LIPSCHITZ BOUNDS FOR INTEGRAL FUNCTIONALS WITH \((p, q)\)-GROWTH CONDITIONS

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Abstract. We study local regularity properties of local minimizer of scalar integral functionals of the form

\[ F[u] := \int_{\Omega} F(\nabla u) - fu \, dx \]

where the convex integrand \( F \) satisfies controlled \((p, q)\)-growth conditions. We establish Lipschitz continuity under sharp assumptions on the forcing term \( f \) and improved assumptions on the growth conditions on \( F \) with respect to the existing literature. Along the way, we establish an \( L^\infty-L^2 \) estimate for solutions of linear uniformly elliptic equations in divergence form which is optimal with respect to the ellipticity contrast of the coefficients.

1. Introduction

In this note, we revisit the question of Lipschitz-regularity for local minimizers of integral functionals of the form

\[ w \mapsto F(w; \Omega) := \int_{\Omega} F(\nabla w) - fw \, dx. \]

We recall in the case \( F(z) = \frac{1}{2}|z|^2 \) local minimizer of (1) satisfy \(-\Delta w = f\). A classic theorem due to Stein [39] implies

\[ -\Delta w = f \in L^{n,1}(\Omega) \Rightarrow \nabla w \in L^\infty_{\text{loc}}(\Omega), \]

where \( L^{n,1}(\Omega) \) denotes the Lorentz space (see below for a definition). In view of [14] the condition \( f \in L^{n,1}(\Omega) \) is optimal in the Lorentz-space scale for the conclusion in (2). In the last decade the implication in (2) was greatly generalized by replacing the linear operator \( \Delta \) by possibly degenerate/singular uniformly elliptic nonlinear operators, see [3, 31, 32, 15]. More recently, those results where extended to a wide range of non-uniformly elliptic variational problems by Beck and Mingione in [6] (see also [20, 22] for related results for non-autonomous or non-convex vector valued problems).

In this paper, we are interested in a specific class of non-uniformly elliptic problems, namely functionals with so-called \((p, q)\)-growth conditions which are described in the following

Assumption 1. Let \( 0 < \nu \leq \Lambda < \infty \), \( 1 < p \leq q < \infty \) and \( \mu \in [0,1] \) be given. Suppose that \( F : \mathbb{R}^n \to [0,\infty) \) is convex, locally \( C^2 \)-regular in \( \mathbb{R}^n \setminus \{0\} \) and satisfies

\[ \begin{aligned}
\nu(\mu^2 + |z|^2)^{\frac{2}{p}} \leq F(z) &\leq \Lambda(\mu^2 + |z|^2)^{\frac{2}{q}} + \Lambda(\mu^2 + |z|^2)^{\frac{2}{q}} \\
|\partial^2 F(z)| \leq \Lambda(\mu^2 + |z|^2)^{\frac{2}{q} - 2} + \Lambda(\mu^2 + |z|^2)^{\frac{2}{q} - 2}, \\
\nu(\mu^2 + |z|^2)^{\frac{2}{q} - 2} |\xi|^2 \leq \langle \partial^2 F(z)\xi, \xi \rangle,
\end{aligned} \]

for every choice of \( z, \xi \in \mathbb{R}^n \) with \( |z| > 0 \).

Regularity properties of local minimizers of (1) in the case \( p = q \) are classical, see, e.g., [26]. A systematic regularity theory in the case \( p < q \) was initiated by Marcellini in [33, 34], see [36] for a recent overview.

Before we state our main result, we recall a standard notion of local minimality in the context of integral functionals with \((p, q)\)-growth.
Definition 1. We call \( u \in W^{1,1}_{\text{loc}}(\Omega) \) a local minimizer of \( F \) given in (1) with \( f \in L^n_{\text{loc}}(\Omega) \) if for every open set \( \Omega' \Subset \Omega \) the following is true:

\[
F(u, \Omega') < \infty
\]

and

\[
F(u, \Omega') \leq F(u + \varphi, \Omega')
\]

for any \( \varphi \in W^{1,1}(\Omega) \) satisfying \( \text{supp} \ \varphi \Subset \Omega' \).

The main result of the present paper is

Theorem 1. Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 3 \) be an open bounded domain and suppose Assumption 1 is satisfied with \( 1 < p < q < \infty \) such that

\[
\frac{q}{p} < 1 + \min \left\{ \frac{2}{n-1}, \frac{4(p-1)}{p(n-3)} \right\}.
\]

Let \( u \in W^{1,1}_{\text{loc}}(\Omega) \) be a local minimizer of the functional \( F \) given in (1) with \( f \in L^n_{\text{loc}}(\Omega) \). Then \( \nabla u \) is locally bounded in \( \Omega \). When \( p \geq 2 - \frac{4}{n-1} \) or when \( f \equiv 0 \) condition (4) can be replaced by

\[
\frac{q}{p} < 1 + \frac{2}{n-1}.
\]

Remark 1. Theorem 1 should be compared to the findings of the recent papers [6] and [8]: In [6], Beck and Mingione proved (among many other things) the conclusion of Theorem 1 under the more restrictive relation

\[
\frac{q}{p} < 1 + \min \left\{ \frac{2}{n-1}, \frac{4(p-1)}{p(n-2)} \right\}.
\]

Hence, we obtain here an improvement in the gap conditions on \( \frac{q}{p} \). In [8], we proved Theorem 1 in the specific case \( f \equiv 0 \), \( p \geq 2 \) and \( \mu = 1 \).

The proof of Theorem 1 closely follows the strategy presented in [6] and relies on careful estimates for certain uniformly elliptic problems. To illustrate this, let us consider the case that \( F \) satisfies Assumption 1 with \( p = 2 < q \) and \( f \equiv 0 \): Let \( u \) be a local minimizer of \( F(\cdot, B_1) \) and assume that \( u \) is smooth. Standard arguments yield

\[
(6) \quad \text{div}(A(x)\nabla \partial_i u) = 0 \quad \text{where} \quad A := \partial^2 F(\nabla u).
\]

Hence, \( \partial_i u \) satisfies a linear elliptic equation where the ellipticity ratio \( R(x) \) of the coefficients, that is the quotient of the highest and lowest eigenvalue of \( A(x) \), is determined by the size of \( |\nabla u(x)| \). More precisely, (3) with \( p = 2 < q \) implies \( R(x) \sim (1 + |\nabla u(x)|^{q-2}) \). By standard theory for uniformly elliptic equations applied to (6), there exists an exponent \( m = m(n) > 0 \) such that

\[
(7) \quad \|\partial_i u\|_{L^\infty(B_{\frac{1}{2}})} \lesssim \|R\|_{L^\infty(B_1)}^{m} \|\nabla u\|_{L^\infty(B_1)} \lesssim (1 + \|\nabla u\|_{L^\infty(B_1)}^{q-2})^m \|\partial_i u\|_{L^2(B_1)}.
\]

Appealing to some well-known iteration arguments, it is possible to absorb the \((1 + \|\nabla u\|_{L^\infty(B_1)}^{q-2})^m\) prefactor on the right-hand side in (7) provided that \((q-2)m < 1\), which yields the restriction \( \frac{q}{p} < 1 + \frac{2}{n-1} \). Once an a priori Lipschitz estimate for smooth minimizer is established, the proof of Theorem 1 follows by a careful regularization and approximation procedure (as in e.g. [6]).

The main technical achievement of the present manuscript – using a method introduced in [7] – is an improvement of the exponent \( m \) in (7) compared to the previous results. This improvement is optimal for \( n \geq 4 \) and essentially optimal for \( n = 3 \). More precisely, we have

Proposition 1. Let \( B = B_{R}(x_0) \subset \mathbb{R}^n \), \( n \geq 3 \) and \( \kappa \in (0, \frac{1}{2}) \). There exists \( c = c(n, \kappa) < \infty \), where \( c = c(n) \) provided \( n \geq 4 \), such that the following is true. Let \( 0 < \nu \leq \lambda < \infty \) and suppose \( a \in L^\infty(B; \mathbb{R}^{n \times n}) \) satisfies that \( a(x) \) is symmetric for almost every \( x \in B \) and uniformly elliptic in the sense

\[
(8) \quad \nu |z|^2 \leq a(x)z \cdot z \leq \Lambda |z|^2 \quad \text{for every} \ z \in \mathbb{R}^n \text{and almost every} \ x \in B.
\]
Let $v \in W^{1,2}(B)$ be a subsolution, that is it satisfies

\begin{equation}
\int_B a \nabla v \cdot \nabla \varphi \leq 0 \quad \text{for all } \varphi \in C^1_c(B) \text{ with } \varphi \geq 0.
\end{equation}

Then,

\begin{equation}
\sup_{\frac{1}{2}B} v \leq c\left(\frac{A}{n}\right)^m \left(\int_B (v_+)^2 \, dx\right)^{\frac{1}{2}}, \quad \text{where } m := \frac{1}{2} \left(\frac{n-1}{1+\kappa}\right) \text{ for } n \geq 4,
\end{equation}

Moreover, for $n \geq 4$ the exponent $m = \frac{n-1}{4}$ is optimal for the estimate in (10) and for $n = 3$ the exponent is essentially optimal in the sense that the estimate in (10) is in general false for $m < \frac{1}{2}$.

While estimate (10) is in some sense optimal, the condition (4) in Theorem 1 is in general not optimal. To see this, we recall a result of [10]: Suppose Assumption 1 is satisfied with $\mu = 1$ and $F(0) = 0$, $\partial F(0) = 0$, then bounded local minimizer of (1) with $f \equiv 0$ are locally Lipschitz provided $q < p + 2$. This can be combined with the recent local boundedness [29], where it is proven that under Assumption 1 local minimizers of (1) (with $f = 0$) are locally bounded provided $\frac{1}{p} - \frac{1}{q} \leq \frac{n-1}{n-4}$ and this condition is sharp in view of [34, Theorem 6.1]. Combining Theorem 1 with the above mentioned results of [10, 29], we deduce that Assumption 1 (together with some mild technical extra assumptions) with $1 < p \leq q$ and

\begin{equation}
\frac{q}{p} < 1 + \max\left\{\frac{2}{n-1}, \min\left\{\frac{2}{p}, \frac{n-1}{n-4}\right\}\right\}
\end{equation}

implies that local minimizer of (1) with $f \equiv 0$ are locally Lipschitz (see also [1] for related discussion). While it is not clear whether (11) is optimal, it strictly improves condition (5) for example in the cases $p = 3$ and $n \geq 5$. Let us mention that condition (5) also appears in [38] in the context of higher integrability results for vectorial problems (where Lipschitz regularity fails even in the case $p = q$, see [40]). While also in that case bounded minimizer enjoy higher gradient integrability under the condition $q < p + 2$, see [13], this result cannot be combined with an a priori local boundedness result which fails in the vectorial case already for $p = q$.

For non-autonomous functionals, i.e., $\int_B f(x, Dv) \, dx$, rather precise sufficient & necessary conditions are established in [25], where the conditions on $p, q$ and $n$ have to be balanced with the (Hölder)-regularity in space of the integrand. Currently, regularity theory for non-autonomous integrands with non-standard growth, e.g. $p(x)$-Laplacian or double phase functionals, are a very active field of research, see, e.g., the recent papers [4, 9, 12, 18, 16, 17, 19, 20, 21, 23, 28, 27, 30, 35] and [2, 11] for related results about the Lavrentiev phenomena.

2. Preliminaries

2.1. Preliminary lemmata. A crucial technical ingredient in the proof of Theorem 1 is the following lemma which can be found in [8, Lemma 3].

**Lemma 1** ([8]). *Fix $n \geq 2$. For given $0 < \rho < \sigma < \infty$ and $v \in L^1(B_\sigma)$ consider*

\[
J(\rho, \sigma, v) := \inf \left\{ \int_{B_\sigma} |v| |\nabla \eta|^2 \, dx \mid \eta \in C^1_0(B_\sigma), \eta \geq 0, \eta = 1 \text{ in } B_\rho \right\}.
\]

*Then for every $\delta \in (0, 1]$

\begin{equation}
J(\rho, \sigma, v) \leq (\sigma - \rho)^{-(\frac{1}{p} + \frac{1}{n} - \frac{1}{\rho})} \left(\int_{S_\rho} |v|^\rho \, d\mathcal{H}^{n-1}\right)^{\frac{1}{\rho}},
\end{equation}

*where*

\[S_\rho := \{x : |x| = \rho\}.
\]

Moreover, we recall here the following classical iteration lemma.
Lemma 2 (Lemma 6.1, [26]). Let $Z(t)$ be a bounded non-negative function in the interval $[\rho, \sigma]$. Assume that for every $\rho \leq s < t \leq \sigma$ it holds
$$Z(s) \leq \theta Z(t) + (t - s)^{-\alpha} A + B,$$
with $A, B \geq 0$, $\alpha > 0$ and $\theta \in [0, 1)$. Then, there exists $c = c(\alpha, \theta) \in [1, \infty)$ such that
$$Z(s) \leq c((t - s)^{-\alpha} A + B).$$

2.2. Non-increasing rearrangement and Lorentz-spaces. We recall the definition and useful properties of the non-increasing rearrangement $f^*$ of a measurable function $f$ and Lorentz spaces, see e.g. [41, Section 22]. For a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, the non-increasing rearrangement is defined by
$$f^*(t) := \inf\{\sigma \in (0, \infty) : |\{x \in \mathbb{R}^n : |f(x)| > \sigma\}| \leq t\}.$$ Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function with $\text{supp} f \subset \Omega$, then it holds for all $p \in [1, \infty)$
$$\int_{\Omega} |f(x)|^p \, dx = \int_0^{[\Omega]} (f^*(t))^p \, dt.$$ A simple consequence of (13) and the fact $f \leq g$ implies $f^* \leq g^*$ is the following inequality
$$\sup_{|A| \leq t} \int_A |f(x)|^p \, dx \leq \int_0^t (f^*_\Omega(t))^p \, dt,$$
where $f^*_\Omega$ denotes the non-increasing rearrangement of $f_\Omega := f \chi_\Omega$ (inequality (14) is in fact an equality but for our purpose the upper bound suffices).
The Lorentz space $L^{n,1}(\mathbb{R}^d)$ can be defined as the space of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ satisfying
$$\|f\|_{L^{n,1}(\mathbb{R}^n)} := \int_0^{[\Omega]} t^\frac{1}{p} f^*(t) \frac{dt}{t} < \infty.$$ Moreover, for $\Omega \subset \mathbb{R}^n$ and a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, we set
$$\|f\|_{L^{n,1}(\Omega)} := \int_0^{[\Omega]} t^\frac{1}{p} f^*_\Omega(t) \frac{dt}{t} < \infty,$$ where $f^*_\Omega$ is defined as above. Let us recall that $L^{n+\varepsilon}(\Omega) \subset L^{n,1}(\Omega) \subset L^n(\Omega)$ for every $\varepsilon > 0$, where $L^{n,1}(\Omega)$ is the space of all measurable functions $f : \Omega \to \mathbb{R}$ satisfying $\|f\|_{L^{n,1}(\Omega)} < \infty$ (here we identify $f$ with its extension by zero to $\mathbb{R}^n \setminus \Omega$).

3. Nonlinear iteration lemma and proof of Proposition 1

In this section, we provide a nonlinear iteration lemma which will eventually be the main workhorse in the proof of Theorem 1:

Lemma 3. Let $B = B_R(x_0) \subset \mathbb{R}^n$, $n \geq 3$, $\kappa \in (0, \frac{1}{2})$ and let $v \in W^{1,2}(B) \cap C(B)$ be non-negative and $f \in L^2(\mathbb{R}^n)$. Suppose there exists $M_1 \geq 1$, $M_2$, $c_m > 0$ and $k_0 \geq 0$ such that for all $k \geq k_0$ and for all $\eta \in C^1(\mathbb{R})$ with $\eta \geq 0$ it holds
$$\int_B |\nabla (v - k) + \eta|^2 \, dx \leq c_m^2 M_1^2 \int_B (v - k)^2 \, dx + c_m^2 \int_{B \cap \{v > k\}} \eta^2 |f|^2.$$ Then there exist $c = c(c_m, n) \in [1, \infty)$ for $n \geq 4$ and $c = c(c_m, \kappa) \in [1, \infty)$ for $n = 3$ such that
$$\|(v)\|_{L^\infty(\frac{1}{4}B)} \leq k_0 + c M_1^{1 + \max\{1, \frac{\kappa - 1}{2}\}} \left( \int_B (v - k_0)^2 \, dx \right)^{\frac{1}{2}} + c M_2^{\max\{1, \frac{\kappa - 1}{2}\}}.$$ Remark 2. (Optimality) The exponent in the factor $M_1^{1 + \max\{1, \frac{\kappa - 1}{2}\}}$ is optimal in dimensions $n \geq 4$ and almost optimal for $n = 3$. Indeed, consider $v(x) := x_n^2 + 1 - \Lambda |x'|^2$, where $x = (x_1, \ldots, x_{n-1}, x_n) =: (x', x_n)$ which clearly satisfies
$$- \nabla \cdot a \nabla v = 0 \quad \text{where} \quad a := \text{diag}(1, \ldots, 1, (n - 1)\Lambda).$$
Hence, by classical computations (using the symmetry of $a$) $v$ satisfies the Caccioppoli inequality (15) with
\begin{equation}
\tag{18}
c^2_m = 4(n-1), \quad M_1 = \Lambda^\frac{1}{k}, \quad f \equiv 0.
\end{equation}
Obviously, we have $\sup_{B_+^r} v \geq v(0) \geq 1$ and
\[
\int_{B_1} (v) \lesssim \int_{-1}^{1} \int_{0}^{1} \Lambda^{-\frac{1}{2}}(1+|x|^2)^{1/2} \sqrt{n-2} x^2 + \Lambda^2 |x|^{n+2} dr dx_n
\lesssim \Lambda^{-\frac{n-1}{2}} \int_{-1}^{1} (1+|x|^2)^{\frac{n+3}{2}} \lesssim \Lambda^{-\frac{n-1}{2}},
\]
where $\lesssim$ means $\leq$ up to a multiplicative constant depending only on $n$. In particular, we have
\[
\frac{\|v\|_{L^\infty(B_+^r)}}{\|v\|_{L^2(B_1)}} \geq c(n) \Lambda^{\frac{n-1}{2}} \quad \text{(18)} \quad \Rightarrow c(n) M_1^{\frac{n-1}{2}}
\]
which matches exactly the scaling in (16) for $n \geq 4$ and almost matches in the case $n = 3$. We mention that $-(v)_+$ appeared already in [37] in the context of optimal dependencies in Krylov-Safanov estimates.

Remark 3. Lemma 3 should be compared with [6, Lemma 3.1], where starting from a similar Caccioppoli inequality as (3) a pointwise bound for $v$ in terms of the $L^2$-norm $v$ and the Riesz potential of $f$ is deduced. The main improvement of Lemma 3 compared to [6, Lemma 3.1] lies in the dependence of $M_1$ (from $M_1^{\max(n,\frac{n-2}{2})}$ in [6] to $M_1^{\max(n,\frac{n-2}{2})}$) which in view of Remark 2 is essentially optimal. Let us remark that variants of [6, Lemma 3.1] also play an important role in [20, 21] where non-autonomous functionals are considered.

Before, we prove Lemma 3 let us observe that Proposition 1 is implied by Lemma 3.

Proof of Proposition 1. By density arguments, we are allowed to use $\phi = \eta^2 (v - k)_+$, $k \geq 0$ and $\eta \in C^1_c(B)$ with $\eta \geq 0$ in (9) which implies
\[
\int_B a \nabla (v - k)_+ \cdot \nabla (v - k)_+ \eta^2 dx \leq -2 \int_B a \nabla (v - k)_+ \cdot (v - k)_+ \eta \nabla \eta dx.
\]
Assumption (8) combined with Cauchy-Schwarz and Young’s inequality imply (15) with $c^2_m = 4$ and $M_1^2 = \Lambda$. The claimed estimate (10) now follows directly from Lemma 3 and the claimed optimality is a consequence of the discussion in Remark 2.

Proof of Lemma 3. Without loss of generality, we only consider the case $B = B_2$.

Throughout the proof we set
\begin{equation}
\tag{19}
2^* = 2^*(n, \kappa) := \begin{cases}
\frac{2(n-1)}{n-1} & \text{if } n \geq 4, \\
2 \frac{n}{n-1} & \text{if } n = 3.
\end{cases}
\end{equation}

Step 1. Optimization Argument. We claim that there exists $c_1 = c_1(n, 2^*) \in [1, \infty)$ such that for all $k > h \geq 0$ and all $\frac{1}{2} \leq \rho < \sigma \leq 1$
\begin{equation}
\tag{20}
\| \nabla (v - k)_+ \|_{L^2(B_{\rho})} \leq c_1 c_m M_1 \frac{\| (v - h) + \| W^{1,2}(B_{\sigma} \setminus B_{\rho}) \|}{(\sigma - \rho)^{\alpha}} + c_m M_2 \omega_f(|A_{k, \sigma}|),
\end{equation}
where $\alpha = \frac{1}{2} + \frac{2^*}{4} > 0$, 
\begin{equation}
\tag{21}
A_{l, r} := B_r \cap \{ x \in \Omega : v(x) > l \} \quad \text{for all } r > 0 \text{ and } l > 0,
\end{equation}
and \( \omega_f : [0, |B|] \to [0, \infty) \) is defined by

\[
\omega_f(t) := \left( \int_0^t ((f \chi_B)^+(s))^2 \, ds \right)^{\frac{1}{2}}.
\]

We optimize the right-hand side of (15) with respect to \( \eta \) satisfying \( \eta \in C^1_0(B_\rho) \) and \( \eta = 1 \) in \( B_\rho \); we use Lemma 1 and \( 1 \leq \frac{c(x) - h}{k - h} \) whenever \( v(x) \geq k \) in the form

\[
\inf_{\eta \in \mathcal{A}_{k, \sigma}} \int_{B_1} |\nabla \eta|^2 (v - k)^2 \, dx 
\leq \frac{1}{(\sigma - \rho)^{1 + \frac{\alpha}{2}}} \left( \int_0^\sigma \left( \int_{S_r \cap \mathcal{A}_{k, \sigma}} (v - k)^2 \, d\mathcal{H}^{n-1} \right) \, dr \right)^{\frac{1}{2}}
\leq \frac{1}{(\sigma - \rho)^{1 + \frac{\alpha}{2}}} \frac{1}{(k - h)^{2^{\frac{n}{2}}} - 2} \left( \int_0^\sigma \left( \int_{S_r \cap \mathcal{A}_{k, \sigma}} (v - h)^{2^{\frac{n}{2}}} \, d\mathcal{H}^{n-1} \right) \, dr \right)^{\frac{1}{2}}
\]

for every \( \delta > 0 \). Appealing to Sobolev inequality on spheres, we find \( c = c(n, 2^*) \in [1, \infty) \) such that for almost every \( r \in [\rho, \sigma] \) it holds

\[
\left( \int_{S_r} (v - h)^{2^{\frac{n}{2}}} \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2^{\frac{n}{2}}}} \leq c \left( \int_{S_r} (v - h)^2 \, dx + |\nabla (v - h)|^2 \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}}.
\]

Inserting (24) in (23) with \( \delta = \frac{2}{n^2} \), we obtain

\[
\inf_{\eta \in \mathcal{A}_{k, \sigma}} \int_{B_1} (v - k)^2 |\nabla \eta|^2 \, dx \leq \frac{c}{(\sigma - \rho)^{2\alpha}} \frac{\|(v - h)_+\|_{W^{1,2}(B_\sigma \setminus B_\rho)}^2}{(k - h)^{2^{\frac{n}{2}}} - 2} \| (v - h)_+ \|_{W^{1,2}(B_\sigma \setminus B_\rho)}^2.
\]

The claim (20) follows from (15) and (25) combined with

\[
\int_{B_1 \cap \{v > k\}} \eta^2 |f|^2 \, dx \leq \sup \left\{ \int_A |f|^2 \, dx : |A| \leq |A_{k, \sigma}|, A \subset B_1 \right\} \left( \int_0^{1_{A_{k, \sigma}}} ((f \chi_B)^+(s))^2 \, ds \right)
\]

and the definition of \( \omega_f \), see (22).

**Step 2.** One-Step improvement.

We claim that there exists \( c_2 = c_2(n) \in [1, \infty) \) such that

\[
J(k, \rho) \leq \frac{c_1 c_m M_1 J(h, \sigma)^{2^*} - 1}{(k - h)^{2^* - 1}} \frac{J(h, \sigma)}{(\sigma - \rho)^{\alpha}} + c_m M_2 \omega_f \left( \frac{c_2 J(h, \sigma)^{2^*}}{(k - h)^{2^*}} \right) + c_2 \frac{J(h, \sigma)^{1 + \frac{2^*}{n}}}{(k - h)^{\frac{2^*}{n}}}
\]

where \( 2_n^* := \frac{2n}{n - 2} \).

\[
J(t, r) := \| (v - t)_+ \|_{W^{1,2}(B_r)} \quad \forall t \geq 0, \, r \in (0, 1]
\]

and \( \omega_f \) is defined in (22). Note that \( k - h < v - h \) on \( A_{k, \sigma} \) for every \( r > 0 \) and thus with help of Sobolev inequality (this time on the \( n \)-dimensional ball \( B_\sigma \))

\[
|A_{k, \sigma}| \leq \int_{A_{k, \sigma}} \left( \frac{v(x) - h}{k - h} \right)^{2_n^*} \, dx \leq \frac{\|(v - h)_+\|_{L^{2^*}(B_\sigma)}^{2_n^*}}{(k - h)^{2_n^*}} \leq c_2 J(h, \sigma)^{2_n^*}
\]

where \( c_2 = c_2(n) \in [1, \infty) \). Combining (28) with (20), we obtain

\[
\| \nabla (v - k)_+ \|_{L^2(B_\rho)} \leq \frac{c_1 c_m M_1 J(h, \sigma)^{2^*}}{(\sigma - \rho)^{\alpha}} + \frac{c_m M_2 \omega_f \left( \frac{c_2 J(h, \sigma)^{2^*}}{(k - h)^{2^*}} \right)}{(k - h)^{\frac{2^*}{n}}}
\]
It remains to estimate \(\|v - k\|_{L^2(B_\rho)}\): A combination of Hölder inequality, Sobolev inequality and (28) yield

\[
\|v - k\|_{L^2(B_\rho)} \leq \|v - h\|_{L^2(B_\rho)} |A_{k,\sigma}|^{\frac{1}{r}} \leq c(n) \frac{J(h, \sigma)^{1 + \frac{2\tau}{r}}}{(k - h)^{\frac{2\tau}{r}}}
\]

Combining (29) and (30), we obtain (26).

**Step 3.** Iteration

For \(k_0 \geq 0\) and a sequence \((\Delta_\ell)_{\ell \in \mathbb{N}} \subset [0, \infty)\) specified below, we set

\[
k_\ell := k_0 + \sum_{i=1}^\ell \Delta_i, \quad \sigma_\ell = \frac{1}{2} + \frac{1}{2\ell + 1}.
\]

For every \(\ell \in \mathbb{N} \cup \{0\}\), we set \(J_\ell := J(k_\ell, \sigma_\ell)\). From (26), we deduce for every \(\ell \in \mathbb{N}\)

\[
J_\ell \leq c_1 c_m M_1 2^{(\ell + 1)\alpha} \left(\frac{J_{\ell - 1}}{\Delta_\ell}\right)^{\frac{2\tau}{r} - 1} J_{\ell - 1} + c_m M_2 \omega_f \left(\frac{c_2 J_{\ell - 1}^{2\tau}}{(\Delta_\ell)^{2\tau}}\right) + c_2 \left(\frac{J_{\ell - 1}}{\Delta_\ell}\right)^{\frac{2\tau}{r}} J_{\ell - 1},
\]

where \(c_1\) and \(c_2\) are as in Step 2. Fix \(\tau = \tau(n, \kappa) \in (0, \frac{1}{2})\) such that

\[
(2\tau)^{\frac{2\tau}{r} - 1} = 2^{-\alpha}.
\]

We claim that we can choose \(\{\Delta_\ell\}_{\ell \in \mathbb{N}}\) satisfying

\[
\sum_{\ell \in \mathbb{N}} \Delta_\ell < \infty
\]

in such a way that

\[
J_\ell \leq \tau^\ell J_0 \quad \text{for all } \ell \in \mathbb{N} \cup \{0\}.
\]

Obviously, (31), (35) and \(\tau \in (0, 1)\) yield boundedness of \(v\) in \(B_\rho\).

For \(\ell \in \mathbb{N}\), we set \(\Delta_\ell = \Delta_\ell^{(1)} + \Delta_\ell^{(2)} + \Delta_\ell^{(3)}\) with

\[
\Delta_\ell^{(1)} = \left(2^{\alpha} 3c_1 c_m M_1 \tau^{-(\frac{2\tau}{r} - 1)}\right) \frac{1}{(\Delta_\ell)^{\frac{2\tau}{r} - 1}} J_0^{2 - \ell}, \quad \Delta_\ell^{(3)} = \left(\frac{3c_2}{\tau}\right)^{\frac{1}{r}} \tau^{\ell - 1} J_0^{1 + \frac{1}{r}}
\]

and \(\Delta_\ell^{(2)}\) being the smallest value such that

\[
\frac{c_m M_2 \omega_f}{(\Delta_\ell^{(2)})^{2\tau}} \leq \frac{1}{3} \tau^\ell J_0
\]

is valid. The choice of \(\tau\) (see (33)), \(\Delta_\ell\) and estimate (32) combined with a straightforward induction argument yield (35). Indeed, assuming \(J_{\ell - 1} \leq \tau^{\ell - 1} J_0\), we obtain

\[
J_\ell \leq \frac{c_1 c_m M_1 2^{(\ell + 1)\alpha}}{2^{\alpha} 3c_1 c_m M_1 \tau^{-(\frac{2\tau}{r} - 1)}} \left(\tau^{\ell - 1} 2^{\ell}\right)^{\frac{2\tau}{r} - 1} \tau^{\ell - 1} J_0 + \frac{1}{3} \tau^\ell J_0 + \frac{c_2 \tau^{\ell - 1}}{3c_3} J_0
\]

\[
= \frac{1}{3} \tau^{2 \alpha \ell} \left(2\tau\right)^{\frac{2\tau}{r} - 1} \tau^{\ell - 1} J_0 + \frac{2}{3} \tau^\ell J_0 \overset{(33)}{=} \tau^\ell J_0.
\]

Using \(\sum_{\ell \in \mathbb{N}} (2^{-\ell} + \tau^\ell) < \infty\), we deduce from (37) and (36)

\[
\sum_{\ell \in \mathbb{N}} \tau^{\ell} \leq \sum_{\ell \in \mathbb{N}} \frac{\frac{1}{3} \tau^{\ell - 1} J_0}{(\omega_f^{\frac{1}{r}}(\frac{\tau^{\ell - 1} J_0}{\tau^\ell J_0}))^{\frac{1}{r}}} + c(1 + M_1^{\frac{1}{r} - 1}) J_0,
\]
where $c = c(n, \kappa, c_m) \in [1, \infty)$. Next, we show that $f \in L^{n,1}(B_1)$ ensures that the first term on the right-hand side of (38) is bounded and thus (34) is valid. Indeed,

$$
\sum_{\ell \in \mathbb{N}} \frac{\tau^\ell J_0(\omega_f^{-1}(\frac{\tau^\ell J_0}{3c_m M_2}))^\frac{1}{2 - \frac{1}{n}}}{\tau \log \tau} \leq \frac{1}{\tau} \int_1^\infty \frac{\tau^\ell J_0(\omega_f^{-1}(\frac{\tau^\ell J_0}{3c_m M_2}))^\frac{1}{2 - \frac{1}{n}}}{dt} \frac{dt}{t} = \frac{1}{\tau \log \tau} \int_0^\infty \frac{t J_0(\omega_f^{-1}(\frac{t J_0}{3c_m M_2}))^\frac{1}{2 - \frac{1}{n}}}{\omega_f^{-1}(\frac{t J_0}{3c_m M_2})^\frac{1}{2 - \frac{1}{n}}} \frac{\omega_f'(s)}{s^\frac{1}{2 - \frac{1}{n}}} ds.
$$

(39)

Recall $\omega_f(t) = \int_0^t ((f \chi_{B_1})^+(s))^2 ds^\frac{1}{2}$ and thus $\omega_f'(t) = \frac{1}{2} \frac{1}{\omega_f(t)} (f \chi_{B_1})^+(t)^2$. Since $(f \chi_{B_1})^+$ is non-increasing, we have $\omega_f(t) \geq t^\frac{1}{2} (f \chi_{B_1})^+(t)$ and

$$
\int_0^\infty \omega_f^{-1}(\frac{\omega_f'(s)}{s^\frac{1}{2 - \frac{1}{n}}}) ds \leq \frac{1}{2} \int_0^\infty \frac{\omega_f'(s)}{s^\frac{1}{2 - \frac{1}{n}}} (f \chi_{B_1})^+(s) ds = \frac{1}{2} \|f\|_{L^{n,1}(B_1)}.
$$

(40)

Notice that (35) implies

$$
\|v - k_0 - \sum_{\ell \in \mathbb{N}} \Delta t_\ell\|_{L^2(B_{\frac{1}{2}})} = 0 \quad \Rightarrow \quad \sup_{B_{\frac{1}{2}}} v \leq k_0 + \sum_{\ell \in \mathbb{N}} \Delta t_\ell.
$$

Hence, appealing to (38)-(40) and $M_1 \geq 1$, we find $c = c(c_m, n, \kappa) \in [1, \infty)$ such that

$$
\sup_{B_{\frac{1}{2}}} v \leq k_0 + cM_1^\frac{1}{n - 1} \|u - k_0\|_{W^{1,2}(B_1)} + cc_m M_2 \|f\|_{L^{n,1}(B_1)}.
$$

The claimed estimate (16) with $B = B_2$, follows by the definition of $2^*$, see (19) and a further application of (15) with $\eta \in C^1_0(B_2)$ with $\eta = 1$ in $B_1$ with $|\nabla \eta| \lesssim 1$.

\[ \square \]

4. A PRIORI ESTIMATE FOR REGULARIZED INTEGRAND

In this section, we derive a priori estimates for regular weak solutions $u \in W^{1,\infty}_{loc}(B)$ of the equation

$$
- \nabla \cdot a_\varepsilon(\nabla u) = f \quad \text{in } B \subset \mathbb{R}^n, \quad f \in L^\infty(\mathbb{R}^n),
$$

where $a_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ is (a possibly regularized version of) $\partial F$ and satisfies

**Assumption 2.** Let $0 < \nu \leq \Lambda < \infty$, $1 < p \leq q < \infty$, $\mu \in [0,1]$ and $\varepsilon, T > 0$ be given. Suppose that $a_\varepsilon \in C^1_{loc}(\mathbb{R}^n, \mathbb{R}^n)$ satisfies

$$
\begin{cases}
\partial_i (a_\varepsilon)_j(z) = \partial_j (a_\varepsilon)_i(z) & \text{for all } z \in \mathbb{R}^n, \text{ and } i, j \in \{1, \ldots, n\}
\end{cases}
$$

(42)

$$
\begin{cases}
g_1(|z|)|\xi|^2 \leq (\partial \alpha_\varepsilon)(z) \xi, \xi \leq 2g_2(\varepsilon)|\xi|^2 & \text{for all } z, \xi \in \mathbb{R}^n \text{ with } |z| \geq T
\end{cases}
$$

where for all $s \geq 0$

$$
g_1(s) := \nu(\mu_2 + s^2)^{\frac{n-2}{2}}, \quad g_2(\varepsilon)(s) := \Lambda(\mu_2 + s^2)\frac{n-2}{2} + \Lambda(\mu_2 + s^2)\frac{n-2}{2} + \varepsilon \Lambda(1 + s^2)^{\frac{n-2}{2}}.
$$

Moreover, we introduce (following [6]) the quantity

$$
G_T(t) := \int_T^{\max(t,T)} g_1(s)ds \quad (T > 0).
$$

(43)
Lemma 4 (Caccioppoli inequality). Suppose Assumption 2 is satisfied for some $\varepsilon, T > 0$. There exists $c = c(n) \in [1, \infty)$ such that the following is true: Let $u \in W^{1,\infty}_\text{loc}(B)$ be a weak solution to (41). Then it holds for all $\eta \in C^1_c(B)$

$$
\int_B |\nabla((G_T(|\nabla u|) - k)_+)|^2 \eta^2 \, dx \leq c \int_B \frac{g_2 \varepsilon(|\nabla u|)}{g_1(|\nabla u|)} (G_T(|\nabla u|) - k)_+^2 |\nabla \eta|^2 \, dx 
+ c \int_{B \cap (G_T(|\nabla u|) > k)} \eta^2 |\nabla u|^2 |f|^2 
$$

(44)

The proof of Lemma 4 follows almost verbatim the proof of [6, Lemma 4.5] where (44) is proven for a specific choice of $\eta$.

Proof. The following computations are essentially a translation of the proof of [6, Lemma 4.5] to the present situation.

Step 0. Preliminaries.

Since we suppose that $u$ is Lipschitz continuous and $a_x \in C^1_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ satisfies the uniform estimate $\langle \partial a_x(z), \xi, \xi \rangle \geq \nu |\xi|^2$ for some $\nu > 0$ for all $z, \xi \in \mathbb{R}^n$ (which follows from (42)), we obtain from standard regularity theory that

$$
u u \in W^{2,2}_\text{loc}(B), \quad u \in C^{1,\alpha}_\text{loc}(B) \text{ for some } \alpha \in (0, 1), \quad a_x(\nabla u) \in W^{1,2}_\text{loc}(B, \mathbb{R}^n).$$

Hence, we can differentiate equation (41) and obtain for all $s \in \{1, \ldots, n\}$ and every $\phi \in W^{1,2}_0(B)$ with compact support in $B$ that

$$
\int_B \langle \partial a_x(\nabla u) \nabla \partial_s u, \nabla \phi \rangle \, dx = - \int_B f \partial_s \phi \, dx.
$$

(45)

We test (45) with $\phi := \varphi_s := \eta^2 (G_T(|\nabla u|) - k)_+ \partial_s u$, $k \geq 0$, $\eta \in C^1_c(B)$.

On the set $\{G_T(|\nabla u|) > k\}$ holds

$$
\nabla \varphi_s = \eta^2 (G_T(|\nabla u|) - k)_+ \nabla \partial_s u + \eta^2 \partial_s u \nabla (G_T(|\nabla u|) - k)_+ + 2 \eta (G_T(|\nabla u|) - k)_+ \partial_s u \nabla \eta
$$

and

$$
\nabla (G_T(|\nabla u|) - k)_+ = \nabla (G_T(|\nabla u|)) = g_1(|\nabla u|) \sum_{s=1}^n \partial_s u \nabla \partial_s u \quad \text{and} \quad g_1(|\nabla u|) > 0.
$$

(47)

We compute,

$$
\sum_{s=1}^n \int_B \langle \partial a_x(\nabla u) \nabla \partial_s u, \nabla \partial_s u \rangle (G_T(|\nabla u|) - k)_+ \eta^2 \, dx
+ \int_B g_1(|\nabla u|)^{-1} \langle \partial a_x(\nabla u) \nabla (G_T(|\nabla u|) - k)_+, \nabla (G_T(|\nabla u|) - k)_+ \rangle \eta^2 \, dx
$$

(47)

$$
= \sum_{s=1}^n \int_B \langle \partial a_x(\nabla u) \nabla \partial_s u, (G_T(|\nabla u|) - k)_+ \nabla \partial_s u \rangle \eta^2 \, dx
+ \sum_{s=1}^n \int_B \langle \partial a_x(\nabla u) \nabla \partial_s u, (\partial_s u) \nabla (G_T(|\nabla u|) - k)_+ \rangle \eta^2 \, dx
$$

(45)

$$
= -2 \sum_{s=1}^n \int_B \langle \partial a_x(\nabla u) \nabla \partial_s u, \nabla \eta \rangle (\partial_s u) (G_T(|\nabla u|) - k)_+ \eta \, dx - \sum_{s=1}^n \int_B f \partial_s \varphi_s \, dx
$$

(47)

$$
= -2 \int_B g_1(|\nabla u|)^{-1} \langle \partial a_x(\nabla u) \nabla ((G_T(|\nabla u|) - k)_+), \nabla \eta \rangle (G_T(|\nabla u|) - k)_+ \eta \, dx - \sum_{s=1}^n \int_B f \partial_s \varphi_s \, dx.
$$
The symmetry of $\partial a_\varepsilon$ and Cauchy-Schwarz inequality yield
\[
-2 \int_B g_1((\nabla u)^{-1}(\partial a_\varepsilon(\nabla u))\nabla((G_T(\nabla u) - k)_+), \nabla \eta)(G_T(\nabla u) - k)_+ \eta \, dx \\
\leq \frac{1}{2} \int_B g_1((\nabla u)^{-1}(\partial a_\varepsilon(\nabla u))\nabla((G_T(\nabla u) - k)_+), \nabla((G_T(\nabla u) - k)_+))\eta^2 \, dx \\
+ 2 \int_B g_1((\nabla u)^{-1}(\partial a_\varepsilon(\nabla u))\nabla \eta, \nabla \eta)((G_T(\nabla u) - k)_+)^2 \, dx.
\]

The previous two (in)equalities combined with (42) imply the existence of $c = c(n) < \infty$ such that
\[
\int_B \left( g_1((\nabla u))(G_T((\nabla u) - k)_+|\nabla^2 u|^2 + |\nabla(G_T(\nabla u) - k)_+|^2 \right)\eta^2 \, dx \\
\leq c \int_B \left( \frac{g_2(\varepsilon(\nabla u))}{g_1(\nabla u)} \right) (G_T((\nabla u) - k)_+^2|\nabla \eta|^2 \, dx + c \sum_{s=1}^n \int_B |f||\nabla \varphi_s| \, dx.
\]

Appealing to Young’s inequality and (46), we estimate the last term on the right-hand side by
\[
\sum_{s=1}^n \int_B |f||\nabla \varphi_s| \, dx \\
\leq \frac{1}{2} \int_B \left( g_1((\nabla u))(G_T((\nabla u) - k)_+|\nabla^2 u|^2 + |\nabla(G_T(\nabla u) - k)_+|^2 \right)\eta^2 \, dx \\
+ c \int_B ((G_T((\nabla u) - k)_+)^2|\nabla \eta|^2 \, dx \\
+ c \int_{B \cap \{G_T((\nabla u)) > k\}} |f|^2 (g_1((\nabla u))^{-1}(G_T((\nabla u) - k)_+ + |\nabla u|^2)\eta^2 \, dx,
\]
where $c = c(n) \in [1, \infty)$. The monotonicity of $s \mapsto g_1(s)s$ yields $G_T(t) \leq g_1(t)(t - T) \leq g_1(t)t^2$ and thus
\[
\int_{B \cap \{G_T((\nabla u)) > k\}} |f|^2 (g_1((\nabla u))^{-1}(G_T((\nabla u) - k)_+ + |\nabla u|^2)\eta^2 \, dx \leq 2 \int_{B \cap \{G_T((\nabla u)) > k\}} |f|^2|\nabla u|^2\eta^2 \, dx.
\]

Combining (48)-(50), we obtain
\[
\int_B \left( g_1((\nabla u))(G_T((\nabla u) - k)_+|\nabla^2 u|^2 + |\nabla(G_T(\nabla u) - k)_+|^2 \right)\eta^2 \, dx \\
\leq c \int_B \left( \frac{g_2(\varepsilon(\nabla u))}{g_1(\nabla u)} + 1 \right) (G_T((\nabla u) - k)_+^2|\nabla \eta|^2 \, dx + c \int_{B \cap \{G_T((\nabla u)) > k\}} |f|^2|\nabla u|^2\eta^2 \, dx
\]
and the claim follows from $1 \leq \frac{g_2(\varepsilon(t))}{g_1(t)}$ for all $t > 0$. \hfill \Box

Combining the Caccioppoli inequality of Lemma 4 with Lemma 3, we obtain the following local $L^\infty$-bound on $G_T(\nabla u)$

**Lemma 5.** Suppose Assumption 2 is satisfied for some $\varepsilon, T \in (0, 1]$ and let $u \in W^{1, \infty}_{\text{loc}}(B)$ be a weak solution to (41). Set
\[
\gamma := \frac{1}{2} \frac{q - p}{p} \max \left\{ \kappa, \frac{n - 3}{2} \right\} + \frac{q}{2p}, \quad \gamma \gamma := \frac{1}{2} \frac{q - p}{p} \max \left\{ \kappa, \frac{n - 3}{2} \right\} + \frac{1}{p}.
\]
and suppose that \( \gamma, \gamma' \in (0, 1) \). Then there exists \( c = c(\gamma, \kappa, \Lambda, n, p, q) \in [1, \infty) \) such that

\[
\|G_T(\nabla u)\|_{L^\infty(\frac{1}{2}B)} \leq c(1 + \frac{\epsilon}{\mu^2 + \mu T})^{\frac{1}{2} + \frac{1}{2}} \cdot \int_B G_T(\nabla u) + c(1 + \frac{\epsilon}{\mu^2 + \mu T})^{\frac{1}{2}} \cdot \int_B (T^2 + \mu^2)^{\frac{1}{2}} + (T^2 + \mu^2)^\gamma \cdot \|G_T(\nabla u)\|_{L^\infty(\frac{1}{2}B)} + c(1 + \frac{\epsilon}{\mu^2 + \mu T})^{\frac{1}{2}} \cdot \int_B \left( (T^2 + \mu^2)^{\frac{1}{2}} + (T^2 + \mu^2)^\gamma \right)^{\gamma - 1} \cdot \|G_T(\nabla u)\|_{L^\infty(\frac{1}{2}B)} + c\|G_T(\nabla u)\|_{L^\infty(\frac{1}{2}B)}.
\]

(52)

**Remark 4.** Note that \( 1 < p \leq q \) and \( \kappa \in (0, \frac{1}{2}) \) imply that \( \gamma \) and \( \tilde{\gamma} \) defined in (51) are positive. Moreover, for \( n \geq 4 \) relation (4) imply \( \gamma, \tilde{\gamma} < 1 \). Indeed, we have

\[
\frac{q}{p} < 1 + \frac{2}{n-1} \quad \Rightarrow \quad \gamma = \frac{1}{2} \left( \frac{q - p n - 3}{2} \right)^\frac{1}{2} + \frac{q}{2p} < \frac{1}{2} \left( \frac{n - 3}{2} \right)^\frac{1}{2} + \frac{1}{2} + \frac{1}{n - 1} = 1
\]

\[
\frac{q}{p} < 1 + \frac{2}{n-1} \quad \Rightarrow \quad \tilde{\gamma} = \frac{1}{2} \left( \frac{q - p n - 3}{2} \right)^\frac{1}{2} + \frac{q}{2p} < \frac{1}{2} \left( \frac{n - 3}{2} \right)^\frac{1}{2} + \frac{1}{2} + \frac{1}{n - 1} = 1.
\]

In dimension \( n = 3 \) and \( \kappa \in (0, \frac{1}{2}) \), a straightforward computation yields that \( \kappa < \frac{2p - q}{q - p} \) implies \( \gamma < 1 \) and \( \kappa < \frac{2p - q}{q - p} \) implies \( \tilde{\gamma} < 1 \).

**Proof of Lemma 5.** Throughout the proof we write \( \lesssim \) if \( \leq \) holds up to a multiplicative constant depending only on \( \kappa, \Lambda, \nu, n, p \) and \( q \).

**Step 0.** Technical estimates.

There exists \( c_1 = c_1(\nu, \Lambda, p, q) \in [1, \infty) \) such that for all \( t \geq T > 0 \) holds

\[
\frac{g_{2c}(t)}{g_1(t)} \leq c_1 \left( G_T(t)^{\frac{q - p}{p - n}} + 1 + \frac{\epsilon}{\mu^2 + \mu T} \right)^{\frac{1}{2}} \quad \text{and} \quad t \leq c_1 G_T(t)^{\frac{1}{p}} + (\mu^2 + T^2)^{\frac{1}{p}}.
\]

(53)

To establish (53), we first compute for all \( t \geq T \)

\[
G_T(t) = \nu \int_T^t (\mu^2 + s^2)^{\frac{p - n}{2}} s \, ds = \nu \left( \frac{1}{p} \left( \mu^2 + t^2 \right)^{\frac{p - 2}{2}} - \nu \left( \mu^2 + T^2 \right)^{\frac{p - 2}{2}} \right)
\]

and thus

\[
\frac{g_{2c}(t)}{g_1(t)} = \frac{\Lambda}{\nu} \left( \mu^2 + t^2 \right)^{\frac{p - n}{2}} + \frac{\Lambda}{\nu} + \epsilon \left( \frac{1}{\mu^2 + t^2} \right)^{\frac{1}{2}} \left( \frac{1}{\mu^2 + t^2} \right)^{\frac{p - n}{2}} \leq \frac{\Lambda}{\nu} \left( \nu G_T(t) + (\mu^2 + T^2)^{\frac{p - n}{2}} \right)^{\frac{1}{p}} + \frac{\Lambda}{\nu} + \epsilon \left( \frac{1}{\mu^2 + t^2} \right)^{\frac{1}{2}} \left( \frac{1}{\mu^2 + t^2} \right)^{\frac{p - n}{2}},
\]

which implies the first estimate of (53) (recall \( \mu \in [0, 1] \) and \( \epsilon, T \in (0, 1) \)). The second estimate of (53) follows from (54) in the form: For \( t \geq T > 0 \) holds

\[
t^p \leq \frac{1}{\nu} G_T(t) + (\mu^2 + T^2)^{\frac{p - n}{2}}
\]

and the second estimate of (53) follows by taking the \( p \)-th root.
Step 1. In this step, we suppose \( B_1 \subseteq B \) and prove  

\[
\|G_T(\nabla u)\|_{L^\infty(B_{\frac{x}{4}})} \leq \|G_T(\nabla u)\|_{L^\infty(B_{\frac{x}{4}})}^\gamma \|G_T(\nabla u)\|_{L^1(B_1)}^{\frac{1}{\gamma}} + \left( 1 + \left( \frac{\varepsilon}{(\mu^2 + T^2)^{-\frac{1}{2}}} \right)^{\frac{\gamma}{\gamma + 1}} \right) \max\{\kappa, \frac{n-\gamma}{\gamma + 1}\} \|G_T(\nabla u)\|_{L^\infty(B_1)}^{\frac{\gamma}{\gamma + 1}} \|G_T(\nabla u)\|_{L^1(B_1)}^{\frac{1}{\gamma + 1}} 
\]

where \( \gamma \) and \( \tilde{\gamma} \) are defined in (51).

A direct consequence of the Caccioppoli inequality of Lemma 4 and the iteration Lemma 3 with the choice  

\[
M_1^2 = \left\| \frac{g_2.(\nabla u)}{g_1(\nabla u)} \right\|_{L^\infty(B_1 \cap \{|\nabla u| \geq T\})} \quad \text{and} \quad M_2^2 = \|\nabla u\|_{L^\infty(B_1)}^2
\]

is the following Lipschitz estimate  

\[
\|G_T(\nabla u)\|_{L^\infty(B_{\frac{x}{4}})} \leq \left( \left\| \frac{g_2.(\nabla u)}{g_1(\nabla u)} \right\|_{L^\infty(B_1 \cap \{|\nabla u| \geq T\})} \right)^{\frac{\gamma}{\gamma + 1}} + \left( 1 + \left( \frac{\varepsilon}{(\mu^2 + T^2)^{-\frac{1}{2}}} \right)^{\frac{\gamma}{\gamma + 1}} \right) \max\{\kappa, \frac{n-\gamma}{\gamma + 1}\} \|G_T(\nabla u)\|_{L^\infty(B_1)}^{\frac{\gamma}{\gamma + 1}} \|G_T(\nabla u)\|_{L^1(B_1)}^{\frac{1}{\gamma + 1}} 
\]

(56)

Estimate (55) follows from (56) in combination with (53) in the form  

\[
\left\| \frac{g_2.(\nabla u)}{g_1(\nabla u)} \right\|_{L^\infty(B_1 \cap \{|\nabla u| \geq T\})} \leq \|G_T(\nabla u)\|_{L^\infty(B_1)}^{\frac{\gamma}{\gamma + 1}} + 1 + \left( \frac{\varepsilon}{(\mu^2 + T^2)^{-\frac{1}{2}}} \right)^{\frac{\gamma}{\gamma + 1}} 
\]

(57)

and  

\[
\|\nabla u\|_{L^\infty(B_1)} \leq c_1 \|G_T(\nabla u)\|_{L^\infty(B_1)}^{\frac{1}{\gamma}} + (\mu^2 + T^2)^{\frac{1}{2}}, 
\]

and the elementary interpolation inequality \( \| \cdot \|_{L^2} \leq (\| \cdot \|_{L^\infty} \cdot \| \cdot \|_{L^1})^{\frac{1}{2}} \).

Step 2. Conclusion

Appealing to standard scaling and covering arguments, we deduce from Step 1 the following: For every \( x_0 \in B \) and \( 0 < \rho < \sigma \) satisfying \( B_\sigma(x_0) \subseteq B \) it holds  

\[
\|G_T(\nabla u)\|_{L^\infty(B_\rho(x_0))} \leq (\sigma - \rho)^{-\frac{n}{2}} \|G_T(\nabla u)\|_{L^\infty(B_\sigma(x_0))} \|G_T(\nabla u)\|_{L^1(B_\sigma(x_0))}^{\frac{1}{2}} 
\]

\[
+ (\sigma - \rho)^{-\frac{n}{2}} \left( 1 + \left( \frac{\varepsilon}{(\mu^2 + T^2)^{-\frac{1}{2}}} \right)^{\frac{\gamma}{\gamma + 1}} \right) \max\{\kappa, \frac{n-\gamma}{\gamma + 1}\} \|G_T(\nabla u)\|_{L^\infty(B_\sigma(x_0))}^{\frac{\gamma}{\gamma + 1}} \|G_T(\nabla u)\|_{L^1(B_\sigma(x_0))}^{\frac{1}{\gamma + 1}} 
\]

\[
+ (1 + \left( \frac{\varepsilon}{(\mu^2 + T^2)^{-\frac{1}{2}}} \right)^{\frac{\gamma}{\gamma + 1}} \max\{\kappa, \frac{n-\gamma}{\gamma + 1}\} \|G_T(\nabla u)\|_{L^\infty(B_\sigma(x_0))}^{\frac{\gamma}{\gamma + 1}} + (\mu^2 + T^2)^{\frac{1}{2}} \|f\|_{L^1(B_\sigma(x_0))} 
\]

(58)
From estimate (58) in combination with Young inequality and assumption $\gamma, \tilde{\gamma} \in (0, 1)$, we obtain the existence of $c = c(\kappa, \nu, \Lambda, n, p, q) \in [1, \infty)$ such that

$$
\|G_T(|\nabla u|)\|_{L^\infty(B_{r}(x_0))} \leq \frac{1}{2}\|G_T(|\nabla u|)\|_{L^\infty(B_{r}(x_0))} + c \frac{\|G_T(|\nabla u|)\|_{L^1(B_{r}(x_0))}}{(\sigma - \rho)^{\frac{1}{\gamma}}}
$$

$$
+ c \left(1 + \frac{\varepsilon}{(\mu^2 + T^2)^{\frac{1}{2}}}ight)^{\max\left\{\frac{\kappa}{\gamma}, \frac{\kappa}{2}\right\}} \|G_T(|\nabla u|)\|_{L^1(B_{r}(x_0))} c \
+ \left(1 + \frac{\varepsilon}{(\mu^2 + T^2)^{\frac{1}{2}}}ight)^{\max\left\{\frac{\kappa}{\gamma}, \frac{\kappa}{2}\right\}} \|G_T(|\nabla u|)\|_{L^1(B_{r}(x_0))} c \
+ c\|f\|_{L^1(B_{r}(x_0))} c.
$$

The claimed inequality (52) (for $B = B_1$) now follows from Lemma 2.

\[ \square \]

5. PROOF OF THEOREM 1

In this section, we prove Theorem 1 together with a suitable gradient estimate. More precisely, we show the following result which obviously contains the statement of Theorem 1.

**Theorem 2.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be an open bounded domain and suppose Assumption 1 is satisfied with $1 < p < q < \infty$ such that (4). Let $u \in W^{1,1}_{loc}(\Omega)$ be a local minimizer of the functional $\mathcal{F}$ given in (1) with $f \in L^{n,1}(\Omega)$. Then $\nabla u$ is locally bounded in $\Omega$. Moreover, for every $\kappa \in (0, \min\left\{\frac{1}{2}, \frac{2p-2q}{q-p}, \frac{2q}{q-p}\right\})$ there exists $c = c(\kappa, \Lambda, \nu, n, p, q) \in [1, \infty)$ such that for all $B \Subset \Omega$ it holds

$$
\|\nabla u\|_{L^\infty(B_{\frac{1}{2}r})} \leq \left(\frac{1}{2}\frac{\kappa}{\gamma} + \frac{\kappa}{\gamma} c\right)\left(1 + \frac{\varepsilon}{(\mu^2 + T^2)^{\frac{1}{2}}}ight)^{\max\left\{\frac{\kappa}{\gamma}, \frac{\kappa}{2}\right\}} \|G_T(|\nabla u|)\|_{L^1(B_{r}(x_0))} c
$$

$$
+ c\|f\|_{L^{n,1}(B_{r}(x_0))} c.
$$

(59)

where

$$
\alpha_n := \begin{cases} 
\frac{2}{0(n+1)(n-1)q} & \text{if } n \geq 4 \quad 2p-q-(q-p)\kappa \\
\frac{4}{2p-q-(q-p)\kappa} & \text{if } n = 3 \quad \beta_n := \begin{cases} 
\frac{4}{2p-q-(q-p)\kappa} & \text{if } n \geq 4 \quad 2p-q-(q-p)\kappa \\
\frac{4}{2p-q-(q-p)\kappa} & \text{if } n = 3 
\end{cases}
\end{cases}
$$

In the case $n \geq 4$ the constant $c$ in (59) is independent of $\kappa$. When $p \geq 2 - \frac{4}{n+1}$ or when $f \equiv 0$ condition (4) can be replaced by (5).

**Proof of Theorem 2.** Throughout the proof we write $\lesssim$ if $\leq$ holds up to a multiplicative constant that depends only on $\kappa, \Lambda, \nu, n, p$ and $q$. We assume that $B_2 \Subset \Omega$ and show

$$
\|\nabla u\|_{L^\infty(B_{\frac{1}{2}r})} \lesssim \left(\frac{1}{2}\frac{\kappa}{\gamma} + \frac{\kappa}{\gamma} c\right)\left(1 + \frac{\varepsilon}{(\mu^2 + T^2)^{\frac{1}{2}}}ight)^{\max\left\{\frac{\kappa}{\gamma}, \frac{\kappa}{2}\right\}} \|G_T(|\nabla u|)\|_{L^1(B_{r}(x_0))} c
$$

$$
+ \left(\frac{1}{2}\frac{\kappa}{\gamma} + \frac{\kappa}{\gamma} c\right)\left(1 + \frac{\varepsilon}{(\mu^2 + T^2)^{\frac{1}{2}}}ight)^{\max\left\{\frac{\kappa}{\gamma}, \frac{\kappa}{2}\right\}} \|G_T(|\nabla u|)\|_{L^1(B_{r}(x_0))} c
$$

$$
+ \|f\|_{L^{n,1}(B_{r}(x_0))} c.
$$

(61)

where $\gamma$ and $\tilde{\gamma}$ are given in (51). Clearly, the conclusion follows from a standard scaling, translation and covering arguments using $\alpha_n = \frac{1}{2p-1-\gamma}$ and $\beta_n = \frac{1}{2p-1-\gamma}$.
Step 0. Preliminaries.
Following [6], we introduce various regularizations on the minimizer \( u \), the integrand \( F \) and the forcing term \( f \): For this we choose a decreasing sequence \( (\varepsilon_m)_{m\in\mathbb{N}} \subset (0,1) \) satisfying \( \varepsilon_m \to 0 \) as \( m \to \infty \). We set \( \overline{u}_m := u \ast \varphi_{\varepsilon_m} \) with \( \varphi := \varepsilon^{-n} \varphi(\frac{x}{\varepsilon}) \) and \( \varphi \) being a non-negative, radially symmetric mollifier, i.e. it satisfies
\[
\varphi \geq 0, \quad \text{supp} \varphi \subset B_1, \quad \int_{\mathbb{R}^n} \varphi(x) \, dx = 1, \quad \varphi(\cdot) = \varphi(|\cdot|) \quad \text{for some} \ \varphi \in C^\infty(\mathbb{R}).
\]
Moreover, we denote by \( f_m \) the truncated forcing \( f_m(x) = \min \{ \max \{ f(x), -m \}, m \} \) and consider the functional
\[
F_m (w, B) := \int_B [F_{\varepsilon_m}(\nabla w) - f_m w] \, dx,
\]
where for all \( z \in \mathbb{R}^n \)
\[
F_{\varepsilon}(z) := \tilde{F}_{\varepsilon}(z) + \varepsilon L_p(z) \quad \text{with} \quad L_p(z) := \frac{1}{2} |z|^2 \quad \text{for} \quad p \geq 2 \quad \text{and} \quad L_p(z) := (1 + |z|^2)^{\frac{p}{2}} - 1 \quad \text{for} \quad p \in (1, 2)
\]
and \( \tilde{F}_{\varepsilon} \) satisfies for all \( \varepsilon \in (0, \varepsilon_0) \) with \( \varepsilon_0 = \varepsilon_0(F, T) \in (0, 1) \)
\[
\tilde{F}_{\varepsilon} \geq 0, \quad \tilde{F}_{\varepsilon} \in C^2_{\text{loc}}(\mathbb{R}^n) \text{ is convex and } \tilde{F}_{\varepsilon} = F \text{ on } \mathbb{R}^n \setminus B_{\varepsilon}^T.
\]
(Obviously \( \tilde{F}_{\varepsilon} \) and thus \( F_{\varepsilon} \) depends also on \( T > 0 \) which is suppressed in the notation) and it holds
\[
\lim_{\varepsilon \to 0} \sup_{z \in T} |\tilde{F}_{\varepsilon}(z) - F(z)| = 0.
\]
In the case \( F \in C^2_{\text{loc}}(\mathbb{R}^n) \), we simply set \( \tilde{F}_{\varepsilon} \equiv F \) and in the case that \( F \) is singular at zero we give (a standard) smoothing and gluing construction in Step 3 below.

Clearly, the functionals \( F_m \) are strictly convex and we denote by \( u_m \in W^{1,1}(B) \) the unique function satisfying
\[
F_m (u_m, B) \leq F_m (v, B) \quad \text{for all} \ v \in \overline{u}_m + W^{1,1}_0(B)
\]
Appealing to [6, Theorem 4.10] (based on [5]), we have \( u_m \in W^{1,\infty}_\text{loc}(B) \). In particular it follows that \( u_m \) satisfies the Euler-Lagrange equation
\[
-\text{div} (\partial F_{\varepsilon_m}(\nabla u_m)) = f_m
\]
and since \( a_{\varepsilon} := \partial F_{\varepsilon_m} \) satisfies Assumption 2 (with \( \varepsilon = \varepsilon_m \)) we can apply Lemma 5. Note that in view of Remark 4, the assumptions on \( p, q \) and \( \kappa \) ensure \( \gamma, \hat{\gamma} \in (0, 1) \).

Step 1. We claim that
\[
\|\nabla u_m\|_{L^\infty(B_\frac{1}{2})}^p \lesssim (1 + \frac{\varepsilon_m}{(\mu^2 + T^2)^{\frac{1}{2}}})^{\max\{\kappa, \frac{n-2}{2}\}} + \left( \int_{B_{1+\varepsilon_m}} \tilde{F}_{\varepsilon_m}(\nabla u) \, dx + \varepsilon_m \int_{B_1} L_p(\nabla \overline{u}_m) \, dx + \|f\|_{L^p(B_1)} + T^p + \mu^p \right)^{\frac{1}{1-n}}
\]
\[
+ \left( \int_{B_{1+\varepsilon_m}} \tilde{F}_{\varepsilon_m}(\nabla u) \, dx + \varepsilon_m \int_{B_1} L_p(\nabla \overline{u}_m) \, dx + \|f\|_{L^p(B_1)} + T^p + \mu^p \right)^{\frac{1}{1-n}}
\]
\[
+ \left( (1 + \frac{\varepsilon_m}{(\mu^2 + T^2)^{\frac{1}{2}}})^{\frac{1}{2}} \max\{\kappa, \frac{n-2}{2}\} \|f\|_{L^p(B_1)} \right)^{\frac{1}{1-n}}
\]
\[
+ \|f\|_{L^p(B_1)}
\]
and
\[
\int_{B_1} F_{\varepsilon_m}(\nabla u_m) \, dx \lesssim \int_{B_{1+\varepsilon_m}} \tilde{F}_{\varepsilon_m}(\nabla u) \, dx + \varepsilon_m \int_{B_1} L_p(\nabla \overline{u}_m) \, dx + \|f\|_{L^p(B_1)} + (\mu^2 + T^2)^{\frac{1}{2}}.
\]
A combination of Hölder and Sobolev inequality with the elementary inequality
\[ \nu|z|^p \leq \nu(\mu^2 + T^2)^{\frac{p}{2}} + F_{\varepsilon_m}(z) \]
(which follows from the definition of \( F_{\varepsilon_m}, (63) \) and (3)) yields
\[ \|f_m(u_m - \overline{u}_m)\|_{L^1(B_1)} \leq \|f_m\|_{L^p(B_1)}\|u_m - \overline{u}_m\|_{L^{\frac{p}{p-1}}(B_1)} \]
\[ \leq c(n, p)\|f_m\|_{L^p(B_1)}\|\nabla(u_m - \overline{u}_m)\|_{L^p(B_1)} \]
\[ \leq c\|f_m\|_{L^p(B_1)} \left( \int_{B_1} F_{\varepsilon_m}(\nabla u_m) + F_{\varepsilon_m}(\nabla \overline{u}_m) \, dx + (\mu^2 + T^2)^{\frac{p}{2}} \right)^{\frac{1}{p}} \]
\[ \leq \frac{1}{p} \left( \int_{B_1} F_{\varepsilon_m}(\nabla u_m) + F_{\varepsilon_m}(\nabla \overline{u}_m) \, dx + (\mu^2 + T^2)^{\frac{p}{2}} \right) + (1 - \frac{1}{p})(c\|f_m\|_{L^p(B_1)})^{\frac{2}{p}} \]
where \( c = c(n, \nu, \Lambda, p, q) \in [1, \infty) \). Combining the above estimate with the minimality of \( u_m \) and the convexity of \( \overline{F}_\varepsilon \) in the form
\[ \int_{B_1} F_{\varepsilon_m}(\nabla u_m) \, dx \leq \int_{B_1} F_{\varepsilon_m}(\nabla \overline{u}_m) - f_m(\overline{u}_m - u_m) \, dx \]
(68) we obtain (67). The claimed Lipschitz-estimate (66) follows from Lemma 5, estimates (53), (67) and
\[ 0 \leq G_T((\nabla u_m)) \leq (\mu^2 + |\nabla u_m|^2)^\frac{p}{2} \lesssim F_{\varepsilon_m}(\nabla u_m) + (\mu^2 + T^2)^{\frac{p}{2}}. \]

**Step 2.** Passing to the limit.

**Substep 2.1.** We claim
\[ \lim_{m \to \infty} \varepsilon_m \int_{B_1} L_p(\nabla \overline{u}_m) \, dx = 0 \]
(69) and
\[ \lim_{m \to \infty} \int_{B_{1+r_m}} \overline{F}_{\varepsilon_m}(\nabla u) \, dx = \int_{B_1} F(\nabla u) \, dx. \]
(70)

We first note that \( F(\nabla u) \in L^1_{\text{loc}}(B_2) \). Indeed, by Definition 1 combined with Hölder and Sobolev inequality, we have for every \( B \subset B_2 \)
\[ \int_{B} F(\nabla u) \, dx = |F(u, \overline{B})| + \int_{\overline{B}} f u \leq |F(u, \overline{B})| + \|f\|_{L^p(\overline{B})}\|u\|_{L^{\frac{p}{p-1}}(\overline{B})} \]
\[ \lesssim |F(u, \overline{B})| + \|f\|_{L^p(\overline{B})}\|u\|_{W^{1,p}(\overline{B})} < \infty \]

For \( p \geq 2 \), equation (69) follows from
\[ \int_{B_1} L_p(\nabla \overline{u}_m) \, dx = \frac{1}{2}\|\nabla \overline{u}_m\|_{L^2(B_1)}^2 \lesssim \|\nabla u\|_{L^2(B_{\frac{1}{2}})}^2 \lesssim \left( \int_{B_{\frac{1}{2}}} F(\nabla u) \, dx \right)^{\frac{2}{p}} < \infty. \]

In the case \( p \in (1, 2) \), equation (69) is a consequence of
\[ \|L_p(\nabla \overline{u}_m)\|_{L^1(B_1)} \lesssim 1 + \|\nabla u\|_{L^p(B_{\frac{1}{2}})}^p \lesssim 1 + \int_{B_{\frac{1}{2}}} F(\nabla u) \, dx < \infty. \]

The argument for (70) follows from the identity
\[ \int_{B_{1+r_m}} \overline{F}_{\varepsilon_m}(\nabla u) \, dx = \int_{B_{1+r_m}} F(\nabla u) \, dx + \int_{B_{1+r_m} \setminus \{|\nabla u| \leq T\}} \overline{F}_{\varepsilon_m}(\nabla u) - F(\nabla u) \, dx. \]

For the uniform convergence (64) and \( F(\nabla u) \in L^1_{\text{loc}}(B_2) \).

**Substep 2.2.** Proof of (61).
From (66), (67) and (69), we deduce the existence of a subsequence and \( \overline{u} \in u + W^{1,1}_0(B_0) \) such that
\[
\begin{align*}
  u_m &\rightharpoonup \overline{u} \quad \text{weakly in } W^{1,p}(B_1) \\
  u_m &\rightharpoonup^* \overline{u} \quad \text{weakly* in } W^{1,\infty}(B_\frac{1}{2})
\end{align*}
\]
In view of (66), (69), (70) and the weak* lower semicontinuity of norms \( \overline{u} \) satisfies
\[
\begin{align*}
  \|\nabla \overline{u}\|_{L^\infty(B_\frac{1}{2})} &\lesssim \left( \int_{B_1} F(\nabla u) \, dx + \|f\|_{L^n(B_1)} + T^p + \mu^p \right)^\frac{1}{p} \\
  &\quad + \left( \int_{B_1} F(\nabla u) \, dx + \|f\|_{L^n(B_1)} + T^p + \mu^p \right)^\frac{1}{p} + \frac{1}{1 - \frac{1}{p}} \\
  &\quad + \left( (T^2 + \mu^2)^\frac{1}{p} + (T^2 + \mu^2)^\frac{1}{p^*} \right) \|f\|_{L^n(B_1)} + \|f\|_{L^n(B_1)}
\end{align*}
\]
(71)
where we use in the last estimate \( \|f\|_{L^n(B_1)} \lesssim \|f\|_{L^n(B_1)} \) and Youngs inequality in the form
\[
((T^2 + \mu^2)^\frac{1}{p} + (T^2 + \mu^2)^\frac{1}{p^*}) \|f\|_{L^n(B_1)} \lesssim ((T^2 + \mu^2)^\frac{1}{p} + \|f\|_{L^n(B_1)} + \|f\|_{L^n(B_1)}).
\]
By the definition of \( F_\varepsilon \) we have
\[
\begin{align*}
  \int_{B_1} F_\varepsilon(m(\nabla u_m)) \, dx &\geq \int_{B_1} F_\varepsilon(m(\nabla u_m)) \, dx \\
  &= \int_{B_1} F(\nabla u_m) \, dx + \int_{B_1 \cap \{|\nabla u| \leq T\}} F_\varepsilon(m(\nabla u_m)) - F(\nabla u_m) \, dx \\
  &\geq \int_{B_1} F(\nabla u_m) \, dx - |B_1| \sup_{z \leq T} |F_\varepsilon(z) - F(z)|.
\end{align*}
\]
Hence, using convexity of \( F \) and the uniform convergence (64), we can pass to the limit (along the above chosen subsequence) in (68) and obtain with help of (69) and (70)
\[
\int_{B_1} F(\nabla u) \, dx \leq \int_{B_1} F(\nabla u) - f(u - \overline{u}) \, dx
\]
and thus
\[
F(\overline{u}, B_1) \leq F(u, B_1).
\]
The above inequality combined with \( \overline{u} \in u + W^{1,p}_0(B_1) \) and the strict convexity of \( F(\cdot, B_1) \) implies \( \overline{u} = u \). The claimed estimate (61) follows by sending \( T \) to 0 in (71) combined with (3) in the form \( \nu \mu^p \leq \int_B F(\nabla u) \, dx \).

**Step 3.** Construction of \( \tilde{F}_\varepsilon \).
Let \( \rho \in C^\infty([0,1]) \) be such that \( \rho \equiv 1 \) on \( (-\infty, \frac{T}{2}) \) and \( \rho \equiv 0 \) on \( (\frac{T}{2}, \infty) \). We set \( \tilde{F}_\varepsilon := F \ast \varphi_\varepsilon \) where \( \varphi_\varepsilon \) is as in Step 1 and \( \tilde{F}_\varepsilon = \rho \tilde{F}_\varepsilon + (1 - \rho)F \). By general properties of the mollification, we have that \( \tilde{F}_\varepsilon \) is smooth, non-negative and convex, and thus \( \tilde{F}_\varepsilon \) is non-negative, locally \( C^2 \) and it holds \( \tilde{F}_\varepsilon \equiv \tilde{F}_\varepsilon ) \) on \( \mathbb{R}^n \setminus B_{T/2} \). Since \( F \) is convex and locally bounded, we also have (64). It remains to show that \( \tilde{F}_\varepsilon \) is convex for \( \varepsilon > 0 \) sufficiently small. For this we observe that
\[
\nabla^2 \tilde{F} = \rho \nabla^2 \tilde{F}_\varepsilon + (1 - \rho) \nabla^2 F + (\tilde{F}_\varepsilon - F_\varepsilon) \nabla^2 \rho + \nabla(\tilde{F}_\varepsilon - F_\varepsilon) \otimes \nabla \rho + \nabla \rho \otimes \nabla(\tilde{F}_\varepsilon - F_\varepsilon)
\]
is strictly positive definite since \( \rho \nabla^2 \tilde{F}_\varepsilon + (1 - \rho) \nabla^2 F \) is strictly positive definite and the remainder tends to zero as \( \varepsilon \to 0 \).
Step 4. The case $f \equiv 0$. It is straightforward to check that the restriction $\gamma \in (0,1)$ is not needed in Lemma 5 if $f \equiv 0$ and since we checked in Remark 4 that (5) suffices to ensure $\gamma \in (0,1)$ the claim follows. □

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