Gravitational Instantons, Confocal Quadrics and Separability of the Schrödinger and Hamilton-Jacobi equations.

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Abstract

A hyperkähler 4-metric with a triholomorphic $SU(2)$ action gives rise to a family of confocal quadrics in Euclidean 3-space when cast in the canonical form of a hyperkähler 4-metric metric with a triholomorphic circle action. Moreover, at least in the case of geodesics orthogonal to the $U(1)$ fibres, both the covariant Schrödinger and the Hamilton-Jacobi equation is separable and the system integrable.

1 Introduction

The work described in what follows was done some time ago, and arose as an attempt to understand in a more conceptual way a calculation given in the appendix of [6]. The reason for returning to it here is two-fold.

• Its relevance to some recent recent work of Dunajski [1] on quadrics in general and integrability. The example discussed below is particularly striking in that respect because one may relate it to a problem of genuine practical significance.

• It’s relevance to the problem of consistent sphere reductions in Kaluza-Klein supergravity and string theories in various dimensions for which the properties of quadrics are essential [2]. The present work is concerned exclusively with particular four-dimensional riemannian metrics which can
be seen to exhibit some of the general features discussed in [2] in a particularly intriguing way.

2 Left and Right invariant vector fields

What follows is a brief summary of some properties of $SU(2)$ which will be needed in the sequel. Suppose that $\sigma^i$ are left-invariant one forms with

$$d\sigma^1 = \sigma^2 \wedge \sigma^3 \text{ etc.}$$

Then the dual basis of left-invariant vector fields $L_i$ satisfies

$$[L_1, L_2] = -L_3 \text{ etc.}$$

The generators of left translations form a right-invariant basis $R_i$ which commutes with the left-invariant basis:

$$[L_i, R_j] = 0,$$

and satisfies:

$$[R_1, R_2] = R_3 \text{ etc.}$$

Now the left and right invariant vector fields are not linearly independent and hence

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix},$$

where

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$$

is an orthogonal matrix which depends upon the Euler angles of $SU(2)$. Thus for example:

$$l_1^2 + l_2^2 + l_3^2 = 1.$$ 

Moreover (3) implies that

$$L_i l_j = -\epsilon_{ijk} l_k,$$

which may also be written as

$$dl_i = -\epsilon_{ijk} l_j \sigma^k.$$
3 BGPP metrics

If SU(2) acts triholomorphically the metric may be cast in the form:

$$ds^2 = ABCd\eta^2 + \frac{BC}{A}(\sigma^1)^2 + \frac{CA}{B}(\sigma^2)^2 + \frac{AB}{C}(\sigma^3)^2$$  \hspace{1cm} (10)

where \(\eta\) labels the orbits of SU(2). If dot indicates differentiation with respect to \(\eta\), then \(A, B, C\) satisfy

$$\dot{A} = BC \hspace{1cm} \text{etc.}$$  \hspace{1cm} (11)

The Kähler forms \(\Omega_i\) are given by

$$\Omega_1 = BCD\eta \wedge \sigma^1 + A\sigma^2 \wedge \sigma^3 \hspace{1cm} \text{etc.}$$  \hspace{1cm} (12)

Note that (11) is equivalent to the closure of the \(\Omega_i\). For future reference, we record that

$$\Omega_1(R_{1_1}) = -BCl_1d\eta + Al_2\sigma^3 - Al_3\sigma^2.$$  \hspace{1cm} (13)

Because \(\langle \sigma^j, L_i \rangle = \delta^j_i\), one easily sees that

$$g(L_1, L_1) = \frac{BC}{A} \hspace{1cm} \text{etc.}$$  \hspace{1cm} (14)

The equations (11) are of a similar form to Euler’s equations for a rigid body but they do not correspond to any set of moments of inertia. Nevertheless the equations may still be integrated completely. In order to avoid elliptic functions we change the variable labelling the orbits of SU(2). Thus we define \(\lambda\) by

$$ABCd\eta^2 = \frac{1}{4}d\lambda^2 \hspace{1cm} \text{etc.}$$  \hspace{1cm} (15)

and find that

$$A = \sqrt{\lambda - \lambda_1} \hspace{1cm} \text{etc.}$$  \hspace{1cm} (16)

4 Canonical form

The canonical form for a self-dual metric with a triholomorphic U(1) action is (see e.g. [5])

$$ds^2 = V^{-1}(d\tau + \omega_i dx^i)^2 + V(dx^2 + dy^2 + dz^2),$$  \hspace{1cm} (17)

where

$$\text{curl } \omega = \text{grad } V.$$  \hspace{1cm} (18)

It follows that the function \(V\) is harmonic on the Euclidean space whose Cartesian coordinates are the three moment maps \(x, y, z\). This Euclidean space is of course just the quotient of the 4-manifold by the triholomorphic circle action.
generated by $\partial/\partial \tau$. Moreover it is clear that given $V$ one may reconstruct the metric.

The three Kähler forms are

$$\Omega_1 = (d\tau + \omega_i dx^i) \wedge dx - V dy \wedge dz \ etc \ (19)$$

Note that the closure of the $\Omega_i$ follows from (18).

From (19) it follows that $(x, y, z)$ are the three moment maps for the Killing field $\partial/\partial \tau$, i.e.

$$\Omega_1\left(\frac{\partial}{\partial \tau}\right) = dx \ etc. \ (20)$$

Moreover we have the obvious relation:

$$V^{-1} = g\left(\frac{\partial}{\partial \tau}, \frac{\partial}{\partial \tau}\right). \ (21)$$

## 5 Comparison

We pick one of the generators, $R_1$ say, of left translations. This generates a $U(1)$ subgroup of $SU(2)$ which we identify with $\partial/\partial \tau$. Now if

$$x = -l_1 A \ etc, \ (22)$$

then by (11) and (9)

$$dx = -l_1 BC d\eta + Al_2 \sigma^3 - Al_3 \sigma^2. \ (23)$$

but using (13) we have

$$dx = \Omega_1\left(R_1, \right). \ (24)$$

Thus we have identified the moment maps. Using (7) and (16) we deduce that

$$\frac{x^2}{\lambda - \lambda_1} + \frac{y^2}{\lambda - \lambda_2} + \frac{z^2}{\lambda - \lambda_3} = 1. \ (25)$$

The cosets $SU(2)/U(1)$ appear as set of confocal quadrics in Euclidean 3-space, labelled by the variable $\lambda$. Each non-degenerate orbit is a circle bundle over the quadric. As bundles they are precisely the Hopf bundle. It remains to find the harmonic function $V$. This is of course given by

$$V^{-1} = g(R_1, R_1). \ (26)$$

Because

$$R_1 = l_1 L_1 + l_2 L_2 + l_3 L_3, \ (27)$$

one has

$$V^{-1} = \sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)}\left(\frac{x^2}{(\lambda - \lambda_1)^2} + \frac{y^2}{(\lambda - \lambda_2)^2} + \frac{z^2}{(\lambda - \lambda_3)^2}\right). \ (28)$$
The result just described was obtained originally in a rather less conceptual way in [6]. Another approach is described in [1].

It is a straightforward task to express this expression in ellipsoidal coordinates and check explicitly that it is indeed a solution of Laplace’s equation. By doing so we obtain a bonus: we discover that, in a special case at least, we may separate the Hamilton-Jacobi equation governing the geodesic flow.

6 Ellipsoidal Coordinates

We begin by introducing ellipsoidal coordinates in three dimensional Euclidean space in the usual way. We assume that \( \lambda_1 < \lambda_2 < \lambda_3 \) and restrict \( \lambda \) to the range \( \lambda_3 \leq \lambda \leq \infty \). Two further coordinates \( \mu \) and \( \nu \) lie in the intervals \( \lambda \leq \mu \leq \lambda_2 \) and \( \lambda_2 \leq \nu \leq \lambda_3 \). The Cartesian coordinates are given by

\[
x^2 = \frac{(\lambda - \lambda_1)(\mu - \lambda_1)(\nu - \lambda_1)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)},
\]

\[
y^2 = \frac{(\lambda - \lambda_2)(\mu - \lambda_2)(\nu - \lambda_2)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)},
\]

\[
z^2 = \frac{(\lambda - \lambda_3)(\mu - \lambda_3)(\nu - \lambda_3)}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)}.
\]

If we define \( R(\lambda) = \sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)} \),

we find, after a short calculation, that

\[
V = \frac{R(\lambda)}{(\lambda - \mu)(\lambda - \nu)}.
\]

By writing out the flat space Laplace operator in ellipsoidal coordinates, one checks that indeed \( V \) is harmonic. In fact \( V \) coincides with the harmonic function \( S_\lambda \) defined in [11].

7 The one-form \( \omega_i dx^i \)

The harmonic property of the function \( V \) guarantees that a local solution of (18) exists, but finding it may not be easy. Rather remarkably, an explicit solution may be found.

If we define

\[
S(\mu) = \sqrt{-(\mu - \lambda_1)(\mu - \lambda_2)(\mu - \lambda_3)},
\]

and

\[
T(\nu) = \sqrt{(\nu - \lambda_1)(\nu - \lambda_2)(\nu - \lambda_3)},
\]

we find...
then, according to Harry Braden one may choose:
\[ \omega_i dx^i = \frac{1}{2} \frac{(\lambda - \nu)}{(\nu - \lambda)} S(\mu) d\nu + \frac{1}{2} \frac{(\lambda - \mu)}{(\nu - \lambda)} T(\nu) d\mu. \] (36)

8 Special cases

If all the constants \( \lambda_i \) are equal we obtain flat Euclidean 4-space. In this case, up to scaling,
\[ V = \frac{1}{r}, \] (37)
where \( r \) is the distance to the origin. If two of the constants coincide, one obtains the Eguchi-Hanson metric. This is a complete metric on the cotangent bundle of the two-sphere, \( T^*(CP^1) \), and is in fact invariant under the action of \( U(2) \). In this case, up to scaling,
\[ V = \frac{1}{r_1} + \frac{1}{r_2}, \] (38)
where \( r_1 \) and \( r_2 \) are the distances from two fixed points.

If the three constants are distinct, we obtain a metric with singularities. A simple description of the potential \( V \) would be interesting. Its singularities lie on the focal conics. Another interesting question is what, if anything, is the relation of these potentials to those described in appendix 15 of Arnold’s textbook on dynamics.

9 Integrability of the geodesic flow

The geodesic flow on the tangent bundle of the general metric with triholomorphic \( U(1) \) can be subjected to a Hamiltonian reduction to get a mechanical system on Euclidean space. This is most simply done using the Hamilton-Jacobi equation. One may separate off the \( \tau \) dependence by writing the action as
\[ S = \epsilon \tau + W(x^i). \] (39)

One gets :
\[ e^2 V + V^{-1}(\nabla_i W - \epsilon \omega_i)^2 = 1. \] (40)

If the conserved momentum \( \epsilon \) vanishes, which means that the geodesics are orthogonal to the \( U(1) \) fibres, then we obtain the Hamilton-Jacobi equation for a particle in an (attractive) gravitational potential \( V \) moving with zero total energy. If the momentum \( \epsilon \) is not zero, then there is an additional contribution to the potential and also a velocity dependent magnetic force given by \( \omega_i \).

It has been known since the time of Euler that the motion of a planet moving around two fixed gravitating centres is completely integrable. It turns out that
the geodesic flow in the Eguchi metric is completely integrable. Thus Euler’s result may be generalized to this case. Moreover the Schrödinger equation also separates [5, 7, 8]. A rather full discussion is given in [7, 8]. Shortly we shall see that at least for a special class of geodesics, the Hamilton-Jacobi equation separates.

Before doing so we shall make some remarks.

• It is easy to see that integrability does not follow just from the $SU(2)$ symmetry, one needs a further commuting quantity.

• Nevertheless, it has been shown that the gravitational instanton constructed from the helicoid using the Jörgens correspondence between solutions of the Monge-Ampère equation and minimal surfaces admits the triholomorphic action of the Bianchi group $VI_0$, i.e. the Poincaré group of two dimensions $E(1, 1)$. Moreover the closely related metric, obtained from the catenoid admits the triholomorphic action of the Bianchi group $VI_0$, that is the Euclidean group in two dimensions $E(2)$. In both cases the authors find that they can separate both the Hamilton-Jacobi equation and the Laplace equation. Since $E(2)$ is a contraction of $SU(2)$ the catenoid metric is a are limiting case of the BGPP metrics. Thus it is not unreasonable to look for separability in our case.

• On the negative side, according to a theorem quoted in the last section of [10] the planar many centre problem with more than two fixed centres is not integrable. However, multi-centre HyperKähler metrics are not, for more than two centres, associated with quadrics.

• According to [9] the general repulsive Coulomb problem motion with freely moving centres is Liouville integrable. Our forces however are attractive and the centres are fixed.

The issue can clearly only be settled by a calculation.

10 Separability of the Hamilton-Jacobi Equation

Consider now the geodesics with vanishing momentum along the $U(1)$ fibres. As explained above, the Hamilton-Jacobi equation becomes

$$|\nabla W|^2 = V.$$  \hspace{1cm} (41)

In ellipsoidal coordinates this becomes

$$\frac{1}{h_\lambda^2} \left( \frac{\partial W}{\partial \lambda} \right)^2 + \frac{1}{h_\mu^2} \left( \frac{\partial W}{\partial \mu} \right)^2 + \frac{1}{h_\nu^2} \left( \frac{\partial W}{\partial \nu} \right)^2 = \frac{R(\lambda)}{(\lambda - \mu)(\lambda - \nu)}. \hspace{1cm} (42)$$
The functions \( h_{\lambda}^2, h_{\mu}^2, h_{\nu}^2 \) appear in the flat Euclidean metric which in ellipsoidal coordinates is given by

\[
\text{ds}^2 = h_{\lambda}^2 d\lambda^2 + h_{\mu}^2 d\mu^2 + h_{\nu}^2 d\nu^2.
\]

(43)

If, as before, we define

\[
S(\mu) = \sqrt{- (\mu - \lambda_1)(\mu - \lambda_2)(\mu - \lambda_3)},
\]

(44)

and

\[
T(\nu) = \sqrt{(\nu - \lambda_1)(\nu - \lambda_2)(\nu - \lambda_3)},
\]

(45)

we have

\[
2h_{\lambda} = \frac{\sqrt{(\lambda - \mu)(\lambda - \nu)}}{R(\lambda)},
\]

(46)

\[
2h_{\mu} = \frac{\sqrt{(\lambda - \mu)(\mu - \nu)}}{S(\mu)},
\]

(47)

and

\[
2h_{\nu} = \frac{\sqrt{(\nu - \mu)(\nu - \lambda)}}{T(\nu)}.
\]

(48)

It is now a simple task to check that the Hamilton-Jacobi equation with vanishing charge \( e \) separates. If however \( e \neq 0 \) the term involving the vector potential enters the equation and so far I have had no success in separating it.

11 Integrability of the Schrödinger Equation

In this section we shall show that not only the Hamilton-Jacobi equation but also the Schrödinger equation is admits separable solutions in the case of vanishing charge \( e \). In fact, by the WKB approximation, this makes the former property almost obvious. In fact since, by virtue of the self-duality property, the metric admits two covariantly constant spinor fields, the eigen functions of other operators may be obtained from the scalar eigen functions in the manner described by Hawking and Pope for the metric on K3 [13]. Thus the separability of the Laplace operator is the key to understanding other separability properties.

The covariant Laplace or Schrödinger equation:

\[
-\nabla^\alpha \nabla_\alpha \Psi = E\Psi
\]

(49)

becomes, when written out for metric (17),

\[
V^{-1}(\nabla_k - \omega_k \frac{\partial}{\partial r})(\nabla^k - \omega^k \frac{\partial}{\partial r})\Psi + V \frac{\partial^2}{\partial r^2} \Psi = -E\Psi.
\]

(50)
We may separate off the $\tau$ dependence by setting $\Psi = \exp(ie\tau)\psi$ and obtain
\[
V^{-1}(\nabla_k - ie\omega_k)(\nabla_k - ie\omega^k)\psi - e^2V\psi = -E\psi. \tag{51}
\]

In the case that the charge $e$ vanishes we get
\[
\nabla_k \nabla^k \psi = -EV\psi. \tag{52}
\]

Separation of variables in ellipsoidal coordinates for equation (52) (without assuming that the function $V$ is harmonic) is discussed in [14] where a sufficient condition is given on the function $V$.

This condition is
\[
V = \frac{f_\lambda}{h_\lambda^2} + \frac{f_\mu}{h_\mu^2} + \frac{f_\nu}{h_\nu^2}, \tag{53}
\]
where $(f_\lambda, f_\mu, f_\nu)$ are arbitrary functions of $(\lambda, \mu, \nu)$ respectively. The sufficient condition is satisfied with the only one of the three functions being non-vanishing:
\[
f_\lambda = \frac{1}{4R(\lambda)}. \tag{54}
\]
in our case.

As far as I am aware, there is no discussion of the separability of the Schrödinger equation with a vector potential.

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