Gyro movement equations

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Abstract. In this work the motion equations of a gyro with essential characteristics were constructed, where the rotor rotates to reason $\psi$ around an axis mounted on a single universal joint, which can rotate freely around the vertical axis. The angle formed by the rotor shaft and the plane of the meridian is denoted with $\theta$, latitude, the angle of the position on earth is denoted $\lambda$ and with $\omega_e$ the angular velocity of the Earth around its axis.

Based on the formulation of Lagrange, the motion equations governing this system were obtained, differential equations were also solved with the standardized Runge-Kutta method of order 4. Also, we present several phase spaces that show the temporal evolution of the system.

1. Introduction

Within the field of nautical and maritime transport there are instruments that help measure the course of a ship, by means of different devices such as: the magnetic compass, the gyro, the GPS, the compass, the satelitarian compass, among others [1]. The gyro is a physical instrument characterized by being oriented to the north – south geographical and remains oriented under the influence of the properties of a gyroscope such as its speed and precession, with respect to the rotation and gravity of the earth. The gyro is a device that uses a set of rings or discs which rotate using high-speed motors and frictional forces to take advantage of earth’s rotation. The gyro was invented in 1906 by the German Hermann Anschütz, arising from the gyroscope which, in turn comes from the spinning top [2], contemplating the rigidity or inertia which is a property of the rotating bodies, moving in a certain direction in such a way that the axes are kept oriented in a fixed direction, in the case of a gyro, as already mentioned, its axis remains toward north true. The precession presented by the gyro is based on the Law of Precession: "When a gyroscope is subjected to an angular force that tries to deflect the direction in which its axis of rotation is, it objects resistance and its axis precedes in a direction"
perpendicular to the applied force, until it places, by the shortest path, the plane and direction of its rotation in the plane and direction of force.” [2]. A 3-degree freedom gyroscope of is the basis of all gyro, since when rotating at a high-speed stiffness causes the rotor shaft to point to a fixed point in space [1, 2].

In this work we analyze a classic mechanical problem: a gyro with basic or essential characteristics, where the m-mass rotor rotates with an angular velocity named \( \dot{\psi} \) on the shaft that is mounted on a joint, which rotates freely around the \( y \) axis and forming an angle between the rotor, on the \( xz \) axis which is denoted as \( \theta \), forming an angle on a line parallel to the Earth axis \( R \) with the plane \( xy \) and denoted as \( \lambda \), this angle corresponds to the latitude of the position of the Earth's axis, this axis has an angular velocity corresponding to that of the Earth denoted as \( \omega_e \) (Figure 1).

![Figure 1. Schematic representation](image)

There are several similar studies in the scientific literature [2,3], but in this work the main task is to determine the motion equations that govern the gyro. First, we present the basic equations in Euler's formalism; we solve the differential equations from numerical methods using the standardized Method Runge-Kutta of order 4, then we present several phase spaces that show the temporal evolution of the system [5, 6].

2. Euler’s equations

As a first step, we did calculate the Angular momentum over the rotor. The angular velocity relative to the rotor's center of mass relative to the Newtonian frame is:

\[
\Omega = \omega_e \hat{k} + \dot{\theta} \hat{j}
\]

With

\[
\hat{k} = -\cos(\lambda)\sin(\theta)\hat{x} + \sin(\lambda)\hat{y} + \cos(\lambda)\cos(\theta)\hat{z}
\]
The angular velocity of the rotor is obtained by adding a turn $\dot{\varphi}$ to $\Omega$ with
\[
\dot{\varphi} + \omega_e \cos(\lambda) \cos(\theta) = \omega_e,
\]
which implies that:
\[
\omega = -\omega_e \cos(\lambda) \sin(\theta) \hat{x} + (\dot{\theta} + \omega_e \sin(\lambda)) \hat{y} + \omega_z \hat{z}
\] (3)

The angular momentum $H_o$ of rotor is expressed like (4), where $I_x = I_y = I'$ and $I_z = I$ are inertia moments of the rotor with respect its symmetry axis and a transversal axis that passes through $O$, respectively:
\[
H_o = -I' \omega_e \cos(\lambda) \sin(\theta) \hat{x} + I'(\dot{\theta} + \omega_e \sin(\lambda)) \hat{y} + \omega_z \hat{z}
\] (4)

When solving problems involving the movement of a rigid body with respect to a fixed point $O$, it is convenient to use the following equation, which removes the reaction components in the support $O$.
\[
\sum M_o = (H_o)_{Oxyz} + \Omega x H_o
\] (5)

because the rotor has to rotate on the $z$ axis and is free to turn on the axis $y$. So that, the components $y$ and $z$ of $\sum M_o$ must be zero, and are obtained from the vector product that gives the component in $x$ resulting in:
\[
I' \dot{\theta} + I \omega_e \omega_z \cos(\lambda) \sin(\theta) - I' \omega_e \omega_z \cos^2(\lambda) \sin(\theta) \cos(\theta) = 0
\] (6)

This is the first movement equation of the gyro. We also get the expression for the second motion equation that governs the gyro:
\[
I' \dot{\omega}_z = 0
\] (7)

3. Numerical results
The numerical solution was obtained using Runge-Kutta’s standardized and well-known order 4 method. The initial conditions were $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$. The numerical results were obtained using the above theoretical expressions and using the data presented in Table I. The initial angle value was randomly generated in the range $[0, 0.4]$ for $\theta$ and $\dot{\theta}$ to get various results to display in the phase space.

This method is applied for simple systems as well as in books of differential equations [3, 4]. The error due to the use of the method is in the order of the step size, which we denote by $\delta h$, raised to 4. The step size is defined as the partition of the integration interval $\delta h = (b - a) / n$, where $b$ is the upper limit of the range, $a$ is the lower limit, $n$ is the number of parts in which the interval is divided, which in fact corresponds to the number of iterations that will need to cover the entire proposed integration interval. For all three cases, the established step is $\delta h = 0.0011$ with a mistake of $\epsilon_{rr} = 1.4 \times 10^{-12}$, this because $n = 90000$ in an interval of 0 to 100.
Table 1. Values of initial condition used in the simulations for all systems working

| Value | Unit   | Value | Unit   |
|-------|--------|-------|--------|
| $I'$  | 0.151 kg m$^2$ | $\omega_e$ | 2 rad s$^{-1}$ |
| $I$   | 0.020 kg m$^2$ | $\omega$  | 4.3 rad s$^{-1}$ |
| $\lambda$ | 10.00 rad |     |       |
| $I'$  | 15.100 kg m$^2$ | $\omega_e$ | 2 rad s$^{-1}$ |
| $I$   | 20.000 kg m$^2$ | $\omega$  | 4.3 rad s$^{-1}$ |
| $\lambda$ | 0.349 rad |     |       |
| $I'$  | 1.100 kg m$^2$ | $\omega_e$ | $7.25 \times 10^{-5}$ rad s$^{-1}$ |
| $I$   | 2.000 kg m$^2$ | $\omega$  | $4.30 \times 10^{-2}$ rad s$^{-1}$ |
| $\lambda$ | 0.349 rad |     |       |

4. Simulation Outcomes.

The following figures show the graphs of position, speed, acceleration, phase plane, and acceleration plane behavior, based on time:

Case 1:
Figure 2. a) Angle behavior $\theta$ depending on time. b) Angle behavior $\dot{\theta}$ depending on time. c) Angle behavior $\ddot{\theta}$ depending on time. d) Phase plane. e) $\dot{\theta}$ vs $\theta$ for the first initial conditions.

Case 2:
Figure 3. a) Angle behavior $\theta$ depending on time. b) Angle behavior $\dot{\theta}$ depending on time. c) Angle behavior $\ddot{\theta}$ depending on time. d) Phase plane. e) $\ddot{\theta}$ vs $\dot{\theta}$ for the second initial conditions.

Case 3:
5. Conclusion
When we talk about gyro, they are a classic theme in the study of physics as it touches on important topics such as angular momentum and spinning top study. There are points or conditions that make them more complicated making these problems more interesting. One point of interest is that the faster they spin, the more stability you have and the more accurate it is for orientation.

Precession is not the most general movement of the gyro when its center of gravity is not above the support point. Accompanying the precession there is usually a pitching or nourishing motion, which is the result of the gyroscope axis trying to reach the slight inclination necessary for the center of gravity of the system to acquire the translation speed due to the precession.

The amplitude of this nutation can be modified by giving the center of masses an initial speed of translation, let us not forget that we all live on a gigantic gyroscope that is the Earth: the fact that the Earth is not completely indeformable, by turning on its axis, it is flattened by the poles and widened by the equator, being of interest and importance for navigation and above all as an important subject in physics.

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