Pre-Geometric Structure of Quantum and Classical Particles in Terms of Quaternion Spinors

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Abstract—It is shown that dyad vectors on a local domain of a complex-number-valued surface, when squared, form a set of four quaternion algebra units. A model of a proto-particle is built by the dyad rotation and stretching; this transformation violates the metric properties of the surface, but the defect is cured by a stability condition for a normalization functional over an abstract space. If the space is the physical one, then the stability condition is precisely the Schrödinger equation; the separated real and imaginary parts of the condition are the mass conservation equation and the Hamilton-Jacoby equation, respectively. A 3D particle (composed of proto-particle parts) should be conceived as a rotating massive point, its Lagrangian automatically becoming that of a relativistic classical particle, with energy and momentum proportional to Planck’s spin term in the Schrödinger equation.

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1. INTRODUCTION

The basic formulas of quantum and relativistic mechanics, such as the energy-frequency link for quantum particles and the space-time interval invariance in special relativity (or the Lagrange function for a relativistic particle) were postulated “in a flash of genius” by Planck and Einstein from empirical and heuristic considerations. These revolutionary discoveries had two serious consequences. First, quantum theory and relativity were separated, their convergence into P. Dirac’s spinor theory and Feynman’s (et al.) quantum electrodynamics required other non-obvious assumptions, while the results appeared not to be so satisfactory and convincing; general relativity still rejects quantization. Second, these theories have not improved, but rather aggravated the understanding of what is going on in the micro-world and what an elementary particle looks like; moreover, quantum theory strictly forbids particle images, Gell-Mann’s model being just a classification attempt with no explanation what a quark is. That is where the notion of pre-geometry arises.

This notion was insistently discussed by Wheeler [1] who was unsatisfied (as many others) by understanding of quantum mechanics, and the pre-geometry was an attempt to incorporate, at least verbally, the extra entities that could match experimental results of the theory with its vague interpretations. Subsequent investigations reveal a way to mathematically describe a structure possibly underlying the physical world. This way implies the use of hypercomplex numbers, in particular, quaternions that constitute the last (in dimension) associative, but non-commutative (in multiplication) division algebra. This set of numbers, overarching the algebras of real and complex numbers, has as well an important property, first noticed by Hamilton, inventor of quaternions [2]: it is “very geometric”, its mathematical magnitudes and relations have clear images of vectors and frames in 3D space, which is frequently associated with that of physical world.

But in this study quaternions emerge only at the second stage because there exist “more elementary” algebraic structures, square roots from 3D space dimensions. Namely, this primitive set is considered here to form the pre-geometry, the fundamental constituent of physical space. In Section 2, an initial pre-geometric structure (a 2D cell) is defined as a local domain of complex-number-valued surface, and it is shown how its elementary basis, a dyad, generates a 3D frame. In Section 3, a condition, preserving the stability of quaternion algebra under the 2D cell’s non-standard transformation, is considered, resulting in derivation of the equation of quantum mechanics and construction of a model of a proto-particle. The respective equation of classical mechanics is derived.

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in Section 4, and a model of a 3D particle is composed with determination of its main physical characteristics and its Lagrangian function. In Section 5, the influence of a vector field on particle propagation is examined; it switches on the quaternion properties of the micro-space, resulting in appearance of Pauli’s spin term in the equation of quantum mechanics. A summery and a short discussion in Section 6 conclude the study.

2. PRE-GEOMETRIC SURFACE AND 3D SPACE

On a smooth 2-dimensional space (surface) endowed with a coordinate system \( x^K \) and a metric \( g_{KM} \), \( K, M, N = 1, 2 \), define a basis \( \{a, b\} \) with vector components \( a^N, b^N \) (and co-vector components \( a_N = g_{NM}a^M, b_N = g_{NM}b^M \)), so that
\[
\begin{align*}
a_{NO}^N &= b_Nb^N = 1, \quad (1) \\
a_{NO}^N &= b_Na^N = 0. \quad (2)
\end{align*}
\]
The metric and its inverse are evidently expressed through the basis \( g_{MN} = a_Ma_N + b_Mb_N, \quad g^{MK} = a^Na^K + b^Nb^K \), while the 2D unit \( I \) (the Kronecker delta \( \delta^M_N \)) is the following square construction:
\[
I \equiv \delta^K_N = g_{MN}g^{MK} = a_Na^K + b_Nb^K. \quad (3)
\]
In general, the 2D space may be curved; locally it is a plane, the simplest basis (and metric) having constant components, e.g.,
\[
\begin{align*}
a^N &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & a_N &= \begin{pmatrix} 1 & 0 \end{pmatrix}, \\
b^N &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & b_N &= \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad (4a)
\end{align*}
\]
\[
g_{KM} = g^{KM} = \delta^K_M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4b)
\]
A local domain of a surface with an initial point of the basis will be called a 2D cell. Apart from the scalar object (3), one can build the following three vector squares: an antisymmetric “vector product”
\[
\mathbf{q}_2|_K^K = a^Mb_K - b^Ma_K, \quad (5)
\]
a symmetric “Lorentz-type vector”
\[
\mathbf{q}_3|_K_N = i(a^Ka_N - b^Kb_N), \quad (6)
\]
and the result of their ordered multiplication [symmetric; Eqs.(1) and (2) are used]
\[
\mathbf{q}_2|_K^M \mathbf{q}_3|_N^K \equiv \mathbf{q}_1|_N^M = -i(a^Mb_N + b^Ma_M). \quad (7)
\]
Together with the scalar unit (3), the three vectors (5)–(7) form a basis of quaternion algebra having the multiplication table
\[
I \mathbf{q}_k = \mathbf{q}_k I, \quad \mathbf{q}_k \mathbf{q}_n = -\delta_{kn} + \varepsilon_{knm}q_m; \quad (8)
\]
small Latin indices are 3-dimensional, 2D-matrix indices are not shown, \( \delta_{kn} \) and \( \varepsilon_{knm} \) are the 3D Kronecker and Levi-Civita symbols. The units \( \mathbf{q}_k \) geometrically constitute a triad of vectors initiating a Cartesian coordinates system in 3D space; thus any 2D cell generates a 3D space. The multiplication rule (8) (hence the quaternion algebra as a whole) evidently holds if the units \( \mathbf{q}_k \) are transformed by matrices \( U \) from the spinor group \( SL(2, \mathbb{C}) \), \( \mathbf{q}_k = U \mathbf{q}_k U^{-1} \), but Eqs. (3), (5)–(7) state that the rule invariance is a consequence of the “more elementary” transformations \( a' = Ua, b' = Ub \), i.e., vectors of any 2D basis are 3D (quaternion) spinors. Transform a basis \( \{a, b\} \) by a spinor matrix \( U = \begin{pmatrix} e^{-ia} & 0 \\ 0 & e^{ia} \end{pmatrix} \in SU(2) \subset SL(2, \mathbb{C}) \) or in 2D cell components \( U^K = e^{iaK}a_K + e^{-iaK}b_K \):
\[
\begin{align*}
a'(a') &= e^{-ia}a_K, & b' &= e^{-ia}b_K, \\
\mathbf{a}' &= (\mathbf{a}^N)^T = e^{-ia}a_N, \\
\mathbf{b}' &= (\mathbf{b}^N)^T = e^{ia}b_N; \quad (9)
\end{align*}
\]
and compute new squares. One discovers that the new scalar unit and vector No. 3 do not change: \( I' = I, \mathbf{q}_3' = \mathbf{q}_3 \), two other new vectors rotate by the angle \( 2\alpha \): \( \mathbf{q}_1' = \cos 2\alpha \mathbf{q}_1 + \sin 2\alpha \mathbf{q}_2, \mathbf{q}_2' = -\sin 2\alpha \mathbf{q}_1 + \cos 2\alpha \mathbf{q}_2 \); all four objects still form a set of quaternion units satisfying Eq. (8). It is important to note that the vector-spinors \( \{a, b\} \) in this case are eigenvectors of \( \mathbf{q}_3 \);
\[
\mathbf{q}_3|_M^K a^M = ia^K, \quad \mathbf{q}_3|_M^K b^M = -ib^K
\]
[Eqs. (1) and (2) are used], and while \( \mathbf{q}_3 \) is not affected by the transformation (9), its constituents acquire a phase.

Let us try to imagine a 2D cell for the basis (9). When \( \alpha = \pi n, n = 0, \pm 1, \pm 2..., \) the components \( a'^K, b'^K \) are real; the respective 2D cell is just a piece of a real plane. As the angle \( \alpha \) changes, the unit vectors \( \{a', b'\} \) “synchronously” rotate in opposite directions in mutually orthogonal complex planes, each perpendicular to the real one [3]. So in general a 2D cell may be conceived as a continuum of real layers (of certain area and non-restricted form) having a thickness of two imaginary units. An infinite set of smoothly bordering 2D cells constitutes a (spinor) surface. This specific 2D entity built of multiple cells, each underlying (and generating) a domain of 3D
space, may be associated with a kind of a screen (“world screen”).

But on the other hand, the spinor-surface model has features perfectly fitting to pre-geometry. First, if geometric forms are attributed to 3D space, then the spinor surface, comprising vectors, “square roots” from a 3D space distance, should logically be assessed as something “previous to geometry”. Second, it is a “very pre-geometric” feature that only two basic vectors on the spinor surface are sufficient to fully describe all dimensions of 3D space. And third, since the notion of pre-geometry emerges in connection with quantum theory, its model discussed here must lead to a mathematical description of quantum phenomena; this claim is verified in the next section.

3. DERIVATION OF SCHRÖDINGER EQUATIONS AND A MODEL OF A 2D PROTO-PARTICLE

In addition to the rotation (9), let us conformally transform the basis on a 2D cell, multiplying the vectors (9) by a real factor \( \sigma \) different from zero and unity; then with the notation

\[ \lambda \equiv \sigma e^{i\alpha} \]  

(10)

new 2D vectors acquire the form

\[ \bar{a}^K \equiv \sigma a^K = \lambda a^K, \quad \bar{b}^K \equiv \sigma b^K = \lambda^* b^K, \]  

(11a)

and the covector-spinors as in Eq. (9) are obtained by Hermitian conjugation,

\[ \bar{a}_M = (\bar{a}^M)^* = \lambda^* \bar{a}_M, \quad \bar{b}_M = (\bar{b}^M)^* = \lambda \bar{b}_M. \]  

(11b)

One immediately notes that the “stretching” (in general sense) defined in Eqs. (11) spoils the dyad, its vector-spinors remain orthogonal \( \bar{a}_N \bar{b}^N = \bar{b}_N \bar{a}^N = 0 \), but are not unit:

\[ \bar{a}_N \bar{a}^N = \bar{b}_N \bar{b}^N = \sigma^2. \]  

(12)

So Eqs. (3), (5)–(7) written with \( \{\bar{a}, \bar{b}\} \) do not give 3D units, the multiplication rule (8) is no more valid, and the quaternion algebra together with the 3D space domain associated with its vector units are destroyed. This trouble emerges because the transformation (11), preserving the Hermitian conjugation for spinors, injects the contraction defect (12) into the metric properties of the 2D cell; as a result, the function \( \sigma^2 \) appears in 3D space.

The 2D cell metric defect may be mathematically smoothed away if the factor (10) is a compact function over a volume of any space (let it be an abstract 3D Euclidean space),

\[ f(\theta) \equiv \int_{V_\xi} \lambda(\xi_k, \theta) \lambda^*(\xi_k, \theta) \, dV_\xi = \int_{V_\xi} \sigma^2(\xi_k, \theta) \, dV_\xi = 1, \]  

(13)

where \( \xi_k \) are (unitless) coordinates, \( \theta \) is a (unitless) free parameter; Eq. (13) evidently retains the properties of the quaternion algebra. The retaining is permanent (in the sense of the parameter \( \theta \)) if \( \partial f/\partial \theta = 0 \), this involves the continuity-type equation

\[ \partial_\theta \lambda \lambda^* + \partial_n(\lambda \lambda^* n) = 0 \]  

(14)

where \( \partial_\theta \equiv \partial/\partial \theta, \partial_n \equiv \partial/\partial \xi_n \), and the propagation vector \( \xi_n \) points at the increasing phase [4],

\[ k_n \equiv \partial_n \alpha = \frac{i}{2} \partial_n \ln \left( \frac{\lambda^*}{\lambda} \right) \]  

\[ = \frac{i}{2} \left( \frac{\partial_n \lambda^*}{\lambda} - \frac{\partial_n \lambda}{\lambda^*} \right). \]  

(15)

Substitution of Eq. (15) into (14), after simple computations, yields a differential equation representing a condition of quaternion algebra stability after curing the 2D cell metric defect (12) with the help of the functional (13):

\[ \partial_\theta \lambda - \frac{i}{2} [\partial_n \partial_n - W(\xi_k, \theta)] \lambda = 0. \]  

(16)

Here \( W(\xi_k, \theta) \) is an arbitrary function. One notes that Eq. (16) has the form of an equation of quantum mechanics. We link the unitless free parameter and the coordinates of physical space, respectively:

\[ \xi_k = x_k/\varepsilon, \quad \theta = t/\tau \]  

(17)

by coefficients built from physical constants (Compton’s length for a spatial distance)

\[ \varepsilon \sim h/mc, \quad \tau \sim \varepsilon/c \sim h/mc^2. \]  

(18)

The coefficients (18) can be taken for standards, the characteristic length and the time interval of a physical micro-scale (e.g. for an electron at rest, \( \varepsilon \sim 3.89 \cdot 10^{-11} \text{ cm}, \tau \sim 1.3 \cdot 10^{-21} \text{ s} \)). We substitute the variables (17) into Eq. (16) to obtain precisely the Schrödinger equation

\[ \left( i\hbar \partial_t + \frac{\hbar^2}{2m} \partial_n \partial_n - U(x_k, t) \right) \lambda = 0, \]  

(19)

with \( U \equiv mc^2W \). It is worth noting that the expressions (18) naturally emerge when a model of a 3D physical particle is constructed (next section) on the basis of a proto-particle model, in fact representing a pre-geometric image of the “state-function” \( \lambda \).

Let us build a model of a proto-particle taking into account the properties of a 2D cell basis described by Eqs. (9)–(11). To endow the normalization condition (13) with a physical meaning, replace the unitless
coordinates in Eq. (13) with physical (Cartesian) coordinates using Eq. (17), then
\[
\int \sigma^2(\xi, \theta)dV_\xi = \frac{1}{\varepsilon^2} \int \sigma^2(x_k, t)dV = 1. \tag{20}
\]
Now put \(\sigma^2 \equiv \rho(x_k, t)/\rho_{\text{mean}}\), where \(\rho(x_k, t)\) is a function of mass density, \(\rho_{\text{mean}}\) is a mean mass density; then Eq. (20), a physical equivalent of the normalization condition, is a definition of mass:
\[
\int \rho(x_k, t)dV = \rho_{\text{mean}}\varepsilon^3 \equiv m, \tag{21a}
\]
the scale factor, the modulus of \(\lambda\), being a square root of the mass distribution \(\sigma = \sqrt{\rho_{\text{mean}}\varepsilon^3/m}\), i.e.,
\[
\sigma \sim \sqrt{\rho(x_k, t)}. \tag{21b}
\]
Eqs. (21) state that the 2D cell becomes “loaded” with a semi-mass distribution \(|\sigma| = g^{1/2}\text{cm}^{-3/2}\) in the CGS system, and while, according to Eqs. (9), the 2D cell’s basic vectors rotate (each in its own complex plane), this semi-mass is “pumped over” from the real section of the 2D cell to its imaginary section, and backwards, depending on the value of the angle \(\alpha(x_k, t)\). The semi-mass distribution in a 2D cell and the process of its “pumping over” between real and imaginary planes comprise the image of a proto-particle “seen by a pre-geometric observer” (provided this one could exist). It is as well a variant of a “physical” (or rather pre-geometric) interpretation of the “state function” \(\lambda\) defined by Eq. (10) and satisfying Eq. (19); so the described image of a proto-particle is in fact that of a quantum particle, never imagined before. Though a question remains, whether the angle \(\alpha\) must be always changing (so that the pumping is permanent) or not; an answer will be given in the next section.

Now (as a step from quantum to classical physics but at the pre-geometric level) substitute the function \(\lambda\) with its form (10) in the mathematical equation (16), thus separating the equation into real and imaginary parts. The real part,
\[
\partial_\theta \sigma + \partial_\alpha \sigma \partial_\theta \alpha + \frac{1}{2} \sigma \partial_\alpha \partial_\alpha \alpha = 0, \tag{22}
\]
is just a square root of the continuity equation (14); it may be treated as the evolution equation for \(\sigma\) if the phase is a known function of 3D space coordinates. The imaginary part has the form
\[
\partial_\theta \alpha + \frac{1}{2}(\partial_\alpha \alpha)(\partial_\theta \alpha) + W_{\text{ext}} = 0, \tag{23}
\]
where
\[
W_{\text{ext}} \equiv W - \partial_\alpha \partial_\alpha \sigma/(2\sigma). \tag{24a}
\]
Eq. (23) is the evolution equation for \(\alpha\); it has the form of the Hamilton-Jacobi equation for a mechanical system where \(W_{\text{ext}}\) is treated as an exterior (alien) “potential energy”. If the term
\[
W_\mu \equiv \partial_\alpha \partial_\alpha \sigma/(2\sigma) \tag{24b}
\]
describes an interior “potential” regulating the distribution of \(\sigma\), then \(W\) is the full “potential”,
\[
W = W_{\text{ext}} + W_\mu. \tag{24c}
\]
The physical Hamilton-Jacobi equation is satisfied by the function of mechanical action; let us find its pre-geometric analogue \(\alpha\) by integrating the full differential expression
\[
d_\theta \alpha(\xi_k, \theta) = d_\theta \alpha + d_\theta \xi_k \partial_\theta \alpha, \tag{25}
\]
where \(d_\theta \equiv d/d\theta \) and \(d_\theta \alpha\) is substituted from Eq. (23):
\[
\alpha = \int_{\theta_1}^{\theta_2} \left\{ d_\theta \xi_k \partial_\theta \alpha - \frac{1}{2}(\partial_\alpha \alpha)(\partial_\theta \alpha) - W_{\text{ext}} \right\} d\theta \equiv \int_{\theta_1}^{\theta_2} L(d_\theta \xi_k, \xi_k) d\theta. \tag{26a}
\]
We demand that the functional (26a), the rotation angle of a 2D cell’s basis, should acquire an extremal (minimal) value on a line \(\xi_k(\theta)\) between fixed points at \(\theta_1\) and \(\theta_2\). Then the variation
\[
\delta \alpha = 0 \tag{26b}
\]
leads to the “Euler-Lagrange” equation
\[
d_\theta \left[ \partial_\alpha \alpha + \frac{\partial(\partial_\alpha \alpha)}{\partial(d_\theta \xi_m)}(d_\theta \xi_k - \partial_\theta \alpha) \right] + \partial_\alpha W_{\text{ext}} = 0, \tag{27}
\]
determining the “law of motion” \(\xi_k(\theta)\) provided the “potential” \(W_{\text{ext}}\) is specified and the gradient \(\partial_\alpha \alpha\) (“momentum”) is known as a function of \(d_\theta \xi_k\) (“velocity”). In the physical world Eq. (27) must become the Newtonian dynamic equation of a massive (material) point.

4. A 3D MODEL OF A SMALL-MASS PARTICLE (MATERIAL POINT)

The above-described pre-geometric 2D model of a quantum proto-particle (“state function”) in the physical world compiles itself into a small mass \(m\) compactly distributed inside a 3D micro-volume \(V\) (the characteristic length is \(\varepsilon \sim V^{1/3}\)) with a “frozen-in” rotating triad. Recalling Sommerfeld’s estimate [5] of the electron’s velocity value \(v^2/h \approx c/137\) (\(c\) is the minimal electric charge), let us assume the
characteristic velocity value of micro-scale processes to be proportional to the fundamental velocity \( c \); then a characteristic micro-scale time interval is \( \tau \sim \varepsilon/c \). And since the micro-particle is regarded as a distributed rotating mass, it has an extra characteristic, the maximum proper angular momentum that by its sense (and units) has a perfect physical equivalent, the Planck constant

\[
mce \sim \hbar, \tag{28}
\]

otherwise Eq. (28) can be treated as the Heisenberg-type uncertainty relation. These are physical grounds of Eqs. (18) and (17). Let this classically imagined particle move in 3D space with a velocity \( u \) along the coordinate \( z \sim ut \). It can be shown that the angle \( \alpha \) has a minimal average value [demanded by Eq. (26b)] if the propagation vector \( k_\alpha \) (hence, the velocity) is parallel to the rotation axis (in this case to \( q_3 \)), i.e., \( q_3 \) is directed along \( z \), while the vector \( q_1 \) (frozen in the particle mass as well as \( q_3 \)) follows the line of a helix with the radius \( \varepsilon/2 \), the ultimate particle’s border. Recalling that the angle of 3D space rotation is \( 2\alpha \), we calculate the small length of the helix line:

\[
dl^2 = dz^2 + (\varepsilon/2)^2 d(2\alpha)^2 = dz^2 + \varepsilon^2 d\alpha^2. \tag{29}
\]

Express the angle’s differential from Eq. (29), the signs implying a right or left helicity:

\[
d\alpha = \pm \frac{1}{\varepsilon} \sqrt{dl^2 - dz^2},
\]

and, taking into account that the radius \( \varepsilon/2 \) is a limit where the micro-process of rotation at the border may occur with the extremal velocity \( c \), we put \( dl = cdt \), while \( dz = u dt \), so that

\[
d\alpha = \pm \frac{cdt}{\varepsilon} \sqrt{1 - \frac{u^2}{c^2}}. \tag{30}
\]

The substitution \( \varepsilon = \hbar/mc \) in Eq. (30) gives, after integration, the expression

\[
\alpha = \pm \frac{mc^2}{\hbar} \int dt \sqrt{1 - \frac{u^2}{c^2}}.
\]

If the minus sign is chosen, then the last equation, rewritten as a definite integral,

\[
\alpha h \equiv S = -mc^2 \int_{t_1}^{t_2} dt \sqrt{1 - \frac{u^2}{c^2}} \equiv -mc \int_{t_1}^{t_2} ds \tag{31}
\]

determines precisely the action \( S \) of a classical particle in relativistic mechanics.

Eq. (31) deserves a short discussion. First, it states that the angle of pre-geometric basis rotation is the action of the respective 3D particle measured in units of Planck’s constant,

\[
\alpha = S/h, \tag{32}
\]

thus endowing the mysterious notion of mechanical action with a “pre-geometric sense”. Second, the “projection” of the angle is found to be negative; this means that the triad frozen in the moving massive particle rotates clockwise, i.e., the propagating particle has a “left helicity”. Third, in the derivation of Eq. (31) only the light-velocity constant has to do with relativity, but this is sufficient to make the equation relativistic with the “space-time interval” \( ds \sim \varepsilon d\alpha \) being in this model a difference between lengths of the helix line and the linear path of the particle; otherwise it is the bordering arc length of the particle at rest (though rotating). From now on, one can think of invariance of the interval in different reference frames, thus arriving at the full theory of relativity. And fourth, Eq. (31) is probably the first one that naturally comprises both relativistic and quantum characteristics.

Now we return to the differentials in Eq. (31) and consider the non-relativistic case \( u \ll c \),

\[
dS = \hbar d\alpha = -mc^2 dt + \frac{mu^2}{2} dt. \tag{33}
\]

In Eq. (33) we add and subtract the term \( mu^2/2 \), substitute \( udt = dz \), and express the full differential of the rotational angle from Eq. (25) through the physical coordinates and time using Eqs. (17):

\[
d\alpha = \omega dt + k_n dx_n = \omega dt + k_z dz,
\]

where \( \omega \) (negative) is the angular frequency of the 2D basis rotation; then Eq. (33) takes the form

\[
dS = \omega hdt + k_z h dz = -\left( mc^2 + \frac{mu^2}{2} \right) dt + mu dz \equiv -E dt + p_z dz. \tag{34}
\]

Eq. (34) simultaneously (and automatically) gives relations of both classical and quantum mechanics, for the particle energy

\[
\frac{\partial S}{\partial t} = -E = -|\omega| h \tag{35a}
\]

and momentum \( \partial S/\partial z = p_z = k_z h \), or

\[
\frac{\partial S}{\partial x_n} = p_n = k_n h. \tag{35b}
\]

Eqs. (35) state that in this model a small classical particle must have quantum properties of the De Broglie’s wave, i.e., for a free particle with \( \omega = \text{const} \), \( k_n = \text{const} \) its state function (10) satisfying Eq. (19) is

\[
\lambda(x_k, t) = \sigma \exp[i (p_n x_n - E t)/\hbar]. \tag{36}
\]

Moreover, Eq. (34) also states that a massive particle at rest \((u = 0)\) must have a non-zero angular frequency \( |\omega| = mc^2/\hbar = 1/\tau \), i.e., the angle \( \alpha \) must
be always changing, thus making the proto-particle’s “pumping over” process permanent; namely, this process is responsible for the existence of the 3D particle’s rest energy.

Let us use the obtained results to put the mathematical equations into a physical form. Eq. (32), inserted together with Eqs. (17) into the pre-geometric equation (23) (for the phase), gives precisely the Hamilton-Jacoby equation of classical mechanics

\[
\partial_t S + \frac{1}{2m}(\partial_n S)(\partial_n S) + U_{\text{ext}}(x_k, t) = 0 \tag{37a}
\]

with \( U_{\text{ext}} \equiv me^2W_{\text{ext}} \), while the “Euler-Lagrange” equation (27) for a low-energy particle in physical variables becomes the Newtonian dynamic equation

\[
d_{tp_k} = -\partial U_{\text{ext}}. \tag{37b}
\]

So Eqs. (37) describe the macro-scale motion of a particle treated as a structureless material point. As to Eq. (22), a similar transition to physical variables in it leads to the equation

\[
\partial_t \rho + \frac{1}{m} \left( \partial_x \rho \partial_x S + \frac{1}{2} \partial_x \partial_x S \right) = 0, \tag{38}
\]
describing the evolution of “semi-density” \( \rho \) in a small domain of 3D space; multiplied by \( \rho \), Eq. (38) becomes the continuity equation \( \partial_t \rho + \partial_x (\rho v_x) = 0 \) with \( v_x \equiv ((1/m)\partial_x S ) \) and Eq. (21b) taken into account. But there is an additional interesting observation concerning Eq. (38). Being linear in \( \rho \), the equation therefore can be written for \( \sqrt{\rho}(x_k, t) \), then each term of this equation is measured in the CGS units \( g^{1/2} cm^{-3/2} S^{-1} \), i.e., those of electric charge density; this fact may be useful for future considerations.

The last essential equation (24b) in physical variables has the form

\[
\partial_n \partial_n \rho - \kappa U_{\mu} \rho = 0, \tag{39}
\]

where \( \kappa \equiv 2m/h^2 \), \( U_{\mu} \equiv me^2W_{\mu} \), it is a static equation determining the distribution of a particle’s “semi-density” (since \( \rho \sim \sqrt{\rho} \)) under the influence of the micro-scale potential \( U_{\mu} \). There is yet no theoretical mechanism providing self-consistency of the potential and the particle’s structure (similar to that of Einstein’s gravity), but one can expect the existence of solutions where both functions, the potential and the density, are nonsingular at any point of 3D space. We can seek a simple nonsingular solution of Eq. (39) in the spherically symmetric case assuming that the particle is pulled together by an elastic force with the potential \( U_{\mu} = B + br^2/2 \), where \( B \) and \( b \) are constants. One can verify that Eq. (39), now having the form

\[
\frac{1}{r^2} \partial_r \left[ r^2 \partial_r (\sqrt{\rho}) \right] = \kappa \left( B + \frac{br^2}{2} \right) (\sqrt{\rho}),
\]
is satisfied by the Gaussian function

\[
\sqrt{\rho}(r) = \sqrt{\rho_0} \exp \left( -\frac{r^2}{2\delta^2} \right), \tag{40}
\]

so that the potential’s constants are linked with the gauge length \( \delta \) as

\[
B = -\frac{3\hbar^2}{2m\delta^2}, \quad b = \frac{\hbar^2}{m\delta^2}.
\]

The constant factor \( \rho_0 \) is found from the normalization condition (20), which in spherical coordinates has the form [Eq. (21a) is taken into account]

\[
\frac{4\pi\rho_0}{m} \int_0^\infty r^2 \exp \left( -\frac{r^2}{\delta^2} \right) dr = 1. \tag{41}
\]

The integral in Eq. (41) can be computed as \( \delta^3 \sqrt{\pi}/4 \) (following Poisson’s method of integration over an infinite plane in polar coordinates), therefore \( \rho_0 = m/(\sqrt{\pi}\delta^3) \). So the solution (40) with the elastic force potential is non-singular and quite “physical”. A preliminary analysis shows that other “good” spherically symmetric solutions of Eq. (39) (including a Yukawa-type mass distribution) can be obtained for different potentials \( U_{\mu} \sim \text{const.}, \sim 1/r, \sim r^2 \), though either a potential or a solution (or both) are singular at the center of symmetry. All such solutions describe a mass distribution inside a static non-rotating particle; if rotation is taken into account, the mass distribution must be different.

It is also important (and mentioned above) that the potential \( U_{\mu} \) emerges in Eq. (39) [and earlier in (16) under the symbol \( W \)] as an arbitrary term preserving equalities, and no special reason is indicated responsible for “stretching” of the 2D cell, thus injecting a semi-mass density and regulating its distribution; this disadvantage partly takes place due to utter simplicity of the model under consideration. Indeed, Eq. (16) is derived for a scalar function (10), while its full form is that of the 2D cell vector-spinor (11); in the simple case of the propagation vector (14) no physical agent affects the spinor constituent of the “wave function”, so it is not needed in the equation. But the simple model can be generalized by introduction various factors of influence; one evident (and important) possibility is suggested in the next section.

5. DERIVATION

OF THE SCHRÖDINGER-PAULI EQUATION

To simplify the subsequent formulas, we use for one spinor function from Eqs. (11) the index-free form

\[
\Psi \equiv a^K = \lambda a^K \equiv \lambda \psi,
\]

GRAVITATION AND COSMOLOGY Vol. 19 No. 2 2013
\[ \Phi \equiv \bar{a}_K = \lambda^* a_K = \lambda^* \varphi, \quad \varphi \psi = 1, \]

then the integral in Eq. (13) is equivalently represented in a more general form:

\[ f(\theta) \equiv \int_V \Phi \Psi dV_\xi = \int \lambda^* \varphi \psi \lambda dV_\xi = \int \sigma^2 dV_\xi = 1, \quad (42) \]

where \( \lambda(\xi_k, \theta) = \sigma e^{i \alpha} \), as in Eq. (10). Let the propagation vector be influenced by a vector field:

\[ k_n \equiv \partial_n \alpha + A_n (\xi_m, \theta). \quad (43) \]

Following Maxwell’s formulation of electrodynamics, we match the vector field \( A_n \) (and any other 3D vector including the gradient \( \partial_n \alpha \)) with a constant quaternion triad \( p_k \equiv \bar{a}_k \), so that the metric of 3D physical world is expressed from Eq. (8) as

\[ \delta_{kn} \equiv \frac{1}{2} (p_k p_n + p_n p_k); \quad (44) \]

then the continuity-type equation (14), providing the stability of Eq. (42), is written as

\[ \partial_\theta (\Phi \Psi) + \delta_{mn} \partial_m (\Phi \Psi k_n) = 0, \]

or in a more detailed form [Eqs. (43) and (44) are used]

\[ \partial_\theta (\varphi \lambda^* \lambda \psi) + \frac{1}{2} (p_m p_n + p_n p_m) \times \partial_m [\varphi \lambda^* \lambda \psi (\partial_n \alpha + A_n)] = 0. \quad (45) \]

Computing the phase gradient as is done in Eq. (15), and following the procedure described therein, one arrives to the following development of Eq. (45):

\[ (\lambda^* \varphi) \times \left[ \left( \partial_\theta - \frac{i}{2} \partial_k \partial_k - i \partial_k A_k + A_k \partial_k \right) + \frac{i}{2} A_k A_k + i \varepsilon_{kmj} p_j \partial_m A_k + i W \right] (\lambda \psi) \]

\[ + \left[ \left( \partial_\theta + \frac{i}{2} \partial_k \partial_k + \frac{1}{2} \partial_k A_k + A_k \partial_k - i \frac{1}{2} A_k A_k \right) \right. \]

\[ + \left. \frac{i}{2} \varepsilon_{kmj} p_j \partial_m A_k - i W \right] (\lambda^* \varphi) \times (\lambda \psi) = 0, \]

where the term

\[ i (\lambda^* \varphi) \times \left( \frac{1}{2} A_k A_k + W \right) \times (\lambda \psi) \]

is added and subtracted. The last equation naturally splits into two parts, one being the Hermitian conjugate of another; the equation satisfied by the vector-spinor \( \Psi \) is

\[ i \partial_\theta - \frac{1}{2} (-i \partial_k + A_k)(-i \partial_k + A_k) \]

with \( B_k \equiv \varepsilon_{kmn} \partial_m A_n \). A transition to physical units with the help of Eqs. (17), (18) converts the unitless equation (45) precisely to the Schrödinger-Pauli equation for an electrically charged particle in an exterior magnetic field

\[ \left[ \frac{i}{2} \partial_\theta - \frac{1}{2 m} \left( -i \partial_k + \frac{e}{c} A_k \right) \left( -i \partial_k + \frac{e}{c} A_k \right) \right. \]

\[ + \frac{e \hbar}{2 mc} p_k B_k - U \right] \Psi = 0, \quad (47) \]

where the magnetic field potential and induction are

\[ \tilde{A}_k \equiv \frac{mc^2}{e} A_k, \quad \tilde{B}_k \equiv \frac{mc^2}{e} B_k, \]

and the scalar (mechanical) potential is \( U \equiv mc^2 W \). It is worth mentioning that the originally empirical Pauli term \( (e \hbar)/(2mc) p_k \tilde{B}_k \) was once deduced theoretically assuming that a charged particle could interact with the micro-space structure influenced by a magnetic field and described by a non-symmetric quaternion metric tensor [6].

6. CONCLUSION

Let us briefly summarize the main points and original results of this study.

(i) It is shown that any pair of unit and orthogonal vectors (a dyad on a 2D surface) creates a set of four unit \( 2 \times 2 \) matrices that form a basis of quaternion algebra; among them, the scalar unit behaves as the metric tensor of the underlying surface, locally plane (2D cell), while the other three vector (imaginary) units are geometrically equivalent to a triad initiating a Cartesian frame in 3D space. So the constituents of the dyad are quaternion spinors.

(ii) A novel type of transformations, conformal stretching of a 2D cell, accompanied by a standard rotation of the dyad is suggested; in general, it violates the multiplication rule of the quaternion algebra, but conditions continuously preserving the rule are offered as integral (normalization-type) and differential (continuity-type) equations for the transforming function over 3D space. In the simplest case of a “propagation” vector, the continuity equation decays into a mutually conjugate pair, each part representing the condition of quaternion algebra stability.

(iii) In physical space, the stability condition becomes precisely the Schrödinger equation; this enables us to imagine a 2D model of a proto-particle, actually endow the wave function with a (pre-)geometric sense.
(iv) Separation of the stability condition into real and imaginary parts over 3D physical space leads to the equation of the particle mass conservation and to the Hamilton-Jacobi equation, respectively. A model of a classical macro-particle (as a rotating massive point) is built on the basis of the proto-particle conjecture; the respective action is proportional to the angle of the 2D cell dyad rotation with the Planck constant as a coefficient and is automatically equal to that of a relativistic particle. A low-energy particle is shown to be described by the proto-particle’s function having the form of a de Broglie wave.

(v) An analysis of free functions arising in the derivation of the stability condition results as well in a static equation for the particle’s mass semi-density (square root from density) in a potential field; physically meaningful solutions for the particle mass distribution at the micro-scale are indicated.

(vi) It is finally shown that introduction of a vector field into the definition of the propagation vector results in a subsequent derivation of the Schrödinger-Pauli equation.

So this study demonstrates that the famous physical (initially empiric) equations of quantum and classical mechanics are naturally contained in the mathematical medium of quaternion numbers and its underlying set of quaternion spinors. It seems important that, apart from many “correct equations” obtained above (hardly an occasional coincidence), the logic of their appearance is also correct: the equations for classical objects are deduced from more fundamental quantum equations. The developed model of a pre-geometric proto-particle seems to firstly offer natural links between some quantum and relativistic physical magnitudes and to endow the habitual but somewhat vague notions of mechanical action and space-time interval with seemingly original geometric meanings. The study also promotes an original method of build physical theories by choosing pre-geometric dyads with different properties and introducing various physical agents into the basic equations saving the quaternion algebra.

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