Non-adiabatic elimination of auxiliary modes in continuous quantum measurements

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When measuring a complex quantum system, we are often interested in only a few degrees of freedom—the plant, while the rest of them are collected as auxiliary modes—the bath. The bath can have finite memory (non-Markovian), and simply ignoring its dynamics, i.e., adiabatically eliminating it, will prevent us from predicting the true quantum behavior of the plant. We generalize the technique introduced by Strunz et. al. [Phys. Rev. Lett. 82, 1801 (1999)], and develop a formalism that allows us to eliminate the bath non-adiabatically in continuous quantum measurements, and obtain a non-Markovian stochastic master equation for the plant which we focus on. We apply this formalism to three interesting examples relevant to current experiments.

Introduction.—Recent developments in techniques of high-precision metrology have allowed quantum-level measurement and control of matters of at all scales, ranging from single atoms [1] to macroscopic mechanical oscillators [2]. In these experiments, the atoms or mechanical oscillators, as objects of interest (or the plant), are usually coupled to auxiliary degrees of freedom (or the bath), e.g., the cavity mode in cavity QED systems, which are in turn coupled to external readout devices. It is often desirable to obtain a self-contained equation for the state of the plant, by eliminating bath degrees of freedom, especially when we want to implement a real-time feedback control.

In the literature, the simplest approach is to ignore the dynamics of bath modes by assuming that they follow the plant's pure state under continuous measurement. This becomes inadequate when bath modes evolve at scales longer than the plant, i.e., when the system becomes non-Markovian.

One way to account for a non-Markovian bath is the Feynman-Vernon influence functional method [3, 4]. Diósi and Strunz et. al. [4–6] developed an equivalent (but much simpler) method by unraveling the bath evolution into possible quantum trajectories. These trajectories are shown to drive a non-Markovian stochastic Schrödinger equation (SSE), which average into the exact non-Markovian master equation. Although their model does not include measurement a priori, the SSE at the Markovian limit can be interpreted as the evolution of the plant's pure state under continuous measurement. In general, however, the physical interpretation of non-Markovian SSE in terms of measurement has yet to be clarified, as discussed by Diosi [7] and Wiseman et al. [8].

Here we consider the non-Markovian measurement process involving a plant-bath system, in which the plant has finite memory, and measurement is done through the bath. We further assume that the incoming probe field for measurement is a quantum Wiener process [9], and that the output field is projectively measured [10]. The setup is shown in Fig. 1. In addition to the usual applied assumptions for the plant-bath interaction—the bath is bosonic and couples linearly to the plant, we assume that the bath is also linearly coupled to the probe field. We do not make assumptions on the plant, nor the plant quantity that couples to the bath. By generalizing the Diosi-Strunz approach, we show that the bath can be eliminated from the full evolution equation, resulting in a non-Markovian Stochastic Master Equation (SME) which governs the density matrix of the plant, and thereby has a distinctive physical meaning.

The prescription we develop here can be applied to a wide range of non-Markovian quantum measurements, thereby laying the foundation for an interesting research direction. The purpose of this Letter is to present the general formalism and highlight three examples of relevance to current experiments, for which analytical forms of the SME can be obtained. Interestingly, in two of the three cases, the non-Markovian dynamics we obtain differs from conventional wisdom.

Model.—The Hamiltonian for our model reads:

\[
\hat{H} = \hat{H}_p + \hat{H}_b + \hat{H}_{\text{int}} + \sum_k \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k + \int_0^t dt \gamma_k \hat{b}_k^\dagger \hat{b}_k(t] + \hat{a}_k^\dagger \hat{b}_k(t],
\]

\[
\hat{H}_b \equiv \sum_k \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k, \quad \hat{H}_{\text{int}} \equiv \sum_k \hbar g_k (\hat{L}^\dagger \hat{a}_k + \hat{L} \hat{a}_k).
\]

(1)

Here \(\hat{H}_p\), \(\hat{H}_b\) and \(\hat{H}_{\text{int}}\) are the plant, bath, and interaction Hamiltonians, respectively: \(\hat{a}_k\) and \(\omega_k\) are the annihilation operators and eigenfrequencies of different bath modes and \([\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}\); the plant operator couples to the bath through \(\hat{L}\), with \(g_k\) its coupling constant to the \(k\)-mode; \(\hat{b}_k(t)\) are annihilation operators for the input probe field at different times and \([\hat{b}_k(t), \hat{b}_{k'}^\dagger (t')] = \delta(t - t')\); \(\gamma_k\) is the coupling strength between the bath and the probe field. We exclude those modes that are not coupled to the probe field, as they will simply in-

![FIG. 1: (color online) Schematics of our measurement process. The plant is coupled to the bath, which in turn couples to an external probe field. The output probe field amplitude is projectively measured by a detector.](image-url)
introduce decoherence, which has already been discussed extensively in the literature. In addition, we only consider one probe field, and can be easily generalized to multiple probe fields.

Conditional dynamics.—At each moment, the output probe field $\hat{b}_{out}(t)$ is projectively measured by a detector, e.g., homodyne detection if the probe field is an optical field. We assume that (phase) quadrature $\hat{b}_2 = [\hat{b}_{out}(t) - \hat{b}_{out}^\dagger(t)]/\sqrt{2}$ is measured with the result at time $t$ being $y(t)$. Given the measurement result, the plant-bath system is projected into a conditional state, with joint wave function $|\psi\rangle$ at $t + dt$ given by

$$|\psi(t + dt)\rangle = \frac{1}{P_{1/2}} (y(t)|\hat{U}(dt)|0\rangle \otimes |\psi(t)\rangle).$$

Here $\hat{U}(dt) = e^{-i\hat{H} dt/\hbar}$ is an evolution operator; we assume that the input probe field (before interaction) is at vacuum state $|0\rangle$ and is separable from the joint plant-bath state; $|y(t)\rangle$ is an eigenstate of $\hat{b}_2(t)$; $P(y)$ is the probability density for the measurement result and

$$P(y) = \text{Tr}_{p+b}(|y(t)\rangle\langle y(t)|).$$

By integrating over the probe field variable, we can obtain the following non-linear Markovian SSE for the plant-bath state:

$$d|\psi\rangle = \frac{-i}{\hbar} (\hat{H}_p + \hat{H}_b + \hat{H}_{\text{int}})|\psi\rangle dt - \sum_{kk'} \sqrt{\gamma_k/2} \left[ \hat{a}_k^\dagger \hat{a}_{k'} \right.\langle \hat{a}_{k'} - \hat{a}_k | |\psi\rangle \langle \psi | \hat{a}_k \rangle + \left. \langle \hat{a}_k - \hat{a}_{k'} | |\psi\rangle \langle \psi | \hat{a}_{k'} \rangle \right] \langle \psi | \hat{a}_{k'} \rangle \langle \psi | \hat{a}_k \rangle + \sum_k i \gamma_k/2 (2\langle \hat{a}_k \rangle \langle \psi | \hat{a}_k \rangle - \langle \hat{a}_k - \hat{a}_k^\dagger | |\psi\rangle \langle \psi | \hat{a}_k \rangle) \langle \psi \rangle dt,$$

and $y(t)dt = \sum_k \sqrt{\gamma_k} \text{Tr}[\hat{a}_k + \hat{a}_k^\dagger]dW + \sqrt{2}(\hat{a} - \hat{a}_k^\dagger)\langle \hat{a}_k^\dagger | y(t)\rangle \langle y(t) | \hat{a}_k \rangle$, where we have introduced:

$$\hat{g}_k = i \int d^2 \alpha e^{-|\alpha|^2} \partial_{\alpha*} |\psi(\alpha^*)\rangle \langle \psi(\alpha^*)|.$$  

Here the non-Markovianity only arises when we eliminate the bath, which has a memory about the plant. Eqs. 1 and 5 will be self-contained SMEs governing the plant and measurement data, if $\hat{g}_k$ can be written in terms of $\hat{\rho}_p$ and other plant operators. To derive $\hat{g}_k$, we use the approach in Ref. [6] by introducing the plant operator $\hat{O}_k$ as follows:

$$\partial_{\alpha*} |\psi(\alpha^*)\rangle \equiv -i \hat{O}_k(t, \alpha^*) |\psi(\alpha^*)\rangle.$$

In the simplest case, $\hat{O}_k$ does not depend on $\alpha^*$ and $\hat{g}_k = \hat{O}_k(t) \hat{\rho}_p$. In general, $\hat{g}_k$ is a super-operator of $\hat{\rho}_p$:

$$\hat{g}_k = \hat{A}_k(t) \hat{\rho}_p + \hat{\rho}_p \hat{A}_k^\dagger(t) + \hat{A}_{2k}(t) \hat{\rho}_p \hat{A}_{2k}(t),$$

where $\hat{A}_k$ are plant operators determined from $\hat{O}_k$. Systematic procedures for deriving $\hat{O}_k$ (without measurement) has been developed by Yu et al. [13], and applied to systems with different plant Hamiltonians. Yu’s method can be generalized to our case by using interaction-picture $|\psi(\alpha^*)\rangle_t = U^{-1}(t) |\psi(\alpha^*)\rangle$ with a non-unitary evolution operator:

$$U(t) = \exp[-i/\hbar (\hat{H}_p + \hat{H}_b - \sum_k \sqrt{\gamma_k} \hat{a}_k^\dagger \hat{a}_k + \hat{\rho}_p \hat{A}_k(t))].$$

In general, $\hat{O}_k$ is difficult to solve for analytically and must be considered case by case. In the following, we shall consider three interesting examples that are closely related to current experiments, and analytical forms of $\hat{O}_k$, or equivalently $\hat{g}_k$, can be obtained.

Atom-cavity interaction.—As shown schematically in Fig 2 we consider the following Hamiltonian:

$$\hat{H} = \hbar \omega_0 \hat{\sigma}_z + \hbar \omega_c \hat{a}^\dagger \hat{a} + \hbar g (\hat{\sigma}_- \hat{a}^\dagger + \hat{\sigma}_+ \hat{a}) + \hbar \sqrt{\gamma} (\hat{b}_\text{in}^\dagger(t)e^{i\omega t} + \hat{b}_\text{in}(t)e^{-i\omega t}).$$

The first three terms describe the Jaynes-Cummings-type interaction with $\omega_0$ the atom transition frequency and $\hat{\sigma}_z$ the cavity transition frequency, which is measured via homodyne detection.

FIG. 2: Schematics showing the atom-cavity system. A two-level atom (or a qubit) interacts with a cavity mode that is coupled to an external continuous optical field which is measured via homodyne detection.
the Pauli matrix, and ωc and ω0 are the cavity resonant frequency and the laser frequency, respectively. In the rotating frame at the laser frequency, the Hamiltonian can be rewritten as: 

$$ \hat{H} = \hbar (\omega_q/2) \hat{\sigma}_z + \hbar \Delta \hat{a}^{\dagger} \hat{a} + \hbar g (\hat{\sigma}_- \hat{a}^{\dagger} + \hat{\sigma}_+ \hat{a}) + \hbar \sqrt{\gamma} [\hat{a} \hat{b}_{in}(t) + \hat{a}^{\dagger} \hat{b}_{in}(t)] $$

with Δ ≡ ωc − ω0. In comparison with the general Hamiltonian in Eq. (1), this corresponds to the case of $L = \hat{\sigma}_-$. and $g_k = g \delta_k$ (the bath has only one cavity mode and we will ignore subscript $k$ afterwards). By using the consistency condition (3), the operator $\hat{O} = f(t) \hat{\sigma}_- \text{ and } \hat{\rho}$ has the following simple form:

$$ \dot{\hat{\rho}} = f(t) \hat{\sigma}_- \hat{\rho}. $$  (10)

Here the time-dependent function $f(t)$ satisfies a Riccati equation, $\dot{f} - i(\omega_q - \Delta + i\gamma)f - g f^2 = g$ with the initial condition $f(0) = 0$, from the assumption that the cavity mode is initially at a vacuum state. The corresponding SME for the atom density matrix reads:

$$ \dot{\hat{\rho}} = -i \left[ \frac{\omega_q}{2} \hat{\sigma}_z + \hbar \frac{3}{2} \{ f \} \hat{\sigma}_+ \hat{\sigma}_- \hat{\rho} \right] dt - g \Re \{ f \} \left[ \hat{\sigma}_+ \hat{\sigma}_- \hat{\rho} + \hat{\rho} \hat{\sigma}_+ \hat{\sigma}_- - 2 \hat{\sigma}_- \hat{\rho} \hat{\sigma}_+ \right] dt + \sqrt{2\gamma} [f \hat{\sigma}_- \hat{\rho} + f^* \hat{\rho} \hat{\sigma}_- - (f \hat{\sigma}_- + f^* \hat{\sigma}_+) \hat{\rho}] \text{dW}. $$  (11)

This equation fully describes non-Markovian dynamics of the atom under continuous measurement. We can also obtain the corresponding master equation if we ignore the measurement result by averaging over dW (mean of dW vanishes), namely,

$$ \dot{\hat{\rho}} = -i \left[ \frac{\omega_q}{2} \hat{\sigma}_z + \hbar \frac{3}{2} \{ f \} \hat{\sigma}_+ \hat{\sigma}_- \hat{\rho} \right] - g \Re \{ f \} \left[ \hat{\sigma}_+ \hat{\sigma}_- \hat{\rho} + \hat{\rho} \hat{\sigma}_+ \hat{\sigma}_- - 2 \hat{\sigma}_- \hat{\rho} \hat{\sigma}_+ \right]. $$  (12)

This gives the exact non-Markovian master equation for a two-level atom coupled to a damped cavity mode—a dissipative environment. Note that it differs from the Markovian SME and master equation, respectively.

The result at the Markovian limit can be recovered by considering the case with the cavity decay rate much larger than the atom-cavity interaction rate and also the atom transition rate, namely $\gamma \gg g$ and $\gamma \gg \omega_q$. The cavity mode memory becomes negligibly short and

$$ f(t) \text{Markovian limit } = g/\gamma, $$  (13)

in which case Eqs (11) and (12) reduce to the usual Markovian SME and master equation, respectively.

To confirm that Eq. (11) is the SME that correctly describes the conditional dynamics of the atom, we numerically solve (i) the Markovian SSE for the joint atom-cavity wave function and (ii) the non-Markovian SME for the atom density matrix to see whether they both give the same conditional mean of $\sigma_x$, $\sigma_y$ and $\sigma_z$. The numerical results are shown in Fig. 4. We have chosen $\omega_q = 1$, $\Delta = 1$ and $\gamma = 2$, and the initial state for the atom and the cavity mode is $\left( |+\rangle_z + |\rangle_z \right) / \sqrt{2} \otimes |0\rangle$. They indeed agree with each other nicely as shown by convergence of their difference.

**Linear optomechanical interaction.**—We now consider another exactly solvable model—the linear optomechanical interaction between a harmonic mechanical oscillator and a cavity mode. The device is shown schematically in Fig. 4 which has been discussed extensively in the literature recently [2]. The Hamiltonian reads [15–17]:

$$ H = \frac{p^2}{2m} + m \omega_m^2 x^2/2 + \hbar \omega_c a^\dagger a + \hbar g \hat{x} a^\dagger a^\dagger + \hbar \sqrt{\gamma} [\hat{a} \hat{b}_{in}(t) e^{i\omega_0 t} + \hat{a}^\dagger \hat{b}_{in}(t) e^{-i\omega_0 t}]. $$  (14)

Here $\hat{x}$ and $\hat{p}$ are the position and momentum of the oscillator with eigenfrequency $\omega_m$. Since the cavity mode usually has a large steady-state amplitude due to coherent pumping by the laser, we can consider perturbations around the steady-state amplitude and linearize the above Hamiltonian. In the rotating frame at the laser frequency, the linearized Hamiltonian is

$$ \hat{H} = \frac{p^2}{2m} + m \omega_m^2 x^2/2 + \hbar \Delta \hat{a}^\dagger \hat{a} + g \hat{g} \hat{x} (\hat{a}^\dagger + \hat{a}) + \hbar \sqrt{\gamma} [\hat{a} \hat{b}_{in}(t) + \hat{a}^\dagger \hat{b}_{in}(t)], $$

where $g' \equiv g \bar{a}$ with $\bar{a}$ the steady-state amplitude of the cavity mode. With the same procedure as the atom-cavity case, $\hat{\rho}$ can be obtained (again the bath has one mode with subscript $k$ ignored):

$$ \dot{\hat{\rho}} = (f_1 \hat{\rho} \hat{\Delta}^\dagger + \hat{\Delta} \hat{\rho}) / (1 - |f_1|^2). $$  (15)

![FIG. 3: The top panel shows numerical results of the time evolution of the conditional means: (σx, σy) and (σz) given a particular realization of dW. The bottom panel shows the convergency of the accumulated numerical difference between the SSE and SME simulation results given different number of grid points for the cavity mode.](image-url)

![FIG. 4: (color online) Schematics showing a typical optomechanical device. The mechanical oscillator is coupled to a cavity mode via radiation pressure force.](image-url)
with \( \hat{A} = e^{-i(\Delta-i\gamma)t}[f_0(t) + f_x(t)\hat{x}] + f_p(t)\hat{p} \). These functions \( f_0, f_1, f_x, \) and \( f_p \) are determined from the consistent condition, and satisfy coupled Riccati equations:

\[
\begin{align*}
\dot{f}_0 &= i\gamma \text{Tr}\{\hat{\sigma} + \hat{\sigma}^\dagger\}f_1 - i\sqrt{2\gamma}f_1\hat{W} - i\hbar g f_0 f_p, \\
\dot{f}_x &= e^{i(\Delta-i\gamma)t}(g' + m\omega_m^2 f_p) - ig' f_1 + h x f_p, \\
\dot{f}_p &= -i(\Delta-i\gamma)f_p - (f_x/m)e^{-i(\Delta-i\gamma)t} - i\hbar g' f_p^2, \\
\dot{f}_1 &= -i(\Delta-i\gamma)f_1 + g' f_p e^{i(\Delta-i\gamma)t} - i\hbar g f_1 f_p.
\end{align*}
\]

These equations can be solved numerically. Similarly, if we average the SME over \( dW \), we will obtain the corresponding non-Markovian master equation. It describes quantum Brownian motion of a harmonic oscillator coupled to a non-Markovian bath with dissipation, which has not yet been fully treated in the literature.

**Weak-coupling limit.**—In the previous cases, we took advantage of the linear interaction. In general, when \( \hat{L} \) is a nonlinear operator of the plant, there is no transparent route that leads to a closed-form solution of \( \hat{\sigma} \). If the plant-bath coupling is weak, namely \( g_k < \gamma_k \), we can perturbatively solve the problem by writing down a hierarchy of equations at different orders of \( g_k/\gamma_k \). The first-order result for the \( \hat{\sigma} \) is very elegant:

\[
\hat{\sigma} = \sum_{k'} \int_0^t \dd{\tau} e^{-i\nu \tau} k_k^* \hat{L}(-\tau)\hat{\rho}
\]

where \( \hat{L}(-\tau) = e^{-iH_r/\hbar} \hat{L} e^{iH_r/\hbar} \) under free evolution.

One interesting application of this result is to study the phonon-counting experiment recently considered in Refs. [18, 21]. The position of a mechanical oscillator is quadratically coupled to a cavity mode, namely \( \hat{H}_{\text{int}} = \hbar g \hat{X}^2 (\hat{a} + \hat{a}^\dagger) \) (\( \hat{X} \) is the position operator normalized by the zero-point uncertainty). If cavity bandwidth \( \gamma \) is less than the mechanical frequency \( \omega_m \), only the time average of \( \hat{X}^2 \)—equivalent to phonon number—is important, and we expect a direct probe of mechanical energy quantization. In the proposed experiment by Thompson et al. [19], the coupling strength \( g \) is smaller than \( \gamma \) [22]; we can therefore use Eq. (20). From \( \hat{X}(-\tau) = \hat{X} \cos \omega_m \tau - \hat{P} \sin \omega_m \tau \), we have

\[
\dot{\hat{\sigma}} = \int_0^t \dd{\tau} e^{-i\gamma \tau} \hat{X}^2(-\tau)\hat{\rho} \approx (g/\gamma) \hat{N}\hat{\rho}
\]

where \( \hat{N} \) is the phonon number, and we have ignored terms proportional to \( e^{-t/\gamma} \), as the characteristic measurement time scale is \( t \sim \gamma^{-1} \). The resulting SME for the mechanical oscillator density matrix reads [cf. Eq. (1)]:

\[
\dot{\hat{\rho}} = -i[\omega_m \hat{N}, \hat{\rho}] \dd{\tau} - g_{\text{eff}}[\hat{X}^2, [\hat{N}, \hat{\rho}]] \dd{\tau} + \sqrt{2g_{\text{eff}}} (\hat{N}, \hat{\rho}) \dd{W} + \mathcal{O}[(g/\gamma)^2]
\]

with \( g_{\text{eff}} = g^2/\gamma \). Note that this does not describe a quantum non-demolition (QND) measurement of the phonon number, as has been argued for above, since the term \( [\hat{X}^2, [\hat{N}, \hat{\rho}]] \) is not in the usual Lindblad form \([\hat{N}, [\hat{N}, \hat{\rho}]]\). It will introduce two-phonon process and cause additional diffusion; we may therefore encounter unexpected features in the actual experiment.

**Conclusions.**—We have reported a formalism that non-adiabatically eliminates bath modes in continuous quantum measurements and yields a self-contained non-Markovian SME for the conditional density matrix of the plant. Conceptually, this formalism is the mathematical embodiment of how memory-induced non-Markovianity arises when we focus on a subsystem of a larger, Markovian system. In practice, if the plant is indeed all we care about, the non-Markovian dynamics obtained here is an exact and the most efficient way of obtaining its evolution, both in terms of analytical and numerical complexity. By averaging over measurement results, the resulting master equation describes the non-Markovian dynamics of the plant coupled to a bath that suffers from additional dissipation, a scenario not yet fully explored in the literature. We have briefly illustrated the powerfulness of this formalism using three examples, and we fully expect that it will find wide theoretical and experimental applications.

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**Note added.**—During the preparation of this draft, we notice that a similar model is considered by Diósi [23].

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