ON REACHABILITY OF MARKOV CHAINS: A MARKOV DECISION PROCESS APPROACH

DANIEL ÁVILA† AND MAURICIO JUNCA†

ABSTRACT. We consider a Markov control model in discrete time with countable both state space and action space. Using the value function of a suitable long-run average reward problem, we study various reachability/controllability problems. First, we characterize the domain of attraction and escape set of the system, and a generalization called $p$-domain of attraction, using the aforementioned value function. Next, we solve the problem of maximizing the probability of reaching a set $A$ while avoiding a set $B$. Finally, we consider a constraint version of the previous problem, where we ask for the probability of reaching the set $B$ to be bounded. In the finite case, we use linear programming formulations to compute the previous solutions. Finally, we apply our results to a small example of an object that navigates under stochastic influence.

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1. INTRODUCTION

Markov Decision Processes (MDP) provide a mathematical framework for modeling decision making problems where outcomes are uncertain, formally, these are discrete time controlled stochastic processes. The most common and studied setting are stochastic systems over a discrete state space with discrete action space, see for example [2,12], but general spaces are also part of the literature, see [3,10,11]. We will focus on the former case. Applications of MDPs range from inventory control and investment planning to economics and behavioral ecology.

Based on the problem there are different planning horizons involved. In finite horizon problems one is interested in the evolution of the process upon a time $T$, while in infinite horizon problems the interest is in the long time behavior of the system. In every case, the objective is to maximize the expected cumulative reward obtained over time, where such reward depends on the state of the system and the action taken at each time (sometimes it also includes the state at the next period).

In this work we will be interested in using the MDP framework to solve some problems concerning the controllability/reachability of a controlled Markov chain. As we will see, such problems do not only evaluate the state of the system at each time, but on the whole evolution of the process. The first problem we aim to solve is to characterize the domain of attraction and escape set of a closed set $A$. The first one refers to the initial states for which there exist a control that takes the system to $A$, while the escape set are the initial states such that no control can take the system to $A$. A somehow related problem was

† Department of Mathematics, Universidad de los Andes, Bogotá, Colombia.
studied in [1], where the idea was to use entropy methods to maximize the number of recurrent states. Let \( X \) be the state space of the Markov model, so we define these sets as follows.

**Definition 1.1.** Given a set \( A \subset X \), let

\[
\Lambda = \left\{ x \in X \left| \lim_{t \to \infty} \inf P^\pi_x(X_t \in A) > 0 \text{ for some policy } \pi \right. \right\},
\]

\[
\Gamma = \left\{ x \in X \left| \lim_{t \to \infty} \inf P^\pi_x(X_t \in A) = 0 \text{ for all policies } \pi \right. \right\}.
\]

The domain of attraction appears in the context of deterministic differential equations when describing the initial states under which the system will approach to a stable point. Such technique is commonly referred in the literature as Zubov’s method, see [9,14]. It allows to characterize the domain of attraction and the escape set in terms of an appropriate value function, which is the solution of a differential equation. In [5], Zubov’s method is generalized for deterministic controlled systems, and in [6] it is further generalized for stochastic differential equations. For the present work we took as guide the constructions made in this last work. Inspired by the literature of stochastic target problem, see [4, 13], we study the \( p \)-domain of attraction for any \( p \in [0, 1] \), defined as follows.

**Definition 1.2.** Given a set \( A \subset X \) and \( p \in (0, 1] \), let

\[
\Lambda_p = \left\{ x \in X \left| \lim_{t \to \infty} \inf P^\pi_x(X_t \in A) \geq p \text{ for some policy } \pi \right. \right\}.
\]

The next problem is to find a control policy that maximize the probability of reaching some set \( A \) while avoiding some set \( B \). Namely, let \( \tau_A \) and \( \tau_B \) be the hitting times of \( A \subset X \) and \( B \subset X \), respectively. Consider an initial distribution \( \nu \) over the state space. Our main objective will be to find a control policy \( \pi \) that solves the problem

\[(P1) \quad \max_{\pi} P^\nu_{\pi}(\tau_A < \tau_B, \tau_A < \infty).\]

Note that the set \( B \) acts as a cemetery set since the evolution of the controlled Markov process is meaningless after this set is reached. This problem has been studied in [7], where the authors consider general state and actions spaces, but include some assumptions in the hitting time that implies irreducibility of the controlled Markov process. A continuous time version of this problem for controlled diffusions and finite time horizon can be found in [8].

Finally, we consider a constrained version of the previous problem as follows

\[(P2) \quad \max_{\pi} P^\nu_{\pi}(\tau_A < \infty) \quad \text{s.t.} \quad P^\nu_{\pi}(\tau_B < \infty) \leq \epsilon.\]

In this case the set \( B \) is no longer a cemetery set, making the evolution of the controlled process different, depending whether the set \( B \) has been reached or not. This fact suggests that this cannot be a Markovian problem. To the best of our knowledge, such problem, or any similar, has not been studied in the literature.

The contributions of this work are the following:
(i) We consider reachability problems in infinite horizon and relate them with long-run average reward problems in the context of MDPs. The main result in this direction is Theorem 2.7 that calculates the probability of reaching closed set in finite time in terms of such reward problems. Later, we use this result to characterize $p$-domains of attraction (Corollary 3.5) and solutions of reach-avoid problems (Theorem 4.2). Our result include the case of reducible Markov chains.

(ii) We consider a reachability problem with a constraint in the probability of hitting a given set of states. In general, stochastic control problems with probabilistic constrains are hard to solve. Using a state space augmentation technique, we are able to formulate the problem in terms of long-run average reward problems (Corollary 5.4). In the case of finite state and action spaces we use linear programming duality to solve the problem (Theorem 6.2).

The paper is organized as follows. Section 2 consists of two parts. In subsection 2.1 we state the framework and known results for Markov Decision Processes, in particular linear programming formulations for long-run average reward problems. Subsection 2.2 presents the result that relates the probability of reaching closed set in finite time with the MDP reward problem. Sections 3, 4 and 5 presents the results for $p$-domains of attraction, Problem (P1) and Problem (P2), respectively. In Section 6 we consider finite state and action spaces and formulate Problem (P2) as a linear program. Finally, in Sections 7 and 8 we present a numerical example to illustrate our findings and some future research directions.

2. Markov control model and closed sets

In this section we establish the framework of Markov Decision Processes (MDP) and some results about closed set that allow to formulate properly the problems described above.

2.1. Markov Decision Processes. We will describe now the decision model, see [11],[12] for details. A Markov control model is a tuple $(\mathbb{X}, \mathbb{U}, \{\mathbb{U}(x)|x \in \mathbb{X}\}, Q, r)$, where:

1. $\mathbb{X}, \mathbb{U}$ are sets corresponding to the state space and control actions respectively; in our work $\mathbb{X}$ and $\mathbb{U}$ are countable.
2. For each $x \in \mathbb{X}$ there is a set $\mathbb{U}(x) \subset \mathbb{U}$ corresponding to the feasible control actions to state $x$.
   The set of feasible states and actions is denoted as $\mathbb{K}$, that is,
   $$\mathbb{K} = \{(x,u)|x \in \mathbb{X}, u \in \mathbb{U}(x)\}.$$
3. $Q$ is a stochastic kernel on $\mathbb{X}$ given $\mathbb{K}$, that is, for each $(x,u) \in \mathbb{K}$, $Q(\cdot|x,u)$ is a probability measure on $\mathbb{X}$ and for each $B \subset \mathbb{X}$, the function $Q(B|\cdot)$ is measurable on $\mathbb{K}$.
4. $r$ is a function $r : \mathbb{K} \rightarrow \mathbb{R}$, called the reward function, which we assume bounded.

A control policy is a sequence $\pi = \{\mu_t\}_{t \geq 0}$ of stochastic kernels on $\mathbb{U}$ given the set of admissible histories up to time $t$, $H_t := \mathbb{K}^t \times \mathbb{X}$, such that
$$\mu_t(\mathbb{U}(x_t)|h_t = (x_0, u_0, \ldots, x_{t-1}, u_{t-1}, x_t)) = 1 \quad \forall h_t \in H_t \quad t \geq 0.$$

We denote by $\Pi$ the set of control policies.
Consider the measurable space \((\Omega, \mathcal{F})\), where \(\Omega := (\mathbb{X} \times \mathbb{U})^\mathbb{N}\) and \(\mathcal{F}\) is the product sigma algebra. Given a policy \(\pi \in \Pi\) and a distribution \(\nu\) over \(\mathbb{X}\) there exist a unique probability measure \(P^\pi\) on \((\Omega, \mathcal{F})\) and a \(\mathbb{X} \times \mathbb{U}\)-valued stochastic process \(\{(X_t, U_t)\}_{t \geq 0}, P^\pi\) such that for each \(B \subset \mathbb{X}\) and \(V \in \mathbb{U}\), (see [11])

\[
\begin{align*}
&\star P^\pi\left(\mathbb{X}^\mathbb{N}\right) = 1 \\
&\star P^\pi(X_0 \in B) = \nu(B) \\
&\star P^\pi(U_t \in V | h_t) = \mu_t(V | h_t) \\
&\star P^\pi(X_{t+1} \in B | h_t, U_t = u) = Q(B | x_t, u).
\end{align*}
\]

We denote by \(E^\pi\nu\) the expected value with respect to \(P^\pi\). When \(\nu = \delta_x\) for \(x \in \mathbb{X}\) we use the notation \(P^\pi_x\) and \(E^\pi_x\). In general the process \(\{X_t\}_{t \geq 0}\) is non-Markovian, so, in order to make the process Markovian we also consider stationary Markovian control policies. In this case we have a constant sequence \(\pi = \{\mu\}_{t \geq 0}\), where \(\mu\) is a stochastic kernel on \(\mathbb{U}\) given only \(\mathbb{X}\). Denote by \(\Pi_M \subset \Pi\) the set of stationary Markovian control policies. In this case the process \(\{X_t\}_{t \geq 0}\) is Markovian and

\[
P^\pi_{x,x'} := P^\pi_{\nu}(X_{t+1} = x' | X_t = x) = \sum_{u \in \mathbb{U}(x)} \mu(u | x)Q(i | x, u).
\]

Finally, if \(\pi \in \Pi_M\) such that \(\mu(u | x) \in \{0, 1\}\) for all \(u \in \mathbb{U}(x)\), then we say that \(\pi\) is a stationary deterministic policy.

We will denote by \(\pi^t\) the history up to time \(t\) of the process \(\{X_t\}\), and similarly for the process \(\{U_t\}\). Hence, admissible histories can be written as \(h_t = (\pi^{t-1}, \pi^{t-1}, x_t)\). We also say that \(\pi^t \notin B\) if none of its components belong to \(B\).

### 2.1.1. Infinite horizon reward problems.

Given a Markov control model there are two infinite horizon control problems studied in the literature: Discounted and long-run average reward problems. In general the average problem is more difficult to analyze than the other. For the remaining of this section we will assume that the reward function \(r\) is bounded.

In the discounted case, there is a penalization \(0 < \gamma < 1\) each period of time. So, given an initial state \(x \in \mathbb{X}\), we will be interested in finding a policy \(\pi \in \Pi\) that maximizes the reward function

\[
v^\pi_\gamma(x) := \mathbb{E}^\pi_x\left[\sum_{t=0}^{\infty} \gamma^t r(X_t, U_t)\right],
\]

and in the average case, we will be interested in finding a policy that maximizes the reward function

\[
v^\pi(x) := \lim_{N \to \infty} \frac{1}{N} \mathbb{E}^\pi_x\left[\sum_{t=0}^{N-1} r(X_t, U_t)\right].
\]

The first important result about these problems is that for any \(\pi \in \Pi\) there exists \(\pi' \in \Pi_M\) such that

\[
(1) \quad P^\pi_{\nu}(X_t = x', U_t = u) = P^\pi_{\nu}(X_t = x', U_t = u)
\]
for any $t \geq 1$, $x' \in X$ and $u \in U(x')$, which implies that the reward functions are the same, see [12]. Therefore, optimal policies can always be found among stationary Markovian policies. In this work we will focus on the long-run average case.

**Long-run average.** Given $x \in X$, let $V(x) = \sup_{\pi \in \Pi} v^\pi(x)$. The existence of an optimal policy relays on the so-called optimality equations for the multi-chain model:

\[(MC1) \quad \max_{u \in U(x)} \left\{ \sum_{x' \in X} P_{x,x'}^u v(x') - v(x) \right\} = 0, \quad \forall x \in X,\]

\[(MC2) \quad \max_{u \in U(x)} \left\{ r(x, u) - v(x) + \sum_{x' \in X} P_{x,x'}^u h(x') - h(x) \right\} = 0, \quad \forall x \in X.\]

**Remark 2.1.** Multi-chain models are those where exist a stationary policy for which the induced Markov chain has at least two recurrent classes. Uni-chain models are easier since there is just one recurrent class and the function $V$ is constant, so there is no need to introduce the first equation. Here, we consider the multi-chain case since we will deal with closed sets.

We have the following theorem that relates the solution of above equations with the average reward function, for a proof see Chapter 9 of [12].

**Theorem 2.2.**

(1) Suppose that the optimality equations have a solution $(v, h)$, then, $v = V$.

(2) Suppose that the state space $X$ is finite and $U(x)$ is finite for all $x \in X$. Then, the optimality equations have a solution.

(3) Suppose that the state space $X$ is finite and $U(x)$ is finite for all $x \in X$. Then, there exist a stationary deterministic policy $\pi$ such that $v^\pi = V$. Moreover, such optimal policy can be defined as $\pi = \{\mu\}$ where

$$\mu(u|x) \in \arg \max_{u \in U(x)} \{ r(x, u) + \sum_{x' \in X} P_{x,x'}^u h(x') \}$$

with the additional condition that $P^\pi v = v$, where $P^\pi$ is the transition matrix induced by $\pi$.

When the state space $X$ is finite and $U(x)$ is finite for all $x \in X$, the optimality equations can also be solved via linear programming. Consider a vector $v \in \mathbb{R}_{\geq 0}^{|X|}$. The linear program is as follows, see [12] for further details:

\[(AvP) \quad \min \sum_{x \in X} v(x)\nu(x)\]

s.t.\n
$$v(x) \geq \sum_{x' \in X} P_{x,x'}^u v(x'), \quad \forall x \in X, u \in U(x)$$

$$v(x) \geq r(x, u) + \sum_{x' \in X} P_{x,x'}^u h(x') - h(x), \quad \forall x \in X, u \in U(x).$$
Also, its dual program is given by

\[(AvD) \quad \max \sum_{x \in X} \sum_{u \in U(x)} r(x, u) \alpha(x, u) \]

\[\text{s.t.} \quad \sum_{u \in U(x)} \alpha(x, u) - \sum_{x' \in X} \sum_{u \in U(x')} P_{x',x}^u \alpha(x', u) = 0, \quad \forall x \in X \]

\[\sum_{u \in U(x)} \alpha(x, u) + \sum_{u \in U(x)} \beta(x, u) - \sum_{x' \in X} \sum_{u \in U(x')} P_{x',x}^u \beta(x', u) = \nu(x), \quad \forall x \in X \]

\[\alpha(x, u) \geq 0, \beta(x, u) \geq 0, \quad \forall x \in X, u \in U(x). \]

The next theorem relates the above programs with the multi-chain optimality equations.

**Theorem 2.3.**

(1) There exist an optimal solution \(v^*\) and \((\alpha^*, \beta^*)\) of the primal and dual linear program, respectively.

(2) Define the stationary policy \(\pi = \{\mu\}\) as

\[\mu(u^+ | x) = \begin{cases} \frac{\alpha^*(x, u^+)}{\sum_{u \in U(x)} \alpha^*(x, u)}, & \text{if } \sum_{u \in U(x)} \alpha^*(x, u) > 0 \\ \frac{\beta^*(x, u^+)}{\sum_{u \in U(x)} \beta^*(x, u)}, & \text{otherwise.} \end{cases} \]

Then, \(v^\pi = V = v^*\).

### 2.2. Closed sets.

Closed subsets of the state space are essential for the correct formulation of the controllability problems described in the introduction. We start with the definition of a closed set under a policy \(\pi\).

**Definition 2.4.** Given a policy \(\pi \in \Pi\) and a set \(A \subset X\), we say \(A\) is closed under \(\pi\) if for all \(t \in \mathbb{N}\) such that \(\overline{x^t} \notin A\) does not hold, we have

\[P^\pi_{x^t}(X_{t+1} \notin A | h_t) = \sum_{u \in U(x_t)} Q(A^c | x_t, u_t) \mu_t(u_t | h_t) = 0.\]

**Remark 2.5.** Note that if \(Q(x' | x, u) = 0\) for all \(x \in A, x' \notin A\) and \(u \in U(x)\), then \(A\) is a closed set under any policy \(\pi \in \Pi\).

Now, given a set \(A \subset X\), the random variable \(\tau_A = \inf\{t \geq 0 \mid X_t \in A\}\) is called the hitting time of \(A\). We have the following result about closed sets, which proof is included in Appendix A.

**Proposition 2.6.** Assume \(A\) is a closed set under \(\pi \in \Pi\), and consider the hitting time of \(A, \tau_A\). Then, any \(x \in X\)

\[\lim_{t \to \infty} P^\pi_x(X_t \in A) = P^\pi_x(\tau_A < \infty).\]
The importance of the previous proposition is that it allows to express the probability of an event that depends on the joint distribution of the process, in terms of probabilities of events that only depend on the marginal distributions. Therefore, we have the following theorem, which is the main result of this section. It relates the probability of the hitting time $\tau_A$ being finite with a particular reward function. Concretely, consider the Markov control model described in Subsection 2.1 with reward function $r = 1_A$, the indicator function of set $A$. Hence, for $x \in \mathbb{X}$ and $\pi \in \Pi$ the associated long-run average reward is given by

$$v^\pi(x) := \limsup_{N \to \infty} \frac{1}{N} \mathbb{E}^\pi_x \left[ \sum_{t=0}^{N-1} 1_A(X_t) \right].$$

**Theorem 2.7.** Let $x \in \mathbb{X}$ and assume that $A$ is a closed set under $\pi \in \Pi$. Then,

$$v^\pi(x) = P^\pi_x(\tau_A < \infty).$$

**Proof.** For any $N \in \mathbb{N}$ we have that

$$1_N \mathbb{E}^\pi_x \left[ \sum_{t=0}^{N-1} 1_A(X_t) \right] = \frac{1}{N} \sum_{t=0}^{N-1} P^\pi_x(X_t \in A).$$

Taking $\limsup$ on both sides we get

$$v^\pi(x) = \limsup_{N \to \infty} \frac{1}{N} \sum_{t=0}^{N-1} P^\pi_x(X_t \in A).$$

Now, recall that given a sequence $\{s_t\}_{t \geq 0}$ such that the limit $L$ exists, then the limit of the average also exists and it is equal to $L$, that is,

$$\lim_{N \to \infty} \frac{s_0 + \ldots + s_{N-1}}{N} = L.$$  

Since $A$ is closed under $\pi$, Proposition 2.6 implies that

$$\lim_{t \to \infty} P^\pi_x(X_t \in A) = P^\pi_x(\tau_A < \infty).$$

Therefore,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=0}^{N-1} P^\pi_x(X_t \in A) = \lim_{t \to \infty} P^\pi_x(X_t \in A) = P^\pi_x(\tau_A < \infty).$$

\[\square\]

3. Domain of attraction

Given a Markov control model $(\mathbb{X}, \mathbb{U}, \{\mathbb{U}(x)|x \in \mathbb{X}\}, Q)$, the first controllability problem that we consider is the characterization of the domain of attraction and the escape sets of a set $A \subset \mathbb{X}$, which are defined in 1.1. To ensure some stability we assume that $A$ is closed under some policy $\pi \in \Pi$. The idea is to characterize both sets in terms of value functions. The first question we ask is whether we can restrict our attention to policies that make $A$ a closed set.
**Definition 3.1.** Let $\Pi_A \subset \Pi$ be the set of control polices that make $A$ a closed set.

The following proposition, proved in Appendix B, is fundamental to answer above question.

**Proposition 3.2.** Given $\pi \in \Pi$ there exists a policy $\hat{\pi} \in \Pi_A$ such that for any $x \in X$ and $t \in \mathbb{N}$

$$P^\pi_X(X_t \in A) \leq P^{\hat{\pi}}_X(\tau \leq t).$$

Furthermore, for any $x \in X$

$$\liminf_{t \to \infty} P^\pi_X(X_t \in A) \leq P^{\hat{\pi}}_X(\tau_A < \infty).$$

As a corollary, we obtain the following description of $\Lambda$ and $\Gamma$.

**Corollary 3.3.**

$$\Lambda = \{ x \in X \mid P^\pi_X(\tau_A < \infty) > 0 \text{ for some policy } \pi \in \Pi_A \},$$

$$\Gamma = \{ x \in X \mid P^\pi_X(\tau_A < \infty) = 0 \text{ for all policies } \pi \in \Pi_A \}.$$

**Proof.** Clearly, by Proposition 2.6

$$\{ x \in X \mid P^\pi_X(\tau_A < \infty) > 0 \text{ for some policy } \pi \in \Pi_A \} \subset \Lambda.$$

Let $x \in \Lambda$, so there exist a policy $\pi \in \Pi$ such that $\liminf_{t \to \infty} P^\pi_X(X_t \in A) > 0$. Let $\hat{\pi} \in \Pi_A$ be the policy given by the previous proposition. Therefore,

$$0 < \liminf_{t \to \infty} P^\pi_X(X_t \in A) \leq P^{\hat{\pi}}_X(\tau_A < \infty).$$

□

Similarly, we can describe $p$-domains of attraction defined in 1.2 as

$$\Lambda_p = \left\{ x \in X \mid P^\pi_X(\tau_A < \infty) \geq p \text{ for some policy } \pi \in \Pi_A \right\}.$$

Then, we can focus on policies that make $A$ a closed set. In fact, if we consider the average reward function $v^\pi$ defined in (2) and define the value function

$$V(x) = \sup_{\pi \in \Pi_A} v^\pi(x),$$

then we can use this function to characterize the sets above.

**Remark 3.4.** Note that $V^*(x) = \sup_{\pi \in \Pi} v^\pi(x)$ since any policy $\pi \in \Pi$ can be majorized by the policy $\pi' \in \Pi_A$ defined in Lemma B.1. In fact, if $x \in A$ it follows clearly. If $x \notin A$, by Proposition 3.2 and Lemma B.1 we obtain that $v^\pi(x) \leq v^{\pi'}(x)$.

**Corollary 3.5.** The following characterization of the sets hold:

$$\Lambda = \left\{ x \in X \mid V^*(x) > 0 \right\},$$

$$\Gamma = \left\{ x \in X \mid V^*(x) = 0 \right\}.$$
and
\[ \Lambda_p = \left\{ x \in \mathbb{X} \mid V^*(x) \geq p \right\}. \]

**Proof.** Let \( x \in \Lambda \), then Corollary 3.3 implies that \( P^\pi_x(\tau_A < \infty) > 0 \) for some policy \( \pi \in \Pi_A \). Since \( A \) is closed under \( \pi \), Theorem 2.7 implies that \( v^\pi(x) > 0 \), as a consequence \( v^*(x) > 0 \). On the other hand, let \( x \in \Gamma \), so by Corollary 3.3 \( P^\pi_x(\tau_A < \infty) = 0 \) for all polices \( \pi \in \Pi_A \), and the result follows again by Theorem 2.7. Similarly, the result holds for the \( p \)-domain of attraction. \( \square \)

4. **Maximizing the Probability of Reaching and Avoiding a Set of States**

The problem described in the previous section can be seen as a feasibility problem. In this section we will solve a related maximization problem, namely the problem \( (P1) \). Consider a given Markov control model \( (\mathbb{X}, U, \{U(x)\mid x \in \mathbb{X}\}, Q) \). Let \( A, B \subset \mathbb{X} \) disjoint sets, with their respective hitting times \( \tau_A, \tau_B \), and an initial distribution \( \nu \) over the state space be given. Recall that the problem is written as:

\[ (P1) \quad \sup_{\pi \in \Pi} P^\nu(\tau_A < \tau_B, \tau_A < \infty). \]

We will show that this problem is equivalent to a long-run average reward problem with a particular reward function. Our first step will be to rewrite the objective function of \( (P1) \). To achieve this, let us define a modified stochastic kernel that make sets \( A \) and \( B \) closed under any policy \( \pi \). Given a pair \((x, u) \in \mathbb{K}\) construct the stochastic kernel \( \tilde{Q} \) as follows:

\[
\begin{align*}
\tilde{Q}(A|x, u) &= 1 \quad \text{if } x \in A \\
\tilde{Q}(B|x, u) &= 1 \quad \text{if } x \in B \\
\tilde{Q}(.|x, u) &= Q(.|x, u) \quad \text{otherwise}
\end{align*}
\]

The measure induced by \( \tilde{Q} \) will be denoted as \( \tilde{P} \) and \( \tilde{E} \) will denote the expectation with respect to \( \tilde{P} \). By Remark 2.5 the sets \( A \) and \( B \) are closed for any control policy in \( \Pi \). Then, we have the following result proved in Appendix C.

**Proposition 4.1.** Given a policy \( \pi \in \Pi \), we have that

\[ P^\nu(\tau_A < \tau_B, \tau_A < \infty) = \tilde{P}^\nu(\tau_A < \infty). \]

Now, we consider the Markov control model \( (\mathbb{X}, U, \{U(x)\mid x \in \mathbb{X}\}, \tilde{Q}, 1_A) \), that is, the given Markov model with the modified kernel and with the characteristic function of set \( A \) as reward function. Hence, Theorem 2.7 implies that for \( x \in \mathbb{X} \) and \( \pi \in \Pi \)

\[ \tilde{P}^\nu(\tau_A < \infty) = \tilde{v}^\pi(x), \]

where \( \tilde{v}^\pi(x) \) is defined in (2) with the modified stochastic kernel, that is

\[ \tilde{v}^\pi(x) = \limsup_{N \to \infty} \frac{1}{N} \sum_{t=0}^{N-1} \tilde{P}^\pi_x(X_t \in A). \]
A straight forward consequence of (4) and (1) is that the problem of maximizing the probability of reaching some set \( A \) while avoiding a set \( B \) over all control policies can be understood as the maximization of an average reward function over stationary Markovian policies.

**Theorem 4.2.**

\[
\sup_{\pi \in \Pi} P^\pi_\nu (\tau_A < \tau_B, \tau_A < \infty) = \sup_{\pi \in \Pi_M} \sum_{x \in \mathcal{X}} \tilde{V}(x) \nu(x) = \sum_{x \in \mathcal{X}} \tilde{V}(x) \nu(x),
\]

where \( \tilde{V}(x) = \sup_{\pi \in \Pi_M} \tilde{v}^\pi(x) \).

5. Maximizing with hitting constraint

In this section we will solve a constrained version of Problem (P1). So, again consider a given Markov control model \((\mathcal{X}, \mathbb{U}, \{ \mathbb{U}(x) | x \in \mathcal{X} \}, Q)\), along with \( A, B \subset \mathcal{X} \) disjoint sets and an initial distribution \( \nu \). Our objective will be to find a control policy that maximize the probability of reaching \( A \) in such a way that the probability of reaching \( B \) is less than some \( \epsilon > 0 \). The formulation is as follows:

\[(P2) \quad \sup_{\pi \in \Pi} P^\pi_\nu (\tau_A < \infty) \quad \text{s.t.} \quad P^\pi_\nu (\tau_B < \infty) \leq \epsilon.\]

In this case, however, a Markovian policy might not be the best control to solve this problem (recall Theorem 4.2). The following example shows this fact. To avoid cumbersome notation, in the sequel we will use \( \pi \) to denote the stochastic kernels associated with such policy.

**Example.** Assume \( \mathcal{X} = \{1, 2, 3, 4\} \), \( \mathbb{U} = \{u_1, u_2\} \) and \( \mathbb{U}(x) = \mathbb{U} \) for all \( x \). The corresponding control matrices are described in Figure 1. Consider the sets \( A = \{4\}, B = \{1, 2\} \). Assuming the uniform initial distribution let us consider the problem

\[
\sup_{\pi \in \Pi} P^\pi_\nu (\tau_A < \infty) \quad \text{s.t.} \quad P^\pi_\nu (\tau_B < \infty) \leq 0.5
\]

First of all, regardless of the kind of policy (Markovian or not) we must have that for any policy \( \pi \),
\[
P^\pi_1 (\tau_B < \infty) = P^\pi_2 (\tau_B < \infty) = 1, \quad P^\pi_3 (\tau_B < \infty) = P^\pi_5 (\tau_B < \infty) = 0.
\]

Thus, to satisfy the restriction we must have that \( P^\pi_3 (\tau_B < \infty) \leq 0.5 \). Moreover, since \( \pi(u_2|3) = P^\pi(X_1 = 2|X_0 = 3) = P^\pi_3 (\tau_B < \infty) \) then we need that \( \pi(u_2|3) \leq 0.5 \). We also have that for any policy,
\[
P^\pi_2 (\tau_A < \infty) = P^\pi (X_1 = 4|X_0 = 2) = \pi(u_2|2),
\]
\[
P^\pi_4 (\tau_A < \infty) = 1, \quad P^\pi_5 (\tau_A < \infty) = 0.
\]

Thus a first obvious choice to maximize \( P^\pi_\nu (\tau_A < \infty) \) is to select \( \pi(u_2|2) = 1 \), so that \( P^\pi_2 (\tau_A < \infty) = 1 \). Let \( \pi \) be a Markovian policy, so that \( P^\pi_1 (\tau_A < \infty) = P^\pi_3 (\tau_A < \infty) \). If we select \( \pi(u_2|3) = 1 \) we would obtain \( P^\pi_1 (\tau_A < \infty) = P^\pi_3 (\tau_A < \infty) = 1 \). However, since \( \pi(u_2|3) > 0.5 \) we can’t select such a policy.
Moreover, note that for any policy for which $\pi(u_2|3) < 1$ we would have $P_1^\pi(\tau_A < \infty) = P_3^\pi(\tau_A < \infty) < 1$.

On the other hand, if we allow non-Markovian policies we can define a policy for which the value of $P_3^\pi(\tau_A < \infty)$ is the same as if we have used a Markovian policy, but $P_1^\pi(\tau_A < \infty) = 1$. Indeed, note that

$$P_1^\pi(\tau_A < \infty) = P_1^\pi(X_1 = 3, X_2 = 4) + P_1^\pi(X_1 = 3, X_2 = 2, X_3 = 4)$$

$$= \sum_{a_0, a_1 \in U} P_1^\pi(U_0 = a_0, X_1 = 3, U_1 = a_1, X_2 = 4)$$

$$+ \sum_{a_0, a_1, a_2 \in U} P_1^\pi(U_0 = a_0, X_1 = 3, U_1 = a_1, X_2 = 2, U_2 = a_2, X_3 = 4).$$

By conditioning, the terms in the first sum can be written as

$$P^\pi(X_2 = 4|h_1, a_1)P^\pi(U_1 = a_1|h_1)P^\pi(X_1 = 3|h_0, a_0)P^\pi(U_0 = a_0|X_0 = 1)$$

$$= Q(4|3, a_1)\pi(a_1|h_1)Q(3|1, a_0)\pi(a_0|1).$$

Similarly, the terms in the second sum can be written as

$$Q(4|2, a_2)\pi(a_2|h_2)Q(2|3, a_1)\pi(a_1|h_1)Q(3|1, a_0)\pi(a_0|1).$$

Thus, by defining $\pi(u_2|1) = \pi(u_2|1, u_2, 3) = \pi(u_2|1, u_2, 3, u_2, 2) = 1$ we obtain that $P_1^\pi(\tau_A < \infty) = 1$. Now, we select $\pi(u_2|3) \leq 0.5$ so that the restriction is satisfied. Repeating the same procedure as before we obtain

$$P_3^\pi(\tau_A < \infty) = P_3^\pi(X_1 = 4) + P_3^\pi(X_1 = 2, X_2 = 4)$$

$$= \sum_{a_0, a_1 \in U} P_3^\pi(U_0 = a_0, X_1 = 4)$$

$$+ \sum_{a_0, a_1 \in U} P_3^\pi(U_0 = a_0, X_1 = 2, U_1 = a_1, X_2 = 4).$$
and
\[
P_3^\pi(U_0 = a_0, X_1 = 4) = Q(4|3, a_0)\pi(a_0|3),
\]
\[
P_3^\pi(U_0 = a_0, X_1 = 2, U_1 = a_1, X_2 = 4) = Q(4|2, a_1)\pi(a_1|h_1)Q(2|3, a_0)\pi(a_0|3).
\]

Thus, by defining \(\pi(a_1|h_1)\) in such a way that only depends of the previous state, we can obtain the same value for \(P_3^\pi(\tau_A < \infty)\) as if we have worked with Markovian policies. Therefore \(P_3^\pi(\tau_A < \infty)\) for such policy is bigger than for any stationary Markovian policy.

The key point is to realize that if the process \(\{X_t\}\) has already gone through the set \(B\), then does not matter if it hits the set again. But instead, if the chain has not gone through \(B\) it is better not to reach the set in order to satisfy the constraint. As a consequence, we will set the problem using control policies which remember whether the process has reached \(B\) or not. Recall that we denote by \(\pi^t\) the history up to time \(t\) of the process \(\{X_t\}\) and we say that \(\pi^t \notin B\) if none of its components belong to \(B\).

Optimization problems with constraints can be rewritten using Lagrange multiplier. Hence the problem above is equivalent to the following problem

\[
sup_{\pi \in \Pi} \inf_{\lambda \geq 0} \mathcal{L}(\pi, \lambda),
\]

where

\[
\mathcal{L}(\pi, \lambda) = P_\nu^\pi(\tau_A < \infty) + \lambda(\epsilon - P_\nu^\pi(\tau_B < \infty)).
\]

The idea will be to write the Lagrangian function \(\mathcal{L}\) as an average reward function with similar ideas as in the previous section. In order to do this, we consider the set \(\tilde{X} = X \times \{0,1\}\). Intuitively a state \((x,0)\) indicates that the process has not reached \(B\), while a state \((x,1)\) indicates that the process has already reached \(B\). Let \(U(x,i) = U(x)\) for \(i = 0, 1\) and \(\tilde{K}\) the corresponding set of feasible states and actions.

We also define the stochastic kernel \(\hat{Q}\) on \(\tilde{X}\) given \(\tilde{K}\) as follows:

\[
\begin{align*}
\hat{Q}((y,0)|(x,0),u) &= Q(y|x,u), & \text{if } y \notin B \\
\hat{Q}((y,0)|(x,0),u) &= 0, & \text{if } y \in B \\
\hat{Q}((y,1)|(x,0),u) &= 0, & \text{if } y \notin B \\
\hat{Q}((y,1)|(x,0),u) &= Q(y|x,u), & \text{if } y \in B \\
\hat{Q}((y,0)|(x,1),u) &= 0, \\
\hat{Q}((y,1)|(x,1),u) &= Q(y|x,u).
\end{align*}
\]

Therefore, we have the augmented Markov control model \(\tilde{\mathcal{X}}, \tilde{U}, \{\tilde{U}(x,i)|(x,i) \in \tilde{K}\}, \tilde{K}, \hat{Q}\). Let \(\tilde{\Pi}\) be the set of policies over the augmented model, \(\tilde{\Pi}_M\) the set of stationary Markovian policies, and for any policy \(\tilde{\pi} \in \tilde{\Pi}\) we denote by \(\tilde{P}^{\tilde{\pi}}\) the measure induced by the policy. The corresponding \(\tilde{X} \times \tilde{U}\)-valued stochastic process is denoted by \(\{\tilde{X}_t, \tilde{I}_t, \tilde{U}_t\}_{t \geq 0}\). Histories in this model are denoted by \(\tilde{h}_t = (h_t, \tilde{\tau}^t)\), with \(\hat{\tau}^t\) the history up to time \(t\) of the process \(\{I_t\}\).

**Definition 5.1.** Given a \(\pi \in \Pi\) we define a policy \(\tilde{\pi} \in \tilde{\Pi}\) as follows: For any \(t \geq 0\)

\[
\tilde{\pi}(u|\tilde{h}_t) = \pi(u|h_t),
\]
whenever \( \hat{h}_t \) satisfies that \( \pi^x \notin B \) and \( i^t \notin \{1\} \), or there is some \( 0 \leq s \leq t \) such that \( \pi^{s-1} \notin B \), \( x_s \in B \), \( \pi^{s-1} \notin \{1\} \) and \( i_r = 1 \) for \( s \leq r \leq t \). Otherwise, we define \( \tilde{\pi}(u|\hat{h}_t) = \delta_{u_0} \) for any \( u_0 \in \mathcal{U}(x_t) \). Note that \( s = 0 \) implies that \( i^t \notin \{0\} \).

The next result allows to express joint distributions of the original model in terms of joint distributions of the augmented model.

**Lemma 5.2.** Let \( \pi \in \Pi \) and consider the policy \( \hat{\pi} \in \hat{\Pi} \) of the previous definition. Given \( h_t \in H_t \) and \( u_t \in \mathcal{U}(x_t) \), let

\[
\hat{\pi}(x,0) = \begin{cases} 
\hat{P}^\pi(x,0)(\hat{h}_t, u_t) & \text{if } x \notin B \\
\hat{P}^\pi(x,1)(\hat{h}_t, u_t) & \text{if } x \in B.
\end{cases}
\]

Then,

\[
P_\pi^\pi(h_t, u_t) = \begin{cases} 
P(\pi^x(h_t), u_t) & \text{if } \pi^x \notin B \\
P(\pi^x(h_t), u_t) & \text{if } \pi^{x-1} \notin B, x_s \in B.
\end{cases}
\]

**Proof.** First, note that

\[
P_\pi^\pi(h_t, u_t) = P_\pi^\pi(x_t, u_t|h_{t-1}, u_{t-1}) \cdots P_\pi^\pi(x_1, u_1|x, u_0)P_\pi^\pi(u_0|x).
\]

Now, let us study the previous conditional probabilities. Note that for \( 0 < r < t \) we have that

\[
P_\pi^\pi(x_r, u_r|h_{r-1}, u_{r-1}) = P_\pi^\pi(u_r|h_r)P_\pi^\pi(x_r|h_{r-1}, u_{r-1})
\]

\[= \pi(u_r|h_r)Q(x_r|x_{r-1}, u_{r-1}). \tag{6}\]

**Case 1:** Suppose \( \pi^x \notin B \). The fact that \( x \notin B \) implies that \( \pi(u_0|x) = \hat{\pi}(u_0|(x, 0)) \), so that \( P_\pi^\pi(u_0|x) = \hat{P}_\pi^\pi(u_0|(x, 0)) \). For \( r \leq t \) define, using equation (6) and the definition of \( \hat{Q} \) and \( \hat{\pi} \) we obtain that

\[
P_\pi^\pi(x_r, u_r|h_{r-1}, u_{r-1}) = \pi(u_r|h_r)Q(x_r|x_{r-1}, u_{r-1})
\]

\[= \hat{\pi}(u_r|h_r)\hat{Q}((x, 0)|(x_{r-1}, 0), u_{r-1})
\]

\[= \hat{P}_\pi^\pi(u_r|h_r)\hat{P}_\pi^\pi((x, 0)|h_{r-1}, u_{r-1})
\]

\[= \hat{P}_\pi^\pi((x, 0), u_r|h_{r-1}, u_{r-1}).
\]

As a consequence \( P_\pi^\pi(h_t, u_t) = \hat{P}_\pi^\pi(x,0)(\hat{h}_t, u_t) \).

**Case 2:** Suppose \( x_s \in B \) for some \( 0 \leq s \leq t \), and \( \pi^{s-1} \notin B \). If \( s = 0 \) then \( x \in B \) and so \( \pi(u_0|x) = \hat{\pi}(u_0|(x, 1)) \) so that \( P_\pi^\pi(u_0|x) = \hat{P}_\pi^\pi(u_0|(x, 1)) \). Otherwise, if \( s > 0 \) we have that \( P_\pi^\pi(u_0|x) = \hat{P}_\pi^\pi(u_0|(x, 0)) \), as in Case 1. Now, for \( 0 < r < b \) using (6) and the definition of \( \hat{Q} \) and \( \hat{\pi} \) we obtain that

\[
P_\pi^\pi(x_r, u_r|h_{r-1}, u_{r-1}) = \hat{P}_\pi^\pi((x, 0), u_r|h_{r-1}, u_{r-1})
\]
For \( r \geq s \), similarly, we obtain that
\[
P^\pi_x(x_r, u_r | h_{r-1}, u_{r-1}) = \pi(u_r | h_r)Q_x(x_r | x_{r-1}, u_{r-1})
\]
\[
= \hat{\pi}(u_r | h_r)\hat{Q}((x_r, 1)|(x_{r-1}, i_{r-1}), u_{r-1}) \quad i_{r-1} = 0 \text{ if } r = s, = 1 \text{ otherwise}
\]
\[
= \hat{P}^{\hat{\pi}}(u_r | h_r)\hat{P}^{\hat{\pi}}((x_r, 1)|h_{r-1}, u_{r-1})
\]
\[
= \hat{P}^{\hat{\pi}}((x_r, 1), u_r | h_{r-1}, u_{r-1})
\]

As a consequence \( P^\pi_x(h_t, u_t) \) equals \( \hat{P}^{\hat{\pi}}(x, 0)(h_t, u_t) \) if \( x \notin B \), and it equals \( \hat{P}^{\hat{\pi}}(x, 1)(h_t, u_t) \) if \( x \in B \).

Note that by definition of \( \hat{Q} \), the set \( \mathbb{X} \times \{1\} \) is closed under every policy in the augmented Markov model. Also, as in Section 4, we can redefine the kernel \( \hat{Q} \) to make the set \( A \times \{0, 1\} \) closed under any policy, that is, \( \hat{Q}(A \times \{i\}|(x, i), u) = 1 \) if \( x \in A \) and \( i = 0, 1 \). With this redefined kernel we consider the Markov model along with the reward function
\[
r(x, i) = (1_{A \times \{0, 1\}} - \lambda 1_{\mathbb{X} \times \{1\}})(x, i).
\]

Given a policy \( \hat{\pi} \in \hat{\Pi} \) and \( (x, i) \in \hat{\mathbb{X}} \), we consider the average reward function given by

\[
(7) \quad v^\hat{\pi}(x, i) := \limsup_{N \to \infty} \frac{1}{N} \mathbb{E}^{\hat{\pi}, \mathbb{X}, \mathbb{I}}(\sum_{t=0}^{N-1} [1_{A \times \{0, 1\}} - \lambda 1_{\mathbb{X} \times \{1\}}](X_t, I_t)).
\]

The next theorem proved in Appendix D shows that the Lagrangean \( \mathcal{L} \) can be written in terms of this function.

**Theorem 5.3.** Let \( \pi \in \Pi \) and \( \hat{\pi} \in \hat{\Pi} \) the policy defined in Definition 5.1. Let \( x \in \mathbb{X} \), then

\[
P^\pi_x(\tau_A < \infty) + \lambda (\epsilon - P^\pi_x(\tau_B < \infty)) = \begin{cases} v^\pi(x, 0) + \lambda \epsilon, & \text{if } x \notin B \\ v^\pi(x, 1) + \lambda \epsilon, & \text{if } x \in B. \end{cases}
\]

Now, since Definition 5.1 does not necessarily recover all policies in \( \hat{\Pi} \), the previous theorem shows that the following is an upper bound for \( P^2 \)

\[
\sup_{\hat{\pi} \in \hat{\Pi}} \inf_{\lambda \geq 0} \lambda \epsilon + \sum_{x \notin B} v^\pi(x, 0)\nu(x) + \sum_{x \in B} v^\pi(x, 1)\nu(x).
\]

Also, note that in the problem above we can consider only stationary Markovian policies by \( (1) \). Furthermore, given a policy \( \hat{\pi} \in \hat{\Pi}_M \) we can define a policy \( \pi \in \Pi \) by

\[
(8) \quad \pi(u|h_t) = \begin{cases} \hat{\pi}(u|x, 0) & \text{if } \pi'^t \notin B \\ \hat{\pi}(u|x, 1) & \text{otherwise}. \end{cases}
\]

Hence, if we denote the set of policies obtained by \( (8) \) as \( \Pi_B \subset \Pi \) we get the following result.
Corollary 5.4. Problem (P2) is equivalent to
\[
\sup_{\pi \in \Pi_B} P^\pi_\nu (\tau_A < \infty) \\
\text{s.t. } P^\pi_\nu (\tau_B < \infty) \leq \epsilon
\]
is optimal value is equal to
\[
(9) \quad \sup_{\hat{\pi} \in \hat{\Pi}_M} \inf_{\lambda \geq 0} \lambda \epsilon + \sum_{x \not\in \mathcal{B}} v^\hat{\pi}(x, 0) \nu(x) + \sum_{x \in \mathcal{B}} v^\hat{\pi}(x, 1) \nu(x)
\]
and any optimal policy of (9) produces an optimal policy of (P2) through (8).

6. LINEAR PROGRAMMING FORMULATIONS: FINITE CASE

In this section we consider the case where the state space \( \mathcal{X} \) and the action space \( \mathcal{U} \) are finite and present some linear programs that solve our controllability problems. We know from Theorem 2.3 and Corollary 3.5 that in order to find the \( p \)-domain of attraction of a given closed set \( \mathcal{A} \), we need to solve the following linear program

\[\text{(ReaP)} \quad \min \sum_{x \in \mathcal{X}} v(x) \]

s.t.
\[
v(x) \geq \sum_{j \in \mathcal{X}} P^u_{x,x'} v(x'), \quad \forall x \in \mathcal{X}, u \in \mathcal{U}(x) \\
v(x) \geq 1_{\mathcal{A}}(x) + \sum_{x' \in \mathcal{X}} P^u_{x,x'} h(x') - h(x), \quad \forall x \in \mathcal{X}, u \in \mathcal{U}(x)
\]

where \( P^u_{x,j} = Q(j | x, u) \).

Similarly, in order to solve Problem (P1), from Theorem 4.2 we need to solve the same linear program as above with \( \tilde{P}^u_{x,j} = \tilde{Q}(j | x, u) \) instead of \( P_{x,j} \), where \( \tilde{Q} \) is defined in (3). In both cases, an optimal stationary Markovian policy can be found by solving the corresponding dual problems, as in Theorem 2.3. The linear program is as follows:

\[\text{(ReaD)} \quad \max \sum_{x \in \mathcal{A}} \sum_{u \in \mathcal{U}(x)} \alpha(x, u) \]

s.t.
\[
\sum_{u \in \mathcal{U}(x)} \alpha(x, u) - \sum_{x' \in \mathcal{X}} \sum_{u \in \mathcal{U}(x')} P^u_{x',x} \alpha(x', u) = 0, \quad \forall x \in \mathcal{X} \\
\sum_{u \in \mathcal{U}(x)} \alpha(x, u) + \sum_{u \in \mathcal{U}(x)} \beta(x, u) - \sum_{x' \in \mathcal{X}} \sum_{u \in \mathcal{U}(x')} P^u_{x',x} \beta(x', u) = 1, \quad \forall x \in \mathcal{X} \\
\alpha(x, u) \geq 0, \beta(x, u) \geq 0, \quad \forall x \in \mathcal{X}, u \in \mathcal{U}(x).
Remark 6.1. Note that in both cases the initial distribution $\nu$ does not play any role in order to find the value functions $V^*$ and $\tilde{V}$, and the optimal policies.

For the case of Problem $\mathbf{(P2)}$, the linear programming formulation is not as straightforward as in the previous problems. In particular, we would like to switch the inf with the sup in (9), that is, we would like to show that it is equivalent to its dual problem. Recall that $\mathbf{(P2)}$ can be written as

$$P^* = \sup_{\pi \in \Pi} \inf_{\lambda \geq 0} P_{\nu}^\pi (\tau_A < \infty) + \lambda (\epsilon - P_{\nu}^\pi (\tau_B < \infty)),$$

which is bounded above by the optimal value of its dual problem

$$D^* = \inf_{\lambda \geq 0} \sup_{\pi \in \Pi} P_{\nu}^\pi (\tau_A < \infty) + \lambda (\epsilon - P_{\nu}^\pi (\tau_B < \infty)).$$

Using Theorems 5.3 and 2.3 we can write the above problem as

(D1) \[
\inf_{\lambda \geq 0} \lambda \epsilon + \max \sum_{x \in A, u \in U(x)} \alpha((x, 0), u) + \alpha((x, 1), u) - \lambda \sum_{x \in X, u \in U(x)} \alpha((x, 1), u)
\]

s.t.

$$\sum_{u \in U(x)} \alpha((x, i), u) - \sum_{(x', i') \in \tilde{X}, u \in U(x')} \hat{P}^u_{(x', i'), (x, i)} \alpha((x', i'), u) = 0, \quad \forall (x, i) \in \tilde{X}$$

$$\sum_{u \in U(x)} \alpha((x, i), u) + \sum_{u \in U(x)} \beta((x, i), u) - \sum_{(x', i') \in \tilde{X}, u \in U(x')} \hat{P}^u_{(x', i'), (x, i)} \beta((x', i'), u) = \hat{\nu}(x, i), \quad \forall (x, i) \in \tilde{X}$$

$$\alpha((x, i), u) \geq 0, \beta((x, i), u) \geq 0, \quad \forall (x, i) \in \tilde{X}, u \in U(x).$$

with $\hat{P}^u_{(x, i), (x', i')} = \hat{Q}((x', i')|(x, i), u)$, where $\hat{Q}$ is defined in (5), and

$$\hat{\nu}(x, i) = \begin{cases} 
\nu(x), & \text{if } x \notin B, i = 0 \\
\nu(x), & \text{if } x \in B, i = 1 \\
0, & \text{otherwise.}
\end{cases}$$
Now, by strong duality of linear programming we further obtain that

\[(D2)\]
\[
D^* = \max_{x \in A, u \in U(x)} \alpha((x, 0), u) + \alpha((x, 1), u)
\]

s.t.
\[
\sum_{u \in U(x)} \alpha((x, i), u) - \sum_{(x', i') \in \mathcal{X}, u \in U(x')} \hat{P}^{u}_{(x', i'), (x, i)} \alpha((x', i'), u) = 0, \quad \forall (x, i) \in \hat{\mathcal{X}}
\]
\[
\sum_{u \in U(x)} \alpha((x, i), u) + \sum_{u \in U(x)} \beta((x, i), u) - \sum_{(x', i') \in \mathcal{X}, u \in U(x')} \hat{P}^{u}_{(x', i'), (x, i)} \beta((x', i'), u) = \nu(x, i),
\]
\[\forall (x, i) \in \hat{\mathcal{X}}\]
\[
\sum_{x \in X, u \in U(x)} \alpha((x, 1), u) \leq \epsilon
\]
\[\alpha((x, i), u) \geq 0, \beta((x, i), u) \geq 0, \quad \forall (x, i) \in \hat{\mathcal{X}}, u \in U(x).\]

If the problem \((D2)\) is infeasible, then \(D^* = -\infty\) and therefore \(P^* = -\infty\), that is, Problem \((P2)\) is infeasible. On the other hand, if \((D2)\) is finite (note that it cannot be unbounded) with optimal solution \((\alpha^*, \beta^*)\), the inf over \(\lambda \geq 0\) in \((D1)\) is attained at some \(\lambda^*\) such that

\[
\lambda^* \left(\epsilon - \sum_{x \in X, u \in U(x)} \alpha^*((x, 1), u)\right) = 0.
\]

Let \(\hat{\pi}^* \in \hat{\Pi}_M\) the optimal policy induced by \((\alpha^*, \beta^*)\) given by Theorem 2.3 and \(\pi^* \in \Pi_B\) its corresponding over \(\mathcal{X}\) given by (8). Therefore

\[
P^* = \sup_{\pi \in \Pi_B} \inf_{\lambda \geq 0} P^*_\nu(\tau_A < \infty) + \lambda (\epsilon - P^*_\nu(\tau_B < \infty))
\]
\[
\geq \inf_{\lambda \geq 0} P^*_\nu(\tau_A < \infty) + \lambda (\epsilon - P^*_\nu(\tau_B < \infty))
\]
\[
= \inf_{\lambda \geq 0} \lambda \epsilon + \sum_{x \in A, u \in U(x)} \alpha^*((x, 0), u) + \alpha^*((x, 1), u) - \lambda \sum_{x \in X, u \in U(x)} \alpha^*((x, 1), u)
\]
\[
= \lambda^* \epsilon + \sum_{x \in A, u \in U(x)} \alpha^*((x, 0), u) + \alpha^*((x, 1), u) - \lambda^* \sum_{x \in X, u \in U(x)} \alpha^*((x, 1), u)
\]
\[= D^*
\]
\[= \sum_{x \in A, u \in U(x)} \alpha^*((x, 0), u) + \alpha^*((x, 1), u)
\]
\[= P^*_\nu(\tau_A < \infty).
\]

Hence, we just proved the following result.
Theorem 6.2. Suppose $\mathbb{X}, \mathbb{U}$ are finite. Then $[\text{P2}]$ satisfies strong duality, that is, it is equivalent to

$$\min\max_{\lambda \geq 0} \max_{\pi \in \Pi} \mathcal{L}(\pi, \lambda).$$

Furthermore, it can be solved by the linear program $[\text{D2}]$ in the augmented model and recover an optimal policy in $\Pi_B$ through $[8]$.

7. Numerical example

To illustrate our results we consider an object that navigates over a grid under the influence of a north-west wind. The object has three controls available as shown in Figure 2. We assume that the states at the upper boundary of the grid are absorbing states and some adjustments are done in the left and right boundaries so that the object does not leave the grid.

7.1. Domain of attraction. In order to show the findings of Section 3, we consider a 100 by 100 grid with a closed set $A$ in the central region of the grid. Figure 3 shows the surface and level sets of the function $V^*(x)$ which defines the $p$-domains $\Lambda_p$. The scape set $\Gamma$ corresponds to the states with value function equal to zero. This function was founded by solving the linear program $[\text{ReaP}]$. We note that no state outside of $A$ belongs to $\Lambda_1$.

7.2. Reach and avoid. For Problem $[\text{P1}]$ we consider a 100 by 20 grid with the set $A$ a portion of the upper boundary of the grid and set $B$ a number of obstacles spread over the grid. Figure 4 shows the level sets of the function $\tilde{V}(x)$ computed by solving the linear program $[\text{ReaP}]$. In Figure 5 we show the paths of 500 simulated trajectories of the object under the optimal policy obtained from the linear program $[\text{ReaD}]$. We choose two different starting states from different level sets according to Figure 4.

7.3. Reach with constraint. For Problem $[\text{P2}]$ we consider the same grid and sets $A$ and $B$ as in the previous problem. It is important to note that the initial distribution $\nu$ plays a key role in the feasibility.
of the problem. Figure 6 shows the level sets of the function
\[ \hat{V}(x) := \max_{\pi \in \Pi_B} P^\pi_x(\tau_A < \infty) \quad \text{s.t.} \quad P^\pi_x(\tau_B < \infty) \leq \epsilon, \]
for different values of $\epsilon$. White regions represents the states for which the problem above is infeasible. As expected, the number of infeasible states decrease with bigger values of $\epsilon$. Figure 7 we show the paths of 1000 trajectories under the optimal policies with different initial distributions and values of $\epsilon$.

8. Future Work

There are various directions for future research. The first one is to apply the ideas presented in this work to a general context that includes general state and actions spaces and continuous time controlled Markov process. The second one is to study the robust counterpart of these reachability problems. The
recent and increasing literature in robust MDPs using different classes of ambiguity sets can be applied in our context. Finally, to consider time dependent problems with moving target and obstacle sets.

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APPENDIX A. PROOF OF PROPOSITION 2.6

To prove the proposition we will first need the two following lemmas.
Lemma A.1. Let \( A \) be closed for a policy \( \pi \in \Pi \) and consider the event
\[
E = \{\exists s, t \in \mathbb{N} \text{ such that } s \leq t, X_s \in A, X_t \notin A\}.
\]
Then \( P_x^\pi(E) = 0 \) for any \( x \in X \).

**Proof.** We can write such an event as \( E = \bigcup_s \bigcup_{s<t} \{X_s \in A, X_t \notin A\} \). Therefore,
\[
P_x^\pi(E) \leq \sum_s \sum_{s<t} P_x^\pi(X_s \in A, X_t \notin A).
\]
Let us see that for all \( s < t \), \( P_x^\pi(X_s \in A, X_t \notin A) = 0 \). We have that,
\[
P_x^\pi(X_s \in A, X_t \notin A) = \sum_{x_1 \in X, x_s \in A, u_i \in U(x_i)} P_x^\pi(u_0, x_1, u_1, \ldots, x_{t-1}, X_t \notin A)
= \sum_{x_1 \in X, x_s \in A, u_i \in U(x_i)} P_x^\pi(X_t \notin A|h_{t-1})P_x^\pi(h_{t-1}|x)
= 0,
\]
where last equality follows from the fact that \( A \) is closed under \( \pi \), and so \( P_x^\pi(X_t \notin A|h_{t-1}) = 0 \). \( \square \)
Lemma A.2. Let $A$ be closed for a policy $\pi \in \Pi$. Then, for any $x \in \mathcal{X}$

$$P^\pi_x(\tau_A \leq t) = P^\pi_x(X_t \in A).$$

**Proof.** Define $B := \{ \omega \in \Omega \mid \forall m > \tau_A(\omega), X_m(\omega) \in A \}$. So that,

$$\{\tau_A \leq t\} = (\{\tau_A \leq t\} \cap B) \cup (\{\tau_A \leq t\} \cap B^c)$$

and

$$\{X_t \in A\} = (\{X_t \in A\} \cap B) \cup (\{X_t \in A\} \cap B^c).$$

Let us see that the events $\{X_t \in A\} \cap B$ and $\{\tau_A \leq t\} \cap B$ are equal. It is clear that $\{X_t \in A\} \cap B \subset \{\tau_A \leq t\} \cap B$. Now, let $\omega \in \{\tau_A \leq t\} \cap B$ so $\tau_A(\omega) \leq t$. Since $\omega \in B$, for all $m \geq \tau_A(\omega)$ we have $X_m(\omega) \in A$, in particular $X_t(\omega) \in A$, so that $\{\tau_A \leq t\} \cap B \subset \{X_t \in A\} \cap B$. Therefore,

$$P^\pi_x(\{\tau_A \leq t\} \cap B^c) = P^\pi_x(\{X_t \in A\} \cap B^c).$$

Finally, consider the event $E = \{3s, t \in \mathbb{N} \text{ such that } j \leq m, X_j \in A, X_m \notin A\}$. Note that, $\{\tau_A \leq t\} \cap B^c \subset E$ and $\{X_t \in A\} \cap B^c \subset E$. Since $A$ is closed, Lemma A.1 implies $P_x(E)^\pi = 0$, so that,

$$0 = P^\pi_x(\{\tau_A \leq t\} \cap B^c) = P^\pi_x(\{X_t \in A\} \cap B^c).$$

Combining both results we obtain the required equality. □

**Proof of Proposition 2.6** The result follows from the previous lemma and Monotone Convergence Thereom. □

**Appendix B. Proof of Proposition 3.2**

In order to prove the proposition we first need the following lemma.

Lemma B.1. Given $\pi \in \Pi$ there exists a policy $\pi' \in \Pi_A$ such that for any $x \notin A$ and $t \in \mathbb{N}$,

$$P^\pi_x(\tau_A = t) = P^\pi'_x(\tau_A = t)$$

**Proof.** Let $\pi = \{\mu_t\}_{t \geq 0} \in \Pi$ and $\pi_A = \{\mu^A_t\}_{t \geq 0} \in \Pi_A$. Define a policy $\pi' = \{\mu'_t\}_{t \geq 0} \in \Pi_A$ as follows:

$$\mu'_t(\cdot|h_t) = \begin{cases} 
\mu_t(\cdot|h_t), & \text{if } \pi^t \notin A \\
\mu^A_t(\cdot|h_t), & \text{otherwise.}
\end{cases}$$

Clearly $\pi' \in \Pi_A$. Let $x \notin A$, note that

$$P^\pi_x(\tau_A = t) = P^\pi_x(X_1 \notin A, \ldots, X_{t-1} \notin A, X_t \in A)$$

$$= \sum_{x_1, \ldots, x_{t-1} \notin A, x_t \in A} P^\pi_x(X_1 = x_1, \ldots, X_t = x_t)$$

$$= \sum_{x_1, \ldots, x_{t-1} \notin A, x_t \in A, u_t \in U(x_t)} P^\pi_x(u_0, x_1, u_1 \ldots x_t, u_t)$$

$$= \sum_{x_1, \ldots, x_{t-1} \notin A, x_t \in A, u_t \in U(x_t)} P^\pi_x(x_t, u_t|h_{t-1}, u_{t-1}) \ldots P^\pi_x(x_1, u_1|h_0, u_0)P^\pi(u_0|x).$$
Since \( x \not\in A \) we have \( P^\pi(u_0|x) = \mu_0(u_0|x) = \mu'_0(u_0|x) = P^{\pi'}(u_0|x) \). Also, for all \( i \in \mathbb{N} \),
\[
P^\pi(x_i, u_i|h_{i-1}, u_{i-1}) = P^\pi(u_i|h_i)P^\pi(x_i|h_{i-1}, u_{i-1}) = \mu_i(u_i|h_i)Q(x_i|x_{i-1}, u_{i-1})
\]
Because of the definition of \( \pi' \) we know that \( \mu_i(u_i|h_i) = \mu'_i(u_i|h_i) \) whenever \( \bar{x}^t \not\in A \). Thus, the fact that \( x, x_1 \ldots x_{t-1} \not\in A \) and previous equality imply that for \( i \leq t \)
\[
P^\pi(x_i, u_i|h_{i-1}, u_{i-1}) = P^{\pi'}(x_i, u_i|h_{i-1}, u_{i-1}).
\]
As a consequence
\[
P^\pi_x(\tau_A = t) = P^{\pi'}_x(\tau_A = t).
\]

**Proof of Lemma 3.2** Let \( \pi' \) be defined as in Lemma B.1. If \( x \in A \), the result follows trivially since \( P^\pi_x(\tau_A \leq t) = 1 \) for all \( t \geq 0 \). Let \( x \not\in A \). By of Lemma B.1 we know that
\[
P^\pi_x(\tau_A \leq t) = P^{\hat{\pi}}_x(\tau_A \leq t),
\]
and since \( \{X_t \in A\} \subset \{\tau_A \leq t\} \), we obtain
\[
P^\pi_x(X_t \in A) \leq P^\pi_x(\tau_A \leq t) = P^{\pi'}_x(\tau_A \leq t).
\]
Therefore, taking \( \lim \inf \) and using Proposition 2.6, we get
\[
\lim \inf_{t \to \infty} P^\pi_x(X_t \in A) \leq \lim_{t \to \infty} P^{\pi'}_x(X_t \in A) = P^{\pi'}_x(\tau_A < \infty).
\]

**Appendix C. Proof of Proposition 4.1**

Equation (4) follows from combining the next two lemmas.

**Lemma C.1.** Given a policy \( \pi \in \Pi \),
\[
P^\pi_{\nu}(\tau_A < \tau_B, \tau_A < \infty) = \tilde{P}^\pi_{\nu}(\tau_A < \tau_B, \tau_A < \infty)
\]

**Proof.** Let \( x \in A \), then
\[
P^\pi_x(\tau_A < \tau_B, \tau_A < \infty) = 1 = \tilde{P}^\pi_x(\tau_A < \tau_B, \tau_A < \infty).
\]
In case \( x \in B \) we obtain,
\[
P^\pi_x(\tau_A < \tau_B, \tau_A < \infty) = 0 = \tilde{P}^\pi_x(\tau_A < \tau_B, \tau_A < \infty).
\]
Let \( x \in \mathbb{X} \setminus \{A \cup B\} \). It’s clear that,
\[
\{\tau_A < \tau_B, \tau_A < \infty\} = \bigcup_{t \in \mathbb{N}} \{\tau_A = t, t < \tau_B\}
\]
Moreover, the events \( \{ \tau_A = t, t < \tau_B \} \) are disjoint, therefore to obtain the desired equality it is enough to show that

\[
P_x^\pi(\tau_A = t, t < \tau_B) = \widetilde{P}_x^\pi(\tau_A = t, t < \tau_B).
\]

Note that,

\[
P_x^\pi(\tau_A = t, t < \tau_B) = P_x^\pi(X_1 \notin A \cup B, \ldots, X_{t-1} \notin A \cup B, X_t \in A)
\]

\[
= \sum_{x_1, \ldots, x_{t-1} \notin A \cup B, x_t \in A} P_x^\pi(x_1, \ldots, x_t)
\]

\[
= \sum_{x_1, \ldots, x_{t-1} \notin A \cup B, x_t \in A, u_t \in U(x_t)} P_x^\pi(u_0, x_1, u_1 \ldots x_t, u_t)
\]

\[
= \sum_{x_1, \ldots, x_{t-1} \notin A \cup B, x_t \in A, u_t \in U(x_t)} P_x^\pi(x_t, u_t|\nu_{t-1}, u_{t-1}) \cdots P_x^\pi(x_1, u_1|\nu_0, u_0) P_x^\pi(u_0|x).
\]

We have \( P^\pi(u_0|x) = \mu_0(u_0|x) = \widetilde{P}_x^\pi(u_0|x) \). Furthermore, for all \( i \in \mathbb{N} \)

\[
P_x^\pi(x_i, u_i|\nu_{i-1}, u_{i-1}) = P_x^\pi(u_i|\nu_i) P_x^\pi(x_i|\nu_{i-1}, u_{i-1})
\]

\[
= \mu_i(u_i|\nu_i) Q(x_i|x_{i-1}, u_{i-1})
\]

Because of the definition of \( \tilde{Q} \) we know that \( Q(\cdot|s, u) = \tilde{Q}(\cdot|s, u) \) whenever \( s \notin A \cup B \). Thus, using the previous equality and the fact that \( x, x_1 \ldots x_{t-1} \notin A \cup B \) we obtain that for \( i \leq t \),

\[
P_x^\pi(x_i, u_i|\nu_{i-1}, u_{i-1}) = \widetilde{P}_x^\pi(x_i, u_i|\nu_{i-1}, u_{i-1}),
\]

and therefore

\[
P_x^\pi(\tau_A = t, t < \tau_B) = \widetilde{P}_x^\pi(\tau_A = t, t < \tau_B).
\]

\[ \square \]

**Lemma C.2.** Given a policy \( \pi \in \Pi \),

\[
\widetilde{P}_\nu^\pi(\tau_A < \tau_B, \tau_A < \infty) = \widetilde{P}_\nu^\pi(\tau_A < \infty).
\]

**Proof.** Consider the event \( \{ \tau_A < \infty, \tau_B < \infty \} \). Let us prove that \( \widetilde{P}_x^\pi(\tau_A < \infty, \tau_B < \infty) = 0 \) for any \( x \in X \). First of all, it is clear that,

\[
\{ \tau_A < \infty, \tau_B < \infty \} = \bigcup_{t, s \in \mathbb{N}} \{ \tau_A = t, \tau_B = s \}.
\]

Note that for all \( s, t \) the events \( \{ \tau_A = t, \tau_B = s \} \) are disjoint and are contained in either one of the following sets:

\[
\{ \exists n, m \in \mathbb{N} \text{ such that } n \leq m, X_n \in A, X_m \notin A \},
\]

\[
\{ \exists n, m \in \mathbb{N} \text{ such that } n \leq m, X_n \in B, X_m \notin B \}.
\]
By the definition of $\tilde{Q}$, the sets $A, B$ are closed under $\pi$. Therefore, Lemma A.1 implies that $\tilde{P}_x^\pi(\tau_A = t, \tau_B = s) = 0$, as a consequence the quantity $\tilde{P}_x^\pi(\tau_A < \infty, \tau_B < \infty)$ is equal to 0. Now, note that

$$\{\tau_A < \tau_B, \tau_A < \infty\} = \{\tau_A < \tau_B, \tau_A < \infty, \tau_B < \infty\} \cup \{\tau_A < \tau_B, \tau_A < \infty, \tau_B = \infty\}$$

and similarly,

$$\{\tau_A < \infty\} = \{\tau_A < \infty, \tau_B < \infty\} \cup \{\tau_A < \infty, \tau_B = \infty\}.$$

Therefore, previous calculations yield,

$$\tilde{P}_x^\pi(\tau_A < \tau_B, \tau_A < \infty) = \tilde{P}_x^\pi(\tau_A < \infty, \tau_B = \infty) = \tilde{P}_x^\pi(\tau_A < \infty).$$

\[\square\]

\textbf{Appendix D. Proof of Theorem 5.3}

The proof of this theorem is divided into several lemmas. The first one allows us to rewrite the average reward function (7) in terms of the probabilities of hitting times being finite. Its proof is analogous to the proof of Theorem 2.7.

\textbf{Lemma D.1.} Let $\hat{\pi}$ be a Markovian policy over $\hat{X}$. Then

$$v^\hat{\pi}(x, i) = \hat{P}_x^\hat{\pi}(\tau_{A \times \{0\}} < \infty) + \hat{P}_x^\hat{\pi}(\tau_{A \times \{1\}} < \infty) - \lambda \hat{P}_x^\hat{\pi}(\tau_{X \times \{1\}} < \infty).$$

In the following lemmas we will always assume that $\hat{\pi} \in \hat{\Pi}$ and that $\pi \in \Pi_B$ is its corresponding policy.

\textbf{Lemma D.2.} If $x \notin B$, then

$$\hat{P}_x^\pi(\tau_{X \times \{1\}} < \infty) = P_x^\pi(\tau_B < \infty).$$

\textbf{Proof.} Note that the event $\{\tau_B < \infty\}$ can be partitioned in disjoint events

$$\{\tau_B < \infty\} = \bigcup_{s \in \mathbb{N}} \{\tau_B = s\}.$$

Therefore, it is enough to restrict our attention to such events. Using Lemma 5.2 we obtain that

$$P_x^\pi(\tau_B = s) = \sum_{x_1, \ldots, x_{s-1} \notin B \atop x_s \in B, u_i \in U(x_i)} P_x^\pi(u_0, x_1, u_1, \ldots, x_s, u_s)$$

$$= \sum_{x_1, \ldots, x_{s-1} \notin B \atop x_s \in B, u_i \in U(x_i)} \hat{P}_x^\pi(u_0, (x_1, 0), u_1, \ldots, (x_{s-1}, 0), u_{s-1}, (x_s, 1), u_s).$$

(10)
Let’s write,
\[
\hat{P}_{(x,0)}^\pi(u_0, (x_1, 0), u_1, \ldots, (x_{s-1}, 0), u_{s-1}, (x_s, 1), u_s)
= \hat{P}_{(x,1)}^\pi((x_s, 1), u_s \mid \hat{h}_{s-1}, u_{s-1}) \cdots \hat{P}_{(x,0)}^\pi((x_1, 0), u_1 \mid \hat{h}_0, u_0) \hat{P}_{(x,0)}^\pi(u_0 | (x, 0))
\]

Because of (6) (which is valid for any Markov model) we know,
\[
\hat{P}_{(x,t), t}^\pi((x_t, i_t), u_t \mid \hat{h}_{t-1}, u_{t-1}) = \hat{P}(u_t \mid \hat{h}_t) \hat{Q}((x_t, i_t) | (x_{t-1}, 0), u_{t-1}).
\]

Consider an element \( x_t \in B \), for \( t < s \), then the definition of \( \hat{Q} \) implies \( \hat{P}_{(x,0)}^\pi((x_t, 0), u_t \mid \hat{h}_{t-1}, u_{t-1}) = 0 \). If \( x_s \notin B \) then the definition of \( \hat{Q} \) implies \( \hat{P}_{(x,1)}^\pi((x_s, 1), u_s \mid \hat{h}_{s-1}, u_{s-1}) = 0 \). Therefore, (10) can be written as
\[
\sum_{x_1, \ldots, x_s \in X \atop u_1 \in U(x_1)} \hat{P}_{(x,0)}^\pi(u_0, (x_1, 0), u_1, \ldots, (x_{s-1}, 0), u_{s-1}, (x_s, 1), u_s)
= \hat{P}_{(x,0)}^\pi((X_1, I_1) \notin X \times \{1\}, \ldots, (X_{s-1}, I_{s-1}) \notin X \times \{1\}, (X_s, I_s) \in X \times \{1\})
= \hat{P}_{(x,0)}^\pi(\tau X \times \{1\} = s).
\]

Next two lemmas prove that whenever \( x \notin B \), we have that
\[
P_x^\pi(\tau_A < \infty) = \hat{P}_{(x,0)}^\pi(\tau_{A \times \{0\}} < \infty) + \hat{P}_{(x,0)}^\pi(\tau_{A \times \{1\}} < \infty).
\]

**Lemma D.3.** If \( x \notin B \), then
\[
\hat{P}_{(x,0)}^\pi(\tau_{A \times \{1\}} < \infty) = P_x^\pi(\tau_A < \infty, \tau_B < \tau_A).
\]

**Proof.** It is enough to prove,
\[
P_x^\pi(\tau_A = t, \tau_B < \tau_A) = \hat{P}_{(x,0)}^\pi(\tau_{A \times \{1\}} = t)
\]

Using Lemma [5.2] we obtain,
\[
P_x^\pi(\tau_A = t, \tau_B < \tau_A) = \sum_{s < t} P_x^\pi(\tau_B = s, \tau_A = t)
= \sum_{1 \leq s < t \atop x_1, \ldots, x_{s-1} \notin B \cup A, u_1 \in U(x_1)} P_x^\pi(u_0, x_1, \ldots, x_t, u_t)
\]

\[
(11) \quad = \sum_{1 \leq s < t \atop x_1, \ldots, x_{s-1} \notin B \cup A, u_1 \in U(x_1)} \hat{P}_{(x,0)}^\pi(u_0, (x_1, 0), u_1, \ldots, (x_{s-1}, 0), u_{s-1}, (x_s, 1), u_s, \ldots (x_t, 1), u_t).
\]
Let’s proceed proving that \( \hat{P}^{\pi}_{(x,0)}(\tau_{A \times \{1\}} = t) \) is equal to \([11]\). We have that,

\[
\hat{P}^{\pi}_{(x,0)}(\tau_{A \times \{1\}} = t) = \hat{P}^{\pi}_{(x,0)}((X_1, I_1) \notin A \times \{1\}, \ldots, (X_{t-1}, I_{t-1}) \notin A \times \{1\}, (X_t, I_t) \in A \times \{1\})
\]

\[
= \sum_{u_0 \in U(j_0), j_1 \notin A \times \{1\}, j_t \in A \times \{1\}} \hat{P}^{\pi}_{(x,0)}(u_0, j_1, u_1, \ldots, j_t, u_t)
\]

\[
= \sum_{u_t \in U(j_t), j_1 \notin A \times \{1\}, j_t \in A \times \{1\}} \hat{P}^{\pi}(j_t, u_t | h_{t-1}, u_{t-1}) \cdots \hat{P}^{\pi}(j_1, u_1 | h_0, u_0) \hat{P}^{\pi}(u_0 | (x, 0)).
\]

We must show that the summation in previous expression can be performed over the indexes showed in \([11]\). To prove this recall that because of \([6]\) we know,

\[
\hat{P}^{\pi}(j_r, u_r | h_{r-1}, u_{r-1}) = \hat{\pi}(u_r | h_r) \hat{Q}(j_r | j_{r-1}, u_{r-1}).
\]

This equation and the definition of \( \hat{Q} \) imply that \( \hat{P}^{\pi}(j_{r+1}, u_{r+1} | h_r, u_r) = 0 \) whenever \( j_{r+1} = (x_{r+1}, 0) \), \( j_r = (x_r, 1) \). Hence, if \( s \) is the first time the process hits the set \( \mathbb{X} \times \{1\} \), it remains there. Note that \( 0 < s < t \) since \( j_t \in A \times \{1\} \). Also, recall that \( A \times \{0,1\} \) is closed for any policy \( \hat{\pi} \) by definition of \( \hat{Q} \). So, it is enough to consider histories of the form,

\[
((x, 0), (x_1, 0), \ldots, (x_{s-1}, 0), (x_s, 1), \ldots, (x_t, 1))
\]

where \( x_1, \ldots, x_{s-1} \notin B \cup A, x_s \in B, x_{s+1}, \ldots, x_{t-1} \notin A \), \( x_t \in A \), and so the summation is equal to

\[
\sum_{1 \leq s < t, x_1, \ldots, x_{s-1} \notin B \cup A, x_s \in B, x_{s+1}, \ldots, x_{t-1} \notin A, x_t \in A} \hat{P}^{\pi}(j_t, u_t | h_{t-1}, u_{t-1}) \cdots \hat{P}^{\pi}(j_1, u_1 | h_0, u_0) \hat{P}^{\pi}(u_0 | (x, 0)),
\]

which is equal to \([11]\). \( \square \)

Similarly, we obtain the next result.

**Lemma D.4.** If \( x \notin B \), then

\[
\hat{P}^{\pi}_{(x,0)}(\tau_{A \times \{0\}} < \infty) = P^{\pi}_{x}(\tau_A < \infty, \tau_A < \tau_B).
\]

The final lemma completes the proof of the theorem by considering the case when the initial state belongs to \( B \).

**Lemma D.5.** If \( x \in B \), then

\[
\hat{u}^{\pi}(x, 1) = P^{\pi}_{x}(\tau_A < \infty) - \lambda
\]

**Proof.** Clearly, it is enough to show that \( P^{\pi}_{x}(\tau_A < \infty) = \hat{P}^{\pi}_{(x,1)}(\tau_{A \times \{1\}} < \infty) \). As before, consider the events \( \{ \tau_A = t \} \). Since \( x \in B \) Lemma 5.2 implies,

\[
\hat{P}^{\pi}_{(x,1)}(\tau_A = t) = \sum_{x_1, \ldots, x_{t-1} \notin A, x_t \in A, u_t \in U(x_t)} P^{\pi}_{x}(u_0, x_1, \ldots, x_t, u_t)
\]

\[
= \sum_{x_1, \ldots, x_{t-1} \notin A, x_t \in A, u_t \in U(x_t)} \hat{P}^{\pi}_{(x,1)}(u_0, (x_1, 1), \ldots, (x_t, 1), u_t).
\]
The fact that $\mathbb{X} \times \{1\}$ is closed implies that the process can not escape from it, as a consequence,

$$\hat{P}_{(x,1)}(\hat{h}_t, u_t) = 0$$

whenever an element $(x_r, 0)$ appears in the history $\hat{h}_t$. Therefore, expression (12) can be written as

$$\sum_{j_1, \ldots, j_{t-1} \notin A \times \{1\}, j_t \in A \times \{1\}, u_t \in U(x_t)} \hat{P}_{(x,1)}(u_0, j_1, \ldots, j_t, u_t)$$

$$= \hat{P}_{(x,1)}((X_1, I_1) \notin A \times \{1\}, \ldots, (X_{t-1}, I_{t-1}) \notin A \times \{1\}, (X_t, I_t) \in A \times \{1\})$$

$$= \hat{P}_{(x,0)}(\tau_{A \times \{1\}} = t).$$

□

REFERENCES

[1] E Arvelo and NC Martins, Maximizing the set of recurrent states of an mdp, subject to convex constraints, Automatica 50 (2014), 994–998.
[2] Dimitri P Bertsekas, Dynamic programming and optimal control, Vol. 1, Athena Scientific, 1995.
[3] Dimitri P Bertsekas and Steven E Shreve, Stochastic optimal control: The discrete time case, Athena Scientific, 1996.
[4] Bruno Bouchard, Romuald Elie, and Nizar Touzi, Stochastic target problems with controlled loss, SIAM Journal on Control and Optimization 48 (2010), no. 5, 3123–3150.
[5] Fabio Camilli, Lars Grüne, and Fabian Wirth, Control Lyapunov functions and Zubov’s method, SIAM Journal on Control and Optimization 47 (2008), no. 1, 301–326.
[6] Fabio Camilli and Paola Loreti, A Zubov’s method for stochastic differential equations, Nonlinear Differential Equations and Applications NoDEA 13 (2006), no. 2, 205–222.
[7] Debasish Chatterjee, Eugenio Cinquemani, and John Lygeros, Maximizing the probability of attaining a target prior to extinction, Nonlinear Analysis: Hybrid Systems 5 (2011), no. 2, 367–381.
[8] Peyman Mohajerin Esfahani, Debasish Chatterjee, and John Lygeros, The stochastic reach-avoid problem and set characterization for diffusions, Automatica 70 (2016), 43–56.
[9] Wolfgang Hahn and Arne P Baartz, Stability of motion, Vol. 138, Springer, 1967.
[10] Onésimo Hernández-Lerma and Jean B Lasserre, Further topics on discrete-time markov control processes, Vol. 42, Springer Science & Business Media, 2012.
[11] Onésimo Hernández-Lerma and Jean Bernard Lasserre, Discrete-time markov control processes: basic optimality criteria (1996).
[12] Martin L Puterman, Markov decision processes: discrete stochastic dynamic programming, John Wiley & Sons, 2014.
[13] H. Mete Soner and Nizar Touzi, Stochastic target problems, dynamic programming, and viscosity solutions, SIAM Journal on Control and Optimization 41 (2002), no. 2, 404–424.
[14] Vladimir Ivanovich Zubov, Methods of AM Lyapunov and their application, P. Noordhoff, 1964.