Accurate Computation of Laplace Eigenvalues by an Analytical Level Set Method

Pavel Grinfeld
February 15th, 2012

Abstract

This purpose of this write-up is to share an idea for accurate computation of Laplace eigenvalues on a broad class of smooth domains. We represent the eigenfunction $u$ as a linear combination of eigenfunctions corresponding to the common eigenvalue $\rho^2$:

$$u(r, \theta) = \sum_{n=0}^{N} P_n J_n (\rho r) \cos n\theta,$$

We adjust the coefficients $P_n$ and the parameter $\rho$ so that the zero level set of $u$ approximates the domain of interest. For some domains, such as ellipses of modest eccentricity, the coefficients $P_n$ decay exponentially and the proposed method can be used to compute eigenvalues with arbitrarily high accuracy.

1 Introduction

The celebrated level set method \[2\] is a numerical method for solving problems with moving interfaces and has been applied to eigenvalue problems \[3\]. The method proposed here also represents interfaces as level sets of functions. However, our level set function is a linear combination of global functions, with the coefficients of the linear combination and possibly additional parameters acting as the degrees of freedom for deforming the level set. When only a small number of functions is required for the accurate computation of some quantity, the proposed method can be viewed as essentially analytical.

The Laplace eigenvalues (with zero or any other type of boundary conditions) are not available analytically for a $2 \times 1$ ellipse. (In fact, eigenvalues for only a handful of shapes have been discovered since Rayleigh’s pioneering ”Theory of Sound” \[4\].) Furthermore, eigenvalues are difficult to compute with high accuracy. The finite element method \[5\] with isoparametric quadratic elements would perhaps be the most common approach. Combined with Richardson extrapolation, it is an impressive 5-th order method. However, in the author’s experience, it fails to deliver the lowest eigenvalue beyond the 13-th digit. Furthermore, the
finite element method is ineffective in computing the high (enough) eigenvalues since the supporting mesh must resolve the high frequency oscillations of the corresponding eigenfunctions.

2 Description of the approach

We propose a method that computes the lowest eigenvalue on a $2 \times 1$ ellipse to arbitrary accuracy. The method can be applied to other shapes and higher eigenvalues. The method has a number of desirable features, including utmost simplicity, exponential accuracy, ease of analysis and the ability to produce an exact solution to an approximate problem.

Represent the domain $\Omega$ with boundary $S$ as the interior $\Omega_u$ of the zero level set $S_u$ of the function $u(r, \theta)$ given by the linear combination

$$u(r, \theta) = \sum_{n=0}^{N} P_n J_n (\rho r) \cos(n\theta), \quad (1)$$

where $J_n$ are Bessel functions. The coefficients $P_n$ and the parameter $\rho$ are used as degrees of freedom in fitting the domain $\Omega_u$ to $\Omega$. Since $u(r, \theta)$ given by (1) satisfies

$$-\Delta u = \rho^2 u, \quad (2)$$

then, by definition, $\rho^2$ is a Laplace eigenvalue on $\Omega_u$ with zero boundary conditions and $u(r, \theta)$ is the corresponding eigenfunction. Note that while $\Omega_u$ is an approximation to $\Omega$, the function $u(r, \theta)$ is an exact eigenfunction on $\Omega_u$.

The objective function measuring the proximity between $S$ and $S_u$ can be defined as

$$\int_S D_u^2(S) \, dS, \quad (3)$$

where $D_u$ is an appropriately defined distance function that captures the discrepancy between $S$ and $S_u$. The Mathematica code below takes a pragmatic approach and defines $D(\alpha)$ as

$$D(\alpha) = r(\alpha) - r_u(\alpha), \quad (4)$$

where $r(\alpha)$ is the polar representation of the ellipse and $r_u(\alpha)$ is the polar representation of $S_u$. The minimization can be carried out effectively in a number of ways, including available generic optimization routines.

As mentioned above, one of the most appealing characteristics of the proposed method is its utmost simplicity. The Mathematica code that computes the lowest eigenvalue on the $2 \times 1$ ellipse is given below in its entirety.

1. Goals = { AccuracyGoal -> 75, PrecisionGoal -> 75, WorkingPrecision -> 120};
2. A = 1/2; B = 1;
3. NumberOfTerms = 30;
4. Shape[theta_] = 1/Sqrt[Cos[theta]^2/A^2 + Sin[theta]^2/B^2];
5. \( \text{lsf}[ho, p][r, \theta] := \text{BesselJ}[0, \rho r] + \sum_{n=1}^{\text{Length}[p]} p[n] \cdot \text{BesselJ}[2 n, \rho r] \cdot \cos[2 n \theta] \); 
6. \( \text{r}[ho, p][\theta] := x ./ \text{FindRoot}[\text{lsf}[ho, p][x, \theta] == 0, \{x, \text{Shape}[	heta]\}] \); 
7. \( \text{ObjectiveFunction}[ho, p : \{?\text{NumericQ}\}] := \sum_{\theta=0}^{2 \pi - 2 \pi/60} (\text{r}[ho, p][\theta] - \text{Shape}[	heta])^2 \); 
8. \( s = \text{FindMinimum}[\text{ObjectiveFunction}[\text{rr}, \text{pp}], \{\{\text{rr}, \text{BesselJZero}[0, 1]\}, \{\text{pp}, \text{ConstantArray}[0, \text{NumberOfTerms}]\}\}, \text{Evaluate}[\text{Goals}], \text{MaxIterations} \rightarrow 500, \text{StepMonitor} :> \text{Print}[\text{rr}, " ", \text{ObjectiveFunction}[\text{rr}, \text{pp}] // \text{N}[, 5] \&", " ", \text{Date}[]] \); 

Code notes:
Line 1 specifies accuracy goals and the working number of digits. Naturally, higher working precision takes more time. On a circa 2007 desktop, the presented code takes about 30 minutes to take the first step and about 10 minutes for each subsequent step. It takes several steps to reduce the error by a factor of 10. These figures are given for computing with 30 terms in series (1) and 125 digits of accuracy. The method is substantially faster when fewer digits are required. For instance, 6 digits of accuracy can be obtained with 3 terms in series (1) in a matter of seconds.

Line 2 specifies the semiaxes of the ellipse.

Line 3 specifies the number of terms in equation (1). Due to the symmetry of the ellipse with respect to the \( y \)-axis, we only use the even terms in the series (1), raising the effective number of terms to 60. The variable \( \text{NumberOfTerms} \) is referenced in Line 8.

Line 4 gives the ellipse in polar coordinates.

Line 5 defines the level set function (lsf) \( u(r, \theta) \). This line specifies that \( u \) also depends on the parameters \( \rho \) and \( P_n \). Since the zero level set remains unchanged when the coefficients \( P_n \) are multiplied by a number, the expansion here assumes that \( P_0 = 1 \) and gives \( u(r, \theta) \) as

\[
  u(r, \theta) = J_0(\rho r) + \sum_{n=1}^{N} P_{2n} J_{2n}(\rho r) \cos 2n\theta.
\]  

Line 6 defines the zero level set in polar coordinates by solving the equation \( u(r, \theta) = 0 \).

Line 7 evaluates the objective function in equation (3). The integral is approximated by a finite sum with 60 terms. More terms are needed when greater accuracy is targeted.

Line 8 finds the optimal configuration. The initial configuration is the unit circle (since all \( P_n = 0 \) for \( n > 1 \)).
3 Further thoughts

3.1 Hadamard acceleration

The level set function \( u(r, \theta) \) is an exact eigenfunction on \( \Omega_u \). This fact can be used to increase the accuracy of the eigenvalue estimate. The Hadamard formula \( \text{[1]} \) gives the rate of change in the eigenvalue \( \lambda \) in response to a deformation of the domain \( \Omega \). Consider a smooth evolution \( S(t) \) parameterized by \( t \). Then the rate of change \( d\lambda/dt \) in the eigenvalue is given by

\[
\frac{d\lambda}{dt} = -\int_{S(t)} C |\nabla \psi|^2 dS,
\]

where \( C \) is the Hadamard velocity of \( S(t) \) and \( \psi \) is the eigenvalue corresponding to \( \lambda \).

Since \( \rho^2 \) is an exact eigenvalue on \( S_u \) and \( u(r, \theta) \) is the corresponding exact eigenfunction, the Hadamard formula can be effectively used by letting \( C \) equal the distance between \( S_u \) and \( S \) along the normal direction to \( S_u \).

3.2 Prescribing \( C \)

Deforming the surface according to a prescribed \( C \) is likely a more effective way of finding the optimal \( u \) than using a generic optimizer. Suppose that \( D_u \) is the signed normal displacement between \( S \) and \( S_u \). Then the differential equation \( C = -D_u \) on the moving interface \( S_u \) corresponds to \( S_u \) approaching \( S \). Therefore, we may vary the parameters \( P_n \) and \( \rho \) in such a way that the resulting velocity of \( S_u \) is as close as possible to \(-D_u \).

Let us calculate the velocity of the interface \( C \) that results when the parameters of the level set function \( u \) are varied in a prescribed way. In a general setting, suppose that the function \( u \) depends on the parameters \( P_n \) and we are now including \( \rho \) among the \( P_n \). Refer \( u \) arbitrary curvilinear coordinates \( Z_i \). Suppose that the parameters \( P_n \) vary according to \( P_n(t) \) and the resulting zero level set has the equation \( Z_i(t, S) \), where \( S \) are arbitrary coordinates on the surface \( S_u \). The functions \( Z_i(t, S) \) satisfy the equation

\[
u(P_n(t), Z_i(t, S)) = 0 \tag{7}\]

Differentiating with respect to \( t \), we find

\[
\frac{\partial F}{\partial P_n} \dot{P}_n + \frac{\partial F}{\partial Z_i} \frac{\partial Z_i}{\partial t} = 0, \tag{8}\]

where \( \dot{P}_n = dP_n/dt \) and summations over \( n \) and \( i \) are implied. Denote the gradient \( \partial F/\partial Z_i \) by \( |\nabla F| N_i \) (where \( N_i \) is the unit normal) and \( \partial Z_i/\partial t \) by \( V_i \):

\[
\frac{\partial F}{\partial P_n} \dot{P}_n + |\nabla F| N_i V_i = 0, \tag{9}\]
Since $\sum N_i V_i = C$, we find

$$C = -\dot{P}_n \frac{\partial F}{\partial P_n} |\nabla F|^{-1}.$$  \hfill (10)

Therefore, $\dot{P}_n$ should be chosen so that the series (10) approximates $C$ as closely as possible – for example, in the least squares sense.

### 3.3 Enriching the family of functions

The presented calculation is based on a very limited set of functions $J_n(\rho r) e^{in\theta}$, all centered at the same origin. Naturally, any Laplace eigenfunction (corresponding to the same eigenvalue) can be added to the mix. For the problem at hand, it is beneficial to consider functions of the form $J_n(\rho r) e^{in\theta}$, shifted other poles. Different geometries may utilize other functions.

### References

[1] J. Hadamard. *Mmoire sur le problme d’analyse relatif l’équilibre des plaques elastiques encastrées*, Oeuvres, tome 2. 1968.

[2] S. J. Osher and R. P. Fedkiw. *Level Set Methods and Dynamic Implicit Surfaces*. Springer, 2002.

[3] S. J. Osher and F. Santosa. Level set methods for optimization problems involving geometry and constraints. *J. Comp. Phys.*, 171:272–288, 2001.

[4] J. Rayleigh. *The Theory of Sound*. Dover, 1945.

[5] G. Strang and G. Fix. *An Analysis of the Finite Element Method, Second edition*. Wellesley-Cambridge Press, 2008.