ARITHMETIC EXPONENT PAIRS FOR ALGEBRAIC TRACE FUNCTIONS AND APPLICATIONS

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Abstract. We study short sums of algebraic trace functions via the \( q \)-analogue of van der Corput method, and develop methods of arithmetic exponent pairs that coincide with the classical case while the moduli has sufficiently good factorizations. As an application, we prove a quadratic analogue of Brun-Titchmarsh theorem on average, bounding the number of primes \( p \leq X \) with \( p^2 + 1 \equiv 0 \pmod{q} \). The other two applications include a larger level of distribution of divisor functions in arithmetic progressions and a sub-Weyl subconvex bound of Dirichlet \( L \)-functions studied previously by Irving.

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1. Introduction

1.1. Background. Given a positive integer \( q \) and \( \Psi : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C} \), a non-trivial bound for the average

\[
S(\Psi; I) = \sum_{n \in I} \Psi(n)
\]

2010 Mathematics Subject Classification. 11T23, 11L05, 11L07, 11N13, 11N36, 11N37, 11M06.
Key words and phrases. \( q \)-analogue of van der Corput method, arithmetic exponent pairs, trace functions of \( \ell \)-adic sheaves, Brun-Titchmarsh theorem, linear sieve.
is highly desired in numerous problems in analytic number theory, where \( I \) is a certain interval. The resolution of such a problem usually depends heavily on some tools from Fourier analysis. A typical example is dated back to the classical estimate for incomplete character sums of Pólya and Vinogradov, who applied a completing method (or equivalently a certain Fourier expansion) to transform the incomplete sum to complete ones and thus obtained non-trivial bounds as long as \( |I| \geq q^{1/2+\varepsilon} \). An ingenious improvement was later realized by Burgess \([Bu1, Bu2]\), who was able to work non-trivially for shorter sums and in particular, the first subconvexity can be derived for Dirichlet \( L \)-functions. While \( \Psi \) is specialized to some other examples such as additive characters, Kloosterman sums, one can also follow the approach of Pólya and Vinogradov, and then succeed roughly in the range \( |I| \geq q^{1/2+\varepsilon} \).

The above \( \frac{1}{2} \)-barrier usually plays a crucial role in applications, and is thus expected to be beaten in many instances. As an important example in history, we recall the pioneer work of Hooley \([Ho5]\) on the greatest prime factors of cubic irreducible polynomials. To seek a positive constant \( \eta \) such that \( P^+(n^3+2) > n^{1+\eta} \) infinitely often, Hooley assumed, for some \( \delta > 0 \), that

\[
\sum_{n \in I} e\left( \frac{an}{q} \right) \ll |I|q^{-\delta}, \quad (a, q) = 1
\]  

holds for all intervals \( I \) with \( |I| > q^{\theta} \) for some \( \theta < \frac{1}{3} \), where \( P^+(n) \) is the largest prime factor of \( n \). However, the completing method of Pólya and Vinogradov barely works for \( \theta > \frac{1}{2} \). The existence of such a positive constant \( \eta \) is nowadays known unconditionally due to the efforts of Heath-Brown \([HB3]\). The approach of Heath-Brown is not devoted to prove a strong estimate such as (1.2), and instead he modified the Chebyshev-Hooley method so that some exponential sums with special features arose. In particular, he was able to allow the modulo \( q \) to have reasonable factorizations, and an estimate of the close strength with (1.2) can be obtained for such special \( q \) by introducing the idea from classical estimates for analytic exponential sums, which is now usually known as \( q \)-analogue of van der Corput method and in what follows we refer it to \( q \)-vdC for short.

In his recent breakthrough on bounded gaps between primes, Zhang \([Zh]\) proved a certain level of distribution of primes in arithmetic progressions that is beyond \( \frac{1}{2} \). A key feather is that he assumed the moduli has only small prime factors and thus allows suitable factorizations. He was able to go beyond the Pólya-Vinogradov barrier in the resultant exponential sums with such special moduli, and the underlying idea can also be demonstrated by \( q \)-vdC. There are many other examples that benefited a lot from \( q \)-vdC, and we will try to present a short list in later discussions.

As in the above instances, one arrives at estimates for certain complete sums over \( \mathbb{Z}/q\mathbb{Z} \) in the last step, and some tools from algebraic geometry enter the picture.
to guarantee square-root cancellations. On the other hand, Fouyty, Kowalski and Michel initiated, from various analytic and geometric points of view, extensive investigations on general trace functions associated to some geometrically isotypic \( \ell \)-adic sheaves on \( \mathbb{A}^1_{\mathbb{F}_p} \) (see [FKM1, FKM2, FKM3] for instance). They are trying to establish a more direct and close relation between analytic number theory and algebraic geometry, where, in most cases, the second one serves as a powerful tool and provide fertile resources for the first, as one can see from the above examples.

1.2. Plan of this paper. In this paper, we study the \( q \)-analogue of van der Corput method for general trace functions, which are composite in the sense that they are defined by suitable products of \( \ell \)-adic sheaves on \( \mathbb{A}^1_{\mathbb{F}_p} \) for a couple of primes \( p \). Roughly speaking, we would like to bound the average (1.1) with \( \Psi \) specialized to such composite trace functions, which contains (1.2) as a special case. In fact, this project was initiated by Polymath [Po] in the improvement to Zhang’s constant. Our observation here allows one to develop a method on arithmetic exponent pairs analogous to those in the classical van der Corput method, from which one can find almost optimal estimates for such averages as long as the moduli has sufficiently good factorizations. On the other hand, one can also develop the multiple exponent pairs that demonstrates how the upper bounds depend on each factor of moduli. We will start from an abstract exponent pair and then produce a series of exponent pairs after applying the \( A \)- and \( B \)-processes in \( q \)-vdC for sufficiently many times.

Three applications of \( q \)-vdC are also derived. On one hand, we can prove a quadratic analogue of Brun-Titchmarsh theorem on primes in arithmetic progressions, for which the linear Rosser-Iwaniec sieve plays a fundamental role and thanks to the efforts of Iwaniec [Iw1], we are able to take full advantage of the well factorizations of remainder terms. On the other hand, we can, using our arithmetic exponent pairs, recover a larger level of Irving on the divisor functions in arithmetic progressions and a sub-Weyl subconvexity for Dirichlet \( L \)-functions.

The ideas and methods of arithmetic exponent pairs are also very powerful on several occasions of the square sieve of Heath-Brown and Jutila’s refinement on the circle method. We will discuss such applications in forthcoming papers.

Due to the special structure of this paper, we cannot state explicitly \( q \)-vdC and arithmetic exponent pairs for algebraic trace functions in the first section; however, we would like to present the three applications mentioned as above.

1.3. Quadratic Brun-Titchmarsh theorem on average. Our first application is devoted to count primes in arithmetic progressions on average.

Let \( q \) be a fixed positive integer and \( (a, q) = 1 \), we are interested in the counting function of primes

\[
\pi(x; q, a) = \sum_{\substack{p \leq x \\ \text{prime} \\ \text{p} \equiv a \pmod{q}}} 1.
\]
Setting $q = x^\theta$, one may expect, as $x \to +\infty$, that
\begin{equation}
\pi(x; q, a) < \left\{ C(\theta) + o(1) \right\} \frac{x}{\varphi(q) \log x}
\end{equation}
holds for $\theta$ as large as possible with some $C(\theta) > 0$. This is called Brun-Titchmarsh theorem since Titchmarsh is the first who proved the existence of such $C(\theta)$ via Brun’s sieve. By virtue of a careful application of Selberg’s sieve, van Lint & Richert [LR] showed that $C(\theta) = 2/(1 - \theta)$ is admissible for $\theta \in (0, 1)$, uniformly in $(a, q) = 1$. This was later sharpened by Motohashi [Mo] for $\theta \in (0, \frac{1}{2})$. Iwaniec [Iw1] introduced his bilinear forms of remainder terms in linear sieves [Iw1] to this problem and obtained $8/(6 - 7\theta)$ for $\theta \in (\frac{2}{3}, \frac{2}{5})$. The progress seems quite slow in this direction and the lastest result going beyond $\frac{1}{2}$, to our best knowledge, is due to Friedlander & Iwaniec [FrI] who gained $2/(1 - \theta) - (1 - \theta)^5/2^{12}$ for $\theta \in (\frac{6}{11}, 1)$.

On the other hand, motivated by the problem on greatest prime factors of shifted primes, Hooley [Ho2, Ho3, Ho4] initiated to bound $\pi(x; q, a)$ from above with extra average over $q$. The subsequent improvement is due to Iwaniec [Iw2], who combined Hooley’s argument with his bilinear remainder terms in linear sieves. It is a comment treatment to transform sums over primes to sums over integers via sieve methods and then to estimates for exponential sums after Poisson summation. One then arrives at Kloosterman sums, so that Weil’s bound does this business as argued by Hooley [Ho2] and Iwaniec [Iw2]. Thanks to the work of Deshouillers & Iwaniec [DI1] on the control of sums of Kloosterman sums, one can do much better on the level of linear sieves, see Deshouillers-Iwaniec [DI3], Fouvry [Fo1, Fo2], Baker-Harman [BH], etc. However, due to the use of the “switching-moduli” trick, the residue class $a$ is usually assumed to be fixed.

We now extend the classical Brun-Titchmarsh theorems to the quadratic case. Let $f \in \mathbb{Z}[X]$ be a fixed quadratic polynomial, and define
\begin{equation}
\pi_f(x; q) := \sum_{\substack{p \leq x \atop f(p)\equiv 0 \pmod{q}}} 1.
\end{equation}

Note that
\begin{equation}
\pi_f(x; q) = \sum_{\substack{a \pmod{q} \atop f(a)\equiv 0 \pmod{q}}} \pi(x; q, a).
\end{equation}

In many situations, the number of solutions to $f(a) \equiv 0 \pmod{q}$ is usually quite small, say $O(q^\varepsilon)$, at least while the leading coefficient of $f$ is coprime to $q$, in which case the estimation for $\pi_f(x; q)$ is thus reduced to the classical Brun-Titchmarsh theorem if $q$ is fixed. Therefore, our concern is to estimate $\pi_f(x; q)$ with extra summation over $q$, for which the residue class is no longer fixed while $q$ varies and we would encounter quite a different problem from the classical situation.
We restrict ourselves to the special case $f(t) = t^2 + 1$ and consider the smoothed sum
\begin{equation}
Q_\ell(X) := \sum_{p \geq 2 \atop p^2+1 \equiv 0 \pmod{\ell}} g\left( \frac{p}{X} \right),
\end{equation}
where $g$ is a non-negative smooth function with compact support in $[1, 2]$. We have the following Quadratic Brun-Titchmarsh Theorem on Average.

**Theorem 1.1.** Let $A > 0$. For sufficiently large $L = X^\theta$ with $\theta \in \left[\frac{1}{2}, \frac{16}{17}\right)$, the inequality
\begin{equation}
Q_\ell(X) \leq \left\{ \frac{2}{\gamma(\theta)} + o(1) \right\} \tilde{g}(0) \frac{\varphi(\ell)}{\varphi(\ell)} \frac{X}{\log X}
\end{equation}
holds for $\ell \in (L, 2L]$ with at most $O_A(L(\log L)^{-A})$ exceptions, where
\begin{equation}
\gamma(\theta) := \begin{cases}
\frac{91 - 89\theta}{62} & \text{if } \theta \in \left[\frac{1}{2}, \frac{64}{97}\right), \\
\frac{86 - 83\theta}{60} & \text{if } \theta \in \left[\frac{64}{97}, \frac{32}{41}\right), \\
\frac{19 - 18\theta}{14} & \text{if } \theta \in \left[\frac{32}{41}, \frac{16}{17}\right].
\end{cases}
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{gamma_theta_graph.png}
\caption{Graph of $\gamma(\theta)$ as a function of $\theta$.}
\end{figure}

The proof of Theorem 1.1 can be summarized as follows. The linear sieve of Rosser-Iwaniec applies to the prime variable in $Q_\ell(X)$, and a routine application
of Fourier analysis will lead us to the Weyl sum

\[
\rho_n(\ell) := \sum_{a \pmod{\ell}, \ a^2 + 1 \equiv 0 \pmod{\ell}} e\left(\frac{an}{\ell}\right).
\] (1.8)

A trivial bound reads \(|\rho_n(\ell)| \leq \rho_0(\ell) = \rho(\ell) \ll \ell^\varepsilon\) for any small \(\varepsilon > 0\), which means one cannot expect any power-savings if each \(\rho_n(\ell)\) is taken into account individually. Fortunately, we may follow the approaches of Hooley [Ho1] and Deshouillers-Iwaniec [DI2], transforming \(\rho_n(\ell)\) to a kind of exponential sums by appealing to the theory of representation of numbers by binary quadratic forms due to Gauß (see Lemma 7.2 below) and a considerable cancellation is possible while summing over \(\ell\).

The above-mentioned exponential sums in [Ho1] and [DI2] are both Kloosterman sums, and the cancellations among such sums can be controlled by appealing to the spectral theory of automorphic forms (see [DI1]). In our current situation, due to the application of linear sieves before Fourier analysis, we are led to some algebraic exponential sums that are not perfectly Kloosterman sums, so that we have to go back to the original approach of Hooley [Ho2]. However, we may invoke the work of Iwaniec [Iw1] on the well-factorable remainder terms in linear sieves. In such way, the moduli of resultant exponential sums allow reasonable factorizations. We can thus employ \(q\)-vdC to capture cancellations although the sums are quite short, the underlying ideas of which are the key observations in our arguments.

Theorem 1.1 is in fact motivated by some arithmetic problems concerning quadratic polynomials at prime arguments. In another joint work [WX], we consider the greatest prime factors and almost prime values of \(p^2 + 1\), as approximations to the conjecture that any given quadratic irreducible polynomial can capture infinitely many prime values at prime arguments, provided that there are no fixed prime factors. One will see that our methods allow us to improve significantly corresponding results in literatures.

Before closing this subsection, we would like to mention that Theorem 1.1 can be extended to general quadratic irreducible polynomials of fixed discriminants.

1.4. **Divisor functions in arithmetic progressions.** For \((a, q) = 1\), define

\[
D(X; q, a) := \sum_{\substack{n \leq X \\ n \equiv a \pmod{q}}} \tau(n).
\]

Put \(q \asymp X^\theta\). It is important to evaluate \(D(X; q, a)\) asymptotically with \(\theta\) as large as possible. As an direct application of Weil’s bound for Kloosterman sums (together
with Fourier analysis), one has, as $X \to +\infty$,

$$D(X; q, a) = \{1 + o(1)\} \frac{1}{\varphi(q)} \sum_{\substack{n \leq X \\ (n, q) = 1}} \tau(n)$$

uniformly in $a$ for any $\theta < \frac{2}{3}$. This was independently obtained by Selberg and Hooley, and is still the best known record for an arbitrary modulo $q$. It is reasonable to expect that (1.9) should hold for all $\theta < 1$.

We would like to mention that Fouvry [Fo3] succeeded in the case $\frac{2}{3} < \theta < 1$ for almost all $q \asymp X^\theta$, but he has to fix the residue class $a$. The gap around $\theta \approx \frac{2}{3}$ was covered by Fouvry and Iwaniec [Fol] for almost all $q$ satisfying certain factorizations.

Quite recently, Irving picked up this problem while $q$ has sufficiently good factorizations, so that the $q$-analogue of van der Corput method applies. His main theorem could be formulated as follows. If $q \asymp X^\theta$ is squarefree and has only prime factors not exceeding $X^\eta$, then (1.9) holds for all $\theta < \frac{2}{3} + \varpi$ with $246\varpi + 18\eta < 1$. In particular, if $\eta$ is sufficiently small, i.e., $q$ is smooth enough, he can take

$$\theta = \frac{2}{3} + \frac{1}{246} + O(\eta).$$

The arithmetic exponent pairs in Section 3 allows us to obtain a slightly larger admissible value of $\theta$. Note that Irving used the exponent pair $BA^3(\frac{1}{2}, \frac{1}{2}) = (\frac{11}{30}, \frac{16}{30})$.

**Theorem 1.2.** Suppose that $q \asymp X^\theta$ is squarefree and has only prime factors not exceeding $q^\eta$ with $\eta > 0$ sufficiently small. Then (1.9) holds for all $\theta$ with

$$\theta \leq \frac{2}{3} + \frac{1}{232},$$

uniformly in $(a, q) = 1$.

1.5. **Subconvexity of Dirichlet $L$-functions to smooth moduli.** Given a positive integer $q$ and a primitive character $\chi (\mod q)$, we are interested in the bound of Dirichlet $L$-functions of the shape

$$L(\frac{1}{2}, \chi) \ll q^{\frac{1}{4} - \delta}$$

for some $\delta > 0$. This is usually called the subconvexity for Dirichlet $L$-functions in the conductor aspect.

Burgess [Bu2] proved that $\delta = \frac{1}{16}$ is admissible in (1.10), and a Weyl bound asserts that any $\delta < \frac{1}{12}$ should be accessible, as a $q$-analogue of Riemann zeta functions. The Weyl bound was already achieved in a few cases: Heath-Brown [HB1] succeeded while $q$ allows certain factorizations; Conrey and Iwaniec [CI] solved the case that $q$ is a prime and $\chi$ is quadratic from quite a different view.

Heath-Brown’s argument relies on the ideas of $q$-vdC as we have mentioned above, and this was developed further by Irving [Ir2], who was able to go beyond
Weyl’s barrier if \( q \) has sufficiently good factorizations. In particular, if \( q \) is square-free and has only prime factors not exceeding \( q^\eta \) for some small \( \eta > 0 \), the one can take

\[
\delta = \frac{7}{82} + O(\eta)
\]

in (1.10). As \( \eta \) becomes sufficiently small, he goes beyond the Weyl bound since

\[
\frac{1}{12} = \frac{7}{84} < \frac{7}{82}.
\]

This coincides with the bound for \( \zeta(\frac{1}{2} + it) \) in the \( t \)-aspect, derived from the classical exponent pair \((\frac{11}{82}, \frac{57}{88})\).

The above assumption of Irving on \( q \) just falls into the application of arithmetic exponent pairs developed in this paper. More precisely, we may conclude the following improvement.

**Theorem 1.3.** Suppose that \( q \) is squarefree and has only prime factors not exceeding \( q^\eta \) with \( \eta > 0 \) sufficiently small. Then, for any primitive Dirichlet character \( \chi \pmod{q} \), we have

\[
L\left(\frac{1}{2}, \chi\right) \ll q^{\frac{1}{4} - 0.085489}.
\]

Note that \( \frac{7}{82} \approx 0.085365 \). The improvement is rather slight, however the proof is quite clear as an immediate consequence of arithmetic exponent pairs.

Before concluding the first section, we would like to announce that the terminology on trace functions will be introduced in Section 2 and we develop the method of arithmetic exponent pairs in Section 3. Theorems 1.1, 1.2 and 1.3 will be proved in Sections 7 and 8. The Mathematica codes can be found at http://gr.xjtu.edu.cn/web/ping.xi/miscellanea or requested from the authors.

**Notation and convention.** As usual, \( \tau, \varphi \) and \( \Lambda \) denote the divisor, Euler and von Mangoldt functions, respectively. The variable \( p \) is reserved for prime numbers. For a real number \( x \), denote by \( \lfloor x \rfloor \) its integral part. Denote by \( \rho(\ell) \) the number of solutions to the congruence equation \( n^2 + 1 \equiv 0 \pmod{\ell} \).

For a function \( f \) defined over \( \mathbb{Z}/q\mathbb{Z} \), the Fourier transform is defined as

\[
\hat{f}(y) := \frac{1}{\sqrt{q}} \sum_{a \pmod{q}} f(a) e\left(-\frac{ay}{q}\right)
\]

where \( e(t) := e^{2\pi it} \). For each \( h \in \mathbb{Z} \) and all \( x \in \mathbb{Z}/q\mathbb{Z} \), define the difference

(1.11)

\[
\Delta_h(f)(x) := f(x)f(x + h).
\]

For a function \( g \in L^1(\mathbb{R}) \), its Fourier transform is defined as

\[
\hat{g}(y) := \int_{\mathbb{R}} g(x)e(-yx)dx.
\]
The multiplicative inverse $\overline{x}$ of $x$ should be defined with respect to some specialized modulo $c$; i.e., $\overline{x}x \equiv 1 \pmod{c}$. Moreover, while $\overline{x}$ appears in fractions, the modulo will be referred implicitly to the denominator, which is assumed to be coprime to $x$ as can be checked on each occasion.

We use $\varepsilon$ to denote a very small positive number, which might be different at each occurrence; we also write $X^\varepsilon \log X \ll X^\varepsilon$. The notation $n \sim N$ means $N < n \leq 2N$.

Acknowledgements. We are grateful to Étienne Fouvry and Philippe Michel for their kind suggestions and to Cécile Dartyge for pointing out an error in an earlier version. The first author is supported in part by IRT1264 from the Ministry of Education of P. R. China and the second author is supported by CPSF (No. 2015M580825) of P. R. China.

2. Basics on algebraic trace functions

This section is devoted to the terminology on trace functions of $\ell$-adic sheaves on $\mathbb{A}^1_{\mathbb{F}_p}$ following the manner of Fouvry, Kowalski and Michel [FKM1, FKM2, FKM3], and $\ell$-adic Fourier transforms will also be discussed after Laumon [La] and Katz [Ka3].

2.1. Trace functions. Let $p$ be a prime and $\ell \neq p$ an auxiliary prime, and fix an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. The functions $K(x)$ modulo $p$ that we consider are the trace functions of suitable constructible sheaves on $\mathbb{A}^1_{\mathbb{F}_p}$ evaluated at $x \in \mathbb{F}_p$. To be precise, we will consider middle-extension sheaves on $\mathbb{A}^1_{\mathbb{F}_p}$ and we refer to the following definition after Katz [Ka2, Section 7.3.7].

**Definition 2.1 (Trace functions).** Let $\mathcal{F}$ be an $\ell$-adic middle-extension sheaf pure of weight zero, which is lisse and of rank $\text{rank}(\mathcal{F})$. The trace function associated to $\mathcal{F}$ is defined by

$$K(x) := \iota((\text{tr} \mathcal{F})(\mathbb{F}_p, x))$$

for $x \in \mathbb{F}_p$.

We need an invariant to measure the geometric complexity of a trace function, which can be given by some numerical invariants of the underlying sheaf.

**Definition 2.2 (Conductor).** For an $\ell$-adic middle-extension sheaf $\mathcal{F}$ on $\mathbb{A}^1_{\mathbb{F}_p}$ of rank $\text{rank}(\mathcal{F})$, we define the (analytic) conductor of $\mathcal{F}$ to be

$$c(\mathcal{F}) := \text{rank}(\mathcal{F}) + \sum_{x \in S(\mathcal{F})} (1 + \text{Swan}_x(\mathcal{F})),$$

where $S(\mathcal{F}) \subset \mathbb{P}^1(\mathbb{F}_p)$ denotes the (finite) set of singularities of $\mathcal{F}$, and $\text{Swan}_x(\mathcal{F}) (\geq 0)$ denotes the Swan conductor of $\mathcal{F}$ at $x$ (see [Ka1]).
We are never lack of practical examples of trace functions in modern analytic number theory. For instance,

- Let \( f \in \mathbb{F}_p[X] \), and \( \psi \) the additive character on \( \mathbb{F}_p \), then \( \psi(f(x)) \) is definitely expected to be a trace function, which is assumed to be zero while meeting a pole of \( f \) at \( x \). Furthermore, one can show that there exists an \( \ell \)-adic middle-extension sheaf modulo \( p \), denoted by \( \mathcal{L}_{\psi(f)} \), such that \( x \mapsto \psi(f(x)) \) is the trace function of \( \mathcal{L}_{\psi(f)} \). The conductor can be bounded in terms of the degree of \( f \), independent of \( p \).

- Let \( f \in \mathbb{F}_p[X] \), and \( \chi \) a multiplicative character of order \( d > 1 \). If \( f \) has no pole or zero of order divisible by \( d \), then one can show that there exists an \( \ell \)-adic middle-extension sheaf modulo \( p \), denoted by \( \mathcal{L}_{\chi(f)} \), such that \( x \mapsto \chi(f(x)) \) is equal to the trace function of \( \mathcal{L}_{\chi(f)} \). The conductor can be bounded in terms of the degree of \( f \), independent of \( p \).

- Another example is the following (normalized) hyper-Kloosterman sum defined, for any fixed positive \( k \), by

  \[
  K_{k}(\cdot, p) : x \mapsto p^{-(k-1)/2} \sum_{x_1, \ldots, x_k \in \mathbb{F}_p} \sum_{x_1 \cdots x_k = x} e\left( \frac{x_1 + \cdots + x_k}{p} \right).
  \]

  Note that \( K_{k}(0, p) = (-1)^{k-1}p^{-(k-1)/2} \). In particular, we have \( K_{1}(x, p) = e(x/p) \), and \( K_{2}(x, p) \) normalizes the classical Kloosterman sum at the invertible point \( x \in \mathbb{F}_p^\times \). According to Deligne, there exists an \( \ell \)-adic middle-extension sheaf \( \mathcal{K}l_{k} \) modulo \( p \), called a Kloosterman sheaf such that

  \[ K_{k}(x) = K_{k}(x, p) \text{ for all } x \in \mathbb{F}_p^\times. \]

  Such a sheaf was constructed by Deligne [De], and extensively studied by Katz [Ka1, Ka2]. Again according to Deligne, \( \mathcal{K}l_{k} \) is geometrically irreducible and is of rank \( k \), and the conductor of which is bounded by \( k + 3 \).

Let \( q \) be a squarefree number. What we will concern is a composite trace function \( K \) modulo \( q \), given by the product

\[
K(n) = \prod_{p \mid q} K_p(n),
\]

where \( K_p \) is a trace function associated to some \( \ell \)-adic middle-extension sheaf on \( \mathbb{A}^1_{\mathbb{F}_p} \). We adopt the convention that \( K(n) = 1 \) for all \( n \) if \( q = 1 \). In practice, the value of \( K_p(n) \) may depend on the complementary divisor \( q/p \).

In the study of trace functions, especially on their analytic properties, one usually needs to control the conductors independently of \( p \), as in the above examples. On the other hand, the following Riemann Hypothesis, proved by Deligne [De], plays an essential role in the practical device, demonstrating the quasi-orthogonality of trace functions of geometrically irreducible sheaves. The following
formulation takes the shape from Fouvry, Kowalski and Michel, see [FKM2] for instance.

Proposition 2.1 (Quasi-orthogonality). Suppose \( \mathcal{F}_1, \mathcal{F}_2 \) are two geometrically irreducible \( \ell \)-adic middle-extension sheaves modulo \( p \), and \( K_1, K_2 \) are the associated trace functions, respectively. Then there exists a complex number \( \alpha \) such that
\[
\left| \sum_{x \in \mathbf{F}_p} K_1(x)K_2(x) - \alpha \cdot p \right| \leq 3c(\mathcal{F}_1)^2c(\mathcal{F}_2)^2\sqrt{p},
\]
where \( \alpha \) vanishes if \( \mathcal{F}_1 \) is not geometrically isomorphic to \( \mathcal{F}_2 \); and \( |\alpha| = 1 \) if \( \mathcal{F}_1 \) is geometrically isomorphic to \( \mathcal{F}_2 \), in which case we have \( K_1(x) = \alpha \cdot K_2(x) \) for all \( x \in \mathbf{F}_p \), and \( c(\mathcal{F}_1) = c(\mathcal{F}_2) \).

2.2. \( \ell \)-adic Fourier transforms. The \( \ell \)-adic Fourier transform over \( \mathbf{F}_p \) starts from a reformulation on the usual Fourier transform over \( \mathbf{Z}/p\mathbf{Z} \) (up to an opposite sign). For a non-trivial additive character \( \psi \) and a function \( f : \mathbf{F}_p \rightarrow \mathbf{C} \), we define the Fourier transform \( \text{FT}_\psi(f) : \mathbf{F}_p \rightarrow \mathbf{C} \) by
\[
\text{FT}_\psi(f)(t) := -p^{-1/2} \sum_{x \in \mathbf{F}_p} f(x)\psi(tx)
\]
for \( t \in \mathbf{F}_p \). The \( \ell \)-adic Fourier transforms are well-defined for Fourier sheaves (see also Katz [Ka2, Definition 8.2.2]).

Definition 2.3 (Fourier sheaf). A middle-extension sheaf \( \mathcal{F} \) over \( \mathbf{F}_p \) is called a Fourier sheaf if none of its geometrically irreducible component is geometrically isomorphic to an Artin-Schreier sheaf \( \mathcal{L}_\psi \) attached to some additive character \( \psi \) of \( \mathbf{F}_p \).

We collect the properties of Fourier transforms of Fourier sheaves of Deligne [De], Laumon [La], Brylinski [Br], Katz [Ka2, Ka3] and Fouvry-Kowalski-Michel [FKM3].

Lemma 2.1. Let \( \psi \) be a non-trivial additive character of \( \mathbf{F}_p \) and \( \mathcal{F} \) a Fourier sheaf on \( \mathbf{A}^1_{\mathbf{F}_p} \). Then there exists an \( \ell \)-adic sheaf \( \mathcal{G} = \text{FT}_\psi(\mathcal{F}) \) called the Fourier transform of \( \mathcal{F} \), which is also an \( \ell \)-adic Fourier sheaf, with the property that
\[
K_{\text{FT}_\psi(\mathcal{F})}(y) = \text{FT}_\psi(K_\mathcal{F})(y) = -p^{-1/2} \sum_{x \in \mathbf{F}_p} K_\mathcal{F}(x)\psi(yx).
\]
Furthermore, we have
(a) The sheaf \( \mathcal{G} \) is pointwise of weight 0, if and only if \( \mathcal{F} \) is;
(b) The sheaf \( \mathcal{G} \) is geometrically irreducible, or geometrically isotypic, if and only if \( \mathcal{F} \) is;
(c) The Fourier transform is involutive, in the sense that we have a canonical arithmetic isomorphism
\[
\text{FT}_\psi(\mathcal{G}) \simeq [\times(-1)]^* \mathcal{F},
\]
where \([\times (-1)]^*\) denotes the pull-back by the map \(x \mapsto -x\);

(d) We have

\[
\mathbf{c}(\text{FT}_\psi(\mathcal{F})) \leq 10\mathbf{c}(\mathcal{F})^2.
\]

Proof. The last claim was proved by Fouvry, Kowalski and Michel [FKM3] using
the theory of local Fourier transforms developed by Laumon [La], and the others
have been established for instance in [Ka3, Theorem 8.4.1]. \(\square\)

The inequality (2.1) is essential in analytic applications, since it implies that if
\(p\) varies but \(\mathcal{F}\) has a bounded conductor, so do the Fourier transforms.

2.3. Amiable trace functions for \(q\)-vdC. Given an average \(S(\Psi; I)\) in (1.1)
while \(\Psi\) specialized to some trace function \(K\), the \(A\)-process in \(q\)-vdC (see Section
3 below) usually produces certain sums involving the difference \(\Delta_h(K_p)\) for some
\(h\). Observe that if

\[K_p(x) = \psi(ax^2 + bx) \quad \text{with} \quad a \in \mathbb{F}_p^\times,
\]
one has \(\Delta_h(K_p)(x) = \psi(-2ahx - bh - ah^2),\) and the resultant sum reveals no
cancellation after one more \(A\)-process since the summand becomes some constant.
As we will see, this phenomenon is essentially the only obstruction to square-root
cancellations. We thus need to determine when a trace function is suitable or the
purpose of our analysis, to which we mean amiable. The similar arguments first
appeared in [Po, Section 6], but a different convention was used therein.

We first formulate the admissibility.

Definition 2.4 (Admissible sheaf). An admissible sheaf over \(\mathbb{F}_p\) is a middle-
extension sheaf on \(\mathbb{A}^1_{\mathbb{F}_p}\) which is pointwise pure of weight 0 (in the sense of Deligne
[De]). A composite trace function \(K(\mod q)\) is called to be admissible, if the re-
duction \(K_p\) is admissible for each \(p \mid q\).

Definition 2.5 (Amiable sheaves and trace functions). For \(d \geq 0\), an admissible
sheaf \(\mathcal{F}\) is said to be \(d\)-amiable if no geometrically irreducible component of \(\mathcal{F}\)
is geometrically isomorphic to an Artin-Schreier sheaf of the form \(L_{\psi(P)}\), where
\(P \in \mathbb{F}_p[X]\) is a polynomial of degree \(\leq d\). In such case, we also say the associated
trace function \(K_p\) is \(d\)-amiable. A composite trace function \(K(\mod q)\) is called to
be \(d\)-amiable if \(K_p\) is \(d\)-amiable for each \(p \mid q\).

In addition, a sheaf (or its associated trace function) is said to be \(\infty\)-amiable if
it is amiable for any fixed \(d \geq 1\).

Remark 1. Given an admissible sheaf \(\mathcal{F}\), we would like to determine if it is
\(\infty\)-amiable while applying \(q\)-vdC along with quite a few iterations. A sufficient
condition is that each geometrically irreducible component of \(\mathcal{F}\) is irreducible of
rank \(\geq 2\), or particularly if \(\mathcal{F}\) itself is geometrically irreducible of rank \(\geq 2\).
Remark 2. In the subsequent study in $B$-process, one has to determine if the Fourier transform of a given Fourier sheaf is amiable, and sometimes the rank does this job. According to Katz [Ka3, Lemma 7.3.9], one has

$$\text{rank}(\text{FT}_\psi(\mathcal{F})) = \sum_{\lambda} \max(0, \lambda - 1) + \sum_x (\text{Swan}_x(\mathcal{F}) + \text{Drop}_x(\mathcal{F})),$$

where $\lambda$ runs over the breaks of $\mathcal{F}(\infty)$ and $x$ over the singularities of $\mathcal{F}$ in $\mathbb{A}^1$. Here $\text{Drop}_x(\mathcal{F}) = \text{rank}(\mathcal{F}) - \text{dim}(\mathcal{F}_x)$, which is at least 1 at each singularity.

For $f_1, f_2 \in \mathbf{F}_p[X]$ with $\deg(f_1) < \deg(f_2) < p$ and $\deg(f_2) \geq 1$, the Artin-Schreier sheaf $\mathcal{F} := \mathcal{L}_\psi(f)$ is of rank 1 and $\infty$-amiable for any primitive additive character $\psi$ of $\mathbf{F}_p$. Note that $\mathcal{F}$ has at least one singularity in $\mathbb{A}^1$, at which the Swan conductor is at least one, it then follows that the rank of the Fourier transform of $\mathcal{F}$ is at least two, i.e.,

$$\text{rank}(\text{FT}_\psi(\mathcal{F})) \geq 2,$$

so that $\text{FT}_\psi(\mathcal{F})$ is also $\infty$-amiable.

Following the above arguments, we may find several examples of trace functions that are $\infty$-amiable in the sense of Definition 2.5.

- $K_p(n) = \psi(f_1(n)f_2(n))$, where $\psi$ is a primitive additive character, $f_1, f_2 \in \mathbf{F}_p[X]$, $\deg(f_1) < \deg(f_2) < p$ and $\deg(f_2) \geq 1$;
- $K_p(n) = \chi(f(n))\psi(g(n))$, where $\chi$ is a primitive multiplicative character mod $p$, $\psi$ is not necessarily primitive, $f, g$ are rational functions and $f$ is not a $d$-th power of another rational function with $d$ being the order of $\chi$;
- $K_p(n) = \text{Kl}_k(n, p)$ as a normalized hyper-Kloosterman sum of rank $k \geq 2$;
- The Fourier transforms of the above examples.

Let $\mathcal{F}$ be an $\ell$-adic middle-extension sheaf on $\mathbb{A}^1_{\mathbf{F}_p}$, denote by $+[a]^*\mathcal{F}$ the pullback of $\mathcal{F}$ under the additive shift $n \mapsto n + a$, where $a \in \mathbf{F}_p$. We also write $\tilde{\mathcal{F}}$ for the the middle-extension dual of $\mathcal{F}$, i.e., given a dense open set $j : U \hookrightarrow \mathbf{P}^1$ where $\mathcal{F}$ is lisse, we have

$$\tilde{\mathcal{F}} = j_*(j^*(\mathcal{F}'))',$$

where $'$ denotes the lisse sheaf of $U$ associated to the contragredient of the representation of the fundamental group of $U$ corresponding to $j^*\mathcal{F}$. If $\mathcal{F}$ is an admissible sheaf, the trace function of $\tilde{\mathcal{F}}$ appears as the complex conjugate of $K$, a trace function of $\mathcal{F}$. In $A$-process, we expect (at least) $n \mapsto \Delta_a(K) = K(n)K(n + a)$ is also an amiable trace function if $a \in \mathbf{F}_p^\times$. This of course requires the independence between the sheaves $\mathcal{F}$ and $+[a]^*\mathcal{F}$.

Lemma 2.2. Let $d$ be a positive integer and $p > d$. Suppose $\mathcal{F}$ is a $d$-amiable admissible sheaf with $c(\mathcal{F}) \leq p$. Then, for each $a \in \mathbf{F}_p^\times$, the sheaf $\mathcal{F} \otimes [a]^*\mathcal{F}$ is
$(d-1)$-amiable. Furthermore, we have
\[ c(\mathcal{F} \otimes [+a]^* \mathcal{F}) \leq c(\mathcal{F})^3. \]

**Proof.** Note that $\mathcal{F}$ can be decomposed into a sum of at most $c(\mathcal{F})$ trace functions of isotypic admissible sheaves, which are also $d$-amiable by the assumptions. The first part of this lemma then follows immediately from [Po, Theorem 6.15]. The second part was stated in [FKM3, Proposition 8.2] in a general setting, but slightly weaker in this special case.

Suppose $\mathcal{F}$ has $n$ singularities and so does $[+a]^* \mathcal{F}$. The total number of singularities of $\mathcal{F} \otimes [+a]^* \mathcal{F}$ is at most $2n$. Note that
\[ \text{Swan}(\mathcal{F} \otimes [+a]^* \mathcal{F}) \leq \text{rank}(\mathcal{F}) \text{rank}([+a]^* \mathcal{F}) (\text{Swan}([+a]^* \mathcal{F})) \]
\[ = 2 \text{rank}(\mathcal{F})^2 \text{Swan}(\mathcal{F}). \]

By the definition of conductors, we have
\[ c(\mathcal{F} \otimes [+a]^* \mathcal{F}) \leq \text{rank}(\mathcal{F})^2 + 2n + 2 \text{rank}(\mathcal{F})^2 \text{Swan}(\mathcal{F}) \leq c(\mathcal{F})^3. \]

This completes the proof of this lemma. \qed

**Remark 3.** For any $\infty$-amiable admissible sheaf and sufficiently large prime $p$, we conclude from Lemma 2.2 that $\mathcal{F} \otimes [+a]^* \mathcal{F}$ is also $\infty$-amiable for each $a \in \mathbb{F}_p^\times$.

---

### 3. $q$-analogue of van der Corput method and arithmetic exponent pairs

#### 3.1. Framework of $q$-analogue of van der Corput method

Given a positive squarefree number $q$ and a composite trace function $K \pmod{q}$, we are interested in the cancellations among the average
\[ \sum_{n \in I} K(n), \]
where $I$ is some interval. While $|I| = q$, the sum is a complete one due to the periodicity of $K$. While $|I| < q$, we are working on an incomplete sum, a non-trivial bound of which is the main objective in many practical problems in analytic number theory. A common treatment is to transform the incomplete sum to complete ones (individual or on average) via Fourier analysis and this can be demonstrated by Lemma 3.2 below. In fact, we have
\[ \sum_{n \in I} K(n) \ll |I| q^{-1/2} (|\widehat{K}(0)| + |I|^{-1} q (\log q) \max_{h \neq 0} |\widehat{K}(h)|) \]
\[ \ll ||\widehat{K}||_\infty (|I| q^{-1} + 1) q^{1/2} \log q. \]
In many situations, we have \( \| \hat{K} \|_{\infty} := \sup_x |\hat{K}(x)| \ll q^\epsilon \) and the above estimate is thus non-trivial for \( |I| > q^{1/2+\epsilon} \). This is what Pólya and Vinogradov have done on estimates for incomplete multiplicative character sums, as we have mentioned in the first section. There seems no universal method to make a power-enlargement to the non-trivial range \( |I| > q^{1/2+\epsilon} \) for general \( K \) and \( q \); a relevant progress was recently made by Fouvry, Kowalski, Michel, et al \([FKM+]\) to cover the gap between \( q^{1/2+\epsilon} \) and \( q^{3/8+\epsilon} \). Nevertheless, Burgess \([Bu1, Bu2]\) succeeded while \( K(n) = \chi(n) \) with general \( q \), non-trivial \( \chi \) and \( |I| > q^{3/8+\epsilon} \), which implies the first subconvexity for Dirichlet \( L \)-functions \( L(1/2, \chi) \). A fascinating phenomenon was discovered by Heath-Brown \([HB1]\), who was able to derive a Weyl-type bound for \( L(1/2, \chi) \) if \( q \) allows reasonable factorizations. This is far earlier than his breakthrough on the greatest prime factors of \( n^3 + 2 \) in \([HB3]\).

As in the classical van der Corput method, estimate for such incomplete sums follows from Weyl differencing and Poisson summation, which are usually called \( A \)-process and \( B \)-process. To formulate the two processes, we start from the general sums \( S(\Psi; I) \) defined as in (1.1).

**Lemma 3.1 (A-process).** Assume \( q = q_1q_2 \) with \( (q_1, q_2) = 1 \) and \( \Psi_i : \mathbb{Z}/q_i \mathbb{Z} \to \mathbb{C} \). Define \( \Psi = \Psi_1 \Psi_2 \), then we have

\[
|S(\Psi; I)|^2 \leq \|\Psi_2\|_{\infty q_2}^2 (|I| + \sum_{0 < \ell \leq |I|/q_2} \left| \sum_{n \in \mathbb{Z}} 1_I(n) 1_I(n + \ell q_2) \Delta_{\ell q_2}(\Psi_1)(n) \right|),
\]

where \( 1_I(\cdot) \) denotes the indicator function of \( I \) and \( \Delta_{\ell q_2}(\Psi_1) \) is defined as in (1.11).

**Lemma 3.2 (B-process).** For \( \Psi : \mathbb{Z}/q \mathbb{Z} \to \mathbb{C} \), we have

\[
S(\Psi; I) \ll \frac{|I|}{\sqrt{q}} \left| \hat{\Psi}(0) \right| + (\log q) \left| \sum_{h \in I} \hat{\Psi}(h)e\left(\frac{ha}{q}\right) \right|
\]

for certain \( a \in \mathbb{Z} \) and some interval \( I \) not containing 0 with \( |I| \leq q/|I| \).

Lemmas 3.1 and 3.2 were stated explicitly by Irving \([Ir2]\) in a slightly different setting. Here we present the proof of Lemma 3.1 in our settings.

**Proof.** Put \( |I| = N \). Assume \( N > q_2 \), otherwise the lemma follows trivially. For any \( \ell \), we have

\[
S(\Psi; I) = \sum_{n \in \mathbb{Z}} 1_I(n + \ell q_2) \Psi(n + \ell q_2).
\]

Summing over \( \ell \leq L \in \mathbb{N} \) with \( 1 \leq L \leq N/q_2 \), we find

\[
S(\Psi; I) = L^{-1} \sum_{\ell \leq L} \sum_{n \in \mathbb{Z}} 1_I(n + \ell q_2) \Psi(n + \ell q_2).
\]
Note that $\Psi(n + \ell q_2) = \Psi_1(n + \ell q_2)\Psi_2(n)$. It then follows that

$$|S(\Psi; I)| \leq \|\Psi_2\|_\infty L^{-1} \sum_{n \in \mathbb{Z}} \left| \sum_{\ell \leq L} 1_I(n + \ell q_2)\Psi_1(n + \ell q_2) \right|.$$ 

Since the outer sum over $n$ is of length at most $N$, by Cauchy inequality we have

$$|S(\Psi; I)|^2 \leq \|\Psi_2\|_\infty^2 L^{-2} N \sum_{n \in \mathbb{Z}} \left| \sum_{\ell \leq L} 1_I(n + \ell q_2)\Psi(n + \ell q_2) \right|^2 \leq \|\Psi_2\|_\infty^2 L^{-1} N \left( N + \sum_{0 < |\ell| \leq L} \left| \sum_{n \in \mathbb{Z}} 1_I(n) 1_I(n + \ell q_2) \Delta_{\ell q_2}(\Psi_1)(n) \right| \right).$$

This completes the proof of the lemma by choosing $L = \lceil N/q_2 \rceil$. □

In practice, the $A$-process is also known as the Weyl differencing and is usually employed with a number of iterations. The resultant sum is roughly of the same length as the original one, but the moduli becomes reasonably smaller, so that more cancellation becomes possible. In contrast to $A$-process, the $B$-process transforms the original sum to a dual form (of different length but with the same moduli), and this is better known as the Poisson summation (or completing method), dated back to Pólya and Vinogradov on the estimate for incomplete character sums.

Different combinations of $A$- and $B$-processes lead to different estimates for incomplete sums. We now recall some pioneer works that benefited from $q$-vdC.

- As mentioned before, Heath-Brown proved that $P^+(n^3 + 2) > n^{1+10^{-303}}$ for infinitely many $n$, where $\Psi(n) = e(f_1(n)f_2(n)/q)$ with $f_1, f_2 \in \mathbb{Z}[X]$ [HB3]; the estimate for such exponential sums was recently used by Dartyge [Da] while studying the greatest prime factors of $n^4 - n^2 + 1$ and by de la Bretêche [dlB] while extending Dartyge’s result to any even unitary irreducible quartic polynomials with integral coefficients and the associated Galois group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$;
- Earlier than the above example, Heath-Brown [HB1] obtained a certain Weyl-type subconvexity for $L(\frac{1}{2}, \chi)$ while the moduli factorizes in a certain way;
- Graham and Ringrose [GR] got an extended zero-free region for Dirichlet $L$-functions with smooth moduli (on average), from which they deduced $\Omega$-results on least quadratic non-residues;
- With the help of the ABC-conjecture, Heath-Brown [HB4] was able to give a sharp estimate of the cubic Weyl sum $\sum_{n \leq N} e(\alpha n^3)$ for any quadratic irrational $\alpha$. The analytic exponential sum can, following a suitable Diophantine approximation to $\alpha$, be transformed to an algebraic one such that the moduli factorizes suitably;
• Pierce [Pi] introduced the ideas of $q$-vdC to the square sieve of Heath-Brown, which enables her to derive the first non-trivial bound for the 3-part of the class group of $\mathbb{Q}(\sqrt{-D})$. In her joint work with Heath-Brown [HBP] on a conjecture of Serre concerning the number of rational points of bounded height on a finite cover of projective space $\mathbb{P}^{n-1}$, they can succeed in the special case of smooth cyclic covers of large degrees invoking the ideas of $q$-vdC to the power sieve.

• The idea of van der Corput method was contained implicitly in the recent breakthrough of Zhang [Zh] on bounded gaps between primes, and this was later highlighted by Polymath [Po] in the subsequent improvement;

• Irving [Ir1] beat the classical barrier of distribution of divisor functions in arithmetic progressions to smooth moduli, which previously follows from the Weil’s bound for Kloosterman sums to general moduli;

• Irving [Ir2] obtained a sub-Weyl bound for $L(\frac{1}{2}, \chi)$ while the moduli is smooth enough;

• Blomer and Miličević [Ir2] evaluated the second moment of twisted modular $L$-functions $L(\frac{1}{2}, f \otimes \chi)$ as $\chi$ runs over primitive characters mod $q$ satisfying certain factorizations, where $f$ is a fixed (holomorphic or Maaß) Hecke cusp form.

The classical van der Corput method was also extended to algebraic exponential and character sums modulo prime powers, in which case one focuses on a fixed prime and the power tends to infinity. This is known as the $p$-adic van der Corput method and the reader is referred to [BM2] and [Mi] for more details.

3.2. Arithmetic exponent pairs. We now restrict our attention in Lemmas 3.1 and 3.2 to composite trace functions and develop the method of arithmetic exponent pairs for incomplete sums of such trace functions, as an analogue of classical exponent pairs of analytic exponential sums initiated by Phillips (see [GK] for instance).

Let $q$ be a positive squarefree number and assume $q$ has the reasonable factorization $q = q_1 q_2 \cdots q_J$ for some $J \geq 1$, where $q_j$’s are not necessarily primes but they are pairwise coprime. To each $q_j$, we associate a (possibly composite) trace function $K(n, q_j)$. Put

$$K(n) = \prod_{1 \leq j \leq J} K(n, q_j).$$

(3.1)

In what follows, we assume $K$ is admissible and, for each $p | q$, $c(\mathcal{F}_p) \leq c$ for some uniform $c > 0$, where $\mathcal{F}_p$ denotes the $\ell$-adic sheaf corresponding to $K_p$.

Let $\delta$ be a fixed positive integer such that $(\delta, q) = 1$ and let $W_\delta \in \ell^2(\mathbb{Z}/\delta\mathbb{Z})$ be an arbitrary function satisfying $\|W_\delta\|_\infty \leq 1$, which we call as a deformation factor roughly.
Keeping the above notation and assumptions, we consider the following sum
\[ S(K, W) := \sum_{n \in I} K(n)W_\delta(n), \]
where \( I = (M, M + N] \) for some \( M \in \mathbb{Z} \). In what follows, we always assume \( N < q\delta \), i.e., we will work on incomplete sums.

For \( J \geq 1 \), let
\[
\begin{bmatrix}
\kappa \\
\lambda \\
\nu
\end{bmatrix}_J := \begin{bmatrix}
(\kappa_1) \\
(\lambda_1) \\
(\nu_1)
\end{bmatrix}, \ldots, \begin{bmatrix}
(\kappa_J) \\
(\lambda_J) \\
(\nu_J)
\end{bmatrix}
\]
be a sequence such that
\[
S(K, W) \lesssim J, \varepsilon, c^J \sum_{1 \leq j \leq J} (q^J + 1 - j)N\kappa_j \delta \| \hat{W}_\delta \|_\infty^{\lambda_j \nu_j},
\]
where the implied constant is allowed to depend on \( J, \varepsilon \) and \( c \).

Let \( (\kappa, \lambda, \nu) \) be a tuple such that
\[
(\Omega) : \quad S(K, W) \lesssim \varepsilon, c^J (qN)^\kappa N^\lambda \delta^\nu \| \hat{W}_\delta \|_\infty,
\]
where the implied constant is allowed to depend on \( \varepsilon \) and \( c \).

We now give some initial choices for the exponents in \( A \)- and \( B \)-processes.

**Proposition 3.1.** If \( K \) is 1-amiable, then \( (\Omega_1) \) holds for
\[
(3.2) \quad \begin{bmatrix}
\kappa \\
\lambda \\
\nu
\end{bmatrix}_1 = \begin{bmatrix}
\frac{1}{2} \\
\frac{1}{2} \\
\frac{1}{2}
\end{bmatrix},
\]
and \( (\Omega) \) holds for
\[
(3.3) \quad (\kappa, \lambda, \nu) = (1, 1, 1).
\]

**Proof.** By Lemma 3.2, we have
\[
S(K, W) \ll \frac{N}{\sqrt{q\delta}} \left( |\hat{K}(0)\hat{W}_\delta(0)| + \log(q\delta) \left| \sum_{h \in J} \hat{K}(h)\hat{W}_\delta(h)e\left( \frac{hM}{q\delta} \right) \right| \right)
\]
for some interval \( J \) of length at most \( q\delta/N \). Note that \( \hat{K}(h) \ll q^\varepsilon \) in view of Chinese Remainder Theorem and Proposition 2.1. Hence we conclude that
\[
S(K, W) \ll N^\varepsilon \sqrt{q\delta} \| \hat{W}_\delta \|_\infty,
\]
which completes the proof of the proposition. \( \square \)
**Remark 4.** The classical van der Corput method for analytic exponential sums starts from the trivial exponent pair $(0, 1)$, and this corresponds to the $q$-analogue

$$S(K, W) \ll N^\varepsilon (q/N)^0 N^1 \delta^1.$$ 

However, we can always employ $B$-process to transform an incomplete sum to a complete one, so that our initial exponent pairs in Proposition 3.1 are in fact compared to $(\frac{1}{2}, \frac{1}{2}) = B \cdot (0, 1)$ in the classical case.

**Remark 5.** Ignoring the contributions from $\delta$, the term $q/N$ takes the place of derivatives of amplitude functions in classical analytic exponential sums. The exponent of $\delta$ is not quite essential in applications since we usually have extra summation of $\delta$ over some sparse sets, so that contributions from $\delta$ are usually bounded on average. One will see that an exponent, such as $\frac{1}{2}$, for $\delta$ is sufficient in our later applications. For this reason, we keep the convention of exponent pairs although $(\kappa, \lambda, \nu)$ and $(\kappa, \lambda, \nu)$ appear as triples.

The next two theorems allow one to produce new exponent pairs from old ones.

**Theorem 3.1** (A-process). Let $J \geq 1$ and assume $K$ is $(J+1)$-amiable. We have

$$A \left[ \begin{array}{c} \kappa \\ \lambda \\ \nu \end{array} \right]_J = \left[ \begin{array}{c} \frac{1}{2} \\ 1 \\ 0 \end{array} , \left( \frac{\kappa}{2} \right) , \frac{\lambda}{2} + 1, \nu_1 \right] , \ldots , \left( \frac{\kappa_J}{2} \right) .$$

(3.4)

**Theorem 3.2** (B-process). Assume $\hat{K}$ is 1-amiable. We have

$$B(\kappa, \lambda, \nu) = \left( \lambda - \frac{1}{2}, \kappa + \frac{1}{2}, \nu + \lambda - \kappa - \frac{1}{2} \right).$$

(3.5)

The following proposition is an immediate consequence of Theorems 3.1, 3.2 and Proposition 3.1.

**Proposition 3.2.** Let $J \geq 2$. If $K$ is $J$-amiable, we have the following exponent pairs

$$\left[ \begin{array}{c} \kappa \\ \lambda \\ \nu \end{array} \right]_J = \left[ \begin{array}{c} 2^{-1} \\ 1 \\ 0 \end{array} , \ldots , \left( 2^{-1} \right) , \left( 1 - 2^{-J} \right) \right].$$

(3.6)

**Remark 6.** As one can see from Theorem 3.1 and Proposition 3.2, the saving against trivial estimates decays exponentially in $J$, so that one cannot make the savings ideally efficient by taking larger value of $J$. In most applications, the choice $J = 3$ or 4 is usually sufficient. We will give explicit estimates for $\tilde{s}(K, W)$ in Section 6 to display the role of each factor of moduli.
3.3. **Optimization of arithmetic exponent pairs.** In certain applications, we may decompose \( q \) freely as a product of several squarefrees that are pairwise coprime. This would produce an estimate of the shape \((\Omega)\) by balancing all these terms in \((\Omega_J)\). In such case, we would like to produce new exponent pairs from old ones by applying suitable combinations of \(A\)- or \(B\)-processes. As a counterpart of Theorem 3.2, we should determine the shape of \(A(\kappa, \lambda, \nu)\).

**Theorem 3.3.** Suppose \( q \) has sufficiently good factorizations in the sense that the size of each divisor can be chosen freely. Assume \( K \) is 1-amiable and \((\Omega)\) holds with some exponent pairs \((\kappa, \lambda, \nu)\), then we have

\[
A(\kappa, \lambda, \nu) = \left( \frac{\kappa}{2(\kappa + 1)}, \frac{\kappa + \lambda + 1}{2(\kappa + 1)}, \frac{\nu}{2} + \frac{1}{4} \right).
\]

(3.7)

**Remark 7.** One can see that Theorems 3.3 and 3.2 coincide with classical exponent pairs for analytic exponential sums up to the exponent in \( \delta \).

While \( q \) has only small prime factors, saying that \( q \) is \( q^\eta \)-smooth or \( q^\eta \)-friable, where \( \eta \) is sufficiently small, the good factorization in Theorem 3.3 is satisfied. We will look into this issue in Section 8 on the distribution of divisor functions in arithmetic progressions and subconvexity of Dirichlet \( L \)-functions. Such an observation also plays an essential role in applications to the quadratic Brun-Titchmarsh theorem in Section 7.

**Remark 8.** In applications of Theorems 3.2 and 3.3 to produce new exponent pairs from old ones, one should determine if \( K \) is amiable enough in the sense of Definition 2.5. Of course, a sufficient condition is that \( K \) is \( \infty \)-amiable. However, in practice, that \( K \) is \( J \)-amiable for a certain large value of \( J \) is usually sufficient since one becomes quite close to optimal exponent pairs after the first several iterations. For instance, the above arguments are applicable to the Weyl sum \( \sum_{n \leq N} e(n^{2016}/q) \) if \( q \) is smooth enough, although the summand is just 2015-amiable.

The following table gives the first several exponent pairs produced by different combinations of \(A\)- and \(B\)-processes to \((\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\). Note that \( \nu \) are omitted in the list since its values are not altered. This can be compared with the table in [Ti, p.117].

| Processes     | \( A \)       | \( A^2 \)        | \( A^3 \)     | \( BA^2 \)     | \( BA^3 \)     | \( ABA^2 \)   | \( A^2BA^2 \) | \( BABA^2 \) |
|---------------|---------------|------------------|--------------|---------------|---------------|---------------|---------------|---------------|
| \((\kappa, \lambda)\) | \((\frac{1}{6}, \frac{2}{3})\) | \((\frac{1}{14}, \frac{11}{14})\) | \((\frac{1}{20}, \frac{26}{30})\) | \((\frac{2}{7}, \frac{4}{7})\) | \((\frac{11}{30}, \frac{16}{30})\) | \((\frac{2}{18}, \frac{13}{18})\) | \((\frac{2}{30}, \frac{33}{30})\) | \((\frac{4}{18}, \frac{11}{18})\) |

**Table 1.** List of (arithmetic) exponent pairs
4. Proof of Theorems 3.1, 3.2 and 3.3

4.1. Proof of Theorem 3.1. Suppose $K$ is $(J+1)$-amiable and $W_\delta$ does not vanish identically. By Lemma 3.1, we have

\begin{equation}
|\mathcal{G}(K,W)|^2 \ll L^{-1}N^2 + L^{-1}N \sum_{\ell \leq L} |\mathfrak{A}(\ell)|,
\end{equation}

where

\[ \mathfrak{A}(\ell) := \sum_{n \in \mathcal{I}} K(n, Q_{J+1}) K(n + \ell q_{J+1}, Q_{J+1}) W_\delta(n) \overline{W_\delta(n + \ell q_{J+1})} \]

with some interval $\mathcal{I} \subset \mathcal{I}$ and $Q_{J+1} = q/q_{J+1}$. Here $K(n, Q_{J+1})$ denotes the reduction of $K$ mod $Q_{J+1}$ as shown in the definition (3.1).

After one $A$-process, the expected estimate follows from the bound for $\mathfrak{A}(\ell)$. Put

\[ q_j' := q_j/\ell, q_j \quad (1 \leq j \leq J), \]

in which case the trace function is a reduced form of $K(n, Q_{J+1}) K(n + \ell q_{J+1}, Q_{J+1})$ mod $Q_{J+1}^* := Q_{J+1}/(\ell, Q_{J+1}) = q_1 q_2 \cdots q_J$. Hence we may write

\[ \mathfrak{A}(\ell) = \sum_{n \in \mathcal{I}} \tilde{K}(n) \tilde{W}(n), \]

where

\[ \tilde{K}(n) := K(n, Q_{J+1}^*) K(n + \ell q_{J+1}, Q_{J+1}^*), \]

and

\[ \tilde{W}(n) := W_\delta(n) W_\delta(n + \ell q_{J+1}) K(n, (\ell, Q_{J+1})) K(n + \ell q_{J+1}, (\ell, Q_{J+1})) \]

is the new deformation factor (mod $\delta(\ell, Q_{J+1})$). According to Lemma 2.2, we find the trace function $\tilde{K}(n)$, which is well-defined mod $Q_j^*$, is $J$-amiable.

**Case I.** We first assume $N < Q_{J+1}^* \delta(\ell, Q_{J+1}) = Q_{J+1}^* \delta$.

By hypothesis, we may deduce that

\begin{equation}
\mathfrak{A}(\ell) \ll N^\varepsilon \sum_{1 \leq j \leq J} \left( \frac{q_{J+1-j}}{N} \right)^{\kappa_j} N^{\lambda_j} (\delta(\ell, Q_{J+1}) \| \hat{W}_{\&} \|_\infty^2)^{\nu_j}
\end{equation}

for each $\ell \leq L$, where

\[ \hat{W}_{\&}(x) := \frac{1}{\sqrt{\delta(\ell, Q_{J+1})}} \sum_{a \pmod{\delta(\ell, Q_{J+1})}} \tilde{W}(a)e \left( \frac{-ax}{\delta(\ell, Q_{J+1})} \right). \]

Trivially, we have

\begin{equation}
\| \hat{W}_{\&} \|_\infty \ll \sqrt{\delta(\ell, Q_{J+1})}.
\end{equation}

Summing over $\ell \leq L$ and taking $L = \lfloor N/q_{J+1} \rfloor$, it follows from (4.1) and (4.2) that

\[ \mathcal{G}(K,W) \ll (N q_{J+1})^{1/2} + N^\varepsilon \sum_{1 \leq j \leq J} \left( \frac{q_{J+1-j}}{N} \right)^{\kappa_j} N^{\frac{1}{2}(\lambda_j + 1)} \delta^{\nu_j} \]
\[ \ll (Nq_{J+1})^{1/2} \| \hat{W}_\delta \|_\infty + N^\varepsilon \sum_{1 \leq j \leq J} \left( \frac{q_{J+1-j}}{N} \right)^{\frac{j}{2}} \| \hat{W}_\delta \|_\infty^{\lambda_j+1} (\delta \| \hat{W}_\delta \|_\infty^2)^{\nu_j} \]

as expected. Here we used the fact that \( \| \hat{W}_\delta \|_\infty \gg 1 \) after Lemma A.4 below.

**Case II.** It remains to consider the case \( N \geq Q_{J+1} \delta \). An alternative estimate for \( S(K, W) \) reads

\[ N^\varepsilon (q\delta)^{1/2} \| \hat{W}_\delta \|_\infty \]

as given by (3.3). This is even sharper since \( (q\delta)^{1/2} \leq (Nq_{J+1})^{1/2} \) by assumption. This completes the proof of Theorem 3.1.

4.2. **Proof of Theorem 3.2.** Suppose \( \hat{K} \) is 1-amiable. Firstly, we put \( \Psi(n) = K(n)W_\delta(n) \) in Lemma 3.2, so that the Fourier transform \( \hat{\Psi} \) of \( F \) over \( \mathbb{Z}/q\delta\mathbb{Z} \) can be expressed by \( \hat{\Psi}(h) = \hat{K}(\delta h)\hat{W}_\delta(\overline{\gamma}h) \), where \( q\overline{q} \equiv 1 \pmod{\delta} \) and \( \delta \overline{\delta} \equiv 1 \pmod{q} \).

From Lemma 3.2, we thus have

\[ \hat{s}(K, W) \ll \frac{N}{q\delta} \left\{ \hat{K}(0)\hat{W}_\delta(0) + (\log q) \left| \sum_{h \in J} \hat{K}(\delta h)\hat{W}_\delta(\overline{\gamma}h)e \left( \frac{hM}{q\delta} \right) \right| \right\} \]

for some interval \( J \) of length at most \( q\delta/N \). By Proposition 2.1, we have

\[ \hat{K}(0) = \prod_{p | q} \hat{K}_p(0) \ll q^\varepsilon. \]

The above inequality reduces to

\[ (4.4) \quad \hat{s}(K, W) \ll N(q\delta)^{-1/2-\varepsilon} \| \hat{W}_\delta \|_\infty + \frac{N \log q}{\sqrt{q\delta}} \left| \sum_{h \in J} \hat{K}(\delta h)\hat{W}_\delta(\overline{\gamma}h)e \left( \frac{hM}{q\delta} \right) \right|. \]

Note that

\[ e \left( \frac{hM}{q\delta} \right) = e \left( \frac{hM\overline{q}}{\delta} \right) e \left( \frac{ha\delta}{q} \right) \]

by the reciprocity law. We are now in a good position to apply the hypothesis for the deformation factor

\[ (4.5) \quad h \mapsto \| \hat{W}_\delta \|_\infty^{-1} \hat{W}_\delta(\overline{\gamma}h)e \left( \frac{ha\overline{q}}{\delta} \right) \]

and the trace function

\[ h \mapsto \hat{K}(\delta h)e \left( \frac{ha\overline{\delta}}{q} \right). \]

By Fourier inversion, the Fourier transform of (4.5) is given by

\[ y \mapsto \| \hat{W}_\delta \|_\infty^{-1} \hat{W}_\delta(q(a - y)), \]
the sup-norm of which bounded by 1 in view of Lemma A.4. Therefore, we derive from (4.4) that
\[
\tilde{s}(K,W) \ll N^{1+\varepsilon}(q\delta)^{-1/2+\varepsilon}\left(\frac{q}{q\delta/N}\right)^\kappa (q\delta/N)^{\lambda\delta^\nu}\|\hat{W}_\delta\|_\infty
\ll N^{\varepsilon}(q/N)^{\lambda-\frac{1}{2}}N^{\kappa+\frac{1}{2}\delta^\nu-\kappa+\lambda-\frac{1}{2}}\|\hat{W}_\delta\|_\infty
\]
as claimed, completing the proof of Theorem 3.2.

4.3. Proof of Theorem 3.3. We now suppose (Ω) holds with some (κ, λ, ν). We also assume \(q = q_1q_2\) with \((q_1, q_2) = 1\) and the sizes of \(q_1, q_2\) can be chosen freely.

Following the similar arguments to those in the proof of Theorem 3.1, we get
\[
|\mathcal{S}(K,W)|^2 \ll Nq_2 + N^{\varepsilon}(q_1/N)^{\kappa}N^{\lambda+1}\delta^\nu+\frac{1}{2}\|\hat{W}_\delta\|_\infty^2.
\]
To balance the two terms, we may choose \(q_1, q_2\) by
\[
(4.6) \quad q_1 = q_1^{\frac{1}{\kappa+1}}N^{\frac{\kappa-2}{\kappa+1}}, \quad q_2 = q_2^{\frac{\kappa}{\kappa+1}}N^{\frac{\lambda-\kappa}{\kappa+1}},
\]
so that the above estimate becomes
\[
\mathcal{S}(K,W) \ll N^{\varepsilon}(q/N)^{\frac{\kappa}{2(\kappa+1)}}N^{\frac{\lambda+1}{2(\kappa+1)}}\delta^{\frac{\nu}{2}+\frac{1}{2}}\|\hat{W}_\delta\|_\infty
\]
as expected, completing the proof of Theorem 3.3.

5. Arithmetic exponent pairs with constraints

In practice, the assumption about good factorizations in Theorem 3.3 is a bit strong; this is usually not satisfied in certain applications. More precisely, the sizes of some factors of \(q\) might be restricted, so that one cannot make balances as freely as in (4.6). Suppose \(q\) has a factor which is of size \(Q\), and the complementary divisor allows sufficiently good factorizations. We can prove an invariant of Theorem 3.3 subject to this constraint.

In fact, we can argue as in the proof of Theorem 3.3. Upon the choice (4.6), one should assume \(q_1 \geq Q\), in which case the iteration can be applied. Note that
\[
A^{-1}(\kappa, \lambda, \nu) = \left(\frac{2\kappa}{1-2\kappa}, \frac{2\lambda - 1}{1 - 2\kappa}, \frac{2\nu - 1}{2}\right)
\]
for \((\kappa, \lambda, \nu) \neq \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\). Hence we obtain an exponent pair \((\kappa, \lambda, \nu)\), if it is produced by \(A\)-processes from \(\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\) and \(q^{1-2\kappa}N^{2\kappa-2\lambda+1} \geq Q\). On the other hand, if the exponent pair \((\kappa, \lambda, \nu)\) is produced by \(A\)-processes with an extra \(B\)-process in the last iteration, one should replace \((\kappa, \lambda)\) by \((\lambda - \frac{1}{2}, \kappa + \frac{1}{2})\) in the above restriction, thus getting the new constraint \(q^{2-2\lambda}N^{2\lambda-2\kappa-1} \geq Q\).
Theorem 5.1. Suppose $q$ has a factor which is of size $Q$, and the complementary divisor allows sufficiently good factorizations. Assume $K$ is $\infty$-amiable. Then (Ω) holds with the exponent pair $(\kappa, \lambda, \nu) \neq (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, provided that one of the following conditions holds:

(a) $(\kappa, \lambda, \nu) = A^k(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ for some $k \geq 1$ and

\[ q^{1-2\kappa N^{2\kappa-2\lambda+1}} \geq Q. \]  

(5.1)

(b) $(\kappa, \lambda, \nu) = B A^k(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ for some $k \geq 1$ and

\[ q^{2-2\lambda N^{2\lambda-2\kappa-1}} \geq Q. \]  

(5.2)

One can also obtain relevant restrictions for other types of $(\kappa, \lambda, \nu)$.

6. Explicit estimates for sums of trace functions

In this section, we give some explicit estimates for averages of algebraic trace functions. In particular, we display the dependence of the upper bound on divisors of $q$. In what follows, we assume $q = q_1 q_2 \cdots q_J$.

The first estimate can be compared with Proposition 3.2.

Theorem 6.1 ($A^k B$-estimate). Assume $K$ is $J$-amiable. For $|I| \ll (q\delta)^O(1)$, we have

\[
\mathcal{G}(K, W) \ll |I|^{1+\varepsilon} \|\hat{W}_\delta\|_\infty \left\{ \frac{\omega_0}{\sqrt{q\delta}} + \sum_{1 \leq j \leq J-1} \left( \frac{(qJ+1-j)}{|I|} \right)^{2-j} + \left( \frac{\delta^2 q_1}{|I|^2} \right)^{2-j} \right\}, \tag{6.1}
\]

where $\omega_0 = 1$ if $|I| > q\delta$ and vanishes otherwise. The implied constant depends on $J, \varepsilon$ and $c$.

Sketch of the proof. Consider the case $\omega_0 = 1$ first. In such case, we may split the summation by periodicity: There are $[|I|/q\delta]$ complete intervals of length $q\delta$, each of which can contribute a complete exponential sum mod $q\delta$. The remaining part is of length $|I| - q\delta[|I|/q\delta]$, which would be equal to $|I|$ if $\omega_0 = 0$. Each above complete sum is equal to

\[
\sum_{a \pmod{q\delta}} K(a) W_\delta(a) = \left\{ \prod_{p|q} \sum_{a \pmod{p}} K_p(a) \right\} \sum_{a \pmod{\delta}} W_\delta(a) \ll (q\delta)^{1/2+\varepsilon} \|\hat{W}_\delta\|_\infty
\]

by Riemann hypothesis for $K_p$ (Proposition 2.1). The total contribution from all complete sums is then

\[
\ll [|I|/(q\delta)][(q\delta)^{1/2+\varepsilon} \|\hat{W}_\delta\|_\infty \ll |I|/(q\delta)^{-1/2+\varepsilon} \|\hat{W}_\delta\|_\infty
\]

as expected.
To complete the proof, it remains to consider the case $\omega_0 = 0$. A direct application of Proposition 3.2 would yield a weaker term

$$\left( \frac{q_1}{|I|^2} \right)^{2^{-J} \delta^{1/2}}$$

in the last one. This matter comes from the trivial estimate (4.3) in each iteration. Following the similar idea, one can even prove a slightly stronger version

$$S(K, W) \ll |I|^{1+\varepsilon} \left\{ \frac{\omega_0}{\sqrt{q\delta}} \|\hat{W}_\delta\|_\infty + \sum_{1 \leq j \leq J-1} \left( \frac{q_{J+1-j}}{|I|} \right)^{2^{-j}} + \left( \frac{\delta^2 q_1}{|I|^2} \right)^{2^{-J}} \right\}$$

by induction, which yields Theorem 6.1 immediately by noting $\|\hat{W}_\delta\|_\infty \gg 1$. \square

**Remark 9.** The case $\delta = 1$ in (6.1) was claimed in [Po, Section 6] without proof. Heath-Brown [HB3] got, in the case $K(n) = e(f_1(n)f_2(n)/q)$ with $f_1, f_2 \in \mathbb{Z}[X]$, a similar estimate with an extra term roughly of the shape $|I|^{1+\varepsilon} q_i^{-J-1} \|\hat{W}_\delta\|_\infty$.

Some alternative estimates could be obtained by making some initial transformations before applying the $A^k B$-estimate.

**Theorem 6.2 (BA^k B-estimate).** Assume the Fourier transform $\hat{K}$ is $J$-amiable. For $|I| \ll (q\delta)^{O(1)}$, we have

$$S(K, W) \ll |I|^{\varepsilon} \|\hat{W}_\delta\|_\infty \sqrt{q\delta} \left\{ \frac{|I|}{q\delta} \frac{\omega_0}{\sqrt{q\delta}} + \sum_{j=1}^{J-1} \left( \frac{|I| q_{J+1-j}}{q^2} \right)^{2^{-j}} + \left( \frac{|I|^2 q_1}{q^2} \right)^{2^{-J}} \right\},$$

where $\omega_0 = 1$ if $|I| > q\delta$ and vanishes otherwise. The implied constant depends on $J, \varepsilon$ and $c$.

**Remark 10.** Irving [Ir2] beat the Weyl bound for Dirichlet $L$-functions using estimates for character sums that are of the same strength with Theorem 6.2 with $K(n) = \chi(n) \chi(n+h)$, $h \neq 0$.

7. **Proof of Theorem 1.1: Quadratic Brun-Titchmarsh Theorem**

We complete the proof of Theorem 1.1 in this section.

7.1. **Linear sieves.** We introduce some conventions and fundamental results on the linear Rosser-Iwaniec sieve.

Let $A = (a_n)$ be a finite sequence of integers and $\mathcal{P}$ a set of prime numbers. Define the sifting function

$$S(A, \mathcal{P}, z) = \sum_{(n,P(z))=1} a_n \quad \text{with} \quad P(z) = \prod_{p<z, p \in \mathcal{P}} p.$$
For squarefree $d$ with all its prime factors belonging to $\mathcal{P}$, we consider subsequence $\mathcal{A}_d = (a_n)_{n \equiv 0 \pmod{d}}$ and the congruence sum

$$A_d = \sum_{n \equiv 0 \pmod{d}} a_n.$$ 

We assume that $\mathcal{A}_d$ is well located in $\mathcal{A}$ in the following sense: There are an appropriate approximation $X$ to $A_1$ and a multiplicative function $g$ supported on squarefree numbers with all its prime factors belonging to $\mathcal{P}$ verifying

$$0 < g(p) < 1 \quad (p \in \mathcal{P})$$

such that

(a) the remainder

$$r(\mathcal{A}, d) = A_d - g(d)X$$

is small on average over $d \mid P(z)$;

(b) there exists a constant $L > 1$ such that

$$\frac{V(z_1)}{V(z_2)} \leq \frac{\log z_2}{\log z_1} \left(1 + \frac{L}{\log z_1}\right)$$

with $V(z) = \prod_{p < z, p \in \mathcal{P}} (1 - g(p))$

for $2 \leq z_1 < z_2$.

Let $F$ and $f$ be the continuous solutions to the system

$$\begin{cases}
sF(s) = 2e^\gamma & \text{for } 0 < s \leq 2, \\
sf(s) = 0 & \text{for } 0 < s \leq 2, \\
(sF(s))' = f(s - 1) & \text{for } s > 2, \\
(sf(s))' = F(s - 1) & \text{for } s > 2,
\end{cases}$$

(7.1)

where $\gamma$ is the Euler constant.

For $k \geq 1$, denote by $\tau_k(n)$ the number of ways of expressing $n$ as the product of $k$ positive integers. An arithmetic function $\lambda(d)$ is of level $D$ and order $k$, if

$$\lambda(d) = 0 \quad (d > D) \quad \text{and} \quad |\lambda(d)| \leq \tau_k(d) \quad (d \leq D).$$

Let $r \geq 2$ be a positive integer. We say that $\lambda$ is well-factorizable of degree $r$, if for every decomposition $D = D_1 D_2 \cdots D_r$ with $D_1, D_2, \ldots, D_r \geq 1$, there exist $r$ arithmetic functions $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that

$$\lambda = \lambda_1 * \lambda_2 * \cdots * \lambda_r$$

with each $\lambda_j$ of level $D_j$ and order $k$.

We now state the following fundamental result of Iwaniec [Iw1].

**Lemma 7.1.** Let $r \geq 2$ and $0 < \varepsilon < \frac{1}{8}$. Under the above hypothesis, we have

$$S(\mathcal{A}, \mathcal{P}, z) \leq XV(z) \left\{ F\left(\frac{\log D}{\log z}\right) + E \right\} + \sum_{\nu \leq T} \sum_{d \mid P(z)} \lambda^{(\nu)}(d)r(\mathcal{A}, d)$$
for all $z \geq 2$, where $F$ is given by the system (7.1), $T$ depends only on $\varepsilon$, $\lambda^{(\nu)}(d)$ is well-factorizable of level $D$, degree $r$ and order $1$, and $E \ll \varepsilon + \varepsilon^{-8} e^{K(\log D)^{-1/3}}$.

As mentioned before, the well-factorization of $\lambda^{(\nu)}(d)$ is essential in estimates for exponential sums arising in the proof of Theorem 1.1. The original statement in [Iw1, Theorem 4] is a bit different, and it implies Lemma 7.1 by combining an iterative application of Lemma 1 therein.

7.2. Quadratic congruences. Before starting the proof of Theorem 1.1, we recall one classical result of Gauß on the representation of numbers by binary quadratic forms. We refer to Disquisitiones Arithmeticae of or Smith’s report [Sm] for a very clear description of this theory in a form suitable for our purpose.

Lemma 7.2. Let $\ell \geq 1$. If

\begin{equation}
(7.2) \quad a^2 + 1 \equiv 0 \pmod{\ell}
\end{equation}

is solvable for $a \pmod{\ell}$, then $\ell$ can be represented properly as a sum of two squares

\begin{equation}
(7.3) \quad \ell = r^2 + s^2, \quad (r, s) = 1, \quad r > 0, \quad s > 0.
\end{equation}

There is a one to one correspondence between the incongruent solutions $a \pmod{\ell}$ to (7.2) and the solutions $(r, s)$ to (7.3) given by

\begin{equation*}
\frac{a}{\ell} = \frac{s}{r} - \frac{s}{r(r^2 + s^2)}.
\end{equation*}

For each integer $d \geq 1$, we put $d = d_1d_2$ with $d_2 = (d, r^\infty)$. Then

\begin{equation}
(7.4) \quad \frac{\overline{d}a}{\ell} \equiv - \frac{rd_2(r^2 + s^2)}{d_1s} + \frac{r}{ds(r^2 + s^2)} - \frac{rd_1s(r^2 + s^2)}{d_2} \pmod{1}.
\end{equation}

In particular, if $d$ is squarefree, we have

\begin{equation*}
\frac{\overline{d}a}{\ell} \equiv - \frac{rd_2(r^2 + s^2)}{d_1s} + \frac{r}{ds(r^2 + s^2)} \pmod{1}.
\end{equation*}

All the mod inverses are well-defined with respect to relevant denominators without special precision.

Proof. It suffices to prove (7.4) and the remaining part can be actually found in [Sm] or [DI2, Lemma 2]. Note that $d_2 = (d, r^\infty)$, thus $(d_1, r\ell) = (d_2, d_1s\ell) = 1$. It follows that

\begin{equation*}
\frac{\overline{d}a}{\ell} \equiv d_2d_1a_{\ell} \pmod{1},
\end{equation*}

where $d_2d_1 \equiv \overline{d_1}d_1 \equiv 1 \pmod{\ell}$. From the Chinese Remainder Theorem, we may choose $d_2, d_1$ such that $d_2d_1 \equiv 1 \pmod{d_1s\ell}$ and $d_1d_1 \equiv 1 \pmod{r\ell}$. On the other
hand, we have
\[
\frac{a}{\ell} \equiv \frac{\overline{\sigma}(r^2 + s^2)}{r(r^2 + s^2)} - \frac{s}{r(r^2 + s^2)} \pmod{1},
\]
where \( s\overline{s} \equiv 1 \pmod{(r(r^2 + s^2))}. \) Combining these yields
\[
\frac{d\alpha}{\ell} \equiv d_2^* \left( \frac{d_1 s(r^2 + s^2)}{r(r^2 + s^2)} - \frac{sd_1}{r(r^2 + s^2)} \right) \pmod{1}.
\]
Using the reciprocity law
\[
\frac{\overline{\sigma}}{u} + \frac{\overline{\nu}}{v} \equiv \frac{1}{uv} \pmod{1}, \quad (u, v) = 1
\]
with tuples \((u, v) = (d_1 s, r(r^2 + s^2))\) and \((u, v) = (d_1, r(r^2 + s^2))\), we get
\[
(7.5) \quad \frac{d\alpha}{\ell} \equiv - \frac{d_2^* \sigma}{d_1 s} + \frac{d_2^* s r(r^2 + s^2)}{d_1} + \frac{d_2^*(r^2 + s^2)}{d_1 rs(r^2 + s^2)} - \frac{d_2^* s}{d_1 r(r^2 + s^2)} \pmod{1}.
\]
Furthermore, we have
\[
- \frac{d_2^* \sigma}{d_1 s} + \frac{d_2^* s r(r^2 + s^2)}{d_1} \equiv - \frac{d_2 r}{d_1 s} + \frac{d_2 r s^2(r^2 + s^2)}{d_1 s} \pmod{1}
\]
\[
\equiv - \frac{rd_2(r^2 + s^2)}{d_1 s} \pmod{1}
\]
and
\[
\frac{d_2^*(r^2 + s^2)}{d_1 rs(r^2 + s^2)} - \frac{d_2^* s}{d_1 r(r^2 + s^2)} \equiv r \left( \frac{1}{d_1 d_2 s(r^2 + s^2)} - \frac{d_1 s(r^2 + s^2)}{d_2} \right) \pmod{1},
\]
which can be simplified to
\[
\equiv \frac{r}{ds(r^2 + s^2)} \pmod{1}
\]
if \( d \) is squarefree, since \( d_2 | r \) in that case. Inserting these to \((7.5)\), we are done. \( \Box \)

7.3. Initial treatment by linear sieves. We now start the proof of Theorem 1.1. The initial step is to transform sums over primes to those over integers via linear sieves. To do so, we consider the congruence sum
\[
A_d(X; \ell) := \sum_{\substack{n \geq 1 \\text{n^2+1\equiv0(mod \ell) \\text{n\equiv0(mod d)}}} g\left(\frac{n}{X}\right) = \sum_{\substack{n \geq 1 \\text{d^2n^2+1\equiv0(mod \ell)}}} g\left(\frac{dn}{X}\right).
\]
By Poisson summation (Lemma A.3), we find

\[
A_d(X; \ell) = \frac{X}{d\ell} \sum_h \hat{g}\left(\frac{hX}{d\ell}\right) \sum_{a \pmod{\ell}} \sum_{a^2+1 \equiv 0 \pmod{\ell}} e\left(\frac{a\bar{d}h}{\ell}\right),
\]

where we have used the implicit condition \((d, \ell) = 1\). The zero-th frequency is expected to produce the main contribution, i.e., \(\hat{g}(0)\rho(\ell)(d\ell)^{-1}X\). Define

\[
r(X, d; \ell) := A_d(X; \ell) - \hat{g}(0)\rho(\ell)(d\ell)^{-1}X,
\]

By the Möbius formula, we have

\[
Q_\ell(X) = \sum_{n^2+1 \equiv 0 \pmod{\ell}} g\left(\frac{n}{X}\right) \sum_{d \mid (n, P(z))} \mu(d) \quad (z = \sqrt{X}).
\]

Let \((\lambda_d)\) be a linear upper bound sieves of level \(D\), so that \(1 \ast \mu \leq 1 \ast \lambda\). Then

\[
Q_\ell(X) \leq \sum_{d \leq D \atop d \mid P(z)} \lambda_d A_d(X; \ell) \leq \hat{g}(0)XV(z)F\left(\frac{\log D}{\log z}\right) + \sum_{d \leq D \atop d \mid P(z)} \lambda_d r(X, d; \ell),
\]

where \(F\) is defined by (7.1) and

\[
V(z) := \prod_{p < z, p \mid \ell} \left(1 - \frac{1}{p}\right) = \frac{\ell}{\varphi(\ell)} \cdot \frac{e^{-\gamma}}{\log z} \left\{1 + O\left(\frac{1}{\log X}\right)\right\}.
\]

Thanks to Lemma 7.1, we may choose well-factorable remainder terms in the above application of linear Rosser-Iwaniec sieve. We thus conclude the following result.

**Proposition 7.1.** Let \(\varepsilon > 0\) and \(D < X\). For any given \(J \geq 2\), we have

\[
Q_\ell(X) \leq \{2 + O(\varepsilon)\}\hat{g}(0)\rho(\ell)\frac{X}{\varphi(\ell) \log D} + \sum_{\nu \leq T(\varepsilon) \atop d \leq D \atop d \mid P(z)} \lambda^{(\nu)}(d)r(X, d; \ell),
\]

where \(T(\varepsilon)\) depends only on \(\varepsilon\), \(\lambda^{(\nu)}(d)\) is well-factorizable of level \(D\), degree \(J\) and order 1.

Given a modulo \(\ell\) of certain size, we hope the level \(D\) could be chosen as large as possible. Theorem 1.1 is then implied by the following key proposition.
Proposition 7.2. Let $J$ be a sufficiently large integer and let $\lambda$ be well-factorizable of degree $J$. With the same notation as Proposition 7.1, for any $\varepsilon > 0$ and $(D, L) := (X^{\gamma(\theta) - \varepsilon}, X^{\theta})$, there exists some $\delta = \delta(\varepsilon) > 0$ such that

$$\sum_{\ell \sim L} \left| \sum_{d \leq D} \mu(d)^2 \lambda(d)r(X, d; \ell) \right| \ll X^{1-\delta},$$

where $\gamma(\theta)$ is given by (1.7) and the implied constant depends on $\varepsilon$ and $J$.

Proposition 7.2 implies, in the ranges (1.7), that

$$\sum_{\nu \leq T(\varepsilon)} \sum_{d \leq D} d |P(z)| \lambda^{(\nu)}(d) r(X, d; \ell) \ll \ell^{-1} X^{1-\delta}$$

save for at most $O(L(\log L)^{-A})$ exceptional values of $\ell \in (L, 2L)$. This, together with Proposition 7.1, yields

$$Q_\ell(X) \leq \{2 + o(1)\} \hat{g}(0) \frac{\rho(\ell)}{\varphi(\ell)} \frac{X}{\log D}$$

for such $\ell$. It thus suffices to prove Propositions 7.2 in order to derive Theorem 1.1.

7.4. Reducing to exponential sums. Recall that

$$r(X, d; \ell) = \frac{X}{d\ell} \sum_{a \pmod{\ell}} \sum_{\substack{h \neq 0 \mod{\ell} \atop a^2 + 1 \equiv 0 \mod{\ell}}} \hat{g} \left( \frac{hX}{d\ell} \right) e \left( \frac{adh}{\ell} \right).$$

From the rapid decay of $\hat{g}$, we get

$$r(X, d; \ell) = \frac{X}{d\ell} \sum_{a \equiv 0 \pmod{\ell}} \sum_{\substack{0 < |h| \leq H \mod{\ell} \atop a^2 + 1 \equiv 0 \mod{\ell}}} \hat{g} \left( \frac{hX}{d\ell} \right) e \left( \frac{adh}{\ell} \right) + O(X^{-100})$$

for $H = DLX^{-1+\varepsilon}$. It then suffices to prove that

$$\sum_{\ell \sim L} \sum_{a \equiv 0 \pmod{\ell}} \left| \sum_{d \leq D} \mu(d)^2 \lambda(d) \sum_{h \leq H} \hat{g} \left( \frac{hX}{d\ell} \right) e \left( \frac{adh}{\ell} \right) \right| \ll DLX^{-\varepsilon'}$$

for some $\varepsilon' > 0$ (depending on $\varepsilon$). Due to the well-factorization of $\lambda$, we may write

$$\lambda = \alpha \ast \beta \quad \text{with} \quad \alpha = \alpha_1 * \cdots * \alpha_{J_1} \quad \text{and} \quad \beta = \beta_1 * \cdots * \beta_{J_2},$$

where $\alpha_j$ and $\beta_j$ are of level $M_j$ and $N_j$, respectively, and

$$M_1 \cdots M_{J_1} = M, \quad N_1 \cdots N_{J_2} = N, \quad MN = D \quad \text{and} \quad J_1 + J_2 = J.$$
By Cauchy inequality, we turn to consider the second moment

\[
\mathcal{B}(M, N) := \sum_{m \sim M} |\alpha(m)| \sum_{\ell \atop a^2 + 1 \equiv 0 \pmod{\ell}} W\left(\frac{\ell}{L}\right) \sum_{n \sim N} \beta(n) \sum_{h \leq H} \hat{\gamma}\left(\frac{hX}{\ell mn}\right) e\left(\frac{ahmn}{\ell}\right),
\]

where \( W \) is a non-negative smooth function with compact support in \( \left[\frac{1}{2}, \frac{5}{2}\right] \) and takes value 1 in \( [1, 2] \). We would like to show, for some small \( \delta > 0 \), that

\[
(7.7) \quad \mathcal{B}(M, N) \ll (ML)^{-1}(DL)^2 X^{-\delta}
\]

holds in the ranges of \( (1.7) \).

Squaring out and switching summations, we get

\[
\mathcal{B}(M, N) = \sum_{n_1, n_2 \sim N} \beta(n_1)\beta(n_2) \sum_{h_1, h_2 \leq H} \sum_{m \sim M} |\alpha_m| \sum_{\ell} \Phi_{\ell}(h; m, n) \rho_{\overline{\nu}(h_1, h_2)}(\ell),
\]

where \( h = (h_1, h_2), \ n = (n_1, n_2), \rho(\ell) \) is defined as in \( (1.8) \) and

\[
\Phi_{\ell}(h; m, n) := W\left(\frac{\ell}{L}\right) \hat{\gamma}\left(\frac{h_1X}{\ell mn_1}\right) \hat{\gamma}\left(\frac{h_2X}{\ell mn_2}\right).
\]

According to Lemma 7.2, for each \( a \pmod{\ell} \) with \( a^2 + 1 \equiv 0 \pmod{\ell} \), we have

\[
\frac{ah_jmn_j}{\ell} \equiv -\frac{h_jr a_j (r^2 + s^2)}{s mn_j/a_j} + \frac{h_jr}{mn_j s (r^2 + s^2)} \pmod{1}
\]

for \( j = 1, 2 \), where \( a_j = (mn_j, r), \ r > 0, \ s > 0, \ (r^2 + s^2, mn_1n_2) = (r, s) = 1 \) and \( r^2 + s^2 = \ell \). Therefore, the exponential sum \( \rho_{\overline{\nu}(h_1, h_2)}(\ell) \) can be rewritten as

\[
\sum_{a_1 | mn_1} \sum_{a_2 | mn_2} e\left(\frac{h_1ra_1 (r^2 + s^2)}{smn_2/a_2} - \frac{h_1ra_1 (r^2 + s^2)}{smn_1/a_1}\right) e\left(\frac{r(h_1/n_1 - h_2/n_2)}{ms(r^2 + s^2)}\right).
\]

Note the Taylor expansion

\[ e\left(\frac{r(h_1/n_1 - h_2/n_2)}{ms(r^2 + s^2)}\right) = 1 + O\left(\frac{Hr}{LMN s}\right), \]

it then follows that

\[
(7.8) \quad \mathcal{B}(M, N) = \sum_{n_1 \sim N} \beta(n_1)\beta(n_2) \sum_{h_1 \leq H} \sum_{m \sim M} |\alpha_m| \sum_{\ell} \sum_{a_1 | mn_1} \sum_{a_2 | mn_2} \Phi_{r^2+s^2}(h; m, n) e\left(\frac{h_2ra_2 (r^2 + s^2)}{smn_2/a_2} - \frac{h_1ra_1 (r^2 + s^2)}{smn_1/a_1}\right) + O(H^3 N).
\]
The innermost sums over $r,s$ can be split into dyadic intervals, $r \sim R, s \sim S$, say, where $R^2 + S^2 \asymp L$ and so that $R, S \ll \sqrt{L}$. The following arguments will focus on estimating the sum over $r$ effectively to gain cancellations and the other sums will be treated trivially. We thus consider the following exponential sum

$$T(\mathbf{a}; m, n, s) := \sum_{r \sim R}^{*} e(\xi(r)),$$

where $\mathbf{a} = (a_1, a_2)$, the symbol $*$ means that $(r, s) = 1$ and $(mn_1n_2, r^2 + s^2) = 1$, and

$$(7.9) \quad \xi(r) \equiv \frac{h_2ra_2(r^2 + s^2)}{smn_2/a_2} - \frac{h_1ra_1(r^2 + s^2)}{smn_1/a_1} \pmod{1}.$$

Thanks to the well-factorizations of $m, n_1, n_2$, we may appeal to $q$-vdC and the method of arithmetic exponent pairs developed in Section 3 to give sharp estimates for $T(\mathbf{a}; m, n, s)$. Note that the restrictions $(mn_1, r) = a_1$ and $(mn_2, r) = a_2$ can be rewritten equivalently as $r \equiv 0 (\text{mod} [a_1, a_2])$ with some extra coprime conditions on $r$. There are many ways to relax the congruence condition; for instance, one can write $r = r'[a_1, a_2]$ with $r' \sim R/[a_1, a_2]$, or the orthogonality of additive characters can be utilized to detect the divisibility. For the economy of presentation, we appeal to the first treatment and write

$$T(\mathbf{a}; m, n, s) = \sum_{r \sim R_0}^{*} e(\xi([a_1, a_2]r)),$$

where $R_0 = R/[a_1, a_2]$. The subsequent argument will devote to the inner sum over $r$, and then sum trivially over $h$.

7.5. Estimates for exponential sums. Let $j = 1, 2$. For $a_j = (mn_j, r)$, we may write $a_j = bc_j$ in a unique way, where $b = (m, r), c_j = (n_j, r)$. Note that $(bc_j, s) = 1$. Put $\tilde{m} := m/b$, so that one can rewrite (7.9) as

$$\xi(r) \equiv \frac{h_2ra_2(r^2 + s^2)}{\tilde{m}(n_2/c_2)s} - \frac{h_1ra_1(r^2 + s^2)}{\tilde{m}(n_1/c_1)s} \pmod{1}.$$

Write $\tilde{n} = [n_1/c_1, n_2/c_2]$, and $\tilde{n} = (n_1/c_1)\sigma_1 = (n_2/c_2)\sigma_2$, so that $(\sigma_1, \sigma_2) = 1$. Thus we may have

$$\xi(r) \equiv r\xi \cdot \frac{a_2(r^2 + s^2)}{\tilde{m}\tilde{n}s} \pmod{1}, \quad \xi := h_2\sigma_2a_1 - h_1\sigma_1a_2.$$

Denote by $(\tilde{m}\tilde{n}s)^b$ and $(\tilde{m}\tilde{n}s)^i$ the squarefree and squarefull parts of $\tilde{m}\tilde{n}s$, respectively. We then have

$$\xi(r) \equiv r\xi \left( \frac{a_2(\tilde{m}\tilde{n}s)^b(r^2 + s^2)}{(\tilde{m}\tilde{n}s)^b} + \frac{a_1a_2(\tilde{m}\tilde{n}s)^b(r^2 + s^2)}{(\tilde{m}\tilde{n}s)^i} \right) \pmod{1}$$
\[
\equiv d \frac{r \xi (\tilde{m} \tilde{n} s)^2 (r^2 + s^2)}{(\tilde{m} \tilde{n} s)^2 / d} + r \xi (\tilde{m} \tilde{n} s)^2 (r^2 + s^2) (\tilde{m} \tilde{n} s)^2 \quad (\text{mod} \ 1),
\]

where \(d := (\xi, (\tilde{m} \tilde{n} s)^2)\).

We would like to apply the method of arithmetic exponent pairs in the case of \(q = (\tilde{m} \tilde{n} s)^2 / d, \delta = (\tilde{m} \tilde{n} s)^2\), the trace function

\[
K : x \mapsto e\left(\frac{x \xi [a_1, a_2] a_1 a_2 (\tilde{m} \tilde{n} s)^2 ([a_1, a_2]^2 x^2 + s^2)}{(\tilde{m} \tilde{n} s)^2 / d}\right)
\]

and the deformation factor

(7.10) \[
W_\delta : x \mapsto e\left(\frac{x \xi [a_1, a_2] a_1 a_2 (\tilde{m} \tilde{n} s)^2 ([a_1, a_2]^2 x^2 + s^2)}{(\tilde{m} \tilde{n} s)^2}\right).
\]

Before stating the precise estimate for \(T(a; m, n, s)\), one has to produce an admissible upper bound for \(\tilde{W}_\delta\). In Appendix B, we will present an almost square-root cancellation for complete algebraic exponential sums, see Theorem B.1 therein, from which we may conclude, if \(W_\delta\) is chosen as (7.10), that

\[
\|\tilde{W}_\delta\|_\infty \leq 2 \cdot 6^{\omega(\delta)} (h_2 \sigma_2 a_1 - h_1 \sigma_1 a_2, (\tilde{m} \tilde{n} s)^2) \cdot \Xi((\tilde{m} \tilde{n} s)^2)^{1/2},
\]

where \(\Xi(\cdot)\) is defined as (A.1) and is bounded on average (see Lemma A.1).

Let \((\kappa, \lambda, \nu)\) be an exponent pair defined by \((\Omega)\). Hence we have

\[
T(a; m, n, s) \ll R^c \|\tilde{W}_\delta\|_\infty \left(\frac{q}{R_0}\right)^\kappa R_0^\lambda \delta^\nu
\]

\[
\ll X^c \cdot \Xi((\tilde{m} \tilde{n} s)^2)^{1/2} \left(\frac{(\tilde{m} \tilde{n} s)^2 / d}{R_0}\right)^\kappa R_0^\lambda \Xi((\tilde{m} \tilde{n} s)^2)^\nu,
\]

where \(c = (h_2 \sigma_2 a_1 - h_1 \sigma_1 a_2, (\tilde{m} \tilde{n} s)^2)\). This estimate is valid for incomplete exponential sums, i.e., \(R < q \delta [a_1, a_2] = \tilde{m} \tilde{n} s [a_1, a_2] / d\), and otherwise, we may appeal to the completing method only, getting

\[
T(a; m, n, s) \ll R^c \left[\frac{R_0}{q \delta}\right] \sqrt{q \delta} \|\tilde{W}_\delta\|_\infty
\]

\[
\ll X^c \cdot \Xi((\tilde{m} \tilde{n} s)^2)^{1/2} \frac{R_0}{\sqrt{\tilde{m} \tilde{n} s / d}}.
\]

Noticing that \(\kappa \leq \frac{1}{2}, \lambda \geq \frac{1}{2}, \nu \leq \frac{1}{2}\), we would like to insert the above two estimates to (7.8), and from Theorem 5.1, we conclude that

\[
\mathcal{B}(M, N) \ll H^3 N + X^c (LH MN + H^2 L^{3/4} M^{1/2}) N
\]

\[
+ L^{(\lambda - \kappa + 1)/2} H MN + (MN^2)^{\kappa + 1} H^2 L^{(\lambda + 1)/2}
\]

(7.11) \[
\ll H^3 N + X^c (HLMN + (MN^2)^{\kappa + 1} H^2 L^{(\lambda + 1)/2}),
\]

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provided that

$$(MN^2)^{1-2\kappa} \geq L^{\lambda-\frac{1}{2}}$$

if $(\kappa, \lambda)$ is of the type $A^k(\frac{1}{2}, \frac{1}{2})$; or that

$$(MN^2)^{2-2\lambda} \geq L^\kappa$$

if $(\kappa, \lambda)$ is of the type $BA^k(\frac{1}{2}, \frac{1}{2})$. Here we have used Lemmas A.1 and A.2. Note that the terms $LHMN$ and $L^{(\lambda-\kappa+1)/2}HMN$ come from the diagonal terms with $h_2 \sigma_2 a_1 = h_1 \sigma_1 a_2$.

7.6. **Concluding Theorem 1.1.** In order to conclude Proposition 7.2, and thus Theorem 1.1, it suffices to check, in view of (7.7), that

$$H^3N + HLMN + (MN^2)^{\kappa+1}(M^2)H^{(\lambda+1)/2}L \ll (ML)^{-1}(DL)^2X^{-\varepsilon'}$$

with the restriction $(MN^2)^{1-2\kappa} \geq L^{\lambda-\frac{1}{2}}$ or $(MN^2)^{2-2\lambda} \geq L^\kappa$. The task is to find the maximum of $D = MN$ while $M, N$ satisfy the above inequalities with some exponent pair $(\kappa, \lambda) \neq (\frac{1}{2}, \frac{1}{2})$. Put

$$M = X^\alpha, \quad N = X^\beta, \quad D = X^\gamma = X^{\alpha+\beta}, \quad L = X^\theta.$$

We are going to maximize $\gamma = \alpha + \beta$ subject to the simultaneous restrictions

$$\alpha > 0, \quad \beta > 0, \quad \alpha + \beta + \theta < \frac{3}{2},$$

$$\alpha + \theta < 1, \quad (\alpha + 2\beta)(\kappa + 1) + \frac{1}{2}\theta(\lambda + 3) + \alpha < 2$$

with an additional constraint

$$(1 - 2\kappa)(\alpha + 2\beta) \geq (\lambda - \frac{1}{2})\theta \quad \text{or} \quad 2(1 - \lambda)(\alpha + 2\beta) \geq \kappa\theta,$$

depending that $(\kappa, \lambda)$ comes from $A^k$- or $BA^k$-process.

Choosing different exponent pairs, we may conclude different maximum of $\gamma$ while $\theta$ is in different ranges. We list the choices as the following table.

| $(\kappa, \lambda)$   | $(\frac{1}{6}, \frac{5}{3}) = A(\frac{1}{2}, \frac{1}{2})$ | $(\frac{1}{14}, \frac{11}{14}) = A^2(\frac{1}{2}, \frac{1}{2})$ | $(\frac{1}{30}, \frac{26}{30}) = A^3(\frac{1}{2}, \frac{1}{2})$ |
|-----------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|
| maximum of $\gamma$   | $\frac{19-18\theta}{14}$                         | $\frac{86-83\theta}{60}$                         | $\frac{91-89\theta}{62}$                         |
| range of $\theta$     | $[\frac{1}{2}, \frac{16}{17}]$                   | $[\frac{1}{2}, \frac{8}{9}]$                     | $[\frac{1}{2}, \frac{112}{131}]$               |

Table 2. Choices of $(\theta, \gamma)$

Proposition 7.2 follows by optimizing the maximum of $\gamma$, and this completes the proof of Theorem 1.1.
8. Proof of Theorems 1.2 and 1.3: Divisor functions in arithmetic progressions and subconvexity of Dirichlet $L$-functions

We now give the sketch of the proof of Theorem 1.2. Following the arguments of Irving, it suffices to estimate the exponential sum

$$S := \sum_{n \sim N} e\left(\frac{hn}{q}\right)$$

while $N \asymp \sqrt{X}$ and $(h, q) = 1$. Let $(\kappa, \lambda)$ be an arithmetic exponent pair. Then

$$S \ll N^{\varepsilon}(q/N)^{\kappa} N^{\lambda} \ll q^{\kappa} X^{(\lambda - \kappa)/2 + \varepsilon}.$$ 

This is dominated by $\varphi(q)^{-1}X \log X$ as long as

$$\theta < \frac{2}{3} + \frac{2 - \kappa - 3\lambda}{6(\kappa + 1)}.$$ 

Using the algorithm for exponent pairs (see [GK, Section 5]), we may choose

$$(\kappa, \lambda) = B A^3 B A^2 B A B A^2 (\frac{1}{2}, \frac{1}{2}) = (\frac{591}{1535}, \frac{808}{1535}),$$

getting

$$\frac{2 - \kappa - 3\lambda}{6(\kappa + 1)} = \frac{55}{12756} \approx \frac{1}{231.92},$$

which completes the proof of Theorem 1.2.

We now turn to the proof of Theorem 1.3. Using the approximate functional equation of $L(\frac{1}{2}, \chi)$, it suffices to consider the incomplete character sum

$$\sum_{M < n \leq M + N} \chi(n).$$

Suppose $(\kappa, \lambda)$ is an exponent pair, we then have

$$\sum_{M < n \leq M + N} \chi(n) \ll q^{\kappa} N^{\lambda - \kappa + O(1)},$$

and

$$L(\frac{1}{2}, \chi) \ll q^{\frac{\lambda - \kappa}{2} + O(1)].$$

According to Rankin (see also [GK, Section 5.4]), the minimal value for $\kappa + \lambda$ should be 0.829021..., which yields the expected exponent in Theorem 1.3.
Appendix A. Several lemmas

A.1. Averages of certain multiplicative functions. For each integer $n \geq 1$, denote by $n^\flat$ and $n^\sharp$ the squarefree and squarefull parts of $n$, respectively. Thus, we may write

$$n = n^\flat \cdot n^\sharp \quad \text{and} \quad (n^\flat, n^\sharp) = 1.$$  

We also consider the multiplicative function

$$\Xi(n) := \prod_{\nu \parallel n, \nu \geq 4} p^\nu.$$ (A.1)

The following lemma will be needed in the proof of Proposition 7.2.

**Lemma A.1.** (a) For any fixed $t \leq \frac{1}{2}$ and all $x \geq 2$, we have

$$\sum_{n \leq x} (n^\sharp)^t \ll x \log x \quad \text{and} \quad \sum_{n \leq x} (n^\flat)^{-t} \ll x^{1-t} \log x.$$ 

(b) For any fixed $t \leq \frac{3}{4}$ and all $x \geq 2$, we have

$$\sum_{n \leq x} \xi(n)^t \ll x \log x.$$ 

**Proof.** Since $n \mapsto n^\sharp$ is multiplicative, for $\Re s > 1$, we have

$$\sum_{n \geq 1} (n^\sharp)^t n^{-s} = \prod_p \left(1 + p^{-s} + p^{-2(s-t)} + p^{-3(s-t)} + \cdots \right) = \zeta(s)\zeta(2s - 2t)A_t(s),$$

where the Euler product $A_t(s)$ converges absolutely for $\Re s > \max\{t + \frac{1}{2}, \frac{1}{2}\}$ and $A_t(s) \ll 1$ for $\Re s \geq \frac{6}{7}$ and $t \leq \frac{1}{2}$. The first statement of assertion (a) follows immediately from a routine application of Perron’s formula. The second inequality is a consequence of the first one.

As before, for $\Re s > 1$, we have

$$\sum_{n \geq 1} \Xi(n)^t n^{-s} = \zeta(s)\zeta(4s - 4t)B_t(s),$$

where $B_t(s)$ is holomorphic for $\Re s > \frac{19}{20}$ and $B_t(s) \ll 1$ for $\Re s \geq \frac{20}{21}$ and $t \leq \frac{3}{7}$. A simple application of Perron’s formula leads to the required inequality. \hfill \Box

We also need the following estimate on the greatest common divisors on average.

**Lemma A.2.** For each $q \geq 1$ and all $x \geq 1$, we have

$$\sum_{n \leq x} (n, q) \leq \tau(q)x.$$
**Proof.** The sum in question can be rewritten as
\[
\sum_{d \mid q} \sum_{n \leq x \atop (n, q) = d} 1 = \sum_{d \mid q} \sum_{n \leq x/d \atop (n, q/d) = 1} 1 \leq \sum_{d \mid q, d \leq x} d \cdot x/d \leq \tau(q)x,
\]
as claimed. \qed

### A.2. Fourier analysis.

**Lemma A.3** (Poisson summation formula). Let \( g \) be a smooth function with compact in \( \mathbb{R} \). For \( X \geq 1 \) and \( q \geq 1 \), we have
\[
\sum_{n \equiv a \pmod{q}} g\left(\frac{n}{X}\right) = \frac{X}{q} \sum_{h \in \mathbb{Z}} \hat{g}\left(\frac{hX}{q}\right) e\left(\frac{ah}{q}\right).
\]

We require a lower bound for Fourier transforms over \( \mathbb{Z}/q\mathbb{Z} \), which follows from the uncertainty principle due to Donoho and Stark [DS].

**Lemma A.4.** Let \( F \in \ell^2(\mathbb{Z}/q\mathbb{Z}) \) be a non-zero function. Then we have
\[
|\text{supp}(F)| \cdot |\text{supp}(\hat{F})| \gtrsim q,
\]
where \( \text{supp}(F) := \{a \in \mathbb{Z}/q\mathbb{Z} : F(a) \neq 0\} \). In particular, we have \( \|\hat{F}\|_\infty \gg 1 \).

**Proof.** The first statement can be found in [DS] and it suffices to prove the second one. Assume there exist a sequence of moduli \( \{q_\nu\}_{\nu=1}^\infty \) and a sequence of functions \( \{F_\nu\}_{\nu=1}^\infty \) such that
\[
F_\nu \in \ell^2(\mathbb{Z}/q_\nu\mathbb{Z}), \quad \|\hat{F}_\nu\|_\infty = o(1),
\]
which implies that \( |\text{supp}(F_\nu)| = o(1) \) as \( \nu \to +\infty \). However, the first statement yields
\[
|\text{supp}(F_\nu)| \gtrsim 1
\]
uniformly in \( \nu \geq 1 \), from which we conclude a contradiction. \qed

## Appendix B. Estimates for complete exponential sums

This appendix is devoted to estimate the complete exponential sum
\[
\Sigma(\lambda, c) := \sum_{a \pmod{c}} e\left(\frac{\lambda(a)}{c}\right),
\]
where \( c \) is a fixed positive integer and \( \lambda = \lambda_1/\lambda_2 \) with \( \lambda_1, \lambda_2 \in \mathbb{Z}[X] \) and \( (\lambda_1, \lambda_2) = 1 \) in \( \mathbb{Z}[X] \). The variable \( a \) such that \( (\lambda_2(a), c) \neq 1 \) is excluded from summation. We define the degree of \( \lambda \) by
\[
d = d(\lambda) = \deg(\lambda_1) + \deg(\lambda_2).
\]
There are many known estimates for complete exponential sums studied extensively by Vinogradov, Hua, Vaughan, Cochrane-Zheng, et al. We here present an alternative estimate that suits well for our applications to Theorem 1.1.

By Chinese Remainder Theorem, we have

\[
\Sigma(\lambda, c) = \prod_{p^\beta \mid c} \Sigma(c/p^\beta \cdot \lambda, p^\beta),
\]

where \(c/p^\beta\) denotes the multiplicative inverse of \(c/p^\beta \pmod{p^\beta}\). Therefore, the evaluation of \(\Sigma(\lambda, c)\) can be reduced to the case of prime power moduli.

The case \(c = p\) can be guaranteed by Weil’s proof on Riemann hypothesis for curves over finite fields. The estimate of a general form can be found, for instance, in [Bo, Theorem 5].

**Lemma B.1.** Let \(d = d(\lambda)\). Then we have

\[
|\Sigma(\lambda, p)| \leq 2dp^{1/2}(\lambda, p)^{1/2}.
\]

Here and in what follows, \((\lambda, c)\) denotes the g.c.d. of \(c\) and the coefficients of non-constant terms in \(\lambda\).

While \((\lambda, p) = 1\) the above bound presents the square-root cancellation among the exponentials, which is best possible in general. The case \(c = p^\beta\) with \(\beta \geq 2\) becomes considerably easier because elementary evaluations are usually sufficient. The following lemma evaluates \(\Sigma(\lambda, p^\beta)\) for \(\beta \geq 2\), from which one can also obtain square-root cancellations up to some extra factor, which can be controlled effectively on average, as we will see later. The detailed proof can be found in [IK, Section 12.3].

**Lemma B.2.** Let \(\alpha \geq 1\). Then we have

\[
\Sigma(\lambda, p^{2\alpha}) = p^\alpha \sum_{y \pmod{p^\alpha}, \lambda'(y) \equiv 0 \pmod{p^\alpha}} e\left(\frac{\lambda(y)}{p^{2\alpha}}\right)
\]

and

\[
\Sigma(\lambda, p^{2\alpha+1}) = p^\alpha \sum_{y \pmod{p^\alpha}, \lambda'(y) \equiv 0 \pmod{p^\alpha}} e\left(\frac{\lambda(y)}{p^{2\alpha}}\right) G_p(y),
\]

where \(G_p(y)\) is a quadratic Gauss sum given by

\[
G_p(y) := \sum_{z \pmod{p}} e\left(\frac{1}{2} \lambda''(y)z^2 + \lambda'(y)p^{-\alpha}z\right).
\]
Using the evaluations as above, we would like to derive a precise estimate for Σ(λ, c) that is applicable in many applications.

Recalling the expression (B.1), we may write

$$\Sigma(\lambda, c) = \prod_{p^\beta \parallel c, \beta \leq 3} \Sigma(c/p^\beta \cdot \lambda, p^\beta) \cdot \prod_{p^\beta \parallel c, \beta \geq 4} \Sigma(c/p^\beta \cdot \lambda, p^\beta) =: \Sigma_1 \cdot \Sigma_2,$$

(say).

For $p \parallel c$, by Lemma B.1, we have

$$|\Sigma(c/p \cdot \lambda, p)| \leq (2d)^{\omega(c)}(\lambda, c_1)^{1/2}c_1^{1/2}$$

with $c_1 := \prod_{p \parallel c} p$,

where $\omega(n)$ is the number of distinct prime factors of $n$. If $p^2 \parallel c$, Lemma B.2 implies

$$|\Sigma(c/p^2 \cdot \lambda, p^2)| \leq p |\{y \pmod{p} : \lambda'(y) \equiv 0 \pmod{p}\}| \leq \deg(\lambda')(\lambda' p) p.$$

Similarly, for $p^3 \parallel c$, it follows from Lemma B.2 that

$$|\Sigma(c/p^3 \cdot \lambda, p^3)| \leq p^{3/2}|\{y \pmod{p} : \lambda'(y) \equiv 0 \pmod{p}\}| \leq \deg(\lambda')(\lambda' p)p^{3/2}.$$

Hence we conclude that

$$|\Sigma_1| \leq c_1^{1/2}(\lambda, c_1)^{1/2}(\lambda', c_1^\dagger)(2d)^{\omega(c_1)}$$

with $c_1 := \prod_{p^\beta \parallel c, \beta \leq 3} p^\beta$, $c_1^\dagger := \prod_{p^\beta \parallel c, \beta \geq 4} p^\beta$.

On the other hand, a trivial estimate gives

$$|\Sigma_2| \leq c_2 := \prod_{p \parallel c, \beta \geq 4} p^\beta = \Xi(c),$$

where $\Xi(c)$ is defined by (A.1).

Collecting the above two estimates, we may conclude the following theorem.

**Theorem B.1.** Let $d = d(\lambda)$. For $c \geq 1$, we have

$$|\Sigma(\lambda, c)| \leq c^{1/2}(\lambda, c^\dagger)^{1/2}(\lambda', c^\dagger)(2d)^{\omega(c)} \cdot \Xi(c)^{1/2},$$

where

$$c^\dagger = \prod_{p^2 \parallel c} p \cdot \prod_{p^3 \parallel c} p.$$

**Remark 11.** We do not intend to seek the strongest estimate for $\Sigma(\lambda, c)$. Our interest here is to present a square-root cancellation up to some harmless factors. As shown in Lemma A.1, we see that $\Xi(c)^{1/2}$ is bounded on average, which is acceptable particularly in our applications to the quadratic Brun-Titchmarsh theorem.
[BH] R. C. Baker & G. Harman, The Brun-Titchmarsh theorem on average, Analytic Number Theory, Vol. I (Allerton Park, IL, 1995), Progr. Math., 138, Birkhäuser Boston, Boston, MA, 1996, 39–103.

[BM1] V. Blomer & D. Miličević, The second moment of twisted modular $L$-functions, GAFA 25 (2015), 453–516.

[BM2] V. Blomer & D. Miličević, $p$-adic analytic twists and strong subconvexity, Ann. Scient. Éc. Norm. Sup. 48 (2015), 561–605.

[Bo] E. Bombieri, On exponential sums in finite fields, Amer. J. Math. 88 (1966), 71–105.

[Br] J.-L. Brylinski, Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques, Astérisque 140-141 (1986), 3–134.

[Bu1] D. A. Burgess, On character sums and $L$-series, Proc. London Math. Soc. 12 (1962), 193–206.

[Bu2] D. A. Burgess, On character sums and $L$-series, II, Proc. London Math. Soc. 13 (1963), 524–536.

[Cl] J. B. Conrey & H. Iwaniec, The cubic moment of central values of automorphic $L$-functions, Ann. of Math. 151 (2000), 1175–1216.

[Da] C. Dartyge, Le problème de Tchébychev pour le douzième polynôme cyclotomique, Proc. London Math. Soc. 111 (2015), 1–62.

[De] P. Deligne, La conjecture de Weil II, Publ. Math. IHÉS 52 (1980), 137–252.

[dlB] R. de la Bretèche, Plus grand facteur premier de valeurs de polynômes aux entiers, Acta Arith. 169 (2015), 221–250 (with an appendix by R. de la Bretèche and J.-F. Mestre).

[DI1] J.-M. Deshouillers & H. Iwaniec, Kloosterman sums and Fourier coefficients of cusp forms, Invent. Math. 79 (1982/83), 171–188.

[DI2] J.-M. Deshouillers & H. Iwaniec, On the greatest prime factor of $n^2 + 1$, Ann. Inst. Fourier (Grenoble) 32 (1982), 1–11.

[DI3] J.-M. Deshouillers & H. Iwaniec, On the Brun-Titchmarsh theorem on average, Topics in Classical Number Theory, Vol. I, II (Budapest, 1981), 319–333, Colloq. Math. Soc. János Bolyai, 34, North-Holland, Amsterdam, 1984.

[DS] D. L. Donoho & P. B. Stark, Uncertainty principles and signal recovery, SIAM J. App. Math. 49 (1989), 906–931.

[Fo1] É. Fouvry, Sur le théorème de Brun-Titchmarsh, Acta Arith. 43 (1984), 417–424.

[Fo2] É. Fouvry, Théorème de Brun-Titchmarsh: application au théorème de Fermat, Invent. Math. 79 (1985), 383–407.

[Fo3] É. Fouvry, Sur le problème des diviseurs de Titchmarsh, J. Reine Angew. Math. 357 (1985), 51–76.

[FoI] É. Fouvry & H. Iwaniec, The divisor function over arithmetic progressions, Acta Arith. LXI.3 (1992), 271–287 (with an appendix by N. M. Katz).

[FKM1] É. Fouvry, E. Kowalski & Ph. Michel, Algebraic trace functions over the primes, Duke Math. J. 163 (2014), 1683–1736.

[FKM2] É. Fouvry, E. Kowalski & Ph. Michel, Trace functions over finite fields and their applications, Colloquium di Giorgi, 2013 and 2014, Vol. 5, 2015, 7–35.

[FKM3] É. Fouvry, E. Kowalski & Ph. Michel, Algebraic twists of modular forms and Hecke orbits, GAFA 25 (2015), 580–657.

[FKM+] É. Fouvry, E. Kowalski, Ph. Michel, C. S. Raju, J. Rivat & K. Soundararajan, On short sums of trace functions, arXiv:1508.00512 [math.NT].
[FrI] J. B. Friedlander & H. Iwaniec, The Brun-Titchmarsh theorem. Analytic Number Theory (Kyoto, 1996), London Math. Soc. Lecture Note Ser., 247, Cambridge Univ. Press, Cambridge, 1997, 85–93.

[GR] S. W. Graham & C. J. Ringrose, Lower bounds for least quadratic nonresidues, Analytic Number Theory (Allerton Park, IL, 1989), Progr. Math. 85, Birkhäuser, Boston, 1990, 269–309.

[HB1] D. R. Heath-Brown, Hybrid bounds for Dirichlet $L$-functions, Invent. Math. 47 (1978), 149–170.

[HB3] D. R. Heath-Brown, The largest prime factor of $X^3 + 2$, Proc. London Math. Soc. 82 (2001), 554–596.

[HB4] D. R. Heath-Brown, Bounds for the cubic Weyl sum, J. Math. Sci. (N. Y.) 171 (2010), 813–823.

[HBP] D. R. Heath-Brown & L. Pierce, Counting rational points on smooth cyclic covers, J. Number Theory 132 (2012), 1741–1757.

[Ho1] C. Hooley, On the greatest prime factor of a quadratic polynomial, Acta Math. 117 (1967), 281–299.

[Ho2] C. Hooley, On the Brun-Titchmarsh theorem, J. Reine Angew. Math. 255 (1972), 60–79.

[Ho3] C. Hooley, On the largest prime factor of $p + a$, Mathematika 20 (1973), 135–143.

[Ho4] C. Hooley, On the Brun-Titchmarsh theorem. II, Proc. London Math. Soc. 30 (1975), 114–128.

[Ho5] C. Hooley, On the greatest prime factor of a cubic polynomial, J. Reine Angew. Math. 303/304 (1978), 21–50.

[Ir1] A. J. Irving, The divisor function in arithmetic progressions to smooth moduli, Int. Math. Res. Not. 15 (2015), 6675–6698.

[Ir2] A. J. Irving, Estimates for character sums and Dirichlet $L$-functions to smooth moduli, Int. Math. Res. Not., DOI: 10.1093/imrn/rnv285.

[Iw1] H. Iwaniec, A new form of the error term in the linear sieve, Acta Arith. 37 (1980), 307–320.

[Iw2] H. Iwaniec, On the Brun-Titchmarsh theorem, J. Math. Soc. Japan 34 (1982), 95–123.

[IK] H. Iwaniec & E. Kowalski, Analytic Number Theory, Amer. Math. Soc. Colloq. Publ., Vol 53, AMS, Providence, RI, 2004.

[Ka1] N. M. Katz, Sommes Exponentielles, Astérisque 79, Société mathématique de France, 1980.

[Ka2] N. M. Katz, Gauss Sums, Kloosterman Sums, And Monodromy Groups, Ann. of Math. Stud., Vol. 116, Princeton University Press, Princeton, NJ, 1988.

[Ka3] N. M. Katz, Exponential Sums and Differential Equations, Ann. of Math. Stud., Vol. 124, Princeton University Press, Princeton, NJ, 1990.

[La] G. Laumon, Transformation de Fourier, constantes d’équations fonctionnelles et conjecture de Weil, Publ. Math. IHÉS 65 (1987), 131–210.

[LR] J. H. van Lint & H.-E. Richert, On primes in arithmetic progressions, Acta Arith. 11 (1965), 209–216.

[Mi] D. Miščević, Sub-Weyl subconvexity for Dirichlet $L$-functions to prime power moduli, Compositio Math., 152, 825–875.

[Mo] Y. Motohashi, On some improvements of the Brun-Titchmarsh theorem. III, J. Math. Soc. Japan 27 (1975), 444–453.
L. Pierce, A bound for the 3-part of class numbers of quadratic fields by means of the square sieve, *Forum Math.* **18** (2006), 677–698.

D. H. L. Polymath, New equidistribution estimates of Zhang type, *Algebra Number Theory* **8** (2014), 2067–2199.

H. J. S. Smith, Report on the Theory of Numbers, Collected Mathematical Papers, Vol. I, reprinted, Chelsea, 1965.

E. C. Titchmarsh, The Theory of the Riemann Zeta-Function, Second edition, Edited and with a preface by D.R. Heath-Brown, The Clarendon Press, Oxford University Press, New York, 1986.

J. Wu & P. Xi, Quadratic polynomials at prime arguments, arXiv:1603.07067 [math.NT].

Y. Zhang, Bounded gaps between primes, *Ann. of Math.* (2) **179** (2014), 1121–1174.

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