Random sampling in weighted reproducing kernel subspaces of $L^p_\nu(\mathbb{R}^d)$

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Abstract: In this paper, we mainly study the random sampling and reconstruction for signals in a reproducing kernel subspace of $L^p_\nu(\mathbb{R}^d)$ without the additional requirement that the kernel function has polynomial decay. The sampling set is independently and randomly drawn from a general probability distribution over $\mathbb{R}^d$, which improves and generalizes the common assumption of uniform distribution on a cube. Based on a frame characterization of reproducing kernel subspaces, we first approximate the reproducing kernel space by a finite dimensional subspace on any bounded domains. Then, we prove that the random sampling stability holds with high probability for all signals in reproducing kernel subspaces whose energy concentrate on a cube when the sampling size is large enough. Finally, a reconstruction algorithm based on the random samples is given for functions in the corresponding finite dimensional subspaces.

Keywords: random sampling; weighted reproducing kernel subspace; sampling stability; probability density function; reconstruction algorithm

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1 Introduction

Random sampling plays an important role in many fields, such as image processing [6], compressed sensing [10] and learning theory [20]. Random sampling has been generally studied for multivariate trigonometric polynomials [2], bandlimited signals [3, 4], signals that satisfy some locality properties in short-time Fourier transform [22], signals with bounded derivatives [25], signals in a shift-invariant space [11, 16, 24, 26] and signals with finite rate of innovation [17]. Recently, ideal random sampling in reproducing kernel subspaces of $L^p(\mathbb{R}^d)$ was given in [19] with random samples taken from a uniform distribution on a bounded domain $[-K, K]^d$. Moreover, [19] used a very strong condition

$$|K(x, y)| \leq \frac{C}{(1 + \|x - y\|_1)\alpha}$$ (1.1)

on the kernel function $K$. Reproducing kernel subspaces have been generally studied in recent years [8, 14, 15, 18, 23] and have potential applications in machine learning [7, 27]. In this paper,
we mainly study the random sampling and reconstruction of signals in a weighted reproducing kernel subspace of $L^p_\nu(\mathbb{R}^d)$ without the condition (1.1), and the random samples are drawn over $\mathbb{R}^d$ from a general probability distribution.

Suppose that $\omega$ is a weight function which is continuous, symmetric, positive and sub-multiplicative,

$$0 < \omega(x + y) \leq \omega(x)\omega(y), \ x, y \in \mathbb{R}^d.$$  \hfill (1.2)

Weight function $\nu$ is said to be $\omega$-moderate, that is, it is continuous, symmetric, positive and satisfies

$$0 < \nu(x + y) \leq C_0 \omega(x)\nu(y), \ x, y \in \mathbb{R}^d$$  \hfill (1.3)

for some positive constant $C_0 > 0$. More details about weight functions can refer to [12].

For $1 \leq p \leq \infty$, $L^p_\nu(\mathbb{R}^d)$ is the Banach space of all weighted $p$-integrable function on $\mathbb{R}^d$,

$$L^p_\nu(\mathbb{R}^d) = \{f : \|f\|_{L^p_\nu} = \|\nu f\|_{L^p} < \infty\}.$$  \hfill (1.4)

Let $K$ be a function defined on $\mathbb{R}^d \times \mathbb{R}^d$ which satisfies

$$\|K\|_W = \sup_{z \in \mathbb{R}^d} |K(\cdot + z, z)|_{L^1_\omega} = \sup_{z \in \mathbb{R}^d} |K(z, \cdot + z)|_{L^1_\omega} < \infty$$  \hfill (1.5)

and

$$\lim_{\delta \to 0} \|\omega_\delta(K)\|_W = 0.$$  \hfill (1.6)

Here, $\omega_\delta(K)$ is the modulus of continuity defined by

$$\omega_\delta(K)(x, y) = \sup_{|x'|, |y'| \leq \delta} |K(x + x', y + y') - K(x, y)|.$$  \hfill (1.7)

Suppose that $T$ is an idempotent ($T^2 = T$) integral operator with kernel $K$,

$$Tf(x) = \int_{\mathbb{R}^d} K(x, y)f(y)dy, \ f \in L^p_\nu(\mathbb{R}^d).$$  \hfill (1.8)

Then its range space

$$V_{K,p} = \{Tf : f \in L^p_\nu(\mathbb{R}^d)\} = \{f \in L^p_\nu(\mathbb{R}^d) : Tf = f\}$$  \hfill (1.9)

is a weighted reproducing kernel subspace of $L^p_\nu(\mathbb{R}^d)$ [15] [23], which means that for any $x \in \mathbb{R}^d$, there exists a $C_x > 0$ such that

$$|f(x)| \leq C_x \|f\|_{L^p_\nu}, \ f \in V_{K,p}.$$  \hfill (1.10)

Let $0 < \delta < 1$ and $C_R = [-R, R]^d$ for $R > 0$. Define a compact subset of $V_{K,p}$ by

$$V_{K,p}(R, \delta) = \left\{f \in V_{K,p} : \int_{C_R} |f(x)\nu(x)|^pdx \geq (1 - \delta) \int_{\mathbb{R}^d} |f(x)\nu(x)|^pdx \right\},$$  \hfill (1.11)
which contains all functions in $V_{K,p}$ whose energy concentrate on the cube $C_R$.

This paper is organized as follows. In section 2, we show that a function $f \in V_{K,p}$ can be approximated by a function $f_N$ in a finite dimensional subspace $V_{K,p}^N$ on any bounded domains. In section 3, we give an estimate for the covering number of normalized $V_{K,p}^N$. In section 4, we prove that the sampling inequality holds with high probability for all functions in $V_{K,p}(R,\delta)$. In section 5, a reconstruction algorithm based on random samples is provided for functions in $V_{K,p}^N$.

# 2 Approximation to $V_{K,p}$

In this section, we will show that $V_{K,p}$ can be approximated by a finite dimensional subspace on any bounded domains. The following definitions of frame is similar to [1, 13, 21].

Definition 2.1 Let $V$ be a Banach subspace of $L^p_\nu(\mathbb{R}^d)$. A family $\Psi = \{\psi_\gamma\}_{\gamma \in \Gamma}$ of functions in $L^{p'}_{1/\nu}(\mathbb{R}^d)$ is a $p$-frame for $V$, if there exist positive constants $A_p$ and $B_p$ such that

$$A_p\|f\|_{L^p_\nu} \leq \|\{\langle f, \psi_\gamma \rangle\}_{\gamma \in \Gamma}\|_{\ell^p_\nu} \leq B_p\|f\|_{L^p_\nu}, \forall f \in V.$$ 

Definition 2.2 Let $V \subset L^p_\nu(\mathbb{R}^d)$ and $W \subset L^{p'}_{1/\nu}(\mathbb{R}^d)$. The $p$-frame $\tilde{\Phi} = \{\tilde{\phi}_\lambda\}_{\lambda \in \Lambda} \subset W$ for $V$ and the $p'$-frame $\Phi = \{\phi_\lambda\}_{\lambda \in \Lambda} \subset V$ for $W$ form a dual pair if the following reconstruction formulae hold:

$$f = \sum_{\lambda \in \Lambda} \langle f, \tilde{\phi}_\lambda \rangle \phi_\lambda \text{ for all } f \in V \quad (2.1)$$

and

$$g = \sum_{\lambda \in \Lambda} \langle g, \phi_\lambda \rangle \tilde{\phi}_\lambda \text{ for all } g \in W. \quad (2.2)$$

The following lemma is a generalization of Theorem A.2 in [13] to the weighted case.

Lemma 2.3 [13] Let $1 \leq p \leq \infty$, $T$ be an idempotent integral operator on $L^p_\nu(\mathbb{R}^d)$ whose kernel $K$ satisfies (1.5) and (1.6), and let $V_{K,p}$ be the range space of $T$. Then there exists a relatively-separated subset $\Lambda = \delta_0 \mathbb{Z}^d$ which is determined by the condition $C_0\|K\|_W \|\omega_{\delta_0}(K)\|_W < 1$, and two families $\Phi = \{\phi_\lambda\}_{\lambda \in \Lambda}$ in $V_{K,p}$ and $\tilde{\Phi} = \{\tilde{\phi}_\lambda\}_{\lambda \in \Lambda}$ in $V_{K,p}^*$ which are defined by

$$\phi_\lambda(x) = \delta_0^{-d/p} \int_{\mathbb{R}^d} \int_{[-\delta_0/2,\delta_0/2]^d} K_{\delta_0}(x,z_1) K(z_1, \lambda + z_2) dz_2 dz_1 \quad (2.3)$$

and

$$\tilde{\phi}_\lambda(x) = \delta_0^{-d+d/p} \int_{[-\delta_0/2,\delta_0/2]^d} K(\lambda + z, x) dz \quad (2.4)$$

such that
(i) Both $\Phi$ and $\tilde{\Phi}$ are localized in the sense that
\[
|\phi_\lambda(x)| + |\tilde{\phi}_\lambda(x)| \leq h(x - \lambda), \quad (2.5)
\]
where $h \in L^1_\nu(\mathbb{R}^d)$.

(ii) $\Phi$ and $\tilde{\Phi}$ form a dual frame pair for $V_{K,p}$ and $V_{K,p}^*$.

(iii) Both $V_{K,p}$ and $V_{K,p}^*$ are generated by $\Phi$ and $\tilde{\Phi}$ in the sense that
\[
V_{K,p} = \left\{ \sum_{\lambda \in \Lambda} c(\lambda)\phi_\lambda : (c(\lambda))_{\lambda \in \Lambda} \in \ell^p(\Lambda) \right\} \quad (2.6)
\]
and
\[
V_{K,p}^* = \left\{ \sum_{\lambda \in \Lambda} \tilde{c}(\lambda)\tilde{\phi}_\lambda : (\tilde{c}(\lambda))_{\lambda \in \Lambda} \in \ell^{p/(p-1)}(\Lambda) \right\}. \quad (2.7)
\]

(iv) $||K_\delta||_W < \infty$ and $\lim_{\delta \to 0} ||\omega_\delta(K_\delta)||_W = 0$.

Based on Lemma 2.3 for a given positive integer $N$, define a finite dimensional subspace
\[
V_{K,p}^N = \left\{ \sum_{\lambda \in \Lambda \cap [-N,N]^d} c(\lambda)\phi_\lambda : c(\lambda) \in \mathbb{R} \right\} \quad (2.8)
\]
of $V_{K,p}$ and its normalization
\[
V_{K,p}^{N,*} = \left\{ f \in V_{K,p}^N : ||f||_{L^p_\nu(\mathbb{R}^d)} = 1 \right\}. \quad (2.9)
\]

In the following, we will show that $V_{K,p}$ can be approximated by $V_{K,p}^N$ on any bounded domains $C_M = [-M, M]^d$ for $M > 0$.

**Lemma 2.4** Let $1 \leq p \leq \infty$ and $p'$ be the conjugate number of $p$. Suppose that $K$ satisfies the assumptions (1.5) and (1.6). If $f \in V_{K,p}$ and $||f||_{L^p_\nu(\mathbb{R}^d)} = 1$, then for given $\varepsilon > 0$, there exist $N = N(\varepsilon, M, f)$ and $f_N \in V_{K,p}^N$ such that
\[
||f - f_N||_{L^p_\nu(\mathbb{R}^d)} \leq \varepsilon \quad \text{and} \quad ||f - f_N||_{L^{p'}_\nu(\mathbb{R}^d)} \leq \frac{\varepsilon}{(2M)^d}. \quad (2.10)
\]

**Proof** For $f = \sum_{\lambda \in \Lambda} (f, \phi_\lambda)\phi_\lambda \in V_{K,p}$, choose
\[
f_N = \sum_{\lambda \in \Lambda \cap [-N,N]^d} (f, \phi_\lambda)\phi_\lambda \in V_{K,p}^N.
\]
Since $||K||_W < \infty$ and $||K_\delta||_W < \infty$, then
\[
\sum_{\lambda \in \Lambda} |\phi_\lambda(x)||\omega(x - \lambda) \leq \delta_0^{d/p} \int_{\mathbb{R}^d} |K_{\delta_0}(x, z)| |\omega(x - z)| \sum_{k \in \mathbb{Z}^d} \int_{\delta_0k + [-\frac{d}{2}, \frac{d}{2}]^d} |K(z_1, z_2)| |\omega(z_1 - \delta_0 k)|dz_2dz_1
\leq \delta_0^{d/p} \left( \max_{x \in [-\frac{d}{2}, \frac{d}{2}]^d} \omega(x) \right) ||K_\delta||_W ||K||_W. \quad (2.11)
\]
Note that \( \Lambda = \delta_0 \mathbb{Z}^d \). For \( k = (k_1, k_2, \cdots, k_d) \in \mathbb{Z}^d \), let \( |k| = \max\{|k_1|, |k_2|, \cdots, |k_d|\} \). Then it follows from (1.2), (1.3) and (2.11) that

\[
\begin{align*}
|f(x) - f_N(x)|\nu(x) &= \left| \sum_{\lambda \in \Lambda \cap \{\mathbb{R}^d \mid [-N,N]^d\}} (f, \tilde{\phi}_\lambda) \phi_\lambda(x) \nu(x) \right| \\
&\leq C_0 \sum_{\lambda \in \Lambda \cap \{\mathbb{R}^d \mid [-N,N]^d\}} |(f, \tilde{\phi}_\lambda)| \nu(\lambda) \cdot |\phi_\lambda(x)| \omega(x - \lambda) \\
&\leq C_0 \left( \sum_{\lambda \in \Lambda \cap \{\mathbb{R}^d \mid [-N,N]^d\}} |(f, \tilde{\phi}_\lambda)| \nu(\lambda) \right)^{1/p'} \left( \sum_{\lambda \in \Lambda} |\phi_\lambda(x)| \omega(x - \lambda) \right)^{1/p} \\
&\leq C_0 \left( \sum_{\lambda \in \Lambda \cap \{\mathbb{R}^d \mid [-N,N]^d\}} |(f, \tilde{\phi}_\lambda)| \nu(\lambda) \right)^{1/p'} \left( \sum_{\lambda \in \Lambda} |\phi_\lambda(x)| \omega(x - \lambda) \right) \\
&\leq C_0 \delta_k^{d-2p} \max_{x \in [-\frac{\delta_k}{2}, \frac{\delta_k}{2}]^d} \omega(x) \|K_{\delta_k}\|_W \|K\|_W \left\{ (f, \tilde{\phi}_\lambda) \right\}_{\lambda = \delta_0 k \in \delta_0 \mathbb{Z}^d, |k| > N} \|\ell_{\nu, p}' \|.
\end{align*}
\] (2.12)

Since \( \tilde{\Phi} = \{\tilde{\phi}_\lambda\}_{\lambda \in \Lambda} \) is a \( p \)-frame of \( V_{K,p} \), by Definition 2.1, one has

\[
\left\| \{ (f, \tilde{\phi}_\lambda) \}_{\lambda \in \Lambda} \right\|_\ell_{\nu, p}' \leq B_p \| f \|_{L_p^\nu} = B_p,
\] (2.13)

which means that \( \lim_{N \to \infty} \left\| \{ (f, \tilde{\phi}_\lambda) \}_{\lambda = \delta_0 k \in \delta_0 \mathbb{Z}^d, |k| > N} \right\|_\ell_{\nu, p}' = 0 \) and the desired result (2.10) follows.

**Lemma 2.5** Suppose that \( K \) satisfies the assumptions (1.5) and (1.6), then there exists a \( C_K > 0 \) such that

\[
\left\| \sum_{\lambda \in \Lambda} c(\lambda) \phi_\lambda \right\|_{L_p^\nu(\mathbb{R}^d)} \leq C_K \left\| (c(\lambda))_{\lambda \in \Lambda} \right\|_{\ell_r(\Lambda)}.\] (2.14)

**Proof** It follow from (2.5) and (2.11) that

\[
\begin{align*}
&\left\| \sum_{\lambda \in \Lambda} c(\lambda) \phi_\lambda \right\|_{L_p^\nu(\mathbb{R}^d)}^p \\
&= \int_{\mathbb{R}^d} \left| \sum_{\lambda \in \Lambda} c(\lambda) \phi_\lambda \right|^p \nu(x) \, dx \\
&\leq C_0^p \int_{\mathbb{R}^d} \left( \sum_{\lambda \in \Lambda} |c(\lambda)| \nu(\lambda) \cdot |\phi_\lambda(x)| \omega(x - \lambda) \right)^p \, dx \\
&\leq C_0^p \left( \sum_{\lambda \in \Lambda} |c(\lambda)| \nu(\lambda) \cdot |\phi_\lambda(x)| \omega(x - \lambda) \right)^p \int_{\mathbb{R}^d} |\phi_\lambda(x)| \omega(x - \lambda) \, dx \\
&\leq C_0^p \delta_k^{d-2p} \max_{x \in [-\frac{\delta_k}{2}, \frac{\delta_k}{2}]^d} \omega(x) \|K_{\delta_k}\|_W \|K\|_W \left\{ c(\lambda) \right\}_{\lambda = \delta_0 k \in \delta_0 \mathbb{Z}^d, |k| > N} \|h\|_{L_1^\nu} \left\| (c(\lambda))_{\lambda \in \Lambda} \right\|_{\ell_r(\Lambda)}^p \\
&=: C_K^p \left\| (c(\lambda))_{\lambda \in \Lambda} \right\|_{\ell_r(\Lambda)}^p.
\end{align*}
\] (2.15)
3 Covering number for $V_{K,p}^{N,*}$

In this section, we discuss the covering number of $V_{K,p}^{N,*}$ with respect to the norm $\| \cdot \|_{L^\infty(\mathbb{R}^d)}$. Let $S$ be a metric space and $\eta > 0$, the covering number $\mathcal{N}(S, \eta)$ is defined to be the minimal integer $m \in \mathbb{N}$ such that there exist $m$ disks with radius $\eta$ covering $S$.

**Lemma 3.1** ([3]) Suppose $E$ is a finite dimensional Banach space with $\dim E = s$. Let $B_\varepsilon := \{x \in E : \|x\| \leq \varepsilon\}$ be the closed ball of radius $\varepsilon$ centered at the origin. Then

$$\mathcal{N}(B_\varepsilon, \eta) \leq \left(\frac{2\varepsilon}{\eta} + 1\right)^s.$$ 

Note that

$$\dim(V_{K,p}^N) \leq \#\{\lambda \in \Lambda : \lambda \in [-N,N]^d\} \leq \left(\frac{2N}{\delta_0} + 1\right)^d. \quad (3.1)$$

Then by Lemma 3.1 we have the following result.

**Lemma 3.2** Let $V_{K,p}^{N,*}$ be defined by (2.2). Then for any $\eta > 0$, the covering number of $V_{K,p}^{N,*}$ concerning the norm $\| \cdot \|_{L_p^\infty(\mathbb{R}^d)}$ is bounded by

$$\mathcal{N}(V_{K,p}^{N,*}, \eta) \leq \exp\left(\left(\frac{2N}{\delta_0} + 1\right)^d \ln \left(\frac{2}{\eta} + 1\right)\right).$$

**Lemma 3.3** Suppose that $K$ satisfies the assumptions (1.5) and (1.6). Then for every $f \in V_{K,p}$, we have

$$\|f\|_{L_p^\infty(\mathbb{R}^d)} \leq C^*\|f\|_{L_p^\infty(\mathbb{R}^d)}, \quad (3.2)$$

where

$$C^* = B_pC_0\delta_0^{-d/p}\left(\max_{x \in [-\frac{N}{2}, \frac{N}{2}]^d} \omega(x)\right)\|K\|_W\|K\|_W. \quad (3.3)$$

**Proof** Suppose that $f \in V_{K,p}$, then it follows from Definition 2.1, Definition 2.2 and Lemma 2.3 that $f = \sum_{\lambda \in \Lambda} \langle f, \tilde{\phi}_\lambda \rangle \phi_\lambda$. Moreover, we can obtain from (2.11) that

$$\|f\|_{L_p^\infty(\mathbb{R}^d)} \leq \sup_{x \in \mathbb{R}^d} \sum_{\lambda \in \Lambda} |\langle f, \tilde{\phi}_\lambda \rangle| |\phi_\lambda(x)| \nu(x)$$

$$\leq C_0 \sup_{x \in \mathbb{R}^d} \sum_{\lambda \in \Lambda} |\langle f, \tilde{\phi}_\lambda \rangle| \nu(\lambda) \cdot |\phi_\lambda(x)| \omega(x - \lambda)$$

$$\leq C_0 \delta_0^{-d/p}\left(\max_{x \in [-\frac{N}{2}, \frac{N}{2}]^d} \omega(x)\right)\|K\|_W\|K\|_W \|\{\langle f, \tilde{\phi}_\lambda \rangle \}_{\lambda \in \Lambda}\|_{L_p^\infty(\mathbb{R}^d)}$$

$$\leq B_pC_0\delta_0^{-d/p}\left(\max_{x \in [-\frac{N}{2}, \frac{N}{2}]^d} \omega(x)\right)\|K\|_W\|K\|_W \|f\|_{L_p^\infty(\mathbb{R}^d)}.$$

**Lemma 3.4** Suppose that $K$ satisfies the assumptions (1.5) and (1.6), then the covering number of $V_{K,p}^{N,*}$ with respect to $\| \cdot \|_{L^\infty(\mathbb{R}^d)}$ is bounded by

$$\mathcal{N}(V_{K,p}^{N,*}, \eta) \leq \exp\left(\left(\frac{2N}{\delta_0} + 1\right)^d \ln \left(\frac{2C^*}{\eta} + 1\right)\right).$$
Proof By Lemma 3.2, the covering number of $V_{K,p}^{N,*}$ with respect to $\| \cdot \|_{L^p(\mathbb{R}^d)}$ satisfies
\[
N\left( V_{K,p}^{N,*}, \frac{\eta}{C_*} \right) \leq \exp \left( \left( \frac{2N}{\delta_0} + 1 \right) d \ln \left( \frac{2C_*}{\eta} + 1 \right) \right). \quad (3.4)
\]

Let $\mathcal{F}$ be the corresponding $\frac{\eta}{C_*}$-net for $V_{K,p}^{N,*}$. It means that for every $f \in V_{K,p}^{N,*}$, there exists a $\tilde{f} \in \mathcal{F}$ such that $\|f - \tilde{f}\|_{L^p(\mathbb{R}^d)} \leq \frac{\eta}{C_*}$. By Lemma 3.3, we have
\[
\|f - \tilde{f}\|_{L^p(\mathbb{R}^d)} \leq C_* \|f - \tilde{f}\|_{L^p(\mathbb{R}^d)} \leq \eta.
\]

Therefore, $\mathcal{F}$ is also a $\eta$-net of $V_{K,p}^{N,*}$ with respect to the norm $\| \cdot \|_{L^\infty(\mathbb{R}^d)}$. Since
\[
\#(\mathcal{F}) \leq \exp \left( \left( \frac{2N}{\delta_0} + 1 \right) d \ln \left( \frac{2C_*}{\eta} + 1 \right) \right),
\]
the desired result is proved.

4 Random sampling inequality of $V_{K,p}(R, \delta)$

Let $X = \{x_j : j \in \mathbb{N}\}$ be a sequence of independent random variables that are drawn from a general probability distribution over $\mathbb{R}^d$ with density function $\rho$ satisfying
\[
0 < c_\rho = \liminf_{x \in \mathbb{R}^d} \rho(x) \text{ and } C_\rho = \limsup_{x \in \mathbb{R}^d} \rho(x) < \infty. \quad (4.1)
\]

Then for any $f \in V_{K,p}$, we introduce the random variables
\[
X_j(f) = |f(x_j)\nu(x_j)|^p - \int_{\mathbb{R}^d} \rho(x)|f(x)\nu(x)|^pdx. \quad (4.2)
\]

It is easy to see that $X_j(f)$ is a sequence of independent random variables with expectation $E[X_j(f)] = 0$. Next, we will give some estimates for $X_j(f)$.

Lemma 4.1 Let $\rho(x)$ be a probability density function over $\mathbb{R}^d$ satisfying (4.1). Then for any $f, g \in V_{K,p}$, the following inequalities hold:
1. $\|X_j(f)\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}^p$.
2. $\|X_j(f) - X_j(g)\|_{L^\infty(\mathbb{R}^d)} \leq 2p \left( \max \left\{ \|f\|_{L^p(\mathbb{R}^d)}, \|g\|_{L^p(\mathbb{R}^d)} \right\} \right)^{p-1} \|f - g\|_{L^p(\mathbb{R}^d)}$.
3. $\text{Var}(X_j(f)) \leq C_\rho \|f\|_{L^p(\mathbb{R}^d)}^p \|f\|_{L^p(\mathbb{R}^d)}^p$.
4. $\text{Var}(X_j(f) - X_j(g)) \leq pC_\rho \left( \max \left\{ \|f\|_{L^p(\mathbb{R}^d)}, \|g\|_{L^p(\mathbb{R}^d)} \right\} \right)^{p-1} \|f - g\|_{L^p(\mathbb{R}^d)}^p + \|g\|_{L^p(\mathbb{R}^d)}^p$.

Proof (1) Direct computation obtains
\[
\|X_j(f)\|_{L^\infty(\mathbb{R}^d)} \leq \sup_{x \in \mathbb{R}^d} \max \left\{ |f(x)\nu(x)|^p, \int_{\mathbb{R}^d} \rho(x)|f(x)\nu(x)|^pdx \right\} \leq \|f\|_{L^p(\mathbb{R}^d)}^p.
\]
(2) By mean value theorem, one has
\[
\|X_j(f) - X_j(g)\|_{\ell^\infty} \leq \sup_{x \in \mathbb{R}^d} \left( \|f(x)\nu(x)\|^p - |g(x)\nu(x)|^p \right) + \int_{\mathbb{R}^d} \rho(x) \left| |f(x)\nu(x)|^p - |g(x)\nu(x)|^p \right| dx
\]
\[
\leq 2 \sup_{x \in \mathbb{R}^d} \|f(x)\nu(x)|^p - |g(x)\nu(x)|^p
\]
\[
= 2p \left( \max \{ \|f\|_{L_\nu^p(\mathbb{R}^d)}, \|g\|_{L_\nu^p(\mathbb{R}^d)} \} \right)^{p-1} \|f - g\|_{L_\nu^p(\mathbb{R}^d)}.
\]

(3) Since \( E[X_j(f)] = 0 \), then
\[
Var(X_j(f)) = E[(X_j(f))^2]
\]
\[
= E[|f(x)\nu(x)\|^p] - \left( \int_{\mathbb{R}^d} \rho(x)|f(x)\nu(x)|^p dx \right)^2
\]
\[
\leq \int_{\mathbb{R}^d} \rho(x)|f(x)\nu(x)|^{2p} dx
\]
\[
\leq C \rho \|f\|_{L_\nu^p(\mathbb{R}^d)} \|f\|_{L_\nu^p(\mathbb{R}^d)}.
\]

(4) Using the similar method as (3), we have
\[
Var(X_j(f) - X_j(g))
\]
\[
= E[(X_j(f) - X_j(g))^2]
\]
\[
\leq C \rho \int_{\mathbb{R}^d} \left( |f(x)\nu(x)|^p - |g(x)\nu(x)|^p \right)^2 dx
\]
\[
\leq C \rho \int_{\mathbb{R}^d} \left( |f(x)\nu(x)|^p - |g(x)\nu(x)|^p \right) \left( |f(x)\nu(x)|^p + |g(x)\nu(x)|^p \right) dx
\]
\[
\leq C \rho \sup_{x \in \mathbb{R}^d} \left( |f(x)\nu(x)|^p - |g(x)\nu(x)|^p \right) \left( \|f\|_{L_\nu^p(\mathbb{R}^d)} + \|g\|_{L_\nu^p(\mathbb{R}^d)} \right)
\]
\[
\leq pC \rho \left( \max \{ \|f\|_{L_\nu^p(\mathbb{R}^d)}, \|g\|_{L_\nu^p(\mathbb{R}^d)} \} \right)^{p-1} \|f - g\|_{L_\nu^p(\mathbb{R}^d)} \left( \|f\|_{L_\nu^p(\mathbb{R}^d)} + \|g\|_{L_\nu^p(\mathbb{R}^d)} \right).
\]

In the following lemma, we will show that a uniform large deviation inequality holds for functions in \( V_{K,p}^N \) by Bernstein’s inequality.

**Lemma 4.2** (Bernstein’s inequality) Let \( X_1, X_2, \ldots, X_n \) be independent random variables with expected values \( E(X_j) = 0 \) for \( j = 1, 2, \ldots, n \). Assume that \( \text{Var}(X_j) \leq \sigma^2 \) and \( |X_j| \leq M_0 \) almost surely for all \( j \). Then for any \( \lambda \geq 0 \),
\[
\text{Prob} \left( \left| \sum_{j=1}^n X_j \right| \geq \lambda \right) \leq 2 \exp \left( -\frac{\lambda^2}{2n\sigma^2 + \frac{3}{4}M_0\lambda} \right).
\]

**Lemma 4.3** Let \( \{x_j : j \in \mathbb{N}\} \) be a sequence of independent random variables that are drawn from a general probability distribution over \( \mathbb{R}^d \) with density function \( \rho \) satisfying (4.1). If \( f \in V_{K,p}^N \), then for \( n \in \mathbb{N} \) and \( \lambda \geq 0 \),
\[
\text{Prob} \left( \sup_{f \in V_{K,p}^N} \left| \sum_{j=1}^n X_j(f) \right| \geq \lambda \right) \leq A \exp \left( -B \frac{\lambda^2}{12nC_\rho + 2\lambda} \right).
\]
where $A$ is of order $\exp(CN^d)$ with $B = \min\{\frac{\sqrt{7}}{25692p(C_*)^p}, \frac{2}{221p(C_*)^p}\}$, $C$ depending on $\Lambda$ and $K$.

**Proof** For given $\ell \in \mathbb{N}$, we construct a $2^{-\ell}$-covering for $V_{K,p}^{N,*}$ with respect to the norm $\|\cdot\|_{L_\infty(\mathbb{R}^d)}$. Let $C_\ell$ be the corresponding $2^{-\ell}$-net for $\ell = 1, 2, \ldots$. Then,

$$\sharp(C_\ell) \leq \mathcal{N}(V_{K,p}^{N,*}, 2^{-\ell}).$$

For given $f \in V_{K,p}^{N,*}$, let $f_\ell$ be the function in $C_\ell$ that is closest to $f$ with respect to the norm $\|\cdot\|_{L_\infty(\mathbb{R}^d)}$. Then, $\|f - f_\ell\|_{L_\infty(\mathbb{R}^d)} \leq 2^{-\ell} \to 0$ when $\ell \to \infty$. Moreover, by Lemma 3.3 and the item (2) of Lemma 4.4, we have

$$X_j(f) = X_j(f_1) + (X_j(f_2) - X_j(f_1)) + (X_j(f_3) - X_j(f_2)) + \cdots.$$  

If $\sup_{f \in V_{K,p}^{N,*}} \left| \sum_{j=1}^{n} X_j(f) \right| \geq \lambda$, the event $\omega_\ell$ must hold for some $\ell \geq 1$, where

$$\omega_1 = \left\{ \text{there exists } f_1 \in C_1 \text{ such that } \left| \sum_{j=1}^{n} X_j(f_1) \right| \geq \frac{\lambda}{2} \right\}$$

and for $\ell \geq 2$,

$$\omega_\ell = \left\{ \text{there exist } f_\ell \in C_\ell \text{ and } f_{\ell-1} \in C_{\ell-1} \text{ with } \|f_\ell - f_{\ell-1}\|_{L_\infty(\mathbb{R}^d)} \leq 3 \cdot 2^{-\ell}, \right.\
\text{such that } \left| \sum_{j=1}^{n} (X_j(f_\ell) - X_j(f_{\ell-1})) \right| \geq \frac{\lambda}{2\ell^2} \right\}.$$  

If this is not the case, then with $f_0 = 0$, we have

$$\left| \sum_{j=1}^{n} X_j(f) \right| \leq \sum_{\ell=1}^{\infty} \left| \sum_{j=1}^{n} (X_j(f_\ell) - X_j(f_{\ell-1})) \right| \leq \sum_{\ell=1}^{\infty} \frac{\lambda}{2\ell^2} = \frac{\pi^2\lambda}{12} \leq \lambda.$$  

Next, we estimate the probability of each $\omega_\ell$. By Lemma 3.3, 4.1 and 4.2 for every fixed $f \in C_1$,

$$\text{Prob} \left( \left| \sum_{j=1}^{n} X_j(f) \right| \geq \frac{\lambda}{2} \right) \leq 2 \exp \left( -\frac{\left(\frac{\pi}{2}\right)^2}{2n\text{Var}(X_j(f)) + \frac{2}{3}\|X_j(f)\|_{L_\infty}\cdot \frac{\lambda}{2}} \right) \leq 2 \exp \left( -\frac{\lambda^2}{8nC_p(C_*)^p + \frac{4}{3}\lambda(C_*)^p} \right).$$

By Lemma 3.3, there are at most

$$\mathcal{N}(V_{K,p}^{N,*}, 1/2) \leq \exp \left( \frac{2N}{\delta_0}^d \ln(4C_* + 1) \right)$$

functions in $C_1$. Thus, the probability of $\omega_1$ is bounded by

$$\text{Prob}(\omega_1) \leq 2 \exp \left( \frac{2N}{\delta_0}^d \ln(4C_* + 1) \right) \exp \left( -\frac{\lambda^2}{8nC_p(C_*)^p + \frac{4}{3}\lambda(C_*)^p} \right) = 2 \exp \left( \frac{2N}{\delta_0}^d \ln(4C_* + 1) \right) \exp \left( -\frac{\lambda^2}{\frac{2}{3}(C_*)^p(12nC_p + 2\lambda)} \right). \quad (4.3)$$
For $\ell \geq 2$, we estimate the probability of $\omega_\ell$ in a similar way. For $f \in C_\ell$, $g \in C_{\ell - 1}$ and $\|f - g\|_{L^\infty(\mathbb{R}^d)} \leq 3 \cdot 2^{-\ell}$, we have

$$
\text{Prob}
\left(
\sum_{j=1}^{n} (X_j(f) - X_j(g)) \geq \frac{\lambda}{2\ell^2}
\right)
\leq 2 \exp\left(-\frac{(\frac{\lambda}{2\ell^2})^2}{2n\text{Var}(X_j(f) - X_j(g)) + \frac{\lambda}{2\ell^2} \|f - g\|_{L^\infty} \cdot \frac{\lambda}{2\ell^2}}\right)
\leq 2 \exp\left(-\nu \frac{2^\ell}{\ell^4}\right),
$$

where $\nu = \frac{\lambda^2}{4p(C^*)^p - (12nC_\rho + 2\lambda)}$. There are at most $\mathcal{N}(V_{K,p}^{N*, 2^{-\ell}})$ functions in $C_\ell$ and $\mathcal{N}(V_{K,p}^{N*, 2^{-\ell+1}})$ functions in $C_{\ell - 1}$. Therefore, we have

$$
\text{Prob}\left(\bigcup_{\ell=2}^{\infty} \omega_\ell\right) \leq \sum_{\ell=2}^{\infty} \mathcal{N}(V_{K,p}^{N*, 2^{-\ell}}) \mathcal{N}(V_{K,p}^{N*, 2^{-\ell+1}}) \sum_{\ell=2}^{\infty} \exp\left(-\nu \frac{2^\ell}{\ell^4}\right)
= C_1 \sum_{\ell=2}^{\infty} \exp\left(C_2 \ell - \frac{\nu 2^\ell}{\ell^4}\right)
= C_1 \sum_{\ell=2}^{\infty} \exp\left(-\nu 2^\frac{\ell}{\ell^4} - \frac{C_2 \ell}{2^\frac{\ell}{\ell^4}}\right),
$$

where $C_1 = 2(2C^* + 1)^2 \left(\frac{2N}{b_0} + 1\right)^d$ and $C_2 = (2\ln 2) \left(\frac{2N}{b_0} + 1\right)^d$.

Let $C_3 := \min_{\ell \geq 2} \frac{2^\frac{\ell}{\ell^4}}{2^\frac{\ell}{\ell^4} + \nu} = \frac{1}{324}$ and $C_4 := \max_{\ell \geq 2} \frac{8p(C^*)^{p-1} \ell \ln 2}{2^\frac{\ell}{\ell^4}} = 6\sqrt{2} p(C^*)^{p-1} \ln 2$. Then

$$
\frac{2^\frac{\ell}{\ell^4} - C_2 \ell}{2^\frac{\ell}{\ell^4} + \nu} = \frac{2^\frac{\ell}{\ell^4} - \frac{8p(C^*)^{p-1} \ell \ln 2}{2^\frac{\ell}{\ell^4}}}{2^\frac{\ell}{\ell^4} + \nu}
\geq \frac{1}{324} - \frac{C_4 \left(\frac{2N}{b_0} + 1\right)^d (12nC_\rho + 2\lambda)}{\lambda^2}.
$$

We first consider the case that

$$
\frac{1}{324} - \frac{C_4 \left(\frac{2N}{b_0} + 1\right)^d (12nC_\rho + 2\lambda)}{\lambda^2} > \frac{1}{648}. \tag{4.4}
$$
Since \( p, a > 0 \), one has \( \sum_{\ell=2}^{\infty} e^{-pa\ell} < \frac{e^{-pa}}{pa\ln a} \) ([20]), then

\[
Prob\left( \bigcup_{\ell=2}^{\infty} \omega_\ell \right) \leq \frac{C_1 \exp\left( -\sqrt{2} v \left( \frac{1}{324} - \frac{C_4 \left( \frac{2N}{\delta_0} + 1 \right)^d (12nC_\rho) }{\lambda^2} \right) \right)}{\sqrt{2} \ln \sqrt{2} \cdot v \left( \frac{1}{324} - \frac{C_4 \left( \frac{2N}{\delta_0} + 1 \right)^d (12nC_\rho) }{\lambda^2} \right)}
\]

\[
= \frac{2(2C^* + 1)^d}{\sqrt{2} \ln \sqrt{2} \cdot v \left( \frac{1}{324} - \frac{C_4 \left( \frac{2N}{\delta_0} + 1 \right)^d (12nC_\rho) }{\lambda^2} \right)}
\]

\[
\times \exp\left( -\sqrt{2} v \left( \frac{1}{324} - \frac{C_4 \left( \frac{2N}{\delta_0} + 1 \right)^d (12nC_\rho) + 2\lambda }{\lambda^2} \right) \right).
\]

Under the condition (4.4), we have

\[
\sqrt{2} \ln \sqrt{2} \cdot v \left( \frac{1}{324} - \frac{C_4 \left( \frac{2N}{\delta_0} + 1 \right)^d (12nC_\rho) + 2\lambda }{\lambda^2} \right) \geq \sqrt{2} \ln \sqrt{2} \cdot C_4 \left( \frac{2N}{\delta_0} + 1 \right)^d
\]

\[
\geq \frac{\sqrt{2} \ln \sqrt{2} C_4 \left( \frac{2N}{\delta_0} + 1 \right)^d}{4p(C^*)^{p-1}}
\]

\[
\geq 3 \ln \sqrt{2} \ln 2.
\]

This together with the probability of \( \omega_1 \) in (4.3) obtains

\[
Prob\left( \sup_{f \in V_{K,p}^N} \left| \sum_{j=1}^{n} X_j(f) \right| \geq \lambda \right) \leq Prob\left( \bigcup_{\ell=1}^{\infty} \omega_\ell \right) \leq A \exp\left( -\frac{B}{12nC_\rho + 2\lambda} \right).
\]

Here, \( A \) is of order \( \exp\left( CN^d \right) \) with \( C = 2^{d+1} \left(1 + \frac{1}{\delta_0} \right)^d \ln(2C^* + 1) \) and \( B = \min\left\{ \frac{\sqrt{2}}{4p(C^*)^{p-1}}, \frac{3}{2(C^*)^p} \right\} \).

Finally, we consider the case that

\[
\frac{1}{324} - \frac{C_4 \left( \frac{2N}{\delta_0} + 1 \right)^d (12nC_\rho) + 2\lambda }{\lambda^2} \leq \frac{1}{648}.
\]

In this case, we can choose \( C \geq 648C_4B2^d \left(1 + \frac{1}{\delta_0} \right)^d \) such that \( A \exp\left( -\frac{B}{12nC_\rho + 2\lambda} \right) \geq 1 \). This completes the proof.

**Lemma 4.4** Let \( X = \{ x_j : j \in \mathbb{N} \} \) be a sequence of independent random variables that are drawn from a general probability distribution over \( \mathbb{R}^d \) with density function \( \rho \) satisfying (4.1). Then for any \( \gamma > 0 \), the sampling inequality

\[
n c_\rho \left( \|f\|_{L^p(C_{0})}^p - \gamma \|f\|_{L^p(\mathbb{R}^d)}^p \right) \leq \sum_{j=1}^{n} |f(x_j)\nu(x_j)|^p \leq n(c_\rho \gamma + C_\rho) \|f\|_{L^p(\mathbb{R}^d)}^p
\]

(4.5)
holds for function \( f \in V_{K,p}^N \) with probability at least
\[
1 - A \exp \left( -B \frac{\gamma^2 n c_p^2}{12 c_p + 2 \gamma c_p} \right),
\]
where \( A \) and \( B \) are as in Lemma 4.3.

**Proof** It is obvious that every \( f \in V_{K,p}^N \) satisfies the inequality (4.5) if and only if \( f \| f \|_{L_p^n(R^d)} \) dose. So we assume that \( f \| f \|_{L_p^n(R^d)} = 1 \), then \( f \in V_{K,p}^N, \ast \). The event
\[
D = \left\{ \sup_{f \in V_{K,p}^N, \ast} \left| \sum_{j=1}^n X_j(f) \right| > \gamma n c_p \right\}
\]
is the complement of
\[
\tilde{D} = \left\{ n \int_{R^d} \rho(x) |f(x)\nu(x)|^p dx - \gamma n c_p \leq \sum_{j=1}^n |f(x_j)\nu(x_j)|^p \right\}
\leq \gamma n c_p + n \int_{R^d} \rho(x) |f(x)\nu(x)|^p dx, \quad \forall f \in V_{K,p}^N, \ast \}
\leq \left\{ n c_p \left( \| f \|_{L_p^n(C_n)}^p - \gamma \| f \|_{L_p^n(R^d)}^p \right) \right\} \leq \sum_{j=1}^n |f(x_j)\nu(x_j)|^p
\leq n (c_p \gamma + C_p) \| f \|_{L_p^n(C_n)}^p, \quad \forall f \in V_{K,p}^N, \ast \}
\]
Using Lemma 4.3 the sampling inequality (4.5) holds for all \( f \in V_{K,p}^N \) with probability
\[
Prob(\tilde{D}) \geq Prob(\tilde{D}) = 1 - Prob(D) \geq 1 - A \exp \left( -B \frac{\gamma^2 n c_p^2}{12 c_p + 2 \gamma c_p} \right).
\]

In the following, we will show that if the sampling size is sufficiently large, the sampling inequality holds with overwhelming probability for functions in \( V_{K,p}(R, \delta) \).

**Theorem 4.5** Let \( X = \{x_j : j \in \mathbb{N}\} \) be a sequence of independent random variables that are drawn from a general probability distribution over \( R^d \) with density function \( \rho \) satisfying (4.1). Suppose that \( M > R \) is a constant such that \( \{x_j : j = 1, 2, \cdots, n\} \subseteq C_M \), then for any \( 0 < \varepsilon, \gamma < 1 \) which satisfy
\[
L(\varepsilon, \gamma) = c_p \left( 1 - \delta - p(1 + \varepsilon)^{p-1} \varepsilon - \gamma \left( B_p C_K \right)^p \right) - p(C^\ast)^{p-1} \frac{\varepsilon}{(2M)^d} > 0,
\]
the sampling inequality
\[
nL(\varepsilon, \gamma) \| f \|_{L_p^n(R^d)}^p \leq \sum_{j=1}^n |f(x_j)\nu(x_j)|^p \leq n U(\varepsilon, \gamma) \| f \|_{L_p^n(R^d)}^p
\]
holds for function \( f \in V_{K,p}(R, \delta) \) with probability at least
\[
1 - A \exp \left( -B \frac{\gamma^2 n c_p^2}{12 c_p + 2 \gamma c_p} \right).
\]
Here, $U(\varepsilon, \gamma) = (c_\rho \gamma + C_\rho)(B_\rho C_K)^p + p(C^*)^{p-1} - \frac{\varepsilon}{(2M)^d}$, $A$ and $B$ are the constants in Lemma 4.3 corresponding to $N = N(\varepsilon, M, f)$.

**Proof** It is obvious that every $f \in V_{K,p}(R, \delta)$ satisfies the inequality (4.7) if and only if $\left\|f\right\|_{L^p_\infty(\mathbb{R}^d)} \leq \delta$. Hence, we assume that $\left\|f\right\|_{L^p_\infty(\mathbb{R}^d)} = 1$.

For random variables $\{x_j : j = 1, 2, \cdots, n\}$, there exists a $M > R$ such that $\{x_j : j = 1, 2, \cdots, n\} \subset C_M$. By Lemma 2.4, for any $\varepsilon > 0$ satisfying (4.6), there exist $N = N(\varepsilon, M, f)$ and $f_N \in V_{K,p}^N$ such that

$$\left\|f - f_N\right\|_{L^p_\infty(C_M)} \leq \varepsilon \quad \text{and} \quad \left\|f - f_N\right\|_{L^\infty(C_M)} \leq \frac{\varepsilon}{(2M)^d}. \tag{4.8}$$

This together with mean value theorem obtains

$$\left|\left\|f\right\|_{L^p_\infty(C_R)}^p - \left\|f_N\right\|_{L^p_\infty(C_R)}^p\right| \leq p(1 + \varepsilon)^{p-1}\varepsilon \tag{4.9}$$

and

$$\left|f(x_j)\nu(x_j)|^p - \left|f_N(x_j)\nu(x_j)|^p\right| \leq p\left(\max\{|f(x_j)\nu(x_j)|, |f_N(x_j)\nu(x_j)|\}\right)^{p-1}\left|f(x_j) - f_N(x_j)\right|\nu(x_j) \leq p(C^*)^{p-1} - \frac{\varepsilon}{(2M)^d}. \tag{4.10}$$

It follows from (4.10) that

$$\sum_{j=1}^n |f_N(x_j)\nu(x_j)|^p - np(C^*)^{p-1} - \frac{\varepsilon}{(2M)^d} \leq \sum_{j=1}^n |f(x_j)\nu(x_j)|^p \leq \sum_{j=1}^n |f_N(x_j)\nu(x_j)|^p + np(C^*)^{p-1} - \frac{\varepsilon}{(2M)^d}. \tag{4.11}$$

For the above $f_N \in V_{K,p}^N$, we know from Lemma 4.3 that

$$nc_\rho\left(\left\|f_N\right\|_{L^p_\infty(C_R)}^p - \gamma\left\|f_N\right\|_{L^p_\infty(\mathbb{R}^d)}^p\right) \leq \sum_{j=1}^n |f_N(x_j)\nu(x_j)|^p \leq n(c_\rho \gamma + C_\rho)\left\|f_N\right\|_{L^p_\infty(\mathbb{R}^d)}^p \tag{4.12}$$

holds with probability at least

$$1 - A \exp\left(-B\frac{\gamma^2 n c_\rho^2}{12C_\rho + 2\gamma c_\rho}\right). \tag{4.13}$$

Then, it follows from (4.9), (4.11) and (4.12) that

$$nc_\rho\left(\left\|f\right\|_{L^p_\infty(C_R)}^p - p(1 + \varepsilon)^{p-1}\varepsilon - \gamma\left\|f_N\right\|_{L^p_\infty(\mathbb{R}^d)}^p\right) - np(C^*)^{p-1} - \frac{\varepsilon}{(2M)^d} \leq \sum_{j=1}^n |f(x_j)\nu(x_j)|^p \leq n\left(c_\rho \gamma + C_\rho\right)\left\|f_N\right\|_{L^p_\infty(\mathbb{R}^d)}^p + np(C^*)^{p-1} - \frac{\varepsilon}{(2M)^d}. \tag{4.14}$$
holds with the same probability as (4.13). Since \( f \in V_{K,p}(R, \delta) \), we have
\[
(1 - \delta)\|f\|_{L_p^p(\mathbb{R}^d)}^{p} \leq \|f\|_{L_p^p(C_R)}^{p},
\]
(4.15)
Moreover, we know from Lemma 2.5 that
\[
\|f_N\|_{L_p^p(\mathbb{R}^d)} \leq C_K \|\left( \langle f, \tilde{\phi}_\lambda \rangle \right)_{\lambda \in \Lambda} \|_{{\ell}_p^p} \leq B_K \|f\|_{L_p^p(\mathbb{R}^d)}. \quad (4.16)
\]
Note that \( \|f\|_{L_p^p(\mathbb{R}^d)} = 1 \). Then, the sampling inequality (4.7) follows from (4.14)-(4.16).

5 Reconstruction algorithm in \( V_{K,p}^N \)

In this section, we consider the reconstruction algorithm of functions in \( V_{K,p}^N \) from the corresponding random samples.

Lemma 5.1 Suppose that there exists some constant \( \zeta > 0 \) such that for any
\[
f = \sum_{\lambda \in \Lambda \cap [-N,N]^d} c(\lambda) \phi_\lambda \in V_{K,p}^N,
\]
we have
\[
\sum_{j=1}^{n} |f(x_j)\nu(x_j)|^p \geq \zeta \|c\|_{\ell_p^p}^p. \quad (5.1)
\]
Then there exist reconstruction functions \( (S_j(x))_{1 \leq j \leq n} \) such that for all \( f \in V_{K,p}^N \),
\[
f(x) = \sum_{j=1}^{n} f(x_j)S_j(x). \quad (5.2)
\]

Proof For \( f = \sum_{\lambda \in \Lambda \cap [-N,N]^d} c(\lambda) \phi_\lambda \in V_{K,p}^N \), we try to solve the system of linear equations
\[
f(x_j) = \sum_{\lambda \in \Lambda \cap [-N,N]^d} c(\lambda) \phi_\lambda (x_j), \quad 1 \leq j \leq n \quad (5.3)
\]
for the coefficients \( (c(\lambda))_{\lambda \in \Lambda \cap [-N,N]^d} \). Define a random matrix
\[
U = (u_{j,\lambda})_{j=1,2,\ldots,n;\lambda \in \Lambda \cap [-N,N]^d}, \quad (5.4)
\]
where \( u_{j,\lambda} = \phi_\lambda(x_j) \). In fact, the column number of \( U \) is less than \( (2N/\delta + 1)^d \). Then the system (5.3) of linear equations can be rewritten as
\[
Uc = f|_{\{x_j; j=1,2,\ldots,n\}}. \quad (5.5)
\]
Since \( \sum_{j=1}^{n} |f(x_j)\nu(x_j)|^p \geq \zeta \|c\|_{\ell_p^p}^p \), we have
\[
\|Uc\|_{\ell_p^p} \geq \zeta^{1/p} \|c\|_{\ell_p^p}, \quad \forall \ c \in \ell_p^p(\Lambda \cap [-N,N]^d). \quad (5.6)
\]
Then $U^TU$ is invertible, which implies that
\[
c = (U^TU)^{-1}U^Tf|_{\{x_j:j=1,2,\ldots,n\}}.
\] (5.7)

Let $\Psi(x) = (\phi_\lambda(x))_{\lambda \in \Lambda \cap [-N,N]^d}^T$ and $(S_j(x))_{1 \leq j \leq n}^T = U(U^TU)^{-1}\Psi$. Then
\[
f(x) = \sum_{j=1}^n f(x_j)S_j(x).
\]

The following theorem presents a reconstruction algorithm for signals in a finite dimensional subspace $V_{K,p}^N$.

**Theorem 5.2** Suppose that there exists a $\alpha_p > 0$ such that for all $c \in \ell^p_\nu(\Lambda \cap [-N,N]^d)$,
\[
\left\| \sum_{\lambda \in \Lambda \cap [-N,N]^d} c(\lambda)\phi_\lambda \right\|_{L^p(C_R)} \geq \alpha_p \|c\|_{\ell^p_\nu}.
\] (5.8)

Let $U$ be as in (5.4), $\Psi(x) = (\phi_\lambda(x))_{\lambda \in \Lambda \cap [-N,N]^d}^T$ and $(S_j(x))_{1 \leq j \leq n}^T = U(U^TU)^{-1}\Psi$. Then for any $0 < \gamma < \frac{\alpha_p}{\epsilon_K}$, the reconstruction formula
\[
f(x) = \sum_{j=1}^n f(x_j)S_j(x)
\] (5.9) holds for all $f \in V_{K,p}^N$ with probability at least
\[
1 - A \exp\left(-B \frac{\gamma^2 n c_p^2}{12 C_p + 2 \gamma c_p}\right),
\]
where $A$ and $B$ are as in Lemma 4.3.

**Proof** For $f = \sum_{\lambda \in \Lambda \cap [-N,N]^d} c(\lambda)\phi_\lambda \in V_{K,p}^N$, it follows from Lemma 4.3 and Lemma 2.5 that
\[
\sum_{j=1}^n |f(x_j)\nu(x_j)|^p \geq n c_p \left( \alpha_p^p \|c\|_{\ell^p_\nu}^p - \gamma \|f\|_{L^p(R^d)}^p \right) \geq n c_p \left( \alpha_p^p - \gamma C_K^p \right) \|c\|_{\ell^p_\nu}^p
\]
holds with probability at least
\[
1 - A \exp\left(-B \frac{\gamma^2 n c_p^2}{12 C_p + 2 \gamma c_p}\right).
\]

Then the desired reconstruction formula (5.9) follows from Lemma 5.1.

For a linear operator $L$ defined on $\ell^p_\nu(\Lambda \cap [-N,N]^d)$, the $(p, \nu)$-norm condition number of $L$ is defined by
\[
\kappa(L, p, \nu) := \max_{a \in \ell^p_\nu(\Lambda \cap [-N,N]^d), a \neq 0} \frac{\|La\|_{\ell^p_\nu}}{\|a\|_{\ell^p_\nu}} \left( \min_{a \in \ell^p_\nu(\Lambda \cap [-N,N]^d), a \neq 0} \frac{\|La\|_{\ell^p_\nu}}{\|a\|_{\ell^p_\nu}} \right)^{-1}.
\] (5.10)
Theorem 5.3 Suppose that random variables $X = \{x_j : j \in \mathbb{N}\}$ and density function $\rho$ are as in Theorem 4.5. If the condition (5.8) holds for all $c \in \ell_p^\nu(\Lambda \cap [-N, N]d)$, then for any $0 < \gamma < \alpha_p^p C_K$, the $(p, \nu)$-norm condition number

$$\kappa(U, p, \nu) \leq \left( \frac{c_p \gamma + C_p}{c_p(\alpha_p^p - \gamma C_K^p)} \right)^{1/p} C_K$$

holds with probability at least

$$1 - A \exp \left( - B \frac{\gamma^2 n c_p^2}{12 C_p + 2 \gamma c_p} \right),$$

where $A$ and $B$ are as in Lemma 4.3.

Proof For every $c \in \ell_p^\nu(\Lambda \cap [-N, N]d)$ and $g = \sum_{\lambda \in \Lambda \cap [-N, N]d} c(\lambda) \phi_\lambda$, it follows from Lemma 4.1 that the sampling inequality

$$nc_p \left( \|g\|_{L_p^\nu(C_K)}^p - \gamma \|g\|_{L_p^\nu(\mathbb{R}^d)}^p \right) \leq \sum_{j=1}^n |g(x_j) \nu(x_j)|^p \leq n(c_p \gamma + C_p) \|g\|_{L_p^\nu(\mathbb{R}^d)}^p$$

holds for all $c \in \ell_p^\nu(\Lambda \cap [-N, N]d)$ with probability at least

$$1 - A \exp \left( - B \frac{\gamma^2 n c_p^2}{12 C_p + 2 \gamma c_p} \right).$$

By Lemma 2.5, $\|g\|_{L_p^\nu(\mathbb{R}^d)} \leq C_K \|c\|_{\ell_p^\nu}$. Furthermore, it can be easily verified from (5.4) that

$$\sum_{j=1}^n |g(x_j) \nu(x_j)|^p = \|Uc\|_{\ell_p^\nu}^p.$$ (5.12)

Then, combining (5.11)-(5.12) obtains

$$nc_p(\alpha_p^p - \gamma C_K^p) \leq \frac{\|Uc\|_{\ell_p^\nu}^p}{\|c\|_{\ell_p^\nu}^p} \leq n(c_p \gamma + C_p) C_K^p.$$ 

This together with (5.10) leads to the desired result.

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