Statefinder Diagnostic for Phantom Model with $V(\phi) = V_0 \exp(-\lambda \phi^2)$

Baorong Chang *, Hongya Liu†, Lixin Xu and Chengwu Zhang

School of Physics & Optoelectronic Technology, Dalian University of Technology, Dalian, 116024, P. R. China

We investigate the phantom field with potential $V(\phi) = V_0 \exp(-\lambda \phi^2)$ and dark matter in the spatially flat FRW model. It has been shown by numerical calculation that there is an attractor solution in this model. We also apply the statefinder diagnostic to this phantom model. It is shown that the evolving trajectories of this scenario in the $s - r$ diagram is quite different form other dark energy models.

Recently, the observations of high redshift type Ia supernovae[1] reveal the speeding up expansion of our universe and the surveys of clusters of galaxies show that the density of matter is very much less than the critical density[2], and the observations of Cosmic Microwave Background (CMB) anisotropies indicate that the universe is flat and the total energy density is very close to the critical one with $\Omega_{\text{total}} \approx 1$[3]. These three tests nicely complement each other and indicate that the dominated component of the present universe is dark energy(DE). Dark energy occupies about 73% of the energy of our universe, while dark matter(DM) about 23%, and the usual baryonic matter occupy about 4%. The accelerating expansion of the present universe is attributed to the dark energy which is an exotic component with negative pressure, such as the cosmological constant $\Lambda$[4, 5] with equation of state $w = -1$, a dynamically evolving scalar field (quintessence) [6,7] with $w > -1$ or the phantom [8] with $w < -1$, meanwhile the accelerating expansion of universe can also be obtained through modified friedmann equation [9] and brane world [10]. The fine tuning problem is considered as one of the most important issues for dark energy models and a good model should limit the fine tuning as much as possible. The dynamical attractors of the cosmological system have been employed to make the late time behaviors of the model insensitive to the initial conditions of the field and thus alleviate the fine tuning problem, which has been studied in many quintessence models [11]. In phantom system, this problem has been studied with cosine potential [12], exponential and inverse power law potential in [13, 14]. In this letter, we study the phantom field $\phi$ with the potential $V(\phi) = V_0 \exp(-\lambda \phi^2)$ and the dark matter in the spatially flat FRW model. The late time behavior of the cosmological equations will give accelerated expansion and a constant ratio between dark matter energy density $\rho_m$ and phantom energy density $\rho_\phi$. This behavior relies on the existence of an attractor solution, and the late time cosmology insensitive to the initial conditions for dark matter and dark energy. Therefore, the fine tuning problem can be alleviated in this model.

In this paper, we will first show the attractor behavior in this scenario and, then we perform a statefinder parameter diagnostic for this model. The statefinder parameter introduced by Sahni et al. [15] are proven to be useful tools to characterize and differentiate between various dark energy models. We show in this paper the evolving trajectory of the $s - r$ diagram is quite different from this of other dark energy models.

We consider a universe model which contains phantom field $\phi$ and the dark matter $\rho_m$. The Friedmann equation in a spatially flat FRW metric can be written as

$$H^2 = \frac{1}{3} (\rho_\phi + \rho_m),$$

the Planck normalization $M_p = 1$ has been used here, $\rho_m$ is the energy density of the dark matter, and the dark matter possesses the equation of state $P_m = (\gamma - 1) \rho_m$. The energy density and pressure of the phantom field $\phi$ are $\rho_\phi$ and $P_\phi$, respectively,

$$\rho_\phi = -\frac{1}{2} \dot{\phi}^2 + V(\phi),$$

$$P_\phi = -\frac{1}{2} \dot{\phi}^2 - V(\phi).$$

where $V(\phi)$ is the phantom field potential, $V(\phi) = V_0 \exp(-\lambda \phi^2)$.

Since the energy of dark energy and dark matter is conserved respectively, the equation of motion for DE and DM can be obtained:

$$\dot{\rho}_\phi + 3H\rho_\phi (1 + w_\phi) = 0,$$

$$\dot{\rho}_m + 3H\rho_m (1 + w_m) = 0.$$  

where the parameter of equation of state for the phantom field is given by:

$$w_\phi = \frac{-\frac{1}{2} \dot{\phi}^2 - V(\phi)}{-\frac{1}{2} \dot{\phi}^2 + V(\phi)}$$

and we can get the equation of motion for the scalar field $\phi$

$$\ddot{\phi} + 3H\dot{\phi} + 2\lambda \phi V(\phi) = 0.$$  

Using the Friedmann equation eq. (1), the eq. (7) can be written as

$$\phi^2 H^2 + \left[1 - \frac{1}{3} \gamma \rho_m + V(\phi)\right] \dot{\phi}^2 = -2\lambda \phi V(\phi).$$

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*changbaorong@student.dlut.edu.cn, telephone:0411-84707869
†Corresponding author: hyliu@dlut.edu.cn, telephone:0411-84707765
FIG. 1: The phase plane for different initial conditions, and $\gamma_{m} = 1$, $\lambda = 2$

where

$$H^2 = \frac{1}{3} \frac{V(\phi) + \rho_m}{1 + \frac{\lambda}{6} \phi^2},$$  \hspace{1cm} (9)$$

primes denote derivatives with respect to $u = \ln(a/a_0) = -\ln(1+z)$, where $z$ is the redshift, and $a_0$ represents the current scalar factor.

It is known that the attractor solution can be found analytically in the exponential potential. However, in our case, it is difficult to obtain the analytical solution of the attractor. The reason is that the late time behavior of the field is not linear in potential of $V(\phi) = V_0 \exp(-\lambda \phi^2)$. Now, we will solve the equation of motion of $\phi$ numerically.

The numerical results show that there exists a stable attractor solution which depends on $\lambda$ while is insensitive to the initial conditions. In Fig. 1, we plot the $(\phi, \dot{\phi})$ phase diagram. It is shown in the phase plane that the lines corresponding to different conditions will converge together to the attractor solution with the cosmological evolution. In Fig. 2, we plot the evolution energy density parameters of dark matter and dark energy with $\gamma_{m} = 1$, $\lambda = 1$. It is exhibited that the dark energy occupies about 76% and dark matter occupies about 24% at present, and the ratio between energy densities of DE and DM will remain constant in the future. At late time, the universe will be dominated by the phantom field alone. From Fig. 3, we can see that the parameter of equation of state of phantom less than $-1$ at present, while it will approach to $-1$ in the future.

The cosmological diagnostic pair $(r, s)$ called statefinder which is introduced by Sahni et al. in [15] and defined as

$$r \equiv \frac{\ddot{a}}{aH^3}, \hspace{1cm} s \equiv \frac{r - 1}{3(q - 1/2)},$$  \hspace{1cm} (10)$$

Here $q$ is the deceleration parameter. The statefinder is a "geometrical" diagnostic in the sense that it depends on the expansion factor and hence on the metric describing spacetime. Since different cosmological models involving dark energy exhibit qualitatively different evolution trajectories in the $s - r$ plane, this statefinder diagnostic can differentiate various kinds of dark energy models. For the spatially flat LCDM cosmological model, the statefinder parameters correspond to a fixed point $(r = 1, s = 0)$. By far some models, including the cosmological constant, quintessence, phantom, quintom, the Chaplygin gas, braneworld models, holographic models, interacting and coupling dark energy models [15, 16, 17], have been successfully differentiated. For example, the quintessence model with inverse power law potential, the phantom model with power law potential and the Chaplygin gas model all tend to approach the LCDM fixed point, but for quintessence and phantom models the trajectories lie in the regions $s > 0$, $r < 1$. We use another form of statefinder parameters in terms of the total energy density $\rho$ and the total pressure $p$ in the universe:

$$r = 1 + \frac{9}{2} \frac{(\rho + p)}{\rho \dot{\rho}}, \hspace{1cm} s = \frac{(\rho + p)}{p} \frac{\dot{\rho}}{\rho}.$$  \hspace{1cm} (11)$$

Since the energy density of DE and DM is conserved, from (4) and (5) we can get:

$$r = 1 - \frac{3}{2} \frac{m}{w_\phi} \Omega_\phi + 6w_\phi (1 + w_\phi) \Omega_\phi,$$  \hspace{1cm} (12)$$

$$s = 1 - \frac{w_\phi}{3w_\phi + w_\phi}.$$  \hspace{1cm} (13)$$

FIG. 2: The energy density parameter of DE and DM versus $u = -\ln(1+z)$. The corresponding parameters are $\gamma_{m} = 1$, $\lambda = 1$.

FIG. 3: The evolution of equation of state of phantom $w$ versus $u = -\ln(1+z)$, in which $\gamma_{m} = 1$, $\lambda = 1.5$ and $\lambda = 2$. 
In summary, we investigate in this letter the attractor solution of the phantom model with potential \( V(\phi) = V_0 \exp(-\lambda \phi^2) \). The statefinder diagnostic have be performed, which is shown that the evolving trajectories of this scenario in the \( s-r \) plane is quite different from other dark energy models. We hope that the future high precision observation will be capable of determining these statefinder parameters and consequently shed light on the nature of dark energy.

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