BV SOLUTION FOR A NON-LINEAR HAMILTON-JACOBI SYSTEM

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Abstract. In this work, we are dealing with a non-linear eikonal system in one dimensional space that describes the evolution of interfaces moving with non-signed strongly coupled velocities. For such kind of systems, previous results on the existence and uniqueness are available for quasi-monotone systems and other special systems in Lipschitz continuous space. It is worth mentioning that our system includes, in particular, the case of non-decreasing solution where some existence and uniqueness results arose for strictly hyperbolic systems with a small total variation. In the present paper, we consider initial data with unnecessarily small BV seminorm, and we use some BV bounds to prove a global-in-time existence result of this system in the framework of discontinuous viscosity solution.

1. Introduction.

1.1. Physical motivation and setting of the problem. In our work, we are looking for solutions of the form $u(t, x) = \left(u^i(t, x)\right)_{i=1,\ldots,d}$ of the following one dimensional Hamilton-Jacobi system

$$
\begin{align*}
\partial_t u^i(t, x) &= \lambda^i(t, x, u) \left| \partial_x u^i(t, x) \right| \quad \text{in } (0, T) \times \mathbb{R}, \\
\ u^i(0, x) &= u_0^i(x) \quad \text{in } \mathbb{R},
\end{align*}
$$

(1)

for $T > 0$ and $i = 1, \ldots, d$, where $d \in \mathbb{N}^*$. The function $u^i$ is real-valued with $\partial_t u^i$ and $\partial_x u^i$ stand for its time and spatial derivatives respectively. Here, the velocity $\lambda^i$ is assumed to satisfy, for all $i = 1, \ldots, d$, the following assumption

$$
\lambda^i \in L^\infty((0, T) \times \mathbb{R} \times \mathbb{K}) \quad \text{for } T > 0 \quad \text{and for all compacts } \mathbb{K} \subset \mathbb{R}^d.
$$

(2)
The goal of this work is to establish the global existence of discontinuous viscosity solutions of system (1) assuming (2) and the following regularity on the initial data

$$u_0^i \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R}),$$

where $BV(\mathbb{R})$ is the space of functions of bounded variations given by

$$BV(\mathbb{R}) = \{ f \in L^1_{loc}(\mathbb{R}) ; \ TV(f) < +\infty \},$$

with $TV(f)$ being the total variation of $f$ defined by

$$TV(f) = \sup \left\{ \int_{\mathbb{R}} f(x)\phi'(x)dx; \ \phi \in C^1(\mathbb{R}) \ and \ ||\phi||_{L^\infty(\mathbb{R})} \leq 1 \right\}.$$

In the following, we take the space $BV(\mathbb{R})$ endowed with the semi-norm $|f|_{BV(\mathbb{R})} = TV(f)$. Note that $BV$ functions are integrable functions whose distributional derivative is a finite Radon measure.

We remark that system (1) can be seen as the “level-set approach” system associated to the motion of the front $\Gamma^i_t := \{ x : u^i(t,x) = 0 \}$ with a normal velocity $\lambda^i(t,x,u)$ (see for instance Barles et al. [6]).

Many existence and uniqueness results were established on similar eikonal equations. Let us mention some of the known results. First, motivated by dislocation dynamics, we can point out the result done by El Hajj and Boudjerada in [8] who were able to prove the global existence of discontinuous viscosity $BV$ solutions for scalar one dimensional non-linear and non-local eikonal equations, including in particular the case $d = 1$ in system (1), where the velocity is independent of the solution. This result has been extended in El Hajj et al. [14] to a more general non-linear and non-local ($2 \times 2$) system where the authors were able to rely on the fact that this ($2 \times 2$) system has a particular local structure, making it possible to verify some quasi-monotonicity properties. This allowed to attain the same estimates obtained in [8] and thus to prove the global existence of discontinuous solutions of the considered particular ($2 \times 2$) system, inspired by the work of Ishii et al. [18, 19].

Also, thanks to those quasi-monotonicity properties and relying on the work of Ishii et al. [18, 19], an existence and uniqueness result of a Lipschitz viscosity solution was proved by El Hajj and Forcadel in [13] for the same ($2 \times 2$) system. In our paper, we will generalize the result established in [8] to the case of non-linear and strongly coupled ($d \times d$) eikonal system (1) but without the quasi-monotonicity condition and with very low regularity on initial data and on the velocity. The idea is based on regularizing the initial data and the velocity in order to preserve the a priori estimates obtained previously in [8] for the scalar equation (independent of $u$), particularly the $BV$ estimate. Then, after freezing the vector field $u = (u^i)_{1 \leq i \leq d}$ that appears in the velocity term $\lambda^i$ inside an appropriate Banach space, we will show, using a fixed point theorem and the $BV$ estimate, the global-in-time existence result of the regularized problem. These estimates also allow us to pass to the limit when the regularization vanishes, in order to get finally the existence of a discontinuous viscosity solutions of (1) in a particular setting, following the definition introduced by Ishii in [18]. In the case of general ($d \times d$) system, it is worth mentioning the result of Ishii, Koike [19] and Ishii [18], who had shown the existence and uniqueness of continuous viscosity solutions for Hamilton-Jacobi systems of the form

$$\begin{cases}
\partial_t u^i + H_i(t,x,u,Du^i) = 0 \\
\text{with } u = (u^1, \ldots, u^d) \in \mathbb{R}^d, \ x \in \mathbb{R}^N, \ and \ t \in (0, +\infty),
\end{cases}$$

$$u^i(0,x) = u_0^i(x) \ for \ x \in \mathbb{R}^N,$$
where the Hamiltonian $H_i$ is quasi-monotone in $u$ (see the definition in Ishii, Koike [19, Th.4.7]). The quasi-monotonicity recommends, in the case of system (1), that the non-diagonal terms of the Jacobian matrix are all nonnegative (i.e. $\frac{\partial H_i}{\partial u_j} \geq 0$ for $j \neq i$). However, in our work, we consider the study of discontinuous solutions without this condition being necessarily fulfilled.

The result presented in this paper allows to give a meaning to system (1) in the framework of discontinuous viscosity solutions with discontinuous initial data and velocity. More precisely, we present a global existence result for the strongly coupled Hamilton-Jacobi system (1) with $BV$ initial data, whose semi-norm is not necessarily small. This result is obtained without sign restrictions on the velocity $\lambda^i$ and with unconditional monotonicity of the solution. We only consider the case when the initial data and the velocity satisfy the assumptions (2) and (3) without any further regularity. However, the state of having non-decreasing initial data is presented as a particular case of our work in Theorem 1.3. In its full generality, the fundamental issue of uniqueness for global solution remains open. This question is related to the fact that system (1) is not only non-linear but it is also non-monotone which means that the comparison principle, which plays a central role in the “level-set approach”, does not hold and thus we cannot directly apply the viscosity solutions theory in order to obtain uniqueness. Nevertheless, according to the work of Ishii et al. [18, 19], it is possible to show a uniqueness result of system (1) in the context of continuous viscosity solutions under some monotonicity and continuity conditions on the velocity to ensure the comparison principle. To do that, it is enough to assume that $\lambda^i$ is Lipschitz continuous and all terms of the Jacobian matrix, except the diagonal, are nonnegative, namely $\frac{\partial \lambda^i}{\partial u_j} \geq 0$ for $j \neq i$. We refer the reader to [3, 9, 10] for a complete overview of viscosity solutions. We also refer to Barles [2] for an interesting counter-example on the uniqueness of discontinuous viscosity solution.

Let us now state the key steps followed to prove our existence result. First, by a classical convolution argument, we regularize the velocity and the initial data in (1). This approximation brings us to consider, for every $0 < \varepsilon \leq 1$ and for $i = 1, \ldots, d$, the following system

\[
\begin{aligned}
&\partial_t u^i_\varepsilon(t, x) = \lambda^i_\varepsilon(t, x, \rho^i_\varepsilon \ast u^i_\varepsilon(t, \cdot)(x)) \left| \partial_x u^i_\varepsilon(t, x) \right| \quad \text{in } (0, T) \times \mathbb{R}, \\
&u^i_\varepsilon(0, x) = u^i_0,\varepsilon(x) \quad \text{in } \mathbb{R},
\end{aligned}
\]  

(4)

where $\lambda^i_\varepsilon$ and $u^i_0,\varepsilon$ are the regularization of the functions $\lambda^i$ and $u^i_0$ respectively, and they are given by

\[
u^i_0(x) = u^i_0 \ast \rho^i_\varepsilon(x) \quad \text{and} \quad \lambda^i_\varepsilon(t, x, w) = \lambda^i \ast \rho^{d+2}_\varepsilon(t, x, w), \quad \forall (t, x, w) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d,
\]  

(5)

with $\lambda^i$ is an extension by 0 in $\mathbb{R}^{d+2}$ of $\lambda^i$, and $\rho^i_\varepsilon$, $n = 1, d + 2$, is the standard mollifier defined as follows

\[
\rho^i_\varepsilon(\cdot) = \frac{1}{\varepsilon^n} \rho^n\left(\frac{\cdot}{\varepsilon}\right), \quad \text{such that} \quad \rho^n \in C^\infty_c(\mathbb{R}^n), \supp \{\rho^n\} \subseteq B(0, 1), \quad \rho^n \geq 0, \quad \text{and} \quad \int_{\mathbb{R}^n} \rho^n = 1.
\]  

(6)

Our result lies initially in proving the global-in-time existence of the solution of the above regularized system, using the fixed point argument and the comparison principle of the associated linear problem obtained by freezing $u_\varepsilon$ in the velocity.
Afterwards, to pass from the solution of the regularized system (4) to that of system (1), we will show that the upper and lower relaxed semi-limits (see Barles and Perthame [4, 5]) which are defined as follows

\[ u_i(t, x) = \limsup_{\varepsilon \to 0} u_i^\varepsilon(t, x) = \limsup_{(s, y) \to (t, x)} u_i^\varepsilon(s, y), \]

and

\[ u_i(t, x) = \liminf_{\varepsilon \to 0} u_i^\varepsilon(t, x) = \liminf_{(s, y) \to (t, x)} u_i^\varepsilon(s, y), \]

are, respectively, discontinuous viscosity sub-solution and super-solution of system (1) in the sense of discontinuous viscosity solutions introduced by Ishii in [18, Definition 2.1] for the Hamilton-Jacobi system and recalled below in Definition 4.1. To reach this result, we pass to the limit in the regularized problem (4) in the viscosity sense, and we assume that the velocity \( \lambda_i \) is non-decreasing with respect to the variable \( u_i \), namely

\[ \lambda^i(t, x, r(u^i)) \leq \lambda^i(t, x, r(v^i)), \]

where \( r(u^i) = (r^1, \ldots, u^i, \ldots, r^d) \) and \( r(v^i) = (r^1, \ldots, v^i, \ldots, r^d) \).

Lastly, leaning on some \( \varepsilon \)-independent a priori estimates, essentially on new BV uniform bound, we come to prove an almost everywhere equality between \( \overline{w}(t, \cdot) \) and \( \underline{w}(t, \cdot) \) in \( \mathbb{R} \), for all \( t > 0 \). These a priori estimates also allow us to construct a function \( u = (u^i)_{i=1, \ldots, d} \), defined as a strong limit of \( u_i^\varepsilon \) in \( \mathbb{R}^d \), such that we have \( \underline{w}(t, \cdot) = \overline{w}(t, \cdot) = \underline{w}(t, \cdot) \) almost everywhere in \( \mathbb{R} \), for all \( t > 0 \) and \( 1 \leq i \leq d \). But due to the discontinuity of the solution, it is not obvious to show, in this framework, that the sub-solution \( \overline{w}^i \) (resp. super-solution \( \underline{w}^i \)) coincides with the upper semi-continuous envelope (resp. lower semi-continuous envelope) of \( w^i \), except in the case of non-decreasing solution. For this reason, we will not be able here to define a classic discontinuous viscosity solution in the case of general solutions (not necessarily non-decreasing), and consequently we will be brought later to introduce a weaker notion of viscosity solution, which requires that the sub-solution and the super-solution only coincide almost everywhere in space.

1.2. **Main results.** In this subsection, we first present, in Theorem 1.2, a global existence result of a special discontinuous viscosity solution of (1). After that, we show in Theorem 1.3, as a consequence of this result, the existence of a classical discontinuous viscosity solution of (1) by taking non-decreasing initial data.

We first show that system (1) admits a BV discontinuous viscosity solution in some weak sense.

1.2.1. **Existence result for eikonal system.** Before announcing our main result let us recall the definition of the continuous viscosity solution corresponding to the regularized system (4).

**Definition 1.1.** (*Continuous viscosity sub-solution, super-solution, and solution*)

Assume (2) and (3). Let \( u_\varepsilon = (u_\varepsilon^i)_{i=1, \ldots, d} \) be a continuous function defined on \((0, T) \times \mathbb{R}\).
(1) (Continuous viscosity sub-solution): We call \( u_\varepsilon \) a continuous viscosity sub-solution of (4) if it satisfies

(i) \( u_\varepsilon^i(0,x) \leq u_{0,\varepsilon}^i(x), \) for every \( i = 1, \ldots, d, \) and \( x \in \mathbb{R} \).

(ii) If whenever \( \phi \in C^1((0,T) \times \mathbb{R}), \) \( i = 1, \ldots, d, \) and \( u_\varepsilon^i - \phi \) attains its local maximum at \( (t_0,x_0) \in (0,T) \times \mathbb{R}, \) then we have

\[
\partial_t \phi(t_0,x_0) - \lambda_i^\varepsilon(t_0,x_0,\rho^\varepsilon_i(t_0,\cdot)(x_0)) |\partial_x \phi(t_0,x_0)| \leq 0.
\]

(2) (Continuous viscosity super-solution): We call \( u \) a continuous viscosity super-solution of (4) if it satisfies

(i) \( u^i(0,x) \geq u_{0,\varepsilon}^i(x), \) for every \( i = 1, \ldots, d, \) and \( x \in \mathbb{R} \).

(ii) If whenever \( \phi \in C^1((0,T) \times \mathbb{R}), \) \( i = 1, \ldots, d, \) and \( u_\varepsilon^i - \phi \) attains its local minimum at \( (t_0,x_0) \in (0,T) \times \mathbb{R}, \) then we have

\[
\partial_t \phi(t_0,x_0) - \lambda_i^\varepsilon(t_0,x_0,\rho^\varepsilon_i(t_0,\cdot)(x_0)) |\partial_x \phi(t_0,x_0)| \geq 0.
\]

(3) (Continuous viscosity solution): A continuous function \( u_\varepsilon \) is a viscosity solution of (4) if and only if it is a viscosity sub- and super-solution of (4).

We have the following result.

**Theorem 1.2. (Global existence result in weak sense)**

Suppose that the assumptions (2) and (3) are satisfied. Then, we have

i) **Global existence of Lipschitz continuous solution**

There exists a unique Lipschitz continuous viscosity solution \( u_\varepsilon = (u_\varepsilon^i)_{i=1,\ldots,d} \) of (4) satisfying, for all \( T > 0 \) and for \( i = 1, \ldots, d, \) the following uniform a priori estimates

\[
\|u_\varepsilon^i\|_{L^\infty((0,T) \times \mathbb{R})} \leq \|u_{0,\varepsilon}^i\|_{L^\infty(\mathbb{R})},
\]

\[
\|\partial_x u_\varepsilon^i\|_{L^\infty((0,T);L^1(\mathbb{R}))} \leq \|u_{0,\varepsilon}^i\|_{BV(\mathbb{R})},
\]

\[
\|\partial_t u_\varepsilon^i\|_{L^{\lambda^i}(0,T);L^1(\mathbb{R})} \leq \|\lambda^i\|_{L^{\infty}(0,T) \times \mathbb{R} \times \mathcal{K}_0} \|u_{0,\varepsilon}^i\|_{BV(\mathbb{R})},
\]

where \( \mathcal{K}_0 = \bigcap_{i=1}^d \left[-\|u_{0,\varepsilon}^i\|_{L^\infty(\mathbb{R})} - 1, \|u_{0,\varepsilon}^i\|_{L^\infty(\mathbb{R})} + 1\right]. \)

ii) **Convergence**

Assume that \( u_\varepsilon^i \), satisfies (10), (11) and (12) for \( i = 1, \ldots, d. \) Then, up to extract a subsequence, the function \( u_\varepsilon^i \) converges, as \( \varepsilon \) goes to zero, to a function

\[
u^i \in L^\infty((0,T) \times \mathbb{R}) \cap L^\infty((0,T); BV(\mathbb{R})) \cap C \left((0,T); L^1_{loc}(\mathbb{R})\right),
\]

strongly in \( C \left((0,T); L^1_{loc}(\mathbb{R})\right) .

Moreover, \( u^i \) satisfies, for all \( T > 0 \) and for \( i = 1, \ldots, d, \) the following inequalities

\[
\|u^i\|_{L^\infty((0,T) \times \mathbb{R})} \leq \|u_{0}^i\|_{L^\infty(\mathbb{R})},
\]

\[
\|u^i\|_{L^{\lambda^i}(0,T);BV(\mathbb{R})} \leq \|u_{0}^i\|_{BV(\mathbb{R})},
\]

\[
\|u^i(t,\cdot) - u^i(s,\cdot)\|_{L^1(\mathbb{R})} \leq \left(\|\lambda^i\|_{L^{\infty}(0,T) \times \mathbb{R} \times \mathcal{K}_0} \|u_{0}^i\|_{BV(\mathbb{R})}\right)|t-s|, \text{ for all } s, t \in [0,T),
\]

and the following equality

\[
u^i(t,\cdot) = \nu^i(t,\cdot), \text{ except at most on a countable set in } \mathbb{R}, \text{ for all } t \in [0,T),
\]
where $\overline{u}^i$ and $\underline{u}^i$ are, respectively, the upper and lower relaxed semi-limits defined in (7) and (8).

iii) **Global existence of weak discontinuous viscosity solution:**

Let $u_\varepsilon$ be the solution of (4), constructed in (i). Suppose that assumption (9) is satisfied. Then $\overline{u} = (\overline{u}^i)_{i=1, \ldots, d}$ and $\underline{u} = (\underline{u}^i)_{i=1, \ldots, d}$ are respectively discontinuous viscosity sub-solution and super-solution of system (1) (in the sense of Definition 4.1) and moreover they satisfy equality (17).

The key point to establish this theorem is the uniform $BV$ estimate (11) on $u_\varepsilon^i$. We first consider the regularized problem of (1) and we show that the smooth solution admits the $L^\infty$ bound (10) and the fundamental $BV$ estimate (11). These estimates will allow us to pass to the limit when the regularization vanishes. Then we will show, from the classical stability properties of viscosity solutions and using (9), that the relaxed semi-limits $\overline{u}$ and $\underline{u}$ are, respectively, sub-solution and super-solution of (1). These estimates also imply that the set of the discontinuous points, with respect to $x$, of the solution $u$ is at most countable. Taking into account the finite speed propagation property of (1) and the time continuous estimate (12), it is then possible to show this property uniformly in time, which proves in particular (17).

**Remark 1.**

1. **(Dimension)** In this paper, we choose to work in dimension 1 since the fundamental $BV$ estimate (11) is only valid in this context.

2. **(Weak discontinuous viscosity solution)** We note that, the solution constructed in the previous theorem, can be seen as a discontinuous viscosity solution but in some weak sense, since it verifies only an almost everywhere equality in space between $\overline{u}^i$ and $\underline{u}^i$, which is reflected by (17). Thanks to this equality, it will be possible to show that the sub-solution $\overline{u}^i$ (resp. super-solution $\underline{u}^i$) coincides with the upper semi-continuous envelope (resp. lower semi-continuous envelope) of $u^i$ only almost everywhere in space. In fact, this is unlike the standard definition of viscosity solutions where this equality is required everywhere. In what follows, a solution verifying this a.e. property in space will be called a weak discontinuous viscosity solution of (1).

Recall that in the framework of non-decreasing solutions, the Hamilton Jacobi system (1) becomes a classical transport system. For such a system, Bianchini and Bressan proved in [7] a striking global existence and uniqueness result for general non-conservative $(d \times d)$ strictly hyperbolic systems, including diagonal systems like system (1). The key step in their proof was an a priori estimate on the total variation of the approximate solutions proved relying on small total variation of initial data. Their existence result is a generalization of Glimm’s result [17], proved in the case of conservation laws. Let us mention that an existence result has also been obtained by LeFloch and Liu [21, 22] in the non-conservative case. After that, El Hajj and Monneau proved in [15] a global existence and uniqueness result for strictly hyperbolic diagonal systems, with the assumption $\frac{\partial N}{\partial x} \leq 0$ ($i = 1, \ldots, d$), by widening the regularity assumption and considering non-decreasing continuous solutions. That was a generalization to the work done by Lax [20] and DiPerna [11, 12], in the case of $(2 \times 2)$ strictly hyperbolic systems, where the global existence was proved for Lipschitz and $L^\infty$ solutions, respectively.

Now, we can state the following theorem as a consequence of Theorem 1.2 in the case of non-decreasing solutions. This result remains valid even in the case where
system (1) is not necessarily strictly hyperbolic and the initial data has a large $BV$ norm.

**Theorem 1.3. (Global existence of non-decreasing discontinuous viscosity solution)**

Assume that (2) and (9) are satisfied. Suppose that $u^i_0 \in L^\infty(\mathbb{R})$ and the function $u^i$ is non-decreasing for $i = 1, \cdots, d$, then system (1) has a discontinuous non-decreasing viscosity solution $u = (u^i)_{i=1, \cdots, d}$, such that for $i = 1, \cdots, d$, $u^i$ satisfies (13), (14), (15) and (16).

**Remark 2.**

1. **(Bounded and non-decreasing data)** In the case of non-decreasing and bounded initial data assumption (3) automatically holds, namely $|u^i_0|_{BV(\mathbb{R})} \leq 2 \|u^i_0\|_{L^\infty(\mathbb{R})}$.

2. **(Weak solution for diagonal transport systems)** Let us also mention that, if we consider the following diagonal hyperbolic systems

$$\begin{cases}
\partial_t u^i(t, x) = \lambda^i(t, x, u) \partial_x u^i(t, x) & \text{in } (0, T) \times \mathbb{R}, \\
u^i(0, x) = u^i_0(x) & \text{in } \mathbb{R},
\end{cases}$$

for the case where the solution is not necessarily non-decreasing, then it is always possible to prove a similar global existence result of weak discontinuous viscosity solution of (18), as that announced in Theorem 1.2. Indeed, with a slight modification of the proof, we can verify that the a priori estimates (10), (11) and (12) remain valid for the solution of the associated regularized problem of (18), and therefore Theorem 1.2 (ii)-(iii) is also true for this diagonal transport system.

**1.3. Organization of the paper.** The paper is organized as follows: Section 2 is devoted to show the global existence of Lipschitz continuous viscosity solution of the regularized system with uniform $BV$ estimate (Theorem 1.2 (i)). In Section 3, we prove Theorem 1.2 (ii) by passing to the limit as $\varepsilon$ goes to 0, using the compactness argument. Finally, in Section 4, we give the proof of Theorem 1.3 and Theorem 1.2 (iii), using the finite speed propagation property and the discontinuity property of non-decreasing functions.

**2. Global Lipschitz continuous viscosity solution of the regularized system.** This section is devoted to prove Theorem 1.2 (i). First, we recall in following subsection some useful results then we give the proof in next subsection.

**2.1. Some useful results.** In this subsection, we recall a global existence and uniqueness result for the following scalar eikonal equation

$$\begin{cases}
\partial_t v(t, x) = c(t, x)|\partial_x v(t, x)| & \text{in } (0, T) \times \mathbb{R}, \\
v(0, x) = v_0(x) & \text{in } \mathbb{R},
\end{cases}$$

(19)

where the initial data and the velocity satisfy the following assumptions

$$v_0 \in W^{1, \infty}(\mathbb{R}) \quad \text{and} \quad c \in W^{1, \infty}(0, T) \times \mathbb{R).}$$

(20)

Before showing this result let us recall the definition of the continuous viscosity solution for (19).
Definition 2.1. (Continuous viscosity sub-solution, super-solution and solution)

A function \( v \in C([0, T) \times \mathbb{R}) \) is a viscosity sub-solution of (19) if and only if \( v(0, x) \leq v_0(x) \) and for any \( \phi \in C^1((0, T) \times \mathbb{R}) \), if \((t_0, x_0) \in (0, T) \times \mathbb{R}\) is a local maximum point of \( v - \phi \), we have

\[
\partial_t \phi(t_0, x_0) - c(t_0, x_0) |\partial_x \phi(t_0, x_0)| \leq 0.
\]

A function \( v \in C([0, T) \times \mathbb{R}) \) is a viscosity super-solution of (19) if and only if \( v(0, x) \geq v_0(x) \) and for any \( \phi \in C^1((0, T) \times \mathbb{R}) \), if \((t_0, x_0) \in (0, T) \times \mathbb{R}\) is a local minimum point of \( v - \phi \), we have

\[
\partial_t \phi(t_0, x_0) - c(t_0, x_0) |\partial_x \phi(t_0, x_0)| \geq 0.
\]

A continuous function \( v \) is a viscosity solution of (19) if and only if it is a sub- and a super-solution of (19).

We have the following existence and uniqueness result for the scalar equation (19).

Theorem 2.2. (Existence and uniqueness of Lipschitz continuous viscosity solution)

Assume that (20) holds, then, for all \( T > 0 \), we have

i) The problem (19) admits a unique Lipschitz continuous viscosity solution on \((0, T) \times \mathbb{R}\), satisfying the following estimates

\[
\|v\|_{L^\infty((0, T) \times \mathbb{R})} \leq \|v_0\|_{L^\infty(\mathbb{R})}, \tag{21}
\]

\[
\|\partial_x v\|_{L^\infty((0, T) \times \mathbb{R})} \leq \|\partial_x v_0\|_{L^\infty(\mathbb{R})} e^{T \|\partial_x c\|_{L^\infty((0, T) \times \mathbb{R})}}. \tag{22}
\]

ii) Assume moreover that

\[
\partial_x v_0 \in L^1(\mathbb{R}), \quad \text{and} \quad \partial_x c \in L^\infty((0, T); L^1(\mathbb{R})). \tag{23}
\]

then \( v \) satisfies the following BV bound

\[
\int_\mathbb{R} |\partial_x v(x, t)| dx \leq \int_\mathbb{R} |\partial_x v_0(x)| dx, \quad \text{for all} \quad t \in [0, T), \tag{24}
\]

and also the following bound

\[
\int_\mathbb{R} |\partial_t v(x, t)| dx \leq \|c\|_{L^\infty((0, T) \times \mathbb{R})} \|\partial_x v_0\|_{L^1(\mathbb{R})}, \quad \text{for all} \quad t \in [0, T). \tag{25}
\]

The point (i) concerning the Lipschitz estimate has been established by Ley in [23]. However, the point (ii) concerning the BV estimate has been proved by El Hajj and Boudjerada in [8].

2.2. Proof of Theorem 1.2 (i). We proceed in two steps.

Step 1. (Local existence and uniqueness) First, we note \( u_{0,\varepsilon} = (u_{0,\varepsilon}^i)_{i=1, \cdots, d} \) and

\[
\mathcal{K}_0 = \prod_{i=1}^d \left[ -\|u_0^i\|_{L^\infty(\mathbb{R})} - 1, \|u_0^i\|_{L^\infty(\mathbb{R})} + 1 \right],
\]

and we introduce the following constants
Now, for uncoupled system (where \( T > 0 \) and \( \epsilon \) small enough. First, we will prove that \( \| \partial_x u_{0,\epsilon} \|_{L^1(\mathbb{R})} = \epsilon, i = 1, \cdots, d \), the set

\[
X_T^{\epsilon,i} = \left\{ \zeta \in L^\infty((0, T) \times \mathbb{R}), \text{ such that } \begin{align*}
\| \zeta \|_{L^\infty((0, T) \times \mathbb{R})} &\leq \| u_{0,i,\epsilon} \|_{L^1(\mathbb{R})} \\
\| \partial_x \zeta \|_{L^\infty((0, T) \times \mathbb{R})} &\leq \Lambda \\
\| \partial_x \zeta \|_{L^\infty((0, T); L^1(\mathbb{R}))} &\leq \| \partial_x u_{0,i,\epsilon} \|_{L^1(\mathbb{R})} \\
\| \partial_t \zeta \|_{L^\infty((0, T); L^1(\mathbb{R}))} &\leq K
\end{align*} \right\},
\]

where \( \rho_1^\epsilon(x) \) and \( \rho_2^{d+2}(t, x, w^1, \cdots, w^d) \) are given by (6). We define, for all \( 0 < \epsilon \leq 1 \), \( T > 0 \) and \( i = 1, \cdots, d \), the set

\[
X_T^{\epsilon,i} = \left\{ \zeta \in L^\infty((0, T) \times \mathbb{R}), \text{ such that } \begin{align*}
\| \zeta \|_{L^\infty((0, T) \times \mathbb{R})} &\leq \| u_{0,i,\epsilon} \|_{L^1(\mathbb{R})} \\
\| \partial_x \zeta \|_{L^\infty((0, T) \times \mathbb{R})} &\leq \Lambda \\
\| \partial_x \zeta \|_{L^\infty((0, T); L^1(\mathbb{R}))} &\leq \| \partial_x u_{0,i,\epsilon} \|_{L^1(\mathbb{R})} \\
\| \partial_t \zeta \|_{L^\infty((0, T); L^1(\mathbb{R}))} &\leq K
\end{align*} \right\},
\]

where

\[
K = \| \lambda^i \|_{L^\infty((0, T) \times \mathbb{R} \times K_0)} \| \partial_x u_{0,i,\epsilon} \|_{L^1(\mathbb{R})} \quad \text{and} \quad \lambda^i = \| \partial_x u_{0,i,\epsilon} \|_{L^\infty((0, T) \times \mathbb{R} \times K_0)}.
\]

Now, for \( v_\epsilon = (v_{\epsilon}^i)_{i=1,\cdots,d} \in X_T^{\epsilon} \), where \( X_T^{\epsilon} = \prod_{i=1}^d X_T^{\epsilon,i} \) we define \( G(v_\epsilon) = u_\epsilon = (u_\epsilon^i)_{i=1,\cdots,d} \) as the unique Lipschitz continuous viscosity solution of the following uncoupled system

\[
\begin{align*}
\partial_t u_{\epsilon}^i(t, x) &= \lambda^i \left( t, x, \rho_1^\epsilon \ast v_\epsilon(t, \cdot)(x) \right) \left| \partial_x u_{\epsilon}^i(t, x) \right| \quad \text{in } (0, T) \times \mathbb{R}, \\
u_{\epsilon}^i(0, x) &= u_{0,i,\epsilon}(x) \quad \text{in } \mathbb{R},
\end{align*}
\]
for \( i = 1, \cdots, d \).

We will show that \( G : X_T^{\epsilon} \rightarrow X_T^{\epsilon} \) is well defined and a strict contraction for \( T \) small enough. First, we will prove that \( G \) is well defined. Indeed, by classical properties of the mollifiers, using Young’s inequality and (3), we know that (see Ambrosio et
al. [1, Th 2.2 (b)] for the third estimate):

\[
\begin{align*}
\|u_{0,\varepsilon}^i\|_{L^{\infty}(\mathbb{R})} & \leq \|u_0^i\|_{L^{\infty}(\mathbb{R})} \\
\|\partial_x u_{0,\varepsilon}^i\|_{L^{\infty}(\mathbb{R})} & \leq \mu_1^i \|u_0^i\|_{L^{\infty}(\mathbb{R})} \quad \text{for } i = 1, \ldots, d,
\end{align*}
\]

(29)

where \(\mu_1^i\) is defined in (26). Similarly, using (2) and (29) we can check that, if \(v_{\varepsilon} \in X^T_T\), then the velocity \(c_{\varepsilon}^i(t, x) = \lambda_{\varepsilon}^i(t, x, \rho_{\varepsilon}^i \ast v_{\varepsilon}(t, \cdot)(x))\) satisfies the following estimates

\[
\begin{align*}
\|c_{\varepsilon}^i\|_{L^{\infty}((0,T) \times \mathbb{R})} & \leq \|\lambda_{\varepsilon}^i\|_{L^{\infty}((0,T) \times \mathbb{R} \times \mathcal{K})} \\
\|\partial_x c_{\varepsilon}^i\|_{L^{\infty}((0,T) \times \mathbb{R})} & \leq \nu_{\varepsilon}^i \|\lambda_{\varepsilon}^i\|_{L^{\infty}((0,T) \times \mathbb{R} \times \mathcal{K})} \quad \text{for } i = 1, \ldots, d,
\end{align*}
\]

(30)

where \(\nu_{\varepsilon}^i\) and \(\nu_{\varepsilon}^j\) are defined in (26).

Estimates (29), (30) and the regularization properties of the mollifiers imply that the velocity \(\lambda_{\varepsilon}^i(t, x, \rho_{\varepsilon}^i \ast v_{\varepsilon}(t, \cdot)(x))\) and the initial data \(u_{0,\varepsilon}^i\) satisfy the assumptions (20) and (23). Therefore, applying Theorem 2.2 with \(v_0 = u_{0,\varepsilon}^i\) and \(c = c_{\varepsilon}^i\) we deduce that system (28) admits a unique Lipschitz continuous viscosity solution \(u_{\varepsilon} = (u_{\varepsilon}^i)_{i=1,\ldots,d}\). Moreover, according to (21), (24), (25), we have for all \(t \in (0,T)\)

\[
\begin{align*}
\|u_{\varepsilon}^i(t, \cdot)\|_{L^{\infty}(\mathbb{R})} & \leq \|u_{0,\varepsilon}^i\|_{L^{\infty}(\mathbb{R})} \\
\|\partial_x u_{\varepsilon}^i(t, \cdot)\|_{L^1(\mathbb{R})} & \leq \|\partial_x u_{0,\varepsilon}^i\|_{L^1(\mathbb{R})} \\
\|\partial_t u_{\varepsilon}^i(t, \cdot)\|_{L^1(\mathbb{R})} & \leq \|\lambda_{\varepsilon}^i\|_{L^{\infty}((0,T) \times \mathbb{R} \times \mathcal{K})} \|\partial_x u_{0,\varepsilon}^i\|_{L^1(\mathbb{R})} 
\end{align*}
\]

(31)

Then, from (29), (30) we can see that \(u_{\varepsilon}^i\) satisfies the \(\varepsilon\)-uniform estimates (10), (11) and (12). Using (22) we can also see that \(u_{\varepsilon}^i\) satisfies the following Lipschitz estimate

\[
\|\partial_x u_{\varepsilon}^i\|_{L^{\infty}((0,T) \times \mathbb{R})} \leq \|\partial_x u_{0,\varepsilon}^i\|_{L^{\infty}(\mathbb{R})} \|\lambda_{\varepsilon}^i\|_{L^{\infty}((0,T) \times \mathbb{R} \times \mathcal{K})}
\]

(32)

\[
\leq \|\partial_x u_{0,\varepsilon}^i\|_{L^1(\mathbb{R})} \|\lambda_{\varepsilon}^i\|_{L^{\infty}((0,T) \times \mathbb{R} \times \mathcal{K})}^T \nu_{\varepsilon}^i
\]

which depends on \(\varepsilon\). This also shows that \(u_{\varepsilon}^i \in X^T_T\) for all \(T > 0\) and so \(G\) is well defined.

It thus remains to show that \(G\) is a contraction. Let \(u_{\varepsilon,\ell} = G(v_{\varepsilon,\ell})\) with \(u_{\varepsilon,\ell} = (u_{\varepsilon,\ell}^i)_{i=1,\ldots,d}\), \(v_{\varepsilon,\ell} = (v_{\varepsilon,\ell}^i)_{i=1,\ldots,d}\), for \(\ell = 1, 2\) and set

\[
L = \|v_{\varepsilon,2} - v_{\varepsilon,1}\|_{L^{\infty}((0,T) \times \mathbb{R})} = \sum_{i=1}^d \|v_{\varepsilon,2}^i - v_{\varepsilon,1}^i\|_{L^{\infty}((0,T) \times \mathbb{R})}.
\]

Then, by the properties of the mollifiers, we can verify that

\[
|\lambda_{\varepsilon}^i(t, x, \rho_{\varepsilon}^i \ast v_{\varepsilon,2}(t, \cdot)(x)) - \lambda_{\varepsilon}^i(t, x, \rho_{\varepsilon}^i \ast v_{\varepsilon,1}(t, \cdot)(x))| \leq \nu_{\varepsilon}^i \|\lambda_{\varepsilon}^i\|_{L^{\infty}((0,T) \times \mathbb{R} \times \mathcal{K})} \|v_{\varepsilon,1} - v_{\varepsilon,2}\|_{L^{\infty}((0,T) \times \mathbb{R})}^d
\]

\[
\leq \nu_{\varepsilon}^i \|\lambda_{\varepsilon}^i\|_{L^{\infty}((0,T) \times \mathbb{R} \times \mathcal{K})} L,
\]
where \( \nu^i_\varepsilon \) is defined in (26). Since \( u_{\varepsilon,2}^i \), for \( i = 1, \cdots, d \), is a viscosity solution of the following equation
\[
\begin{cases}
\partial_t u_{\varepsilon,2}^i(t, x) = \lambda_2^i \left( t, x, \rho_1^i \ast v_{\varepsilon,2}(t, \cdot) (x) \right) \left| \partial_x u_{\varepsilon,2}^i(t, x) \right| & \text{in} \ (0, T) \times \mathbb{R}, \\
u^i_\varepsilon(0, x) = u^i_{0, \varepsilon}(x) & \text{in} \ \mathbb{R},
\end{cases}
\]
we can remark, by adding and subtracting \( \lambda_2^i \left( t, x, \rho_1^i \ast v_{\varepsilon,1}(t, \cdot) (x) \right) \left| \partial_x u_{\varepsilon,2}^i(t, x) \right|, \)
that \( u_{\varepsilon,2}^i \) is a viscosity sub-solution of
\[
\begin{cases}
\partial_t u_{\varepsilon,2}^i(t, x) = \lambda_2^i \left( t, x, \rho_1^i \ast v_{\varepsilon,1}(t, \cdot) (x) \right) \left| \partial_x u_{\varepsilon,2}^i(t, x) \right| \\
& \quad + \nu^3_\varepsilon L_\varepsilon(T) \left| \lambda_i \right|_{L^\infty((0, T) \times \mathbb{R} \times \mathcal{K}_\varepsilon)} L_t \text{in} \ (0, T) \times \mathbb{R}, \\
u^i_\varepsilon(0, x) = u^i_{0, \varepsilon}(x) & \text{in} \ \mathbb{R}.
\end{cases}
\]
In addition, \( u_{\varepsilon,1}^i + \nu^3_\varepsilon L_\varepsilon(T) \left| \lambda_i \right|_{L^\infty((0, T) \times \mathbb{R} \times \mathcal{K}_\varepsilon)} L_t \) is a viscosity solution of the same equation. By comparison principle applied to the scalar eikonal equation (see Barles [3]), we deduce that
\[ u_{\varepsilon,2}^i \leq u_{\varepsilon,1}^i + \nu^3_\varepsilon L_\varepsilon(T) \left| \lambda_i \right|_{L^\infty((0, T) \times \mathbb{R} \times \mathcal{K}_\varepsilon)} L_t. \]
Interchanging \( u_{\varepsilon,1}^i \) and \( u_{\varepsilon,2}^i \) we get
\[
\left\| u_{\varepsilon,1}^i - u_{\varepsilon,2}^i \right\|_{L^\infty((0, T) \times \mathbb{R})} \leq \nu^3_\varepsilon L_\varepsilon(T) \left| \lambda_i \right|_{L^\infty((0, T) \times \mathbb{R} \times \mathcal{K}_\varepsilon)} T \left\| v_{\varepsilon,1} - v_{\varepsilon,2} \right\|_{(L^\infty((0, T) \times \mathbb{R}))^d}.
\]
Then
\[
\left\| u_{\varepsilon,1}^i - u_{\varepsilon,2}^i \right\|_{L^\infty((0, T) \times \mathbb{R})} \leq \nu^3_\varepsilon \Lambda L_\varepsilon(T) \left\| v_{\varepsilon,1} - v_{\varepsilon,2} \right\|_{(L^\infty((0, T) \times \mathbb{R}))^d},
\]
where \( \Lambda \) is defined in (27). This shows that, for \( T \) small enough, \( G \) is a contraction on \( X_T^\varepsilon \) which is a closed set. So, by fixed point theorem, there exists a unique Lipschitz continuous viscosity solution of (4) in \( X_T^\varepsilon \), for all \( T > 0 \), such that \( T \leq T_1 \), where
\[
T_1 = \min \left( \frac{1}{4 \nu^3_\varepsilon \Lambda \left\| \partial_x u_{0, \varepsilon}^i \right\|_{(L^\infty(\mathbb{R}))^d}}, \frac{\ln(2)}{\nu^3_\varepsilon \Lambda} \right).
\]

**Step 2. (Global existence and uniqueness)** We are going to prove that the local time solution obtained in Step 1 can be extended to global one. We argue by contradiction. Assume that there exists a maximum time \( T_{max} \) such that, we have the existence of solutions of the system (4) in the function space \( W^{1,\infty}(\mathbb{R}) \).

For every \( \delta > 0 \), we consider the system (4) with the initial conditions
\[
u_{\varepsilon,1}^i(x) = u^i_\varepsilon(T_{max} - \delta, x).
\]
We apply for the second time the same techniques of Step 1 to deduce that there exists a time \( T_{\delta}^* \) such that system (4) admits Lipschitz continuous viscosity solution defined until the time
\[
T_0 = (T_{max} - \delta) + T_{\delta}^*,
\]
with
\[
T_{\delta}^* = \min \left( \frac{1}{4 \nu^3_\varepsilon \Lambda \left\| \partial_x u_{0, \varepsilon}^i \right\|_{(L^\infty(\mathbb{R}))^d}}, \frac{\ln(2)}{\nu^3_\varepsilon (1 + \mu^2_\varepsilon \left\| \partial_x u_{0, \varepsilon}^i \right\|_{L^1(\mathbb{R})} L_t) \Lambda} \right),
\]
where \( u_{0, \varepsilon}^i = (u_{\varepsilon,1}^{\delta, i})_{i=1, \cdots, d} \). According to (31) and (32), we can verify that, for \( i = 1, \cdots, d \), we have
\[
\left\| \partial_x u_{0, \varepsilon}^i \right\|_{L^1(\mathbb{R})} = \left\| \partial_x u^i_\varepsilon(T_{max} - \delta, \cdot) \right\|_{L^1(\mathbb{R})} \leq \left\| \partial_x u_{0, \varepsilon}^i \right\|_{L^1(\mathbb{R})},
\]
and
\[
\|\partial_x u_{0,i}^\epsilon \|_{L^\infty(\mathbb{R})} = \|\partial_x u_{0}^\epsilon(T_{\max} - \delta, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|\partial_x u_{0}^\epsilon\|_{L^\infty((0,T_{\max}) \times \mathbb{R})} \\
\leq \|\partial_x u_{0,i}^\epsilon\|_{L^\infty(\mathbb{R})} e^{\lambda^i T_{\max}} \|\lambda\|_{L^\infty((0,T_{\max}) \times \mathbb{R} \times K_0)} T_{\max}.
\]
These estimates show that \(\partial_x u_{0,i}^\epsilon\) is \(\delta\)-uniformly bounded in \((L^\infty(\mathbb{R}))^d \cap (L^1(\mathbb{R}))^d\) and therefore there exists a constant
\[
C(\epsilon, \Lambda, T_{\max}, \|\partial_x u_{0,i}^\epsilon\|_{L^1(\mathbb{R})}) \geq \delta > 0,
\]
and therefore there exists a constant
\[
C(\epsilon, \Lambda, T_{\max}, \|\partial_x u_{0,i}^\epsilon\|_{L^1(\mathbb{R})}) \geq \delta > 0.
\]

3. Passage to the limit when \(\epsilon\) tends to 0. In this section we give the proof of Theorem 1.2 (ii) using the following compactness lemma.

**Lemma 3.1. (Simon's Lemma [24, Corollary 4])**

Let \(X, B\) and \(Y\) be three Banach spaces, where \(X \hookrightarrow B\) with compact embedding and \(B \hookrightarrow Y\) with continuous embedding. If \((\theta_n)_n\) is a sequence uniformly bounded in \(L^\infty((0,T);X)\) and \((\theta_n)_n\) is uniformly bounded in \(L^r((0,T);Y)\) for \(r > 1\), then, \((\theta_n)_n\) is relatively compact in \(C((0,T);B)\).

**Proof of Theorem 1.2 (ii):** For the sake of a clear presentation, we proceed in two steps.

**Step 1. (Convergence)** Let \(u_{0,i} = (u_{0,i}^\epsilon)_{i=1,\ldots,d}\) be the solution of (4), constructed in Theorem 1.2 (i). From estimates (10), (11) and (12), we can say that, for all compact \(K_0 \subset \mathbb{R}\), \((u_{0,i}^\epsilon)\) is uniformly bounded in \(L^\infty((0,T);BV(K_0)) \cap L^\infty((0,T) \times K_0)\) and \((\partial_t u_{0,i}^\epsilon)\) is uniformly bounded in \(L^\infty((0,T);L^1(K_0))\). Using Simon’s lemma in the particular case \(X = BV(K_0), B = Y = L^1(K_0)\) and the following compact embedding \(BV(K_0) \hookrightarrow L^1(K_0)\), we can extract a subsequence, denoted by \((u_{0,i}^\epsilon)_{\epsilon_n,K_0}\) that converges strongly in \(L^\infty((0,T);L^1(K_0))\) to some limit \(u_{0,i}\) as \(n \to +\infty\). By a standard diagonalization procedure, we can extract a subsequence \((u_{0,i}^\epsilon)_{\epsilon_n}\) (independent of \(i\) and \(K\)) that converges to the limit \(u_{0,i}\) strongly in \(C((0,T);L^1(K))\) for all compact \(K \subset \mathbb{R}\). Now, thanks to estimates (10) and (11) we can extract a subsequence, still denoted by \((u_{0,i}^\epsilon)_{\epsilon_n}\), satisfying the following convergence
\[
\begin{align*}
\|u_{0,i}^\epsilon\|_{L^\infty((0,T) \times \mathbb{R})}, & \quad \text{for all compact } K \subset \mathbb{R}, \\
|u_{0,i}^\epsilon|_{L^\infty((0,T);BV(\mathbb{R}))}.
\end{align*}
\]

Taking the \(\liminf\) in estimates (10), (11) and using the lower semi-continuity of \(\|\cdot\|_{L^\infty(\mathbb{R})}\) and \(\|\cdot\|_{BV(\mathbb{R})}\), we can prove that \(u_{0,i}\) satisfies (13), (14) and (15). Moreover, using estimate (12), we can remark that \(u_{0,i}^\epsilon\) satisfies the following \(L^1\)-Lipschitz estimate in time
\[
\|u_{0,i}^\epsilon(t, \cdot) - u_{0,i}^\epsilon(s, \cdot)\|_{L^1(\mathbb{R})} \leq (\|\lambda\|_{L^\infty((0,T) \times \mathbb{R} \times K_0)} \|u_{0}^\epsilon\|_{BV(\mathbb{R})}) |t - s| \quad \text{for all } t, s \in [0,T).
\]
Which implies, according to the strong convergence in \(C((0,T);L^1_{\text{loc}}(\mathbb{R}))\), that \(u_{0,i}\) satisfies also the \(L^1\)-Lipschitz estimate in time (16).
Step 2. (Set of discontinuity points) It remains to show equality (17). Indeed, since \( u^i(t, \cdot) \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R}) \), we know that this function coincides with a right-continuous function, almost everywhere in \( \mathbb{R} \) and consequently in \( L^1_{\text{loc}}(\mathbb{R}) \). This is a direct consequence of the fact that the \( BV \)-functions are continuous except at most on a countable set. Now, we can prove as in [8, Section 6.3] that there exists a countable set \( D^i \) (independent of \( t \)) such that
\[
\tag{34}
u^i(t, \cdot) = \varpi(t, \cdot) = \bar{u}^i(t, \cdot), \quad \text{in } \mathbb{R} \setminus D^i, \quad \text{for all } t \in [0, T),
\]
where \( \varpi \) and \( \bar{u}^i \) are defined in (7) and (8), respectively. This result is obtained based on estimates (10), (11), (12) and on the finite speed propagation property of the equation satisfied by \( u^i, \) proved below in Lemma 4.2.

4. Global discontinuous viscosity solution of (1). In this section we prove the two global existence results of discontinuous viscosity solution of system (1), announced in Theorem 1.2 (iii) and Theorem 1.3. Before doing that, we recall the definition of discontinuous viscosity solution of system (1) introduced by Ishii in [18, Definition 2.1].

We denote by \( f^* \) and \( f_* \) the respective upper and lower semi-continuous envelopes of a locally bounded function \( f \) defined on an open domain in \( \mathbb{R}^n \) and given by
\[
f^*(X) = \limsup_{Y \to X} f(Y) \quad \text{and} \quad f_*(X) = \liminf_{Y \to X} f(Y). \tag{35}
\]
For a vector \( u = (u^1, \cdots, u^d) \) locally bounded on \( [0, T) \times \mathbb{R} \) for all \( T > 0 \), we write \( u^* = ((u^1)^*, \cdots, (u^d)^*) \) and \( u_* = (u^1_*, \cdots, u^d_*) \). Also we define \( \bar{U} : (0, T) \times \mathbb{R} \to 2^{\mathbb{R}^d} \), the graph closure of \( u \), by
\[
\bar{U}(t, x) = \{ r \in \mathbb{R}^d : \text{ there is a sequence } \{ (t_n, x_n) \} \subset (0, T) \times \mathbb{R} \text{ such that } (t_n, x_n) \to (t, x) \text{ and } u(t_n, x_n) \to r \}. 
\]
It should be remarked that \( \bar{U}(t, x) \) is closed; i.e., for all sequence \( \{ r_m \} \subset \mathbb{R}^d \), if \( r_m \in \bar{U}(t, x) \) and \( r_m \to r \) for some \( r \in \mathbb{R}^d \), then \( r \in \bar{U}(t, x) \).

Definition 4.1. (Discontinuous viscosity sub-solution, super-solution and solution)

Assume that \( \lambda^i \) is locally bounded on \( (0, T) \times \mathbb{R} \times \mathbb{R}^d \) and \( u_0 = (u_0^i)_{i=1,\ldots,d} \) is locally bounded on \( \mathbb{R} \). Let \( u = (u^i)_{i=1,\ldots,d} \) be a locally bounded function defined on \( [0, T) \times \mathbb{R} \).

1. (Discontinuous viscosity sub-solution) We call \( u \) a discontinuous viscosity sub-solution of (1) if it satisfies
   \begin{enumerate}[(i)]
   \item \( (u^i)^*(0, x) \leq (u_0^i)^*(x) \), for all \( i = 1, \cdots, d \) and \( x \in \mathbb{R} \).
   \item If whenever \( \phi \in C^1((0, T) \times \mathbb{R}) \), \( i = 1, \cdots, d \) and \( (u^i)^* - \phi \) attains its local maximum at \( (t_0, x_0) \in (0, T) \times \mathbb{R} \), then we have
     \[
     \min \left\{ \partial_x \phi(t_0, x_0) - (\lambda^i)^*(t_0, x_0, r) \right\}_{r \in \bar{U}(t_0, x_0), \text{ such that } r = (u^i)^*(t_0, x_0)} \leq 0. \tag{36}
     \]
   \end{enumerate}

2. (Discontinuous viscosity super-solution) Similarly, we call \( u \) a discontinuous viscosity super-solution of (1) if it satisfies
   \begin{enumerate}[(i)]
   \item \( (u^i)_*(0, x) \geq (u_0^i)_*(x) \), for all \( i = 1, \cdots, d \) and \( x \in \mathbb{R} \).
   \end{enumerate}
(ii) If whenever $\phi \in C^1((0, T) \times \mathbb{R})$, $i = 1, \cdots, d$ and $(u^i)_*$ attains its local minimum at $(t_0, x_0) \in (0, T) \times \mathbb{R}$, then we have
\[
\max \left\{ \partial_t \phi(t_0, x_0) - (\lambda^i)_*(t_0, x_0, r) \left| \partial_x \phi(t_0, x_0) \right| : r \in \mathcal{U}(t_0, x_0), r^i = (u^i)_*(t_0, x_0) \right\} \geq 0.
\]

(3) **(Discontinuous viscosity solution)**

Finally, we call $u$ a discontinuous viscosity solution of (1) if it is both a discontinuous viscosity sub-solution and super-solution of (1).

Note that the minimum and the maximum in (36) and (37) are attained, since the sets
\[
\left\{ r \in \mathbb{R}^d : r \in \mathcal{U}(t_0, x_0), r^i = (u^i)_*(t_0, x_0) \right\}
\]
and
\[
\left\{ r \in \mathbb{R}^d : r \in \mathcal{U}(t_0, x_0), r^i = (u^i)_*(t_0, x_0) \right\}
\]
are non-empty and compact. Also note that $(\lambda^i)_*$ and $(\lambda^i)_*$ are upper and lower semi-continuous, respectively.

To prove Theorem 1.2 (iii) and Theorem 1.3, we need also to establish in the following subsection some preliminary results.

4.1. **Preliminary results.** We start this subsection by showing the following finite speed propagation property, valid for continuous viscosity solutions of system (4).

**Lemma 4.2. (Finite speed propagation property)**

Under assumptions (2) and (3), if $u^\varepsilon = (u^\varepsilon_i)_{i=1,\ldots,d}$ is the unique continuous viscosity solution of (4), given by Theorem 1.2 (i), then $u^\varepsilon_i$ satisfies, for all $h \geq 0$, the following estimate
\[
\inf_{|y-x| \leq t\Lambda} u^\varepsilon_i(h, y) \leq u^\varepsilon_i(t+h, x) \leq \sup_{|y-x| \leq t\Lambda} u^\varepsilon_i(h, y), \quad \text{for all } (t, x) \in [0, T-h] \times \mathbb{R},
\]
where $\Lambda$ is defined in (27).

**Proof of Lemma 4.2:** Let us start by proving the right hand side of (38), in viscosity sense, namely
\[
u_i^\varepsilon(t+h, x) \leq \sup_{|y-x| \leq t\Lambda} u^\varepsilon_i(h, y).
\]

Let $u^\varepsilon_{i,h}(t, x) = u^\varepsilon_i(t+h, x)$. Then, we can see that
\[
\partial_t u^\varepsilon_{i,h}(t, x) = \partial_t u^\varepsilon_i(t+h, x) = \lambda^\varepsilon_i(t+h, x, \rho^\varepsilon_i(t+h, \cdot)(x)) \mid \partial_x u^\varepsilon_i(x, t+h)\]

\[
= \lambda^\varepsilon_i(t, x, [u^\varepsilon_i]) \mid \partial_x u^\varepsilon_{i,h}(t, x).\]

Now, form (30) we have $\lambda^\varepsilon_i(t, x, [u^\varepsilon_i]) \leq \| \lambda^\varepsilon_i \|_{L^\infty((0, T) \times \mathbb{R} \times K^\varepsilon)} \leq \Lambda$, where $\Lambda$ is defined in (27). This implies that $u^\varepsilon_{i,h}$ is viscosity sub-solution of the following equation
\[
\partial_t w = \Lambda \mid \partial_x w \mid, \quad w(0, x) = u^\varepsilon_i(h, x).
\]

Furthermore, if we note
\[
\alpha(t, x) = \sup_{|y-x| \leq t\Lambda} u^\varepsilon_i(h, y),
\]
then, by Lax-Oleinik formula (See [3, Lemma 2.1]), we know that $\alpha$ is the unique continuous viscosity solution of (40). Using the comparison principle (see [6, Th 1.1]), we deduce that

$$u_{\varepsilon}(t + h, x) \leq \alpha(t, x), \quad \text{on } (0, T) \times \mathbb{R},$$

which implies (39). The same proof is done for the inequality

$$\inf_{|y-x| \leq t \Lambda} u_{\varepsilon}(h, y) \leq u_{\varepsilon}(t + h, x),$$

by considering the following equation $\partial_t w = -\Lambda |\partial_x w|$. \hfill \Box

Now, we show a local estimate valid on sequences of non-decreasing functions converging locally and strongly in $L^1(\mathbb{R})$.

**Lemma 4.3. (Sequences of non-decreasing functions)**

Let $(\phi_{\varepsilon})_{\varepsilon}$ be a sequence of defined functions on $[0, T) \times \mathbb{R}$ such that, for all $t \in [0, T)$, the function $\phi_{\varepsilon}(t, \cdot)$ is non-decreasing on $\mathbb{R}$. Assume, moreover, that $\phi_{\varepsilon} \to \phi$ strongly in $C([0, T); L^1_{\text{loc}}(\mathbb{R}))$, as $\varepsilon \to 0$, with, for all $t \in (0, T)$, the function $\phi(t, \cdot)$ is defined and non-decreasing on $\mathbb{R}$. Then, for all $a > 0$ and $0 < \delta \leq \frac{a}{2}$, there exists $\varepsilon_{a,T} > 0$, such that, for every $0 < \varepsilon \leq \varepsilon_{a,T}$, the following estimate holds

$$-\delta + \phi(t, x - \delta) \leq \phi_{\varepsilon}(t, x) \leq \delta + \phi(t, x + \delta), \quad \forall x \in [-a, a], \forall t \in [0, T).$$

For the proof of this lemma see [8, Lemma 6.2].

We end up by proving the following technical lemma.

**Lemma 4.4. (Envelopes in non-decreasing case)**

Assume (2) is satisfied. Suppose that $u^0 \in L^\infty(\mathbb{R})$ and the function $u^0$ is non-decreasing for $i = 1, \ldots, d$. Let $u = (u^i)_{i=1,\ldots,d}$ be the limit of $u_{\varepsilon_n} = (u^i_{\varepsilon_n})_{i=1,\ldots,d}$ constructed in Section 3, where $u_{\varepsilon_n} = (u^i_{\varepsilon_n})_{i=1,\ldots,d}$ is the unique continuous viscosity solution of (4) given by Theorem 1.2 (i). Then, for $i = 1, \ldots, d$, we have

$$\bar{u}^i(t, x) = (u^i)^*(t, x) \quad \text{and} \quad \underline{u}^i(t, x) = u^i_\varepsilon(t, x) \quad \text{for all } (t, x) \in [0, T) \times \mathbb{R},$$

where $\bar{u}^i$ and $\underline{u}^i$ are defined in (7) and (8) respectively.

**Proof of Lemma 4.4:** We will only show the proof of the first equality, the second can be proved in a similar way.

**Step 1.** We will prove, for $i = 1, \ldots, d$, the following inequality

$$\bar{u}^i(t, x) \leq (u^i)^*(t, x).$$

Let $a > 0$, $x \in [-\frac{a}{2}, \frac{a}{2}]$ and $t \in [0, T)$. In fact, by the definition of $\bar{u}^i$, we know that there exists a sequence $(\varepsilon_m, t_{\varepsilon_m}, x_{\varepsilon_m}) \to (0, t, x)$, when $m \to +\infty$, such that

$$\bar{u}^i(t, x) = \lim_{m \to +\infty} u^i_{\varepsilon_m}(t_{\varepsilon_m}, x_{\varepsilon_m}).$$

For all $\alpha > 0$, we can state that, there exists $m_\alpha > 0$, such that, for all $m \geq m_\alpha$, we have

$$|x_{\varepsilon_m} - x| \leq \alpha \quad \text{and} \quad |t_{\varepsilon_m} - t| \leq \alpha.$$

Using Lemma 4.2, with

$$h_\alpha = \begin{cases} t - \alpha & \text{if } t > 0, \\ 0 & \text{if } t = 0, \end{cases}$$

then the inequality

$$\bar{u}^i(t, x) \leq (u^i)^*(t, x)$$

holds for all $i = 1, \ldots, d$, and for all $t \in [0, T)$, as $\varepsilon_m \to 0$. Since $u^0 \in L^\infty(\mathbb{R})$, we get

$$\bar{u}^i(t, x) \leq (u^i)^*(t, x)$$

for all $i = 1, \ldots, d$, and for all $t \in [0, T)$. This completes the proof.
we deduce that, for all \( m \geq m_\alpha \) and \( \alpha > 0 \) such that \( h_\alpha \geq 0 \),
\[
u_{\epsilon m}^i(t_{\epsilon m}, x_{\epsilon m}) \leq \sup_{|y-x_{\epsilon m}| \leq (t_{\epsilon m} - h_\alpha)\Lambda} u_{\epsilon m}^i(h_\alpha, y)
\]
\[
\leq \sup_{|y-x_{\epsilon m}| \leq 2\alpha} u_{\epsilon m}^i(h_\alpha, y)
\]
\[
\leq \sup_{|y-x| \leq \alpha(2\Lambda+1)} u_{\epsilon m}^i(h_\alpha, y).
\]
Moreover, from the maximum principle of (4) and since the initial data is non-decreasing, we know that \( u_{\epsilon m}^i \) is non-decreasing (with respect to \( x \)) and therefore, for all \( m \geq m_\alpha \),
\[
u_{\epsilon m}^i(t_{\epsilon m}, x_{\epsilon m}) \leq u_{\epsilon m}^i(h_\alpha, x + \alpha(2\Lambda + 1)).
\]
Now, as we have indicated in Section 3, since \( u_{\epsilon m}^i \) satisfies estimates (10), (11), (12) we can extract a subsequence, still denoted by \((u_{\epsilon m}^i)_{\epsilon m}\), that converges in the sense of (33) to a function \( u^i \). Since \( u_{\epsilon m}^i \) is non-decreasing (with respect to \( x \)), then for all \( t \in [0, T] \) the limit \( u^i(t, \cdot) \) can be considered non-decreasing and defined on \( \mathbb{R} \).
For all \( 0 < \alpha \leq \alpha_\alpha \), where
\[
\alpha_\alpha(t, \|c\|_{L^\infty(\mathbb{R} \times (0, T))}) = \begin{cases} \min(t, \frac{a}{2(2\Lambda+1)}) & \text{if } t > 0, \\ \frac{a}{2(2\Lambda+1)} & \text{if } t = 0, \end{cases}
\]
and using the previous inequality and Lemma 4.3, we conclude that there exists \( m_{a,T}^\alpha > 0 \), such that, for every \( m \geq m_{a,T}^\alpha \), we have
\[
u_{\epsilon m}^i(t_{\epsilon m}, x_{\epsilon m}) \leq u^i(h_\alpha, x + 2\alpha(\Lambda + 1)) + \alpha.
\]
Passing to the limit \( m \to +\infty \) and then \( \alpha \to 0 \), we obtain (42).

**Step 2.** It remains to show that
\[
(u^i)^*(t, x) \leq \tilde{w}(t, x).
\]
Consider \( a > 0, x \in [-\frac{a}{2}, \frac{a}{2}] \) and \( t \in [0, T] \). In fact, from the definition of \((u^i)^*\) we know that there exists a sequence \((t_{\epsilon m}, x_{\epsilon m}) \to (t, x)\), when \( m \to +\infty \), such that
\[
(u^i)^*(t, x) = \lim_{m \to +\infty} u^i(t_{\epsilon m}, x_{\epsilon m}).
\]
Similarly, as in Step 1, we can state that, for all \( \alpha > 0 \), there exists \( m_{\alpha} \) such that, for all \( m \geq m_{\alpha} \), we have
\[
|x_{\epsilon m} - x| \leq \alpha \quad \text{and} \quad |t_{\epsilon m} - t| \leq \alpha.
\]
However, using Lemma 4.3, we know that, for all \( 0 < \alpha \leq \frac{a}{2} \), there exists \( k_{a,T}^\alpha > 0 \) and a subsequence \( 0 < \varepsilon_{\alpha k} \leq \alpha \) such that, for every \( k \geq k_{a,T}^\alpha \),
\[
u^i(t_{\epsilon m}, x_{\epsilon m}) \leq u_{\epsilon m}^i(t_{\epsilon m}, x_{\epsilon m} + \alpha) + \alpha \leq \sup_{\varepsilon_{\alpha k} \leq \alpha, |s-t| \leq \alpha} u_{\epsilon m}^i(s, y) + \alpha.
\]
Passing to the limit \( m \to +\infty \) and then \( \alpha \to 0 \), we obtain (43). \( \square \)

In the next subsection, we will give meaning to the initial data of the discontinuous viscosity sub-solution and super-solution introduced in Theorem 1.2 (iii) and Theorem 1.3, that is reflected by the points (1)-(i) and (2)-(i) in Definition 4.1.
4.2. **Meaning of the initial data.** The purpose of this subsection is to make sense of the initial data of the viscosity sub-solution and super-solution announced in Theorem 1.2 (iii) and Theorem 1.3. We proceed in two steps. First, we treat the case of Theorem 1.2 (iii) and then the case of the Theorem 1.3.

**Step 1. (Initial data in Theorem 1.2 (iii))** We only prove the result for the sub-solution case, the super-solution case is proved analogously. Let $u_i$ be the solution of (4), constructed in Theorem 1.2 (i). We have to prove that the relaxed semi-limit $(\bar{\pi}^i)^* = \bar{\pi}^i$ satisfies (1)-(i) in Definition 4.1. It is sufficient to prove the following inequality

$$\bar{\pi}^i(0, x) \leq (u^i_0)^*(x) \quad \text{for all} \quad x \in \mathbb{R}, \ i = 1, \ldots, d. \quad (44)$$

From the definition of $\bar{\pi}^i$, we know that there exists a sequence $(\varepsilon_n, t_{\varepsilon_n}, x_{\varepsilon_n}) \to (0, 0, x)$ as $n \to +\infty$, such that

$$\bar{\pi}^i(0, x) = \lim_{n \to +\infty} u^i_{\varepsilon_n}(t_{\varepsilon_n}, x_{\varepsilon_n}).$$

Using Lemma 4.2 with $h = 0$ and $t = t_{\varepsilon_n}$, we get

$$u^i_{\varepsilon_n}(x_{\varepsilon_n}, t_{\varepsilon_n}) \leq \sup_{|y-x_{\varepsilon_n}| \leq t_{\varepsilon_n} \Lambda} u^i_{\varepsilon_n}(0, y)$$

$$\leq \sup_{|y-x_{\varepsilon_n}| \leq t_{\varepsilon_n} \Lambda} \left( \int_{\mathbb{R}} u^1_0(z) \rho_{\varepsilon_n}(y-z) \, dz \right)$$

$$\leq \sup_{|y-x_{\varepsilon_n}| \leq t_{\varepsilon_n} \Lambda} \left( \sup_{|z-y| \leq \varepsilon_n} u^1_0(z) \right),$$

where we have used in the second and the third lines the definition of the functions $u^1_0, \varepsilon_n$ in (5) and also the classical properties of the mollifiers. Furthermore, the convergence as $n \to +\infty$ of $(\varepsilon_n, t_{\varepsilon_n}, x_{\varepsilon_n})_n$ to $(0, 0, x)$ implies that for all $\alpha > 0$, there exists $n_\alpha > 0$ such that, for all $n \geq n_\alpha$, we have

$$\varepsilon_n \leq \alpha, \quad |x_{\varepsilon_n} - x| \leq \alpha \quad \text{and} \quad t_{\varepsilon_n} \leq \alpha.$$

Thus, for every $\alpha > 0$ and $n \geq n_\alpha$, we get

$$u^i_{\varepsilon_n}(t_{\varepsilon_n}, x_{\varepsilon_n}) \leq \sup_{|z-x| \leq \alpha(2+\Lambda)} u^1_0(z). \quad (45)$$

First, we pass to the limit, $n \to +\infty$, in the previous inequality to obtain

$$\bar{\pi}^i(0, x) \leq \sup_{|z-x| \leq \alpha(2+\Lambda)} u^1_0(z).$$

Then, we pass to the limit, $\alpha \to 0$, to complete the proof of (44).

**Step 2. (Initial data in Theorem 1.3)** In the framework of non-decreasing initial data, we have from (41), $\bar{\pi}^i(0, x) = (u^i)^*(0, x)$ and $u^i(0, x) = (u^i)_*(0, x)$. Taking into account what was done before in Step 1, we get that the functions $(u^i)^*$ and $(u^i)_*$ satisfy, respectively (1)-(i) and (2)-(i) in Definition 4.1.

4.3. **Proofs of the main results Theorem 1.2 (iii) and Theorem 1.3.** After giving meaning to the initial data in the previous sub-section, it remains to show that the sub-solution and super-solution introduced in Theorem 1.2 (iii) and Theorem 1.3, satisfy the points (1)-(ii) and (2)-(ii) in Definition 4.1, respectively. To do that, we need the following lemma.
Lemma 4.5. (Envelopes of the velocity)
Assume that \( \lambda^i \) is locally bounded on \((0, T) \times \mathbb{R} \times \mathbb{R}^d \) for all \( T > 0 \). Let \( \lambda^i \) be the standard regularization of the functions \( \lambda^i \) defined in (5). Noting
\[
\bar{\lambda}^i(t, x, r) = \limsup_{\varepsilon \to 0} \lambda^i(s, y, w), \quad \text{and} \quad \underline{\lambda}^i(t, x, r) = \liminf_{\varepsilon \to 0} \lambda^i(s, y, w).
\]
Then, we have
\[
\bar{\lambda}^i(t, x, r) \leq (\lambda^i)^*(t, x, r) \quad \text{and} \quad (\lambda^i)^*(t, x, r) \leq \underline{\lambda}^i(t, x, r)
\]
for all \((t, x, r) \in [0, T) \times \mathbb{R} \times \mathbb{R}^d\), where \((\lambda^i)^*\) and \((\lambda^i)^+\) are respectively the upper and lower semi-continuous envelopes of \( \lambda^i \) defined in (35).

Proof of Lemma 4.5: We only show the proof of the first inequality, the second is proved similarly. Indeed, we know that there exists a sequence \((\varepsilon_n, t_{\varepsilon_n}, x_{\varepsilon_n}, r_{\varepsilon_n}) \to (0, t, x, r)\), as \( n \to +\infty \), such that
\[
\bar{\lambda}^i(t, x, r) = \lim_{n \to +\infty} \lambda^i_n(t_{\varepsilon_n}, x_{\varepsilon_n}, r_{\varepsilon_n}).
\]
From (5), we can see that
\[
\lambda^i_n(t_{\varepsilon_n}, x_{\varepsilon_n}, r_{\varepsilon_n}) = \int_{(0, T) \times \mathbb{R} \times \mathbb{R}^d} \lambda^i(\tau, y, w) \rho^{d+2}_\varepsilon(t_{\varepsilon_n} - \tau, x_{\varepsilon_n} - y, r_{\varepsilon_n} - w) \, dyd\tau dw
\]
\[
\leq \sup_{|y - x_{\varepsilon_n}| \leq \varepsilon_n, |\tau - t_{\varepsilon_n}| \leq \varepsilon_n} \lambda^i(\tau, y, w),
\]
where we have used the fact that \( \rho^{d+2}_\varepsilon \geq 0 \) and \( \int_{\mathbb{R}^{d+2}} \rho^{d+2}_\varepsilon = 1 \). Thanks to the convergence, as \( n \to +\infty \), of \((\varepsilon_n, t_{\varepsilon_n}, x_{\varepsilon_n}, r_{\varepsilon_n})\) to \((0, t, x, r)\), we can deduce, as in (45), that for every \( \alpha > 0 \) there exists \( n_\alpha > 0 \), such that, for all \( n \geq n_\alpha \), we have
\[
\lambda^i_n(t_{\varepsilon_n}, x_{\varepsilon_n}, r_{\varepsilon_n}) \leq \sup_{|y - x| \leq 2\alpha, |\tau - t| \leq 2\alpha} \lambda^i(\tau, y, w).
\]
Now, we pass to the limit as \( n \to +\infty \) first and then as \( \alpha \to 0 \) to get \( \bar{\lambda}^i(t, x, r) \leq (\lambda^i)^*(t, x, r) \). \( \square \)

4.3.1. Proof of Theorem 1.3. We start by showing that \( u^* = ((u^i)^*, \cdots, (u^d)^*) \), satisfies (1)-(ii) in Definition 4.1. Indeed, let \( \phi \in C^4((0, T) \times \mathbb{R}) \). Suppose that, for \( i = 1, \cdots, d \), the function \( (u^i)^* - \phi \) attains its local maximum at \((t_0, x_0) \in (0, T) \times \mathbb{R}\). According to Lemma 4.4, \((t_0, x_0)\) is also local maximum of \( \bar{\pi}^i - \phi \). Then, \((t_0, x_0)\) is strict local maximum of \( \bar{\pi}^i - \phi \), where \( \bar{\phi}(t, x) = \phi(t, x) + |t - t_0|^2 + |x - x_0|^2 \). By a usual technique used in the theory of viscosity solutions (see Barles [3, Lemma 4.2]), we can say that there exists a subsequence \((\varepsilon^i_m, t_{\varepsilon^i_m}, x_{\varepsilon^i_m}) \to (0, t_0, x_0)\) when \( m \to +\infty \), such that \((t_{\varepsilon^i_m}, x_{\varepsilon^i_m})\) is local maximum of \( u_{\varepsilon^i_m}^j - \phi \) and
\[
\bar{\pi}^j(t_0, x_0) = \lim_{m \to +\infty} u_{\varepsilon^i_m}^j(t_{\varepsilon^i_m}, x_{\varepsilon^i_m}).
\]
Moreover, from Theorem 1.2 (i), we know that \( u_{\varepsilon^i_m} = (u_{\varepsilon^i_m}^j)_{j=1,\cdots,d} \) is a continuous viscosity solution of system (4), thus
\[
\begin{align*}
\partial_\tau \bar{\phi}(t_{\varepsilon^i_m}, x_{\varepsilon^i_m}) &- \lambda^i_{\varepsilon_m}(t_{\varepsilon^i_m}, x_{\varepsilon^i_m}, v_{\varepsilon^i_m}^1(t_{\varepsilon^i_m}, x_{\varepsilon^i_m}), \cdots, v_{\varepsilon^i_m}^d(t_{\varepsilon^i_m}, x_{\varepsilon^i_m})) |\partial_\tau \bar{\phi}(t_{\varepsilon^i_m}, x_{\varepsilon^i_m})| &\leq 0,
\end{align*}
\]
where
\[ v^j(t,x) = \rho^j \ast u^j(t,\cdot)(x) \quad \text{for} \quad j = 1, \cdots, d. \]

It’s easy to verify that \( v^i_{\varepsilon_m} \) is non-decreasing in the case of non-decreasing solution and therefore using assumption (9), we obtain
\[
\partial_t \tilde{\phi}(t_{\varepsilon_m}, x_{\varepsilon_m}) - \lambda^i_{\varepsilon_m}(t_{\varepsilon_m}, x_{\varepsilon_m}, v^i_{\varepsilon_m}(t_{\varepsilon_m}, x_{\varepsilon_m}), \cdots, v^d_{\varepsilon_m}(t_{\varepsilon_m}, x_{\varepsilon_m})) \partial_x \phi(t_{\varepsilon_m}, x_{\varepsilon_m}) \leq 0.
\]

By the definition of \( v^i_{\varepsilon_m} \), we can also verify that in the case of non-decreasing solution we have
\[
u^i_{\varepsilon_m}(t_{\varepsilon_m}, x_{\varepsilon_m}) \leq v^i_{\varepsilon_m}(t_{\varepsilon_m}, x_{\varepsilon_m} + \varepsilon_m) \leq u^i_{\varepsilon_m}(t_{\varepsilon_m}, x_{\varepsilon_m} + 2\varepsilon_m). \tag{46}
\]

Since \( v^j \) are uniformly bounded for \( j = 1, \cdots, d \), we can extract a subsequence (independent on \( j \)), still noted \( \varepsilon_m \), such that
\[
\lim_{m \to +\infty} v^j_{\varepsilon_m}(t_{\varepsilon_m}, x_{\varepsilon_m}) = r^j \quad \text{for} \quad j \neq i,
\]
\[
\lim_{m \to +\infty} v^i_{\varepsilon_m}(t_{\varepsilon_m}, x_{\varepsilon_m} + \varepsilon_m) = r^i = \mathfrak{v}^i(t_0, x_0) = (u^i)^*(t_0, x_0),
\]

where we have used inequality (46) and Lemma 4.4 in the second line. Now, passing to the inferior limit as \( m \to +\infty \) in the previous inequality satisfied by \( \tilde{\phi} \), we get
\[
\partial_t \phi(t_0, x_0) - \mathfrak{v}^i(t_0, x_0, r^1, \cdots, r^d) |\partial_x \phi(t_0, x_0)| \leq 0 \quad \text{with} \quad r^i = (u^i)^*(t_0, x_0).
\]

Which proves, using Lemma 4.5, that
\[
\min \{ \partial_t \phi(t_0, x_0) - (\lambda^i)^*(t_0, x_0, r) |\partial_x \phi(t_0, x_0)| : r \in U(t_0, x_0), r^i = (u^i)^*(t_0, x_0) \} \leq 0.
\]

and therefore \( u^* = (u^1)^*, \cdots, (u^d)^* \) is viscosity sub-solution of (1). Similarly, we can verify that \( u_* \) satisfies (37).

\[\square\]

4.3.2. Proof of Theorem 1.2 (iii). We will prove that \( \mathfrak{v} = (\mathfrak{v}^1, \cdots, \mathfrak{v}^d) = (\mathfrak{v}^i)^* \) satisfies (1)-(ii) in Definition 4.1. Indeed, let \( \phi \in C^1((0,T) \times \mathbb{R}) \) and suppose that for \( i = 1, \cdots, d \) the function \( \mathfrak{v}^i - \phi \) attains its local maximum at \((t_0, x_0) \in (0, T) \times \mathbb{R}\). As in the proof of Theorem 1.3 we can say that there exists a subsequence \( (\varepsilon_m, t_{\varepsilon_m}, x_{\varepsilon_m}) \to (0, t_0, x_0) \) when \( m \to +\infty \), such that \( (t_{\varepsilon_m}, x_{\varepsilon_m}) \) is local maximum of \( u^i_{\varepsilon_m} - \tilde{\phi} \) and
\[
\mathfrak{v}^i(t_0, x_0) = \lim_{m \to +\infty} u^i_{\varepsilon_m}(t_{\varepsilon_m}, x_{\varepsilon_m}).
\]

Thus
\[
\partial_t \tilde{\phi}(t_{\varepsilon_m}, x_{\varepsilon_m}) - \lambda^i_{\varepsilon_m}(t_{\varepsilon_m}, x_{\varepsilon_m}, v^i_{\varepsilon_m}(t_{\varepsilon_m}, x_{\varepsilon_m}), \cdots, v^d_{\varepsilon_m}(t_{\varepsilon_m}, x_{\varepsilon_m})) \partial_x \phi(t_{\varepsilon_m}, x_{\varepsilon_m}) \leq 0.
\]

Using the fact that \( \rho^i_{\varepsilon_m} \geq 0, \int_{\mathbb{R}} \rho^i_{\varepsilon_m} = 1 \) and the convergence, as \( m \to +\infty \), of \( (\varepsilon_m, t_{\varepsilon_m}, x_{\varepsilon_m}) \) to \( (0, t, x) \), we can deduce, as in Lemma 4.5, that for every \( \alpha > 0 \) there exists \( m_\alpha > 0 \), such that, for all \( m \geq m_\alpha \), we have
\[
v^i_{\varepsilon_m}(t_{\varepsilon_m}, x_{\varepsilon_m}) \leq \sup_{\varepsilon_k \leq \alpha} u^i_{\varepsilon_k}(\tau, y) := u^i_{\alpha}(t_0, x_0),
\]
which implies using assumption (9), that
\[
\partial_t \phi(t_{e_{m}^i}, x_{e_{m}^i}) - \lambda^i_{e_{m}^i}(t_{e_{m}^i}, x_{e_{m}^i}, v_{e_{m}^i}^i(t_{e_{m}^i}, x_{e_{m}^i}), \ldots, u^{i, \alpha}(t_0, x_0) \cdots,
\]
\[
v_{e_{m}^i}^i(t_{e_{m}^i}, x_{e_{m}^i})) \mid \partial_r \phi(t_{e_{m}^i}, x_{e_{m}^i}) \mid \leq 0.
\]

Now, we proceed as in Theorem 1.3, passing to the limit as \( m \to +\infty \) first and then as \( \alpha \to 0 \) we get
\[
\partial_t \phi(t_0, x_0) - \lambda^i(t_0, x_0, r^1, \ldots, r^i, \ldots, r^d) \mid \partial_x \phi(t_0, x_0) \mid \leq 0 \quad \text{with} \quad r^i = \overline{u}(t_0, x_0).
\]

This proves, using Lemma 4.5, that
\[
\min \left\{ \partial_t \phi(t_0, x_0) - (\lambda^i)^r(t_0, x_0, r) \mid \partial_x \phi(t_0, x_0) \mid : r \in \overline{U}(t_0, x_0), r^i = \overline{u}(t_0, x_0) \right\} \leq 0.
\]

and therefore \( \overline{u} = (\overline{u}^1, \ldots, \overline{u}^d) \) is viscosity sub-solution of (1). In the same way, we prove that \( \underline{u} = (\underline{u}^1, \ldots, \underline{u}^d) \) is viscosity super-solution of (1), under assumption (9). \( \square \)

REFERENCES

[1] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variations and Free Discontinuity Problems*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2000.

[2] G. Barles, Discontinuous viscosity solutions of first-order Hamilton-Jacobi equations: A guided visit, *Nonlinear Anal.*, 20 (1993), 1123–1134.

[3] G. Barles, *Solutions de Viscosité Des Équations de Hamilton-Jacobi*, vol. 17 of Mathématiques et Applications (Berlin), Springer-Verlag, Paris, 1994.

[4] G. Barles and B. Perthame, Exit time problems in optimal control and vanishing viscosity method, *SIAM J. Control Optim.*, 26 (1988), 1133–1148.

[5] G. Barles and B. Perthame, Comparison principle for Dirichlet-type Hamilton-Jacobi equations and singular perturbations of degenerated elliptic equations, *Appl. Math. Optim.*, 21 (1990), 21–44.

[6] G. Barles, H. M. Soner and P. E. Souganidis, Front propagation and phase field theory, *SIAM J. Control Optim.*, 31 (1993), 439–496.

[7] S. Bianchini and A. Bressan, Vanishing viscosity solutions of nonlinear hyperbolic systems, *Ann. Math.*, 161 (2005), 223–342.

[8] R. Boudjerada and A. El Hajj, Global existence results for eikonal equation with BV initial data, *Nonlinear Differ. Equ. Appl.*, 22 (2015), 947–978.

[9] M. G. Crandall, H. Ishii and P.-L. Lions, User’s guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc. (N.S.)*, 27 (1992), 1–67.

[10] M. G. Crandall and P.-L. Lions, Viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.*, 277 (1983), 1–42.

[11] R. J. DiPerna, Convergence of approximate solutions to conservation laws, *Arch. Ration. Mech. Anal.*, 82 (1983), 27–70.

[12] R. J. DiPerna, Compensated compactness and general systems of conservation laws, *Trans. Amer. Math. Soc.*, 292 (1985), 383–420.

[13] A. El Hajj and N. Forcadel, A convergent scheme for a non-local coupled system modelling dislocation densities dynamics, *Math. Comp.*, 77 (2008), 789–812.

[14] A. El Hajj, H. Ibrahim and V. Rizik, Global BV solution for a non-local coupled system modeling the dynamics of dislocation densities, *J. Differential Equations*, 264 (2018), 1750–1785.

[15] A. El Hajj and R. Monneau, Uniqueness results for diagonal hyperbolic systems with large and monotone data, *J. Hyper. Differ. Equ.*, 10 (2013), 461–494.

[16] A. El Hajj and R. Monneau, Global continuous solutions for diagonal hyperbolic systems with large and monotone data, *J. Hyper. Differ. Equ.*, 7 (2010), 139–164.

[17] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Commun. Pure Appl. Math.*, 18 (1965), 697–715.
[18] H. Ishii, Perron’s method for monotone systems of second-order elliptic partial differential equations, *Differential Integral Equations*, **5** (1992), 1–24.

[19] H. Ishii and S. Koike, Viscosity solution for monotone systems of second-order elliptic PDEs, *Comm. Partial Differential Equations*, **16** (1991), 1095–1128.

[20] P. D. Lax, Hyperbolic systems of conservation laws and the mathematical theory of shock waves, *CBMS Regional Conference Series in Mathematics*, Vol. **11** (SIAM, Philadelphia, 1973).

[21] P. LeFloch, Entropy weak solutions to nonlinear hyperbolic systems under nonconservative form, *Commun. Partial Differential Equations*, **13** (1988), 669–727.

[22] P. LeFloch and T.-P. Liu, Existence theory for nonlinear hyperbolic systems in nonconservative form, *Forum Math.*, **5** (1993), 261–280.

[23] O. Ley, Lower-bound gradient estimates for first-order Hamilton-Jacobi equations and applications to the regularity of propagating fronts, *Adv. Differential Equations*, **6** (2001), 547–576.

[24] J. Simon, Compacts sets in the space $L^p(0; T; B)$, *Ann. Mat. Pur. Appl.*, **146** (1987), 65–96.

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