Quasibound states for a scalar field under the influence of an external magnetic field in the near-horizon geometry of the BTZ black hole with torsion

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Abstract: We consider a charged scalar field under the effect of an external uniform magnetic field in the near horizon of the Banados–Teitelboim–Zanelli black hole with torsion. By solving the corresponding Klein–Gordon equation, we determine quasi-stationary states of the system in question. We observe that the real and damped oscillations of the system depend on the strength of the external magnetic field as well as the parameters of the spacetime background. We see that the magnitude of the real oscillations decreases and the decay time of the damped modes becomes longer as the strength of the external magnetic field increases. Additionally, the results indicate that the background is stable under such a perturbation field.

Keywords: Quasibound states; Quasi-stationary states; BTZ black hole; Klein–Gordon equation; Magnetic field; Torsion

1. Introduction

General relativity in 2+1 dimensions has gained great interest after the seminal papers of Deser, Jackiw, ’t Hooft [1–3] and Witten [4, 5] were announced [6]. After these research works, the 2+1-dimensional gravity has been regarded as a very useful laboratory to discuss some conceptual issues. At that times, it was believed that 2+1-dimensional gravity models cannot give much insight into 3+1-dimensional real gravitating physical systems [6]. This is because three-dimensional general relativity has no propagating degrees of freedom and has no Newtonian limit [6, 7]. However, it has been shown that solution of the standard Einstein equations in asymptotically anti-de Sitter spacetime in 2+1 dimensions [8–10] provides a black hole solution, the Banados–Teitelboim–Zanelli (BTZ) black hole (BH). This 2+1-dimensional BH is characterized by mass, angular momentum, and charge. That is, the BTZ BH is similar to its 3+1-dimensional counterparts [8] since it has an event horizon and an inner horizon (in the rotating case). However, the BTZ BH differs from the Kerr and Schwarzschild BHs in some aspects. For example, it has no curvature singularity at the spatial origin and is not asymptotically flat [6, 8]. The BTZ BH has attracted great attention [11–19] and, moreover, has played a crucial role in many developments in string theory [20–22]. The BHs are generic predictions of Einstein’s general relativity, but it was shown that these objects are not just mathematical objects. Hawking has shown us that the BHs may emit radiation from the horizon [23, 24], and this provides an opportunity for testing the theories of gravity. This can be acquired by determining the characteristic oscillations of the BH spacetimes since it is known that the BH spacetime undergoes damped oscillations when it is perturbed [25–28]. Determining quasinormal modes (QNMs) [9] for the BHs, described by some parameters such as the mass, charge, and angular momentum, is very important since these modes carry information about the BHs [17, 18]. The QNMs can include long-lived trapped modes called quasibound states (QBSs). These states, known also as quasi-stationary levels or resonance spectra, are localized in the BH potential well and tend to zero at spatial infinity. This means that there exist two boundary conditions associated with the QBSs [29]. QBSs for BHs have been widely studied. The quasi-spectrum of resonant frequencies in the Schwarzschild acoustic BH spacetime [30], QBSs of a massive scalar field for dilatonic charged BHs [31],

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resonance modes for the near-extremal Kerr BH spacetime [32], and QBS test field configurations of scalar fields in the background of the Reissner–Nordström BH [33] can be considered among such investigations. Additionally, one of the fundamental results of Kerr/CFT correspondence is that the near horizon of a BH encodes many important pieces of information [34]. This means that determining the evolution of test fields in the near horizon of the BHs (with and without torsion) can be very important. Here, it may be also useful to give some information about the role of torsion in the gravity theories [10]. In Einstein’s general relativity, there is no torsion in the spacetime generated by the source(s). However, it has been shown that there can be several new possibilities when the presence of the torsion or vorticity is considered (see also Refs. [35–39]). For example, dark matter may represent a manifestation of non-trivial torsion [10, 40–42], or dark energy and dark matter may be replaced by non-trivial torsion [10, 43].

On the other hand, it is known that magnetic fields exist at almost every point in the universe, and these fields can magnetize the universe [43–45]. Even though the origin of intra-cluster, galactic, and cosmological magnetic fields is not yet exactly known, it is thought that dynamo effects in turbulent fluids can amplify (exponentially) the seed fields [43–45]. The magnetic fields can be responsible for many interesting phenomena in the universe, and determining the effects of magnetic fields on the quantum mechanical systems has attracted great attention [46–51]. At high energies, determining the dynamics of quantum mechanical systems in curved spaces can be acquired by solving the Lorentz-invariant wave equations, such as the Dirac equation, the Duffin–Kemmer–Petiau equation, vector boson equation, fully covariant many-body equations, and the Klein–Gordon equation (KGE). Several applications of these equations can be found in Refs. [52–59]. The KGE is used to describe the relativistic dynamics of spinless particles [60] and has been studied many times to determine the characteristic oscillations of the BH spacetimes [61–65] (for more details see the reviews in [9, 29]). We think that it can be useful to determine the QBSs for a charged scalar field under the influence of an external magnetic field in the near-horizon geometry of the BTZ BH with torsion obtained by solving an analytical solution of the KGE (see also [66]). The QBSs are obtained through solving the wave equations for ongoing waves at the exterior event horizon. These modes tend to zero at spatial infinity, and the resonant frequency spectrum is related to the decay of the perturbation. That is, they are related to the damped oscillations. Such an investigation may allow us to discuss the effect of an external magnetic field on both the real oscillation frequency of the modes and their decay time. In this manuscript, we investigate the evolution of a test scalar field (massive and charged) exposed to an external uniform magnetic field in the near-horizon geometry of the BTZ BH with torsion (see [10]). This manuscript is structured as follows: In Sect. 2, we write the generalized KGE and obtain the corresponding wave equation for a massive KGE particle (with charge $e$) exposed to an external uniform magnetic field in the near-horizon geometry of the BTZ BH with torsion. In Sect. 3, we obtain complex spectra for the test field in question by analytically solving the associated wave equation. In Sect. 4, we give a summary and discuss the results. In this manuscript, we use the units $\hbar = c = 1$.

### 2. Mathematical procedure

In this part of the paper, we introduce the generalized KGE and obtain the corresponding form of this equation for a spinless particle (massive and charged) exposed to an external magnetic field in the near-horizon geometry of the BTZ BH with torsion. The generalized form of the KGE can be written as follows [67–71]:

$$
\left[ \frac{1}{\sqrt{|-g|}} D_\nu \left( \sqrt{-g} g^{\nu \lambda} D_\lambda \right) - \mu^2 \right] \chi(x) = 0,
$$

(1)

where,

$$
D_\nu = \partial_\nu + ieA_\nu, \quad D_\nu = \partial_\nu + ieA_\nu, \quad \lambda, \nu = 0, 1, 2.
$$

Here, $g = det(g_{\mu \nu})$, $g^{\nu \lambda}$ is the contravariant metric tensor, $e$ is the electrical charge of the particle, $\mu$ is the KGE field with mass of $\mu$, $A_\nu$ is the electromagnetic vector potential, and $x$ is the spacetime position vector of the particle. The near-horizon geometry of the rotating BTZ BH with torsion was investigated to see the role of torsion in the near-horizon geometry, and it was shown that this spacetime background corresponds to the generalization of the AdS self-dual orbifold possessing non-trivial torsion. More detailed discussion about this spacetime structure can be found in Ref. [10]. The mentioned geometric background can be represented through the following 3D line element with signature $+,-,-$ [10]

$$
\text{d}s^2 = \frac{4\rho \ell}{\ell} \text{d}t \text{d}\phi - \ell^2 \text{d}r^2 - r_0^2 \text{d}\phi^2,
$$

(2)

for which $det(g_{\mu \nu}) = r_0^2$. Here, $r_0$ is the value of the radius of the event horizon, and $\ell$ (inverse) have been associated with the torsion in the geometric background [10]. According to Eq. (2), the contravariant metric tensor is determined as follows:

$$
\hat{g}^{\nu \lambda} = \begin{pmatrix}
\frac{\ell^2}{4\tau^2} & 0 & \frac{\ell}{2\tau r_0^2} \\
0 & -\frac{4\tau^2}{4\tau r_0^2} & 0 \\
\frac{\ell}{2\tau r_0^2} & 0 & 0
\end{pmatrix}.
$$

(3)

Here, it is useful to underline that we consider a charged scalar field under the effect of an external uniform
magnetic field in the background geometry represented through the metric in Eq. (2) without discussing the origin of the external field. This field is taken into account through the angular component of the electromagnetic vector potential as \( A_\varphi = \beta \varphi \) [72], in which \( B_0 \) is the amplitude of the external uniform magnetic field and \( r \) is the radial coordinate of the particle in question. According to Eq. (2), we can factorize the wave function \( \chi(x) \) as
\[
\chi(x) = e^{-i(\omega t - m \varphi)} \tilde{\chi}(r),
\]
in which \( \omega \) and \( m \) represent the relativistic frequency and the azimuthal quantum number, respectively. At that rate, one can obtain the following wave equation:
\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \frac{\mathcal{K} - \omega \mathcal{K}_1}{r^2} - \frac{\omega \mathcal{K}_2}{r^3} + \frac{\omega^2 \mathcal{K}_3}{r^4} \right] \tilde{\chi}(r) = 0, \tag{4}
\]
where
\[
\mathcal{K} = \frac{\mu^2 \ell^2}{4}, \quad \mathcal{K}_1 = \frac{B \ell^3}{4r_0}, \quad \mathcal{K}_2 = \frac{m \ell^3}{4r_0}, \quad \mathcal{K}_3 = \frac{\ell^4}{16},
\]
and \( B = \frac{\ell^3}{r_0} \). In the next step, we will solve this equation and obtain frequency spectra for the considered perturbation field.

3. Quasibound states

By considering a new change of variable, \( \rho = \frac{2i \omega \sqrt{\mathcal{K}}}{r} \) (note that \( \rho \to 0 \) when \( r \to \infty \)), the wave equation in Eq. (4) can be clarified as follows:
\[
\left[ \frac{d^2}{d\rho^2} + \frac{i \mathcal{K}_2}{2 \sqrt{\mathcal{K}_3} \rho} + \frac{\mathcal{K} - \omega \mathcal{K}_1}{\rho^2} - \frac{1}{4} \right] \chi(\rho) = 0. \tag{5}
\]
This equation has an irregular singular point at spatial infinity and a regular singular point at spatial origin [73]. Here, it is clear that information about the frequency of the system in question is lost when \( B_0 = 0 \). This also means that the presence of the external uniform magnetic field allows us to determine the frequency modes for the system under scrutiny. If \( B_0 \neq 0 \), solution functions are obtained in terms of the confluent hypergeometric \((\mathcal{C}\mathcal{H})\) functions, \( _1\mathcal{F}_1 \), as follows [73]:
\[
\chi(\rho) = e^{-\frac{1}{2} \rho^2} \left[ C_1 \rho^{\beta} _1\mathcal{F}_1 \left( \frac{1}{2} - \alpha + \beta, 1 + 2 \beta; \rho \right) + C_2 \rho^{-\beta} _1\mathcal{F}_1 \left( \frac{1}{2} - \alpha - \beta, 1 + 2 \beta; \rho \right) \right], \tag{6}
\]
where
\[
\alpha = \frac{i \mathcal{K}_2}{2 \sqrt{\mathcal{K}_3}}, \quad \beta = \sqrt{1 + 4(\omega \mathcal{K}_1 - \mathcal{K})},
\]
and \( C_1 \) and \( C_2 \) are the normalization constants. In the near horizon, the functions in Eq. (6) become
\[
\chi(\rho) = C_1 \rho^{\beta - \frac{1}{2}} + C_2 \rho^{-\beta - \frac{1}{2}}. \tag{7}
\]
However, the solution function must be finite near the horizon. Hence, we take \( C_2 = 0 \). By using the asymptotic forms of the \( \mathcal{C}\mathcal{H} \) functions [73], the solution functions can be written as follows:
\[
\chi(\rho) \to C_1 \left( e^{\rho^2} \rho^{-1-2} \frac{\Gamma(1+2\beta)}{\Gamma(\frac{1}{2}+\beta-\alpha)} + e^{-3\rho^2} \rho^{-1+2} (-1)^{-\frac{1}{2}+\beta+\frac{1}{2}} \right). \tag{8}
\]
The first term in Eq. (8) corresponds to asymptotic ingoing waves, and the second term corresponds to asymptotic outgoing waves. The resonance condition, which requires \( \frac{1}{2} + \beta - \alpha = -n \) [73] where \( n = 0, 1, 2, \ldots \) is the overtone number, guarantees that the solution function becomes polynomial of degree \( n \) with respect to the \( \rho \). This leads to the following spectra [47]:
\[
\omega = \omega_{Re} - i \omega_{Im}, \tag{9}
\]
where
\[
\omega_{Re} = \frac{\mu^2 m^2 r_0 - 4r_0}{2eB_0 \ell^3} + \frac{2r_0 n^2 - 8m^2}{eB_0 r_0 \ell^2}, \quad \omega_{Im} = \frac{4mn^2}{eB_0 \ell^2}, \quad n^2 = (n + 1)^2,
\]
which tell us what the dependence of the modes is on the external magnetic field and spacetime parameters. It can be seen that the spectrum consists of real \((\omega_{Re})\) and imaginary \((\omega_{Im})\) parts. Here, it is very important to say that our aim is to find discrete frequency modes of the system. Also, it is worth mentioning that we consider a charged scalar particle exposed to an external uniform magnetic field in the near-horizon geometry of the BTZ BH with torsion. The line element in Eq. (2) describes only the near-horizon geometry of the BTZ BH with torsion, and hence the frequency expression given by Eq. (9) cannot give all of the information about the extremal BTZ BH. Our results describe only the interaction of a charged scalar particle exposed to an external uniform magnetic field with the near-horizon geometry of the BTZ BH with torsion. However, we think that the near-horizon geometry of the BHs encodes important information about the BH. Therefore, our results may give some useful information about the BTZ BH with torsion. The presence of a nonvanishing inward current at the BH horizon implies that these states cannot be steady states. These states, called quasibound states, include real and imaginary parts. The imaginary part gives the exponential behavior, that is the field decays for \( \omega_{Im} > 0 \) and grows for \( \omega_{Im} < 0 \) in time. The \( \omega_{Im} < 0 \) corresponds to unstable mode, whereas \( \omega_{Im} > 0 \) corresponds to the stable mode. Physically, positive \( \omega_{Im} \)
implies that such a charged particle has probabilities to permeate into the BH for the nonvanishing strength of the external magnetic field. Here, we see that the real and imaginary parts of the spectra are inversely proportional to the magnetic field. As we mentioned before, our results are valid only for nonvanishing values of the strength of the external magnetic field (see also Refs. [68–71]). The real part \( \omega_{Re} \) gives the real oscillation frequency of the modes and the imaginary part \( \omega_{Im} \) relates to their decay rates [9]. In Eq. (9), we observe that real oscillation frequency of the modes and the decay times [9] \( (\tau, \tau \propto \frac{1}{\omega_{Im}}) \) depend on the strength of the external uniform magnetic field as well as the parameters of the spacetime background, \( r_0 \) and \( \ell \) [10]. According to the spectrum in Eq. (9), we obtain the following expression for the decay time of the modes associated with the test field in question

\[
\tau = \frac{eB_0 \ell^2}{4m}\.\nonumber\]

This result indicates that the decay times of the modes become longer as the strength of the external magnetic field increases. It is known that the decay time of such modes is related to the time to reach thermal equilibrium of the BHs [9]. Hence, one can infer that the existence of such an external magnetic field may change the thermalization process of the BHs. Here, we should also underline that the decay time for the modes corresponding to excited states of the considered perturbation test field is smaller than the decay time corresponding to the ground state \((n = 0)\) mode provided that \( m \neq 0 \) (note that the sign of \( \omega_{Im} \) remains unchanged when \( m \to -m \) and \( e \to -e \) and \( \omega_{Im} = 0 \) when \( m = 0 \)). Our results have also shown that the amplitude of the real oscillation modes decreases as the strength of the external magnetic field increases.

4. Results and discussion

In this contribution, we have analyzed the evolution of the test scalar field (with mass \( \mu \), and charge \( e \)) exposed to an external uniform magnetic field in the near-horizon geometry of the BTZ BH with torsion. To acquire this, we have analytically solved the corresponding \( KGE \). First of all, we have obtained the corresponding form of the \( KGE \) for the system in question, which is given by Eq. (4). We have determined the solution function of this equation and, accordingly, have arrived at a frequency expression including real \((\omega_{Re})\) and imaginary \((\omega_{Im})\) parts in the form of \( \omega = \omega_{Re} - i\omega_{Im} \) (see Eq. 9). We observe that it does not seem to be possible to determine the characteristic oscillations of the considered geometric background under test scalar field perturbation in the absence of the external magnetic field. Our results show that the presence of an external uniform magnetic field can provide an opportunity to acquire information about the geometric background. In Eq. (9), \( \omega_{Re} \) gives the real oscillation frequency of the modes and \( \omega_{Im} \) (inverse) is proportional to their decay time. Our results have shown that only real oscillations seem possible when \( m = 0 \) because the imaginary part becomes \( \omega_{Im} = 0 \) for this case. Provided that \( m \neq 0 \), real oscillation frequency of the modes and their decay time depend on the strength of the external uniform magnetic field \((\propto B_0)\) as well as the parameters of the spacetime background. This fact allows us to analyze the effects of the external magnetic field on the real and damped modes.

5. Conclusions

Here, we observe that the amplitude of the real oscillation modes decreases as the strength of the external magnetic
field increases (see Fig. 1). Our results show also that the decay time of the modes increases linearly as the strength of the external magnetic field increases (see Fig. 2). In our results, $\omega_{00}$ is positive and this means that the spacetime background is stable under the considered perturbation field. One can also see that the decay time of the modes does not depend on the value of the $r_0$ even though the real oscillations depend explicitly on the $r_0$. Our results imply that the existence of the external magnetic field in the near horizon of the BTZ BH with torsion may affect the time to reach thermal equilibrium of the BH (see [9]).

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**Declarations**

**Conflict of interest** There is no conflict of interest declared by the authors.

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