Time reversal and Lanczos iterations

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Abstract. We describe a new time-reversal algorithm capable of focusing on multiple scatterers in a relatively small number of iterations. We recognize that the traditional time-reversal method is based on utilizing power iterations to determine the dominant eigenpairs of the time-reversal operator. The convergence properties of these iterations are known to be sub-optimal. Motivated by this we propose and describe a new time-reversal method based on Lanczos iterations. We consider a simplified scatterer identification problem and compare the performance of the Lanczos and power iterations based methods. We conclude that for the same number of transmitted and received signals, the Lanczos iterations based approach is substantially more accurate.

1. Introduction

Time reversal-methods are used in several applications that involve the targeting of inhomogeneities or scatterers in a propagating medium [1]. These include medical applications such as overcoming hypothermia using focused ultrasound and lithotripsy in which kidney stones are broken up using ultrasound waves. These methods also find applications in the detection of voids and flaws in solid materials, in the detection of mines and other objects buried in sediments underwater, and in communications in reverberant environments.

The basic idea in time-reversal is rather simple and is derived from the symmetry in time of the second order wave equation [2]. It may be easily explained in the context of a series of transmitters/receivers that are surrounded by an inhomogeneous medium. An arbitrary signal is transmitted by each transducer of the array and the corresponding scattered signal is collected by all the receivers. The collected signal is then reversed in time and this new signal is then transmitted into the medium. This process of transmission, time-reversal and retransmission is repeated several times. It can be shown that with each iteration of the process the transmitted signal selectively focuses on the strongest scatterer in the medium.

The time-reversal iterations described above are effective when there is one dominant scatterer in the medium. Their usefulness is limited when multiple scatterers of comparable strength are present. In the multiple scatterer case the iterations converge to the strongest scatterer with a rate that is proportional to the ratio of the strength of the second strongest scatterer to the strongest scatterer. Clearly, in the case of multiple scatterers of similar strength, the convergence can be poor. Further the recovery of weak scatterers is cumbersome, even when
their strengths are very different. In order to locate weaker scatterers, the strongest scatterer needs to be identified first. Thereafter, starting from a new initial transmitted signal, time-reversal iterations have to be repeated and care has to be exercised to subtract the transmitted signal corresponding to the strongest scatterer at each instance. This can lead to a large number of transmitting and receiving cycles if several scatterers are to be identified.

The time-reversal method described above can be interpreted as the power iteration method applied to the time-reversal operator $H$. This operator is related to the scattering operator $G$ through $H = G^T G$, where $G^T$ is the transpose of $G$ in the time domain. Each iteration in the time-reversal method corresponds to one iteration of the power method, that is the evaluation of the product $Hv$ for a signal $v$. The power iterations converge to the eigenvector corresponding to the largest eigenvalue of $H$. This eigenvector is the time-reversed scattered field, restricted to the measurement array. Clearly when this field is transmitted, it focuses on the strongest scatterer. Indeed, the power method performs so poorly in cases of multiple scatterers, that many authors advocate simply measuring the entire scattering operator in these cases, and then performing an SVD offline.

When compared with other Krylov subspace methods, the convergence properties of power iterations are rather poor, especially when multiple eigenvalues and eigenvectors are desired [3, 4]. Motivated by this observation we consider the application of Lanczos iterations to the time-reversal operator and propose a new time-reversal method that uses Lanczos iterations in an experiment, just as the current version of the time-reversal method uses power iterations. This new method inherits the properties of Lanczos iterations and its advantages over power iterations. In particular, we demonstrate that when compared with the power iterations based method it requires fewer iterations to converge to an accurate answer. This observation directly translates to fewer transmitting and receiving cycles in an experimental setting. It is worth reemphasizing here that the method described below offers rapid convergence to multiple eigenvectors with few transmission cycles. The method does not depend upon measuring the entire scattering operator.

The layout of the remaining part of this manuscript is as follows. In the following section the traditional power iterations based time-reversal method is analyzed. Thereafter the new Lanczos iterations based method is described. In Section 3 representative numerical calculations are presented demonstrating the improved convergence of the Lanczos iterations based method. Conclusions are drawn in Section 4.

2. Time reversal using Lanczos iterations

2.1. Preliminaries

Time reversal concepts are conveniently described in terms of scattering and time-reversal operators. These operators act on functions that are defined in a space-time domain and generate functions that are also typically defined in the same domain. Depending on the application, the space-time domain for the functions may be discretized in space, or in time or in both space and time. For example if one considers $N$ receivers/transmitters located at $x_i, i = 1, \cdots, N$ each capable of transmitting and receiving signals in time, then if the distance between these transmitters is large it is useful to consider the spatial domain to be discrete and the time domain to be continuous. In the development below we leave both the space and time domains as continuous for generality. The development for discrete versions is identical.

We consider a surface in $\mathbb{R}^3$ denoted by $\Gamma$ that is continuously embedded with transmitters and receivers. The time domain of interest is $[0, T]$. Thus in our problem transmitted and received signals are functions of $x \in \Gamma$ and $t \in [0, T]$. We require that they be somewhat smooth, that is any signal $v \in \mathcal{V} \equiv L_2(\Omega \times [0, T])$. We denote the $L_2$ inner product on our space-time domain by $(\cdot, \cdot)$ and the corresponding norm by $|| \cdot ||$.

Let the kernel of the scattering operator of the problem of interest be denoted by $g(x, t; x', t')$. The time-reversal operator $H$ is given by $H = G^T G$, where $G^T$ is the transpose of $G$ in the time domain. Each iteration in the time-reversal method corresponds to one iteration of the power method, that is the evaluation of the product $Hv$ for a signal $v$. The power iterations converge to the eigenvector corresponding to the largest eigenvalue of $H$. This eigenvector is the time-reversed scattered field, restricted to the measurement array. Clearly when this field is transmitted, it focuses on the strongest scatterer. Indeed, the power method performs so poorly in cases of multiple scatterers, that many authors advocate simply measuring the entire scattering operator in these cases, and then performing an SVD offline.

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Let the transmitted signal be given by $v(x, t)$. Then the received signal is given by

$$r(x, t) = \int_{\Gamma} \int_{0}^{T} g(x, t; x', t') v(x', t') dx' dt'$$

(1)

This can be written succinctly as

$$r(x, t) = G[v(x, t)]$$

(2)

by defining the scattering operator $G : V \rightarrow V$.

It is also useful to define the kernel of the time-reversal operator as

$$h(x, t; x', t') = \int_{\Gamma} \int_{0}^{T} g(x, t; x'', t'') g(x'', T - t''; x', t') dx'' dt''$$

(3)

Note that the operation of the time-reversal operator on the signal $v$ can be written as

$$r(x, t) = \int_{\Gamma} \int_{0}^{T} h(x, t; x', t') v(x', t') dx' dt'$$

(4)

This can be written succinctly as

$$r(x, t) = H[v(x, t)]$$

(5)

by defining the time-reversal operator $H : V \rightarrow V$.

Let $\{s^{(i)}, \phi^{(i)}(x, t)\}$ be the $i$th eigenpair for $G$, arranged such that $|s^{(1)}| \geq |s^{(2)}| \geq \cdots$. We assume that the eigenvectors have unit norm, that is $(\phi^{(i)}, \phi^{(j)}) = \delta_{ij}$. Then from (3) above and due to the reversibility in time of the wave equations we conclude that the eigenpairs for the time-reversal operator are given by $\{\lambda^{(i)}, \phi^{(i)}(x, t)\}$, where $\lambda^{(i)} = |s^{(i)}|^2$. Note that the time-reversal operator $H$ has the same eigenvectors as the scattering operator $G$.

Under certain circumstances it can be shown that if the eigenvector of the scattering operator is used as the transmitted signal, the wave field selectively focuses on the scatterer associated with the corresponding eigenvalue. Thus determining the eigenvectors of $G$ or $H$ allows an experimentalist to target specific scatterers in the propagation domain. In the following subsection we describe how the power iterations may be used to determine these eigenvectors. This corresponds to the traditional time-reversal method. Thereafter we describe the use of Lanczos iterations.

### 2.2. Power Iterations

We define the following sequence as a single time-reversal iteration.

(i) Transmit $v^{(0)}(x, t)$.

(ii) Receive $v^{(1/2)}(x, t) = G[v^{(0)}(x, t)]$

(iii) Transmit $v^{(1/2)}(x, T - t)$

(iv) Receive $v^{(1)}(x, t) = G[v^{(1/2)}(x, T - t)]$

Using the definition of the scattering operator and the time-reversal operator, the entire sequence above can be written as

$$v^{(1)}(x, t) = H[v^{(0)}(x, t)]$$

(6)

In the time-reversal method based on power iterations the sequence described above is repeated several times. That is $v^{(n)} = H \cdot H \cdots H[v^{(0)}]$. Using the spectral decomposition of $H$ it is
easy to show that as \( n \) increases \( v^{(n)} \) converges to the eigenvector corresponding to the largest eigenvalue of \( H \). That is \( v^{(n)}/\|v^{(n)}\| \approx \phi^{(1)} \), for \( n \) large.

The second eigenvector (which corresponds to the second strongest scatterer) can be determined once the first eigenvector has been determined. This is accomplished by performing power iterations with \( v^{(n)} - (v^{(n)}, \phi^{(1)})\phi^{(1)} \), where \( \phi^{(1)} \) is the previously determined estimate of the first eigenvector. That is at every iteration the component of the signal in the direction of \( \phi^{(1)} \) is annihilated. This way the power iterations will converge to \( \phi^{(2)} \). This process can be repeated to locate weaker scatterers [5].

2.3. Lanczos Iterations

In this section we describe how Lanczos iterations may be used to determine the eigenvectors of the time-reversal operator. For a detailed description of these iterations the reader is referred to either one of the two texts [3, 4].

(i) Begin with an arbitrarily chosen signal \( v^{(1)} \) such that \( (v^{(1)}, v^{(1)}) = 1 \).
(ii) Set \( v^{(0)} = 0 \) and \( \beta_1 = 0 \).
(iii) For \( j = 1, \cdots, n \)

(a) \( w^{(j)} = H[v^{(j)}] - \beta_j v^{(j-1)} \)
(b) \( \alpha_j = (w^{(j)}, v^{(j)}) \)
(c) \( w^{(j)} = w^{(j)} - \alpha_j v^{(j)} \)
(d) \( \beta_{j+1} = \sqrt{(w^{(j)}, w^{(j)})} \)
(e) \( v^{(j+1)} = w^{(j)}/\beta_{j+1} \)

After \( n \) iterations we will have created the \( n \times n \) tridiagonal matrix

\[
T = \begin{bmatrix}
\alpha_1 & \beta_2 & 0 & 0 & 0 & \cdots \\
\beta_2 & \alpha_3 & \beta_3 & 0 & 0 & \cdots \\
0 & \beta_3 & \alpha_4 & \beta_4 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\] (7)

Let \( \{\omega^{(i)}, \psi^{(i)}\} \) be the \( i \)th eigenpair for \( T \). Then we have

(i) \( \lambda^{(i)} \approx \omega^{(i)} \). That is the eigenvalues of \( T \) approximate the eigenvalues of \( H \).
(ii) \( \phi^{(i)} \approx \sum_{j=1}^{m} \psi^{(i)} v^{(j)} \). The eigenvectors of \( H \) can be determined from the eigenvectors of \( T \) and the Lanczos vectors \( v^{(i)} \).

There are several comments to be made here. First, in implementing this algorithm in a physical experiment one would have to store the Lanczos vectors \( v^{(i)}(x, t) \). This is an added memory cost over power iterations. Note, however that this is not a big penalty as these vectors may be stored on the hard disk of a computer. Second, like the power iterations this algorithm also involves a time-reversal step (step 3a). Thus its implementation in an experimental step is similar to that of power iterations, except that every time-reversal step is followed by some additional signal processing. Finally, at the completion of \( n \) iterations approximations to \( n \) eigenpairs may be evaluated. Out of these a small fraction (say about 20%) can be expected to be accurate. The accuracy of these estimates may be increased by increasing the number of iterations.
3. Numerical examples

In this section we compare the performance of time-reversal methods based on power iterations and Lanczos iterations for a simplified problem. We consider a medium with several ($N$) sound-hard point scatterers located at $x_j$, $j = 1, \ldots, N$ clustered about the origin within a sphere of radius $\delta$. The transmitters/receivers are continuously embedded in the $y - z$ plane located at $x = D$, with $D \gg \delta$. We denote this plane by $\Gamma$. We consider the time-harmonic case with frequency $\omega$ and wavenumber $k = \omega/c$, where $c$ is the sound speed in the medium. Further, following [2] we neglect multiple scattering events and assume that scatterers are ideally separated. That is it is possible to focus on one of the scatterers without directing any energy to another scatterer. For this case, the scattering operator $G : L_2(\Gamma) \rightarrow L_2(\Gamma)$ is given by

$$G[v] = \sum_{j=1}^{N} s^{(j)} \phi^{(j)}(y, z) \int_{\Gamma} v(y', z') \phi^{(j)}(y', z') dy' dz'$$

(8)

where $\phi^{(j)}(y, z) = g^{(j)}(D, y, z)/\|g^{(j)}(D, y, z)\|$. In this expression $\| \cdot \|$ denotes the $L_2$ norm on $\Gamma$ and $g^{(j)}$ is given by

$$g^{(j)}(x) = \frac{e^{ik|x-x_j|}}{4\pi|x-x_j|}.$$ 

(9)

In (8) $s^{(j)}$ is a parameter that is proportional to the strength of the $j$th scatterer. The assumption of ideal separation implies that the functions $\phi^{(j)}$ are orthonormal in $\Gamma$, that is

$$\langle \phi^{(i)}, \phi^{(j)} \rangle = \delta_{ij}.$$ 

(10)

Hence $\{s^{(j)}, \phi^{(j)}(y, z)\}$ is the $j$th eigenpair of the scattering operator. The time-reversal operator is given by $H[v] = G^*[G[v]]$. Note that the complex conjugate in this expression is a result of the fact that in the time-harmonic case, the time-reversal of a field corresponds to taking its complex conjugate. Using this definition of $H$ and making use of (8) and (10) yields the following expression

$$H[v] = \sum_{j=1}^{N} \lambda^{(j)*} \phi^{(j)}(y, z) \int_{\Gamma} v(y', z') \phi^{(j)}(y', z') dy' dz'$$

(11)

where $\lambda^{(j)} = |s^{(j)}|^2$. Our goal is to use time-reversal methods to determine the functions $\phi^{(j)}$. Once these functions are known, they can be used to either focus the acoustic energy on to the $j$th scatterer or be used to determine its location.

3.1. Two scatterers

First we consider the simple case of two scatterers, that is $N = 2$. For this case the solution of the time-reversal iterations can be evaluated analytically. Let us start with an initial arbitrary signal $v^{(0)}(y, z)$. Further let $\rho_j = \int_{\Gamma} \phi^{(j)} v^{(0)} dydz$ be the components of this signal along $\phi^{(j)}$. Then after $n$ steps of time-reversal iterations using the power method, the signal $v^{(n)}$ will be given by

$$\frac{v^{(n)}(y, z)}{\rho_1 |s^{(1)}|^2n} = \phi^{(1)*}(y, z) + \rho_2 \frac{s^{(2)}}{\rho_1 |s^{(1)}|^2n} 2^n \phi^{(2)*}(y, z).$$

(12)

In the time-reversal method this expression will be used as an approximation for $\phi^{(1)*}$. From the expression above it is clear that the error is given by $\epsilon_1 = \frac{\rho_2 |s^{(2)}|^2}{\rho_1 |s^{(1)}|^2n}$. Assuming that the
initial guess is arbitrary and thus not aligned with either $\phi^{(1)}$ or $\phi^{(2)}$ we expect $\epsilon_1 = O(1)$. Thus we conclude that the error is determined by the ratio of the strength of the scatterers. Let us consider the case of scatterers of similar strength that is, $s^{(2)} = 0.98s^{(1)}$. Then using this expression, after 10 time-reversal iterations the error is $\epsilon_1 \approx 0.98^{20} = 0.66$. That is about 66% error. Clearly for scatterers with disparate strengths, the error will be smaller. For example, say $s^{(2)} = 0.70s^{(1)}$ this error will be about 0.08% after 10 iterations.

On the other hand, when using Lanczos iterations in the time-reversal method, for this case we recover the functions $\phi^{(1)}$ and $\phi^{(2)}$ in two iterations regardless of the relative strength of the scatterers.

3.2. 200 scatterers

Next we consider the case of 200 scatterers with $s^{(j)}$ selected from a normal distribution of mean 0.5 and standard deviation 1. We use time-reversal methods to determine the eigenfunctions corresponding to the two strongest scatterers. For the power iterations we use the first 10 iterations to determine a guess for $\phi^{(1)}$, then use another 10 iterations in order to determine a guess for $\phi^{(2)}$. This corresponds to a total to 20 time-reversal iterations. In the Lanczos iterations based method also we apply a total of 20 iterations.

In order to obtain reliable indicators of the performance of the methods we consider a total of 10,000 realizations of the 200 scatterer problem. For each realization we apply the time-reversal method (using power and Lanczos iterations) as described above and determine the normalized error in estimating the first two eigenfunctions. Thereafter we plot a histogram of the log of this error across all realizations. This gives us an estimate of the pdf of the error for each method. In Figure 1, we have shown this error for the first eigenfunction estimated using power iterations. In Figure 2, we have shown the error for the second eigenfunction estimated using power iterations. Figures 3 and 4 display errors in estimating the first and second eigenfunctions, respectively, using Lanczos iterations.

![Figure 1](image1.png)  ![Figure 2](image2.png)

**Figure 1.** Error distribution of power iterations estimate of the first eigenfunction.  **Figure 2.** Error distribution of power iterations estimate of the second eigenfunction.

From Figure 1 we observe that the peak in the histogram is around $-0.2$ which corresponds to about 60% error in estimating the first eigenfunction using power iterations. For Lanczos iterations, the corresponding error is close to $10^{-6}$% as seen in Figure 3. This is a remarkable improvement. For the second eigenfunction when using power iterations there appear to be two peaks (see Figure 2). The first is at 0, which corresponds to 100% error, thereby indicating that the results carry no useful information. The second is at $-0.2$ corresponding to about 60%
error. For the second eigenfunction estimated using Lanczos iterations the peak is observed at an error of $10^{-3}\%$. Once again a significant improvement. We also observe that while the Lanczos iterations produce good results, there are several cases for the second eigenfunction (around 200 out of 10,000) where they too have not produced a viable answer. This shortcoming needs to be investigated.

4. Conclusions
We have presented a new time-reversal method for efficiently identifying the eigenvectors of a scattering operator. The eigenvector can be used (e.g. by DORT [6], MUSIC [7], or other methods) to focus on or locate strong scatterers in a propagating medium. Our method makes use of Lanczos iterations in order to converge to the eigenvalues and eigenfunctions of the time-reversal operator associated with these scatterers. When compared to the traditional approach which is based on power iterations, the new approach is shown to have remarkably better convergence properties for the same number of total transmit and receive operations. This property is verified numerically for the simple case of ideally separated point scatterers. Thinking ahead to an experimental validation, the additional costs associated with the new approach (over the traditional approach) are storage of several (as many as time-reversal iterations) time signals for each transmitter on a hard drive and some extra numerical processing.

References
[1] Fink M, Cassereau D, Derode A, Prada C, Roux P, Tanter M, Thomas J and Wu F 2000 Rep. Prog. Phys 63 1995
[2] Prada C, Thomas J and Fink M 1995 The Journal of the Acoustical Society of America 97 62
[3] Golub G and Van Loan C 1996 Matrix computations 3rd ed (Philadelphia, USA: Johns Hopkins University Press)
[4] Saad Y 1995 Iterative methods for sparse linear systems 1st ed (Boston, MA: PWS Publishing Company)
[5] Montaldo G, Tanter M and Fink M 2004 The Journal of the Acoustical Society of America 115 776
[6] Mordant N, Prada C and Fink M 1999 The Journal of the Acoustical Society of America 105 2634
[7] Devaney A, Marengo E and Gruber F 2005 The Journal of the Acoustical Society of America 118 3129