Dynamical Correlations
for Vicious Random Walk
with a Wall

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Abstract

A one-dimensional system of nonintersecting Brownian particles is constructed as the diffusion scaling limit of Fisher’s vicious random walk model. \( N \) Brownian particles start from the origin at time \( t = 0 \) and undergo mutually avoiding motion until a finite time \( t = T \). Dynamical correlation functions among the walkers are exactly evaluated in the case with a wall at the origin. Taking an asymptotic limit \( N \to \infty \), we observe discontinuous transitions in the dynamical correlations. It is further shown that the vicious walk model with a wall is equivalent to a parametric random matrix model describing the crossover between the Bogoliubov-deGennes universality classes.

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1 Introduction

The vicious walk model was first introduced by Fisher and applied to wetting and melting phenomena\[11,2\]. It recently attracts renewed interest in statistical and mathematical physics, since intimate connections were established to other research fields, such as Young tableaux in combinatorics\[3\,4\,5\], asymmetric exclusion process (ASEP) in nonequilibrium statistical mechanics\[6\], Kardar-Parisi-Zhang (KPZ) universality in surface growth process\[7\,8\,9\] and the theory of random matrices\[10,11\]. In the context of random matrix theory, the ensembles of vicious walkers in one dimension correspond to discretizations of random matrix ensembles.

Suppose that there are $N$ walkers on a lattice $\mathbb{Z} = \{ \cdots, -2, -1, 0, 1, 2, \cdots \}$. Starting from $N$ distinct (even integer) positions $2s_1 < 2s_2 < \cdots < 2s_N$, at each time step each walker moves to the left or right one lattice site with equal probability. Let us denote the position of the $j$-th walker at time $k \geq 0$ by $R_{k}^{s_j}$. Walkers are “vicious” so that two or more walkers are prohibited to arrive at the same site simultaneously. That is, the nonintersecting condition

$$R_{k}^{s_1} < R_{k}^{s_2} < \cdots < R_{k}^{s_N}, \quad 1 \leq k \leq K$$

(1.1)

is imposed. In this paper, we further impose a condition

$$R_{k}^{s_j} \geq 0, \quad 1 \leq k \leq K,$$  

(1.2)

which implies that there is a wall at the origin. Let us define $V(R_{K}^{s_j} = 2d_j)$ as the realization probability that the vicious walkers arrive at (even integer) positions $2d_1 < 2d_2 < \cdots < 2d_N$ at an even integer time $K$. Utilizing the Lindström-Gessel-Viennot theorem and the reflection principle of random walks, we find an explicit formula\[2\]

$$V(R_{K}^{s_j} = 2d_j) = 2^{-KN} \det \left[ \left( \frac{K}{K/2 + s_k - d_j} \right) - \left( \frac{K}{K/2 + s_k + d_j + 1} \right) \right]_{j,k=1,2,\ldots,N}.$$  

(1.3)

This realization probability can be simplified in the diffusion scaling limit\[12,13,14,15\]. Let us introduce a positive number $L$ and set $K = Lt$, $s_j = \sqrt{Lt}x_j/2$ and $d_j = \sqrt{Lt}y_j/2$. Then we can easily find

$$f(t; y_1, y_2, \cdots, y_N \mid x_1x_2, \cdots, x_N) \equiv \lim_{L \to \infty} \left( \frac{\sqrt{L}}{2} \right)^N V(R_{Lt}^{\sqrt{Lt}x_j/2} = \sqrt{Lt}y_j)$$

$$= (2\pi t)^{-N/2} \det \left[ \exp \left\{ \frac{-1}{2t} (x_j - y_k)^2 \right\} - \exp \left\{ \frac{-1}{2t} (x_j + y_k)^2 \right\} \right].$$  

(1.4)

This function gives the nonintersecting probability of the Brownian particles on the rescaled lattice $\mathbb{Z}/(\sqrt{L}/2)$ up to times $t$ depending on the initial positions $\{x_j\}$ and final positions $\{y_j\}$. Therefore the probability amplitude that the vicious walkers are located at $x_0^j, x_2^j, \cdots, x_N^j$ at times $t_j, j = 0, 1, 2, \cdots, M$, is given by

$$P(t_0; x_0^0, x_0^2, \cdots, x_0^N; t_1; x_1^1, x_1^2, \cdots, x_1^N; \cdots; t_M; x_M^1, x_M^2, \cdots, x_M^N)$$

$$= \prod_{j=0}^{M-1} \varphi^T(t_j; x_j^1, x_j^2, \cdots, x_j^N; t_{j+1}; x_{j+1}^1, x_{j+1}^2, \cdots, x_{j+1}^N),$$  

(1.5)
where

\[
\varphi^T(s; x_1, x_2, \ldots, x_N; t; y_1, y_2, \ldots, y_N) = f(t - s; y_1, y_2, \ldots, y_N \mid x_1, x_2, \ldots, x_N) \mathcal{N}(T - t; y_1, y_2, \ldots, y_N) \mathcal{N}(T - s; x_1, x_2, \ldots, x_N).
\]

(1.6)

\[
\mathcal{N}(t; x_1, x_2, \ldots, x_N) = \int_{0<y_1<y_2<\ldots<y_N<\infty} dy_1 dy_2 \cdots dy_N f(t; y_1, y_2, \ldots, y_N \mid x_1, x_2, \ldots, x_N).
\]

(1.7)

The dynamical correlation functions among the walkers at times \(t_1, t_2, \ldots, t_M\) are defined as

\[
\rho(x_1^1, \ldots, x_{n_1}^1; x_1^2, \ldots, x_{n_2}^2; \ldots; x_1^M, \ldots, x_{n_M}^M)
= \frac{1}{\prod_{i=1}^M (N - n_i)!} \int_0^\infty \prod_{j=1}^N dx_j^0 \int_0^\infty \prod_{j=n_{i+1}}^N dx_j \cdots \int_0^\infty \prod_{j=n_{M+1}}^N dx_j^M
\times p_0(\{x_j^0\}) \prod_{m=0}^{M-1} \varphi^T(t_m, \{x_j^m\}; t_{m+1}; \{x_j^{m+1}\}).
\]

(1.8)

Here \(p_0(\{x_j^0\})\) is the initial distribution at \(t_0 = 0\). Let us suppose that all the Brownian particles start at the origin so that we can set \(p_0(\{x_j^0\}) = \prod_j \delta(x_j^0)\). Making replacements

\[
x_j^1 \rightarrow \sqrt{y_j}, \quad x_j^2 \rightarrow \sqrt{y_j^2}, \ldots, x_j^M \rightarrow \sqrt{y_j^M}, \quad x_j^{M+1} \rightarrow \sqrt{y_j^{M+1}},
\]

we rewrite the dynamical correlation functions as

\[
\rho(y_1^1, \ldots, y_{n_1}^1; y_1^2, \ldots, y_{n_2}^2; \ldots; y_1^M, \ldots, y_{n_M}^M)
\propto \int_0^\infty \prod_{j=n_{i+1}}^N dy_j^1 \int_0^\infty \prod_{j=n_{i+1}}^N dy_j^2 \cdots \int_0^\infty \prod_{j=n_{M+1}}^N dy_j^M
\times \prod_{j>k}^N (y_j^1 - y_k^1) \prod_{j>k}^N \text{sgn}(y_j^{M+1} - y_k^{M+1}) \prod_{m=1}^M \det \left[ g^m(y_j^m, y_k^{m+1}) \right]_{j,k=1,2,\ldots,N}.
\]

(1.10)

Here

\[
g^1(x, y) = \frac{1}{2\sqrt{2\pi t_1}} \frac{1}{\sqrt{2\pi(t_2 - t_1)y}} \exp \left\{ -\frac{1}{2t_1} x \right\}
\times \left[ \exp \left\{ -\frac{1}{2(t_2 - t_1)} (\sqrt{x} - \sqrt{y})^2 \right\} - \exp \left\{ -\frac{1}{2(t_2 - t_1)} (\sqrt{x} + \sqrt{y})^2 \right\} \right]
\]

(1.11)

and

\[
g^m(x, y) = \frac{1}{2\sqrt{2\pi(t_{m+1} - t_m)y}}
\times \left[ \exp \left\{ -\frac{1}{2(t_{m+1} - t_m)} (\sqrt{x} - \sqrt{y})^2 \right\} - \exp \left\{ -\frac{1}{2(t_{m+1} - t_m)} (\sqrt{x} + \sqrt{y})^2 \right\} \right],
\]

\(2 \leq m \leq M\).
In the case with no wall, the vicious walk model in the diffusion scaling limit is equivalent to the eigenvalue dynamics of parametric random matrices belonging to the standard symmetry class \[13, 14\]. Similarly, in the presence of a wall, it will be shown that the vicious walk model and parametric random matrices with the Bogoliubov-deGennes symmetry are equivalent. The Bogoliubov-deGennes matrix model was proposed by Altland and Zirnbauer as an effective model of mesoscopic normal-conducting-superconducting hybrid structures \[16, 17\].

This paper is organized as follows. In §2, quaternion determinant expressions for the dynamical correlation functions are presented. In §3, we rewrite the quaternion determinant expressions in terms of the Laguerre polynomials. In §4, asymptotic forms of the dynamical correlation functions are evaluated in the limit \(N \to \infty\). In §5, an equivalence between the vicious walk and the Bogoliubov-deGennes matrix model is demonstrated.

2 Dynamical Correlation Functions

2.1 Quaternion Determinant Expressions

We begin with the definition of a quaternion determinant \[18\]. A quaternion is defined as a linear combination of four basic units \(\{1, e_1, e_2, e_3\}\):

\[
q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3. \tag{2.1}
\]

Here \(q_0, q_1, q_2\) and \(q_3\) are real or complex numbers. We call \(q_0\) the scalar part of \(q\). The quaternion multiplication is associative but in general not commutative. The multiplication rule of the four basic units are given by

\[
1 \cdot 1 = 1, \quad 1 \cdot e_j = e_j \cdot 1 = e_j, \quad j = 1, 2, 3, \tag{2.2}
\]

\[
e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1. \tag{2.3}
\]

We define a dual \(\hat{q}\) a quaternion \(q\)

\[
\hat{q} = q_0 - q_1 e_1 - q_2 e_2 - q_3 e_3. \tag{2.4}
\]

A dual is an analogue of the complex conjugate of a complex number. For a matrix \(Q\) with quaternion elements \(q_{ij}\), we can also define a dual matrix \(\hat{Q} = [\hat{q}_{ij}]\). The quaternion units can be represented as \(2 \times 2\) matrices

\[
1 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_1 \rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},
\]

\[
e_2 \rightarrow \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \quad e_3 \rightarrow \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}. \tag{2.5}
\]

Let us now introduce a quaternion determinant \(\text{Tdet}\). For a self-dual \(Q\) ( i.e., \(Q = \hat{Q}\)), it is defined as

\[
\text{Tdet } Q = \sum_p (-1)^{N-p} \prod_{l=1}^l (q_{ab} q_{bc} \cdots q_{da})_0. \tag{2.6}
\]
Here $P$ denotes any permutation of the indices $(1, 2, \cdots, N)$ consisting of $l$ exclusive cycles of the form $(a \rightarrow b \rightarrow c \rightarrow \cdots \rightarrow d \rightarrow a)$ and $(-1)^{N-l}$ is the parity of $P$. The subscript 0 means that the scalar part of the product is taken over each cycle. If all the elements of $Q$ have only the scalar parts, every element is commutable so that a quaternion determinant becomes an ordinary determinant.

In both cases $N$ even and odd, we define quaternion matrices $B^{\mu\nu}$, $\mu, \nu = 1, 2, \cdots, M$ in terms of the following $2 \times 2$ representations of the quaternion elements:

\[
B_{jl}^{\mu\nu} = \begin{bmatrix}
\tilde{S}_{jl}^{\mu\nu} & \tilde{I}_{jl}^{\mu\nu} \\
D_{jl}^{\mu\nu} & \tilde{S}_{lj}^{\mu\nu}
\end{bmatrix}, \quad j, l = 1, 2, \cdots, N.
\] (2.7)

The matrices $\tilde{S}_{jl}^{\mu\nu}, \tilde{I}_{jl}^{\mu\nu}$ and $D_{jl}^{\mu\nu}$ are defined in the following subsections. Applying the integration technique developed in Ref. [19] to the integral (1.10), we find the quaternion determinant expression of the dynamical correlation functions

\[
\rho(y_{1}, \cdots, y_{n_1}; y_{1}^{2}, \cdots, y_{n_2}; \cdots; y_{1}^{M}, \cdots, y_{n_{M}}) = T \det[B^{\mu\nu}(n_{\mu}, n_{\nu})],
\]

\[
\mu, \nu = 1, 2, \cdots, M,
\] (2.8)

where each block $B^{\mu\nu}(n_{\mu}, n_{\nu})$ is obtained by removing the $n_{\mu} + 1, n_{\mu} + 2, \cdots, N$-th rows and $n_{\nu} + 1, n_{\nu} + 2, \cdots, N$-th columns from $B^{\mu\nu}$.

## 2.2 Skew Orthogonal Polynomials

In order to define the matrices $\tilde{S}_{jl}^{\mu\nu}, \tilde{I}_{jl}^{\mu\nu}$ and $D_{jl}^{\mu\nu}$, we need to first introduce skew orthogonal polynomials. In terms of

\[
G_{mn}^{\mu}(x, y) = \begin{cases}
\delta(x - y), & m = n, \\
g_{m}^{\mu}(x, y), & m = n - 1, \\
\int_{0}^{\infty} dy_{m+1} dy_{m+2} \cdots dy_{n-1} \times g_{m}^{\mu}(x, y_{m+1}) g_{m+1}(y_{m+1}, y_{m+2}) \cdots g_{n-1}(y_{n-1}, y), & m < n - 1,
\end{cases}
\] (2.9)

we define

\[
F_{mn}^{\mu}(x, y) = \int_{0}^{\infty} dz' \int_{0}^{z'} dz \{ G_{m}^{\mu M+1}(x, z) G_{n}^{\mu M+1}(y, z') - G_{m}^{\mu M+1}(y, z) G_{n}^{\mu M+1}(x, z') \}.
\] (2.10)

In terms of an antisymmetric inner product

\[
\langle f(x), g(y) \rangle_{m} = \frac{1}{2} \int_{0}^{\infty} dx \int_{0}^{\infty} dy F_{mn}^{\mu}(y, x) [f(y) g(x) - f(x) g(y)],
\] (2.11)

monic polynomials $R_{l}^{k}(x) = x^{k} + \cdots$ of degree $k$ are constructed so that they satisfy the skew orthogonality relations:

\[
\langle R_{l}^{k}(x), R_{l+1}^{k}(y) \rangle_{1} = -\langle R_{l+1}^{k}(x), R_{l}^{k}(y) \rangle_{1} = r_{j} \delta_{jl},
\]
Defining a set of functions $R^m_k(x)$, $m = 2, 3, \cdots, M+1$ as

\[ R^m_k(x) = \int_0^\infty dy R^1_k(y) G^{1m}(y, x), \]

we can immediately find the following skew orthogonality relations for $m = 1, 2, \cdots, M+1$:

\[ \langle R^m_{2j}(x), R^m_{2j+1}(y) \rangle = 0, \quad \langle R^m_{2j+1}(x), R^m_{2j+1}(y) \rangle = 0. \]  \hspace{1cm} (2.12)

2.3 The Case $N$ Even

Let us now introduce matrices $D^{mn}$, $I^{mn}$, $S^{mn}$, $F^{mn}$ and $G^{mn}$ as

\[ D^{mn}_{jl} = \sum_{k=0}^{(N/2) - 1} \frac{1}{r_k} [R^m_{2k}(y_j^m) R^m_{2k+1}(y_l^n) - R^m_{2k+1}(y_j^m) R^m_{2k}(y_l^n)], \]  \hspace{1cm} (2.15)

\[ I^{mn}_{jl} = - \sum_{k=0}^{(N/2) - 1} \frac{1}{r_k} [\Phi^m_{2k}(y_j^m) \Phi^n_{2k+1}(y_l^n) - \Phi^m_{2k+1}(y_j^m) \Phi^n_{2k}(y_l^n)], \]  \hspace{1cm} (2.16)

\[ S^{mn}_{jl} = \sum_{k=0}^{(N/2) - 1} \frac{1}{r_k} [\Phi^m_{2k}(y_j^m) R^n_{2k+1}(y_l^n) - \Phi^m_{2k+1}(y_j^m) R^n_{2k}(y_l^n)], \]  \hspace{1cm} (2.17)

where

\[ \Phi^m_k(x) = \int_0^\infty F^{mn}(y, x) R^m_k(y) dy. \]  \hspace{1cm} (2.18)

Further we define

\[ F^{mn}_{jl} = F^{mn}(y_j^m, y_l^n), \]  \hspace{1cm} (2.19)

\[ G^{mn}_{jl} = 0, \quad m \geq n, \]  \hspace{1cm} (2.20)

\[ G^{mn}_{jl} = G^{mn}(y_j^m, y_l^n), \quad m < n \]  \hspace{1cm} (2.21)

and

\[ \tilde{I}^{mn}_{jl} = I^{mn}_{jl} + F^{mn}_{jl}, \quad \tilde{S}^{mn}_{jl} = S^{mn}_{jl} - G^{mn}_{jl}. \]

2.4 The Case $N$ Odd

In terms of skew orthogonal functions $R^m_k(x)$, another set of functions are defined as

\[ \hat{R}^m_k(x) = R^m_k(x) - \frac{\sigma_n}{\sigma_{N-1}} R^m_{N-1}(x), \quad n = 0, 1, \cdots, N-2, \]  \hspace{1cm} (2.22)

\[ \hat{R}^m_{N-1}(x) = R^m_{N-1}(x), \]  \hspace{1cm} (2.23)

where

\[ \sigma_k = \int_0^\infty dx f^M(x) R^M_k(x) \]  \hspace{1cm} (2.24)
with

\[ f^m(x) = \int_0^\infty dy G^{mn}(x, y). \] (2.25)

Using the definitions

\[ f^m = f^m(y_j) \] (2.26)

and

\[ R_j^m = \hat{R}_j^m(y_j^m) \frac{1}{\sigma_{N-1}}, \] (2.27)

we introduce the matrices \( D_{mn} \), \( I_{mn} \) and \( S_{mn} \) in the case \( N \) odd as

\[ D_{jl}^{(N-3)/2} = \sum_{k=0}^{(N-3)/2} \frac{1}{r_k} \left[ \hat{R}_2^{(m)}(y_j) \hat{R}_2^{(n)}(y_l) - \hat{R}_2^{(m)}(y_j) \hat{R}_2^{(n)}(y_l) \right], \] (2.28)

\[ I_{jl}^{(N-3)/2} = - \sum_{k=0}^{(N-3)/2} \frac{1}{r_k} \left[ \hat{R}_2^{(m)}(y_j) \hat{R}_2^{(n)}(y_l) - \hat{R}_2^{(m)}(y_j) \hat{R}_2^{(n)}(y_l) \right] \]

\[ + \left[ \frac{1}{\sigma_{N-1}} \hat{F}_j^{(m)} - \frac{1}{\sigma_{N-1}} \hat{F}_j^{(n)} \right] \] (2.29)

and

\[ S_{jl}^{(N-3)/2} = \sum_{k=0}^{(N-3)/2} \frac{1}{r_k} \left[ \hat{R}_2^{(m)}(y_j) \hat{R}_2^{(n)}(y_l) - \hat{R}_2^{(m)}(y_j) \hat{R}_2^{(n)}(y_l) \right] + f_j^m R_j^n, \] (2.30)

where

\[ \hat{F}_j^{(m)}(x) = \int_0^\infty F^{mn}(y, x) \hat{R}_k^{(m)}(y) dy. \] (2.31)

Then the matrices \( \hat{I}_{mn} \) and \( \hat{S}_{mn} \) in the case \( N \) odd are given by

\[ \hat{I}_{jl}^{mn} = I_{jl}^{mn} + f_{jl}^{mn}, \quad \hat{S}_{jl}^{mn} = S_{jl}^{mn} - G_{jl}^{mn}, \] (2.32)

where matrices \( F^{mn} \) and \( G^{mn} \) are defined in eqs. (2.19) and (2.20).

### 3 Description in terms of the Laguerre Polynomials

In order to derive the asymptotic behavior of the dynamical correlation functions, it is convenient to rewrite the quaternion determinant formula presented in §2 in terms of the Laguerre polynomials. Note that \( (t_{M+1} \equiv T) \)

\[ G^{ln}(x, y) = \frac{1}{2\sqrt{2\pi t_1}} \frac{1}{\sqrt{2\pi (t_n - t_1)y}} \]

\[ \times \left[ \exp \left\{ -\frac{1}{2t_1} x \right\} - \exp \left\{ -\frac{1}{2(t_n - t_1)} (\sqrt{x} - \sqrt{y})^2 \right\} \right] \]

\[ = \frac{1}{\sqrt{2\pi t_1}} \frac{1}{\sqrt{2\pi (t_n - t_1)^2 y}} \]

\[ \times \exp \left\{ -\frac{1}{2} \left( \frac{1}{t_1} + \frac{1}{t_n - t_1} \right) x \right\} \exp \left\{ -\frac{1}{2(t_n - t_1)} y \right\} \]

\[ \times \left[ \exp \left\{ \frac{1}{t_n - t_1} \sqrt{xy} \right\} - \exp \left\{ -\frac{1}{t_n - t_1} \sqrt{xy} \right\} \right]. \] (3.1)
Let us define
\[ c_n = \frac{t_n(2T - t_n)}{T} \] (3.2)

and rescale \( x \) and \( y \) as
\[
G^{1n}(c_1 \lambda, c_n \lambda') = \frac{1}{\sqrt{2\pi t_1}} \frac{1}{\sqrt{2\pi (t_n - t_1)} \sqrt{c_n \lambda'}} \exp \left\{ -\frac{1}{2} \frac{1}{1 - z_n} (\lambda + \lambda') \right\} \exp \left\{ \frac{t_n}{2T} \lambda' \right\} 
\times \left[ \exp \left\{ \frac{2\sqrt{z_n \lambda' \lambda}}{1 - z_n} \right\} - \exp \left\{ \frac{-2\sqrt{z_n \lambda' \lambda}}{1 - z_n} \right\} \right] 
\times \frac{1}{\sqrt{2\pi t_1}} \frac{1}{\sqrt{2\pi (t_n - t_1)} \sqrt{c_n \sqrt{z_n(1 - z_n)}}} \lambda^{1/2} e^{-\lambda} \exp \left[ -\left(1 - \frac{t_n}{2T}\right) \lambda' \right] \sum_{j=0}^{\infty} \frac{j! z_n^j}{\Gamma(j + (3/2))} L_j^{(1/2)}(\lambda) L_j^{(1/2)}(\lambda'). \] (3.3)

Here
\[ z_n = \frac{t_1}{2T - t_1} \frac{2T - t_n}{t_n} \] (3.4)

and \( L_j^{(1/2)}(x) \) are the Laguerre polynomials (with a parameter 1/2). For \( m > 1 \), we can similarly derive
\[
G^{mn}(c_m \lambda, c_n \lambda') = \frac{1}{\sqrt{c_m c_n}} \sqrt{\frac{2\pi(t_m - t_1)}{2\pi(t_n - t_1)}} \frac{z_n(1 - z_n)}{z_m(1 - z_m)} \lambda^{1/2} e^{-\lambda} \exp \left[ -\left(1 - \frac{t_n}{2T}\right) \lambda' \right] \sum_{j=0}^{\infty} \frac{(z_n/z_m)^j}{\Gamma(j + (3/2))} \frac{j! z_n^j}{\Gamma(j + (3/2))} L_j^{(1/2)}(\lambda) L_j^{(1/2)}(\lambda'). \] (3.5)

Let us now define monic orthogonal polynomials
\[ C_n(x) = (-1)^n n! L_n^{(1/2)}(x) = x^n + \cdots \] (3.6)
satisfying
\[ \int_0^\infty x^{1/2} e^{-x} C_j(x) C_l(x) dx = h_j \delta_{jl} \] (3.7)
with
\[ h_j = (j + (1/2))! j!. \] (3.8)

We then introduce skew orthogonal polynomials
\[ \tilde{C}_n^m(x) = (\chi_m)^{-k} \sum_{r=0}^{k} \alpha_{kr} C_r(x) (\chi_m)^r, \] (3.9)
where
\[
\alpha_{2k} r = (-1)^r \left( \frac{2k!}{(2k - r)!} \right)^{1/2} \left( \frac{2k - r - (3/2)}{r!} \right),
\alpha_{2k+1} r = (-1)^r \left( \frac{2k + 1}{4} \right) \left( \frac{2k - r + (3/4)}{(2k - r + 1)!} \right)^{1/2} \left( \frac{2k - r - (3/2)}{(2k - r + 1)!} \right)^{1/2}. \] (3.10)
and
\[ \chi_m = \frac{z_m}{z_{M+1}}. \] (3.11)

It is known that \( \bar{R}_k^m(x) \) satisfy skew orthogonality relations
\[ \langle\langle \bar{R}_k^m(x), \bar{R}_{k+1}^m(y) \rangle\rangle_m = -\langle\langle \bar{R}_{k+1}^m(x), \bar{R}_k^m(y) \rangle\rangle_m = \bar{r}_k^m \delta_{kj}, \]
\[ \langle\langle \bar{R}_k^m(x), \bar{R}_{k}^m(y) \rangle\rangle_m = 0, \quad \langle\langle \bar{R}_{k+1}^m(x), \bar{R}_{k+1}^m(y) \rangle\rangle_m = 0 \] (3.12)

with
\[ \bar{r}_k^m = 4(2j + 1)!(2j)!(\chi_m)^{-4j-1}. \] (3.13)

Here the antisymmetric inner product is defined as
\[ \langle\langle f(x), g(y) \rangle\rangle_m = \frac{1}{2} \int_0^\infty dx \int_0^\infty dy x^{1/4} y^{1/4} e^{-x/2} e^{-y/2} F^{mm}(y, x) \]
\[ \times \left[ f(y)g(x) - f(x)g(y) \right], \] (3.14)

where
\[ F^{mm}(x, y) = \int_0^\infty dz' \int_0^{z'} \frac{1}{(z')^{1/4}} g^m M+1(x, z') g^m M+1(y, z) \] (3.15)

with
\[ g^{mn}(x, y) = x^{1/4} y^{1/4} e^{-x/2} e^{-y/2} \sum_{j=0}^\infty \frac{C_j(x)C_j(y)}{h_j} \left( \frac{z_m}{z_m} \right)^j. \] (3.16)

We can readily find \((m > 1)\)
\[ G^1 M+1(c_1 \lambda, c_{M+1} \lambda') = \frac{1}{\sqrt{2\pi T}} \frac{1}{2T - t_1} \left( \frac{\lambda}{\lambda'} \right)^{1/4} e^{-\lambda/2} g^1 M+1(\lambda, \lambda'), \]
\[ G^m M+1(c_m \lambda, c_{M+1} \lambda') = \frac{1}{\sqrt{c_m c_{M+1}}} \sqrt{\frac{t_m - t_1}{t_{M+1} - t_1}} \frac{z_{M+1}(1 - z_{M+1})}{z_m(1 - z_m)} \] (3.17)
\[ \times \left( \frac{\lambda}{\lambda'} \right)^{1/4} \exp \left[ \left( 1 - \frac{t_m}{T} \right) \frac{\lambda}{2} \right] g^m M+1(\lambda, \lambda'), \]

so that
\[ F^{11}(c_1 \lambda, c_1 \lambda') = \frac{1}{2\pi T} \left( \frac{t_1}{c_1} \right)^2 (\lambda \lambda')^{1/4} e^{-\lambda/2} e^{-\lambda' /2} F^{11}(\lambda, \lambda'), \]
\[ F^{mm}(c_m \lambda, c_m \lambda') = \frac{C_{M+1}}{c_m} \frac{t_m - t_1}{t_{M+1} - t_1} \frac{z_{M+1}(1 - z_{M+1})}{z_m(1 - z_m)} \] (3.18)
\[ \times (\lambda \lambda')^{1/4} \exp \left[ \left( 1 - \frac{t_m}{T} \right) \frac{\lambda}{2} \right] \exp \left[ \left( 1 - \frac{t_m}{T} \right) \frac{\lambda'}{2} \right] F^{mm}(\lambda, \lambda'). \]

It is now straightforward to see that the polynomials
\[ R_k^1(x) = (c_1)^k \bar{R}_k^1(x/c_1) \] (3.19)
satisfy the skew orthogonality relation (2.12) with

\[ r_j = \frac{2}{\pi} (2j + 1)!(2j)! \left( \frac{(t_1)^2}{T} \right)^{4j+2} \],

(3.20)
since

\[ \langle \langle R_j^m(x), R_i^n(y) \rangle \rangle_m = 2\pi \frac{T}{(t_1)^2} (c_1 z_m)^{-j-l} \langle \langle R_j^m(x), R_i^n(y) \rangle \rangle_m. \]

(3.21)

Then, from the definition, we obtain \((m > 1)\)

\[ R_i^n(x) = (z_m)^k(c_1)^{k+1} \frac{1}{\sqrt{2\pi t_1}} \frac{1}{\sqrt{2\pi(t_m - t_1)}} \sqrt{\frac{\pi}{c_m}} \sqrt{z_m(1 - z_m)} \times \exp \left[ -\left(1 - \frac{t_m}{2T} \right) \frac{x}{c_m} \right] \Phi_k^m(x/c_m). \]

(3.22)

Putting the above results into the quaternion determinant formula yields

\[ \rho(y_1, \ldots, y_{n_1}; y_1, \ldots, y_{n_2}; \ldots; y_1^M, \ldots, y_{n_M}) = \text{Tdet}[B^{\mu\nu}(n_\mu, n_\nu)] = \prod_{l=1}^M (c_l)^{-n_l} \text{Tdet}[B^{\mu\nu}(n_\mu, n_\nu)], \]

(3.23)

where each block \(\bar{B}^{\mu\nu}(n_\mu, n_\nu)\) is obtained by removing the \(n_\mu + 1, n_\mu + 2, \ldots, N\)-th rows and \(n_\nu + 1, n_\nu + 2, \ldots, N\)-th columns from \(\bar{B}^{\mu\nu}\). The quaternion elements \(\bar{B}^{\mu\nu}_{jl}\) are represented in terms of the scaled variables

\[ \lambda_j^m = y_j^m/c_m \]

(3.24)
as

\[ \bar{B}^{\mu\nu}_{jl} = \begin{bmatrix} \bar{S}^{\mu\nu}(\lambda_j^\mu, \lambda_l^\nu) & \bar{I}^{\mu\nu}(\lambda_j^\mu, \lambda_l^\nu) \\ \bar{D}^{\mu\nu}(\lambda_j^\mu, \lambda_l^\nu) & \bar{S}^{\nu\mu}(\lambda_j^\nu, \lambda_l^\mu) \end{bmatrix}, \quad j, l = 1, 2, \ldots, N. \]

The definitions of \(\bar{S}^{\mu\nu}(x, y)\), \(\bar{I}^{\mu\nu}(x, y)\) and \(\bar{D}^{\mu\nu}(x, y)\) are (for even \(N\))

\[ \bar{D}^{mn}(x, y) = x^{1/4} e^{-x/2} y^{1/4} e^{-y/2} \sum_{k=0}^{(N/2)-1} \frac{1}{\bar{r}_k^{MN+1}} \left[ (\lambda_m)^{2k} \bar{R}^m_{2k}(x)(\lambda_n)^{2k+1} \bar{R}^n_{2k+1}(y) \right. \]

\[ - \left. (\lambda_m)^{2k+1} \bar{R}^m_{2k+1}(x)(\lambda_n)^{2k} \bar{R}^n_{2k}(y) \right], \]

(3.26)

\[ \bar{I}^{mn}(x, y) = - \sum_{k=0}^{(N/2)-1} \frac{1}{\bar{r}_k^{MN+1}} \left[ (\lambda_m)^{2k} \bar{F}^m_{2k}(x)(\lambda_n)^{2k+1} \bar{F}^n_{2k+1}(y) \right. \]

\[ - \left. (\lambda_m)^{2k+1} \bar{F}^m_{2k+1}(x)(\lambda_n)^{2k} \bar{F}^n_{2k}(y) \right] + F^{mn}(x, y), \]

(3.27)

\[ \bar{S}^{mn}(x, y) = y^{1/4} e^{-y/2} \sum_{k=0}^{(N/2)-1} \frac{1}{\bar{r}_k^{MN+1}} \left[ (\lambda_m)^{2k} \bar{G}^m_{2k}(x)(\lambda_n)^{2k+1} \bar{G}^n_{2k+1}(y) \right. \]

\[ - \left. (\lambda_m)^{2k+1} \bar{G}^m_{2k+1}(x)(\lambda_n)^{2k} \bar{G}^n_{2k}(y) \right] - \bar{G}^{mn}(x, y), \]

(3.28)
where
\[ \bar{\Phi}_k^m(x) = \int_0^\infty \hat{F}^{mm}(y, x)y^{1/4}e^{-y/2}\bar{R}_k^m(y)dy \] (3.29)
and
\[
\begin{align*}
G^{mn}(x, y) &= 0, \quad m \geq n, \\
\tilde{G}^{mn}(x, y) &= g^{mn}(x, y), \quad m < n.
\end{align*}
\] (3.30)

## 4 Asymptotic Correlations

Let us consider the asymptotic limit \( N \to \infty \) of the dynamical correlation functions. A new result should be searched in the neighborhood of the origin, since, in the regions far from the origin, the asymptotic behavior of the dynamical correlations should not be changed by the presence of the wall. In order to see the asymptotic correlations around the origin, we define scaled temporal and spatial variables \( \upsilon_m \) and \( X_{j}^{m} \) as
\[
\begin{align*}
t_m &= \left( 1 - \frac{\upsilon_m}{2N} \right) T, \\
y_{j}^{m} &= \chi_{j}^{m} = \frac{X_{j}^{m}}{N}.
\end{align*}
\] (4.1)

Taking the expansion (3.9) and putting an asymptotic formula
\[
\lim_{n \to \infty} L_{n}^{(a)}(x) \sim e^{x/2}\left( \frac{n}{x} \right)^{a/2} J_{a}(2\sqrt{nx})
\] (4.2)
\((J_{a}(x)\) is the Bessel function) which holds uniformly for \( x = O(1/n) \), we can readily derive
\[
\bar{R}_{2k}^{m}(x/N)\left( \chi_{m} \right)^{2k} = \frac{N\theta^{3/4}}{(2k)!} \int_0^1 d\eta \frac{\eta^{1/4}}{(1 - \eta)^{1/2}} J_{1/2}(2\sqrt{\theta\eta x})e^{\theta \upsilon_m},
\] (4.3)
where
\[
\theta = \frac{2k}{N}.
\] (4.4)

The skew orthogonal polynomials with odd order are rewritten as
\[
\bar{R}_{2k+1}^{m}(x/N)\left( \chi_{m} \right)^{2k+1} = -\frac{(2k + (1/2))!}{\sqrt{\pi}(2k + 1)!} L_{0}^{(1/2)}(x) - \frac{(2k - (1/2))!}{\sqrt{\pi}(2k)!} L_{1}^{(1/2)}(x) \chi_{m}
\]
\[= \frac{1}{\sqrt{\pi}} \sum_{r=2}^{2k+1} \frac{(2k - r + (1/2))!}{(2k - r + 1)!}[L_{r}^{(1/2)}(x)(\chi_{m})^{r} - L_{r-2}^{(1/2)}(x)(\chi_{m})^{r-2}].
\] (4.5)

Here the contribution to the asymptotic form comes from the last term and is evaluated as
\[
\bar{R}_{2k+1}^{m}(x/N)\left( \chi_{m} \right)^{2k+1} \sim -\frac{2}{\sqrt{\pi}(x\theta)^{1/4}} \int_0^1 d\eta \frac{1}{(1 - \eta)^{1/2}} \left[ \eta^{1/4} J_{1/2}(2\sqrt{\theta\eta x})e^{\theta \upsilon_m} \right].
\] (4.6)
We substitute the asymptotic forms of $\mathbf{R}_k^m(x)$ into (3.26) to derive

$$
\bar{D}^{mn}(x/N, y/N) \sim -\frac{N^{3/2}}{4\pi} \int_0^1 d\theta \theta^{1/2} \times \left[ \int_0^1 d\eta \eta^{1/4} J_{1/2}(2\sqrt{\theta\eta}) e^{\eta \nu m} \int_0^1 d\xi \frac{1}{(1-\xi)^{1/2}} \frac{d}{d\xi} \left\{ \xi^{1/4} J_{1/2}(2\sqrt{\theta\xi}) e^{\xi \nu n} \right\} \right].
$$

(4.7)

We now introduce the inverse matrix $[\beta_{jl}]_{j,l=1,\ldots,k}$ of $[\alpha_{jl}]_{j,l=1,\ldots,k}$ as

$$
C_k(x) = (\chi_m)^{-k} \sum_{r=0}^k \beta_{kr} \bar{R}_r^m(x)(\chi_m)^r,
$$

(4.8)

where

$$
\beta_{2r} = \frac{(-1)^{k+1}(k - 2r - (3/2))!}{2\sqrt{\pi} (k - 2r)!} \frac{k!}{(2r)!},
$$

$$
\beta_{2r+1} = \frac{(-1)^k k!}{2\sqrt{\pi} (2r + 1)!} \sum_{l=0}^r \frac{\{k-(2r-1)/2\} (k - 2r - 2l - (5/2))!}{(k - 2r - 2l - 1)!}.
$$

(4.9)

In terms of the inverse matrix $\beta_{jl}$, the function $\bar{\Phi}_k^m(x) \chi^2$ is written as

$$
\bar{\Phi}_k^m(x)(\chi_m)^{2k+1} = \frac{1}{2} e^{-x/2} \bar{R}_k^m \sum_{\nu=2k}^\infty \frac{C_{\nu}(x)(\chi_m)^{-\nu}}{h_{\nu}} \beta_{\nu 2k},
$$

$$
\bar{\Phi}_k^m(x)(\chi_m)^{2k} = \frac{1}{2} e^{-x/2} \bar{R}_k^m \sum_{\nu=2k+1}^\infty \frac{C_{\nu}(x)(\chi_m)^{-\nu}}{h_{\nu}} \beta_{\nu 2k+1}.
$$

(4.10)

Using the above expression and identities

$$
\sum_{l=0}^n \frac{(2l - 1/2)!}{(2l + 1)!} = \sqrt{2\pi} - \sum_{l=n+1}^\infty \frac{(2l - 1/2)!}{(2l + 1)!},
$$

$$
\sum_{l=0}^n \frac{(2l - 3/2)!}{(2l)!} = -\sqrt{2\pi} - \sum_{l=n+1}^\infty \frac{(2l - 3/2)!}{(2l)!},
$$

(4.11)

we can rewrite the function $\bar{\Phi}_k^m(x)$ as

$$
\bar{\Phi}_k^m(x)(\chi_m)^{2k} = \frac{1}{2} e^{-x/2} \bar{R}_k^m \sum_{r=k+1}^\infty \frac{(2r)!}{(2r + (1/2))!} \frac{1}{L^{(1/2)}_{2r}(x)(\chi_m)^{-2r}} \sum_{l=r-k}^\infty \frac{(2l - (1/2))!}{(2l + 1)!} \frac{1}{L^{(1/2)}_{2r-1}(x)(\chi_m)^{2r+1}} \frac{1}{(2l)!}.
$$
Putting the expansion (4.8) into (3.15) and using (4.10) yields
\[ + 2^{3/2} \sum_{r=k+1}^{\infty} \left[ \frac{(2r)!}{(2r + (1/2))!} \frac{L_{2r}^{(1/2)}(x)}{(\chi_m)^{2r}} - \frac{(2r - 1)!}{(2r - (1/2))!} \frac{L_{2r-1}^{(1/2)}(x)}{(\chi_m)^{2r-1}} \right]. \]  

The first and second terms contribute to the asymptotic limit and give
\[ \Phi_{2k}^m(x/N)(\chi_m)^{2k} \frac{(2k)!}{(2k+1)!} \sim - \frac{2}{\sqrt{\pi}} (N\theta)^{1/4} \int_1^{\infty} \frac{1}{\eta^{1/4}(\eta - 1)^{1/2}} J_{1/2}(2\sqrt{\theta\eta}x)e^{-\theta\eta r^2}. \]  

Then we put \( \Phi_{2k+1}^m(x) \) in a form
\[ \Phi_{2k+1}^m(x)(\chi_m)^{2k+1} \frac{\sqrt{\pi}}{4} x^{-1/4} e^{x/2} \]
\[ = \sum_{r=k+1}^{\infty} \frac{(2r - 2k - (5/2))!}{(2r - 2k - 2)!} \frac{(2r - 2)!}{(2r - (1/2))!} \times (\chi_m)^{-2r+1} \left[ (2r - 1)L_{2r-1}^{(1/2)}(x) - \left( \frac{2r - 1}{2} \right) L_{2r-2}^{(1/2)}(x) \right] \]
\[ + \sum_{r=k+1}^{\infty} \frac{(2r - 2k - (3/2))!}{(2r - 2k - 1)!} \frac{(2r - 1)!}{(2r + (1/2))!} \times (\chi_m)^{-2r+2} \left[ (2r - 3)L_{2r-3}^{(1/2)}(x) - \left( \frac{2r - 3}{2} \right) L_{2r-4}^{(1/2)}(x) \right] \]
\[ + \sum_{r=k+1}^{\infty} \frac{(2r - 2k - (7/2))!}{(2r - 2k - 3)!} \frac{(2r - 2)!}{(2r - (3/2))!} \frac{L_{2r-2}^{(1/2)}(x)}{(\chi_m)^{-2r+2} - (\chi_m)^{-2r+3}} \]
\[ + \sum_{r=k+1}^{\infty} \frac{(2r - 2k - (1/2))!}{(2r - 2k - 2)!} \frac{(2r - 2)!}{(2r - (3/2))!} \frac{L_{2r-2}^{(1/2)}(x)}{(\chi_m)^{-2r+1} - (\chi_m)^{-2r+2}} \]

and substitute (4.12) to derive
\[ \Phi_{2k+1}^m(x/N)(\chi_m)^{2k+1} \frac{(2k+1)!}{(2k+1)!} \sim \frac{4}{\sqrt{\pi}} \frac{1}{(N\theta)^{3/4}} \int_1^{\infty} \frac{1}{\eta^{1/4}(\eta - 1)^{1/2}} \frac{d\eta}{\eta} \int_1^{\infty} \frac{1}{\eta^{1/4} J_{1/2}(2\sqrt{\theta\eta}x)e^{-\theta\eta r^2}}. \]  

Putting the expansion (4.8) into (3.15) and using (4.10) yields
\[ \bar{F}^{mn}(x, y) = \sum_{k=0}^{\infty} \frac{1}{r^M+1} \left[ (\chi_m)^{2k} \Phi_{2k}^m(x)(\chi_n)^{2k+1} \Phi_{2k+1}^n(y) \right] - (\chi_m)^{2k+1} \Phi_{2k+1}^m(x)(\chi_n)^{2k+1} \Phi_{2k+1}^n(y), \]

so that
\[ \bar{I}^{mn}(x, y) = \sum_{k=N/2}^{\infty} \frac{1}{r^M+1} \left[ (\chi_m)^{2k} \Phi_{2k}^m(x)(\chi_n)^{2k+1} \Phi_{2k+1}^n(y) \right] - (\chi_m)^{2k+1} \Phi_{2k+1}^m(x)(\chi_n)^{2k+1} \Phi_{2k+1}^n(y). \]

Asymptotic forms (4.13) and (4.15) are substituted into the above expression and yield
\[ \bar{I}^{mn}(x/N, y/N) \sim - \frac{N^{1/2}}{\pi} \int_1^{\infty} \frac{d\theta}{\theta^{1/2}}. \]
Moreover we put (4.3), (4.6), (4.13) and (4.15) into (3.28) and find

\[ \bar{S}_{mn}(x/N, y/N) + \bar{G}_{mn}(x/N, y/N) \sim N \int_0^1 d\theta \]

\[ \times \left[ \int_1^\infty d\eta \frac{J_{1/2}(2\sqrt{\eta x})e^{-\theta \eta \nu_m}}{\eta^{1/4}(\eta - 1)^{1/2}} \int_1^\infty d\xi \frac{\xi^{1/4}J_{1/2}(2\sqrt{\xi x})e^{-\theta \xi \nu_m}}{(\xi - 1)^{1/2}} \right] \]

\[ \times \int_1^\infty d\eta \left( \frac{1}{\eta^{1/4}} \int_1^\infty d\xi \frac{\xi^{1/4}J_{1/2}(2\sqrt{\xi x})e^{-\theta \xi \nu_m}}{(\xi - 1)^{1/2}} \right) \]

\[ \times \int_0^1 d\xi \left\{ \xi^{1/4}J_{1/2}(2\sqrt{\xi x})e^{\theta \xi \nu_n} \right\} \int_1^\infty d\eta \left( \frac{1}{\eta^{1/4}} \int_1^\infty d\xi \frac{\xi^{1/4}J_{1/2}(2\sqrt{\xi y})e^{\theta \xi \nu_n}}{(\xi - 1)^{1/2}} \right) \].

(4.18)

The function \( \bar{G}_{mn}(x, y) \) is identical to \( g_{mn}(x, y) \) when \( m < n \). The asymptotic limit of \( g_{mn}(x, y) \) is derived from (3.16) and (4.2) as

\[ g_{mn}(x/N, y/N) \sim N \int_0^1 ds J_{1/2}(2\sqrt{sx})J_{1/2}(2\sqrt{sy})e^{-(v_m - v_n)s}. \]

(4.20)

Substituting (4.7), (4.18), (4.19) and (4.20) into (3.23), we can see how the dynamical correlation functions asymptotically depend on the scaled variables \( v_m \) and \( X_j^m \).

### 5 Bogoliubov-deGennes Matrix Model

In this last section we show an equivalence relation between the vicious walk model with a wall and the Bogoliubov-deGennes matrix model describing the symmetry crossover \( CI \to C \). The Bogoliubov-deGennes matrix model was proposed by Altland and Zirnbauer as a model of normal conducting-superconducting hybrid structures in mesoscopic physics [16, 17]. It is a part of a classification scheme of random matrix ensembles in terms of the Lie algebra [22, 23]. In the class \( C \), the spin is conserved while the time reversal symmetry is broken. Then the (reduced) system Hamiltonian has a structure

\[ \mathcal{H}_C = \begin{pmatrix} a & b \\ b^\dagger & -a^T \end{pmatrix} \]

(5.1)

with an \( N \times N \) hermitian \( a \) and an \( N \times N \) complex symmetric \( b \). On the other hand, in the class \( CI \), the system is symmetric with respect to both spin rotations and time reversal. Then the Hamiltonian matrix structure is

\[ \mathcal{H}_{CI} = \begin{pmatrix} a & b \\ b & -a^T \end{pmatrix}, \]

(5.2)

where \( a \) and \( b \) are both \( N \times N \) real symmetric matrices.
Dyson proposed Brownian motion models for parametric random matrices[24]. In his prescription, the time evolution of a combination
\[ \mathcal{H} = e^{-\tau}(\mathcal{H}_{CI} + \sqrt{e^{2\tau} - 1}\mathcal{H}_C) \]
(5.3)
describes the crossover \( CI \rightarrow C \). The matrix \( \mathcal{H} \) is identical to \( \mathcal{H}_{CI} \) at \( \tau = 0 \) while \( \mathcal{H} \) approaches \( \mathcal{H}_C \) as \( \tau \) goes to infinity. Let us redefine \( a \) and \( b \) as
\[ \mathcal{H} = \begin{pmatrix} a & b \\ b^\dagger & -a^\top \end{pmatrix} \]
(5.4)
so as to reproduce (5.1) in the limit \( \tau \rightarrow \infty \). We can diagonalize \( \mathcal{H} \) as
\[ \mathcal{H} = U^\dagger \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} U, \]
(5.5)
where \( U \) is a \( 2N \times 2N \) unitary matrix and \( \omega \) is a real diagonal matrix
\[ \omega = \begin{pmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \omega_N \end{pmatrix}. \]
(5.6)
Assuming that \( \mathcal{H}_C \) is distributed according to the Gaussian distribution, we obtain the probability distribution function for \( \mathcal{H} \) as
\[ P(\mathcal{H} ; \tau)d\mathcal{H} = A_\tau \exp \left[ -\frac{\text{Tr}(\mathcal{H} - e^{-\tau}\mathcal{H}_{CI})^2}{2(1 - e^{-2\tau})} \right] d\mathcal{H}, \]
(5.7)
where
\[ d\mathcal{H} = \prod_{j=1}^N da_{jj} db^R_{jj} db^I_{jj} \prod_{j<l}^N da_{jl} db^R_{jl} db^I_{jl}. \]
(5.8)
Here the normalization constant \( A_\tau \) is evaluated as
\[ A_\tau = 2^{N(N-1)/2} \pi^{-N^2-(N/2)} (1 - e^{-2\tau})^{-N^2-(N/2)}. \]
(5.9)

It is easy to calculate the differentiations of \( P(\mathcal{H} ; \tau) \) and find the Fokker-Planck equation
\[ \frac{\partial P}{\partial \tau} = \Delta P + \sum_{j=1}^N \left( \frac{\partial}{\partial a_{jj}} (a_{jj} P) + \frac{\partial}{\partial b^R_{jj}} (b^R_{jj} P) + \frac{\partial}{\partial b^I_{jj}} (b^I_{jj} P) \right) \]
\[ + \sum_{j<l}^N \left[ \frac{\partial}{\partial a_{jl}} (a_{jl} P) + \frac{\partial}{\partial a^I_{jl}} (a^I_{jl} P) + \frac{\partial}{\partial b^R_{jl}} (b^R_{jl} P) + \frac{\partial}{\partial b^I_{jl}} (b^I_{jl} P) \right], \]
(5.10)
where \( \Delta \) is the Laplace-Beltrami operator
\[ \Delta = \frac{1}{2} \sum_{j=1}^N \left[ \frac{\partial^2}{\partial (a_{jj})^2} + \frac{\partial^2}{\partial (b^R_{jj})^2} + \frac{\partial^2}{\partial (b^I_{jj})^2} \right] + \frac{1}{4} \sum_{j<l}^N \left[ \frac{\partial^2}{\partial (a_{jl})^2} + \frac{\partial^2}{\partial (a^I_{jl})^2} + \frac{\partial^2}{\partial (b^R_{jl})^2} + \frac{\partial^2}{\partial (b^I_{jl})^2} \right]. \]
(5.11)
The Laplace-Beltrami operator on a Riemannian manifold defined by the line element
\[ ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu \] (5.12)
is given by
\[ \Delta = \sum_{\mu\nu} \frac{1}{\sqrt{|\det g|}} \frac{\partial}{\partial x^\mu} (g^{-1})_{\mu\nu} \sqrt{|\det g|} \frac{\partial}{\partial x^\nu}. \] (5.13)

In our case the corresponding line element is
\[ ds^2 = 2 \sum_{j=1}^{N} \left[ (da_{jj})^2 + (db_{jj}^R)^2 + (db_{jj}^I)^2 \right] + 4 \sum_{j<l}^{N} \left[ (da_{jl}^R)^2 + (da_{jl}^I)^2 + (db_{jl}^R)^2 + (db_{jl}^I)^2 \right] \]
\[ = 2 \sum_{j=1}^{N} \sum_{l=1}^{N} \left[ da_{jl} da_{jl}^* + db_{jl} db_{jl}^* \right]. \] (5.14)

Let us introduce \( N \times N \) matrices \( u_1, u_2, u_3 \) and \( u_4 \) as
\[ U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \] (5.15)
and rewrite \( da_{jl} \) and \( db_{jl} \) as
\[ da_{jl} = \sum_{k=1}^{N} \left[ (u_1)^*_k (u_1)_{kl} - (u_3)^*_k (u_3)_{kl} \right] d\omega_k + dA_{jl}, \]
\[ db_{jl} = \sum_{k=1}^{N} \left[ (u_1)^*_k (u_2)_{kl} - (u_3)^*_k (u_4)_{kl} \right] d\omega_k + dB_{jl}, \] (5.16)
where
\[ dA_{jl} = \sum_{k=1}^{N} \left[ (du_1)^*_k (u_1)_{kl} + (u_1)^*_k (du_1)_{kl} - (du_3)^*_k (u_3)_{kl} - (u_3)^*_k (du_3)_{kl} \right] \omega_k, \]
\[ dB_{jl} = \sum_{k=1}^{N} \left[ (du_1)^*_k (u_2)_{kl} + (u_1)^*_k (du_2)_{kl} - (du_3)^*_k (u_4)_{kl} - (u_3)^*_k (du_4)_{kl} \right] \omega_k. \]

Using the unitarity of the matrix \( U \) (\( U^\dagger U = UU^\dagger = I \)), we can readily see that
\[ ds^2 = 2 \sum_{k=1}^{N} (d\omega_k)^2 + 2 \sum_{j=1}^{N} \sum_{l=1}^{N} \left[ dA_{jl} dA_{jl}^* + dB_{jl} dB_{jl}^* \right], \] (5.17)
which, by means of (5.12) and (5.13), yields
\[ \Delta = \frac{1}{2J} \sum_{j=1}^{N} \frac{\partial}{\partial \omega_j} \left( J \frac{\partial}{\partial \omega_j} \right) + \Delta_U. \] (5.18)

Here an operator \( \Delta_U \) involves derivatives with respect to the variables associated with \( U \). The Jacobian \( J = \sqrt{|\det g|} \) was evaluated in Ref. [16, 17] as
\[ J = \prod_{j<l} \left| \omega_j^2 - \omega_l^2 \right| \prod_{j=1}^{N} \left| \omega_j \right|^2. \] (5.19)
Assuming that \( P(H; \tau) \) depends only on the radial variables \( \omega_k \) and a time variable \( \tau \), we can rewrite the Fokker-Planck equation as

\[
\frac{\partial P}{\partial \tau} = \frac{1}{2J} \sum_{j=1}^{N} \frac{\partial}{\partial \omega_j} \left( J \frac{\partial P}{\partial \omega_j} \right) + \sum_{j=1}^{N} \omega_j \frac{\partial P}{\partial \omega_j} + (2N^2 + N)P. \tag{5.20}
\]

Substituting \( P = p/J \) gives

\[
\frac{\partial p}{\partial \tau} = \mathcal{L}p, \quad \mathcal{L} = \sum_{j=1}^{N} \frac{\partial}{\partial \omega_j} \left( \frac{\partial W}{\partial \omega_j} + \frac{1}{2} \frac{\partial}{\partial \omega_j} \right) = \frac{1}{2} \sum_{j=1}^{N} \frac{\partial}{\partial \omega_j} e^{2W} \frac{\partial}{\partial \omega_j} e^{2W}, \tag{5.21}
\]

where

\[
W = \frac{1}{2} \sum_{j=1}^{N} \omega_j^2 - \frac{1}{2} \log J. \tag{5.22}
\]

In order to transform the Fokker-Planck operator into a form in which all the variables are separated, let us consider

\[
- \frac{1}{2} (H - E_0) = e^W \mathcal{L} e^{-W} \tag{5.23}
\]

(\( E_0 \) is a constant) and find

\[
H = - \sum_{j=1}^{N} \frac{\partial^2}{\partial \omega_j^2} + \sum_{j=1}^{N} \omega_j^2. \tag{5.24}
\]

For the imaginary time Schrödinger equation

\[
\frac{\partial \psi}{\partial \tau} = - \frac{1}{2} (H - E_0) \psi, \tag{5.25}
\]

we call \( \psi = G^{(H)}(\omega_1^{(0)}, \cdots, \omega_N^{(0)}; \omega_1, \cdots, \omega_N; \tau) \) the Green function solution if it satisfies the initial condition

\[
G^{(H)}(\omega_1^{(0)}, \cdots, \omega_N^{(0)}; \omega_1, \cdots, \omega_N; 0) = \prod_{j=1}^{N} \delta(\omega_j - \omega_j^{(0)}). \tag{5.26}
\]

Since the imaginary time Schrödinger equation (5.25) describes the dynamics of free fermions, the Green function solution is given by

\[
G^{(H)}(\omega_1^{(0)}, \cdots, \omega_N^{(0)}; \omega_1, \cdots, \omega_N; \tau) = \det[g^{(H)}(\omega_j^{(0)}, \omega_l; \tau)]_{j,l=1,\ldots,N}, \tag{5.27}
\]

where \( g^{(H)}(\omega_j^{(0)}, \omega; \tau) \) is the Green function solution of (5.25) with \( N = 1 \).

Let us denote the Green function solution of the Fokker-Planck equation (5.20) as \( G^{(FP)}(\omega_1^{(0)}, \cdots, \omega_N^{(0)}; \omega_1, \cdots, \omega_N; \tau) \). Then we can readily see that

\[
G^{(FP)}(\omega_1^{(0)}, \cdots, \omega_N^{(0)}; \omega_1, \cdots, \omega_N; \tau) = e^{\tau E_0/2} \frac{e^{-W(\omega_1, \cdots, \omega_N)}}{e^{W(\omega_1^{(0)}, \cdots, \omega_N^{(0)})}} G^{(H)}(\omega_1^{(0)}, \cdots, \omega_N^{(0)}; \omega_1, \cdots, \omega_N; \tau)
\]

\[
= e^{\tau E_0/2} \prod_{j=1}^{N} \left( 4 \omega_j \right)^{1/2} \prod_{j=1}^{N} \left( (\omega_j)^2 - (\omega_j^{(0)})^2 \right) \prod_{j=1}^{N} \left( (\omega_j^{(0)})^2 - (\omega_j^{(0)})^2 \right) \det[g((\omega_j^{(0)})^2, (\omega_j^{(0)})^2; \tau)]_{j,l=1,\ldots,N}. \tag{5.28}
\]
where
\[ g(x, y; \tau) = x^{1/4} y^{1/4} e^{-x/2} e^{-y/2} \sum_{j=0}^{\infty} \frac{1}{h_j} C_j(x) C_j(y) e^{-2j\tau}. \]  
\text{(5.29)}

Here \( C_j(x) \) and \( h_j \) are defined in (3.6) and (3.7), respectively. Note that the above result gives the Harish-Chandra integral \([25]\) in the case of the Bogoliubov-deGennes symmetry. We introduce new variables \( \epsilon_j = \omega_j^2 \) and the corresponding Green functions as
\[ G(\epsilon_0, \cdots, \epsilon_N; \epsilon, \cdots, \epsilon_N; \tau) \prod_{j=1}^{N} d\epsilon_j = G^{(FP)}(\omega_0, \cdots, \omega_N; \omega, \cdots, \omega_N; \tau) \prod_{j=1}^{N} d\omega_j \]  
\text{(5.30)}

so that
\[ G(\epsilon_0, \cdots, \epsilon_N; \epsilon, \cdots, \epsilon_N; \tau) = e^{\tau E_0/2} \frac{\prod_{j=1}^{N} e^{1/4} e^{-\epsilon_j/2} \prod_{j>l}^{N} (\epsilon_j - \epsilon_l)}{\prod_{j=1}^{N} (\epsilon_j^{(0)} + 1/4) e^{-\epsilon_j^{(0)/2}} \prod_{j>l}^{N} (\epsilon_j^{(0)} - \epsilon_l^{(0)})} \det [g(\epsilon_j^{(0)}, \epsilon_l; \tau)]_{j,l=1,\cdots,N}. \]  
\text{(5.31)}

The probability distribution functions for the eigenparameters \( \epsilon_j^n \) at times \( \tau_n \) can be evaluated from the Green function as
\[ p(\epsilon_1, \cdots, \epsilon_N; \tau_1; \epsilon_1, \cdots, \epsilon_N; \tau_2; \cdots; \epsilon_1^M, \cdots, \epsilon_N^M; \tau_M) \]
\[ = \frac{1}{N!} \int_0^{\infty} d\epsilon_1 \cdots \int_0^{\infty} d\epsilon_N p_0(\epsilon_0, \cdots, \epsilon_N) \prod_{l=1}^{M} G(\epsilon_0^{(l)-1}, \cdots, \epsilon_N^{(l)-1}; \epsilon_1^{l}, \cdots, \epsilon_N^{l}; \tau_l - \tau_{l-1}), \]  
\text{(5.32)}

where \( p_0(\epsilon_1, \cdots, \epsilon_N) \) is the initial probability distribution function at \( \tau_0 = 0 \). The corresponding multilevel dynamical correlation functions are given by
\[ \rho_{BdG}(\epsilon_1, \cdots, \epsilon_{m_1}; \epsilon_2, \cdots, \epsilon_{m_2}; \cdots; \epsilon_1^M, \cdots, \epsilon_N^M) \]
\[ = \frac{1}{C_N \prod_{l=1}^{M} (N - m_l)!} \int_0^{\infty} d\epsilon_1 \cdots \int_0^{\infty} d\epsilon_N \cdots \int_0^{\infty} d\epsilon_{m_{M+1}} \cdots \int_0^{\infty} d\epsilon_{N} \times p(\epsilon_1, \cdots, \epsilon_N; \tau_1; \epsilon_1, \cdots, \epsilon_N; \tau_2; \cdots; \epsilon_1^M, \cdots, \epsilon_N^M; \tau_M). \]  
\text{(5.33)}

Here we define the normalization constant \( C_N \) as
\[ C_N = \int_0^{\infty} d\epsilon_1 \cdots \int_0^{\infty} d\epsilon_N \cdots \int_0^{\infty} d\epsilon_1^M \cdots \int_0^{\infty} d\epsilon_N^M \times p(\epsilon_1, \cdots, \epsilon_N; \tau_1; \epsilon_1, \cdots, \epsilon_N; \tau_2; \cdots; \epsilon_1^M, \cdots, \epsilon_N^M; \tau_M). \]  
\text{(5.34)}

The initial eigenparameter distribution for the Gaussian random matrices with the symmetry \([5.22]\) can be written as
\[ p_0(\omega_1, \cdots, \omega_N) d\omega_1 \cdots d\omega_N \propto \prod_j^{N} \frac{\omega_j^{1/2}}{\prod_{j<l}^{N} |\omega_j - \omega_l|} \prod_{j=1}^{N} |\omega_j| d\omega_1 \cdots d\omega_N \]  
\text{(5.35)}

or, equivalently,
\[ p_0(\epsilon_1, \cdots, \epsilon_N) d\epsilon_1 \cdots d\epsilon_N \propto \prod_j^{N} e^{-\epsilon_j/2} \prod_{j<l}^{N} |\epsilon_j - \epsilon_l| d\epsilon_1 \cdots d\epsilon_N. \]  
\text{(5.36)}
A parameter $\alpha$ determines the variance of the Gaussian distribution for the matrix elements and we set $\alpha = 1$.

We now see that multilevel dynamical correlation functions defined in (5.33) with the initial condition (5.36) have the same forms as the dynamical correlation functions for vicious walkers (1.10). Therefore we can similarly rewrite them in quaternion determinant forms\cite{26}

$$
\rho_{Bog}(\epsilon_1^1, \ldots, \epsilon_1^{m_1}; \epsilon_2^1, \ldots, \epsilon_2^{m_2}; \ldots; \epsilon_1^M, \ldots, \epsilon_M^{m_M}) = T \det[\tilde{B}^{\mu\nu}(n_\mu, n_\nu)],
$$

$$
\mu, \nu = 1, 2, \ldots, M.
$$

(5.37)

Here the quaternion determinant is identical to that in (3.23) if we adopt a correspondence

$$
\epsilon_j^l = \lambda_j^{M-l+1}, \quad m_l = n_{M-l+1}, \quad e^{2\tau_l} = \chi_{M-l+1}.
$$

(5.38)

Therefore all the dynamical correlation functions are shared by the matrix model and the vicious walk model. We have thus shown the equivalence of the vicious walk model with a wall in the diffusion scaling limit and the parametric Bogoliubov-deGennes matrix model. This equivalence holds for finite $N$ and also in the asymptotic limits $N \to \infty$.

Although we have here presented the equivalence only for even $N$, we can similarly and straightforwardly prove it for odd $N$.

The eigenparameter distributions of the Bogoliubov-deGennes matrix model in the limits $\tau \to \infty$ and $\tau = 0$ are known to be equivalent to the eigenvalue distributions of the Laguerre unitary and orthogonal ensembles\cite{27, 28} of random matrices, respectively. Let us consider the limit $\tau_m \to \infty$ with the time differences $\tau_m - \tau_n$ fixed. We can readily see that in this limit the quaternion determinant is reduced to an ordinary determinant. The resulting determinant expressions describe temporally homogeneous dynamical correlations within the $C$ universality class. It follows from the rescaling\cite{41} that the $C$ universality class survives until time $t$ very close to $T$: only when $T - t \sim O(N^{-1})$, the transition to $CI$ class occurs. Therefore we can conclude that the transition from $C$ to $CI$ class is discontinuous in the limit $N \to \infty$. The asymptotic correlation functions describing the $CI$ universality class is obtained by putting $\upsilon_m = \upsilon_n = 0$ (equivalent to $\tau_{M-m+1} = \tau_{M-n+1} = 0$) in (4.7), (4.18) and (4.19). Using the Bessel function identities, we can easily confirm that they are identical to Nagao and Slevin’s result\cite{28} for the Laguerre orthogonal ensemble.

6 Conclusion

In this paper we have analyzed the vicious walk model with a wall in the diffusion scaling limit. It was shown that all the dynamical correlation functions are written in the forms of quaternion determinants. Using the quaternion determinant formulas we were able to derive the asymptotic formulas for the correlation functions. Finally we showed that the vicious walk model in the diffusion scaling limit was equivalent to the parametric Bogoliubov-deGennes matrix model. As the equivalence to the matrix model is so far established only in the diffusion scaling limit, it is interesting to consider how the matrix model should be generalized corresponding to the discrete vicious walk.
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