ALGEBRAIC CONSTRUCTIONS FOR JACOBI-JORDAN ALGEBRAS

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Abstract. For a given Jacobi-Jordan algebra $A$ and a vector space $V$ over a field $k$, a non-abelian cohomological type object $H^2_A(V, A)$ is constructed: it classifies all Jacobi-Jordan algebras containing $A$ as a subalgebra of codimension equal to $\dim_k(V)$. Any such algebra is isomorphic to a so-called unified product $A \bowtie V$. Furthermore, we introduce the bicrossed (semi-direct, crossed, or skew crossed) product $A \bowtie V$ associated to two Jacobi-Jordan algebras as a special case of the unified product. Several examples and applications are provided: the Galois group of the extension $A \subseteq A \bowtie V$ is described as a subgroup of the semidirect product of groups $\text{GL}_k(V) \rtimes \text{Hom}_k(V, A)$ and an Artin type theorem for Jacobi-Jordan algebra is proven.

Introduction

In the jungle of non-associative algebras, Jacobi-Jordan algebras (JJ algebras for short) are rather special objects. A JJ algebra is a vector space $A$ together with a bilinear map $[-,-]: A \times A \to A$ such that $[a, b] = [b, a]$ and $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$, for all $a, b, c \in A$. According to [20], where a detailed history of JJ algebras is given and several conjectures are proposed, this family of algebras was first defined in [15] and since then they have been studied independently in various papers [5, 6, 7, 8, 9, 12, 13, 14, 16] under different names such as Lie-Jordan algebras, Jordan algebras of nil index 3, pathological algebras, mock-Lie algebras or Jacobi-Jordan algebras. Throughout, we shall adopt the name JJ algebras. Although at first sight JJ algebras are very close to Lie algebras having only the skew-symmetry condition replaced by the symmetry condition, we will see that in fact this class of algebras exhibits rather different properties. Indeed, for instance two classical theorems in Lie algebra theory, namely Ado’s theorem and the Poincaré–Birkhoff–Witt theorem, fail for JJ algebras [20]. However, JJ algebras have an interesting and rich structure theory which deserves to be developed further. This is the staring point of this paper which is organised as follows. The first section fixes notations and conventions used throughout and recalls some basic concepts in the context of JJ algebras. Section 2 is devoted to the study of the extending structures problem (ES-problem), introduced in [1] for arbitrary categories. In the context of JJ algebras it comes down to the following question:

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Let $A$ be a JJ algebra and $E$ a vector space containing $A$ as a subspace. Describe and classify up to an isomorphism that stabilizes $A$ (i.e. acts as the identity on $A$) the set of all JJ algebra structures that can be defined on $E$ such that $A$ becomes a subalgebra of $E$.

If we fix $V$ a complement of $A$ in the vector space $E$ then the ES-problem asks for the description and classification of all JJ algebras containing and stabilizing $A$ as a subalgebra of codimension equal to $\dim(V)$. Following the strategy we previously developed in [2, 4] the approach we will use for studying the ES-problem is the following: we start by constructing in Theorem 2.3 the unified product $A \diamond V$ associated to a JJ algebra $A$ and a vector space $V$ connected through two actions and a cocycle. Next we show in Theorem 2.4 that a JJ algebra structure $(E, [-, -]_E)$ on $E$ contains $A$ as a subalgebra if and only if there exists an isomorphism of JJ algebras $(E, [-, -]_E) \cong A \diamond V$. Finally, the theoretical answer to the ES-problem is given in Theorem 2.7: a non-abelian cohomological type object $H^2_A(V, A)$ is explicitly constructed; it parameterizes and classifies all JJ algebras containing and stabilizing $A$ as a subalgebra of codimension equal to $\dim(V)$.

The unified product is a general construction containing as special cases the bicrossed product, semi-direct product, crossed product or skew crossed product associated to JJ algebras. Section 3 describes in detail all these special cases, highlighting the role and scope of the subsequent problem associated to each such product. For instance, in Definition 3.1 we introduce matched pairs of JJ algebras and the corresponding bicrossed product: these are the JJ counterparts of similar constructions performed for Lie algebras [11, Theorem 4.1]. Corollary 3.3 proves that the bicrossed product of two JJ algebras is the object responsible for the factorization problem and is the JJ algebra version of [10, Theorem 3.9]. If $A \bowtie V$ is the bicrossed product associated to a matched pair $(A, V, <, \triangleright)$ of JJ algebras, then the Galois group of the extension $A \subseteq A \bowtie V$ is explicitly computed in Corollary 3.4 as a subgroup of the semidirect product of groups $\text{GL}_k(V) \rtimes \text{Hom}_k(V, A)$. The crossed product of two JJ algebras is also a special case of the unified product: it was introduced and studied in [3] related to Hilbert’s extension problem. Here we highlight a new application of crossed products as the main characters in our strategy for classifying finite dimensional supersolvable JJ algebras (Proposition 3.8). Skew crossed products are also introduced and used in Theorem 3.11 in order to prove an Artin type theorem for JJ algebras which gives the reconstruction of a JJ algebra $A$, on which the finite group $G$ acts, from its subalgebra of invariants. Computing the classifying object $H^2_A(V, A)$ constructed in Theorem 2.7, for a given JJ algebra $A$ and a vector space $V$ is, in general, a very difficult problem. As the starting point in achieving this goal we develop in Section 4 a general strategy for explicitly computing $H^2_A(k, A)$ (Theorem 4.8).

1. Preliminaries

All vector spaces, linear or bilinear maps are over an arbitrary field $k$. A bilinear map $f : W \times W \to V$ is called symmetric if $f(x, y) = f(y, x)$, for all $x, y \in W$. For a vector space $V$ its dual is denoted by $V^* := \text{Hom}_k(V, k)$ and $\text{GL}_k(V) := \text{Aut}_k(V)$ is the group of all linear automorphisms of $V$. 
A **JJ algebra** is a vector space $A$ together with a bilinear map $[-,-]: A \times A \to A$, called multiplication, such that for any $a, b, c \in A$:

$$[a, b] = [b, a], \quad [a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 \quad (1)$$

that is, $[-,-]$ is commutative/symmetric and satisfies the Jacobi identity. Throughout, when describing the multiplication of a certain JJ algebra we will only write down the non-zero products. The concepts of morphism of JJ algebras, subalgebra, ideal, derivation etc. are defined in the obvious way. We denote by $\text{Aut}_{\text{JJ}}(A)$ the automorphism group of the JJ algebra $A$. $\sum(c)$ stands for the circular sum: for example, $\sum(c)[a, [b, c]] = [a, [b, c]] + [b, [c, a]] + [c, [a, b]]$. A JJ algebra $A$ is called abelian if it has trivial multiplication, i.e. $[a, b] = 0$ for all $a, b \in A$. Over a field of characteristic $\neq 2, 3$, any JJ algebra is a Jordan algebra [7] and Jordan algebras of nilpotent index 3 are JJ algebras [20]. If $A$ is a JJ algebra and $B$ a commutative associative algebra, then the tensor product $A \otimes B$ can be endowed with a JJ algebra structure whose multiplication is given by the formula:

$$[a \otimes b, a' \otimes b'] := [a, a'] \otimes b b'$$

for all $a, a' \in A$ and $b, b' \in B$. Following [19], we call this object a current JJ algebra. Several examples of JJ algebras are given in [3, 6, 7, 20]. An **antiderivation** of a JJ algebra $A$ is a linear map $D: A \to A$ such that for any $a, b \in A$:

$$D([a, b]) = -[D(a), b] - [a, D(b)]$$

Unfortunately, unlike Lie algebras, the space of all derivations (resp. antiderivations) of a JJ algebra does not carry a canonical JJ algebra structure.

A **left JJ $A$-module** [3, Definition 1.4] is a vector space $V$ equipped with a bilinear map $\triangleright: A \times V \to V$, called action, such that for any $a, b \in A$ and $x \in V$:

$$(a, b) \triangleright x = -a \triangleright (b \triangleright x) - b \triangleright (a \triangleright x) \quad (2)$$

We denote by $\mathcal{M}_A$ the category of all (left) JJ $A$-modules with action preserving linear maps as morphisms. A **right JJ $A$-module** is a vector space $V$ equipped with a bilinear map $\triangleleft: V \times A \to V$ such that for any $a, b \in A$ and $x \in V$:

$$x \triangleleft [a, b] = -(x \triangleleft a) \triangleleft b - (x \triangleleft b) \triangleleft a \quad (3)$$

Since $A$ is in particular a commutative algebra there exists an isomorphism of categories $\mathcal{M}_A \cong \mathcal{M}_A$. Nevertheless, we shall use both categories throught the paper. We can easily see that $A$ and the linear dual $A^* := \text{Hom}_k(A, k)$ are left JJ $A$-modules via the canonical actions:

$$a \triangleright x := [a, x], \quad (a \triangleright a^*)(x) := a^*[a, x] \quad (4)$$

for all $a, x \in A$ and $a^* \in A^*$.

### 2. Extending structures problem

This section deals with the extending structures problem for JJ algebras. First we introduce the following:
**Definition 2.1.** Let $A$ be a JJ algebra and $E$ a vector space containing $A$ as a subspace. Two JJ algebra structures on $E$, $\{ -, - \}$ and $\{ -, - \}'$, both containing $A$ as a subalgebra, are called *equivalent*, and we denote this by $(E, \{ -, - \}) \equiv (E, \{ -, - \}')$, if there exists a JJ algebra isomorphism $\varphi : (E, \{ -, - \}) \to (E, \{ -, - \}')$ which stabilizes $A$, that is $\varphi(a) = a$, for all $a \in A$. We denote by $\text{Extd}(E, A)$ the set of all equivalence classes on the set of all JJ algebras structures on $E$ containing $A$ as a subalgebra via the equivalence relation $\equiv$.

Extd $(E, A)$ as defined above is the classifying object of the extending structures problem. We shall prove that Extd $(E, A)$ is parameterized by a cohomological type object, denoted by $\mathcal{H}^2_A(V, A)$, which will be explicitly constructed in this section, where $V$ is a complement of $A$ in $E$, that is $E = A + V$ and $A \cap V = 0$.

**Definition 2.2.** Let $A$ be a JJ algebra and $V$ a vector space. An *extending datum of $A$ through $V$* is a system $\Omega(A, V) = (\triangleleft, \triangleright, \{ -, - \})$ consisting of four bilinear maps

$$\triangleleft : V \times A \to V, \quad \triangleright : V \times A \to A, \quad f : V \times V \to A, \quad \{ -, - \} : V \times V \to V.$$ 

Let $\Omega(A, V) = (\triangleleft, \triangleright, \{ -, - \})$ be an extending datum. We denote by $A \triangleright \Omega(A, V) V = A \triangleright V$ the vector space $A \times V$ together with the bilinear map $\{ -, - \} : (A \times V) \times (A \times V) \to A \times V$ defined by:

$$[(a, x), (b, y)] := ([a, b] + x \triangleright b + y \triangleright a + f(x, y), \{ x, y \} + x \triangleleft b + y \triangleleft a) \quad (5)$$

for all $a, b \in A$ and $x, y \in V$. The object $A \triangleright V$ is called the *unified product* of $A$ and $V$ if it is a JJ algebra with the multiplication given by (5). In this case the extending datum $\Omega(A, V) = (\triangleleft, \triangleright, f, \{ -, - \})$ is called a *JJ extending structure of $A$ through $V$*. The maps $\triangleleft$ and $\triangleright$ are called the *actions* of $\Omega(A, V)$ and $f$ is called the *cocycle* of $\Omega(A, V)$.

Let $\Omega(A, V)$ be an extending datum of $A$ through $V$. Then, the following relations, very useful in computations, hold in $A \triangleright V$:

$$[(a, 0), (b, y)] = ([a, b] + y \triangleright a, y \triangleleft a) \quad (6)$$

$$[(0, x), (b, y)] = (x \triangleright b + f(x, y), x \triangleleft b + \{ x, y \}) \quad (7)$$

for all $a, b \in A$ and $x, y \in V$.

**Theorem 2.3.** Let $\Omega(A, V) = (\triangleleft, \triangleright, f, \{ -, - \})$ be an extending datum of a JJ algebra $A$ through a vector space $V$. The following statements are equivalent:

1. $A \triangleright V$ is a unified product;
2. The following compatibilities hold for any $a, b \in A$, $x, y, z \in V$:
   a. $f : V \times V \to A$ and $\{ -, - \} : V \times V \to V$ are symmetric maps;
   b. $(V, \triangleleft)$ is a right JJ $A$-module;
   c. $x \triangleright [a, b] = -[x \triangleright a, b] - [a, x \triangleright b] - (x \triangleleft a) \triangleright b - (x \triangleleft b) \triangleright a$;
   d. $\{ x, y \} \triangleleft a = -\{ x, y \triangleleft a \} - \{ x \triangleleft a, y \} - x \triangleleft (y \triangleright a) - y \triangleleft (x \triangleright a)$;
(E5) \( \{x, y\} \triangleright a = -x \triangleright (y \triangleright a) - y \triangleright (x \triangleright a) - [a, f(x, y)] - f(x, y \triangleleft a) - f(x \triangleleft a, y) \); 

(E6) \( \sum_{(c)} f(x, \{y\}) + \sum_{(c)} x \triangleright f(y, z) = 0 \);

(E7) \( \sum_{(c)} \{x, \{y, z\}\} + \sum_{(c)} x \triangleleft f(y, z) = 0 \).

**Proof.** The proof is based on a rather long and laborious but straightforward computation. We restrict to indicating only the main steps of the proof. First, we can easily prove that the multiplication defined by (5) is commutative if and only if \( f : V \times V \to A \) and \( \{-, -\} : V \times V \to V \) are both symmetric maps, i.e. (E1) holds. From now on we will assume that (E1) holds. Thus \( A^\sharp V \) is a JJ algebra if and only if Jacobi’s identity holds, i.e.:

\[
\sum_{(c)} [(a, x), [(b, y), (c, z)]] = 0 \tag{8}
\]

for all \( a, b, c \in A \) and \( x, y, z \in V \). Since in \( A^\sharp V \) we have \( (a, x) = (a, 0) + (0, x) \) it follows that (8) holds if and only if it holds for all generators of \( A^\sharp V \), i.e. for the set \( \{(a, 0) \mid a \in A\} \cup \{(0, x) \mid x \in V\} \). Since (8) is invariant under circular permutations we are left with only three cases to study. First, using (6) we can easily notice that (8) holds for the triple \((a, 0), (b, 0), (c, 0)\). Next, we can prove that (8) holds for \((a, 0), (b, 0), (0, x)\) if and only if (E2) and (E3) hold. Secondly, we can prove that (8) holds for \((a, 0), (0, x), (0, y)\) if and only if (E4) and (E5) hold. Finally, (8) holds for \((0, x), (0, y), (0, z)\) if and only if (E6) and (E7) hold and the proof is finished. \( \square \)

For a given JJ algebra \( A \) a vector space \( V \), we denote by \( \mathcal{J}(A, V) \) the set of all JJ extending structures of \( A \) through \( V \), i.e. all systems \( \Omega(A, V) = \{\triangleleft, \triangleright, f, \{-, -\}\} \) satisfying the compatibility conditions (E1)-(E7) of Theorem 2.3. Several examples of extending structures will be given in Section 3 and Section 4. Notice that the set \( \mathcal{J}(A, V) \) is nonempty: it contains the extending structure \( \Omega(A, V) = \{\triangleleft, \triangleright, f, \{-, -\}\} \) for which all bilinear maps are trivial. In this case the associated unified product \( A^\sharp V = A \times V \), the direct product between \( A \) and the abelian JJ algebra \( V \).

Let \( \Omega(A, V) = \{\triangleleft, \triangleright, f, \{-, -\}\} \in \mathcal{J}(A, V) \) be a JJ extending structure and \( A^\sharp V \) the associated unified product. Then the canonical inclusion

\[ i_A : A \to A^\sharp V, \quad i_A(a) = (a, 0) \]

is an injective JJ algebra map. Therefore, we can see \( A \) as a JJ subalgebra of \( A^\sharp V \) through the identification \( A \cong i_A(A) \cong A \times \{0\} \). Conversely, we will prove that any JJ algebra structure on a vector space \( E \) containing \( A \) as a JJ subalgebra is isomorphic to a unified product.

**Theorem 2.4.** Let \( A \) be a JJ algebra, \( E \) a vector space containing \( A \) as a subspace and \( \{-, -\} \) a JJ algebra structure on \( E \) such that \( A \) is a JJ subalgebra in \( (E, \{-, -\}) \). Then there exists a JJ extending structure \( \Omega(A, V) = \{\triangleleft, \triangleright, f, \{-, -\}\} \) of \( A \) through a subspace \( V \) of \( E \) and an isomorphism of JJ algebras \( (E, \{-, -\}) \cong A^\sharp V \) that stabilizes \( A \).
Proof. Since we work over a field $k$, there exists a linear map $p : E \to A$ such that $p(a) = a$, for all $a \in A$. Then $V := \ker(p)$ is a subspace of $E$ and a complement of $A$ in $E$. We can now define the extending datum of $A$ through $V$ as follows:

$$\triangleright = \triangleright_p : V \times A \to A, \quad x \triangleright a := p([x, a])$$

$$\triangleleft = \triangleleft_p : V \times A \to V, \quad x \triangleleft a := [x, a] - p([x, a])$$

$$f = f_p : V \times V \to A, \quad f(x, y) := p([x, y])$$

for any $a \in A$ and $x, y \in V$. First of all, it is straightforward to see that the above maps are well defined bilinear maps: $x \triangleleft a \in V$ and $\{x, y\} \in V$, for all $x, y \in V$ and $a \in A$. We will show that $\Omega(A, V) = (\triangleleft, \triangleright, f, \{-, -\})$ is a JJ extending structure of $A$ through $V$ and $\varphi : A \bowtie V \to E$, $\varphi(a, x) := a + x$ is an isomorphism of JJ algebras that stabilizes $A$. The strategy we use, relaying on Theorem 2.3, is the following: $\varphi : A \times V \to E$, $\varphi(a, x) := a + x$ is a linear isomorphism between the JJ algebra $E$ and the direct product of vector spaces $A \times V$ with the inverse given by $\varphi^{-1}(y) := (p(y), y - p(y))$, for all $y \in E$. Thus, there exists a unique JJ algebra structure on $A \times V$ that $\varphi$ is an isomorphism of JJ algebras and this unique multiplication on $A \times V$ is given for any $a, b \in A$ and $x, y \in V$ by:

$$[(a, x), (b, y)] := \varphi^{-1}([\varphi(a, x), \varphi(b, y)])$$

We are now left to prove that the above multiplication coincides with the one associated to the system $(\triangleleft_p, \triangleright_p, f_p, \{-, -\}_p)$ as defined by (5). Indeed, for any $a, b \in A$ and $x, y \in V$ we have:

$$[(a, x), (b, y)] = \varphi^{-1}([\varphi(a, x), \varphi(b, y)]) = \varphi^{-1}([a, b] + [a, y] + [x, b] + [x, y])$$

$$= (p([a, b]), [a, b] - p([a, b])) + (p([a, y]), [a, y] - p([a, y])) + (p([x, b]), [x, b] - p([x, b])) + (p([x, y]), [x, y] - p([x, y]))$$

$$= \left( [a, b] + p([a, y]) + p([x, b]) + p([x, y]), [a, b] + [a, y] + [x, b] + [x, y] - p([x, b]) - p([x, y]) \right)$$

$$= \left( [a, b] + [y \triangleright a + x \triangleright b + f(x, y), [x, y] + x \triangleleft b + y \triangleleft a \right)$$

as desired. Note that the commutativity of $[-, -]$ was intensively used in the above computations. Moreover, the following diagram

$$\begin{array}{ccc}
A & \overset{i_A}{\longrightarrow} & A \bowtie V \\
Id & \downarrow & \varphi \\
A & \overset{i}{\longrightarrow} & E
\end{array}$$

is obviously commutative which shows that $\varphi$ stabilizes $A$ and this finishes the proof. \qed
Using Theorem 2.4, the classification of all JJ algebra structures on $E$ that contain $A$ as a subalgebra, reduces to the classification of all unified products $A \circ V$, associated to all JJ extending structures $\Omega(A, V) = (\langle, \triangleright, f, \{-, -\})$, for a given complement $V$ of $A$ in $E$. In order to construct a cohomological type object $H^2_A(V, A)$ which will parameterize the classifying sets $\text{Ext}d(E, A)$ defined in Definition 2.1, we introduce the following:

**Lemma 2.5.** Let $\Omega(A, V) = (\langle, \triangleright, f, \{-, -\})$ and $\Omega'(A, V) = (\langle', \triangleright', f', \{-, -\})$ be two JJ algebra extending structures of $A$ through $V$ and $A \circ V$, respectively $A \circ V$, the associated unified products. Then there exists a bijection between the set of all morphisms of JJ algebras $\psi : A \circ V \to A \circ V$ which stabilize $A$ and the set of pairs $(r, v)$, where $r : V \to A$, $v : V \to V$ are two linear maps satisfying the following compatibility conditions for any $a \in A$, $x, y \in V$:

(M1) $v(x) \triangleleft a = v(x \triangleleft a)$, i.e. $v$ is a morphism of right JJ $A$-modules;
(M2) $v(x) \triangleright a = r(x \triangleleft a) + x \triangleright a - [a, r(x)];$
(M3) $v\{x, y\} = \{v(x), v(y)\} + v(x) \triangleleft r(y) + v(y) \triangleleft r(x);$ 
(M4) $r\{x, y\} = [r(x), r(y)] + v(x) \triangleright r(y) + v(y) \triangleright r(x) + f'(v(x), v(y)) - f(x, y)$

Under the above bijection the morphism of JJ algebras $\psi = \psi_{(r,v)} : A \circ V \to A \circ V$ corresponding to $(r, v)$ is given for any $a \in A$ and $x \in V$ by:

$$\psi(a, x) = (a + r(x), v(x))$$

Moreover, $\psi = \psi_{(r,v)}$ is an isomorphism if and only if $v : V \to V$ is bijective.

**Proof.** A linear map $\psi : A \circ V \to A \circ V$ which makes the following diagram commutative:

$$\begin{array}{ccc}
A & \xrightarrow{i_A} & A \circ V \\
\downarrow{Id_A} & & \downarrow{\psi} \\
A & \xrightarrow{i_A} & A \circ V 
\end{array}$$

is uniquely determined by two linear maps $r : V \to A$, $v : V \to V$ such that $\psi(a, x) = (a + r(x), v(x))$, for all $a \in A$, and $x \in V$. Indeed, if we denote $\psi(0, x) = (r(x), v(x)) \in A \times V$ for all $x \in V$, we obtain:

$$\psi(a, x) = \psi((a, 0) + \psi(0, x)) = \psi(a, 0) + \psi(0, x)$$

$$= (a, 0) + (r(x), v(x)) = (a + r(x), v(x))$$

Let $\psi = \psi_{(r,v)}$ be such a linear map, i.e. $\psi(a, x) = (a + r(x), v(x))$, for some linear maps $r : V \to A$, $v : V \to V$. We will prove that $\psi$ is a morphism of JJ algebras if and only if the compatibility conditions (M1)-(M4) hold. To this end, it is enough to prove that the compatibility

$$\psi\{[(a, x), (b, y)]\} = [\psi(a, x), \psi(b, y)]$$

(9)

holds on all generators of $A \circ V$. Again, we skip the detailed computations and indicate only the key steps of the process. First, it is easy to see that (9) holds for the pair $(a, 0)$, $(b, 0)$, for all $a, b \in A$. Secondly, we can prove that (9) holds for the pair $(a, 0), (0, x)$ if and only if (M1) and (M2) hold. Finally, (9) holds for the pair $(0, x), (0, y)$ if and only if (M3) and (M4) hold. The last statement follows immediately by noticing that
if \( v : V \to V \) is bijective, then \( \psi_{(r,v)} \) is an isomorphism of JJ algebras with the inverse given for any \( b \in A \) and \( y \in V \) by:
\[
\psi^{-1}_{(r,v)}(b, y) = (b - r(v^{-1}(y)), v^{-1}(y))
\]
The proof is now finished.

For classification purposes we introduce the following:

**Definition 2.6.** Let \( A \) be a JJ algebra and \( V \) a vector space. Two JJ algebra extending structures of \( A \) by \( V \), \( \Omega(A, V) = (\triangleright, \triangleright', f, \{\cdot, -\}) \) and \( \Omega'(A, V) = (\triangleright', \triangleright'', f', \{\cdot, -\}') \) are called equivalent, and we denote this by \( \Omega(A, V) \equiv \Omega'(A, V) \), if there exists a pair of linear maps \((r, v)\), where \( r : V \to A \) and \( v \in \text{Aut}_k(V) \) such that \((\triangleright', \triangleright'', f', \{\cdot, -\}')\) is defined via \((\triangleright, \triangleright', f, \{\cdot, -\})\) using \((r, v)\) as follows:
\[
\begin{align*}
x \triangleleft a &= v(v^{-1}(x) \triangleright a) \\
x \triangleleft' a &= r(v^{-1}(x) \triangleright a) + v^{-1}(x) \triangleright' a - [a, r(v^{-1}(x))] \\
f'(x, y) &= f(v^{-1}(x), v^{-1}(y)) + [r(v^{-1}(x), v^{-1}(y))] + [r(v^{-1}(x)), r(v^{-1}(y))] \\
&- r(v^{-1}(x)) \triangleright r(v^{-1}(y)) - v^{-1}(x) \triangleright r(v^{-1}(y)) - r(v^{-1}(y)) \triangleright r(v^{-1}(x)) \\
&- v^{-1}(y) \triangleright r(v^{-1}(x)) \\
\{x, y\}' &= v(v^{-1}(x), v^{-1}(y)) - v(v^{-1}(x) \triangleright r(v^{-1}(y))) - v(v^{-1}(y) \triangleright r(v^{-1}(x)))
\end{align*}
\]
for all \( a \in A, x, y \in V \).

We summarize the results of this section in the following result which provides the answer to the extending structures problem for JJ algebras:

**Theorem 2.7.** Let \( A \) be a JJ algebra, \( E \) a vector space that contains \( A \) as a subspace and \( V \) a complement of \( A \) in \( E \). Then:

1. \( \equiv \) is an equivalence relation on the set \( \mathcal{J} \mathcal{J}(A, V) \) of all JJ algebra extending structures of \( A \) through \( V \). We denote by \( \mathcal{H}^2_A(V, A) := \mathcal{J} \mathcal{J}(A, V) / \equiv \), the quotient set.

2. The map
\[
\mathcal{H}^2_A(V, A) \to \text{Extd}(E, A), \quad (\triangleright, \triangleright', f, \{\cdot, -\}) \to (g \triangleright V, [-, -])
\]
is bijective, where \((\triangleright, \triangleright', f, \{\cdot, -\})\) is the equivalence class of \((\triangleright, \triangleright', f, \{\cdot, -\}) \) via \( \equiv \).

**Proof.** The proof follows from Theorem 2.3, Theorem 2.4 and Lemma 2.5 once we observe that \( \Omega(A, V) \equiv \Omega'(A, V) \) in the sense of Definition 2.6 if and only if there exists an isomorphism of JJ algebras \( \psi : A \triangleright V \to A \triangleright' V \) which stabilizes \( A \). Therefore, \( \equiv \) is an equivalence relation on the set \( \mathcal{J} \mathcal{J}(A, V) \) of all JJ algebra extending structures \( \Omega(A, V) \) and the conclusion follows from Theorem 2.4 and Lemma 2.5.

### 3. Special cases of unified products. Applications

In this section we consider the most important special cases of unified products of JJ algebras, namely bicrossed/semidirect/crossed/skew crossed products, and we will provide applications for each of these products. We consider the following convention: if
one of the maps $\triangleleft, \triangleright, f$ or $\{-, -\}$ of an extending datum $\Omega(A, V) = (\triangleleft, \triangleright, f, \{-, -\})$ is trivial then we will omit it from the quadruple $(\triangleleft, \triangleright, f, \{-, -\})$.

**Matched pairs and bicrossed products.** Let $\Omega(A, V) = (\triangleleft, \triangleright, f, \{-, -\})$ be an extending datum of the JJ algebra $A$ through a vector space $V$ such that $f$ is the trivial map, i.e. $f(x, y) = 0$ for all $x, y \in V$. Then, using Theorem 2.3 we obtain that $\Omega(A, V) = (\triangleleft, \triangleright, \{-, -\})$ is a JJ extending structure of $A$ through $V$ if and only if $(V, \{-, -\})$ is a JJ algebra and the following compatibilities hold for all $a, b \in A$, $x, y \in V$:

1. $(V, \triangleleft)$ is a right JJ $A$-module, i.e. $x \triangleleft [a, b] = -(x \triangleleft a) \triangleleft b - (x \triangleleft b) \triangleleft a$;
2. $(A, \triangleright)$ is a left JJ $V$-module, i.e. $\{x, y\} \triangleright a = -x \triangleright (y \triangleright a) - y \triangleright (x \triangleright a)$;
3. $x \triangleright [a, b] = -(x \triangleright a, b) - [a, x \triangleright b] - (x \triangleleft a) \triangleright b - (x \triangleleft b) \triangleright a$;
4. $\{x, y\} \triangleleft a = -(x, y \triangleleft a) - \{x \triangleleft a, y\} - x \triangleleft (y \triangleright a) - y \triangleleft (x \triangleright a)$.

Following [11, Theorem 4.1] we introduce the following concept:

**Definition 3.1.** Let $A = (A, \{-, -\})$ and $V = (V, \{-, -\})$ be two JJ algebras. Then $(A, V, \triangleleft, \triangleright)$ is called a matched pair of JJ algebras if $(V, \triangleleft)$ is a right JJ $A$-module, $(A, \triangleright)$ is a left JJ $V$-module and the following compatibilities hold for all $a, b \in A$, $x, y \in V$:

1. $x \triangleright [a, b] = -(x \triangleright a, b) - [a, x \triangleright b] - (x \triangleleft a) \triangleright b - (x \triangleleft b) \triangleright a$;
2. $\{x, y\} \triangleleft a = -(x, y \triangleleft a) - \{x \triangleleft a, y\} - x \triangleleft (y \triangleright a) - y \triangleleft (x \triangleright a)$.

If $(A, V, \triangleleft, \triangleright)$ is a matched pair of JJ algebras then the associated unified product $A \ast_{\Omega(A, V)} V$ will be denoted by $A \bowtie V$ and will be called the bicrossed product of the matched pair $(A, V, \triangleleft, \triangleright)$. Thus, $A \bowtie V = A \times V$ as a vector space with multiplication given by:

$$[(a, x), (b, y)] := ([a, b] + x \triangleright b + y \triangleright a, \{x, y\} + x \triangleleft b + y \triangleleft a)$$

for all $a, b \in A$ and $x, y \in V$.

**Example 3.2.** Let $(A, V, \triangleleft, \triangleright)$ be a matched pair of JJ algebras such that $\triangleleft$ is the trivial map. Then the associated bicrossed product $A \bowtie V$ will be denoted by $A \bowtie V$ and was first introduced in [3] under the name of semidirect product. Explicitly, the semidirect product $A \bowtie V$ is associated to a left JJ $V$-module structure $(A, \triangleright)$ such that for any $a, b \in A$ and $x \in V$:

$$x \triangleright [a, b] = -(x \triangleright a, b) - [a, x \triangleright b]$$

or equivalently the map $x \triangleright - : A \to A$ is an antiderivation of $A$, for all $x \in V$.

The bicrossed product of two JJ algebras is the construction responsible for the so-called factorization problem.
Let $A$ and $V$ be two given JJ algebras. Describe and classify all JJ algebras $E$ that factorize through $A$ and $V$, i.e. $E$ contains $A$ and $V$ as JJ subalgebras such that $E = A + V$ and $A \cap V = \{0\}$.

Indeed, using Theorem 2.4 we can prove the JJ algebra version of [10, Theorem 3.9]:

**Corollary 3.3.** A JJ algebra $E$ factorizes through two given JJ algebras $A$ and $V$ if and only if there exists a matched pair of JJ algebras $(A, V, \triangleright, \triangleleft)$ such that $E \cong A \rtimes V$.

**Proof.** First observe that $A \cong A \times \{0\}$ and $V \cong \{0\} \times V$ are JJ subalgebras of $A \rtimes V$ and of course $A \rtimes V$ factorizes through $A \times \{0\}$ and $\{0\} \times V$. Conversely, assume that a JJ algebra $E$ factorizes through two JJ subalgebras $A$ and $V$. Since $V$ is a subalgebra of $E$, the cocycle $f = f_p : V \times V \rightarrow A$ constructed in the proof of Theorem 2.4 is just the trivial map $f_p(x, y) = 0$, for all $x, y \in V$. Thus, the unified product $A \rtimes_G (A \rtimes V) = A \rtimes V$ coincides with the bicrossed product of the JJ algebras $A$ and $V := \text{Ker}(p)$. \qed

Based on Corollary 3.3 we can restate the factorization problem as follows: Let $A$ and $V$ be two given JJ algebras. Describe the set of all matched pairs $(A, V, \triangleright, \triangleleft)$ and classify up to an isomorphism all bicrossed products $A \rtimes V$.

Due to its important applications to the theory of JJ algebras we will consider this problem separately in greater detail in a forthcoming paper.

In what follows we compute the Galois group of the JJ algebra extension $A \subseteq A \rtimes V$. Given a matched pair of JJ algebras $(A, V, \triangleright, \triangleleft)$ we define the *Galois group* $\text{Gal}(A \rtimes V/A)$ of the extension $A \subseteq A \rtimes V$, as the subgroup of $\text{Aut}_{JJ}(A \rtimes V)$ of all JJ algebra automorphisms of $A \rtimes V$ that stabilize $A$:

$$\text{Gal}(A \rtimes V/A) := \{\sigma \in \text{Aut}_{JJ}(A \rtimes V) \mid \sigma(a) = a, \forall a \in A\}$$

As a straightforward consequence of Lemma 2.5 we obtain a bijection between the set of all elements $\psi \in \text{Gal}(A \rtimes V/A)$ and the set of all pairs $(\sigma, r) \in \text{GL}_k(V) \times \text{Hom}_k(V, A)$, satisfying the following compatibility conditions for any $a \in A$, $x, y \in V$:

- (G1) $v(x) \triangleleft a = v(x \triangleleft a)$;
- (G2) $v(x) \triangleright a = r(x \triangleright a) + x \triangleright a - [a, r(x)]$;
- (G3) $r(\{x, y\}) = \{v(x), v(y)\} + v(x) \triangleleft r(y) + v(y) \triangleleft r(x)$;
- (G4) $r(\{x, y\}) = [r(x), r(y)] + v(x) \triangleright r(y) + v(y) \triangleright r(x)$

The bijection is such that $\psi = \psi_{(\sigma, r)} \in \text{Gal}(A \rtimes V/A)$ corresponding to $(\sigma, r) \in \text{GL}_k(V) \times \text{Hom}_k(V, A)$ is given by $\psi(a, x) := (a + r(x), \sigma(x))$, for all $a \in A$ and $x \in V$.

We point out that $\psi_{(\sigma, r)}$ is indeed an element of $\text{Gal}(A \rtimes V/A)$ with the inverse given by $\psi_{(\sigma, r)}^{-1}(a, x) = (a - r(\sigma^{-1}(x)), \sigma^{-1}(x))$, for all $a \in A$ and $x \in V$.

We denote by $\mathbb{G}_A^{V}(\triangleleft, \triangleright)$ the set of all pairs $(\sigma, r) \in \text{GL}_k(V) \times \text{Hom}_k(V, A)$ satisfying the compatibility conditions (G1)-(G4). It can be easily seen that $\mathbb{G}_A^{V}(\triangleleft, \triangleright)$ is a subgroup of the semidirect product of groups $\text{GL}_k(V) \times \text{Hom}_k(V, A)$ with the group structure defined as follows:

$$(\sigma, r) \cdot (\sigma', r') := (\sigma \circ \sigma', r \circ \sigma' + r') \quad (11)$$
for all \( \sigma, \sigma' \in \text{GL}_k(V) \) and \( r, r' \in \text{Hom}_k(V, A) \). Now, for any \((\sigma, r)\) and \((\sigma', r')\) \(\in \mathbb{G}^V_A(\lhd, \rhd)\), \(a \in A\) and \(x \in V\) we have:
\[
\psi_{(\sigma, r)} \circ \psi_{(\sigma', r')}(a, x) = (a + r'(x) + r(\sigma'(x)), \sigma(\sigma'(x))) = \psi_{(\sigma \circ \sigma', r \circ r' + r)}(a, x)
\]
i.e. \(\psi_{(\sigma, r)} \circ \psi_{(\sigma', r')} = \psi_{(\sigma \circ \sigma', r \circ r' + r)}\). To summarize, we have proved the following:

**Corollary 3.4.** Let \((A, V, \lhd, \rhd)\) be a matched pair of JJ algebras. Then there exists an isomorphism of groups defined as follows:
\[
\Omega : \mathbb{G}^V_A(\lhd, \rhd) \to \text{Gal}(A \bowtie V/A), \quad \Omega(\sigma, r)((a, x)) := (a + r(x), \sigma(x)) \tag{12}
\]
for all \((\sigma, r) \in \mathbb{G}^V_A(\lhd, \rhd)\), \(a \in A\) and \(x \in V\). In particular, there exists an embedding \(\text{Gal}(A \bowtie V/A) \hookrightarrow \text{GL}_k(V) \times \text{Hom}_k(V, A)\), where the right hand side is the semidirect product of groups defined by (11).

**Crossed products and supersolvable algebras.** Let \(\Omega(A, V) = (\lhd, \rhd, f, \{-, -\})\) be an extending datum of the JJ algebra \(A\) through a vector space \(V\) such that \(\lhd\) is trivial, i.e. \(x \lhd a = 0\), for all \(x \in V\) and \(a \in A\). Then, \(\Omega(A, V) = (\rhd, f, \{-, -\})\) is a JJ extending structure of \(A\) through \(V\) if and only if \((V, \{-, -\})\) is a JJ algebra and the following compatibilities hold for all \(a, b \in A\) and \(x, y, z \in V\):

(CP1) \(f : V \times V \to A\) is a symmetric map;

(CP2) \(x \rhd [a, b] = -[x \rhd a, b] - [a, x \rhd b]\), i.e. \(x \rhd - : A \to A\) is an antiderivation of \(A\);

(CP3) \(\{x, y\} \rhd a = -x \rhd (y \rhd a) - y \rhd (x \rhd a) - [a, f(x, y)];\)

(CP4) \[\sum_{(c)} f(x, \{y, z\}) + \sum_{(c)} x \rhd f(y, z) = 0\]

A system \((A, V, \rhd, f)\) consisting of two JJ algebras \(A, V\) and two bilinear maps \(\rhd : V \times V \to A\), \(f : V \times V \to A\) satisfying the above four compatibility conditions was called a **crossed system** of \(A\) and \(V\) in [3, Proposition 2.2]. In this case, the associated unified product \(A \bowtie \Omega(A, V) V = A \#_{\Omega(A, V)} V\) is the **crossed product** of the JJ algebras \(A\) and \(V\) and is defined as follow: \(A \#_{\Omega(A, V)} V = A \times V\) with the multiplication given for any \(a, b \in A\) and \(x, y \in V\) by:
\[
[(a, x), (b, y)] := [(a, b) + x \rhd b + y \rhd a + f(x, y), \{x, y\}] \tag{13}
\]
If \((A, V, \rhd, f)\) is a crossed system of two JJ algebras then, \(A \cong A \times \{0\}\) is an ideal in \(A \#_{\Omega(A, V)} V\) since \([\{0\}, (a, y)] := [(a, b) + y \rhd a, 0]\). Conversely, crossed products describe all JJ algebra structures on a vector space \(E\) such that a given JJ algebra \(A\) becomes an ideal of \(E\).

**Corollary 3.5.** Let \(A\) be a JJ algebra, \(E\) a vector space containing \(A\) as a subspace. Then any JJ algebra structure on \(E\) that contains \(A\) as an ideal is isomorphic to a crossed product of JJ algebras \(A \#_{\Omega} V\).

**Proof.** Let \([-, -]\) be a JJ algebra structure on \(E\) such that \(A\) is an ideal in \(E\). In particular, \(A\) is a subalgebra of \(E\) and hence we can apply Theorem 2.4. In this case the
action $\prec = \prec_p$ of the JJ extending structure $\Omega(A, V) = (\prec_p, \succ_p, f_p, \{-, -\}_p)$ constructed in the proof of Theorem 2.4 is trivial since for any $x \in V$ and $a \in A$ we have $[x, a] \in A$ and hence $p([x, a]) = [x, a]$. Thus, $x \prec_p a = 0$, i.e. the unified product $A \sharp_{\Omega(A, V)} V = A \# f \nabla V$ is the crossed product of the JJ algebras $A$ and $V := \ker(p)$. □

The crossed product of JJ algebras was studied in detail in [3] related to Hilbert’s extension problem. We consider here a new application: using Corollary 3.5 we show that crossed products play a key role in the classification of finite dimensional supersolvable JJ algebras.

Definition 3.6. Let $n$ be a positive integer. An $n$-dimensional JJ algebra $E$ is called supersolvable if there exists a finite chain of ideals of $E$

$$0 = I_0 \subset I_1 \subset \cdots \subset I_n = E$$

such that $I_j$ has codimension 1 in $I_{j+1}$, for all $j = 0, \cdots, n - 1$.

All finite dimensional supersolvable JJ algebras can be classified by a recursive method based on Corollary 3.5. Indeed, the key step of the process consist in describing all crossed products $A \# f \nabla V$, for a 1-dimensional vector space $V$ and a given JJ algebra $A$. To this end, we introduce the following:

Definition 3.7. Let $A$ be a JJ algebra. A supersolvable datum of $A$ is a pair $(D, a_0) \in \text{End}_k(A) \times A$ such that:

(S1) $D : A \to A$ is an antiderivation of $A$ and $3D(a_0) = 0$;

(S2) $2D^2(a) = -[a, a_0]$, for all $a \in A$.

We denote by $S(A)$ the set of all supersolvable data of $A$.

Proposition 3.8. Let $k$ be a field of characteristic $\neq 3$, $A$ a JJ algebra and $V$ a vector space of dimension 1 with basis $\{x\}$. Then there exists a bijection between the set of all crossed systems of $A$ and $V$ and the set $S(A)$ of all supersolvable data of $A$. Through the above bijection, the crossed system $(\succ, f, \{-, -\})$ corresponding to $(D, a_0) \in S(A)$ is defined as follows:

$$x \succ a = D(a), \quad f(x, x) = a_0, \quad \{x, x\} = 0$$

for all $a \in A$.

Proof. Since $k$ is a field of characteristic $\neq 3$, the only JJ algebra structure on $V := kx$ is the abelian one, i.e. $\{x, x\} = 0$. Moreover, as $V$ has dimension 1 the set of all bilinear maps $\succ : V \times A \to A$, $f : V \times V \to A$ is in bijection with the set of all pairs $(D, a_0) \in \text{End}_k(A) \times A$ and the bijection is given such that (15) holds. We are left to prove that the compatibilities (CP2)-(CP4) are equivalent to $(D, a_0) \in S(A)$. Indeed, (CP2) is equivalent to the fact that $D$ is an antiderivation of $A$, (CP3) is equivalent to the fact that (S2) holds and (CP4) is equivalent to $3D(a_0) = 0$. □
Now let \((D, a_0) \in \mathcal{S}(A)\) be a supersolvable datum of \(A\). The crossed product \(A \#_{kx} f\) associated to the crossed system \((15)\) will be denoted by \(A_{(D, a_0)} := A \times kx\) and has the multiplication given as follows:

\[
[(a, x), (b, x)] = ([a, b] + D(a) + D(b) + a_0, 0)
\]

for all \(a, b \in A\). Using Corollary 3.5 and Proposition 3.8 we obtain:

**Corollary 3.9.** Let \(k\) be a field of characteristic \(\neq 3\) and \(A\) a JJ algebra. Then a JJ algebra \(E\) contains \(A\) as an ideal of codimension 1 if and only if there exists a pair \((D, a_0) \in \mathcal{S}(A)\) such that \(E \cong A_{(D, a_0)}\).

**Remark 3.10.** The concept of supersolvable JJ algebras was introduced by analogy with the Lie algebra theory. In that context, the classical Lie theorem proves that over an algebraically closed field of characteristic zero, any finite dimensional solvable Lie algebra is supersolvable. Consequently, we ask the following question:

*Let \(k\) be an algebraically closed field of characteristic zero and \(A\) a finite dimensional solvable JJ algebra. Is \(A\) supersolvable?*

One of the reasons for asking this question is the following: Zelmanov and Skosyrskii [18, Theorem 1 and Corollary 1] proved that any JJ algebra \(A\) without elements of order \(\leq 5\) in its additive group \((A, +)\) is solvable. In particular, if \(k\) is a field of characteristic \(\neq 2, 3\) or \(5\), then any JJ algebra is solvable.

Now, a positive answer to the above question collaborated with the aforementioned result will show that, over an algebraically closed field of characteristic zero, any finite dimensional JJ algebra is supersolvable. Therefore, the classification of all finite dimensional JJ algebras could be reduced to the recursive method described above.

**Skew crossed products and an Artin type theorem.** Let \(A\) be a JJ algebra, \(V\) a vector space and \(\Omega(A, V) = (\triangleleft, \triangleright, f, \{−, −\})\) an extending datum of \(A\) through \(V\) such that \(\triangleright\) is trivial, i.e. \(x \triangleright a = 0\), for all \(x \in V\) and \(a \in A\). Then, using Theorem 2.3 we obtain that \(\Omega(A, V) = (\triangleleft, f, \{−, −\})\) is a JJ extending structure of \(A\) through \(V\) if and only if the following compatibilities hold for all \(a, b \in A\) and \(x, y, z \in V\):

- (SC1) \(f : V \times V \to A\) and \(\{−, −\} : V \times V \to V\) are symmetric maps;
- (SC2) \((V, \triangleleft)\) is a right JJ \(A\)-module;
- (SC3) \(\{x, y\} \triangleleft a = −\{x, y \triangleleft a\} − \{x \triangleleft a, y\}\);
- (SC4) \([a, f(x, y)] + f(x, y \triangleleft a) + f(x \triangleleft a, y) = 0\);
- (SC5) \(\sum_{(c)} f(x, \{y, z\}) = 0\);
- (SC6) \(\sum_{(c)} \{x, \{y, z\}\} + \sum_{(c)} x \triangleleft f(y, z) = 0\).

A system \((A, V, \triangleleft, f)\) satisfying the above compatibility conditions will be called a *skew crossed system* of \(A\) through \(V\) while the associated unified product \(A \#_{\Omega(A, V)} V\) will be
denoted by $A \#^\ast V$ and called the skew crossed product of $A$ and $V$. Thus $A \#^\ast V = A \times V$ with the multiplication given as follows:

\[
[(a, x), (b, y)] := ([a, b] + f(x, y), \{x, y\} + x \triangleleft b + y \triangleleft a)
\]

for all $a, b \in A$ and $x, y \in V$.

Skew crossed products will be used to prove an Artin type theorem for JJ algebras, which provides a way of reconstructing a JJ algebra from its subalgebra of invariants:

**Theorem 3.11.** Let $G$ be a finite group of invertible order in $k$ acting on a JJ algebra $A$ by means of a group morphism $\varphi : G \to \text{Aut}_J(A)$, $\varphi(g)(a) = g \cdot a$, for all $g \in G$ and $a \in A$. Let $A^G := \{a \in A \mid g \cdot a = a, \forall g \in G\} \subseteq A$ be the subalgebra of invariants and $V$ a complement of $A^G$ in $A$.

Then there exists a skew crossed system $\Omega(A^G, V) = (a, f, \{-, -\})$ of $A^G$ through $V$ and an isomorphism of JJ algebras $A \cong A^G \#^\ast V$.

**Proof.** First we note that $A^G$ is indeed a subalgebra of $A$ and $g \cdot [a, b] = [g \cdot a, g \cdot b]$, for all $g \in G$, $a, b \in A$. We define the map $t$ as follows for all $x \in A$:

\[
t : A \to A^G, \quad t(x) := |G|^{-1} \sum_{g \in G} g \cdot x
\]

We observe that $t(x) \in A^G$ and for all $a \in A^G$ we obtain:

\[
t([x, a]) = |G|^{-1} \sum_{g \in G} g \cdot [x, a] = |G|^{-1} \sum_{g \in G} [g \cdot x, g \cdot a] = [t(x), a]
\]

Secondly, we note that the trace map $t : A \to A^G$ is a linear retraction of the canonical inclusion $A^G \hookrightarrow A$, i.e. $t(a) = a$, for all $a \in A^G$. Now, if we compute the canonical extending structure of $A^G$ through $V := \text{Ker}(t)$ associated to the trace map $t$, using the formulas from the proof of Theorem 2.4, we obtain that for all $x \in V$ and $a \in A^G$ we have:

\[
x \triangleright t a = t([x, a]) = [t(x), a] = 0,
\]

i.e. the action $\triangleright$ is the trivial one. Thus, the extending structure of $A^G$ through $V$ associated to the trace map $t$ reduces to a skew crossed system and applying once again Theorem 2.4 we obtain that the map defined for all $a \in A^G$ and $x \in V$ by:

\[
\vartheta : A^G \#^\ast V \to A, \quad \vartheta(a, x) := a + x
\]

is an isomorphism of JJ algebras. This finishes the proof.

**Remark 3.12.** In the context of Theorem 3.11, if $G = \langle g \rangle$ is a finite cyclic group generated by $g$, then we can easily prove that the kernel of the trace map defined by (18) has a nice description, namely: $\text{Ker}(t) = \{a - g \cdot a \mid a \in A\}$.

4. **Flag extending structures**

Theorem 2.7 offers the theoretical answer to the extending structures problem. However, computing the classifying object $H^2_A(V, A)$, for a given JJ algebra $A$ and a vector space $V$ is a highly non-trivial task. In this section we shall explicitly compute $H^2_A(k, A)$. 
**Definition 4.1.** Let $A$ be a JJ algebra and $E$ a vector space containing $A$ as a subspace. A JJ algebra structure on $E$ such that $A$ is a subalgebra is called a **flag extending structure** of $A$ to $E$ if there exists a finite chain of subalgebras of $E$

$$E_0 := A \subset E_1 \subset \cdots \subset E_m = E$$

such that $E_i$ has codimension 1 in $E_{i+1}$, for all $i = 0, \ldots, m - 1$.

All flag extending structures of $A$ to $E$ can be completely described and classified by a recursive method. The key step of this process is the case $m = 1$ which describes and classifies all unified products $A \natural V_1$, for a 1-dimensional vector space $V_1$. We can now continue the process by replacing the initial JJ algebra $A$ with such a unified product $A \natural V_1$. The latter product can be described in terms of $A$ only and iterating the process we obtain the description of all flag extending structures of $A$ to $E$ after $m = \dim_k(V)$ steps. We start by introducing the following concept:

**Definition 4.2.** A **flag datum** of a JJ algebra $A$ is a system $(D, \lambda, a_0, \alpha_0) \in \text{End}_k(A) \times A^* \times A \times k$ such that:

(F1) $\lambda([a, b]) + 2\lambda(a)\lambda(b) = 0$;

(F2) $D([a, b]) = -[D(a), b] - [a, D(b)] - \lambda(a)D(b) - \lambda(b)D(a)$;

(F3) $[a, a_0] + \alpha_0 D(a) + 2D^2(a) + 2\lambda(a)a_0 = 0$;

(F4) $3\lambda(a)a_0 + 2\lambda(D(a)) = 0$;

(F5) $3D(a_0) + 3\alpha_0a_0 = 0$;

(F6) $3\lambda(a_0) + 3\alpha_0^2 = 0$.

for all $a, b \in A$. The set of all flag data of $A$ will be denoted by $\mathcal{F}(A)$.

**Examples 4.3.** 1. If $D$ is an antiderivation of $A$ with $D^2 = 0$, then $(D, \lambda := 0, a_0 := 0, \alpha_0 := 0)$ is a flag datum of $A$.

2. Assume that $k$ is a field of characteristic $\neq 2, 3$ and let $A$ be the abelian JJ algebra, i.e. $[a, b] = 0$, for all $a, b \in A$. Then the set $\mathcal{F}(A)$ of all flag data of $A$ is in bijection with the set of all pairs $(D, a_0) \in \text{End}_k(A) \times A$, such that $D^2 = 0$ and $D(a_0) = 0$.

We shall prove now that the set of all JJ extending structures $\mathcal{J}(A, V)$ of a JJ algebra $A$ through a 1-dimensional vector space $V$ is parameterized by $\mathcal{F}(A)$.

**Proposition 4.4.** Let $A$ be a JJ algebra and $V$ a vector space of dimension 1 with basis $\{x\}$. Then there exists a bijection between the set $\mathcal{J}(A, V)$ of all JJ extending structures of $A$ through $V$ and the set $\mathcal{F}(A)$ of all flag data of $A$. Through the above bijection, the JJ extending structure $\Omega(A, V) = (\langle, \rangle, f, \{-, -\})$ corresponding to $(D, \lambda, a_0, \alpha_0) \in \mathcal{F}(A)$ is given for all $a \in A$ by:

$$x \triangleleft a = \lambda(a)x, \quad x \triangleright a = D(a), \quad f(x, x) = a_0, \quad \{x, x\} = \alpha_0 x$$

(21)
Proof. Since $V := kx$ has dimension 1, the set of all bilinear maps $:\delta : V \times A \to V$, $\triangleright : V \times V \to A$, $f : V \times V \to A$ and $\{ - , - \} : V \times V \to V$ is in bijection with the set of all systems $(D, \lambda, a_0, \alpha_0) \in \text{End}_k(A) \times A^* \times A \times k$ and the bijection is given such that (21) hold. The only thing left to prove is that the compatibility conditions (E1)-(E7) from Theorem 2.3 are equivalent to (F1)-(F6) from Definition 4.2. This follows by a straightforward computation.

\begin{remark}
Let $(D, \lambda, a_0, \alpha_0) \in \mathcal{F}(A)$ be a flag datum of $A$. The unified product $A_{\sharp} kx$ associated to the JJ extending structure (21) will be denoted by $A_{(D, \lambda, a_0, \alpha_0)}$ and has the multiplication given for all $a, b \in A$ by:

$$[(a, x), (b, x)] = [(a, b) + D(a) + D(b) + a_0, (\lambda(a) + \lambda(b) + \alpha_0)x]$$

Explicitly, if $\{e_i \mid i \in I\}$ is a $k$-basis of $A$ then $A_{(D, \lambda, a_0, \alpha_0)}$ is the JJ algebra having $\{x, e_i \mid i \in I\}$ as a $k$-basis and the multiplication given by:

$$[e_i, e_j] := [e_i, e_j]_A, \quad [e_i, x] := D(e_i) + a_0 + \lambda(e_i)x, \quad [x, x] := a_0 + \alpha_0 x$$

Using Theorem 2.4 and Proposition 4.4 we obtain:

\begin{corollary}
Let $A$ be a JJ algebra. Then a JJ algebra $E$ contains $A$ as a subalgebra of codimension 1 if and only if there exists $(D, \lambda, a_0, \alpha_0) \in \mathcal{F}(A)$ a flag datum of $A$ such that $E \cong A_{(D, \lambda, a_0, \alpha_0)}$.
\end{corollary}

Now, an easy computation shows that the equivalence relation defined in Definition 2.6 applied for the set $\mathcal{F}(A)$ via formulas (21) of Proposition 4.4 takes the following form:

\begin{definition}
Two flag data $(D, \lambda, a_0, \alpha_0)$ and $(D', \lambda', a_0', \alpha_0') \in \mathcal{F}(A)$ of a JJ algebra $A$ are called equivalent and we denote this by $(D, \lambda, a_0, \alpha_0) \equiv (D', \lambda', a_0', \alpha_0')$ if $\lambda = \lambda'$ and there exists a pair $(r, u) \in A \times k^*$ such that for all $a \in A$ we have:

$$D(a) = u D'(a) + [a, r] - \lambda'(a)r$$

$$a_0 = u a_0' + 2\lambda'(r)$$

$$a_0 = u^2 a_0' + [r, r] + 2u D'(r) - u a_0' r - 2\lambda'(r)r$$

Next, we classify all JJ algebras $A_{(D, \lambda, a_0, \alpha_0)}$ by computing the cohomological type object $\mathcal{H}_A^2(V, A)$, where $V$ is a 1-dimensional vector space. This is the first explicit classification result of the extending structures problem for JJ algebras and the key step in the classification of all flag extending structures.

\begin{theorem}
Let $A$ be a JJ algebra of codimension 1 in the vector space $E$ and $V$ a complement of $A$ in $E$. Then, $\equiv$ is an equivalence relation on the set $\mathcal{F}(A)$ of all flag data of $A$ and

$$\text{Ext}(E, A) \cong \mathcal{H}_A^2(V, A) \cong \mathcal{F}(A)/\equiv$$

The bijection between $\mathcal{F}(A)/\equiv$ and $\text{Ext}(E, A)$ is given by:

$$\{D, \lambda, a_0, \alpha_0\} \mapsto A_{(D, \lambda, a_0, \alpha_0)}$$

where $\{D, \lambda, a_0, \alpha_0\}$ is the equivalence class of $(D, \lambda, a_0, \alpha_0)$ via the relation $\equiv$ from Definition 4.7 and $A_{(D, \lambda, a_0, \alpha_0)}$ is the JJ algebra constructed in (23).
\end{theorem}
\textbf{Proof.} Follows from Theorem 2.7 and the results of this section. \hfill \Box

\textbf{Remark 4.9.} In fact, the method described above can be used for classifying all finite dimensional JJ algebras over a field of characteristic \( \neq 2, 3 \) or 5. Indeed, using \cite[Corollary 1]{[18]} we obtain that, over such a field, any JJ algebra is solvable and in particular is a flag extending structure of \( \{0\} \). Thus, by starting the recursive method described above with \( A := 0 \) yields the description of all JJ algebras of a given finite dimension.

Next we provide two explicit examples for the above results by computing \( \mathcal{H}^2_A(V, A) \) and then describing all JJ algebra structures which extend the JJ algebra structure from \( A \) to a vector space of dimension \( 1 + \dim_k(A) \). The detailed computations are rather long but straightforward and will be omitted.

\textbf{Example 4.10.} Let \( k \) be a field of characteristic \( \neq 2, 3 \) and \( A := k^n \), the abelian JJ algebra of dimension \( n \), i.e. \( [a, b] = 0 \), for all \( a, b \in k^n \). Then

\[ \mathcal{H}^2_{k^n}(k, k^n) \cong \{ (D, a_0) \in \text{End}_k(k^n) \times k^n \mid D^2 = 0, \ D(a_0) = 0 \}/ \cong \]

where \( (D, a_0) \equiv (D', a_0') \) if and only if there exists a pair \( (r, u) \in k^n \times k^* \) such that

\[ D(a) = u D'(a), \quad a_0 = u^2 a_0' + 2u D'(r) \]

for all \( a \in k^n \). Indeed, using Example 4.3 the set of all flag data \( \mathcal{F}(k^n) \) identifies with the set \( \{ (D, a_0) \in \text{End}_k(k^n) \times k^n \mid D^2 = 0, \ D(a_0) = 0 \} \). The conclusion follows from Theorem 4.8.

Let \( \{e_i \mid i = 1, \cdots, n\} \) be the canonical basis of \( k^n \) and \( (D, a_0) \in \mathcal{F}(k^n) \). Then \( k^n_{(D, a_0)} \) is the \((n + 1)\)-dimensional JJ algebra with multiplication given for all \( i, j = 1, \cdots, n \):

\[ [e_i, e_j] := 0, \quad [e_i, e_{n+1}] = [e_{n+1}, e_i] := D(e_i), \quad [e_{n+1}, e_{n+1}] := a_0 \]

Any \((n + 1)\)-dimensional JJ algebra containing \( k^n \) as an abelian subalgebra is isomorphic to \( k^n_{(D, a_0)} \).

\textbf{Example 4.11.} Let \( k \) be a field of characteristic \( \neq 2, 3 \) and \( A := \mathfrak{h}(3, k) \) the Heisenberg JJ algebra \cite{[7]} having \( \{e_1, e_2, e_3\} \) as a \( k \)-basis and the multiplication defined by \( [e_1, e_2] = [e_2, e_1] = e_3 \).

Then, a straightforward computation shows that the set of all flag data \( \mathcal{F}(\mathfrak{h}(3, k)) \) is in bijection with the set of triples \( (\alpha, \beta, \gamma) \in k^3 \) which satisfy \( \alpha \gamma = 0 \).

The bijection is defined such that the flag datum \((D, \lambda, a_0, a_0) \in \mathcal{F}(\mathfrak{h}(3, k)) \) corresponding to \((\alpha, \beta, \gamma) \) is given by:

\[ \lambda \equiv 0, \quad a_0 = 0, \quad a_0 = \alpha e_2 \]

\[ D(e_1) = \beta e_3, \quad D(e_2) = \gamma e_3, \quad D(e_3) = 0 \]

The compatibility condition \( \alpha \gamma = 0 \) imposes a discussion on whether \( \alpha = 0 \) or \( \alpha \neq 0 \). If \( \alpha = 0 \), the corresponding flag datum defines the family of JJ algebras denoted by \( \mathfrak{h}(3, k)_{(\beta, \gamma)} \) with the following multiplication:

\[ [e_1, e_2] = [e_2, e_1] = e_3, \quad [e_1, x] = [x, e_1] = \beta e_3, \quad [e_2, x] = [x, e_2] = \gamma e_3 \]
It can be easily seen that two flag datums induced by \((\beta, \gamma)\) and respectively \((\beta', \gamma')\) are equivalent in the sense of Definition 4.7 if and only if there exists \(u \in k^*\) such that 
\[ \beta \gamma = \beta' \gamma' u^2. \]
We denote by \(\cong_1\) the equivalence relation on \(k \times k\) defined as follows:
\[ (\beta, \gamma) \cong_1 (\beta', \gamma') \iff \exists u \in k^* \text{ such that } \beta \gamma = \beta' \gamma' u^2. \]

If \(\alpha \neq 0\), the corresponding flag datum defines the family of JJ algebras denoted by \(h(3, k)^\alpha\beta\) with the following multiplication:
\[
[e_1, e_2] = [e_2, e_1] = e_3, \ [e_1, x] = [x, e_1] = \beta e_3 + \alpha e_2, \\
[e_2, x] = [x, e_2] = \alpha e_3, \ [x, x] = \alpha e_2.
\]

Now two flag datums induced by \((\alpha, \beta)\) and respectively \((\alpha', \beta')\) are equivalent in the sense of Definition 4.7 if and only if there exists \(u \in k^*\) such that \(\alpha = \alpha' u^2\). In particular, this shows that given \(\beta \in k\) the flag datum induced by \((\alpha, \beta)\) is equivalent to the flag datum induced by \((\alpha, 0)\). We denote by \(\cong_2\) the equivalence relation on \(k\) defined as follows:
\[ \alpha \cong_2 \alpha' \iff \exists u \in k^* \text{ such that } \alpha = \alpha' u^2. \]

Furthermore, a flag datum induced by a triple with \(\alpha = 0\) is never equivalent to a flag datum induced by a triple with \(\alpha \neq 0\). This leads to the description of \(\mathcal{H}^2_{h(3, k)}(k, h(3, k))\) as the following coproduct of sets:
\[ \mathcal{H}^2_{h(3, k)}(k, h(3, k)) \cong ((k \times k)/ \cong_1) \sqcup (k^*/ \cong_2) \]

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