Quantum particle on a quantum circle *

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Abstract

We describe a $q$-deformed dynamical system corresponding to the quantum free particle moving along the circle. The algebra of observables is constructed and discussed. We construct and classify irreducible representations of the system.

1 Introduction

Non-commutative geometry [3, 5] has attracted much attention of theoretical physicists. It is based on the idea, that the commutative algebra of functions on a manifold can be replaced by an abstract non-commutative algebra. This

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is in the similarity to the usual quantisation procedure, when one considers a non-commutative algebra of operators in place of the commutative algebra of real-valued functions as a set of observables. In particular usual quantum mechanics can be understood as a non-commutative symplectic geometry [4]. Another point of view is presented in [6] where quantum dynamics is treated as a non-commutative differential calculus (quantum deRham complex). The ideas of non-commutative geometry lead to the concept of non-commutative or $q$-deformed physics recently realised as a number of simple quantum mechanical models with the $q$-deformed phase-space structure [1, 2, 6, 7, 8].

In this paper we construct a simple toy model of the non-commutative quantum mechanics, i.e. we describe the motion of the quantum particle on a quantum circle. Our goal is to describe the unitary time evolution in the non-commutative phase-space. The similar problem has been stated in [6], where the motion of the particle on quantum line has been considered. There are two possible schemes of the construction of quantum mechanical non-commutative models described by the so called Faddeev’s rectangle. One can begin with the classical system, quantise it and then deform or one can first perform deformation and then quantisation. Both procedures do not necessary lead to the same quantum mechanical system. In our case however, the Faddeev’s rectangle appears to be commutative.

In the quantum theory one constructs the algebra of observables $\mathcal{H}(I, x, p)$ which in our case is generated by the hermitean angular momentum $p$ and unitary position operator $x$, interpreted as $x = e^{-i\phi}$ where $\phi$ is the angle. They obey the Heisenberg commutation relation $[x, p] = \hbar x$. Having defined the algebra of observables one can construct the notions of states, measurement, mean value etc. which are related to the irreducible Hilbert space representations of the algebra of observables. The dynamics of the system is given by the unitary time evolution which is provided by the Heisenberg equations of motion $\dot{\Omega} = \frac{i}{\hbar}[H, \Omega] + \partial_t \Omega$. The possible convenient description of the dynamics is given by the suitable deRham complex. The demanding of the unitary time evolution forces the Heisenberg equations of motion to be unchanged on the $q$-deformed level too. This implies that the $q$-deformation leaves the probabilistic interpretation of the system unchanged. The only part in that scheme which can be deformed is the algebra of observables. This is in remarkable contrast to [2] where the Heisenberg equations of motion were deformed. In our case we can demand that the phase-space is given by the quantum cylinder rather than the classical one.
The paper is organized as follows. In Section 2 we describe the usual quantum particle on a circle from the algebraic point of view. In Section 3 we deform the algebra of observables of the particle on circle and we solve the Heisenberg equations of motion showing that the unitary time evolution is possible in the case of a free particle. Section 4 is devoted to the construction and classification of the irreducible representations of the deformed algebra of observables. Finally in Section 5 we discuss the classical limits and the invariance of constructed algebra under space and time inversions.

2 Quantum mechanics of the particle on a circle

In the standard approach, quantum free particle on a circle is described by the unitary operator \( x \), corresponding to the position of the particle and an hermitean operator \( p \)—canonical momentum (angular momentum). The dynamics of the system is given by the hermitean Hamiltonian \( H = \frac{p^2}{2B} \), where \( B \) denotes the moment of inertia of the particle. The algebra of observables \( \mathcal{H}(I, x, p) \) can be defined as

\[
\mathcal{H}(I, x, p) = \mathbb{C}[I, x, p]/J(I, x, p).
\]

Here \( \mathbb{C}[I, x, p] \) is an associative, involutive (i.e. equipped with \( * \) structure, which is represented as the hermitean conjugation on Hilbert space) free algebra over \( \mathbb{C} \) generated by \( I, x, p \) (\( I \) is the identity) and \( J(I, x, p) \) is a two-sided ideal in \( \mathbb{C}[I, x, p] \) generated by the relation:

\[
 xp = px + \hbar x.
\]

Notice that the parameters of the theory (e.g. \( B \) or \( \hbar I \)) can be treated as independent of time operators belonging to the center of the algebra of observables. Namely we can extend the algebra \( \mathcal{H} \) to the algebra \( \mathcal{H}' \) defined by

\[
\mathcal{H}' = \mathbb{C}[I, x, p, K, \Lambda]/J(I, x, p, K, \Lambda)
\]

where the new generators \( \Lambda, K \) are hermitean and \( J(I, x, p, K, \Lambda) \) is a two-sided ideal defined by the relations

\[
xp = px + \hbar \Lambda x
\]
\[ xA = Ax \]
\[ pA = Ap \]
\[ xK = Kx \]
\[ pK = Kp \]
\[ AK = KA. \]  
(4)

\( \Lambda \) and \( K \) are assumed both invertible and \( \Lambda \) is positive definite. The Hamiltonian reads
\[ H = p^2 K^2 \]  
i.e. \( K^2 \) is related to the moment of inertia \( B \). Now the irreducibility demanded on the representation level implies that \( \Lambda \) and \( K \) are multiplies of the identity. To obtain the standard quantum-mechanical limit we can choose
\[ \Lambda = I, \quad K = \frac{1}{\sqrt{2B}}I. \]  
(6)

The Hamiltonian form of the Heisenberg equations of motion reads:
\[ \dot{\Lambda} = \dot{K} = 0 \]
\[ \dot{x} = -\frac{i}{2B} x(2p - \hbar) \]  
\[ \dot{p} = 0 \]  
(7)

Eqs. (7) have the well known solution
\[ p(t) = p_0, \quad x(t) = x_0 e^{-\frac{i}{\hbar}(2p_0 - \hbar)t} \]  
(8)

where \( p_0 \) denotes initial angular momentum and \( x_0 \) denotes the initial position of a particle. Equations (4) as well as (5) are just identical to the equations obtained from considerations of the algebra \( \mathcal{H} \), therefore the algebras \( \mathcal{H}' \) and \( \mathcal{H} \) describe the same physical situation. In the next section however we will see that the algebra \( \mathcal{H}' \) will be suitable to the non-commuative extension of the described quantum mechanical model.

3 \textit{q-deformed quantum particle on a circle}

Quantum cylinder is defined as a free involutive algebra generated by the identity \( I \), unitary \( x \) and hermitean \( p \) modulo the relations
\[ xp = qpx, \quad x^* = x^{-1}, \quad p^* = p, \]  
(9)
where \( q \) is a positive real number. This space can be considered as a phase-space of the \( q \)-deformed particle on a circle. To quantise this system we have to replace the first of equations (9) by the equation

\[
x p = q p x + \hbar \Lambda x.
\]  

(10)

It means that we have to deform consistently the extended algebra of observables \( \mathcal{H}' \) to the algebra \( \mathcal{H}_{q\varepsilon\xi} \) i.e. we have to deform the ideal \( J(I, x, p, K, \Lambda) \) to the ideal \( J_{q\varepsilon\xi}(I, x, p, K, \Lambda) \) in such a way that both \( K \) and \( \Lambda \) are no longer commutative in the algebra \( \mathcal{H}_{q\varepsilon\xi} = C[I, x, p, K, \Lambda]/J_{q\varepsilon\xi} \). This can be done by the replacement of equations (4) by the set of the following relations

\[
\begin{align*}
    x p &= q p x + \hbar \Lambda x \\
    x \Lambda &= \varepsilon \Lambda x \\
    p \Lambda &= \Lambda p \\
    x K &= \xi K x \\
    p K &= K p \\
    K \Lambda &= \Lambda K,
\end{align*}
\]  

(11)

where all parameters \( q, \varepsilon, \xi \) are real and positive. We use the same structure as in the commutative case, i.e. \( p, K, \Lambda \) are hermitean and \( x \) is unitary. Now we can consider the unitary time evolution of the system with the Hamiltonian:

\[
H = p^2 K^2 + V(K, \Lambda)
\]  

(12)

where \( V \) is an arbitrary element of \( \mathcal{H}_{q\varepsilon\xi} \) constructed only from \( K \) and \( \Lambda \). It leads to the Hamilton equations of the form

\[
\begin{align*}
    \dot{\Lambda} &= \dot{K} = 0 \\
    \dot{x} &= ix[-2\varepsilon^{-1} \Lambda K^2 p + \hbar \varepsilon^{-2}(\Lambda K)^2 + \hbar^{-1}(V(qK, \varepsilon^{-1}A) - V(K, \Lambda))] \\
    \dot{p} &= 0.
\end{align*}
\]  

(13)

Here we have used the natural condition, that \( \dot{x} \) is linear in the momentum \( p \). This condition is satisfied if

\[
\xi = q^{-1}.
\]  

(14)
This reduces algebra $\mathcal{H}_{q\varepsilon}$ to the algebra $\mathcal{H}_{q\varepsilon}$ given by the following relations:

\[
\begin{align*}
  xp &= qpx + hAx \\
  xA &= \varepsilon Ax \\
  pA &= Ap \\
  xK &= q^{-1}Kx \\
  pK &= Kp \\
  KA &= \Lambda K.
\end{align*}
\]

(15)

Note that the function $V(K, \Lambda)$ which plays a role of a scale of energy in the non-deformed case now comes into equations of motion, giving a correction to the velocity of a particle. In principle this correction can be quite large since it is proportional to the inverse of the Planck constant. If we demand the existence of classical limit ($\hbar = 0$) we have to estimate $q = 1 + c_1\hbar + O(\hbar^2)$, $\varepsilon = 1 + c_2\hbar + O(\hbar^2)$. One can interpret this term as an internal or not related to the motion contribution to the angular velocity. We will see in Section 5 that the dependence of $\dot{x}$ on $V(K, \Lambda)$ vanishes, when we assume that the Hamiltonian is invariant under the time inversion. Equations (13) have the solution

\[
p(t) = p_0, \quad x(t) = x_0 e^{i[-2\varepsilon^{-1}\Lambda K^2 p_0 + h\varepsilon^{-2}(\Lambda K)^2 + h^{-1}(V(qK, \varepsilon^{-1}A) - V(K, \Lambda))]t}
\]

(16)

from what we see immediately, that the time evolution of the system is given by the unitary operator $U = \exp(\frac{i}{\hbar}Ht)$.

## 4 Representations of $\mathcal{H}_{q\varepsilon}$ in the Hilbert space

In this section we will find the Hilbert space of the representations of the $q$-deformed algebra of observables $\mathcal{H}_{q\varepsilon}$. The simplest way to do this is to define the action of the operators $x, p, A$ and $K$ on the orthonormal set of vectors. The operation $*$ is represented by the hermitean conjugation. Since $K, p, A$ are hermitean and commute, they can be diagonalised simultaneously. Let us denote the eigenvectors of $p, K, A$ by $|k, \kappa, \lambda\rangle$ where $k, \kappa, \lambda$ belong to the spectrum of $p, K, A$ respectively, i.e.

\[
\begin{align*}
  p|k, \kappa, \lambda\rangle &= k|k, \kappa, \lambda\rangle \\
  K|k, \kappa, \lambda\rangle &= \kappa|k, \kappa, \lambda\rangle \\
  A|k, \kappa, \lambda\rangle &= \lambda|k, \kappa, \lambda\rangle.
\end{align*}
\]

(17)
By means of equations (15) we see that

\[ x^n |k, \kappa, \lambda\rangle = |q^n (k - n \hbar \varepsilon^{-1} \lambda), q^n \kappa, \varepsilon^{-n} \lambda\rangle \]

are again eigenvectors of \( p, K \) and \( \Lambda \). Therefore the Hilbert space of representations of the described system is spanned by a lattice of vectors defined by (18). Let us now classify representations of \( \mathcal{H}_{q\varepsilon} \). We normalize \( K \) in such a way that it becomes \( I \) in a non-deformed case. Assume first that \( q \geq 1 \) and \( \varepsilon \geq 1 \). Then each set of numbers \( \kappa_0, \lambda_0, k_0 \) such that

\[ q^{-\frac{1}{2}} \leq \kappa_0 < q^{\frac{1}{2}}, \quad \varepsilon^{-\frac{1}{2}} \leq \lambda_0 < \varepsilon^{\frac{1}{2}}, \quad 0 \leq k_0 < \hbar \varepsilon^{-1} \lambda_0 \]

(19)

defines the irreducible representation of \( \mathcal{H}_{q\varepsilon} \). Similarly if we assume that \( q < 1 \) and \( \varepsilon \geq 1 \), we obtain that the irreducible representations of \( \mathcal{H}_{q\varepsilon} \) can be labelled by the numbers \( k_0, \kappa_0, \lambda_0 \) such that

\[ q^{\frac{1}{2}} \leq \kappa_0 < q^{-\frac{1}{2}}, \quad \varepsilon^{-\frac{1}{2}} \leq \lambda_0 < \varepsilon^{\frac{1}{2}}, \quad -\hbar \varepsilon^{-1} \lambda_0 < k_0 \leq 0 \]

(20)

The same classification can be repeated for \( \varepsilon < 1 \) but now \( \varepsilon^{\frac{1}{2}} \leq \lambda_0 < \varepsilon^{-\frac{1}{2}} \).

Immediately from the equation (18) we see that the angular momentum \( p \) is quantised. Moreover the classification of irreducible representations suggests that in general free particle on a \( q \)-deformed circle has anyonic rather than usual Bose-Fermi statistics.

Let us now put \( V(K, \Lambda) = 0 \). Using equation (18) we can obtain the energy spectrum of a free particle moving along a quantum circle

\[ E_n = (k_0 - n \hbar \varepsilon^{-1} \lambda_0)^2 \kappa_0^2. \]

5 Symmetries

It is an easy exercise to check that the algebra \( \mathcal{H}_{q\varepsilon} \) is invariant under the transformation \( x \rightarrow -x, \ p \rightarrow p, \ K \rightarrow K, \ \Lambda \rightarrow \Lambda \). This transformation corresponds to the space inversion in the non-deformed case. The representation (18) is obviously invariant under this transformation. In the non-deformed case however the algebra (2) is also invariant under the time inversion \( x \rightarrow x^{-1}, \ p \rightarrow -p \) which is no longer a symmetry of \( \mathcal{H}_{q\varepsilon} \). We can try to
find the symmetry of $H_{q\varepsilon}$ which generalizes the time-inversion. To do this let us consider the general, anti-unitary transformation $T, T^2 = I$ defined as

$$TxT^{-1} = x^{-1}, \quadTpT^{-1} = -pf(K, \Lambda),$$

$$TKT^{-1} = g(K, \Lambda), \quadT\Lambda T^{-1} = h(K, \Lambda)$$

where $f, g, h$ are arbitrary elements of $H_{q\varepsilon}$ depending only on $K, \Lambda$. The algebra $H_{q\varepsilon}$ is invariant under this transformation if and only if $\varepsilon = q$ and $T$ has the form

$$TxT^{-1} = x^{-1}, \quadTpT^{-1} = -pf(K, \Lambda)$$

$$TKT^{-1} = f^{-1}(K, \Lambda)K, \quadT\Lambda T^{-1} = f(K, \Lambda)\Lambda$$

where function $f$ fulfils the condition

$$q^2 f(q^{-1}K, q\Lambda) = f(K, \Lambda).$$

The algebra $H_{q\varepsilon}$ reduces now to $H_q$ with only one deformation parameter $q$ and the commutation relations

$$xp = qpx + hAx$$

$$xA = qAx$$

$$pA = A p$$

$$xK = q^{-1}Kx$$

$$pK = Kp$$

$$KA = AK.$$

From the quations (25) it follows that $I$ and $KA$ generate the center of $H_q$. Focusing on the irreducible representations of $H_q$ we can put

$$K = cA^{-1},$$

where the constant $c$ can be derived from the classical limit, i.e. $c = (2B)^{-\frac{1}{2}}$. Using this identification and the classical limit we can easily solve equation (24), namely

$$f(K) = \Lambda^{-2},$$

8
\[ T_x T^{-1} = x^{-1}, \quad T_p T^{-1} = -p\Lambda^{-2}, \quad T\Lambda T^{-1} = \Lambda^{-1}. \]  

(28)

In this case the action of the position operator \( x \) on the basis \( |k, \lambda \rangle \) takes the form

\[ x^n |k, \lambda \rangle = |q^{-n}(k - nhq^{-1}\lambda), q^{-n}\lambda\rangle. \]  

(29)

An irreducible representation of \( \mathcal{H}_q \) contains vectors numbered by the knots of the lattice generated by \( x^n \),

\[ \{q^{-n}(k - nhq^{-1}\lambda), q^{-n}\lambda\} \]

with integer \( n \). Now

\[ T|k, \lambda \rangle = |-\lambda^{-2}k, \lambda^{-1}\rangle \]

thus the lattice should contain point \((q^{-n}(k - nhq^{-1}\lambda), q^{-n}k)\) together with the point \((-q^{-n}\lambda^{-2}(k - nhq^{-1}), q^n\lambda^{-1})\). It means that we have two kinds of irreducible representation. One is given by the choice \( \lambda_0 = 1, \ k_0 = 0 \) while the second one can be generated from \( \lambda_0 = q^{\frac{1}{2}}, \ k_0 = \frac{1}{2}q^{\frac{1}{2}}h \). They correspond to Bose and Fermi statistics respectively like in the non-deformed case.

If we now consider the motion given by the Hamiltonian (12) and demand that \( THT^{-1} = H \) we will see that \( \dot{V}(\Lambda) = \text{const.} \) and the equations of motion will take the form

\[ \dot{\Lambda} = 0, \quad \dot{x} = ix[-2q^{-1}AK^2p + hq^{-2}(AK)^2], \quad \dot{p} = 0. \]  

(30)

i.e. the additional term \( V \) does not contribute to the angular velocity of the particle.

If we now introduce new variables:

\[ P = qp\Lambda^{-1}, \quad X = x \]

(31)

then we obtain the canonical commutation relation \([X, P] = hX\) of the standard quantum mechanics on a circle. This is in a contrast to the case \( \varepsilon \neq q \), where such a reparametrization is not possible. Note however that even in the case \( q = \varepsilon \) the deformed theory can not be treated just as a standard quantum mechanics with non-commutative moment of inertia, since transformation (31) is not unitary.
6 Conclusions

We described the $q$-deformation of the quantum mechanics of a particle on a circle. To describe the unitary time evolution of the system we had to deform the algebra of observables leaving the Heisenberg equations of motion unchanged. We were able to reduce number of free parameters requiring the existence of classical limit and finally the symmetry of irreducible representations on the time-inversion. In this last case we found a non-unitary transformation of variables which allowed us to replace deformed canonical commutation relations by the standard ones. This possibility was the straightforward consequence of the additional symmetry (time-inversion) in a contrast to the situation described in [3].

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