Graph Reinforcement Learning for Wireless Control Systems: Large-Scale Resource Allocation over Interference Channels
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Abstract—Modern control systems routinely employ wireless networks to exchange information between spatially distributed plants, actuators and sensors. With wireless networks defined by random, rapidly changing transmission conditions that challenge assumptions commonly held in the design of control systems, proper allocation of communication resources is essential to achieve reliable operation. Designing resource allocation policies, however, is challenging, motivating recent works to successfully exploit deep learning and deep reinforcement learning techniques to design resource allocation and scheduling policies for wireless control systems. As the number of learnable parameters in a neural network grows with the size of the input signal, deep reinforcement learning algorithms may fail to scale, limiting the immediate generalization of such scheduling and resource allocation policies to large-scale systems. The interference and fading patterns among plants and controllers in the network, on the other hand, induce a time-varying communication graph that can be used to construct policy representations based on graph neural networks (GNNs), with the number of learnable parameters now independent of the number of plants in the network. That invariance to the number of nodes is key to design scalable and transferable resource allocation policies, which can be trained with reinforcement learning. Through extensive numerical experiments we show that the proposed graph reinforcement learning approach yields policies that not only outperform baseline solutions and deep reinforcement learning-based policies in large-scale systems, but that can also be transferred across networks of varying size.

Index Terms—Wireless Control Systems, Resource Allocation, Graph Neural Networks, Constrained Reinforcement Learning.

I. INTRODUCTION

Modern control systems routinely employ wireless networks to exchange information between spatially distributed plants, actuators and sensors. Transmission conditions in wireless channels, however, vary rapidly, and information packets addressed to different users are further subject to destructive interference [2]. Those issues can be alleviated by designing proper allocation policies to distribute the limited communication resources across devices sharing a communication network, prioritizing devices with better transmission conditions, for example. Usually, the objective of such resource allocation problems is to optimize some communication metric while satisfying constraints on resource utilization, cf. e.g. [3]–[5]. Although resource allocation problems are often nonconvex and of infinite dimensionality [6], making them hard to solve, recent advances in machine learning motivated the design of model-free approaches for resource allocation policies in wireless systems [7]–[10].

Here we consider wireless control systems (WCSs), in which a collection of plants shares a wireless communication network to communicate with remote controllers. The use of wireless networks instead of wired communication in this setting has the potential to reduce costs by making the installation of components easier and maintenance more flexible, but also adds particular challenges to the design of control and communication policies [11], [12]. Compared to the standard wireless setting, the allocation of network resources here must take into account not only communication metrics but also performance requirements of each plant in order to achieve good performance. Design of control policies and stability analysis for networked control systems under general communication models have a rich literature and are considered in [13]–[16], among others; resource allocation and scheduling problems in WCSs are studied in [17]–[20], to name a few. As in the pure wireless setting, finding optimal resource allocation policies in WCSs usually leads to a hard optimization problem, motivating the use of heuristics, and, more recently, (deep) reinforcement learning, to design resource allocation and scheduling policies, cf., for example, [21]–[25] and our previous work [26]. The scheduling algorithms proposed in [22] and [23], for example, rely on Deep Q-Network (DQN), a value-based deep RL algorithm. In [24], the authors rely on actor-critic algorithms to learn communication and control policies in event-triggered WCSs. The algorithm learns a triggering policy for communication of the control signal and although frequent communication is penalized in the objective function, the problem does not take resource constraints explicitly into account. In our previous work [26], on the other hand, we relied on constrained reinforcement learning algorithms to find resource allocation and control policies for WCSs subject to long-term constraints on, e.g., resource utilization or performance of individual plants.

These approaches rely on neural networks to parametrize resource allocation policies, and hence cannot be immediately applied to large-scale WCSs. As neural networks consist of successive computational layers made up of pointwise nonlinearities on top of linear combinations, the number of parameters to be learned quickly grows with the dimensionality of the problem. We can, however, turn our attention to the regularity of the underlying communication graph and rely on graph neural networks (GNNs) to parameterize resource allocation policies in wireless [27]–[30] and wireless control systems, as we propose here. Training GNNs corresponds
to optimizing the coefficients of the graph filters used to aggregate information from the network nodes — instead of the weights of the linear combinations at each hidden unit as in multilayer neural networks. This not only reduces the number of overall parameters the algorithm must learn, but also makes that number independent of the number of plants, allowing us to design scalable and transferable resource allocation policies.

The rest of the paper is structured as follows. We first describe the resource allocation problem for WCSs in Section II. Allocation decisions are taken by a remote base station (BS) based on estimates of current plant states and channel conditions. The wireless communication model takes into account fading and interference conditions in the network, defining a time-varying interference graph that allows us to use random edge GNNs (Section III-A) to parameterize the resource allocation policy. As random edge GNNs are equivariant under permutation [27], we establish conditions under which the optimal solution of the resource allocation problem is invariant to permutations (Section III-C). That is, the optimal filter coefficients learned for a particular network will also be optimal for its permutations. To solve the resource allocation problem in a model-free manner, we further propose a practical primal-dual graph reinforcement learning algorithm (Section IV). Extensive numerical experiments in Section V illustrate the use of the proposed approach and show that the proposed graph RL approach yields resource allocation policy that not only outperform heuristic and deep RL solutions, but that can also be transferred across networks of varying size. Throughout the paper uppercase letters refer to matrices and lowercase letters to vectors. Positive (semi)definiteness of a matrix is indicated by $X(\geq) > 0$. $\mathbb{R}$ and $\mathbb{N}$ stand for the set of real and natural numbers.

II. RESOURCE ALLOCATION IN WCSs

Consider a system made up by $m$ independent control loops sharing a common wireless medium to communicate with remote base stations — which in turn communicate with each other over a wired connection, as shown in Figure 1. Each control plant $i$ can be described by a discrete, time-invariant model $f(i)(\cdot, \cdot) : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}^p$ mapping a current state vector $x^{(i)}_t \in \mathbb{R}^p$ and corresponding control input $u^{(i)}_t \in \mathbb{R}^q$ to the next state of the system. Furthermore, each of those plants is affected by some zero-mean random noise $w^{(i)}_t \in \mathbb{R}^p$ with covariance matrix $W^{(i)} \in \mathbb{R}^{p \times p}$ standing for eventual disturbances and unmodeled dynamics, leading to

$$x^{(i)}_{t+1} = f(i)(x^{(i)}_t, u^{(i)}_t) + w^{(i)}_t.$$  \hspace{1cm} \hspace{1cm} (1)

Actuation signals $g(x^{(i)}_t)$ are computed at the corresponding remote controller co-located with the base stations, based on observations of states sent by the plants,

$$\tilde{x}^{(i)}_t = x^{(i)}_t + u^{(i)}_t,$$  \hspace{1cm} \hspace{1cm} (2)

with $w^{(i)}_t \in \mathbb{R}^q$ standing for a zero-mean observation noise with covariance matrix $W^{o} \in \mathbb{R}^{q \times q}$. As wireless networks are prone to packet drops, the control signal $u^{(i)}_t$ received by plant $i$ will depend on the channel transmission conditions. If transmission conditions are favorable and the transmission of the packet sent by the remote controller is successful, the plant executes its intended control action. When the transmission fails, however, we assume the plant does not execute any action.

In other words, significant noise in the wireless channel will cause the system to operate in open loop, in which it cannot be directly controlled. The probability of closing the feedback loop in this setting will depend upon the the current fading state in the wireless channel, the transmission power allocated to the control signal sent by the base station, as well as interference caused by transmissions made by other control loops sharing the wireless medium. We may directly or indirectly control power and interference levels in response to the changes in the fading state, which is itself out of our control.

More precisely, fading stands for the rapidly changing transmission conditions in a wireless channel [2, ch. 2]. Fading conditions in channel $i$ can be represented by a random variable $h^{(i)} \in \mathcal{H} \subseteq \mathbb{R}^+$ drawn from some probability distribution $\chi(h)$. Due to multiple plants communicating over the same wireless network, significant interference may be caused by concurrent transmissions. The fading state of the interference link between the transmitter associated with plant $i$ and the base station associated with plant $j$ is given by random variables $h^{(ij)}$. The overall fading conditions in the network can be aggregated in a matrix $H$ describing the state of the wireless network at a given point in time,

$$H = \begin{bmatrix} h^{(11)} & \ldots & h^{(1m)} \\ \vdots & \ddots & \vdots \\ h^{(m1)} & \ldots & h^{(mm)} \end{bmatrix}.$$ \hspace{1cm} (3)

The reliability of the wireless channel does not depend only on the wireless network state, $H$, but also on the power levels used in each of the transmissions. Based on the observations received from the plants (2), as well as current fading and interference conditions in the wireless network, summarized in the interference matrix, base stations compute a downlink power allocation decision $\alpha_t$ sampled from a downlink power allocation policy parametrized by some parameter vector $\theta$,

$$\alpha_t = \alpha(\tilde{x}_t, H_t; \theta).$$ \hspace{1cm} (4)

We denote by $\alpha^{(i)} := [\alpha_t]_i \in \mathbb{R}^+$ a power level used in the transmission of the control packet sent to plant $i$. In particular, for the control signal sent to plant $i$, we can define the signal to interference plus noise ratio (SNIR) $\xi^{(i)} : \mathbb{R}^m_+ \times \mathcal{H}^m \rightarrow \mathbb{R}_+$,

$$\xi^{(i)} = \frac{h^{(ii)} \alpha^{(i)}}{\sigma^2 + \sum_{j \neq i} h^{(ij)} \alpha^{(j)}},$$ \hspace{1cm} (5)

where $\sigma^2$ is the variance of a standard AWGN channel. The SNIR $\xi^{(i)}$ gives the effective channel quality of the link between plant $i$ and its remote controller after the effects of transmission power and interference are taken into account. Observe that higher power levels increase the SNIR in the direct link, but cause greater interference for other transmissions in the network. Mitigating the interference across many devices through power control remains an active problem in wireless communications.
From the control system standpoint, the effective channel quality as measured by the SNIR directly specifies current performance of a wireless control loop. In particular, we define a general function $v : \mathbb{R}_{++}^m \to [0, 1]^m$ that, given the SNIR $\xi^{(i)}$ in a wireless channel, returns the probability of plants closing the feedback loop. This is typically modeled as a sigmoidal-shaped function, but we do not assume such a model is known. Thus, we search for a resource allocation policy as the average power consumption, the fundamental objective in this setting is to keep the plants in desirable states without violating the available communication resources. In particular, we aim to keep plants operating around an equilibrium point, which we assume to be zero without loss of generality. The objective of the resource allocation policy can then be represented by a quadratic cost that penalizes large deviations of the plant states from the equilibrium point. As fading and interference conditions in the wireless network impact the transmission probabilities and stability of the plants is directly tied to their control states, it is paramount that the resource allocation policy take into account both the fading states in the communication network and the states of the control plants. Thus, we search for a resource allocation policy $\alpha(\hat{x}_t, H_t; \theta) : \mathbb{R}^{m_p} \times \mathbb{R}^{m^2} \to \mathcal{A} \subseteq \mathbb{R}^m$ that, given current channel conditions aggregated in $H_t := [h_t^{(11)}, h_t^{(1m)}, \ldots, h_t^{(mm)}]$ and estimates of plant states $\hat{x}_t := [\hat{x}_t^{(1)}, \ldots, \hat{x}_t^{(m)}]$, returns an allocation vector $\alpha \in \mathcal{A} \subseteq \mathbb{R}^m$. The resource allocation problem consists then in finding the allocation policy that minimizes the cost of operating the plants over a finite horizon $T$ starting from some initial state $x_0$, while respecting constraints $l(\alpha) : \mathbb{R}^m \to \mathbb{R}$ on the power used to send control signals back to the plants,

$$u^{(i)}_t = \begin{cases} g(x^{(i)}_t), & \text{w.p. } v(\xi^{(i)}), \\ 0, & \text{w.p. } 1 - v(\xi^{(i)}). \end{cases}$$

(6)

While the aim of resource allocation policies in standard wireless networks is to optimize communication metrics such as the average power consumption, the fundamental objective in this setting is to keep the plants in desirable states without violating the available communication resources. In particular, we aim to keep plants operating around an equilibrium point, which we assume to be zero without loss of generality. The objective of the resource allocation policy can then be represented by a quadratic cost that penalizes large deviations of the plant states from the equilibrium point. As fading and interference conditions in the wireless network impact the transmission probabilities and stability of the plants is directly tied to their control states, it is paramount that the resource allocation policy take into account both the fading states in the communication network and the states of the control plants. Thus, we search for a resource allocation policy $\alpha(\hat{x}_t, H_t; \theta) : \mathbb{R}^{m_p} \times \mathbb{R}^{m^2} \to \mathcal{A} \subseteq \mathbb{R}^m$ that, given current channel conditions aggregated in $H_t := [h_t^{(11)}, h_t^{(1m)}, \ldots, h_t^{(mm)}]$ and estimates of plant states $\hat{x}_t := [\hat{x}_t^{(1)}, \ldots, \hat{x}_t^{(m)}]$, returns an allocation vector $\alpha \in \mathcal{A} \subseteq \mathbb{R}^m$. The resource allocation problem consists then in finding the allocation policy that minimizes the cost of operating the plants over a finite horizon $T$ starting from some initial state $x_0$, while respecting constraints $l(\alpha) : \mathbb{R}^m \to \mathbb{R}$ on the power used to send control signals back to the plants,

$$\theta := \arg \min_{\theta} J(\theta) = \mathbb{E}_{x_0}^{\alpha(\cdot; \theta)} \sum_{t=0}^{T} \gamma^t c(x_t),$$

s.t. $L(\theta) = \mathbb{E} \left[ \sum_{t=0}^{T} \gamma^t l(\alpha) \right] \leq 0,$

(7)

with $c(x_t) = \sum_{i=1}^{m} x_t^{(i)^T} Q x_t^{(i)}$ the one-step cost, $\gamma \in [0, 1]$ a discount factor and $Q \geq 0$ a weight matrix. We note that, although performance of a controller is often measured by a quadratic cost that penalizes both large deviations from the equilibrium point and large control efforts, as in the classical linear quadratic control formulation, the control policy here is known a priori. Thus, there is no need to additionally penalize large control efforts in the quadratic cost given in (7). We further note the set $\mathcal{A}$, which defines the space of possible allocation decisions as given by the communication model.

The optimization problem in (7) involves finding the resource allocation function $\alpha(x, H)$ that minimizes the operation cost of the plants while satisfying the resource constraints. As the optimization problem has infinite dimensionality, it is generally intractable to find optimal solutions even if we restrict our attention to systems with a low number of plants and with short optimization horizons. Moreover, finding an optimal policy directly in (7) necessarily requires explicit knowledge of the plant dynamics and communication models in (6), which are often unavailable in practice. The challenging nature of the problem motivated recent works [22]–[26] to use deep reinforcement learning to design resource allocation policies for wireless control systems.

III. GRAPH NEURAL NETWORKS

Reinforcement learning (RL) is a mathematical framework to handle sequential decision problems. At each time step, an
agent executes some action \( a_t \in A \) sampled from a stochastic policy \( \pi(a|s) \in \Pi \), observes the resulting state of the system, \( s_t \in S \), receives a one-step cost \( r_t \) from the environment and then tries to find a policy that minimizes the cumulative cost of those transitions [31]. As looking for policies directly is usually infeasible, one approximates the stochastic policy with some parametrization, with that parametrization corresponding to a neural network — high capability approximators [32] — in the field of deep reinforcement learning. The success of deep reinforcement learning, in turn, motivated the use of deep RL algorithms to design resource allocation policies for wireless control systems, cf., e.g., [22]–[26].

A standard neural network consists of a series of computational layers where each unit in layer \( l \) computes a linear combination of the outputs of layer \( l-1 \), and then applies a pointwise nonlinear transformation on top of that linear combination. Each hidden layer \( l \) is composed of hidden units than can be computed by

\[
z_l = \phi(C_l z_{l-1} + b_l),
\]

with \( \phi(\cdot) \) a nonlinearity and the matrices \( C_l, b_l \) aggregating the weights of the linear combination in that layer. Combining the successive computational layers, the output of the neural network is then given by

\[
y_{NN}(z_0) = \phi(C_L \phi(\ldots(\phi(C_1 z_0 + b_1) + b_L)).
\]

For the allocation policy in (7), the input \( z_0 \) corresponds to, e.g., estimates of the plants states and interference conditions in the network, whereas the output of the neural network characterizes the corresponding allocation policy.

Multilayer neural networks are known near-universal approximators [32], but that high representability capacity comes at the cost of a rather large number of learnable parameters. As evidenced by equations (8) - (9), the learnable parameters correspond to the linear combinations at each hidden unit in the network, that is, \( \theta = [C_1; b_1; \ldots; C_L; b_L] \). As the dimension of the input of the neural network grows, so does the number of learnable parameters. In particular, let \( K_l \) be the number of hidden units in layer \( l \). For the resource allocation problem, the overall number of learnable parameters is given by

\[
r_{NN} = m(m + n_f)K_1 + \sum_{l=2}^L K_l K_{l-1}.
\]

scaling linearly with the dimension of the input associated with each plant, \( n_f \), and quadratically with the number of plants in the network, \( m \). Although the resource allocation problem (7) lacks the type of spatial or temporal regularity that allows convolutional neural networks to circumvent this dimensionality issue in applications such as image processing, the communication model (3) does define an underlying graph structure for the optimization problem. This suggests the use of graph neural networks (GNNs) to parameterize the resource allocation policies, as we discuss next.

### A. Graph Neural Networks

GNNs can be viewed as a generalization of the popular convolutional neural network (CNN) model. In CNNs, the linear operations used in standard deep neural networks are replaced with linear convolutional filters. This more controlled structure significantly reduces the number of overall parameters the model must learn during training, since the algorithm now learns the coefficients of the corresponding convolutional filters and not the weights of the linear combinations computed at every neuron in the network. Moreover, the dimensionality of the filters being learned is invariant to the size of the input data, making the CNN attractive for large scale applications. While the convolutions employed by CNNs are naturally suited for processing of temporal or spatial data, the same does not hold true for inputs without such a regular structure. The WCS architecture considered here, however, nonetheless contains structure embedded in the fading and interference patterns \( H \) that can be incorporated into the policy parameterization — namely, this structure may be represented by a graph.

GNNs generalize the CNN model by replacing the standard convolutional filter with a graph convolutional filter [33]. For a graph \( G = (V, E, \mathcal{W}) \) with node and edge sets \( V = \{ 1, \ldots, N \} \), \( E = \{ (i,j) ; i,j \in V \} \) and weight function \( \mathcal{W} : E \rightarrow \mathbb{R} \), let then the graph shift operator (GSO) be defined as a matrix \( S \in \mathbb{R}^{N \times N} \) that reflects the sparsity of the graph, that is, \( S_{ij} = 0 \) if \( i \neq j \) and \( (i,j) \notin E \). Let also \( y = [y^{(1)}, \ldots, y^{(N)}] \) a graph signal with components \( y^{(i)} \in \mathbb{R}^{n_f}, i \in V \). A graph convolution can then be defined as a weighted sum of shifted versions of the graph signal,

\[
z = \sum_{k=0}^{K-1} S^K y^k,
\]

producing another graph signal \( z \in \mathbb{R}^{m \times g} \) with \( g \) features. The matrix \( \Psi_k \in \mathbb{R}^{n_f \times g} \) aggregates the filter taps \( \Psi_k^f g = \psi_{fg}^k \) used to modulate information received by the \( k \)-hop neighborhood of each node. Note that the graph convolution in (11) reflects the local structure of the graph: \( S^K \neq 0 \) only if node \( j \) is a \( k \)-hop neighbor of \( i \) [33].

Combining graph convolutions and pointwise nonlinear operations yields graph neural networks. Each layer \( l \) in a GNN takes as input a graph signal \( y_l \) produced by the previous layer and outputs a graph signal \( y_{l+1} \) computed by a graph
convolution followed by a nonlinear operation,
\[ y_{l+1} = \phi_l \left( \sum_{k=0}^{K_l-1} S^k y_l \Psi_{lk} \right). \]  
(12)

The nonlinear operation \( \phi_l \) may be any function that respects the local structure of the GSO \( S \) [33].

The interference matrix representing the state of the wireless network underlying the control system at some time instant \( t \), \( H_t \), can be used to define a graph \( G_t \) with nodes \( V = \{1, \ldots, m\} \) given by the \( m \) plants and edges corresponding to the interference between the transmitter associated to plant \( i \) and the receiver associated to plant \( j \), that is, \( \mathcal{W}_t((i,j)) := h_t^{(ij)} \) — see Figure 3. The GSO here corresponds to the interference matrix itself, \( S_t = H_t \). Observe that while in standard applications of GNNs, e.g. [33], the graph \( G \) is fixed, here the graph defined by the interference model in (3) is randomly distributed. Hence we make use of the notion of random edge GNNs (REGNNs) introduced by [27]. If we denote by \( z_0 \) the input graph signal — made up, for example, by the current state of the plants in the wireless control system — then we can define the REGNN as

\[ y_{\text{GNN}}(z_0) = \phi_L \left( \sum_{k=0}^{K_L-1} H_t^k \left( \ldots \phi_1 \left( \sum_{k=0}^{K_1-1} H_t^k z_0 \Psi_{1k} \right) \ldots \right) \Psi_{Lk} \right). \]  
(13)

This output may then be used to parameterize the policy distribution in (4) — e.g. success probability of a Bernoulli distribution — where \( \theta := [\Psi_1, \ldots, \Psi_L] \) contains the filter coefficients that define the REGNN. One needs to learn only the coefficients of the graph filters used at each hidden layer; letting \( K_l \) the filter length and \( F_l \) the number of features at each layer \( l \), the overall number of learnable parameters is

\[ r_{\text{GNN}} = \sum_{l=1}^{L} K_l F_l F_{l-1}, \]  
(14)

with \( F_0 = n_f \), the number of features of the input signal, and \( F_L \) the number of features of the output signal. Observe in (14) that, contrary to standard neural networks (10), the parameter dimension \( r_{\text{GNN}} \) is independent of the number of plants, \( m \), and thus structurally suitable to large control networks both in terms of its scalable dimensionality and possible transferability to alternative networks of varying size. This latter case is of particular interest in the practical design of learning solutions, which we explore in greater detail in the proceeding section.

B. Permutation Equivariance of (RE)GNNs

The smaller number of learnable parameters — independent of the size of the network — makes GNNs naturally scalable. But the lower dimensionality of the learning space is not the only appeal behind the use of GNNs. Training GNNs boils down to learning the filter coefficients \( \psi_l \) used at each layer of the GNN. As the filter coefficients do not depend on a particular graph, they can be trained in a given graph but applied to any other graph shift operator \( S \) to construct a graph convolution. Hence, a trained GNN can be deployed on other networks, making them also transferable. Naturally, transferability of the filter taps does not imply necessary transferability of the GNN output. Neither does it mean that we will necessarily observe the same performance on the new graph. While we investigate more general instances of transferability with numerical experiments in Section V, we first turn our attention into permutations or node reorderings. As node reorderings do not alter the structure of the graph, it is not unreasonable to expect that filters learned in a graph can be applied to its permutations without loss of performance.

To see if that intuition actually holds here, let then a permutation matrix be defined as a matrix in the set

\[ \mathcal{P} = \{ P \in \{0,1\}^{m \times m} : P \mathbf{1} = 1, P^T \mathbf{1} = 1 \}. \]  
(15)

For any matrix \( P \) in \( \mathcal{P} \), the product \( P^T v \) reorders the entries of vector \( v \), and, accordingly, the product \( P^T M P \) reorders the rows and columns of matrix \( M \). One can show that GNNs are permutation equivariant, in the sense that node reorderings lead to a similar permutation of the output of a GNN [33]. Initially discussed for GNNs with fixed topology, the notion of permutation equivariance was later extended to random edge GNNs, as shown in [27].

**Proposition 1** (Eisen et al [27, Prop. 2]). Consider graphs \( H \) and \( \hat{H} \) along with signals \( z \) and \( \hat{z} \) such that for some permutation matrix \( P \) we have \( \hat{H} = P^T H P \) and \( \hat{z} = P^T z \). The outputs of a REGNN with filter tensor \( \theta \) to the pairs \((H, z)\) and \((\hat{H}, \hat{z})\) are such that

\[ y(\hat{H}, \hat{z}; \theta) = P^T y(H, z; \theta). \]  
(16)

**Proposition 1** states that, if we reorder the nodes of the graph with some permutation matrix \( P \), the output of the GNN will be reordered accordingly. Hence, we can expect permutations not to affect the performance of a GNN. Although it is not immediately clear if the permutation equivariance structure of (RE)GNNs is shared by the resource allocation problem, conditions under which that holds are discussed next.

C. Permutation Invariance of Resource Allocation in WCS

According to Proposition 1, REGNNs are equivariant to node reorderings; permutation equivariant architectures, however, do not necessarily imply that optimal filters learned for a given graph will achieve a similar performance when applied to permutations of the original graph. For invariance of the optimal filter taps to hold under permutation, let us first introduce some assumptions on the structure of the WCS that make the resource allocation problem symmetric.

**Assumption 1.** The one-step cost \( c(\cdot) \) in (7) is permutation invariant, that is for any \( P \in \mathcal{P} \) and any \( x \in \mathbb{R}^{m p} \),

\[ c(P^T x) = c(x). \]  
(17)

**Assumption 2.** The constraint function \( l(\alpha) \) is permutation invariant in the sense that for any \( P \in \mathcal{P} \) and \( \alpha \in \mathbb{R}^m \),

\[ l(P^T \alpha) = l(\alpha). \]  
(18)
Assumption 3. The control plants have similar dynamics,
\[ f^{(1)}(x,u) = \cdots = f^{(m)}(x,u), \]  
and similar control policies,
\[ g^{(1)}(x_t) = \cdots = g^{(m)}(x_t). \]

Assumption 4. The random variables \( w_t^{(i)} \) in (1) are independent and identically distributed (i.i.d.).

Assumption 5. The initial states \( x_0^{(i)} \) in (1) are independent and identically distributed (i.i.d.).

Assumption 1 is easily satisfied in the quadratic case as long as the weight matrix \( Q \) is permutation invariant — which is equivalent to penalizing the deviation of all plants from the equilibrium point with the same weight. Requiring all plants to have similar dynamics is a somewhat restrictive but reasonable assumption in industrial applications. If all plants have similar dynamics, the resource allocation policy can observe similar outcomes for different plants given certain channel conditions and plant states, and we can then expect the learned policy to assign resources based only on those inputs, regardless of the particular ordering of the nodes. Given (19), condition (20) is mild and holds for example under the assumption that the controller is a quadratic regulator based on a locally valid linear model of the plants. Under Assumptions 1 — 5, one can construct a global function mapping previous plant states to current ones that remains equivariant under node reorderings. Combined with requirements on the underlying probability distribution characterizing the behavior of the wireless network, this allows us to show that problem (7) is invariant to permutations.

Proposition 2. Consider the resource allocation problem (7) and wireless networks defined by probability distributions \( \chi(H) \) and \( \hat{\chi}(\hat{H}) \). Assume hypotheses 1 — 5 hold and that
\[ \hat{\chi}(\hat{H}) = \hat{\chi}(P^{\top}HP) = \chi(H). \]  
Moreover assume that, for the same permutation matrix, the resource allocation vectors \( \alpha \) are such that
\[ \alpha(H, \hat{x}) = P^{\top} \alpha(H, x). \]  
Then the optimal cost, \( J(\theta^*) \), and the corresponding constraint, \( L(\theta^*) \), are invariant under permutations, that is,
\[ J_{\hat{\chi}(\hat{H})}(\theta^*) = J_{\chi(H)}(\theta^*), \]
\[ L_{\hat{\chi}(\hat{H})}(\theta^*) = L_{\chi(H)}(\theta^*). \]

Proof. See appendix A.

The proposition states that, under equivariance assumptions on the probability distributions characterizing the wireless networks, and similar assumptions that make the resource allocation policy and the structure of the WCS symmetric, the optimal cost of the resource allocation problem is invariant under permutations. That is, reordering the plants — and the communication graph accordingly — will not affect the expected optimal cost of the problem. The requirement on the equivariance of the resource allocation policy depends on the parameterization in hand and holds for example if the allocation policy is parameterized with REGNNs. The equivariance of REGNNs (Proposition 1) and the invariance of the optimal cost (Proposition 2) support transferability of optimal filters among permutations of the WCS.

Theorem 1. Consider WCSs operating over wireless networks characterized by probability distributions \( \chi(H) \) and \( \hat{\chi}(\hat{H}) \). Assume there exists \( P \in \mathcal{P} \) such that, if \( H = P^{\top}HP \) and \( \hat{x}_0 = P^{\top}x_0 \), the probability distributions satisfy
\[ \hat{\chi}(\hat{H}) = \hat{\chi}(P^{\top}HP) = \chi(H) \]  
and Assumptions 1 — 5 hold. Then, the optimal GNN filter coefficients for the wireless control systems characterized by distributions \( \chi(H) \) and \( \hat{\chi}(\hat{H}) \) are the same.

Proof. Since \( \theta^* \) solves (7) for \( \chi(H) \), we have
\[ \theta^* = \arg\min_{\theta} J(\theta) \]
\[ \text{s.t. } \mathbb{E} \left[ \sum_{l=0}^L \gamma^l(\alpha) \right] \leq 0 \]
\[ \alpha \in \mathcal{A}. \]  
If \( \hat{\chi}(\hat{H}) = \hat{\chi}(P^{\top}HP) = \chi(H) \), Proposition 1 implies
\[ y(\hat{H}, \hat{x}; \theta) = P^{\top} y(H, z; \theta) \]
and then Proposition 2 holds. Thus \( \theta^* \) induces a feasible solution to the allocation problem defined over \( \hat{\chi}(\hat{H}) \) and
\[ J(\theta^*)_{\hat{\chi}(\hat{H})} = J(\theta^*)_{\chi(H)}. \]

Following similar arguments, Propositions 1 — 2 yield
\[ J(\theta^*)_{\hat{\chi}(\hat{H})} = J(\theta^*)_{\chi(H)}. \]

Since \( \hat{\theta}^* \) solves (7) for \( \hat{\chi}(\hat{H}) \),
\[ J(\hat{\theta}^*)_{\hat{\chi}(\hat{H})} \leq J(\theta^*)_{\hat{\chi}(\hat{H})} = J(\theta^*)_{\chi(H)}. \]

But \( \theta^* \) is optimal for the problem defined over \( \chi(H) \), and thus
\[ J(\theta^*)_{\chi(H)} \leq J(\hat{\theta}^*)_{\chi(H)} = J(\theta^*)_{\hat{\chi}(\hat{H})}. \]

For both inequalities to be satisfied, equality must hold; hence,
\[ J(\theta^*)_{\chi(H)} = J(\theta^*)_{\hat{\chi}(\hat{H})}. \]

Similar arguments show that the same holds for the constraint \( L(\theta) \), that is, \( L(\theta^*)_{\chi(H)} = L(\theta^*)_{\hat{\chi}(\hat{H})} \). Finally, this implies that \( \theta^* \) is optimal for the resource allocation problem defined over \( \hat{\chi}(\hat{H}) \), and that \( \theta^* \) is optimal for the problem defined over \( \chi(H) \), from which it follows that
\[ \hat{\theta}^* = \theta^*. \]
IV. CONSTRAINED GRAPH REINFORCEMENT LEARNING

As the actuation signals in the dynamics of the WCS described in equations (1) — (6) depend only on the current value of the estimates of the plants states, \(x_t\), and we restrict the allocation policy to be Markovian, the transition of the system to a new state will depend only on the current state of the system, resource allocation decisions and state estimates. We can thus see the resource allocation problem described in equations (1) — (7) as a Markov decision process (MDP), and employ RL algorithms to solve it in a model-free manner. To account for the constraint on the resource allocation policy, cf. (7), we first introduce a dual variable \(\lambda \in \mathbb{R}^p\) to formulate the Lagrangian of the constrained RL problem,

\[
\mathcal{L}(\theta, \lambda) := J(\theta) + \lambda^T \mathbb{E}_{s_0} \left[ \sum_{t=0}^{T} \gamma^t l(\alpha_t) \right]
\]

(33)

with \(l(\alpha)\) standing for the resource allocation constraint in (7) and \(\tilde{r}_t\) the penalized one-step cost,

\[
\tilde{r}_t = c(x_t) + \lambda l(\alpha_t).
\]

(34)

The dual optimization problem can then be defined as

\[
D^*_\theta = \max_{\lambda} \min_{\theta \geq 0} \mathcal{L}(\theta, \lambda).
\]

(35)

With a fixed \(\lambda\), the inner optimization problem in (35) can be seen as a standard MDP with objective corresponding to the Lagrangian \(\mathcal{L}(\theta, \lambda)\) [26]. The inner optimization problem can then be solved used standard RL algorithms, while the convexity of the outer optimization problem indicates that the dual variable can be updated via approximate gradient ascent. Here, we consider a single reinforcement learning agent learning the allocation policy \(\alpha(\cdot; \theta)\).

As standard policy gradient algorithms can exhibit high variance and slow convergence and value based algorithms are unsuitable to learn continuous allocation decisions, we turn out attention to actor-critic algorithms. In particular, we rely on the Proximal Policy Optimization (PPO) algorithm [34], with both the actor and the critic networks parametrized with REGNNs. The actor network is trained to optimize a surrogate cost function measuring the policy performance while restricting how much the policy can change at each update,

\[
\theta^* := \arg\min_{\theta} J_{\text{PPO}}(\theta),
\]

\[
J_{\text{PPO}}(\theta) = \mathbb{E} \left[ \min \left( R_i(\theta) \hat{A}_t, \text{clip}(r_i(\theta), 1 - \epsilon, 1 + \epsilon) \hat{A}_t \right) \right]
\]

(36)

with \(\hat{A}_t\) an estimate of the advantage function, defined as the difference between the state-value function \(Q_{\alpha(\cdot; \theta)}(s, a)\) — the expected cost-to-go from state \(s\) given action \(a\) — and the expected cost \(J_{\alpha(\cdot; \theta)}(s)\) from following the current stochastic policy; \(\epsilon\) a clipping hyperparameter; and the probability ratio

\[
r_i(\theta) = \frac{\pi_\theta(a_t|s_t)}{\pi_{\theta*}(a_t|s_t)}.
\]

(37)

The parameters of the critic network are optimized to minimize a squared error loss between the value predicted by the current parameters and the estimated cost-to-go depending on the penalized one-step costs,

\[
\eta^* := \arg\min_{\eta} \mathbb{E}_{s_0} (\tilde{R}_t - \rho(s; \eta))^2,
\]

(38)

with \(\tilde{R}_t = \sum_{\tau=t}^{\bar{t}} \gamma^{\tau-t} \tilde{r}_\tau + \gamma^{\tau-t_{\max}} R_t, t = \bar{t} - 1, \ldots, \bar{t} - t_{\max}\), and \(\bar{t} = kt_{\max}, k \geq 1\), and \(t_{\max}\) the interval between updates. \(R_t = 0\), if \(\bar{t} = T\), and \(R_t = \rho(s_t; \eta)\) otherwise [34]. The dual variable \(\lambda\) can be updated via approximate gradient descent,

\[
\lambda_{i+1} = [\lambda_i + \beta_i \nabla_{\lambda} \mathcal{L}(\theta, \lambda)]_+,
\]

(39)

As described in Algorithm 1, we rely on \(N\) simultaneous realizations of the WCS with fixed horizon \(T\). Initial states of the control plants are sampled from a standard normal distribution, and fading states representing the transmission conditions of the wireless channel are sampled from \(\chi(H)\). At each time step, the allocation decisions for each realization of the WCS are sampled from the agent’s policy \(\pi(\cdot; \theta)\), the plants states evolve according to equations (1) — (6), and updated fading states are sampled from \(\chi(H)\). Every \(t_{\max}\) time steps, \(Nt_{\max}\) transitions are then used to update the policy and value networks according to (36) and (38), respectively. At the end of each training episode, \(NT\) transitions are used to estimate the dual gradient, \(\nabla_{\lambda} \mathcal{L}(\theta, \lambda)\), and the dual variable \(\lambda\) is updated according to (39).

**Remark 1.** Training can be facilitated by initially teaching the allocation policy to imitate a heuristic policy using the Dataset Aggregation (DAgger) imitation learning algorithm [35].

V. NUMERICAL EXPERIMENTS

Next we present numerical experiments to compare the performance of the graph RL approach presented in Section IV against standard deep RL techniques and heuristics commonly used to design resource allocation policies for WCSs. We created custom environments on OpenAI Gym [36] and customized the Stable Baselines 3 PPO implementation [37] to account for the dual learning step in Algorithm 1. Unless indicated otherwise, graph RL policies were parametrized...
with a REGNN consisting of 3 graph convolutional layers with 5 filter taps and 10 features per layer, and deep RL policies were parametrized with a standard multilayer neural network made up by two hidden layers with 64 hidden units each. To facilitate training, the resource allocation policies receive the norm of the observations of each plant as inputs. Policies were trained over $N = 16$ simultaneous realizations of the WCS, with a discount factor $\gamma = 0.95$, imitation learning rate $\beta_{RL} = 5 \times 10^{-4}$, reinforcement learning step size $\beta_{RL} = 5 \times 10^{-5}$ and dual variable step size $\beta_{\lambda} = 1 \times 10^{-5}$. Hyperparameters were tuned heuristically.

We consider that the plants in the WCS are all linear and open-loop unstable, and have the same dynamics,

$$x_{t+1}^{(i)} = Ax_{t}^{(i)} + Bu_{t}^{(i)} + w_{t},$$

with

$$A = \begin{bmatrix} 1.05 & 0.2 & 0.2 \\ 0 & 1.05 & 0.2 \\ 0 & 0 & 1.05 \end{bmatrix}; \quad B = I. \quad (41)$$

Plants are spatially distributed around the remote base stations. The path loss coefficient, $p_l$, and the distance $d_i$ between a plant and its controller determine the slow fading component, $h_s = d_i^{-p_l}$. The fast fading component $h_f$ is randomly sampled from a Rayleigh distribution $\chi(h)$ with parameter $\lambda_h$, leading to $h_{ij} = h_s^{(i)} \times h_f^{(j)}$. Given resource allocation decisions $\alpha$ and fading and interference conditions $H$, the probability of each plant successfully closing its feedback loop — unknown to the agent — is given by

$$v(\xi^{(i)}(\alpha, H)) = 1 - \exp(-\xi^{(i)}(\alpha, H)),$$

with $\xi^{(i)}$ the SNIR for plant $i$.

### A. Multi-Cellular Networks

In this first experiment we consider a multi-cellular network with $n = 5$ base stations and $k = 6$ users per base station. The base stations are evenly spaced in a line, with the distance between consecutive base stations given by $m/n$. The plants are randomly placed around the base stations, with the vertical position of a plant sampled uniformly from an interval $[-k, k]$ around the corresponding base station, and its horizontal position sampled uniformly from an interval $[-m/2n, m/2n]$ around the corresponding base station. In this scenario, the resource allocation policy must decide whether to send or not the control signal to a particular plant. The total power budget available at each time step, $m p_0$, is divided equally between the transmitting signals. Here, we take $Q = I$, $\lambda_h = 2$, $p_l = 1.5$, $p_0 = 2.5$ and $T = 50$. We do not use imitation learning to pre-train the agents in this scenario. The deep RL and the graph RL agents are trained over 10000 episodes each. Figure 4 shows the mean and standard deviation of the finite horizon cost per training episode for the graph RL (blue) and the deep RL (orange) policies over the $N = 16$ parallel realizations of the WCS. Although both policies achieve a similar performance, the REGNN policy converges after about 2000 episodes, while the standard deep RL resource allocation policy requires almost 5 times as many episodes to converge.

After training, we next compare the performance of the graph RL and deep RL allocation policies against some commonly used heuristics, namely

1) Equal power: sends control signals to all plants;
2) Control-aware: chooses $m/3$ plants currently further away from equilibrium point to transmit;
3) Round robin: schedules $m/3$ plants to transmit at each time instant;
4) Random access: randomly chooses $m/3$ plants to transmit;
5) WMMSE: the weighted minimum mean squared error algorithm [38], commonly used to maximize the weighted sum rate (WSR) in wireless channels subject to a power constraint.

To make comparisons between the learned allocation policies and the solutions outlined above fair, the total transmitting power, $m p_0$, is divided equally between the transmitting signals. We consider a longer horizon, $T = 80$, than the one seen by the agents during training. The finite horizon cost achieved by each solution, scaled by the number of plants in the network, $m$, is shown in Figure 5. Each test point corresponds to the mean over 10 realizations, with each realization using
different random seeds. The graph RL and the deep RL policies achieve the same performance in this scenario, and both outperform the heuristic solutions mentioned above. The resource allocation policy parametrized by REGNNs, however, requires a significantly smaller number of training episodes to reach this performance, and is transferable across scale — as we show in Section V-C.

We next consider a scenario similar to the first experiment, but with the allocation decisions now distributed between the base stations. More precisely, each base station now chooses one of the plants in its vicinity to transmit to. Figure 6 shows the mean and standard deviation of the finite horizon cost per training episode for the graph RL (blue) and the deep RL (orange) policies in this scenario. Here, not only does the resource allocation policy parametrized by REGNNs converge faster, but it also achieves a smaller operation cost than the one achieved by the deep RL solution.

We adapt the heuristic solutions to this scenario, and once again the total transmitting power, \( mp_0 \), is divided equally between the transmitting signals. We consider a longer horizon, \( T = 80 \), than the one seen by the agents during training. The finite horizon cost achieved by each solution (scaled by the number of plants in the network) is shown in Figure 7. In this scenario, the graph RL policy and the control-aware heuristic outperform the other solutions.

B. Ad-Hoc Networks

We now consider an ad-hoc network (cf. Figure 2) with \( m = 30 \) pairs of plants and remote controllers. The remote controllers are evenly spaced in a line, with the distance between consecutive remote controllers equals to 4. The plants are randomly place in an \( [-m/10, m/10]^2 \) area around the corresponding controller. Here, \( p_0 = 2.5 \), \( \mu = 2 \), and \( p_l = 1.5 \), and the agents aim to stabilize the control plants while respecting a sum-power constraint on the power used by all the remote controllers to communicate with the plants,

\[
E \left[ \sum_{t=0}^{T} \gamma^t \left( \sum_{i=1}^{m} \alpha_t^{(i)} - mp_0 \right) \right] \leq 0.
\]

Each training episode has a fixed horizon \( T = 30 \). First, the resource allocation policies are trained to imitate the WMMSE algorithm using the DAgger algorithm [35] for a total of 1000 episodes. After the imitation learning phase, we then use reinforcement learning updates to train the resource allocation policies parametrized by REGNNs and by standard neural networks, with the training curves shown in Figures 8 and 9. The mean and standard deviation of the quadratic objective per training episode is shown in Figure 8, where one can see that the graph RL approach converges faster than the deep RL policy. The graph RL resource allocation policy also quickly converges to a feasible solution, as shown in Figure 9.

After training, we next compare the performance of the graph RL and deep RL allocation policies against heuristics adapted to this setting, with each of the scheduling heuristics choosing \( m/3 \) plants to transmit at each time instant. To make comparisons fair, the total transmitting power, \( mp_0 \), is divided equally between the transmitting signals for the heuristic solutions, and we use a softmax layer to ensure that the total power used by the learned solutions at each time step is equal to \( mp_0 \). We consider a longer horizon, \( T = 80 \), than the one seen by the agents during training. The finite horizon cost achieved by each solution, scaled by the number of plants in the network, \( m \), is shown in Figures 10 and 11. The REGNN policy outperforms both the deep RL and the heuristic solutions.

For a larger network with \( m = 60 \) pairs of plants and remote controllers, the performance of the deep RL approach degrades, but the graph RL approach still yields a resource allocation policy that outperforms the heuristic solutions, see Figures 12 — 14. As in the previous scenario, the remote controllers are evenly spaced in a line, with the distance between consecutive remote controllers equals to 4. The plants are randomly place in an \( [-m/10, m/10]^2 \) area around the corresponding controller and we have \( p_0 = 5 \), \( \mu = 2 \), and \( p_l = 1.5 \). As shown in Figures 12 and 13, both approaches still find feasible allocation policies, but the resource allocation policy parametrized with REGNNs achieves a better training objective than the resource allocation policy parametrized with standard neural networks. The runtime simulations (Figure 14)
also consider a longer horizon ($T = 80$) than the one seen during training ($T = 30$).

C. Transference

The trainable parameters of a REGNN correspond to the filter taps used to modulate the signal received from successive $k$-hop neighborhoods of each node, as one can see in equations (11) - (13). Given a time-varying GSO $S_t$, the filter taps of a REGNN do not depend on the number of users in the network, and can thus be transferred to networks of varying size — as long as the number of features of the input signal remains the same. That implies that one can train a resource allocation policy on a network of a certain size, and transfer the learned parameters to be executed on a larger network. To evaluate how well the performance of a REGNN resource allocation policy transfers at scale to larger networks, we revisit the ad-hoc scenario with $m = 60$ pairs of plants and remote controllers, and draw successively larger ad-hoc networks with controllers evenly spaced in a line and the distance between consecutive remote controllers equals to 4. The plants are randomly placed in an $[-6, 6]^2$ area around the corresponding controller, and we take $p_0 = 5$, $\mu = 2$, and $p_1 = 1.5$, as in the previous scenario. We then compare the performance of the REGNN policy against the heuristic solutions over a runtime horizon $T = 80$, and present the simulation results in Figure 15. Each test point shows the mean and standard deviation of 20 realizations of the WCS over a network of a certain size. As shown in Figure 15, the REGNN allocation policy consistently outperforms the heuristic solutions on networks up to 10 times larger the network over which it was trained. Note that a standard neural network is not transferable at scale, hence we do not evaluate its performance in this scenario.

VI. Conclusion

This paper presents a primal-dual graph reinforcement learning approach to design resource allocation policies for large-scale wireless control systems. Resource allocation problems are challenging, even in the absence of policies tailored to the specific application in mind, such as wireless control systems. To tackle this problem and learn or improve upon allocation policies, previous works have successfully relied on deep learning and deep reinforcement learning. As neural networks are made up of successive linear computational layers followed by pointwise nonlinearities, however, those
approaches do not scale well. In this paper, we then propose the use of reinforcement learning and graph neural networks to design scalable resource allocation policies for wireless control systems. Extensive numerical experiments demonstrate that the proposed approach yields resource allocation policies that routinely outperform deep RL and heuristic solutions, and are transferable across networks of varying size.

**APPENDIX A**

**PROOF OF PROPOSITION 2**

The invariance of the optimal cost is a direct consequence of the equivariance of the resource allocation policy and the equivariance of the WCS. For the permuted WCS defined over the probability distribution \( \hat{\chi}(\hat{H}) \) resulting from the permuted graph, the overall cost is given by

\[
J_{\hat{\chi}(\hat{H})}(\theta) = \mathbb{E}^{\hat{\chi}(\hat{H});\theta} \left[ \sum_{t=0}^{T} \gamma^t c(\hat{x}_t) \right]
\]

\[
= \int_{(\hat{S} \times \hat{A})^T} \left( \sum_{t=0}^{T} \gamma^t c(\hat{x}_t) \right) \hat{\pi}_\alpha(\hat{s}, \hat{a}) d\hat{s} d\hat{a},
\]

with \( \hat{s} = (\hat{s}_0, \hat{s}_1, \ldots) \) and \( \hat{a} = (\hat{a}_0, \hat{a}_1, \ldots) \). From Markov’s property it follows that

\[
\hat{\pi}_\alpha(\hat{s}, \hat{a}) = \prod_{u=1}^{T} \hat{\pi}(\hat{s}_u | \hat{a}_{u-1} \hat{a}_u) \hat{\pi}(\hat{s}_0) \hat{\pi}(\hat{a}_0 | \hat{s}_0) \hat{\pi}(\hat{a}_0 | \hat{s}_0).
\]

Now introduce the change of variables \( \hat{x} = P^T x, \hat{H} = P^T HP \). To simplify the notation, let \( s_t = [x_t, H_t] \) and, with a slight abuse of notation, \( P^T s_t = [P^T x_t, P^T H_t P] \) in the following. Since the initial states are independent and identically distributed according to Assumption 5 and the wireless network characterizing the permuted system satisfies (21), we have that

\[
\hat{\pi}_\alpha(\hat{s}_0) = \hat{\pi}(P^T s_0) = p(s_0).
\]

Similarly, it follows from the assumption on the permutation equivariance of the resource allocation policy that

\[
\pi(\hat{a}_0 | \hat{s}_0) = \pi(P^T a_0 | P^T s_0) = \pi(a_0 | s_0),
\]
and, similarly,
\[ \pi(\alpha_t | \tilde{s}_t) = \pi(P^T \alpha_t | P^T s_t) = \pi(\alpha_t | s_t). \] (48)

The kernel \( \hat{p}(\tilde{s}_t | \tilde{s}_{t-1}, \alpha_{t-1}) \) representing a one-step transition of the permuted WCS is given by
\[ \hat{p}(\tilde{s}_t | \tilde{s}_{t-1}, \alpha_{t-1}) = \hat{p}(\tilde{x}_t | \tilde{x}_{t-1}, \tilde{H}_{t-1}, \alpha_{t-1}) = \hat{p}(\tilde{x}_t | \tilde{x}_{t-1}, \tilde{H}_{t-1}, \alpha_{t-1}) \chi(H_t) \] (49)

since \( \tilde{H}_t \) is independent of \( \tilde{x}_{t-1}, \tilde{H}_{t-1}, \) and \( \alpha_{t-1}, \) and thus,
\[ \hat{p}(\tilde{s}_t | P^T s_{t-1}, P^T \alpha_{t-1}) = \hat{p}(\tilde{x}_t | P^T x_{t-1}, P^T H_{t-1} P, P^T \alpha_{t-1}) \chi(H_t) \] (50)

by (21). Now, let
\[ \hat{f}(\tilde{x}, \tilde{u}) := [\hat{f}(1)(\tilde{x}(1), \tilde{u}(1)), \ldots, \hat{f}(m)(\tilde{x}(m), \tilde{u}(m))] \]
represent a one-step transition of the states of the plants given the control signals received by the plants under the communication model (6). Under Assumption 3, the mapping \( \hat{f}(\cdot) \) is equivariant under permutations, that is,
\[ \hat{f}(P^T x, P^T u) = P^T \hat{f}(x, u). \]

Now, let \( \text{I}_X \) represent the indicator function for a region \( X \subset X' \), and \( \xi(\tilde{x}(t), P^T H P) = \xi(\alpha, H) \). Then,
\[ \hat{p}(\tilde{x}_{t+1} | P^T X | P^T s_t, P^T \alpha_t) = \mathbb{E} \left[ \mathbb{I}_{P^T X} \left( \hat{f}(P^T x_t, u(g(P^T x_t), v(\xi))) + P^T w \right) \right] \]
\[ = \mathbb{E} \left[ \mathbb{I}_{P^T X} \left( \hat{f}(P^T x_t, u(P^T g(x_t), v(\xi))) + P^T w \right) \right] \] (51)

since the collection of Bernoulli random variables representing the packet drops over the communication channels for the WCS are independent, the control policies are all the same and the control system noise is i.i.d. according to Assumptions 3 and 4. Thus,
\[ \hat{p}(\tilde{x}_{t+1} | P^T X | P^T s_t, P^T \alpha_t) = \int \mathbb{I}_{P^T X} \left( \hat{f}(P^T x_t, P^T u_t) + P^T w \right) p(P^T u | P^T x_t) \mu(w) dw \]
\[ = \int \mathbb{I}_{P^T X} \left( P^T \hat{f}(x_t, u_t) + P^T w \right) p(P^T u | P^T x_t) \mu(w) dw \]
\[ = \int \mathbb{I}_{X} (f(x_t, u_t) + w) p(u | \xi) \mu(w) dw = p(x_{t+1} | X, s_t, \alpha_t) \] (52)

Combining (45) — (50) and (52), we have
\[ \hat{p}_\alpha(P^T \hat{s}, P^T \hat{\alpha}) = \prod_{u=1}^{T} \hat{p}(P^T s_u | P^T \alpha_u) \pi(P^T \alpha_u | P^T s_u) \hat{p}(P^T s_0 | \alpha_0 | P^T s_0). \]

By the invariance of the one-step cost in Assumption 1. Now let \( L(\hat{\alpha}, \hat{H}) \) be the constraint achieved by the resource allocation policy. Following similar arguments, we have
\[ J(\hat{\alpha}, \hat{H}) = \mathbb{E} \left[ \sum_{t=0}^{T} \gamma^t \hat{I}(\hat{\alpha}_t, \hat{H}_t; \theta) \right] \]
\[ = \int \mathbb{I}_{\theta} \left( \sum_{t=0}^{T} \gamma^t \hat{I}(\hat{\alpha}_t) \right) p(\hat{\alpha}, \hat{H}) d\hat{\alpha} d\hat{H} \]
\[ = \int \mathbb{I}_{\theta} \left( \sum_{t=0}^{T} \gamma^t \hat{I}(\hat{\alpha}_t) \right) p(\hat{\alpha}, \hat{H}) d\hat{\alpha} d\hat{H} \]
\[ = L(\alpha, H) \] (54)

Finally, note that the relations above hold for any parameterization that renders the resource allocation policy equivariant under permutations. In particular, it also holds for the parameter set that solves (7), from which (20) follows.

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