Quasi-stationary analysis of the contact process on annealed scale-free networks

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We present an analysis of the quasi-stationary (QS) state of the contact process (CP) on annealed scale-free networks using a mapping of the CP dynamics in a one-step processes and analyzing numerically and analytically the corresponding master equation. The relevant QS quantities determined via the master equation exhibit an excellent agreement with direct QS stochastic simulations of the CP. The high accuracy of the resulting data allows to probe the strong corrections to scaling present in both the critical and supercritical regions, corrections that mask the correct finite size scaling obtained analytically by applying an exact heterogeneous mean-field approach. Our results represent a promising starting point for a deeper understanding of the contact process and absorbing phase transitions on real (quenched) complex networks.

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I. INTRODUCTION

Complex network theory represents a general unifying formalism under which is possible to understand and rationalize the intricate connectivity and interaction patterns of many natural and man-made systems. Thus, a systematic statistical analysis of large scale dataset has allowed to unveil the existence of apparently universal topological features, shared by a large number of different systems from the technological, social or biological domains [1–3]. Among these characteristics, probably the most intriguing is the discovery of the apparently ubiquitous scale-free (SF) nature of the connectivity described by a probability $P(k) \sim k^{-\gamma}$ that an element (vertex) is connected to other $k$ elements (has degree $k$), with a degree exponent usually in the range $2 < \gamma < 3$ [1–3]. These and other discoveries have promoted a large modeling activity, aimed at understanding the origin and nature of the observed topological features [2–3]. In recent years, the research community has also devoted a great deal of attention to the study of the dynamical processes on complex networks [1–7], which can have important implications in the real understanding of complex systems such as the spread of epidemics in social systems [6] or traffic in technological systems as the Internet [7] or transport infrastructures [8].

The theoretical understanding of dynamical processes on complex networks is based in the application of mean-field approaches that are essentially based in the annealed network approximation [5]. Any non-weighted network is fully characterized by its adjacency matrix $a_{ij}$, taking the value 1 when vertices $i$ and $j$ are connected by an edge, and zero otherwise. In real (quenched) networks, the values of the adjacency matrix are fixed and do not change with time. When a dynamical process takes place on top of such a network, one is considering the network as frozen, with respect to the characteristic time scale $\tau_D$ of the dynamics. Other networks, however, are dynamical objects, changing in time over a characteristic time scale $\tau_N$. In this case, the adjacency matrix is defined only in a statistical sense, and a complete description of the network can be given in terms of its probability distribution $P(k)$ and its degree correlations $P(k|k')$ [9]. In the limit $\tau_N < \tau_D$, when all edges are completely reshuffled within a small time dynamical time steps, the resulting network is called annealed [10]. Annealed networks represent an extremely important theoretical tool, because mean-field predictions for dynamical processes turn out to be exact in this kind of substrates [5].

Dynamical processes with absorbing configurations constitute a subject of outstanding interest in non-equilibrium Statistical Physics [11–12] that have also found a place in network science, as representative models of practical problems ranging from epidemic spreading [13–14], infrastructure’s resilience [15–16], or activated dynamics [17], to mention just a few. The simplest lattice model allowing absorbing configurations is the classical contact process (CP) [18], whose universality class and mean-field (MF) description have been discussed along the last few years [10–19–22]. In the CP defined in an arbitrary network, vertices can be in two different states, either empty or occupied. The dynamics includes the spontaneous annihilation of occupied vertices, which become empty at unitary rate, and the self-catalytic occupation of an empty vertex $i$ with rate $\lambda n_i/k_i$, where $n_i$ is the number of occupied neighbors of $i$ and $k_i$ is its degree. The model is thus characterized by a phase transition at a value of the control parameter $\lambda = \lambda_c$, separating an active from an absorbing phase devoid of active vertices. Despite of its simplicity, the CP on SF networks exhibits a very complex critical behavior even if studied in an annealed substrate [10–21–22]. This case is particularly interesting, since in annealed networks all connections are rewired at a rate much larger than the typical rates involved in the dynamical process, implying that dynamical correlations are absent [10]. In this case,
The configuration in which all vertices are empty plays a very particular role, since once the system has fallen into this state, the dynamics becomes frozen. For this reason, these states are called absorbing and constitute a central feature in the analysis of finite size systems since, in this case, the single actual stationary state is the absorbing one \([11]\). Finite size and absorbing states must therefore be handled using suitable strategies, concomitantly with an ansatz for the finite size scaling (FSS) \([23]\). A widely adopted procedure is the so-called quasi-stationary (QS) state \([21, 25]\), in which the absorbing configuration is suitably excluded from the dynamics.

In this work, we present a study of the QS state of the CP on SF annealed networks, combining the QS numerical approach developed in Ref. \([24]\), suitably extended to complex networks, with the theoretical analysis of an approximated one-step process derived from mean-field theory \([21]\). Our analysis allows us to determine the probability distribution of activity both close to the critical point and in the off-critical regime, as well as to obtain high quality data for relevant QS quantities, such as the density of active sites or the characteristic times. This last information is used to check the finite-size scaling forms derived from mean-field theory, which turn out to be loaded with very strong corrections to scaling.

The results presented in our paper provide a deeper understanding of the nontrivial dynamics of the contact process in annealed networks. Moreover, they open the path to extension of the QS approach to the analysis of other dynamic processes with absorbing phase transitions taking place on more complex and/or realistic substrates such as, for example, quenched networks and small-world topologies \([26]\), where edges are never rewired and dynamic correlations are usually present.

We have organized our paper as follows. In Sec. \([1]\) we review the necessary background for QS analysis and numerical simulations, while in Sec. \([11]\) we summarize the main results of MF theory for the CP on annealed networks. A master equation approach for the QS state is developed in Sec. \([4]\). Section \([X]\) is devoted to discuss the finite-size scaling forms of the relevant QS quantities, as well as the corrections to scaling at criticality. The off-critical analysis is discussed in Sec. \([VI]\). Finally, our concluding remarks are presented in Sec. \([VII]\).

II. FINITE SIZE AND THE QUASI-STATIONARY STATE

In finite systems, the absorbing state is a fixed point that can be visited even in the supercritical phase due to stochastic fluctuations. Numerical simulations of finite systems are particularly sensitive to absorbing states and therefore suitable simulation strategies are required. The standard procedure consists in restricting the averages to those runs that did not visit the absorbing configuration \([11]\), the so-called surviving averages. From a mathematical point of view, it is useful to define the quasi-stationary (QS) state that consists of the ensemble of states accessed by the original dynamical process at long times restricted to those not trapped into an absorbing one \([24]\). The intensive quantities in a QS ensemble must converge to the stationary ones in the thermodynamic limit. Thus, in the active phase, the lifespan grows exponentially fast with the system size and the QS state becomes identical to the stationary one. In the sub-critical phase, on the other hand, the activity in the QS state corresponds to complex networks, with the theoretical analysis of an alternative simulation strategy, in which every time the system visits the absorbing state, this configuration is replaced by an active one randomly taken from the history.
of the simulation [24]. For this task, a list with $M$ active configurations is stored and constantly updated. An update consists in randomly choosing a configuration in the list and replacing it by the present active configuration with a probability $p_r$. After a relaxation time $t_r$, the QS distributions are determined during an averaging time $t_a$. The improved QS method has been successfully applied to accurately determine the universality class of several models with absorbing configurations [24, 27, 28].

III. CP IN ANNEALED NETWORKS

The network in which dynamics takes place is assumed to be an annealed and therefore completely characterized by the degree distribution $P(k)$, the probability that a randomly chosen vertex has a degree $k$, and the degree correlation function $P(k'|k)$ defined as the conditional probability that a vertex of degree $k$ is connected to a vertex of degree $k'$. The number of vertices of the network is denoted by $N$ and its maximum degree (cutoff) by $k_c$ [29]. In an annealed framework, the MF rate equation for the density of occupied vertices in the degree class $k$ (i.e. the probability that a vertex of degree $k$ is occupied) is given by [21]

$$\frac{d}{dt} \rho_k(t) = -\rho_k(t) + \lambda k [1 - \rho_k(t)] \sum_{k'} P(k'|k) \rho_{k'}$$ \hspace{1cm} (4)

The first term represents the spontaneous annihilation and the second one the creation inside the compartment $k$ due to the interaction with all compartments under the hypothesis that there are no dynamical correlations. A simple linear stability analysis allows to show the presence of a phase transition, located at the value $\lambda_c = 1$ and independent of the degree distribution and degree correlations, separating an active from an absorbing phase with $\rho_k = 0$ [10]. Considering in addition uncorrelated networks with $P(k'|k) = k' P(k')/\langle k' \rangle$ [2], the overall density $\rho = \sum_k \rho_k P(k)$ obeys the equation

$$\frac{d}{dt} \rho(t) = -\rho(t) + \lambda \rho \left( 1 - \langle k \rangle^{-1} \sum_k k P(k) \rho_k(t) \right)$$ \hspace{1cm} (5)

In Ref. [21], it was realized that the low density regime of Eq. (5) can be understood as a one-step process (biased random walk) with transition rates

$$W(n - 1, n) = n, \quad W(n + 1, n) = \lambda n \left[ 1 - \langle k \rangle^{-1} \sum_k k P(k) \rho_k(t) \right]$$ \hspace{1cm} (6)

where $W(m, n)$ corresponds to the transition from a state with $n$ particles to one with $m$ particles. The stationary state $\partial_t \rho_k = 0$ of Eq. (4) reads as

$$\tilde{\rho}_k = \frac{\lambda k \rho k}{1 + \lambda k \rho}$$ \hspace{1cm} (7)

Close to the criticality, when the density at long times is sufficiently small such that $\rho k_c \ll 1$, equation (7) becomes $\rho k \approx \lambda k \rho / k$. Substituting in the transition rates, the first order approximation for the one-step process is

$$W(n - 1, n) = n, \quad W(n + 1, n) = \lambda n (1 - \lambda / \gamma N)$$ \hspace{1cm} (8)

in which $\gamma = \langle k^2 \rangle / \langle k \rangle^2$. Based on numerical evidences and scaling arguments, later confirmed by more rigorous means in Refs. [10] [22], the authors proposed that the critical characteristic time $\tau$ and stationary density $\tilde{\rho}$ scale as [21]

$$\tau \sim (N/g)^{1/2} \quad (9)$$

and

$$\tilde{\rho} \sim (Ng)^{-1/2} \quad (10)$$

respectively. For a network with degree exponent $\gamma$ and a cutoff scaling with the system size as $k_c \sim N^{1/\gamma}$, where $\omega$ is an arbitrary positive parameter, the factor $g$ scales for large $N$ as $g \sim k_c^\omega$ for $2 < \gamma < 3$ and $g \sim \text{const}$ for $\gamma > 3$. Therefore the critical QS density scales as $\rho \sim N^{-\alpha}$ where

$$\alpha = \frac{1}{2} + \max \left( \frac{3 - \gamma}{2\omega}, 0 \right)$$ \hspace{1cm} (11)

Similarly, the characteristic time follows $\tau \sim N^{-\hat{\alpha}}$ with exponent

$$\hat{\alpha} = \frac{1}{2} - \max \left( \frac{3 - \gamma}{2\omega}, 0 \right)$$ \hspace{1cm} (12)

The MF supercritical density for an infinite system was found to vanish at criticality as $\rho \sim \Delta^\beta$ where $\beta = 1/(\gamma - 2)$ [21] and $\Delta = \lambda - \lambda_c$. For finite systems, the QS density has an anomalous cutoff-dependent FSS given by [10]

$$\tilde{\rho}(\Delta, N) = \frac{1}{\sqrt{gN}} G \left( \Delta \sqrt{\frac{N}{g}} \right) \quad \text{for} \quad \frac{\Delta}{g} \ll \frac{\lambda (k_c)}{k_c}$$ \hspace{1cm} (13)

where $G(x) \sim x$ for $x \gg 1$ and $G(x)$ is constant for $x \ll 1$. The anomaly lies on the supercritical density dependence on the system size through the factor $g$ given by $\rho \sim \Delta / g$ if $\Delta > \sqrt{g/N}$ [10].

IV. MASTER EQUATION APPROACH OF THE QS STATE

In order to gain analytical information about the QS distribution not far away from the critical point, we can consider the one-step process approximation described by the transition rates in Eq. (6). Starting from them, it is possible to write down a master equation (ME) for
the evolution of the number of particles $P_n(t)$, taking the standard form

$$\dot{P}_n = \sum_m W(n, m)P_m(t) - \sum_m W(m, n)P_n(t).$$

(14)

In the long time limit we have $\dot{P}_n \approx 0$ and, consequently, equation (7) can be applied resulting the ME

$$\dot{P}_n = (n + 1)P_{n+1} + u_{n-1}P_{n-1} - (n + u_n)P_n,$$

(15)

with $u_n = \lambda n(1 - \Theta)$ and $\Theta$ given by

$$\Theta[\rho] = \frac{\lambda n}{\langle k \rangle^2} \sum_k k^2 P(k) \frac{1}{1 + \lambda n^2 k/\langle k \rangle}.$$

(16)

Substituting now $P_n(t) = P_s(t)\bar{P}_n$ and using $dP_s/dt = -\dot{P}_1 P_s$, the following recurrence relation is obtained

$$\bar{P}_n = \frac{1}{n}[(u_{n-1} + n - 1 - \bar{P}_1)\bar{P}_{n-1} - u_{n-2}\bar{P}_{n-2}],$$

(17)

where $n = 2, \ldots, N$ and $\bar{P}_0 \equiv 0$. The QS distributions are completely determined since $P_1$, the initial condition to iterate (17), is given by the normalization $\sum_{n\geq1} \bar{P}_n = 1$.

Full information of the QS distribution can be obtained from the ME by solving it numerically. A numerical recipe to iterate the recurrence relation is as follows [25]: Start with a guess for $\bar{P}_1^{(0)}$ and iterate (17) to find $\bar{P}_n^{(0)}$, $n = 2, \ldots, N$. Repeat the procedure using $\bar{P}_1^{(j+1)} = \bar{P}_1^{(j)}/\sum_n \bar{P}_n^{(j)}$ as new guess until the normalization be reached. Suitable truncations can be used to speed up the numerical process and to prevent instabilities. The truncations at finite densities are justifiable since the Central Limit Theorem guarantees that fluctuations much larger or much smaller than the average are exponentially negligible.

In order to explore the properties of the QS state, we have performed extensive Monte Carlo simulations of the CP on annealed networks. We use a random neighbor network (RNN) [10] with degree distribution $P(k) \sim k^{-\gamma}$, degree correlations $P(k'|k) = k'|P(k')/\langle k \rangle$, a degree cut-off $k_c = N^{1/\gamma}$, and a fixed minimum degree $k_0 = 2$. The single effect in increasing the minimum degree $k_0$ is a shift to higher densities which does not affect the critical properties. In an annealed approach, all links are redefined between any two time steps in such a way that the neighbor of a given vertex is selected by randomly choosing a vertex of the network with a probability $k'|P(k')/\langle k \rangle$. We perform stochastic simulations using the usual scheme [11]: At each time step, an occupied vertex is chosen at random and the time updated as $t \rightarrow t + \Delta t$, where $\Delta t = 1/[(1 + \lambda)n(t)]$ and $n(t)$ is the number of occupied vertices at time $t$. With probability $p = 1/(1 + \lambda)$, the occupied vertex becomes vacant. With complementary probability $1 - p = \lambda/(1 + \lambda)$, one of its neighbors (following RNN rules) is selected and, if empty, occupied. If the selected neighbor is already occupied the simulation goes to the next step. QS states were simulated using the method described in Sec. [11] with $M = 10^5$, $p = 0.02\Delta t$ and $t_a = t_f = 10^6$. Due the short distance between vertices, the relaxation times are very short if compared with critical relaxation on regular lattices. The network sizes were varied from $10^3$ to $10^5$ and $50 - 500$ network samples were used in the averages (the larger the size of the network, the smaller the number of samples). During the averaging interval, the current configuration is counted in the QS distribution with a probability proportional to its lifespan.

The QS probability distributions for obtained in simulations of the critical CP are shown in Fig. 1 for different values of the degree exponent $\gamma$ and network size $N$ and compared with the results of the numerical solution of the recursion relation, equation (17). A remarkable agreement between the simulations and the one-step process approach is achieved even for the asymptotic (Gaussian) tail as one can see in the inset A of Fig. 1. A good accordance, which is improved as size increases, is observed even for sizes as small as $10^3$, while neat discrepancies appears for $N \sim 10^2$ (inset B of Fig. 1). Actually, we can show that the one-step and the Langevin approach developed in Ref. [10] are equivalent in the low density limit (see Appendix) and, consequently, the lower the density the better the one-step mapping.

V. THE QS STATE AT CRITICALITY

A. Analytical approximation at criticality

At criticality, where the densities at long times are very low, we have $u_n = \lambda n(1 - \lambda n/\Omega)$ where $\Omega = N/g$. In this limit, equation (15) corresponds exactly to the ME of the CP on a complete graph of size $\Omega$, for which the QS
analysis was already worked out elsewhere [23]. Analytical insights about the criticality can be obtained through a van Kampen’s expansion [30] of the recurrence relation [17]. Let us consider the scaling solution of Eq. (17)

$$P_n = \frac{1}{\sqrt{\Omega}} f \left( \frac{n}{\sqrt{\Omega}} \right),$$

where $f(x)$ is a scaling function to be determined. Plugging Eq. (18) into (17), and performing a Taylor expansion up to second order, the result up to order $\Omega^{-1}$ is

$$x \frac{d^2 f}{dx^2} + (2 + x^2) \frac{df}{dx} + 2xf = -f_0 f$$

where $f_0 = P_1 \Omega^{1/2} = f(0)$ must be chosen to impose the normalization condition $\int_0^\infty f(y)dy = 1$. Dickman and Vidgal [25] analyzed Eq. (19) numerically and checked the agreement with the recurrence relation for the CP on the complete graph. We complement the analysis by obtaining the asymptotic behaviors analytically. It is straightforward to see that the distribution decays linearly by obtaining the asymptotic behaviors analytically. It is straightforward to see that the distribution decays linearly as $f(x) \simeq f_0 (1 - x f_0/2)$ for $x \ll 1$. A correction to this initial behavior can be obtained discarding the term $xf''(x)$ (a low curvature approximation) and the solution satisfying the boundary condition $f(0) = f_0$ is

$$f(x) \simeq \frac{2f_0}{2 + x^2} \exp \left[ -\frac{f_0}{\sqrt{2}} \arctan \left( \frac{x}{\sqrt{2}} \right) \right], \quad x \ll 1.$$  

For $x \gg 1$ the zeroth order terms are discarded and Eq. (19) turns to $f'' + xf' + 2f \simeq 0$. The solution satisfying the boundary condition $xf(x) \rightarrow 0$ for $x \rightarrow \infty$ ($(x)$ is finite) is

$$f(x) \sim \exp(-x^2/2), \quad x \gg 1,$$

implying a Gaussian tail.

In Fig. 2 we compare the results of numerical simulations of the CP at criticality ($\lambda = 1$) on a network with $N = 160000$ nodes, degree exponent $\gamma = 2.25$ and cutoff scaling exponent $\omega = 2$ with the corresponding analytical approximations. Despite of the lack of rigor, the analytical result is in good agreement with numerical simulations. The approximation [21] agrees with numerical analysis even for $x \approx 1$ or equivalently for a number of active vertices $n \approx \sqrt{\Omega}$. For the particular network of Fig. 2 we have $\sqrt{\Omega} \approx 147$. The accordance still holds for small networks ($\sim 10^3$) independently of the exponent degree and cutoff scaling.

The scaling function [15] encloses the FSS form of the CP at criticality. In fact, the mean number of occupied vertices at the QS regime is given by

$$\bar{n} = \sum_{n=1}^N n \bar{P}_n = \sum_{n=1}^N \frac{n}{\sqrt{\Omega}} f \left( \frac{n}{\sqrt{\Omega}} \right),$$

where $\Omega = N/g$. Letting $x = n/\sqrt{\Omega}$ and $\Delta x = 1/\sqrt{\Omega}$, the sum can be approximated by a continuous integration when $N \rightarrow \infty$, namely

$$\bar{n} = \Omega^{1/2} \sum_{x=m/\sqrt{\Omega}}^{N/\sqrt{\Omega}} x f(x) \Delta x \approx \Omega^{1/2} \int_0^\infty x f(x) dx \sim \Omega^{1/2}.$$ 

Therefore, the critical QS density is simply $\bar{\rho} \equiv \bar{n}/N \sim (gN)^{-1/2}$, recovering the result firstly presented in Ref. [21]. Analogously, the characteristic time $\tau$ is also directly obtained by the present QS analysis by noticing that, since $P_1 = f_0 \Omega^{-1/2}$, we have $\tau = 1/P_1 \sim \Omega^{1/2} = (N/g)^{1/2}$.

B. Scaling at criticality

The analysis of the QS state, either by direct QS simulations or of the iterative solution of the corresponding approximate ME, allows us to obtain high-quality data for the characteristic quantities at criticality, namely the stationary density $\bar{\rho}$ and the characteristic time $\tau$. At criticality, these quantities are expected to exhibit a scaling with system size of the form $\bar{\rho} \sim N^{-\nu}$ and $\tau \sim N^{\nu}$, with exponents given by Eqs. (11) and (12) and depending on the cutoff scaling exponent $\omega$. QS critical densities as functions of the network size $N$, computed for several degree exponents and $\omega = 2$ are shown in the main plot of Fig. 3. Again, an incontestable agreement between QS simulations and the numerical ME approach is observed. Similar agreement is obtained for the analysis of characteristic time (data not shown). However, if one tries to recover the theoretical scaling exponents, taking the values $\hat{\nu} = \max[1/2, (5 - \gamma)/4]$ and $\hat{\alpha} = \min[1/2, (\gamma - 1)/4]$, by means of a direct power law
regression (insets in Fig. 3), a very poor agreement with the expected analytical exponents is observed, as already noted in Ref. [31]. Indeed, if we look carefully at the data we can observe that, even though a pretty good linear fit can be resolved for data range corresponding to $10^3 \leq N \leq 10^7$ as shown in Fig. 2, the actual regressions are a little bit curved. Indeed, a careful analysis of the numerical data can resolve a slight downward (negative) curvature for $\gamma < 5/2$ and an8 slight upward (positive) curvature for $\gamma > 5/2$ at $\lambda = \lambda_c$. In a QS analysis these behaviors usually indicate a system slightly out of the critical point, being sub and supercritical for down and upward curvatures, respectively. But this is not the case for the data shown in Figs. 3 and 4 since the MF result is exact for the critical CP on annealed networks. Similar behaviors occur for plots of $\tau$ versus $N$.

As noted in [10, 21], the origin of this poor agreement between theory and simulations lies in the implicit dependence on $N$ of the $g$ factor defining the size scaling of $\bar{\rho}$ and $\tau$. In fact, the scaling forms $\bar{\rho} \sim N^{-\nu}$ and $\tau \sim N^{\alpha}$ make only sense in the limit of very large $N$, when $g$ has achieved its true asymptotic form. For intermediate values of $N$, instead, one should keep the scaling forms with the simultaneous dependence on $g$ and $N$ [21]. If we want instead to make explicit the scaling with network size, we must consider that $g = \langle k^2 \rangle / \langle k \rangle^2$ behaves in the continuous degree limit, as

$$g = \frac{(\gamma - 2)k_0^{\gamma-1}}{(\gamma - 1)(3 - \gamma)} \frac{(1 - \xi^{-1})(1 - \xi^{3-\gamma})}{(1 - \xi^{\gamma-2})^2} k_c^{\xi-\gamma}, \quad (24)$$

where $\xi = k_0 / k_c < 1$. So, when $N \to \infty$, $g \sim N^{(3-\gamma)/\omega}$ for $2 < \gamma < 3$ and $g \sim \text{const}$ for $\gamma > 3$.

From Eq. (24) it is possible to work out the explicit form of the corrections to scaling in a direct analysis of the QS quantities as functions of the network size. So, considering $2 < \gamma < 3$ and performing an expansion to leading order in $\xi$, equation (24) yields

$$g \simeq \text{const} \times (1 - \xi^{3-\gamma}) \left( 1 + 2\xi^{-2} + \cdots \right) k_c^{3-\gamma}. \quad (25)$$

Substituting into the MF scaling result $\bar{\rho} \sim (gN)^{-1/2}$, we obtain the expression for the stationary density

$$\ln \rho = C - \bar{\rho} \ln N + \frac{1}{2} \frac{k_0^{3-\gamma}}{N^{3-\gamma}} - \frac{k_0^{-2}}{N^{2-\gamma}}. \quad (26)$$

Notice that the corrections do not introduce any parameters to be fitted. Similar expressions are found for $\gamma \geq 3$. 

FIG. 3: QS critical density for the CP on annealed SF networks with cutoff $k_c = N^{1/2}$. In main plot, the densities obtained with the numerical solution of the ME (lines) and QS simulations (symbols) are shown. Inset A shows the exponent of the scaling law $\bar{\rho} \sim N^{-\nu}$ obtained analytically for asymptotically large systems ($N \to \infty$), performing a direct fit to a power-law form (PL) of the data in main figure, and performing a fit to a power with corrections to scaling. Inset B shows the same analysis for the characteristic time $\tau \sim N^{\alpha}$.

FIG. 4: Power law regressions (solid lines) of the critical QS densities (symbols) for CP on annealed SF networks with cutoff $k_c = N^{1/2}$ and degree exponents $\gamma = 2.25$ and $\gamma = 3.00$. Inset shows the QS densities for $\gamma = 2.25$ rescaled by pure power law (PL), $\zeta = \bar{\rho}^0.6875$, and a PL with correction to scaling, $\zeta = \bar{\rho}^0.6875(1 + 2\times 10^{-0.125} N^{-0.25} + 3\times 10^{-0.5} N^{-0.25})^{0.5}$.

FIG. 5: Critical QS characteristic times for CP on annealed SF networks with cutoff $k_c = N^{1/2}$, $N^{1/\omega}$ and degree exponents $\gamma = 2.25$ and $\gamma = 3.00$. Legends as in Fig. 3. 

[31]
Equation (26) explains the deviations from the power law regime $\rho \propto N^{-\gamma}$ observed for the CP on annealed SF networks. It is easy to see that the leading term for $2 < \gamma \leq 5/2$ is negative and causes a downward curvature, in the same way that the leading term for $\gamma > 5/2$ bends the curve upwardly. Even though the correction vanishes for $N \to \infty$, it may occur extremely slowly due to the small exponents involved. For $\gamma \approx 3$ and $\gamma \approx 2$ the corrections are logarithmic and are thus relevant for any finite size.

Introducing the corrections given in Eq. (26) in the form

$$\ln \rho' = \ln \rho - \left( \frac{1}{2} \frac{k_0^{3-\gamma}}{N^\omega} - \frac{k_0^{\gamma-2}}{N^{\omega\gamma-2}} \right) = C - \hat{\nu} \ln N$$

and performing a linear fit, the asymptotic exponent $\hat{\nu}$ is recovered as one can see in the inset A in Fig. 3. Equivalent corrections can be easily obtained for $\ln \tau_{eq}$. $\ln N$ and the expected exponent $\hat{\alpha}$ is recovered as shown in the inset B in Fig. 3. Additional proof of the strong finite size corrections is provided in the inset of Fig. 1 in which the critical QS density is rescaled by the predicted power law (PL) with exponent $\hat{\nu}$ and by this same PL with the correction (25). The first case is clearly size dependent while the second is flat. It is worth to note that the corrections are so strong for $\gamma = 2.25$ and $\omega = 2$ that keeping only the leading term $O(N^{-0.125})$ was not enough to account for the deviation.

We additionally performed the analysis for $\omega = \gamma - 1$, which corresponds to the natural cutoff that emerges in the absence of a structural cutoff [5]. For sake of simplicity, a hard cutoff was adopted such that connectivities larger than $k_c = N^{1/(\gamma-1)}$ are forbidden. For this cutoff, the scaling exponents for the critical density and characteristic time are $\hat{\nu} = \max[1, 2, 1/(\gamma - 1)]$ and $\hat{\alpha} = \max[1, (\gamma - 2)/(\gamma - 1)]$, respectively. Exactly as in the $\omega = 2$ case, the QS analysis via ME agrees with simulations and the correct scaling exponents are obtained if corrections to the scaling are considered. Figure 5 shows the characteristic time and the insets therein the scaling exponent analysis for $\omega = \gamma - 1$. Comparing the insets in Figs. 4 and 5 we can observe that the relative deviation between the exponents obtained using a simple and a corrected PL is smaller for the natural ($\omega = \gamma - 1$) than for the cutoff $k_c = N^{1/2}$. Indeed, equation (26) tells that the larger the cutoff exponent $\omega$, the stronger the corrections to the scaling. However, even if the cutoffs are not imposed, the natural one emerges spontaneously in networks with power-law degree distributions [5]. Consequently, these corrections to the scaling may be also present in the CP and other dynamical processes in SF substrates including the quenched case.

The previous results may have a remarkable impact in the analysis of absorbing phase transitions in complex networks. The usual QS analysis assumes a power law dependence of the order parameters with the size at criticality. Such assumption is commonly used as a criterion to determine the critical point of absorbing phase transitions in regular lattices [11] and has been extended to quenched complex networks [19, 20]. Corrections to scaling in the form $1 - const \times N^{-\gamma}$ were already observed in QS simulations of the directed percolation universality class in hypercubic lattices, including the contact process [24, 32]. Since these corrections decay with a large exponent, they are significant only for small systems. In complex networks, the scenario is quite different since the corrections, which emerge from the intrinsic SF nature of the substrate, vanish very slowly and are important even for large systems ($N \sim 10^7$ in the present work).

VI. OFF-CRITICAL QS ANALYSIS

The analysis performed for the critical CP in the previous sections is expected to work also for the off-critical phase if the densities are still sufficiently small. In Fig. 6 we compare the QS density obtained by ME iterative solution and numerical computed for networks with different degree exponents. In the plot we explore the supercritical regime with rates $\lambda = 1.004$ and $1.040$ (0.4% and 4% above the critical point, respectively). The one-step ME predicts QS densities very accurately even for a substantial distance from the critical point, corroborating that the approach is also suitable for the supercritical phase. The anomalous FSS form in the supercritical regime, equation (15), depending on $\Delta$, $N$ and $g$ simultaneously, is checked in Fig. 7 where we present the collapses of the data obtained from iterative solutions of the ME. Degree exponents $\gamma = 2.25$ and $2.75$ using structural ($\omega = 2$) and natural ($\omega = \gamma - 1$) cutoffs are shown. Excellent collapses are obtained in all cases, in agreement with Refs. [10, 22].

The strong size dependence observed in Fig. 6 for SF networks ($\gamma = 2.25$ and $2.75$) but not for the homogeneous one ($\gamma = 3.25$) calls for an anomalous dependence
of $\rho$ on $g$, as pointed out in the Langevin [10] and random walk mapping [22] approaches. Our numerical approach allows a more detailed investigation of this issue. Inspired by the anomalous FSS prediction [10],

$$\tilde{\rho} \sim \Delta/g, \quad \Delta > \sqrt{g/N},$$  

we have performed a further test, analyzing the behavior of the QS density, rescaled as $g^{\gamma} \tilde{\rho}$, as a function of $N$, as shown in Fig. 8. The theoretical exponent $b = 1$, which corresponds to the anomalous scaling in Eq. (28), does not show up as a plateau in the plots of $g^{\gamma} \tilde{\rho}$ versus $N$. Interestingly, plateaus are observed if an exponent $b < 1$ is used instead. For $\lambda = 1.040$, plateaus are observed for $b = 0.88$ and $b = 0.53$ for $\gamma = 2.25$ and $\gamma = 2.75$, respectively. In an analysis performed closer to critical point, for $\lambda = 1.004$, the plateaus are observed with larger exponents $b = 0.95$ and $b = 0.56$ for $\gamma = 2.25$ and $\gamma = 2.75$, respectively. Notice that a scaling consistent with $b = 1$ was obtained for $\gamma = 2.25$ but not for $\gamma = 2.75$. Actually, the anomalous scaling [25], derived from Eq. (13), is valid for $\sqrt{g/N} < \Delta < g(k)/k_c$. For $\omega = 2$, the right and left sides of this inequality scale as $g(k)/k_c \simeq c_\gamma N^{-(\gamma-2)/2}$ and $\sqrt{g/N} \simeq \tilde{c}_\gamma N^{-(\gamma-1)/4}$, respectively, where $c_\gamma = k_0^{-\gamma}/(3 - \gamma)$ and $\tilde{c}_\gamma = k_0^{-\gamma-1}(\gamma - 1)(\gamma - 2)^2/(3 - \gamma)$ are constants of the same order and $c_\gamma > \tilde{c}_\gamma$. If $\gamma$ is close to $3$, the exponents involved in the lower and upper bounds of $\Delta$ are very close and we cannot make $\Delta$ sufficiently small to fulfill the upper bound and still larger than the lower one, except for numerically unaccessible large systems. Therefore, this anomalous scaling can be clearly seen only for $\gamma$ close to 2. The scaling forms with $b < 1$ are thus metastable crossovers between the regimes $\tilde{\rho} \sim (gN)^{-1/2}$ and $\tilde{\rho} \sim \Delta/g$, that can last for decades, and could only be resolved by simulations in much larger systems sizes than those considered in this work (up to $N = 10^9$ in the ME solutions).

**VII. CONCLUDING REMARKS**

The contact process (CP) on scale-free networks shows remarkably rich features, even in the simple case of random annealed topologies. In the present work we have explored the quasi-stationary (QS) properties of this problem by combining QS numerical simulations and a master equation approach applied to an approximate mapping to a one-step process. The resulting master equation, apart from providing quite accurate analytical approximations for the asymptotic shape of the QS activity distribution at the critical point, can be very efficiently solved numerically. The QS distribution and the relevant QS quantities (density of active sites and characteristic time) determined in this way show an excellent agreement with direct QS numerical simulations of the contact process, both at criticality and in the supercritical regime.

The high accuracy of our data allows to identify strong corrections to the scaling in the critical quantities that mask the correct finite size scaling exponents obtained analytically by means of an exact mean-field solution. Both critical density and characteristic time show tem-
rect exponents if a simple power law decay is assumed. In annealed networks, for which the critical point is exactly known, we can determine the corrections to scaling analytically and thus recover the theoretical exponents in the finite size analysis, including the abrupt change when the network loses its SF property. The analysis of the supercritical region, on the other hand, hints that those finite size corrections are also relevant for very large network sizes. Indeed, the asymptotic scaling is observable only for extremely large values of \( N \), much larger than those possibly attainable with present day computers.

It is worth noticing that the QS analysis presented in this work is equivalent to the Langevin approach developed in Ref. [10] in the limit of very low densities (i.e. in the critical region, see Appendix). The main difference is that the former starts from an approximation of the original dynamical processes [Eq. (1)], while the latter represents an exact Langevin approach in the coarse-grained limit (density approximated by a continuous variable). The critical advantage of the present approach lies on the fact that our ME analysis allows to determine the critical properties in a very intuitive way, as well as to easily obtain highly accurate results, free from statistical errors, for all quantities of interest.

Our work opens the path to a more detailed characterization of absorbing phase transitions on scale-free networks in general, and the CP in particular. In the more realistic framework of quenched networks, in which edges are frozen and do not change, this goal may be however hindered by the interplay between corrections to scaling and the usual lack of knowledge about the true position of the critical point. In fact, the standard characterization of the QS state by the usual procedure assuming a simple power law of the system size at the critical point may be misleadingly done off the critical point and/or be affected by important scaling corrections. Further work in this direction, following the proposed lines, might thus help to throw light on the numerical assessment of the correct critical scaling of absorbing phase transitions on heterogeneous networks.

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Appendix

The connection between the Langevin approach developed in Ref. [10] and the one-step process [6] can be established by analyzing the respective Fokker-Planck (FP) equations. The general form of a FP equation for a stochastic variable \( x \) is [30]

\[
\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x} A(x)P(x,t) + \frac{\partial^2}{\partial x^2} D(x)P(x,t)
\]

where \( A(x) \) and \( D(x) \) are the drift and the diffusion terms, respectively. The Langevin analysis in Ref. [10] yielded \( A(n) = n[\lambda - 1 - \lambda \Theta(n/N)] \) and \( D(n) = 2\lambda n\Lambda(n/N) \), where \( \Theta \) is given by Eq. (16) and

\[
\Lambda(\rho) = \sum_k kP(k)\langle 1 + \lambda k\rho/\langle k\rangle \rangle^\rho.
\]

In turn, the drift and diffusion terms of the FP equation for an arbitrary one-step process is given by \( A(n) = W(n+1,n) - W(n-1,n) \) and \( D(n) = W(n+1,n) + W(n-1,n) \) [30], respectively. Equation (1) results in exactly the same drift obtained in the Langevin approach, while the diffusion term takes the form \( D(n) = (1 + \lambda)n + \lambda \Theta(n/N) \).

At low densities, one can expand \( \Theta \) and \( \Lambda \) to leading order and obtain \( D(n) \approx 2n + O(\Theta) \) in both cases. Therefore, Langevin and ME approaches are equivalent in the low density limit.

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