Rigorous Analysis and Dynamics of Hibler’s Sea Ice Model

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Abstract
This article develops for the first time a rigorous analysis of Hibler’s model of sea ice dynamics. Identifying Hibler’s ice stress as a quasilinear second-order operator and regarding Hibler’s model as a quasilinear evolution equation, it is shown that a regularized version of Hibler’s coupled sea ice model, i.e., the model coupling velocity, thickness and compactness of sea ice, is locally strongly well-posed within the $L^q$-setting and also globally strongly well-posed for initial data close to constant equilibria.

Keywords Hibler’s sea ice model · Local and global well-posedness · Viscous–plastic stress tensor · Stability of equilibria

Mathematics Subject Classification 35Q86 · 35K59 · 86A05 · 86A10

1 Introduction

Sea ice is a material with a complex mechanical and thermodynamical behavior. Freezing sea water forms a composite of pure ice, liquid brine, air pockets and solid salt. The
details of this formation depend on the laminar or turbulent environmental conditions, see e.g., Hibler (1979), Feltham (2008) and Golden (2015). This composite responds differently to heating, pressure or mechanical forces than for example the (salt-free) glacial ice of ice sheets.

The evolution of sea ice has attracted much attention in climate science due to its role as a hot spot in global warming. The state of the art concerning the modeling of sea ice is described in the very recent survey article (Golden et al. 2020) in the Notices of the American Mathematical Society.

Somewhat surprisingly, the field of sea ice dynamics forms a terra incognita to rigorous mathematical analysis. In contrast to atmospheric or oceanic models, see e.g., the work of Lions et al. (1992a, b) and Cao and Titi (2007) for the primitive equations as well as the work of Majda (2003) and Benacchio and Klein (2019) for atmospheric flows, rigorous analysis of sea ice models is essentially non-existent.

The governing equations of large-scale sea ice dynamics that form the basis of virtually all sea ice models in climate science are suggested in a seminal paper by Hibler (1979). Sea ice is here modeled as a material with a very specific constitutive law based on viscous–plastic rheology. This model has been investigated numerically by various communities (see e.g., Mehlmann et al. 2021; Mehlmann and Korn 2021; Mehlmann 2019; Mehlmann and Richter 2017; Seinen and Khouider 2018; Danilov et al. 2015; Kimmrich et al. 2015; Bouchat and Tremblay 2014; Lemieux and Tremblay 2009), but it seems it never was studied from a rigorous analytical point of view. In fact, even the existence of weak solutions to Hibler’s sea ice model seems to be unknown until today.

Moreover, fundamental questions in this respect such as thermodynamical consistency of the Hibler model with the second law of thermodynamics, as well as existence, uniqueness and regularity properties of solutions of this sea ice PDE system, seem to be open problems. Under distinct simplifications, certain submodels were, however, considered in Gray (1999) and Guba et al. (2013) within the context of hyperbolic systems. The authors postulate ill-posedness of these simplified submodels.

As stated above, Hibler’s sea ice model was already investigated numerically by many authors. All of these approaches are based on various regularizations of the underlying ice stress tensor. For example, the original viscous–plastic equations have been regularized by means of additional artificial elasticity, see e.g., the work of Hunke–Dukowicz (1997), in order to improve computational efficiency. This elastic–viscous–plastic approach has been implemented then in many sea ice and climate models. Let us emphasize that simulations of sea ice behavior show a distinct discrepancy whether the original Hibler or the regularized sea ice PDEs are being used, see Losch and Danilov (2012). For a thorough numerical study based on a regularization of the original viscous–plastic ice stress, we refer to the work of Mehlmann and Korn (2021), Mehlmann and Richter (2017) and Mehlmann (2019).

The modeling of sea ice dynamics is a very active and dynamic field of research (Golden et al. 2020). In particular, we refer to alternative models e.g., by Moritz and Ukita (2000), Wilchinsky and Feltham (2004) and Eisenman and Wettlaufer (2009) as well as to the ones described in the survey paper (Golden et al. 2020).

In this article for the first time, we rigorously prove the existence and uniqueness of a strong solution to Hibler’s sea ice model. Our approach is based on the theory of
quasilinear parabolic evolution equations and a regularization of Hibler’s original ice stress $\sigma$. This regularization has been used already in various numerical approaches by Mehlmann, Richter and Korn, see Mehlmann (2019), Mehlmann and Richter (2017) and Mehlmann and Korn (2021).

A key point of our analysis is the understanding of the term $\text{div}\,\sigma$ as a strongly elliptic quasilinear operator $A$ within the $L_q$-setting. We show that its linearization, subject to Dirichlet boundary conditions, fulfills the Lopatinskii–Shapiro condition yielding the maximal $L_q$-regularity property of the linearized Hibler operator. The latter property is then extended to the coupled system, described precisely later on in (2.10), consisting of the momentum equation for the velocity $u$ and the two balance laws for the mean ice thickness $h$ and the ice compactness $a$. Regarding this model as a quasilinear evolution equation, we obtain strong well-posedness of the fully coupled system. For background information on linear and quasilinear evolution equations, we refer to Arendt et al. (2011), Denk et al. (2003, 2004, 2007), Kunstmann and Weis (2004), Prüss and Simonett (2016) and Hieber et al. (2020).

In our first main result, we prove the existence and uniqueness of a local strong solution to (2.10) for suitably chosen initial data and show that this solution depends continuously on the data, exists on a maximal time interval and regularizes instantly in time. Secondly, we show that this solution extends uniquely to a global strong solution provided the initial data are close to the equilibria $v_* = (0, h_*, a_*)$, where $h_*$ and $a_*$ denote constants and the external forces vanish.

To put our result in perspective, note that the existence of a weak solution to Hibler’s sea ice model (2.10) is not known until today. It is also interesting to observe that Hibler’s sea ice stress tensor is related to the stress tensor of certain non-Newtonian fluids, as described in Bothe and Prüss (2007). It was shown recently by Burczak et al. (2021) that under certain assumptions weak solutions to these non-Newtonian fluid models are highly non unique.

This article is organized as follows: Sect. 2 presents Hibler’s model as well as our main well-posedness results for this system. In Sect. 3, we rewrite Hibler’s operator as a second-order quasilinear operator, whose linearization will be investigated in Sect. 4. There we show that the linearization of Hibler’s operator is a strongly elliptic operator within the $L_q$-setting and that this operator subject to Dirichlet boundary conditions satisfies the maximal $L_q$-regularity property. After a short section on functional analytic properties of Hibler’s operator, in Sect. 6, we present the proof of our local well-posedness result. Finally, the proof of our global well-posedness result is given in Sect. 7.

After having finished our article, we became aware of the work by Liu et al. (2021), which studies a related problem.

### 2 Hibler’s Viscous–Plastic Sea Ice Model and Main Results

Hibler (1979) proposed a rheology model for sea ice dynamics, which has become since then the standard sea ice dynamics model and serves until today as a basis for many numerical studies in this field. Roughly speaking, pack ice consists of rigid plates which drift freely in open water or are closely packed together in areas of high
ice concentration. Although individual ice floes may have very different sizes, pack ice may be considered as a highly fractured two-dimensional continuum.

The momentum balance in this model is given by the two-dimensional equation:

\[
m(\dot{u} + u \cdot \nabla u) = \text{div} \sigma - mc_{\text{cor}} n \times u - mg \nabla H + \tau_{\text{atm}} + \tau_{\text{ocean}},
\]

where \( u : (0, \infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) denotes the horizontal ice velocity and \( m \) the ice mass per unit area. Moreover, \(-mc_{\text{cor}} n \times u\) represents the Coriolis force with Coriolis parameter \( c_{\text{cor}} > 0 \) and unit vector \( n : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) normal to the surface, while \(-mg \nabla H\) describes the force arising from changing sea surface tilt with sea surface dynamic height \( H : (0, \infty) \times \mathbb{R}^2 \rightarrow (0, \infty) \) and gravity \( g \). The terms \( \tau_{\text{atm}} \) and \( \tau_{\text{ocean}} \) describe atmospheric wind and oceanic forces given by

\[
\tau_{\text{atm}} = \rho_{\text{atm}} C_{\text{atm}} |U_{\text{atm}}| R_{\text{atm}} U_{\text{atm}},
\]

\[
\tau_{\text{ocean}} = \rho_{\text{ocean}} C_{\text{ocean}} |U_{\text{ocean}} - u| R_{\text{ocean}} (U_{\text{ocean}} - u),
\]

where \( U_{\text{atm}} \) and \( U_{\text{ocean}} \) denote the velocity of the surface winds and current, respectively. Furthermore, \( C_{\text{atm}} \) and \( C_{\text{ocean}} \) are air and ocean drag coefficients, \( \rho_{\text{atm}} \) and \( \rho_{\text{ocean}} \) denote the densities for air and sea water, and \( R_{\text{atm}} \) and \( R_{\text{ocean}} \) are rotation matrices acting on wind and current vectors. For results on fluids driven by wind forces, we refer to Bresch and Simon (2001).

Following Hibler (1979), the viscous–plastic rheology is given by a constitutive law that relates the internal ice stress \( \sigma \) and the deformation tensor \( \varepsilon = \varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T) \) through an internal ice pressure \( P \) and nonlinear bulk and shear viscosities, \( \zeta \) and \( \eta \), such that the principal components of the stress lie on an elliptical yield curve with the ratio of major to minor axes \( e \). This constitutive law is given by

\[
\sigma = 2\eta(\varepsilon, P)\varepsilon + [\zeta(\varepsilon, P) - \eta(\varepsilon, P)] \text{tr}(\varepsilon) I - \frac{P}{2} I.
\]

The pressure \( P \) measures the ice strength, depending on the thickness \( h \) and the ratio \( a \) of thick ice per unit area, and is explicitly given by

\[
P = P(h, a) = p^* h \exp(-c(1-a)),
\]

where \( p^* > 0 \) and \( c > 0 \) are given constants. The bulk and shear viscosities \( \zeta \) and \( \eta \) increase with pressure and decreasing deformation tensor and are given by

\[
\zeta(\varepsilon, P) = \frac{P}{2\Delta^2(\varepsilon)} \quad \text{and} \quad \eta(\varepsilon, P) = e^{-2} \zeta(\varepsilon, P),
\]

where

\[
\Delta^2(\varepsilon) := (\varepsilon_{11}^2 + \varepsilon_{22}^2) \left( 1 + \frac{1}{e^2} \right) + \frac{4}{e^2} \varepsilon_{12}^2 + 2\varepsilon_{11}\varepsilon_{22} \left( 1 - \frac{1}{e^2} \right).
\]
and $e$ as described above is the ratio of the long axis to the short axis of the elliptical yield curve. The above law represents an idealized viscous–plastic material, whose viscosities, however, become singular if $\Delta$ tends to zero.

For this reason, already Hibler proposed to regularize this behavior by bounding the viscosities when $\Delta$ is getting small and by defining maximum values $\zeta_{\text{max}}$ and $\eta_{\text{max}}$ for $\zeta$ and $\eta$. Then, $\zeta$ and $\eta$ become

$$
\zeta' = \min\{\zeta, \zeta_{\text{max}}\} \quad \text{and} \quad \eta' = \min\{\eta, \eta_{\text{max}}\}.
$$

This formulation of the viscosities leads, however, to non-smooth rheology terms. To enforce smoothness, several regularizations have been considered in the literature, see e.g., Mehlmann and Richter (2017), Lemieux and Tremblay (2009). For example, Lemieux and Tremblay (2009) replaced $\zeta$ by

$$
\zeta = \zeta_{\text{max}} \tanh\left(\frac{P}{(2\Delta\zeta_{\text{max}})}\right).
$$

An elastic–viscous–plastic stress tensor was introduced by Hunke and Dukowicz (1997). Starting from the observation that the relation (2.4) for $\sigma$ can be rewritten as

$$
\frac{1}{2\eta} \sigma + \frac{\eta - \zeta}{4\eta\zeta} \text{tr} \sigma + \frac{P}{4\zeta} I = \epsilon,
$$

one recovers the elasticity equation $\frac{1}{E} \partial_t \sigma = \epsilon$. In the pure plastic case, the compressive stress $\sigma_d = \text{tr} \sigma$ and the shear stress $\sigma_s = ((\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2)^{1/2}$ are linked by the relation $(\sigma_d + P)^2 + e^2 \sigma_s^2 = P^2$, which leads to an elliptical yield curve.

Following Mehlmann and Korn (2021), see also Kreyscher et al. (2000), we consider for $\delta > 0$ the regularization

$$
\Delta_\delta(\epsilon) := \sqrt{\delta + \Delta^2(\epsilon)}.
$$

We then set $\zeta_\delta = \frac{P}{2\Delta_\delta(\epsilon)}$ and $\eta_\delta = e^{-2} \zeta_\delta$ as well as

$$
\sigma_\delta := 2\eta_\delta \epsilon + [\zeta_\delta - \eta_\delta] \text{tr}(\epsilon) I - \frac{P}{2} I.
$$

We consider in the following the above momentum equation (2.1) in a bounded domain $\Omega \subset \mathbb{R}^2$ with boundary of class $C^2$. It is coupled to two balance equations for the mean ice thickness

$$
\dot{h} + \text{div}(uh) = S_h + d_h \Delta h,
$$

and the ice compactness $a : J \times \Omega \to \mathbb{R}$ with $a \geq \alpha$ for some $\alpha \in \mathbb{R}$ given by

$$
\dot{a} + \text{div}(ua) = S_a + d_a \Delta a.
$$
Here, \( \kappa \) is a small parameter that indicates the transition to open water in the sense that for \( m = \rho_{\text{ice}}h \) a value of \( h(t, x, y) \) less than \( \kappa \) means that at \( (x, y) \in \Omega \) and at time \( t \) there is open water. Furthermore, let \( J = (0, T) \) for \( 0 < T \leq \infty \), let \( \Delta \) be the Laplacian, \( d_h > 0 \) and \( d_a > 0 \) be constants and for an arbitrary \( f \in C^1([0, \infty); \mathbb{R}) \), for example the one considered by Hibler (1979), define the terms \( S_h \) and \( S_a \) by

\[
S_h = f \left( \frac{h}{a} \right) a + (1 - a) f(0), \quad (2.8)
\]

\[
S_a = \begin{cases} 
\frac{f(0)}{\kappa} (1 - a), & \text{if } f(0) > 0 \\
0, & \text{if } f(0) < 0 \\
0, & \text{if } S_h > 0, \\
da \frac{S_h}{\kappa}, & \text{if } S_h < 0.
\end{cases} \quad (2.9)
\]

The system is finally completed by Dirichlet boundary conditions for \( u \) and Neumann boundary conditions for \( h \) and \( a \).

Given \( 0 < T \leq \infty \) and \( J = (0, T) \), the complete set of equations describing sea ice dynamics by Hibler’s model then reads as

\[
\begin{aligned}
&m(\dot{u} + u \cdot \nabla u) = \text{div} \sigma_\delta - mc_{\text{cor}} n \times u \\
&\quad - mg \nabla H + \tau_{\text{atm}} + \tau_{\text{ocean}}, \quad x \in \Omega, \ t \in J, \\
&\dot{h} + \text{div} (uh) = S_h + d_h \Delta h, \quad x \in \Omega, \ t \in J, \\
&\dot{a} + \text{div} (ua) = S_a + d_a \Delta a, \quad x \in \Omega, \ t \in J, \\
&u = \frac{\partial h}{\partial v} = \frac{\partial a}{\partial v} = 0, \quad x \in \partial \Omega, \ t \in J, \\
&u(0, x) = u_0(x), \quad h(0, x) = h_0(x), \quad x \in \Omega, \\
&a(0, x) = a_0(x), \quad x \in \Omega.
\end{aligned} \quad (2.10)
\]

Note that \( m = \rho_{\text{ice}}h \) and \( h \) is subject to (2.7).

To formulate our main well-posedness result for the system (2.10), we first rewrite it as a quasilinear evolution equation and introduce a setting as follows. Denoting the principle variable of the system by \( v = (u, h, a) \), we rewrite system (2.10) as a quasilinear evolution equation of the form

\[
v' + A(v)v = F(v), \quad t > 0, \quad v(0) = v_0. \quad (2.11)
\]

Here, \( v \) belongs to the ground space \( X_0 \) defined by

\[
X_0 = L_q(\Omega; \mathbb{R}^2) \times L_q(\Omega) \times L_q(\Omega),
\]

where \( 1 < q < \infty \). The regularity space will be

\[
X_1 = \{ u \in H_q^2(\Omega; \mathbb{R}^2) : u = 0 \text{ on } \partial \Omega \} \times \{ h \in H_q^2(\Omega) : \partial_v h = 0 \text{ on } \partial \Omega \} \\
\times \{ a \in H_q^2(\Omega) : \partial_v a = 0 \text{ on } \partial \Omega \}.
\]
Furthermore, the quasilinear operator \( A(v) \) is given by the upper triangular matrix

\[
A(v) = \begin{pmatrix}
\frac{1}{\rho_{\text{ice}}} A_D^H(\nabla u, P(h, a)) & \frac{\partial_h P(h, a)}{2\rho_{\text{ice}}h} \nabla & \frac{\partial_a P(h, a)}{2\rho_{\text{ice}}h} \nabla \\
0 & -d_h \Delta_N & 0 \\
0 & 0 & -d_a \Delta_N
\end{pmatrix}.
\] (2.12)

Here, \( A_D^H \) denotes the realization of Hibler’s operator subject to Dirichlet boundary conditions on \( L_q(\Omega; \mathbb{R}^2) \), introduced and defined precisely in Sect. 3, and \( \Delta_N \) the Neumann Laplacian on \( L_q(\Omega) \) defined by \( \Delta_N = \{ h \in H^2(\Omega) : \partial_{\nu} h = 0 \text{ on } \partial \Omega \} \). The semilinear part \( F(v) \) is defined by

\[
F(v) = \begin{pmatrix}
-u \cdot \nabla u - c_{\text{cor}} n \times u - g \nabla H \\
- \text{div} (u h) + S_h \\
- \text{div} (u a) + S_a \\
\end{pmatrix}
+ \begin{pmatrix}
\frac{c_1}{h} |U_{\text{atm}}| U_{\text{atm}} + \frac{c_2}{h} |U_{\text{ocean}} - u|(U_{\text{ocean}} - u) \\
0 \\
0
\end{pmatrix},
\] (2.13)

where \( c_1 = \rho_{\text{atm}} C_{\text{atm}} R_{\text{atm}} \rho_{\text{ice}}^{-1} \) and \( c_2 = \rho_{\text{ocean}} C_{\text{ocean}} R_{\text{ocean}} \rho_{\text{ice}}^{-1} \) and \( U_{\text{atm}} \) and \( U_{\text{ocean}} \) are given functions.

We consider solutions \( v \) within the class

\[
v \in H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1) =: \mathbb{E}_1(J),
\]

where \( J = (0, T) \) as above is an interval and \( \mu \in (1/p, 1] \) indicates a time weight. More precisely,

\[
v \in H^k_{p,\mu}(X_l) \iff t^{1-\mu} v \in H^k_p(X_l), \quad k, l = 0, 1.
\]

The time trace space of this class is given by

\[
X_{y,\mu} = (X_0, X_1)_{\mu-1/p, p} = D_{A_D^H(v_0)}(\mu - 1/p, p) \times D_{\Delta_N}(\mu - 1/p, p) \times D_{\Delta_N}(\mu - 1/p, p) \quad (2.14)
\]

provided \( p \in (1, \infty) \) and \( \mu \in (1/p, 1] \). Note that [see e.g., Section 7 of Adams and Fournier (2003)]

\[
X_{y,\mu} \hookrightarrow B_{qp}^{2(\mu-1/p)}(\Omega)^4 \hookrightarrow C^1(\overline{\Omega})^4 \quad (2.15)
\]

provided

\[
\frac{1}{2} + \frac{1}{p} + \frac{1}{q} < \mu \leq 1. \quad (2.16)
\]
It is well-known that for $\mu$ satisfying (2.16), the above real interpolation spaces can be characterized as

\[
 u \in D_{\Delta_0}^{\mu}(v_0)(\mu - 1/p, p) \iff u \in B_{2q}^{\mu - 2/p}(\Omega)^2, u = 0 \text{ on } \partial \Omega,
\]

\[
 h \in D_{\Delta_0}^{\mu}(\mu - 1/p, p) \iff h \in B_{2q}^{2\mu - 2/p}(\Omega), \partial_\nu h = 0 \text{ on } \partial \Omega.
\]

For brevity, we set $X_\gamma := X_{\gamma,1}$. Moreover, let $V_\mu$ be an open subset of $X_{\gamma, \mu}$ such that all

\[
 (u, h, a) \in V_\mu \text{ satisfy } h \geq \kappa \text{ for some } \kappa > 0 \text{ and } a \in [0, 1]. \tag{2.17}
\]

**Theorem 2.1** (Well-posedness of Hibler’s sea ice model)

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary of class $C^2$, and for $\delta > 0$, let $\sigma_\delta$ be defined as in (2.6). Assume that $1 < p, q < \infty$ and that $\mu \in (1/p, 1]$ are subject to (2.16) and let $v_0 \in V_\mu$, where $V_\mu$ is as in (2.17).

(a) Then there exist $\tau = \tau(v_0) > 0$ and $r = r(v_0) > 0$ with $B_{X_{\gamma, \mu}}(v_0, r) \subset V_\mu$ such that Eq. (2.11), i.e., Eqs. (2.10), (2.2), (2.3), (2.5), (2.8) and (2.9), has a unique solution

\[
 v(\cdot, v_1) \in H^1_{p, \mu}(0, \tau; X_0) \cap L_{p, \mu}(0, \tau; X_1) \cap C([0, \tau); V_\mu)
\]

for each initial value $v_1 \in B_{X_{\gamma, \mu}}(v_0, r)$. Moreover, there exists $C = C(v_0)$ such that

\[
 \|v(\cdot, v_1) - v(\cdot, v_2)\|_{E_1(0, \tau)} \leq C\|v_1 - v_2\|_{X_{\gamma, \mu}}
\]

for all $v_1, v_2 \in B_{X_{\gamma, \mu}}(v_0, r)$. In addition,

\[
 t \partial_t v \in H^1_{p, \mu}(0, \tau; X_0) \cap L_{p, \mu}(0, \tau; X_1),
\]

i.e., the solution regularizes instantly in time. In particular,

\[
 v \in C^1([b, \tau]; X_{\gamma,1}) \cap C^{1 - 1/p}([b, \tau]; X_1)
\]

for any $b \in (0, \tau)$.

(b) The solution $v = v(v_0)$ exists on a maximal time interval $J(v_0) = [0, t^+(v_0))$, which is characterized by the following alternatives:

(i) global existence, i.e., $t^+(v_0) = \infty$,

(ii) $\lim_{t \to t^+(v_0)} \text{dist}_{X_{\gamma, \mu}}(v(t), \partial V_\mu) = 0$,

(iii) $\lim_{t \to t^+(v_0)} v(t)$ does not exist in $X_{\gamma, \mu}$.

**Remark 2.2** (a) Assuming $1 < p, q < \infty$ and $\mu \in (1/p, 1]$ subject to (2.16), the smoothness condition required for the initial data $v_0 = (u_0, h_0, a_0)$ in Theorem 2.1
can be characterized as
\[ u_0 \in B_{2q}^{2\mu-2/p}(\Omega)^2, u_0 = 0 \text{ on } \partial\Omega, \quad h_0 \in B_{2q}^{2\mu-2/p}(\Omega), \partial_v h_0 = 0 \text{ on } \partial\Omega, \]
\[ a_0 \in B_{2q}^{2\mu-2/p}(\Omega), \partial_v a_0 = 0 \text{ on } \partial\Omega. \]

(b) These conditions are in particular satisfied if \((u_0, h_0, a_0) \in H_1^{1+2/q+s}(\Omega)^4\) for \(s > 0\) satisfy the above boundary conditions.

Assuming that \(h_*\) and \(a_*\) are constant in time and space, we observe that \((0, h_*, a_*)\) are trivial equilibria for Eq. (2.11) subject to vanishing forcing terms. We proceed by showing that the equilibrium \((0, h_*, a_*)\) is stable in \(X_{\gamma,\mu}\), and the unique solution of (2.11) exists globally for initial data close to the aforementioned equilibrium provided \(\delta\) is chosen small enough and the external forces vanish.

**Theorem 2.3** There exists \(\delta^* > 0\) such that for all \(\delta \in (0, \delta^*)\) and for \(h_*\) and \(a_*\) as above, the equilibrium \(v_* = (0, h_*, a_*)\) is stable in \(X_{\gamma,\mu}\), and there exists \(r > 0\) such that the unique solution \(v\) of (2.11) without forcing terms and with initial value \(v_0 \in X_{\gamma,\mu}\) fulfilling \(\|v_0 - v_*\|_{X_{\gamma,\mu}} < r\) exists on \(\mathbb{R}_+\) and converges at an exponential rate in \(X_{\gamma,\mu}\) to some equilibrium \(v_\infty\) of (2.11) as \(t \to \infty\).

**Remark 2.4** Equation (2.10) shows that without forcing, the solution tends to the equilibria \(h_* = \frac{1}{|\Omega|} \int_\Omega h(0)dx\) and \(a_* = \frac{1}{|\Omega|} \int_\Omega a(0)dx\) determined by the initial mean values of \(h\) and \(a\), respectively.

**3 Hibler’s Ice Stress Viewed as a Second Order Quasilinear Operator**

In this section, we interpret the term \(\text{div } \sigma\) as a quasilinear second-order operator. To this end, denote by \(\varepsilon = (\varepsilon)_{ij}\) the deformation or rate of strain tensor and define the map \(S: \mathbb{R}^{2 \times 2} \to \mathbb{R}^{2 \times 2}\) in such a way that
\[
S\varepsilon = \begin{pmatrix}
(1 + \frac{1}{e^2})\varepsilon_{11} + (1 - \frac{1}{e^2})\varepsilon_{22} & \frac{1}{e^2}(\varepsilon_{12} + \varepsilon_{21}) \\
\frac{1}{e^2}(\varepsilon_{12} + \varepsilon_{21}) & (1 - \frac{1}{e^2})\varepsilon_{11} + (1 + \frac{1}{e^2})\varepsilon_{22}
\end{pmatrix}.
\]

If \(\varepsilon \in \mathbb{R}^{2 \times 2}\) is identified with the vector \((\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{21}, \varepsilon_{22})^T \in \mathbb{R}^4\), \(S\) corresponds to the symmetric positive semi-definite matrix
\[
S = (S_{ij}^{kl}) = \begin{pmatrix}
1 + \frac{1}{e^2} & 0 & 0 & 1 - \frac{1}{e^2} \\
0 & \frac{1}{e^2} & \frac{1}{e^2} & 0 \\
0 & \frac{1}{e^2} & \frac{1}{e^2} & 0 \\
1 - \frac{1}{e^2} & 0 & 0 & 1 + \frac{1}{e^2}
\end{pmatrix},
\]
and we obtain

\[ \triangle^2(\varepsilon) = \varepsilon^T S\varepsilon = \sum_{i,j,k,l=1}^{2} \varepsilon_{ik} \varepsilon_{kl} \]

\[ = (\varepsilon_{11} + \varepsilon_{22})^2 + \frac{1}{\varepsilon^2} (\varepsilon_{11} - \varepsilon_{22})^2 + \frac{1}{\varepsilon^2} (\varepsilon_{12} + \varepsilon_{21})^2. \] (3.1)

The stress tensor \( \sigma = \sigma(\varepsilon, P) \) can then be represented as

\[ \sigma(\varepsilon, P) = S(\varepsilon, P) - \frac{P}{2} I, \quad \text{where} \quad S(\varepsilon, P) := \frac{P}{2} \frac{S\varepsilon}{\triangle(\varepsilon)}. \] (3.2)

As explained in Sect. 2, for \( \delta > 0 \), we then substitute \( S \) by

\[ S_\delta = S_\delta(\varepsilon, P) := \frac{P}{2} \frac{S\varepsilon}{\delta^2}, \] (3.3)

and define Hibler’s operator as

\[ \mathcal{A}^H u := - \text{div} S_\delta(u) = - \text{div} \left( \frac{P}{2} \frac{S\varepsilon}{\sqrt{\delta + \varepsilon^T S\varepsilon}} \right). \]

Employing product and chain rule as well as symmetries of \( S \), we infer that

\[ \text{div} \left( \frac{P}{2} \frac{S\varepsilon}{\sqrt{\delta + \varepsilon^T S\varepsilon}} \right)_i = \sum_{j,k,l=1}^{2} \frac{P}{2} \frac{1}{\Delta_\delta(\varepsilon)} \left( S_{kl}^{ij} - \frac{1}{\Delta_\delta^2(\varepsilon)} (S\varepsilon)_{ik} (S\varepsilon)_{lj} \right) \partial_k \varepsilon_{jl} \]

\[ + \frac{1}{2\Delta_\delta(\varepsilon)} \sum_{j=1}^{2} (\partial_j P)(S\varepsilon)_{ij} \]

for \( i = 1, 2 \). Exploiting symmetries of \( S \) and \( \varepsilon \) once again, we conclude that

\[ (\mathcal{A}^H u)_i = \sum_{j,k,l=1}^{2} \frac{P}{2} \frac{1}{\Delta_\delta(\varepsilon)} \left( S_{kl}^{ij} - \frac{1}{\Delta_\delta^2(\varepsilon)} (S\varepsilon)_{ik} (S\varepsilon)_{lj} \right) D_k D_l u_j \]

\[ - \frac{1}{2\Delta_\delta(\varepsilon)} \sum_{j=1}^{2} (\partial_j P)(S\varepsilon)_{ij} \] (3.4)

for \( i = 1, 2 \) and \( D_m = -i\partial_m \).

We denote the coefficients of the principal part of \( \mathcal{A}^H \) by

\[ a_{kl}^{ij}(\nabla u, P) := \frac{P}{2} \frac{1}{\Delta_\delta(\varepsilon)} \left( S_{kl}^{ij} - \frac{1}{\Delta_\delta^2(\varepsilon)} (S\varepsilon)_{ik} (S\varepsilon)_{lj} \right). \] (3.5)
In view of the symmetries of $S$ and $S\epsilon$, we conclude that
\[
a_{ij}^{kl} = a_{kj}^{il} = a_{ij}^{lk} = a_{kj}^{il} = a_{ij}^{il} = a_{kj}^{lj}.
\] (3.6)

For given $v_0 = (u_0, h_0, a_0) \in V_\mu$ with $\mu > \frac{1}{2} + \frac{1}{p} + \frac{1}{q}$, let
\[
[A^H(v_0)u] = \sum_{j,k,l=1}^2 a_{ij}^{kl}(\nabla u_0, P(h_0, a_0))D_kD_lu_j
- \frac{1}{\Delta_z(v_0)} \sum_{j=1}^2 (\partial_j P(h_0, a_0))(S\epsilon(u))_{ij}
\] (3.7)

be Hibler’s operator with frozen coefficients. The representation in (3.5) shows that the principal coefficients $a_{ij}^{kl}(\nabla u_0, P(h_0, a_0))$ of $A^H(v_0)$, as well as lower-order terms, depend smoothly on $u_0$, $h_0$ and $a_0$ with respect to the $C^1$-norm. Moreover, the embedding (2.15) yields that they lie in $C(\bar{\Omega})$.

4 Hibler’s Operator: Ellipticity and Maximal Regularity

In this section, we show that Hibler’s operator $A^H(v_0)$ given as in (3.7) defines a strongly elliptic operator and, when subject to Dirichlet boundary conditions, satisfies the Lopatinskii–Shapiro condition. This implies then that the $L_q$-realization $A_{D}^H(v_0)$ of $A^H(v_0)$ given by
\[
[A_{D}^H(v_0)] := [A^H(v_0)]u,
\]
\[
u \in D(A_{D}^H(v_0)) := \{u \in H^2_q(\Omega; \mathbb{R}^2) : u = 0 \text{ on } \partial \Omega\}
\] (4.1)
satisfies the maximal $L_q$-regularity property and furthermore that $A_{D}^H(v_0)$ admits a bounded $H^\infty$-calculus on $L_q(\Omega; \mathbb{R}^2)$.

For $\theta \in (0, \pi)$, let $\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\}$ be a sector in the complex plane, and let $D = -i(\partial_1, \ldots, \partial_n)$. We start by recalling from Denk et al. (2003) that for $x \in \mathbb{R}^n$ an operator $B$ of the form $B(x, D) = \sum_{|\alpha| \leq 2} b_\alpha(x)D^\alpha$ with continuous top-order coefficients $b_\alpha \in L(E)$, $E$ an arbitrary Banach space, is said to be parameter-elliptic of angle $\phi \in (0, \pi]$ if the spectrum $\sigma(B_#(x, \xi))$ of the symbol of the principal part $B_#(x, \xi) = \sum_{|\alpha| = 2} b_\alpha(x)\xi^\alpha$ satisfies
\[
\sigma(B_#(x, \xi)) \subset \Sigma_\phi
\]
for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^n$ with $|\xi| = 1$. We call $\phi_B = \inf\{\phi : \sigma(B_#(x, \xi)) \subset \Sigma_\phi\}$ the angle of ellipticity of $B$. Moreover, the operator $B(x, D)$ is called normally elliptic if it is parameter-elliptic of angle $\phi_B < \pi/2$. If $E$ is a Hilbert space, an operator $B$ of the above form $B(x, D) = \sum_{|\alpha| = 2} b_\alpha(x)D^\alpha$ is called strongly elliptic if there exists a constant $c > 0$ such that
\[
\text{Re}(B_#(x, \xi)w|w)_E \geq c\|w\|_E^2
\] (4.2)
for all $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$ and all $w \in E$. Here, $(-\cdot)_E$ denotes the inner product on $E$. To understand this condition, let $n(L)$ be the numerical range of a bounded linear operator on $E$, i.e., $n(L)$ is the closure of the set consisting of all $z \in \mathbb{C}$ such that $z = (Lw|w)_E$ for some $w \in E$ with $\|w\|_E = 1$. Since $\sigma(L) \subset n(L)$, we see that every strongly elliptic operator $B$ is parameter–elliptic of angle $\phi_B < \pi/2$, hence even normally elliptic.

Consider now the special case of homogeneous differential operators acting on $\mathbb{C}^n$-valued functions as

$$[B(x, D)v(x)]_i := \sum_{j,k,l=1}^n b_{ij}^{kl}(x) D_k D_l v_j(x), \quad x \in \Omega.$$ 

Here, $\Omega \subset \mathbb{R}^n$ denotes a domain with boundary of class $C^2$. Its symbol is defined as

$$\left(B_\#(x, \xi)\right)_{ij} := \sum_{k,l=1}^n b_{ij}^{kl}(x) \xi_k \xi_l, \quad x \in \Omega.$$ 

We now show that $A^H(v_0)$ is strongly elliptic provided $v_0 \in V_\mu$.

**Proposition 4.1** Let $p, q \in (1, \infty)$ and $\mu \in (\frac{1}{p}, 1]$ such that (2.16) holds. Then, for fixed $v_0 \in V_\mu$, the principal part of Hibler’s operator $A^H(v_0)$ defined as in (3.7) is strongly elliptic, and moreover parameter–elliptic of angle $\phi_{A^H(v_0)} = 0$.

**Proof** Recall that the principal part of $A^H(v_0)$ is given by

$$A^H_\#(x, \xi) := \sum_{k,l=1}^2 a_{ij}^{kl}(x) \xi_k \xi_l, \quad x \in \Omega,$$

with

$$a_{ij}^{kl}(\nabla u_0, P_0) := \frac{P_0}{2} \frac{1}{\Delta_\delta(\varepsilon)} \left( \xi_{ij}^{kl} - \frac{1}{\Delta_\delta(\varepsilon)} (\xi_{ik} (\xi_{j} \varepsilon)_{ji}) \right)$$

as in (3.5) and $P_0 = P(h_0, a_0)$. Taking into account the underlying symmetries, we see that the symbol of the principal part of $A^H(v_0)$ is given by

$$A^H_\#(x, \xi) = \begin{pmatrix}
a_{11}^0 \xi_1^2 + 2a_{11}^2 \xi_1 \xi_2 + a_{12}^2 \xi_2^2 & a_{11}^2 \xi_1^2 + (a_{12}^0 + a_{11}^2) \xi_1 \xi_2 + a_{12}^2 \xi_2^2 \\
a_{12}^2 \xi_1^2 + (a_{12}^0 + a_{11}^2) \xi_1 \xi_2 + a_{12}^2 \xi_2^2 & a_{22}^0 \xi_2^2 + 2a_{12}^2 \xi_1 \xi_2 + a_{22}^2 \xi_2^2
\end{pmatrix}.$$ (4.3)

For given $d \in \mathbb{R}^{2 \times 2}$, we use the notation

$$d_1 := d_{11} + d_{22}, \quad d_{11} := d_{11} - d_{22}, \quad d_{111} := \frac{d_{12} + d_{21}}{2}.$$
To verify condition (4.2), first recall that for any given $d \in \mathbb{R}^{2 \times 2}$, [see (3.1)],

$$d^T S d = d_I^2 + \frac{1}{e^2} (d_{II}^2 + 4 d_{III}^2) =: \Delta^2 (d). \quad (4.4)$$

Furthermore, using Young’s inequality, we estimate

$$(d^T (S \varepsilon))^2 = \left( d_I \varepsilon_I + \frac{d_{II} \varepsilon_{II}}{e^2} + \frac{4 d_{III} \varepsilon_{III}}{e^2} \right)^2$$

$$\leq d_I^2 \left( \varepsilon_I^2 + \frac{\varepsilon_{II}^2}{e^2} + \frac{4 \varepsilon_{III}^2}{e^2} \right)$$

$$+ \frac{1}{e^2} \left( d_{II}^2 + 4 d_{III}^2 \right) \left( \varepsilon_I^2 + \frac{\varepsilon_{II}^2}{e^2} + \frac{4 \varepsilon_{III}^2}{e^2} \right)$$

$$= \Delta^2 (d) \Delta^2 (\varepsilon). \quad (4.5)$$

Due to our assumptions on $v_0$, the function $\frac{P_0}{2 \Delta^2 (\varepsilon)^3}$ is real-valued, bounded, continuous and positively bounded from below by a constant $c_{\delta, \kappa, \alpha} > 0$. Thus, combining (4.4) and (4.5), for all $d \in \mathbb{R}^{2 \times 2}$, we obtain

$$\sum_{i,j,k,l=1}^{2} a_{ij}^{kl} d_{ik} d_{jl} = \frac{P_0}{2 \Delta^2 (\varepsilon)^3} \left( \Delta^2 (\varepsilon)^2 d^T S d - (d^T S \varepsilon)^2 \right) \geq c_{\delta, \kappa, \alpha} \delta \Delta^2 (d). \quad (4.6)$$

We can now verify condition (4.2). Given $\xi \in \mathbb{R}^2$ and $\eta \in \mathbb{C}^2$ with $|\xi| = |\eta| = 1$, set $\eta_i =: x_i + i y_i$ for $i = 1, 2$. Because of the symmetries of $(a_{kl}^{ij})$ as pointed out in (3.6) and using (4.6), we derive

$$\text{Re} (\mathcal{A}_H^H (x, \xi) \eta | \eta) = \text{Re} \sum_{i,j,k,l=1}^{2} a_{ij}^{kl} (\xi \otimes \eta)_{ji} (\overline{\xi} \otimes \overline{\eta})_{ik}$$

$$= \sum_{i,j,k,l=1}^{2} a_{ij}^{kl} (\xi \otimes x)_{ji} (\overline{\xi} \otimes x)_{ik} + a_{ij}^{kl} (\xi \otimes y)_{ji} (\overline{\xi} \otimes y)_{ik}$$

$$\geq c_{\delta, \kappa, \alpha} \delta \left( \Delta^2 (\xi \otimes x) + \Delta^2 (\xi \otimes y) \right).$$

Moreover, using $|\xi| = 1$,

$$\Delta^2 (\xi \otimes x) = (\xi \cdot x)^2 + \frac{1}{e^2} \|x\|^2,$$

so using $|\eta| = 1$,

$$\Delta^2 (\xi \otimes x) + \Delta^2 (\xi \otimes y) \geq \frac{1}{e^2},$$

and thus $\mathcal{A}_H^H (v_0)$ is strongly elliptic with an ellipticity constant $\geq \frac{c_{\delta, \kappa, \alpha} \delta}{e^2}$. 
To prove parameter–ellipticity of $A^H(v_0)$, we first note that due to (4.3), strong ellipticity implies normal ellipticity, and by symmetry of $A^H$, we conclude that \( \sigma(A^H)(x, \xi) \subset \mathbb{R}_+ \) is valid for every \( x \in \overline{\Omega} \) and \( \xi \in \mathbb{R}^2 \) with \( |\xi| = 1 \). This implies parameter–ellipticity of $A^H(v_0)$ with $\phi_{A^H(v_0)} = 0$. \( \square \)

The assertion of the following lemma will be crucial in the proof of the fact that the linearized Hibler operator $A^H(v_0)$ subject to Dirichlet boundary conditions satisfies the Lopatinskii–Shapiro condition.

**Lemma 4.2** Let $p, q \in (1, \infty)$ and $\mu \in (\frac{1}{p}, 1]$ such that (2.16) holds. For fixed $v_0 \in V_\mu$, let $a^{kl}_{ij}$ be the coefficients of the principal part of Hibler's operator $A^H(v_0)$ defined as in (3.5). Assume that $x \in \partial \Omega$, \( \xi, \nu \in \mathbb{R}^2 \) with \( |\xi| = |\nu| = 1 \) and \( (\xi | \nu) = 0 \) as well as $u, v \in \mathbb{C}^2$. Then,

\[
\text{Re} \left( \sum_{i,j,k,l=1}^{2} a^{kl}_{ij} (\xi_l u_j - \nu_l v_j)(\xi_k u_i - \nu_k v_i) \right) \geq 0 \quad \text{and} \quad (4.7)
\]

\[
\text{Re} \left( \sum_{i,j,k,l=1}^{2} a^{kl}_{ij} (\xi_l u_j - \nu_l v_j)(\xi_k v_i - \nu_k v_i) \right) > 0 \quad \text{provided} \quad \text{Im}(u|v) \neq 0. \quad (4.8)
\]

**Proof** Let $x \in \partial \Omega$, \( \xi, \nu \in \mathbb{R}^2 \) with \( |\xi| = |\nu| = 1 \) and \( (\xi | \nu) = 0 \) as well as $u, v \in \mathbb{C}^2$.

We introduce the notation $u_i = x_i + i y_i$ and $v_i = \tilde{x}_i + i \tilde{y}_i$, $i = 1, 2$. Using the symmetries of $a^{kl}_{ij}$ as in (3.6) and the estimate (4.6), we obtain

\[
\text{Re} \left( \sum_{i,j,k,l=1}^{2} a^{kl}_{ij} (\xi_l u_j - \nu_l v_j)(\xi_k u_i - \nu_k v_i) \right)
= \sum_{i,j,k,l=1}^{2} a^{kl}_{ij} (\xi_l x_j - \nu_l \tilde{x}_j)(\xi_k x_i - \nu_k \tilde{x}_i)
+ \sum_{i,j,k,l=1}^{2} a^{kl}_{ij} (\xi_l y_j - \nu_l \tilde{y}_j)(\xi_k y_i - \nu_k \tilde{y}_i)
\geq c_{\delta, x, \alpha} \delta \left( (\Delta^2(\xi \otimes x - \nu \otimes \tilde{x}) + \Delta^2(\xi \otimes y - \nu \otimes \tilde{y}) \right) \geq 0. \quad (4.9)
\]

Thus, condition (4.7) is satisfied. To verify condition (4.8), it remains to consider the case $= 0$ in the last line and deduce

\[
\text{Im}(u|v) = \tilde{x}_1 y_1 - x_1 \tilde{y}_1 + \tilde{x}_2 y_2 - x_2 \tilde{y}_2 = 0. \quad (4.10)
\]

For general $d \in \mathbb{R}^{2 \times 2}$, $\Delta^2(d) = 0$ implies $d_{11} = d_{22} = 0$, so from $= 0$ in (4.9), we obtain

\[
\xi_1 x_1 - \nu_1 \tilde{x}_1 = \xi_2 x_2 - \nu_2 \tilde{x}_2 = \xi_1 y_1 - \nu_1 \tilde{y}_1 = \xi_2 y_2 - \nu_2 \tilde{y}_2 = 0. \quad (4.11)
\]
Due to $|\xi| = 1$, either $\xi_1 \neq 0$ or $\xi_2 \neq 0$. Assume $\xi_1 \neq 0$. In view of $(\xi|\nu) = 0$ and $|\nu| = 1$, this implies $\nu_2 \neq 0$. Thus, from (4.11), we obtain

$$
x_1 = \frac{\nu_1}{\xi_1} x_1, \quad \tilde{x}_2 = \frac{\xi_2}{\nu_2} x_2, \quad y_1 = \frac{\nu_1}{\xi_1} \tilde{y}_1, \quad \tilde{y}_2 = \frac{\xi_2}{\nu_2} y_2.
$$

Plugging this into (4.10) immediately yields the claim. The case $\xi_2 \neq 0$ follows analogously. 

We proceed by showing that Hibler’s operator subject to Dirichlet boundary conditions fulfills the Lopatinskii–Shapiro condition. For the formulation of the latter condition in the context of parabolic boundary value problems subject to general boundary conditions, see e.g., Denk et al. (2003, 2004, 2007).

In our context of Hibler’s operator subject to Dirichlet boundary conditions, the Lopatinskii–Shapiro condition reads as follows: For all $x_0 \in \partial \Omega$, all $\xi \in \mathbb{R}^2$ with $(\xi, \nu(x_0)) = 0$, and all $\lambda \in \mathbb{C}$ satisfying $\text{Re} \lambda \geq 0$ and $|\xi| + |\lambda| \neq 0$, any solution $w \in C_0(\mathbb{R}_+; \mathbb{C}^2)$ of the ordinary differential equation in $\mathbb{R}_+$

$$
(\lambda + A_H^H(x_0, \xi - \nu(x_0)D_y))w(y) = 0, \quad y > 0,
$$

$$
w(0) = 0,
$$

equals zero.

**Proposition 4.3** Let $p, q \in (1, \infty)$ and $\mu \in (\frac{1}{p}, 1]$ such that (2.16) holds. Then, for fixed $v_0 \in V_\mu$, the principal part of Hibler’s operator $A_H(v_0)$ subject to homogeneous Dirichlet boundary conditions satisfies the Lopatinskii–Shapiro condition.

**Proof** Taking the inner product of the above equation with a solution $w$, we obtain

$$0 = \lambda \langle w(y) | w(y) \rangle + \langle A_H^H(x_0, \xi - \nu(x)D_y)w(y) | w(y) \rangle.
$$

Integrating over $\mathbb{R}_+$ and integrating by parts yields

$$
0 = \lambda \|w\|_2^2
$$

$$
+ \int_0^\infty \sum_{i,j,k,l=1}^2 d_{ijkl}^k (\xi_l - \nu_l(x)D_y) w_j(y) (\xi_k - \nu_k(x)D_y) w_i(y) dy.
$$

Our aim is to deduce from (4.13) that $w \equiv 0$ for each solution $w \in H^2_2(\mathbb{R}_+; \mathbb{C}^2)$ and thus for $w \in C^1_b(\mathbb{R}_+; \mathbb{C}^2)$.

Taking real parts in (4.13), we see by (4.7) that

$$
\int_0^\infty \text{Re} \sum_{i,j,k,l=1}^2 d_{ijkl}^k (\xi_l w_j(y) - \nu_l(x)D_y w_j(y))(\xi_k w_i(y) - \nu_k(x)D_y w_i(y)) dy = 0.
$$
Assuming \( \frac{d}{dy}|w(y)|^2 = 0 \) for all \( y > 0 \) yields that \( |w(y)| \) is constant on \( \mathbb{R}_+ \) and that consequently \( w(y) = 0 \) on \( \mathbb{R}_+ \).

We calculate
\[
\frac{d}{dy}|w(y)|^2 = 2 \text{Re} \left( \frac{d}{dy} w(y) w(y) \right) = -2 \text{Im} (D_y w(y) |w(y)|).
\]

Suppose now that there exists \( y_0 > 0 \) such that \( \frac{d}{dy}|w(y)|^2 \) does not vanish at \( y_0 \). Then, by smoothness of \( w \), there exists a neighborhood \( U \subset \mathbb{R}_+ \) of \( y_0 \) with \( \frac{d}{dy}|w(y)|^2 \neq 0 \) for all \( y \in U \). Then also \( \text{Im}(D_y w(y) |w(y)|) \neq 0 \) for all \( y \in U \). Setting \( u := D_y w(y) \in \mathbb{C}^2 \) and \( v := w(y) \in \mathbb{C}^2 \) for \( y \in U \) we see that \( \text{Im}(u, v) \neq 0 \) for all \( y \in U \) as well. Consequently, (4.8) and (4.7) yield
\[
\int_0^\infty \text{Re} \sum_{i,j,k,l=1}^2 a_{ij}^k l(w_j(y) - v_i(x) D_y w_j(y))(\xi_k w_i(y) - v_k(x) D_y w_i(y)) \, dy > 0.
\]

Combining this with relation (4.7) contradicts, however, condition (4.14). Thus, it holds that \( w \equiv 0 \).

We recall that for \( 1 < r < \infty \), the \( L_r \)-realization \( A^H_D(v_0) \) of \( A^H(v_0) \) subject to Dirichlet boundary conditions is given by
\[
[A^H_D(v_0)]u = [A^H(v_0)]u,
\]
\[
u \in D(A^H_D(v_0)) := \{u \in H^2_1(\Omega; \mathbb{R}^2) : u = 0 \text{ on } \partial\Omega \}.
\]

We will now prove the maximal \( L_s \)-regularity property for \( A^H_D(v_0) \) in the \( L_r \)-setting, where \( 1 < s, r < \infty \).

**Theorem 4.4** Let \( p, q, r, s \in (1, \infty) \) and \( \mu \in \left( \frac{1}{r}, 1 \right] \) such that (2.16) holds and let \( v_0 \in V_\mu \) be fixed. Then, there exists \( \omega_0 \in \mathbb{R} \) such that for all \( \omega > \omega_0 \)
(a) \( A^H_D(v_0) + \omega \) has the property of maximal \( L_s \{0, \infty) \)-regularity on \( L_r(\Omega; \mathbb{R}^2) \),
(b) \( A^H_D(v_0) + \omega \) admits a bounded \( H^\infty \)-calculus on \( L_r(\Omega; \mathbb{R}^2) \).

**Proof** By Proposition 4.1, for fixed \( v_0 \in V_\mu \), the principal part of Hibler’s operator \( A^H_D(v_0) \) is a parameter-elliptic operator with continuous and bounded coefficients on \( \overline{\Omega} \) having angle of ellipticity \( \phi_{A^H_D(v_0)} = 0 \). Furthermore, Proposition 4.3 tells us that the principal part of Hibler’s operator \( A^H_D(v_0) \) subject to homogeneous Dirichlet boundary conditions satisfies the Lopatinskii–Shapiro condition. Since the coefficients of the lower-order terms of \( A^H_D(v_0) \) are smooth, the first assertion follows from the results in Denk et al. (2003, 2007).

The second assertion follows by the results in Denk et al. (2004) provided the top-order coefficients of \( A^H_D(v_0) \) are Hölder continuous. The latter condition is satisfied due to the embedding \( B^2_{q, p} \left( \Omega \right) \hookrightarrow C^{1,\alpha} \left( \overline{\Omega} \right) \).

\( \square \)

### 5 Functional Analytic Properties of Hibler’s Operator

Hibler’s operator enjoys many interesting properties, some of which we collect in the following.
Proposition 5.1 Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with boundary of class $C^2$, $1 < p, q, r < \infty$, $\mu \in (\frac{1}{p}, 1]$ such that (2.16) is satisfied, and for $v_0 \in V_\mu$, let Hibler’s operator $A^H_D(v_0)$ on $L_r(\Omega; \mathbb{R}^2)$ with domain $D(A^H_D(v_0))$ be defined as in (4.15). Then,

(a) $-A^H_D(v_0)$ generates an analytic semigroup $e^{-tA^H_D(v_0)}$ on $L_r(\Omega; \mathbb{R}^2)$,
(b) $A^H_D(v_0)$ is an operator with compact resolvent,
(c) the spectrum $\sigma(A^H_D(v_0))$ of $A^H_D(v_0)$ viewed as an operator on $L_r(\Omega)^2$ is $r$-independent,
(d) for $\alpha \in (0, 1)$ and $\omega$ as in Theorem 4.4

\[ D((A^H_D(v_0) + \omega)^{\alpha}) \simeq [L_r(\Omega; \mathbb{R}^2), D(A^H_D(v_0))]_{\alpha} = \begin{cases} \{u \in H_r^{2\alpha}(\Omega) : u|_{\partial\Omega} = 0, \alpha \in (1/2r, 1], \\ H_r^{2\alpha}(\Omega) : \alpha \in [0, 1/2r), \end{cases} \]

(e) The Riesz transform $\nabla(A^H_D(v_0) + \omega)^{-1/2}$ of $A^H_D(v_0)$ is bounded on $L_r(\Omega)^2$.

Proof Assertions (a) follows by standard arguments. The compact embedding $D(A^H_D(v_0)) \hookrightarrow L_r(\Omega)^2$ implies that $A^H_D(v_0)$ has compact resolvent and that thus $\sigma(A^H_D(v_0))$ is independent of $r \in (1, \infty)$. Assertion (d) follows from the fact that $A^H_D(v_0) + \omega$ admits a bounded $H^\infty$-calculus on $L_r(\Omega)^2$ by Theorem 4.4. Finally, assertion (e) is obtained by noting that $D((A^H_D(v_0) + \omega)^{1/2}) \subset H_r^1(\Omega)$. \qed

6 Proof of Theorem 2.1

We recall from Sect. 2 that the ground space $X_0$ for $q \in (1, \infty)$ is given by

\[ X_0 = L_q(\Omega; \mathbb{R}^2) \times L_q(\Omega) \times L_q(\Omega) =: X_0^u \times X_0^h \times X_0^a. \]

The regularity space $X_1$ is defined as

\[ X_1 = \{u \in H_q^2(\Omega; \mathbb{R}^2) : u = 0 \text{ on } \partial\Omega\} \times \{h \in H_q^2(\Omega) : \partial_v h = 0 \text{ on } \partial\Omega\} \times \{a \in H_q^2(\Omega) : \partial_v a = 0 \text{ on } \partial\Omega\}. \]

Since we are considering solutions within the class

\[ v \in H_{p, \mu}^1(J; X_0) \cap L_p(J; X_1), \]

where $J = (0, T)$ with $0 < T \leq \infty$ is an interval and $\mu \in (1/p, 1]$ indicates a time weight, the time trace space of this class is given by

\[ X_{\gamma, \mu} = (X_0, X_1)_{\mu-1/p, p} = D_{\Delta_N}(\mu - 1/p, p) \times D_{\Delta_N}(\mu - 1/p, p) \times D_{\Delta_N}(\mu - 1/p, p) =: X^u_{\gamma, \mu} \times X^h_{\gamma, \mu} \times X^a_{\gamma, \mu} \] (6.1)
provided \( p \in (1, \infty) \) and \( \mu \in (1/p, 1] \). Note that
\[
X_{\gamma, \mu} \hookrightarrow B_{q_p}^{2(\mu-1/p)}(\Omega)^4 \hookrightarrow C^1(\overline{\Omega})^4
\]
provided (2.16) is satisfied.

For \( \omega \) as in Theorem 4.4, we now consider the operator \( A_{\omega}(v_0) \) on \( X_0 \) with domain \( X_1 \) given by the upper triangular matrix
\[
A_{\omega}(v_0) = \begin{pmatrix}
\frac{1}{\rho_{\text{ice}} h_0} (A_{D}^H(v_0) + \omega) & \frac{1}{2 \rho_{\text{ice}} h_0} \nabla \cdot \frac{\partial h P(h_0, a_0)}{\partial h} \\
0 & -d_h \Delta_N \\
0 & -d_a \Delta_N
\end{pmatrix}, \tag{6.2}
\]
as well as
\[
F_{\omega}(v) = F(v) + \frac{1}{\rho_{\text{ice}} h} (\omega, 0, 0)^T, \tag{6.3}
\]
where \( F \) is given as in (2.13).

**Lemma 6.1** Let \( p, q \in (1, \infty), \mu \in (1/p, 1] \) such that (2.16) is satisfied and assume that \( v_0 = (u_0, h_0, a_0) \in V_\mu \). Let \( J = [0, T) \) for some \( 0 < T < \infty \). Then, \( A_{\omega}(v_0) \) has maximal \( L_p(J) \)-regularity on \( X_0 \).

**Proof** By assumption, we have \( h_0 \geq \kappa \) for some \( \kappa > 0 \) and (2.16) implies that \( 1/h_0 \in C^1(\overline{\Omega}) \). Since \( \Delta_N \) as well as \( A_{D}^H(v_0) + \omega \) and \( \frac{1}{\rho_{\text{ice}}}(A_{D}^H(v_0) + \omega) \) have the maximal \( L_p(J) \)-regularity property on \( X_0 \) by Theorem 4.4, the upper triangular structure of \( A_{\omega}(v_0) \) implies that also \( A_{\omega}(v_0) \) has the maximal \( L_p(J) \)-regularity property on \( X_0 \).

We now show that \( (A_{\omega}, F_{\omega}) \in C^1-(V_\mu; L(X_1 \times X_0)) \) for \( \mu \in (1/p, 1] \) satisfying (2.16) and where \( A \) and \( F \) are defined as in (6.2) and (6.3). Recall that \( V_\mu \) is an open subset of \( X_{\gamma, \mu} \) such that all \( (u, h, a) \in V_\mu \) satisfy \( h \geq \kappa \) for some \( \kappa > 0 \).

**Lemma 6.2** Let \( p, q \in (1, \infty), \mu \in (1/p, 1] \) such that (2.16) is satisfied. Suppose that \( A_{\omega} \) and \( F_{\omega} \) are defined as in (6.2) and (6.3) and let \( v_0 = (u_0, h_0, a_0) \in V_\mu \). Then, there exists \( r_0 > 0 \) and a constant \( L > 0 \) such that \( B_{X_{\gamma, \mu}}(v_0, r_0) \subset V_\mu \) and
\[
\| A_{\omega}(v_1) w - A_{\omega}(v_2) w \|_{X_0} \leq L \| v_1 - v_2 \|_{X_{\gamma, \mu}} \| w \|_{X_1},
\]
\[
\| F_{\omega}(v_1) - F_{\omega}(v_2) \|_{X_0} \leq L \| v_1 - v_2 \|_{X_{\gamma, \mu}}.
\]
for all \( v_1, v_2 \in B_{X_{\gamma, \mu}}(v_0, r_0) \) and all \( w \in X_1 \).
Choose \( r_0 > 0 \) small enough such that \( v_1, v_2 \in \overline{B}_{X_{γ,μ}}(v_0, r_0) \subset V_μ \). For \( w = (u, h, a) \in X_1 \), we then obtain

\[
\|A_ω(v_1)w - A_ω(v_2)w\|_{X_0} = \left\| \frac{1}{ρ_{cc} h_1} [A_0^H (\nabla u_1, P(h_1, a_1))u + ωu - δ_h P(h_1, a_1)∇h - δ_a P(h_1, a_1)∇a] \right. \\
\left. - \frac{1}{ρ_{cc} h_2} [A_0^H (\nabla u_2, P(h_2, a_2))u + ωu - δ_h P(h_2, a_2)∇h - δ_a P(h_2, a_2)∇a] \right\|_{L_q(Ω; \mathbb{R}^2)} \\
\leq L\|v_1 - v_2\|_{C^1(\|D_i D_j u\|_q + \|u\|_q)} + L\|(h_1, a_1)^T - (h_2, a_2)^T\|_∞(∥(∇h\|_q + ∥∇a\|_q) ≤ L\|v_1 - v_2\|_{X_{γ,μ}}\|w\|_{X_1}.
\]

To prove the assertion for \( F_ω \), we start with the convective term \( u \nabla u \). Hölder’s inequality and the embedding \( X_{γ,μ} \hookrightarrow L_s \cap H^1_r \) for \( s = qr \) and \( s = qr' \) imply

\[
\|u_1 \nabla u_1 - u_2 \nabla u_2\|_{L_q} \leq \|u_1 - u_2\|_{L_{qr'}} \|u_1\|_{H^1_{qr'}} + \|u_1 - u_2\|_{H^1_{qr'}} \|u_2\|_{L_{qr}} \\
\leq 2Cr_0\|u_1 - u_2\|_{X_{γ,μ}^r}.
\]

A similar argument shows that

\[
\|\text{div}(u_1 h_1) - \text{div}(u_2 h_2)\|_{X_0^h} \leq C\|v_1 - v_2\|_{X_{γ,μ}} \quad \text{and} \\
\|\text{div}(u_1 a_1) - \text{div}(u_2 a_2)\|_{X_0^a} \leq C\|v_1 - v_2\|_{X_{γ,μ}}.
\]

Furthermore, note that \( τ_{atm} \) is constant in \( v \), and thus \( \frac{τ_{atm}}{ρ_{cc} h_0} \) is Lipschitz continuous in \( v \). Concerning \( τ_{ocean} \), we may assume that \( U_{atm} = 0 \) (otherwise consider \( u + U_{atm} \)). It thus suffices to show that \( v \mapsto \frac{1}{h} u|u| \) is Lipschitz continuous viewed as a mapping from \( V_μ \) to \( L_q \). The term \( \frac{1}{ρ_{cc} h_0} ωu \) is treated in the same way.

Finally, we consider the terms \( S_h \) and \( S_a \) defined as in (2.8) and (2.9), respectively. By assumption, \( f \in C^1 \) and hence \( S_h \) as well as \( S_a \) are Lipschitz continuous in \( v \). □

The assertion of Theorem 2.1 follows hence by the local existence theorem for quasilinear evolution equation as described in Thm. 5.1.1 in Prüss and Simonett (2016).

7 Proof of Theorem 2.3

Throughout this section, we consider \( p, q \in (1, ∞) \) and \( μ \in (\frac{1}{p}, 1] \) such that (2.16) holds. Moreover, we abbreviate \( P(h_*, a_*) \) by \( P_* \), i.e.,

\[
P_* = p^h h_* \exp(-c(1 - a_*)).
\]

We study equilibria in the case that no external forces are present in the momentum equation, i.e.,

\[
-g \nabla H = \frac{c_1}{h} |U_{atm}| U_{atm} = \frac{c_2}{h} |U_{ocean} - u|(U_{ocean} - u) = 0,
\]
and neglect external freezing and melting effects by setting

\[ S_h = S_a = 0. \]

For \( A \) as in (2.12) and the simplified semilinear right-hand side \( F_s \) given by

\[
F_s(v) = \begin{pmatrix}
-u \cdot \nabla u - c_{\text{cor}} (n \times u) \\
- \text{div}(uh) \\
- \text{div}(ua)
\end{pmatrix},
\]

we prove similarly as in Sect. 6 that there is an open set \( V \subset V_\mu \subset X_{\gamma,\mu} \) such that

\[
(A, F_s) \in C^1(V, \mathcal{L}(X_1, X_0) \times X_0). \tag{7.1}
\]

We denote by \( E \subset V \cap X_1 \) the set of equilibrium solutions of

\[
v' + A(v)v = F_s(v), \quad t > 0, \quad v(0) = v_0. \tag{7.2}
\]

An equilibrium solution \( v \in E \) is characterized by \( v \in V \cap X_1 \) and \( A(v)v = F_s(v) \).

For \( h_* \geq \kappa \) and \( a_* \geq 0 \) constant in time and space, \( v_* = (0, h_*, a_*) \in V \cap X_1 \) is an equilibrium solution of (7.2) due to \( A(v_*)v_* = 0 = F_s(v_*) \).

To prove Theorem 2.3, we aim to apply the generalized principle of linearized stability, see Prüss et al. (2009) or Prüss and Simonett (2016). Note that we already verified that \( v_* \in V \cap X_1 \) is an equilibrium of (7.2) and that \( (A, F_s) \) satisfy (7.1).

Consider next the linearization of (7.2) at \( v_* \) which reads as

\[
A_0 v = A(v_*)v + (A'(v_*)v)v_* - F'_s(v_*)v, \quad v \in X_1.
\]

Computing \( A_0 v \), we see first that \( A(v_*)v \) is given by

\[
A(v_*)v = \begin{pmatrix}
\frac{1}{\rho_{\text{ice}}h_*} A^H(v_*)u + \frac{\partial h}{\partial \rho_{\text{ice}}h_*} \nabla h + \frac{\partial a}{\partial \rho_{\text{ice}}h_*} \nabla a \\
-d_h \Delta_N h \\
-d_a \Delta_N a
\end{pmatrix},
\]

where

\[
(A^H(v_*)u)_i = -\frac{P_*}{2\delta^{1/2}} \sum_{j,k,l=1}^2 \mathcal{G}_{ijkl} \partial_i h \partial_j u. \]

Secondly, we deduce that \( (A'(v_*)v)v_* = 0 \) for all \( v \in X_1 \) and that

\[
F'_s(v_*)v = \begin{pmatrix}
-c_{\text{cor}} (n \times u) \\
-h_* \text{div}(u) \\
-a_* \text{div}(u)
\end{pmatrix}.
\]
The linearization $A_0$ hence becomes

$$A_0 v = A(v_*) v - F'_*(v_*)v = \left( \frac{1}{\rho_{ice} h_*} A_D^D(v_*)u + \frac{\partial_h P_*}{2 \rho_{ice} h_*} \nabla h + \frac{\partial_a P_*}{2 \rho_{ice} h_*} \nabla a - c_{\text{cor}} (n \times u) \right).$$

**Lemma 7.1** If $v_*$ is as above, then there exists $\delta_* > 0$ such that $\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+$ holds for all $0 < \delta < \delta_*$. Furthermore, $0$ is a semi-simple eigenvalue of $A_0$ and $N(A_0)$ has dimension 2.

**Proof** To locate the spectrum of $A_0$, we test the equation $(\lambda + A_0)v = 0$ by $v = (u, h, a)$ and use integration by the vector parts, which leads to

$$0 = \lambda \| v \|^2_{L^2(\Omega)^4} + \frac{1}{\rho_{ice} h_*} \int_{\Omega} A_D^D(v_*) u \cdot u \, dx$$

$$+ \frac{\partial_h P_*}{2 \rho_{ice} h_*} \int_{\Omega} \nabla h \cdot u \, dx + \frac{\partial_a P_*}{2 \rho_{ice} h_*} \int_{\Omega} \nabla a \cdot u \, dx$$

$$+ h_* \int_{\Omega} h \, \text{div} (u) \, dx + d_h \| \nabla h \|^2_{L^2(\Omega)^2} + a_* \int_{\Omega} a \, \text{div} (u) \, dx + d_a \| \nabla a \|^2_{L^2(\Omega)^2}. \quad (7.3)$$

Thanks to the Dirichlet boundary condition for $u$ and employing (4.4) as well as Korn’s and Poincaré’s inequality, we deduce that

$$\frac{1}{\rho_{ice} h_*} \int_{\Omega} A_D^D(v_*) u \cdot u \, dx$$

$$= - \frac{P_*}{2 h_* \rho_{ice} \sqrt{\delta}} \int_{\Omega} \varepsilon_{ij} \partial_k \partial_l u_j u_i \, dx = \frac{P_*}{2 h_* \rho_{ice} \sqrt{\delta}} \int_{\Omega} \varepsilon_{ij} \partial_k u_j \partial_l u_i \, dx$$

$$= \frac{P_*}{2 h_* \rho_{ice} \sqrt{\delta}} \int_{\Omega} \Delta^2 (\nabla u) \, dx \geq \frac{2}{h_* \rho_{ice} \sqrt{\delta}} \frac{\| \varepsilon(u) \|^2}{\varepsilon} \geq \frac{C_*}{\sqrt{\delta}} \| u \|_{H^1}^2 \quad (7.4)$$

for some constant $C_* > 0$ independent of $\delta$ and $u$. Now, the remaining terms in (7.3) can be absorbed: First determine $\gamma_h, \gamma_a > 0$ depending in particular on $h_*, a_*, d_h$ and $d_a$ such that

$$\frac{\partial_h P_*}{2 \rho_{ice} h_*} \int_{\Omega} \nabla h \cdot u \, dx + h_* \int_{\Omega} h \, \text{div} (u) \, dx = \left( \frac{\partial_h P_*}{2 \rho_{ice} h_*} - h_* \right) \int_{\Omega} \nabla h \cdot u \, dx$$

$$\geq - \frac{d_h}{2} \| \nabla h \|^2_{L^2(\Omega)^2} - \gamma_h \| u \|_{L^2(\Omega)^2},$$

and, similarly, such that

$$\frac{\partial_a P_*}{2 \rho_{ice} h_*} \int_{\Omega} \nabla a \cdot u \, dx + a_* \int_{\Omega} a \, \text{div}(u) \, dx \geq - \frac{d_a}{2} \| \nabla a \|^2_{L^2(\Omega)^2} - \gamma_a \| u \|_{L^2(\Omega)^2}.$$
Then, choose \( \delta_* > 0 \) sufficiently small to ensure that \( \gamma_h + \gamma_a < \frac{C_*}{\sqrt{\delta_*}} \). In particular, this implies that for all \( \delta < \delta_* \), there exists \( C_\delta > 0 \) such that

\[
0 \geq \lambda \|v\|_{L^2(\Omega)}^2 + C_\delta \left( \|u\|_{H^1(\Omega)}^2 + \|\nabla h\|_{L^2(\Omega)}^2 + \|\nabla a\|_{L^2(\Omega)}^2 \right).
\] (7.5)

The relation in (7.5) can only hold provided that \( \lambda \) is real and that \( \lambda \leq 0 \). Hence, \( \sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+ \). For \( \lambda = 0 \), we infer that \( u = 0 \) and \( h \) as well as \( a \) are constant. This implies that \( 0 \) is a semi-simple eigenvalue of \( A_0 \) and that \( N(A_0) \) has dimension 2.

\[\square\]

**Lemma 7.2** Near \( v_* \), the set of equilibria \( \mathcal{E} \) is a \( C^1 \)-manifold in \( X_1 \), and the tangent space of \( \mathcal{E} \) at \( v_* \) is isomorphic to \( N(A_0) \).

**Proof** Consider equilibria \( v \in V \cap X_1 \) such that \( \|v - v_*\|_{X^1} < r \) for given \( r > 0 \). The resulting equation for such \( v \) is

\[
0 = \left( A_D^H(v)u + \frac{\nabla P(v)}{2} + \rho_{\text{ice}} h u \cdot \nabla u - c_{\text{cor}} (n \times u) \right) \div(uh) - d_h \Delta_N h
+ \left( \div(ua) - d_a \Delta_N a \right).
\]

We set the constants

\[
C_h := \frac{p^* \exp(-c(1-a_*))}{2h_*} \quad \text{and} \quad C_a := \begin{cases} \frac{c p_* h_* \exp(-c(1-a_*))}{2a_*}, & \text{if } a_* > 0, \\ 1, & \text{if } a_* = 0, \end{cases}
\] (7.6)

and test the above equation with \( (u, Ch h, C_a a) \) to obtain

\[
0 = \int_\Omega A_D^H(v)u \cdot u \, dx + \int_\Omega \frac{\nabla P(v)}{2} \cdot u \, dx
+ \int_\Omega \rho_{\text{ice}} h(u \cdot \nabla u) \cdot u \, dx + C_h \int_\Omega \div(uh) h \, dx
+ d_h C_h \|\nabla h\|_{L^2(\Omega)}^2 + C_a \int_\Omega \div(ua) a \, dx + d_a C_a \|\nabla a\|_{L^2(\Omega)}^2.
\] (7.7)

Using the symmetry of \( S \), the estimate \( P(v) \geq P_* \kappa \exp(-c(1-\alpha)) \), the estimate \( \Delta_2^2(\varepsilon) \leq C \varepsilon r \) and Korn’s and Poincare’s inequalities, the first term on the right-hand side satisfies

\[
\int_\Omega A_D^H(v)u \cdot u \, dx = - \int_\Omega \div \left( \frac{P}{2} \frac{S \varepsilon}{\Delta_2(\varepsilon)} \right) \cdot u \, dx = \int_\Omega \frac{P}{2} \frac{S \varepsilon}{\Delta_2(\varepsilon)} \, dx \\
\geq \frac{C_V}{\sqrt{\delta + C r}} \|u\|_{H^1(\Omega)}^2
\]

for some constant \( C_V > 0 \) independent of \( \delta, r \) and \( v \). We show how terms without sign in (7.7) can now be absorbed. We first discuss the case \( a_* \neq 0 \) and remark on the
case \( a_* = 0 \) below. First note that using \( \| v - v_\# \|_{X_{\gamma,\mu}} < r \) and any bound on \( r > 0 \),

\[
\int_{\Omega} \rho_{\text{ice}} h(u \cdot \nabla u) \cdot u \, dx \leq C_* r \| u \|_{H^1(\Omega)},
\]

for a suitable constant \( C_* > 0 \) that is independent of \( \delta, r \) and \( v \). Secondly, we calculate

\[
\int_{\Omega} \frac{\nabla P(v)}{2} \cdot u \, dx = \int_{\Omega} \left( \frac{\partial_h P}{2} - C_h h \right) \nabla h \cdot u \, dx + C_h \int_{\Omega} h \nabla h \cdot u \, dx + C_a \int_{\Omega} a \nabla a \cdot u \, dx \tag{7.8}
\]

and use that part of this expression cancels with the terms

\[
C_h \int_{\Omega} \text{div}(uh)h \, dx = -C_h \int_{\Omega} h(\nabla h \cdot u) \, dx \quad \text{as well as} \quad C_a \int_{\Omega} \text{div}(ua)a \, dx
\]

\[
= -C_a \int_{\Omega} a(\nabla a \cdot u) \, dx
\]

in (7.7). It remains to check that due to the particular choice of \( C_h, C_a \) in (7.6), we find that for a (possibly increased) constant \( C_* > 0 \),

\[
\| \frac{\partial_h P}{2} - C_h h \|_\infty \leq \frac{p^*}{2} \left( \| \exp(ca) - \exp(ca_\#) \|_\infty + h_\# \exp(-c(1-a_\#)) \| h - h_\# \|_\infty \right) \leq C_* r
\]

and similarly

\[
\| \frac{\partial_a P}{2} - C_a a \|_\infty \leq C_* r,
\]

and hence the terms

\[
\int_{\Omega} \left( \frac{\partial_h P}{2} - C_h h \right) \nabla h \cdot u \, dx \geq -C_* r \left( \| \nabla h \|^2_{L^2(\Omega)^2} + \| u \|^2_{L^2(\Omega)^2} \right)
\]

and

\[
\int_{\Omega} \left( \frac{\partial_a P}{2} - C_a a \right) \nabla a \cdot u \, dx \geq -C_* r \left( \| \nabla a \|^2_{L^2(\Omega)^2} + \| u \|^2_{L^2(\Omega)^2} \right)
\]

are controlled.

In summary, inserting the above estimates into Eq. (7.7), we conclude that

\[
0 \geq \left( \frac{C_v}{\sqrt{\delta + C_{fr}}} - C_* r \right) \| u \|^2_{H^1(\Omega)^2} + \left( d_h C_h - C_* r \right) \| \nabla h \|^2_{L^2(\Omega)^2} + \left( d_a C_a - C_* r \right) \| \nabla a \|^2_{L^2(\Omega)^2}.
\]
Hence, if \( r > 0 \) is sufficiently small, then (7.7) implies
\[
0 \geq \| u \|^2_{H^1(\Omega)^2} + \| \nabla h \|^2_{L^2(\Omega)^2} + \| \nabla a \|^2_{L^2(\Omega)^2}.
\] (7.9)
This shows that for \( v = (u, h, a) \in V_\ast \) with \( \| v - v_\ast \|_{X_{\gamma, \mu}} < r \), we have \( u = 0 \) and \( h \) as well as \( a \) must be constant. In particular, \( \mathcal{E} = N(A_0) \) is valid in a neighborhood of \( v_\ast \).

The case \( a_\ast = 0 \) can be included by a slight adjustment of the argument. Replace (7.8) by
\[
\int \nabla P(v) \cdot a \, dx = \int \left( \frac{\partial h}{2} - C_h h \right) \nabla v \cdot a \, dx + C_h \int h \nabla h \cdot a \, dx + \int \frac{\partial a}{2} P \nabla a \cdot a \, dx,
\]
and directly estimate
\[
\int \frac{\partial a}{2} P \nabla a \cdot a \, dx \geq -C_a \| a \|_\infty \int \nabla a \cdot a \, dx \geq -C_a r \left( \| \nabla a \|^2_{L^2(\Omega)^2} + \| a \|^2_{L^2(\Omega)^2} \right)
\]
as well as
\[
\int \text{div}(ua) a \, dx = -\int a (\nabla a \cdot u) \, dx \geq -C_a r \left( \| \nabla a \|^2_{L^2(\Omega)^2} + \| a \|^2_{L^2(\Omega)^2} \right),
\]
to conclude as before.

\[ \Box \]

**Lemma 7.3** For \( v_\ast \) as above, \( A(v_\ast) \) has the property of maximal \( L_s \)-regularity on \( L_r(\Omega; \mathbb{R}^2) \).

**Proof** We know from Theorem 4.4 that there is \( \omega_0 \in \mathbb{R} \) such that \( A^H_D(v_\ast) + \omega \) has the maximal \( L_s \)-regularity on \( L_r(\Omega; \mathbb{R}^2) \) for all \( \omega > \omega_0 \). Considering the eigenvalue equation for \( A^H_D(v_\ast) \) it follows by (7.4) that
\[
0 = \lambda \| u \|^2 + \frac{1}{\rho \cdot \mu} \int \frac{A^H_D(v_\ast)}{\mu} \cdot u \, dx \geq \lambda \| u \|^2 + \frac{C}{\rho \cdot \mu} \| u \|^2_{H^1}.
\]
Thus, \( s(-A^H_D(v_\ast)) < 0 \) and \( A^H_D(v_\ast) \) is invertible in \( L^2(\Omega)^2 \). Due to compact embeddings, \( A^H_D(v_\ast) \) has compact resolvent and hence the spectrum of \( A^H_D(v_0) \) is \( r \)-independent, and we see that \( s(-A^H_D(v_\ast)) = s(-A^H_D(v_0)) \) < 0. It can be shown that \( \omega_0 \) can be chosen to be equal to the spectral bound \( s(-A^H_D(v_\ast)) \) of \( A^H_D(v_\ast) \), i.e., \( \omega_0 = s(-A^H_D(v_\ast)) \), which implies that \( A^H_D(v_\ast) \) has the maximal \( L_s \)-regularity on \( L_r(\Omega; \mathbb{R}^2) \). The triangular structure of \( A(v_\ast) \) implies that this the latter property holds also for \( A(v_\ast) \).

Summarizing we see that Lemmas 7.1, 7.2 and 7.3 imply that the assumptions of the principle of linearized stability described as in Prüss et al. (2009) or Prüss and Simonett (2016) are fulfilled. The assertion of Theorem 2.3 follows thus by this principle.

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