A note on the potentials of probabilistic and fuzzy logic

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Abstract—This paper mainly focuses on (1) a generalized treatment of fuzzy sets of type \( n \), where \( n \) is an integer larger than or equal to 1, with an example, mathematical discussions, and real-life interpretation of the given mathematical concepts; (2) the potentials and links between fuzzy logic and probability logic that have not been discussed in one document in literature; (3) representation of real-life random and fuzzy uncertainties and ambiguities that arise in data-driven real-life problems, due to uncertain mathematical and vague verbal terms in datasets.

Index Terms—Type-\( n \) fuzzy sets; fuzzy logic; probability theory.

I. INTRODUCTION

In this paper an extensive treatment of type-2 fuzzy sets and membership functions is presented that is more general than the ones that can be found in literature. Moreover, the potentials of fuzzy logic and probability theory are discussed, as well as how real-life random and fuzzy uncertainties may be represented mathematically. The paper is organized as follows: Section II provides an extensive discussion about fuzzy logic and the potential of this logic in representing linguistic uncertainties. Section III discusses the probability theory. Section IV represents various formulations of random and fuzzy uncertainties.

II. FUZZY LOGIC

Two main concepts that mathematical logic deals with are sets and propositions [1]. Fuzzy logic was introduced by Zadeh in the 1960s [2], [3] to extend the concepts of set theory and propositional calculus, which by then were analyzed only through classical logic. Fuzzy logic is a continuous multi-valued logic system. Fuzzy logic may be considered as a generalization to the classical logic. Linguistic variable, a concept unique to fuzzy logic, allows this logic to serve as a basis for computations based on verbal information [4]–[6]. Moreover, inspired by the procedure of reasoning of humans upon imprecise information, FL maps an imprecise concept into one with a higher precision [7].

A set in classical logic is a crisp concept: a mathematical object (e.g., a number, partition, matrix, variable, ...) either “belongs to” the set, i.e., the degree of membership of the mathematical object to the set is 1, or “does not belong to” the set, i.e., the degree of membership of the mathematical object to the set is 0. Therefore, a crisp set \( C \) may be expressed as a collection of mathematical objects, e.g.,

\[
C=\{ x_1, x_2, \ldots, x_n \}.
\]

Similarly, a proposition in classical logic is either “true” (may be quantified by a crisp value 1) or “false” (may be quantified by a crisp value 0).

In fuzzy logic, sets are fuzzy concepts [8]. Our main focus is on the general case of type-\( n \) fuzzy sets, with \( n=1,2,3,\ldots \). To motivate this, we start with an example.

Opening example

1) Suppose that based on the information “Felix is 27”, we would like to answer to the question: (To what extent) is Felix young? Since young is a qualitative concept, while 27 is quantitative, these concepts should be bridged by quantifying young and corresponding the quantitative information about Felix’s age with the quantified definition of young.

One may define young with the graph represented in Figure 1, which corresponds to a definition that categorizes people in two crisp sets, young and not young, i.e., people at an age below 40 belong to the set young and otherwise they are not young. Then, in quantitative terms, “Felix belongs to the set young with a membership degree of 1”, and in qualitative terms, “Felix is certainly young”.

Alternatively, young may be quantified via the graph shown in Figure 2, where instead of certainly young and certainly old, ages vary in a spectrum, i.e., the degree of membership to the set young varies in \([0,1]\) instead of \([0,1]\). Then, quantitatively, “Felix belongs to the set young with a membership degree of 0.9”. Qualitatively, “Felix is to a high extent young”. Young is then quantified using a type-1 fuzzy set.

Next, suppose that the border of the curve that defines young is not strictly known. Figure 3 is an example where the intensity of the color black indicates our certainty about where any point at the border of the separating curve may lie in the given 2-dimensional plane. By just looking at the vertical axis (which is called the primary membership degree), one may reply “Felix belongs to the set young with a primary membership degree in the interval [0.57,0.98]”, or “Felix may to a high extent be young”. Now consider a third dimension that quantifies the intensity of the color black with real values between 0 and 1, i.e., black corresponds to 1 and white corresponds to 0 (see figure 4). This dimension represents the secondary membership degree. Then one can respond “Felix is young with a primary membership degree varying in the interval [0.57,0.98], and a secondary membership degree varying in the interval [0,1]. For instance, “Felix’s age corresponds to a secondary membership degree of 0.83 for the primary membership degree of 0.88”. A type-2 fuzzy set (see [9]–[11] for more details on this type of fuzzy sets) has been used to quantify young for this case.

More generally, if one interprets the word young with a curve with blurry borders that extend in 3 (instead of 2)
dimensions, i.e., the graph of young can be re-illustrated in 4 dimensions considering primary, secondary, and tertiary membership degrees, the qualitative word young has actually been quantified using a type-3 fuzzy set. This concept can further be extended to type-n fuzzy sets for which the corresponding membership functions can be illustrated in n+1 dimensions, and there are n membership degrees involved.

**Mathematical Discussion**

Next, the concept of type-n fuzzy sets will be detailed mathematically. In contrast to a crisp set, a mathematical object belongs to any type-n fuzzy set, with n=1,2,..., with a primary, secondary, ... membership degree varying in [0,1]. Similarly, in fuzzy logic a proposition may “to a certain extent” be true or false. To link the two theories of fuzzy propositions and fuzzy sets, one may consider two fuzzy sets, the “set of true propositions” and the “set of false propositions”. Any new proposition, which corresponds to one or several mathematical variables called membership degrees, can belong to both these sets at the same time with certain degrees of membership.

Generally, a membership function \( f^n: \text{dom}(f^n) \rightarrow [0,1] \) together with its domain, \( \text{dom}(f^n) \), which itself can be a fuzzy set, characterizes a fuzzy set. For a type-1 fuzzy set, the corresponding type-1 membership function \( \mu_i \) generates the degree of membership of mathematical objects \( x_i \) within \( C \) to the fuzzy set \( F \), i.e.,

\[
\mu_i: x_1, x_2, ..., x_n \rightarrow [0,1]: x_i \mapsto \mu_i^n
\]

or equivalently,

\[
\mu_i^n = f_i^n(x_i), \quad i=1,2,...,n.
\]

In comparison with a crisp set, a type-1 fuzzy set \( F \) is composed of pairs of mathematical objects and their degrees of membership to \( F \) (see (2)).

**Remark 1:** One may generalize the above discussion by re-defining the crisp set \( C \) in (1) as a special case of a type-1 fuzzy set, where \( \mu_i^n = 1 \), i.e., \( i=1,2,...,n \). This means that the corresponding type-1 membership function of \( C \) is the unit function. This definition may even further be generalized (see the following discussions for more details), i.e., a crisp set may be re-defined as a type-n fuzzy set, with n=1,2,..., with all the membership functions of order 1,...,n the unit function.

For data-driven approaches in engineering problems, large amount of information that are expressed in the human language, may be used to generate fuzzy sets. The degree to which a fuzzy set can handle the uncertainties and vagueness that exist in this information, depends on the type of the fuzzy set. The main difference between type-1 fuzzy sets and fuzzy sets of type 2 and larger is in the domain of their corresponding membership functions, i.e., the domain of a type-1 membership function is a crisp set \( C \) (see (2)), or equivalently, a type-1 fuzzy set with the unit function as its membership function, while the domain of a type-n fuzzy set with \( n=2,3,... \) is a union of various fuzzy sets of type \( n-1 \).

More specifically, a type-2 fuzzy set \( F^{[2]} \),

\[
F^{[2]} = \left\{ \left( x_1, \mu_{i_1,1} \right), \left( x_2, \mu_{i_2,1} \right), ..., \left( x_n, \mu_{i_n,1} \right) \right\}
\]

\( \left\{ \left( x_1, \mu_{i_1,2} \right), \left( x_2, \mu_{i_2,2} \right), ..., \left( x_n, \mu_{i_n,2} \right) \right\}
\]

\( \left\{ \left( x_1, \mu_{i_1,n} \right), \left( x_2, \mu_{i_2,n} \right), ..., \left( x_n, \mu_{i_n,n} \right) \right\}
\]

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\]

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\]

\( \left\{ \left( x_1, \mu_{i_1,n} \right), \left( x_2, \mu_{i_2,n} \right), ..., \left( x_n, \mu_{i_n,n} \right) \right\}
\]

corresponds to a type-2 membership function \( f^{[2]} \), which generates the secondary degree of membership of any math-
emathematical object \( (x_i, \mu_{i,j}) \) within the type-1 fuzzy set \( C \times \mathcal{R} \) to the type-2 fuzzy set \( \mathcal{F}^2 \), where \( \mathcal{R} \subseteq [0,1] \), and \( i=1,\ldots,n \), \( j=1,\ldots,m_i \). Therefore,

\[
\mathcal{F}^2 := \left\{ (x_1, \mu_{1,1}), (x_2, \mu_{2,1}), \ldots, (x_n, \mu_{n,1}) \right\} 
\]

... 

\[
\left\{ (x_n, \mu_{n,1}), (x_n, \mu_{n,2}), \ldots, (x_n, \mu_{n,m_n}) \right\} \rightarrow [0,1]:
\]

\[
\left\{ (x_i, \mu_{i,j}) \right\} \rightarrow \mu_{2,j},
\]

or equivalently,

\[
\mu_{2,j} = \mu_{2,j}(x_i), \quad i=1,\ldots,n, \quad j=1,\ldots,m_i.
\]

We can reformulate the domain of \( \mathcal{F}^2 \) as:

\[
\text{dom} \left( \mathcal{F}^2 \right) := \left\{ x_1, x_2, \ldots, x_n, \mu_{1,1}, \ldots, \mu_{n,m_n} \right\} \cup \left\{ \ldots, \right\} \cup \left\{ \ldots, \right\} 
\]

\[
\left\{ (x_1, \mu_{2,1}), (x_2, \mu_{2,1}), \ldots, (x_n, \mu_{2,1}) \right\} \cup \left\{ (x_1, \mu_{2,2}), (x_2, \mu_{2,2}), \ldots, (x_n, \mu_{2,2}) \right\} \cup 
\]

\[
\vdots 
\]

\[
\vdots 
\]

\[
\left\{ (x_1, \mu_{m_1,1}), (x_2, \mu_{m_1,1}), \ldots, (x_n, \mu_{m_1,1}) \right\} \cup 
\]

\[
\vdots 
\]

\[
\left\{ (x_{k,m_{2}},1), \ldots, (x_{k,m_{2}},m_n) \right\} = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \left\{ (x_i, \mu_{j,i}) \right\},
\]

assuming that

\[
\arg\max_{i} \{ m_i \} = k.
\]

This shows the domain of the type-2 membership function \( \mathcal{F}^2 \) is the union of \( m_k \) type-1 fuzzy sets. Therefore, a type-2 fuzzy set also corresponds to \( m_k \) type-1 membership functions \( \mu_{j,i} \) that generate the primary membership degrees \( \mu_{j,i} \), i.e.,

\[
\mu_{j,i} = \mu_{j,i}(x_i), \quad i=1,\ldots,n, \quad j=1,\ldots,m_i.
\]

The domains of these type-1 membership functions are \( C \) or a subset of \( C \).

In summary, to identify a type-2 fuzzy set \( \mathcal{F}^2 \) on a domain \( C \) of the independent variables \( x_i \), one needs to know the corresponding type-2 membership function \( \mathcal{F}^2 \) and the domain of this function. Equivalently, one should know \( \mathcal{F}^2 \) and the \( m_k \) type-1 membership functions \( \mu_{j,i} \). From (5), a type-2 fuzzy set \( \mathcal{F}^2 \) is composed of pairs, where each pair itself includes a pair consisting of the independent variable \( x_i \) and a primary membership degree \( \mu_{j,i} \) of \( x_i \), and a secondary membership degree.

Generally speaking, a type-\( n \) fuzzy set \( \mathcal{F}^n \) includes mathematical objects of the form:

\[
\left\{ (x_1, \mu_{1,j_1}), (x_2, \mu_{2,j_1}), \ldots, (x_n, \mu_{n,j_1}) \right\}
\]

with \( j_1 \in \{ 1,\ldots,m_{i_1} \} \), \ldots, \( j_{n-1} \in \{ 1,\ldots,m_{i_{n-1}} \} \), and corresponds to a type-\( n \) membership function \( \mathcal{F}^n \), which generates the \( n^{th} \) membership degrees \( \mu_{j_{n-1},\ldots,j_{1},i} \) of any mathematical object that belongs to the union of \( m_{k_0} \) fuzzy sets of type \( n-1 \), with

\[
k_n = \arg\max_{i} \{ m_{i,n-1} \}.
\]

A type-\( n \) fuzzy set corresponds to \( \arg\max_{i} \{ m_{i,n-1} \} \) membership functions of type \( n-1 \), to \( \arg\max_{i} \{ m_{i,n-2} \} \) membership functions of type \( n-2 \), \ldots, and to \( \arg\max_{i} \{ m_{i,1} \} \) type-1 membership functions. One should know all these membership functions, as well as the unique type-\( n \) membership function to identify \( \mathcal{F}^n \).

**Interpretation for real-life problems**

Based on the discussions given above, the following conclusions can be made. For an independent variable \( x_i \in C \), the *extent* that it belongs to a type-1 fuzzy set \( \mathcal{F}^1 \) is characterized by one uncertainty, which is specified by the degree of membership of \( x_i \) to \( \mathcal{F}^1 \).

Any type-2 fuzzy set \( \mathcal{F}^2 \) corresponds to \( m_k \) type-1 membership functions and hence, the \( m_k \) corresponding type-1 fuzzy sets. Therefore, \( \forall x_i \in C \), the *extent* that \( x_i \) belongs to \( \mathcal{F}^2 \) is characterized by two uncertainties:

- Uncertainty about the extent to which \( x_i \) belongs to the union or either of the \( m_k \) corresponding type-1 fuzzy sets. This is quantified by the primary membership degrees.
- Uncertainty about the extent to which \( x_i \), which belongs with specific primary membership degrees to the type-1 fuzzy sets, belongs to \( \mathcal{F}^2 \). This is quantified by the secondary membership degree.

Generally speaking, the *extent* that \( x_i \in C \) belongs to a type-\( n \) fuzzy set \( \mathcal{F}^n \) is characterized by \( n \) uncertainties:

- Uncertainty about the extent to which \( x_i \) belongs to the union or either of the corresponding \( \arg\max_{i} \{ m_{i,1} \} \) type-1 fuzzy sets. This is quantified by the primary membership degrees.
- Uncertainty about the extent to which \( x_i \) belongs to the union or either of the corresponding \( \arg\max_{i} \{ m_{i,n-1} \} \) fuzzy sets of type \( n-1 \). This is quantified by the \( (n-1)^{th} \) membership degrees.
- Uncertainty about the extent to which \( x_i \) belongs to \( \mathcal{F}^n \). This is quantified by the \( n^{th} \) membership degree.

**III. Probabilistic Logic**

In this section, probabilistic logic [12] versus fuzzy logic is discussed. With the analogy explained before, a proposition in mathematical logic may correspond to a mathematical object that can belong to either (crisp or fuzzy) sets of true or false propositions. For the sake of brevity, the discussions of this section are represented using the concept of sets only. To notify the correspondence of sets and propositions, the notations \( T \) (referring to *true*) and \( \mathcal{F} \) (referring to *false*) are used for the sets that build up the possible world\(^2\), \( \mathcal{W} \), of an event \( E \).

A main difference between fuzzy logic and probabilistic logic is that for the former, \( \mathcal{F} \) and \( T \) are fuzzy sets, while for the latter, they are crisp sets. Therefore, the borders of \( \mathcal{F} \) and \( T \) in fuzzy logic are exposed to uncertainties. Hence,

\(^2\)Possible world of the event \( E \) is a world set that embraces all possible realizations of \( E \).
The sets of true $\mathcal{T}$ and false $\mathcal{F}$ propositions in fuzzy logic may have an overlap, i.e., $\mathcal{T} \cap \mathcal{F} \neq \emptyset$. Moreover, some mathematical objects that belong to $\mathcal{F} \setminus \mathcal{T}$ or $\mathcal{T} \setminus \mathcal{F}$ may not necessarily have a membership degree of 1.

In real life, an event $E$ may be prone to various types of imprecision and hence, uncertainties.

Uncertainties before realization of an event

Suppose that the world set $\mathcal{W}$ of possible realizations of an event $E$ is known, i.e., we know and can measure (or estimate the value of) all the characteristics that can define and distinguish any specific position within $\mathcal{W}$ (this is called a state in real-life engineering problems).

Before $E$ is realized, there is always uncertainty about which possible realization is going to occur. If the realization of $E$ is measurable (e.g., a temperature, which can be measured directly using a thermostat), the realization is precise in value, is determined by the same measurable characteristics as those that define the world set, and hence its position in the world set can strictly be distinguished (see the left-hand side plot in Figure 7). Therefore, the uncertainty is one-fold, i.e., uncertainty about the possible measurements from the world set that will be realized in occurrence of $E$. In this case, if one knows and can measure (or estimate the value of) all the effective factors (these are called the controllable and uncontrollable inputs in real-life engineering problems) that may play a role in any realization of $E$ (or equivalently in any realization of the characteristics that define the world set $\mathcal{W}$), then probabilistic logic can determine the probability of occurrence of any possible realization. In short, when all the concepts and characteristics involved in the procedure of realization of an event are measurable or precise in value, then probabilistic logic can handle the uncertainties.

In case the realization of $E$ is non-measurable (e.g., the comfort, which cannot be measured directly using a measurement device), this realization is imprecise in value. Then the uncertainty is two-fold, i.e., uncertainty about which possible non-measurable realization in the world set is going to occur, and uncertainty about the exact position of the realization in the world set. Systems in real life work with measurable concepts. For instance, the non-measurable instruction “when the room’s comfort is low, decrease the temperature”, should be transformed into a measurable instruction for an air conditioning system, e.g., “when the room’s temperature is between 23 and 25 degrees of Celsius, decrease the temperature for 3 degrees of Celsius”. This transforms a concept that is imprecise in value into one that is precise in meaning, and reduces the original uncertainty about the exact position of the concept in the world set $\mathcal{W}$ to an uncertainty about the exact position of the concept in a known subset of $\mathcal{W}$. Fuzzy logic can transform a realization that is imprecise in value into one that is precise in meaning by assigning a membership function of $x$ to the fuzzy sets $\mathcal{F}$ and $\mathcal{T}$, respectively.

To generalize, the summation $\sum_{i=1}^{s_c} \pi_i$ of the probabilities $\pi_i$ of $x$ belonging to $s_c$ crisp sets that build up the possible world $\mathcal{W}$ of the event $E$ is necessarily 1, while the sum $\sum_{i=1}^{s_f} \mu_i$ of the degrees of membership of $x$ to $s_f$ fuzzy sets that build up $\mathcal{W}$ may be smaller than, equal to, or larger than 1.

IV. MATHEMATICAL FORMULATION OF UNCERTAINTIES IN REAL-LIFE EVENTS

In real life, an event $E$ may be prone to various types of imprecision and hence, uncertainties.
Fig. 7: When the realization of an event is measurable, it will take an exact position (illustrated by the black dot in the left-hand side plot) in the world set. When the realization of an event is non-measurable, it will take position in a subset (colored in grey in the right-hand side plot) of the world set, while its exact position is uncertain. The corresponding subsets are defined to make the non-measurable realization, precise in meaning. Although the subsets illustrated in this plot are crisp, they may in general be fuzzy.

(of, generally, type \( n \)) to the characteristics that are imprecise in value.

It is important to note that although the transformation from a realization that is imprecise in value into one that is precise in meaning is with the aim of reducing the uncertainty from the entire world set to a subset of it, further uncertainties may or may not exist in the exact position of the borders of these subsets. For instance, when a specific type of search-and-rescue robot is deployed to a burning building, the domains of temperature for which the robot is functional and dysfunctional should be determined. Knowing the materials, sensors, ... used in the construction of the robot, one can determine the temperature at which this robot or any other robot of this type will become dysfunctional. In this case, the subsets functional and dysfunctional are crisp. In this case the transformation to a realization that is precise in meaning using membership functions (fuzzy logic) is identical to the transformation using probability functions (probabilistic logic). Distinguishing the exact borders of these two subsets for a human search-and-rescuer is more challenging, since humans are not as homogeneous as identical robots produced in a factory. Therefore, these borders may vary from person to person and the resulting subsets of functional and dysfunctional can become fuzzy. Then the only right tool for transforming the non-measurable realization into one that is precise in meaning is fuzzy logic.

Uncertainties after realization of an event

After \( E \) is realized, in case the realization of \( E \) is measurable, e.g., a temperature, which can be measured directly using a thermostat, the realization is \textit{precise in value} and its position in the world set can strictly be distinguished. When the realization of \( E \) is non-measurable, e.g., the comfort, which cannot be measured directly using a measurement device, this realization is imprecise in value. Therefore, there is uncertainty about its position in the world set. The realization should first be quantified to become precise in meaning, and only then, a subset of the world set that embeds all the possible positions of the realization in the world set, together with the degree to which the realization is positioned at any of these possible positions can be determined.