A conserved parity operator

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Abstract. The symmetry of Nature under a Space Inversion is described by a Parity operator. Contrary to popular belief, the Parity operator is not unique. The choice of the Parity operator requires several arbitrary decisions to be made. It is shown that alternative, equally plausible, choices leads to the definition of a Parity operator that is conserved by the Weak Interactions. The operator commonly known as CP is a more appropriate choice for a Parity operator.

Pacs 11.30.Er, Charge conjugation, parity, time reversal, and other discrete symmetries
1. Introduction

All the classical laws of Nature are symmetric under a Parity operation. However, in particle physics the weak interactions seemed to show some asymmetry between left and right. The experiments, and the theory that described the interactions, were taken as clear evidence of an inexplicable asymmetry in Nature. The ramifications of the experiments extend far beyond the weak interactions and particle physics; they color our view of the Universe and are widely referred to in textbooks that discuss the nature of space and time.

This paper shows that parity violation, is simply a consequence of the arbitrary assumptions that are made in the choice of Parity operator.

Symmetry under space inversion is described by a Parity operator $P$. Under space inversion we have by definition:

$$
\begin{align*}
  x &\rightarrow -x \\
  y &\rightarrow -y \\
  z &\rightarrow -z \\
  t &\rightarrow t
\end{align*}
$$

(1)

It follows from the definition of $P$ that velocity and acceleration vectors also change sign. Given a mathematical object defined in terms of co-ordinates and derivatives, the transformation properties can be determined (eg angular velocity $r \times \dot{r}$ is unchanged). However this is not sufficient to define the parity operator because we do not know the transformation properties of other entities that appear in the equations of Physics. In most cases there is a choice of mathematical representations and the most obvious choice is not necessarily the most appropriate.

2. Electrodynamics

Electrodynamics offers a neat illustration of the choices that need to be made when defining a Parity operation - and the importance of making the best choice. Choosing a parity operator is equivalent to choosing mathematical representations for the physical quantities that appear in the equations.

Maxwell’s equations have an Electric Field, $E$, and a Magnetic field, $B$. A simple mathematical representation would be to treat them both as vectors. Both $E$ and $B$ would then change sign under a parity operation. Maxwell’s equation would not be invariant under this parity operation. The Lorentz equation:

$$
\ddot{r} = \frac{q(E + \dot{r} \times B)}{m}
$$

(2)

Would be composed of a parity violating term $q\dot{r} \times B/m$ and a parity conserving term $qE$.

Electrodynamics would be inherently lefthanded. A symmetrical experimental arrangement such as an electron beam moving forward in the $x$-direction and a magnetic
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field pointing in the z-direction would result in electrons moving to the left. The mirror image would be contrary to experimental observations.

The predictions of electromagnetism would be unchanged by the choice of parity operator. Since a parity operation cannot be performed in the laboratory, there could be no direct test of the transformation properties of \( E \) and \( B \). The most that experiment could do is confirm the two different parts of the Lorentz equation. Particle motions would still be correctly described regardless of the transformation properties (under parity) of \( E \) and \( B \).

Although the choice of operator is of no experimental significance, it is certainly of philosophical interest for it describes, an asymmetric Universe. But the choice of operator is far more significant than just philosophical. The choice of operator clarifies (or confuses) a theoretical understanding of the Physics, which is vital for progress. It is particularly important when relating electrodynamics to relativity.

Either \( E \) or \( B \), or both, can be axial vectors that do not change sign under a parity operation. If they are both axial vectors then electrodynamics does not conserve Parity. But if only \( B \) is an axial vector then the parity operator transforms \( E \) to \(-E\) and leaves \( B \) unchanged. With this definition Electrodynamics conserves parity.

The different transformation properties of the Electric and Magnetic fields seems puzzling and arbitrary. However if Maxwell’s equations are described using the Faraday two-form (antisymmetric second rank tensor) the Magnetic field is seen as the space-space components while the Electric field is the space-time components of the same mathematical entity.

\[
F = \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & B_z & -B_y \\
E_y & -B_z & 0 & B_y \\
E_z & B_y & -B_x & 0
\end{pmatrix}
\] (3)

The dual of the Faraday two form is the Maxwell two form defined by \( M = \ast F \). The dual interchanges the position of the electric and magnetic fields. The source-free Maxwell’s equations can then be written in the most elegant form \( dF = 0 \) and \( dM = 0 \).

The different nature of the electric and magnetic fields is also evident if they are described using a four-vector potential \( A_\mu \) with:

\[
E = -\nabla A_t - \partial_t (A_x, A_y, A_z) \tag{4}
\]
\[
B = \nabla \times (A_x, A_y, A_z) \tag{5}
\]

From which it is clear that \( E \) is a vector and \( B \) is an axial vector.

The definition of the Parity operator assumed that mass, \( m \), in the Lorentz equation (2) was unchanged. The charge \( q \) is unchanged but this is not a separate assumption because the charge, \( q \), in a volume can be defined from the surface integral of electric field evaluated over a spherical surface that encloses the volume:

\[
P : \quad q = \oint E \cdot \hat{n} \, dS \rightarrow \oint (-E) \cdot (-\hat{n}) \, dS = q \tag{6}
\]
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A similar definition of magnetic charge $Q_m$ is:

$$ P: Q_m = \oint B \hat{n} \, dS \to \oint B(-\hat{n}) \, dS = -Q_m $$  \hspace{1cm} (7)

From which it is clear that a Parity operation will change the sign of a magnetic charge. Under a reflection a North pole changes to a South pole and vice versa. The intimate relation between charges and fields is such that the Parity operator can equally well be defined through the effect on the charges (magnetic and electric). Conventionally electric charge is represented as a scalar quantity and the magnetic charge as pseudoscalar.

It is convention to choose the magnetic field to be an axial vector, but it is well known [11, 3] that the equations of electrodynamics are invariant under a duality rotation that interchanges the role of the $E$ and $B$ fields. With this unconventional representation, Parity is still conserved because the sign of the electric charge also changes:

$$ P: \dot{\mathbf{r}} = \frac{q(E + \dot{\mathbf{r}} \times B)}{m} \to \left(-q\right)(E - \dot{\mathbf{r}} \times -B)/m = -\dot{\mathbf{r}} $$ \hspace{1cm} (8)

Classical electrodynamics cannot determine the transformation of the physical quantities that appear in the equations. Under a parity operation four possible options exist corresponding to scalar or pseudoscalar character for the electric and/or magnetic charges. Two options conserve parity and two violate parity. For aesthetic reasons we may favor the options that conserve parity. However the strongest reason for parity conservation in electromagnetism is that it is a consequence of a unified mathematical representation of the fields (3) - that gives two possible options both of which conserve parity.

3. A conserved Parity operator in Weak Interactions

Returning to particle physics. The definition of a Parity transformation has to include the transformations of mass, $m$, Lepton numbers, $L_i$, Baryon number, $B$, Strangeness, $s$, charm, $c$, top, $t$, and bottom, $b$. The origin of all these quantum numbers is unknown and hence the appropriate transformation properties can only be guessed at. The convention is to define an operator which has no effect on all these properties. Mathematically, these quantum numbers are represented by scalars. The result is Parity violation in the Weak Interactions and the puzzle of an asymmetric Universe.

Text books that introduce the Parity operator treat it as a uniquely defined operator without mentioning the arbitrary choices that have been made (see for example [2, 5]). The experimental evidence requires a more careful interpretation than is received in most texts. Ballentine [4] is one of the few authors who notes that the inference of parity violation from experimental evidence requires additional non-trivial assumptions about the symmetry of the initial state - assumptions which he justifies.

An alternative definition of the Parity operator is to have $m$ unchanged as before but all the other quantum numbers, $L_i, B, s, c, t, b$ negated (mathematically they are pseudo-scalars) and $q$ is also negated. Note that this definition requires the electric rather than
the magnetic field to be represented by an axial vector, which is unconventional but consistent with electrodynamics as described above. Within the Weinberg and Salam model of the weak interaction, such an operator is well-known. It is commonly called \( CP \). It is conserved by the weak and strong interactions. If this is recognized as being the appropriate operator for space inversion then the mystery of left-right asymmetry in Weak Interactions is removed.

The requirement for Parity conservation in the model of the weak interactions also leads to a unique parity conserving operator for the electrodynamics. The Electric and magnetic fields must have transformation properties opposite to the conventional assignments so that magnetic fields and not electric fields change sign under the Parity operator.

Without a deeper model of the elementary particles and their quantum numbers, the assignment of their transformation properties is only a matter of aesthetics. However the motivation for this analysis came from just such models. The author is working on geometric models of elementary particles and quantum theory using classical general relativity \[6,7\]. Some exciting results have been achieved (see for example \[8\]), but a clear prediction of such models is that parity is conserved. General relativity conserves parity. For every structure that displays handedness the opposite structure is also a valid solution. General relativity combined with symmetric boundary conditions must lead to a parity conserving model of elementary particles. Far from being a contradiction with experiment the analysis throws new light on the definition of parity and challenges the conventions that are universally and uncritically adopted.

4. Conclusion

The operator corresponding to space inversion is not unique. Making different, but equally valid assumptions leads to a Parity operator conserved in the weak and electromagnetic interactions. It is the operator normally denoted \( CP \).

When making calculations using the weak interactions the new identification for the Parity operator is irrelevant and inconsequential. But the Parity operator and the supposed violation of Parity are used far more widely (\[9,10\] are examples). The correct choice of operator is vital to our understanding of Nature. Countless textbooks published each year make statements about the asymmetry of Nature based on the assumption that the conventional parity operator is the unique representation of a space inversion. Philosophers are left puzzling over the meaning of the asymmetry and theoreticians looking for a unifying theory and have the doubly difficult task of creating an asymmetric theory and explaining why the mirror image theory is not seen.

Accepting that a few quantum numbers can be pseudo-scalars rather than scalars is sufficient to restore symmetry.

The violation of \( CP \) in the neutral Kaon system remains perplexing. But with the new definition of Parity, it is clear that the Universe itself has left-right asymmetry due to the predominance of positive Lepton and Baryon number. While such a fact is a long
way short of being an explanation, the observation of an asymmetry in the presence of asymmetric boundary conditions is rather less surprising than Parity violation in a symmetric Universe.

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