Light Front Nuclear Physics: Toy Models, Static Sources and Tilted Light Front Coordinates

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Abstract

The principles behind the detailed results of a light-front mean field theory of finite nuclei are elucidated by deriving the nucleon mode equation using a simple general argument, based on the idea that a static source in equal time coordinates corresponds to a moving source in light front coordinates. This idea also allows us to solve several simple toy model examples: scalar field in a box, 1+1 dimensional bag model, three-dimensional harmonic oscillator and the Hulthén potential. The latter provide simplified versions of momentum distributions and form factors of relevance to experiments. In particular, the relativistic correction to the mean square radius of a nucleus is shown to be very small. Solving these simple examples suggests another more general approach— the use of tilted light front coordinates. The simple examples are made even simpler.
I. INTRODUCTION

Light front techniques [1]-[9] have long been used to analyze high energy experiments involving nuclear and nucleon targets. In the parton model ratio $k^+/p^+$ where $k^+ = k^0 + k^3$ is the plus-momentum of the struck quark and $p^+$ is the plus-momentum of the target turns out to be equal to the Bjorken variable $x_{Bj}$. Quark distributions represent the probability that a quark has a plus-momentum fraction $x_{Bj} = k^+/p^+$. In nuclear physics we are often concerned with the distribution functions which describe the plus-momentum carried by the nucleons within the nucleus. Such distributions, which depend on $k^+$ of the struck nucleon, are needed to analyze deep inelastic lepton-nucleus scattering, and enter also in the analysis of high-momentum transfer nuclear quasi-elastic reactions such as $(e,e')$, $(e,e'p)$, $(p,2p)$. If the light front formalism is used, these distributions are simply related to the square of the nuclear ground state wave functions. Many nuclear high momentum transfer are planned, so that it is necessary to derive a relativistic formulation which uses $k^+$ variables closely related to experiments and which incorporates the full knowledge of nuclear dynamics.

If the relevant nuclear wave functions depend on $k^+$, the canonical spatial variable is $x^- = x^0 - x^3$. This leaves $x^+ = x^0 + x^3$ to be used as a time variable, with the light front Hamiltonian ($x^+$ development operator) as $P^- = P^0 - P^3$. These are the light front variables of Dirac. There are clear advantages in using these variables, but a principal problem arises because the use of $x^+$ as “time” and $x^-, x_\perp$ as “space” involves the loss of manifest rotational invariance. This is especially important in nuclear physics because the understanding of magic numbers rests on the $2j + 1$ degeneracy of single particle orbitals of good angular momentum. Our previous papers [10] showed how light front techniques could be used to derive a mean field theory for finite-sized nuclei with results that respect rotational invariance.

There were three features that emerged from that detailed mean field theory treatment:

- The variational principle which leads to the field equations involves minimizing the expectation value of $P^-$ subject to the constraint that the expectation value of $P^+$ is the same as that of $P^-$. 

- The meson field equations of the light front (LF) formalism are the same as those for the equal time (ET) formulation, except for the replacement $z \rightarrow -x^-/2$ in going from the ET to the LF formulation.

- The approximate solution of the LF nucleon mode equation, was a phase factor times a solution of the ET nucleon mode equation, again evaluated with $z \rightarrow -x^-/2$.

The purpose of the present paper is to explain these features in a more general way and to present some examples of solved problems using light front techniques. The latter is intended to build up the reader’s confidence in the idea that many different problems can be solved using light front techniques. The essential feature that allows a simple explanation of the three features is that in each case the mode equation involves external potentials that do not depend on $x^+$. We show that such problems are simply related to cases of static potentials in the ET formulation. Since using the ET formulation is natural, our basic logic begins by noticing that static sources in the usual ET formulation (position $(x,y,z)$ fixed for all time
correspond to sources moving with constant velocity in the LF formulation (because \( z + t \) changes with respect to \( z - t \) as \( t \) changes. But static sources depend on the variable \( x^+ \) in a manner simple enough to be easily removable with a simple transformation to a source which is static in light front coordinates. The result is a LF theory in which \( x^+ \) is absent, and in which the operator \( P^+ \) is a part of the \( x^+ \) development operator. The transformation is applied to the Dirac equation in Sect. II. The same technique is used to solve four problems: scalar field in a box, 1+1 dimensional bag model, three dimensional harmonic oscillator and the Hulthén wave function in Sect. III. We also consider the resulting light front momentum distribution function as a function of \( p^+ \) and \( p_\perp \), and also study the related electromagnetic form factors. For the examples we consider, the only difference between our present approach and the full calculations in Ref. 10 is in the support properties. In the complete theory the wave functions (of positive energy solutions) have no components with \( k^+ < 0 \). This is not kept in the present simple approach, and the sizes of errors introduced depend strongly on the mass of the particle involved. For nucleons, the errors are shown to be very small indeed.

The transformation to a light front static source suggests the use of a new set of coordinates--tilted light front coordinates: “time” \( \tau \equiv x^0 + x^3 \), “space” \( \zeta \equiv -x^3 \). With these coordinates a static source in the ET formulation is also a static source in the tilted LF formulation, and the requirement of the constrained variational principle that the operator \( P^+ \) be part of the “time” development operator is satisfied from the very beginning. Tilted coordinates allow us to easily compare LF and ET calculations in problems where a single particles moves in an external potential (fixed source, mean field type problems). We introduce these coordinates in Sect. IV and demonstrate some utility in Sect. V by applying these to solve the simple problems of Sect. III. A brief discussion of the results is presented in Sect. VI.

In this paper, our use of tilted coordinates is restricted to fixed source, mean field problems in which one uses quantum mechanics and not field theory. This is done to provide the reader with some intuitive ideas about these coordinates and to illustrate their practical use. However, this does by no means exhaust the potential applications of tilted coordinates. There are many genuine quantum field theory problems that do involve also external fields, such as heavy-light systems in QCD (e.g. B-mesons) or a QFT treatment of large nuclei. For such systems, the relation between ET and LF is more complicated than the situations discussed here because the microscopic degrees of freedom (the quanta) in terms of which the theory is formulated are different in these two approaches and the LF approach can provide new insights into observables that probe light-like correlations. The use of tilted coordinates is that they provide a significant shortcut towards constructing the LF Hamiltonian.

II. ALTERNATE DERIVATION OF THE NUCLEON MODE EQUATION

In Ref. 10 the constrained minimization (of \( P^- \)) led to a new equation for the single-nucleon modes (Eq. (4) in the short paper, (3.28) of the long paper). In order to illuminate the physics we present another, more heuristic, derivation of the nucleon single particle wave equation. The first step towards formulating the mean field approximation in light front LF coordinates is to develop the formalism for static potentials (static in a rest frame!) on
the LF. For this purpose, we start from the Dirac equation for a static potential in normal coordinates

\[ i\gamma^\mu \partial_\mu - m - g_S V_S(\vec{r}) - g_V \gamma^0 V_0(\vec{r}) \] \psi' = 0. \quad (2.1) 

Since \( \gamma^0 = (\gamma^+ + \gamma^-)/2 \) and since couplings using the “bad” component \( \gamma^- \) are difficult to handle in the LF framework, we perform the transformation

\[ \psi' = e^{igV(\vec{r})} \tilde{\psi}'. \quad (2.2) \]

where

\[ \partial_3 \Gamma = V_0, \quad (2.3) \]

and \( \Gamma \) does not depend on time. Using the transformation (2.2), one finds that \( \tilde{\psi}' \) couples to the vector field only via \( \gamma^0 + \gamma^3 \) (the LF-\( \gamma^+ \) component!)

\[ 0 = \left[ i\gamma^\mu \partial_\mu + i\gamma^\perp \cdot (\vec{\partial}_\perp \Gamma) - m - g_S V_S(\vec{r}) - g_V \left( \gamma^0 + \gamma^3 \right) V_0(\vec{r}) \right] \tilde{\psi}'. \quad (2.4) \]

Simply rewriting Eq. (2.1) in LF coordinates, with \( \gamma^\pm \equiv (\gamma^0 \pm \gamma^3) \) and \( x^\pm \equiv (x^0 \pm x^3) \), yields

\[ \left[ \frac{1}{2} \left( i\gamma^+ \partial^- + i\gamma^- \partial^+ \right) + i\gamma^\perp \cdot \left( \vec{\partial}_\perp + ig_0 \vec{\partial}_\perp \Gamma (\vec{r}_\perp, \frac{x^+ - x^-}{2}) \right) - m \right. \]

\[ \left. - g_S V_S(\vec{r}_\perp, \frac{x^+ - x^-}{2}) - g_V \gamma^+ V_0(\vec{r}_\perp, \frac{x^+ - x^-}{2}) \right] \tilde{\psi}' = 0. \quad (2.5) \]

Even though the potential is static in the equal time formulation, the Dirac equation for the same potential in light-front coordinates is LF-“time”, i.e. \( x^+ \), dependent. The physical origin for this result is that a static source in a rest-frame corresponds to a uniformly moving source on the light-front. Given that the time dependence of the external fields is only due to a uniform translation, it should be easy to transform Eq. (2.3) into a form which contains only static (with respect to \( x^+ \)) potentials. For this purpose, we consider the equation of motion satisfied by Dirac fields which are obtained by an \( x^+ \) (LF-time) dependent translation

\[ \tilde{\psi}'(\vec{x}_\perp, x^-, x^+) \equiv e^{-ix^+P^+/2} \psi(\vec{x}_\perp, x^-, x^+) \quad (2.6) \]

Using \( P^+ = -i2\frac{\partial}{\partial x^-} \), we find

\[ e^{ix^+P^+/2} f \left( \frac{x^+ - x^-}{2} \right) e^{-ix^+P^+/2} = f \left( \frac{-x^-}{2} \right) \]

\[ e^{ix^+P^+/2} \partial^- e^{-ix^+P^+/2} = \partial^- - \partial^+ \quad (2.7) \]

so that the equation of motion for \( \psi \) takes the form:

\[ \left[ \frac{1}{2} i\gamma^+(\partial^- - \partial^+) + \frac{1}{2} i\gamma^- \partial^+ + i\gamma^\perp \cdot (\vec{\partial}_\perp + ig_0 \Gamma (\vec{r}_\perp, \frac{x^-}{2})) - m \right. \]

\[ \left. - g_S V_S(\vec{r}_\perp, \frac{x^-}{2}) - g_V \gamma^+ V^- (\vec{r}_\perp, \frac{x^-}{2}) \right] \psi = 0, \quad (2.8) \]
with $V^- = V_0$. The translated fields satisfy an equation of motion with potentials that do not depend on $x^+$. Moreover, the static potentials evaluated at $\vec{r}$ correspond to light front potentials evaluated at $(\vec{r}_{\perp} - \vec{x}^-)$. A simple derivation of this can be obtained from evaluating $z = (x^+ - x^-)/2$ at $x^+ = 0$.

That the result (2.8) is the same as the equations for $\psi_\pm$ in Ref. [10], can be seen by making a decomposition into a dynamical and a constraint equation. Multiplication of Eq. (2.8) by $\gamma^+$ from the left yields a constraint equation (no $x^+$-derivative !)

$$i\partial^+ \psi_\mp = [i\vec{\alpha}_{\perp} \cdot (\vec{\partial}_{\perp} + igV(\vec{\partial}_{\perp} \Gamma)) + \beta m + V_S] \psi_\pm$$

(2.9)

where, as usual, $\psi_\pm \equiv \frac{1}{2} \gamma^0 \gamma^\pm \psi$. Multiplication of Eq. (2.8) by $\gamma^-$ from the left yields an equation for $\psi_+$:

$$i(\partial^- - \partial^+ - igV)\psi_+ = [i\vec{\alpha}_{\perp} \cdot (\vec{\partial}_{\perp} + igV(\vec{\partial}_{\perp} \Gamma)) + \beta m + V_S] \psi_-.$$  

(2.10)

One may use the constraint equation (2.9) to eliminate $\psi_- \in$ Eq. (2.10) to obtain the equation of motion for the dynamical degrees of freedoms. The results (2.9) and (2.10) are the desired equations.

### III. SIMPLE PROBLEMS

In order to illustrate the application of the LF formalism for static external potentials, let us consider a few simple examples: a scalar field in a box with vanishing boundary conditions, a Dirac field in a “bag” with bag boundary conditions, the 3 dimensional (scalar) harmonic oscillator, and the 3 dimensional Hulthén potential.

#### A. Scalar Field in a Box

Let us first consider a box of length $L$ (extending from 0 to $L$) with vanishing boundary conditions and determine the eigenstates in the equal time framework. The two degenerate solutions to the free Klein-Gordon equation are left and right moving plane waves $\phi_\pm(x, t) = e^{-iEt}e^{\pm ikz}$ where $E = \sqrt{k^2 + m^2}$. Using the boundary conditions to match coefficients and wave number in the superposition of these two solutions one obtains the familiar solution

$$\phi_n(z, t) = e^{-iE_n t} \sin(k_n z),$$

(3.1)

where

$$k_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, ...$$

$$E_n^2 = m^2 + k_n^2.$$  

(3.2)

In a light front calculation the boundary conditions depend on $x^+$ because $z = (x^+ - x^-)/2$. This dependence can be eliminated using transformation of the form of (2.6). We start with the Klein-Gordon equation
\[(\partial^+ \partial^- + m^2)\tilde{\phi}(x^-, x^+) = 0, \quad (3.3)\]

and make the transformation
\[
\tilde{\phi} = e^{-ix^+ P^+/2} \phi \quad (3.4)
\]
to find
\[
\left(\partial^+ (\partial^- - \partial^+) + m^2\right)\phi(x^-, x^+) = 0, \quad (3.5)
\]
with (according to the translation 2.7) the boundary condition
\[
\phi(x^- = 0, -2L, x^+) = 0. \quad (3.6)
\]
We find solutions of the form
\[
\phi(x^-, x^+) = e^{-ik^- x^+ /2} e^{-ik^- x^- /2}, \quad (3.7)
\]
with the dispersion relation:
\[
k^- = \frac{m^2}{k^+} + k^+. \quad (3.8)
\]
Before taking the boundary condition into account, there are two linearly independent solutions \(e^{-ik_a^+ x^- /2}\) and \(e^{-ik_b^+ x^- /2}\), each of which has the value of \(k^-\) provided by Eq. (3.8). Imposing the boundary condition at \(x^- = 0\) leads to the form:
\[
\phi(x^+, x^-) = e^{-ik^- x^+ /2} \left( e^{ik_a^+ x^- /2} - e^{ik_b^+ x^- /2} \right). \quad (3.9)
\]
The vanishing of \(\phi\) at the other boundary \((x^- = -2L)\) implies
\[
k^+_a - k^+_b = n\frac{2\pi}{L}, \quad n = 1, 2, ... \quad (3.10)
\]
which constrains the allowed energy eigenvalues \(k^-_n\). Using Eq. (3.8) for \(k^-_a = k^-_b = k^-_n\) shows that the two independent solutions to Eq. (3.8) are related by
\[
k^+_b = \frac{m^2}{k^+_a} \quad (3.11)
\]
and thus the quantization condition for the LF momenta in a stationary box can be written as \(k^+_a, n - \frac{m^2}{k^+_a, n} = n\frac{2\pi}{L}\). Hence the quantized energies satisfy
\[
k^-_n^2 \equiv \left( \frac{m^2}{k^+} + k^+ \right)^2 = \left( \frac{m^2}{k^+} - k^+ \right)^2 + 4m^2
\[
= \left( n\frac{2\pi}{L} \right)^2 + 4m^2 = 4 \left[ \left( n\frac{\pi}{L} \right)^2 + m^2 \right], \quad (3.12)
\]
which is consistent with Eq. (3.2) because \(k^-_n\) is identified with \(2E_n\).
The fact that the LF-ansatz Eq. (3.9) consists of two waves — both moving in the same direction (both LF-momenta positive!) — has a very intuitive interpretation. In ordinary coordinates, the stationary solutions for a particle in a box with hard walls is described by a superposition of plane waves moving in opposite directions. Since the LF momentum of a plane wave is given by the sum of the particle’s momentum in the z-direction and its energy, the two waves moving in opposite directions will have different LF-momenta. However, both LF momenta are positive, since the ET energy is larger than the absolute value of the momentum.

B. 1+1 Dimensional Bag Model

As a second example, let us consider the bag model. For simplicity, we will only consider the 1+1 dimensional case here. In 1+1 dimensions, Dirac matrices are $2 \times 2$ matrices and we choose to work in the chiral representation

$$
\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
$$

In order to solve the Dirac equation for a “bag”

$$
i (\gamma_0 \partial_t - \gamma_1 \partial_x) \psi = m\psi \quad \text{(inside)}
$$

$$
n^\mu \gamma_\mu \psi = i\psi \quad \text{(at boundary)},
$$

where $n^\mu$ is normal to the surface, we make a stationary wave ansatz

$$
\psi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} e^{-iEt},
$$

yielding

$$
E\chi_2 - i\chi_2' = m\chi_1 \quad \text{(inside)}
$$

$$
E\chi_1 + i\chi_1' = m\chi_2 
$$

$$
\chi_2 = i\chi_1 \quad \text{($x = L$)}
$$

$$
\chi_2 = -i\chi_1 \quad \text{($x = 0$)}.
$$

The case $m = 0$ is particularly easy to solve, and one finds

$$
\chi_1^n = e^{-iE_n x}
$$

$$
\chi_2^n = -ie^{iE_n x}
$$

$$
E_n = \left(n + \frac{1}{2}\right) \frac{\pi}{L} \quad n = 0, 1, ...
$$

Turn now to the light front calculation. We use the same representation for the $\gamma$-matrices as above (3.13), yielding

$$
\gamma^+ = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \quad \gamma^- = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.
$$
Our version of the Dirac equation is given in 1+1 dimensions (after the transformation (2.6)) leads to the 1+1 dimensional version of (2.8) as

\[
i \partial^- \psi_+ = i \partial^+ \psi_+ + m \psi_-
\]
\[
i \partial^+ \psi_- = m \psi_+.
\]  
(3.19)

We look for a solution of the form:

\[
\psi(x) = e^{-ip^-x^+/2} \psi(x^-),
\]  
(3.20)

which gives for (3.19):

\[
p^- \psi_+ = i \partial^+ \psi_+ + m \psi_-
\]
\[
i \partial^+ \psi_- = m \psi_+,
\]  
(3.21)

and therefore

\[
p^- \psi_+ = \left( \frac{m^2}{i \partial^+} + i \partial^+ \right) \psi_+.
\]  
(3.22)

Using plane wave solutions of the form \( \psi_+ = e^{-ik^+ x^-/2} \) or \( \psi_+ = e^{-ik_b^+ x^-/2} \), leads immediately to \( p^- = \frac{m^2}{k_{a,b}^+} + k_b^+ \), so that once again \( k_b^+ = m^2/k_a^+ \). The boundary conditions of Eq. (3.14) are given in light front coordinates as

\[
\psi_- = i \psi_+ \quad (x^- = -2L)
\]
\[
\psi_- = -i \psi_+ \quad (x^- = 0).
\]  
(3.23)

For \( m = 0 \) the solutions to Eqs. (3.21) are given by

\[
\psi_+ = e^{-ip^-x^-/2}
\]
\[
\psi_- = const,
\]  
(3.24)

so the boundary conditions (3.23) lead immediately to

\[
p^-_n = \left( n + \frac{1}{2} \right) \frac{2\pi}{L},
\]  
(3.25)

which is \( 2E_n \).

C. Three Dimensional Harmonic Oscillator

In ordinary coordinates, the Klein-Gordon equation for a 3 dimensional relativistic oscillator

\[
E^2 \phi = \left[ \vec{p}^2 + m^2 + \kappa \vec{x}^2 \right] \phi
\]  
(3.26)

closely resembles the equation for a nonrelativistic harmonic oscillator and we can thus immediately write down its energy eigenvalues
\[ E_n^2 = m^2 + \omega \left( n + \frac{3}{2} \right), \]  
(3.27)

where \( \omega^2 = 4\kappa \).

The light front version of the Helmholtz equation is

\[ \left( \partial^+ \partial^+ - \nabla_\perp^2 + m^2 + \kappa(x_\perp^2 + (x^+ - x^-)^2/4) \right) \phi = 0. \]  
(3.28)

Once again we see a familiar pattern. The static potential in ET coordinates becomes a ‘time” or \( x^+ \) dependent potential in LF coordinates. Once again this dependence is of a simple form; it can be transformed away using

\[ \phi = e^{-i\frac{x^+p^+}{2}} \chi. \]  
(3.29)

The use of Eqns. (2.6) and (2.7) then leads to the result

\[ \left( (\partial^- \partial^+) \partial^+ - \nabla_\perp^2 + m^2 + \kappa(x_\perp^2 + (\frac{x^-}{2})^2) \right) \chi = 0. \]  
(3.30)

In the absence of interactions the \( P^- \) operator would consist of the usual term plus an additional \( p^+ \) operator. We look for a solution of the form:

\[ \chi(x) = e^{-ip_n x^+/2} \chi(x^-, \vec{x}_\perp). \]  
(3.31)

This, along with completing the square, leads to the result

\[ \left( -(\partial^+ + ip_n^-/2)^2 - \nabla_\perp^2 + m^2 + \kappa(x_\perp^2 + (\frac{x^-}{2})^2) \right) \chi = (p_n^-/2)^2 \chi. \]  
(3.32)

One converts the operator \( (\partial^+ + ip_n^-/2)^2 \) to \( (\partial^+)^2 \) using yet another transformation:

\[ \chi(x^-, \vec{x}_\perp) = e^{-ip_n^- x^-/4} F(x^-, \vec{x}_\perp) \]  
(3.33)

and to find

\[ \left( -(\partial^+)^2 - \nabla_\perp^2 + m^2 + \kappa(x_\perp^2 + (\frac{x^-}{2})^2) \right) F = (p_n^-/2)^2 F. \]  
(3.34)

This is the same form as the equation in the equal time coordinates and \( p_n^-/2 \) takes on the values of \( E_n \).

We seem to be getting the same results as in the ET development. So one might wonder why we are doing the light front at all. The point is that we are able to compute the light front wave functions that depend on \( x^- \), or in momentum space depend on \( p^+ \). The wave functions of the ground state is given by

\[ \chi_0(x^-, \vec{x}_\perp) = e^{ip_n^- x^-/4} N_0 \exp \left( -\frac{1}{2} \sqrt{\kappa} \left( x_\perp^2 + \frac{x^-}{4} \right) \right). \]  
(3.35)
The number density \( n_0(p_\perp, p^+) \) is defined as the square of the momentum space version of \( \chi_0 \). This quantity is accessible in high energy proton and electron nuclear quasi-elastic reactions. It is useful to define the light front variable

\[
\alpha \equiv p^+ / (\vec{p}_0 / 2).
\]  

(3.36)

Then one may easily determine that

\[
n_0(p_\perp, \alpha) = \tilde{N}_0 e^{-\frac{p^2}{\sqrt{\kappa}}} e^{-\frac{(p_0^+(\alpha-1))^2}{4\sqrt{\kappa}}}. \tag{3.37}
\]

Note that one finds the same \( p_\perp \) distribution for each value of the variable \( \alpha \). This is not a general feature of light-front wave functions, as we show in the next sub-section.

In an exact calculation, the number density should vanish for values of \( \alpha \) that are not between 0 and 1. This is referred to as the support, which we examine here. How large can the value of \( \alpha \) be? For large values of the particles mass \( m \), \( (m \gg \kappa^{1/4}) \) (which corresponds to the situation of nuclear physics in which the product of the nucleon mass and the nuclear radius is a very large number) the value of \( \alpha \) must always be close to unity. If \( m \to 0 \) then the behavior is \( \sim e^{-3(\alpha-1)^2} \). We may better understand this factor by considering an extremely simple model—the nucleon consists of three massless quarks moving in a harmonic oscillator potential. We shall compute the quark distribution function \( q(x) \), which here is the same as \( n_0 \) integrated over \( p_\perp \) and evaluated as a function of the Bjorken variable, \( x \). For massless quarks, Eq. (3.27) gives

\[
\left( \frac{\vec{p}_0}{2} \right)^2 = 3\sqrt{\kappa}. \tag{3.38}
\]

Furthermore, in the target rest frame \( P_N^+ \) is the mass of the nucleon \( 3\vec{p}_0/2 \). Thus using Eq. (3.38) \( \alpha = 3p^+/P_N^+ = 3x \). The last equality is from the parton model in which the ratio of the quark and target plus momenta is \( \frac{Q^2}{2m_{\text{NN}}\nu} = x \). Thus, we find

\[
q(x) \propto e^{-27(x-1/3)^2}. \tag{3.39}
\]

In this limit of massless quarks, the value of \( x \) (in this un-evolved) quark distribution function is constrained to be very close to 1/3. One would naively expect the value of \( x \) to easily exceed unity, since we have used a mean field model for a three-quark system. This does not occur. If \( x = 1 \) the value of the \( q \) is \( e^{-12} \) its peak value, which is reasonably small. If \( x \) approaches 0, one finds a factor \( e^{-3} \), instead of the required 0 so there is an inherent inaccuracy of some 5%. If one included a non-zero value of the quark mass, the support properties would be improved because

\[
q(x) \propto e^{-\left(\frac{2m^2}{\sqrt{\kappa}} + 27\right)(x-1/3)^2}. \tag{3.40}
\]

In constituent quark models \( m^2 = \sqrt{\kappa} \), so that at \( x = 0 \) one finds a factor of \( e^{-4} \). In nuclear physics \( m^2 \) is of the order of \( (5R_A/\text{Fm})^2 \) which taking \( R_A = 4 \) Fm, yields \( e^{-400} \) at \( x = 0 \). In that case, there is no problem with the support.
It is of interest to compute the electromagnetic form factor of the ground state. This quantity has been often measured in elastic electron scattering, and the sizes of nuclei have been determined as one of the classic achievements of nuclear physics. Interest in this topic has been revised because of a recent proposal to Jefferson Laboratory to use parity-violating electron scattering to measure the neutron radius [11]. A high precision is needed and can be obtained provided one knows the proton distribution. Therefore one needs to examine the influence of small effects such as relativistic corrections. One works using a reference frame in which the plus component of the four vector \( q^\mu \) of the virtual photon vanishes, so that \( Q^2 = -q^2 = q^2_\perp \). In this case the form factor \( F(Q^2) \) (matrix element of the plus component of the electromagnetic current operator) is given by

\[
F(Q^2) = \int d^2 p_\perp d\alpha \chi_0(\alpha, p_\perp) \chi_0(\alpha, p_\perp - \alpha q_\perp),
\]

in which the influence of relativity appears in the integral over \( \alpha \) and the factor \( \alpha \). For the harmonic oscillator ground state we find:

\[
F(Q^2) = N \int d^2 p_\perp d\alpha e^{-\left(\frac{p_\perp^2 + \frac{Q^2}{4\sqrt{\kappa}}}{\sqrt{\kappa}}\right)} e^{-\frac{p_0^{-2}}{4} \alpha^2 (\alpha - 1)^2},
\]

(3.42)

where

\[
\frac{p_0^{-2}}{4} = m^2 + 3\sqrt{\kappa}.
\]

(3.43)

Our purpose here is the study of nuclear physics, so we are interested in the non-relativistic limit and the corrections to it. To this end, we define a variable \( p_z \) using

\[
\alpha = 1 + \frac{p_z}{m},
\]

(3.44)

The non-relativistic limit of (3.42) is obtained by letting \( m \) approach infinity. Then \( \frac{p_0^{-2}}{4} = m^2 \) and we find

\[
F_{NR}(Q^2) = N_{NR} \int \frac{d^3 p}{m} e^{-\left(\frac{p_\perp^2 + \frac{Q^2}{4\sqrt{\kappa}}}{\sqrt{\kappa}}\right)} e^{-\frac{p_0^{-2}}{4} \alpha^2 (\alpha - 1)^2}.
\]

(3.45)

The mean square radius \(-6 \frac{dF(Q^2)}{dQ^2} |_{Q^2=0}\), is given by

\[
R_{NR}^2 = 3 \frac{1}{2\sqrt{\kappa}}.
\]

(3.46)

The leading corrections to this will be of order \( p_z^2/m^2 \sim \sqrt{\kappa}/m^2 \). We define a semi-relativistic limit \( SR \) via the use of (3.42) and keeping the leading correction terms. Performing the straightforward evaluations leads to the result:

\[
\delta \equiv \frac{R_{SR}^2 - R_{NR}^2}{R_{NR}^2} = \frac{3}{2} \frac{\sqrt{\kappa}}{\sqrt{\pi} m^2},
\]

(3.47)

or

\[
\delta \approx \frac{\sqrt{\pi}}{12A^{2/3}}.
\]

(3.48)

This corresponds to very small (0.004) effects for large nuclei \( A \sim 200 \).
D. Light Front Hulthéen Wave Function

Any static potential of the form $V(x^2 = x_\perp^2 + z^2)$ can be solved on the light front. The transformation (2.6) corresponds to including the $p^+$ term in the $x^+$ development operator and a simple prescription of replacing $z$ by $-x^-/2$ in $V$. We present here the solution for the Hulthéen potential. This allows us to demonstrate an interesting contrast between the implications of different forms of potentials. In the equal time formulation we have the wave equation:

$$E^2\phi = \left[\vec{p}^2 + m^2 + V^H(\vec{x}^2)\right]\phi,$$

in which $V^H$ is chosen so that the lowest energy solution is

$$\phi(r) = N(e^{-ar} - e^{-br}),$$

where $b > a$. The eigenenergy is given by

$$E = \sqrt{m^2 - a^2}.$$

The light front version of the wave equation is

$$\frac{p^-}{4}\chi(x^-, \vec{x}_\perp) = \left[\left(-2i\frac{\partial}{\partial x^-}\right)^2 + p^2_\perp + V^H \left(x_\perp^2 + \frac{x^2}{4}\right)\right]\chi(x^-, \vec{x}_\perp).$$

The lowest value of $p^-/2$ is clearly the same as $E$ of Eq. (3.51), and the wave function is given by

$$\chi_0(x^-, \vec{x}_\perp) = e^{ip^-x^-/4}N^H_0 \left[\exp\left(-a\sqrt{x_\perp^2 + \frac{x^2}{4}}\right) - \exp\left(-b\sqrt{x_\perp^2 + \frac{x^2}{4}}\right)\right].$$

The momentum distribution $n^H_0(p_\perp, p^+)$ obtained here provides an interesting contrast with that of the harmonic oscillator (3.37). We find

$$n^H_0(p_\perp) = \tilde{N}^H_0 \left[\frac{1}{((a^2 + p^2_\perp + (p^-_\perp/2)^2(\alpha - 1))^2 - ((b^2 + p^2_\perp + (p^-_\perp/2)^2(\alpha - 1))^2}\right].$$

It is clear that one finds a different $p_\perp$ distribution for each value of $\alpha$. The distribution is a broader function of $p_\perp$ for larger values of $\alpha$; see the Fig. [1]. The results in the figure are obtained using $m=.94$ GeV, $E=.932$ GeV and $b = 5a$. An experimental hint of such a behavior has been found recently [12].
FIG. 1. Light front momentum distribution as a function of $\alpha$ and $p_{\perp}$.

IV. TILTED LIGHT-FRONT COORDINATES

We have worked out several examples involving potentials that are independent of time in equal time coordinates. This leads to an $x^+$-dependent interaction when light front coordinates are used. However this dependence is very simple, and in each case we have removed this by using a version of Eqs. (2.6). In each case we have derived a light front wave equation in which the kinetic energy has the same form as the standard LF-kinetic energy plus an additional term linear in the momentum which is identical to the recoil term in the static source formalism on the LF! It is worthwhile to see if there is a more general way to remove this dependence, once and for all, by finding a set of coordinates in which static sources in equal time coordinates are also described by static sources in a coordinate system that is very much like that of the light front.

For this purpose, we introduce new “tilted” LF-coordinates

\[
\begin{align*}
\tau &\equiv x^0 + x^3 \\
\zeta &\equiv -x^3,
\end{align*}
\]

i.e.

\[
\begin{align*}
x^0 &= \tau + \zeta \\
x^3 &= -\zeta
\end{align*}
\]

with $\vec{x}_\perp$ as usual. These coordinates very much resemble LF-coordinates, since (Fig. 2)
• surfaces of constant $\tau$ are the usual LF hypersurfaces and thus quantization is very similar to LF quantization.

• correlation functions at $\tau = 0$ in the $\zeta$ direction yield the usual LF distributions. Therefore, a lot of the familiar LF-phenomenology (e.g. structure functions and wave functions) can still be used.

However, in contradistinction to light-front coordinates, static sources in a rest-frame are also described by static sources in above tilted coordinates, since

$$
\begin{align*}
\partial_0 &= \partial_\tau \\
\partial_3 &= \partial_\tau - \partial_\zeta,
\end{align*}
$$

i.e. if $\partial_0 V = 0$ then $\partial_\tau V = 0$. For a static ($\partial_t V = 0$) potential, one thus finds for the static potential $V(\vec{x}_\perp, x^3)$

$$
V(\vec{x}_\perp, x^3) = V(\vec{x}_\perp, -\zeta)
$$

Furthermore, using Eq. (4.3), one finds for the longitudinal part of the kinetic energy operator

$$
\partial_0^2 - \partial_3^2 = 2\partial_\tau \partial_\zeta - \partial_\zeta^2.
$$

The Lagrangian for a scalar field interacting with an external field can then be written as

$$
\mathcal{L} = \frac{1}{2} \left[ \partial_{\mu} \phi \partial^{\mu} \phi - \left( m^2 + V(\vec{x}) \right) \phi^2 \right] = \partial_{\tau} \phi \partial_{\zeta} \phi - \frac{1}{2} \left( \partial_{\zeta} \phi \right)^2 - \frac{1}{2} \left( \partial_{\perp} \phi \right)^2 - \frac{m^2 + V(\vec{x})}{2} \phi^2
$$
in tilted coordinates, the canonical momenta are the longitudinal “space”-derivatives of the fields

\[ \Pi \equiv \frac{\partial L}{\partial (\partial_\tau \phi)} = \partial_\zeta \phi \equiv \partial_- \phi \]  \quad (4.7)

— just like in genuine LF-coordinates. This should not come as a surprise, since the constant “time” surfaces in our tilted coordinates and in LF coordinates are identical (Fig. 2).

The kinetic energy operator differs from the one in LF-coordinates. Using Eq. (4.3) in the form

\[ 2\mathcal{E}\mathcal{P} - \mathcal{P}^2 - \mathcal{P}_\perp^2 - m^2 = 0, \]

where \( \mathcal{E} \) is conjugate to \( \tau \) and \( \mathcal{P} \) is conjugate to \( \zeta \), one finds the kinetic energy of a free particle in tilted coordinates:

\[ \mathcal{E} = \frac{m^2 + \mathcal{P}_\perp^2}{\mathcal{P}} + \mathcal{P}. \]  \quad (4.8)

For non-interacting systems, this is identical to the expression derived in LF-coordinates including the “Lagrange multiplier” [10].

The point of using the tilted LF coordinates is that the term \( \mathcal{P} \) is included automatically. The energies \( \mathcal{E} \) obtained using tilted LF coordinates are the same as \( P^- \) obtained in the usual LF dynamics including the Lagrange multiplier term. We expect that the \( \mathcal{E} \) of tilted LF must be the same as the \( P^- \) of LF because both formulations use the same quantization (i.e. equal ‘time’) hypersurfaces. The only place where they differ is the space direction, where the difference between the spatial coordinates \( x^- \) and \( \zeta \) is a shift \( x^0 \). This is a simple translation — hence the term linear in the momentum in the tilted coordinates Hamiltonian. However, it turns out that in the mean field approach one needs to introduce a Lagrange multiplier term to fix the momentum, and the two Hamiltonians have an identical form.

We shall see how this works out by solving the same simple problems as above using the tilted LF dynamics.

**V. SIMPLE PROBLEMS ON THE TILTED FRONT**

In order to illustrate the application of the LF formalism for static external potentials, let us consider a few simple examples: a scalar field in a box with vanishing boundary conditions, the 3 dimensional (scalar) harmonic oscillator and a Dirac field in a “bag” with bag boundary conditions.

**A. Scalar field in a box**

In order to derive the tilted LF solution for this example, we first note that vanishing boundary conditions are frame independent, i.e. a stationary box with vanishing boundary condition in an equal time framework corresponds also in tilted LF coordinates to a stationary box with vanishing boundary condition. Before taking the boundary condition into account, there are two linearly independent solutions \( e^{-ik_\perp \zeta} \) and \( e^{-ik_\parallel \zeta} \) for a given energy, which is the same according to Eqs. (4.8) as that of Eq. (3.8).
The only superposition which satisfies the requirement that the wave function vanishes at \( \zeta = 0 \) is given by

$$\phi(\zeta) = e^{-ik_a^+ \zeta} - e^{-ik_b^+ \zeta}. \tag{5.1}$$

Vanishing of \( \phi(x^-) \) at the other boundary (\( \zeta = -L \)) implies

$$k_a^+ - k_b^+ = n \frac{2\pi}{L} \quad n = 1, 2, \ldots \tag{5.2}$$

which constrains the allowed energy eigenvalues \( k^-_n \). Using Eq. (4.8) for \( k_a^- = k_b^- = k^-_n \) shows that the two independent solutions to Eq. (3.8) are related by

$$k_b^+ = \frac{m^2}{k_a^+} \tag{5.3}$$

and thus the quantization condition for the LF momenta in a stationary box can be written as \( k_a^+ - \frac{m^2}{k_a, n} = n \frac{2\pi}{L} \). Hence the quantized energies satisfy Eq. (3.12), which is consistent with Eq. (3.2) provided one identifies \( k^-_n \) with \( 2E_n \) (4.3).

### B. 1+1 Dimensional Bag Model

Making a stationary wave ansatz

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} e^{-i\mathcal{E} \tau/2}, \tag{5.4}$$

one thus writes the equation of motion for a free Dirac particle (3.14), in tilted coordinates by multiplication by either \( \gamma^+ \) or \( \gamma^- \). This gives

$$\mathcal{E}\psi_+ - i\partial_{\zeta}\psi_+ = m\psi_-$$

$$-i\partial_{\zeta}\psi_- = m\psi_+, \tag{5.5}$$

where \( \partial_{\zeta} \equiv \frac{\partial}{\partial\zeta} \). We use the second of Eqs. (5.5) in the first to to obtain the “dynamical” component:

$$\mathcal{E}\psi_+ = \left[ \frac{m^2}{i\partial_{\zeta}} + i\partial_{\zeta} \right] \psi_+. \tag{5.6}$$

As is the case for free scalars, one finds in general two linearly independent plane wave solutions for each energy \( \psi_+ = e^{-ik_a^+ \zeta} \) and \( \psi_+ = e^{-ik_b^+ \zeta} \) where \( \mathcal{E} = \frac{m^2}{k_{a,b}^+} + k_{n,n}^+ \) and \( k_{a,b}^+ = \frac{m^2}{k_a} \). Note that the boundary condition (3.14) mixes \( \psi_+ \) and \( \psi_- \)

$$\psi_- = i\psi_+ \quad (\zeta = -L)$$

$$\psi_- = -i\psi_+ \quad (\zeta = 0). \tag{5.7}$$

This mixing should not come as a surprise, since the boundary condition arises from assuming an infinite mass for the fermion outside the bag.
For $m = 0$ the solutions to Eqs. (5.3) read
\[ \psi_+ = e^{-iE\zeta} \]
\[ \psi_- = \text{const.} \]  
(5.8)
and thus from the boundary conditions (5.7)
\[ E_n = \left(n + \frac{1}{2}\right) \frac{2\pi}{L}. \]  
(5.9)
Using again $E_n = 2E_n$, we find that the spectrum obtained in tilted LF coordinates is consistent with Eq. (3.17).

**C. Three Dimensional Harmonic Oscillator**

In order to solve the Helmholtz equation in tilted coordinates
\[ \mathcal{PE} \phi = \left[m^2 + \kappa (\vec{x}^2_\perp + \zeta^2) + \vec{p}^2_\perp + \mathcal{P}^2\right] \phi, \]  
(5.10)
we first complete the square, yielding
\[ \frac{\mathcal{E}^2}{4} \phi = \left(\mathcal{P} - \mathcal{E}\right)^2 \phi + \left[m^2 + \kappa (\vec{x}^2_\perp + \zeta^2) + \vec{p}^2_\perp\right] \phi. \]  
(5.11)
First of all, we note that a shift in $\mathcal{P}$, included by multiplying the wave function $\phi$ by a suitably chosen phase factor can be made to absorb the dependence of the r.h.s. of Eq. (5.11) on $\mathcal{E}$. Furthermore, the transverse and longitudinal dynamics in Eq. (5.11) completely separates, yielding
\[ \frac{\mathcal{E}^2}{4} = m^2 + \omega_L \left(n_L + \frac{1}{2}\right) + \omega_\perp (n_\perp + 1), \]  
(5.12)
where $\omega_L = \omega_\perp = 4\kappa = \omega$. Taking into account that $\mathcal{E}_n = 2E_n$ we thus find that the relativistic harmonic oscillator formulated in an equal time framework and in tilted LF coordinates yield identical spectra.

**D. Dirac Equation with Static Potential**

We express Eq. (2.5) in tilted coordinates to find:
\[ \left[i\gamma^+ \frac{\partial}{\partial \tau} - i\gamma^3 \frac{\partial}{\partial \zeta}\right] + i\vec{\gamma}_\perp \cdot \left(\vec{\sigma}_\perp + g_\sigma \vec{\sigma}_\perp \mathbf{\Gamma}(\vec{r}_\perp, -\zeta)\right) - m \]
\[ -V_S(\vec{r}_\perp, -\zeta) - \gamma^+ V_0(\vec{r}_\perp, -\zeta) \right] \psi' = 0. \]  
(5.13)
The potentials $V_S$ and $V_0$ are functions of $r^2_\perp + \zeta^2$, so we expect rotational invariance to be immediate. Indeed, the above is just the standard Dirac equation, except for the appearance of $\gamma^+$ instead of $\gamma^0$. This can immediately be removed by using the transformation of Eq. (3.44) of Ref. [10]. Thus the spectra of the Dirac equation in tilted coordinates is going to be the same as that of the usual equal time formulation.
VI. SUMMARY

The three features discussed in the introduction emerge naturally from a more general notion that static sources in the usual ET formulation become static sources in the LF formulation if one makes an appropriate transformation (2.6). The resulting LF formulation for scalar interactions can be summarized as simply including a $p^+$ in the $x^+$ development operator, and evaluating the potentials using the replacement $z \rightarrow -x^-/2$. The resulting simplicity allows us solve any problem involving a spherically symmetric static source. An application of this in Sect. V.D enables us to obtain a momentum distribution which increases in width as a function of $p_\perp$ for increasing values of $p^+$. A hint of such behavior has recently been observed [12].

We note that the spectrum condition—the feature that light front mode functions have support only for positive values of plus-momentum—has not been maintained in any of the solutions we present here. The desire to maintain this consistency with the original field theory led us to implement a new numerical procedure in Ref. [10]. In the case of the harmonic oscillator, we see that for the general conditions of nuclear physics, the spectrum condition is maintained, even though the solution procedure ignores this condition. The detailed study of the how important maintaining the spectrum condition, for models other than the harmonic oscillator, is will be a topic of future investigation. The numerical results of Ref. [10], for nucleons in nuclei, strongly indicate that there is no strong need to maintain the spectrum condition. The presence of a 5% effect for $q(x = 0)$ indicates that the maintaining the spectrum condition is important for massless quarks.

The simplicity of the results for the simple modeled considered, along with the need to automate the transformation procedures, suggests the use of a new set of coordinates—tilted coordinates in which the $p^+$ term is in from the beginning and in which static sources in the ET formulation are also static sources in the tilted LF formulation. For the case of the simple models considered here, this new formulation is actually easier to apply than the usual LF formulation. Whether or not this simplicity survives more detailed problems is a matter for future investigation.

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