UNIVERSAL FINITE-SIZE-SCALING FUNCTIONS

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ABSTRACT

The idea of universal finite-size-scaling functions of the Ising model is tested by Monte Carlo simulations for various lattices. Not only regular lattices such as the square lattice but quasiperiodic lattices such as the Penrose lattice are treated. We show that the finite-size-scaling functions of the order parameter for various lattices are collapsed on a single curve by choosing two nonuniversal scaling metric factors. We extend the idea of the universal finite-size-scaling functions to the order-parameter distribution function. We pay attention to the effects of boundary conditions.

1. Introduction

Quite recently, Hu et al. have discussed the universal scaling functions of the finite-size scaling, studying the two-dimensional (2D) percolation problems. They have shown that the data of the existence probability and the percolation probability of bond and site percolation on various lattices fall on the same universal scaling functions. The idea of the universal finite-size-scaling functions was first proposed by Privman and Fisher.

In this article, the finite-size-scaling functions of the Ising model are studied by Monte Carlo simulations for various lattices. We deal with not only regular lattices such as the square lattice, the triangular lattice, etc., but quasiperiodic lattices such as the Penrose lattice and its dual lattice. We show that the finite-size-scaling functions of order parameter, $m$, as a function of temperature, $T$, for various lattices are

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collapsed on the universal curve by choosing two scaling metric factors. Attention is paid to boundary conditions in connection with surface effects. The idea of the universal finite-size-scaling functions is extended to the direction of external field, $H$. In the phase plane of temperature and field, we discuss a universal curved surface for the finite-size scaling. Generalizing the idea of the universal finite-size-scaling functions, we argue the scaling of the order-parameter distribution function.

2. **Universal scaling functions for the equation of state**

We study the Ising model on various lattices by Monte Carlo simulations. The simulation is performed at the critical point, $T = T_c$ and $H = 0$, and all the information near the critical point is obtained with the trick of a histogram method.

The temperature dependence of the second moments of order parameter for the square lattice and the Penrose lattice, as examples, are shown in Fig. 1. The data for various sizes with periodic and free boundary conditions are plotted with a unit of $J$.

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Fig. 1. Temperature dependence of the order parameter for the square (sq) lattice and the Penrose (Pen) lattice. Both the data for periodic and free boundary conditions are plotted.

One comment should be made here on the Penrose lattice. We use the ‘periodic’ Penrose lattice, which has been previously studied by Okabe and Niiyeki. The irrational golden ratio, $(1 + \sqrt{5})/2$, is approximated by a consecutive pair of the Fibonacci numbers. Thus, the number of spins, $N$, of the ‘periodic’ Penrose lattice are restricted to 76, 199, 521, 1364, and so on. The unit cell of the present system is represented by a large rhombus.

The finite-size-scaling plots for the order parameter are given in Fig. 2. Following the standard finite-size scaling, the temperature and the order parameter are scaled as $tL^{1/\nu}$ and $\langle m^2 \rangle L^{23/14\nu}$, where $t = T - T_c$ with $T_c$ being the critical temperature, and
Fig. 2. Universal finite-size-scaling plots of the order parameter. The values of the metric factors, $C_1$ and $C_2$ are given in Table 1.

$1/\nu$ and $\beta/\nu$ are the thermal and magnetic critical exponents, respectively. For the 2D Ising model, it is well known that $\nu = 1$ and $\beta = 1/8$. The linear dimension of the system is denoted by $L$, and for the Penrose lattice, $L$ is determined by $L = \sqrt{N}$. In Fig. 2, we have introduced two metric factors, $C_1$ and $C_2$. Thus, the scaling form of the equation reads

$$C_2^2 \langle m^2 \rangle L^{2\beta/\nu} = f(C_1 t L^{1/\nu}),$$

We have chosen $C_1 = C_2 = 1$ for the square lattice as a standard. The metric factors of the Penrose lattice are determined so as to get the best fit. The data for other lattices are also collapsed on the same curves with choosing appropriate metric factors. The estimated metric factors together with $T_c$ are tabulated in Table 1. The data for the triangular lattice, the honeycomb lattice, and the dual Penrose lattices are also shown in Table 1. It should be emphasized that the obtained $C_1$ and $C_2$ are the same for the periodic and free boundary conditions.

Table 1. Metric factors for the Ising model on various lattices.

| lattice         | $T_c$      | $C_1$     | $C_2$     | $C_3$     |
|-----------------|------------|-----------|-----------|-----------|
| square          | 2.269...   | 1         | 1         | 1         |
| Penrose         | 2.393±0.002| 0.86±0.02 | 1.03±0.02 | 0.95±0.02 |
| dual Penrose    | 2.150±0.002| 1.04±0.02 | 0.97±0.02 | 1.05±0.02 |
| triangular      | 3.641...   | 0.60±0.02 | 1.02±0.02 | 0.98±0.02 |
| honeycomb       | 1.519...   | 1.50±0.03 | 0.98±0.02 | 1.02±0.02 |
The Binder parameter,\(^5\)

\[ g = \frac{1}{2} (3 - \frac{\langle m^4 \rangle}{\langle m^2 \rangle^2}), \tag{2} \]

is often used in the finite-size-scaling analysis of critical phenomena. The raw data for the Binder parameter are shown in Fig. 3 for the square lattice and the Penrose lattice.

Fig. 3. Temperature dependence of the Binder parameter for the square (sq) lattice and the Penrose (Pen) lattice. Both the data for periodic and free boundary conditions are plotted.

The universal finite-size-scaling plots for the Binder parameter are given in Fig. 4. Here the same metric factor \( C_1 \) is used as that in Fig. 2. Thus, we have shown that the finite-size-scaling plot of the Binder parameter becomes universal among various lattices by choosing appropriate metric factors. However, the Binder parameter strongly depends upon boundary conditions. It means that the Binder parameter reflects the finite-size effects due to fluctuations. The universality of the Binder parameter at the critical point has been pointed out by Bruce.\(^6\) Moreover, Kamieniarz and Blöte\(^7\) have argued the aspect-ratio dependence of the universal value of the Binder parameter for rectangular systems using a transfer-matrix technique.

Next turn to the system with an external field. The second-order cumulants of the order parameter, \( \langle m^2 \rangle - \langle m \rangle^2 \), for the square lattice and the Penrose lattice are shown in Fig. 5. The system size is fixed for each lattice but the data for various \( h = H/T \) are plotted. Here, we restrict ourselves to the case of the periodic boundary conditions to escape from the complication of the figure.

Let us try the universal finite-size-scaling plot,

\[ C_2^2 (\langle m^2 \rangle - \langle m \rangle^2) L^{2\beta/\nu} = f_2(C_1 t L^{1/\nu}, C_3 h L^{\delta \beta/\nu}). \tag{3} \]

Here, \( \beta \delta \) is the so-called gap exponent, and \( \delta = 15 \) for the 2D Ising model. We show the finite-size-scaling plot in Fig. 6, where \( C_1 \) and \( C_2 \) are the same as before. We
Fig. 4. Universal finite-size-scaling plots of the Binder parameter.

Fig. 5. Temperature dependence of the second-order cumulant of the order parameter for various external fields, $h = H/T$ with the periodic boundary conditions.
have also introduced a new numeric factor $C_3$. In Fig. 6, the data for various sizes at different temperatures with different fields fall on a universal curved surface.

In Fig. 6, Universal finite-size-scaling plots of the second-order cumulant of the order parameter. For the square lattice, the data of the system sizes $L=24$, 40 and 64 are shown. For the Penrose lattice, those of the sizes $N=521$, 1364 and 3571 are shown. The numeric factors $C_1$ and $C_2$ are the same as in Figs. 2 and 4. The values of the new numeric factor $C_3$ are given in Table 1.

In this way, we have shown the universal finite-size-scaling plot for the order parameter in the phase plane of temperature and field. It is a generalization of the scaling equation of state for the bulk system.

### 3. Finite-size scaling of order-parameter distribution function

In this section, we extend the idea of the universal finite-size scaling to the probability distribution function of the order parameter, $P(m)$. The finite-size scaling of the distribution function has been frequently used in the analysis of Monte Carlo data, for example, for the spin-glass problem.

In Fig. 7, we show one example of the data for $P(m)$ off the critical point. Here, the temperature and the field are chosen under the conditions, $C_1 t L = 1.0$ and $C_3 h L^{1.875} = 0.8$, for the systems of different lattices with various sizes. The numeric factors $C_1$ and $C_3$ are those determined before. Figure 8 shows the universal finite-size-scaling plot of the order-parameter distribution function; all the data fall on the same universal curve.

\[
P(m; T, h; L) = C_2 L^{3/\nu} P_2(C_2 m L^{\beta/\nu}; C_1 t L^{1/\nu}; C_3 h L^{3\delta/\nu}).
\]  

The equation (5) is the general universal finite-size-scaling function of the distribution function.
Fig. 7. Order-parameter probability distribution function $P(m)$ for the square lattice and the Penrose lattice, where temperature and field are chosen under the conditions, $C_1 t L = 1.0$ and $C_3 h L^{1.875} = 0.8$.

Fig. 8. Universal finite-size-scaling plot of the distribution function of the order parameter.
4. Summary and Discussions

We have shown that the equations of state of 2D Ising model for various lattices are
described by the universal finite-size-scaling function with a small number of numeric
factors. The above statement holds not only for the scaling along the temperature
axis but in the whole $t - h$ plane in the finite-size-scaling critical region.

We have introduced three numeric factors, $C_1$, $C_2$ and $C_3$. However, all three
numeric factors are not independent. The relation $C_3 = C_2^{-1}$ is expected from the
scaling argument. Actually, this relation appears to hold from Table 1 within statistical
errors. Moreover, these numeric factors are related to the relative ratio of the
critical amplitudes of bulk systems for different lattices. The detailed analysis of
this problem will be discussed elsewhere.

The universal finite-size-scaling functions depend on boundary conditions. Actually,
the difference of the order parameter between periodic and free boundary condi-
tions is related to the so-called surface magnetization, $m_s$, in the study of surface
effects in critical phenomena:

$$ m_s = \frac{m_{\text{free}} - m_{\text{periodic}}}{L}. \quad (5) $$

We should also note that the finite-size-scaling functions are also sensitive to the
lattice anisotropy, which will be left to a future study.

We have generalized the concept of the universal finite-size-scaling function for
the order-parameter probability distribution; we have shown that the probability
distribution function is described by the general universal finite-size-scaling function.

Finally, we should make a comment. Through the present study, as a by-product,
we have shown again that the Ising model in quasicrystals belongs to the same uni-
versality class as that of regular lattices.

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