Strong barriers for weighted quasilinear equations

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Abstract. In potential theory, use of barriers is one of the most important techniques. We construct strong barriers for weighted quasilinear elliptic operators. There are two applications: (i) solvability of Poisson-type equations with boundary singular data, and (ii) a geometric version of Hardy inequality. Our construction method can be applied to a general class of divergence form elliptic operators on domains with rough boundary.

Painollisten kvasilineaaristen yhtälöiden vahvat esteratkaisut

Tiivistelmä. Esteiden käyttö on potentiaaliteorian tärkeimpiä menetelmiä. Tässä työssä rakennetaan vahvoja esteratkaisuita painollisille kvasilineaarisille elliptisille operaattoreille. Tällä on kaksi sovellusta: (i) Poissonin-tyyppisten yhtälöiden ratkeavuus singulaarisilla reuna-arvoilla ja (ii) Hardyn epäyhtälön geometrinen muotoilu. Työssä esitetyt menetelmä soveltuu yleiseen luokkaan karkeareunaisten alueiden lähdemuotoisia elliptisiä operaattoreita.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$ ($n \geq 1$) be an open set with nonempty boundary, and let $1 < p < \infty$. We consider elliptic differential equations of the type

\begin{equation}
\begin{cases}
- \text{div } \mathcal{A}(x, \nabla u(x)) = f(x) & \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

where $\text{div } \mathcal{A}(x, \nabla \cdot)$ is a weighted $(p, w)$-Laplacian type elliptic operator, $w$ is a doubling weight on $\mathbb{R}^n$ which admit a $p$-Poincaré inequality (see (2.1) and (2.2) for detail), and $f$ is a locally integrable function on $\Omega$ such that $f/w$ is locally bounded. The most simple example of $w$ is $w \equiv 1$, and the reason for considering weighted equations will be explained later. The precise assumptions on $\mathcal{A}: \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ are as follows: For each $z \in \mathbb{R}^n$, $\mathcal{A}(\cdot, z)$ is measurable, for each $x \in \Omega$, $\mathcal{A}(x, \cdot)$ is continuous, and there exists $1 \leq L < \infty$ such that

\begin{align}
&\mathcal{A}(x, z) \cdot z \geq w(x)|z|^p, \\
&|\mathcal{A}(x, z)| \leq Lw(x)|z|^{p-1}, \\
&(\mathcal{A}(x, z_1) - \mathcal{A}(x, z_2)) \cdot (z_1 - z_2) > 0, \\
&\mathcal{A}(x, tz) = t|t|^{p-2} \mathcal{A}(x, z)
\end{align}

for all $x \in \Omega$, $z, z_1, z_2 \in \mathbb{R}^n$, $z_1 \neq z_2$ and $t \in \mathbb{R}$. When $\mathcal{A}(x, z) = w(x)|z|^{p-2}z$, the operator $\text{div } \mathcal{A}(x, \nabla u)$ is called as the $(p, w)$-Laplacian. In particular, if $\mathcal{A}(x, z) = z$, then the operator coincides with the classical Laplacian.

The aim of this paper is to provide an existence result of weak solutions to (1.1) for boundary singular data. The study of equation (1.1) has long history, and the

https://doi.org/10.54330/afm.147579

2020 Mathematics Subject Classification: Primary 35J92, 35J25, 31C15, 31C45.

Key words: Potential theory, Hardy inequality, $p$-Laplacian, quasilinear elliptic equation, boundary value problem, boundary regularity.

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The standard approach to solve this problem is the variational method and its generalization (see, [Lio69]). However, this method yields only solutions with finite energy. On the other hand, we can confirm existence of infinite energy solutions for boundary singular data by considering Poisson's equation (in the classical sense) or ordinary differential equations. To obtain such solutions, we divide the problem into three steps: (i) find finite energy solutions to approximate problems, (ii) prove compactness of solutions, and (iii) derive a uniform bound for solutions. The part (i) is trivial in our setting, and we can find sufficient results for (ii) in prior work (e.g., [BM92, HKM06, TW02]). This sequel from [Har21, Har22] proposes a new perspective for (iii). More precisely, we construct supersolutions called strong barriers.

If there are known supersolutions to (1.1), they are an effective tool to control boundary behavior of solutions. For concrete problems, direct calculations for the distance from the boundary often yield sharp estimates (see, e.g., [Tol83, MS00, BK06, AKSZ07, FMT07]). Unfortunately, this method cannot be applied to general elliptic equations on domains with rough boundary. When conditions of the form (1.2)–(1.5) have to be considered, such as in applications to homogenization problems (e.g., [BLP78]), it is needed to consider a construction method of supersolutions itself.

Ancona [Anc86] defined strong barriers for linear elliptic operators and constructed them under capacity density type conditions. In addition, a Hardy inequality was proved as one application of them. Since other proofs by Lewis [Lew88] and Wannebo [Wan90], many authors have proved more general Hardy-type inequalities under capacity density conditions (see, §2). However, another application of strong barriers, to the Dirichlet problem of the type (1.1), seems not to be discussed sufficiently.

We construct strong barriers for quasilinear operators (Theorem 5.1) and apply the result to (1.1) (Theorem 6.3). Specifically, we make auxiliary functions by a boundary Hölder estimate in the De Giorgi–Nash–Moser theory and construct a global function by gluing them. These results can be regarded as extensions of [Anc86, Theorem 1 and Remark 6.2] or [Har22, Corollary 4.4] as well as analogs of known facts for the \( p \)-Laplacian on \( C^2 \) domains (see, [GT01, Problem 6.6], [MS00, Theorem 1]). In addition, we prove a Hardy-type inequality (Corollary 5.3) by combining the results and the Picone inequality (see [AH98]). For connection with prior work on Hardy-type inequalities, we consider weighted operators borrowing a framework in [HKM06] (see also [BB11, KLV21] for recent progress). Known results for Hardy-type inequalities that seem to be particularly relevant to this study will be discussed in §2. Throughout the paper, we assume only (1.2)–(1.5), (2.1)–(2.2) and \((p,w)\)-capacity density conditions. The quantitative statements in results are new even for unweighted linear equations (compare with [Anc86, Remark 5.2]). Our method seems to work for Cheeger differential equations on metric measure spaces (see, [Che99, BMS01], [BB11, Chapter B.2], [Har18]). On the other hand, there are large gaps in its direct application to minimizers of variational problems.

Organization of the paper. In §2, we confirm our problem and pick up related results on weighted Sobolev spaces. In §3, we define weak solutions to (1.1) and prove a Kato-type inequality. In §4, we state regularity estimates that will be used in §5. Proofs of two lemmas will be provided in §A. In §5, we construct strong barriers for (1.1). In §6, we apply the result in §5 to (1.1) and achieve our goals.

**Notation.** Let \( \Omega \subseteq \mathbb{R}^n \) be an open set.

- \( 1_E(x) := \) the indicator function of a set \( E \).
• \( C^\infty_c(\Omega) \) := the set of all infinitely-differentiable functions with compact support in \( \Omega \).
• \( L^p(\Omega; \mu) \) := the \( L^p \) space with respect to a measure \( \mu \) on \( \Omega \).
• \( f_+ := \max\{f, 0\} \) and \( f_- := -\min\{f, 0\} \).

For a closed set \( \Gamma \subset \mathbb{R}^n \), we denote by \( \delta_\Gamma \) the distance from \( \Gamma \). For a ball \( B = B(x, R) = \{y: \text{dist}(x, y) < R\} \) and \( \lambda > 0 \), we denote \( B(x, \lambda R) \) by \( \lambda B \). The letters \( c \) and \( C \) denote various constants with and without indices.

2. Setting and related work

2.1. Admissible weights. Let \( 1 < p < \infty \) be a fixed number. A function \( w \in L^1_{\text{loc}}(\mathbb{R}^n; dx) \) is called the weight if \( w(x) > 0 \) a.e. in \( \mathbb{R}^n \). We write \( w(E) = \int_E w \, dx \) for a Lebesgue measurable set \( E \subset \mathbb{R}^n \). Throughout the below, we assume that \( w \) satisfying the doubling condition

\[
(2.1) \quad w(2B) \leq C_D w(B)
\]

and the \( p \)-Poincaré inequality

\[
(2.2) \quad \int_B |u - u_B| \, dw \leq C_P \text{diam}(B) \left( \int_{\lambda B} |\nabla u|^p \, dw \right)^{1/p}, \quad \forall u \in C^\infty_c(\mathbb{R}^n),
\]

where \( B \) is an arbitrary ball in \( \mathbb{R}^n \), \( f_B := w(B)^{-1} \int_B u \, dw \) and \( C_D, C_P \) and \( \lambda \geq 1 \) are constants. A weight \( w \) satisfying (2.1) and (2.2) is said to be \( p \)-admissible.

It is well-known that (2.1) and (2.2) yield the following Sobolev type inequality:

\[
(2.3) \quad \left( \int_B |u|^\chi p \, dw \right)^{1/\chi p} \leq C \text{diam}(B) \left( \int_B |\nabla u|^p \, dw \right)^{1/p}, \quad \forall u \in C^\infty_c(B),
\]

where \( C \) and \( \chi > 1 \) are constants depending only on \( p, C_D, C_P \) and \( \lambda \). For detail, we refer to [HKM06, Chapter 15] and the references cited therein.

Muckenhoupt \( A_p \)-weights are one typical example of \( p \)-admissible weights ([HKM06, Chapter 15]). The power function \( |x|^\mu \) is an \( A_p \)-weight on \( \mathbb{R}^n \) if and only if \(-n < \mu < n(p - 1)\). It seems to be known conventionally that if \( \Omega \) is a bounded Lipschitz domain, then \( w(x) = \delta(x)^\mu \) is an \( A_p \)-weight on \( \mathbb{R}^n \) for \(-1 < \mu < p - 1 \). Finer results can be found in [Hor89, Hor91, DLG10, DIL'19] and [KLV21, Chapter 10]. Roughly speaking, if \( \Gamma \) is an \( s \)-dimensional set with \( 0 < s < n \), then \( w(x) = \delta_\Gamma(x)^\mu \) is an \( A_p \)-weight on \( \mathbb{R}^n \) for \(-(n - s) < \mu < (n - s)(p - 1)\).

2.2. Sobolev spaces and capacities. The weighted Sobolev space \( H^{1,p}(\Omega; w) \) is the closure of \( C^\infty(\Omega) \) with respect to the norm

\[
\|u\|_{H^{1,p}(\Omega; w)} := \left( \int_\Omega |u|^p + |\nabla u|^p \, dw \right)^{1/p}.
\]

where \( \nabla u \) is the gradient of \( u \) in the sense of\( \mathbb{R}^n \). The corresponding local space \( H^{1,p}_{\text{loc}}(\Omega; w) \) is defined in the usual manner. We denote by \( H^{1,p}_0(\Omega; w) \) the closure of \( C^\infty_c(\Omega) \) in \( H^{1,p}(\Omega; w) \). It is well-known that if \( u, v \in H^{1,p}_{\text{loc}}(\Omega; w) \), then \( \min\{u, v\} \in H^{1,p}_{\text{loc}}(\Omega; w) \).

Let \( O \subset \mathbb{R}^n \) be open, and let \( K \subset O \) be compact. The \((p, w)\)-capacity \( \text{cap}_{p,w}(K, O) \) of the condenser \((K, O)\) is defined by

\[
(2.4) \quad \text{cap}_{p,w}(K, O) := \inf \left\{ \|\nabla u\|_{L^p(O; w)}^p : u \geq 1 \text{ on } K, \ u \in C^\infty_c(O) \right\}.
\]
For a boundary point $\xi \in \partial \Omega$ (more generally, for $\xi \in \mathbb{R}^n \setminus \Omega$), we consider the following $(p, w)$-capacity density condition: There exists $\gamma > 0$ such that

$$\frac{\text{cap}_{p,w}(B(\xi, R) \setminus \Omega, B(\xi, 2R))}{\text{cap}_{p,w}(B(\xi, R), B(\xi, 2R))} \geq \gamma.$$  \tag{2.5}

If (2.5) holds for all small $R > 0$, then $\xi$ is a regular point of the corresponding Dirichlet problem (see Lemma 4.3 below, [HKM06, Theorem 6.31] and the references cited therein). Sufficient conditions for (2.5) via exterior corkscrew-type conditions can be found in [HKM06, Theorem 6.31] and [BB11, Corollary 11.25]. In particular, every boundary point of a ball satisfies (2.5) for all $R > 0$.

When (2.5) holds for all $\xi \in \mathbb{R}^n \setminus \Omega$ and $R > 0$, the set $\mathbb{R}^n \setminus \Omega$ is called as uniformly $(p, w)$-fat, and this condition is closely related to Hardy-type inequalities. The study of Hardy-type inequalities on uniformly fat sets started with Ancona’s work [Anc81].

Three proofs have been proposed that differ in their use of the uniform assumption.

(i) Ancona [Anc86] constructed strong barriers using decay of harmonic measures.
(ii) Lewis [Lew88] developed theory of self-improvement property of uniformly $(p, 1)$-fat sets using Havin–Maz’ya potentials. (iii) Wannebo [Wan90] proved higher order Hardy inequalities using an estimate for the overlap of Whitney-type cubes. Later, Mikkonen [Mik96] proved self-improvement property of uniformly $(p, w)$-fat sets using a boundary Hölder estimates of $(p, w)$-harmonic functions. See also [BMS01] for the extension to the metric measure space setting. We also refer to [Haj99, KM97, Leh08, KLT11] for further properties of uniformly fat sets. Wannebo’s method was revisited in [EHS05, BK04], and Korte, Lehrbäck and Tuominen [KLT11] gave a new proof of Mikkonen’s result. Further related work can be found in [KLV21, Chapter 6] and [BEL15, Chapter 3]. However, the proof in [Anc86] is not very similar to any of the others.

The study of cases where (2.5) holds on a part of the boundary is more limited. The most similar known result to Corollary 5.3 below is [LTV17, Proposition 5.4] (see also [KLV21, Theorem 7.31]). They have used Wannebo’s idea. In [EHDR15], another form of Hardy inequality and its applications are discussed under assumptions on Sobolev extensibility of domains.

### 3. Quasilinear elliptic operator

Assume that $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies (1.2)–(1.5). For simplicity, we denote the following extended function $\overline{A}$ by $A$ again:

$$\overline{A}(x, z) = \begin{cases} A(x, z) & x \in \Omega, \\ w(x)|z|^{p-2}z & \text{otherwise}, \end{cases} \quad z \in \mathbb{R}^n.$$  

Let $f \in L^1_{\text{loc}}(\Omega)$. A function $u \in H^{1,p}_{\text{loc}}(\Omega; w)$ is called weak (super-, sub-)solution to $- \text{div} A(x, \nabla u) = f$ in $\Omega$ if

$$\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi \, dx = (\geq, \leq) \int_{\Omega} \varphi f \, dx$$  \tag{3.1}

for any nonnegative $\varphi \in C_c^\infty(\Omega)$.

If $u$ is a weak supersolution to $- \text{div} A(x, \nabla u) = 0$ in $\Omega$, then, by the Riesz representation theorem, there is a unique Radon measure $\nu[u]$ in $\Omega$ such that

$$\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi \, dx = \int_{\Omega} \varphi \, d\nu[u].$$
for any $\varphi \in C^{\infty}_c(\Omega)$. The measure $\nu[u]$ is called the Riesz measure of $u$. We refer to [HKM06, Chapter 21] for further detail.

By (1.2), if $u$ is a solution to $-\text{div} \mathcal{A}(x, \nabla u) = f$ in $\Omega$, then its truncation $\min\{u, k\}$ is a supersolution to $-\text{div} \mathcal{A}(x, \nabla u) = \min\{f, 0\}$ in $\Omega$. The monotonicity condition (1.4) yields the following more general result.

**Lemma 3.1.** Assume that $u$ and $v$ are weak supersolutions to $-\text{div} \mathcal{A}(x, \nabla u) = 0$ in $\Omega$. Then,

\[
\int_{\Omega} \varphi \, d\nu[\min\{u, v\}] \geq \int_{\Omega} \varphi \, d\nu[u] + \int_{\Omega} \varphi \, d\nu[v]
\]

for any nonnegative $\varphi \in C^\infty(\Omega)$. Moreover, if $u_1, \ldots, u_k$ are weak supersolutions to $-\text{div} \mathcal{A}(x, \nabla u) = 0$ in $\Omega$, and if there is a Radon measure $\sigma$ such that $\nu[u_k] \ll \sigma$ for all $k$, then

\[
\nu[\min\{u_1, \ldots, u_k\}] \geq \min\{f_1, \ldots, f_k\} \sigma
\]

in the sense of distributions, where $f_k$ is the Radon–Nikodym derivative of $\nu[u_k]$ with respect to $\sigma$.

**Proof.** Note that $u, v \in H^{1,p}_{\text{loc}}(\Omega; w)$. For $\epsilon > 0$, consider the functions

\[
\Phi_\epsilon(u - v) := \epsilon \frac{\varphi}{(u - v)_+ + \epsilon} \quad \text{and} \quad \Psi_\epsilon(u - v) := 1 - \Phi_\epsilon(u - v).
\]

These functions are globally Lipschitz continuous with respect to $u - v$; therefore, $\Phi_\epsilon(u - v), \Psi_\epsilon(u - v) \in H^{1,p}_{\text{loc}}(\Omega; w)$. We also note that $\Phi_\epsilon(u - v)(x) \to 1_{\{u \leq v\}}(x)$ for all $x$. Fix a nonnegative function $\varphi \in C^\infty(\Omega)$. Then,

\[
\int_{\Omega} \varphi \, d\nu[\min\{u, v\}] = \int_{\Omega} \varphi \Phi_\epsilon(u - v) \, d\nu[\min\{u, v\}] + \int_{\Omega} \varphi \Psi_\epsilon(u - v) \, d\nu[\min\{u, v\}].
\]

By the definition of $\nu[\min\{u, v\}]$, we have

\[
\int_{\Omega} \varphi \Phi_\epsilon(u - v) \, d\nu[\min\{u, v\}] = \int_{\{u \leq v\}} \mathcal{A}(x, \nabla u) \cdot \nabla (\varphi \Phi_\epsilon(u - v)) \, dx
\]

\[
+ \int_{\{u > v\}} \mathcal{A}(x, \nabla u) \cdot \nabla (\varphi \Phi_\epsilon(u - v)) \, dx.
\]

Note that $(\mathcal{A}(x, \nabla v) - \mathcal{A}(x, \nabla u)) \cdot \nabla \Phi_\epsilon(u - v) \geq 0$ by (1.4). Therefore,

\[
\int_{\{u > v\}} \mathcal{A}(x, \nabla u) \cdot \nabla (\varphi \Phi_\epsilon(u - v)) \, dx + I_\epsilon \geq \int_{\{u > v\}} \mathcal{A}(x, \nabla u) \cdot \nabla (\varphi \Phi_\epsilon(u - v)) \, dx,
\]

where

\[
I_\epsilon := \int_{\{u > v\}} (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla v)) \cdot \nabla \varphi \Phi_\epsilon(u - v) \, dx.
\]

Adding the two inequalities, we get

\[
\int_{\Omega} \varphi \Phi_\epsilon(u - v) \, d\nu[\min\{u, v\}] + I_\epsilon \geq \int_{\Omega} \varphi \Psi_\epsilon(u - v) \, d\nu[u].
\]

Similarly, since

\[
\int_{\Omega} \varphi \Psi_\epsilon(u - v) \, d\nu[\min\{u, v\}] = \int_{\{u \leq v\}} \mathcal{A}(x, \nabla u) \cdot \nabla (\varphi \Psi_\epsilon(u - v)) \, dx
\]

\[
+ \int_{\{u > v\}} \mathcal{A}(x, \nabla u) \cdot \nabla (\varphi \Psi_\epsilon(u - v)) \, dx
\]
and \((\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla v)) \cdot \nabla \Psi_\epsilon(u - v) \geq 0\), we have

\[
(3.5) \quad \int_\Omega \varphi \Psi_\epsilon(u - v) \, d\nu[\min\{u, v\}] + \Pi_\epsilon \geq \int_\Omega \varphi \Psi_\epsilon(u - v) \, d\nu[v],
\]

where

\[
\Pi_\epsilon := \int_{\{u \leq v\}} (\mathcal{A}(x, \nabla v) - \mathcal{A}(x, \nabla u)) \cdot \nabla \varphi \Psi_\epsilon(u - v) \, dx.
\]

Combining (3.4) and (3.5), we obtain

\[
\int_\Omega \varphi \, d\nu[\min\{u, v\}] + I_\epsilon + \Pi_\epsilon \geq \int_\Omega \varphi \Phi_\epsilon(u - v) \, d\nu[u] + \int_\Omega \varphi \Psi_\epsilon(u - v) \, d\nu[v].
\]

Take the limit \(\epsilon \to 0\). By the dominated convergence theorem, the right-hand side of this inequality goes to the right-hand of (3.2). By the same reason, \(I_\epsilon, \Pi_\epsilon \to 0\). Therefore, (3.2) holds. The latter statement is a consequence of induction. \(\square\)

4. Regularity estimates

By standard techniques in the De Giorgi–Nash–Moser theory, the Sobolev inequality (2.3) yields the following global \(L^\infty\) estimate and weak Harnack inequality. Abbreviated proofs of them will be provided in \(\S A\).

**Lemma 4.1.** Let \(u \in H^{1,p}(\Omega; w)\) be a weak subsolution to \(-\text{div} \mathcal{A}(x, \nabla u) = f\) in \(\Omega\). Let \(F_+ = (\text{diam}(\Omega)^p \|f_+/w\|_{L^\infty(\Omega)})^{1/(p-1)}\). Then,

\[
\text{ess sup}_\Omega u \leq \sup_{\partial \Omega} u + CF_+,
\]

where \(\sup_{\partial \Omega} u := \inf\{k \in \mathbb{R}^n: (u-k)_+ \in H^{1,p}_0(\Omega; w)\}\) and \(C\) is a constant depending only on \(p, C_D, \{C_p, \lambda\}\). In particular, there is a constant \(c_1 = c_1(p, C_D, \{C_p, \lambda\})\) such that if \(\|f_+/w\|_{L^\infty(\Omega)} \leq c_1 \text{diam}(\Omega)^{-p}\), then \(\text{ess sup}_\Omega u \leq \sup_{\partial \Omega} u + 1/4\).

**Lemma 4.2.** Let \(u \in H^{1,p}(2B; w)\) be a nonnegative weak supersolution to \(-\text{div} \mathcal{A}(x, \nabla u) = f\) in \(2B\), and let \(F_- = (R^p \|f_-/w\|_{L^\infty(2B)})^{1/(p-1)}\). Then, (i) for each \(0 < s < \chi(p-1)\), there exists a constant \(C\) depending only on \(p, C_D, \{C_p, \lambda\}, L\) and \(s\) such that

\[
(4.1) \quad \left(\int_B u^s \, dw\right)^{1/s} \leq C \left(\text{ess inf}_B u + F_-\right).
\]

(ii) there exists a constant \(C\) depending only on \(p, C_D, \{C_p, \lambda\}\) and \(L\) such that

\[
(4.2) \quad R^{p-1} \int_B |
abla u|^{p-1} \, dw \leq C \left(\text{ess inf}_B u + F_-\right)^{p-1}.
\]

It is well-known that Lemma 4.2 yields a local Hölder estimate (see, e.g., [GT01, Theorem 8.24]). Below, we always assume that \(f/w\) is locally bounded. Under this assumption, any weak solution \(u\) to \(-\text{div} \mathcal{A}(x, \nabla u) = f\) can be regard as a locally Hölder continuous function.

We use the exterior condition (2.5) via the following boundary Hölder estimate. The proof below is the same as [GT01, pp. 206–209] (see also [GZ77] and [Str84]).

**Lemma 4.3.** Let \(B\) be a ball centered at \(\xi \in \partial \Omega\) with radius \(R\). Assume that (2.5) holds. Let \(u \in H^{1,p}(\Omega; w) \cap L^\infty(\Omega)\) satisfy \(-\text{div} \mathcal{A}(x, \nabla u) = f\) in \(\Omega \cap 4B\). Let
Note that

\[ \nabla u \leq \left( 1 - \frac{\gamma^{1/(p-1)}}{C} \right) \nabla u + \frac{\gamma^{1/(p-1)}}{C} \nabla u + F^+ + F^-, \]

where \( C \) is a constant depending only on \( p, C_D, \{C_P, \lambda\} \) and \( L \) and \( \text{osc} u := \sup u - \inf u \).

**Proof.** We first assume that \( u \) is a nonnegative supersolution to \(-\div \mathcal{A}(x, \nabla u) = f\) in \( \Omega \cap 4B \). Let \( m = \inf_{\partial \Omega \cap B} u \), and let

\[ u_m^-(x) = \begin{cases} \min \{u(x), m\} & x \in \Omega, \\ m & \text{otherwise.} \end{cases} \]

Note that \( 0 \leq u_m^- \leq m \) and \( u_m^- = m \) on \( \overline{\Omega} \setminus \Omega \). Take \( \eta \in C_c^\infty(2B) \) such that \( 0 \leq \eta \leq 1 \) in \( 2B \), \( \eta = 1 \) on \( B \) and \( |\nabla \eta| \leq C/\rho \). Then, (2.5) gives

\[ \int_{2B} |\nabla (u_m^- \eta)|^p \rho \leq C \rho \int_{2B} |\nabla u_m^- \eta|^p \rho \, dw. \]

By the product rule, \( \nabla (u_m^- \eta) = \nabla u_m^- \eta + u_m^- \nabla \eta \) a.e. Therefore,

\[ \int_{2B} |\nabla (u_m^- \eta)|^p \rho \leq C \rho \int_{2B} (u_m^-)^p \rho \, dw + \int_{2B} |\nabla u_m^-|^p \nabla \eta^p \rho \, dw. \]

By Lemma 4.2 (i), the former term on the right-hand side is estimated by

\[ \int_{2B} (u_m^-)^p \rho \, dw \leq m \int_{2B} (u_m^-)^p \rho \, dw \leq C m \left( \inf_B u_m^- + F_- \right)^{p-1}. \]

Consider the test function \( (m-u_m^-)^p \eta^p \). Since \( u_m^- \) is a supersolution to \(-\div \mathcal{A}(x, \nabla u) = \min\{f,0\} \) in \( \Omega \), we have

\[ \int_{\Omega} \mathcal{A}(x, \nabla u_m^-) \cdot \nabla (m-u_m^-)^p \eta \, dx + p \int_{\Omega} \mathcal{A}(x, \nabla u_m^-) \cdot \nabla \eta \eta^{p-1} (m-u_m^-) \, dx \]

\[ \geq \int_{\Omega} (m-u_m^-)^p \eta \frac{\min\{f,0\}}{w} \, dw. \]

By (1.2) and (1.3), this inequality yields

\[ \int_{2B} |\nabla u_m^-|^p \eta \, dw = \int_{2B} |\nabla (m-u_m^-|^p \eta \, dw \]

\[ 
\leq C m \left( R^{-1} \int_{2B} |\nabla u_m^-|^p \, dw + \int_{2B} F_m^{-1} \, dw \right). \]

By Lemma 4.2 (ii), the former term on the right-hand side is estimated by

\[ R^{-1} \int_{2B} |\nabla u_m^-|^p \, dw \leq C R^{-p} \left( \inf_B u_m^- + F_- \right)^{p-1}. \]

Combining (3.3), (4.4) and (4.5), we obtain

\[ m \leq \frac{C}{\gamma^{1/(p-1)}} \left( \inf_B u_m^- + F_- \right). \]

Let \( M(R) = \sup_{\partial \Omega \cap B} u \) and \( m(R) = \inf_{\partial \Omega \cap B} u \). Applying (4.6) to \( M(4R) - u \) and \( u - m(4R) \), we obtain

\[ M(4R) - \sup_{\partial B \cap \Omega} u \leq \frac{C}{\gamma^{1/(p-1)}} (M(4R) - M(R) + F_+), \]
\[ \inf_{\partial \Omega \cap B} u - m(4R) \leq \frac{C}{\gamma^{1/(p-1)}} (m(R) - m(4R) + F). \]

Adding the two inequalities, we arrive at the desired estimate. \( \square \)

Using (4.7) in Lemma 4.3 iteratively, we get the following lemma.

**Lemma 4.4.** Let \( B \) be a ball centered at \( \xi \in \partial \Omega \) with radius \( R_0 \). Assume that (2.5) holds for all \( 0 < R \leq R_0 \). Then, there are positive constants \( c_2 \) and \( \theta \) in \((0,1)\) depending only on \( p, C_D, \{C_F, \lambda\}, L \) and \( \gamma \) such that if \( u \in H^{1,p}_0(\Omega;w) \) is a nonnegative bounded weak subsolution to \(-\text{div} \mathcal{A}(x, \nabla u) = f \) in \( \Omega \cap B \), and if \( \|f_+ / w\|_{L^{\infty}(\Omega)} \leq c_2 R_0^{-p} (\sup_{\partial \Omega \cap B} u)^{p-1} \), then \( \sup_{\partial \Omega \cap B} u \leq (1/4) \sup_{\Omega \cap B} u \).

5. Construction of strong barriers

We prove Theorem 5.1 using Lemma 5.2 infinitely many times. Lemma 5.2 itself is a consequence of regularity estimates in the previous section. Also, we prove Corollary 5.3 as a corollary of Theorem 5.1.

**Theorem 5.1.** Let \( \Gamma \subset \partial \Omega \) be a closet set, and assume that (2.5) holds for all \( \xi \in \Gamma \) and \( R > 0 \). Then, there exists a nonnegative function \( s_\Gamma \in H^{1,p}_{\text{loc}}(\Omega;w) \cap C(\Omega) \) satisfying

\[ -\text{div} \mathcal{A}(x, \nabla s_\Gamma) \geq c_H \frac{s_\Gamma(x)^{p-1}}{\delta_\Gamma(x)^p} w(x) \quad \text{in } \Omega \]

and

\[ \delta_\Gamma(x)^\alpha \leq s_\Gamma(x) \leq 30 \delta_\Gamma(x)^\alpha \]

for all \( x \in \Omega \), where \( c_H \) and \( \alpha > 0 \) are positive constants depending only on \( p, C_D, \{C_F, \lambda\}, L \) and \( \gamma \).

**Lemma 5.2.** Let \( B \) be a ball centered at \( \xi \in \partial \Omega \) with radius \( R_0 \). Assume that (2.5) holds for all \( 0 < R \leq R_0 \). Let \( c_3 := \min\{c_1, c_2\} \), where \( c_1 \) and \( c_2 \) are constants in Lemmas 4.1 and 4.4. Then, there exists a function \( u_B \in H^{1,p}(\Omega;w) \cap C(\Omega) \) satisfying

\[ -\text{div} \mathcal{A}(x, \nabla u_B) = c_3 R_0^{-p} w \quad \text{in } \Omega \cap B, \]

\[ \frac{1}{4} \leq u_B \leq \frac{5}{4} \quad \text{in } \Omega, \]

\[ u_B = 1 \quad \text{on } \Omega \setminus B, \quad u_B \leq \frac{1}{2} \quad \text{on } \Omega \cap \partial B, \]

where \( \theta \) is a constant in Lemma 4.4.

**Proof.** We follow the method in [Lio69, p. 177]. Take \( \eta_B \in C^\infty(\overline{\Omega}) \) such that

\[ \eta_B = \frac{1}{4} \text{ on } \Omega \cap B/2, \quad \eta_B = 1 \text{ on } \overline{\Omega \setminus B}, \quad \frac{1}{4} \leq \eta_B \leq 1 \text{ in } \overline{\Omega}. \]

Consider the Dirichlet problem

\[ \begin{cases} -\text{div} \mathcal{A}(x, \nabla (v_B + \eta_B)) = c_3 R_0^{-p} w & \text{in } \Omega \cap B, \\ v_B \in H^{1,p}_0(\Omega \cap B;w). \end{cases} \]

By (2.3), the right-hand side belongs to the dual of \( H^{1,p}_0(\Omega \cap B;w) \). Therefore, by the Minty–Browder theorem, there exists a unique solution \( u_B \in \eta_B + H^{1,p}_0(\Omega \cap B;w) \) to (5.3). By Lemma 4.2, \( u_B \) is continuous in \( \Omega \cap B \). Meanwhile, if \( x \in \Omega \cap \partial B \), then \( u_B \) is continuous at \( x \) by Lemma 4.3. Consequently, \( u_B \) is continuous in \( \Omega \). By the
comparison principle, \(u_B \geq 1/4\) in \(\Omega\). Also, by Lemma 4.1, \(u_B \leq \sup_{\partial \Omega} u + 1/4 \leq 5/4\) in \(\Omega\). The latter bound in (5.5) follows from Lemma 4.4.

**Proof of Theorem 5.1.** First, we construct a function \(s\) by the following two steps. (i) For \(k \in \mathbb{Z}\), set \(E_k = \{x \in \Omega : \delta \Gamma(x) \leq (\theta/2)^k\}\), where \(\theta\) is a constant in Lemma 5.2. Choosing \(\{\xi_j\}_{j \in J_k} \subset \Gamma\), we construct a locally finite covering \(\{B(\xi_j, (\theta/2)^{k+1})\}_{j \in J_k}\) of \(E_{k+1}\). Note that

\[
E_{k+1} \subset D_k := \Omega \cap \bigcup_{j \in J_k} B(\xi_j, (\theta/2)^k) \subset E_k.
\]

Using Lemma 5.2, we define a function \(v_k \in H_{\text{loc}}^1(\Omega; w) \cap C(\Omega)\) by

\[
v_k(x) = \min_{\xi_j \in J_k} u_{B(\xi_j, (\theta/2)^k)}(x) \quad \text{for } x \in D_k
\]

and \(v_k = 1\) on \(\Omega \setminus D_k\). By (5.4) and (5.5), we have

\[
1/4 \leq v_k \leq 5/4 \quad \text{in } \Omega
\]

and

\[
v_k \leq 1/2 \quad \text{on } E_{k+1}.
\]

Moreover, by Lemma 3.1, we have

\[
- \text{div } A(x, \nabla v_k) \geq c_3 \left(\frac{2}{\theta}\right)^{kp} w \quad \text{in } D_k.
\]

(ii) Define a function \(s\) on \(\Omega\) by

\[
s(x) = \inf_{E_k \ni x} \left(\frac{3}{4}\right)^k v_k(x).
\]

By (5.6) and the inequality \((3/4)^6 \leq 1/5 < (3/4)^5\),

\[
\left(\frac{3}{4}\right)^{k-6} v_{k-6}(x) \geq \left(\frac{3}{4}\right)^{k-6} \frac{1}{4} \geq \left(\frac{3}{4}\right)^k \frac{5}{4} \geq \left(\frac{3}{4}\right)^k v_k(x)
\]

for any \(x \in E_k\). Therefore,

\[
s(x) = \min \left\{\left(\frac{3}{4}\right)^{k-5} v_{k-5}(x), \ldots, \left(\frac{3}{4}\right)^{k} v_k(x)\right\}
\]

for all \(x \in E_k \setminus E_{k+1}\). In particular, \(s \in H_{\text{loc}}^1(\Omega; w) \cap C(\Omega)\).

Next, we claim that

\[
- \text{div } A(x, \nabla s) \geq c_3 \left(\frac{3}{4}\right)^{(k-1)p} \left(\frac{2}{\theta}\right)^{(k-5)p} w
\]

in an open neighborhood of \(E_k \setminus E_{k+1}\). By Lemma 3.1 and (5.8), this inequality holds in \(D_k \setminus E_{k+1}\). Meanwhile, by (5.7), we have

\[
v_{k-1} \leq \frac{1}{2} < \frac{3}{4} = \frac{3}{4} v_k \quad \text{in } E_k \setminus D_k.
\]
By continuity of $v_k$ and $v_{k-1}$, we can take an open set $O_k$ such that $v_{k-1} < (3/4)v_k$ in $O_k$. Note that

$$s(x) = \min \left\{ \left( \frac{3}{4} \right)^{k-5} v_{k-5}(x), \ldots, \left( \frac{3}{4} \right)^{k-1} v_{k-1}(x) \right\}$$

for all $x \in O_k$. Since $v_{k-5}, \ldots, v_{k-1}$ satisfy (5.8) in $D_{k-1}$, (5.10) holds in $O_k \cap D_{k-1}$ by the same reason as above. The open set $(D_k \setminus E_{k+1}) \cup (O_k \cap D_{k-1})$ has the desired property.

Finally, we consider pointwise behavior of $s$. Fix $x \in E_k \setminus E_{k+1}$. By (5.9), we have

$$\frac{1}{4} \left( \frac{3}{4} \right)^k \leq s(x) \leq \frac{5}{4} \left( \frac{4}{3} \right)^5 \left( \frac{3}{4} \right)^k.$$  

By the latter inequality and the definition of $E_k$, the right-hand side of (5.10) is estimated from below by

$$c_3 \left( \frac{3}{4} \right)^{k(p-1)} \left( \frac{2}{\theta} \right)^{(k-5)p} w(x) \geq s^{p-1} \frac{c_H}{\delta_T(x)^p} w(x),$$

where $c_H = c_3 \{(5/4)(4/3)^5\}^{1-p} (\theta/2)^{6p}$. Therefore, $s$ satisfies (5.1). Take $\alpha > 0$ such that $3/4 = (\theta/2)^{\alpha}$. By (5.11), we have

$$1/4 \delta_T(x)^\alpha \leq s(x) \leq 5/4 \left( \frac{4}{3} \right)^6 \delta_T(x)^\alpha \leq 30/4 \delta_T(x)^\alpha$$

for all $x \in \Omega$. Thus, the function $s_T := 4s$ has the desired properties. \hfill \Box

**Corollary 5.3.** Assume that $w$ is a $p$-admissible weight on $\mathbb{R}^n$. Let $\Gamma \subset \partial \Omega$ be a nonempty closed set, and assume that (2.5) holds for all $\xi \in \Gamma$ and $R > 0$. Let $c_H = c_H(p, C_D, \{C_P, \lambda\}, L = 1, \gamma) > 0$ be the constant in Theorem 5.1. Then,

$$c_H \int_\Omega \frac{|\varphi|^p}{\delta_T} \, dw \leq \int_\Omega |\nabla \varphi|^p \, dw$$

for all $\varphi \in C_c^\infty(\Omega)$.

**Proof.** Applying Theorem 5.1 to $A(x, z) = w(x)|z|^{p-2}z$, we get a nonnegative function $s_T \in H_{\text{loc}}^1(\Omega; w) \cap C(\Omega)$ satisfying

$$-\Delta_{p,w} s_T \geq c_H \frac{s_T(x)^{p-1}}{\delta_T(x)^p} w(x) \quad \text{in } \Omega.$$

Take $\varphi \in C_c^\infty(\Omega)$ such that $\varphi \geq 0$. Then,

$$c_H \int_\Omega \frac{\varphi^p}{\delta_T^p} \, dw = c_H \int_\Omega \frac{\varphi^p}{s_{\Gamma}^{p-1}} \frac{s_T^{p-1}}{\delta_T^p} \, dw \leq \int_\Omega \varphi^p \frac{d\nu[s_T]}{s_T^{p-1}},$$

where $\nu[s_T]$ is the Riesz measure of $s_T$ with respect the $(p, w)$-Laplacian. Applying the Picone inequality ([AH98, Theorem 1.1]) to the right-hand side (for detail, see also [HS20, Lemma 3.2]), we get

$$\int_\Omega \frac{\varphi^p}{s_T^{p-1}} \frac{d\nu[s_T]}{s_T^{p-1}} \leq \int_\Omega |\nabla \varphi|^p \, dw.$$  

Combining the two inequalities, we obtain the desired inequality. \hfill \Box
6. Applications to Dirichlet problems

We discuss the substitutions into concave functions of the barriers constructed in the previous section. Taking them as super-solutions, we derive Theorem 6.3.

**Lemma 6.1.** Assume that there exists a function $s_\Gamma$ on $\Omega$ satisfying (5.1) and (5.2). Let $h: (0, \infty) \to (0, \infty)$ be a continuously differentiable nondecreasing concave function such that

$$g(s) := \int_0^s h(t) \frac{dt}{t} < \infty$$

for some $s > 0$. Then, $v(x) := g(s_\Gamma(x)) \in H^{1,p}_{\text{loc}}(\Omega; w) \cap C(\Omega)$ is a nonnegative weak supersolution to

$$- \text{div} \, A(x, \nabla v) \geq c_H \frac{h(\delta_\Gamma(x)^\alpha)}{\delta_\Gamma(x)^p} w(x) \quad \text{in } \Omega.$$  

Moreover, $g(\delta_\Gamma(x)^\alpha) \leq v(x) \leq g(30 \delta_\Gamma(x)^\alpha)$ for all $x \in \Omega$.

**Proof.** For the sake of simplicity, we denote $s_\Gamma$ by $s$ and the derivative of $f(t)$ by $f'(t)$. By assumption, $h'(t)t \leq h(t)$ for all $t \in (0, \infty)$; therefore, $g$ is increasing and concave on $(0, \infty)$. Fix a nonnegative function $\varphi \in C^\infty_c(\Omega)$. Since $g'' \leq 0$, we have

$$\int_\Omega A(x, \nabla s) \cdot \nabla (g'(s)^{p-1}) \varphi \, dx \leq 0.$$  

By the chain rule, (1.5) and (5.1), we get

$$\int_\Omega A(x, \nabla g(s)) \cdot \nabla \varphi \, dx = \int_\Omega A(x, g'(s)\nabla s) \cdot \nabla \varphi \, dx$$

$$\geq \int_\Omega A(x, \nabla s) \cdot \nabla (g'(s)^{p-1} \varphi) \, dx$$

$$\geq c_H \int_\Omega \frac{s^{p-1}}{\delta_\Gamma(x)^{p-1}} (g'(s)^{p-1} \varphi) \, dw.$$  

Since $h$ is nondecreasing, (5.2) gives

$$s^{p-1}g'(s)^{p-1} = h(s)^{p-1} \geq h(\delta_\Gamma(x)^\alpha)^{p-1}.$$  

The latter statement is a consequence of the monotonicity of $g$. \qed

**Proposition 6.2.** Let $\Omega$ be a bounded open set, and let $\Gamma \subset \partial \Omega$ be a nonempty closed set satisfying (2.5) for all $\xi \in \Gamma$ and $R > 0$. Let $f \in L^1_{\text{loc}}(\Omega)$, and assume that there exists a function $h$ satisfying the assumption in Lemma 6.1 and

$$|f(x)| \leq c_H \frac{h(\delta_\Gamma(x)^\alpha)^{p-1}}{\delta_\Gamma(x)^p} w(x)$$

for almost every $x \in \Omega$, where $c_H$ and $\alpha$ are constants in Theorem 5.1. Then, there exists a nonnegative unique weak solution $u \in H^{1,p}_{\text{loc}}(\Omega; w) \cap C(\Omega)$ to (1.1) in the sense that (i) the zero extension of $u$ belongs to $H^{1,p}_{\text{loc}}(\mathbb{R}^n \setminus \Gamma; w)$, and (ii) $\lim_{x \to \xi} u(x) = 0$ for all $\xi \in \Gamma$.

**Proof.** Using Theorem 5.1 and Lemma 6.1, we get a nonnegative supersolution $v(x) = g(s(x))$ to $-\text{div} \, A(x, \nabla u) = |f|$ in $\Omega$. Set $D_k = \{x \in \Omega: \text{dist}(x, \Gamma) > 1/k\}$,
and consider the sequence of weak solutions \( \{v_k\}_{k=1}^{\infty} \subset H_0^{1,p}(\Omega; \omega) \cap C(\Omega) \) to
\[
\begin{cases}
- \text{div } A(x, \nabla v_k) = |f|1_{D_k} & \text{in } \Omega, \\
v_k = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Since \( \Omega \) is bounded, the right-hand side belongs to the dual of \( H_0^{1,p}(\Omega; \omega) \). By the comparison principle in [Har22, Theorem 3.5],
\[
(6.1) \quad 0 \leq v_k(x) \leq v(x)
\]
for all \( x \in \Omega \). Also, consider the sequence of weak solutions \( \{u_k\}_{k=1}^{\infty} \subset H_0^{1,p}(\Omega; \omega) \cap C(\Omega) \) to
\[
\begin{cases}
- \text{div } A(x, \nabla u_k) = f1_{D_k} & \text{in } \Omega, \\
u_k = 0 & \text{on } \partial \Omega.
\end{cases}
\]
By the comparison principle for weak solutions, we have
\[-v_k(x) \leq u_k(x) \leq v_k(x)
\]
for all \( x \in \Omega \); therefore,
\[
(6.2) \quad |u_k(x)| \leq v(x)
\]
for all \( x \in \Omega \). Fix \( j \geq 1 \), and take a nonnegative function \( \eta \in C^\infty(\mathbb{R}^n) \) such that \( \eta \equiv 1 \) on \( D_j \) and \( \eta \equiv 0 \) on \( \Omega \setminus D_{j+1} \). Testing the equation of \( u_k \) with \( u_k \eta^p \), we get
\[
\int_{\Omega} A(x, \nabla u_k) \cdot \nabla u_k \eta^p \, dx + p \int_{\Omega} A(x, \nabla u_k) \cdot \nabla \eta \eta^p - 1 u_k \, dx = \int_{\Omega} f u_k \eta^p \, dx.
\]
By (1.2) and (1.3), this inequality yields
\[
(6.3) \quad \int_{D_j} |\nabla u_k|^p \, dw \leq C \int_{D_{j+1}} |u_k|^p \, dw + C \left( \left| f \right|_w^{p/(p-1)} w(D_{j+1}) \right).
\]
By (6.2) and assumption on \( f \), the right-hand side is bounded with respect to \( k \). Meanwhile, Lemma 4.2 yields a local Hölder estimate of \( u_k \). Therefore, for each \( j \geq 1 \), there are constants \( C_j \) and \( \alpha \in (0, 1) \) independent of \( k \) such that
\[
\|u_k\|_{C^\alpha(D_j)} + \|\nabla u_k\|_{L^p(D_j; w)} \leq C_j.
\]
Take a subsequence of \( \{u_k\}_{k=1}^{\infty} \) and a function \( u \) on \( \Omega \) such that \( u_k \to u \) locally uniformly in \( \Omega \) and \( \nabla u_k \rightharpoonup \nabla u \) weakly in \( L^p(D_j; w) \) for all \( j \). Let \( \eta \) be the function defined before. By the product rule, we have
\[
\int_{\Omega} |\nabla (u_k \eta)|^p \, dw \leq C \left( \int_{D_{j+1}} |\nabla u_k|^p \eta^p \, dw + \int_{D_{j+1}} |u_k|^p |\nabla \eta|^p \, dw \right).
\]
The left-hand side belong to \( H_0^{1,p}(\Omega; w) \), and thus, \( u \eta \in H_0^{1,p}(\Omega; w) \). This implies that the zero extension of \( u \) belongs to \( H_0^{1,p}(\mathbb{R}^n \setminus \Gamma; w) \). Using the test function \( u \eta - u_k \eta \in H_0^{1,p}(\Omega; w) \), we obtain
\[
(6.4) \quad \int_{\Omega} A(x, \nabla u_k) \cdot \nabla (u - u_k) \eta \, dx = - \int_{\Omega} A(x, \nabla u_k) \cdot \nabla \eta (u - u_k) \, dx + \int_{\Omega} f (u - u_k) \eta \, dx.
\]
Take the limit $k \to \infty$. The latter term on the right-hand side goes to zero. Moreover, by (1.3), the former term is estimated by
\[
\left| \int_{\Omega} A(x, \nabla u_k) \cdot \nabla \eta (u - u_k) \, dx \right| 
\leq C \left( \int_{D_{j+1}} |\nabla u_k|^p \, dw \right)^{(p-1)/p} \left( \int_{D_{j+1}} |u - u_k|^p \, dw \right)^{1/p}.
\]
The right-hand side goes to zero because $u_k \to u$ uniformly in $D_{j+1}$. Thus, the left-hand side of (6.4) goes to zero. Meanwhile, by the weak convergence of $\nabla u_k$ in $L^p(D_{j+1}; w)$, we have
\[
(6.5) \quad \int_{\Omega} A(x, \nabla u) \cdot \nabla (u - u_k) \eta \, dx \to 0.
\]
Combining (6.4), (6.5) and (1.4), we find that
\[
\int_{D_j} A(x, \nabla u) - A(x, \nabla u_k) \cdot \nabla (u - u_k) \eta \, dx \to 0.
\]
It follows from [HKM06, Lemma 3.73] that $u$ satisfies $-\text{div} A(x, \nabla u) = f$ in $\Omega$. Interior regularity of $u$ follows from Lemma 4.2. If $\xi \in \Gamma$, then $u$ is continuous at $\xi$ by the upper bound (6.2).

Let $u, v \in H^{1,p}_{\text{loc}}(\Omega; w) \cap C(\Omega)$ be weak solutions to (1.1) satisfying the Dirichlet boundary condition in the statement. Assume that $u(x) > v(x)$ for some $x \in \Omega$. Then, $D = \{x \in \Omega: u(x) > v(x) + \epsilon\}$ is a nonempty bounded open set for some $\epsilon > 0$. If $\text{dist}(\overline{D}, \Gamma) = 0$, then there is a boundary point $\xi \in \overline{D} \cap \Gamma$. This contradicts to assumption because $(u - v)(\xi) = 0$ and $\inf_D (u - v) \geq \epsilon > 0$. Therefore, $\text{dist}(\overline{D}, \Gamma) > 0$ and $u, v \in H^{1,p}(D; w)$. By density,
\[
\int_D (A(x, \nabla u) - A(x, \nabla v)) \cdot \nabla \varphi \, dx = 0
\]
for all $\varphi \in H^1_0(D; w)$. Testing this equation with $\varphi = u - v - \epsilon$ and using (1.4), we find that $u = v$ on $D$. This contradicts to the assumption. \hfill \square

**Theorem 6.3.** Assume that $\Omega$ is a bounded open set and that $\mathbb{R}^n \setminus \Omega$ is uniformly $(p, w)$-fat. Let $f \in L^1_{\text{loc}}(\Omega)$, and assume that there are constants $K > 0$ and $0 < \beta \leq \alpha$ such that
\[
|f(x)| \leq K \delta_{\partial \Omega}(x)^{\beta(p-1)-p} w(x)
\]
for almost every $x \in \Omega$, where $\alpha$ is a positive number in Theorem 5.1. Then, there exists a unique weak solution $u \in H^{1,p}_{\text{loc}}(\Omega; w) \cap C(\overline{\Omega})$ to (1.1). Moreover,
\[
(6.6) \quad |u(x)| \leq CK^{1/(p-1)} \delta_{\partial \Omega}(x)^{\beta}
\]
for all $x \in \Omega$, where $C = c_H^{1-p} \delta_{\partial \Omega}(\alpha/\beta)$.

**Proof.** Applying to Proposition 6.2 to $\Gamma = \partial \Omega$ and $h(t) = (K/c_H)^{1/(p-1)} t^{\beta/\alpha}$, we obtain the desired unique weak solution $u$. The upper bound (6.6) follows from (6.2). \hfill \square
Appendix A. Proofs of Lemmas 4.1 and 4.2

The proofs below are combinations of [KS80, p. 63, Lemma B.2], [GT01, Theorem 8.18] and [HKM06, Theorems 3.59 and 7.46]. See also [Str84, Chapter 3.1.0].

Proof of Lemma 4.1. Fix any $k > \sup_{\Omega} u$, and consider the test function $(u - k)_+ \in H^1_0(\Omega; w)$. By (1.2), we have

$$\int_{\Omega} |\nabla (u - k)_+|^p w \, dx \leq \int_{\Omega} A(x, \nabla u) \cdot (u - k)_+ \, dx = \int_{\Omega} f(u - k)_+ \, dx.$$ 

By Hölder's inequality, the right-hand side is estimated by

$$\int_{\Omega} f(u - k)_+ \, dx \leq \left( \int_{\Omega} (u - k)_+^{\chi p} \, dx \right)^{1/\chi} \left( \frac{f_+}{w} \right)_{L^\infty(\Omega)}^{1/(\chi p)} w([u > k])^{1-1/\chi}.$$ 

Meanwhile, by taking a ball $B$ with radius $diam(\Omega)$ such that $\Omega \subset B$, (2.3) yields

$$\left( \frac{1}{w(B)} \int_{\Omega} (u - k)_+^{\chi p} \, dx \right)^{1/\chi} \leq C \frac{\text{diam}(\Omega)^p}{w(B)} \int_{\Omega} |\nabla (u - k)_+|^p w \, dx.$$ 

Combining the three inequalities, we obtain

$$\frac{1}{w(B)} \int_{\Omega} (u - k)_+^{\chi p} \, dx \leq A \left( \frac{w([u > k])}{w(B)} \right)^{(1-1/\chi p)/(p-1)},$$

where $A = C \frac{\text{diam}(\Omega)^p}{(h-k)^p} \left( \frac{w([u > h])}{w(B)} \right)^{(p-1)/(p-1)}$, for any $h > k$. From [KS80, p. 63, Lemma B.1], the desired estimate follows. □

Proof of Lemma 4.2. Set $\bar{u} = u + F_-$. Consider the test function $\bar{u}^\beta \eta^p$, where $\beta < 0$ and $\eta \in C^\infty_c(2B)$. Since $\nabla \bar{u} = \nabla u$, we have

$$\int_{2B} A(x, \nabla \bar{u}) \cdot \nabla \bar{u}^\beta \eta^p \, dx + \int_{2B} A(x, \nabla \bar{u}) \cdot \nabla \eta \bar{u}^\beta \eta^p \, dx \geq \int_{2B} f \bar{u}^\beta \eta^p \, dx.$$ 

By the definition of $F_-$, the right-hand side is estimated from below by

$$\int_{2B} f \bar{u}^\beta \eta^p \, dx \leq C \frac{R^p}{R^p} \int_{2B} \bar{u}^{p-1+\beta} \eta^p \, dw.$$ 

Hence, by (1.2) and (1.3), we obtain

$$\int_{2B} |\nabla \bar{u}|^p \bar{u}^{\beta-1} \eta^p \, dw \leq C \left( |\beta|^{1-p} \|
abla \eta\|^{p}_{L^\infty(2B)} + R^{-p} \right) \int_{\supp \eta} \bar{u}^{p-1+\beta} \, dw.$$ 

Fix a ball $B(x, r)$ such that $B(x, 4\lambda r) \subset 2B$, where $\lambda$ is a constant in (2.2). Assume that $\beta \neq 1 - p$. Let $r \leq r_1 < r_2 \leq 2r$. By the chain rule, (A.2) and (2.3), we have

$$\int_{B(x, r_1)} \bar{u}^{\chi(p-1+\beta)} \, dw \leq \frac{C \cdot c(\beta)}{(r_2 - r_1)^p} \int_{B(x, r_2)} \bar{u}^{p-1+\beta} \, dw,$$

where $c(\beta) = |(p - 1 + \beta)/p|(|\beta|^{1-p} + |\beta|^{-1})$. First, we consider the case $\beta > 1 - p$. Using (A.3) finitely many times, we get

$$\int_{B(x, r)} \bar{u}^s \, dw \leq C \left( \int_{B(x, 2r)} \bar{u}^{s_0} \, dw \right)^{1/s_0},$$
where \( s > s_0 > 0 \) and \( C \) is a constant depending also on \( s_0 \). Next, for \( k = 0,1,2,\ldots \), we consider \( \beta_k = (1 - p) - \chi^k s_0 \) and the sequence of balls \( B^k = B(x,(1 + 2^{-k})r) \). Then, (A.3) gives

\[
\left( \int_{B^{k+1}} \bar{u} \chi^{k+1} s_0 \, dw \right)^{-1/\chi^{k+1} s_0} \geq \frac{1}{C_1 C_2^{1/\chi^k}} \left( \int_{B^k} \bar{u} \chi^k s_0 \, dw \right)^{-1/\chi^k s_0},
\]

where \( C_1 \) and \( C_2 \) are constants depending also on \( s_0 \), but independent of \( k \). Since \( \sum_k 1/\chi^k \) and \( \sum_k k/\chi^k \) are finite, we obtain

\[
\text{ess inf} \bar{u} \geq \frac{1}{C} \left( \int_{B(x,2r)} \bar{u}^{-s_0} \, dw \right)^{-1/s_0}, \tag{A.5}
\]

Finally, we consider \( \beta = 1 - p \). Then, (A.2) yields

\[
\int_{B(x,2r)} |\nabla \log \bar{u}|^p \, dw \leq Cr^{-p}w(B(x,4\lambda r)).
\]

Applying (2.2) to the left-hand side, we obtain

\[
\int_{B(x,2\lambda r)} |\log \bar{u} - c|^p \, dw \leq Cr^p \int_{B(x,2\lambda r)} |\nabla \log \bar{u}|^p \, dw \leq C,
\]

where \( c = \int_{B(x,2r)} \log \bar{u} \, dw \). Combining (A.4), (A.5) and (A.6) and using the John-Nirenberg lemma in [HKM06, Chapter 19] (see also [SC02, Lemma 2.2.6]), we arrive at

\[
\left( \int_{B(x,r)} \bar{u}^s \, dw \right)^{1/s} \leq C \text{ess inf} \bar{u}.
\]

The desired inequality (4.1) follows from this inequality and a covering argument. The latter inequality (4.2) is a consequence of (4.1) and (A.2). \( \square \)

Acknowledgments. This work was supported by JST CREST (doi:10.13039/501100003382) Grant Number JPMJCR18K3 and JSPS KAKENHI (doi:10.13039/501100001691) Grant Numbers JP17H01092 and JP23H03798.

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Received 13 January 2023 • Revision received 13 July 2024 • Accepted 1 August 2024

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