Wavelet preconditioning for EIT

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Abstract. In this paper the forward problem of Electrical Impedance Tomography (EIT) is considered. To achieve a flexible treatment of boundaries, the original two-dimensional domain is extended to a simple square one and essential boundary conditions are imposed by means of Lagrange multipliers, resulting in a saddle point system. For discretisation, we employ a biorthogonal B-Spline wavelet basis for which it can be shown that in the suggested forward EIT formulation the condition number of the system matrix is asymptotically uniformly bounded, and therefore iterative solvers converge with a speed independent of the discretisation level.

1. Introduction

The forward problem in EIT is typically solved by iterative methods due to the large number of unknowns coupled with noise contaminated measurements. The speed of iterative (approximate) solvers, in turn, is determined by the spectral condition number, defined as \( \kappa(A) := \|A\| \|A^{-1}\| \), of the system matrix \( A \) and the operator norm with respect to the Euclidean norm. Unfortunately, in modern numerical computations due to an increasing demand in the discretisation precision, large-scale systems are encountered when standard finite elements (conventional piecewise polynomials of local support) are employed. Notably, the obtained condition numbers are inversely proportional to the effectively shrinking pitch (grid-spacing). Thus, it is essential to counteract for the increasing condition number without, of course, spoiling the overall computational complexity. This procedure is called preconditioning and, ideally, yields an equivalent system with a uniformly bounded condition number of the system matrix. Here, we propose a conceptually different to the well-established optimal preconditioner approach (Multigrid). We follow the Domain Embedding Method (DEM) paradigm and embed the original domain \( \Omega \) into a ‘fictitious’, simpler one, e.g., a unit square (denoted as \( \square \)) to allow for simpler yet efficient (uniform grid) discretisation. Essential boundary conditions are appended by Lagrange multipliers, leading to a saddle point problem. For its discretisation, we employ biorthogonal B-spline wavelets which provide an optimal preconditioning: norm equivalences between the underlying Sobolev spaces and the vector spaces of expansion coefficients show that a simple diagonal scaling achieves a uniform bound of the condition number with respect to the discretisation level. In addition, the multiscale structure of wavelets allows for a posteriori steered adaptive solution schemes (if desired) for nonsmooth solutions.

3 We gratefully acknowledge financial support by: the SFB/TR 32 “Pattern in Soil-Vegetation-Atmosphere Systems: Monitoring, Modelling, Data Assimulation” www.tr32.de (funded by the Deutsche Forschungsgemeinschaft (DFG) and by the DFG Research Training Group “Scientific Computing”, PaSCo), the EPSRC (grant EP/G061580/1) and the State Scholarships Foundation of Greece (IKY).
2. The forward EIT problem

The forward problem in EIT is essentially a generalised Laplacian of the form \( \nabla \cdot (\sigma \nabla u) = 0 \) subject to boundary conditions to ensure existence and uniqueness of the solution. Following [1] in the formulation of the Complete Electrode Model (CEM), let a bounded domain \( \Omega \subset \square \subset \mathbb{R}^2 \) be equipped with \( L \) electrodes attached on its Lipschitz boundary surface \( \partial \Omega \). Denote by \( \Gamma \subset \partial \Omega \) the union of areas under each electrode, assumed to be open connected subsets \( \bigcup_{\ell=1}^L \Gamma_\ell = \Gamma \), whose closures are disjoint, \( \bigcap_{\ell=1}^L \overline{\Gamma}_\ell = \emptyset \), and \( \overline{\gamma} := \partial \Omega \setminus \Gamma \) is the union of the remaining areas. Boundary conditions on \( \Gamma \) are defined as

\[
\sigma \nabla u \cdot \nu = \nu \quad \text{on} \ \Gamma, \quad \text{and} \quad u + z_\ell \sigma \nabla u \cdot \nu = U_\ell \quad \text{on} \ \Gamma_\ell
\]

where \( \sigma, u, U_\ell, \nu, \nu, z_\ell \) are the admittivity, the interior potential distribution, the surface potential on the \( \ell \)-th electrode, the outward unit normal vector, the current density and the surface impedance, respectively. Additional boundary conditions for the interelectrode gaps require that

\[
\sigma \nabla u \cdot \nu = 0 \quad \text{on} \ \overline{\gamma}.
\]

We denote the associated Sobolev space for \( u \) as \( H_{1}^1(\Omega) := H^1(\Omega) \oplus \mathbb{C}^L \) with dual space \( (H_{1}^1(\Omega))' \).

Defining the sesquilinear form

\[
a_\Omega((v, V), (w, W)) := \int_{\Omega} \sigma \nabla v \cdot \nabla w \, d\Omega + \sum_{\ell=1}^L \int_{\Gamma_\ell} \frac{1}{z_\ell}(v - V_\ell)(w - W_\ell) \, ds_{\Gamma_\ell},
\]

the weak formulation of the EIT problem on the original domain \( \Omega \) can be stated as follows: Given the stimulation patterns (right hand sides) \( I = (I_1, \ldots, I_L)^T \in \mathbb{R}^L \), with \( I_\ell \) denoting the current applied to the \( \ell \)-th electrode, find \((u, V) \in H_{1}^1(\Omega)\) such that

\[
a((u, U), (v, V)) = \sum_{\ell=1}^L I_\ell V_\ell \quad \text{for all} \ (v, V) \in H_{1}^1(\Omega).
\]

2.1. Saddle point formulation in the fictitious domain \( \square \)

In order to allow for a flexible treatment of boundaries for the EIT problem, we extend the original domain \( \Omega \) to a fictitious, simpler one, denoted as \( \square \), see, e.g., [2]. Assuming suitable extensions for all functions from \( \Omega \) to \( \square \), the extension of \( a_\Omega(\cdot, \cdot) \) to \( \square \) is defined as

\[
a_\square((v, V), (w, W)) := \int_{\square} \sigma \nabla v \cdot \nabla w \, d\square + \sum_{\ell=1}^L \int_{\Gamma_\ell} \frac{1}{z_\ell}(v - V_\ell)(w - W_\ell) \, ds_{\Gamma_\ell}.
\]

It is crucial here that the resulting weak problem formulations, using either \( a_\Omega(\cdot, \cdot) \) or \( a_\square(\cdot, \cdot) \), are not equivalent, as the ‘natural’ boundary condition (1) translates into an ‘essential’ boundary condition for \( \square \) and, therefore, needs to be explicitly satisfied. Using Lagrange multipliers as in [3], an equivalent EIT formulation for (3) in \( \square \) can be recasted as a functional minimisation problem

\[
\inf_{(v, V) \in H_{1}^1(\square)} \sup_{q \in (H^{1/2}(\overline{\gamma}))'} \frac{1}{2} a_\square((v, V), (v, V)) - \sum_{\ell=1}^L I_\ell V_\ell + b(v, q)
\]

with \( b(v, q) := \int_{\square} q \sigma \nabla v \cdot \nu \, ds \). The standard first order optimality conditions are then

\[
a_\square((u, U), (v, V)) + b(v, p) = \sum_{\ell=1}^L I_\ell V_\ell
\]

\[
b(u, q) = 0.
\]
In operator form, this reads

\[
\begin{bmatrix}
A & B^T \\
B & 0
\end{bmatrix}
\begin{bmatrix}
[u, U]^T \\
p
\end{bmatrix} = \begin{bmatrix}
\sum_{\ell=1}^L H^\ell \tilde{V}_\ell \\
0
\end{bmatrix}
\tag{6}
\]

where \( A : H^1(\Box) \to (H^1(\Box))^\prime \) is self-adjoint, \( B : H^1(\Box) \to (H^{-1/2}(\Box))^\prime \) is the trace operator and \( B^T \) its adjoint. We have established existence and uniqueness of the solution of (6) on \( \Box \) along with the equivalence between the associated solutions in \( \Omega \) and \( \Box \). We will publish these results together with remarks about the space \((H^{-1/2}(\Box))^\prime\) elsewhere.

2.2. Discretisation of the continuous problem using wavelets

The wavelets we consider here are biorthogonal Riesz basis functions of the involved function spaces. This means that they allow a representation of the continuous saddle point problem (6) without having to specify a certain finest resolution level. Thus, the system is discretised but not yet made finite-dimensional. This makes it particularly apparent to explain our wavelet preconditioning. Our choice of wavelets are tensor products of the biorthogonal spline-wavelets from [4], adapted to the interval as in [5]. They are defined in terms of scaling wavelet basis functions (for various resolution levels \( j \) and translations \( k \)). For notational convenience, we rewrite the fundamental two-scale relation

\[
\phi(2^{j}x) = \sum_{m} c_{m} \phi(2^{j}x - m)
\]

in a compact matrix-vector form for resolution pitch 2^{-j} as \( \Phi_{j} = M_{j,0}^T \Phi_{j+1} \) and \( \Psi_{j} = M_{j,1}^T \Phi_{j+1} \) where the matrices \( M_{j,0}, M_{j,1} \) contain appropriately scaled coefficients \( c_{m} \), and \( d_{n} \) for the wavelets. Let \( \Psi = \{ \Psi_{j} \}_{j=j_{0}}^{\infty} \), \( \tilde{\Psi} = \{ \tilde{\Psi}_{j} \}_{j=j_{0}}^{\infty} \) be the collection of all functions on all levels, often called a multiresolution basis. Now every function \( v \in H^{1}_{\Box} \) can be expanded as

\[
v = d^T \Psi =: \sum_{j=j_{0}-1}^{\infty} d^T_{j} \Phi_{j} := c^T_{j_0} \Phi_{j_0} + \sum_{j=j_{0}}^{\infty} d^T_{j} \Psi_{j}
\tag{7}
\]

whose coefficients \( c_{j_0}, d_{j} \) correspond to inner products defined with respect to the duals \( \Phi_{j_0}, \tilde{\Phi}_{j} \), \( \Psi_{j} \), and \( j_0 \) is the coarsest (minimum) level. From the Riesz basis property, one can infer that

\[\|d^T \Psi\|_{H^{1}(\Box)} \sim \|Dd\|_{\ell_{2}},\]

where \( D := 2^{j} \delta_{j,j'} \delta_{n,n'} \) is a diagonal matrix and \( a \sim b \) means that \( a, b \) are equivalent up to constants.

2.3. Proposed multilevel preconditioning for EIT and numerical results

For the EIT setting in saddle point form (6), \( A \) is an isomorphism on the kernel of \( B \), and the corresponding relation \( \|Au\|_{H^{-1}_{\Box}} \sim \|u\|_{H^{1}_{\Box}} \) can be expanded using \( \tilde{\Psi} \) as in [3],

\[
\|Au\|_{H^{-1}_{\Box}} = \|\langle Au, \Psi \rangle \tilde{\Psi} \|_{H^{0}_{\Box}} \sim \|D^{-1}(Au, \Psi)\|_{\ell_{2}},
\tag{8}
\]

where \( \langle \cdot, \cdot \rangle \) denotes the dual form between \( H^{1}_{\Box} \) and its dual. Expanding \( u \) in (8) as in (7) yields

\[
\|D^{-1}(Au, \Psi)\|_{\ell_{2}} \sim \|D^{-1}(A\Psi, \Psi)d\|_{\ell_{2}} = \|D^{-1}(A\Psi, \Psi)D^{-1}Dd\|_{\ell_{2}}
\tag{9}
\]

where \( A_{\Psi} := (A\Psi, \Psi) \) is the representation of the operator \( A \) in terms of the infinite wavelet basis \( \Psi \). The main norm equivalence result is now

\[
\|Au\|_{H^{-1}_{\Box}} \sim \|u\|_{H^{1}_{\Box}} \sim \|d^T \tilde{\Psi}\|_{H^{0}_{\Box}} \sim \|D^{-1}A_{\Psi}D^{-1}Dd\|_{\ell_{2}} \sim \|Dd\|_{\ell_{2}}.
\tag{10}
\]

A similar relation for \( A^{-1} \) can be shown. As a result of both, one obtains a uniformly bounded spectral condition number for the resulting stiffness matrix [3],

\[
\kappa(A_{\Psi}^{-1}) = O(1), \quad \text{where } A_{\Psi} := D^{-1}A_{\Psi}D^{-1}.
\tag{11}
\]
Table 1. Spectral condition numbers for the boundary operator $B_\Psi$ with 8 interelectrode gaps.

| $l$ | $j = 07$, $\zeta = 03$ | $j = 07$, $\zeta = 04$ | $j = 07$, $\zeta = 05$ |
|-----|-----------------|-----------------|-----------------|
| 01  | 2.69e+000       | 8.22e+000       | 4.90e+016       |
| 02  | 2.69e+000       | 8.22e+000       | 3.80e+016       |
| 03  | 2.12e+000       | 4.27e+000       | 2.48e+016       |
| 04  | 2.12e+000       | 4.26e+000       | 1.38e+016       |
| 05  | 2.72e+000       | 8.07e+000       | 1.42e+016       |
| 06  | 2.71e+000       | 8.08e+000       | 7.22e+015       |
| 07  | 2.12e+000       | 4.26e+000       | 3.38e+016       |
| 08  | 2.12e+000       | 4.27e+000       | 3.14e+016       |

Of course, for numerical simulations the operator $A_\Psi^D$ has to be made finite-dimensional. From (11) one can immediately conclude that also any finite-dimensional portion of $A_\Psi^D$ has a uniformly bounded condition number.

As for the boundary operator $B$, the well-posedness of the saddle point formulation (6) requires that the condition

$$\inf_{q \in (H^{1/2}(\gamma))^\prime} \sup_{v \in H_0^1} \frac{\langle Bv, q \rangle}{\| v \|_{H_0^1} \| q \|_{(H^{1/2}(\gamma))^\prime}} \geq \beta \quad (12)$$

holds for some constant $\beta > 0$. Assuming that $B$ is discretised with wavelet basis functions for $\Box$ and for $\gamma$, for convenience abbreviated as $B_\Psi$, a similar yet slightly more complicated analysis for preconditioning holds where the scaling operator $D$ is adjusted according to the underlying domain and trace spaces. However, condition (12) does not hold any more automatically for any finite-dimensional subspaces of $H_0^1$ and $(H^{1/2}(\gamma))^\prime$. The corresponding, so-called ‘LBB-condition’ [3, 6] is a stability condition which requires, in the setting considered here, that the discretisation level $\zeta$ on the boundary is not too high when compared to the discretisation level $j$ on $\Box$. We have tested the behaviour of the operator $B$ in wavelet coordinates for the case of $\Omega$ a disk of radius $R = 1/4$, centered at $(.5,.5)$ with 8 electrodes attached ($z_\ell = 10$ Ohm), embedded into a fictitious unit square. We present the resulting spectral condition numbers of $B_\Psi$ in Table 1 for different values of $j$ and $\zeta$, and $l$ denotes the interelectrode gaps (arcs). It is evident from Table 1 that the LBB condition holds numerically as long as $\zeta + 2 < j$ (first and second column in Table 1). Finally, it can be shown that optimal preconditioning together with nested iteration yields an optimal $O(N)$ algorithm, $N$ being the total amount of unknowns [2].

3. Conclusions

In this paper, a saddle point formulation using CEM and optimal preconditioning by wavelets was proposed for the solution of the forward EIT problem. It was shown that a diagonal scaling applied to operators in wavelet coordinates, $A_\Psi$ and $B_\Psi$, yield uniformly bounded condition numbers, allowing for efficient iterative solvers. Moreover, for the boundary operator $B$ and for $\zeta + 2 < j$, the computed condition number of $B_\Psi$ attained very low absolute values, i.e., stability.

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