QUASILOCAL ENERGY IN KERR SPACETIME

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Abstract. In this work we study the quasilocal energy as in [11] for a constant radius surface in Kerr spacetime in Boyer-Lindquist coordinates. We show that under suitable conditions for isometric embedding, for a stationary observer the quasilocal energy defined in [11] for constant radius in a Kerr like spacetime is exactly equal to the Brown-York quasilocal energy [2]. By some careful estimations, we show that for a constant radius surface in the Kerr spacetime which is outside the ergosphere the embedding conditions for the previous result are satisfied. We discuss extremal solutions as described in [14]. We prove a uniqueness result. We find all extremal solutions in the Minkowski spacetime. Finally, we show that near the horizon of the Kerr spacetime for the small rotation case the extremal solutions are trivial.

1. Introduction

In this work, we want to discuss the quasi-local energy (QLE) as in [11, 12, 13, 14] for some spacelike surfaces on a Kerr-like spacetime. Let us recall the formulation of such a QLE. From the covariant Hamiltonian formalism [5, 6], the conserved current is defined by the Hamiltonian 3-form $\mathcal{H}(N)$ (on the space-like hypersurface) under the infinitesimal diffeomorphism generated by $N$. The conserved quantity is then the integration over the space-like region $\Omega$ which reduces to the boundary integration of the total derivative term in $\mathcal{H}(N)$ on shell. The field equations are preserved under any modification of the total derivative term, which changes the boundary term and then changes the value of the conserved quantity. Modifying the boundary term implies a different boundary condition (corresponding to a different pseudotensor expression) and the choice of reference (corresponding to the frame choice of the pseudotensor). For a specific boundary expression, the choice of reference is still arbitrary. The difficulty comes from choosing a reasonable reference. There are several different strategies...
for choosing the reference. We will follow the 4D isometric matching as in [7, 8, 11, 12, 13]. The idea of the 4D isometric matching \( g_{\mu\nu} \doteq \bar{g}_{\mu\nu} \) is that the metric \( \bar{g}_{\mu\nu} \) of the background spacetime matches the physical metric \( g_{\mu\nu} \) on the quasi-local 2-boundary. One can imagine that there is an observer in a specific spacetime who measures the conserved quantity of the physical world. The matching of the metric may be regarded as the calibration of the measurement on the 2-surface. Consequently, the conserved quantity is obtained by the specific displacement vector field \( N \), e.g. energy conservation for a time-like displacement; linear momentum for a space-like transition; angular momentum for a rotation and the center-of-mass moment for a boost displacement.

We will focus on the quasi-local energy of the Kerr spacetime. The background spacetime is chosen to be the Minkowski spacetime and \( N \) to be the time-like Killing vector of the background. The reference choice appears as the Jacobian of the background coordinate system, which also determines the displacement.

The 4D matching \( g_{\mu\nu} \doteq \bar{g}_{\mu\nu} \) gives 10 constraints on the 12 independent unknowns of the Jacobian, and it reduces to two freedoms: one corresponds to the 2-surface isometric embedding (we will call it “the embedding freedom”) and the other to the displacement, or in other words, the observer dependence (we will call it “the boost freedom”). The 2-surface isometric embedding is unique up to one free function, which is called “the admissible \( \tau \)” in [9]. The other freedom is from the remaining 7 constraints of the 8 unknowns. The Hamiltonian boundary expression is then a functional of these two free choices. There is no unique value of energy because different observers have different measurements. One could find the critical value via the variation with respect to the free choices [14, 13].

Consider the physical spacetime \((M, g)\) so that the metric is a Kerr-like metric:

\[
g = F dt^2 + 2G dt d\varphi + H d\varphi^2 + R^2 d\tau^2 + \Sigma^2 d\theta^2
\]

where \( F, G, H, R, \Sigma \) are functions of \( r, \theta \) only. The background we choose is the Minkowski spacetime \((\bar{M}, \bar{g})\):

\[
\bar{g} = -dT^2 + dX^2 + dY^2 + dZ^2.
\]

We use the Chen-Nester-Tung (CNT) quasi-local expression in [12, eq. (4)], which is

\[
\mathcal{B}(N) = \frac{1}{2\kappa} (\Delta \Gamma^{\alpha}_{\beta\gamma} \wedge \epsilon_N \eta_{\alpha}^{\beta} + \bar{D}_\beta N^\alpha \Delta \eta_{\alpha}^{\beta}),
\]
where \( \kappa = 8\pi \), \( \epsilon_\alpha \eta^\alpha = \sqrt{-g} \mathcal{N}^\mu_{\alpha} g^{\beta\gamma} \epsilon_{\alpha\gamma\mu
u} dx^\nu \) and \( \Delta \alpha := \alpha - \bar{\alpha} \) is the difference of the physical field and the reference one. Note that the second term includes \( \Delta \eta^\alpha_{\beta} := \frac{1}{2}(\sqrt{g} g^{\beta\gamma} - \sqrt{\bar{g}} \bar{g}^{\beta\gamma}) \epsilon_{\alpha\gamma\mu
u} dx^\mu \wedge dx^\nu \), which vanishes for the 4D matching condition \( g_{\mu\nu} = \bar{g}_{\mu\nu} \). The quasilocal energy can then be simplified to

\[
E(N, \Omega) = \oint_{\partial \Omega} \mathcal{B}(N) = \oint_{S} \frac{1}{2\kappa} \Delta \Gamma^\alpha_{\beta} \wedge \epsilon_\alpha \eta^\alpha.
\]

In this work, we will consider the spacelike two surfaces \( S(r_0) \) in \( M \) given by \( r = r_0, t = t_0 \) where \( r_0, t_0 \) are constants. We consider embeddings in \( \bar{M} \) of the form:

\[
T = T(t, r, \theta), \quad X = \rho \cos(\phi + \Phi), \quad Y = \rho \sin(\phi + \Phi), \quad Z = Z(t, r, \theta)
\]

where \( \rho \) and \( \Phi \) are functions of \( (t, r, \theta) \). As mentioned above, the 4D matching has two free choices, which will be chosen to be \( x = T_r, y = T_\theta \). We choose \( N = \partial_r \). The next step is to find the critical value by varying \( x, y \). One then obtains (11) (50) and (51)

\[
E(N, \Omega(r); x, y) = -\int_{0}^{2\pi} \left( \int_{0}^{\pi} \left( \frac{AH_{\theta} xy}{D} + A \frac{\alpha}{D} + C \frac{1}{D} \right) d\theta \right) d\varphi
\]

where

\[
\begin{align*}
A &= \Sigma(H\Sigma^2)_r, \\
C &= 2HR^2(H\Sigma - H\Sigma_\theta - 2\Sigma^2) \\
D &= 16\pi H^\frac{1}{2} R^2 \Sigma \\
\alpha &= (x^2\Sigma^2 + R^2(y^2 + \Sigma^2))^{\frac{1}{2}} \\
\beta &= (-H_\theta^2 + 4H(y^2 + \Sigma^2))^{\frac{1}{2}} \\
\ell &= y^2 + \Sigma^2.
\end{align*}
\]

Here \( \Omega(r) \) is the domain in the time slice \( t = t_0 \) with boundary \( S(r) \) and \( E \) depends on the choice of \( x, y \).

It is obvious that (1.6) has the trivial solution \( x \equiv 0, y \equiv 0 \), which may be considered to correspond to a stationary observer. Our first result is:
Suppose $S(r)$ has positive Gaussian curvature and can be isometrically embedded in $\mathbb{R}^3$ as a surface of revolution, then $E(N, \Omega(r); 0, 0)$ is exactly equal to the Brown-York QLE in [2].

It is easy to see that for the Kerr metric, in the slow rotation case, $S(r)$ satisfies the above embedding condition. In [11], using the slow rotation approximation of Martinez [4], it was proved the $E(N, \Omega(r); 0, 0)$ is equal to the Brown-York QLE in the slow rotation approximation. It was also proved that $E(N, \Omega(r); 0, 0)$ is the Brown-York QLE for the Schwarzschild metric. Our result says that in fact for the slow rotation case in the Kerr spacetime, they are exactly the same.

One question is whether this is still true for a general rotation. Direct computations show that in the extremal case for the Kerr metric, the horizon has negative Gaussian curvature somewhere. In fact, it is known [1] that the Kerr horizon cannot be embedded globally in $\mathbb{R}^3$ whenever $a > \sqrt{3}m/2$. Here $a$ is the angular momentum per unit mass and $m$ is the mass. We always assume that $a \leq m$. Physically one would like to consider $S(r)$ which is outside the ergosphere. Our second result is:

For the Kerr spacetime, if $r \geq 2m$, then $S(r)$ has positive Gaussian curvature and can be isometrically embedded in $\mathbb{R}^3$ as a surface of revolution.

Hence our first result applies to this situation. In particular, we obtain an explicit formula for the Brown-York quasi-local energy for $S(r)$.

Our next result is to consider solutions of (1.6) for Kerr spacetime. For Minkowski spacetime, i.e. $a = m = 0$, then one can find nontrivial solutions to (1.6), see [14]. In fact, one can find all the solutions in the Minkowski spacetime. This follows from an uniqueness result. Namely, we prove:

For the Kerr spacetime, two sets of solutions to (1.6) with bounded derivatives are equal if they are equal at $\theta = 0$ or at $\theta = \pi$. In particular, the Minkowski spacetime, the solutions are of the form $x = -k \cos \theta$, $y = kr \sin \theta$, where $k$ is a constant.

Hence for the Minkowski spacetime, $E(N, \Omega(r); x, y) = 0$ for all solutions $x, y$ to (1.6).

On the other hand, if there is a horizon, then we may only have trivial solutions. Let $r_+ = m + (m^2 - a^2)^{1/2}$. We prove that:

If $0 \leq a \ll m$ and if $r > r_+$ which is close enough to $r_+$, then any solution $x, y$ to (1.6) with $|x_\theta|, |y_\theta|$ being bounded must be trivial, i.e. $x \equiv 0, y \equiv 0$. 
The organization of the paper is as follows: In section 2, we will prove that under suitable conditions \( E(N, \Omega(r); 0, 0) \) is equal to the Brown-York QLE. In section 3, we will discuss the embedding problem for surfaces \( S(r) \) in the Kerr spacetime. In section 4, we will discuss the solutions to the system (1.6).

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2. CNT QLE and Brown-York QLE

Let \((M, g)\) be a Kerr like spacetime with \(g\) given as in (1.1), with \(0 \leq \theta \leq \pi, 0 \leq \varphi < 2\pi\). We assume that the slice \(\hat{M} = \{t = t_0\}\) is spacelike, where \(t_0\) is a constant. Then the induced metric on \(\hat{M}\) is:

\[
\hat{g} = H^2 d\varphi^2 + R^2 dr^2 + \Sigma^2 d\theta^2.
\]

Let \(S(r_0)\) be the surface \(r = r_0\) in \(\hat{M}\), where \(r_0\) is a constant. The induced metric on \(S(r_0)\) is given by

\[
d\sigma^2 = \Sigma^2 d\theta^2 + H d\varphi^2.
\]

We assume this is a closed surface.

**Lemma 2.1.** Let \(\kappa\) be the mean curvature of \(S(r_0)\) with respect to the unit normal \(\nu = R^{-1} \partial_r\). Then

\[
\frac{1}{8\pi} \int_{S(r_0)} \kappa d\sigma = \int_0^{2\pi} \left( \int_0^\pi \frac{A\alpha}{D\ell} \, d\theta \right) \, d\varphi,
\]

where \(A, \alpha, D, \ell\) are as in (1.8) with \(\alpha\) being evaluated at \(x = 0, y = 0\), and \(d\sigma\) is the area element on \(S(r_0)\).

**Proof.** Note that \(H, R, \Sigma\) are independent of \(\varphi\). Let \(\nabla\) be the covariant derivative of the slice. Then the mean curvature is

\[
\kappa = - \left( H^{-1} \langle \nabla_{\partial_r} \partial_\varphi, \nu \rangle + \Sigma^{-2} \langle \nabla_{\partial_\theta} \partial_\varphi, \nu \rangle \right).
\]

(2.1)
On the other hand,
\[
\langle \nabla_{\partial_{\varphi}} \partial_{\varphi}, \nu \rangle = \frac{1}{R} \langle \nabla_{\partial_{\varphi}} \partial_{\varphi}, \partial_r \rangle
\]
\[
= - \frac{1}{R} \langle \partial_{\varphi}, \nabla_{\partial_{\varphi}} \partial_r \rangle
\]
\[
= - \frac{1}{R} \langle \partial_{\varphi}, \nabla_{\partial_{\varphi}} \partial_{\varphi} \rangle
\]
\[
= - \frac{1}{2R} \partial_r \langle \partial_{\varphi}, \partial_{\varphi} \rangle
\]
\[
= - \frac{H_r}{2R}.
\]

Similarly,
\[
\langle \nabla_{\partial_{\theta}} \partial_{\theta}, \nu \rangle = - \frac{\left(\Sigma^2\right)_r}{2R}.
\]

Hence
\[
\kappa = \frac{H_r}{2HR} + \frac{\left(\Sigma^2\right)_r}{2\Sigma^2R} = \frac{\Sigma H_r + 2H\Sigma_r}{2HR\Sigma} = \frac{A}{2HR\Sigma^3}.
\]

(2.2)

Hence
\[
\int_{S(r_0)} \kappa d\sigma = \int_0^{2\pi} \left( \int_0^{\pi} A \frac{1}{2HR\Sigma} H^\frac{1}{2}\Sigma d\theta \right) d\varphi
\]
\[
= \int_0^{2\pi} \left( \int_0^{\pi} A \frac{1}{2H^\frac{1}{2}R\Sigma^2} d\theta \right) d\varphi
\]
(2.3)

On the other hand, at \(x = 0, y = 0\),
\[
\int_0^{\pi} A \frac{\alpha}{D} d\theta = \int_0^{\pi} A \frac{\alpha}{16\pi H^\frac{1}{2}R^2\Sigma \cdot \Sigma^2} d\theta
\]
\[
= \int_0^{\pi} A \frac{1}{16\pi H^\frac{1}{2}R^2\Sigma^2} d\theta
\]

From the above two relations, the result follows. \(\square\)

By direct computations, we have:

**Lemma 2.2.** The Gaussian curvature \(K\) of \(S(r_0)\) is given by
\[
K = \frac{1}{4} \cdot \frac{-H (2H_{\theta\theta}\Sigma^2 - H_\theta(\Sigma^2)_{\theta} - 4\Sigma^4) - (4H\Sigma^2 - H_{\theta})\Sigma^2}{\Sigma^4 H^2}.
\]

We want to isometrically embed the surface \((S(r_0), d\sigma^2)\) in \(\mathbb{R}^3\). Consider the plane curve in the \(xz\)-plane, \((\eta(\theta), 0, \xi(\theta))\), with
\[
\eta(\theta) = H^\frac{1}{2}
\]
(2.4)
\[
(2.5) \quad \xi(\theta) = \int_{\frac{\pi}{2}}^{\theta} \left( \frac{-H_\theta^2 + 4H\Sigma^2}{4H} \right)^{\frac{1}{2}} d\theta.
\]

Again we are fixing \( t = t_0, r = r_0 \). Assume that \( \eta(0) = \eta(\pi) = 0 \) and assume that when we rotate the curve about \( z \)-axis we have a smooth embedded closed surface in \( \mathbb{R}^3 \). Then the surface is given by
\[
(2.6) \quad X(\theta, \varphi) = (x(\theta, \varphi), y(\theta, \varphi), z(\theta, \varphi)) = (\eta(\theta) \cos \varphi, \eta(\theta) \sin \varphi, \xi(\theta)).
\]

It is easy to see that
\[
\langle X_\theta, X_\theta \rangle = \Sigma^2, \langle X_\theta, X_\varphi \rangle = 0, \langle X_\varphi, X_\varphi \rangle = H.
\]

Hence \( X \) is an isometric embedding of \( (S(r_0), d\sigma^2) \) in \( \mathbb{R}^3 \).

**Lemma 2.3.** Under the above assumptions and notations, we have

\[
\frac{1}{8\pi} \int_{S(r_0)} \kappa_0 d\sigma = -\int_0^{2\pi} \left( \int_0^{\pi} C \frac{1}{D} \frac{d\theta}{\beta} \right) d\varphi
\]

where \( \kappa_0 \) is the mean curvature with respect to the unit outward normal of \( S(r_0) \) when it is isometrically embedded in \( \mathbb{R}^3 \), and \( \beta \) is evaluated at \( y = 0 \).

**Proof.** Let \( s \) be the arc length of the curve \((\eta, 0, \xi)\). Since the embedded surface is a surface of revolution, one of the eigenvalues \( \lambda_1 \) of the second fundamental form is \( \mathbb{R} \) section 3.3]:

\[
\lambda_1 = \frac{d\xi}{ds} \frac{1}{\eta} = \frac{\xi'}{H_\theta^2 (d\theta)^{-1}} = \frac{1}{2H\Sigma} \left( -H_\theta^2 + 4H\Sigma^2 \right)^{\frac{1}{2}}.
\]
Using the expression for the Gaussian curvature $K$ in Lemma 2.2, the mean curvature in $\mathbb{R}^3$ is

$$\kappa_0 = \lambda_1 + \lambda_2 = \lambda_1 + \frac{K}{\lambda_1} = \frac{1}{\lambda_1} (\lambda_1^2 + K)$$

Using the expression

$$\frac{1}{\lambda_1} \left[ -\frac{H_0^2}{4H^2\Sigma^2} + \frac{1}{4} \cdot \frac{-H (2H_0\Sigma^2 - H_0(\Sigma^2)_{\theta} - 4\Sigma^4) - (4H\Sigma^2 - H_0^2)\Sigma^2}{\Sigma^4H^2} \right]$$

$$= -\frac{H (2H_0\Sigma^2 - H_0(\Sigma^2)_{\theta} - 4\Sigma^4)}{4\cdot \frac{1}{2\Sigma^2} (-H_0^2 + 4H\Sigma^2)^{\frac{1}{2}} H^2\Sigma^4}$$

$$= \frac{- (2H_0\Sigma^2 - H_0(\Sigma^2)_{\theta} - 4\Sigma^4)}{2\Sigma^3 (-H_0^2 + 4H\Sigma^2)^{\frac{1}{2}}}$$

Hence

$$\int_{S(y_0)} \kappa_0 d\sigma = \int_0^{2\pi} \left( \int_0^{\pi} \frac{- (2H_0\Sigma^2 - H_0(\Sigma^2)_{\theta} - 4\Sigma^4)}{2\Sigma^3 (-H_0^2 + 4H\Sigma^2)^{\frac{1}{2}}} H^2\Sigma d\theta \right) d\phi$$

$$= \int_0^{2\pi} \left( \int_0^{\pi} \frac{-H \Sigma^2 (2H_0\Sigma^2 - H_0(\Sigma^2)_{\theta} - 4\Sigma^4)}{2\Sigma^2 (-H_0^2 + 4H\Sigma^2)^{\frac{1}{2}}} d\theta \right) d\phi$$

At $y = 0$,

$$\frac{C}{D} = \frac{1}{16\pi\Sigma^2} \frac{H^2(2H_0\Sigma^2 - H_0(\Sigma^2)_{\theta} - 4\Sigma^4)}{(-H_0^2 + 4H\Sigma^2)^{\frac{1}{2}}}.$$

From these two relations, we conclude that the lemma is true. $\square$

**Theorem 2.1.** For the Kerr like metric (1.1), suppose the embedding (2.4) is defined for the surface $S$ with $r =$constant, $t =$constant, we have

$$E(N, \Omega(r); 0, 0) = m_{\text{BY}}(S(r))$$

where $m_{\text{BY}}(S(r))$ is the Brown-York mass (with respect to the slice $t =$constant) of $S(r)$. 
3. ISOMETRIC EMBEDDING FOR SURFACES IN THE KERR SPACETIME

In this section, we will concentrate on the Kerr metric. The Kerr-like metric (1.1) now becomes:

\[
\begin{align*}
F &= -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma^2}, \\
G &= -\frac{4mar \sin^2 \theta}{\Sigma^2}, \\
H &= \frac{\sin^2 \theta \left( (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta \right)}{\Sigma^2}, \\
R^2 &= \Sigma^2, \\
\Sigma^2 &= r^2 + a^2 \cos^2 \theta, \\
\Delta &= r^2 - 2mr + a^2.
\end{align*}
\]

Our aim is to discuss where the surface \( r = \text{constant}, \ t = \text{constant} \) has positive Gaussian curvature and can be isometrically embedded in \( \mathbb{R}^3 \) as a surface of revolution. We always fix \( t = t_0 \) and denote the surface \( r = \text{constant} \) on the slice \( t = t_0 \) by \( \mathcal{S}(r) \) as before.

In the rest of this section, we assume that \( m = 1, \ 0 \leq a \leq m \) and \( r \geq r_+ \) which is the largest root of \( \Delta = 0 \). We also use the following notations:

\[
\begin{align*}
s &= \sin \theta; \\
c &= \cos \theta; \\
\epsilon &= \frac{a}{r}; \\
p &= \frac{2\epsilon^2 s^2}{r(1 + \epsilon^2 c^2)}; \\
q &= \frac{1 + \epsilon^2}{(1 + \epsilon^2 c^2)}.
\end{align*}
\]

Then

\[
\begin{align*}
\Sigma^2 &= r^2(1 + \epsilon^2 c^2) \\
\Delta &= r^2(1 + \epsilon^2 - \frac{2}{r}) \\
H &= r^2 s^2(1 + \epsilon^2 + p)
\end{align*}
\]

Lemma 3.1. We have

(i) \((\Sigma^2)_{\theta} = -2r^2 \epsilon^2 cs;\)

(ii) \(\frac{1}{2r^2} H_{\theta} = cs \left[ (1 + \epsilon^2) + p(1 + q) \right]; \) and
\( \frac{1}{2r^2} H_{\theta \theta} = (c^2 - s^2) \left[ (1 + \epsilon^2) + p(1 + q) \right] + 2c^2 \left[ pq(1 + q) + \frac{s^2 \epsilon^2}{(1 + \epsilon^2) pq^2} \right] \).

Proof. Note that

\[
p_\theta = \frac{2c^2}{r} \left( \frac{2cs}{1 + \epsilon^2 c^2} + \frac{\epsilon^2 s^2 \cdot 2cs}{(1 + \epsilon^2 c^2)^2} \right)
= \frac{4cs \epsilon^2}{r} \cdot \frac{1 + \epsilon^2}{(1 + \epsilon^2 c^2)^2}.
\]

So

\[
s^2 p_\theta = 2cspq; \quad q_\theta = \frac{2cs \epsilon^2}{(1 + \epsilon^2)} q^2.
\]

Hence

\[
\frac{1}{r^2} H_\theta = 2cs \left( 1 + \epsilon^2 + p \right) + s^2 p_\theta
= 2cs \left[ (1 + \epsilon^2) + p(1 + q) \right].
\]

Hence

\[
\frac{1}{2r^2} H_{\theta \theta} = (c^2 - s^2) \left[ (1 + \epsilon^2) + p(1 + q) \right] + cs \left[ p_\theta(1 + q) + pq_\theta \right]
= (c^2 - s^2) \left[ (1 + \epsilon^2) + p(1 + q) \right] + 2c^2 \left[ pq(1 + q) + \frac{s^2 \epsilon^2}{(1 + \epsilon^2) pq^2} \right].
\]

\[\square\]

From (2.5), it is easy to see that a necessary condition so that the spacelike surface \( r = \text{constant}, t = \text{constant} \) can be isometrically embedded in \( \mathbb{R}^3 \) is: \( 4H \Sigma^2 - H_\theta^2 > 0 \). To estimate this expression, we have the following: (In the rest of the paper, \( E \) always denotes a quantity which is bounded in absolute value by a constant which is independent of \( r, a, \theta \) provided \( r \geq 1 \). Its meaning may vary from line to line).

**Lemma 3.2.**

\[ 4H \Sigma^2 - H_\theta^2 = 4r^4 s^2 \left\{ (1 + \epsilon^2) s^2 + \frac{2c^2 s^2}{r} - 2c^2 p(1 + \epsilon^2)(1 + q) - c^2 p^2 (1 + q)^2 \right\}. \]

Moreover,

(i) If \( r \geq 2 \), then

\[ 4H \Sigma^2 - H_\theta^2 \geq r^4 s^4 \frac{15}{64}. \]

(ii) If \( r \geq 1 \), then

\[ 4H \Sigma^2 - H_\theta^2 = 4r^4 s^4 (1 + E \epsilon^2). \]
Proof.

\[ 4H \Sigma^2 - H^2 \]

\[ = 4r^4 s^2 \left[ (1 + \epsilon^2) + p \right] \left( 1 + \epsilon^2 c^2 \right) - 4r^4 c^2 s^2 \left[ (1 + \epsilon^2) + p(1 + q) \right]^2 \]

\[ = 4r^4 s^2 \left\{ \left[ (1 + \epsilon^2)(1 + \epsilon^2 c^2) + \frac{2\epsilon^2 s^2}{r} \right] - \epsilon^2(1 + \epsilon^2)^2 - 2c^2p(1 + \epsilon^2)(1 + q) - c^2p^2(1 + q)^2 \right\} \]

\[ = 4r^4 s^2 \left\{ (1 + \epsilon^2)s^2 + \frac{2\epsilon^2 s^2}{r} - 2c^2p(1 + \epsilon^2)(1 + q) - c^2p^2(1 + q)^2 \right\}. \]

This proves the first part of the lemma. (ii) follows from this immediately.

To prove (i), suppose \( r \geq 2 \), then \( \epsilon \leq \frac{1}{2} \).

\[
(1 + \epsilon^2)s^2 + \frac{2\epsilon^2 s^2}{r} - 2c^2p(1 + \epsilon^2)
\]

\[= (1 + \epsilon^2)s^2 + \frac{2\epsilon^2 s^2}{r} - \frac{4c^2 \epsilon^2 s^2 (1 + \epsilon^2)}{r(1 + \epsilon^2 c^2)} \]

\[= \frac{s^2}{1 + \epsilon^2 c^2} \left[ (1 + \epsilon^2)(1 + \epsilon^2 c^2) + \frac{2c^2}{r} (s^2 - \epsilon^2 c^2 - c^2) \right] \]

\[\geq \frac{s^2}{1 + \epsilon^2 c^2} \left[ (1 + \epsilon^2)(1 + \epsilon^2 c^2) + \epsilon^2 (-\epsilon^2 c^2 - c^2) \right] \]

\[= s^2q.\]

Since \( p \leq \frac{1}{4}s^2; 1 \leq q \leq \frac{5}{4} \), we have

\[
(1 + \epsilon^2)s^2 + \frac{2\epsilon^2 s^2}{r} - 2c^2p(1 + \epsilon^2)(1 + q) - c^2p^2(1 + q)^2
\]

\[\geq s^2q - 2c^2p(1 + \epsilon^2)q - c^2p^2(1 + q)^2 \]

\[\geq s^2q(1 - 2 \cdot \frac{1}{4} \cdot \frac{5}{4}) - \frac{81}{256}s^2 \]

\[\geq \frac{15}{256}s^2. \]

From this the lemma follows.

\[ \square \]

Using (2.4), (2.5) :

**Corollary 3.1.** Suppose \( r \geq 2 \), or \( r \geq r_+ \) and \( \epsilon \) is small enough, then \( S(r) \) can be isometrically embedded in \( \mathbb{R}^3 \) as a surface revolution.
Next we want to estimate the Gaussian curvature of $S(r)$. We will use the expression in Lemma 2.2. First, we have the following:

**Lemma 3.3.**

$$\frac{1}{4r^4} \left\{ 2H_{\theta\theta}\Sigma^2 - H_\theta(\Sigma^2)_\theta - 4\Sigma^4 \right\}$$

$$= -2s^2 - \epsilon^2 s^2 - \epsilon^2 c^2 s^2 + p(1 + q) \left[ 3c^2 + 2\epsilon^2 c^2 + \epsilon^2 c^4 - s^2 \right]$$

$$+ 2\epsilon^2 c^2 s^2 pq.$$

**Proof.**

\begin{align*}
\frac{1}{4r^4} \left\{ 2H_{\theta\theta}\Sigma^2 - H_\theta(\Sigma^2)_\theta - 4\Sigma^4 \right\} \\
= \left\{ (c^2 - s^2) \left[ (1 + \epsilon^2) + p(1 + q) \right] + 2c^2 \left[ pq(1 + q) + \frac{s^2 \epsilon^2}{(1 + \epsilon^2) pq^2} \right] \right\} (1 + \epsilon^2 c^2) \\
+ \epsilon^2 c^2 s^2 \left[ (1 + \epsilon^2) + p(1 + q) \right] - (1 + \epsilon^2 c^2)^2 \\
= \left\{ -2s^2 - \epsilon^2 s^2 + (c^2 - s^2)p(1 + q) + 2c^2 \left[ pq(1 + q) + \frac{s^2 \epsilon^2}{(1 + \epsilon^2) pq^2} \right] \right\} (1 + \epsilon^2 c^2) \\
+ \epsilon^2 c^2 s^2 \left[ (1 + \epsilon^2) + p(1 + q) \right] \\
= \left\{ -2s^2 - \epsilon^2 s^2 \right\} (1 + \epsilon^2 c^2) + p(1 + q)(c^2 - s^2)(1 + \epsilon^2 c^2) + 2c^2 p(1 + q)(1 + \epsilon^2) + 2\epsilon^2 c^2 s^2 pq \\
+ \epsilon^2 c^2 s^2 (1 + \epsilon^2) + \epsilon^2 c^2 s^2 p(1 + q) \\
= \left\{ -2s^2 - \epsilon^2 s^2 \right\} (1 + \epsilon^2 c^2) + p(1 + q) \left\{ (c^2 - s^2)(1 + \epsilon^2 c^2) + 2c^2 (1 + \epsilon^2) + \epsilon^2 c^2 s^2 \right\} \\
+ 2\epsilon^2 c^2 s^2 pq + \epsilon^2 c^2 s^2 (1 + \epsilon^2) \\
= -2s^2 - \epsilon^2 s^2 - \epsilon^2 c^2 s^2 + p(1 + q) \left[ 3c^2 + 2\epsilon^2 c^2 + \epsilon^2 c^4 - s^2 \right] \\
+ 2\epsilon^2 c^2 s^2 pq.
\end{align*}

\[ \square \]

**Theorem 3.1.** Suppose $r \geq 2$, or $r \geq r_+$ and $\epsilon$ is small then the Gaussian curvature of $S(r)$ is positive.

**Proof.** By Lemma 2.2

$$K = \frac{1}{4} \frac{-H (2H_{\theta\theta}\Sigma^2 - H_\theta(\Sigma^2)_\theta - 4\Sigma^4) - (4H\Sigma^2 - H_\theta^2)\Sigma^2}{\Sigma^4 H^2}.$$
By Lemmas 3.1, 3.2 and 3.3 we have

\begin{equation}
\tilde{K} := \frac{1}{4r^6s^2} \left\{ -H \left( 2H_{\theta\theta} \Sigma^2 - H_{\theta} (\Sigma^2)_{\theta} - 4\Sigma^4 \right) - (4H\Sigma^2 - H^2_{\theta})\Sigma^2 \right\}
\end{equation}

\begin{align}
&\geq (1 + \epsilon^2 + p) \left\{ 2s^2 + \epsilon^2 s^2 + \epsilon^2 c^2 s^2 - c^2 p(1 + q) \left[ 3 + 2\epsilon^2 + \epsilon^2 c^2 \right] \\
&- 2\epsilon^2 c^2 s^2 pq \right\} \\
&- (1 + \epsilon^2 c^2) \left\{ (1 + \epsilon^2) s^2 + \frac{2\epsilon^2 s^2}{r} - 2c^2 p(1 + \epsilon^2)(1 + q) - c^2 p^2(1 + q)^2 \right\}
\end{align}

It is easy to see that if \( \epsilon \ll 1 \) and if \( r \geq 1 \), then \( K > 0 \).

Suppose \( r \geq 2 \), then \( p \leq \frac{1}{4}, q \leq \frac{5}{4} \). So

\begin{equation}
\epsilon^2 c^2 s^2 - 2\epsilon^2 c^2 s^2 pq = \epsilon^2 c^2 s^2 \left( 1 - 2pq \right)
\end{equation}

\begin{align}
&\geq \epsilon^2 c^2 s^2 \left( 1 - \frac{5}{8} \right) \\
&= \frac{3}{8} \epsilon^2 c^2 s^2.
\end{align}

Now

\begin{align}
&-c^2 p(1 + q)(1 + \epsilon^2 + p) \left( 3 + 2\epsilon^2 + \epsilon^2 c^2 \right) + 2c^2 p(1 + q)(1 + \epsilon^2 c^2)(1 + \epsilon^2) \\
= &c^2 p(1 + q) \left[ (1 + \epsilon^2)(2 + 2\epsilon^2 c^2 - 3 - 2\epsilon^2 - \epsilon^2 c^2) - p \left( 3 + 2\epsilon^2 + \epsilon^2 c^2 \right) \right] \\
= &c^2 p(1 + q) \left[ (1 + \epsilon^2)(\epsilon^2 c^2 - 1 - 2\epsilon^2) - p \left( 3 + 2\epsilon^2 + \epsilon^2 c^2 \right) \right]
\end{align}

Hence

\begin{equation}
-c^2 p(1 + q)(1 + \epsilon^2 + p) \left( 3 + 2\epsilon^2 + \epsilon^2 c^2 \right) + 2c^2 p(1 + q)(1 + \epsilon^2 c^2)(1 + \epsilon^2) \\
+ c^2 p^2(1 + q)^2(1 + \epsilon^2 c^2) \\
= c^2 p(1 + q)(1 + \epsilon^2)(c^2 c^2 - 1 - 2\epsilon^2) + c^2 p^2(1 + q) \left[ (1 + q)(1 + \epsilon^2 c^2) - (3 + 2\epsilon^2 + \epsilon^2 c^2) \right] \\
= c^2 p(1 + q)(1 + \epsilon^2)(-\epsilon^2 s^2 - 1 - \epsilon^2) - c^2 p^2(1 + q)(1 + \epsilon^2) \\
= c^2 p(1 + q)(1 + \epsilon^2)(-\epsilon^2 s^2 - 1 - \epsilon^2 - p).
By (3.4)–(3.7), using the facts that $r \geq 2$, $p \leq \epsilon^2 s^2 \leq \frac{1}{4} \epsilon^2 \leq \frac{1}{4}$ and $q \leq \frac{5}{4}$, we have

$$\tilde{K} \geq (1 + \epsilon^2 + p)(\epsilon^2 s^2 + \frac{3}{8} \epsilon^2 c^2 s^2) + \left(\frac{15}{16} + \epsilon^2 s^2 + 2p\right)s^2$$

$$- (1 + \epsilon^2 c^2)\epsilon^2 s^2 - c^2 p(1 + q)(1 + \epsilon^2)(\epsilon^2 s^2 + 1 + \epsilon^2 + p)$$

$$\geq \frac{3}{8} \epsilon^2 c^2 s^2(1 + \epsilon^2) + \epsilon^2 s^2(1 + \epsilon^2) + \left(\frac{15}{16} + \epsilon^2 s^2\right)s^2 + s^2 p(\epsilon^2 + \frac{3}{8} \epsilon^2 c^2 + 2)$$

$$- (1 + \epsilon^2 c^2)\epsilon^2 s^2 - c^2 p(1 + q)(1 + \epsilon^2)^2 - c^2 p(1 + q)(1 + \epsilon^2)(\epsilon^2 s^2 + p)$$

$$\geq \frac{3}{8} \epsilon^2 c^2 s^2(1 + \epsilon^2) + \left(\frac{15}{16} + \epsilon^2 s^2\right)s^2 - c^2 p(1 + q)(1 + \epsilon^2)^2$$

$$+ p\epsilon^2 c^2 s^2 \left(1 + \frac{3}{8} + \frac{9}{4} \cdot \frac{5}{4} \cdot \frac{1}{2} \right)$$

$$\geq \epsilon^2 c^2 s^2 \left(\frac{3}{8} + \frac{14}{16} \cdot \frac{9}{4} \cdot \frac{5}{4}\right) + \frac{1}{16} s^2$$

$$\geq \epsilon^2 c^2 s^2 \left(\frac{3}{8} + \frac{14}{16} \cdot \frac{9}{4} \cdot \frac{5}{4}\right) + \frac{1}{16} s^2$$

$$\geq \frac{1}{16} s^2,$$

because $\epsilon^{-2} \geq 4$. From this it is easy to see that $K > 0$.  

□

By Theorems 2.1, 3.1 and Corollary 3.1 we have the following:

**Corollary 3.2.** For $r \geq 2$, or $r \geq r_0$ and $\epsilon \ll 1$, the spacelike surface $S : r = \text{constant}$, $t = \text{constant}$, the Brown-York quasilocal energy of $S$ is given by

$$m_{BY}(S(r)) = E(N, \Omega(r); 0, 0).$$

4. CRITICAL SOLUTIONS TO (1.6)

As described in the introduction, we obtain the QLE in $E(N, \Omega(r) ; x, y)$ by finding critical value of $x, y$. Under the embedding of the form (1.5), for the Kerr metric, $x, y$ should satisfy (1.6). In this section, we want to discuss the problems of existence and uniqueness of this system for the Kerr metric. First of all, let us rewrite the system. Recall the system (1.6) is:

$$\begin{align*}
  y_\theta &= -\frac{(\Sigma^2 H)_r}{2HR^2} x + \left(\frac{\Sigma_\theta}{\Sigma} - \frac{H_\theta}{2H}\right) y \\
  x_\theta &= \frac{R_\theta}{R} x + \left(\frac{(\Sigma^2 H)_r}{2H\Sigma^2} - \frac{\alpha \beta + xy H_\theta}{2H \ell}\right) y.
\end{align*}$$

(4.1)
We consider the Kerr metric with $0 \leq a \leq m$. We will discuss the system for $r > 2m$ because of the results in section 3. As in that section, let $s = \sin \theta$, $c = \cos \theta$, $\epsilon = a/r$, $p = (2me^2s^2)/(r(1 + e^2c^2))$, $q = (1 + \epsilon^2)/(1 + e^2c^2)$. Let

\[
\begin{align*}
\tilde{H} &= r^{-2}H = s^2(1 + \epsilon^2 + p); \\
\tilde{\Sigma}^2 &= r^{-2}\Sigma^2 = 1 + \epsilon^2c^2; \\
\tilde{\Delta} &= r^{-2}\Delta = 1 + \epsilon^2 - \frac{2m}{r}; \\
\tilde{x} &= \tilde{\Delta}^{\frac{1}{2}}x; \\
\tilde{y} &= r^{-1}y; \\
\tilde{\ell} &= r^{-2}\ell = \tilde{y}^2 + 1 + \epsilon^2c^2; \\
\tilde{\alpha} &= r^{-1}\ell \tilde{\Delta}^{\frac{1}{2}}\alpha = [(1 + \epsilon^2c^2)(\tilde{x}^2 + \tilde{y}^2 + 1 + \epsilon^2c^2)]^\frac{1}{2}; \\
\tilde{\beta} &= r^{-2}\beta = 2s \left[(1 + \epsilon^2 + p)\tilde{y}^2 + (1 + \epsilon^2 + \frac{2me^2}{r})s^2 - c^2p(1 + q)(2(1 + \epsilon^2) + p(1 + q))\right]^\frac{1}{2}
\end{align*}
\]

where we have used Lemmas 3.1 and 3.2. Note that $\tilde{\alpha}, \tilde{\beta}, \tilde{\ell}$ depend also on the functions $\tilde{x}, \tilde{y}$. Direct computations show that

\[
\frac{(\Sigma^2H)_r}{2HR^2} = r P \tilde{\Delta}
\]

where $P = (1 + \frac{1}{2}p + q)/(1 + \epsilon^2 + p)$. Also,

\[
\frac{(\Sigma^2H)_r}{2H\Sigma^2} = \frac{1}{r}P, \quad \frac{R_\theta}{R} = -\frac{\epsilon^2cs}{1 + \epsilon^2c^2}
\]

Using Lemmas 3.1 one can check that (4.1) is equivalent to

\[
\begin{align*}
\tilde{y}_\theta &= -\tilde{\Delta}^{\frac{1}{2}}P \tilde{x} - \left(\frac{\epsilon^2cs}{1 + \epsilon^2c^2} + \frac{c}{s} \left(1 + \frac{pq}{1 + \epsilon^2 + p}\right)\right)\tilde{y} \\
\tilde{x}_\theta &= \tilde{\Delta}^{\frac{1}{2}}P \tilde{y} - \frac{\epsilon^2cs}{1 + \epsilon^2c^2} \tilde{x} - \frac{\alpha\tilde{\beta} + \tilde{x}\tilde{y}\tilde{H}_\theta}{2\tilde{H}\tilde{\ell}}\tilde{y}.
\end{align*}
\]

Since $c/s$ and $\tilde{H}_\theta/H$ will become infinite when $\theta \to 0$ or $\pi$, we cannot use the apply standard theory to discuss the system. However, we still have the following uniqueness result.

**Theorem 4.1.** Let $\tilde{x}_i, \tilde{y}_i$, $i = 1, 2$ be two sets of solutions to (4.3) in $[0, \pi]$ with bounded derivatives. Then $\tilde{y}_i(0) = \tilde{y}_i(\pi) = 0$, $i = 1, 2$. Moreover, if $\tilde{x}_1(0) = \tilde{x}_2(0)$ (or $\tilde{x}_1(\pi) = \tilde{x}_2(\pi)$), then $\tilde{x}_1 = \tilde{x}_2, \tilde{y}_1 = \tilde{y}_2$.

**Proof.** It is easy to see that $P$ and $\tilde{\Delta}$ are bounded below away from 0 in $[0, \pi]$. By the first equation of the system, we conclude that $|\tilde{y}_i| \leq c_1 s$ for some constant $c_1, i = 1, 2$. In particular, $\tilde{y}_i(0) = \tilde{y}_i(\pi) = 0$. 


Suppose $\tilde{x}_1(0) = \tilde{x}_2(0)$ we want to prove that:

\begin{equation}
\frac{1}{2}[(\tilde{y}_1 - \tilde{y}_2)^2 + (\tilde{x}_1 - \tilde{x}_2)^2]_\theta \leq c_2[(\tilde{y}_1 - \tilde{y}_2)^2 + (\tilde{x}_1 - \tilde{x}_2)^2]
\end{equation}

for some constant $c_2$ in $[0, \pi/2]$. Suppose this is true, then one can conclude that $[(\tilde{y}_1 - \tilde{y}_2)^2 + (\tilde{x}_1 - \tilde{x}_2)^2] = 0$ on $[0, \pi/2]$. Since the system is well-behaved in $(0, \pi)$ one can apply the uniqueness of solutions of ODE to conclude that the proposition is true for this case. The case that $\tilde{x}_1(\pi) = \tilde{x}_2(\pi)$ is similar.

To prove (4.4), by the first equation of the system, in $[0, \pi/2]$, we have

\begin{equation}
\frac{1}{2}[(\tilde{y}_1 - \tilde{y}_2)^2]_\theta = -\Delta^\frac{1}{2} P(\tilde{x}_1 - \tilde{x}_2)(\tilde{y}_1 - \tilde{y}_2) - \left(\frac{\epsilon^2 cs}{1+\epsilon^2 c^2} + \frac{c}{s} \left(1 + \frac{pq}{1+\epsilon^2 + p}\right)\right)(\tilde{y}_1 - \tilde{y}_2)^2 \\
\leq c_3 [(\tilde{x}_1 - \tilde{x}_2)^2 + (\tilde{y}_1 - \tilde{y}_2)^2]
\end{equation}

for some constant $c_3 > 0$ because $c = \cos \theta \geq 0$ in $[0, \pi/2]$. To estimate $\frac{1}{2}[(\tilde{x}_1 - \tilde{x}_2)^2]_\theta$, let us denote $\tilde{\alpha}(\tilde{x}_i, \tilde{y}_i, \theta)$ by $\tilde{\alpha}_i$, and define $\tilde{\beta}_i, \tilde{\ell}_i$ similarly, $i = 1, 2$.

One can check that

\begin{equation}
|\tilde{\alpha}_1 - \tilde{\alpha}_2| = \frac{|\tilde{\alpha}_1^2 - \tilde{\alpha}_2^2|}{\tilde{\alpha}_1 + \tilde{\alpha}_2} \\
= \frac{|(1 + \epsilon^2 c^2)(\tilde{x}_1^2 + \tilde{y}_1^2 - \tilde{x}_2^2 - \tilde{y}_2^2)|}{\tilde{\alpha}_1 + \tilde{\alpha}_2} \\
\leq c_4 (|\tilde{x}_1 - \tilde{x}_2| + |\tilde{y}_1 - \tilde{y}_2|)
\end{equation}

for some constant $c_4$ which may also depend on the bounds of the solutions $\tilde{x}_i, \tilde{y}_i$, where we have used the fact that $\tilde{\alpha}_i \geq 1$. Similarly,

\begin{equation}
\left|\frac{1}{\ell_1} - \frac{1}{\ell_2}\right| = \frac{|\tilde{y}_1^2 - \tilde{y}_2^2|}{\ell_1 \ell_2} \\
\leq c_5 (|\tilde{y}_1 - \tilde{y}_2|)
\end{equation}

for some constant $c_5$ because $\tilde{\ell}_i \geq 1$. By Lemma 3.2 we have

$$\tilde{\beta}_i = r^{-2} \beta_i \geq \left(\frac{15}{64}\right)^{\frac{1}{2}} s^2$$
for some constant $c_5$ for $r > 2m$. So we have

$$\left| \tilde{\beta}_1 - \tilde{\beta}_2 \right| = \left| \frac{\tilde{\beta}_1^2 - \tilde{\beta}_2^2}{\tilde{\beta}_1 + \tilde{\beta}_2} \right|$$

$$= \frac{4s^2(1 + \epsilon^2c^2) |\tilde{y}_1^2 - \tilde{y}_2^2|}{\tilde{\beta}_1 + \tilde{\beta}_2}$$

$$= \frac{4s^2(1 + \epsilon^2c^2) |(\tilde{y}_1 + \tilde{y}_2)(\tilde{y}_1 - \tilde{y}_2)|}{\tilde{\beta}_1 + \tilde{\beta}_2}$$

$$\leq c_6s |\tilde{y}_1 - \tilde{y}_2|$$

for some constant $c_6 > 0$, where we have used the fact that near $\theta = 0, \pi, |\tilde{y}_i| \leq c_1 s$ for some constant $c_1$. By the second equation of (4.3), and by (4.6)–(4.8), we have

$$\frac{1}{2}[(\tilde{x}_1 - \tilde{x}_2)^2]_\theta \leq c_7[(\tilde{y}_1 - \tilde{y}_2)^2 + (\tilde{x}_1 - \tilde{x}_2)^2] + (\tilde{x}_1 - \tilde{x}_2) \left( -\frac{\tilde{\alpha}_1\tilde{\beta}_1 + \tilde{x}_1\tilde{y}_1\tilde{H}_\theta}{2\tilde{H}_1} \tilde{y}_1 + \frac{\tilde{\alpha}_2\tilde{\beta}_2 + \tilde{x}_2\tilde{y}_2\tilde{H}_\theta}{2\tilde{H}_2} \tilde{y}_2 \right)$$

$$= c_7[(\tilde{y}_1 - \tilde{y}_2)^2 + (\tilde{x}_1 - \tilde{x}_2)^2] - \frac{\tilde{\alpha}_1\tilde{\beta}_1 + \tilde{x}_1\tilde{y}_1\tilde{H}_\theta}{2\tilde{H}_1}(\tilde{y}_1 - \tilde{y}_2)(\tilde{x}_1 - \tilde{x}_2)$$

$$+ \left( \frac{\tilde{\alpha}_2\tilde{\beta}_2 + \tilde{x}_2\tilde{y}_2\tilde{H}_\theta}{2\tilde{H}_2} - \frac{\tilde{\alpha}_1\tilde{\beta}_1 + \tilde{x}_1\tilde{y}_1\tilde{H}_\theta}{2\tilde{H}_1} \right)(\tilde{x}_1 - \tilde{x}_2)\tilde{y}_2$$

$$\leq c_8[(\tilde{y}_1 - \tilde{y}_2)^2 + (\tilde{x}_1 - \tilde{x}_2)^2]$$

for some constants $c_7, c_8$, where we have used the facts that $|\tilde{y}_i| \leq c_1 s$, $|\tilde{\beta}_1| \leq c_6 s^2$ for some constant $c_6$. Combining the above inequality with (4.3), we conclude that (4.4) is true. This completes the proof of the theorem.

We apply the theorem to the Minkowski spacetime. In this case, $a = m = 0$ and (4.3) becomes:

$$\begin{align*}
\tilde{y}_\theta &= -2\tilde{x} - \frac{c}{s}\tilde{y} \\
\tilde{x}_\theta &= 2\tilde{y} - \left( \frac{(\tilde{x}^2 + \tilde{y}^2 + 1)^{\frac{3}{2}}(s^2 + \tilde{y}^2)^{\frac{1}{2}}}{s(1 + \tilde{y}^2)} + \frac{\tilde{x}\tilde{y}c}{s(1 + \tilde{y}^2)} \right) \tilde{y}.
\end{align*}$$

For this system, nontrivial solution exists: $\tilde{x} = -k \cos \theta, \tilde{y} = k \sin \theta, k = \text{constant}$, (see [11] [14]). So $x = -k \cos \theta, y = kr \sin \theta$ solve (4.4).
The nontrivial solution corresponds to the *inertial observer*. One can look at the displacement vector (see (58)). For the trivial solution ($k = 0$), $N = \partial_t$ is the *static observer*; for constant $k$, the displacement vector is

$$N = \sqrt{1 + k^2}\partial_t + k \cos \theta \partial_r - r - \frac{k}{\sqrt{1 + k^2}}\partial_\theta,$$

for the coordinate transformation $x' = r \cos \theta$, $y' = r \sin \theta$. It is a Lorentz transformation in the $t - x'$ plane of the static observer with constant velocity $-k/\sqrt{1 + k^2}$ in the $x'$ direction. One can check that the $E(N, \Omega(r); x, y) = 0$ in this case. This reflects the fact that each inertial observer is equivalent and measures zero energy for Minkowski spacetime. By Theorem 4.1, these are the only solutions for the system (4.1).

**Corollary 4.1.** *Let $x, y$ be solutions to the system (4.1). Suppose the derivatives of $x, y$ with respect to $\theta$ are bounded, then $x = -k \cos \theta$, $y = kr \sin \theta$.*

*Proof.* This follows from Theorem 4.1 by letting $k = -x(0)$. □

Next we want to prove that in case a horizon exists, then (4.1) may only have trivial solutions for a fixed $r$. With the same notations as before. We still normalize so that $0 \leq a \leq m = 1$. Also, $r_+ = 1 + (1 - a^2)^{\frac{1}{2}}$ is the larger root of $r^2 - 2r + a^2 = 0$. We have:

**Theorem 4.2.** *In the Kerr spacetime, for $8/3 > r > r_+$, if $\epsilon \ll 1$, then any solution to (4.1) with $|x_\theta|, |y_\theta|$ being bounded must be the trivial solution: $x \equiv 0, y \equiv 0$.*

Before we prove the theorem, we need the following:

**Lemma 4.1.** *For $8/3 > r > r_+$, if $\epsilon \ll 1$, then $\frac{(\Sigma^2 H)_{r}}{2H\Sigma^2} = \frac{\alpha \beta}{2H\ell} < 0$ in $(0, \pi)$ for any $x, y$.*

*Proof.* If $\epsilon$ is small, then

$$\frac{\alpha \beta}{\ell} \geq \frac{R \beta}{\ell^2} = R \left(-\frac{H_\theta^2}{\ell} + 4H \right)^{\frac{1}{2}} \geq R \left(-\frac{H_\theta^2}{\Sigma^2} + 4H \right)^{\frac{1}{2}} = R \Sigma \cdot 2r^2s^2(1 + E_1 \epsilon^2).$$
Here and below $E_i$ will denote a quantity which is bounded by a constant independent of $r, \theta, a$ provided $r \geq 1$. Since,

\begin{equation}
(\Sigma^2 H)_r = 2r^3 s^2 \left( 2(1 + \epsilon^2) - \left( 1 - \frac{1}{r} \right) \epsilon^2 s^2 \right),
\end{equation}

for $\frac{8}{3} > r > r_+$, if $\epsilon \ll 1$, then we have

\[ 1 + \frac{a^2}{r^2} - \frac{2}{r} < \frac{1}{4}. \]

Hence

\begin{equation}
\frac{(\Sigma^2 H)_r}{2H\Sigma^2} - \frac{\alpha\beta}{2H\ell} \leq \frac{1}{2H\Sigma^2} \left( 4r^3 s^2 (1 + E_2 \epsilon^2) - 2r^2 s^2 R(1 + E_1 \epsilon^2) \right)
\end{equation}

\begin{equation}
= \frac{2r^3 s^2}{2H\Sigma^2} \left( 2(1 + E_2 \epsilon^2) - \frac{1 + E_3 \epsilon^2}{(1 + \frac{a^2}{r^2} - \frac{2}{r})^2} \right)
\end{equation}

\begin{equation}
< 0
\end{equation}

for $\theta \in (0, \pi)$. This completes the proof of the lemma.

Proof of Theorem 4.2. Suppose $x, y$ are solutions to the system in $[0, \pi]$ so that $x_\theta, y_\theta$ are bounded in $[0, \pi]$. Then $|y| \leq c_1 s$ for some constant $c_1$ near $\theta = 0, \pi$ by the proof of Theorem 4.1. By Lemma 4.1, for $8/3 > r > r_+$, if $\epsilon \ll 1$, then we have:

\begin{equation}
(xy)_\theta = -\frac{(\Sigma^2 H)_r}{2HR^2} - \frac{\Sigma H_\theta - 2H\Sigma \theta}{2H\Sigma} xy + \frac{R_\theta}{R} xy + \left( \frac{(\Sigma^2 H)_r}{2H\Sigma^2} - \frac{\alpha\beta + xy H_\theta}{2H\ell} \right) y^2
\end{equation}

\begin{equation}
\leq \left( \frac{R_\theta}{R} - \frac{\Sigma H_\theta - 2H\Sigma \theta}{2H\Sigma} - H_\theta \cdot \frac{y^2}{2H\ell} \right) xy
\end{equation}

\begin{equation}
= -P(r, \theta) xy - \frac{H_\theta}{2H} xy
\end{equation}

where in the second line we have used Lemma 4.1 and the fact that $(\Sigma^2 H)_r \geq 0$ by (4.10). Here

\[ -P = \frac{R_\theta}{R} + \frac{\Sigma \theta}{\Sigma} - \frac{y^2 H_\theta}{2H\ell} \]

which is bounded on $[0, \pi]$. Let $Q = \int_0^\pi P d\theta$. We need to be careful here because $H_\theta / H$ is not integrable. However, $H_\theta y / H$ is integrable because $|y| \leq c_1 s$ near $\theta = 0, \pi$. So $Q$ is continuous on $[0, \pi]$. Hence we have

\begin{equation}
(xy \exp Q)_\theta \leq -\frac{H_\theta}{2H} (xy \exp Q).
\end{equation}
Let $\Psi = xy \exp Q$. Then the above inequality is:

$$\Psi_\theta \leq -\frac{H_\theta}{2H} \Psi.$$  

Suppose $\Psi(\theta_0) \leq -c_{10}$ for some $c_{10} > 0$ for some $\pi > \theta_0 > \pi/2$. Then $H_\theta \Psi \geq 0$ on $[\theta_0, \theta_0 + \delta] \subset [\pi/2, \pi)$ for some $\delta > 0$ because $H_\theta(\theta_0) < 0$. Hence $\Psi$ is decreasing on this interval, which implies $\Psi \leq -c_{10}$ in this interval. Continuing in this way, we conclude that $\Psi(\pi) \leq -c_{10}$. This is impossible because $y(\pi) = 0$ which implies $\Psi(\pi) = 0$. Hence we conclude that $\Psi \geq 0$ on $[\pi/2, \pi]$.

Similarly, one can prove that $\Psi \leq 0$ on $[0, \pi/2]$. In particular, $\Psi(\pi/2) = 0$.

On the other hand, let

$$W(\theta) = \int_{\frac{\pi}{2}}^{\theta} \frac{H_\theta}{2H} d\theta$$

which is well defined on $(0, \pi)$. Then we have

$$(\Psi \exp W)_\theta \leq 0$$

on $(0, \pi)$. Since $\Psi \exp W = 0$ at $\pi/2$, we have $\Psi \exp W \leq 0$ on $[\pi/2, \pi)$. In particular, we have $\Psi \leq 0$ on $[\pi/2, \pi)$. Since $\Psi \geq 0$ on $[\pi/2, \pi)$, we have $\Psi \equiv 0$ on $[\pi/2, \pi)$. Similarly, one can prove that $\Psi \equiv 0$ on $(0, \pi/2)$. To summarize, we have $\Psi \equiv 0$. This implies that $xy \equiv 0$.

Suppose $x$ is never zero on $(0, \pi)$, then we must have $y \equiv 0$. By the first equation of (4.1), we conclude that $x \equiv 0$. This is a contradiction. Hence $x(\theta_0) = 0$ for some $\theta_0 \in (0, \pi)$. Since $xy \equiv 0$, by the second equation in (4.1), we have

$$\frac{1}{2}(x^2)_\theta = \frac{R_\theta}{R} x^2,$$

which implies that $x^2 \equiv 0$. By the second equation again, we have $y \equiv 0$ because $xy \equiv 0$ and by Lemma 4.1 if $\theta \in (0, \pi)$

$$\frac{(\Sigma^2 H)_r}{2H\Sigma^2} - \frac{\alpha \beta}{2H \ell} < 0.$$  

This completes the proof of the theorem. \qed
Since we have the boundary expression (see \[11\] (48))

\[
B(\partial_T) = -\frac{\alpha (H\Sigma^2)_r}{2\kappa \sqrt{H} R^2 \Sigma^2} \\
- \frac{\sqrt{H}}{\kappa} \left( \frac{H_{\theta\theta} - 2\ell}{\beta} + \frac{R_\theta x y}{R\alpha} - \frac{xy^3 \beta + H_\theta \alpha \Sigma^2}{\ell \alpha \Sigma} \Sigma_{\theta} \right) \\
+ \frac{\sqrt{H} y}{\kappa \alpha} x_\theta + \frac{\sqrt{H} y (H_\theta \alpha - xy \beta)}{\kappa \ell \alpha \beta} y_\theta,
\]

we substitute (1.6) into (5.1) which becomes

\[
B(\partial_T) = -\frac{\alpha (H\Sigma^2)_r}{2\kappa \sqrt{H} R^2 \Sigma^2} \\
- \frac{\sqrt{H}}{\kappa} \left( \frac{H_{\theta\theta} - 2\ell}{\beta} + \frac{R_\theta x y}{R\alpha} - \frac{xy^3 \Sigma_{\theta}}{\ell \alpha \Sigma} \Sigma_{\theta} \right) \\
- \frac{R_\theta}{R\alpha} x y - \frac{(\Sigma^2 H)_r y^2}{2H \Sigma^2 \alpha} - \frac{\beta y^2}{2H \ell} - \frac{xy^3 H_\theta}{2H \ell \alpha} \\
\left( -\frac{H_\theta y}{\ell \beta} + \frac{xy^2}{\ell \alpha} \right) \left( -\frac{(\Sigma^2 H)_r x}{2H R^2} - \frac{\Sigma H_\theta - 2H \Sigma_{\theta}}{2H \Sigma} y \right)
\]

\[
= -\frac{\alpha (H\Sigma^2)_r}{2\kappa \sqrt{H} R^2 \Sigma^2} \\
- \frac{\sqrt{H}}{\kappa} \left( \frac{H_{\theta\theta} - 2\ell}{\beta} - \frac{H_\theta \Sigma_{\theta}}{\ell \beta} \right) \\
+ \frac{\beta y^2}{2H \ell} + \frac{H_\theta y^2}{2H \ell \beta} - \frac{H_\theta \Sigma_{\theta} y^2}{\ell \beta \Sigma} \\
- \frac{(\Sigma^2 H)_r y^2}{2H \Sigma^2 \alpha} + \frac{H_\theta (\Sigma^2 H)_r}{\ell \beta \Sigma} xy - \frac{x^2 y^2 (\Sigma^2 H)_r}{\ell \alpha \Sigma} 2H R^2,
\]

where the terms with \(xy^3\) are canceled. By using \(\beta^2 = 4H \ell - H_\theta^2\), the boundary term becomes

\[
B(\partial_T) = -\frac{\alpha (H\Sigma^2)_r}{2\kappa \sqrt{H} R^2 \Sigma^2} \\
- \frac{\sqrt{H}}{\kappa} \left( \frac{H_{\theta\theta} - 2\ell + 2y^2}{\beta} - \frac{H_\theta \Sigma_{\theta} (\Sigma^2 + y^2)}{\ell \beta \Sigma} \right) \\
- \frac{(\Sigma^2 H)_r y^2}{2H \Sigma^2 \alpha} + \frac{H_\theta (\Sigma^2 H)_r}{\ell \beta \Sigma} xy - \frac{x^2 y^2 (\Sigma^2 H)_r}{\ell \alpha \Sigma} 2H R^2.
\]
By using $\ell = y^2 + \Sigma^2$ and $\alpha^2 = x^2\Sigma^2 + R^2\ell$, it is further simplified to be

$$
B(\partial_T) = \frac{(H\Sigma^2)_r}{2\kappa \sqrt{H} R^2 \Sigma^2} \left( -\alpha + \frac{R^2 y^2}{\alpha} + \frac{x^2 \Sigma^2 y^2}{\ell \alpha} \right)
$$

$$
- \frac{\sqrt{H}}{\kappa} \left[ \frac{H_{\theta \theta} - 2 \Sigma^2}{\beta} - \frac{H_{\theta \Sigma}}{\Sigma \beta} + \frac{H_{\theta} (\Sigma^2 H)_r}{\ell \beta} \right]
$$

$$
= - \frac{\Sigma(H \Sigma^2)_r}{2\kappa \sqrt{H} R^2 \Sigma} \left( \frac{\alpha}{\ell} \right)
- \frac{1}{2\kappa \sqrt{H} R^2 \Sigma} \left[ \frac{2H R^2 (\Sigma H_{\theta \theta} - 2 \Sigma^3 - H_\theta \Sigma)}{\beta} + \frac{\Sigma H_{\theta} (\Sigma^2 H)_r}{\ell \beta} \right]
$$

$$
= - \left( \frac{A \alpha}{D \ell} + \frac{C 1}{D \beta} + \frac{A H_{\theta} xy}{D \beta \ell} \right).
$$

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