SO(5) Superconductors in a Zeeman Magnetic Field

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The generic symmetry of a system under a uniform Zeeman magnetic field is $U(1) \times U(1)$. However, we show that SO(5) models in the presence of a finite chemical potential and a finite Zeeman magnetic field can have an exact $SU(2) \times U(1)$ symmetry. This principle can be used to test SO(5) symmetry at any doping level.

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A fundamental question one can ask in connection with high $T_c$ superconductors is whether they are in the same universality class of conventional $d$ wave BCS superconductors. While many aspects of high $T_c$ superconductors are anomalous and quantitatively different from conventional BCS superconductors, no sharp distinction based on symmetry has been made so far. In the absence of a external magnetic field and spin anisotropy, the symmetry of the Hamiltonian is $SU(2) \times U(1)$, where the $U(1)$ charge symmetry is spontaneously broken in the superconducting state.

A notable exception is the idea of SO(5) symmetry between antiferromagnetism (AF) and superconductivity (SC) \cite{1}. This theory predicts a finite temperature bicritical point with an enlarged SO(5) symmetry at the transition point between AF and SC. It also predicts a spin triplet $\pi$ resonance \cite{2} in the SC state which can be interpreted as the pseudo Goldstone mode associated with the spontaneous symmetry breaking. However, in the presence of a finite chemical potential, the explicit symmetry of the Hamiltonian is still a direct product of the spin $SU(2)$ and the charge $U(1)$ symmetry, which is not different from that of a conventional BCS system.

In this paper, we point out a remarkable symmetry property of SO(5) symmetric Hamiltonians. In the presence of a finite chemical potential $\mu$ and a finite Zeeman magnetic field $B$, the original SO(5) symmetry is broken to $U(1) \times U(1)$. Here the first $U(1)$ group describes the spin rotation symmetry in a plane perpendicular to the applied magnetic field and the second $U(1)$ group is the usual charge symmetry. In fact, any generic spin invariant Hamiltonian in the presence of a finite Zeeman field would have the same $U(1) \times U(1)$ symmetry. From that point of view, SO(5) symmetric models do not seem to be different from any generic models once a chemical potential or a magnetic field is applied. However, we will show that for a special combination where $B = \mu$, the SO(5) symmetric models enjoy an enlarged $SU(2) \times U(1)$ symmetry, which is not shared by generic models. Furthermore, this special $SU(2) \times U(1)$ symmetry at $B = \mu$ is equivalent to the original SO(5) symmetry in the absence of these fields. This gives a powerful new tool to test the SO(5) symmetry at any doping level. The original SO(5) symmetry exists only at a particular doping level where the AF to SC transition occurs. This point is very difficult to reach in high $T_c$ superconductors because of complicated doping chemistry, and has not yet been identified experimentally. Under the new proposal, however, the SO(5) symmetry can be revealed at any doping level, provided one applies a Zeeman magnetic field. This new test can give a sharp symmetry distinction between a SO(5) superconductor and a conventional BCS superconductor, and it can also distinguish various explanations of the $\pi$ resonance.

Let us consider the following Hamiltonian

$$H = H_{SO(5)} - \mu Q - BS_z$$

where $H_{SO(5)}$ is a SO(5) symmetric Hamiltonian which commutes with the ten SO(5) symmetry generators $L_{ab}$. (For notations and definitions please see ref. \cite{1}). $Q$ and $S_z$ are members of SO(5) symmetry generators $L_{ab}$, they generate charge rotation and spin rotation in the $xy$ plane perpendicular to the external Zeeman field. Since SO(5) is a rank two algebra, one can choose $Q$ and $S_z$ as the two mutually commuting generators. For $B = 0$, the generic symmetry of $H$ is $SU(2) \times U(1)$, while for non-vanishing values of $B$, the original SO(5) symmetry of $H_{SO(5)}$ is broken explicitly to the $U(1) \times U(1)$, generated by $Q$ and $S_z$.

However, $H$ has a exact enlarged symmetry $SU(2) \times U(1)$ at $B = \mu$. At this point, both the chemical potential and the Zeeman term can be combined as $-\mu Q\uparrow$, where $Q\uparrow$ and $Q\downarrow$ measure the number of up spin and down spin electrons respectively. Furthermore, we can define a $SU(2)$ subalgebra of the original SO(5) algebra generated by

$$\pi\downarrow = \sum_k \text{sgn}(\cos k_x - \cos k_y)c_{Q+k\downarrow}c_{Q-k\downarrow}, \quad \pi\uparrow, \quad Q\downarrow. \quad (2)$$

It is easy to see that they form a closed $SU(2)$ algebra,

$$J_1 = \frac{1}{2}(\pi + \pi^\uparrow), \quad J_2 = \frac{i}{2}(\pi - \pi^\uparrow),$$

$$J_3 = \frac{1}{2}Q\downarrow, \quad [J_\alpha, J_\beta] = i\epsilon_{\alpha\beta\gamma}J_\gamma \quad (3)$$

Since the generators of this subalgebra are formed by linear combinations of the original SO(5) generators $L_{ab}$, they all commute with $H_{SO(5)}$. Furthermore, since they only involve down spin electrons, they commute with...
\(-\mu Q_T\). Therefore, we have proven that at \(B = \mu\), \(H\) has a \(SU(2) \times U(1)\) symmetry, generated by the \(SU(2)\) algebra defined by \(\hat{R}\) and the \(U(1)\) generator \(Q_T\).

Mathematically, the new symmetry \(SU(2)\) at \(B = \mu\) is related to the isomorphism between the \(SO(5)\) and the \(SP(4)\) Lie algebras. The root diagrams of these two algebras can be obtained from each other through a 45 degree rotation. This exactly corresponds to going from the \((Q, S)\) basis for the root diagram to the \((Q_T, T)\) basis. In the original basis, the \(-\mu Q_T\) breaks the \(SO(5)\) symmetry into a \(SU(2) \times U(1)\) symmetry. In the new basis, it is then obvious that the \(-\mu Q_T\) should also break the \(SP(4)\) symmetry into a remaining \(SU(2) \times U(1)\) symmetry.

Now we proceed to analyze the collective modes associated with this new symmetry. For this purpose, it is useful to first see how the new symmetry emerge in the Lagrangian formalism. The effective Lagrangian with exact \(SO(5)\) symmetry can be expressed as:

\[
\mathcal{L} = \chi (\partial_t n_\alpha)^2 - \rho (\partial_k n_\alpha)^2 - V(n), \tag{4}
\]

where \(V(n) = -\frac{1}{4} n_\alpha^2 + \frac{M}{2} |n|^4\). We can introduce a magnetic field and a chemical potential simultaneously in the above Lagrangian by applying the following transformation:

\[
\partial_t n_\alpha \rightarrow \partial_t n_\alpha - i\epsilon_{\alpha\beta\gamma} B_{\beta} n_\gamma, \alpha = 2, 3, 4;
\partial_{\alpha} n_\alpha \rightarrow \partial_{\alpha} n_\alpha + i\epsilon_{\alpha\beta\gamma} M_{\beta} n_\gamma, i, j = 1, 5. \tag{5}
\]

Choosing \(\hat{B} = (0,0,-B)\), the Lagrangian becomes

\[
\mathcal{L} = \chi (\partial_t n_\alpha)^2 - \rho (\partial_k n_\alpha)^2 - 2i\chi(B_{n_2}\partial_t n_2 - B_{n_3}\partial_t n_3 - \mu_1\partial_t n_5 + \mu_2\partial_t n_4)
+ \chi |B|^2 (n_2^2 + n_3^2 + \mu^2 (n_4^2 + n_5^2)) - V(n). \tag{6}
\]

Denoting \(\hat{M} = (n_1, n_2, n_5, n_3)\), and taking \(B = \mu\), we can rewrite the above equation into the following form:

\[
\mathcal{L} = \chi (\partial_t n_\alpha)^2 - \rho (\partial_k n_\alpha)^2 + 2i\chi \mu \hat{M} \partial_t \hat{M}^T
+ \chi \mu^2 \hat{M}^2 - V(n), \tag{7}
\]

where \(\hat{R}\) is a four dimensional matrix,

\[
\hat{R} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.
\]

Now we discuss the symmetry of above Lagrangian. Obviously, except the third term in above equation, all other terms have a exact \(SO(4)\) symmetry in the \(\hat{M}\) space. However not all of rotation will keep the invariance of the third term. If \(\hat{O}\) denotes a rotation matrix in the \(\hat{M}\) space, then it must satisfy

\[
\hat{O}^T \hat{O} = 1; \quad \hat{O}^T \hat{R} \hat{O} = \hat{R}. \tag{8}
\]

in order to keep the Lagrangian \(\bar{\mathcal{L}}\) invariant. Since \(SO(4) \cong SU(2) \times SU(2)\), we immediately find one of the \(SU(2)\) subgroup whose generators are defined by the following matrix:

\[
G_1 = \frac{1}{2} \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}, \quad G_2 = \frac{1}{2} \begin{pmatrix} 0 & -i\sigma_x \\ i\sigma_x & 0 \end{pmatrix},
\]

\[
G_3 = \frac{1}{2} \begin{pmatrix} 0 & i\sigma_z \\ -i\sigma_z & 0 \end{pmatrix}, \quad [G_\alpha, G_\beta] = i\epsilon_{\alpha\beta\gamma} G_\gamma; \tag{9}
\]

These matrices also have the following properties

\[
[G_\alpha, R] = 0.
\]

Therefore, \(G_\alpha\), together with \(R\), generates a symmetry \(SU(2) \times U(1)\). The Lagrangian \(\bar{\mathcal{L}}\) is invariant under above \(SU(2) \times U(1)\) transformations. By Noether’s theorem, each internal symmetry is associated to a conserved charge. From the infinitesimal variations of \(\hat{M}\),

\[
\delta \hat{M}^T = iG_\alpha \hat{M}^T \delta \phi_\alpha + RM \delta \phi_R,
\]

we obtain the following conserved currents

\[
\begin{align*}
\hat{j}_t^R &= 2\chi \partial_t \hat{M} \hat{R}^T + 2\chi \mu \hat{M} \hat{M}^T \\
\hat{j}_k^R &= 2\rho \partial_k \hat{M} \hat{R}^T \\
\hat{j}_t^R &= 2i\chi \partial_t \hat{M} G_\alpha \hat{M}^T - 2i\chi \mu \hat{M} R G_\alpha \hat{M}^T \\
\hat{j}_k^R &= 2i\rho \partial_k \hat{M} G_\alpha \hat{M}^T, \\
0 &= \partial_t \hat{j}_t^R + \partial_k \hat{j}_k^R.
\end{align*} \tag{10}
\]

The associated conserved charges can be directly related to the symmetry generators \(\bar{\mathcal{L}}\) in the Hamiltonian formalism:

\[
J_\alpha = \int dx \hat{j}_t^\alpha; \quad Q_\uparrow = \int dx \hat{j}_t^R. \tag{11}
\]

Since the static potential is explicitly broken from \(SO(5)\) to \(SO(4)\), one might expect three massless Goldstone modes and one massive mode for this kind of symmetry breaking. However, there are two massless modes and two massive modes in this case, because the total Lagrangian \(\bar{\mathcal{L}}\) has lower \(SU(2) \times U(1)\) symmetry than the static potential. We can pick one of the direction in \(\hat{M}\) space and linearize the mode equation around this direction, say \(n_1\) (superconductor phase):

\[
\begin{align*}
\chi \partial_t^2 n_2 &= \rho \partial_k^2 n_2 - 2\mu \partial_t n_3 \\
\chi \partial_t^2 n_3 &= \rho \partial_k^2 n_3 + 2\mu \partial_t n_2 \\
\chi \partial_t^2 n_5 &= \rho \partial_k^2 n_5 \\
\chi \partial_t^2 n_4 &= \rho \partial_k^2 n_4 - \mu^2 n_4.
\end{align*} \tag{12}
\]

The last equation describes the massive modes with energy \(\omega_5 = \mu\), which is associated with the explicit symmetry breaking (from \(SO(5)\) to \(SO(4)\)) of the static potential. The third equation describes the usual Goldstone massless mode (sound mode) of the superconductor with linear dispersion \(\omega_5 = (\rho/\chi)k\). The first two equations predict a new doublet-spin wave modes. One is massless,
the other is massive. In the long wavelength limit, the energies of the modes are \( \omega_2 = \nu k^2, \omega_3 = 2\mu \). Therefore, there are always two gapless modes, one with linear dispersion and the other with quadratic dispersion, independent of the orientation of superspin.

It is also interesting to investigate the case where the SO(5) symmetry is explicitly broken, but a projected SO(5) symmetry defined in Ref. [3] and Ref. [4] is present. We can add a term \(-g(n_2^2 + n_3^2 + n_4^2)\) to the SO(5) symmetric potential \( V(n) \), and choose \( g > 0 \) so that AF is favored at half-filling where \( \mu = 0 \). In this case, the effective potential in the presence of \( B \) and \( \mu \) is given by

\[
V_{\text{eff}}(n) = V(n) - g(n_2^2 + n_3^2 + n_4^2) - \chi [B^2(n_2^2 + n_3^2) + \mu^2(n_2^2 + n_3^2)]
\]  

(13)

For \( B = 0 \), there is a AF to SC transition at \( \mu_c = \sqrt{g/\chi} \). For \( \mu > \mu_c \), the system is in a SC state. This SC state has a \( \pi \) resonance mode with frequency

\[
\omega_0 = \sqrt{\mu^2 - \mu_c^2}
\]  

(14)

A finite magnetic field \( B \) causes a triplet Zeeman splitting of this \( \pi \) mode, where the lower mode vanishes at a critical value

\[
B_c = \sqrt{\mu^2 - \mu_c^2}
\]  

(15)

of the Zeeman field. On the other hand, from Eq. (13), we see that a finite Zeeman magnetic field \( B \) induces a SC to AF transition when \( B \) exceeds the same critical value \( B_c \) as given by Eq. (15). At \( B = B_c \), the effective potential \( V_{\text{eff}} \) as given in Eq. (13) is exactly SO(4) invariant in the \( \tilde{M} = (n_1, n_2, n_5, n_3) \) space. The kinetic terms further break this symmetry to \( SU(2) \times U(1) \). Summarizing above discussions we conclude that both exact and projected SO(5) symmetric models have a exact quantum \( SU(2) \times U(1) \) symmetry at a critical value of the Zeeman magnetic field, which is the energy of the \( \pi \) resonance mode measured in the units of the magnetic field.

From above discussions we see that there are only two remaining massless modes at the \( B = \mu \) point. It would be interesting to formulate a low energy theory where the two other massive modes are explicitly projected out. In the Lagrangian formalism, this can be accomplished by dropping the \( n_4 \) degree of freedom, and discarding the second time derivative terms in equation (7). This gives us the possibility of testing the SO(5) symmetry of the original model at any doping. Starting from a SC state at zero magnetic field, the superspin lies in the \((n_1, n_5)\) plane. Within the SO(5) model, the only effect of a applied Zeeman magnetic field is to split the \( \pi \) triplet resonance mode. The intensity and commensurability of each member of the triplet remain the same. At a critical field \( B_c \), there is a first order transition from the SC state into the AF state where the superspin lies in the \((n_2, n_3)\) plane. At the same time, one of the \( \pi \) mode softens to zero energy at \( B_c \). The exact coincidence of mode softening transition and a first order transition is the signature of the new symmetry. As we shall see, in a generic system, either the first order transition occurs before the mode softens to zero energy, or the mode softening occurs before the first order transition, in which case the system will have two separate second order phase transitions.

All above discussions are based on the assumption where the original model has a exact or projected SO(5) symmetry. In order to see the physical signature of the SO(5) symmetry, it is useful to study the effects of a finite chemical potential and Zeeman magnetic field on models without SO(5) symmetry. A general Landau-Ginzburg potential a approximate SO(5) model in the presence of a finite Zeeman magnetic field \( B \) and chemical potential \( \mu \) can be expressed as

\[
V = -\delta_x \frac{x}{2} - \delta_y \frac{y}{2} - \delta_z \frac{z}{2} + W_x x^2 + W_y (y + z)^2 + W_0 \frac{y^2}{2} (y + z)
\]  

(18)

where \( n_1^2 + n_2^2 = x, n_2^2 + n_3^2 = y, n_4^2 = z, \delta_x = 2\chi, \mu^2 + \delta \) and \( \delta_y = 2\chi, B^2 + \delta \). There are two kinds of generic phase diagrams described by this effective potential. The first type of phase diagram is realized for \( W_0^2 > W_x W_y \) and is depicted in Fig. 1. In this case, the Zeeman magnetic field induces a first order phase transition from the SC state to the AF state at a critical value of the magnetic field \( B_c \). However, the \( \pi \) mode is still massive at \( B_c \), which clearly distinguishes this from the SO(5) symmetric case. The first order line terminates at a bi-critical point \( T_{bc} \)
where all static properties have a emergent $SO(4)$ symmetry and all dynamic properties have a $SU(2) \times U(1)$ symmetry. The second type of phase diagram is realized for $W_0^2 < W_c W_s$, it describes two second order phase transitions, with a intervening mixed phase region where both SC and AF orders coexist, as shown in Fig. 2. The mixed region shrinks to zero at a finite temperature tetra-critical point $T_{bc}$. In the mixed phase, there are also two gapless modes and two massive modes. However, there is a major difference for the gapless modes between exact and approximate $SO(5)$ models. The two gapless modes in this approximate $SO(5)$ model both have linear dispersion in mixed phase. In an exact $SO(5)$ symmetry model, as what we pointed out before, there is one gapless mode with quadratic dispersion, leading to a system with infinite compressibility at the transition point $T_{bc}$.

In conclusion we have discovered a new symmetry of $SO(5)$ models in the presence of a finite Zeeman magnetic field $B$ and chemical potential $\mu$. At the special point $B_c = \mu$, the static potential has a exact $SO(4)$ symmetry and the full Hamiltonian has a exact $SU(2) \times U(1)$ symmetry. These considerations also generalize to the projected $SO(5)$ model, where the critical magnetic field is shifted to $B_c = \sqrt{\mu^2 - \mu_c^2}$, as given by equation (15). This observation gives the possibility to experimentally test the $SO(5)$ symmetry at any doping level. The Zeeman magnetic field can be experimentally realized by applying a magnetic in the two dimensional plane [5], so that the orbital effects can be minimized. Below the critical value $B_c$, our theory predicts that the Zeeman magnetic field will only split the resonance energy, but not change the intensity of the $\pi$ resonance mode. The $\pi$ mode should also remain commensurate at momentum $(\pi, \pi)$. The critical value of magnetic field needed for reaching the exact $SU(2) \times U(1)$ symmetry point can also be expressed as $B_c = \omega_0/g\mu_B$, where $\omega_0$ is the neutron resonance energy, $g$ is the electronic $g$ factor, and $\mu_B$ is the Bohr magneton. Unfortunately, this value exceeds $100T$ for all high $T_c$ superconductors where neutron resonance has been discovered. While it is not realistic to reach such a high magnetic field, one could imagine starting from sufficiently underdoped materials where the neutron resonance energy is much lower, or one can perform the proposed experiments on other materials [6] where the intrinsic energy scales are much lower.

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