On the characteristic of homomorphisms on cyclically ordered groups

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Abstract. For cyclically ordered groups $G$, $G'$, the mapping $f: G \rightarrow G'$ is called a homomorphism, if $f$ is a homomorphism with respect to the group operation, and whenever $x,y,z$ in $G$ such that $[x,y,z]$, and $f(x)$, $f(y)$, $f(z)$, are distinct, then $[f(x)$, $f(y)$, $f(z)]$. In this paper, it will be given some conditions related to group homomorphisms.

1. Introduction
The concept of cyclically ordered group (cog) was originally introduced by Rieger [1]. Actually, the concept is closely related to the concepts of (linearly) ordered groups, and ordered sets, in general. A linear order is a relation involved two elements in a set, meanwhile a cyclic order involved three. It is interesting to observe the connection between a binary relation and a ternary one in the same set. The reader may refer to [2] for a comprehensive elaboration about many aspects of the concept.

One of most important fact about cyclically ordered groups is its relation to linearly ordered groups. Precisely, a cyclic order can be obtained naturally from the linear one. Roughly speaking, every linearly ordered group is a cyclically ordered group. In a natural way, Świerczkowski [3] used and developed this aspect to construct a new cyclically ordered groups from the previous ones.

In line with the work of Świerczkowski, some studies devoted to analyzed a sufficient condition of cyclically ordered groups to be a linearly ordered groups. In [6], Jakubik presented such sufficient conditions. Moreover, Jakubik also used the result to give some conditions of existence of linearly ordered subgroup of cyclically ordered group. The study about this aspect also recently conducted by Rosjanuardi et.al. (see [4]).

One interesting fact is that one may connect the theory of cyclically ordered (semi)groups to the operator theory. The important point to note here is that we have to identify the existence of the positive cone of the set, from which we may generate a Hilbert space of operators from the positive cone to complex field. The reader may realize that the existence of a positive cone is essential. Recently, Rosjanuardi et.al. conduct research on this topic, and for the result, one may find it in [4].

In this paper, we interested in connection between cyclically ordered groups. Some investigators have considered the homomorphisms between these cyclically ordered groups. The original definition and some results regarding this topic may be found in [3]. In line with this, we try to observe the necessary condition of a group homomorphism to be a homomorphism, in the sense of cog homo.
2. Preliminaries

For the convenience of the reader, in this part we repeat the relevant material from [2]. We begin by the definition of cog. Let G be a group with the operation +. The relation \([x,y,z]\) in G is called a cyclic order if \(x, y, z\), are different elements, and for all \(a, b\) in G, the followings hold:

i. It is exactly must be \([x, y, z]\) or \([z, y, x]\).

ii. \([x, y, z]\) \(\Rightarrow [y, z, x]\).

iii. \([x, y, z]\) and \([y, u, z]\) implies \([x, u, z]\).

iv. \([x, y, z]\) implies \(a + x + b, a + y + b, a + z + b\).

The group G with this ternary relation is called a cyclically ordered group. Note that the term cyclic refers to the condition (ii). The third condition says that the order is transitive. The last condition suggests that the order compatible with the group operation.

Here is the first example. Let G be a linearly ordered group with respect to the relation <. For elements \(x, y, z\) in G, the relation \([x, y, z]\) holds if one of the following is true:

\[x < y < z, \quad y < z < x, \quad \text{or} \quad z < x < y.\]

In this case, the cyclic order in G is induced by its linear order. As stated before, every linearly ordered group is a cyclically ordered group.

We know that the set of integers, \(\mathbb{Z}\), is a linearly ordered group with respect to the usual order <. Hence, \(\mathbb{Z}\) is a cyclically ordered group. Based on this fact, from now we assume \(\mathbb{Z}\) as a cyclically ordered group related to its linear order.

The following is an example of finite cyclically ordered group. Let G be a cyclic group of order \(n\) with the generator \(a\). So, we can write \(G = \{a^i : 0 < i < n-1\}\). For \(x, y, z\) in G, the order \([x, y, z]_G\) is defined if \(i < j < k\), \(j < k < i\), or \(k < i < j\), where \(x = a^i\), \(y = a^j\), \(z = a^k\). [2].

It should be noted that the different generator may give the different cyclic order. Consider the group \(\mathbb{Z}_7 = \{2\}\). The generator 2 gives the order, say \([...,]_2\), and the generator 3 gives \([...,]_3\). It can be verified that elements 4, 1, 5 satisfy the relation \([4, 1, 5]_3\), but not the \([4, 1, 5]_2\).

Suppose that \(G\) and \(G'\) are groups. Recall that, a map \(f: G \rightarrow G'\) is a homomorphism from \(G\) to \(G'\) if it satisfies \(f(ab) = f(a)f(b)\). Of course, the operation of \(ab\) may be different to \(f(a)f(b)\).

3. Results

Now, we come to the main part of the paper. Here, we present some fact about homomorphisms from a cyclically ordered group to the other one. The mapping is no other than a group homomorphism with an additional condition, namely that it preserves the cyclic order.

Let \(G, G'\) be cyclically ordered groups. The mapping \(f: G \rightarrow G'\) is called a homomorphism if \(f\) is a homomorphism with respect to the group operation, and satisfies the condition: if \(x, y, z\) in G such that \([x, y, z]_G\), and \(f(x), f(y), f(z)\) are distinct, then \([f(x), f(y), f(z)]_{G'}\).

Example.

As stated in the previous section, \(\mathbb{Z}\) is a cyclically ordered group. Let \(f\) be the mapping \(f: Z \rightarrow Z\) with \(f(x) = 2x\). It is clear that \([x, y, z]_{\mathbb{Z}}\) implies \([f(x), f(y), f(z)]_{\mathbb{Z}}\). So, \(f\) is a homomorphism in the sense of group and cyclically ordered group.

Of course, in this case, \(f(x) = cx\) is also a homomorphism for every positive integer \(c\). Meanwhile, if \(c\) is negative or zero, \(f\) is not a homomorphism.

Proposition.

Let \(G, G'\) be linearly ordered groups. If \(f: G \rightarrow G'\) is a group homomorphism and that \(f\) preserves the linear order, then \(f\) is a homomorphism.
Proof.
Assume that \(x, y, z\) in \(G\) with condition \([x, y, z]_G\). This means \(x < y < z\), \(y < z < x\), or \(z < x < y\). Since \(f\) preserves the linear order, we have the following \(f(x) < f(y) < f(z)\), \(f(y) < f(z) < f(x)\), or \(f(z) < f(x) < f(y)\). This is equivalent to \([f(x), f(y), f(z)]_{G'}\).

Proposition.
Let \(G\) be a cyclically ordered group with the order \([...,]_G\), and \(G'\) be a group. If \(f: G \rightarrow G'\) is a group isomorphism, then \(G'\) is a cyclically ordered group.

Proof.
Let \(a, b, c\) in \(G'\), and assume that \(f(a) = a, f(b) = b, f(c) = c\), where \(x, y, z\) in \(G\). We can define a relation \([a, b, c]_{G'}\), which corresponds to the condition \([x, y, z]_G\).

As an illustration, consider the isomorphism \(f\) from \(R\), the group of real numbers, to \(R^*\), the group of positive real numbers via \(f(x) = 3^x\). The relation \([x, y, z]\) in \(R^*\) given by the condition that \(f(a) = x, f(b) = y, f(c) = z\), and \([a, b, c]\) in \(R\).

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