Spectral analysis on pseudo-Riemannian locally symmetric spaces

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Abstract: We summarize and announce some recent results initiating spectral analysis on pseudo-Riemannian locally symmetric spaces $\Gamma \backslash G/H$, beyond the classical setting where $H$ is compact (e.g. theory of automorphic forms for arithmetic $\Gamma$) or $\Gamma$ is trivial (e.g. Plancherel-type formula for semisimple symmetric spaces).

Key words: Locally symmetric space; pseudo-Riemannian manifold; discontinuous group; invariant differential operator; branching law; spherical variety.

1. Introduction. A pseudo-Riemannian manifold is a smooth manifold $M$ equipped with a smooth, nondegenerate symmetric bilinear tensor $g$ of signature $(p,q)$. It is called Riemannian if $q=0$, and Lorentzian if $q=1$. As in the Riemannian case, the metric $g$ induces a Radon measure on $M$ and a second-order differential operator $\Box_M = \text{div grad}$

called the Laplacian. It is a symmetric operator on the Hilbert space $L^2(M)$. The Laplacian $\Box_M$ is not an elliptic differential operator if $p,q > 0$.

A semisimple symmetric space $X$ is a homogeneous space $G/H$ where $G$ is a semisimple Lie group and $H$ an open subgroup of the group of fixed points of $G$ under some involutive automorphism. The manifold $X$ carries a $G$-invariant pseudo-Riemannian metric induced by the Killing form of the Lie algebra $\mathfrak{g}$ of $G$. The group $G$ acts on $X$ by isometries, and the $C_\ast$-algebra $\mathcal{D}(X)$ of $G$-invariant differential operators on $X$ is commutative.

In this note we consider quotients $X_\Gamma = \Gamma \backslash X$ of a semisimple symmetric space $X = G/H$ by discrete subgroups $\Gamma$ of $G$ acting properly discontinuously and freely on $X$ (“discontinuous groups for $X$”). Such quotients are called pseudo-Riemannian locally symmetric spaces. They are complete $(G,X)$-manifolds in the sense of Ehresmann and Thurston, and they inherit a pseudo-Riemannian structure from $X$. Any $G$-invariant differential operator $D$ on $X$ induces a differential operator $D_\Gamma$ on $X_\Gamma$ via the covering map $p_\Gamma: X \to X_\Gamma$. For instance, the Laplacian $\Box_X$ on $X$ is $G$-invariant, and $(\Box_X)_\Gamma = \Box_{X_\Gamma}$. As in [7,8], we think of

$$\mathcal{P} := \{D_\Gamma : D \in \mathcal{D}(X)\}$$

as the set of “intrinsic differential operators” on the locally symmetric space $X_\Gamma$. It is a subalgebra of the $C_\ast$-algebra $\mathcal{D}(X_\Gamma)$ of differential operators on $X_\Gamma$:

$$D_\Gamma f = \lambda(D)f \quad \text{for all } D \in \mathcal{D}(X).$$

For a $C_\ast$-algebra homomorphism $\lambda: \mathcal{D}_G(X) \to \mathbb{C}$, we denote by $C^\infty(X_\Gamma; \mathcal{M}_\lambda)$ the space of smooth functions $f$ on $X_\Gamma$ (joint eigenfunctions) satisfying the following system of partial differential equations:

$$(\mathcal{M}_\lambda) \quad D_\Gamma f = \lambda(D)f$$

Let $L^2(X_\Gamma; \mathcal{M}_\lambda)$ be the space of square-integrable functions on $X_\Gamma$ satisfying $(\mathcal{M}_\lambda)$ in the weak sense. It is a closed subspace of the Hilbert space $L^2(X_\Gamma)$. We are interested in the following problems.

Problems 1. For intrinsic differential operators on $X_\Gamma = \Gamma \backslash G/H$,

1. construct joint eigenfunctions on $X_\Gamma$;
2. find a spectral theory on $L^2(X_\Gamma)$.

In the classical setting where $H$ is a maximal compact subgroup $K$ of $G$, i.e. $X_\Gamma$ is a Riemannian locally symmetric space, a rich and deep theory has been developed over several decades, in particular, in connection with automorphic forms when $\Gamma$ is arithmetic. For compact $H$, the spectral decompo-
sition of $L^2(X_F)$ is closely related to a disintegration of the regular representation of $G$ on $L^2(\Gamma \backslash G)$:

$$L^2(\Gamma \backslash G) \simeq \int_G m_!(\pi) \pi \, d\sigma(\pi),$$

where $d\sigma$ is a Borel measure on the unitary dual $\widehat{G}$ and $m_!: \widetilde{G} \to \mathbb{N} \cup \{\infty\}$ a measurable function called multiplicity. There is a natural isomorphism

$$L^2(X_F) \simeq L^2(\Gamma \backslash G)^H$$

and the Hilbert space $L^2(X_F)$ is decomposed as

$$L^2(X_F) \simeq \int_{\widehat{G}/H} m_!(\pi) \pi^H \, d\sigma(\pi),$$

where $\pi^H$ denotes the space of $H$-invariant vectors in the representation space of $\pi$ and

$$(\widehat{G}/H) := \{ \pi \in \widehat{G} : \pi^H \neq \{0\} \}.$$ 

Since the center $3(\mathfrak{g}_C)$ of the universal enveloping algebra $U(\mathfrak{g}_C)$ acts on the space of smooth vectors of $\pi$ as scalars for every $\pi \in \widehat{G}$, the decomposition (1.4) respects the actions of $\mathfrak{g}_C(X)$ and $3(\mathfrak{g}_C)$ via the natural $C$-algebra homomorphism $\delta_!: 3(\mathfrak{g}_C) \to \mathfrak{g}_C(X)$. This homomorphism is surjective e.g. if $G$ is a classical group.

The situation changes drastically beyond the aforementioned classical setting, namely, when $H$ is not compact anymore. New difficulties include:

1. (Representation theory) If $H$ is noncompact, then $L^2(\Gamma \backslash G)^H = \{0\}$ (because the fact that $\Gamma$ acts properly on $X = G/H$ implies that $H$ acts properly on $\Gamma \backslash G$), and so (1.3) fails:

$$L^2(X_F) \not\simeq L^2(\Gamma \backslash G)^H$$

and the irreducible decomposition (1.2) of the regular representation $L^2(\Gamma \backslash G)$ of $G$ does not yield a spectral decomposition of $L^2(X_F)$.

2. (Analysis) In contrast to the usual Riemannian case (see [22]), the Laplacian $\Box_{X_F}$ is not elliptic anymore, and thus even the following subproblems of Problem 1.(2) are open in general for $X_F = \Gamma \backslash G/H$ with $H$ noncompact.

Questions 2.

1. Does the Laplacian $\Box_{X_F}$, defined on $C_c^\infty(X_F)$, extend to a self-adjoint operator on $L^2(X_F)$?

2. Does $L^2(\Gamma \backslash G) \supset M_\lambda$ contain real analytic functions as a dense subspace?

3. Does $L^2(X_F)$ decompose discretely into a sum of subspaces $L^2(X_F; M_\lambda)$ when $X_F$ is compact?

Detailed proofs of Theorems 9, 10, 11, 15, and 16 below will appear elsewhere.

2. Standard quotients. We observe that a discrete group of isometries on a pseudo-Riemannian manifold $X$ does not always act properly discontinuously on $X$, and the quotient space $X_F = \Gamma \backslash X$ is not necessarily Hausdorff. In fact, some semisimple symmetric spaces $X$ do not admit infinite discontinuous groups of isometries (Calabi–Markus phenomenon [2,11]), and thus it is not obvious a priori whether there are interesting examples of pseudo-Riemannian locally symmetric spaces $X_F$ beyond the classical Riemannian case.

Fortunately, there exist semisimple symmetric spaces $X = G/H$ admitting “large” discontinuous groups $\Gamma$, e.g. such that $X_F$ is compact or of finite volume. Let us recall a useful idea for finding such $X$ and $\Gamma$. Suppose a Lie subgroup $L$ of $G$ acts properly on $X$. Then the action of any discrete subgroup $\Gamma$ of $L$ on $X$ is automatically properly discontinuous, and this action is free whenever $\Gamma$ is torsion-free. Moreover, if $L$ acts cocompactly (e.g. transitively) on $X$, then $\text{vol}(X_F) < +\infty$ if and only if $\text{vol}(\Gamma \backslash L) < +\infty$.

Definition 3 (Standard quotient $X_F$, see [8, Def. 1.4]). A quotient $X_F = \Gamma \backslash X$ of $X = G/H$ by a discrete subgroup of $G$ is called standard if $\Gamma$ is contained in a reductive subgroup $L$ of $G$ acting properly on $X$.

A criterion on triples $(L, G, H)$ of reductive Lie groups for $L$ to act properly on $X = G/H$ was established in [11], and a list of irreducible symmetric spaces $G/H$ admitting proper and cocompact actions of reductive subgroups $L$ was given in [18]. Recently, Tojo [23] announced that the list in [18] exhausts all such triples $(L, G, H)$ with $L$ maximal.

3. Construction of discrete spectrum. Let $X = G/H$ be a semisimple symmetric space. Let $j$ be a maximal semisimple abelian subspace in the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the Killing form, and let $W$ be the Weyl group for the root system $\Sigma(\mathfrak{g}_C, j\mathfrak{C})$. The Harish-Chandra isomorphism $\Psi: S(j\mathfrak{C})^W \xrightarrow{\sim} \mathfrak{g}_C(X)$ (see [6]) induces a bijection

$$\Psi^*: \text{Hom}_{\mathfrak{g}_C}(\mathfrak{g}_C(X), \mathfrak{C}) \xrightarrow{\sim} j\mathfrak{C}/W.$$ 

The dimension of $j$ is called the rank of the symmetric space $X = G/H$. Let $K$ be a maximal compact subgroup of $G$ such that $H \cap K$ is a maximal compact subgroup of $H$. Assume that $G$
is connected without compact factor and that the following rank condition is satisfied:

(3.2) \[ \text{rank } G/H = \text{rank } K/(H \cap K). \]

Then we can take \( j \) to be a subspace of \( t \). We fix compatible positive systems \( \Sigma^+(\mathfrak{g}_C, i_C) \) and \( \Sigma^+(\mathfrak{r}_C, i_C) \), denote by \( \rho \) and \( \rho_\mathfrak{r} \) the corresponding half sums of positive roots counted with multiplicities, and set

\[ \Lambda := 2\rho_\mathfrak{r} - \rho + \mathbb{Z}\text{-span}(\text{highest weights of } (K)_{H\cap K}). \]

For \( C \geq 0 \), we consider the countable set

\[ \Lambda_C := \{ \lambda \in \Lambda : \langle \lambda, \alpha \rangle > C \text{ for all } \alpha \in \Sigma^+(\mathfrak{g}_C, i_C) \}. \]

**Fact 4** (Flensted-Jensen [5]). If the rank condition (3.2) holds, then there exists \( C > 0 \) such that

\[ L^2(X; \mathcal{M}_\lambda) \neq \{0\} \quad \text{for all } \lambda \in \Lambda_C. \]

In fact one can take \( C = 0 \) [19]. We now turn to locally symmetric spaces \( X_\Gamma \):

**Theorem 5** ([7], [8, Th. 1.5]). Under the rank condition (3.2), for any standard quotient \( X_\Gamma \) with \( \Gamma \) torsion-free, there exists \( C_\Gamma > 0 \) such that

\[ L^2(X_\Gamma; \mathcal{M}_\lambda) \neq \{0\} \quad \text{for all } \lambda \in \Lambda_{C_\Gamma}. \]

Thus the discrete spectrum \( \text{Spec}_d(X_\Gamma) \), which is by definition the set of \( \lambda \in \text{Hom}_{\text{alg}}(\mathcal{D}_G(X), \mathcal{C}) \) such that \( L^2(X_\Gamma; \mathcal{M}_\lambda) \neq \{0\} \), is infinite.

Theorem 5 applied to \( (G \times \{1\}, G \times G, \text{Diag } G) \) instead of \( (L, G, H) \) (group manifold case) implies:

**Example 6.** Suppose rank \( G = \text{rank } K \). For any torsion-free discrete subgroup \( \Gamma \) and any discrete series representation \( \pi_\lambda \) of \( G \) with sufficiently regular Harish-Chandra parameter \( \lambda \),

(3.3) \[ \text{Hom}_C(\pi_\lambda, L^2(\Gamma \backslash G)) \neq \{0\}. \]

This sharpens and generalizes classical results asserting that if \( \Gamma \) is an arithmetic subgroup of \( G \), then (3.3) holds after replacing \( \Gamma \) by a finite-index subgroup \( \Gamma' \) (possibly depending on \( \pi_\lambda \)), see Borel–Wallach [1], Clozel [3], DeGeorge–Wallach [4], Kazhdan [10], Rohlfs–Speh [20], and Savin [21].

**Remark 7.** (1) Theorem 5 extends to a more general setting where \( X_\Gamma \) is not necessarily standard: namely, the conclusion still holds as long as the action of \( \Gamma \) on \( X \) satisfies a strong properness condition called *sharpness* [8, Th. 3.8].

(2) The rank condition (3.2) is necessary for \( \text{Spec}_d(X) \) to be nonempty (see Matsuki–Oshima [19]), in which case Fact 4 applies. On the other hand, \( \text{Spec}_d(X_\Gamma) \) may be nonempty even if (3.2) fails. This leads us to the notion of discrete spectrum of type \( \mathcal{I} \) and \( \mathcal{II} \), see Definition 12 below.

4. Spectral decomposition of \( L^2(X_\Gamma) \). In this section, we discuss spectral decomposition on standard quotients \( X_\Gamma \). We do not impose the rank condition (3.2), but require that \( L \) act spherically on \( X_C \), i.e. a Borel subgroup of \( L_C \) has an open orbit in \( X_C \). To be precise, our setting is as follows:

**Setting 8.** We consider a symmetric space \( X = G/H \) with \( G \) noncompact and simple, a reductive subgroup \( L \) of \( G \) acting properly on \( X \) such that \( X_C = G_C/H_C \) is \( L_C \)-spherical, and a torsion-free discrete subgroup \( \Gamma \) of \( L \).

For compact \( H \), we can take \( L = G \). However, our main interest is for noncompact \( H \), in which case the proper action of \( L \) in the setting 8 forces \( L \neq G \) (see [11, Th. 4.1] for a properness criterion).

In Theorems 9 and 10 below, we allow the case where \( \text{vol}(X_\Gamma) = +\infty \).

**Theorem 9** (Spectral decomposition). In the setting 8, there exist a measure \( d\mu \) on \( \text{Hom} := \text{Hom}_{\text{alg}}(\mathcal{D}_G(X), \mathcal{C}) \) and a measurable family \( \{\mathcal{F}_\lambda\}_{\lambda \in \text{Hom}} \) of linear maps, with

\[ \mathcal{F}_\lambda : C^\infty_c(X_\Gamma) \longrightarrow C^\infty(X_\Gamma; \mathcal{M}_\lambda), \]

such that any \( f \in C^\infty_c(X_\Gamma) \) can be expanded into joint eigenfunctions on \( X_\Gamma \) as

(4.1) \[ f = \int_{\text{Hom}} \mathcal{F}_\lambda f \ d\mu(\lambda), \]

with a Parseval–Plancherel type formula

\[ \|f\|^2_{L^2(X_\Gamma)} = \int_{\text{Hom}} \|\mathcal{F}_\lambda f\|^2_{L^2(X_\Gamma)} \ d\mu(\lambda). \]

The measure \( d\mu \) can be described via a “transfer map” discussed in Section 5, see (5.4). In particular, we see that (4.1) is a discrete sum if \( X_\Gamma \) is compact, answering Question 2.3(3) in our setting. The proof of Theorem 9 gives an answer to Questions 2.1(1)–(2):

**Theorem 10.** In the setting 8,

(1) the pseudo-Riemannian Laplacian \( \square_{X_\Gamma} \) defined on \( C^\infty_c(X_\Gamma) \) is essentially self-adjoint on \( L^2(X_\Gamma) \);

(2) any \( L^2 \)-eigenfunction of the Laplacian \( \square_{X_\Gamma} \) can be approximated by real analytic \( L^2 \)-eigenfunctions.

**Theorem 11.** In the setting 8, the discrete spectrum \( \text{Spec}_d(X_\Gamma) \) is infinite whenever \( \Gamma \) is cocompact or arithmetic in the subgroup \( L \).
Let \( \mathcal{D}'(X) \) be the space of distributions on \( X \), endowed with its standard topology. Let \( p^*_\lambda: L^2(X_\Gamma) \to \mathcal{D}'(X) \) be the pull-back by the projection \( p_\lambda: X \to X_\Gamma \). For \( \lambda \in \Spec_d(X_\Gamma) \), we denote by \( L^2(X_\Gamma; \mathcal{M}_\lambda) \) the preimage under \( p^*_\lambda \) of the closure in \( \mathcal{D}'(X) \) of \( L^2(X_\Gamma; \mathcal{M}_\lambda) \), and by \( L^2(X_\Gamma; \mathcal{M}_\lambda)_\Pi \) its orthogonal complement in \( L^2(X_\Gamma; \mathcal{M}_\lambda) \).

**Definition 12.** For \( i = 1 \) or \( \Pi \), the discrete spectrum of type \( i \) of \( X_\Gamma \) is the subset \( \Spec_d(X_\Gamma)_i \) of \( \Spec_d(X_\Gamma) \) consisting of those elements \( \lambda \) such that \( L^2(X_\Gamma; \mathcal{M}_\lambda) \neq \{0\} \).

By construction, \( \Spec_d(X_\Gamma)_\Pi \) is contained in \( \Spec_d(X_\Gamma) \), hence it is nonempty only if (3.2) holds (Remark 7.(2)); in this case \( \Spec_d(X_\Gamma)_\Pi \) is actually infinite for standard \( X_\Gamma \) by Theorem 5. On the other hand, Theorem 11 has the following refinement.

**Theorem 13.** In the setting 8, \( \Spec_d(X_\Gamma)_\Pi \) is infinite whenever \( \Gamma \) is cocompact or arithmetic in \( L \).

**Example 14.** For any compact standard anti-de Sitter 3-manifold \( M = \Gamma \backslash \text{SO}(2,2)/\text{SO}(2,1) \), both \( \Spec_d(X_\Gamma)_1 \) and \( \Spec_d(X_\Gamma)_\Pi \) are infinite, and

\[
\Spec_d(X_\Gamma)_1 \subset [0, +\infty), \quad \Spec_d(X_\Gamma)_\Pi \subset (-\infty, 0].
\]

**5. Transfer maps.** Let \( L \) be a reductive subgroup of \( G \) acting properly on \( X = G/H \) and \( \Gamma \) a discrete subgroup of \( L \). In Section 1 we considered spectral analysis on the standard locally symmetric space \( X_\Gamma \) through the algebra \( \mathcal{P} \) of intrinsic differential operators on \( X_\Gamma \). Another \( \mathcal{C} \)-algebra \( \mathcal{Q} \) of differential operators on \( X_\Gamma \) is obtained from the center \( \mathcal{Z}(\mathcal{G}) \) of the enveloping algebra \( U(\mathcal{G}) \): indeed, \( \mathcal{Z}(\mathcal{G}) \) acts on smooth functions on \( X \) by differentiation, yielding a \( \mathcal{C} \)-algebra of \( L \)-invariant differential operators on \( X \), hence a \( \mathcal{C} \)-algebra of differential operators on \( X_\Gamma = \Gamma \backslash X \) since \( \Gamma \subset L \). In general, there is no inclusion relation between \( \mathcal{P} \) and \( \mathcal{Q} \). In order to compare the roles of \( \mathcal{P} \) and \( \mathcal{Q} \), we highlight a natural homomorphism \( \mathcal{Z}(\mathcal{G}) \to \mathcal{P} \) and a surjective one \( \text{d}!; \mathcal{Z}(\mathcal{G}) \to \mathcal{Q} \). Loosely speaking, the algebras \( \mathcal{Z}(\mathcal{G}) \) and \( \mathcal{Z}(\mathcal{G}) \) separate irreducible representations of the groups \( G \) and \( L \), respectively, hence it is important to understand how irreducible representations of \( G \) behave when restricted to the subgroup \( L \) (branching problem) in order to utilize the algebra \( \mathcal{Q} \) for the spectral analysis on \( X_\Gamma \) via the algebra \( \mathcal{P} \) (see [15,16]). We shall return to this point in Theorem 15 below.

Now assume the proper action of \( L \) on \( X = G/H \) is also transitive, so that \( X \simeq L/L_H \) where \( L_H := L \cap H \) is compact. Up to conjugation, we may assume that \( L_K := L \cap K \) is a maximal compact subgroup of \( L \) containing \( L_H \). Then the pseudo-Riemannian symmetric space \( X \) fibers over the Riemannian symmetric space \( Y = L/L_K \) with fiber \( F := L_K/L_H \), and this induces a fibration for the quotients by \( \Gamma \):

\[
(5.1) \quad F \longrightarrow X_\Gamma \simeq \Gamma \backslash L/L_H \longrightarrow Y_\Gamma = \Gamma \backslash L/L_K.
\]

To expand functions on \( X_\Gamma \) along the fiber \( F \), we define an endomorphism \( p_\tau \) of \( C^\infty(X_\Gamma) \) by

\[
(p_\tau f)(\cdot) := \frac{1}{\dim \tau} \int_K f(\cdot,k) \text{Trace}(\tau(k)) \, dk
\]

for every \( \tau \in \hat{L}_K \). Then \( p_\tau \) is an idempotent, namely, \( p_\tau^2 = p_\tau \). The \( \tau \)-component of \( C^\infty(X_\Gamma) \) is defined by

\[
C^\infty(X_\Gamma)_\tau := \text{Image}(p_\tau) = \text{Ker}(p_\tau - \text{id})
\]

We note that \( C^\infty(X_\Gamma)_\tau \neq \{0\} \) if and only if \( \tau \) has a nonzero \( L_K \)-invariant vector, i.e. \( \tau \in \hat{L}_K \mid L_K \). It is easy to see that the projection \( p_\tau \) commutes with any element in \( \mathcal{Q} \) (\( \simeq \text{d}(\mathcal{Z}(\mathcal{G})) \)), but not always with “intrinsic differential operators” \( D_f \in \mathcal{P} \) (\( \simeq \mathcal{D}(X) \)), and consequently it may well happen that

\[
p_\tau(C^\infty(X_\Gamma; \mathcal{M}_\lambda)) \not\subset C^\infty(X_\Gamma; \mathcal{M}_\lambda).
\]

To make a connection between the two subalgebras \( \mathcal{P} \) and \( \mathcal{Q} \), we introduce a third subalgebra \( \mathcal{R} \) of \( \mathcal{D}(X_\Gamma) \), coming from the fiber \( F \) in (5.1). Namely, \( \mathcal{R} \) is isomorphic to the \( \mathcal{C} \)-algebra \( \mathcal{D}_{L_H}(F) \) of \( L_K \)-invariant differential operators \( D \) on \( F \), and obtained by extending elements of \( \mathcal{D}_{L_H}(F) \) to \( L \)-invariant differential operators on \( X \), yielding differential operators on the quotient \( X_\Gamma \).

Suppose now that we are in the setting 8. The subgroup \( L \) acts transitively on \( X \) by [17, Lem. 4.2] and [12, Lem. 5.1]. Moreover, we can prove [9] that

\[
(5.2) \quad \mathcal{Q} \subset \langle \mathcal{P}, \mathcal{R} \rangle
\]

where \( \langle \mathcal{P}, \mathcal{R} \rangle \) denotes the subalgebra of \( \mathcal{D}(X_\Gamma) \) generated by \( \mathcal{P} \) and \( \mathcal{R} \). This implies the following strong constraints on the restriction of representations:

**Theorem 15.** In the setting 8, any irreducible \( (\mathfrak{g}, K) \)-module occurring in \( C^\infty(X) \) is discretely decomposable as an \( (\mathfrak{l}, L \cap K) \)-module.
See [12–14] for a general theory of discretely decomposable restrictions of representations. See also [16] for a discussion on Theorem 15 when dropping the assumption that L acts properly on X.

In addition to (5.2), the quotient fields of $\mathcal{P}$ and $\langle Q, R \rangle$ coincide [9, Th. 1.3 & §6.9], and we obtain:

**Theorem 16 (Transfer map).** In the setting 8, for any $\tau \in (\tilde{L}_K)^{\mathbb{Z}}$, there is an injective map $\nu(\cdot, \tau): \text{Hom}_{\text{alg}}(\mathcal{D}(X), \mathbb{C}) \rightarrow \text{Hom}_{\text{alg}}(3(\mathbb{C}), \mathbb{C})$ such that for any $\lambda \in \text{Hom}_{\text{alg}}(\mathcal{D}(X), \mathbb{C})$, any $f \in C^\infty(X_1; \mathcal{M}_\lambda)$, and any $z \in 3(\mathbb{C})$, $d\ell(z)(p,f) = \nu(\lambda, \tau)(z) p, f$.

We write $\lambda(\cdot, \tau)$ for the inverse map of $\nu(\cdot, \tau)$ on its image. We call $\nu(\cdot, \tau)$ and $\lambda(\cdot, \tau)$ transfer maps, as they “transfer” eigenfunctions for $\mathcal{P}$ to those for $Q$, and vice versa, on the $\tau$-component $C^\infty(X_\tau)$.

For an explicit description of transfer maps, let

$$\Phi^*: \text{Hom}_{\text{alg}}(3(\mathbb{C}), \mathbb{C}) \xrightarrow{\sim} t_C/W(\mathbb{C})$$

be the Harish-Chandra isomorphism as in (3.1), where $W(\mathbb{C})$ denotes the Weyl group of the root system $\Delta(\mathbb{C}, t_C)$ with respect to a Cartan subalgebra $t_C$ in $\mathbb{C}$. We note that there is no natural inclusion relation between $i_C$ and $t_C$.

For each $\tau \in (\tilde{L}_K)^{\mathbb{Z}}$, we find an affine map $S_\tau: i_C \rightarrow t_C$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{S_\tau} & t_C \\
\downarrow \rho_C/W & & \downarrow \rho_C/W(\mathbb{C}) \\
\Phi^* & & \Phi^* \\
\end{array}
$$

$\text{Hom}_{\text{alg}}(\mathcal{D}(X), \mathbb{C}) \xrightarrow{\nu(\cdot, \tau)} \text{Hom}_{\text{alg}}(3(\mathbb{C}), \mathbb{C})$.

A closed formula for the transfer map $\nu(\cdot, \tau)$ is derived from that of the affine map $S_\tau$, which was determined explicitly in [9, §6–7] for the complexifications of the triples $(L, G, H)$ in the setting 8.

Via the transfer maps, we can utilize representations of the subgroup L efficiently for the spectral analysis on $X_1$, as follows. As in (1.2), let

$$L^2(\Gamma\backslash L) \simeq \int_{\mathcal{F}_\lambda} m_1(\vartheta) \vartheta \, d\sigma(\vartheta)$$

be a disintegration of the regular representation $L^2(\Gamma\backslash L)$ of the subgroup $L$. Then the transform $\mathcal{F}_\lambda$ in Theorem 9 can be built naturally by using (5.3) and the expansion of $C^\infty(X\tau)$ along the fiber $F$ in (5.1). Consider the map

$$\Lambda: (\tilde{L}_H)^{\mathbb{Z}} \rightarrow \text{Hom}_{\text{alg}}(\mathcal{D}(X), \mathbb{C})$$

and the expansion of $C^\infty(X\tau)$ along the fiber $F$ in (5.1). Consider the map

$$\Lambda: (\tilde{L}_H)^{\mathbb{Z}} \rightarrow \text{Hom}_{\text{alg}}(\mathcal{D}(X), \mathbb{C}),$$

$$(\vartheta, \tau) \mapsto \chi_\vartheta(\chi_\vartheta, \tau),$$

where $\chi_\vartheta \in \text{Hom}_{\text{alg}}(3(\mathbb{C}), \mathbb{C})$ is the infinitesimal character of $\vartheta \in L$. Then the Plancherel measure $d\mu$ on $\text{Hom}_{\text{alg}}(\mathcal{D}(X), \mathbb{C})$ in Theorem 9 can be defined by

$$d\mu = \Lambda_*(d\sigma)_{(\tilde{L}_H)^{\mathbb{Z}}} \times (\tilde{L}_H)^{\mathbb{Z}}.$$

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