On measures of classicality/quantumness in quasiprobability representations of finite-dimensional quantum systems

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Abstract

In the present report we discuss measures of classicality/quantumness of states of finite-dimensional quantum systems, which are based on a deviation of quasiprobability distributions from true statistical distributions. Particularly, the dependence of the global indicator of classicality on the assigned geometry of a quantum state space is analysed for a whole family of Wigner quasiprobability representations. General considerations are exemplified by constructing the global indicator of classicality/quantumness for the Hilbert-Schmidt, Bures and Bogoliubov-Kubo-Mori ensembles of qubits and qutrits.

1 Background and motivation

A centenary history of development of quantum theory shows a persistent request for a genuine unification of basic quantum-mechanical principles with concepts of classical statistical physics. The primary difficulties on this way are due to a fundamental ban originating from the Heisenberg canonical commutation relations, $[q, p] = i \hbar$, between the phase-space variables $(q, p)$. The non-vanishing Plank constant $\hbar$ impedes the existence of a function $W_\rho(p, q)$, playing the role of a proper joint probability distribution of coordinates $q$ and momenta $p$ associated with a given quantum state $\rho$. In the early years of development of quantum theory, rejection of a complete statistical
description allowed Weyl and Wigner to formulate a phase-space representation of quantum mechanics with a quasiprobability function $W_\varrho(p, q)$, such that the corresponding marginals are true probability distributions of the canonically conjugate coordinates $q$ and $p$ \[1, 2\]. However, in contrast to proper distributions the function $W_\varrho(p, q)$ is not everywhere non-negative for all quantum states and thus it can only be interpreted as a quasiprobability distribution function\[1\]. Today, despite this drawback, a description of quantum systems using the technique of quasiprobability distribution became an important source of our understanding of quantum phenomena (see e.g. \[3, 4, 5\] and references therein). Furthermore, perceiving the inevitability of the existence of negative values in quasiprobability representation of states as a reflection of the real “quantumness” of physical systems, studies move to a practical context, where the negativity of states is taken as a basis for building corresponding measures of nonclassicality (e.g., \[6\] and references therein). However, elaborating this idea we are faced with a serious complexity. Indeed, introducing an admissible indicator of classicality/quantumness, the following general requirements should be taken into account:

(I) independence of an indicator from a representation of quantum states;

(II) independence of an indicator from a quasiprobability representation of states.

While satisfying the first requirement is a relatively simple issue, it is enough to assume that an indicator is a function of a state unitary invariants, the second task is a highly nontrivial one. There exist infinitely many quasiprobability distributions and a quantum state can be negative in one representation and positive in another. In \[7\] it was argued that the positivity in one representation is neither a necessary nor a sufficient condition for classical description, nor the negativity of a specific representation is sufficient for nonclassicality. Considering any one of these quasiprobability representations we are not able to determine absolute criteria for the classicality/quantumness. Ideally, in order to quantify a state classicality/quantumness we need to determine characteristics which are unique for a complete family of such representations.

In the present report we will discuss both issues, (I) and (II), constructing a classicality/quantumness measure for a family of the Wigner quasiprobability representation of finite-dimensional quantum systems. We will follow approach \[9, 10, 11\] to the construction of the Wigner quasiprobability distributions $W^{(\nu)}_{\varrho}(\Omega_N)$ of an $N$–level quantum system via the dual pairing,

$$W^{(\nu)}_{\varrho}(\Omega_N) = \text{tr} \left[ \varrho \Delta(\Omega_N | \nu) \right], \quad (1)$$

\[1\]Certainly, for some states there exists such a true statistical distribution. For example, according to the Hudson’s theorem \[8\], a Gaussian wave function is the only pure state corresponding to a positive Wigner function.
of the density matrix \( \varrho \in \mathcal{P}_N \) from the quantum state space \( \mathcal{P}_N \):

\[
\mathcal{P}_N = \{ X \in M_N(\mathbb{C}) \mid X = X^\dagger, \; X \geq 0, \; \text{tr}(X) = 1 \}, \quad (2)
\]

and the Stratonovich-Weyl (SW) kernel \( \Delta(\Omega_N | \nu) \in \mathcal{P}_N^* \) from the dual space \( \mathcal{P}_N^* \):

\[
\mathcal{P}_N^* = \{ X \in M_N(\mathbb{C}) \mid X = X^\dagger, \; \text{tr}(X) = 1, \; \text{tr}(X^2) = N \}. \quad (3)
\]

Analysing algebraic equations (3), one can conclude that,

a) The phase-space \( \Omega_N \) can be identified as a complex flag manifold, \( \Omega_N \rightarrow \mathbb{R}^N_{d_1,d_2,...,d_s} = U(N)/H \), where \( (d_1,d_2,...,d_s) \) is a sequence of positive integers with sum \( N \), such that \( k_1 = d_1 \) and \( k_{i+1} = d_{i+1} - d_i \) with \( d_{s+1} = N \). The corresponding SW kernel has the isotropy group \( H = U(k_1) \times U(k_2) \times U(k_{s+1}) \);

b) The isotropy group \( H \) of SW kernel provides the existence of a family of Wigner distributions. The corresponding moduli space \( \mathcal{P}_N \) represents a spherical polyhedron on \( (N-2) \)-dimensional sphere \( S_{N-2}(1) \) of radius one. Further in the text, the \( s \)-dimensional moduli parameter \( \nu = (\nu_1, \nu_2, \ldots, \nu_s) \), \( s \leq N-2 \) will be used to enumerate the Wigner distributions (see details in [10],[11]).

The representation independent characteristic of the classicality can be constructed by averaging over the moduli space \( \mathcal{P}_N(\nu) \):

\[
\langle Q \rangle = \frac{1}{\text{Vol}(\mathcal{P}_N)} \int_{\mathcal{P}_N} d\mathcal{P}_N(\nu) \; Q_N[g | \nu] \quad (4)
\]

of the global indicator of classicality/quantumness \( Q_N \) defined as the relative volume ratio of the subspace \( \mathcal{O}[\mathcal{P}_N^{(+)}] \) of the orbit space \( \mathcal{O}[\mathcal{P}_N] = \mathcal{P}_N/SU(N) \) of the state space \( \mathcal{P}_N \), where the Wigner function is non-negative [12]:

\[
Q_N[g | \nu] = \frac{\int \cdots \int_{\mathcal{O}[\mathcal{P}_N^{(+)}]} d\mathcal{P}_N(g | \nu)}{\int \cdots \int_{\mathcal{O}[\mathcal{P}_N]} d\mathcal{P}_N(g | \nu)} . \quad (5)
\]

The total orbit space \( \mathcal{O}[\mathcal{P}_N] \) can be realised as the ordered \( (N-1) \)-simplex in the space of eigenvalues \( r^\downarrow = \{ r_1, r_2, \ldots, r_N \} \) of a density matrix \( g \):

\[
\mathcal{C}^{(N-1)} = \{ r \in \mathbb{R}^N \mid \sum_{i=1}^{N} r_i = 1, \; 1 \geq r_1 \geq r_2 \geq \cdots \geq r_{N-1} \geq r_N \geq 0 \} , \quad (6)
\]
while according to [12] the subspace $\mathcal{O}[\mathfrak{P}_N^{(+)}]$ represents a dual cone of $\mathcal{O}[\mathfrak{P}_N]$:

$$\mathcal{O}[\mathfrak{P}_N^{(+)}] = \{ \pi \in \text{spec} (\Delta(\Omega_N)) \mid (r^\dagger, \pi^\dagger) \geq 0, \quad \forall r \in \mathcal{O}[\mathfrak{P}_N] \},$$

with the cone defined via the dual pairing $(r^\dagger, \pi^\dagger) = r_1\pi_N + r_2\pi_{N-1} + \cdots + r_N\pi_1$, of $r$ and the $N$-tuple $\pi$ of increasing eigenvalues of SW kernel $\Delta(\Omega_N | \nu)$.

The suggested measure of classicality (4) fulfils both conditions, (I) and (II). By the averaging procedure in (4) we fulfil requirement (II) and since the global indicator of classicality/nonclassicality is defined on the orbit space of a quantum system and therefore provides an unitary invariant measure, the requirement (I) is satisfied as well. However, the indicator (4) depends on metrical characteristics of the moduli space $\mathcal{P}_N$ and the orbit space $\mathcal{O}[\mathfrak{P}_N]$. Since the moduli space is represented by an $(N-2)$—dimensional spherical polyhedron, we suppose that the corresponding measure corresponds to uniform distribution on $\mathbb{S}_{N-1}(1)$, while the measure on the orbit space, $d\mathbb{P}_N(g | r) = \sqrt{\det ||g||} dr_1 \wedge dr_2 \wedge \cdots \wedge dr_N$, is induced from the Riemannian metric $g$ on the state space $\mathfrak{P}_N$. In the remaining part of the report we will discuss the dependence of the suggested indicator of classicality on the metric of a quantum state space. From a wide variety of special Riemannian metrics commonly used in the Quantum Statistics and Information Theory, we will analyse the Hilbert-Schmidt metric and two monotone metrics [13, 14], the Bures and the Bogoliubov-Kubo-Mori metrics. Detailed discussion of the indicator of classicality for qubit ($N = 2$) and qutrit ($N = 3$) will be given.

## 2 Riemannian geometry of state space $\mathfrak{P}_N$ and classicality/quantumness indicator $Q_N$

In this section the results of our studies of the dependence of the indicator (5) on the metric of a quantum state space will be presented. We will consider a basic, Hilbert-Schmidt (HS) metric and two representatives of the family of the so-called monotone metrics, the Bures (B) and Bogoliubov-Kubo-Mori (BKM) metrics.

• **The Hilbert-Schmidt metric** • For an $N$—dimensional quantum system the infinitesimal version of the Hilbert-Schmidt distance is given by the expression:

$$g_{\text{HS}} = 4 \text{tr} (d\varrho \otimes d\varrho).$$

(8)

For further computational aims it is convenient to rewrite (8) in terms of SVD of a density matrix, $\varrho = UDU^\dagger$, where $U \in SU(N)$, and $D = \text{diag} ||r_1, r_2, \ldots, r_N||$ with descending order of eigenvalues from the $(N-1)$-simplex (3). Here we assume that the spectrum of $\varrho$ is generic and thus the arbitrariness of $U$ is given by the isotropy group represented by the torus $T$ of $SU(N)$. In terms of SVD coordinates, the volume form factorizes into the measure $\omega_{SU(N)/T}$ on the coset $SU(N)/T$ induced from the Haar
measure on the $U(N)$ group manifold and the “radial part” factor which depends on the spectrum of a state only. The latter represents the Hilbert-Schmidt measure $dP(g_{\text{hs}})$ on the orbit space [6],

$$dP_N(g_{\text{hs}}|r) = c_{\text{hs}} \delta\left(\sum_{i=1}^{N} r_i - 1\right) \prod_{i<j} (r_i - r_j)^2 dr_1 \wedge dr_2 \wedge \cdots \wedge dr_N , \quad (9)$$

where $c_{\text{hs}}$ is a normalization constant.

• The Bures and Bogoliubov-Kubo-Mori metrics • Using the SVD decomposition of elements of $\mathfrak{P}_N$, the stochastically monotone metrics can be written in the following form:

$$g_f = \frac{1}{4} \sum_{i=1}^{N} \frac{dr_i \otimes dr_i}{r_i} + \frac{1}{2} \sum_{i<j} c_f(r_i, r_j)(r_i - r_j)^2 \left(U^\dagger dU\right)_{ij} \otimes \left(U^\dagger dU\right)_{ij} , \quad (10)$$

where $c(x, y)$ is the so-called Morozova-Chentsov function; $c_f(x, y) = \frac{1}{y f(x/y)}$ is given by the operator monotone function $f(t)$. For the Bures and BKM metrics these functions are $f_{BW}(t) = (1 + t)/2$ and $f_{\text{bkm}}(t) = (t - 1)/\ln t$, respectively. Having these representations, we are in a position to compare the indicators $Q$ for the simplest, two- and three-level systems endowed with the above described metrics.

2.1 Qubit

From (3) it follows that the spectrum of SW kernel of a 2-level system is unique, $\text{spec}(\Delta_2) = \{(1 + \sqrt{3})/2, (1 - \sqrt{3})/2\}$. Its dual pairing with a 2-level density matrix $\rho = \frac{1}{2} [I_2 + (\xi, \sigma)]$, characterized by the Bloch vector $\xi \in \mathbb{R}^3$, gives the Wigner quasiprobability distribution of the qubit defined on a 2-sphere:

$$W_\phi(n) = \frac{1}{2} + \frac{\sqrt{3}}{2} (\xi, n) , \quad n \in S^2 .$$

All mixed states belong to the Bloch ball, $(\xi, \xi) \leq 1$, while the positivity cone $(\xi, \xi) < 1/3$.

• The Hilbert-Schmidt metric • Taking into account the positivity domain (7) and using the expression (9) for $N = 2$, the indicator $Q_2$ of the Hilbert-Schmidt qubit reduces to the ratio of two simple integrals,

$$Q_2[g_{\text{hs}}] = \frac{\int_0^{1/\sqrt{3}} r^2 dr}{\int_0^{1} r^2 dr} = \frac{1}{3 \sqrt{3}} \approx 0.19245 . \quad (11)$$
Figure 1: A qubit probability $Q(r)$, calculated for the Hilbert-Schmidt, Bures and the Bogoliubov-Kubo-Mori metrics.

- **Bures and BKM metric** - Similar calculation for the Bures and BKM ensemble of qubits give,

$$Q_2^{[g_B]} = \frac{\text{Vol}_B\left(\frac{1}{\sqrt{3}}\right)}{\text{Vol}_B(1)} = \frac{2}{\pi} \left[ \arcsin \frac{1}{\sqrt{3}} - \frac{\sqrt{2}}{3} \right] \approx 0.09172, \quad (12)$$

$$Q_2^{[g_{BKM}]} = \frac{\text{Vol}_{BKM}\left(\frac{1}{\sqrt{3}}\right)}{\text{Vol}_{BKM}(1)} = \frac{2}{\pi} \left[ \arcsin \frac{1}{\sqrt{3}} - \sqrt{\frac{2}{3}} \operatorname{arcoth} \sqrt{3} \right] \approx 0.0495506,$$

where $\text{Vol}_X(r)$ denotes the volume of the Bloch ball of radius $r$ in metric “X”. The corresponding probability $Q(r)$ to find a qubit state $\rho$ with positive WF within the Bloch ball of radius $r$ is depicted in Fig. 1.

2.2 Qutrit

- **The Hilbert-Schmidt metric** - According to (3), the Wigner quasiprobability representation of a 3-level system is one-parametric. The spectrum of SW kernel can be parametrized by the apex angle $\zeta \in [0, \pi/3]$ of a unit circle segment [10]:

$$\text{spec}(\Delta_3) = \left\{ \frac{1}{3} + \frac{2}{\sqrt{3}} \sin \zeta + \frac{2}{3} \cos \zeta, \frac{1}{3} - \frac{2}{\sqrt{3}} \sin \zeta + \frac{2}{3} \cos \zeta, \frac{1}{3} - \frac{4}{3} \cos \zeta \right\}. \quad (13)$$

Decomposing a qutrit density matrix spectrum via the polar coordinates $(r, \varphi)$,

$$\text{spec}(\rho) = \left\{ \frac{1}{3} - \frac{2r}{\sqrt{3}} \cos \frac{\varphi + 2\pi}{3}, \frac{1}{3} - \frac{2r}{\sqrt{3}} \cos \frac{\varphi + 4\pi}{3}, \frac{1}{3} - \frac{2r}{\sqrt{3}} \cos \frac{\varphi}{3} \right\}. \quad (14)$$
a qutrit orbit space and its subspace of WF positivity reads \((r \geq 0, \varphi \in [0, \pi])\):

\[
\mathcal{O}[\mathbb{P}_3] : \cos \left( \frac{\varphi}{3} \right) \leq \frac{1}{2\sqrt{3}r}, \quad \mathcal{O}[\mathbb{P}_3^{(+)}] : \cos \left( \frac{\varphi}{3} + \frac{\pi}{3} \right) \leq \frac{1}{4\sqrt{3}r}.
\] (15)

Taking into account the expression for the Hilbert-Schmidt measure on the orbit space \(\mathcal{O}[\mathbb{P}_3]\), \(\omega_{\mathcal{O}[\mathbb{P}_3]} = r^7 \sin^2 \varphi \, dr \wedge d\varphi\), we derive the global indicator of classicality of the Hilbert-Schmidt qutrit as function of the moduli parameter \(\zeta\) [12]:

\[
Q_3(\zeta) = \int_0^\pi d\varphi \int_0^{\frac{1}{2\sqrt{3} \cos \frac{\pi}{6}}} r^7 \sin^2(\varphi) dr = 1 \frac{1 + 20 \cos^2 \left( \zeta - \frac{\pi}{6} \right)}{128 \left( -1 + 4 \cos^2 \left( \zeta - \frac{\pi}{6} \right) \right)^{\frac{5}{2}}}. \] (16)

Note, that the indicator \(Q_3(\zeta)\) attains at a qutrit moduli parameter \(\zeta = \pi/6\) the absolute minimum, \(\min_{\zeta \in [0, \frac{\pi}{3}]} Q_3(\zeta) \approx 0.000675\), corresponding to SW kernel with the spectrum: \(\text{spec} (\Delta_3) |_{\zeta = \frac{\pi}{6}} = \|1 + 2\sqrt{3} \frac{3}{3}, \frac{1}{3}, \frac{1 - 2\sqrt{3}}{3}\|\).

3 Final remarks

As it was outlined in the first part of our report, a true classicality/quantumness measure, being universal for different quasiprobability representations, may be sensitive to the geometry of a state space. Our calculations of the average of the global indicator \(Q(\zeta)\) over a qutrit moduli space support this supposition,

\[
\langle Q_{\text{HS}} \rangle_\zeta = 0.00136368, \quad \langle Q_B \rangle_\zeta = 0.00019165, \quad \langle Q_{\text{BKM}} \rangle_\zeta = 0.00002762. \] (17)

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