Model of the AdS/QFT duality

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It is observed and illustrated in a greatly simplified example that the idea of AdS/QFT duality can be considered a special case of the Ehrenfest’s correspondence principle between classical and quantum mechanics in the context of relativistic dynamics of fields and renormalization group procedure for effective particles.

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I. INTRODUCTION

The Maldacena conjecture of AdS/CFT duality [1] has been referred to by Polchinsky and Strassler [2, 3] to propose that hadrons may correspond to classical fields in a five-dimensional space-time. The fields can be approximated by a product of a four-dimensional plane wave and a nontrivial function of the fifth coordinate. The nontrivial function is a solution of the simple differential equation that depends on the plane-wave four-momentum squared and hence is capable of determining squares of masses of hadrons through an adjustment of boundary conditions in the fifth dimension. Irrespective of the AdS/CFT conjecture and its M/string-theory motivation, the question arises of how a solution to quantum field theory (QFT) could reduce to a solution of a simple, one-dimensional differential equation. We address this issue using light-front (LF) holography discovered by Brodsky and de Téramond [4–6].

Brodsky and de Téramond observed that the hadron form factor formula proposed by Polchinsky and Strassler matches the formula for hadron form factors which one obtains in QFT when the latter is developed in the front form (FF) of Hamiltonian dynamics. The unique utility of the FF as potentially helpful in understanding duality is a fairly new finding in comparison with the fact that the FF of Hamiltonian dynamics as an intriguing alternative to the commonly used instant form (IF) had been discovered a long time ago by Dirac [7].

In this paper we find that the time-honored Ehrenfest correspondence principle between quantum and classical mechanics [8] can be used in combination with the LF holography to explain the possibility and actually suggest the necessity of the existence of the reduction of the QFT description of observables, such as hadronic masses squared or form factors, to the formulas that resemble the formulas inspired by the duality conjecture.

Brodsky and de Téramond argue that the FF valence Fock sector wave function for a meson depends on the transverse separation between the quark and antiquark precisely in the same way that the AdS hadron field depends on the fifth-dimension coordinate. The specific soft-wall alteration of the AdS metric [9] is meant to correspond to a harmonic oscillator potential between the two valence quarks in the direction transverse to the z axis in the infinite momentum frame (IMF).

The interpretation of LF holography that the Ehrenfest correspondence principle provides does not distinguish the Fock-space valence wave function. Instead of focusing on the valence Fock sector, the Ehrenfest correspondence relies on averaging over constituents in all Fock sectors in QFT. As a result, solutions to the Ehrenfest equation are expected to describe the charge distribution in hadrons in a scale-independent way, i.e., in a way which does not rely on the Fock-space decomposition at any arbitrarily selected renormalization-group scale.

Regarding the renormalization-group scale dependence, it is known that the renormalization of Hamiltonians faces severe complications in the FF of dynamics [10, 11]. Therefore, we assume in our line of reasoning that in order to define the Fock-space decomposition of hadron states one can use the renormalization group procedure for effective particles (RGPEP) [12]. The most succinct description of the RGPEP can be found in Appendix C of Ref. [13].

The paper is organized in the following way. Our Ehrenfest equation is introduced in QFT in Sec. II. Hadron form factors are discussed in Sec. III. The general considerations of Secs. II and III are subsequently illustrated in Sec. IV with an example that specifies key details of the required reasoning, ignoring spin and all other such quantum numbers of all constituents, i.e., treating all of them as indistinguishable scalar bosons. Section V concludes the paper with a summary of the connection between the Ehrenfest picture and LF holography.

II. THE EHRENFEST EQUATION FOR MASSES

We assume that the FF of dynamics in a QFT, such as QCD, allows one to represent physical states such as hadrons in a suitably defined Fock space once a theory is renormalized using the RGPEP. The RGPEP framework is close in its rules to the similarity renor-
nalization group procedure introduced in Ref. [10] and used in Ref. [11], but it includes an advantage of the nonperturbative operator calculus recently described in Refs. [12, 13] in terms of simple examples.

For brevity and simplicity, we ignore below all quantum numbers of a hadron except its momentum. All the other quantum numbers are irrelevant to the idea of the Ehrenfest correspondence that we focus on. For the same reason, it is also useful to omit all quantum numbers of hadron constituents except their momenta. Despite these simplifications, we do continue to employ the word hadron as signifying a well-defined quantum state. Despite these simplifications, we do continue to employ the word hadron as signifying a well-defined quantum state. Despite these simplifications, we do continue to employ the word hadron as signifying a well-defined quantum state.

For example, in the proton, one would have a combination of three quark components |uud⟩, |udu⟩, |udd⟩, |udgg⟩, ... Thus, in baryons in QCD one has n_min = 3. In nonexotic mesons, n_min = 2.

We shall define below a quantity which is an expectation value that describes a hadron and takes the form of a function that we shall call the Ehrenfest function. Our Ehrenfest function describes the motion of an averaged active parton, for example, the quark that absorbs a photon in a photon-hadron interaction described by the hadron form factor, with respect to spectators, averaged over all the Fock components. The averaging that we describe below yields the Ehrenfest equation that does not depend on λ and takes the form of an eigenvalue problem for just one function ψ(ūk),

\[
\hat{M}^2 \psi(\vec{k}) = \left[ \hat{k}^2 + (m_{\text{active}} + m_{\text{core}})^2 + U_{\text{eff}} \right] \psi(\vec{k}) = M^2 \psi(\vec{k}). \tag{2}
\]

The precise meaning of the three-dimensional variable \(\vec{k}\) will follow from our analysis of motion of the active constituent with respect to the rest of a hadron. The effective Ehrenfest potential \(U_{\text{eff}}\) shall be discussed at length.

In order to outline the procedure of averaging over \(\lambda\)-dependent Fock components and over motion of all spectator partons in them, this article sketches the scenario in which three different equations appear: the Schrödinger-picture Hamiltonian eigenvalue equation in QFT in the FF of dynamics, the Ehrenfest equation that results from our averaging, and the form factor formula that Brodsky and de Téramond use in their holographic picture with a soft-wall, or a harmonic oscillator potential. As a result, we suggest that the classical five-dimensional equation with a potential such as in the soft-wall model [9] corresponds to our Ehrenfest equation that resembles the Schrödinger equation for one particle in a corresponding potential.

### A. Explanation of Eq. (1)

The hadron state can be written in terms of the FF wave functions that depend on the transverse momenta of the constituents, \(p_i^\perp = (p_i^\perp)'\), and ratios of longitudinal momenta of constituents to the hadron longitudinal momentum, \(x_i = p_i^\perp / P^\perp\). In our notation, bold symbols will always refer to a set of variables that describe a set of constituents. For every constituent, its FF fraction \(x_i\) matches its parton-model Bjorken variable \(x\) in the IMF.

Let us ignore all details and assume that all constituents have the same mass \(m = m(\lambda)\) so that one can define their free minus momentum components using \(p_i' = p_i - p_i^2 = m^2\). For integrating over kinematical momentum variables, we use notation

\[
\int [p^\perp] = \prod_{i=1}^{n} \frac{dp_i^\perp}{(2\pi)^2} \quad \text{and} \quad \int [x] = \prod_{i=1}^{n} \int \frac{dp_i^\perp}{(2\pi)^2p_i^\perp}. \tag{3}
\]

In order to model QFT such as QCD, we assume that for \(\lambda \sim \Lambda_{QFT}\), where \(\Lambda_{QFT}\) stands for a characteristic physical parameter of a theory (such as \(\Lambda_{QCD}\) in a specific renormalization scheme, which in our case is the RGPEP), a model of a hadron can be constructed using a small number of constituents. For example, in a proton the sector with \(n = 3\) and \(\lambda \sim \Lambda_{QCD}\) almost saturates the entire sum over \(n\), gluons contributing to the bulks of constituent quarks, e.g., see Ref. [12]. In great contrast, the expansion into Fock components corresponding to \(\lambda \gg \Lambda_{QFT}\) extends over a whole range of \(n\)-particle states with large numbers \(n\) and large relative momenta.

Our model of a hadron state is normalized using the condition

\[
\langle \text{Hadron} : P^\perp, P^\parallel | \text{Hadron} : P^\perp, P^\parallel \rangle = 2P^\perp (2\pi)^3 \delta(P^\perp - P^\perp) \delta^2(P^\perp - P^\parallel). \tag{4}
\]

The relative momenta are separated from the total momentum by writing

\[
\psi^{(n)}_P (p^\perp, x; \lambda) = 2(2\pi)^3 \delta \left( \sum_{i=1}^{n} x_i - 1 \right) \delta^2 \left( P^\perp - \sum_{i=1}^{n} p_i^\perp \right) \times \psi^{(n)} (k^\perp, x; \lambda), \tag{5}
\]

where \(k^\perp = (k_i^\perp)'\) for \(i = 1, \ldots, n\) and \(k_i^\perp = p_i^\perp - x_i P^\perp\) is the transverse relative momentum of the \(i\)th constituent with respect to the center of mass of constituents
treated as free particles of mass $m$. The wave function $\psi(n)(k^\perp, x; \lambda)$ always appears in conjunction with the $\delta$ functions such as in $\psi(n)(p^\perp, x; \lambda)$ in Eq. (5). Therefore, its domain is actually limited by the conditions

$$\sum_{i=1}^{n} x_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} k_i^\perp = 0. \quad (6)$$

Thus, only $n - 1$ of the $n$ three-dimensional arguments of $\psi(n)(k^\perp, x; \lambda)$ are independent.

In every sector, we distinguish one constituent and we focus on the description of its motion with respect to other constituents. The selected constituent is described using variables with subscript $n$ or without any subscript. The other constituents are named spectators or a core, depending on the context, and they are described using variables with subcripts $i = 1, \ldots, n - 1$.

### B. Eigenvalue problem for a hadron state

In QFT, the FF Hamiltonian $\hat{P}^-$ determines the structure of a composite system, as a solution to its eigenvalue equation. The eigenvalue is expressible in terms of the components $P^+$ and $P^\perp$ of the total kinematical momentum of the system and its mass squared, $M^2$. We call such a solution a hadron when the eigenvalue $M^2$ takes one of the several smallest possible values for a system with the hadron quantum numbers. The limitation to smallest masses is meant to exclude from consideration the scattering and bound states of entire hadrons. So,

$$\hat{P}^-[\text{Hadron}; P^+, P^\perp] = \frac{M^2 + P^\perp}{P^+}[\text{Hadron}; P^+, P^\perp] \quad (7)$$

$\hat{P}^-$ is a sum of a free part $\hat{P}_0^-$ ascribing free FF energies to the system constituents and the interaction part $\hat{P}_I^-$. Therefore, we can write the expectation value of $P^+\hat{P}^--P^\perp$ in a hadron in the form

$$M^2 = \sum_n \int [\kappa, x] 2(2\pi)^3 \delta \left( \sum_{j=1}^{n-1} \kappa_j - 1 \right) \delta(2) \left( \sum_{j=1}^{n-1} \kappa_j \right) \times \int_{k, x} \psi(n) \frac{k_j^\perp}{x(1-x)} + m_n^2 \frac{M^2}{1-x} \psi(n) \quad (8)$$

The integration symbol $\int_{k, x}$ appears here and later on universally in the entire paper to denote the integration over subscript arguments $k^\perp$ and $x$ in the Fock-space wave functions of a hadron state, $\psi(n)$, as required, and over arguments of our Ehrenfest function, see Eq. (15). In all the cases,

$$\int_{k, x} = \int \frac{d^2k^\perp dx}{2(2\pi)^3 x(1-x)}. \quad (9)$$

For every $n$, the set of variables denoted by symbols $k^\perp$ and $x$ defined previously for $n$ constituents is split in Eq. (8) into $k^\perp = k^\perp_n$ and $x = x_n$ for the active constituent and a set of variables denoted by $k^\perp = (k^\perp_n)_{n=1}^{n-1}$ and $x = (x_n)_{n=1}^{n-1}$ for the $n - 1$ spectators that form the core. The constituents' relative momentum variables internal to the core are defined by $\kappa_j = k_j^\perp + x_j k^\perp$.

Note the plus sign in front of the terms $\chi_j k^\perp$ which correspond to $x_i P^\perp$ in the definition of $k^\perp_j$ below Eq. (5). The change in sign results from the core being made of spectators carrying as a whole $-k^\perp$ and $1 - x$ when the active constituent carries $k^\perp$ and $x$.

The functions $\psi(n)_{k, x}$ depend on $x$ and $k^\perp$ in a way that we separate in our notation from their dependence on the variables $\kappa^\perp$ and $\chi$. The function

$$M_{n-1}^2(\kappa^\perp, \chi) = \sum_{j=1}^{n-1} \frac{\kappa_j^\perp + m_n^2}{\chi_j} \quad (10)$$

is an invariant mass squared of the $n - 1$ spectators that form a core in the sector number $n$, treated as a set of free particles of mass $m_n$.

Both the value of $m_n^2$ and the presence of connected interactions in Eq. (8) require an explanation. The masses result from adding the self-interactions generated in the eigenvalue equation to the renormalized mass squared, $m^2$, that one obtains in $\hat{P}^-$ at scale $\lambda$ using the RGPEP. The connected interactions are the ones that are left after the constituent self-interactions are included in their mass terms.

Consider first a simple theory in which the Hamiltonian contains $m^2$ and its eigenstate is built from only a two- and a three-constituent sector, so that

$$|\psi\rangle = |2\rangle + |3\rangle. \quad (11)$$

The Ehrenfest expectation value of the Hamiltonian $\hat{P}^- = \hat{P}_0^- + \hat{P}_I^-$ has the form

$$\langle \psi | \hat{P}^- | \psi \rangle = \langle 3 | \hat{P}_I^- | 3 \rangle + \langle 2 | \hat{P}_I^- | 2 \rangle \quad (12)$$

In the matrix elements that are diagonal in the number of constituents, the interaction terms change the momenta of at least two constituents on at least one side of the matrix element, because $m^2$ in $\hat{P}_0^-$ includes all terms that contribute to single-particle energies in $\hat{P}^-$. The matrix elements that involve a change of the number of constituents can be expressed in terms of diagonal matrix elements using the eigenvalue equation for the state $|\psi\rangle$. In the simple model, one can assume that the interactions in the component $|3\rangle$ are negligible in comparison to relevant eigenvalues of $\hat{P}_0^-$ and write

$$|3\rangle \simeq \frac{P_3}{P^- - P_0^-} |2\rangle, \quad (13)$$
where $P_3$ denotes projection on the three-body component. The presence of an eigenvalue in the denominator can be nearly ignored for a whole set of lightest hadron masses $M$ while the total momentum eigenvalues $P^+$ and $P^{-}$ drop out irrespective of the value of $M$. Assuming the above approximations, the expectation value of Eq. (12) takes the form

$$ \langle \psi | \hat{P}^- | \psi \rangle \simeq \langle 3 | \hat{P}^-_0 | 3 \rangle + (2 | \hat{P}^-_1 | 2 \rangle + 2 (2 | \hat{P}^-_1 P_3^{-} P_0^{-} \hat{P}^- | 2 \rangle .$$

The last term contains interaction effects due to the exchange of field quanta between two constituents and the self-interaction terms that change $m^2$ to $m_2^2$ in the sector $| 2 \rangle$ and make $m_2^2$ differ from $m_3^2 = m^2$ in the sector $| 3 \rangle$.

Now, in the case of a complex theory, the number of effective-particle sectors and the number of interaction terms in a $\hat{P}^-$ corresponding to any finite value of $\lambda$ will both be in principle infinite, although finite $\lambda$ typically implies a quick decrease of wave functions with an increase of the number of massive constituents and their invariant mass (this is a general feature of Hamiltonians one generates using the RGPEP). In any case, $\hat{P}^-$ in theories of physical interest contains the terms that change a single constituent to more than one of them or turn more than one constituent into just one. All these terms will result in sector-dependent single-particle mass-squared terms, which combine with $m^2$ and are denoted in Eqs. (8) and (10) by $m_n^2$.

C. Explanation of Eq. (2)

Equation (8) indicates that QFT quite generally yields an averaged equation of motion of an active constituent with respect to the core formed by spectators. We call this averaged equation the Ehrenfest equation, Eq. (2), because of an analogy to the Ehrenfest equation for expectation values of observables in quantum mechanics. The original Ehrenfest equation for expectation values in quantum mechanics corresponds to the Newton equations of classical mechanics [8]. In our case, the eigenvalue equation for a Hamiltonian in QFT implies an expectation-value equation which results from averaging over all numbers, momenta, and other quantum numbers of virtual constituents. Our Ehrenfest equation thus explains in what sense a QFT can be dual to a classical theory and how it may render a holographic picture of hadrons.

Since the expectation value of Eq. (8) involves averaging over the structure and dynamics inside the core and between the active constituent and the core, our Ehrenfest equation describes the motion of the active constituent in the effective potential that describes the net result of all connected interactions in the hadron.

The effective potential is denoted below and in Eq. (2) by $U_{eff}$.

We first focus on the kinetic and mass terms that result from averaging in Eq. (8). These terms help us define the variables that are suitable for writing our Ehrenfest equation in a simple form. Namely, the first term on the right-hand side of Eq. (8) is written as

$$ \left\langle \int k, x \psi_{k,x}^\dagger \left[ \frac{k_1^2}{x(1-x)} + \frac{m_n^2}{x} + \frac{M_{n-1}^2}{1-x} \right] \psi_{k,x}^\dagger \right\rangle \psi_{k,x}^\dagger \psi(k_{\perp}, x) .$$

The brackets $\langle \rangle$ denote integrating over the relative motion of constituents and summing over all sectors that form a hadron. We call the function $\psi(k_{\perp}, x)$ the Ehrenfest function. It depends on three momentum variables, two transverse, $k_{\perp} = (k_{1}, k_{2})$, and one + fraction, $x$, no matter how many constituents of any kind a true hadron contains in any of its Fock components in any QFT.

The same averaging produces all the quantities that appear in our Ehrenfest equation. In the model example of Sec. IV, the averaging will be defined in detail using Eq. (39). Here we begin our description of the averaging by stating that it satisfies the following conditions:

$$ \left\langle \int k, x \psi_{k,x}^\dagger \left[ \frac{k_1^2}{x(1-x)} + \frac{m_n^2}{x} + \frac{M_{n-1}^2}{1-x} \right] \psi_{k,x}^\dagger \right\rangle \psi_{k,x}^\dagger = \kappa^2 , (16a) $$

$$ \left\langle \int k, x \psi_{k,x}^\dagger m_n^2 \psi_{k,x} \right\rangle = m_{active}^2 , (16b) $$

$$ \left\langle \int k, x \psi_{k,x}^\dagger M_{n-1}^2 \psi_{k,x} \right\rangle = m_{core}^2 , (16c) $$

The width $\kappa$ in Eq. (16a) corresponds to the half-width of the Ehrenfest function, whereas the quantities $\kappa_n$ correspond to the half-widths of the Fock-space wave functions $\psi_{k,n}$. Below, we show in Eq. (28) that $\kappa$ is an observable corresponding to the inverse of a hadron size. The hadron size depends on all the Fock sectors, whose wave functions depend on $\lambda$, but the hadron size is an observable measurable in terms of form factors and it does not depend on $\lambda$. Hence, $\kappa$ must not depend on our RGPEP scale $\lambda$. The mass terms in Eqs. (16b) and (16c) will be discussed later.

In addition to the kinetic and mass terms, we have to include the expectation value of all the connected interactions that yield the effective interaction potential between the active constituent and the core. We call this potential the Ehrenfest effective potential, or just the Ehrenfest potential. We write

$$ U_{eff} = \left\langle \text{connected interactions} \right\rangle . (17) $$

This completes our introductory description of how we evaluate the relevant expectation values.
Variation of the Ehrenfest expectation value with respect to $\psi(k^\perp, x)$, keeping the norm of $\psi(k^\perp, x)$ fixed, yields Eq. (2). We now proceed to the description of our averaging procedure in greater detail.

D. Averaging denoted by $\langle \rangle$

First of all, we see no reason for the Ehrenfest function $\psi(k^\perp, x)$ to depend on the RGPEP scale $\lambda$, since its modulus squared turns out to describe the measurable charge distribution in a hadron (e.g., see Sec. III below). As so closely related to observables, the Ehrenfest function may only involve some width parameter $\kappa$ that characterizes the QFT, in which conformal symmetry is naturally broken.

For example, a natural breaking of conformal symmetry occurs in the case of asymptotic freedom, as in QCD, where $\kappa$ must be in a one-to-one correspondence to $\Lambda_{QCD}$. Since the canonical QCD Lagrangian density does not contain $\Lambda_{QCD}$, one needs to consider renormalization in order to see its presence in the FF of QFT, one may introduce $\Lambda_{QCD}$ using the RGPEP). We do not consider renormalization in its full generality below but do address the issue of dependence of effective theories on the RGPEP scale $\lambda$. It is in this dependence where one expects the parameters such as $\kappa$ or $\Lambda_{QCD}$ to appear as reference quantities. These reference parameters allow us to tell the magnitude of $\lambda$ and consider nontrivial functions of $\lambda$.

The averaging over Fock sectors implies that the Ehrenfest function is not equal to the Fock-space wave function in the sector with a smallest admissible number of constituents in a hadron, which we denote by $\psi_k$ with $n = n_{\text{min}}$. The function $\psi_k$ changes with $\lambda$ considerably. In contrast, our Ehrenfest function is independent of $\lambda$.

Regarding the issue of connection with local QFT, we wish to stress that our Ehrenfest function $\psi(k^\perp, x)$ is not identifiable with $\psi_k$ in the canonical theory that is regularized and supplied with a complete set of counterterms. Such theory only forms the initial condition for the RGPEP evolution of effective theory with the scale parameter $\lambda$, at the initial value of $\lambda = \infty$. In such initial-condition theory, the canonical quanta appear in hadrons in great numbers and with great relative momenta. The canonical quanta could also be called the current quanta, in analogy with quarks in the formal current algebra. The canonical quanta could also be thought about as quarks and gluons considered in the perturbative QCD and parton models.

In contrast, our Ehrenfest function is a relatively soft function of $k^\perp$ and $x$ over a dominant range of these variables. It satisfies a simple wave equation with a soft potential. Thus, the Ehrenfest function for a hadron can only be seen as, graphically speaking, the function whose modulus squared provides an averaged parton charge distribution. It does not report on any details of potentially violent interactions of quanta for any large $\lambda$, i.e., much larger than the parameters such as $\kappa$, $\Lambda_{QCD}$, or quark masses in the constituent quark model.

On the other hand, we wish to clarify that when $\lambda$ is comparable with quark masses in the constituent quark model, the effective-particle sector with $n = n_{\text{min}}$ may saturate to a large extent the probability distribution of constituents in a hadron in the effective-particle Fock-space basis. In this case, our Ehrenfest function in the nonrelativistic domain of $k^\perp$ and $x$ may be related to $\psi_k$ in the relatively simple way that involves mainly averaging over motion and numbers of a small set of effective constituent components in a hadron. Thus, one can expect a close resemblance between our Ehrenfest function and the quantum mechanical wave functions in constituent quark models in the domain of nonrelativistic relative motion of an effective quark with respect to other effective constituents, when all these effective particles correspond to a small value of $\lambda$.

We also wish to stress that despite the simplicity of our Ehrenfest function, it is compatible with the hadron form factors being dependent on all Fock components. We shall see below that the Ehrenfest function includes contributions from all sectors and sums them up in the form factor formulas as required by the renormalized theory with an arbitrary value of $\lambda$, for as long as the current operators evolve with $\lambda$ without involvement of significant form factors for the constituents themselves.

The Ehrenfest interpretation of holography suggested in this paper is thus alternative to the one that is based on the valence Fock-space wave function, i.e., actually, the wave function with $n = n_{\text{min}} = 2$ mesons. The valence interpretation was suggested in Refs. [14, 15] using Ref. [16]. The latter work attempts to reduce the hadron eigenvalue problem to some equation for the valence Fock wave function in the canonical QFT. For this purpose, Ref. [16] employs Eq. (2.7) of Ref. [17]. But Eq. (2.7) of Ref. [17] is used there to replace the eigenvalue in a hadron eigenvalue problem by a combination of free energies of arbitrarily moving constituents, when one eliminates Fock sectors trying to obtain an equation for the valence sector alone. Since the canonical parton energies in QFT such as QCD can differ from the hadron eigenvalue by arbitrarily large amounts, it is not clear how the hadron eigenvalue problem could be reduced to its valence component alone in QFT using the approach proposed in Refs. [16, 17].

Considering the alternative interpretation of LF holography proposed in this paper, one also needs to take into account that the valence component is generally not sufficient to evaluate form factors; all Fock sectors contribute. This means that the valence, or just two-parton component of a hadron cannot appear alone in the exact representation of hadron form factors that is claimed valid on the basis of duality arguments [2, 3]. Therefore, our Ehrenfest interpretation of holography and duality through averaging over parton numbers and motion of spectators within the RG-
PEP appears to resolve the conceptual difficulty with the identification of $\psi(k^\pm, x)$ with the Fock-space wave function for $n = n_{\text{min}}$ in QFT.

In its most simple version at small $\lambda$, the need for our alternative interpretation is evident in the case of baryons, where $n_{\text{min}} = 3$ and one has to perform averaging over spectators in the form factor formula in order to obtain an expression that only involves a function of just three variables $k^\pm$ and $x$, instead of the six momentum variables that form the arguments of the valence wave function for constituent quarks in baryons in QCD with small $\lambda$ (see Ref. [12] for the relevant discussion using the RGPEP). Note that the constituent picture at small $\lambda$ incorporates the entire Fock-space composition of a hadron in a canonical theory within the structure of the small-$\lambda$ constituent quarks.

Now we proceed to an explanation of the averaging that renders mass parameters. The quantities $m_{\text{active}}^2$ and $m_{\text{core}}^2$ in Eqs. (16b) and (16c) are the expectation values of the active constituent’s and core’s masses squared, respectively. For example, in QCD, for mesons built from lightest quarks, one may expect $m_{\text{active}} \sim m_{\text{core}} \sim \Lambda_{\text{QCD}}$. This is expected by analogy with the constituent quark model, in accordance with the idea that the constituent quark mass corresponds to $m$ at $\lambda \sim \Lambda_{\text{QCD}}$, at which $\lambda$ the effective quark self-interactions are small. By the same token, in the light baryons $m_{\text{core}}$ is expected to be about twice larger than $m_{\text{active}}$. For heavy quarks, variation of the mass $m$ with $\lambda$ is not as significant as for the light ones and the averaged masses are expected similar to the ones assumed in QCD.

Following [12], we introduce the third component of relative momentum of the active constituent with respect to the core by writing

$$m_{\text{active}}^2 \frac{x}{1-x} + m_{\text{core}}^2 \frac{1}{1-x} = \frac{[m_{\text{core}} x - m_{\text{active}} (1-x)]^2}{x(1-x)} + (m_{\text{active}} + m_{\text{core}})^2,$$

and identifying in the first term on the right-hand side the square of

$$k^3 = m_{\text{core}} x - m_{\text{active}} (1-x).$$

This variable is introduced as an intermediate step in defining $k_x$, $k_y$, and $k_z$ in the Ehrenfest equation by the formula

$$\vec{k} = (k_x, k_y, k_z) = \frac{(k^+, k^\perp)}{\sqrt{x(1-x)}}.$$  

Note that the free invariant mass squared of the active constituent and core on the right-hand side of Eq. (15) can be written in terms of $\vec{k}$ as

$$\frac{k^+ 2}{x(1-x)} + m_{\text{active}}^2 \frac{x}{1-x} + m_{\text{core}}^2 \frac{1}{1-x} = \vec{k}^2 + (m_{\text{active}} + m_{\text{core}})^2.$$  

The key feature of the variable $\vec{k}$ is that a simple rescaling relates it to the Jacobi relative momentum of an effective active constituent with respect to the corresponding effective core in the nonrelativistic domain of their relative motion around the minimum of their potential energy described by $U_{\text{eff}}$ in Eq. (2). The required rescaling factor is the ratio of the reduced mass $\mu = m_{\text{active}} m_{\text{core}} / (m_{\text{active}} + m_{\text{core}})$, to the sum of masses, $m_{\text{active}} + m_{\text{core}}$. Thus, the Jacobi relative momentum of constituents in nonrelativistic two-body models of hadrons is $\sqrt{\beta(1-\beta) \vec{k}}$ where $\beta = \mu / (m_{\text{active}} + m_{\text{core}})$.

We mention the rescaling because in interpreting our result for the relativistic Ehrenfest function, such as in Eqs. (40), (41), or (47) below, the factor $\sqrt{\beta(1-\beta)}$ must be taken into account in order to connect the relativistic FF theory with nonrelativistic models of hadrons for $|\vec{k}|$ smaller than the masses $m_{\text{active}}$ and $m_{\text{core}}$. Physically, the rescaling is required due to the fact that the nonrelativistic expression for energy of a system of two free objects at rest as a whole is $M = m_{\text{active}} + m_{\text{core}} + \vec{k}^2 / (2\mu)$ and leads to $M^2 = (m_{\text{active}} + m_{\text{core}})^2 + \vec{k}^2 / [\beta(1-\beta)]$ with corrections order $\vec{k}^4$ that are neglected in the nonrelativistic models. In contrast, the FF expression for $M^2$ given in Eq. (21) is exact for arbitrary values of $\vec{k}$.

The sum of masses $m_{\text{active}} + m_{\text{core}}$ that appears squared in the second term in Eq. (18), differs from the mass eigenvalue, $M$. Besides the relative motion of the active constituent and core, the difference results also from the potential energy $U_{\text{eff}}$ that is obtained from averaging of the interactions, Eq. (17). The averaging of the interactions is carried out excluding the disconnected self-interactions that we have already included in the masses $m_n$. The effective potential is further described using Eq. (22) below.

Regarding the shape of $U_{\text{eff}}$ as a function of the average distance between the active constituent and the core, one can expect that in the ground states and lowest excited states of hadrons it is quadratic. The quadratic potential is expected as a result of the standard reasoning in which one assumes that every constituent moves in a field produced by many others, such as in the case of a nucleon in a nucleus. The constituents interact with each other and every one of them sees the potential produced by others. One can think about the self-consistent potential that binds every constituent. This potential has a minimum, and around a physically reasonable minimum a potential is most likely quadratic. Precise mathematical reasoning would require modeling of a lot of details as in nuclear physics. Let us consider instead a simple model of a chargeless hadron in which in every Fock component all constituents interact through a Coulomb potential. One can imagine a small parton of charge $Q$ moving in a relatively large, uniformly charged sphere of spectators with total charge $-Q$ and density $\rho$. Then, the
Coulomb potential the parton sees at a distance $r$ from the spectator-sphere center is the charge in the sphere of radius $r$, $Q_r = \rho$ times $4\pi r^3/3$, divided by $r$. Such potential is of the harmonic oscillator type. This picture could correspond to large $\lambda$ in the RGPEP so that the Fock components with many partons are dominant. At small $\lambda$, one may rather think in terms of two large constituents that overlap nearly entirely [12]. If these constituents are each described as Gaussian charge densities of opposite signs and with their centers separated by a distance much smaller than their individual sizes, the potential between them is also of the harmonic oscillator type as long as the opposite-sign charge densities attract each other via any reasonable potential, including the Coulomb one. Thus, at both ends of the RG trajectory in $\lambda$ the oscillator potential appears reasonably realistic if the Coulomb potential is considered realistic. The lack of scale dependence of the final Ehrenfest oscillator must result from summing contributions from all sectors at every scale $\lambda$, leading to the same averaged $U_{\text{eff}}$ no matter what value $\lambda$ has.

Irrespective of the details introduced in the above examples, the bottom-line argument for the harmonic oscillator potential between the active constituent and the core in the ground states and lowest excited states of hadrons is that the potential should describe the constituent motion around a minimum of their potential energy. This principle appears to us sufficient for suggesting the quadratic form for $U_{\text{eff}}$ because we cannot identify any reason for the minimum to be described by the fourth or higher even power of the distance. Hence, $U_{\text{eff}}$ around its minimum should have the form

$$U_{\text{eff}} = -\kappa^4 \left( \frac{\partial}{\partial \vec{k}} \right)^2 - B.$$  \hfill (22)

The spherical symmetry is required because general principles forbid that a relativistic QFT produces an effective equation for a hadron mass squared outside a multiplet representing rotational symmetry in the Minkowski space. Accordingly, the $x$, $y$, and $z$ components of $\vec{k}$ in Eq. (20) combine to a three-dimensional quantity that is capable of supporting a representation of the rotational symmetry in a generally valid eigenvalue formula for hadron masses in a relativistic theory. This happens irrespective of the fact that the FF of dynamics is developed using a specific choice of the lightlike vector that defines the front $x^+ = 0$.

It will become clear below that the parameter $\kappa$ in $U_{\text{eff}}$ must be the same as the $\kappa$ in Eq. (16a). The unknown constant $B$ represents the expectation value of interactions among constituents within the core.

The quadratic Ehrenfest potential appears to be in harmony with the expectation that the potential energy as an ingredient of the hadron mass $M$ in the IF of dynamics increases linearly with a distance between static quarks in the absence of pair creation. Since the Hamiltonian eigenvalue in the FF of dynamics is $M^2$, instead of $M$ of the IF of dynamics, the FF potential should be quadratic if the IF potential is linear [12]. Since the string picture that leads to Regge trajectories and is supported by results of lattice simulations is meant to be valid for large masses, our argument for quadratic $U_{\text{eff}}$ can be considered applicable also to highly excited hadron states for which the dual soft-wall model with a quadratic potential could be relevant. A much less commonly known argument for a quadratic potential is based on the role of conformal symmetry in introducing a scale in a quantum theory [18], especially in the context of LF holography [15, 19]. The theoretical argument suggests through holography that on the gravity side of duality the soft-wall model with quadratic potential deserves attention even if it is not perfect phenomenologically and requires corrections in QCD.

Our observation of correspondence between QFT and the Ehrenfest equation that resembles quantum mechanics of a single particle in a quadratic potential, appears to provide a direct link between the two sides of AdS/QFT duality. The link is the claim that these alternative versions of the theory describe the same motion around a multidimensional potential energy minimum using different variables.

III. THE EHRENFEST EQUATION FOR FORM FACTORS

In this section we calculate a form factor of a hadron, $F(q^2)$, where $q$ denotes the four-momentum transferred to a hadron by an external electroweak or gravitational probe. Our result suggests that the Brodsky-deTéramond holographic density corresponds to the modulus squared of our Ehrenfest function.

We consider the form factor defined in terms of a matrix element of the current $J^+(x = 0)$ with $q^+ = 0$. Namely,

$$\langle \text{Hadron}: P^+, P^0, q^\perp | J^+(0) | \text{Hadron}: P^+, P^\perp \rangle, \hfill (23)$$

where $Q_{\text{Hadron}}$ denotes the relevant charge and $F(0) = 1$. The current $J^+$ is expressed in terms of the fields that correspond to the same scale $\lambda$ with which the Fock-space decomposition is constructed. We arbitrarily simplify the theory by assuming that the effective constituents can be considered pointlike in the entire experimentally accessible range of momentum transfers to a hadron in elastic scattering processes and thus we do not introduce any significant constituent form factors (we set them to 1). The FF form factor formula...
The sector contribution is not indicated below. So, one can choose a value of \( \psi \) such that one obtains the formula

\[
\sum_{i=1}^{n} \chi_i - 1 \delta^{(2)} \left( \sum_{j=1}^{n} \kappa_j - P \right) \times \psi^{(n)}(p_j - x P; \lambda) \psi^{(n)}(p_j - x P; \lambda),
\]

where \( e_j \) is the ratio of charges carried by the active constituent to \( Q_{\text{Hadron}} \), the latter assumed not zero. The momenta of \( n \) constituents in the outgoing hadron of momentum \( P' = P + q \) are denoted by \( p_j = (p_j^+ = 0, q_j, p_j^+, \lambda) \), where \( q_j \) is the momentum of the \( j \)-th constituent. Using the Fourier transforms of the Fock-space wave functions \( \psi^{(n)}(k^+, x; \lambda) \), one obtains the formula

\[
F(q^2) = \sum_n \int [\kappa^+ \chi, \lambda]^{2(2\pi)^3} \delta^{(2)} \left( \sum_{j=1}^{n} \chi_j - 1 \right) \delta^{(2)} \left( \sum_{j=1}^{n} \kappa_j - P \right) \times \psi^{(n)}(k^+, x; \lambda) \psi^{(n)}(k^+, x; \lambda),
\]

where \( k^+ = k^+ + (1 - x) q^+ \) and \( \kappa^+ = (k^+ + \chi_j k^+, j = 1, ..., n) \) are the relative momenta of spectators treated as constituents of the core in their own center-of-mass frame.

Using the Fourier transforms of the Fock-space wave functions \( \psi^{(n)}(k^+, x; \lambda) \) in the variable \( k^+ \), i.e., in the variable that stands in the subscripts, and employing the notation

\[
\int_{\eta, x} = \int \frac{d^2 \eta^+}{4\pi x (1 - x)},
\]

one obtains the formula

\[
F(q^2) = \sum_n \int_{\eta} e^{i(1-x) \eta^+ q^+} \left| \psi^{(n)}(\kappa^+, \chi; \lambda) \right|^2,
\]

where \( \left| \psi^{(n)}(\eta^+, x) \right|^2 \) contributes to the Brodsky-de Téramond probability density. In terms of our Ehrenfest function, the form factor of Eq. (27) reads

\[
F(q^2) = \left\langle \int_{\eta, x} e^{i(1-x) \eta^+ q^+} \left| \psi^{(n)}(\kappa^+, \chi; \lambda) \right|^2 \right\rangle.
\]

where the Fourier transform of the Ehrenfest function \( \psi(k^+, x) \) denoted by \( \tilde{\psi}(\eta^+, x) \)

\[
\tilde{\psi}(\eta^+, x) = \int \frac{d^2 k}{(2\pi)^2} e^{-i\eta^+ k^+} \psi(k^+, x).
\]
son model of an atom where an electron is attracted to the center of a distribution of a positive charge. Another intuitive analogy is provided by the model of a nucleus where a single nucleon moves in a potential created by other nucleons. These old intuitive pictures must now be extended by a new element which is introduced by averaging over the presumably infinitely large collection of effective Fock components, which are defined using the RGPEP and which only together satisfy the FF Hamiltonian eigenvalue equation for the whole hadron.

A. Transverse position variables

Let us focus first on the transverse distribution of constituents because the transverse arguments of the Ehrenfest function play the key role in the form factor formula for \( q^+ = 0 \). Let the absolute transverse position of the \( i \)th constituent be denoted by \( r_i^\perp \). Consequently, the same transverse position of a hadron as a whole in every sector with any number of constituents denoted by \( n \) is described by (cf. Ref. [22])

\[
R^\perp = \sum_{i=1}^{n} x_i r_i^\perp ,
\]

while the position of the center of mass of the spectators accompanying the constituent \( j \) is defined by

\[
R_j^\perp = \sum_{i \neq j} x_i r_i^\perp \sum_{i \neq j} x_i .
\]

We denote by \( \eta_i^\perp \) the relative distance between the \( i \)th constituent and the hadron position,

\[
\eta_i^\perp = r_i^\perp - R^\perp = (1 - x_i)(r_i^\perp - R_i^\perp) .
\]

The latter equality reminds us that \( \eta_i^\perp \) is proportional to the transverse distance between the \( i \)th constituent and the mass center of spectators.

B. Symmetric wave functions

Details of the calculations that follow require a specific model of the basis states in the Fock space and the corresponding hadron wave functions, both of them defined at some scale \( \lambda \). The mathematically simplest model we can imagine is built assuming that all constituents are electrically charged scalar quanta that are bound by strong interactions irrespective of their charges. This means that all the constituents appear identical from the point of view of the strong dynamics that describes their binding and we can neglect differences in charges of the constituents when building their Fock-space wave functions. Thus, we assume that the basis states \( |n : p^\perp, xP^+ ; \lambda \rangle \) in Eq. (1) are of the form

\[
|n : p^\perp, xP^+ ; \lambda \rangle = \frac{1}{\sqrt{n!}} \prod_{i=1}^{n} a_{p_i}^\dagger (\lambda)|0\rangle
\]

and the wave functions \( \psi^{(n)}(p^\perp, x; \lambda) \) are fully symmetric functions of their arguments under permutations of numbers \( i = 1, \ldots, n \).

Certainly, the theory of such charged scalar quanta does not appear useful as far as explaining true hadron observables is concerned. However, it is useful as an example that helps in seeking the rules of approximating theories that do include the necessary other types of quanta. It is only the latter type of utility that we try to exploit by building the example discussed below.

Thus, one can say that the goal of the example is only to illustrate the idea of Ehrenfest correspondence between QFT and its holographic approximation in terms of a plausible sketch, making it as simple as possible but including the difficulty of dealing with a renormalized theory in the FF Fock space.

Let the hadron ground-state wave function in the Fock component of \( n \) effective constituents correspond to motion around a potential energy minimum. We identify the concept of such minimum using the transverse position of a constituent with respect to the center of mass of a hadron \( \eta_i^\perp \) defined in Eq. (32). The issue is then precisely what function of the variable \( \eta_i^\perp \) we should use for modeling the motion of constituents around the relevant minimum. The ambiguity concerns only the coefficient of \( \eta_i^{-2} \) in the exponent of a suitable Gaussian wave function. It will be demonstrated below that the choice corresponding to the Ehrenfest function that matches the Brodsky-deTéramond holography has the form

\[
\tilde{\psi}^{(n)}(\eta^\perp, x; \lambda) = \kappa_n \frac{n}{\sqrt{n!}} A_n(\lambda)
\]

\[
\times \exp \left\{ -\frac{1}{2} \sum_{i=1}^{n} \left[ x_i \eta_i^2 \eta_i^\perp + \frac{m_n^2}{x_i \sqrt{x_i}} \right] \right\}
\]

where \( \tilde{A}_n(\lambda) \) is related to the probability amplitude for finding in a hadron the \( n \) constituents corresponding to the scale \( \lambda \). The parameter \( \kappa_n = \kappa_n(\lambda) \) determines the width of the \( n \)-particle wave function in the variables \( \sqrt{x_i} \eta_i^\perp, i = 1, \ldots, n \). Our demonstration below will provide an explanation of the factor \( \sqrt{x_i} \) on the basis of the argument that the motion of constituents corresponds to the motion around the minimum of a suitably defined potential.

The Fourier-transformed wave function of momentum variables has the form shown in Eq. (5) and contains the
factor \( \psi^{(n)}(k^\perp, x; \lambda) \) given by the formula

\[
\psi^{(n)}(k^\perp, x; \lambda) = \frac{A_n(\lambda)}{\kappa_n^n} \times \exp \left\{ -\frac{1}{2 \kappa_n^n} \sum_{i=1}^{n} \left[ \frac{k_i^\perp + m_i^2}{x_i} \right] \right\},
\]

where the relative momenta \( k^\perp = (k_i^\perp)_{i=1,\ldots,n} \) are defined below Eq. (5) and satisfy constraints in Eq. (6). The coefficient \( A_n \) is related to \( \tilde{A}_n \) by the Fourier transform.

C. Expectation values

In this section we define our averaging procedure that has the properties described by Eqs. (16a)-(16c) and (17). The normalization condition of Eq. (4) for a hadron state given in Eq. (1) in terms of wave functions defined in Eq. (36), yields

\[
1 = \sum_n \int |k^\perp, \chi|^2 (2\pi)^3 \delta \left( \sum_{j=1}^{n-1} \chi_j - 1 \right) \delta^{(2)} \left( \sum_{j=1}^{n-1} \kappa_j^\perp \right) 
\times \int_{k, x} |A_n|^{2} \exp \left\{ -\left[ \frac{k^\perp}{x(1-x)} + \frac{m^2}{x} + \frac{M^2_{n-1}}{1-x} \right] / \kappa_n^2 \right\},
\]

where \( M^2_{n-1} \) is given by Eq. (10).

In normalization Eq. (37), the entire internal structure of the core manifests itself only through the mass squared, which varies from \([n-1]m_n^2 \to \infty \). This great simplification is a property of our symmetric Gaussian example. However, we recall that the motion of any system around a quadratic minimum of potential energy is described by some Gaussian and the lowest excited states are also generally expected to resemble some harmonic oscillator pattern. Therefore, even if we do not know the Gaussian widths \( \kappa_n \) and amplitudes \( A_n \) for all relevant values of \( n \) at any given \( \lambda \), we can still trace the consequences of averaging various quantities in our example and derive the Ehrenfest formulas that are likely to be valid for hadrons in QFT irrespective of the errors introduced by our minimal, Gaussian approximation.

Knowing that the spectator dependence in Eq. (37) is reduced to dependence on \( M_{n-1} \), we can replace the integration over all variables \( k^\perp \) and \( \chi \) for spectators in every component by an integral over a single variable \( M^2 \) with the phase space density \( \rho_n \) defined by

\[
\rho_n(M^2) = \int |k^\perp, \chi|^2 \delta \left[ M^2 - M^2_{n-1}(k^\perp, \chi) \right] 
\times 2(2\pi)^3 \delta \left( \sum_{j=1}^{n-1} \chi_j - 1 \right) \delta^{(2)} \left( \sum_{j=1}^{n-1} \kappa_j^\perp \right),
\]

The hadron expectation value of any one-particle quantity \( X_n \), i.e., any quantity \( X_n \) that in all the Fock components depends only on \( k^\perp \), \( x \), and \( M_{n-1} \), is given by the formula

\[
\left\langle \int_{k, x} \psi^{(n)}_{k, x} X_n \psi^{(n)}_{k, x} \right\rangle = \sum_n \int dM^2 \rho_n(M^2) \int_{k, x} X_n(k^\perp, x, M) 
\times |A_n|^2 \exp \left\{ -\left[ \frac{k^\perp}{x(1-x)} + \frac{m^2}{x} + \frac{M^2_{n-1}}{1-x} \right] / \kappa_n^2 \right\},
\]

D. The Ehrenfest function

Using Eq. (39) as a definition for the expectation values in Eqs. (16a)-(16c) and (17), one arrives at Eq. (15). Using the concept of motion around a minimum, one considers the Ehrenfest potential given in Eq. (22). Subsequently, variation of the hadron expectation value of a FF Hamiltonian for any RGPEP parameter \( \lambda \), yields the Ehrenfest Eq. (2) with the oscillator potential. Its ground-state solution is

\[
\psi(k^\perp, x) = \mathcal{N} e^{-\vec{k}^2/(2 \kappa^2)},
\]

where the variable \( \vec{k} \) is defined in Eq. (20) and the constant \( \mathcal{N} \) denotes the normalization factor that can be calculated, for example, from the normalization of a hadron charge distribution.

An alternative appearing fully relativistic form of the same Ehrenfest function derived using Eq. (21) is

\[
\psi(\vec{k}) = \mathcal{N} e^{-(s_0-s_n)/(2 \kappa^2)},
\]

where the \( s \)-channel Mandelstam invariants are defined by

\[
s_0 = (m_{\text{active}} + m_{\text{core}})^2, \quad s_k = (p_{\text{active}} + p_{\text{core}})^2 = \frac{k^\perp + m_{\text{active}}^2}{x} + \frac{k^\perp + m_{\text{core}}^2}{1-x}.
\]

This result means that the Ehrenfest active constituent and core can be considered on-mass-shell particles with masses \( m_{\text{active}} \) and \( m_{\text{core}} \), while the Ehrenfest function depends only on their total invariant mass squared.

One should remember at this point that the FF of Hamiltonian dynamics requires that the difference of the four-momenta of a hadron with mass squared eigenvalue \( M^2 \) and its Ehrenfest constituents is lightlike,

\[
p^{\mu}_{\text{active}} + p^{\mu}_{\text{core}} = P^{\mu} + \frac{s_k - M^2}{P^{\nu}} n^{\nu},
\]

where the null-vector \( n \) defines the front \( x^+ = 0 \) by the condition \( n x = 0 \) and its only nonzero component
is \( n^- = 2 \). On the other hand, Eq. (45) shows that in the IMF, where the parton model is defined in the limit \( P^+ \to \infty \), the Ehrenfest active constituent has the interpretation of the Feynman parton [23]. The active constituent carries the fraction \( x \) and the core carries the fraction \( 1 - x \) of the hadron momentum. The limit \( P^+ \to \infty \) for the Ehrenfest constituents exists because the averaged effective dynamics does not allow \( s_k \) to take large values, see Eqs. (40) and (41). In particular, the dominant part of the transverse motion of the Ehrenfest partons is soft, since it is limited by the scale \( \lambda_{QFT} \). Only the large-\( k^\perp \) tail of the transverse distribution may be sensitive to the hard processes driven by the underlying local theory.

The Ehrenfest function interpretation discussed above closely resembles the two-body quantum wave function interpretation of LF holography in the case \( m_{\text{active}} = m_{\text{core}} = 0 \) [5]. However, the phenomenologically useful holographic wave functions, for example, see Refs. [24–26], often operate with a generalization to constituents with nonzero masses. The Ehrenfest function provides a natural explanation of the presence of masses in the parton densities in the forms advocated on phenomenological grounds. The masses enter through the dynamics in the \( z \) direction, which complements the transverse dynamics in our model in agreement with the requirement of rotational symmetry as a part of the Poincaré symmetry in QFTs and their duals.

V. CONCLUSION

When the renormalized Schrödinger Hamiltonian eigenvalue equation for a “hadron” state in QFT in the FF of dynamics is reduced to its expectation value in any of the eigenstates corresponding to smallest masses for a fixed set of other quantum numbers, one obtains the Ehrenfest-like equation that describes a semiclassical function \( \psi(\vec{k}) \), or its Fourier transform \( \tilde{\psi}(\vec{q}) \) that describes the charge density in a hadron according to the formula

\[
\rho(\vec{q}) = Q_{\text{Hadron}} |\tilde{\psi}(\vec{q})|^2. \tag{46}
\]

This interpretation follows from the Ehrenfest formula for hadron form factors.

The Ehrenfest form factor formula matches the Brodsky-deTéramond LF holography formula for the form factors. Thus, the principle of correspondence between quantum and classical dynamics discovered by Ehrenfest [8], appears to provide a link between the quantum field theoretic Fock-space picture of ground states or slightly excited states of hadrons and the semiclassical gravitational picture suggested for them by the AdS/QFT duality [1–3]. While the duality based on some \( M^* \) or string theory requires investigation, the Ehrenfest correspondence between quantum and classical theories is already well established in other areas of physics in the low-energy domain and should be falsifiable in particle physics in the high-energy domain.

If the Ehrenfest function is expected to capture in its shape the parton distributions for all choices of the momentum transfer from an external probe to an active constituent, the function must fall off at large momenta in a way that is sensitive to the Fock components with constituents of large virtuality, while for small \( \vec{k} \) it may still have the shape resembling Gaussian. In particular, for small momenta \( \vec{k} \) one may expect resemblance of \( \psi(\vec{k}) \) to the constituent quark model wave functions (quark-antiquark for mesons and quark-diquark in baryons), while for the large \( \vec{k} \) one must expect the corrections to a Gaussian shape that reflect the influence of high-energy FF Fock-space wave functions on the Ehrenfest expectation value. The high-energy behavior of wave functions can be studied using perturbative methods [27].

It should also be remembered, as mentioned in Sec. II D, that the observables such as elastic form factors result from coupling of the external probes to the effective constituents in a way that depends on the momentum scale of external probes. Only in the approximation that the current operators of effective constituents are independent of the momentum transfer from external probes, as we assumed for simplification in this paper, one can ignore the corrections that are not of the Gaussian type. This also means that one should expect corrections to the soft-wall models with quadratic potential in the bulk on the gravity side of duality concerning QCD.

The Ehrenfest description of hadrons in Gaussian approximation and LF holography may be explicitly related to each other using the resemblance between the three-dimensional Ehrenfest equation with harmonic potential and the Brodsky-deTéramond two-dimensional eigenvalue equation with a transverse harmonic oscillator potential associated with a corresponding AdS dilaton warping. Separating variables in Eq. (2), one obtains solutions in the form

\[
\psi(\vec{k}) = \phi(k_x, k_y) H_{n_z} \left( \frac{k_z}{\chi} \right) e^{-k_z^2/2\chi^2}, \tag{47}
\]

where \( H_{n_z} \) denotes a Hermite polynomial. Assuming that the transverse motion corresponds to the angular momentum projection on the \( z \) axis equal \( l_z \), one arrives at

\[
\left[ -\left( \frac{\partial}{\partial \zeta} \right)^2 - \frac{1}{\zeta} \frac{\partial}{\partial \zeta} + \frac{1}{\zeta^2} l_z^2 + (2n_z + 1) \chi^2 + \chi^2 \zeta^2 \right] \tilde{\phi}(\zeta) = M^2 \tilde{\phi}(\zeta). \tag{48}
\]

The function \( \tilde{\phi}(\zeta) \) is the radial factor in the Fourier transform of \( \phi(k_x, k_y) \) and \( \zeta = |\zeta^\perp| \) is the length of
the Brodsky-deTéramond holography transverse position variable understood as

\[
\zeta^+ = \sqrt{x(1-x)} \left( r_{\text{active}}^+ - r_{\text{core}}^+ \right),
\]

where variables \( r_{\text{active}} \) and \( r_{\text{core}} \) denote the transverse positions of the parton and core on the front. Thus, \( \zeta^+ \) is the relative transverse position of the active parton and hadron core rescaled by \( \sqrt{x(1-x)} \), where \( x \) is the hadron-momentum fraction carried by the active parton in the IMF.

Equation (48) precisely matches the holography equations, e.g., Eq. (11) in Ref. [6] or Eq. (33) in Ref. [14], after rescaling \( \hat{\phi} (\zeta) \) by \( \sqrt{\zeta} \) and adjusting the additive constant \( (m_{\text{active}} + m_{\text{core}})^2 - B \).

Unfortunately, our simple consideration does not uniquely predict the value of the constant \( B \) for various hadrons and instead merely indicates a need for its existence. Since the constant \( \propto \) in the Ehrenfest \( U_{\text{eff}} \) may be thought of as related to the gluon condensate inside hadrons [12], one may expect that an explanation of the constant \( B \) in \( U_{\text{eff}} \) also requires understanding of the dynamics that involves distances comparable with the size of a hadron.

Explanation of the Ehrenfest harmonic potential in a full theory must account for the interaction between the active constituent and core including a form factor of the core, say \( f(q^2) \), where \( \vec{q} = \vec{k}' - \vec{k} \) is the momentum transfer between the active constituent and the core. The core form factor is to describe the core structure. The validity of this picture is expected by analogy with approximations such as the Hartree-Fock approximation to many-body dynamics. However, in the case of QFT the situation is greatly complicated due to the interactions that are capable of violent changes of energies and extensive mixing of various numbers of virtual constituents. Nevertheless, for sufficiently big numbers of effective constituents, which means sufficiently large \( \lambda \), the core is similar in its strong-interaction charge or inertia distribution to the hadron itself. So, in the first approximation,

\[
f(q^2) \sim F(q^2),
\]

where \( F(q^2) \) is the measurable hadron form factor and the proportionality refers to the proper charge coefficient. Although \( F(q^2) \) does not describe the distribution of neutral constituents, such as gluons in the case of electromagnetic form factors of a proton, while \( f(q^2) \) describes the density of all strongly interacting constituents that form the core, which includes the gluons, the average distributions are likely to be of essentially similar shape. Establishing that they are not would constitute a new insight into the structure of hadrons.

Assuming a considerable range of validity of the Gaussian approximation, the main limitation in the accuracy of representing the QFT theory solutions with the Ehrenfest function stems from the fact that the averaging involved in evaluating expectation values misses the interference effects that can be fully described only using the Fock-space wave functions. On the other hand, it can be checked to what extent the full Ehrenfest function may describe the parton distributions in more detail than may its modulus squared alone. If it did, it would offer a semiclassical single-parton picture that would stand a chance of approximating the physics of hadrons a bit more accurately than the entirely probabilistic description of the parton-model type, or of the type of quantum mechanical models that are not directly related to QFT.

We hasten to conclude that the duality of an AdS-like picture to QFT can be interpreted in terms of the Ehrenfest function. The behavior of the Ehrenfest potential could thus correspond to the behavior of potentials in duality models such as in Ref. [9]. Needless to say, similar considerations should apply in principle to the discussion of many QFTs.

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