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To cite this article: Walter Smilga 2019 J. Phys.: Conf. Ser. 1194 012100

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Abstract. Linear and angular momenta are well-known examples of conserved quantities. Therefore, it seems strange that the perturbation algorithm of quantum electrodynamics explicitly ensures the conservation of the linear momentum at each vertex by appropriate δ functions but does not make similar provisions for the angular momentum. I address the specific role of the orbital angular momentum in relativistic multi-particle systems and explain how the conserved angular momentum determines the basic structures of quantum electrodynamics and quantum gravity.

1. Introduction
The Poincaré group (inhomogeneous Lorentz group) is the basic symmetry group of particle physics. Its irreducible representations describe individual elementary particles.

If we consider multi-particle systems, which are composed of two or more elementary particles, their spins together with the orbital angular momentum of the particles relative to each other form the total angular momentum of the system. As a consequence of Poincaré symmetry, the total angular momentum and also the total linear momentum of an isolated multi-particle system are conserved quantities.

The careful isolation from external influences is a natural prerequisite for the experimental investigation of multi-particle systems. Comparable care is required in the theoretical modelling of multi-particle systems. This means that the conservation not only of the total linear moment but also of the total angular momentum should be ensured in any realistic theory of isolated multi-particle systems. All the more so since the explicit inclusion of system constants usually contributes significantly to the validity and predictive power of physical models.

Strangely enough, the most successful theory of particle physics, quantum electrodynamics (QED), does not seem to sufficiently fulfil these criteria. Although its perturbation algorithm (expressed by the Feynman rules) carefully ensures the conservation of the linear momentum at each vertex by appropriate δ functions, it does not seem to care about the orbital angular momentum. This is at least irritating and raises the question of how relativistic particle theories deal or should deal with the orbital angular momentum of multi-particle systems.

2. Irreducible representations of Poincaré group
The commutation relations of the infinitesimal generators of the Poincaré group are (see, for example, [1])

\[ [M_{\mu\nu},p_{\sigma}] = i (g_{\nu\sigma}p_{\mu} - g_{\mu\sigma}p_{\nu}), \]  

(1)
\[ [M_{\mu\nu}, M_{\rho\sigma}] = -i \left( g_{\mu\rho} M_{\nu\sigma} - g_{\nu\rho} M_{\mu\sigma} + g_{\mu\sigma} M_{\rho\nu} - g_{\nu\sigma} M_{\rho\mu} \right), \] (2) 
\[ [p_{\mu}, p_{\nu}] = 0 \] (3)

(Greek indices = 0,1,2,3).

Here \( p_{\mu} \) are the generators of translations in time and space, and also the observables of energy and momentum; \( M_{\mu\nu} \) are the generators of rotations and boost operations. The generators of rotations \( M_{ik} \) (\( i, k = 1, 2, 3 \)) are the observables of the angular momentum.

Different irreducible representations are distinguished by the values of the two Casimir operators 
\[ P = p_{\mu} p_{\mu} \text{ and } W = -w_{\mu} w_{\mu} \], with \( w_{\sigma} = \frac{1}{2} \epsilon_{\sigma\mu\nu\lambda} M^{\mu\nu} p^\lambda \) (4)
\( (\epsilon_{\sigma\mu\nu\lambda} \text{ is the antisymmetric tensor of rank four}).

Within an irreducible representation these operators are multiples of the unit operator. Their values determine mass and spin of the different elementary particles.

According to the axioms of quantum mechanics (see, for example, \[2\]), a composed system of independent particles is described by a product representation of the Poincaré group. By decomposing a product representation into their irreducible components, we obtain irreducible representations describing isolated multi-particle systems. The fixed values of the Casimir operators then correspond to the effective mass and the total angular momentum of a system. Besides spin, the angular momentum now includes the orbital angular momentum of the particles relative to each other. While the first Casimir operator \( P \), in the form of a Klein–Gordon equation, describes the (trivial) centre of mass kinematics, the second Casimir operator \( W \) has a structure forming function: it forces the particles into a common orbital angular momentum, giving the multi-particle system a distinct geometric structure in space and time. The particles are, therefore, no longer independent, but correlated by the common orbital angular momentum.

3. Irreducible two-particle representations

The simplest multi-particle system is a two-particle system described by an irreducible two-particle representation. As with any irreducible representation, and also with an irreducible two-particle representation, there is a basis of eigenstates \( |p, m\rangle \) of the total 3-momentum with eigenvalue \( p \) and of a component of \( M^{\mu\nu} \) with eigenvalue \( m \)
\[ |p, m\rangle = \int_{\Omega} \delta^3 p_1 d^3 p_2 \delta (p - p_1 - p_2) c_{p,m}(p_1, p_2) |p_1, p_2\rangle \] (5)

(see, for example, \[1\]). Here \( p_1 \) and \( p_2 \) are the individual particle momenta. The states \( |p_1, p_2\rangle \) are two-particle product states, normalised by

\[ \langle p_1, p_2 | p_1', p_2' \rangle = \delta(p_1 - p_1') \delta(p_2 - p_2'). \] (6)

The coefficients \( c_{p,m}(p_1, p_2) \) are the analogues of the Clebsch–Gordan coefficients, as known from the coupling of angular momenta.

If we ignore the spin variables, then these states are eigenstates of (a component of) the orbital angular momentum. As such, they are invariant under the corresponding rotations, which means that they must be a superposition of product states, such that along with any pure product state, the rotated versions of this state also contribute to the eigenstate. This necessarily gives the basis states a rotational symmetric momentum-entangled structure.

While the basis states are eigenstates of the total linear momentum, they are not eigenstates of the individual particle momenta. This follows directly from the commutation relation (1) of the Poincaré group which is equal to 0 only if \( \sigma \neq \mu, \nu \). If the total momentum \( p \) points in the
direction $\sigma$, then $M^{\mu\nu}$ commutes with $p$, but does not commute with $p_1$ and $p_2$, unless $p_1$ and $p_2$ are parallel or anti-parallel to $p$.

After integration over $p_1$ or $p_2$, the states (5) take the form

$$|p,m\rangle = \int_{\Omega} d^3k \frac{1}{2} c_{p,m}(k) |p - k\rangle c_{p,m}(k) |p + k\rangle,$$

which offers a familiar physical interpretation: these states describe two particles, coupled by the field $c_{p,m}(k)$, which controls ('causes') a virtual exchange of momentum $k$ between the particles. This equals a description of the interaction mechanism of the Standard Model, in which the field is understood as a gauge field and the exchanged momenta as virtual gauge bosons. Since all basis states have this structure, 'virtual exchange of momentum' is a general property of irreducible two-particle representations of the Poincaré group. Mathematically, this property expresses the correlation between the individual particle states; physically, it can be interpreted as an interaction between the particles.

3.1. Classical limit
It is well-known that for large quantum numbers the eigenfunctions of the orbital angular momentum preserve its rotationally symmetric structure [3, 4]. Consequently, in the classical limit, two particles in an eigenstate of the orbital angular momentum behave like two particles forced by a potential into a closed rotationally symmetric orbit. Because of the rotational symmetry, this potential depends only on the absolute value of the distance between the particles. If the mass of one of the particles is much larger than the other, then this potential becomes a central potential. According to Bertrand’s theorem [5], there are only two central potentials with closed orbits: (1) an inverse-square central force such as the Coulomb potential and (2) the radial harmonic oscillator potential. If we make the reasonable assumption that at large distances the particles behave like free particles, then the harmonic oscillator potential is ruled out and we have to conclude that in the classical limit these particles behave like two charged particles linked by a Coulomb potential.

The following thought experiment will substantiate this conclusion.

3.2. A scattering experiment
We prepare two independent incoming particles with momenta $p_1$ and $p_2$. They are described by the product state $|p_1, p_2\rangle$. The particles are then assumed to pass through a measuring device that, without changing the total momentum, analysles the two-particle state by measuring the orbital angular momentum, i.e. a component of the two-particle observable $M_{\mu\nu}$. This measurement leaves the particles in an eigenstate $|p, m\rangle$ of the total linear momentum with eigenvalue $p = p_1 + p_2$ and the orbital angular momentum with eigenvalue $m$. Finally, we measure the outgoing momenta $p_1'$ and $p_2'$. Since we have not yet normalised the intermediate state $|p, m\rangle$, we have to add the normalisation factor $\omega$. This gives the quantum mechanical scattering amplitude

$$S = \omega^2 \langle p_1', p_2' | p, m\rangle (p, m | p_1, p_2).$$

Because of the entangled structure of the intermediate state, the information about the incoming particle momenta is lost. Therefore, the outgoing particle momenta will, in general, be different from the incoming momenta. In other words, we will observe a scattering with an amplitude determined by $\omega^2$, which therefore acts like a coupling constant.

3.3. Normalisation factor and coupling constant
The numerical value of $\omega^2$ has been determined in [6], see also [7]. The following is a summary of the calculation.
According to the normalisation rules of quantum mechanics, the correct normalisation factor \( \omega \) of the two-particle states (5) is the reciprocal of the square root of the volume of the integration area \( \Omega \)

\[
\omega = V(\Omega)^{-\frac{1}{2}}.
\]  

(9)

The integration area itself is a bounded sub-domain of the two-particle mass shell, which is generated by the actions of \( SO(3,1) \) on the total momentum and a moving \( SO(2) \), which rotates the particle momenta around the total momentum as the axis. The mass shell is a fibre space with circular fibres over the mass hyperboloid of the total momentum. This is a subspace of the symmetric space \( D^5 = SO(5,2)/(SO(5) \times SO(2)) \), also known as the Lie ball in five dimensions. This allows the volume of \( \Omega \) to be expressed by a combination of known volumes

\[
\omega^2 = 8\pi V(D^5)\frac{1}{2} / (V(Q^5) V(S^4)),
\]  

(10)

where \( D^5 \) is the unit Lie ball, \( Q^5 \) its boundary, and \( S^4 \) the unit sphere in five dimensions. Inserting the numerical values of their volumes, taken from [8], gives the result

\[
\omega^2 = \frac{9}{8\pi^4} \left( \frac{\pi^5}{2^4 5!} \right)^{\frac{1}{4}} = 1/137.03608245.
\]  

(11)

The resulting value of \( \omega^2 \) equals the electromagnetic fine-structure constant \( \alpha \) with the ‘official’ value \( 1/137.03599914 \) [9].

The volume expression (10) has become known as ‘Wyler’s formula of \( \alpha \)’ [10]. What Wyler, a Swiss mathematician, could not have known is that his more or less empirically founded formula represents the reciprocal value of the volume of the integration area \( \Omega \) of two-particle states of irreducible two-particle representations of the Poincaré group.

Whereas the structural similarity of two-particle states with elements of gauge theories could still be regarded as a curious coincidence, the close agreement of the calculated coupling constant with the experimental value of \( \alpha \) shows that the two-particle states of an irreducible representation of the Poincaré group describe an interaction that is equal in structure and strength with the electromagnetic interaction.

It is important to understand that this interaction and the associated coupling constant (and ‘charges’) are not physical properties of the individual particles but are mathematical properties of two-particle states; more precisely, of eigenstates of the orbital angular momentum.

### 3.4. Comparison with quantum electrodynamics

These results suggest that the perturbation algorithm of QED is an attempt to model, through virtually exchanged gauge bosons, the virtual exchange of momentum, which is characteristic of the structure of two-particle states of irreducible two-particle representations of the Poincaré group. However, the integration specifications of the Feynman rules are quite obviously not compatible with the mathematical structure of these two-particle states. While in two-particle states the correct area of integration, indicated previously by \( \Omega \), is strictly bounded, the Feynman rules specify integration over unbounded domains; namely, the full 3-dimensional momentum space of ‘virtual particles’. This is known to lead to the divergent integrals typical of QED and other gauge theories. These divergences need to be removed ‘manually’ by ‘renormalising’ the diverging integrals.

### 4. Irreducible N-particle representations

The insights gained from two-particle systems arouse curiosity as to what insights an \( N \)-particle system may offer. (The particles in the following may be neutral atoms or even macroscopic
objects.) In fact, the representation of eigenstates of the (total) orbital angular momentum by virtually exchanged momenta (as in Equation (7)) can easily be extended to $N$-particle systems described by irreducible $N$-particle representations of the Poincaré group. The number $N$ and all quantum numbers are now assumed to be very large. This allows us to treat the system in a quasi-classical way. The modelling of the state space by virtual momentum exchange then leads to classical trajectories of the individual particles that are curved by absorption or emission of virtual momenta (cf. Section 3.1). In a general relativistic covariant formulation, this results in a curved space-time structure.

In the two-particle states (7), the virtually exchanged quanta $k$ are part of the particle momenta. Depending on the sign of $k$, they can be understood as ‘emitted’ or ‘absorbed’ by the particle momenta, which therefore act as sources or sinks of the virtually exchanged momenta. In a covariant quasi-classical formulation, the momentum, or rather, the flux of momentum in space-time, is described by the momentum tensor $P_{\mu\nu}$, which is the traceless part of the energy-momentum (or stress-energy) tensor $T_{\mu\nu}$.

In contrast, in general relativity, Einstein postulated the full energy-momentum tensor as the source of the gravitational field and justified his choice with the ‘gravitational effect of masses’ [11].

Corresponding to the traceless part of the energy-momentum tensor, the generated metric of space-time can be expected to be described by the traceless part $C_{\lambda\mu\nu\kappa}$ of the Riemann tensor $R_{\lambda\mu\nu\kappa}$, which is known as the Weyl tensor.

### 4.1. Conformal gravity

Following Mannheim [12], a relation between momentum tensor and Weyl tensor is obtained through the principle of least action. With the Weyl tensor, a conformal-invariant action

$$I_W = \int C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} \sqrt{-g} d^4x$$

(12)

can be formed. Variation with respect to the metric tensor of this action together with the matter action

$$I_M = 4\kappa \int g_{\mu\nu} P^{\mu\nu} \sqrt{-g} d^4x$$

(13)

then leads to the field equations of Conformal Gravity

$$W^{\mu\nu} + \kappa P^{\mu\nu} = 0,$$

(14)

where

$$W^{\mu\nu} = \frac{1}{2} g^{\mu\nu} (R^\alpha_\alpha)^{\beta\gamma}_\beta \gamma + R^{\mu\nu\beta}_\beta - R^{\mu\beta\nu}_\beta - 2R^{\mu\beta}_\beta R^\nu_\beta - R^{\mu\beta}_\beta R^{\nu}_\beta - \frac{1}{2} g^{\mu\nu} (R^\alpha_\alpha)^{\beta\gamma}_\beta \gamma + \frac{2}{3} (R^\alpha_\alpha)^{\mu\nu}_\alpha + \frac{2}{3} R^\alpha_\alpha R^{\mu\nu}_\alpha - \frac{1}{6} g^{\mu\nu} (R^\alpha_\alpha)^2.$$  

(15)

At first glance, the field equations (14) look like Einstein’s field equations but here both tensors, $W^{\mu\nu}$ and $P^{\mu\nu}$, are traceless and the coupling constant $\kappa$ is dimensionless.

Looking at the determination of the coupling constant in Section 3.3, which was based on the inverse of a volume on the two-particle mass shell, we can expect that the strength of the gravitational interaction is related to a corresponding volume on the $N$-particle mass shell. With the assumed very large value of $N$, this would result in a very small value of $\kappa$. 


An outstanding property of Conformal Gravity is that it is able to describe the observed galactic rotation curves without the aid of dark matter. (Note that dark matter is the reification of discrepancies between Einstein–Newton gravity and the observation data.) Details can be found in Mannheim’s work (see, for example, [12]).

4.2. Quantum gravity

The basis of these considerations is an irreducible $N$-particle representation of the Poincaré group, which can therefore be considered to be a quantum mechanical basis of Conformal Gravity. This basis is just elementary quantum mechanics of multi-particle systems—there is no specific quantum gravity. This releases us from the problematic task of ‘quantising’ Conformal Gravity. Consequently, we have neither a compatibility problem between quantum mechanics and gravity, nor a renormalisation problem, nor ghost states as observed in the canonical quantisation of Conformal Gravity.

5. Conclusions

Despite its close connection to the Poincaré group and its well-established role in non-relativistic quantum mechanics, the orbital angular momentum has largely been ignored in relativistic particle theories. Instead, its role as a structure-forming element was attempted to be modelled by additional assumptions, such as specially adapted interaction terms or the postulate of gauge invariance.

My analysis identifies the orbital angular momentum as a key element in the quantum mechanical description of relativistic multi-particle systems. It provides a ‘missing link’ between quantum mechanics and relativity theories, and gives answers to long-standing questions that the Standard Model was unable to answer, such as:

- Why does the fine-structure constant have the value it has?
- What is the reason for the divergences of QED?
- How do we find a quantum theory of gravitation?
- Why is the gravitational interaction so weak?
- What is the nature of black matter?

The answers to these questions result from the hardly disputable statement that isolated multi-particle systems are described by irreducible multi-particle representations of the Poincaré group.

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