LOCAL WELL-POSEDNESS FOR THE (N+1) - DIMENSIONAL MAXWELL-KLEIN-GORDON EQUATIONS IN TEMPORAL GAUGE

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Abstract. This is an extension of the paper [14] by the author for the 2+1 dimensional Maxwell-Klein-Gordon equations in temporal gauge to the n+1 dimensional situation for $n \geq 3$. They are shown to be locally well-posed for low regularity data, in 3+1 dimensions even below energy level improving a result by Yuan. Fundamental for the proof is a partial null structure of the nonlinearity which allows to rely on bilinear estimates in wave-Sobolev spaces, in 3+1 dimensions proven by d’Ancona, Foschi and Selberg, on an $(L^{2+1}_x L^2_t)$ - estimate for the solution of the wave equation, and on the proof of a related result for the Yang-Mills equations by Tao.

1. Introduction and main results

Consider the Maxwell-Klein-Gordon equations

\[ \partial^a F_{\alpha \beta} = -Im(\phi D_{\beta} \phi) \quad (1) \]
\[ D^\mu D_\mu \phi = m^2 \phi \quad (2) \]

in Minkowski space $\mathbb{R}^{1+n} = \mathbb{R}_x \times \mathbb{R}_t^n$ with metric $\text{diag}(-1, ..., 1)$. Greek indices run over $\{0, 1, ..., n\}$, Latin indices over $\{1, ..., n\}$, and the usual summation convention is used. Here $m \in \mathbb{R}$ and

\[ \phi : \mathbb{R} \times \mathbb{R}^n \to C \,, \quad A_\alpha : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \,, \quad F_{\alpha \beta} = \partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha} \,, \quad D_\mu = \partial_\mu + i A_\mu \,.
\]

$A_\mu$ are the gauge potentials, $F_{\mu \nu}$ is the curvature. We use the notation $\partial_\mu = \frac{\partial}{\partial x_\mu}$, where we write $(x^0, x^1, ..., x^n) = (t, x^1, ..., x^n)$ and also $\partial_0 = \partial_t$.

Setting $\beta = 0$ in (1) we obtain the Gauss-law constraint

\[ \partial^0 F_{00} = -Im(\phi D_0 \phi) \quad (3) \]

The system (1), (2) is invariant under the gauge transformations

\[ A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi \,, \quad \phi \to \phi' = e^{i\chi} \phi \,, \quad D_\mu \to D'_\mu = \partial_\mu + i A'_\mu \,.
\]

This allows to impose an additional gauge condition. We exclusively consider the temporal gauge

\[ A_0 = 0 \quad (4) \]

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In this gauge the system (1), (2) is equivalent to
\[
\begin{align*}
    \partial_t \phi^j A_j &= 1m(\phi \partial_t \phi) \\
    \Box A_j &= \partial_j (\partial^k A_k) - 1m(\phi \partial_j \phi) + A_j |\phi|^2 \\
    \Box \phi &= -i(\partial^k A_k) \phi - 2iA^j \partial_k \phi + A^k A_k \phi + m^2 \phi,
\end{align*}
\]
where \( \Box = -\partial_t^2 + \Delta \) is the d’Alembert operator.

Other choices of the gauge are the Coulomb gauge \( \partial^j A_j = 0 \) and the Lorenz gauge \( \partial^\mu A_\mu = 0 \).

The classical (3+1)-dimensional Maxwell-Klein-Gordon system has been studied by Klainerman and Machedon [8] where the existence of global solutions for data in energy space and above in Coulomb gauge was shown. Uniqueness in a suitable subspace was also shown. For the temporal gauge they also showed a similar result by using a suitable gauge transformation applied to the solution constructed in Coulomb gauge. They made use of a null structure for the main bilinear term to achieve this result. Local well-posedness in Coulomb gauge for data for the Sobolev space \( H^s \) and for \( A \) in \( H^r \) with \( r = s > 1/2 \), i.e., almost down to the critical space with respect to scaling, was shown by Machedon and Sterbenz [10]. Global well-posedness below energy space (for \( r = s > \sqrt{3}/2 \)) in Coulomb gauge was shown by Keel, Roy and Tao [6].

The problem in Lorenz gauge was considered by Selberg and Tesfahun [18], who detected a null structure also in this case, and proved global well-posedness in energy space, especially also unconditional uniqueness in this space. The author [12] proved local well-posedness for \( s = \frac{3}{4} + \epsilon \) and \( r = \frac{1}{2} + \epsilon \).

The problem in temporal gauge was treated by Yuan [21] directly in \( X^{s,b} \)-spaces. He stated local well-posedness in \( X^{s,b} \)-spaces for large data for \( \phi \) in \( H^s \) and for \( A \) in \( H^r \) with \( r = s > 3/4 \), where he just referred to the estimates given for Tao’s small data local well-posedness results [20] in the Yang-Mills case. As a consequence he proved existence of a global solution in energy space and also uniqueness in subspaces of \( X^{s,b} \)-type. Unconditional uniqueness in the natural solution space in the finite energy case was shown by the author [13]. These results in temporal gauge rely on a similar result by Tao [20] for the Yang-Mills equations and small data.

All these results were given in the (3+1)-dimensional case.

In 2+1 dimensions local well-posedness in Lorenz gauge for \( s = \frac{3}{4} + \epsilon \) and \( r = \frac{1}{2} + \epsilon \) was shown by the author [12]. In Coulomb gauge local well-posedness for \( s = r = \frac{5}{8} + \epsilon \) and also for \( s = \frac{5}{8} + \epsilon , r = \frac{1}{2} + \epsilon \) was obtained by Czubak and Pikula [4], which was slightly improved to the case \( s = \frac{3}{4} + \epsilon , r = \frac{1}{2} + \epsilon \) in [14]. In the temporal gauge in [14] local well-posedness was shown for data under the minimal smoothness assumption \( s = r = \frac{1}{2} + \frac{1}{12} + \epsilon \).

In the present paper we consider the (n+1)-dimensional case in the temporal gauge and lower down the minimal regularity assumptions on the data further using similar methods as in the (2+1)-dimensional case in [14]. We prove local well-posedness for data for \( \phi \) in \( H^s \) and \( A \) in \( H^r \), where \( s > \frac{3}{2} - \frac{1}{4} \) and \( r > \frac{3}{2} - 1 \), where uniqueness holds in \( X^{s,b} \)-spaces (for more precise assumptions cf. Theorem 1[3]). The critical case with respect to scaling is \( r = s = \frac{3}{2} - 1 \), which we almost reach with respect to \( r \). For technical reasons it is necessary to assume in a first step that the curl-free part of \( A(0) \) vanishes (cf. Proposition 3[1]). This condition is removed by a suitable gauge transformation afterwards, which preserves the regularity of the solution. We need the null structure of some of the nonlinearities, the bilinear estimates for wave-Sobolev spaces \( X^{s,b}_{r=|\xi|} \), which were formulated by d’Ancona, Foschi and Selberg [1] in arbitrary dimensions and proven in the case
forms. An elementary calculation namely shows that $X$ and in product Sobolev spaces $X_{s}^{l}$ and product Sobolev spaces $X_{s=0}^{l}$ (cf. the definition of the spaces below) which have to be generalized from the special case $n = 3$ and $l = s + \frac{1}{2}$. Moreover we need an appropriate generalization of the estimates for the terms which fulfill a null condition. Of fundamental importance is an $(L_{x}^{2} \cap L_{t}^{\infty})$ - estimate for the solution of the wave equation which goes back to Tataru [17] and Tao [20].

We denote both the Fourier transform with respect to space and time and with respect to space by $\hat{\cdot}$ or $\mathcal{F}$. The operator $D^{\alpha}$ is defined by $(\mathcal{F}(D^{\alpha} f))(\xi) = |\xi|^\alpha (\mathcal{F} f)(\xi)$ and similarly $\Lambda^{\alpha}$ by $(\mathcal{F}(\Lambda^{\alpha} f))(\xi) = \langle \xi \rangle^{\alpha} (\mathcal{F} f)(\xi)$, where we define $\langle \cdot \rangle := (1 + |\cdot|^{2})^{\frac{1}{2}}$. The inhomogeneous Sobolev spaces are denoted by $H^{s,p}$. For $p = 2$ we simply denote them by $H^{s}$. We repeatedly use the Sobolev embeddings $H^{s,p} \hookrightarrow L^{q}$ for $\frac{1}{p} \geq \frac{1}{2} \geq \frac{1}{p} - \frac{1}{2}$ and $1 < p \leq q < \infty$. We also use the notation $a_{\pm} := a \pm \epsilon$ for a sufficiently small $\epsilon > 0$.

The standard space $X_{s}^{l,b}$ of Bourgain-Klainerman-Machedon type (which were already considered by M. Beals [3]) belonging to the half waves is the completion of the Schwarz space $\mathcal{S}(\mathbb{R}^{n+1})$ with respect to the norm

$$\|u\|_{X_{s}^{l,b}} = \|\langle \xi \rangle^{s} (\tau \pm |\xi|) \hat{u}(\tau, \xi)\|_{L_{\tau}^{2}}.$$  

The wave-Sobolev space $H^{s,b}$ is the completion of the Schwarz space $\mathcal{S}(\mathbb{R}^{n+1})$ with respect to the norm

$$\|u\|_{H^{s,b}} = \|\langle \xi \rangle^{s} (|\tau| - |\xi|) \hat{u}(\tau, \xi)\|_{L_{\tau}^{2}}$$

and also $X_{\tau=0}^{s,b}$ with norm

$$\|u\|_{X_{\tau=0}^{s,b}} = \|\langle \xi \rangle^{s} \hat{u}(\tau, \xi)\|_{L_{\tau}^{2}}.$$

We also define $X_{s}^{l,b}[0, T]$ as the space of the restrictions of functions in $X_{s}^{l,b}$ to $[0, T] \times \mathbb{R}^{n}$ and similarly $H^{s,b}[0, T]$ and $X_{s=0}^{l,b}[0, T]$. We frequently use the estimate $\|u\|_{X_{s=0}^{l,b}} \leq \|u\|_{H^{s,b}}$ for $b \leq 0$ and the reverse estimate for $b \geq 0$. This allows to replace the spaces $X_{s}^{l,b}$ by $H^{s,b}$ in the nonlinear estimates.

We decompose $A = (A_{1}, ..., A_{n})$ into its divergence-free part $A^{df}$ and its curl-free part $A^{cf}$:

$$A = A^{df} + A^{cf},$$  

(8)

where

$$A_{j}^{df} = R_{i}^{k}(R_{i}A_{k} - R_{k}A_{i}) \quad A_{j}^{cf} = -R_{i}R_{k}A_{j},$$  

(9)

and $R_{k} := D^{-1}\partial_{k}$ are the Riesz operators. Let $PA := A^{df}$ denote the projection operator onto the divergence free part. Then we obtain the equivalent system

$$\partial_{t} A^{cf} = -D^{-2}\nabla Im(\phi \nabla \phi)$$  

(10)

$$\Box A^{df} = -P(Im(\phi \nabla \phi) + iA_{i}\phi_{i}^{2})$$  

(11)

$$\Box \phi = i(\partial_{i} A_{j}^{cf})\phi + 2iA_{j}^{df}\phi_{i} + 2iA_{j}^{cf}\phi_{i} + A_{j} A_{j} \phi,$$  

(12)

where $A$ is replaced by $\mathcal{S}$. Klainerman and Machedon detected that $A^{df} \cdot \nabla \phi$ and $P(Im(\phi \nabla \phi))_{k}$ are null forms. An elementary calculation namely shows that

$$2A_{i}^{cf}\phi_{i} = Q_{ij}(\phi, \nabla^{-1}(R^{i}A^{l} - R^{l}A^{i}))$$  

(13)

and

$$P(Im(\phi \nabla \phi))_{k} = -2R^{i} \nabla^{-1}Q_{kj}(Re \phi, Im \phi)$$  

(14)

$n \leq 3$, a generalization of a special case to higher dimensions by [15], and Tao’s hybrid estimates [20] for the product of functions in wave-Sobolev spaces $X_{s}^{l,b}$ and in product Sobolev spaces $X_{s=0}^{l,b}$ (cf. the definition of the spaces below) which have to be generalized from the special case $n = 3$ and $l = s + \frac{1}{2}$. Moreover we need an appropriate generalization of the estimates for the terms which fulfill a null condition. Of fundamental importance is an $(L_{x}^{2} \cap L_{t}^{\infty})$ - estimate for the solution of the wave equation which goes back to Tataru [17] and Tao [20].
where the null form \( Q_{ij} \) is defined by

\[
Q_{ij}(u, v) := \partial_i u \partial_j v - \partial_j u \partial_i v.
\]

Defining

\[
\phi_{\pm} = \frac{1}{2}(\phi \pm i\Lambda^{-1}\partial_t \phi) \iff \phi = \phi_+ + \phi_- , \quad \partial_t \phi = i\Lambda(\phi_+ - \phi_-)
\]

\[
A_{df}^{ij} = \frac{1}{2}(A_{df}^{ij} \pm i\Lambda^{-1}\partial_t A_{df}^{ij}) \iff A_{df}^{ij} = A_{df}^{ij}_+ + A_{df}^{ij}_- , \quad \partial_t A_{df}^{ij} = i\Lambda(A_{df}^{ij}_+ - A_{df}^{ij}_-)
\]

we can rewrite (10), (11), (12) as

\[
\partial_t A_{df}^{ij} = -D^{-2}\nabla \text{Im}(\partial_t \phi)
\]

(15)

\[
(-i\partial_t \pm \Lambda)A_{df}^{ij}_\pm = \mp 2^{-1}\Lambda^{-1}(2R^1 D^{-1} Q_{kj}(\text{Re} \phi, \text{Im} \phi) + iA_j |\phi|^2 - A_{df}^{ij}_j)
\]

(16)

\[
(-i\partial_t \pm \Lambda)\phi_{\pm} = \mp 2^{-1}\Lambda^{-1}(i(\partial^j A_{df}^{ij}) \phi + iQ_{kj}(\phi, |\nabla|^{-1}(R^k A^j - R^j A^k)) + 2iA_{df}^{ij}_j \partial_t \phi + A^j A_j \phi - \phi).
\]

(17)

The initial data are transformed as follows:

\[
\phi_{\pm}(0) = \frac{1}{2}(\phi(0) \pm i^{-1}\Lambda^{-1}(\partial_t \phi)(0))
\]

(18)

\[
A_{df}^{ij}(0) = \frac{1}{2}(A_{df}(0) \pm i^{-1}\Lambda^{-1}(\partial_t A_{df})(0)).
\]

(19)

Our main result is preferably formulated in terms of the system (5), (6), (7).

**Theorem 1.1.** Let \( n \geq 3 \).

1. Assume \( a_2^2 - \frac{1}{2}a_1^2 \geq r > \frac{s-1}{2} , s > \frac{n}{2} - \frac{3}{4} , 2r - s > \frac{n}{2} - \frac{3}{4} , r \geq s - 1 , l > \frac{a_1-a_2}{2} , 1 \leq l + s , l < 2s - \frac{n}{2} + 1 \). Let \( \phi_0 \in H^s(\mathbb{R}^n), \phi_1 \in H^{s-1}(\mathbb{R}^n), a_0 \in H^r(\mathbb{R}^n) , a_1 \in H^{r-1}(\mathbb{R}^n) \) be given, which satisfy the compatibility condition

\[
\partial_t a_1^2 = \text{Im}(\phi_0 \overline{\phi}_1)
\]

(20)

Then there exists \( T > 0 \), such that [5], [3], [4] with initial conditions \( \phi(0) = \phi_0 , \) \( (\partial_t \phi)(0) = \phi_1 , A(0) = a_0 , \) \( (\partial_t A)(0) = a_1 \) has a unique local solution

\[
\phi = \phi_+ + \phi_- , \quad A = A_+ + A_- + \tilde{A}
\]

with

\[
\phi_{\pm} \in X^{s, \frac{3}{2}+\epsilon}[0, T] , \quad A_{\pm} \in X^{s, \frac{5}{2}-\epsilon}[0, T] , \quad \tilde{A} \in X^{1, \frac{5}{2}+\epsilon}[0, T],
\]

where \( \epsilon > 0 \) is sufficiently small.

2. This solution satisfies

\[
\phi \in C^0([0, T], H^s(\mathbb{R}^n)) \cap C^1([0, T], H^{s-1}(\mathbb{R}^n)) ,
\]

\[
A \in C^0([0, T], H^r(\mathbb{R}^n)) \cap C^1([0, T], H^{r-1}(\mathbb{R}^n)) .
\]

**2. Basic tools**

Fundamental for us are the following estimates. We frequently use the classical Sobolev multiplication law in dimension \( n \):

\[
\|uv\|_{H^{-n}} \lesssim \|u\|_{H^r_1} \|v\|_{H^{r_2}} ,
\]

(21)

if \( s_0 + s_1 + s_2 \geq \frac{n}{2} \) and \( s_0 + s_1 + s_2 \geq \max(s_0, s_1, s_2) \), where at most one of these inequalities is an equality.

The corresponding bilinear estimates in wave-Sobolev spaces were formulated in arbitrary dimension \( n \geq 2 \) and proven by d’Ancona, Foschi and Selberg in the case \( n = 3 \) in [1] and also proven in the case \( n = 2 \) in [2] in a form which includes some more limit cases which we do not need.
Proposition 2.1. For $s_0, s_1, s_2, b_0, b_1, b_2 \in \mathbb{R}$ and $u, v \in S(\mathbb{R}^{n+1})$ the estimate

$$\|uv\|_{H^{-s_0-s_1}} \lesssim \|u\|_{H^{s_0+b_1}} \|v\|_{H^{s_2+b_2}}$$

holds, provided the following conditions are satisfied:

- $b_0 + b_1 + b_2 > \frac{1}{2}$
- $b_0 + b_1 + b_2 \geq 0$
- $b_0 + b_2 \geq 0$
- $b_1 + b_2 \geq 0$

$$s_0 + s_1 + s_2 > \frac{n+1}{2} - (b_0 + b_1 + b_2)$$

$$s_0 + s_1 + s_2 > \frac{n}{2} - \min(b_0 + b_1, b_0 + b_2, b_1 + b_2)$$

$$s_0 + s_1 + s_2 > \frac{n-1}{2} - \min(b_0, b_1, b_2)$$

$$s_0 + s_1 + s_2 > \frac{n+1}{4}$$

$$s_0 + b_0 + 2s_1 + 2s_2 > \frac{n}{2}$$

$$2s_0 + (s_1 + b_1) + 2s_2 > \frac{n}{2}$$

$$2s_0 + 2s_1 + (s_2 + b_2) > \frac{n}{2}$$

$$s_1 + s_2 \geq \max(0, -b_0), \quad s_0 + s_2 \geq \max(0, -b_1), \quad s_0 + s_1 \geq \max(0, -b_2).$$

The proof of the following special case in higher dimensions and its Corollary was given in [15], Prop. 3.6 and Cor. 3.1.

Proposition 2.2. Assume $n \geq 4$ and

$$s_0 + s_1 + s_2 > \frac{n-1}{2}, \quad (s_0 + s_1 + s_2) + s_1 + s_2 > \frac{n}{2},$$

$$s_0 + s_1 \geq 0, \quad s_0 + s_2 \geq 0, \quad s_1 + s_2 \geq 0.$$

The following estimate holds:

$$\|uv\|_{H^{-s_0-s_1}} \lesssim \|u\|_{H^{s_0+b_1}} \|v\|_{H^{s_2+b_2}}.$$

Corollary 2.1. Under the assumptions of Prop. 2.2

$$\|uv\|_{H^{-s_0-s_1}} \lesssim \|u\|_{H^{s_0+b_1}} \|v\|_{H^{s_2+b_2}}.$$

Moreover we need the standard Strichartz type estimates for the wave equation given in the next proposition.

Proposition 2.3. If $n \geq 2$ and

$$2 \leq q \leq \infty, \quad 2 \leq r < \infty, \quad \frac{2}{q} \leq (n-1) \left(\frac{1}{2} - \frac{1}{r}\right),$$

then the following estimate holds:

$$\|u\|_{L_t^q L_x^r} \lesssim \|u\|_{H^{\frac{n}{2} - \frac{1}{q} + \frac{1}{r}}},$$

especially

$$\|u\|_{L_{t,x}^{q+1}} \lesssim \|u\|_{H^{\frac{n}{2} + \frac{1}{q}}}, \quad (22)$$

Proof. This the Strichartz type estimate, which can be found for e.g. in [5], Prop. 2.1, combined with the transfer principle.

Essential for us is the following estimate, which essentially goes back to Tataru [7] and Tao [19].
Proposition 2.4. The following estimates hold:

\[ \|u\|_{L^2_x L_t^{n+1}} \lesssim \|u\|_{H^{\frac{n-1}{2}}} + \frac{1}{4}, \]
\[ \|u\|_{L^2_x L_t^{2(n+1)}} \lesssim \|u\|_{H^{\frac{n-1}{2}}} + \frac{1}{4}. \]

Proof. In the case \( n = 3 \) one may simply refer to [19], Prop. 4.1. Alternatively by [7], Thm. B2 we obtain \( \|F_t u\|_{L^2_x L_t^{2(n+1)}} \lesssim \|u_0\|_{H^{\frac{n-1}{2}}} \), if \( u = e^{itD}u_0 \) and \( F_t \)

denotes the Fourier transform with respect to time. This implies by Plancherel and Minkowski’s inequality

\[ \|u\|_{L^2_x L_t^{2(n+1)}} = \|F_t u\|_{L^2_x L_t^{2(n+1)}} \lesssim \|u_0\|_{H^{\frac{n-1}{2}}}. \]

The transfer principle [10], Prop. 8 implies

\[ \|u\|_{L^2_x L_t^{2(n+1)}} \lesssim \|u\|_{H^{\frac{n-1}{2}}} + \frac{1}{4}. \]

Interpolation with (22) gives

\[ \|u\|_{L^2_x L_t^{2(n+1)}} \lesssim \|u\|_{H^{\frac{n-1}{2}}} + \frac{1}{4}. \]

Lemma 2.1.

\[ Q_{ij}(\phi, \psi) \lesssim D^\frac{1}{2} \left( D^\frac{1}{2} \phi D^\frac{1}{2} \psi + D^\frac{1}{2} (D^\frac{1}{2} \phi D^\frac{1}{2} \psi) + D^\frac{1}{2} (D^\frac{1}{2} \phi D^\frac{1}{2} D^\frac{1}{2} \psi) \right) \]
\[ + D^\frac{1}{2} (D^\frac{1}{2} + 2r \phi D^\frac{1}{2} + 2r \psi) + D^\frac{1}{2} (D^\frac{1}{2} + 2r \phi D^\frac{1}{2} + 2r \psi) \]
\[ + D^\frac{1}{2} (D^\frac{1}{2} + 2r \phi D^\frac{1}{2} + 2r \psi) \]
\[ = Q_{ij}(\phi, \psi) \lesssim D^1 (D^\frac{1}{2} + 2r \phi D^\frac{1}{2} + 2r \psi) + D^\frac{1}{2} (D^\frac{1}{2} + 2r \phi D^\frac{1}{2} + 2r \psi) \]

for \( 0 \leq \epsilon \leq \frac{1}{4} . \)

Proof. (23) is proven in [4], whereas (24) and (25) follow by interpolation with the trivial estimate \( Q_{ij}(\phi, \psi) \lesssim D\phi D\psi \).

3. PROOF OF THEOREM 1.1

For the proof it is essential to show that we may assume in a first step that the initial data satisfy \( a_0^{ij} = 0 \) and that is is possible to cancel this condition in a second step by using a suitable gauge transformation.

Proposition 3.1. Let \( n \geq 3 \).

1. Assume \( \frac{n}{2} - \frac{1}{2} \geq r > \frac{n}{2} - 1 , s > \frac{n}{2} - \frac{3}{2} , 2r - s > \frac{n}{2} - \frac{3}{2} , r \geq s - 1 , l > \frac{n-1}{2} \), \( \frac{l}{2} - 1 < 2s - \frac{n}{2} + 1 \). Let \( \phi_0 \in H^s(\mathbb{R}^n) \), \( \phi_1 \in H^{s-1}(\mathbb{R}^n) \), \( a_0 \in H^r(\mathbb{R}^n) \), \( a_1 \in H^{r-1}(\mathbb{R}^n) \) be given, which satisfy the compatibility condition

\[ \partial_j a_1^j = \text{Im}(\phi_0 \phi_1) \]

and

\[ a_0^{ij} = 0 . \]

Then there exists \( T > 0 \), such that [10], [17], [12] with initial conditions \( \phi(0) = \phi_0 \), \( (\partial_t \phi)(0) = \phi_1 \), \( A(0) = a_0 \), \( (\partial_t A)(0) = a_1 \) has a unique local solution

\[ \phi = \phi_+ + \phi_- , \quad A = A_+ + A_- + A, \]

\[ A = A^{ij} + A_{ij} + A^{ij} \]

\[ A = A^{ij} + A_{ij} + A \]
with
\[ \phi_{\pm} \in X^s_{\pm} \oplus [0, T], \ A^{q\pm}_{\pm} \in X^{r,q}_{\pm} \oplus [0, T], \ A^{c\pm} \in X^1_{\tau=0} \oplus [0, T], \]
where \( \epsilon > 0 \) is sufficiently small.

2. This solution satisfies
\[ \phi_{\pm} \in C^0([0, T], H^s(\mathbb{R}^n)), \ A^{q\pm}_{\pm} \in C^0([0, T], H^t(\mathbb{R}^n)), \]
\[ A^{c\pm} \in C^0([0, T], H^l(\mathbb{R}^n))^n \cap C^1([0, T], H^{l-1}(\mathbb{R}^n)). \]

Proof of Proposition 3.1: Proof of part 2: We assume for the moment that part 1 is true. The compatibility condition \([20]\), which is necessary in view of \([3]\), determines \( a^{c\pm}_1 \) as \( a^{c\pm}_1 = -(-\Delta)^{-1}\nabla(Im(\phi_1)) \).

Is is not difficult to see that \( a^{c\pm}_1 \) fulfills \( a^{c\pm}_1 \in H^{l-1}(\mathbb{R}^n) \). One only has to show that
\[ ||D^{-1}(\phi_0\phi_1)||_{H^{l-1}} \lesssim ||\phi_0||_{H^s} ||\phi_1||_{H^{l-1}}. \]

By duality this is equivalent to
\[ ||\phi_0\phi_2||_{H^{l-s}} \lesssim ||\phi_0||_{H^s} ||D\phi_2||_{H^{l-1}}. \]

In the case of high frequencies of \( \phi_2 \) this follows from the Sobolev multiplication law \([21]\) using \( 2s - l \geq \frac{n}{2} + 1 \), and the low frequency case can be easily handled using \( s > \frac{1}{2} \). In the same way we also obtain from \([15]\): \( \partial_t A^{c\pm} \in C^0([0, T], H^{l-1}(\mathbb{R}^n)). \)

Proof of part 1: By a contraction argument the local existence and uniqueness proof is reduced to suitable multilinear estimates for the right hand sides of \([15] , [13] , [17] \). For \([13] \), e.g., we make use of the following well-known estimate for a solution of the linear equation \((-i\partial_t \mp \Lambda) A^{q\pm}_{\pm} = G\), namely
\[ ||A^{q\pm}_{\pm}||_{X^{s,b}_{\pm}[0, T]} \lesssim ||A^{q\pm}_{\pm}(0)||_{H^s} + T^{b'-b}[G]_{X^{s,b'}_{\pm}[0, T]}, \]
which holds for \( k \in \mathbb{R} , \frac{1}{2} < b < b' < 1 \) and \( 0 < T \leq 1 \).

Thus the local existence and uniqueness for large data (in which case we have to choose \( b < b' \) \), in the regularity class
\[ \phi_{\pm} \in X^s_{\pm} \oplus [0, T], \ A^{q\pm}_{\pm} \in X^{r,q}_{\pm} \oplus [0, T], \ A^{c\pm} \in X^1_{\tau=0} \oplus [0, T], \]
can be reduced to the following estimates, if we take the assumption \( a^{c\pm} = 0 \) into account (remark that we do not want to assume \( a^{c\pm} \in H^l \) later):
\[ ||D^{-1}(\phi_0\phi_1)||_{X^{s,b}_{\pm}} \lesssim ||\phi_1||_{H^{s+b}}, \]
\[ ||D^{-1}Q_{ij}(\phi_1, \phi_2)||_{H^{s+b-j+r-1+2s}} \lesssim ||\phi_1||_{H^{s+b}}, \]
\[ ||\nabla A\phi||_{H^{s+b-j+r+1+2s}} \lesssim ||\phi||_{X^{s+1}_{\tau=0}}, \]
\[ ||A^1 A^2||_{H^{s+b-j+r+1+2s}} \lesssim \min(||A||_{H^{s+b-j+r+1+2s}}, ||A||_{X^{s+1}_{\tau=0}}) \prod_{i=1}^2 ||\phi||_{H^{s+b}}. \]

Proof of (31): We use \([26]\) and reduce the claim to the following estimates:
\[ ||uu||_{H^{s+b-j+2r}, b} \lesssim ||u||_{H^{s+b-j+2r}, \infty} ||v||_{H^{s+b-j+2r}, \infty}, \]
\[ ||uu||_{H^{s+b-j+2r}, b} \lesssim ||u||_{H^{s+b-j+2r}, \infty} ||v||_{H^{s+b-j+2r}, \infty}, \]
\[ ||uu||_{H^{s+b-j+2r}, b} \lesssim ||u||_{H^{s+b-j+2r}, \infty} ||v||_{H^{s+b-j+2r}, \infty}, \]
\[ ||uu||_{H^{s+b-j+2r}, b} \lesssim ||u||_{H^{s+b-j+2r}, \infty} ||v||_{H^{s+b-j+2r}, \infty}. \]
where \( b = \frac{r}{2} - r + \epsilon \).

The estimate (35) follows from Prop. 2.21 and Prop. 2.11 where we remark
that \( b > \frac{1}{2} \), because \( r < \frac{n}{2} - \frac{1}{2} \) by assumption. Here the parameters are given by
\( s_0 = \frac{1}{2} - s + 2\epsilon, \quad b_0 = 0, \quad s_1 = s - \frac{1}{2} - 2\epsilon, \quad b_1 = \frac{r}{2} + \epsilon, \quad s_2 = r + \frac{1}{2} - 2\epsilon, \quad b_2 = b \),
so that \( s_0 + s_1 + s_2 = r + \frac{1}{2} - 2\epsilon > \frac{n-1}{2} \) and \( s_0 + s_1 + s_2 + s_1 + s_2 > \frac{n}{2} \) under our
assumption \( r > \frac{n}{2} - 1 \).

(36) is by duality equivalent to

\[
\|uw\|_{L^\frac{1}{2} - \frac{r}{2} + 2\epsilon, 0} \lesssim \|v\|_{H^{r + \frac{1}{2} - 2\epsilon, 0}} \|w\|_{H^{\frac{r}{2} - \frac{r}{2} + 2\epsilon}}.
\]

We use Prop. 2.11 and Cor. 2.11 with parameters \( s_0 = \frac{1}{2} - s + 2\epsilon, \quad b_0 = 0, \quad s_1 = r + \frac{1}{2} - 2\epsilon, \quad b_1 = b, \quad s_2 = \frac{1}{2} - s + 2\epsilon, \quad b_2 = \frac{1}{2} - 2\epsilon \), so that \( s_0 + s_1 + s_2 = r + \frac{1}{2} - 2\epsilon > \frac{n-1}{2} \)
and \( s_0 + s_1 + s_2 + s_1 + s_2 = 2r - s - 2\epsilon + \frac{1}{2} > \frac{n}{2} \) under our assumption \( 2r - s > \frac{n}{2} - \frac{3}{2} \).

The estimate (37) is equivalent to

\[
\int \frac{\widetilde{u}_1(\xi_1, \tau_1)}{(\xi_1)^{r + \frac{1}{2} - 2\epsilon} (\tau_1 - |\xi_1|)^{\frac{r}{2} + \epsilon}} \frac{\widetilde{u}_2(\xi_2, \tau_2)}{(\xi_2)^{r + \frac{1}{2} - 2\epsilon} (\tau_2 - |\xi_2|)^{b - \frac{r}{2} + 2\epsilon}} \frac{\widetilde{u}_3(\xi_3, \tau_3)}{(\tau_3 - |\xi_3|)^{\frac{1}{2} - 2\epsilon}} d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L^s_t L^r_x}.
\]

The Fourier transforms are nonnegative without loss of generality. Here * denotes
integration over \( \sum_{i=1}^3 \xi_i = 0, \sum_{i=1}^3 \tau_i = 0 \) and \( d\xi d\tau = d\xi_1 d\xi_2 d\xi_3 d\tau \).

Case 1: \( |\xi_1| \leq |\xi_1| \). The left hand side of (38) is estimated by

\[
\int \frac{\widetilde{u}_1(\xi_1, \tau_1)}{(|\tau_1| - |\xi_1|)^{\frac{r}{2} + \epsilon}} \frac{\widetilde{u}_2(\xi_2, \tau_2)}{(\xi_2)^{r + \frac{1}{2} - 2\epsilon} (\tau_2 - |\xi_2|)^{b - \frac{r}{2} + 2\epsilon}} \frac{\widetilde{u}_3(\xi_3, \tau_3)}{(\tau_3 - |\xi_3|)^{\frac{1}{2} - 2\epsilon}} d\xi d\tau
\]

\[
\lesssim \|F^{-1}(\widetilde{u}_1 / (|\tau_1| - |\xi_1|)^{\frac{r}{2} + \epsilon})\|_{L^s_t L^r_x} \|F^{-1}(\xi_2)^{r + \frac{1}{2} - 2\epsilon} (\tau_2 - |\xi_2|)^{b - \frac{r}{2} + 2\epsilon})\|_{L^s_t L^r_x}
\]

\[
\lesssim \|F^{-1}(\xi_2)^{r + \frac{1}{2} - 2\epsilon} (\tau_2 - |\xi_2|)^{b - \frac{r}{2} + 2\epsilon})\|_{L^s_t L^r_x} \lesssim \prod_{i=1}^3 \|u_i\|_{L^s_t L^r_x}.
\]

For the second factor we interpolate the Strichartz estimate

\[
\|u\|_{L^s_t L^r_x} \lesssim \|u\|_{H^{\frac{r}{2} - \frac{r}{2} + \frac{1}{2}}}
\]

and the Sobolev estimate

\[
\|u\|_{L^s_t L^r_x} \lesssim \|u\|_{H^{\frac{r}{2} - \frac{r}{2} + \frac{1}{2}}},
\]

which gives

\[
\|u\|_{L^s_t L^r_x} \lesssim \|u\|_{H^{r + \frac{1}{2} - 3\epsilon + n - 2\epsilon}}
\]

using our assumption \( \frac{n}{2} - 1 < r < \frac{n-1}{2} \). This implies immediately by Sobolev

\[
\|u\|_{L^s_t L^r_x} \lesssim \|u\|_{H^s_t H^{r,-2\epsilon} x} \lesssim \|u\|_{H^{r + \frac{1}{2} - 3\epsilon + n - 2\epsilon}}
\]

by our choice \( b = \frac{r}{2} - r + \epsilon \), as desired.

Case 2: \( |\xi_1| \gg |\xi_1| \), thus \( |\xi_2| \sim |\xi_1| \gg |\xi_1| \). The left hand side of (38) is estimated
by
\[
\int \frac{\tilde{u}_1(\xi_1, \tau_1)}{(\xi_1)^{\frac{3}{2}} - r + \varepsilon + (|\tau_1| - |\xi_1|)^{\frac{3}{2}} + 2(\xi_2 - |\tau_2| - |\xi_2|)^{\frac{3}{2}}}
\frac{\tilde{u}_2(\xi_2, \tau_2)}{(\xi_2)^{\frac{3}{2}} - r + \varepsilon + (|\tau_2| - |\xi_2|)^{\frac{3}{2}} + 2(\xi_3 - |\tau_3| - |\xi_3|)^{\frac{3}{2}}}
\frac{\tilde{u}_3(\xi_3, \tau_3)}{(\xi_3)^{\frac{3}{2}} - r + \varepsilon}
\frac{d\xi_1 d\tau_1}{d\xi_2 d\tau_2} \frac{d\xi_3 d\tau_3}{d\xi_4 d\tau_4} \frac{d\xi_5 d\tau_5}{d\xi_6 d\tau_6}
\]
\[
\leq \int \frac{\tilde{u}_1(\xi_1, \tau_1)}{(\xi_1)^{\frac{3}{2}} - r + \varepsilon + (|\tau_1| - |\xi_1|)^{\frac{3}{2}} + 2(\xi_2 - |\tau_2| - |\xi_2|)^{\frac{3}{2}}}
\frac{\tilde{u}_2(\xi_2, \tau_2)}{(\xi_2)^{\frac{3}{2}} - r + \varepsilon + (|\tau_2| - |\xi_2|)^{\frac{3}{2}} + 2(\xi_3 - |\tau_3| - |\xi_3|)^{\frac{3}{2}}}
\frac{\tilde{u}_3(\xi_3, \tau_3)}{(\xi_3)^{\frac{3}{2}} - r + \varepsilon}
\frac{d\xi_1 d\tau_1}{d\xi_2 d\tau_2} \frac{d\xi_3 d\tau_3}{d\xi_4 d\tau_4} \frac{d\xi_5 d\tau_5}{d\xi_6 d\tau_6}
\]
\[
\lesssim \|F^{-1}(\frac{\tilde{u}_1}{(\xi_1)^{\frac{3}{2}} - r + \varepsilon + (|\tau_1| - |\xi_1|)^{\frac{3}{2}} + 2(\xi_2 - |\tau_2| - |\xi_2|)^{\frac{3}{2}}} \|_{L_t^2 L_x^\infty} \|F^{-1}(\frac{\tilde{u}_2}{(\xi_2)^{\frac{3}{2}} - r + \varepsilon + (|\tau_2| - |\xi_2|)^{\frac{3}{2}} + 2(\xi_3 - |\tau_3| - |\xi_3|)^{\frac{3}{2}}} \|_{L_t^2 L_x^\infty} \leq \prod_{i=1}^3 \|u_i\|_{L_t^2 L_x^\infty}
\]
where we used Strichartz estimate and Sobolev for the first factor similarly as in Case 1 under the condition that $\frac{n}{2} - 1 < r < \frac{n-1}{2}$, and recalling our choice of $b$. This completes the proof of (31).

**Proof of (30):** We control $Q_{ij}(u, v)$ by (27) which reduces (30) by symmetry to the following estimates:
\[
\|uv\|_{H^{0, -1, 0}} \lesssim \|u\|_{H^{0, -1, 0}} \|v\|_{H^{0, -1, 0}}, \quad (39)
\]
\[
\|uv\|_{H^{\frac{n}{2} - 1, 0}} \lesssim \|u\|_{H^{\frac{n}{2} - 1, 0}} \|v\|_{H^{\frac{n}{2} - 1, 0}}, \quad (40)
\]
where $b = \frac{n}{2} - r + \varepsilon$. If we use (20) instead we may replace (39) by
\[
\|uv\|_{H^{\frac{n}{2} - 1, 0}} \lesssim \|u\|_{H^{\frac{n}{2} - 1, 0}} \|v\|_{H^{\frac{n}{2} - 1, 0}}, \quad (41)
\]
If $n = 3$ we prove (41) by use of Prop. 2.1 (which holds in this case) with parameters $s_0 = \frac{n}{2} - r$, $b_0 = \frac{n}{2} - b$, $s_1 = s_2 = s = \frac{n}{2}$, $b_1 = b_2 = \frac{n}{2} + \varepsilon$. The conditions of Prop. 2.1 are satisfied, because $s_0 + s_1 + s_2 = 2s - r + \frac{n}{2} > n - 1 - r + \varepsilon = \frac{n-1}{2} - b_0$ for $s > \frac{n}{2} - \frac{n}{2}$. Moreover $s_0 + s_1 + s_2 > \frac{n}{2} - \frac{n}{2} \geq \frac{n+1}{2}$ for $n \geq 3$ and $r < \frac{n-1}{2}$, and also $s_0 + s_1 + s_2 + s_3 + b_0 > n - 1 - r + 2s - 1 + \frac{n}{2} + \frac{n}{2} + r > \frac{n+1}{2} n - 3 \geq \frac{n}{2}$ for $s > \frac{n}{2} - \frac{n}{2}$ and $n \geq 3$. Furthermore $s_0 + s_1 + s_2 + s_3 + b_0 = 3s - 2r + 2 + \varepsilon > \frac{n}{2}$ under our assumptions $s > \frac{n}{2} - \frac{n}{2}$ and $r < \frac{n-1}{2}$, and finally $s_0 + s_1 + s_2 > -b_0 \Leftrightarrow 2s + r > \frac{n-1}{2}$, which holds for $n \geq 3$.

If $n \geq 4$ we now prove (40) by use of Prop. 2.2 with parameters $s_0 = b + 1 + r + s_1 = s_2 = s - b - s$ and so $s_0 + s_1 + s_2 = 2s - r + 1 - b - s = 2s + b - \varepsilon > \frac{n+1}{2}$ under our assumption $s > \frac{n}{2} - \frac{n}{2}$. Moreover $s_0 + s_2 = 2s - 2b - s = 2s - n - 2r - 2 + \varepsilon > \frac{n}{2}$ under our assumptions on $s$ and $r$, so that $n \geq 4$, so that $s_0 + s_2 + s_1 + s_2 > \frac{n}{2}$.

It remains to prove (40). First we consider the case $r = \frac{n}{2} - 1 - \varepsilon$, so that $b + 1 + 0 = 0$ and $2s - 1 + \frac{n}{2} > \frac{n}{2}$ for $s > \frac{n}{2} - \frac{n}{2}$. The estimate follows immediately by the Sobolev multiplication law (21). Next let $r = \frac{n}{2} - 1 - \varepsilon$, so that we have to prove
\[
\|uv\|_{H^{\frac{n}{2} - 2, 0}} \lesssim \|u\|_{H^{\frac{n}{2} - 2, 0}} \|v\|_{H^{\frac{n}{2} - 2, 0}}.
\]
This follows from Cor. 2.3 in the case $n \geq 4$ and Prop. 2.1 in the case $n = 3$ with parameters $s_0 = s - \frac{1}{2}, s_1 = s - \frac{1}{2}, s_2 = 2 - \frac{n}{2}$, so that $s_0 + s_1 + s_2 = 2s + 1 - \frac{n}{2} > \frac{n-1}{2}$ and $s_1 + s_2 > \frac{n}{2}$. The general case $\frac{n}{2} - 1 + \varepsilon < r < \frac{n-1}{2}$ follow by interpolation of these two cases, as one easily checks.

**Proof of (29):** We first remark that the singularity of $D^{-1}$ is harmless in $n \geq 3$ dimensions (19), Cor. 8.2) and it can be replaced by $A^{-1}$. As a first step we use Sobolev’s multiplication law (21) and obtain
\[
|\int \int u_1 u_2 u_3 dx dt| \lesssim \|u_1\|_{X^{\frac{1}{2}, 0}_{r=0}} \|u_2\|_{X^{\frac{1}{2}, 0}_{r=0}} \|u_3\|_{X^{\frac{1}{2}, 0}_{r=0}}.
\]
provided that \( t < 2s - \frac{n}{p} + 1 \) and \( s \geq l - 1 \), which is fulfilled under our assumptions. This implies taking the time derivative into account

\[ \|\Lambda^{-1}(\phi_t \phi_0)\|_{X^{s+3}_{t=0}} \lesssim \|\phi_0\|_{X^{s+1}_{t=0}} \|\phi_2\|_{X^{s+4}_{t=0}}. \]  

(42)

In a second step we want to prove

\[ \|\Lambda^{-1}(\phi_t \phi_t)\|_{X^{s+3}_{t=0}} + \|\Lambda^{-1}(\phi_t \phi_t)\|_{X^{s+3}_{t=0}} \lesssim \|\phi_0\|_{X^{s+1}_{t=0}} \|\phi_2\|_{X^{s+4}_{t=0}}. \]  

(43)

If \( \phi_t(\xi, \tau) \) is supported in \( |\tau_3| - |\xi_3| \gtrsim |\xi_3| \) we have the trivial bound

\[ \|\phi_1\|_{X^{s+3}_{t=0}} \lesssim \|\phi_1\|_{X^{s+1}_{t=0}}, \]  

(44)

so that (43) follows from (42). Assuming from now on \( |\tau_3| - |\xi_3| \ll |\xi_3| \) we have to prove

\[ \int m(\xi_1, \xi_2, \xi_3, \tau_1, \tau_2, \tau_3) \prod_i \hat{u}_i(\xi_i, \tau_i) d\xi d\tau \lesssim \prod_i \|u_i\|_{L^2_x}. \]  

(45)

where

\[ m = \frac{\langle \xi_1 \rangle \langle |\tau_1| \rangle \chi_{|\tau_3| - |\xi_3| \subseteq |\xi_3|}}{\langle \xi_1 \rangle \langle |\tau_1| \rangle \chi_{|\tau_3| - |\xi_3| \subseteq |\xi_3|}}. \]  

Since \( \langle \tau_3 \rangle \sim \langle \xi_3 \rangle \) and \( \tau_1 + \tau_2 + \tau_3 = 0 \) we have

\[ |\tau_2| + |\tau_3| \lesssim \langle \tau_1 \rangle^{\frac{1}{2} - \epsilon} (\tau_2)\frac{1}{2} + \epsilon (\xi_3)^{\frac{1}{2} + \epsilon} + \langle \tau_2 \rangle^{\frac{1}{2} + \epsilon} (\xi_3)^{\frac{1}{2} - \epsilon}. \]  

(46)

For the first term on the r.h.s. we have to show

\[ \left| \int \int uvw dxdz \right| \lesssim \|u\|_{X^{s+1}_{t=0}} \|v\|_{X^{s+1}_{t=0}} \|w\|_{X^{s+1}_{t=0}}, \]  

which follows from Sobolev’s multiplication law (4.1). For the other terms we use \( l \geq 1 \) so that \( \langle \xi_1 \rangle^{l-1} \lesssim \langle \xi_2 \rangle^{l-1} + \langle \xi_3 \rangle^{l-1} \) and the second term on the r.h.s. reduces to the following estimates:

\[ \left| \int \int uvw dxdz \right| \lesssim \|u\|_{X^{s+1}_{t=0}} \|v\|_{X^{s+1}_{t=0}} \|w\|_{X^{s+1}_{t=0}} \right] \]  

(47)

\[ \left| \int \int uvw dxdz \right| \lesssim \|u\|_{X^{s+1}_{t=0}} \|v\|_{X^{s+1}_{t=0}} \|w\|_{X^{s+1}_{t=0}} \right] \]  

(48)

First we prove (47). We obtain

\[ \left| \int \int uvw dxdz \right| \lesssim \|u\|_{L^p_x} \|v\|_{L^q_x} \|w\|_{L^r_x} \lesssim \|u\|_{L^q_x} \right] \]  

(49)

where \( \frac{1}{p} = \frac{m}{2(m+1)} - \frac{m}{n+1} \), where \( m = s - \frac{1}{2} - \frac{n-1}{2(n+1)} - \epsilon \) and \( \frac{1}{q} = \frac{1}{2} - \frac{1}{p} \), so that by Sobolev \( H^\epsilon x \frac{m}{2(m+1)} \lesssim L^p_x \). This implies by Prop. 2.3

\[ \|w\|_{L^q_x} \lesssim \|u\|_{H^\epsilon x \frac{m}{2(m+1)} L^p_x} \lesssim \|u\|_{H^\epsilon x \frac{m}{2(m+1)} \frac{1}{2} + \epsilon} = \|w\|_{H^\epsilon x \frac{1}{2} + \epsilon}. \]  

Moreover an easy calculation shows that \( \frac{1}{q} \geq \frac{1}{2} - \frac{n-1}{n+1} \Leftrightarrow \leq 2s - \frac{n}{2} + 1 - \epsilon \), which holds by assumption, so that by Sobolev \( H^\epsilon x \frac{m}{2(m+1)} \lesssim L^q_x \) and thus

\[ \|v\|_{L^q_x} \lesssim \|v\|_{X^{s+1}_{t=0}}. \]  

This implies (47).

Next we prove (48). We apply (46) with the choice \( \frac{1}{p} = \frac{1}{2} - \frac{1}{n} \) and \( \frac{1}{q} = \frac{2}{n} \). This implies by Sobolev \( H^\epsilon x \frac{2(n+1)}{n} \lesssim L^q_x \), if \( k = \frac{n(n-1)}{2(n+1)} - s \). One easily checks that
\[
k + \frac{n-1}{2(n+1)} \leq s - l + \frac{1}{2} - \epsilon \iff l \leq 2s - \frac{n+1}{2} + 1 - \epsilon
\]

, which we assumed. Consequently, by Prop. 2.4 we obtain
\[
\|w\|_{L^2 L^2_t} \lesssim \|w\|_{H^\epsilon} \times \frac{2(n+1)}{n+1} \lesssim \|w\|_{H^{\epsilon+\frac{n-1}{2(n+1)}}} \lesssim \|w\|_{H^{\epsilon+\frac{1}{2}}} \lesssim \|w\|_{H^{\epsilon+\frac{1}{2}}}.
\]

and by Sobolev \(\|v\|_{L^p L^q_t} \lesssim \|v\|_{X^{s, q}_{r=0}}\), so that (33) is proven.

For the third term on the r.h.s. of (10) we have to show
\[
\left| \int \int uvwdxdt \right| \lesssim \|u\|_{L^3 L^6_t} \|v\|_{X^{s, q}_{r=0}} \|w\|_{H^{\epsilon+\frac{1}{2}}},
\]

\[
\left| \int \int uvwdxdt \right| \lesssim \|u\|_{L^3 L^6_t} \|v\|_{X^{s, q}_{r=0}} \|w\|_{H^{\epsilon+\frac{1}{2}}},
\]

The first estimate follows from
\[
\left| \int \int uvwdxdt \right| \lesssim \|u\|_{L^3 L^6_t} \|v\|_{X^{s, q}_{r=0}} \|w\|_{L^3_t L^6_t}.
\]

exactly as for the proof of (17) with (up to an \(\epsilon\)) the same parameters. The second estimate follows similarly by choosing \(p\) and \(q\) (up to an \(\epsilon\)) as for the proof of (18).

We now come to the proof of (20) and remark that we may assume now that both functions \(\phi_1\) and \(\phi_2\) are supported in \(|\tau| - |\xi| \ll |\xi|\), because otherwise (20) is an immediate consequence of (13) and (11). Thus (20) follows if we can prove the following estimate:
\[
\int m(\xi_1, \xi_2, \xi_3, \tau_1, \tau_2, \tau_3) \prod_{i=1}^3 \mathcal{G}_i(\xi_1, \tau_1) d\xi d\tau \lesssim \|u\|_{L^3_t L^3_t},
\]

where
\[
m = \frac{|\tau_1|^{1/2} - |\xi_2| |\xi_3|}{|\xi_1|^{1/2} - |\xi_2| |\xi_3|}.\]

Since \(\langle \tau_3 \rangle \sim \langle \xi_3 \rangle\), \(\langle \tau_2 \rangle \sim \langle \xi_2 \rangle\) and \(\tau_1 + \tau_2 + \tau_3 = 0\) we obtain
\[
|\tau_3| \lesssim \langle \xi_1 \rangle^{1/2} - \langle \xi_2 \rangle^{1/2} + \langle \xi_3 \rangle^{1/2}.
\]

The first term is taken care of by the estimate
\[
\int \int uvwdxdt \lesssim \|u\|_{X^{s, q}_{r=0}} \|v\|_{H^{\epsilon+\frac{1}{2}}} \|w\|_{H^{\epsilon+\frac{1}{2}}},
\]

which is equivalent to
\[
\|vw\|_{H^{\epsilon+\frac{1}{2}}} \lesssim \|v\|_{H^{\epsilon+\frac{1}{2}}} \|w\|_{H^{\epsilon+\frac{1}{2}}},\]

This is true by Prop. 2.2 and Prop. 2.1 with the parameters \(s_0 = 1 - l\), \(s_1 = 1\), \(s_2 = 1 - l\), which are such that \(s_0 + s_1 + s_2 = 2s - l + \frac{1}{2} - \epsilon > \frac{n+1}{2} \leq \frac{n+2}{2}\) under our assumption \(l < 2s - \frac{n+1}{2},\) and also \(s_1 + s_2 > \frac{n}{2}\).

In order to treat the second term on the right hand side we assume w.l.o.g. \(|\xi_3| \geq |\xi_3|\), so that \(\langle \xi_3 \rangle^{-1} \lesssim \langle \xi_3 \rangle^{-1}\), so that it suffices to show:
\[
\int \int uvwdxdt \lesssim \|u\|_{X^{s, q}_{r=0}} \|v\|_{H^{\epsilon+\frac{1}{2}}} \|w\|_{H^{\epsilon+\frac{1}{2}}},
\]

This is shown as follows:
\[
\int \int uvwdxdt \lesssim \|u\|_{L^3_t L^3_t} \|v\|_{L^3_t L^3_t} \|w\|_{L^3_t L^3_t},
\]

We choose \(q_1\) such that \(\frac{1}{q_1} \geq \frac{n-1}{2(n+1)} - \frac{1}{2n}\) with \(k_1 + \frac{n-1}{2(n+1)} = s - l + \frac{1}{2}\), which is equivalent to \(\frac{1}{q_1} \geq \frac{n-1}{2n} - \frac{1}{2n-2}\). This implies \(H^{k_1} \subseteq L^{q_1}\), so that by Prop. 2.1 we obtain
\[
\|v\|_{L^3_t L^3_t} \lesssim \|v\|_{H^{k_1} \times \frac{2(n+1)}{n+1} \lesssim \|v\|_{H^{\epsilon+\frac{1}{2}}} \|w\|_{H^{\epsilon+\frac{1}{2}}}.\]
Moreover we want to choose $q_2$ such that \( \frac{1}{q_2} \geq \frac{n-1}{2(n+1)} - \frac{k_2}{n} \) with \( k_2 + \frac{n-1}{2(n+1)} < \frac{s}{2} \), which means that \( \frac{1}{q_2} > \frac{n-1}{2m} - \frac{k_2}{n} \), thus as before
\[
\|v\|_{L_{q_2}^1 L_2^\infty} \lesssim \|v\|_{H_\tau^{q_2, \frac{m+1}{m} + \frac{k_2}{n}}} \|v\|_{H^{-\frac{1}{2}, \frac{1}{2}}}.
\]
This choice of the parameters $q_1$ and $q_2$ is possible, if \( \frac{1}{q_2} = \frac{1}{m} + \frac{1}{q_2} > \frac{n-1}{2m} - \frac{k_2}{n} \), which is equivalent to our assumption \( l < 2s - \frac{n}{2} + 1 \).

This completes the proof of \((32)\).

**Proof of \((32)\):** This proof is similar to a related estimate for the Yang-Mills equation given by Tao \cite{20}. We have to show
\[
\int_\mathbb{R} m(\xi_1, \xi_2, \xi_3, \tau_1, \tau_2, \tau_3) \prod_{i=1}^3 \tilde{a}_i(\xi_i, \tau_i) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L_{q_1}^3},
\]
where
\[
m = \frac{(|\xi_2| + |\xi_3|)|\xi_1|^{s-1}}{(|\tau_1| - |\xi_1|)^{1/2-\varepsilon} (|\tau_2| - |\xi_2|)^{1/2+\varepsilon} (|\tau_3| - |\xi_3|)^{1/2+\varepsilon}}.
\]

Case 1: \(|\xi_2| \leq |\xi_1| \Rightarrow |\xi_2| + |\xi_3| \leq |\xi_1|\).

We ignore the factor \((|\tau_1| - |\xi_1|)^{1/2-\varepsilon}\) and use the averaging principle \((19), \text{Prop. } 5.1\) to replace $m$ by
\[
m' = \frac{(|\xi_2| + |\xi_3|)|\xi_1|^{s-1}}{(|\xi_2|)^s (|\xi_3|)^s}.
\]

Let now $\tau_2$ be restricted to the region $\tau_2 = T + O(1)$ for some integer $T$. Then $\tau_1$ is restricted to $\tau_1 = -T + O(1)$, because $\tau_1 + \tau_2 + \tau_3 = 0$, and $\xi_2$ is restricted to $|\xi_2| = |\tau_1 + \tau_2| + O(1)$. The $\tau_1$-regions are essentially disjoint for $T \in \mathbb{Z}$ and similarly the $\tau_2$-regions. Thus by Schur’s test \((19), \text{Lemma } 3.11\) we only have to show
\[
\sup_{T \in \mathbb{Z}^+} \int_{\mathbb{R}} \frac{(|\xi_2| + |\xi_3|)|\xi_1|^{s-1}}{(|\tau_1| - |\xi_1|)^{1/2-\varepsilon} (|\tau_2| - |\xi_2|)^{1/2+\varepsilon} (|\tau_3| - |\xi_3|)^{1/2+\varepsilon}} \prod_{i=1}^3 \tilde{a}_i(\xi_i, \tau_i) d\xi d\tau 
\lesssim \prod_{i=1}^3 \|u_i\|_{L_{q_1}^3}.
\]

The $\tau$-behaviour of the integral is now trivial, thus we reduce to
\[
\sup_{T \in \mathbb{N}} \int \frac{(|\xi_2| + |\xi_3|)|\xi_1|^{s-1}}{(T)^s (|\xi_2|)^s (|\xi_3|)^s} \tilde{f}_1(\xi_1) \tilde{f}_2(\xi_2) \tilde{f}_3(\xi_3) d\xi \lesssim \prod_{i=1}^3 \|f_i\|_{L_2^3}. \quad (50)
\]

It only remains to consider the following two cases:

Case 1.1: \(|\xi_1| \sim |\xi_3| \geq T\). We have to show
\[
\sup_{T \in \mathbb{N}} \int \frac{|\xi_2| + |\xi_3| + O(1)}{T^{s+1}} \tilde{f}_1(\xi_1) \tilde{f}_2(\xi_2) \tilde{f}_3(\xi_3) d\xi \lesssim \prod_{i=1}^3 \|f_i\|_{L_2^3}.
\]

The l.h.s. is bounded by
\[
\sup_{T \in \mathbb{N}} \frac{1}{T} \|f_1\|_{L_2^3} \|f_3\|_{L_2^3} \|F^{-1}(\chi_{|\xi_1| = T + O(1)} \tilde{f}_2)\|_{L_x^\infty(\mathbb{R}^3)} 
\lesssim \sup_{T \in \mathbb{N}} \frac{1}{T} \|f_1\|_{L_2^3} \|f_3\|_{L_2^3} \|\chi_{|\xi_1| = T + O(1)} \tilde{f}_2\|_{L_1^1(\mathbb{R}^3)}
\lesssim \sup_{T \in \mathbb{N}} \frac{T^{s+1}}{T} \prod_{i=1}^3 \|f_i\|_{L_2^3} \lesssim \prod_{i=1}^3 \|f_i\|_{L_2^3}.
\]
An elementary calculation shows that the l.h.s. is bounded by

\[ \sup_{T \in \mathbb{N}} \int_{\xi = 0}^{\xi_1} \frac{\chi(\xi + T + O(1))}{(\xi_1)^{\frac{1}{2}}} f_1(\xi_1) f_2(\xi_2) f_3(\xi_3) d\xi \leq \prod_{i=1}^{3} ||f_i||_{L^2}. \]

An elementary calculation shows that the l.h.s. is bounded by

\[ \sup_{T \in \mathbb{N}} ||\chi(\xi + T + O(1))||_{L^{\infty}(\mathbb{R}^3)} \prod_{i=1}^{3} ||f_i||_{L^2} \leq \prod_{i=1}^{3} ||f_i||_{L^2}, \]

using that \( l > \frac{n-1}{2} \).

The proof of (32) is complete.

**Proof of (33):** We estimate by Sobolev’s multiplication law (21), Prop. 2.1 and Prop. 2.2, using \( s > \frac{n}{2} - \frac{1}{4} \):

\[ \|A \phi_1 \phi_2\|_{H^{s-1}} \|H^r\|_{L^2} \|H^s\|_{L^2} \leq \|A \phi_1 \phi_2\|_{L^2} \|H^{s-1}\|_{L^2} \|H^s\|_{L^2} \]

\[ \leq \|A\|_{H^{\frac{1}{2}+\epsilon}} \|\phi_1\|_{H^{\frac{s}{2}+\epsilon}} \|\phi_2\|_{H^{s+\epsilon}}. \]

Similarly we also obtain for \( r \leq \frac{n}{2} - \frac{1}{4} \) and \( s > \frac{n}{2} - \frac{3}{4} \):

\[ \|A \phi_1 \phi_2\|_{L^2} \|H^{s-1}\|_{L^2} \leq \|A\|_{L^2} \|\phi_1\|_{H^{s-1}} \|\phi_2\|_{H^{s+\epsilon}}. \]

**Proof of (34):** By Sobolev’s multiplication rule (21) and \( l \geq \frac{n-1}{2} \) we obtain

\[ \|A_1 A_2 \phi\|_{H^{s-1}} \leq \|A_1 A_2\|_{L^2} \|H^r\|_{L^2} \|H^s\|_{L^2} \]

\[ \leq \|A_1\|_{L^2} \|H^r\|_{L^2} \|A_2\|_{L^2} \|H^s\|_{L^2} \]

\[ \leq \|A_1\|_{H^{s-1}} \|A_2\|_{H^{r-1}} \|\phi\|_{H^r}. \]

Next for \( \frac{n}{2} - \frac{1}{4} > r > \frac{n}{2} - 1 \) Prop. 2.1 or Prop. 2.2 imply:

\[ \|A_1 A_2 \phi\|_{H^{s-1}} \leq \|A_1 A_2\|_{H^{r-1}} \|\phi\|_{H^r}. \]

Finally we also obtain

\[ \|A_1 A_2\|_{H^{r-1}} \leq \|A_1 A_2\|_{L^2} \|H^r\|_{L^2} \|H^s\|_{L^2} \]

\[ \leq \|A_1\|_{H^{s-1}} \|A_2\|_{H^{r-1}} \|\phi\|_{H^r}. \]

by (21) under our assumptions \( l > \frac{n-1}{2} \) and \( r < \frac{n}{2} - \frac{1}{4} \).

This completes the proof of (34) and part 1 of Proposition 3.1.

Now we eliminate the assumption \( a_{0f} = 0 \).

**Proof of Theorem 3.1** We use Proposition 3.1 to construct a unique solution \((\phi', A')\) of the Cauchy problem for (10), (11), (12) with initial conditions \( \phi'(0) = e^{-i\chi} \phi_0 \), \( (\partial_t \phi)(0) = e^{-i\phi} \phi_1 \), \( A'(0) = a_{0f} \), \( (\partial_t A)(0) = a_1 \), where \( a_0 \in H^r \), \( a_1 \in H^{r-1} \), \( \phi_0 \in H^s \), \( \phi_1 \in H^{s-1} \) and the compatibility condition (20) is satisfied. Here \( \chi := -(-\Delta)^{-1} \text{div} a_0 \) is chosen such that \( \nabla \chi = a^2 f(0) \). The assumptions for the data in Prop. 3.1 are now shown to be satisfied. It is immediately clear
Lemma 3.1. Thus by (21) using \( r > 0 \) so that the regularity of \( \chi \) it certainly preserves the temporal gauge, because \( \psi \). This leads to a solution \((A, \varphi)\) of (10), (11), (12) with initial conditions \( A(0) = a_0^f + \nabla \chi = a_0^f + a_0^f = a_0 \), \( \partial_t A)(0) = a_1 \), \( \phi(0) = \phi_0 \), \( \partial_t \phi(0) = \phi_1 \). What remains to be shown is that the regularity of the solution is preserved. It is easy to see that \( A \) has the same regularity as \( A' \). Let now \( \psi \) be a smooth function with \( \psi(t) = 1 \) for \( 0 \leq t \leq T \) and \( \psi(t) = 0 \) for \( t \geq 2T \). By Lemma 3.1 below and (51) we obtain for \( s > \frac{n}{2} - \frac{3}{4} \) and \( r > \frac{n}{2} - 1 \):

\[
\|e^{i\chi} \psi'\|_{X^{s+\frac{1}{2}+[0,T]}} \leq \|\nabla(e^{i\chi})\psi\|_{X^{s+\frac{1}{2}+[0,T]}} \leq \|\nabla(e^{i\chi})\|_{H^s} \|\phi'\|_{X^{s+[0,T]}} \leq c_1^{-1} \exp(c_1 \|\nabla \chi\|_{H^s}) < \infty,
\]

so that the regularity of \( \psi \) is also preserved. \( \square \)

In the last proof we used the following lemma.

Lemma 3.1. The following estimate holds for \( r + 1 \geq s > \frac{n}{2} - \frac{3}{4} \), \( r > \frac{n}{2} - 1 \) and \( \epsilon > 0 \) sufficiently small:

\[
\|uv\|_{X^{s+\epsilon+[0,T]}} \leq \|\nabla u\|_{X^{s+\epsilon+[0,T]}} \|v\|_{X^{s+\epsilon+[0,T]}}.
\]

Proof. By Tao [19], Cor. 8.2 we may replace \( \nabla \) by \( A \) so that it suffices to prove

\[
\|uv\|_{X^{s+\epsilon+[0,T]}} \leq \|u\|_{X^{s+1+[0,T]}} \|v\|_{X^{s+[0,T]}}.
\]

We start with the elementary estimate

\[
|\tau_1 + \tau_2| + |\xi_1 + \xi_2| \leq |\tau_1| + |\xi_1| + |\tau_2| + |\xi_2| + 2|\xi_1| + |\xi_2| - |\xi_1 + \xi_2|.
\]

Assume now w.l.o.g. \( |\xi_2| \geq |\xi_1| \). We have

\[
|\xi_1| + |\xi_2| - |\xi_1 + \xi_2| \leq |\xi_1| + |\xi_2| + |\xi_1| - |\xi_2| = 2|\xi_1|,
\]

so that

\[
|\tau_1 + \tau_2| + |\xi_1 + \xi_2| \leq |\tau_1| + |\xi_1| + |\tau_2| + |\xi_2| + 2\min(|\xi_1|, |\xi_2|).
\]
Using Fourier transforms by standard arguments it thus suffices to show the following three estimates:

\[
\|uv\|_{X^{s,0}} \lesssim \|u\|_{X^{r+1,0}} \|v\|_{X^{s,0}} \\
\|uv\|_{X^{s,0}} \lesssim \|u\|_{X^{r+1,4}} \|v\|_{X^{s,0}} \\
\|uv\|_{X^{s,0}} \lesssim \|u\|_{X^{r+1,4}} \|v\|_{X^{s,0}}
\]

The first and second estimate easily follow from SML (21), whereas the last one is implied by Prop. 2.1 and Prop. 2.2 with the parameters

\[
s_0 = -s, \quad s_1 = r + \frac{1}{2} - \epsilon, \quad s_2 = s, \quad \text{so that } s_0 + s_1 + s_2 > \frac{n-5}{2} \text{ and } s_1 + s_2 = s + r + \frac{1}{2} - \epsilon > n - \frac{5}{2} > \frac{3}{2}.
\]

□

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