Lower bound on the radius of analyticity of solution for fifth order KdV–BBM equation

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Abstract. We show that the uniform radius of spatial analyticity \( \sigma(t) \) of solution at time \( t \) for the fifth order KdV–BBM equation cannot decay faster than \( 1/t \) for large \( t > 0 \), given initial data that is analytic with fixed radius \( \sigma_0 \). This significantly improves a recent result by Carvajal and Panthee (On the radius of analyticity for the solution of the fifth order KdV–BBM model, 2020. arXiv:2009.09328), where they established an exponential decay of \( \sigma(t) \) for large \( t \).

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1. Introduction

We consider the fifth order KdV–BBM type equation of the form

\[
\partial_t \eta + \partial_x \eta - \gamma_1 \partial_t \partial_x^2 \eta + \gamma_2 \partial_x^3 \eta + \delta_1 \partial_t \partial_x^4 \eta + \delta_2 \partial_x^5 \eta = -\frac{3}{4} \partial_x (\eta^2) - \gamma \partial_x^3 (\eta^2) + \frac{7}{48} \partial_x (\eta_x^2) + \frac{1}{8} \partial_x (\eta^3),
\]

where the unknown function is

\[\eta : \mathbb{R}^{1+1} \to \mathbb{R}.\]

The parameters \( \gamma_1, \gamma_2, \delta_1, \delta_2, \gamma \) are constants that satisfy certain constraints; see [1,7] for more details.

The fifth-order PDE (1) describes the unidirectional propagation of water waves, and this was recently introduced by Bona et al. [1] by using the second order approximation in the two-way model, the so-called \( abcd\text{-system} \) derived in [2,3].

We complement (1) with initial data

\[
\eta(x, 0) = \eta_0(x).
\]
In the case $\gamma = 7/48$, the energy

$$E[\eta(t)] = \frac{1}{2} \int_{\mathbb{R}} (\eta^2 + \gamma \eta_x^2 + \delta_1 \eta_{xx}^2) \, dx$$

is conserved by the flow of (1), i.e.,

$$E[\eta(t)] = E[\eta_0] \quad \text{for all } t.$$  \hfill (3)

In the case $\gamma \neq 7/48$, the corresponding energy has no positive sign, and therefore not useful to prove global well-posedness of (1)–(2).

Local well-posedness of the Cauchy problem (1)–(2) with data in the Sobolev spaces $H^s(\mathbb{R})$ for $s \geq 1$ was established by Bona et al in [1]. When $\gamma_1, \delta_1 > 0$ and $\gamma = 7/48$, the authors [1] used the energy conservation (3) to prove global well-posedness of (1)–(2) for data in $H^s(\mathbb{R}), s \geq 2$. Furthermore, the authors used the method of high-low frequency splitting to obtain global well-posedness for data with Sobolev regularity $3/2 \leq s < 2$. This global well-posedness result was further improved in [6] for initial data with Sobolev regularity $s \geq 1$.

Recently, Carvajal and Panthee [7] studied the property of spatial analyticity of the solution $\eta(x,t)$ to (1)–(2), given that the initial data $\eta_0(x)$ is real-analytic with uniform radius of analyticity $\sigma_0$, so there is a holomorphic extension to a complex strip

$$S_{\sigma_0} = \{x + iy \in \mathbb{C} : |y| < \sigma_0\}.$$  

The authors proved that, for short times, the radius of analyticity $\sigma(t)$ of the solution $\eta(x,t)$ remains at least as large as the initial radius, i.e. one can take $\sigma(t) = \sigma_0$. On the other hand, for large times, they proved that $\sigma(t)$ decays exponentially in $t$. In the present paper, we use the idea introduced in [19] (see also [20,22]) to improve this result significantly showing $\sigma(t)$ cannot decay faster than $1/t$ for large $t$.

For earlier studies concerning properties of spatial analyticity of solutions for a large class of nonlinear partial differential equations, see for instance [4,5,8–14,16–23].

A class of analytic function spaces suitable to study analyticity of solution is the Gevrey class, denoted $G^{\sigma,s}(\mathbb{R})$, which are defined by the norm

$$\|f\|_{G^{\sigma,s}} = \|e^{\sigma|D_x|/\langle D_x \rangle^s}f\|_{L^2_x},$$

where $D_x = -i\partial_x$ and $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$.

The reason for considering initial data in the space $G^{\sigma,s}$ is due to the analyticity properties of Gevrey functions, which are detailed in the following theorem:

**Paley–Wiener Theorem** Let $\sigma > 0$ and $s \in \mathbb{R}$. Then the following are equivalent:

(a) $f \in G^{\sigma,s}(\mathbb{R})$,
(b) $f$ is the restriction to $\mathbb{R}$ of a function $F$ which is holomorphic in the strip $S_{\sigma} = \{x + iy \in \mathbb{C} : |y| < \sigma\}$.
Moreover, the function $F$ satisfies the estimates
\[
\sup_{|y| < \sigma} \|F(\cdot + iy)\|_{H^s} < \infty.
\]
A proof can be found in [15] in the case $s = 0$; the general case follows from a simple modification. The quantity $\sigma$ is known as the radius of analyticity.

We remark that Gevrey spaces satisfy the embeddings
\[
\|f\|_{G^{\sigma,s}} \leq C \|f\|_{G^{\sigma',s'}}
\]
for any $s, s' \in \mathbb{R}, \sigma < \sigma'$ and some constant $C > 0$. This implies
\[
\|f\|_{H^s} \leq C \|f\|_{G^{\sigma',s'}}
\]
for all $s, s' \in \mathbb{R}, \sigma' > 0$.

As a consequence of property (5) and the existing well-posedness theory in $H^s(\mathbb{R})$ (see [1]), we conclude that the Cauchy problem (1)–(2) (with $\gamma_1, \delta_1 > 0$ and $\gamma = 7/48$) has a unique, smooth solution for all time, given initial data $\eta_0 \in G^{\sigma_0, s}$ for all $\sigma_0 > 0$ and $s \in \mathbb{R}$.

Our main result is as follows:

**Theorem 1.** Assume $\gamma_1, \delta_1 > 0$ and $\gamma = 7/48$. Suppose that $\eta$ is the global solution of (1)–(2) with $\eta_0 \in G^{\sigma_0,2}$ for $\sigma_0 > 0$. Then
\[
\eta(t) \in G^{\sigma(t),2} \quad \text{for all} \quad t > 0,
\]
with the radius of analyticity $\sigma(t)$ satisfying the asymptotic lower bound
\[
\sigma(t) \geq \frac{c}{t} \quad \text{as} \quad t \to +\infty,
\]
where $c > 0$ is a constant depending on the initial data norm $\|\eta_0\|_{G^{\sigma_0,2}}$.

Thus, the solution at any time $t$, $\eta(t)$, is analytic in the strip $S_{\sigma(t)}$.

**Notation** For any positive numbers $a$ and $b$, the notation $a \lesssim b$ stands for $a \leq cb$, where $c$ is a positive constant that may change from line to line. Moreover, we denote $a \sim b$ when $a \lesssim b$ and $b \lesssim a$.

## 2. Local well-posedness in $G^{\sigma_0,s}$

We outline the argument in [7] that enables the authors to obtain the local well-posedness result for (1)–(2) in $G^{\sigma_0, s}$ with $s \geq 1, \sigma_0 > 0$.

Taking the spatial Fourier transform of (1) we obtain
\[
i\partial_t \hat{\eta} - \phi(\xi) \hat{\eta} = \pi(\xi) \hat{\eta}^2 - \frac{1}{8} \psi(\xi) \hat{\eta}^3 - \frac{7}{48} \psi(\xi) \hat{\eta}_x^2,
\]
where
\[
\phi(\xi) = \frac{\xi (1 - \gamma_2 \xi^2 + \delta_2 \xi^4)}{\varphi(\xi)}, \quad \psi(\xi) = \frac{\xi}{\varphi(\xi)}, \quad \tau(\xi) = \frac{\xi (3 - 4\gamma \xi^2)}{4 \varphi(\xi)}
\]
with
\[
\varphi(\xi) = 1 + \gamma_1 \xi^2 + \delta_1 \xi^4.
\]
Defining the Fourier multipliers
\[ \phi(D_x)f = \mathcal{F}^{-1}[\varphi(\xi)f], \quad \psi(D_x)f = \mathcal{F}^{-1}[\psi(\xi)f], \quad \tau(D_x)f = \mathcal{F}^{-1}[\tau(\xi)f], \]
we can rewrite (6) in an operator form as
\[ i\partial_t \eta - \phi(D_x)\eta = F(\eta), \quad (7) \]
where
\[ F(\eta) = \tau(D_x)\eta^2 - \frac{1}{8}\psi(D_x)\eta^3 - \frac{7}{48}\psi(D_x)\eta_x^2. \]

Then the integral equation for (7)–(8) with initial data (2) is given by
\[ \eta(t) = e^{-it\phi(D_x)}\eta_0 - i\int_0^t e^{-i(t-t')\phi(D_x)}F(\eta)(t')\,dt'. \quad (9) \]

Combining the estimates in [7, Lemma 2.2–2.4], we obtain the following nonlinear estimate.

**Lemma 1.** [7, Lemma 2.2–2.4] Let \( F(\eta) \) be defined as in (8). Then for \( s \geq 1, \sigma > 0 \), we have
\[ \|F(\eta)\|_{G^{\sigma,s}} \lesssim [1 + \|\eta\|_{G^{\sigma,s}}]\|\eta\|^2_{G^{\sigma,s}} \quad (10) \]
for all \( \eta \in G^{\sigma,s} \).

Now applying the contraction mapping argument to the integral equation (9) and using Lemma 1 yields the following local result.

**Theorem 2.** [7] Let \( s \geq 1, \sigma_0 > 0 \) and \( \eta_0 \in G^{\sigma_0,s} \). Then there exists a unique solution
\[ \eta \in C([0, T]; G^{\sigma_0,s}(\mathbb{R})) \]
of the Cauchy problem (1)–(2), where the existence time is
\[ T \sim (1 + \|\eta_0\|)^{-2}_{G^{\sigma_0,s}}. \quad (11) \]
Moreover,
\[ \|\eta\|_{L^\infty_t G^{\sigma_0,s}} \lesssim \|\eta_0\|_{G^{\sigma_0,s}}. \quad (12) \]

Here we use the notation
\[ L^\infty_t G^{\sigma_0,s} = L^\infty_t G^{\sigma_0,s}([0, T] \times \mathbb{R}). \]

3. Almost conservation law and proof of Theorem 1

We fix \( \gamma_1, \delta_1 > 0 \) and \( \gamma = 7/48 \). Let
\[ v(x, t) := \Lambda_{\sigma}\eta(x, t), \quad \text{where} \quad \Lambda_{\sigma} = e^{\sigma|D_x|}. \]
Then \( \eta = \Lambda_{-\sigma}v \). Note also that \( v_0 := v(x, 0) = \Lambda_{\sigma}\eta_0 \).

Now define the modified energy
\[ \mathcal{E}_{\sigma}[v(t)] = \frac{1}{2} \int_{\mathbb{R}} \left( v^2 + \gamma_1 v_x^2 + \delta_1 v_{xx}^2 \right) \, dx. \]
Observe that for $\sigma = 0$, we have $v = \eta$, and therefore the energy conserved, i.e., $E_0[v(t)] = E_0[v_0]$ for all $t$. However, this fails to hold for $\sigma > 0$. In what follows we will nevertheless prove the approximate conservation
\[
\sup_{0 \leq t \leq T} E_\sigma[v(t)] = E_\sigma[v_0] + \sigma \cdot \mathcal{O} \left( \left[ 1 + (E_\sigma[v_0])^{\frac{1}{2}} \right] (E_\sigma[v_0])^{\frac{3}{2}} \right)
\]
for $T$ as in Theorem 2. Thus, in the limit as $\sigma \to 0$, we recover the conservation $E_0[v(t)] = E_0[v_0]$.

### 3.1. Almost conservation law

Applying the operator $\Lambda_\sigma$ to equation (1) we obtain
\[
\partial_t v + \partial_x v - \gamma_1 \partial_t \partial_x^2 v + \gamma_2 \partial_x^3 v + \delta_1 \partial_t \partial_x^4 v + \delta_2 \partial_x^5 v = -\left( \frac{3}{4} + \gamma \partial_x^2 \right) \partial_x (v^2) + \gamma \partial_x (v_x^2) + \frac{1}{8} \partial_x (v^3) + N(v),
\]
where
\[
N(v) = \left( \frac{3}{4} + \gamma \partial_x^2 \right) \partial_x N_1(v) - \gamma \partial_x N_2(v) - \frac{1}{8} \partial_x N_3(v)
\]
with
\[
N_1(v) = v^2 - \Lambda_\sigma \left[ (\Lambda_{-\sigma} v)^2 \right], \\
N_2(v) = v_x^2 - \Lambda_\sigma \left[ (\Lambda_{-\sigma} v_x)^2 \right], \\
N_3(v) = v^3 - \Lambda_\sigma \left[ (\Lambda_{-\sigma} v)^3 \right].
\]

Using integration by parts\footnote{Assuming that the solution is sufficiently regular.} and (13)–(15) we compute
\[
\frac{d}{dt} E_\sigma[v(t)] = \int_\mathbb{R} (vv_t + \gamma_1 vv_x v_{xt} + \delta_1 vv_x v_{xxt}) \, dx \\
= \int_\mathbb{R} v \left( \partial_t v - \gamma_1 \partial_t \partial_x^2 v + \delta_1 \partial_t \partial_x^4 v \right) \, dx \\
= -\int_\mathbb{R} v \left( \partial_x v + \gamma_2 \partial_x^3 v + \delta_2 \partial_x^5 v + \frac{3}{4} \partial_x (v^2) + \gamma \partial_x^3 (v^2) \\
-\gamma \partial_x (v_x^2) - \frac{1}{8} \partial_x (v^3) \right) \, dx \\
+ \int_\mathbb{R} v N(v) \, dx.
\]
The integral on the third line is zero due to the identities
\[
v \partial_x v = \frac{1}{2} (v^2)_x, \quad v \partial_x^3 v = (vv_{xx})_x - \frac{1}{2} (v_x^2)_x, \\
v \partial_x^2 v = (v \partial_x^4 v)_x - (\partial_x v \partial_x^3 v)_x + \frac{1}{2} (v_{xx}^2)_x.
\]
\[
\begin{align*}
\v v \partial_x (v^2) &= \frac{2}{3} (v^2)_x, \\
\v v \partial_x (v^3) &= \frac{3}{4} (v^4)_x, \\
\v v \partial^3_x (v^2) &= 2 (v^2 v_{xx})_x + v (v_x^2)_x.
\end{align*}
\]

Therefore,
\[
\frac{d}{dt} \mathcal{E}_\sigma[v(t)] = \int_\mathbb{R} v \mathcal{N}(v) \, dx.
\]

Consequently,
\[
\begin{align*}
\mathcal{E}_\sigma[v(t)] &= \mathcal{E}_\sigma[v(0)] + \int_0^t \frac{d}{ds} \mathcal{E}_\sigma[v(s)] \, ds \\
&= \mathcal{E}_\sigma[v_0] + \int_0^t \int_\mathbb{R} v(x, s) \mathcal{N}(v(x, s)) \, dx ds.
\end{align*}
\]

Now we state a key estimate that will be proved in the last section.

**Lemma 2.** We have
\[
\left| \int_\mathbb{R} v \mathcal{N}(v) \, dx \right| \leq C_\sigma \left[ 1 + \|v\|_{H^2} \right] \|v\|^3_{H^2} \tag{17}
\]
for all \( v \in H^2 \).

So in view of (16) and (17), we have the a priori energy estimate
\[
\sup_{0 \leq t \leq T} \mathcal{E}_\sigma[v(t)] = \mathcal{E}_\sigma[v_0] + \sigma T \cdot \mathcal{O} \left( \left[ 1 + \|v\|_{L_T^\infty H^2} \right] \|v\|^3_{L_T^\infty H^2} \right), \tag{18}
\]
where
\[
L_T^\infty H^2 := L_t^\infty H^2([0, T] \times \mathbb{R}).
\]

We combine this estimate with the local existence theory in Theorem 2 above to obtain an almost conservation law to the modified energy.

**Lemma 3.** [Almost conservation law] Let \( \eta_0 \in G^{\sigma, 2} \). Suppose that \( \eta \in C \left([0, T]; G^{\sigma, 2}\right) \) is the local-in-time solution to the Cauchy problem (1)–(2) that is constructed in Theorem 2. Then
\[
\sup_{0 \leq t \leq T} \mathcal{E}_\sigma[v(t)] = \mathcal{E}_\sigma[v_0] + \sigma \cdot \mathcal{O} \left( \left[ 1 + \left( \mathcal{E}_\sigma[v_0] \right)^{\frac{1}{2}} \right] \left( \mathcal{E}_\sigma[v_0] \right)^{\frac{3}{2}} \right). \tag{19}
\]

**Proof.** By Theorem 2 we have the bound
\[
\|v\|_{L_T^\infty H^2} = \|\eta\|_{L_T^\infty G^{\sigma, 2}} \leq C \|\eta_0\|_{G^{\sigma, 2}} = C \|v_0\|_{H^2}; \tag{20}
\]
where \( T \) is as in (11). On the other hand, for fixed constants \( \gamma_1, \delta_1 > 0 \), we have
\[ E_{\sigma_0}[v_0] = \frac{1}{2} \int_{\mathbb{R}} \left( v_0^2 + \gamma_1 (v_0')^2 + \delta_1 (v_0'')^2 \right) \, dx \]
\[ \sim \|v_0\|_{H^2}^2. \] (21)

Then combining (20)–(21) with (18) yields the desired estimate (19). \( \square \)

3.2. Proof of Theorem 1

Suppose that \( \eta_0 \in G^{\sigma_0,2} \) for some \( \sigma_0 > 0 \). From the local theory there is a unique solution

\[ \eta \in C \left( [0,T]; G^{\sigma_0,2}(\mathbb{R}) \right) \]

of (1), (2) constructed in Theorem 2 with existence time \( T \) as in (11).

Note that since \( v_0 = e^{\sigma_0 |D_x|} \eta_0 \in H^2 \)

we have

\[ E_{\sigma_0}[v_0] \sim \|v_0\|_{H^2}^2 < \infty. \]

Now following the argument in [20,22] we can construct a solution on \([0,T^*]\) for arbitrarily large time \( T^* \) by applying the approximate conservation law in Lemma 3, (19), so as to repeat the above local result on successive short time intervals of size \( T \) to reach \( T^* \) by adjusting the strip width parameter \( \sigma \) according to the size of \( T^* \). Doing so, we establish the bound

\[ \sup_{t \in [0,T^*]} E_{\sigma}[v(t)] \leq 2E_{\sigma_0}[v_0] \] (22)

for \( \sigma \) satisfying

\[ \sigma(t) \geq C/T^*. \] (23)

Thus, \( E_{\sigma}(t) < \infty \) for \( t \in [0,T^*] \), which in turn implies

\[ \eta(t) \in G^{\sigma(t),2} \quad \text{for all} \quad t \in [0,T^*]. \]

4. Proof of Lemma 2

Estimate (17) reduces to

\[ \left| \int_{\mathbb{R}} v N(v) \, dx \right| \leq C \sigma \|v\|_{H^2} \|v\|_{H^2}^3, \] (24)

where \( v \) can be regarded as a function of only \( x \) (since \( t \) is fixed).

Using (14)–(15), Plancherel and Cauchy–Schwarz, we get

\[ \int_{\mathbb{R}} v N(v) \, dx = \int_{\mathbb{R}} v \left( \frac{3}{4} + \gamma \partial_x^2 \right) \partial_x N_1(v) \, dx - \gamma \int_{\mathbb{R}} v \partial_x N_2(v) \, dx \]
\[ - \frac{1}{8} \int_{\mathbb{R}} v \partial_x N_3(v) \, dx \]
\[ = \int_{\mathbb{R}} \left( \frac{3}{4} + \gamma \partial_x^2 \right) v \partial_x N_1(v) \, dx + \gamma \int_{\mathbb{R}} \partial_x v \cdot N_2(v) \, dx \]
So (24) follows from the following estimates:

\[
\begin{align*}
\|\partial_x N_1(v)\|_{L_x^2} & \lesssim \sigma \|v\|_{H^2}^2 \\
\|N_2(v)\|_{L_x^2} & \lesssim \sigma \|v\|_{H^2}^2 \\
\|N_3(v)\|_{L_x^2} & \lesssim \sigma \|v\|_{H^2}^3.
\end{align*}
\]

4.1. Proof of (25)

By taking the FT, we write

\[
\hat{\partial_x N_1}(\xi) = i \int_{\xi_1 + \xi_2} \xi \left( e^{\sigma(|\xi_1| + |\xi_2|)} - e^{\sigma|\xi|} \right) \hat{\eta}(\xi_1) \hat{\eta}(\xi_2) \, d\xi_1 d\xi_2
\]

where

\[
p_\sigma(\xi_1, \xi_2) = 1 - \exp \left( -\sigma \left( |\xi_1| + |\xi_2| \right) \right).
\]

Since \(1 - e^{-r} \leq r\) for all \(r \geq 0\), we have

\[
|p_\sigma(\xi_1, \xi_2)| \leq \sigma \left( |\xi_1| + |\xi_2| \right) - |\xi_1 + \xi_2|
\]

\[
= \sigma \frac{|\xi_1| + |\xi_2|}{|\xi_1| + |\xi_2|} - \frac{|\xi_1 + \xi_2|^2}{|\xi_1| + |\xi_2|}
\]

\[
\leq 2\sigma \min \left( |\xi_1|, |\xi_2| \right)
\]

By symmetry, we may assume \(|\xi_1| \leq |\xi_2|\). This implies \(|\xi| \leq 2|\xi_2|\). Now let

\[
V = F_{\chi}^{-1}(\hat{v}).
\]

Then by (28)

\[
|\partial_x N_1(v)(\xi)| \leq 4\sigma \int_{\xi_1 + \xi_2} |\xi_1| |\hat{\nu}(\xi_1)| \cdot |\xi_2| |\hat{\nu}(\xi_2)| \, d\xi_1 d\xi_2
\]

\[
= 4\sigma \int_{\xi_1 + \xi_2} |\xi_1| |\hat{V}(\xi_1)| \cdot |\xi_2| |\hat{V}(\xi_2)| \, d\xi_1 d\xi_2
\]

\[
= 4\sigma |D_\chi| V \cdot |D_\chi| V(\xi).
\]

Therefore, using Plancherel, Hölder and Sobolev inequalities we get
By taking the FT, we write
\[
\|\partial_x N_1(U)\|_{L^2_x} \leq 4\sigma\|D_x |V| \|_{L^2_x} \\
\leq 4\sigma\|D_x |V|\|_{L^2_x} \|D_x |V|\|_{L^\infty_x} \\
\lesssim \sigma\|V\|_{H^2}^2 \sim \sigma\|v\|_{H^2}^2
\]
as desired.

4.2. Proof of (26)
By taking the FT, we write
\[
\widehat{N}_2(v)(\xi) = \int_{\xi = \xi_1 + \xi_2} \left( e^{\sigma(|\xi_1|+|\xi_2|)} - e^{\sigma|\xi|} \right) \hat{\eta}_x(\xi_1)\hat{\eta}_x(\xi_2) \, d\xi_1 d\xi_2 \\
= -\int_{\xi = \xi_1 + \xi_2} \xi_1 \xi_2 p_\sigma(\xi_1, \xi_2) \hat{v}(\xi_1)\hat{v}(\xi_2) \, d\xi_1 d\xi_2,
\]
where \(p_\sigma(\xi_1, \xi_2)\) as in the preceding subsection.
Assuming \(|\xi_1| \leq |\xi_2|\), by symmetry, we have by (28)
\[
|\xi_1 \xi_2 p_\sigma(\xi_1, \xi_2)| \leq 2\sigma|\xi_1|^2|\xi_2|.
\]
Then
\[
|\widehat{N}_2(v)(\xi)| \leq 2\sigma\int_{\xi = \xi_1 + \xi_2} |\xi_1|^2|\hat{v}(\xi_1)| \cdot |\xi_2|^2|\hat{v}(\xi_2)| \, d\xi_1 d\xi_2 \\
= 2\sigma\int_{\xi = \xi_1 + \xi_2} |\xi_1|^2\hat{V}(\xi_1) \cdot |\xi_2|^2\hat{V}(\xi_2) \, d\xi_1 d\xi_2 \\
= 2\sigma \mathcal{F}_x \|D_x |^2V \cdot |D_x |V\|_2(\xi).
\]
Therefore, by Plancherel, Hölder and Sobolev inequalities we get
\[
\|N_2(U)\|_{L^2_x} \leq 2\sigma\|D_x |^2V \cdot |D_x |V\|_{L^2_x} \\
\leq 2\sigma\|D_x |^2V\|_{L^2_x} \|D_x |V\|_{L^\infty_x} \\
\lesssim \sigma\|V\|_{H^2}^2 \sim 2\sigma\|v\|_{H^2}^2
\]
as desired.

4.3. Proof of (27)
\[
\widehat{N}_3(v)(\xi) = \int_{\xi = \xi_1 + \xi_2 + \xi_3} \left( e^{\sum_{j=1}^3 \sigma_j|\xi_j|} - e^{\sigma|\xi|} \right) \hat{\eta}(\xi_1)\hat{\eta}(\xi_2)\hat{\eta}(\xi_3) \, d\xi_1 d\xi_2 d\xi_3 \\
= \int_{\xi = \xi_1 + \xi_2 + \xi_3} q_\sigma(\xi_1, \xi_2, \xi_3) \hat{v}(\xi_1)\hat{v}(\xi_2)\hat{v}(\xi_3) \, d\xi_1 d\xi_2 d\xi_3,
\]
where
\[
q_\sigma(\xi_1, \xi_2, \xi_3) = 1 - \exp \left( -\sigma \left[ \sum_{j=1}^3 \xi_j \right] - \left[ \sum_{j=1}^3 \xi_j \right] \right).
\]
We estimate
\[ |q_σ(ξ_1, ξ_2, ξ_3)| \leq σ \left[ \sum_{j=1}^{3} |ξ_j| - \sum_{j=1}^{3} |ξ_j| \right] \]
\[ = \sigma \left( \sum_{j=1}^{3} |ξ_j|^2 - \sum_{j=1}^{3} |ξ_j|^2 \right) \]
\[ \leq 12σ \text{med} (|ξ_1|, |ξ_2|, |ξ_3|). \] (29)

By symmetry, we may assume \(|ξ_1| \leq |ξ_2| \leq |ξ_3|\). Then by (29)
\[ |\hat{N}_3(v)(ξ)| \leq 12σ \int_{ξ=ξ_1+ξ_2+ξ_3} \hat{V}(ξ_1) \cdot |ξ_2|\hat{V}(ξ_2) \cdot |ξ_3|\hat{V}(ξ_3) \, dξ_1 \, dξ_2 \, dξ_3 \]
\[ = 12σ \int_{ξ=ξ_1+ξ_2+ξ_3} \hat{V}(ξ_1) \cdot |ξ_2|\hat{V}(ξ_2) \cdot \hat{V}(ξ_3) \, dξ_1 \, dξ_2 \, dξ_3 \]
\[ = 12σ \mathcal{F}_x [V \cdot |D_x|V \cdot V](ξ). \]

Therefore, using Plancherel, Hölder and Sobolev inequalities we get
\[ \|N_3(v)\|_{L^2_x} \leq 12σ \|V \cdot |D_x|V \cdot V\|_{L^2_x} \]
\[ \leq 12σ \|V\|_{L^∞_x} \|D_x|V\|_{L^2_x} \|V\|_{L^∞_x} \]
\[ \lesssim σ\|V\|_{H^2}^3 \sim σ\|v\|_{H^2}^3 \]
as desired.

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