An FBSDE approach to market impact games with stochastic parameters

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Abstract

We analyze a market impact game between n risk averse agents who compete for liquidity in a market impact model with permanent price impact and additional slippage. Most market parameters, including volatility and drift, are allowed to vary stochastically. Our first main result characterizes the Nash equilibrium in terms of a fully coupled system of forward-backward stochastic differential equations (FBSDEs). Our second main result provides conditions under which this system of FBSDEs has indeed a unique solution, which in turn yields the unique Nash equilibrium. We furthermore obtain closed-form solutions in special situations and analyze them numerically.

1 Introduction

Market impact games analyze situations in which several agents compete for liquidity in a market impact model or try to exploit the price impact generated by competitors. In this paper, we follow Carlin et al. [6], Schöneborn and Schied [23], Carmona and Yang [7], Schied and Zhang [19], Casgrain and Jaimungal [8], and others by analyzing a market impact game in the context of the Almgren–Chriss market impact model. In [6, 23], all agents are risk-neutral and market parameters are constant, which leads to deterministic Nash equilibria. Deterministic open-loop equilibrium strategies are also obtained in [19], where agents maximize mean variance functionals or CARA utility. In [7] closed-loop equilibria are studied numerically in a similar setup, and it is found by means of simulations that then equilibrium strategies may no longer be deterministic. The approach in [8] is the closest to ours. There, the authors analyze the infinite-agent, mean-field limit of a market impact game for heterogeneous, risk-averse agents in a model with constant coefficients and partial information, and they characterize the mean-field game through a forward-backward stochastic differential equation (FBSDE). In addition, there are several papers that study market impact games in other price impact models, including models with linear transient price impact; see, e.g., [14, 20, 15, 13].

Our contribution to this literature is twofold. First, on the mathematical side, we completely solve the problem of determining an open-loop Nash equilibrium with stochastic model parameters and risk aversion for arbitrary numbers of agents. Our solution relies on a characterization of the equilibrium strategies in terms of a fully coupled systems of forward-backward stochastic differential equations (FBSDEs). This characterization is given in Theorem 4.1. In the subsequent Theorem 4.2, we give sufficient conditions that guarantee the existence of a unique solution. The main restriction is a lower bound on the volatility. Then we analyze the case of constant coefficients and the case in which all agents share the same parameters but have different initial inventories. Numerical simulations are provided for the case of constant coefficients which work for many agents.

Our second contribution consists in a modification of the traditional setup of the interaction term in a market impact game with Almgren–Chriss-style price impact. The Almgren–Chriss model has two price impact components, one permanent and one temporary. It is clear that permanent price impact must affect the execution prices of all agents equally, and in [6, 23, 7, 19, 8] the same is assumed of the transient price impact. This assumption can sometimes lead to counterintuitive results. For instance, if the temporary price impact is large in comparison with the permanent price impact, then, in the presence of a large seller, it can be beneficial to build up a long position in the stock, because a cessation of the trading activities of the large seller will lead to an immediate upwards jump of

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the expected price \[23\]. In the price impact literature, it is however not consensus that “temporary price impact” is of the same nature as permanent price impact. For instance, Almgren et al. \[3\] write about temporary impact:

This expression is a continuous-time approximation to a discrete process. A more accurate description would be to imagine that time is broken into intervals such as, say, one hour or one half-hour. Within each interval, the average price we realise on our trades during that interval will be slightly less favorable than the average price that an unbiased observer would measure during that time interval. The unbiased price is affected on previous trades that we have executed before this interval (as well as volatility), but not on their timing. The additional concession during this time interval is strongly dependent on the number of shares that we execute in this interval.

Likewise, Gatheral \[10, p. 751\] writes:

The second component of the cost of trading corresponds to market frictions such as effective bid-ask spread that affect only our execution price: We refer to this component of trading cost as slippage (temporary impact in the terminology of Huberman and Stanzl).

Based on these interpretations of “temporary price impact” as slippage, it appears to be more natural that only the trades of the executing agent and not the trades of the other market participants are affected by the resulting cost. In our paper, we therefore keep a term for “temporary price impact”, but it only affects the execution costs of the corresponding agent and not of the other agents.

The paper is organized as follows. In Section 2, we set up our model on portfolio liquidation in the Almgren-Chriss framework. Single agent optimization is studied in Section 3, where the corresponding existence, uniqueness and characterization results for the optimal liquidation strategy are stated. Section 4 is dedicated to present the characterization result for Nash equilibrium and investigates the solvability of the characterizing FBSDE. Some explicit solutions for Nash equilibria are analyzed in Section 5.

2 Preliminaries and problem formulation

2.1 Frequently used notation

Let \(W = (W_t)_{t \geq 0}\) be a \(d\)-dimensional Brownian motion on a probability space \((\Omega, \mathcal{F}, P)\) and denote by \((\mathcal{F}_t)_{t \geq 0}\) the complete filtration generated by \(W\). Throughout, we fix a finite time horizon \(T > 0\). We endow \(\Omega \times [0, T]\) with the predictable \(\sigma\)-algebra \(\mathcal{P}\) and \(\mathbb{R}^n\) with its Borel \(\sigma\)-algebra \(\mathcal{B}(\mathbb{R}^n)\). Equalities and inequalities between random variables and processes are understood in the \(P\)-a.s. and \(P \otimes dt\)-a.e. sense, respectively. The Euclidean norm is denoted by \(|\cdot|\). For \(m \in [1, \infty]\) and \(k \in \mathbb{N}\), we denote by \(\|Y\|_{S^m(\mathbb{R}^k)} := \sup_{0 \leq t \leq T} |Y_t|^m < \infty\), and by \(\|Z\|_{H^m(\mathbb{R}^k)} = \left\| \left( \int_0^T |Z_s|^2 ds \right)^{\frac{1}{2}} \right\|_m < \infty\),

The space \(\text{BMO}(\mathbb{R}^k)\) consists of all predictable \(\mathbb{R}^k\)-valued processes \(Z\) such that

\[
\|Z\|_{\text{BMO}(\mathbb{R}^k)} = \sup_{\tau \in \mathcal{T}} \left\| \mathbb{E} \left[ \left( \int_\tau^T |Z_s|^2 ds \right)^{\frac{1}{2}} \right| \mathcal{F}_\tau \right\|_\infty < \infty
\]

where \(\mathcal{T}\) is the set of all stopping times with values in \([0, T]\).

2.2 Model setup

We consider \(n\) financial agents who are active in a financial market of Almgren–Chriss-type and whose trading strategies interact via permanent price impact. More precisely, we adapt the continuous-time setting of \(\Pi\), where
each agent $i$ has initial inventory $Q^i_0$ at time $t = 0$ and subsequently uses a trading strategy whose trading rate is given by a process $q^i \in H^2(\mathbb{R})$. That is, at time $t \in [0, T]$, the inventory of agent $i$ is given by

$$Q^i_t = Q^i_0 + \int_0^t q^i_s ds.$$  

This trading strategy impacts the price of the risky asset by means of permanent price impact. It is usually assumed that this permanent price impact is linear in the traded inventory (see, e.g., the discussion in Section 3 of [11]). Thus, we assume that the price at which shares of the risky assets can be traded at time $t$ is given by

$$S^i_t = S_0 + \int_0^t \mu_s ds + a \sum_{i=1}^n \int_0^t q^i_s ds + \int_0^t \sigma_s dW_s,$$  \hspace{1cm} (2.1)  

where $\mu \in S^\infty(\mathbb{R})$ is a generic drift, $\sigma \in S^\infty(\mathbb{R}^d)$ is a volatility process, and, for a fixed price impact parameter $a > 0$, the term $a \sum_{i=1}^n \int_0^t q^i_s ds$ describes the cumulative price impact generated by the strategies of all agents.

At time $t$, the $i^{th}$ agent sells $-q^i_t$ dt shares at price $S^i_t$. The implementation shortfall, i.e., the difference between book value and liquidation proceeds, is therefore given by $Q^i_0 S_0 - Q^i_t S^i_t + \int_0^T q^i_t ds dt$. In addition, the trading strategy $q^i$ generates “slippage”, including transaction costs, instantaneous price impact effects etc., modeled by the cost functional $b \int_0^T (q^i_t)^2 dt$; see, e.g., [1] and the discussion in the introduction. Moreover, any inventory held at time $t > 0$ gives rise to financial risk. We assume that this risk is measured by the expectation of the term

$$\alpha_i \left( Q^i_T \right)^2 + \int_0^T \lambda_i \sigma^2_i \left( Q^i_t \right)^2 dt$$  \hspace{1cm} (2.2)  

where $\alpha_i$ and $\lambda_i$ are nonnegative constants. The first term in (2.2) is clearly a penalty term penalizing any inventory here, $q^{-i} := (q^1, \ldots, q^{i-1}, q^{i+1}, \ldots, q^n)$ denotes the collection of the strategies of all other agents.

Our goal in this paper is to discuss the existence, uniqueness and structure of Nash equilibria for the cost criterion described above. As usual, a collection $q^* = (q^1, \ldots, q^n) \in H^2(\mathbb{R}^n)$ of strategies will be called a Nash equilibrium if, for $i = 1, \ldots, n$,

$$\min_{q^i \in H^2(\mathbb{R})} E \left[ C^i_T(Q^i_0, q^i, q^{-i}) \right] = E \left[ C^*_T(Q^0, q^*, q^{-*}) \right].$$

### 3 Single-agent optimization

In preparation for the discussion of Nash equilibria defined at the end of Section 2.2, we analyze first the optimization problem for a fixed agent $i$ when the strategies of all other agents are fixed. A variety of methods has been used to solve similar and related problems; see, e.g., [2] [9] [24] [17] [12] [4]. Here, our goal is to represent solutions in terms of a BSDE in Theorem 3.1.

First, plugging formula (2.1) for $S^i_t$ into our expression (2.3) of the cost-risk functional $C^i_T(Q^i_0, q^i, q^{-i})$ and integrating by parts, we obtain the alternative expression

$$C^i_T(Q^i_0, q^i, q^{-i}) = \frac{a}{2} (Q^i_0)^2 - \int_0^T Q^i_t \left( \mu_t + a \sum_{j \neq i} q^j_t \right) dt - \int_0^T Q^i_t \sigma_t dW_t$$

$$+ \int_0^T b (q^i_t)^2 dt + \left( \alpha_i - \frac{a}{2} \right) (Q^i_T)^2 + \int_0^T \lambda_i \sigma^2_i (Q^i_t)^2 dt.$$
In the following, we will denote \( \beta \) Denoting BSDE admits a unique solution \( A \) in terms of component \((\beta_1, \beta_2, \beta_3)\). Suppose that \( \sigma \) is well defined and given by \( \lambda_i \sigma_i^2 \left( Q_i^\theta \right)^2 \). Our next goal is to obtain a representation of \((Q_t, q_t)\) if, at time \( t \), agent \( i \) starts using the strategy \( q_t \) with the inventory \( Q_t^\theta \), i.e.,

\[
C_{t,T}^i(Q_t^\theta, q_t, q_t^{-i}) = a \left( Q_t^\theta \right)^2 - \int_t^T Q_t^\theta \left( \mu_u + a \sum_{j \neq i} q_j \right) du + \int_t^T \lambda_i \sigma_i^2 \left( Q_t^\theta \right)^2 du
\]

Let

\[
\Phi_t^i \left( Q_t^\theta \right) : = \essinf_{\bar{q} \in H^2(\mathbb{R})} \mathbb{E} \left[ C_{t,T}^i \left( Q_t^\theta, \bar{q}, q_t^{-i} \right) \bigg| F_t \right]
\]

Our next goal is to obtain a representation of

\[
\Phi_t^i \left( Q_t^\theta \right) := \Phi_t^i \left( Q_t^\theta \right) - a \left( Q_t^\theta \right)^2.
\]

in terms of component \((A^i, B^i, C^i)\) of a solution of a three-dimensional BSDE, which will be discussed in the following proposition.

**Proposition 3.1** Suppose that \( \beta_1 \geq 0 \) and \( q^i \in H^2(\mathbb{R}) \), \( j \neq i \), then the following BSDE

\[
\begin{cases}
A^i_t = \beta_i - \int_t^T \frac{1}{\frac{1}{4b}} (A^i_s)^2 - \lambda_i \sigma_i^2 ds - \int_t^T Z^{A^i}_s dW_s,
B^i_t = 0 - \int_t^T \frac{1}{\frac{1}{4b}} A^i_s B^i_s + \mu_s + a \sum_{j \neq i} q_j^i ds - \int_t^T Z^{B^i}_s dW_s
\end{cases}
\]

admits a unique solution \((A^i, B^i, Z^{A^i}, Z^{B^i}) \in S^\infty(\mathbb{R}) \times S^2(\mathbb{R}) \times \text{BMO}(\mathbb{R}^d) \times H^2(\mathbb{R}^d)\). Moreover, the solution of the BSDE

\[
dC^i_t = \frac{1}{4b} \left( B^i_t \right)^2 dt + Z^{C^i}_t dW_t,
C^i_T = 0,
\]

is well defined and given by

\[
C^i_t = 0 - \int_t^T \frac{1}{4b} \left( B^i_t \right)^2 ds - \int_t^T Z^{C^i}_s dW_s.
\]

**Proof.** Denoting \( M = \beta_i + \lambda_i ||\sigma||_\infty^2 T \), it follows from Pardoux and Peng [15] that BSDE

\[
A^i_t = \beta_i - \int_t^T \frac{1}{\frac{1}{6b}} \left( (-M) \vee A^i_s \wedge M \right)^2 - \lambda_i \sigma_i^2 ds - \int_t^T Z^{A^i}_s dW_s
\]
admits a unique solution \((A^i, Z^{A^i}) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)\). Moreover, we have the following estimate for \(A^i\),
\[
A^i_t \leq E \left[ \beta_i - \int_t^T \left( \frac{1}{b} \left( (-M) \lor A^i_s \land M \right)^2 - \lambda_i \sigma^2_s \right) du \right] F_t \leq \beta_i + \lambda_i \|\sigma\|^2_\infty (T - t).
\]

Meanwhile by denoting \(\xi_t = \frac{(-M)\lor A^i_t \land M}{b}\), it holds that
\[
e^{-\int_0^t \xi_s ds} A^i_t = e^{-\int_0^t \xi_s ds} \beta_i + \int_t^T e^{\int_0^s \xi_u du} \left( \lambda_i \sigma^2_u + \xi_u A^i - b \xi_s^2 \right) ds - \int_t^T e^{-\int_0^s \xi_u du} Z^{A^i}_s dW_s
\]
\[
\geq e^{-\int_0^t \xi_s ds} \beta_i - \int_t^T e^{-\int_0^s \xi_u du} Z^{A^i}_s dW_s.
\]

Therefore, we have
\[
A^i_t \geq E \left[ e^{-\int_0^T \xi_s ds} \beta_i \right] F_t \geq \beta_i e^{-\frac{M(T-t)}{b}}.
\]

Hence, \((A^i, Z^{A^i}) \in \mathcal{S}^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)\) and satisfies
\[
A^i_t = \beta_i - \int_t^T \left( \frac{1}{b} \left( A^i_s \right)^2 - \lambda_i \sigma^2_s \right) ds - \int_t^T Z^{A^i}_s dW_s.
\]

It is easy to check that \(Z^{A^i} \in \text{BMO}(\mathbb{R}^d)\). On the other hand, if
\[
A^i_t = \beta_i - \int_t^T \left( \frac{1}{b} \left( A^i_s \right)^2 - \lambda_i \sigma^2_s \right) ds - \int_t^T Z^{A^i}_s dW_s
\]
admits a solution \((A^i, Z^{A^i}) \in \mathcal{S}^\infty(\mathbb{R}) \times \text{BMO}(\mathbb{R}^d)\), we have
\[
A^i_t \leq E \left[ \beta_i - \int_t^T \left( \frac{1}{b} \left( A^i_s \right)^2 - \lambda_i \sigma^2_s \right) ds \right] F_t \leq \beta_i + \lambda_i \|\sigma\|^2_\infty (T - t)
\]
and
\[
e^{-\int_0^t \frac{A^i_s}{b} ds} A^i_t = e^{-\int_0^t \frac{A^i_s}{b} ds} (\beta_i) + \int_t^T e^{-\int_0^s \frac{A^i_u}{b} ds} \lambda_i \sigma^2_u ds - \int_t^T e^{-\int_0^s \frac{A^i_u}{b} ds} Z^{A^i}_s dW_s
\]
\[
\geq e^{-\int_0^t \frac{A^i_s}{b} ds} (\beta_i) - \int_t^T e^{-\int_0^s \frac{A^i_u}{b} ds} Z^{A^i}_s dW_s.
\]

Therefore, we have
\[
A^i_t \geq E \left[ e^{-\int_0^T \frac{A^i_s}{b} ds} \beta_i \right] F_t \geq \beta_i e^{-\frac{M(T-t)}{b}}.
\]

Hence, \((A^i, Z^{A^i})\) satisfies
\[
A^i_t = \beta_i - \int_t^T \left( \frac{1}{b} \left( (-M) \lor A^i_s \land M \right)^2 - \lambda_i \sigma^2_s \right) ds - \int_t^T Z^{A^i}_s dW_s.
\]

Again, it follows from Pardoux and Peng [15] that
\[
B^i_t = 0 - \int_t^T \left( \frac{1}{b} A^i_s B^i_s + \mu_s + a \sum_{j \neq i} q^i_s \right) du - \int_t^T Z^{B^i}_s dW_s
\]

admits a unique solution \((B^i, Z^{B^i}) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)\). The rest is clear. \(\square\)
Theorem 3.1 Suppose that $\beta_i \geq 0$ and $q^j \in \mathcal{H}^2(\mathbb{R})$, $j \neq i$, then $\Phi^i_t(Q^q_t)$ is given by

$$
\Phi^i_t(Q^q_t) = A^i_t \left( Q^q_t \right)^2 + B^i_t Q^q_t + C^i_t
$$

where $A^i_t, B^i_t, C^i_t$ are given as in Proposition 3.1. The unique optimal strategy for the agent $i$ is given in feedback form by

$$
q^i_t = -\frac{1}{b} \left( A^i_t Q^q_t + \frac{1}{2} B^i_t \right).
$$

Proof. By denoting

$$
\left\{
\begin{align*}
V^q_t &= A^i_t \left( Q^q_t \right)^2 + B^i_t Q^q_t + C^i_t, \\
g^A_t &= \frac{1}{b} (A^i_t)^2 - \lambda_i \sigma^2_t, \\
g^B_t &= \frac{1}{b} A^i_t B^i_t + \mu_i + a \sum_{j \neq i} q^j_t, \\
g^C_t &= \frac{1}{b} (B^i_t)^2,
\end{align*}
\right.
$$

and applying Itô’s formula, we have

$$
dV^q_t = 2A^i_t Q^q_t q^i_t dt + \left( Q^q_t \right)^2 dA^i_t + B^i_t q^i_t dt + Q^q_t dB^i_t + g^C_t dt + Z^C_t dW_t
$$

$$
= 2A^i_t Q^q_t q^i_t dt + \left( Q^q_t \right)^2 g^A_t dt + \left( Q^q_t \right)^2 Z^A_t dW_t + g^C_t dt + Z^C_t dW_t + B^i_t q^i_t dt + Q^q_t g^B_t dt + Q^q_t Z^B_t dW_t
$$

$$
= \left( 2A^i_t Q^q_t q^i_t + \left( Q^q_t \right)^2 g^A_t + B^i_t q^i_t + Q^q_t g^B_t + g^C_t \right) dt + \left( Q^q_t \right)^2 Z^A_t + Q^q_t Z^B_t + Z^C_t dW_t.
$$

Therefore it holds

$$
dV^q_t + \left( Q^q_t \right)^2 \mu_i + a \sum_{j \neq i} q^j_t - \lambda_i \sigma^2_t Q^q_t + b q^i_t dt
$$

$$
= \left( 2A^i_t Q^q_t q^i_t + \left( Q^q_t \right)^2 g^A_t + B^i_t q^i_t + Q^q_t g^B_t + g^C_t \right) dt + \left( Q^q_t \right)^2 Z^A_t + Q^q_t Z^B_t + Z^C_t dW_t.
$$

and rearranging the drift terms, one can see

$$
dV^q_t + \left( Q^q_t \right)^2 \mu_i + a \sum_{j \neq i} q^j_t - \lambda_i \sigma^2_t Q^q_t + b q^i_t dt
$$

$$
= \left( 2A^i_t Q^q_t q^i_t + \frac{1}{b} \left( Q^q_t \right)^2 (A^i_t)^2 + B^i_t q^i_t + \frac{1}{b} Q^q_t A^i_t B^i_t + \frac{1}{4b} (B^i_t)^2 + b q^i_t \right) dt
$$

$$
+ \left( \left( Q^q_t \right)^2 Z^A_t + Q^q_t Z^B_t + Z^C_t \right) dW_t
$$

$$
= \frac{1}{b} \left( A^i_t Q^q_t + b q^i_t + \frac{1}{2} B^i_t \right)^2 dt + \left( Q^q_t \right)^2 Z^A_t + Q^q_t Z^B_t + Z^C_t dW_t.
$$

Hence, it holds that

$$
V^q_t = V^q_t + \int_t^T -\left( Q^q_s \left( \mu_s + a \sum_{j \neq i} q^j_s - \lambda_i \sigma^2_s Q^q_s \right) + b (q^i_s)^2 \right) ds
$$

$$
- \int_t^T \frac{1}{b} \left( A^i_t Q^q_s + b q^i_s + \frac{1}{2} B^i_s \right)^2 ds - \int_t^T \left( \left( Q^q_s \right)^2 Z^A_s + Q^q_s Z^B_s + Z^C_s \right) dW_s.
$$
Therefore, for any \(q^i, \bar{q}^i \in \mathcal{H}^2(\mathbb{R})\) and \(t \in [0, T]\), by taking \(\bar{q}^i_t = q^i_s 1_{s \leq t} + q^i_s 1_{s > t}\) for all \(s \in [0, T]\), we have
\[
V_t^{q^i} = V_t^{\bar{q}^i} = \beta_i Q_{t,T}^{\bar{q}^i} + \int_t^T \left(-Q_{t,s}^{\bar{q}^i} \left(\mu_s + a \sum_{j \neq i} q^j_s - \lambda_i \sigma^2 Q^j_{t,s}\right) + b (q^i_s)^2\right) ds
- \int_t^T \frac{1}{b} \left(A^i_s Q_{t,s}^{\bar{q}^i} + b q^i_s + \frac{1}{2} B^i_s\right)^2 ds - \int_t^T \left(\left(Q_{t,s}^{\bar{q}^i}\right)^2 Z^A_{t,s} + Q_{t,s}^{\bar{q}^i} Z^B_{t,s} + Z^C_{t,s}\right) dW_s
\]
which implies that
\[
V_t^{\bar{q}^i} \leq E \left[\beta_i Q_{t,T}^{\bar{q}^i} + \int_t^T \left(-Q_{t,s}^{\bar{q}^i} \left(\mu_s + a \sum_{j \neq i} q^j_s - \lambda_i \sigma^2 Q^j_{t,s}\right) + b (q^i_s)^2\right) ds \mid \mathcal{F}_t\right] 
= \hat{\Phi}_i^i \left(Q_t^i\right).
\]

On the other hand, for any \(t \in [0, T]\) and \(q^i \in \mathcal{H}^2(\mathbb{R})\), the following random ODE
\[
Q^i_s = Q^i_t - \frac{1}{b} \int_t^s \left(A^i_u Q^i_u + \frac{1}{2} B^i_u\right) du, \quad s \in [t, T]
\]
admits a unique solution \(Q^i_s \in \mathcal{S}^2(\mathbb{R})\) on \([t, T]\). Therefore, by taking \(q^i_s = q^i_s 1_{s \leq t} + q^i_s 1_{s > t}\) with
\[
q^i_s = -\frac{1}{b} \left(A^i_s Q^i_s + \frac{1}{2} B^i_s\right),
\]
we have
\[
V_t^{q^i} = V_t^{\bar{q}^i} = \beta_i Q_{t,T}^{q^i} + \int_t^T \left(-Q_{t,s}^{q^i} \left(\mu_s + a \sum_{j \neq i} q^j_s - \lambda_i \sigma^2 Q^j_{t,s}\right) + b (q^i_s)^2\right) ds
- \int_t^T \frac{1}{b} \left(A^i_s Q_{t,s}^{q^i} + b q^i_s + \frac{1}{2} B^i_s\right)^2 ds - \int_t^T \left(\left(Q_{t,s}^{q^i}\right)^2 Z^A_{t,s} + Q_{t,s}^{q^i} Z^B_{t,s} + Z^C_{t,s}\right) dW_s
\]
which implies that
\[
V_t^{q^i} = E \left[\beta_i Q_{t,T}^{q^i} + \int_t^T \left(-Q_{t,s}^{q^i} \left(\mu_s + a \sum_{j \neq i} q^j_s - \lambda_i \sigma^2 Q^j_{t,s}\right) + b (q^i_s)^2\right) ds \mid \mathcal{F}_t\right] 
\geq \hat{\Phi}_i^i \left(Q_t^i\right).
\]
Therefore, it holds that
\[
\hat{\Phi}_i^i \left(Q_t^i\right) = A^i_t \left(Q_t^i\right)^2 + B_i Q_t^i + C_i.
\]
It is easy to verify that the unique optimal strategy (feedback form) for the agent \(i\) is given by
\[
q^i_s = -\frac{1}{b} \left(A^i_s Q^i_s + \frac{1}{2} B^i_s\right).
\]
\(\square\)
3.1 Characterization of the optimal strategy in terms of an FBSDE

In this section, we show that the optimal strategy for agent $i$ can be given by the unique solution of an FBSDE.

**Theorem 3.2** Suppose that $\beta_i \geq 0$ and $q^i \in \mathcal{H}^2(\mathbb{R})$, $j \neq i$, then $\left(Q^{q^i, q^j}, Q^{q^i, Z^{A^i}}_{b} + Z^{b^i}_{2b}ight)$ is the unique solution of the following FBSDE

$$
\begin{align*}
Q^{q^i}_t &= Q^{q^i}_0 + \int_t^T q^i_s ds,
q^i_t &= -\frac{\beta_i}{b} Q^{q^i}_T + \int_t^T \left( -\lambda_i \sigma^2_i Q^{q^i}_s + \frac{\mu_s + a \sum_{j \neq i} q^j_s}{2} \right) ds + \int_t^T Z^q_s dW_s.
\end{align*}
$$

in $\mathcal{S}^2(\mathbb{R}) \times \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$.

**Proof.** By denoting

$$
\Lambda^i_t := e^{-\int_t^T \frac{1}{2} A^i_s ds},
$$

it is easy to deduce that:

$$
d \left( \Lambda^i_t Q^{q^i}_t A^i_t \right) = \Lambda^i_t Q^{q^i}_t dA^i_t + \Lambda^i_t q^i_t A^i_t dt + \frac{\Lambda^i_t Q^{q^i}_t}{b} dt - \lambda_i \sigma^2_i \Lambda^i_t Q^{q^i}_t dt - \frac{\Lambda^i_t Q^{q^i}_t}{b} dt + \Lambda^i_t Q^{q^i}_t Z^{A^i}_t dW_t
$$

Therefore, it holds that

$$
\Lambda^i_t Q^{q^i}_t A^i_t = \beta_i \Lambda^i_t Q^{q^i}_T - \int_t^T \Lambda^i_s q^{i^*_s} A^i_s ds + \int_t^T \lambda_i \sigma^2_i \Lambda^i_s Q^{q^i}_s ds - \int_t^T \Lambda^i_s Q^{q^i}_s Z^{A^i}_s dW_s.
$$

Noting that

$$
\Lambda^i_t B^i_t = -\int_t^T \Lambda^i_s \left( \mu_s + a \sum_{j \neq i} q^j_s \right) ds - \int_t^T \Lambda^i_s Z^{B^i}_s dW_s,
$$

one has

$$
\Lambda^i_t q^{i^*_s} = -\frac{\beta_i}{b} \Lambda^i_t Q^{q^i}_T + \int_t^T \left( \Lambda^i_s q^{i^*_s} A^i_s - \lambda_i \sigma^2_i \Lambda^i_s Q^{q^i}_s + \frac{\Lambda^i_s \left( \mu_s + a \sum_{j \neq i} q^j_s \right)}{2} \right) ds
$$

$$
+ \int_t^T \left( \frac{\Lambda^i_s Q^{q^i}_s Z^{A^i}_s}{b} + \frac{\Lambda^i_s Z^{B^i}_s}{2b} \right) dW_s.
$$

Therefore, it holds that

$$
dq^{i^*_s} = d \left( (\Lambda^i)^{-1} \Lambda^i_t q^{i^*_s} \right)
$$

$$
= \frac{A^i_t}{b} q^{i^*_s} dt - \left( \frac{\Lambda^i_t}{b} \right)^{-1} \left( \frac{\Lambda^i_t q^{i^*_s} A^i_t - \lambda_i \sigma^2_i \Lambda^i_t Q^{q^i}_t + \frac{\Lambda^i_t \left( \mu_s + a \sum_{j \neq i} q^j_s \right)}{2} \right) dt
$$

$$
+ \left( \frac{\Lambda^i_t Q^{q^i}_s Z^{A^i}_s}{b} + \frac{\Lambda^i_t Z^{B^i}_s}{2b} \right) dW_t
$$

$$
= \frac{1}{b} \left( \lambda_i \sigma^2_i Q^{q^i}_s - \frac{\left( \mu_t + a \sum_{j \neq i} q^j_t \right)}{2} \right) dt - \left( \frac{Q^{q^i}_s Z^{A^i}_t}{b} + \frac{Z^{B^i}_t}{2b} \right) dW_t.
$$
It is easy to check that \( \left( Q^{q_i}, q^i, \frac{Q^{q_i} + Z^A}{b} + \frac{Z^{B^i}}{2b} \right) \) is in \( S^2(\mathbb{R}^n) \times S^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{d}) \). We now prove the uniqueness. Suppose that FBSDE (3.1) admits another solution \((Q^q, q, Z) \in S^2(\mathbb{R}^n) \times S^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{d}) \). Then, we have

\[
\begin{align*}
Q^q_t - Q^q_t &= \int_0^t (q^q_s - q^q_s) \, ds, \\
q^q_t - q^q_t &= -\frac{\beta_t}{b} \left( Q^q_T - Q^q_T \right) + \int_t^T \frac{1}{b} \left( -\lambda_t \sigma_s^2 (Q^q_s - Q^q_s) \right) ds + \int_t^T (Z^q_s - Z^q_s) \, dW_s.
\end{align*}
\]

Therefore, it holds that

\[
(q^q_t - q^q_t) (Q^q_t - Q^q_t) = -\frac{\beta_t}{b} (Q^q_T - Q^q_T)^2 + \int_t^T \frac{1}{b} \left( -\lambda_s \sigma_s^2 (Q^q_s - Q^q_s) \right)^2 ds \\
- \int_t^T (q^q_s - q^q_s)^2 ds + \int_t^T (Q^q_t - Q^q_t) (Z^q_s - Z^q_s) \, dW_s.
\]

Thus, it holds that

\[
0 = E \left[ -\frac{\beta_t}{b} (Q^q_T - Q^q_T)^2 - \int_0^T \frac{\lambda_s \sigma_s^2}{b} (Q^q_s - Q^q_s)^2 ds - \int_0^T (q^q_s - q^q_s)^2 ds \right] \leq 0
\]

which implies uniqueness.

\[\square\]

4 Characterization and existence of a Nash equilibrium

We first provide a characterizing result of a Nash equilibrium in terms of a system of FBSDE.

**Theorem 4.1** Suppose that \( \beta_i \geq 0 \), if the following FBSDE:

\[
\begin{align*}
Q^q_i &= Q^i_0 + \int_0^t q^q_s ds, \quad i = 1, \ldots, n \\
q^q_i &= -\frac{\beta_t}{b} Q^q_t + \int_t^T \frac{1}{2b} \left( -\lambda_t \sigma_s^2 Q^q_s + \frac{\mu_s + \sum_{j \neq i} q^j_s}{2} \right) ds + \int_t^T Z^q_s dW_s, \quad i = 1, \ldots, n.
\end{align*}
\]

admits a solution \((Q^q, q, Z) \in S^2(\mathbb{R}^n) \times S^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n \times d}) \), then \( q \) is a Nash equilibrium. On the other hand, if \( q \in \mathcal{H}^2(\mathbb{R}^{n \times d}) \) is a Nash equilibrium, then \((Q^q, q, Z) \) is a solution of FBSDE (4.1) in \( S^2(\mathbb{R}^n) \times S^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n \times d}) \), where \( Z \) is given by

\[
Z = \left( \frac{Q^q_i Z^A_i}{b} + \frac{Z^{B_i}}{2b}, \ldots, \frac{Q^q_n Z^A_n}{b} + \frac{Z^{B_n}}{2b} \right),
\]

where for \( i = 1, \ldots, n \), \( Z^A, Z^B \) are given as in Theorem 3.4 and \( M^t \) denotes the transpose of the matrix \( M \).

**Proof.** The result follows directly from Theorem 3.4 and Theorem 3.2. \[\square\]

In order to get a Nash equilibrium, it is sufficient to have the existence of solution for FBSDE (4.1). In this section, we will investigate the solvability for FBSDE (4.1). An existence and uniqueness result for small time horizon is due to Antonelli [5]. Under some assumptions, we get a unique global solution for FBSDE (4.1) which is stated in the following theorem.

**Theorem 4.2** Suppose that \( \beta_i > 0 \), \( \lambda_i > 0 \) and \( \lambda_i \sigma_i^2 > \frac{1}{10} a^2 b(n-1) \) for all \( i = 1, \ldots, n \) and \( t \in [0, T] \), then FBSDE (4.1) admits a unique solution \((Q^q, q, Z) \in S^2(\mathbb{R}^n) \times S^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n \times d}) \).

**Proof.** Denoting \( \tilde{q}^q_i = -q^q_i \), we have

\[
\begin{align*}
Q^q_i &= Q^q_0 - \int_0^t \tilde{q}^q_s ds, \quad i = 1, \ldots, n \\
q^q_i &= -\frac{\beta_t}{b} Q^q_t - \int_t^T \frac{1}{b} \left( -\lambda_t \sigma_s^2 Q^q_s + \frac{\nu}{2} \right) ds + \int_t^T Z^q_s dW_s, \quad i = 1, \ldots, n.
\end{align*}
\]
Since it holds that
\[ \sum_{i=1}^{n} \frac{\beta_i}{b} |x_i|^2 \geq \min_{1 \leq i \leq n} \frac{\beta_i}{b} |x|^2 \]
and
\[ \sum_{i=1}^{n} \left( -\frac{\lambda_i \sigma_i^2}{b} |x_i|^2 - |y_i|^2 - \frac{a}{2} \sum_{j \neq i} y_j x_i \right) \leq \sum_{i=1}^{n} \left( -\frac{\lambda_i \sigma_i^2}{b} |x_i|^2 - |y_i|^2 + \frac{a^2(n-1)}{16} |x_i|^2 + \sum_{j \neq i} \frac{1}{n-1} |y_j|^2 \right) \]
\[ = \sum_{i=1}^{n} \left( -\frac{\lambda_i \sigma_i^2}{b} + \frac{a^2(n-1)}{16} \right) |x_i|^2 \]
\[ \leq -\min_{1 \leq i \leq n} \inf_{0 \leq t \leq T} \left( \frac{\lambda_i \sigma_i^2}{b} - \frac{a^2(n-1)}{16} \right) |x|^2, \]
the monotonicity condition in Peng-Wu [16] is satisfied. Therefore, the solvability follows.

As a direct consequence of Theorem 4.1 and Theorem 4.2, we have the following corollary on the existence and uniqueness of a Nash equilibrium.

**Corollary 4.1.** Suppose that \( \beta_i > 0, \lambda_i > 0 \) and \( \lambda_i \sigma_i^2 > \frac{1}{16} a^2 b(n-1) \) for all \( i = 1, \ldots, n \) and \( t \in [0, T] \), then there exists a unique Nash equilibrium.

### 4.1 A Riccati-type equation

Since FBSDE [4.1] is linear, we will investigate it’s solvability through Riccati equations. Indeed, FBSDE [4.1] could be rewritten as

\[
\begin{aligned}
Q_t^q &= Q_0 + \int_0^t q_s ds, \\
q(t) &= GQ_T^p + \int_t^T \left( \hat{A}_s Q_s^q + \left( -\frac{a}{2b} I_n + \frac{2a}{b} \hat{B} \right) q_s + \hat{C}_s \right) ds + \int_t^T \hat{Z}_s dW_s
\end{aligned}
\]

where \( G \) is \( n \times n \) diagonal matrix with diagonal elements \(-\frac{a}{2b}\), \( \hat{A}_s \) is \( n \times n \) diagonal matrix with diagonal elements \(-\lambda_i \sigma_i^2\), \( I_n \) is \( n \times n \) identity matrix, \( \hat{B} \) is \( n \times n \) matrix whose elements are all equal to 1 and \( \hat{C}_s = (\frac{a}{2b}, \ldots, \frac{a}{2b})^T \).

Suppose that the following holds:
\[ q_t = P_t Q_t^q + p_t, \quad t \in [0, T], \]
with \( (P, \lambda) \) and \( (p, \eta) \) being the adapted solutions of the following BSDEs respectively:

\[
\begin{aligned}
dP_t &= \Gamma_t dt + \Lambda_t dW_t, \\
P_T &= G
\end{aligned}
\]

and

\[
\begin{aligned}
dp_t &= \xi_t dt + \eta_t dW_t, \\
p_T &= 0
\end{aligned}
\]

where \( \Gamma \) and \( \xi \) will be chosen later. Applying Itô’s formula, we have the following

\[ \left( \Gamma_t Q_t^q + P_t q_t + \xi_t \right) dt + \left( \Lambda_t Q_t^q + \eta_t \right) dW_t \]
\[ = dq_t = -\left( \hat{A}_t Q_t^q + \left( -\frac{a}{2b} I_n + \frac{a}{2b} \hat{B} \right) q_t + \hat{C}_t \right) dt - Z_t dW_t. \]

Comparing drift and diffusion terms, we should have

\[
\begin{aligned}
&\left( \Gamma_t + P_t^2 \right) Q_t^q + P_t q_t + \xi_t = -\left( \hat{A}_t + \left( -\frac{a}{2b} I_n + \frac{a}{2b} \hat{B} \right) P_t \right) Q_t^q - \left( \left( -\frac{a}{2b} I_n + \frac{a}{2b} \hat{B} \right) p_t + \hat{C}_t \right), \\
&\Lambda_t Q_t^q + \eta_t = -Z_t.
\end{aligned}
\]

Therefore, we will take

\[
\begin{aligned}
\Gamma &= -\hat{A}_t - \left( -\frac{a}{2b} I_n + \frac{a}{2b} \hat{B} \right) P_t - P_t^2 \\
\xi_t &= -\left( -\frac{a}{2b} I_n + \frac{a}{2b} \hat{B} + P_t \right) p_t - \hat{C}_t
\end{aligned}
\]

Thus, we obtain the following result.
Proposition 4.1 Suppose that the following BSDE
\[
\begin{cases}
    dP_t = \left(-\dot{A}_t - \left(-\frac{\rho}{2b} I_n + \frac{\rho}{2b} \dot{B}\right) P_t - P_t^2\right) dt + \Lambda_t dW_t, \\
    P_T = G
\end{cases}
\] (4.2)
admits an adapted solution \((P, \Lambda) \in S^m(\mathbb{R}^{n \times n}) \times H^m(\mathbb{R}^{n \times n \times d})\) for all \(m \geq 1\). Suppose moreover that the following BSDE
\[
\begin{cases}
    dp_t = \left(-\left(-\frac{\rho}{2b} I_n + \frac{\rho}{2b} \dot{B} + P_t\right) p_t - \dot{C}_t\right) dt + \eta_t dW_t, \\
    p_T = 0
\end{cases}
\] (4.3)
admits a unique adapted solution \((p, \eta) \in S^m(\mathbb{R}^n) \times H^m(\mathbb{R}^{n \times d})\) for all \(m \geq 1\) and that the unique solution of following random ODE,
\[
Q'' = \dot{A}Q' - \dot{A}Q - \dot{C} \quad \text{equivalent to} \quad \Lambda' = MA + N
\]
where
\[
\Lambda = \begin{bmatrix} Q' \\ Q \end{bmatrix}, \quad M = \begin{bmatrix} \dot{A} & -\dot{A} \\ I_n & 0 \end{bmatrix}, \quad N = \begin{bmatrix} -\dot{C} \\ 0 \end{bmatrix}
\]
Since \(M\) is invertible, the solution is given by
\[
\Lambda = \exp(tM) \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \int_0^t \exp(sM) N ds = \exp(tM) \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + (\exp(tM) - I_{2n}) M^{-1} N
\]
where \(\xi_1, \xi_2 \in \mathbb{R}^{2n}\) is a vector to be determined by the conditions:
\[
\Lambda[n+1, 2n](0) = Q(0) = Q_0, \quad \text{and} \quad \Lambda[1, n](T) = GQ(T)
\]
It follows that \(\xi_2 = Q_0\). Hence the second condition is given by
\[
\exp(TM) \begin{bmatrix} \xi_1 \\ Q_0 \end{bmatrix} + (\exp(TM) - I_{2n}) M^{-1} N
\]
equivalent to
\[
\begin{bmatrix} I_n & -G \end{bmatrix} \exp(TM) \begin{bmatrix} \xi_1 \\ Q_0 \end{bmatrix} = \begin{bmatrix} -I_n & G \end{bmatrix} (\exp(TM) - I_{2n}) M^{-1} N
\]
Hence, denoting by
\[
\exp(TM) = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}
\]
it follows that \(\xi_1\) is given by
\[
\xi_1 = (E_1 - GE_3)^{-1} \left[ -I_n \quad G \right] (\exp(TM) - I_{2n}) M^{-1} N - (E_2 - GE_4) Q_0.
\]
4.2.1 Numerical results

Throughout we consider the following set of parameters

- Market parameters
  - drift: $\mu = 2\%$
  - vol: $\sigma = 20\%$
  - Maturity: $T = 1$
  - price impact: $a = 1\%$
  - slippage: $b = 1\%$

- 3 traders:
  - Risk aversion: $\alpha = (1, 0.5, 0.25)$, $\lambda = (1, 0.5, 0.25)$,
  - Position to liquidate: $Q = (1, 1, 0.5)$

- Dependence on the drift

![Figure 1: Plot of the three agents' inventory for different drift values](image)

As the drift increases, the agents tend to liquidate slowly or even start buying at the beginning to benefit from the future mean return which will compensate to the liquidation cost.

- Dependence on the volatility

![Figure 2: Plot of the three agents' inventory for different volatility values](image)

As the volatility increases, the agents tend to liquidate quickly at the beginning to reduce the liquidation risk.

- Dependence on $a$
As the permanent market impact increases, the agents tend to liquidate quickly at the beginning to reduce the liquidation cost. For high permanent market impact, the agent with smaller initial inventory tend to short sell and reliquidate to make profit which will compensate to the liquidation cost.

- Dependence on $b$

For small slippage, the agent with smaller initial inventory tend to vary between liquidation and purchasing to make profit which will compensate to the liquidation cost.

- Dependence on $\alpha$ joint magnitude

- Dependence on $\alpha$ different for the first agent
Figure 6: Plot of the three agents’ inventory for different values of risk aversion on terminal value

• Dependence on $\lambda$ joint magnitude

Figure 7: Plot of the three agents’inventory for different values of risk aversion on continuous trading

• Dependence on $\lambda$ first agent

Figure 8: Plot of the three agents’inventory for different values of risk aversion on continuous trading for one agent

• Dependence on $Q$ first agent

Figure 9: Plot of the three agents’inventory for different start value of first agent’s inventory.
Dependence on $Q$ first agent with two arbitrageurs

Figure 10: Plot of the three agents’ inventory for different start value of fist agent’s inventory with two arbitrageurs.

When the initial position of the first agent is small, arbitrageurs tend to first buy and then liquidate to benefit from the future mean return. When the initial position of the first agent is high, arbitrageurs tend to first short sell and then buy to make profit from the price differences. Moreover, The arbitrageurs will not use very aggressive strategies.

4.3 Case for similar agents

**Theorem 4.3** Suppose that $\beta_1 = \ldots = \beta_n = \beta \geq 0$ and $\lambda_1 = \ldots = \lambda_n = \lambda \geq 0$. Then

\[
\begin{align*}
\tilde{Q}_t &= \sum_{i=1}^n Q_0^i + \int_0^t \tilde{q}_i ds, \\
\tilde{q}_t &= -\frac{1}{\alpha} QT + \int_t^T \left( -\lambda_0 Q_s + \sum_{i=1}^n \alpha_i \right) ds + \int_0^T Z_s dW_s
\end{align*}
\]

admits a unique solution $(\tilde{Q}, \tilde{q}, \tilde{Z}) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$ and

\[
\begin{align*}
Q_t^i &= Q_0^i + \int_0^t q_s^i ds, \quad i = 1, \ldots, n \\
q_t^i &= -\frac{1}{\alpha} Q_T^i + \int_t^T \left( -\lambda_0 Q_s^i + \alpha_i - \sum_{i=1}^n \alpha_i \right) ds + \int_t^T Z^i_s dW_s, \quad i = 1, \ldots, n
\end{align*}
\]

admits a unique solution $(Q^i, q^i, Z^i) \in \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n \times d})$. In addition, it holds that

\[
\begin{align*}
\tilde{Q} &= \sum_{i=1}^n Q^i, \\
\tilde{q} &= \sum_{i=1}^n q^i, \\
\tilde{Z} &= \sum_{i=1}^n Z^i.
\end{align*}
\]

Moreover, $(Q^i, q^i, Z^i)$ is the unique solution of FBSDE (4.1).

**Proof.** We will divide the proof into several steps.

**Step 1:** Denoting $M = e^{\frac{1}{2b} \alpha T} + e^{\frac{-1}{2b} \alpha T} \lambda \sigma^2 \|S\|^2$ it follows from Pardoux and Peng [15] that BSDE

\[
P_t = -\frac{\beta}{b} + \int_t^T \left( -\lambda_0 + \frac{\alpha}{2b} - \frac{\beta}{b} \right) P_s + \left( -\lambda M \right) dW_s
\]

admits a unique solution $(P, \Lambda) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$. Moreover, we have the following a priori estimate for $P$. Since

\[
e^{\frac{1}{2b} \alpha T} P_t = e^{\frac{1}{2b} \alpha T} \left( -\lambda_0 + \frac{\alpha}{2b} - \frac{\beta}{b} \right) + \int_t^T \left( -\lambda M \right) dW_s
\]

it holds that

\[
e^{\frac{1}{2b} \alpha T} P_t \geq E\left[ e^{\frac{1}{2b} \alpha T} \left( -\lambda M \right) dW_s \bigg| F_t \right]
\]

\[
\geq -e^{\frac{1}{2b} \alpha T} \left( -\lambda M \right) \frac{\sigma^2}{2b} (T-t)
\]
\[ P_t \geq -e^{\frac{(n-1)aT}{2b}} \frac{\beta}{b} - e^{\frac{(n-1)aT}{2b}} \frac{\lambda \| \sigma \|^2}{2b} T \]

Meanwhile by denoting \( \xi_t = (-M) \lor P_t \land M \), it holds that
\[
e^{\int_0^t (\xi_s + \frac{(n-1)a}{2b}) ds} \beta \frac{P}{b} + \int_t^T e^{\int_0^s (\xi_u + \frac{(n-1)a}{2b}) du} \left( -\frac{\lambda \sigma_s^2}{2b} - \xi_s P_s + \beta \right) ds
\]
\[- \int_t^T e^{\int_0^s (\xi_u + \frac{(n-1)a}{2b}) du} \Lambda_s dW_s
\]
\[\leq -e^{\int_0^t (\xi_s + \frac{(n-1)a}{2b}) ds} \beta \frac{P}{b} - \int_t^T e^{\int_0^s (\xi_u + \frac{(n-1)a}{2b}) du} \Lambda_s dW_s.\]

Therefore, we have
\[ P_t \leq E \left[ -e^{\int_t^T (\xi_s + \frac{(n-1)a}{2b}) ds} \beta \frac{P}{b} \mid \mathcal{F}_t \right] \leq -\frac{\beta}{b} e^{-M(T-t)}.\]

Hence, \((P, \Lambda) \in S^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)\) and satisfies
\[ P_t = -\frac{\beta}{b} + \int_t^T \left( -\frac{\lambda \sigma_s^2}{b} + \frac{(n-1)a}{2b} P_s + P_s^2 \right) ds - \int_t^T \Lambda_s dW_s \]

On the other hand, if
\[ P_t = -\frac{\beta}{b} + \int_t^T \left( -\frac{\lambda \sigma_s^2}{b} + \frac{(n-1)a}{2b} P_s + P_s^2 \right) ds - \int_t^T \Lambda_s dW_s \]

admits a solution \((P, \Lambda) \in S^\infty(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)\), we have
\[ e^{\frac{(n-1)aT}{2b}} P_t = -e^{\frac{(n-1)aT}{2b}} \frac{\beta}{b} + \int_t^T \left( -e^{\frac{(n-1)aT}{2b}} \frac{\lambda \sigma_s^2}{2b} + e^{\frac{(n-1)aT}{2b}} P_s^2 \right) ds - \int_t^T e^{\frac{(n-1)aT}{2b}} \Lambda_s dW_s \]

and
\[ e^{\int_0^t (\xi_s + \frac{(n-1)a}{2b}) ds} P_t = -e^{\int_0^t (\xi_s + \frac{(n-1)a}{2b}) + P_s \frac{ds}{2b}} + \int_t^T \left( -e^{\int_0^s (\xi_u + \frac{(n-1)a}{2b}) du} \frac{\lambda \sigma_s^2}{2b} \right) ds - \int_t^T e^{\int_0^t (\xi_s + \frac{(n-1)a}{2b}) + P_s du} \Lambda_s dW_s \]

Therefore, we have
\[ P_t \geq E \left[ -e^{\frac{(n-1)a(T-t)}{2b}} \frac{\beta}{b} + \int_t^T \left( -e^{\frac{(n-1)a(T-(t+s))}{2b}} \frac{\lambda \sigma_s^2}{2b} \right) ds \mid \mathcal{F}_t \right] \]
\[\geq -e^{\frac{(n-1)aT}{2b}} \frac{\beta}{b} - e^{\frac{(n-1)aT}{2b}} \frac{\lambda \| \sigma \|^2}{2b} T\]

and
\[ P_t \leq E \left[ -e^{\int_t^T (P_s + \frac{(n-1)a}{2b}) \frac{ds}{b}} \mid \mathcal{F}_t \right] \leq -\frac{\beta}{b} e^{-M(T-t)}.\]

Hence, \((P, \Lambda)\) satisfies
\[ P_t = -\frac{\beta}{b} + \int_t^T \left( -\frac{\lambda \sigma_s^2}{b} + \frac{(n-1)a}{2b} P_s + (-M) \lor P_s \land M \right)^2 ds - \int_t^T \Lambda_s dW_s \]
Again, it follows from Pardoux and Peng [15] that

\[ p_t = \int_t^T \left( \left( \frac{(n-1)a}{2b} + P_s \right) p_s + \frac{n\mu_s}{2b} \right) ds - \int_t^T \eta_s dW_s \]

admits a unique solution \((p, \eta) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)\). Moreover, one could easily check that \(p \in \mathcal{S}^\infty(\mathbb{R}), \eta \in \text{BMO}(\mathbb{R}^d)\) and \(\Lambda \in \text{BMO}(\mathbb{R}^d)\). Hence, from standard theory of SDEs, SDE (4.4) admits a unique strong solution \(\tilde{Q} \in \mathcal{S}^\infty(\mathbb{R})\).

Therefore, according to Proposition 4.1, FBSDE (4.5) admits a unique solution \((\tilde{Q}, \tilde{q}, \tilde{Z}) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)\).

We now prove the uniqueness. Suppose that FBSDE (4.5) admits another solution \((\bar{Q}, \bar{q}, \bar{Z}) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)\). Then, we have

\[
\begin{aligned}
\left\{ \begin{array}{l}
\tilde{Q}_t - \bar{Q}_t = \int_t^T (\tilde{q}_s - \bar{q}_s) ds, \\
\tilde{q}_t - \bar{q}_t = -\frac{a}{b} (\tilde{Q}_T - \bar{Q}_T) + \int_t^T \frac{1}{b} \left( -\lambda \sigma_s^2 \left( \tilde{Q}_s - \bar{Q}_s \right) + \frac{(n-1)a}{2} (\tilde{q}_s - \bar{q}_s) \right) ds + \int_t^T \left( \tilde{Z}_s - \bar{Z}_s \right) dW_s
\end{array} \right.
\end{aligned}
\]

Therefore, it holds that

\[
(\tilde{q}_t - \bar{q}_t) (\tilde{Q}_t - \bar{Q}_t) = -\frac{a}{b} (\tilde{Q}_T - \bar{Q}_T)^2 + \int_t^T \frac{1}{b} \left( -\lambda \sigma_s^2 (\tilde{Q}_s - \bar{Q}_s)^2 + \frac{(n-1)a}{2} (\tilde{q}_s - \bar{q}_s) (\tilde{Q}_t - \bar{Q}_t) \right) ds
\]

\[
- \int_t^T (\tilde{q}_s - \bar{q}_s)^2 ds + \int_t^T (\tilde{Q}_t - \bar{Q}_t) (\tilde{Z}_s - \bar{Z}_s) dW_s
\]

Hence, we have

\[
e^{\frac{(n-1)at}{2}} (\tilde{q}_t - \bar{q}_t) (\tilde{Q}_t - \bar{Q}_t) = -\frac{a}{b} e^{\frac{(n-1)at}{2}} (\tilde{Q}_T - \bar{Q}_T)^2 - \int_t^T \frac{1}{b} \left( -\lambda \sigma_s^2 (\tilde{Q}_s - \bar{Q}_s)^2 + \frac{(n-1)a}{2} (\tilde{q}_s - \bar{q}_s) (\tilde{Q}_t - \bar{Q}_t) \right) ds
\]

\[
- \int_t^T e^{\frac{(n-1)as}{2}} (\tilde{q}_s - \bar{q}_s)^2 ds + \int_t^T e^{\frac{(n-1)as}{2}} (\tilde{Q}_t - \bar{Q}_t) (\tilde{Z}_s - \bar{Z}_s) dW_s
\]

Thus, it holds that

\[
0 = E \left[ -\frac{a}{b} e^{\frac{(n-1)at}{2}} (\tilde{Q}_T - \bar{Q}_T)^2 - \int_t^T \frac{1}{b} \left( -\lambda \sigma_s^2 (\tilde{Q}_s - \bar{Q}_s)^2 + \frac{(n-1)a}{2} (\tilde{q}_s - \bar{q}_s) (\tilde{Q}_t - \bar{Q}_t) \right) ds
\]

\[
- \int_t^T e^{\frac{(n-1)as}{2}} (\tilde{q}_s - \bar{q}_s)^2 ds \right] \leq 0
\]

which implies uniqueness.

**Step 2:** Noting that \(q \in \mathcal{S}^\infty(\mathbb{R})\), following a similar technique as in **Step 1**, FBSDE (4.6) admits a unique solution \((Q^i, q, Z) \in \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^d)\). Moreover, it holds that

\[
\begin{aligned}
\left\{ \begin{array}{l}
\sum_{i=1}^n Q^i_t = \sum_{i=1}^n Q^i_0 + \int_0^t \sum_{i=1}^n q^i_s ds, \\
\sum_{i=1}^n q^i_t = -\frac{a}{b} \sum_{i=1}^n Q^i_T + \int_t^T \frac{1}{b} \left( -\lambda \sigma_s^2 \sum_{i=1}^n Q^i_s + \frac{na}{2} - \sum_{i=1}^n \frac{a\mu^i_s}{2} + \frac{a\eta^i_s}{2} \right) ds + \int_t^T \sum_{i=1}^n Z^i_s dW_s
\end{array} \right.
\end{aligned}
\]

Therefore, we have

\[
\begin{aligned}
\left\{ \begin{array}{l}
\sum_{i=1}^n Q^i_t - \tilde{Q}_t = \int_0^t \left( \sum_{i=1}^n q^i_s - \tilde{q}_s \right) ds, \\
\sum_{i=1}^n q^i_t - \tilde{q}_t = -\frac{a}{b} \left( \sum_{i=1}^n Q^i_T - \tilde{Q}_T \right) + \int_t^T \frac{1}{b} \left( -\lambda \sigma_s^2 \left( \sum_{i=1}^n Q^i_s - \tilde{Q}_s \right) - \frac{a}{2} \left( \sum_{i=1}^n q^i_s - \tilde{q}_s \right) \right) ds
\end{array} \right.
\]

\[
+ \int_t^T \left( \sum_{i=1}^n Z^i_s - \tilde{Z}_s \right) dW_s
\]

It follows from the uniqueness part of **Step 1** that

\[
\begin{aligned}
\tilde{Q} = \sum_{i=1}^n Q^i, \\
\tilde{q} = \sum_{i=1}^n q^i, \\
\tilde{Z} = \sum_{i=1}^n Z^i.
\end{aligned}
\]

**Step 3:** The last statement follows immediately from the uniqueness of solutions of FBSDEs (4.5) and (4.6). \(\square\)
4.3.1 Asymptotic property

If we scale the permanent market impact by the number of agents $n$ or equivalently the permanent market impact is generated by the average of liquidation strategy of all agents, the FBSDE characterizing the Nash equilibrium turns to be the following FBSDE

\[
\begin{aligned}
Q^i_t &= Q^i_t = Q^i_0 + \int_0^t g^i_s ds, \quad i = 1, \ldots, n \\
q^i_t &= -\alpha \frac{\sigma}{b} Q^i_T + \frac{1}{b} \left( -\lambda \sigma^2 Q^i_s + \frac{(\mu + \sigma^2)}{2} \right) ds + \int_t^T Z_s, dW_s,
\end{aligned}
\]

Then we have the following theorem.

**Theorem 4.4** Suppose that $\alpha = \alpha > 0$, $\lambda = \lambda \geq 0$ for all $i \in \mathbb{N}$ and $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n Q^i_0 = Q^i_0 \in \mathbb{R}$. Then

\[
\begin{aligned}
Q^i_t &= Q^i_0 + \int_0^t q^i_s ds, \\
q^i_t &= -\alpha \frac{\sigma}{b} Q^i_T + \frac{1}{b} \left( -\lambda \sigma^2 Q^i_s + \frac{(\mu + \sigma^2)}{2} \right) ds + \int_t^T Z_s, dW_s.
\end{aligned}
\]

admits a unique solution $(Q^i, q^i, Z^i) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$ and

\[
\begin{aligned}
Q^i_t &= Q^i_0 + \int_0^t \tilde{q}^i_s ds, \\
\tilde{q}^i_t &= -\alpha \frac{\sigma}{b} Q^i_T + \frac{1}{b} \left( -\lambda \sigma^2 \tilde{Q}^i_s + \frac{(\mu + \sigma^2)}{2} \right) ds + \int_t^T \tilde{Z}_s, dW_s.
\end{aligned}
\]

admits a unique solution $(Q^i, \tilde{q}^i, \tilde{Z}^i) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$ for all $i \in \mathbb{N}$. Let $n$ be large enough such that $\alpha \geq \frac{s}{2n}$ and $(Q^i, q^i, Z^i) \in \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n \times d})$ be the unique solution of FBSDE [4.7]. Then it holds that

\[
\frac{1}{n} \sum_{i=1}^n Q^i, - Q^i, S^2(\mathbb{R}) + \frac{1}{n} \sum_{i=1}^n q^i, - q^i, S^2(\mathbb{R}) + \frac{1}{n} \sum_{i=1}^n Z^i, - Z^i, H^2(\mathbb{R}^d) \to 0 \text{ as } n \to \infty
\]

and

\[
\frac{1}{n} \sum_{i=1}^n Q^i, - Q^i, S^2(\mathbb{R}) + \frac{1}{n} \sum_{i=1}^n q^i, - q^i, S^2(\mathbb{R}) + \frac{1}{n} \sum_{i=1}^n Z^i, - Z^i, H^2(\mathbb{R}^d) \to 0 \text{ for all } 1 \leq i \leq n, \text{ as } n \to \infty
\]

**Proof.** It follows from a similar technique as in Theorem 4.3 FBSDE [4.8] admits a unique solution $(Q^i, q^i, Z^i) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$ and FBSDE [4.9] admits a unique solution $(Q^i, \tilde{q}^i, \tilde{Z}^i) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$ for all $i \in \mathbb{N}$. Let $n$ be large enough such that $\alpha \geq \frac{s}{2n}$, it follows from Theorem 4.3 that FBSDE [4.7] admits a unique solution $(Q^i, q^i, Z^i) \in \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n \times d})$. Moreover, one could check that there exists a constant $M$ which does not depend on $n$ such that

\[
\frac{1}{n} \sum_{i=1}^n Q^i, S^2(\mathbb{R}) + \frac{1}{n} \sum_{i=1}^n q^i, S^2(\mathbb{R}) + \frac{1}{n} \sum_{i=1}^n Z^i, H^2(\mathbb{R}^d) \leq M, \text{ for all } 1 \leq i \leq n.
\]

In addition, we have

\[
\begin{aligned}
\frac{1}{n} \sum_{i=1}^n Q^i_t &= \frac{1}{n} \sum_{i=1}^n Q^i_0 + \int_0^t \frac{1}{n} \sum_{i=1}^n g^i_s ds, \\
\frac{1}{n} \sum_{i=1}^n q^i_t &= \frac{1}{n} \sum_{i=1}^n Q^i_T + \frac{1}{b} \left( -\lambda \sigma^2 Q^i_s + \frac{(\mu + \sigma^2)}{2} \right) ds + \int_t^T Z^i, dW_s.
\end{aligned}
\]

Therefore, it holds that

\[
\begin{aligned}
\frac{1}{n} \sum_{i=1}^n Q^i_t - Q^i_t &= \frac{1}{n} \sum_{i=1}^n Q^i_0 - Q^i_0 + \int_0^t \frac{1}{n} \sum_{i=1}^n g^i_s ds, \\
\frac{1}{n} \sum_{i=1}^n q^i_t - q^i_t &= -\alpha \frac{\sigma}{b} \frac{1}{n} \sum_{i=1}^n Q^i_T + \frac{1}{b} \left( -\lambda \sigma^2 Q^i_s + \frac{(\mu + \sigma^2)}{2} \right) ds + \int_t^T \frac{1}{b} \left( \frac{1}{n} \sum_{i=1}^n g^i_s - \frac{n}{2n^2} \sum_{i=1}^n g^i_s \right) ds + \int_t^T \frac{1}{n} \sum_{i=1}^n Z^i, dW_s.
\end{aligned}
\]

Thus, we get

\[
\left( \frac{1}{n} \sum_{i=1}^n q^i_t - q^i_0 \right) = \left( \frac{1}{n} \sum_{i=1}^n Q^i_t - Q^i_0 \right)
\]
Hence, it holds that
\[
\| \frac{1}{n} \sum_{i=1}^{n} Q_{i,T}^{i,n} - Q_T^* \|_{S^2(\mathbb{R})} + \| \frac{1}{n} \sum_{i=1}^{n} q_{i,n}^i - q_T^* \|_{S^2(\mathbb{R})} + \| \frac{1}{n} \sum_{i=1}^{n} Z_{i,n}^i - Z_T^* \|_{\mathcal{H}^2(\mathbb{R}^d)} \to 0 \text{ as } n \to 0
\]
Similarly, it holds that
\[
\| Q_{i,n}^{i,n} - \hat{Q}_i^i \|_{S^2(\mathbb{R})} + \| q_{i,n}^i - \hat{q}_i^i \|_{S^2(\mathbb{R})} + \| Z_{i,n}^i - \hat{Z}_i \|_{\mathcal{H}^2(\mathbb{R}^d)} \to 0 \text{ for all } 1 \leq i \leq n, \text{ as } n \to 0
\]

\[\square\]

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