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Coherence effects on estimating two-point separation: supplementary material

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This document provides supplementary material to "Coherence effects on estimating two-point separation," https://doi.org/10.1364/OPTICA.403497. A derivation for the density matrix representing the two-point separation problem is provided. This is followed by the process of obtaining a set of spanning orthonormal kets required for the calculation of the quantum Fisher information matrix.

1. DENSITY MATRIX FOR TWO POINT SOURCES

The derivation of the density matrix $\hat{\rho}$ follows that of Tsang’s work (see Appendix B of Ref. 1). To summarize, one begins with the coherent state (Sudarshan-Glauber) representation of the density operator

$$\hat{\rho} = \int \Phi(v)|v\rangle\langle v| \, d^2Mv,$$  \hspace{1cm} (S1)

where the integral is over the entire complex phase space in which the coherent state is defined and $v = [v_1, \ldots, v_M]\textsuperscript{T}$ is a column vector of complex field (coherent) amplitudes for $M$ optical space modes on the image plane. That is, $|v\rangle$ is a multimode coherent state with (vector) amplitude $v$. The probability of having $j$ total photons, $p_j$, is then given by $p_j = \text{Tr}(\hat{\rho}|j\rangle\langle j|)$, where $|j\rangle$ is the $j$ photon multimode state.

Several reasonable assumptions are now considered. The average number of photons arriving at the image plane during the coherence time of the source, $\epsilon$, is considered to be much smaller than 1. That is,

$$\epsilon \triangleq \langle \hat{n} \rangle = \sum_{m=0}^{M} \text{Tr}(\hat{\rho}\hat{n}_m) \ll 1,$$  \hspace{1cm} (S2)

where $\hat{n}$ is the multimode photon number operator, $\hat{n}_m$ is the photon number operator for the $m$-th mode and $\langle \cdot \rangle$ denotes an expectation value with respect to the representation in Eq. (S1). One can rewrite $\epsilon$, using $\hat{n} = \hat{a}^\dagger \hat{a}$ as

$$\epsilon = \langle \hat{a}^\dagger \hat{a} \rangle = \langle |v|^2 \rangle = \int \Phi(v)|v|^2 \, d^2Mv \ll 1,$$  \hspace{1cm} (S3)

where $\hat{a}^\dagger$ and $\hat{a}$ are the multimode creation and annihilation operators. In the last step of Eq. (S3), the operators are replaced with their corresponding eigenvalues since Eq. (S1) is a coherent state representation.

The condition in Eq. (S3) implies several more useful simplifications. In particular, note that $p_j$ is given by

$$p_j = \int \Phi(v)|v|\langle j\rangle|v\rangle\langle j\rangle \, d^2Mv = \frac{1}{j!} \left| \langle j| \right|^2 \left( \exp(-|v|^2) \right),$$  \hspace{1cm} (S4)

which, upon Taylor expansion of the exponential, gives

$$p_j = \frac{1}{j!} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left| \langle j| \right|^2 \left( |v|^2(1+i\epsilon) \right)^k = \frac{1}{j!} \left[ \langle |v|^2 \rangle - \langle |v|^2(1+i\epsilon) \right] + O(e^{j+2}).$$  \hspace{1cm} (S5)

Since $\epsilon \ll 1$, only terms up to linear order in $\epsilon$ are significant. This can only happen for $j = 0$ and $j = 1$. That is, we arrive at

$$p_0 = \langle 1 \rangle - \langle |v|^2 \rangle + O(e^2) = 1 - \epsilon + O(e^2),$$  \hspace{1cm} (S6)

$$p_1 = \langle |v|^2 \rangle - \langle |v|^4 \rangle + O(e^3) = \epsilon + O(e^2),$$  \hspace{1cm} (S7)

$$p_{j\geq 2} = O(e^2).$$  \hspace{1cm} (S8)

Note that we used the fact that $\langle 1 \rangle = 1$, which results from $\text{Tr}(\hat{\rho}) = 1$. Equations (S6) - (S8) indicate that only that multi-photon events are insignificant when compared to the zero-photon and one-photon events. Moreover, since the zero-photon event (vacuum state) provides no information regarding measurements, it is actually only the one-photon event, corresponding to $p_1$, that should be examined.
This particular event corresponds to an element of $\hat{\rho}$ in its Fock (number) state representation. That is, one can consider

$$\hat{\rho} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle m | \hat{\rho} | n \rangle \langle m | \langle n \rangle, \quad (S9)$$

where the elements $\langle m | \hat{\rho} | n \rangle$ can be found through Eq. (S1). Note that $\hat{\rho}$ is not necessarily diagonal when represented in the multi-mode Fock basis; nevertheless the preceding discussion regarding the sole significance of the one-photon event allows us to approximate, using $p_1 \approx \epsilon$

$$\hat{\rho} \approx \left( \frac{1}{\epsilon} \right) \langle 1 | \langle 1 \rangle = \epsilon \hat{p}_1, \quad (S10)$$

where $\hat{p}_1$ is the one-photon multi-mode Fock state. Note that, in order for $\hat{\rho}$ to maintain unit trace, an additional $\epsilon^{-1}$ factor had to be introduced to the definition of $\hat{p}_1$ in relation to $|1\rangle \langle 1\rangle$.

At this point, through the choice of only considering one-photon events, we shift our focus from the entire $\hat{\rho}$ to just $\hat{p}_1$. That is, the density matrix we are after is $\hat{p}_1 = \hat{\rho} / \epsilon$. Although subtle, this choice is important due to the normalization of density matrices, which is detailed later. Note that $\hat{p}_1$ can be decomposed into a sum of single-mode one-photon states by considering the one-photon basis kets $|1_m\rangle$, where $m = 1, \ldots, M$. That is,

$$\hat{p}_1 \approx \frac{1}{\epsilon} \sum_{j=1}^{M} \sum_{k=1}^{M} \langle 1_j | \hat{p}_1 | 1_k \rangle \langle 1_j | \langle 1_k \rangle, \quad (S11)$$

where

$$\langle 1_j | \hat{p}_1 | 1_k \rangle = \int \Phi(v) \langle 1_j | v \rangle \langle v | 1_k \rangle d^2M_v \approx \frac{1}{\epsilon} \int \Phi(v) \exp\left(-\frac{|v|^2}{2}\right) |v|^2 \rho_k^2 d^2M_v,$$

and in the final line, only the zeroth order Taylor series term for the exponential was retained [in accordance to terms of $O(\epsilon^2)$ being insignificant]. Note then, that Eq. (S12) is, by definition, the $(j,k)$-th element of the image-plane mutual coherence matrix $\hat{C}$. With this identification,

$$\hat{p}_1 \approx \frac{1}{\epsilon} \sum_{j=1}^{M} \sum_{k=1}^{M} \hat{C}_{jk} | 1_j \rangle \langle 1_k \rangle, \quad (S13)$$

and now it remains to determine $\hat{C}$. For an imaging system, the image-plane mutual coherence matrix is related to the object-plane mutual coherence, $\hat{C}_0$, through

$$\hat{C} = \hat{S} \hat{C}_0 \hat{S}^†, \quad (S14)$$

where $\hat{S}$ is the system’s field scattering matrix [often not unitary and closely related to the well-known point spread function (PSF) in classical optics]. Assuming that the imaging system operates under paraxial conditions, it is possible to use localized wave-packet modes as a basis. In other words, the modes $| 1_j \rangle$ can be replaced with $| x_j \rangle = | 1_j \rangle / \sqrt{d\chi}$, with $d\chi$ the spacing in the position space. These are discrete position kets whose position eigenvalues are given by $x_j = x_0 + jdx$, where $x_0$ is an arbitrary origin. At this point, we specialize to the case where the object plane consists of two point sources located at $w_+$ and $w_-$. The object-plane mutual coherence is given by

$$\hat{C}_0 = \epsilon_0 \left[ \delta_{w_+, w_-} + A \delta_{w_+, \Gamma} + \delta_{w_-, \Gamma} \right], \quad (S15)$$

where $A$ is the (relative to $w_+$) intensity of the point source at $w_-$, $\Gamma$ is the unnormalized coherence parameter between the two point sources, and $\delta_{vw}$ is the Kronecker delta symbol. Using Eq. (S14), we find that

$$\hat{C}_{jk} = \epsilon_0 \left( S_{jw_+} S_{kw_+}^† + A S_{jw_+} S_{kw_-}^† + \Gamma S_{jw_-} S_{kw_-}^† + \Gamma S_{jw_-} S_{kw_+}^† \right). \quad (S16)$$

We return now to $\epsilon$, the average number of photons within a coherence time. It is related to the the scattering matrix elements and the coherence parameter $\Gamma$ through

$$\epsilon = \text{Tr} \left( \sum_{j=1}^{M} \sum_{k=1}^{M} \hat{C}_{jk} | 1_j \rangle \langle 1_k \rangle \right) \epsilon_0 \eta_s, \quad (S17)$$

where $\eta_s = \sum_{s} | S_{js} |^2$, with $s \in \{+, -\}$, is the quantum efficiency (assumed to be equal for both point sources). Note that this value of $\epsilon$ ensures that $\text{Tr}(\hat{p}_1) = 1$ and explicitly demonstrates that the number of photons arriving at the image plane, for a fixed object plane photon number, depends on the possibly unknown parameters of $\{ A, \eta_s \}$. In order to express $\hat{p}$ in the familiar basis of two shifted PSFs, we consider the following relations:

$$| \psi_s \rangle = \sum_{j=1}^{M} \sum_{k=1}^{M} S_{js} | x_j \rangle \sqrt{d\chi} \quad \text{and} \quad \psi_s(x_j) = S_{js} / \sqrt{d\chi}, \quad (S18)$$

one can then take the continuous-space limit of $d\chi \to 0$ (and hence $M \to \infty$) to arrive at

$$\hat{p}_1 = \frac{1}{(1 + A)(1 + d^2(s) + 4d\epsilon_s \text{Re}(\Gamma))} \left[ \begin{array}{c} 1 \\ \Gamma \\ A \end{array} \right] \text{no}, \quad (S19)$$

which is in the not-orthogonal basis of $\{ | \psi_+ \rangle, | \psi_- \rangle \}$, as desired. Note that in the main body, for simplicity, this $\hat{p}_1$ is labeled as just $\hat{\rho}$.

2. OBTAINING AN ORTHONORMAL BASIS

The explicit process for obtaining a set of orthonormal vectors that spans the space of $\hat{\rho}$, given by (in the orthogonal $\{ | \psi_+ \rangle, | \psi_- \rangle \}$ basis)

$$\hat{\rho} = \left[ \begin{array}{c} 1 + Ad^2 + 2d\cos(\phi) \\ 1 - d^2 (Ad + re^{i\phi}) \end{array} \right] \times \left[ \begin{array}{c} 1 \\ A(1 - d^2) \\ [1 + A + 2d\cos(\phi)]^{-1} \end{array} \right], \quad (S20)$$

and $\delta_{j\rho}$ where $j \in P = \{ A, r, \phi, s \}$, is shown here. First, it is straightforward to diagonalize $\hat{\rho}$ in order to obtain two (normalized) eigenvectors $| e_1 \rangle$ and $| e_2 \rangle$, which automatically span $\hat{\rho}$. Of course, once diagonalized, $\hat{\rho}$ can be expressed simply as

$$\hat{\rho} = \sum_{i=1}^{2} \lambda_i | e_i \rangle \langle e_i |, \quad (S21)$$

where $\lambda_i$ are the eigenvalues that correspond to $| e_i \rangle$ for $i = 1, 2$. 

The next step is to find the eigenvectors to $\partial_j \hat{\rho}$, where
\[
\partial_j \hat{\rho} = \sum_{i=1}^{2} \left[ \partial_j \lambda_i |e_i\rangle \langle e_i| + \lambda_i \left( |f_i\rangle \langle e_i| + |e_i\rangle \langle f_i| \right) \right],
\] (S22)
where
\[
|f_i\rangle \triangleq \partial_j |e_i\rangle.
\] (S23)

First, the case of $j \neq s$, which turns out to be the simpler case, is analyzed. Given Eq. (S22), it is desirable to rewrite the expression of $\partial_j \hat{\rho}$ in terms of the $\hat{\rho}$-spanning eigenvectors $|e_i\rangle$. In order to do this, we first note that these eigenvectors can be expressed in terms of the non-orthogonal basis kets $|\psi_{\pm}\rangle$ through
\[
|e_i\rangle = F_{ik} |\psi_k\rangle,
\] (S24)
where $k = 1$ and $k = 2$ correspond to $+$ and $-$, respectively. The transformation matrix elements $F_{ik}$ can be easily obtained in the diagonalization process of $\hat{\rho}$ and relating the kets $\{|\psi_{+}\rangle, |\psi_{-}\rangle\}$ back to $\{|\psi_{+}\rangle, |\psi_{-}\rangle\}$. Using Eqs. (S23) and (S24), we find that the second term in Eq. (S22) can be expressed as
\[
\lambda_i |f_i\rangle \langle e_i| + \text{H.C.} = \lambda_i |f_i\rangle \langle e_i| + \text{H.C.},
\] (S25)
where $B_{jl}^i \triangleq (\partial_j F_{ik} (F^{-1})_{kl})$. It turns out that, for $j \neq s$, the diagonal terms $B_{j1}^i$ and $B_{j2}^i$ are purely imaginary and therefore do not contribute further. Using this fact, the matrix $\partial_j \hat{\rho}$, for $j \neq s$, can be written in the set of $\{|e_i\rangle, |e_j\rangle\}$ basis (indicated by a subscript $e$) as
\[
\partial_j \hat{\rho} = \left[ \begin{array}{cc}
\lambda_1 B_{11}^1 + \lambda_2 B_{12}^1 & \lambda_1 B_{12}^1 \langle e_i| \langle e_i| + \lambda_2 B_{21}^1 \\
\lambda_1 B_{12}^1 + \lambda_2 B_{21}^1 & \lambda_2 B_{21}^1 \langle e_j| \langle e_j| + \lambda_1 B_{11}^1 \end{array} \right],
\] (S26)
where the Hermiticity of $\partial_j \hat{\rho}$ is readily apparent (recall that $\lambda_1$ and $\lambda_2$ are real). Note that the $\partial_j \hat{\rho}$ remains spanned by $\{|e_i\rangle, |e_j\rangle\}$ for $j \neq s$. This was expected because the original basis states $\{|\psi_{+}\rangle, |\psi_{-}\rangle\}$ do not depend on $j \neq s$.

We now look at the remaining case of $j = s$, which, as noted in the discussion after Eq. (S26), is complicated by the fact that the original basis states themselves depend on $s$ through the point spread function $\psi$:
\[
|\psi_{\pm}\rangle \triangleq \int_{-\infty}^{\infty} dx \psi \left( x \pm \frac{z}{2} \right) |x\rangle.
\] (S27)
Because of this, $\partial_j \hat{\rho}$ is insufficiently spanned by $\{|e_i\rangle, |e_j\rangle\}$ and additional kets. Evidently, these additional kets are $\{|f_i\rangle, |f_j\rangle\}$. For the case of $s = \Psi$, both of these additional kets are automatically through the definition in Eq. (S23) orthogonal to $\{|e_i\rangle, |e_j\rangle\}$ and the construction of an orthonormal basis that spans $\hat{\rho}$ and $\partial_j \hat{\rho}$ is completed through the normalization of $\{|f_i\rangle, |f_j\rangle\}$. However, in the more general setting explored in this work, this simplifying fact is not true. Nevertheless, it is still relatively straightforward (through the Gram-Schmidt process) to compute the additional basis kets $|e_3\rangle$ and $|e_4\rangle$, needed to span $\partial_j \hat{\rho}$. That is, we take
\[
|e_3\rangle \triangleq N_3 \left( |f_i\rangle - \sum_{p=1}^{2} \langle e_p| f_i\rangle |e_p\rangle \right),
\] (S28)
\[
|e_4\rangle \triangleq N_4 \left( |f_j\rangle - \sum_{p=1}^{3} \langle e_p| f_j\rangle |e_p\rangle \right),
\] (S29)

3. COMPARISON TO DIRECT INTENSITY MEASUREMENTS

For a real PSF, $\psi$, the photon probability density at the image plane from two partially coherent, equal intensity, point sources is given by
\[
P_{\text{DI}}(x) = \frac{\psi_2^2(x) + \psi_2^2(x) + 2\text{Re}(\Gamma) \psi_+(x) \psi_-(x)}{2[1 + \text{Re}(\Gamma) d(x)]},
\] (S30)
where $x$ is the image plane coordinate, $\Gamma = \text{Re}^{(i)}$ is the degree of coherence, and $d(s)$ is the overlap between $\psi_+(x)$ and $\psi_-(x)$. In order to calculate the classical Fisher information (FI) with respect to the separation $s$, one must construct a classical FI matrix, whose elements are given by
\[
F_{ij}(P_u) = \int_{-\infty}^{\infty} \frac{1}{P_{\text{DI}}(x)} |\partial_i P_{\text{DI}}(x)| |\partial_j P_{\text{DI}}(x)| dx,
\] (S31)
where $i, j \in P_u$, the set of unknown parameters. The classical FI for the parameter $s$ is then given by
\[
F_{ss} = \{ F(P_u)^{-1} \}_{ss}^{-1}
\] (S32)
and $F_{ss}$ reduces to
\[
F_{ss} = \int_{-\infty}^{\infty} \frac{1}{P_{\text{DI}}(x)} |\partial_i P_{\text{DI}}(x)|^2 dx.
\] (S33)
A plot of $F_{ss}$ is shown in Fig. S1. Note that $F_{ss} = 0$ over the $s = 0$ transverse disk, which indicates the traditional Rayleigh’s curse of direct intensity measurements.

One can also consider scenarios where there are additional unknown parameters. For instance, consider $P_u = \{ r, s \}$, which corresponds to the case when the magnitude of the coherence parameter is unknown in addition to the separation. The classical FI
Fig. S2. The classical FI, $F_{ss}$, for direct intensity measurement is shown as a function of $s$ and complex $\Gamma$. Here, the set of unknown parameters is $P_u = \{r, s\}$.

$F_{ss}$ for this case is shown in Fig. S2. Evidently, the inclusion of an additional unknown parameter drastically lowers the classical FI for direct intensity measurements. In particular, we note that $F_{ss}$ is smaller than the quantum FI for the case of $P_u = \{r, \phi, s\}$, which is shown in Fig. 7 of the primary manuscript. This indicates that, even though both situations have vanishing FI over the $s = 0$ transverse disk, the quantum FI calculations still suggest a possible advantage in terms of how the information scales with $s$.

REFERENCES

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