CROSSED MODULES, NON-ABELIAN EXTENSIONS OF ASSOCIATIVE CONFORMAL ALGEBRAS AND WELLS EXACT SEQUENCES

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Abstract. In this paper, we introduce the notions of crossed module of associative conformal algebras, 2-term strongly homotopy associative conformal algebras, and discuss the relationship between them and the 3-th Hochschild cohomology of associative conformal algebras. We classify the non-abelian extensions by introducing the non-abelian cohomology. We show that non-abelian extensions of an associative conformal algebra can be viewed as Maurer-Cartan elements of a suitable differential graded Lie algebra, and prove that the Deligne groupoid of this differential graded Lie algebra corresponds one to one with the non-abelian cohomology. Based on this classification, we study the inducibility of a pair of automorphisms about a non-abelian extension of associative conformal algebras, and give the fundamental sequence of Wells in the context of associative conformal algebras. Finally, we consider the extensibility of a pair of derivations about an abelian extension of associative conformal algebras, and give an exact sequence of Wells type.

1. Introduction

The notion of a conformal algebra encodes an axiomatic description of the operator product expansion of chiral fields in conformal field theory. The theory of Lie conformal algebras appeared as a formal language describing the algebraic properties of the operator product expansion in two-dimensional conformal field theory ([8, 33, 34]). The structure of a Lie conformal algebra gives an axiomatic description of the operator product expansion of chiral fields in conformal field theory. Associative conformal algebras naturally come from representations of Lie conformal algebras. Moreover, some Lie conformal algebras appeared in physics are embeddable into an associative one [46, 47]. The structure theory and representation theory of associative conformal algebras have attracted the attention of many scholars and achieved a series of results, see [9, 10, 35, 36, 44, 45].

The concept of crossed modules was introduced by J.H.C. Whitehead in the late 1940s [56]. Crossed modules of Lie algebras first appeared in the work of Gerstenhaber [25]. Later, the crossed modules of various algebraic structures have been extensively studied [7, 12, 13, 15, 37, 54]. They have therefore been used in homotopy theory, homological algebra, non-abelian cohomology, algebraic K-theory, ring theory, combinatorial group theory and applications of the related algebra [11, 29, 40]. Crossed modules of groups and Lie algebras turn up in the book of homological algebra as interpretations of cohomology classes of cohomological degree 3. It is well-known that the category of crossed modules of Lie algebras is equivalent to the category of strict Lie 2-algebras [2]. Here we introduce crossed modules of associative conformal algebras and consider conformal analogue of these results.

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Extension problem is a famous and still open problem. The non-abelian extension theory is a relatively general one in various kinds of extensions. In [22], Eilenberg and MacLane first considered non-abelian extensions of abstract groups. Subsequently, the non-abelian extension theory has been widely studied in various fields of mathematics, to which a vast literature was devoted, such as Lie groups [43], Leibniz algebras [14, 59], Lie algebras [23, 24, 31], associative algebras [1, 27], 3-Lie algebras [48]. Lie 2-algebras [51], Hom-Lie algebras [49, 50], Rota-Baxter algebras [17, 18], groupoids [41], etc. Another interesting study related to extensions of algebraic structures is given by the inducibility of pair of automorphisms. Such study was first initiated by Wells in extensions of abstract groups [55]. In [55], the author defined a map, known as Wells map, and constructed a short exact sequence (the fundamental short exact sequence of Wells) connecting various automorphism groups. The Wells map and the inducibility of a pair of automorphisms about an abelian extension of Lie algebras and associative algebras. W e introduce the notion of crossed modules of associative conformal algebras. Moreover, we also relate the 3-th Hochschild cohomology of associative conformal algebras and crossed modules of associative conformal algebras. Moreover, we also relate the 3-th Hochschild cohomology of associative conformal algebras by means of a particular kind of crossed modules. Second, we consider non-abelian extensions of associative conformal algebras. For given two associative conformal algebras $A$ and $B$, we define the non-abelian 2-cocycle on $B$ with values in $A$, and give an equivalence relation on the set $\mathcal{Z}^2_{\text{nab}}(B, A)$ of all the non-abelian 2-cocycle on $B$ with values in $A$. The non-abelian cohomology of $B$ with values in $A$, denoted by $\text{HH}^2_{\text{nab}}(B, A)$, is the quotient of $\mathcal{Z}^2_{\text{nab}}(B, A)$ by this equivalence relation. By giving a 1-1 correspondence between the equivalence class of non-abelian extension of $B$ by $A$ and the equivalence class of $\mathcal{Z}^2_{\text{nab}}(B, A)$, we prove that the non-abelian extensions of $B$ by $A$ can be classified by $\text{HH}^2_{\text{nab}}(B, A)$. For this classification, we can also use the Maurer Cartan elements of a suitable differential graded Lie algebra to characterize. For the direct product conformal algebra $A \oplus B$, there is a differential graded Lie algebra $(C^{*+1}(A \oplus B, A \oplus B), [-, -], d_*)$ on the Hochschild type complex of $A \oplus B$. We construct a differential graded Lie subalgebra $\mathcal{L}$ of $(C^{*+1}(A \oplus B, A \oplus B), [-, -], d_*)$, and show that the Maurer Cartan elements of $\mathcal{L}$ control the non-abelian extensions of $B$ by $A$. Based on this results, we define the Wells map $\mathcal{W} : \text{Aut}(A) \times \text{Aut}(B) \to \text{HH}^2_{\text{nab}}(B, A)$, and give a necessary and sufficient condition for a pair of automorphisms in $\text{Aut}(A) \times \text{Aut}(B)$ to be inducible by the Wells map. Moreover, using the Wells map, we obtain the Wells short exact sequence connecting of various automorphism groups and cohomology in the context of associative conformal algebras. Since the non-abelian extension is the generalization of abelian extension, here we also give the corresponding results of abelian
extension for each obtained results of non-abelian extension. Moreover, for an abelian extension of associative conformal algebras, we also define a Wells map, give a necessary and sufficient condition for a pair of derivations to be extensible, and obtain an exact sequence of Wells type.

The paper is organized as follows. In Section 2, we recall the notions of associative conformal algebras, and the differential graded Lie algebra structure on the Hochschild complex of associative conformal algebras. In Section 3, we introduce the notions of crossed module of associative conformal algebras and 2-term strongly homotopy associative conformal algebras. We will discuss the relationship between them and the 3-th Hochschild cohomology of associative conformal algebras with coefficient in bimodules. In Section 4, we define a non-abelian cohomology group, and show that the non-abelian extensions can be classified by the non-abelian cohomology group. In Section 5, we construct a differential graded algebra $L$ for two associative conformal algebras $A$ and $B$. We show that the non-abelian cocycles are in bijection with the Maurer-Cartan elements of $L$, and get that the equivalence relation on $\mathbb{Z}^2_{nab}(B,A)$ can be interpreted as gauge equivalence relation on the set $MC(L)$ of Maurer Cartan elements of $L$. In Section 6, we study the inducibility of a pair of automorphisms about a non-abelian extension of associative conformal algebras by the Wells map, and obtain the fundamental sequence of Wells in the context of associative conformal algebras. In Section 7, for an abelian extension of associative conformal algebras, we consider the extensibility of a pair of derivations about this extension, and give an exact sequence of Wells type.

Throughout this paper, we fix $\mathbb{C}$ an algebraical closed field and characteristic zero (for example, the field of complex numbers). All vector spaces are $\mathbb{C}$-vector spaces, all linear maps and bilinear maps are $\mathbb{C}$-linear, all tensor product are over $\mathbb{C}$, unless otherwise specified. For any vector space $V$ and variable $\lambda$, we use $V[\lambda]$ to denote the set of polynomials of $\lambda$ with coefficients in $V$.

2. ASSOCIATIVE CONFORMAL ALGEBRAS AND HOCHSCHILD COHOMOLOGY

We recall the notions of associative conformal algebras, conformal bimodule over associative conformal algebras, and the Hochschild cohomology of an associative conformal algebra with coefficients in a bimodule. For the details see [3, 19, 4, 30].

Definition 2.1. A conformal algebra $A$ is a $\mathbb{C}[\partial]$-module endowed with a bilinear map $\cdot : (\cdot) : A \times A \to A[\lambda]$, $(a, b) \mapsto a (\lambda) b$ satisfying

$$\partial a (\lambda) b = -\lambda a (\lambda) b, \quad a (\lambda) \partial b = (\partial + \lambda) a (\lambda) b,$$

for any $a, b \in A$. An associative conformal algebra $A$ is a conformal algebra satisfying

$$(a (\lambda) b) (\lambda + \mu) c = a (\lambda) (b (\mu) c),$$

for any $a, b, c \in A$.

Let $\langle A, \cdot \rangle$, $\langle B, \cdot \rangle$ be two associative conformal algebras. A $\mathbb{C}[\partial]$-module homomorphism $f : A \to B$ is called a homomorphism of associative conformal algebras if for any $a, b \in A$, $f(a (\lambda) b) = f(a) (\lambda) f(b)$. A homomorphism $f : A \to B$ is said to be an isomorphism if $f$ is a bijection. For an associative conformal algebra $A$, we denote the automorphism group by $\text{Aut}(A)$. 
Example 2.2. Let \((A, \cdot)\) be an associative algebra. Then \(\text{Cur}(A) = \mathbb{C}[\partial] \otimes A\) is an associative conformal algebra with the following \(\lambda\)-product:
\[
(p(\partial)a)_{(\lambda)} (q(\partial)b) = p(-\lambda) q(\lambda + \partial)(a \cdot b),
\]
for any \(p(\partial), q(\partial) \in \mathbb{C}[\partial]\) and \(a, b \in A\).

Now we recall that definition of left (or right) modules over an associative conformal algebra.

Definition 2.3. A (conformal) left module \(M\) over an associative conformal algebra \(A\) is a \(\mathbb{C}\)-bilinear map \(A \times M \to M[\lambda]\), \((a, v) \mapsto a \triangleright_{(\lambda)} v\), satisfying the following axioms:
\[
(\partial a) \triangleright_{(\lambda)} v = -\lambda a \triangleright_{(\lambda)} v, \quad a \triangleright_{(\lambda)} (\partial v) = (\partial + \lambda)(a \triangleright_{(\lambda)} v),
\]
\[
(a \triangleright_{(\lambda)} b) \triangleright_{(\lambda+\mu)} v = a \triangleright_{(\lambda)} (b \triangleright_{(\mu)} v),
\]
for any \(a, b \in A\) and \(v \in M\). We denote it by \((M, \triangleright)\).

A (conformal) right module \(M\) over an associative conformal algebra \(A\) is a \(\mathbb{C}\)-bilinear map \(M \times A \to M[\lambda]\), \((v, a) \mapsto v \triangleleft_{(\lambda)} a\), satisfying:
\[
(\partial v) \triangleleft_{(\lambda)} a = -\lambda v \triangleleft_{(\lambda)} a, \quad v \triangleleft_{(\lambda)} (\partial a) = (\partial + \lambda)(v \triangleleft_{(\lambda)} a),
\]
\[
(v \triangleleft_{(\lambda)} a) \triangleleft_{(\lambda+\mu)} b = v \triangleleft_{(\lambda)} (a \triangleleft_{(\mu)} b),
\]
for any \(a, b \in A\) and \(v \in M\). We denote it by \((M, \triangleleft)\).

A (conformal) \(A\)-bimodule is a triple \((M, \triangleright, \triangleleft)\) such that \((M, \triangleright)\) is a left \(A\)-module, \((M, \triangleleft)\) is a right \(A\)-module, and they satisfy
\[
(a \triangleright_{(\lambda)} v) \triangleleft_{(\lambda+\mu)} b = a \triangleright_{(\lambda)} (v \triangleleft_{(\mu)} b),
\]
for any \(a, b \in A\) and \(v \in M\).

Let \(A\) be an associative conformal algebra. Define two bilinear maps \(\triangleright^{A}, \triangleleft^{A} : A \otimes A \to A\) by \(a \triangleright^{A}_{(\lambda)} b = a \triangleright_{(\lambda)} b\) and \(b \triangleleft^{A}_{(\lambda)} a = b \triangleleft_{(\lambda)} a\) for all \(a, b \in A\). Then \((A, \triangleright^{A}, \triangleleft^{A})\) is a bimodule of \(A\), and it is called the regular bimodule of \(A\).

Next, we recall the Hochschild type cohomology of an associative conformal algebra. In 1999, Bakalov, Kac and Voronov first gave the definition of Hochschild cohomology of associative conformal algebras [4]. This definition is a conformal analogue of Hochschild cohomology of associative algebras. In [19], for the case of Lie conformal algebras, the definition was improved by taking \(n - 1\) variables. Following this idea, we define the Hochschild cohomology for an associative conformal algebra \(A\) by a bimodule \(M\). We denote \(C^{0}(A, M) = M/\partial M, C^{1}(A, M) = \text{Hom}_{\mathbb{C}[\partial]}(A, M)\), the set of \(\mathbb{C}[\partial]\)-module homomorphisms from \(A\) to \(M\), and for \(n \geq 2\), the space of \(n\)-cochains \(C^{n}(A, M)\) consists of all conformal sesquilinear maps from \(A^\otimes n\) to \(M[\lambda_{1}, \ldots, \lambda_{n-1}]\), i.e., the \(\mathbb{C}\)-linear maps
\[
\varphi_{\lambda_{1}, \ldots, \lambda_{n-1}} : A^\otimes n \longrightarrow M[\lambda_{1}, \ldots, \lambda_{n-1}],
\]
such that
\[
\varphi_{\lambda_{1}, \ldots, \lambda_{n-1}}(a_{1}, \ldots, \partial a_{i}, \ldots, a_{n}) = -\lambda_{i} \varphi_{\lambda_{1}, \ldots, \lambda_{n-1}}(a_{1}, \ldots, a_{n}),
\]
for \(i = 1, 2, \ldots, n - 1\) and
\[
\varphi_{\lambda_{1}, \ldots, \lambda_{n-1}}(a_{1}, \ldots, a_{n-1}, \partial a_{n}) = (\partial + \lambda_{1} + \cdots + \lambda_{n-1}) \varphi_{\lambda_{1}, \ldots, \lambda_{n-1}}(a_{1}, \ldots, a_{n}).
\]
The differentials are defined by $d_0 : M/\partial M \to \text{Hom}_{\mathbb{C}[\partial]}(A, M)$,
\[ d_0(v + \partial M)(a) = (a \triangleright_{(-1)\partial} v - v \triangleleft_{(1)\partial} a) |_{a=0}, \]
and for $n \geq 1$,
\[ d_n(\varphi)_{a_1, \ldots, a_n}(a_1, \ldots, a_{n+1}) = a_1 \triangleright_{(1)} \varphi_{a_2, \ldots, a_n}(a_2, \ldots, a_{n+1}) \]
\[ + \sum_{i=1}^{n} (-1)^i \varphi_{a_1, \ldots, a_{i-1}, a_i+\partial, a_i+\partial, a_i+\partial, \ldots, a_n}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, a_{i+2}, \ldots, a_{n+1}) \]
\[ + (-1)^{n+1} \varphi_{a_1, \ldots, a_{n+1}}(a_1, \ldots, a_n) \triangleleft_{(1)+\partial} a_{n+1}. \]
One can verify that the operator $d_n$ preserves the space of cochains and $d_{n+1} \circ d_n = 0$. The cochains of an associative conformal algebra $A$ with coefficients in a bimodule $M$ form a complex $(C(A, M), d_n)$, called the Hochschild complex. We denote the space of $n$-cocycles by $Z^n(A, M) = \{ \varphi \in C^n(A, M) | d_n(\varphi) = 0 \}$, and the space of $n$-coboundaries by $B^n(A, M) = \{ d_{n-1}(\varphi) | \varphi \in C^{n-1}(A, M) \}$. The $n$-th Hochschild cohomology of $A$ with coefficients in $M$ is defined by
\[ HH^n(A, M) = Z^n(A, M)/B^n(A, M). \]
In particular, if $M = A$ as conformal bimodule, we denote $HH^n(A) := HH^n(A, A)$, and call the $n$-th Hochschild cohomology of $A$. For the Hochschild cohomology of an associative conformal algebra $A$, we have
\[ \text{(i) Im} \, d_0 = \text{Inn}(A, M), \text{ where Inn}(A, M) = \{ f_0 \in \text{Hom}_{\mathbb{C}[\partial]}(A, M) | v \in M, f_0(a) = a \text{ (}\partial\text{)} v - v\text{ (}0\text{)} a \}. \]
\[ \text{(ii) Ker} \, d_1 = \text{Der}(A, M), \text{ where Der}(A, M) = \{ f \in \text{Hom}_{\mathbb{C}[\partial]}(A, M) | f(a \text{ (}1\text{)} b) = a \text{ (}1\text{)} f(b) + f(a) \triangleleft_{(1)} b \}. \]
\[ \text{(iii) If } A \text{ as } \mathbb{C}[\partial]\text{-module is projective, the equivalence classes of } \mathbb{C}[\partial]\text{-split abelian extensions of } A \text{ by bimodule } M \text{ is projective correspond bijectively to } HH^2(A, M). \text{ For the details see [20, 38].} \]

In [30], for an associative conformal algebra $A$, we have shown that there is a Gerstenhaber algebra structure on $HH^*(A) = \bigoplus_{n \geq 0} HH^n(A)$. First, for the Hochschild complex $(C^*(A, A), d_*)$, we define the Gerstenhaber bracket as following:
\[ [f, g] = f \bullet g - (-1)^{(m-1)(n-1)} g \bullet f \]
for any $f \in C^m(A, A)$ and $g \in C^n(A, A)$, where $f \bullet g = \sum_{i=0}^{m-1} (-1)^{(n-1)i} f \cdot_i g$, and
\[ (f \cdot_i g)_{a_0, \ldots, a_{m+n-3}}(a_0, a_1, \ldots, a_{m+n-2}) = f_{a_0, \ldots, a_{i-1}, a_i, \ldots, a_{m+n-3}}(a_0, \ldots, a_{i-1}, g_{a_{i+1}, \ldots, a_{m+n-3}}(a_i, \ldots, a_{i+n}), a_{i+n+1}, \ldots, a_{m+n-2}). \]

**Proposition 2.4 ([30]).** Let $A$ be an associative conformal algebra. The triple $(C^*+1(A, A) = \bigoplus_{n \geq 0} C^{n+1}(A, A), [+ -], d_*)$ is a differential graded Lie algebra.

For a differential graded Lie algebra $(L = \bigoplus_{n \geq 0} L^n, [+ -], d)$, recall that an element $c \in L^1$ is called a Maurer Cartan element of this differential graded Lie algebra if $d(c) + \frac{1}{2}[c, c] = 0$. We denote the set of all the Maurer Cartan elements of $(L = \bigoplus_{n \geq 0} L^n, [+ -], d)$ by $MC(L)$. If $L^0$ is abelian, there is an equivalence relation on $MC(L)$ which is called gauge equivalence relation...
(see [26]). Let $c$ and $\check{c}$ be two elements in $MC(L)$, they are called gauge equivalent if and only if there exists $\xi \in L^0$ such that

\[
\check{c} = e^{ad_{\xi}}(c) - e^{ad_{\xi}}d(\xi).
\]

The set of the gauge equivalence classes of $MC(L)$ is denoted by $MC(L)$.

Denote the element in $C^2(A,A)$ corresponding to the conformal multiplication by $m_A$, i.e.,

\[m_A(a_1, a_2) = a_1 \cdot_{\lambda} a_2 \text{ for } a_1, a_2 \in A.\]

Then $m_A$ is a Maurer-Cartan element of $(C^{*+1}(A,A), [-,-])$, and for any $f \in C^m(A,A)$, $d_n(f) = (-)^n - 1 \cdot [m_A, f]$. Moreover, this bracket induces a degree $-1$ bracket $[-,-]$ on $HH^*(A) = \bigoplus_{n \geq 0} HH^n(A)$, such that there is a Gerstenhaber algebra structure on $HH^*(A)$. For the details see [30]. Define $\bar{d}_n = (-)^{n-1}d_n$ for $n \geq 0$, i.e., $\bar{d}_n(f) = [m_A, f]$ for any $f \in C^m(A,A)$. Then we get a new differential graded Lie algebra $(C^{*+1}(A,A), [-,-], \bar{d}_n)$. In Section 5 we will use the Maurer-Cartan elements of a subalgebra of this kind of differential graded Lie algebra to characterize the non-abelian extensions of associative conformal algebras.

3. CROSSED MODULES AND 2-TERM STRONGLY HOMOTOPY ASSOCIATIVE CONFORMAL ALGEBRAS

In this section, we introduce the notions of crossed module of associative conformal algebras and 2-term strongly homotopy associative conformal algebras. We will discuss the relationship between them and the 3-th Hochschild cohomology of associative conformal algebras with coefficient in bimodules.

**Definition 3.1.** A 2-term strongly homotopy associative conformal algebra consists of a complex of $C(\partial)$-modules $A_1 \xrightarrow{d} A_2$, conformal sesquilinear maps $m^2 : A_i \times A_j \to A_{i+j}[\lambda]$, $i, j \in \{0, 1\}$, and a sesquilinear map $m^3 : A_0 \times A_0 \times A_0 \to A_1[\lambda_1, \lambda_2]$, such that for any $a, a_1, a_2, a_3, a_4 \in A_0$ and $b, b_1, b_2 \in A_1$,

\[
m^3_i(b_1, b_2) = 0,
\]

\[
d(m^2_i(a, b)) = m^2_i(a, d(b)),
\]

\[
d(m^2_i(b, a)) = m^2_i(d(b), a),
\]

\[
m^3_j(d(b_1), b_2) = m^3_j(b_1, d(b_2)),
\]

\[
d(m^3_{i_1, i_2}(a_1, a_2, a_3)) = m^3_{i_1, i_2}(m^2_{i_1}(a_1, a_2), a_3) - m^2_{i_1}(a_1, m^3_{i_2}(a_2, a_3)),
\]

\[
m^3_{i_1, i_2}(d(b_1), a_2, a_3) = m^3_{i_1, i_2}(m^2_{i_1}(b_1, a_2), a_3) - m^2_{i_1}(b_1, m^3_{i_2}(a_2, a_3)),
\]

\[
m^3_{i_1, i_2}(a_1, d(b_2), a_3) = m^3_{i_1, i_2}(m^2_{i_1}(a_1, b_2), a_3) - m^2_{i_1}(a_1, m^3_{i_2}(b_2, a_3)),
\]

\[
m^3_{i_1, i_2}(d(a_1, a_2, d(b_3))) = m^3_{i_1, i_2}(m^2_{i_1}(a_1, a_2), b_3) - m^2_{i_1}(a_1, m^3_{i_2}(a_2, b_3)),
\]

\[
m^3_{i_1, i_2, i_3}(m^2_{i_1, i_2}(a_1, a_2, a_3), a_4) + m^3_{i_1}(a_1, m^3_{i_2, i_3}(a_2, a_3, a_4)) = m^3_{i_1, i_2, i_3}(m^2_{i_1}(a_1, a_2), a_3, a_4) - m^3_{i_1, i_2, i_3}(m^2_{i_1}(a_2, a_3), a_4) + m^3_{i_1, i_2, i_3}(a_1, a_2, m^3_{i_1}(a_3, a_4)).
\]

We denote this 2-term strongly homotopy associative conformal algebra by $(A_1 \xrightarrow{d} A_0, m^2, m^3)$, and it is said to be skeletal if $d = 0$, is said to be strict if $m^3 = 0$.

The strongly homotopy associative algebras, in particular, the 2-term strongly homotopy associative algebras were studied in [16, 22]. The 2-term strongly homotopy associative conformal algebras can be viewed as a conformal analogue of 2-term strongly homotopy associative
algebras. Following, we will simply write “2-term strongly homotopy associative conformal algebra” by “2-term SHAC-algebra”.

**Example 3.2.** Let $A$ be an associative conformal algebra and $f : M_1 \to M_0$ be a homomorphism between bimodules over $A$. We consider the 2-term chain complex of $\mathbb{C}[\partial]$-modules: $M_1 \xrightarrow{d} A \oplus M_0$. It can be given the structure of a strict 2-term SHAC-algebra by setting

$$m_3((a, u), (b, v)) = (a \triangleright (b \triangleright (u + v), b),$$

$$m_3(v', (a, u)) = a \triangleright (v', a),$$

$$d(v') = (0, f(v')),$$

and $m^3 = 0$, for any $a, b \in A, u, v \in M_0$ and $v' \in M_1$.

Let $A = (A_1 \xrightarrow{d} A_0, m^2, m^3)$ and $A' = (A_1' \xrightarrow{d'} A_0', m'^2, m'^3)$ be two 2-term SHAC-algebras. A morphism $f = (f^0, f^1, f^2) : A \to A'$ consists of a chain map $(f^1, f^0)$ from $A_1 \xrightarrow{d} A_0$ to $A'_1 \xrightarrow{d'} A'_0$, and a conformal sesquilinear map $f^2 : A_0 \times A_0 \to A'_1[\lambda]$ such that for any $a, a_1, a_2, a_3 \in A_0$ and $b \in A_1$,

$$d'(f^2_1(a_1, a_2)) = f^0(m^2_1(a_1, a_2)) - m^2_1(f^0(a_1), f^0(a_2)),$$

$$f^2_1(a, d(b)) = f^1(m^2_1(a, b)) - m^2_1(f^0(a), f^1(b)),$$

$$f^2_1(d(b), a) = f^1(m^2_1(b, a)) - m^2_1(f^1(b), f^0(a)),$$

$$f^2_1(a_1, a_2) = f^1(m^2_1(a_1, a_2)) - m^2_1(f^0(a_1), f^1(a_2), f^0(a_3)) + m^2_1(f^0(a_1), f^1(a_2), f^0(a_3)) + m^2_1(f^0(a_1), f^1(a_2), f^0(a_3)).$$

If $f = (f^0, f^1, f^2) : A \to A'$ and $g = (g^0, g^1, g^2) : A' \to A''$ be two morphism of 2-term SHAC-algebras, their composition $g \circ f : A \to A''$ is defined by $(g \circ f)^0 = g^0 \circ f^0$, $(g \circ f)^1 = g^1 \circ f^1$, and

$$(g \circ f)^2_1(a_1, a_2) = g^2_1(f^0(a_1), f^0(a_2)) + g^1_1(f^2_1(a_1, a_2), a_2),$$

for any $a_1, a_2 \in A_0$. For any 2-term SHAC-algebra $A$, there is an identity morphism $\text{id}_A = (\text{id}_{A_0}, \text{id}_{A_1}, 0)$.

**Proposition 3.3.** The collection of 2-term SHAC-algebras and morphisms between them form a category.

Here we mainly consider skeletal 2-term SHAC-algebras, strict 2-term SHAC-algebras, and the relationship between cohomology of associative conformal algebras and them.

**Definition 3.4.** Let $A = (A_1 \xrightarrow{d} A_0, m^2, m^3)$ and $A' = (A_1' \xrightarrow{d'} A_0', m'^2, m'^3)$ be two skeletal 2-term SHAC-algebras on the same chain complex of $\mathbb{C}[\partial]$-modules. They are said to be equivalent if $m^2 = m'^2$ and there exists a conformal sesquilinear map $\sigma : A_0 \times A_0 \to A_1[\lambda]$ such that

$$m^3 = m^3 + d_2(\sigma),$$

where $d_2$ is the differential in the Hochschild type cohomology complex.

The following theorem characterizes the skeletal 2-term SHAC-algebras, and it also gives a method to construct skeletal 2-term SHAC-algebras.
**Theorem 3.5.** There is a 1-1 correspondence between equivalence classes of skeletal 2-term SHAC-algebras and 3-th Hochschild cohomologies of associative conformal algebras with coefficient in bimodules.

**Proof.** Let $A = (A_1 \rightarrow A_0, m^2, \mathfrak{m}^3)$ be a skeletal 2-term SHAC-algebra. Then $(A_0, \mathfrak{m}^2)$ is an associative conformal algebra, $\mathfrak{m}^2 : A_0 \times A_1 \rightarrow A_1[\lambda]$ and $\mathfrak{m}^3 : A_1 \times A_0 \rightarrow A_1[\lambda]$ give an $A_0$-bimodule structure on $A_1$. By the equation (8) in Definition 3.1 we get $\mathfrak{m}^3$ is a 3-cocycle of $A_0$ with coefficients in bimodule $A_1$.

Conversely, given a quintuple $(A, M, \triangleright, \triangleleft, \zeta)$, where $A$ is an associative conformal algebra, $(M, \triangleright, \triangleleft)$ is a bimodule over $A$, and $\zeta$ is a 3-cocycle of $A$ with coefficients in bimodule $M$. We denote $A_0 := A, A_1 := M$, and define conformal sesquilinear maps $\mathfrak{m}^2 : A_1 \times A_j \rightarrow A_{j+1}[\lambda]$ and $\mathfrak{m}^3 : A_0 \times A_0 \times A_0 \rightarrow A_1[\lambda_1, \lambda_2]$ by

$$\mathfrak{m}^2(a, b) = a \triangleright_i b, \quad \mathfrak{m}^2(a, v) = a \triangleright_i v,$$

$$\mathfrak{m}^3(v, a) = v \triangleleft_i a, \quad \mathfrak{m}^3(a, b, c) = \zeta_i(a, b, c),$$

for any $a, b, c \in A$ and $v \in M$. Then it is easy to verify that $(A_1 \rightarrow A_0, m^2, \mathfrak{m}^3)$ is a skeletal 2-term SHAC-algebra, and the above two correspondences are inverses to each other.

If $A = (A_1 \rightarrow A_0, m^2, \mathfrak{m}^3)$ and $A' = (A_1' \rightarrow A_0, m'^2, \mathfrak{m}'^3)$ are two equivalent skeletal 2-term SHAC-algebras, where the equivalence is given by a conformal sesquilinear map $\sigma : A_0 \times A_0 \rightarrow A_1[\lambda]$, i.e., $\mathfrak{m}^3 = \mathfrak{m}'^3 + d_2(\sigma)$. Then $[\mathfrak{m}^3] = [\mathfrak{m}'^3]$, where $[\mathfrak{m}^3]$ means the equivalence class of $\mathfrak{m}^3$ in $\text{HH}^3(A_0, A_1)$, where the $A_0$-bimodule structure on $A_1$ is given by $\mathfrak{m}^2$. We denote by $\text{Skel}(A_0, A_1, m^2)$ the set of equivalent classes of skeletal 2-term SHAC-algebra structures on $A_1 \rightarrow A_0$ with same $m^2$. Then there is a map

$$\Upsilon : \text{Skel}(A_0, A_1, m^2) \rightarrow \text{HH}^3(A_0, A_1), \quad [(A_1 \rightarrow A_0, m^2, \mathfrak{m}^3)] \mapsto [\mathfrak{m}^3].$$

Conversely, let $A_0$ be an associative conformal algebra and $A_1$ be an $A_0$-bimodule. It is easy to see that any two representatives of an element in $\text{HH}^3(A_0, A_1)$ give two equivalent skeletal 2-term SHAC-algebras on $A_1 \rightarrow A_0$ with same $m^2$, where $m^2$ is given by the module action of $A_0$ on $A_1$ and the multiplication of $A_0$. Thus, the map $\Upsilon$ induces a 1-1 correspondence between equivalence classes of skeletal 2-term SHAC-algebras and the elements in 3-th Hochschild cohomology of associative conformal algebras with coefficient in bimodules. □

Next, we consider the crossed module of associative conformal algebras.

**Definition 3.6.** Let $(X, \cdot X_{(i)})$, $(Y, \cdot Y_{(i)})$ be two associative conformal algebras. If there are two linear maps $\triangleright : X \times Y \rightarrow Y[\lambda]$ and $\triangleleft : Y \times X \rightarrow Y[\lambda]$ such that $(Y, \triangleright, \triangleleft)$ is a bimodule over $X$ and for any $x \in X$ and $y_1, y_2 \in Y$,

$$(x \triangleright_{(i)} y_1)_{(\lambda + \mu)} y_2 = x \triangleright_{(i)} (y_1_{(\mu)} y_2),$$

$$(y_1 \triangleleft_{(j)} x)_{(\lambda + \mu)} y_2 = y_1_{(i)} (x \triangleright_{(\mu)} y_2),$$

$$(y_1 y_{(j)} y_2)_{(\lambda + \mu)} x = y_1 y_{(i)} (y_2 \triangleleft_{(\mu)} x),$$

then we call $X$ acts on $Y$ by $(\triangleright, \triangleleft)$, or simply $X$ acts on $Y$.

Following, we will simply denote $x_1 X_{(i)} x_2$ by $x_1 (i) x_2$ and denote $y_1 Y_{(i)} y_2$ by $y_1 (i) y_2$ for any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, if it does not affect understanding.
Definition 3.7. Let $X$, $Y$ be two associative conformal algebras. A crossed module of associative conformal algebras is a quintuple $(X, Y, \rho, \triangleright, \lhd)$, where $(\triangleright, \lhd)$ is an action of $X$ on $Y$, $\rho : Y \to X$ is a homomorphism of $\mathbb{C}[\partial]$-modules such that for any $x \in X$ and $y, y_1, y_2 \in Y$,

$$\rho(x \triangleright (y)) = x \cdot \rho(y),$$

$$\rho(y \mathbin{\lhd} (x)) = \rho(y) \cdot x,$$

$$\rho(y_1) \triangleright (y_2) = y_1 \cdot \rho(y_2).$$

The crossed module of groups, associative algebras, Lie algebras and Hopf algebras have been extensively studied. See literature [54] for details. The crossed module of associative conformal algebras can be viewed as a conformal analogue of crossed module of associative algebras. We will just say “crossed module” instead of “crossed module of associative conformal algebras” except when emphasis is needed, and we remark that from the above crossed module conditions one get $\rho$ is an associative conformal algebra homomorphism: $\rho(y_1 \cdot (y_2) = \rho(\rho(y_1) \cdot (y_2)) = \rho(y_1) \cdot \rho(y_2)$.

Example 3.8. Let $I$ be a two sided ideal of an associative conformal algebra $A$. Then $(I, A, i, \triangleright, \lhd)$ is a crossed module, where $i$ is the canonical inclusion map and the action $(\triangleright, \lhd)$ is given by the multiplication of $A$. In particular, if $I = 0$ or $I = A$, then $(0, A, 0, 0, 0)$ and $(A, A, id, \cdot \triangleright, \cdot \lhd)$ are crossed modules.

Theorem 3.9. There is a 1-1 correspondence between strict 2-term SHAC-algebras and crossed modules of associative conformal algebras.

Proof. Let $A = (A_1 \xrightarrow{d} A_0, m^2, m^3)$ be a strict 2-term SHAC-algebra. Let $X := A_0$. Then, by the equation (4) in Definition 3.1, $X$ with the multiplication $x_1 \cdot (x_2 = m^2_{\lambda}(x_1, x_2), x_1, x_2 \in X$, is an associative conformal algebra. Set $Y := A_1$ and $y_1 \cdot (y_2 = m^2_{\lambda}(d(y_1), y_2) = m^2_{\lambda}(y_1, d(y_2))$, Then for any $y_1, y_2, y_3 \in Y$, since $m^3 = 0$, by the equation (7), we get

$$y_1 \cdot (y_2 \cdot (y_3) = (y_1 \cdot (y_2) \cdot (y_3)$$

$$= m^2_{\lambda}(d(y_1), m^2_{\lambda}(d(y_2), y_3)) - m^2_{\lambda}(d(y_1), m^2_{\lambda}(d(y_2), y_3))$$

$$= m^3_{\lambda}(d(y_1), d(y_2), y_3)) = 0.$$

This means that $Y$ is an associative conformal algebra. Moreover, by equations (5)-(7), we obtain that $\triangleright := m^2 : X \times Y \to Y[\lambda]$ and $\lhd := m^2 : Y \times X \to Y[\lambda]$ give an action of $X$ on $Y$. Finally, take $\rho = d : Y \to X$. Then $\rho$ is a homomorphism of $\mathbb{C}[\partial]$-modules and the equations (1)-(3) are just the conditions in Definition 3.7. Thus the quintuple $(X, Y, \rho, \triangleright, \lhd)$ is a crossed module.

Conversely, given a crossed module $(X, Y, \rho, \triangleright, \lhd)$, we can construct a strict 2-term SHAC-algebra as follows. Set $A_0 = X, A_1 = Y, d = \rho$, and conformal sesquilinear maps $m^2 : A_i \times A_j \to A_{i+j}[\lambda]$ by

$$m^2_{\lambda}(x_1, x_2) = x_1 \cdot (x_2), \\
m^2_{\lambda}(y, x) = y \cdot (x), \\
m^2_{\lambda}(y_1, y_2) = 0.$$
Definition 3.10. Let $A$ be an associative conformal algebra and $(M, \triangleright, \downarrow)$ be a bimodule over $A$. A crossed extension of $A$ by $M$ is an exact sequence of associative conformal algebras

$$S : 0 \longrightarrow M \overset{\alpha}{\longrightarrow} Y \overset{\beta}{\longrightarrow} X \overset{\gamma}{\longrightarrow} A \longrightarrow 0,$$

such that there is an action $(\triangleright, \downarrow)$ of $X$ on $Y$ and $(X, Y, \beta, \triangleright, \downarrow)$ is a crossed module, where the multiplication of associative conformal algebra $M$ is trivial.

The crossed extension $S$ is said to be split if there are $\mathbb{C}[\partial]$-module homomorphisms $\varrho: A \rightarrow X$ and $\varsigma: \text{Im}(g) \rightarrow Y$ such that $\gamma \circ \varrho = \text{id}_A$, $\beta \circ \varsigma = \text{id}_{\text{Im}(g)}$. Here $\varrho$ is called a section of $S$.

Given a crossed module $(X, Y, \rho, \triangleright, \downarrow)$, since $\rho$ is a homomorphism of associative conformal algebras, there is an exact sequence of associative conformal algebras

$$S' : 0 \longrightarrow M \overset{i}{\longrightarrow} Y \overset{\rho}{\longrightarrow} X \overset{\pi}{\longrightarrow} A \longrightarrow 0,$$

where $M = \text{Ker}(\rho)$ and $A = \text{Coker}(\rho)$. Note that for any $u, v \in M$, $u \overset{\rho}{\longrightarrow} v = \rho(u) \overset{\triangleright}{\longrightarrow} v = 0$ since $\rho(u) = 0$, we get the multiplication of $M$ is trivial. If there is a $\mathbb{C}[\partial]$-module homomorphism $\varrho$ such that $\pi \circ \varrho = \text{id}_A$, there is an $A$-bimodule structure on $M$ as follows. For any $a \in A$ and $v \in M$, we define $a \overset{\triangleright}{\longrightarrow} v := \varrho(a) \overset{\triangleright}{\longrightarrow} m$ and $v \overset{\downarrow}{\longleftarrow} a := v \overset{\varsigma}{\longleftarrow} \varrho(a)$, then one can check that $(M, \triangleright, \downarrow)$ is an $A$-bimodule which is called the $A$-bimodule structure on $M$ induced by $\varrho$, and it does not depend on the choice of $\varrho$.

Following, each crossed extension of $A$ by $M$ refers to a crossed extension such that the $A$-module structure on $M$ coincides with the $A$-bimodule structure induced by the section.

Definition 3.11. Let $0 \longrightarrow M \overset{\alpha}{\longrightarrow} Y \overset{\beta}{\longrightarrow} X \overset{\gamma}{\longrightarrow} A \longrightarrow 0$ and $0 \longrightarrow M \overset{\alpha'}{\longrightarrow} Y' \overset{\beta'}{\longrightarrow} X' \overset{\gamma'}{\longrightarrow} A \longrightarrow 0$ be two split crossed extensions of $A$ by $M$. If there are homomorphisms of associative conformal algebras $\varphi: Y \rightarrow Y'$ and $\psi: X \rightarrow X'$ such that the following diagram is commutative

$$
\begin{array}{ccc}
0 & \longrightarrow & M \\
\vert & \downarrow & \vert \\
0 & \longrightarrow & M
\end{array}
\begin{array}{ccc}
\alpha & \longrightarrow & Y \\
\vert & \downarrow & \vert \\
\alpha' & \longrightarrow & Y'
\end{array}
\begin{array}{ccc}
\beta & \longrightarrow & X \\
\vert & \downarrow & \vert \\
\beta' & \longrightarrow & X'
\end{array}
\begin{array}{ccc}
\gamma & \longrightarrow & A \\
\vert & \downarrow & \vert \\
\gamma' & \longrightarrow & A
\end{array}
$$

we call $(\varphi, \psi)$ is a morphism of crossed extensions of $A$ by $M$.

This two crossed extensions of $A$ by $M$ are said to be equivalent if there is a morphism from one to the other. Denote by $\text{Ext}^2(A, M)$ the set of equivalence classes of split crossed extensions of $A$ by $M$.

Proposition 3.12. For any associative conformal algebra $A$ and a bimodule $M$ over $A$, there is a canonical map

$$\Theta : \text{Ext}^2(A, M) \longrightarrow \text{HH}^3(A, M).$$
Proof. Given a split crossed extension $S : 0 \rightarrow M \xrightarrow{\alpha} Y \xrightarrow{\beta} X \xrightarrow{\gamma} A \rightarrow 0$ with a section $\varrho : A \rightarrow X$ and $\mathbb{C}[\partial]$-module homomorphism $\varsigma : \text{Im}(\beta) \rightarrow Y$ such that $\beta \circ \varsigma = \text{id}_{\text{Im}(\beta)}$. We have $\varrho(a) (\lambda) \varrho(b) - \varrho(a (\lambda), b) \in \text{Ker}(\gamma)[\lambda] = \text{Im}(\beta)[\lambda]$, for any $a, b \in A$. Denote $g_1(a, b) = \varsigma(\varrho(a) (\lambda) \varrho(b) - \varrho(a (\lambda), b))$ and define

$$f_{\lambda_1, \lambda_2}(a, b, c) = \varrho(a) \mapsto_{(\lambda_1)} \bar{g}_{\lambda_2}(b, c) - g_{\lambda_1 + \lambda_2}(a (\lambda_1), b, c) + g_{\lambda_1}(a, b) \circ_{(\lambda_1 + \lambda_2)} \varrho(c),$$

for any $a, b, c \in A$. Then one can check that $f$ is a conformal sesquilinear map, and equation $\beta(f_{\lambda_1, \lambda_2, \lambda_3}(a, b, c)) = 0$ holds, since $(X, Y, \beta, \triangleright, \triangleleft)$ is a crossed module. Moreover, one can obtain $d_3(f) = 0$ by routine calculations, where $d_3$ is a differential in the Hochschild cohomology complex of $A$ with coefficients in $M$. We define $\Theta(S) = [f]$. Following we need to show that the map $\Theta$ is well-defined.

First, we are going to show that $\Theta$ does not depend on the choice of $\varrho$. Let $\tilde{\varrho}$ be another section of $S$ and $\tilde{f}$ be the 3-cocycle defined using $\tilde{\varrho}$ instead of $\varrho$. Then, there exists a $\mathbb{C}[\partial]$-module homomorphism $\eta : A \rightarrow Y$ such that $\beta \circ \eta = \tilde{\varrho} - \varrho$. Denote by $\tilde{g}_1(a, b) = \varsigma(\tilde{\varrho}(a) (\lambda) \tilde{\varrho}(b) - \tilde{\varrho}(a (\lambda), b))$. Then, by crossed modules properties, we have

$$(\tilde{f} - f)_{\lambda_1, \lambda_2}(a, b, c) = \varrho(a) \mapsto_{(\lambda_1)} (\tilde{g}(\tilde{g} - g))_{\lambda_2}(b, c) - (g - \bar{g})_{\lambda_1 + \lambda_2}(a (\lambda_1), b, c) + \eta(a) (\lambda_1) \tilde{g}_{\lambda_2}(b, c) - \tilde{g}_{\lambda_2}(a, b) \circ_{(\lambda_1 + \lambda_2)} \eta(c),$$

for any $a, b, c \in A$. Define conformal sesquilinear map $\tilde{g} : A \times A \rightarrow Y[\lambda]$ by

$$\tilde{g}_1(a, b) = \tilde{\varrho}(a) \mapsto_{(\lambda_1)} \eta(b) + \eta(a) \circ_{(\lambda_1)} \tilde{\varrho}(b) - \eta(a (\lambda_1), b) - \eta(a) (\lambda_1) \eta(b),$$

for $a, b \in A$. Then it is easy to see that $\beta \circ \tilde{g} = \beta \circ (\tilde{g} - g)$. That is $\tilde{g} - g - \tilde{g} \in \mathcal{C}^2(A, M)$. We denote

$$\tilde{f}_{\lambda_1, \lambda_2}(a, b, c) := \varrho(a) \mapsto_{(\lambda_1)} \tilde{g}_{\lambda_2}(b, c) - \tilde{g}_{\lambda_1 + \lambda_2}(a (\lambda_1), b, c) + \tilde{g}_{\lambda_1}(a, b) \circ_{(\lambda_1 + \lambda_2)} \eta(c).$$

Then, by the crossed modules properties, we obtain $\tilde{f}_{\lambda_1, \lambda_2}(a, b, c) = 0$. Note that $\tilde{f} - f - \tilde{f} = d_2(\tilde{g} - g - \tilde{g})$, we get $[\tilde{f} - f] = 0$ in $\text{HH}^2(A, M)$, i.e., $[\tilde{f}] = [f]$ in $\text{HH}^2(A, M)$.

Let $S' : 0 \rightarrow M \xrightarrow{\alpha'} Y' \xrightarrow{\beta'} X' \xrightarrow{\gamma'} A \rightarrow 0$ be another split crossed extension of $A$ by $M$, and $(\varphi, \psi)$ be a morphism from $S$ to $S'$. Denote by $(X', Y', \beta', \triangleright', \triangleleft')$ the corresponding crossed module of $S'$. Then $\varrho' = \psi \circ \varrho$ is a section of $S'$, and for any $v \in M, a \in A, \varrho'(a) \mapsto' v = \varrho(a) \mapsto v, v \circ \varrho(a) = v \circ' \varrho'(a)$. Let $\varsigma' : \text{Im}(\beta') \rightarrow Y'$ be an arbitrary $\mathbb{C}[\partial]$-module homomorphism such that $\beta' \circ \varsigma' = \text{id}_{\text{Im}(\beta')}$. As discussed above, we can use $\varrho$ and $\varsigma$ to define $f$, use $\varrho'$ and $\varsigma'$ to define $f'$. Now we define a conformal sesquilinear map $h : A \times A \rightarrow M[\lambda]$ by

$$h_1(a, b) = (\varphi \circ \varsigma - \varsigma' \circ \psi)(\varrho(a) (\lambda) \varrho(b) - \varrho(a (\lambda), b)).$$
for $a, b \in A$. Then, for any $a, b, c \in A$, we have

$$
(f - f')_{\delta_1, \delta_2}(a, b, c) = \theta(a, b, c) \cdot \varphi(a, b, c) - \varphi(a, b, c) \cdot \theta(a, b, c) - \varphi(a, b, c) \cdot \theta(a, b, c) - \varphi(a, b, c) \cdot \theta(a, b, c)
$$

That is to say, $[f] = [f']$ in $\text{HH}^3(A, M)$. By the arbitrariness of $\varphi$, we also get that $\Theta$ does not depend on the choice of $\varphi$. Thus, the map $\Theta$ is well-defined.

As in the classical case, we would like to construct an isomorphism between $\text{Ext}^2(A, M)$ and $\text{HH}^3(A, M)$. But now we cannot construct a canonical example of split crossed extension for a given cohomology class in $\text{HH}^3(A, M)$. We need to further consider the conditions for map $\Theta$ to be injective and surjective.

4. Non-abelian extension and cohomology

In this section, we consider the non-abelian extensions of associative conformal algebras, define a non-abelian cohomology group, and show that the non-abelian extensions can be classified by the non-abelian cohomology group. First, we give the definition of non-abelian extensions of associative conformal algebras.

**Definition 4.1.** Let $A$ and $B$ be two associative conformal algebras. A non-abelian extension $E$ of $B$ by $A$ is a short exact sequence

$$
E : 0 \rightarrow A \xrightarrow{\alpha} E \xrightarrow{\beta} B \rightarrow 0,
$$

where $E$ is an associative conformal algebra, $\alpha, \beta$ are homomorphisms of associative conformal algebras, and this sequence is split in the category of $\mathbb{C}[\partial]$-modules. Let $E$ and $E'$ be two extensions of $B$ by $A$. They are called equivalent if there exists a homomorphism of associative conformal algebras $\theta : E \rightarrow E'$ such that the following diagram commutes

$$
\begin{array}{ccc}
E : & 0 & \rightarrow & A & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & B & \rightarrow & 0 \\
E' : & 0 & \rightarrow & A & \xrightarrow{\alpha'} & E' & \xrightarrow{\beta'} & B & \rightarrow & 0.
\end{array}
$$

**Remark 4.2.** The non-abelian extension of associative conformal algebra defined here is also a $\mathbb{C}[\partial]$-split extension. In this regard, we can require associative conformal algebra $B$ to be projective as $\mathbb{C}[\partial]$-module. It is a generalization of $\mathbb{C}[\partial]$-split abelian extension of associative conformal algebras.

We now define a non-abelian cohomology and show that the non-abelian extensions are classified by the non-abelian cohomology.
Definition 4.3. Let $A$ and $B$ be two associative conformal algebras. A non-abelian 2-cocycle on $B$ with values in $A$ is a triplet $(\triangleright,\triangleleft,\chi)$ of $\mathbb{C}[\partial]$-bilinear maps $\chi : B \times B \to A[\lambda], (b_1, b_2) \mapsto \chi_\lambda(b_1, b_2)$, $\triangleright : B \times A \to A[\lambda], (b, a) \mapsto b \triangleright (\lambda) a$, and $\triangleleft : A \times B \to A[\lambda], (a, b) \mapsto a \triangleleft (\lambda) b$, satisfying the following properties:

\begin{align}
(9) & \quad b_1 \triangleright (\lambda) (b_2 \triangleleft (\mu) a) = (b_1 \triangleright (\lambda) b_2) \triangleright (\lambda+\mu) a + \chi_\lambda(b_1, b_2) \triangleright (\lambda+\mu) a, \\
(10) & \quad (a \triangleleft (\lambda) b_1) \triangleleft (\lambda+\mu) b_2 = a \triangleleft (\lambda) (b_1 \triangleleft (\mu) b_2) + a \triangleleft (\lambda) \chi_\mu(b_1, b_2), \\
(11) & \quad b_1 \triangleright (\lambda) (a \triangleleft (\mu) b_2) = (b_1 \triangleright (\lambda) a) \triangleleft (\lambda+\mu) b_2, \\
(12) & \quad (a_1 \triangleleft (\lambda) a_2) \triangleleft (\lambda+\mu) b = a_1 \triangleleft (\lambda) (a_2 \triangleleft (\mu) b), \\
(13) & \quad b \triangleright (\lambda) (a_1 \triangleleft (\mu) a_2) = (b \triangleright (\lambda) a_1) \triangleleft (\lambda+\mu) a_2, \\
(14) & \quad (a_1 \triangleleft (\lambda) b) \triangleleft (\lambda+\mu) a_2 = a_1 \triangleleft (\lambda) (b \triangleright (\mu) a_2), \\
(15) & \quad b_1 \triangleright (\lambda) \chi_\mu(b_2, b_3) + \chi_{\lambda+\mu}(b_1, b_2 \triangleleft (\mu) b_3) = \chi_{\lambda+\mu}(b_1 \triangleright (\lambda) b_2, b_3) + \chi_\lambda(b_1, b_2) \triangleleft (\lambda+\mu) b_3,
\end{align}

for any $a, a_1, a_2 \in A$ and $b, b_1, b_2, b_3 \in B$. One denotes by $Z^2_{\mathrm{nab}}(B, A)$ the set of theses cocycles.

Moreover, $(\triangleright, \triangleleft, \chi)$ and $(\triangleright, \triangleleft, \tilde{\chi})$ are said to be equivalent if there exists a $\mathbb{C}[\partial]$-module homomorphism $\delta : B \to A$ satisfying:

\begin{align}
(16) & \quad b \triangleleft (\lambda) a - b \triangleright (\lambda) a = \delta(b) \triangleleft (\lambda) a, \\
(17) & \quad a \triangleleft (\lambda) b - a \triangleright (\lambda) b = a \triangleleft (\lambda) \delta(b), \\
(18) & \quad \tilde{\chi}_\lambda(b_1, b_2) - \chi_\lambda(b_1, b_2) = b_1 \triangleleft (\lambda) \delta(b_2) - \delta(b_1 \triangleright (\lambda) b_2) + \delta(b_1) \tilde{\chi}_\lambda(b_2) - \delta(b_1) \triangleleft (\lambda) \delta(b_2),
\end{align}

for any $a \in A$ and $b, b_1, b_2 \in B$. In this case, we denote $(\triangleright, \triangleleft, \chi) \approx (\triangleright, \triangleleft, \tilde{\chi})$. The non-abelian cohomology $\mathrm{HH}^2_{\mathrm{nab}}(B, A)$ is the quotient of $Z^2_{\mathrm{nab}}(B, A)$ by this equivalence relation.

Let $A$ and $B$ be two associative conformal algebras. Let $E$ be a non-abelian extension of $B$ by $A$, i.e., there is a short exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0,$$

which is split in the category of $\mathbb{C}[\partial]$-modules. Thus, there is a $\mathbb{C}[\partial]$-module homomorphism $\gamma : B \to E$ such that $\beta \circ \gamma = \text{id}_B$, which is called a section of $E$. We define $\mathbb{C}[\partial]$-bilinear maps $\chi^\gamma : B \times B \to A[\lambda], \triangleright^\gamma : B \times A \to A[\lambda]$ and $\triangleleft^\gamma : A \times B \to A[\lambda]$ by the multiplication of $E$ as following:

\begin{align*}
(\text{a}) & \quad b \triangleright^\gamma (\lambda) a = \gamma(b) (\lambda) a, \\
(\text{b}) & \quad a \triangleleft^\gamma (\lambda) b = a (\lambda) \gamma(b), \\
(\text{c}) & \quad \chi^\gamma_\lambda(b_1, b_2) = \gamma(b_1) (\lambda) \gamma(b_2) - \gamma(b_1 \triangleright (\lambda) b_2),
\end{align*}

for any $a \in A$ and $b, b_1, b_2 \in B$.

Lemma 4.4. With the above notations, the triplet $(\triangleright^\gamma, \triangleleft^\gamma, \chi^\gamma)$ is a non-abelian 2-cocycle on $B$ with values in $A$.

Proof. By direct calculation, for any $a \in A$ and $b_1, b_2 \in B$, we have

\begin{align*}
b_1 \triangleright^\gamma (\lambda) (b_2 \triangleright^\gamma (\mu) a) & = \gamma(b_1) (\lambda) (\gamma(b_2) (\mu) a) \\
& = \left(\gamma(b_1) (\lambda) \gamma(b_2) - \gamma(b_1 \triangleright (\lambda) b_2)\right) (\lambda+\mu) a + \gamma(b_1) (\lambda) b_2 (\lambda+\mu) a \\
& = (b_1 \triangleright (\lambda) b_2) \triangleright^\gamma (\lambda+\mu) a + \chi^\gamma_\lambda(b_1, b_2) (\lambda+\mu) a.
\end{align*}
This means the equation 9 in Definition 4.3 is satisfied. Similarly, one can check that the equations (10)-(14) are satisfied too. Considering the equation (15), we have, for any $b_1, b_2, b_3 \in B$,

$$b_1 \triangleright (b_2, b_3) - \chi_1^\gamma(b_1, b_2, b_3) + \chi_1^\gamma(b_1, b_2, b_3) - \chi_1^\gamma(b_1, b_2) \triangleright (b_3) = 0.$$ 

Thus the triplet $(\triangleright, \triangleright^\gamma, \chi^\gamma)$ is a non-abelian 2-cocycle on B with values in A. □

For a split sequence in category of $\mathbb{C}[\partial]$-modules, the section is not unique in general. For two different sections, we have the following lemma.

**Lemma 4.5.** Let $E : 0 \longrightarrow A \xrightarrow{\alpha} E \xrightarrow{\beta} B \longrightarrow 0$ be a non-abelian extension of associative conformal algebra B by associative conformal algebra A. For any two sections $\gamma$ and $\gamma'$ of $E$, we have $(\triangleright, \triangleright^\gamma, \chi^\gamma) \approx (\triangleright, \triangleright^\gamma', \chi^\gamma')$.

**Proof.** For the sections $\gamma$ and $\gamma'$, we define $\delta = \gamma' - \gamma : B \rightarrow A$. Then for any $a \in A$ and $b \in B$, we have

$$b \triangleright (b, a) = \gamma(b) (a) a + \gamma'(b) (a) a + \gamma - \gamma'(b)(a) (a) a = b \triangleright (b, a) - \delta(b) (a) a.$$ 

Similarly, one can check that the equations (17) and (18) are true. Thus, $(\triangleright, \triangleright^\gamma, \chi^\gamma)$ and $(\triangleright, \triangleright^\gamma', \chi^\gamma')$ are equivalent. □

Therefore, we get a map from the set of all the non-abelian extensions of associative conformal algebra B by associative conformal algebra A to $\text{HH}_2^{nab}(B, A)$. Next we show that this map keeps the equivalence relation.

**Lemma 4.6.** Let A and B be two associative conformal algebras. If $E$ and $E'$ are two equivalent non-abelian extensions of B by A, $\gamma$ and $\gamma'$ are the sections of $E$ and $E'$ respectively. With the above notations, we get $(\triangleright, \triangleright^\gamma, \chi^\gamma) \approx (\triangleright, \triangleright^\gamma, \chi^\gamma')$.

**Proof.** Since $E$ and $E'$ are two equivalent non-abelian extensions of B by A, there exists a homomorphism of associative conformal algebras $\theta : E \rightarrow E'$ and a commutative diagram:

$$
\begin{array}{ccc}
0 & \longrightarrow & A \\
\Bigg| & & \Bigg| \\
\alpha & \longrightarrow & E \xrightarrow{\beta} B \longrightarrow 0 \\
\Bigg| & & \Bigg| \\
0 & \longrightarrow & A \\
\end{array}
$$

Consider the sections $\gamma : B \rightarrow E$ and $\gamma' : B \rightarrow E'$, we define $\delta = \gamma - \theta^{-1} \circ \gamma' : B \rightarrow A$. Then for any $a \in A$ and $b \in B$,

$$b \triangleright (b, a) = \gamma'(b)(a) a + \gamma - \gamma'(b)(a) a = b \triangleright (b, a) - \delta(b)(a) a.$$ 

Similarly, one can check the equations (17)-(18) are true. Thus $(\triangleright, \triangleright^\gamma, \chi^\gamma) \approx (\triangleright, \triangleright^\gamma', \chi^\gamma')$. □
Let $A$ and $B$ be two associative conformal algebras. Denote by $\text{Ext}_{nab}(B,A)$ the set of all the equivalence classes of non-abelian extensions of $B$ by $A$, by $[\mathcal{E}]$ the equivalence class of $\mathcal{E}$ for each extension $\mathcal{E}$. According to the above discussion, we get a map

$$\Phi : \text{Ext}_{nab}(B,A) \rightarrow \text{HH}_2^{nab}(B,A)$$

$$[\mathcal{E}] \mapsto [(\gamma, \xi, \chi^\gamma)].$$

Conversely, for any non-abelian 2-cocycles $(\gamma, \xi, \chi)$ on $B$ with values in $A$, we define a conformal multiplication on $\mathbb{C}[\partial]$-module $A \oplus B$ by

$$(a_1, b_1) (a_2, b_2) = (a_1 \cdot a_2 + b_1 \triangleright a_2 + a_1 \triangleright a_2 + a_1 \triangleleft b_2 + \chi_4(a_1 b_2), b_1 \triangleright a_2),$$

for any $(a_1, b_1), (a_2, b_2) \in A \oplus B$. Then we get an associative conformal algebra, denote by $A \oplus_{(\gamma, \xi, \chi)} B$. By this associative conformal algebra, we obtain a non-abelian extension of $B$ by $A$ as follows:

$$\mathcal{E}_{(\gamma, \xi, \chi)} : 0 \rightarrow A \overset{a}{\rightarrow} A \oplus_{(\gamma, \xi, \chi)} B \overset{\beta}{\rightarrow} B \rightarrow 0,$$

where $\alpha(a) = (a, 0), \beta(a, b) = b$. Moreover, we have the following lemma.

**Lemma 4.7.** Let $A$ and $B$ be two associative conformal algebras. Then two non-abelian 2-cocycles $(\gamma, \xi, \chi)$ and $(\gamma', \xi', \chi')$ of $B$ by $A$ are equivalent if and only if the corresponding non-abelian extensions $\mathcal{E}_{(\gamma, \xi, \chi)}$ and $\mathcal{E}_{(\gamma', \xi', \chi')}$ are equivalent.

**Proof.** If two non-abelian 2-cocycles $(\gamma, \xi, \chi)$ and $(\gamma', \xi', \chi')$ are equivalent, there is a linear map $\delta : B \rightarrow A$ such that the equations (16)-(18) hold. Define $\theta : A \oplus_{(\gamma, \xi, \chi)} B \rightarrow A \oplus_{(\gamma', \xi', \chi')} B$ by

$$\theta(a, b) = (a - \delta(b), b).$$

It is easy to see that $\theta$ is a linear bijection with inverse $\theta^{-1}(a, b) = (a + \delta(b), b)$. For any $(a_1, b_1), (a_2, b_2) \in A \oplus_{(\gamma, \xi, \chi)} B$, by equations (16)-(18), we have

$$\theta((a_1, b_1) (a_2, b_2)) = \theta(a_1 (a_2 + b_1 \triangleright a_2 + a_1 \triangleright a_2 + a_1 \triangleleft b_2 + \chi_4(b_1, b_2), b_1 \triangleright a_2))$$

$$= (a_1 \cdot a_2 + b_1 \triangleright a_2 + a_1 \triangleright a_2 + a_1 \triangleleft b_2 + \chi_4(b_1, b_2) - \delta(b_1 \triangleright a_2), b_1 \triangleright a_2)$$

$$= (a_1 - \delta(b_1)) (a_2 - \delta(b_2)) + b_1 \triangleright (a_2 - \delta(b_2))$$

$$+ (a_1 - \delta(b_1)) \triangleleft b_2 + \chi_4(b_1, b_2), b_1 \triangleright a_2).$$

Thus $\theta$ is an isomorphism of associative conformal algebras. Moreover, one can check that the diagram:

$$(\bullet)$$

is commutative. This means that $\mathcal{E}_{(\gamma, \xi, \chi)}$ and $\mathcal{E}_{(\gamma', \xi', \chi')}$ are equivalent.

Conversely, if $\mathcal{E}_{(\gamma, \xi, \chi)}$ and $\mathcal{E}_{(\gamma', \xi', \chi')}$ are equivalent, by the calculation above, it is not hard to see that there is a linear map $\theta$ such that the diagram (\bullet) is commutative if and only if there exists a uniquely determined linear map $\delta : B \rightarrow A$ such that $\theta(a, b) = (a - \delta(b), b)$ for any $a \in A$ and
Let $A$ and $B$ be two associative conformal algebras. There is a 1-1 correspondence between classes of non-abelian extensions of $B$ by $A$ and elements of $\text{HH}_n(B,A)$. In other words, $\text{HH}_n(B,A)$ classifies non-abelian extensions of $B$ by $A$.

Proof. Given a non-abelian extension $E : 0 \to A \xrightarrow{a} E \xrightarrow{\beta} B \to 0$ of $B$ by $A$ and a section $\gamma$ of $E$, we get a non-abelian 2-cocycle $(\gamma, \epsilon, \xi, \chi)$. By the 2-cocycle, we obtain a non-abelian extension $\mathcal{E}_{(\gamma, \epsilon, \xi, \chi)}$. We define $\theta : A \oplus_{(\gamma, \epsilon, \xi, \chi)} B \to E$ by

$$\theta(a, b) = a + \gamma(b)$$

for $a \in A$ and $b \in B$. Then $\theta$ is a bijection and for any $(a_1, b_1), (a_2, b_2) \in A \oplus_{(\gamma, \epsilon, \xi, \chi)} B$,

$$\theta((a_1, b_1) \cdot (a_2, b_2)) = \theta((a_1 + a_2 + b_1 \gamma_{(\lambda)} a_2 + a_1 \xi_{(\lambda)} b_2 + \chi_{(\lambda)}(b_1, b_2), b_1 \epsilon_{(\lambda)} b_2))$$

$$= a_1 + a_2 + \gamma(b_1 + b_2) + a_1 \gamma(b_2) + \gamma(b_1) \epsilon(b_2)$$

$$= \theta(a_1, b_1) \cdot \theta(a_2, b_2).$$

Moreover, one can check that the diagram:

$$\begin{array}{c}
0 \to A \xrightarrow{a'} A \oplus_{(\gamma, \epsilon, \xi, \chi)} B \xrightarrow{\beta'} B \to 0 \\
\Downarrow \quad \Downarrow \quad \Downarrow \theta \\
0 \to A \xrightarrow{a} E \xrightarrow{\beta} B \to 0
\end{array}$$

is commutative. This means that $\Psi \circ \Phi(E)$ and $E$ are equivalent. Thus $\Psi$ is surjective, and so that it is a bijection. □

At the end of this section, we apply the conclusion of Theorem 4.8 to the $\mathbb{C}[\partial]$-split abelian extension of associative conformal algebras. If $B$ as $\mathbb{C}[\partial]$-module is projective and the multiplication in $A$ is trivial, the Definition 4.3 is just the definition of $\mathbb{C}[\partial]$-split abelian extension of $B$ by $A$ in [20]. In this case, for an extension $E : 0 \to A \to E \to B \to 0$, the corresponding non-abelian 2-cocycle $(\gamma, \epsilon, \xi, \chi)$ induces a bimodule structure on $A$ by equations (9)-(11), and gives an element $\chi \in \mathcal{Z}_2(B,A)$ by equation (15). The equation (18) means that two extensions $E$ and $\tilde{E}$ are equivalent if and only if $\chi - \tilde{\chi} \in \mathcal{B}_2(B,A)$. Thus we obtain the following corollary.

Corollary 4.9 ([20]). Let $B$ be an associative conformal algebra, which is projective as $\mathbb{C}[\partial]$-module. For any bimodule $A$ over $B$, the Hochschild cohomology $\text{HH}_n(B,A)$ classifies abelian extensions of $B$ by $A$. 
5. Non-abelian extensions in terms of Deligne groupoid

In this section, we construct a subalgebra \( \mathcal{L} \) of differential graded algebra \((\mathcal{C}^{*+1}(A \oplus B, A \oplus B), [-,-], d)\) for two associative conformal algebras \( A \) and \( B \). We show that the non-abelian cocycles are in bijection with the Maurer-Cartan elements of \( \mathcal{L} \), and get that the equivalence relation on \( \mathcal{Z}_{\text{nab}}^2(B, A) \) can be interpreted as gauge equivalence relation on \( \text{MC}(\mathcal{L}) \).

Let \( \mathcal{E} : 0 \rightarrow A \xrightarrow{\alpha} E \xrightarrow{\beta} B \rightarrow 0 \) be a non-abelian extension of associative conformal algebra \( B \) by associative conformal algebra \( A \). Suppose \( \gamma \) is a section of \( \mathcal{E} \). Then in the category of \( \mathbb{C}[\partial] \)-modules, we have a commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & A \\
\downarrow & & \downarrow \\
\gamma & \rightarrow & A \oplus B \\
\downarrow & & \downarrow \\
0 & \rightarrow & B \\
\end{array}
\]

where \( \gamma(a) = (a, 0) \), \( \rho_B(a, b) = b \), \( \bar{A} \) is the \( \mathbb{C}[\partial] \)-module of image of \( A \) under \( \gamma \), \( \bar{B} \) is the \( \mathbb{C}[\partial] \)-module of image of \( B \) under \( \gamma \). Using these isomorphisms, we can transfer the associative conformal algebra structure of \( A, B \) and \( E \) to \( \bar{A}, \bar{B} \) and \( \bar{A} \oplus \bar{B} \), such that this commutative graph holds in category of associative conformal algebras. Denote the multiplication in \( \bar{A} \oplus \bar{B} \) (or \( E \)) by \( \mathcal{M} \), i.e., \( \mathcal{M}(a_1, b_1), (a_2, b_2)) = (a_1, b_1) \oplus (a_2, b_2) \). Then we have \( \mathcal{M} = \mathcal{M}_{\bar{A} \bar{A}} + \mathcal{M}_{\bar{A} \bar{B}} + \mathcal{M}_{\bar{B} \bar{A}} + \mathcal{M}_{\bar{B} \bar{B}} \), where \( \mathcal{M}_{\bar{A} \bar{A}}((a_1, b_1), (a_2, b_2)) = \rho_\bar{A}((a_1, 0)) \oplus (0, b_2) \), and others can be defined similarly. For these components, we have the following lemma.

**Lemma 5.1.** With the above notations, we have the following conclusions.

1. One can identify \( \mathcal{M}_{\bar{B} \bar{B}} \) with multiplication in \( B \), identifies \( \mathcal{M}_{\bar{A} \bar{A}} \) with multiplication in \( A \), and \( \mathcal{M}_{\bar{A} \bar{B}} = \mathcal{M}_{\bar{B} \bar{A}} = \mathcal{M}_{\bar{B} \bar{B}} = 0 \).

2. Denote by \( \bullet, \ll, \chi \) the non-abelian 2-cocycle on \( B \) with values in \( A \) corresponding to \( \mathcal{E} \). Then one can identify \( \mathcal{M}_{\bar{A} \bar{A}}, \mathcal{M}_{\bar{A} \bar{B}}, \mathcal{M}_{\bar{B} \bar{B}} \) with \( \bullet, \ll, \chi \) respectively.

**Proof.** (i) Since \( \beta \circ \gamma = \text{id}_B \), we can identify \( \mathcal{M}_{\bar{B} \bar{B}} \) with multiplication in \( B \). And similarly for \( \mathcal{M}_{\bar{A} \bar{A}} \) and multiplication in \( A \). Moreover, since \( \rho_B \) is a morphism of associative conformal algebras and \( \rho_B(a, 0) = 0 \), for any \((a_1, b_1), (a_2, b_2) \in \bar{A} \oplus \bar{B} \),

\[
(\mathcal{M}_{\bar{B} \bar{B}})((a_1, b_1), (a_2, b_2)) = \rho_B \circ \mathcal{M}(0, b_1), (a_2, 0)) = \rho_B((a_1, 0)) \oplus \rho_B(a_2, 0) = 0.
\]

Similarly, \( \mathcal{M}_{\bar{B} \bar{A}} = \mathcal{M}_{\bar{A} \bar{B}} = 0 \).

(ii) If \( \ll, \chi \) is the non-abelian 2-cocycle on \( B \) with values in \( A \) corresponding to \( \mathcal{E} \). Then the multiplication in \( \bar{A} \oplus \bar{B} \) (or \( E \)) is given by

\[
\mathcal{M}(a_1, b_1), (a_2, b_2)) = (a_1 \ll a_2 + b_1 \ll a_2 + a_1 \ll b_2 + \chi(b_1, b_2), b_1 \ll b_2)\]

for any \( a_1, a_2 \in A, b_1, b_2 \in B \). Thus,

\[
(\mathcal{M}_{\bar{A} \bar{B}})((0, b_1), (a_2, 0)) = \rho_\bar{A} \circ \mathcal{M}(0, b_1), (a_2, 0)) = b_1 \ll a_2.
\]

Hence we can identify \( \mathcal{M}_{\bar{B} \bar{A}} \) with \( \ll \). Similarly, we identify \( \mathcal{M}_{\bar{A} \bar{B}} \) with \( \chi \). \( \square \)

Next, we consider the associator \( \mathcal{A}_{\text{ss}} \) of the multiplication \( \mathcal{M} \) in \( \bar{A} \oplus \bar{B} \), i.e., \( \mathcal{A}_{\text{ss}} = \mathcal{M} \otimes \text{id} - \text{id} \otimes \mathcal{M} \). Consider the components of \( \mathcal{A}_{\text{ss}} \), we have the following lemma.
Proposition 5.2. With the above notations, we have

(i) $\text{Ass}^b_{BBB} = 0$ is equivalent to the multiplication in $B$ is associative;
(ii) $\text{Ass}^1_{AAA} = 0$ is equivalent to the multiplication in $A$ is associative;
(iii) $\text{Ass}^b_{BBA} = 0$ is equivalent to the equation (9);
(iv) $\text{Ass}^1_{ABB} = 0$ is equivalent to the equation (10);
(v) $\text{Ass}^b_{ABB} = 0$ is equivalent to the equation (11);
(vi) $\text{Ass}^1_{ABB} = 0$ is equivalent to the equation (12);
(vii) $\text{Ass}^b_{BAA} = 0$ is equivalent to the equation (13);
(viii) $\text{Ass}^b_{ABA} = 0$ is equivalent to the equation (14);
(ix) $\text{Ass}^b_{BBB} = 0$ is equivalent to the equation (15).

Proof. Using the conclusion of Lemma 5.1, this proposition can be obtained by direct calculation. The details are as follows. Let $a_1, a_2, a_3 \in A$ and $b_1, b_2, b_3 \in B$.

(i) Note that

$$\text{Ass}^b_{BBB}(a_1, b_1), (a_2, b_2), (a_3, b_3)$$

$$= (\text{Ass}^b_{BBB})(a_1, b_1), (a_2, b_2), (a_3, b_3)$$

$$= (\text{Ass}^b_{BBB})(a_1, b_1), (a_2, b_2), (a_3, b_3)$$

Thus $\text{Ass}^b_{BBB} = 0$ is equivalent to the multiplication in $B$ is associative. Similarly, we get (ii).

(iii) Since

$$\text{Ass}^b_{BBA}(a_1, b_1), (a_2, b_2), (a_3, b_3)$$

$$= (\text{Ass}^b_{BBA})(a_1, b_1), (a_2, b_2), (a_3, b_3)$$

That is to say, $\text{Ass}^b_{BBB} = 0$ is equivalent to equation (9). Similarly, we can obtain (iv)-(ix). \qed

Remark 5.3. By direct calculation, we have $\mathcal{A}_{ABB}^b = \mathcal{A}_{ABB}^b = \mathcal{A}_{ABA}^b = \mathcal{A}_{ABA}^b = \mathcal{A}_{AAA}^b = 0$.

Given two associative conformal algebras $A$ and $B$, we get an associative conformal algebra structure on the direct sum of $\mathbb{C}[\partial]$-module $A \oplus B$ by the conformal multiplication $m_{AB}$:

$$(m_{AB})(a_1, b_1), (a_2, b_2) = (a_1, b_1) \circ (a_2, b_2)$$

for $a_1, a_2 \in A$ and $b_1, b_2 \in B$, where $m_A$ and $m_B$ are the conformal multiplications of $A$ and $B$ respectively. Considering the Hochschild complex of $A \oplus B$, we get a differential graded Lie algebra $(C^{**}(A \oplus B), \lbrack -,-\rbrack, \partial)$, where $\lbrack -,-\rbrack$ is the Gerstenhaber bracket, $\partial = $
if and only if

Let $A$ and $B$ be two associative conformal algebras on $\mathbb{C}[\partial]$-modules $\tilde{A}$ and $\tilde{B}$ respectively. Then we have

$$[m_{A\otimes B}, -] = [m_A + m_B, -].$$

View $A$ as an $A \oplus B$-module via the action of $A$ on itself, and consider the Hochschild complex of $A \oplus B$ with coefficients in $A$, we define

$$L = \bigoplus_{i \geq 0} L^i,$$

$$L^i = \bigoplus_{m+n=i, m \geq 0, n > 0} L^{m,n},$$

where $L^{m,n}$ is the set all conformal sesquilinear maps from $(A \otimes m \otimes B \otimes n) \oplus (A \otimes (m-1) \otimes B \otimes A \oplus B \otimes A \otimes \ldots \otimes (B \otimes A \otimes \ldots \otimes A \otimes m))$ to $A$. Then clearly, $L^i \subseteq C^{i+1}(A \oplus B, A) \subseteq C^{i+1}(A \oplus B, A \oplus B)$ and $C^{i+1}(A \oplus B, A) = L^i \otimes C^{i+1}(A, A)$. We also denote the restriction of Gerstenhaber bracket and differential on $L$ by $[-, -]$ and $\bar{\partial}$.

**Proposition 5.4.** With the above notations, the triples $(L, [\cdot, \cdot], \bar{\partial})$ is a sub-differential graded Lie algebra of $(C^{i+1}(A \oplus B, A \oplus B), [-, -], \bar{\partial})$. Its degree 0 part is abelian.

**Proof.** (i) $L$ is closed under the Gerstenhaber bracket. Denote by $L_{i,k}$ the set all conformal sesquilinear maps from $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ to $\mathbb{C}$, where $A$ appears $l$-times and $B$ appears $k$-times in $X_1, X_2, \ldots X_n$ and $l + k = n$. Then any $f \in L_{i,n}$ can be decomposed as the sum of $f_{i,j}$, where $f_{i,j} \in L_{i,j}, i + j = m + n, j > 0$. For any $f \in L_{i,n}$ and $g \in L_{m,n}$, we get

$$[f, g] \in L_{i+m+1,n+1},$$

since $[f_{i,j}, g_{i',j'}] \in L_{i+i'-1,j+j'}$ by the definition of the Gerstenhaber bracket. Thus, $L$ is closed under the Gerstenhaber bracket.

(ii) $L$ is closed under the differential. For any $f \in L_{i,n} \subseteq L^i$, $\bar{\partial}(f) = [m_A + m_B, f]$. Note that $f$ can be decomposed as the sum of $f_{i,j}$, $i + j = m + n, j > 0$, and

$$[m_A, f_{i,j}] \in L_{i+1,j}, \quad [m_B, f_{i,j}] \in L_{i,j+1},$$

$L_{i+1,j} \subseteq L^{i+1}$ and $L_{i,j+1} \subseteq L^{i+1}$, we get $\bar{\partial}(f_{i,j}) \in L^{i+1}$, and so that $\bar{\partial}(f) \in L^{i+1}$. That is to say, $L$ is closed under the differential.

Finally, note that $L^0$ is the set of all the $\mathbb{C}[\partial]$-module homomorphisms from $B$ to $A$, we get $L^0$ is abelian under the Gerstenhaber bracket. 

**Lemma 5.5.** Let $A$ and $B$ be two associative conformal algebras on $\mathbb{C}[\partial]$-modules $\tilde{A}$ and $\tilde{B}$ respectively. Then we have

$$\epsilon := \mathcal{A} \mathcal{S} \tilde{A} \tilde{B} + \mathcal{A} \mathcal{S} \tilde{B} \tilde{A} + \mathcal{A} \mathcal{S} \tilde{A} \tilde{A} \mathcal{B} + \mathcal{A} \mathcal{S} \tilde{B} \tilde{A} \mathcal{A} + \mathcal{A} \mathcal{S} \tilde{A} \tilde{A} \mathcal{B} = 0$$

if and only if $\epsilon := \mathfrak{M} \tilde{A} \tilde{B} + \mathfrak{M} \tilde{B} \tilde{A} + \mathfrak{M} \tilde{A} \tilde{B} \tilde{A} \tilde{B} \mathcal{MC}(L)$.

**Proof.** For any $e_1 = (a_1, b_1)$, $e_2 = (a_2, b_2)$, $e_3 = (a_3, b_3) \in A \oplus B$, we need compute $(\bar{\partial}_2(\epsilon) + \frac{1}{2}[\epsilon, \epsilon])_{\mu}(e_1, e_2, e_3)$. Note that $\bar{\partial}_2(\epsilon) = [m_A, \epsilon] + [m_B, \epsilon]$, and we can identify $\mathfrak{M} \tilde{B} \tilde{A}$ and $\mathfrak{M} \tilde{A} \tilde{A}$ with $m_B$ and $m_A$ respectively, we have

$$([m_A, \epsilon])_{\mu}(e_1, e_2, e_3) = (\mathfrak{M} \tilde{A} \tilde{A}),_{\lambda+\mu}(\mathfrak{M} \tilde{B} \tilde{B} \tilde{A}),_{\lambda}(e_1, e_2, e_3) - (\mathfrak{M} \tilde{A} \tilde{A}),_{\lambda}(e_1, (\mathfrak{M} \tilde{B} \tilde{B} \tilde{A})),_{\lambda}(e_2, e_3) + (\mathfrak{M} \tilde{A} \tilde{A}),_{\mu}(\mathfrak{M} \tilde{B} \tilde{B} \tilde{A}),_{\lambda}(e_1, e_2, e_3) - (\mathfrak{M} \tilde{A} \tilde{A}),_{\lambda}(e_1, (\mathfrak{M} \tilde{B} \tilde{B} \tilde{A})),_{\lambda}(e_2, e_3) + (\mathfrak{M} \tilde{A} \tilde{A}),_{\mu}(\mathfrak{M} \tilde{B} \tilde{B} \tilde{A}),_{\lambda}(e_1, e_2, e_3) - (\mathfrak{M} \tilde{A} \tilde{A}),_{\lambda}(e_1, (\mathfrak{M} \tilde{B} \tilde{B} \tilde{A})),_{\lambda}(e_2, e_3) + (\mathfrak{M} \tilde{A} \tilde{A}),_{\mu}(\mathfrak{M} \tilde{B} \tilde{B} \tilde{A}),_{\lambda}(e_1, e_2, e_3) - (\mathfrak{M} \tilde{A} \tilde{A}),_{\lambda}(e_1, (\mathfrak{M} \tilde{B} \tilde{B} \tilde{A})),_{\lambda}(e_2, e_3).$$
Proposition 5.6.\[ ([m_B, c])_{\lambda, \mu}(e_1, e_2, e_3) = (\mathfrak{m}^B_{AB})_{\lambda, \mu}(\mathfrak{m}^B_{BB})(e_1, e_2, e_3) - (\mathfrak{m}^B_{BB})(e_1, (\mathfrak{m}^B_{BB})_{\lambda, \mu}(e_2, e_3)) + (\mathfrak{m}^B_{BA})_{\lambda, \mu}(\mathfrak{m}^B_{BB})(e_1, e_2, e_3) - (\mathfrak{m}^B_{BA})(e_1, (\mathfrak{m}^B_{BB})_{\lambda, \mu}(e_2, e_3)),\]

and

\[ ([\bar{1}/2, c])_{\lambda, \mu}(e_1, e_2, e_3) = (\mathfrak{m}^B_{AB})_{\lambda, \mu}(\mathfrak{m}^B_{BB})(e_1, e_2, e_3) - (\mathfrak{m}^B_{AB})(e_1, (\mathfrak{m}^B_{BB})_{\lambda, \mu}(e_2, e_3)) + (\mathfrak{m}^B_{BA})_{\lambda, \mu}(\mathfrak{m}^B_{BB})(e_1, e_2, e_3) - (\mathfrak{m}^B_{BA})(e_1, (\mathfrak{m}^B_{BB})_{\lambda, \mu}(e_2, e_3)).\]

Compared with the calculation of Proposition 5.2, we get

\[ ([m_A, c] + [m_B, c] + \frac{1}{2}([\bar{1}/2, c])_{\lambda, \mu}(e_1, e_2, e_3) = e_{\lambda, \mu}(e_1, e_2, e_3).\]

Thus we obtain the lemma. \[\square\]

Now we can give a 1-1 correspondence between the non-abelian 2-cocycles on \(B\) with values in \(A\) and Maurer-Cartan elements of \(L\).

**Proposition 5.6.** Let \(A\) and \(B\) be two associative conformal algebras, and \(L\) be the differential graded Lie algebra defined as above. Then we have a 1-1 correspondence

\[ Z^2_{nab}(B, A) \leftrightarrow \text{MC}(L) \]

\[ (\triangleright, \lhd, \chi) \leftrightarrow \chi + \triangleright + \lhd. \]

**Proof.** By Theorem 4.8, we know that an element in \(Z^2_{nab}(B, A)\) can be view as a non-abelian extension of \(B\) by \(A\). Given a non-abelian extension of \(B\) by \(A\), since the extension algebra is associative, we get the associator \(\mathcal{A} = \mathcal{A}^A_{AA} + \mathcal{A} + \mathcal{A}^B_{BB} = 0\). Since \(A\) and \(B\) are associative, i.e., \(\mathcal{A}^A_{AAA} = 0\) and \(\mathcal{A}^B_{BBB} = 0\), we have \(\mathcal{A} = 0\). By Lemma 5.5, there exists a unique Maurer-Cartan element \(c = \mathfrak{m}^A_{BB} + \mathfrak{m}^A_{BA} + \mathfrak{m}^A_{AB} = \chi + \triangleright + \lhd\) of \(L\) such that \(e = 0\) Thus we obtain the 1-1 correspondence. \(\square\)

Next we show that the 1-1 correspondence in the proposition above also keeps the equivalence relations on \(Z^2_{nab}(B, A)\) and \(\text{MC}(L)\). We then give the main theorem of this section.

**Theorem 5.7.** Let \(A\) and \(B\) be two associative conformal algebras, and \(L\) be the differential graded Lie algebra defined as above. Then we have a 1-1 correspondence

\[ \text{HH}^2_{nab}(B, A) \cong \text{MC}(L). \]

**Proof.** We need show that equivalence relation on non-abelian 2-cocycles coincides with gauge relation on \(\text{MC}(L)\). Recall that two elements \(c\) and \(\tilde{c}\) in \(\text{MC}(L)\) are equivalent if there exists \(\xi \in L^0 \subseteq \text{Hom}_{\mathbb{C}[\partial]}(A \oplus B, A)\) such that

\[ \tilde{c} = e^{ad_{\xi}}c - \frac{e^{ad_{\xi}} - 1}{ad_{\xi}}d_1(\xi). \]

We consider \(c = \chi + \triangleright + \lhd = \mathfrak{m}^A_{BB} + \mathfrak{m}^A_{BA} + \mathfrak{m}^A_{AB}\). Since \(L^0\) is the set of all \(\mathbb{C}[\partial]\)-module homomorphisms from \(B\) to \(A\), \(\xi\) is ad-nilpotent. Thus for any \(e_1 = (a_1, b_1), e_2 = (a_2, b_2) \in A \oplus B\),
we have

\[ e^{\text{ad}_\xi}(\cdot, \cdot)_{\cdot}(e_1, e_2) = \left( \xi \cdot \left[ \xi, \gamma \right] + \frac{1}{2} \left\{ \xi, \left[ \xi, \gamma \right] \right\} + \cdots \right)_{\cdot}(e_1, e_2) \]

\[ = b_1 \triangleright_{(4)} a_2 + a_1 \triangleleft_{(4)} b_2 + \chi_{\cdot}(b_1, b_2) + ([\xi, \gamma])_{\cdot}(e_1, e_2). \]

Note that \([\xi, \gamma] = 0\), since \(\xi\) and \(\gamma\) take values in \(A\) and \(A[\cdot]\), and

\[
\begin{align*}
([\xi, \gamma])_{\cdot}(e_1, e_2) &= \xi(\gamma(e_1)) - b_1 \triangleright_{(4)} \cdot \xi(b_2), \\
([\xi, \gamma])_{\cdot}(e_1, e_2) &= \xi(\gamma(e_1)) - \chi_{\cdot}(b_1) (b_2) - b_1 \triangleright_{(4)} \cdot \xi(b_2). 
\end{align*}
\]

Next, since \(\tilde{d}_1(\xi)_{\cdot}(e_1, e_2) = ([m_A + m_B, \gamma])_{\cdot}(e_1, e_2)\) and

\[
\begin{align*}
([m_A, \gamma])_{\cdot}(e_1, e_2) &= (m_A)_{\cdot}(\gamma(e_1), e_2) + (m_A)_{\cdot}(e_1, \gamma(e_2)) - \xi((m_A)_{\cdot}(e_1, e_2)) \\
&= \xi(b_1) \triangleright_{(4)} a_2 + a_1 \triangleleft_{(4)} \xi(b_2), \\
([m_B, \gamma])_{\cdot}(e_1, e_2) &= (m_B)_{\cdot}(\gamma(e_1), e_2) + (m_B)_{\cdot}(e_1, \gamma(e_2)) - \xi((m_B)_{\cdot}(e_1, e_2)) \\
&= -\xi(b_1) \triangleright_{(4)} \xi(b_2),
\end{align*}
\]

we obtain

\[
\tilde{d}_1(\xi)_{\cdot}(e_1, e_2) = \xi(b_1) \triangleright_{(4)} a_2 + a_1 \triangleleft_{(4)} \xi(b_2) - \xi(b_1) \triangleright_{(4)} \xi(b_2),
\]

and \([\xi, \tilde{d}_1(\xi)] = [\xi, m_A \circ (\xi, -)] + m_A \circ (-, \xi) - \xi \circ m_B\). By direct calculation, we have

\[
\begin{align*}
([m_A \circ (\xi, -)]_{\cdot}(e_1, e_2) &= \xi((m_A)_{\cdot}(e_1, e_2)) - (m_A)_{\cdot}(\xi(e_1), e_2) - (m_A)_{\cdot}(e_1, \xi(e_2)) \\
&= -\xi(b_1) \triangleright_{(4)} \xi(b_2), \\
([m_A \circ (-, \xi)]_{\cdot}(e_1, e_2) &= \xi((m_A)_{\cdot}(e_1, e_2)) - (m_A)_{\cdot}(\xi(e_1), e_2) - (m_A)_{\cdot}(e_1, \xi(e_2)) \\
&= -\xi(b_1) \triangleright_{(4)} \xi(b_2),
\end{align*}
\]

and \(([\xi, \gamma]_{\cdot}(e_1, e_2) = 0\). That is to say,

\[
([\xi, \tilde{d}_1(\xi)])_{\cdot}(e_1, e_2) = -2\xi(b_1) \triangleright_{(4)} \xi(b_2). \]

Note that \((\text{ad}_\xi)^n = 0\) for all \(n \geq 2\), since \(\xi\) takes values in \(A\), and

\[
\frac{e^{\text{ad}_\xi} - 1}{\text{ad}_\xi} \tilde{d}_1(\xi)_{\cdot}(e_1, e_2) = \sum_{n=0}^{\infty} \frac{1}{(n + 1)!} (\text{ad}_\xi)^n \tilde{d}_1(\xi),
\]

we get

\[
\frac{e^{\text{ad}_\xi} - 1}{\text{ad}_\xi} \tilde{d}_1(\xi)_{\cdot}(e_1, e_2) = \tilde{d}_1(\xi)_{\cdot}(e_1, e_2) + \frac{1}{2} ([\xi, \tilde{d}_1(\xi)])_{\cdot}(e_1, e_2) \\
= \xi(b_1) \triangleright_{(4)} a_2 + a_1 \triangleleft_{(4)} \xi(b_2) - \xi(b_1) \triangleright_{(4)} \xi(b_2). 
\]
Therefore an element \( \tilde{c} := \tilde{\chi} + \tilde{\eta} + \tilde{\xi} \in \text{MC}(L) \) is equivalent to \( c \) if
\[
\tilde{c}(e_1, e_2) = \langle \tilde{\chi} + \tilde{\eta} + \tilde{\xi} \rangle (e_1, e_2)
\]
\[
= \tilde{c}(e_1, e_2) - b_1 \triangleright (a) \xi(b_2) - \xi(b_1) \pretriangleright (a) b_2 - \xi(b_1) (a) a_2
\]
\[
- d_1 (a) \xi(b_2) + \xi(b_1) b_2 + \xi(b_1) (a) \xi(b_2)
\]
\[
= (b_1 \triangleright (a) a_2 - \xi(b_1) (a) a_2) + (a_1 \pretriangleright (a) b_2 - a_1 (a) \xi(b_2))
\]
\[
+ (\chi_1 (b_1, b_2) - b_1 \triangleright (a) \xi(b_2) - \xi(b_1) \pretriangleright (a) b_2 + \xi(b_1) (a) \xi(b_2)).
\]
Thus we get that two non-abelian 2-cocycles (16)-(18) are satisfied, if and only if the equation above is satisfied, if and only if \( \tilde{c} \) and \( \tilde{\chi} + \tilde{\eta} + \tilde{\xi} \) are gauge equivalent.

Thus, for any associative conformal algebras \( A \) and \( B \), we have the following 1-1 correspondences
\[
\text{Ext}_{\text{non}}(B, A) \leftrightarrow \text{HH}_{\text{non}}^2(B, A) \leftrightarrow \text{MC}(L)
\]
\[
[E(\nabla, \xi, \eta)] \leftrightarrow [(\nabla, \xi, \eta)] \leftrightarrow [\nabla + \xi + \eta],
\]
where \([E(\nabla, \xi, \eta)]\) means the equivalent class of \( E(\nabla, \xi, \eta) \).

Finally, we consider the abelian extensions of associative conformal algebras. Let \( B \) be an associative conformal algebra and \( A \) is an associative conformal algebra with trivial multiplication. Then a non-abelian extension \( E : 0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0 \) is just an abelian extension of \( B \) by \( A \). Denote the corresponding non-abelian 2-cocycle by \((\nabla, \xi, \eta)\). Then the differential graded Lie algebra \((L, [-, -], d)\) degenerates to \((\mathfrak{C}^{*+1}(B, A), [-, -], d)\), where the bracket \([-,-]_G\) is trivial and \(d\) is given in Section 2. In this case, the Maurer-Cartan element of \((\mathfrak{C}^{*+1}(B, A), [-, -], d)\) corresponding to extension \( E \) is just \( \chi \in \mathcal{Z}^2(B, A) \), and two extensions \( E \) and \( \tilde{E} \) are equivalent if and only if \( \chi - \tilde{\chi} \in \mathcal{B}^2(B, A) \) by equation (18). Thus, in this case, \( \text{MC}(L) \) is exactly \( \text{HH}^2(B, A) \), and so that Theorem 4.8 and Theorem 5.7 are the same.

6. THE INDUCIBILITY PROBLEM OF AUTOMORPHISMS

In this section, we study the inducibility of a pair of automorphisms about a non-abelian extension of associative conformal algebras, and give the fundamental sequence of Wells in the context of associative conformal algebras.

Let \( A \) and \( B \) be two associative conformal algebras, \( E : 0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0 \) be a non-abelian extension of \( B \) by \( A \) with a section \( \gamma \). Denote the corresponding non-abelian 2-cocycle of \( E \) by \((\nabla, \xi, \eta)\). From the conclusion of Theorem 4.8, one can often identify \( E \) with \( A \oplus B \) under the multiplication defined in Lemma 4.7, identify \( \alpha \) and \( \beta \) with an injection and a projection respectively. Denote
\[
\text{Aut}_A(E) := \{ f \in \text{Aut}(E) \mid f(A) = A \}.
\]
Then, \( f|_A \in \text{Aut}(A) \) if \( f \in \text{Aut}_A(E) \). For any \( f \in \text{Aut}_A(E) \), we define a \( \mathbb{C}[[\partial]] \)-module homomorphism
\[
\hat{f} := \beta \circ f \circ \gamma : B \rightarrow B.
\]
Then one can check that \( \hat{f} \) is independent of the choice of the section \( \gamma \). Moreover, since \( \beta \) is regarded as the projection onto \( B \) and \( f \) preserves \( B \), we get \( \hat{f} \) is a bijection. For any \( b_1, b_2 \in B \),
we have
\[
\hat{f}(b_1 (a) b_2) = \beta \circ f \circ \gamma(b_1 (a) b_2)
\]
\[
= \beta \circ f(\gamma(b_1 (a) \gamma(b_2) - \chi_A(b_1, b_2))
\]
\[
= \beta \circ f(\gamma(b_1) \gamma(b_2))
\]
\[
= \hat{f}(b_1) \hat{f}(b_2).
\]

This means \(\hat{f} \in \text{Aut}(B)\). Note that \(g \circ f = \beta \circ g \circ f \circ \gamma = \beta \circ g \circ \gamma \circ \beta \circ f \circ \gamma = \hat{g} \circ \hat{f}\) for \(f, g \in \text{Aut}_A(E)\), we obtain a group homomorphism
\[
\kappa : \text{Aut}_A(E) \rightarrow \text{Aut}(A) \times \text{Aut}(B), \quad f \mapsto (f|_A, \hat{f}).
\]

**Definition 6.1.** Let \(A\) and \(B\) be two associative conformal algebras, \(E : 0 \rightarrow A \overset{\alpha}{\rightarrow} E \overset{\beta}{\rightarrow} B \rightarrow 0\) be a non-abelian extension of \(B\) by \(A\), \((\triangleright, \triangleleft, \chi)\) be the corresponding non-abelian 2-cocycle. A pair \((g, h) \in \text{Aut}(A) \times \text{Aut}(B)\) of automorphisms is said to be inducible if there exists a map \(f \in \text{Aut}_A(E)\) such that \((g, h) = (f|_A, \hat{f})\).

Given a pair \((g, h) \in \text{Aut}(A) \times \text{Aut}(B)\) and a non-abelian 2-cocycle \((\triangleright, \triangleleft, \chi)\), we define a new triple \((\triangleright^{g,h}, \triangleleft^{g,h}, \chi^{g,h})\) with \(\triangleright^{g,h} : B \times A \rightarrow A[\lambda]\), \(\triangleleft^{g,h} : A \times B \rightarrow A[\lambda]\) and \(\chi^{g,h} : B \times B \rightarrow A[\lambda]\) by
\[
\triangleright^{g,h}(b, a) = g(h^{-1}(b) \triangleright (a) g^{-1}(a)),
\]
\[
\triangleleft^{g,h}(a, b) = g^{-1}(a) \triangleleft (a h^{-1}(b)),
\]
\[
\chi^{g,h}_A(b_1, b_2) = g \circ \chi_A(h^{-1}(b_1), h^{-1}(b_2)),
\]
for \(a \in A\) and \(b, b_1, b_2 \in B\). Then one can check that \((\triangleright^{g,h}, \triangleleft^{g,h}, \chi^{g,h})\) is also a non-abelian 2-cocycle on \(B\) with values in \(A\).

**Theorem 6.2.** Let \(E : 0 \rightarrow A \overset{\alpha}{\rightarrow} E \overset{\beta}{\rightarrow} B \rightarrow 0\) be a non-abelian extension of associative conformal algebra \(B\) by associative conformal algebra \(A\), \((\triangleright, \triangleleft, \chi)\) be the corresponding non-abelian 2-cocycle. With the above notations, a pair \((g, h) \in \text{Aut}(A) \times \text{Aut}(B)\) is inducible if and only if the non-abelian 2-cocycles \((\triangleright, \triangleleft, \chi)\) and \((\triangleright^{g,h}, \triangleleft^{g,h}, \chi^{g,h})\) are equivalent.

**Proof.** If the pair \((g, h)\) is inducible, there exists an element \(f \in \text{Aut}_A(E)\) such that \(f|_A = g\) and \(\hat{f} = \beta \circ f \circ \gamma = h\), where \(\gamma\) is a section of \(E\). Note that for any \(b \in B\),
\[
\beta(f \circ \gamma \circ h^{-1}(b) - \gamma(b)) = b - b = 0,
\]
we get \( f \circ \gamma \circ h^{-1}(b) - \gamma(b) \in \text{Ker}(\beta) \cong A \). Thus we can define a \( \mathbb{C}[\partial]\)-module homomorphism \( \omega : B \to A, b \mapsto \gamma(b) - f \circ \gamma \circ h^{-1}(b) \). Therefore, for any \( a \in A \) and \( b, b_1, b_2 \in B \), we have

\[
(\chi_A - \chi_A^{g,h})(b_1, b_2) = \chi_A(b_1, b_2) - g \circ \chi_A(h^{-1}(b_1), h^{-1}(b_2))
\]

\[
= -f(\gamma \circ h^{-1}(b_1)(\_a)) \gamma \circ h^{-1}(b_2) - \gamma(h^{-1}(b_1)(\_a)) h^{-1}(b_2))
+ \gamma(b_1)(\_a) \gamma(b_2) - \gamma(b_1)(\_a) b_2)
\]

\[
= f \circ \gamma \circ h^{-1}(b_1) \_a) \gamma(b_2) - f \circ \gamma \circ h^{-1}(b_2))
+ (\gamma(b_1) - f \circ \gamma \circ h^{-1}(b_1)) \_a) f \circ \gamma \circ h^{-1}(b_2) - \gamma(b_1)(\_a) b_2)
- (\gamma(b_1) - f \circ \gamma \circ h^{-1}(b_1)) \_a) \gamma(b_2) - f \circ \gamma \circ h^{-1}(b_2))
- (\gamma - f \circ \gamma \circ h^{-1})(b_1(\_a) b_2)
= b_1 \_a) \omega(b_2) + \omega(b_1) \_a) b_2 - \omega(b_1)(\_a) b_2 - \omega(b_1)(\_a) \omega(b_2),
\]

and

\[
(\_a) - \_a^{g,h}(b, a) = b \_a) a - g(h^{-1}(b) \_a) g^{-1}(a))
= \gamma(b)(\_a) a - f(\gamma \circ h^{-1}(b)(\_a) g^{-1}(a))
= (\gamma(b) - f \circ \gamma \circ h^{-1}(b))(\_a) a
= \omega(b)(\_a) a.
\]

Similarly, \( (\_a) - \_a^{g,h}(a, b) = a(\_a) \omega(b) \). Thus, it follows from Definition 4.3 that \( (\_a, \_a, \chi) \) and \( (\_a^{g,h}, \_a^{g,h}, \chi^{g,h}) \) are equivalent.

On the other hand, if the non-abelian 2-cocycles \( (\_a, \_a, \chi) \) and \( (\_a^{g,h}, \_a^{g,h}, \chi^{g,h}) \) are equivalent, there exists a \( \mathbb{C}[\partial]\)-module homomorphism \( \omega : B \to A \) satisfying the equations (16)-(18), and there is a commutative diagram:

\[
0 \longrightarrow A \overset{a}{\longrightarrow} A \oplus (\_a, \_a, \chi) \overset{\beta}{\longrightarrow} B \longrightarrow 0
\]

\[
0 \longrightarrow A \overset{\tilde{a}}{\longrightarrow} A \oplus (\_a^{g,h}, \_a^{g,h}, \chi^{g,h}) \overset{\tilde{\beta}}{\longrightarrow} B \longrightarrow 0,
\]

where \( \tilde{\theta}(a, b) = (a - \omega(b), b) \). Since \( E \cong A \oplus (\_a, \_a, \chi) \) as associative conformal algebras, we denote each element in \( E \) by \( (a, b) \), and define a \( \mathbb{C}[\partial]\)-module homomorphism \( f : E \to E \) by

\[
f(a_1, b_1)(\_a) f(a_2, b_2)
= \tilde{\theta}(g(a_1), h(b_1)(\_a) g(a_2), h(b_2)(\_a))
= \tilde{\theta}(g(a_1)(\_a) g(a_2) + b_1(\_a^{g,h}) g(a_2) + g(a_1)(\_a^{g,h}) b_2 + \chi^{g,h}(a_1, b_1) h(b_2), b_1(\_a)(\_a) \omega(b), b_2)
= f(a_1(\_a) a_2 + b_1(\_a) a_2 + a_1(\_a) b_1 + \chi^{g,h}(a_1, b_1), b_1(\_a) b_2)
= f((a_1, b_1)(\_a) (a_2, b_2)).
\]
This means $f \in \text{Aut}(E)$. Clearly, $f|_A = g$ and $\hat{f} = \beta \circ f \circ \gamma = h$. Hence the pair $(g, h)$ is inducible.

From the equations (16)-(18), we can directly get the following corollary.

**Corollary 6.3.** Let $\mathcal{E} : 0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$ be a non-abelian extension of associative conformal algebra $B$ by associative conformal algebra $A$, $(\bullet, \bigast, \chi)$ be the corresponding non-abelian 2-cocycle. A pair $(g, h) \in \text{Aut}(A) \times \text{Aut}(B)$ is inducible if and only if there exists a $\mathbb{C}[\partial]$-module homomorphism $\omega : B \rightarrow A$ such that for any $a \in A$ and $b, b_1, b_2 \in B$,

\[
\begin{array}{l}
g(b \bigast_{(\lambda)} a) = h(b) \bigast_{(\lambda)} g(a) - \omega \circ h(b) \bigast_{(\lambda)} g(a), \\
g(a \bigast_{(\lambda)} b) = g(a) \bigast_{(\lambda)} h(b) - g(a) \bigast_{(\lambda)} \omega \circ h(b), \\
g \circ \chi_a(b_1, b_2) = \chi_a(h(b_1), h(b_2)) - h(b_1) \bigast_{(\lambda)} \omega \circ h(b_2) + \omega(h(b_1)) \bigast_{(\lambda)} \omega \circ h(b_2) \\
\end{array}
\]

Next, we will interpret the above theorem in terms of the Wells map in the context of associative conformal algebras. Let $\mathcal{E} : 0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$ be a non-abelian extension of $B$ by $A$, $(\bullet, \bigast, \chi)$ be the corresponding non-abelian 2-cocycle. We define a map $\mathcal{W} : \text{Aut}(A) \times \text{Aut}(B) \rightarrow \text{HH}^2_{nab}(B, A)$ by

\[
\mathcal{W}(g, h) = [(\bullet^{g, h}, \bigast^{g, h}, \chi^{g, h}) - (\bullet, \bigast, \chi)].
\]

Then one can check that $\mathcal{W}$ is a group homomorphism, which is called Wells map. Therefore, we can directly obtain the following corollary by Theorem 6.2.

**Corollary 6.4.** Let $\mathcal{E} : 0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$ be a non-abelian extension of associative conformal algebra $B$ by associative conformal algebra $A$. A pair $(g, h) \in \text{Aut}(A) \times \text{Aut}(B)$ of automorphisms is inducible if and only if $\mathcal{W}(g, h) = 0$.

Let $\mathcal{E} : 0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$ be a non-abelian extension of $B$ by $A$. Next, we define

\[
\text{Aut}^\text{id}_A(E) := \{ f \in \text{Aut}_A(E) \mid \kappa(f) = (\text{id}_A, \text{id}_B) \}.
\]

Then $\text{Aut}^\text{id}_A(E)$ is a subgroup of $\text{Aut}_A(E)$ and there exists an exact sequence as following.

**Corollary 6.5.** (Fundamental sequence of Wells) With the above notations, there is an exact sequence

\[
1 \longrightarrow \text{Aut}^\text{id}_A(E) \overset{\iota}{\longrightarrow} \text{Aut}_A(E) \overset{\kappa}{\longrightarrow} \text{Aut}(A) \times \text{Aut}(B) \overset{\mathcal{W}}{\longrightarrow} \text{HH}^2_{nab}(B, A),
\]

where $\iota$ is the inclusion map.

**Proof.** We need show this sequence is exact at $\text{Aut}_A(E)$ and $\text{Aut}(A) \times \text{Aut}(B)$. First, note that an element $f \in \text{Aut}_A(E)$ such that $f \in \text{Ker}(\kappa)$ if and only if $\kappa(f) = (\text{id}_A, \text{id}_B)$, i.e., $f \in \text{Aut}^\text{id}_A(E)$, we obtain the sequence is exact at $\text{Aut}_A(E)$. Second, by Corollary 6.4 we get a pair $(g, h) \in \text{Aut}(A) \times \text{Aut}(B)$ satisfying $(g, h) \in \text{Im}(\kappa)$, i.e., $(g, h)$ is inducible, if and only if $\mathcal{W}(g, h) = 0$. This means the sequence is exact at $\text{Aut}(A) \times \text{Aut}(B)$.

More generally, if define

\[
\text{Aut}^A_A(E) := \{ f \in \text{Aut}_A(E) \mid f|_A = \text{id}_A \},
\]

\[
\text{Aut}^B_A(E) := \{ f \in \text{Aut}_A(E) \mid f = \text{id}_B \},
\]

Then $\text{Aut}^A_A(E)$ and $\text{Aut}^B_A(E)$ are subgroups of $\text{Aut}_A(E)$ and the sequence above is also exact at $\text{Aut}^A_A(E)$ and $\text{Aut}^B_A(E)$.
we obtain two morphisms of groups \( \kappa_B : \text{Aut}^A_A(E) \rightarrow \text{Aut}(B) \), \( f \mapsto f|_A \), and \( \kappa_A : \text{Aut}^B_B(E) \rightarrow \text{Aut}(A) \), \( f \mapsto f|_A \). Define maps \( \mathcal{W}_A : \text{Aut}(A) \rightarrow \text{HH}^{2}_{\text{hab}}(B, A) \) and \( \mathcal{W}_B : \text{Aut}(B) \rightarrow \text{HH}^{2}_{\text{hab}}(B, A) \) by

\[
\mathcal{W}_A(g) := [(\lambda, \text{id}^A) \circ \chi^A, \chi^A \circ \lambda] = (\lambda, \chi),
\]

\[
\mathcal{W}_B(h) := [(\lambda^A, h), \chi^A \circ \lambda^A] = (\lambda, \chi).
\]

Then similar to Corollary 6.5, we have two exact sequences of groups.

**Proposition 6.6.** Let \( \mathcal{E} : 0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0 \) be a non-abelian extension of associative conformal algebra \( B \) by associative conformal algebra \( A \). With the above notations, there are two exact sequences

\[
\begin{array}{ccc}
1 & \rightarrow & \text{Aut}^A_A(E) \xrightarrow{\iota_A} \text{Aut}^B_B(E) \xrightarrow{\kappa_A} \text{Aut}(A) \xrightarrow{\mathcal{W}_A} \text{HH}^{2}_{\text{hab}}(B, A) , \\
1 & \rightarrow & \text{Aut}^A_A(E) \xrightarrow{\iota_B} \text{Aut}^A_A(E) \xrightarrow{\kappa_B} \text{Aut}(B) \xrightarrow{\mathcal{W}_B} \text{HH}^{2}_{\text{hab}}(B, A) ,
\end{array}
\]

where \( \iota_A \) and \( \iota_B \) are inclusion maps.

By using the exact sequences in this proposition, we can give some results of the lifting problem of automorphism in a non-abelian extension.

**Corollary 6.7.** Let \( \mathcal{E} : 0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0 \) be a non-abelian extension of associative conformal algebra \( B \) by associative conformal algebra \( A \).

(i) An automorphism \( g \in \text{Aut}(A) \) can be extended to an automorphism of \( E \) inducing the identity on \( B \) if and only if \( \mathcal{W}_A(g) = 0 \).

(ii) An automorphism \( h \in \text{Aut}(B) \) can be lifted to an automorphism of \( E \) fixing \( B \) pointwise if and only if \( \mathcal{W}_B(h) = 0 \).

At the end of this section, let’s go back to the abelian extension. Let \( B \) be an associative conformal algebra, \( A \) be a bimodule over \( B \), and \( \mathcal{E} : 0 \rightarrow A \xrightarrow{\alpha} E \xrightarrow{\beta} B \rightarrow 0 \) be an abelian extension of \( B \) by \( A \), where the \( B \)-bimodule structure on \( A \) is given by \( (\lambda, \psi) \). Denote by \( \text{Aut}(A) \) the set of all \( \mathbb{C}[\partial] \)-module automorphisms, and by \( \text{Aut}_{\mathcal{E}}(A, B) \) the set of all pairs \( (g, h) \in \text{Aut}(A) \times \text{Aut}(B) \) satisfying

\[
g(b \lambda a) = h(b) \lambda g(a), \quad g(a \lambda b) = g(a) \lambda h(b),
\]

for \( a \in A \) and \( b \in B \). Then \( \text{Aut}_{\mathcal{E}}(A, B) \) is a subgroup of \( \text{Aut}(A) \times \text{Aut}(B) \). Let \( \gamma \) be a section of \( \mathcal{E} \), and \( \chi \) be the 2-cocycle corresponding to \( \mathcal{E} \). For a pair \( (g, h) \in \text{Aut}(A) \times \text{Aut}(B) \), we define \( \chi^{g,h} : B \times B \rightarrow A[\lambda] \) by

\[
\chi^{g,h}_{(b_1, b_2)}(b_1, b_2) = g \circ \chi(h^{-1}(b_1), h^{-1}(b_2)),
\]

for \( b_1, b_2 \in B \). If \( (g, h) \in \text{Aut}_{\mathcal{E}}(A, B) \), we have

\[
d_2(\chi^{g,h}, \alpha_1 \beta_2 (b_1, b_2, b_3)) = g \circ d_2(\chi)(\alpha_1 \beta_2 (h^{-1}(b_1), h^{-1}(b_2), h^{-1}(b_3))) = 0,
\]

for any \( b_1, b_2, b_3 \in B \), since \( d_2(\chi) = 0 \). That is, \( \chi^{g,h} \in Z^2(B, A) \). From Theorem 6.2 and Corollary 6.3, we obtain following theorem.
Corollary 6.10. category of groups. Thus we have the following corollary.

\[ \iota \] semidirect product

\[ A \] as groups.

\[ \beta \] (2-cocycles \( \chi \) and \( \chi^{\gamma,h} \) are equivalent, if and only if there exists a \( \mathbb{C}[\partial] \)-module homomorphism \( \omega : B \to A \) such that for any \( b_1, b_2 \in B \),

\[ g \circ \chi_A(b_1, b_2) = \chi_A(h(b_1), h(b_2)) - h(b_1) \rhd_{(4)} \omega \circ h(b_2) + \omega(h(b_1) \rhd_{(4)} h(b_2)) - \omega \circ h(b_1) \rhd_{(4)} h(b_2). \]

We next define the Wells map for the abelian extensions by

\[ \tilde{W} : \text{Aut}_{\mathcal{E}}(A, B) \to \text{HH}^2(B, A) \quad (g, h) \mapsto [\chi^{\gamma,h} - \chi]. \]

Let \( \gamma \) be a section of \( \mathcal{E} \). For any \( f \in \text{Aut}_A(E) \), we define \( \hat{f} := \beta \circ f \circ \gamma \). Then we obtain a group homomorphism

\[ \tilde{k} : \text{Aut}_A(E) \to \text{Aut}_{\mathcal{E}}(A, B), \quad f \mapsto (f|_A, \hat{f}). \]

The fundamental sequence of Wells can be obtained as the following form.

Corollary 6.9. Let \( \mathcal{E} : 0 \to A \to E \to B \to 0 \) be an abelian extension of associative conformal algebra \( B \) by bimodule \( A \). With the above notations, there is an exact sequence

\[ 1 \to \text{Aut}_A^\mathcal{E}(E) \overset{i}{\to} \text{Aut}_A(E) \overset{k}{\to} \text{Aut}_{\mathcal{E}}(A, B) \overset{\tilde{W}}{\to} \text{HH}^2(B, A), \]

where \( i \) is the inclusion map.

In particular, if the abelian extension \( \mathcal{E} : 0 \to A \overset{\alpha}{\to} E \overset{\beta}{\to} B \to 0 \) is split in the category of associative conformal algebras, then the associative conformal algebra \( E \) is isomorphic to the semidirect product \( A \rtimes B \), where the conformal multiplication on \( A \oplus B \) is given by

\[ (a_1, b_1) \rhd_{(4)} (a_2, b_2) = (a_1 \rhd_{(4)} b_2 + b_1 \rhd_{(4)} a_2, b_1 \rhd_{(4)} b_2), \]

for \( (a_1, b_1), (a_2, b_2) \in A \oplus B \), and the section \( \gamma : B \to E \), is given by \( \gamma(b) = (0, b) \) for \( b \in B \). In this case the corresponding 2-cocycle \( \chi \) of \( \mathcal{E} \) is zero, and the Wells map vanishes identically. Thus we have an exact sequence

\[ 1 \to \text{Aut}_A^\mathcal{E}(E) \overset{i}{\to} \text{Aut}_A(E) \overset{k}{\to} \text{Aut}_{\mathcal{E}}(A, B) \to 1. \]

We define map \( \varrho : \text{Aut}_{\mathcal{E}}(A, B) \to \text{Aut}_A(E) \) by

\[ \varrho(g, h)(a, b) = (g(a), h(b)), \]

for any \( (g, h) \in \text{Aut}_{\mathcal{E}}(A, B) \) and \( (a, b) \in E \). Then \( \tilde{k} \circ \varrho(g, h) = (\varrho(g, h)|_A, \tilde{\gamma}(g, h)) = (g, h), \) since \( \beta \circ \varrho(g, h) \circ \gamma(b) = h(b) \) for any \( b \in B \). That is to say, the exact sequence above is split in the category of groups. Thus we have the following corollary.

Corollary 6.10. Let \( \mathcal{E} : 0 \to A \to E \to B \to 0 \) be a split abelian extension of associative conformal algebra \( B \) by bimodule \( A \). Then

\[ \text{Aut}_A(E) \cong \text{Aut}_A^\mathcal{E}(E) \times \text{Aut}_{\mathcal{E}}(A, B) \]

as groups.
7. The extensibility problem of derivations

For an abelian extension of associative conformal algebras, we construct a Lie algebra such that the 2-th cohomology group is a representation over it. Using this representation, we study the extensibility of a pair of derivations about an abelian extension of associative conformal algebras, and give an exact sequence of Wells type.

Let \( B \) be an associative conformal algebra. Recall that a derivation of \( B \) is a \( \mathbb{C}[\partial] \)-module homomorphism \( D : B \to B \) such that \( D(b_1 \cdot b_2) = D(b_1) \cdot b_2 + b_1 \cdot D(b_2) \), for any \( b_1, b_2 \in B \), where the first \( D \) is extended canonically to a \( \mathbb{C}[\partial] \)-module homomorphism from \( B[\partial] \) to \( B[\partial] \), i.e., \( D(\sum b_i \partial^i) = \sum D(b_i) \partial^i \). We denote the set of all derivations of \( B \) by \( \text{Der}(B) \). Then \( \text{Der}(B) \) is a Lie algebra over \( \mathbb{C} \) with respect to the commutator. Let \( A \) be a (conformal) \( B \)-bimodule, and

\[
E : \quad 0 \longrightarrow A \xrightarrow{\alpha} E \xrightarrow{\beta} B \longrightarrow 0
\]

be an abelian extension of \( B \) by \( A \). Then there is an associative conformal algebra structure on \( \mathbb{C}[\partial] \)-module \( A \oplus B \), which is given by

\[
(a_1, b_1) \cdot (a_2, b_2) = (a_1 \cdot_a b_2 + b_1 \cdot_b a_2, b_1 \cdot_a b_2),
\]

for any \( a_1, a_2 \in A \) and \( b_1, b_2 \in B \), where \( \cdot_a, \cdot_b \) is the \( B \)-bimodule action on \( A \). Here the module action of \( B \) on \( A \) is also given by the multiplication of \( E \). Indeed, let \( \gamma : B \to E \) be a \( \mathbb{C}[\partial] \)-module homomorphism such that \( \beta \circ \gamma = \text{id}_B \). Then \( a_1 \cdot_b b_2 = a_1 \cdot_a \gamma(b_2), b_1 \cdot_a a_2 = \gamma(b_1) \cdot_a a_2 \), and this action does not depend on the choice of \( \gamma \). We denote this associative conformal algebra by \( A \rtimes B \). We view \( A \) as a trivial associative conformal algebra. Then \( \text{Der}(A) \) just the set of all \( \mathbb{C}[\partial] \)-module endomorphisms of \( A \). Then by direct calculation, we have the following lemma.

**Lemma 7.1.** Let \( (\mathcal{D}_A, \mathcal{D}_B) \in \text{Der}(A) \times \text{Der}(B) \). Then \( (\mathcal{D}_A, \mathcal{D}_B) \in \text{Der}(A \rtimes B) \) if and only if

\[
\begin{align*}
\mathcal{D}_A(b \cdot_b a) &= b \cdot_b \mathcal{D}_A(a), \\
\mathcal{D}_A(a \cdot_a b) &= a \cdot_a \mathcal{D}_B(b) + \mathcal{D}_A(a) \cdot_a b,
\end{align*}
\]

for any \( a \in A \) and \( b \in B \).

Denote by

\[ g(A, B) := \{ (\mathcal{D}_A, \mathcal{D}_B) \in \text{Der}(A) \times \text{Der}(B) \mid (\mathcal{D}_A, \mathcal{D}_B) \text{ satisfies equations (19) and (20)} \}. \]

Then one can check that \( g(A, B) \) is a Lie subalgebra of \( \text{Der}(A \rtimes B) \). Now we define an action of \( g(A, B) \) on the space of 2-cochains \( C^2(B, A) \) by

\[
\Theta(\mathcal{D}_A, \mathcal{D}_B)(f)_i(b_1, b_2) = \mathcal{D}_A(f_i(b_1, b_2)) - f_i(\mathcal{D}_B(b_1), b_2) - f_i(b_1, \mathcal{D}_B(b_2)),
\]

for any \( (\mathcal{D}_A, \mathcal{D}_B) \in g(A, B), \ f \in C^2(B, A) \) and \( b_1, b_2 \in B \). Then \( d_2(\Theta(\mathcal{D}_A, \mathcal{D}_B)(f)) = 0 \) for any \( f \in Z^2(B, A) \), and \( \Theta(\mathcal{D}_A, \mathcal{D}_B)(d_1(f)) = d_1(\mathcal{D}_A \circ f - f \circ \mathcal{D}_B) \) for any \( f \in C^1(B, A) \). Thus we get a linear map

\[
\Theta : g(A, B) \longrightarrow \text{gl}(\text{HH}^2(B, A)), \quad \Theta(\mathcal{D}_A, \mathcal{D}_B)([f]) = [\Theta(\mathcal{D}_A, \mathcal{D}_B)(f)],
\]

for any \( (\mathcal{D}_A, \mathcal{D}_B) \in g(A, B) \) and \([f] \in \text{HH}^2(B, A) \). Moreover, one can show that \( \Theta \) is a homomorphism of Lie algebras as following.

**Proposition 7.2.** Then map \( \Theta \) gives a representation of Lie algebra \( g(A, B) \) on \( \text{HH}^2(B, A) \).
Proof. For any \((\mathcal{D}_A, \mathcal{D}_B), (\mathcal{D}_A', \mathcal{D}_B') \in \text{Der}(A) \times \text{Der}(B)\) and \([f] \in \text{HH}^2(B, A)\), note that
\[
\Theta((\mathcal{D}_A, \mathcal{D}_B) \circ \Theta(\mathcal{D}_A', \mathcal{D}_B'))([f])
= \Theta(\mathcal{D}_A, \mathcal{D}_B)([\mathcal{D}_A \circ f - f \circ (\mathcal{D}_B \otimes \text{id}_B) - f \circ (\text{id}_B \otimes \mathcal{D}_B')])
= \Theta(\mathcal{D}_A) \circ \Theta(\mathcal{D}_A') \circ \Theta(\mathcal{D}_B) \circ \Theta(\mathcal{D}_B')([f])
= \Theta((\mathcal{D}_A, \mathcal{D}_B), (\mathcal{D}_A, \mathcal{D}_B))([f]).
\]
we get
\[
\Theta((\mathcal{D}_A, \mathcal{D}_B), \Theta(\mathcal{D}_A', \mathcal{D}_B'))([f])
= \Theta((\mathcal{D}_A, \mathcal{D}_B) \circ \Theta(\mathcal{D}_A', \mathcal{D}_B'))([f]) - \Theta(\Theta(\mathcal{D}_A, \mathcal{D}_B) \circ \Theta(\mathcal{D}_A, \mathcal{D}_B'))([f])
= \Theta(\mathcal{D}_A \circ \mathcal{D}_A' - \mathcal{D}_A \circ \mathcal{D}_B, \mathcal{D}_A \circ \mathcal{D}_B' - \mathcal{D}_B \circ \mathcal{D}_B)([f])
= \Theta((\mathcal{D}_A, \mathcal{D}_B), (\mathcal{D}_A', \mathcal{D}_B'))([f]).
\]
Thus \(\Theta\) is a homomorphism of Lie algebras. \qed

Now we consider the extensibility of a pair of derivations about an abelian extension of associative conformal algebras.

**Definition 7.3.** Let \(E : 0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} B \to 0\) be an abelian extension of associative conformal algebra \(B\) by bimodule \(A\). A pair \((\mathcal{D}_A, \mathcal{D}_B) \in \text{Der}(A) \times \text{Der}(B)\) is called an extensible pair if there is a derivation \(\mathcal{D}_E \in \text{Der}(E)\) such that \(\mathcal{D}_E \circ \alpha = \alpha \circ \mathcal{D}_A\) and \(\mathcal{D}_B \circ \beta = \beta \circ \mathcal{D}_E\).

For the abelian extension \(\mathcal{E}\), there is a \(\mathbb{C}[\partial]\)-module homomorphism \(\gamma : B \to E\) such that \(\beta \circ \gamma = \text{id}_B\). Define \(\chi : B \times B \to A\mathbb{C}[\lambda]\) by
\[
\chi(b_1, b_2) = \gamma(b_1) \cdot \gamma(b_2) - \gamma(b_1) \cdot b_2,
\]
for any \(b_1, b_2 \in B\). By Section 4, we get \(\chi\) is a 2-cocycle in \(\mathbb{Z}^2(B, A)\), and the cohomology class \([\chi] \in \text{HH}^2(B, A)\) does not depend on the choice of \(\gamma\). Thus \(\Theta(\mathcal{D}_A, \mathcal{D}_B)([\chi])\) does not depend on the choice of \(\gamma\). Using \(\Theta(\mathcal{D}_A, \mathcal{D}_B)([\chi])\), we can define a Wells map associated to \(\mathcal{E}\) as follows.

**Definition 7.4.** Let \(E : 0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} B \to 0\) be an abelian extension of associative conformal algebra \(B\) by bimodule \(A\). The map \(\mathcal{W} : \mathfrak{g}(A, B) \to \text{HH}^2(B, A)\),
\[
\mathcal{W}(\mathcal{D}_A, \mathcal{D}_B) = \Theta(\mathcal{D}_A, \mathcal{D}_B)([\chi])
\]
is called the Wells map associated to \(\mathcal{E}\).

If the abelian extension \(\mathcal{E}\) is split, i.e., there is a homomorphism of associative conformal algebras \(\gamma : B \to E\) such that \(\beta \circ \gamma = \text{id}_B\), then \(\chi = 0\) and \(\mathcal{W} = 0\). More generally, we have the following theorem.

**Theorem 7.5.** Let \(E : 0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} B \to 0\) be an abelian extension of associative conformal algebra \(B\) by bimodule \(A\). A pair \((\mathcal{D}_A, \mathcal{D}_B) \in \text{Der}(A) \times \text{Der}(B)\) is extensible if and only if \((\mathcal{D}_A, \mathcal{D}_B) \in \mathfrak{g}(A, B)\) and \(\mathcal{W}(\mathcal{D}_A, \mathcal{D}_B) = [0]\).
Proof. Let $\gamma$ be a section of $E$. We identify $A$ as a $\mathbb{C}[\partial]$-submodule of $E$. Then the map $\tau : A \oplus B \to E$, $(a, b) \mapsto a + \gamma(b)$ gives an isomorphism of $\mathbb{C}[\partial]$-modules. For convenience, denote an element in $E$ by $a + \gamma(b)$ for $a \in A$ and $b \in B$.

If $(\mathcal{D}_A, \mathcal{D}_B) \in \mathfrak{g}(A, B)$ and $W(\mathcal{D}_A, \mathcal{D}_B) = \{0\}$, there exists a map $f \in \text{Hom}_{\mathbb{C}[\partial]}(B, A)$ such that $\Omega(\mathcal{D}_A, \mathcal{D}_B) = [d_1(f)]$. We define a map $\mathcal{D}_E : E \to E$ by

$$\mathcal{D}_E(a + \gamma(b)) = \mathcal{D}_A(a) + f(b) + \gamma(\mathcal{D}_B(b)),$$

for any $a \in A$ and $b \in B$. It is easy to see that $\mathcal{D}_E$ is a homomorphism of $\mathbb{C}[\partial]$-modules, and

$$\mathcal{D}_E(\alpha(a)) = \mathcal{D}_E(a) = \mathcal{D}_A(a) = \alpha(\mathcal{D}_A(a)),$$

$$\beta(\mathcal{D}_E(a + \gamma(b))) = \beta(\mathcal{D}_A(a) + f(b) + \gamma(\mathcal{D}_B(b))) = \mathcal{D}_B(b) = \mathcal{D}_B(\beta(a + \gamma(b))),$$

for any $a \in A$ and $b \in B$. Moreover, for any $b_1, b_2 \in B$, by the definition of $\mathcal{D}_E$, we have

$$\gamma(\mathcal{D}_B(b_1)) \gamma(b_2) = \mathcal{D}_E(\gamma(b_1)) \gamma(b_2) = \gamma(-f(b_1) + \gamma(b_2),$$

$$\gamma(b_1) \gamma(\mathcal{D}_B(b_2)) = \mathcal{D}_E(\gamma(b_2)) = \gamma(\mathcal{D}_B(b_2)) = \gamma(b_1) \gamma(\mathcal{D}_B(b_2)) - \gamma(b_1) \gamma(f(b_2)).$$

Thus,

$$\mathcal{D}_E(\gamma(b_1) \gamma(b_2)) = \mathcal{D}_E(\chi_a(b_1, b_2) + \gamma(b_1) \gamma(b_2))$$

$$= \mathcal{D}_A(\chi_a(b_1, b_2) + f(b_1 + \gamma(b_2)) + \mathcal{D}_B(b_2))$$

$$= \mathcal{D}_A(\chi_a(b_1, b_2) + f(b_1 + \gamma(b_2)) + \mathcal{D}_B(b_2))$$

$$- \chi_a(\mathcal{D}_B(b_1), \mathcal{D}_B(b_2)) + \gamma(b_1) \gamma(\mathcal{D}_B(b_1)) - \chi_a(b_1, \mathcal{D}_B(b_2))$$

$$= \mathcal{D}_E(\gamma(b_1)) \gamma(b_2) + \gamma(b_1) \gamma(\mathcal{D}_B(b_2))$$

$$+ \Theta(\mathcal{D}_A, \mathcal{D}_B)(\chi_a(b_1, b_2) - d_1(f)(b_1, b_2))$$

$$= \mathcal{D}_E(\gamma(b_1)) \gamma(b_2) + \gamma(b_1) \gamma(\mathcal{D}_B(b_2)) + \Theta(\mathcal{D}_A, \mathcal{D}_B)(\chi_a(b_1, b_2) - d_1(f)(b_1, b_2)).$$

Therefore, by the equations \[(19)\] and \[(20)\], we obtain $\mathcal{D}_E \in \text{Der}(E)$. Hence the pair $(\mathcal{D}_A, \mathcal{D}_B) \in \text{Der}(A) \times \text{Der}(B)$ is extensible.

Conversely, if there exists a derivation $\mathcal{D}_E \in \text{Der}(E)$ such that $\mathcal{D}_E \circ \alpha = \alpha \circ \mathcal{D}_A$ and $\beta \circ \mathcal{D}_E = \mathcal{D}_B \circ \beta$, then $\mathcal{D}_A = \mathcal{D}_E|_A$. Since $\beta(\mathcal{D}_E(\gamma(b)) - \gamma(\mathcal{D}_B(b))) = 0$ for any $b \in B$, we get a $\mathbb{C}[\partial]$-module homomorphism $\delta : B \to A,$

$$\delta(b) = \mathcal{D}_E(\gamma(b)) - \gamma(\mathcal{D}_B(b)),$$

for $b \in B$. Then, for any $a \in A$ and $b \in B$,

$$\mathcal{D}_E(\gamma(b) \gamma(a) \gamma(b)) = \mathcal{D}_E(\gamma(b) \gamma(a) \gamma(b)) = \mathcal{D}_E(\gamma(b) \gamma(a) \gamma(b)) = \gamma(\mathcal{D}_B(b)) + \delta(b) \gamma(a)$$

$$= \gamma(\mathcal{D}_B(b)) \gamma(a).$$

This means the equation \[(19)\] holds. Similarly, one can check that the equation \[(20)\] also holds. Hence $(\mathcal{D}_A, \mathcal{D}_B) \in \mathfrak{g}(A, B)$. Finally, we show that $\Theta(\mathcal{D}_A, \mathcal{D}_B)(\chi) = d_1(\delta)$. For any $b_1, b_2 \in B$, we
have
\[ \mathcal{D}_E(\gamma(b_1) \circ \gamma(b_2)) = \mathcal{D}_E(\gamma(b_1) b_2 + \chi_1(b_1, b_2)) = \gamma(\mathcal{D}_B(b_1) b_2) + \delta(b_1) b_2 + \mathcal{D}_A(\chi_1(b_1, b_2)), \]
and
\[ \mathcal{D}_E(\gamma(b_1)(\gamma(b_2)) = \mathcal{D}_E(\gamma(b_1)) \gamma(b_2) + \gamma(b_1) \mathcal{D}_E(\gamma(b_2)). \]

Since \( \mathcal{D}_E \) is a derivation, i.e., \( \mathcal{D}_E(\gamma(b_1)(\gamma(b_2))) = \mathcal{D}_E(\gamma(b_1)) \gamma(b_2) + \gamma(b_1) \mathcal{D}_E(\gamma(b_2)), \) we get
\[ \mathcal{D}_E(\chi_1(b_1, b_2)) = \chi_1(\mathcal{D}_B(b_1), b_2) - \chi_1(b_1, \mathcal{D}_B(b_2)) = \gamma(b_1) \delta(b_2) - \delta(b_1) \gamma(b_2). \]
That is to say, for any \( b_1, b_2 \in B, \)
\[ \Theta(\mathcal{D}_A, \mathcal{D}_B)(\chi_1(b_1, b_2)) = \mathcal{D}_E(\chi_1(b_1, b_2)) = \chi_1(\mathcal{D}_B(b_1), b_2) - \chi_1(b_1, \mathcal{D}_B(b_2)) = d_1(\delta)(b_1, b_2). \]

Hence \( \mathcal{W}(\mathcal{D}_A, \mathcal{D}_B) = [0]. \) The proof is finished. \( \square \)

Now we consider the exact sequence of Wells about derivations. Let \( B \) be an associative con-
formal algebra, \( A \) be a \( B \)-bimodule, and \( E : 0 \rightarrow A \xrightarrow{\alpha} E \xrightarrow{\beta} B \rightarrow 0 \) be an abelian extension
of \( B \) by \( A \) with a section \( \gamma. \) Denote
\[ \text{Der}_A(E) := \{ \mathcal{D}_E \in \text{Der}(E) \mid \mathcal{D}_E(A) \subseteq A \}. \]

Then \( \text{Der}_A(E) \) is a Lie subalgebra of \( \text{Der}(E). \) Any \( \mathcal{D}_E \in \text{Der}_A(E), \) induces two maps \( \mathcal{D}_E|_A \in \text{Der}(A) \) and \( \hat{\mathcal{D}}_E = \beta \circ \mathcal{D}_E \circ \gamma : B \rightarrow B. \) Then one can check that \( \hat{\mathcal{D}}_E \in \text{Der}(B) \) by direct
Calculation and it does not depend on the choice of \( \gamma. \) Thus we get a linear map
\[ \kappa : \text{Der}_A(E) \rightarrow \text{Der}(A) \times \text{Der}(B), \quad \mathcal{D}_E \mapsto (\mathcal{D}_E|_A, \hat{\mathcal{D}}_E). \]

**Lemma 7.6.** With the above notations, \( \text{Im}(\kappa) \subseteq g(A, B), \) and \( \kappa \) induces a Lie algebra homomor-
phism \( \tilde{\kappa} : \text{Der}_A(E) \rightarrow g(A, B). \)

**Proof.** Let \( (\mathcal{D}_A, \mathcal{D}_B) \in \text{Im}(\kappa). \) There exists a derivation \( \mathcal{D}_E \in \text{Der}(E) \) such that \( \mathcal{D}_A = \mathcal{D}_E|_A \) and \( \mathcal{D}_B = \hat{\mathcal{D}}_E. \) Then clearly, \( \mathcal{D}_E \circ \alpha = \alpha \circ \mathcal{D}_A. \) Moreover, for any \( a \in A \) and \( b \in B, \) we have
\[ \hat{\mathcal{D}}_E(\beta(a + \gamma(b))) = \hat{\mathcal{D}}_E(b) = \beta(\mathcal{D}_E(a + \gamma(b))). \]
That is, \( \mathcal{D}_B \circ \beta = \beta \circ \mathcal{D}_E. \) Hence \( \text{Im}(\kappa) \subseteq g(A, B), \) and so that \( \kappa \) induces a linear map \( \tilde{\kappa} : \text{Der}_A(E) \rightarrow g(A, B). \) We now show that \( \tilde{\kappa} \) is a homomorphism of Lie algebras. Let \( \mathcal{D}_E, \mathcal{D}'_E \in \text{Der}_A(E). \) For any \( b \in B, \) there exist \( a_1, a_2 \in A \) and \( b_1, b_2 \in B \) such that
\[ \mathcal{D}_E(\gamma(b)) = a_1 + \gamma(b_1), \quad \mathcal{D}'_E(\gamma(b)) = a_2 + \gamma(b_2). \]
Then \( \hat{D}_E(b) = b_1, \hat{D}'_E(b) = b_2 \), and hence
\[
[D_E, D'_E]\{b\} = \beta(D_E \circ D'_E - D'_E \circ D_E)(\gamma(b)) = \beta(D_E(a_2 + \gamma(b_2)) + \beta(D'_E(a_1 + \gamma(b_1))) = \hat{D}_E(b_2) - \hat{D}'_E(b_1)
\]
\[
= [\hat{D}_E, \hat{D}'_E]\{b\}.
\]
Therefore,
\[
k([D_E, D'_E]) = ([D_E, D'_E]\{0\}, [\hat{D}_E, \hat{D}'_E]) = ([\hat{D}_E, \hat{D}'_E]) = [\hat{k}(D_E), \hat{k}(D'_E)]
\]
Thus \( \hat{k} \) is a homomorphism of Lie algebras.
\[\square\]

Consider the image and kernel of \( \hat{k} \), we get an exact sequence of Wells type.

**Theorem 7.7.** Let \( E : 0 \to A \xrightarrow{\alpha} E \xrightarrow{\beta} B \to 0 \) be an abelian extension of associative conformal algebra \( B \) by bimodule \( A \). There is an exact sequence of vector spaces
\[
0 \to \mathcal{Z}^1(B, A) \xrightarrow{\iota} \text{Der}_A(E) \xrightarrow{\hat{k}} \mathfrak{g}(A, B) \xrightarrow{\mathcal{W}} \text{HH}^2(B, A),
\]
where \( \iota \) is the inclusion map.

**Proof.** If \( D_E \in \text{Ker}(\hat{k}) \), then \( \hat{D}_E = 0 \) and \( D_E|_A = 0 \). For any \( b \in B \), we get \( \beta(D_E(\gamma(b))) = \hat{D}_E(b) = 0 \), that is, \( D_E(\gamma(b)) \in A \). Thus there is a linear map
\[
\phi : \text{Ker}(\hat{k}) \to C^1(B, A) = \text{Hom}_{C[\partial]}(B, A), \quad D_E \mapsto D_E \circ \gamma.
\]
Then one can check that \( \phi \) does not depend on the choice of \( \gamma \), and \( \phi(D_E) \in \mathcal{Z}^1(B, A) \) for any \( D_E \in \text{Ker}(\hat{k}) \). Thus \( \phi \) induces a linear map \( \hat{\phi} : \text{Ker}(\hat{k}) \to \mathcal{Z}^1(B, A) \). First, if \( D_E \in \text{Ker}(\hat{k}) \) and \( \hat{\phi}(D_E) = 0 \), for any \( a \in A \) and \( b \in B \), we get
\[
D_E(a + \gamma(b)) = D_E|_A(a) + \hat{\phi}(D_E)(b) = 0.
\]
This means that \( \hat{\phi} \) is injective. Second, for any \( f \in \mathcal{Z}^1(B, A) \), we define a \( C[\partial] \)-module homomorphism
\[
D_E : E \to E, \quad D_E(a + \gamma(b)) = f(b),
\]
for any \( a \in A \) and \( b \in B \). Clearly, \( D_E|_A = 0, \hat{D}_E(b) = \beta(f(b)) = 0 \) and \( \hat{\phi}(D_E)(b) = D_E(\gamma(b)) = f(b) \), for any \( b \in B \). Moreover, one can check that \( D_E \in \text{Der}(E) \), and so that \( D_E \in \text{Ker}(\hat{k}) \). Thus \( \hat{\phi} \) is surjective, and is an isomorphism of vector spaces.

Next, we show that the sequence is exact at \( \mathfrak{g}(A, B) \). For any \( (D_A, D_B) \in \text{Im}(\hat{k}) \), which is extensible, we get \( \mathcal{W}(D_A, D_B) = [0] \) by Theorem 7.5. Thus \( \text{Im}(\hat{k}) \subseteq \text{Ker}(\mathcal{W}) \). Conversely, for any \( (D_A, D_B) \in \text{Ker}(\mathcal{W}) \), that is, there exists a element \( f \in C^1(B, A) \) such that \( \Theta(D_A, D_B)(\chi) = [d_1(f)] \). Similar to the proof of Theorem 7.5 we define \( D_E : E \to E \) by
\[
D_E(a + \gamma(b)) = D_A(a) + f(b) + \gamma(D_B(b)),
\]
for any \( a \in A \) and \( b \in B \). Then \( \mathcal{D}_E \in \text{Der}(E) \) and \( \hat{k}(\mathcal{D}_E) = (\mathcal{D}_A, \mathcal{D}_B) \). That is, \( \text{Ker}(W) \subseteq \text{Im}(\hat{k}) \). Therefore, we get the sequence which is an exact sequence. \( \square \)

Finally, we consider a special case, that is, when the abelian extension of associative conformal algebras is split. In this case, we can see that there is a close relationship between the Lie algebra \( \text{Der}_A(E) \) and \( g(A, B) \).

**Corollary 7.8.** Let \( \mathcal{E} : 0 \rightarrow A \xrightarrow{\alpha} E \xrightarrow{\beta} B \rightarrow 0 \) be a split abelian extension of associative conformal algebra \( B \) by bimodule \( A \), i.e., there exists a homomorphism of associative conformal algebras \( \gamma : B \rightarrow E \) such that \( \beta \circ \gamma = \text{id}_B \). Then

\[
\text{Der}_A(E) \cong g(A, B) \times \mathcal{Z}^1(B, A)
\]

as Lie algebras, where \( \mathcal{Z}^1(B, A) \) is regarded as a trivial Lie algebra.

**Proof.** Since \( \mathcal{E} \) is split, by Theorem 7.7, we get a short exact sequence of Lie algebras

\[
(\ast) \quad 0 \longrightarrow \mathcal{Z}^1(B, A) \xrightarrow{i} \text{Der}_A(E) \xrightarrow{\hat{k}} g(A, B) \xrightarrow{W} 0,
\]

if we regard \( \mathcal{Z}^1(B, A) \) as a trivial Lie algebra. For any \( (\mathcal{D}_A, \mathcal{D}_B) \in g(A, B) \), we define

\[
\mathcal{D}_E : E \rightarrow E, \quad \mathcal{D}_E(a + \gamma(b)) = \mathcal{D}_A(a) + \gamma(D_B(b)),
\]

for any \( a \in A \) and \( b \in B \). Note that in this case the cohomology class \( [\chi] = 0 \), by the proof of Theorem 7.3, we get \( \mathcal{D}_E \in \text{Der}(E) \). Thus we obtain a linear map

\[
\eta : g(A, B) \rightarrow \text{Der}(E), \quad (\mathcal{D}_A, \mathcal{D}_B) \mapsto \mathcal{D}_E.
\]

It is easy to see \( \hat{k} \circ \eta(\mathcal{D}_A, \mathcal{D}_B) = \hat{k}(\mathcal{D}_E) = (\mathcal{D}_E|_A, \hat{D}_E) = (\mathcal{D}_A, \mathcal{D}_B) \). That is, \( \hat{k} \circ \eta = \text{id}_{g(A,B)} \). Moreover, by direct calculation, one can show that \( \eta \) is a homomorphism of Lie algebras. Thus the sequence (\( \ast \)) is split in the category of Lie algebras. Hence we get \( \text{Der}_A(E) \cong g(A, B) \times \mathcal{Z}^1(B, A) \) as Lie algebras. \( \square \)

**Remark 7.9.** In the context of associative conformal algebras, there is a class of derivation, called conformal derivation (see [4]). All conformal derivations of an associative conformal algebra form a Lie conformal algebra. But the cohomology of associative conformal algebra we use here has only a vector space structure, no \( \mathbb{C}[\partial] \)-module structure. This cohomology cannot form a representation of a Lie conformal algebra, and no relevant conclusions can be obtained. Therefore, here we still consider derivations on associative conformal algebras instead of conformal derivations.

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**References**

[1] A.L. Agore, G. Militaru, Hochschild products and global non-abelian cohomology for algebras. Applications, J. Pure Appl. Algebra 221, 2017, 366–392.

[2] J.C. Baez, A.S. Crans, Higher-dimensional algebra VI: Lie 2-algebras, Theory Appl. Categ. 12, 2004, 492–538.

[3] B. Bakalov, A. D’Andrea, V.G. Kac, Theory of finite pseudoalgebras, Adv. Math. 162, 2001, 1–140.
[4] B. Bakalov, V.G. Kac, A. Voronov, Cohomology of conformal algebras. Comm. Math. Phys. 200, 1999, 561–589.
[5] V.G. Bardakov, M. Singh, Extensions and automorphisms of Lie algebras, J. Algebra Appl. 16, 2017, 1750162, 15pp.
[6] M. Barati, F. Saeedi, Derivations of abelian Lie algebra extensions, Note Mat. 39, 2019, 71–86.
[7] J.H. Baues, E.G. Minian, Crossed extensions of algebras and Hochschild cohomology, Homol. Homot. Appl. 4, 2002, 63–82.
[8] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, Nuclear Phys. 241, 1984, 333–380.
[9] L.A. Bokut, Y. Fong, W.F. Ke, Grobner-Shirshov bases and composition lemma for associative conformal algebras: an example, Contemp. Math. 264, 2000, 63–90.
[10] L.A. Bokut, Y. Fong, W.F. Ke, Composition-Diamond lemma for associative conformal algebras, J. Algebra 272, 2004, 739–774.
[11] R. Brown, J. Huebschmann, Identities among relations, in: Low-Dimensional Topology, London Math. Soc. Lecture Notes, Vol. 46, Cambridge Univ. Press, Cambridge, 1982, pp. 153–202.
[12] J. M. Casas, M. Ladra, T. Pirashvili, Crossed modules for Lie–Rinehart algebras, J. Algebra 274, 2004, 192–201.
[13] J. M. Casas, E. Khmaladze, M. Ladra, Crossed modules for Leibniz n-algebras, Forum Math. 20, 2008, 841–858.
[14] J.M. Casas, E. Khmaladze, M. Ladra, Low-dimensional non-abelian Leibniz cohomology, Forum Math. 25, 2013, 443-469.
[15] J.M. Casas, X. García-Martínez, Abelian extensions and crossed modules of Hom-Lie algebras, J. Pure Appl. Algebra 224, 2020, 987–1008.
[16] A. Das, Hom-associative algebras up to homotopy, J. Algebra 556, 2020, 836–878.
[17] A. Das, S.K. Hazra, S.K. Miskra Non-abelian extensions of Rota-Baxter Lie algebras and induciblity of automorphisms, ArXiv: 2204.01060.
[18] A. Das, N. Rathee, A unified extension theory of Rota-Baxter algebras, dendriform algebras, and a fundamental sequence of Wells, ArXiv: 2206.09019.
[19] A. D’Andrea, V.G. Kac, Structure theory of finite conformal algebras, Sel. Math., New Ser. 4, 1998, 377–418.
[20] I.A. Dolguntseva, The Hochschild cohomology for associative conformal algebras, Algebra Logic 46, 2007, 373–384.
[21] L. Du, Y. Tan, Wells sequences for abelian extensions of Lie coalgebras, J. Algebra Appl. 20, 2021, No. 2150149, 41 pp.
[22] S. Eilenberg, S. Maclane, Cohomology theory in abstract groups. II. Group extensions with non-abelian kernel, Ann. of Math. 48, 1947, 326–341.
[23] A. Fialowski, M. Penkava, Extensions of (super) Lie algebras, Commun. Contemp. Math. 11, 2009, 709–737.
[24] Y. Fréjgier, Non-abelian cohomology of extensions of Lie algebras as Deligne groupoid, J. Algebra 398, 2014, 243–257.
[25] M. Gerstenhaber, A uniform cohomology theory for algebras, Pros. Nat. Acad. Sci. U.S.A. 51, 1964, 626–629.
[26] J.M.W. Goldman, The deformation theory of representations of fundamental groups of compact Kahler manifolds, Publ. Math. IHES 67, 1988, 43–96.
[27] J. Gouray, A differential graded Lie algebra approach to non abelian extensions of associative algebras, ArXiv: 1802.04641.
[28] S.K. Hazra, A. Habib, Wells exact sequence and automorphisms of extensions of Lie superalgebras, J. Lie Theory, 30, 2020, 179–199.
[29] D.H. Holt, Cohomology groups $H^n(G; M)$, J. Algebra 60, 1979, 307–320.
[30] B. Hou, Z. Shen, J. Zhao, Gerstenhaber algebra of the Hochschild cohomology of an associative conformal algebra, ArXiv: 2209.08715.
[31] N. Inassaridze, E. Khmaladze, M. Ladra, Non-abelian cohomology and extensions of Lie algebras, J. Lie Theory 18 2008, 413–432.
[32] P. Jin, H. Liu, The Wells exact sequence for the automorphism group of a group extension, J. Algebra 324, 2010, 1219–1228.
[33] V.G. Kac, Vertex algebras for beginners, 2nd Edition, Amer. Math. Soc. Providence, RI, 1998.
[34] V.G. Kac, Formal distribution algebras and conformal algebras, XII-th Int. Congr. Math. Phys. Int. Press, Cambridge, MA 1999, pp. 80–97.
[35] P.S. Kolesnikov, Simple associative conformal algebras of linear growth, J. Algebra 295, 2006, 247–268.
[36] P.S. Kolesnikov, Associative conformal algebras with finite faithful representation, Adv. Math. 202, 2006, 602–637.
[37] H. Lang, Z. Liu, Crossed modules for Lie 2-algebras, Appl. Categor. Struct. 24, 2016, 53–78.
[38] J.I. Liberati, Cohomology of associative H-pseudoalgebras, arXiv: 2012.13832.
[39] J. Liu, Y. Sheng, Q. Wang, On non-abelian extensions of Leibniz algebras, Commun. Algebra 64, 2018, 574–587.
[40] J.L. Loday, Cohomologie et groupe de Steinberg relatifs, J. Algebra 54, 1978, 178–202.
[41] N. Machado, J. Öinert, S. Wagner, Non-abelian extensions of groupoids and their groupoid rings, arXiv: 2207.03369.
[42] S. Mescher, A primer on A-infinite algebras and their Hochschild homology, arXiv: 1601.03963.
[43] K.H. Neeb, A primer on A-infinite algebras and their Hochschild homology, arXiv: 1601.03963.
[44] A. Retakh, Non-abelian extensions of infinite-dimensional Lie groups, Ann. Inst. Fourier (Grenoble) 57, 2007, 209–271.
[45] A. Retakh, On associative conformal algebras of linear growth II, J. Algebra 304, 2006, 543–556.
[46] M. Roitman, Universal enveloping conformal algebras, Sel. Math. New Ser. 6, 2000, 319–345.
[47] M. Roitman, On embedding of Lie conformal algebras into associative conformal algebras, J. Lie theory 15, 2005, 575-588.
[48] L. Song, A. Makhlouf, R. Tang, On non-Abelian extensions of 3-Lie algebras, Commun. Theor. Phys. 69, 2018, 347–356.
[49] L. Song, R. Tang, Cohomological characterizations of diagonal non-abelian extensions of regular Hom-Lie algebras, Symmetry 9, 2017, 297.
[50] L. Song, R. Tang, Derivation Hom-Lie 2-algebras and non-abelian extensions of regular Hom-Lie algebras, J. Algebra Appl. 16, 2018, No. 1850081, 17 pp.
[51] R. Tang, Y. Sheng, Cohomological characterizations of non-abelian extensions of strict Lie 2-algebras, J. Geom. Phys. 144, 2019, 294–307.
[52] Y. Tan, S. Xu, The Wells map for abelian extensions of 3-Lie algebras, Czech. Math. J. 69 2019, 1133—1164.
[53] Y. Tan, S. Xu, On a Lie algebraic approach to abelian extensions of associative algebras, Cana. Math. Bull. 64 2021, 25—38.
[54] F. Wagemann, Crossed modules, Walter de Gruyter GmbH, Berlin/Boston, 2021.
[55] C. Wells, Automorphisms of group extensions, Trans. Amer. Math. Soc. 155, 1971, 189–194.
[56] J.H.C. Whitehead, Combinatorial homotopy II, Bull. Amer. Math. Soc. 55, 1949, 453–496.
[57] J. Zhao, B. Hou, Non-abelian extensions of Lie conformal algebras. In preparation.

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