A Γ-STRUCTURE ON LAGRANGIAN GRASSMANNIANS

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Abstract. For $n$ odd the Lagrangian Grassmannian of $\mathbb{R}^{2n}$ is a Γ-manifold.

1. Introduction and statement of the result

We denote by $(\mathbb{R}^{2n}, \omega)$ the standard symplectic vector space. The (unoriented) Lagrangian Grassmannian $\mathcal{L}$ is the space of all Lagrangian subspaces of $\mathbb{R}^{2n}$. It is a homogeneous space $\mathcal{L} \cong U(n)/O(n)$, see [AG01, MS98]. Every Lagrangian subspace can be identified with the fixed point set of a linear orthogonal anti-symplectic involution. Using this identification, we define a smooth map

$$\Theta : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$$

by

$$(R, S) \mapsto RSR,$$

which we think of as a product. On every space there are products such as constant maps and projections to one factor. In [Hop41] Hopf introduced the notion of Γ-manifolds which rules these trivial products out. The purpose of this paper is to prove that the above product gives the Lagrangian Grassmannian $\mathcal{L}$ the structure of a Γ-manifold for $n$ odd.

Definition 1.1. A closed, connected, orientable manifold $M$ carries the structure of a Γ-manifold if there exists a map

$$\Theta : M \times M \to M$$

such that the maps

$$x \mapsto \Theta(x, y_0) \quad \text{and} \quad y \mapsto \Theta(x_0, y)$$

have non-zero mapping degree for one and thus all pairs $(x_0, y_0) \in M \times M$.

It is well-known that $\mathcal{L}$ is orientable if and only if $n$ odd, see [Fuk68]. The main result of this article is the following theorem.

Theorem 1.2. If $n$ is odd, then $(\mathcal{L}, \Theta)$ is a Γ-manifold.

Using Hopf’s theorem [Hop41, Satz 1], we get a new proof of the following Corollary due to Fuks [Fuk68].

Corollary 1.3 ([Fuk68]). For $n$ odd, the rational cohomology ring of $\mathcal{L}$ is an exterior algebra on generators of odd degree.

Remark 1.4. The cohomology ring of the oriented and unoriented Lagrangian Grassmannian was computed by Borel and Fuks for all $n$, see [Bor53a, Bor53b, Fuk68]. A nice summary of these results can be found in the book of Vassilyev [Vas88, Chapter 22].
The above situation fits into the following general framework. It is well-known that \( \mathcal{L} \) embeds into \( U(n) \) as the set \( U(n) \cap \text{Sym}(n) \), i.e. the symmetric unitary matrices. Thus, \( \mathcal{L} \) can be interpreted as the fixed point set of the involutive anti-isomorphism \( A \mapsto A^T \) of \( U(n) \).

On any Lie group \( G \) we can define a new product: \( (g,h) \mapsto gh^{-1}g \). If \( I : G \to G \) is an involutive anti-isomorphism then this new product restricts to a product on the fixed point set \( \text{Fix}(I) \). This is precisely the situation for the Lagrangian Grassmannian, namely the map \( \Theta \) corresponds under the embedding of \( \mathcal{L} \) into \( U(n) \) to \( (g,h) \mapsto gh^{-1}g \).

For general Lie groups this new product does not always give rise to a \( \Gamma \)-structure for various reasons. For example, if we take \( G = O(n) \) resp. \( G = U(n) \) and \( I(A) := A^{-1} \), then \( \text{Fix}(I) \) can be identified with \( \cup_k G(k,n) \), the union of all real resp. complex Grassmannians, which is not connected. Another example is \( G = SU(n) \) with \( I = \text{transposition} \). Then for \( n = 2 \) we can identify \( \text{Fix}(I) \cong S^2 \). But by Hopf’s theorem \([\text{Hop41}, \text{Satz} 1]\) \( S^2 \) is not a \( \Gamma \)-manifold.

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2. Proof of Theorem 1.2

We recall that the (unoriented) Lagrangian Grassmannian \( \mathcal{L} \) is the homogeneous space

\[ \mathcal{L} \cong U(n)/O(n), \]

see \([\text{AG01}, \text{MS98}]\). Since \( n \) is odd, \( \mathcal{L} \) is a closed connected orientable manifold \([\text{Fuk68}]\). The space \( \mathcal{L} \) is naturally identified with the space of linear orthogonal anti-symplectic involutions of \( \mathbb{R}^{2n} \) with the standard symplectic structure. Using this identification, we define the map

\[ \Theta : \mathcal{L} \times \mathcal{L} \to \mathcal{L} \]

by \( (R,S) \mapsto RSR \). In order to prove Theorem 1.2 it suffices to show for one choice of basepoint \( R_0 \) that the mapping degrees of

\[ S \mapsto \Theta(R_0, S) \quad \text{and} \quad S \mapsto \Theta(S, R_0) \]

are non-zero. Since

\[ S \mapsto \Theta(R_0, S) = R_0SR_0 \mapsto \Theta(R_0, R_0SR_0) = R_0R_0SR_0R_0 = S, \]

the first map is an involution and therefore has mapping degree \( \pm 1 \). The non-trivial case is to compute the mapping degree of

\[ \Theta_0(S) := \Theta(S, R_0) = S R_0 S. \]

Theorem 1.2 follows immediately from the following proposition.

Proposition 2.1. The mapping degree of \( \Theta_0 \) equals

\[ \deg \Theta_0 = 2^{m+1} \]

where \( n = 2m + 1 \).
Proof. Identify \( \mathbb{R}^{2n} = \mathbb{C}^n \) in the standard way. Denote by \( \tau : \mathbb{C}^n \to \mathbb{C}^n \) the map given by complex conjugation of all coordinates simultaneously. It is a standard fact, see for instance [MS98], that an orthogonal symplectic map \( \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) corresponds to a unitary map \( \mathbb{C}^n \to \mathbb{C}^n \). It follows that an orthogonal anti-symplectic map \( R : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) can be written as the composition \( A \circ \tau : \mathbb{C}^n \to \mathbb{C}^n \) for \( A \) a unitary linear map. The condition \( R^2 = \text{Id} \) then translates to \( A\overline{A} = \text{Id} \). So, we identify 
\[
\mathcal{L} = \{ A \in U(n) | A\overline{A} = \text{Id} \}.
\]
Under this identification, the map \( \Theta \) is given by 
\[
\Theta(A, B) = ABA.
\]
Let \( B_0 \) be the unitary matrix corresponding to \( R_0 \). Then the map \( \Theta_0 \) is given by 
\[
\Theta_0(A) = \Theta(A, B_0) = A\overline{B_0}A.
\]
In the following, we take \( B_0 = B \), the diagonal matrix with entries \( b_{jk} = e^{i\theta_j} \delta_{jk} \) where 
\[
0 < \theta_j < 2\pi, \quad \theta_1 < \theta_2 < \cdots < \theta_n.
\]
For this choice of \( B_0 \), we show that \( \text{Id} \) is a regular value of \( \Theta_0 \) and compute the signed cardinality of \( \Theta_0^{-1}(\text{Id}) \).

Indeed, if \( \Theta_0(A) = \text{Id} \), then \( A\overline{B_0}A = \text{Id} \), and therefore \( AB = A \). Throughout this paper, we do not use the Einstein summation convention. Letting \( a_{jk} \) denote the matrix entries of \( A \), we have 
\[
\bar{a}_{jk}e^{i\theta_j} = a_{jk}.
\]
Write \( a_{jk} = r_{jk}e^{i\psi_{jk}} \), where \( r_{jk} \in \mathbb{R} \) and \( 0 \leq \psi_{jk} < \pi \). So, 
\[
e^{2i\psi_{jk}} = a_{jk}/\bar{a}_{jk} = e^{i\theta_j},
\]
and therefore \( \psi_{jk} = \theta_k/2 \). Writing the unitary condition for \( A \) in terms of \( r_{jk} \) and \( \psi_{jk} \), we have 
\[
\delta_{jl} = \sum_k a_{jk}\bar{a}_{lk} = \sum_k r_{jk}r_{lk}e^{i(\psi_{jk} - \psi_{lk})} = \sum_k r_{jk}r_{lk}.
\]
Thus \( r_{jk} \) is an orthogonal matrix. Furthermore, the condition \( A\overline{A} = \text{Id} \) translates to 
\[
\delta_{jl} = \sum_k a_{jk}\bar{a}_{kl} = \sum_k r_{jk}r_{kl}e^{i(\theta_k - \theta_l)/2}.
\]
In particular, taking \( j = l \), we obtain 
\[
1 = \sum_k r_{jk}r_{kj} \cos((\theta_k - \theta_j)/2).
\]
Writing \( r'_{jk} = \cos((\theta_k - \theta_j)/2)r_{jk} \), we can reformulate the preceding equation in terms of the inner product of the row and column vectors \( r'_j \) and \( r_j \). Namely, 
\[
\langle r'_j, r_j \rangle = 1. \tag{2.1}
\]
On the other hand, since \( r_{jk} \) is unitary, \( |r_{jk}| = 1 \) and 
\[
|r'_j|^2 = \sum_k r'_{jk}^2 \cos^2((\theta_k - \theta_j)/2) \leq \sum_k r_{jk}^2 = |r_j|^2 = 1,
\]
with equality only if \( r_{jk} = 0 \) when \( k \neq j \). Applying Cauchy-Schwartz to equation (2.1), we have
\[
1 \leq |r_{jk}| = |r_{jk}^T| \leq 1.
\]
Thus equality must hold, and the matrix \( r_{jk} \) is diagonal. Moreover, orthogonality implies that \( r_{jk} = \pm \delta_{jk} \). Summing up, \( A \in \Theta_0^{-1}(\text{Id}) \) if and only if we have \( A = A^\epsilon \), where
\[
eq (\epsilon_1, \ldots, \epsilon_n), \quad \epsilon_k \in \{0,1\},
\]
and \( A^\epsilon \) is the matrix with elements
\[
a_{jk}^\epsilon = e^{i(\theta_k/2 + \epsilon_k \pi)}.
\]
In particular, \( \Theta_0^{-1}(\text{Id}) \) has unsigned cardinality \( 2^n \).

It remains to show that \( \text{Id} \) is a regular value and compute the signs. Let \( \text{Sym}(n) \) denote the space of real \( n \times n \) symmetric matrices. It is easy to see that the tangent space to \( \mathcal{L} \) at \( A = \text{Id} \) is given by
\[
T_{\text{Id}} \mathcal{L} = \{ T \in \mathfrak{u}(n) | T + T^T = 0 \} = \{ iQ | Q \in \text{Sym}(n) \}.
\]
Recall that \( U(n) \) acts on \( \mathcal{L} \) by \( A \mapsto UAU^{-1} \). Thus, if \( A = UU^{-1} \), we have an isomorphism
\[
\kappa_U : T_{\text{Id}} \mathcal{L} \xrightarrow{\sim} T_A \mathcal{L}
\]
given by \( T \mapsto UTU^{-1} \). Since \( \mathcal{L} \) is a \( U(n) \) homogeneous space, the isomorphism \( \kappa_U \) preserves orientation. Moreover, for \( T \in T_A \mathcal{L} \) we have
\[
d\Theta_0|_{A^\epsilon}(T) = T\overline{A^\epsilon} + A^n\overline{T} = T\overline{A} + \overline{A} T.
\]
If \( U^\epsilon \in U(n) \) satisfies
\[
A^\epsilon = U^\epsilon(\overline{U}^\epsilon)^{-1},
\]
then \( A^\epsilon \) is a regular point of \( \Theta_0 \) if the linear map
\[
\alpha^\epsilon = d\Theta_0|_{A^\epsilon} \circ \kappa_U : T_{\text{Id}} \mathcal{L} \to T_{\text{Id}} \mathcal{L}
\]
is invertible, and in that case the sign of \( A^\epsilon \) is sign \( \det(\alpha^\epsilon) \). Explicitly,
\[
\alpha^\epsilon(T) = U^\epsilon T(\overline{U}^\epsilon)^{-1} \overline{A} + \overline{A} U^\epsilon T(\overline{U}^\epsilon)^{-1} = U^\epsilon T(U^\epsilon)^{-1} + \overline{U}^\epsilon T(U^\epsilon)^{-1} = U^\epsilon T(U^\epsilon)^{-1} - \overline{U}^\epsilon T(U^\epsilon)^{-1} = 2i \text{Im}(U^\epsilon T(U^\epsilon)^{-1}).
\]
Writing \( T = iQ \), we can think of \( \alpha^\epsilon \) as the map \( \text{Sym}(n) \to \text{Sym}(n) \) given by
\[
\alpha^\epsilon(Q) = 2 \text{Re}(U^\epsilon Q(U^\epsilon)^{-1}).
\]
For convenience, we take \( U^\epsilon \) to be the unitary linear map given by
\[
u_{jk}^\epsilon = e^{i(\theta_j/4 + \epsilon_k \pi/2)} \delta_{jk}.
\]
Then, denoting by \( q_{jk} \) the matrix elements of \( Q \), we have
\[
\alpha^\epsilon(Q)_{jk} = 2 \text{Re} \left( e^{i((\theta_j - \theta_k)/4 + (\epsilon_j - \epsilon_k) \pi/2)} \right) q_{jk} = 2 \cos \left( ((\theta_j - \theta_k)/4 + (\epsilon_j - \epsilon_k) \pi/2) \right) q_{jk}.
\]
Since $Q$ is a symmetric matrix, it is determined by $q_{jk}$ for $j \leq k$. Thus
\[
\det(\alpha^\epsilon) = \prod_{j \leq k} 2 \cos \left( \frac{(\theta_j - \theta_k)}{4} + \frac{(\epsilon_j - \epsilon_k)\pi}{2} \right) q_{jk}.
\]
We need to show this determinant does not vanish and compute its sign. For $j = k$, clearly
\[
\cos \left( \frac{(\theta_j - \theta_k)}{4} + \frac{(\epsilon_j - \epsilon_k)\pi}{2} \right) = 1.
\]
For $j < k$, by assumption, $0 < \theta_j < \theta_k < 2\pi$, so
\[
-\frac{\pi}{2} < \frac{\theta_j - \theta_k}{4} < 0.
\]
It follows that for all $j \leq k$, we have
\[
\cos \left( \frac{(\theta_j - \theta_k)}{4} + \frac{(\epsilon_j - \epsilon_k)\pi}{2} \right) \neq 0.
\]
Therefore $\det(\alpha^\epsilon) \neq 0$ for all $\epsilon$ and $\text{Id}$ is a regular value. Moreover,
\[
\cos \left( \frac{(\theta_j - \theta_k)}{4} + \epsilon_j - \epsilon_k \right) \pi/2 < 0 \iff \epsilon_j = 0, \epsilon_k = 1.
\]
Let $\Upsilon_n$ be the set of all binary sequences $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$. For $\epsilon \in \Upsilon_n$ define $\text{sign}(\epsilon)$ to be the number modulo 2 of pairs $j < k$ such that $\epsilon_j = 0$ and $\epsilon_k = 1$. The upshot of the preceding calculations is that
\[
\text{sign det}(\alpha^\epsilon) = \text{sign}(\epsilon),
\]
therefore
\[
\deg \Theta_0 = \sum_{\epsilon \in \Upsilon_n} (-1)^{\text{sign}(\epsilon)}.
\]
A combinatorial argument given below in Lemma 2.2 then implies the theorem. □

**Lemma 2.2.** For $n = 2m + 1$, we have
\[
d_n := \sum_{\epsilon \in \Upsilon_n} (-1)^{\text{sign}(\epsilon)} = 2^{m+1}.
\]

**Proof.** Let $M_n$ denote the number of $\epsilon \in \Upsilon_n$ such that $\text{sign}(\epsilon) = 0$. Then
\[
d_n = M_n - (2^n - M_n) = 2M_n - 2^n.
\]
For $\epsilon \in \Upsilon_n$ denote by $\text{par}(\epsilon)$ the parity of $\epsilon$, or in other words the number modulo 2 of $j$ such that $\epsilon_j = 1$. Let $P_n$ denote the number of $\epsilon \in \Upsilon_n$ such that $\text{sign}(\epsilon) + \text{par}(\epsilon) = 0$. By analyzing what happens when we adjoin either 1 or 0 to the beginning of a sequence $\epsilon \in \Upsilon_{n-1}$, we find that
\[
M_n = M_{n-1} + P_{n-1}, \quad P_n = (2^{n-1} - P_{n-1}) + M_{n-1}.
\]
Iterating these recursions twice, we obtain
\[
M_n = M_{n-2} + P_{n-2} + 2^{n-2} - P_{n-2} + M_{n-2} = 2M_{n-2} + 2^{n-2}.
\]
Clearly $M_1 = 2$, so $d_1 = 2$. Using the preceding recursion for $M_n$, we obtain
\[
d_n = 2(2M_{n-2} + 2^{n-2}) - 2^n = 2(2M_{n-2} - 2^{n-2}) = 2d_{n-2}.
\]
The lemma follows by induction. □
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