AN ALGEBRAIC $C_2$-EQUIVARIANT BÉZOUT’S THEOREM

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Abstract. Bézout’s theorem, nonequivariantly, can be interpreted as a calculation of the Euler class of a sum of line bundles over complex projective space, expressing it in terms of the rank of the bundle and its degree. We give here a generalization to the $C_2$-equivariant context, using the calculation of the cohomology of a $C_2$-complex projective space from an earlier paper. We use ordinary $C_2$-cohomology with Burnside ring coefficients and an extended grading necessary to define the Euler class, which we express in terms of the equivariant rank of the bundle and the degrees of the bundle and its fixed subbundles. We do similar calculations using constant $\mathbb{Z}$ coefficients and Borel cohomology and compare the results.

CONTENTS

Introduction 1
1. The cohomology of $\mathbb{P}(\mathbb{C}^p+q\sigma)$ 4
1.1. Ordinary cohomology 4
1.2. The cohomology of projective space 5
2. The algebraic equivariant Bézout theorem 9
3. Comparison with constant $\mathbb{Z}$ coefficients 16
4. Comparison with Borel cohomology 18
References 20

INTRODUCTION

Suppose that we have $n$ nonzero homogeneous polynomials $f_i$, $1 \leq i \leq n$, in $N$ variables, where $n < N$, and let $d_i$ be the degree of $f_i$. If $\mathbb{P}^{N-1}$ is the complex projective space, we can consider each $f_i$ as giving a section of the complex line bundle $O(d_i)$, the $d_i$-fold tensor power of the dual of the tautological line bundle over $\mathbb{P}^{N-1}$. Each $f_i$ determines a hypersurface $H_i \subset \mathbb{P}^{N-1}$, its zero locus. In this context, the (nonequivariant) Bézout theorem, as given in [5], for example, can be stated in several ways. One is a purely algebraic statement: In the cohomology ring

$$H^*(\mathbb{P}^{N-1}; \mathbb{Z}) \cong \mathbb{Z}[\hat{c}]/(\hat{c}^N),$$

we have that the Euler class of $F = O(d_1) \oplus O(d_2) \oplus \cdots \oplus O(d_n)$ is

$$e(F) = \Delta \hat{c}^n,$$

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P over the two-element group C

set bundle F over C and let ∆ denote the nontrivial representation. If p ≥ 0 and q ≥ 0 are integers, let Cp+qσ be the sum of p copies of C and q copies of Cσ, and let P(Cp+qσ) be its (complex) projective space, a C2-space. Using the equivariant ordinary cohomology with extended grading defined in [4], we computed the cohomology of P(Cp+qσ) in [3] with Burnside ring coefficients. We also gave the zero-dimensional version of an equivariant Bézout theorem, showing that the equivariant Euler class in equivariant ordinary cohomology allows us to determine the finite C2-set in P(Cp+qσ) given by the intersection of p + q − 1 equivariant hypersurfaces.

Our goal in this paper is to generalize the algebraic part of this C2-Bézout theorem to higher dimensions. In a followup paper we will discuss the geometric meaning of the Euler class that we calculate here.

Let us set up the context more precisely. If F is a nonequivariant vector bundle over P N, its Euler class has the form e(F) = ∆ e n, where n is the rank of F and we set ∆ = 0 if n ≥ N. We call ∆ the degree of F. By the nonequivariant Bézout theorem, if F is a sum of line bundles of the form O(d) and n < N, then ∆ is the product of the degrees d.

Now suppose that we have n < p + q C2-line bundles over P(Cp+qσ) with direct sum F. We let ∆ be the nonequivariant degree of F. We can also consider the fixed-set bundle F C2 over P(Cp+qσ)C2 = P(Cp) ∪ P(Cqσ). Let n0 denote the rank of the restriction of F C2 to P(Cp) and let ∆0 be its degree. We know that n0 ≤ n, and, to keep the situation geometrically meaningful, we would like the generic intersection of the corresponding hypersurfaces in P(Cp) to have dimension no more than the dimension of the intersection of all the hypersurfaces in P(Cp+qσ). For that, we require that p − n0 − 1 ≤ p + q − n − 1, that is, n0 ≥ n − q. Similarly, let n1 denote the rank of F C2 over P(Cqσ) and let ∆1 be its degree; we require that n1 ≥ n − p.

We record these notations and conditions for later reference.

Bézout Context 0.1. F is the sum of n C2-line bundles over P(Cp+qσ) and ∆ is its nonequivariant degree. The restriction of F C2 to P(Cp) has rank n0 and degree ∆0, while its restriction to P(Cqσ) has rank n1 and degree ∆1. We assume that

\[ n < p + q \]
\[ n − q ≤ n0 ≤ n \]
\[ n − p ≤ n1 ≤ n. \]

We call the triple (∆, ∆0, ∆1) the C2-degrees of F.

Bézout’s Theorem, Part I. In the context above, the Euler class e(F) is completely determined by the ranks (n, n0, n1) and the degrees (∆, ∆0, ∆1). Moreover,
these ranks and degrees can be recovered from $e(F)$. The ranks are additive and the
degrees are multiplicative.

This will be proved as Theorem 2.11. When we say that the degrees are multi-
plicative, we really mean the following: Suppose that we have two such bundles
$F$ and $F'$, with ranks $(n, n_0, n_1)$ and $(n', n'_0, n'_1)$, respectively, and corresponding
degrees. We assume that $F \oplus F'$ still satisfies the conditions of the Bézout context
above. This allows the possibility that $n_0 + n'_0 \geq p$, in which case the corre-
sponding degree is not $\Delta_0 \Delta'_0$ but 0, and similarly if $n_1 + n'_1 \geq q$.

The nonequivariant Bézout theorem can also be viewed as expressing $e(F)$ in
terms of the basis of the cohomology of $\mathbb{P}^{N-1}$ given by the powers of $\hat{c}$. In any
given grading, there is at most one such power, so there is only one coefficient to
specify, which turns out to be the degree $\Delta$. Equivariantly, the result is more com-
plicated. The cohomology of $\mathbb{P}(\mathbb{C}^{p+q})$ is free over the $RO(C_2)$-graded equivariant
cohomology of a point, but the cohomology of a point is no longer concentrated in
grading 0. As shown in [3], in any given grading of the cohomology of $\mathbb{P}(\mathbb{C}^{p+q})$,
there are up to $p + q$ basis elements that can contribute, so an element poten-
tially requires a $(p + q)$-tuple of coefficients to specify. Our second main result is
summarized as follows.

Bézout’s Theorem, Part II. In the context above, the Euler class $e(F)$ is the
linear combination of at most three basis elements.

This is proved as Theorem 2.12, which also gives the details as to which three
basis elements are involved and what their coefficients are. The three basis ele-
ments are determined by $(p$ and $q$ and) the ranks $(n, n_0, n_1)$. The coefficients are
determined by the degrees $(\Delta, \Delta_0, \Delta_1)$, but are not simply equal to them.

This paper is structured as follows. In Section 1, we review the cohomology
of $\mathbb{P}(\mathbb{C}^{p+q})$ as computed in [3], including our preferred basis. In Section 2 we
give the main results, proving the two theorems above. There are two other equi-
variant ordinary cohomology theories in common use, cohomology with constant $\mathbb{Z}$
coefficients and Borel cohomology. In Section 3 we discuss how the computation
changes if we use constant $\mathbb{Z}$ coefficients rather than Burnside ring coefficients, and
in Section 4 we discuss the similar computation in Borel cohomology. There are
maps from cohomology with Burnside ring coefficients to cohomology with con-
stant $\mathbb{Z}$ coefficients, and from that theory to Borel cohomology, both respecting
Euler classes, and we will see that the Euler classes in the latter two theories carry
less information than the Euler class in cohomology with Burnside ring coefficients.
In particular, we cannot recover the degrees $\Delta_0$ and $\Delta_1$ from the Euler class in
cohomology with constant $\mathbb{Z}$ coefficients or the class in Borel cohomology.

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Universität Wuppertal.
1. The Cohomology of $\mathbb{P}(C^+\eta\tau)$

1.1. Ordinary cohomology. We will use $C_2$-equivariant ordinary cohomology with the extended grading developed in [4]. This is an extension of Bredon’s ordinary cohomology to be graded on representations of the fundamental groupoids of $C_2$-spaces. We review here some of the notation and computations we will be using. A more detailed description of this theory can be found in [3].

For an ex-$C_2$-space $Y$ over $X$, we write $H_{C_2}^{RO(IX)}(Y; \mathcal{T})$ for the ordinary cohomology of $Y$ with coefficients in a Mackey functor $\mathcal{T}$, graded on $RO(IX)$, the representation ring of the fundamental groupoid of $X$. Through most of this paper we will use the Burnside ring Mackey functor $\mathcal{A}$ as the coefficients, and write simply $H_{C_2}^{RO(IX)}(Y)$.

In [4] and [3] we considered cohomology to be Mackey functor–valued, which is useful for many computations, and wrote $H_{C_2}^{RO(IX)}(Y)$ for the resulting theory. In this paper we will be concentrating on the values at level $C_2/C_2$, and write $H_{C_2}^{RO(IX)}(Y) = H_{C_2}^{RO(IX)}(Y)/C_2(C_2/C_2)$. However, we will still refer to the restriction functor $\rho$ from equivariant cohomology to nonequivariant cohomology, and the transfer map $\tau$ going in the other direction.

For all $X$ and $Y$, $H_{C_2}^{RO(IX)}(Y)$ is a graded module over

$$\mathbb{H} = H_{C_2}^{RO(C_2)} = H_{C_2}^{RO(C_2)}(S^0),$$

the cohomology of a point. The grading on the latter is just $RO(C_2)$, the real representation ring of $C_2$, which is free abelian on 1, the class of the trivial representation $\mathbb{R}$, and $\sigma$, the class of the sign representation $\mathbb{R}^\sigma$. The cohomology of a point was calculated by Stong in an unpublished manuscript and first published by Lewis in [6]. We can picture the calculation as in Figure 1, in which a group in grading $a + b\sigma$ is plotted at the point $(a, b)$, and the spacing of the grid lines is 2 (which is more convenient for other graphs we will give). The box at the origin is a copy of $A(C_2)$, the Burnside ring of $C_2$, closed circles are copies of $\mathbb{Z}$, and open circles are copies of $\mathbb{Z}/2$.

Recall that $A(C_2)$ is the Grothendieck group of finite $C_2$-sets, with multiplication given by products of sets. Additively, it is free abelian on the classes of the orbits of $C_2$, for which we will write $1 = [C_2/C_2]$ and $g = [C_2/e]$. The multiplication is given by $g^2 = 2g$. We will also write $\kappa = 2 - g$. Other important elements are shown in the figure: The group in degree $\sigma$ is generated by an element $e$, while the group in degree $-2 + 2\sigma$ is generated by an element $\xi$. The groups in the second quadrant are generated by the products $e^m\xi^n$, with $2e\xi = 0$. We have $g\xi = 2\xi$ and $ge = 0$. The groups in gradings $-m\sigma$, $m \geq 1$, are generated by elements $e^{-m}\kappa$, so named because $e^m \cdot e^{-m}\kappa = \kappa$. We also have $g^{-m}\kappa = 0$.

To explain $\tau(e^{-n})$, we think for moment about the nonequivariant cohomology of a point. If we grade it on $RO(C_2)$, we get $H^{RO(C_2)}(S^0; \mathbb{Z}) \cong \mathbb{Z}[t^{\pm 1}]$, where $\deg t = -1 + \sigma$. (Nonequivariantly, we cannot tell the difference between $\mathbb{R}$ and $\mathbb{R}^\sigma$.) We have $\rho(1) = t^2$ and $\tau(t^2) = g\xi = 2\xi$. Note also that $\tau(1) = g$. In the fourth quadrant we have that the group in grading $n(1 - \sigma)$, $n \geq 2$, is generated by $\tau(e^{n-1})$. The remaining groups in the fourth quadrant will not concern us here. For more details, see [2] or [3].
1.2. The cohomology of projective space. As described in the introduction, the form of Bézout’s theorem we shall give expresses the Euler class of a bundle over \( \mathbb{P}(\mathbb{C}^{p+q}) \) in terms of a basis of its cohomology. We now review the structure of that cohomology as calculated in [3].

Write \( B = \mathbb{P}(\mathbb{C}^{\infty+\infty}) \). Its fixed set is

\[
B^{C_2} = \mathbb{P}(\mathbb{C}^p) \sqcup \mathbb{P}(\mathbb{C}^{q}) = B^0 \sqcup B^1,
\]

where we use the indices 0 and 1 to evoke the trivial and nontrivial representation of \( C_2 \), respectively. (We will use this convention throughout, that a subscript 0 refers to something related to \( B^0 \) and subscript 1 refers to something related to \( B^1 \).) Representations of \( \Pi B \) are determined by their restrictions to \( B^0 \) and \( B^1 \), which are elements of \( RO(C_2) \) that must have the same nonequivariant rank and the same parity for the ranks of their fixed point representations. As a result, we can write

\[
RO(\Pi B) = \mathbb{Z}\{1, \sigma, \Omega_0, \Omega_1\}/\langle \Omega_0 + \Omega_1 = 2\sigma - 2 \rangle,
\]

where \( \Omega_0 \) is the representation whose value on \( B^0 \) is \( 2\sigma - 2 \) and on \( B^1 \) is 0; while \( \Omega_1 \) is the representation whose value on \( B^0 \) is 0 and on \( B^1 \) is \( 2\sigma - 2 \). For any \( \alpha \in RO(\Pi B) \), write \( |\alpha| \in \mathbb{Z} \) for its underlying nonequivariant rank, and \( \alpha_0 \) and \( \alpha_1 \in RO(C_2) \) for its restrictions to \( B^0 \) and \( B^1 \), respectively. What we said above can be phrased as: \( \alpha \) is completely determined by the triple of ranks \((|\alpha|, |\alpha_0|, |\alpha_1|)\), where the latter two ranks have the same parity.

We think of the finite projective spaces as spaces over \( B \) by the evident inclusions \( \mathbb{P}(\mathbb{C}^{p+q}) \to \mathbb{P}(\mathbb{C}^{\infty+\infty}) \), so will grade their cohomologies on \( RO(\Pi B) \). Let \( \omega \) denote the tautological line bundle over \( B \), let \( \omega^\vee \) be its dual bundle, let \( \chi \omega = \omega \otimes_{\mathbb{C}} \mathbb{C}^\sigma \), and let \( \chi\omega^\vee \) be the dual of \( \chi\omega \). We will also use the notation from algebraic geometry in which \( \omega = O(-1) \) and \( \omega^\vee = O(1) \); we write \( \chi O(-1) = \chi\omega \) and \( \chi O(1) = \chi\omega^\vee \).

Associated to any bundle over \( B \) is a representation in \( RO(\Pi B) \) that we think of as the equivariant rank of the bundle; this representation is given by the fiber
representations over $B^0$ and $B^1$. In the case of $\omega$ and $\chi\omega$, we have

$$\omega = 2 + \Omega_1$$

$$\chi\omega = 2 + \Omega_0,$$

where we write $\omega$ and $\chi\omega$ again for the associated elements of $RO(\Pi B)$.

Let $\widehat{c}_\omega$ and $\widehat{c}_{\chi\omega}$ denote the Euler classes of $\omega^\vee$ and $\chi\omega^\vee$, respectively. The cohomology of $P$ is infinitely divisible by $\chi\omega$.

Theorem 1.1. $H^{RO(\Pi B)}_{C_2}(B_+) \subset \mathbb{H}$ generated by the Euler classes $\widehat{c}_\omega$ and $\widehat{c}_{\chi\omega}$ together with classes $\zeta_0$ and $\zeta_1$. These elements live in gradings $\text{grad } \widehat{c}_\omega = \omega$ \quad $\text{grad } \widehat{c}_{\chi\omega} = \chi\omega$

$$\text{grad } \zeta_1 = \omega - 2 \quad \text{grad } \zeta_0 = \chi\omega - 2$$

They satisfy the relations

$$\zeta_0\zeta_1 = \xi$$

and these relations completely determine the algebra. Moreover, $H^{RO(\Pi B)}_{C_2}(B_+)$ is free as a module over $\mathbb{H}$. \hfill \square

Pulling back along the inclusion $P(\mathcal{O}^{p+q}) \hookrightarrow P(\mathcal{O}^{\infty+\infty})$, the cohomology of $P(\mathcal{O}^{p+q})$ contains elements we will also call $\widehat{c}_\omega$, $\widehat{c}_{\chi\omega}$, $\zeta_0$, and $\zeta_1$. In [3], we showed the following.

Theorem 1.2 ([3, Theorem A]). Let $0 \leq p < \infty$ and $0 \leq q < \infty$ with $p + q > 0$. Then $H^{RO(\Pi B)}_{C_2}(P(\mathcal{O}^{p+q})_+) \subset \mathbb{H}$ is a free module over $\mathbb{H}$. As a (graded) commutative algebra over $\mathbb{H}$, the ring $H^{RO(\Pi B)}_{C_2}(P(\mathcal{O}^{p+q})_+)$ is generated by $\widehat{c}_\omega$, $\widehat{c}_{\chi\omega}$, $\zeta_0$, and $\zeta_1$, together with the following classes: $\widehat{c}_\omega^p$ is infinitely divisible by $\zeta_0$, meaning that, for $k \geq 1$, there are unique elements $\zeta_0^{k-1}\widehat{c}_\omega^p$ such that

$$\zeta_0^{k-1}\widehat{c}_\omega^p \cdot \zeta_0^p \cdot \widehat{c}_\omega^p = \widehat{c}_\omega^p.$$ 

Similarly, $\widehat{c}_{\chi\omega}^q$ is infinitely divisible by $\zeta_1$, meaning that, for $k \geq 1$, there are unique elements $\zeta_1^{k-1}\widehat{c}_{\chi\omega}^q$ such that

$$\zeta_1^{k-1}\widehat{c}_{\chi\omega}^q \cdot \zeta_1^q \cdot \widehat{c}_{\chi\omega}^q = \widehat{c}_{\chi\omega}^q.$$ 

The generators satisfy the following further relations:

$$\zeta_0\zeta_1 = \xi$$

$$\zeta_1\widehat{c}_{\chi\omega} = (1 - \kappa)\zeta_0\widehat{c}_\omega + e^2$$

and

$$\widehat{c}_\omega^p\widehat{c}_{\chi\omega}^q = 0.$$ 

We also gave an explicit basis for $H^{RO(\Pi B)}_{C_2}(P(\mathcal{O}^{p+q})_+)$ over $\mathbb{H}$, which we can describe as follows. We define sets $F_{p,q}(m)$, recursively on $p$ and $q$, that give bases for $H^{\infty+RO(C_2)}_{C_2}(P(\mathcal{O}^{p+q})_+)$. For $m \in \mathbb{Z}$, let

$$F_{p,0}(m) := \{c_0^m, c_1^{m-1}c_\omega, c_1^{m-2}c_\omega^2, \ldots, c_1^{m-p+1}c_\omega^{p-1}\}$$

and

$$F_{0,q}(m) := \{\zeta_0^m, \zeta_0^{m-1}c_{\chi\omega}, \zeta_0^{m-2}c_{\chi\omega}^2, \ldots, c_0^{m-q+1}\zeta_{\chi\omega}^{q-1}\}.$$
(Note that $\zeta_1$ is invertible in the first case and $\zeta_0$ is invertible in the second.) For $p, q > 0$ we then define

\[ F_{p,q}(m) := \begin{cases} \{\zeta_1^m\} \cup i_! F_{p-1,q}(m-1) & \text{if } m \geq 0 \\ \{\zeta_0^{|m|}\} \cup j_! F_{p,q-1}(m+1) & \text{if } m < 0, \end{cases} \]

where $i: \mathbb{P}(\mathbb{C}^{p-1+q\sigma}) \to \mathbb{P}(\mathbb{C}^{p+q\sigma})$ and $j: \mathbb{P}(\mathbb{C}^{p+(q-1)\sigma}) \to \mathbb{P}(\mathbb{C}^{p+q\sigma})$ are the inclusions. The pushforward $i_!$ is given algebraically by multiplication by $\hat{c}_\omega$ and $j_!$ is multiplication by $\hat{c}_\chi$. It is possible from this description to write down the bases explicitly, but the results are messy, having to be broken down by cases depending on where $m$ falls in relation to $p$ and $q$; this is done in [3, Proposition 4.7]. However, we can make the following general statements.

1. For fixed $p$, $q$, and $m$, there are exactly $p + q$ basis elements lying in $H_{\mathbb{C}_2}^{m\omega + RO}(\mathbb{P}(\mathbb{C}^{p+q\sigma})).$
2. Those basis elements have gradings of the form $m(\omega - 2) + 2a_i + 2b_i\sigma$, $0 \leq i \leq p + q - 1$, where $a_i + b_i = i$.
3. The basis element with grading $m(\omega - 2) + 2a + 2b\sigma$ restricts to the nonequivariant class $\hat{c}_a + b$, where $\hat{c}$ is the first nonequivariant Chern class of $O(1)$.
4. For a given integer $k$, there are at most two indices $i$ such that $a_i = k$.

The following illustrate, in the case of $\mathbb{P}(\mathbb{C}^{4+5\sigma})$, how the basis elements can be arranged for various values of $m$. In each case, the basis element with grading $m(\omega - 2) + 2a + 2b\sigma$ is marked by a dot at coordinates $(a, b)$.
For ease of reference, we will write the bases as

\[ F_{p,q}(m) = \{ P_{i}^{(m)} \} \]

where \( P_{i}^{(m)} \) is the basis element in \( H_{C_{2}}^{m+RO(C_{2})}(\mathbb{F}(\mathbb{C}^{p+q\sigma})_{+}) \) restricting to the element \( \hat{c}^{i} \) nonequivariantly. When \( m \) is understood, we will simply write \( P_{i} \) for \( P_{i}^{(m)} \).

We can also say that \( P_{i} \) is the basis element in grading \( m(\omega - 2) + 2a + 2b\sigma \) with \( a + b = i \), as illustrated in the following diagram with \( m = 0 \).

\[ \begin{array}{c}
\text{P}_{0} \\
\text{P}_{1} \\
\text{P}_{2} \\
\text{P}_{3} \\
\text{P}_{4} \\
\text{P}_{5} \\
\text{P}_{6} \\
\text{P}_{7} \\
\text{P}_{8}
\end{array} \]

**Definition 1.3.** Given any element \( x \in H_{C_{2}}^{m+RO(C_{2})}(\mathbb{F}(\mathbb{C}^{p+q\sigma})_{+}) \), we can write \( x \) uniquely as

\[ x = \sum_{i=0}^{p+q-1} \alpha_{i} P_{i}^{(m)} \]

with each coefficient \( \alpha_{i} \in \mathbb{H} \). We call the \((p+q)\)-tuple \((\alpha_{i})\) the coefficient vector of \( x \).

Because elements of \( \mathbb{H} \) lie in a restricted set of gradings, the number of nonzero coefficients possible for a given \( x \) may be limited, depending on the grading of \( x \), though there are elements \( x \) for which all coefficients are nonzero.

There are two restriction maps we will use,

\[ \rho: H_{C_{2}}^{p}(\mathbb{F}(\mathbb{C}^{p+q\sigma})_{+}) \rightarrow H_{[i]}^{p}(\mathbb{F}(\mathbb{C}^{p+q\sigma})_{+}) \]

restriction to nonequivariant cohomology, and

\[ (\neg)_{C_{2}}: H_{C_{2}}^{p}(\mathbb{F}(\mathbb{C}^{p+q\sigma})_{+}) \rightarrow H_{C_{2}}^{0}(\mathbb{F}(\mathbb{C}^{p})_{+}) \oplus H_{C_{2}}^{1}(\mathbb{F}(\mathbb{C}^{q\sigma})_{+}) \]

the fixed-point map. These are ring maps and their values on the multiplicative generators are given by the following.

\[
\begin{align*}
\rho(\zeta_{0}) &= 1 \\
\rho(\hat{c}_{\omega}) &= \hat{c} \\
\zeta_{0}^{C_{2}} &= (0, 1) \\
\hat{c}_{\omega}^{C_{2}} &= (\hat{c}, 1) \\
\rho(\zeta_{1}) &= 1 \\
\rho(\hat{c}_{\chi\omega}) &= \hat{c} \\
\zeta_{1}^{C_{2}} &= (1, 0) \\
\hat{c}_{\chi\omega}^{C_{2}} &= (1, \hat{c})
\end{align*}
\]
We also need the values of the similar restriction maps

\[ \rho: \mathbb{H}^\alpha \rightarrow H^{|\alpha|}(S^0), \]
\[ (-)^C_2: \mathbb{H}^\alpha \rightarrow H^\alpha C_2(S^0). \]

The particular values we will need are

\[ \rho(\tau(i^{2k})) = 1 \]
\[ \rho(e^{-k}\kappa) = 0 \]
\[ \rho(e^{k}) = 0 \]
\[ \tau(i^{2k})C_2 = 0 \]
\[ (e^{-k}\kappa)C_2 = 2 \]
\[ (e^{k})C_2 = 1. \]

2. The algebraic equivariant Bézout theorem

It is possible to take the calculation of Euler classes in [3] and, by brute force, work out their expression in terms of the basis for the cohomology of \( \mathbb{P}(\mathbb{C}^{p+q}) \) discussed in the preceding section. Instead, we will take advantage of some features of the cohomology of a point to give a more conceptual approach that shows better why the calculation works the way it does.

Definition 2.1.

- Let \( T \subset \mathbb{H} \) consist of the elements \( a\tau(i^{2\ell}) \) for \( a \in \mathbb{Z} \) and \( \ell \in \mathbb{Z} \), \( ae^{-m}\kappa \) for \( a \in \mathbb{Z} \) and \( m \geq 1 \), and the elements \( ae^m \) for \( a \in \mathbb{Z} \) and \( m \geq 1 \), and all of \( A(C_2) = \mathbb{H}^0 \).
- Let \( I_\varepsilon \subset T \) consist of the elements \( a\tau(i^{2\ell}) \) for \( a \in \mathbb{Z} \) and \( \ell \in \mathbb{Z} \), \( ae^m\kappa \) for \( a \in \mathbb{Z} \) and \( m \in \mathbb{Z} \), and the elements \( a + bg \in A(C_2) \) such that \( a \) is even.

Note that \( e^m\kappa = 2e^m \) if \( m > 0 \).

Proposition 2.2. \( I_\varepsilon \) is an ideal of \( \mathbb{H} \).

Proof. This is a straightforward check from the known structure of \( \mathbb{H} \), as given in [3]. □

On the other hand, \( T \) is not an ideal, because \( e\xi \notin T \) while \( e \in T \). But \( T \) is an additive subgroup.

An important fact about \( T \) is that, as shown in the following diagram, all of its elements lie in gradings of the form \( n\sigma \) or \( 2n(1 + \sigma) \), that is, on the vertical line through the origin or the diagonal through the origin with slope \(-1\). Closed circles
indicate copies of \( \mathbb{Z} \), while the box at the origin is \( A(C_2) \). \( T \) is a free \( \mathbb{Z} \)-module.

Another fact that follows from the known structure of \( \mathbb{H} \) is that the quotient ring \( \mathbb{H}/I_e \) is all 2-torsion.

**Remark 2.3.** The ideal \( I_e \) is almost, but not quite, the kernel of the restriction map \( \mathbb{H} = H_{C_2}^{RO}(C_2)(S^0; A) \to H_{C_2}^{RO}(C_2)(S^0; \mathbb{Z}/2) \). That kernel would not contain all the elements \( a\tau(i^{-2m}) \) for \( m \geq 1 \), but only those of the form \( 2a\tau(i^{-2m}) \). Either ideal would serve our purpose here, but we chose to use the one that is slightly simpler to describe.

**Definition 2.4.**
- Let \( \tilde{T} \subset H_{C_2}^{RO}(\Pi B) \subset H_{C_2}^{RO}(C_2)(P(C^p+q\sigma)_+) \) denote the set of linear combinations of elements of our preferred basis of \( H_{C_2}^{RO}(\Pi B) \), with coefficients in \( T \).
- Let \( J_e \) be the ideal defined by \( J_e = I_e \cdot H_{C_2}^{RO}(\Pi B) \subset H_{C_2}^{RO}(\Pi B) \subset H_{C_2}^{RO}(C_2)(P(C^p+q\sigma)_+) \).

Every element of \( J_e \) is a linear combination of elements from our preferred basis with coefficients in \( I_e \) (and this would be true for any basis we used). Because \( J_e \subset \tilde{T} \), the following facts about \( \tilde{T} \) apply to \( J_e \) as well.

**Lemma 2.5.** Every element \( x \in \tilde{T} \) is a linear combination of at most three basis elements: If \( x \) lies in grading \( m(\omega-2) + a + b\sigma \), the only basis elements that can contribute to \( x \) are the one (if any) lying on the same diagonal as \( x \), that is, in a grading \( m(\omega-2) + a' + b'\sigma \) with \( a' + b' = a + b \), and the two (at most) lying in the same vertical line as \( x \), that is, in gradings \( m(\omega-2) + a + b'\sigma \).

**Proof.** This follows from the description of the locations of the basis elements given in the preceding section together with the locations of the elements of \( T \). \( \square \)

See the example in Remark 2.14 below for an illustration of this lemma.

**Proposition 2.6.** If \( x \in \tilde{T} \), then \( x \) is determined by its restrictions \( \rho(x) \) and \( x^{C_2} \).
Proof. By the preceding lemma, \( x \) can be written as a linear combination of at most three elements from our standard basis. There are various cases that should be considered. Suppose, for example, that \( x \) lies on the same diagonal as a basis element \( P_n \) and lies above two basis elements \( P_k \) and \( P_{k-1} \). Then we can write
\[
x = \alpha (i^{2k}) P_n + \beta e^{m} P_k + \gamma e^{m+2} P_{k-1}
\]
for some integers \( \alpha, \beta, \gamma, \ell, \) and \( m \). We now appeal to [3, 4.6], where we showed that our standard basis restricts to a nonequivariant basis for \( \mathbb{P}(\mathbb{C}^{p+q\sigma}) \) and a nonequivariant basis for \( \mathbb{P}(\mathbb{C}^{p+q\sigma})^{C_2} \). We have \( \rho(x) = 2\alpha \rho(P_n) \), so \( \alpha \) is determined by \( \rho(x) \). On the other hand, \( x^{C_2} = \beta P_k^{C_2} + \gamma P_{k-1}^{C_2} \), so \( \beta \) and \( \gamma \) are determined by \( x^{C_2} \).

There are other cases, for example, where \( x \) lies below two basis elements rather than above, or where it lies in the same grading as a basis element. Each of these cases can be handled in the same way as the case above.

Note that this is not true for general elements of \( H_{C_2}^{RO(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+) \) because there are elements of \( \mathbb{H} \) that vanish under both \( \rho \) and \( (-)^{C_2} \).

For any \( x \in H_{C_2}^{RO(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+) \), we have
\[
\rho(x) \in H^Z(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+),
\]
so \( \rho(x) = \Delta \tilde{c}^k \) for some integers \( \Delta \) and \( k \), or is 0, in which case we set \( \Delta = 0 \). We also have
\[
x^{C_2} \in H^Z(\mathbb{P}(\mathbb{C}^p)_+) \oplus H^Z(\mathbb{P}(\mathbb{C}^q)_+),
\]
so \( x^{C_2} = (\Delta_0 \tilde{c}^i, \Delta_1 \tilde{c}^j) \) for some integers \( \Delta_0, \Delta_1, i, \) and \( j \). (Again, we set \( \Delta_0 = 0 \) if \( \Delta_0 \tilde{c}^i = 0 \) and \( \Delta_1 = 0 \) if \( \Delta_1 \tilde{c}^j = 0 \).)

**Definition 2.7.** We call the triple of integers \((\Delta, \Delta_0, \Delta_1)\) determined as above the \( C_2 \)-degrees of \( x \).

**Corollary 2.8.** If \( x \in \tilde{T} \), then \( x \) is determined by its grading and its \( C_2 \)-degrees.

**Proof.** Suppose that \( x \) lies in grading \( m(\omega - 2) + a + b\sigma \) and that the degrees of \( x \) are \((\Delta, \Delta_0, \Delta_1)\). By the structure of \( \tilde{T} \) and the locations of the basis elements, we can assume that \( a \) is even. Then we have
\[
\rho(x) = \begin{cases} 
\Delta \tilde{c}^{(a+b)/2} & \text{if } b \text{ is even} \\
0 & \text{otherwise} 
\end{cases}
\]
and
\[
x^{C_2} = (\Delta_0 \tilde{c}^{a/2}, \Delta_1 \tilde{c}^{a/2-m})
\]
Thus, the grading of \( x \) and its degrees determine \( \rho(x) \) and \( x^{C_2} \), so the result follows from the preceding proposition.

In order to apply these results to derive the two parts of Bézout’s theorem, we need to know a little more about the line bundles that are the summands of \( F \) as in the Bézout context 0.1. In [3] we showed that the line bundles over \( \mathbb{P}(\mathbb{C}^{p+q\sigma}) \) all have the form \( O(d) \) or \( \chi O(d) \). It is useful to further break these down into four types:

**Type I:** bundles of the form \( O(2d + 1) \)

**Type II:** bundles of the form \( O(2d) \)

**Type III:** bundles of the form \( \chi O(2d + 1) \)
Type IV: bundles of the form $\chi O(2d)$

The fixed points $O(2d+1)^{C^2}$ of a bundle of type I have fiber $\mathbb{C}$ over $\mathbb{P}(\mathbb{C}^p)$ and 0 over $\mathbb{P}(\mathbb{C}^{p\sigma})$, while the reverse is true for a bundle of type III. The fixed points $O(2d)^{C^2}$ of a bundle of type II have fiber $\mathbb{C}$ over both components of $\mathbb{P}(\mathbb{C}^{p+q\sigma})^{C^2}$, while the fixed points of a bundle of type IV have fiber 0 over both components.

In [3] for $\dagger \in \{I, II, III, IV\}$ we wrote $n_1$ for the number of summands of type $\dagger$ and $d_i$ for the products of their degrees. These are related to the ranks and $C_2$-degrees of $F$ by

$$n = n_1 + n_{III} + n_{IV}$$
$$n_0 = n_1 + n_{II}$$
$$n_1 = n_{II} + n_{III}$$
$$\Delta = d_ID_ID_{III}D_{IV}$$

(2.9)

$$\Delta_0 = \begin{cases} d_ID_{II} & \text{if } n_0 < p \\ 0 & \text{if } n_0 \geq p \end{cases}$$

(2.10)

$$\Delta_1 = \begin{cases} d_ID_{III} & \text{if } n_1 < q \\ 0 & \text{if } n_1 \geq q. \end{cases}$$

Now, $d_I$ and $d_{III}$ are always odd, and $d_{II}$ and $d_{IV}$ are even if and only if there is a summand of type II or IV, respectively. Notice that, when $n_{II} > 0$, the quantities $\Delta, \Delta_0, \text{and } \Delta_1$ will all be even. If $n_{II} = 0$, then $n_0 + n_1 \leq n$, which implies that

$$n_0 \leq n - n_1 \leq n - (n - p) = p$$

and $n_1 \leq q$, similarly, with equality possible only if $n_{IV} = 0$. So, if $n_{II} = 0$ but $n_{IV} > 0$, we will have $\Delta$ even and both $\Delta_0$ and $\Delta_1$ odd. When $n_{II} = 0$ and $n_{IV} = 0$, we will have $\Delta$ odd while $\Delta_0$ and $\Delta_1$ will be odd if nonzero.

**Theorem 2.11 (Bézout’s Theorem, Part I).** Let $F$ be as in the Bézout context 0.1. Then $e(F)$ lies in $\tilde{T}$, hence is determined by its grading, which is

$$(n_0 - n_1)(\omega - 2) + 2n_0 + 2(n - n_0)\sigma,$$

and its $C_2$-degrees, which are $(\Delta, \Delta_0, \Delta_1)$. Moreover, the grading and degrees can be recovered from $e(F)$. The ranks $(n, n_0, n_1)$ are additive while the degrees are multiplicative.

**Proof.** The additivity of the grading and the multiplicativity of the degrees are clear (but see the caveat about multiplicativity given in the Introduction).

Given that $n$ is the nonequivariant (complex) rank of $F$ and $n_0$ and $n_1$ are the ranks of the restriction of $F^{C_2}$ to $\mathbb{P}(\mathbb{C}^p)$ and $\mathbb{P}(\mathbb{C}^{p\sigma})$, respectively, $e(F)$ must lie in the grading given, which is the grading $\alpha$ with $|\alpha| = n$, $\alpha_0 = 2n_0 + 2(n - n_0)\sigma$, and $\alpha_1 = n_1 + 2(n - n_1)\sigma$.

Conversely, if $e(F)$ lies in grading $m(\omega - 2) + 2a + 2b\sigma$, then we can recover $n = a + b$, $n_0 = a$, and $n_1 = a - m$.

The degrees $(\Delta, \Delta_0, \Delta_1)$ are, by the nonequivariant Bézout theorem, given by

$$\rho(e(F)) = \Delta \tilde{e}^n$$
$$e(F)^{C_2} = (\Delta_0 \tilde{e}^{n_0}, \Delta_1 \tilde{e}^{n_1}).$$
using the fact that $\rho$ and $(-)^{C_2}$ preserve Euler classes. Thus, we can recover the degrees from $e(F)$.

It remains to show that $e(F)$ is determined by its grading and $C_2$-degrees.

Recall the discussion before the theorem of the four types of line bundles over $\mathbb{P}(\mathbb{C}^{p+q})$. In [3, Proposition 6.5] we computed their Euler classes, which are

\[
e(O(2d+1)) = \hat{\omega} + d(\tau(1) \hat{\omega} + e^{-2} \kappa \hat{\omega} \hat{\chi}_\omega) \equiv \hat{\omega} \pmod{J_e},
\]

\[
e(O(2d)) = d(\tau(i^{-2}) \zeta_0 \hat{\omega} + e^{-2} \kappa \hat{\omega} \hat{\chi}_\omega) \equiv 0 \pmod{J_e},
\]

\[
e(\chi O(2d+1)) = \hat{\chi}_\omega + d(\tau(1) \hat{\chi}_\omega + e^{-2} \kappa \zeta_0 \hat{\omega} \hat{\chi}_\omega) \equiv \hat{\chi}_\omega \pmod{J_e},
\]

\[
e(\chi O(2d)) = e^2 + d(\tau(1) \zeta_0 \hat{\omega} \hat{\chi}_\omega) \equiv e^2 \pmod{J_e}.
\]

From (2.9) and (2.10), we see that $\Delta_0$ and $\Delta_1$ are both even if and only if $F$ contains at least one summand of the form $O(2d)$ (type II). If $F$ does not contain such a summand, then $n_0$ is the number of summands of the form $O(2d+1)$ and $n_1$ is the number of summands of the form $\chi O(2d+1)$, and we will have $n_0 + n_1 \leq n$. From the congruences above we have, modulo $J_e$, that

\[
e(F) \equiv \begin{cases} 0 & \text{if } \Delta_0 \text{ and } \Delta_1 \text{ are even} \\ e^{2(n-n_0-n_1)} \hat{\omega}^{n_0} \hat{\chi}_\omega^{n_1} & \text{if } \Delta_0 \text{ or } \Delta_1 \text{ is odd.} \end{cases}
\]

When $\Delta_0$ or $\Delta_1$ is odd we have that $n_0 \leq p$ and $n_1 \leq q$, with at least one of the inequalities being strict, so $\hat{\omega}^{n_0} \hat{\chi}_\omega^{n_1}$ is a basis element and $e^{2(n-n_0-n_1)} \hat{\omega}^{n_0} \hat{\chi}_\omega^{n_1} \in \hat{T}$. It follows that $e(F) \in \hat{T}$, and then the fact that $e(F)$ is determined by its grading and $C_2$-degrees follows from Corollary 2.8.

By Lemma 2.5, the Euler class $e(F)$ can be written as a linear combination of just three basis elements. We next work out the explicit expression, which, by Theorem 2.11, is determined by the grading of $e(F)$ and its $C_2$-degrees.

**Theorem 2.12 (Bézout’s Theorem, Part II).** Let $F$ be as in the Bézout context 0.1. Then we can write

\[
e(F) = \alpha P_n^{(m)} + \beta P_k^{(m)} + \gamma P_{k-1}^{(m)}
\]

for some $1 \leq k < p+q$ and some coefficients $\alpha$, $\beta$, and $\gamma$ in $\mathbb{H}$, so the coefficient vector of $e(F)$ has at most three nonzero components. Allowing for the possibility that $n = k$ or $n = k - 1$, we can arrange that the coefficient $\alpha$ is always an integer multiple of $\tau(i^{2i})$ for some $i \in \mathbb{Z}$, and the coefficients $\beta$ and $\gamma$ are always integer multiples of $e^{2i}$ or $e^{-2i} \kappa$ for some $i \geq 0$.

Use the briefer notation $P_n$ and write $\epsilon = 0$ or 1 for the remainder on dividing $n + n_0 + n_1$ by 2. We have

\[
P_n = \begin{cases} c_0^{-(n-n_0-n_1-2p)} \hat{\omega}^{n_0} \hat{\chi}_\omega^{n_1+p} & \text{if } n + n_0 - n_1 > 2p \\ c_1^{-(n-n_0-n_1-2q)} \hat{\omega}^{n_0-n_1+q} & \text{if } n - n_0 + n_1 > 2q \\ c_2^{-(n-n_0-n_1+1)/2} \hat{\omega}^{(n-n_0+n_1-1)/2} & \text{otherwise,} \end{cases}
\]

\[
P_k = \begin{cases} c_0^{n_0+n_1+1} \hat{\omega}^{n_0} \hat{\chi}_\omega^{n_1} & \text{if } n_0 < p \\ c_1^{-(n_0-n_0-p)} \hat{\omega}^{n_0} \hat{\chi}_\omega^{n_1} & \text{if } n_0 \geq p, \end{cases}
\]

and

\[
P_{k-1} = \begin{cases} c_0^{n_0} \hat{\omega}^{n_0} \hat{\chi}_\omega^{n_1} & \text{if } n_1 < q \\ c_1^{-(n_1-n_0-q)} \hat{\omega}^{n_0} \hat{\chi}_\omega^{n_1} & \text{if } n_1 \geq q. \end{cases}
\]
The coefficient \( \alpha \) will be an integer multiple of
\[
\tau_n = \begin{cases} 
\tau(2^{n-n_1-p}) & \text{if } n + n_0 - n_1 > 2p \\
\tau(2^{n_0-n_0-q}) & \text{if } n - n_0 + n_1 > 2q \\
\tau(\nu^{-n_0-n_1}) & \text{otherwise.}
\end{cases}
\]

Finally, write \( \bar{\nu}_0 = \min\{n_0, p-1\} \) and \( \bar{\nu}_1 = \min\{n_1, q\} \). Then, we break the result into the following cases:

1. If \( \Delta \) is even, then
   \[
   \alpha = \frac{\Delta}{2} \tau_n, \quad \beta = \frac{\Delta_1 - \Delta_0}{2} e^{-2(\bar{\nu}_0 + \bar{\nu}_1 - n + 1)\kappa}, \\
   \gamma = \frac{\Delta_0}{2} e^{-2(\bar{\nu}_0 + \bar{\nu}_1 - n)\kappa} \quad \text{and} \quad k = \bar{\nu}_0 + \bar{\nu}_1 + 1.
   \]

2. If \( \Delta \) is odd and \( \Delta_0 \neq 0 \), then
   \[
   \alpha = \frac{\Delta - \Delta_0}{2} \tau(1), \quad \beta = \frac{\Delta_1 - \Delta_0}{2} e^{-2\kappa}, \\
   \gamma = \Delta_0 \quad \text{and} \quad k = n + 1.
   \]

3. If \( \Delta \) is odd and \( \Delta_0 = 0 \), then
   \[
   \alpha = \frac{\Delta - \Delta_1}{2} \tau(1), \quad \beta = 0, \\
   \gamma = \Delta_1 \quad \text{and} \quad k = n + 1.
   \]

Proof. Theorem 2.11 and Lemma 2.5 imply the first claim, that we can write \( e(F) \) in terms of just three basis elements.

To determine \( P_n, P_k, \) and \( P_{k-1} \), we recall from [3, Proposition 4.7] that the basis elements take one of six possible forms:
\[
\begin{align*}
\zeta_a^m \hat{c}_\omega^a & \quad m > 1, \ a < p \\
\zeta^b_a \hat{c}_\omega^b & \quad a \leq p, \ b \leq q \\
\zeta^a_0 \hat{c}_\omega^b & \quad a \leq p, \ b < q \\
\zeta^a_0 \hat{c}_\omega^b & \quad a < p, \ b < q \\
\zeta^a_1 \hat{c}_\omega^b & \quad m > 0, \ b < q \\
\zeta^b_1 \hat{c}_\omega^a & \quad m > 0, \ a < p
\end{align*}
\]

where we recall that \( \hat{c}_\omega^a \hat{c}_\omega^a = 0 \), so we do not have \( a = p \) and \( b = q \) above.

We noted earlier that \( e(F) \) lies in grading
\[
\text{grad } e(F) = (n_0 - n_1)(\omega - 2) + 2n_0 + 2(n - n_0)\sigma.
\]

\( P_n \) is the unique basis element having grading in \((n_0 - n_1)(\omega - 2) + RO(C_2)\) restricting to \( \hat{c}_a^n \), and we can check that the formula given in the statement of the theorem has those properties. Similarly, \( P_k \) and \( P_{k-1} \) are the (at most) two basis elements having gradings of the form \((n_0 - n_1)(\omega - 2) + 2n_0 + 2b\sigma\) and we can check that the formulas given have that property. The coefficient \( \tau_n \) is the element of the form \( \tau(i^{2n}) \) such that \( \tau_n P_n \) lies in the same grading as \( e(F) \). The terms of the form \( e^m \kappa \) multiplying \( P_k \) and \( P_{k-1} \) in the formulas for \( e(F) \) are determined similarly.

We should point out some abuses of notation we are indulging in. The formulas for \( P_k \) and \( P_{k-1} \) evaluate to 0, not basis elements, when both \( n_0 \geq p \) and \( n_1 \geq q \). In the case \( n_0 < p - 1 \) and \( n_1 \geq q \), the formula for \( P_k \) is not a basis element, but we know that its coefficient will be a multiple of \( e^m \kappa \) for some integer \( m \), and the product \( e^m \kappa P_k = 0 \) in that case, because of the relations in the cohomology
of $\mathbb{P}(\mathbb{C}^{p+q})$. A similar vanishing happens in the case of $P_{k-1}$ when $n_0 > p$ and $n_1 < q$. Finally, the formulas for $P_k$ and $P_{k-1}$ coincide when $n_0 = p$ and $n_1 < q$, but in that case $\Delta_0 = 0$ so only one copy of this basis element appears in the formula for $e(F)$.

To verify the coefficients of $P_n$, $P_k$, and $P_{k-1}$, we use the fact that $e(F)$ is determined by the nonequivariant elements

$$\rho(e(F)) = \Delta \hat{c}^n$$

and

$$e(F)^{C_2} = (\Delta_0 \hat{c}^{n_0}, \Delta_1 \hat{c}^{n_1}),$$

so we simply need to check that the formulas of the theorem have the correct values on applying these restriction maps.

First note that, regardless of which case we fall in, we will always have

$$\rho(\tau_n P_n) = 2 \hat{c}^n$$

$$(\tau_n P_n)^{C_2} = (0, 0).$$

For $P_k$ and $P_{k-1}$ we have

$$\rho(P_k) = \hat{c}^k$$

$$P_k^{C_2} = (0, \hat{c}^{n_1})$$

$$\rho(P_{k-1}) = \hat{c}^{k-1}$$

$$P_{k-1} = (\hat{c}^{n_0}, \hat{c}^{n_1})$$

which includes the possibility that $P_{k-1} = (\hat{c}^{n_0}, 0)$ if $n_1 \geq q$.

Now, when $\Delta$ is even, in the formulas given, $\beta$ and $\gamma$ each have a factor of the form $e^m \kappa$, and we have $\rho(e^m \kappa) = 0$ and $(e^m \kappa)^{C_2} = 2$. Combined with the formulas above, this verifies case (1) of the theorem, except that we should say something about the parities of $\Delta_0$ and $\Delta_1$. From the discussion before Theorem 2.11, because $\Delta$ is even, $\Delta_0$ and $\Delta_1$ have the same parity. There is a possibility that $\Delta_0$ is odd, but this can happen only when $n_{11} = 0$ and $n_{IV} > 0$, in which case $n_0 < p$, $n_1 < q$, and $n_0 + n_1 < n$. The coefficient $\gamma$ in that case is

$$\gamma = \frac{\Delta_0}{2} e^{-2(n_0 + n_1 - n) \kappa} = \frac{\Delta_0}{2} e^{2(n-n_0-n_1)}$$

which we interpret as $\Delta_0 e^{2(n-n_0-n_1)}$ by another abuse of notation. (The abuse is that division by 2 is not well-defined in $\mathbb{H}$.) We then use that $\rho(e^m) = 0$ and $(e^m)^{C_2} = 1$ for $m > 0$.

If $\Delta$ is odd, then $n = n_0 + n_1$, $n_0 \leq p$, and $n_1 \leq q$. IF $\Delta_0$ and $\Delta_1$ are both nonzero, then $n_0 < p$ and $n_1 < q$, $P_n = P_{k-1}$, and the formula in case (2) of the theorem is easily verified.

If $\Delta_0 \neq 0$ but $\Delta_1 = 0$, then we have $n_0 < p$ and $n_1 = q$. In this case, we have

$$e^{-2 \kappa P_k} = e^{-2 \kappa \hat{c}^{n_0+1} \hat{c}_\omega^q} = 0,$$

so we allow the abuse of notation that $\Delta_1 - \Delta_0$ is odd in the formula for $\beta$. With that caveat, the verification of case (2) can be completed.

In Case (3), since $\Delta_0 = 0$ we must have $\Delta_1 \neq 0$ and odd. The verification is then just as for the previous cases.

The asymmetries in these formulas comes from an asymmetry in our preferred basis regarding $\hat{c}_\omega$ vs $\hat{c}_\chi$.

Remark 2.13. Theorems 2.11 and 2.12 give us two related ways of determining $e(F)$: It is determined by the ranks $(n, n_0, n_1)$ and the $C_2$-degrees $(\Delta, \Delta_0, \Delta_1)$, and it is also determined by its triple of nonzero coefficients. The advantage of using
the degrees is that they are multiplicative. This is simpler to calculate with and also parallels the result of the nonequivariant Bézout theorem, that degrees are multiplicative under intersection of projective varieties.

**Remark 2.14.** The summary of Theorem 2.12 is that $e(F)$ can be expressed in terms of at most three basis elements. This is not a restriction imposed by the locations of the basis elements. As an example, consider $\mathbb{P}(\mathbb{C}^{5}+5\sigma)$ and the bundle $F = 4\chi_{O}(2)$, the sum of 4 copies of $\chi_{O}(2)$, so $n = 4$ and $n_0 = n_1 = 0$. This Euler class lives in grading

$$(n_0 - n_1)(\omega - 2) + 2n_0 + 2(n - n_0)\sigma = 8\sigma.$$  

The following diagram shows the location of $e(F)$, the “×” at $8\sigma$, and the locations of the basis elements in the $RO(C_2)$-grading:

![Diagram](image)

The five basis elements within the shaded area have nonzero multiples in degree $8\sigma$, so could conceivably contribute to $e(F)$, but the theorem says that it can be written in terms of just three of them, the one on the same diagonal as $e(F)$, $P_4$, and the two below it, $P_0$ and $P_1$.

In fact, we are in case (1) of Theorem 2.12, with $\Delta = 8$ and $\Delta_0 = \Delta_1 = 1$, so

$$e(4\chi_{O}(2)) = 8\tau(i^4)P_4 + 0 \cdot P_1 + e^8P_0 = 8\tau(i^4)\hat{c}_2\hat{c}_2\omega + e^8.$$  

As it happens, $P_1$ does not actually contribute in this example.

**Remark 2.15.** In [3], we looked in detail at the case $n = p + q - 1$, so that the hypersurfaces associated with the line bundle summands of $F$ intersect generically in a $C_2$-set of points in $\mathbb{P}(\mathbb{C}^{p+q}\sigma)$. In that case, we showed that the explicit formula for $e(F)$ can be read as telling us how that collection of points breaks down as free orbits versus fixed points in each of the components of $\mathbb{P}(\mathbb{C}^{p+q}\sigma)^{C_2}$. In a followup to this paper, we will show how the Euler class more generally gives us geometric information about the intersection of hypersurfaces.

### 3. Comparison with constant $\mathbb{Z}$ coefficients

Another equivariant cohomology theory commonly used is ordinary cohomology with coefficients in $\mathbb{Z}$, the constant-$\mathbb{Z}$ Mackey functor. We calculate the Euler class $e(F)$ with $\mathbb{Z}$ coefficients and compare it to the class obtained with Burnside ring coefficients.

As shown in [2], $H^{RO(C_2)}_{C_2}(S^0;\mathbb{Z})$ is obtained from $\mathbb{H}$ by setting $\kappa = 0$. This has the effect of removing the elements $e^{-n}\kappa$ and making $2e = 0$. Since $\kappa = 2 - g$, it also has the effect of setting $g = 2$. Put another way, this theory cannot distinguish between a free orbit and two fixed points.

Because the cohomology of $\mathbb{P}(\mathbb{C}^{p+q}\sigma)$ with $A$ coefficients is free over the cohomology of a point, we obtain the cohomology with $\mathbb{Z}$ coefficients by setting $\kappa = 0$. The result is the following.
Theorem 3.1 ([3, Corollary 5.4]). Let $0 \leq p < \infty$ and $0 \leq q < \infty$ with $p + q > 0$. Then $H^{RO(IB)}_{C_2}(P(C^p+q^q)_+; \mathbb{Z})$ is a free module over $H^{RO(C_2)}_{C_2}(S^0; \mathbb{Z})$. Its structure as a graded commutative algebra over $H^{RO(C_2)}_{C_2}(S^0; \mathbb{Z})$ is described as in Theorem 1.2, except that the relation $\zeta_1 \tilde{c}_\omega = (1 - \kappa)\zeta_0 \tilde{c}_\omega + e^2$ is replaced by the relation

$$\zeta_1 \tilde{c}_\omega = \zeta_0 \tilde{c}_\omega + e^2.$$ 

Setting $\kappa = 0$ in Theorem 2.12, remembering that $e^m$ is 2-torsion, and paying attention to the abuses of notation mentioned in the proof of that theorem, we get the following.

Theorem 3.2 (Bézout's Theorem for Constant $\mathbb{Z}$ Coefficients). Let $F$ be as in the Bézout context 0.1. Then the Euler class $e_{\mathbb{Z}}(F) \in H^{RO(IB)}_{C_2}(P(C^p+q^q)_+; \mathbb{Z})$ is given by

$$e_{\mathbb{Z}}(F) = \begin{cases} \frac{\Delta}{2} \tau_n P_n^{(m)} & \text{if } \Delta, \Delta_0, \text{ and } \Delta_1 \\
 & \text{are even}\n \frac{\Delta}{2} \tau_n P_n^{(m)} + e^{2(n-n_0-n_1)} P_{k-1}^{(m)} & \text{if } \Delta \text{ is even and } \\
 & \Delta_0 \text{ or } \Delta_1 \text{ is odd}\n \Delta T_n^{(m)} & \text{if } \Delta \text{ is odd} \end{cases}$$

where, writing $\epsilon = 0$ or 1 for the remainder on dividing $n + n_0 + n_1$ by 2, we set

$$P_n^{(m)} = \begin{cases} \zeta_0^{-n_0-n_1+2p} \tilde{c}_\omega \tilde{c}_{n-p} & \text{if } n + n_0 - n_1 > 2p \\
 \zeta_0^{-n_0-n_1+2q} \tilde{c}_\omega \tilde{c}_{n-q} & \text{if } n - n_0 + n_1 > 2q \\
 \zeta_0^{n_0-n_1+2q} \tilde{c}_\omega \tilde{c}_{n_0-n_1+2q} & \text{otherwise.} \end{cases}$$

$$\tau_n = \begin{cases} \tau(2n-n_1-p) & \text{if } n + n_0 - n_1 > 2p \\
 \tau(2n-n_0-q) & \text{if } n - n_0 + n_1 > 2q \\
 \tau(n-n_0-n_1-\epsilon) & \text{otherwise.} \end{cases}$$

and, when $\Delta$ is even and $\Delta_0$ or $\Delta_1$ is odd,

$$P_{k-1}^{(m)} = \tilde{c}_{n_0} \tilde{c}_{n_1}.$$ 

While this result has the benefit of relative simplicity, it carries significantly less information than Theorem 2.12. In particular, we cannot reconstruct $\Delta_0$ and $\Delta_1$ from $e_{\mathbb{Z}}(F)$. This follows from the formula in the theorem, but we can also look again at the fixed-point map $(-)^{C_2}$ to see why this must happen. As defined in [4], the fixed-point map takes $G$-equivariant cohomology with coefficients in a Mackey functor $T$ to nonequivariant cohomology with coefficients in $T^G$. In the case of the group $C_2$, we have

$$T^{C_2} = T(C_2/C_2)/\tau(T(C_2/e)).$$

This gives $A^{C_2} = \mathbb{Z}$, but $\mathbb{Z}^{C_2} = \mathbb{Z}/2$. We then get the following.

Corollary 3.3. With $F$ as in the Bézout context 0.1, we have

$$e_{\mathbb{Z}}(F)^{C_2} = (\Delta_0 \tilde{c}_n, \Delta_1 \tilde{c}_m^n) \in H^{2a}(P(C^p)_+; \mathbb{Z}/2) \oplus H^{2(n-m)}(P(C^p)_+; \mathbb{Z}/2),$$

where

$$A^{C_2} = \mathbb{Z},$$

and

$$\mathbb{Z}^{C_2} = \mathbb{Z}/2.$$
so
\[ e_Z(F)^{C_2} = \begin{cases} (0, 0) & \text{if } \Delta_0 \text{ and } \Delta_1 \text{ are even} \\ (c^{n_0}, c^{n_1}) & \text{if } \Delta_0 \text{ or } \Delta_1 \text{ is odd.} \end{cases} \]

**Proof.** From the commutativity of the diagram
\[
\begin{array}{ccc}
H_2^{RO(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q})_+; \mathbb{A}) & \xrightarrow{(-)^{C_2}} & H_2^{RO(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q})_+; \mathbb{Z}) \\
\downarrow & & \downarrow \quad \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
H^Z(\mathbb{P}(\mathbb{C}^{p+q})^{C_2}_+; \mathbb{Z}) & \xrightarrow{(-)^{C_2}} & H^Z(\mathbb{P}(\mathbb{C}^{p+q})^{C_2}_+; \mathbb{Z}/2),
\end{array}
\]
where the horizontal arrows are given by change of coefficients, \( e_Z(F)^{C_2} \) is just the reduction of \( e(F)^{C_2} \) modulo 2.

Thus, from this Euler class we cannot recover \( \Delta_0 \) and \( \Delta_1 \), only their parities. This goes back to the fact that, because \( q = 2 \), cohomology with \( \mathbb{Z} \) coefficients cannot distinguish between a free orbit and two fixed points, hence retains only parity information about fixed points.

For example, in the case \( n = p + q - 1 \) discussed in detail in [3], we can think of the Euler class in terms of the finite \( C_2 \)-set given by the zero locus of a section of \( F \), or the intersection of the hypersurfaces given by the zero loci of sections of the line bundles making up \( F \). The Euler class with Burnside ring coefficients completely determines this \( C_2 \)-set, including how many fixed points lie in each component of \( \mathbb{P}(\mathbb{C}^{p+q})^{C_2} \). The Euler class with constant \( \mathbb{Z} \) coefficients can tell us only the parity of the number of fixed points in each component.

4. **Comparison with Borel cohomology**

Borel cohomology was the first theory thought of as equivariant ordinary cohomology, but is a considerably weaker theory than Bredon cohomology. (See, for example, May’s discussion in [7].) There is a map from ordinary cohomology with \( \mathbb{Z} \) coefficients to Borel cohomology, so the latter is also weaker than cohomology with \( \mathbb{Z} \) coefficients. To see how much weaker, let us look at the calculation of \( e(F) \) in Borel cohomology.

We take Borel cohomology to be the \( RO(C_2) \)-graded theory defined on based \( C_2 \)-spaces by
\[
BH^{RO(C_2)}(X) = H^{RO(C_2)}((EC_2)_+ \wedge X),
\]
where, as usual, we use Burnside ring coefficients on the right, but suppress them from the notation. (Because \( EC_2 \) is free, and \( A \to \mathbb{Z} \) is an isomorphism at the \( C_2/e \) level, we could instead use \( \mathbb{Z} \) coefficients and get naturally isomorphic results.) This is the usual Borel cohomology with \( \mathbb{Z} \) coefficients but we have expanded the grading from the common \( \mathbb{Z} \) to \( RO(C_2) \). As shown in [2], the Borel cohomology of a point is \( \mathbb{H} \) with \( \xi \) inverted:
\[
BH^{RO(C_2)}(\mathbb{S}^0) \cong \mathbb{Z}[e, \xi, \xi^{-1}]/\langle 2e \rangle,
\]
where \( \deg e = \sigma \) and \( \deg \xi = 2\sigma - 2 \) as before. In the map \( \mathbb{H} \to BH^{RO(C_2)}(\mathbb{S}^0) \), \( \kappa \) goes to 0. As with cohomology with \( \mathbb{Z} \) coefficients, Borel cohomology cannot tell the difference between \( q \) and 2.

Note that, if we restrict to the \( \mathbb{Z} \) grading, as is usually done, we get a polynomial algebra in \( e^2\xi^{-1} \) modulo \( 2e^2\xi^{-1} = 0 \), a copy of the group cohomology of \( C_2 \) with \( \mathbb{Z} \).
coefficients. If we restrict the grading to $\sigma + \mathbb{Z}$, we see the group cohomology of $C_2$ with twisted $\mathbb{Z}$ coefficients. That the twisted and untwisted cohomologies can be combined in a single algebra like this seems to have been first observed by Čadek in [1].

Because the ordinary $C_2$-cohomology of $\mathbb{P}^{(p+q\sigma)}$ is free over the cohomology of a point, we obtain its Borel cohomology also by inverting $\xi$. On doing so, the elements $\zeta_0$ and $\zeta_1$ become invertible, with the result that, if we continued to grade on $RO(P)$, the groups outside the $RO(C_2)$ grading would all be isomorphic to groups in the $RO(C_2)$ grading via multiplication by an appropriate power of, say, $\zeta_0$. So we lose nothing by considering the $RO(C_2)$-graded part only. To give the explicit result, let $\hat{c}$ be the image of $\zeta_0 \hat{c}$ in $BH^{C_2}(\mathbb{P}^{(p+q\sigma)}_+)$. The following is then a corollary of Theorem 1.2.

**Corollary 4.1.** Let $0 \leq p < \infty$ and $0 \leq q < \infty$ with $p + q > 0$. Then $BH^{RO(C_2)}(\mathbb{P}^{(p+q\sigma)}_+)$ is a free module over $BH^{RO(C_2)}(S^0)$. As a (graded) commutative algebra over $BH^{RO(C_2)}(S^0)$, $BH^{RO(P)}(\mathbb{P}^{(p+q\sigma)}_+)$ is generated by $\hat{c}$ in degree $2\sigma$, which satisfies the single relation

$$\hat{c}^p(\hat{c} + e^2)^q = 0.$$ 

□

Of course, we could also use as a generator the element $c' = \xi^{-1} \hat{c}$ in degree 2, but the relation is then

$$(c')^p(c' + e^2)^q = 0.$$ 

For the simplicity of the relation, and to keep the generator more closely related to an element from ordinary cohomology, we prefer to use $\hat{c}$.

We view $\hat{c}$ as the Euler class of $\omega^\vee$. The Euler class of $\chi \omega^\vee$ is then $\hat{c} + e^2$, the image of $\zeta_1 \hat{c} \omega$. In doing this, we are choosing to say that $\omega$ is a rank $2\sigma$ bundle over $EC_2 \times \mathbb{P}(\mathbb{C}^{p+q\sigma})$. Because $EC_2$ is free, we are as free to say $\omega$ has rank $2\sigma$ as to say it has rank 2.

Another way of seeing that $e(\chi \omega) = \hat{c} + e^2$ is to recall that $\chi \omega = \omega \otimes_{C} C^\sigma$, then use the additive formal group law of nonequivariant ordinary cohomology and the fact that $e(C^\sigma) = e^2$.

Now consider the Euler classes of the bundles $O(d)$ and $\chi O(d)$, all of which we will think of as having rank $2\sigma$. As a corollary of [3, 6.5], or as a consequence of the formal group law for nonequivariant cohomology, we have the following.

**Proposition 4.2.** In the Borel cohomology of $\mathbb{P}^{(p+q\sigma)}$, we have

$$e(O(d)) = d \hat{c}$$

and

$$e(\chi O(d)) = d \hat{c} + e^2$$

for every $d \in \mathbb{Z}$. □
Theorem 4.3 (Bézout’s Theorem for Borel Cohomology). Let $F$ be as in the Bézout context 0.1. The Euler class of $F$ in the Borel cohomology of $\mathbb{P}(\mathbb{C}^{p+q})$ is

$$e_{BH}(F) = \begin{cases} 
\Delta \hat{c}^n & \text{if } \Delta, \Delta_0, \text{ and } \Delta_1 \\
\Delta \hat{c}^n + e^{2(n-n_0-n_1)} \hat{c}^{n_0} (\hat{c} + e^2)^{n_1} & \text{if } \Delta \text{ is even and } \\
\Delta \hat{c}^{n_0} (\hat{c} + e^2)^{n-n_0} & \text{if } \Delta \text{ is odd.}
\end{cases}$$

Proof. These formulas can be derived from the preceding proposition or from Theorem 3.2, using the fact that $\tau(1) = 2$ in Borel cohomology. □

As we saw with ordinary cohomology with $\mathbb{Z}$ coefficients, the Euler class in Borel cohomology contains significantly less information than the one in ordinary cohomology with Burnside ring coefficients. The fixed point map would be

$$(-)^{C_2} \colon H_{C_2}^{RO(C_2)}((EC_2)_+ \wedge \mathbb{P}(\mathbb{C}^{p+q})_+) \rightarrow H^Z(((EC_2)_+ \wedge \mathbb{P}(\mathbb{C}^{p+q})_+)^{C_2}; \mathbb{Z}) = H^Z(\ast; \mathbb{Z}) = 0.$$ 

Thus, Borel cohomology contains no information at all about fixed points.

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