Twisted $sl(3, C)^{(2)}_k$ Current Algebra: Free Field Representation and Screening Currents

Xiang-Mao Ding $^a,b$ Mark D. Gould $^a$ and Yao-Zhong Zhang $^a$

$^a$ Center of Mathematical Physics, Department of Mathematics, University of Queensland, Brisbane, Qld 4072, Australia
$^b$ Institute of Applied Mathematics, Academy of Mathematics and System Sciences; Chinese Academy of Sciences, P.O.Box 2734, 100080, China.

Abstract

Motivated by application of twisted current algebra in description of the entropy of $AdS_3$ black hole, we investigate the simplest twisted current algebra $sl(3, C)^{(2)}_k$. Free field representation of the twisted algebra, and the corresponding twisted Sugawara energy-momentum tensor are obtained by using three $(\beta, \gamma)$ pairs and two scalar fields. Primary fields and two screening currents of the first kind are presented.

1 Introduction

Virasoro algebra and affine algebras are algebraic structures in conformal field theories (CFT) in two dimensional spacetime $[1, 2, 3]$. They also play a central role in the study of string theory $[4]$. The free field realization is a common approach used in both conformal field theories and representation theory of affine Lie algebras $[5]$. The free field representations for untwisted affine algebra have been extensively studied. The simplest case $sl(2, C)^{(1)}$ was first treated in $[5]$, and generalization to $sl(n, C)^{(1)}$ was given in $[6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]$. For twisted cases, however, little has been known. The recently study shows that twisted affine algebras are useful in the description of the entropy of $AdS_3$ black hole $[17]$. So it is desirable to further investigate the free field representations of twisted affine algebras. Let us remark that throughout this paper we are dealing with algebras over the complex field.
In this letter we consider the simplest twisted affine algebra $sl(3, C)^{(2)}$, and construct the free field representation for this algebra. Moreover we also give the screening currents and primary fields. We remark that our result is different from the one obtained in [18].

2 Notation: twisted $sl(3, C)^{(2)}$ affine currents

Let us start with some basic notations of twisted affine algebras [2]. Let $g$ be a simple finite-dimensional Lie algebra and $\sigma$ be an automorphism of $g$ satisfying $\sigma^r = 1$ for a positive integer $r$,

then $g$ can be decomposed into the form:

$$g = \bigoplus_{j \in \mathbb{Z}/r\mathbb{Z}} g_j,$$

(2.1)

where $g_j$ is the eigenspace of $\sigma$ with eigenvalue $e^{2j\pi i/r}$, and $[g_i, g_j] \subset g(i+j) \mod r$, then $r$ is called the order of the automorphism.

For our purpose, we consider here only the simplest twisted affine algebra $A_2^{(2)}$, the 2-order twisted affine algebra of $A_2 = sl(3, C)$. Let $\tilde{\alpha}_1 = \epsilon_1 - \epsilon_2$, $\tilde{\alpha}_2 = \epsilon_2 - \epsilon_3$ be the two simple roots of $sl(3, C)$ with normalization $\tilde{\alpha}_1^2 = \tilde{\alpha}_2^2 = 2$, $\tilde{\alpha}_1 \cdot \tilde{\alpha}_2 = -1$, and $\theta^0 = \tilde{\alpha}_1 + \tilde{\alpha}_2$. Set $\alpha_1 = \frac{1}{2}(\tilde{\alpha}_1 + \tilde{\alpha}_2)$ and $\alpha_2 = \frac{1}{2}(\tilde{\alpha}_1 - \tilde{\alpha}_2)$. Then we have $\alpha_1^2 = \frac{1}{2}$, $\alpha_2^2 = \frac{3}{2}$, and $\alpha_1 \cdot \alpha_2 = 0$. We can write

$$g = g_0 \oplus g_1$$

(2.2)

where $g_0$ is a fixed point subalgebra under the automorphism, while $g_1$ is a five dimensional representation of $g_0$, $g_0$ and $g_1$ satisfy $[g_i, g_j] \subset g(i+j) \mod 2$. Let $\tilde{e}_{ij}$ are the matrix with entry 1 at the $i$-th row and $j$-th column, and zero elsewhere. We choose the basis of $g_0$ as,

$$e = \sqrt{2}(\tilde{e}_{12} + \tilde{e}_{23}) = \sqrt{2}(\tilde{e}_1 + \tilde{e}_2); \quad f = \sqrt{2}(\tilde{e}_{21} + \tilde{e}_{32}) = \sqrt{2}(\tilde{f}_1 + \tilde{f}_2);$$

$$h = 2(\tilde{e}_{11} - \tilde{e}_{33}) = 2(\tilde{h}_1 + \tilde{h}_2).$$

(2.3)

The basis for $g_1$ is taken to be

$$\tilde{e} = \sqrt{2}(\tilde{e}_{12} - \tilde{e}_{23}) = \sqrt{2}\tilde{e}_1 - \tilde{e}_2, \quad \tilde{f} = \sqrt{2}(\tilde{e}_{21} - \tilde{e}_{32}) = \sqrt{2}(\tilde{f}_1 - \tilde{f}_2),$$

$$\tilde{E} = -2(\tilde{e}_1\tilde{e}_2 - \tilde{e}_2\tilde{e}_1) = -2\tilde{e}_{13}, \quad \tilde{F} = -2(\tilde{f}_1\tilde{f}_2 - \tilde{f}_2\tilde{f}_1) = -2\tilde{e}_{31},$$

$$\tilde{h} = 2(\tilde{e}_{11} - 2\tilde{e}_{22} + \tilde{e}_{33}) = 2(\tilde{h}_1 - \tilde{h}_2).$$

(2.4)

Then we have the following relations:

$$[h, e] = 2e; \quad [h, f] = -2f;$$
\[ [\tilde{e}, \tilde{F}] = 2f; \quad [\tilde{f}, \tilde{E}] = -2e; \]
\[ [\tilde{h}, \tilde{e}] = 6e; \quad [\tilde{h}, \tilde{f}] = -6f; \]
\[ [e, f] = h = [\tilde{e}, \tilde{f}]; \quad [\tilde{E}, \tilde{F}] = 2h; \]
\[ [e, \tilde{e}] = 2\tilde{E}; \quad [f, \tilde{f}] = -2\tilde{F}; \]
\[ [\tilde{f}, \tilde{E}] = 2\tilde{e}; \quad [e, \tilde{F}] = -2\tilde{e}; \]
\[ [\tilde{h}, e] = 6e; \quad [\tilde{h}, f] = -6f; \]
\[ [h, \tilde{e}] = 2\tilde{e}; \quad [h, \tilde{f}] = -2\tilde{f}; \]
\[ [h, \tilde{E}] = 4\tilde{E}; \quad [h, \tilde{F}] = -4\tilde{F}; \]
\[ [e, \tilde{f}] = \tilde{h} = -[f, \tilde{e}]. \]

All other commutators are zero. It is easy to verify that the following operator is the quadratic Casimir of \(sl(3, \mathbb{C})\) in the above basis

\[ C_2 = h^2 + \frac{1}{3} \tilde{h}^2 + 4fe + 4\tilde{f}\tilde{e} + 4\tilde{E}\tilde{F}. \]  

This quadratic Casimir element is useful in sequel to construct the twisted energy-momentum stress tensor.

The commutators of \(sl(3, \mathbb{C})^{(2)}_k\) can be expressed as

\[ [z^m \otimes X, z^n \otimes Y] = z^{m+n} \otimes [X, Y] + 2km\delta_{m+n,0} \frac{(X[Y])}{2}. \]  

Denote the currents corresponding to \(e, h, f\) by \(j^+(z)\), \(j^0(z)\), \(j^-(z)\), and to \(\tilde{e}, \tilde{h}, \tilde{f}, \tilde{E}, \tilde{F}\) by \(J^+(z)\), \(J^0(z)\), \(J^-(z)\), \(J^{++}(z)\), \(J^{--}(z)\), respectively. Then (2.7) can be written in terms of the following OPE’s:

\[ j^+(z)j^-(w) = \frac{4k}{(z-w)^2} + \frac{1}{(z-w)^3}j^0(w) + \ldots; \]
\[ j^0(z)j^\pm(w) = \frac{\pm 2}{(z-w)}j^\pm(w) + \ldots; \]
\[ j^0(z)j^0(w) = \frac{8k}{(z-w)^2} + \ldots; \]
\[ J^+(z)J^-(w) = \frac{4k}{(z-w)^2} + \frac{1}{(z-w)^3}j^0(w) + \ldots; \]
\[ J^{++}(z)J^{--}(w) = \frac{4k}{(z-w)^2} + \frac{2}{(z-w)^3}j^0(w) + \ldots; \]
\[ j^0(z)j^\pm(w) = \frac{\pm 6}{(z-w)}j^\pm(w) + \ldots; \]
\[ J^0(z)J^\pm(w) = \frac{\pm 6}{(z-w)}J^\pm(w) + \ldots; \]
\[ J^+(z)J^-(w) = \frac{2}{(z-w)^2}j^-(w) + \ldots; \]
\[ J^-(z)J^{++}(w) = \frac{-2}{(z-w)^2}j^+(w) + \ldots; \]
\[ j^+(z)J^+(w) = \frac{2}{(z-w)}J^{++}(w) + \ldots; \]
\[ j^-(z)J^-(w) = \frac{-2}{(z-w)}J^{--}(w) + \ldots; \]
\[ j^+(z)J^-(w) = \frac{1}{(z-w)}J^{0}(w) + \ldots; \]
\[ J^+(z)j^-(w) = \frac{1}{(z-w)}J^{0}(w) + \ldots; \]
\[ j^+(z)J^{--}(w) = \frac{-2}{(z-w)}J^{--}(w) + \ldots; \]
\[ j^-(z)J^{++}(w) = \frac{2}{(z-w)}J^{++}(w) + \ldots; \]
\[ j^0(z)J^\pm(w) = \frac{\pm 2}{(z-w)}J^\pm(w) + \ldots; \]
\[ j^0(z)J^{\pm\pm}(w) = \frac{\pm 4}{(z-w)}J^{\pm\pm}(w) + \ldots; \]
\[ J^0(z)J^0(w) = \frac{24k}{(z-w)^2} + \ldots. \]

All other OPE’s contain trivial regular terms only. Here and throughout ”…” stands for regular terms.

3 Wakimoto realization of the twisted affine currents

To obtain a free field realization of the twisted \( sl(3, \mathbb{C})^{(2)}_k \) currents, we first construct a Fock space of \( sl(3, \mathbb{C}) \) in the basis given in section 2. The Fock space is constructed by the repeated actions of \( f, \tilde{f}, \tilde{F} \) on the highest weight state \( v_\Lambda \), which is determined by

\[ e v_\Lambda = \bar{e} v_\Lambda = \bar{E} v_\Lambda = 0; \]
\[ h v_\Lambda = (\Lambda, \alpha_1) v_\Lambda, \quad \tilde{h} v_\Lambda = (\Lambda, \alpha_2) v_\Lambda. \]

(3.9)

Set \( |l, m, n> = f^l \tilde{F}^m \tilde{f}^n v_\Lambda \). We find

\[ f|l, m, n> = |l + 1, m, n>; \]
\[ \tilde{f}|l, m, n> = 2|l - 1, m + 1, n > +|l, m, n + 1>; \]
\[ \tilde{F}|l, m, n> = |l, m + 1, n>; \]
\[ h|l, m, n> = -[2l + 4m + 2n - (\Lambda, \alpha_1)] |l, m, n>; \]
\[ \tilde{h}|l, m, n> = -6l|l - 1, m, n + 1 > -6l(l - 1)|l - 2, m + 1, n > -6n|l + 1, m, n - 1 > -6n(n - 1)|l, m + 1, n - 2 > +(\Lambda, \alpha_2)|l, m, n> \]

(3.10)

For other generators, we obtain

\[ e|l, m, n> = -l [(l - 1) + 4m + 2n - (\Lambda, \alpha_1)] |l - 1, m, n> \]
we obtain a realization of the non-affine algebra in terms of the differential operators.

Now we use the above representation as a tool to construct, via the induced procedure [12], free field representations of the twisted $sl(3, \mathbb{C})^{(2)}_k$ current algebra. In the Fock space, if we regard $\gamma_0, \gamma_1, \gamma_2$ as $f, \tilde{f}, \tilde{E}$, and $\beta_0, \beta_1, \beta_2$ as $-\frac{\partial}{\partial f}, -\tilde{\partial}_f, -\frac{\partial}{\partial \tilde{E}}$, respectively, then we obtain a realization of the non-affine algebra in terms of the differential operators:

\[
\begin{align*}
\gamma &= -2m|l, m-1, n+1 > -3n(n-1)|l+1, m, n-2 > \\
&-4n(n-1)(n-2)|l, m+1, n-3 > \\
&+n(\Lambda, \alpha_2)|l, m, n-1 >;
\end{align*}
\]

\[
\tilde{\gamma}|l, m, n > = -n [6l + (n-1) - (\Lambda, \alpha_1)]|l, m, n-1 > +2m|l+1, m-1, n > -6ln(n-1)|l-1, m+1, n-2 > \\
&-2l(l-1)(l-2)|l-3, m+1, n > +l(\Lambda, \alpha_2)|l-1, m, n-1 > \\
&-3l(l-1)|l-2, m, n-1 >;
\]

\[
\tilde{E}|l, m, n > = -2m [2l + 2n + 2(m-1) - m(\Lambda, \alpha_1)]|l, m-1, n > +2ln [(n-1) + 3(l-1) - (\Lambda, \alpha_1)]|l-1, m, n-1 > \\
&+6l(l-1)n(n-1)|l-2, m+1, n-2 > +2l(l-1)(l-2)|l-3, m, n+1 > \\
&+l(l-1)(l-2)(l-3)|l-4, m+1, n > -2m(m-1)(m-2)|l+1, m, n-3 > \\
&-3m(m-1)(m-2)(m-3)|l, m+1, n-4 > \\
&-l(l-1)(\Lambda, \alpha_2)|l-2, m, n > \\
&+m(m-1)(\Lambda, \alpha_2)|l, m, n-2 >.
\]

\[
e = -\gamma_0 \left( \beta_0^2 + 3\beta_1^2 \right) - 2\gamma_1 \left( \beta_0 \beta_1 - \beta_2 \right) \\
-4\gamma_2 \left( \beta_0 \beta_2 - \beta_1^3 \right) - (\Lambda, \alpha_1) \beta_0 - (\Lambda, \alpha_2) \beta_1;
\]

\[
h = 2\beta_0 \gamma_0 + 2\beta_1 \gamma_1 + 4\beta_2 \gamma_2 + (\Lambda, \alpha_1);
\]

\[
f = \gamma_0(z);
\]

\[
\tilde{E} = 2\gamma_0 \left( \beta_1^3 - 2\beta_0 \beta_2 - 3\beta_0^2 \beta_1 \right) \\
-2\gamma_1 \left( \beta_1^3 + \beta_0 \beta_1^2 + \beta_1 \beta_2 \right) \\
+\gamma_2 \left( \beta_1^4 - 3\beta_1^2 + 6\beta_0^2 \beta_1^2 - 4\beta_0^2 \right) \\
-2(\Lambda, \alpha_1) \left( \beta_0 \beta_1 + \beta_2 \right) - (\Lambda, \alpha_2) \left( \beta_0^2 - \beta_1^2 \right);
\]

\[
\tilde{e} = -2\gamma_0 \left( 3\beta_0 \beta_1 + \beta_2 \right) - \gamma_1 \left( 3\beta_0^2 + \beta_1^2 \right) \\
+2\gamma_2 \left( \beta_0^3 + 3\beta_0 \beta_1^2 \right) - (\Lambda, \alpha_1) \beta_1(z) - (\Lambda, \alpha_2) \beta_0;
\]

\[
\tilde{h} = 6\gamma_0 \beta_1 + 6\gamma_1 \beta_0 - 6\gamma_2 \left( \beta_0^2 + \beta_1^2 \right) + (\Lambda, \alpha_2);
\]

\[
\tilde{f} = \gamma_1 - 2\gamma_2 \beta_0;
\]

\[
\tilde{F} = \gamma_2;
\]
It turns out that to get a free field realization of the twisted currents, we have to interchanging the $\beta_i$ with $\gamma_i$ in the above expressions, and at the same time interchange $\bar{e}$ with $f$. Now we introduce three $\beta\gamma$ pairs and two scalar fields $\phi_a$, $a = 1$, 2. ($\beta_i; \gamma_i$) pairs have conformal dimension $(1; 0)$.

$$
\beta_i(z)\gamma_j(w) = -\gamma_j(z)\beta_i(w) = -\frac{\delta_{ij}}{z-w}, \quad i, j = 0, 1, 2
$$

$$
\phi_a(z)\phi_b(w) = -2\delta_{ab}\ln(z-w), \quad a, b = 0, 1
$$

(3.12)

Introduce the notation $\vec{e}_1 = \frac{1}{2}(1, 1); \vec{e}_2 = \frac{\sqrt{3}}{2}(1, -1); \text{and } \vec{\Phi} = (\phi_0, \phi_1)$. Then we have

$$
\vec{e}_1 \cdot \vec{\Phi}(z)\vec{e}_1 \cdot \vec{\Phi}(w) = -\ln(z-w); \quad \vec{e}_2 \cdot \vec{\Phi}(z)\vec{e}_2 \cdot \vec{\Phi}(w) = -3\ln(z-w);
$$

$$
\vec{e}_1 \cdot \vec{\Phi}(z)\vec{e}_2 \cdot \vec{\Phi}(w) = 0.
$$

With the help of the differential operator representation of the non-affine algebra, we find the Wakimoto realization of $sl(3, \mathbb{C})_k^{(2)}$ in terms of the eight free fields:

$$\begin{align*}
    j^+(z) &= \beta_0(z); \\
    j^0(z) &= 2\beta_0(z)\gamma_0(z) + 2\beta_1(z)\gamma_1(z) + 4\beta_2(z)\gamma_2(z) + \frac{1}{\alpha_+}(\vec{e}_1 \cdot i\partial\vec{\Phi}(z)); \\
    j^- &= -\beta_0(z)\left(\gamma_0^2(z) + 3\gamma_1^2(z)\right) - 2\beta_1(z)\left(\gamma_0(z)\gamma_1(z) - \gamma_2(z)\right) - 4\beta_2(z)\left(\gamma_0(z)\gamma_2(z) - \gamma_1^2(z)\right) - 4k\partial\gamma_0(z) \\
    &\quad - \frac{1}{\alpha_+}(\vec{e}_1 \cdot i\partial\vec{\Phi}(z))\gamma_0(z) - \frac{1}{\alpha_+}(\vec{e}_2 \cdot i\partial\vec{\Phi}(z))\gamma_1(z); \\
    J^{++}(z) &= \beta_2(z); \\
    J^+(z) &= \beta_1(z) - 2\beta_2(z)\gamma_0(z); \\
    J^0(z) &= 6\beta_0(z)\gamma_1(z) + 6\beta_1(z)\gamma_0(z) - 6\beta_2(z)\left(\gamma_0^2(z) + \gamma_1^2(z)\right) \\
    &\quad + \frac{1}{\alpha_+}(\vec{e}_2 \cdot i\partial\vec{\Phi}(z)); \\
    J^- &= -2\beta_0(z)\left(3\gamma_0(z)\gamma_1(z) + \gamma_2(z)\right) - \beta_1(z)\left(3\gamma_0^2(z) + \gamma_1^2(z)\right) \\
    &\quad + 2\beta_2(z)\left(\gamma_0^3(z) + 3\gamma_0(z)\gamma_1^2(z)\right) - 4(k+1)\partial\gamma_1(z) \\
    &\quad - \frac{1}{\alpha_+}(\vec{e}_1 \cdot i\partial\vec{\Phi}(z))\gamma_1(z) - \frac{1}{\alpha_+}(\vec{e}_2 \cdot i\partial\vec{\Phi}(z))\gamma_0(z); \\
    J^{--} &= 2\beta_0(z)\left(\gamma_1^3(z) - 2\gamma_0(z)\gamma_2(z) - 3\gamma_0^2(z)\gamma_1(z)\right) \\
    &\quad - 2\beta_1(z)\left(\gamma_0^3(z) + \gamma_0(z)\gamma_1^2(z) + 2\gamma_1(z)\gamma_2(z)\right) \\
    &\quad + \beta_2(z)\left(\gamma_0^4(z) - 3\gamma_1^4(z) + 6\gamma_0^2(z)\gamma_1^2(z) - 4\gamma_2^2(z)\right) \\
    &\quad - \frac{2}{\alpha_+}(\vec{e}_1 \cdot i\partial\vec{\Phi}(z))\left(\gamma_0(z)\gamma_1(z) + \gamma_2(z)\right) \\
    &\quad - \frac{1}{\alpha_+}(\vec{e}_2 \cdot i\partial\vec{\Phi}(z))\left(\gamma_0^2(z) - \gamma_1^2(z)\right) \\
    &\quad - 8(k+1)\gamma_0(z)\partial\gamma_1(z) - 4k\partial\gamma_2(z).
\end{align*}$$
Here $\alpha_+ = 1/\sqrt{8k+24}$, and normal ordering is implied in the expressions. It is straightforward to check that the above currents satisfy the OPEs given in last section.

Note that the twisted currents have the following mode expansions:

$$ j^a(z) = \sum_{n \in \mathbb{Z}} j_n^a z^{-n-1}; \quad J^a(z) = \sum_{n \in \mathbb{Z}+1/2} J_n^a z^{-n-1}. \quad (3.14) $$

From the expressions of the currents, we find that the following relations hold

$$ J^-(z) = \frac{1}{2} \frac{\partial}{\partial \gamma_0(z)} J^-(z); \quad J^0(z) = -\frac{1}{3} \frac{\partial}{\partial \gamma_0(z)} J^-(z); $$

$$ J^+(z) = \frac{1}{2} \frac{\partial}{\partial \gamma_0(z)} J^0(z); \quad J^{++}(z) = -\frac{1}{2} \frac{\partial}{\partial \gamma_0(z)} J^+(z); \quad (3.15) $$

Now let’s examine the condition for our representation to be unitary. We introduce the conjugate operation on the modes of the currents:

$$ T^a_n = T^{-a}_{-n}, \quad k^\dagger = k; \quad (3.16) $$

where $T^a$ stands for $j^0, j^\pm, J^0, J^\pm$ or $J^{\pm\pm}$ and $T^{-a}$ for $j^0, j^\mp, J^0, J^\mp$ or $J^{\mp\mp}$, respectively. From the OPEs (2.8) we know that

$$ T^{-a}_{1/b}, \quad T^a_{-1/b}, \quad 4k - bj_0^0, \quad (3.17) $$

where $a = +$ or $++$ and $-a = -$ or $-+$, $b = 1$ for $T^\pm = j^\pm$ and $b = 2$ for $T^\pm = J^\pm$ or $J^{\pm\pm}$. Obviously, for a given $a$ and $b$, $T^{-a}_{1/b}, T^a_{-1/b}$ and $4k - bj_0^0$ form the subalgebra of $su(2)$ subalgebra of $sl(3, \mathbb{C}_k^{(2)})$. Firstly, the eigenvalues of $4k - bj_0^0$ must be integral for an unitary representation. Applying this to the state $v_\Lambda$, one obtains

$$ j_0^0 v_\Lambda = (\Lambda, \alpha_1) v_\Lambda, \quad (3.18) $$

which implies that

$$ 4k - b(\Lambda, \alpha_1) \in \mathbb{Z}. \quad (3.19) $$

As $v_\Lambda$ is a vacuum state, i.e.

$$ T^{-a}_{1/b} v_\Lambda = 0, \quad (3.20) $$

where $a = +, ++$, thus we have the norm

$$ (T^a_{1/b} v_\Lambda, T^a_{-1/b} v_\Lambda) = (v_\Lambda, [T^{-a}_{1/b}, T^a_{-1/b}] v_\Lambda) = (v_\Lambda, T^{-a}_{1/b} T^a_{-1/b} v_\Lambda) = 4k - b(\Lambda, \alpha_1)(v_\Lambda, v_\Lambda). \quad (3.21) $$

Hence $4k \geq b(\Lambda, \alpha_1)$ with $b = 1$ or $2$ corresponding to $a = +$ or $a = ++$, respectively. We take $\Lambda$ to be dominant. Then $4k \geq 2(\Lambda, \alpha_1) \geq 0$. So the condition for our representation to be unitary is $4k \in \mathbb{Z}$ and $4k \geq 2(\Lambda, \alpha_1) \geq 0$. 

4 Twisted stress energy tensor

It is well known that Virasoro algebras are related to currents algebras via the so called Sugawara construction. In the present case, the twisted Sugawara construction of the energy-momentum tensor is given by

\[
T(z) = \frac{1}{8(k+3)} \left[ \frac{1}{2} j^0(z) j^0(z) + \frac{1}{6} j^0(z) J^0(z) + 2 j^+(z) j^+(z) \right. \\
\left. + 2 J^-(z) J^+(z) + 2 J^-(z) J^+(z) \right], \tag{4.22}
\]

in which \( : \) implies the normal ordering. The above expression can be rephrased through the \( \beta \gamma \) pairs and the scalar field \( \Phi \). We obtain

\[
T(z) = - \left[ \beta_0(z) \partial \gamma_0(z) + \beta_1(z) \partial \gamma_1(z) + \beta_2(z) \partial \gamma_2(z) \right] : \\
+ \frac{1}{2} : \left( \vec{e}_1 \cdot i \partial \vec{\Phi}(z) \right)^2 : + \frac{1}{6} : \left( \vec{e}_2 \cdot i \partial \vec{\Phi}(z) \right)^2 : \\
- 4 \alpha_+ (\vec{e}_1 \cdot i \partial^2 \vec{\Phi}(z)). \tag{4.23}
\]

Following the standard practice, we get the OPE of the energy-momentum tensor,

\[
T(z) T(w) = \frac{c/2}{(z-w)^4} \left( z-w \right)^2 + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots, \tag{4.24}
\]

where \( c = 8k/(k+3) \) is the central charge for the Virasoro algebra.

5 Twisted screening currents

An important object in the free field approach is the screening current. Screening currents are a primary fields with conformal dimension 1, and their integration give the screening charges. They commute with the affine currents up to a total derivative. These properties ensure that screening charges may be inserted into correlators while the conformal or affine ward identities remain intact. For the present case, we find the following screening currents

\[
S_\pm(z) = \left[ 2 \beta_2(z) \gamma_1(z) - \beta_0(z) \pm \beta_1(z) \right] \tilde{S}_\pm(z), \tag{5.25}
\]

where

\[
\tilde{S}_\pm(z) = e^{-2 \alpha_+ (\vec{e}_1 \cdot i \vec{\Phi}(z) \pm 2 \alpha_+ \vec{e}_2 \cdot i \vec{\Phi}(z)). \tag{5.26}
\]

The OPE of the twisted screening currents with the twisted affine currents are
\[
T(z)S_{\pm}(w) = \partial_w \left( \frac{1}{z-w} S_{\pm}(w) \right) + \ldots;
\]
\[
j^+(z)S_{\pm}(w) = \ldots; \quad j^0(z)S_{\pm}(w) = \ldots;
\]
\[
j^-(z)S_{\pm}(w) = \partial_w \left( -\frac{1}{2\alpha_+^2} \frac{1}{z-w} \tilde{S}_{\pm}(w) \right) + \ldots;
\]
\[
J^+(z)S_{\pm}(w) = \ldots; \quad J^0(z)S_{\pm}(w) = \ldots;
\]
\[
J^-(z)S_{\pm}(w) = \partial_w \left( \pm \frac{1}{2\alpha_+^2} \frac{1}{z-w} \tilde{S}_{\pm}(w) \right) + \ldots;
\]
\[
J^{++}(z)S_{\pm}(w) = \ldots;
\]
\[
J^{--}(z)S_{\pm}(w) = \partial_w \left( \pm \frac{1}{\alpha_+^2} \frac{1}{z-w} [\gamma_0(w) \pm \gamma_1(w)] \tilde{S}_{\pm}(w) \right) + \ldots;
\]
\[
J_{-+}(z)S_{\pm}(w) = \partial_w \left( \pm \frac{1}{\alpha_+^2} \frac{1}{z-w} [\gamma_0(w) \pm \gamma_1(w)] \tilde{S}_{\pm}(w) \right) + \ldots;
\]
\[
(5.27)
\]

The screening currents obtained here are the twisted versions of the first kind screening currents [6].

6 Twisted primary fields

Primary fields are fundamental objects in conformal field theories. A primary field \( \Psi \) has the following OPE with the energy-momentum tensor:
\[
T(z)\Psi(w) = \frac{h_\Psi}{(z-w)^2} \Psi(w) + \frac{\partial_w \Psi(w)}{z-w} + \ldots,
\]
where the \( h_\Psi \) is the conformal dimension of \( \Psi \). Moreover the most singular part in the OPE of \( \Psi \) with the affine currents only has a single pole. A special kind of the primary fields is highest weight state. In the present case, a highest weight state is one that is annihilated by currents \( j^+(z), J^+(z) \) and \( J^{++}(z) \). So the highest state has the form
\[
V(z) = e^{a\vec{e}_1 \cdot i\Phi(z) + b\vec{e}_2 \cdot i\Phi(z)}.
\]
It has conformal dimension \( \Delta(V_{a,b}) = \frac{1}{2} a (a + 8\alpha_+) + \frac{3}{2} b^2 \). Obviously \( \Delta(V_{a,b}) = \Delta(V_{a,-b}) \).
We find that only when
\[
V(z) = V_{j,\pm}(z) = e^{\alpha_+ \left( \frac{1}{\alpha_+^2} \vec{e}_1 \cdot i\Phi(z) \pm \frac{1}{\alpha_+^2} \vec{e}_2 \cdot i\Phi(z) \right)} = e^{\alpha_+ \left( 2j\vec{e}_1 \cdot i\Phi(z) \pm \frac{1}{2} j\vec{e}_2 \cdot i\Phi(z) \right)},
\]
the number of fields produced by the repeated action of \( j^-, J^-, J^{--} \) currents on this highest state are finite. The conformal dimensions corresponding to (6.30) are
\[
\Delta_j = \frac{8}{3} j(j+3)\alpha_+^2 = \frac{j(j+3)}{3(k+3)}.
\]
From this highest weight state, we obtain the following fields,
\[
\Phi_{j;\pm}^m(z) = \left[ (-\gamma_0(z) \pm \gamma_1(z))^{j-m} + \Sigma_{n=1}(-1)^{j-m-n} \left( j - \frac{1}{2} (j - \frac{3}{2}) \right) \cdots \right] \\
\left( j - \frac{(2n - 1)}{2} \right)^{-1} \times \frac{m(m - 1) \cdots (m - 2n + 1)}{2 \times 4 \times \cdots \times 2n} \\
\times (\gamma_0(z) \pm \gamma_1(z))^{j-m-2n}(\gamma_1^2(z) \mp \gamma_2(z))^n\right] V_{j;\pm}(z) \quad m - 2n \geq 0. (6.32)
\]

They are primary fields in the sense that their OPE with the energy-momentum tensor is

\[
T(z) \Phi_{j;\pm}^m(w) = \frac{\Delta_j}{(z - w)^2} \Phi_{j;\pm}^m(w) + \frac{\partial_w \Phi_{j;\pm}^m(w)}{z - w} + \ldots, \quad (6.33)
\]

and the OPE's with the fixed point subalgebra of \(sl(3)^{(2)}_k\) are

\[
\begin{align*}
J^+(z) \Phi_{j;\pm}^m(w) &= \frac{1}{z - w} \left[ \pm (j - m) \Phi_{j;\pm}^{m+1}(w) \mp \frac{(j - m)(j - m - 1)}{j - 1/2} \Phi_{j;\pm}^{j-1}(w) \Phi_{j;\pm}^{j-1}(w) \right] + \ldots; \\
J^-(z) \Phi_{j;\pm}^m(w) &= \frac{1}{z - w} \left[ \pm 2j \Phi_{j;\pm}^{j-1}(w) \Phi_{j;\pm}^m(w) \mp 2(j - m)(j - 1/2) \Phi_{j;\pm}^{j-2}(w) \Phi_{j;\pm}^m(w) \right] + \ldots; \\
J^0(z) \Phi_{j;\pm}^m(w) &= \frac{1}{z - w} \left[ \pm 2j \Phi_{j;\pm}^m(w) \mp 6(j - m) \Phi_{j;\pm}^{j-1}(w) \Phi_{j;\pm}^{j+1}(w) \right] + \ldots; \quad (6.35)
\end{align*}
\]

From the expression of \(\Phi_{j;\pm}^m(z)\), we see that \(j\) can only take integer values. The OPE's of \(\Phi_{j;\pm}^m\) with other currents are much more involved:

\[
\begin{align*}
J^+(z) \Phi_{j;\pm}^m(w) &= \frac{1}{z - w} \left[ \pm (j - m) \Phi_{j;\pm}^{m+1}(w) \mp \frac{(j - m)(j - m - 1)}{j - 1/2} \Phi_{j;\pm}^{j-1}(w) \Phi_{j;\pm}^{j-1}(w) \right] + \ldots; \\
J^-(z) \Phi_{j;\pm}^m(w) &= \frac{1}{z - w} \left[ \pm 2j \Phi_{j;\pm}^{j-1}(w) \Phi_{j;\pm}^m(w) \mp 2(j - m)(j - 1/2) \Phi_{j;\pm}^{j-2}(w) \Phi_{j;\pm}^m(w) \right] + \ldots; \\
J^0(z) \Phi_{j;\pm}^m(w) &= \frac{1}{z - w} \left[ \pm 2j \Phi_{j;\pm}^m(w) \mp 6(j - m) \Phi_{j;\pm}^{j-1}(w) \Phi_{j;\pm}^{j+1}(w) \right] + \ldots; \quad (6.35)
\end{align*}
\]

\[
J^{++}(z) \Phi_{j;\pm}^m(w) = \frac{\mp 1}{z - w} \left[ \frac{(j - m)(j - m - 1)}{2(j - 1/2)} \Phi_{j;\pm}^{m+1}(w) \right] + \ldots; \\
J^{--}(z) \Phi_{j;\pm}^m(w) = \frac{1}{z - w} \left[ \pm 4(j - 1)m \left( \frac{\Phi_{j;\pm}^{j+1}(w)}{\Phi_{j;\pm}^m(w)} \right)^2 \Phi_{j;\pm}^m(w) \right]
\]
\[ + 4(j - 1/2)m \left( \frac{\Phi_j^{-2}(w)}{\Phi_j^{j;\pm}(w)} \right) \Phi_j^{m;\pm}(w) \]
\[ - \frac{(j - m)(j - m - 1)}{2(j - 1/2)} \left( \frac{\Phi_j^{-1}(w)}{\Phi_j^{j;\pm}(w)} \right)^4 \Phi_j^{m+1;\pm}(w) \]
\[ \pm 2(j - m)(j - m - 1)(j - 1/2) \times \left[ \left( \frac{\Phi_j^{j;\pm}(w)}{\Phi_j^{j;\pm}(w)} \right) - \left( \frac{\Phi_j^{j;\pm}(w)}{\Phi_j^{j;\pm}(w)} \right)^2 \right] \Phi_j^{m+1;\pm}(w) \] + \ldots.

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