(1, q = −1) Model as a Topological Description of 2d String Theory

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Abstract

We study the (1, q = −1) model coupled to topological gravity as a candidate to describing 2d string theory at the self-dual radius. We define the model by analytical continuation of q > 1 topological recursion relations to q = −1. We show that at genus zero the q = −1 recursion relations yield the $W_{1+\infty}$ Ward identities for tachyon correlators on the sphere. A scheme for computing correlation functions of q = −1 gravitational descendants is proposed and applied for the computation of several correlators. It is suggested that the latter correspond to correlators of discrete states of the c = 1 string. In a similar manner to the q > 1 models, we show that there exist topological recursion relations for the correlators in the q = −1 theory that consist of only one and two splittings of the Riemann surface. Using a postulated regularized contact, we prove that the genus one q = −1 recursion relations for tachyon correlators coincide with the $W_{1+\infty}$ Ward identities on the torus. We argue that the structure of these recursion relations coincides with that of the $W_{1+\infty}$ Ward identities for any genus.

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1 Introduction

Non-critical string theory in general and the $c \leq 1$ string models in particular attracted much attention in recent years. They provide a framework for studying fundamental questions of string theory and quantum gravity such as non-perturbative structure. The most challenging of these class of models is the $c = 1$ string. It differs from the $c < 1$ string models by that it has a space-time description and a propagating massless degree of freedom, the tachyon. Furthermore, BRST analysis in the continuum description of the theory revealed the existence of special states at discrete values of momentum \cite{1, 2, 3, 4, 5}, one of which was argued to correspond to a discrete graviton describing the two-dimensional black hole of \cite{6}.

$c < 1$ non-critical string models constructed by coupling $c < 1$ conformal matter to two-dimensional gravity, have been studied using a variety of methods reflecting their integrable and topological structures. Their topological phase is realized by $(1, q)$ minimal topological matter coupled to topological gravity \cite{4, 5}, from which they can be reached by appropriate perturbations. Intersection theory interpretation of the correlators has been given in \cite{3, 10}.

A $q^{th}$ KdV integrable hierarchy underlies the $(1, q)$ topological models and the corresponding $(p, q)$ non-critical strings \cite{11}. This together with the string equation provides a complete description of the theories. The generating functions for correlators of the theories are $\tau$–functions of the corresponding integrable hierarchies \cite{12, 13}. Matrix integral representations of the generating functions have been suggested by \cite{14, 15, 16}. For the case $q = 2$, that is pure topological gravity, they reduce to the Kontsevich integral \cite{14}. However, while the latter integral has a nice interpretation in terms of cell decomposition of the moduli space of Riemann surfaces, the geometrical meanings of the general integrals are less clear. Nevertheless, these matrix integrals provide a link between the integrable and the topological structures. A manifestation of the integrable and topological structures is also provided by a set of Ward identities given by $W_q$ constraints on the partition function of the theory. This is argued to be equivalent to specifying the integrable hierarchy together with a string equation \cite{16} and to a set of topological recursion relations \cite{17} as we discuss in the sequel.

Another relation between the integrable and the topological structures of the theories has been discovered using the topological Landau-Ginzburg formulation \cite{18}. The integrable hierarchy appears in the Landau-Ginzburg description in its Lax formulation. The Landau-Ginzburg superpotential is identified with the Lax operator in the dispersionless
There exist two types of topological recursion relations for correlators of \((1, q)\) theories. The basic strategy behind both is analysis of contributions to the correlators from the boundary of the moduli space. One set of recursion relations has been derived, first for pure topological gravity in [19, 20] and later generalized to topological matter coupled to topological gravity [11]. The second set has been derived for pure topological gravity in [21]. In the latter the concept of contact algebra was introduced, which together with the requirement of the invariance of the correlators under the interchange of operators provides a complete solution of the theory. The, thus derived, topological recursion relations coincide with the Virasoro constraints on the partition function of the theory. The uniqueness of the contact algebra underlying the theory has been proved in [22].

The generalization of the contact algebra to \((1, q)\) models coupled to topological gravity was suggested in [17], and was argued to correspond to the \(W_q\) constraints on the partition function. A key role in this topological procedure is played by multicontacts whose importance was realized in [8, 12, 22]. Correlators of general \((1, q)\) models on the sphere, as well as those of \((1, 3)\) model on higher genus, were consistently calculated using the topological scheme. The computation of genus \(g\) correlators for \(q > 3\) requires a consistent regularization scheme which is still lacking. However, as shown in [23], one can overcome this problem since the recursion relations can be recast in a form that involves only one and two contacts.

The topological description of the \(c = 1\) string is conjectured to correspond to \((1, q)\) model coupled to topological gravity, analytically continued to \(q = -1\) or equivalently, to \(N = 2\) twisted minimal model coupled to topological gravity. Indeed the cohomology of the latter, realized an \(SU(2)/U(1)\) coset at level \(k = -3\), coincides with that of the \(c = 1\) string [24, 25]. Interpretation of intersection theory calculations of correlators at \(k = -3\) as correlators in \(c = 1\) string theory is in agreement with matrix model results. First, the partition functions were observed to be identical [10]. Second, the four-point correlator computed in [10], when analytically to \(k = -3\) was shown to be the tachyons four-point function [24]. Third, \(1 \rightarrow n\) amplitudes as well as five-point function at various kinematic regions agree [24].

The integrable hierarchy underlying the \(c = 1\) string is the Toda-lattice hierarchy [27]. The generating function for tachyon correlators is a \(\tau\)–function of the hierarchy. There exist a matrix integral representation of the latter [27], generalizing that of the \(c < 1\) case.
Recently the string equations were constructed \([30, 31]\).

The topological Landau-Ginzburg description of the theory is constructed as the \(A_{k+1}\) model at \(k = -3\) \([32, 33]\). A relation between the integrable and the topological structures is established by that the Landau-Ginzburg superpotential is the Baker-Akhiezer wave function of the Toda lattice hierarchy \([32]\). This parallels the identification of the superpotential with the Lax operator in the \(c < 1\) cases. There exist \(W_{1+\infty}\) Ward identities for tachyon correlators in the \(c = 1\) string \([28, 27]\). These relations determine completely the tachyon dynamics. The \(W_{1+\infty}\) algebra of constraints on the partition function is probably the \(c = 1\) analog of the \(W_q\) algebra of constraints in the \(c < 1\) cases. These Ward identities can be derived from the Toda lattice and the string equations \([30, 31]\). On the other hand they coincide with period integrals in the topological Landau-Ginzburg formulation of the theory \([32]\). These provide another link between the topological and the integrable structures of the \(c = 1\) string.

As described, the integrable and topological structures of the \(c = 1\) case parallel those of the \(c < 1\) cases. One of the main differences, however, is the lack of topological recursion relations in the former. It is straightforward to verify that Witten’s topological recursion relations are no longer correct for the \(c = 1\) string, and should be modified. However, the required modification is still unknown. Our aim in this paper is to generalize the second set of topological recursion relations as proposed in \([17]\) for the \(c < 1\) models, and provide a topological procedure for computation of correlators in \(c = 1\). The idea will be to analytically continue the \((1, q)\) recursion relations to \(q = -1\). We will show explicitly for genus zero and one that the analytically continued topological recursion relations coincide with the \(W_{1+\infty}\) Ward identities for tachyon correlators, and will argue that their structures coincide for any genus. We will further use the topological recursion relations to compute correlators of \(q = -1\) gravitational descendants which are suggested to correspond to discrete states of the \(c = 1\) string. Similarly to the \(c < 1\) cases, we show that the topological recursion relations can be recast in a form that involves only one and two splittings of the Riemann surface. This provides us with another set of Ward identities for the theory.

The paper is organized as follows. In section two we review the \((1, q)\) topological models and the topological procedure to compute their correlators. In section three the \(W_{1+\infty}\) Ward identities of 2d string theory at the self-dual radius are described. In section four we pose the rules for the analytic continuation of the topological recursion relations

\footnote{A two-matrix integral representation of the generating function for tachyon correlators has been proposed in \([29]\).}
from $q > 1$ to $q = -1$. We use the analytically continued scheme to compute up to five point tachyon correlators at genus zero. The results are in agreement with $c = 1$ matrix model calculations. The equivalence between the $(1, q = -1)$ recursion relations and the $W_{1+\infty}$ Ward identities at genus zero is proven in section five. In section six we propose a topological scheme for computation of correlators of $q = -1$ gravitational descendants, which we interpret as discrete states and compute several correlators. Topological recursion relations in terms of one and two splittings are written in section seven. Section eight is devoted to the study of higher genus recursion relations. We define a regularized contact and use it to prove the equivalence between the recursion relations and the $W_{1+\infty}$ Ward identities on the torus. Two and three point tachyon correlators on the torus are computed via the topological procedure. Section nine is devoted to discussion and conclusions. In appendix A we include an example of computation of three-point tachyon correlator on the sphere as well as two-point tachyon correlator on the torus using the new recursion relations introduced in section seven.

2 (1, q) topological models

(1, q) models form a special sub sector of the $(p, q)$ minimal models. Having zero physical fields, they are not well defined conformal field theories. However, they make sense as topological field theories, the so called topological minimal models [4, 8]. The observables consist of $q - 1$ primary fields $P_{0,\alpha}, \alpha = 1, ..., q - 1$. When coupled to topological gravity, a family of gravitational descendants $P_{k,\alpha}$ is associated with each primary field $P_{0,\alpha}$, where $k$ takes positive integer values. For $q = 2$ we get topological gravity with $P_{0,1}$ as the puncture operator. $P_{k,1}$ correspond in this case to the Mumford, Morita and Miller cohomology classes on the compactified moduli space $\bar{\mathcal{M}}_{g,s}$.

An integer ghost number is attributed to the fields:

$$gh(P_{k,\alpha}) = (k - 1)q + (\alpha - 1).$$

(2.1)

From the viewpoint of the integrable structure the ghost number basically corresponds to the power of the KP Lax operator associated with the field [34], while from the viewpoint of the topological structure it corresponds to the degree of the form on the moduli space associated with it [3, 11].

The ghost number conservation law for the genus $g$ correlator $\langle \prod_{i=1}^{s} P_{k_i,\alpha_i} \rangle_g$ reads

$$\sum_{i=1}^{s} gh(P_{k_i,\alpha_i}) = 2(g - 1)(1 + q).$$

(2.2)
The ghost number conservation follows from the requirement of having a residue in the KdV computational scheme, and from the demand that the total form associated with the correlator be a top form on the moduli space in the corresponding intersection theory.

An equivalent description, which is naturally analytically continued to the $q = -1$ case, is the following: $P_{k,\alpha} \to P_n$ where $n = kq + \alpha$. $P_n$ are in $1-1$ correspondence with $P_{k,\alpha}$:

$$\alpha = n \mod q, \quad k = \frac{(n - \alpha)}{q}. \quad (2.3)$$

The conservation law for the correlator $\langle \prod_{i=1}^{s} P_{n_i} \rangle_g$ is

$$\sum_{i=1}^{s} n_i = (s + 2g - 2)(q + 1). \quad (2.4)$$

A topological procedure to calculate correlation functions of the model which is equivalent to the $W_q$ constraints has been developed in [17]. In contrast to the $W_q$ Ward identities which are complicated and are not known in general, the topological procedure consists of simple topological rules. The idea behind the scheme is that the correlators can be determined by contacts between the operators and between them and the degenerations of the Riemann surface. This procedure yields topological recursion relations generalizing those proposed by [21] for topological gravity, i.e. the $(1,2)$ model. Let us briefly review it.

The first thing to notice is that the metric on the space of physical fields, defined by the genus zero two-point function, vanishes, since

$$\eta_{ij} \equiv \langle P_i P_j \rangle_0 = |i|\delta_{i+j,0}, \quad (2.5)$$

while the physical fields of the theory are $P_n$ with $n$ being positive integer charges. In order to overcome this difficulty, auxiliary unphysical fields with negative charges are introduced. These fields appear in the metric and decouple from higher point functions. Note that the definition of the metric (2.5) is differs from the standard one [33]: $\eta_{i,j} = \langle P_1 P_i P_j \rangle_0$ with $P_1$ being the puncture operator and $P_i, P_j$ are primary operators. The metric (2.5) is defined on the space of both primaries and descendants.

An identity operator to be inserted in degenerations is constructed in the usual way:

$$I = \sum_{i,j} |P_i\rangle \eta^{ij} \langle P_j| \quad (2.6)$$

As a consequence of introducing negative charge fields there exists a one point function that does not vanish on the sphere

$$\langle P_{-q-1} \rangle_0 = -q. \quad (2.7)$$
Consider now the genus $g$ correlation function $\langle P_n \prod_{i=1}^{m} P_{n_i} \rangle_g$. Denote $P_n$ as the marked operator, that is the operator that performs contacts in this procedure. It has contacts with $\alpha = n \mod q$ operators. The contact algebra reads:

$$P_n P_{\alpha_1} \ldots P_{\alpha_\alpha} = P_n + \sum_{k=1}^{\alpha} i_k - \alpha (q+1),$$

where over brace means contact. Contacts and degenerations are the ingredients for computing correlators in this scheme. At each degeneration one inserts a complete set of states. The topological procedure of [17] is summarized by the degeneration equation:

$$\sum_\Delta \langle P_n \prod_{i=1}^{m} P_{i} \rangle = 0,$$

where $\sum_\Delta$ means summation over all the degenerations with the first operator, i.e. $P_n$, performing the contacts. The contributions to the degeneration equation come from the boundary of the moduli space and are of three types: splitting, pinching of dividing cycles and pinching of nontrivial homology cycles.

The degeneration equation yields Ward identities for the $(1, q)$ models coupled to topological gravity. As an example to that consider the $(1, 2)$ model. The degeneration equation for $\langle P_n \prod_{i=1}^{m} P_{n_i} \rangle_g$ reads

$$\sum_j \langle \hat{P}_n \hat{P}_j \prod_{i=1}^{m} P_{n_i} \rangle_g \langle P_{-j} \rangle_0 + \sum_{j,k=1}^{m} \langle \hat{P}_n \hat{P}_j \prod_{k \neq i=1}^{m} P_{n_i} \rangle_g \langle P_{-j} P_{n_k} \rangle_0 +$$

$$\sum_{g' \neq j} \langle \hat{P}_n \hat{P}_j \prod_{k \in S_1} P_{n_k} \rangle_{g'} \langle P_{-j} \prod_{l \in S_2} P_{n_l} \rangle_{g-g'} +$$

$$\sum_{j} \langle \hat{P}_n \hat{P}_j \prod_{i=1}^{m} P_{n_i} \rangle_{g-1} = 0,$$

with $S_1 \cup S_2 = (1 \ldots m)$.

As shown in Fig.1, the first two terms in (2.10) correspond to splitting, the third to pinching of a dividing cycle and the last term corresponds to pinching a nontrivial homology cycle. Note that the second term in (2.10) may be considered from the degeneration equation viewpoint as a special case of the third term.
Using (2.5), (2.7) and (2.8) the degeneration equation (2.10) takes the form

\[
\langle P_n \prod_{i=1}^m P_{n_i} \rangle_g = \frac{1}{2} \left[ \sum_{j=1}^{m} n_j \langle P_{n+n_j-3} \prod_{j \neq i=1}^m P_{n_i} \rangle_g + \sum_{j \neq j'} (\sum_{k \in S_1} P_{n+k} \langle P_{n+j-3} \prod_{i \in S_2} P_{n_i} \rangle_{g'-g} + \sum_{j} \langle P_{n+j-3} \prod_{i=1}^m P_{n_i} \rangle_{g-1} \right],
\]

where we require the correlators with auxiliary fields to vanish. Equation (2.11) is the Verlinde’s recursion relation for topological gravity [21], with a difference of notation due to the difference between the definition of the ghost number (2.1) and the degree of the Mumford-Morita-Miller cohomology classes. It should be stressed, however, that we derive the recursion relations via a different procedure than that of [21].
Our aim is to relate the topological phase of 2d string theory to the \((1, q = -1)\) model coupled to topological gravity, defined via analytical continuation of the \(q > 1\) degeneration equation to \(q = -1\). This will provide us with topological recursion relations, which, as we shall argue, reduce for tachyon correlators to the \(W_{1+\infty}\) Ward identities of 2d string theory at the self-dual radius.

3 \(W_{1+\infty}\) Ward identities of 2d string theory

Tachyon dynamics in 2d string theory has been studied in the continuum \([36]\) as well as in the matrix formulation of the theory \([37]\). The full scattering matrix was computed \([38]\) and a set of \(W_{1+\infty}\) constraints on the amplitudes was derived \([27, 28]\). The latter form Ward identities that determine the tachyon correlators completely. In this section we review these \(W_{1+\infty}\) Ward identities and derive various formulas that will be needed later.

Introduce the notation:

\[
\langle O \rangle \equiv \sum_{g \geq 0} \frac{1}{\mu^{2g-2}} \langle O \rangle_g ,
\]

(3.1)

where expansion in \(\frac{1}{\mu^2}\) corresponds to genus expansion, and \(\langle O \rangle_g\) is the genus \(g\) correlator of \(O\).

The \(W_{1+\infty}\) Ward identities of 2d string theory read \([27]\):

\[
\langle\langle T_n \rangle\rangle \equiv \langle T_n \exp[\sum_{k=\infty}^{\infty} t_k T_k] \rangle = \bar{W}_{-n}^{(n+1)} Z ,
\]

(3.2)

where

\[
Z \equiv \langle \exp[\sum_{k=\infty}^{\infty} t_k T_k] \rangle .
\]

(3.3)

\(T_n\) is the tachyon of integer momentum \(n\), and \(t_n\) is the time associated with it. \(\bar{W}_{-n}^{(n+1)}\) is the \(-n\) mode of a spin \(n + 1\) current \(\bar{W}^{(n+1)}(x)\) and is given by \([27]\)

\[
\bar{W}_{-n}^{(n+1)} = \int dx \frac{(i\mu)^{-n+1}(x)}{n+1} : e^{-i\mu\varphi(x)} \partial_x^{n+1} e^{i\mu\varphi(x)} : ,
\]

(3.4)

where

\[
\partial\varphi(x) = \frac{1}{x} \left[ 1 + \sum_{k>0} t_{-k} x^k + \frac{1}{\mu^2} \sum_{k>0} k x^{-k} \partial_{-k} \right] ,
\]

(3.5)

with \(\partial_{-k} \equiv \frac{\partial}{\partial t_{-k}}\).
Evaluating (3.2) we get

\[
\begin{align*}
\langle\langle T_n \rangle\rangle_0 &= \frac{1}{n(n+1)}\text{res}(\bar{W}_n^{n+1}) , \\
\langle\langle T_n \rangle\rangle_1 &= \frac{1}{n}\text{res}(\bar{W}_0^n \bar{W}_1) - \frac{1}{24}(n-1)\text{res}(\bar{W}_0^{n-2} \bar{W}_0'') , \\
\langle\langle T_n \rangle\rangle_g &= \frac{1}{n}\text{res}(\bar{W}_0^n \bar{W}_g + n\bar{W}_0^{n-1} \bar{W}_1 \bar{W}_{g-1} + ...) \quad g > 1 ,
\end{align*}
\]

(3.6)

where \(\text{res}\) means picking the \(x^{-1}\) term in the Laurent expansion, \(\text{prime}\) denotes a derivative with respect to \(x\) and

\[
\begin{align*}
\bar{W}_0 &= \frac{1}{x}[1 + \sum_{k>0} t_{-k} x^k + \sum_{k>0} kx^{-k}\langle\langle T_{-k} \rangle\rangle_0] , \\
\bar{W}_g &= \sum_{k>0} kx^{-k-1}\langle\langle T_{-k} \rangle\rangle_g \quad g \geq 1 .
\end{align*}
\]

(3.7)

Define \(\Phi_{n}^{(g)} = \partial_n \bar{W}_g\), then

\[
\begin{align*}
\Phi_{n}^{(0)} &= \Theta(-n)x^{-n-1} + \sum_{k>0} x^{-k-1}\langle\langle T_n T_{-k} \rangle\rangle_0 , \\
\Phi_{n}^{(g)} &= \sum_{k>0} x^{-k-1}\langle\langle T_n T_{-k} \rangle\rangle_g .
\end{align*}
\]

(3.8)

In the topological Landau-Ginzburg formulation of the theory \(\bar{W}_0\) and \(\Phi_{n}^{(0)}\) correspond to the superpotential and to the Landau-Ginzburg field respectively \[32\].

For the explicit expansion of the \(W_{1+\infty}\) we will need the following formula:

\[
\partial_{n_1} \ldots \partial_{n_{m-1}} \Phi_{n_m}^{(g)}(t = 0) = (\sum_{i=1}^{m} n_i)\Theta(\sum_{i=1}^{m} n_i)x^{-1-\sum_{i=1}^{m} n_i}\langle\langle T_\sum_{i=1}^{m} n_i \prod_{i=1}^{m} T_m \rangle\rangle_g .
\]

(3.9)

In the sequel we will also need:

\[
\begin{align*}
\partial_{-n} \bar{W}_0''(t = 0) &= (n-1)(n-2)x^{n-3} , \\
\partial_{n_1} \ldots \partial_{n_m} \bar{W}_g''(t = 0) &= (\sum_{i=1}^{n} n_i)(1 + \sum_{i=1}^{m} n_i)(2 + \sum_{i=1}^{m} n_i) \\
\Theta(\sum_{i=1}^{m} n_i)x^{-3-\sum_{i=1}^{m} n_i}\langle\langle T_\sum_{i=1}^{m} n_i \prod_{i=1}^{m} T_m \rangle\rangle_g \quad g \geq 0 .
\end{align*}
\]

(3.10)

4 Genus zero tachyon correlators via \((1, q = -1)\) theory

The approach that we take in order to define the \((1, q = -1)\) theory is to analytically continue the \(q > 1\) degeneration equation. This will provide us with a set of topological
recursion relations for 2d string theory at the self-dual radius. The rules for the analytical continuation are the following: (i) we consider correlators of physical operators in $q > 1$ models, that is $P_n$ with $n$ positive and allow negative values only at the final analytically continued recursion relations. (ii) The argument of the Heaviside function $\Theta(x)$ that appears in the $q > 1$ degeneration equation, due to the decoupling of the auxiliary fields, will change sign at $q = -1$. The reasoning for this will be given in the sequel.

In order to demonstrate the analytic continuation procedure let us compute the tachyon correlators up to the five point function on the sphere.

Consider first the two-point function

$$\langle P_n P_{-n} \rangle = n ,$$

(4.1)

where $P_n$ being a primary operator. At $q = -1$ we identify the primary operator $P_n$ as the positive momentum tachyon $T_n$ and the auxiliary operator $P_{-n}$ as the negative momentum tachyon $T_{-n}$, thus

$$\langle T_n T_{-n} \rangle = -\frac{1}{n} .$$

(4.2)

Equation (4.2) differs by sign from the conventions of 2d string matrix model [27]. This overall sign difference will persist for all the tachyons correlators.

We have not used the degeneration equation yet, but it is used already for computing the tachyons three point function. We assume in the following that the marked operator is primary. In section 6 we will consider the case when the marked operator is a gravitational descendant. Consider the correlator $\langle P_n P_{n_1} P_{n_2} \rangle$, with all the operators in the correlator being physical. Thus in the framework of the $(1, q)$ models, the charges $n_i$ are positive, and we allow negative values in the final analytically continued formula.

Taking $P_n$ as the marked operator, the degeneration equation reads:

$$(n + 1)(-q)^n \langle P_n P_{n_1} P_{n_2} \rangle + (n + 1)n(-q)^{n-1} \langle P_{-n_1} P_{n_1} \rangle \langle P_{-n_2} P_{n_2} \rangle = 0 .$$

(4.3)

with the different terms depicted in Fig. 2.

1From now on, unless explicitly stated, we consider genus zero correlators.
Fig. 2: The degeneration equation for three-point function on the sphere

Solving for the required three-point function we get:

\[ \langle P_n P_{n_1} P_{n_2} \rangle = \frac{n n_1 n_2}{q} \]  (4.4)

After normalizing we have at \( q = -1 \)

\[ \langle T_n T_{n_1} T_{n_2} \rangle = -1 \]  (4.5)

Note that we can get the same answer by taking the marked operator to be \( P_{-n} \) and formally performing \(-n\) contacts. This parity invariance property of the procedure implies that we may think of \( T_{-n} \) as primaries with negative charges. We will further discuss this issue in section 6.

Consider now the correlator \( \langle P_n P_{n_1} P_{n_2} P_{n_3} \rangle \). The degeneration equation reads

\[
(-q)^n \langle P_n P_{n_1} P_{n_2} P_{n_3} \rangle + n(-q)^{n-1} \sum_{i=1}^{3} \langle P_{n+n_i-(q+1)} \prod_{i \neq j=1}^{3} P_{n_j} \rangle = 0 ,
\]

\[
\langle P_{-n_i} P_{n_i} \rangle + n(n-1)(-q)^{n-2} \prod_{i=1}^{3} \langle P_{-n_i} P_{n_i} \rangle = 0 ,
\]  (4.6)
functions. In a formulation that is more adequate for the $W_q$ models with $q > 1$ such a correlator is proportional to $\Theta(n + n_i - (q+1))$ since auxiliary fields should decouple from all the correlators besides the two-point functions. In a formulation that is more adequate for the $W_q$ constraints approach this can also be stated as $\Theta((k + k_i - 1) + \frac{(\alpha + n_i - 1)}{q})$. Thus, when we analytically continue to $q = -1$ the correlator becomes proportional to $\Theta(-n - n_i)$. This $\Theta$ term implies that there is no contact between two positive momentum tachyons as has been found in the topological Landau-Ginzburg description of the $c = 1$ string [22, 33].

Taking the fields in (4.6) as primaries or their auxiliary analogs and setting $q = -1$ we get the tachyons four point function

$$\langle T_n T_{n_1} T_{n_2} T_{n_3} \rangle = -(n - 1) + \sum_{i=1}^{3} (n + n_i) \Theta(-n - n_i) .$$

(4.7)

As a final example consider the five-point function $\langle P_n \prod_{i=1}^{4} P_{n_i} \rangle$. The degeneration equation reads

$$(-q)^n \langle P_n \prod_{i=1}^{4} P_{n_i} \rangle + n(-q)^{n-1} \sum_{i=1}^{4} \langle P_{n+n_i-(q+1)} \prod_{i \neq j=1}^{4} P_{n_j} \rangle \langle P_{-n_i} P_{n_i} \rangle$$

$$+ n(-q)^{n-1} \sum_{i,j=1,i \neq j}^{4} \langle P_{n+n_i+n_j-2(q+1)} \prod_{i,j \neq k=1}^{4} P_{n_k} \rangle \langle P_{-n_i-n_j+(q+1)} P_{n_i} P_{n_j} \rangle$$

$$+ n(n - 1)(-q)^{n-2} \sum_{i,j=1,i \neq j}^{4} \langle P_{n+n_i+n_j-2(q+1)} \prod_{i,j \neq k=1}^{4} P_{n_k} \rangle \langle P_{-n_i} P_{n_i} \rangle \langle P_{-n_j} P_{n_j} \rangle$$

$$+ n(n - 1)(n - 2)(-q)^{n-3} \prod_{i=1}^{4} \langle P_{-n_i} P_{n_i} \rangle = 0 .$$

(4.8)

Following the same procedure as before we get at $q = -1$ the tachyons five point function,

$$\langle T_n \prod_{i=1}^{4} T_{n_i} \rangle = -(n - 1)(n - 2) +$$

$$(n - 1) \sum_{i,j=1,i \neq j}^{4} (n + n_i + n_j) \Theta(-n - n_i - n_j) \langle T_{n+n_i+n_j} \prod_{i,j \neq k=1}^{4} T_{n_k} \rangle$$

$$+ \sum_{i=1}^{4} (n + n_i) \Theta(-n - n_i) \langle T_{n+n_i} \prod_{i \neq j=1}^{4} T_{n_j} \rangle .$$

(4.9)

Note that the splitting to two three-point functions in (4.8) vanishes when we analytically continue to $q = -1$. This simplification of the five point function will not persist for higher correlators.
The above computations describe the scheme for calculating tachyon correlators via the $q = -1$ model by analytical continuation of the degeneration equation. Positive momentum tachyons are identified with primary operators, while negative momentum tachyons are identified with their auxiliary analogs. The latter decouple for $q > 1$ but not at $q = -1$. They provide us with the negative times that are needed in order to pass from the KP integrable hierarchy underlying the minimal models coupled to gravity to the Toda lattice hierarchy underlying 2d string theory [27, 32, 30, 31].

5 The equivalence between the $(1, q = -1)$ degeneration equation and the $W_{1+\infty}$ Ward identities for tachyon correlators at genus zero

Our aim in this section is to prove that the genus zero $(1, q = -1)$ degeneration equation for tachyon correlators is identical to the genus zero $W_{1+\infty}$ Ward identities for tachyon correlators in 2d string theory.

5.1 The $(1, q = -1)$ degeneration equation for genus zero $n$ tachyons correlator

Consider the genus zero correlator $\langle P_n \prod_{i=1}^m P_{n_i} \rangle$, where $P_n$ is a primary operator. Taking $P_n$ as the marked operator, the degeneration equation reads:

\[
(-q)^n \langle P_n \prod_{i=1}^m P_{n_i} \rangle + n(-q)^{n-1} \sum_{i=1}^m \langle P_{n+n_i-(q+1)} \prod_{i \neq j=1}^m P_{n_j} \rangle \langle P_{-n_i} P_{n_i} \rangle +
\]

\[
n(-q)^{n-1} \sum_{i,j=1; i \neq j}^m \langle P_{n+n_i+n_j-2(q+1)} \prod_{i,j \neq k=1}^m P_{n_k} \rangle \langle P_{-n_i-n_j+(q+1)} P_{n_i} P_{n_j} \rangle +
\]

\[
n(n-1)(-q)^{n-2} \sum_{i,j=1; i \neq j}^m \langle P_{n+n_i+n_j-2(q+1)} \prod_{i,j \neq k=1}^m P_{n_k} \rangle \langle P_{-n_i} P_{n_i} \rangle \langle P_{-n_j} P_{n_j} \rangle +
\]

\[
\ldots + \frac{\Gamma(n+1)}{\Gamma(n-k+1)} (-q)^{n-k} \sum_{i_1,..,i_k=1; i_j \neq i_l}^m \langle P_{n+n_{i_1}+..+n_{i_k}-(q+1)} \prod_{i_1,..,i_k \neq i_l=1}^m P_{n_i} \prod_{l=1}^k \langle P_{-n_{i_l}} P_{n_{i_l}} \rangle \rangle
\]

\[
\ldots + \frac{\Gamma(n+1)}{\Gamma(n-m+3)} (-q)^{n-m+2} \sum_{j,k=1; j \neq k}^m \langle P_{n+\sum_{j,k \neq i=1}^m n_i-(m-2)(q+1)} P_{n_j} P_{n_k} \rangle
\]

\[
\prod_{j,k \neq l=1}^m \langle P_{-n_l} P_{n_l} \rangle + \ldots + \frac{\Gamma(n+1)}{\Gamma(n-m+2)} (-q)^{n-m} \prod_{i=1}^m \langle P_{-n_i} P_{n_i} \rangle.
\]

(5.1)
A general term in (5.1) as depicted in Fig. 3 is, up to a combinatorial factor, of the form

\[ \langle P_n P_{i_1} \ldots P_{i_n} \rangle \prod_{i \in S} P_{n_i} \langle P_{-i_1} \prod_{j \in S_1} P_{n_{j_1}} \rangle \ldots \langle P_{-i_n} \prod_{j_n \in S_n} P_{n_{j_n}} \rangle, \]  

(5.2)

where the sets \( S, S_1 \ldots S_n \) are disjoint, possibly empty, and satisfy \( S \cup S_1 \cup \ldots \cup S_n = (1 \ldots m) \).

In order to derive (5.1) we used (2.8). Note that if \( n \) is less than \( m \) then some of the terms in equation (5.1) vanish.

Taking the operators to be primaries or their auxiliary analogs we get at \( q = -1 \) a Ward identity for tachyon correlators:

\[
\langle T_n \prod_{i=1}^m T_{n_i} \rangle = \sum_{i=1}^m (n + n_i) \Theta(-n - n_i) \langle T_{n+n_i} \prod_{i \neq j=1}^m T_{n_j} \rangle + \\
\sum_{i,j=1; i \neq j}^m (n + n_i + n_j) \Theta(-n - n_i - n_j) \langle n_i + n_j \rangle \Theta(n_i + n_j) \langle T_{n+n_i+n_j} \prod_{i,j \neq k=1}^m T_{n_k} \rangle + \\
(n - 1) \sum_{i,j=1; i \neq j}^m \Theta(-n - n_i - n_j) \langle T_{n+n_i+n_j} \prod_{i,j \neq k=1}^m T_{n_k} \rangle + \ldots + \\
\frac{\Gamma(n)}{\Gamma(n-k+1)} \sum_{i_1 \ldots i_k=1; i_j \neq i_l}^m (n_{i_1} + \ldots + n_{i_k}) \Theta(n_{i_1} + \ldots + n_{i_k}) \langle T_{n+n_{i_1}+\ldots+n_{i_k}} \prod_{i_1 \ldots i_k \neq j=1}^m T_{n_j} \rangle
\]

Fig. 3: A general term in the genus zero degeneration equation
+\ldots + \frac{\Gamma(n)}{\Gamma(n - m + 3)} \sum_{j,k=1;j \neq k}^{m} (n + \sum_{j,k \neq i=1}^{m} n_i) \Theta(-n - \sum_{j,k \neq i=1}^{m} n_i) \langle T_{n+\sum_{j,k \neq i=1}^{m} n_i} T_{n_j} T_{n_k} \rangle \\
+\ldots - \frac{\Gamma(n)}{\Gamma(n - m + 2)} . 

(5.3)

The last piece is recognized as the $1 \to m$ amplitude and the rest are contributions from other kinematic regions. This recursion relations are highly non-linear with a general term consisting of a product of tachyon correlators.

5.2 $W_{1+\infty}$ Ward identities for genus zero $n$ tachyons correlator

Consider the genus zero correlator $\langle T_n \prod_{i=1}^{m} T_{n_i} \rangle$. Using the Ward identities (3.6) we have

$$
\langle T_n \prod_{i=1}^{m} T_{n_i} \rangle \equiv \partial_{n_1} \ldots \partial_{n_m} \langle \langle T_n \rangle \rangle (t = 0) = \frac{1}{n(n+1)} \partial_{n_1} \ldots \partial_{n_m} \text{res}(W_0)^{n+1} = \\
\text{res}\left[ \frac{\Gamma(n)}{\Gamma(n - m + 2)} \Phi_{n_1}^{(0)} \ldots \Phi_{n_m}^{(0)} W_0^{n-m+1} + \right.
\frac{\Gamma(n)}{\Gamma(n - m + 3)} (\partial_{n_1} (\Phi_{n_2}^{(0)} \ldots \Phi_{n_m}^{(0)}) + \Phi_{n_1}^{(0)} \partial_{n_2} (\Phi_{n_3}^{(0)} \ldots \Phi_{n_m}^{(0)}) + \ldots + \Phi_{n_1}^{(0)} \ldots \partial_{n_{m-1}} \Phi_{n_m}^{(0)}) W_0^{n-m+2} \\
+\ldots + \frac{\Gamma(n)}{\Gamma(n - k + 1)} (\partial_{n_1} \ldots \partial_{n_{m-k-1}} (\Phi_{n_{m-k}}^{(0)} \ldots \Phi_{n_m}^{(0)}) + \ldots) W_0^{n-k} \\
+\ldots + (n - 1) (\partial_{n_1} \ldots \partial_{n_{m-2}} (\Phi_{n_{m-1}}^{(0)} \Phi_{n_m}^{(0)}) + \ldots) W_0^{n-2} + \\
+(\partial_{n_1} \ldots \partial_{n_{m-1}} \Phi_{n_m}^{(0)} + \ldots) W_0^{n-1} + \frac{1}{n} \partial_{n_1} \ldots \partial_{n_{m-1}} \Phi_{n_m}^{(0)} W_0^{n},
$$

(5.4)

A general term in (5.4) is, up to a combinatorial factor, of the form

$$
\text{res}[W_0^{n+1-p} \prod_{i \in S} \partial_{n_i} W_0 \prod_{j_1 \in S_1} \partial_{n_{j_1}} W_0 \ldots \prod_{j_n \in S_n} \partial_{n_{j_n}} W_0],
$$

(5.5)

where the sets $S, S_1..S_n$ are disjoint, possibly empty, and satisfy $S \cup S_1.. \cup S_n = (1..m)$. $p$ is the number of empty sets. The general term (5.5) is the analog in the $W_{1+\infty}$ Ward identities to (5.2) in the degeneration equation.

Using (3.9) we get

$$
\langle T_{-n} \prod_{i=1}^{m} T_{n_i} \rangle = \frac{\Gamma(n)}{\Gamma(n - m + 2)} - \\
\frac{\Gamma(n)}{\Gamma(n - m + 3)} \sum_{j,k=1;j \neq k}^{m} (n + \sum_{j,k \neq i=1}^{m} n_i) \Theta(-n - \sum_{j,k \neq i=1}^{m} n_i) \langle T_{n+\sum_{j,k \neq i=1}^{m} n_i} T_{n_j} T_{n_k} \rangle - \\
-\frac{\Gamma(n)}{\Gamma(n - k + 1)} \sum_{i_1..i_k=1;i_j \neq i}^{m} (n_{i_1} + \ldots + n_{i_k}) \Theta(n_{i_1} + \ldots + n_{i_k}) \langle T_{n+i_1} \ldots_{i_k} T_{n_{j+1}} \ldots T_{n_k} \rangle \\
\prod_{i_1..i_k \neq j=1}^{m} T_{n_j},
$$

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\(-\ldots - (n-1) \sum_{i,j=1}^{m} (n + n_i + n_j) \Theta(-n - n_i - n_j)\langle T_{n+n_i+n_j} \prod_{k \neq i,j} T_{n_k} \rangle - \sum_{i,j=1; i \neq j}^{m} (n + n_i + n_j) \Theta(-n - n_i - n_j)(n_i + n_j) \Theta(n_i + n_j)\langle T_{n+n_i+n_j} \prod_{i,j \neq k=1}^{m} T_{n_k} \rangle - \sum_{i=1}^{m} (n + n_i) \Theta(-n - n_i)\langle T_{n+n_i} \prod_{i \neq j=1}^{m} T_{n_j} \rangle \). \tag{5.6}

The last term in (5.4) is proportional to \(\Theta(\sum_{i=1}^{m} n_i)\) and it vanishes since \(\sum_{i=1}^{m} n_i = -n\) is negative.

The Ward identity (5.6) is identical to the \((1, q = -1)\) degeneration equation (5.3), thus establishing the required equivalence. This result parallels the equivalence between the Virasoro constraints and the Verlinde’s recursion relations for topological gravity[21], and more generally between the Montano-Rivlis degeneration equation[17] for \((1, q)\) model and \(W_q\) constraints. We have shown the equivalence at genus zero and will return to higher genera at section 8.

6 Discrete states as gravitational descendants

6.1 The spectrum of \((1, q = -1)\) model versus 2d string theory

The spectrum of 2d string theory consists of the tachyon as well as discrete states, i.e. states at special quantized values of momentum[?]. As we saw, in the topological framework the positive momentum tachyons are most naturally identified with primary fields. The non-positive momentum tachyons correspond to additional fields introduced into the calculational scheme of minimal topological models and are for \(q > 1\) the auxiliary fields analogous to the primary fields. This however does not characterize them as primaries or descendants in the topological phase of the 2d string. It has been argued [32, 33] that negative momentum tachyons are actually gravitational descendants of the zero momentum tachyons. The natural conjecture is to identify the discrete states as gravitational descendants of positive momentum tachyons [32, 33].

We will take this viewpoint in the following and use the degeneration equation analytically continued to \(q = -1\) in order to compute their correlators. In analogy with the tachyons, we identify positive momentum discrete states as gravitational descendants and
negative ones as their auxiliary analogs. Thus,

\[ \mathcal{Y}^{+}_{J,m} \rightarrow \frac{P_{k,\alpha}}{\alpha + kq}, \quad \alpha = J + m, k = J - m, \]  

(6.1)

with \( J, m = 0, \frac{1}{2}, 1, \ldots \) and \( 0 \leq m \leq J \), while \( \mathcal{Y}^{+}_{J,-m} \) are considered as their auxiliary analogs. This picture implies that the operators \( P_{n} = kq + \alpha \) should decouple for \( k > \alpha \) at \( q = -1 \), since negative momentum discrete states are identified as the auxiliary analogs of the positive momentum ones. This is indeed the case as we will verify in the next section.

### 6.2 Correlators of \((1, q = -1)\) gravitational descendants

Our aim is to establish a consistent scheme for calculating correlators of discrete states viewed as gravitational descendants of \((1, q = -1)\) theory by analytically continuing the degeneration equation with gravitational descendants of \((1, q)\) minimal models to \( q = -1 \). In order to pose the rules for the analytical continuation let us perform some sample calculations.

Consider first the two-point function \( \langle \mathcal{Y}^{+}_{J_{1},m_{1}} \mathcal{Y}^{+}_{J_{2},m_{2}} \rangle \). Using (2.4) and (2.5) we expect that the only non-vanishing two-point functions are \( \langle \mathcal{Y}^{+}_{J,m} \mathcal{Y}^{+}_{J,-m} \rangle \). However, since we do not derive this result from the degeneration equation we cannot exclude the possibility that other two-point functions that satisfy momentum conservation do not vanish. The choice of the non-vanishing two-point functions with gravitational descendants is basically part of the definition of the \((1, q = -1)\) theory. Different choices do not affect the correlators of the tachyons as they do not appear in their degeneration equation. They may show up, however, in the \((1, q = -1)\) degeneration equation for gravitational descendants. Since we consider the latter as an analytical continuation of the \( q > 1 \) equation in which those correlators do not appear, they will actually have no effect on the higher point functions of descendants.

The simplest correlator for which we have to use the degeneration equation is the three-point function. Thus consider the correlator \( \langle P_{n} P_{n_{1}} P_{n_{2}} \rangle \), where, for instance, \( P_{n_{1}} \) and \( P_{n_{2}} \) are primaries while \( P_{n} \) is a descendant with \( n = kq + \alpha \). Such a correlator corresponds in \( q = -1 \) to \( \langle \mathcal{Y}^{+}_{J,m_{1}} T_{n_{1}} T_{n_{2}} \rangle \) with \( J = \frac{\alpha + k}{2}, m = \frac{\alpha - k}{2} \). There are two different ways to compute this correlator using the degeneration equation (1.3) since one can choose the marked operator to be either the descendant or one of the primary fields. Evidently, the result must be independent of the choice. In the \( q > 1 \) degeneration equation scheme an operator has as many contacts as his primary index. Using (1.3) we see that the value of the correlator is \( \frac{\alpha_{	ext{min}}}{q} \) when \( P_{n} \) is the marked operator and \( \frac{\alpha_{	ext{min}} + k}{q} \) when one of the
primary fields is chosen as the marked operator. The result is explicitly dependent on the choice and consistency requires that the correlator vanishes. Indeed the ghost number conservation law (2.4) reads in this case

\[ kq + \alpha + n_1 + n_2 = q + 1 \]

(6.2)

and thus \( k \) must be zero. The same problem with consistency occurs if we take two different descendants as marked operators. Thus there is no non-vanishing three point function with a descendant in \((1, q)\) models. In order to get a consistent scheme at \( q = -1 \) via analytical continuation of the \( q > 1 \) degeneration equation we must require that three point functions with discrete states vanish, although momentum conservation does not forbid such correlators. Thus we conclude that

\[ \langle Y^+_{J_1, m_1} Y^+_{J_2, m_2} Y^+_{J_3, m_3} \rangle = 0 \]

(6.3)

unless all the three fields are tachyons.

Consider now the four point function \( \langle P_n P_{n_1} P_{n_2} P_{n_3} \rangle \), where \( P_n \) is a descendant and the rest are primaries. The ghost number conservation fixes \( n = q + \alpha \), that is \( P_n \) is the first descendant of the primary field \( P_\alpha \). Using (4.6) we get:

\[ \langle P_{q+\alpha} P_{n_1} P_{n_2} P_{n_3} \rangle = \frac{\alpha(q + \alpha)n_1 n_2 n_3}{q^2} \]

(6.4)

Unlike the correlator of four primaries, the correlator (6.4) does not include \( \Theta \) terms and thus does not depend on the choice of the kinematic region.

The constraint \( n = q + \alpha \) on the descendant as a consequence of the ghost number conservation law is also necessary for consistency. If the constraint is not satisfied, that is \( n = kq + \alpha, k > 1 \), the value of the correlator depends on the choice of the marked operator. In such a case, when the descendant is chosen as the marked operator we get for the correlator \( \frac{\alpha(\alpha-1)n_1 n_2 n_3}{q^2} \) while if one of the primaries, for instance \( P_{n_1} \), is taken to be the marked operator we get \( \frac{n_1(\alpha-1)(kq+\alpha)n_2 n_3}{q^2} \).

Consistency of the analytical continuation to \( q = -1 \) requires that the only non-vanishing four point functions with discrete states are

\[ \langle Y^+_{J,J-1} T_{n_1} T_{n_2} T_{n_3} \rangle = 2J - 1 \]

(6.5)

where we factored out the \( \delta \)-function enforcing the momentum conservation \( 2(J - 1) + n_1 + n_2 + n_3 = 0 \) and used the normalization (6.1).
In our picture \( \mathcal{Y}_{1,0}^+ \) corresponds to \( P_{q+1} \) which is the dilaton operator, i.e. the first descendant of the puncture. Thus, for \( J = 1 \) equation (5.3) is a special case of the dilaton equation on the sphere. Let us derive the general genus \( g \) dilaton equation. Thus, consider the genus \( g \) correlator \( \langle P_{q+1} \prod_{i=1}^s P_{n_i} \rangle_g \). The degeneration equation reads:

\[
(-q) \langle P_{q+1} \prod_{i=1}^s P_{n_i} \rangle_g + \sum_{i=1}^s (P_{q+1} P_k \prod_{i \neq j=1}^s P_{n_j}) \langle P_{-k} P_{n_i} \rangle_0 = 0 .
\]

(6.6)

Using (2.8) and (2.4) we get

\[
\langle P_{q+1} \prod_{i=1}^s P_{n_i} \rangle_g = \frac{q + 1}{q} (s + 2g - 2) \langle \prod_{i=1}^s P_{n_i} \rangle_g .
\]

(6.7)

At \( q = -1 \) we have

\[
\langle \mathcal{Y}_{1,0}^+ \prod_{i=1}^s \mathcal{Y}_{J_i,m_i}^+ \rangle_g = (2 - 2g - s) \langle \prod_{i=1}^s \mathcal{Y}_{J_i,m_i}^+ \rangle_g .
\]

(6.8)

Equation (6.8) is expected from the topological viewpoint since the dilaton measures the Euler characteristic of the Riemann surface with \( s \) punctures, that is \( 2 - 2g - s \).

The correlator (6.4) belongs to a family of correlators that do not include \( \Theta \) terms, namely \( \langle \prod_{i=1}^n P_{n_i} \rangle \), \( n_i = k_i q + \alpha_i \) such that \( \sum_{i=1}^n k_i = n - 3 \). They read

\[
\langle \prod_{i=1}^n P_{n_i} \rangle = \frac{1}{q^{n-2}} \frac{(n-3)!}{\prod_{i=1}^n k_i!} \prod_{i=1}^n \prod_{j_i=0}^{k_i} (j_i q + \alpha_i) .
\]

(6.9)

Note that, as expected from the picture described in the previous section, the correlator for the normalized operators \( P_{n_i}^{\alpha_i+k_i q} \) vanishes if \( k_i > \alpha_i \). As each descendant \( P_{n=kq+\alpha} \) in a correlator seems to contribute a multiplicative factor

\[
\prod_{j=0}^{k} (jq + \alpha) ,
\]

(6.10)

it implies the vanishing of the correlator at \( q = -1 \) in these cases.

At \( q = -1 \) (6.9) corresponds to

\[
\langle \prod_{i=1}^n \mathcal{Y}_{J_i,m_i}^+ \rangle = (-1)^n (n-3)! \prod_{i=1}^n \frac{\prod_{j_i=0}^{J_i-m_i-1} (J_i + m_i - j_i)}{(J_i - m_i)!} ,
\]

(6.11)

where \( \sum_{i=1}^n (J_i - m_i) = n - 3 \).

\footnote{We checked this formula analytically as well as numerically, but we do not have a full proof for it.}
The first correlator with discrete states that exhibits dependence on the choice of kinematic region is the five point. The ghost number selection rule allows various possibilities. We can have in the correlator one descendant field $P_n$ with $k = 1, 2$ or two descendants with $k = 1$. Two of these possibilities satisfy the condition $\sum_{i=1}^5 k_i = 2$ and thus belong to the family of correlators that do not have $\Theta$ terms as described above and are given by (6.9). The third possibility is the correlator $\langle P_n \prod_{i=1}^4 P_{n_i} \rangle$ where $n = q + \alpha$ and $P_{n_i}$ are primaries. We can compute this correlator using the degeneration equation (4.8) in two ways, by choosing either the descendant or one of the primaries as the marked operator. Taking the descendant field as the marked operator yields

$$\langle P_n \prod_{i=1}^4 P_{n_i} \rangle = \frac{\alpha}{q} \sum_{i=1}^4 n_i \langle P_{\alpha+n_i-1} \prod_{i \neq j=1}^4 P_{n_j} \rangle - \frac{\alpha(\alpha - 1)}{q^2} \sum_{i,j=1; i \neq j}^4 n_i n_j \Theta(\alpha + n_i + n_j - q - 2) \langle P_{\alpha+n_i+n_j-q-2} \prod_{i,j,k=1}^4 P_{n_k} \rangle + \frac{\alpha(\alpha - 1)(\alpha - 2)}{q^3} \langle \prod_{i=1}^4 n_i \rangle. \quad (6.12)$$

It is straightforward to verify that one gets the same result if one of the primary fields is taken as the marked operator. Analytical continuation of (6.12) to $q = -1$ proceeds in the same manner as that of the tachyon correlators, namely $\Theta(x) \rightarrow \Theta(-x)$. We get

$$\langle Y_{j_1,j_2,\ldots,j_l}^+ \prod_{i=1}^4 T_{n_i} \rangle = -\sum_{i=1}^4 \frac{(2J-1)(n_i + 2J - 2)}{2J - 2} \langle T_{2(J-1)+n_i} \prod_{i,j=1}^4 T_{n_j} \rangle - \sum_{i,j=1; i \neq j}^4 (2J-1)(n_i + n_j + 2J - 2) \Theta(-n_i - n_j - 2J + 2) \langle T_{2(J-1)+n_i+n_j} \prod_{i,j,k=1}^4 T_{n_k} \rangle - (2J-1)(2J-3). \quad (6.13)$$

Let us turn now to the general case. Consider the correlator $\langle \prod_{i=1}^n P_{n_i} \rangle$ where $n_i = k_i q + \alpha_i$. From the ghost number conservation law one gets the constraint

$$\sum_{i=1}^n k_i \leq n - 3. \quad (6.14)$$

This constraint (6.14) is also necessary for the consistency of the whole calculational scheme. Consider, for instance, the case where $k_1 \geq n - 2$ and all the other $k_i$ zero. If we take $P_{n_1}$ as the marked operator we get by induction:

$$\langle \prod_{i=1}^n P_{n_i} \rangle = \frac{(-1)^{n-1}}{q^{n-2}} \prod_{i=2}^n n_i \frac{\Gamma(\alpha_i + 1)}{\Gamma(\alpha_i - n + 3)}. \quad (6.15)$$
On the other hand, if we take one of the primary fields for instance $P_{n_2}$ we get

$$\langle \prod_{i=1}^{n} P_{n_i} \rangle = \frac{(-1)^{n-1}}{q^{n-2}} \prod_{i \neq 2}^{n} n_i \frac{\Gamma(n_2 + 1)}{\Gamma(n_2 - n + 3)} ,$$

(6.16)

and it clearly differs from (6.15).

Thus, although the constraint (6.14) does not follow from momentum conservation in the $q = -1$ model we must impose it in the analytical continuation scheme in order to have a consistent framework. An immediate consequence of the constraint is that there is no non-vanishing $n-$ point function with more than $n - 3$ descendants. Evidently, the constraint is necessary for consistency but is it sufficient? For the $q > 1$ models the degeneration equation is equivalent to the $W-$ constraints and thus to the KdV calculational scheme and is therefore consistent. We strongly believe that the analytic continuation process to $q = -1$ does not spoil this property. It is indeed the case for all the computations that we made but we do not have a complete proof for that.

Finally, let us consider the Ward identity for the momentum one tachyon $T_1$. Topological and integrable evidence imply that it corresponds to the puncture operator in the topological phase of 2d string theory [32]. Consider the genus $g$ correlator $\langle P_1 \prod_{i=1}^{s} P_{n_i} \rangle_g$ The degeneration equation at $q > 1$ reads

$$(-q)\langle P_1 \prod_{i=1}^{s} P_{n_i} \rangle_g + \sum_{i=1}^{s} \langle P_1 P_k \prod_{i \neq j=1}^{s} P_{n_j} \rangle_g \langle P_{-k} P_{n_i} \rangle_0 = 0 ,$$

(6.17)

or

$$\langle P_1 \prod_{i=1}^{s} P_{n_i} \rangle_g = \frac{1}{q} \sum_{i=1}^{s} n_i \Theta(n_i + 1 - (q + 1)) \langle P_{n_i+1-(q+1)} \prod_{i \neq j=1}^{s} P_{n_j} \rangle_g .$$

(6.18)

Thus, the puncture operator shifts the momentum of a gravitational descendant and does not affect a primary. At $q = -1$ we have

$$\langle T_1 \prod_{i \in S_1} \mathcal{Y}_{J_i,m_i}^{+} \prod_{j \in S_2} \mathcal{Y}_{J_j,-J_j}^{+} \prod_{k \in S_3} \mathcal{Y}_{J_k,J_k}^{+} \rangle_g =$$

$$\sum_{l \in S_1} (J_j - m_l - 1) \langle \mathcal{Y}_{J_l,m_l}^{+} \prod_{i \neq l \in S_1} \mathcal{Y}_{J_i,m_i}^{+} \prod_{j \in S_2} \mathcal{Y}_{J_j,-J_j}^{+} \prod_{k \in S_3} \mathcal{Y}_{J_k,J_k}^{+} \rangle_g -$$

$$\sum_{l \in S_2} (2J_l - 1) \langle \mathcal{Y}_{J_l,-J_l}^{+} \prod_{i \neq l \in S_1} \mathcal{Y}_{J_i,m_i}^{+} \prod_{j \in S_2} \mathcal{Y}_{J_j,-J_j}^{+} \prod_{k \in S_3} \mathcal{Y}_{J_k,J_k}^{+} \rangle_g ,$$

(6.19)

with $S_1 \cup S_2 \cup S_3 = (1...s)$.

Considering (6.13) as the puncture equation in the topological phase of 2d string theory implies that indeed the negative momentum tachyons and all the discrete states
correspond to gravitational descendants while non-negative momentum tachyons correspond to gravitational primaries. Note, however, that we cannot take a negative tachyon to be the marked operator within the degeneration equation framework while considering him as a gravitational descendant of the zero momentum tachyon, since it will have zero number of contacts. On the other hand we saw that we can take it to be the marked operator with a formal negative number of contacts. The latter is probably related to the fact that gravitational descendants can be described via matter degrees of freedom [39].

7 (1, q = −1) topological recursion relations via one and two splittings

In [23] it was shown that correlation functions of any (1, q > 1) model can be computed using recursion relations derived not from the \( W_q \) constraints but rather from the \( W_3 \) constraints on the partition function. In terms of the topological procedure this means that only terms with one and two splittings of the Riemann surface are needed in order to reproduce the results of the (1, q) degeneration equation.

It is natural to ask whether a similar statement can be made about the topological procedure for the \( q = −1 \) model. As we shall see in this section, this is indeed the case. This will provide us with new recursion relations for tachyon correlation functions, as well as a support for the consistency of the \( (1, q = −1) \) degeneration equation for gravitational descendants.

Let us first briefly summarize the method of [23] for the \( q > 1 \) minimal topological models. The idea is to write down an algorithm for computing genus \( g \) correlators \( \langle \prod_{i=1}^{N} P_{k_i, \alpha_i} \rangle_g \) based only on one and two splittings. Consider the following relations

\[
\langle P_{0,2}^{\alpha_1-2} P_{k_1+\alpha_1-2,2} \prod_{i=2}^{N} P_{k_i, \alpha_i} \rangle_g = c_1 \langle \prod_{i=1}^{N} P_{k_i, \alpha_i} \rangle_g + \Delta_1 \\
\langle P_{k_1+\alpha_1-2,2} P_{0,2}^{\alpha_1-2} \prod_{i=2}^{N} P_{k_i, \alpha_i} \rangle_g = c_2 \langle \prod_{i=1}^{N} P_{k_i, \alpha_i} \rangle_g + \Delta_2 ,
\]

(7.1)

where \( \alpha_1 = \min\{\alpha_i\} \) and \( c_1 \) and \( c_2 \) are some coefficients. \( \Delta_1 \) and \( \Delta_2 \) are each a sum of correlators which follow from the contacts of \( P_{0,2} \) and \( P_{k_1+\alpha_1-2,2} \) respectively. The particular choice of the left hand sides (l.h.s) of equations (7.1) is made so that the following properties are obeyed: (i) The degeneration equation of only one and two splittings is required; (ii) \( c_1 - c_2 \neq 0 \); (iii) all the terms in \( \Delta_1 \) and \( \Delta_2 \) are either on genus \( g' < g \) or on...
genus $g$ and contain less operators then $N$ or contain $N$ operators with $\min\{\alpha_i\} < \alpha_1$. The second property was proven by invoking certain novel polynomial identities which are generalizations of Abel’s identity \[23\] while the third one was shown to hold using induction procedure. By subtracting the two equations (7.1) we derive the following recursion relation
\[\langle \prod_{i=1}^{N} P_{k_i,\alpha_i} \rangle_g = \frac{\Delta_2 - \Delta_1}{c_1 - c_2}.\] (7.2)
We would like to emphasize that since $\Delta_1$ and $\Delta_2$ include only “lower correlators”, (7.2) is a recursion algorithm for determining the original correlator. Does this procedure hold also for the $q = -1$ description of the $c = 1$ string model? It seems that since in the derivation of (7.2) we have used only the degeneration equation together with the consistency of the calculational scheme, it should hold also for the $(1, q = -1)$ model. Let us examine things more carefully. Unlike the $q > 1$ cases, at $q = -1, c_1$ which is given by \[23\]
\[c_1 = \left(\frac{2}{q}\right)^{\alpha_1 - 2} \prod_{i=1}^{\alpha_1 - 2} (k_1 + i)q + 2\], (7.3)
vanishes for certain values of $k_1$ and $\alpha_1$: $k_1 = 0$, $\alpha_1 \geq 4$ and $k_1 = 1$, $\alpha_1 \geq 3$. Note, however, that $c_2$ is different from zero. The difference $c_1 - c_2$ is independent of $q$ and is given by \[23\]
\[c_1 - c_2 = 2^{\alpha_1 - 2} \prod_{i=1}^{\alpha_1 - 2} (k_1 + i).\] (7.4)
This obviously implies that even for the cases where $c_1 = 0$ the recursion relation (7.2) still holds: $\langle \prod_{i=1}^{N} P_{k_i,\alpha_i} \rangle_g = -\frac{\Delta_2}{c_2}$.

We have thus found that for the whole range of values of $k_1, \alpha_1$ one can express the correlators in terms of “lower correlators” using the degeneration equation with only one and two splittings, thus obtaining a new set of “Ward identities”.

The main advantage of these new “Ward identities” for $q > 1$ is that they provide a practical algorithm to compute correlators on higher genus Riemann surfaces. Let us briefly explain why the topological procedure of \[17\] fails in these cases. When inserting (2.6) in a degeneration of the sphere, no more than one field of (2.6) contributes to the summation due to the ghost number conservation. On higher genus surfaces things change. When pinching a handle we insert both fields of (2.6) on the same surface, thus the ghost number is conserved for an infinite number of fields. The $\Theta$ functions that are inserted in order to avoid contributions from unphysical operators restrict the sum and prevent infinities. Two point functions, however, do not include these $\Theta$ functions and therefore divergences due to the infinite sum occur. For $q = 2$ the regularization
for these infinities is known, and for \( q = 3 \) no new infinities occur. Thus, the new set of “Ward identities” provide a regularized method for calculating correlators on higher genus Riemann surfaces. For the case of interest \( q = -1 \) this algorithm enables us to compute correlators of gravitational descendants on \( g > 0 \) surfaces. Furthermore one may use it to deduce the “postulated regularization” which we use in the next section to retrieve the genus one tachyon correlators derived from the \( W_{1+\infty} \) Ward identities. In the appendix an explicit evaluation of \( < P_4 P_6 P_{-10} > \) on the sphere as well as \( < P_4 P_{-4} > \) on the torus are presented.

8 \((1, q = -1)\) model on higher genus Riemann surfaces

We have seen that at genus zero, that the topological recursion relations for tachyon correlators implied by the degeneration equation analytically continued to the \((1, q = -1)\) theory coincide with the \( W_{1+\infty} \) matrix model Ward identities of the \( c = 1 \) string. In this section we will show that the equivalence between these two sets of Ward identities persists beyond genus zero. Since the topological recursion relations for genus greater than zero receive contributions from pinching of handles, infinities are encountered due to the propagation of infinite number of fields in the degeneration and should be properly regularized. We will prove explicitly the equivalence for the genus one case, and discuss the equivalence for genus \( g \) surfaces.

8.1 Tachyon correlators on the torus via \((1, q = -1)\) theory

Let us begin by computing the two and three point tachyon correlators on the torus \( \langle T_n T_{-n} \rangle_1 \). The degeneration equation reads:

\[
\langle P_n P_{-n} \rangle_1 + n\langle \overrightarrow{P_n P_i} \overrightarrow{P_{-i} P_{-n}} \rangle_0 + \sum_{i \geq 0} (n - 1) \langle \overrightarrow{P_n P_i P_{-i} P_{-n}} \rangle_0 \\
+ \sum_{i \geq 0} (n - 1)(n - 2) \langle \overrightarrow{P_n P_j P_{-i} P_{-n}} \rangle_0 \langle P_{-j} P_{-n} \rangle_0 = 0 .
\]

The first two terms come from splitting and the rest from pinching of the handle as shown in Fig.4. Note that in the latter terms the marked operator contacts two operators.
inserted around the pinching point. This implies that one has to insert $n - 2$ additional splitted spheres.

![Diagram](image)

**Fig. 4: The degeneration equation for two-point function on the torus**

The second term vanishes since it is proportional to $\langle P_{-1} \rangle_1 \equiv 0$, while the last two terms need to be regularized. We will now postulate a regularization for the contact operation which we justify, as we will show, by that it yields the correct results for general tachyon correlators on the torus. We also checked for several cases that it reproduces the correct results for minimal models.

The regularization reads:

\[
\text{reg}\left[ \sum_{i \geq 0} \langle P_n P_i P_{-i} \prod_{j=1}^{m} P_{n_j} \rangle_0 \right] \equiv \frac{n}{24},
\]

\[
\text{reg}\left[ \sum_{i \geq 0} \langle P_n P_i P_{-i} \prod_{j=1}^{m} P_{n_j} P_{n_{m+1}} \rangle_0 \right] \equiv \frac{n}{24} (1 + n_{m+1})(2 + n_{m+1}) \langle P_{-m+1} P_{n_{m+1}} \rangle_0,
\]

\[
\text{reg}\left[ \sum_{i \geq 0} \langle P_n P_i P_{-i} \prod_{j=1}^{m} P_{n_j} \prod_{j=m+1}^{s} P_{n_j} \rangle_0 \right] \equiv \frac{n}{24} (1 + \sum_{j=m+1}^{s} n_j)(2 + \sum_{j=m+1}^{s} n_j).
\]
\[ \Theta(\sum_{j=m+1}^{s} n_j)(P_{n+m+1}^j \prod_{j=m+1}^{s} P_{n_j})_0 . \] (8.2)

Using the regularized contact (8.2) in equation (8.1) we get
\[ \langle T_n T_{-n} \rangle_1 = \frac{1}{24} (n-1)(n+1)(n-2) , \] (8.3)
which is the correct result at the self-dual radius [40]. As in the genus zero case we have a sign difference between the tachyon correlators derived from the \( q = -1 \) degeneration equation and those that were derived from the matrix model. The second example in the appendix demonstrates how using the one and two splitting method one finds an identical result to that derived from the postulated regularization.

In order to see the reasoning behind the definition of the regularized contact consider the \( W_{1+\infty} \) Ward identities for the two-point function on the torus. Using (3.6) we get
\[ \langle T_n T_{-n} \rangle_1 = \text{res} [ \bar{W}_{n-1} \partial_{-n} \bar{W}_0 \bar{W}_1 + \frac{1}{n} \bar{W}_0^n \partial_{-n} \bar{W}_1 - \frac{1}{24} (n-1)(n-2) \bar{W}_0^{n-3} \partial_{-n} \bar{W}_0 \bar{W}_1^\prime \prime ] . \] (8.4)
The second term in (8.4) vanishes, as can be seen from (3.7). The other terms of (8.4) coincide with those of (8.1) if we use (3.10) and the definition of the regularized contact (8.2).

Consider next the three point function of tachyons on the torus \( \langle T_n T_{n_1} T_{n_2} \rangle_1 \), where \( n + n_1 + n_2 = 0 \). The degeneration equation reads:
\[ \langle P_{n+P_{n_1}P_{n_2}} \rangle_1 + (n\langle \bar{P}_{n+P_{n_1}P_{n_2}} \rangle_1 \langle P_{-i}P_{n_1} \rangle_0 + (n_1 \leftrightarrow n_2)) + ((n-1)(n-2)\langle \bar{P}_{n+P_{n_1}P_{n_2}} \rangle_0 \langle P_{-j}P_{n_1} \rangle_0 + (n_1 \leftrightarrow n_2)) + (n-1)(n-2)(n-3)\langle \bar{P}_{n+P_{n_1}P_{n_2}} \rangle_0 \langle P_{-j}P_{n_1} \rangle_0 \langle P_{-k}P_{n_2} \rangle_0 = 0 , \] (8.5)
where we have written only the non-vanishing terms. The first three terms in (8.5) arise from splitting and the rest from pinching. We can use now the regularization (8.2) together with the appropriate \( \Theta \) terms and get, up to a sign, the right three point function for tachyons on the torus [40]
\[ \langle T_n T_{n_1} T_{n_2} \rangle_1 = \frac{1}{24} (n-1)(n-2)(n_1^2 + n_2^2 - n - 2) . \] (8.6)
It is more instructive, however, to compare the degeneration equation (8.5) with the \( W_{1+\infty} \) Ward identities for the three point function. Using (3.6) we get:
\[ \langle T_n T_{n_1} T_{n_2} \rangle_1 = \text{res} [ \bar{W}_0^{n-1}(\partial_{n_1} \bar{W}_0 \partial_{n_2} \bar{W}_1 + (n_1 \leftrightarrow n_2)) - \frac{1}{24} (n-1)(n-2) \bar{W}_0^{n-3} \]
\[ (\partial_{n_1} \bar{W}_0 \partial_{n_2} \bar{W}_0'' + (n_1 \leftrightarrow n_2)) - \frac{1}{24} (n-1)(n-2)(n-3) \bar{W}_0'' \partial_{n_1} \bar{W}_0 \partial_{n_2} \bar{W}_0 ] (8.7) \]

where, similarly to equation (8.5), we have written only the non-vanishing terms. Comparing (8.5) and (8.7) we see that by associating the pinching of a handle with \( W'' \), namely

\[ \sum_{i \geq 0} (P_1 P_{i-1} P_j P_k)_{0} \leftrightarrow \frac{n}{24} \bar{W}_0'' \]

and using (3.7), (3.10) the two equations are identical. The generalization of (8.8) is clear,

\[ \sum_{i \geq 0} (P_n P_{j-1} \prod_{j \in S} P_{n_j} \prod_{k \in S'} P_{n_k})_{0} \leftrightarrow \frac{n}{24} \partial_{n} \bar{W}_0'' . \] (8.9)

### 8.2 The equivalence between the \((1, q = -1)\) degeneration equation and the \( W_{1+\infty} \) Ward identities on the torus

Our aim is to prove for the torus that the topological recursion relations of the \((1, q = -1)\) theory are identical to the \( W_{1+\infty} \) Ward identities of 2d string theory.

Consider the correlator \( \langle T_n \prod_{i=1}^{m} T_{n_i} \rangle \). Using the Ward identities we have

\[ \langle T_n \prod_{i=1}^{m} T_{n_i} \rangle \equiv \partial_{n_1} \ldots \partial_{n_m} \langle \langle T_n \rangle \rangle (t = 0) = \frac{1}{n} \partial_{n_1} \ldots \partial_{n_m} \text{res} (\bar{W}_0^n W_1) - \frac{1}{24} (n-1) \partial_{n_1} \ldots \partial_{n_m} \text{res} (\bar{W}_0^{n-2} \bar{W}_0'') . \] (8.10)

We will distinguish between two types of terms, those that include a genus one correlator and arise topologically from splittings and those that include only genus zero correlators and arise topologically from pinching the handle of the torus. The former are derived from the first term in (8.10) and the latter from the second term.

Consider first the terms coming from the process of splitting. They read:

\[
\text{res} \sum_{i=1}^{m} \frac{\Gamma(n)}{\Gamma(n-m+2)} \Phi_{n_1}^{(0)} \ldots \Phi_{n_i}^{(1)} \ldots \Phi_{n_m}^{(0)} \bar{W}_0^{n-m+1} + \\
\frac{\Gamma(n)}{\Gamma(n-m+3)} (\partial_{n_1} (\Phi_{n_2}^{(0)} \ldots \Phi_{n_i}^{(1)} \ldots \Phi_{n_m}^{(0)})) + \Phi_{n_1}^{(0)} \partial_{n_2} (\Phi_{n_3}^{(0)} \ldots \Phi_{n_i}^{(1)} \ldots \Phi_{n_m}^{(0)})) + \\
+ \ldots + \Phi_{n_1}^{(0)} \ldots \Phi_{n_{m-1}}^{(1)} \partial_{n_m} \Phi_{n_m}^{(0)} W_0^{n-m+2} +
\]

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we get

\[ + (n - 1) \langle \partial_{n_1} \partial_{n_{m-3}} \left( \Phi_{n_{m-2}} \Phi_{n_{m-2}} \Phi_{n_{m-1}} \Phi_{n_{m}} \right) + ... \rangle \bar{W}_{0}^{n-2} + \\
+ \langle \partial_{n_1} \partial_{n_{m-2}} \left( \Phi_{n_{m-1}} \Phi_{n_{m}} \right) + ... \rangle \bar{W}_{0}^{n-1} + \frac{1}{n} \partial_{n_1} \partial_{n_{m-1}} \Phi_{n_m} \bar{W}_{0}^{n} \].

(8.11)

This has the same structure as (5.4) with one of the $\Phi$ being $\Phi^{(1)}$ and the rest being $\Phi^{(0)}$. A general term in (8.11) takes the form of (5.3) with one of the $\bar{W}_0$ replaced by $\bar{W}_1$. Using (3.3) we get

\[
\langle T_{-n} \prod_{i=1}^{m} T_{n_i} \rangle_1 = \sum_{i=1}^{m} \frac{\Gamma(n)}{\Gamma(n - m + 2)} n_i \langle T_{n_i} T_{-n_i} \rangle_1 - \\
\frac{\Gamma(n)}{\Gamma(n - m + 3)} \left( \sum_{i \neq j=1}^{m} (n + m) \Theta(-n - \sum_{i,j \neq l=1}^{m} n_l) \langle T_{n_i} T_{n_j} \rangle_1 - \\
\sum_{i \neq j,k=1}^{m} (n + m) \Theta(-n - \sum_{j,k \neq l=1}^{m} n_l) n_i \langle T_{n_i} T_{n_j} T_{n_k} \rangle_0 \langle T_{n_i} T_{-n_i} \rangle_1 - \\
\langle T_{n_i} T_{-n_i} \rangle_1 \cdots - (n + n_i) \Theta(-n - n_i) n_i \langle T_{n_i} \prod_{i \neq j=1}^{m} T_{n_j} \rangle_0 \langle T_{n_i} T_{-n_i} \rangle_1 \right) .
\]

(8.12)

Consider now the correlator $\langle P_n \prod_{i=1}^{m} P_{n_i} \rangle_1$, where $P_n$ is a primary operator. Taking $P_n$ as the marked operator, the degeneration equation reads:

\[
(-q)^n \langle P_n \prod_{i=1}^{m} P_{n_i} \rangle_1 + n(-q)^{n-1} \sum_{i=1}^{m} \langle P_{n+n_i-(q+1)} \prod_{i,j=1; i \neq j}^{m} P_{n_j} \rangle_1 \langle P_{-n_i} P_{n_i} \rangle_0 + \\
\frac{n(-q)^{n-2}}{\Gamma(n - k + 1)} \sum_{i,j=1; i \neq j}^{m} \langle P_{n+n_i+n_j-2(q+1)} \prod_{i,j \neq k=1}^{m} P_{n_k} \rangle_1 \langle P_{-n_i-n_j+(q+1)} P_{n_j} P_{n_j} \rangle_0 + \\
n(n-1)(-q)^{n-2} \sum_{i,j=1; i \neq j}^{m} \langle P_{n+n_i+n_j-2(q+1)} \prod_{i,j \neq k=1}^{m} P_{n_k} \rangle_1 \langle P_{-n_i} P_{n_i} \rangle_0 \langle P_{-n_j} P_{n_j} \rangle_0 + ... \\
+ \frac{\Gamma(n+1)}{\Gamma(n - k + 1)} (-q)^{n-k} \sum_{i_1 \ldots i_k=1}^{m} \langle P_{n+n_i+\ldots+n_i_k-k(q+1)} \prod_{i_1 \ldots i_k \neq i=1}^{m} P_{n_i} \rangle_1 \langle P_{-n_i} P_{n_i} \rangle_0 \prod_{l=1}^{k} \langle P_{-n_l} P_{n_l} \rangle_0 + \\
+ \cdots + \frac{\Gamma(n+1)}{\Gamma(n - m + 3)} (-q)^{n-m+2} \sum_{j,k=1; j \neq k}^{m} \langle P_{n+n_j+\ldots+n_i_k-(m-2)(q+1)} P_{n_j} P_{n_k} \rangle_1 .
\]

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\[ \prod_{j,k \neq i=1}^m \langle P_{-n_i} P_{n_i} \rangle_0 + \ldots + \frac{\Gamma(n+1)}{\Gamma(n-m+2)} (-q)^{n-m} \sum_{i=1}^m \prod_{i \neq j=1}^m \langle P_{-n_i} P_{n_i} \rangle_1 \langle P_{-n_j} P_{n_j} \rangle_0 \] (8.13)

A general term in (8.13) takes the form of (5.2) with one of the correlators is on the torus and the rest on the sphere. Passing to \( q = -1 \) it is easy to see that to each term in (8.12) correspond an identical term in (8.13) and vice versa.

Let us turn now to the contributions from surfaces with pinched handles. The degeneration equation yields

\[ (n-1)(n-2) \sum_{i \geq 0} \langle P_n P_{n_j} P_{n_i} \rangle_0 \prod_{j \neq k=1}^m \langle P_{n_j} P_{n_i} \rangle_0 + \]

\[ (n-1)(n-2)(n-3) \sum_{i \geq 0} \langle P_n P_{n_j} P_{n_k} P_{n_i} \rangle_0 \prod_{j,k \neq l=1}^m \langle P_{n_j} P_{n_i} \rangle_0 \langle P_{n_k} P_{n_i} \rangle_0 \]

\[ + \ldots , \] (8.14)

with a general term depicted in Fig.5

\[ \sum_{i \geq 0} \langle P_n P_{n_i} \ldots P_{n_{i-2}} P_{n_i} \rangle_0 \prod_{i \in S} \langle P_{-i} \rangle_0 \prod_{j_1 \in S_1} \langle P_{n_{j_1}} \rangle_0 \ldots \langle P_{-i_{n-2}} \rangle_0 \prod_{j_{n-2} \in S_{n-2}} \langle P_{n_{j_{n-2}}} \rangle_0 , \] (8.15)

\[ \sum_{i \geq 0} \langle P_n P_{n_i} \ldots P_{n_{i-2}} P_{n_i} \rangle_0 \prod_{i \in S} \langle P_{-i} \rangle_0 \prod_{j_1 \in S_1} \langle P_{n_{j_1}} \rangle_0 \ldots \langle P_{-i_{n-2}} \rangle_0 \prod_{j_{n-2} \in S_{n-2}} \langle P_{n_{j_{n-2}}} \rangle_0 \]
where the sets $S, S_1..S_{n-2}$ are disjoint, possibly empty, and satisfy $S \cup S_1..S_{n-2} = (1...m)$.

The corresponding terms in the $W_{1+\infty}$ Ward identities are

$$\frac{1}{24} \text{res}[(n - 1)(n - 2)\partial_{n_j} \bar{W}_0 \prod_{j \neq k=1}^{m} \partial_{n_k} \bar{W}_0'' + (n - 1)(n - 2)(n - 3)\partial_{n_j} \bar{W}_0 \partial_{n_k} \bar{W}_0 \prod_{j,k \neq l=1}^{m} \partial_{n_l} \bar{W}_0' + ...] , \quad (8.16)$$

with a general term of the form

$$\frac{1}{24} \text{res}\prod_{i \in S} \partial_{n_i} \bar{W}_0'' \prod_{j_1 \in S_1} \partial_{n_{j_1}} \bar{W}_0... \prod_{j_{n-2} \in S_{n-2}} \partial_{n_{j_{n-2}}} \bar{W}_0] . \quad (8.17)$$

Using (8.2) and (8.8) we establish the equivalence of equations (8.14) and (8.16).

### 8.3 The $(1, q = -1)$ degeneration equation versus $W_{1+\infty}$ Ward identities for genus $g$ Riemann surfaces

Consider a general genus $g$ correlator $\langle P_n \prod_{i=1}^{m} P_{n_i} \rangle_g$ in $(1, q = -1)$ theory. The degeneration equation for this correlator receives contributions, as depicted in Fig.6, from splittings, for instance

$$\langle P_{n+\sum_{i \in S} n_i} \prod_{j \in S'} P_{n_j} \rangle_g \langle P_{-n_{i_k} - n_{i_l}} P_{n_{i_k}} P_{n_{i_l}} \rangle_{0...} , \quad (8.18)$$

with $S \cup S' = (1...m)$, from pinching a dividing cycle such as

$$\langle P_{n+\sum_{i \in S} n_i} \prod_{j \in S'} P_{n_j} \rangle_{g-g_1} \langle P_{-n_{i_k} - n_{i_l}} P_{n_{i_k}} P_{n_{i_l}} \rangle_{g_2...} , \quad (8.19)$$

and from pinching $h$ non-trivial homology cycles

$$\sum_{j_1...j_h \geq 0} \langle P_{n+\sum_{i \in S} n_i} P_{j_1}^{-1} P_{-j_1} P_{j_2}^{-1} ... P_{j_h}^{-1} P_{-j_h} \prod_{j \in S'} P_{n_j} \rangle_{g-h} \langle P_{-n_{i_k} - n_{i_l}} P_{n_{i_k}} P_{n_{i_l}} \rangle_{0...} . \quad (8.20)$$
Consider now a genus $g$ tachyons correlator $\langle T_n \prod_{i=1}^n T_{n_i} \rangle_g$. From the $W_{1+\infty}$ Ward identities we expect terms with a similar structure to (8.18), (8.19) and (8.20). The term that corresponds to the splitting (8.18) is

$$res[\prod_{i \in S'} \partial_{n_i} \bar{W}_g \partial_{n_{ik}} \partial_{n_{il}} \bar{W}_0 \ldots] ,$$

(8.21)

the term that corresponds to the pinching of a dividing cycle (8.19) is

$$res[\prod_{i \in S'} \partial_{n_i} \bar{W}_{g-g_i} \partial_{n_{ik}} \partial_{n_{il}} \bar{W}_{g_2} \ldots] ,$$

(8.22)

and the term that corresponds to the pinching of a non-trivial cycle (8.20)

$$res[\prod_{i \in S'} \partial_{n_i} \bar{W}_{g-h}^{(2h)} \partial_{n_{ik}} \partial_{n_{il}} \bar{W}_0 \ldots] .$$

(8.23)

Evidently, the contribution from reduction of the genus by pinching a non-trivial homology cycle needs to be properly regularized in the framework of the degeneration equation in order to be identified with (8.23).
The general structures of the degeneration equation and the $W_{1+\infty}$ Ward identities for any genus seem to be equivalent, thus it is plausible to conjecture that they coincide. However more work is needed in order to prove this in details.

9 Summary and Conclusions

We studied in this paper the $(1, q = -1)$ model coupled to topological gravity. This model is supposed to describe $2d$ string theory at the self-dual radius, that is at its topological phase. We defined the model by analytical continuation of the $q > 1$ degeneration equation to $q = -1$. We have shown that at genus zero the $q = -1$ degeneration equation yields the genus zero $W_{1+\infty}$ Ward identities for tachyon correlators. The positive momentum tachyons were identified as primary fields while the negative ones as their analytically continued $q > 1$ auxiliary analogs. By defining the discrete states of $2d$ string theory as the gravitational descendants of $(1, q = -1)$ model we proposed a scheme for computing their correlators and computed some of them. Unfortunately we could not compare the results since these correlators have not been computed by other means. We derived the puncture equation as well as the dilaton equation, with the puncture operator being the momentum one tachyon, and the dilaton operator being, as usual, its first descendant. The puncture equation supports the identification of the negative momentum tachyons as well as the discrete states as gravitational descendants.

We showed that in a similar manner to the $q > 1$ models, there exist recursion relations for the correlators that consist of only one and two splittings. Unlike the $W_{1+\infty}$ Ward identities which consist only of tachyon fields, the new recursion relations involve $(1, q = -1)$ gravitational descendants. This provides us with other means to compute tachyon correlators and supports the consistency of the whole computational scheme. However, an open problem is to fully prove the consistency of the $(1, q = -1)$ degeneration equation for gravitational descendants. A related question is what is the algebra underlying the $(1, q = -1)$ degeneration equation. For $q > 1$ the algebra is $W_q$, while for $q = -1$ on the space of tachyon times the algebra is $W_{1+\infty}$, but the algebra on the full phase space that includes also $q = -1$ gravitational descendants is not known. It is plausible to conjecture that it is also a $W_{1+\infty}$ algebra.

As we discussed in section 7, on $g > 0$ Riemann surfaces one encounters infinities in the degeneration equation framework that should be regularized. A proper regularization is known for the $q = 2, 3$ models, that is for one and two splittings of the Riemann surface.
Thus, when using the recursion relations (7.2) analytically continued to $q = -1$, we can calculate any genus $g$ correlator of primaries and descendants. However, the nature of this procedure is numerical and it does not easily yield closed and explicit expressions for correlators. One example of this type is described in the appendix. We therefore took another route. We postulated a regularized contact and used it to prove that the $q = -1$ degeneration equation for tachyon correlators coincides with the $W_{1+\infty}$ Ward identities on the torus. Derivation of the regularized contact from the recursion relations (7.2) is important and may lead to a regularized degeneration equation for $q = -1$ gravitational descendants. This is currently in study. As we showed, the structure of the $q = -1$ degeneration equation for tachyon correlators seems to parallel that of the $W_{1+\infty}$ Ward identities for any genus. It is important to show that this is indeed correct in details, since it will provide us with a topological interpretation of the $W_{1+\infty}$ Ward identities and thus with another link between the topological and integrable structures of 2d string theory.

There are several open problems that require further study. First, there are specific questions related to the present work. (i) Is the analytical continuation of the $q > 1$ models to $q = -1$ unique? (ii) Can one prove that the $q = -1$ model, as described in this paper, indeed describes 2d string theory at the self-dual radius without the need to compare all correlation functions calculated in the present approach to those derived via other methods.

Second, there are questions related to more general implications

1. The relation between the degeneration equation and intersection theory for $q > 1$ as well as for $q = -1$.
2. Can we consistently analytically continue the degeneration equation to other domains of definition of $q$ such as complex values, and are there corresponding physical systems.
3. Is it possible to write topological recursion relations based on contacts and degenerations for any topological string theory?. Computations done in \[41\] seem to fall into this category.
4. Finally, other generalizations defined by the assignment of different charges, e.g. with non-abelian structure, and different anomalous conservation laws may lead to topological recursion relations describing interesting theories.
Appendices

A  Recursion relations via one and two splittings, a numerical example

In this appendix we present two numerical computations of correlators using the topological procedure of section seven, namely, via iterative use of one and two splittings of the Riemann surface.

Consider first the correlator \( \langle P_4P_6P_{-10} \rangle_0 \) which corresponds to the tachyons three point function on the sphere. The degeneration equation for this correlator consists of at least four splittings, which is the case when we choose \( P_4 \) as the marked operator. Its numerical value at \( q = -1 \), before normalization, is 240 as given by (4.4). Following (7.1), let us consider the correlator \( \langle P_2P_2P_6P_{-10} \rangle_0 \), which vanishes by (6.9). Choosing \( P_{2,2} \) to be the marked operator we have a degeneration equation with one and two splittings

\[
0 \equiv \langle P_{2,2}P_2P_6P_{-10} \rangle_0 = \frac{2}{q} [4 \langle P_2P_{1,3}P_6P_{-10} \rangle_0 + 6 \langle P_{1,7}P_2P_2P_{-10} \rangle_0 \\
-10 \langle P_{1,-9}P_2P_6 \rangle_0] - \frac{2}{q^2} \left[ 4 \langle P_4P_6P_{-10} \rangle_0 + 12 \langle P_2P_8P_{-10} \rangle_0 \\
-40 \langle P_2P_6P_{-8} \rangle_0 - 60 \langle P_2P_2P_{-4} \rangle_0 \right] \\
+ \frac{2}{q} \left[ \langle P_2P_2P_{-4} \rangle_0 \langle P_4P_6P_{-10} \rangle_0 + 2 \langle P_2P_6P_{-8} \rangle_0 \langle P_2P_8P_{-10} \rangle_0 \right],
\]

(A.1)

where we used (2.8) and (3.1). Note that we do not have \( \Theta \) terms in (A.1). The reason is that the operator \( P_{2,2} \) can perform a contact with two physical operators and the result is still physical:

\[
\overrightarrow{P_{2,2}} P_n P_m = P_{n+m}.
\]

Since the analytical continuation is defined such that we allow in the correlators operators \( P_n \) with \( n \) negative only at the final \( q = -1 \) recursion relations, there are no \( \Theta \) terms in a five-point function with \( P_{2,2} \) as the marked operator. Note in contrast that a five-point function with \( P_{1,1} \) as the marked operator includes \( \Theta \) terms (3.12).

The r.h.s of equation (A.1) consist of the required correlator plus other terms, all of which can be computed via one and two splittings. Note that the required correlator for primaries is expressed via correlators of primaries and descendants. Using the degeneration equation for these terms we get \( \langle P_4P_6P_{-10} \rangle_0 = 240 \) as expected.

Consider next the correlator \( \langle P_4P_{-4} \rangle_1 \) which corresponds to the tachyons two-point
function on the torus. The degeneration equation for this correlator consists of four split-
tings, and its value is $-20$. Following (7.1), let us consider the correlator $\langle P_{2,2}P_2P_{2-4}\rangle_1$ 
which vanishes by the arguments of section six. Taking $P_{2,2}$ as the marked operator we get

$$0 \equiv \langle P_{2,2}P_2P_2P_{4-4}\rangle_1 = \frac{2}{q} [4 < P_2P_{1,3}P_{4-4}\rangle_1 - 4\langle P_{1,-3}P_2P_2\rangle_1]$$

$$-\frac{2}{q^2} [4\langle P_4P_{4-4}\rangle_1] + \frac{2}{q} \langle P_2P_2P_{4-4}\rangle_0 \langle P_4P_{4-4}\rangle_1$$

$$-\frac{1}{q^2}[-4\langle P_2P_2P_{-2}P_{-2}\rangle_0 - 8\langle P_2P_2P_{-1}P_{-3}\rangle_0] . \quad (A.3)$$

Using $P_2$ as the marked operator for the computation of the first two correlators on the 
r.h.s of (A.3) we have

$$\langle P_4P_{4-4}\rangle_1 = 4\langle P_2P_{-1}P_{-1}\rangle_0 - 8\langle P_{1,3}P_{-2}\rangle_1$$

$$+\frac{1}{2}\langle P_2P_2P_{-2}P_{-2}\rangle_0 + \langle P_2P_2P_{-1}P_{-3}\rangle_0 . \quad (A.4)$$

In order to evaluate (A.4) we have to compute the correlator $\langle P_{1,3}P_{-2}\rangle_1$, which we calculate 
by considering the correlator $\langle P_{2,2}P_{2-2}\rangle_1$. We get $\langle P_{1,3}P_{-2}\rangle_1 = -1$, and using (A.4) we 
arrive at the required result

$$\langle P_4P_{4-4}\rangle_1 = -20 . \quad (A.5)$$

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Weak Coupling

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