ON THE SYMMETRY STRUCTURE OF THE MINIMAL SURFACE EQUATION

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Abstract. An infinite sequence of commuting nonpolynomial contact symmetries of the two-dimensional minimal surface equation is constructed. Local and nonlocal conservation laws for \( n \)-dimensional minimal area surface equation are obtained by using the Noether identity.

Introduction. In this paper, we analyze the symmetry properties of the Euler equation

\[ \mathcal{E}_{\min, \Sigma} = \left\{ E_u \left( \left( 1 + (\text{grad} u)^2 \right)^{1/2} \, dx \right) = 0 \right\} \]

whose solutions describe the minimal area \( n \)-dimensional surfaces \( \Sigma = \{ x^0 = u(x^1, \ldots, x^n) \} \subset \mathbb{R}^{n+1} \). We obtain variational conservation laws (either local or nonlocal) for the equation \( \mathcal{E}_{\min, \Sigma} \) in case \( n \geq 1 \) is arbitrary and construct a denumerable set of nonpolynomial contact symmetries of the minimal surface equation if \( n = 2 \) (note that the equation \( \mathcal{E}_{\min, \Sigma} \) is the zero mean curvature equation in this case). We use the nonparametric representation of the minimal surfaces \( \Sigma \) such that at any regular point within an open domain in the ambient manifold \( \mathbb{R}^{n+1} \) there is the fibre bundle \( \pi: \mathbb{R}^{n+1} \to \mathbb{R}^n \) and any surface \( \Sigma \) is the graph of a certain section in this bundle. The minimal surface equation \( \mathcal{E}_{\min, \Sigma} \) is a restriction upon these sections. The latter equation itself is a submanifold \( \mathcal{E}_{\min, \Sigma} \subset J^2(\pi) \) of the second jet space of the fibre bundle \( \pi \), see [3] for notation, definitions, and details. From now on, we assume that the symmetries, conservation laws, and other structures are restricted onto the infinite prolongation \( \mathcal{E}_{\min, \Sigma}^{\infty} \subset J^\infty(\pi) \); the superscript \( \infty \) will be omitted by default.

The analytic and topological properties of the minimal surfaces (i.e., either the minimal area surfaces of the zero mean curvature surfaces) are considered in the vastest literature ([8]). Recently, classical results and concepts on this topic were enlarged by their applications in mathematical physics (e.g., the harmonic maps theory and the \( \sigma \)-models, see [4]). Nevertheless, the symmetry structure of the minimal surface equation itself remains rather unveiled. In the paper [2], seven polynomial symmetries of the two-dimensional equation \( \mathcal{E}_{\min, \Sigma} \) (namely, the shift, two translations, three rotations, and the dilatation) were obtained — and a denumerable set of nonpolynomial contact symmetries ([6]) was missed at all.

The aim of this paper is to fill in the apparent gap in description of the symmetry algebra for the two-dimensional minimal surface equation. Also, we reconstruct several
conservation laws for the $n$-dimensional equation $\mathcal{E}_{\min, \Sigma}$. The knowledge of local and nonlocal conservation laws for the minimal surface equation is expected to contribute to further studies of the recursion operators for (not necessarily local part of) the symmetry algebra of this equation.

1. **Conservation Laws of the $n$-Dimensional Equation**

First, we recall known symmetries of the minimal area surface equation.

**Proposition 1** ([2]). *The generators of the Lie algebra of point symmetries for the two-dimensional minimal surface equation

$$\mathcal{E}_{\min, \Sigma} = \{(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0\}$$

are

$$\begin{align*}
\varphi_1 &= 1 \\
\varphi_2^i &= u_x^i \\
\varphi_3^{ij} &= y u_x^i - x u_y^j \\
\varphi_3^i &= x^i + u u_x^i \\
\varphi_4 &= u - \langle \text{grad } u, x \rangle.
\end{align*}$$

*Remark 1.* The sections $\varphi_1$, $\varphi_2^i$, $\varphi_3^{ij}$, $\varphi_3^i$, and $\varphi_4 = u - \langle \text{grad } u, x \rangle$ are symmetries of the $n$-dimensional minimal area surface Euler equation $\mathcal{E}_{\min, \Sigma} = \{E_u(\mathcal{L}_{\min, \Sigma}) = 0\}$ if $n \geq 1$ is arbitrary and $1 \leq i, j \leq n$; here the Lagrangian is $\mathcal{L}_{\min, \Sigma} = [L_{\min, \Sigma} \, dx]$ and $L_{\min, \Sigma} = \sqrt{\sum(D_i)^2}.$

Also, we note that the sections $\varphi_1$, $\varphi_2^i$, $\varphi_3^{ij}$, and $\varphi_3^i$ generate the $\frac{1}{2}(n + 1)(n + 2)$-dimensional algebra of the Killing vector fields on the Euclidean space $\mathbb{R}^{n+1} \supset \Sigma$.

Now we assign continuity equations to the symmetries that preserve the Lagrangian.

*Remark 2.* The resulting conserved currents are either purely local or involve nonlocalities. This is owing to three possible ways for a Lagrangian $L$ to be conserved w.r.t. the symmetries of the corresponding Euler equation $\mathcal{E}_E$:

- If the functional $\mathcal{L}$ is conserved on the whole jet space $J^\infty(\pi) \supset \mathcal{E}_E$ along a symmetry $\varphi \in \text{sym } \mathcal{E}_E$, then $\varphi$ is a *Noether symmetry* of the Lagrangian $\mathcal{L}$ and the local conservation law is assigned to $\varphi$ by the first Noether theorem ([1], [3]).

Then we use the following lemmas.

**Lemma 2** (Noether’s identity, [5], [7]). *Let $\mathcal{E}_E = \{F \equiv E_u(\mathcal{L}) = 0\}$ be the Euler equation assigned to a Lagrangian $\mathcal{L} = [L \, dx]$ and let $\varphi$ be a Noether symmetry: $\mathcal{E}_{\varphi}(\mathcal{L}) = d_h(\mu)$, where $d_h = \sum_{i=1}^n dx^i \otimes D_i$ is the standard horizontal differential. Then $\varphi$ is the generating section of the conservation law $[\omega] = [\mu - \nu - \lambda]$: $d_h(\omega) = \Box(F) \, dx$ such that $\varphi = \Box^*(1)$; here $\mathcal{E}_{\varphi}(\mathcal{L}) = \langle \varphi, F \rangle + d_h(\nu)$ and $\langle 1, \Box(F) \rangle = \langle \varphi, F \rangle + d_h(\lambda)$.

*Proof.* We have

$$d_h(\mu) = \mathcal{E}_\varphi(\mathcal{L}) = \langle 1, \ell^*_L(\varphi) \rangle = \langle \ell^*_L(1), \varphi \rangle + d_h(\nu),$$

where $\ell^*_L(1) = E_u(\mathcal{L})$ by definition. Therefore, $d_h(\mu - \nu) = \langle 1, \Box(F) \rangle + d_h(\lambda)$ such that $\varphi = \Box^*(1)$.\qed
Lemma 3 ([5]). In local coordinates, the Noether identity is

\[ \Theta_\varphi = \varphi \cdot E_u + \sum_{i=1}^{n} D_i \circ Q_{\varphi,i}, \]  

(3)

where

\[ Q_{\varphi,i} = \sum_{\tau} \sum_{\rho+\eta=\tau} (-1)^{\eta} D_\rho(\varphi) \cdot D_\eta \circ \partial/\partial u_{\tau+1}. \]

Proof. Fix a multi-index \( \sigma \); then the coefficient of \( \partial/\partial u_{\sigma} \) in the r.h.s. of Eq. (3) is

\[ \varphi \cdot (-1)^{\sigma} D_\sigma + \sum_{i=1}^{n} D_i \circ \left[ \sum_{\tau+1_i=\sigma} \sum_{\rho+\eta=\tau} (-1)^{\eta} D_\rho(\varphi) \cdot D_\eta \right] = \]

\[ = \varphi \cdot (-1)^{\sigma} D_\sigma + \sum_{\rho'+\eta=\sigma \atop |\rho'|>0} (-1)^{\eta} D_\rho(\varphi) \cdot D_{\eta'} = \]

\[ = \varphi \cdot (-1)^{\sigma} D_\sigma + D_\sigma(\varphi) - (-1)^{\sigma} \varphi \cdot D_\sigma = D_\sigma(\varphi). \]

This completes the proof. \( \square \)

- If \( \mathcal{L} \) is conserved on the equation \( \mathcal{E}_{EL} \) only (i.e., \( [\Theta_\varphi(\mathcal{L})] = \text{const} \cdot \mathcal{L} \)), then \( \varphi \) is a variational symmetry of \( \mathcal{E}_{EL} \) and the conservation law reconstructed by \( \varphi \) involves nonlocalities.

Indeed, we have

\[ d_h(\mu) + \text{const} \cdot \mathcal{L} = \Theta_\varphi(\mathcal{L}) = \langle E_u(\mathcal{L}), \varphi \rangle + d_h(\nu) = \langle 1, \Box(F) \rangle + d_h(\nu + \lambda), \]

where \( \varphi = \Box^s(1) \). Therefore, if a set of nonlocalities \( s \) trivializes the Lagrangian: \( d_h(\eta) = \mathcal{L} \), here we put \( \eta = \sum_i s^i \cdot (\partial/\partial x^i) \, d\mathbf{x} \), then we obtain a nonlocal conservation law \( \Omega = \mu + \eta - \nu - \lambda \) whose generating section is \( \varphi \). Generally, the compatibility conditions \( s_{x^j x^k} = s_{x^j x^k} \) determine an infinite set of new variables.

- Finally, if the Lagrangian \( \mathcal{L} \) is not conserved neither on the jet space \( J^\infty(\pi) \) nor on the equation \( \mathcal{E}_{EL} \), then there is no conservation law for the symmetry \( \varphi \).

We obtain the following assertion.

Proposition 4.

(1) Under the notation of Proposition 1 and Remark 1, the point symmetries \( \varphi_1, \varphi_2, \) and \( \varphi_3^{ij} \) are Noether’s symmetries of the Lagrangian \( \mathcal{L}_{\text{min}} \). Indeed, the equality \( [\Theta_{\varphi_k}(\mathcal{L}_{\text{min}})] = 0 \) holds on the jet space \( J^\infty(\pi) \) for \( 1 \leq k \leq 3 \). The conserved currents assigned to these symmetries are such that the corresponding continuity equations are, respectively,

\[ \sum_{i=1}^{n} \bar{D}_i(\partial L/\partial u_{x^i}) = 0, \]

\[ \bar{D}_i(u_{x^i} \cdot \partial L/\partial u_{x^i} - L) + \sum_{j \neq i} \bar{D}_j(u_{x^j} \cdot \partial L/\partial u_{x^j}) = 0, \]

\[ \bar{D}_i(x^j L - \varphi_3^{ij} u_{x^j}/L) - \bar{D}_j(x^i L + \varphi_3^{ij} u_{x^i}/L) - \sum_{k \neq i,j} \bar{D}_k(\varphi_3^{ij} \cdot u_{x^k}/L) = 0. \]
(2) The dilatation $\varphi_4$ preserves the Lagrangian $L_{\min \Sigma}$ on the equation $E_{\min \Sigma}$:
$$\left[ \mathcal{D} \varphi_4 (L_{\min \Sigma}) \right] = n L_{\min \Sigma},$$
and hence
$$E_u (\mathcal{D} \varphi_4 (L_{\min \Sigma})) = n E_u (L_{\min \Sigma}) = 0 \quad \text{on} \quad E_{\min \Sigma} = \{ E_u (L_{\min \Sigma}) = 0 \}.$$

Let $s^i, 1 \leq i \leq n$, be nonlocal variables such that their derivatives be $s^i x^i = L_{\min \Sigma}$. Then the nonlocal continuity equation
$$\sum_i \tilde{D}_i \left( s^i - x^i L - \frac{u u_i}{\sqrt{1 + \sum_j u_j^2}} + \frac{u_i \cdot \sum_j x^j u_j}{\sqrt{1 + \sum_k u_k^2}} \right) = 0$$
is assigned to the symmetry $\varphi_4$.

The proof is straightforward.

2. CONTACT SYMMETRIES OF THE TWO-DIMENSIONAL EQUATION

From now on we assume $n = 2$. We claim that the symmetries $\varphi_1, \ldots, \varphi_4$ do not exhaust the set of generators of the contact symmetry Lie algebra for the minimal surface equation $E_{\min \Sigma}$, see Eq. (2). Indeed, we have

**Proposition 5.** The Lie algebra of contact symmetries for the minimal surface equation $E_{\min \Sigma}$ is generated by the solutions $\varphi(u_x, u_y)$ of the equation

$$(1 + u_x^2) \frac{\partial^2 \varphi}{\partial (u_x)^2} + 2 u_x u_y \frac{\partial^2 \varphi}{\partial u_x \partial u_y} + (1 + u_y^2) \frac{\partial^2 \varphi}{\partial (u_y)^2} = 0$$

(e.g., the shift $\varphi_1 = 1$ and the translations $\varphi_2^i = u_x^i$ are solutions to Eq. (4)) and the sections $\varphi_3^1, \varphi_3^2$, and $\varphi_4$; here $i = 1, 2$.

The proof is elementary by using the analytic transformations software ([7]) and therefore omitted. We also note that Eq. (4) and its solutions except the point symmetries $\varphi_1$ and $\varphi_2^i$ were missed in [2].

Now we construct a denumerable set ([6]) contact symmetries of the minimal surface equation [2]. Consider Eq. (4) and suppose that $\varphi(u_x, u_y)$ is polynomial in $u_y$ (of course, all reasonings are preserved by the symmetry transformation $x \leftrightarrow y$). Then we obtain two distinct cases: the degree $K$ of the polynomial at hand can be either even ($K = 2k$) or odd ($K = 2k + 1$). In what follows, we treat these cases separately.

**Case 1 ($K = 2k$).** We assume $\varphi_{\text{even}} = \sum_{\ell=0}^k f_{\ell} (u_x) u_{y}^{2\ell}$. Then from Eq. (4) we obtain the following chain of equations:

1. The homogeneous equation

$$(1 + u_x^2) f_k'' + 4 k u_x f_k' + 2 k (2k - 1) f_k = 0$$
at the highest power $u_{y}^{2k}$.

2. The nonhomogeneous equations

$$(1 + u_x^2) f_{\ell}'' + 4 \ell u_x f_{\ell}' + 2 \ell (2\ell - 1) f_{\ell} = -(2\ell + 2)(2\ell + 1) f_{\ell+1}$$
at the intermediate powers $u_{y}^{2\ell}, 1 \leq \ell < k$; here the homogeneous components are Eq. (5) with the index shift.
(3) The terminal relation at the zero power $u_0^0$:

$$(1 + u_x^2) f_0'' = -2f_1. \quad (7)$$

The solutions of system (5–7) are the following.

1. The solution of homogeneous equation (5) is

$$f_k(u_x) = C_1 \cdot (i + u_x)^{−2k+1} + C_2 \cdot (−i + u_x)^{−2k+1},$$

where the constants $C_1, C_2$ are arbitrary. The equivalent basis of real solutions to Eq. (5) is

$$f_1^k = (1 + u_x^2)^{−2k+1} \cdot \frac{1}{2k-1} \sum_{\ell=0}^{k-1} (-1)^\ell \left(\frac{2k-1}{2\ell}\right) u_x^{2k-2\ell-1},$$

$$f_2^k = (1 + u_x^2)^{−2k+1} \cdot \frac{1}{2\ell-1} \sum_{\ell=1}^{k} \left(\frac{2k-1}{2\ell}\right) u_x^{2k-2\ell}.$$

2. The solutions to the intermediate nonhomogeneous equations (6) are

$$f_\ell(u_x) = \left(2\ell + 1\right) \cdot \int \frac{f_{\ell+1}^1 f_\ell^2 \, du_x}{f_\ell^1 (f_{\ell+1}^2)' - (f_\ell^1)' f_\ell^2} - f_\ell^2(u_x) \cdot \int \frac{f_{\ell+1}^1 f_\ell^1 \, du_x}{f_\ell^1 (f_{\ell+1}^2)' - (f_\ell^1)' f_\ell^2}.$$

All integration constants can be omitted since they originate from the homogeneous equations that provide the symmetries polynomial in $u_y$ whose degrees are less than $K = 2k$.

3. The quadrature for terminal equation (7) is

$$f_0(u_x) = -2 \int_{u_x}^{u_y} d\xi \int_{\xi}^{\eta} f_1(\eta) \, d\eta + \alpha u_x + \beta, \quad \alpha, \beta \in \mathbb{R}.$$

The coefficients of the integration constants $\alpha, \beta$ provide the translation $u_x$ and the shift $1 \in \text{sym} \mathcal{E}_{\text{min}}$, respectively.

Case 2 ($K = 2k + 1$). Now we have $\varphi_k^{\text{odd}} = \sum_{\ell=0}^{k} g_\ell(u_x) u_y^{2\ell+1}$. From Eq. (11) we obtain the chain of equations:

1. The homogeneous equation

$$(1 + u_x^2) g_k'' + 2(2k + 1)ku_x g_k' + 2k(2k + 1) g_k = 0 \quad (8)$$

at the highest power $u_y^{2k+1}$.

2. The nonhomogeneous equations

$$(1 + u_x^2) g_\ell'' + 2(2\ell + 1)u_x g_\ell' + 2\ell(2\ell + 1) g_\ell = -(2\ell + 3)(2\ell + 2) g_{\ell+1} \quad (9)$$

at the intermediate powers $u_y^{2\ell+1}, 1 \leq \ell < k$; again, the homogeneous components are Eq. (3) with the index shift.

3. The terminal relation at $u_y$:

$$(1 + u_x^2) g_0'' + 2u_x g_0' = -6g_1. \quad (10)$$

The solutions of system (8–10) are the following.
(1) The solution of homogeneous equation (8) is
\[ g_k(u_x) = C_3 \cdot (1 + u_x^2)^{-k} \cdot \left( \frac{i + u_x}{-i + u_x} \right)^k + C_4 \cdot (1 + u_x^2)^{-k} \cdot \left( \frac{-i + u_x}{i + u_x} \right)^k, \]
where \( C_3, C_4 \) are arbitrary constants. The equivalent basis of real solutions to Eq. (8) is
\[ g_k^1 = (1 + u_x^2)^{-2k} \cdot \sum_{\ell=0}^{k-1} (-1)^\ell \binom{2k}{2\ell} u_x^{2k-2\ell}, \]
\[ g_k^2 = (1 + u_x^2)^{-2k} \cdot \sum_{\ell=1}^{k} \binom{2k}{2\ell-1} u_x^{2k-2\ell+1}. \]

(2) The solutions to the intermediate nonhomogeneous equations (9) are
\[ g_\ell(u_x) = (2\ell + 3)(2\ell + 2) \cdot \left\{ g_\ell^1(u_x) \cdot \int \frac{g_{\ell+1} g_\ell^2 \, du_x}{g_\ell^1(g_\ell^3) - (g_\ell^1)' g_\ell^2} - g_\ell^2(u_x) \cdot \int \frac{g_{\ell+1} g_\ell^1 \, du_x}{g_\ell^1(g_\ell^3) - (g_\ell^1)' g_\ell^2} \right\}. \]
Again, the integration constants can be omitted since they originate from the homogeneous equations that provide the symmetries which are polynomial in \( u_y \) and whose degrees are less than \( K = 2k + 1. \)

(3) The quadrature for terminal equation (10) is
\[ g_0(u_x) = -6 \int_{-\infty}^{u_x} \frac{d\xi}{1 + \xi^2} \int_{-\infty}^{\xi} g_1(\eta) \, d\eta + \gamma + \delta \arctan u_x, \quad \gamma, \delta \in \mathbb{R}. \]
The coefficients of the constants \( \gamma, \delta \) are the translation \( u_y \) and the first non-polynomial symmetry \( \varphi_5 = u_y \arctan u_x, \) respectively.

The above reasonings provide two sequences of the symmetries \( \varphi_k^{\text{even, odd}} \in \text{sym} \mathcal{E}_{\text{min}} \) that are polynomial in \( u_y \) and nonpolynomial (except three starting terms) in \( u_x. \)

**Example 1.** The initial terms of these sequences are
\[ \varphi_1 = 1, \quad \varphi_2^1 = u_x, \quad \varphi_2^2 = u_y, \quad \varphi_5 = u_y \arctan u_x, \]
\[ \varphi_6 = \frac{u_x u_y^2}{1 + u_x^2} + \arctan u_x, \quad \varphi_7 = \frac{u_y^2}{1 + u_x^2} - u_x \arctan u_x, \]
\[ \varphi_8 = \frac{u_x u_y^3}{(1 + u_x^2)^2} + \frac{3}{2} \frac{u_x u_y}{1 + u_x^2}, \quad \varphi_9 = \frac{u_x^2 - 1}{(1 + u_x^2)^2} \cdot u_y^3 - \frac{3u_y}{1 + u_x^2}. \]

**Remark 3.** Analogous reasonings can be applied to the formal power series solutions
\[ \varphi_k^{\text{even}} = \sum_{\ell \leq k} f_\ell(u_x) u_y^{2\ell}, \quad \varphi_k^{\text{odd}} = \sum_{\ell \leq k} g_\ell(u_x) u_y^{2\ell+1}, \]
where \( k > 0. \) The symmetries \( \varphi_k^{\text{even}} \) are therefore polynomial in \( u_y^{-1}. \)

**Remark 4.** There are two different types of recursion operators that act on the contact symmetries \( \varphi_k(u_x, u_y). \) Let \( k \geq 0, \ i = 1, 2, \) and suppose \( \varphi_{k,i} \) is the symmetry such that \( f_k^i \) (resp., \( g_k^i \)) is the coefficient of the highest power of \( u_y. \) Then we have
• the operator $\Delta: \varphi_{k,1} \mapsto \varphi_{k,2}$ that swaps the elements of the basis if the highest power that depends on $k$ is fixed;
• the operator $\nabla: \varphi_{k,i} \mapsto \varphi_{k+1,i}$ that proliferates the symmetry heads along the sequence by the rules

\[
\nabla^{\text{even}}: f_k^1 \mapsto \frac{1}{(i + u_x)^2} f_k^1 = f_{k+1}^1, \quad \nabla^{\text{even}}: f_k^2 \mapsto \frac{1}{(i + u_x)^2} f_k^2 = f_{k+1}^2,
\]
\[
\nabla^{\text{odd}}: g_k^1 \mapsto \frac{1}{(i + u_x)^2} g_k^1 = g_{k+1}^1, \quad \nabla^{\text{odd}}: g_k^2 \mapsto \frac{1}{(i + u_x)^2} g_k^2 = g_{k+1}^2.
\]

**Lemma 6.** Assume that $\varphi'(u_x, u_y)$ and $\varphi''(u_x, u_y)$ are the generating sections of evolutionary vector fields $\mathcal{E}_\varphi$ and $\mathcal{E}_\varphi'$. Then their Jacobi bracket $\{\varphi', \varphi''\}$ is always trivial.

**Proof.** In local coordinates, the Jacobi bracket $\{\varphi', \varphi''\} = \mathcal{E}_\varphi'(\varphi'') - \mathcal{E}_\varphi''(\varphi')$ is

\[
\{\varphi', \varphi''\} = \frac{\partial \varphi'}{\partial u_x} u_{xx} \cdot \frac{\partial \varphi''}{\partial u_x} + \frac{\partial \varphi'}{\partial u_y} u_{xy} \cdot \frac{\partial \varphi''}{\partial u_x} + \frac{\partial \varphi'}{\partial u_x} u_{xy} \cdot \frac{\partial \varphi''}{\partial u_y} + \frac{\partial \varphi'}{\partial u_y} u_{yy} \cdot \frac{\partial \varphi''}{\partial u_y} - \text{v. v.} = 0.
\]

\[\square\]

**Corollary 7.** The contact symmetries $\varphi_k(u_x, u_y)$ of the minimal surface equation $\mathcal{E}_{\min, \Sigma}$ commute.

**Proposition 8.** Suppose that a minimal surface $\Sigma$ is invariant w.r.t. a contact symmetry $\varphi(u_x, u_y)$. Then $\Sigma$ is a plane.

**Proof.** Consider the constraint $\varphi = 0$. By the implicit function theorem, we have $u_y = \phi(u_x)$ almost everywhere. Therefore $u_{yy} = (\phi'(u_x))^2 \cdot u_{xx}$ and from Eq. (2) we obtain the equation

\[
\left(1 + u_x^2\right) \cdot (\phi'(u_x))^2 - 2u_x \phi(u_x) \cdot \phi'(u_x) + (1 + \phi^2(u_x)) \right) \cdot u_{xx} = 0.
\]

Hence either $u_{xx} = 0$ and we have $u_x = \text{const}$, $u_y = \phi(u_x) = \text{const}$, or $u_x$ is subject to the algebraic equation such that its solutions are $u_x = \text{const}$ and therefore $u_y = \phi(u_x) = \text{const}$ again. □

We conjecture that none of the contact symmetries $\varphi_k$, $k \geq 5$ (see Example [1]), are Noether.

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