COHOMOLOGICAL INVARIANTS OF A VARIATION OF FLAT CONNECTIONS

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ABSTRACT. In this paper, we apply the theory of Chern-Cheeger-Simons to construct canonical invariants associated to a $r$-simplex whose points parametrize flat connections on a smooth manifold $X$. These invariants lie in degrees $(2p-r-1)$-cohomology with $\mathbb{C}/\mathbb{Z}$-coefficients, for $p > r \geq 1$. This corresponds to a homomorphism on the higher homology groups of the moduli space of flat connections, and taking values in $\mathbb{C}/\mathbb{Z}$-cohomology of the underlying smooth manifold $X$.

CONTENTS

1. Introduction 1
2. Preliminaries: original Chern-Cheeger-Simons theory 4
3. Pushforward of Chern-Simons forms 6
4. (Co)Homological invariants of the simplicial set of flat connections 8
5. Cohomological invariants of moduli space of flat connections 12
6. Questions on rationality of (higher) Cheeger-Simons invariants 13
References 14

1. Introduction

Suppose $X$ is a smooth manifold and $E$ is a smooth complex vector bundle on $X$ of rank $n$. Consider a smooth connection $\nabla$ on $E$. The Chern-Weil theory defines the Chern forms $P_p(\nabla^2, \ldots, \nabla^2)$, of degree $2p$ and for $p \geq 1$. Here $P_p$ denotes a $GL_n$-invariant polynomial of degree $p$ such that $P_p(\nabla^2, \ldots, \nabla^2) = tr(\nabla^2)$ and $\nabla^2$ denotes the curvature form of the connection form $\nabla$ (see §2.1). The Chern forms are closed and correspond to the de Rham Chern classes $c_p(E) \in H^{2p}_{dR}(X, \mathbb{C})$. These are the primary invariants of $E$, and are independent of the connection. In particular when $\nabla$ is flat, i.e., when $\nabla^2 = 0$, the Chern-Cheeger-Simons theory [Ch-Sm], [Cg-Sm] defines cohomology classes

$$\hat{c}_p(E, \nabla) \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z}(p)),$$

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for $p \geq 1$. These are the secondary invariants of $(E, \nabla)$. These invariants are known to be rigid in a family of flat connections, when $p \geq 2$ \cite{Cg-Sm, Proposition 2.9, p.61}. In other words, the cohomology class remains the same in a variation of flat connections.

The proof of the rigidity is obtained by looking at the eta-differential form:

$$\eta_p := p \int \nabla^1_t, \ldots, \nabla^2_t).dt.$$

The difference of the Chern-Simons differential character $\hat{c}_p(E, \nabla_1) - \hat{c}_p(E, \nabla_0)$ is just the linear functional defined by the degree $(2p - 1)$-form $\eta_p$ (see \cite{Cg-Sm, Proposition 2.9, p.61}). This form also gives the rigidity of the Chern-Simons classes for flat connections, in degrees at least two.

M. Kontsevich asked whether the eta-form is related to certain lift inings in Chow theory. Later, C. Simpson remarked that, there should be maps given on the higher homotopy groups of the moduli space $M_{X(n)}$ of flat connections.

We investigate these remarks and thoughts in this paper. We are motivated by Karoubi’s simplicial invariants in \cite{Kb}, p.68, where he considers sequences of smooth connections on a smooth vector bundle. His construction provides maps on the homology groups of moduli space of connections and taking values in the cohomology of the underlying manifold (see §4.1). The maps are provided by transgression of Chern-Weil forms and using integration along fibres of differential forms.

We would like to extend his methods, to obtain suitable maps on homology of moduli space of flat connections, by applying the Chern-Simons theory and integration along fibres.

Integration along fibres on differential cohomology have been defined by several authors, see \cite{Fr, Ho-Sg}. In our situation, we need to apply for the projection $X \times \Delta^r \rightarrow X$, $r \geq 1$. However $X \times \Delta^r$ is a manifold with boundary and corners. C. Baer and C. Becker (see \cite{Ba-Be}) have defined 'integration along fibres' on differential cohomology for manifolds with boundary and C. Becker has adapted their proof to the projection $X \times \Delta^r \rightarrow X$. These constructions however seem inadequate, for our purpose. Hence we restrict the range on degrees where the pushforward on odd-degree $\mathbb{C}/\mathbb{Z}$-cohomology could be applied. We make this precise as follows.

In §4.2, we first consider the simplicial set $\mathcal{D}(E)$ of sequences of flat connections on a vector bundle $E$ on $X$. Using methods similar to \cite{Kb, §3}, we deduce the following.

**Theorem 1.1.** There is a non-zero homomorphism on the homology of $\mathcal{D}(E)$, for $r \geq 1$:

$$\rho_r : \mathbb{H}_r(\mathcal{D}(E)) \rightarrow \bigoplus_{p > r} \hat{H}^{2p - r}(X).$$

When the points of a $r$-simplex parametrize flat connections, then the above construction yields a well-defined Chern-Simons class, see Proposition 4.3. We apply the
pushforward map on the $\mathbb{C}/\mathbb{Z}$-cohomology of $X \times \Delta^r$, to obtain well-defined classes,

$$CS(\tilde{D})_{p,r} \in H^{2p-r-1}(X, \mathbb{C}/\mathbb{Z}(p)),$$

when $r < p$.

In turn, the homomorphism in (2) restricts on the homology groups of moduli space of flat connection, and takes value in $\mathbb{C}/\mathbb{Z}$-cohomology. More precisely, we have

**Theorem 1.2.** There is a non-zero homomorphism, for $r \geq 1$:

$$\rho'_r : H_r(M_X(n)) \to \bigoplus_{p > r} H^{2p-r-1}(X, \mathbb{C}/\mathbb{Z}(p)).$$

We briefly explain the construction of the maps $\rho_r$ and $\rho'_r$.

We consider a smooth connection $\tilde{D} := t_0D_0 + \ldots + t_rD_r$, with $\sum t_i = 1$ and $t_i \geq 0$, on $q^*E$. Here $q : X \times \Delta^r \to X$ is the first projection and $\Delta^r$ is a $r$-simplex with vertices $D_0, \ldots, D_r$.

To define the map $\rho_r$, we consider the integration of Chern forms, denoted by

$$TP(\tilde{D})_{p,r} := \int_{\Delta^r} P_p(\tilde{D}^2, \ldots, \tilde{D}^2).$$

This is a $(2p-r)$-differential form on $X$. We now want to obtain a well-defined canonical lift of this form, as a differential character on $X$.

We start by defining a $2p - r - 1$-form $\omega(\tilde{D})_{p,r}$ on $X$, and which satisfies the identity:

$$d\omega(\tilde{D})_{p,r} = -\sum_{i=0}^r (-1)^r \omega(\tilde{D}_i)_{p,r-1} - (-1)^{2p-r-1}TP(\tilde{D})_{p,r}.$$

See (10).

This identity helps us to obtain a canonical lift of $TP(\tilde{D})_{p,r}$ as a well-defined differential character on $X$, if we replace $\tilde{D}$ by a cycle in $D(E)$. The details are provided in §6.

The above constructions could also be carried out replacing the Chern forms by $GL_n$-invariant polynomial forms.

In the final section §6, we extend the Cheeger-Simons question on rationality of above maps.

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2. Preliminaries: original Chern-Cheeger-Simons theory

Suppose $X$ is a smooth (finite dimensional) manifold, and $E$ is a smooth complex vector bundle on $X$. Our aim is to construct a canonical lifting of the differential form $\Omega$, in the differential cohomology, defined by Cheeger and Simons in [Cg-Sm]. We start by recalling the original constructions.

2.1. Conventions. Denote by $\mathbb{Z}(p)$ the subgroup of $\mathbb{C}$ generated by $(2\pi i)^p$. For each subgroup $L$ of $\mathbb{C}$, set

$$L(p) = L \otimes \mathbb{Z}(p).$$

The isomorphism $\mathbb{Z} \to \mathbb{Z}(p)$ that takes 1 to $(2\pi i)^p$ induces a canonical isomorphism

$$H^\bullet(X, \mathbb{Z}) \to H^\bullet(X, \mathbb{Z}(p)).$$

The $p$-th Chern class of a complex vector bundle lies in $H^{2p}(X, \mathbb{Z}(p))$ which is the image of the usual topological Chern classes, under the isomorphism given by (3).

Suppose $(E, \nabla)$ is a complex vector bundle of rank $n$ with a connection on $X$. Then the Chern forms $c_k(E, \nabla) \in A^{2k}_{cl}(X, \mathbb{Z})$ for $0 \leq k \leq \text{rank}(E)$, are defined using the universal Weil homomorphism [Ch-Sm p.50]. There is a $GL_n$-invariant, symmetric, homogeneous and multilinear polynomial $P_k$ of degree $k$ in $k$ variables on the Lie algebra $\mathfrak{gl}_n$ such that if $\Omega$ is the curvature of $\nabla$ then

$$c_k(E, \nabla) = P_k(\frac{-1}{2\pi i} \Omega, \ldots, \frac{-1}{2\pi i} \Omega).$$

Here $P_k$ are defined as follows;

$$\det(I_r + x) = 1 + P_1(x) + P_2(x) + \ldots + P_r(x), \quad x \in \mathfrak{gl}_n(\mathbb{C}).$$

When $x_i = x$ for each $i$, then $P_k(x, \ldots, x) = \text{trace}(\wedge^k x)$ (see [Gr-Ha p.403]), however the wedge product here is taken in the variable $\mathbb{C}^n$, not the wedge of forms on the base. If $x$ is a diagonal matrix with eigenvalues $\lambda_1, \ldots, \lambda_r$ then $P_k(x, \ldots, x) = \sum I \lambda_{i_1} \cdots \lambda_{i_k}$. We can also express $P_k$ in terms of the traces of products of matrices. In this expression, the highest order term of $P_k$ is the symmetrization of $Tr(x_1 \cdots x_k)$ multiplied by a constant, the lower order terms are symmetrization’s of $Tr(x_1 \cdots x_i)Tr(\cdots)\cdots Tr(x_{i_a+1} \cdots x_k)$, with suitable constant coefficients.

Remark 2.1. The following constructions and proofs also hold if we replace the polynomial $P_p$ by any $GL_n$-invariant polynomial of degree $p$, in particular we could also take the polynomial which gives the $p$-th Chern character term.

We recall the convention on forms, from [Ch-Sm p.50]: denote $\bigwedge^{k,l}(E)$, $k$-forms on $E$ taking values in $\mathfrak{gl}_n^\otimes l$. We have the usual exterior differential

$$d : \bigwedge^k(E) \to \bigwedge^{k+1}(E).$$
If $\phi \in \bigwedge^{k,l}(E)$, $\psi \in \bigwedge^{k',l'}(E)$, then define

$$\phi \wedge \psi \in \bigwedge^{k+k',l+l'}(E),$$

$$\phi \wedge \psi(x_1, \ldots, x_{k+k'}) = \sum_{\pi \text{ shuffle}} \sigma(\pi) \phi(x_{\pi_1}, \ldots, x_{\pi_k}) \otimes \psi(x_{\pi_{k+1}}, \ldots, x_{\pi_{k+k'}}).$$

If $P$ is a $GL_n$-invariant polynomial of degree $l$ and $\phi \in \bigwedge^{k,l}(E)$. Then $P(\phi) = P \circ \phi$ is a real valued $k$-form on $E$. The following identities hold:

$$P(\phi \land \psi \land \rho) = (-1)^{kk'} P(\psi \land \phi \land \rho),$$

$$d(P(\phi)) = P(d\phi),$$

$$P(\phi \land \psi \land \rho) = (-1)^{kk'} P(\psi \land \phi \land \rho).$$

Here $\phi \in \bigwedge^{k,l}, \psi \in \bigwedge^{k',l'}, \rho \in \bigwedge^{k'',l''}.$

### 2.2. Preliminaries. [Cg-Sm]:

Suppose $X$ is a smooth manifold and $(E, \nabla)$ is a flat connection of rank $n$ on $X$. Recall the differential cohomology:

$$\widehat{H}^p(X) := \{(f, \alpha) : f : Z_{p-1}(X) \to \mathbb{C}/\mathbb{Z}, \partial(f) = \alpha, \alpha \text{ is a closed form and integral valued}\}.$$

Here $Z_{p-1}(X)$ is the group of $(p-1)$-dimensional cycles, and $\partial$ is the coboundary map on cochains. The linear functional $\partial(f)$ is defined by integrating the form $\alpha$ against $p$-cycles, and it takes integral values on integral cycles.

In [Cg-Sm], the functional $f$ is $\mathbb{R}/\mathbb{Z}$-valued. We could also take $\mathbb{C}/\mathbb{Z}$-valued functionals, since we hope that our applications will utilise $\mathbb{C}/\mathbb{Z}$-characters.

This cohomology fits in an exact sequence:

$$0 \to H^{p-1}(X, \mathbb{C}/\mathbb{Z}) \to \widehat{H}^p(X) \to A^p_2(X)_{cd} \to 0.$$  \hspace{1cm} (4)

Here $A^p_2(X)_{cd}$ denotes the group of closed complex valued $p$-forms with integral periods.

Given a smooth connection $(E, \nabla)$ on $X$. The Chern forms $c_p(E, \nabla)$ are degree $2p$-differential forms on $X$. Let $P_p$ denote the degree $p$ polynomial defining the $p$-Chern form, i.e., $c_p(E, \nabla) := P_p(\nabla^2, \ldots, \nabla^2)$ (see §2.1). Here $\nabla^2$ denotes the curvature form, and we say that $\nabla$ is flat if $\nabla^2$ is identically zero.

We recall the following construction, to motivate the construction of tertiary classes in the next section.

**Theorem 2.2. [Cg-Sm]** The Cheeger-Simons construction defines a differential character, for any smooth connection $(E, \nabla)$:

$$\hat{c}_p(E, \nabla) \in \widehat{H}^{2p}(X), \ p \geq 0.$$  

Moreover, if $\nabla$ is flat then

$$\hat{c}_p(E, \nabla) \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z}(p)).$$
This is the same as the Chern-Simons class $CS_p(E, \nabla)$.

**Proof.** Let $P_p$ denote the degree $p$, $GL_n$-invariant polynomial, defined in (2.1). The curvature form of $\nabla$ is $\Theta := \nabla^2$. The Chern form in degree $2p$ is given as $P_p(\Theta, \ldots, \Theta)$. The Chern-Weil theory says that the Chern form is closed and that it has integral periods. Hence it gives an element in the group $A^2(X)$. Now use the universal smooth connection [Na-Ra], which lies on a Grassmannian $G(r, N)$, for large $N$. The differential cohomology of $G(r, N)$ is just the group $A^2(X)$, since the odd degree cohomologies of $G(r, N)$ are zero. Hence the differential character $\hat{c}_p$ is the Chern form of the universal connection. The pullback of this element via a classifying map of $(E, \nabla)$ defines the differential character $\hat{c}_p(E, \nabla) \in \hat{H}^{2p}(X)$.

Suppose $\nabla$ is flat, i.e., $\Theta = 0$. Then the Chern form $P_p(\Theta, \ldots, \Theta) = 0$. Hence the differential character lies in the group $H^{2p-1}(X, \mathbb{C}/\mathbb{Z}(p))$.

3. **Pushforward of Chern-Simons forms**

3.1. **Rigidity of Chern-Cheeger-Simons classes and the eta-form.** As in the previous section, suppose $(E, \nabla)$ is a smooth connection on a smooth manifold. Suppose $\gamma := \{\nabla_t\}_{t \in I}$ is a one parameter family of smooth connections on $E$ and $\nabla = \nabla_0$. Here $I := [0, 1]$ is the closed unit interval.

The family $\{\nabla_t\}_{t \in I}$ gives us a family of differential characters:

$$\hat{c}_p(E, \nabla_t) \in \hat{H}^{2p}(X)$$

for $p \geq 1$ and $t \in [0, 1]$.

The difference of the differential characters when $t = 0$ and when $t = 1$, is given by the following variational formula.

**Proposition 3.1.** With notations as above, we have the following equality:

$$\hat{c}_p(E, \nabla_1) - \hat{c}_p(E, \nabla_0) = p \int_0^1 P_p(\frac{d}{dt} \nabla_t, \nabla_t^2, \ldots, \nabla_t^2) dt$$

for $p \geq 1$.

**Proof.** See [Cg-Sm] p.61, Proposition 2.9].

In particular, if $\{\nabla_t\}_{t \in I}$ is a family of flat connections then the degree $(2p - 1)$-form

$$\eta_p := p \int_0^1 P_p(\frac{d}{dt} \nabla_t, \nabla_t^2, \ldots, \nabla_t^2) dt$$

is identically zero, when $p \geq 2$. This implies that

$$\hat{c}_p(E, \nabla_1) = \hat{c}_p(E, \nabla_0).$$
This gives the rigidity of Chern-Simons classes
\[ CS_p(E, \nabla_1) = CS_p(E, \nabla_0) \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z}) \]
whenever \( p \geq 2 \).

### 3.2. Canonical invariants lifting the difference of Chern-Simons classes.

Recall the coefficient sequence:
\[ 0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}/\mathbb{Z}(p) \to 0. \]
The associated long exact cohomology sequence is given by:
\[ \to H^k(X, \mathbb{C}/\mathbb{Z}(p)) \to H^{k+1}(X, \mathbb{Z}(p)) \to H^{k+1}(X, \mathbb{C}) \to H^{k+1}(X, \mathbb{C}/\mathbb{Z}(p)) \to \).

When \( k+1 = 2p \), and \((E, \nabla)\) is a flat connection then the Chern-Simons class \( CS_p(E, \nabla) \) is a canonical lifting of the (torsion) integral Chern class \( c_p(E) \). Here we note that the de Rham Chern class \( c_p(E, \nabla) \in H^{2p}(X, \mathbb{C}) \) is identically zero, for \( p \geq 1 \). We noticed in proof of Theorem 2.2, that the differential cohomology is introduced as an intermediate object in the above cohomology sequence where the canonical invariants lie and they lift the de Rham Chern form.

When \( k+1 = 2p-1 \), and suppose we are given a path of flat connection \( \gamma := \{ \nabla_t \}_t \in I \).

The difference of Chern-Simons classes, using the rigidity formula in (6), gives us
\[ CS_p(E, \nabla_1) - CS_p(E, \nabla_0) = 0 \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z}(p)). \]

This difference is characterised by the eta-form \( \eta_p \), in (5). Hence, we would like to construct a canonical lifting in \( H^{2p-2}(X, \mathbb{C}/\mathbb{Z}(p)) \) whose image in \( H^{2p-1}(X, \mathbb{C}) \) is the difference of the Chern-Simons class, via differential characters. The differential character should be a canonical lifting of the eta form \( \eta_p \), upto an exact form. We do not write a formula for the lift but rather consider suitable lifting of closely associated forms \( TP_p \), which we will define in the next section.

### 3.3. Integration along fibres.

Suppose \( \pi : X \times T \to X \) and \( E \) is a vector bundle on \( X \). Suppose \( \hat{\Theta} \) is a smooth connection on \( \pi^*E \), \( T \) is a smooth compact manifold of dimension \( l \). Then there is a map given by integration along the fibres
\[ B : A^k(X \times T) \to A^{k-l}(X), \alpha \mapsto \int_T \alpha. \]

Similarly, there is a map (given by proper pushforward)
\[ B : H^k(X \times T, \mathbb{C}/\mathbb{Z}) \to H^{k-l}(X, \mathbb{C}/\mathbb{Z}). \]

We note that the above two maps in (7), (8) give a differential character \( \hat{B}(\hat{f}) \) on \( X \) for a differential character \( \hat{f} \) on \( X \times T \), under some conditions. See [Fr, Proposition 2.1], [Ba-Be].

However, in our situation \( T \) is a \( r \)-simplex and \( X \times \Delta^r \) is a manifold with boundary and corners, and the maps on differential cohomology defined by above authors seem
inadequate. Hence, we restrict to an appropriate range where the Chern-forms on $X \times \Delta^r$ vanish identically, and we need to only apply the map in (8).

4. (Co)Homological invariants of the simplicial set of flat connections

We start by recalling a remark by M. Karoubi [Kb, Remarque 3.6, p.68].

4.1. Karoubi’s simplicial invariants. Suppose $E$ is a vector bundle on $X$. Let $D(E)$ denote the simplicial set formed by sequence of connections under the Gauge group transformation. It is interesting to consider the topology of $D(E)$. When $E$ is trivial there is an application

$$B\mathbb{G}^\delta \rightarrow D(E)$$

where $\mathbb{G}^\delta$ denotes the Gauge group with discrete topology. In particular, if $\mathbb{G}$ is the infinite linear group $GL(\mathbb{C}) := \cup_m GL_m(\mathbb{C})$ and if $P$ is the degree $n$ component of Chern character, the composed homomorphism

$$K_r(A) \simeq \pi_r(B\mathbb{G}^{\delta+}) \rightarrow H_r(B\mathbb{G}^\delta) \rightarrow H_r(D(E)) \rightarrow H_{2n-r}(X)$$

with $A = C^\infty(X)$ is identified with the definition using cyclic homology. In particular there is a factorisation

$$K_r(A) \rightarrow K_r^{\text{top}} \simeq K^{-r}(X) \xrightarrow{\text{ch}} H_{2n-r}(X).$$

We now consider a similar set-up where the vertices of the simplices are flat connections. We would like to deduce cohomological invariants of moduli space of flat connections, using this approach.

4.2. Simplicial set $D(E)$ of sequence of flat connections. Now suppose we form the simplicial set $D(E)$ formed by sequences of flat connections, under the Gauge group transformation. As above, we would like to understand the topology of $D(E)$, in terms of giving suitable maps on homology groups of $D(E)$.

Given $E$ on $X$, we associate a simplicial abelian group $M_\ast$: let $M_r$ be the free abelian group generated by the sequences $(D^0, D^1, ..., D^r)$ where $D^i$ are smooth flat connections on $E$. If $\phi: [s] \rightarrow [r]$ is an increasing function, the map

$$\phi^*: M_r \rightarrow M_s$$

is defined by

$$\phi^*(D^0, ..., D^r) = (\nabla^0, \nabla^1, ..., \nabla^s)$$

with $\nabla^i := D^{\phi(i)}$. The gauge group $\mathbb{G}$ of automorphisms of $E$ acts on $M_\ast$ as follows:

$$\phi^*(D^0, ..., D^r).g = (g^*D^0, g^*D^1, ..., g^*D^r).$$

The quotient $N_\ast := M_\ast/\mathbb{G}$ is also a simplicial abelian group. We denote $\mathbb{H}_r(E)$ the homology groups of $N_\ast$.

We pose the following question:
Question 4.1. Is there a non-zero group homomorphism, for each \( r \geq 1 \):
\[
\rho_{p,r} : \mathbb{H}_r(\mathcal{D}(E)) \to \widehat{H}^{2p-r}(X).
\]

This question is motivated by the following computations.

Let \( \Delta^r \) be the standard \( r \)-simplex with the variables \( t = (t_0, t_1, \ldots, t_r) \) such that \( t = \sum_{i=0}^r t_i \). Let \( \tilde{D} \) be the connection
\[
\tilde{D} := t_0 D^1 + t_1 D^1 + \ldots + t_r D^r
\]
on \( \pi^* E \) where \( \pi : \Delta^r \times X \to X \) is the second projection.

For \( r \geq 1 \) and \( r \leq p \), denote
\[
TP(\tilde{D})_{p,r} := \int_{\Delta^r} P_p(\tilde{D}^2, \ldots, \tilde{D}^2).
\]
This is a differential form of degree \( 2p - r \) on \( X \).

We recall the following formula (see [Kb, Theoreme 3.3, p.67]):
\[
dTP(\tilde{D})_{p,r} = -\sum_{i=0}^r (-1)^i TP(\tilde{D}_i)_{p,r-1}.
\]

Here \( \tilde{D}_i := (D^0, \ldots, \hat{D}^i, \ldots, D^r) \), with the \( i \)-th coordinate omitted, i.e. the \( i \)-th face. When \( r = 1 \), then we have
\[
dTP(\tilde{D})_{p,1} = c_p(D^1) - c_p(D^0).
\]
This is the difference of Chern forms. Since \( D^0 \) and \( D^1 \) are flat connections, the Chern forms are zero. In other words, \( TP(\tilde{D})_{p,1} \) is a closed form. If a suitable ‘integration along fibres’ exists on the differential cohomology of \( X \times \Delta^r \) then we could get a canonical differential character \( TP(\tilde{D})_{p,1} \) lifting the form \( TP(D^i)_{p,1} \).

When \( r > 1 \), it is not immediate that the form \( TP_{p,r} \) is a closed form, unless all \( TP_{p,r-1} \) are zero or cancel out (using (9)). To rectify this situation, we define a canonical differential character lifting \( TP(D_i)_{p,1} \), in the next subsection.

We now proceed to answer Question 4.2.

4.3. Higher homotopy maps.

Theorem 4.2. With notations as in previous subsection, and when \( p > r \geq 1 \), there are well-defined maps
\[
\rho_{p,r} : \mathbb{H}_r(E) \to \widehat{H}^{2p-r}(X).
\]

Proof. With notations as in §4.2 we define the ‘fiber integration’ element of the differential character of \( (q^* E, \tilde{D}) \), on \( X \):
\[
TP(\tilde{D})_{p,r} = (tp(\tilde{D})_{p,r}, TP(\tilde{D})_{p,r})
\]
This element is well-defined if \( tp(\tilde{D})_{p,r} : Z_{2p-r-1}(X) \to \mathbb{C}/\mathbb{Z}(p) \) is a degree \( (2p - r - 1) \) cochain whose coboundary is the cochain corresponding to the degree \( (2p - r) \) form.
$TP(\tilde{D})_{p,r}$. In particular, we will show that if $\sigma$ is a cycle in $\mathcal{D}(E)$, then the differential character $\overline{TP(\sigma)}$ lifting the associated differential form $TP(\sigma)$, is well-defined.

For this purpose we now write

$$tp(\tilde{D})_{pr} : Z_{2p-r-1}(X) \to \mathbb{C}/\mathbb{Z}(p)$$

as the map $c' \mapsto \int_c \omega(\tilde{D})_{p,r}$, for a canonical form $\omega(\tilde{D})_{p,r} \in A^{2p-r-1}(X)$, such that $d\omega(\tilde{D})_{p,r} = TP(\tilde{D})_{p,r}$ (up to certain forms depending on the $r-1$-faces of $\Delta^r$).

We proceed to define the form $\omega(\tilde{D})_{p,r}$.

When $r = 1$, we can make this explicit by choosing a path $\psi_s := \{\tilde{D}^s\}$ of 1-simplices associated to length one sequences, such that when $s = 1$, $\psi_1 = \tilde{D}$ and when $s = 0$, $\psi_0 = \tilde{D}^0$. Here $\tilde{D}^0$ is the trivial 1-simplex. Then define, when $p > r$:

$$\omega(\tilde{D})_{p,1} := \int_I TP(\tilde{D}^s)_{p,1} = \int_I \int_{\Delta^r} P_p(\tilde{D}^s, ..., \tilde{D}^r).$$

By Stokes theorem, we have

$$d\omega(\tilde{D})_{p,r} = TP(\tilde{D})_{p,1} - TP(\tilde{D}^0)_{p,1} = TP(\tilde{D})_{p,1}.$$

In fact when $p > r \geq 1$, we similarly define:

$$\omega(\tilde{D})_{p,r} := \int_I TP(\tilde{D}^s)_{p,r}$$

where $\psi_s := \{\tilde{D}^s\}$ is a path of $r$-simplices, where $\psi_1 = \tilde{D}$ and $\psi_0$ is the trivial $r$-simplex.

Recall that the $i$-th face of the $r$-simplex $\tilde{D}^s$ is denoted by $\tilde{D}^s_i$, which is a $(r-1)$-simplex. Then notice that

$$d\omega(\tilde{D})_{p,r} = \int_I dTP(\tilde{D}^s)_{p,r} - (-1)^{2p-r-1}[TP(\tilde{D})_{p,r} - 0]$$

$$= - \int_I \sum_{i=0}^r (-1)^i TP(\tilde{D}^s_i)_{p,r-1} - (-1)^{2p-r-1}TP(\tilde{D})_{p,r}, \text{ use (9)}$$

$$= - \sum_{i=0}^r (-1)^i \int_I TP(\tilde{D}^s_i)_{p,r-1} - (-1)^{2p-r-1}TP(\tilde{D})_{p,r}.$$
then we have the equality:

\[ < c, \partial \tilde{D} > = - \int c \sum_{i=0}^{r} (-1)^{r} \omega(\tilde{D}_i)_{p,r-1} \]

\[ = \int_c d\omega(\tilde{D})_{p,r} + (-1)^{2p-r-1} TP(\tilde{D})_{p,r}, \text{ use } [10] \]

\[ = -\int_{\partial c} \omega(\tilde{D})_{p,r} + \int_c (-1)^{2p-r-1} TP(\tilde{D})_{p,r}, \text{ by Stokes theorem} \]

\[ = -< \partial c, \tilde{D} > + \int_c (-1)^{2p-r-1} TP(\tilde{D})_{p,r}. \]

Now replacing \( \tilde{D} \) by a \( \sigma \), which is a finite linear combination of \( r \)-simplices \( \tilde{D} \), we deduce,

1) if \( \sigma \) is a cycle in \( \mathcal{D}(E) \) then the singular cochain \( c \mapsto < c, \sigma > \) has coboundary \( (-1)^{2p-r-1} TP(\sigma)_{p,r} \) on \( X \). In other words, it corresponds to the differential character \( (\langle ., \sigma \rangle, TP(\sigma)_{p,r}) \).

2) if \( \sigma \) is a boundary \( \sigma = \partial \sigma' \), then the singular cochain \( c \mapsto < c, \sigma > \) is equal to

\[ < c, \partial \sigma' > = - < \partial c, \sigma' > + \int_c (-1)^{2p-r-1} TP(\partial \sigma') \]

\[ = - < \partial c, \sigma' > + \int_{\partial c} TP(\sigma'), \text{ use } [9] \]

\[ = - < \partial c, \sigma' > + \int_{\partial c} TP(\sigma'), \text{ by Stokes theorem.} \]

If \( c \) is a cycle then the quantity on the right hand side is zero.

Hence from 1) and 2) we deduce a group homomorphism, for \( p > r \geq 1 \):

\[ \rho_{p,r} : \mathbb{H}_r(\mathcal{D}(E)) \to \widehat{H}^{2p-r}(X). \]

It was pointed out by Deligne that when \( r < p \) we could apply integration along fibres, directly on \( H^{2p-1}(\Delta^{r} \times X, \mathbb{C}/\mathbb{Z}(p)) \), since the Chern forms of \( \tilde{D} \) vanish in this range. We make this precise as follows.

**Proposition 4.3.** Assume for each \( t \) as above the connection \( \tilde{D}^t \) is a flat connection. (In particular, this assumption will hold when \( \Delta^{r} \) is a \( r \)-simplex mapping into the moduli space \( M_X(k) \) of rank \( k \) flat connections on \( X \).) Then there are well-defined cohomology classes, for \( r < p \):

\[ CS(\tilde{D})_{p,r} \in H^{2p-r-1}(X, \mathbb{C}/\mathbb{Z}(p)). \]

**Proof.** Consider the degree \( p \), \( GL_n \)-invariant polynomial \( P_p \) (see §2.1).
By assumption, for each \( t \in \Delta^r \), \( \tilde{D} \) is flat and hence for \( p > r \geq 1 \), the \( p \)-th Chern form
\[
P_p(\tilde{D}^2, ..., \tilde{D}^2) = 0.
\]
Locally, this is easy to check since each term of this form will have a \( dx \wedge dx \).

Hence we can apply the pushforward map (see (8)) on the Chern-Simons class
\[
CS_p(q^*E, \tilde{D}) \in H^{2p-1}(\Delta^r \times X, \mathbb{C}/\mathbb{Z}(p)),
\]
to obtain a well-defined class
\[
CS(\tilde{D})_{p,r} \in H^{2p-r-1}(X, \mathbb{C}/\mathbb{Z}(p)).
\]
\[\square\]

We note that in the above situation the form \( TP(\tilde{D})_{p,r} \) is zero. This property will enable us to identify the restriction of the map \( \rho_r \) in Theorem 4.2, to the simplices parametrizing flat connections.

5. COHOMOLOGICAL INVARIANTS OF MODULI SPACE OF FLAT CONNECTIONS

Suppose \( M_X(n) \) denotes the moduli space of flat connections of rank \( n \), on \( X \).

The rigidity property of the original Chern-Simons classes can be thought of as giving a map:
\[
\pi_0(M_X(n)) \to \bigoplus_{p \geq 2} H^{2p-1}(X, \mathbb{C}/\mathbb{Z}(p)),
\]
on the connected components of the moduli space \( M_X(n) \).

We would like to express the construction of the higher invariants associated to a variation of flat connections (see previous section), as a map on the higher homology groups of the moduli space.

As a consequence of Proposition 4.3 and Theorem 4.2, we obtain the following map, on the homology groups of \( M_X(n) \).

**Theorem 5.1.** There is a group homomorphism, for \( r \geq 1 \):
\[
\rho'_r : \mathbb{H}_r(M_X(n)) \to \bigoplus_{p > r} H^{2p-r-1}(X, \mathbb{C}/\mathbb{Z}(p)).
\]

**Proof.** Consider the simplicial set \( S := S(M_X(n)) \) associated to the moduli space \( M_X(n) \). This means, for each \( r \geq 0 \), we have
\[
S_r := \{ f : \Delta^r \to M_X(n) \}
\]
together with the usual face operators \( d_i \) and degeneracy operators \( s_i \), for \( 0 \leq i \leq n \), satisfying the simplicial identities.
We now relate the homology of $\mathcal{S}$ with the homology of $D(E)$ from §4.2. If $D(E)_{fl} \subset D(E)$ denotes the sub-simplicial set of sequences of flat connections, such that $D^t := t_0D_0 + \ldots + t_rD_r$ is a flat connection for each $t = (t_0, \ldots, t_r)$, then there is a map

$$H_r(D(E)_{fl}) \to H_r(D(E)) = H_r(E).$$

Composed with the map in Theorem 4.2, we obtain a map:

$$\rho_{p,r,fl} : H_r(D(E)_{fl}) \to \hat{H}^{2p-r}(X).$$

We note that this map factors via the homology $H_r(\mathcal{S})$ which is the same as the homology $H_r(M_X(n))$ of the moduli space $M_X(n)$.

For $f \in \mathcal{S}_r$, there is a projection $\pi : \Delta^r \times X \to X$, together with a universal connection $\tilde{\nabla}$ on $\pi^*E$, such that $\tilde{\nabla}_t$ is a flat connection for any $t \in \Delta^r$.

Using a further subdivision (if necessary), we note that the map $H_r(D(E)_{fl}) \to H_r(\mathcal{S})$ is an isomorphism. Hence, we deduce a map

$$(11) \quad \rho'_r : H_r(M_X(n)) \to \hat{H}^{2p-r}(X).$$

However, using Proposition 4.3, we note that the $(2p-r)$-degree form

$$TP(\tilde{D})_{p,r} = 0$$

whenever $1 \leq r < p$ and $\tilde{D}$ parametrises flat connections.

This implies that the map $\rho'_r$ in (11) actually maps into the $\mathbb{C}/\mathbb{Z}(p)$-cohomology, i.e., $\rho_r$ factorises as follows:

$$\rho'_r : H_r(M_X(n)) \to H^{2p-r-1}(X, \mathbb{C}/\mathbb{Z}(p))$$

whenever $1 \leq r < p$.

We refer to $\rho'_r$, as the higher Cheeger-Simons invariants, in degree $r$.

6. Questions on rationality of (higher) Cheeger-Simons invariants

Recall, that when $r = 0$, and $X$ is a smooth manifold we have the map:

$$\rho_0 : \pi_0(M_X(n)) \to H^{2p-1}(X, \mathbb{C}/\mathbb{Z}(p)).$$

Cheeger and Simons [Cg-Sm], raised the following question:

Question 1: Since the moduli space of representations has countably many connected components, are the classes

$$\tilde{c}_p(E, \nabla) \in H^{2p-1}(X, \mathbb{C}/\mathbb{Z}(p))$$

torsion ? i.e., takes value in $\mathbb{Q}/\mathbb{Z}$, whenever $p \geq 2$.

For reasonable smooth manifolds $X$, the homology groups of $M_X(n)$ are countably generated, and hence the above question can be extended as follows:
**Question 2:** When \( r \geq 1 \) and \( r < p \), does the image of the map \( \rho_r \) lie in the torsion subgroup of \( \mathbb{C}/\mathbb{Z}(p) \)-cohomology of \( X \)?

We note that when \( X \) is a smooth complex projective variety, Question 1 (i.e., when \( r = 0 \)) was answered positively by A. Reznikov \([Rz]\), and partial results in the quasi-projective case was obtained by Simpson and the author \([Iy-Si]\).

We hope to investigate the above questions in a future work.

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