IMPROVEMENT OF THE BERNSTEIN-TYPE THEOREM FOR SPACE-LIKE ZERO MEAN CURVATURE GRAPHS IN LORENTZ-MINKOWSKI SPACE USING FLUID MECHANICAL DUALITY

S. AKAMINE, M. UMEHARA AND K. YAMADA

Abstract. Calabi’s Bernstein-type theorem asserts that a zero mean curvature entire graph in Lorentz-Minkowski space $L^3$ which admits only space-like points is a space-like plane. Using the fluid mechanical duality between minimal surfaces in Euclidean 3-space $E^3$ and maximal surfaces in Lorentz-Minkowski space $L^3$, we give an improvement of this Bernstein-type theorem. More precisely, we show that a zero mean curvature entire graph in $L^3$ which does not admit time-like points (namely, a graph consists of only space-like and light-like points) is a plane.

1. Introduction

Consider a 2-dimensional barotropic steady flow on a simply connected domain $D$ in the $xy$-plane $R^2$ whose velocity vector field is $v = (u, v)$, with density $\rho$ and pressure $p$. We assume there are no external forces. Then

- the flow is a foliation of the integral curve of $v$,
- $\rho$ is a scalar field on $R^2$,
- $p: R \to R$ is a monotone function of $\rho$,
- $c := \sqrt{p'(\rho)}$ ($p' := dp/d\rho$) is called the local speed of sound.

The following Euler’s equation of motion holds

$$dp + \frac{\rho}{2} d(|v|^2) = 0. \tag{1.1}$$

We also assume the flow is irrotational, that is,

$$0 = \text{rot}(v) = v_x - u_y, \tag{1.2}$$

where $v_x := \partial v/\partial x$, $u_y := \partial u/\partial y$. Here, ‘the equation of continuity’ is equivalent to the fact that

$$0 = \text{div}(\rho v) = (\rho u)_x + (\rho v)_y. \tag{1.3}$$

By (1.2), there exists a function $\Phi: D \to R$, called the potential of the flow, such that $\nabla \Phi = v$, where $\nabla \Phi := (\Phi_x, \Phi_y)$. Since $p$ is a function of $\rho$, the fact $c^2 = p'(\rho)$ and (1.1) yield that

$$\rho_x = -\frac{\rho(uu_x + vv_x)}{c^2}, \quad \rho_y = -\frac{\rho(uu_y + vv_y)}{c^2}. \tag{1.4}$$

By (1.3), one can easily check that

$$0 = (c^2 - \Phi_x^2)\Phi_{xx} - 2\Phi_x \Phi_y \Phi_{xy} + (c^2 - \Phi_y^2)\Phi_{yy}. \tag{1.5}$$

Date: June 25, 2019.

Umehara was partially supported by the Grant-in-Aid for Scientific Research (A) No. 26247005, and Yamada by (B) No. 17H02839 from Japan Society for the Promotion of Science.
On the other hand, by (1.3), there exists a function \( \Psi: D \to \mathbb{R} \), called the stream function of the flow, such that

\[
\Psi_x = -\rho v, \quad \Psi_y = \rho u.
\]

If we set \( \xi := \rho u \) and \( \eta := \rho v \), (1.3) can be written as

\[
(\rho^2 c^2 - \xi^2 - \eta^2)(\rho_x, \rho_y) = -\rho(\xi \xi_x + \eta \xi_y, \xi \xi_y + \eta \eta_y).
\]

Since

\[
0 = v_x - u_y = \frac{\eta_x}{\rho} - \frac{\xi_y}{\rho} = \frac{\eta \rho_x}{\rho^2} + \frac{\xi \rho_y}{\rho^2},
\]

the identity \( 0 = \rho(\xi^2 + \eta^2 - \rho^2 c^2)(v_x - u_y) \) yields that

\[
(\rho^2 c^2 - \xi^2 - \eta^2)(\rho_x, \rho_y) = -\rho(\xi \xi_x + \eta \xi_y, \xi \xi_y + \eta \eta_y).
\]

A flow satisfying

\[
\rho c = 1
\]

is called a Chaplygin gas flow. For a given stream function \( \Psi: D \to \mathbb{R} \) of the Chaplygin gas flow, we set

\[
B_\Psi := 1 - \Psi_x^2 - \Psi_y^2.
\]

In this paper, for the sake of simplicity, we abbreviate ‘zero mean curvature’ by ‘ZMC’. Under the condition (1.8), the equation (1.7) for the stream function \( \Psi \) reduces to

\[
(1 - \Psi_y^2)\Psi_{xx} + 2 \Psi_x \Psi_y \Psi_{xy} + (1 - \Psi_x^2)\Psi_{yy} = 0.
\]

We call this the ZMC-equation in Lorentz-Minkowski 3-space \( L^3 \) of signature (+ + −). In fact, at the point where \( B_\Psi \neq 0 \) (cf. (1.9)) the mean curvature function \( H \) of the graph \( t = \Psi(x, y) \) is well-defined, where \((x, y, t)\) are the canonical coordinates of \( L^3 \). Then (1.10) is equivalent to the condition that \( H = 0 \).

**Definition 1.1.** A surface in \( L^3 \) whose image can be locally expressed as the graph of a certain \( \Psi \) satisfying (1.10) after a suitable motion in \( L^3 \) is called a ZMC-surface. A point where \( B_\Psi > 0 \) (resp. \( B_\Psi < 0 \), \( B_\Psi = 0 \)) is said to be space-like (resp. time-like, light-like).

A ZMC-surface consisting only of space-like points is called a maximal surface. On the other hand, a surface in \( L^3 \) consisting only of light-like points is called a light-like surface.

It is known that the identity \( B_\Psi = 0 \) implies (1.10) (see [21 Proposition 2.1]). In particular, any light-like surfaces are ZMC-surfaces in our sense.

If \( \rho c = 1 \), then we have \( 1/\rho^2 = c^2 = dp/d\rho \), that is, \( dp = d\rho/\rho^2 \) is obtained. Substituting this into (1.4), we get \( d(|v|^2 - 1/\rho^2) = 0 \) and so there exists a constant \( \mu \) such that

\[
|v|^2 + \mu = \frac{1}{\rho^2}(= c^2).
\]

By (1.6), we can rewrite this as

\[
B_\Psi = \mu \rho^2.
\]

By (1.11) and (1.12), the sign change of \( B_\Psi \) corresponds to the type change of the Chaplygin gas flow from sub-sonic (\(|v| < c\)) to super-sonic (\(|v| > c\)), that is, the sub-sonic part satisfies \( B_\Psi > 0 \). If \( \mu = 0 \), then \( B_\Psi \) vanishes identically, and the graph of \( \Psi \) gives a light-like surface. Such surfaces are discussed in the appendix,
and we now consider the case $\mu \neq 0$. Since $B_\Psi$ and $\mu$ have the same sign (cf. (1.12)), we can write

$$\rho = \frac{1}{\sqrt{|v|^2 + \mu}} = \sqrt{\frac{1 - \Psi_x^2 - \Psi_y^2}{\mu}}.$$  

By (1.11) and the fact $|v|^2 = \Phi_x^2 + \Phi_y^2$, (1.13) can be written as

$$(\mu + \Phi_y^2)\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + (\mu + \Phi_x^2)\Phi_{yy} = 0.$$  

We set

$$(1.15) \quad \varphi(x, y) := \tilde{\mu}\Phi(\tilde{\mu}x, \tilde{\mu}y) \quad (\tilde{\mu} := 1/\sqrt{|\mu|}).$$

If $\mu > 0$, then (1.14) reduces to

$$(1.16) \quad (1 + \varphi_y^2)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + (1 + \varphi_x^2)\varphi_{yy} = 0,$$

which is known as the condition that the graph of $\varphi(x, y)$ gives a minimal surface in the Euclidean 3-space $E^3$. On the other hand, if $\mu < 0$, then (1.14) reduces to

$$(1.17) \quad (1 - \varphi_y^2)\varphi_{xx} + 2\varphi_x\varphi_y\varphi_{xy} + (1 - \varphi_x^2)\varphi_{yy} = 0,$$

which is the ZMC-equation (cf. (1.10)). It can be easily checked that the graph of $\varphi$ is a time-like ZMC-surface in $L^3$. In both of the two cases, it can be easily checked that ($\epsilon := \text{sign}(\mu) \in \{1, -1\}$)

$$(\psi_x \psi_y) = \frac{1}{\sqrt{\varphi_x^2 + \varphi_y^2 + \epsilon}} (-\varphi_y) \quad \text{holds, where } \psi := (\tilde{\mu}\Phi(\tilde{\mu}x, \tilde{\mu}y)/\tilde{\mu}. \quad \text{Note that } \Psi \text{ satisfies (1.10) if and only if } \psi \text{ satisfies (1.10).}$$

Moreover, one can easily check that

$$(1.18) \quad (\hat{\rho} :=) \frac{1}{\sqrt{\varphi_x^2 + \varphi_y^2 + \epsilon}} = \sqrt{\epsilon(1 - \psi_x^2 - \psi_y^2)}.$$  

This means that $\varphi \leftrightarrow \psi$ corresponds to the duality between potentials and stream functions of Chaplygin gas flow such that

- $\mu = \pm 1(= \epsilon)$,
- the density $\hat{\rho}$ is given as (1.18), and
- $p = p_0 - 1/\hat{\rho}$ for some constant $p_0$.

This gives a correspondence between graphs of minimal surfaces in $E^3$ and graphs of maximal surfaces in $L^3$ (resp. an involution on the set of graphs of time-like ZMC-surfaces in $L^3$) which we call the fluid mechanical duality.

A part of the above dualities is suggested in the classical book [4]. Calabi [5] also recognized this duality for $\mu > 0$, and pointed out the following:

**Fact 1.2** (Calabi’s Bernstein-type theorem). *Suppose that the graph of a function $\psi: R^2 \to R$ gives a maximal surface (that is, a surface consisting only of space-like points whose mean curvature function vanishes identically). Then $\psi - \psi(0, 0)$ is linear.*

This is an analogue of the classical Bernstein theorem for minimal surfaces in $E^3$. Moreover, Calabi [5] obtained the same conclusion for entire space-like ZMC-graphs in $L^{n+1}$ ($n \leq 4$), and Cheng and Yau [6] extended this result for complete maximal hypersurfaces in $L^{n+1}$ for $n \geq 5$. The assumption that the graph consists only of space-like points is crucial. Entire ZMC-graphs which are not planar actually exist. Typical such examples are of the form

$$(1.19) \quad \psi_0(x, y) := y + g(x),$$
where \( g: \mathbb{R} \to \mathbb{R} \) is a \( C^\infty \)-function of one variable. A point \( p = (x_0, y_0) \in \mathbb{R}^2 \) is a light-like point of \( \psi_0 \) if and only if \( g'(x_0) = 0 \). However, \( \psi_0 \) does not contain any space-like points. The potential function \( \varphi_0 \) corresponding to \( \psi_0 \) is given by

\[
\varphi_0(x, y) = y - \int_0^x \frac{du}{g'(u)}
\]

up to a constant. On the other hand, Osamu Kobayashi [13] pointed out the existence of entire graphs of ZMC-surfaces with space-like points, light-like points and time-like points all appearing. Such a surface is called of mixed type. Recently, many such examples are constructed in [9].

By definition, any entire ZMC-graph of mixed type has at least one light-like point. So we give the following definition:

**Definition 1.3.** A light-like point \( p \) of the function \( \psi \) (i.e. \( B_\psi(p) = 0 \)) is said to be non-degenerate (resp. degenerate) if \( \nabla B_\psi \) does not vanish (resp. vanishes) at \( p \).

At each non-degenerate light-like point, the graph of \( \psi \) changes its causal type from space-like to time-like. This case is now well-understood. In fact, under the assumption that the surface is real analytic, it can be re-constructed from a real analytic null regular curve in \( L^3 \) (cf. Gu [12] and also [11, 17, 16]).

On the other hand, there are several examples of ZMC-surfaces with degenerate light-like points (cf. [1, 2, 10, 14]). Moreover, a local general existence theorem for maximal surfaces with degenerate light-like points is given in [21]. For such degenerate light-like points, we need a new approach to analyze the behavior of \( \psi \) and \( \varphi \). The following fact was proved by Klyachin [17] (see also [21]).

**Fact 1.4 (The line theorem for ZMC-surfaces).** Let \( D \) be a domain of \( \mathbb{R}^2 \) and \( F: D \to L^3 \) a \( C^3 \)-differentiable ZMC-immersion such that \( o \in D \) is a degenerate light-like point. Then, there exists a light-like line segment \( \hat{\sigma} \) passing through \( F(o) \) of \( L^3 \) such that \( o \) does not coincide with one of the two end points of \( \hat{\sigma} \) and \( F(\Sigma) \) contains \( \hat{\sigma} \), where \( \Sigma \) is the set of degenerate light-like points of \( F \).

Recently, Fact 1.4 was generalized to a much wider class of surfaces, including constant mean curvature surfaces in \( \mathbb{E}^3 \), see [21, 22]. (In [21], the general local existence theorem for surfaces which changes their causal types along degenerate light-like lines was also shown.) The asymptotic behavior of \( \psi \) along the line \( l \) consisting of degenerate light-like points is discussed in [21].

The purpose of this paper is to prove the following assertion:

**Theorem A.** An entire \( C^3 \)-differentiable ZMC-graph which is not a plane admits a non-degenerate light-like point if its space-like part is non-empty.

This assertion is proved in Section 2 using the fluid mechanical duality and the half-space theorem for minimal surfaces in \( \mathbb{E}^3 \) given by Hoffman-Meeks [15]. It should be remarked that the half-space theorem does not hold for time-like ZMC-surfaces. In fact, the graph of \( \varphi(x, y) := y + \log(\tan x) \) \((x \in (0, \pi/2))\) gives a properly embedded time-like ZMC-surface lying between two parallel vertical planes. In Section 2, we give further examples, and provide a few questions related to Theorem A. As an application, we give the following improvement of Calabi’s Bernstein-type theorem:

**Corollary B.** An entire \( C^3 \)-differentiable ZMC-graph which does not admit any time-like points is a plane.

In fact, if the ZMC-graph admits a space-like point, then the assertion immediately follows from Theorem A. So it remains to show the case that the graph consists only of light-like points. However, such a graph must be a plane, as shown in the appendix (see Theorem A.1).
2. Proof of Theorem A

In this section, we prove Theorem A in the introduction. We let \( \psi : \mathbb{R}^2 \to \mathbb{R} \) be a \( C^2 \)-function satisfying the ZMC-equation \([1,10]\). We assume \( \psi \) admits a space-like point \( q_0 \in \mathbb{R}^2 \), but admits no non-degenerate light-like points. By Calabi’s Bernstein-type theorem (cf. Fact \([1,2]\), \( \psi \) has at least one degenerate light-like point. We set

\[
F_\psi(x,y) := (x, y, \psi(x,y)),
\]

which gives the ZMC-graph of \( \psi \). We denote by \( ds^2 \) the positive semi-definite metric which is the pull-back of the canonical Lorentzian metric of \( L^3 \) by \( F_\psi \). The line theorem (cf. Fact \([1,3]\)) yields that the image of \( F_\psi \) contains a light-like line segment \( \hat{\sigma} \). Then the projection of \( \hat{\sigma} \) is a line segment \( \sigma \) on the \( xy \)-plane \( \mathbb{R}^2 \). Then \( \sigma \) lies on a line \( l \) on \( \mathbb{R}^2 \). If \( \sigma \neq l \), then there exists an end point \( p \) of \( \sigma \) on \( l \). Since \( p \) is the limit point of degenerate light-like points, \( p \) itself is also a degenerate light-like point. By applying the line theorem, there exists a light-like line segment \( \sigma' \) containing \( F_\psi(p) \) as its interior point. We denote by \( \sigma' \) the projection of \( \hat{\sigma}' \) to the \( xy \)-plane. Since the null direction at \( p \) with respect to the metric \( ds^2 \) is uniquely determined, \( \sigma' \) also lies on the line \( l \). Thus, the entire graph contains a whole light-like line containing \( \hat{\sigma} \). In particular, degenerate light-like points on the graph consist of a family of straight lines in \( \mathbb{R}^2 \).

Let \( l \) and \( l' \) are two such straight lines. Then \( l' \) never meets \( l \). In fact, if not, then there is a unique intersection point \( q \in l \cap l' \). By Fact \([1,4]\), two lines \( l, l' \) can be lifted to two light-like lines \( \hat{l} \) and \( \hat{l}' \) in \( L^3 \) passing through \( F_\psi(q) \). The tangential directions of \( \hat{l} \) and \( \hat{l}' \) are linearly independent light-like vectors at \( F_\psi(q) \). Then by \([19]\) Lemma 27 in \( \text{Section 5} \), \( p \) is a time-like point, a contradiction.

Thus, the set of degenerate light-like points of \( F_\psi \) consists of a family of parallel lines in the \( xy \)-plane. Without loss of generality, we may assume that these lines are vertical and one of them is the \( y \)-axis. Then we can find a domain \( (\Delta \in (0, \infty]) \)

\[
\Omega := \{(x,y) : 0 < x < 2\Delta\}
\]

such that \( q_0 \in \Omega \) and \( F_\psi \) has no light-like points on \( \Omega \) and both of the lines \( l = \{x = 0\} \) and \( l' = \{x = 2\Delta\} \) consist of light-like points unless \( \Delta = \infty \). Since there are no light-like points on \( \Omega \), the potential function \( \varphi : \Omega \to \mathbb{R} \) is induced by \( \psi \) as the fluid mechanical dual. The graph of \( \varphi \) is a minimal surface in \( E^3 \). In particular, \( \varphi \) is real analytic. If we succeed to prove that the map \( F_\varphi(x,y) := (x, y, \varphi(x,y)) \) is proper, then Theorem A follows. In fact, by the half-space theorem given in \([15]\) the image \( F_\varphi(\Omega) \) lies in a plane in \( E^3 \). Then the map \( F_\varphi(x,y) \) also lies in a plane \( \Pi \) in \( L^3 \) on \( \mathbb{T} \). Since \( F_\varphi(l) \) is light-like, the plane \( \Pi \) must be light-like, contradicting the the fact \( q_0 \in \Omega \).

To prove the properness of \( F_\varphi \), it is sufficient to show the following:

**Lemma 2.1.** Let \( \{p_n\}_{n=1}^{\infty} \) be a sequence of points in \( \Omega \) accumulating to a point on \( l \) or \( l' \). Then \( \{|\varphi(p_n)|\}_{n=1}^{\infty} \) diverges.

**Proof.** By switching the roles of \( l \) and \( l' \) if necessary, it is sufficient to consider the case that \( \{p_n\}_{n=1}^{\infty} \) accumulates to a point on \( l \). Taking a subsequence and using a suitable translation of the \( xy \)-plane, we may assume that \( \{p_n\}_{n=1}^{\infty} \) converges to the origin \((0,0) \in \mathbb{L} \) and \( p_n = (x_n, y_n) \) \((n = 1, 2, 3, \ldots) \) satisfy the following properties:

- there exists \( \varepsilon > 0 \) such that \( |y_n| < \varepsilon \) for each \( n = 1, 2, \ldots \) and
- there exists \( (\delta,0) \in \Omega \) \((\delta > 0) \) such that

\[
\delta > x_1 > x_2 > \cdots > x_n > x_{n+1} > \cdots.
\]
Since \( l \) consists of degenerate light-like points, there exists a neighborhood \( U \) of \((0,0)\) such that (see [10] or [21 (6.1)])
\[
\psi(x,y) = y + x^2 h(x,y) \quad ((x,y) \in U),
\]
where \( h(x,y) \) is a \( C^1 \)-differentiable function defined on \( U \) (see [21 Appendix A]).

Taking \( \epsilon, \delta \) where \( B \)
\[
\text{where } k \text{ is non-negative on the closure } (0,0) \text{ such that } (\text{see } [10] \text{ or } [21, (6.1)])
\]

\[V := \{(x,y) \in \Omega : |x| \leq \delta, |y| < \epsilon \} \subset U.\]

Since \( B_\psi > 0 \), the potential function \( \varphi \) associated to \( \psi \) satisfies (cf. (1.15))
\[
\varphi_x = \frac{\psi_y}{\rho}, \quad \rho = \sqrt{1 - \psi_x^2 - \psi_y^2}.
\]

Since
\[
1 - \psi_x^2 - \psi_y^2 = -x^2 \left( (2h + xh_x)^2 + 2h_y + x^2 h_y^2 \right)
\]
is non-negative on the closure \( \bar{V} \) of \( V \), we can write
\[
(2.1)
\]
where \( k(x,y) \) is a non-negative continuous function defined on \( \bar{V} \) such that \( k \) is positive-valued on \( V \).

We set \( p_0 := (\delta,0) \), and consider the path \( \gamma_n : [0,1] \to V \) defined by \( \gamma_n(s) := (\delta,2sy_n) \) if \( 0 \leq s \leq 1/2 \) and
\[
\gamma_n(s) := (2(x_n - \delta)s - x_n + 2\delta, y_n)
\]
if \( 1/2 \leq s \leq 1 \), which starts at \( p_0 \) and terminates at \( p_n \). This curve \( \gamma_n \) is the union of the vertical subarc \( \gamma_{n,1} \) and the horizontal subarc \( \gamma_{n,2} \). So we can write
\[
\varphi(p_n) - \varphi(p_0) = \int_{\gamma_n} \varphi_x dx + \varphi_y dy
\]
\[
= \int_{\gamma_{n,2}} \varphi_x dx + \int_{\gamma_{n,1}} \varphi_y dy.
\]

Since \([-\epsilon, \epsilon] \ni y \mapsto \varphi_y(\delta, y) \in \mathbb{R} \) is a continuous function, we have that
\[
\left| \int_{\gamma_{n,1}} \varphi_y dy \right| \leq \int_{\gamma_{n,1}} \left| \varphi_y(\delta, 2t y_n) \right| |dy|
\]
\[
\leq \epsilon \max_{|y| \leq \epsilon} \left| \varphi_y(\delta, y) \right| < \infty.
\]

So to prove the lemma, it is sufficient to show that \( \int_{\gamma_{n,2}} \varphi_x dx \) diverges as \( n \to \infty \).

We set
\[
m := \max_{x \in [0,\delta], |y| \leq \epsilon} k(x,y) \quad (\geq 0),
\]
where \( k \) is the continuous function given in \( (2.1) \). On the other hand, we can take a constant \( m'(>0) \) such that
\[
\psi_y = 1 + x^2 h_y(x,y) > m' \quad (x \in [0,\delta], |y| \leq \epsilon),
\]
since \( \epsilon, \delta \) can be chosen to be sufficiently small. Since \( \varphi_x = \psi_y/\rho \), we have
\[
\left| \int_{\gamma_{n,2}} \varphi_x dx \right| = \int_{x_n}^{\delta} \frac{1 + x^2 h_y(x,y)}{x^2 k^2(x,y)} dx
\]
\[
> \frac{m'}{m} \int_{x_n}^{\delta} \frac{dx}{x^2} = \frac{m'}{m} \left( \frac{1}{x_n} - \frac{1}{\delta} \right) \to \infty
\]
proving the assertion. \( \square \)
Remark 2.2. In the above proof, we showed that $F_\psi(\Omega)$ lies in a plane using the fluid mechanical duality. We remark here that this can be proved by a different method. In fact, $\psi$ satisfies the assumption of Ecker [7, Theorem G] or is an $\text{PS}$-graph on the convex domain $\Omega$ in the sense of Fernandez and Lopez [8]. Thus, we can conclude that $\psi(\Omega)$ lies in a light-like plane.

Figure 1. The ZMC-surfaces in Example 2.3 (left) and in Example 2.4 (right), where the white lines indicate light-like points.

In [1], the first author constructed several ZMC-surfaces foliated by circles and at most countably many straight lines. At the end of this paper, we pick up two important examples of them which contain degenerate light-like points. (In [1], these two examples are not precisely indicated. Here we show their explicit parametrization and embeddedness.)

Example 2.3 ([1, Figure 5]). We set
\[ F(u, v) := (u + a \cos v, a \sin v, u), \]
where $a > 0$ and $(u, v) \in \mathbb{R} \times [0, 2\pi)$. Then the image of $F$ contains two parallel degenerate light-like lines which correspond to the special values $\theta = \pm \pi/2$ (see Figure 1 left). The image of $F$ can be characterized by the implicit function $(x - t)^2 + y^2 = a^2$. This ZMC-surface is properly embedded and is not simply connected.

Example 2.4 ([1, Figure 2]). We set
\[ F(r, \theta) := \left( r + \frac{1}{2a} \log \left( \frac{ar - 1}{ar + 1} \right), r \cos \theta, r \sin \theta, \frac{1}{2a} \log \left( \frac{ar - 1}{ar + 1} \right) \right), \]
where $a > 0$ and $\theta \in [0, 2\pi)$. This map is defined for $r > 1/a$ or $r < -1/a$, and the closure of the image of $F = (x, y, t)$ can be expressed as
\[ \Psi := a \sinh(at) \left( (x - t)^2 + y^2 \right) + 2(x - t) \cosh(at) = 0. \]
It can be checked that $(\Psi_x, \Psi_y, \Psi_t)$ never vanishes along $\Psi = 0$. So the closure of $F$ gives a properly embedded ZMC-surface in $L^3$ (see Figure 1 right).

Regarding our main result, we state a few open problems:

(Question 1.) Does a properly embedded ZMC-surface which consists only of space-like or light-like points coincide with a plane?

If this question is affirmative, then Corollary B follows as a corollary. Suppose that we can find such a non-planar ZMC-surface $S$, it must contain a light-like line. In fact, if $S$ consists only of space-like points, then $S$ is complete, and such a surface must be a plane (see [20, Remark 1.2]). So $S$ has a light-like point $p$. If $p$ is non-degenerate, then $S$ has a time-like point near $p$, so $p$ must be degenerate. By the line theorem (Fact 1.4), $S$ must contain a light-like line consisting of degenerate light-like points.
Acknowledgement. The authors would like to express their gratitude to Atsufumi Honda for fruitful discussions.

(Assumption 2.) Are there entire ZMC-graphs of mixed type containing degenerate light-like points?

This question needs to consider ZMC-graphs of mixed type. In fact, if we choose a function \( g(x) \) satisfying \( g'(0) = 0 \) as in [19], then the \( y \)-axis consists of the degenerate light-like points. If we weaken 'entire ZMC-graphs' to 'properly embedded ZMC-surfaces of mixed type' the answer is 'yes'. In fact, Example 2.4 gives a properly embedded ZMC-surface of mixed type which contains a degenerate light-like line \( L \). Although the space-like points never accumulate to \( L \) in the case of this example, one can show the existence of a function \( \psi : U \rightarrow R \) defined on a domain \( U \) in \( R^2 \) containing the \( y \)-axis such that

- the \( y \)-axis corresponds to a degenerate light-like line,
- \( \psi \) is of mixed type, or consisting only of space-like points except along the \( y \)-axis.

See [3] for details. Also, the following question arises:

(Question 3.) Are there entire ZMC-graphs of mixed type which are not obtained as analytic extensions of Kobayashi surfaces given as in [9]?

In fact, all known examples of entire ZMC-graphs of mixed type are obtained as analytic extensions of Kobayashi surfaces (cf. [9]), and they admit only non-degenerate light-like points.

APPENDIX A. A PROPERTY OF LIGHT-LIKE SURFACES IN \( L^3 \)

It can be easily checked that an embedded surface \( S(\subset L^3) \) is light-like if and only if the restriction of the canonical Lorentzian metric on \( L^3 \) to the tangent space \( T_p S \) of each \( p \in S \) is positive semi-definite but not positive definite. The purpose of this appendix is to prove the following:

Theorem A.1. If an entire \( C^2 \)-differentiable graph of \( \psi : R^2 \rightarrow R \) gives a light-like surface in \( L^3 \), then \( \psi - \psi(0,0) \) is a linear function.

We set \( F(x, y) = (x, y, \psi(x, y)) \). Since \( F \) is a light-like surface, \( \psi_x^2 + \psi_y^2 = 1 \) holds on \( R^2 \). Differentiating this with respect to \( x \) and \( y \), we get two equations. Since \( F \) is light-like, \( (\psi_x, \psi_y) \) \( \neq (0,0) \). By thinking \( \psi_x, \psi_y \) are unknown variables of these two equations, the determinant \( \psi_{xx} \psi_{yy} - \psi^2_{xy} \) vanishes identically. So the Gaussian curvature of \( F \) with respect to the Euclidean metric of \( R^3 \) vanishes identically. Then, by the Hartman-Nirenberg cylinder theorem, \( F \) must be a cylinder. (The proof of the cylinder theorem in [13] needs only \( C^2 \)-differentiability). That is, there exist a non-zero vector \( a \), a plane \( \Pi \) which is not parallel to \( a \) and a regular curve \( \gamma : R \rightarrow \Pi \) such that \( F(u, v) := \gamma(u) + va \) gives a new parametrization of \( F \). If \( F \) is not a plane, there exists \( u_0 \in R \) such that \( \gamma'(u_0) \) and \( \gamma''(u_0) \) are linearly independent. Then the point \( (u, v) = (u_0, 0) \) is not an umbilical point of \( F \). Since the asymptotic direction is uniquely determined at each non-umbilical point on a flat surface, the line theorem (cf. Fact [24]) yields that \( a \) is a light-like vector. By a suitable homothetic transformation and an isometric motion in \( L^3 \), we may set \( a := (1, 0, 1) \). Then it holds that

\[
0 = \gamma' \cdot a = x' - t'.
\]

Since \( \gamma' \cdot \gamma'' = 0 \), we have \( y' = 0 \). So, with out loss of generality, we may assume \( y(u) = 0 \). Differentiating (A.1), we have \( x'' - t'' = 0 \), contradicting the fact that \( \gamma'(u_0) \) and \( \gamma''(u_0) \) are linearly independent. Thus \( F \) is a plane. \( \square \)

Acknowledgement. The authors would like to express their gratitude to Atsufumi Honda for fruitful discussions.
References

[1] S. Akamine, Causal characters of zero mean curvature surfaces of Riemann type in Lorentz-Minkowski 3-space, Kyushu J. Math., 71 (2017), 211-249.

[2] S. Akamine and R.K. Singh, Wick rotations of solutions to the minimal surface equation, the zero mean curvature equation and the Born-Infeld equation, to appear in Proc. Indian Acad. Sci. Math. Sci.

[3] S. Akamine, M. Umehara and K. Yamada, Space-like zero mean curvature surface containing a complete degenerate light-like line in the Lorentz-Minkowski 3-space, preprint.

[4] L. Bers, Mathematical aspects of subsonic and transonic gas dynamics, John Wiley & Sons, 1958.

[5] E. Calabi, Examples of Bernstein problems for some nonlinear equations in Global Analysis, (Proc. Sympos. Pure Math., Vol. XV, Berkeley, CA, 1968), Amer. Math. Soc., Providence, RI, 1970, 223–230.

[6] S. Y. Cheng and S. T. Yau, Maximal space-like hypersurfaces in the Lorentz-Minkowski spaces, Ann. Math. 104 (1976) 407–419.

[7] K. Ecker, Area minimizing hypersurfaces in Minkowski space, Manuscripta Math. 56 (1986), 375–397.

[8] I. Fernandez and F. J. Lopez, On the uniqueness of the helicoid and Enneper’s surface in the Lorentz-Minkowski space $R^3_1$, Trans. Amer. Math. Soc. 363 (2011), 4693–4650.

[9] S. Fujimori, Y. Kawakami, M. Kokubu, W. Rossman, M. Umehara, K. Yamada, Entire zero mean curvature graphs of mixed type in Lorentz-Minkowski 3-space, The Quarterly J. Math. 67 (2016), 801–837.

[10] S. Fujimori, Y.-W. Kim, S.-E. Koh, W. Rossman, H. Shin, H. Takahashi, M. Umehara, K. Yamada and S.-D. Yang, Zero mean curvature surfaces in $L^3$ containing a light-like line, C.R. Acad. Sci. Paris. Ser. I. 350 (2012), 975–978.

[11] S. Fujimori, Y. W. Kim, S.-E. Koh, W. Rossman, H. Shin, M. Umehara, K. Yamada and S.-D. Yang, Zero mean curvature surfaces in Lorentz-Minkowski 3-space and 2-dimensional fluid mechanics, Math. J. Okayama Univ. 57 (2015), 173–200.

[12] C.H. Gu, The extremal surfaces in the 3-dimensional Minkowski space, Acta Math. Sinica (N.S.) 1 (1985), 173–180.

[13] P. Hartman, and L. Nirenberg, On spherical image whose Jacobians do not change sign, Amer. J. Math. 81 (1959), 901–920.

[14] K. Hashimoto and S. Kato, Bicomplex extensions of zero mean curvature surfaces in $R^{2,1}$ and $R^{2,2}$, J. Geom. Phys. 138 (2019), 223–240.

[15] D. Hoffman and W. Meeks, The strong half-space theorem for minimal surfaces, Invent. Math. 101 (1990), 373–377.

[16] Y. W. Kim, S.-E. Koh, H. Shin and S.-D. Yang, Spacelike maximal surfaces, timelike minimal surfaces, and Björling representation formulae, J. Korean Math. Soc. 48 (2011), 1083–1100.

[17] V. A. Klyachin, Zero mean curvature surfaces of mixed type in Minkowski space, Izv. Math. 67 (2003), 209–224.

[18] O. Kobayashi, Maximal surfaces in the 3-dimensional Minkowski space $L^3$, Tokyo J. Math., 6 (1983), 297–309.

[19] B. O. Neill, Semi-Riemannian Geometry, Academic Press 1983, USA.

[20] M. Umehara and K. Yamada, Maximal surfaces with singularities in Minkowski space, Hokkaido Math. J. 35 (2006), 13–40.

[21] M. Umehara and K. Yamada, Surfaces with light-like points in Lorentz-Minkowski space with applications, in “Lorentzian Geometry and Related Topics”, Springer Proceedings of Mathematics & Statics 211, 253-273, 2017.

[22] M. Umehara and K. Yamada, Hypersurfaces with light-like points in a Lorentzian manifold, to appear in J. Geom. Anal. (arXiv:1806.09233)

(Shintaro Akamine) Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, JAPAN
E-mail address: s-akamine@math.nagoya-u.ac.jp

(Masaaki Umehara) Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo 152-8552, JAPAN
E-mail address: umehara@is.titech.ac.jp

(Kotaro Yamada) Department of Mathematics, Tokyo Institute of Technology, Tokyo 152-8551, JAPAN
E-mail address: kotaro@math.titech.ac.jp