On group classification of evolution equations admitting non-local symmetries

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Abstract

We prove that any evolution equation admitting a potential symmetry can always be reduced to another evolution equation such that the potential symmetry in question maps into the group of its contact symmetries. Based on this fact is our group approach to classification of evolution equations possessing non-local symmetries. We present several examples of classifications of second-order evolution equations admitting potential symmetries.

1 Introductory Remarks

We study evolution type equations in one spatial dimension

\[ u_t = F(t, x, u, u_1, u_2, \ldots, u_n), \quad n \geq 2, \tag{1} \]

where \( u = u(t, x) \) is a real-valued function of two real variables \( t, x \), \( u_i = \partial^i u / \partial x^i \), \( i = 1, 2, \ldots, n \), and \( F \) is an arbitrary smooth real-valued function.

We say that partial differential equation (1) admits a Lie symmetry if there is a one-parameter family of maps \( \mathbb{R}^3 \to \mathbb{R}^3 \)

\[ t' = T(t, x, u, \theta), \quad x' = X(t, x, u, \theta), \quad u = U(t, x, u, \theta), \tag{2} \]

that transform the solution set of equation (1) into itself. One of the fundamental results of Sophus Lie are his celebrated theorems revealing group

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nature of such maps and establishing the infinitesimal criterium for any partial differential equations to admit a Lie symmetry see (e.g., [1]-[4]).

If we allow for the right-hand sides of (2) to depend on the first derivatives, namely,

\[
\begin{align*}
t' &= T(t, x, u, u_t, u_x, \theta), \\
x' &= X(t, x, u, u_t, u_x, \theta), \\
\frac{du}{dt} &= U(t, x, u, u_t, u_x, \theta), \\
\frac{du}{dx} &= V(t, x, u, u_t, u_x, \theta), \\
\end{align*}
\]

requiring that the first-order tangency condition

\[
du - u_t dt - u_x dx = 0
\]

is invariant under the action of (3), then we arrive at the concept of contact transformation group [2].

Following Lie’s methodology we rewrite (3) in the infinitesimal form

\[
\begin{align*}
t' &= t + \epsilon \tau(t, x, u, u_t, u_x), \\
x' &= x + \epsilon \xi(t, x, u, u_t, u_x), \\
\frac{du}{dt} &= u + \epsilon \eta(t, x, u, u_t, u_x), \\
\frac{du}{dx} &= u + \epsilon \rho(t, x, u, u_t, u_x).
\end{align*}
\]

Here

\[
\begin{align*}
\tau &= \left. \frac{dT}{d\theta} \right|_{\theta=0}, & \xi &= \left. \frac{dX}{d\theta} \right|_{\theta=0}, & \eta &= \left. \frac{dU}{d\theta} \right|_{\theta=0}, & \zeta &= \left. \frac{dV}{d\theta} \right|_{\theta=0}, & \rho &= \left. \frac{dW}{d\theta} \right|_{\theta=0}
\end{align*}
\]

and \( \epsilon \) is an infinitesimal parameter.

It is a common knowledge that the coefficients \( \tau, \ldots, \rho \) are expressed in terms of the generating function \( \varphi = \varphi(t, x, u, u_t, u_x) \) [2]

\[
\begin{align*}
\tau &= -\frac{\partial \varphi}{\partial u_t}, & \xi &= -\frac{\partial \varphi}{\partial u_x}, \\
\eta &= \varphi - u_t \frac{\partial \varphi}{\partial u_t} - u_x \frac{\partial \varphi}{\partial u_x}, \\
\zeta &= \frac{\partial \varphi}{\partial t} + u_t \frac{\partial \varphi}{\partial u}, & \rho &= \frac{\partial \varphi}{\partial x} + u_x \frac{\partial \varphi}{\partial u}.
\end{align*}
\]

Magadeev proves [5] that the most general form of the contact transformation group admitted by evolution equation (11) under \( n \geq 2 \) reads as

\[
\begin{align*}
t' &= T(t, \theta), \\
x' &= X(t, x, u, u_x, \theta), \\
\frac{du}{dt} &= U(t, x, u, u_x, \theta).
\end{align*}
\]
Lie symmetries plays the central role in the modern theory of differential equations. The main reason is that the most successful mathematical models of the physical, chemical and biological processes do admit nontrivial Lie symmetries. One can argue that it is this very property which singles out the Maxwell equations from the set of hyperbolic type systems of first-order linear partial differential equations \[7\]. This type of speculations were the major motivation for the so called symmetry selection principle. It says, that those equations should be considered as realistic models of the nature processes which admit nontrivial Lie symmetries (for further discussion of this matter, see \[8\]). It is customary to call this procedure group classification of differential equations.

There is a host of papers devoted to group classification of different subclasses of the class of equations (1) (see, e.g., \[9\]-\[11\] and references therein). They provide a very clear picture of what we can do and what we cannot achieve by utilizing the classical Lie symmetries. Since Lie symmetries cannot be answers to all challenges of the modern theory of nonlinear differential equations, one has always been looking for a way to generalize the notions of Lie and contact symmetries. A natural approach is allowing for the right-hand sides of (5) to depend on higher derivatives of \(u\), which yields the concept of Lie-Bäcklund symmetry \[2\].

A more radical generalization would be to allow for dependance on integrals of \(u\), which is the way the non-local symmetries arose. The major problem with non-local symmetries is that unlike the case of Lie or Lie-Bäcklund symmetries there is no generic algorithm for computing these. Almost each specific class of partial differential equations requires special, often unique treatment \[2\]-\[7\]. The most well studied is the case of non-local symmetries of linear partial differential equations (see \[6\]-\[7\] and the references therein).

Much less is known about non-local symmetries of nonlinear partial differential equations. One of the possible approaches has been suggested by Bluman \[12\]-\[13\]. The idea is presenting evolution equation (1) in the form of conservation law

\[
\frac{\partial}{\partial t} \left( f(t, x, u) \right) = \frac{\partial}{\partial x} \left( g(t, x, u, u_1, \ldots, u_{n-1}) \right),
\]

(6)

where \(f, g\) are some smooth real-valued functions. Next, one introduces the new dependant variable \(v = v(t, x)\) and rewrite (6) as the system of two differential equations

\[
v_t = g(t, x, u, u_1, \ldots, u_n), \quad v_x = f(t, x, u).
\]

(7)
Suppose now that system (7) admits a Lie symmetry
\begin{align*}
t' &= T(t, x, u, v, \theta), \quad x' = X(t, x, u, v, \theta), \\
u' &= U(t, x, u, v, \theta), \quad v' = V(t, x, u, v, \theta),
\end{align*}
and what is more, at least one of the functions $T, X, U, V$ depends on the non-local variable $v$. Then (7) is the non-local symmetry of the initial equation (1) called also its potential symmetry.

There is another approach to constructing non-local symmetries which is to apply a non-local transformation to an equation admitting non-trivial Lie symmetries. Then some of symmetries of the initial equation will remain Lie symmetries of the transformed equation, while the others become non-local ones. They are called quasi-local symmetries \cite{14, 15}.

Recently, we developed the regular group approach to classification of systems of evolution equations that admit quasi-local symmetries \cite{16, 17}. It has been conjectured that there is a link between quasi-local and potential symmetries. The principal aim of this paper is to prove that the connection does exist and, in fact, any potential symmetry is a quasi-local one. Namely, there exists a non-local change of variables that maps any equation possessing potential symmetry into an equation, which admits a contact symmetry. The connection between contact and potential symmetries provides a convenient tool for group classification of equations admitting non-local symmetries. We give several examples of such classification.

\section{Theoretical Background}

Let us introduce new dependent variable $w(t, x) = f(t, x, u(t, x))$ and rewrite equations (6), (7) as follows
\begin{equation}
v_t = \tilde{g}(t, x, w, w_1, \ldots, w_n), \quad v_x = w.
\end{equation}
Equation (1) now reads as
\begin{equation*}
w_t = \frac{\partial}{\partial x} \left( \tilde{g}(t, x, w, w_1, \ldots, w_{n-1}) \right)
\end{equation*}
and its Lie symmetry group (8) takes the form
\begin{align*}
t' &= \tilde{T}(t, x, w, v, \theta), \quad x' = \tilde{X}(t, x, w, v, \theta), \\
u' &= \tilde{U}(t, x, w, v, \theta), \quad v' = \tilde{V}(t, x, w, v, \theta).
\end{align*}
Note, that the above transformation group by definition preserve the first-order tangency conditions
\[ dv - v_t dt - v_x dx = 0, \quad dw - w_t dt - w_x dx = 0. \]

Eliminating \( w \) from the first equation of system (9) yields
\[ v_t = \tilde{g}(t, x, v_1, \ldots, v_n). \] (11)

By construction, transformation group (10) maps the solution set of (11) into itself. Consequently, this group is a Lie symmetry of evolution equation (11). Moreover, using the second equation from (9) we can eliminate the function \( w \) from the relations (10) thus getting
\[ t' = \tilde{T}(t, x, v, x, v) \]
\[ x' = \tilde{X}(t, x, v, x, v) \]
\[ u' = \tilde{U}(t, x, v, x, v) \]
\[ v' = \tilde{V}(t, x, v, x, v) \] (12)

Again, by construction, this group transforms the solution set of equation (11) into itself and therefore is the symmetry group of (11). Next, as group (10) preserves the tangency condition \( dv - v_t dt - v_x dx = 0 \), so does group (12). Hence, transformation group (12) is the group of contact transformations.

Thus we proved the following assertion.

**Theorem 1** Let (1) be an arbitrary evolution type differential equation possessing potential symmetry. Then there is a change of variables that transforms potential symmetry into a contact symmetry group of the appropriately transformed evolution equation (11).

**Note 1.** In the case under consideration the transformed evolution equation is given by (11), while its contact symmetry group reads as (12). The change of variables reducing (1) to (11) is given by the formula \( v_x = f(t, x, u) \).

**Note 2.** Transformed evolution equation (11) admits Lie symmetry \( \partial_u \). This important observation will be used to classify potential symmetries associated with a given evolution equation.

We established that any potential symmetry of an evolution equation is quasi-local symmetry in a sense that it can be reduced to a local (contact) symmetry group. We are going to prove that the inverse assertion is also true. Suppose that evolution equation (1) admits contact symmetry group, which has, at least, a one-parameter subgroup
\[ t' = t, \quad x' = \tilde{X}(t, x, u, u_x, \theta), \]
\[ u' = \tilde{U}(t, x, u, u_x, \theta), \quad u_x' = \tilde{U}(t, x, u, u_x, \theta) \] (13)
preserving the temporal variable \( t \). Then there is a contact transformation

\[
\begin{align*}
\bar{t} &= t, \quad \bar{x} = X(t, x, u, u_x, \theta), \\
\bar{u} &= U(t, x, u, u_x, \theta), \quad \bar{u}_x = V(t, x, u, u_x, \theta)
\end{align*}
\]

reducing (13) to the one-parameter groups of translations by \( \bar{u} \). The corresponding evolution equation (11) takes the form

\[
u_t = f(t, x, u_1, \ldots, u_n) .
\]

(15)

Note that we dropped the bars.

Now following [17, 18] (see, also [19]) we differentiate (15) with respect to \( x \) and introduce the new dependent variable \( v(t, x) = \partial u / \partial x \) thus getting

\[
v_t = \frac{\partial}{\partial x} \left( f(t, x, v, v_1, \ldots, v_{n-1}) \right) .
\]

(16)

So we arrive at the equation written in the form of conservation law. Let us emphasize that the reason for an initial equation (11) to be transformable to the form (16) is the contact symmetry group (13) admitted by (1).

Now, provided evolution equation (15) admits an additional contact transformation group \( \mathcal{G} \)

\[
\begin{align*}
t' &= T(t, \theta), \quad x' = X(t, x, u, u_x, \theta), \\
u' &= U(t, x, u, u_x, \theta), \quad u'_x = V(t, x, u, u_x, \theta),
\end{align*}
\]

the latter is mapped into the symmetry group of evolution equation (16) by the non-local change of the dependent variable \( v = \partial u / \partial x \). The type of symmetry, Lie or non-Lie, is determined by the coefficients \( T, X, U, V \).

If

\[
\frac{\partial T}{\partial u} = \frac{\partial X}{\partial u} = \frac{\partial U}{\partial u} = \frac{\partial V}{\partial u} = 0,
\]

then the group \( \mathcal{G} \) is mapped into the Lie symmetry

\[
\begin{align*}
t' &= T(t, x, v, \theta), \quad x' = X(t, x, v, \theta), \quad v' = V(t, x, v, \theta)
\end{align*}
\]

of equation (15). Next, provided

\[
\left( \frac{\partial T}{\partial u} \right)^2 + \left( \frac{\partial X}{\partial u} \right)^2 + \left( \frac{\partial U}{\partial u} \right)^2 + \left( \frac{\partial V}{\partial u} \right)^2 \neq 0,
\]

6
is transformed into the non-local (potential) symmetry
\[ t' = T(t, x, \partial_x^{-1} v, v, \theta), \quad x' = X(t, x, \partial_x^{-1} v, v, \theta), \quad v' = V(t, x, \partial_x^{-1} v, v, \theta). \]

Hereafter, $\partial_x^{-1}$ is the inverse of $\partial_x$, so that $\partial_x \partial_x^{-1} \equiv \partial_x^{-1} \partial_x \equiv 1$.

Rewriting the above listed constraints through infinitesimals of the group $G$ we get the following assertion.

**Theorem 2** Let evolution equation (15) be invariant under the group of contact transformations
\[
\begin{align*}
t' &= t + \epsilon \tau(t) \quad x' = x + \epsilon \xi(t, x, u, u_x), \\
u' &= u + \epsilon \eta(t, x, u, u_x), \quad u' &= u_x + \epsilon \rho(t, x, u, u_x) .
\end{align*}
\]

Then differentiating (15) with respect to $x$ and making the non-local change of variables $v = \partial u/\partial x$ yield an evolution equation in a form of conservation law (16), while group (17) is mapped into

- **Lie symmetry of equation (16)**, provided
  \[ \frac{\partial \tau}{\partial u} = \frac{\partial \xi}{\partial u} = \frac{\partial \eta}{\partial u} = \frac{\partial \rho}{\partial u} = 0; \]

- **non-local (potential) symmetry of equation (16)**, provided
  \[ \left( \frac{\partial \tau}{\partial u} \right)^2 + \left( \frac{\partial \xi}{\partial u} \right)^2 + \left( \frac{\partial \eta}{\partial u} \right)^2 + \left( \frac{\partial \rho}{\partial u} \right)^2 \neq 0. \]

It immediately follows from Theorem 1 that the problem of constructing of all possible evolution equations that admit potential symmetries is equivalent to classification of equations of the form (16) invariant under groups of contact transformations
\[
\begin{align*}
t' &= t + \epsilon \tau(t, x, u, u_x) \quad x' &= x + \epsilon \xi(t, x, u, u_x), \\
u' &= u + \epsilon \eta(t, x, u, u_x), \quad u' &= u_{xx} + \epsilon \rho(t, x, u, u_x)
\end{align*}
\]
with
\[ \left( \frac{\partial \tau}{\partial u} \right)^2 + \left( \frac{\partial \xi}{\partial u} \right)^2 + \left( \frac{\partial \eta}{\partial u} \right)^2 + \left( \frac{\partial \rho}{\partial u} \right)^2 \neq 0. \]
Indeed, Theorem 1 states that any evolution type equation (16) that is presented in the form of conservation law can be reduced to the form (15). What is more, its potential symmetry transforms into contact symmetry (18) of evolution equation (15). Based on this observation is our algorithm for group classification of evolution equations admitting potential symmetries.

1. We compute the maximal symmetry group $G$ of contact transformations leaving differential equation (1) invariant.

2. We classify inequivalent one-parameter subgroups of $G$ and select subgroups $G_1, \ldots, G_p$ of the form (13).

3. For each subgroup, $G_i$, we construct change of variables (14) reducing the corresponding subgroup to the group of translations by $u$, which leads to evolution equations of the form (16).

4. Since the invariance group, $\bar{G}$, admitted by (15) is isomorphic to $G$, we can utilize the results of subgroup classification of $G$. For each of the one-parameter subgroups of $\bar{G}$ we check whether its infinitesimals (18) satisfy (19). This yields the list of evolution equations that can be reduced to those admitting potential symmetries.

5. Performing the non-local change of variables $v = \partial u/\partial x$ yields evolution equations (16) admitting potential symmetries.

The above algorithm takes especially simple form for the case when the maximal group, $G$, admitted by evolution equation (1) contains Lie symmetries only. Suppose that evolution equation (1) admits at least a two-parameter Lie symmetry group $\mathfrak{g}$. Furthermore, we suppose that this group has a one-parameter subgroup leaving the variable $t$ invariant. With this condition in hand we can reduce equation under study to the form (15). The transformed equation (15) admits at least two Lie symmetries $e_1 = \partial_u$ and $e_2 = \tau(t)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u$. Now, if the infinitesimals $\tau, \xi, \eta$ satisfy one of the inequalities

\begin{align}
\frac{\partial \xi}{\partial u} &\neq 0, \\
\frac{\partial \xi}{\partial u} = 0, \quad \left(\frac{\partial^2 \eta}{\partial u \partial x}\right)^2 + \left(\frac{\partial^2 \eta}{\partial u \partial u}\right)^2 &\neq 0,
\end{align}
then (15) reduces to the conservation law form (16), and what is more, the Lie symmetry $e_2$ is mapped into the potential symmetry of the obtained evolution equation (see, also [17]).

3 Examples.

As shown in [9] the maximal Lie symmetry algebra of the second-order evolution equation

$$u_t = u_{xx} - uu_x + \lambda u_x^{3/2}, \quad \lambda \in \mathbb{R}, \quad \lambda \neq 0$$

(22)

is spanned by the following infinitesimal operators

$$e_1 = \partial_t, \quad e_2 = \partial_x, \quad e_3 = t\partial_x + \partial_u, \quad e_4 = 2t\partial_t + x\partial_x - u\partial_u.$$  (23)

This algebra has two inequivalent one-dimensional subalgebras that preserve the variable $t$, namely, $A_1 = \langle \partial_x \rangle$ and $A_2 = \langle t\partial_x + \partial_u \rangle$. Consider, first, the algebra $A_1$. As a first step of the algorithm we need to transform the basis operator of $A_1$ to the canonic form. This is achieved by the following change of variables

$$\bar{t} = t, \quad \bar{x} = u, \quad \bar{u} = x.$$  (24)

Now algebra (23) reads as

$$e_1 = \partial_{\bar{t}}, \quad e_2 = \partial_{\bar{u}}, \quad e_3 = t\partial_{\bar{u}} + \partial_{\bar{x}}, \quad e_4 = 2t\partial_{\bar{t}} + u\partial_{\bar{u}} - x\partial_{\bar{x}}$$

(we dropped the bars). Evidently, the coefficients of the operators $e_1,e_3,e_4$ do not obey the conditions (20), (21), which means that the operator $e_2 = \partial_x$ does not yield an equation possessing potential symmetries.

Turn now to the second algebra $A_2$. The change of variables

$$\bar{t} = t, \quad \bar{x} = x - tu, \quad \bar{u} = u$$  (24)

reduces its basis element $t\partial_x + \partial_u$ to the canonic form $\partial_{\bar{u}}$. After making in (23) the above change of variables, we get

$$e_1 = \partial_{\bar{t}} - u\partial_{\bar{x}}, \quad e_2 = \partial_{\bar{x}}, \quad e_3 = \partial_{\bar{u}}, \quad e_4 = 2t\partial_{\bar{t}} + u\partial_{\bar{u}} - x\partial_{\bar{x}}$$

(as earlier, we drop the bars). Now the coefficients of the operator $e_1$ satisfy (20) and, consequently, change of variables (24) transforms equation (22) to the one admitting the potential symmetry.
Indeed, being written in the new variables equation (22) takes the form
\[ u_t = (1 + tu_x)^{-2}u_{xx} + \lambda(1 + tu_x)^{-1/2}u_{x}^{3/2}. \]
Differentiating this equation with respect to \( x \) and introducing the new dependant variable \( v(t, x) = u_{x} \), we get
\[ v_t = \frac{\partial}{\partial x}\left((1 + tv_x)^{-2}v_x + \lambda(1 + tv)^{-1/2}v_{x}^{3/2}\right). \]
This is the evolution equation represented in the form of conservation law and it admits the following one-parameter group of non-local transformations
\[ t' = t + \theta, \quad x' = x - \theta \partial^{-1} x, \quad v' = v. \]
Here \( \theta \) is the group parameter.

Consider now applying our algorithm to the following \( sl(2, \mathbb{R}) \) invariant evolution equation [18]:
\[ u_t = xu_x f(t, \omega), \quad \omega = x^{-5}u_{x}^{-3}u_{xx} + 2x^{-6}u_{x}^{-2}. \] (25)
Under arbitrary \( f \) the maximal symmetry algebra admitted by equation (25) reads as
\[ e_1 = \partial_u, \quad e_2 = 2u\partial_u - x\partial_x, \quad e_3 = -u^2\partial_u + xu\partial_x. \]
It is a common knowledge that there exist three inequivalent one-dimensional subalgebras of the algebra \( sl(2, \mathbb{R}) \), namely, \( \langle e_1 \rangle \), \( \langle e_2 \rangle \), \( \langle e_1 + e_3 \rangle \).

Consider first the algebra \( \langle e_1 \rangle \). Its basis operator is already in the canonic form. Since the coefficients of the basis element \( e_3 \) satisfy (21), it leads to the evolution equation that admits a potential symmetry. The equation in question is obtained if we differentiate (25) with respect \( x \) and introduce the new dependant variable \( v = u_{x} \)
\[ v_t = \frac{\partial}{\partial x}\left(xvf(t, \omega)\right), \quad \omega = x^{-5}v^{-3}v_{x} + 2x^{-6}v^{-2}. \]

Turn now to the algebra \( e_2 \). Making the change of variables
\[ \bar{t} = t, \quad \bar{x} = x^2u, \quad \bar{u} = \frac{1}{2}\ln u \] (26)
reduces \( e_2 \) to the canonic form \( \partial_{u} \). The operators \( e_1 \) and \( e_3 \) take the form
\[ e_1 = x\exp(-2u)\partial_x + \frac{1}{2}\exp(-2u)\partial_u, \quad e_2 = -x\exp(2u)\partial_x + \frac{1}{2}\exp(2u)\partial_u. \]
Note that in the above formulas we drop the bars. Since the coefficients of the operators $e_1, e_3$ satisfy constraints (21) they both lead to the potential symmetries of the evolution equation obtained from (25) by making the change of variables (26), then differentiating the obtained relation with respect to $\bar{x}$ and, finally, replacing $u_x$ with $v$. As a result, we get the differential equation

$$v_t = \frac{\partial}{\partial x} \left( 2xvf(t, \omega) \right), \quad \omega = 2x^{-3}v^{-3}v_x + 3x^{-4}v^{-2} - 4x^{-2},$$

which admits the two-parameter group of potential symmetries.

Finally, consider the algebra $\langle e_1 + e_3 \rangle$. The change of variables

$$\bar{t} = t, \quad \bar{x} = x^2(1 - u^2), \quad \bar{u} = \frac{1}{2} \ln \frac{1 - u}{1 + u}$$  \hspace{1cm} (27)

reduces the operator $e_1 + e_3$ to the canonic form $\partial_{\bar{u}}$. And what is more, the basis operators $e_1, e_2$ now read as

$$e_1 = x \sinh 2u \partial_x - \cosh^2 u \partial_u, \quad e_3 = -2x \cosh 2u \partial_x + \sinh 2u \partial_u.$$

As usual, we drop the bars. Making change of variables (27) in the initial evolution equation, differentiating with respect to $x$ and replacing $u_x$ with $v$ yield

$$v_t = \frac{\partial}{\partial x} \left( -2xvf(t, \omega) \right), \quad \omega = 2x^{-3}v^{-3}v_x + 6x^{-4}v^{-2} - 4x^{-2}.$$

Since the coefficients of the operators $e_1$ and $e_3$ satisfy constraints (21), the above evolution equation admits the two-parameter group of potential symmetries.

4 Concluding Remarks

Group approach to classification of evolution equations developed in the present paper can easily be modified to become applicable to systems of evolution equations in the same way as it has been done for quasi-local symmetries in [17]. The essential difference is that for the case of systems of evolution equations the potential symmetries always correspond to Lie symmetries of transformed system of evolution equation. The reason is that if a system of partial differential equations admits a group of contact symmetries, the latter is always a first prolongation of a Lie symmetry group [2].
Hence it follows, in particular, that the problem of classification of systems of evolution type equations admitting potential symmetries can be solved completely within the framework of point Lie symmetries.

In the case of a single evolution equation we should go beyond Lie symmetry in order to recover all possible equations admitting potential symmetries. One needs to classify inequivalent finite-dimensional groups of contact transformations realized as symmetry groups of \([1]\). While the problem of group classification of equations \([1]\) that admit Lie symmetry is well understood and there are powerful methods to handle it, systematic exploration of contact symmetries of evolution equations is yet to be done. Some partial results on this topic can be found in \([19]-[21]\).

We intend to devote one of our future publication to application of the approach developed in the present paper to utilize contact symmetries of evolution equations in order to classify their non-local symmetries.

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