Kirillov–Reshetikhin crystals $B^{7,s}$ for type $E_7(1)$

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ABSTRACT

We construct a combinatorial crystal structure on the Kirillov–Reshetikhin crystal $B^{7,s}$ in type $E_7(1)$, where 7 is the unique node in the orbit of 0 in the affine Dynkin diagram. We then describe the combinatorial $R$-matrix $R : B^{7,s} \otimes B^{7,s} \rightarrow B^{7,s} \otimes B^{7,s}$.

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1. Introduction

An important class of finite-dimensional representations for affine Lie algebras are the Kirillov–Reshetikhin (KR) modules, which are characterized by their Drinfel’d polynomials [4, 5]. We denote a KR module by $W^{r,s}$, where $r$ is a node of the classical Dynkin diagram and $s$ is a positive integer. KR modules have been well-studied and have many interesting properties. For example, their characters (resp. $q$-characters) are solutions of the Q-system (resp. T-system) [12] (see also [20] and references therein). Moreover, graded (Demazure-type) characters of tensor products of single-column KR modules are (nonsymmetric) Macdonald polynomials at $t = 0$ for untwisted affine types [22–24].

One significant aspect of a KR module $W^{r,s}$ is that it (conjecturally) admits a crystal base [10, 11] despite not being a highest weight module. The corresponding crystal of $W^{r,s}$ is called a Kirillov–Reshetikhin (KR) crystal and denoted by $B^{r,s}$. KR crystals have been shown to exist in all nonexceptional types in [32], types $G_2^{(1)}$ and $D_4^{(3)}$ in [30], for a number of nodes in exceptional types [1, 31], and for $r$ being in the orbit of or adjacent to 0 in all affine types from the general theory [15, 16]. An open problem is to determine a uniform model for KR crystals. This has been achieved for $B^{1,1}$ by using Kashiwara’s construction of projecting an extremal level-zero module/crystal [18]. This was done explicitly by Naito and Sagaki using Lakshmibai–Seshadri (LS) paths [27–29]. The construction of Kashiwara was also shown to partially extend to general $B^{r,s}$ in nonexceptional affine types (conjecturally in all affine types) [25]. In contrast, the models in [6, 14, 19, 37] are all type-dependent, but are given for $B^{r,s}$ all $s$. 

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KR crystals are connected with mathematical physics. For instance, tensor products of KR modules are used to describe certain vertex models and are related with Heisenberg spin chains by the \(X=M\) conjecture of \([10, 11]\). The \(X=M\) conjecture implies a fermionic formula for the graded characters of a tensor product of KR crystals; see \([33, 35]\) for recent progress. Furthermore, tensor products of KR crystals describe the dynamics of soliton cellular automata, a generalization of the Takehashi–Satsuma box-ball system (which is an ultradiscrete version of the Korteweg–de Vries (KdV) equation). We refer the reader to \([13, 26]\) for more details. Another important (conjectural) property of KR crystals is that they are perfect \([7, 15, 16, 19, 37]\), a technical condition that allows highest weight crystals to be modeled using a semi-infinite tensor product known as the Kyoto path model \([16]\).

In this note, we give a combinatorial model for the KR crystal \(B^{7,s}\) in type \(E_7^{(1)}\), where 7 is the unique node in the orbit of 0 in the Dynkin diagram (see Figure 1 below). We achieve this by considering the (Levi) decomposition of the classical (type \(E_7\)) highest weight crystal \(B(s\omega_7)\) into \(A_6\) highest weight crystals, which is multiplicity free. From this, we reconstruct the \(A_7\) decomposition of \(B^{7,s}\) since \(B^{7,s} \cong B(s\omega_7)\) as \(E_7\) crystals and the decomposition of \(A_7\) highest weight crystals into \(A_6\) crystals is multiplicity free. We note that the KR crystal \(B^{7,s}\) exists since 7 is a minuscule node, so \(B^{7,s}\) is irreducible as a classical crystal. The novelty of our approach is doing a further Levi decomposition and reconstructing the affine action to a type \(A_7\) crystal rather than through the classical decomposition. We then give an explicit description of the combinatorial \(R\)-matrix \(R : B^{7,s} \otimes B^{7,s} \rightarrow B^{7,s} \otimes B^{7,s}\). We note that the local energy function is given by \([35, \text{Thm. 7.5}]\).

As a potential application of our results, the combinatorial \(R\)-matrix allows us to study soliton cellular automata of \(B^{7,1}\) using different techniques from \([25]\). Moreover, our results could potentially be used to show that \(B^{7,s}\) is a perfect crystal of level \(s\).

This paper is organized as follows. In Section 2, we give the necessary background. In Section 3, we give our main results. In Section 4, we give a conjecture about the decomposition of \(B^{7,1}\) into \(A_7\) crystals in an effort to prove \([14, \text{Conj. 3.26}]\).

2. Background

Let \(\mathfrak{g}\) be an affine Kac–Moody Lie algebra with index set \(I\), Cartan matrix \((A_{ij})_{i,j \in I}\), simple roots \((\alpha_i)_{i \in I}\), fundamental weights \((\omega_i)_{i \in I}\), and simple coroots \((\alpha_i')_{i \in I}\). Let \(U_q(\mathfrak{g})\) denote the corresponding (Drinfel’d–Jimbo) quantum group, and we will be using \(U_q'(\mathfrak{g}) := U_q([\mathfrak{g}, \mathfrak{g}])\), which has weight lattice \(P = \sum_{i \in I} \mathbb{Z} \omega_i\). Let \(P^+\) denote the positive weight lattice. Let \(Q\) be the root lattice with \(Q^+\) being the positive root lattice. We denote the canonical pairing \(\langle , \rangle : P^+ \times P \rightarrow \mathbb{Z}\), which is given by \(\langle \alpha_i', \alpha_j \rangle = A_{ij}\).

Recall that \(\text{lev}(\lambda) := \langle c, \lambda \rangle\) is the level of the weight \(\lambda\), where \(c\) is the canonical central element of \(\mathfrak{g}\). In particular, for \(\mathfrak{g}\) of type \(E_7^{(1)}\), we have

\[
\begin{align*}
\text{lev}(\omega_0) &= 1, \quad \text{lev}(\omega_1) = 2, \quad \text{lev}(\omega_2) = 2, \quad \text{lev}(\omega_3) = 3, \\
\text{lev}(\omega_4) &= 4, \quad \text{lev}(\omega_5) = 3, \quad \text{lev}(\omega_6) = 2, \quad \text{lev}(\omega_7) = 1.
\end{align*}
\]

We denote the dominant weights of level \(\ell\) by \(P^+_{\ell}\).
Let $g_0$ denote the canonical simple Lie algebra given by the index set $I_0 = I \setminus \{0\}$, and $U_q(g_0)$ the corresponding quantum group. Let $P_0$ and $Q_0$ be the weight and root lattice of $g_0$, and let $\omega_i$ be the natural projection of the fundamental weight $\omega_i$ onto $P_0$. Let $W_0$ be the Weyl group of $g_0$.

### 2.1. Crystals

An abstract $U_q(\mathfrak{g})$-crystal is a set $B$ endowed with crystal operators $e_i, f_i : B \to B \cup \{0\}$, for $i \in I$, and weight function $\mathrm{wt} : B \to P$ that satisfy the following conditions:

1. $\varphi_i(b) = e_i(b) + (\varphi_i', \mathrm{wt}(b))$, for all $b \in B$ and $i \in I$,
2. $f_i b = b'$ if and only if $b = e_i b'$, for $b, b' \in B$ and $i \in I$,
3. $\mathrm{wt}(f_i b) = \mathrm{wt}(b) - \alpha_i$ if $f_i b \neq 0$;

where the statistics $e_i, \varphi_i : B \to \mathbb{Z}_{\geq 0}$ are defined by

$$e_i(b) := \max\{k \mid e_i^k b \neq 0\} , \quad \varphi_i(b) := \max\{k \mid f_i^k b \neq 0\} .$$

**Remark 2.1.** The definition of an abstract crystal given in this paper is sometimes called a regular or seminormal abstract crystal in the literature. See, e.g., [2] for the more general definition.

Using the axioms, we identify $B$ with an $I$-edge colored weighted directed graph whose vertices are $B$ and having an $i$-colored edge $b \to b'$ if and only if $f_i b = b'$. Therefore, we can depict an entire $i$-string through an element $b \in B$ diagrammatically by

$$e_i^j(b) \to \cdots \to e_i b \to f_i b \to f_i^2 b \to \cdots \to f_i^{\varphi_i(b)} b .$$

Let $J \subseteq I$. An element $b \in B$ is $J$-highest (resp. lowest) weight if $e_i b = 0$ (resp. $f_i b = 0$) for all $i \in J$. When $J = I$, we simply say $b$ is highest (resp. lowest) weight.

For abstract $U_q(\mathfrak{g})$-crystals $B_1, B_2, \ldots, B_L$, the action of the crystal operators on the tensor product $B_1 \otimes \cdots \otimes B_2 \otimes B_1$, which equals the Cartesian product $B_1 \times \cdots \times B_2 \times B_1$ as sets, can be defined by the signature rule. Let $b := b_1 \otimes \cdots \otimes b_2 \otimes b_1 \in B$, and for $i \in I$, we write

$$\ldots \otimes e_i(b_1) \otimes \ldots \otimes e_i(b) \otimes \ldots \otimes e_i(b_1) \otimes \ldots .$$

Then by successively deleting consecutive $++$-pairs (in that order), we obtain a sequence

$$\text{sgn}_i(b) := \ldots - \ldots + \ldots ,$$

called the reduced signature. If there does not exist a $+$ (resp. $-$) in $\text{sgn}_i(b)$, then $e_i b = 0$ (resp. $f_i b = 0$). Otherwise, suppose $1 \leq j_- \leq j_+ \leq L$ are such that $b_{j_-}$ contributes the rightmost $-$ in $\text{sgn}_i(b)$ and $b_{j_+}$ contributes the leftmost $+$ in $\text{sgn}_i(b)$. Then, we have

$$e_i b := b_1 \otimes \cdots \otimes b_{j_-+1} \otimes e_i b_{j_-} \otimes b_{j_-+1} \otimes \cdots \otimes b_1 ,$$

$$f_i b := b_L \otimes \cdots \otimes b_{j_-+1} \otimes f_i b_{j_-} \otimes b_{j_-+1} \otimes \cdots \otimes b_1 .$$

**Remark 2.2.** Our tensor product convention follows [2], which is opposite to that of Kashiwara [17].

Let $B_1$ and $B_2$ be two abstract $U_q(\mathfrak{g})$-crystals. A crystal morphism $\psi : B_1 \to B_2$ is a map $B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ with $\psi(0) = 0$, such that the following properties hold for all $b \in B_1$ and $i \in I$:

1. if $\psi(b) \in B_2$, then $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b), e_i(\psi(b)) = e_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$.
(2) we have $\psi(e, b) = e, \psi(b)$ if $\psi(e, b) \neq 0$ and $e, \psi(b) \neq 0$ ;
(3) we have $\psi(f, b) = f, \psi(b)$ if $\psi(f, b) \neq 0$ and $f, \psi(b) \neq 0$ .

An embedding (resp. isomorphism) is a crystal morphism such that the induced map $B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ is an embedding (resp. bijection).

An abstract crystal $B$ is a $U_q(g)$-crystal if $B$ is the crystal basis of some $U_q(g)$-module. Kashiwara [17] has shown that the irreducible highest weight module $V(\lambda)$, for $\lambda \in P^+$, admits a crystal basis denoted $B(\lambda)$. The highest weight crystal $B(\lambda)$ is generated by a unique highest weight element $u_\lambda$ that satisfies $\text{wt}(u_\lambda) = \lambda$.

### 2.2. Minuscule crystals

We say a highest weight $U_q(g_0)$-crystal $B(\lambda)$ is minuscule if $W_0$ acts transitively on $B(\lambda)$. In other words, there exists a bijection between $B(\lambda)$ and $W_0^\lambda$, the set of minimal length coset representatives of $W_0/\text{stab}_{W_0}(\lambda)$ (recall $\text{stab}_{W_0}(\lambda) := \{w \in W_0 \mid w\lambda = \lambda\}$ is the stabilizer of $\lambda$ and a parabolic subgroup of $W_0$). Indeed, consider a minimal length coset representative $w \in W_0^\lambda$ with a reduced expression $s_{i_1}s_{i_2}\ldots s_{i_k}$, then the corresponding element is $u_{w\lambda} := f_{i_1}f_{i_2}\ldots f_{i_k}u_\lambda$. We note that the element $u_{w\lambda}$ is independent of the choice of reduced expression.

For a minuscule representation $B(\lambda)$, we can characterize the elements in $B(s\lambda)$ as follows. Recall that since $B(\lambda)$ is a highest weight crystal, it can be considered as the Hasse diagram of a poset with $u_\lambda$ being the smallest element.

**Proposition 2.3** ([35, Prop. 7.29]). Let $B(\lambda)$ be a minuscule representation. The crystal $B(s\lambda)$ is isomorphic to the set of semistandard tableaux whose shape is a single row of length $s$

$$T = \left[ \begin{array}{c} x_1 \ x_2 \ \cdots \ \ x_s \end{array} \right]$$

with $x_1 \leq x_2 \leq \cdots \leq x_s$ in $B(\lambda)$, and the crystal structure is given by considering $T$ as the element $x_1 \otimes x_2 \otimes \cdots \otimes x_s \in B(\lambda)^{\otimes s}$.

### 2.3. Type $A_n$ Crystals

In this section, we consider the Lie algebra of type $A_n$, which is $\mathfrak{sl}_{n+1}$. We denote the fundamental weights of type $A_n$ by $\{\eta_i \mid 1 \leq i \leq n\}$. Recall that we have a natural bijection between $P^+$ and partitions of length at most $n$ by $\eta_i$ corresponding to a column of height $i$. Let $B(\eta_1)$ denote the crystal of the vector representation of $\mathfrak{sl}_{n+1}$:

$$\begin{array}{cccccccc}
1 & 1 & \rightarrow & 2 & 2 & \rightarrow & 3 & 3 & \cdots & n-1 & \rightarrow & n & n & \rightarrow & n+1
\end{array}$$

Furthermore, the crystal $B(\eta)$ can be described by semistandard Young tableaux (SSYT), written in English convention, whose entries are at most $n+1$. The crystal structure is given by embedding a SSYT $T \in B(\eta)$ into $B(\eta_1)^{\otimes |\eta|}$ by the reverse Far-Eastern reading word: reading bottom-to-top and left-to-right.

Next, we recall the Levi branching rule $\mathfrak{sl}_{n+1} \setminus \mathfrak{sl}_n$ given at the level of crystals. In terms of the crystal graph, we simply remove all $n$-colored edges. For the SSYT, this amounts to fixing all $n+1$’s that appear. Since any $n+1$ must be the bottom entry of every column and the largest entry in a given row, we obtain the following statement (which is well-known to experts).

**Proposition 2.4.** As $U_q(\mathfrak{sl}_n)$-crystals, we have

$$B(\eta) \cong \bigoplus_{\mu} B(\mu),$$

where the sum is taken over all $\mu$ such that $\eta/\mu$ is a horizontal strip (i.e. a skew partition that does not contain a vertical domino).
Note that the decomposition of Proposition 2.4 is multiplicity-free.

We recall that there exists a natural order 2 diagram automorphism \( \sigma \) on type \( A_n \) crystals given by \( i' = n + 1 - i \). Indeed, we define an automorphism of the weight lattice, which we abuse notation and also denote by \( \sigma : P \to P \), by \( \eta_i \mapsto \eta_{i'} = \eta_{n+1-i} \); in particular, we note that \( \eta \mapsto -w_0\eta \). Next, we define a map also denoted by \( \sigma : B(\eta) \to B(-w_0\eta) \) given by

\[
\sigma(f_{i_1} \cdots f_{i_k} \eta) = f_{i_1} \cdots f_{i_k} u_{-w_0\eta}.
\]

We recall from [34] that \( \sigma = \triangledown \circ \ast \), where \( \triangledown \) denotes the contragredient dual map and \( \ast \) is the Lusztig involution (which equals the Schützenberger involution [21]).

In the sequel, we require the \( A_7 \) Levi subalgebra \( \mathfrak{g}_2 \) of the type \( E_7^{(1)} \) affine Lie algebra \( \mathfrak{g} \) given by the index set \( I_7 := I \setminus \{2\} \). Therefore, the fundamental weights correspond by

\[
\begin{align*}
\eta_1 &= \omega_7, \\
\eta_2 &= \omega_6, \\
\eta_3 &= \omega_5, \\
\eta_4 &= \omega_4, \\
\eta_5 &= \omega_3, \\
\eta_6 &= \omega_1, \\
\eta_7 &= \omega_0.
\end{align*}
\]

We let \( I_{0.2} := J \setminus \{0, 2\} \) be index set of the \( A_6 \) Levi subalgebra of \( \mathfrak{g}_6 \) (of type \( E_7 \)).

3. Results

In this section, we give our main results.

We follow [14, Fig. 3] and label an element \( b \in B(\bar{\omega}_7) \) as a word in \( I_7 \) and \( I_0 := \{1, \ldots, 7\} \), where for \( i \in b \) such that \( i \in I_0 \) (resp. \( i \in I_0 \)), we have \( \varphi_i(b) = 1 \) (resp. \( \varepsilon_i(b) = 1 \)). See Figure 2 for the crystal \( B(\bar{\omega}_7) \) with this labeling convention.

3.1. Multiplicity freeness of \( B(s\bar{\omega}_7) \)

We first prove that when we decompose the \( E_7 \) crystals \( B(s\bar{\omega}_7) \) into the Levi subalgebra of type \( A_6 \), we obtain a multiplicity free decomposition.

The \( I_{0.2} \)-highest weight elements of \( B(s\bar{\omega}_7) \) are single row tableaux of size \( k \) whose entries consist of

\[
\begin{align*}
x_1 &= 7, \\
x_2 &= 65, \\
x_3 &= 423, \\
x_4 &= \frac{12}{23}, \\
x_5 &= \frac{124}{23}, \\
x_6 &= 26, \\
x_7 &= 21.
\end{align*}
\]

This can be seen by a direct computation using Proposition 2.3, the crystal graph \( B(\bar{\omega}_7) \), and the signature rule. Note that in the crystal graph of \( B(\bar{\omega}_7) \), we have

\[
\begin{tikzpicture}
  \node (x1) at (0,0) {\(x_1\)}; 
  \node (x2) at (1,0) {\(x_2\)}; 
  \node (x3) at (2,0) {\(x_3\)}; 
  \node (x4) at (3,0) {\(x_4\)}; 
  \node (x5) at (4,0) {\(x_5\)}; 
  \node (x6) at (5,0) {\(x_6\)}; 
  \node (x7) at (6,0) {\(x_7\)}; 

  \draw[->] (x1) -- (x2); 
  \draw[->] (x2) -- (x3); 
  \draw[->] (x3) -- (x4); 
  \draw[->] (x4) -- (x5); 
  \draw[->] (x5) -- (x6); 
  \draw[->] (x6) -- (x7); 
  \draw[->] (x4) -- (x4'); 
\end{tikzpicture}
\]

(3.1)

and so, nearly all of the elements \( x_i \) are comparable except \( x_4 \) with \( x_4' \).

We recall from [14, Def. 3.10] that a (reduced) composition graph \( G_k(\bar{\omega}) \) is essentially the smallest acyclic digraph with loops whose vertices are elements of \( B(\bar{\omega}) \) such that for any \( s > 0 \) and every \( (I_7 \setminus \{k\}) \)-highest weight element \( b_1 \otimes \cdots \otimes b_s \in B(s\bar{\omega}) \subseteq B(\bar{\omega})^{\otimes s} \), the elements \( (b_1, \ldots, b_s) \) occurs as a subsequence of a directed path in \( G_k(\bar{\omega}) \). We remark by reversing the arrows and adding loops to every vertex in (3.1), we obtain the composition graph \( G_1(\bar{\omega}_7) \). Code to compute composition graphs is provided in Appendix A.

Note that for a semistandard tableau \( T \in B(s\bar{\omega}_7) \), we have

\[
\text{wt}(T) = \sum_{i \in I} (a_i - a_i) \bar{\omega}_i.
\]
where \( a_i \) (resp. \( a_i \)) equals the number if \( i \)'s (resp. \( \bar{i} \)'s) that appear in \( T \). Therefore, from (3.1) and the signature rule, we have the following.

**Lemma 3.1.** Let \( m_i \) denote the number of occurrences of \( x_i \) in \( T \in B(\bar{s}\bar{\omega}_7) \). Then \( T \) is a \( I_{0,2} \)-highest weight element if and only if entries in \( T \) consist of \( \{x_1, x_2, x_3, x_4, x_4', x_5, x_6, x_7\} \) and

\[
m_2 \leq m_6, \quad m_3 \leq m_5, \quad m_4 + m_5 \leq m_7, \quad \min(m_4, m_4') = 0,
\]

with \( \sum_i m_i = s \). Moreover, if \( T \in B(\bar{s}\bar{\omega}_7) \) is a \( I_{0,2} \)-highest weight element, then

\[
\text{wt}(T) = (m_7 - m_4 - m_5)\bar{\omega}_1 + (m_3 + m_4 - m_4' - m_5 - m_6 - m_7)\bar{\omega}_2 + (m_3 + m_4')\bar{\omega}_3 + (m_5 - m_3)\bar{\omega}_4 + m_2\bar{\omega}_5 + (m_6 - m_2)\bar{\omega}_6 + m_1\bar{\omega}_7.
\] (3.2)

We note that the condition \( \min(m_4, m_4') = 0 \) is precisely the fact that \( x_4 \) and \( x_4' \) cannot simultaneously appear in an element of \( B(\bar{s}\bar{\omega}_7) \).
Proposition 3.2. The decomposition of $B(s\varpi_7)$ into type $A_6$ crystals is given by

$$B(s\varpi_7) \cong \bigoplus_{\mu} B(\mu),$$

where

$$\mu = (m_7 - m_4 - m_5) \eta_6 + (m_3 + m_4') \eta_5 + (m_5 - m_3) \eta_4 + m_2 \eta_3 + (m_5 - m_2) \eta_2 + m_1 \eta_1$$

such that $m_1, \ldots, m_7$ satisfy

$$m_2 \leq m_6, \quad m_3 \leq m_5, \quad m_4 + m_5 \leq m_7,$$

$$\min(m_4, m_4') = 0, \quad s = m_1 + m_2 + m_3 + m_4 + m_4' + m_5 + m_6 + m_7.$$

Moreover, this decomposition is multiplicity free.

**Proof.** The first claim follows immediately from Lemma 3.1 and relabeling the fundamental weights. For the second claim, consider a weight $\mu = \sum_{i=1}^{6} a_i \eta_i$ such that $B(\mu)$ appears in the $A_6$ decomposition of $B(s\varpi_7)$. Thus, we have

$$m_1 = a_1, \quad m_2 = a_3,$n_6 = a_2 + m_2 = a_2 + a_3, \quad m_3 = a_5 - m_4', \quad m_5 = a_4 + m_3 = a_4 + a_5 - m_4' - m_4,$$  

$$m_7 = a_6 + m_4 + m_5 = a_4 + a_5 + a_6 + m_4 - m_4'.$$

Since $\min(m_4, m_4') = 0$ with $m_4, m_4' \geq 0$, there exists a unique $m_4$ and $m_4'$ such that $m_4 - m_4' = C$ for any constant $C$. Next, we have

$$s = m_1 + m_2 + m_3 + m_4 + m_4' + m_5 + m_6 + m_7$$

$$= a_1 + a_3 + (a_5 - m_4') + m_4 + m_4' + (a_4 + a_5 - m_4') + (a_2 + a_3) + (a_4 + a_5 + a_6 + m_4 - m_4')$$

$$= a_1 + a_2 + 2a_3 + 2a_4 + 3a_5 + a_6 + 2(m_4 - m_4').$$

Hence, we have

Figure 3. The composition graph $G_2(\varpi_7)$ to compute the $(\{0 \setminus \{2\})$-highest weight elements. We have suppressed the loops that occur at every node except $32 \otimes 23$. 

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Therefore, there is a unique \( m_1, \ldots, m_7 \) that yields the weight \( \mu \). \( \square \)

### 3.2. Reconstructing the \( A_7 \) crystals

In this section, we continue to use the notation of Proposition 3.2.

**Lemma 3.3.** Let \( T \in B^{7,5} \) be a \( I_{0,2} \)-highest weight element. Then

\[
-(x_0^\mu, \text{wt}(T)) = m_1 + m_2 + m_3 + m_4.
\]

**Proof.** This follows from Equation (3.2), that all elements in \( B^{7,5} \) are of level 0, and (2.1). \( \square \)

Since \( (x_0^\mu, \text{wt}(T)) \leq 0 \) for all \( I_{0,2} \)-highest weight elements \( T \in B^{7,5} \), we know that any potential \( I_2 \)-highest weight element must have \( m_1 = m_2 = m_3 = m_4 = 0 \). Hence, the possible \( I_2 \)-dominant weights are

\[
\mu = (m_7 - m_4 - m_5)\eta_6 + m_5\eta_4 + m_6\eta_2
\]

(i.e., \( (x_0^\mu, \mu) \geq 0 \) for all \( i \in I_2 \)) such that

\[
m_4 + m_5 \leq m_7, \quad m_4 + m_5 + m_6 + m_7 = s.
\]

Because we can only remove horizontal strips for the branching rule from \( A_7 \setminus A_6 \) (Proposition 2.4), each of such \( I_2 \)-dominant weights \( \mu \) must correspond to a \( I_2 \)-highest weight component \( B(\mu) \). Hence, we obtain the following.

**Proposition 3.4.** The decomposition of \( B(s\tilde{o}_7) \) into type \( A_7 \) crystals is given by

\[
B(s\tilde{o}_7) \cong \bigoplus_{\mu} B(\mu),
\]

where

\[
\mu = (m_7 - m_4 - m_5)\eta_6 + m_5\eta_4 + m_6\eta_2
\]

such that \( m_4, m_5, m_6, m_7 \) satisfy

\[
m_4 + m_5 \leq m_7, \quad s = m_4 + m_5 + m_6 + m_7.
\]

Moreover, Proposition 2.4 states that for any element \( T \) expressed as an \( A_6 \) tableau, we add in a horizontal strip to \( T \) with every entry an 8 such that every column has even height to obtain the representation as an \( A_7 \) tableau. In particular, for a \( I_{0,2} \)-highest weight element \( b \), we have

\[
e_0(b) = m_1 + m_2 + m_3 + m_4', \quad \varphi_0(b) = 0.
\]

Now we combine this to form a combinatorial crystal structure on \( B^{7,5} \) by extending the \( E_7 \) crystal structure on \( B(s\tilde{o}_7) \) as follows. Let \( \tilde{\psi} : B(s\tilde{o}_7) \rightarrow \bigoplus_\mu B(\mu) \) be the \( I_{0,2} \)-crystal (i.e., type \( A_6 \)) isomorphism given by Proposition 3.2. From Proposition 3.4 and Proposition 2.4, we can uniquely extend the image of \( \tilde{\psi} \) to highest weight crystals of type \( A_7 \). Therefore, we define

\[
e_0 := \tilde{\psi}^{-1} \circ e_0^A \circ \tilde{\psi}, \quad f_0 := \tilde{\psi}^{-1} \circ f_0^A \circ \tilde{\psi},
\]

where \( e_0^A \) and \( f_0^A \) are the crystal operators from this extended type \( A_7 \) crystal. Let \( B^{7,5} \) denote the corresponding crystal.

In order to show this is the combinatorial structure of KR crystal, we need the following uniqueness theorem. The proof is similar to [14, Thm. 3.15] with \( K = I_{0,2} \) and using Proposition 3.2 instead of [14, Lemma 3.12].
Theorem 3.5. Let $B$ and $B'$ be two affine type $E_7^{(1)}$ crystals such that there exists a $I_2$-crystal (i.e., type $A_7$) isomorphism and $I_0$-crystal (i.e., type $E_7$) isomorphism

$$
\Psi_{1b} : B|_{I_1} \to B'|_{I_1} \cong \bigoplus_{\mu} B(\mu) \quad \Psi_{0b} : B|_{I_0} \to B'|_{I_0} \cong B(s\omega_7),
$$

where the direct sum is over $\mu$ given in Proposition 3.4. Then, we have $\Psi_{1b}(b) = \Psi_{0b}(b)$ for all $b \in B$. Moreover, there exists an $I$-crystal isomorphism $\Psi : B \to B'$.

Corollary 3.6. We have $B'^{7,s} \cong B^{7,s}$.

Proof. We have that $B'^{7,s} \cong B(s\omega_7)$ as $I_0$-crystals since the corresponding KR module is irreducible as a $U_q(y_0)$-module [3]. Hence, the KR crystal $B^{7,s}$ exists by [15, 16]. We can then decompose the crystal $B(s\omega_7)$ into $I_0,2$-crystals according to Proposition 3.2, and so the $I_2$-crystal decomposition of $B^{7,s}$ is given by Proposition 3.4. Hence, we have isomorphisms between $B^{7,s}$ and $B'^{7,s}$ as $I_0$-crystals and $I_2$-crystals because these decompositions are multiplicity free.

We have verified that $B^{7,s}$ is a perfect crystal of level $s$ for all $s \leq 4$ using the implementation of $B^{7,s}$ in SAGEMATH [36] done by the second author.

3.3. Combinatorial isomorphism

Let $\langle b \rangle$ denote the $I_0,2$-subcrystal generated by an element $b$. Consider the subcrystals

$$
C_1 = \langle x_1 \rangle, \quad C_2 = \langle x_6 \rangle, \quad C_3 = \langle x_2 \otimes x_6 \rangle, \\
C_4 = \langle x_5 \otimes x_7 \rangle, \quad C_5 = \langle x_3 \otimes x_5 \otimes x_7 \rangle, \quad C_6 = \langle x_7 \rangle.
$$

It is clear that we have $I_0,2$-crystal isomorphisms $\psi_i : B(\eta_i) \rightarrow C_i$ for all $i$. (Note that we could have $C_7 = \langle x_4 \otimes x_5 \rangle$, which would correspond to $m_4$.) Let $C_5 = \langle x_q \rangle$, and we also have an $I_0,2$-crystal isomorphism $\psi_5 : B(\eta_5) \rightarrow C'_5$.

Therefore, we can construct a bijection between SSYT for $B(\mu)$ with the $A_6$ component

$$
C_5^{\otimes m} \otimes C_4^{\otimes m} \otimes C_3^{\otimes m} \otimes C_2^{\otimes m} \otimes C_1^{\otimes m}
$$

by applying $\psi_i$ on each column of height $i$, with possibly $\psi_5'$ on columns of height 5, of a SSYT (of type $A_6$). Note that the result is “out-of-order” in that it does not result is $x_1 \otimes \cdots \otimes x_6$ with $i_1 \leq \cdots \leq i_6$ (i.e., a semistandard tableau given by Proposition 2.3). Thus it would remain to “sort” these elements to obtain an honest component of $B(s\omega_7)$. However, we will instead do the reverse process: we refine the isomorphisms $\psi_i^{-1}$ to apply them to elements of $B(\omega_7)$, which we combine to apply to an element of $B(s\omega_7)$.

Using the tensor product rule, we obtain the following defining crystal isomorphism with a tensor product of $A_6$ columns:

$$
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
$$

We need one additional step to decouple $x_5 \otimes x_7$, which we do by
We could have arrived at this refined isomorphism by using the $A_6$ decomposition of $B(\bar{\omega}_7)$. Now using our refined isomorphism and jeu-de-taquin, we obtain a crystal isomorphism $\Psi$ from elements of $B(s\bar{\omega}_7)$ to a direct sum of $A_6$ SSYT given by Proposition 3.2. Recall that jeu-de-taquin is a crystal isomorphism.

Indeed, performing jeu-de-taquin (see, e.g., [9]) on a highest weight element of the tensor product of single column SSYT, we obtain the SSYT

where row $i$ is filled with only $i$. Note that we have included an entry from both $m_4$ and $m'_4$ for illustrative purposes, but both cannot appear. Augmenting the result with a horizontal strip of 8’s to obtain all columns of even height (which agrees with Proposition 3.2 after ignoring the columns of height 8), we have

Note that the 8’s in the even rows can overlap any of the blocks depending on $m_1$, $m_2$, and $m_3 + m'_4$ compares with $m'_4 + m_5 + m_6 + m_7, m'_4 + m_5 + m_7,$ and $m_7$ respectively.

### 3.4. Diagram automorphism

We can construct the type $E_6$ crystal decomposition as follows. Let $I_{0,7} := I_0 \setminus \{7\}$. We construct the $I_{0,7}$-highest weight elements by considering semistandard tableaux in $B(s\bar{\omega}_7)$ consisting of the elements of
The computation is similar to (3.1). Similarly, by reversing the arrows and adding loops at every vertex, we obtain the composition graph $G_7(\tilde{\omega}_7)$. Thus, the $I_{0,7}$-highest weight elements in $B(s\tilde{\omega}_7)$ are given by

$$7^\otimes m_7 \otimes \tilde{76}^\otimes m_{76} \otimes 71^\otimes m_{71} \otimes \tilde{7}^\otimes m_7$$

(3.3)

(recall that we naturally identify this with a semistandard tableau) with $m_7 + m_{76} + m_{71} + m_7 = s$.

**Proposition 3.7.** Let $m_b$ be the number of occurrences of $b$ in a single row tableau $R \in B(s\tilde{\omega}_7)$. Then $R$ is an $I_{0,7}$-highest weight element. Moreover, as $E_6$ crystals, we have

$$B(s\tilde{\omega}_7) \cong \bigoplus_{m_{76} + m_{71} + m_7 \leq s} B(m_{76}\tilde{\omega}_6 + m_{71}\tilde{\omega}_1).$$

Note that the decomposition into $E_6$ crystals from Proposition 3.7 is not multiplicity free; indeed, the multiplicity of $B(m_{76}\omega_6 + m_{71}\tilde{\omega}_1)$ is $s - m_7 - m_{76} - m_7$ (the number of distinct values $m_i$ can take). However, we can distinguish each of the $I_{0,7}$-highest weight elements by using the extra information (in addition to the weight) of $\langle \text{wt}(b), x_7^\vee \rangle = m_7 - m_7 - m_{76} - m_7$. Furthermore, by the level-zero condition, we have $\langle \text{wt}(b), x_7^\vee \rangle = m_7 - m_7 - m_{76} - m_7$. By also using $m_7 + m_{76} + m_{71} + m_7 = s$, we thus obtain the following.

**Proposition 3.8.** The map $\Phi : B^{7,s} \to B^{7,s'}$ given by

$$\Phi(7^\otimes a \otimes \tilde{76}^\otimes b \otimes 7\otimes c \otimes \tilde{7}^\otimes d) = 7^\otimes a \otimes \tilde{76}^\otimes b \otimes 71^\otimes c \otimes \tilde{7}^\otimes d$$

and extended as a twisted $I_{0,7}$-crystal isomorphism is a twisted $I_{0,7}$-crystal isomorphism. Moreover $\Phi$ is an twisted crystal involution that is induced from the order 2 diagram automorphism of $E_7^{(1)}$.

By restricting to $I_{0,2}$-highest weight elements, we have that $\Phi = \psi^{-1} \circ \sigma \circ \psi$, where $\psi$ is the isomorphism from Proposition 3.4. This can be seen by equating the weight and $\langle \text{wt}(b), x_7^\vee \rangle$. Thus, $\Phi$ when restricted to the $A_7$ crystal of $B^{7,s}$ is the order 2 diagram automorphism of $A_7$.

As a consequence, we have that $B^{7,s}$ satisfies [8, Assumption 1], and so we have the following by [8, Thm. 4.7]. (We refer the reader to [8] for the precise definitions.)

**Corollary 3.9.** The tensor product $B = (B^{7,s})^\otimes m \otimes \{u_{s_0}\}$ is isomorphic to a Demazure subcrystal $B_w(s_0\omega_{(0)})$, where $t_\mu = \omega_\mu$ in the extended affine Weyl group with $\mu = -m\omega_7$. Moreover, $B$ and $(B^{7,1})^\otimes m$ are connected.

### 3.5. Combinatorial R-matrix

We give an explicit description of the combinatorial $R$-matrix $R : B^{7,s} \otimes B^{7,s'} \to B^{7,s'} \otimes B^{7,s}$ by noting the classical decomposition of $B^{7,s} \otimes B^{7,s'}$ is multiplicity free as $E_7$ crystals. We may assume $s \leq s'$ without loss of generality as the combinatorial $R$-matrix is an involution. Thus it is sufficient to consider the $I_{0,7}$-highest weight elements of $B^{7,s}$, which is given by (3.3). Therefore, the $I_{0}$-highest weight elements have weight

$$(m_7 - m_7 - m_{76} - m_{71} - s')\omega_0 + m_7\omega_1 + m_{76}\omega_6 + (s' + m_7 - m_{76} - m_{71} - m_7)\omega_7.$$ 

The fact that the decomposition into $E_7$ crystals is multiplicity free is exactly the same reasoning as in the proof of Proposition 3.8. Since the combinatorial $R$-matrix must map classical components to classical components, we have the following.

**Theorem 3.10.** For $s \leq s'$, the combinatorial $R$-matrix $R : B^{7,s} \otimes B^{7,s'} \to B^{7,s'} \otimes B^{7,s}$ is defined by

$$(7^\otimes m_7 \otimes \tilde{76}^\otimes m_{76} \otimes 71^\otimes m_{71} \otimes \tilde{7}^\otimes m_7) \otimes 7^{\otimes s'} \to (7^\otimes (s'-s) + m_7 \otimes \tilde{76}^\otimes m_{76} \otimes 71^\otimes m_{71} \otimes \tilde{7}^\otimes m_7) \otimes 7^\otimes s$$

and extended as an $I_0$-crystal isomorphism.
4. Conjectures for $B^{1,s}$

We conclude with a conjectural decomposition of $B^{1,s}$ into $A_7$ crystals. Recall that $A_7$ has a natural diagram symmetry $\sigma$ that respects the $E_7^{(1)}$ diagram symmetry. Thus, proving this conjecture using $E_7$ crystal decomposition of $B^{1,s} \cong \bigoplus_{k=0} B(k\sigma_1)$ could possibly lead to a proof of [14, Conj. 3.26].

**Conjecture 4.1.** Let $a, b, c, d \in \mathbb{Z}_{\geq 0}$ such that $a + 2b + 3c + d \leq s$. Then we have

$$B^{1,s} \cong \bigoplus B(a(\eta_1 + \eta_7) + b(\eta_2 + \eta_6) + c(\eta_3 + \eta_5) + d\eta_4)^{\bigoplus m_a, b, c, d},$$

as $A_7$ crystals, where the multiplicities are $m_{a, b, c, d} = m_{d, s - a - 2b - 3c}$, where

$$m_{d, s'} = \sum_{i=M}^{d-1} \left\lfloor \frac{i-1}{2} \right\rfloor$$

with $M = \max(d + 1 - (s' - d), 0)$.

We compute the multiplicity of $\eta_4$ in $B^{1,s}$ using SAGEMATH [36]:

```sage
sage: for s in range(10):
    ....:     [sum(ceil(i/2) for i in range(max(0,2*d+1-s),d+1))
    ....:      for d in range(s+1)]
[1]
[1, 1]
[1, 2, 2]
[1, 2, 3, 2]
[1, 2, 4, 4, 3]
[1, 2, 4, 5, 5, 3]
[1, 2, 4, 6, 7, 6, 4]
[1, 2, 4, 6, 8, 8, 7, 4]
[1, 2, 4, 6, 9, 10, 10, 8, 5]
[1, 2, 4, 6, 9, 11, 12, 11, 9, 5]
```

We compute the decomposition of $B^{7,s}$ into $A_7$ crystals using SAGEMATH:

```sage
sage: def compute_branching(s):
    ....:     A7 = WeylCharacterRing(['A',7], style="coroots")
    ....:     E7 = WeylCharacterRing(['E',7], style="coroots")
    ....:     La = E7.fundamental_weights()
    ....:     chi = sum(E7(k*La[1]) for k in range(s+1))
    ....:     return chi.branch(A7, rule="extended")

sage: compute_branching(1)
A7(0,0,0,0,0,0) + A7(0,0,1,0,0,0) + A7(1,0,0,0,0,0)

sage: compute_branching(2)
2*A7(0,0,0,0,0,0,0) + 2*A7(0,0,0,1,0,0,0) + A7(0,0,0,2,0,0,0)

sage: compute_branching(3)
2*A7(0,0,0,0,0,0,0) + 2*A7(0,0,0,1,0,0,0) + 2*A7(0,0,0,2,0,0,0)
```
We have verified Conjecture 4.1 for \( s \leq 9 \) by using a heavily optimized version of the branching rule code in SAGEMath [36].

One way to construct the \( A_7 \) highest weight elements would be to use the \( A_6 \) highest weight elements, which we can compute from the composition graph in Figure 3 (equivalently Figure 4). From there, we will want the elements invariant (as a set) under the diagram automorphism from [14] as highest weight elements of weight \( \eta \) must map to a highest weight element of weight \( \omega_0 \) under the \( A_7 \) diagram automorphism \( \sigma \).

As a step toward proving Conjecture 4.1, we show the analog of [14, Prop. 2.13] that characterizes the elements in \( B(2\omega_1) \subseteq B(\omega_1)^{\otimes 2} \).

**Proposition 4.2.** We have

\[
(b_1 \otimes c_1) \otimes (b_2 \otimes c_2) \in B(2\omega_1) \subseteq B(\omega_1)^{\otimes 2} \subseteq B(\omega_7)^{\otimes 4}
\]

if and only if \( b_1 \leq b_2 \) and \( c_1 \leq c_2 \) (the comparisons are in \( B(\omega_7) \)) and \( (b_1 \otimes c_1) < (b_2 \otimes c_2) \) (the comparison is in \( B(\omega_1) \)).

**Proof.** This is a finite computation that can be done, e.g., by using the following SageMath [36] code:

```sage
sage: L = crystals.Letters(['E', 7])
sage: x = L.highest_weight_vector().f_string([7,6,5,4,2,3,4,5,6,7])
sage: A = tensor([x,L.highest_weight_vector()]).subcrystal()
sage: S = tensor([A.module_generators[0].value,
               ....:            A.module_generators[0].value]).subcrystal()
sage: all((P.le(x.value[0][1], x.value[1][1]) for x in S) # Check b_i condition
sage: all((P.le(x.value[0][0], x.value[1][0]) for x in S) # Check c_i condition
sage: P = Poset(L.digraph())
sage: PA = Poset(A.digraph())
sage: data = [[x.value[0], x.value[1]] for x in S]
sage: T = tensor([A, A])
sage: all((not PA.le(A(x[0].value), A(x[1].value))) or x[0].value == x[1].value
sage:     ....:      for x in T if P.le(x[0].value[0], x[1].value[0])
sage:     ....:      and P.le(x[0].value[1], x[1].value[1])
sage:     ....:      and (x[0].value, x[1].value) not in data)
```

**Appendix A: SAGEMath code for composition graphs**

To compute the composition graph in Figure 3, we first do some setup:

```sage
sage: L = crystals.Letters(['E',7])
sage: x = L.highest_weight_vector().f_string([7,6,5,4,2,3,4,5,6,7])
sage: A = tensor([x,L.highest_weight_vector()]).subcrystal()
sage: TA = tensor([A.module_generators[0], A.module_generators[0]]).subcrystal()
sage: _ = A.list()
```

To compute the composition graph \( G_2(\omega_1) \), we run the following functions:
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```python
def check_le(x, y):
    return tensor([x, y]) in TA

def composition_graph(J):
    I = A.index_set()
    ImJ = sorted(set(I) - set(J))
    G = DiGraph([[b for b in A if b.is_highest_weight(ImJ)], check_le])
    num_verts = 0
    while num_verts != G.num_verts():
        num_verts = G.num_verts()
        verts = set(G.vertices())
        for b in A:
            ep = set([i for i in I if b.epsilon(i) > 0])
            Jplus = set(list(J) + [i for i in ImJ
                                    if any(bp.phi(i) > 0 and check_le(b, bp)
                                           for bp in verts)]
                        )
            if ep.issubset(Jplus):
                G.add_vertex(b)
                for bp in verts:
                    if check_le(b, bp):
                        G.add_edge(b, bp)
        loops = G.loops()
    G = G.transitive_reduction()
    for l in loops:
        G.add_edge(l)
        G.set_latex_options(format='dot2tex')
    return G
```

Figure 4. The composition graphs of Figure 3 with every node written in “compact form,” where a $k$ adds 1 to $\varphi_k(b)$ and $\overline{k}$ adds 1 to $\epsilon_k(b)$. Recall that the only vertex that does not have a loop is 22.
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