ON THE BOUNDEDNESS OF BERGMAN PROJECTION

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Abstract. The main purpose of this survey is to gather results on the boundedness of the Bergman projection. First, we shall go over some equivalent norms on weighted Bergman spaces $A^p_\omega$ which are useful in the study of this question. In particular, we shall focus on a decomposition norm theorem for radial weights $\omega$ with the doubling property $\int_1^r \omega(s) \, ds \leq C \int_1^{1+r} \omega(s) \, ds$.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ be the space of all analytic functions in the unit disc $\mathbb{D} = \{ z : |z| < 1 \}$. If $0 < r < 1$ and $f \in \mathcal{H}(\mathbb{D})$, set

$$M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|.$$

For $0 < p \leq \infty$, the Hardy space $H^p$ consists of functions $f \in \mathcal{H}(\mathbb{D})$ such that $\|f\|_{H^p} = \sup_{0 < r < 1} M_p(r, f) < \infty$. A nonnegative integrable function $\omega$ on the unit disc $\mathbb{D}$ is called a weight. It is radial if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. For $0 < p < \infty$ and a weight $\omega$, the weighted Bergman space $A^p_\omega$ is the space of $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z) \, dA(z) < \infty,$$

where $dA(z) = \frac{dx \, dy}{\pi}$ is the normalized Lebesgue area measure on $\mathbb{D}$. That is, $A^p_\omega = L^p_\omega \cap \mathcal{H}(\mathbb{D})$ where $L^p_\omega$ is the corresponding weighted Lebesgue space. As usual, we write $A^p_\alpha$ for the standard weighted Bergman space induced by the radial weight $(1-|z|^2)^\alpha$, where $-1 < \alpha < \infty$ \([15, 19, 37]\). We denote $dA_\alpha = (\alpha + 1)(1-|z|^2)^\alpha \, dA(z)$ and $\omega(E) = \int_E \omega(z) \, dA(z)$ for short. The Carleson square $S(I)$ based on an interval $I \subset \mathbb{T}$ is the set $S(I) = \{ re^{it} \in \mathbb{D} : e^{it} \in I, 1-|I| \leq r < 1 \}$, where $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{T}$. We associate to each $a \in \mathbb{D} \setminus \{0\}$ the interval $I_a = \{ e^{i\theta} : |\arg(ae^{-i\theta})| \leq \frac{1-|a|}{2} \}$, and denote $S(a) = S(I_a)$.

If the norm convergence in the Bergman space $A^2_\omega$ implies the uniform convergence on compact subsets, then the point evaluations $L_z$ (at the point $z \in \mathbb{D}$)

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are bounded linear functionals on $A^2_{\omega}$. Therefore, there are reproducing kernels $B^\omega_z \in A^2_{\omega}$ with $\|L_z\| = \|B^\omega_z\|_{A^2_{\omega}}$ such that

$$L_z f = f(z) = \langle f, B^\omega_z \rangle_{A^2_{\omega}} = \int_{\mathbb{D}} f(\zeta) \overline{B^\omega_z(\zeta)} \omega(\zeta) dA(\zeta), \quad f \in A^2_{\omega}. $$

Since $A^2_{\omega}$ is a closed subspace of $L^2_{\omega}$, we may consider the orthogonal Bergman projection $P_{\omega}$ from $L^2_{\omega}$ to $A^2_{\omega}$, that is usually called the Bergman projection. It is the integral operator

$$P_{\omega}(f)(z) = \int_{\mathbb{D}} f(\zeta) \overline{B^\omega_z(\zeta)} \omega(\zeta) dA(\zeta). $$

The main purpose of these lectures is to gather results on the inequality

$$\|P_{\omega}(f)\|_{L^p_{\omega}} \leq C\|f\|_{L^p_{\omega}}. \quad (1.1)$$

We shall also provide a collection of equivalent norms on $A^p_{\omega}$ which have been used to study this problem. A solution for (1.1) is known for the class of standard weights $\omega(z) = (1 - |z|^2)^{p}$ and $1 < \alpha < \infty;$$

$$P_{\alpha}(f)(z) = (\alpha + 1) \int_{\mathbb{D}} \frac{f(\zeta)(1 - |\zeta|^2)^{\alpha}}{(1 - z\bar{\zeta})^{2a}} dA(\zeta), \quad \alpha > -1,$$

is bounded on $L^p_{\omega}$ if and only if $\frac{u(z)}{|1 - |z|^2|^p}}$ belongs to the Bekollé-Bonami class $B_p(\alpha)$ [7, 8]. We remind the reader that $v \in B_p(\alpha)$ if

$$B_{p,\alpha}(v) = \sup_{I \subset \mathbb{T}} \left( \int_{S(I)} v(z) dA_\alpha(z) \right) \left( \int_{S(I)} \frac{v'(z)}{v(z)} dA_\alpha(z) \right)^{\frac{1}{p'}} < \infty. \quad (1.2)$$

It is worth mentioning that the above result remains true replacing $P_{\alpha}$ by its sublinear positive counterpart

$$P_{\alpha}^+(f)(z) = (\alpha + 1) \int_{\mathbb{D}} \frac{|f(\zeta)|(1 - |\zeta|^2)^{\alpha}}{|1 - z\bar{\zeta}|^{2a}} dA(\zeta).$$

Roughly speaking, this means that cancellation does not play an essential role in this question.

The situation is completely different when $\omega$ is not a standard weight, because of the lack of explicit expressions for the Bergman reproducing kernels $B^\omega_z$. If $\omega$ is a admissible radial weight, then the normalized monomials $\frac{z^n}{\sqrt{2 \pi \omega(r)}}$, $n \in \mathbb{N} \cup \{0\}$, form the standard orthonormal basis of $A^2_{\omega}$ and then [37, Theorem 4.19] yields

$$B^\omega_z(\zeta) = \sum_{n=0}^{\infty} \frac{(\zeta \bar{\zeta})^n}{2 \int_0^r \sqrt{2 \pi \omega(r)} dr}, \quad z, \zeta \in \mathbb{D}. \quad (1.3)$$

This formula and a decomposition norm theorem has been used recently in order to obtain precise estimates for the $L^p_{\omega}$-integral of $B^\omega_z$, see Theorem [13] below. This is a key to tackle the two weight inequality (1.1) when $\omega$ and $v$ belong to a certain class of radial weights [32].

If $\omega$ is not necessarily radial, the theory of weighted Bergman spaces is at its early stages, and plenty of essential properties such as the density of polynomials (polynomials may not be dense in $A^p_{\omega}$ if $\omega$ is not radial, [30, Section 1.5] or [15, p. 138]) have not been described yet. Because of this fact, from now on we shall be mainly focused on Bergman spaces induced by radial weights.
Throughout the paper $\frac{1}{p} + \frac{1}{p'} = 1$. Further, the letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \leq Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we will write $a \approx b$.

2. Background on radial weights

We shall write $\hat{D}$ for the class of radial weights such that $\hat{\omega}(z) = \int_1^1 |z| \omega(s) \, ds$ is doubling, that is, there exists $C = C(\omega) \geq 1$ such that $\hat{\omega}(r) \leq C \hat{\omega}(1 + r^2)$ for all $0 \leq r < 1$. We call a radial weight $\omega$ regular, denoted by $\omega \in R$, if $\omega \in \hat{D}$ and $\omega(r)$ behaves as its integral average over $(r, 1)$, that is,

$$\omega(r) \approx \int_r^1 \omega(s) \, ds \frac{1}{1 - r}, \quad 0 \leq r < 1.$$ 

As to concrete examples, we mention that every standard weight as well as those given in [4, (4.4)–(4.6)] are regular. It is clear that $\omega \in R$ if and only if for each $s \in [0, 1)$ there exists a constant $C = C(s, \omega) > 1$ such that

$$C^{-1} \omega(t) \leq \omega(r) \leq C \omega(t), \quad 0 \leq r \leq t \leq r + s(1 - r) < 1,$$

and

$$\int_r^1 \omega(s) \, ds \frac{1}{1 - r} \lesssim \omega(r), \quad 0 \leq r < 1.$$ 

The definition of regular weights used here is slightly more general than that in [30], but the main properties are essentially the same by Lemma 1 below and [30, Chapter 1].

A radial continuous weight $\omega$ is called rapidly increasing, denoted by $\omega \in I$, if

$$\lim_{r \to 1^-} \frac{\int_r^1 \omega(s) \, ds}{\omega(r)(1 - r)} = \infty.$$ 

It follows from [30 Lemma 1.1] that $I \subset \hat{D}$. Typical examples of rapidly increasing weights are

$$v_\alpha(r) = \left((1 - r) \left(\log \frac{e}{1 - r}\right)^\alpha\right)^{-1}, \quad 1 < \alpha < \infty.$$ 

Despite their name, rapidly increasing weights may admit a strong oscillatory behavior. Indeed, the weight

$$\omega(r) = \left|\sin \left(\log \frac{1}{1 - r}\right)\right| v_\alpha(r) + 1, \quad 1 < \alpha < \infty,$$

belongs to $I$ but it does not satisfy [241, 30, p. 7].

A radial continuous weight $\omega$ is called rapidly decreasing if $\lim_{r \to 1^-} \frac{\int_r^1 \omega(s) \, ds}{\omega(r)(1 - r)} = 0$. The exponential type weights $\omega_{\gamma,\alpha}(r) = (1 - r)^\gamma \exp \left(\frac{\alpha}{1 - r}\right)$, $\gamma \geq 0$, $\alpha, c > 0$, are rapidly decreasing. For further information on these classes see [30 Chapter 1] and the references therein.

The following characterizations of the class $\hat{D}$ will be frequently used from here on.

**Lemma 1.** Let $\omega$ be a radial weight. Then the following conditions are equivalent:
(i) \( \omega \in \hat{D} \);
(ii) There exist \( C = C(\omega) > 0 \) and \( \beta_0 = \beta_0(\omega) > 0 \) such that
\[
\hat{\omega}(r) \leq C \left( \frac{1-r}{1-t} \right)^\beta \hat{\omega}(t), \quad 0 \leq r \leq t < 1,
\]
for all \( \beta \geq \beta_0 \);
(iii) There exist \( C = C(\omega) > 0 \) and \( \gamma_0 = \gamma_0(\omega) > 0 \) such that
\[
\int_0^t \left( \frac{1-t}{1-s} \right)^\gamma \omega(s) \, ds \leq C \hat{\omega}(t), \quad 0 \leq t < 1,
\]
for all \( \gamma \geq \gamma_0 \);
(iv) There exists \( C = C(\omega) > 0 \) such that
\[
\int_0^t s^{\frac{1}{1-t}} \omega(s) \, ds \leq C \hat{\omega}(t), \quad 0 \leq t < 1.
\]
(v) There exists \( C = C(\omega) > 0 \) such that
\[
\hat{\omega}(r) \leq Cr^{\frac{1}{1-t}} \hat{\omega}(t), \quad 0 \leq r \leq t < 1.
\]
(vi) The asymptotic equality
\[
\omega_x = \int_0^1 s^x \omega(s) \, ds \asymp \hat{\omega} \left( 1 - \frac{1}{x} \right),
\]
is valid for any \( x \geq 1 \).

Proof. We are going to prove (i) \( \iff \) (ii) \( \iff \) (iii) \( \implies \) (iv) \( \implies \) (v) \( \implies \) (i) and (iv) \( \iff \) (vi).

Let \( \omega \in \hat{D} \). If \( 0 \leq r \leq t < 1 \) and \( r_n = 1 - 2^{-n} \) for all \( n \in \mathbb{N} \cup \{0\} \), then there exist \( k \) and \( m \) such that \( r_k \leq r < r_{k+1} \) and \( r_m \leq t < r_{m+1} \). Hence
\[
\hat{\omega}(r) \leq \hat{\omega}(r_k) \leq C \hat{\omega}(r_{k+1}) \leq \cdots \leq C^{m-k+1} \hat{\omega}(r_{m+1}) \leq C^{m-k+1} \hat{\omega}(t) = C^2 2^{(m-k-1) \log_2 C} \hat{\omega}(t) \leq C \left( \frac{1-r}{1-t} \right)^\log_2 C \hat{\omega}(t), \quad 0 \leq r \leq t < 1.
\]

On the other hand, it is clear that (ii) implies that \( \omega \in \hat{D} \). So, we have proved (i) \( \iff \) (ii).

If (ii) is satisfied and \( \gamma > \beta \), then
\[
\int_0^t \left( \frac{1-t}{1-s} \right)^\gamma \omega(s) \, ds \leq C^{\frac{\gamma}{\beta}} \int_0^t \left( \frac{\hat{\omega}(t)}{\hat{\omega}(s)} \right)^{\frac{\gamma}{\beta}} \omega(s) \, ds = C^{\frac{\gamma}{\beta}} \hat{\omega}(t)^{\frac{\gamma}{\beta}} \int_0^t \omega(s) \, ds \leq \frac{\beta}{\gamma - \beta} C^{\frac{\gamma}{\beta}} \hat{\omega}(t), \quad 0 \leq t < 1.
\]
Conversely, if (iii) is satisfied, then an integration by parts yields
\[
C \hat{\omega}(t) \geq \int_{0}^{t} \left( \frac{1 - t}{1 - s} \right)^{\gamma} \omega(s) \, ds \\
= -\hat{\omega}(t) + (1 - t)^{\gamma}\hat{\omega}(0) + \gamma(1 - t) \int_{0}^{t} \frac{\hat{\omega}(s)}{(1 - s)^{\gamma+1}} \, ds \\
\geq -\hat{\omega}(t) + (1 - t)^{\gamma}\hat{\omega}(0) + \gamma(1 - t) \int_{0}^{r} \frac{ds}{(1 - s)^{\gamma+1}} \\
= -\hat{\omega}(t) + (1 - t)^{\gamma}(\hat{\omega}(0) - \hat{\omega}(r)) + \left( \frac{1 - t}{1 - r} \right)^{\gamma} \hat{\omega}(r) \\
\geq \left( \frac{1 - t}{1 - r} \right)^{\gamma} \hat{\omega}(r) - \hat{\omega}(t), \quad 0 \leq r \leq t < 1,
\]
therefore (ii)\(\Leftrightarrow\)(iii).

The proof of [30, Lemma 1.3] shows that (iii) implies (iv), we include a proof for the sake of completeness. A simple calculation shows that for all \(s \in (0, 1)\) and \(x > 1\),
\[
s^{x-1}(1 - s)^{\gamma} \leq \left( \frac{x - 1}{x - 1 + \gamma} \right)^{x-1} \left( \frac{\gamma x}{x - 1 + \gamma} \right) \leq \left( \frac{\gamma x}{x - 1 + \gamma} \right).
\]
Therefore (iii), with \(t = 1 - \frac{r}{x}\), yields
\[
\int_{0}^{1 - \frac{1}{x}} s^{x\omega(s)} \, ds \leq \left( \frac{\gamma x}{x - 1 + \gamma} \right) \int_{0}^{1 - \frac{1}{x}} \frac{\omega(s)}{s^{\gamma}(1 - s)^{\gamma}} \, ds \\
\leq \int_{1 - \frac{1}{x}}^{1} \omega(s) \, ds, \quad x > 1,
\]
which is equivalent to (iv).

On the other hand, if (iv) is satisfied and \(0 \leq r \leq t < 1\), then an integration by parts yields
\[
C \hat{\omega}(t) \geq \int_{0}^{t} \frac{s^{\frac{1}{x\gamma}} \omega(s)}{t^{\frac{1}{x\gamma}}} \, ds = -\hat{\omega}(t)t^{\frac{1}{x\gamma}} + \frac{1}{1 - t} \int_{0}^{t} \hat{\omega}(s)s^{\frac{1}{x\gamma}} \, ds \\
\geq -\hat{\omega}(t)t^{\frac{1}{x\gamma}} + \frac{1}{1 - t} \int_{0}^{r} \hat{\omega}(s)s^{\frac{1}{x\gamma}} \, ds \\
\geq -\hat{\omega}(t)t^{\frac{1}{x\gamma}} + \frac{\hat{\omega}(r)}{1 - t} \int_{0}^{r} s^{\frac{1}{x\gamma}} \, ds = -\hat{\omega}(t)t^{\frac{1}{x\gamma}} + r^{\frac{1}{x\gamma}} \hat{\omega}(r),
\]
and thus
\[
r^{\frac{1}{x\gamma}} \hat{\omega}(r) \leq \left( C + t^{\frac{1}{x\gamma}} \right) \hat{\omega}(t), \quad 0 \leq r \leq t < 1.
\]
This implies (v), and by choosing \(t = \frac{1 + r}{2}\) in (v), we deduce \(\omega \in \hat{D}\). Finally, it is clear that (iv) is equivalent to (vi).

\[\Box\]

3. Equivalent norms

In this section we shall present several equivalent norms on weighted Bergman spaces. In particular we shall give a detailed proof of a decomposition norm theorem for \(A_{\omega}^{p}\) when \(\omega \in \hat{D}\) and \(1 < p < \infty\).
It is well-known that a choice of an appropriate norm is often a key step when solving a problem on a space of analytic functions. For instance, in the study of the integration operators

\[ T_g(f)(z) = \int_0^z f(\zeta) g'(\zeta) \, d\zeta, \quad z \in \mathbb{D}, \quad g \in \mathcal{H}(\mathbb{D}), \]

one wants to get rid of the integral symbol, so one looks for norms in terms of the first derivative. It is worth mentioning that the operator \( T_g \) began to be extensively studied after the appearance of the works by Aleman, Cima and Siskakis \([1, 4]\). A description of its resolvent set on Hardy and standard Bergman spaces is strongly connected with the classical theory of the Muckenhoupt weights and the Bekollé-Bonami weights \([2, 3]\).

3.1. Norms in terms of the derivative. Following Siskakis \([34]\), for a radial weight \( \omega \) we define its distortion function by

\[ \psi_\omega(z) = \frac{1}{\omega(|z|)} \int_{|z|}^1 \omega(s) \, ds, \quad z \in \mathbb{D}. \]

For a large class of radial weights, which includes any differentiable decreasing weight and all the standard ones, the most appropriate way to obtain a useful norm involving the first derivative is to establish a kind of Littlewood-Paley type formula \([28, \text{Theorem 1.1}]\).

**Theorem 2.** Suppose that \( \omega \) is a radial differentiable weight, and there is \( L > 0 \) such that

\[ \sup_{0 < r < 1} \frac{\omega'(r)}{\omega(r)^2} \int_r^1 \omega(x) \, dx \leq L. \]

Then, for each \( p \in (0, \infty) \)

\[ \int_{\mathbb{D}} |f(z)|^p \omega(z) \, dA(z) \asymp |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p \psi_\omega^p(z) \omega(z) \, dA(z), \quad f \in \mathcal{H}(\mathbb{D}). \]

If \( \omega \in \mathcal{I} \) and \( p \neq 2 \), a result analogous to Theorem 4 cannot be obtained in general \([30, \text{Proposition 4.2}]\).

**Proposition 3.** Let \( p \neq 2 \). Then there exists \( \omega \in \mathcal{I} \) such that, for any function \( \varphi : [0, 1) \to (0, \infty) \), the relation

\[ \|f\|_{A_p^\varphi} \asymp \int_{\mathbb{D}} |f'(z)|^p \varphi(|z|)^p \omega(z) \, dA(z) + |f(0)|^p \]

cannot be valid for all \( f \in \mathcal{H}(\mathbb{D}) \).

As for a Littlewood-Paley formula for \( A_p^\varphi \), the following result was proved in \([2, \text{Theorem 3.1}]\).

**Theorem 4.** Suppose that \( \omega \) is a weight such that \( \frac{\omega(z)}{(1-|z|)^\eta} \) satisfies the Bekollé-Bonami condition \( B_{p_0}(\eta) \) for some \( p_0 > 0 \) and some \( \eta > -1 \). Then, for each \( p \in (0, \infty) \)

\[ \int_{\mathbb{D}} |f(z)|^p \omega(z) \, dA(z) \asymp |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|)^\eta \omega(z) \, dA(z), \quad f \in \mathcal{H}(\mathbb{D}). \]
We remark that whenever \( \omega \in C^1(D) \) and
\[
(1 - |z|)|\nabla \omega(z)| \lesssim \omega(z), \quad z \in D,
\]
(3.1) is equivalent to a Bekollé-Bonami condition [2, Theorem 3.1].

Now, we consider the non-tangential approach regions
\[
\Gamma(\zeta) = \left\{ z \in D : |\theta - \arg z| < \frac{1}{2} \left( 1 - \frac{|z|}{r} \right) \right\}, \quad \zeta = re^{i\theta} \in \mathbb{D} \setminus \{0\}
\]
and the related tents \( T(z) = \{ \zeta \in D : z \in \Gamma(\zeta) \} \).

Whenever \( \omega \) is a radial weight, \( A^p_\omega \) can be equipped with other norms which are inherited from the classical Fefferman-Stein estimate [16] and the Hardy-Stein-Spencer identity [18] for the \( H^p \)-norm. Here \( \omega^*(z) = \int_{|z|}^1 \omega(s) \log \frac{s}{|z|} ds, \quad z \in \mathbb{D} \setminus \{0\} \).

**Theorem 5.** Let \( 0 < p < \infty \), \( n \in \mathbb{N} \) and \( f \in \mathcal{H}(D) \), and let \( \omega \) be a radial weight. Then
\[
\|f\|_{A^p_\omega} = p^2 \int_D \left| f(z) \right|^{p-2} |f'(z)|^2 \omega^*(z) dA(z) + \omega(D) \|f(0)\|^p,
\]
and
\[
\|f\|_{A^p_\omega}^p \propto \int_D \left( \left( \int_{\Gamma(u)} |f^{(n)}(z)|^2 \left( 1 - \frac{|z|}{u} \right)^{2n-2} dA(z) \right)^{\frac{p}{2}} \omega(u) dA(u)
\]
\[+ \sum_{j=0}^{n-1} |f^{(j)}(0)|^p,
\]
where the constants of comparison depend only on \( p, n \) and \( \omega \). In particular,
\[
\|f\|_{A^p_\omega}^2 = 4 \|f'\|_{A^p_\omega^*}^2 + \omega(D) \|f(0)\|^2.
\]

Next, we present an equivalent norm for weighted Bergman spaces which has been very recently used to describe the \( q \)-Carleson measures for \( A^p_\omega \) when \( \omega \in \hat{D} \) [31].

Let \( f \in \mathcal{H}(D) \), and define the non-tangential maximal function of \( f \) in the (punctured) unit disc by
\[
N(f)(u) = \sup_{z \in \Gamma(u)} |f(z)|, \quad u \in \mathbb{D} \setminus \{0\}.
\]

**Lemma 6.** [30, Lemma 4.4] Let \( 0 < p < \infty \) and let \( \omega \) be a radial weight. Then there exists a constant \( C > 0 \) such that
\[
\|f\|_{A^p_\omega} \leq \|N(f)\|_{L^p_\omega} \leq C \|f\|_{A^p_\omega}
\]
for all \( f \in \mathcal{H}(D) \).

**Proof.** It follows from [18, Theorem 3.1 on p. 57] that there exists a constant \( C > 0 \) such that the classical non-tangential maximal function
\[
f^*(\zeta) = \sup_{z \in \Gamma(\zeta)} |f(z)|, \quad \zeta \in \mathbb{T},
\]
satisfies
\[
\|f^*\|_{L^p(\mathbb{T})} \leq C \|f\|_{H^p}.
\]
for all $0 < p < \infty$ and $f \in \mathcal{H}(D)$. Therefore
\[
\|f\|_{A_p^\omega}^p \leq \|N(f)\|_{L_p^\omega}^p = \int_D (N(f)(u))^p \omega(u) \, dA(u)
= \int_0^1 \omega(r)r \int_T ((f_\alpha)^* (\zeta))^p |d\zeta| \, dr \leq C \int_0^1 \omega(r)r \int_T f(r\zeta)^p |d\zeta| \, dr = C\|f\|_{A_p^\omega}^p,
\]
and the assertion is proved. \hfill \square

3.2. Decomposition norm theorems. The main purpose of this section is to extend [29] Theorem 4 to the case of when $\omega \in \hat{D}$. Decomposition norm theorems have been obtained previously in [24, 25, 26] for several type of mixed norm spaces. For $0 < p \leq \infty$, $0 < q < \infty$, and a radial weight $\omega$, the mixed norm space $H(p,q,\omega)$ consists of those $g \in \mathcal{H}(D)$ such that
\[
\|g\|^q_{H(p,q,\omega)} = \int_0^1 M_p^q (r,g) \omega(r) \, dr < \infty.
\]
If in addition $-\infty < \beta < \infty$, we will denote $g \in H(p,\infty,\hat{\omega}^\beta)$, whenever
\[
\|g\|_{H(p,\infty,\hat{\omega}^\beta)} = \sup_{0<r<1} M_p(r,g) \hat{\omega}(r)^\beta < \infty.
\]
It is clear that $H(p,p,\omega) = A_p^\omega$. The mixed norm spaces play an essential role in the closely related question of studying the coefficient multipliers and the generalized Hilbert operator
\[
\mathcal{H}_g(f)(z) = \int_0^1 f(t)g'(tz) \, dt, \quad g \in \mathcal{H}(D),
\]
on Hardy and weighted Bergman spaces [6, 17, 29].

In order to give the precise statement of the main result of this section, we need to introduce some more notation. To do this, let $\omega$ be a radial weight such that $\int_0^1 \omega(r) \, dr = 1$. For each $\alpha > 0$ and $n \in \mathbb{N} \cup \{0\}$, let $r_n = r_n(\omega, \alpha) \in [0,1)$ be defined by
\[
\hat{\omega}(r_n) = \int_{r_n}^1 \omega(r) \, dr = \frac{1}{2n\alpha}.
\]
(3.2)
Clearly, $\{r_n\}_{n=0}^\infty$ is an increasing sequence of distinct points on $[0,1)$ such that $r_0 = 0$ and $r_n \to 1^-$, as $n \to \infty$. For $x \in [0,\infty)$, let $E(x)$ denote the integer such that $E(x) \leq x < E(x) + 1$, and set $M_n = E\left(\frac{1}{1-r_n}\right)$ for short. Write
\[
I(0) = I_{\omega,\alpha}(0) = \{k \in \mathbb{N} \cup \{0\} : k < M_1\}
\]
and
\[
I(n) = I_{\omega,\alpha}(n) = \{k \in \mathbb{N} : M_n \leq k < M_{n+1}\}
\]
for all $n \in \mathbb{N}$. If $f(z) = \sum_{n=0}^\infty a_n z^n$ is analytic in $D$, define the polynomials $\Delta_n^{\omega,\alpha} f$ by
\[
\Delta_n^{\omega,\alpha} f(z) = \sum_{k \in I_{\omega,\alpha}(n)} a_k z^k, \quad n \in \mathbb{N} \cup \{0\}.
\]

**Theorem 7.** Let $1 < p < \infty$, $0 < \alpha < \infty$ and $\omega \in \hat{D}$ such that $\int_0^1 \omega(r) \, dr = 1$, and let $f \in \mathcal{H}(D)$. 
(i) If \(0 < q < \infty\), then \(f \in H(p, q, \omega)\) if and only if
\[
\sum_{n=0}^{\infty} 2^{-n\alpha} \| \Delta_n^\omega f \|_{H_p}^q < \infty.
\]
Moreover,
\[
\|f\|_{H(p, q, \omega)} \asymp \left( \sum_{n=0}^{\infty} 2^{-n\alpha} \| \Delta_n^\omega f \|_{H_p}^q \right)^{1/q}.
\]
(ii) If \(0 < \beta < \infty\), then \(f \in H(p, \infty, \hat{\omega}^\beta)\) if and only if
\[
\sup_n 2^{-n\alpha\beta} \| \Delta_n^\omega f \|_{H_p} < \infty.
\]
Moreover,
\[
\|f\|_{H(p, \infty, \hat{\omega}^\beta)} \asymp \sup_n 2^{-n\alpha\beta} \| \Delta_n^\omega f \|_{H_p}.
\]

The proof of Theorem 7 follows that of [30, Theorem 4], and it only distinguishes from it because of the technicalities of broadening the class \(\mathcal{R} \cup \mathcal{I}\) to \(\hat{\mathcal{D}}\). Some previous results are needed. Recall that a function \(h\) is called essentially decreasing if there exists a positive constant \(C = C(h)\) such that \(h(x) \leq Ch(y)\) whenever \(y \leq x\). Essentially increasing functions are defined in an analogous manner.

**Lemma 8.** Let \(\omega \in \hat{\mathcal{D}}\) such that \(\int_{0}^{1} \omega(r) \, dr = 1\). For each \(\alpha > 0\) and \(n \in \mathbb{N} \cup \{0\}\), let \(r_n = r_n(\omega, \alpha) \in [0, 1)\) be defined by (3.2). Then the following assertions hold:

(i) For each \(\gamma > 0\), there exists \(C = C(\alpha, \gamma, \omega) > 0\) such that
\[
\eta_\gamma(r) = \sum_{n=0}^{\infty} 2^{n\gamma} r_nM_n \leq C \hat{\omega}(r)^{-\frac{\gamma}{\alpha}}, \quad 0 \leq r < 1.
\]
(ii) For each \(0 < \beta < 1\), there exists \(C = C(\alpha, \beta, \omega) > 0\) such that
\[
2^{-n\alpha\beta} \int_{0}^{1} r^{-1} M_n \omega(r) \, dr \leq C \int_{0}^{1} r^{-1} M_n \omega(r) \, dr.
\]

**Proof.** (i). We will begin with proving (3.3) for \(r = r_N\), where \(N \in \mathbb{N}\). To do this, note first that
\[
\sum_{n=0}^{N} 2^{n\gamma} r_nM_n \leq \frac{2^{\gamma}}{2^{\gamma} - 1} \hat{\omega}(r_N)^{-\frac{\gamma}{\alpha}}
\]
by (3.2). To deal with the remainder of the sum, we apply Lemma 1(ii) and (3.2) to find \(\beta = \beta(\omega) > 0\) and \(C = C(\beta, \omega) > 0\) such that
\[
\frac{1 - r_n}{1 - r_{n+j}} \geq C \left( \frac{\hat{\omega}(r_n)}{\hat{\omega}(r_{n+j})} \right)^{1/\beta} = C 2^{i\tilde{p}}, \quad n, j \in \mathbb{N} \cup \{0\}.
\]
This, the inequality \(\log \frac{1}{x} \geq 1 - x, \quad 0 < x \leq 1\), and (3.2) give
\[
\sum_{n=N+1}^{\infty} 2^{n\gamma} r_nM_n \leq 2^{N\gamma} \sum_{j=1}^{\infty} 2^{j\gamma} e^{-r x_{N+j}} \frac{1 - x_n}{1 - x_{N+j}} \leq 2^{N\gamma} \sum_{j=1}^{\infty} 2^{j\gamma} e^{-x r} 2^{i\tilde{p}}
\]
\[= C(\beta, \alpha, \gamma, \omega) \hat{\omega}(r_N)^{-\frac{\gamma}{\alpha}}.\]
Since \(\beta = \beta(\omega)\), this together with (3.5) gives (3.3) for \(r = r_N\), where \(N \in \mathbb{N}\). Now, using standard arguments, it implies (3.3) for any \(r \in (0, 1)\).
(ii). Let us write \( \tilde{\omega}(r) = \frac{\omega(r)}{\omega(r)^\beta} \). Clearly,

\[
2^{-\alpha \beta} \int_0^r r^{M_n} \tilde{\omega}(r) \, dr \leq \frac{2^{-\alpha \beta}}{\omega(r)^\beta} \int_0^r r^{M_n} \omega(r) \, dr \leq \int_0^1 r^{M_n} \omega(r) \, dr. \tag{3.6}
\]

Moreover, \([30] \text{ Lemma 1.4 (iii)}\) yields

\[
2^{-\alpha \beta} \int_{r_n}^1 r^{M_n} \tilde{\omega}(r) \, dr \leq 2^{-\alpha \beta} \omega(r_n) \psi(r_n) = \frac{2^{-\alpha \beta}}{1 - \beta} \tilde{\omega}(r_n) \psi(r_n)
\]

\[= \frac{1}{1 - \beta} \int_{r_n}^1 \omega(r) \, dr \leq C(\beta, \alpha, \omega) \int_{r_n}^1 r^{M_n} \omega(r) \, dr. \tag{3.7}
\]

By combining (3.6) and (3.7) we obtain (ii). \( \square \)

We now present a result on power series with positive coefficients. This result will play a crucial role in the proof of Theorem 7.

**Proposition 9.** Let \( 0 < p, \alpha < \infty \) and \( \omega \in \mathcal{D} \) such that \( \int_0^1 \omega(r) \, dr = 1 \). Let

\[ f(r) = \sum_{k=0}^\infty a_k r^k, \]

where \( a_k \geq 0 \) for all \( k \in \mathbb{N} \cup \{ 0 \} \), and denote \( t_n = \sum_{k \in \mathcal{I}_{\omega,\alpha}} a_k \). Then there exists a constant \( C = C(p, \alpha, \omega) > 0 \) such that

\[
\frac{1}{C} \sum_{n=0}^\infty 2^{-\alpha \beta} t_n^p \leq \int_0^1 f(r)^p \omega(r) \, dr \leq C \sum_{n=0}^\infty 2^{-\alpha \beta} t_n^p, \tag{3.8}
\]

**Proof.** We will use ideas from the proof of [23] Theorem 6]. The definition (3.2) yields

\[
\int_0^1 f(r)^p \omega(r) \, dr \geq \sum_{n=0}^\infty \int_{r_n}^{r_{n+2}} \left( \sum_{k=0}^n t_k r^{M_k+1} \right)^p \omega(r) \, dr
\]

\[\geq \sum_{n=0}^\infty \left( \sum_{k=0}^n t_k r^{M_k+1} \right)^p \int_{r_n}^{r_{n+2}} \omega(r) \, dr
\]

\[\geq \left( 1 - \frac{1}{2^\alpha} \right) \sum_{n=0}^\infty t_n^p r_n^{M_{n+1}} 2^{(-n-1)\alpha} \geq C \sum_{n=0}^\infty t_n^p 2^{-\alpha \beta},
\]

where \( C = C(p, \alpha, \omega) > 0 \) is a constant. This gives the first inequality in (3.8).

To prove the second inequality in (3.8), let first \( p > 1 \) and take \( 0 < \gamma < \frac{\alpha}{p-1} \). Then Hölder’s inequality gives

\[ f(r)^p \leq \left( \sum_{n=0}^\infty t_n r^{M_n} \right)^p \leq \eta_\gamma(r)^{p-1} \sum_{n=0}^\infty 2^{-\alpha \gamma (p-1)} t_n^p r^{M_n}. \]

Therefore, by (3.3) and (3.4) in Lemma 8 and Lemma 11 (vi) there exist constants \( C_1 = C_1(\alpha, \gamma, p, \omega) > 0 \), \( C_2 = C_2(\alpha, \gamma, p, \omega) > 0 \) and \( C_3 = C_3(\alpha, \gamma, p, \omega) > 0 \) such
that
\[ \int_0^1 f(r)^p \omega(r) \, dr \leq \sum_{n=0}^{\infty} 2^{-n\gamma(p-1)} t_n^p \int_0^1 r^{M_n \gamma}(r)^{p-1} \omega(r) \, dr \]
\[ \leq C_1 \sum_{n=0}^{\infty} 2^{-n\gamma(p-1)} t_n^p \left( \frac{1}{\omega(r)} \right)^{\frac{1}{\gamma(p-1)}} \int_0^1 r^{M_n} \omega(r) \, dr \]
\[ \leq C_2 \sum_{n=0}^{\infty} t_n^p \int_0^1 r^{M_n} \omega(r) \, dr \]
\[ \leq C_3 \sum_{n=0}^{\infty} t_n^p \hat{\omega}(r_n) = C_3 \sum_{n=0}^{\infty} t_n^p 2^{-n\alpha}. \]

Since \( \gamma = \gamma(\alpha, p) \), this gives the assertion for \( 1 < p < \infty \).

If \( 0 < p \leq 1 \), then
\[ f(r)^p \leq \left( \sum_{n=0}^{\infty} t_n r^{M_n} \right)^p \leq \sum_{n=0}^{\infty} t_n^p r^{M_n p}, \]
so using Lemma \[ \text{(vi)} \] and (ii), there exists a constant \( C_1 = C_1(\alpha, \gamma, p, \omega) > 0 \) such that
\[ \int_0^1 f(r)^p \omega(r) \, dr \leq \sum_{n=0}^{\infty} t_n^p \int_0^1 r^{pM_n} \omega(r) \, dr \]
\[ \leq C_1 \sum_{n=0}^{\infty} t_n^p \hat{\omega}(r_n) = C_1 \sum_{n=0}^{\infty} t_n^p 2^{-n\alpha}. \]

This finishes the proof. \( \square \)

Next, for \( g(z) = \sum_{k=0}^{\infty} b_k z^k \in \mathcal{H}(\mathbb{D}) \) and \( n_1, n_2 \in \mathbb{N} \cup \{0\} \), we set
\[ S_{n_1, n_2} g(z) = \sum_{k=n_1}^{n_2-1} b_k z^k, \quad n_1 < n_2. \]

The chain of inequalities
\[ r^{n_2} \| S_{n_1, n_2} g \|_{H^p} \leq M_p(r, S_{n_1, n_2} g) \leq r^{n_1} \| S_{n_1, n_2} g \|_{H^p}, \quad 0 < r < 1, \]
follows from \[ [24] \text{Lemma 3.1}. \]

**Lemma 10.** Let \( 0 < p \leq \infty \) and \( n_1, n_2 \in \mathbb{N} \) with \( n_1 < n_2 \). If \( g(z) = \sum_{k=0}^{\infty} c_k z^k \in \mathcal{H}(\mathbb{D}) \), then
\[ \| S_{n_1, n_2} g \|_{H^p} \asymp M_p \left( 1 - \frac{1}{n_2}, S_{n_1, n_2} g \right). \]

**Proof of Theorem 7** (i). By the M. Riesz projection theorem and (3.9),
\[ \| f \|_{H^p(q, \omega, \cdot)} \asymp \sum_{n=0}^{\infty} \| \Delta_n^{\omega, \alpha} f \|_{H^p}^q \int_{r_{n+1}}^{r_{n+2}} r^{qM_{n+1}} \omega(r) \, dr \]
\[ \asymp \sum_{n=0}^{\infty} \| \Delta_n^{\omega, \alpha} f \|_{H^p}^q \int_{r_{n+1}}^{r_{n+2}} \omega(r) \, dr \asymp \sum_{n=0}^{\infty} 2^{-n\alpha} \| \Delta_n^{\omega, \alpha} f \|_{H^p}^q. \]
On the other hand, Minkowski’s inequality and (3.9) give
\[ M_p(r, f) \leq \sum_{n=0}^{\infty} M_p(r, \Delta_n^{\omega, \alpha} f) \leq \sum_{n=0}^{\infty} r^{M_n} \| \Delta_n^{\omega, \alpha} f \|_{H^p}, \] (3.10)
and hence Proposition 9 yields
\[ \| f \|_{H(p,q,\omega)} \leq \int_0^1 \left( \sum_{n=0}^{\infty} r^{M_n} \| \Delta_n^{\omega, \alpha} f \|_{H^p} \right)^q \omega(r) \, dr \simeq \sum_{n=0}^{\infty} 2^{-n\alpha} \| \Delta_n^{\omega, \alpha} f \|_{H^p}^q. \]

(ii). Using again the M. Riesz projection theorem and (3.9) we deduce
\[ \sup_{0<r<1} M_p(r, f) \Delta(r)^\beta \gtrsim r^{M_{n+1}} \| \Delta_n^{\omega, \alpha} f \|_{H^p} 2^{-n\alpha \beta}, \quad n \in \mathbb{N} \cup \{0\}, \]
and hence
\[ \| f \|_{H(p,\infty,\omega)} \gtrsim \sup_n 2^{-n\alpha \beta} \| \Delta_n^{\omega, \alpha} f \|_{H^p}. \]
Conversely, assume that \( M = \sup_n 2^{-n\alpha \beta} \| \Delta_n^{\omega, \alpha} f \|_{H^p} < \infty \). Then (3.10) and Lemma 8(i) yield
\[ M_p(r, f) \leq \sum_{n=0}^{\infty} r^{M_n} \| \Delta_n^{\omega, \alpha} f \|_{H^p} \leq M \sum_{n=0}^{\infty} 2^{n\alpha \beta} r^{M_n} \lesssim M \Delta(r)^{-\beta}. \]
This finishes the proof.
\[ \square \]
It is worth mentioning that Theorem 7 does not remain valid for \( 0 < p \leq 1 \). But the part that is true in this case is contained in the next result.

**Proposition 11.** Let \( 0 < p \leq 1, 0 < \alpha < \infty \) and \( \omega \in \hat{D} \) such that \( \int_0^1 \omega(r) \, dr = 1 \).

(i) If \( 0 < q < \infty \), then
\[ \| f \|_{H(p,q,\omega)} \lesssim \left( \sum_{n=0}^{\infty} 2^{-n\alpha} \| \Delta_n^{\omega, \alpha} f \|_{H^p}^q \right)^{1/q}, \quad f \in \mathcal{H}(\mathbb{D}). \]

(ii) If \( 0 < \beta < \infty \), then
\[ \| f \|_{H(p,\infty,\omega)} \lesssim \sup_n 2^{-n\alpha \beta} \| \Delta_n^{\omega, \alpha} f \|_{H^p}, \quad f \in \mathcal{H}(\mathbb{D}). \]

Proposition 11 follows from the inequality
\[ M_p^p(r, f) \leq \sum_{n=0}^{\infty} M_p^p(r, \Delta_n^{\omega} f) \leq \sum_{n=0}^{\infty} r^{pM_n} \| \Delta_n^{\omega} f \|_{H^p}, \] (3.9) and Proposition 9. See also [32, Lemma 8].

4. **Bergman Projection**

4.1. **One weight inequality.** The boundedness of projections on \( L^p \)-spaces is an intriguing topic which has attracted a lot attention in recent years [5, 11, 12, 13, 19, 32, 35, 37]. In fact, as far as we know, to characterize those radial weights for which \( P_\omega : L^p_\omega \to L^p_\omega \) is bounded, is still an open problem [12, p. 116].

For the class of standard weights, the Bergman projection \( P_\alpha \) (as well as \( P_\alpha^+ \)) is bounded on \( L^p_\omega \) if and only if \( 1 < p < \infty \) [37, Theorem 4.24]. As for \( p = \infty \), \( P_\alpha \) is bounded and onto from \( L^\infty \) to \( \mathcal{B} \). Here \( \mathcal{B} \) [37, Chapter 5] denotes the Bloch space that consists of \( f \in \mathcal{H}(\mathbb{D}) \) such that
\[ \| f \|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2) + |f(0)| < \infty. \]
These results have been recently extended to the class of regular weights [32].

**Theorem 12.** Let \(1 < p < \infty\).

(i) If \(\omega \in \mathcal{R}\), then \(P_\omega^+ : L_p^\omega \to L_p^\omega\) is bounded. In particular, \(P_\omega : L_p^\omega \to A_p^\omega\) is bounded.

(ii) If \(\omega \in \mathcal{R}\), then \(P_\omega : L^\infty(\mathbb{D}) \to \mathcal{B}\) is bounded.

In the original source [32], Theorem 12 (i) is obtained as a consequence of Theorem 17 below. Here, we shall offer a simple proof of this result. Both arguments use strongly Lemma 1, in particular the description of the class \(\hat{D}\) in terms of the moments of the weights

\[
\int_0^1 s^x \omega(s) ds \asymp \hat{\omega}\left(1 - \frac{1}{x}\right), \quad x \in [1, \infty).
\]

Now, we offer a simple proof of the one weight inequality for regular weights.

**Proof of Theorem 12 (i).** Let \(1 < p < \infty\) and \(\omega \in \mathcal{R}\). Let \(h = \hat{\omega}^{-\frac{1}{p'}}\), where \(\frac{1}{p} + \frac{1}{p'} = 1\). Since \(p > 1\), [30, Lemma 1.4(iii)] shows that \(h\psi\omega\) is a weight with \(\psi_{h\psi\omega} = \frac{r}{p-1}\psi_\omega\), and thus \(h\psi\omega \in \mathcal{R}\). Since \(\omega \in \hat{D}\), by Lemma 1(ii) there exists \(\beta = \beta(\omega)\) such that \(\hat{\omega}(1-s)^{-\beta}\) is essentially increasing on \([0,1]\). On the other hand, since \(\omega \in \mathcal{R}\) there is \(\alpha = \alpha(\omega) > 0\) with \(\alpha \leq \beta\) such that \(\hat{\omega}(1-s)^{-\alpha}\) is essentially decreasing, see [30 (ii) p. 10]. By using this and \(h\psi\omega \in \mathcal{R}\) we deduce

\[
\int_0^r \frac{h\psi\omega(s)}{\hat{\omega}(s)(1-s)} ds \asymp \int_0^r \frac{ds}{\hat{\omega}(s)^{\frac{1}{p'}}(1-s)} \asymp \frac{1}{\hat{\omega}(r)^{\frac{1}{p'}}} = h\psi\omega(r), \quad r \geq \frac{1}{2}.
\]

By symmetry, a similar reasoning applies when \(p\) is replaced by \(p\), and hence we may use Theorem 13(ii) and [41,1] to deduce

\[
\int_{\mathbb{D}} |B^\omega(z,\zeta)|h\psi\omega(\zeta)\omega(\zeta) dA(\zeta) \asymp h\psi\omega(z), \quad z \in \mathbb{D},
\]

and

\[
\int_{\mathbb{D}} |B^\omega(z,\zeta)|h\psi\omega(z)\omega(z) dA(z) \asymp h\psi\omega(\zeta), \quad \zeta \in \mathbb{D}.
\]

It follows from Schur’s test [37, Theorem 3.6] that \(P_\omega^+ : L_p^\omega \to L_p^\omega\) is bounded. □

The situation is different for \(\omega \in \mathcal{I}\) because then \(P_\omega^+\) is not bounded on \(L_p^\omega\) [32]. This result points out that many finer function-theoretic properties of \(A_p^\alpha\) do not carry over to \(A_p^\alpha\) induced by \(\omega \in \mathcal{I}\).

Concerning rapidly decreasing weights, Dostanic [12] proved that the Bergman projection is bounded on \(L_p^\omega\) only for \(p = 2\) in the case of Bergman spaces with
the exponential type weights $w(r) = (1 - r^2)^A \exp\left(\frac{-B}{1 - r^2}\right)$, $A \in \mathbb{R}$, $B, \alpha > 0$.

The next result proves that it is a general phenomenon which holds for rapidly decreasing and smooth weights [11, 35].

**Proposition 14.** Assume that $\omega(r) = e^{-\frac{\alpha}{1 - r^2}}$ is a radial weight such that $\varphi$ is a positive $C^\infty$-function, $\varphi'$ is positive on $[0, 1)$, $\lim_{r \to 1^-} \varphi(r) = \lim_{r \to 1^-} \varphi'(r) = +\infty$ and

$$\lim_{r \to 1^-} \frac{\varphi^{(n)}(r)}{(\varphi'(r))^n} = 0, \quad \text{for any } n \in \mathbb{N} \setminus \{1\}.$$ (4.2)

Then, the Bergman projection is bounded from $L^p_\omega$ to $L^p_\omega$ only for $p = 2$.

Consequently, if $\omega$ a rapidly decreasing weight, it is natural to look for a substitute for the boundedness of the Bergman projection $P_\omega$. Inspired by the Fock space setting, the following result has been proved for a canonical example [11].

**Theorem 15.** Let $\omega(r) = \exp\left(-\frac{\alpha}{1 - r^2}\right)$, $\alpha > 0$, and $1 \leq p < \infty$. Then, the Bergman projection $P_\omega$ is bounded from $L^p_{\alpha/2}$ to $A^p_{\alpha/2}$.

The approach to prove this result relies on an instance of Schur’s test and accurate estimates for the integral means of order one of the corresponding Bergman reproducing kernel [11, Proposition 5].

**Proposition 16.** Let $\omega(r) = \exp\left(-\frac{\alpha}{1 - r^2}\right)$, $\alpha > 0$, and let $K(z) = \sum_{n=0}^\infty \frac{z^n}{n! \omega(2n+1)}$. Then, there is a positive constant $C$ such that

$$M_1(r, K) \asymp \exp\left(-\frac{\alpha}{1 - r^2}\right), \quad r \to 1^-.$$

These estimates are obtained by using two key tools: the sharp asymptotic estimates obtained in [21] for the moments of the weight in terms of the Legendre-Fenchel transform, and an upper estimate of $M_1(r, K)$ by the $L^1$-norm of the $H^1$-norms of the Hadamard product of $K$, with certain smooth polynomials.

Finally, we mention that a generalization of Theorem 14 for a class of not necessarily radial weights has been achieved in [11, Theorem 4.1]. Their approach is different from that of [11], it uses ideas from [22] and Hörmander-Berndtsson $L^2$-estimates for solutions of the $\bar{\partial}$-equation [10, 20].

We refer to [13, 36] for other results concerning the particular case $\omega = v$ in (1.1).

4.2. Two weight inequality. As it has been commented before, the weights $v$ satisfying (1.1) when $\omega$ is an standard weight and $1 < p < \infty$, were characterized by Bekollé and Bonami [7, 8]. Recently [33], it has been proved the following quantitative version of this result

$$\|P_{\alpha}^+(f)\|_{L^p_{v(1-|a|^2)^\alpha}} \leq C(p, \alpha)B_{p, \alpha}(v) \|f\|_{L^p_{v(1-|a|^2)^\alpha}},$$

where $B_{p, \alpha}(v)$ was defined in (1.2). With regard to the case $p = 1$, we define the weighted maximal function

$$M_\alpha(v)(z) = \sup_{z \in S(a)} \frac{v(S(a))}{A_\alpha(S(a))}, \quad z \in \mathbb{D}.$$
It is known \[8,9\] that the weak \((1, 1)\) inequality holds (and its analogue replacing \(P_\alpha\) by \(P_\alpha^+\))

\[
\omega \left\{ z \in \mathbb{D} : |P_\alpha(f)(z)| > \lambda \right\} \leq C_{\alpha, \omega} \frac{||f||_{L^p}}{\lambda}
\]

if and only if the weighted maximal function satisfies

\[
M_\alpha(\omega)(z) \leq C \frac{\omega(z)}{(1 - |z|)^\alpha}, \quad z \in \mathbb{D}.
\]

As far as we know, apart from Bekollé-Bonami’s results \[7, 8\] on the standard Bergman projection \(P_\alpha\) very little is known about \((1.1)\) when \(\omega \neq v\). We note that \[5, \text{Theorem } 4.1\] or \[11, \text{Theorem } 1\] may be seen as positive examples for \((1.1)\) in the context of rapidly increasing weights. A recent result \[32\] describes those regular weights \(\omega\) and \(v\) for which \((1.1)\) holds for \(1 < p < \infty\).

**Theorem 17.** Let \(1 < p < \infty\) and \(\omega, v \in \mathcal{R}\). Then the following conditions are equivalent:

(a) \(P_\omega^+ : L^p_v \to L^p_v\) is bounded;
(b) \(P_\omega : L^p_v \to L^p_v\) is bounded;
(c) \(\sup_{0 < r < 1} \frac{\omega(r)^{p/2} \left( \int_r^1 \left( \frac{\omega(s)}{v(s)} \right)^{p'} v(s) ds \right)^{1/p'}}{\hat{v}(r)^{p/2}} < \infty\);
(d) \(\sup_{0 < r < 1} \frac{\omega(r)^p (1 - r)^{p-1} \int_r^1 \frac{v(s)}{\omega(s)^p (1 - s)^p} ds}{\hat{v}(r)} < \infty\);
(e) \(\sup_{0 < r < 1} \left( \int_0^r \frac{v(s)}{\omega(s)^p (1 - s)^p} ds \right)^{1/p'} \left( \int_r^1 \left( \frac{\omega(s)}{v(s)} \right)^{p'} v(s) ds \right)^{1/p} < \infty\);
(f) \(\sup_{0 < r < 1} \frac{\omega(r)^{1/2} \int_r^1 \frac{\omega(s)}{(1 - s)^{1/2} v(s)^{1/2}} ds}{\hat{v}(r)} < \infty\);
(g) \(\sup_{0 < r < 1} \frac{\omega(r)^{1/2} \hat{v}(r)}{v(r)^{1/2}} \int_0^r \frac{v(s)^{1/2}}{\omega(s)(1 - s)^{1/2}} ds < \infty\).

It is worth noticing that condition (f) above makes sense also for \(p = 1\), and it turns out to be the condition that describes those regular weights such that \(P_\omega\) is bounded on \(L^1_v\) \[32\].

**Theorem 18.** Let \(\omega, v \in \mathcal{R}\). Then the following conditions are equivalent:

(a) \(P_\omega : L^1_v \to L^1_v\) is bounded;
(b) \(P_\omega^+ : L^1_v \to L^1_v\) is bounded;
(c) \(\sup_{0 < r < 1} \frac{\omega(r)}{v(r)} \int_0^r \frac{\hat{v}(s)}{\omega(s)(1 - s)} ds < \infty\);
(d) \(\sup_{0 < r < 1} \frac{\omega(s)}{\hat{v}(s)} \int_r^1 \frac{\omega(s)}{v(s)(1 - s)} ds < \infty\).

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