A new integrable 3+1 dimensional generalization of the Burgers equation

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March 30, 2022

Abstract

A new nonlinear 3+1 dimensional evolution equation admitting the Lax pair is presented. In the case of one spatial dimension, the equation reduces to the Burgers equation. A method of construction of exact solutions, based on a class of discrete symmetries of the former equation is developed. These symmetries reduce to the Cole-Hopf transformation in one-dimensional limit. Some exact solutions are analyzed, in the physical context of spatial dissipative structures and shock wave dressing.

Keywords: Integrable PDEs, Lax pairs, Darboux transformation, Burgers equation, Cole-Hopf transformation

1. Introduction

The vast majority of known completely integrable nonlinear evolution PDEs is 1+1 dimensional [1], [10]. Dealing with more than a single spatial dimension, one faces fundamental algebraic and geometric obstructions. This fact accounts for the scarcity of integrable \( d+1 \) dimensional systems, for \( d \geq 2 \), available today.

A notable exception is the Burgers equation, which in 1+1 dimensions is

\[
 u_t + uu_x - \nu u_{xx} = 0. \tag{1}
\]

It generalizes to \( d \geq 2 \) spatial dimensions as follows

\[
 u_t + u \cdot \nabla u - \nu \nabla^2 u = 0, \quad u = -\nabla \Phi, \quad \nabla = (\partial_{x_1}, \ldots, \partial_{x_d}). \tag{2}
\]

from the physical point of view, the equation describes the balance between nonlinearity and dissipation, [2], rather than dispersion, which is more characteristic of integrable evolution equations. It comes as a model in various problems of continuous media dynamics, condensed matter physics cosmology, etc., see for instance [4], [3], [11].

Mathematically, the Burgers equation is special, as it can be fully linearized via the Cole-Hopf ([5], [7]) substitution

\[
 u = -2\nu \nabla \log \Theta, \tag{3}
\]

AMS subject classification 35Q53, 35Q58
which reduces it to the heat equation for $\Theta$, with diffusivity $\nu$. For the state of the art on the pure mathematical side of the Burgers equation, see for instance [12] and the references contained therein.

The principal content of this note is as follows. Apart from the natural integrable higher dimensional generalization (2) of the Burgers equation (1), the latter can be used as the foundation to construct higher dimensional integrable PDEs, are not fully linearizable, yet their solutions can be found via Lax pairs. In the recent work [13], the 2+1 dimensional BLP (Boiti, Leon and Pempinelli) system was studied and shown to reduce to an integrable two dimensional generalization of the Burgers equation (1). Here we present a 3+1 dimensional nonlinear equation, which contains dissipative additives and has the following properties:

1. It is a scalar second order evolution PDE with quadratic nonlinearity.
2. In one dimensional limits this equation reduces to the Burgers equation. However, unlike the $d+1$ dimensional Burgers equation, our equation is non-isotropic, nor it is linearizable via the Cole-Hopf substitution.
3. It allows for an explicit Lax pair.
4. It possesses a class of Darboux-transformation-like discrete symmetries, and to take advantage of these symmetries one has to solve the Lax pair equations. The symmetries generate a rich spectrum of exact solutions of the equation. In particular they enable one to fulfill a 3+1 dimensional dressing of particular solutions of the 1+1 dimensional Burgers equation. Conversely, in one dimensional limits the symmetries reduce to the Cole-Hopf transformation.

2. General result

Consider the following equation:

$$ K[u] \equiv u_t + a_1 \left( u_x^2 - u_{xx} \right) + a_2 \left( u_z^2 - u_{zz} \right) + b_1 \left( u_x u_y - u_{xy} \right) + b_2 \left( u_x u_z - u_{xz} \right) - \rho u_x - \mu u_y - \lambda u_z = 0, $$

(4)

where $u = u(t, x, y, z, t)$, and all other parameters are constants.

In one dimensional limits equation (4) reduces to the dissipative Burgers equation. Indeed, if $u = u(x, t)$, defining

$$ \xi(x, t) = u_x(x, t), $$

(5)

we obtain

$$ \xi_t - \rho \xi_x - a_1 \xi_{xx} + 2a_1 \xi_x = 0. $$

(6)

The latter equation boils down to the Burgers equation after the change $t \to t'$, such that

$$ \partial_t' = \partial_t - \rho \partial_x, $$

or simply letting $\rho = 0$.

In the same fashion, if $u = u(z, t)$ and $\eta(z, t) = u_z(z, t)$, one has

$$ \eta_t - \lambda \eta_z - a_2 \eta_{zz} + 2a_2 \eta_z = 0, $$

the analog of (6).
Finally, the reduction $u = u(y,t)$ results in a linear equation

$$u_t - \mu u_y = 0.$$ 

In view of the above, equation (4) can be viewed as a special non-isotropic three dimensional extension of the Burgers equation. To emphasize this, let $w = u_x$, consider $\mu = \rho = \lambda = 0$ and rewrite (4) as follows:

$$w_t + 2a_1 w w_x + b_1 w w_y + b_2 w w_z - a_1 w_{xx} - a_2 w_{zz} - b_1 w_{xy} - b_2 w_{xz} - \mu w_y +$$

$$+ b_1 w_x w_x + b_2 w_x w_z + 2a_2 w_z w_z = 0. \quad (8)$$

Our main result is the following theorem

**Theorem 1** Let $u(x,y,z,t)$ be a particular solution of equation (4) and $\psi = \psi(x,y,z,t)$ be a solution of the following linear equation:

$$\psi_t = a_1 \psi_{xx} + a_2 \psi_{zz} + b_1 \psi_{xy} + b_2 \psi_{xz} + (\rho - 2a_1 u_x - b_2 u_z - b_1 u_y) \psi_x + (\mu - b_1 u_x) \psi_y +$$

$$+ (\lambda - 2a_2 u_z - b_2 u_x) \psi_z \equiv A[u] \psi. \quad (9)$$

Then any $\tilde{u}_{klm} = \tilde{u}_{klm}(x,y,z,t)$, defined by the formula

$$\tilde{u}_{klm} = u - \log \left( (\partial_x - u_x)^k (\partial_y - u_y)^l (\partial_z - u_z)^m \psi \right), \quad (k,l,m) \in \mathbb{Z}_+^3. \quad (10)$$

is also a solution of equation (4).

Theorem 1 rests on the following fact.

**Proposition 2** Equation (4) admits the following Lax pair: $\psi_t = A(u)\psi$, cf. (2), and

$$\psi_{xyz} = u_z \psi_{xy} + u_y \psi_{xz} + u_x \psi_{yz} + (u_{yz} - u_y u_z) \psi_x + (u_{xz} - u_x u_z) \psi_y + (u_{xy} - u_x u_y) \psi_z +$$

$$(u_{xyz} - u_y u_x u_z - u_x u_y u_z) \psi_x + u_x u_y u_z \psi, \quad (11)$$

Verification of this proposition a direct calculation. Further in the note we shall refer to equations (11) and (9) as the Lax, or LA-pair for equation (4), and $u$ as a potential. Observe that the spectral equation (11) of the Lax pair can be rewritten in a more compact form:

$$\left( \partial_x - u_x \right) \left( \partial_y - u_y \right) \left( \partial_z - u_z \right) \psi \equiv L_1[u]L_2[u]L_3[u] \psi \equiv L[u] \psi = 0. \quad (12)$$

Also observe that if we redefine the operator $A[u] \rightarrow A'[u] = A[u] + K[u]$, then the compatibility condition of the Lax pair equations (11) and (9) will be reduced to a an identity. Namely the operators $L[u]$ and $B'[u] = \partial_t - A'[u]$ will commute: $[L[u], B'[u]] = 0$.

Theorem 1 implies the following corollary, which follows after successive iteration of (11).

**Corollary 3** If $\{\psi_i\}, i = 1, \ldots, N$ is a set of particular solutions of the Lax pair (11), (9), given the potential $u$, satisfying equation (4), new solutions of (4) are generated by the following rule:

$$u = u - \log \left( \prod_{i=1}^N L_1^{k_i}[u] L_2^{l_i}[u] L_3^{m_i}[u] \psi_i \right), \quad (k_i, l_i, m_i) \in \mathbb{Z}_+^3, \forall i = 1, \ldots, N. \quad (13)$$

As we have indicated earlier, the Burgers equation (6) results from a 1+1 dimensional reduction of equation (4). Conversely, formulae (10), (13) yield a bona fide generalization of the Cole-Hopf substitution (3). Indeed, setting $u \equiv 0$, $k = l = m = 0$ in (10), after differentiation in $x$, we obtain the one dimensional Cole-Hopf transformation, cf. (3). Clearly, the same can be said about the $z$ variable reduction as well.
Proof of Theorem 11

The theorem will follow from the following lemma.

Lemma 4 Let ψ be a solution of the Lax pair equations (11), (14) with the potential u, which is a solution of equation (4). Then the function

\[ \tilde{\psi}_{klm} = L^1_i[u] L^2_{ij}[u] L^3_{ijm}[u] \psi, \quad (k, l, m) \in \mathbb{Z}^3_+ \]  

(14)

also satisfies (11), (14), with the same potential u.

To prove the lemma, observe the validity of the following commutator relations, for \( i, j = 1, 2, 3 \):

\[ [L_i[u], L_i[u]] = [L_i[u], B[u]] = [L_i[u], L_j[u]] = 0. \]

Lemma 4 then follows by induction. □

Now, to prove Theorem 11, let us introduced three intertwining operators

\[ D_i = f_i \partial_i - g_i, \quad i = 1, 2, 3, \]

(15)

with the quantities \( f_i, g_i \) to be found (naturally, \( \partial_{1,2,3} = \partial_{x,y,z} \)), respectively, such that the operators \( D_i \) have the following property: for some \( u_i = u_i(x, y, z, t) \),

\[ L(u_i) D_i = D_i L[u], \quad B(u_i) D_i = D_i B[u]. \]

(16)

The commutation relations (16) determine the maps \( u \to u_i \), which come from substitution of (15) into (16). The explicit form of the operators \( D_i \) is found as follows.

Substituting (15) into (16) and equating the components at the same partial derivatives results in a system of nonlinear equations (which is not quoted because of its bulk) whence it follows:

\[ D_i = e^{-v} (L_i[u] - c_i), \quad u_i = \bar{u} = u - v, \]

(17)

where \( c_i \) are constants, which will be further assigned zero values. The quantity \( v = v(x, y, z, t) \) is a solution of the following nonlinear equation:

\[ v_i = a_1 (v_{xx} + v_z^2) + a_2 (v_{xz} + v_z v_x) + b_1 (v_{xy} + v_x v_y) + b_2 (v_{xz} + v_x v_z) + \rho - 2a_1 u_x - b_2 u_z - b_1 u_y) v_x + (\mu - b_1 u_x) v_y + (\lambda - 2a_2 u_z - b_2 u_x) v_z \]

(18)

Therefore (17) or explicitly (18) indicate that for \( u \equiv 0 \), the function “−v” satisfies equation (14).

Then automatically the quantity \( \psi = e^{-u} \) will satisfy the Lax pair equations (11) and (14). In fact, the L-equation, cf. (11), is satisfied as the identity. The A-equation (9) however is satisfied only if \( u \) is a solution of (14). On the other hand, by Lemma 4, the functions of the form \( \tilde{\psi}_{klm} \) defined via relation (14) are also solutions of (11) and (14), with the same potential \( u \). Rewriting them as \( \tilde{\psi}_{klm} = \exp(v_{klm}) \) and substituting into (9) one verifies that the quantities \( v_{klm} \) are indeed solutions of equation (14). Theorem 1 and formula (10) now follow from the second relation from (17). □

Remark. Formula (10) has a countenance similar to the Darboux transformation, which is a standard tool for construction of exact solutions of nonlinear PDEs (usually 1+1, more rarely 2+1 dimensional) which admit Lax pairs, see e.g. [9] for the general theory, applications and references. However (10) does not represent a bona fide Darboux transform for two following reasons.
1. Darboux transforms, representing discrete symmetries of a Lax pair, possess a non-trivial kernel on the solution space of the pair. In other words, there always exists some Lax pair solution which zeroes the transform. This is the property which enabled one Crum, [6], to write down the determinant formulae for successive Darboux transforms. Transformation (10) however does not have this property.

It is known that in addition to the Darboux transform, fairly rich spectral problems, such as the Zakharov-Shabat problem for the Nonlinear Schrödinger equation or its two-dimensional generalization for the Davey-Stewartson equations, admit another discrete symmetry, namely the Schlesinger transform, [8]. The difference between (10) and the latter transformations lies in the fact that for the Schlesinger transform, the potential transformation rules can be locally defined without using the Lax pair solution \( \psi \), while (10) certainly does so. This feature is shared by (10) and the standard Darboux transformation.

2. To construct exact solutions of nonlinear PDEs via the Darboux transform, one has to take advantage of the solution of the full Lax pair as a system of equations. In order to get (10) however, we have used the solution of the A-equation (9) only. The L-equation (11) has been used only as a tool to prove Theorem 1. To this effect, transformation (10) combines essential features of the Darboux and Cole-Hopf transformations.

4. Some exact solutions

Let us use the above formalism in order to construct some exact solutions of equation (8) (which is the equation for \( u_x \), where \( u \) is a solution of equation (4) with \( \mu = \lambda = \rho = 0 \)). We consider equation (8), because it appears to be a closer relation of the Burgers equation and is likely to be interesting from the physics point of view.

**Example 4.1.** As the first example let us consider dressing on the vacuum background \( u \equiv 0 \). In this case the function

\[
\psi(x, y, z, t) = a^2 x^2 + b^2 y^2 + c^2 z^2 + 2 (a_1 a^2 + a_2 c^2) t + s^2,
\]

where \( a, b, c, s \) are some real constants, is clearly a solution of the Lax pair equations (11) and (9). Substituting (19) in (10) we derive the \( \tilde{u}_{klm} \). After differentiating it with respect to \( x \) and choosing \( k = l = m = 0 \), we obtain a solution \( w \) of equation (8) as follows:

\[
w(x, y, z, t) = -\frac{2a^2 x}{a^2 x^2 + b^2 y^2 + c^2 z^2 + 2 (a_1 a^2 + a_2 c^2) t + s^2}.
\]

Physically, solution (20) describes a rationally localized impulse, vanishing as \( t \to +\infty \). To ensure that (20) is non-singular for \( t \geq 0 \), one should impose the inequality \( a_2 \geq -a^2 a_1 / c^2 \) on the coefficients. Moreover, if \( a_2 = -a^2 a_1 / c^2 \), the solution in question is rationally localized and stationary.

**Remark.** The fact that there exists a localized stationary solution in an equation containing dissipative terms may appear somewhat paradoxical from the point of view of physics. However solution (20) is stationary only if \( a_1 a_2 < 0 \). One can see that these constants appear in the dissipative terms of equation (8), \( a_1 \) characterizing the dissipation along the \( x \) and \( a_2 \) along the \( z \) axes. The fact that \( a_1 \) and \( a_2 \) should have different signs implies that dissipation in the direction of one axis is compensated by instability in the direction of the other. The balance of these two effects results in the stationary solution, which can be regarded as a three-dimensional dissipative structure. A similar situation occurs with two-dimensional stationary solutions of the BLP equation, cf. [13].
Example 4.2. Another solution of (11), (9) when \( u \equiv 0 \) is
\[
\psi(x,y,z,t) = c_1 e^{(\alpha a_1 + \beta b_1)t} \cosh(\alpha x + \beta y) + c_2 e^{(\alpha a_2 + \beta b_2 + \alpha b_2)t} \cosh(ax + bz) + c_3 e^{a_2c^2t} \cosh(cz),
\]
where \( \alpha, \beta, a, b, c_1, c_2, c_3 \) are some real constants. Choosing them such that
\[
\beta = -\frac{\alpha a_1}{b_1}, \quad b = \frac{-b_2 \pm \sqrt{b_2^2 - 4a_1a_2}}{2a_2}
\]
and using (21) and (22) in the same way as in Example 4.1 above (20), we obtain another solution:
\[
w(x,y,z,t) = -\frac{\alpha c_1 \sinh(\alpha x + \beta y) + ac_2 \sinh(ax + bz)}{c_1 \cosh(\alpha x + \beta y) + c_2 \cosh(ax + bz) + c_3 e^{a_2c^2t} \cosh(cz)}. \quad (23)
\]

Example 4.3. Developing the analog with [13] further, one can develop a procedure of construction of exact solutions of the three-dimensional equation (8) which are based initially on the solutions of the 1+1 dimensional Burgers equation. Consider equation (6) for the unknown \( \xi \).

Suppose \( \lambda = \mu = \rho = 0 \), let \( a_1 = \nu \) in (6). Clearly the quantity
\[
U(x,t) = 2\nu \xi(x,t) \quad \text{solves the one dimensional Burgers equation}
\]
\[
U_t + U U_x - \nu U_{xx} = 0. \quad (24)
\]
As a starting point let us take a shock wave solution of (24), e.g.
\[
\xi = U_x = \frac{v - \nu a}{2\nu} + \frac{a}{1 + e^{a(x-vt)}}, \quad (25)
\]
where \( a \) and \( v \) are constants. Seek a solution of (9) as a superposition
\[
\psi = \sum_{k=1}^{N} A_k(\eta) e^{\beta_k y + \gamma_k z}, \quad (26)
\]
where \( \eta = x - vt \), while the \( 2N \) quantities \( \beta_k \) and \( \gamma_k \) can in general be functions of \( \eta \). For simplicity however let us render them constants to be determined.

Substitution of (26) into (9) yields \( N \) linear equations for \( A_k(\eta) \):
\[
\nu \ddot{A} + (v + \sigma - 2\nu \xi) \dot{A} + (a_2 \gamma^2 - \sigma \xi) A = 0. \quad (27)
\]
In equation (27) the indices for the quantities \( A_k, \beta_k, \gamma_k \) and \( \sigma_k \equiv b_1 \beta_k + b_2 \gamma_k \) have been omitted, while the quantity \( \xi \) came from (25), dot standing for differentiation with respect to \( \eta \).

Equation (27) can be simplified further. To do so, let us introduce a new independent variable:
\[
q = \xi(x,t) - \xi_0, \quad e^{\alpha \eta} = \frac{a}{q} - 1, \quad \xi_0 = \frac{v - \nu a}{2\nu}.
\]
In terms of \( q \) equation (27) becomes
\[
\nu (q^2 - aq)^2 A''(q) + \sigma (q^2 - aq) A'(q) + (\delta - \sigma q) A(q) = 0, \quad (28)
\]
where $\delta = a_2 \gamma^2 - \sigma \xi_0$, while prime stands for differentiation in $q$. The latter equation can be simplified further via a substitution

$$A(q) = W(q) \left( \frac{q}{q - a} \right)^{\sigma/(2\nu a)},$$

reducing (28) to the following equation for $W(q)$:

$$\frac{W''(q)}{W(q)} = \frac{2a\sigma\nu - 4\delta\nu + \sigma^2}{4\nu^2(q - a)^2q^2}. \tag{30}$$

In a particular case of the dependence

$$\beta_k = -b_2 \gamma_k - \nu \pm \sqrt{\nu^2 + 4a_2 \nu \gamma_k^2},$$

one can solve (29) explicitly as

$$\psi = \sum_{k=1}^{N} (W_k q + V_k) \left( \frac{q}{q - a} \right)^{\sigma_k/(2\nu a)} e^{\beta_k y + \gamma_k z}, \tag{31}$$

where $W_k$ and $V_k$ are arbitrary constants. Substitution of (31) into (10) yields exact solutions of equations (4), (8). The described procedure can be regarded as the shock wave dressing.

Acknowledgement: Research has been partially supported by EPSRC Grant GR/S13682/01.

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