Incarnations of Instantons

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Abstract

Yang-Mills instantons in a pure Yang-Mills theory in four Euclidean space can be promoted to particle-like topological solitons in $d = 4 + 1$ dimensional space-time. When coupled to Higgs fields, they transform themselves in the Higgs phase into Skyrmions, lumps and sine-Gordon kinks, with trapped inside a non-Abelian domain wall, non-Abelian vortex and monopole string, respectively. Here, we point out that a closed monopole string, non-Abelian vortex sheet and non-Abelian domain wall in $S^1$, $S^2$ and $S^3$ shapes, respectively, are all Yang-Mills instantons if their $S^1$, $S^2$ and $S^3$ moduli, respectively, are twisted along their world-volumes.

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I. INTRODUCTION

Recent discovery of non-Abelian vortices [1, 2] and non-Abelian domain walls [3–5] has been revealing relations among different topological solitons in diverse dimensions [6–8]. When a ’t Hooft-Polyakov monopole [9] is put into the Higgs phase, the magnetic fluxes from it are squeezed into the form of magnetic vortices, becoming a confined monopole [10–13]. This configuration can be regarded as a kink inside a vortex. In other words, the monopole turns to the kink when it resides in the vortex. In the Higgs phase, Yang-Mills instantons are unstable to shrink in the bulk. However, they can stably exist as lumps (or sigma model instantons) [14] when trapped inside a non-Abelian vortex [12, 15], while they can stably exist as Skyrmions when trapped inside a non-Abelian domain wall [4]. Recently, it has been also found that instantons transform themselves into sine-Gordon kinks when trapped inside a monopole string in a certain situation [16]. Vortices or lumps become sine-Gordon kinks [17–19] when trapped inside a $CP^1$ domain wall [20, 21]. Skyrmions become lumps or baby Skyrmions [22, 23] inside a non-Abelian domain wall [24], and more generally $N$ dimensional Skyrmions become $N−1$ dimensional Skyrmions inside a non-Abelian domain wall [25]. Among those, one of the most successful applications to field theory may be made by confined monopoles [11, 12] which explain the coincidence of BPS spectra in 3+1 and 1+1 dimensions [26].

The $CP^1$ model in $d = 1 + 1$ dimensions offers a toy model of Yang-Mills theory in $d = 3 + 1$ dimensions. The presence of instantons [14] is one of such similarities between them. Sigma model instantons can be promoted to lumps in $d = 2 + 1$ dimensions. With a mass term admitting two vacua, the $CP^1$ model allows a domain line with a $U(1)$ modulus [20, 21]. If the $U(1)$ modulus winds around a straight domain line, a sine-Gordon kink is formed on it corresponding to a lump in the bulk [17], which is a lower dimensional analogue of trapped instantons. The dynamics of such domain wall Skyrmions were studied [19].

Then, a question arises. Do all instantons have to be trapped into some host solitons to become composite states in theories in the Higgs phase? The answer is no. If one makes a closed domain line with the $U(1)$ modulus wound along it, such a twisted closed domain line is nothing but an isolated lump [27]. This is stabilized against shrinkage and becomes a baby Skyrmion if one adds a four derivative Skyrme term in the original theory [18]. An alternative way to stabilize a twisted closed domain line is to give a linear time dependence...
TABLE I: Host solitons of trapped instantons in $d = 2 + 1$ (a) and in $d = 4 + 1$ (b)–(d). The shape of the world-volume can be noncompact $\mathbb{R}^n$ or compact $S^n$, corresponding to trapped and untrapped instantons, respectively. NA denotes “non-Abelian.”

| host solitons     | bulk dim | codim. | moduli | w.v. shape | w.v. soliton | homotopy |
|-------------------|----------|--------|--------|------------|--------------|----------|
| (a) $\mathbb{C}P^1$ domain wall | $2 + 1$  | 1      | $S^1$  | $\mathbb{R}^1$ or $S^1$ | SG kink | $\pi_1(S^1)$ |
| (b) NA domain wall | $4 + 1$  | 1      | $S^3$  | $\mathbb{R}^3$ or $S^3$ | Skyrmion | $\pi_3(S^3)$ |
| (c) NA vortex sheet | $4 + 1$  | 2      | $S^2$  | $\mathbb{R}^2$ or $S^2$ | lump | $\pi_2(S^2)$ |
| (d) monopole string | $3$      | 3      | $S^1$  | $\mathbb{R}^1$ or $S^1$ | SG kink | $\pi_1(S^1)$ |

FIG. 1: Incarnations of instantons in $d = 2 + 1$ (a) and in $d = 4 + 1$ (b)–(d). (a)–(d) correspond to those of Table I. The arrows schematically denote points in the $S^1$, $S^2$ and $S^3$ moduli.

on the $U(1)$ modulus, which results in a Q-lump. This situation is summarized in Table I (a) and is illustrated in Fig. I(a). Thus, we have both trapped and untrapped instantons, where the domain line world-volume is $\mathbb{R}^1$ and $S^1$, respectively. We may call the latter as
incarnations of instantons.

Here, we propose higher dimensional analogues of this phenomenon for Yang-Mills instantons in Yang-Mills-Higgs theories in \(d = 4 + 1\) dimensions. In this dimensionality, instantons are particle-like solitons, while a vortex and monopole are a sheet (membrane) and string, respectively. In order to demonstrate our idea, we take the gauge group as \(U(2) = [SU(2) \times U(1)]/\mathbb{Z}_2\) but generalizations to \(U(N) = [SU(N) \times U(1)]/\mathbb{Z}_N\) or other groups are straightforward, since vortices in arbitrary gauge groups \([29]\) such as \(SO(N)\) and \(USp(2N)\) \([30]\) were already constructed. We put the system into the Higgs phase where the \(U(2)\) gauge group is spontaneously broken completely, by introducing some doublet Higgs fields with the common \(U(1)\) charges. Unlike the case of \(d = 2 + 1\), there are three possibilities of incarnations of instantons. In Table I (b)–(d), we summarize host solitons with world-volume \(\mathbb{R}^{n,1}\) in which instantons can reside stably, \(i.e.,\) a non-Abelian domain wall \((n = 3)\), non-Abelian vortex sheet \((n = 2)\), and monopole string \((n = 1)\) of codimensions one, two and three, respectively. These solitons have internal moduli \(S^3, S^2\) and \(S^1\), respectively, as localized Nambu-Goldstone zero modes in addition to translational moduli. When these moduli wind in the spatial world-volumes \(\mathbb{R}^n\) of the host solitons according to the homotopy groups \(\pi_n(S^n) \simeq \mathbb{Z}\), there appear Skyrmions, lumps and sine-Gordon kinks in the world-volumes of the domain wall, vortex sheet, and monopole string, respectively. They are all Yang-Mills instantons in the bulk point of view. Here, we have the following relation among the dimensionality and the number of moduli:

\[
\#(\text{codim}) + \#(\text{moduli}) = \text{spatial dim (bulk)},
\]

for both \(d = 2 + 1\) and \(4 + 1\) dimensions.

In this paper, we point out that when world-volumes of the host solitons are closed as \(S^n\) instead to be flat and infinite, they can be regarded as incarnations of instantons, that is, untrapped instantons in the bulk, if the moduli \(S^n\) wind along the world-volumes \(S^n\), as in Fig. \(\text{I}\). Since the world-volumes are closed, the total topological charge of the host soliton is canceled out, and there remains only the instanton charge. We need higher derivative terms for the stability of these solitons in the same spirit with the Skyrme model, which has been demonstrated explicitly for a twisted closed domain line as a baby Skyrmeon in \(d = 2 + 1\) \([18]\).

They are all higher dimensional generalizations of vortons. When a vortex string has a
$U(1)$ modulus in $d = 3 + 1$ dimensions, one can consider a vorton, which is a closed vortex string with the $U(1)$ modulus twisted along the string \cite{31,33}. To enhance the stability, one usually considers a linear time dependence on the $U(1)$ modulus. The stability of the vorton in $d = 3 + 1$ dimensions has been a longstanding problem for decades after the proposal, and has been established recently \cite{34}.

This paper is organized as follows. In Sect. II, we introduce three models of $U(2)$ gauge theories coupled with some Higgs doublets. In Sect. III, we present instantons trapped into host solitons, a domain wall, vortex-sheet and monopole-string. In Sect. IV we point out that closed host solitons are instantons when moduli are twisted along their world-volumes. Sect. V is devoted to summary and discussion.

II. YANG-MILLS-HIGGS THEORIES

We consider the following three models, $U(2)$ gauge theories coupled with $N_F$ doublet Higgs fields with the common $U(1)$ charge in $d = 4 + 1$ dimensions. The field contents are the $U(2)$ gauge field $A_A$ ($A, B = 0, 1, 2, 3, 4$), a two by two real matrix of adjoint scalar fields $\Sigma$, and a $2 \times N_F$ matrix of complex scalar fields $H$. The Lagrangians which we consider are of the form

$$
\mathcal{L} = -\frac{1}{4g^2} \text{tr} F_{AB} F^{AB} + \frac{1}{2g^2} \text{tr} (D_A \Sigma)^2 + \text{tr} D_A H^\dagger D^A H - V, \tag{2}
$$

where $g$ is the gauge coupling, and the covariant derivatives are $D_A \Sigma = \partial_A \Sigma - ig[A_A, \Sigma]$ and $D_A H = \partial_A H - igA_A H$. We consider the following three models in order to trap instantons into a domain wall, vortex, and monopole string.

1. The theory 1 contains four Higgs doublets with the common $U(1)$ charge ($N_F = 4$) summarized as a two by four matrix $H$. The potential term is (see, e.g.,Ref. \cite{7})

$$
V = g^2 \text{tr} (HH^\dagger - v^2 1_2)^2 + \text{tr} [\Sigma H - HM]^2, \tag{3}
$$

with the mass matrix $M = \text{diag.}(m, m, -m, -m)$. The flavor symmetries are $SU(2)_L \times SU(2)_R \times U(1)$ acting on the first and the last two flavors, respectively, for $m \neq 0$, and $SU(4)$ for $m = 0$. 

5
2. The theory 2 is consisting of two Higgs doublets with the common $U(1)$ charge ($N_F = 2$) summarized as a two by two matrix $H$. The potential term is (see, e.g., Ref. [7])

$$V = g^2 \text{tr} (HH^\dagger - v^2 1_2)^2. \quad (4)$$

We do not introduce mass parameters. The flavor symmetry is $SU(2)$.

3. The theory 3 contains two Higgs doublets with the common $U(1)$ charge ($N_F = 2$) in a two by two matrix $H$. The potential term is [16]

$$V = g^2 \text{tr} (HH^\dagger - v^2 1_2)^2 + \text{tr} [H(\Sigma - M)^2 H^\dagger] + \frac{\beta^2}{v^2} \text{tr} (H\sigma_3 H^\dagger), \quad (5)$$

with the mass matrix $M = \text{diag}(m, -m)$. The case $m = \beta = 0$ becomes the theory 2. The flavor symmetry is $U(1)$ for $\beta = 0, m \neq 0$ and no flavor symmetry for $\beta \neq 0, m \neq 0$.

These Lagrangians can be made $\mathcal{N} = 2$ supersymmetric (i.e., with eight supercharges) by suitably adding fermions except for the case of $\beta \neq 0$ in the theory 3. The constant $v^2$ giving a VEV to $H$ is called the Fayet-Iliopoulos parameter in the context of supersymmetry. In the limit of vanishing $v^2$, the systems go to the unbroken phase of the gauge group where $H$’s decouple in the vacuum.

### III. INSTANTONS INSIDE HOST SOLITONS WITH FLAT WORLD-VOLUME

#### A. Theory 1: Instantons inside a flat domain wall

In the massless case $m = 0$, the vacuum can be taken to be

$$H = (v 1_2, 0_2), \quad \Sigma = 0_2 \quad (6)$$

by using the flavor symmetry. This is the so-called color-flavor locked vacuum. The moduli space of vacua is the Grassmannian manifold $Gr_{4,2} \simeq SU(4)/[SU(2) \times SU(2) \times U(1)]$, see, e.g., Ref. [35]. In the massive case, $m \neq 0$, the vacuum is split into three disjoint vacua

$$H = (v 1_2, 0_2), \quad \left( \begin{array}{ccc} v & 0 & 0 \\ 0 & 0 & v \end{array} \right), \quad \text{or} \quad (0_2, v 1_2), \quad \text{and} \quad \Sigma = v 1_2, \quad (7)$$

with the following unbroken symmetries, respectively:

$$SU(2)_{C+L}, \quad U(1) \times U(1), \quad \text{or} \quad SU(2)_{C+R}. \quad (8)$$
A non-Abelian domain wall solution interpolating between the first and third vacua, which is perpendicular to the \(x^4\) coordinate, is given by \[ 3–5, 36 \]

\[
H_{\text{wall},0} = \frac{1}{\sqrt{1 + |u_{\text{wall}}|^2}} (1_2, u_{\text{wall}}1_2), \quad u_{\text{wall}}(x^4) = e^{\mp m(x^4 - X) + i\varphi},
\]

\[
\Sigma_{\text{wall},0} = v^{-2}HMH^\dagger, \quad A_{4,\text{wall},0} = iv^{-2}(H\partial_4 H^\dagger - \partial_4 H \cdot H^\dagger),
\]

in the strong gauge coupling limit. The general solution is

\[
H_{\text{wall}} = VH_{\text{wall},0} \begin{pmatrix} V^\dagger & 0 \\ 0 & V \end{pmatrix} = \frac{1}{\sqrt{1 + |e^{\mp m(x^4 - X)}|^2}} \begin{pmatrix} 1_2, e^{\mp m(x^4 - X)}U \end{pmatrix},
\]

\[
\Sigma_{\text{wall}} = V\Sigma_{\text{wall},0}V^\dagger, \quad A_{4,\text{wall}} = VA_{4,\text{wall},0}V^\dagger,
\]

with \(V \in SU(2)\) and \(U \equiv V^2 e^{i\varphi} \in U(2)\). Therefore the domain wall has \(\mathbb{R} \times U(2)\) moduli. By using the moduli approximation \[37, 38\], let us construct the effective theory of the domain wall with promoting the moduli \(X\) and \(U\) to fields \(X(x^i)\) and \(U(x^i)\), respectively \((i = 0, 1, 2, 3)\) on the world volume of the domain wall \[3–5\]:

\[
\mathcal{L}_{\text{wall}} = -\frac{v^2}{4m} \text{tr} \left( U^\dagger \partial_i U U^\dagger \partial^i U \right) + \frac{v^2}{2m} \partial_4 X \partial^4 X.
\]

By substituting the solution to gauge kinetic term, we have the Skyrme term \[4\]

\[
\mathcal{L}_{\text{wall}}^{(4)} = c \text{tr} \left[ U^\dagger \partial_i U, U^\dagger \partial_j U \right]^2
\]

with a numerical constant \(c\).

Since \(\pi_3[U(3)] \simeq \mathbb{Z}\), one can construct Skyrmions on the domain wall. One can confirm that Skyrmion solutions in the domain wall effective theory are instantons in the bulk \[4\].

**B. Theory 2: Instantons inside a flat vortex sheet**

The system is in the unique color-flavor locked vacuum

\[
H = v1_2, \quad \Sigma = 0_2,
\]

where the unbroken symmetry is the color-flavor locked symmetry \(SU(2)_{C+F}\). This model admits a non-Abelian \(U(2)\) vortex solution \[1\], \(H = \text{diag.}(f(r) e^{i\theta}, v)\), where \((r, \theta)\) are polar coordinates in the \(x^3-x^4\) plane, where the vortex world-volume has the coordinates \((x^0, x^1, x^2)\).
The transverse width of the vortex is $1/gv$. The vortex solution breaks the vacuum symmetry $SU(2)_{\text{C+F}}$ into $U(1)$ in the vicinity of the vortex, and consequently there appear $\mathbb{C}P^1 \simeq U(2)_{\text{C+F}}/U(1)$ Nambu-Goldstone modes localized around the vortex. The vortex solutions have the orientational moduli $\mathbb{C}P^1$ in addition to the translational (position) moduli $Z$. By promoting the moduli to the fields depending on the world-volume coordinates $(x^0, x^1, x^2)$, the low-energy effective theory of these modes can be constructed to yield the $\mathbb{C}P^1$ model in $d = 2 + 1$ dimensional vortex effective theory \cite{1, 2, 10, 11, 15, 37} ($\mu = 0, 1, 2$)

$$\mathcal{L}_{\text{vort.eff.}} = 2\pi v^2 |\partial_\mu Z|^2 + \frac{4\pi}{g^2} \left[ \frac{\partial_\mu u^* \partial^\mu u}{(1 + |u|^2)^2} \right].$$

Here $Z(x^\mu), u(x^\mu) \in \mathbb{C}$ represent the position and orientational moduli (the projective coordinate of $\mathbb{C}P^1$), respectively. The points $u = 0$ and $u = \infty$ corresponding to the north and south poles of the target space $\mathbb{C}P^1$.

Since $\pi_2(\mathbb{C}P^1) \simeq \pi_2(S^2) \simeq \mathbb{Z}$, lumps can exist on the vortex world-volume. The BPS equation for lumps in the $\mathbb{C}P^1$ model (with $z \equiv x^1 + ix^2$) and the solutions are

$$\partial_z u = 0,$$

$$u_{\text{lump}} = \sum_{a=1}^{k} \frac{\lambda_a}{z - z_a}$$

respectively. The lump charge and energy are

$$T_{\text{lump}} \equiv \int d^2x \frac{i(\partial_i u^* \partial_i u - \partial_i u^* \partial_i u)}{(1 + |u|^2)^2} = 2\pi k,$$

$$E_{\text{lump}} = \frac{4\pi}{g^2} T_{\text{lump}} = \frac{4\pi}{g^2} \times 2\pi k = \frac{8\pi^2}{g^2} k = E_{\text{inst}},$$

showing that the lumps inside the vortex are instantons in the bulk.

C. Theory 3: Instantons inside a straight monopole string

In this case, we need two different host solitons, a vortex sheet and monopole string. We embed a configuration of a kink inside a monopole string into a non-Abelian vortex sheet. We start to deform the vortex theory in the previous subsection by mass $m$. To this end, we consider the limit $\beta = 0$ for a while. In the presence of mass, $m \neq 0$, the $SU(2)_{\text{C+F}}$ symmetry is explicitly broken and the system is in the color-flavor locked vacuum

$$H = v\mathbf{1}_2, \quad \Sigma = \text{diag.}(m, -m).$$
vortex sheet

\( d = 2 + 1 \) world-volume

(\( gv \))

instanton particle
= lump in \( d = 2 + 1 \) vortex sheet

FIG. 2: An instanton trapped inside a non-Abelian vortex sheet in \( d = 4 + 1 \). In the \( 2 + 1 \) dimensional world-volume of the vortex sheet, it is realized as a \( \mathbb{C}P^1 \) lump. The arrows denote points in the \( S^2 \) moduli.

Let us consider the non-Abelian vortex as in the theory 2. Considering a regime \( m \ll gv \) of small mass, it induces the mass in the \( d = 2 + 1 \) dimensional vortex effective theory \([1, 10, 15, 37]\) (

\[
\mathcal{L}_{\text{vort.eff.}} = 2\pi v^2 |\partial_\mu Z|^2 + \frac{4\pi}{g^2} \left[ \frac{\partial_\mu u^* \partial_\mu u - m^2 |u|^2}{(1 + |u|^2)^2} \right].
\]

(20)

This is the massive \( \mathbb{C}P^1 \) model, which can be made supersymmetric with fermions \([20, 21]\).

A monopole solution can be constructed as a domain wall interpolating the two vacua \( u = 0 \) and \( u = \infty \) \([20, 21]\) in the vortex effective theory (20). The solution

\[
u_{\text{mono.}}(x^2) = e^{\mp m(x^2 - Y) + i\varphi},
\]

(21)

is placed perpendicular to the \( x^2 \)-coordinate, where \( \mp \) represents a monopole and an antimonopole with the width \( 1/m \). Here, \( Y \) and \( \varphi \) are moduli parameters representing the position and \( U(1) \) phase of the (anti-)monopole. The domain wall tension \( E_{\text{wall}} = \frac{4\pi}{g^2} \times m = E_{\text{mono.}} \) coincides with the monopole mass \( E_{\text{mono.}} \) and the monopole charge in the bulk theory, showing that the wall in the vortex theory is a monopole string in the bulk \([10]\). The effective theory of the monopole string is obtained by promoting the moduli \( Y \) and \( \varphi \) to fields \( Y(x^i) \) and \( \varphi(x^i) \) \([37, 38]\) on the string as \([21]\).

\[
\mathcal{L}_{\text{mono.eff.}} = \frac{4\pi}{g^2} \frac{1}{2m} \left[ (\partial_i Y)^2 + (\partial_i \varphi)^2 \right],
\]

(22)
which is a free theory, a sigma model with the target space \( \mathbb{R} \times U(1) \).

Since \( \pi_1[U(1)] \simeq \mathbb{Z} \) there can exist a phase kink on the monopole string. However, a phase kink is unstable against the expansion and is diluted along the string. It can be stabilized by turning on \( \beta \) perturbatively (\( \beta \ll m v \)), which deforms the vortex effective Lagrangian by the potential term \[ \Delta L_{\text{vort.eff.}} = -c\beta^2 \frac{u + u^*}{1 + |u|^2}, \quad c = \sqrt{2\pi} \int_0^\infty dr \, r (v^2 - f^2) \] (23)

with the vortex profile function \( f \), and the monopole effective Lagrangian by \[ \Delta L_{\text{mono.eff.}} = c\beta^2 \int_{-\infty}^{+\infty} dy \frac{e^{my+i\varphi} + e^{my-i\varphi}}{1 + e^{2my}} = \frac{\pi c^2 m^2}{m} \cos \varphi. \] (24)

We thus obtain the sine-Gordon model \( L_{\text{mono.eff.}} + \Delta L_{\text{mono.eff.}} \) with the additional field \( Y \).

We construct an instanton as a sine-Gordon kink. The BPS equation and its a one-kink solution are

\[ \partial_i \varphi \pm \tilde{\beta} \sin \frac{\varphi}{2} = 0, \quad \tilde{\beta}^2 \equiv \frac{1}{2} \tilde{c}\beta^2 \] (25)

\[ \varphi = 4 \arctan \exp \frac{\tilde{\beta}}{4} (x - X) + \frac{\pi}{2}, \] (26)

respectively, with the width \( \Delta x \sim 1/\tilde{\beta} \). The topological charge and energy are

\[ T_{\text{SG}} = \frac{4\tilde{\beta}}{m}, \quad E_{\text{SG}} = \frac{2\pi g^2 m}{g^2 m} T_{\text{SG}} = \frac{8\pi \tilde{\beta}}{g^2 m^2}, \] (27)

respectively. One can confirm that \( k \) sine-Gordon kinks can be identified with \( k \mathbb{CP}^1 \) lumps with the topological charge \( k \in \pi_2(\mathbb{CP}^1) \) \[ \text{in } d = 2 + 1 \text{ dimensional vortex world-volume, by explicitly calculating a lump charge } T_{\text{lump}} = 2\pi k \] \[ \text{in Eq. (18). We conclude that the sine-Gordon kink on the monopole string corresponds to a Yang-Mills instanton in the bulk point of view, as schematically illustrated in Fig. 3 (a). This has been checked by taking various limits \[ \text{in Eq. (18). For instance, in the limit } m, \beta \to 0 \text{ with keeping } \beta/m^2 \text{ fixed, the configuration becomes the instanton trapped inside a non-Abelian vortex sheet in Fig. 2 in the theory 2.} \]}

IV. INSTANTONS AS TWISTED CLOSED SOLITONS

A. Theory 3: Instantons as twisted closed monopole strings

First, we assume a closed monopole string as a background. By making a closed loop, the total monopole charge is canceled out. Since the closed monopole string has a finite
FIG. 3: (a) An instanton trapped inside a monopole string in $d = 4 + 1$. In the $1 + 1$ dimensional world-volume of the monopole string, it is realized as a sine-Gordon kink. The arrows denote points in the $S^1$ modulus. (b) An instanton as twisted closed monopole string.

world-volume, we do not need to localize the phase kink to the form of a sine-Gordon kink. Therefore, we set $\beta = 0$ and the phase gradient becomes uniform along the closed monopole string as in Fig. [1](d). In order to stabilize the closed monopole string, there are two possible ways. One is giving a time dependence and the other is to add higher derivative terms in the Lagrangian in the same spirit with the Skyrme model.

Let us discuss the first possibility. In this case, in fact, we do not need the help of the vortex, so we take the limit $v = 0$ in which the vortex is diluted and eventually disappears. In taking this limit, we keep $m \neq 0$ with non-vanishing VEVs for $\Sigma = \text{diag}(m, -m)$. The closed monopole string with the $U(1)$ phase twisted along the string and the linear time dependence on the $U(1)$ phase is known as a dyonic instanton [40]. This is a BPS state and is stable in supersymmetric gauge theory.

This is a higher dimensional analogue of a vorton, which is a closed vortex string in $d = 3 + 1$ dimensions with the $U(1)$ phase twisted along the string and the linear time dependence on the $U(1)$ phase. The stability of such a soliton has been established recently [34].

We can also consider the configuration inside a non-Abelian vortex sheet for $v \neq 0$ and $m \neq 0$, see Fig. [3](b). The dyonic instanton inside a non-Abelian vortex is a twisted closed domain line with a linear time dependence on the $U(1)$ phase. This is nothing but a Q-lump
The Q-lump is a $1/2$ BPS state in the massive $\mathbb{C}P^1$ model and the total configuration of the Q-lump inside the vortex is a $1/4$ BPS state in supersymmetric gauge theories [6].

Higher derivative corrections to the vortex effective theory was studied in Refs. [41, 42]. If we calculate the next-to-leading order to the moduli approximation, we obtain four derivative terms in the $\mathbb{C}P^{N-1}$ Lagrangian [41]. However, this form for the BPS vortices is not suitable for stabilizing the lumps in the vortex. Then, let us discuss the possibility of adding higher derivative terms in the original Lagrangian. The Lagrangian containing possible four derivative terms can be written as

$$L^{(4)} = -\frac{1}{g^2} \text{tr} F_{AB}^3 + c_1 \text{tr} (D_A \Sigma D^A \Sigma D_B \Sigma D^B \Sigma) + c_2 [\text{tr} (D_A \Sigma D^A \Sigma)]^2$$

$$+ d_1 \text{tr} (D_A H^\dagger D^A H D_B H^\dagger D^B H) + d_2 [\text{tr} (D_A H^\dagger D^A H)]^2 + O(\partial^6).$$

(28)

In order to stabilize the configuration, we need at least four derivative terms in Eq. (28) in the original Lagrangian from the beginning. These terms would induce more general four derivative terms in the vortex effective theory [42]. Then, lumps become baby Skyrmions in $\mathbb{C}P^1$ model with four derivative term [22, 23]. With the mass term which we are considering, these lumps are in the form of twisted domain wall rings [18, 23]. Only the first term contain the time derivatives in the second order; the rests contain the fourth order time derivatives and may be unsuitable for stabilization of solitons.

In the end, let us consider the limit $v \to 0$ in which the Higgs fields $H$ decouple from other fields and the vortex disappears. We expect that the twisted closed monopole string remains stable without the help of the vortex in a certain parameter region because of the presence of the four derivative terms for the adjoint scalar fields $\Sigma$ in the first line in Eq. (28). This remains as a challenging future problem, if one notes that the stability of the vorton in $d = 3 + 1$ dimensions has been a longstanding problem until recently [34] for decades after the proposal [31].

In $d = 3 + 1$ dimensions, one can interpret it as an instanton in the following way by identifying one axis in Fig. 1(d) as the time direction. First, a pair of a monopole and an anti-monopole is created with the same $S^1$ moduli, say the down arrow. Subsequently, the $S^1$ moduli of the monopole and anti-monopole gradually change in time clockwise and counterclockwise, respectively toward the opposite point of $S^1$ the up arrow. Finally, they annihilate each other with the same $S^1$ moduli.
B. Theory 2: Instantons as twisted closed vortex sheets

An instanton trapped in a non-Abelian vortex sheet with a flat and infinite world-volume $\mathbb{R}^{2,1}$ is schematically drawn in Fig. 2. Since the infinities of the spatial world-volume $\mathbb{R}^2$ are identified, the configuration is topologically $S^2$ by one point compactification. We now physically compactify the world-volume $\mathbb{R}^2$ to $S^2$ by a stereographic map to obtain a twisted closed vortex sheet drawn in Fig. 1(c). In this case, the host soliton, the non-Abelian vortex sheet, has $S^2$ moduli, for which a dyonic extension is impossible. Only possibility to stabilize the closed vortex sheet is to consider higher derivative terms in Eq. (28).

In $d = 3 + 1$ dimensions, one can interpret this configuration as an instanton in the following way by identifying the vertical axis in Fig. 1(c) as the time direction. First, a closed vortex string with the same $U(1)$ phase, say the south pole, is created at one point. Subsequently, the $S^2$ moduli gradually change directions along the closed string toward the opposite point of the $S^2$, the north pole. Finally, the closed vortex string shrinks with the same $S^2$ moduli, the north pole, to be annihilated at one point.

C. Theory 1: Instantons as twisted closed domain walls

Finally, we consider a non-Abelian domain wall having $S^3$ moduli. When the domain wall has a flat world-volume $\mathbb{R}^{3,1}$, the infinities of the spatial world-volume $\mathbb{R}^3$ are identified, and the configuration is topologically $S^3$ by one point compactification. We physically compactify $\mathbb{R}^3$ to $S^3$ and consider a Skyrmion on it as in Fig. 1(b). Since the non-Abelian domain wall has the $S^3$ moduli, a dyonic extension is again impossible. Only possibility to stabilize these solitons is to consider higher derivative terms in Eq. (28).

We can interpret this configuration as an instanton in $d = 3 + 1$ dimensions by identifying the vertical axis in Fig. 1(b) as the time direction as before. After a closed domain wall of an $S^2$ shape with the same $S^3$ moduli is created at one point, it grows and shrinks at the other point, where the $S^3$ moduli wind as before.

V. SUMMARY AND DISCUSSION

Although Yang-Mills instantons are usually unstable to shrink in the Higgs phase, they can stably live inside some host solitons, such as a domain wall, vortex sheet, or monopole.
string, as Skyrmions, lumps, or sine-Gordon kinks, respectively. We have pointed out instantons can exist as twisted closed domain walls, vortex sheet or monopole string, when world-volumes of these host solitons are compact, \( S^3 \), \( S^2 \), or \( S^1 \), respectively, and the moduli \( S^3 \), \( S^2 \), or \( S^1 \) are wound around the world-volume, as summarized in Table I and Fig 1. Maps from world-volume of the host solitons to the moduli space of the host solitons are nontrivial; \( \pi_3(S^3) \), \( \pi_2(S^2) \) and \( \pi_1(S^1) \) for a closed domain wall, vortex sheet and monopole string, respectively. We have called these solitons as incarnations of instantons. They are all higher dimensional generalizations of a vorton, vortex ring with the \( U(1) \) modulus twisted. The stability of these solitons will need higher derivative terms as in Skyrmions in the Skyrme model. While the lower dimensional version of incarnation of instantons (twisted closed domain walls) was constructed numerically \[18\], construction of numerical solutions for twisted solitons in 4+1 dimensions remains as an important future problem.

A spherical domain wall with the \( S^2 \) moduli twisted as a Skyrmion has been numerically constructed in a modified Skyrme model \[43\]. There, the conventional Skyrme term has been taken to be negative and the sixth order term has been added, in order to have a stable spherical world-volume of the domain wall. The same may be needed for spherical solutions in our case; a gauge kinetic term \( \text{tr} F_{AB}^2 \) with the negative sign and a higher order term \( \text{tr} F_{AB}^3 \).

Without higher derivative terms, all models which we considered can be made supersymmetric with eight supercharges (the term \( \beta(\neq 0) \) in the theory 3 breaks supersymmetry, which is needed for the stability of sine-Gordon kink on a straight monopole-string, but that term is not needed for a closed monopole string, where supersymmetry is preserved). Higher derivative terms may be added preserving supersymmetry which is a future problem.

In this paper, we have considered shapes of world-volumes are \( S^n \). One may also consider toroidal shape \( T^n \) or other geometries. For instance, toroidal domain walls in \( d = 3 + 1 \) admit two cycles along which the \( U(1) \) modulus of the domain wall can be twisted with two winding numbers. In this case, they carry a Hopf charge instead of the instanton charge, see e.g., Refs. \[44\]. Relations between world-volume geometry and topological charge should be clarified.

There is a \( d = 2 + 1 \) dimensional version, where a lump is a twisted domain wall as in Fig 1(a). There may be higher dimensional version of incarnation of instantons. For instance, six dimensional instantons \[45, 46\] may be realized as an \( S^5 \) domain wall, \( S^4 \) vortex, \( S^3 \),
monopole, and so on.

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