Absorbing Phase Transitions with Coupling to a Static Field and a Conservation Law

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The coupling of branching-annihilating random walks to a static field with a local conservation law is shown to change the scaling properties of their phase transitions to absorbing states. In particular, we find that directed-percolation-class transitions give rise to a new universality class distinct from that characterizing the depinning of the so-called linear interface model.

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Among the many works aiming at an understanding of universality in out-of-equilibrium critical phenomena, those on transitions to absorbing states play a leading role because these phase transitions have no equilibrium counterparts and even occur in one-dimensional (1D) systems [1,2]. In the simple and general case of the reaction and diffusion of identical particles A without site occupation restriction, recent numerical progress (awaiting analytical confirmation) has led to a global picture involving four basic universality classes [3]. Consider reactions of the type \[ mA \rightarrow (m + k)A, \] \[ nA \rightarrow (n - l)A \] where \( m, n, k, \) and \( l \) are positive integers. Outside the prominent directed percolation (DP) class, whose simplest representatives are given by single-particle reactions \( (m, n = 1) \), the PCPD and TCPD classes (for “pair/triplet contact process with diffusion” — a vocable largely used for historical reasons) are respectively characterized by reactions involving two \( (m, n = 2) \) and three \( (m, n = 3) \) particles [4]. The fourth class has been considered to be defined by the conservation of the parity of the number of particles, hence its usual name PC (for “parity-conserving”) [4], but it is now clear that this is not its defining feature [5]. Nevertheless, the relevance of conservation laws is ascertained at equilibrium, and it remains important to explore it within absorbing phase transitions (APT), since conserved quantities abound in physical situations.

As a matter of fact, it was argued that DP-class problems where the order parameter is coupled to an auxiliary (diffusive) field with conservation generally show non-DP critical properties [6]. The particular case where the auxiliary field is static was conjectured recently to lead to yet another class of APT [7,8]. (Note that the corresponding models then possess infinitely-many non-connected absorbing states and strong memory effects.) Numerical simulations confirmed only partially the above ideas. The situation, in our view, remains unsatisfactory for the static case: while in 1D no definite conclusions could be reached [8], in 2D and 3D the numerically-estimated critical exponents, when self-consistent, were found to roughly coincide with those of the depinning transition of the so-called linear interface model (LIM) [10]. A significant departure from DP-scaling was thus found, but the existence of a separate universality class remains in question. Meanwhile, no rigorous result is available, although a heuristic mapping on the dynamics of LIM was advocated in [11].

In this Letter, we investigate the relevance of conservation laws to APT within the more general framework of four “basic” universality classes briefly recalled above, but restricting ourselves to the coupling to a static auxiliary field: now the creation of active, diffusive particles \( A \) is conditioned to the presence of passive, static particles \( B \) in such a way that the total number of particles is locally conserved. We thus study the reactions \[ mA + kB \rightarrow (m + k)A, \] \[ nA \rightarrow (n - l)A + lB, \] and find evidence for the existence of new universality classes. We revisit some of the models studied in [7,8,9], but we mostly deal with two-species reaction-diffusion models derived from those introduced in [6]. We show the existence of a specific “C-DP” (“DP with conservation”) class distinct from both DP and LIM in space dimensions 1 to 3, and point at a crucial difference with the LIM which was overlooked in [11]. We also show that a specific C-PCPD class exists, while no significant departure from TCPD and PC scaling could be observed in the presence of the coupling to a static field with conservation.

Extending the notation introduced in [6], we encode the rules by the order of the branching and annihilation reactions using letters \( s, p, t \) (standing for singleton, pair, triplet) for \( m, n = 1, 2, 3 \), followed by integers \( k \) and \( l \). For instance, reactions \[ 2A + B \rightarrow 3A, \] \[ 2A \rightarrow 2B, \] and \( A \rightarrow B \) are coded as “conserved PCPD rule” C-pp12. Our two-species bosonic models are updated in two parallel substeps: \( A \) particles move to one of their nearest neighbors (strong diffusion), then on-site reactions take place, involving the \( n_A \) and \( n_B \) particles present locally.

We first present our results on the implementation of the C-ss11 rule studied in [6]. In this case, the branching reaction \( A + B \rightarrow 2A \) is performed for each of the \( n_A \) particles present with probability \( p(n_B) = 1 - 1/2^{n_B} \), while the annihilation reaction \( A \rightarrow B \) is performed every \( A \) particle with probability \( q \). Note that since \( p(0) = 0 \), branching is indeed conditioned to the presence of \( B \) particles. Unlike in [6], where the total density of particles is used as a control parameter, here we vary \( q \), which
allows to start from perfectly homogeneous initial conditions (typically one A particle on each site). At large q, annihilation dominates and a static configuration of B particles is quickly reached, while at small q frequent branching ensures a stationary density of A particles.

Our numerical methodology is standard: monitoring various order parameters (e.g. \( \rho_A \) the density of A particles), we first determine the critical point \( q_c \), separating decay to an absorbing state from sustained activity, expecting then an algebraic law \( \rho_A \sim t^{-\delta} \) with \( \delta = \beta/\nu_A \).

We record also, during these runs, the mean squared local gradient \( \langle (\nabla h)^2 \rangle \) of the interface \( h(x,t) \), where \( h(x,t) \) is the time integral of the local activity at site \( x \) until time \( t \) (\( h \to h + 1 \) whenever \( n_A > 0 \), with \( h = 0 \) initially). First introduced in [12] for DP-class models, this fictitious interface was shown to have “anomalous scaling” in the form of local gradients diverging algebraically at the critical point: \( \langle (\nabla h)^2 \rangle \sim t^{2\kappa} \). (Note that it is also the interface conjectured to behave like in the LIM by Alava and Muñoz for C-DP critical points [11].) After this first series of runs, we estimate the decay of the (stationary) order parameter with the distance to the previously-estimated threshold, yielding an estimate of exponent \( \beta \). Finally, exponent \( z \) is estimated via the finite-size scaling of the lifetime of activity at threshold.

Table I summarizes our findings for the C-ss11 rule. For usual DP models were measured by ourselves with conditions (typically one \( \rho_A \) initially). Finally, if \( n_A = 1 \), the branching reaction can occur (with probability \( p \) and if \( n_B > 0 \)).

Rule C-ss11, implemented as above, was investigated in

![FIG. 1: 1D data from the C-ss11 rule implemented as in [8] and from the Leschhorn automaton for the LIM [13]. Near-critical decay from fully active homogeneous initial conditions (system size \( 2^{12} \) sites). (a): density of active particles \( \rho \) vs time (top 3 curves: Leschhorn model with \( p = 0.80085, 0.8008, 0.80075 \); bottom 5 curves: C-ss11 model with \( 1 - q = 0.82858, 0.8286, 0.82861, 0.82863, 0.82865 \). Top inset: \( \rho \times t^{0.14} \) vs time. Bottom inset: \( \rho \times t^{0.125} \) vs time. (b): growth of \( y^2 = \langle (\nabla h)^2 \rangle \) during the same runs at criticality for the C-ss11 model (1 − \( q_c = 0.82861 \), top curve) and for the Leschhorn automaton (\( p_c = 0.8008 \), bottom curve).]
1, 2, and 3 dimensions, yielding exponent values equal, to numerical accuracy, to those reported in Table III. Rules C-ss21, C-sp12, and C-ps11 were also studied thoroughly in 1D with, again, exponent values fully compatible with those found for rules C-ss11 (Fig. 3). All these results give credence to the existence of specific C-DP universality class, distinct from both the DP and the LIM classes.

We now report on our results on conserved PCPD, TCPD, and PC systems, which we only investigated in 1D. For rules C-pp12 and C-pp22, our estimates of critical exponents reveal an identical departure from the PCPD class which we interpret as testifying to the existence of a C-PCPD class. In particular, the critical decay experiments allow to clearly rule out the PCPD value of the δ exponent (Fig. 4). We find, for both rules studied: \( \delta = 0.17(1), \beta = 0.32(2) \) and \( z = 1.55(5) \), whereas our estimates for the PCPD class are \( \delta = 0.20(1), \beta = 0.37(1) \) and \( z = 1.70(5) \). The careful study of rules C-tt12 and C-tt22 did not reveal any significant departure from TCPD scaling for these C-TCPD rules. Similarly, we found only small influence of the coupling to a static field for generalized voter (PC) rules C-mp22 and C-mp42. We have, at present, no reason to believe that the C-PC and C-TCPD classes “do not exist”.

Before our main conclusions, we come back to the suggested equivalence between the LIM and C-DP classes which is contradicted by our numerical results. In the Leschhorn automaton, used here for our estimates of LIM class exponents, an integer-height interface evolves in parallel according to:

\[
    h_i \rightarrow h_i + 1 \quad \text{iff} \quad \nabla^2 h_i + f_i > 0
\]

where \( i \) is a lattice index and \( \nabla^2 h_i \) is the discrete Laplacian calculated with the nearest-neighbors of \( i \). A key ingredient is the quenched bimodal random “force” \( f_i \) (\( f_i = \pm 1 \) with probability \( p \) and \( 1 - p \)) which is redrawn every time the interface advances.

The evolution of the fictitious interfaces from which exponent \( \kappa \) is estimated in C-DP models can be cast into an equation very similar to (1) (11). This is perhaps best seen when considering the Manna or “fixed energy”...
sandpile, a much-studied model of the C-DP type which, unfortunately, suffers from severe corrections to scaling \[^{12}\]\[^{16}\]. There, if the number \(z_i\) of “sand grains” present at lattice site \(i\) is equal to or exceeds a threshold \(z_c\) usually taken to be 2, then \(z_c\) grains independently hop to a randomly-chosen nearest-neighbor. This defines active and passive sites, corresponding to the above transitions with coupling to an auxiliary field requires, this should now be tested against the general framework put forward here. Similarly, the case of a diffusive field with conservation should be extended beyond the DP-class. Both directions are currently being investigated.

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