STABILITY OF HOPF BIFURCATIONS IN TIME-DELAYED FULLY-CONNECTED PLL NETWORKS

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Abstract

Dynamics in delayed differential equations (DDEs) is a well studied problem mainly because DDEs arise in models in many areas of science including biology, physiology, population dynamics and engineering. The change of nature in the solutions in the parameter space for a network of Phase-Locked Loop oscillators was studied in Symmetric bifurcation analysis of synchronous states of time-delayed coupled Phase-Locked Loop oscillators. Communications in Nonlinear Science and Numerical Simulation, Elsevier BV, Volume 22, Issues 13, May 2015, Pages 793-820, where the existence of Hopf bifurcations for both cases, symmetry-preserving and symmetry-breaking synchronization was well stablished. In this work we continue the analysis exploring the stability of small-amplitude periodic solutions emerging near Hopf bifurcations in the Fixed-point subspace, based on the reduction of the infinite-dimensional space onto a two-dimensional center manifold. Numerical simulations are presented in order to confirm our analytical results. Although we explore network dynamics of second-order oscillators, results are extendable to higher order nodes.

INTRODUCTION

Networks of oscillators have been studied for decades because their models can represent dynamics in a very wide range of fields, as astronomy, biology, neurology, economics, and the stock market, [10, 17, 15, 21, 18]. Much of the research has focused to understand the influence that changes in the parameter space have over the global dynamics. In that sense, the stability of the synchronization in the network have been explored from many approaches [10, 21, 20, 0]. In this contribution, we study the stability of small-amplitude periodic so-
lutions which emerge near Hopf bifurcations of the symmetry-preserving solutions for a N-node PLL network, using the centre manifold theorem extended to functional differential equations, and the normal form in the center space. This work is mainly based on previous results obtained on the existence of symmetry-preserving and symmetry-breaking Hopf bifurcations in a N-node second-order PLL network, by Ferruzzo et al. [6].

It is important to note that when the lag in the communications between nodes is considered, the ordinary differential equation (ode) system which describe the network dynamics, becomes a delayed differential equation (dde), whose solution lies, in general, in the function space, and its characteristic equation has infinitely many roots. This particular kind of functional differential equations appear in many engineering problems [15, 19, 8].

We consider the Full Phase model introduced in [5] to analyse stability of small-amplitude periodic orbits near Hopf bifurcations emerging in the parameter space $(\mu, \tau)$ at the Fixed-point subspace, for the non degenerative case $(K > 1)$. It has been shown that these bifurcations can occur when a pair of complex conjugate eigenvalues crosses the imaginary axis in either direction, from the left to the right and from the right to the left. The main approach used for the analysis is the decomposition of the infinite-dimensional space into a 2-dimensional center space spanned by the eigenvectors corresponding to the simple imaginary eigenvalues $\lambda = \pm i\omega$, $\omega > 0$, and into an infinite-dimensional space “orthogonal” to the first one (the orthogonality condition will be defined below). We will follow closely the theory and procedures presented in [15, 17, 22, 19].

The Full-phase model

The general model for a $N$-node, fully-connected, second-order oscillator network, in terms of the $i$-th node output phase $\phi_i(t)$, is:

$$\ddot{\phi}_i(t) + \mu \dot{\phi}_i(t) - \mu - \frac{K\mu}{N-1} \sum_{j \neq i}^N f(\phi_i, \phi_j) = 0, \quad i = 1, \ldots, N,$$

(1)

where:

$$f(\phi_i, \phi_j) = \sin(\phi_j(t-\tau) - \phi_i(t)) + \sin(\phi_j(t-\tau) + \phi_i(t)).$$

(2)

The equilibria $\phi^\pm$, in equation (1), are:

$$\phi^+(n) = \frac{1}{2} \left( \arcsin \left(-\frac{1}{K}\right) + 2n\pi \right),$$

$$\phi^-(n) = \frac{1}{2} \left( \pi - \arcsin \left(-\frac{1}{K}\right) + 2n\pi \right),$$

(3)

$n \in \mathbb{Z}$, $K \geq 1$. For our analysis, we consider three main assumptions:

(a) The critical eigenvalue $\lambda$ of the linearization of (1) at equilibria crosses the imaginary axis with non vanishing velocity, i.e. $\text{Re}(\lambda'(\phi^\pm)) \neq 0$. 

(b) The purely imaginary eigenvalue $\lambda = i\omega$ is simple.

(c) The linearization of (1) at equilibria, has no eigenvalues of the form $i\omega$, $k \in \mathbb{Z} - \{1, -1\}$.

The Taylor expansion of (1) at equilibria is:

$$\delta \ddot{\phi}_i + \mu \delta \dot{\phi}_i - \frac{K\mu}{N-1} \sum_{j \neq i}^N \sum_{r=1}^\infty \frac{1}{r!} \delta \phi_i \frac{\partial}{\partial \phi_j} \left. f(\phi_i, \phi_j) \right|_{\phi_j = \phi^\pm} = 0.$$ 

(4)

where $\phi_j := \phi_j(t-\tau)$. Truncate the series up to the third-order term:

$$\ddot{\phi}_i + \mu \dot{\phi}_i = \frac{K\mu}{N-1} \sum_{j \neq i}^N \left( (\phi_j - \phi_i) + (\phi_j + \phi_i) \cos 2\phi^\pm \right. 

- \frac{1}{2} (\phi_j + \phi_i) \sin 2\phi^\pm - \frac{1}{6} \left[ (\phi_j - \phi_i)^3 + (\phi_j + \phi_i)^3 \cos 2\phi^\pm \right].$$ 

(5)
\[ i = 1, \ldots, N, \text{ here, for the sake of notation we changed } \delta \phi_1 \to \phi_1. \]

The vector field form \( \dot{x} = G(x_1, x; \eta) \), \( G : \mathbb{R}^{2N} \times \mathbb{R} \to \mathbb{R}^{2N} \), can be obtained by choosing \( x_1^{(i)} = \phi_1 \) and \( x_2^{(i)} = \phi_1 \), then, the restriction \( G|_i \), or, \( \dot{x}^{(i)} = G^{(i)}(x_1, x_1; \eta) \) gives:

\[
\begin{align*}
\dot{x}_1^{(i)} &= x_2^{(i)} \\
\dot{x}_2^{(i)} &= -\mu x_2^{(i)} + \frac{K \mu}{N - 1} \sum_{j=1}^{N} \left\{ -1 + \cos 2\phi j^{(i)} \right\} x_1^{(i)} \\
&\quad + \frac{K \mu}{N - 1} \sum_{j=1}^{N} \left\{ -1 + \cos 2\phi j^{(i)} \right\} x_1^{(i)} - \frac{1}{2} \left( x_1^{(i)} + x_1^{(i)} \right)^2 \sin 2\phi j^{(i)} \\
&\quad - \frac{1}{6} \left[ \left( x_1^{(i)} - x_1^{(i)} \right)^3 + \left( x_1^{(i)} + x_1^{(i)} \right)^3 \cos 2\phi j^{(i)} \right].
\end{align*}
\]

\[ i = 1, \ldots, N. \tag{6} \]

Following [11], we can represent the dynamics in (6) by the abstract differential equation:

\[
\frac{d}{dt} x_t(\theta) = A(\theta)x_t(\theta) + F(x_t(\theta), \eta). \tag{7}
\]

We define \( \mathcal{X} := C([-\tau, 0], \mathbb{R}^{2N}) \), the Banach space of continuous functions from \([-\tau, 0]\) into \( \mathbb{R}^{2N} \), equipped with the usual norm

\[ ||\theta|| = \sup_{-\tau \leq \theta \leq 0} |\theta(\theta)|, \quad \theta \in C([-\tau, 0]), \]

\( x_t, \) in (7), lies in \( \mathcal{X} \) and satisfies \((T(t))\theta(\theta) = (x_t(\theta))(\theta) = x_t(t + \theta)\), where \( T(t) \) is a semigroup of family of operators, \( \theta \in [-\tau, 0], \) and \( \eta \) is a vector of parameters. The linear operator \( A(\theta) \in \text{Mat}(2N) \) is:

\[
(A(\theta))\theta = \left\{ \begin{array}{ll}
\frac{\partial \psi}{\partial \theta} & , -\tau < \theta \leq 0 \\
A_0(\eta)\theta(0) + A_\tau(\eta)\theta(-\tau) & , \theta = 0
\end{array} \right.
\]

where \( A_0(\eta) := \frac{\partial G}{\partial x_2}|_{\phi_{\pm}} \), \( A_\tau(\eta) := \frac{\partial G}{\partial x_1}|_{\phi_{\pm}} \) and,

\[
(F(x))\theta = \left\{ \begin{array}{ll}
\frac{\partial f}{\partial \theta} & , -\tau \leq \theta < 0 \\
F(x(0), x(-\tau), \eta) & , \theta = 0
\end{array} \right.
\]

\[ F = (f^{(1)}, \ldots, f^{(N)})^T, \quad f^{(i)} = (f_1^{(i)}, f_2^{(i)}), \quad f_1^{(i)} = 0, \] and \( f_2^{(i)} = \frac{K \mu}{N - 1} \sum_{j=1}^{N} \left\{ -\frac{1}{2} \left( x_1^{(j)} + x_1^{(i)} \right)^2 \sin 2\phi j^{(i)} - \frac{1}{6} \left[ \left( x_1^{(i)} - x_1^{(i)} \right)^3 + \left( x_1^{(i)} + x_1^{(i)} \right)^3 \cos 2\phi j^{(i)} \right] \right\}. \]

In order to build the decomposition of the infinite-dimensional space, we need the adjoint operator associated to the linear part of the linearization and a inner product, via a bilinear form. Associated to the linear part of (7), the formal adjoint equation is

\[
\frac{dy}{dt}(t, \eta) = A_0^T(\eta)y(t, \eta) + A_\tau^T(\eta)y(t + \tau, \eta), \tag{10}
\]

The strongly continuous semigroup \((T^*(t))\psi(\theta) = (y_t(\psi))(\theta) = y(t + \theta), \) defines the infinitesimal generator:

\[
(A^*(\eta)\psi) = \left\{ \frac{\partial \psi}{\partial \theta} \right\} , 0 < \theta \leq \tau, \tag{11}
\]

such that \( \frac{d}{dt} T^*(t)\psi = A^*T^*(t)\psi, \psi \in \mathcal{X}^* := C([0, \tau], \mathbb{R}^{2N}). \) The natural inner product has the form [12]:

\[
\langle x, y \rangle = \bar{x}^T(0)y(0) + \int_{-\tau}^{0} \bar{x}^T(s + \tau) A_\tau(\eta)y(s)ds,
\]

\( x \in \mathcal{X} \) and \( y \in \mathcal{X}^*; \) thus, we have [7]:

1. \( \lambda \) is an eigenvalue of \( A(\eta) \) if and only if \( \bar{\lambda} \) is and eigenvalue of \( A^*(\eta) \).

2. If \( \varphi_1, \ldots, \varphi_d \) is a basis for the eigenspace of \( A(\eta) \) and \( \psi_1, \ldots, \psi_d \) is a basis for the eigenspace of \( A^*(\eta) \), construct the matrices \( \Phi = (\varphi_1, \ldots, \varphi_d) \) and \( \Psi = (\psi_1, \ldots, \psi_d) \). Define the bilinear form:

\[
\langle \Psi, \Phi \rangle = I \tag{12}
\]

**The Fixed Point space \( S_N \)**

Due to the \( S_N \)-symmetry of [11] the space where solutions \( \phi_i \) lie can be decomposed into the Fixed-point subspace where symmetry-preserving solutions
emerge and a subspace with symmetry-breaking solutions, this was shown in [3]. We analyze stability of the small-amplitude periodic solutions near Hopf bifurcations in the Fixed point space, these bifurcations satisfy assumptions (i)-(c) for $K > 1$. In this subspace, equation (10) has the form:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\mu x_2 + K\mu (-1 + \cos 2\phi^\pm) x_1 + K\mu \left( (1 + \cos 2\phi^\pm) x_{1\tau} - \frac{1}{2} (x_{1\tau} + x_1)^2 \sin 2\phi^\pm - \frac{1}{6} [(x_{1\tau} - x_1)^3 + (x_{1\tau} + x_1)^3 \cos 2\phi^\pm] \right),
\end{align*}
\]

(13)

then matrices $A_0(\eta)$ and $A_\tau(\eta)$ in (11) become:

\[
A_0(\eta) = \begin{pmatrix}
0 & 1 \\
K\mu (-1 + \cos 2\phi^\pm) & -\mu
\end{pmatrix},
\]

(14)

\[
A_\tau(\eta) = \begin{pmatrix}
0 & 0 \\
K\mu (1 + \cos 2\phi^\pm) & 0
\end{pmatrix},
\]

(15)

and $F$ in (10) takes the form $F = (f_1 f_2)^T$, with $f_1 = 0$, and $f_2$:

\[
\begin{align*}
 f_2(x_1, \eta) &= K\mu \left\{ -\frac{1}{2} (x_{1\tau} + x_1)^2 \sin 2\phi^\pm - \frac{1}{6} [(x_{1\tau} - x_1)^3 + (x_{1\tau} + x_1)^3 \cos 2\phi^\pm] \right\},
\end{align*}
\]

(16)

We need the complex eigenfunctions $A\phi(\vartheta) = \omega s(\vartheta)$, $A^* n(\vartheta) = \omega n(\vartheta)$, associated to the critical eigenvalues $\lambda = \omega$, and $\bar{\lambda} = -\omega$ with $s(\vartheta) = s_1(\vartheta) + is_2(\vartheta)$ and $n(\vartheta) = n_1(\vartheta) + in_2(\vartheta)$. These eigenfunctions can be computed solving the boundary value problem $\frac{d}{d\vartheta}s_{1,2} = \pm \omega s_{2,1}(\vartheta)$, and $\frac{d}{d\vartheta}n_{1,2} = \pm \omega n_{2,1}(\vartheta)$, which, after substituting the operator $A(\eta)$, becomes:

\[
\begin{align*}
 A_0(\eta)s_1(0) + A_\tau(\eta)s_1(-\tau) &= -\omega s_2(0) \\
 A_0(\eta)s_2(0) + A_\tau(\eta)s_2(-\tau) &= \omega s_1(0)
\end{align*}
\]

(17)

and

\[
\begin{align*}
 A_0^T(\eta)n_1(0) + A_\tau^T(\eta)n_1(-\tau) &= \omega n_2(0) \\
 A_0^T(\eta)n_2(0) + A_\tau^T(\eta)n_2(-\tau) &= -\omega n_1(0)
\end{align*}
\]

(18)

with general solutions:

\[
\begin{align*}
 s_1(\vartheta) &= \cos(\omega \vartheta)c_1 - \sin(\omega \vartheta)c_2 \\
 s_2(\vartheta) &= \sin(\omega \vartheta)c_1 + \cos(\omega \vartheta)c_2 \\
 n_1(\vartheta) &= \cos(\omega \vartheta)d_1 - \sin(\omega \vartheta)d_2 \\
 n_2(\vartheta) &= \sin(\omega \vartheta)d_1 + \cos(\omega \vartheta)d_2
\end{align*}
\]

(19)

The coefficients $c_1 = [c_{11} c_{12}]^T$, $c_2 = [c_{21} c_{22}]^T$, $d_1 = [d_{11} d_{12}]^T$, $d_2 = [d_{21} d_{22}]^T$ can be obtained by considering the boundary conditions,

\[
\begin{align*}
 (A_0(\eta) + \cos(\omega \tau)A_\tau(\eta)) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= 0, \\
 (A_0^T(\eta) + \cos(\omega \tau)A_\tau^T(\eta)) \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} &= 0
\end{align*}
\]

(20)

the “orthonormality” condition $\langle s, n \rangle = I$, and setting $c_{11} = 1$ and $c_{21} = 0$, see [15, 13] for more details.

It is also possible to decompose the solution $x_i(\vartheta)$ to equation (17) into $x_i(\vartheta) = y_1(t)s_1(\vartheta) + y_2(t)s_2(\vartheta) + w_i(\vartheta)$, where $y_1$ and $y_2$ lie in the center subspace, such that $y_{1,2}(t) = \langle n_{1,2}(0), x_i(0) \rangle$, and $w_i$ in the infinite-dimensional component subspace, thus, we have:

\[
\begin{align*}
 \dot{y}_1 &= \omega y_2 + n_1^T(0) F \\
 \dot{y}_2 &= -\omega y_1 + n_2^T(0) F
\end{align*}
\]

(21)

\[
\dot{w} = A(\eta)w_t + F(x_\tau, \eta) - n_1^T(0) F s_1 - n_2^T(0) F s_2
\]

(22)

where

\[
F = \begin{cases}
0 & , \vartheta \in [-\tau, 0) \\
F(y_1(t)s_1(0) + y_2(t)s_2(0) + w(t)(0)) & , \vartheta = 0.
\end{cases}
\]

(23)

**The center manifold**

Following [14, 15, 22], we know that $w$ can be approximated by the second-order expansion:

\[
w(y_1, y_2)(\vartheta) = \frac{1}{2}(h_1(\vartheta)y_1^2 + 2h_2(\vartheta)y_1y_2 + h_3(\vartheta)y_2^2).
\]

(24)
thus, by differentiating and substituting equation (22) keeping up to second order terms, we obtain:

$$\dot{w} = -\omega h_2 y_2^2 + \omega(h_1 - h_3) y_1 y_2 + \omega h_2 y_2^2 + O(y^3),$$

(25)

and from equation (22),

$$\frac{dw}{dt} = A(\eta)w + F(w + y_1 s_1 + y_2 s_2) - (d_{12}s_1 + d_{22}s_2)f_2.$$  

(26)

From the definition of $A(\eta)$, equivalent to (11), we see that

$$A(\eta)w = \begin{cases} \frac{1}{2}(h_1 y_1^2 + 2 h_2 y_1 y_2 + h_3 y_2^2), & \theta \in [-\tau, 0] \\ A_0(\eta)w(0) + A_\tau(\eta)w(-\tau), & \theta = 0, \end{cases}$$

(27)

then, from equation (24), (25), (26), and (27), we can obtain the unknown coefficients $h_1$, $h_2$, and $h_3$, solving:

$$\begin{align*}
\dot{h}_1 &= 2(-\omega h_2 + f_2^{20}(d_{12}s_1(\vartheta) + d_{22}s_2(\vartheta))), \\
\dot{h}_2 &= \omega(h_1 - h_3) + f_1^{11}(d_{12}s_1(\vartheta) + d_{22}s_2(\vartheta))), \\
\dot{h}_3 &= 2(\omega h_2 + f_2^{22}(d_{12}s_1(\vartheta) + d_{22}s_2(\vartheta))).
\end{align*}$$

(28)

and,

$$A_0(\eta)h_3(0) + A_\tau(\eta)h_1(-\tau) = 2(-\omega h_2(0) + f_2^{20}(d_{12}s_1(0) + d_{22}s_2(0))),$$

$$A_0(\eta)h_2(0) + A_\tau h_2(-\tau) = \omega(h_1(0) - h_3(0)) + f_1^{11}(d_{12}s_1(0) + d_{22}s_2(0))),$$

$$A_0(\eta)h_3(0) + A_\tau(\eta)h_3(-\tau) = 2(\omega h_2(0) + f_2^{22}(d_{12}s_1(0) + d_{22}s_2(0))),$$

(29)

where

$$f_2^{20} = \frac{\partial^2 f}{\partial y_1^2} \bigg|_0, \quad f_1^{11} = \frac{\partial^2 f}{\partial y_1 \partial y_2} \bigg|_0, \quad \text{and} \quad f_2^{22} = \frac{1}{2} \frac{\partial^2 f}{\partial y_2^2} \bigg|_0.$$  

Equation (28) is written as the inhomogenous differential equation:

$$\frac{dh}{d\vartheta} = Ch + p \cos(\omega \vartheta) + q \sin(\omega \vartheta)$$

(30)

where

$$h := \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}, \quad C := \omega \begin{pmatrix} 0 & -2I & 0 \\ I & 0 & -I \\ 0 & 2I & 0 \end{pmatrix},$$

$$p := \begin{pmatrix} f_2^{20} p_0 \\ f_2^{21} p_0 \end{pmatrix}, \quad q := \begin{pmatrix} f_2^{20} q_0 \\ f_2^{21} q_0 \end{pmatrix},$$

$$p_0 := \begin{pmatrix} d_{12} \\ e_{22} d_{22} \end{pmatrix}, \quad q_0 := \begin{pmatrix} d_{22} \\ -e_{22} d_{12} \end{pmatrix},$$

with general solution:

$$h(\vartheta) = e^{C \vartheta} K + M \cos(\omega \vartheta) + N \sin(\omega \vartheta).$$

(31)

After substituting the general solution into (30) we solve for $M$ and $N$, and then from the boundary value problem we solving for $K$,

$$\begin{pmatrix} C & -\omega I \\ \omega I & C \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix} = -\begin{pmatrix} p \\ q \end{pmatrix}$$

(32)

where

$$P := \begin{pmatrix} A_0 & 0 & 0 \\ 0 & A_0 & 0 \\ 0 & 0 & A_0 \end{pmatrix} - C,$$

$$Q := \begin{pmatrix} A_\tau & 0 & 0 \\ 0 & A_\tau & 0 \\ 0 & 0 & A_\tau \end{pmatrix},$$

and $r := (0 \quad f_2^{20} \quad 0 \quad f_2^{21} \quad 0 \quad f_2^{22})^T$. The expressions for $w_1(0)$ and $w_1(-\tau)$, necessary in (28), are:

$$w_1(0) = \frac{1}{2} \left( (M_1 + K_1) y_1^2 + 2(M_3 + K_3) y_1 y_2 + (M_5 + K_5) y_2^2 \right),$$

$$w_1(-\tau) = \frac{1}{2} \left( (e^{-C \tau} K_1 + M_1 \cos(\omega \tau) - N_1 \sin(\omega \tau)) y_1^2 + 2(e^{-C \tau} K_3 + M_3 \cos(\omega \tau) - N_3 \sin(\omega \tau)) y_1 y_2 + (e^{-C \tau} K_5 + M_5 \cos(\omega \tau) - N_5 \sin(\omega \tau)) y_2^2 \right).$$

(35)
note that we only need \( w_1(\nu) \) since the nonlinear function in \([16]\) only depends on \( x_1 \); then by substituting \((35)\) into \((21)\), we obtain:

\[
\begin{align*}
\dot{y}_1 &= \omega y_2 + g_1(y_1, y_2; \eta) \\
\dot{y}_2 &= -\omega y_1 + g_2(y_1, y_2; \eta)
\end{align*}
\]

or

\[
\begin{align*}
\dot{y}_1 &= \omega y_2 + a_{20}y_1^2 + a_{11}y_1y_2 + a_{02}y_2^2 + a_{30}y_1^3 \\
&\quad + a_{21}y_1y_2 + a_{12}y_1y_2^2 + a_{03}y_2^3, \\
\dot{y}_2 &= -\omega y_1 + b_{20}y_1^2 + b_{11}y_1y_2 + b_{02}y_2^2 + b_{30}y_1^3 \\
&\quad + b_{21}y_1y_2 + b_{12}y_1y_2^2 + b_{03}y_2^3.
\end{align*}
\]

In \([9]\) is computed the coefficient \( a \), which determines stability of the normal form \((37)\),

\[
a = \frac{1}{16} \left[ g_{20}^0 + g_{21}^0 + g_{11}^2 + g_{10}^{02} \right] + \frac{1}{16\omega} \left[ g_{21}^1 (g_{20}^2 + g_{20}^2) \\
- g_{11}^1 (g_{10}^{02} + g_{10}^{02}) - g_{20}^{02} g_{10}^{02} + g_{20}^{02} g_{20}^{02} \right],
\]

where \( g_{ij}^{kl} = \frac{\partial^{i+j}}{\partial y_1^i \partial y_2^j} g_r(0, 0) \). Periodic orbits near Hopf bifurcation at the critical eigenvalue \( \lambda = \omega \), will be stable if \( a < 0 \) and unstable if \( a > 0 \).

**Numerical Results**

We reproduced some of the computations for the Hopf bifurcations in the Fixed point space for the case \( K > 1 \) presented in \([5]\), because we will compute stability for these bifurcation curves using results obtained in the previous section. In figure 1 (part of figure 10, in \([4]\) are shown the symmetry-preserving bifurcation curves in the parameter space \((\mu, \tau)\) for \( K = 1.05 \), for both cases: bifurcations with \( \text{Re}(\lambda') > 0 \) in black color, and with \( \text{Re}(\lambda') < 0 \) in red color; we also choose three testing point for numerical simulation \( A = (\mu, \tau) = (0.15, 7.46) \), \( B = (0.3, 11) \), and \( C = (0.421, 7.10) \).

In figure 2 is shown the coefficient \( a \) computed using equation \((38)\), in the parameter space \((\mu, \tau)\), for \( K = 1.05 \), related to the Hopf bifurcations curves shown in figure 1. The black curve corresponds to stability of periodic orbits near Hopf bifurcations with \( \text{Re}(\lambda') > 0 \) (black curves in figure 1), as we can see, these periodic solutions are all stable \((a < 0)\). The red curve corresponds to stability of periodic orbits near Hopf bifurcations with \( \text{Re}(\lambda') < 0 \) (red curves in figure 1), these periodic orbits are unstable for \( \mu < \mu^*(K) \approx 0.386 \), and stable for \( \mu > \mu^* \). Are also shown point \( A \), \( B \) and \( C \). At points \( A \) and \( C \), small amplitude periodic orbits are stable, whilst at point \( B \), they are unstable.

In order to confirm our results, we computed branches of periodic solutions near the Hopf bifurcations points \( A, B \), and \( C \), using DDE-BIFTOOL \([3]\) \([3]\), along with the Floquet multipliers for a specific periodic solution chosen in the branch.

A branch of periodic solutions with small amplitude, emerging from the Hopf bifurcation point \( A = (\mu, \tau) = (0.15, 7.46) \) is shown in figure 2(a). In 2(b) it is shown the periodic solution profile \( psol \) at \( \tau = 7.5315 \). The Floquet multipliers related to \( psol \) are shown in figure 2(c). It is clear that this periodic solution is stable, since there is no Floquet multiplier outside the unity circle.

For the point \( B = (\mu, \tau) = (0.3, 11) \), the branch of
periodic solutions is shown in figure 4-(a), in figure 4-(b), it is shown the profile for the periodic solution \( psol \) chosen at \( \tau = 11.3744 \), these solution is unstable, because there is a Floquet multiplier outside the unity circle, see figure 4-(c).

Finally, the branch of periodic solutions near the Hopf bifurcation point \( C = (\mu, \tau) = (0.421, 7.10) \) is shown in figure 5-(a). The periodic solution chosen in the branch is at \( \tau = 7.00 \), its profile is shown in figure 5-(b). All the Floquet multipliers shown in figure 5-(c) are within the unity circle, therefore the solution is stable.

**Conclusions**

The reduction of the infinite-dimensional space onto the center manifold in normal form, was applied to the Fixed point space for the Full-phase model in order to analyse the stability of small-amplitude periodic orbits near simple Hopf bifurcations, in both cases, for \( \text{Re}(\lambda') > 0 \) and \( \text{Re}(\lambda') < 0 \), we found that in the first case periodic orbits which are stable \( (a < 0) \) can emerge, and, in the other case, unstable \( (a > 0) \) periodic orbits can emerge for \( \mu < \mu^*(K) \), and stable periodic orbits for \( \mu > \mu^*(K) \). The numerics show that the analytical results are correct.
Figure 4: (a) BRANCH OF PERIODIC SOLUTIONS EMERGING FROM POINT $B = (\mu, \tau) = (0.3, 11.001518)$. (b) PERIODIC SOLUTION PROFILE AT $\mu = 0.3, \tau = 11.3744, T = 12.8506$ seg, (POINT $psol$). (c) FLOQUET MULTIPLIERS FOR THE PERIODIC SOLUTION $psol$.

Figure 5: (a) BRANCH OF PERIODIC SOLUTIONS EMERGING FROM POINT $C = (\mu, \tau) = (0.421, 7.101329)$. (b) PERIODIC SOLUTION PROFILE AT $\mu = 0.421, \tau = 7.00, T = 8.8704$ seg, (POINT $psol$). (c) FLOQUET MULTIPLIERS FOR THE PERIODIC SOLUTION $psol$. 
Although, we computed the coefficient $a$ for a specific value of $K$, the procedure shown is valid for all the parameter space where simple Hopf bifurcations appear.

Finally, it is important to spotlight some points for further research: First, what is the nature of the solutions at the special point $\mu = \mu^*(K)$, at which the coefficient $a$ changes sign. Second, analyze stability of the degenerate Hopf bifurcations at the Fixed point space for $K = 1$, which are codimension 2, pure imaginary eigenvalue and zero eigenvalue; and third, the stability of the symmetry-breaking degenerate Hopf bifurcations which have multiplicity $N - 1$.

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