On the spectral problem of $\mathcal{N} = 4$ SYM with orthogonal or symplectic gauge group

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Abstract

We study the spectral problem of $\mathcal{N} = 4$ SYM with gauge group $SO(N)$ and $Sp(N)$. At the planar level, the difference to the case of gauge group $SU(N)$ is only due to certain states being projected out, however at the non-planar level novel effects appear: While $\frac{1}{N}$-corrections in the $SU(N)$ case are always associated with splitting and joining of spin chains, this is not so for $SO(N)$ and $Sp(N)$. Here the leading $\frac{1}{N}$-corrections, which are due to non-orientable Feynman diagrams in the field theory, originate from a term in the dilatation operator which acts inside a single spin chain. This makes it possible to test for integrability of the leading $\frac{1}{N}$-corrections by standard (Bethe ansatz) means and we carry out various such tests. For orthogonal and symplectic gauge group the dual string theory lives on the orientifold $AdS_5 \times \mathbb{RP}^5$. We discuss various issues related to semi-classical strings on this background.
1 Introduction

Whereas the planar spectral problem of $\mathcal{N} = 4$ SYM seems to be close to resolution \cite{1, 2, 3, 4, 5, 6, 7, 8, 9}, much less has been achieved in the non-planar case. Non-planar corrections, when studied perturbatively in $1/N$, lead to a breakdown of the spin chain picture which was the key to the progress at the planar level. More precisely, $1/N$-corrections to the dilatation generator lead to interactions which split and join spin chains \cite{10}. This enormously enlarges the Hilbert space of states and, furthermore, implies that excitations on different chains can interact, rendering the standard tools of integrable spin chains inapplicable and leaving little hope for the existence of a Bethe ansatz in the usual sense\footnote{The situation is the same in the three-dimensional ABJM and ABJ theories \cite{11, 12}.}

In order to gain further insight into $1/N$-corrections we will study $\mathcal{N} = 4$ SYM with gauge groups $SO(N)$ and $Sp(N)$. At the planar level, the only essential difference of these theories from the traditionally studied $SU(N)$ case is that certain states are projected out. However, at the non-planar level new effects arise. Namely, for orthogonal and symplectic gauge group the leading non-planar corrections originate from non-orientable Feynman diagrams with a single cross-cap \cite{13}. At the level of the dilatation generator these leading non-planar corrections are described by an operator which acts entirely inside a single spin chain. This implies that restricting oneself to the leading $1/N$-corrections one does not face the problems mentioned above. The Hilbert space of states remains the same as on the planar level and all interactions take place inside a single spin chain. Thus the existence of a usual Bethe ansatz is not a priori excluded and one may test for integrability using standard methods.

In the AdS/CFT correspondence, changing the gauge group on the field theory side translates into a modification of the background geometry on the string theory side. For orthogonal and symplectic gauge groups the relevant geometry becomes that of the orientifold $AdS_5 \times \mathbb{R}P^5$ where the case of $Sp(N)$ differs from that of $SO(N)$ by the presence of an additional $B$-field \cite{14}. In the case of $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ the leading non-planar effects on the string theory side have their origin in string diagrams of genus one but in the case of orthogonal and symplectic gauge groups the leading non-planar corrections should be associated with non-orientable string worldsheets with a single cross-cap. At least naively, it
seems easier to deal with cross-caps than higher genus surfaces so our study might open new avenues for comparison of gauge and string theory beyond the planar limit.

Our main focus will be on the gauge theory side where we will study in depth the one-loop dilatation generator. We start in section 2 by explaining the reduction of the space of states compared to the theory with gauge group $SU(N)$ and subsequently write down the one-loop dilatation generator including all non-planar corrections. In section 4 we determine analytically the leading $\frac{1}{N}$-correction to the anomalous dimension of two-excitation states, thereby providing a prediction for the dual string theory. After that, in section 5 we search for integrability in the non-planar spectrum in various ways. We look for unexpected degeneracies and for conserved charges. In addition, we put forward various possible modifications of the planar Bethe equations which would produce the correct $\frac{1}{N}$-correction for two-excitation states and test numerically if these equations also work for higher numbers of excitations. Unfortunately, the outcome of these tests is negative. In section 6 we discuss the dual string theory picture and, in particular, mention a number of interesting open problems. Finally, section 7 contains our conclusion.

2 $\mathcal{N} = 4$ SYM with gauge group $SO(N)$

In this section we will study non-planar effects in the spectrum of $\mathcal{N} = 4$ SYM with gauge group $SO(N)$. Before doing so, it is useful to briefly recall how this theory arises as a suitable projection of the $SU(N)$ theory. As is well known, in string theory the latter is constructed by taking the low-energy limit of a stack of $N$ D3-branes in ten-dimensional Minkowski space. The group $SU(N)$ arises because the matrices $\lambda^i_j$ encoding the Chan-Paton factors of the open strings stretching between the D3-branes are hermitian. In order to obtain an orthogonal gauge group, one performs an orientifold projection which, on bosonic states, amounts to relating the Chan-Paton matrices to their transpose matrices as $\eta$

$$\lambda = -\eta^{-1} \lambda^T \eta$$

where $\eta$ is a symmetric matrix which can simply be taken to be unity. The Chan-Paton matrices are thus restricted to be antisymmetric $N \times N$ matrices, which generate the adjoint representation of the group $SO(N)$. As explained in $\eta\eta$, in order to ensure that this procedure does not break $\mathcal{N} = 4$
supersymmetry one has to combine it with a spacetime identification of the six transverse to the brane coordinates $X^i$ as $X^i \rightarrow -X^i$. This procedure leaves us with $\mathcal{N} = 4$ SYM with gauge group $SO(N)$.

We will restrict ourselves to considering the $SU(2)$ sub-sector of the theory, consisting of multi-trace operators built from two complex fields, say $\phi$ and $Z$, i.e. operators of the form

$$O = \text{Tr}(Z \ldots Z \phi \ldots \phi Z \ldots \phi \ldots Z \ldots)$$  \hspace{1cm} (2)

The adjoint fields $Z$ and $\phi$, being elements of the algebra of $SO(N)$, fulfill

$$\phi^T = -\phi, \quad Z^T = -Z.$$  \hspace{1cm} (3)

The dilatation generator of the $SU(2)$ sub-sector at one and two-loops can formally be written in the same way as for the $SU(N)$ case [2]. At one loop order it reads

$$\hat{D} = -g_{\text{YM}}^2 \frac{8\pi^2}{\pi^2} \text{Tr}[\phi, Z][\Phi, \tilde{Z}] \equiv g_{\text{YM}}^2 \frac{8\pi^2}{\pi^2} \hat{H}.$$  \hspace{1cm} (4)

Here $\tilde{Z}$ is an operator which acts on a field $Z$ by contraction of $SO(N)$ indices, i.e.

$$\tilde{Z}_{\alpha \beta} Z_{\gamma \epsilon} = \frac{1}{2}(\delta_{\alpha \epsilon} \delta_{\beta \gamma} - \delta_{\alpha \gamma} \delta_{\beta \epsilon}),$$  \hspace{1cm} (5)

and similarly for $\tilde{\phi}$.

In the analysis of $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ the concept of parity played a central role. In a spin chain context, parity is the operation which inverts the order of operators inside a given trace, i.e. [16]

$$\hat{P} \text{Tr}(X_{i_1} X_{i_2} \ldots X_{i_L}) = \text{Tr}(X_{i_L} X_{i_{L-1}} \ldots X_{i_1}).$$  \hspace{1cm} (6)

Parity commutes with $\hat{H}$ which means that eigenstates of $\hat{H}$ can be chosen to be states with definite parity. (The same is the case for ABJM theory, whereas for ABJ theory parity is broken at the non-planar level [11, 12].) In general, for $\mathcal{N} = 4$ SYM with gauge group $SU(N)$, for a given length $L$ the spectrum will then contain operators of positive as well as negative parity. However, since the group generators for gauge group $SO(N)$ are antisymmetric, a state is related to its parity conjugate in the following way:

$$\hat{P} \text{Tr}(X_{i_1} X_{i_2} \ldots X_{i_L}) = (-1)^L \text{Tr}(X_{i_1} X_{i_2} \ldots X_{i_L}).$$  \hspace{1cm} (7)

\footnote{We chose to keep the normalization of generators $\text{Tr} T^a T^b = \delta^{ab}$ when passing from $SU(N)$ to $SO(N)$.}
In other words, parity has been gauged. We thus see that, compared to the case of $SU(N)$, the $SO(N)$ theory has a lot fewer states: For even length only positive parity states survive whereas for odd length only negative parity states survive. When acting on operators of the type (2), the one-loop dilatation generator $\hat{H}$ can be usefully decomposed as

$$\hat{H} = N \hat{H}_0 + \hat{H}_+ + \hat{H}_- + \hat{H}_{flip}. \quad (8)$$

Here $\hat{H}_0$ is the planar part which, up to a factor of two, is the same as for $SU(N)$, i.e.

$$\hat{H}_0^{SO(N)} \equiv \hat{H}_0 = \frac{1}{2} \sum_{i=1}^{L} (1 - P_{i,i+1}) = \frac{1}{2} \hat{H}_0^{SU(N)} . \quad (9)$$

In particular, this means that the information about the planar anomalous dimensions in the case of gauge group $SO(N)$ is encoded in the same Heisenberg spin chain Bethe equations as for $SU(N)$. However, due to the fact that certain states are projected out, some of the other information encoded in these equations becomes redundant.

For single trace operators consisting of $M$ fields of type $\phi$ and $(L - M)$ fields of type $Z$, where $M \leq L/2$, the Bethe equations are expressed in terms of $M$ rapidities $\{u_k\}_{k=1}^M$ and read

$$\left( \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^L = \prod_{j=1,j\neq k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}. \quad (10)$$

The rapidity $u$ is related to the momentum $p$ via

$$u = \frac{1}{2} \cot \left( \frac{p}{2} \right), \quad (11)$$

and the eigenvalues of $\hat{H}_0$ are given by

$$E_0 = \frac{1}{2} \sum_{k=1}^{M} u_k^2 + \frac{1}{4} = 2 \sum_{k=1}^{M} \sin^2 \left( \frac{p_k}{2} \right). \quad (12)$$

\footnote{The relative factor of $\frac{1}{2}$ in the hamiltonian arises because of our normalisation of the gauge group generators.}
The momenta have to satisfy the condition
\[ \sum_k p_k = 0, \]  
which reflects the cyclicity of the trace. The Bethe equations, the cyclicity constraint and the expression for the energy are all invariant under \( u_k \to -u_k \). This implies that for any solution, \( \{u_k\} \), either \( \{-u_k\} = \{u_k\} \) or \( \{-u_k\} \) is a partner solution of the same energy. Following \[17, 18\], we will refer to the first type of solutions as unpaired solutions and the second type as paired. In \( SU(N) \) terminology, the two solutions in a pair are each other’s parity conjugates. The values of the higher conserved charges for the two states are identical for even charges and differ by a sign for odd charges. Unpaired states have vanishing odd charges. Considering gauge group \( SO(N) \) instead of \( SU(N) \), the two states in a pair get identified via eqn. (7) and the odd charges lose their meaning. An unpaired state survives the projection if it has parity \((-1)^L\) where \( L \) is its length. The reduction procedure is hence clear on the level of solutions. It would be neat, however, if it could be formulated at the level of the Bethe equations.\[4\]

At the non-planar level the dilatation operator contains the three terms \( \hat{H}_+, \hat{H}_- \) and \( \hat{H}_{\text{flip}} \). The operators \( \hat{H}_+ \) and \( \hat{H}_- \) respectively increase and decrease the trace number by one and have analogues in the case of \( SU(N) \). The operator \( \hat{H}_{\text{flip}} \) is trace conserving and does not have any analogue in the case of \( SU(N) \). In the language of string theory the operators \( \hat{H}_+ \) and \( \hat{H}_- \) correspond to string splitting and joining whereas \( \hat{H}_{\text{flip}} \) corresponds to the insertion of a cross-cap on the string worldsheet. It is well-known that for gauge theories with orthogonal or symplectic gauge group the topological expansion includes Feynman diagrams which correspond to non-orientable surfaces, i.e. surfaces with cross-caps \[13\]. Each occurrence of a cross-cap is associated with a factor of \( \frac{1}{N} \) whereas a handle as usual gives rise to a factor of \( \frac{1}{N^2} \), see Fig. 1. Acting with \( \hat{H}_{\text{flip}} \) on a single trace operator gives a

\[ \left( \frac{u_k + i \frac{1}{2}}{u_k - i \frac{1}{2}} \right)^{L-1} = \prod_{j=1, j \neq k}^{M/2} \frac{u_k - u_j + i u_k + u_j + i}{u_k - u_j - i u_k + u_j - i} \]

which is similar to the (not completely unrelated) case of open strings \[20, 21, 22, 23\].

\[4\]One can show that the surviving unpaired states always have \( L \) and \( M \) even \[19\]. For these states, one can hence directly see that the Bethe equations will take a form like

\[ \left( \frac{u_k + i \frac{1}{2}}{u_k - i \frac{1}{2}} \right)^{L-1} = \prod_{j=1, j \neq k}^{M/2} \frac{u_k - u_j + i u_k + u_j + i}{u_k - u_j - i u_k + u_j - i} \]

which is similar to the (not completely unrelated) case of open strings \[20, 21, 22, 23\].
contribution for each pair of fields of type $\phi, Z$ that the operator contains. This contribution is most conveniently described in the following way

$$\hat{H}_{\text{flip}} \text{Tr}(\phi XZY) = \frac{1}{2} \text{Tr}(X^T Y[Z, \phi]) + \frac{1}{2} \text{Tr}(Y X^T[Z, \phi]).$$ (14)

Here $X$ and $Y$ are arbitrary operators, and it is understood that the $\hat{Z}$ and $\hat{\phi}$ in $\hat{H}_{\text{flip}}$ are contracted with the explicitly written $Z$ and $\phi$ in $\text{Tr}(\phi XZY)$. The operator $\hat{H}_{\text{flip}}$ hence cuts out a piece of the operator and reinserts it with the opposite orientation. Since this piece can be of arbitrary length, we see that all sites in the chain are involved in the interaction. So, although $\hat{H}_{\text{flip}}$ takes single-trace operators to single-trace operators, and can thus be interpreted as a spin-chain interaction, in contrast with the planar part of the dilatation operator its action on the spin chain is highly non-local.

Up to a factor of 2, the operator $\hat{H}_+$ takes the same form for $SU(N)$ and $SO(N)$ whereas the operator $\hat{H}_-$ has extra terms for $SO(N)$. More precisely

$$\hat{H}^{SO(N)}_+ = \frac{1}{2} \hat{H}^{SU(N)}_+$$

$$\hat{H}^{SO(N)}_- \text{Tr}(\phi X)\text{Tr}(ZY) = \frac{1}{2} \hat{H}^{SU(N)}_- \text{Tr}(\phi X)\text{Tr}(ZY)$$

$$+ \frac{1}{2} \text{Tr}(X^T Y[Z, \phi]) + \frac{1}{2} \text{Tr}(Y X^T[Z, \phi]),$$ (16)

where the notation is as above and where $\hat{H}^{SU(N)}_\pm$ can be found in [10]. The extra terms in $\hat{H}^{SO(N)}_- \pm$ are natural since for non-orientable surfaces there are two possible ways of gluing objects together. We notice that in a basis of planar eigenstates the perturbations $\hat{H}_+$ and $\hat{H}_-$ are always off-diagonal. Only
\( \hat{H}_{\text{flip}} \) can have diagonal matrix elements in such a basis. Treating the energy corrections perturbatively in \( \frac{1}{N} \), \( \hat{H}_+ \) and \( \hat{H}_- \) will thus generically give corrections to the energy of order \( \frac{1}{N^2} \) whereas \( \hat{H}_{\text{flip}} \) can give corrections already at order \( \frac{1}{N} \). The expansion of the anomalous dimensions hence generically takes the form
\[
\mathcal{E} = \frac{g_{YM}^2 N}{8\pi^2} \left( E_0 + \frac{1}{N} E_1 + \frac{1}{N^2} E_2 + \mathcal{O}\left( \frac{1}{N^3} \right) \right),
\]
(17)
where the contribution \( E_1 \) is mainly due to \( \hat{H}_{\text{flip}} \). It should be noticed, however, that if there are degeneracies in the planar spectrum, energy corrections induced by \( \hat{H}_+ \) and \( \hat{H}_- \) can also be of order \( \frac{1}{N^3} \). This phenomenon does not occur for strong coupling where the closed string perturbation theory taking into account string splitting and joining always gives rise to an expansion in \( \frac{1}{N} \). The \( \frac{1}{N} \) corrections to the energies induced by \( \hat{H}_+ \) and \( \hat{H}_- \) are hence expected to vanish for strong coupling (and only arise here due to an order of limits issue). Assuming this to be true we can thus study corrections to the string energy induced by cross-caps by considering only the corrections coming from \( \hat{H}_{\text{flip}} \).

3 \( \mathcal{N} = 4 \) SYM with gauge group \( Sp(N) \).

We now consider the case of \( \mathcal{N} = 4 \) SYM with gauge group \( Sp(N) \), the group of \( N \times N \) symplectic matrices. The construction of this theory in terms of an orientifold projection is also well known [15]: The projection in this case relates the Chan-Paton matrices of open-string states as
\[
\lambda = -J^{-1} \lambda^T J
\]
(18)
where \( J \) is an antisymmetric matrix satisfying \( J^2 = -1_{N \times N} \), which can be taken to be (\( N \) is even):
\[
J = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)_{N \times N}.
\]
(19)
The Chan-Paton matrices in this case turn out to be symmetric, and generate the adjoint representation of \( Sp(N) \). Combining this with the identification \( X^i \to -X^i \) of the \( \mathcal{N} = 4 \) SYM scalars leads to \( \mathcal{N} = 4 \) SYM theory with gauge group \( Sp(N) \) [14].
In $Sp(N)$, indices are raised and lowered with the matrix $J$, and adjoint fields with both indices down are symmetric. Thus an adjoint field $Z_\alpha^\beta = J^{\alpha\gamma} Z_{\gamma\beta}$ behaves in the following way under transposition

$$Z^T = JZJ.$$  \hfill (20)

This in particular implies that a single trace operator is again related to its parity conjugate as given in eqn. (7) and parity is gauged in the same way as before. Furthermore, for gauge group $Sp(N)$ the one-loop dilatation generator of $\mathcal{N} = 4$ SYM can again formally be expressed in exactly the same form as for $SU(N)$, cf. eqn. (4). Only the contraction rules are different. More precisely one has

$$\tilde{Z}_{\alpha\beta} Z_{\gamma\epsilon} = \frac{1}{2} (\delta_{\alpha\epsilon} \delta_{\beta\gamma} - J_{\alpha\gamma} J_{\beta\epsilon}).$$  \hfill (21)

Again one finds that the Hamiltonian can be written in the form given in (8). The action of $\hat{H}_{flip}^{Sp(N)}$ can be presented in the following way

$$\hat{H}_{flip}^{Sp(N)} \text{Tr}(\phi X Z Y) = \frac{1}{2} \text{Tr}(JX^T JY[Z, \phi]) + \frac{1}{2} \text{Tr}(Y JX^T J[Z, \phi]).$$  \hfill (22)

We notice that the result differs from that of $SO(N)$ by $X^T$ being replaced by $JX^T J$. This difference amounts to a shift of sign as we have for an operator $X$ of length $L$

$$SO(N) : \quad X^T = (-1)^L \hat{P} X,$$

$$Sp(N) : \quad JX^T J = (-1)^{L+1} \hat{P} X,$$  \hfill (23, 24)

where $\hat{P}$ is the parity operator. This is in full accordance with the general result that $SO(N)$ can be understood as $Sp(-N)$ \cite{24, 25}. Notice that this sign difference need not explicitly manifest itself in the off-diagonal terms $\hat{H}_+$ and $\hat{H}_-$ since these will generically give rise to energy corrections of order $1/N$.

For $Sp(N)$ we again find that the operator $\hat{H}_+$ differs from that of $SU(N)$ only by a factor of $\frac{1}{2}$ whereas the operator $\hat{H}_-$ has extra terms compared to the corresponding operator for $SU(N)$. More precisely

$$\hat{H}_+^{Sp(N)} = \frac{1}{2} \hat{H}_+^{SU(N)}$$  \hfill (25)

$$\hat{H}_-^{Sp(N)} \text{Tr}(\phi X) \text{Tr}(Z Y) = \frac{1}{2} \hat{H}_-^{SU(N)} \text{Tr}(\phi X) \text{Tr}(Z Y) + \frac{1}{2} \text{Tr}(JX^T JY[Z, \phi]) + \frac{1}{2} \text{Tr}(Y JX^T J[Z, \phi]).$$  \hfill (26)
The difference between the extra terms for $Sp(N)$ and $SO(N)$ is that $X^T$ is replaced by $JX^TJ$, cf. eqn (16), which as before amounts to a change of sign.

4 Analysis of BMN operators

BMN operators are operators consisting of a background of $Z$ fields and a finite number of excitations in the form of $\phi$-fields. We will restrict ourselves to discussing the simplest operators of this type, i.e. those having two excitations. Two-excitation BMN operators always have positive parity and therefore in the case of gauge group $SO(N)$ exist only for even length. At the planar level a basis for the two-excitation states can be chosen as

$$O_{J}^p = \text{Tr}(\phi Z^p \phi Z^{J-p}), \quad 0 \leq p \leq J.$$  

(27)

In terms of these the eigenstates of $\hat{H}_0$ read

$$|n\rangle \equiv O_{n}^J = \frac{1}{J+1} \sum_{p=0}^J \cos \left( \frac{\pi n (2p+1)}{J+1}\right) O_{p}^J, \quad 0 \leq n \leq \frac{J}{2},$$  

(28)

and the corresponding eigenvalues are

$$E_{n}^0 = 4 \sin^2 \left( \frac{\pi n}{J+1}\right).$$  

(29)

The inverse transformation giving $O_{p}^J$ in terms of $|n\rangle$ takes the form

$$O_{p}^J = |0\rangle + 2 \sum_{n=1}^{J/2} \cos \left( \frac{\pi n (2p+1)}{J+1}\right) |n\rangle.$$  

(30)

The energy correction induced by the perturbation $\hat{H}_{flip}$ is simply given by the expression from first order quantum mechanical perturbation theory, i.e.

$$E_{1}^n = \langle n|\hat{H}_{flip}|n\rangle.$$  

(31)

In order to determine this quantity we first evaluate $\hat{H}_{flip} O_{p}^J$ where $J$ is assumed to be even. We find (after some manipulations)

$$\hat{H}_{flip} O_{p}^J = \frac{1}{4}(1 - (-1)^p) \left\{ 2O_{p}^J - O_{p-1}^J - O_{p+1}^J \right\}$$

$$-\frac{1}{2}(-1)^p \left\{ O_{0}^J + O_{J}^J + 2 \sum_{k=1}^{J-1} (-1)^k O_{k}^J \right\}.$$  

(32)
Having this expression, it is straightforward to determine the general matrix element of \( \hat{H}_{\text{flip}} \) as all sums involved are geometric sums. The result reads

\[
\langle m | \hat{H}_{\text{flip}} | n \rangle = -\frac{1}{J+1} \sin^2 \left( \frac{\pi m}{J+1} \right) \left\{ \delta_{n,m}(J+1) - \frac{1}{\cos \left( \frac{\pi(n-m)}{J+1} \right)} - \frac{1}{\cos \left( \frac{\pi(n+m)}{J+1} \right)} \right\} \\
-\frac{2}{J+1} \tan^2 \left( \frac{\pi n}{J+1} \right) \cos \left( \frac{\pi n}{J+1} \right) \cos \left( \frac{\pi m}{J+1} \right). \tag{33}
\]

We notice that \( \hat{H}_{\text{flip}} \) is not hermitian but this phenomenon is well-known \[26, 10\]: The operator \( \hat{H}_{\text{flip}} \) is related to its hermitian conjugate by a similarity transformation. For \( n = m \) the expression (33) reduces to

\[
E_1^n = \langle n | \hat{H}_{\text{flip}} | n \rangle = -\frac{2}{J+1} \tan^2 \left( \frac{\pi n}{J+1} \right) - \frac{1}{J+1} \sin^2 \left( \frac{\pi n}{J+1} \right) \left( J - \frac{1}{\cos \left( \frac{2\pi n}{J+1} \right)} \right). \tag{34}
\]

This should correspond to the energy correction to a closed string state resulting from the insertion of a cross-cap on its worldsheet. Defining \( \lambda' = \frac{g_s^2 N}{J^2} \) and \( g_2 = \frac{J^2}{N} \), the anomalous dimensions of BMN operators were originally believed to have a double expansion in \( \lambda' \) and \( g_2 \) in the limit \( \lambda, J, N \rightarrow \infty \) with \( \lambda', g_2 \) fixed \[27, 28, 29\]. This double expansion worked for BMN operators in \( \mathcal{N} = 4 \) SYM with gauge group \( SU(N) \) for the first few terms in \( \lambda' \) and \( g_2 \) and led to some success in reproducing the first non-planar correction on the gauge theory side from LCSFT, for a review see \[30\]. Later it was understood that planar BMN scaling breaks down at four loop order in the gauge theory \[31, 4, 32\]. Furthermore, on the string theory side a BMN expansion would involve half-integer powers of \( \lambda' \) starting at one-loop order \[33\]. Here the first few terms of the expansion in powers of \( \lambda' \) and \( g_2 \) for the anomalous dimension in eqn. (35) read

\[
\mathcal{E}^n = \frac{\lambda'}{2} \left( n^2 - g_2 \frac{n^2}{4J^2} \right), \tag{35}
\]

meaning that the first non-planar contribution would not survive the above mentioned limit. Still it would be interesting to analyse the cross-cap scenario in the pp-wave geometry by some version of LCSFT.
5 Search for integrability at finite $N$

For gauge group $SU(N)$ an important concept in the search for integrability was the occurrence of so-called planar parity pairs, i.e. pairs of operators which at the planar level had the same anomalous dimension but opposite parity. The existence of such parity pairs could be traced back to the existence of an extra conserved charge commuting with the Hamiltonian but anti-commuting with parity [2]. When splitting and joining of traces were taken into account the degeneracy between the operators in a parity pair disappeared and this was taken as an indication that integrability was lost beyond the planar level [2]. The situation was the same for ABJM theory [11]. In the case of gauge group $SO(N)$ where parity is gauged one obviously does not even have planar parity pairs. Thus one has to invent other means to test for integrability.

One option is to look for other types of degeneracies in the spectrum which could survive the non-planar corrections. One such type of degeneracy is that between anomalous dimensions of certain single- and multi-trace operators, for instance between BMN operators with different number of traces, i.e. operators of the type

$$\mathcal{O}_n^{J_0;J_1,...,J_k} \equiv \mathcal{O}_n^{J_0} \text{Tr}(Z^{J_1})\text{Tr}(Z^{J_2})\ldots\text{Tr}(Z^{J_k}),$$

with anomalous dimension

$$\mathcal{E}_{n;J_0}^{J_0;J_1,...,J_k} = 4\sin^2\left(\frac{\pi n}{J_0 + 1}\right).$$

These degeneracies between BMN states with different numbers of traces were what rendered the non-planar problem of $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ intractable. The degeneracies are less pronounced in the case of gauge group $SO(N)$ due to the gauging of the parity symmetry. The first case of planar degenerate BMN states in the $SO(N)$ case is the degeneracy between the states $\mathcal{O}_3^8$ and $\mathcal{O}_3^{2;4}$. The second case is the degeneracy between the operators $\mathcal{O}_3^{14}$ and $\mathcal{O}_3^{8;6}$. Using the full Hamiltonian we can easily check if the first non-planar correction which is of order $1/N$ lifts the degeneracy in these two cases and it turns out that it does. There is thus no hint of non-planar integrability from this analysis.

Another option to test for integrability is to directly try to construct conserved charges commuting with the Hamiltonian. In the higher loop analysis of $\mathcal{N} = 4$ SYM it was found that such conserved charges could be constructed.
order by order in the coupling constant, $\lambda$ \cite{2}. More generally one can generate perturbatively integrable long range spin chains with GL(K) symmetry starting from chains with nearest neighbour interactions \cite{34, 23}. The construction can be elegantly described in terms of a master symmetry \cite{35} or a boost operator \cite{36} and leads to a large family of long range perturbatively integrable spin chains \cite{37, 38}. These techniques do unfortunately not immediately apply to our case as they require that the spin chain length exceeds the range of the interaction. Nevertheless, we will discuss the possibility of constructing higher conserved charges perturbatively in $1/N$. For spin chains with local interactions integrability follows as soon as a single additional charge commuting with the Hamiltonian can be found \cite{39, 40}. Again, this does not necessarily apply to our type of spin chain.

Since, as discussed earlier, the odd charges lose their meaning in our setting, where parity is gauged, at planar level the next higher conserved charge after the hamiltonian $\hat{H} = \hat{Q}_2$ is the even charge $\hat{Q}_4$. If we expand to first order in $1/N$,

$$\hat{H} = \hat{H}_0 + \frac{1}{N}\hat{H}_{flip}, \quad \hat{Q}_4 = \hat{Q}_4^{(0)} + \frac{1}{N}\hat{Q}_4^{(1)},$$

our task is to determine a suitable $\hat{Q}_4^{(1)}$ such that

$$[\hat{H}_0, \hat{Q}_4^{(1)}] + [\hat{H}_{flip}, \hat{Q}_4^{(0)}] = 0.$$  \hspace{1cm} (39)

Since $\hat{H}_{flip}$ only acts within a single trace, we can assume the same about $\hat{Q}_4^{(1)}$. At the planar level, the higher charges can be constructed iteratively starting from the Hamiltonian by means of the boost operator $\hat{B}$ \cite{41}, i.e.

$$[\hat{B}, \hat{Q}_n^{(0)}] = \hat{Q}_{n+1}^{(0)},$$

where $\hat{B}$ is a moment of the Hamiltonian:

$$\hat{B} = \frac{1}{2i}\sum_{j=1}^{L} j \sigma_j \cdot \sigma_{j+1},$$ \hspace{1cm} (41)

with the $\sigma$’s being the Pauli matrices.

Ignoring constants and terms commuting with $\hat{H}^{(0)}$, this gives\footnote{This matches the expression for $\hat{Q}_4^{(0)}$ given in \cite{42}, up to the terms mentioned.}

$$\hat{Q}_4^{(0)} = \sum_{i=1}^{L} (-8[P_{i,i+3}P_{i+1,i+2} - P_{i,i+2}P_{i+1,i+3}] + 4P_{i,i+3} - 4P_{i,i+2}).$$ \hspace{1cm} (42)
Lacking a constructive way of extending this expression beyond the planar level, we have tried to guess a possible form by first rewriting all the permutation operators in terms of nearest-neighbour ones:

\[
P_{i,i+3} = P_{i+2,i+3}P_{i+1,i+2}P_{i+1,i+2}P_{i+2,i+3} \quad \text{and} \quad P_{i,i+2} = P_{i+1,i+2}P_{i,i+1}P_{i+1,i+2}
\]

and then using the relation \( P_{i,i+1} = I_{i,i+1} - 2H^{(0)}_{i,i+1} \) (cf. eqn. 9) to rewrite \( \hat{Q}^{(0)}_4 \) in terms of the planar Hamiltonian. Having done this (with the caveat that the rewritings in (43) are not unique), it is then natural to introduce a dependence on \( \hat{H}^{\text{flip}} \) by perturbing as:

\[
H^{(0)}_{i,i+1} \rightarrow H^{(0)}_{i,i+1} + \frac{1}{N}H^{\text{flip}}_i,
\]

where we have decomposed \( \hat{H}^{\text{flip}} \) as

\[
\hat{H}^{\text{flip}} = \sum_{i=1}^{L} H^{\text{flip}}_i.
\]

More precisely, we define \( H^{\text{flip}}_i \) by

\[
H^{\text{flip}}_i = \sum_{j=1}^{L} H^{\text{flip}}_{ij},
\]

with \( H^{\text{flip}}_{ij} \) acting on sites \( i \) and \( j \) of a periodic chain of length \( L \) as, (cf. eqn. 14)

\[
H^{\text{flip}}_{ij} (\mathcal{M}_{L-j+1,L-j+i-1} \otimes a_i \otimes \mathcal{N}_{1,j-i-1} \otimes b_j \otimes \mathcal{M}_{1,L-j})
= -\frac{1}{2} \left( (\mathcal{N}^T \otimes \mathcal{M})_{L-i,L-2} \otimes [a_i, b_j]_\otimes \otimes (\mathcal{N}^T \otimes \mathcal{M})_{1,L-i-1} \right) - \frac{1}{2} \left( (\mathcal{M} \otimes \mathcal{N}^T)_{L-i,L-2} \otimes [a_i, b_j]_\otimes \otimes (\mathcal{M} \otimes \mathcal{N}^T)_{1,L-i-1} \right).
\]

\(^6\)Note that there is an ambiguity in the location of the index \( i \) on the chain after the action of \( H^{\text{flip}}_i \), which we have fixed by cyclically shifting the resulting chain by a suitable number of sites, such that the first term of the commutator \( [a_i, b_j] \) always ends up at position \( i \). Keeping track of \( i \) is important when deforming the higher charges, since in a typical term \( H^{\text{flip}}_i \) will be preceded or followed by e.g. \( H^{(0)}_{i,i+1} \) or \( H^{(0)}_{i+1,i+2} \) and the sum over \( i \) is performed only at the end.
Here we have defined \( M_{k,l} = m_k \otimes m_{k+1} \cdots m_{l-1} \otimes m_l \) and similarly for \( N \).

The expression for \( \hat{Q}^{(1)}_4 \) obtained by inserting (44) into (42) is too long to be reproduced here, but with the help of computer algebra we can check whether (39) is satisfied. This turns out not to be the case for our naive guess for \( \hat{Q}^{(1)}_4 \). Given the amount of ambiguity involved in obtaining \( \hat{Q}^{(1)}_4 \), this is perhaps not surprising, and outlines the need for a more systematic approach.

A third way to look for integrability is to see if the first few non-planar corrections can be reproduced from a perturbative Bethe ansatz as was the case in the higher loop analysis of \([2, 18]\). The most obvious way to check this is to simply try and derive a set of Bethe equations, for instance using the coordinate space approach. This direct approach is, however, not straightforward. First, it is not clear how to implement the gauging of parity in a convenient way in this language. Secondly, it is obvious that our spin chain does not have an asymptotic regime since, as soon as we go beyond the planar limit, all sites of the chain interact with each other. Therefore, we will take a more naive approach.

Let us recall the perturbative Bethe equation for \( \mathcal{N} = 4 \) SYM with gauge group \( SU(N) \). For operators of length \( L \) containing \( M \) \( \phi \)-fields and \((L-M)\) \( Z \)-fields (with \( M \leq L/2 \)) it reads

\[
\left( \frac{x(u_k + \frac{i}{2})}{x(u_k - \frac{i}{2})} \right)^L = \prod_{j \neq k}^M \frac{u_k - u_j + i}{u_k - u_j - i}, \tag{48}
\]

where

\[
x(u) = \frac{1}{2}u + \frac{1}{2}\sqrt{u^2 - 2g^2} \equiv u(1 - g^2f(u)), \tag{49}
\]

and where \( g^2 = \frac{g_{YM}^2}{8\pi^2} \). Here \( u \) is related to the momentum \( p \) via

\[
e^{ip} = \frac{x^+(u)}{x^-(u)}, \tag{50}
\]

with

\[
x^\pm(u) = x(u \pm \frac{i}{2}). \tag{51}
\]

For later convenience we notice that purely algebraic arguments pertaining to the symmetry properties of the full \( \mathcal{N} = 4 \) SYM (and not just its \( SU(2) \)-sector) imply that one needs \([13]\)

\[
x^+ + \frac{g^2}{2x^+} - x^- - \frac{g^2}{2x^-} = i, \tag{52}
\]
which is of course fulfilled by the function $x(u)$ given above. Furthermore, we have the cyclicity constraint \[13\] and the energy is given as

$$E = \sum_k \frac{1}{g^2} \left( \sqrt{1 + 8g^2 \sin^2 \left( \frac{p_k}{2} \right)} - 1 \right). \quad (53)$$

For BMN states with two excitations we have $M = 2$, $L = J + 2$. Following \[18\] and expanding the Bethe root $u \equiv u_1 = -u_2$ as

$$u = u_0 + g^2 \delta u, \quad (54)$$

we find from the Bethe equation to order $g^2$

$$\delta u = \frac{u_0}{u_0^2 + \frac{1}{4}} \left( \frac{J + 2}{J + 1} \right), \quad (55)$$

and consequently, with $E = E_0 + g^2 \delta E$,

$$\delta E_{SU(N)} = -16 \sin^4 \left( \frac{n\pi}{J + 1} \right) - 64 \frac{1}{J + 1} \cos^2 \left( \frac{n\pi}{J + 1} \right) \sin^4 \left( \frac{n\pi}{J + 1} \right). \quad (56)$$

where the first term comes from the correction to the dispersion relation and the second one from the correction of the momenta. Let us rewrite the first $\frac{1}{N}$-correction to the BMN states of the $SO(N)$ gauge theory in a similar way

$$\delta E_{SO(N)} = - \sin^2 \left( \frac{n\pi}{J + 1} \right) \quad (57)$$

$$- \frac{1}{J + 1} \left\{ 2 \tan^2 \left( \frac{n\pi}{J + 1} \right) - \frac{1}{2} \tan^2 \left( \frac{2n\pi}{J + 1} \right) \right\}. \quad (57)$$

From this expression it is clear that if this were to arise from a Bethe system the first term would have to originate from a correction of the dispersion relation and the second one from a correction of the rapidities, i.e. a correction of the Bethe equations. The needed correction of the rapidities would be

$$\delta u = - \frac{1}{J + 1} \frac{4u_0^2 + 1}{64u_0^3 (4u_0^2 - 1)}. \quad (58)$$

There are of course many possible ways to deform the Bethe equations so that we would get the rapidity corrections for two-excitation states appearing in \[18\]. Given a plausible deformation one can test if it gives the correct
answer for the energy of states with more excitations which we can of course
again compute using quantum mechanical perturbation theory. Let us illus-
trate this with a simple example. Parametrising the function \( x(u) \) as

\[
x(u) = u(1 - \frac{1}{N} f(u)),
\]

we find that in order to correctly reproduce the \( \frac{1}{N} \)-correction to the energies
of the two-excitation states the function \( f(u) \) needs to fulfill the following
equation

\[
f_-(u) \equiv f(u + \frac{i}{2}) - f(u - \frac{i}{2}) = -i \frac{1}{16u^3(4u^2 - 1)}.
\]

This implies that \( f(u) \) can neither be written as a Taylor expansion nor
as a Laurent expansion in \( u \). Notice, however, that to solve the modified
Bethe equations perturbatively we would only need to know \( f_-(u) \). We have
checked whether the Bethe equations with the expression for the \( x(u) \) given in
eqn. (59) and the dispersion relation corrected by the first term in eqn. (57)
correctly reproduce the energy of states with four excitations and length
eight, cf. Appendix A. We found that the simple modification of the Bethe
ansatz described above does not lead to the correct non-planar correction to
the energy of any of these states. Now, one may ask whether the algebraic
arguments which led to (52) and (53) are valid for the non-planar case as
well. It follows from the analysis of reference [43] that the dispersion relation
can indeed be modified to include a correction which would lead to the first
term in the relation (57). However, the relation (52) to leading order in \( \lambda \)
simply becomes \( x^+(u) - x^-(u) = i \) which leads to the following constraint
on the function \( f(u) \)

\[
f(u + \frac{i}{2}) + f(u - \frac{i}{2}) = 2iu \left[ f(u + \frac{i}{2}) - f(u - \frac{i}{2}) \right].
\]

This constraint is unfortunately incompatible with the relation (60). Thus
the naive proposal for the modification of the Bethe ansatz would anyway
not have a chance to work for the full \( \mathcal{N} = 4 \) SYM theory.

Obviously, there are many other possible ways to deform the Bethe ansatz.
In particular, there is the possibility of including a phase factor [44]. This
would, in the simplest possible approach, mean modifying the Bethe ansatz
to
\[
\left( \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^L = \prod_{j \neq k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i} \left( 1 + \frac{i}{N} h(u_k - u_j) \right).
\]
(62)

Here we have for simplicity assumed that the phase factor depends only on the difference of rapidities and that the modification of the Bethe equations is due to the appearance of a phase factor alone. Demanding again the modification of rapidities to be given by (58) we find for the function \( h(u) \)
\[
h(u) = \frac{1}{2u^3(u^2 - 1)}. \tag{63}
\]

Note the non-trivial fact that \( h(u) \) is real for real \( u \) and that \( h(u) \) does not depend on the length of the spin chain. We have checked if the modified Bethe equation (62) correctly reproduce the energy correction for length eight and four excitations. Unfortunately, this is not the case. Needless to say that the tests performed here do not exclude the existence of a modified Bethe ansatz.

6 Comments on the string theory side

As discussed in the previous sections, the spectral problem of \( SO(N) \) and \( Sp(N) \) \( \mathcal{N} = 4 \) SYM theory exhibits several interesting differences compared to the \( SU(N) \) case. In this section we make some preliminary observations on how these differences manifest themselves on the string theory side.

In sections 2 and 3 we sketched how the \( \mathcal{N} = 4 \) SYM theory with orthogonal or symplectic gauge group can be obtained by performing an orientifold operation on a stack of D3-branes. Taking the near-horizon limit we find that the AdS/CFT dual gravity background should be given by an orientifold of \( \text{AdS}_5 \times S^5 \) \cite{14}. Embedding the sphere in \( \mathbb{R}^6 \) as
\[
\sum_{i=1}^{6} (X^i)^2 = 1,
\]
(64)
this orientifold is a combination of the \( \mathbb{Z}_2 \) action \( X^i \rightarrow -X^i \) and the world-sheet orientation reversal \( \sigma \rightarrow 2\pi - \sigma \). Note that the \( \mathbb{Z}_2 \) acts without fixed points on \( S^5 \) and thus there is no orientifold plane. Consequently, there is no need for additional branes to cancel the orientifold plane charge, and thus no
open string sector. Therefore, this setting still corresponds to an $\mathcal{N} = 4$ theory\footnote{Orientifolds of $\mathcal{N} = 4$ SYM with fixed planes, which lead to $\mathcal{N} = 2$ conformal theories with additional flavours, have been considered in an integrability context in \cite{45,20,21}.}. The dual geometry is now $\text{AdS}_5 \times \mathbb{RP}^5$, and the difference between the $SO(N)$ and $Sp(N)$ projections lies in the presence of an additional B-field.

As discussed in \cite{46}, in the strict planar (free string) limit all correlation function calculations in the orientifolded theory can be reduced, up to trivial rescalings, to those in the oriented one. We thus do not expect our picture of planar integrability to be modified in a major way. Of course, any spinning string solutions on $S^5$ not invariant under the orientifold procedure will be projected out.

Therefore, in the planar limit the differences to the $S^5$ case are relatively minor and arise only because some spinning string solutions on $S^5$ are not invariant under the orientifold transformation and are projected out. This corresponds to the fact, discussed in section 2, that certain gauge theory operators are projected out, depending on their length and parity. Unfortunately, since the semi-classical string solutions have large length, the distinction between odd and even length is not as apparent as on the gauge theory side. It would be interesting to do a thorough analysis of spinning strings on $\text{AdS}_5 \times \mathbb{RP}^5$ along the lines of \cite{47,48,49} and we hope to return to this problem in the future.

For the moment, however, we will confine ourselves to the straightforward observation that, by analogy with other contexts involving orientifolds, one can obtain invariant solutions by extending known ones with the addition of mirror strings. Let us demonstrate this for the $SU(2)$ sector, in which classical string solutions can be described in terms of their profile on an $S^2$ inside $S^5$. This $S^2$ is defined by $\sum_{i=1}^{3} (x^i)^2 = 1$, where we have written the coordinates of $S^5$ as $X_1 \pm iX_4 = x^1 \exp(\pm i\phi_1)$, etc. Then the orientifold projection can be taken to act on the coordinates of this $S^2$ as $x^i \rightarrow -x^i$, resulting in the real projective space $\mathbb{RP}^2$. Now, given any string solution with a profile $x^i(\sigma)$ for $0 \leq \sigma < 2\pi$ on $S^2$, we can construct a “doubled” solution on $\mathbb{RP}^2$ by taking the profile to be $x^i(\sigma)$ for $0 \leq \sigma < \pi$ and $-x^i(\sigma)$ for $\pi \leq \sigma < 2\pi$. See Fig. 2 for a drawing of such a solution on $\mathbb{RP}^2$. Note that, despite appearances, the string in the figure is a closed string, since antipodal points are identified on $\mathbb{RP}^2$. The energy of such strings is always quadratic in $x^i(\sigma)$, so it will be exactly the same as the solution on $S^2$.\footnote{For the purpose of comparing with weak coupling results, it might thus be more appropriate to use a different normalisation of the $SU(N)$ and $SO(N)$ generators in the gauge
Figure 2: A closed string solution on $\mathbb{R}P^2$ which is invariant under the orientifold. The configuration $X(\sigma = 0) = x_A$, $X(\sigma = \pi) = x_B = x_C \sim -x_B$, $X(\sigma = 2\pi) = x_D$ is invariant under $X^i \rightarrow -X^i$ and $\sigma \rightarrow 2\pi - \sigma$.

Arguing in this way, it seems that any solution which in the original AdS$_5 \times S^5$ geometry is confined to a half $S^2$ (the fundamental domain of $\mathbb{R}P^2$) inside the $S^5$, can be extended to a solution in AdS$_5 \times \mathbb{R}P^5$ by superimposing it with its mirror under the transformation $X_i \rightarrow -X_i$ and $\sigma \rightarrow 2\pi - \sigma$. This includes for instance the giant magnon solution [50] and the folded spinning string solution [47].

Things become more interesting when considering $\frac{1}{N}$-corrections, which correspond to turning on string interactions. Recall that the analogue of a spin chain splitting–and–joining operation is a process where a string decays into two strings, which later recombine, creating a worldsheet of genus one. Such processes are not well understood, even in the pp-wave geometry, the main obstacle coming from the necessity of summing over the infinite number of intermediate states (see [30] for a discussion). A simple model for splitting and joining of semi-classical strings in AdS$_5 \times S^5$ was presented in [55]. However, as discussed (in a simplified model) in [50], semi-classical splitting–and-joining does not seem to capture all of the relevant physics.

In our SO($N$) case, apart from the splitting–and–joining terms $\hat{H}_+$ and $\hat{H}_-$, the dilatation operator contains an additional term which we have denoted by $\hat{H}_{\text{flip}}$. What is the analogue of this term on the string side? It

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9Giant magnon solutions on $\mathbb{R}P^2$ have previously appeared in the context of the AdS$_4 \times$ CP$^3$ dual of ABJM theory, where the $\mathbb{R}P^2$ in that context arises as a suitable subspace of CP$^3$ [51, 52, 53, 54]. The main difference in our case is that, since we are dealing with an orientifold, we additionally need to implement the worldsheet identification $\sigma \rightarrow 2\pi - \sigma$.  

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will clearly be related to the fact that, due to the orientifold operation, one should now also consider non-orientable string worldsheets, or in other words worldsheets with cross-caps. Recall the weighting of a worldsheet with $b$ boundaries (each with $N$ Chan-Paton factors), $c$ cross-caps and $g$ handles:

$$
(N g_s)^b g_c g_s^{2g-2} = \lambda^{2g-2+2b+c} N^{-c-2g+2},
$$

where on the right-hand side we have rewritten the result in terms of gauge theory quantities, where the 't Hooft coupling is $\lambda = g_{YM}^2 N = g_s N$. We see that a cross-cap weights the amplitude by a factor of $1/N$ compared to the oriented amplitude, while a handle by a factor of $1/N^2$. See Fig. 3. The cross-cap contribution thus, as expected, appears at the same order as the leading contribution from $\hat{H}_{\text{flip}}$ on the gauge theory side and it is natural to identify the two. Intuitively, it is also clear that $\hat{H}_{\text{flip}}$ is associated with cross-caps since the operator acts by cutting out a piece of an operator and gluing it back in with the opposite orientation. Since it does not require summation over all intermediate states, the cross-cap calculation on the string theory side could be expected to be simpler than the genus-one case.

It would be very interesting to perform such a non-oriented string calculation and compare with the gauge theory side. Especially using a pp-wave geometry one might be able to compare with our gauge theory results for BMN operators, cf. section 4.
7 Conclusion

We have studied a number of features which distinguish the spectral problem of $\mathcal{N} = 4$ SYM with gauge group $SO(N)$ or $Sp(N)$ from that of $\mathcal{N} = 4$ SYM with gauge group $SU(N)$. Of particular interest to us was the difference in the leading non-planar corrections. For orthogonal and symplectic gauge groups the leading non-planar corrections define a novel type of spin chain interaction of highly non-local nature which cuts out a piece of the chain and re-inserts it with the opposite orientation. Unlike the case of gauge group $SU(N)$, the leading non-planar corrections a priori could fit into the standard framework of integrability. However, the resulting spin chain did not show any signs of integrability when studied by usual methods. In particular, our attempts to describe the diagonalization problem for $\hat{H}_{\text{flip}}$ by means of a Bethe ansatz were unsuccessful. However, given that the spin chain described by this Hamiltonian seems to lack an asymptotic regime (since all sites of the chain are involved in the interaction) it could still be that integrability, if present, simply cannot be formulated in terms of a Bethe ansatz.

Just as $\mathcal{N} = 4$ SYM with orthogonal or symplectic gauge group is much less studied than its $SU(N)$ cousin, the same holds for the dual string theories. Here we briefly discussed some issues related to studying the spectrum of type IIB string theory on the $\text{AdS}_5 \times \mathbb{RP}^5$ background. We mentioned some features of spinning string solutions and discussed how the leading non-planar corrections to anomalous dimensions on the gauge theory side should originate from non-oriented string worldsheets with a single cross-cap. By considering such worldsheets, one might hope to reproduce the leading non-planar corrections for two-excitation states that we found from the gauge theory side. More generally, as cross-caps might be easier to handle than higher genus surfaces, this might open new possibilities for comparing gauge and string theories beyond the planar limit.

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A Numerical tests of Bethe equations.

We specify here the details of the numerical tests we performed. We focused on the (single trace) states of length eight with four excitations. There are three such highest weight states. At one loop order at the planar level they can be described in terms of the corresponding roots of the Bethe equations given in (10). The three sets of roots \( \{u^1_i\}, \{u^2_i\} \) and \( \{u^3_i\}, i \in \{1, 2, 3, 4\} \) read\(^{10}\)

\[
\begin{align*}
\{u^1_i\} &= \{\pm 0.525, \pm 0.129\}, \\
\{u^2_i\} &= \{\pm 0.0413, \pm 1.026i\}, \\
\{u^3_i\} &= \{\pm 0.463 \pm 0.502i\},
\end{align*}
\]

and the corresponding planar one-loop energies, \( E^j_0, j = 1, 2, 3 \) are the roots of the polynomial

\[ -x^3 + 10x^2 - 29x + 200 = 0. \]

By direct diagonalization of \( H_0 + \frac{1}{N} \hat{H}_{flip} \) we find the \( \frac{1}{N} \)-corrections to the energies, \( E^i_1 \) to be\(^{11}\)

\[
E^1_1 = 1.618, \quad E^2_1 = -6.75, \quad E^3_1 = -19.85.
\]

On the other hand solving the Bethe ansatz (48) with \( x(u) \) given by (59) and (60) we find the following \( \frac{1}{N} \)-correction to the rapidities

\[
\begin{align*}
\delta u^1_1 &= \{\pm 0.0255 \pm 0.000893i\}, \\
\delta u^2_2 &= \{\pm 47.6, \pm 138.4i\}, \\
\delta u^3_3 &= \{\pm 3.65, \pm 10.74\},
\end{align*}
\]

which leads to the following \( \frac{1}{N} \)-correction to the energies

\[
E^1_1 = -0.43, \quad E^2_1 = -504, \quad E^3_1 = -26.6.
\]

These values clearly differ from the exact ones given in eqn. (70).

\(^{10}\)These roots as well as others can be found in references [17, 18].

\(^{11}\)We remark that the operators considered here do not exhibit degeneracy with any multi-trace states and thus there are no further corrections to their energies of order \( \frac{1}{N} \).
Using instead the deformed Bethe ansatz given by (62) and (63) the $\frac{1}{N}$-correction to the Bethe roots are

\[
\{\delta u^1_i\} = \{\pm 1.146 \pm 0.0327i\}, \quad (75)
\]
\[
\{\delta u^2_i\} = \{\pm 5.96, \pm 17.29i\}, \quad (76)
\]
\[
\{\delta u^3_i\} = \{\pm 0.799, \pm 1.045\}, \quad (77)
\]

and the energy corrections, $E^i_1$ become

\[
E^1_1 = -2.07, \quad E^2_1 = -63.6, \quad E^3_1 = -8.25. \quad (78)
\]

These values also fail to agree with the exact ones given in eqn. (70).

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