RANDOM $k$-NONCROSSING PARTITIONS

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Abstract. In this paper, we introduce polynomial time algorithms that generate random $k$-noncrossing partitions and 2-regular, $k$-noncrossing partitions with uniform probability. A $k$-noncrossing partition does not contain any $k$ mutually crossing arcs in its canonical representation and is 2-regular if the latter does not contain arcs of the form $(i, i + 1)$. Using a bijection of Chen et al. [2, 4], we interpret $k$-noncrossing partitions and 2-regular, $k$-noncrossing partitions as restricted generalized vacillating tableaux. Furthermore, we interpret the tableaux as sampling paths of a Markov-processes over shapes and derive their transition probabilities.

1. Introduction

Recently Chen et al. [3] studied a computational paradigm for $k$-noncrossing RNA pseudoknot structures. The authors regarded $k$-noncrossing matchings as oscillating tableaux [2] and interpreted the latter as stochastic processes over Young tableaux of less than $k$ rows. According to which, they generated $k$-noncrossing RNA pseudoknot structures [3] with uniform probability. Since the generating function of oscillating lattice walks in $\mathbb{Z}^{k-1}$, that remain in the interior of the dominant Weyl chamber is given as a determinant of Bessel functions [3], it is $D$-finite. As a result, the transition probabilities of the Markov-process can be derived in polynomial time as a pre-processing step and each $k$-noncrossing RNA pseudoknot structure can be generated in linear time.

The analogue of the above result is for $k$-noncrossing partitions much more involved. Only for $k = 3$, Bousquet-Méou and Xin [1] compute the generating function and conjecture that $k$-noncrossing partitions are not $P$-recursive for $k \geq 4$. Since the enumerative result are unavailable for $k \geq 4$, it

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is important to construct an algorithm which can uniformly generate \( k \)-noncrossing partitions in order to obtain statistic results.

In this paper, we show that \( k \)-noncrossing partitions and 2-regular, \( k \)-noncrossing partitions can be uniformly generated in linear time. We remark that there does not exist a general framework for the uniform generation of elements of a non-inductive combinatorial class. In this context relevant work has been done by Wilf [9] as well as [7].

The paper is organized as follows: in Section 2 we present all basic facts on \( k \)-noncrossing partitions, vacillating tableaux and lattice walks. In Section 3 we compute the number of lattice walks ending at arbitrary \( \nu \in Q_{k-1} \) from the construction and prove Theorem 3.2, the uniform generation result for \( k \)-noncrossing partitions. In Section 4 we show how to uniformly generate 2-regular, \( k \)-noncrossing partitions. To this end, we establish a bijection between lattice walks associated to partitions and braids [4]. We will show that walks associated with 2-regular partitions correspond to walks associated to loop-free braids. The latter can easily be dealt with via the inclusion-exclusion principle. In Theorem 4.4 we present an algorithm that uniformly generates 2-regular, 3-noncrossing partitions in linear time. Finally, in Section 5 we show the particular algorithms in case of \( k = 4 \).

We remark that the generation of random 2-regular, \( k \)-noncrossing partitions has important applications in computational biology. The latter allow to represent so called base triples in three dimensional RNA structures, see [8] for details of the framework. The uniform generation then facilitates energy-based, \textit{ab initio} folding algorithms along the lines of [3]. This paper is accompanied by supplemental materials containing MAPLE implementations of the algorithms, available at www.combinatorics.cn/cbpc/par.html.

2. Some basic facts

A \textit{set partition} \( P \) of \( [N] = \{1, 2, \ldots, N\} \) is a collection of nonempty and mutually disjoint subsets of \( [N] \), called blocks, whose union is \( [N] \). A \( k \)-noncrossing partition is called \( m \)-\textit{regular}, \( m \geq 1 \), if for any two distinct elements \( x, y \) in the same block, we have \( |x - y| \geq m \). A \textit{partial matching} and a \textit{matching} is a particular type of partition having block size at most two and exactly two,
respectively. Their standard representation is a unique graph on the vertex set \([N]\) whose edge set consists of arcs connecting the elements of each block in numerical order, see Fig.1.

Given a (set) partition \(P\), a \(k\)-crossing is a set of \(k\) edges \(\{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}\) such that \(i_1 < i_2 < \ldots < i_k < j_1 < j_2 < \ldots < j_k\), see Fig.1. A \(k\)-noncrossing partition is a partition without any \(k\)-crossings. We denote the sets of \(k\)-noncrossing partitions and \(m\)-regular, \(k\)-noncrossing partitions by \(\mathcal{P}_k(N)\) and \(\mathcal{P}_{k,m}(N)\), respectively. For instance, the set of 2-regular, 3-noncrossing partitions are denoted by \(\mathcal{P}_3,2(N)\).

A (generalized) vacillating tableau \([4]\) \(V^{2N}_\lambda\) of shape \(\lambda\) and length \(2N\) is a sequence \(\lambda^0, \lambda^1, \ldots, \lambda^{2N}\) of shapes such that (1) \(\lambda^0 = \emptyset\), \(\lambda^{2N} = \lambda\) and (2) for \(1 \leq i \leq N\), \(\lambda^{2i-1}, \lambda^{2i}\) are derived from \(\lambda^{2i-2}\) by elementary moves (EM) defined as follows: \((\emptyset, \emptyset)\): do nothing twice; \((-\square, \emptyset)\): first remove a square then do nothing; \((\emptyset, +\square)\): first do nothing then add a square; \((\pm\square, \pm\square)\): add/remove a square at the odd and even steps, respectively. We use the following notation: if \(\lambda_{i+1}\) is obtained from \(\lambda_i\) by adding, removing a square from the \(j\)-th row, or doing nothing we write \(\lambda_{i+1} \setminus \lambda_i = +\square_j\), \(\lambda_{i+1} \setminus \lambda_i = -\square_j\) or \(\lambda_{i+1} \setminus \lambda_i = \emptyset\), respectively, see Fig.2.

A braid over \([N]\) can be represented via introducing loops \((i, i)\) and drawing arcs \((i, j)\) and \((j, \ell)\) with \(i < j < \ell\) as crossing, see Fig.3(B1). A \(k\)-noncrossing braid is a braid without any \(k\)-crossings. We denote the set of \(k\)-noncrossing braids over \([N]\) with and without isolated points by \(\mathcal{B}_k(N)\) and \(\mathcal{B}^*_k(N)\), respectively. Chen et al. [2] have shown that each \(k\)-noncrossing partition corresponds uniquely to a vacillating tableau of empty shape, having at most \((k - 1)\) rows, obtained via the EMs \{\((-\square, \emptyset), (\emptyset, +\square), (\emptyset, \emptyset), (-\square, +\square)\}\), see Fig.3(A2). In [4], Chen et al. proceed by proving...
that vacillating tableaux of empty shape, having at most \((k - 1)\)-rows which are obtained by the EMs \(\{(-\Box, \emptyset), (\emptyset, +\Box), (\emptyset, \emptyset), (+\Box, -\Box)\}\) correspond uniquely to \(k\)-noncrossing braids.

In the following, we consider \(k\)-noncrossing partition or a 2-regular, \(k\)-noncrossing partition of \(N\) vertices, where \(N \geq 1\). To this end, we introduce two \(\mathbb{Z}_{k-1}\) domains \(Q_{k-1} = \{\nu = (\nu_1, \nu_2, \ldots, \nu_{k-1}) \in \mathbb{Z}_{k-1} \mid \nu_1, \nu_2, \ldots, \nu_{k-1} \geq 0\}\) and \(W_{k-1} = \{\nu = (\nu_1, \nu_2, \ldots, \nu_{k-1}) \in \mathbb{Z}_{k-1} \mid \nu_1 > \nu_2 > \cdots > \nu_{k-1} \geq 0\}\). Set \(D_{k-1} \in \{Q_{k-1}, W_{k-1}\}\), \(e_i\) be the unit of \(i\)-th axis, where \(i \in \{1, 2, \ldots, k - 1\}\) and \(e_0 = \mathbf{0}\) be the origin. A \(P_{D_{k-1}}\)-walk is a lattice walk in \(D_{k-1}\) starting at \(e = (k - 2, k - 1, \ldots, 0)\) having
steps $\pm e_i$, such that even steps are $+e_i$ and odd steps are either $-e_i$, where $i \in \{0, 1, \ldots , k-1\}$. Analogously, a $B_{D_{k-1}}$-walk is a lattice walk in $D_{k-1}$ starting at $\epsilon$ whose even steps are either $-e_i$ and whose odd steps are $+e_i$, where $i \in \{0, 1, \ldots , k-1\}$. By abuse of language, we will omit the subscript $D_{k-1}$.

Interpreting the number of squares in the rows of the shapes as coordinates of lattice points, we immediately obtain

**Theorem 2.1.** [4]

1. The number of $k$-noncrossing partitions over $[N]$ equals the number of $P_{W_{k-1}}$-walks from $\epsilon$ to itself of length $2N$.
2. The number of $k$-noncrossing braids over $[N]$ equals the number of $B_{W_{k-1}}$-walks from $\epsilon$ to itself of length $2N$.

Recall $\delta = (k-2, k-1, \ldots , 0)$, Bousquet-Mélou and Xin [1] according to the reflection principle, arrive at the following proposition:

**Proposition 2.2.** For any ending points $\mu \in W_{k-1}$, set $\omega_{k,\ell}^{\mu}$ and $a_{k,\ell}^{\mu}$ denote the number of $W_{k-1}$-walks and $Q_{k-1}$ going from $\delta$ to $\mu$ of length $\ell$, respectively. Then we have

\[
\omega_{k,\ell}^{\mu} = \sum_{\pi \in S_{k-1}} \text{sgn}(\pi) a_{k,\ell}^{\pi(\mu)},
\]

where $S_{k-1}$ is the permutation group of $[k-1]$, $\text{sgn}(\pi)$ is the sign of $\pi$ and

\[
\pi(\mu) = \pi(\mu_1, \mu_2, \ldots , \mu_{k-1}) = (\mu_{\pi(1)}, \mu_{\pi(2)}, \ldots , \mu_{\pi(k-1)}).
\]

3. **Random $k$-noncrossing partitions**

In this section, we establish an algorithm to uniformly generate a $k$-noncrossing partition of $[N]$ employing a Markov-process as an interpretation of a vacillating tableau.

**Lemma 3.1.** (a) Suppose $1 \leq \ell \leq N-1$ and $\nu = (\nu_1, \nu_2, \ldots , \nu_{k-1}) \in Q_{k-1}$, then set $a_{k,2\ell+1}^{\nu} = f_{k,\ell}^{\nu}$, accordingly we have

\[
a_{k,s}^{\nu} = \begin{cases} 
    f_{k,\ell}^{\nu} & \text{for } s = 2\ell + 1, \\
    \sum_{i=0}^{k-1} f_{k,\ell}^{\nu-e_i} & \text{for } s = 2\ell + 2.
\end{cases}
\]
where \( f_{k,\ell}^\nu = 0 \) for \( \sum_i \nu_i \geq \ell + \frac{(k-1)(k-2)}{2} \) and for \( \ell = 0 \),
\[
 f_{k,0}^\nu = \begin{cases} 
 1 & \text{for } \nu = \lambda; \\
 0 & \text{otherwise.} 
\end{cases}
\]

(b) \( f_{k,\ell}^\nu \) satisfies the recursion
\[
 f_{k,\ell}^\nu = \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} f_{k,\ell-1}^{\nu_i+e_i-e_j}.
\]

Proof. Assertion (a) is obvious by construction. To prove assertion (b), indeed by definition of \( f_{k,\ell}^\nu \), we have
\[
 a_{k,2\ell+1} = f_{k,\ell}^\nu. 
\]
Furthermore, since each lattice walk ending at \( \nu \) of length \( 2\ell + 1 \) can be obtained from a lattice walk ending at \( (\nu - e_i + e_j) \in W_{k-1} \) of length \( 2\ell - 1 \) via first \( +e_i \) then \( -e_j \). I.e. we obtain eq. (3.2) and the proof of the lemma is complete. \( \square \)

In the following, we consider a vacillating tableau as the sampling path of a Markov-process, whose transition probabilities can be calculated via the terms \( \omega_{k,\ell}^\nu \).

**Theorem 3.2.** A random \( k \)-noncrossing partition can be generated, in \( O(N^k) \) pre-processing time and \( O(N^k) \) space complexity, with uniform probability in linear time. Each \( k \)-noncrossing partition is generated with \( O(N) \) space and time complexity, see Algorithm 1.

Proof. The main idea is to interpret tableaux of \( k \)-noncrossing partitions as sampling paths of a stochastic process, see Fig. for the case \( k = 3 \). We label the \((i+1)\)-th shape, \( \lambda_{i+1}^\alpha \) by \( \alpha = \lambda^{i+1} \setminus \lambda^i \) and \( \alpha \in \{+\square_j, -\square_j, \emptyset\}_{j=1}^{k-1} \), specifying the particular transition from \( \lambda^i \) to \( \lambda^{i+1} \).

Let \( V_k(\lambda_{i+1}^\alpha, 2N - (i + 1)) \) denote the number of vacillating tableaux of length \((i + 1)\) such that \( \lambda^{i+1} \setminus \lambda^i = \alpha \). We remark here that if \( \lambda^{i+1} \) has \((\nu_i - k - i + 3)\) boxes in the \( i \)-th row, then
\[
 V_k(\lambda_{i+1}^{\nu_i-k+i+3}, 2N - (i + 1)) = \omega_{k,2N-(i+1)}^\nu \text{, where } \nu = (\nu_1, \nu_2, \ldots, \nu_{k-1}).
\]

Let \((X^i)_{i=0}^{2N} \) be given as follows:

- \( X^0 = X^{2N} = \emptyset \) and \( X^i \) is a shape having at most \( k - 1 \) rows;
- for \( 1 \leq i \leq N - 1 \), we have \( X_{2i+1} \setminus X_{2i} \in \{\emptyset, -\square_j\}_{j=1}^{k-1} \) and \( X_{2i+2} \setminus X_{2i+1} \in \{\emptyset, +\square_j\}_{j=1}^{k-1} \);
- for \( 1 \leq i \leq 2N - 1 \), we have
\[
 \mathbb{P}(X^{i+1} = \lambda^{i+1} | X^i = \lambda^i) = \frac{V(\lambda_{(i+1)}^{\nu_i-k+i+3}, 2N - i - 1)}{V_k(\lambda^i, 2N - i)}.
\]

In the following, we consider a vacillating tableau as the sampling path of a Markov-process, whose transition probabilities can be calculated via the terms \( \omega_{k,\ell}^\nu \).

**Theorem 3.2.** A random \( k \)-noncrossing partition can be generated, in \( O(N^k) \) pre-processing time and \( O(N^k) \) space complexity, with uniform probability in linear time. Each \( k \)-noncrossing partition is generated with \( O(N) \) space and time complexity, see Algorithm 1.
Algorithm 1 Uniform generation of $k$-noncrossing partitions

1: $i=1$
2: \(\text{Tableaux}\) (Initialize the sequence of shapes, \(\{\lambda^i\}_{i=0}^{2N}\))
3: \(\lambda^0 = \emptyset, \lambda^{2N} = \emptyset\)
4: while \(i < 2N\) do
5: \(\text{if} \ i \ \text{is even then}\)
6: \(X[0] \leftarrow V_k(\lambda^{i+1}_{2N}, 2N - (i + 1))\)
7: \(\text{for} \ 1 \leq j \leq k-1 \ \text{do}\)
8: \(X[j] \leftarrow V_k(\lambda_{i+1}^{i+1+j}, 2N - (i + 1))\)
9: \(\text{end for}\)
10: \(\text{end if}\)
11: \(\text{if} \ i \ \text{is odd then}\)
12: \(X[0] \leftarrow V_k(\lambda_{i+1}^{i+1}, 2N - (i + 1))\)
13: \(\text{for} \ 1 \leq j \leq k-1 \ \text{do}\)
14: \(X[j] \leftarrow V_k(\lambda_{i+1}^{i+1-j}, 2N - (i + 1))\)
15: \(\text{end for}\)
16: \(\text{end if}\)
17: \(\text{sum} \leftarrow (\sum_{j=1}^{k-1} X[j]) + X[0]\)
18: \(\text{Shape} \leftarrow \text{Random}(\text{sum})\) (Random generates the random shape \(\lambda_{i+1}^{i+1} + \square_j\) with probability \(X[j]/\text{sum}\) or \(\lambda_{i+1}^{i+1-j}\) with probability \(X[0]/\text{sum}\))
19: \(i \leftarrow i + 1\)
20: Insert \(\text{Shape}\) into \(\text{Tableau}\) (the sequence of shapes).
21: \(\text{end while}\)
22: \(\text{Map(\text{Tableau})}\) (maps \(\text{Tableau}\) into its corresponding $k$-noncrossing partition)

In view of eq. (3.3), we immediately observe

(3.4) \[ \prod_{i=0}^{2N-1} \mathbb{P}(X^{i+1} = \lambda^{i+1} \mid X^i = \lambda^i) = \frac{V_k(\lambda^{2N} = \emptyset, 0)}{V_k(\lambda^0 = \emptyset, N)} = \frac{1}{V_k(\emptyset, 2N)}. \]

Consequently, the process \((X^i)_{i=0}^{2N}\) generates random $k$-noncrossing partitions with uniform probability in $O(N)$ time and space.

As for the derivation of the transition probabilities, suppose we are given a shape $\lambda^h$ having exactly $(\nu_i - k - i + 3)$ boxes in the $i$-th row, where $i \in \{1, 2, \ldots, k-1\}$. According to Lemma 3.1, set $\nu = (\nu_1, \nu_2, \ldots, \nu_{k-1})$, then we reduce the problem obtain $\omega_\beta^\nu$ to the calculation of several coefficients $f_\beta^\nu$ for some fixed $\beta \in W_{k-1}$ related to $\nu$ and $1 \leq \ell \leq N - 1$. 
Figure 4. Sampling paths: we display all tableaux sequences where $\lambda^1 = \emptyset$, $\lambda^2 = \lambda^3 = \Box$. We display the three possible $\lambda^4$-shapes (blue), induced by $\emptyset$, $+\Box_1$, $+\Box_2$ and all possible continuations (grey).

Furthermore, according to Lemma 3.1 (b), the coefficients $f^\beta_\ell$ for all $\beta \in W_{k-1}$ and $1 \leq \ell \leq N - 1$, can be computed in $O(N^k)$ time and $O(N^k)$ space complexity via $k$-nested For-loops.

Given the coefficients $f^\beta_\ell$ for all $\beta, \ell$, we can derive the transition probabilities in $O(1)$ time. Accordingly, we obtain all transition probabilities $V_k(\lambda^\alpha_i, 2N - i)$ in $O(N^k)$ time and $O(N^k)$ space complexity.

4. RANDOM 2-REGULAR, $k$-NONCROSSING PARTITIONS

In this section, we generate random 2-regular, $k$-noncrossing partitions employing a bijection between the set of 2-regular, $k$-noncrossing partitions over $[N]$ and the set of $k$-noncrossing braids without loops over $[N - 1]$.

Lemma 4.1. Set $k \in \mathbb{N}$, $k \geq 3$. Suppose $(\lambda^i)_{i=0}^{2\ell+1}$ in which $\lambda^0 = \emptyset$ is a sequence of shapes such that $\lambda^{2j} \setminus \lambda^{2j-1} \in \{+\Box_h\}_{h=1}^{h=k-1}, \emptyset\}$ and $\lambda^{2j-1} \setminus \lambda^{2j-2} \in \{-\Box_h\}_{h=1}^{h=k-1}, \emptyset\}$. Then $(\lambda^i)_{i=0}^{2\ell+1}$ induces a unique sequence of shapes $(\mu^i)_{i=0}^{2\ell}$ with the following properties

(4.1) $\mu^{2j+1} \setminus \mu^{2j} \in \{+\Box_h\}_{h=1}^{h=k-1}, \emptyset\}$ and $\mu^{2j+2} \setminus \mu^{2j+1} \in \{-\Box_h\}_{h=1}^{h=k-1}, \emptyset\}$,

(4.2) $\mu^{2j} = \lambda^{2j+1}$

(4.3) $\mu^{2j+1} \neq \lambda^{2j+2} \implies \mu^{2j+1} \in \{\lambda^{2j+1}, \lambda^{2j+3}\}$. 
**Figure 5.** Illustration of Lemma 4.1. We show how to map the $P_{W_2}$-walk induced by the 3-noncrossing partition $P$, of Fig. 3 (A), into the $B_{W_2}$-walk. The latter corresponds to the 3-noncrossing braid of Fig. 3 (B). Here each vertex, $i$, is aligned with the triple of shapes $(\lambda_{2i}, \lambda_{2i-1}, \lambda_{2i-2})$ in the corresponding tableaux.

**Proof.** Since $\lambda^1 = \emptyset$, $(\lambda^j)_{j=1}^{2\ell+1}$ corresponds to a sequence of pairs $((x_i, y_i))_{i=1}^{\ell}$ given by $x_i = \lambda_{2i} \setminus \lambda_{2i-1}$ and $y_i = \lambda_{2i+1} \setminus \lambda_{2i}$ such that

$$\forall 1 \leq i \leq \ell; \quad (x_i, y_i) \in \{(\emptyset, \emptyset), (+\Box h, \emptyset), (\emptyset, -\Box h), (+\Box h, -\Box j)\}.$$  

Let $\varphi$ be given by

$$\varphi((x_i, y_i)) = \begin{cases} (x_i, y_i) & \text{for } (x_i, y_i) = (+\Box h, -\Box j) \\ (y_i, x_i) & \text{otherwise}, \end{cases}$$

and set $\varphi(x_i, y_i))_{i=1}^{\ell} = (a_i, b_i))_{i=1}^{\ell}$. Note that $(a_i, b_i) \in \{(-\Box h, \emptyset), (\emptyset, +\Box h), (\emptyset, \emptyset), (+\Box h, -\Box j)\}$, where $1 \leq h, j \leq k - 1$, see Fig. 5. Let $(\mu^j)_{j=1}^{\ell}$ be the sequence of shapes induced by $(a_i, b_i))_{i=1}^{\ell}$ according to $a_i = \mu_{2i} \setminus \mu_{2i-1}$ and $b_i = \mu_{2i+1} \setminus \mu_{2i}$ initialized with $\mu_0 = \emptyset$. Now eq. (4.1) is implied by eq. (4.4) and eq. (4.2) follows by construction. Suppose $\mu^{2j+1} \neq \lambda^{2j+2}$ for some $0 \leq j \leq \ell - 1$. By definition of $\varphi$, only pairs containing “$\emptyset$” in at least one coordinate are transposed from which we can conclude $\mu^{2j+1} = \mu^{2j}$ or $\mu^{2j+1} = \mu^{2j+2}$, whence eq. (4.3). I.e. we have the following
situation

\[
\begin{align*}
\mu^{2j} & \quad \mu^{2j+1} & \quad \mu^{2j+2} \\
\lambda^{2j+1} & \quad \lambda^{2j+2} & \quad \lambda^{2j+3}
\end{align*}
\]

or

\[
\begin{align*}
\mu^{2j} & \quad \mu^{2j+1} & \quad \mu^{2j+2} \\
\lambda^{2j+1} & \quad \lambda^{2j+2} & \quad \lambda^{2j+3}
\end{align*}
\]

and the lemma follows. \(\square\)

Lemma 4.1 establishes a bijection between \(P_{W_{k-1}}\)-walks of length \(2\ell+1\) and \(B_{W_{k-1}}\)-walks of length \(2\ell\), where \(W_{k-1} = \{(a_1, a_2, \ldots, a_{k-1}) \mid a_1 > a_2 > \cdots > a_{k-1}\}\).

Corollary 4.2. Let \(P_k(N)\) and \(B_k(N-1)\) denote the set of \(k\)-noncrossing partitions on \([N]\) and \(k\)-noncrossing braids on \([N-1]\). Then

(a) we have a bijection

\[(4.6) \quad \vartheta: P_k(N) \rightarrow B_k(N-1).\]

(b) \(\vartheta\) induces by restriction a bijection between 2-regular, \(k\)-noncrossing partitions on \([N]\) and \(k\)-noncrossing braids without loops on \([N-1]\).

Proof. Assertion (a) follows from Lemma 4.1 since a partition is completely determined by its induced \(P_{W_{k-1}}\)-walk of shape \(\lambda^{2n-1} = \emptyset\). (b) follows immediately from the fact that, according to the definition of \(\varphi\) given in Lemma 4.1 any pair of consecutive EMs \((\emptyset, +\Box_1), (-\Box_1, \emptyset)\) induces an EM \((+\Box_1, -\Box_1)\). Therefore, \(\vartheta\) maps 2-regular, \(k\)-noncrossing partitions into \(k\)-noncrossing braids without loops. We illustrate Corollary 4.2(b) in Fig. 6. \(\square\)

Lemma 4.3. The number of \(B_{W_{k-1}}\)-walks ending at \(\nu \in W_{k-1}\) of length \(2\ell\), where \(1 \leq \ell \leq N\) is given by

\[(4.7) \quad \sigma_{\nu, 2\ell}^{\nu, R} = \sum_h (-1)^h \binom{\ell}{h} \omega_{2(\ell-h)+1}.\]

Furthermore, we obtain the recurrence of \(\sigma_{\nu, 2\ell-1}^{\nu, R}\) given by

\[(4.8) \quad \sigma_{\nu, 2\ell-1}^{\nu, R} = \sum_{j=0}^{k-1} \sigma_{\nu, 2\ell+2}^{\nu, R}.\]
\textbf{Figure 6.} Mapping 2-regular, 3-noncrossing partitions into loop-free braids: illustration of Corollary 4.2(b).

**Proof.** According to Lemma 4.1 any \( P_{W_{k-1}} \)-walk of length \( 2\ell + 1 \) corresponds to a unique \( B_{W_{k-1}} \)-walk of length \( 2\ell \), whence \( \sigma_{k,2\ell} = \omega_{k,2\ell+1} \). Let \( A_{2\ell}(h) \) denote the set of \( B_{W_{k-1}} \)-walks of length \( 2\ell \) in which there exist at least \( h \) pairs of shapes \((\mu_{2q-1}, \mu_{2q})\) induced by the EM \((+e_1, -e_1)\). Since the removal of \( h \) EMs, \((+e_1, -e_1)\), from a \( B_{W_{k-1}} \)-walk, results in a \( B_{W_{k-1}} \)-walk of length \( 2(\ell - h) \), we derive \( A_{2\ell}(h) = \binom{\ell}{h} \sigma_{k,2\ell-2h} \). Using the inclusion-exclusion principle, we arrive at

\begin{equation}
\sigma_{k,2\ell}^{\nu, *} = \sum_{h} (-1)^{h} \binom{\ell}{h} \sigma_{k,2(\ell - h)}^{\nu, *} = \sum_{h} (-1)^{h} \binom{\ell}{h} \omega_{k,2(\ell - h)+1}.
\end{equation}

By construction, an odd step in a \( B_{W_{k-1}} \)-walk is \(+e_j\), where \( j \in \{0, 1, \ldots, k-1\} \), whence

\begin{equation}
\sigma_{k,2\ell-1}^{\nu, *} = \sum_{j=0}^{k-1} \sigma_{k,2\ell-2}^{\nu - e_j, *},
\end{equation}

\( \square \).
Via Lemma 4.3, we have explicit knowledge about the numbers of \( B_{W_k} \)-walks, i.e. \( \sigma_{k,i}^{2,s} \) for all \( \nu \in W_{k-1} \) and \( 1 \leq s \leq 2N \). Accordingly, we are now in position to generate 2-regular, \( k \)-noncrossing partitions with uniform probability via \( k \)-noncrossing, loop-free braids.

**Theorem 4.4.** A random 2-regular, \( k \)-noncrossing partition can be generated, in \( O(N^{k+1}) \) preprocessing time and \( O(N^k) \) space complexity, with uniform probability in linear time. Each 2-regular, \( k \)-noncrossing partition is generated with \( O(N) \) space and time complexity, see Algorithm 2, 3.

**Proof.** We interpret \( B_{W_k} \)-walks as sampling paths of a stochastic process. To this end, we again label the \((i+1)\)-th shape, \( \lambda_{s+1}^{i} \) by \( \alpha = \lambda^{i+1} \setminus \lambda^{i}, \alpha \in \{ \{+\square\}_{j=1}^{j=k-1}, \{-\square\}_{j=1}^{j=k-1}, \emptyset \} \), where the labeling specifies the transition from \( \lambda^{i} \) to \( \lambda^{i+1} \). In the following, we distinguish even and odd labeled shapes.

For \( i = 2s \), we let \( W_k^*(\lambda^{2s}, 2N - 2s) \) denote the number of \( B_{W_k} \)-walks such that \( \alpha = \lambda^{2s} \setminus \lambda^{2s-1} \).

By construction, assume \( \lambda^{2s} \) has \( \nu_1 - k + 3 \) boxes in \( i \)-th row, respectively, then we have

\[
W_k^*(\lambda^{2s}, 2N - 2s) = \sigma_{k,2N-2s}^{2,s}
\]

i.e. \( W_k^*(\lambda^{2s}, 2N - 2s) \) is independent of \( \alpha \) and we write \( W_k^*(\lambda^{2s}, 2N - 2s) = W_k^*(\lambda^{2s}, 2N - 2s) \).

For \( i = 2s + 1 \), let \( V_k^*(\lambda^{2s+1}, 2N - (2s + 1)) \) denote the number of \( B_{W_k} \)-walks of shape \( \lambda^{2s+1} \) of length \( 2N - (2s + 1) \) where \( \lambda^{2s+1} \setminus \lambda^{2s} = \alpha \).

Then we have setting \( u = 2N - 2s - 2 \)

\[
V_k^*(\lambda^{2s+1}, u + 1) = \begin{cases} 
\sum_{j=2}^{k-1} W_k^*(\lambda^{2s+1}_j, u) & \text{for } \alpha = +\square_1; \\
\sum_{i \neq 0,j} W_k^*(\lambda^{2s+1}_j, u) + \frac{1}{2}(1 + (-1)^j) W_k^*(\lambda^{2s+2}_j, u) & \text{for } \alpha = +\square_j, j \neq 0, 1; \\
\sum_{i=0}^{k-1} W_k^*(\lambda^{2s+2}_j, u) & \text{for } \alpha = \emptyset; \\
W_k^*(\lambda^{2s+2}_0, u) & \text{for } \alpha = -\square_j, j \neq 0, \\
\end{cases}
\]

in which,

\[
\beta = \begin{cases} 
1 & W_k^*(\lambda^{2s+1}, u + 1) = 0, \\
0 & W_k^*(\lambda^{2s+1}, u + 1) \neq 0.
\end{cases}
\]

We are now in position to specify the process \( (X^i)_{i=1}^{i=2N} \):

- \( X^0 = X^{2N} = \emptyset \) and \( X^i \) is a shape with at most \( k - 1 \) rows
- for \( 1 \leq i \leq N - 1 \), \( (X^{2i+1} \setminus X^{2i}, X^{2i+2} \setminus X^{2i}) \in \{ (-\square, \emptyset), (\emptyset, +\square), (\emptyset, \emptyset), (+\square, -\square) \} \)
- there does not exist any subsequence \( (X^{2i}, X^{2i+1}, X^{2i+2}) \) such that \( (X^{2i+1} \setminus X^{2i}, X^{2i+2} \setminus X^{2i}) = (+\square_1, -\square_1) \).
Algorithm 2 Tableaux: Uniform generation of 2-regular, $k$-noncrossing partitions.

1: *Tableaux* (Initialize the sequence of shapes to be a list $\{\lambda^0\}_{i=0}^{2N}$)
2: $\lambda^0 \leftarrow \emptyset$, $\lambda^{2N} \leftarrow \emptyset$, $\lambda^{2N-1} \leftarrow \emptyset$, $i \leftarrow 1$
3: while $i < 2N - 1$ do
4:   if $i$ is even then
5:      $X[0] \leftarrow V_k^*(\lambda^i+1, 2N - (i + 1))$
6:      for $j = 1$ to $k - 1$ do
7:         $X[2j - 1] \leftarrow V_k^*(\lambda^i+1, 2N - (i + 1))$, $X[2j] \leftarrow V_k^*(\lambda^i+1, 2N - (i + 1))$
8:      end for
9:   else
10:      $X[0] \leftarrow W^*(\lambda^i+1, 2N - (i + 1))$
11:      for $j = 1$ to $k - 1$ do
12:         $X[2j - 1] \leftarrow W_k^*(\lambda^i+1, 2N - (i + 1))$, $X[2j] \leftarrow W_k^*(\lambda^i+1, 2N - (i + 1))$
13:      end for
14:      FLAG(flag0, flag1, flag2, flag3)
15:      $sum \leftarrow \sum_{t=0}^{2(k-1)} X[t]$
16:   end if
17:   $Shape \leftarrow Random(sum)$ (Random generates the random shape $\lambda^i+1$ with probability $X[2j-1]/sum$ or $\lambda^i+1$ with probability $X[2j]/sum$)$\leftarrow \lambda^i+1\emptyset$ with probability $X[0]/sum$)
18:   flag0 $\leftarrow 1$, flag1 $\leftarrow 1$, flag2 $\leftarrow 1$, flag3 $\leftarrow 1$
19:   if $i$ is even and $Shape=\lambda^i+1\emptyset$ then
20:      flag1 $\leftarrow 0$
21:   end if
22:   for $1 \leq s \leq k - 1$ do
23:      if $i$ is even and $Shape=\lambda^i+1\emptyset$ then
24:         flag2 $\leftarrow 0$
25:      end if
26:   end for
27:   for $2 \leq s \leq k - 1$ do
28:      if $i$ is even and $Shape=\lambda^i+1\emptyset$ then
29:         flag3 $\leftarrow 0$
30:      end if
31:   end for
32:   if $i$ is even and $Shape=\lambda^i+1\emptyset$ then
33:      flag0 $\leftarrow 0$
34:   end if
35:   Insert $Shape$ into $Tableaux$ (the sequence of shapes)
36:   $i \leftarrow i + 1$
37: end while
38: Map($Tableaux$) (maps $Tableaux$ into its corresponding 2-regular, $k$-noncrossing partition)
Algorithm 3 FLAG: Distinguish the last odd step

1: FLAG(flag0, flag1, flag2, flag3)
2: if flag0 = 0 then
3:     for $j = 1$ to $k - 1$ do
4:         $X[2j] \leftarrow 0$
5:     end for
6: end if
7: if flag1 = 0 then
8:     $X[0] \leftarrow 0, X[2] \leftarrow 0$
9:     for $j = 1$ to $k - 1$ do
10:        $X[2j - 1] \leftarrow 0$
11:     end for
12: end if
13: if flag2 = 0 then
14:     for $j = 1$ to $k - 1$ do
15:        $X[2j - 1] \leftarrow 0, X[2j] \leftarrow 0$
16:     end for
17: end if
18: if flag3 = 0 then
19:     for $j = 1$ to $k - 1$ do
20:        $X[2j - 1] \leftarrow 0$
21:     end for
22: end if

• the transition probabilities are given as follows

(1) for $i = 2\ell$, we obtain

$$
\mathbb{P}(X^{i+1} = \lambda^{i+1} | X^i = \lambda^i) = \frac{V_k^*(\lambda^{i+1}, 2N - i - 1)}{W_k^*(\lambda^i, 2N - i)}.
$$

(2) for $i = 2\ell + 1$, we have

$$
\mathbb{P}(X^{i+1} = \lambda^{i+1} | X^i = \lambda^i) = \frac{W_k^*(\lambda^{i+1}, 2N - i - 1)}{V_k^*(\lambda^i, 2N - i)}.
$$

By construction,

$$
\mathbb{P}(X^{i+1} = \lambda^{i+1} | X^i = \lambda^i) = \frac{W_k^*(\lambda^{2N}, \emptyset)}{W_k^*(\emptyset, 2N)} = \frac{1}{W_k^*(\emptyset, 2N)},
$$

(4.13)
whence \((X^i)_{i=0}^{2N}\) generates random 2-regular, \(k\)-noncrossing partitions with uniform probability in \(O(N)\) time and space.

As for the computation of the transition probabilities, according to Theorem 3.2 the terms \(\omega_{k,\ell}^\nu\) for \(\nu \in W_{k-1}\) and \(1 \leq \ell \leq 2N + 1\) can be calculated in \(O(N^{k+1})\) time and \(O(N^k)\) space.

We claim that for fixed indices \(\nu = (\nu_1, \nu_2, \ldots, \nu_{k-1})\) where \(\nu \in W_{k-1}\) and \(1 \leq s \leq 2N\), \(\sigma_{k,s}^\nu\) can be computed in \(O(N)\) time. There are two cases: in case of \(s = 2\ell_1\), using a \texttt{For}-loop summing over the terms \((-1)^{h} f_{k,2(\ell_1-h)+1}\) we derive \(\sigma_{k,2\ell_1}^{\nu^s}\). In case of \(s = 2\ell_1 + 1\), we calculate \(\sigma_{k,2\ell_1+1}^{\nu^s}\), where \(i \in \{0, 1, \ldots, k - 1\}\) via eq. (19). Then \(\sigma_{k,2\ell_1+1}^{\nu^s}\) follows in view of \(\sigma_{k,2\ell_1+1}^{\nu^s} = \sum_{j=0}^{k-1} \sigma_{k,2\ell_1}^{\nu^j}\).

Furthermore, using \(k\) nested \texttt{For}-loops for \(\nu\), we derive \(\sigma_{k,s}^{\nu^s}\) for arbitrary \(\nu\) and \(s\). Consequently, we compute \(\sigma_{k,s}^{\nu^s}\) for all \(\nu\), \(s\) with \(O(N^k) + O(N \times N) = O(N^{k+1})\) time and \(O(N^k)\) space complexity. Once the terms \(\sigma_{k,s}^{\nu^s}\) for \(\nu\) and \(s\) are calculated, we can compute the transition probabilities in \(O(1)\) time. Therefore we obtain the transition probabilities \(W^*(\lambda^i, 2N - i)\) in \(O(N^{k+1})\) time and \(O(N^k)\) space complexity and the theorem follows.

\[\square\]

**Remark** We remark here it is feasible to generate a 2-regular, \(k\)-noncrossing partition in \(O(N^k)\) time complexity if we obtain the transition probabilities for \(B_{W_{k-1}}\)-walks follow the similar routine of \(P_{W_{k-1}}\)-walks. The reason we show a different routine is we prefer to compute transition probabilities via the relation between two different combinatorial objects.

5. Example: in case of \(k = 4\)

In the following, we omit \(k = 4\) in the subscripts of the notations.

### 5.1. Example: random 4-noncrossing partitions

Set \(a_{i,j,r}^{s}\) denote the number of \(P_{Q_s}\)-walks of length \(s\) starting at \((2,1,0)\), ending at \((i,j,r)\) in \(Q_3\) and set \(f_{i,j,r}^{s} = a_{2s+1}^{i,j,r}\).

**Lemma 5.1.** (a) Suppose \(1 \leq \ell \leq N - 1\) and \((i,j,r)\) \(\in W_3\), then

\[\omega_{s}^{i,j,r} = \begin{cases} \frac{f_{i,j,r}^{s} - f_{i,j,r}^{s+1} - f_{i,j,r}^{s+2} + f_{i,j,r}^{s+3}}{f_{i,j,r}^{s} - f_{i,j,r}^{s+1}} & \text{for } s = 2\ell + 1, \\ \frac{f_{i,j,r}^{s} - f_{i,j,r}^{s+1} - f_{i,j,r}^{s+2} + f_{i,j,r}^{s+3}}{f_{i,j,r}^{s} - f_{i,j,r}^{s+1}} & \text{for } s = 2\ell + 2. \end{cases}\]
where for \( a, b, c \in \mathbb{Z} \), \( f_{\ell}^{a,b,c} = 0 \) for \( a + b + c \geq \ell + 3 \) and for \( \ell = 0 \),
\[
\begin{cases}
1 & \text{for } i = 2, j = 1, r = 0; \\
0 & \text{otherwise}.
\end{cases}
\]

(b) \( f_{\ell}^{p,q,s} \) satisfies the recursion
\[
f_{\ell}^{p,q,s} = f_{\ell-1}^{p,q+1,s} + \sum_{h=1}^{\ell-1} (-1)^h \binom{\ell}{h} f_{\ell-1-h}^{p,q-1,s} \cdot \omega_{2h}^{i,j,r} .
\]

Proof. We first prove assertion (a). Indeed, by definition of \( f_{\ell}^{i,j,r} \), we have
\[
a_{2\ell+1}^{i,j,r} = f_{\ell}^{i,j,r} \quad \text{and} \quad a_{2\ell}^{i,j,r} = f_{\ell-1}^{i,j,r} + f_{\ell-1}^{i,j-1,r} + f_{\ell-1}^{i,j,r-1},
\]
whence eq. (5.1) follows from Proposition 2.2 for the case \( k = 4 \). I.e.
\[
\omega_{\ell}^{i,j,r} = a_{\ell}^{i,j,r} - a_{\ell}^{i,j-1,r} - a_{\ell}^{i,j,r-1} + a_{\ell}^{i,j-1,r-1} + a_{\ell}^{i,j,r-1}.
\]
Next, by construction, whence (b) and the proof of the lemma is complete. □

Once \( \omega_{\ell}^{i,j,r} \) for arbitrary \((i, j, r)\) \( \in \mathbb{W}_3 \) can be calculated follow the routine given in Lemma 5.1, we arrive at the following algorithm:

**Corollary 5.2.** A random 4-noncrossing partition can be generated, in \( O(N^4) \) pre-processing time and \( O(N^4) \) space complexity, with uniform probability in linear time. Each 4-noncrossing partition is generated with \( O(N) \) space and time complexity, see Algorithm 4.

5.2. Example: 2-regular, random 4-noncrossing partitions.

**Lemma 5.3.** The number of \( \mathbb{B}_{\mathbb{W}_3}^{i,j,r} \)-walks ending at \( \nu \in \mathbb{W}_3 \) of length \( 2\ell \), where \( 1 \leq \ell \leq N \) is given by
\[
\sigma_{2\ell}^{i,j,r} = \sum_{h=1}^{\ell} (-1)^h \binom{\ell}{h} \omega_{2(\ell-h)+1}^{i,j,r} .
\]
Furthermore, we obtain the recurrence of \( \sigma_{2\ell-1}^{i,j,r} \) given by
\[
\sigma_{2\ell-1}^{i,j,r} = \sigma_{2\ell-2}^{i,j,r} + \sigma_{2\ell-2}^{i,j-1,r} + \sigma_{2\ell-2}^{i,j-1,r} + \sigma_{2\ell-2}^{i,j,r-1} .
\]
Algorithm 4 Uniform generation of 4-noncrossing partitions

1: Tableaux (Initialize the sequence of shapes, \(\{\lambda^i\}_{i=0}^{2N}\))
2: \(\lambda^0 \leftarrow \emptyset, \lambda^{2N} \leftarrow \emptyset, i \leftarrow 1\)
3: while \(i < 2N\) do
4: \(\text{if } i \text{ is even then}\)
5: \(X[0] \leftarrow \text{V}(\lambda^{i+1}_\emptyset, 2N - (i + 1))\)
6: \(X[1] \leftarrow \text{V}(\lambda^{i+1}_{-\Box_1}, 2N - (i + 1))\)
7: \(X[2] \leftarrow \text{V}(\lambda^{i+1}_{-\Box_2}, 2N - (i + 1))\)
8: \(X[3] \leftarrow \text{V}(\lambda^{i+1}_{-\Box_3}, 2N - (i + 1))\)
9: \(\text{end if}\)
10: \(\text{if } i \text{ is odd then}\)
11: \(X[0] \leftarrow \text{V}(\lambda^{i+1}_\emptyset, 2N - (i + 1))\)
12: \(X[1] \leftarrow \text{V}(\lambda^{i+1}_{+\Box_1}, 2N - (i + 1))\)
13: \(X[2] \leftarrow \text{V}(\lambda^{i+1}_{+\Box_2}, 2N - (i + 1))\)
14: \(X[3] \leftarrow \text{V}(\lambda^{i+1}_{+\Box_3}, 2N - (i + 1))\)
15: \(\text{end if}\)
16: sum \(\leftarrow X[0] + X[1] + X[2] + X[3]\)
17: Shape \(\leftarrow \text{Random}(\text{sum})\) (Random generates the random shape \(\lambda^{i+1}_{+\Box_j}(or-\Box_j)\) with probability \(X[j]/\text{sum}\) or \(\lambda^{i+1}_\emptyset\) with probability \(X[0]/\text{sum}\))
18: \(i \leftarrow i + 1\)
19: Insert Shape into Tableau (the sequence of shapes).
20: end while
21: Map(Tableau) (maps Tableau into its corresponding 4-noncrossing partition)

Corollary 5.4. A random 2-regular, 4-noncrossing partition can be generated, in \(O(N^5)\) preprocessing time and \(O(N^4)\) space complexity, with uniform probability in linear time. Each 2-regular, 4-noncrossing partition is generated with \(O(N)\) space and time complexity, see Algorithm 5, 6.

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Algorithm 5 Uniform generation of 2-regular, 4-noncrossing partitions.

1: Tableaux (Initialize the sequence of shapes to be a list $\{\lambda_i\}_{i=0}^{2N}$)
2: $\lambda^0 \leftarrow \emptyset$, $\lambda^{2N} \leftarrow \emptyset$, $\lambda^{2N-1} \leftarrow \emptyset$, $i \leftarrow 1$
3: while $i < 2N - 1$ do
4:  if $i$ is even then
5:    $X[0] \leftarrow V^*(\lambda^{i+1}_0, 2N - (i + 1))$
6:    $X[1] \leftarrow V^*(\lambda^{i+1}_{-1}, 2N - (i + 1))$, $X[2] \leftarrow V^*(\lambda^{i+1}_1, 2N - (i + 1))$
7:    $X[3] \leftarrow V^*(\lambda^{i+1}_2, 2N - (i + 1))$, $X[4] \leftarrow V^*(\lambda^{i+1}_{-2}, 2N - (i + 1))$
8:    $X[5] \leftarrow V^*(\lambda^{i+1}_{-1}, 2N - (i + 1))$, $X[6] \leftarrow V^*(\lambda^{i+1}_3, 2N - (i + 1))$
9:  else
10:   $X[0] \leftarrow W^*(\lambda^{i+1}_0, 2N - (i + 1))$
11:   $X[1] \leftarrow W^*(\lambda^{i+1}_{-1}, 2N - (i + 1))$, $X[2] \leftarrow W^*(\lambda^{i+1}_1, 2N - (i + 1))$
12:   $X[3] \leftarrow W^*(\lambda^{i+1}_2, 2N - (i + 1))$, $X[4] \leftarrow W^*(\lambda^{i+1}_{-2}, 2N - (i + 1))$
13:   $X[5] \leftarrow W^*(\lambda^{i+1}_{-1}, 2N - (i + 1))$, $X[6] \leftarrow W^*(\lambda^{i+1}_3, 2N - (i + 1))$
14:   FLAG(flag0, flag1, flag2, flag3)
15:   $sum \leftarrow X[0] + X[1] + X[2] + X[3] + X[4] + X[5] + X[6]$
16: end if
17: Shape ← Random(sum) (Random generates the random shape $\lambda^{i+1}_{i+j}$ or with probability $X[2j]/sum$ or $X[2j]/sum$ or $X[0]/sum$)
18:  flag0 ← 1, flag1 ← 1, flag2 ← 1, flag3 ← 1
19:  if $i$ is even and Shape=λ^{i+1}_1 then
20:    flag1 ← 0
21: end if
22:  if $i$ is even and Shape=λ^{i+1}_{-1} then
23:    flag2 ← 0
24: end if
25:  if $i$ is even and Shape=λ^{i+1}_{-3} then
26:    flag2 ← 0
27: end if
28:  if $i$ is even and Shape=λ^{i+1}_3 then
29:    flag2 ← 0
30: end if
31:  if $i$ is even and Shape=λ^{i+1}_2 then
32:    flag3 ← 0
33: end if
34:  if $i$ is even and Shape=λ^{i+1}_3 then
35:    flag3 ← 0
36: end if
37:  if $i$ is even and Shape=λ^{i+1}_{-2} then
38:    flag0 ← 0
39: end if
40:  Insert Shape into Tableaux (the sequence of shapes)
41:  $i \leftarrow i + 1$
42: end while
43: Map(Tableaux) (maps Tableaux into its corresponding 2-regular, 4-noncrossing partition)
Algorithm 6 FLAG: Distinguish the last odd step

1: \( \text{FLAG}(\text{flag0}, \text{flag1}, \text{flag2}, \text{flag3}) \)
2: if \( \text{flag0} = 0 \) then
3: \( X[2], X[4], X[6] \leftarrow 0 \)
4: end if
5: if \( \text{flag1} = 0 \) then
6: \( X[0], X[1], X[2], X[3], X[5] \leftarrow 0 \)
7: end if
8: if \( \text{flag2} = 0 \) then
9: \( X[1], X[2], X[3], X[4], X[5], X[6] \leftarrow 0 \)
10: end if
11: if \( \text{flag3} = 0 \) then
12: \( X[1], X[3], X[5] \leftarrow 0 \)
13: end if

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