New Equation of Nonrelativistic Physics and Theory of Dark Matter

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Two infinite sets of Galilean invariant equations are derived using the irreducible representations of the orthochronous extended Galilean group. It is shown that one set contains the Schrödinger equation, which is the fundamental equation for ordinary matter, and the other set has a new asymmetric equation, which is proposed to be the fundamental equation for dark matter. Using this new equation, a theory of dark matter is developed and its profound physical implications are discussed. This theory explains the currently known properties of dark matter and also predicts a detectable gravitational radiation.

Keywords: Galilean groups; Galilean invariance; equations of physics; dark matter

1. Introduction

There is strong astronomical evidence for the existence of Dark Matter (DM) in the Universe [1-3]. DM is a concept whose physical meaning is currently unknown despite numerous theories that have attempted to explain it. The theories have postulated different particles [4-6] but so far none of these particles have been found experimentally [7-9]. To overcome this impasse, a novel theoretical approach is presented. In this approach, a new equation of non-relativistic physics is discovered and this paper reports on a theory of DM based on this equation.

Searching for new equations is conducted within the framework of Galilean space and time. The Galilean Principle of Relativity (GPR) is used to define a class of Galilean observers [10]. According to GPR, all Galilean observers agree that physical laws are the same in all inertial frames of reference. Moreover, the observers also obey the Principle of Analyticity (PA) [11], which requires that only analytic ($C^\infty$) functions are used.

Dynamical equations describing the space and time evolution of a scalar wavefunction are derived using the irreducible representations (irreps) of the orthochronous extended Galilean group [12]. The derived equations can be classified as symmetric and asymmetric (in space and time) equations with constant coefficients. It is shown that none of the symmetric equations is Galilean invariant and that among all asymmetric equations, there are only two infinite sets of Galilean invariant equations. One of these sets contains Schrödinger-like equations, and the
elements of the other set are newly discovered second-order equations, which are called here the new asymmetric equations.

The prominent feature of these two sets of the Galilean invariant equations is their dependence on constant coefficients, which must be determined. In this paper, an equation is called fundamental if, and only if, it is local, Galilean invariant, its Lagrangian exists and its coefficient is determined uniquely by the properties of either Ordinary Matter (OM) or Dark Matter (DM).

Among the two sets of derived equations, one fundamental equation for each set was identified. It is shown that one of the Schrödinger-like equations becomes the fundamental Schrödinger equation for OM, and that one of the new asymmetric equations is proposed to be considered as the fundamental equation for DM. If this choice of the fundamental equation for DM is correct, it has significant physical implications as it requires a new constant of Nature valid for DM only. The constant is called the quanta of energy, and its presence suggests that quantization of the energy of DM is frequency independent. The physical meaning of this new constant is discussed and a model of DM is constructed. It is also suggested that the fundamental equation for DM, with interactions included, may be used to formulate a quantum theory of DM, which would become the non-relativistic limit of a future relativistic theory of DM.

The paper is organized as follows: in Section 2, Galilean groups and the eigenvalue equations are described; symmetric and asymmetric equations are given in Section 3; Galilean invariant equations are presented in Section 4; the fundamental equations for ordinary and dark matter are identified in Section 5; and conclusions are given in Section 6.

2. Galilean groups and eigenvalue equations

The Galilean group of the metric $G$ is the group of Galilean space and time, and its structure is $G = [T(1) \otimes O(3)] \otimes_s [T(3) \otimes B(3)]$, where $T(1)$, $O(3)$, $T(3)$ and $B(3)$ are subgroups of translations in time, rotations, translations in space, and boosts, respectively [12]. The general structure of the extended Galilean group is $G_e = [O(3) \otimes_s B(3)] \otimes_s [T(3 + 1) \otimes U(1)]$, where $T(3 + 1)$ is an invariant Abelian subgroup of combined translations in space and time, and $U(1)$ is a one-parameter unitary subgroup [13]. Note that $G_e$ does not include the space and time inversions. Observers who use the group $G_e$ are called the extended Galilean observers and they must also agree on the square of the wavefunction in all inertial frames [12].

The group $G_e$ is known to have scalar and spinor irreducible representations (irreps), which are physical; that is, they allow uniquely defining elementary particles by scalar (Schrödinger equation) [10] or spinor (Lévy-Leblond equation) [13,14] wavefunctions. The vector and tensor representations are not physical because they do not allow particles to be localized [12]. The invariant subgroup $T(3 + 1) = T(3) \otimes T(1) \subset G_e$ has the irreps, and a scalar wavefunction $\phi$ transforms as one of these irreps if, and only if, the following eigenvalue equations [10,15] are
New Equation of Nonrelativistic Physics and Theory of Dark Matter

satisfied
\[ i \frac{\partial}{\partial t} \phi(t, x) = \omega \phi(t, x) , \] (1)
and
\[ -i \nabla \phi(t, x) = k \phi(t, x) , \] (2)
where \( \omega \) and \( k \) are labels of \( \phi(t, x) \), which is an eigenfunction of the generators of the invariant Abelian subgroup \( T(3 + 1) \).

3. Local symmetric and asymmetric equations

In general, the equations derived from the eigenvalue equations can be divided into two separate families, namely, the symmetric equations, with the same order of space and time derivatives, and the asymmetric equations, with different orders of space and time derivatives. Since only local (either first or second-order) equations are derived, the higher derivative equations [15] are not considered in this paper.

By combing the eigenvalue equations (Eqs. 1 and 2), the following symmetric first-order equation is obtained
\[ \left[ \frac{\partial}{\partial t} - C_s k \cdot \nabla \right] \phi(t, x) = 0 , \] (3)
where \( C_s = \omega / k^2 \). Since the labels \( \omega \) and \( k \) of the irreps may have any real value, there is an infinite number of \( C_s \) and first-order equations.

Similarly, the symmetric second-order equation can be written as
\[ \left[ \frac{\partial^2}{\partial t^2} - C_w \nabla^2 \right] \phi(t, x) = 0 , \] (4)
where \( C_w = \omega^2 / k^2 \) with \( k^{2n} = (k \cdot k)^n \). With \( C_w \) being arbitrary, there is an infinite set of second-order equations. In a special case \( C_w = v_w^2 \), where \( v_w = \text{const} \) is a characteristic wave speed, Eq. (4) becomes the regular wave equation of classical physics.

There are also two infinite sets of asymmetric second-order equations:
\[ \left[ i \frac{\partial}{\partial t} + C_s \nabla \right] \phi(t, x) = 0 , \] (5)
and
\[ \left[ \frac{\partial^2}{\partial t^2} - i C_w k \cdot \nabla \right] \phi(t, x) = 0 . \] (6)
Following [10,15], who originally derived the infinite set of equations given by Eq. (5), these equations are called the Schrödinger-like equations. However, Eq. (6) represents a set of new asymmetric equations derived in this paper.

All local equations that can be obtained from the eigenvalue equations, which are based on the irreps of the orthochrous extended Galilean group, are given by Eqs. (3) through (6). Since the derivation does not guarantee that these equations are Galilean invariant, their invariance must be verified.
4. Galilean invariant equations

4.1. Symmetric equations

Having derived the local symmetric and asymmetric equations, their invariance with respect to the Galilean transformations of $G_e$, which involve translations in space and time, rotations and boosts, is now verified.

Let $S$ and $S'$ be two intertial frames of reference in Galilean space and time, and let $v = \text{const}$ be the velocity with which $S'$ moves with respect to $S$. Then, a boost is given by $x = x' + vt$ and $t' = t$, which can be written as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - v \cdot \nabla',$$

and $\nabla = \nabla'$ [16]. Applying these transformations to the eigenvalue equations (Eqs. 1 and 2), it is seen that the transformed equations have different forms, which means that they are not Galilean invariant.

It can be shown that all symmetric and asymmetric equations are invariant with respect to the Galilean translations and rotations, which are the subgroups of $G_e$. However, their boost invariance must be investigated independently by using the following transformation law for the wavefunction $\phi(t, x) = \chi(t', x') \phi'(t', x')$, where $\chi(t', x')$ is to be determined [15]. The results previously obtained show that none of the symmetric equations given by Eqs. 3 and 4 is Galilean invariant [15].

4.2. Schrödinger-like equations

Galilean invariance of the Schrödinger-like equations given by Eq. 5 requires that the coefficient $C_s$ is the same as the coefficient $C_s'$ in the following transformed Schrödinger-like equations

$$\left[ i \frac{\partial}{\partial t'} + C_s' \nabla'^2 \right] \phi'(t', x') = 0,$$

where $C_s' = \omega'/k'^2$.

The additional requirement for the Galilean invariance is that the wavefunction $\phi(t, x)$ transforms as

$$\phi(t, x) = \phi(t', x' + vt') = \phi'(x', t') \chi(t', x')$$

$$= \phi'(x', t') e^{i\eta(t', x')},$$

where the phase factor $\eta(t', x') = (v \cdot x' + v^2 t'/2) / (2C_s')$ [16,17]; this standard phase factor for Galilean transformations appears also in the Galilean invariant Newton laws of dynamics [18] from which it can be removed in a self-consistent way [19].

The above two conditions are necessary for all Schrödinger-like equations with arbitrary $C_s$ to be Galilean invariant, which means that all extended Galilean observers agree upon the form of the dynamical equation and its wavefunction in their inertial frames of reference.
In previous work \cite{10,15}, it was assumed that $C_s = C'_s$ in all inertial frames; a formal proof of this is now presented in Proposition 1.

**Proposition 1:** The Schrödinger-like equations given by Eq. (5) are Galilean invariant if, and only if, $C_s = \omega/k^2 = C'_s = \omega'/k'^2$ in all inertial frames of reference.

**Proof:** The general solutions to Eq. (5) can be written as

$$\phi(t,x) = \phi_0 e^{-i(\omega t - k \cdot x)}, \quad (10)$$

and

$$\phi'(t',x') = \phi'_0 e^{-i(\omega' t' - k' \cdot x')}, \quad (11)$$

where $\phi_0$ and $\phi'_0$ are constant amplitudes. Combining these solutions with Eq. (9) and assuming that the amplitudes are the same, $\phi_0 = \phi'_0$, then, the result yields

$$k \cdot x - \omega t = k' \cdot x' - \omega' t' + \frac{1}{2C'_s}v \cdot x' + \frac{1}{4C'_s}v^2 t'. \quad (12)$$

Using the Galilean transformations $t' = t$ and $x' = x - vt$, and also substituting $C'_s = C_s$, Eq. (12) becomes

$$\left( k' - k + \frac{1}{2C_s}v \right) \cdot x + \left( \omega - \omega' - k' \cdot v + \frac{1}{4C_s}v^2 \right) t = 0, \quad (13)$$

which gives

$$k' = k - \frac{1}{2C'_s}v, \quad (14)$$

and, taking this result into account, the expression for $\omega'$ becomes

$$\omega' = \omega - k \cdot v + \frac{1}{4C'_s}v^2. \quad (15)$$

Therefore,

$$\frac{\omega'}{k'^2} = \frac{\omega - k \cdot v + v^2/4C_s}{k^2 - k \cdot v/C_s + (v/2C_s)^2} = \frac{\omega}{k^2}, \quad (16)$$

and

$$\left( \frac{\omega}{C_s} - k^2 \right) \left( \frac{v^2}{4C_s} - k \cdot v \right) = 0. \quad (17)$$

Since the second term on the LHS of this equation is non-zero, then $C_s = \omega/k^2$. This concludes the proof.

**Corollary 1:** The value of $C_s$ in the Schrödinger-like equations (Eq. 5) has the same value in all inertial frames of reference.

According to Corollary 1, the constant $C_s$ plays the same role in Galilean Relativity as the speed of light in the Special Theory of Relativity; this means that $C_s$ remains the same for all Galilean observers.

The main result is that there is an infinite set of Schrödinger-like equations that are Galilean invariant and that they only differ by their different values of $C_s$. The significance of this constant in the physics of OM and DM is discussed in Sect. 5.
Z.E. Musielak

4.3. New asymmetric equations

The new asymmetric equations given by Eq. (6) also form an infinite set because the constant \( C_w \) may have any real value. Typically, Galilean invariance requires a phase factor in the wavefunction [13,15-17]. However, the results of the following proposition show that no phase factor cannot be defined to make these asymmetric equations Galilean invariant.

**Proposition 2:** Let \( \phi(t, x) \) be the wavefunction of Eq. (8), \( \phi'(x', t') \) be the transformed wavefunction, and \( \eta(t', x') \) its phase factor given by Eq. (9). Then, no \( \eta(t', x') \) exists to guarantee Galilean invariance of the new asymmetric equations.

**Proof:** After performing the Galilean transformations, Eq. (6) becomes

\[
\left[ \frac{\partial^2}{\partial t'^2} - iC_w^2 \mathbf{k}' \cdot \nabla' \right] \phi(t', x')
- \left[ 2(\mathbf{v} \cdot \nabla') \frac{\partial}{\partial t'} - (\mathbf{v} \cdot \nabla')^2 \right] \phi(t', x') = 0 ,
\]  

(18)

This equation has two extra terms when compared to Eq. (6), and therefore it is not Galilean invariant.

Introducing the phase factor \( \eta(t', x') \) in Eq. (9) and some algebraic manipulations, Eq. (18) can be written as

\[
\left[ \frac{\partial^2}{\partial t'^2} - iC_w^2 \mathbf{k}' \cdot \nabla' \right] \phi'(t', x')e^{i\eta(t', x')}
- \left[ 2(\mathbf{v} \cdot \nabla') \frac{\partial}{\partial t'} - (\mathbf{v} \cdot \nabla')^2 \right] \phi'(t', x')e^{i\eta(t', x')} = 0 ,
\]  

(19)

which is of the same form as Eq. (18), except the exponential term with the phase factor. The two extra terms remain; thus, there is no \( \eta(t', x') \) that can make Eq. (19) Galilean invariant or of the same form as Eq. (6). This concludes the proof.

**Corollary 2:** The results of Proposition 2 can be generalized to \( \phi(t, x') = \phi'(x', t') \chi(t', x') \), where \( \chi \) is any differentiable function.

Since Galilean invariance cannot be guaranteed by introducing a phase factor, a different method is now presented in the following proposition.

**Proposition 3:** The new asymmetric equation given by Eq. (10) becomes Galilean invariant if, and only if, \( \phi(x, t) = \phi(r) \) and \( \phi'(x', t') = \phi'(r') \), where \( r = x + vt/2 \) and \( r' = x' + vt'/2 \).

**Proof:** Comparison of Eq. (10) and Eq. (19) shows that these equations are of the same form if

\[
\left[ 2(\mathbf{v} \cdot \nabla') \frac{\partial}{\partial t'} - (\mathbf{v} \cdot \nabla')^2 \right] \phi'(t', x') = 0 ,
\]  

(20)

where \( \phi'(t', x') = \phi'(r') \) and \( \eta(t', x') = 0 \). Using Eq. (20), it can be shown that Eq. (19) is of the same form as Eq. (6).
It must be also noted that after \( \phi(x, t) = \phi(r) \) is substituted into Eq. (6), its form remains the same as that of Eq. (19); this concludes the proof.

The results of Proposition 3 can be used to derive the following equation for \( \phi(x, t) = \phi(r) \)

\[
\frac{d^2 \phi}{d(k \cdot r)^2} - iC_{w,v} \frac{d\phi}{d(k \cdot r)} = 0 ,
\]

or after the integration

\[
\frac{d\phi}{d(k \cdot r)} - iC_{w,v} \phi = C_0 ,
\]

where \( r = x + vt/2, C_{w,v} = 4C_w/v^2 \) and \( C_0 \) is an integration constant.

Similarly, the corresponding equation for \( \phi(x', t') = \phi'(r') \) becomes

\[
\frac{d^2 \phi'}{d(k' \cdot r')^2} - iC_{w,v}' \frac{d\phi'}{d(k' \cdot r')} = 0 ,
\]

or after the integration

\[
\frac{d\phi'}{d(k' \cdot r')} - iC_{w,v}' \phi = C_0' ,
\]

where \( r' = x' + v't'/2, C_{w,v}' = 4C_{w,v}'/v^2 \) and \( C_0' \) is an integration constant.

Equations (21), (22), (23) and (24) are of the same form if, and only if, \( C_{w,v} = C_{w,v}' \) and \( C_0 = C_0' \) have the same values for all extended Galilean observers.

Finding the solutions to Eqs. (21) and (23) is straightforward; they can be written as

\[
\phi(k \cdot r) = -i \frac{C_1}{C_{w,v}} e^{iC_{w,v}(k \cdot r)} + C_2 ,
\]

and

\[
\phi'(k' \cdot r') = -i \frac{C_1'}{C_{w,v}'} e^{iC_{w,v}'(k' \cdot r')} + C_2' ,
\]

where \( C_1, C_2, C_1' \) and \( C_2' \) are integration constants. The solutions to Eqs. (22) and (24) are of the same form as those given by Eqs. (25) and (26) if, and only if, \( C_2 = iC_0/C_{w,v} \) and \( C_2' = iC_0'/C_{w,v}' \).

For these solutions to be of the same form, it is required that \( C_1 = C_1', C_2 = C_2', C_{w,v} = C_{w,v}' \) and \( C_{w,v}' = C_{w,v} \). The following proposition presents the necessary conditions to guarantee Galilean invariance of Eqs. (21) and (23).

**Proposition 4:** The conditions for the Galilean invariance \( C_w = \omega^2/k^2 = C_{w,v}' = \omega^2/k'^2 \), \( C_1 = C_1' \) and \( C_2 = C_2' \) are satisfied if, and only if

\[
k' = \xi(x, t)k ,
\]

where

\[
\xi(x, t) = \left( x + \frac{1}{2} vt \right)^2 \left( x^2 - \frac{1}{4} v^2 t^2 \right)^{-1} ,
\]
Proof: According to Proposition 3, there is no phase factor; thus, 
\[ \phi(k \cdot r) = \phi'(k' \cdot r') \]
and Eqs. (25) and (26) give
\[ -i \frac{C_1}{C_{w,v}} e^{i C_{w,v}(kr)} + C_2 = -i \frac{C'_1}{C'_{w,v}} e^{i C'_w(k'r')} + C'_2. \]  
(29)
Taking \( C_1 = C'_1 \), \( C_2 = C'_2 \) and \( C_w = C'_w \), Eq. (29) simplifies to
\[ k \cdot \left( x + \frac{1}{2} vt \right) = k' \cdot \left( x' + \frac{1}{2} vt' \right). \]  
(30)
Using the Galilean transformations \( x' = x - vt \) and \( t' = t \), Eq. (30) becomes
\[ k \cdot \left( x + \frac{1}{2} vt \right) = k' \cdot \left( x - \frac{1}{2} vt \right), \]  
(31)
or
\[ k' = \left( x + \frac{1}{2} vt \right)^2 \left( x^2 - \frac{1}{4} vt^2 \right)^{-1} k. \]  
(32)
This concludes the proof.

Corollary 3: Since \( v = \text{const} \), the condition \( C_{w,v} = C'_{w,v} \) reduces to \( C_w = C'_w \), or more specifically to
\[ C_w = \frac{\omega^2}{k^2} = \frac{\omega'^2}{k'^2} = C'_w. \]  
(33)
The obtained results demonstrate that the new asymmetric equations are Galilean invariant if, and only if, the argument of the wavefunction is \((x + vt/2)\), and the wavevector and the transformed wavevector are related to each other by Eq. (33). The method to make equations Galilean invariant presented in Propositions 4 and 5 can also be applied to other second-order equations.

5. Physical implications

5.1. Fundamental equations of physics

The main results of this paper are the infinite sets of Galilean invariant Schrödinger-like and new asymmetric equations given by Eqs. (5) and (6), respectively, which are local and their Lagrangians are known.

The coefficients \( C_s \) and \( C_w \) in the Schrödinger-like and new asymmetric equations, respectively, are called here the Galilean constants as they are the same for all extended Galilean observers; in other words, they resemble the constant speed of light in Special Theory of Relativity but their physical meaning is different, as shown next.

Since the main differences between the equations in each set are their constant coefficients, it is suggested that only after specific values of these constants are determined by physical properties of matter, such equations with fixed constants are called the fundamental equations of physics. After introducing this new definition,
the fundamental equations of physics for ordinary and dark matter are now identified.

5.2. Fundamental equation for ordinary matter

The two infinite sets of Galilean invariant equations for the scalar wavefunction cannot be used to describe classical particles of OM as their motion is governed by Newton’s equations, which are Galilean invariant, if the force in the second law of dynamics is also Galilean invariant [11,13,19]. Previous work (e.g., [20-22]) showed some applications of the Schrödinger equation to classical waves. Similar applications of the Schrödinger-like and new asymmetric equations are also possible but they will not be considered here.

Instead, this study concentrates on the microscopic structure of OM and the quantum description of elementary particles, which requires taking into account the wave-particle duality [17]. In the Galilean Relativity the energy-momentum relationship is given by $E = p^2/2m$, where $E$, $p$ and $m$ are the kinetic energy, momentum and mass of the free particles, respectively. Based on this relationship, let $C_s = \omega/k^2 = \alpha_0/2m$, with $\alpha_0$ being a constant of Nature to be determined from the physical properties of OM.

Applying the de Broglie relationship $p = \hbar k$ to $C_s$, one obtains

$$\omega = \frac{\alpha_0 k^2}{2m} = \frac{\alpha_0 p^2}{2m\hbar^2} = \frac{\alpha_0}{\hbar^2} E = \frac{\alpha_0}{\hbar} \omega.$$  \hspace{1cm} (34)

This expression is valid if, and only if, $\alpha_0 = \hbar$, or $\alpha_0$ is the same as the Planck constant. Then, Eq. (7) with $C_s = \alpha_0/2m = \hbar/2m$ becomes the fundamental Schrödinger equation of Quantum Mechanics [17].

Since there are also infinitely many new asymmetric equations, it remains to be determined whether one of them can also describe the microscopic properties of OM; this problem is discussed in Sect. 5.3, where the equations are applied to DM.

5.3. Fundamental equation for dark matter

As of today, no electromagnetic radiation of any kind has been detected from DM [4-9]. The NASA Bullet Cluster observations show a different behavior of DM when compared to OM [23]. This may imply that DM does not have the same quantum structure as OM and that the fundamental Schrödinger equation with $C_s = \hbar/2m$ is not the fundamental equation for DM; nevertheless, other Schrödinger-like equations with different $C_s$ are now explored.

Let $C_s = \alpha_1/2m$, where $\alpha_1$ is a constant of Nature whose value is different than the Planck constant, namely, either $\alpha_1 < \hbar$ or $\alpha_1 > \hbar$. If this constant exists for DM, then the resulting Schrödinger-like equation would be the fundamental equation for DM. The main problem with this idea is that the dimension of $\alpha_1$ is the same ($J\cdot s$) as the dimension $\alpha_0 = \hbar$ which, from a physical point of view, is highly unlikely that there are two constants of Nature with identical dimensions but different values.
Based on this argument, it is concluded that no other Schrödinger-like equation can become the fundamental equation for DM.

The only other possibility is that such fundamental equation for DM is among the set of new asymmetric equations discovered in this paper. The coefficient $C_{w,v}$ is given by

$$C_{w,v} = \frac{4}{v^2} C_{w} = \frac{4}{v^2} \frac{\omega^2}{k^2},$$

and for $C_{w}$ see Eq. (35).

For a DM non-relativistic elementary particle of mass $m$, the coefficient can be written as

$$C_{w} = \frac{\omega^2}{k^2} = \frac{\beta_0}{2m},$$

(36)

where $\beta_0$ is a constant of Nature for DM only. Since $\beta_0$ is measured in $J$, which is different than the Planck constant, let $\beta_0 = \varepsilon_o$ be a quanta of energy of DM. Then

$$C_{w,v} = \frac{\varepsilon_o}{2m} \frac{4}{v^2} \frac{2m}{\varepsilon_v} = \frac{\varepsilon_o}{\varepsilon_v},$$

(37)

where $\varepsilon_v = \frac{mv^2}{2}$ is the kinetic energy of DM particles confined to the inertial frame moving with the velocity $v = |v|$. The energy $\varepsilon_v$ normalizes the quanta of energy of DM, so that the coefficient in the fundamental equation for DM (see Eq. 21) and in its solution (see Eq. 25) is dimensionless.

Based on the above results, it is proposed that the fundamental equation for DM is the new asymmetric equation given by

$$\left[ \frac{\partial^2}{\partial t^2} - i \frac{\varepsilon_o}{2m} \frac{k \cdot \nabla}{\varepsilon_v} \right] \phi(t,\mathbf{x}) = 0,$$

(38)

whose Galilean invariant form is

$$\frac{d^2 \phi}{d(k \cdot r)^2} - i \frac{\varepsilon_o}{\varepsilon_v} \frac{d\phi}{d(k \cdot r)} = 0,$$

(39)

and its solution is

$$\phi(k \cdot r) = -i \frac{C_1}{\varepsilon_o} e^{i \varepsilon_v(k \cdot r)/\varepsilon_v} + C_2.$$

(40)

This shows that the fundamental equations for OM and DM are different and that they depend on different constants of Nature. As a result, the Schrödinger equation describes the quantum structure of OM, whereas the quantum structure of DM is described by the new asymmetric equation (see Eqs. 39 and 40).

5.4. Implications for dark matter

In the proposed fundamental equation for DM (see Eq. 6 or Eq. 21), the coefficient $C_{w}$ depends on the new universal constant of Nature $\varepsilon_o$, which represents the quanta of energy for DM. The role of $\varepsilon_o$ is similar to the Planck constant but its physical
meaning is different, namely, while the energy for OM is given by $E = \hbar \omega$, the energy for DM is simply $E_0 = \varepsilon_0$. In other words, the main difference between OM and DM is that quantization of DM can only be done using the quanta $\varepsilon_0$; this may explain the lack of observed radiation [4-9] and a significantly different behavior of DM than OM [23].

The obtained results imply that DM may be considered as a collection of DM elementary particles with mass $m$ and that these particles may exchange their energy by emitting or absorbing the quanta of energy $\varepsilon_0$. Assuming that the particles interact only gravitationally, the process of quanta exchange may produce a detectable gravitational radiation.

6. Conclusions

Among the two infinite sets of derived Galilean invariant equations, the fundamental equations of matter are identified. It is shown that in one set, the fundamental equation is the Schrödinger equation for ordinary matter, and in the other set, a new asymmetric equation is proposed to be the fundamental equation for dark matter. The obtained results demonstrate that the equation for dark matter requires a new constant of Nature, which is called a quanta of energy. The constant is only valid for dark matter and its physical implication is that quantization of dark matter energy is independent from frequency. The resulting frequency independence may be responsible for the lack of electromagnetic radiation from dark matter, for its different behavior compare with ordinary matter, and it may also be responsible for the generation of detectable gravitational radiation.

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