Remarks on a Paper by Y. Caro and R. Yuster on Turán Problem

(Revised)

Oleg Pikhurko
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213-3890
Web: http://www.math.cmu.edu/~pikhurko

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Abstract

For a graph $F$ and a function $f : \mathbb{N} \to \mathbb{R}$, let $e_f(F) = \sum_{x \in V(F)} f(d(x))$ and let $\text{ex}_f(n, F)$ be the maximum of $e_f(G)$ over all $F$-free graphs $G$ with $n$ vertices.

Suppose that $f$ is a non-decreasing function with the property that for any $\varepsilon > 0$ there is $\delta > 0$ such that for any $n \leq m \leq (1+\delta)n$ we have $f(m) \leq (1+\varepsilon)f(n)$. Under this assumption we prove that the asymptotics of $\text{ex}_f(n, F)$, where $F$ is a fixed non-bipartite graph and $n$ tends to the infinity, can be computed by considering complete $(\chi(F) - 1)$-partite graphs only.

This research was motivated by a paper of Y. Caro and R. Yuster [Electronic J. Combin., 7 (2000)] who studied the case when $f : x \mapsto x^\mu$ is a power function.

1 Introduction

Let $\mathbb{N}$ denote the set of non-negative integers and $\mathbb{R}$ the set of reals. Let $f : \mathbb{N} \to \mathbb{R}$ be an arbitrary function.

For a graph $F$ define $e_f(F) = \sum_{x \in V(F)} f(d(x))$, where $d(x)$ denotes the degree of a vertex $x$. For example, for $f : x \mapsto \frac{x}{2}$ we have $e_f(F) = e(F)$; thus $e_f(F)$ can be viewed as a generalization of the size of $F$.

Define $\text{ex}_f(n, F)$ to be the maximal value of $e_f(G)$ over all $F$-free graphs $G$ of order $n$. This mimics the definition of the usual Turán function $\text{ex}(n, F)$. The special case when $f$ is the power function $P_\mu : x \mapsto x^\mu$, with integer $\mu \geq 1$, appears in the paper of Caro and Yuster [2] which was the main motivation for the present research.
Let $\text{ex}'_f(n, F)$ be the maximum of $e_f(H)$ over all complete $(\chi(F) - 1)$-partite graphs of order $n$. Clearly, we have

$$\text{ex}'_f(n, F) \leq \text{ex}_f(n, F).$$

Caro and Yuster [2] proved that $\text{ex}_{P_\mu}(n, K_r) = \text{ex}'_{P_\mu}(n, K_r)$ for $r \geq 3$. However, they incorrectly claimed that the Turán graph $T_{r-1}(n)$, which has part sizes almost equal, is always optimal. (A revised version [3] of their paper appeared.) This mistake also appears in the first version of the current paper.

First of all, let us observe that the proof from [2] remains true for a much wider class of functions $f$. Namely, a function $f$ is called non-decreasing if for any $m \leq n$ we have $f(m) \leq f(n)$.

**Theorem 1** For any $n \geq 0$, $r \geq 3$ and non-decreasing $f : \mathbb{N} \to \mathbb{R}$ we have

$$\text{ex}_f(n, K_r) = \text{ex}'_f(n, K_r).$$

**Proof.** By (1) we have to prove the ‘$\leq$’-inequality only. Let an $F$-free graph $G$ achieve $\text{ex}_f(n, F)$. By a theorem of Erdős [4] there is an $(r - 1)$-partite graph $H$ on the same vertex set $V$ such that $d_H(x) \geq d_G(x)$ for every $x \in V$. We have

$$\text{ex}_f(n, K_r) = \sum_{x \in V} f(d_G(x)) \leq \sum_{x \in V} f(d_H(x)) \leq \text{ex}'_f(n, K_r),$$

establishing the required.

Caro and Yuster [2, Conjecture 6.1] posed the problem of computing $\text{ex}_{P_\mu}(n, F)$ for an arbitrary graph $F$. Here we show that if $F$ is a fixed graph of chromatic number $r \geq 3$ and $f$ is a ‘nice’ function (including power functions $P_\mu$ for $\mu \geq 0$) then in order to compute $\text{ex}_f(n, F)$ asymptotically it is enough to consider complete $(r - 1)$-partite graphs only. Determining the optimal sizes of the $r - 1$ parts may be a difficult analytical task. (Bollobás and Nikiforov [11] investigate this problem for the power function $P_\mu$.) But, combinatorially, the problem of computing the asymptotics of $\text{ex}_f(n, F)$ for such $f, F$ may be considered as solved.

**2 Main Result**

Let us call a non-decreasing function $f : \mathbb{N} \to \mathbb{R}$ log-continuous if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any $m, n \in \mathbb{N}$ with $n \leq m \leq (1 + \delta)n$ we have

$$f(m) \leq (1 + \varepsilon)f(n).$$

For example, $P_\mu$ is log-continuous for any $\mu > 0$ while the exponent $x \mapsto e^x$ is not.
Theorem 2 Let $F$ be a fixed non-bipartite graph. Let $f : \mathbb{N} \to \mathbb{R}$ be an arbitrary non-decreasing and log-continuous function. Then, as $n \to \infty$, 

$$
ex_f(n, F) = (1 + o(1)) e'_f(n, F).
$$

Proof. Let $r = \chi(F) \geq 3$, $\varepsilon > 0$ be arbitrary, $n$ be large, and $G$ achieve $e_f(n, F)$.

We will need the following result of Erdős, Frankl and Rödl [5, Theorem 1.5].

Theorem 3 For every $\varepsilon > 0$ and a graph $F$, there is a constant $n_0 = n_0(\varepsilon, F)$ with the following property. Let $G$ be a graph of order $n \geq n_0$ that does not contain $F$ as a subgraph. Then $G$ contains a set $E'$ of less than $\varepsilon n^2$ edges such that the subgraph $H = G - E'$ has no $K_r$, where $r = \chi(F)$. 

Theorem 3 is proved by applying Szemerédi’s Regularity Lemma so the lower bounds on $n_0$ are huge. Also, $\delta = \delta(\varepsilon)$ in [3] is implicit. Therefore, in what follows we make no attempt to optimize the constants. In the proof we choose positive constants $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. The Reader can check that indeed we can choose $\gamma_i$sufficiently small (depending on $F, f, \varepsilon, \gamma_1, \ldots, \gamma_{i-1}$) so that all inequalities are true for all large $n$.

First, let us observe that by the assumptions on $f$

$$
e'_f(n, F) \geq e_f(T_{r-1}(n)) \geq n f(\lfloor n(r-1)/r \rfloor) \geq \gamma_1 nf(n). \quad (4)
$$

Let $V = V(G)$. Define $A = \{x \in V : d_G(x) \leq 2\gamma_2 n\}$ and $B = V \setminus A$. The subgraph $G' = G[B]$ spanned by $B$ is of course $F$-free. Theorem 3 gives us a $K_r$-free subgraph $H' \subset G'$ with $e(G') - e(H') \leq \gamma_4 n^2$. By the theorem of Erdős [4] there is an $(r - 1)$-partite graph on $B$ majorizing the degrees of $H'$. Extend it to a complete $(r - 1)$-partite graph on $V$ by arbitrarily splitting $A$ into $r - 1$ almost equal parts.

The proof will be complete if we show that

$$
e_f(G) - e_f(H) \leq \varepsilon e_f(G). \quad (5)
$$

To prove (5) we estimate the contribution to $e_f(G) - e_f(H)$ by various sets of vertices.

If, for example, $|A| \geq 2\gamma_2 n$, then for any $x \in A$ we have $d_H(x) \geq \lceil 2\gamma_2 n \rceil \geq \gamma_2 n \geq d_G(x)$, i.e., the contribution of $A$ to the left-hand side of (5) is non-positive. Otherwise, the contribution is at most

$$
\sum_{x \in A} f(d_G(x)) \leq |A| f(n) \leq \frac{\varepsilon \gamma_1}{4} nf(n) \leq \frac{\varepsilon}{4} e_f(G).
$$

(In the last inequality we used (4).)
Let $C = \{x \in B : |\Gamma_G(x) \cap A| \geq \frac{1}{2}|A|\}$. By counting the degrees in $A$, we obtain $|C| \leq 2\gamma_2 n$ and thus the $C$-contribution is also at most $\frac{\varepsilon}{4} e_f(G)$. Notice that any vertex of $D = B \setminus C$ has at least as many $A$-neighbors in $H$ as it has in $G$.

Define $K = \{x \in D : d_{H'}(x) \leq d_{G'}(x) - \gamma_3 n\}$. Clearly, 

$$|K| \leq \frac{2(e(G') - e(H'))}{\gamma_3 n} \leq \frac{2\gamma_4}{\gamma_3} n.$$ 

Again, $|K|$ is so small that $K$ contributes at most $\frac{\varepsilon}{4} e_f(G)$ to (5).

Let us estimate the contribution of $K = D \setminus L$. For any vertex $x$ of $D$ we have 

$$d_G(x) - d_H(x) \leq d_{G'}(x) - d_{H'}(x) \leq \gamma_3 n.$$ 

All vertices in $K \subset B$ has $G$-degree at least $\gamma_2 x$. As $\gamma_3$ is small compared with $\gamma_2$, we can assume by the log-continuity of $f$ that $f(d_G(x)) \leq (1 + \frac{\varepsilon}{4}) f(d_H(x))$ for any $x \in K$. This completes the proof of (5) and the theorem. □

Remark. Taking $f : x \mapsto \log x$ we can also solve the problem of maximizing $\prod_{x \in V(G)} d(x)$ over all $F$-free graphs $G$ of order $n$. (However, please notice that the relative error here will not be $1 + o(1)$ but becomes such after taking the logarithm.) More generally, we can maximize $\prod_{x \in V(G)} f(d(x))$ for any non-decreasing $f$ such that $\log(f(x))$ is log-continuous; in particular, this is true if $f$ itself is log-continuous.

3 Some Negative Examples

In Theorem 2 we do need some condition bounding the rate of growth of $f$. For example, if $f$ grows so fast that $e_f(G)$ is dominated by the contribution from the vertices of degree $n - 1$, then the conclusion of Theorem 2 is no longer true: for example, for $K_3(2)$ (the blown-up $K_3$ where each vertex of $K_3$ is duplicated) the value $\text{ex}_f(n, K_3(2)) = (3 + o(1)) f(n - 1)$ cannot be achieved by a bipartite graph.

In fact, one can get refuting examples of $f$ with moderate rate of growth. For example, for any constant $c < 1$ there is a non-decreasing $f$ such that 

$$\frac{f(n + 1)}{f(n)} \leq 1 + n^{-c}$$

for any $n$ and yet the conclusion of Theorem 2 does not hold for this $f$. Let us demonstrate the above claim.

Let $c > 0$. Choose $t$ such that for all large $n$ there is an $K_{t,t}$-free graph $G_n$ of order $n$ with almost all vertices having degree at least $n^c$ each. This $t$ exists by a construction of Kollár, Rónyai and Szabó [3].
Let $F = K_3(2t - 1)$ be a blown-up $K_3$. Take an arbitrary function $f$ satisfying (6) and the additional property that there is an infinite sequence $n_1 < n_2 < \ldots$ such that for any $k$ we have

$$f(n_k + m_k) = f(n_k + m_k + 1) = f(n_k + m_k + 2) = \cdots = f(2n_k),$$

while $f(n_k) \leq \frac{1}{2} f(n_k + m_k)$, where $m_k = \lfloor \frac{1}{2} n^c \rfloor$. Such an $f$ exists: choose the numbers $n_k$ spaced far apart (with $n_1$ being sufficiently large), let $f(n + 1) = f(n)$ except for $n_k \leq n \leq n_k + m_k$ we let $f(n + 1) = 2^{1/m_k} f(n)$. Note that $2^{1/m_k} < 1 + \frac{1}{m_k} < 1 + n^{-c}$ so our $f$ does satisfy (6).

On one hand, we have

$$\text{ex}_f (2n_k, F) \geq (2 + o(1)) n_k f(n_k + m_k).$$

(7)

Indeed, Let $G$ be obtained from the complete bipartite graph $K_{n_k,n_k}$ by adding to each part the $K_{t,t}$-free graph $G_{n_k}$ defined above. It is easy to see that $G \not\supset F$. Almost all vertices of $G$ have degree at least $n_k + m_k$, giving (7).

On the other hand, for any bipartite graph $H$ of order $2n_k$ at least $n_k$ vertices will have degree at most $n_k$ and thus

$$e_f(H) \leq n_k f(2n_k - 1) + n_k f(n_k) \leq \frac{3}{2} n_k f(n_k + m_k).$$

We obtain by (7) that $\text{ex}_f (n, F)$ cannot always be approximated by bipartite graphs.

References

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