**Enumeration of ramified coverings of the sphere and 2-dimensional gravity**

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July 20, 2018

**Abstract**

Let $\mathcal{A}$ be the algebra generated by the power series $\sum n^{-1} q^n / n!$ and $\sum n^n q^n / n!$. We prove that many natural generating functions lie in this algebra: those appearing in graph enumeration problems, in the intersection theory of moduli spaces $\overline{\mathcal{M}}_{g,n}$ and in the enumeration of ramified coverings of the sphere.

We argue that ramified coverings of the sphere with a large number of sheets provide a model of 2-dimensional gravity. Our results allow us to compute the asymptotic of the number of coverings as the number of sheets goes to infinity. The leading terms of such asymptotics are the values of certain observables in 2-dimensional gravity. We prove that they coincide with the values provided by other models. In particular, we recover a solution of the Painlevé I equation and the string solution of the KdV hierarchy.

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1 Introduction

Denote by $A$ the subalgebra of the algebra of power series in one variable, generated by the series

$$\sum_{n \geq 1} \frac{n^{n-1}}{n!} q^n \quad \text{and} \quad \sum_{n \geq 1} \frac{n^n}{n!} q^n.$$  

We wish to show that this algebra plays an important role in the intersection theory of moduli spaces $\overline{M}_{g,n}$ of stable curves and in the problem of enumeration of ramified coverings of the sphere.

SECTION 2 contains a more explicit description of the algebra $A$. In this section we also prove some relations between $A$ and the combinatorics of Cayley trees (= trees with numbered vertices).

SECTION 3 is devoted to the problem of enumerating the ramified coverings of the sphere with specified ramification types.

Consider a holomorphic map $f : C \to \mathbb{C}P^1$ of degree $n$ from a smooth complex curve $C$ to the Riemann sphere. Such maps will be called ramified coverings with $n$ sheets.

A ramification point of $f$ is a point of the target Riemann sphere that has less than $n$ distinct preimages.
For each ramification point $y$ of a ramified covering, we are going to single out several simple preimages of $y$.

**Definition 1.1** A marked ramified covering is a ramified covering with a choice, for every ramification point $y$, of a subset of the set of simple preimages of $y$.

Consider a partition $\mu = 1^{a_1}2^{a_2} \ldots$ of an integer $m \leq n$. (Here we use multiplicative notation for partitions: the partition $\mu$ contains $a_1$ parts equal to 1, $a_2$ parts equal to 2, and so on, $\sum ia_i = m$.) Suppose that a point $y \in \mathbb{C}P^1$ has $a_1$ marked simple preimages, $a_2$ double preimages, and so on. (Consequently, $y$ also has $n - m$ unmarked simple preimages.) We then say that $y$ is a ramification point of $f$ of multiplicity $r = a_2 + 2a_3 + 3a_4 + \ldots$ and of ramification type $\mu = 1^{a_1}2^{a_2} \ldots$. Sometimes the number $r$ will also be called the degeneracy of the partition $\mu$.

**Definition 1.2** A Hurwitz number $h_{n;\mu_1,\ldots,\mu_k}$ is the number of connected $n$-sheeted marked ramified coverings of $\mathbb{C}P^1$ with $k$ ramification points, whose ramification types are $\mu_1, \ldots, \mu_k$. Every such covering is counted with weight $1/|\text{Aut}|$, where $|\text{Aut}|$ is the number of automorphisms of the covering.

Note that the genus $g$ of the covering surface can be reconstituted from the data $(n; \mu_1, \ldots, \mu_k)$ using the Riemann-Hurwitz formula: if the degeneracy of $\mu_i$ equals $r_i$, then

$$2 - 2g = 2n - r_1 - \ldots - r_k.$$ 

Fix $k$ nonempty partitions $\mu_1, \ldots, \mu_k$ with degeneracies $r_1, \ldots, r_k$. Let $r$ be the sum $r = r_1 + \ldots + r_k$.

**Notation 1.3** Denote by $h_{g,n;\mu_1,\ldots,\mu_k}$ the number of $n$-sheeted marked ramified coverings of $\mathbb{C}P^1$ by a genus $g$ surface, with $k$ ramification points of types $\mu_1, \ldots, \mu_k$ and, in addition, $c(n) = 2n + 2g - 2 - r$ simple (= of multiplicity 1) ramification points. Each covering is counted with weight $1/|\text{Aut}|$.

**Theorem 1** Fix any $g \geq 0$, $k \geq 0$. If $g = 1$, we suppose that $k \geq 1$. Then for any partitions $\mu_1, \ldots, \mu_k$, the series

$$H_{g;\mu_1,\ldots,\mu_k}(q) = \sum_{n \geq 1} \frac{h_{g,n;\mu_1,\ldots,\mu_k}}{c(n)!} q^n$$
lies in the algebra $A$.

The only known proof of this theorem involves a surprising detour by the intersection theory on moduli spaces of stable curves. Using the Ekedahl-Lando-Shapiro-Vainshtein (or “ELSV”) formula, one can express the Hurwitz numbers for $k = 1$ as integrals of some cohomology classes over these moduli spaces. In [8] the ELSV formula is used to express the generating functions for Hurwitz numbers in the case $k = 1$ as rational functions of the series $Y(q)$. This essentially proves the theorem for $k = 1$. After that, we proceed by induction on $k$.

Among other things, the theorem allows one to find the asymptotic of the coefficients of $H_{g,\mu_1,\ldots,\mu_k}$ as $n \to \infty$, knowing only the several first coefficients.

SECTION 4 describes a model of 2-dimensional gravity obtained by counting ramified coverings of the sphere. We show that one can extract from the asymptotic of Hurwitz numbers a solution of the Painlevé I equation and the string solution of the Korteweg - de Vries equation. The same solutions are obtained in other models of 2-dimensional gravity (by counting quadrangulations or using integrals over moduli spaces of curves).

We also compare the enumerative problems concerning ramified coverings of the sphere and those of the torus. While the former are related to the intersection theory on $\mathcal{M}_{g,n}$ and give rise to the algebra $A$, the latter are related to volumes of spaces of abelian differentials on Riemann surfaces and give rise to the algebra of quasi-modular forms.

Acknowledgments The author is grateful to J.-M. Bismut, F. Labourie, M. Kontsevich, S. Natanzon, Ch. Okonek, A. Okounkov, D. Panov, J.-Y. Welschinger, D. Zagier, A. Zorich and A. Zvonkin for useful discussions and remarks. A special thank to Sergei Lando, with whom we proved together some of the results of the last section, and to M. Kazarian for sharing his own work on the same subject. I would also like to thank for their interest the participants of the mathematical physics seminar at the ETH Zürich and of the mathematical seminar at the ENS Lyon, as well as the participants of the Luminy conference on billiards and Teichmüller spaces.

This work was partially supported by EAGER - European Algebraic Geometry Research Training Network, contract No. HPRN-CT-2000-00099 (BBW) and by the RFBR grant 02-01-22004.
Notation. Here we summarize some notation that we use consistently throughout the paper.

\( n \) The number of sheets of a covering. The power of the variable \( q \) in generating series. The number of marked points on a Riemann surface is sometimes \( n \) and sometimes \( n - r \).

\( q \) The variable in generating series (to a sequence \( s_n \) we usually assign the series \( \sum s_n q^n / n! \)).

\( g \) The genus of a Riemann surface.

\( \mu \) A partition.

\( p \) The number of parts of a partition \( \mu \).

\( a_i \) The number of parts of a partition \( \mu \) that are equal to \( i \).

\( b_i \) The parts of a partition \( \mu \) are denoted by \( b_1, \ldots, b_p \).

\( r \) The degeneracy of a partition \( \mu \) defined by \( r = \sum (b_i - 1) \). The multiplicity of a ramification point.

\( k \) The number of partitions. If \( k > 1 \), the partitions are denoted by \( \mu_1, \ldots, \mu_k \), their degeneracies by \( r_1, \ldots, r_k \), while \( r \) is the total degeneracy \( r = \sum r_i \).

\( c(n) \) The number of simple ramification points in a ramified covering.

\( \psi_i \) The first Chern class \( c_1(L_i) \) of the line bundle \( L_i \) over \( \overline{M}_{g,n} \).

\( d_i \) The power of the class \( \psi_i \) in the intersection numbers we consider.

2 The algebra \( \mathcal{A} \) of power series

The algebra of power series

\[
\mathcal{A} = \mathbb{Q} \left[ \sum_{n \geq 1} \frac{n^{n-1}}{n!} q^n, \sum_{n \geq 1} \frac{n^n}{n!} q^n \right]
\]

plays a central role in this paper. Here we give an explicit description of \( \mathcal{A} \) and show its relation with the combinatorics of Cayley trees. Many of the results below are known, but have probably never been put together. As far as we know, the algebra \( \mathcal{A} \) itself was first discovered by D. Zagier several years ago (unpublished), and then independently introduced in our paper [21], where most of the results of Section 2.1 are given. Various series from \( \mathcal{A} \) also appear in [8].
2.1 How to make computations in \( \mathcal{A} \)

Denote by \( Y \) and \( Z \) the generators of \( \mathcal{A} \)

\[
Y = \sum_{n \geq 1} \frac{n^{n-1}}{n!} q^n, \quad Z = \sum_{n \geq 1} \frac{n^n}{n!} q^n.
\]

Denote by \( D \) the differential operator \( D = q \frac{\partial}{\partial q} \). Thus \( Z = DY \).

Note that both \( Y \) and \( Z \) have a radius of convergence of \( 1/e \). Therefore the same is true of all series in \( \mathcal{A} \). The function \( Y(q) \), more precisely, \( -Y(-q) \), was considered by J. H. Lambert \([13]\) in 1758\(^1\). The relations that follow can be deduced from the Lagrange inversion theorem applied to the equation \( Y(q) = q e^{Y(q)} \) or from the Abel identities (see \([6]\), Section 1.2).

**Proposition 2.1** We have

\[
Y = q e^Y.
\]

**Proof.** \( Y \) is the exponential generating series for rooted Cayley trees (Definition 2.9). Therefore \( e^Y \) is the exponential generating series for forests of rooted Cayley trees. Add a new vertex \( * \) to such a forest and join \( * \) to the root of each tree. We obtain a Cayley tree with root \( * \). This operation is a one-to-one correspondence, hence \( Y = q e^Y \).

\( \diamond \)

**Corollary 2.2** On the disc \( |q| < 1/e \), the function \( Y(q) \) is the inverse of the function \( q(Y) = Y/e^Y \).

**Proposition 2.3** We have \( (1 - Y)(1 + Z) = 1 \).

**Proof.**

\[
Z = DY = D(qe^Y) = qe^Y + qe^Y DY = qe^Y (1 + Z) = Y(1 + Z).
\]

Hence \( (1 - Y)(1 + Z) = 1 \).

\( \diamond \)

**Corollary 2.4** As an abstract algebra, \( \mathcal{A} \) is isomorphic to \( \mathbb{Q}[X, X^{-1}] \), where \( X = 1 - Y \).

\(^1\)We thank N. A’Campo for this reference.
Proposition 2.5 We have

\[ Y^k = k \sum_{n \geq 1} \frac{(n-1)\ldots(n-k+1)n^{n-k}}{n!} q^n = k \sum_{n \geq k} \frac{n^{n-k-1}}{(n-k)!} q^n. \]

Proof. Induction on \( k \). For \( k = 1 \) the assertion is true. To go from \( k \) to \( k + 1 \), one uses the equality

\[ D \left( \frac{Y^{k+1}}{k+1} \right) - \frac{Y^k}{k} = (Y^k - Y^{k-1})DY = Y^{k-1}(Y - 1)Z = -Y^k. \]

It is compatible with our expressions for \( Y^k \) and \( Y^{k+1} \), which determines \( Y^{k+1} \) up to a constant. But the constant term of \( Y^{k+1} \) vanishes. ◇

Now we study the powers of \( Z \).

Definition 2.6 Denote by \( A_n \) the sequence of integers

\[ A_n = \sum_{\substack{p+q=n \\ \ p,q \geq 1}} \frac{n!}{p!q!} p^p q^q. \]

Its first terms are 0, 2, 24, 312, 4720, ... We have

\[ Z^2 = \sum_{n \geq 1} \frac{A_n}{n!} q^n. \]

One can show that

\[ A_n = n! \sum_{k=0}^{n-2} \frac{n^k}{k!} \sim \sqrt{\pi/2} n^{n+\frac{1}{2}}. \]

As far as we know, there is no simple expression for the powers of \( Z \). However, we can prove that they are linear combinations of the series

\[ D^k Z = \sum_{n \geq 1} \frac{n^{n+k}}{n!} q^n \quad \text{and} \quad D^k (Z^2) = \sum_{n \geq 1} \frac{n^k A_n}{n!} q^n. \]

Proposition 2.7 For any integer \( k \geq 0 \), the power series \( D^k Z \) and \( D^k (Z^2) \) are polynomials in \( Z \) with positive integer coefficients, of degrees \( 2k + 1 \) and \( 2k + 2 \) respectively:

\[ D^k (Z) = (2k - 1)!! Z^{2k+1} + \text{lower order terms}, \]

\[ D^k (Z^2) = (2k)!! Z^{2k+2} + \text{lower order terms}. \]
**Proof.** Applying $D$ to both sides of the equality $(1 - Y)(1 + Z) = 1$ we get

$$-Z(1 + Z) + (1 - Y) \cdot DZ = 0.$$ 

Thus

$$DZ = \frac{Z(1 + Z)}{1 - Y} = Z(1 + Z)^2.$$ 

Hence

$$D(Z^2) = 2Z^2(1 + Z)^2.$$ 

Now we proceed by induction on $k$. 

**Corollary 2.8** For any positive integer $k$, the power series $Z^k$ is a linear combination with rational coefficients of the first $k$ series from the list $Z, Z^2, DZ, D(Z^2), D^2Z, D^2(Z^2), \ldots$.

From Proposition 2.5 and Corollary 2.8 we deduce the following theorem.

**Theorem 2** [21] The algebra $\mathcal{A}$ is spanned over $\mathbb{Q}$ by the power series

$$1, \quad \sum_{n \geq 1} \frac{n^{n+k}}{n!} q^n, \quad k \in \mathbb{Z}, \quad \sum_{n \geq 1} \frac{n^k A_n}{n!} q^n, \quad k \in \mathbb{N}.$$ 

Note that the Stirling formula together with the asymptotic for the sequence $A_n$ allows one to determine the leading term of the asymptotic for the coefficients of any series in $\mathcal{A}$. We have

$$\frac{n^n}{n!} \sim \frac{1}{\sqrt{2\pi n}} e^n, \quad \frac{A_n}{n!} \sim \frac{1}{2} e^n.$$ 

Note also that if, for some series $F \in \mathcal{A}$, we know in advance its degree in $Y$ and in $Z$, then we can reconstitute the series $F$ using only a finite number of its initial terms – a very useful property for computer experiments.

Combining both remarks, we see that initial terms of the sequence of coefficients of $F$ determine the asymptotic of the sequence.

### 2.2 Dendrology

**Definition 2.9** A *Cayley tree* is a tree with numbered vertices.
It is well-known (Cayley theorem) that there are \(n^{n-2}\) Cayley trees with \(n\) vertices. Note that the corresponding exponential generating function

\[
\sum_{n \geq 1} \frac{n^{n-2}}{n!} q^n
\]

lies in the algebra \(\mathcal{A}\).

Consider a Cayley tree \(T\) with two marked vertices \(a\) and \(b\). Denote by \(l(T)\) the distance between these vertices, i.e., the number of edges in the shortest path joining them.

**Definition 2.10** Denote by \(m_{n,k}\) and \(p_{n,k}\) the sums

\[
m_{n,k} = \sum_{T} l(T)^k, \quad p_{n,k} = \sum_{T} \frac{l(T)(l(T) - 1) \ldots (l(T) - k + 1)}{k!}
\]

where the sum is taken over all Cayley trees \(T\) with \(n\) vertices, two of which are marked.

For instance, \(m_{2,1} = p_{2,1} = 2\). Note that if we consider \(l(T)\) as a random variable, then \(m_{n,k}\) is its \(k\)th moment.

**Theorem 3** For any \(k\), the power series

\[
\sum_{n \geq 1} \frac{m_{n,k}}{n!} q^n \quad \text{and} \quad \sum_{n \geq 1} \frac{p_{n,k}}{n!} q^n
\]

lie in \(\mathcal{A}\).

**Example 2.11** It follows from the proof below that \(p_{n,1} = m_{n,1} = A_n\). This number is called the total height of Cayley trees and was introduced in [18].

**Proof of Theorem 3** It is sufficient to prove the theorem for \(p_{n,k}\).

Fix \(k\). There is a natural bijection between the following sets of objects. \(E_n\) is the set of Cayley trees with \(n\) vertices, on which one has marked two vertices by \(a\) and \(b\) and chosen \(k\) distinct edges on the shortest path from \(a\) to \(b\). The number of elements in \(E_n\) equals \(p_{n,k}\).

\(F_n\) is the set of ordered \((k + 1)\)-tuples of trees with \(n\) vertices in whole; the vertices are numbered from 1 to \(n\) and, in addition, two vertices \(a_i\) and \(b_i\), \(1 \leq i \leq k + 1\), are marked on each tree.
The bijection is established as follows. Take a forest from the set \( F_n \). Draw new edges \((b_1, a_2), (b_2, a_3), \ldots, (b_k, a_{k+1})\). We obtain a tree with \( k \) marked edges lying on the path between \( a_1 \) and \( b_{k+1} \), i.e., a tree from the set \( E_n \).

Now, the trees with two marked vertices are enumerated by the series \( Z \), therefore the exponential generating series for the sequence \( |F_n| \) is \( Z^{k+1} \).

3 Counting ramified coverings of the sphere

This section is devoted to the enumeration of ramified coverings of the sphere by surfaces of a fixed genus \( g \) and to a proof of Theorem 1.

3.1 The ELSV formula

Curiously, the most difficult part of the proof of Theorem 1 is the case with only one multiple ramification point, \( k = 1 \). We know no other way to prove it than to use the intersection theory on moduli spaces. The main ingredient of the proof is a theorem by T. Ekedahl, S. K. Lando, M. Shapiro, and A. Vainshtein that we formulate below after introducing some notation.

Let \( \mu = 1^{a_1}2^{a_2}\ldots \) be a partition with degeneracy \( r \). We define \(|\text{Aut}(\mu)|\) to be \(|\text{Aut}(\mu)| = a_1!a_2!\ldots\). For the formulation of the theorem it is more convenient to switch to using the additive notation for the partition \( \mu \), \( \mu = (b_1, \ldots, b_p) \), the \( b_i \) being the parts of \( \mu \). The Hurwitz number \( h_{g,n;\mu} \) is defined in Notation 1.3.

We denote by \( \mathcal{M}_{g,n} \) the moduli space of smooth genus \( g \) curves with \( n \) marked and numbered distinct points.

Further, \( \overline{\mathcal{M}}_{g,n} \) is the Deligne-Mumford compactification of this moduli space; in other words, \( \overline{\mathcal{M}}_{g,n} \) is the space of stable genus \( g \) curves with \( n \) marked points.

We denote by \( L_i \), \( 1 \leq i \leq n \), the \( i \)th tautological line bundle over \( \overline{\mathcal{M}}_{g,n} \); consider a point \( x \in \overline{\mathcal{M}}_{g,n} \) and the corresponding stable curve \( C_x \); then the fiber of \( L_i \) over \( x \) is the cotangent line to \( C_x \) at the \( i \)th marked point. The first Chern class of \( L_i \) is denoted by \( c_1(L_i) = \psi_i \).

We will use the expression

\[
\frac{1}{1 - \psi_i} = 1 + \psi_i + \psi_i^2 + \ldots \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).
\]
Further, we introduce the Hodge vector bundle $W$ over $\overline{M}_{g,n-r}$. The fiber of $W$ over a smooth curve is the set of holomorphic 1-forms on this curve. The fiber of $W$ over a general stable curve is the set of global sections of its dualizing sheaf. We do not give the details here (see the paper [4] itself). Suffice it to note that $W$ is a vector bundle of rank $g$.

Now we can write down the ELSV formula.

**Theorem 4 (The ELSV formula, [4])** For any $g$, $n$, and $\mu$ such that $2 - 2g - (n - r) < 0$, we have

$$h_{g,n;\mu} = \frac{(2n + 2g - 2 - r)!}{|\text{Aut}(\mu)|} \prod_{i=1}^{p} \frac{b_i^{b_i}}{b_i!} \times \frac{1}{(n - p - r)!} \int_{\overline{M}_{g,n-r}} c(W^*) \prod_{j=1}^{p} (1 - b_j \psi_j') \prod_{l=1}^{n-r} (1 - \psi_l + 1) \prod_{l=p+1}^{2g-2+p} (1 - \psi_{p+l})$$

3.2 Proof of Theorem 1

We are going to prove that all generating series $H_{g;\mu_1,\ldots,\mu_k}$ for the Hurwitz numbers $h_{g,n;\mu_1,\ldots,\mu_k}$ (see Notation 1.3) with respect to the number of sheets $n$ lie, once again, in the algebra $A$. In Section 4 we give a motivation for considering these particular generating series.

First consider the case of just one partition $k = 1$. This case was essentially covered in [8]. Recently, M. Kazarian [10] suggested an improvement of the theorem for $k = 1$, giving an explicit expression for the series $H_{g;\mu}$ in terms of the generators $Y$ and $Z$ of $A$.

**Theorem 5 [Kazarian]** Consider a partition $\mu = (b_1, \ldots, b_p)$, and let $m = \sum b_i$. Then we have

$$H_{g;\mu} = \frac{1}{|\text{Aut}(\mu)|} \prod_{i=1}^{p} \frac{b_i^{b_i}}{b_i!} \cdot Y^m (Z + 1)^{2g-2+p} \varphi(Z),$$

where $\varphi(Z)$ is the polynomial

$$\varphi(Z) = \sum_{l \geq 0} \frac{Z^l}{l!} \int_{\overline{M}_{g,p+l}} c(W^*) \prod_{j=1}^{p} \frac{1}{(1 - b_j \psi_j') \prod_{l=p+1}^{2g-2+p} (1 - \psi_{p+l})}.$$

Note that the last sum only goes up to $l = 3g - 3 + p$, otherwise the integral equals 0 for dimension reasons.
**Sketch of a proof** (borrowed from [10]). Introduce the series \( F = F_{g,b_1,...,b_p} \) in an infinite number of variables

\[ F(t_0, t_1, \ldots) = \sum_{t_{d_1}, \ldots, t_{d_l}} \frac{t_{d_1} \ldots t_{d_l}}{l!} \int_{\mathcal{M}_{g,p+l}} \frac{c(W^*) \psi_{d_1}^{d_1} \ldots \psi_{d_l}^{d_l}}{(1 - b_1 \psi_1) \ldots (1 - b_p \psi_p)} . \]

We have \( H_{g,b_1,...,b_p}(q) = q^m F(q,q,q,\ldots) \), where \( m = \sum b_i \). On the other hand, \( F \) satisfies the string and the dilaton equations (see [20]):

\[
\begin{align*}
\frac{\partial F}{\partial t_0} &= m F + \sum_{i \geq 0} t_{i+1} \frac{\partial F}{\partial t_i}, \\
\frac{\partial F}{\partial t_1} &= \chi F + \sum_{i \geq 0} t_i \frac{\partial F}{\partial t_i}, \quad \chi = 2g - 2 + p .
\end{align*}
\]

The theorem now follows from the following fact, obtained by a manipulation of PDEs. For any series \( F \) satisfying the above string and dilaton equations, let \( \varphi(q) = F(0,0,q,q,q,\ldots) \). Then we have

\[ q^m F(q,q,q,\ldots) = Y^m (1 + Z)^\chi \varphi(Z). \]

\[ \diamond \]

Now we deduce the general case from the case \( k = 1 \).

**Theorem 2** Fix any \( g \geq 0, k \geq 0 \). If \( g = 1 \), we suppose that \( k \geq 1 \). Then for any partitions \( \mu_1, \ldots, \mu_k \), the series

\[ H_{g,\mu_1,\ldots,\mu_k}(q) = \sum_{n \geq 1} \frac{h_{g,n;\mu_1,\ldots,\mu_k}}{c(n)!} q^n \]

lies in the algebra \( \mathcal{A} \).

**Proof of Theorem 2.** The theorem is proved by induction on the number \( k \) of partitions.

**Base of induction.** For \( k = 0,1 \), the result is obtained by a direct application of Theorem 5. In the case \( k = 0 \), we must use Theorem 5 with an empty partition \( \mu \).
There are three exceptional cases in which Theorem 5 cannot be applied: 
\( g = 0, k = 0; \) \( g = 0, k = 1, p \leq 2; \) \( g = 1, k = 0. \) These cases are discussed in Remark 3.1 below. It turns out that the assertion of Theorem 1 fails only if \( g = 1, k = 0, \) as stated in the formulation.

**Step of induction.** The step of induction is an almost exact repetition of the proof of Theorem 2 from our previous work [21]. We only give a short summary of the argument here. The proof goes in the spirit of [7]. A similar proof, using the formalism of colored permutations, is given in [10].

It is easy to see that there is only a finite number of possible cycle structures for a permutation that can be obtained as a product of two permutations with given cycle structures \( \mu_1 \) and \( \mu_2. \)

Let \( \mu_1 \) and \( \mu_2 \) be two partitions from the list \( \mu_1, \ldots, \mu_k. \) We can move the two corresponding ramification points on \( \mathbb{C}P^1 \) towards each other until they collapse. We obtain a new (not necessarily connected) ramified covering. Its monodromy at the new ramification point is the product of the monodromies of the two points that have collapsed.

Let us choose one of the possible cycle structures of the product monodromy and also one of the possible ways in which the covering can split into connected components. By the induction assumption, we obtain a series from the algebra \( \mathcal{A} \) assigned to each connected component of the covering. Indeed, each connected component is itself a ramified covering of the sphere as in Theorem 1 but with \( k - 1 \) fixed ramification types instead of \( k. \) We obtain the generating series for the number of nonconnected ramified coverings by multiplying the series that correspond to the connected components. Since it is a finite product of series lying in \( \mathcal{A}, \) we obtain again a series from \( \mathcal{A}. \)

Finally, we must add the generating series described above for all choices of types of nonconnected coverings. Since the number of choices is finite, we obtain, once again, a series from \( \mathcal{A}. \)

\[ \diamond \]

**Remark 3.1** Let us consider the exceptional cases \( g = 0, k = 0, 1 \) and \( g = 1, k = 0. \)

In the genus zero case, the ELSV formula transforms into a much simpler Hurwitz formula [9] [19], which turns out to be applicable even if the multiple ramification point has only 1 or 2 preimages.
We have, using the notation of Theorem 4 and Notation 1.3,
\[ h_{0,n;\mu} = \frac{(2n - 2 - r)!}{|\text{Aut}(\mu)|} \prod_{i=1}^{p} \frac{b_i^h}{b_i!} \cdot \frac{n^{n-r-3}}{(n-p-r)!}. \]

This formula is true for any \( n \geq p + r \) and for any partition \( \mu \) (including even the empty partition). We see that the corresponding generating series always lies in the algebra \( \mathcal{A} \).

The case \( g = 1, k = 0 \) is covered by the ELSV formula with an empty partition \( \mu \). Consider the moduli space \( \overline{M}_{1,1} \). Denote by \( \beta \) the 2-cohomology class of \( \overline{M}_{1,1} \) whose integral over the fundamental homology class equals 1. One can prove that the Hodge bundle over \( \overline{M}_{1,1} \) is a line bundle with first Chern class \( \beta/24 \). Therefore we obtain
\[ h_{1,n;\emptyset} = (2n)! \cdot \frac{1}{n!} \int_{\overline{M}_{1,n}} \frac{1 - \frac{1}{24} \beta}{(1 - \psi_1) \ldots (1 - \psi_n)}. \]

From this we get
\[ \sum_{n \geq 1} \frac{h_{1,n;\emptyset}}{(2n)!} q^n = \frac{1}{24} \sum_{n \geq 1} \frac{A_n}{n} q^n. \]

This series does not lie in \( \mathcal{A} \) (and constitutes the only exception to the general rule). It suffices to consider the partition \( \mu = (1) \), which amounts to distinguishing one sheet in the ramified covering, to obtain the series
\[ \frac{1}{24} \sum_{n \geq 1} \frac{A_n}{n!} q^n \in \mathcal{A}. \]

### 4 Random metrics and 2-dimensional gravity

In this section we propose a model of 2-dimensional gravity via the enumeration of ramified coverings. We show that the “free energy” function and the values of “observables” coincide with those obtained in other models.

We also draw a parallel between the study of spaces of Riemannian metrics using ramified coverings of the sphere and the study of spaces of abelian differentials using ramified coverings of the torus.
4.1 Models of 2-dimensional gravity

Here we explain what sort of questions about ramified coverings arise in 2-dimensional gravity and why. Precise mathematical results are given below in Sections 4.2 and 4.3.

In every problem of statistical physics one starts with introducing a space of states and by assigning an energy to every state.

In 2-dimensional gravity, a state is a 2-dimensional compact oriented real not necessarily connected surface endowed with a Riemannian metric. Two surfaces like that are equivalent, i.e., correspond to the same state, if they are isometric.

Consider a surface $S$ with a Riemannian metric. Let $\chi(S)$ be its Euler characteristic and $A$ its total area. To such a surface one assigns an energy $E = \lambda A + \mu \chi(S)$.

Here $\lambda$ and $\mu$ are two constants called the cosmological constant and the gravitational constant, respectively. Note that $\chi(S)$ is actually the integral over $S$ of the scalar curvature of the metric. The fact that this integral takes such a simple form is special to dimension 2.

Now the first thing to do is to compute the partition function $z(\lambda, \mu)$ or, equivalently, the free energy $f(\lambda, \mu)$

$$z(\lambda, \mu) = \int_{\text{states}} e^{-E}, \quad f(\lambda, \mu) = \ln z(\lambda, \mu) = \sum_{g \geq 0} \int_{\text{metrics}} e^{-E}.$$ 

The free energy is the sum of contributions of connected surfaces, while the partition function is the sum of contributions of all surfaces.

Neither of the above integrals is well-defined mathematically, but we would still like to compute them. To do that, physicists introduced a discrete model of Riemannian metrics, replacing them by quadrangulations [2, 20] (see also [14], Chapter 3 for a mathematical description). In this model, instead of considering Riemannian metrics, one considers metrics obtained by gluings of squares of area $\varepsilon$. Our goal is to show that the ramified coverings of the sphere provide a new (maybe more natural) discrete model of Riemannian metrics.

Fix a positive number $\varepsilon$. Consider a sphere with the standard (round) Riemannian metric of total area $\varepsilon$. On this sphere, choose at random $2n + 2g - 2$ points. Now chose a random connected $n$-sheeted covering of the sphere
with simple ramifications over the $2n + 2g - 2$ chosen points. The covering surface $S$ will automatically be of genus $g$. The metric on the sphere can be lifted to $S$, which will give us a metric with constant positive curvature except at the critical points, where it has conical singularities with angles $4\pi$. This metric is, of course, not Riemannian. However, one can argue that if $\varepsilon$ is very small and the number of sheets very large, a random metric obtained in this way looks similar to a random Riemannian metric (unless we look at them through a microscope to reveal the difference). We do not know any rigorous statement that would formalize this intuitive explanation, but the same argument is used by physicists to justify the usage of quadrangulations.

Using our discrete model of metrics, one can write the free energy for the 2-dimensional gravity in the following way:

$$f(\lambda, \mu) = \sum_{g,n} \frac{\varepsilon^{2n+2g-2}}{(2n + 2g - 2)!} h_{g,n;\emptyset} e^{-\lambda \varepsilon - \mu (2-2g)}.$$ 

Here the factor $\varepsilon^{2n+2g-2}/(2n + 2g - 2)!$ is the volume of the space of choices of $2n + 2g - 2$ unordered points on the sphere of area $\varepsilon$, while $n\varepsilon$ in the exponent is the area of the covering surface.

In Section 4.2 we show that if we let $n \to \infty$ while $g$ remains fixed, we have

$$\frac{h_{g,n;\emptyset}}{(2n + 2g - 2)!} \sim \varepsilon^n n^{\frac{5}{2}(g-1)-1} b_g$$

for some constants $b_g$. Thus the coefficients of $f$ have the following asymptotic:

$$\frac{\varepsilon^{2n+2g-2}}{(2n + 2g - 2)!} h_{g,n;\emptyset} e^{-\lambda \varepsilon - \mu (2-2g)} \sim b_g e^{-(\lambda \varepsilon - 2 \ln \varepsilon + 1)n} (\varepsilon e^\mu)^{2g-2} n^{\frac{5}{2}(g-1)-1}.$$ 

Now we make the final step by letting $\varepsilon$ tend to 0 in the expression of $f$. To obtain an interesting limit for the free energy, we must make $\lambda$ and $\mu$ depend on $\varepsilon$. We want to use

$$\sum_{n \geq 1} n^{\gamma-1} e^{-\delta n} \sim \frac{\Gamma(\gamma)}{\delta^\gamma} \quad \text{as} \quad \delta \to 0.$$ 

Therefore we set $\gamma = \frac{5}{2}(g - 1)$ and we let $\delta = \lambda \varepsilon - 2 \ln \varepsilon - 1$ tend to 0, while

$$y = \frac{(\varepsilon e^\mu)^{4/5}}{\delta} = \frac{(\varepsilon e^\mu)^{4/5}}{\lambda \varepsilon - 2 \ln \varepsilon - 1}.$$
remains fixed. This gives us the final expression of the free energy, now depending on only one variable $y$:

$$f(y) = \Gamma(-5/2) b_0 y^{5/2} - b_1 \ln y + \sum_{g \geq 2} \Gamma\left(\frac{5(g-1)}{2}\right) b_g y^{5(1-g)/2}.$$  

The coefficients $\Gamma\left(\frac{5(g-1)}{2}\right) b_g$ are rational for odd $g$ and rational multiples of $\sqrt{2}$ for even $g$.

Our above treatment is parallel to E. Witten’s treatment of the quadrangulation model in [20]. Denote by $Q_{g,n}$ the number of ways to divide a surface of genus $g$ into $n$ squares. Then the study of the quadrangulation model involves the asymptotic of $Q_{g,n}$, which is given by

$$Q_{g,n} \sim 12^g n^{\frac{5}{2}(g-1)-1} b'_g,$$

for another sequence of constants $b'_g$. This sequence was studied using matrix integrals, and it is known that a generating function for the sequence $b'_g$ satisfies the Painlevé I equation. In the next section we show a similar result for the constants $b_g$. This implies that the functions $f$ obtained in the two models coincide up to a rescaling of the variable $y$; more precisely, we have

$$b'_g = 2^{\frac{5}{2}(g-1)+1} b_g.$$

In the treatment of the quadrangulation model in [20], Witten also introduced observables that correspond to counting quadrangulations with “impurities”, that is, the number of ways to divide a surface into a large number of squares and a fixed number of given polygons. Each observable $\tau_d$ is represented by a formal linear combination of a 2-gon, a 4-gon, and so on, up to a $(2d+2)$-gon. The values of these observables combine into a generating function $F(t_0, t_1, \ldots)$ that can be studied using matrix integrals. It turns out that $\partial^2 F / \partial t_0^2$ is a solution of the Korteweg–de Vries (KdV) hierarchy. This solution is called the “string solution”.

From now on the notation $\langle \cdot \rangle$ or $\langle \cdot \rangle_g$ will mean

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_g = \langle \tau_{d_1} \ldots \tau_{d_n} \rangle = \int_{\mathcal{H}_{g,n}} \psi_{d_1}^{d_1} \ldots \psi_{d_n}^{d_n}.$$
It turns out that the generating series
\[ F(t_0, t_1, \ldots) = \sum_{n \geq 1} \sum_{d_1, \ldots, d_n} \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \frac{t_{d_1} \ldots t_{d_n}}{n!}. \]

coincides with the series \( F \) obtained from the quadrangulation model. In particular, its second derivative \( U = \partial^2 F / \partial t_0^2 \), satisfies the KdV equation:
\[
\frac{\partial U}{\partial t_1} = U \frac{\partial U}{\partial t_0} + \frac{1}{12} \frac{\partial^3 U}{\partial t_0^3}.
\]

This was conjectured by E. Witten in [20] and proved by M. Kontsevich in [12].

In Section 4.3 we show that the numbers \( \langle \tau_{d_1} \ldots \tau_{d_n} \rangle \) can also be obtained in the model of ramified coverings. Each observable \( \tau_d \) is represented by a formal linear combination of a noncritical point, a simple critical point, and so on, up to a \( d \)-tuple critical point.

Recently, M. Kazarian and S. K. Lando [11] found an independent proof of the fact that the function \( U \) arising in the enumeration of Hurwitz numbers satisfies the KdV equation. This has lead them to a new proof of Witten’s conjecture.

### 4.2 The Painlevé I equation

The results of this section were obtained in common with S. Lando.

**Proposition 4.1** For a fixed \( g \), we have
\[
\frac{h_{g,n,\emptyset}}{(2n+2g-2)!} \sim e^{n^2 \frac{5}{2} (g-1)^{-1}} b_g \quad \text{as} \quad n \to \infty,
\]

where
\[
b_g = \frac{\langle \tau_2^{3g-3} \rangle}{(3g-3)! \cdot 2^{4(g-1)} \cdot \Gamma \left( \frac{5}{2} (g-1) \right)}.
\]

**Proof.** From Theorem [5] we see that \( H_{g,\emptyset} \) is a polynomial in \( Z \) with leading term
\[
\frac{\langle \tau_2^{3g-3} \rangle}{(3g-3)!} Z^{5g-5}.
\]
Indeed, if \( l = 3g - 3 \), the degree of the class \( \psi_1^2 \ldots \psi_l^2 \) is exactly the dimension of \( \overline{M}_{g,l} \) (both are equal to \( 6g - 6 \)). Therefore the classes \( c(W^*) \) and \( 1/(1 - \psi) \) do not contribute. On the other hand, by Proposition 2.7, the coefficients of the series \( Z^l \) grow as

\[
\frac{n^{l-1} e^n}{\Gamma(l/2) 2^{l/2}}.
\]

Multiplying the coefficient of the leading term

\[
\frac{\langle r_2^{3g-3} \rangle}{(3g - 3)!} Z^{5g-5}
\]

of \( H_g \) by the asymptotic of coefficients of \( Z^{5g-5} \) we obtain the leading term of the asymptotic of \( h_{g,n} \), in particular, the constant \( b_g \).

The first values of the constants \( b_g \) are

\[
\begin{align*}
b_0 &= \frac{1}{\sqrt{2\pi}}, & b_1 &= \frac{1}{2^4 \cdot 3}, & b_2 &= \frac{1}{\sqrt{2\pi}} \frac{7}{2^5 \cdot 3^3 \cdot 5}, \\
b_3 &= \frac{5 \cdot 7^2}{2^1 \cdot 3^5}, & b_4 &= \frac{1}{\sqrt{2\pi}} \frac{7 \cdot 5297}{2^{11} \cdot 3^8 \cdot 5^2 \cdot 11 \cdot 13}.
\end{align*}
\]

The above expression for \( b_g \) allows us to rewrite the function \( f''(y) = u(y) \) in the following way:

\[
u(y) = -\sqrt{2y} + \frac{1}{12} (2y)^{-2} + \sum_{g \geq 2} (5 - 5g)(3 - 5g) \frac{\langle r_2^{3g-3} \rangle}{(3g - 3)!} (2y)^{\frac{1}{2} - \frac{5g}{2}}.
\]

(2)

**Proposition 4.2** The function \( u(y) \) satisfies the Painlevé I equation

\[
u^2(y) + \frac{1}{6} u''(y) = 2y.
\]

**Proof.** By extracting the coefficient of \( t_2^{3g-1} \) in the KdV equation (I); we obtain, for every \( g \geq 1 \),

\[
\frac{\langle r_0 r_1 r_2^{3g-1} \rangle_{g}}{(3g - 1)!} = \sum_{g' + g'' = g, g' \geq 1, g'' \geq 0} \frac{\langle r_0^2 r_2^{3g'-1} \rangle_{g'}}{(3g' - 1)!} \cdot \frac{\langle r_0^3 r_2^{3g''} \rangle_{g''}}{(3g'')!} + \frac{1}{12} \frac{\langle r_0^5 r_2^{3g-1} \rangle_{g-1}}{(3g - 1)!}.
\]

(3)
Now we use the string and the dilaton equations to kill the $\tau_0$ and $\tau_1$ factors in all the brackets of (3). We obtain the following identities (with some exceptions for low genus):

\[
\frac{\langle \tau_0^2 \tau_1 \tau_2^{3g-1} \rangle_g}{(3g-1)!} = (5g - 5)(5g - 3)(5g - 1) \frac{\langle \tau_2^{3g-3} \rangle_g}{(3g - 3)!}, \quad \frac{\langle \tau_0^2 \tau_1 \tau_2^2 \rangle_1}{2!} = \frac{1}{3}.
\]

\[
\frac{\langle \tau_0^2 \tau_2^{3g'-1} \rangle_{g'}}{(3g'-1)!} = (5g - 5)(5g - 3) \frac{\langle \tau_2^{3g'-3} \rangle_{g'}}{(3g' - 3)!}, \quad \frac{\langle \tau_0^2 \tau_2^2 \rangle_1}{2!} = \frac{1}{12}.
\]

\[
\frac{\langle \tau_0^3 \tau_2^{3g''} \rangle_{g''}}{(3g'')!} = (5g - 5)(5g - 3)(5g - 1) \frac{\langle \tau_2^{3g''-3} \rangle_{g''}}{(3g'' - 3)!}.
\]

\[
\langle \tau_0^3 \rangle_0 = 1, \quad \frac{\langle \tau_0^3 \tau_2 \rangle_1}{3!} = \frac{1}{3}.
\]

\[
\frac{\langle \tau_0^5 \tau_2^{3g-1} \rangle_{g-1}}{(3g-1)!} = (5g - 10)(5g - 8)(5g - 6)(5g - 4)(5g - 2) \frac{\langle \tau_2^{3g-6} \rangle_{g-1}}{(3g - 6)!},
\]

\[
\frac{\langle \tau_0^5 \tau_2 \rangle_0}{2!} = 3, \quad \frac{\langle \tau_0^5 \tau_2^2 \rangle_1}{5!} = 16.
\]

Substitute these expressions in (3) and compare to the expression (2) of $u$. Taking into account the exceptional starting terms we obtain

\[
\sqrt{2y} \left( u'(y) + \frac{1}{\sqrt{2y}} \right) = \left( u(y) + \sqrt{2y} \right) u'(y) + \frac{1}{12} u''(y).
\]

We rewrite this as

\[ u(y) u'(y) + \frac{1}{12} u''(y) = 1 \]

and integrate it once to obtain

\[ u^2(y) + \frac{1}{6} u''(y) = 2y. \]
4.3 The KdV hierarchy

Now we show how to obtain the numbers \( \langle \tau_{d_1} \ldots \tau_{d_p} \rangle \) as leading term coefficients of the asymptotic of Hurwitz numbers.

Note that Okounkov and Pandharipande \[17\] also obtained the numbers \( \langle \tau_{d_1} \ldots \tau_{d_p} \rangle \) using asymptotics of Hurwitz numbers. However their asymptotics are different from ours and have no direct physical interpretation.

The Hurwitz numbers involved are those that count ramified coverings with many simple ramification points, but only one multiple ramification point that has \( p \) marked preimages. Each marked preimage corresponds to a factor \( \tau \) in the bracket. Using the notation from Section \( 3 \), the numbers we are interested in are

\[
|\text{Aut}(\mu)| \cdot h_{g,n;\mu}, \text{ with } \mu = (b_1, \ldots, b_p).
\]

The factor \( |\text{Aut}(\mu)| \) is due to the fact that the marked preimages are numbered.

Denote by \( b \) a \( b \)-tuple preimage of the special ramification point. Our result is then best described by the following symbolic formula:

\[
\tau_d = \frac{1}{0! (d+1)d} \cdot \frac{d+1}{Y^{d+1}} - \frac{1}{1! d^{d-1}} \cdot \frac{d}{Y^d} + \ldots + (-1)^d \frac{1}{d! 1^0} \cdot \frac{1}{Y}.
\]

The recipe for obtaining the number \( \langle \tau_{d_1} \ldots \tau_{d_p} \rangle \) is the following.

1. Replace each \( \tau_d \) by the right-hand side of the above symbolic equality.
2. Expand the product to obtain a linear combination of terms of the form

\[
\text{const} \cdot \frac{b_1 \ldots b_p}{Y^{b_1 + \ldots + b_p}}.
\]

3. To each such term assign the partition \( \mu = (b_1, \ldots, b_p) \) and the series

\[
\text{const} \cdot |\text{Aut}(\mu)| H_{g;\mu}.
\]

4. Add all the series thus obtained. This gives a series in \( \mathcal{A} \) that we denote by \( H[\tau_{d_1} \ldots \tau_{d_p}](q) \).

**Theorem 6** We have

\[
H[\tau_{d_1} \ldots \tau_{d_p}] = \langle \tau_{d_1} \ldots \tau_{d_p} \rangle (Z + 1)^{2g-2+l}.
\]

The asymptotic of the coefficient of \( q^n \) (as \( n \to \infty \)) in \( H[\tau_{d_1} \ldots \tau_{d_p}](q) \) equals

\[
\frac{\langle \tau_{d_1} \ldots \tau_{d_p} \rangle}{2^{2g-2+p} \Gamma \left( \frac{2g-2+p}{2} \right)} e^n n^{2g-2+p-1}.
\]
Proof. Here again we will use Theorem 5. The crucial part of this proposition is the polynomial
\[ \varphi_{b_1,\ldots,b_p}(Z) = \sum_{l \geq 0} \frac{Z^l}{l!} \int_{\mathcal{M}_{g,p+l}} \frac{1}{(1 - b_1 \psi_1) \ldots (1 - b_p \psi_p)} \frac{c(W^*) \psi_{p+1}^2 \ldots \psi_{p+l}^2}{(1 - \psi_{p+1}) \ldots (1 - \psi_{p+l})}. \]

We are going to consider linear combinations of such polynomials for different \( b_i \)'s. Our goal is to obtain a cancellation of all higher order terms in \( Z \) leaving only a constant term \( (l = 0) \). This constant term will turn out to be \( \langle \tau_{d_1} \ldots \tau_{d_p} \rangle \).

First of all, here is a linear combination of the series \( 1/(1 - \psi), 1/(1 - 2\psi), \ldots, 1/(1 - (d + 1)\psi) \) whose terms up to \( \psi^{d-1} \) vanish:
\[ \frac{1}{d!} \sum_{b=1}^{d+1} \frac{(-1)^{d+1-b} \binom{d}{b-1}}{1 - b\psi} = \psi^d + O(\psi^{d+1}). \]

In the expression
\[ |\text{Aut}(\mu)| H_{g,n} = (Z + 1)^{2g-2+p} Y^{b_1 + \ldots + b_p} \prod_{i=1}^{p} \frac{b_i^{b_i}}{b_i!} \cdot \varphi(Z), \]

the integrand \( 1/(1 - b_i \psi_i) \) appears with an additional factor \( Y^{b_i} b_i^{b_i}/b_i! \) in front of the integral. To compensate for this factor, we multiply the coefficients of \( \varphi \) by its inverse, which gives
\[ \sum_{b=1}^{d+1} \frac{(-1)^{d+1-b} \binom{d}{b-1}}{(d+1-b)! b^{d-1}} \cdot \frac{1}{Y^b}. \]

This is precisely the formula that we gave for \( \tau_d \).

Multiplying such expressions for \( d = d_1, \ldots, d_p \) and adding them up we obtain
\[ H[\tau_{d_1} \ldots \tau_{d_n}] = (Z + 1)^{2g-2+p} \sum_{l \geq 0} \frac{Z^l}{l!} \int_{\mathcal{M}_{g,p+l}} (\psi_1^{d_1} \ldots \psi_p^{d_p} + \text{h.o.t.}) \frac{c(W^*) \psi_{p+1}^2 \ldots \psi_{p+l}^2}{(1 - \psi_{p+1}) \ldots (1 - \psi_{p+l})}, \]

where “h.o.t.” means “higher order terms”.
The above integral vanishes for dimension reasons whenever $l > 0$. For $l = 0$, the factor $c(W^*)$ contributes only by $c_0(W^*) = 1$. Thus $H[\tau_{d_1} \ldots \tau_{d_p}] = \langle \tau_{d_1} \ldots \tau_{d_p} \rangle \cdot (Z + 1)^{2g-2+p}$ as claimed.

The second assertion of the theorem follows from the first one and from the asymptotic of the coefficients of $Z^k$ (Proposition 2.7).

Remark 4.3 (Kazarian) Equality (4) can be used to express the numbers $\langle \tau_{d_1} \ldots \tau_{d_n} \rangle$ as finite linear combinations of Hurwitz numbers, without considering any asymptotics. More precisely, we have

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle = \sum_{b_1,\ldots,b_p} \left( \prod_{i=1}^p \frac{(-1)^{d_i+1-b_i}}{(d_i + 1 - b_i)! b_i^{b_i-1}} \right) \frac{\left| \text{Aut}(b_1,\ldots,b_p) \right| h_{g,n;b_1,\ldots,b_p}}{c(n)!}.$$ 

Here the sum is over $1 \leq b_i \leq d_i + 1$, the number of sheets is $n = \sum b_i$ and $c(n) = n + p + 2g - 2$.

4.4 Ramified coverings of a torus and abelian differentials

Fix an integer $g \geq 1$ and a list of $p$ nonnegative integers $b_1,\ldots,b_p$ with the condition $\sum b_i = 2g - 2$. We consider the space $D_{g;b_1,\ldots,b_p}$ of abelian differentials on Riemann surfaces of genus $g$, with zeroes of multiplicities $b_1,\ldots,b_p$. More precisely, $D_{g;b_1,\ldots,b_p}$ is the space of triples $(C,\{x_1,\ldots,x_p\},\alpha)$, where $C$ is a smooth complex curve, $x_1,\ldots,x_p \in C$ are distinct marked points, and $\alpha$ is an abelian (= holomorphic) differential on $C$ whose zero divisor is precisely $b_1x_1 + \ldots + b}px_p$.

It turns out that the space $D_{g;b_1,\ldots,b_p}$ has a natural integer affine structure. This means that it can be covered by charts of local coordinates in such a way that the transition functions are affine maps with integer coefficients. Such local coordinates are introduced as follows. Fix a basis $l_1,\ldots,l_{2g+p-1}$ of the relative homology group $H_1(C,\{x_1,\ldots,x_p\},\mathbb{Z})$. Then the integrals of $\alpha$ over the cycles $l_i$ are the local coordinates we need. The area function

$$A : (C,\alpha) \mapsto \frac{i}{2} \int_C \alpha \wedge \bar{\alpha}$$

is a quadratic form with respect to the affine structure.

The integer affine structure allows one to define a volume measure on the space $D_{g;b_1,\ldots,b_p}$. It is then a natural question to find the total volume of the
part of the space $D_{g; b_1, \ldots, b_p}$ defined by $A \leq 1$ (the volume of the whole space being infinite).

A. Eskin and A. Okounkov [5] obtained an effective way to calculate these volumes using the asymptotic for the number of ramified coverings of a torus. Consider the elliptic curve obtained by gluing the opposite sides of the square $(0, 1, i, 1+i)$ endowed with the abelian differential $dz$. Given a ramified covering of this elliptic curve with critical points of multiplicities $b_1, \ldots, b_p$, we can lift the abelian differential to the covering curve and obtain a point of $D_{g; b_1, \ldots, b_p}$. One can then easily show that such points are densely and uniformly distributed in $D_{g; b_1, \ldots, b_p}$ if one considers coverings with a big number of sheets. Moreover, R. Dijkgraaf [3] and S. Bloch and A. Okounkov [1] showed that the generating series for the ramified coverings of the torus that arise in this study are quasi-modular forms. In other words, they lie in the algebra

$$\mathbb{Q}[E_2, E_4, E_6],$$

where $E_{2k}$ are the Eisenstein series

$$E_{2k}(q) = \frac{1}{2} \zeta(1-2k) + \sum_{n \geq 1} \left( \sum_{d|n} d^{2k-1} \right) q^n.$$

We conclude with the following comparison between the counting of ramified coverings of a sphere and of a torus.

**Sphere:** The generating series enumerating the ramified coverings lie in the algebra $\mathcal{A}$.

**Torus:** The generating series enumerating the ramified coverings lie in the algebra of quasi-modular forms.

**Sphere:** The coefficients of a generating series grow as $e^n \cdot n^{\gamma-1} \cdot c$. The exponent $\gamma$ is a half-integer. The number $-\gamma$ is called the string susceptibility. The constant $c$ is an observable in 2-dimensional gravity.

**Torus:** The sum of the first $n$ coefficients of a generating series grows as $n^d \cdot c$. The number $d$ is the complex dimension of the corresponding space of abelian differentials. The constant $c$ is its volume.

**Sphere:** The observables can be arranged into a generating series that is a solution of the KdV hierarchy.

**Torus:** As far as we know, nobody has tried to arrange the volumes of the spaces of abelian differentials into a unique generating series.
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