Tachyonic instabilities in 2 + 1 dimensional Yang–Mills theory and its connection to number theory

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Abstract
We consider the 2 + 1 dimensional Yang–Mills theory with gauge group SU(N) on a flat 2-torus under twisted boundary conditions. We study the possibility of phase transitions (tachyonic instabilities) when N and the volume vary and certain chromomagnetic flux associated to the topology of the bundle can be adjusted. Under natural assumptions about how to match the perturbative regime and the expected confinement, we prove that the absence of tachyonic instabilities is related to some problems in number theory, namely the Diophantine approximation of irreducible fractions by other fractions of smaller denominator.

Keywords: large N gauge theories, flux through torus, Diophantine approximation, volume independence, twisted boundary conditions

(Some figures may appear in colour only in the online journal)

1. Introduction

Yang–Mills field theory plays a fundamental role in our understanding of elementary particles and their interactions. This theory represents a challenge to physicists and mathematicians since, under a relatively simple formulation, it is believed to encompass a large number of phenomena which are yet to be fully mastered and proved. Many of these phenomena escape control of the standard perturbative techniques which physicists use to perform calculations in quantum field theory. One is then faced with the rather limited set of procedures to address this type of problems. The most important of these follow from lattice gauge theories. This amounts to a precise definition of the corresponding path integral formulation of these fields,
and provides also a source for numerical calculational techniques of some of the most important non-perturbative quantities involving these fields. Most notably we should cite the string tension \( \sigma \) which determines the long-distance behaviour of the force induced between two sources which couple to these fields. In the case of electrodynamics the force falls at large distances like the inverse distance \( d \) square with a coefficient given by the electric charge of the source \( F \sim e/d^2 \). In the case of non-abelian Yang–Mills fields, the force tends to a constant \( F \sim \sigma \). The phenomenon is believed to explain the impossibility of separating the quarks that make up a proton or a meson, and hence called quark confinement.

On the mathematical side, Yang–Mills fields (non-abelian gauge fields) are connections on vector bundles or their associated principal fiber bundles. The classical and quantum dynamics of these fields is determined by a real valued functional on the space of connections called the Yang–Mills functional and given by a volume integral

\[
Y(A_\mu) = \frac{1}{2g^2} \int \text{Tr}(F \wedge *F)
\]

where \( g \) is the coupling constant and \( F \) the bundle curvature 2-form. Typically one considers the bundle group to be a direct product of various SU\((N)\) and the base space \( \mathbb{R} \times \mathbb{R}^3 \) with the Minkowski metric. The most physically interesting case being that of quantum chromodynamics involving an SU\((3)\) gauge group in \( 3 + 1 \) dimensions, and generally accepted to be the fundamental force underlying the strong interactions of elementary particles. However, in the Physics literature more general cases are considered, either because they represent a simplification of the model or because they occur within beyond the standard model scenarios, early universe studies, etc.

The present paper relates to problems in which many of these generalizations are simultaneously employed. We will be dealing with Yang–Mills fields with SU\((N)\) gauge group with large \( N \), and base manifold \( \mathbb{R} \times T^2 \) and euclidean metric (\( T^2 \) stands for the two-dimensional flat torus). Descending down to three space dimensions from the canonical \( 3 + 1 \) Minkowski space, produces an important simplification. However, a full understanding of this case is not available and could serve as a necessary first step in achieving the same in four dimensions. The use of the rank of the matrices \( N \) as a variable parameter in the game could also be an important source of information. Indeed, when taking \( N \) to infinity crucial simplifications occur, though the theory still eludes complete understanding. The limit lies also at the crux of the connection of quantum field theory with string theory. Another ingredient in the situation that we are studying is the formulation of the theory on a two dimensional compact manifold. This brings in new variables into the game. The non-trivial topology of the base manifold is inherited by the bundle. This might turn out to be a bonus since the topology can be used as a probe of the dynamics.

The different ingredients described in the previous paragraph are intimately tied to one another. The notion of volume independence [1] introduced the idea that under certain circumstances, in the \( N \rightarrow \infty \) limit the physical quantities become independent of the torus size \( l \). Thus, compactness, group rank and bundle topology are connected. Understanding the way in which this happens was the goal of the work published in [2, 3]. The work employed both perturbative as non-perturbative lattice techniques to achieve this goal. This study focused on observables which are the eigenvalues of the Hamiltonian of the system. A conclusion was that the main scale of the problem is a combination of the different parameters in the game: \( x = g^2N^2I/(4\pi) \). Thus when taking the limit of large \( N \) at \( \lambda = g^2N \) fixed, the size \( l \) only appears in this combination. The topology of the gauge fields, as embodied in the boundary conditions for the gauge fields, plays an important role in this behaviour. ‘t Hooft [4] realized that, since Yang–Mills fields transform with the adjoint representation of the gauge group
SU(N), one can consider certain topological sectors in the space of gauge fields, which he called *twist sectors*. These sectors are characterized by an integer parameter $k$ defined modulo $N$, representing a certain discrete flux traversing the 2-dimensional torus. The value of this integer plays an important role in the way in which the large $N$ limit is approached. In particular, volume independence is lost if $k = 0$ [5]. The range of values of $k$ at which this property is preserved is still subject of investigation and was one of the main goals of [2].

One can calculate the spectrum of the Hamiltonian to a given order in perturbation theory. This expansion is expected to be a good approximation at small $x$. If $k$ is coprime with $N$, the theory has a discrete spectrum with a mass gap: a unique ground state separated from excited states with energies proportional to $1/x$. The big question is whether this separation is preserved as $x$ grows. The answer becomes doubtful since perturbation theory predicts that the gap gets reduced as $x$ grows. Indeed, taking next-to-leading order perturbation theory at face value it does indeed predict that the gap shrinks to zero at a finite value of $x$. If this happens we could describe this phenomenon as a *tachyonic instability*, representing a certain phase transition in which the dynamical system behaves qualitatively distinct in different regions of $x$. Does this transition occur? This is the main physics problem to which our present work is connected. The conjecture, made by one of the authors in the context of the 4 dimensional field theory [6], is that there is an appropriate choice of flux $k = k(N)$ for which this transition can be avoided at each value of $N$. This result is confirmed by numerical simulations [2] at several values of $N$. The analysis of these numerical cases provides an understanding of how the instability is avoided. This puts us in the right setting to formulate the question whether the result would continue to hold at arbitrary large values of $N$. It is at this stage that the problem can be translated into a problem in number theory.

In the next section we will explain the essentials of the physics problem. For a full explanation of the context we remit the reader to [2, 3] and references therein. Then we will reformulate the problem in a more adequate mathematical fashion, which will allow us to use known results in number theory to address it. The paper closes with a summary of the conclusions and several comments about possible extensions and improvements.

In writing the paper we have kept in mind the possibility of a wide spectrum of readers, according to the interdisciplinary nature of its contents. Furthermore, applications of number theory to active physical problems are scarce. The work itself serves to exemplify the potential fruitfulness of these cross collaborations.

2. Statement of the problem

In this section we will formulate in a more precise fashion what the physical problem that we are addressing is about. As mentioned earlier we are considering SU(N) Yang–Mills fields living in a 2-dimensional torus times the real line $\mathbb{R} \times T^2$. For simplicity we take this torus to be a square torus with euclidean metric and having the same length $l$ in both directions. The non-periodic direction can be considered the euclidean time direction. At the classical level gauge fields are connections in an SU(N) vector bundle. The bundle is given in terms of patches and transition matrices. As is typical in the Physics literature we will work with a single coordinate patch covering the full torus and a trivialization of the bundle. The gauge fields are then specified by the connection 1-form in this patch: the vector potential $A_\mu(x_0, x_1, x_2)$, which

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3 In a recent paper [7] we have learned that there are also interesting number theoretical connections when one studies the self-energy in the context of the $\phi^4$ theory on the noncommutative torus. Our problem is related to that of non-commutative field theory as explained in [2, 3], although at this stage we do not know if that relation also affects the role of number theory.
takes values in the Lie algebra of the group. As usual, the time coordinate is labeled with 0 and placed in the first place. The non-trivial character of the bundle arises through the boundary conditions imposed on the vector potential:

\[ A_\mu(x_0, x_1, x_2 + l) = \Gamma_1 A_\mu(x_0, x_1, x_2) \Gamma_1^\dagger \] (2.1)

\[ A_\mu(x_0, x_1, x_2 + l) = \Gamma_2 A_\mu(x_0, x_1, x_2) \Gamma_2^\dagger \] (2.2)

where \( \Gamma_i \) are SU(N) matrices satisfying

\[ \Gamma_1 \Gamma_2 = e^{2\pi i k/N} \Gamma_2 \Gamma_1 \] (2.3)

with \( k \) an integer defined modulo \( N \). These relations derive from the most general twisted boundary conditions satisfied by the transition matrices \( \Omega_1, \Omega_2 \) (see [4, 8])

\[ \Omega_j(x + L\vec{e})\Omega_l(x) = e^{2\pi i n_{jl}/N} \Omega_l(x + L\vec{e})\Omega_j(x) \quad \text{where} \quad n_{jl} = -n_{lj}. \] (2.4)

We will restrict ourselves to the case in which \( k = n_{12} \) is coprime with \( N \). The different values of \( k \) label topologically inequivalent sectors, called twist sectors by 't Hooft. He also explained the physical interpretation of this quantity as a certain discrete chromomagnetic flux. The reader is addressed to the appropriate references [4, 8, 9, 10] for a more detailed explanation of the physical and mathematical interpretation of the twist. The solutions \( \Omega_1(x) = \Gamma_1, \Omega_2(x) = \Gamma_2 \) of equation (2.4) with \( \Gamma_1 \) and \( \Gamma_2 \) constant matrices as in equation (2.3) are called twist eaters (see [11] and [12] for their general form).

Our interest is focused in the quantum version of this field theory. According to the postulates of quantum physics the set of (gauge-invariant) states of the system are given by rays in a Hilbert space \( \mathcal{H} \). Our main interest is the study of the spectrum of the Hamiltonian operator \( H \), which acts on this space. In the alternative path-integral quantization the spectrum of the Hamiltonian determines the exponential fall off of correlation functions of gauge invariant observables at different euclidean times

\[ \langle O(0)|O'(x_0)\rangle \rightarrow \langle O(0)|O'(0)\rangle + Ae^{-x_0E} + \ldots \] (2.5)

where \( E \) is an eigenvalue of the Hamiltonian. It is worth mentioning here that it is enough to consider as our gauge invariant observables the algebra generated by spatial Wilson loops: the trace of the parallel transport matrices along a closed path \( \gamma \) on the 2-torus.

As first understood by 't Hooft [4], the formulation of the theory on the torus introduces a new ingredient in the game: center symmetry. This is a set of transformations, forming a group isomorphic to \( \mathbb{Z}_N^2 \), which commute with the Hamiltonian. Under these transformations the Wilson loops associated to paths with non-zero winding transform multiplicatively. The irreducible representations of this group are labelled by a two dimensional vector of integers defined modulo \( N \): \( \vec{e} = (e_1, e_2) \). Hence, the Hilbert space can be decomposed into a direct sum of spaces associated to each irreducible representation:

\[ \mathcal{H} = \bigoplus_{\vec{e}} \mathcal{H}_{\vec{e}}. \]

The integer-vector \( \vec{e} \) is called the (chromo-)electric flux 2-vector. It is clear that the Hamiltonian operator does not change the value of the electric flux. We may then write

\[ H_{\vec{e}} : \mathcal{H}_{\vec{e}} \rightarrow \mathcal{H}_{\vec{e}} \]

and our interest is centered on the study of the spectrum of the Hamiltonian \( H_{\vec{e}} \) in each of these sectors.
After this introduction we are now in position of explaining what is the main goal of our physics program. We would like to compute and interpret the lowest energy levels of the Hamiltonians \( H_{\varepsilon} \). These are given by the following real quantities: \( E_p(\varepsilon, \lambda, l, N, k) \), where \( p \) is a non-negative integer index listing the eigenvalues in each sector in increasing order. A very important part of our program, connected with the interpretation goal, is that of determining the dependence of the energies \( E_p \) on the two real parameters \( \lambda \) and \( l \) and the two discrete parameters \( k \) and \( N \).

By definition the ground state in the \( \varepsilon = 0 \) sector is called the vacuum and its energy is set to 0. This is part of the renormalization program, necessary to make sense of our computational program. The lowest eigenvalue of \( H_{\varepsilon} \) in the remaining sectors is called the energy of the electric flux sectors. Its value depends on all the parameters that describe the system \( E_0(\varepsilon, \lambda, l, N, k) \), where \( \lambda \) is ’t Hooft coupling constant: \( \lambda = g^2 N \). It is important to realize that in \( 2 + 1 \) dimensions \( \lambda \) has dimensions of energy. Thus, if we apply dimensional analysis one must have

\[
E_0(\varepsilon, \lambda, l, N, k) = \lambda \mathcal{E}(\varepsilon, \lambda, l, N, k),
\]

Our goal is to determine the function \( \mathcal{E} \). In addition, we are also interested in the first excited energy (the second lowest eigenvalue) of \( H_{\varepsilon=0} \): \( E_1(0, \lambda, l, N, k) \). This is called the lowest glueball mass.

There is only one way in which physicists know how to compute the energy levels without making uncontrolled assumptions: lattice gauge theory [13]. This, so-called first principles calculation, is based on formulating the problem as that of computing expectation values in a probability measure over the space of a finite number of SU(\( N \)) matrices. The philosophy is to discretize the \( \mathbb{R} \times T_2 \) space into a cubic lattice in which the points are separated by a distance \( a \): the lattice spacing. The number of points in the spatial directions is given by \( L = \frac{\lambda}{la} \). In principle there are infinite points after discretizing the non-compact directions, however the energy levels can be determined with arbitrary precision even if we consider this direction finite as well, provided its length is large enough. The quantum field theory results, including our desired energies, are obtained after taking the limit \( a \rightarrow 0 \). This limit has to be accompanied by an appropriate tuning of the probability density, a process known as renormalization. At the end of the day physical results are obtained which do not depend on the details of the discretization procedure. This statement has not been proved rigorously, so that from the mathematical standpoint can be considered a conjecture. However, there is a large number of results of all types that gives credibility to the conjecture. Indeed, proving this conjecture can be considered a possible ingredient in solving one of the millennium problems stated by the Clay foundation.

In previous papers, one of the present authors and collaborators used lattice gauge theory to compute the \( E_0(\varepsilon, \lambda, l, N, k) \) quantities for certain values of the arguments [2, 3]. The study of the glueball masses \( E_1(0, \lambda, l, N, k) \) is currently underway [14]. Remember, however, that our goal is also to interpret and understand the results, and for that purpose the dependence on the different parameters is crucial.

As stated earlier the dependence on the size of the torus comes in the combination \( l \lambda \), which is a dimensionless parameter. A well-known calculational technique in quantum field theory is perturbation theory, which provides an (asymptotic) expansion of the physical quantities in powers of \( \lambda \). Obviously the expansion becomes a better approximation for small \( l \lambda \). The leading term is very simple to obtain, since at that order the system behaves like a quantum system of free massless bosons: the gluons. Each gluon has an energy given by \( |\vec{p}| \), the modulus of its...
momentum $\vec{p} = (p_1, p_2)$. Because of the boundary conditions the momentum of the gluons is quantized as follows:

$$p_j = \frac{2\pi n_j}{Nl}$$

(2.6)

where the $n_j$ are integers. There is a connection between these integers and the electric flux number quantum as follows:

$$\vec{n} = k\vec{e}_\perp + N\vec{m}$$

(2.7)

where $k$ is the magnetic flux, $\vec{e}_\perp = (-e_2, e_1)$ and $\vec{m}$ is a 2-vector of integers. Hence, the minimal energy is given by $2\pi/(Nl)$ and corresponds to the electric flux sectors $(k, 0)$ and $(0, k)$, where $kk = \pm 1 \mod N$ with $|k| \leq N/2$. If we introduce the common notation in number theory $||x||$ to denote the distance of the real number $x$ to the nearest integer, then $N||k/N||$ is the representative of the class of $k$ or $-k$ modulo $N$ that fits the interval $[0, N/2]$. With this notation, for a generic value of the electric flux $\vec{e}$, the minimal energy is given by

$$\frac{2\pi}{Nl} \sqrt{N^2||ke_1/N||^2 + N^2||ke_2/N||^2 + ||ke_1/N||^2} \equiv \frac{2\pi}{T} \frac{2}{N} ||\vec{e}/N||_2$$

(2.8)

where in the last equality we have introduced a new rather natural notation.

For large values of $l\lambda$ we enter a non-perturbative domain where we lack calculational techniques other than the lattice. Nevertheless, this is also the regime of non-compact space-times ($l \to \infty$) in which most physical results concentrate. The concept of quark confinement leads to the expectation that the energies of the non-zero fluxes grow linearly with the torus length $l$. Combining this with dimensional analysis, it leads to

$$\mathcal{E}(\vec{e}, \lambda, N, k) \to \sigma(\vec{e}, N)\lambda l$$

(2.9)

where we used the idea that for large sizes the magnetic flux value $k$ (entering only in the boundary conditions) becomes irrelevant. The quantity $\sigma(\vec{e}, N)$ is the well-known string tension measured in $\lambda^2$ units. The name reflects the interpretation of the phenomenon, as the formation of a chromo-electric flux tube (a thick string) carrying a certain energy per unit length. This phenomenon being the electric-magnetic dual of the Meissner effect in superconductors.

Now the problem is focused upon understanding how the energies evolve from the $1/l$ behaviour of perturbation theory to the linear $l$ dependence typical of large volumes. The $N$ dependence could serve to clarify this transition. One crucial question would be whether in the large $N$ limit the energies become discontinuous or non-analytic in $l\lambda$. This type of situation is labelled as a large $N$ phase transition in the literature. The purpose of the present work is precisely that of advancing in the resolution of this question.

When studying the large $N$ behaviour of the system a crucial ingredient is that of investigating the phenomenon of volume independence [1]. This means that in the limit of large $N$ the physical results become independent on the torus size $l$. Whether this is true and how this situation is approached is still a subject of debate. From early times it is known that using twisted boundary conditions is crucial for the implementation of this idea [15, 16]. In the 4 dimensional euclidean setting it has been argued recently [6] that the phenomenon indeed takes place provided the magnetic flux $k$ and its congruency inverse $\bar{k}$ are scaled linearly with $\sqrt{N}$, as $N$ grows. Our previous work on the ground state energies of the non-vanishing electric flux sectors of the $2+1$ dimensional system provided numerical evidence that a similar situation (with $N$ replacing $\sqrt{N}$) also occurs here [2, 3].
What are the reasons suggesting the possibility of a large $N$ phase transition? These come mainly from the next-to-leading order calculation of the energies in perturbation theory. The contribution turns out to be negative and physically can be interpreted as a self-energy for the gluon. Following the notation of [2] we will write the perturbative contribution as follows:

$$E^2(\vec{e}, \lambda, N, k) = \frac{\phi_0(\vec{e}, N, k)}{4x^2} - \frac{G(\vec{e}, N)}{x} + \ldots$$  \hspace{1cm} (2.10)

where

$$x = \frac{\lambda N}{4\pi}$$  \hspace{1cm} (2.11)

$$\phi_0 = N^2(||ke_1/N||^2 + ||ke_2/N||^2) = N^2||k\vec{e}/N||^2$$  \hspace{1cm} (2.12)

and the self-energy is given by

$$G(\vec{e}, N) = -\frac{1}{16\pi^2} \int_0^\infty \frac{dt}{\sqrt{t}} (\theta_3^2(0, it) - \theta_3(e_1/N, it)\theta_3(e_2/N, it) - \frac{1}{t})$$  \hspace{1cm} (2.13)

where $\theta_3(z, \tau)$ is Jacobi theta function, defined by the series

$$\theta_3(z, \tau) = \sum_{n=-\infty}^{\infty} \exp(\pi in^2\tau + 2\piinz).$$  \hspace{1cm} (2.14)

For the numerical evaluation of the self-energy one can use this series when $t$ is large. For small values it is advantageous to use the modular relation

$$\theta_3(z, \tau) = (-i\tau)^{1/2}e^{\pi i z^2/\tau}\theta_3(z, \tau)$$  \hspace{1cm} (2.15)

that follows easily from Poisson summation formula [17].

We recall that the first term on the right-hand side in equation (2.10) is the momentum square of the gluon in $\lambda$ units. Thus, by analogy with the relativistic dispersion relation, the remaining terms can be thought of as giving the mass square. If $G$ is positive this would mean a negative mass square, a situation often described as tachyonic. Furthermore, if we neglect all the higher corrections appearing as dots in the right-hand side of equation (2.10), then for $x \geq \bar{x}$ given by

$$\bar{x} = \frac{\phi_0(\vec{e}, N, k)}{4G(\vec{e}, N)}$$  \hspace{1cm} (2.16)

the energy square becomes negative. This situation makes no sense and it signals the breakdown of perturbation theory and the system entering a different phase in which the vacuum has a non-vanishing condensate. This is precisely the situation that is described as tachyonic instability. There is numerical evidence that this instability does indeed take place for some cases of $\vec{e}$, $N$ and $k$. Our main goal can now be stated clearly: we want to see if for every $N$ there are choices of $k$ for which the instability can be avoided for all $\vec{e}$. Previous numerical work obtained at certain values of $(k, N)$ suggests that this is the case. However, we would like to know if the situation survives the large $N$ limit.

To analyze this problem further, let us examine the previous arguments suggesting the instability. The weak point is the fact that we neglected corrections which have higher positive powers of $x$ and dominate for large $x$. There is one case, however, in which the argument
in favour of instabilities becomes compelling and that is when $\bar{x}$ becomes very small as $N$ goes to infinity. This is actually happening in several cases because $G(\vec{e},N)$ has indeed a pole singularity

$$G(\vec{e},N) \sim \frac{1}{16\pi^2\|\vec{e}/N\|^2} + R(\vec{e}/N)$$

(2.17)

where $R$ is a positive definite regular function which vanishes at the origin and numerically rather small. Indeed, it is bounded as follows

$$0.01 < \frac{R(\vec{e}/N)}{\|\vec{e}/N\|^2} < 0.02.$$  

(2.18)

Hence, for the sake of studying the possible instability it is a good approximation to consider only the pole part in estimating $\bar{x}$:

$$\bar{x} \sim 4\pi^2N^2 \frac{\|ke/N\|^2}{\|\vec{e}/N\|^2}.$$  

(2.19)

Now given the definition it is easy to see that the right-hand side is bounded from below by

$$4\pi^2N^2 \left(\|ke_1/N\|^2 \|e_1/N\| + \|ke_2/N\|^2 \|e_2/N\|\right).$$

(2.20)

Hence, if we want to find a lower bound of $\bar{x}$ for all values of $\vec{e}$ it is enough to find the minimum for electric fluxes of the form ($\vec{e},0$). Our goal then is that of tuning $k$ in order to maximize this minimum. The result will be called $a(N,2)$

$$a(N,2) = \max_k \min_{\vec{e}} N^2\|ke/N\|^2\|e/N\|.$$  

(2.21)

This, sets our first mathematical goal, that of studying whether $a(N,2)$ can be bounded from below uniformly as $N$ runs over all integers beyond a certain one.

Obtaining such a bound eliminates the argument in favour of a tachyonic instability in the perturbative region. However, it could still happen that at finite values of $x$ one of the minimum flux energies crosses zero. To eliminate this possibility we need to have some control about the possible additional contributions for higher values of $x$. We have already commented about the behaviour at large $x$ arising from the phenomenon of quark confinement. Indeed, it is also possible to describe the expected subleading terms that govern the approach towards the confinement regime. These follow from the effective string picture description of the flux tube formation. The leading correction has a universal character and is often referred as Lüscher term [18, 19]. Thus, using the same notation as for the perturbative part we can express the expected behaviour of the flux energies for large values of $x$

$$E^2(\vec{e},\lambda,N,k) = \frac{1}{4} \tau^2 \tilde{\sigma}^2(\vec{e}/N)\chi^2 - \frac{\tau}{24} \chi(\vec{e},N,k) + \ldots$$

(2.22)

where we have parameterized the string tension as follows

$$\tilde{\sigma}(\vec{e},N) = \frac{\tau N}{8\pi} \tilde{\sigma}(\vec{e}/N)$$

(2.23)

where $\tau$ is a numerical coefficient whose value has been determined numerically to be close to 1, as predicted by Nair [20] (see also [21, 22]). The function $\tilde{\sigma}(\vec{e}/N)$ describes what is known as the $k$-string spectrum and for small value of the argument goes like $||\vec{e}/N||$. The correction term is $x$-independent. The dependence on $\vec{e}$ is controlled by the function $\chi(\vec{e},N,k)$ which
goes to 1 as $\|\vec{e}/N\|_2$ goes to zero. The normalization is fixed by the numerical value of the Lüscher term for the string.

Some comments about equation (2.22) are interesting. The first is that higher order string-like corrections vanish for the particular case of the Nambu–Goto string. The second comment is that the constant (Lüscher) term has the same $x$ dependence as a possible $\lambda^2$ contribution in perturbation theory. Finally, we point out that, given that $\tilde{\sigma}(\vec{e}/N)$ is an increasing function of its argument, the right-hand side of equation (2.22) acquires a certain magnitude at a lower value of $x$ the bigger the value of the corresponding electric flux. The numerical results of [2] also show an earlier (in terms of $x$) departure from the perturbative behaviour for higher electric fluxes. This suggested the authors to try to compare the numerical data on the energies with the formula

$$E^2(\vec{e}, \lambda, N, k) = \frac{\phi_0(\vec{e}, N, k)}{4x^2} - \frac{G(\vec{e}, N)}{x} - \frac{\tau}{24} \tilde{\chi}(\vec{e}, N, k) + \frac{1}{4} \tau^2 \tilde{\sigma}^2(\vec{e}/N)x^2$$

(2.24)

obtained by adding the perturbative terms, dominant for small $x$, with the confining terms, valid for large ones. Curiously the formula describes rather well the behaviour of the data even at intermediate values of $x$. This is exemplified in figure 1, in which the energies for various electric fluxes $\vec{e} = (e, 0)$ for $N = 17$ and $k = 3$ are plotted as a function of $x$. The data points are from [2, 3, 14]. The continuous lines having the same colour as the points result from using equation (2.24) with $G(\vec{e}, N)$ given by the pole, $\frac{\tau}{24} \tilde{\chi}(\vec{e}, N, k) = 0.03$, and $\tau\tilde{\sigma}(\vec{e}/N) = 1.2 \sin(\pi e/N)/\pi$. It is clear that the energies for each of the electric fluxes follow the pattern given by the equation. The qualitative agreement is not spoilt by small variations of $\tau$ or the constant term. The same pattern is present for other values of $k$ and $N$. However, all the tested values of $N$ are prime numbers. This is done for simplicity as well as to avoid the existence of non-trivial subgroups of $\mathbb{Z}_N$. It is unclear if the formula also describes the data when $e$ and $N$ have common divisors. Hence, at least for prime $N$, we have a rationale to understand the behaviour of the energies which allows us to predict whether tachyonic instabilities would develop at intermediate values of $x$.

Now the question becomes: assuming the validity of the formula, can we prove that for every $N$ there is a value of $k$ such that all energies $E^2(\vec{e}, \lambda, N, k)$ are larger than zero? To work out the analytical conditions following from this assumption we first realize that the condition can be translated into a condition on the quantity $Z = \tilde{\sigma}(\vec{e}/N)\sqrt{\phi_0(\vec{e}, N, k)}$. As a matter of fact we can approximate $\tilde{\sigma}(x) = \|x\|_2$, so that we can write

$$Z = N\|\vec{e}/N\|_2\|k\vec{e}/N\|_2.$$  

(2.25)

Defining $y = \sqrt{\phi_0(\vec{e}, N, k)}/x$, the energy square can be written

$$E^2(\vec{e}, \lambda, N, k) = \frac{\chi^2}{4} - \frac{y}{16\pi^2Z} - \frac{\tau}{24} \tilde{\chi}(\vec{e}, N, k) + \frac{\tau^2Z^2}{4\chi^2}$$

(2.26)

where we have used the pole form of the self-energy. Now we can minimize this function with respect to $y$ and compute the value of the function at the minimum. This can be done analytically since the equation for the minimum is a quartic equation, whose single real and positive root can be substituted back into the function of the energy. The result for the minimum energy takes the form

$$E^2 = f(Z, \tau) - \frac{\tau}{24} \tilde{\chi}(\vec{e}, N, k).$$

(2.27)
All that we need to know about the function $f$ is that it is a monotonously increasing function of $Z$ in the region of interest and with a slight dependence on $\tau \sim 1$. Indeed, $\chi(\hat{\epsilon},N,k)$ is also approximately equal to 1, although it does not play any role in our reasoning. The minimum energy would then cross zero at a specific value of $Z$, which we will call $Z_0$. Varying the parameters within reasonable limits we obtain values of $0.09 \leq Z_0 \leq 0.12$. The conclusion then is that we would be able to avoid tachyonic instabilities at a given value of $N$ and $k$ provided $Z > Z_0$ for all values of the electric flux. Again this minimum condition is saturated by electric fluxes of the type $(\epsilon,0)$. Hence, our original problem then reduces to proving that for any $N$, one has

$$Z_0 < a(N,1) = \max_k \min_{\epsilon} \max N \left\| k e/N \right\| \left\| e/N \right\|.$$  

(2.28)

It is this condition that will be explored in the next section using known results in number theory.

3. Connection to number theory results

According to our previous considerations, the existence of tachyonic instabilities is related to the behavior of the sequence $\{ a(N,n) \}_{N=1}^{\infty}$ defined as

$$a(N,n) = \max_k \min_{\epsilon} N^n \left\| k e/N \right\| \left\| e/N \right\|$$  

(3.1)

where $k$ runs over integers relatively prime to $N$ and $\epsilon \neq 0 \mod N$. As we have seen in the previous section, the value $n = 2$ appears when studying tachyonic instabilities in the perturbative regime and $n = 1$ corresponds to the general case.
Some of the most natural problems regarding this sequence and their physical motivations are in the following list:

1. If a universal lower bound holds for \( n = 2 \), then the existence of instabilities at the level of perturbation theory is disproved.
2. If \( a(N,n) \) has a large enough universal lower bound for \( n = 1 \) as \( N \to \infty \), then one can show that it is possible to avoid the existence of instabilities in the model.
3. An algorithm to find a \( k \) in equation (3.1) perhaps not reaching the maximum but establishing a good lower bound for \( a(N,n) \) would give a way of finding for each \( N \) a twist \( k \) to avoid instabilities.
4. The possibility of bounds of this kind for a sequence \( N_j \to \infty \) (and the corresponding \( k_j \)) would establish a form of defining a large \( N \) limit of the model.
5. An algorithm to restrict the possibilities for \( k \) and \( e \) in equation (3.1) would be very convenient to carry out numerical studies of the stability of the model.

In what follows we will address the points listed above. For that purpose we will introduce some terminology which will allow us to reformulate our problem in a number theoretical fashion.

First we recall that the Farey sequence \( F_N \) (rather a set in our case) is the set of irreducible fractions in \([0, 1]\) with denominator at most \( N \). Then we can rewrite

\[
a(N,n) = N^{n-1} \max_{\alpha} \min_{p/q \leq 1/2} \left| \alpha - \frac{p}{q} \right|^n \quad \text{where} \quad \alpha \in F_N - F_{N-1} \quad \text{and} \quad \frac{p}{q} \in F_{N-1}. \tag{3.2}
\]

In other words, \( \alpha \) is an irreducible fraction with denominator exactly \( N \) while \( p/q \) has a denominator \( q \) strictly smaller.

To deduce equations (3.2) from (3.1) we should identify \( \|e/N\| = q/N \) and \( \alpha = \|k/N\| = l/N \). Then one has

\[
a(N,n) = \max_l \min_q N^{n-1} q \left\| \frac{lq}{N} \right\|^n \tag{3.3}
\]

which shows that \( a(N,n) \) is invariant under

\[
k \mapsto N - k \quad \text{and} \quad e \mapsto N - e. \tag{3.4}
\]

Hence, we can restrict ourselves to \( 0 < k,e < N/2 \), and then the integers \( l \) and \( q \) would coincide with \( k \) and \( e \) respectively. In addition one can rewrite

\[
\left\| \frac{lq}{N} \right\| = \min_p \left| \frac{lq}{N} - p \right| = q \min_p \left| \alpha - \frac{p}{q} \right| \tag{3.5}
\]

which rather easily leads to equation (3.2)

The previous formulation drives us to the question of approximating a real number by an irreducible fraction (with fixed denominator), and this calls for the use of continued fractions. This constitutes a classical topic in number theory. See for instance the reference [23] for the properties stated below.

Any rational number can be expressed as a continued fraction

\[
[a_0; a_1, a_2, \ldots, a_M] := a_0 + \frac{1}{(a_1 + 1/(a_2 + 1/(a_3 + \ldots)))} \quad \text{where} \quad a_0 \in \mathbb{Z} \quad \text{and} \quad a_1, a_2, \ldots, a_M \in \mathbb{Z}^+.
\tag{3.6}
\]

In fact this representation is unique for nonintegral numbers imposing \( a_M > 1 \). For instance, \( 5/11 = [0, 2, 5] \) and \( 77/103 = [0, 1, 2, 1, 25] \).
The fractions
\[
\frac{p_n}{q_n} = [a_0; a_1, a_2, a_3, \ldots, a_n]
\] (3.7)
are called convergents of the continued fractions (assumed irreducible). Notice that if \( M \) is the number of terms in the continued fraction then \( p_M = l \) and \( q_M = N \). Very often it is also defined \( p_{-1} = 1 \) and \( q_{-1} = 0 \). In this way, we have the recurrence formulas
\[
p_j = a_j p_{j-1} + p_{j-2} \quad \text{and} \quad q_j = a_j q_{j-1} + q_{j-2} \quad \text{for } j \in \mathbb{Z}^+.
\] (3.8)
These are the same formulas that those for the Euclidean algorithm. One can prove that the so-called partial quotients \( a_j \) are actually the successive quotients obtained when one applies the Euclidean algorithm to the numerator and denominator of the initial rational number in its irreducible form.

The key property that we are going to use is that the convergents give optimal approximations. It means that if \( p_j/q_j \) are the convergents of \( \alpha \), then
\[
|q_j \alpha - p_j| = \min_{p/q \in F_0} |q \alpha - p| \quad \text{for any } q_j < Q < q_{j+1}.
\] (3.9)
This is valid, even if \( \alpha \in \mathbb{R} \) is not rational, in this case we have an infinite continued fraction. Since \( q \geq q_j \) this also holds for \( q^{j+1} | \alpha - p/q|^\alpha \) in the same interval.

Thus equation (3.9) proves that we can restrict ourselves to consider \( p/q \) as a convergent of \( \alpha \) in equation (3.2). This is a major advance in the fifth point of the list above with respect to a brute force search for the minimum. The number of steps of the Euclidean algorithm is \( O(\log m) \) where \( m \) is the minimum of the initial numbers. Then the computation of \( a(N,n) \) examining all the convergents requires at most \( O(N \log N) \) steps. Recall that \( k = N \alpha \) and \( \epsilon = q \), then once the magnetic flux \( k \) is fixed, there are very few electric fluxes to be checked.

On the other hand, if we expect a bound \( a(N,n) > c_0 \) then it is not necessary to consider \( \alpha \) such that \( |\alpha - p/q| \leq c_0^{1/n} q^{-1/n} N^{-1+1/n} \). In other words, we can omit the values of \( k \) in the intervals \( [k - Np/q, k] \leq (c_0 N)^{1/n} q^{-1/n} \). The interval is larger for small values of \( q \). A method in the direction of the fifth point of the list is to perform a preliminary sieve of the values of \( p/q \) choosing \( p/q \in F_0 \) with \( Q \) not very large. Indeed, we have used the methods described above to make a scan of the minimum values of \( a(N,n) \) for the first few thousands prime values of \( N \). We will comment on the results later.

It is possible to give an alternative formula [23, section 7.5] for the left hand of equation (3.9):
\[
q_j \alpha - p_j = \frac{(-1)^j}{\alpha_j'; q_j'; q_{j-1}} \quad \text{with } \alpha_j' = [a_j; a_{j+1}, \ldots, a_M],
\] (3.10)
where, as before, \([a_0; a_1, a_2, \ldots, a_M]\) is the continued fraction of \( \alpha \). By equations (3.2) and (3.9)
\[
a(N,n) = N^{n-1} \max_{\alpha} \min_{j < M} \left( \frac{q_j}{\alpha_j' q_j + q_{j-1}} \right)^n \quad \text{with } \alpha_j' = [a_j; a_{j+1}, \ldots, a_M].
\] (3.11)
After this long introduction we are now ready to address the first two points of our previous list. Let us begin with the \( n = 2 \) case.
3.1. The case $n = 2$ case

For prime values of $N > 2$ (the $N = 2$ case is trivial) we are going to prove that

$$a(N, 2) > \frac{3}{\pi^2} (1 - N^{-1}).$$  \hspace{1cm} (3.12)

The method below gives a slightly better bound.

Restricting $k$ to the interval $(0, N/2)$, we can define a function $F(k)$ and re-express equation (3.2) in terms of it as follows:

$$F(k) = N \min_{p/q} \left| \frac{k}{N} - \frac{p}{q} \right|^2$$

so that

$$a(N, 2) = \max_{0 < k < N/2} F(k).$$  \hspace{1cm} (3.13)

We can also express $F(k)$ as follows

$$F(k) = \min_{0 < m < N/2} m^2 q(k, m) \equiv \min_{0 < m < N/2} \left\| \frac{km}{N} \right\|$$

where we recall that $\bar{k} = 1 \mod N$.

Given $k$, say that the minimum appearing in (3.14) is attained for a particular value of $m$.

We define the sets

$$C_m = \left\{ 0 < k < N/2 : q(k, m) = \frac{NF(k)}{m^2} \right\}. \hspace{1cm} (3.15)$$

Clearly, all the values of $k$ must belong to at least one of the sets so that

$$\sum_m \# C_m \geq \# \left( \bigcup_m C_m \right) \geq \# \{ 0 < k < N/2 \} = (N - 1)/2,$$  \hspace{1cm} (3.16)

where $\# A$ stands for the cardinality of the set $A$. On the other hand, if $k \in C_m$ then

$$q(k, m) = \frac{NF(k)}{m^2} \leq \frac{Na(N, 2)}{m^2}. \hspace{1cm} (3.17)$$

Let us now define the sets

$$D_m = \left\{ 0 < q < N/2 : \exists k \in C_m / q(k, m) = q \right\}. \hspace{1cm} (3.18)$$

From equation (3.17) we conclude that \# $D_m \leq Na(N, 2)/m^2$.

The last step is to realize $D_m$ has the same cardinality as $C_m$. To see this we notice that $k = \bar{m} q \mod N$, where $m$ is the congruent inverse of $m$. From these observations and equation (3.16), we get

$$(N - 1)/2 \leq \sum_{0 \neq m < N/2} \frac{N}{m^2} a(N, 2) < a(N, 2) N \sum_{m=1}^{\infty} \frac{1}{m^2} = a(N, 2) \zeta(2).$$  \hspace{1cm} (3.19)

Substituting the value of $\zeta(2)$ gives equation (3.12). Our proof succeeds in avoiding the existence of tachyonic instabilities in the perturbative region for prime $N$.

A similar analysis in the case $n > 2$, produces

$$\frac{N - 1}{2} \leq \sum_{0 \neq m < N/2} \frac{N}{m^n} a(N, n) \leq \zeta(n) Na(N, n).$$  \hspace{1cm} (3.20)
On the other hand, the last convergent of \( \alpha \) different from itself and \( \alpha \) are consecutive Farey fractions. If we take it as \( \frac{p}{q} \) in equation (3.2), we have \( |\alpha - \frac{p}{q}| = (qN)^{-1} \) and \( a(n,N) < 1/2 \) because \( q < N/2 \). In this way, we have the upper and lower bounds

\[
\frac{1}{2}\zeta(n)(1 - N^{-1}) < a(n) < \frac{1}{2}
\]

(3.21)

If \( N \) and \( n \) go to infinity, \( a(n) \) tends to be 1/2. For \( n = 2 \), we have \( 2\zeta(n) = \pi^2/3 \) that gives equation (3.12).

3.2. The \( n = 1 \) case

Firstly we are going to show that in the case \( n = 1 \) the optimal values of \( N \) to avoid tachyonic instabilities are the Fibonacci numbers.

Recall the (extended) Fibonacci sequence \( \{F_k\}_{k=0}^{\infty} = (0, 1, 1, 2, 3, 5, 8, \ldots) \) defined by the recurrence \( F_{j+2} = F_{j+1} + F_j \). Assume that \( N \geq 5 \) is a Fibonacci number, say \( N = F_j \). Thanks to equation (3.8), it is easy to see that

\[
\frac{J-5 \text{ times}}{0; 2, 1, \ldots, 1, 2} = \frac{F_{j-2}}{F_j} = \frac{F_{j-2}}{N}.
\]

(3.22)

Let us call this number \( \alpha_0 \). We are going to check that \( a(N, 1) = \alpha_0 \). The convergents are \( 0/1, 1/2, 1/3, 2/5, \ldots, F_{j-1}/F_{j-2} \) and \( \alpha_0 \). In the same way, \( \alpha'_1 = 1/\alpha_0 \), \( \alpha'_j = F_{j-1}/F_{j-2} \).

Using the properties of the Fibonacci sequence, namely \([24, p 89, 47, r = 1]\) and \([24, p 88, 18 n = 2]\), we have for \( 0 < j \leq J - 4 \)

\[
q_j - \alpha_0 (\alpha'_{j+1} q_j + q_{j-1}) = q_j - \alpha_0 \left( \frac{F_{j-1}}{F_{j-2}} \frac{F_{j+2}}{F_{j+1}} + \frac{j}{F_{j+1}} \right) = F_{j+2} - \frac{F_{j-2}}{N} \frac{F_{j-2}}{F_{j-2}} = \frac{F_{j-2} - \alpha_0 F_j}{F_{j-2}} \geq 0
\]

(3.23)

which is still valid for \( j = 0 \) with equality. Hence for \( \alpha = \alpha_0 \) the minimum in equation (3.11) is reached for \( j = 0 \) giving \( a(N, 1) \geq \alpha_0 \). To deduce \( a(N, 1) = \alpha_0 \), it remains to prove that for any \( \alpha \neq \alpha_0 \) (with denominator \( N \)) there exists \( j_0 \) such that

\[
q_{j_0} \leq (\alpha'_{j_0+1} q_{j_0} + q_{j_0-1}) \alpha_0.
\]

(3.24)

If \( \alpha < \alpha_0 \) holds trivially for \( j_0 = 0 \) because \( \alpha'_1 = 1/\alpha > 1/\alpha_0 \). We consequently assume \( \alpha > \alpha_0 \) and then \( a_1 = 2 \). If \( \alpha \) has a partial quotient which is at least 3 then equation (3.24) holds true because \( \alpha_0 > 1/3 \) and we can take \( \alpha'_{j_0+1} = [3; \ldots] \geq 3 \). Otherwise, the partial quotients of the \( \alpha'_j \) are 1 or 2. Take \( j_0 \) such that \( a_j = 1 \) for \( 1 < j \leq j_0 \), then by the recurrence formulas equation (3.8) \( q_{j_0} = F_{j_0+2} + q_{j_0+1} = F_{j_0+1} \). Clearly \( \alpha'_{j_0+1} \) is of the form \([2, 1, \ldots, 1], [2; 2, \ldots] \) or \([2; 2] = [2, 1, 1] \). In any of these cases \( \alpha'_{j_0+1} > 7/3 \) and the right hand side of equation (3.24) is greater than

\[
\alpha_0 \left( \frac{7}{3} F_{j_0+2} + F_{j_0+1} \right) = \alpha_0 q_{j_0} \left( \frac{7}{3} + \frac{F_{j_0+1}}{F_{j_0+2}} \right) \geq \alpha_0 q_{j_0} \left( \frac{7}{3} + \frac{F_3}{F_5} \right) = \frac{17}{6} \alpha_0 q_{j_0}
\]

(3.25)

and (3.24) follows.

Revising the proof, one notes that the assumption \( N = F_j \) was only employed to compute the value at \( \alpha = \alpha_0 \). The proof still applies for \( N > F_j \) except that \( \alpha = \alpha_0 \) is not a valid value in equation (3.2). Then we have proved

\[
a(N, 1) \leq \frac{F_{j-2}}{F_j} \quad \text{if} \quad N \geq F_j, \quad \text{with equality if} \quad N = F_j.
\]

(3.26)
The well-known asymptotic $F_j \sim r^j$ with $r$ the golden ratio, gives the asymptotic bound $a(N,1) \leq r^{-2} = 0.381966\ldots$, $N \to \infty$. This value is well within the safe region equation (2.28) in which there are no tachyonic instabilities. The values of $N$ reaching the bound equation (3.26) appear in the numerical data as outliers because of the exponential growth.

As a final comment we translate our result to the original Physics notation by saying that for $N = F_j$ the optimal value corresponds to $k = k = F_{j-2}$ and electric flux either $e = 1$ or $e = k$ (any of them). If we restrict ourselves to $N$ being a prime number, one should look only at Fibonacci numbers having this property. Indeed, taking $J$ a small prime number, $N = F_j$ is also prime (it holds for $J < 19$, $N < 4181$), which could be of practical use, but we do not know of a general rule which would provide a solution to the fourth point in our list. In fact the existence of infinitely many Fibonacci primes, as other exponential problems (Mersenne primes or Fermat primes), is considered out of reach with current knowledge in number theory.

We turn now to the most important property of our list (point number 2). The problem of finding general lower bounds is connected with Zaremba’s conjecture. This is a problem posed in 1971 (see an overview in [25]) that remains open yet. It claims the existence of a positive integer $A$ with the following property:

For every $N \in \mathbb{Z}^+$, there exist $a_1, a_2, \ldots, a_j \leq A$ such that $[0; a_1, a_2, \ldots, a_j] = p_j/N$.  \hspace{1cm} (3.27)

In a more elementary way, it means that the recurrence $q_{j+1} = a_{j+1}q_j + q_{j-1}$, $q_0 = 0$, $q_1 = 1$ can capture any positive integer with a judicious choice of the $a_j$.

Indeed the lower bound for $a(N,1)$ is a straightforward consequence of the formula equation (3.11) under equation (3.27), because

$$a(N,1) > \max_{\alpha} \min_j \frac{1}{\alpha_{j+1} + 1} \geq \frac{1}{A + 2}. \hspace{1cm} (3.28)$$

A kind of converse is also true: using $\alpha_{j+1}' \geq a_j$, if $a(N,1) > \epsilon$ then we could take $A = \lfloor \epsilon^{-1} \rfloor$ in equation (3.27) for that $N$, where $\lfloor x \rfloor$ means the integral part.

Recently there was a breakthrough on equation (3.27). In [26] it has been proved that $A = 50$ is valid for any $N$ except for a zero density set (i.e. if there are $E_N$ exceptions less than $N$, then $E_N/N \to 0$ as $N \to \infty$). Unfortunately the resulting bound $1/52$ is small enough not to be conclusive about the absence of tachyonic instabilities. Nonetheless, the value was reduced later to $A = 5$ [28] and it is thought that $A = 2$ for a certain (large value of) $N$ onwards. If this is true, one argument that we do not reproduce here (essentially bounding $q_j/q_{j-1} > 5/4$) would lead to $a(N,1) > 5/19 \sim 0.26315\ldots$ for large $N$. That would clearly suffice to exclude the necessity of tachyonic instabilities in the large $N$ limit. We must add that we have also explored the question numerically for $N$ one of the first 4000 prime numbers ($N < 37831$) and found $a(N,1) > 0.22779$. Hence, there are reasons to be optimistic.

4. Conclusions and outlook

In this paper we have applied results and methods arising from the mathematical area of number theory to a problem that appears when studying the behaviour of SU($N$) Yang–Mills fields in $2 + 1$ dimensions where the spatial dimensions are compactified in a 2-torus with twisted boundary conditions. The main point is to determine whether it is possible to choose the integer flux $k$, characterizing these boundary conditions, in such a way as to avoid a large $N$ phase transition appearing at specific torus sizes. In the absence of this transition one can continuously connect the small size region, where perturbation theory applies, to the large size
region, where confinement takes place. The question has been analyzed numerically in [2] for various values of $N$ suggesting that the problem can indeed be avoided. However, building on the understanding of the system provided by the numerical work, we faced here the goal of trying to determine whether the result will continue to hold at arbitrarily large values of $N$.

The possibility of a phase transition emerges when studying the self-energy contribution to the energy levels of the system in non-vanishing electric flux sectors. This contribution is negative and indeed it can be seen that for several values of the chromo-magnetic flux $k$ the system will develop a phase transition with condensation of certain modes, a phenomenon named as tachyonic instability. Next to leading order of perturbation theory predicts this to happen for all values of $k$, but the conclusion is only trustworthy whenever the problem occurs at sufficiently small values of the coupling constant. Our first result, presented in the previous paragraph, has been to show that for any prime value of $N$, it is always possible to choose $k$ in such a way as to make the perturbative prediction of a phase transition inconclusive. This is based on the lower bound obtained for the quantity $a(N,2)$ defined in the text.

We then proceeded to study the problem in general. This requires a certain understanding of the behaviour of the system in the non-perturbative region. For that purpose we built on the results obtained in previous numerical studies in which the behaviour of the energies at all torus sizes is well described by a function involving all the parameters of the problem. This function allows us to extrapolate our analysis to arbitrary values of $N$ and $k$. On the basis of this, we studied the necessary condition to be able to avoid any instability occurring at any value of the torus size and $N$. The condition takes the form of a lower bound on $a(N,1)$. The existence of such a lower bound and its actual value turns out to be related to a conjecture in number theory, formulated by Zaremba in 1971 [27] (see also [25]), and which remains to be proven. The situation is aggravated by the necessity that the bound is large enough to guarantee the avoidance of instabilities. However, on the positive side we must mention that, from the Physics perspective, it would be enough if the bound holds only for large enough $N$, or just for a sequence of values of $N$ running up to infinity. Indeed, one of the results of this paper was to show that there exist an optimal sequence given by the Fibonacci numbers for which the instability can be avoided. Nonetheless, if we insist that $N$ should be a prime number, to avoid other potential problems, we are faced with the question of existence of an infinite number of primes in the Fibonacci sequence.

To summarize, we can say that although the issue has not been definitely settled, our work has established a connection with interesting open problems in number theory, which might eventually lead to a full understanding of this and other related physical results.

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