Comment on

SMOOTH AND DISCONTINUOUS SIGNATURE TYPE CHANGE IN GENERAL RELATIVITY

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ABSTRACT

Kossowski and Kriele [1] derived boundary conditions on the metric at a surface of signature change. We point out that their derivation is based not only on certain smoothness assumptions but also on a postulated form of the Einstein field equations. Since there is no canonical form of the field equations at a change of signature, their conclusions are not inescapable. We show here that a weaker formulation is possible, in which less restrictive smoothness assumptions are made, and (a slightly different form of) the Einstein field equations are satisfied. In particular, in this formulation it is possible to have a bounded energy-momentum tensor at a change of signature without satisfying their condition that the extrinsic curvature vanish.

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1. INTRODUCTION

If a “spacetime” contains regions of both Lorentzian and Riemannian signature, then the metric must be degenerate at the boundary between them. One way this can occur is for the metric to be continuous, but have vanishing determinant at the boundary, while another possibility is for the metric to be discontinuous at the boundary; in both cases we assume that the metric is piecewise smooth. We will refer to these two possibilities as the continuous metric and discontinuous metric versions of signature change, respectively. (These correspond to Kossowski and Kriele’s type changing spacetimes and type changing spacetimes with jump, respectively [1].)

One wishes to consider Einstein’s field equations for such signature-changing metrics, but there is an immediate problem: The derivation of these equations assumes that the metric is nondegenerate. This deserves emphasis: There are no canonically defined “Einstein field equations” in the presence of signature change.

In the continuous metric case, one reasonable way to proceed is as follows: Formally compute the Einstein tensor, and investigate the resulting set of singular differential equations. One can then ask what smoothness assumptions must be placed on the metric in order for these equations to be well-defined. Kossowski and Kriele [1] give one such smoothness condition; we show below that a weaker condition is also possible.

In the discontinuous metric case, one possibility, adopted by Kossowski and Kriele, is to postulate that the discontinuous metric field equations be obtained by formally substituting the discontinuous metric into the continuous metric field equations. They then investigate the conditions needed for these equations to make sense. While they correctly construct such a class of solutions, it must be emphasized that there is no way to derive the field equations themselves within this class; they must be postulated separately. Because of this, Kossowski and Kriele’s Remark 2 criticizing Ellis et al. [2] is not valid, as the latter have not assumed the same form of the field equations.

In our approach, we require that the discontinuous metric field equations be obtained as a suitable limit of the continuous metric field equations, which results in a slightly different form of the Einstein tensor for discontinuous metrics. We show here that these equations can be satisfied under weaker assumptions than those made by Kossowski and Kriele. We previously showed [3] that, at a surface of signature change, these weaker assumptions lead to a jump in the Einstein tensor and a surface effect in the conservation law. Which set of assumptions to make depends on what problem one is solving, and ultimately on the as yet ambiguous notion of what it means for there to be, or more precisely for there not to be, a surface layer at a boundary at which the metric signature changes. (One approach to this problem has recently been discussed in [4], where a notion of surface layer is derived from a piecewise Einstein-Hilbert action. This generalizes one of the approaches discussed by Embacher [5], who considered the implications of several different actions for signature change.)

2. EINSTEIN TENSOR

To emphasize the fundamental differences between the two approaches, we will discuss here the construction of the Einstein tensor for a particular class of signature-changing spacetimes. Consider a homogeneous isotropic universe with scale factor $a(t)$ and squared
lapse function $N(t)$, with metric

$$ds^2 = -N(t)\, dt^2 + a^2(t) \, h_{ij} \, dx^i dx^j$$  \hspace{1cm} (1)$$

where

$$h_{ij} \, dx^i dx^j = \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta \, d\chi^2)$$  \hspace{1cm} (2)$$

Considering for simplicity the case $k = 0$, formal calculation leads to the Einstein tensor

$$G^t_t = -\frac{3\, a'^2}{Na^2}$$  \hspace{1cm} (3)$$
$$
$$G^r_r = \frac{N' a'}{N^2 a} - \frac{(2aa'' + a'^2)}{Na^2} = G^\theta_\theta = G^\chi_\chi$$  \hspace{1cm} (4)$$

where $a' := \partial a/\partial t$.

Let $\Sigma$ denote the hypersurface $\{t = 0\}$, and suppose that $N|_\Sigma = 0$ and $dN|_\Sigma \neq 0$. We initially make no demands on $a$ other than that it be piecewise smooth. We define proper time $\tau$ by

$$\tau = \int_0^t \sqrt{\varepsilon N} \, dt$$ \hspace{1cm} (5)$$

and we introduce the notation

$$\dot{a} = \frac{da}{d\tau} = \frac{a'}{\sqrt{\varepsilon N}}$$  \hspace{1cm} (6)$$

so that for $t \neq 0$

$$\ddot{a} = \varepsilon \left( \frac{a''}{N} - \frac{a' N'}{2N^2} \right)$$  \hspace{1cm} (7)$$

where $\varepsilon := \text{sgn}(N)$. Inserting these expressions into (3) and (4) results in

$$G^t_t = -3\varepsilon \frac{\dot{a}^2}{a^2}$$  \hspace{1cm} (8)$$
$$
$$G^r_r = -\varepsilon \frac{2a\ddot{a} + \dot{a}^2}{a^2}$$  \hspace{1cm} (9)$$

away from $t = 0$.

For discontinuous metrics, Kossowski and Kriele simply take equations (3) and (4), derived for continuous metrics, change the form of $N$, and postulate that the result also holds for discontinuous metrics. They then show that, provided $a(\tau)$ is $C^2$, the resulting Einstein tensor is bounded if and only if the extrinsic curvature of the boundary is zero, i.e. if and only if $da/d\tau|_\Sigma = 0$.

We postulate here an alternative form of the Einstein tensor for discontinuous metrics simply by noting that the above form (8), (9) of the Einstein tensor does not contain $N$. For discontinuous metrics, the rest is easy: We now assume that $a(\tau)$ is $C^2 -$ (i.e. the
second derivative exists but may be discontinuous). This is essentially the Darmois junction conditions, and implies that \( \ddot{a} \) may contain a step function but no (Dirac) distribution. But this means that (7) and (9) are valid everywhere! We thus postulate Einstein’s equations as relating (8) and (9) to the appropriate components of the energy-momentum tensor. There are no further restrictions. There are no distributional terms in the Einstein tensor, which is bounded.

For continuous metrics, we still require \( a(\tau) \) to be \( C^2 \), but this requirement now takes the form

\[
-\infty < \lim_{t \to 0^-} \frac{a_+'}{\sqrt{-N}} = \lim_{t \to 0^+} \frac{a_-'}{\sqrt{+N}} < \infty, \quad \lim_{t \to 0^\pm} \left( \frac{\ddot{a}_+ - \ddot{a}_- N'}{N^2} \right) < \infty
\]  

(10)

The first of conditions (10) is precisely the Darmois boundary condition, and because of (10), the particular combination of derivatives of \( a \) occurring in (3) and (4) is well-behaved, so that the Einstein tensor is at worst discontinuous. However, as outlined below, this requires a particular choice of the measure with respect to which distributions are to be defined (cf. [6]).

If we write

\[
a = a_-(1 - \Theta) + a_+ (\Theta)
\]

(11)

where \( a_\pm \) are smooth and where \( \Theta \) is the Heaviside function, then the distributional part of the field equations occurs in the \( a'' \) term of (4). The key observation is to note that in order to interpret the distribution

\[
D = \frac{[a']}{\varepsilon N} \Theta'
\]

(12)

(where \( [a'] := \lim_{t \to 0^+} a_+ - \lim_{t \to 0^-} a_- \)), one must first give the measure with respect to which distributions are to be defined. The choice of measure corresponds to deciding whether \( \Theta' \) or \( \dot{\Theta} \) is “the” Dirac distribution. We choose \( \dot{\Theta} \) because it is defined using proper time \( \tau \), so that (12) must be rewritten as

\[
D = [\dot{a}] \dot{\Theta}
\]

(13)

But the vanishing of \( D \) is just the Darmois junction condition, which we are assuming anyway. The term \( a''/\varepsilon N - a' N'/2\varepsilon N^2 \) is thus at worst discontinuous, and can be multiplied with the discontinuous function \( \varepsilon \); there is no (Dirac) distributional term in the Einstein tensor.

3. DISCUSSION

Kossowski and Kriele propose (3) and (4) as the Einstein equations, where \( a \) is a \( C^2 \) function of \( t \). They show that \( T^\mu_\nu \) is bounded if and only if \( da/d\tau|_{\Sigma} = 0 \), so that the extrinsic curvature must vanish at \( \Sigma \). We propose (8) and (9) as the Einstein equations, where \( a \) is a \( C^2 \) function of \( \tau \). In this case, \( T^\mu_\nu \) is bounded automatically, since \( \ddot{a} \) is finite (but possibly discontinuous) at \( \Sigma \), and \( da/d\tau|_{\Sigma} \neq 0 \) in general, so that the extrinsic curvature does not need to vanish at \( \Sigma \). This supports the position of Ellis et al. [2], and contradicts Kossowski and Kriele’s Remark 2 [1].
In the absence of a derivation of the distributional, signature-changing Einstein field equations from first principles, one should be careful not to claim that a particular form of these equations is “the” field equation. Rather, one must investigate and compare the properties of alternative definitions [4,5]. (We note that it has recently been shown [7] that the difference in results obtained by various authors in signature change calculations may be interpreted as stemming from whether or not the effective proper time coordinate becomes imaginary in the region with Euclidean signature.)

Our approach is analogous to that of Dray, Manogue, and Tucker for the scalar field [8], and that of Ellis [2] and Hellaby and Dray [3], whereas Kossowski and Kriele’s approach is a rigorous version of Hayward’s point of view [9]. For discontinuous metrics, the essential difference is in the Einstein tensors used, which differ by a distributional term. For continuous metrics, the essential difference is in the required smoothness of the fields: They require $a$ to be a smooth ($C^3$!) function of coordinate time $t$, whereas we consider distributional solutions in which $a$ is a smooth ($C^{2-}$) function of proper time $\tau$; our solutions are only $C^1$ as functions of $t$ (but satisfy additional conditions, namely (10)). Neither of these two function spaces contains the other. Kossowski and Kriele show that to obtain a bounded Einstein tensor in their approach one must impose an additional boundary condition, namely $a|_\Sigma = 0$. With this condition, their solution space turns out to be a subspace of ours. In the absence of any intrinsic criteria to select a preferred version of the distributional Einstein field equations in the presence of signature change, both theories are reasonable.

It is important to note that $t$ and $\tau$ are not both admissible coordinates on the same manifold. One must thus make a choice at the beginning between the corresponding differentiable structures, which amounts to deciding between the continuous metric and discontinuous metric approaches. This raises the question of whether these two approaches should be, in an appropriate sense, equivalent. We emphasize that (8) and (9) are equivalent to (3) and (4) for continuous $N$, whereas Kossowski and Kriele’s approach for discontinuous metrics contains an extra distributional term. Our choice of $\Theta$ as the Dirac distribution in the continuous case may be mathematically non-standard, but is motivated by a preference for working with physically measurable quantities — proper times and distances. This choice also serves to unify the continuous and discontinuous approaches; if one rejects it, then there does appear to be a significant difference between them.

In conclusion, we reiterate that Kossowski and Kriele’s theorems on the necessary conditions for the energy-momentum tensor to be bounded are valid, provided one is willing to make their more restrictive assumptions. Our approach leads to less restrictive assumptions, but nevertheless results in a theory with bounded energy-momentum. Both theories are viable.

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