon |\partial_t u_\varepsilon|^2 + (\nabla \cdot \xi) \partial_T (u_\varepsilon) \cdot \partial_t u_\varepsilon + (\nabla \cdot \xi) H \cdot \nabla \psi_\varepsilon + H_\varepsilon \cdot H |\nabla u_\varepsilon| \\
= -\varepsilon |\partial_t u_\varepsilon|^2 + \frac{1}{2\varepsilon} \left( |\nabla u_\varepsilon|^2 - 2(\nabla \cdot \xi) \partial_T (u_\varepsilon) \cdot \partial_t u_\varepsilon + (\nabla \cdot \xi)^2 |\partial_T (u_\varepsilon)|^2 \right) \\
+ \frac{1}{2\varepsilon} |H_\varepsilon|^2 - 2\varepsilon |\nabla u_\varepsilon| H \cdot H + \varepsilon^2 |\nabla u_\varepsilon|^2 |H|^2 \\
+ \frac{1}{2\varepsilon} \left( |H_\varepsilon|^2 + \varepsilon^2 |\nabla u_\varepsilon|^2 |H|^2 \right) \\
= -\varepsilon |\partial_t u_\varepsilon|^2 - (\nabla \cdot \xi) \partial_T (u_\varepsilon) |H_\varepsilon|^2 - \frac{1}{2\varepsilon} |H_\varepsilon - \varepsilon |\nabla u_\varepsilon| H|^2 - \frac{1}{2\varepsilon} |\varepsilon |\partial_t u_\varepsilon|^2 + \frac{1}{2\varepsilon} |H_\varepsilon|^2 \\
+ \frac{1}{2\varepsilon} \left( |\nabla \cdot \xi|^2 |\partial_T (u_\varepsilon)|^2 + 2\varepsilon (\nabla \cdot \xi) \nabla \psi_\varepsilon \cdot H + \varepsilon \Pi u_\varepsilon \nabla u_\varepsilon |H|^2 \right) \\
+ \frac{\varepsilon}{2} (|\nabla u_\varepsilon|^2 - |\Pi u_\varepsilon \nabla u_\varepsilon|^2) |H|^2.
\]

Using (2.39a) and the chain rule (2.41), the terms above form the last missing square. Integrating over the domain $\Omega$ and substituting into (A.8) we arrive at (A.2).
Proof of Proposition 2.9. We first estimate the right-hand side of (A.2) by \( E_\varepsilon[u_\varepsilon|\Sigma] \) up to a constant that only depends on \( \Sigma_t \). We start with (A.2a): it follows from the triangle inequality that

\[
\int \left| \varepsilon^{-1/2}(\nabla \cdot \xi) \partial \varepsilon F(u_\varepsilon) |n_\varepsilon + \sqrt{\varepsilon} |\Pi_{u_\varepsilon} \nabla u_\varepsilon| \right|^2 dx \\
\leq \int \left| (\nabla \cdot \xi) \left( \varepsilon^{-1/2} |\partial \varepsilon F(u_\varepsilon)| - \sqrt{\varepsilon} |\Pi_{u_\varepsilon} \nabla u_\varepsilon| \right) n_\varepsilon \right|^2 dx \\
+ \int \left| (\nabla \cdot \xi) \sqrt{\varepsilon} |\Pi_{u_\varepsilon} \nabla u_\varepsilon| (n_\varepsilon - \xi) \right|^2 dx \\
+ \int \left| (\nabla \cdot \xi) \xi + H \right| \sqrt{\varepsilon} |\Pi_{u_\varepsilon} \nabla u_\varepsilon| \right|^2 dx.
\]

The first integral on the right-hand side of the above inequality is controlled by (2.45c). Due to the elementary inequality \(|\xi - n_\varepsilon|^2 \leq 2(1 - n_\varepsilon \cdot \xi)\), the second integral is controlled by (2.45d). The third integral can be treated using the relation \( H = (H \cdot \xi) \xi + O(d_\Sigma(x,t)) \) and (2.33a). So it can be controlled by (2.45e).

The integrals in (A.2b) can be controlled using (2.45b) and (2.45d). The integrals in (A.2c) is controlled by (2.45a). The first term in (A.2d) can be controlled using (2.45d). It remains to estimate (A.3) and (A.4). The last two terms defining \( J_\varepsilon \) can be bounded using (2.45b). Therefore,

\[
\int J_\varepsilon^1 dx \lesssim \int \nabla H : (n_\varepsilon \otimes (n_\varepsilon - \xi)) (|\nabla \psi_\varepsilon| - \varepsilon |\nabla u_\varepsilon|) dx \\
+ \int (\xi \cdot \nabla) H \cdot n_\varepsilon (|\nabla \psi_\varepsilon| - \varepsilon |\nabla u_\varepsilon|) dx + CE_\varepsilon[u_\varepsilon|\Sigma] \leq \int |n_\varepsilon - \xi| (|\varepsilon |\nabla u_\varepsilon|^2 - \varepsilon |\Pi_{u_\varepsilon} \nabla u_\varepsilon|) dx \\
+ \int |n_\varepsilon - \xi| |\varepsilon |\Pi_{u_\varepsilon} \nabla u_\varepsilon| - |\nabla \psi_\varepsilon| dx \\
+ \int \min(d_\varepsilon^2, 1)(|\nabla \psi_\varepsilon| + \varepsilon |\nabla u_\varepsilon|) dx + E_\varepsilon[u_\varepsilon|\Sigma].
\]

The first and the third integrals in the last display can be estimated using (2.45b) and (2.45e) respectively. Then we employ (2.41) and yield

\[
\int J_\varepsilon^1 dx \lesssim \int |n_\varepsilon - \xi| |\varepsilon |\Pi_{u_\varepsilon} \nabla u_\varepsilon| - |\nabla \psi_\varepsilon| dx + E_\varepsilon[u_\varepsilon|\Sigma] \\
= \int |n_\varepsilon - \xi| \sqrt{\varepsilon} |\Pi_{u_\varepsilon} \nabla u_\varepsilon| \left| \varepsilon^{-1/2} |\Pi_{u_\varepsilon} \nabla u_\varepsilon| - \varepsilon^{-1/2} |\partial \varepsilon F(u_\varepsilon)| \right| dx + E_\varepsilon[u_\varepsilon|\Sigma].
\]

Finally applying the Cauchy-Schwarz inequality and then (2.45c) and (2.45d), we obtain \( \int J_\varepsilon^1 \lesssim E_\varepsilon[u_\varepsilon|\Sigma] \). As for \( J_\varepsilon^2 \), we employ (2.33c) and (2.45a) to obtain \( \int J_\varepsilon^2 \lesssim E_\varepsilon[u_\varepsilon|\Sigma] \). Altogether, we prove that the right-hand side of (A.2) is bounded by \( E_\varepsilon[u_\varepsilon|\Sigma] \) up to a multiplicative constant which only depends on \( \Sigma_t \).

\[\Box\]

**References**

[1] N. D. Alikakos, P. W. Bates, and X. Chen. Convergence of the Cahn-Hilliard equation to the Hele-Shaw model. *Arch. Rational Mech. Anal.*, 128(2):165–205, 1994.

[2] L. Ambrosio and G. Dal Maso. A general chain rule for distributional derivatives. *Proc. Amer. Math. Soc.*, 108(3):691–702, 1990.

[3] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.
[4] L. Bronsard and B. Stoth. The singular limit of a vector-valued reaction-diffusion process. Trans. Amer. Math. Soc., 350(12):4931–4953, 1998.
[5] D. Burago, Y. Burago, and S. Ivanov. A course in metric geometry, volume 33 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001.
[6] X. Chen. Global asymptotic limit of solutions of the Cahn-Hilliard equation. J. Differential Geom., 44(2):262–311, 1996.
[7] Y. G. Chen, Y. Giga, and S. Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. J. Differential Geom., 33(3):749–786, 1991.
[8] Y. M. Chen and M. Struwe. Existence and partial regularity results for the heat flow for harmonic maps. Math. Z., 201(1):83–103, 1989.
[9] P. De Mottoni and M. Schatzman. Geometrical evolution of developed interfaces. Trans. Amer. Math. Soc., 347(5):1533–1589, 1995.
[10] L. C. Evans and R. F. Gariepy. Measure theory and fine properties of functions. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.
[11] L. C. Evans, H. M. Soner, and P. E. Souganidis. Phase transitions and generalized motion by mean curvature. Comm. Pure Appl. Math., 45(9):1097–1123, 1992.
[12] L. C. Evans and J. Spruck. Motion of level sets by mean curvature. I. J. Differential Geom., 33(3):635–686, 1991.
[13] M. Fei, F. Lin, W. Wang, and Z. Zhang. Matrix-valued Allen–Cahn equation and the Keller–Rubinstein–Sternberg problem. Inventiones mathematicae, 2023.
[14] M. Fei, W. Wang, P. Zhang, and Z. Zhang. On the Isotropic-Nematic phase transition for the liquid crystal. Peking Math. J., 1(2):141–219, 2018.
[15] J. Fischer and S. Hensel. Weak-strong uniqueness for the Navier-Stokes equation for two fluids with surface tension. Arch. Ration. Mech. Anal., 236(2):967–1087, 2020.
[16] J. Fischer, T. Laux, and T. M. Simon. Convergence rates of the Allen-Cahn equation to mean curvature flow: a short proof based on relative entropies. SIAM J. Math. Anal., 52(6):6222–6233, 2020.
[17] I. Fonseca and L. Tartar. The gradient theory of phase transitions for systems with two potential wells. Proc. Roy. Soc. Edinburgh Sect. A, 111(1-2):89–102, 1989.
[18] J. E. Hutchinson and Y. Tonegawa. Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory. Calc. Var. Partial Differential Equations, 10(1):49–84, 2000.
[19] T. Ilmanen. Convergence of the Allen-Cahn equation to Brakke’s motion by mean curvature. J. Differential Geom., 38(2):417–461, 1993.
[20] R. L. Jerrard and D. Smets. On the motion of a curve by its binormal curvature. J. Eur. Math. Soc. (JEMS), 17(6):1487–1515, 2015.
[21] S. G. Krantz and H. R. Parks. The implicit function theorem. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2013. History, theory, and applications, Reprint of the 2003 edition.
[22] T. Laux and Y. Liu. Nematic-isotropic phase transition in liquid crystals: a variational derivation of effective geometric motions. Arch. Ration. Mech. Anal., 241(3):1785–1814, 2021.
[23] T. Laux and T. M. Simon. Convergence of the Allen-Cahn equation to multiphase mean curvature flow. Comm. Pure Appl. Math., 71(8):1597–1647, 2018.
[24] F. Lin and C. Wang. Isotropic-nematic phase transition and liquid crystal droplets. Comm. Pure Appl. Math., to appear.
[25] F.-H. Lin, X.-B. Pan, and C.-Y. Wang. Phase transition for potentials of high-dimensional wells. Comm. Pure Appl. Math., 65(6):833–888, 2012.
[26] F.-H. Lin and C.-Y. Wang. Harmonic maps in connection of phase transitions with higher dimensional potential wells. Chin. Ann. Math. Ser. B, 40(5):781–810, 2019.
[27] Y. Liu. Phase transition of anisotropic Ginzburg–Landau equation. arXiv preprint arXiv:2111.115067, 2021.
[28] L. Modica. A gradient bound and a Liouville theorem for nonlinear Poisson equations. Comm. Pure Appl. Math., 38(5):679–684, 1985.
[29] L. Modica and S. Mortola. Un esempio di Γ’-convergenza. Boll. Un. Mat. Ital. B (5), 14(1):285–299, 1977.
[30] A. Pisante and F. Punzo. Allen-Cahn approximation of mean curvature flow in Riemannian manifolds I, uniform estimates. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 15:309–341, 2016.
[31] M. Röger and R. Schätzle. On a modified conjecture of De Giorgi. Math. Z., 254(4):675–714, 2006.
[32] J. Rubinstein, P. Sternberg, and J. B. Keller. Fast reaction, slow diffusion, and curve shortening. SIAM J. Appl. Math., 49(1):116–133, 1989.
[33] J. Rubinstein, P. Sternberg, and J. B. Keller. Reaction-diffusion processes and evolution to harmonic maps. SIAM J. Appl. Math., 49(6):1722–1733, 1989.
[34] N. Sato. A simple proof of convergence of the Allen-Cahn equation to Brakke’s motion by mean curvature. *Indiana Univ. Math. J.*, 57(4):1743–1751, 2008.

[35] H. M. Soner. Ginzburg-Landau equation and motion by mean curvature. I. Convergence. *J. Geom. Anal.*, 7(3):437–475, 1997.

[36] P. Sternberg. The effect of a singular perturbation on nonconvex variational problems. *Arch. Rational Mech. Anal.*, 101(3):209–260, 1988.

[37] Y. Tonegawa. Integrality of varifolds in the singular limit of reaction-diffusion equations. *Hiroshima Math. J.*, 33(3):323–341, 2003.

[38] W. Wang. On the minimal pair condition for phase transition with high dimensional wells. *in preparation.*

NYU Shanghai, 567 Yangsi W road, Pudong, Shanghai 200126, China, and NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai, 3663 Zhongshan Road North, Shanghai, 200062, China

*Email address: y167@nyu.edu*