Symmetry analysis for a charged particle in a certain varying magnetic field

Karmadeva Maharana*;†

*Department of Mathematics,
Massachusetts Institute of Technology, Cambridge MA 02139, USA
email: maharana@math.mit.edu
and
†Physics Department, Utkal University, Bhubaneswar 751 004, India
email: karmadev@iopb.res.in

Abstract

We analyze the classical equations of motion for a particle moving in the presence of a static magnetic field applied in the z direction, which varies as $\frac{1}{x^2}$. We find the symmetries through Lie’s method of group analysis. In the corresponding quantum mechanical case, the method of spectrum generating $su(1, 1)$ algebra is used to find the energy levels for the Schroedinger equation without explicitly solving the equation. The Lie point symmetries are enumerated. We also find that for specific eigenvalues the vector field contains $\frac{1}{x} \frac{\partial}{\partial x}$ and $\frac{1}{x^2} \frac{\partial}{\partial x}$ type of terms and a finite Lie product of the generators do not close.

*Present address
†Permanent address
1 Introduction

We use some techniques from group theory to analyse the model physical situation of a charged particle in magnetic field $B_x = 0$, $B_y = 0$, and $B_z = \frac{1}{x^2}$. Usually the symmetries of such systems are found with the use of Noether’s theorem by setting up a Lagrangian. However, a Lagrangian formulation of the problem sometimes becomes difficult, as in Witten’s example [2] of a classical Wess Zumino model[1], representing the motion of a charged particle in the presence of a magnetic monopole and constrained to move on the surface of the sphere. The equations of motion in that case cannot be obtained from a Lagrangian as no Lagrangian can be written whose variation will give the desired equation of motion, if we restrict ourselves not going to a higher dimensional theory. So in this case the usual Noether’s theorem is difficult to apply and the generalised method has to be followed to find the symmetries from the classical equation of motion [3],[4]. Finding the generators of the symmetries will inform us which symmetries exist and which of the symmetries are broken when the magnetic field is applied.

We also consider the related quantum mechanical case. The method of spectrum generating algebra is applied to obtain the energy eigenvalues without solving the eigenvalue equation.

The quantum case is dealt first in a manner analogous to the treatment followed by Landau to explain the diamagnetism arising out of conduction electrons. The Schrödinger equation separates to a second order differential equation involving $x$. The energy eigenvalues for this equation can be obtained by mapping this problem to that of the spectrum generating algebra of $SU(1,1)$ by identifying the generators of the algebra with the (differential ) operator realization of the algebra, and calculating the Casimir invariant.

We try to find the Lie point symmetries of this differential equation and it is found that, when certain relations are satisfied by the coefficients of the different terms,some symmetry exist. The vector fields representing the symmetries do not close under the Lie product. The fact that the Schrödinger equation can have solutions when certain symmetries exist is verified in three cases.
The quantum eigenvalues

The Schrödinger equation for a spinless charged particle in the presence of a magnetic field is obtained by the usual procedure of replacing $p^a$ by $p^a + \frac{e}{c} A^a$ in the free particle equation \[^5\]. For the time being we consider any electrostatic potential to be absent. Here $A^a$ is the vector potential and we choose a particular gauge where it is given by $A = \{ A_x = 0, A_y = \frac{\mathbf{B} \times x}{x}, A_z = 0 \}$ so that the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ is $\{ B_x = 0, B_y = 0, B_z = -\frac{\mathbf{B} \times x}{x^2} \}$. Here $\mathbf{B}$ is a constant. Thus the Hamiltonian operator becomes

$$H = \frac{1}{2m} \left( p + \frac{e}{c} \mathbf{A} \right)^2 = -\frac{\hbar^2}{2m} \nabla^2 + \frac{\hbar}{i mc} \mathcal{B} \frac{1}{x} \frac{\partial}{\partial y} + \frac{e^2 \mathbf{B}^2}{2mc^2 x^2}.$$ \hspace{1cm} (2.1)

Denoting

$$\frac{e\mathbf{B}}{2mc} = \omega_B$$ \hspace{1cm} (2.2)

the Schrödinger equation becomes

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 + \frac{\hbar}{i mc} \omega_B \frac{1}{x} \frac{\partial}{\partial y} + 2m \omega_B^2 \frac{1}{x^2} \right] \phi = E \phi.$$ \hspace{1cm} (2.3)

The second and third terms have only explicit $x$ dependence and we may use

$$\phi(x, y, z) = u(x) e^{ik_y y + k_z z}$$ \hspace{1cm} (2.4)

to separate out the $x$ dependent part. We thus obtain, for $u(x)$, the equation

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dx^2} + \frac{1}{2m} \left( \hbar k_y + \frac{2m \omega_B^2}{x} \right)^2 u = \left( E - \frac{\hbar^2 k_z^2}{2m} \right) u.$$ \hspace{1cm} (2.5)

It is interesting to note that the Schrödinger equation for Kratzer’s molecular potential is a special case of the above equation. The Kratzer molecular potential is of the form, in our notation,

$$V(x) \propto \left( \frac{a}{x} - \frac{1}{2} \frac{a^2}{x^2} \right)$$ \hspace{1cm} (2.6)

and the eigenvalue equation is solved by the usual series expansion method. The eigenfunctions turn out to be a general type of Kummer’s hypergeometric function $\mathbf{1}_1 F_1 \[6\]$. Also other (electrostatic) potentials of the form $\frac{1}{x}$ and $\frac{1}{x^2}$ can be included in the above equation, that will just change the appropriate coefficients C, D and E.
and Ghirardi have calculated the energy eigenvalues for such cases using the spectrum generating algebra method [7]. We write the equation (2.5) as

$$\frac{d^2u}{dx^2} + \left(\frac{C}{x^2} + \frac{D}{x} + \hat{E}\right)u = 0$$

(2.7)

where

$$C = -\frac{4m^2\omega_B^2}{\hbar^2}, \quad D = -\frac{4kym\omega_B}{\hbar}, \quad \hat{E} = \frac{2mE}{\hbar^2} - (k_y^2 + k_z^2).$$

(2.8)

The generation of the spectrum associated with a second order differential equation of the form

$$\frac{d^2R}{ds^2} + f(s)R = 0$$

(2.9)

where

$$f(s) = \frac{a}{s^2} + bs^2 + c$$

(2.10)

has been analysed by several authors [8]. To bring equation (2.7) to the above form we set

$$x = s^2, \quad u(x) = s^\frac{\alpha}{2} R(s)$$

(2.11)

to get

$$\frac{d^2R}{ds^2} + \left[\left(\frac{16C - 3}{4}\right)\frac{1}{s^2} + 4\hat{E}s^2 + 4D\right]R = 0.$$  

(2.12)

We indicate in brief the procedure to obtain the eigenvalues. The Lie algebra of non-compact groups $SO(2,1)$ and $SU(1,1)$ can be realized in terms of a single variable by expressing the generators [9]

$$\Gamma_1 = \frac{\partial^2}{\partial s^2} + \frac{\alpha}{s^2} + \frac{s^2}{16}$$

(2.13)

$$\Gamma_2 = -\frac{i}{2}(s \frac{\partial}{\partial s} + \frac{1}{2})$$

(2.14)

$$\Gamma_3 = \frac{\partial^2}{\partial s^2} + \frac{\alpha}{s^2} - \frac{s^2}{16}$$

(2.15)

so that the $\Gamma$’s satisfy the standard algebra

$$[\Gamma_1, \Gamma_2] = -i\Gamma_3, \quad [\Gamma_2, \Gamma_3] = i\Gamma_1, \quad [\Gamma_3, \Gamma_1] = i\Gamma_2.$$  

(2.16)
The existence of the Casimir invariant for $su(1,1)$

$$\Gamma^2 = \Gamma_3^2 - \Gamma_1^2 - \Gamma_2^2$$

(2.17)

is exploited to obtain the explicit form of $\Gamma_i$'s. The second order differential operator in equation (2.9) in terms of the $su(1,1)$ generators is now

$$\frac{\partial^2}{\partial s^2} + \frac{a}{s^2} + bs^2 + c = \left( \frac{1}{2} + 8b \right) \Gamma_1 + \left( \frac{1}{2} - 8b \right) \Gamma_3 + c$$

(2.18)

and (2.9) becomes

$$[(\frac{1}{2} + 8b) \Gamma_1 + (\frac{1}{2} - 8b) \Gamma_3 + c] \mathcal{R} = 0.$$  (2.19)

Next a transformation involving $e^{-i \theta \Gamma_2}$ can be performed on $\mathcal{R}$ and the $\Gamma$'s. A choice of $\theta$ such that

$$\tanh \theta = -\frac{\frac{1}{2} + 8b}{\frac{1}{2} - 8b}$$

(2.20)

will diagonalize the compact operator $\Gamma_3$ and the discrete eigenvalues may be obtained. The arguments of the standard representation theory then leads to the result,

$$4n + 2 + \sqrt{1 - 4a} = \frac{c}{\sqrt{-b}}, \quad n = 0, 1, 2, \ldots$$

(2.21)

Substitution of the corresponding values from equation (2.12)

$$a = (\frac{16C - 3}{4}), \quad b = 4 \hat{E}, \quad c = 4D$$

(2.22)

gives

$$\hat{E} = -\frac{16m^2 \omega_B^2 k_y^2}{\hbar^2 \left[ (2n + 1) + \left( 1 + \frac{16m^2 \omega_B^2}{\hbar^2} \right) \frac{1}{2} \right]^2}.$$  (2.23)

Having obtained the energy eigenvalues, an analysis in the context of two dimensional confined systems on the lines of de Haas - van Alphen effect and quantum Hall effects would be of much interest from the physics point of view.

### 3 Group Analysis

Since symmetries and symmetry breakings play such a fundamental role in physics, we are interested in finding the symmetry generators in the presence and in the absence of the fields.
We try to find the symmetries of the classical electrodynamics for a charged particle moving in the presence of a magnetic field.

The equation of motion of a charged particle with charge $e$ and mass $m$ in a magnetic field $B$ is the Lorentz force equation

$$\ddot{q}^a = \frac{e}{m} \varepsilon^{abc} \dot{q}^b B^c$$  \hspace{1cm} (3.24)

where $a = 1, 2, \text{ and } 3$, and $q^1 = x$, $q^2 = y$, and $q^3 = z$,

with a dot representing differentiation with respect to time. $B$ in our case is given by

$$B = \{B_x, B_y, B_z\} = \{0, 0, -\frac{B_x}{x^2}\}$$  \hspace{1cm} (3.25)

It is convenient to write the set of coupled equations in the following form to perform the group analysis.

$$\ddot{q}^a = \frac{e}{m} \omega^a(q^i, \dot{q}^i, t)$$  \hspace{1cm} (3.26)

where $a, i = 1, 2, \text{ and } 3$, and $\omega^a$ is equal to the right hand side of equation (3.24). Henceforth we follow the notation and method of Stephani to find the symmetry generators.

These set of equations can be analysed by means of one parameter groups by infinitesimal transformations. We demand the equation to be invariant under infinitesimal changes of the explicit variable $t$, as well as simultaneous infinitesimal changes of the dependent functions $q^a$ in the following way,

$$t \rightarrow t_1 = t + \epsilon \xi(t, q^1, q^2, q^3) + O(\epsilon^2),$$

$$q^a \rightarrow q_1^a = q^a + \epsilon \eta^a(t, q) + O(\epsilon^2).$$  \hspace{1cm} (3.27)

Under $t \rightarrow t_1$ and $q^a \rightarrow q_1^a$, the equation changes to,

$$\ddot{q}_1^a = \omega^a(t_1, q_1^i, \dot{q}_1^i)$$  \hspace{1cm} (3.28)

where we have set $\frac{e}{m}$ as unity. To illustrate the procedure consider the simple case in one space dimension. We express the above equation in terms of $t$ and $q$ by using the transformation (3.27). Then the invariance condition implies that an expression
containing various partial derivatives of $\xi$ and $\eta$ is obtained which equates to zero. For example, we get
\[
\frac{dq_1}{dt} = \frac{dq}{dt} + \epsilon \left( \frac{\partial \eta}{\partial q} \frac{dq}{dt} + \frac{\partial \xi}{\partial q} \right) + O(\epsilon^2). 
\] (3.29)
and now relate the left hand side with $\frac{dq}{dt}$ by using binomial theorem for the denominator to obtain
\[
\frac{dq_1}{dt} = \frac{dq}{dt} + \epsilon \left( \frac{\partial \eta}{\partial q} - \frac{\partial \xi}{\partial t} \right) \frac{dq}{dt} - \frac{\partial \xi}{\partial q} \frac{dq}{dt} + O(\epsilon^2). 
\] (3.30)
A similar procedure is followed to express $\frac{d^2 q_1}{dt^2}$ likewise. By substituting equations (3.27) - (3.30) for a given explicit expression for $\omega_1$ and remembering that $\frac{d^2 q}{dt^2} - \omega$ is zero, we obtain the desired partial differential equation whose solution would determine $\xi(t, q)$ and $\eta(t, q)$. In our case, of course, we have to find $\xi(t, q^1, q^2, q^3)$ and $\eta(a, t, q^1, q^2, q^3)$’s.

To relate these to the generators of the infinitesimal transformations we write
\[
t_1(t, q^i; \epsilon) = t + \epsilon \xi(t, q^i) + \cdots = t + \epsilon X t + \cdots 
\] (3.31)
\[
q^a_1(t, q^i; \epsilon) = q^a + \epsilon \eta(t, q^i) + \cdots = q^a + \epsilon X q^a + \cdots 
\] (3.32)
where the functions $\xi$ and $\eta^a$ are defined by
\[
\xi(t, q^i) = \frac{\partial t_1}{\partial \epsilon} |_{\epsilon = 0}, 
\] (3.33)
\[
\eta^a(t, q^i) = \frac{\partial q^a_1}{\partial \epsilon} |_{\epsilon = 0} 
\] (3.34)
and the operator $X$ is given by
\[
X = \xi(t, q^i) \frac{\partial}{\partial t} + \eta^a(t, q^i) \frac{\partial}{\partial q^a}. 
\] (3.35)
Following Stephani [3], we will consider the equation having the symmetry generated by $X$ and its extension
\[
\dot{X} = \xi \frac{\partial}{\partial t} + \eta^a \frac{\partial}{\partial q^a} + \dot{\eta}^a \frac{\partial}{\partial q^a} 
\] (3.36)
and the symmetry condition determines $\dot{\eta}^a$. The symmetry condition is given by
\[
\xi \omega^a, t + \eta^b \omega^a, b + (\eta^b, t + \dot{q}^b \eta^b, c - \dot{q}^b \dot{q}^c \xi, c) \frac{\partial \omega^a}{\partial q^b} \\
+ 2 \omega^a (\xi, t + \dot{q}^b \xi, b) + \omega^b (\dot{q}^a \xi, b - \eta^a, b) + \dot{q}^b \dot{q}^c \xi, bc \\
+ 2 q^a c \xi, tc - q^b \dot{q}^b \eta^a, bc + \dot{q}^a \xi, tt - 2 \dot{q}^a \eta^a, tb - \eta^a, tt = 0 
\] (3.37)
where \( f, t = \frac{\partial f}{\partial t} \) and \( f, q = \frac{\partial f}{\partial q} \). By herding together coefficients of the terms with cubic, quartic, and linear in \( \dot{q}^a \), and the ones independent of \( \dot{q}^a \) separately, and equating each of these to zero we obtain an over determined set of partial differential equations and solve for \( \xi \) and \( \eta^a \). The condition (3.37) for \( a = 1, 2 \) gives

\[
\xi = \lambda, \quad \eta^1 = 0, \quad \eta^2 = \sigma,
\]  

(3.38)

and for \( a = 3 \) we obtain

\[
\eta^3 = \rho + q^3
\]  

(3.39)

where \( \lambda, \sigma, \) and \( \rho \) are constants. This gives rise to the vector fields

\[
X_\xi = \lambda \frac{\partial}{\partial t}, \quad X_{\eta^2} = \sigma \frac{\partial}{\partial q^2}, \quad X_3 = \rho \frac{\partial}{\partial q^3}, \quad X_{\eta^3} = q^3 \frac{\partial}{\partial q^3},
\]  

(3.40)

that forms a solvable Lie algebra.

This may be compared with the results for the cases where no magnetic field is present and for a magnetic field proportional to \( q^a \). In the case of no magnetic field present, the equations trivially decouple. Each equation

\[
\ddot{q}^a = 0
\]  

(3.41)

has the eight parameter symmetry generator of the general projective transformation

\[
X = [a_1 + a_2 t + a_3 q^a + a_4 t q^a + a_5 t^2] \frac{\partial}{\partial t} + [a_6 + a_7 t + a_8 q^a + a_9 t q^a + a_4 (q^a)^2] \frac{\partial}{\partial q^a}.
\]  

(3.42)

For \( B^a = q^a \) the five generators are [10]

\[
X_{q^a} = \varepsilon^k_a k^h \frac{\partial}{\partial q^k}, \quad X_\xi = \frac{\partial}{\partial t}, \quad X_5 = t \frac{\partial}{\partial t} - q^a \frac{\partial}{\partial q^a}.
\]  

(3.43)

(3.44)

(3.45)

In order to find the Lie point symmetries of the corresponding quantum case we go back to the Schrödinger equation (2.7). We write this equation as

\[
u'' = \omega(x, u, u')
\]  

(3.46)
where
\[ \omega(x, u, u') = -\left(\frac{C}{x^2} + \frac{D}{x} + \hat{E}\right)u(x). \] (3.47)

The infinitesimal generator of the symmetry under which the differential equation does not change is given by the vector field
\[ X = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u} \]
(3.48)
and for a second order differential equation, \( \xi \) and \( \eta \) are to be determined from
\[ \omega(\eta, u - 2\xi, x, 3u\xi, u) - \omega_x \xi - \omega_u \eta - \omega_{u'}[\eta, x] + u'(\eta, u - \xi, x) - u^2 \xi, u] 
+ \eta_{xx} + u'2\eta_{xu} - \xi_{xx} + u^2(\eta_{uu} - 2\xi, xu) - u^3 \xi, uu = 0 \]
(3.49)
where a prime denotes differentiation with respect to \( x \), the partial derivative of a function by a comma followed by the variable with respect to which the derivation has been performed.

The symmetry condition (3.49) for our equation (3.46) is
\[ -\left[\left(\frac{C}{x^2} + \frac{D}{x} + \hat{E}\right)u(x)\right]\eta, u - \xi, xu + \left(\frac{3C}{x^2} + \frac{D}{x} + \hat{E}\right)\eta + \eta_{xx} 
- u'\left[3\left(\frac{C}{x^2} + \frac{D}{x} + \hat{E}\right)\xi, u + 2\eta_{xu} - \xi_{xx}\right] + u^2(\eta_{uu} - 2\xi, xu) - u^3 \xi, uu = 0. \]
(3.50)
Equating to zero the coefficients of \( u^3 \) and \( u^2 \) we get
\[ \xi, uu = 0, \qquad \eta, uu = 2\xi, xu \]
(3.51)
which are satisfied for
\[ \xi = u\alpha(x) + \beta(x), \qquad \eta = u^2\alpha'(x) + u\gamma(x) + \delta(x). \]
(3.52)
Using these and equating to zero the coefficient of \( u' \) one obtains
\[ 3u[\alpha''(x) + \left(\frac{C}{x^2} + \frac{D}{x} + \hat{E}\right)\alpha(x)] + 2\gamma'(x) - 2\beta''(x) = 0. \]
(3.53)
This shows that either
\[ \alpha = 0 \]
(3.54)
or \( \alpha(x) \) satisfies the same equation as \( u(x) \) does and
\[ 2\gamma'(x) = \beta''(x). \]
(3.55)
This integrates to
\[ \gamma(x) = \frac{\beta'(x)}{2} + \kappa \]  
(3.56)

where \( \kappa \) is a constant. The rest of the equation, after using the fact that \( \alpha(x) \) satisfies the same equation as \( u(x) \) does, boils down to
\[ \beta'''(x) + 4\left(\frac{C}{x^2} + \frac{D}{x} + \hat{E}\right)\beta'(x) - \left(\frac{4C}{x^3} + \frac{2D}{x^2}\right)\beta(x) = 0 \]
(3.57)

with \( \delta(x) \) also obeying the same equation as \( u(x) \) does. The simplest nontrivial solution to this equation is
\[ \beta(x) = \frac{p}{x} + q \]
(3.58)

where \( p \) and \( q \) are constants and \( C, D, \) and \( \hat{E} \) must satisfy
\[ q = 2pD, \quad C = -\frac{3}{4}, \quad D^2 = -\hat{E}. \]
(3.59)

One sees that these conditions are in principle possible to be satisfied. It is further worth noting that the condition \( D^2 = -\hat{E} \) is satisfied for \( C = -\frac{3}{4} \) if we take the square root to be only negative while evaluating \( \sqrt{1 - 4a} \) and for \( n = 1 \). Also note that the symmetry is present for a particular value of the energy. It may be recalled that \( n \) must be zero or a positive integer for our analysis if any meaningful eigenvalue is to be obtained. The above result indicates that only for \( n = 1 \) there is the symmetry corresponding to \( \beta = \frac{p}{x^2} + q \).

So the original equation (3.46) has the symmetry represented by the vector field
\[ X = \left[u\alpha(x) + \frac{p}{x} + 2pD\right] \frac{\partial}{\partial x} + \left[u^2\alpha'(x) - u\frac{p}{x^2} + u\kappa + \delta(x)\right] \frac{\partial}{\partial u}. \]
(3.60)

Here \( u, \alpha, \) and \( \delta \) are related to the hypergeometric functions \( _1F_1 \) which are solutions to generalised Kummer type of equations mentioned in section 2. We immediately observe that the generators do not close under the Lie product.

We also find that
\[ \beta(x) = \frac{g_2}{x^2} + \frac{g_1}{x} + g_0 \]
(3.61)

will satisfy the equation (3.57) and everything will be consistent if
\[ g_0 = -\frac{2\hat{E}}{D}g_1, \quad g_1 = 2Dg_2, \quad C = -2. \]
(3.62)
This would imply
\[ \hat{E} = -\frac{d^2}{4}. \] (3.63)
This can only be obtained if we again take the negative root of \( \sqrt{1 - 4a} \) and \( n = 2 \). Hence there is again an enhancement of symmetry at another possible eigenvalue and for another value of \( n \). The vector-field for this case is
\[
\mathbf{X} = \left[ u\alpha(x) + \left( \frac{1}{4Ex^2} - \frac{2\hat{E}}{Dx} \right) g_0 \right] \frac{\partial}{\partial x} \\
+ \left[ u^2\alpha'(x) - u\left\{ \frac{1}{2Ex^3} + \frac{2\hat{E}}{Dx^2} \right\} g_0 - \kappa \right] \frac{\partial}{\partial u}
\] (3.64)
and again all Lie products do not close.

Taking
\[ \beta(x) = \frac{g_3}{x^3} + \frac{g_2}{x^2} + \frac{g_1}{x} + g_0 \] (3.65)
the relation between \( g_i \)'s with \( l = 0, 1, 2, \) and \( 3 \), becomes
\[
g_3 = \frac{3}{2D}g_2, \quad g_2 = \frac{12D}{9\hat{E} + 5D^2}g_1, \quad g_1 = \frac{5}{[\frac{32DE}{9E+5D^2} + 2D]}g_0, \quad C = -\frac{15}{4}. \] (3.66)
All the previous considerations apply in this case with \( n = 3 \). The vector field can be easily determined.

Hence, in all above cases, we find that the symmetry gets enhanced at particular eigenvalues.

We expect that similar analysis can be performed by including the higher negative powers of \( x \), such as,
\[ \beta(x) = \sum_n g_n x^{-n}, \quad n = 0, 1, 2, \ldots \] (3.67)
as a solution of (3.57).

In the absence of any magnetic field, equation (3.46) reduces to
\[ u'' + u = 0 \] (3.68)
where we have scaled \( u \) so that \( \hat{E} \) becomes equal to unity. For comparison, the vector field in this case is
\[
\mathbf{X} = \left[ a_1 u \sin (x + a_2) + a_7 \sin 2x + a_8 \right] \frac{\partial}{\partial x} \\
+ \left[ a_1 u^2 \cos (x + a_2) + u \left\{ a_7 \sin (2x + a_8) + a_3 \right\} + a_3 \sin (x + a_4) \right] \frac{\partial}{\partial u}.
\] (3.69)
However, the algebra we obtain, (3.60), looks more interesting.
4 Conclusion

The physical system of a charged particle under the influence of a constant magnetic field, which was considered early after the foundation of quantum mechanics, has recent ramifications in quantum Hall effects and in the gauge theories on noncommutative spaces in connection with the quantization of D-branes[11]. One would expect that the study of the system under a space dependent magnetic field may result in more pronounced and interesting behaviour. We consider a simple situation where only the $y$ - component of the vector potential exists in the form $B_z$ proportional to $\frac{1}{x^2}$. Using the method of group analysis for the classical equation of motion, we find that the symmetry is drastically reduced in this case. Such analysis has also been carried out for magnetic field proportional to coordinates, $B^a \propto q^a$. More symmetric and interesting results were obtained [10] in that case.

In the present case a simple group theoretic calculation gives the eigenvalues for the quantum mechanical problem. Further, the group analysis of the differential equation shows how the symmetries are enhanced at certain values of energy and correspondingly terms get added to the vectorfield characterising the symmetry. For the lowest values of $n$ we also obtain vector fields that do not close under Lie product. The structure of the vector fields reminds us of the algebras of Kac - Moody - Virasoro type [12].

Acknowledgement

I would like to thank Professor David S. Jerison and Professor David A. Vogan for kind invitation to visit Massachusetts Institute of Technology, where this work was carried out. I am also grateful to Professor D. A. Vogan for discussions. It is a pleasure to thank Professor B. Boghosian and Professor P. Love for helpful comments.

References

[1] J. Wess and B. Zumino, Phys. Lett. B37, 95 (1971).

[2] E. Witten, Nucl. Phys. B 223, 422 (1983).

[3] H. Stephani, Differential equations, their solution using symmetries, Cambridge University Press, Cambridge,(1989);
[4] P. J. Olver, *Application of groups to differential equations*, Springer, Berlin (1986); L. V. Ovsiannikov, *Group analysis of differential equations*, Academic Press, New York (1982); J.M. Hill, *Solution of differential equations by means of one parameter groups*, Pitman Advanced Publishing Program, Boston (1982); N. H. Ibragimov, (ed.) *CRC handbook of Lie group analysis of differential equations*, CRC Press, Boca Raton (1994).

[5] L. Landau and E. M. Lifshitz, *Quantum Mechanics*, Pergamon Press, Oxford, (1977); L. I. Schiff, *Quantum Mechanics*, McGraw Hill, New York (1968); A. Haug, *Theoretical Solid state Physics*, Pergamon Press, Oxford, (1972).

[6] S. Flügge, *Practical Quantum Mechanics*, Springer Verlag, Berlin (1994).

[7] P. Cordero and G. C. Ghirardi, Nuovo Cimento *2A*, 217 (1971).

[8] B. G. Wybourne, *Classical Groups for Physicists*, John Wiley, New York, (1972).

[9] J. Lanik, Nucl. Phys. *B5*, 263 (1967).

[10] K. Maharana, [hep-th/0106198](http://arxiv.org/abs/hep-th/0106198).

[11] D. Bigatti and L. Susskind, Phys. Rev. *D62*, 066004 (2000) [hep-th/9908056](http://arxiv.org/abs/hep-th/9908056).

[12] P. Goddard and D. Olive, *Kac - Moody and Virasoro algebras*, World Scientific, Singapore (1988).