Invariants and Coherent States for Nonstationary Fermionic Forced Oscillator

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Abstract
The most general form of Hamiltonian that preserves fermionic coherent states stable in time is found in the form of nonstationary fermion oscillator. Invariant creation and annihilation operators and related Fock states and coherent states are built up for the more general system of nonstationary forced fermion oscillator.

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1 Introduction
The time evolution of coherent states (CS) has attracted a great deal of attention since the introduction of Glauber’s CS of the harmonic oscillator [1][2][3]. Of particular interest has been the determination of the Hamiltonian operator for which an initial coherent state remains coherent under time evolution. It is established that this Hamiltonian has the form of the nonstationary bosonic forced oscillator Hamiltonian [4][5][6][7][8]:

\[ H_{cs} = \omega(t)a^{\dagger}a + f(t)a^{\dagger} + f^{*}(t)a + \beta(t), \]

(1)
where $\omega(t)$ and $\beta(t)$ are arbitrary real functions of time $t$, and $f(t)$ is arbitrary complex function.

Our purpose in the present article is to study the dynamical invariants and time evolution of CS for general (one mode) fermionic Hamiltonian and to establish the most general form of Hamiltonian which preserves the fermionic CS under the time evolution.

The organization of the article is as follows. We start with a review in Sec. II of some main results of time evolution of bosonic forced harmonic oscillator. In Sec. III we study the temporal stability of fermionic CS and we show, by using the fermionic analog of the invariant boson ladder operator method [9,10,11], that the most general form of Hamiltonian that preserves fermionic CS stable in time is in the form of nonstationary fermion oscillator. In Sec. IV we treat the more general system of nonstationary forced fermion oscillator (FFO), which is shown to be the most general one mode fermionic Hamiltonian system. Following the scheme related to the boson system [9] we construct the dynamically invariant fermion ladder operators and related Lewis-Riensenfeld Hermitian invariant [12]. Using these invariants, we construct fermionic Fock states and CS for FFO system, which can represent (under appropriate initial conditions) the exact time-evolution of initial canonical CS. Finally the relation of the invariant ladder operators method [9,10] to the Lewis-Riesenfeld method [12] is briefly described on the example of FFO. The paper ends with concluding remarks.

2 Canonical CS and their temporal stability

The standard boson coherent states (CS) (called also Glauber CS, or canonical CS) are defined as the right eigenstates of the boson (photon) annihilation operator $a$ [11,2,3]

$$a|z\rangle = z|z\rangle$$

the eigenvalue $z$ being a complex number. The annihilation and creation operators $a$ and $a^\dagger$ satisfy the commutation relations

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = 1.$$ 

The normalized CS $|z\rangle$ can be constructed in the form of displaced ground state $|0\rangle$ [11,2,3],

$$|z\rangle = D(z) |0\rangle, \quad D(z) = e^{(za^\dagger - z^*a)},$$

and their expansion in terms of the number states $|n\rangle$ reads

$$|z\rangle = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$$
The problem of temporal stability of bosonic CS is solved by Glauber [4] and Mehta and Sudarshan [5] (in the case of one mode CS, and for n-mode CS - by Mehta et al. [6]). The result is that the most general Hamiltonian that preserves an initial CS $|z⟩$ stable in later time is of the form of the nonstationary forced oscillator Hamiltonian $H_{cs}$, eq. (1). The Hamiltonian (1) that preserves CS stable is shortly called coherence Hamiltonian. Thus the boson coherence Hamiltonian takes the form of a non-stationary forced oscillator Hamiltonian. Here “stable” means that the time evolved state $|z;t⟩$,

$$i\frac{d}{dt}|z;t⟩ = H_{cs}|z;t⟩.$$  

remains eigenstate of $a$ possibly with a time-dependent eigenvalue $z(t)$,

$$a |z;t⟩ = z(t) |z;t⟩$$  

From the latter equation one deduces that, up to a time-dependent phase factor $exp(i\varphi(t))$, the time-evolved CS $|z;t⟩$ depends on time $t$ through $z(t)$, that is

$$|z;t⟩ = e^{i\varphi(t)}|z(t)⟩, \quad |z(t)⟩ = e^{a^†z(t) - z^*(t)a}|0⟩$$

One says that for system with Hamiltonian (1) an initial CS remains CS all the later time [4, 5] (or remains temporally stable). For the Hamiltonian system (1) the time dependent eigenvalue value $z(t)$ obeys the equation [4, 5]

$$i\dot{z} = \omega(t)z + f(t)$$

the solution of which takes the explicit form ($z = z(0)$)

$$z(t) = \tilde{\beta}(t)z + \tilde{\gamma}(t), \quad \tilde{\beta}(t) = e^{-i\int_0^t \omega(\tau)d\tau'}, \quad \tilde{\gamma}(t) = -i\left(\int_0^t e^{\int_0^\tau \omega(\tau')d\tau'} f(\tau')d\tau\right) e^{-i\int_0^t \omega(\tau')d\tau'}$$

In the particular case of constant $\omega$ we have

$$z(t) = e^{-i\omega_0 t}\left(z - i\int_0^{t} e^{i\omega_0 t'} F(t')dt'\right).$$

The forced oscillator system (1) admits linear in terms of $a$ and $a^†$ invariant boson annihilation operator $A_c(t)$, $[A(t), A^†(t)] = 1$,

$$A(t) = U(t)aU^†(t) = \beta(t)a + \gamma(t) \equiv A_c,$$
where \( U(t) \) is the unitary evolution operator, and

\[
\beta(t) = e^{\int_0^t \omega(t') dt'} = \tilde{\beta}^{-1}(t), \quad \gamma(t) = i \int_0^t f(t') e^{\int_0^t \omega(t') dt'} dt' = -\tilde{\gamma}(t).
\]

For any system the time-evolved CS \(|z; t⟩\) are eigenstates of the corresponding invariant annihilation operator \( A(t) \) with constant eigenvalues \( z \), \( A(t)|z; t⟩ = z|z; t⟩ \), and can be represented in the form of invariantly displaced time-evolved ground state \(|0; t⟩ = U(t)|0⟩\),

\[
|z; t⟩ = D(z, A(t))|0; t⟩, \quad D(z, A(t)) = e^{A\dagger(t)z - z^* A(t)}. \tag{10}
\]

If \( A(t) \) is invariant then \( A\dagger(t) \) also is, and any other combination of them is also invariant. In particular \( A\dagger(t)A(t) \) and \( D(z, A(t)) \) are also invariant operators of the forced oscillator \( H \). Invariant operators are very useful, since they transform solutions into solutions, as demonstrated in \( (10) \).

The invariant boson ladder operator \( \mathbf{(9)} \) is a simple particular case of linear invariants of general quadratic quantum system, constructed first in \( \mathbf{[9, 10]} \). For the nonstationary quantum oscillator Hermitian quadratic in \( a \) and \( a\dagger \) invariant was constructed and studied by Lewis and Riesenfeld \( \mathbf{[12]} \). Using these properties of the invariants it was shown \( \mathbf{[11]} \) that a given Hamiltonian \( H \) preserves the temporal stability of CS \(|z⟩\) if and only if it admits invariant of the form \( A_c = \beta(t)a + \gamma(t) \). The general form of such Hamiltonian coincides with Glauber-Mehta-Sudarshan coherence Hamiltonian \( \mathbf{[1]}. \)

3 Temporal stability of canonical fermion CS

Fermion coherent states (CS) are defined (see \( \mathbf{[13, 14, 15, 16]} \)) as eigenstates of the fermion annihilation operator \( b \),

\[
b|ζ⟩ = ζ|ζ⟩, \tag{11}
\]

where the eigenvalue \( ζ \) is a Grassmannian variable: \( ζ^2 = 0, \quad ζζ^* + ζ^*ζ = 0 \). Recall the fermion algebra:

\[
\{b, b\dagger\} \equiv bb\dagger + b\dagger b = 1, \quad b^2 = b\dagger^2 = 0. \tag{12}
\]

For definiteness eigenstates of fermion ladder operator \( b \) should be called canonical fermion CS. This is in analogy to the eigenstates of boson annihilation operator \( a \), which are known as Glauber CS and canonical boson CS as well. In terms of the Grassmann eigenvalues \( ζ \) many of the properties of \(|ζ⟩\)
repeat the corresponding ones of the bosonic CS $|z\rangle$ [16]. In particular one has
\[ |\zeta\rangle = D(\zeta) |0\rangle = e^{-\frac{i}{2} \zeta^* \zeta} (|0\rangle - \zeta |1\rangle) . \]  
\[ \int d\zeta^* d\zeta |\zeta\rangle \langle \zeta| = 1, \]  
where $D(\zeta) = \exp(b^\dagger \zeta - \zeta^* b)$, $|0\rangle$ is the fermionic vacuum, $b |0\rangle = 0$, and $|1\rangle$ is the one-fermion state, $|1\rangle = b^\dagger |0\rangle$. The integrations over $\zeta$ and $\zeta^*$ are performed according to the Berezin rules [16]
\[ \int d\zeta d\zeta^* \zeta \zeta^* = 1, \int d\zeta d\zeta^* \zeta = \int d\zeta^* d\zeta \zeta = \int d\zeta^* d\zeta \zeta^* = \int d\zeta d\zeta^* 1 = 0. \]  
The temporal stability of the canonical fermion CS is defined in analogy to the temporal stability of canonical boson CS, namely the evolution of an initial $|\zeta\rangle$ is stable if the time-evolved state $|\zeta; t\rangle = U(t)|\zeta\rangle$ ($U(t)$ being the evolution operator of the system) remains eigenstate of $b$ in all later time,
\[ b |\zeta; t\rangle = \zeta(t) |\zeta; t\rangle . \]
It is clear that the time-evolved states $|\zeta; t\rangle$ also obey the overcompleteness relation [14] and are eigenstates of the invariant ladder operator $B(t) = U(t)bU^\dagger(t)$. This means that the $B(t)$ and $b$ should commute (we suppose that $\zeta(t)$ and $\zeta$ commute). The general form of a fermionic operator is a (complex) linear combination of $b$, $b^\dagger$ and $b^\dagger b$. Such a combination will commute with $b$ under certain simple restrictions. Taking into account that the invariants $B(t)$ and $B^\dagger(t)$ have to obey the fermion algebra [12] we derive that $[b, B(t)] = 0$ if and only if $B(t)$ is proportional to $b$, $B(t) = \beta(t)b$. Thus the fermion coherence Hamiltonian should admit invariant of the form
\[ B_c(t) = \beta(t)b , \]  
where $\beta(t)$ may be arbitrary complex function of time. As we have already noted at the end of the preceding section, similar form of the ladder operator invariant $A_c$, eq. [9], is required in the case of boson systems [11]. To obtain now the general fermion coherence Hamiltonian $H_{fCS}$ we apply the defining requirement for quantum time-dependent invariants $B(t)$,
\[ \frac{\partial}{\partial t} B(t) - i[B(t), H] = 0 \]  
to the operator [17]. The general form of fermionic (one-mode) Hamiltonian is a Hermitian linear combination of $b$, $b^\dagger$ and $b^\dagger b$,
\[ H_f = \omega(t)b^\dagger b + f(t)b^\dagger + f^*(t)b + g(t) , \]  
\[ \int d\zeta^* d\zeta \zeta \zeta^* = 1, \int d\zeta^* d\zeta \zeta = \int d\zeta^* d\zeta \zeta^* = \int d\zeta^* d\zeta 1 = 0. \]
where $\omega(t)$ and $g(t)$ are real functions of time. The substitution of this $H_f$ into (18) for $B_c(t)$ produces the two conditions

$$
\dot{\beta} = i\beta \omega, \quad 0 = \beta f.
$$

These simple conditions are readily solved, $f(t) = 0$, $\beta(t) = \exp\left(i \int_0^t \omega(\tau) d\tau\right)$, leading to Hamiltonian

$$
H_{fCS} = \omega(t) b^\dagger b + g(t),
$$

which is the most general form of fermion coherence Hamiltonian. If the evolution of an initial CS $|\zeta\rangle$ is governed by $H_{fCS}$, then the time-evolved state $|\zeta; t\rangle$ remains eigenstate of $b$ with eigenvalue

$$
\zeta(t) = \beta^{-1}(t) \zeta = e^{-i \int_0^t \omega(\tau) d\tau} \zeta.
$$

The results (21) and (22) are similar in form, but not identical, to those for the boson systems (1) and (7). The fermion coherence Hamiltonian (21) is of the form of an oscillator with time dependent frequency (nonstationary fermion oscillator), while the boson coherence Hamiltonian (11) is of the more general form of the nonstationary forced oscillator. In the next section we find the exact evolution of fermion CS and fermion number states, governed by the nonstationary forced oscillator Hamiltonian using the time-dependent integrals of motion method [9, 10, 12].

4 FFO and invariant ladder operators

We consider the single nonstationary fermionic forced oscillator (FFO) described by the Hamiltonian (19). As we have noted this in fact is the most general Hamiltonian of single fermion system. The fermion number operator is defined as $N = b^\dagger b$. It obey the relation $N^2 = N$ and the three operators $b$, $b^\dagger$ and $N$ close under commutation the algebra

$$
[b, N] = b, \quad [b^\dagger, N] = b^\dagger, \quad [b, b^\dagger] = 1 - 2N.
$$

The Hilbert space $\mathcal{H}$ of the single-fermion system is spanned by the two eigenstates $\{|0\rangle, |1\rangle\}$ of $N$:

$$
b^\dagger b |n\rangle = n |n\rangle, \quad n = 0, 1
$$

The operators $b$ and $b^\dagger$ allow transitions between the states as

$$
b |0\rangle = 0, \quad b |1\rangle = |0\rangle, \quad b^\dagger |1\rangle = 0, \quad b^\dagger |0\rangle = |1\rangle.
$$
Let us also note that linear combinations of $b^\dagger$, $b$ and $N$ produce the half-spin operators $J_i$,

$$
J_1 = \frac{1}{2}(b^\dagger + b), \quad J_2 = \frac{1}{2i}(b^\dagger - b), \quad J_3 = b^\dagger b - \frac{1}{2},
$$
closing the $su(2)$ algebra: $[J_k, J_l] = i\epsilon_{klm}J_m$. It is clear that the fermion forced oscillator Hamiltonian $H_{FCS}$ belongs to the central extension of $su(2)$ (is a linear combination of $J_i$ plus free C-number term).

It is convenient to use raising and lowering operators $J_\pm = J_1 \pm iJ_2$ which satisfy the following commutation relation: $[J_+, J_-] = 2J_3$, $[J_3, J_\pm] = \pm J_\pm$, where $J_+ = b^\dagger$, $J_- = b$. So that in terms of these half spin operators the Hamiltonian (19) takes the form

$$
H_f(t) = \omega(t)J_3 + f(t)J_+ + f^*(t)J_- + g(t) + \frac{\omega(t)}{2}.
$$

Our task is the construction of the time-dependent invariants for the system (19), (25). The defining equation of the invariant operator $B(t)$ for a quantum system with Hamiltonian $H(t)$ is (18). Formal solutions to this equation are operators $B(t) = U(t)B(0)U^\dagger(t)$, where $U(t)$ is the evolution operator of the system, $U(t) = T\exp[-i \int_0^t H(t')dt']$. In our case of FFO (19), (25) we look for the non-Hermitian invariants of the form of linear combination of the $SU(2)$ generators (25),

$$
B(t) = \nu_-^*(t)J_+ + \nu_+^*(t)J_- + \nu_3^*(t)J_3,
$$

$$
B^\dagger(t) = \nu_-^*(t)J_- + \nu_+^*(t)J_+ + \nu_3^*(t)J_3,
$$

where $\nu_\pm(t)$, $\nu_3(t)$ may be complex functions of the time. Hermitian invariants then can be easily built up as Hermitian combinations of $B$ and $B^\dagger$. In particular if $B$ is a non-Hermitian invariant the operator

$$
I = B^\dagger B - \frac{1}{2}
$$

is an Hermitian invariant, the fermion analog of the Lewis-Riesenfeld quadratic invariant [12].

Substituting (26), (25) into (18), we find the following system of differential equations for the parameter functions of $B(t)$

$$
\dot{\nu}_3 = 2i(\nu_+ f^* - \nu_- f), \quad (28)
$$

$$
\dot{\nu}_+ = i(\nu_3 f - \nu_+ \omega), \quad (29)
$$

$$
\dot{\nu}_- = i(\nu_- \omega - \nu_3 f^*). \quad (30)
$$

The solutions of the above linear system of first order equations are uniquely determined by the initial conditions $\nu_\pm(0) = \nu_{0,\pm}$, $\nu_3(0) = \nu_{0,3}$. If we want
the invariants $B(t)$ and $B^\dagger(t)$ be again a fermion ladder operator, i.e. to obey the conditions
\[ B^2(t) = 0, \quad \{B(t), B^\dagger(t)\} = 1, \]
we have to take $\nu_{0,\pm}$ and $\nu_{0,3}$ satisfying
\[ \nu_{0,3}^2 = -4\nu_{0,+}\nu_{0,-}, \quad |\nu_{0,-}| + |\nu_{0,+}| = 1. \] \[ (32) \]
Indeed, for $B^2(t)$ and $\{B(t), B^\dagger(t)\}$ we find
\[ B^2(t) = \nu_+\nu_- + \frac{1}{4}\nu_3^2 \equiv \lambda_1(\nu_\pm, \nu_3), \]
\[ \{B(t), B^\dagger(t)\} = |\nu_-|^2 + |\nu_+|^2 + \frac{1}{2}|\nu_3|^2 \equiv \lambda_2(\nu_\pm, \nu_3). \] \[ (33) \]
The quantities $\lambda_1(\nu_\pm, \nu_3), \lambda_2(\nu_\pm, \nu_3)$ turned out to be two different 'constants of motion' for the system \([28]-[30]\), their time derivatives being vanishing:
\[ \frac{d}{dt}\lambda_1 \equiv \frac{d}{dt}(\nu_+\nu_- + \frac{1}{4}\nu_3^2) = 0, \]
\[ \frac{d}{dt}\lambda_2 \equiv \frac{d}{dt}(|\nu_-|^2 + |\nu_+|^2 + \frac{1}{2}|\nu_3|^2) = 0. \] \[ (34) \]
Therefore we can fix the values of these constants as $\lambda_1 = 0, \lambda_2 = 1$, i.e.
\[ \nu_+\nu_- + \frac{1}{4}\nu_3^2 = 0, \quad |\nu_-|^2 + |\nu_+|^2 + \frac{1}{2}|\nu_3|^2 = 1, \] \[ (35) \]
and satisfy the conditions \([31]\). If the initial conditions are taken as
\[ \nu_-(0) = 1, \quad \nu_+(0) = 0 = \nu_3(0), \] \[ (36) \]
then $B(0) = b$.

Let us first note that in the particular case of the free oscillator, $f(t) \equiv 0$, the solution of the above system of equations is readily obtained in the form
\[ \nu_{\pm}(t) = \nu_{0,\pm}e^{\pm i\int_0^t \omega(\tau)d\tau}, \quad \nu_3 = \nu_{0,3}, \] \[ (37) \]
where $\nu_{0,\pm}, \nu_{0,3}$ are constants. For this solution the expressions $\lambda_1(\nu_\pm, \nu_3), \lambda_2(\nu_\pm, \nu_3)$ are readily seen to be constant in time as expected: $\lambda_1 = \nu_{0,-}\nu_{0,+} + \nu_{0,3}^2/4, \lambda_2 = |\nu_{0,-}|^2 + |\nu_{0,+}|^2 + |\nu_{0,3}|^2/2$. Then from \([33], [35]\) we see that the invariant fermion annihilation operator $B(t)$ now takes the form $B_{so}(t),$
\[ B_{so}(t) = \nu_{0,-}e^{-i\varphi(t)}b + \nu_{0,+}e^{i\varphi(t)}b^\dagger + 2\sqrt{-\nu_{0,-}\nu_{0,+}}(b^\dagger b - \frac{1}{2}), \] \[ (38) \]
where $\varphi(t) = \int_0^t \omega(\tau)d\tau$ and $|\nu_{0,-}| + |\nu_{0,+}| = 1$, the phases of $\nu_{0,\pm}$ remaining arbitrary.

Consider now in greater detail the nonstationary forced oscillator with nonvanishing $f(t)$: $f(t) \neq 0$. 

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For this system we can express all the three parameter functions \( \nu_\pm(t) \), \( \nu_3(t) \) in terms of one of them, which has to obey more simple second order differential equation. Let for example, express \( \nu_3(t) \) and \( \nu_-(t) \) in terms of \( \nu_+(t) \) and its derivatives. We have

\[
\nu_3 = -\frac{i}{f} (\nu_+ + i \nu_\omega),
\]

\[
\nu_- = \frac{1}{2f^2} \left( \dot{\nu}_+ + (i \omega - \frac{f}{2}) \nu_+ + \left( 2ff^* + i \omega - i \frac{f\dot{f}}{f^2} \right) \nu_+ \right). \tag{39}
\]

Taking the time derivative of both sides of (40) and using eqs. (30) and (39) we arrive to the third order equation for \( \nu_+(t) \),

\[
\frac{1}{2f^2} \dddot{\nu}_+ = \frac{3f}{2f^2} \ddot{\nu}_+ - \left( 2f^* + \frac{\omega^2}{2f^2} + \frac{i \omega}{2f} + \frac{3f\dot{f}}{2f^3} - i \frac{\omega \dot{f}}{f^2} \right) \dot{\nu}_+ - \left( i \frac{f}{f} + \frac{\omega \dot{f}}{2f^2} - \frac{\omega^2}{2f^2} - i \frac{f\dot{f}}{f^2} - i \frac{3\omega \dot{f}}{2f^2} + i \frac{i \omega \dot{f}}{f^2} + i \frac{3f \dot{f}}{2f^2} \right) \nu_+. \tag{41}
\]

Furthermore if we could find a first integral of the equation (41) then we can express \( \nu_- \) in terms of \( \nu_+ \), \( \dot{\nu}_+ \), eliminating the second derivative in eq. (40). One can check that the following expression of \( \nu_+ \), \( \dot{\nu}_+ \) and \( \dddot{\nu}_+ \) is a first integral of equation (41) (that is \( \lambda \lambda = 0 \)),

\[
\lambda = \frac{4}{f^2} \left[ 2\nu_+ \dddot{\nu}_+ - \nu_+^2 - 2\nu_+ \dot{\nu}_+ \dot{f} + 4\nu_+^2 \left( |f|^2 + \frac{\omega^2}{4} + i \frac{\omega}{2} - i \frac{\omega^3}{2f} \right) \right]. \tag{42}
\]

We regard this formula as second order equation for \( \nu_+ \), depending on an arbitrary constant \( \lambda \). Using this, and supposing that \( \nu_+ \neq 0 \), we obtain for \( \nu_- \) the more compact expression in terms of \( \nu_+ \) and \( \dot{\nu}_+ \),

\[
\nu_- = \frac{\lambda}{16\nu_+} - \frac{1}{4f^2\nu_+} (\omega \nu_+ - i \nu_\omega)^2, \tag{43}
\]

and we see that \( \nu_- = (\lambda/4 - \nu_3^2)/4\nu_+ \). The first integral \( \lambda \) of eq. (41) is proportional to the constant of motion \( \lambda_1 \) of system (28) - (30): \( \lambda_1 = \lambda/16 \).

Thus the operators \( B(t), B^+(t) \), eq.(26), are invariant for the forced oscillator (19), (25) if \( \nu_+(t) \) is a nonvanishing solution of the second order equation (42) with any constant \( \lambda \) and \( \nu_3 \) and \( \nu_\omega \) being given by eqs. (39) and (43) respectively. They will obey the fermionic ladder operator conditions (31) if \( \lambda_1 = 0 = \lambda \) and \( \lambda_2 = 1 \). Instead of fixing \( \lambda_2 = 1 \) we can redefine \( B(t) \to B(t)/\sqrt{\lambda_2} \), i.e. take the invariant fermion annihilation operator of the form valid for any nonnegative constant of motion \( \lambda_2 = |\nu_-|^2 + |\nu|^2 + |\nu_3|^2/2 \),

\[
B(t) = \frac{1}{\sqrt{\lambda_2}} \left[ \frac{1}{f} (\nu_+ \omega - i \nu_-) (b^\dagger b - \frac{1}{2}) + \nu_+ b^\dagger - \frac{1}{4f^2\nu_+} (\omega \nu_+ - i \nu_\omega)^2 b \right], \tag{44}
\]

where \( \nu_+ \) is a solution to the equation (42) with \( \lambda = 0 \).
In this case the equations (41), (43) and (39) can be greatly simplified if we put
\[ \nu_+ (t) = \frac{1}{2} \epsilon^2 (t). \] (45)
The result is
\[ \nu_- = -\frac{1}{2 f^2} \left( \frac{\omega}{2} \epsilon - i \dot{\epsilon} \right)^2, \]
\[ \nu_3 = \frac{1}{f} \left( \frac{\omega}{2} \epsilon^2 - i \dot{\epsilon} \epsilon \right), \] (46)
where \( \epsilon(t) \) satisfies the equation
\[ \ddot{\epsilon} - \frac{i}{f} \dot{\epsilon} + \Omega(t) \epsilon = 0, \] (47)
\[ \Omega(t) = |f(t)|^2 + \frac{1}{4} \omega^2(t) + \frac{i}{2} \dot{\omega} - \frac{i}{2} \omega \frac{\dot{f}}{f}. \] (48)
This latter equation admits \( \lambda_2 \), eq. (33), as a first integral,
\[ \lambda_2 = |\epsilon|^4 \left( 1 + \frac{2}{|f|^2} |\frac{\omega}{2} \epsilon - i \dot{\epsilon}|^2 + \frac{1}{|f|^4} |\frac{\omega}{2} \epsilon - i \dot{\epsilon}|^4 \right). \] (49)
In (47) the term proportional to the first derivative can be eliminated by the substitution
\[ \epsilon = \epsilon' \exp \left( \frac{1}{2} \int \frac{t}{f} \dot{f}(\tau) d\tau / f(\tau) \right), \] (50)
which leads to the more simple equation
\[ \ddot{\epsilon}' + \Omega'(t) \epsilon' = 0. \] (51)
\[ \Omega'(t) = |f(t)|^2 + \frac{1}{4} \omega^2(t) + \frac{i}{2} \dot{\omega} - \frac{i}{2} \omega \frac{\dot{f}}{f} + \frac{1}{2} \left( \frac{\dot{f}}{f} \right)^2 - \frac{3}{4} \left( \frac{\dot{f}}{f^2} \right)^2. \] (52)
It is worth noting at this point that the invariant ladder operators for the boson nonstationary forced oscillator have been obtained \cite{9, 10} in terms of solutions to the same classical equation (51).

5 CS for the fermion forced oscillator

We define coherent states (CS) for a given fermion system as eigenstates of the corresponding invariant fermion annihilation (or creation) operator \( B(t) \).

Since the most general fermion one mode Hamiltonian operator is of the form of (nonstationary) forced oscillator \( (25) \), the one-mode fermion CS are defined as eigenstates of the invariant ladder operator \( B(t) \), eq. \( (44) \) (or eqs. \( (26), (31) \)):
\[ B(t) |\zeta; t\rangle = \zeta |\zeta; t\rangle. \] (53)
Since $B(t)$ is invariant operator for the FFO, the eigenvalue $\zeta$ does not depend on time $t$. In terms of the $\zeta, B(t), B^\dagger(t)$ and the $B(t)$-vacuum $|0; t\rangle$ we have for $|\zeta; t\rangle$ the same formulas as for the canonical fermion CS $|\zeta\rangle$, eq. (11), (13)-(15), in terms of $\zeta, b, b^\dagger$ and the $b$-vacuum $|0\rangle$. In particular

$$|\zeta; t\rangle = e^{-\frac{1}{2}\zeta^* \zeta} \left(|0; t\rangle - \zeta B^\dagger(t)|0; t\rangle\right).$$

It remains therefore to construct the (normalized) new ground state $|0; t\rangle$ according to its defining equations

$$B(t)|0; t\rangle = 0,$$
$$i \frac{d}{dt}|0; t\rangle = H_f|0; t\rangle.$$

We put

$$|0; t\rangle = \alpha_0(t)|0\rangle + \alpha_1(t)|1\rangle,$$

and after some tedious calculations find

$$\alpha_1(t) = \alpha_0(t) \frac{\nu^*_+(t)}{2\nu^*_+(t)};$$
$$\alpha_0(t) = \sqrt{\nu^*_+(t)} \exp \left[-\frac{i}{2} \left( \varphi_{\nu^+_+}(t) + \int^t (2g(\tau) + \omega(\tau)) d\tau \right) \right],$$

where $\varphi_{\nu^+_+}$ is the phase of $\nu^+_+(t)$. The state $|\zeta; t\rangle$ will represent the exact time evolution of an initial canonical CS $|\zeta\rangle$ if the initial conditions (36) are imposed: $|\zeta; 0\rangle = |\zeta\rangle$. In this case, as we have shown in section 3, the time evolved state $|\zeta; t\rangle$ could be again an eigenstate of $b$ if the oscillator is not 'forced', i.e. $f(t) = 0$. Let us note that the time-dependence of the constructed states is obtained in terms of solutions to the classical system (28)-(30), or equivalently to the classical equation (51). The latter is the same equation that appeared in the time evolution of the CS of bosonic FFO [9, 10]

Our method of construction of dynamical invariants differs slightly from the Lewis-Riesenfeld method [12] (developed for bosonic oscillators). Lewis and Riesenfeld used to first construct Hermitian invariant, which then is represented as a product of normally ordered ladder operators. To make connection to their approach let us suppose that we first succeeded to construct the Hermitian invariant $N(t)$ and to find some ladder operators $\tilde{B}(t), \tilde{B}^\dagger(t)$ that factorize it: $N(t) = \tilde{B}^\dagger(t)\tilde{B}(t)$. It is clear that $\tilde{B}(t)$ may differ from our non-Hermitian invariant $B(t)$ in a phase factor: $\tilde{B}(t) = e^{i\varphi(t)} B(t)$. We can then in a standard way construct normalized eigenstates of $N(t)$,

$$N(t)|\overline{0}; t\rangle = 0 \quad \text{and} \quad N(t)|\overline{1}; t\rangle = |\overline{1}; t\rangle,$$
and of $\widetilde{B}(t)$,

$$\widetilde{B}(t)\langle \zeta; t \rangle = \zeta \langle \zeta; t \rangle,$$

$$\langle \zeta; t \rangle = (1 - \frac{1}{2}\zeta^\ast \zeta) \left[ \langle 0; t \rangle - \zeta \langle 1; t \rangle \right]$$

which however do not obey the Schrödinger equation since, in general $\widetilde{B}(t)$ may not be invariant. To obtain solutions $|n; t\rangle$ and $|\zeta; t\rangle$ the above eigenstates $|n; t\rangle$, $n = 0, 1$, should be multiplied by phase factors,

$$|n; t\rangle = e^{i\phi_n(t)}|n; t\rangle, \quad n = 0, 1,$$

$$|\zeta; t\rangle = (1 - \frac{1}{2}\zeta^\ast \zeta) \left[ e^{i\phi_0(t)}|0; t\rangle - \zeta e^{i\phi_1(t)}|1; t\rangle \right]$$

which should obey the equations

$$\frac{d}{dt}\phi_n = \langle n; t|i\frac{\partial}{\partial t} - H|n; t\rangle.$$  

Evidently the state (62) is eigenstate of $\widetilde{B}(t)$ with time dependent eigenvalue $\zeta(t) = \exp(i\varphi(t))$, $\varphi(t) = \phi_1(t) - \phi_0(t)$. The phase $\varphi(t) = \phi_1(t) - \phi_0(t)$ consists of two parts - geometrical one $\varphi^G$, and dynamical one $\varphi^D = \varphi - \varphi^G$ [17],

$$\varphi^G(t) = i \int_0^t \left( \langle 1; t'|\frac{\partial}{\partial t'}|1; t'\rangle - \langle 0; t'|\frac{\partial}{\partial t'}|0; t'\rangle \right) dt'$$

$$= \varphi(t) + \int_0^t \left( \langle 1; t'|H|1; t'\rangle - \langle 0; t'|H|0; t'\rangle \right) dt'$$

Concluding Remarks

In this article, we have extend the earlier results of the boson coherence Hamiltonian and boson invariant ladder operators to the fermion coherence Hamiltonian and fermion invariant ladder operators. We have pointed out that unlike the boson coherence Hamiltonian, which is of the more general form of the nonstationary forced oscillator, the fermion coherence Hamiltonian is of more simple form of (nonstationary) fermion oscillator.

For the more general (in fact most general) fermionic system of nonstationary forced oscillator we have constructed invariant ladder operators and the related Fock and coherent states. We succeeded to express these invariants and the time evolution of the corresponding states in terms of the same classical equation, that describe the evolution of coherent states of the boson nonstationary forced oscillator [9] [10]. The relation of the invariant ladder operators method to the Lewis-Riesenfeld method [12] was briefly described on the example of nonstationary fermion systems.
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