Diagonalizable Thue equations: revisited

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Abstract
Let \( r, h \in \mathbb{N} \) with \( r \geq 7 \) and let \( F(x, y) \in \mathbb{Z}[x, y] \) be a binary form such that
\[
F(x, y) = (\alpha x + \beta y)^r - (\gamma x + \delta y)^r,
\]
where \( \alpha, \beta, \gamma, \) and \( \delta \) are algebraic constants with \( \alpha \delta - \beta \gamma \neq 0 \). We establish upper bounds for the number of primitive solutions to the Thue inequality \( 0 < |F(x, y)| \leq h \), improving an earlier result of Siegel and of Akhtari, Saradha, and Sharma.

Keywords Diagonalizable forms · Thue equations · Primitive solutions

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1 Introduction
Let \( F(x, y) \) be a binary form with integer coefficients, having at least three pairwise non-proportional linear factors in its factorization over \( \mathbb{C} \), and let \( h \) be a non-zero integer. Thue \cite{18} proved that the equation
\[
F(x, y) = h
\]
has only a finite number of integer solutions. Such equations are called Thue equations. The problem of bounding the number of solutions of such equations, when the discriminant of \( F \) has large absolute value, has garnered significant interest. See, for example, [1, 2, 4–6, 8, 10–12, 14–17, 19].

Consider a diagonalizable binary form \( F(x, y) \in \mathbb{Z}[x, y] \) given by

\[
F(x, y) = (\alpha x + \beta y)^r - (\gamma x + \delta y)^r,
\]

where the constants \( \alpha, \beta, \gamma, \) and \( \delta \) satisfy

\[ j = \alpha \delta - \beta \gamma \neq 0. \]

It turns out (see [13, Lemma 4.1]) that either \( \alpha, \beta, \gamma, \delta \in \mathbb{Q} \) or \( [\mathbb{Q}(\beta/\alpha) : \mathbb{Q}] = 2 \), \( \delta/\gamma \) is the algebraic conjugate of \( \beta/\alpha \) over \( \mathbb{Q} \), \( \alpha' \in \mathbb{Q}(\beta/\alpha) \), and \( -\gamma' \) is the algebraic conjugate of \( \alpha' \) over \( \mathbb{Q} \). Hence

\[
u v = (\alpha x + \beta y)(\gamma x + \delta y) = \chi(Ax^2 + Bxy + Cy^2)
\]

for some \( A, B, C \in \mathbb{Z} \) and a constant \( \chi \), where \( \chi' \in \mathbb{Q} \). Let

\[
D = D(F) = B^2 - 4AC.
\]

Then

\[
j^2 = \chi^2 D.
\]

Thus \( D \neq 0 \). We denote the discriminant of \( F(x, y) \) by \( \Delta = \Delta(F) \). Put

\[
\Delta' = \frac{|\Delta|}{2r^2 - r^2 h^2 - 2r}. \tag{1}
\]

Let \( h \in \mathbb{N} \) and consider the Thue inequality

\[
0 < |F(x, y)| \leq h. \tag{2}
\]

We are interested in counting the number of non-zero tuples \((x, y) \in \mathbb{Z} \times \mathbb{Z}\) satisfying this inequality. Such a solution \((x, y)\) is said to be primitive if \( \gcd(x, y) = 1 \). We count \((x, y)\) and \((-x, -y)\) as one solution. Let \( N_F(h) \) denote the number of primitive solutions to the inequality (2).

In 1970, Siegel [16] proved the following theorem.

**Theorem 1.1** Suppose that

\[
\Delta' > (r^4 h)^{cr^{2-\epsilon}}.
\]
where \( r \geq 6 - \ell, \ell = 1, 2, 3, \)
\[
c_1 = 45 + \frac{593}{913}, \quad c_2 = 6 + \frac{134}{4583}, \quad \text{and} \quad c_3 = 75 + \frac{156}{167}.
\]

Then
\[
N_F(h) \leq \begin{cases} 
2\ell r & \text{if } D < 0, \\
4\ell & \text{if } D > 0, \; r \text{ is even and } F \text{ is indefinite,} \\
2\ell & \text{if } D > 0, \; r \text{ is odd and } F \text{ is indefinite,} \\
1 & \text{if } D > 0, \; \text{and } F \text{ is definite.}
\end{cases}
\]

In 2018, Akhtari et al. [3] improved the above result as follows. See [3, Theorems 1.3 and 1.4].

**Theorem 1.2**

(i) Let \( r \geq 6 \) and
\[
\Delta' \geq r^{13r^2(r-1)/(r^2-5r-2)} h^{4(r-1)(r^2-r+2)/(r^2-5r-2)}.
\]

Then
\[
N_F(h) \leq \begin{cases} 
2r + 1 & \text{if } D < 0, \\
5 & \text{if } D > 0, \; r \text{ is even and } F \text{ is indefinite,} \\
3 & \text{if } D > 0, \; r \text{ is odd and } F \text{ is indefinite,} \\
1 & \text{if } D < 0 \; \text{and } F \text{ is definite.}
\end{cases}
\]

(ii) Let \( r \geq 5 \) and
\[
\Delta' \geq r^{7r^2(r-1)/((r-1)m^2-2r-1)} h^{(r-1)(r^2+r+2)/((r-1)m^2-2r-1)}.
\]

Then for any \( m \geq 3 \) we have
\[
N_F(h) \leq \begin{cases} 
rm & \text{if } D < 0, \\
2m & \text{if } D > 0, \; r \text{ is even and } F \text{ is indefinite,} \\
m & \text{if } D > 0, \; r \text{ is odd and } F \text{ is indefinite,} \\
1 & \text{if } D < 0 \; \text{and } F \text{ is definite.}
\end{cases}
\]

We refer to [3, Table 1] for a comparison of Theorem 1.2 with Theorem 1.1. In short, Theorem 1.2(ii) is better than Theorem 1.1 as far as the lower bound for \( \Delta' \) is concerned and gives the same upper bound for \( N_F(h) \) by taking \( m = 2\ell, \ell = 2, 3 \). For the case \( \ell = 1 \) Theorem 1.2 is better than Theorem 1.1 with respect to \( \Delta' \). But in the estimate for \( N_F(h) \), we overshot by 1. This discrepancy seems to be coming out of a technical glitch while counting the number of solutions of (2) that are related to an \( r \)th root of unity. See Sect. 3 for details. In this paper, we correct this discrepancy and improve Theorem 1.2 as follows.
Theorem 1.3 Let \( r \geq 7 \) and
\[
\Delta' \geq r^{13 r^2 (r-1)/(r^2-5 r-2)} h^{4 (r-1) (r^2-r+2)/(r^2-5 r-2)},
\]
(5)
Then
\[
N_F(h) \leq \begin{cases} 
2r & \text{if } D < 0, \\
4 & \text{if } D > 0, r \text{ is even and } F \text{ is indefinite}, \\
2 & \text{if } D > 0, r \text{ is odd and } F \text{ is indefinite}, \\
1 & \text{if } D < 0 \text{ and } F \text{ is definite}.
\end{cases}
\]
Note that the bound for \( \Delta' \) in (5) is better than that of Siegel’s in Theorem 1.1 when \( \ell = 1 \). The technical condition \( r \geq 7 \) is needed in the symbolic computation done in Lemma 4.1. (See also Remark 5.1.)

2 Notation and preliminaries

Let \((x, y) \in \mathbb{Z}^2\) be a generic primitive solution of (2) i.e., the inequality
\[
0 < |F(x, y)| \leq h.
\]
If \( f \) is any function of \((x, y)\) then we write
\[
f = f(x, y).
\]
While enumerating the solutions of (2) as \((x_0, y_0), (x_1, y_1), \ldots\) we denote by
\[
f_i = f(x_i, y_i), \quad i \geq 0.
\]
We follow this notation throughout the paper without further mention. Define the functions \( u, v, \xi, \eta, \mu, Z, \zeta \) of \((x, y)\) as follows.
\[
u = \alpha x + \beta y, \quad v = \gamma x + \delta y, \quad \xi = u^r, \quad \eta = v^r.
\]
From [3, Lemmata 3.1 & 3.2], it follows that when \( D < 0, u \) and \( v \) are complex conjugates, giving \(|u| = |v|\). When \( D > 0, \xi \) and \( \eta \) are real with \( u \) and \( v \) as algebraic conjugates. Note that
\[
F(x, y) = \xi - \eta = u^r - v^r.
\]
Define
\[
\mu = \frac{\eta}{\xi} = \frac{v^r}{u^r}, \quad Z = \max(|u|, |v|), \quad \zeta = \frac{|F|}{Z^r}.
\]
Note that when $D > 0$, $\mu$ is real and

$$\zeta = \begin{cases} 1 - \mu & \text{if } 0 < \mu < 1, \\ 1 - \mu^{-1} & \text{if } \mu > 1, \\ 1 + |\mu| & \text{if } -1 < \mu < 0, \\ 1 + |\mu^{-1}| & \text{if } \mu < -1. \end{cases}$$ (6)

3 Solutions related to an $r$th root of unity

Write

$$F = \xi - \eta = \prod_{k=1}^{r} (u - ve^{2\pi ik/r}).$$

Let $\omega$ be an $r$th root of unity. We say that the solution $(x, y)$ is related to $\omega$ if

$$|u - v\omega| = \min_{1 \leq k \leq r} |u - ve^{2\pi ik/r}|.$$

Assume that

$$\frac{u}{v} = \frac{|u|}{v} e^{i\theta},$$

where

$$\frac{2(h-1)\pi}{r} \leq \theta < \frac{2h\pi}{r}$$

for some integer $h$ with $1 \leq h < r$ i.e., $u/v$ lies in the arc subtended by the rays passing through the two $r$th roots of unity viz., $e^{2(h-1)\pi/r}$ and $e^{2h\pi/r}$. The perpendicular bisector of the chord joining these two roots of unity is $z = e^{h\pi/r}$. So if $u/v$ lies below this line, then $\omega = e^{2(h-1)\pi/r}$. If $u/v$ lies above this line, then $\omega = e^{2h\pi/r}$. If $u/v$ lies on this line then it is at equal distance from both the roots of unity. In this case, as a convention we take $\omega = e^{2(h-1)\pi/r}$. Thus we see that every solution is uniquely related to an $r$th root of unity. We denote by $S$ the set of all primitive solutions of (2) and by $S_\omega$ the set of all primitive solutions of (2) that are related to $\omega$. Then

$$S = \cup S_\omega,$$

where $\omega$ ranges over all the $r$th roots of unity. Thus it is enough to estimate $|S_\omega|$. In the following lemmas we restrict to the set $S_\omega$. Let us enumerate all the solutions in $S_\omega$ as $(x_1, y_1), (x_2, y_2), \ldots$ so that $\cdots \leq \zeta_3 \leq \zeta_2 \leq \zeta_1$. Here we re-do many of the lemmas from [3, Sect. 5]. We point out that many of these lemmas are proved under
the assumption that the relevant \( \zeta < 1 \). Below we avoid this assumption. This is the technical glitch referred to in the Introduction.

**Lemma 3.1** Let \( D > 0 \). Suppose \( (x, y) \in S_\omega \). Then

\[
\frac{u}{v} = |\mu - 1|^{1/r} e^{\pi i \epsilon/r} \omega,
\]

where

\[
\epsilon = \begin{cases} 
0 & \text{if } \mu > 0 \\
1 & \text{if } \mu < 0.
\end{cases}
\]

**Proof** As noted earlier, when \( D > 0 \), \( \mu - 1 \) is real. So we write

\[
\frac{u^r}{v^r} = |\mu - 1| e^{\pi i \epsilon}.
\]

Then

\[
\frac{u}{v} = |\mu - 1|^{1/r} e^{\pi i (\epsilon + 2h)/r}
\]  \hspace{1cm} (7)

for some integer \( h \) with \( 0 \leq h < r \). Since \( (x, y) \in S_\omega \), we have

\[
\left| |\mu - 1|^{1/r} e^{\pi i \epsilon/r} - \omega e^{-2\pi i h/r} \right| = \min_{0 \leq k < r} \left| |\mu - 1|^{1/r} e^{\pi i \epsilon/r} - e^{2\pi i (k - h)/r} \right|.
\]  \hspace{1cm} (8)

As \( k \) varies from 0 to \( r - 1 \), \( k - h \) varies over a complete residue system \( \pmod{r} \). So

\[
\min_{0 \leq k < r} \left| |\mu - 1|^{1/r} e^{\pi i \epsilon/r} - e^{2\pi i (k - h)/r} \right| = \min_{0 \leq k < r} \left| |\mu - 1|^{1/r} e^{\pi i \epsilon/r} - e^{2\pi i k/r} \right|.
\]

Now \( |\mu - 1|^{1/r} e^{i \pi \epsilon/r} \) is a complex number in the first quadrant with argument \( \pi \epsilon/r \). So by our convention, the \( r \)th root of unity nearest to it is 1. Hence we conclude in (8) that \( \omega e^{-2\pi i h/r} = 1 \) giving

\[
e^{2\pi i h/r} = \omega,
\]

which proves the result.

\( \square \)

**Lemma 3.2** Let \( (x, y) \in S_\omega \). Also assume that \( \epsilon = 0 \) if \( D > 0 \). Then

\[
\left| \frac{u}{v} - \omega \right| \leq \frac{Z}{|v|} \zeta.
\]
Proof Let $D < 0$. Then from [3, Lemma 5.5, (37)], we have for $r \geq 3$,

$$\left| \frac{u}{v} - \omega \right| \leq \frac{\pi}{2r} \xi \leq \xi \leq \frac{Z}{|v|} \xi.$$ 

Now we take $D > 0$. Since $\epsilon = 0$, by Lemma (3.1), we get

$$\frac{u}{v} = |\mu|^{-1} |^{1/r} \omega.$$ 

Observe that

$$\begin{cases} 0 < |\mu|^{-1}|^{1/r} < 1 & \text{if } |u| \leq |v| \\ |\mu|^{-1}|^{1/r} > 1 & \text{if } |u| > |v|. \end{cases}$$

So it follows from Lemma 3.1 that

$$\left| \frac{u}{v} - \omega \right| = |\mu|^{-1}|^{1/r} - 1 = \begin{cases} 1 - |\mu|^{-1}|^{1/r} & \text{if } |u| \leq |v| \\ |\mu|^{-1}|^{1/r} - 1 & \text{if } |u| > |v|. \end{cases}$$

Let $|u| \leq |v|$. Then

$$\left| \frac{u}{v} - \omega \right| \leq 1 - |\mu|^{-1} = \frac{|F|}{Z^r} = \xi.$$ 

Let $|u| > |v|$. Then

$$\left| \frac{u}{v} - \omega \right| = |\mu|^{-1}|^{1/r} - 1 = |\mu|^{-1}|^{1/r} (1 - |\mu|^{1/r})$$

$$\leq |\mu|^{-1}|^{1/r} (1 - |\mu|) = \frac{|u|}{|v|} \left( 1 - \frac{|v|^r}{|u|^r} \right)$$

$$= \frac{|u|}{|v|} \frac{|F|}{Z^r} = \frac{Z}{|v|} \xi.$$ 

The proof of the lemma is complete.

Lemma 3.3 Let $(x, y) \neq (x_*, y_*)$ be two primitive solutions of (2) in $S_\omega$ with $\zeta_* \leq \zeta$. Then

$$Z_* \geq \frac{|j|}{2h^{1/r}}.$$ 

Proof Consider

$$uv_* - u_*v = (\alpha \delta - \beta \gamma) (xy_* - yx_*) = j(xy_* - yx_*) \neq 0.$$ (9)
Hence

$$|j| \leq |uv_*| + |u_*v| \leq 2ZZ_*.$$  \hfill (10)

Thus

$$Z_* \geq \frac{|j|\zeta^{1/r}}{2|F|^{1/r}}$$

which gives the assertion if \( \zeta \geq 1 \), since \(|F| \leq h\). So let us assume that \( \zeta < 1 \). If \( D > 0 \), then we have \( \mu > 0 \), \( \xi_* < 1 \), and \( \mu_* > 0 \) so that Lemma 3.2 is applicable. Thus from (9) and Lemma 3.2, we get

$$|j| \leq |vv_*| \left| \frac{u}{v} - \frac{u_*}{v_*} \right| \leq |vv_*| \left( \left| \frac{u}{v} - \omega \right| + \left| \frac{u_*}{v_*} - \omega \right| \right) \leq |vv_*| \left( \frac{Z\zeta}{|v|} + \frac{Z_*\xi_*}{|v_*|} \right) \leq 2ZZ_*\zeta.$$  

Hence

$$|j| \leq 2h^{1/r} \zeta^{(r-1)/r}Z_* \leq 2h^{1/r}Z_*$$ \hfill (11)

which proves the assertion for \( \zeta < 1 \). \hfill \(\Box\)

**Note.**

From (10),

$$|j| \leq 2h^{2/r}\zeta^{-1/r}\xi_*^{-1/r} \leq 2h^{2/r}\zeta_*^{-2/r}$$
giving

$$\xi_* \leq 2^{r/2}h|j|^{-r/2}.$$  

Thus if \(|j| > 2^{1+2\nu/r}h^{2/r}\), then \(\xi_* < 2^{-\nu}\). In particular, if \(|j| > 2h^{2/r}\), then \(\xi_* < 1\). As before, let us arrange the solutions in \(S_\omega\) as \((x_1, y_1), (x_2, y_2), \ldots\), so that

$$\xi_1 \geq \xi_2 \geq \xi_3 \geq \cdots.$$  

By our observation above, we see that

$$\xi_i < 1 \text{ for } i \geq 2 \text{ if } |j| > 2h^{2/r}.$$ \hfill (12)

We will use this arrangement of the solutions in \(S_\omega\) from now on.
Lemma 3.4 Let \((x_1, y_1), (x_2, y_2), \ldots, (x_t, y_t)\) be in \(S_\omega\) with \(t \geq 3\). Assume that
\[
|j| > 2^{1+(r-2)/(r(R(t-1)-1))}h^{2/r}.
\]
Then
\[
\zeta_{t-1} < 1/2.
\]

Proof Applying Lemma 3.3 with \((x, y) = (x_1, y_1)\) and \((x_*, y_*) = (x_2, y_2)\) we get
\[
\zeta_2 \leq \frac{2^r h^2}{|j|^r}.
\]
Thus
\[
\zeta_2 \leq H^r,
\]
where
\[
H = 2h^{2/r}|j|^{-1}.
\]
Again applying Lemma 3.3 with \((x, y) = (x_2, y_2)\) and \((x_*, y_*) = (x_3, y_3)\), we get
\[
\zeta_3 \leq H^r \zeta_2^{r-1} \leq H^r (1+(r-1)).
\]
Proceeding inductively, we obtain that
\[
\zeta_{t-1} \leq H^r (1+(r-1)+\cdots+(r-1)^{t-3}).
\]
Thus \(\zeta_{t-1} < 1/2\) if
\[
|j|^{r(R(t-1)-1)/(r-2)} > 2^{1+r(R(t-1)-1)/(r-2)} h^{2(R(t-1)-1)/(r-2)}.
\]
Hence \(\zeta_{t-1} < 1/2\) if
\[
|j| > 2^{1+(r-2)/(r(R(t-1)-1))}h^{2/r}.
\]
\[\square\]

Lemma 3.5 Let \((x_{i-1}, y_{i-1}), (x_i, y_i)\) be in \(S_\omega\) with \(i \geq 2\). Assume that \(|j| > 2h^{2/r}\) if \(D > 0\). Then
\[
Z_i \geq \frac{|j| Z_{i-1}^{r-1}}{2h}.
\]
\textbf{Proof} We always have
\[ \zeta_i \leq \zeta_{i-1}. \]

Note that
\[ |j| \leq |u_{i-1}v_i - u_iv_{i-1}|. \]

So it is enough to show that
\[ |u_{i-1}v_i - u_iv_{i-1}| \leq 2Z_{i-1}Z_i\zeta_{i-1} \tag{13} \]

since \( \zeta_{i-1} \leq h/Z_{i-1}^r \). First let \( D < 0 \). Then by Lemma 3.2
\[ |u_{i-1}v_i - u_iv_{i-1}| \leq |v_{i-1}v_i| \left( |\mu_{i-1}^{-1}|^{1/r} e^{\pi i \epsilon} \right) \leq Z_{i-1}Z_i (\zeta_{i-1} + \zeta_i) \leq 2Z_{i-1}Z_i\zeta_{i-1}. \]

Next let \( D > 0 \). Then by (12), \( \zeta_i < 1 \). By Lemma 3.1 we have
\[ \frac{u_{i-1}}{v_{i-1}} = |\mu_{i-1}^{-1}|^{1/r} e^{\pi i \epsilon}; \quad \frac{u_i}{v_i} = |\mu_i^{-1}|^{1/r} e^{\pi i \epsilon}. \]

So
\[ |u_{i-1}v_i - u_iv_{i-1}| = \begin{cases} (i) |u_{i-1}u_i| |\mu_i|^{1/r} - |\mu_{i-1}|^{1/r} e^{-\pi i \epsilon / r} |, \\ (ii) |u_{i-1}v_i| \left( 1 - |\mu_i|^{1/r} |\mu_{i-1}|^{1/r} e^{-\pi i \epsilon / r} \right), \\ (iii) |v_{i-1}u_i| \left( 1 - |\mu_i|^{1/r} |\mu_{i-1}|^{1/r} e^{\pi i \epsilon / r} \right), \\ (iv) |v_{i-1}v_i|, |\mu_i|^{1/r} |\mu_{i-1}|^{1/r} e^{\pi i \epsilon / r}. \end{cases} \tag{14} \]

First we deal with the case \( \epsilon = 0 \). Then \( \mu_{i-1} > 0 \). Since \( \zeta_i < 1 \), by (6) we get \( \mu_i > 0 \). We need to consider 8 cases depending on the signs and the values of \( \mu_{i-1} \) and \( \mu_i \). In each case we show that (13) holds.

\textit{Case 1} Let \( 0 < \mu_i, \mu_{i-1} < 1 \).

Then
\[ 0 < 1 - \mu_{i-1}^{1/r} < 1 - \mu_{i-1} = \zeta_{i-1}; \quad 0 < 1 - \mu_i^{1/r} < 1 - \mu_i = \zeta_i. \]

By (14)(i),
\[ |u_{i-1}v_i - u_iv_{i-1}| = |u_{i-1}u_i| \left( 1 - |\mu_i|^{1/r} \right) \left( 1 - |\mu_{i-1}|^{1/r} \right) \]
\[
\leq |u_{i-1}u_i|(\zeta_{i-1} + \zeta_i)
\]
\[
\leq 2Z_{i-1}Z_i\zeta_{i-1}.
\]

Case 2 Let \(0 < \mu_i < 1, \mu_{i-1} > 1\).
Then
\[
0 < 1 - \mu_i^{-1/r} < 1 - \mu_{i-1}^{-1} = \zeta_{i-1}; \quad 0 < 1 - \mu_i^{1/r} < 1 - \mu_i = \zeta_i
\]
and
\[
0 < 1 - \mu_i^{-1/r} \mu_i^{1/r} < 1 - \mu_i^{-1}\mu_i = 1 - (1 - \zeta_{i-1})(1 - \zeta_i) < 2\zeta_{i-1}.
\]

By (14)(iii),
\[
|u_{i-1}v_i - u_i v_{i-1}| = |v_{i-1}u_i| \left| 1 - \mu_i^{-1/r} \mu_i^{-1} \right|^{1/r}
\]
\[
\leq 2|v_{i-1}u_i|\zeta_{i-1}
\]
\[
\leq 2Z_{i-1}Z_i\zeta_{i-1}.
\]

Case 3 Let \(\mu_i > 1, 0 < \mu_{i-1} < 1\).
Then
\[
\zeta_{i-1} = 1 - |\mu_{i-1}|; \quad 0 < 1 - \mu_i^{-1/r} < 1 - \mu_i^{-1} = \zeta_i.
\]

By (14)(ii),
\[
|u_{i-1}v_i - u_i v_{i-1}| = |u_{i-1}v_i| \left| 1 - |\mu_i|^{-1/r} |\mu_{i-1}| \right|^{1/r}
\]
\[
\leq |u_{i-1}v_i|(1 - |\mu_{i-1}\mu_i^{-1}|)
\]
\[
\leq |u_{i-1}v_i|(1 - |\mu_{i-1}|(1 - \zeta_i))
\]
\[
\leq |u_{i-1}v_i|(1 + \zeta_i) \leq 2Z_{i-1}Z_i\zeta_{i-1}.
\]

Case 4 Let \(\mu_i > 1, \mu_{i-1} > 1\).
Then
\[
0 < 1 - \mu_{i-1}^{-1/r} < 1 - \mu_{i-1}^{-1} = \zeta_{i-1}; \quad 0 < 1 - \mu_i^{-1/r} < 1 - \mu_i^{-1} = \zeta_i.
\]

By (14)(iv),
\[
|u_{i-1}v_i - u_i v_{i-1}| = |v_{i-1}v_i| \left| |\mu_i|^{-1/r} - |\mu_{i-1}| \right|^{1/r}
\]
\[
\leq |v_{i-1}v_i|(1 - |\mu_{i-1}|^{1/r} + |1 - \mu_i^{-1}|)
\]
\[
\leq |v_{i-1}v_i|(\zeta_{i-1} + \zeta_i)
\]
\[ \leq 2|v_{i-1}v_i|\zeta_{i-1} \leq 2Z_{i-1}Z_i\zeta_{i-1}. \]

This completes all the cases when \( \epsilon = 0 \).

Next we take \( \epsilon = 1 \). Then \( \mu_{i-1} < 0 \) and so \( \zeta_{i-1} > 1 \). Recall that \( \mu_i > 0 \). There are four cases to consider, viz.,

(a) \( 0 < \mu_i \leq 1, -1 \leq \mu_{i-1} < 0 \)  
(b) \( 0 < \mu_i \leq 1, \mu_{i-1} < -1 \);  
(c) \( \mu_i > 1, -1 \leq \mu_{i-1} < 0 \)  
(d) \( \mu_i > 1, \mu_{i-1} < -1 \).

Using (i), (iii), (ii), and (iv), respectively, for (a), (b), (c), and (d), we find that

\[ |u_{i-1}v_i - u_iv_{i-1}| \leq 2Z_{i-1}Z_i < 2Z_{i-1}Z_i\zeta_{i-1} \]

since \( \zeta_{i-1} > 1 \).

\[ \square \]

4 Condition on \( |j| \)

Using Padé approximation, in [3, Lemma 7.3] certain algebraic numbers were constructed. This construction was used along with symbolic computation and Lemma 3.5 to get an iterative bound in [3, Lemma 8.1] which we state below. As the Lemmas in Sect. 3 are valid without any condition on the \( \zeta \)'s, we are able to state this lemma for \( S_\omega \) rather than \( S'_\omega \) (see [3, (40)]). We shall also restrict to \( |S_\omega| = 3 \).

**Lemma 4.1** Let \( r \geq 7 \). Assume that \( |S_\omega| = 3 \). Suppose that

\[ |j| \geq 2r^{i_7/r}h^{i_8/r} \]  

with

\[ i_7 = \frac{13r^2}{r^2 - 5r - 2}, \quad i_8 = \frac{2(3r - 1)(r - 2)}{r^2 - 5r - 2}. \]  

Then for every integer \( n \geq 1 \), we have

\[ Z_3 \geq \frac{Z_2^{nr}}{2n+4r(3nr+2)/(r-2)|j|(nr+2)/(r-2)h^{2n+1}}. \]  

Note that the exponent of \( Z_2 \) in the above estimate is \( nr \) while in Lemma 8.1 of [3], it was \( (n + 1)r - 1 \). This weaker estimate is sufficient for our purpose.

**Proof** We follow the proof of Lemma 8.1 of [3]. We give details wherever necessary. Let us denote the solutions in \( S_\omega \) as

\( (x_1, y_1), (x_2, y_2), (x_3, y_3) \) with \( \zeta_1 \geq \zeta_2 \geq \zeta_3 \).
Then

\[ Z_3 \geq \frac{|j|}{2h} Z_2^{r-1}, \quad Z_2 \geq \frac{|j|}{2h^{1/r}} \]  \hspace{1cm} (18)

giving

\[ Z_3 \geq \frac{|j|^r}{2^r h^{2-1/r}}. \]

Now we follow the argument as in the proof of Theorem 1.3 of [3] to get

\[ Z_3 \geq Z_r(n + 1 - g) - 1 + g, \quad n \geq 1, \quad g \in \{0, 1\}. \]  \hspace{1cm} (19)

if

\[ Z_3^{r-1} \geq 2^{3n+4} r^{(2g+3n)2/(r-2)}|j|^{(r(g+n)+2)/(r-2)} h^{2n+1-g}. \]  \hspace{1cm} (20)

For a given integer \( n \geq 1 \) and \( g \in \{0, 1\} \) let \( a_i = a_i(n, r, g) \), \( 1 \leq i \leq 5 \), we say that property \( P[a_1, a_2, a_3, a_4, a_5] \) holds if

\[ Z_3 \geq \frac{Z_2^{a_1}}{2^{a_2} h^{a_3} |j|^{a_4} h^{a_5}}. \]  \hspace{1cm} (21)

Let us now assume that \( P[a_1, a_2, a_3, a_4, a_5] \) holds with \( a_2 + a_4 \geq 0 \). Then (20) is valid if

\[ Z_2^{a_1(r-1) - nr - 1 + g} \geq 2^{a_2(r-1) + 3n + 4} r^{a_3(r-1) + (2g+3n)2/(r-2)} |j|^{a_4(r-1) + (r(g+n)+2)/(r-2)} h^{a_5(r-1) + 1}. \]

By (18), the above inequality is valid if

\[ |j|^{A_1} \geq 2^{A_1 + a_2(r-1) + 3n + 4} r^{a_3(r-1) + (2g+3n)2/(r-2)} |j|^{a_4(r-1) + (r(g+n)+2)/(r-2)} h^{A_1/r + a_5(r-1) + 1}, \]  \hspace{1cm} (22)

where

\[ A_1 = a_1(r - 1) - nr - 1 + g. \]

We set

\[ B_1 = A_1 - a_4(r - 1) - (r(g + n) + 2)/(r - 2), \]

\[ B_2 = A_1 + a_2(r - 1) + 3n + 4, \]
We implement the induction procedure given in [3, Sect. 8] with the above values of $B_1, \ldots, B_4$ and $a_1, \ldots, a_5$ as already given in the procedure. If the conditions

\begin{align*}
(i) & \quad A_1 > 0 \quad (ii) B_1 > 0 \quad (iii) B_1 \times \frac{13r^2}{r^2 - 5r - 2} \geq r(B_3 + (B_2 - B_1)/2) \\
(iv) & \quad B_1 \times \frac{2(3r - 1)(r - 2)}{r^2 - 5r - 2} \geq rB_4
\end{align*}

are satisfied then (22) holds and hence (19) holds. To begin with, by (21), $P[r - 1, 1, 0, -1, 1]$ holds. Let $(n, g) = (1, 0)$. The conditions (i)–(iv) are satisfied for $a_1 = r - 1, a_2 = 1, a_3 = 0, a_4 = -1, a_5 = 1$. If $\Sigma_{1,0} \neq 0$ (see [3, Sect. 7] for the definition of $\Sigma_{n,g}$), we get that $P[2r - 1, 5, (3r + 2)/(r - 2), (r + 2)/(r - 2), 3]$ is valid. Thus (17) is valid with $n = 1$ since

$$Z_{r-1}^2 \geq 1$$

by (15),(16), and (13). If $\Sigma_{1,0} = 0$, then $\Sigma_{1,1}, \Sigma_{2,1}$ are non-zero. Fixing $(n, g) = (1, 1)$ and using the same parameters as in [3, Lemma 8.1], we find that $P[r, 5, (5r + 2)/(r - 2), (2r + 2)/(r - 2), 2]$ is valid. Using this and taking $(n, g) = (2, 1)$, we get that

$$Z_3 \geq \frac{Z_2^{2r}}{2^6r(8r+2)/(r-2)|j|^{(3r+2)/(r-2)}h^4}.$$

Thus (17) is valid with $n = 1$ provided

$$\frac{Z_2^r}{2^{5r/(r-2)}|j|^{2r/(r-2)}h} \geq 1.$$  \hspace{1cm} (23)

This is again satisfied by (15),(16), and (13). Now we proceed by induction as in [3, Lemma 8.1] to complete the lemma. \hfill \Box

**5 Proof of Theorem 1.3**

Assume that (3) holds. Since

$$\Delta = (-1)^{\frac{(r-1)(r+2)}{2}} r^r j^r (r-1),$$

we get (15). Now, by [3, Theorem 1.3], we may assume that

$$N_F(h) = |S| = 2r + 1.$$
Our arguments are similar to Theorem 1.3 of [3]. We give the details wherever it is necessary. By our assumption on $N_F(h)$, there exists some $\omega$, say $\omega_1$ with

$$|S_{\omega_1}| \geq 3.$$ 

Suppose there is another $\omega_2$ with $|S_{\omega_2}| \geq 3$. Then the solution $(x_0, y_0)$ with $\zeta_0$ largest can belong to at most only one of these two sets. Hence there exists a set $S_{\omega}$, $\omega = \omega_1$ or $\omega_2$ such that $S_{\omega}$ does not contain $(x_0, y_0)$ and $|S_{\omega}| \geq 3$. Then $S_{\omega} = S_{\omega''}$, where $S_{\omega''}$ is as in the proof of Theorem 1.3 in [3]. So we may follow the argument therein to get a contradiction. Hence there exists exactly one $\omega$, say $\omega_1$ such that

$$|S_{\omega_1}| = 3 \quad \text{and} \quad |S_{\omega}| = 2 \quad \text{for } \omega \neq \omega_1.$$ 

Now by Lemma 4.1 applied to $S_{\omega_1}$ we get, with the same notation as in there, that

$$Z_3 \geq \frac{Z_2^{nr}}{2n+4r(3nr+2)/(r-2)|j|^{(nr+2)/(r-2)}h^{2n+1}} \quad \text{(24)}$$

holds for all $n \geq 1$. But the right-hand side approaches infinity as $n$ approaches infinity, which gives the final contradiction. \hfill \Box

**Remark 5.1** In Lemma 4.1, we have used (13) i.e.,

$$Z_3 \geq \frac{|j|}{2h}Z_2^{r-1}, \quad Z_2 \geq \frac{|j|}{2h^{1/r}}.$$ 

The bound for $Z_2$ which is achieved (irrespective of $\zeta_2 < 1$ or $> 1$) in Lemma 3.3 is weaker than the following bound, which was used in the proof of [3, Theorem 1.3] (see [3, (72),(73)])

$$Z_3 \geq \frac{|j|}{2h}Z_2^{r-1}, \quad Z_2 \geq \frac{|j|}{2h^{1/r}} \quad \text{with } Z_1 \geq \frac{|j|^{1/2}}{2^{1/2}h^{1/r}}.$$ 

As already pointed out in the Introduction, using the above inequalities, we had obtained a bound for $N_F(h)$ which was 1 more than Siegel’s bound. In this paper, we use the weaker bound (13) and implement the induction procedure in Lemma 4.1. We need to check the conditions $(i) - (iv)$ at every stage. In the initial stage, when $\Sigma_{1,0} = 0$, $P[r, 5, (5r+2)/(r-2), (2r+2)/(r-2), 2]$ holds and $(n, g) = (2, 1)$ we find

$$B_1 = \frac{r^3 - 7r^2 + 2r}{r - 2} < 0 \quad \text{if } r = 6.$$ 

Thus the whole induction procedure fails for $r = 6$. So the proof of Theorem 1.3 which depends heavily on Lemma 4.1 is valid for $r \geq 7$. May we say, a loss in profit? Strengthening Lemma 3.3 is a possible remedy for this. At present, we are unable to do it.
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References

1. Akhtari, S.: The method of Thue-Siegel for binary quartic forms. Acta. Arith. 141(1), 1–31 (2010)
2. Akhtari, S., Bengoechea, P.: Representation of integers by sparse binary forms. Trans. Am. Math. Soc. 374(3), 1687–1709 (2020)
3. Akhtari, S., Saradha, N., Sharma, D.: Thue’s inequalities and the hypergeometric method. Ramanujan J. 45(2), 521–567 (2018)
4. Bennett, M.A.: Rational approximation to algebraic numbers of small height: the Diophantine equation $|ax^n - by^n| = 1$. J. Reine Angew. Math. 535, 1–49 (2001)
5. Bennett, M.A.: On the representation of unity by binary cubic forms. Trans. Am. Math. Soc. 353, 1507–1534 (2001)
6. Bennett, M.A., de Weger, B.M.M.: On the Diophantine equation $|ax^n - by^n| = 1$. Math. Comput. 67, 413–438 (1998)
7. Corrigendum, ibid. 86/3-4 , 503–504 (2015)
8. Dethier, C.: Diagonalizable quartic Thue equations with negative discriminant. Acta Arith. 193(3), 235–252 (2020)
9. Erratum. Invent. Math. 75(2), 379 (1984)
10. Evertse, J.H.: On the representation of integers by binary cubic forms of positive discriminant. Invent. Math. 73(1), 117–138 (1983); Erratum. Invent. Math. 75(2), 379 (1984)
11. Győry, K.: Thue inequalities with a small number of primitive solutions. Period. Math. Hungar. 42, 199–209 (2001)
12. Győry, K.: On the number of primitive solutions of Thue equations and Thue inequalities, Paul Erdős and his mathematics I. Bolyai Soc. Math. Stud. 11, 279–294 (2002)
13. Paul, M.: Voutier, Thue’s Fundamental theorem, I: the general case. Acta Arith. 143(2), 101–144 (2010)
14. Saradha, N., Sharma, D.: Number of representations of integers by binary forms. Publ. Math. Debrecen 85(1–2), 233–255 (2014); Corrigendum, ibid. 86/3-4 , 503–504 (2015)
15. Saradha, N., Sharma, D.: Number of solutions of cubic Thue inequalities with positive discriminant. Acta Arith. 171(1), 81–95 (2015)
16. Siegel, C.L.: Einige Erläuterungen zu Thues Untersuchungen über Annäherungswerte algebraischer Zahlen und diophantische Gleichungen. Nach. Akad. Wissen Göttingen Math-phys 169–195 (1970)
17. Stewart, C.L.: On the number of solutions of polynomial congruences and Thue equations. J. Am. Math. Soc. 4, 793–835 (1991)
18. Thue, A.: Über Annäherungswerte algebraischer Zahlen. J. Reine Angew. Math. 135, 284–305 (1909)
19. Wakabayashi, I.: Cubic Thue inequalities with negative discriminant. J. Number Theory 97(2), 222–251 (2002)

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