Ode to commutator operators

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Abstract
These brief remarks have been prepared in connection with a conference in honor of my thesis advisor, Richard Rochberg.

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1 It was a dark and stormy night

Or maybe it was just another pleasant springtime evening in the suburbs of Stockholm. The professor was sharing a house on the institute grounds with a couple of messy bachelor mathematicians, and numerous empty beer bottles were scattered about the room. The professor was enjoying a cool swedish brew too while his earnest doctoral student listened intently. “People have been studying the linear terms for fifty years”, or something like that. This was in the context of nonlinear problems in which functions are themselves variables for some other functional. Linearizations in simpler situations have been studied
for a much longer period of time. Now it would be time to pursue the quadratic and higher-order terms, as in the calculus of Newton and Leibniz.

2 Commutators

If \( A \) is an associative algebra and \( a, b \in A \), then the \textit{commutator} of \( a, b \in A \) is defined by
\[
[a, b] = ab - ba. \tag{2.1}
\]

Commutators automatically satisfy the Leibniz rule for products,
\[
[a, bc] = abc - bca = [a, b]c + b[a, c]. \tag{2.2}
\]

Thus a commutator is like a derivative.

A typical situation of interest would be an algebra of linear operators on a vector space of functions. Suppose that \( D \) is a first-order differential operator acting on functions on some space, which satisfies the ordinary Leibniz rule
\[
D(f_1, f_2) = (Df_1)f_2 + f_1(Df_2) \tag{2.3}
\]
with respect to pointwise multiplication of functions. Let \( b \) be a function on the same space, and let \( M_b \) be the corresponding operator of multiplication by \( b \), \( M_b(f) = bf \). In this case, the commutator of \( D \) and \( M_b \) is the same as the operator of multiplication by \( Db \).

3 The Calderón commutator

Let \( H \) be the Hilbert transform on the real line \( \mathbb{R} \), and let \( D \) be the usual differentiation operator \( d/dx \). Suppose that \( A \) is a Lipschitz function on \( \mathbb{R} \), so that the derivative \( a = A' \) of \( A \) is bounded. If \( M_A \) again denotes the operator of multiplication by \( A \), then a celebrated theorem of Calderón [10] states that the commutator \([M_A, HD]\) determines a bounded operator on \( L^2(\mathbb{R}) \). This would be trivial without the Hilbert transform, since multiplication by a bounded function is a bounded operator on \( L^2 \).

Observe that
\[
D[M_A, H] = D M_A H - D H M_A = M_a H + M_A D H - D H M_A = M_a H + [M_A, HD]. \tag{3.1}
\]

Hence the boundedness of \([M_A, HD]\) and \( D[M_A, H] \) on \( L^2(\mathbb{R}) \) are equivalent when \( A \) is Lipschitz, since the Hilbert transform is bounded too. The boundedness of \( D[M_A, H] \) on \( L^2(\mathbb{R}) \) can be reformulated as the boundedness of \([M_A, H]\) as a mapping from \( L^2(\mathbb{R}) \) to the Sobolev space of functions on the real line with first derivative in \( L^2 \). This suggests many interesting questions about analogous operators on other metric spaces, where it may be easier to make sense of something like \(|Df|\) than \(Df\).
4 Connes’ noncommutative geometry

In Connes’ theory [21], the commutator \([T, M_b]\) between a singular integral operator \(T\) and the operator \(M_b\) of multiplication by a function \(b\) represents a sort of derivative of \(b\). Situations in which \(T^2 = I\) are of particular interest. More precisely, one may work with operators acting on vector-valued functions here. For instance, \(T\) could be made up of Riesz transforms.

As analogues of classical integrals of products of functions, Connes studies certain types of traces of products of commutators. These products of commutators are normally not quite of trace class, and so an extension of the trace due to Dixmier [27] is employed. The one-dimensional case where such a commutator might be of trace class is somewhat exceptional in this context. The Dixmier trace of a trace class operator is equal to 0, and involves asymptotic behavior of an operator in general.

For that matter, ordinary derivatives involve limits. This is an important part of localization. That is a natural feature to try to have.

5 Commutators as derivatives

Commutators often occur as derivatives of nonlinear functionals. This is a familiar theme in mathematics, where the nonlinear functional involves some form of conjugation. As a basic example, the commutator \([M_b, T]\) of the operator \(M_b\) of multiplication by \(b\) and another linear operator \(T\) can be seen as the derivative in \(b\) at the origin of the conjugation \(M_e^b T M_e^{-b}\) of \(T\) by multiplication by \(e^b\). This was studied in [18] in connection with weighted norm inequalities for singular integral operators and bounded mean oscillation.

The Calderón commutators correspond to derivatives of the Cauchy integral operator on Lipschitz graphs, with respect to the Lipschitz function being graphed. As in [15, 16], this is closely related to the conjugation of the Hilbert transform by a change of variables. More precisely, the Cauchy integral operator can be obtained as an analytic continuation of the conjugation of the Hilbert transform by a change of variables, by interpreting it as a conjugation of the Hilbert transform by a change of variables that extends into the complex plane.

6 Projections

Let \(\mathcal{H}\) be a Hilbert space, and let \(P\) be a bounded linear operator on \(\mathcal{H}\) which is a projection. Thus \(P^2 = P\). Let \(V_0\) be the set of \(v \in \mathcal{H}\) such that \(P(v) = 0\), and let \(V_1\) be the set of \(v \in \mathcal{H}\) such that \(P(v) = v\). These are closed linear subspaces of \(\mathcal{H}\), and \(P\) is the projection of \(\mathcal{H}\) onto \(V_1\) with kernel \(V_0\). Consider \(T = 2P - I\). Equivalently, \(T\) is characterized by the conditions that \(T(v) = -v\) when \(v \in V_0\) and \(T(v) = v\) when \(v \in V_1\). In particular, \(T^2 = I\). Note that \(T\) is self-adjoint if and only if \(P\) is, which happens exactly when \(V_0\) and \(V_1\) are orthogonal. If \(P\) is the orthogonal projection of \(L^2\) on the real line or unit circle onto the Hardy space of functions of analytic type, then \(T\) corresponds to
the Hilbert transform. Examples that are not self-adjoint can be obtained from Cauchy integrals and using weights.

7 Toeplitz operators

Let \( X \) be a space equipped with a topology and a Borel measure, and perhaps a metric. As a Hilbert space \( \mathcal{H} \), consider \( L^2(X) \). Suppose that \( P \) is a bounded linear operator on \( \mathcal{H} \) which is a projection. It might also be nice if \( T = 2P - I \) is something like a singular integral operator. A key point is for \( [M_b, T] \) to be compact when \( b \) is a continuous function on \( X \) which is continuous at infinity if \( X \) is not compact. Because compact operators form a closed linear subspace of the space of bounded linear operators on \( \mathcal{H} \), it suffices to check this for a dense class of functions \( b \). For instance, \( [M_b, T] \) is more regular when \( b \) is a Lipschitz function with compact support. Under these conditions, one can consider Toeplitz operators \( P M_b \) on \( V_1 = P(L^2(X)) \). These are the classical Toeplitz operators when \( X \) is the unit circle or real line and \( P \) is the orthogonal projection onto the associated Hardy space.

In some cases, it may be convenient to allow vector-valued functions on \( X \). For example, one might be interested in functions on \( \mathbb{R}^n \) with values in a Clifford algebra. Using Clifford analysis, one gets a Hardy space and a projection defined in terms of singular integral operators. In the classical situation, the product of holomorphic functions is homomorphic. This does not work for Clifford holomorphic functions. Thus some of the usual structure is not available. One way to deal with this is to restrict one’s attention to scalar-valued functions \( b \). This ensures that \( b \) commutes with elements of the Clifford algebra, so that the commutator \( [M_b, T] \) still behaves well. Alternatively, one can be more careful about the way in which a multiplication operator acts, from the left or the right. Because of noncommutativity, Clifford holomorphicity can also be defined from the left or right. The product of a Clifford holomorphic function and a constant in the Clifford algebra is Clifford holomorphic, when the constant is on the appropriate side. Similarly, commutators of Clifford-valued multiplication operators and singular integral operators also behave well when the operators involve multiplication on opposite sides.

8 Extensions

Let \( X \) be a compact metric space, and let \( \mathcal{C}(X) \) be the Banach algebra of continuous complex-valued functions on \( X \) with the supremum norm. Let \( \mathcal{H} \) be a complex Hilbert space, and let \( \mathcal{B}(\mathcal{H}) \) be the Banach algebra of bounded linear operators on \( \mathcal{H} \). Suppose that \( \mathcal{E} \) is a \( \mathcal{C}^* \)-subalgebra of \( \mathcal{B}(\mathcal{H}) \) and that \( \phi \) is a \( \mathcal{C}^* \)-algebra homomorphism of \( \mathcal{E} \) onto \( \mathcal{C}(X) \) whose kernel is contained in the ideal \( \mathcal{K}(\mathcal{H}) \) of compact operators on \( \mathcal{H} \). If \( T \in \mathcal{E} \) and \( \phi(T) \) is an invertible continuous function on \( X \), then \( T \) is a Fredholm operator on \( \mathcal{H} \). For if \( R \in \mathcal{E} \) and \( \phi(R) = \phi(T)^{-1} \), then \( RT - I \) and \( TR - I \) are compact operators. It may
be that $T$ is a compact perturbation of an invertible operator. However, this is not possible when the index of $T$ is nonzero.

As in [7, 8, 9, 28], these are interesting circumstances to be in. Classical Toeplitz operators in one or more complex variables and pseudodifferential operators of order 0 provide basic examples of this type of situation. From the point of view of this theory, it is very natural to consider Toeplitz operators associated to multiplication by scalar-valued functions in the context of complexified Clifford analysis. One can also compress these Toeplitz operators to smaller spaces of Clifford-holomorphic functions.

9 Abstract elliptic operators

Let $X$ be a compact metric space equipped with a positive Borel measure $\mu$. As in [3], it is interesting to look at bounded linear operators on $L^2(X)$ whose commutators with multiplication by continuous functions on $X$ are compact. In particular, Fredholm operators of this type generalize classical elliptic pseudodifferential operators of order 0 on compact smooth manifolds. One can also consider operators acting on vector-valued functions, and other extensions, as explained in [3].

The Hilbert transform on the unit circle is a basic example. Fractional integral operators of imaginary order are also examples. These may be given as imaginary powers of unbounded nonnegative self-adjoint operators, like Laplace operators.

10 Fredholm indices

As in [3], Fredholm operators with compact commutators with operators of multiplication by continuous functions lead to other Fredholm operators, using vector bundles on the underlying metric space. Thus one gets indices for all of these operators. One also gets Fredholm operators and hence indices associated to invertible continuous complex-valued functions in the context of Section 8. This can be extended to matrix-valued functions, since matrices of complex functions lead to matrices of operators which can be interpreted as operators on other spaces. Of course, these two theories are closely related to each other. In connection with Toeplitz-type operators defined using Clifford analysis, one should be careful about the precise domains and ranges of the operators. At any rate, nontrivial indices can be a good indication of the presence of significant geometric or other structure.

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