Tunneling between chiral magnets: Spin current generation without external fields

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Magnons can generate a spin current, and the standard generating mechanism requires at least one external field. Since this mechanism is often applied to a multilayer system including a magnet and a paramagnetic metal, the system can possess not only the charge current induced by the spin current but also the charge current induced by the external field. The latter is an unnecessary accompaniment. Here we show that the tunneling of a magnon pair between chiral magnets can generate a spin current even without external fields. This phenomenon originates from a phase difference between magnon pairs of separate, weakly coupled chiral magnets, and is essentially different from the mechanism using the angle degree of freedom of the magnon Bose-Einstein condensates. The pair’s tunneling is possible in chiral magnets due to lack of the Goldstone-type gapless excitations. This phenomenon opens the door to spintronics not requiring any external field and using the magnon pair tunneling.

I. INTRODUCTION

Spin transport phenomena use a spin current\[\textsuperscript{1}\]. The spin current is a flow of the spin angular momentum. Typically, its carriers for metals or semiconductors are conducting electrons, and the carriers for insulators are magnons; magnons are bosonic quasiparticles describing collective excitations in a magnet, a magnetically ordered insulator\[\textsuperscript{2}\]. The spin current for magnets can often flow over a larger distance than for metals or semiconductors\[\textsuperscript{3}\]. In addition, magnets have another advantage: charge transport is never accompanied due to lack of charge degree of freedom. Because of those advantages, the spin transport phenomena using a magnet have been studied extensively\[\textsuperscript{4}\]–\[\textsuperscript{14}\].

To generate the spin current in a magnet, we often apply an external field. For example, we consider a ferromagnet\[\textsuperscript{7,13}\]. If an external magnetic field fulfills the resonance conditions\[\textsuperscript{15}\], it causes precession of the magnetization, and then this precession induces the spin current. This spin current is detectable, for example, by measuring the voltage due to the inverse spin Hall effect in a bilayer system of a ferromagnet and a metal\[\textsuperscript{2}\]. This is because the spin current in the ferromagnet layer pumps the spin current in the metal layer, and the latter can cause the charge current perpendicular. An external field is necessary even for an antiferromagnet\[\textsuperscript{16}\].

The standard method of generating the spin current in a magnet possesses an unnecessary accompaniment. The standard method uses a multilayer system and at least one external field; the multilayer system includes at least one magnet layer and one metal layer; the external field is applied to the whole system. Thus, the external field can induce not only spin transport but also charge transport in the metal layer; this charge transport is distinct from the charge transport induced by the spin current. For example, in a bilayer system, the external magnetic field can induce the charge Hall effect in combination with the charge current induced by the spin current as long as that field is not parallel to the charge current. Furthermore, the accompanied charge transport may contribute to the final output. Actually, the voltage generation using a temperature gradient in a bilayer system includes both the contribution of the spin Seebeck effect in the magnet layer and inverse spin Hall effect in the metal layer\[\textsuperscript{8,14}\]. The electronic structure of the metal layer determines whether or not the latter is negligible, and whether the latter decreases or increases the total.

A method not requiring external fields may be useful to utilize an advantage of magnets, lack of charge transport, as much as possible in a multilayer system. This is because such a method is free from charge transport accompanied by the external field.

Here we show that the tunneling between chiral magnets can generate the spin current without external fields (see Fig. \[\textsuperscript{1}\]). This originates from the tunneling of a magnon pair, which is present in chiral magnets but absent in ferromagnets and antiferromagnets. This phenomenon is analogous to the Josephson effect\[\textsuperscript{16,17}\], which originates from the tunneling of a Cooper pair.

II. BASIC CONCEPT

We start the basic concept of this phenomenon. The chiral magnet is a magnet having the spin scalar chirality, \[\langle \hat{S}_i \cdot (\hat{S}_j \times \hat{S}_k) \rangle\], where \(\hat{S}\) are spin operators, and \(i, j, k\) are site indices. This magnet also has finite three compo-
ments of $\langle S^x_k \rangle$ for $\alpha = x, y, z$, and the number of the finite components is invariant under global rotations in the spin space. This property results in a characteristic property of chiral magnets, the absence of the Goldstone-type gapless excitations of magnons, which holds even without magnetic anisotropy terms. This property contrasts with the presence in ferromagnets and antiferromagnets [compare Figs. 2(a) and 2(b)]; for those nonchiral magnets, the global rotation can change the number of the finite $\langle S^x_k \rangle$. (Here we consider only the chiral or nonchiral magnet which is most stable without quantum fluctuations for a certain model, i.e., do not consider the magnets which are not most stable, because the stable solution of the magnon properties can be obtained only for the most stable state.) No Goldstone-type gapless excitations in the absence of magnetic anisotropy terms mean the finite expectation values of a magnon pair because the creation and annihilation operators of a magnon pair in the Hamiltonian act like the operators of the superconducting gap [compare Eqs. (1) and (3)]. The magnon pair is analogous to a Cooper pair in a superconductor. This close relationship can be understood if we recall two properties: the superconductivity cannot be regarded as the BEC of Cooper pairs due to the Pauli principle of electrons, and the expectation cannot be regarded as the BEC of Cooper pairs due to the Pauli principle of electrons. The absence of the Goldstone-type gapless excitation is absent in panel (a), and present in panel (b). The Goldstone type gapless excitation is in panel (a), and present in panel (b).

### III. DERIVATION

To demonstrate the phenomenon predicted above, we derive the spin current due to the tunneling of a magnon pair in a trilayer system. This system consists of two layers of a chiral magnet and one intermediate layer of a paramagnetic metal or insulator or semiconductor, as shown in Fig. 1. We describe the tunneling in this system using the following Hamiltonian: $H = H_L + H_R + V$. $H_L$ and $H_R$ are the Hamiltonians of a chiral magnet, and $V$ is the tunneling Hamiltonian. For simplicity’s sake, we choose the linearized-spin-wave Hamiltonians as $H_L$ and $H_R$: $H_a = \sum q_s H_{LSW}(q_a) (a = L, R)$ with

$$H_{LSW}(q) = S \sum_{l,l'=1}^{N_{sub}} \left[A_{ll'}(q)b^*_q l b^*_q l' + A_{l'l'}^{*}(-q)b^*_q l b^*_q l' + B_{ll'}^{*}(-q)b^*_q l b^*_q l'\right] + S \sum_{l,l'=1}^{N_{sub}} \left[B_{ll'}^{*}(q)b^*_q l b^*_q l' + B_{l'l'}^{*}(q)b^*_q l b^*_q l'\right]. \quad (2)$$

Here $q_L$ and $q_R$ are momenta in the left and right layers, $S$ is the total spin per site, $l$ and $l'$ are sublattice indices, and $b^*_q l$ and $b^*_q l'$ are annihilation and creation operators of a magnon. If we compare Eqs. (2) and (1), we can see that the terms of $B_{ll'}(q)$ and $B_{l'l'}^{*}(q)$ correspond to the terms of $\Delta_k$ and $\Delta^*_k$. We can express any exchange interactions of a magnet as Eq. (2) by using the Holstein-Primakoff transformation if we neglect the terms of magnon-magnon interaction. It is however necessary to keep in mind that Eq. (2) should be derived for the most stable ordered state of a nonperturbative spin Hamiltonian without quantum fluctuations. Note that the linearized-spin-wave Hamiltonian for an antiferromagnet with two sublattices can be expressed without the $B_{ll'}(q)$ and $B_{l'l'}^{*}(-q)$ terms if we set $b^*_q l = a_q l$ for $l = A$ and $b^*_q l = b^*_q l$ for $l = B$. Then, we choose $V$ as

$$\hat{V} = 2S \sum_{q_L,q_R} \sum_{l=1}^{N_{sub}} \left[T_{q_L,q_R} l b^*_q l b^*_q l + T_{q_L,q_R}^{*} l b^*_q l b^*_q l\right]. \quad (3)$$

We assume that this term is finite and small, and then treat it as a second-order perturbation. This assumption
physically means that the intermediate layer fulfills two conditions: its width is so thin that the overlap between the wave functions of a magnon in the left and right layers is finite; the width is so thick that the energy scale of the overlap is smaller than that of exchange interactions.

The \( \hat{V} \) changes the magnon numbers in the left and right layers, and then induces the spin current through the intermediate layer. The changes of the magnon numbers are determined by the Heisenberg equations of motion, \( \dot{N}_a = (i/\hbar) [\hat{V}, \hat{N}_a] \) with \( \hat{N}_a = \sum_{q_{ln}} \hat{b}_{a q_{ln}}^\dagger \hat{b}_{a q_{ln}} \). The change for the right layer is given by

\[
\langle \dot{N}_R \rangle_V = -\frac{4S}{\hbar} \sum_{q_{ln}} \sum_{l, l'} \frac{1}{4\pi} \text{Im} [T_{q_{ln}} (\hat{b}_{l q_{ln}}^\dagger (t) \hat{b}_{l' q_{ln}} (t))] V
\]

\[
\approx \frac{4S}{\hbar} \sum_{q_{ln}} \sum_{l, l'} \int_{-\infty}^{t} dt' e^{i\alpha t'} \text{Re} [T_{q_{ln}} (\hat{b}_{l q_{ln}}^\dagger (t') \hat{b}_{l' q_{ln}} (t'))],
\]

(4)

The \( J_S^{(1)} \) and \( J_S^{(2)} \) originate from, respectively, the tunneling of a single magnon and tunneling of a magnon pair. This is because the former includes \( \langle \hat{b}_l^\dagger \hat{b}_l \rangle \) and \( \langle \hat{b}^\dagger \hat{b} \rangle \), and the latter includes \( \langle \hat{b}_l^\dagger \hat{b}_{l'}^\dagger \rangle \) and \( \langle \hat{b}_{l'}^\dagger \hat{b}_l \rangle \). Hereafter, we consider only the \( J_S^{(2)} \) because the single-magnon’s tunneling is possible even for nonchiral magnets, and the pair’s tunneling is possible only for chiral magnets. Furthermore, the analogy with the Josephson effect suggests only the \( J_S^{(2)} \) can be finite even without external fields.

For further insight into the \( J_S^{(2)} \), we rewrite Eq. (4) in a simpler form. In a similar way for the Josephson effect, we can rewrite Eq. (4) in terms of the eigenvalues and eigenfunctions of \( \hat{H}_L \) and \( \hat{H}_R \) (for the details, see Appendix A):

\[
J_S^{(2)} = -\frac{16S^2}{\hbar} \sum_{q_{ln}, q_{ln}} \sum_{l, l'} \sum_{l, l'} \left| T_{q_{ln}} \right|^2 \times P \left( \frac{n_{\epsilon_{l'} (q_{ln})} - n_{\epsilon_{l'} (q_{ln})} + n_{\epsilon_{l'} (q_{ln})}}{\epsilon_{l'} (q_{ln}) + \epsilon_{l'} (q_{ln})} \right) \times \text{Im} [P_{l'}^\dagger (q_{ln}) P_{l'}^\dagger (q_{ln})].
\]

(7)

This is similar to the charge current due to the tunneling of a Cooper pair except for the differences in the coefficients and quasiparticles; the quasiparticles are fermions in superconductors.

Due to the difference in quasiparticles, the spin current is zero at zero temperature. In Eq. (7), the eigenvalues, \( \epsilon_{l'} (q_{ln}) \), and eigenfunctions, \( P_{l'} (q_{ln}) \), are given by \( \epsilon_{l'} (q_{ln}) = \sum_{l, l'} [P_{l'}^\dagger (q_{ln}) A_{l'} (q_{ln}) P_{l'} (q_{ln}) + P_{l'}^\dagger (q_{ln}) B_{l'} (q_{ln}) P_{l'} (q_{ln})] + \sum_{l, l'} [P_{l'}^\dagger (q_{ln}) A_{l'} (q_{ln}) P_{l'} (q_{ln}) + P_{l'}^\dagger (q_{ln}) B_{l'} (q_{ln}) P_{l'} (q_{ln})] \), and \( n_{\epsilon_{l'} (q_{ln})} \) is the Bose-Einstein distribution function at temperature \( T \), \( n_{\epsilon_{l'} (q_{ln})} = (e^{\frac{\epsilon_{l'} (q_{ln})}{k_B T}} - 1)^{-1} \). Because of \( P_{l'} (q_{ln}) P_{l'} (q_{ln}) = \langle \nu | \hat{b}_{q_{ln}}^\dagger \hat{b}_{q_{ln}} | \nu \rangle = C_{q_{ln}} e^{i\phi_{q_{ln}}' |l' \rangle \nu} \) and \( P_{l'}^\dagger (q_{ln}) P_{l'}^\dagger (q_{ln}) = \langle \nu | \hat{b}_{q_{ln}}^\dagger \hat{b}_{q_{ln}} | \nu \rangle = C_{q_{ln}} e^{-i\phi_{q_{ln}}' |l' \rangle \nu} \), we can express Eq. (7) using the phase difference between the wave functions of a magnon pair in the left and right layers:

\[
J_S^{(2)} = \sum_{q_{ln}, q_{ln}} \sum_{l, l'} J_{l'^l l'^l} (q_{ln}, q_{ln}) \sin (\phi_{q_{ln}} - \phi_{q_{ln}}'),
\]

(8)

where \( J_{l'^l l'^l} (q_{ln}, q_{ln}) = -\frac{16S^2}{\hbar} \sum_{q_{ln}, q_{ln}} \sum_{l, l'} \left| T_{q_{ln}} \right|^2 [P_{l'}^\dagger (q_{ln}) - n_{\epsilon_{l'} (q_{ln})}] \langle \epsilon_{l'} (q_{ln}) - \epsilon_{l'} (q_{ln}) \rangle C_{q_{ln}} e^{-i\phi_{q_{ln}}' |l' \rangle \nu} \). This phase difference can induce the spin current without external fields.

IV. DISCUSSION

First, we discuss the validity of our theory. We used the linearized-spin-wave approximation for a general Hamil-
tonian of exchange interactions to treat magnons in a chiral magnet. This is valid if the ground state under the hard-spin constraint is nondegenerate, and if the temperature is so low that the interaction terms, neglected in this approximation, are negligible. The former condition is necessary because this approximation treats collective motions of spins as fluctuations against a most stable ground state under the hard-spin constraint. The latter is necessary because this approximation is a low-order expansion of the spin operators in terms of $\hat{b}\hat{b}^\dagger/2S$, which is small at low temperature. We also used Eq. (3) and its second-order perturbation to treat the tunneling between chiral magnets. This is valid if the left and right layers are weakly coupled due to the overlap between the wave functions of a magnon. Such weak coupling is probably reliable in experiments. Thus, our theory is valid for analyzing the tunneling between chiral magnets coupled weakly at low temperature.

Next, we argue that our mechanism is distinct from the mechanism for generating the spin current in the magnon BEC (e.g., Refs. [12]). In the magnon BEC, $(\hat{n}_q) = O(N^{1/2})$ is fulfilled due to coherence of the wave function. Such coherence is preserved even in the presence of the magnon-magnon interactions as long as the interaction-induced damping is negligible compared with temperature. To generate the spin current in the magnon BEC, the angle degree of freedom of $(\hat{b}_j)$ (or $(\hat{b}_j^\dagger)$), $\theta_j$, of $(\hat{b}_j) = B_je^{-i\theta_j}$, is used; the spin current is proportional to $\nabla\theta_j$. (For its example, see Appendix B.) While $(\hat{b}\hat{b}) = \langle \hat{b}\hat{b}\rangle$ for noninteracting magnons, $(\hat{b}\hat{b}) \neq \langle \hat{b}\hat{b}\rangle$ for interacting magnons. In particular, $(\hat{b}\hat{b})$ and $(\hat{b}\hat{b}^\dagger)$ can be finite even in the absence of the magnon BEC. Namely, our mechanism can generate the spin current not only at low temperature, where the magnon BEC is present, but also at high temperature, where the magnon BEC is absent. Thus, our mechanism and the mechanism using the magnon BEC are essentially different.

Then, we argue a method of testing our phenomenon by experiment. In our phenomenon, the spin current is generated in a multilayer system even without external fields. We can observe this spin current, for example, by using the inverse spin Hall effect in a paramagnetic metal of the intermediate layer in a similar way to Ref. [7]. This is because near the interface between C and P in Fig. 1 part of the magnon spin current can be converted into the electron spin current using the interfacial exchange interactions, and because chiral magnets can possess the non-degenerate magnon energy dispersion (i.e., we can avoid vanishing of the spin current due to the degeneracy of the magnon energies). However, there are several main differences between this and the standard phenomenon. Our phenomenon neither needs any external field nor has a charge current induced by it. As chiral magnets, we can use, for example, all-in/all-out, two-in-two-out, and three-in-one-out chiral magnets in pyrochlore oxides. While controlling the phase difference in our phenomenon is more difficult than controlling an external field, an analogy with the Josephson effect suggests that the phase difference can be finite using chiral magnets whose gap structures have different momentum dependence. (For the Josephson effect, one of the simple ways of obtaining the phase difference is to use superconductors whose gap structures are different, e.g., s-wave and d-wave superconductors.) Thus, our phenomenon is experimentally testable.

Finally, we discuss implications of our phenomenon. Our phenomenon provides a method of generating the spin current in an insulator. Moreover, we can extend the formulation to study the high-temperature properties. At high temperature, we need to consider the magnon-magnon interactions, which cause spin-Coulomb drag, a characteristic dissipation of a nonconserved quantity. Thus, our phenomenon provides an opportunity of studying the spin-Coulomb drag. Then, a similar phenomenon is possible even for chiral metals, magnetically ordered metals with the spin scalar chirality: the tunneling between chiral metals can generate the spin current without external fields. There is however a significant difference: chiral metals have spin and charge degrees of freedom. It is thus desirable to reveal roles of charge degrees of freedom in the tunneling between chiral metals. In addition, knowledge of the Josephson effect implies some research. It is known that the Josephson effect in the presence of external dc and ac voltages is useful to determine the value of $e/h$. Also, it is known that the Josephson effect in the presence of an external magnetic field is useful to distinguish a s-wave gap and a d-wave gap. Thus, our phenomenon in the presence of some external field may be useful to determine the value of $1/h$ (not including $e$) and the differences between the gap structures of different chiral magnets. Moreover, our phenomenon implies a similar phenomenon for phonons or photons. For example, the tunneling of a phonon pair may generate a charge current even without external fields.

V. SUMMARY

In summary, we have proposed that the tunneling of a magnon pair between chiral magnets can generate the spin current even without external fields. We presented this proposal by deriving the spin current tunneling through the intermediate layer of the trilayer system of Fig. 1. In this derivation, we treated magnons in a chiral magnet using the linearized-spin-wave approximation. We also treated the weak tunneling between chiral magnets using the second-order perturbation. Our treatments are valid for the tunneling between weakly coupled chiral magnets at low temperature. Our phenomenon opens the door to spintronics not requiring any external field and using the magnon pair tunneling.
Appendix A: Derivation of Eq. (7)

We explain the detail of the derivation of Eq. (7). We first explain a method of obtaining the eigenvalues and eigenfunctions of $H_L$ and $H_R$. Since its detailed explanations have been provided, for example, in Refs. [24] and [15] we here provide the brief explanation. We can diagonalize the Hamiltonian of a chiral magnet in the linearized-spin-wave approximation as follows:

$$
\hat{H}_a = S \sum_{q, l, l'} \sum_{\nu=1}^{\text{N}_{\text{sub}}} \left( \hat{b}_{q, l}^\dagger \hat{b}_{-q, l} \right) \left( \begin{array}{cc} A_{ll'}(q) & B_{ll'}(q) \\ B_{ll'}(-q) & A_{ll'}(-q) \end{array} \right) \left( \begin{array}{c} \hat{b}_{q, l} \\ \hat{b}_{-q, l}^\dagger \end{array} \right) 
+ \sum_{q, l, l'} \sum_{\nu=1}^{\text{N}_{\text{sub}}} \left( \hat{b}_{q, l}^\dagger \hat{b}_{-q, l} \right) \left( \begin{array}{c} \epsilon_{\nu}(q) \\ 0 \end{array} \right) \left( \begin{array}{c} \hat{b}_{q, l} \\ 0 \end{array} \right) 
+ \sum_{q, l, l'} \sum_{\nu=1}^{\text{N}_{\text{sub}}} \left( \hat{b}_{q, l}^\dagger \hat{b}_{-q, l} \right) \left( \begin{array}{c} 0 \\ \epsilon_{\nu+N_{\text{sub}}}(q) \end{array} \right) \left( \begin{array}{c} 0 \\ \hat{b}_{q, l} \end{array} \right).
$$

(A1)

Here $\hat{b}$ and $\hat{b}'$ are connected in the following equation:

$$
\left( \begin{array}{c} \hat{b}_{q, l} \\ \hat{b}_{q, l}^\dagger \end{array} \right) = \sum_{\nu=1}^{\text{N}_{\text{sub}}} \left( P_{ll'}(q) P_{l+N_{\text{sub}}l'}(q) P_{l+N_{\text{sub}}l'+N_{\text{sub}}}(q) \right) \left( \begin{array}{c} \hat{b}_{q, l} \\ \hat{b}_{q, l}^\dagger \end{array} \right),
$$

(A2)

where $P$ is a $(2N_{\text{sub}} \times 2N_{\text{sub}})$ paraunitary matrix. Note that paraunitary matrices should fulfill $P_y P_0 = g$, where $g$ is the $(2N_{\text{sub}} \times 2N_{\text{sub}})$ paraunitary matrix. Due to this property, the $P$ fulfills

$$
\delta_{l, l'} = \sum_{\nu=1}^{\text{N}_{\text{sub}}} P_{l, l'} P_{l, l'}^\dagger - \sum_{\nu=1}^{\text{N}_{\text{sub}}} P_{l, l'} P_{l+N_{\text{sub}}, l'+N_{\text{sub}}},
$$

(A3)

$$
0 = \sum_{\nu=1}^{\text{N}_{\text{sub}}} P_{l, l'} P_{l', l} P_{l, l'}^\dagger - \sum_{\nu=1}^{\text{N}_{\text{sub}}} P_{l, l'} P_{l+N_{\text{sub}}, l'+N_{\text{sub}}},
$$

(A4)

$$
0 = \sum_{\nu=1}^{\text{N}_{\text{sub}}} P_{l+N_{\text{sub}}, l}, P_{l', l+N_{\text{sub}}}, P_{l, l'}^\dagger - \sum_{\nu=1}^{\text{N}_{\text{sub}}} P_{l+N_{\text{sub}}, l'+N_{\text{sub}}}, P_{l, l'}^\dagger ,
$$

(A5)

$$
-\delta_{l, l'} = \sum_{\nu=1}^{\text{N}_{\text{sub}}} P_{l+N_{\text{sub}}, l, l'}^\dagger P_{l, l} - \sum_{\nu=1}^{\text{N}_{\text{sub}}} P_{l+N_{\text{sub}}, l', l}^\dagger P_{l, l+N_{\text{sub}}},
$$

(A6)

Combining Eqs. (A1) and (A2), we obtain the eigenvalues, $\epsilon_{\nu}(q)$, and the eigenfunctions, $P_{l, l'}(q)$:

$$
\epsilon_{\nu}(q) = \sum_{l, l'=1}^{\text{N}_{\text{sub}}} P_{l, l'}^\dagger(q) A_{ll'}(q) P_{l, l'}(q)
+ \sum_{l, l'=1}^{\text{N}_{\text{sub}}} P_{l, l'}^\dagger(q) B_{ll'}(q) P_{l, l'+N_{\text{sub}}}(q)
+ \sum_{l, l'=1}^{\text{N}_{\text{sub}}} P_{l, l'}^\dagger(q) A_{ll'}(q) P_{l, l'+N_{\text{sub}}}(q)
+ \sum_{l, l'=1}^{\text{N}_{\text{sub}}} P_{l, l'}^\dagger(q) B_{ll'}(q) P_{l+N_{\text{sub}}, l'}(q).
$$

(A7)

Note that $\epsilon_{\nu+N_{\text{sub}}}(q)$ is given by $\epsilon_{\nu+N_{\text{sub}}}(q) = \epsilon_{\nu}(q)$.

Then, we rewrite Eq. (6) by using the eigenvalues and eigenfunctions of $H_L$ and $H_R$. For that purpose, we take five steps. First, we express the magnon operators in Eq. (6) in terms of $\hat{b}'$ and $\hat{b}'^\dagger$. We obtain the expressions by using the relations,

$$
\hat{b}_{q, l}(t) = \sum_{\nu=1}^{\text{N}_{\text{sub}}} P_{l, l'}(q) b_{q, l}^\dagger e^{-i \epsilon_{\nu}(q)t} + \sum_{\nu=1}^{\text{N}_{\text{sub}}} P_{l+N_{\text{sub}}, l'}(q) b_{-q, l}^\dagger e^{i \epsilon_{\nu}(q)t},
$$

(A8)

$$
\hat{b}_{-q, l}^\dagger(t) = \sum_{\nu=1}^{\text{N}_{\text{sub}}} P_{l+N_{\text{sub}}, l}(q) b_{q, l}^\dagger e^{i \epsilon_{\nu}(q)t} + \sum_{\nu=1}^{\text{N}_{\text{sub}}} P_{l+N_{\text{sub}}, l'}(q) b_{q, l'}^\dagger e^{i \epsilon_{\nu}(q)t}.
$$

(A9)

Namely, by using the above equations, we obtain
Here we have used the relations, such as \( \langle \hat{b}^\dagger_{qL}(t) \hat{b}_{qR}(t') \rangle = \delta_{q,q'} \delta_{\nu,\nu'} n[\epsilon_\nu(q)] \). Second, we perform the integration about \( t' \) in Eq. (6). For the integration, it is sufficient to use the following results:

\[
\int_{-\infty}^{t} dt' e^{\alpha t'} e^{i[\epsilon_\nu(q_L)/\hbar - \epsilon_{\nu'}(q_R)/\hbar](t-t')} = i P \left( \frac{\hbar}{\epsilon_\nu(q_L) - \epsilon_{\nu'}(q_R)} \right) + \pi \delta \left( \frac{\epsilon_\nu(q_L) - \epsilon_{\nu'}(q_R)}{\hbar} \right),
\]

\[
-\int_{-\infty}^{t} dt' e^{\alpha t'} e^{-i[\epsilon_\nu(q_L)/\hbar - \epsilon_{\nu'}(q_R)/\hbar](t-t')} = i P \left( \frac{\hbar}{\epsilon_\nu(q_L) - \epsilon_{\nu'}(q_R)} \right) - \pi \delta \left( \frac{\epsilon_\nu(q_L) - \epsilon_{\nu'}(q_R)}{\hbar} \right),
\]

\[
-\int_{-\infty}^{t} dt' e^{\alpha t'} e^{i[\epsilon_\nu(q_L)/\hbar + \epsilon_{\nu'}(q_R)/\hbar](t-t')} = -i P \left( \frac{\hbar}{\epsilon_\nu(q_L) + \epsilon_{\nu'}(q_R)} \right) - \pi \delta \left( \frac{\epsilon_\nu(q_L) + \epsilon_{\nu'}(q_R)}{\hbar} \right),
\]

\[
\int_{-\infty}^{t} dt' e^{\alpha t'} e^{-i[\epsilon_\nu(q_L)/\hbar + \epsilon_{\nu'}(q_R)/\hbar](t-t')} = -i P \left( \frac{\hbar}{\epsilon_\nu(q_L) + \epsilon_{\nu'}(q_R)} \right) + \pi \delta \left( \frac{\epsilon_\nu(q_L) + \epsilon_{\nu'}(q_R)}{\hbar} \right).
\]

After the integration, we took the limit \( \alpha \to 0 \). Third, we combine Eq. (6) with Eqs. (A10)–(A13) and (A14)–(A17). As the result, Eq. (6) becomes...
the ferromagnetic Heisenberg interaction, and its spin state due to the nonperturbative Hamiltonian, such as to (\hat{\theta}_j) for the commensurate ferromagnetic order is proportional to \(\hat{b}_j + \hat{b}^\dagger_j\), this perturbation leads to \(\hat{b}_j \neq 0\), which is site-dependent. Thus, the perturbation induces the site-dependent angle of \(\hat{\theta}_j = B_j e^{-i\theta_j}\), and then \(\nabla \theta_j\) generates the spin current. This spin current generation using the magnon BEC is essentially the same as the mass flow generation in a superfluid. 

Appendix B: Example of the spin current generation using the magnon BEC

We explain an example of how to generate the spin current using the magnon BEC. As an example, we consider a ferromagnet. We first assume that the commensurate ferromagnetic order becomes the most stable ground state due to the nonperturbative Hamiltonian, such as the ferromagnetic Heisenberg interaction, and its spin structure is given by \(\langle \hat{S}_i^x \rangle = \langle \hat{S}_i^y \rangle = 0\) and \(\langle \hat{S}_i^z \rangle = S\) for all \(i\). Then, we consider a perturbation, such as an external magnetic field or the magnetic anisotropy, which induces

\[
J_S^{(2)} = -\frac{8S^2}{\hbar} \sum_{\mathbf{q}_L, \mathbf{q}_R} \sum_{l, l'}^{N_{\text{sub}}} \sum_{\nu, \nu'}^{N_{\text{sub}}} |T_{\mathbf{q}_L, \mathbf{q}_R}|^2 \left( n[\epsilon_\nu(\mathbf{q}_L)] - n[\epsilon_\nu(\mathbf{q}_R)] \right) P \left( \frac{1}{\epsilon_\nu(\mathbf{q}_L) - \epsilon_\nu'(\mathbf{q}_R)} \right) \times \left\{ \text{Im} \left[ P^\dagger_{\nu\nu}(\mathbf{q}_L) P_{\nu\nu + N_{\text{sub}}} (\mathbf{q}_R) P_{\nu'\nu + N_{\text{sub}}} (\mathbf{q}_R) \right] \right\},
\]

\[
\text{and Eq. (A5) shows}
\[
P_{\nu\nu}(\mathbf{q}) P_{\nu\nu + N_{\text{sub}}}(\mathbf{q}) = P_{\nu\nu + N_{\text{sub}}}^\dagger(\mathbf{q}) P_{\nu\nu + N_{\text{sub}}} (\mathbf{q}).
\]

Fifth, we substitute Eqs. (A19) and (A20) into Eq. (A18):

\[
J_S^{(2)} = -\frac{16S^2}{\hbar} \sum_{\mathbf{q}_L, \mathbf{q}_R} \sum_{l, l'}^{N_{\text{sub}}} \sum_{\nu, \nu'}^{N_{\text{sub}}} |T_{\mathbf{q}_L, \mathbf{q}_R}|^2 \text{Im} \left[ \left( P_{\nu\nu + N_{\text{sub}}}^\dagger(\mathbf{q}_L) P_{\nu\nu + N_{\text{sub}}}(\mathbf{q}_R) P_{\nu'\nu + N_{\text{sub}}}(\mathbf{q}_R) \right) \right] \times \left[ P \left( \frac{n[\epsilon_\nu(\mathbf{q}_L)] - n[\epsilon_\nu'(\mathbf{q}_R)]}{\epsilon_\nu(\mathbf{q}_L) - \epsilon_\nu'(\mathbf{q}_R)} \right) \right] \times \left[ P \left( \frac{1 + n[\epsilon_\nu(\mathbf{q}_L)] + n[\epsilon_\nu'(\mathbf{q}_R)]}{\epsilon_\nu(\mathbf{q}_L) + \epsilon_\nu'(\mathbf{q}_R)} \right) \right].
\]

This is Eq. (7) because the Bose-Einstein distribution function fulfills \(1 + n(\epsilon) + n(\epsilon') = n(\epsilon) - n(-\epsilon')\).
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