STRICTLY ERGODIC MODELS AND THE CONVERGENCE OF NON-CONVENTIONAL POINTWISE ERGODIC AVERAGES

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Abstract. The well-known Jewett-Krieger’s Theorem states that each ergodic system has a strictly ergodic model. Strengthening the model by requiring that it is strictly ergodic under some group actions, and building the connection of the new model with the convergence of pointwise non-conventional ergodic averages we prove that for an ergodic system \((X, \mathcal{X}, \mu, T)\), \(d \in \mathbb{N}\), \(f_1, \ldots, f_d \in L^\infty(\mu)\), the averages

\[
\frac{1}{N^2} \sum_{(n,m) \in F_N} f_1(T^n x)f_2(T^{n+m} x)\ldots f_d(T^{n+(d-1)m} x)
\]

converge \(\mu\) a.e. We remark that the same method can be used to show the pointwise convergence of ergodic averages along cubes which was firstly proved by Assani and then extended to a general case by Chu and Franzikinakis.

1. Introduction

In the introduction we will state the main results of the paper and give some backgrounds.

1.1. Main results. Throughout this paper, by a topological dynamical system (t.d.s. for short) we mean a pair \((X, T)\), where \(X\) is a compact metric space and \(T\) is a homeomorphism from \(X\) to itself. A measurable system (m.p.t. for short) is a quadruple \((X, \mathcal{X}, \mu, T)\), where \((X, \mathcal{X}, \mu)\) is a Lebesgue probability space and \(T : X \to X\) is an invertible measure preserving transformation.

Let \((X, \mathcal{X}, \mu, T)\) be an ergodic m.p.t. We say that \((\hat{X}, T)\) is a topological model (or just a model) for \((X, \mathcal{X}, \mu, T)\) if \((\hat{X}, T)\) is a t.d.s. and there exists an invariant probability measure \(\hat{\mu}\) on the Borel \(\sigma\)-algebra \(\mathcal{B}(\hat{X})\) such that the systems \((X, \mathcal{X}, \mu, T)\) and \((\hat{X}, \mathcal{B}^{(d)}(\hat{X}), \hat{\mu}, T)\) are measure theoretically isomorphic.

The well-known Jewett-Krieger’s theorem [28, 29] states that every ergodic system has a strictly ergodic model. We note that one can add some additional properties to the topological model. For example, in [30] Lehrer showed that the strictly ergodic model can be required to be a topological (strongly) mixing system in addition.

Let \((\hat{X}, T)\) be a t.d.s. Write \((x, \ldots, x) \ (2^d \text{ times})\) as \(x^{[d]}\). Let \(F^{[d]}, G^{[d]}\) and \(Q^{[d]}(\hat{X})\) be the face group of dimension \(d\), the parallelepiped group of dimension \(d\) and the
dynamical parallelepiped of dimension \( d \) respectively (see Section 2 for definitions). The orbit closure of \( x^{[d]} \) under the face group action will be denote by \( F^{[d]}(x^{[d]}) \). It was shown by Shao and Ye [34] that if \((\hat{X}, T)\) is minimal then \((\hat{F}^{[d]}(x^{[d]}), F^{[d]}(x^{[d]}) \) is minimal for all \( x \in \hat{X} \) and \((\hat{Q}^{[d]}(\hat{X}), G^{[d]}(\hat{X})) \) is minimal.

In this paper we will strengthen Jewett-Krieger’s theorem in another direction. Namely, we have the following Theorem A and Theorem B.

**Theorem A:** Let \((X, \mathcal{X}, \mu, T)\) be an ergodic m.p.t. and \( d \in \mathbb{N} \). Then

1. it has a strictly ergodic model \((\hat{X}, T)\) such that \((\hat{F}^{[d]}(x^{[d]}), F^{[d]}(x^{[d]}) \) is strictly ergodic for all \( x \in \hat{X} \).
2. it has a strictly ergodic model \((\hat{X}, T)\) such that \((\hat{Q}^{[d]}(\hat{X}), G^{[d]}(\hat{X})) \) is strictly ergodic.

Now let \( \tau_d = T \times \ldots \times T \) \((d \text{ times})\) and \( \sigma_d = T \times \ldots \times T^d \). The group generated by \( \tau_d \) and \( \sigma_d \) is denoted \( \langle \tau_d, \sigma_d \rangle \). For any \( x \in \hat{X} \), let \( N_d(\hat{X}, x) = \mathcal{O}((x, \ldots, x), \langle \tau_d, \sigma_d \rangle) \), the orbit closure of \((x, \ldots, x) \) \((d \text{ times})\) under the action of the group \( \langle \tau_d, \sigma_d \rangle \). We remark that if \((\hat{X}, T)\) is minimal, then all \( N_d(\hat{X}, x) \) coincide, which will be denoted by \( N_d(\hat{X}) \). It was shown by Glasner [17] that if \((\hat{X}, T)\) is minimal, then \((N_d(\hat{X}), \langle \tau_d, \sigma_d \rangle) \) is minimal.

**Theorem B:** Let \((X, \mathcal{X}, \mu, T)\) be an ergodic m.p.t. and \( d \in \mathbb{N} \). Then it has a strictly ergodic model \((\hat{X}, T)\) such that \((N_d(\hat{X}), \langle \tau_d, \sigma_d \rangle) \) is strictly ergodic.

We note that we have formulas to compute the unique measure in Theorems A and B. Particularly, when \((X, \mathcal{X}, \mu, T)\) is weakly mixing, the unique measure is nothing but the product measure. Moreover, for small \( d \) we also have explicit description of the unique measure.

Surprisingly, Theorems A and B are closely related the pointwise convergence of non-conventional multiple ergodic averages. That is, we can show Theorems C and D as applications of Theorems A and B respectively.

**Theorem C:** Let \((X, \mathcal{X}, \mu, T)\) be an ergodic m.p.t. and \( d \in \mathbb{N} \). Then

1. for functions \( f_\epsilon \in L^\infty(\mu), \epsilon \in \{0, 1\}^d, \epsilon \neq (0, \ldots, 0), \) the averages

\[
\frac{1}{N^d} \sum_{n \in \{0, 1, \ldots, N-1\}^d} \prod_{\epsilon \neq (0, \ldots, 0) \in \{0, 1\}^d} f_\epsilon(T^{n+\epsilon}x)
\]

converge \( \mu \) a.e..

2. for functions \( f_\epsilon \in L^\infty(\mu), \epsilon \in \{0, 1\}^d, \) the averages

\[
\frac{1}{N^{d+1}} \sum_{n \in \{0, 1, \ldots, N-1\}^d} \prod_{\epsilon \in [d]} f_\epsilon(T^{n+\epsilon}x)
\]

converge \( \mu \) a.e..
Theorem D: Let $(X, \mathcal{X}, \mu, T)$ be an ergodic m.p.t. and $d \in \mathbb{N}$. Then for $f_1, \ldots, f_d \in L^\infty(\mu)$ the averages

\begin{equation}
\frac{1}{N^2} \sum_{n,m \in [0,N-1]} f_1(T^n x) f_2(T^{n+m} x) \ldots f_d(T^{n+(d-1)m} x)
\end{equation}

converge $\mu$ a.e.

As we said above we have formulas to compute the limits. For example the limit is Theorem D is $\int_{N_d(X)} \otimes f_i d\lambda_{\tau,\sigma,d}$, where $\lambda_{\tau,\sigma,d}$ is defined in (5.16) and we assume $(X, \mathcal{X}, \mu, T)$ itself is the model defined in Theorem B.

1.2. Backgrounds. In this subsection we will give backgrounds of our research.

1.2.1. Topological model. The pioneering work on topological model was done by Jewett in [28]. He proved the theorem under the additional assumption that $T$ is weakly mixing and conjectured that if the condition of being weakly mixing is replaced by that of being ergodic, the theorem would still be valid. Jewett’s conjecture was proved by Krieger in [29] soon. This was followed by the papers of Hansel and Raoult [21] and Denker [10], giving different proofs of the theorem in the general ergodic case (see also [11]). Bellow and Furstenberg [4] showed how with an additional piece of information the Key Lemma in Jewett’s paper – and hence Jewett’s whole proof – carries over to the general ergodic case. One can add some additional properties to the topological model. For example, in [30] Lehrer showed that the strictly ergodic model can be required as a topological (strongly) mixing system in addition. Our Theorems A and B strengthen Jewett-Krieger Theorem in other direction, i.e. we can require the model to be well behavioral under some group actions.

It is well known that each m.p.t. has a topological model [15]. There are universal models, models for some group actions and models for some special classes. Weiss [38] showed the following nice result: There exists a minimal t.d.s. $(X, T)$ with the property that for every aperiodic ergodic m.p.t. $(Y, \mathcal{Y}, \nu, S)$ there exists a $T$-invariant Borel probability measure $\mu$ on $X$ such that the systems $(Y, \mathcal{Y}, \nu, S)$ and $(X, \mathcal{B}(X), \mu, T)$ are measure theoretically isomorphic. Note that there exists universal model for all ergodic m.p.t. with entropy less than or equal to a given number $t > 0$ [32] and it is interesting that there is no such a model for zero entropy m.p.t. [33]. Weiss [37] showed that Jewett-Krieger Theorem can be generalized from $\mathbb{Z}$-actions to commutative group actions (in [37] there is only an outline of a proof, and the exposition of his proof can be found in [40], more details can be found in [18, 20]). An ergodic system has a doubly minimal model if and only if it has zero entropy [39] (other topological models for zero entropy systems can be found in [22, 12]); and an ergodic system has a strictly ergodic, UPE (uniform positive entropy) model if and only if it has positive entropy [19].

We say that $\hat{\pi} : \hat{X} \rightarrow \hat{Y}$ is a topological model for $\pi : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, T)$ if $\hat{\pi}$ is a topological factor map and there exist measure theoretical isomorphisms $\phi$
and $\psi$ such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & \hat{X} \\
\pi & \downarrow & \downarrow \hat{\pi} \\
Y & \xrightarrow{\psi} & \hat{Y}
\end{array}
\]

is commutative, i.e. $\hat{\pi}\phi = \psi\pi$. Weiss \[37\] generalized the theorem of Jewett-Krieger to the relative case. Namely he proved that if $\pi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, T)$ is a factor map with $(X, \mathcal{X}, \mu, T)$ ergodic and $(\hat{Y}, \hat{\mathcal{Y}}, \hat{\nu}, T)$ is a uniquely ergodic model for $(Y, \mathcal{Y}, \nu, T)$, then there is a uniquely ergodic model $(\hat{X}, \hat{\mathcal{X}}, \hat{\mu}, T)$ for $(X, \mathcal{X}, \mu, T)$ and a factor map $\hat{\pi} : \hat{X} \to \hat{Y}$ which is a model for $\pi : X \to Y$. We will refer this theorem as Weiss’s Theorem. We note that in \[37\] Weiss pointed that the relative case holds for commutative group actions.

1.2.2. Ergodic averages. In this subsection we recall some results related to pointwise ergodic averages.

The first pointwise ergodic theorem was proved by Birkhoff in 1931. Followed from Furstenberg’s work in 1977, problems concerning the convergence of multiple ergodic averages (in $L^2$ or pointwisely) become a very important part of the study of ergodic theory.

The convergence of the averages

\[
\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) \cdots f_d(T^{dn} x)
\]

in $L^2$ norm was established by Host and Kra \[24\] (see also Ziegler \[42\]). We note that in their proofs, the characteristic factors play a great role. The multiple ergodic average for commuting transformations was obtained by Tao \[35\] using finitary ergodic method, see \[3, 23\] for more traditional ergodic proofs. Recently, convergence of multiple ergodic averages for nilpotent group actions was obtained by Walsh \[36\].

The first breakthrough on pointwise convergence of (1.4) for $d > 1$ is due to Bourgain, who showed in \[8\] that for $d = 2$, the limit in (1.4) exists a.e. for all $f_1, f_2 \in L^\infty$. It is a big open question if the same holds for $d > 2$. Very recently, Assani claimed the convergence for weakly mixing transformations \[2\].

The study of the limiting behavior of the averages along cubes was initiated by Bergelson in \[5\], where convergence in $L^2(\mu)$ was shown in dimension 2. Bergelson’s result was later extended by Host and Kra for cubic averages of an arbitrary dimension $d$ in \[24\]. More recently in \[1\], Assani established pointwise convergence for cubic averages of an arbitrary dimension $d$. Chu and Franzikinakis \[9\] extended the result to a very general case, i.e. they showed that for measure preserving transformations $T_\epsilon : X \to X$, functions $f_\epsilon \in L^\infty(\mu)$, $(0, \ldots, 0) \neq \epsilon \in \{0, 1\}^d$, the averages

\[
\frac{1}{N^d} \sum_{n \in [0, N-1]^d} \prod_{\epsilon \neq \epsilon \in \{0, 1\}^d} f_\epsilon(T_\epsilon^n x)
\]
converge $\mu$ a.e.. Moreover, they obtained in the same paper that
\[
\frac{1}{Nb(N)} \sum_{1 \leq m \leq N, 1 \leq n \leq b(N)} f_1(T^{m+n}x)f_2(T^{m+2n}x)\ldots f_d(T^{m+dn}x)
\]
converges pointwisely, where $b(N)/N^{1/d} \rightarrow 0$ as $N \rightarrow \infty$.

We remark that our method to prove Theorem D does not apply the general case as shown by Chu and Franzikinakis in [9]. The advantage of our method is that we can give formulas for the limits, meanwhile this can not obtained in [1, 9].

1.3. Main ideas of the proofs. Now we describe the main ideas and ingredients in the proof of Theorem A (the proof of Theorem B will follow by the similar idea). The first fact we face is that for an ergodic m.p.t. $(X, \mathcal{X}, \mu, T)$, not every strictly ergodic model is its $F[d]$-strictly ergodic model. For example, let $(X, \mathcal{X}, \mu, T)$ be a Kronecker system. By Jewett-Krieger’ Theorem, we may assume that $(X, T)$ is a topologically weakly mixing minimal system and strictly ergodic. By [34, Theorem 3.11] $(F[d](x), F[d])$ is minimal for all $x \in X$ and $F[d](x) = \{x\} \times X^{[d]}$. It is easy to see that $\delta_x \times \mu \otimes 2^{d-1}$ and $\mu^{[d]}_x$ are two different invariant measures on it (see Section 2 for the definitions). This indicates that to obtain Theorem A, Jewett-Krieger’ Theorem is not enough for our purpose. Fortunately, we find that Weiss’s Theorem [37] is a right tool.

Precisely, let $\pi_d : X \rightarrow Z_d$ be the factor map from $X$ to its $d$-step nilfactor $Z_d$. By definition, $Z_d$ may be regarded as a topological system in the natural way. By Weiss’s Theorem there is a uniquely ergodic model $(\hat{X}, \hat{\mathcal{X}}, \hat{\mu}, T)$ for $(X, \mathcal{X}, \mu, T)$ and a factor map $\hat{\pi}_d : \hat{X} \rightarrow Z_d$ which is a model for $\pi_d : X \rightarrow Z_d$.

We then show (though it is difficult) that $(\hat{X}, T)$ is what we need. To do this we heavily use the theory of joinings (for a reference, see [18]) and some facts related to $d$-step nilsystems. Once Theorem A (resp. B) is proven, Theorem C (resp. D) will follow by an argument using some well known theorems related to pointwise convergence for $Z^d$ actions by and for uniquely ergodic systems.

We remark that currently we do not know how to prove the pointwise convergence of (1.4) using similar ideas.

1.4. Organization of the paper. In Section 2 we give basic notions and facts about dynamical parallelepipeds and characteristic factors. In Section 3 we define $F$ and $G$-strictly ergodic models and prove that each ergodic system has $F$ and $G$-strictly ergodic model. Moreover, we build the connection between $F$ and $G$-strictly ergodic models with pointwise convergence of averages along cubes and faces, and deduce the existence of the limit of the averages. In the two sections followed, we study arithmetic progression models and prove pointwise ergodic theorem along arithmetic progressions.
2. Dynamical parallelepipeds and characteristic factors

In this section we introduce basic knowledge about dynamical parallelepipeds and characteristic factors. For more details, see [24, 25, 26] etc.

2.1. Ergodic theory and topological dynamics. In this subsection we introduce some basic notions in ergodic theory and topological dynamics. For more information, see Appendix.

2.1.1. Measurable systems. For a m.p.t. \((X, \mathcal{X}, \mu, T)\) we write \(\mathcal{I} = \mathcal{I}(T)\) for the \(\sigma\)-algebra \(\{A \in \mathcal{X} : T^{-1}A = A\}\) of invariant sets. A m.p.t. is ergodic if all the \(T\)-invariant sets have measure either 0 or 1. \((X, \mathcal{X}, \mu, T)\) is weakly mixing if the product system \((X \times X, \mathcal{X} \times \mathcal{X}, \mu \times \mu, T \times T)\) is ergodic.

A homomorphism from m.p.t. \((X, \mathcal{X}, \mu, T)\) to \((Y, \mathcal{Y}, \nu, S)\) is a measurable map \(\pi : X_0 \to Y_0\), where \(X_0\) is a \(T\)-invariant subset of \(X\) and \(Y_0\) is an \(S\)-invariant subset of \(Y\), both of full measure, such that \(\pi_*\mu = \mu \circ \pi^{-1} = \nu\) and \(S \circ \pi(x) = \pi \circ T(x)\) for \(x \in X_0\). When we have such a homomorphism we say that \((Y, \mathcal{Y}, \nu, S)\) is a factor of \((X, \mathcal{X}, \mu, T)\). If the factor map \(\pi : X_0 \to Y_0\) can be chosen to be bijective, then we say that \((X, \mathcal{X}, \mu, T)\) and \((Y, \mathcal{Y}, \nu, S)\) are (measure theoretically) isomorphic (bijective maps on Lebesgue spaces have measurable inverses). A factor can be characterized (modulo isomorphism) by \(\pi^{-1}(\mathcal{Y})\), which is a \(T\)-invariant sub- \(\sigma\)-algebra of \(\mathcal{X}\), and conversely any \(T\)-invariant sub-\(\sigma\)-algebra of \(\mathcal{X}\) defines a factor. By a classical result abuse of terminology we denote by the same letter the \(\sigma\)-algebra \(\mathcal{Y}\) and its inverse image by \(\pi\). In other words, if \((Y, \mathcal{Y}, \nu, S)\) is a factor of \((X, \mathcal{X}, \mu, T)\), we think of \(\mathcal{Y}\) as a sub-\(\sigma\)-algebra of \(\mathcal{X}\).

2.1.2. Topological dynamical systems. A t.d.s. \((X, T)\) is transitive if there exists some point \(x \in X\) whose orbit \(O(x, T) = \{T^n x : n \in \mathbb{Z}\}\) is dense in \(X\) and we call such a point a transitive point. The system is minimal if the orbit of any point is dense in \(X\). This property is equivalent to saying that \(X\) and the empty set are the only closed invariant sets in \(X\). \((X, T)\) is topologically weakly mixing if the product system \((X \times X, T \times T)\) is transitive.

A factor of a t.d.s. \((X, T)\) is another t.d.s. \((Y, S)\) such that there exists a continuous and onto map \(\phi : X \to Y\) satisfying \(S \circ \phi = \phi \circ T\). In this case, \((X, T)\) is called an extension of \((Y, S)\). The map \(\phi\) is called a factor map.

2.1.3. \(M(X)\) and \(M_T(X)\). For a t.d.s. \((X, T)\), denote by \(M(X)\) the set of all probability measure on \(X\). Let \(M_T(X) = \{\mu \in M(X) : T_*\mu = \mu \circ T^{-1} = \mu\}\) be the set of all \(T\)-invariant measure of \(X\). It is well known that \(M_T(X) \neq \emptyset\).

Definition 2.1. A t.d.s. \((X, T)\) is called uniquely ergodic if there is a unique \(T\)-invariant probability measure on \(X\). It is called strictly ergodic if it is uniquely ergodic and minimal.

2.2. Cubes and faces.
2.2.1. Let $X$ be a set, let $d \geq 1$ be an integer, and write $[d] = \{1, 2, \ldots, d\}$. We view $\{0, 1\}^d$ in one of two ways, either as a sequence $\epsilon = \epsilon_1 \ldots \epsilon_d$ of 0's and 1's, or as a subset of $[d]$. A subset $\epsilon$ corresponds to the sequence $(\epsilon_1, \ldots, \epsilon_d) \in \{0, 1\}^d$ such that $i \in \epsilon$ if and only if $\epsilon_i = 1$ for $i \in [d]$. For example, $0 = (0, 0, \ldots, 0) \in \{0, 1\}^d$ is the same to $\emptyset \subset [d]$.

Let $V_d = \{0, 1\}^d = [d]$ and $V_d^* = V_d \setminus \{0\} = V_d \setminus \{\emptyset\}$. If $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $\epsilon \in \{0, 1\}^d$, we define

$$n \cdot \epsilon = \sum_{i=1}^d n_i \epsilon_i.$$ 

If we consider $\epsilon$ as $\epsilon \subset [d]$, then $n \cdot \epsilon = \sum_{i \in \epsilon} n_i$.

2.2.2. We denote $X^{2d}$ by $X^d$. A point $x \in X^d$ can be written in one of two equivalent ways, depending on the context:

$$x = (x_\epsilon : \epsilon \in \{0, 1\}^d) = (x_\epsilon : \epsilon \subset [d]).$$

Hence $x_0 = x_\emptyset$ is the first coordinate of $x$. As examples, points in $X^2$ are like

$$(x_{00}, x_{10}, x_{01}, x_{11}) = (x_\emptyset, x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}}).$$

For $x \in X$, we write $x^{[d]} = (x, x, \ldots, x) \in X^d$. The diagonal of $X^d$ is $\Delta^d = \{x^{[d]} : x \in X\}$. Usually, when $d = 1$, denote diagonal by $\Delta_X$ or $\Delta$ instead of $\Delta^1$.

A point $x \in X^d$ can be decomposed as $x = (x', x'')$ with $x', x'' \in X^{d-1}$, where $x' = (x_{\epsilon_0} : \epsilon \in \{0, 1\}^{d-1})$ and $x'' = (x_{\epsilon_1} : \epsilon \in \{0, 1\}^{d-1})$. We can also isolate the first coordinate, writing $X^d = X^{2d-1}$ and then writing a point $x \in X^d$ as $x = (x_\emptyset, x_*)$, where $x_* = (x_\epsilon : \epsilon \neq \emptyset) \in X^*_d$.

2.2.3. The faces of dimension $r$ of a point in $x \in X^d$ are defined as follows. Let $J \subset [d]$ with $|J| = d - r$ and $\xi \in \{0, 1\}^{d-r}$. The elements $(x_\epsilon : \epsilon \in \{0, 1\}^d, \epsilon_J = \xi)$ of $X^{[r]}$ are called faces of dimension $r$ of $x$, where $\epsilon_J = (\epsilon_i : i \in J)$. Thus any face of dimension $r$ defines a natural projection from $X^d$ to $X^{[r]}$, and we call this the projection along this face.

2.3. Dynamical parallelepipeds.

**Definition 2.2.** Let $(X, T)$ be a topological dynamical system and let $d \geq 1$ be an integer. We define $Q^d(X)$ to be the closure in $X^d$ of elements of the form

$$(T^{n_{1\epsilon_1}}x, T^{n_{1\epsilon_2}}x, \ldots, T^{n_{d\epsilon_d}}x : \epsilon = (\epsilon_1, \ldots, \epsilon_d) \in \{0, 1\}^d),$$

where $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $x \in X$. When there is no ambiguity, we write $Q^d$ instead of $Q^d(X)$. An element of $Q^d(X)$ is called a (dynamical) parallelepiped of dimension $d$.

As examples, $Q^2$ is the closure in $X^2 = X^4$ of the set

$$\{(x, T^mx, T^nx, T^{m+n}x) : x \in X, m, n \in \mathbb{Z}\}$$

and $Q^3$ is the closure in $X^3 = X^8$ of the set

$$\{(x, T^mx, T^nx, T^{m+n}x, T^px, T^{m+p}x, T^{n+p}x, T^{m+n+p}x) : x \in X, m, n, p \in \mathbb{Z}\}.$$
Definition 2.3. Let $\phi : X \to Y$ and $d \in \mathbb{N}$. Define $\phi^d : X^d \to Y^d$ by $(\phi^d x)_\epsilon = \phi x_\epsilon$ for every $x \in X^d$ and every $\epsilon \subset [d]$. Let $(X, T)$ be a system and $d \geq 1$ be an integer. The \textit{diagonal transformation} of $X^d$ is the map $T^d$.

Definition 2.4. \textbf{Face transformations} are defined inductively as follows: Let $T^{[0]} = T$, $T^{[1]}_1 = \text{id} \times T$. If $\{T^{[d-1]}_j\}_{j=1}^{d-1}$ is defined already, then set
\begin{equation}
T^{[d]}_j = T^{[d-1]}_j \times T^{[d-1]}_j, \quad j \in \{1, 2, \ldots, d - 1\},
\end{equation}
and
\begin{equation}
T^{[d]}_d = \text{id}^{[d-1]} \times T^{[d-1]}.
\end{equation}

The \textit{face group} of dimension $d$ is the group $\mathcal{F}^d(X)$ of transformations of $X^d$ spanned by the face transformations. The \textit{parallelepiped group} of dimension $d$ is the group $\mathcal{G}^d(X)$ spanned by the diagonal transformation and the face transformations. We often write $\mathcal{F}^d$ and $\mathcal{G}^d$ instead of $\mathcal{F}^d(X)$ and $\mathcal{G}^d(X)$, respectively. For $\mathcal{G}^d$ and $\mathcal{F}^d$, we use similar notations to that used for $X^d$: namely, an element of either of these groups is written as $S = (S_\epsilon : \epsilon \in \{0, 1\}^d)$. In particular, $\mathcal{F}^d = \{S \in \mathcal{G}^d : S_0 = \text{id}\}$.

For convenience, we denote the orbit closure of $x \in X^d$ under $\mathcal{F}^d$ by $\overline{O}(x, \mathcal{F}^d)$, instead of $\overline{O}(x, \mathcal{F}^d)$. It is easy to verify that $Q^d$ is the closure in $X^d$ of
\[\{S x^d : S \in \mathcal{F}^d, x \in X\}.
\]

If $x$ is a transitive point of $X$, then $Q^d$ is the closed orbit of $x^d$ under the group $\mathcal{G}^d$.

2.4. \textbf{Measure $\mu^k$.}

2.4.1. \textbf{Notation.} When $f_\epsilon, \epsilon \in V_k = \{0, 1\}^d$, are $2^k$ real or complex valued functions on the set $X$, we define a function $\bigotimes_{\epsilon \in V_k} f_\epsilon$ on $X^k$ by
\[\bigotimes_{\epsilon \in V_k} f_\epsilon(x) = \prod_{\epsilon \in V_k} f_\epsilon(x_\epsilon).
\]

2.4.2. We define by induction a $T^k$-invariant measure $\mu^k$ on $X^k$ for every integer $k \geq 0$.

Set $X^{[0]} = X$, $T^{[0]} = T$ and $\mu^{[0]} = \mu$. Assume that $\mu^k$ is defined. Let $\mathcal{I}^k$ denote the $T^k$-invariant $\sigma$-algebra of $(X^k, \mu^k, T^k)$. Identifying $X^{[k+1]}$ with $X^k \times X^k$ as explained above, we define the system $(X^{[k+1]}, \mu^{[k+1]}, T^{[k+1]})$ to be the relatively independent joining of two copies of $(X^k, \mu^k, T^k)$ over $\mathcal{I}^k$. That is,
\[\mathcal{I}^k = \{A \subset X^k : T^k A = A\},
\]and
\[\mu^{[k+1]} = \mu^k \times \mu^k_{\mathcal{I}^k}.
\]
Equivalently, for all bounded function $f_\epsilon, \epsilon \in V_{k+1}$ of $X$,
\begin{equation}
\int_{X^{[k+1]}} \bigotimes_{\epsilon \in V_{k+1}} f_\epsilon \, d\mu^{[k+1]} = \int_{X^k} \mathbb{E}\left( \bigotimes_{\eta \in V_k} f_\eta \big| \mathcal{I}^k \right) \mathbb{E}\left( \bigotimes_{\eta \in V_k} f_\eta \big| \mathcal{I}^k \right) \, d\mu^k.
\end{equation}
Since \((X, \mu, T)\) is ergodic, \(\mathcal{T}^{[0]}\) is the trivial \(\sigma\)-algebra and \(\mu^{[1]} = \mu \times \mu\). If \((X, \mu, T)\) is weakly mixing, then by induction \(\mathcal{T}^{[k]}\) is trivial and \(\mu^{[k]}\) is the \(2^k\) Cartesian power \(\mu \otimes 2^k\) of \(\mu\) for \(k \geq 1\).

We now give an equivalent formulation of the definition of these measures. For an integer \(k \geq 1\), let \((\Omega_k, P_k)\) be the system corresponding to the \(\sigma\)-algebra \(\mathcal{T}^{[k]}\) and let

\begin{equation}
(2.3) \quad \mu^{[k]} = \int_{\Omega_k} \mu^{[k]}_\omega \, dP_k(\omega)
\end{equation}

denote the ergodic decomposition of \(\mu^{[k]}\) under \(T^{[k]}\). Then by definition

\begin{equation}
(2.4) \quad \mu^{[k+1]} = \int_{\Omega_k} \mu^{[k]}_\omega \times \mu^{[k]}_\omega \, dP_k(\omega).
\end{equation}

We generalize this formula. For \(k, l \geq 1\), the concatenation of an element \(\alpha\) of \(V_k\) with an element \(\beta\) of \(V_l\) is the element \(\alpha \beta\) of \(V_{k+l}\). This defines a bijection of \(V_k \times V_l\) onto \(V_{k+l}\) and gives the identification \((X^{[k]}|[l] = X^{[k+l]}\). By [24, Lemma 3.1.]

\begin{equation}
(2.5) \quad \mu^{[k+l]} = \int_{\Omega_k} (\mu^{[k]}_\omega | [l]) \, dP_k(\omega).
\end{equation}

### 2.5. Characteristic factors \((Z_k, \mu_k)\).

2.5.1. Notice that in [24], \(G^k\) and \(\mathcal{F}^{[k]}\) are denoted by \(\mathcal{T}^{[k]}_{k-1}\) and \(\mathcal{T}^{[k]}_s\) respectively. Let \(\mathcal{J}^{[k]}\) denote the \(\sigma\)-algebra of sets on \(X^{[k]}\) that are invariant under the group \(\mathcal{F}^{[k]}\). On \((X^{[k]}, \mu^{[k]}\), the \(\sigma\)-algebra \(\mathcal{J}^{[k]}\) coincides with the \(\sigma\)-algebra of sets depending only on the coordinate \(0\) ([24, Proposition 3.4]).

**Proposition 2.5.** [24] For all \(k \in \mathbb{N}\), \((X^{[k]}, \mu^{[k]}\) is ergodic for the group of side transformations \(G^{[d]}\). And \((\Omega_k, P_k)\) is ergodic under the action of the group \(\mathcal{F}^{[k]}\).

We consider the \(2^k - 1\)-dimensional marginals of \(\mu^{[k]}\). Recall that \(V^*_k = V_k \setminus \{0\}\). Consider a point \(x \in X^{[k]}\) as a pair \((x_0, x_*)\), with \(x_0 \in X\) and \(x_* \in X^{[k]}\). Let \(\mu^*_k\) denote the measure on \(X^*_k\), which is the image of \(\mu^{[k]}\) under the natural projection \(x \mapsto x_*\) from \(X^{[k]}\) onto \(X^*_k\).

All the transformations belonging to \(G^{[k]}\) factor through the projection \(X^{[k]} \to X^*_k\) and induce transformations of \(X^*_k\) preserving \(\mu^*_k\). This defines a measure-preserving action of the group \(G^{[k]}\) and of its subgroup \(\mathcal{F}^{[k]}\) on \(X^*_k\). The measure \(\mu^*_k\) is ergodic for the action of \(G^{[k]}\).

On the other hand, all the transformations belonging to \(G^{[k]}\) factor through the projection \(x \mapsto x_0\) from \(X^{[k]}\) to \(X\), and induce measure-preserving transformations of \(X\). The transformation \(T^{[k]}\) induces the transformation \(T\) on \(X\), and each transformation belonging to \(\mathcal{F}^{[k]}\) induces the trivial transformation on \(X\). This defines a measure-preserving ergodic action of the group \(G^{[k]}\) on \(X\), with a trivial restriction to the subgroup \(\mathcal{F}^{[k]}\).
2.5.2. A system of order \( k \). Let \( J^k_\sigma \) denote the \( \sigma \)-algebra of subsets of \( X^k \) which are invariant under the action of \( F^k \). Since the \( \sigma \)-algebra \( J^k_\sigma \) coincides with the \( \sigma \)-algebra of sets depending only on the coordinate \( \theta \) ([24, Proposition 3.4]). Hence there exists a \( \sigma \)-algebra \( Z_{k-1} \) of \( X \) such that \( Z_{k-1} \) is isomorphic to \( J^k_\sigma \). To be precise, for each \( A \in J^k_\sigma \), there is unique \( B \in Z_{k-1} \) such that \( 1_B(x_0) = 1_A(x_\sigma) \) for \( \mu^k \)-almost every \( x = (x_0, x_\sigma) \in X^k \).

**Definition 2.6.** The \( \sigma \)-algebra \( Z_k \) is invariant under \( T \) and so defines a factor of \( (X, \mu, T) \) written \( (Z_k(X), \mu_k, T) \), or simply \( (Z_k, \mu_k, T) \). The factor map \( X \rightarrow Z_k \) is written by \( \pi_k \).

\( (Z_k, Z_k, \mu_k, T) \) is called a system of order \( k \).

\( (Z_k, Z_k, \mu_k) \) has a very nice structure:

**Theorem 2.7.** [24] Let \( (X, \mathcal{X}, \mu, T) \) be an ergodic system and \( k \in \mathbb{N} \). Then the system \( (Z_k, Z_k, \mu_k, T) \) is a (measure theoretic) inverse limit of \( k \)-step nilsystems.

**Remark 2.8.** In this section we follows from the treatment of Host and Kra. Ziegler has a different approach, see [42]. For more details about the difference between these two methods, see Leibman's notes in the appendix in [6].

2.5.3. Properties about \( Z_k \). The following properties may be useful in the next section.

**Theorem 2.9.** [24, 25] Let \( k \geq 2 \) is an integer and \( (X = G/\Gamma, \mu, T) \) be an ergodic \((k - 1)\)-step nilsystem.

1. The measure \( \mu^k \) is the Haar measure of a sub-nilmanifold \( X_k^k \) of \( X^k \). \( (Q^k, \mu^k, G^k) \) is strictly ergodic.
2. Let \( X_{k^*} \) be the image of \( X_k \) under the projection \( x \mapsto x_\sigma \) from \( X^k \) to \( X_{k^*}^k = X^{k^* - 1} \). There exists a smooth map \( \Phi : X_{k^*} \rightarrow X_k \) such that \( X_k = \{(\Phi(x_\sigma), x_\sigma) : x \in X_{k^*}\} \).
3. For every \( x \in X \), let \( W_{k,x} = \{x \in X_k : x_0 = x\} \). Then \( W_{k,x} = F^k(\sigma^k(x^k)) \) and it is uniquely ergodic under \( F^k \).
4. For every \( x \in X \), let \( \rho_{k,x} \) be the invariant measure of \( W_{k,x} \). Then for every \( x \in X \) and \( g \in G \), \( \rho_{k,gx} \) is the image of \( \rho_{k,x} \) under the translation by \( g^k = (g, g, \ldots, g) \).

We need the following result replacing \( d \)-step nilmanifold with \( d \)-step nilsystem.

**Theorem 2.10.** Let \( k \geq 2 \) is an integer and \( (X, T, \mu) \) is an ergodic \((k - 1)\)-step nilsystem.

1. The measure \( \mu^k \) is an invariant measure of \( Q^k \). \( (Q^k, \mu^k, G^k) \) is strictly ergodic.
2. For every \( x \in X \), let \( W_{k,x} = \{x \in Q^k : x_0 = x\} \). Then \( W_{k,x} = F^k(\sigma^k(x^k)) \) and it is uniquely ergodic under \( F^k \).
3. For every \( x \in X \), let \( \rho_{k,x} \) be the invariant measure of \( W_{k,x} \). Then for every \( x \in X \), \( \rho_{k,Tx} \) is the image of \( \rho_{k,x} \) under the translation by \( T^k = (T, T, \ldots, T) \).

**Proof.** By [24] \((X, T, \mu) \) is an inverse limit of \((X_j = G_j/\Gamma_j, \mu_j, T) \) of \( d \)-step nilsystems. Then the result follows. \( \square \)
3. Deducing Theorems C and D from Theorems A and B

In this section we show how we obtain Theorem C (resp. D) from Theorem A (resp. B). The proof of Theorem A will be carried out in the next section and the proof of Theorem B will be presented in Section 5. Moreover, we will use Furstenberg-Weiss’ almost one-to-one Theorem to get a $d$-step almost automorphic model.

3.1. The proof of Theorem D assuming Theorem B. To simplify some statements, we introduce the following definition. Recall that
\[ \tau_d = T \times \ldots \times T \] 
\[ \sigma_d = T \times \ldots \times T^d \] 
and \[ \langle \tau_d, \sigma_d \rangle \] is the group generated by $\tau_d$ and $\sigma_d$. Moreover,
\[ N_d(\hat{X}) = O(\Delta_d(\hat{X}), \sigma_d) = O((x, \ldots, x), \langle \tau_d, \sigma_d \rangle) \] 
when $(\hat{X}, T)$ is minimal.

**Definition 3.1.** Let $(X, \mathcal{X}, \mu, T)$ be an ergodic m.p.t. and $(\hat{X}, T)$ be its model.

1. For $d \in \mathbb{N}$, $(\hat{X}, T)$ is called an $F[d]$-strictly ergodic model for $(X, \mathcal{X}, \mu, T)$ if $(\hat{X}, T)$ is a strictly ergodic model and $(\mathcal{F}[d](\hat{X}), \mathcal{F}[d])$ is strictly ergodic for all $x \in \hat{X}$.
2. For $d \in \mathbb{N}$, $(\hat{X}, T)$ is called a $G[d]$-strictly ergodic model for $(X, \mathcal{X}, \mu, T)$ if $(\hat{X}, T)$ is a strictly ergodic model and $(Q[d], G[d])$ is strictly ergodic.
3. For $d \in \mathbb{N}$, $(\hat{X}, T)$ is called a $\langle \tau_d, \sigma_d \rangle$-strictly ergodic model for $(X, \mathcal{X}, \mu, T)$ if $(\hat{X}, T)$ is a strictly ergodic model and $(N_d(\hat{X}), \langle \tau_d, \sigma_d \rangle)$ is strictly ergodic.

To obtain the connection between Theorems A (resp. B) and C (resp. D), we need the following formula which is easy to be verified.

**Lemma 3.2.** Let \( \{a_i\}, \{b_i\} \subseteq \mathbb{C} \). Then
\[
\prod_{i=1}^{k} a_i - \prod_{i=1}^{k} b_i = (a_1 - b_1)b_2 \ldots b_k + a_1(a_2 - b_2)b_3 \ldots b_k + a_1 \ldots a_{k-1}(a_k - b_k).
\]

We will show Theorem D can be deduced from Theorem B. The proofs of Theorem C assuming Theorem A follows similarly.

**The proof of Theorem D assuming Theorem B:** Since $(X, \mathcal{X}, \mu, T)$ has a $\langle \tau_d, \sigma_d \rangle$-strictly ergodic model, we may assume that $(X, T)$ itself is a minimal t.d.s. and $\mu$ is its unique measure such that $(N_d(\hat{X}), \langle \tau_d, \sigma_d \rangle)$ is uniquely ergodic with the unique measure $\lambda_{\tau, \sigma, d}$ defined in (5.16).

Let $\delta > 0$. Without loss of generality, we assume that for all $1 \leq j \leq d$, $\|f_j\|_\infty \leq 1$. Choose continuous functions $g_j$ such that $\|g_j\|_\infty \leq 1$ and $\|f_j - g_j\|_1 < \delta/d$ for all
$1 \leq j \leq d$. We have

$$
\frac{1}{N^2} \sum_{n \in [0,N-1]} \sum_{m \in [0,N-1]} \prod_{j=1}^{d} f_j(T^{n+(j-1)m}x) - \int_{N(X)} \otimes_{j=1}^{d} f_j d\lambda_{\tau,\sigma,d} \leq \frac{1}{N^2} \sum_{n \in [0,N-1]} \sum_{m \in [0,N-1]} \prod_{j=1}^{d} f_j(T^{n+(j-1)m}x) - \frac{1}{N^2} \sum_{n \in [0,N-1]} \sum_{m \in [0,N-1]} \prod_{j=1}^{d} g_j(T^{n+(j-1)m}x) + \frac{1}{N^2} \sum_{n \in [0,N-1]} \sum_{m \in [0,N-1]} \prod_{j=1}^{d} g_j(T^{n+(j-1)m}x) - \int_{N(X)} \otimes_{j=1}^{d} g_j d\lambda_{\tau,\sigma,d} + \int_{N(X)} \otimes_{j=1}^{d} f_j d\lambda_{\tau,\sigma,d} - \int_{N(X)} \otimes_{j=1}^{d} g_j d\lambda_{\tau,\sigma,d} \Bigg|.
$$

(3.2)

Now by Pointwise Ergodic Theorem for $\mathbb{Z}^2$ i.e. Theorem [3.2](applying to $(n, m) \mapsto T^{n+(j-1)m}$) we have that for all $1 \leq j \leq d$

$$
\frac{1}{N^2} \sum_{n \in [0,N-1]} \sum_{m \in [0,N-1]} \left| f_j(T^{n+(j-1)m}x) - g_j(T^{n+(j-1)m}x) \right| \rightarrow \| f_j - g_j \|_1, \text{ a.e. } N \rightarrow \infty.
$$

Hence by Lemma [3.2] when $N$ is large

$$
\frac{1}{N^2} \sum_{n \in [0,N-1]} \sum_{m \in [0,N-1]} \prod_{j=1}^{d} f_j(T^{n+(j-1)m}x) - \frac{1}{N^2} \sum_{n \in [0,N-1]} \sum_{m \in [0,N-1]} \prod_{j=1}^{d} g_j(T^{n+(j-1)m}x) \leq \sum_{j=1}^{d} \left[ \frac{1}{N^2} \sum_{n \in [0,N-1]} \sum_{m \in [0,N-1]} \left| f_j(T^{n+(j-1)m}x) - g_j(T^{n+(j-1)m}x) \right| \right] \leq 2 \sum_{j=1}^{d} \| f_j - g_j \|_1 \leq 2\delta, \text{ a.e.}
$$

(3.3)

Note that

$$
\frac{1}{N^2} \sum_{n \in [0,N-1]} \sum_{m \in [0,N-1]} \prod_{j=1}^{d} g_j(T^{n+(j-1)m}x) = \frac{1}{N^2} \sum_{n \in [0,N-1]} \sum_{m \in [0,N-1]} g_1(T^nx)g_2(T^{n+m}x) \cdots g_d(T^{n+(d-1)m}x) = \frac{1}{N^2} \sum_{n \in [0,N-1]} \sum_{m \in [0,N-1]} g_1 \otimes \cdots \otimes g_d \left( \tau_d^n \sigma_d^m(x, x, \ldots, x) \right).
$$
Since $g_1 \otimes \ldots \otimes g_d : X^d \to \mathbb{R}$ is continuous and $(N_d(X), \langle \tau_d, \sigma_d \rangle)$ is uniquely ergodic, by Theorem C.1, $\frac{1}{N^2} \sum_{n \in [0,N-1]} \prod_{j=1}^{d} g_j(T^{n+(j-1)m}x)$ converges pointwisely to $\int_{N(X)} \otimes_{j=1}^{d} g_j d\lambda_{\tau,\sigma;d}$. So when $N$ is large

$$\left| \frac{1}{N^2} \sum_{n \in [0,N-1]} \prod_{j=1}^{d} g_j(T^{n+(j-1)m}x) - \int_{N(X)} \otimes_{j=1}^{d} g_j d\lambda_{\tau,\sigma;d} \right| \leq \delta. \tag{3.4}$$

By Lemma 3.2

$$\left| \int_{N(X)} \otimes_{j=1}^{d} g_j d\lambda_{\tau,\sigma;d} - \int_{N(X)} \otimes_{j=1}^{d} f_j d\lambda_{\tau,\sigma;d} \right| \leq \sum_{j=1}^{d} \int_{N(X)} |g_j - f_j| d\lambda_{\tau,\sigma;d} \leq \delta. \tag{3.5}$$

So combining (3.2)-(3.5), when $N$ is large, we have

$$\left| \frac{1}{N^2} \sum_{n \in [0,N-1]} \prod_{j=1}^{d} f_j(T^{n+(j-1)m}x) - \int_{N(X)} \otimes_{j=1}^{d} f_j d\lambda_{\tau,\sigma;d} \right| \leq 4\delta, a.e.$$  

This clearly implies that $f_1(T^n x) f_2(T^{n+m} x) \ldots f_d(T^{n+(d-1)m} x)$ converge $\mu$ a.e.. The proof is completed.

Remark 3.3. It is easy to see that if (1.1) holds for all $d$, then we have (1.2) holds for all $d$. That is, (1.1) is more fundamental. For example, if we want to get $\mathcal{G}^{[1]}$-case:

$$\frac{1}{N^2} \sum_{0 \leq n_1, n_2 \leq N-1} f_0(T^{n_1} x) f_1(T^{n_1+n_2} x),$$

then what need do is in the $\mathcal{F}^{[2]}$-case

$$\frac{1}{N^2} \sum_{0 \leq n_1, n_2 \leq N-1} f_{01}(T^{n_1} x) f_{10}(T^{n_2} x) f_{11}(T^{n_1+n_2} x)$$

by setting $f_{00} = f_0, f_{10} = 1$ and $f_{11} = f_1$.

3.2. $d$-step almost automorphic systems. $d$-step almost automorphic systems were defined and studied in [27] which are the generalization of Veech’s almost automorphic systems.

Definition 3.4. Let $(X, T)$ be a minimal t.d.s. and $d \in \mathbb{N}$. $(X, T)$ is called a $d$-step almost automorphic system if it is an almost one-to-one extension of a $d$-step nilsystem.
See [27] for more discussion about $d$-step almost automorphy. In this subsection we will show that in Theorem A we can also require the models are $d$-step almost automorphic systems. To do so, first we state Furstenberg-Weiss’s almost one-to-one Theorem.

**Theorem 3.5 (Furstenberg-Weiss).** [16] Let $(Y, T)$ be a non-periodic minimal t.d.s., and let $\pi' : X' \to Y$ be an extension of $(Y, T)$ with $(X', T)$ topologically transitive and $X'$ a compact metric space.

\[
\begin{array}{ccc}
X' & \xrightarrow{\theta} & X \\
\pi' \downarrow & & \pi \\
Y & \xrightarrow{\pi} & Y
\end{array}
\]

Then there exists an almost 1-1 minimal extension $\pi : (X, T) \to (Y, T)$, a Borel subset $X'_0 \subseteq X'$ and a Borel measurable map $\theta : X'_0 \to X$ satisfying:

1. $\theta \circ T = T \circ \theta$;
2. $\pi \circ \theta = \pi'$;
3. $\theta$ is a Borel isomorphism of $X'_0$ onto its image $X_0 = \theta(X'_0) \subseteq X$;
4. $\mu(X'_0) = 1$ for any $T$-invariant measure $\mu$ on $X'$.
5. if $(X', T)$ is uniquely ergodic, then $(X, T)$ can be chosen to be uniquely (hence strictly) ergodic.

**Remark 3.6.** In [16] Theorem 1], (1)-(4) are stated. From the proof of the theorem given in [16], we have (5), which is pointed out in [19].

Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system with non-trivial nil-factors (non-triviality here means infinity) and $d \in \mathbb{N}$. Let $\pi_d : X \to Z_d$ be the factor map from $X$ to its $d$-step nilfactor $Z_d$. By definition, $Z_d$ may be regarded as a t.d.s. in the natural way. By Weiss’s theorem [37], there is a uniquely ergodic model $(\hat{X}', \hat{\mathcal{X}}, \hat{\mu}, T)$ for $(X, \mathcal{X}, \mu, T)$ and a factor map $\hat{\pi}'_d : \hat{X}' \to Z_d$ which is a model for $\pi_d : X \to Z_d$. Let

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & \hat{X}' \\
\downarrow \pi_d & & \downarrow \hat{\pi}'_d \\
Z_d & \xrightarrow{\pi_d} & Z_d
\end{array}
\]

Now by Theorem 3.5, $\hat{\pi}'_d : \hat{X}' \to Z_d$ may be replaced by $\hat{\pi}_d : \hat{X} \to Z_d$, where $\hat{\pi}_d$ is almost 1-1 and $\hat{X}'$ and $\hat{X}$ are measure theoretically isomorphic. In particular, $(\hat{X}, T)$ is a strictly ergodic model for $(X, \mathcal{X}, \mu, T)$.

As we described in the introduction, once we have a model $\hat{\pi} : \hat{X} \to Z_d$ then it is $\mathcal{F}^{[d]}$ and $\mathcal{G}^{[d]}$ models. Hence combining above discussion with Theorem A, we have

**Theorem 3.7.** Let $d \in \mathbb{N}$. Then every ergodic m.p.t. with a non-trivial $d$-step nilfactor has an $\mathcal{F}^{[d]}$ and $\mathcal{G}^{[d]}$ strictly ergodic model $(X, T)$ which is a $d$-step almost automorphic system.
4. Proof of Theorem A

In this section we give a proof for Theorem A. To make the idea of the proof clearer before going into the proof for the general case we show the cases when $d = 1$ and $d = 2$ first. We also give a proof for weakly mixing systems for independent interest. Finally we show the general case by induction.

4.1. Case when $d = 1$. By Jewett-Krieger’s Theorem, every ergodic system has a strictly ergodic model. Now we show this model is $F[1]$-strictly ergodic. Let $(X,T)$ be a strictly ergodic system and let $\mu$ be its unique $T$-invariant measure. Note that $F[1] = \langle id \times T \rangle$. Hence for all $x \in X$,
\[ F[1](x[1]) = \{x\} \times X. \]
Since $(X, T)$ is uniquely ergodic, $\delta_x \times \mu$ is the only $F[1]$-invariant measure of $F[1](x[1])$. In this case Theorem A(1) is nothing but Birkhorff pointwise ergodic theorem.

Now consider $Q[1]$. Since $G[1] = \langle T \times T, id \times T \rangle$, it is easy to see that $Q[1] = X \times X$. Let $\lambda$ be a $G[1]$-invariant measure of $(X[1], X[1]) = (X \times X, X \times X)$. Since $\lambda$ is $T \times T$-invariant, it is a self-joining of $(X, X, \mu, T)$ and has $\mu$ as its marginal. Let
\[ \lambda = \int_X \delta_x \times \lambda_x \, d\mu(x) \]
be the disintegration of $\lambda$ over $\mu$. Since $\lambda$ is id $\times$ $T$-invariant, we have
\[ \lambda = id \times T \lambda = \int_X \delta_x \times T \lambda_x \, d\mu(x). \]
The uniqueness of disintegration implies that
\[ T \lambda_x = \lambda_x, \mu \text{ a.e.} \]
Since $(X, X, T)$ is uniquely ergodic, $\lambda_x = \mu, \mu$ a.e. Thus by (4.1) one has that
\[ \lambda = \int_X \delta_x \times \lambda_x \, d\mu(x) = \int_X \delta_x \times \mu \, d\mu(x) = \mu \times \mu. \]
Hence $(Q[1], G[1])$ is uniquely ergodic, and $\mu[1] = \mu \times \mu$ is its unique $G[1]$-invariant measure.

4.2. Weakly mixing systems. In this subsection we show Theorem A holds for weakly mixing systems. This result relies on the following proposition.

**Proposition 4.1.** Let $(X, T)$ be uniquely ergodic, $(X, \mathcal{X}, \mu, T)$ be weakly mixing and $d \in \mathbb{N}$. Then

1. $(X[d], G[d])$ is uniquely ergodic with the unique measure $\mu[d] = \mu \times \ldots \times \mu$, $2^d$ times
2. $(X^*[d], F[d])$ is uniquely ergodic with the unique measure $\mu^*[d] = \mu \times \ldots \times \mu$, $2^d - 1$ times
Proof. We prove the result inductively. First we show the case when $d = 1$. In this case $\mathcal{F}^1 = (\mathrm{id} \times T)$ and $\mathcal{G}^1 = (\mathrm{id} \times T, T \times T)$. Hence $(X^1, \mathcal{X}^1, \mathcal{F}^1) = (X, \mathcal{X}, T)$, and it follows that $\mu_1 = \mu$ is the unique $T$-invariant measure. Let $\lambda$ be a $\mathcal{G}^1$-invariant measure of $(X^1, \mathcal{X}^1) = (X \times X, \mathcal{X} \times \mathcal{X})$. By the argument in subsection 1.1 we know that $\lambda = \mu^1 = \mu \times \mu$.

Now assume the statements hold for $d - 1$, and we show the case for $d$. Let $\lambda$ be a $\mathcal{G}^d$-invariant measure of $(X^d, \mathcal{X}^d)$. Let
\begin{align*}
p_1 : (X^d, \mathcal{G}^d) &\to (X^{d-1}, \mathcal{G}^{d-1}); \quad x = (x', x'') \mapsto x' \\
p_2 : (X^d, \mathcal{G}^d) &\to (X^{d-1}, \mathcal{G}^{d-1}); \quad x = (x', x'') \mapsto x''
\end{align*}
be the projections. Then $(p_2)_*(\lambda)$ is a $\mathcal{G}^{d-1}$-invariant measure of $X^{d-1}$. By inductive assumption, $(p_2)_*(\lambda) = \mu^{d-1}$. Let
\begin{equation}
(4.2) \quad \lambda = \int_{X^{d-1}} \lambda_x \times \delta_x \, d\mu^{d-1}(x)
\end{equation}
be the disintegration of $\lambda$ over $\mu^{d-1}$. Since $\lambda$ is $T^d = \mathrm{id}^{d-1} \times T^{d-1}$-invariant, we have
\begin{align*}
\lambda &= \mathrm{id}^{d-1} \times T^{d-1} \lambda = \int_{X^{d-1}} \lambda_x \times T^{d-1} \delta_x \, d\mu^{d-1}(x) \\
&= \int_{X^{d-1}} \lambda_x \times \delta_T^{d-1} \delta_x \, d\mu^{d-1}(x) \\
&= \int_{X^{d-1}} \lambda_{(T^{d-1})^{-1} x} \times \delta_x \, d\mu^{d-1}(x).
\end{align*}
The uniqueness of disintegration implies that
\begin{equation}
(4.3) \quad \lambda_{(T^{d-1})^{-1} x} = \lambda_x, \quad \mu^{d-1} \text{ a.e. } x \in X^{d-1}.
\end{equation}

Define
\begin{equation}
F : (X^{d-1}, \mathcal{X}^{d-1}, T^{d-1}) \longrightarrow M(X^{d-1}) : \ x \mapsto \lambda_x.
\end{equation}
By (4.3), $F$ is a $T^{d-1}$-invariant $M(X^{d-1})$-value function. Since $(X, \mathcal{X}, \mu, T)$ is weakly mixing, $(X^{d-1}, \mathcal{X}^{d-1}, T^{d-1})$ is ergodic and hence $\lambda_x = \nu$, $\mu^{d-1}$ a.e. for some $\nu \in M(X^{d-1})$. Thus by (1.2) one has that
\begin{equation}
\lambda = \int_{X^{d-1}} \lambda_x \times \delta_x \, d\mu^{d-1}(x) = \int_{X^{d-1}} \nu \times \delta_x \, d\mu^{d-1}(x) = \nu \times \mu^{d-1}.
\end{equation}
Then we have that $\nu = (p_1)_*(\lambda)$ is a $\mathcal{G}^{d-1}$-invariant measure of $X^{d-1}$. By inductive assumption, $\mu^{d-1}$ is the only $\mathcal{G}^{d-1}$-invariant measure of $X^{d-1}$ and hence $\nu = (p_1)_*(\lambda) = \mu^{d-1}$. Thus $\lambda = \mu^{d-1} \times \mu^{d-1} = \mu^d$. That is, $(X^d, \mathcal{X}^d, \mu^d, \mathcal{G}^d)$ is uniquely ergodic.

Now we show that $(X^d, \mathcal{X}^d, \mu^d, \mathcal{F}^d)$ is uniquely ergodic. The proof is similar. Let $\lambda$ be a $\mathcal{F}^d$-invariant measure of $(X^d, \mathcal{X}^d)$. Let
\begin{align*}
q_1 : (X^d, \mathcal{F}^d) &\to (X^{d-1}, \mathcal{F}^{d-1}); \quad x = (x'_d, x'_d) \mapsto x'_d \\
q_2 : (X^d, \mathcal{F}^d) &\to (X^{d-1}, \mathcal{G}^{d-1}); \quad x = (x'_d, x''_d) \mapsto x''_d
\end{align*}
be the projections. Then \((q_2)_*(\lambda)\) is a \(G^{[d-1]}\)-invariant measure of \(X^{[d-1]}\). By inductive assumption, \((q_2)_*(\lambda) = \mu^{[d-1]}\). Let

\[
(4.4) \quad \lambda = \int_{X^{[d-1]}} \lambda_x \times \delta_x \, d\mu^{[d-1]}(x)
\]

be the disintegration of \(\lambda\) over \(\mu^{[d-1]}\). Since \(\lambda\) is \(T^{(d)} = \text{id}^{[d-1]} \times T^{[d-1]}\)-invariant, we have

\[
\lambda = \text{id}^{[d-1]} \times T^{[d-1]} \lambda = \int_{X^{[d-1]}} \lambda_x \times T^{[d-1]} \delta_x \, d\mu^{[d-1]}(x)
\]

\[
= \int_{X^{[d-1]}} \lambda_x \times \delta_{T^{[d-1]}x} \, d\mu^{[d-1]}(x)
\]

\[
= \int_{X^{[d-1]}} \lambda_{(T^{[d-1]})^{-1}x} \times \delta_x \, d\mu^{[d-1]}(x).
\]

The uniqueness of disintegration implies that

\[
(4.5) \quad \lambda_{(T^{[d-1]})^{-1}x} = \lambda_x, \quad \mu^{[d-1]} \text{ a.e.}
\]

Define

\[
F : (X^{[d-1]}, \mathcal{X}^{[d-1]}, T^{[d-1]}) \longrightarrow M(X^{[d-1]}_*) : x \mapsto \lambda_x.
\]

By \((4.4)\), \(F\) is a \(T^{[d-1]}\)-invariant \(M(X^{[d-1]}_*)\)-value function. Since \((X, \mathcal{X}, \mu, T)\) is weakly mixing, \((X^{[d-1]}, \mathcal{X}^{[d-1]}, T^{[d-1]})\) is ergodic and hence \(\lambda_x = \nu, \mu^{[d-1]} \text{ a.e. for some } \nu \in M(X^{[d-1]}_*)\). Thus by \((4.4)\) one has that

\[
\lambda = \int_{X^{[d-1]}} \lambda_x \times \delta_x \, d\mu^{[d-1]}(x) = \int_{X^{[d-1]}} \nu \times \delta_x \, d\mu^{[d-1]}(x) = \nu \times \mu^{[d-1]}.
\]

Then we have that \(\nu = (q_1)_*(\lambda)\) is a \(\mathcal{F}^{[d-1]}\)-invariant measure of \(X^{[d-1]}_*\). By inductive assumption, \(\mu^{[d-1]}_*\) is the only \(\mathcal{F}^{[d-1]}\)-invariant measure of \(X^{[d-1]}_*\) and \(\nu = (q_1)_*(\lambda) = \mu^{[d-1]}_*\). Thus \(\lambda = \mu^{[d-1]}_* \times \mu^{[d-1]} = \mu^{[d]}_*\). Hence \((X^{[d]}, \mathcal{X}^{[d]}, \mu^{[d]}_*, \mathcal{F}^{[d]})\) is uniquely ergodic. The proof is completed. \(\square\)

**Theorem 4.2.** If \((X, \mathcal{X}, \mu, T)\) is a weakly mixing m.p.t., then it has an \(\mathcal{F}^{[d]}\) and \(G^{[d]}\) strictly ergodic model for all \(d \in \mathbb{N}\).

**Proof.** By Jewett-Krieger' Theorem, \((X, \mathcal{X}, \mu, T)\) has a uniquely ergodic model. Without loss of generality, we assume that \((X, T)\) itself is a minimal t.d.s. and \(\mu\) is its unique \(T\)-invariant measure. By \([34]*\) Theorem 3.11.], \((Q^{[d]} = X^{[d]}, G^{[d]}))\) is minimal, and for all \(x \in X\), \((\mathcal{F}^{[d]}(x^{[d]}), \mathcal{F}^{[d]})\) is minimal and \(\mathcal{F}^{[d]}(x^{[d]}) = \{x\} \times X^{[d]} = \{x\} \times X^{2d-1}\). By Proposition \([4.1]*\), \((Q^{[d]}, G^{[d]})\) and \((\mathcal{F}^{[d]}(x^{[d]}), \mathcal{F}^{[d]}))\) (for all \(x \in X\)) are uniquely ergodic. Hence it has an \(\mathcal{F}^{[d]}\) and \(G^{[d]}\) strictly ergodic model for all \(d \in \mathbb{N}\). \(\square\)

**4.3. Case when \(d = 2\).** In this case we can give the explicit description of the unique measure. Since the proof is long, we put it in Appendix \([4]*\). People familiar with the materials can read the proof for the general case directly.
4.4. General case. In this section we prove Theorem A in the general case. We prove it by induction on \( d \). \( d = 1 \) and \( d = 2 \) is showed in subsection 4.1 and Appendix H. Now we assume \( d \) and show the case when \( d + 1 \).

4.4.1. Notations. Recall that \( \mathcal{I}^{[d]} \) is the \( T^{[d]} \)-invariant \( \sigma \)-algebra of \( (X^{[d]}, \mu^{[d]}, T^{[d]}) \) and
\[
\mu^{[d+1]} = \mu^{[d]} \times \mu^{[d]}.
\]
Let
\[
(X^{[d]}, \mu^{[d]}) \xrightarrow{\phi} (\Omega_d, \mathcal{I}^{[d]}, P_d); \ x \mapsto \phi(x)
\]
be the factor map. Let
\[
\mu^{[d]} = \int_{\Omega_d} \mu^{[d]}_\omega \ dP_d(\omega)
\]
denote the ergodic decomposition of \( \mu^{[d]} \) under \( T^{[d]} \). Then by definition
\[
\mu^{[d+1]} = \int_{\Omega_d} \mu^{[d]}_\omega \times \mu^{[d]}_\omega \ dP_d(\omega).
\]

4.4.2. A property about \( Z_d \).

**Proposition 4.3.** [24 Proposition 4.7.] Let \( d \geq 1 \) be an integer.

1. As a joining of \( 2^d \) copies of \( (X, \mu) \), \( (X^{[d]}, \mu^{[d]}) \) is relatively independent over the joining \((Z^{[d-1]}_d, \mu^{[d-1]}_d)\) of \( 2^d \) copies of \((Z_{d-1}, \mu_{d-1})\).
2. \( Z_d \) is the smallest factor \( Y \) of \( X \) so that the \( \sigma \)-algebra \( \mathcal{I}^{[d]} \) is measurable with respect to \( Y^{[d]} \).

We say that a factor map \( \pi : (X, \mathcal{X}, \mu, T) \rightarrow (Y, \mathcal{Y}, \nu, T) \) is an ergodic extension if every \( T \)-invariant \( \mathcal{X} \)-measurable function is \( \mathcal{Y} \)-measurable, i.e. \( \mathcal{I}(X, T) \subset \mathcal{Y} \). Thus Proposition 4.3 implies that
\[
\pi^{[d]} : (X^{[d]}, \mu^{[d]}, T^{[d]}) \rightarrow (Z^{[d]}_d, \mu^{[d]}_d, T^{[d]})
\]
is \( T^{[d]} \)-ergodic. That means that \( \mathcal{I}^{[d]}(X) = \mathcal{I}^{[d]}(Z_d) \), and hence \((\Omega_d(X), \mathcal{I}^{[d]}(X), P_d) = (\Omega_d(Z_d), \mathcal{I}^{[d]}(Z_d), P_d)\). So we can denote the ergodic decomposition of \( \mu^{[d]}_d \) under \( T^{[d]} \) by
\[
\mu^{[d]}_d = \int_{\Omega_d} \mu^{[d]}_{d, \omega} \ dP_d(\omega).
\]
Then by definition
\[
\mu^{[d+1]}_d = \int_{\Omega_d} \mu^{[d]}_{d, \omega} \times \mu^{[d]}_{d, \omega} \ dP_d(\omega).
\]
This property is crucial in the proof. Combining (4.6) and (4.9), one has factor maps
\[
(X^{[d]}, \mu^{[d]}) \xrightarrow{\pi^{[d]}_d} (Z^{[d]}_d, \mu^{[d]}_d) \xrightarrow{\psi} (\Omega_d, P_d)
\]
Note that \( \phi = \psi \circ \pi^{[d]}_d \).
4.4.3. \( \mathcal{G} \)-action. Now we assume that Theorem A(2) holds for \( d \geq 1 \). In this subsection we show the existence of \( \mathcal{G}^{[d+1]} \)-model.

Let \( \pi_d : X \rightarrow Z_d \) be the factor map from \( X \) to its \( d \)-step nilfactor \( Z_d \). By definition, \( Z_d \) may be regarded as a topological system in the natural way. By Weiss’s Theorem, there is a uniquely ergodic model \((\hat{X}, \hat{\mu}, T)\) for \((X, \mathcal{X}, \mu, T)\) and a factor map \( \hat{\pi}_d : \hat{X} \rightarrow Z_d \) which is a model for \( \pi_d : X \rightarrow Z_d \).

\[
\begin{align*}
X & \longrightarrow \hat{X} \\
\pi_d & \downarrow \quad \downarrow \hat{\pi}_d \\
\quad \quad Z_d & \longrightarrow Z_d
\end{align*}
\]

Hence for simplicity, we may assume that \((\hat{X}, \hat{\mathcal{X}}, \hat{\mu}, T) = (X, \mathcal{X}, \mu, T) \) and \( \pi_d = \hat{\pi}_d \).

Now we show that \((\mathcal{Q}^{[d+1]}(X), \mu^{[d+1]}, \mathcal{G}^{[d+1]})\) is uniquely ergodic.

Let \( \lambda \) be a \( \mathcal{G}^{[d+1]} \)-invariant measure of \( \mathcal{Q}^{[d+1]} = \mathcal{Q}^{[d+1]}(X) \). Let

\[
\begin{align*}
p_1 : (\mathcal{Q}^{[d+1]}, \mathcal{G}^{[d+1]}) & \rightarrow (\mathcal{Q}^{[d]}, \mathcal{G}^{[d+1]}); \quad \mathbf{x} = (\mathbf{x}', \mathbf{x}'') \mapsto \mathbf{x}' \\
p_2 : (\mathcal{Q}^{[d+1]}, \mathcal{G}^{[d+1]}) & \rightarrow (\mathcal{Q}^{[d]}, \mathcal{G}^{[d+1]}); \quad \mathbf{x} = (\mathbf{x}', \mathbf{x}'') \mapsto \mathbf{x}''
\end{align*}
\]

be the projections. Then \( (p_2)_*(\lambda) \) is a \( \mathcal{G}^{[d+1]} \)-invariant measure of \( \mathcal{Q}^{[d]} \). Note that \( \mathcal{G}^{[d+1]} \) acts on \( \mathcal{Q}^{[d]} \) as \( \mathcal{G}^{[d]} \) actions. By the induction hypothesis, \( (p_2)_*(\lambda) = \mu^{[d]} \).

Hence let

\[
(4.12) \quad \lambda = \int_{\mathcal{Q}^{[d]}} \lambda_\mathbf{x} \times \delta_\mathbf{x} \; d\mu^{[d]}(\mathbf{x})
\]

be the disintegration of \( \lambda \) over \( \mu^{[d]} \). Since \( \lambda \) is \( T^{[d+1]} = \text{id}^{[d]} \times T^{[d]} \)-invariant, we have

\[
\begin{align*}
\lambda & = \text{id}^{[d]} \times T^{[d]} \lambda = \int_{\mathcal{Q}^{[d]}} \lambda_\mathbf{x} \times T^{[d]} \delta_\mathbf{x} \; d\mu^{[d]}(\mathbf{x}) \\
& = \int_{\mathcal{Q}^{[d]}} \lambda_\mathbf{x} \times \delta_{T^{[d]}(\mathbf{x})} \; d\mu^{[d]}(\mathbf{x}) \\
& = \int_{\mathcal{Q}^{[d]}} \lambda_{(T^{[d]})^{-1}(\mathbf{x})} \times \delta_\mathbf{x} \; d\mu^{[d]}(\mathbf{x}).
\end{align*}
\]

The uniqueness of disintegration implies that

\[
(4.13) \quad \lambda_{(T^{[d]})^{-1}(\mathbf{x})} = \lambda_\mathbf{x}, \quad \mu^{[d]} \; \text{a.e.} \; \mathbf{x} \in \mathcal{Q}^{[d]}.
\]

Define

\[
F : (\mathcal{Q}^{[d]}, T^{[d]}) \longrightarrow M(\mathcal{X}^{[d]}) : \; \mathbf{x} \mapsto \lambda_\mathbf{x}.
\]

By (4.13), \( F \) is a \( T^{[d]} \)-invariant \( M(\mathcal{X}^{[d]}) \)-value function. Hence \( F \) is \( T^{[d]} \)-measurable, and hence \( \lambda_\mathbf{x} = \lambda_{\phi(\mathbf{x})}, \quad \mu^{[d]} \; \text{a.e.} \), where \( \phi \) is defined in (4.6).
Thus by (4.12) one has that
\[
\lambda = \int_{Q^d} \lambda_x \times \delta_x \, d\mu^d(x) = \int_{Q^d} \lambda_{\phi(x)} \times \delta_x \, d\mu^d(x)
\]
\[
= \int_{\Omega_d} \int_{Q^d} \lambda_\omega \times \delta_x \, d\mu^d_\omega(x) \, dP_d(\omega)
\]
\[
= \int_{\Omega_d} \lambda_\omega \times (\int_{Q^d} \delta_x \, d\mu^d_\omega(x)) \, dP_d(\omega)
\]
\[
= \int_{\Omega_d} \lambda_\omega \times \mu^d_\omega \, dP_d(\omega)
\]

Let \(\pi_{d+1}^d : (Q^{d+1}(X), G^{d+1}) \to (Q^d(Z_d), G^d)\) be the natural factor map. By Theorem 2.9, \((Q^d(Z_d), G^d)\) is uniquely ergodic. Hence
\[
\pi_{d+1}^d(\lambda) = \mu_{d+1} = \int_{\Omega_d} \mu^d_\omega \times \mu^d_\omega \, dP_d(\omega).
\]
So
\[
(4.14) \quad \pi_{d*}^d(\lambda_\omega) = \pi_{d*}^d(\mu^d_\omega) = \mu^d_\omega.
\]
Note that we have that
\[
(p_1)_*(\lambda) = (p_2)_*(\lambda) = \mu^d,
\]
and hence we have
\[
(4.15) \quad \mu^d = \int_{\Omega_d} \lambda_\omega \, dP_d(\omega) = \int_{\Omega_d} \mu^d_\omega \, dP_d(\omega).
\]
But by (4.14) and (4.11) we have
\[
\phi_*(\lambda_\omega) = \phi_*(\mu^d_\omega) = \psi_{d*}(\mu^d_\omega) = \delta_\omega.
\]
Hence by the uniqueness of disintegration and (4.15), we have that \(\lambda_\omega = \mu^d_\omega\), \(P_d\) a.e. \(\omega \in \Omega_d\). Thus we have
\[
(4.16) \quad \lambda_{Q^{d+1}} = \int_{\Omega_d} \lambda_\omega \times \mu^d_\omega \, dP_d(\omega) = \int_{\Omega_d} \mu^d_\omega \times \mu^d_\omega \, dP_d(\omega) = \mu^{d+1}.
\]
That is, \((Q^{d+1}, \mu^{d+1}, G^{d+1})\) is uniquely ergodic. The proof of Theorem A(2) for \(G\) is completed.

4.4.4. \(F\)-actions. Now we assume that Theorem A(1) holds for \(d \geq 1\). In this subsection we show the existence of \(F^{d+1}\)-model. We use the same model as in the previous subsection.

Let \(\lambda\) be a \(F^{d+1}\)-invariant measure of \(\overline{F^{d+1}}(x^{d+1})\). Let
\[
p_1 : (\overline{F^{d+1}}(x^{d+1}), F^{d+1}) \to (\overline{F^d}(x^d), F^{d+1}); \quad x = (x', x'') \mapsto x'
\]
\[
p_2 : (\overline{F^{d+1}}(x^{d+1}), F^{d+1}) \to (Q^d, F^{d+1}); \quad x = (x', x'') \mapsto x''
\]
be the projections. Note that
\[
(\overline{F^d}(x^d), F^{d+1}) \simeq (\overline{F^d}(x^d), F^d) \quad \text{and} \quad (Q^d, F^{d+1}) \simeq (Q^d, G^d).
\]
Then $(p_2)_*(\lambda)$ is a $G^{[d]}$-invariant measure of $Q^{[d]}$. By subsection 4.3, $(p_2)_*(\lambda) = \mu^{[d]}$. Hence let

\[ \lambda = \int_{Q^{[d]}} \lambda_x \times \delta_x \, d\mu^{[d]}(x) \tag{4.17} \]

be the disintegration of $\lambda$ over $\mu^{[d]}$. Since $\lambda$ is $T^{[d]+1}_{d+1} = \text{id}^{[d]} \times T^{[d]}$-invariant, we have

\[
\begin{align*}
\lambda &= \text{id}^{[d]} \times T^{[d]} \lambda = \int_{Q^{[d]}} \lambda_x \times T^{[d]} \delta_x \, d\mu^{[d]}(x) \\
&= \int_{Q^{[d]}} \lambda_x \times \delta_{T^{[d]}(x)} \, d\mu^{[d]}(x) \\
&= \int_{Q^{[d]}} \lambda_{(T^{[d]}-1)(x)} \times \delta_x \, d\mu^{[d]}(x).
\end{align*}
\]

The uniqueness of disintegration implies that

\[ \lambda_{(T^{[d]}-1)(x)} = \lambda_x, \quad \mu^{[d]} \text{ a.e.} \tag{4.18} \]

Define

\[ F : Q^{[d]} \to M\left(\overline{\mathcal{F}^{d}(x^{[d]})}\right) : x \mapsto \lambda_x. \]

By (4.18), $F$ is a $T^{[d]}$-invariant $M(\overline{\mathcal{F}^{d}(x^{[d]})})$-value function. Hence $F$ is $T^{[d]}$-measurable, and hence $\lambda_x = \lambda_{\phi(x)}$, $\mu^{[d]}$ a.e. $x \in Q^{[d]}$, where $\phi$ is defined in (4.6).

Thus by (4.17) one has that

\[ \lambda = \int_{Q^{[d]}} \lambda_x \times \delta_x \, d\mu^{[d]}(x) = \int_{Q^{[d]}} \lambda_{\phi(x)} \times \delta_x \, d\mu^{[d]}(x) \]

\[ = \int_{\Omega_d} \int_{Q^{[d]}} \lambda_\omega \times \delta_x \, d\mu^{[d]}_\omega(x) \, dP_d(\omega) \]

\[ = \int_{\Omega_d} \lambda_\omega \times \left( \int_{Q^{[d]}} \delta_x \, d\mu^{[d]}_\omega(x) \right) \, dP_d(\omega) \]

\[ = \int_{\Omega_d} \lambda_\omega \times \mu^{[d]}_\omega \, dP_d(\omega). \]

Since $(\overline{\mathcal{F}^{d}(x^{[d]})}, \mathcal{F}^{[d]})$ is uniquely ergodic by assumption, and we let $\nu^{[d]}_x$ be the unique measure. Then

\[ (p_1)_*(\lambda) = \nu^{[d]}_x, \quad \text{and} \quad (p_2)_*(\lambda) = \mu^{[d]}, \]

and hence we have

\[ \nu^{[d]}_x = \int_{\Omega_d} \lambda_\omega \, dP_d(\omega). \tag{4.19} \]

Note that we have a factor map $\pi^{[d]}_d : (\overline{\mathcal{F}^{d}(x^{[d]})}, \mathcal{F}^{[d]}, \nu^{[d]}_x) \to (\overline{\mathcal{F}^{d}(\hat{x}^{[d]}), \mathcal{F}^{[d]}, \rho_{d,\hat{x}}})$, where $\hat{x} = \pi_d(x)$ and $\rho_{d,\hat{x}}$ as in Theorem 2.2. For each $z \in \overline{\mathcal{F}^{d}(\hat{x}^{[d]})}$, let $\eta_z$ be the unique $T^{[d]}$-invariant measure on $\overline{\mathcal{O}(z, T^{[d]})}$. Then the map

\[ \overline{\mathcal{F}^{d}(\hat{x}^{[d]})} \to M(Q^{[d]}(Z_d)), \quad z \mapsto \eta_z. \]
is a measurable map. This fact follows from that $z \mapsto \frac{1}{N} \sum_{n<N} \delta_{T^n z}$ is continuous and $\frac{1}{N} \sum_{n<N} \delta_{T^n z}$ converges to $\eta_z$ weakly. Hence we have

$$\mu_d^{[d]} = \int_{\mathcal{F}^d(\hat{x}[d])} \eta_z \, d\rho_{d,\hat{z}}(z).$$

In fact, it is easy to check that $\int_{\mathcal{F}^d(\hat{x}[d])} \eta_z \, d\rho_{d,\hat{z}}(z)$ is $\mathcal{G}^{[d]}$-invariant and hence it is equal to $\mu_d^{[d]}$ by the uniqueness. Note that (4.20) is the “ergodic decomposition” of $\mu_d^{[d]}$ under $T^{[d]}$, except that it happens that $\eta_z = \eta_{z'}$ for some $z \neq z'$. Hence via map $\psi$, we have a factor map

$$\Psi : (\mathcal{F}^{[d]}(\hat{x}[d]), \rho_{d,\hat{z}}) \to (\Omega_d, P_d).$$

And (4.20) can be rewritten as

$$\mu_d^{[d]} = \int_{\mathcal{F}^d(\hat{x}[d])} \eta_z \, d\rho_{d,\hat{z}}(z) = \int_{\Omega_d} \eta_\omega \, dP_d(\omega) = \int_{\Omega_d} \mu_d^{[d]} \, dP_d(\omega).$$

Since we have

$$\left(\mathcal{F}^{[d]}(x[d]), \nu_d^{[d]}\right) \xrightarrow{\pi_d^{[d]}} \left(\mathcal{F}^d(\hat{x}[d]), \rho_{d,\hat{z}}\right) \xrightarrow{\Psi} (\Omega_d, P_d)$$

we assume that

$$\nu_d^{[d]} = \int_{\Omega_d} \nu_\omega \, dP_d(\omega)$$

is the disintegration of $\nu_d^{[d]}$ over $\Omega_d$.

Let $\pi_d^{[d+1]} : (\mathcal{F}^{[d+1]}(x[d+1]), \mathcal{F}^{[d+1]}(\hat{x}[d+1])) \to (\mathcal{F}^{[d]}(x[d+1]), \mathcal{F}^{[d+1]})$ be the natural factor map. By Theorem 2.9, $\left(\mathcal{F}^{[d+1]}(\hat{x}[d+1]), \rho_{d+1,\hat{z}}\right)$ is uniquely ergodic. Let

$$(\pi_d^{[d+1]})_* (\lambda) = \rho_{d+1,\hat{z}} = \int_{\mathcal{F}^d(\hat{x}[d])} \delta_z \times \eta_z \, d\rho_{d,x}(z).$$

be the disintegration of $\rho_{d+1,x}$ over $\mathcal{F}^d(\hat{x}[d])$. By (4.22), we have

$$(\pi_d^{[d+1]})_* (\lambda) = \rho_{d+1,\hat{z}} = \int_{\mathcal{F}^d(\hat{x}[d])} \delta_z \times \eta_z \, d\rho_{d,x}(z) = \int_{\Omega_d} \rho_\omega \times \mu_d^{[d]} \, dP_d(\omega),$$

where $\rho_{d,\hat{z}} = \int_{\Omega_d} \rho_\omega \, dP_d(\omega)$ is the disintegration of $\rho_{d,\hat{z}}$ over $P_d$. Then

$$\left(\pi_d^{[d]}\right)_* (\lambda_\omega) = \rho_\omega; \quad \text{and} \quad \left(\pi_d^{[d]}\right)_* (\mu_d^{[d]}) = \mu_d^{[d]}.$$ 

Since $(\pi_d^{[d]})(\nu_d^{[d]}) = \rho_{d,\hat{z}}$, by (4.23) we have $(\pi_d^{[d]})(\nu_\omega) = \rho_\omega$. Hence by the uniqueness of disintegration, we have that $\lambda_\omega = \nu_\omega$, $P_d$ a.e. Thus

$$\lambda_{\mathcal{F},d+1} = \lambda = \int_{\Omega_d} \lambda_\omega \times \mu_d^{[d]} \, dP_d(\omega) = \int_{\Omega_d} \nu_\omega \times \mu_d^{[d]} \, dP_d(\omega).$$

That is, $\lambda$ is unique and hence $(\mathcal{F}^{[d+1]}(x[d+1]), \mathcal{F}^{[d+1]})$ is uniquely ergodic. The proof is completed. \qed
5. Proof of Theorem B

In this section we show Theorem B. We start from the case when \((X, \mathcal{X}, \mu, T)\) is weakly mixing.

5.1. Preparation. Let \(T : X \to X\) be a map and \(d \in \mathbb{N}\). Set
\[
\tau_d = T \times \ldots \times T (d \text{ times}),
\]
\[
\sigma_d = T \times \ldots \times T^d
\]
and
\[
\sigma'_d = id \times T \times \ldots \times T^{d-1} = id \times \sigma_{d-1}.
\]
Note that \(\langle \tau_d, \sigma_d \rangle = \langle \tau_d, \sigma'_d \rangle\). For any \(x \in \hat{X}\), let \(N_d(\hat{X}, x) = \overline{O((x, \ldots, x), \langle \tau_d, \sigma_d \rangle)}\), the orbit closure of \((x, \ldots, x) (d \text{ times})\) under the action of the group \(\langle \tau_d, \sigma_d \rangle\). We remark that if \((\hat{X}, T)\) is minimal, then all \(N_d(\hat{X}, x)\) coincide, which will be denoted by \(N_d(\hat{X})\). It was shown by Glasner [17] that if \((\hat{X}, T)\) is minimal, then \((N_d(\hat{X}), \langle \tau_d, \sigma_d \rangle)\) is minimal. Hence if \((N_d(X), \langle \tau_d, \sigma_d \rangle)\) is uniquely ergodic, then it is strictly ergodic.

Definition 5.1. Let \((X, T)\) be a minimal system with \(\mu \in M_T(X)\) and \(d \geq 1\). If \((N_d(X), \langle \tau_d, \sigma_d \rangle)\) is uniquely ergodic, then we denote the unique measure by \(\mu^{(d)}\), and call it the Furstenberg self joining.

Since \((N_d(X), \langle \tau_d, \sigma_d \rangle)\) is uniquely ergodic, it is easy to see that
\[
\frac{1}{N} \sum_{n=0}^{N-1} \sigma^n_d \mu^d_\Delta \longrightarrow \mu^{(d)}, \ N \to \infty, \ \text{weakly in } M(X^d),
\]
where \(\mu^d_\Delta\) is the diagonal measure on \(X^d\) as defined in [14], i.e. it is defined on \(X^d\) as follows
\[
\int_{X^d} f_1(x_1)f_2(x_2)\ldots f_d(x_d) \, d\mu^d_\Delta(x_1, x_2, \ldots, x_d) = \int_X f_1(x)f_2(x)\ldots f_d(x) \, d\mu(x).
\]
Note that if \((N_d(X), \langle \tau_d, \sigma_d \rangle)\) is not uniquely ergodic, we still can define \(\mu^{(d)}\), i.e. generally one may define \(\mu^{(d)}\) as a weak limit point of sequence \(\{\frac{1}{N} \sum_{n=0}^{N-1} \sigma^n_d \mu^d_\Delta\}\) in \(M(X^d)\). In this case one may have lots of choices for \(\mu^{(d)}\).

5.2. Weakly mixing systems. In this section we show Theorem B holds for weakly mixing systems.

Proposition 5.2. Let \((X, \mathcal{X}, \mu, T)\) be a weakly mixing dynamical system and \(d \in \mathbb{N}\). If \((X, \mathcal{X}, \mu, T)\) is uniquely ergodic, then \((X^d, \mathcal{X}^d, \mu^d, \langle \tau_d, \sigma_d \rangle)\) is uniquely ergodic, where \(\mu^d = \mu \times \ldots \times \mu\) \(d\ \text{times}.

Proof. We prove the result inductively. It is trivial when \(d = 1\), since \(\tau_1 = \sigma_1 = T\).

Now assume the statements hold for \(d-1 \ (d \geq 2)\), and we show the case for \(d\).
Let \(\lambda\) be a \(\langle \tau_d, \sigma_d \rangle\)-invariant measure of \((X^d, \mathcal{X}^d)\). Let
\[
p_1 : X^d = X \times X^{d-1} \to X; \ x = (x_1, x') \mapsto x_1
\]
\[
p_2 : X^d = X \times X^{d-1} \to X^{d-1}; \ x = (x_1, x') \mapsto x'
\]
be the projections. Note that \((p_2)_*(\lambda)\) is a \((\tau_{d-1}, \sigma_{d-1})\)-invariant measure of \(X^{d-1}\). By inductive assumption, \((p_2)_*(\lambda) = \mu^{d-1}\). Let

\[
(5.1) \quad \lambda = \int \lambda_x \times \delta_x \, d\mu^{d-1}(x)
\]

be the disintegration of \(\lambda\) over \(\mu^{d-1}\). Since \(\lambda\) is \(\sigma_d' = \text{id} \times \sigma_{d-1}\)-invariant, we have

\[
\lambda = \sigma_d' \lambda = \text{id} \times \sigma_{d-1} \lambda = \int \lambda_x \times \delta_{\sigma_{d-1}x} \, d\mu^{d-1}(x)
= \int \lambda_x \times \delta_{\sigma_{d-1}x} \, d\mu^{d-1}(x)
= \int \lambda_{(\sigma_{d-1})^{-1}x} \times \delta_x \, d\mu^{d-1}(x).
\]

The uniqueness of disintegration implies that

\[
(5.2) \quad \lambda_{(\sigma_{d-1})^{-1}x} = \lambda_x, \quad \mu^{d-1} \text{ a.e.}
\]

Define

\[
F : (X^{d-1}, \mathcal{X}^{d-1}, \sigma_{d-1}) \longrightarrow M(X) : x \mapsto \lambda_x.
\]

By (5.2), \(F\) is a \(\sigma_{d-1}\)-invariant \(M(X)\)-value function. Since \((X, \mathcal{X}, \mu, T)\) is weakly mixing, \((X^{d-1}, \mathcal{X}^{d-1}, \sigma_{d-1}) = (X^{d-1}, \mathcal{X}^{d-1}, T \times T^2 \times \ldots \times T^{d-1})\) is ergodic and hence \(\lambda_x = \nu, \mu^{d-1} \text{ a.e. for some } \nu \in M(X)\). Thus by (5.1) one has that

\[
\lambda = \int \lambda_x \times \delta_x \, d\mu^{d-1}(x) = \int \nu \times \delta_x \, d\mu^{d-1}(x) = \nu \times \mu^{d-1}.
\]

Then we have that \(\nu = (p_1)_*(\lambda)\) is a \(T\)-invariant measure of \(X\). By assumption, \(\nu = (p_1)_*(\lambda) = \mu\). Thus \(\lambda = \mu \times \mu^{d-1} = \mu^d\). Hence \((X^d, \mathcal{X}^d, \mu^d, (\tau_d, \sigma_d))\) is uniquely ergodic. The proof is completed. \(\square\)

**Theorem 5.3.** If \((X, T)\) is a t.d.s. and \((X, \mathcal{X}, \mu, T)\) is weakly mixing, then it has a \((\tau_d, \sigma_d)\)-strictly ergodic model for all \(d \in \mathbb{N}\).

**Proof.** By Jewett-Krieger’s Theorem, \((X, \mathcal{X}, \mu, T)\) has a uniquely ergodic model. Without loss of generality, we may assume that \((X, T)\) itself is a topological minimal system and \(\mu\) is its unique \(T\)-invariant measure. By Proposition 5.2 \((X^d, (\tau_d, \sigma_d))\) is uniquely ergodic for all \(d \in \mathbb{N}\). Hence it has a \(d\)-arithmetic progression strictly ergodic model. \(\square\)

5.3. **Nilsystems under action** \((\tau_d, \sigma_d)\). Before going on, we need some results on nilsystems under action \((\tau_d, \sigma_d)\).

5.3.1. **Basic properties.** In this subsection \(d \geq 2\) is an integer, and \((X = G/\Gamma, \mu_{d-1}, T)\) is an ergodic \((d-1)\)-step nilsystem and the transformation \(T\) is translation by the element \(t \in G\). Let

\[
N_d = N_d(X) = \overline{O(\Delta_d(X), \sigma_d)} = \overline{O((x, \ldots, x), (\tau_d, \sigma_d))} \subset X^d
\]

and

\[
N_d[x] = \overline{O((x, \ldots, x), \sigma_d^d)},
\]

where \(x \in X\). Then we have
Theorem 5.4. [7, 11] With the notations above, we have

1. The \((N_d, \langle \tau_d, \sigma_d \rangle)\) is ergodic (and thus uniquely ergodic) with some measure \(\mu_{d-1}^{(d)}\).
2. For \(\mu\)-almost every \(x \in X\), the system \((N_d[x], \sigma'_d)\) is uniquely ergodic with some measure \(\mu_{d-1,x}^{(d)}\).
3. \(\mu_{d-1}^{(d)} = \int_X \delta_x \times \mu_{d-1,x}^{(d)} \, d\mu(x)\).
4. (Ziegler) Let \(f_1, f_2, \ldots, f_{d-1}\) be continuous functions on \(X\) and let \(\{M_i\}\) and \(\{N_i\}\) be two sequences of integers such that \(N_i \to \infty\). For \(\mu\)-almost every \(x \in X\),

\[
\frac{1}{N_i} \sum_{n=M_i}^{N_i+M_i-1} f_1(T^n x) f_2(T^{2n} x) \cdots f_{d-1}(T^{(d-1)n} x)
\]

\(\to \int f_1(x_1) f_2(x_2) \cdots f_{d-1}(x_{d-1}) \, d\mu_{d-1,x}^{(d)}(x_1, x_2, \ldots, x_{d-1})\)

as \(i \to \infty\).

5.3.2. The ergodic decomposition of \(\mu_{d-1}^{(d)}\) under \(\sigma_d\). Now we study the ergodic decomposition of \(\mu_{d-1}^{(d)}\) under \(\sigma_d\). For each \(x \in X\), let \(\nu_{d-1,x}^{(d)}\) be the unique \(\sigma_d\)-invariant measure on \(O(x^d, \sigma_d)\), where \(x^d = (x, x, \ldots, x) \in X^d\). Then

\(\varphi : X \to M(N_d); \ x \mapsto \nu_{d-1,x}^{(d)}\)

is a measurable map. This fact follows from that \(x \mapsto \frac{1}{N} \sum_{n<N} \delta_{\sigma^1_n x^d}\) is continuous and \(\frac{1}{N} \sum_{n<N} \delta_{\sigma^1_n x^d}\) converges to \(\nu_{d-1,x}^{(d)}\) weakly. Hence we have

\(\mu_{d-1}^{(d)} = \int_X \nu_{d-1,x}^{(d)} \, d\mu(x)\).

In fact, it is easy to check that \(\int_X \nu_{d-1,x}^{(d)} \, d\mu(x)\) is \(\langle \tau_d, \sigma_d \rangle\)-invariant and hence it is equal to \(\mu_{d-1}^{(d)}\) by the uniqueness. Now we show that \(\nu_{d-1,x}^{(d)}\) is the “ergodic decomposition” of \(\mu_{d-1}^{(d)}\) under \(\sigma_d\). It is left to show that \(\nu_{d-1,x}^{(d)} \neq \nu_{d-1,y}^{(d)}\) whenever \(x \neq y\). This result will follows from the following fact: \(O(x^d, \sigma_d) \cap O(y^d, \sigma_d) = \emptyset\) for all \(x \neq y\). In fact, if \(O(x^d, \sigma_d) \cap O(y^d, \sigma_d) \neq \emptyset\), then \(y^d \in O(x^d, \sigma_d)\) since both \(O(x^d, \sigma_d)\) and \(O(y^d, \sigma_d)\) are minimal. This means that \((x, y, y, \ldots, y) \in Q^d(X)\). Hence \(x = y\) by [26, Theorem 1.2].

To sum up, we have

Proposition 5.5. The algebra \(\mathcal{I}(Z_{d-1,1}^d, Z_{d-1,1}^d, \mu_{d-1}^{(d)}, \sigma_d)\) of invariant sets under \(\sigma_d\) is isomorphic to \(Z_{d-1,1}^d\).

5.4. Proof of Theorem B. Let \((X, T)\) be a strictly ergodic system and let \(\mu\) be its unique \(T\)-invariant measure.

5.4.1. Case when \(d = 1\). Now \(X^1 = X, \tau_1 = T, \sigma_1 = T\) and \(\sigma'_1 = \text{id}\). It is trivial in this case.
5.4.2. Case when $d = 2$. In this case $X^2 = X \times X$, $\tau_2 = T \times T$, $\sigma_2 = T \times T^2$ and $\sigma'_2 = \text{id} \times T$. Note that $\langle \tau_2, \sigma_2 \rangle = G^{|1|}$. Hence it is the same to subsection 5.1. In this case $N^2(X) = X \times X$, and its $\langle \tau_2, \sigma_2 \rangle$-uniquely ergodic measure is $\mu \times \mu$.

5.4.3. Case when $d = 3$. Let $\pi_1 : X \to Z_1$ be the factor map from $X$ to its Kronecker factor $Z_1$. Since $Z_1$ is a group rotation, it may be regarded as a topological system in the natural way. By Weiss’s Theorem, there is a uniquely ergodic model $(\hat{X}, \hat{\mathcal{X}}, \hat{\mu}, T)$ for $(X, \mathcal{X}, \mu, T)$ and a factor map $\hat{\pi}_1 : \hat{X} \to Z_1$ which is a model for $\pi_1 : X \to Z_1$.

$$
\begin{align*}
X & \longrightarrow \hat{X} \\
\pi_1 & \downarrow \quad \quad \hat{\pi}_1 \\
Z_1 & \longrightarrow Z_1
\end{align*}
$$

Hence for simplicity, we may assume that $(\hat{X}, \hat{\mathcal{X}}, \hat{\mu}, T) = (X, \mathcal{X}, \mu, T)$ and $\pi_1 = \hat{\pi}_1$. Now we show that $(N^3(X), \langle \tau_3, \sigma_3 \rangle)$ is uniquely ergodic.

Before continuing we need some properties about the Kronecker factor $(Z_1(X), t_1)$ of the ergodic system $(X, \mu, T)$. Recall that $\mu_1$ is the Haar measure of $Z_1$.

For $s \in Z_1$, let $\xi_{1,s}$ denote the image of the measure $\mu_1$ under the map $z \mapsto (z, sz^2)$ from $Z_1$ to $Z_1^2$. This measure is invariant under $\sigma_2 = T \times T^2$ and is a self-joining of the rotation $(Z_1, t_1)$. Let $\xi_s$ denote the relatively independent joining of $\mu$ over $\xi_{1,s}$. This means that for bounded measurable functions $f$ and $g$ on $X$,

$$
\int_{Z_1 \times Z_1} f(x_0) g(x_1) \, d\xi_s(x_0, x_1) = \int_{Z_1} \mathbb{E}(f|Z_1)(z) \mathbb{E}(g|Z_1)(sz^2) \, d\mu_1(z),
$$

where we view the conditional expectations relative to $Z_1$ as functions defined on $Z_1$.

Claim: The invariant $\sigma$-algebra $\mathcal{I}(\sigma_2) = \mathcal{I}(T \times T^2)$ of $(X \times X, \mu \times \mu, T \times T)$ is isomorphic to $Z_1$.

Proof of Claim: This is a classical result. Here we give a sketch of a proof and later we will give another proof when we deal with the general case. First by Theorem A.3 we have $K(T^2) = K(T)$, and hence $Z_1$ is the Kronecker factor for both $(X, \mathcal{X}, \mu, T)$ and $(X, \mathcal{X}, \mu, T^2)$. Let $q_1 : (X, \mathcal{X}, \mu, T) \to (Z_1, \mathcal{Z}_1, \mu_1, T)$ and $q_2 : (X, \mathcal{X}, \mu, T^2) \to (Z_1, \mathcal{Z}_1, \mu_1, T^2)$ be the factor maps. By Theorem A.5 if $F \in L^2(X \times X, \mu \times \mu)$ is invariant under $T \times T^2$, then there exists a function $\Phi \in L^2(Z_1 \times Z_1, \mu_1 \times \mu_1)$ so that $F(x, y) = \Phi(q_1(x), q_2(y))$. That means $\mathcal{I}(X \times X, T \times T^2)$ is measurable with respect to $Z_1 \times Z_1$. Hence $\mathcal{I}(X \times X, T \times T^2) = \mathcal{I}(Z_1 \times Z_1, T \times T^2)$, which is isometric to $Z_1$. This ends the proof of Claim.

Let $\phi : (X \times X, \mathcal{X} \times \mathcal{X}) \to (\Omega_1, \mathcal{I}^{|1|}, P_1)$ be the factor map and let $\psi : (\Omega_1, \mathcal{I}^{|1|}, P_1) \to (Z_1, \mathcal{Z}_1, \mu_1)$ be the isomorphic map. Hence we have

$$
(X \times X, \mathcal{X} \times \mathcal{X}) \xrightarrow{\phi} (\Omega_1, \mathcal{I}^{|1|}, P_1) \xleftarrow{\psi} (Z_1, \mathcal{Z}_1, \mu_1)
$$

$$
(x, y) \mapsto \phi(x, y) \longleftrightarrow s = \psi(\phi(x, y))
$$
From this, it is not difficult to deduce that the ergodic decompositions of $\mu_1 \times \mu_1$ and $\mu \times \mu$ under $\sigma_2 = T \times T^2$ can be written as

\begin{equation}
\mu_1 \times \mu_1 = \int_{Z_1} \xi_{1,s} \, d\mu_1(s); \quad \mu \times \mu = \int_{Z_1} \xi_s \, d\mu_1(s).
\end{equation}

In particular, for $\mu_1$-almost every $s$, the measure $\xi_s$ is ergodic for $\sigma_2 = T \times T^2$.

Now we continue our proof for $d = 3$. Let $\lambda$ be a $\langle \tau_3, \sigma_3 \rangle$-invariant measure of $N_3(X)$. Let

\begin{align*}
p_1 : (N_3(X), \langle \tau_3, \sigma_3 \rangle) &\to (X, T); \quad (x_1, x_2, x_3) \mapsto x_1 \\
p_2 : (N_3(X), \langle \tau_3, \sigma_3 \rangle) &\to (N_2(X), \langle \tau_2, \sigma_2 \rangle); \quad (x_1, x_2, x_3) \mapsto (x_2, x_3)
\end{align*}

be the projections. Then $(p_2)_*(\lambda)$ is a $\langle \tau_2, \sigma_2 \rangle$-invariant measure of $N_2(X) = X \times X$. By the case $d = 2$, $(p_2)_*(\lambda) = \mu \times \mu$. Hence let

\begin{equation}
\lambda = \int_{X^2} \lambda(x,y) \times \delta(x,y) \, d(\mu \times \mu)(x, y)
\end{equation}

be the disintegration of $\lambda$ over $\mu \times \mu$. Since $\lambda$ is $\sigma'_3 = \text{id} \times \sigma_2 = \text{id} \times T \times T^2$-invariant, we have

\begin{align*}
\lambda &= \text{id} \times \sigma_2 \lambda = \int_{X^2} \lambda(x,y) \times \sigma_2 \delta(x,y) \, d\mu \times \mu(x, y) \\
&= \int_{X^2} \lambda(x,y) \times \delta_{\sigma_2(x,y)} \, d\mu \times \mu(x, y) \\
&= \int_{X^2} \lambda(\sigma_2^{-1}(x,y)) \times \delta(x,y) \, d\mu \times \mu(x, y).
\end{align*}

The uniqueness of disintegration implies that

\begin{equation}
\lambda(\sigma_2^{-1}(x,y)) = \lambda(x,y), \quad \mu \times \mu \text{ a.e.}
\end{equation}

Define

\begin{equation*}
F : (N_2(X) = X \times X, \sigma_2 = T \times T^2) \longrightarrow M(X) : \lambda(x,y) \mapsto \lambda(x,y).
\end{equation*}

By (5.8), $F$ is a $\sigma_2 = T \times T^2$-invariant $M(X)$-value function. Hence $F$ is $T(\sigma_2)$-measurable, and hence $\lambda(x,y) = \lambda(\phi(x,y)) = \lambda, \mu \times \mu$ a.e., where $\phi$ is defined in (5.5).

Thus by (5.7) one has that

\begin{align*}
\lambda &= \int_{X^2} \lambda(x,y) \times \delta(x,y) \, d\mu \times \mu(x, y) = \int_{X^2} \lambda(\phi(x,y)) \times \delta(x,y) \, d\mu \times \mu(x, y) \\
&= \int_{Z_1} \int_{X^2} \lambda_s \times \delta(x,y) \, d\xi_s(x,y) d\mu_1(s) \\
&= \int_{Z_1} \lambda_s \times \left( \int_{X^2} \delta(x,y) \, d\xi_s(x,y) \right) d\mu_1(s) \\
&= \int_{Z_1} \lambda_s \times \xi_s \, d\mu_1(s)
\end{align*}
Let \( \pi_1^3 : (N_3(X), \langle \tau_3, \sigma_3 \rangle) \rightarrow (N_3(Z_1), \langle \tau_3, \sigma_3 \rangle) \) be the natural factor map. By Theorem 5.4, \((N_3(Z_1), \langle \tau_3, \sigma_3 \rangle, \mu_1^{(3)})\) is uniquely ergodic. Hence
\[
\pi_1^3(\lambda) = \mu_1^{(3)} = \int_{Z_1} \delta_s \times \mu_1^{(3)} \ d\mu_1(s).
\]
And
\[
\pi_1^s(\lambda_s) = \delta_s, \text{ and } (\pi_1 \times \pi_1)^s(\xi_s) = \mu_1^{(3)}.
\]
Note that we have that
\[
(p_1)^s(\lambda) = \mu, \text{ and } (p_2)^s(\lambda) = \mu \times \mu,
\]
and hence we have
\[
\mu = \int_{Z_1} \lambda_s \ d\mu_1(s).
\]
Let \( \mu = \int_{Z_1} \rho_s \ d\mu_1(s) \) be the disintegration of \( \mu \) over \( \mu_1 \). Note that \( \pi_1^s(\lambda_s) = \pi_1^s(\rho_s) = \delta_s, \mu_1, a.e. \). Hence by the uniqueness of disintegration, we have that \( \lambda_s = \rho_s, \mu_1 a.e. \). Thus
\[
\lambda = \int_{Z_1} \lambda_s \times \xi_s \ d\mu_1(s) = \int_{Z_1} \rho_s \times \xi_s \ d\mu_1(s).
\]
That is, \((N_3(X), \langle \tau_3, \sigma_3 \rangle)\) is uniquely ergodic.

5.4.4. Some preparations. Before going into the proof of the general case, we need some preparations. Recall the definition of \( \mu^{(d)} \) after Definition 5.1.

**Lemma 5.6.** Let \((X, \mathcal{X}, \mu, T)\) be an ergodic system and \( d \geq 1 \) be an integer. Assume that \( f_1, \ldots, f_d \in L^\infty(X, \mu) \) with \( \|f_j\|_\infty \leq 1 \) for \( j = 1, \ldots, d \). Then
\[
\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x_1) f_2(T^{2n} x_2) \ldots f_d(T^{dn} x_d) \right\|_{L^2(\mu^{(d)})} \leq \min_{1 \leq l \leq d} \{l \cdot \|f_l\| d\}
\]

**Proof.** We proceed by induction. For \( d = 1 \), by the Ergodic Theorem,
\[
\left\| \frac{1}{N} \sum_{n=0}^{N-1} T^n f_1 \right\|_{L^2(\mu)} \rightarrow \left| \int f_1 d\mu \right| = \| f_1 \|_1.
\]
Let \( d \geq 1 \) and assume that (C.1) holds for \( d \). Let \( f_1, \ldots, f_{d+1} \in L^\infty(\mu) \) with \( \|f_j\|_\infty \leq 1 \) for \( j = 1, \ldots, d+1 \). Choose \( l \in \{2, 3, \ldots, d+1\} \). (The case \( l = 1 \) is similar). Write
\[
\xi_n = \bigotimes_{j=1}^{d+1} T^j f_j = f_1(T^n x_1) f_2(T^{2n} x_2) \ldots f_{d+1}(T^{(d+1)n} x_{d+1}).
\]

By the van der Corput lemma (Lemma F.1),
\[
\lim_{N \to \infty} \sup_{H} \left| \frac{N}{N} \sum_{n=0}^{N-1} \xi_n \right|_{L^2(\mu^{(d+1)})} \leq \lim_{H \to \infty} \sup_{h=0}^{H-1} \left| \frac{1}{N} \sum_{n=0}^{N-1} \xi_{n+h} : \xi_n d\mu^{(d+1)} \right|.
\]
Letting $M$ denote the last lim sup, we need to show that $M \leq l^2 \| f_l \|_{d+1}^2$. For any $h \geq 1$,

$$
\left| \frac{1}{N} \sum_{n=0}^{N-1} \int \xi_{n+h} \cdot \xi_n \, d\mu^{(d+1)} \right|
\leq \left| \int (f_1 \cdot T^hf_1) \otimes \frac{1}{N} \sum_{n=0}^{N-1} (\sigma_d)^n \otimes f_j \cdot T^{jh} f_j \mu^{(d+1)}(x_1, \ldots, x_{d+1}) \right|
\leq \left\| f_1 \cdot T^h f_1 \right\|_{L^2(\mu^{(d+1)})} \cdot \left\| \frac{1}{N} \sum_{n=0}^{N-1} (\sigma_d)^n \otimes f_j \cdot T^{jh} f_j \right\|_{L^2(\mu^{(d+1)})}
= \left\| f_1 \cdot T^h f_1 \right\|_{L^2(\mu)} \cdot \left\| \frac{1}{N} \sum_{n=0}^{N-1} (\sigma_d)^n \otimes f_j \cdot T^{jh} f_j \right\|_{L^2(\mu)}
$$

and by the inductive assumption,

$$
\left| \frac{1}{N} \sum_{n=0}^{N-1} \int \xi_{n+h} \cdot \xi_n \, d\mu^{(d+1)} \right| \leq l \left\| f_1 \cdot T^h f_1 \right\|_{d}.
$$

We get

$$
M \leq l \limsup_{H \to \infty} \frac{1}{H} \sum_{h=0}^{H-1} \left\| f_1 \cdot T^h f_1 \right\|_{d} \leq l^2 \limsup_{H \to \infty} \frac{1}{H} \sum_{h=0}^{H-1} \left\| f_1 \cdot T^h f_1 \right\|_{d}
\leq l^2 \limsup_{H \to \infty} \left( \frac{1}{H} \sum_{h=0}^{H-1} \left\| f_1 \cdot T^h f_1 \right\|_{d}^2 \right)^{1/2}
= l^2 \left\| f_1 \right\|_{d+1}^2.
$$

The last equation follows from Lemma E.1. The proof is completed. \(\square\)

**Lemma 5.7.** Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system and $d \in \mathbb{N}$. Assume that $f_1, \ldots, f_d \in L^\infty(X, \mu)$. Then

$$
E \left( \bigotimes_{j=1}^d f_j \big| \mathcal{I}(X^d, \mu^{(d)}, \sigma_d) \right) = E \left( \bigotimes_{j=1}^d E(f_j \big| \mathcal{Z}_{d-1}) \big| \mathcal{I}(X^d, \mu^{(d)}, \sigma_d) \right).
$$

**Proof.** By Lemma 3.2, it suffices to show that

$$
E \left( \bigotimes_{j=1}^d f_j \big| \mathcal{I}(X^d, \mu^{(d)}, \sigma_d) \right) = 0
$$
whenever $\mathbb{E}(f_k Z_{d-1}) = 0$ for some $k \in \{1, 2, \ldots, d\}$. This condition implies that $\|f_k\|_d = 0$. By the Ergodic Theorem and Lemma \ref{lem:ergodic}, we have

$$
\left| \mathbb{E}\left( \bigotimes_{j=1}^{d} f_j \mathcal{I}(X, \mathcal{F}, \mu), \sigma_d \right) \right| \\
= \lim_{N \to \infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x_1) f_2(T^{2n} x_2) \ldots f_d(T^{dn} x_d) \right| \leq k \cdot \|f_k\|_d = 0.
$$

So the lemma follows.

**Proposition 5.8.** Let $(X, \mathcal{X}, \mu, T)$ be ergodic and $d \in \mathbb{N}$. Then the $\sigma$-algebra $\mathcal{I}(X, \mathcal{F}, \mu)$ is measurable with respect to $Z_{d-1}$.

**Proof.** Every bounded function on $X^d$ which is measurable with respect to $\mathcal{I}(X, \mathcal{F}, \mu)$ can be approximated in $L^2(\mu)$ by finite sums of functions of the form $\mathbb{E}(\bigotimes_{j=1}^{d} f_j \mathcal{I}(X, \mathcal{F}, \mu), \sigma_d)$ where $f_1, \ldots, f_d$ are bounded functions on $X$. By Lemma \ref{lem:ergodic}, one can assume that these functions are measurable with respect to $Z_{d-1}$. In this case $\bigotimes_{j=1}^{d} f_j$ is measurable with respect to $Z_{d-1}$. Since this $\sigma$-algebra $Z_{d-1}$ is invariant under $\sigma_d$, $\mathbb{E}(\bigotimes_{j=1}^{d} f_j \mathcal{I}(X, \mathcal{F}, \mu), \sigma_d)$ is also measurable with respect to $Z_{d-1}$. Therefore $\mathcal{I}(X, \mathcal{F}, \mu)$ is measurable with respect to $Z_{d-1}$.

**Corollary 5.9.** Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system and $d \in \mathbb{N}$. Then the factor map $\pi_{d-1} : (X^d, \mathcal{F}, \mu) \to (Z_{d-1}^d, \mathcal{F}, \mu)$ is ergodic.

5.4.5. **General case.** Now we show the general case. Assume that Theorem B holds for $d \geq 1$. We show it also holds for $d + 1$.

Let $\pi_{d-1} : X \to Z_{d-1}$ be the factor map from $X$ to its $d - 1$-step nilfactor $Z_{d-1}$. By definition, $Z_{d-1}$ may be regarded as a topological system in the natural way. By Weiss’s Theorem, there is a uniquely ergodic model $(\hat{X}, \hat{\mathcal{X}}, \hat{\mu}, T)$ for $(X, \mathcal{X}, \mu, T)$ and a factor map $\hat{\pi}_{d-1} : \hat{X} \to Z_{d-1}$ which is a model for $\pi_{d-1} : X \to Z_{d-1}$.

$$
\begin{array}{ccc}
X & \longrightarrow & \hat{X} \\
\pi_{d-1} & \downarrow & \hat{\pi}_{d-1} \\
Z_{d-1} & \longrightarrow & Z_{d-1}
\end{array}
$$

Hence for simplicity, we may assume that $(\hat{X}, \hat{\mathcal{X}}, \hat{\mu}, T) = (X, \mathcal{X}, \mu, T)$ and $\pi_{d-1} = \hat{\pi}_{d-1}$. Now we show that $(X^{d+1}, \{\tau_{d+1}, \sigma_{d+1}\})$ is uniquely ergodic. Recall that $(X^d, \{\tau_d, \sigma_d\})$ is uniquely ergodic by the inductive assumption, and we denote its unique measure by $\mu^{(d)}$.

By Corollary \ref{cor:ergodic}, the factor map $\pi_{d-1}^{(d)} : (X^d, \mu^{(d)}, \sigma_d) \to (Z_{d-1}^d, \mu^{(d)}_{d-1}, \sigma_d)$ is ergodic. Hence $\mathcal{I}(X^d, \mu^{(d)}, \sigma_d) = \mathcal{I}(Z_{d-1}^d, \mu^{(d)}_{d-1}, \sigma_d)$. By \eqref{eq:ergodicity},

$$
\mu^{(d)}_{d-1} = \int_{Z_{d-1}} \nu^{(d)}_{d-1,x} \, d\mu_{d-1}(x)
$$
is the ergodic decomposition of \( \mu^{(d)}_{d-1} \) under \( \sigma_d \). Hence \((X^d, I(X^d, \mu^{(d)}; \sigma_d))\) is isomorphic to \((Z_{d-1}, \mathcal{Z}_{d-1}, \mu_{d-1})\). Let
\[
(X^d, \mathcal{X}^d, \mu^{(d)}) \xrightarrow{\phi} (X^d, I(X^d, \mu^{(d)}; \sigma_d), \mu^{(d)}) \xleftarrow{\psi} (Z_{d-1}, \mathcal{Z}_{d-1}, \mu_{d-1})
\]
\[
\mathbf{x} \xrightarrow{} \phi(\mathbf{x}) \leftrightarrow s = \psi(\phi(\mathbf{x}))
\]

From this, we can denote the ergodic decompositions of \( \mu^{(d)} \) under \( \sigma_d \) by
\[
\mu^{(d)} = \int_{Z_{d-1}} \nu^{(d)}_s \, d\mu_{d-1}(s).
\]

Now we continue our proof for \( d + 1 \). Let \( \lambda \) be a \( (\tau_{d+1}, \sigma_{d+1}) \)-invariant measure of \( N_{d+1}(X) \). Let
\[
p_1 : (N_{d+1}(X), (\tau_{d+1}, \sigma_{d+1})) \rightarrow (X, T); (x_1, \mathbf{x}) \mapsto x_1
\]
\[
p_2 : (N_{d+1}(X), (\tau_{d+1}, \sigma_{d+1})) \rightarrow (N_d(X), (\tau_d, \sigma_d)); (x_1, \mathbf{x}) \mapsto \mathbf{x}
\]
be the projections. Then \( (p_2)_*(\lambda) \) is a \( (\tau_d, \sigma_d) \)-invariant measure of \( N_d(X) \). By the assumption on \( d \), \( (p_2)_*(\lambda) = \mu^{(d)} \). Hence let
\[
\lambda = \int_{X^d} \lambda_x \times \delta_x \, d\mu^{(d)}(\mathbf{x})
\]
be the disintegration of \( \lambda \) over \( \mu^{(d)} \). Since \( \lambda \) is \( \sigma_{d+1}' = \text{id} \times \sigma \)-invariant, we have
\[
\lambda = \text{id} \times \sigma_d \lambda = \int_{X^d} \lambda_x \times \sigma_d \delta_x \, d\mu^{(d)}(\mathbf{x})
\]
\[
= \int_{X^d} \lambda_x \times \delta_x \, d\mu^{(d)}(\mathbf{x})
\]
\[
= \int_{X^d} \lambda_{(\sigma_d)^{-1}(x)} \times \delta_x \, d\mu^{(d)}(\mathbf{x}).
\]
The uniqueness of disintegration implies that
\[
\lambda_{(\sigma_d)^{-1}(x)} = \lambda_x, \quad \mu^{(d)} \text{ a.e.}
\]

Define
\[
F : (X^d, \mu^{(d)}, \sigma_d) \rightarrow M(X) : (x, y) \mapsto \lambda(x, y).
\]
By \((5.15)\), \( F \) is a \( \sigma_d \)-invariant \( M(X) \)-value function. Hence \( F \) is \( \mathcal{I}(X^d, \mu^{(d)}, \sigma_d) \)-measurable, and hence \( \lambda_x = \lambda_{(x)} = \lambda_y, \quad \mu^{(d)} \text{ a.e.} \), where \( \phi \) is defined in \((5.12)\).

Thus by \((5.14)\) one has that
\[
\lambda = \int_{X^d} \lambda_x \times \delta_x \, d\mu^{(d)}(\mathbf{x}) = \int_{X^d} \lambda_{\phi(x)} \times \delta_x \, d\mu^{(d)}(\mathbf{x})
\]
\[
= \int_{Z_{d-1}} \int_{X^d} \lambda_s \times \delta_x \, d\nu^{(d)}_s(\mathbf{x}) \, d\mu_{d-1}(s)
\]
\[
= \int_{Z_{d-1}} \lambda_s \times \left( \int_{X^d} \delta_x \, d\nu^{(d)}_s(\mathbf{x}) \right) \, d\mu_{d-1}(s)
\]
\[
= \int_{Z_{d-1}} \lambda_s \times \nu^{(d)}_s \, d\mu_{d-1}(s)
\]
Let $\pi^{d+1} : (N_{d+1}(X), (\tau_{d+1}, \sigma_{d+1})) \to (N_{d+1}(Z_{d-1}), (\tau_{d+1}, \sigma_{d+1}))$ be the natural factor map. By Theorem 5.4, $(N_{d+1}(Z_{d-1}), (\tau_{d+1}, \sigma_{d+1}), \mu_d)$ is uniquely ergodic. Hence by Theorem 5.4

$$\pi^{d+1}_*(\lambda) = \mu_d = \int_{Z_{d-1}} \lambda_s \, d\mu_{d-1}(s).$$

And

$$\pi_*(\lambda_s) = \delta_s, \text{ and } (\pi^d)_*(\nu^{(d)}_s) = \mu^{(d+1)}_{d-1,s}.$$ Note that we have that

$$(p_1)_*(\lambda) = \mu, \text{ and } (p_2)_*(\lambda) = \mu^{(d)},$$

and hence we have

$$\mu = \int_{Z_{d-1}} \lambda_s \, d\mu_{d-1}(s).$$

Let $\mu = \int_{Z_{d-1}} \theta_s \, d\mu_{d-1}(s)$ be the disintegration of $\mu$ over $\mu_{d-1}$. Note that $\pi_*(\lambda_s) = \pi_*(\theta_s) = \delta_s, \, \mu_{d-1}, \text{ a.e.}$ Hence by the uniqueness of disintegration, we have that $\lambda_s = \theta_s, \, \mu_{d-1} \text{ a.e.}$. Thus

$$(5.16) \lambda_{\tau, \sigma_{d+1}} = \lambda = \int_{Z_{d-1}} \lambda_s \times \nu_s^{(d)} \, d\mu_{d-1}(s) = \int_{Z_{d-1}} \theta_s \times \nu_s^{(d)} \, d\mu_{d-1}(s).$$

That is, $(N_{d+1}(X), (\tau_{d+1}, \sigma_{d+1}))$ is uniquely ergodic. The whole proof is completed.

\[\square\]

**APPENDIX A. Background on Ergodic Theory**

In this Appendix we try to cover notions and results in ergodic theory which are used in the article. Let $(X, \mathcal{X}, \mu, T)$ be a measurable system.

**A.0.6. Ergodicity and weak mixing.** First we list some equivalent conditions for ergodicity and weak mixing.

**Theorem A.1.** Let $(X, \mathcal{X}, \mu, T)$ be a measurable system. Then the following conditions are equivalent:

1. $T$ is ergodic.
2. Every measurable function $f$ from $X$ to some Polish Space $P$ satisfying $f \circ T = f$ a.e. is of form $f \equiv p$ a.e. for some point $p \in P$.
3. $\lim_{N \to \infty} \sum_{n=0}^{N-1} \int f \circ T^n \cdot g \, d\mu = \int f \, d\mu \int g \, d\mu, \text{ for all } f, g \in L^2(\mu) \text{ (or } L^1(\mu)).$

**Theorem A.2.** Let $(X, \mathcal{X}, \mu, T)$ be a measurable system. Then the following conditions are equivalent:

1. $T$ is weakly mixing.
2. 1 is the only eigenvalue of $T$ and the geometric multiplicity of eigenvalue 1 is 1.
3. The product system with any ergodic system is still ergodic.
A.0.7. Conditional expectation. If \( \mathcal{Y} \) is a \( T \)-invariant sub-\( \sigma \)-algebra of \( \mathcal{X} \) and \( f \in L^1(\mu) \), we write \( \mathbb{E}(f|\mathcal{Y}) \), or \( \mathbb{E}_\mu(f|\mathcal{Y}) \) if needed, for the conditional expectation of \( f \) with respect to \( \mathcal{Y} \). The conditional expectation \( \mathbb{E}(f|\mathcal{Y}) \) is characterized as the unique \( \mathcal{Y} \)-measurable function in \( L^2(Y,\mathcal{Y},\nu) \) such that
\[
\int_Y g \mathbb{E}(f|\mathcal{Y})d\nu = \int_X g \circ \pi fd\mu
\]
for all \( g \in L^2(Y,\mathcal{Y},\nu) \). We will frequently make use of the identities
\[
\int \mathbb{E}(f|\mathcal{Y}) \ d\mu = \int f \ d\mu \quad \text{and} \quad T\mathbb{E}(f|\mathcal{Y}) = \mathbb{E}(Tf|\mathcal{Y}).
\]
We say that a function \( f \) is orthogonal to \( \mathcal{Y} \), and we write \( f \perp \mathcal{Y} \), when it has a zero conditional expectation on \( \mathcal{Y} \). If a function \( f \in L^1(\mu) \) is measurable with respect to the factor \( \mathcal{Y} \), we write \( f \in L^1(Y,\mathcal{Y},\nu) \).

The disintegration of \( \mu \) over \( \nu \), written as \( \mu = \int \mu_y \ d\nu(y) \), is given by a measurable map \( y \mapsto \mu_y \) from \( Y \) to the space of probability measures on \( X \) such that
\[
\mathbb{E}(f|\mathcal{Y})(y) = \int_X f d\mu_y
\]
\( \nu \)-almost everywhere.

A.0.8. Ergodic decomposition. Let \( x \mapsto \mu_x \) be a regular version of the conditional measures with respect to the \( \sigma \)-algebra \( \mathcal{I} \). This means that the map \( x \mapsto \mu_x \) is \( \mathcal{I} \)-measurable, and for very bounded measurable function \( f \) we have \( E_\mu(f|\mathcal{I})(x) = \int f \ d\mu_x \) for \( \mu \)-almost every \( x \in X \). Then the ergodic decomposition of \( \mu \) is \( \mu = \int \mu_x d\mu(x) \). The measures \( \mu_x \) have the additional property that for \( \mu \)-almost every \( x \in X \) the system \( (X,\mathcal{X},\mu_x,T) \) is ergodic.

A.0.9. Inverse limit. We say that \( (X,\mathcal{X},\mu,T) \) is an inverse limit of a sequence of factors \( (X,\mathcal{X}_j,\mu,T) \) if \( (\mathcal{X}_j)_{j \in \mathbb{N}} \) is an increasing sequence of \( T \)-invariant sub-\( \sigma \)-algebras such that \( \bigvee_{j \in \mathbb{N}} \mathcal{X}_j = \mathcal{X} \) up to sets of measure zero.

A.0.10. Group rotation. All locally compact groups are implicitly assumed to be metrizable and endowed with their Borel \( \sigma \)-algebras. Every compact group \( G \) is endowed with its Haar measure, denoted by \( m_G \).

For a compact abelian group \( Z \) and \( t \in Z \), we write \( (Z,t) \) for the probability space \( (Z,m_Z) \), endowed with the transformation given by \( z \mapsto tz \). A system of this kind is called a rotation.

A.0.11. Joining and conditional product measure. Let \( (X_i,\mu_i,T_i), i = 1,\ldots,k \), be measurable systems, and let \( (Y_i,\nu_i,S_i) \) be corresponding factors, and \( \pi_i : X_i \rightarrow Y_i \) the factor maps. A measure \( \nu \) on \( Y = \prod_i Y_i \) defines a joining of the measures on \( Y_i \) if it is invariant under \( S_1 \times \cdots \times S_k \) and maps onto \( \nu_j \) under the natural map \( \prod_i Y_i \rightarrow Y_j \).

Let \( \nu \) be a joining of the measures on \( Y_i, i = 1,\ldots,k \), and let \( \mu_i = \int \mu_{x_i,y_i} \ d\nu_i(y_i) \) represent the disintegration of \( \mu_i \) with respect to \( \nu_i \). Let \( \mu \) be a measure on \( X = \prod_i X_i \) with
\[ \prod X_i \text{ defined by} \]
\[
(A.3) \quad \mu = \int_Y \mu_{X_1,Y_1} \times \mu_{X_2,Y_2} \times \cdots \times \mu_{X_k,Y_k} \, d\nu(y_1, y_2, \ldots, y_k).
\]

Then \( \mu \) is called the conditional product measure with respect to \( \nu \).

Equivalently, \( \mu \) is conditional product measure relative to \( \nu \) if and only if for all \( k \)-tuple \( f_i \in L^\infty(X_i, \mu_i), i = 1, \ldots, k \)
\[
(A.4) \quad \int_X f_1(x_1)f_2(x_2)\cdots f_k(x_k) \, d\mu(x_1, x_2, \ldots, x_k)
\]
\[
= \int_Y \mathbb{E}(f_1|Y_1)(y_1)\mathbb{E}(f_2|Y_2)(y_2)\cdots \mathbb{E}(f_k|Y_k)(y_k) \, d\nu(y_1, y_2, \ldots, y_k).
\]

A.0.12. Relatively independent joining. Let \( (X_1, X_1', \mu_1, T), (X_2, X_2', \mu_2, T) \) be two systems and let \( (Y, \mathcal{Y}, \nu, S) \) be a common factor with \( \pi_i : X_i \to Y \) for \( i = 1, 2 \) the factor maps. Let \( \mu_i = \int \mu_{i,y} \, d\nu(y) \) represent the disintegration of \( \mu_i \) with respect to \( Y \). Let \( \mu_1 \times_Y \mu_2 \) denote the measure defined by
\[
\mu_1 \times_Y \mu_2(A) = \int_Y \mu_{1,y} \times \mu_{2,y} \, d\nu(y),
\]
for all \( A \in \mathcal{X}_1 \times \mathcal{X}_2 \). The system \((X_1 \times X_2, \mathcal{X}_1 \times \mathcal{X}_2, \mu_1 \times_Y \mu_2, T \times T)\) is called the relative product of \( X_1 \) and \( X_2 \) with respect to \( Y \) and is denoted \( X_1 \times_Y X_2 \). \( \mu_1 \times_Y \mu_2 \) is also called relatively independent joining of \( X_1 \) and \( X_2 \) over \( Y \).

A.0.13. Isometric extensions. Let \( \pi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, S) \) be a factor map. The \( L^2(X, \mathcal{X}, \mu) \) norm is denoted by \( \| \cdot \| \) and the \( L^2(X, \mathcal{X}, \mu_\nu) \) norm by \( \| \cdot \|_{\nu} \) for \( \nu \)-almost every \( y \in Y \). Recall \( \{ \mu_y \}_{y \in Y} \) is the disintegration of \( \mu \) relative to \( \nu \).

A function \( f \in L^2(X, \mathcal{X}, \mu) \) is almost periodic over \( \mathcal{Y} \) if for every \( \epsilon > 0 \) there exist \( g_1, \ldots, g_l \in L^2(X, \mathcal{X}, \mu) \) such that for all \( n \in \mathbb{Z} \)
\[
\min_{1 \leq j \leq l} \| T^n f - g_j \|_y < \epsilon
\]
for \( \nu \) almost every \( y \in Y \). One writes \( f \in AP(\mathcal{Y}) \). Let \( K(X|Y, T) \) be the closed subspace of \( L^2(X) \) spanned by the almost periodic functions over \( \mathcal{Y} \). When \( \mathcal{Y} \) is trivial, \( K(X, T) = K(X|Y, T) \) is the closed subspace spanned by eigenfunctions of \( T \).

\( X \) is an isometric extensions of \( Y \) if \( K(X|Y, Y) = L^2(X) \) and it is a relatively weak mixing extension of \( Y \) if \( K(X|Y, T) = L^2(Y) \).

Theorem A.3. [14, Lemma 6.7.] For all \( n \in \mathbb{N} \), we have \( K(X|Y, T^n) = K(X|Y, T) \).

Theorem A.4. [14, Theorem 7.1.] \( K(X_1 \times X_2|Y, T) = K(X_1|Y, T) \otimes K(X_2|Y, T) \).

Theorem A.5. [14, Theorem 9.5.] Let \( k \in \mathbb{N} \). Assume \((X_i, \mathcal{X}_i, \mu_i, T_i)\) is an extension of \((Y_i, \mathcal{Y}_i, \nu_i, T_i)\) and each \( T_i \) has only finitely many ergodic components for all \( i \in \{1, 2, \ldots, k\} \). Let \( \mu \) is a conditional product measure with respect to a joining \( \nu \) over \( Y_i \). Let \((Y_i', \mathcal{Y}_i', \nu_i', T_i)\) be the largest isometric extension of \( Y_i \) in \( X_i \), and \( \pi_i' : X_i \to Y_i' \) be the factor map for all \( i \). Then almost all ergodic components of \( \mu \) are conditional product measures relative to \( Y' = \prod Y_i' \).
Equivalently, if \( F \in L^2(\mu) \) is invariant under \( T_1 \times T_2 \times \ldots \times T_k \), then there exists a function \( \Phi \in L^2(\prod Y'_i, \nu') \) for \( \nu' \) the image of \( \mu \), so that
\[
F(x_1, x_2, \ldots, x_k) = \Phi(\pi'_1(x_1), \ldots, \pi'_k(x_k)).
\]

Appendix B. The pointwise ergodic theorem for amenable groups

B.0.14. Amenability has many equivalent formulations; for us, the most convenient definition is that a locally compact group \( G \) is amenable if for any compact \( K \subset G \) and \( \delta > 0 \) there is a compact set \( F \subset G \) such that
\[
|F \Delta KF| < \delta |F|,
\]
where we use both \( |\cdot| \) and \( m \) to denote the left Haar measure on \( G \) (for discrete \( G \), we take this to be the counting measure on \( G \)). Such a set \( F \) will be called \((K, \delta)\)-invariant. A sequence \( F_1, F_2, \ldots \) of compact subsets of \( G \) will be called a Følner sequence if for every compact \( K \) and \( \delta > 0 \), for all large enough \( n \) we have that \( F_n \) is \((K, \delta)\)-invariant. Here all groups are assumed to be locally compact second countable.

B.0.15. Suppose now that \( G \) acts bi-measurably from the left by measure preserving transformations on a Lebesgue space \((X, \mathcal{B}, \mu)\) with \( \mu(X) = 1 \). We will use for any \( f : X \to \mathbb{R} \) the symbol \( A(F, f)(x) = A_F(f) \) to denote the average
\[
A(F, f)(x) = \frac{1}{|F|} \int_{F} f(gx) \, dm(g).
\]

Definition B.1. A Følner sequence \( F_n \) will be said to be tempered if for some \( C > 0 \) and all \( n \)
\[
\left| \bigcup_{k \leq n} F_k^{-1} F_n \right| \leq C |F_n|.
\]

Theorem B.2 (Lindenstrauss [31]). Let \( G \) be an amenable group acting on a measure space \((X, \mathcal{B}, \mu)\) by measure preserving transformation, and let \( F_n \) be a tempered Følner sequence. Then for any \( f \in L^1(\mu) \), there is a \( G \)-invariant \( f^* \in L^1(\mu) \) such that
\[
\lim_{n \to \infty} A(F_n, f)(x) = f^*(x) \quad \text{a.e.}
\]
In particular, if the \( G \) action is ergodic,
\[
\lim_{n \to \infty} A(F_n, f)(x) = \int f(x) \, d\mu(x) \quad \text{a.e.}
\]

Appendix C. Uniquely ergodic systems

In this section we give some conditions for unique ergodicity under \( \mathbb{Z}^d \) actions \((d \in \mathbb{N})\). For completeness a proof is given.

Theorem C.1. Let \((X, \Gamma)\) be a topological system, where \( \Gamma = \mathbb{Z}^d \). The following conditions are equivalent.
(1) \((X, \Gamma)\) is uniquely ergodic.
(2) For every continuous function \( f \in C(X) \) the sequence of functions
\[
A_N f(x) = \frac{1}{N^d} \sum_{\gamma \in [0,N-1]^d} f(\gamma x)
\]
converges uniformly to a constant function.

(3) For every continuous function \( f \in C(X) \) the sequence of functions \( A_N f(x) \)
converges pointwise to a constant function.

(4) There exists a \( \mu \in M_\Gamma(X) \) such that for all continuous function \( f \in C(X) \) and all \( x \in X \) the sequence of functions
\[
A_N f(x) \rightarrow \int f \, d\mu, \quad N \to \infty.
\]

Proof. “(2) \Rightarrow (3)” is obvious.

“(3) \Rightarrow (4)”:

Define a functional \( \Phi : C(X) \to \mathbb{C} \) by
\[
f \mapsto \lim_{N \to \infty} A_N f(x) = \lim_{N \to \infty} \frac{1}{N^d} \sum_{\gamma \in [0,N-1]^d} f(\gamma x)
\]
Since \( \left| \frac{1}{N^d} \sum_{\gamma \in [0,N-1]^d} f(\gamma x) \right| \leq \|f\|_\infty \), it is easy to see that \( \Phi \) a continuous linear positive operator. By Riesz Representation Theorem, there is some \( \mu \in M(X) \) such that
\[
\Phi(f) = \int f \, d\mu.
\]
Since \( \Phi(f \circ \gamma) = \Phi(f) \) for all \( \gamma \in \Gamma \), we have
\[
\int f \, d\gamma \mu = \int f \, d\mu
\]
for all \( f \in C(X) \).

Thus \( \gamma \mu = \mu \) for all \( \gamma \in \Gamma \) and hence \( \mu \in M_\Gamma(X) \).

“(4) \Rightarrow (1)”:

Let \( \nu \in M_\Gamma(X) \). We will show that \( \nu = \mu \). By assumption for all \( x \in X \),
\[
A_N f(x) = \frac{1}{N^d} \sum_{\gamma \in [0,N-1]^d} f(\gamma x) \rightarrow \int f \, d\mu, \quad N \to \infty.
\]
By Dominated Convergence Theorem
\[
\int f \, d\nu = \lim_{N \to \infty} \int \frac{1}{N^d} \sum_{\gamma \in [0,N-1]^d} f(\gamma x) \, d\nu = \int \int f \, d\mu \, d\nu = \int f \, d\mu,
\]
for all \( f \in C(X) \). Thus \( \nu = \mu \).

“(1) \Rightarrow (2)”:

If (2) does not hold, then there is some \( g \in C(X) \) and \( \epsilon > 0 \) such that for any \( N \in \mathbb{N} \) there is some \( n > N \) and \( x_n \in X \) such that
\[
\left| \frac{1}{n^d} \sum_{\gamma \in [0,n-1]^d} g(\gamma x_n) - \int g \, d\mu \right| \geq \epsilon.
\]
Let \( \mu_n = \frac{1}{n^d} \sum_{\gamma \in [0,n-1]^d} \delta_{\gamma x_n} = \frac{1}{n^d} \sum_{\gamma \in [0,n-1]^d} \gamma \delta_{x_n} \). Then rewrite (C.3) as
\[
\left| \int g \, d\mu_n - \int g \, d\mu \right| \geq \epsilon.
\]
Take a limit point \( \mu_\infty \) of \( \{\mu_n\} \) in \( M(X) \). Then it is easy to check that \( \mu_\infty \in M_\Gamma(X) \) and by (C.4) \( \mu_\infty \neq \mu \). This contradicts \( M_\Gamma(X) = \{\mu\} \). The proof is completed. □
D.0.16. Let \( G \) be a group. For \( g, h \in G \), we write \([g, h] = ghg^{-1}h^{-1}\) for the commutator of \( g \) and \( h \) and we write \([A, B]\) for the subgroup spanned by \([a, b] : a \in A, b \in B\). The commutator subgroups \( G_j, j \geq 1 \), are defined inductively by setting \( G_1 = G \) and \( G_{j+1} = [G_j, G] \). Let \( k \geq 1 \) be an integer. We say that \( G \) is \( k \)-step nilpotent if \( G_{k+1} \) is the trivial subgroup.

D.0.17. Let \( G \) be a \( k \)-step nilpotent Lie group and \( \Gamma \) a discrete cocompact subgroup of \( G \). The compact manifold \( X = G/\Gamma \) is called a \( k \)-step nilmanifold. The group \( G \) acts on \( X \) by left translations and we write this action as \( (g, x) \mapsto gx \). The Haar measure \( \mu \) of \( X \) is the unique probability measure on \( X \) invariant under this action. Let \( \tau \in G \) and \( T \) be the transformation \( x \mapsto \tau x \) of \( X \). Then \((X, T, \mu)\) is called a \( k \)-step nilsystem.

D.0.18. For every integer \( j \geq 1 \), the subgroup \( G_j \) and \( \Gamma G_j \) are closed in \( G \). It follows that the group \( \Gamma_j = \Gamma \cap G_j \) is cocompact in \( G_j \).

D.0.19. Here are some basic properties of nilsystems:

**Theorem D.1.** Let \((X = G/\Gamma, \mu, T)\) be a \( k \)-step nilsystem with \( T \) the translation by the element \( t \in G \). Then:

1. \((X, T)\) is uniquely ergodic if and only if \((X, \mu, T)\) is ergodic if and only if \((X, T)\) is minimal if and only if \((X, T)\) is transitive.

2. Let \( Y \) be the closed orbit of some point \( x \in X \). Then \( Y \) can be given the structure of a nilmanifold, \( Y = H/\Lambda \), where \( H \) is a closed subgroup of \( G \) containing \( t \) and \( \Lambda \) is a closed cocompact subgroup of \( H \).

Assume furthermore that \( G \) is spanned by the connected component of the identity and the element \( t \). Then:

3. The groups \( G_j, j \geq 2 \), are connected.

4. The nilsystem \((X, \mu, T)\) is ergodic if and only if the rotation induced by \( t \) on the compact abelian group \( G/G_2\Gamma \) is ergodic.

5. If the nilsystem \((X, \mu, T)\) is ergodic then its Kronecker factor is \( Z = G/G_2\Gamma \) with the rotation induced by \( t \) and with the natural factor map \( X = G/\Gamma \to G/G_2\Gamma = Z \).

**Theorem D.2.** Let \( X = G/\Gamma \) be a nilmanifold with Haar measure \( \mu \) and let \( t_1, \ldots, t_k \) be commuting elements of \( G \). If the group spanned by the translations \( t_1, \ldots, t_k \) acts ergodically on \((X, \mu)\), then \( X \) is uniquely ergodic for this group.

**Appendix E. HK-seminorms**

Let \((X, \mu, T)\) be an ergodic system and \( k \in \mathbb{N} \). We write \( C : \mathbb{C} \to \mathbb{C} \) for the conjugate map \( z \mapsto \overline{z} \). Let \( |\epsilon| = \epsilon_1 + \ldots + \epsilon_k \) for \( \epsilon \in V_k = \{0, 1\}^k \). It is easy to verify that for all \( f \in L^\infty(\mu) \) the integral \( \int_{X^{|k|}} \bigotimes_{\epsilon \in V_k} C^{(|\epsilon|} f(x) d\mu^{(|k|} \) is real and nonnegative. Hence we can define

\[
\|f\|_k = \left( \int_{X^{|k|}} \bigotimes_{\epsilon \in V_k} C^{(|\epsilon|} f(x) d\mu^{(|k|} \right)^{1/2k}.
\]
As $X$ is assumed to be ergodic, the $\sigma$-algebra $\mathcal{I}^{[0]}$ is trivial and $\mu^{[1]} = \mu \times \mu$. We therefore have

$$\|f\|_1 = \left( \int_{X^2} f(x_0)f(x_1)d\mu \times \mu(x_0,x_1) \right)^{1/2} = \left| \int f d\mu \right|.$$

It is showed in [24] that $\| \cdot \|_k$ is a seminorm on $L^\infty(\mu)$, and for all $f_\epsilon \in L^\infty(\mu), \epsilon \in V_k$,

$$\left| \int \bigotimes_{\epsilon \in V_k} f_\epsilon d\mu^{[k]} \right| \leq \prod_{\epsilon \in V_k} \|f_\epsilon\|_k.$$

The following lemma follows immediately from the definition of the measures and the Ergodic Theorem.

**Lemma E.1.** For every integer $k \geq 0$ and every $f \in L^\infty(\mu)$, one has

$$\|f\|_{k+1} = \left( \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \|f \cdot T^n f\|_k^{2^k} \right)^{1/2^{k+1}}. \tag{E.2}$$

An important property is

**Proposition E.2.** For $f \in L^\infty(\mu)$, $\|f\|_k = 0$ if and only if $E(f|\mathcal{Z}_{k-1}) = 0$.

**APPENDIX F. The van der Corput lemma**

**Lemma F.1.** Let $\{x_n\}$ be a bounded sequence in a Hilbert space $\mathcal{H}$ with norm $\| \cdot \|$ and inner product $\langle \cdot, \cdot \rangle$. Then

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} x_n^2 \leq \limsup_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \sum_{n=1}^{\infty} \langle x_n, x_{n+h} \rangle.$$

**APPENDIX G. Invariant algebra of $T \times T^2 \times \ldots \times T^d$**

**Lemma G.1.** Let $(X,\mathcal{X},\mu,T)$ be an ergodic system, $d \geq 1$ be an integer and let $\lambda$ be any $d$-fold self-joining of $X$. Assume that $f_1, \ldots, f_d \in L^\infty(X,\mu)$ with $\|f_j\|_\infty \leq 1$ for $j = 1, \ldots, d$. Then

$$\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^nx_1)f_2(T^{2n}x_2) \ldots f_d(T^{dn}x_d) \right\|_{L^2(X,\lambda)} \leq \min_{1 \leq i \leq d} \{t \cdot \|f_i\|_d\}. \tag{G.1}$$

**Proof.** We proceed by induction. For $d = 1$, the only self-joining $\lambda$ is $\mu$. So by the Ergodic Theorem,

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} T^n f_1 \right\|_{L^2(\mu)} \to \left| \int f_1 d\mu \right| = \|f_1\|_1.$$

Let $d \geq 1$ and assume that (G.1) holds for $d$ and any $d$-fold self-joining of $X$. Let $f_1, \ldots, f_{d+1} \in L^\infty(\mu)$ with $\|f_j\|_\infty \leq 1$ for $j = 1, \ldots, d+1$. Let $\lambda$ be any $d+1$-fold self-joining of $X$. Choose $l \in \{2,3,\ldots,d+1\}$. (The case $l = 1$ is similar). Write

$$\xi_n = \bigotimes_{j=1}^{d+1} T^{i_j} f_j = f_1(T^nx_1)f_2(T^{2n}x_2) \ldots f_{d+1}(T^{(d+1)n}x_{d+1}).$$
By the van der Corput lemma (Lemma E.1),
\[
\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} \xi_n \right|_{L^2(\lambda)}^2 \leq \limsup_{H \to \infty} \frac{1}{H} \sum_{h=0}^{H-1} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} \int \xi_{n+h} \cdot \xi_n d\lambda \right|.
\]
Letting \( M \) denote the last lim sup, we need to show that \( M \leq l^2 \| f_i \|_{d+1}^2 \). For any \( h \geq 1 \),
\[
\left| \frac{1}{N} \sum_{n=0}^{N-1} \int \xi_{n+h} \cdot \xi_n d\lambda \right| = \left| \int (f_1 \cdot T^h f_1) \otimes \left( \frac{1}{N} \sum_{n=0}^{N-1} (\sigma_d)^n \otimes f_j \cdot T^{jh} f_j \right) d\lambda(x_1, \ldots, x_{d+1}) \right|
\leq \left\| f_1 \cdot T^h f_1 \right\|_{L^2(\lambda)} \cdot \left( \frac{1}{N} \sum_{n=0}^{N-1} (\sigma_d)^n \otimes f_j \cdot T^{jh} f_j \right)_{L^2(\lambda)}
= \left\| f_1 \cdot T^h f_1 \right\|_{L^2(\mu)} \cdot \left( \frac{1}{N} \sum_{n=0}^{N-1} (\sigma_d)^n \otimes f_j \cdot T^{jh} f_j \right)_{L^2(\lambda)}^d,
\]
where \( \lambda' \) is the image of \( \lambda \) to the last \( d \) coordinates. It is clear \( \lambda' \) is a \( d \)-fold self-joining of \( X \), and by the inductive assumption,
\[
\left| \frac{1}{N} \sum_{n=0}^{N-1} \int \xi_{n+h} \cdot \xi_n d\lambda \right| \leq l \cdot \left\| f_i \cdot T^{th} \right\|_d.
\]
We get
\[
M \leq l \cdot \limsup_{H \to \infty} \frac{1}{H} \sum_{h=0}^{H-1} \left\| f_i \cdot T^{th} f_i \right\|_d \leq l^2 \cdot \limsup_{H \to \infty} \frac{1}{H} \sum_{h=0}^{H-1} \left\| f_i \cdot T^h f_i \right\|_d
\leq l^2 \cdot \limsup_{H \to \infty} \left( \frac{1}{H} \sum_{h=0}^{H-1} \left\| f_i \cdot T^h f_i \right\|_d^2 \right)^{1/2}
= l^2 \cdot \left\| f_i \right\|_{d+1}^2.
\]
The last equation follows from Lemma E.1. The proof is completed. \( \square \)

**Lemma G.2.** Let \( (X, X, \mu, T) \) be an ergodic system and \( d \in \mathbb{N} \). Suppose that \( \lambda \) is a \( d \)-fold self-joining of \( X \) and it is \( \sigma_d \)-invariant. Assume that \( f_1, \ldots, f_d \in L^\infty(X, \mu) \). Then
\[
(\text{G.2}) \quad \mathbb{E} \left( \bigotimes_{j=1}^{d} f_j \big| \mathcal{I}(X^d, \lambda, \sigma_d) \right) = \mathbb{E} \left( \bigotimes_{j=1}^{d} \mathbb{E}(f_j | \mathcal{Z}_{d-1}) \big| \mathcal{I}(X^d, \lambda, \sigma_d) \right).
\]

**Proof.** By Lemma G.2, it suffices to show that
\[
(\text{G.3}) \quad \mathbb{E} \left( \bigotimes_{j=1}^{d} f_j \big| \mathcal{I}(X^d, \lambda, \sigma_d) \right) = 0
\]
whenever $\mathbb{E}(f_k | Z_{d-1}) = 0$ for some $k \in \{1, 2, \ldots, d\}$. This condition implies that $\|f_k\|_d = 0$. By the Ergodic Theorem and Lemma \ref{lem: ergodic theorem} we have

$$\left| \mathbb{E} \left( \bigotimes_{j=1}^{d} f_j | \mathcal{I}(X^d, \lambda, \sigma_d) \right) \right| = \lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x_1) f_2(T^{2n} x_2) \cdots f_d(T^{dn} x_d) \right\|_{L^2(X^d, \mu)} \leq k \cdot \|f_k\|_d = 0.$$ 

So the lemma follows.

\begin{proposition}
Let $(X, \mathcal{X}, \nu, T)$ be an ergodic system and $d \in \mathbb{N}$. Suppose that $\lambda$ is a $d$-fold self-joining of $X$ and it is $\sigma_d$-invariant. Then the $\sigma$-algebra $\mathcal{I}(X^d, \lambda, \sigma_d)$ is measurable with respect to $Z_{d-1}^{(d)}$.
\end{proposition}

\begin{proof}
Every bounded function on $X^d$ which is measurable with respect to $\mathcal{I}(X^d, \lambda, \sigma_d)$ can be approximated in $L^2(X^d, \lambda)$ by finite sums of functions of the form $\mathbb{E}(\otimes_{j=1}^{d} f_j | \mathcal{I}(X^d, \lambda, \sigma_d))$ where $f_1, \ldots, f_d$ are bounded functions on $X$. By Lemma \ref{lem: measurable functions}, one can assume that these functions are measurable with respect to $Z_{d-1}^{(d)}$. In this case $\otimes_{j=1}^{d} f_j$ is measurable with respect to $Z_{d-1}^{(d)}$. Since this $\sigma$-algebra $Z_{d-1}^{(d)}$ is invariant under $\sigma_d$, $\mathbb{E}(\otimes_{j=1}^{d} f_j | \mathcal{I}(X^d, \lambda, \sigma_d))$ is also measurable with respect to $Z_{d-1}^{(d)}$. Therefore $\mathcal{I}(X^d, \lambda, \sigma_d)$ is measurable with respect to $Z_{d-1}^{(d)}$.
\end{proof}

\begin{corollary}
Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system and $d \in \mathbb{N}$. Suppose that $\lambda$ is a $d$-fold self-joining of $X$ and it is $\sigma_d$-invariant. Then the factor map $\pi_{d-1}^d : (X^d, \lambda, \sigma_d) \to (Z_{d-1}^{d}, \tilde{\lambda}, \sigma_d)$ is ergodic, where $\tilde{\lambda}$ is the image of $\lambda$.

In particular, one has that $\mathcal{I}(X^d, \lambda, \sigma_d)$ is isomorphic to $\mathcal{I}(Z_{d-1}^{d}, \tilde{\lambda}, \sigma_d)$.
\end{corollary}

\begin{theorem}
Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system and $d \in \mathbb{N}$. Suppose that $\lambda$ is a $d$-fold self-joining of $X$ and it is $(\tau_d, \sigma_d)$-ergodic. Then $\mathcal{I}(X^d, \lambda, \sigma_d)$ is isomorphic to $Z_{d-1}^{(d)}$.
\end{theorem}

\section*{Appendix H. The proof when $d=2$ in Theorem A}

\begin{itemize}
\item[H.0.20] \textit{Graph joinings.} Let $\phi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, T)$ be a homomorphism of ergodic systems. Let $\id \times \phi : X \to X \times Y, x \mapsto (x, \phi(x))$. Define

$$\text{gr}(\mu, \phi) = \int_X \delta_x \times \delta_{\phi(x)} \, d\mu(x) = (\id \times \phi)_*(\mu).$$

It is called a \textit{graph joining} of $\phi$. Equivalently, $\text{gr}(\mu, \phi)$ is defined by

$$\text{gr}(\mu, \phi)(A \times B) = \mu(A \cap \phi^{-1}B), \ \forall A \in \mathcal{X}, B \in \mathcal{Y}.$$

\item[H.0.21] \textit{Kronecker factor $Z_1$.} The Kronecker factor of the ergodic system $(X, \mu, T)$ is an ergodic rotation and we denote it by $(Z_1(X), t_1)$, or more simply $(Z_1, t_1)$. Let $\mu_1$ denote the Haar measure of $Z_1$, and $\pi_{X,1}$ or $\pi_1$ denote the factor map $X \to Z_1$.

For $s \in Z_1$, let $\mu_{1,s}$ denote the image of the measure $\mu_1$ under the map $z \mapsto (z, sz)$ from $Z_1$ to $Z_1^2$, i.e. $\mu_{1,s} = \text{gr}(\mu_1, s)$. This measure is invariant under $T^{[1]} = T \times T$ and is a self-joining of the rotation $(Z_1, t_1)$. Let $\mu_s$ denote the relatively independent
joining of \( \mu \) over \( \mu_{1,s} \). This means that for bounded measurable functions \( f \) and \( g \) on \( X \),

\[
(H.3) \quad \int_{Z_1 \times Z_1} f(x_0)g(x_1) \, d\mu_s(x_0, x_1) = \int_{Z_1} \mathbb{E}(f|Z_1)(z)\mathbb{E}(g|Z_1)(sz) \, d\mu_1(z).
\]

where we view the conditional expectations relative to \( Z_1 \) as functions defined on \( Z_1 \).

It is a classical result that the invariant \( \sigma \)-algebra \( \mathcal{I}^{[1]} \) of \((X \times X, \mu \times \mu, T \times T)\) consists in sets of the form

\[
(H.4) \quad \{(x, y) \in X \times X : \pi_1(x) - \pi_1(y) \in A\}
\]

where \( A \in \mathcal{Z}_1 \). Hence \( \mathcal{I}^{[1]} \) is isomorphic to \( \mathcal{Z}_1 \). Let \( \phi : (X \times X, \mathcal{X} \times \mathcal{X}) \to (\Omega_1, \mathcal{I}^{[1]}, P_1) \) be the factor map and let \( \psi : (\Omega_1, \mathcal{I}^{[1]}, P_1) \to (Z_1, \mathcal{Z}_1, \mu_1) \) be the isomorphic map. Hence we have

\[
(H.5) \quad (X \times X, \mathcal{X} \times \mathcal{X}) \xrightarrow{\phi} (\Omega_1, \mathcal{I}^{[1]}, P_1) \xleftarrow{\psi} (Z_1, \mathcal{Z}_1, \mu_1)
\]

\[
(x, y) \quad \mapsto \quad \phi(x, y) \quad \mapsto \quad s = \psi(\phi(x, y))
\]

From this, it is not difficult to deduce that the ergodic decomposition of \( \mu \times \mu \) under \( T \times T \) can be written as

\[
(H.6) \quad \mu \times \mu = \int_{Z_1} \mu_s \, d\mu_1(s).
\]

In particular, for \( \mu_1 \)-almost every \( s \), the measure \( \mu_s \) is ergodic for \( T \times T \). For an integer \( d > 0 \) we have

\[
(H.7) \quad \mu^{[d+1]} = \int_{Z_1} (\mu_s)^{[d]} \, d\mu_1(s).
\]

Especially, we have

\[
(H.8) \quad \mu^{[2]} = \int_{Z_1} \mu_s \times \mu_s \, d\mu_1(s).
\]

**H.0.22. \( G^{[2]} \)-actions.** Let \( \pi_1 : X \to Z_1 \) be the factor map from \( X \) to its Kronecker factor \( Z_1 \). Since \( Z_1 \) is a group rotation, it may be regarded as a topological system in the natural way. By Weiss’s Theorem, there is a uniquely ergodic model \((\hat{X}, \hat{X}, \hat{\mu}, T)\) for \((X, \mathcal{X}, \mu, T)\) and a factor map \( \pi_1 : X \to Z_1 \) which is a model for \( \pi_1 : X \to Z_1 \).

\[
\begin{array}{ccc}
X & \xrightarrow{\pi_1} & \hat{X} \\
Z_1 & \xrightarrow{\pi_1} & Z_1
\end{array}
\]

Hence for simplicity, we may assume that \((\hat{X}, \hat{X}, \hat{\mu}, T) = (X, \mathcal{X}, \mu, T)\) and \( \pi_1 = \pi_1 \).

Now we show that \((Q^{[2]}, \mu^{[2]}, G^{[2]})\) is uniquely ergodic.

Let \( \lambda \) be a \( G^{[2]} \)-invariant measure of \( Q^{[2]} \). Let

\[
p_1 : (Q^{[2]}, G^{[2]}) \to (Q^{[1]}, G^{[2]}); \quad x = (x', x'') \mapsto x'
\]

\[
p_2 : (Q^{[2]}, G^{[2]}) \to (Q^{[1]}, G^{[2]}); \quad x = (x', x'') \mapsto x''
\]
be the projections. Then \((p_2)_*(\lambda)\) is a \(G^2\)-invariant measure of \(Q^1 = X^1\). Note that \(G^2\) acts on \(Q^1\) as \(G^1\) actions. By subsection 4.1, \((p_2)_*(\lambda) = \mu^1 = \mu \times \mu\).

Hence let
\[
\lambda = \int_{X^2} \lambda(x,y) \times \delta_{(x,y)} \, d\mu \times \mu(x,y)
\]
be the disintegration of \(\lambda\) over \(\mu^1\). Since \(\lambda\) is \(T^2\) = \(id^1 \times T^1\)-invariant, we have
\[
\lambda = \int_{X^2} \lambda(x,y) \times T^1 \delta_{(x,y)} \, d\mu \times \mu(x,y)
\]
\[
= \int_{X^2} \lambda(x,y) \times \delta_{T^1(x,y)} \, d\mu \times \mu(x,y)
\]
\[
= \int_{X^2} \lambda(T^1(x,y))^{-1} \times \delta_{(x,y)} \, d\mu \times \mu(x,y).
\]

The uniqueness of disintegration implies that
\[
\lambda(T^1(x,y))^{-1} = \lambda(x,y), \quad \mu^1 = \mu \times \mu \, a.e.
\]

Define
\[
F : (Q^1 = X^1, T^1) \longrightarrow M(X^1) : (x, y) \mapsto \lambda(x,y).
\]
By (H.10), \(F\) is a \(T^1\)-invariant \(M(X^1)\)-value function. Hence \(F\) is \(T^1\)-measurable,
and hence \(\lambda(x,y) = \lambda(\phi(x,y)) = \lambda_s, \mu^1 \, a.e.,\) where \(\phi\) is defined in (H.5).

Thus by (H.9) one has that
\[
\lambda = \int_{X^2} \lambda(x,y) \times \delta_{(x,y)} \, d\mu \times \mu(x,y) = \int_{X^2} \lambda(x,y) \times \delta_{(x,y)} \, d\mu \times \mu(x,y)
\]
\[
= \int_{Z_1} \int_{X^2} \lambda_s \times \delta_{(x,y)} \, d\mu_s(x,y)d\mu_1(s)
\]
\[
= \int_{Z_1} \lambda_s \times \left( \int_{X^2} \delta_{(x,y)} \, d\mu_s(x,y) \right)d\mu_1(s)
\]
\[
= \int_{Z_1} \lambda_s \times \mu_s \, d\mu_1(s)
\]

Let \(\pi^2_1 : (Q^2(X), G^2) \longrightarrow (Q^2(Z_1), G^2)\) be the natural factor map. By Theorem 2.9 \((Q^2(Z_1), \mu^1)\) is uniquely ergodic. Hence
\[
\pi^2_1(\mu^1) = \mu^1 = \int_{Z_1} \mu_1 \, d\mu_1(s).
\]
So
\[
(\pi \times \pi)_*(\lambda_s) = (\pi \times \pi)_*(\mu_s) = \mu_1 \, a.e.
\]

Note that we have that
\[
(p_1)_*(\lambda) = (p_2)_*(\lambda) = \mu^1 = \mu \times \mu,
\]
and hence we have
\[
\mu \times \mu = \int_{Z_1} \lambda_s \, d\mu_1(s) = \int_{Z_1} \mu_s \, d\mu_1(s).
\]
Hence by the uniqueness of disintegration, we have that $\lambda_s = \mu_s$, $\mu_1$ a.e. More precisely, if $\lambda_s \neq \mu_s$, $\mu_1$ a.e., then $\mu_1(\{s \in Z_1 : \lambda_s \neq \mu_s\}) > 0$. So there is some function $f \in C(X \times X)$ such that

$$\mu_1(\{s : \lambda_s(f) > \mu_s(f)\}) > 0.$$ 

Let $A = \{s : \lambda_s(f) > \mu_s(f)\}$. By (H.11), we can consider $A$ as a subset of $X \times X$:

$$A = \{s : \lambda_s(f) > \mu_s(f)\} = \{(x, y) \in X \times X : \lambda_{(x,y)}(f) > \mu_{(x,y)}(f)\}.$$ 

Hence by $\mu \times \mu = \int_{Z_1} \lambda_s \, d\mu_1(s)$ we have

$$\mu \times \mu(f \cdot 1_A) = \int_{X^2} f \cdot 1_A \, d\mu \times \mu$$

$$= \int_{Z_1} \int_{X^2} f \cdot 1_A \, d\lambda_s(x, y) \, d\mu_1(s)$$

$$= \int_{Z_1} 1_A \int_{X^2} f \, d\lambda_s(x, y) \, d\mu_1(s)$$

$$= \int_A \lambda_s(f) \, d\mu_1(s)$$

Similarly, by $\mu \times \mu = \int_{Z_1} \mu_s \, d\mu_1(s)$ we have

$$\mu \times \mu(f \cdot 1_A) = \int_A \mu_s(f) \, d\mu_1(s)$$

Thus

$$0 = \int_A \lambda_s(f) \, d\mu_1(s) - \int_A \mu_s(f) \, d\mu_1(s) = \int_A (\lambda_s(f) - \mu_s(f)) \, d\mu_1(s) > 0,$$

a contradiction! Hence $\lambda_s = \mu_s$, $\mu_1$ a.e., and

$$\lambda = \int_{Z_1} \lambda_s \times \mu_s \, d\mu_1(s) = \int_{Z_1} \mu_s \times \mu_s \, d\mu_1(s) = \mu^{[2]}.$$ 

That is, $(Q^{[2]}, \mu^{[2]}, G^{[2]})$ is uniquely ergodic. The proof is completed.

H.0.23. $F^{[2]}$-actions. We use the same model as in the proof of Proposition H.0.22. Let $\lambda$ be a $F^{[2]}$-invariant measure of $F^{[2]}(x^{[2]})$. Let

$$p_1 : (F^{[2]}(x^{[2]}), F^{[2]}) \to (F^{[1]}(x^{[1]}), F^{[2]}); \ x = (x', x'') \mapsto x'$$

$$p_2 : (F^{[2]}(x^{[2]}), F^{[2]}) \to (Q^{[1]}, F^{[2]}); \ x = (x', x'') \mapsto x''$$

be the projections. Note that

$$(F^{[1]}(x^{[1]}), F^{[2]}) \cong (X, T) \text{ and } (Q^{[1]}, F^{[2]}) \cong (X \times X, G^{[1]}).$$

Then $(p_2)_*(\lambda)$ is a $G^{[1]}$-invariant measure of $Q^{[1]} = X^{[1]}$. By subsection 4.1 $(p_2)_*(\lambda) = \mu^{[1]} = \mu \times \mu$. Hence let

$$(H.11) \quad \lambda = \int_{X^2} \lambda_{(x,y)} \times \delta_{(x,y)} \, d(\mu \times \mu)(x, y)$$
be the disintegration of $\lambda$ over $\mu^1$. Since $\lambda$ is $T^2_2 = \text{id}^1 \times T^1$-invariant, we have
\[
\lambda = \text{id}^1 \times T^1 \lambda = \int_{X^2} \lambda_{(x,y)} \times T^1 \delta_{(x,y)} \, d\mu \times \mu(x, y)
\]
\[
= \int_{X^2} \lambda_{(x,y)} \times \delta_{T^1(x,y)} \, d\mu \times \mu(x, y)
\]
\[
= \int_{X^2} \lambda_{(T^1)^{-1}(x,y)} \times \delta_{(x,y)} \, d\mu \times \mu(x, y).
\]
The uniqueness of disintegration implies that
\[
\lambda_{(T^1)^{-1}(x,y)} = \lambda_{(x,y)}, \quad \mu^1 = \mu \times \mu \text{ a.e.}
\]
Define
\[
F : (Q^1 = X^1, T^1) \rightarrow M(X) : (x, y) \mapsto \lambda_{(x,y)}.
\]
By (H.12), $F$ is a $T^1$-invariant $M(X)$-value function. Hence $F$ is $I^1$-measurable, and hence $\lambda_{(x,y)} = \lambda_{\phi(x,y)} = \lambda_s$, $\mu^1$ a.e., where $\phi$ is defined in (H.5).
Thus by (H.11) one has that
\[
\int_{Z_1} \lambda_{(T^1)^{-1}(x,y)} \times \delta_{(x,y)} \, d\mu \times \mu_1(s) = \int_{Z_1} \lambda_s \times \mu_1(s) \, d\mu_1(s)
\]
Let $\pi^2_1 : (\mathcal{F}^2([x])^2, \mathcal{F}^2) \rightarrow (\mathcal{F}^2([\pi_1(x)])^2, \mathcal{F}^2)$ be the natural factor map. By Theorem 2.9 $\mathcal{F}^2([\pi_1(x)])^2$ is uniquely ergodic. Hence
\[
\pi^2_1(\lambda) = \int_{Z_1} \mu_1 \times \mu_1 s \, d\mu_1(s) = \mu_1^3.
\]
And
\[
\pi_1 s(\lambda) = \mu_1, \text{ and } (\pi_1 \times \pi_1)_s(\mu_s) = \mu_1 s.
\]
Note that we have that
\[
(p_1)_s(\lambda) = \mu, \text{ and } (p_2)_s(\lambda) = \mu^1 = \mu \times \mu,
\]
and hence we have
\[
\mu = \int_{Z_1} \lambda_s \, d\mu_1(s).
\]
Let $\mu = \int_{Z_1} \nu_s \, d\mu_1(s)$ be the disintegration of $\mu$ over $\mu_1$. Hence by the uniqueness of disintegration, we have that $\lambda_s = \nu_s$, $\mu_1$ a.e.. Thus
\[
\lambda = \int_{Z_1} \lambda_s \times \mu_s \, d\mu_1(s) = \int_{Z_1} \nu_s \times \mu_s \, d\mu_1(s).
\]
That is, $(\mathcal{F}^2([x])^2, \mathcal{F}^2)$ is uniquely ergodic. The proof is completed.
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