Opérateurs bornés inférieurement sur les groupes de Lie gradués

Véronique Fischer a, Michael Ruzhansky b

a Universita degli studi di Padova, DMIMMSA, Via Trieste 63, 35121 Padova, Italy
b Department of Mathematics, Imperial College London, 180 Queen’s Gate, London SW7 2AZ, United Kingdom

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ABSTRACT

In this note we present a symbolic pseudo-differential calculus on any graded (nilpotent) Lie group and, as an application, a version of the sharp Gårding inequality. As a corollary, we obtain lower bounds for positive Rockland operators with variable coefficients as well as their Schwartz-hypoellipticity.

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RÉSUMÉ

Dans cette note nous présentons un calcul pseudo-différentiel symbolique sur tous les groupes de Lie (nilpotents) gradués et, comme application, une version de l’inégalité de Gårding. En découlent des bornes inférieures pour des opérateurs de Rockland positifs à coefficients variables ainsi que leur hypo-ellipticité Schwartz.

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Version française abrégée

Dans cette note nous présentons un calcul pseudo-différentiel sur les groupes de Lie gradués. Comme application, nous obtenons une inégalité de Gårding avec gain d’une dérivée ainsi que l’hypo-ellipticité de Schwartz pour des opérateurs dans ce contexte.

Nous noterons G le groupe gradué. Par définition, le groupe de Lie G est connexe et simplement connexe et son algèbre de Lie g admet une graduation g = ⊕∞n=1 g[n] satisfaisant [g[ℓ], g[l']] ⊂ g[ℓ+l'] pour tout ℓ, ℓ' ∈ N, où les g[ℓ], ℓ = 1, 2,..., sont des sous-espaces vectoriels de g presque égaux à zéro. Cela implique que le groupe G est nilpotent.

Soient {X1, ..., Xn} une base de g[1], {Xn+1, ..., Xn+n2} une base de g[2], et ainsi de suite jusque’à produire une base X1, ..., Xn de g. Grâce à l’application exponentielle, cela permet d’identifier les éléments (x1, ..., xn) de ℝn avec ceux x = exp(x1X1 + ... + xnXn) du groupe G. Nous notons dx la mesure de Haar correspondante sur G. Nous définissons aussi les espaces de fonctions de Schwartz S(G) et des distributions tempérées S′(G) comme ceux de g. On note xj la fonction x = (x1, ..., xn) ∈ G ⟷ xj ∈ ℝ. Plus généralement, on définit xα = x[α1]1 ... x[αn]n pour tout multi-indice α = (α1, ..., αn) ∈ ℕn. De la même façon, nous posons Xα = X[α1]1 ... X[αn]n dans l’algèbre enveloppante de g.

Pour tout r > 0, nous définissons l’application linéaire Dr : g → g par DrX = rαX pour tout X ∈ g[α], ℓ ∈ ℕ. Ainsi, on a équipé l’algèbre de Lie g de la famille de dilatations {Dr : r > 0} et g devient une algèbre de Lie homogène au sens de [7]. Les poids de ces dilatations sont les entiers ν1, ..., νn donnés par DrXj = r[νj]Xj, j = 1,...,n. Les dilatations de groupe associées sont définis par r · x := (r[ν1]x1, r[ν2]x2, ..., r[νn]xn), x = (x1, ..., xn) ∈ G, r > 0. De manière canonique, cela conduit à
Définition 0.1. Un symbole \( \sigma = (\sigma(x, \pi)) : x \in \mathbb{R}, \pi \in \hat{G} \) est une famille d'opérateurs telle que

1. pour tout \( x \in \mathbb{R} \), la famille \( \{\sigma(x, \pi) : \pi \in \hat{G}\} \) est un champ d'opérateurs \( \mathcal{H}_G^\infty \to \mathcal{H}_\pi \) qui est \( \mu \)-mesurable;
2. il existe \( \gamma_1, \gamma_2 \in \mathbb{R} \) tels que pour tout \( x \in \mathbb{R} \), les opérateurs \( \pi(I + R)^{\gamma_1}(x, \pi)\pi(I + R)^{\gamma_2} \) sont bornés sur \( \mathcal{H}_\pi \) uniformément dans \( \pi \in \hat{G} \);
3. pour tout \( \pi \in \hat{G} \) et \( u, v \in \mathcal{H}_\pi \), la fonction \( G \ni x \mapsto (\sigma(x, \pi)u, v) \) est \( C^\infty \) sur \( G \).

Ici, les opérateurs \( \pi(I + R)^{\gamma} \) sont définis par le théorème spectral pour \( \pi(R) \). L'existence de \( \gamma_1, \gamma_2 \) dans la seconde condition garantit que la formule suivante a un sens : \( T f(x) = \int f(\pi) \text{trace}(\pi(x)\sigma(x, \pi)) \mu(\pi) \), \( f \in \mathcal{S}(G) \), \( x \in \mathbb{R} \). De plus, l'opérateur \( T = \text{Op}(\sigma) \) est bien défini et continu \( \mathcal{S}(G) \to \mathcal{S}'(G) \). Nous remarquons que, si l'opérateur \( T \) est invariant à gauche, alors le symbole est indépendant de \( x \).

Définition 0.2. Les opérateurs de différence \( \Delta^\alpha \), \( \alpha \in \mathbb{N}_0^n \), sont densément définis sur l'algèbre \( C^* \) du groupe \( G \) via :

\[ (\Delta^\alpha f)(\pi) := \hat{f}(\alpha) \hat{\sigma}(\pi) - f(\pi), \quad f \in \mathcal{S}(G). \]

Soient \( m \in \mathbb{R} \) et \( \rho, \delta \in \mathbb{R} \) tels que \( 0 \leq \delta \leq \rho \leq 1, \delta \neq 1 \).

Définition 0.3. La classe de symboles \( S^m_{\rho, \delta} \) est définie comme l'ensemble des symboles \( \sigma \) satisfaisant, pour tout \( \alpha, \beta \in \mathbb{N}_0^n \) et \( \gamma \in \mathbb{R} \):

\[ \sup_{x \in \mathbb{R}, \pi \in \hat{G}} \left\| \pi(I + R)^{\frac{\alpha(1 - \rho) - \beta(1 - \delta) + \gamma}{\rho}} \pi(I + R)^{-\gamma} \right\|_{op} \leq \infty. \]

(Le supremum sur \( \pi \) est en fait le supremum essentiel par rapport à la mesure de Plancherel \( \mu \).)

Il est facile de voir que si \( m_1 \leq m_2 \), \( \delta_1 \leq \delta_2 \) et \( \rho_1 \geq \rho_2 \), alors \( S^m_{\rho_1, \delta_1} \subset S^m_{\rho_2, \delta_2} \). De plus, les expressions (1) conduisent à une topologie de Fréchet sur l'espace vectoriel \( S^m_{\rho, \delta} \).

Dans le cas abélien, c'est-à-dire \( \mathbb{R}^n \) équipé de la loi additive et \( \mathcal{R} = -L \), \( L \) l'opérateur de Laplace, \( S^m_{\rho, \delta} \) coïncide avec la classe de symboles de Hörmander sur \( \mathbb{R}^n \). Cependant, notre motivation initiale ne provient pas du cas abélien : nous voulons définir des opérateurs de différence et des classes de symboles analogues à ceux définis dans [12] sur les groupes de Lie compacts. Dans ce cas-ci, une définition similaire à (1) donnerait formellement les mêmes classes de symboles que [12], car l'opérateur \( \pi(I + R) \) est scalaire, \( \mathcal{R} \) ayant été choisi comme \( \mathcal{R} = -L \), où \( L \) est l'opérateur de Laplace–Béltrami. Dans notre cas, celui de G groupe gradué non abélien, l'opérateur \( \mathcal{R} \) n'est même pas central et nous avons introduit \( \gamma \) dans (1) afin que \( \bigcup_{m \in \mathbb{R}} S^m_{\rho, \delta} \) devienne une algèbre (cf. (1) dans le Théorème 0.4).

Nous avons obtenu les propriétés suivantes pour les classes d'opérateurs \( \Psi^m_{\rho, \delta} \) définies grâce à la procédure de quantification \( \sigma \mapsto \text{Op}(\sigma) \) décrite ci-dessus.

Théorème 0.4. Soient \( 0 \leq \delta \leq \rho \leq 1 \) avec \( \rho 
eq 0 \) et \( \delta \neq 1 \). Nous avons les propriétés suivantes :

1. La classe de symboles \( \bigcup_{m \in \mathbb{R}} S^m_{\rho, \delta} \) est une algèbre d'opérateurs stable par l'application adjointe. Chaque espace vectoriel \( S^m_{\rho, \delta} \) ne dépend pas du choix de l'opérateur de Rockland positif \( \mathcal{R} \).
2. Pour \( \rho 
eq 0 \), la classe d'opérateurs \( \bigcup_{m \in \mathbb{R}} \Psi^m_{\rho, \delta} \) est une algèbre d'opérateurs stable par l'application adjointe.
3. Pour tout \( \alpha \in \mathbb{N}_0^n \) avec \( \alpha \neq 0, \Psi^{\alpha}_{1,0} \) est une algèbre d'opérateurs.
4. \( (1 + R)^{\frac{\alpha}{\rho}} \in \Psi^m_{1,0} \), \( (1 + R)^{\frac{\alpha}{\rho}} \in \Psi^m_{1,0} \).
5. Si \( \rho \in [0, 1) \), alors les opérateurs dans \( \Psi^{0}_{\rho, \rho} \) sont continus sur \( L^2(G) \).
Théorème 0.5. Soient 0 ≤ δ ≤ ρ ≤ 1 avec ρ ≠ 0 et δ ≠ 1, et soient T ∈ Ψ^m_{ρ,δ} et σ = (σ(π(x)), π) son symbole. Nous supposons que chaque σ(π(x), π) est positif au sens des opérateurs sur H_q et qu’il commute avec la mesure spectrale de π(R) pour tout x ∈ G et µ-presque tout π ∈ G. Alors il existe une constante C > 0 telle que l’on a : Re(Tf)_{L^2(G)} ≥ −C∥f∥_{W^{−(ρ−δ)/2}(G)}^2, f ∈ S(G).

Cette classe d’opérateurs inclut :

- les opérateurs de Rockland à coefficients variables de la forme a(x)R avec a(x) ≥ 0 satisfaisant Xαa ∈ L^∞(G) pour tout α ∈ N^0,
- les multiplicateurs de la forme φ(R) où la fonction φ : [0, +∞) → [0, +∞) est C^∞ et satisfait :

\[∀α ∈ N_0 ∃ C = C_0 > 0 ∀ λ ≥ 0 \left| \partial_α^0 φ(λ) \right| ≤ C(1 + λ)^{−(ρ−δ)} \]  

(2)

- plus généralement les opérateurs dont les symboles ont la forme σ(x, π) = φ_σ(π(R)) où la fonction (x, λ) → φ_σ(λ) est positive ou nulle ainsi que régulière sur G × [0, +∞) et, pour chaque x ∈ G, la fonction φ_σ satisfait (2), avec une constante C indépendante de x.

La condition de commutation avec la mesure spectrale de π(R) semble raisonnable. En effet, dans la version correspondante de l’inégalité de Gårding sur R^n abélien ou sur les groupes compacts [11], cette condition est automatiquement satisfaite, car l’opérateur de Laplace est central. Dans le cadre fixé ci-dessus, nous avons la possibilité de choisir l’opérateur R.

Dans [8], Helffer et Nourrigot ont montré que R et de manière équivalente I + R sont hypo-elliptiques. Enfin, nous énonçons la version Schwartz de ce résultat.

Théorème 0.6. L’opérateur I + R est hypo-elliptique au sens de Schwartz : pour tout f ∈ S(G), la condition (I + R)f ∈ S(G) implique f ∈ S(G).

En fait, notre calcul symbolique nous permet de construire une paramétrix pour l’opérateur I + R ; les propriétés d’hypo-ellipticité usuelle (c’est-à-dire C^∞) et Schwartz en découlent.

1. Introduction

In this note we present a symbolic pseudo-differential calculus on any graded Lie group. As applications, we obtain a version of the sharp Gårding inequality and results on the Schwartz-hypoellipticity for operators in this context.

In the usual Euclidean setting, the positivity of the full symbol is required for the sharp Gårding inequality as well as, for instance, for the Fefferman–Phong inequality. This contrasts with the Gårding or Melin–Hörmander inequalities, for example, where knowing the principal (or subprincipal) symbol is sufficient. Thus the latter inequalities can be proved on manifolds using the standard Hörmander theory of pseudo-differential operators together with the usual Kohn–Nirenberg quantisation on R^n. Since the geometric control of the full symbol of an operator is impossible with these tools in general, the study of sharp Gårding inequalities appears limited. However, the sharp Gårding inequality on compact Lie groups was recently established in [11]. This approach uses the notion of a full matrix-valued symbol defined in terms of the representation...
theory of the group. In this note we explain how we follow the same strategy in the case of the Heisenberg group or more general nilpotent Lie groups.

The pseudo-differential calculus that we define is different from the several pseudo-differential calculi already developed on the Heisenberg group: see, e.g., Taylor [13] and [14] for symbol classes coming from standard Hörmander classes though the exponential mapping; Bahouri, Fermanian-Kammerer and Gallagher [1] for classes defined in terms of explicit formulæ coming from the Schrödinger representation; or Beals and Greiner [2] or Ponge [9] for different types of analysis on the Heisenberg manifolds. On more general nilpotent groups, Christ, Geller, Glowacki and Polin [3] proposed an approach to pseudo-differential operators, however based on the properties of kernels and not on a symbolic calculus. Following Ruzhansky and Turunen [12] and [10], we define symbol classes directly on the group. As such, our approach can be extended for general graded nilpotent Lie groups, and by developing the symbolic calculus and the Friedrichs approximation on the group, we obtain the corresponding sharp Gårding inequality and Schwartz-hypoellipticity results.

While the symbol classes in [10] are based on the spectral theory of the Laplace–Beltrami operator, here, it is not available and it becomes natural to use the sub-Laplacian on stratified groups or more general Rockland operators on graded groups. Moreover, surpassing [11], since a dilation structure is present, we establish the sharp Gårding inequality for suitable \((\rho, \delta)\) classes of operators by establishing a Calderón–Vaillancourt type theorem in this context. This is, in fact, the best known lower bound available in the \((\rho, \delta)\)-setting already on \(\mathbb{R}^n\).

The appearing operators are Calderón–Zygmund in the sense of Coifman and Weiss [4, Ch. III], so that \(L^p\) results follow as well. In Section 2 we fix the notation concerning Lie groups that we are working on. In Section 3 we formulate the results.

2. Preliminaries

Let us first briefly recall the necessary notions and set some notation. In general, we will be concerned with graded Lie groups \(G\) which means that \(G\) is a connected and simply connected Lie group (of step \(s > 1\)) with the gradation of its Lie algebra \(g\) given by \(g = \bigoplus_{\mathbb{N}} g_{\ell}\) with \([g_{\ell}, g_{\ell'}] \subset g_{\ell+\ell'}\) for every \(\ell, \ell' \in \mathbb{N}\), where the \(g_{\ell}, \ell = 1, 2, \ldots,\) are vector subspaces of \(g\), almost all equal to \(0\). This implies that the group \(G\) is nilpotent. If \(g\) is generated by \(g_1\) in this way, the group \(G\) is said to be stratified.

Let \(\{X_1, \ldots, X_n\}\) be a basis of \(g_1\) (which is possibly reduced to \(\{0\}\)), \(\{X_{1+1}, \ldots, X_{n+1}\}\) a basis of \(g_2\) and so on, so that we obtain a basis \(X_1, \ldots, X_n\) of \(g\) adapted to the gradation. Via the exponential mapping \(\exp : g \rightarrow G\), we identify the points \((x_1, \ldots, x_n) \in \mathbb{R}^n\) with the points \(x = \exp(x_1X_1 + \cdots + x_nX_n)\) in \(G\). This leads to a corresponding Lebesgue measure on \(g\) and the Haar measure \(d\mu\) on the group \(G\). We define the spaces of Schwartz functions \(S(G)\) and tempered distributions \(S'(G)\) of the group \(G\) as those on \(\mathbb{R}^n\). The coordinate function \(x = (x_1, \ldots, x_n) \in G \mapsto x_j \in \mathbb{R}\) is denoted by \(x_j\). More generally we define for every multi-index \(\alpha \in \mathbb{N}^n\), \(x^\alpha := x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}\), as a function on \(G\). Similarly we set \(X^\alpha = X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_n^{\alpha_n}\) in the universal enveloping Lie algebra of \(g\).

For any \(r > 0\), we define the linear mapping \(D_r : g \rightarrow g\) by \(D_rX = r^jX\) for every \(X \in g_\ell\), \(\ell \in \mathbb{N}\). Then the Lie algebra \(g\) is endowed with the family of dilations \(\{D_r, r > 0\}\) and becomes a homogeneous Lie algebra in the sense of [7]. The weights of the dilations are the integers \(\nu_1, \ldots, \nu_n\) given by \(D_rX_j = r^{\nu_j}X_j, j = 1, \ldots, n\). The associated group dilations are defined by \(r \cdot x := (r^{\nu_1}x_1, \ldots, r^{\nu_n}x_n), x = (x_1, \ldots, x_n) \in G, r > 0\). In a canonical way this leads to the notions of homogeneity for functions and operators. For instance the degree of homogeneity of \(x^\alpha\) and \(X^\alpha\), viewed respectively as a function and a differential operator on \(G\), is \([\alpha] = \sum_j \nu_j \alpha_j\). Indeed, let us recall that a vector of \(g\) defines a left-invariant vector field on \(G\) and more generally that the universal enveloping Lie algebra of \(g\) is isomorphic with the left-invariant differential operators; we keep the same notation for the vectors and the corresponding operators.

The dimension of \(G\) is \(n = \sum \nu_i\ell_i\) while its homogeneous dimension is \(Q = \sum \nu_i\ell_i + \nu_1 + \nu_2 + \cdots + \nu_n\).

We denote by \(\hat{G}\) the set of equivalence classes of (continuous) irreducible unitary representations of \(G\). We will often identify a representation of \(G\) with its equivalence class. We will also keep the same notation for the corresponding infinitesimal representation. For \(\pi \in \hat{G}\), we denote by \(\hat{\pi}\) the representation space of \(\pi\) and by \(\hat{C}_\alpha^0\) its subspace of smooth vectors. For \(f \in L^1(G)\), we define its Fourier transform at \(\pi \in \hat{G}\) by \(\hat{f}(\pi) = f \int f(\pi \eta(\varphi))d\mu(\pi)\), with the integral understood in the Bochner sense. Denoting by \(\mu\) the Plancherel measure on \(\hat{G}\), the inverse Fourier formula holds:

\[
\hat{f}(\pi) = \int \frac{1}{\hat{\varphi}} \text{trace}(\pi(\varphi)\hat{f}(\pi))d\varphi(\pi) \quad \text{when} \quad \int \frac{1}{\hat{\varphi}} \text{trace}(\hat{f}(\pi))d\mu(\pi) < \infty.
\]

Let \(R\) be a positive (left) Rockland operator on \(G\); this means that \(R\) is a left-invariant differential operator, homogeneous of degree \(\nu\) necessarily even, positive in the operator sense, and such that for every non-trivial \(\pi \in \hat{G}\) the operator \(\pi(R)\) is injective on \(\hat{C}_\alpha^0\). The operator \(R\) admits an essentially self-adjoint extension on \(C_0^\infty(G)\) (see [7]), and we will still denote this extension by \(R\). Examples of such operators are given in the stratified case by \(R = -\mathcal{L}\) where \(\mathcal{L} = \sum_{1 \leq j \leq \ell} X_j^2\) is a Kohn–sub-Laplacian, and in the graded case by the operators \(\sum_{1 \leq j \leq \ell} (-1)^{v_0/\nu_j} X_j^{v_0/\nu_j}\) and \(\sum_{1 \leq j \leq \ell} X_j^{v_0/\nu_j}\), where \(v_0\) denotes some common multiple of \(\nu_1, \ldots, \nu_n\). In fact our class of operators do not depend on the choice of such an operator \(R\).
3. Results

We aim at defining the symbol classes in terms of the operators $\mathcal{R}$ as above.

**Definition 3.1.** A symbol is a family of operators $\sigma = \{\sigma(x, \pi) : x \in G, [\pi] \in \hat{G}\}$, such that

1. for each $x \in G$, the family $\{\sigma(x, \pi), \pi \in \hat{G}\}$ is a $\mu$-measurable field of operators $\mathcal{H}_\pi^\infty \to \mathcal{H}_\pi$;
2. there exist $\gamma_1, \gamma_2 \in \mathbb{R}$ such that for every $x \in G$, the operator $\pi(I + \mathcal{R})^{\gamma_1} \sigma(x, \pi) \pi(I + \mathcal{R})^{\gamma_2}$ is bounded on $\mathcal{H}_\pi$ uniformly in $\pi \in \hat{G}$;
3. for any $\pi \in \hat{G}$ and any $u, v \in \mathcal{H}_\pi$, the scalar function $x \mapsto (\sigma(x, \pi)u, v)_{\mathcal{H}_\pi}$ is smooth on $G$.

Here, the powers $\pi(I + \mathcal{R})^\gamma$ are defined by the spectral theorem for the positive operator $\pi(\mathcal{R})$. The existence of $\gamma_1, \gamma_2$ in the second condition is used to guarantee that the following formula makes sense:

$$Tf(x) = \int\limits_{\hat{G}} \text{trace}(\pi(x)\sigma(x, \pi)\hat{f}(\pi)) d\mu(\pi), \quad f \in \mathcal{S}(G), \ x \in G;$$

indeed such operator $T = \text{Op}(\sigma)$ is well-defined and continuous $\mathcal{S}(G) \to \mathcal{S}'(G)$. We note that if the operator $T$ is left-invariant, then its symbol is independent of $x$.

**Definition 3.2.** The difference operators $\Delta^\alpha$, $\alpha \in \mathbb{N}_0^\mathbb{N}$, are densely defined on the $C^*$-algebra of the group via $(\Delta^\alpha \hat{f})(\pi) := (x^\alpha \hat{f})(\pi)$, $f \in \mathcal{S}(G)$.

Let now $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$ with $\delta \neq 1$.

**Definition 3.3.** The symbol class $\mathcal{S}^m_{\rho, \delta}$ is defined as the set of symbols $\sigma$ satisfying for all $\alpha, \beta \in \mathbb{N}_0^\mathbb{N}$ and every $\gamma \in \mathbb{R}$:

$$\sup_{x \in G, \pi \in \hat{G}} \|\pi(I + \mathcal{R})^{\rho_1 - \delta} \sigma(x, \pi) \pi(I + \mathcal{R})^{-\rho_2} \Delta^\alpha \sigma(x, \pi)\pi(I + \mathcal{R})^{-\gamma} \|^\gamma_{op} < \infty. \quad (3)$$

(The supremum over $\pi$ is in fact the essential supremum over the Plancherel measure $\mu$.)

It is easy to see that if $m_1 \leq m_2$, $\delta_1 \leq \delta_2$ and $\rho_1 \geq \rho_2$, then $\mathcal{S}^{m_1}_{\rho_1, \delta_1} \subset \mathcal{S}^{m_2}_{\rho_2, \delta_2}$. Furthermore, the expressions in (3) define a Fréchet topology on the linear space $\mathcal{S}^m_{\rho, \delta}$.

In the Abelian case, that is, $\mathbb{R}^n$ endowed with the addition law and $\mathcal{R} = -L$, $L$ being the Laplace operator, $\mathcal{S}^m_{\rho, \delta}$ boils down easily to the usual Hörmander classes. However our initial motivation did not come from the Abelian case: we wanted to define the difference operators and the symbol classes in analogy with the ones defined in [12] on compact Lie groups. In this case, a definition similar to (3) would formally give the same classes of symbols defined in [12] since, $\mathcal{R} = -L$, $L$ being the Laplace–Beltrami operator, the operator $\pi(I + \mathcal{R})$ is scalar. In our case, i.e. $G$ being a graded non-Abelian Lie group, the operator $\mathcal{R}$ is not even central and the introduction of $\gamma$ in (3) assures that $\bigcup_{m \in \mathbb{R}} \mathcal{S}^m_{\rho, \delta}$ is an algebra.

We have the following properties for the operator classes $\Psi^m_{\rho, \delta} := \text{Op}(\mathcal{S}^m_{\rho, \delta})$ defined using the quantisation procedure $\sigma \mapsto \text{Op}(\sigma)$ described above.

**Theorem 3.4.** Let $0 \leq \delta \leq \rho \leq 1$. We have the following properties:

1. The symbol classes is an algebra of operators $\bigcup_{m \in \mathbb{R}} \mathcal{S}^m_{\rho, \delta}$ stable by taking the adjoint. Each vector space $\mathcal{S}^m_{\rho, \delta}$ does not depend on the choice of the positive Rockland operator $\mathcal{R}$.
2. For $\rho \neq 0$, the operator class $\bigcup_{m \in \mathbb{R}} \mathcal{S}^m_{\rho, \delta}$ is an algebra stable by taking the adjoint.
3. For any $\alpha \in \mathbb{N}_0^\mathbb{N}$, we have $X^\alpha \in \Psi^1_{1,0}$.
4. For any positive Rockland operator of homogeneous degree $\nu$, we have $(I + \mathcal{R})^\nu \in \Psi^m_{1,0}$.
5. If $\rho \in [0, 1)$ then the operators in $\Psi^0_{\rho, \rho}$ are continuous on $L^2(G)$.
6. Let $\rho \neq 0$. The integral kernel $K(x, y)$ of an operator $T \in \Psi^m_{\rho, \delta}$ is smooth on $(G \times G) \setminus \{(x, y) : x = y\}$. It is of Calderón–Zygmund type in the sense of Coifman and Weiss [4, Ch. III]. It decreases rapidly as $|xy^{-1}| \to \infty$ (here we have fixed a homogeneous norm $|\cdot|$ on $G$, i.e. a continuous function, homogeneous of degree one and vanishing only at $0$): i.e. for any $M > 0$ there exists $C_M > 0$ such that $|xy^{-1}| \geq 1 \implies |K(x, y)| \leq C_M |xy^{-1}|^{-M}$. At the diagonal it satisfies $|xy^{-1}| \leq 1 \implies |K(x, y)| \leq C |xy^{-1}|^{-\frac{m}{2}}$.

By [4, Ch. III, Théorème 2.4] Property (6) implies that the operators in $\Psi^0_{\rho, \delta}$, $1 \geq \rho \geq \delta > 0$, $\rho \neq 0$, $\delta \neq 1$, are continuous on $L^p(G)$, $1 < p < \infty$. 

By Properties (2) and (4), any operator in $\Psi^m_{\rho, \delta}$, $1 \geq \rho \geq \delta > 0$, $\rho \neq 0$, $\delta \neq 1$, is continuous on the natural Sobolev spaces (denoted by $L^2(G)$) associated with the dilations and the loss of derivatives is controlled by the order $m$. The Sobolev space $L^2(G)$ is defined as the set of tempered distributions $f \in \mathcal{S}(G)$ such that $(I + R)^{t} f \in L^2(G)$ but does not depend on the choice of $\mathcal{R}$. These Sobolev spaces enjoy properties similar to the stratified case proved by Folland [6], in particular for interpolation (see [5]).

We now give the sharp Gårding inequality.

**Theorem 3.5.** Let $0 \leq \delta \leq \rho \leq 1$, $\rho \neq 0$, $\delta \neq 1$, and let $T \in \Psi^m_{\rho, \delta}$ with symbol $\sigma = [\sigma(x, \pi)]$. Assume that each $[\sigma(x, \pi)]$ of $T$ is non-negative on $\mathcal{H}_\pi$ (in the operator sense). Assume also that there exists a Rockland operator $\mathcal{R}$ such that each $[\sigma(x, \pi)]$ commutes with the spectral measure of $\pi(\mathcal{R})$ for every $x \in G$ and almost every $\pi \in \hat{G}$. Then there exists $C > 0$ such that for every $f \in \mathcal{S}(G)$ we have $\text{Re}(Tf, f)_{L^2(G)} \geq -C \|f\|_{L^2(G)}^{2(\rho-\delta)\rho^{-1}}$.

The class includes:

- the variable coefficient Kohn–sub-Laplacians and Rockland operators of the form $a(x)\mathcal{R}$, with $a(x) \geq 0$ satisfying $X^\alpha a \in L^\infty(G)$ for all $\alpha \in \mathbb{N}_0^n$,
- the multipliers $\phi(\mathcal{R})$ for a smooth function $\phi : [0, \infty) \mapsto [0, \infty)$ satisfying
  \[ \exists C > 0 \forall \lambda \geq 0 \quad \left| a_\phi^{\alpha} \phi(\lambda) \right| \leq C (1 + \lambda)^{\frac{\rho-\delta}{\rho}}, \tag{4} \]
- more generally the operators with symbols given by $\left[ \sigma(x, \pi) = \phi_\delta(\pi(\mathcal{R})) \right]$ with $(x, \lambda) \mapsto \phi_\lambda$ being non-negative and smooth on $G \times [0, \infty)$, and $\phi_\lambda$ satisfying (4) at each $x$ with a constant $C$ independent of $x$.

The condition on the commutation with the spectral measure of $\pi(\mathcal{R})$ seems to be reasonable: in the corresponding version of the sharp Gårding inequality on compact Lie groups in [11] or in the Abelian case, this condition is automatically satisfied there because if $\mathcal{R} = -\mathcal{L}$ and $\mathcal{L}$ is the Laplacian then $\pi(\mathcal{R})$ is central.

In [8], Helffer and Nourrigat proved that $\mathcal{R}$ and, equivalently, $I + \mathcal{R}$ are hypoelliptic. We finally give the result stating that the Schwartz version of such hypoellipticity is also true.

**Theorem 3.6.** The operator $I + \mathcal{R}$ is Schwartz-hypoelliptic, i.e. for $f \in \mathcal{S}(G)$, the condition $(I + \mathcal{R})f \in \mathcal{S}(G)$ implies $f \in \mathcal{S}(G)$.

In fact, our symbolic calculus allows the construction of a parametrix for the operator $I + \mathcal{R}$ from which both the hypoellipticity and the Schwartz-hypoellipticity follow.

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