Fluctuations and scaling of inverse participation ratios in random binary resonant composites

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Abstract

We study the statistics of local field distribution solved by the Green’s-function formalism (GFF) [Y. Gu et al., Phys. Rev. B 59 12847 (1999)] in the disordered binary resonant composites. For a percolating network, the inverse participation ratios (IPR) with $q = 2$ are illustrated, as well as the typical local field distributions of localized and extended states. Numerical calculations indicate that for a definite fraction $p$ the distribution function of IPR $P_q$ has a scale invariant form. It is also shown the scaling behavior of the ensemble averaged $\langle P_q \rangle$ described by the fractal dimension $D_q$. To relate the eigenvectors correlations to resonance level statistics, the axial symmetry between $D_2$ and the spectral compressibility $\chi$ is obtained.

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The statistics of eigenfunctions of the random matrix with orthogonal, unitary and symplectic symmetry has been well reviewed by Guhr\textsuperscript{1} and Mirlin\textsuperscript{2}. A strong fluctuation of eigenfunctions at the critical point is one of the prominent hallmarks of Anderson metal-insulator transition. These fluctuations can be represented by a set of inverse participation ratios (IPR) \( P_q = \int d^d r |\Psi(r)|^2 q \). Wegner put forward from the renormalization group treatment of the supersymmetry \( \delta \) model in \( 2 + \epsilon \) dimensions that IPR show the multifractal scaling\textsuperscript{3,4} with respect to the system size \( L \), \( P_q = AL^{-D_q(q-1)} \), in a good metal, while in an insulator, \( P_q \propto L^0 \). For the critical power-law random bounded matrix (PRBM) ensemble\textsuperscript{5} with \( \beta = 1 \), not only the distribution function of the critical \( P_q \) was found to have a scale invariant form, but also the scaling exponent \( D_q \) was obtained from the shift of the distribution of IPR with the size \( N \). At the criticality, the multifractality of the ensemble-averaged \( \langle P_q \rangle \) has been derived analytically in the limits of weak and strong couplings, as well as calculated numerically in the full range of couplings\textsuperscript{6}. Recently, local field distribution of resonant composites has attracted great interest. Various optical eigenmode localizations, such as surface-plasmon modes in the colloid clusters\textsuperscript{7}, localized dipolar excitations on the roughly nanostructured surfaces\textsuperscript{8}, and selective photomodification in the fractal aggregates of colloidal particles\textsuperscript{9}, were reported to be relevant to the specific filed distribution. It was found that the local field at the percolation threshold fluctuates from one optical eigenmode to another\textsuperscript{10,11}. Some attempts to investigate the scaling of the successive local electric field moments have been made\textsuperscript{11−13}. However, it is difficult to take into account the structure sensitivity to local field distribution when resonance happens. In order to study the statistics of resonant composites, one must answer the following questions. How do the optical eigenmodes distribute in the resonant area? For each mode, how about its local field distribution? And how to understand the fluctuations and scaling of the local field distribution? It is known that the resonance spectrum and local field distribution for each eigenmode can be solved analytically using the Green’s-function formalism (GFF)\textsuperscript{14,15}. The first question has been answered in view of level spacing statistics\textsuperscript{14,16}. In this work, local field distributions of typical localized and extended eigenstates are illustrated, and to answer the third
question, the statistics of local field distribution is numerically studied from the random matrix theory (RMT).

In this work, a binary network is considered where the impurity bonds with admittance $\epsilon_1$ are employed to replace the bonds in an otherwise homogeneous network of identical admittance $\epsilon_2$. The admittance of each bond is generally complex and frequency-dependent. All the impurity bonds construct the clusters subspace. Using the GFF, the eigenvalues and eigenvectors of Green’s-matrix $M$ can be solved analytically\textsuperscript{14,15}. Because $M$ maps the geometric configuration of the clusters subspace, it is a random matrix where RMT can set in. Its solutions summarize all the geometric resonances of the clusters subject to the external sources and in the quasistatic limit. The element of $M$ is defined as $M_{x,y} = \sum_{z \in C(y)}(G_{x,y} - G_{x,z})$ in which $z \in C(y)$ means that the jointing points $z$ and $y$ belong to the clusters subspace and are the nearest neighbors, and $G_{x,y}$ is the Green’s function of Laplace operator on the infinite square, i.e., $-\Delta G_{x,y} = \delta_{x,y}$ with $G_{x,x} = 0$. More clearly, $M_{x,y}$ describes the interaction between $x$ and $y$, and is closely related to the “environment” or the nearest neighbors of $y$. Different from the quantum systems, local field distribution is not directly equal to the eigenvector of $M$. When resonance happens, the local field $V_x$ can be expressed as a separate product of one sum of right eigenvectors of $M$ and another sum of left eigenvectors that depends on the source term. Motivated by that local field distribution may have the general statistical features, this paper mainly devotes to the fluctuations and scaling of the IPR.

In the following, we start from solving Green’s-matrix $M$ numerically in the disordered binary networks. The distribution functions of the IPR are calculated for system size ranging from $L = 16$ to $L = 38$ and for various values of $p$. For the ensemble averaging over 1000 samples are used in the percolating case at $L = 30$, for each of which there are more than 750 nontrivial eigenstates to be produced. In the calculation of the scaling behavior of the IPR, $\langle P_q \rangle$ is averaged over $7 \times 10^5$ right eigenvectors. The difference between using 1000 samples of $30 \times 30$ system and using a large system, say $300 \times 300$, is that there is a minor correction in the calculations. Numerical calculations indicate that the distribution function of IPR $P_q$
has a scale invariant form for a definite $p$ within the interval $[0.3, 0.5]$. It is also found the scaling behavior of the ensemble averaged $\langle P_q \rangle$ described by the fractal dimension $D_q$. The multifractal scaling of $\langle P_q \rangle$ with respect to $L$ is obtained for $p = 0.5$ and $p = 0.4$, as well as the fractal behavior of $\langle P_q \rangle$ for $p = 0.3$. Note that here $L$ represents the scale of square network, while in Refs. [3] and [4], $N$ is the size of $N \times N$ matrix. To relate the level statistics to eigenvectors correlations, the symmetry between $D_2$ and the spectral compressibility $\chi$ is obtained.

In order to have a direct observation, figure 1 displays the values of IPR of right eigenvectors with $q = 2$ for a $40 \times 40$ percolating network. The peaks represent the localized states and valleys the extended states. It is shown that peaks and valleys randomly distribute in the resonant area. The localized states incline to accumulate near $s \to 0$ and $s \to 1.0$, while at about $s \to 0.5$, more extended states are found. Various optical excitations$^7$-$^9$, optical nonlinear enhancements$^{13,17}$, and eigenmode localizations$^{18}$ are typical localized states with the high values of IPR. For a localized state, it is found that the value of $P_2$ of right eigenvector is always high, as well as the IPR of potentials and electric fields$^{14}$. Therefore, it is reasonable to represent the localization or extension of eigenstates only by the IPR of right eigenvectors. When resonance happens, the inhomogeneous local field around the impurity metallic clusters leads to a large enhancement in the effective linear and nonlinear optical responses. It is expected that the high values of IPR, or the localized states, correspond to the strong nonlinear optical enhancements. There are an anomalous absorption in the infrared area in percolating metal-dielectric thin films$^{11,20}$ and a large optical nonlinear enhancement in the region of high frequencies$^{14}$. In ref. [14], the separation of the absorption peak from nonlinear enhancement peak is discussed. Different from the causes in the dilute anisotropic networks$^{19}$, it originates in the self-similar percolating structure.

Then, corresponding to the different values of IPR in Fig. 1, local field distributions of typical localized and extended states are illustrated in the forms of 3D plots. Fig. 2(a) shows a very localized state with the high value of $P_2$. It is seen that when one eigenmode is excited, the rest of the volume remains almost unexcited$^7$. For simplicity, those strong
localized fields are called “hot spots”. In this figure, the brightest “hot spot” is found. In Fig. 2(b), an extended state with the low value of $P_2$ is plotted, where local fields are not uniform and many smaller and weaker “hot spots” are found. In Ref. [14], we also shown an intermediate state with the average value of $P_2$, where we can see more “hot spots” and local fields are very inhomogeneous. “Hot spots” are found to be sensitive to the admittance ratio $h(=\epsilon_1/\epsilon_2)$ and external fields. They also lead to the photomodification\(^9\) and huge enhanced nonlinear responses\(^{13,17}\).

When we plot the IPR of several samples with different size, it is intuitive that the general feature of the distribution function of the IPR should exist. Figure 3 displays the distributions of $\ln(P_q/\langle P_q \rangle)$ for $p = p_c = 0.5$, where $P_q$ is normalized by $\langle P_q \rangle$ and $q$ is ranged from 2 to 8. It is seen that the shape of the distribution function changes with increasing $q$. It is also shown that for the distribution function of $\ln(P_q/\langle P_q \rangle)$ the change of $p$ leads to the shift and deformation of the shape\(^{14}\). This is different from the recent argument of Mirlin in which different parameter $b$ leads to the changes of the distribution function of $\ln(P_q)$ for the critical PRBM ensemble\(^6\).

Figure 4 displays the distributions of $\ln(P_2)$ with the system size ranged from 16 to 32 stepped by 4 for $p = p_c = 0.5$, i.e., the percolating case. While for $p = 0.4$ and $p = 0.3$, the distribution functions of $\ln(P_2)$ with the different system size are also calculated\(^{14}\). A scale invariant form of the IPR distribution is found for the critical case $p_c = 0.5$, as well as for the noncritical $p = 0.4$ and $p = 0.3$. Note that in the Refs. \[1\] and \[2\], only at the criticality of PRBM ensemble, a scale invariant form of the distribution of the IPR is investigated. The shift of the curves implies a scaling of distribution function of the IPR with respect to $L$. There exist different scaling exponents $D_q$ for the different values of $q$ at $p = 0.5$, 0.4, and 0.3, as verified by the ensemble averaging $\langle P_q \rangle$ in ref. \[3\]. All curves are not strictly overlapped when we shift them by $\ln \langle P_q \rangle$, especially for the large $q$. One possible reason is that the sample is not large enough. Therefore, from Figures 3 and 4 we can conclude that for a definite $p$ the fluctuations of the IPR have the general statistical features.

Different from the eigenfunctions of the Anderson metal-insulator Hamiltonian and
PRBM ensemble, the right eigenvectors of Green’s-matrix $M$ are the main product of the local field $V$ for the binary composites. In ref. [14], we found the scaling of $\langle P_q \rangle$ with respect to the size $L$ for $p_c = 0.5$, $p = 0.4$ and $p = 0.3$. By the relation, $\langle P_q \rangle = L^{-\tau_q}$, the scaling exponent $\tau_q$ can be calculated through the slope of $\ln \langle P_q \rangle$ with $\ln(L)$. The results are shown in Fig. 5. For $p = 0.5$ and $p = 0.4$, we observe the multifractal scaling of $\langle P_q \rangle$. Note that for $p = 0.4$ it is not a critical case. While in Ref. [21] and Ref. [3], the multifractal scaling is investigated only in the critical ensembles. However, for $p = 0.3$, only the fractal behavior of the ensemble averaged $\langle P_q \rangle$ is found and for $q > 2$ the extended line of $\tau_q$ can be approximately approaching the original point $(0, 0)$. By the relation $\tau_q = D_q \times (q - 1)$, the values of $D_q$ are obtained. Here $D_2$ is not closely related to the spatial geometry as claimed in Ref. [21].

Finally, to connect with the level spacing statistics, the scaling exponent $D_2$ and spectral compressibility $\chi$ are shown in Fig. 6, and the axial symmetry according to their mean values is found when $p$ is ranged in the interval $[0.3, 0.5]$. The result apparently disagrees with the claim of Ref. [4], where $D_2$ and $\chi$ satisfy $D_2 + 2\chi = 1$ for the large $b$, and $2D_2 + \chi = 1$ for the small $b$. It is also violated with the recent statement [21], where between $D_2$ and $\chi$ the exact relation holds, $\chi = (d - D)/2d$.

In a summary, we have not only presented a detailed study on the local field distribution near resonance, but also the statistics of local field distribution by means of the IPR for the disordered binary networks. On the qualitative level, the main findings are summarized as follows: 1) For the percolating composites, the IPR of the right eigenvectors are illustrated, as well as the typical local field distributions of the localized and extended states in the forms of 3D plots. When resonance happens, the fields are localized within the impurity metallic clusters. For a percolating network, a large enhancement in the effective linear and nonlinear optical responses is also found [14]. 2) The distribution functions of the IPR $P_q$ have a scale invariant form in the limit of the large system size $L$ for the critical case, $p_c = 0.5$, as well as for the noncritical case, $p = 0.4$ or $p = 0.3$. 3) The multifractal scaling of the ensemble averaged $\langle P_q \rangle$ with the system size $L$ is obtained for $p_c = 0.5$ and $p = 0.4$, while
for $p = 0.3$, only the fractal behavior is found. 4) To relate the eigenvectors correlations to level spacing statistics, the axial symmetry between $D_2$ and $\chi$ is obtained. Finally, it is worthwhile to emphasize on the extent of the universality of the IPR distribution at the noncritical case, such as $p = 0.4$ or $p = 0.3$. In the previous works$^{5,6}$, only at the criticality, the fluctuations and scaling of the IPR were investigated.
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FIGURES

FIG. 1. IPR of right eigenvectors with \( q = 2 \) at the percolating threshold \( p_c \). Here we use a 40 \times 40 sample with 1331 resonances.

FIG. 2. The typical local field distributions of localized and extended eigenstates. \( a \) with a high value of IPR; \( b \) with a low value of IPR.

FIG. 3. Distribution functions of \( \ln(P_q/\langle P_q \rangle) \) with respect to \( \ln(P_q/\langle P_q \rangle) \) for \( q \) ranged from 2 to 8 and for the percolating networks. Here \( L = 32 \) and the distributions are averaged over 900 samples.

FIG. 4. Scale invariant form of distribution functions \( \ln(P_2) \) with respect to \( \ln(P_2) \) for \( p_c = 0.5 \).

FIG. 5. Diagram of multifractal scaling \( \tau_q \) for \( p = 0.5, 0.4, \) and 0.3. For \( p = 0.3 \), the extended line of \( \tau_q \) can be approaching to the point \( (0, 0) \) when \( q > 2 \), therefore only fractal behavior is found.

FIG. 6. The fractal dimension \( D_2(\text{circles}) \) and the spectral compressibility \( \chi(\text{squares}) \) as a function of the parameter \( p \).
Distribution of $\ln(p_q/\langle p_q \rangle)$
Distribution of $\ln(p_2)$
