Photon polarization tensor in a magnetized plasma system

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Abstract

We investigate the photon polarization tensor at finite temperatures in the presence of a static and homogeneous external magnetic field. In our scheme, the summing of the Matsubara frequency is performed after Poisson resummation, which is easily completed and converges quickly. Moreover, the behaviors of finite Landau levels are presented explicitly. It shows a convergence while summing infinite Landau levels. Consequently, there is no necessity to truncate the Landau level in a numerical estimation. At zero temperature, the lowest Landau level (LLL) approximation is analytically satisfied for the vacuum photon polarization tensor. However, we examine that the LLL approximation is not enough for the thermal polarization tensor. The thermal tensor obtains non-trivial contributions from the finite-\(n\) Landau levels. And, photon spectra gains a large imaginary contribution in thermal medium, which is the so-called Landau damping. Finally, it is argued that the summation of Matsubara frequency is not commuted with Landau level ones, such conjecture is excluded in our calculations.

Keywords: magnetic field, thermal field theory, photon polarization, Landau damping

1. Introduction

At the Relativistic Heavy Ion Collider and the Large Hadron Collider, strong magnetic fields with the magnitude of \(10^{13} - 10^{20} \text{ G}\) (corresponding to \(eB \sim (0.1 - 1.0 \text{ GeV})^2\)) can be generated through non-central heavy-ion collisions \([1, 2]\). In recent years, studies of quantum chromodynamics (QCD) phase structure and phase diagram under strong external magnetic fields have attracted lots of interests \([3–5]\), and it is found that QCD matter under strong magnetic fields exhibits some novel properties, e.g. the chiral magnetic effect \([6–8]\), the magnetic catalysis (MC) \([9–11]\) and inverse magnetic catalysis (IMC) \([12–14]\), a possible formation of the vacuum superconductor \([15, 16]\) and its survival at high temperature \([17, 18]\), etc. With such strong magnetic fields, quantum electrodynamics (QED) will also be modified, such as photon decay into an electron-positron pair via Schwinger mechanism \([19–21]\), vacuum birefringence of a photon \([22, 23]\), photon splitting and so on \([24, 25]\).

The strong magnetic field is generated in the early stage of non-central heavy-ion collisions, and then fast decays. It is intriguing to know whether and how properties of the created quark-gluon-plasma (QGP) will be modified by the strong magnetic field, and whether it can be measured in experiment. Dileptons and photons are produced through the bulk of the matter and throughout the entire history of the collision, thus can be regarded as penetrating probes of QGP. Recently, experimental evidences of photon anisotropy from the PHENIX Collaboration \([26]\) challenge existing theoretical models, and it was proposed in \([27]\) that the photon anisotropy might be induced by a large anisotropic magnetic field in heavy ion collisions. This raises the interests of study the photon and dilepton production in the presence of an external magnetic field \([28, 29]\).

The photon polarization tensor carries the fundamental information of magnetized vacuum or medium \([30–34]\). A complete description of the vacuum polarization tensor is particularly complicated to approach, since it is expressed in terms of a double summation of infinite series with respect to two Landau levels occupied by virtual charged particles. Most works were focusing on the strong filed limit with an assumption of lowest Landau level (LLL) \([31, 35]\). In \([36]\) we obtained a full description of vacuum polarization tensor in response to all the Landau levels at any field strength of \(B\) for...
the first time, and we found out that the imaginary part of the photon polarization tensor becomes non-zero at the time-like momenta region $Q^2 > 4(M^2 + 2neB)$ at $T = 0$. We confirmed that the LLL approximation is analytically satisfied [32, 36].

It is not fully understood of the imaginary parts of thermal photon polarization tensor in a magnetized media. The main purpose of this paper is to investigate whether the above conclusion will be influenced by temperatures. At finite temperature, in our scheme, the Matsubara frequency is summed after applying Poisson summation formula, which completed easily and convergent quickly under the help of proper time representation. It was argued that the summation of Matsubara frequency is not commuted with Landau level numbers. However, in the early works, one has to test by a numerical way to find out the cutoff of the Landau level. In our work, the dependence of Landau level is expressed in an obviously manner. Therefore, one is able to truncate the Landau level via a systematic consideration while proceeding numerical calculation.

The paper is organized as follows. We introduce the vacuum photon polarization tensor $\Pi_{\mu\nu}$ at external $B$ in section 2. In section 3, we present the magnetized photon polarization tensor $\Pi_{\mu\nu}$ at finite temperature. Due to the additional vector of the velocity of the plasma, the decomposition of the photon polarization tensor will become more complicated. The full expression will be examined in section 4. We discuss the limiting behaviors of thermal photon polarization tensor $\Pi_{\mu\nu}$ and physical explanations at different kinematics regimes in section 5. We end up with the summary and future applications in section 6.

2. Vacuum photon polarization tensor

Following [35], the decomposed fermion propagator $\mathcal{J}(k)$ in a static and homogeneous external magnetic field can be written as:

$$\mathcal{J}(k) = i\exp(2k_z^2) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathcal{B}_n(eB, k),$$

(1)

with

$$\mathcal{B}_n(eB, k) = 2(k_1 + M)\mathcal{O}^+ L_n(-4k_z^2)$$

$$- 2(k_1 + M)\mathcal{O}^- L_{n-1}(-4k_z^2) - 4(k_1 L_{n-1}^-(-4k_z^2),$$

(2)

where $k_\perp = k_z^2/(2eB)$. Note here, we normalize all the energy scale to dimensionless, where $\hat{q}^2 = q^2/(2eB)$, $\hat{M}^2 = M^2/(2eB)$, and so on. $L^\pm_n(\xi)$ are the generalized Laguerre polynomials, with $L_n^\pm(0) = 0$ if $n < k$. $\mathcal{O}^\pm = (1 \pm i\gamma^1 \gamma_5)/2$ are the projecting operators corresponding to the spin states of charged particle parallelising or anti-parallelising to the direction of external field $B$, and they satisfy following commutation relations:

$$\mathcal{O}^\pm \gamma^\mu \mathcal{O}^\pm = \mathcal{O}^\pm \gamma_\perp^\mu,$$

$$\mathcal{O}^\pm \gamma^\mu \mathcal{O}^\mp = \mathcal{O}^\mp \gamma_\perp^\mu.$$  

(3)

Here, the metric convention $g^\mu\nu$ is decomposed into two orthogonal subspaces:

$$g^\parallel_{\mu\nu} = \text{diag}(1, 0, 0, -1), \quad g^\perp_{\mu\nu} = \text{diag}(0, -1, -1, 0).$$

(4)

Similar decompositions are adopted for four-dimensional momentums $k^{\parallel} = k^{\parallel}_\perp + k^{\parallel}_\perp$, with

$$k^{\parallel}_\perp = (k_0, 0, 0, k^3), \quad k^{\parallel}_\perp = (0, k^1, k^2, 0)$$

(5)

and Dirac matrices

$$\gamma^{\parallel}_\mu = (\gamma^0, 0, 0, \gamma^3), \quad \gamma^{\perp}_\mu = (0, \gamma^1, \gamma^2, 0).$$

(6)

In vacuum, the photon polarization tensor is expressed as [36]:

$$\Pi_{\mu\nu}(q) = -i e^2 \text{Tr}[\mathcal{J}(k)\gamma^\mu \mathcal{P}(p)\gamma^\nu],$$

(7)

with $p = k + q$. In general, the vacuum polarization tensor $\Pi_{\mu\nu} \sim I_{\mu\nu}$ and

$$I_{\mu\nu} = 2\text{Tr}[\mathcal{J}^\Gamma_{\mu\nu} \gamma^0 \mathcal{P}(L_{\perp} L_{\perp} - L_{\parallel} L_{\parallel} - L_{\parallel} L_{\parallel} - L_{\perp} L_{\perp})$$

$$- 2\text{Tr}[\mathcal{J}^\Gamma_{\mu\nu} \gamma^0 \mathcal{P}(L_{\perp} L_{\parallel} - L_{\parallel} L_{\perp} - L_{\perp} L_{\perp} - L_{\parallel} L_{\parallel})$$

$$- 4\text{Tr}[\mathcal{J}^\Gamma_{\mu\nu} \gamma^0 \mathcal{P}(L_{\perp} L_{\perp} - L_{\parallel} L_{\parallel} - L_{\perp} L_{\perp} - L_{\parallel} L_{\parallel})$$

$$+ 16\text{Tr}[\mathcal{J}^\Gamma_{\mu\nu} \gamma^0 \mathcal{P}(L_{\perp} L_{\perp} - L_{\parallel} L_{\perp} - L_{\perp} L_{\perp} - L_{\parallel} L_{\parallel})].$$

(8)

A detailed description of the above notation will be explained in the next section. We emphasize here, a naive counting of the independent tensor structures is five at zero temperature. However, after applying the transversal requirement and Ward identity, the number will reduce to three. It means that some above terms will combine together. Therefore, we will determine the orthogonal thermal tensor basis before stepping into the main calculation, which will present in section 4.

3. Photon polarization tensor at finite temperature

Temperature breaks Lorentz invariance and separates the zeroth component of the thermal photon polarization tensor. The form of photon polarization tensor presents as:

$$\Pi_{\mu\nu} = \int d\Gamma I_{\mu\nu} \exp[-(\hat{M}^2 - \eta \hat{q}^2)^2 + nx + m(1 - x)]$$

$$- (k_0 + (1 - x) \hat{q}_0^2 + k_3 \pi)],$$

(9)

where $\eta = x(1 - x)$ with Feynman parameter $x$ and the volume space is:

$$d\Gamma = -i e^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{n+m} \int_0^1 \xi \int_0^\infty d\tau \int_0^{\xi} \frac{d^2 k_\perp}{(2\pi)^2} \tau \exp[2k_z^2 + 2\hat{p}_z^2].$$

(10)
It is well known shifting \( k \to k - (1 - x)q \) can simplify the calculations. But, as pointed out before, because the external magnetic field breaks the transverse space, it is only allowed to shift \( k_0 \to k_0 - (1 - x)q_0 \) along the direction of the \( B \)-field as usual.

The transverse momentums \( k_i \) have to be shifted to \( k_i - q_i \beta / (\alpha + \beta) \), which will show later by the explicit expression of equation (14). The detail notations \( \alpha, \beta \) are described below. We remark that here is nothing on the zero component since the integration with respect to \( k_0 \) is replaced by summing over Matsubara frequencies \( 2\pi i T \sum_q \).

And then, the tensor structure of thermal polarization \( \Pi^{\mu\nu} \) is presented as:

\[
\Pi^{\mu\nu} = 2 \text{Tr} [k_i^2 \gamma^\mu \gamma^\nu (L_m + L_{n-1}L_{m-1})] \\
- 2 \text{Tr} [k_i^2 \gamma^\mu \gamma^\nu (L_{n-1}L_{m-1} + L_{n-1}L_m)] \\
- 4 \text{Tr} [k_i^2 \gamma^\mu \gamma^\nu L_{n-1}L_{m-1}^{i1} - L_{m-1}^{i1}L_{n-1}] \\
- 4 \text{Tr} [k_i^2 \gamma^\mu \gamma^\nu L_{n-1}L_{m-1}^{i1} - L_{m-1}^{i1}L_{n-1}] \\
+ 16 \text{Tr} [k_i^2 \gamma^\mu \gamma^\nu L_{n-1}L_{m-1}^{i1} - L_{m-1}^{i1}L_{n-1}] \\
+ 2 \text{Tr} [k_i^2 \gamma^\mu \gamma^\nu L_{n-1}L_{m-1}^{i1} - L_{m-1}^{i1}L_{n-1}] \\
+ 4 \text{Tr} [k_i^2 \gamma^\mu \gamma^\nu L_{n-1}L_{m-1}^{i1} - L_{m-1}^{i1}L_{n-1}],
\]

(11)

where

\[
k_i^2 = (k_0 + q_0)\gamma^0 - (k_3 + xq_3)\gamma^3 + M,
\]

\[
k_i^2 = k_0\gamma^0 - (k_3 + xq_3)\gamma^3 + M,
\]

\[
k_i^2 = k_i + q_i \beta / (\alpha + \beta),
\]

\[
k_i^2 = k_i - q_i \beta / (\alpha + \beta).
\]

Under the help of generating function of Laguerre polynomials [38]:

\[
\sum_{n=0}^{\infty} t^n \frac{a^n}{n!} \xi^n = \frac{1}{(1 - t)^n} \exp \left[ \frac{-it\xi}{1 - t} \right],
\]

for \(|t| < 1\), we are able to evaluate the summation of Landau levels in a direct manner. We obtain:

\[
\exp[2\tilde{k}_j^2 + 2\tilde{p}_j^2] \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{n+m} \times \exp[-(n+m)(1-x)]L_{n-m}^b(-4\tilde{k}_j^2)L_{n-m}^b(-4\tilde{p}_j^2) \\
= \frac{t^i_1 t^j_2}{(1 - t)^{n+1}} \exp \left[ \frac{\alpha\beta}{\alpha + \beta} \tilde{q}^2 \right] \times \exp \left[ (\alpha + \beta) \left( \tilde{k}_j + \frac{\beta}{\alpha + \beta} \tilde{q}_j \right)^2 \right].
\]

(14)

The last exponential term explains the unusual shifting of transverse momentums which are early taken in equation (8). Here, \( t_1 = -e^{-\tau}, t_2 = -e^{-(1-\tau)} \), \( \alpha = 2(1 + t_1)/(1 - t_1) \) and \( \beta = 2(1 + t_2)/(1 - t_2) \).

4. Tensor structures at finite temperature

In a hot plasma, the velocity of the fluid \( u \) is introduced. It combines with the momentum of particles to create three second order tensors \( u \otimes u, u \otimes q \) and \( q \otimes q \). Including the metric \( g^\mu\nu \), the electromagnetic tensor \( F_{\mu\nu} \) and dual tensor \( \tilde{F}_{\mu\nu} \), we have six independent second order tensors. Because of the requirement of Ward identity \( \Pi^{\mu\nu} q_{\mu} = 0 \) and the without loss of generality choice \( w^u u_\mu = 1 \), we mean we are able to separate \( \Pi^{\mu\nu} \) into four independent structures. From the definition \( \Pi^{\mu\nu} = -ie^2 \text{Tr} \left( k(k + q)\gamma^\mu \gamma^\nu (k - q)(k - q)\gamma^\nu \right) = -ie^2 \text{Tr} \left( k(k + q)\gamma^\mu \gamma^\nu (k + q)\gamma^\nu \right) \), it draws a conclusion that \( \Pi^{\mu\nu}(q) = \Pi^{\mu\nu}(-q) \) [24, 25]. Therefore, \( \Pi^{\mu\nu} \) contains symmetry parts made by even powers of four momentum \( q \) and antisymmetric parts which formed by odd powers of \( q \). To find out the subspace, we set up four mutual orthogonal vectors:

\[
x_0 = q^0; \quad x_1 = \tilde{F}^{\mu\nu} q_{\nu}; \quad x_2 = F^{\mu\nu} q_{\nu};
\]

\[
x_3 = u^\mu - x_1^\mu /\sqrt{2} - x_2^\mu /\sqrt{2} - x_3^\mu /\sqrt{2},
\]

(15)

where the fluid velocity \( u = (1, 0, 0, 0) \). Hence, the associated transversed symmetric tensors are:

\[
P^{\mu\nu}_1 = \frac{x_1^\mu x_1^\nu}{x_1^2}; \quad P^{\mu\nu}_2 = \frac{x_2^\mu x_2^\nu}{x_2^2}; \quad P^{\mu\nu}_3 = \frac{x_3^\mu x_3^\nu}{x_3^2},
\]

(16)

which satisfy the following relationship:

\[
P^{\mu\nu}_1 = P^{\nu\mu}_1; \quad P^{\mu\nu}_2 = P^{\nu\mu}_2; \quad P^{\mu\nu}_3 = P^{\nu\mu}_3;
\]

\[
P^{\mu\nu}_i P^{\nu\mu}_i = g^{\mu\nu} - q^\mu q^\nu / q^2;
\]

(17)

with \( i = 1, 2, 3 \) and \( i \neq j \). Involving fluid velocity \( u \), the antisymmetric tensors must contain the form structures of \( u^\mu x_1^\nu - x_1^\mu u^\nu \) for \( i = 0, 1, 2 \), the natural candidates \( F^{\mu\nu} \) or \( \tilde{F}^{\mu\nu} \). In fact, we obtain two independent structure forms, which satisfy Ward identity as required. But, one of them is totally null in our simple electromagnetic tensor environment. Another one is \( P^{\mu\nu}_3(q) = P^{\mu\nu}_3(-q) = u^\mu x_3^\nu - x_3^\mu u^\nu + (u \cdot q) F^{\mu\nu} \).

We normalize it to a conjugate dimensionless projector that

\[
P^{\mu\nu}_4 = (P^{\mu\nu}_4) = \frac{ie^2 x_1^\mu x_1^\nu - ie^2 x_2^\mu x_2^\nu + iF^{\mu\nu}}{u \cdot q},
\]

(18)

which satisfy \( P^4_4 = 0 \) for \( i = 1, 2, 3 \). Similar analytic structures of gluon polarization tensor in a magnetized medium can be found in [39, 40]. In work [40], four possible independent tensors are constructed same as here. In work [39], nine independent tensor structures are calculated since the authors explore the ghost and tadpole contributions for color-magnetic background.

Then, after summing over Landau levels, \( \Pi^{\mu\nu} \) takes the form of

\[
\Pi^{\mu\nu} = \sum_{i=1}^{4} P^{\mu\nu}_i \tilde{v}_i
\]

\[
= \sum_{i=1}^{4} P^{\mu\nu}_i \int d\Gamma \frac{4\tau e^{-\tau}}{(1 - h)(1 - t_2)} \exp \left[ h(x, \tau)\tilde{q}_j^2 \right] T_i,
\]

(19)
where $v = \frac{M^2}{\hbar^2} - \eta q_0^2$ and $h(x, \tau) = \alpha \beta / (\alpha + \beta)$. The scalar functions $\pi^\alpha$ are developed by the inner product of $(P_{\mu}^\nu; \Pi^\mu)$, which are in the form of as below:

$$I_1 = 2(M^2 + \Theta_1^3)(1 + t_2) + \frac{4\alpha \beta q_0^2}{(\alpha + \beta)(1 - t_2)};$$

$$I_2 = 2\Theta_2^2(1 + t_2) + \frac{4q_0^2(\alpha \beta q_0^2 - q_0^\alpha)}{(\alpha + \beta)(1 - t_2)};$$

$$I_3 = \frac{2q_0^2}{q^2} \{ \{M^2 - \Theta_1^3\}(1 + t_2) + \frac{4q_0^2[\alpha \beta(q_0^2 - q_0^\alpha) - k_0^2(\alpha + \beta)]}{q^2(\alpha + \beta)(1 - t_2)}
\nonumber + \frac{4q_0^2}{q^2(1 - t_2)}[(k_0 q_0 - (1 - x)q_0^2)(1 - t_2)^2 t_2
\nonumber - (k_0 q_0 + xq_0^2)(1 - t_2)^2 t_2] + \frac{2q_0^2}{q^2}(1 + t_2);$$

$$I_4 = 4\Theta_2^2(1 - t_2) - \frac{4q_0^2}{q_0(1 - t_2)}[(k_0 - (1 - x)q_0^2)
\nonumber \times (1 + t_2)^2 t_2 + (k_0 + xq_0)(1 + t_2)^2 t_2],$$

where $\Theta_1 = \eta q_0^2 - (q_0^2 + q_0^\alpha)(k_0^2 + k_0^\alpha) + (1 - 2x)k_0 q_0$, $\Theta_2 = k_0^2 - (1 - 2x)k_0 q_0 - k_0 - M^2 - \eta q_0^2$, and $k_0 = k_0 + (1 - x)q_0$. The identity $(1 - t)(1 - t_2)(\alpha + \beta) = 4(1 - t_t t_2)$ has been plugged in. The corresponding integral volume space is changed to:

$$d\Gamma = e^2 \int_0^1 dx \int_0^\infty d\tau \int \frac{d\Gamma_k}{(2\pi)^2} \int \frac{d\Gamma_k}{(2\pi)^2},$$

where

$$\int d\Gamma_k = \tilde{T} \sum_{l = -\infty}^{\infty} \exp[-(\omega_l - i(1 - x)q_0^2)\tau] \times \int d^2k \exp[-k^2 \tau];$$

$$\int d\Gamma_k = \int d^2k \exp[-(\alpha \beta)(k_1^2 + k_2^2)].$$

Note here that $k_0$ has been replaced by $i\omega_l$ with $\omega_l^2 = (2l + 1)^2 \pi^2 T^2 / (2eB)$.

With respect to the momenta $k^2$, the integrand of $\Pi^\mu$ is in a simple Gaussian form. Hence, we are able to write down the polarization tensors by the simple scalar functions and these scalar functions $\Xi$, are denoted as below:

$$\Xi_3 = \frac{1}{8\pi^3} \sum_{l = -\infty}^{\infty} \int_0^\infty \frac{k^2 \tilde{T} \tau}{1 - e^{-\tau}} \times \exp[-(v + k_3^2 + (l + \omega)^2 \tilde{T}^2) \tau + h(x, \tau)q_0^2].$$

(24)

Here $\tilde{T} = (2\pi T) / \sqrt{2eB}$ and $\omega = \frac{1}{2} - i(1 - x)q_0 / (2\pi T)$. Plus, the other scalar functions are in the form as:

$$\Xi_4 = \frac{1}{8\pi^3} \sum_{l = -\infty}^{\infty} \int_0^\infty \frac{\tilde{T} \tau_1 (\tilde{T})^{(0,1)}(l + \omega)^{(0,1)}}{(1 - e^{-\tau})^2} \times \exp[-(v + 1 + k_3^2 + (l + \omega)^2 \tilde{T}^2) \tau + h(x, \tau)q_0^2].$$

(25)

Poisson summation formula have been applied in equations (23)–(25) via

$$\sum_{l = -\infty}^{\infty} \exp[-(l + \omega)^2 \pi / \sigma] = \sigma^2 + 2 \sum_{l = 1}^{\infty} \sigma^2 \exp[-l^2 \pi / \sigma] \cos(2l\pi \sigma);$$

$$\sum_{l = -\infty}^{\infty} (l + \omega) \exp[-(l + \omega)^2 \pi / \sigma] = 2 \sum_{l = 1}^{\infty} l \sigma^2 \exp[-l^2 \pi \sigma] \sin(2l\pi \sigma);$$

$$\sum_{l = -\infty}^{\infty} (l + \omega)^2 \exp[-(l + \omega)^2 \pi / \sigma] = \frac{\sigma^2}{2 \pi} + 2 \sum_{l = 1}^{\infty} \left( \frac{\sigma^2}{2 \pi} - l^2 \pi / \sigma \right) \exp[-l^2 \pi \sigma] \cos(2l\pi \sigma),$$

(26)

and so on. In this work, $\sigma = \pi / (\tau \tilde{T}^2)$.

Obviously, comparing to the conventional Matsubara summation technique, the trick of Poisson summation is applied to split $l$ into positive values and zero, i.e. the parts of thermal, $\Xi_3^T$ (here $T$ denotes temperature) and vacuum, $\Xi_3^V$, respectively. All the results of vacuum contribution have been discussed in [36]. In this paper, we focus on the left thermal term.

5. Branch cuts and Landau damping

At finite temperatures, the modified polarization tensor is the necessary ingredient not only in calculating the related thermodynamic behaviors beyond tree level but also presenting the radiation spectra of photons. In this section, we will evaluate the magnetized photon self-energy in an analytical approach.
As discussed above, the extracting thermal scalar functions are written as:

\[ \Xi^2_i(v) = \frac{1}{16\pi^2} \sum_{n=0}^{\infty} \sum_{s=\pm 1} \sum_{l=1}^{\infty} \rho_o \exp \left[ -(v+n)\tau \right] \]

\[ - \frac{\pi q^2}{T^2 \tau} + h(x, \tau) q^2 + \frac{sl}{T}(1-x)q_0 \cos(\tau \pi). \]

\[ \text{(27)} \]

We have rephrased the above equations via the geometric tricks

\[ \Xi_{\text{loc}}(v) = \frac{1}{16\pi^2} \sum_{n=0}^{\infty} \sum_{s=\pm 1} \sum_{l=1}^{\infty} \rho_o \exp \left[ -(v+n)\tau \right] \]

\[ - \frac{\pi q^2}{T^2 \tau} + h(x, \tau) q^2 + \frac{sl}{T}(1-x)q_0 \cos(\tau \pi). \]

\[ \text{(28)} \]

As discussed in [36], \( h(x, \tau) \) locates in the interval \([0, 1]\). It limits to \( \eta \) as \( \tau \to 0 \) and behaves as \( \tan h \frac{\pi}{4} \approx 1 \) as \( \tau \gg 1 \).

Without including \( h(x, \tau) q^2 \), the rest integral with respect to \( \tau \) can be written in terms of a modified Bessel function of second kind, \( K_n \), where

\[ \int_0^\infty \tau^{\nu-1} \exp \left( -\lambda \tau - \frac{\chi}{\tau} \right) d\tau = 2 \frac{\chi}{\lambda} K_n(2\sqrt{\lambda \chi}), \]

\[ \text{(29)} \]

for \( \Re \lambda > 0 \) and \( \Re \chi > 0 \). From equation (27), we have \( \lambda = v + n \) and \( \chi = 4e^{-\beta_\nu} / T^2 \). The most contributed integration with respect to \( \tau \) is from the regime where \( \tau \sim (\chi/\lambda)^{\nu} \sim 2eB/(MT) \) for \( n = 0 \) and \( \sim \sqrt{2eB / T} \) for \( n \gg 1 \). In this work, we focus on the region of temperature where \( T \leq (eB)^\nu \) and \( h(x, t) \) is simply determined by the ratio of \( M \) and \( (eB)^\nu \). It means we are able to set

\[ h(x, t) \approx \eta \tau \sec \delta + \theta(\tau) \tan h \delta, \]

\[ \text{(30)} \]

where \( \delta = 2eB/(MT) \). From now on, \( q^2 = q_0^2 + q_2^2 \sec \delta \) and \( \hat{q}^2 = q_0^2 + q_2^2 \sec \delta \) in this work without description.

More precisely, as \( q^2 > 4M_0^2 \), equation (29) is not applicable since \( \lambda \) is not positive defined. The non-positive \( \lambda \) induces instability in exponential integration and contributes additional imaginary value. To include it, we express the result by the Cauchy's principal value method, and

\[ \text{Im} \left( \int_0^\infty \tau^{\nu-1} \exp \left( -\lambda \tau - \frac{\chi}{\tau} \right) d\tau \right) = \pi \frac{\chi}{\lambda} J_{\nu}(2\sqrt{\lambda \chi}), \]

\[ \text{(31)} \]

where \( J_{\nu}(z) \) is the Bessel functions of the first kind for \( \Re \lambda > 0 \) and \( \Re \chi > 0 \). According to the formula \( J_{\nu}(e^{i\pi/2}z) = e^{i\pi\nu/2}J_{\nu}(z) \) for integer \( n \), one gets \( J_{\nu}(z)|_{\nu=\nu_0+1} = -J_{\nu}(z)|_{\nu=\nu_0-1} \) for half integer \( \nu \).

At \( \nu = \frac{1}{2} \), one has \( J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z \). Therefore, we obtain

\[ \text{Im} (\Xi_0^2) = \frac{(eB)^2 T}{16\pi^2} \int dk_3 \sin (\eta q^2 - E^2) \]

\[ \times \int \frac{d^2 q_0}{q^2} \left( \sum_{n=0}^{\infty} \sum_{s=\pm 1} \sum_{l=1}^{\infty} (-1)^l e^{i(l-x)q_0} \right) \]

\[ \times \exp \left[ i \nu \left( \frac{q_0^2}{\eta (q^2 - E^2)} \right) \right] \]

\[ \times \sum_{l=1}^{\infty} \left( -1 \right)^l \bar{\xi}_l (q^2 - E^2)^{l/2}, \]

\[ \text{(32)} \]

where \( \xi_l = \sqrt{1 - 4e^2 / q^2} \) and \( \bar{\xi}_l = \sqrt{q^2 / 4 - M_0^2} \) with \( E_n^2 = M_n^2 + k_n^2 \) and \( M_n^2 = M^2 + 2neB \). Here we have impose an abbreviation that \( f = \int d^2 k_3 \sin (\eta q^2 - E^2) \). Taylor expansion of \( \sin (\eta q) / x = 1 - \frac{\eta q^2}{2} \) has been applied. One notices that the integrand is exponentially suppressed in the low temperature limit for \( \eta q \gg T \). Hence, we will render only the result at high temperatures. Obviously, in strong magnetic limit, the allowed momentum region \( q_0^2 > q^2 > 2neB \gg T^2 \) is extremely precluded for finite \( n \). The LLL is left and contributes an imaginary part in the time-like momenta, which is similar to the result at zero temperature.

Back to the kinetic region where equation (29) is applicable, for \( \nu = \frac{1}{2} \), the modified Bessel function at the second kind can be rewritten as \( K_{\nu}(z) = \sqrt{\frac{z}{\pi}} e^{-z} \). We apply the saddle point method to integrate with respect to \( x \), which read as:

\[ I(N) = \int_a^b g(z) \exp[Nf(z)] dz \]

\[ \approx e^{\nu \phi} \left( \frac{2\pi}{\sqrt{-NF^2(z_0)}} \right)^{\nu} g(z_0) \exp[Nf(z_0)]. \]

\[ \text{(33)} \]

Employing the above formula, the summation of \( l \) is easily obtained, we get

\[ \int_0^\infty \Xi^2_i(v) dv \approx -\frac{(eB)^2 T^2}{16\pi^2} \sum_{n=0}^{\infty} \sum_{s=\pm 1} e^{i\nu \phi} \]

\[ \times \left( \frac{2\pi}{\sqrt{\left| \frac{\nu}{\nu+1} \right|}} \right)^{\nu} \frac{e^{i\nu \phi}}{ \sqrt{E_n^2 - x_0 (1-x_0)q^2}}, \]

\[ \text{(34)} \]
We will extend \( q_0 \) to \( q_0 + i \epsilon \) to determine the value of \( \phi \) at below.

To obtain a real root of \( x_0 \) locating in the interval \([0, 1]\), it requires time-like kinetic regime \( q^2 > 4E_n^2 \geq q^2 / q_0^2 \geq 0 \) or space-like region where \( q^2 < 4E_n^2 < q^2 / q_0^2 \). In the latter case \( q^2 \) must be a space-like momentum since \( q^2 q_0^2 \) is always larger than \( q^2 \) for positive \( q^2 \).

Precisely, the allowed regions are \( 4E_n^2 < q^2 < 2E_n^2 + 2\sqrt{E_n^2 (q_0^2 - q^2)} < 4E_n^2 + q^2 \) and \( q^2 < 2E_n^2 - 2\sqrt{E_n^2 (q_0^2 - q^2)} < 0 \). In these two regions, the result of equation (34) becomes pure imaginary.

For \( q^2 > 4E_n^2 \), the saddle point of \( x_0 \) locates

\[
x_0 = \frac{1}{2} - \frac{s \text{sign}(q_0)}{2} \sqrt{q_0^2 - 4E_n^2}.
\]

(35)

It gives

\[
\sqrt{E_n^2 - x_0 (1 - x_0) q_0^2} = \sqrt{q^2 - 4E_n^2 q_0^2} / (2|q|) \text{, }
\]

where \( s e^{\phi^q} (- x_0) = 2 \frac{|q|}{|q|} \sqrt{|q^2 - 4E_n^2 q_0^2|} - i \text{sign}(q_0) \epsilon \).

Generally speaking, while \( x_0 \) is out of the range \((1/2, 1)\), the integration with respect to \( x \) in equation (34) reduces to equation (32).

Moreover, the interval \((1 - \frac{s}{2}, 1 + \frac{s}{2})\) shrinks to one point at \( q^2 = 4E_n^2 \) with \( x_0 = 1/2 \). Indeed, there is no determined root of equation \( \partial x_0 / \partial q_0 |_{x_0} = 0 \) at this point. We shall not include it but refer its contribution rendered by equation (32). To avoid this double counting, we set a cutoff on \( k_3 \) to require \( q^2 - 4E_n^2 \geq g^2 T^2 \) and \( g \ll 1 \) in high temperature limit.

Like pointed out in [29], the interval of \( q^2 \in (4E_n^2, 4E_n^2 + q^2) \) is belong to the collinear region, which enhances the infrared singularity. Finally, for \( q^2 > 4E_n^2 \), one has:

\[
\text{Im} \langle \Xi_0^j \rangle = \frac{\text{sign}(q_0) (2eB) T_0 \gamma}{16 \pi^2 |q|^2 |q_0|^2} \sum_{n=0}^{j} \sum_{i=1}^{\Lambda_1} \left( q^2 - 4E_n^2 \right)^{rac{1}{2}} \times \frac{1}{\partial k_3^2} \left[ \exp \left( -s x_0 + \frac{i}{4} \sqrt{q^2 - 4E_n^2} \right) \right] \text{d}E_n^2,
\]

(36)

where \( j = \left| q^2 / 4 - M^2 \right| \), \( \Lambda_1 = \frac{q^2}{4} - g^2 T^2 \) and \( \Lambda_2 = \max \left( \frac{q^2}{4}, M_0^2 \right) \).

For \( q^2 < 0 \), one has:

\[
x_0 = \frac{1}{2} - \frac{s \text{sign}(q_0)}{2} \sqrt{\frac{q_0^2 q^2 - 4E_n^2}{q^2}}.
\]

(37)

It leads \( e^{\phi^q} (- x_0) = 2 \frac{|q|}{|q|} \sqrt{|q^2 - 4E_n^2 q_0^2|} + i \text{sign}(q_0) \epsilon \) and

\[
\text{Im} \langle \Xi_0^j \rangle = \frac{\text{sign}(q_0) (2eB) T_0 \gamma}{16 \pi^2 |q|^2 |q_0|^2} \sum_{n=0}^{j} \sum_{i=1}^{\Lambda_1} \left( q^2 - 4E_n^2 q_0^2 \right)^{rac{1}{2}} \times \frac{1}{\partial E_n^2} \left[ \exp \left( -s x_0 + \frac{i}{4} \sqrt{q^2 - 4E_n^2 q_0^2} \right) \right] \text{d}E_n^2,
\]

(38)

where \( j = \left| q^2 / (8eB q_0^2) - M^2 \right| \).

It is easily to see we have two new branch cuts along the real axis of \( q_0 \). After extended them to the complex plane, one has

\[
\text{Disc} \pi(q_0) = \pi(q_0 + i \epsilon) - \pi(q_0 - i \epsilon) = 2i \text{Im} \pi(q_0).
\]

We will write down the discontinuities explicitly later.

Representing by equation (32), the first branch cut, \( q_0^2 > 4M_0^2 + q^2 \), is due to the conventional process \( \gamma = \psi + \psi \) and its final result is same as the vacuum contribution. On the other hand, the second branch cut, \( 4M_0^2 < q_0^2 < 4M_0^2 + q^2 \), is infrared divergent inducing by the soft particles moving in thermal field. To subtract the divergence, in the chiral limit \( M \to 0 \), we impose an angular cutoff in Feynman integrals and obtain:

\[
\text{Disc} \pi(q_0) \approx 2 \text{Im} \langle (\eta q_0^2) \Xi_0^j \rangle - (2eB) \langle \Xi_0^j \rangle \rangle
\]

\[
= \frac{(2eB) |q_0|^3}{2^{31} \cdot T^2};
\]

\[
\text{Disc} \pi(q_0) = 0;
\]

\[
\text{Disc} \pi(q_0) = - \frac{q_0^2}{q^2} \text{Disc} \pi(q_0);
\]

\[
\text{Disc} \pi(q_0) = 0.
\]

(39)

The third branch cut, \( q_0^2 < q_0^2 + 2M_0^2 \), is developed at finite temperatures, which is corresponding to the process of \( \gamma = \psi + \psi \). One has:

\[
\text{Disc} \pi(q_0) \approx 2 \text{Im} \langle (M^2 + \eta q_0^2) \Xi_0^j \rangle + (2eB) \langle \Xi_0^j \rangle \rangle + (2eB) \langle \Xi_0^j \rangle \rangle + \Xi_0^j (v + 1)
\]

\[
= \sum_{n=0}^{j} \sum_{i=1}^{\Lambda_1} \left( 1 - \frac{1}{2} (2eB) q_0^2 + 4q_0^2 M_0^2 \right)^{\frac{1}{2}}
\]

\[
\times \text{Li}_{-1} (- e^{- T^2 q_0^2})
\]

\[
\text{Disc} \pi(q_0) \approx \frac{q_0^2}{2eB} \text{Disc} \pi(q_0);
\]

\[
\text{Disc} \pi(q_0) \approx - \frac{q_0^2}{3q^2} \text{Disc} \pi(q_0);
\]

\[
\text{Disc} \pi(q_0) = 0.
\]

(40)

Finally, Disc \( \Pi^{\mu \nu} (q_0) \) is represented by summation

\[
\sum_{n=0}^{N} \pi^{\mu \nu} (q_0) \text{Disc} \pi(q_0) \text{ as denoted before. Roughly speaking, the finite-} n \text{ Landau level contribution is exponentially suppressed by } e^{-n}, \text{ which is the baseline to apply the LLL approximation in strong B-field. But, in a strict manner, the } n \text{th Landau level presents as } L_n (\alpha) e^{-n}. \text{ Indeed, the Laguerre polynomial was neglected improperly in lots of early works. For large } n, \text{ the asymptotic behavior of } L_n (\alpha) \text{ is limit to } n^2 e^{-n} \text{ which can not be ignored. Hence, } L_n (\alpha) e^{-n} \text{ is characterized by a non-monotonic behavior of } n \text{ when } \alpha \geq 1. \text{ In other words, the LLL approximation will never able to get similar results like equation (40) and this approximation is not enough to}.
\]
describe the thermal system when the transverse part, i.e. $L_n^{(1)}$, plays an important role in the allowed kinetic regimes.

6. Conclusion and discussions

In this work, we have completed a calculation of the photon polarization tensor at finite temperature in the media of a static and homogeneous magnetic field. In our process, the summation of Matsubara frequency is applied elegantly, since its concise formulation limits well and achieves easily in the approach of proper time representation. Meanwhile, it was argued that the summation of Matsubara frequency is not commuted with summing Landau levels [37], such problem does not occur in our calculations. As expected, without an additional divergent environment included, the summation of Landau levels is convergent. Before, one has to cut out the cutoff of the Landau levels via a test of numerical method. Unlike such cumbersome way, we truncate the Landau levels in a systematic consideration since the dependence of Landau levels is expressed in an obviously manner.

It is known that the self-energies of gauge bosons take the same forms for both QED and QCD plasmas in the limit of long wavelength [41]. It takes place in the strong magnetic field limit $(2\epsilon B) \gg q_0^2$, as well. In the space-like momenta regime $q_2^2 \gg q_0^2$, it is allowed us to define $(2\epsilon B) \sim \lambda T^2$ and $\lambda > 1$. The classification of the energy scale is similar to the hard-loop action, where loop momenta $k \sim M_n$ for finite-$n$ Landau levels while as the external momenta $q_1 \sim \lambda^{-2} T^2 M_n^2$, $q_0 \sim \lambda \sim T$. Described by the result of equation (40), $\text{Disc} \, n$ is at the order of $\lambda^2 \text{Li}_{-1}(\epsilon^{-\lambda})$, which is not monotonically decreasing as $\lambda$ increasing. Physically, such unique feature is essentially same as other gauge theories governed plasma systems. This large imaginary part only arises at finite temperatures, which is the so called Landau damping. It demonstrates the absorption of soft fields by hard plasma constituents. Such kind of processes gain the contribution not only from the lowest (soft) Landau level but also up to the finite-$n$ (hard) levels. Eventually, we conclude that the LLL approximation is suit at zero temperature but not well described at the finite temperatures. In particular, the hard-loop approximation is taking control for the magnetized plasma systems, whose constituent of typical momentums are much larger than the probing wave vectors.

Our results have two-fold applications in the next future. One is that the catalyzing finite Landau levels contribution in thermal medium has the potential to solve the IMC which observed in the laboratory simulation. The second is that the modification of the photon production rate is a new effect which has to be considered for discussing photon $v_2$ puzzle in heavy ion collisions.

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