Some finiteness properties of regular vertex operator algebras

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Abstract

We give a natural extension of the notion of the contragredient module for a vertex operator algebra. By using this extension we prove that for regular vertex operator algebras, Zhu’s $C_2$-finiteness condition holds, fusion rules (for any three irreducible modules) are finite and the vertex operator algebras themselves are finitely generated.

1 Introduction

In this paper, we study certain finiteness properties of vertex operator algebras, motivated by a conjecture of Zhu about the relation between the rationality and the $C_2$-finiteness condition for a vertex operator algebra $V$.

To prove the convergence of a trace function Zhu [Z] made a technical assumption on the vertex operator algebra, called finiteness condition $C$, so that his beautiful results hold for a rational vertex operator algebra that satisfies the finiteness condition $C$. Zhu’s finiteness condition $C$ consists of what we call $C_2$-finiteness condition in the present paper and the condition that $V$ is a sum of lowest weight modules for the Virasoro algebra. We say that a vertex operator algebra $V$ satisfies $C_2$-finiteness condition if $C_2(V)$ is finite-codimensional in $V$ where $C_2(V)$ is the subspace of $V$ linearly spanned by elements $u_{-2}v$ for $u, v \in V$. (Note that as one of the results in [DLM3], the convergence of trace functions was proved under only the $C_2$-finiteness condition.) It was proved in [Z] (see also [DLM3]) that the familiar rational vertex operator algebras satisfy the $C_2$-finiteness condition and it was conjectured that the rationality in the sense of [Z] (defined in Section 2) implies the $C_2$-finiteness condition.

In [DLM2], with various motivations we proved that for the familiar rational vertex operator algebras, any weak module (defined in Section 2) is a direct sum of irreducible (ordinary) modules. Consequently, any irreducible weak module is an (ordinary) module. Vertex operator algebras with this property are said to be regular. It was conjectured in [DLM2] that the notions of rationality and regularity are equivalent. The combination of the two conjectures gives rise to a third conjecture: Regularity implies the $C_2$-finiteness condition. Since regularity clearly implies rationality, the third conjecture is also a weak version of Zhu’s conjecture.

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In this paper, as one of our main results we prove the third conjecture, which is that regularity implies the $C_2$-finiteness condition. Moreover, we prove that if $V$ is regular, the fusion rules are finite and $V$ is finitely generated as a vertex operator algebra.

In [Z], Zhu associated an associative algebra $A(V)$ (a certain quotient space of $V$) to any vertex operator algebra $V$ and established natural functors between the category of $\mathbb{N}$-gradable weak $V$-modules (defined in Section 2) and the category of $A(V)$-modules. Furthermore, these functors give rise to a one-to-one correspondence between the set of equivalence classes of irreducible $A(V)$-modules and the set of equivalence classes of irreducible $\mathbb{N}$-gradable weak $V$-modules. Using these functors Zhu was able to prove that $A(V)$ is (finite-dimensional) semisimple if $V$ is rational. Although the quotient space $Z_2(V) = V/C_2$ has a natural commutative associative algebra structure [Z], there are no appropriate functors between the category of $Z_2(V)$-modules and the category of $\mathbb{N}$-gradable weak $V$-modules. Notice that in the notion of module or weak module $V$ is clearly a weak $\mathbb{N}$-gradable weak $V$-module again. Thus there is a unique maximal weak $V$-module where $V$ are subspaces of $W$ and the condition $V = \bigoplus_{n=0}^{\infty} V(n)$ with $V(0) = \mathfrak{c}1$, then the nil-radical of $Z_2(V)$ is $1$-codimensional (because $Z_2(V)$ is an $\mathbb{N}$-graded algebra). Thus, in principle there are no desired functors between the category of $V$-modules and the category of $Z_2(V)$-modules. This indicates that a different technique is needed.

In the following we give an account of the main stream of this paper. Let $W = \bigcap_{h \in \mathbb{C}} W(h)$ be a $V$-module. Then we generalize the definition of $C_2(V)$ to define $C_2(W)$ in the obvious way. View $(W/C_2(W))^*$ as a natural subspace of $W^*$. Since $C_2(W)$ is a graded subspace, $C_2(W)$ is finite-codimensional if and only if $(W/C_2(W))^* \subseteq W'$ ($= \bigcap_{h} W_{(h)}^*$, the restricted dual of $W$). To use the rationality or the regularity, first of all one needs to relate $(W/C_2(W))^*$ to a certain weak $V$-module.

A fundamental result proved in [FHL] is that $(W', Y')$ carries the structure of a $V$-module where

$$\langle Y'(v, x) f, w \rangle = \langle f, Y(e^{xL(1)}(-x^{-2})L(0)v, x^{-1})w \rangle$$

for $v \in V, f \in W', w \in W$. One notices that the action of $Y'(v, x)$ on $W'$ obviously extends to $W^*$, say $Y^*(v, x)$. But $W^*$ fails to be even a weak $V$-module. If there is a weak $V$-module $M$ in $W^*$ containing $(W/C_2(W))^*$, on which $L(0)$ acts semisimply, then it is easy to prove that $M \subseteq W'$, so that $(W/C_2(W))^* \subseteq W'$. This would solve our problem. Notice that $L(0)$ acts semisimply on any weak (gradable weak) module for a regular (rational) vertex operator algebra. Now, the question is whether there exists a weak or an $\mathbb{N}$-gradable weak $V$-module in $W^*$ containing $(W/C_2(W))^*$ as a subspace. If $S$ and $T$ are subspaces of $W^*$ such that $(S, Y^*)$ and $(T, Y^*)$ are weak $V$-modules, then $(S + T, Y^*)$ is clearly a weak $V$-module again. Thus there is a unique maximal weak $V$-module in $W^*$ with the vertex operator map $Y^*$. Then we are naturally led to this weak module to see whether it contains $(W/C_2(W))^*$.

Notice that in the notion of module or weak module $W$, the first axiom is the truncation condition: $Y(v, x)w \in W((x))$ for $v \in V, w \in W$, and that without this condition the Jacobi identity cannot make sense. With this in mind, we define $D(W)$ to be the subspace of $W^*$ consisting of $\alpha$ such that

$$Y^*(v, x)\alpha \in W^*((x)) \quad \text{for every } v \in V.$$
Following the proof of Theorem 5.2.1 of [FHL] closely, we easily see that FHL actually proved that $D(W)$ is a weak $V$-module. It is clear that $D(W)$ is the maximal weak $V$-module inside $W^*$ with $Y = Y^*$. Next, we prove that $(W/C_2(W))^* \subseteq D(W)$ (Proposition 3.6). Therefore, any regular vertex operator algebra satisfies Zhu’s $C_2$-finiteness condition. Furthermore, by exploiting a result of Frenkel and Zhu [FZ] on fusion rules in terms of $A(V)$-bimodules we prove that all fusion rules are finite. We also define a graded subspace $C_1(V)$ of $V$ and prove that any graded subspace of $V$ complementary to $C_1(V)$ generates $V$ as a vertex operator algebra (Proposition 3.3). As a corollary we prove that any regular vertex operator algebra is finitely generated.

In [L5], the notion of $D(W)$ was extended further to the notion of $D(W)$ consisting of what we called representative functionals on $W$. The space $D(W)$ was proved to have a natural $V$-bimodule structure where the $V$-bimodule actions were not exactly $Y^*$, but certain analytic continuations of $Y^*$. Using the notion $D(W)$ we were able to prove some results of Peter-Weyl type for vertex operator algebras.

This paper is organized as follows: In Section 2, we associate a canonical weak $V$-module $D(W)$ inside $W^*$ to any weak $V$-module $W$ and we prove that $D(W) = W'$ for a certain class $A$ of vertex operator algebras. In Section 3, we prove the $C_2$-finiteness condition and the finiteness of fusion rules for vertex operator algebras of class $A$. We also prove that vertex operator algebras of class $A$ are finitely generated.

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## 2 The contragredient module and an extension

In this section, we first review some basic notions and the contragredient module [FHL] and we then give a natural extension of the contragredient module (to a weak module).

We shall use standard definitions and notions in [FHL] and [FLM] such as the notions of vertex operator algebra, module, intertwining operator and fusion rule, which will not be given here. We shall also use certain concepts which we recall next.

Let $V$ be a vertex operator algebra. A *weak* $V$-module [DLM2] is a pair $(W, Y_W)$ satisfying all the axioms except those involving the grading for a $V$-module given in [FHL] and [FLM]. It has been noticed in [DLM2] that the $L(-1)$-derivative property: $Y_W(L(-1)v, x) = \frac{d}{dx} Y_W(v, x)$ for $v \in V$, follows from the other axioms, so that one need not check this axiom for having a module or weak module. A *generalized* $V$-module [HL] is a weak $V$ module on which $L(0)$ acts semisimply. An $\mathbb{N}$-gradable weak $V$-module is a weak $V$-module $W$ on which there exists an $\mathbb{N}$-grading $W = \bigoplus_{n \in \mathbb{N}} W(n)$ such that

$$ v_m W(n) \subseteq W(wtv + n - m - 1) \quad (2.1) $$

for homogeneous $v \in V$ and for $m \in \mathbb{Z}, n \in \mathbb{N}$, where by convention $W(n) = 0$ for $n < 0$. (This notion was essentially introduced by Zhu in [Z].) It is clear that the sum of $\mathbb{N}$-gradable weak $V$-modules is again an $\mathbb{N}$-gradable weak $V$-module.

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Let \( M = \bigoplus_{h \in \mathbb{C}} M_{(h)} \) be a \( V \)-module. By definition ([FHL], [FLM]), for any \( h \in \mathbb{C} \), \( M_{(h+n)} = 0 \) for sufficiently small integer \( n \). For any \( h \in \mathbb{C} \), it is clear that \( \bigoplus_{m \in \mathbb{Z}} M_{(m+h)} \) is a submodule of \( M \). Furthermore, let \( k \in \mathbb{Z} \) be such that \( M_{(h+k)} \neq 0 \) and \( M_{(h+m)} = 0 \) for \( m < k \). Then \( \bigoplus_{m \in \mathbb{Z}} M_{(m+h)} = \sum_{n \in \mathbb{N}} M_{(h+k+n)} \) is an \( \mathbb{N} \)-gradable \( V \)-module with \( \left( \bigoplus_{m \in \mathbb{Z}} M_{(m+h)} \right)(n) = M_{(n+k+h)} \) for \( n \in \mathbb{N} \). It follows that \( M \) is a direct sum of \( \mathbb{N} \)-gradable \( V \)-modules, so that \( M \) is an \( \mathbb{N} \)-gradable \( V \)-module.

A vertex operator algebra \( V \) is said to be rational if any \( \mathbb{N} \)-gradable weak \( V \)-module is a direct sum of irreducible \( \mathbb{N} \)-gradable weak \( V \)-modules, and \( V \) is said to be regular [DLM2] if any weak \( V \)-module is a direct sum of irreducible (ordinary) \( V \)-modules. If \( V \) is rational, it was proved in [DLM1] that \( V \) has only finitely many inequivalent irreducible modules and that each irreducible \( \mathbb{N} \)-gradable weak \( V \)-module is a module, so that this notion of rationality is the same as the one defined in [Z]. Note that there are different notions of rationality (see for example [HL]).

Examples of rational vertex operator algebras are: \( V_L \) associated to a positive definite even lattice \( L \) ([B], [D1], [FLM]); \( L(\ell, 0) \) associated to a finite-dimensional simple Lie algebra \( \mathfrak{g} \) and a positive integer \( \ell \) ([DL], [FZ], [L2]); \( L(cp, q, 0) \) associated to the Virasoro algebra and a rational number \( cp, q \) ([FZ], [DMZ], [W]); \( V^2 \), Frenkel, Lepowsky and Meurman’s Moonshine module ([B], [D2], [FLM]); tensor products of vertex operator algebras from above. It was proved in [DLM2] that all these rational vertex operator algebras are also regular.

It is well known (cf. [B], [FFR], [L1], [L3], [MP]) that there is a Lie algebra \( g(V) \) associated to any vertex operator algebra \( V \). More precisely,

\[
g(V) = \hat{V}/d\hat{V}
\]  

(2.2)

where

\[
\hat{V} = V \otimes \mathbb{C}[t, t^{-1}] \quad \text{and} \quad d = L(-1) \otimes 1 + 1 \otimes \frac{d}{dt},
\]

with the following bracket formula:

\[
[u(m), v(n)] = \sum_{i=0}^{\infty} \binom{m}{i} (u_i v)(m+n-i)
\]

(2.3)

for \( u, v \in V, m, n \in \mathbb{Z} \), where \( u(m) = u \otimes t^m + d\hat{V} \). Furthermore, \( g(V) \) is a \( \mathbb{Z} \)-graded Lie algebra where \( \text{deg} \ u(m) = wt u - m - 1 \) for homogeneous \( u \) and for \( m \in \mathbb{Z} \). Let \( g(V)_\pm \) be the subalgebras of \( g(V) \) linearly spanned by homogeneous elements of positive degrees (negative degrees). A \( g(V) \)-module \( W \) is said to be restricted if for any \( v \in V, w \in W, v(m)w = 0 \) for \( m \) sufficiently large. It is easy to see that any weak \( V \)-module \( W \) is a restricted \( g(V) \)-module where \( v(n) \) for \( v \in V, n \in \mathbb{Z} \) is represented by \( v_n \).

Let \( V \) be a vertex operator algebra, let \( W = \bigoplus_{h \in \mathbb{C}} W_{(h)} \) be a \( V \)-module, and let \( W' = \bigoplus_{h \in \mathbb{C}} W^*_{(h)} \) be the restricted dual of \( W \). For \( v \in V, w' \in W' \), we define

\[
\langle Y'(v, x)w', w \rangle = \langle w', Y \left( e^{xL(1)} (-x^{-2})^{L(0)} v, x^{-1} \right) w \rangle.
\]

(2.4)
The following fundamental result was due to Frenkel, Huang and Lepowsky ([FHL], Theorem 5.2.1 and Proposition 5.3.1).

**Proposition 2.1** The pair \((W', Y')\) carries the structure of a \(V\)-module and \((W'', Y'') = (W, Y)\).

This module is called the *contragredient* module of \(W\). Proposition 2.1 is analogous to the fact in the classical Lie theory that for any Lie algebra \(g\) and any \(g\)-module \(U\), \(U^*\) is a \(g\)-module where

\[
(a f)(u) = -f(au) \quad \text{for } a \in g, u \in U, f \in U^*.
\]

In the following we consider a natural extension of the contragredient module (in general to a weak module).

Let us start with a weak \(V\)-module \(W\) (without grading). For \(v \in V, \alpha \in W^*\) we define

\[
\langle Y^*(v, x)\alpha, w \rangle = \langle \alpha, Y \left(e^{xL(1)} \left(-x^{-2}\right)^{L(0)}v, x^{-1}\right) w \rangle. \tag{2.5}
\]

If \(W\) is an (ordinary) \(V\)-module, then \(Y^*\) extends \(Y'\). It follows from Proposition 5.3.1 of [FHL] that

\[
\langle \alpha, Y(v, x)w \rangle = \left\langle Y^* \left(e^{xL(1)} \left(-x^{-2}\right)^{L(0)}v, x^{-1}\right) \alpha, w \right\rangle. \tag{2.6}
\]

**Remark 2.2** Since \(e^{xL(1)} \left(-x^{-2}\right)^{L(0)}v\) is a finite sum and

\[
Y(u, x^{-1})w \in W((x^{-1})) \quad \text{for any } u \in V, w \in W,
\]

by (2.5) we have

\[
\langle Y^*(v, x)\alpha, w \rangle \in \mathbb{C}((x^{-1}))). \tag{2.7}
\]

That is,

\[
Y^*(v, x)\alpha \in \text{Hom}(W, \mathbb{C}((x^{-1}))). \tag{2.8}
\]

For \(v \in V\), we set

\[
Y^*(v, x) = \sum_{n \in \mathbb{Z}} v_n^* x^{-n-1}. \tag{2.9}
\]

Let \(v \in V_{(h)}, i.e., L(0)v = hv\). Then

\[
\langle v_n^* \alpha, w \rangle = \left\langle \alpha, (-1)^h \sum_{i \in \mathbb{N}} \frac{1}{i!} (L(1)v)_{2h-n-i-2}w \right\rangle. \tag{2.10}
\]
Furthermore, if $v$ is quasi-primary, i.e., $L(1)v = 0$, then

$$\langle v^*_n \alpha, w \rangle = \langle \alpha, (-1)^h v_{2h-n-2}w \rangle. \quad (2.11)$$

It was observed in [HL] that FHL ([FHL], Proposition 2.1) in fact proves the following opposite Jacobi identity:

$$x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_W(e^{x_2 L(1)}(-x_2^{-2})L(0)v, x_2^{-1})Y_W(e^{x_1 L(1)}(-x_1^{-2})L(0)u, x_1^{-1})$$

$$-x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y_W(e^{x_1 L(1)}(-x_1^{-2})L(0)u, x_1^{-1})Y_W(e^{x_2 L(1)}(-x_2^{-2})L(0)v, x_2^{-1})$$

$$= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y_W(e^{x_2 L(1)}(-x_2^{-2})L(0)Y(u, x_0)v, x_2^{-1}) \quad (2.12)$$

for $u, v \in V$. Although this observation was made in [HL] for a module $W$, obviously this is true if $W$ is a weak module. (Notice that the symbol $Y^*(v, x)$ was used in [HL] for $Y(e^{xL(1)}(-x^{-2})L(0)v, x^{-1})$, which is different from ours.)

Taking $\text{Res}_{x_0}$ from (2.12) and then using (2.5) we obtain

$$[Y^*(u, x_1), Y^*(v, x_2)] = \text{Res}_{x_0} x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y^*(Y(u, x_0)v, x_2). \quad (2.13)$$

The same proof of Theorem 5.2.1 of [FHL] shows that $Y^*(L(-1)v, x) = d/dx Y^*(v, x)$. Then we have proved:

**Proposition 2.3** Let $W$ be any weak $V$-module. Then $W^*$ is a $g(V)$-module. \(\Box\)

To obtain a weak $V$-module out of $(W^*, Y^*)$ we consider the Jacobi identity for $Y^*$. Notice that in the definition of a (weak) module, the truncation condition, in this case which is $Y^*(v, x)\alpha \in W^*((x))$, is necessary for the Jacobi identity to make sense. For example, the first term

$$x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y^*(u, x_1)Y^*(v, x_2)\alpha$$

of the Jacobi identity may not exist algebraically if $Y^*(v, x_2)\alpha$ involves infinitely many negative powers of $x_2$. Having known this fact, we consider a certain subspace of $W^*$.

**Definition 2.4** Let $W$ be a weak $V$-module. Then we define $D(W)$ to be the subspace of $W^*$ consisting of vectors $\alpha$ such that for every $v \in V$

$$Y^*(v, x)\alpha \in W^*((x)), \quad (2.14)$$

i.e., $v^*_n \alpha = 0$ for $n$ sufficiently large.

If $W$ is an (ordinary) $V$-module, it is clear that $W' \subseteq D(W)$. The following proposition gives a characterization of elements of $D(W)$.
Proposition 2.5 Let $W$ be a weak $V$-module and let $\alpha \in W^*$. Then $\alpha \in D(W)$ if and only if for any $v \in V$, there exists $k \in \mathbb{Z}$ such that
\[ v_m W \subseteq \ker \alpha \quad \text{for } m \leq k, \tag{2.15} \]
or equivalently, there exists $r \in \mathbb{Z}$ such that
\[ x^r \langle \alpha, Y(v, x)w \rangle \in \mathbb{C}[x^{-1}] \tag{2.16} \]
for all $w \in W$.

Proof. Notice that \( e^{xL(1)}(-x^{-2})L(0)v \) for $v \in V$ is a finite sum. Then it follows from (2.5) and (2.6) immediately. 

As a corollary we have:

Corollary 2.6 Let $W$ be a weak $V$-module and let $U$ be a subspace of $W$. Then $(W/U)^* \subseteq D(W)$ if and only for any $v \in V$ there exists $k \in \mathbb{Z}$ such that $v_m W \subseteq U$ for $m \leq k$, where $(W/U)^*$ is viewed as a subspace of $W^*$ in the natural way.

The following lemma establishes the stability of the action of $g(V)$ on $D(W)$.

Lemma 2.7 Let $W$ be a weak $V$-module. Then $D(W)$ is a restricted $g(V)$-submodule of $W^*$.

Proof. Since $W^*$ is a $g(V)$-module, for $u, v \in V, m \in \mathbb{Z}, \alpha \in D(W)$, we have
\[ Y^*(u, x)v_m^* \alpha = v_m^* Y^*(u, x)\alpha - \sum_{i \in \mathbb{N}} \binom{m}{i} x^{m-i} Y^*(v_i u, x)\alpha. \tag{2.17} \]
Because $Y^*(v_i u, x)\alpha \in W^*((x))$ for each $i \in \mathbb{N}$ and $v_i u = 0$ for all but finitely many $i \in \mathbb{N}$, we have
\[ Y^*(u, x)v_m^* \alpha \in W^*((x)). \]
Thus $v_m^* \alpha \in D(W)$. Therefore $D(W)$ is a $g(V)$-submodule of $W^*$. From the definitions, $D(W)$ is restricted. 

Furthermore we have:

Proposition 2.8 Let $W$ be a weak $V$-module. Then $D(W)$ is a weak $V$-module.

Proof. Notice that the existences of the three main terms in the Jacobi identity for $Y^*$ are guaranteed by the truncation condition (2.14). Then it follows from Lemma 2.7 and (2.12) immediately. 

Remark 2.9 Because of the truncation axiom in the notion of (weak) module, it is clear that $D(W)$ is the maximal weak $V$-module in $W^*$ with $Y = Y^*$. 

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Let $W$ be a weak $V$-module. For any $h \in \mathbb{C}$, we set
\[ W_{(h)} = \{ w \in W \mid L(0)w = hw \}. \] (2.18)
Furthermore we set
\[ W^0 = \prod_{h \in \mathbb{C}} W_{(h)}. \]
Then it is clear that $W^0$ is a submodule of $W$.

**Proposition 2.10** Let $W$ be an (ordinary) $V$-module. Then $D(W)^0 = W'$.

**Proof.** Let $\alpha \in D(W)_{(h_1)}$, $w \in W_{(h_2)}$. Then
\[ h_1 \langle \alpha, w \rangle = \langle L(0)\alpha, w \rangle = \langle \alpha, L(0)w \rangle = h_2 \langle \alpha, w \rangle. \]
It follows that
\[ \langle D(W)_{(h_1)}, W_{(h_2)} \rangle = 0 \quad \text{for } h_1 \neq h_2. \]
Thus $D(W)_{(h)} \subseteq W^*_{(h)}$ for $h \in \mathbb{C}$. Because $W^*_{(h)} \subseteq D(W)_{(h)}$ we get $D(W)_{(h)} = W^*_{(h)}$ for $h \in \mathbb{C}$. Thus $D(W)^0 = W'$. □

By definition $D(W) = D(W)^0$ if and only if $L(0)$ acts semisimply on $D(W)$. Then as an immediate corollary we have:

**Corollary 2.11** Let $V$ be a vertex operator algebra such that $L(0)$ acts semisimply on any weak module. Then for any $V$-module $W$ we have $D(W) = W'$. □

Suppose that $V$ is regular. Then any weak module is a direct sum of irreducible $V$-modules, so that $L(0)$ acts semisimply. Then we immediately have:

**Corollary 2.12** Let $V$ be a regular vertex operator algebra and let $W$ be a $V$-module. Then $D(W) = W'$. □

Motivated Corollary 2.11 we define $\mathcal{A}$ to be the class of vertex operator algebras satisfying the condition that $L(0)$ acts semisimply on any weak module. Let $V$ be a vertex operator algebra containing a vertex operator subalgebra (with the same Virasoro element) $V^0$ of class $\mathcal{A}$. Since any weak $V$-module is a weak $V^0$-module, $L(0)$ (the same for both $V$ and $V^0$) acts semisimply on any weak $V$-module. Thus $V$ is also of class $\mathcal{A}$. Since the class $\mathcal{A}$ contains all regular vertex operator algebras, any vertex operator algebra that has a regular vertex operator subalgebra (with the same Virasoro element) is of class $\mathcal{A}$. In the next section, we shall study some finiteness properties for this class of vertex operator algebras.

Notice that in the definition of $D(W)$, it was required that $Y^*(v, x)\alpha \in W^*((x))$ for all $v \in V$. However, for vertex operator algebras of certain types such as those associated to affine Lie algebras or the Virasoro algebra we only need to check this for each $v$ of a (usually finite-dimensional) subspace.
Proposition 2.13 Let $V$ be a vertex operator algebra such that $V = \prod_{n \geq 0} V(n)$ (without negative weights) with $V(0) = \mathbb{C}1$ and let $U$ be a graded subspace of $\prod_{n \geq 1} V(n)$ ($\subseteq V$) such that $V$ is linearly spanned by elements

$$1, \ u_{-n_1} \cdots u_{-n_r} 1, \quad (2.19)$$

where $u^i \in U, n_1, \ldots, n_r \geq 1$. Let $W$ be a weak $V$-module and let $\alpha \in W^*$. If for any $u \in U$, there exists $k \in \mathbb{Z}$ such that for all $w \in W$,

$$\langle \alpha, u_m w \rangle = 0 \quad \text{whenever } m \leq k. \quad (2.20)$$

Then $\alpha \in D(W)$.

**Proof.** Let $B$ be the subspace of $V$ consisting of each $v$ such that there exists $k \in \mathbb{Z}$ such that

$$x^k \langle \alpha, Y(v, x)w \rangle \in \mathbb{C}[x^{-1}] \quad \text{for all } w \in W.$$ 

Then by Proposition 2.3 $\alpha \in D(W)$ if and only if $V \subseteq B$. In the following we shall prove by induction that

$$\bigoplus_{i=0}^n V(i) \subseteq B \quad \text{for } n \in \mathbb{N}.$$ \hspace{1cm} 

First, from the assumption we have $U \subseteq B$. Since $V(0) = \mathbb{C}1$, it is clear that $V(0) \subseteq B$. Assume that $\bigoplus_{i=0}^n V(i) \subseteq B$ for some $n \in \mathbb{N}$. Let $a = u_{-m} v \in V(n+1)$ be a homogeneous element where $u \in U, v \in V, m \geq 1$. Note that for any $w \in W$, by taking $\text{Res}_{x_1} \text{Res}_{x_0} x_{-m}^n$ from the Jacobi identity we obtain

$$\langle \alpha, Y(a, x_2)w \rangle = \text{Res}_{x_1} (x_1 - x_2)^{-m} \langle \alpha, Y(u, x_1)Y(v, x_2)w \rangle$$

$$-((-x_2 + x_1)^{-m} \langle \alpha, Y(v, x_2)Y(u, x_1)w \rangle. \quad (2.21)$$

Since $n + 1 = wta = wt u + m - 1 + wt v$ and $wt u \geq 1$, we have $wt v \leq n$ so that $v \in B$ (by the inductive assumption). Then there is $k_1 \in \mathbb{Z}$ such that

$$\text{Res}_{x_1} x_{-k_1}^n (-x_2 + x_1)^{-m} \langle \alpha, Y(v, x_2)Y(u, x_1)w \rangle \in \mathbb{C}[x_2^{-1}] \quad (2.22)$$

for all $w \in W$. Since $u \in U$, by assumption there is $r \in \mathbb{N}$ such that $\langle \alpha, u_n W \rangle = 0$ for $n \leq -m - r$. Then

$$\text{Res}_{x_1} (x_1 - x_2)^{-m} \langle \alpha, Y(u, x_1)Y(v, x_2)w \rangle$$

$$= \sum_{i=0}^r (-1)^i \binom{-m}{i} x_2^i \langle \alpha, u_{-m-i} Y(v, x_2)w \rangle$$

$$= \sum_{i=0}^r (-1)^i \binom{-m}{i} x_2^i \langle \alpha, Y(v, x_2)u_{-m-i} w \rangle$$

$$+ \sum_{i=0}^r \sum_{j \in \mathbb{N}} (-1)^i \binom{-m}{i} \binom{-m-i}{j} x_2^{-m-j} \langle \alpha, Y(u_j v, x_2)w \rangle. \quad (2.23)$$
Since $w tv = wtu + wtv - j - 1 < wtu + wtv + m - 1 = n + 1$, $u_jv \in B$ for $j \in \mathbb{N}$. Because $v, u_jv \in B$ and $u_jv \neq 0$ only for finitely many $j \in \mathbb{N}$, by (2.23) there is $k_2 \in \mathbb{Z}$ such that
\[
\text{Res}_{x_1} x_2^{k_2} (x_1 - x_2)^{-m} \langle \alpha, Y(u, x_1)Y(v, x_2)w \rangle \in \mathbb{C}[x_2^{-1}]
\] (2.24)
for all $w \in W$. Combining (2.22) with (2.24) we get
\[
x_2^k \langle \alpha, Y(a, x_2)w \rangle \in \mathbb{C}[x_2^{-1}]
\] for all $w \in W$, where $k = \min\{k_1, k_2\}$. Thus $a \in B$. By the spanning property (2.19), $V_{(n+1)}$ is linearly spanned by elements like $a$. Thus $V_{(n+1)} \subseteq B$. Therefore $V \subseteq B$. This proves that $\alpha \in D(W)$. $\square$

Proposition 2.13 will be useful if one wants to determine $D(W)$ explicitly for certain vertex operator algebras.

3 The $C_2$-finiteness condition and the finiteness of fusion rules

This section is the core of the paper. In this section we define subspaces $C_n(W)$ for $n \geq 1$ and for any weak $V$-module, generalizing Zhu’s $C_2(V)$ subspace defined in [Z]. We prove that $(W/C_n(W))^* \subseteq D(W)$ and that $V$ is finitely generated if $C_1(V)$ is finite-codimensional. By applying Corollaries 2.11 and 2.12 we then prove the $C_2$-finiteness condition, the finiteness of fusion rules and the finite generating property for the class $\mathcal{A}$ of vertex operator algebras defined in Section 2.

Let $W$ be a weak $V$-module and let $n \geq 2$. We define $C_n(W)$ to be the linear span of elements of type
\[
v_{-n}w, \quad \text{for } v \in V, w \in W.
\] (3.1)
Since $(L(-1)v)_{-n} = nv_{-n-1}$, we have:
\[
v_{-m}w \in C_n(W) \quad \text{for } v \in V, w \in W, m \geq n.
\]
Thus
\[
\cdots \subseteq C_{n+1}(W) \subseteq C_n(W) \subseteq \cdots \subseteq C_2(W).
\]
Note that $C_2(V)$ is exactly the one defined in [Z]. A vertex operator algebra $V$ is said to satisfy $C_2$-finiteness condition ([Z], [DLM3]) if $V/C_2(V)$ is finite-dimensional. (Note that Zhu defined the concept of finiteness condition $C$ by requiring that $V/C_2(V)$ is finite-dimensional and that $V$ is a sum of lowest weight modules for the Virasoro algebra.)

Let $W$ be a generalized $V$-module, i.e., a weak module on which $L(0)$ acts semisimply.
Set
\[
W_+ = \bigoplus_{\mathbb{C}, \mathbb{R} e h > 0} W(h).
\] (3.2)
Then we define $C_1(W)$ to be the subspace of $W$ linearly spanned by

$$v_{-1}W_+, \quad L(-1)W \quad \text{for } v \in V_+.$$  \hfill (3.3)

(The subspace $C_1(W)$ is designed to make the proof of Proposition 3.3 work. However, one may define $C_1(W)$ differently for other purposes.) Notice that if we had defined $C_1(V)$ to be the linear span of elements of type (3.1) with $n = 1$, then we would have $C_1(V) = V$ because $v = v_{-1}1 \in C_1(V)$ for $v \in V$. That is not what we want.

Since $k!v_{-k-1} = (L(-1)^k)v_{-1}$ for $v \in V, k \in \mathbb{N}$, we have

$$v_{-k-1}w \in C_1(W) \quad \text{for } v \in V_+, w \in W_+, k \in \mathbb{N}.$$  \hfill (3.4)

More generally, we have:

**Lemma 3.1** Let $W$ be a generalized $V$-module, let $v \in V, w \in W$ be homogeneous and let $r, s \in \mathbb{N}$ be such that $L(-1)^r v \in V_+$ and $L(-1)^s w \in W_+$. Then

$$v_{-r-s-k}w \in C_1(W) \quad \text{for } k \geq 1.$$  \hfill (3.5)

**Proof.** Since $L(-1)^r v \in V_+, L(-1)^s w \in W_+$ by (3.4) we have

$$(L(-1)^r v)_{-k} L(-1)^s w \in C_1(W) \quad \text{for } k \geq 1.$$  

Since $[L(-1), u_m] = -mu_{m-1}$ for $u \in V, m \in \mathbb{Z}$ and $L(-1)W \subseteq C_1(W)$, we get

$$(L(-1)^r v)_{-k-s} w \in C_1(W).$$

Then

$$v_{-r-s-k} w \in C_1(W) \quad \text{for } k \geq 1. \quad \square$$

**Remark 3.2** In general, $C_2(W)$ is not a subspace of $C_1(W)$. However, if $V = \bigsqcup_{n \in \mathbb{N}} V_n$ (without negative weights) such that $V_{(0)} = \mathbb{C}1$, then $C_2(V) \subseteq C_1(V)$ because $v_{-2}1 = L(-1)v$ and $1_{-2}v = 0$ for any $v \in V$.

The following proposition gives a way to find a relatively small generating subspace of $V$ as a vertex operator algebra by using $C_1(V)$.

**Proposition 3.3** Let $V$ be a vertex operator algebra and $U$ be a graded subspace of $V$ such that $V = U + C_1(V)$ and $\oplus_{n \leq 0} V_{(n)} \subseteq U + \mathbb{C}1$. Then $V$ is linearly spanned by elements of type

$$u_{n_1}^1 \cdots u_{n_r}^r 1,$$  \hfill (3.6)

where $r \in \mathbb{N}, u^i \in U, n_i \in \mathbb{Z}$ for $1 \leq i \leq r$, i.e., $U$ generates $V$ as a vertex operator algebra.
Proof. Let \( \langle U \rangle \) be the vertex operator subalgebra of \( V \) generated by \( U \). We shall prove by induction that for any \( n \in \mathbb{N} \),
\[
\oplus_{i \leq n} V((i)) \subseteq \langle U \rangle.
\]
By the assumption, we have \( \oplus_{i \leq 0} V((i)) \subseteq U + C1 \subseteq \langle U \rangle \). Suppose that \( \oplus_{i \leq n} V((i)) \subseteq \langle U \rangle \) for some \( n \in \mathbb{N} \). Let \( a \in V(n+1) \). Since \( V(n+1) = C1(V) \cap V(n+1) + U \cap V(n+1) \), we have
\[
a = u^1 v^1 + \cdots + u^r v^r + L(-1)u + b
\]
for some homogeneous \( u^i, v^i \in V, b \in U \) with \( wtu^i v^i = wt u^i + wt v^i = n + 1, wt u^i > 0, wt v^i > 0 \) and \( wt u = n \). Then \( wt u^i, wt v^i \leq n \) for all \( i \). By inductive assumption, we have
\[
u^i, v^i, u \in \langle U \rangle \quad \text{for all } i,
\]
so that
\[
u_{-1}^i v^i \in \langle U \rangle \quad \text{for all } i
\]
and \( L(-1)u = u_{-2} 1 \in \langle U \rangle \). Thus \( a \in \langle U \rangle \). This proves that \( V(n+1) \subseteq \langle U \rangle \). Therefore \( V = \langle U \rangle \). □

As a refinement of Proposition 3.3, it was proved in [KL] that \( V \) is linearly spanned by elements in (3.6) with \( n_1, \ldots, n_r < 0 \) in a fixed lexicographical order.

As an immediate corollary of Proposition 3.3 we have:

**Corollary 3.4** Let \( V \) be a vertex operator algebra such that \( C_1(V) \) is finite-codimensional. Then \( V \) is finitely generated. □

Furthermore, by Remark 3.2 we have:

**Corollary 3.5** Suppose that \( V = \bigoplus_{n \in \mathbb{N}} V(n) \) (without negative weights) and \( V(0) = C1 \). Then the \( C_2 \)-finiteness condition on \( V \) implies that \( V \) is finitely generated. □

In the following we shall prove that the \( C_2 \)-finiteness condition holds for vertex operator algebras of class \( \mathcal{A} \). For convenience, we set \( Z_n(W) = W/C_n(W) \).

**Proposition 3.6** Let \( W \) be a weak \( V \)-module and let \( n \geq 2 \). Then
\[
Z_n(W)^* = \{ \alpha \in W^* \mid v^m \alpha = 0 \quad \text{for homogeneous } v \in V, m \geq 2wt v + n - 2 \}.
\]
In particular,
\[
Z_n(W)^* \subseteq D(W).
\]

**Proof.** Let \( \alpha \in Z_n(W)^* \), \( v \in V \). Then
\[
\langle \alpha, v_m w \rangle = 0 \quad \text{for } v \in V, w \in W, m \geq n.
\]
Let $u \in V$ be homogeneous. Then for any $m \geq 2\text{wt} u + n - 2$

$$\langle u^*_m \alpha, w \rangle = \left\langle \alpha, \sum_{i \in \mathbb{N}} \frac{(-1)^{\text{wt} u}}{i!} (L(1)^i u)_{2\text{wt} u - i - m - 2} \right\rangle = 0$$

(3.10)

for any $w \in W$. Thus

$$u^*_m \alpha = 0 \quad \text{for } m \geq 2\text{wt} u + n - 2.$$  

(3.11)

Conversely, let $\alpha \in W^*$ be such that (3.11) holds and let $v \in V(k)$ for $k \in \mathbb{Z}$. Then

$$\langle \alpha, v_{-m} w \rangle = \sum_{i \in \mathbb{N}} (-1)^k \frac{1}{i!} \left\langle (L(1)^i v)^*_0, v_{2k - i - 2 + m} \alpha, w \right\rangle = 0$$

(3.12)

for $m \geq n$ (noticing that $2k - i - 2 + m = 2\text{wt}(L(1)^i v) + i + m$). Thus $\alpha \in Z_n(W)^*$. Then the proof is complete. $\Box$

Now we present our key result of this paper:

**Proposition 3.7** Let $V$ be a vertex operator algebra of class $\mathcal{A}$ and let $W$ be a $V$-module. Then $Z_n(W)$ is finite-dimensional for $n \geq 2$ and $Z_1(W)$ is also finite-dimensional if the real parts of the weights of $W$ are bounded from below.

**Proof.** We first notice that a graded subspace $U$ of $W$ is finite-codimensional if and only if $(W/U)^* \subseteq W'$, where $(W/U)^*$ is viewed as a natural subspace of $W^*$. It is clear that $C_n(W)$ is a graded subspace. Then $\dim Z_n(W) < \infty$ if and only if $Z_n(W)^* \subseteq W'$. Furthermore, since $D(W) = W'$ (Corollary 2.11), $\dim Z_n(W) < \infty$ if and only if $Z_n(W)^* \subseteq D(W)$.

If $n \geq 2$, it follows immediately from Proposition 3.6. For $n = 1$, since the real parts of the weights of $W$ are bounded from below, there is a nonnegative integer $s$ such that $L(-1)^s W \subseteq W_+$. Then it follows from Lemma 3.1 and Corollary 2.6 that $Z_1(W)^* \subseteq D(W)$. This completes the proof. $\Box$

Combining Corollary 3.4 with Proposition 3.7, we immediately have:

**Theorem 3.8** Any vertex operator algebra of class $\mathcal{A}$ satisfies the $C_2$-finiteness condition and it is finitely generated. In particular, any regular vertex operator algebra $V$ satisfies the $C_2$-finiteness condition and it is finitely generated. $\Box$

Next we shall prove the finiteness of fusion rules for vertex operator algebras of class $\mathcal{A}$. To do this we shall use Frenkel and Zhu’s $A(V)$-bimodule theory [FZ].

Recall $A(V)$ and $A(W)$ from [FZ] and [Z]. For any weak $V$-module $W$, let $O(W)$ be the linear span of elements of type

$$\sum_{i \in \mathbb{N}} \binom{\text{wt} v}{i} v_{i - 2} w \quad = \text{Res}_{x^2} \frac{(1 + x)^{\text{wt} v}}{x^2} Y(v, x) w$$

(3.13)
for homogeneous \( v \in V \) and for \( w \in W \). It was proved ([Z], Lemma 2.1.1) that

\[
\sum_{i \in \mathbb{N}} \binom{wt v}{i} v_{i-m} w \in O(W)
\]

for homogeneous \( v \in V \) and for \( w \in W, m \geq 2 \). Set \( A(W) = W/O(W) \). Then we have [Z]:

**Proposition 3.9** (a) The space \( A(V) \) is an associative algebra with the product:

\[
(u + O(V))(v + O(V)) = \sum_{i \in \mathbb{N}} \binom{wt u}{i} (u_{i-1} v + O(V))
\]

for homogeneous \( u, v \in V \).

(b) For any \( \mathbb{N} \)-gradable weak \( V \)-module \( W = \coprod_{n \in \mathbb{N}} W(n) \), \( W(0) \) is an \( A(V) \)-module where \( v + O(V) \) acts on \( W(0) \) as \( v_{wt v-1} \) for homogeneous \( v \in V \).

(c) There is a one-to-one correspondence between the set of equivalence classes of irreducible \( A(V) \)-modules and the set of equivalence classes of irreducible \( \mathbb{N} \)-gradable weak \( V \)-modules.

Furthermore, we have [FZ]:

**Proposition 3.10** Let \( W \) be a weak \( V \)-module. Then \( A(W) \) is an \( A(V) \)-bimodule with the following left and right actions:

\[
(v + O(V))(w + O(W)) = \sum_{i \in \mathbb{N}} \binom{wt v}{i} (v_{i-1} w + O(W))
\]

\[
(w + O(W))(v + O(V)) = \sum_{i \in \mathbb{N}} \binom{wt v - 1}{i} (v_{i-1} w + O(W))
\]

for homogeneous \( v \in V \) and for \( w \in W \).

Let \( W_1, W_2, W_3 \) be irreducible \( V \)-modules and let \( I \left( \frac{W_3}{W_1 W_2} \right) \) be the space of intertwining operators of the indicated type. Then we have ([L4], Proposition 2.10, and [FZ]):

**Proposition 3.11** Let \( W_1, W_2, W_3 \) be irreducible \( V \)-modules. Then

\[
\dim I \left( \frac{W_3}{W_1 W_2} \right) \leq \dim \text{Hom}_{A(V)}(A(W_1) \otimes_{A(V)} W_2(0), W_3(0)),
\]

where \( W_2(0) \) and \( W_3(0) \) are the lowest weight subspaces of \( W_2 \) and \( W_3 \), respectively.

The following lemma was proved in [DLM3] (Proposition 3.6) (see also [Z], Lemma 4.4.1).
Lemma 3.12 Let $W$ be a $V$-module such that $\dim Z_2(W) < \infty$. Then $\dim A(W) < \infty$.

Combining Lemma 3.12 with Propositions 3.9 and 3.7 we obtain

Theorem 3.13 Let $V$ be a vertex operator algebra of class $A$. Then there are only finitely many inequivalent irreducible $\mathbb{N}$-gradable weak $V$-modules and the fusion rule for any three irreducible modules is finite. In particular, the assertions hold if $V$ is regular. \(\square\)

Remark 3.14 Motivated by Proposition 3.6, one may also consider the subspace $A(W)^*$ of $W^*$ for a (weak) $V$-module $W$. By using the proof of Proposition 3.6 for $A(W)^*$, one can see that in general $A(W)^*$ may not be a subspace of $D(W)$. However, it was proved in [L5] that $A(W)^*$ is a subspace of a canonical weak $V$-bimodule $D(W)$ where $D(W)$ is the space of what we call representative functionals on $W$ containing $D(W)$ as a subspace.

Remark 3.15 Note that $\dim A(W_1) < \infty$ is a sufficient condition for the fusion rule $\dim I\left(\frac{W_3}{W_1^2W_2}\right)$ to be finite. However, by no means it is necessary. As a matter of fact, if $A(W_1)$ is a finitely generated $A(V)$-bimodule, the fusion rule $\dim I\left(\frac{W_3}{W_1^2W_2}\right)$ is finite even though $A(W_1)$ may be infinite-dimensional. Indeed, if $S$ is a finite-dimensional subspace of $A(W_1)$ which generates $A(W_1)$ as an $A(V)$-bimodule, then

$$\dim \text{Hom}_{A(V)}(A(W_1) \otimes_{A(V)} W_2(0), W_3(0)) \leq \dim \text{Hom}_{C}(S \otimes W_2(0), W_3(0)).$$

By Proposition 3.11, we get $\dim I\left(\frac{W_3}{W_1^2W_2}\right) < \infty$.

For the rest of this section we shall give a sufficient condition for $A(W)$ to be a finitely generated $A(V)$-bimodule. For convenience, we assume that $V = \coprod_{n \in \mathbb{N}} V_n$ with $V_{(0)} = \mathbb{C}1$. For any $V$-module $W$, we define $B(W)$ to be the subspace of $W$ linearly spanned by

$$u_{-1}W \quad \text{for } u \in V_+, \quad (3.19)$$

$$v_0W \quad \text{for homogeneous } v \quad \text{with } \text{wt } v \geq 2. \quad (3.20)$$

That is, $B(W) = g(V)_+ W$, where $g(V)_+$ is the subalgebra of $g(V)$ linearly spanned by homogeneous elements of positive degrees.

Proposition 3.16 Let $W$ be a $V$-module such that $W = \coprod_{n \in \mathbb{N}} W_{(h+n)}$ for some $h \in \mathbb{C}$ and let $W^0$ be a graded subspace of $W$ such that $W = W^0 + B(W)$. Then $(W^0 + O(W))/O(W)$ generates $A(W)$ as an $A(V)$-bimodule. In particular, if $\dim W/B(W) < \infty$, $A(W)$ is a finitely generated $A(V)$-bimodule.
**Proof.** Let $E$ be the $A(V)$-bimodule generated by $W^0 + O(W)$. We shall prove by induction that $W_{(h+n)} + O(W) \subseteq E$ for $n \in \mathbb{N}$. Since $W^0$ and $B(W)$ are graded and $B(W) \cap W_{(h)} = 0$, we have $W_{(h)} \subseteq W^0$. Suppose that

$$(\bigoplus_{i=0}^n W_{(n+i)}) + O(W) \subseteq E$$

for some $n \in \mathbb{N}$. Let $w \in W_{(h+n+1)}$. Then

$$w = u_{-1} w^1 + \cdots + u_{-1} w^r + v_0 w^{r+1} + \cdots v_0 w^{r+s} + w'$$

(3.21)

for some homogeneous $u^i, v^j \in V_+ \subseteq W$ such that $wtu^i \geq 1$, $wtv^j \geq 2$. Since $wtu_{-1} \mid wtv_0 \geq 1$, we have $w^k \in \bigoplus_{i=0}^n W_{(t+k)}$, so that by the inductive assumption we have $w^k + O(W) \subseteq E$ for $1 \leq k \leq r + s$. Then (recall (3.16) and (3.17))

$$(u^i + O(V)) * (w^i + O(W)) = \sum_{p=0}^{wtu^i} \binom{wtu^i}{p} u_{p-1} w^i + O(W) \subseteq E, \quad (3.22)$$

$$(v^j + O(V)) * (w^{i+r} + O(W)) = (u^{i+r} + O(W)) * (v^j + O(V))$$

$$= \sum_{p=0}^{wtv^j} \binom{wtv^j}{p} v_p w^{i+r} + O(W) \subseteq E. \quad (3.23)$$

Since $u_{p-1} w^i \in \bigoplus_{i=0}^n W_{(h+t)}$, by the inductive assumption we have $u_{p-1} w^i + O(W) \subseteq E$ for $p \geq 1, 1 \leq i \leq r$. Then

$$(u_{-1} w^i + O(W)) = u^i * w^i \sum_{p=1}^{wtu^i} \binom{wtu^i}{p} u_{p-1} w^i + O(W) \subseteq E.$$

Similarly, we have $v_0 w^{r+j} + O(W) \subseteq E$ for $1 \leq j \leq s$. Then $w + O(W) \subseteq E$. This completes the induction. Therefore $E = A(W)$. \(\square\)

As an immediate corollary of Proposition 3.16 and Remark 3.15, we have:

**Corollary 3.17** Let $W_1, W_2, W_3$ be irreducible $V$-modules such that $\dim W_1/B(W_1) < \infty$. Then $\dim I(W_3_{W_1W_2}) < \infty$. \(\square\)

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