Abstract. Let $G$ be a group and write $\text{Perm}(G)$ for its symmetric group. The holomorph $\text{Hol}(G)$ of $G$ is defined to be the normalizer of the subgroup of left translations in $\text{Perm}(G)$. Its normalizer $\text{NHol}(G)$ in $\text{Perm}(G)$ in turn is called the multiple holomorph of $G$. We shall show that the quotient $\text{NHol}(G)/\text{Hol}(G)$ has order two when $G$ is finite and almost simple.

Contents

1. Introduction 1
2. Preliminaries on the multiple holomorph 4
3. Descriptions of regular subgroups in the holomorph 5
4. Basic properties of almost simple groups 8
5. Proof of Theorem 1.2 9
References 12

1. Introduction

Let $G$ be a group and write $\text{Perm}(G)$ for its symmetric group. Recall that a subgroup $N$ of $\text{Perm}(G)$ is said to be regular if the map

$$\xi_N : N \rightarrow G; \quad \xi_N(\eta) = \eta(1_G)$$

is bijective, or equivalently, if the $N$-action on $G$ is both transitive and free. For example, both $\lambda(G)$ and $\rho(G)$ are regular subgroups of $\text{Perm}(G)$, where

$$\begin{align*}
\lambda : G &\rightarrow \text{Perm}(G); \quad \lambda(\sigma) = (x \mapsto \sigma x), \\
\rho : G &\rightarrow \text{Perm}(G); \quad \rho(\sigma) = (x \mapsto x\sigma^{-1}),
\end{align*}$$

denote the left and right regular representations of $G$, respectively. Plainly, we have $\lambda(G)$ are $\rho(G)$ are equal precisely when $G$ is abelian. Further, recall

Date: April 23, 2019.
that the holomorph of $G$ is defined to be

\begin{equation}
\text{Hol}(G) = \rho(G) \rtimes \text{Aut}(G).
\end{equation}

Alternatively, it is not hard to verify that

$$\text{Norm}_{\text{Perm}(G)}(\lambda(G)) = \text{Hol}(G) = \text{Norm}_{\text{Perm}(G)}(\rho(G)).$$

Then, it seems natural to ask whether $\text{Perm}(G)$ has other regular subgroups whose normalizer is equal to $\text{Hol}(G)$. Note that for any regular subgroup $N$ of $\text{Perm}(G)$, the bijection $\xi_N$ induces an isomorphism

\begin{equation}
\Xi_N : \text{Perm}(N) \longrightarrow \text{Perm}(G); \quad \Xi_N(\pi) = \xi_N \circ \pi \circ \xi_N^{-1}
\end{equation}

under which $\lambda(N)$ is sent to $N$. Thus, in turn $\Xi_N$ induces an isomorphism

$$\text{Hol}(N) \simeq \text{Norm}_{\text{Perm}(G)}(N),$$

and so we have

$$\text{Norm}_{\text{Perm}(G)}(N) = \text{Hol}(G) \implies \text{Hol}(N) \simeq \text{Hol}(G).$$

In general, non-isomorphic groups (of the same order) can have isomorphic holomorphs. But let us restrict only to the $N$ isomorphic to $G$, and consider

$$\mathcal{H}_0(G) = \left\{ \text{regular subgroups } N \text{ of } \text{Perm}(G) \text{ isomorphic to } G \quad \text{and such that } \text{Norm}_{\text{Perm}(G)}(N) = \text{Hol}(G) \right\}.$$ 

In [13], G. A. Miller defined the multiple holomorph of $G$ to be

$$\text{NHol}(G) = \text{Norm}_{\text{Perm}(G)}(\text{Hol}(G)),$$

which clearly acts on $\mathcal{H}_0(G)$ via conjugation, and he showed that this action is transitive so the quotient group

$$T(G) = \frac{\text{NHol}(G)}{\text{Hol}(G)}$$

acts regularly on $\mathcal{H}_0(G)$; also see Section 2 below for a proof. In particular, this group $T(G)$ has the same cardinality as $\mathcal{H}_0(G)$. The structure of $T(G)$ has been computed for certain families of groups $G$; see [2, 3, 4, 10, 12]. The purpose of this paper is to compute $T(G)$ for a new family of groups $G$. 

To explain our motivation, first observe that elements of $\mathcal{H}_0(G)$ are normal subgroups of Hol$(G)$; this is known and also see Section 2 below for a proof. Instead of $\mathcal{H}_0(G)$, let us consider the possibly larger sets

$$\mathcal{H}_1(G) = \{\text{normal and regular subgroups of Hol}(G)\},$$

$$\mathcal{H}_2(G) = \{\text{regular subgroups of Hol}(G) \text{ isomorphic to } G\}.$$ 

Then, we have the inclusions

$$\mathcal{H}_0(G) \subset \mathcal{H}_1(G) \text{ and } \mathcal{H}_0(G) \subset \mathcal{H}_2(G).$$

This last set has been computed for various families of groups $G$ because of its connections with Hopf-Galois structures as well as set-theoretic solutions of the Yang-Baxter equation; see [6, Chapter 2] and [9]. For example, if $G$ is finite and non-abelian simple, then by [5, Theorem 4], we have

$$\mathcal{H}_2(G) = \{\lambda(G), \rho(G)\},$$

which in turn implies that

$$(1.3) \quad \mathcal{H}_0(G) = \{\lambda(G), \rho(G)\} \text{ and so } T(G) \simeq \mathbb{Z}/2\mathbb{Z}.$$ 

Let us now consider three generalizations of non-abelian simple groups.

**Definition 1.1.** A group $G$ is said to be

1. **quasisimple** if $G = [G, G]$ and $G/Z(G)$ is simple, where $[G, G]$ and $Z(G)$ denote the commutator subgroup and the center of $G$, respectively;
2. **characteristically simple** if it has no non-trivial proper characteristic subgroup, which for finite $G$, is equivalent to $G = T^n$ for some simple group $T$ and natural number $n$;
3. **almost simple** if $\text{Inn}(T) \leq G \leq \text{Aut}(T)$ for some non-abelian simple group $T$, where $\text{Inn}(T)$ denotes the inner automorphism group of $T$.

If $G$ is finite and quasisimple, then by [15, Theorem 1.3], we have

$$\mathcal{H}_2(G) = \{\lambda(G), \rho(G)\},$$

and hence (1.3) holds as above. However, if $G$ is finite and non-abelian characteristically simple or almost simple, then $\mathcal{H}_2(G)$ contains lots of elements
in general; see [16] and [5, Theorem 5]. Nevertheless, if $G$ is finite and non-abelian characteristically simple, then it follows from [3, Theorem 7.7] that

$$\mathcal{H}_1(G) = \{\lambda(G), \rho(G)\},$$

and so (1.3) holds as well. Our result is that if $G$ is finite and almost simple, then the same phenomenon occurs. More specifically, we shall prove:

**Theorem 1.2.** For any finite almost simple group $G$, we have

$$\mathcal{H}_1(G) = \{\lambda(G), \rho(G)\}.$$

In particular, the statement (1.3) holds.

Therefore, for any finite group $G$ which is quasisimple, non-abelian characteristically simple, or almost simple, the quotient group $T(G)$ is always cyclic of order two.

2. Preliminaries on the Multiple Holomorph

In this section, we shall give a proof of the fact that the action of $\mathrm{NHol}(G)$ on the set $\mathcal{H}_0(G)$ via conjugation is transitive, and the fact that elements of $\mathcal{H}_0(G)$ are normal subgroups of $\mathrm{Hol}(G)$. Both of them are already known in the literature and are consequences of the next simple observation.

**Lemma 2.1.** Isomorphic regular subgroups of $\mathrm{Perm}(G)$ are conjugates.

*Proof.* Let $N_1$ and $N_2$ be any two isomorphic regular subgroups of $\mathrm{Perm}(G)$. Let $\varphi : N_1 \rightarrow N_2$ be an isomorphism and note that the isomorphism

$$\Xi_{\varphi} : \mathrm{Perm}(N_1) \rightarrow \mathrm{Perm}(N_2); \quad \Xi_{\varphi}(\pi) = \varphi \circ \pi \circ \varphi^{-1}$$

sends $\lambda(N_1)$ to $\lambda(N_2)$. For $i = 1, 2$, recall that the isomorphism $\Xi_{N_i}$ defined as in (1.2) sends $\lambda(N_i)$ to $N_i$. It follows that $\Xi_{N_2} \circ \Xi_{\varphi} \circ \Xi_{N_1}^{-1}$ maps $N_1$ to $N_2$. We then deduce that $N_1$ and $N_2$ are conjugates via $\xi_{N_2} \circ \varphi \circ \xi_{N_1}^{-1}$. \hfill \Box

Lemma 2.1 implies that regular subgroups of $\mathrm{Perm}(G)$ isomorphic to $G$ are precisely the conjugates of $\lambda(G)$. For any $\pi \in \mathrm{Perm}(G)$, we have

$$\mathrm{Norm}_{\mathrm{Perm}(G)}(\pi \lambda(G) \pi^{-1}) = \pi \mathrm{Hol}(G) \pi^{-1},$$
which is equal to Hol(G) if and only if \( \pi \in \text{NHol}(G) \). Hence, we have

\[
\mathcal{H}_0(G) = \{ \pi \lambda(G) \pi^{-1} : \pi \in \text{NHol}(G) \}.
\]

(2.1)

With this alternative description of \( \mathcal{H}_0(G) \), we then deduce that:

**Corollary 2.2.** Via conjugation, the multiple holomorph \( \text{NHol}(G) \) acts transitively on \( \mathcal{H}_0(G) \), and hence the quotient \( T(G) \) acts regularly on \( \mathcal{H}_0(G) \).

**Proof.** The fact that \( \text{NHol}(G) \) acts transitively on \( \mathcal{H}_0(G) \) follows immediately from (2.1). By definition, the stabilizer of any \( N \) in \( \mathcal{H}_0(G) \) under this action is equal to \( \text{Hol}(G) \), and so \( T(G) \) acts regularly on \( \mathcal{H}_0(G) \). \( \square \)

**Corollary 2.3.** Elements of \( \mathcal{H}_0(G) \) are normal subgroups of \( \text{Hol}(G) \).

**Proof.** For any \( \pi \in \text{NHol}(G) \), we have

\[
\pi \lambda(G) \pi^{-1} \subset \pi \text{Hol}(G) \pi^{-1} \quad \text{and} \quad \pi \text{Hol}(G) \pi^{-1} = \text{Hol}(G).
\]

By (2.1), this implies that elements of \( \mathcal{H}_0(G) \) are subgroups of \( \text{Hol}(G) \). The fact that they are normal in \( \text{Hol}(G) \) is clear from the definition of \( \mathcal{H}_0(G) \). \( \square \)

3. Descriptions of regular subgroups in the holomorph

In this section, let \( \Gamma \) be a group of the same cardinality as \( G \). Plainly, the regular subgroups of \( \text{Hol}(G) \) isomorphic to \( \Gamma \) are the images of the homomorphisms in the set

\[
\text{Reg}(\Gamma, \text{Hol}(G)) = \{ \text{injective } \beta \in \text{Hom}(\Gamma, \text{Hol}(G)) : \beta(\Gamma) \text{ is regular} \}.
\]

Note that for \( G \) and \( \Gamma \) finite, the map \( \beta \) is automatically injective when \( \beta(\Gamma) \) is regular. Below, we shall give two different ways of describing this set, and it shall be helpful to recall the definition of \( \text{Hol}(G) \) given in (1.1).

The first description uses bijective crossed homomorphisms.

**Definition 3.1.** Given \( \tilde{f} \in \text{Hom}(\Gamma, \text{Aut}(G)) \), a map \( g \in \text{Map}(\Gamma, G) \) is said to be a **crossed homomorphism with respect to** \( \tilde{f} \) if

\[
g(\gamma \delta) = g(\gamma) \cdot \tilde{f}(\gamma)(g(\delta)) \quad \text{for all } \gamma, \delta \in \Gamma.
\]
Write \( Z_1^f(\Gamma, G) \) for the set of all such maps. Also, let \( Z_1^f(\Gamma, G)^* \) and \( Z_1^f(\Gamma, G)^* \), respectively, denote the subsets consisting of those maps which are bijective and injective. Note that these two subsets coincide when \( G \) and \( \Gamma \) are finite.

**Proposition 3.2.** For \( f \in \text{Map}(\Gamma, \text{Aut}(G)) \) and \( g \in \text{Map}(\Gamma, G) \), define 
\[
\beta_{(f,g)} : \Gamma \longrightarrow \text{Hol}(G); \quad \beta_{(f,g)}(\gamma) = \rho(g(\gamma)) \cdot f(\gamma).
\]

Then, we have 
\[
\text{Map}(\Gamma, \text{Hol}(G)) = \{ \beta_{(f,g)} : f \in \text{Map}(\Gamma, \text{Aut}(G)), \ g \in \text{Map}(\Gamma, G) \},
\]
\[
\text{Hom}(\Gamma, \text{Hol}(G)) = \{ \beta_{(f,g)} : f \in \text{Hom}(\Gamma, \text{Aut}(G)), \ g \in Z_1^f(\Gamma, G) \},
\]
\[
\text{Reg}(\Gamma, \text{Hol}(G)) = \{ \text{injective } \beta_{(f,g)} : f \in \text{Hom}(\Gamma, \text{Aut}(G)), \ g \in Z_1^f(\Gamma, G)^* \}.
\]

**Proof.** This follows easily from (1.1); see [15, Proposition 2.1] for a proof and note that the argument there does not require \( G \) and \( \Gamma \) to be finite. \( \square \)

The second description uses fixed point free pairs of homomorphisms. The use of such pairs was observed by N. P. Byott and L. N. Childs in [1]; similar ideas also appeared in [5, 7].

**Definition 3.3.** For any groups \( \Gamma_1 \) and \( \Gamma_2 \), a pair \((f, g)\) of homomorphisms from \( \Gamma_1 \) to \( \Gamma_2 \) is said to be fixed point free if the equality \( f(\gamma) = g(\gamma) \) holds precisely when \( \gamma = 1_{\Gamma_1} \).

Let \( \text{Out}(G) \) denote the outer automorphism group of \( G \) and write 
\[
\pi_G : \text{Aut}(G) \longrightarrow \text{Out}(G); \quad \pi_G(\varphi) = \varphi \cdot \text{Inn}(G)
\]
for the natural quotient map. Given \( f \in \text{Hom}(\Gamma, \text{Aut}(G)) \), define 
\[
\text{Hom}_f(\Gamma, \text{Aut}(G)) = \{ h : h \in \text{Hom}(\Gamma, \text{Aut}(G)) : \pi_G \circ h = \pi_G \circ f \},
\]
\[
\text{Hom}_f(\Gamma, \text{Aut}(G))^* = \{ h : h \in \text{Hom}_f(\Gamma, \text{Aut}(G)) : (f, h) \text{ is fixed point free} \}.
\]

In view of Proposition 3.2, it is enough to consider \( Z_1^f(\Gamma, G)^* \), which in turn is equal to \( Z_1^f(\Gamma, G)^* \) when \( G \) and \( \Gamma \) are finite. The next proposition gives an alternative description of this latter set in the case that \( G \) has trivial center. Let us remark that it is motivated by the arguments in [5, pp. 83–84] and is also a generalization of [15, Proposition 2.4].
Proposition 3.4. Let $f \in \mathrm{Hom}(\Gamma, \mathrm{Aut}(G))$. For $g \in Z^1_f(\Gamma, G)$, define

$$h_{(f,g)} : \Gamma \longrightarrow \mathrm{Aut}(G); \quad h_{(f,g)}(\gamma) = \text{conj}(g(\gamma)) \cdot f(\gamma).$$

where $\text{conj}(\cdot) = \lambda(\cdot)\rho(\cdot)$. Then, the map $h_{(f,g)}$ is a homomorphism, and

\begin{align}
Z^1_f(\Gamma, G) &\longrightarrow \mathrm{Hom}_f(\Gamma, \mathrm{Aut}(G)); \quad g \mapsto h_{(f,g)} \\
Z^1_f(\Gamma, G)^* &\longrightarrow \mathrm{Hom}_f(\Gamma, \mathrm{Aut}(G))^*; \quad g \mapsto h_{(f,g)}
\end{align}

are well-defined bijections when $G$ has trivial center.

Proof. First, let $g \in Z^1_f(\Gamma, G)$. For any $\gamma, \delta \in \Gamma$, we have

$$h_{(f,g)}(\gamma \delta) = \text{conj}(g(\gamma \delta)) \cdot f(\gamma \delta)
= \text{conj}(g(\gamma))f(\gamma) \cdot f(\gamma)^{-1}\text{conj}(f(\gamma)(g(\delta)))f(\gamma) \cdot f(\delta)
= \text{conj}(g(\gamma))f(\gamma) \cdot \text{conj}(g(\delta))f(\delta)
= h_{(f,g)}(\gamma)h_{(f,g)}(\delta)$$

and so $h_{(f,g)}$ is a homomorphism. This means that (3.1) is well-defined.

Now, suppose that $G$ has trivial center, in which case $\text{conj} : G \longrightarrow \text{Inn}(G)$ is an isomorphism. Given $h \in \mathrm{Hom}_f(\Gamma, \mathrm{Aut}(G))$, define

$$g : \Gamma \longrightarrow G; \quad g(\gamma) = \text{conj}^{-1}(h(\gamma)f(\gamma)^{-1}),$$

where $h(\gamma)f(\gamma)^{-1} \in \text{Inn}(G)$ since $\pi_G \circ h = \pi_G \circ f$. For any $\gamma, \delta \in \Gamma$, we have

$$\text{conj}(g(\gamma \delta)) = h(\gamma \delta)f(\gamma \delta)^{-1}
= h(\gamma)f(\gamma)^{-1} \cdot f(\gamma)h(\delta)f(\delta)^{-1}f(\gamma)^{-1}
= \text{conj}(g(\gamma)) \cdot f(\gamma)\text{conj}(g(\delta))f(\gamma)^{-1}
= \text{conj}(g(\gamma)) \cdot \text{conj}(f(\gamma)(g(\delta)))
= \text{conj}(g(\gamma)) \cdot f(\gamma)(g(\delta))$$

and so $g$ is a crossed homomorphism with respect to $f$. Clearly $h = h_{(f,g)}$ and so this shows that (3.1) is surjective. Let $g_1, g_2, g \in Z^1_f(\Gamma, G)$. For any $\gamma \in \Gamma$, since $\text{conj}$ is an isomorphism, we have

$$g_1(\gamma) = g_2(\gamma) \iff \text{conj}(g_1(\gamma)) = \text{conj}(g_2(\gamma)) \iff h_{(f,g_1)}(\gamma) = h_{(f,g_2)}(\gamma)$$
and so (3.1) is also injective. For any $\gamma_1, \gamma_2 \in \Gamma$, similarly
\[
g(\gamma_1) = g(\gamma_2) \iff \text{conj}(g(\gamma_1)) = \text{conj}(g(\gamma_2)) \iff h_{(f,g)}(\gamma_1^{-1}\gamma_2) = f(\gamma_1^{-1}\gamma_2)
\]
and this implies that (3.2) is a well-defined bijection as well. \hfill \Box

Finally, we shall give a necessary condition for a subgroup of $\text{Hol}(G)$ to be normal in terms of the notation of Propositions 3.2 and 3.4.

**Proposition 3.5.** Let $f \in \text{Hom}(\Gamma, \text{Aut}(G))$ and $g \in Z^1_f(\Gamma, G)$. If $\beta_{(f,g)}(\Gamma)$ is a normal subgroup of $\text{Hol}(G)$, then both $f(\gamma)$ and $h_{(f,g)}(\gamma)$ are normal subgroups of $\text{Aut}(G)$.

*Proof.* Suppose that $\beta_{(f,g)}(\Gamma)$ is normal in $\text{Hol}(G)$ and let $\varphi \in \text{Aut}(G)$. Then, for any $\gamma \in \Gamma$, there exists $\gamma_\varphi \in \Gamma$ such that
\[
\varphi \beta_{(f,g)}(\gamma) \varphi^{-1} = \beta_{(f,g)}(\gamma_\varphi).
\]
Rewriting this equation yields
\[
\rho(\varphi(g(\gamma))) \cdot \varphi f(\gamma) \varphi^{-1} = \rho(g(\gamma_\varphi)) \cdot f(\gamma_\varphi).
\]
Since (1.1) is a semi-direct product, this in turn gives
\[
\varphi(g(\gamma)) = g(\gamma_\varphi) \text{ and } \varphi f(\gamma) \varphi^{-1} = f(\gamma_\varphi).
\]
The latter shows that $f(\Gamma)$ is normal in $\text{Aut}(G)$. The above also implies that
\[
\varphi h_{(f,g)}(\gamma) \varphi^{-1} = \text{conj}(\varphi(g(\gamma))) \cdot \varphi f(\gamma) \varphi^{-1} = \text{conj}(g(\gamma_\varphi)) \cdot f(\gamma_\varphi) = h_{(f,g)}(\gamma_\varphi),
\]
and hence $h_{(f,g)}(\Gamma)$ is normal in $\text{Aut}(G)$ as well. \hfill \Box

4. **Basic properties of almost simple groups**

In this section, let $S$ be an almost simple group and let $T$ be a non-abelian simple group such that $\text{Inn}(T) \leq S \leq \text{Aut}(T)$. Notice that $\text{Inn}(T)$ is normal in $\text{Aut}(T)$ and hence is normal in $S$ as well. Recall the known fact, which is easily verified, that for any $\varphi \in \text{Aut}(T)$, we have
\[
\varphi \circ \psi = \psi \circ \varphi \text{ for all } \psi \in \text{Inn}(T) \text{ implies } \varphi = \text{Id}_T.
\]
This implies the next three basic properties of $S$ which we shall need.
Lemma 4.1. The center of $S$ is trivial.

Proof. This follows directly from (4.1). □

Lemma 4.2. Every non-trivial normal subgroup of $S$ contains $\text{Inn}(T)$.

Proof. Let $R$ be any normal subgroup of $S$ such that $R \not\supset \text{Inn}(T)$, or equivalently $R \cap \text{Inn}(T) \neq \text{Inn}(T)$. Then, since $R \cap \text{Inn}(T)$ is normal in $\text{Inn}(T)$ and $\text{Inn}(T) \cong T$ is simple, this means that $R \cap \text{Inn}(T) = 1$. For any $\varphi \in R$, since both $R$ and $\text{Inn}(T)$ are normal in $\text{Aut}(T)$, we have

$$\psi \circ \varphi \circ \psi^{-1} \circ \varphi^{-1} \in R \cap \text{Inn}(T)$$

for all $\psi \in \text{Inn}(T)$. We then deduce from (4.1) that $\varphi = \text{Id}_T$ and so $R$ is trivial. □

Lemma 4.3. There is an injective homomorphism $\text{Aut}(S) \rightarrow \text{Aut}(T)$ such that its image contains the subgroup $\text{Inn}(T)$.

Proof. Put $T^# = \text{Inn}(T)$, which is the socle of $S$ by Lemma 4.2, and hence is a characteristic subgroup of $S$. Thus, there is a well-defined homomorphism

(4.2) $\text{Aut}(S) \rightarrow \text{Aut}(T^#); \quad \theta \mapsto \theta|_{T^#}.$

Its image clearly contains $\text{Inn}(T^#)$. Now, suppose that $\theta$ is in its kernel. For any $\varphi \in S$, we then have

$$\theta(\varphi) \circ \psi \circ \theta(\varphi)^{-1} = \theta(\varphi \circ \psi \circ \varphi^{-1}) = \varphi \circ \psi \circ \varphi^{-1}$$

for all $\psi \in T^#$. From (4.1), we deduce that $\theta(\varphi) = \varphi$, and thus $\theta = \text{Id}_S$. This shows that the map (4.2) is injective. Since $T^# \cong T$, the claim now follows. □

5. Proof of Theorem 1.2

Our proof relies on the following consequences of the classification theorem of finite simple groups. Recall that for any group $\Gamma$, an endomorphism $\varphi$ on $\Gamma$ is said to be fixed point free if the equality $\varphi(\gamma) = \gamma$ holds precisely when $\gamma = 1_\Gamma$, or in other words, if the pair $(\varphi, 1_\Gamma)$ is fixed point free in the sense of Definition 3.3.

Lemma 5.1. Let $T$ be a finite non-abelian simple group. Then:
(a) There is no fixed point free automorphism on $T$.
(b) The outer automorphism group $\text{Out}(T)$ of $T$ is solvable.
(c) The inequality $|T|/|\text{Out}(T)| \geq 30$ holds.

Proof. They are all consequences of the classification theorem of finite simple groups. See [8, Theorems 1.46 and 1.48] for parts (a) and (b). See [14, Lemma 2.2] for part (c). □

Lemma 5.1 (c) in particular implies the following corollaries.

Corollary 5.2. Let $T$ be a finite non-abelian simple group. Then, any finite group $S$ of order less than $30|\text{Aut}(T)|$ cannot have subgroups $T_1$ and $T_2$, both of which are isomorphic to $T$, such that $T_1 \cap T_2 = 1$.

Proof. Suppose that $S$ is a finite group having subgroups $T_1$ and $T_2$, both of which are isomorphic to $T$, such that $T_1 \cap T_2 = 1$. Then, we have

$$|T_1T_2| = |T_1||T_2| = |\text{Inn}(T)||T| = |\text{Aut}(T)||T|/|\text{Out}(T)|.$$ 

Since $T_1T_2$ is a subset of $S$, from Lemma 5.1 (c), it follows that $S$ must have order at least $30|\text{Aut}(T)|$. □

Corollary 5.3. Let $T$ be a finite non-abelian simple group. Then, the inner automorphism group $\text{Inn}(T)$ is the only subgroup of $\text{Aut}(T)$ isomorphic to $T$.

Proof. Let $R$ be any subgroup of $\text{Aut}(T)$ isomorphic to $T$. Since $\text{Inn}(T) \cap R$ is normal in $R$, and it cannot be trivial by Corollary 5.2, it must be equal to the entire $R$. It follows that $R \subset \text{Inn}(T)$, and we have equality because these are finite groups of the same order. □

We are now ready to prove our main theorem.

Proof of Theorem 1.2. Let $G$ be a finite almost simple group, say

$$\text{Inn}(T) \leq G \leq \text{Aut}(T),$$

where $T$ is a finite non-abelian simple group. From Lemma 4.3, we see that the group $\text{Aut}(G)$ is also almost simple, as well as that

$$\text{Inn}(T^\#) \leq \text{Aut}(G) \leq \text{Aut}(T^\#),$$
where $T^#$ is a group isomorphic to $T$. Note that $Z(G) = 1$ by Lemma 4.1.

Now, let us consider a regular subgroup $N$ of Hol($G$). By Proposition 3.2, we may write it as

$$N = \{ \rho(g(\gamma)) \cdot f(\gamma) : \gamma \in \Gamma \},$$

where $f \in \text{Hom}(\Gamma, \text{Aut}(G))$, $g \in Z_f^1(\Gamma, G)^*$

and $\Gamma$ is a group isomorphic to $N$. By Proposition 3.4, we may define

$$h \in \text{Hom}(\Gamma, \text{Aut}(G)); \quad h(\gamma) = \text{conj}(g(\gamma)) \cdot f(\gamma),$$

and the pair $(f, h)$ is fixed point free. Further, observe that

$$\begin{cases} N \subset \rho(G) & \text{if } f(\Gamma) \text{ is trivial,} \\ N \subset \lambda(G) & \text{if } h(\Gamma) \text{ is trivial,} \end{cases}$$

and we must have equalities because $N$ is regular.

In what follows, assume that both $f(\Gamma)$ and $h(\Gamma)$ are non-trivial. Suppose also for contradiction that $N$ is normal in Hol($G$). Then, both $f(\Gamma)$ and $h(\Gamma)$ are normal in Aut($G$) by Proposition 3.5, whence

$$(5.1) \quad \text{Inn}(T^#) \leq f(\Gamma), h(\Gamma) \leq \text{Aut}(T^#)$$

by Lemma 4.2. This means that $f(\Gamma)$ and $h(\Gamma)$ are almost simple as well.

(a) **Suppose that both $\ker(f)$ and $\ker(h)$ are non-trivial.**

Note that $\ker(f) \cap \ker(h) = 1$ because $(f, h)$ is fixed point free. This means that $f$ restricts to an injective homomorphism

$$\text{res}(f) : \ker(h) \longrightarrow \text{Aut}(G), \quad \text{and } f(\ker(h)) \text{ is non-trivial.}$$

Since $f(\ker(h))$ is normal in $f(\Gamma)$, from Lemma 4.2 and (5.1), we see that

$$\text{Inn}(T^#) \leq f(\ker(h)),$$

and so there is a subgroup $\Delta_f$ of $\ker(h)$ isomorphic to $T$. Similarly, there is a subgroup $\Delta_f$ of $\ker(h)$ isomorphic to $T$. But

$$\Delta_f \cap \Delta_h \subset \ker(f) \cap \ker(h) \text{ and so } \Delta_f \cap \Delta_h = 1.$$ 

This contradicts Corollary 5.2 because $\Gamma$ has the same order as $G$, which is contained in Aut($T$) by assumption.
(b) Suppose that \( \ker(f) \) is trivial but \( \ker(h) \) is non-trivial.

Note that \( f \) is injective, so \( f \) induces an isomorphism

\[
\frac{\Gamma}{\ker(h)} \cong \frac{f(\Gamma)}{f(\ker(h))}, \quad \text{and } f(\ker(h)) \text{ is non-trivial.}
\]

(5.2)

On the one hand, the first quotient group in (5.2) is isomorphic to \( h(\Gamma) \), which is insolvable by (5.1). On the other hand, since \( f(\ker(h)) \) is normal in \( f(\Gamma) \), by Lemma 4.2 and (5.1), we have \( \text{Inn}(T^\#) \leq f(\ker(h)) \). There are natural homomorphisms

\[
\begin{align*}
\frac{f(\Gamma)}{\text{Inn}(T^\#)} & \quad \text{surjective} \quad \frac{f(\Gamma)}{f(\ker(h))} \quad \text{and} \quad \\
\frac{f(\Gamma)}{\text{Inn}(T^\#)} & \quad \text{injective} \quad \frac{f(\Gamma)}{\text{Inn}(T^\#)} \quad \rightarrow \quad \text{Out}(T^\#).
\end{align*}
\]

Since \( \text{Out}(T^\#) \) is solvable by Lemma 5.1 (b), the second quotient group in (5.2) is also solvable, and this is a contradiction.

(c) Suppose that \( \ker(h) \) is trivial but \( \ker(f) \) is non-trivial.

By symmetry, we obtain a contradiction as in case (b).

(d) Suppose that both \( \ker(f) \) are \( \ker(h) \) are trivial.

Note that \( \Gamma \cong f(\Gamma) \), so \( \Gamma \) contains a unique subgroup \( \Delta \) isomorphic to \( T \) by Corollary 5.3 and (5.1). Since both \( f \) and \( h \) are injective, they restrict to isomorphisms

\[
\text{res}(f) : \Delta \rightarrow \text{Inn}(T^\#) \quad \text{and} \quad \text{res}(h) : \Delta \rightarrow \text{Inn}(T^\#).
\]

But then \( \text{res}(f)^{-1} \circ \text{res}(h) \) is a fixed point free automorphism on \( \Delta \), which contradicts Lemma 5.1 (a).

We have thus shown that in order for \( N \) to be normal in \( \text{Hol}(G) \), either \( f(\Gamma) \) or \( h(\Gamma) \) must be trivial, and consequently \( N \) is equal to \( \lambda(G) \) or \( \rho(G) \). Hence, indeed \( \lambda(G) \) are \( \rho(G) \) are the only elements of \( \mathcal{H}_1(G) \), as desired. \( \square \)

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