Abstract. In this paper we study a class of Hausdorff–transformed power series whose convergence is extremely slow for large values of the argument. We perform a Watson–type resummation of these expansions, and obtain, by the use of the Pollaczek polynomials, a new representation whose convergence is much faster. We can thus propose a new algorithm for the numerical evaluation of these expansions, which include series playing a relevant role in the computation of the partition function in statistical mechanics. By the same procedure we obtain also a solution of the classical Hausdorff moment problem.

1. Introduction

Following Fuchs, Rogosinski [6, 13] and Hardy [9] we say that $f(x)$ is summable to the value $S$ by the continuous Hausdorff method (or $f(x)$ is $H$–summable to $S$) if

$$g(x) = \int_0^1 f(xt) d\chi(t) \quad (x > 0)$$

(1)
tends to $S$ as $x \to \infty$. Integral (1) is a Lebesgue–Stieltjes integral, $f(x)$ is Borel summable and bounded in every finite interval $[0, x]$, and $\chi(t)$ is of bounded variation in $[0, 1]$. Next, in Hardy [9] the following theorem is proved.

**Theorem 1** (Hardy). In order that the transformation (1) should be regular, i.e., that $f(x) \to S \,(\text{for } x \to \infty)$ should imply $g(x) \to S \,(\text{for } x \to \infty)$, it is necessary and sufficient that $\chi(0^+) = \chi(0) = 0$ and $\chi(1) = 1$.

Let us now suppose that $f(x)$ is a function of $x$ regular on the positive real axis, and so expressible in the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} a_n x^n. \quad (2)$$

Then, we substitute expansion (2) in integral (1) and, in view of the uniform convergence of the Taylor series, we can exchange the sum with the integral and obtain

$$g(x) = \int_0^1 \sum_{n=0}^{\infty} a_n (xt)^n d\chi(t) = \sum_{n=0}^{\infty} a_n x^n \int_0^1 t^n d\chi(t). \quad (3)$$

The last integral at the r.h.s. of (3) represents the **Hausdorff moment** $\mu_n$, i.e.,

$$\mu_n = \int_0^1 t^n d\chi(t) = \int_0^1 t^n u(t) dt. \quad (4)$$

In formula (4), $\chi(t)$ is supposed to be a real function of bounded variation in $t \in [0, 1]$, and the numbers $\mu_n$ are called **moment constant**, of rank $n$, of $\chi$. If we
suppose, without loss of generality, that \( \chi(0) = 0, \chi(1) = 1, \) and \( \chi(0^+) = \chi(0) = 0, \) so that \( \chi(t) \) is continuous at the origin, then \( \mu_n \) is called a regular moment constant (see Theorem 1 above). Moreover, the following theorem can be proved [9].

**Theorem 2** (Hardy). Sums, differences and products of moment constants are themselves moment constants. The product of two regular moment constants is a regular moment constant.

Two relevant examples of regular Hausdorff transformations are the following [9, Theorem 200]:

\[(i) \quad \mu_n = \ell \int_0^1 t^n (1-t)^{(\ell-1)} \, dt = \left( \frac{n+\ell}{\ell} \right)^{-1} \quad (\ell > 0), \quad (5)\]

which corresponds to the Cesaro transformation \( C(\ell) \);

\[(ii) \quad \mu_n = \frac{1}{\Gamma(\ell)} \int_0^1 t^n \left( \log \frac{1}{t} \right)^{(\ell-1)} \, dt = \frac{1}{(n+1)^\ell} \quad (\ell > 0), \quad (6)\]

which corresponds to the Hölder transformation \( H^{(\ell)} \).

From (3) we are naturally led to consider expansions of the following form:

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \mu_n x^n, \quad (7)
\]

where the terms \( \mu_n \) are Hausdorff moments. If we suppose that \( f^{(n)}(0) = (-1)^n \), we obtain expansions which read:

\[
g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mu_n x^n. \quad (8)
\]

These series may be slowly convergent, for values of \( x \) sufficiently large. We thus face a serious problem of numerical analysis, which is quite relevant in view of the fact that sums like expansion (8) occur in several problems, including some of physical interest. For instance,

(a) The Laplace transform of the functions of compact support gives rise to sums of the form (8), if we expand in series the exponential \( e^{-xt} \). This case appears in statistical mechanics, where the partition function is the Laplace transform of the density of states. If the latter is a function of compact support, as in the case of harmonic crystals, then we obtain a representation of the partition function in terms of a power series of the type (8) [14].

(b) Confluent hypergeometric function of the following type:

\[
\Phi(1, \ell + 1; -x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{n+\ell}{\ell} \right)^{-1} x^n \quad (\ell > 0), \quad (9)
\]

are expansions of the form (8) since the terms \( \mu_n = \left( \frac{n+\ell}{\ell} \right)^{-1} \) form a Hausdorff sequence (see [5]).

(c) Hausdorff sequences can be constructed as follows [19]. Consider a sequence \( \{\mu_n\}_0^{\infty} \) of (real) numbers, and denote by \( \Delta \) the forward difference operator:
\[ \Delta \mu_n = \mu_{n+1} - \mu_n. \] Then we have
\[ \Delta^k \mu_n = \Delta \times \Delta \times \cdots \times \Delta \mu_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} \mu_{n+k-m} \quad (k = 0, 1, 2, \ldots), \] (10)
\( \Delta^0 \) is the identity operator, by definition. Now, suppose that there exists a positive constant \( M \) such that
\[ (n + 1)^{(p-1)} \sum_{i=0}^{n} \left( \begin{array}{c} n \\ 0 \\ i \end{array} \right) (-1)^{n-i} \Delta^{n-i} \mu_i \right|^p < M \quad (n = 0, 1, 2; \ldots; p > 1). \] (11)
It can be proved [19] that condition (11) is necessary and sufficient to represent the sequence \( \{\mu_n\}_0^\infty \) as follows: \( \mu_n = \int_0^1 t^n u(t) \, dt \) (see formula (4)), where \( u \in L_p[0, 1] \). Thus we can say that the set \( \{\mu_n\}_0^\infty \), constrained by the condition (11), forms a Hausdorff sequence.

The main purpose of the present paper consists in performing a Watson–type resummation of expansions of type \[ 5, \] where the set of numbers \( \{\mu_n\}_0^\infty \) is assumed to be a Hausdorff sequence generated by a function \( u(t) \) (see formulae (4) and (11)) which belongs to \( L^2(\mathbb{C}) \setminus (0, 1) \) \( (\epsilon > 0) \). In this case we can regard the sequence \( \{\mu_n\}_0^\infty \) as the restriction to the integers of a function \( \mu(z) (z \in \mathbb{C}) \), which belongs to the Hardy space \( H^2(\mathbb{C}) \setminus (0, 1/2) \), and which is the unique Carlesonian interpolation \[ 3 \] of the numbers \( \{\mu_n\}_0^\infty \). We can thus perform the Watson–type resummation of expansion \[ 5, \] and finally obtain another representation whose numerical handling is much more convenient and effective.

The paper is organized as follows. In Section 2 we study the Carlesonian interpolation of the Hausdorff moments \( \{\mu_n\}_0^\infty \), and expand the function \( \mu(iy - 1/2) \) in terms of the so– called Pollaczek functions. In Section 3 we perform a Watson–type resummation of expansion \[ 5. \] In Section 4 we study an appropriate truncation procedure of the new representation obtained in Section 3. In Section 5 we solve the Hausdorff moment problem by the use of the Pollaczek polynomials \[ 2, 10, \] and show the connection between this problem and the Watson resummation of expansion \[ 2. \] Finally, Section 6 is devoted to numerical analysis and examples.

2. Interpolation of Hausdorff moments and Hardy spaces

We prove the following theorem.

Theorem 3. Let the sequence \( \{\mu_n\}_0^\infty \) satisfy condition (11) with \( p \geq 2 + \epsilon \) \( (\epsilon > 0) \). Then there exists a unique Carlesonian interpolation of the numbers \( \mu_n \), denoted by \( \mu(z) (z \in \mathbb{C}, \mu(n) = \mu_n) \), that satisfies the following conditions:

(i) \( \mu(z) \) is holomorphic in the half–plane \( \Re z > -1/2 \), continuous at \( \Re z = -1/2 \);
(ii) \( \mu(z) \) belongs to \( L^2(-\infty, +\infty) \) for any fixed value of \( \Re z \equiv x \geq -1/2 \);
(iii) \( \mu(z) \) tends uniformly to zero as \( z \) tends to infinity inside any fixed half–plane \( \Re z \geq \delta > -1/2 \);

Proof. If the sequence \( \{\mu_n\}_0^\infty \) satisfies condition (11) with \( p \geq 2 + \epsilon \) \( (\epsilon > 0) \), then
\[ \mu_n = \int_0^1 t^n u(t) \, dt, \] (12)
with \( u \in L^{2+\epsilon}[0,1] \). Next, set \( t = e^{-s} \) in formula (12), and obtain

\[
\mu_n = \int_0^\infty e^{-ns}e^{-s}u(e^{-s}) \, ds \quad (n = 0, 1, 2, \ldots).
\]

Therefore the numbers \( \mu_n \) can be regarded as the restriction to the integers of the following Laplace transform:

\[
\mu(z) = \int_0^\infty e^{-(z+1/2)s}e^{-s/2}u(e^{-s}) \, ds.
\]

Indeed, one has \( \mu(n) = \mu_n \). By applying the Paley–Wiener theorem to equality (14), and recalling that the function \( e^{-s/2}u(e^{-s}) \) belongs to \( L^2[0,\infty) \), we can conclude that \( \mu(z) \) belongs to the Hardy space \( H^2(\mathbb{C}_{-1/2}) \). Consequently, property (ii) holds true for any fixed \( z > -1/2 \) (see Ref. [13]). We can thus state that \( \mu(z) \) is holomorphic in the half-plane \( \text{Re} \, z > -1/2 \), and tends uniformly to zero as \( z \) tends to infinity inside any fixed half-plane \( \text{Re} \, z \geq \delta > -1/2 \). We can then apply the Carlson theorem, and say that \( \mu(z) \) is the unique Carlsonian interpolation of the numbers \( \mu_n \). Furthermore, in view of the fact that \( e^{-s/2}u(e^{-s}) \) belongs to \( L^2[0,\infty) \), then \( \mu(-1/2 + iy) \) belongs to \( L^2(-\infty,\infty) \), and, consequently, property (ii) holds true for any fixed value of \( \text{Re} \, z \equiv x \geq -1/2 \). Finally, let us note that the function \( e^{-s/2}u(e^{-s}) \) belongs to \( L^1[0,\infty) \); in fact, \( \int_0^\infty |e^{-s/2}u(e^{-s})| \, ds = \int_0^1 |u(t)/\sqrt{t}| \, dt < \infty \) since \( u \in L^{2+\epsilon}[0,1] \) (\( \epsilon > 0 \)). Therefore, in view of the Riemann–Lebesgue theorem applied to representation (14), it follows that the function \( \mu(-1/2 + iy) \) \((y \in \mathbb{R})\) is continuous, and thus property (i) is proved.

Let us now introduce the following set of functions:

\[
\Psi_n(y) = \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{1}{2} + iy \right) P_n^{(1/2)}(y),
\]

where \( \Gamma(\cdot) \) denotes the Euler gamma function, and \( P_n^{(1/2)}(\cdot) \) denote the Pollaczek polynomials \( P_n^{(\lambda)}(\cdot) \), with \( \lambda = 1/2 \) (see the appendix). These polynomials (in what follows the superscript \( \lambda = 1/2 \) will be omitted) are orthonormal in \((\infty,\infty)\) with weight function \( \omega(y) = \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{1}{2} + iy \right) \)

\[
\int_\infty^\infty \omega(y) \Psi_n(y) \Psi_m(y) \, dy = \delta_{n,m}.
\]

Therefore the orthonormality condition reads:

\[
\int_\infty^\infty w(y) \Psi_n(y) \Psi_m(y) \, dy = \delta_{n,m}.
\]

It can be proved that the functions \( \{\Psi_n(y)\}_0^\infty \) form a complete basis in the space \( L^2(-\infty,\infty) \). Therefore the function \( \mu(-1/2 + iy) \), which belongs to \( L^2(-\infty,\infty) \) (see Theorem 9), can be expanded in terms of this basis. We can state the following proposition.

**Proposition 1.** If the sequence \( \{\mu_n\}_0^\infty \) satisfies condition (17) with \( p \geq 2 + \epsilon \) \((\epsilon > 0)\), then

\[
\mu \left( -\frac{1}{2} + iy \right) = \sum_{n=0}^\infty c_n \Psi_n(y),
\]

where \( c_n \) are the coefficients of the expansion.

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\int_\infty^\infty \omega(y) \Psi_n(y) \Psi_m(y) \, dy = \delta_{n,m}.
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**Proposition 1.** If the sequence \( \{\mu_n\}_0^\infty \) satisfies condition (17) with \( p \geq 2 + \epsilon \) \((\epsilon > 0)\), then

\[
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\]

where \( c_n \) are the coefficients of the expansion.
which converges in the $L^2$–norm. The coefficients $c_n$ are given by

$$c_n = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mu \left( -\frac{1}{2} + iy \right) \Gamma \left( \frac{1}{2} - iy \right) P_n(y) \, dy. \quad (19)$$

Proof. The sequence $\{\mu_n\}_{n=0}^{\infty}$ is a Hausdorff sequence satisfying condition (11) with $p \geq 2 + \epsilon$ ($\epsilon > 0$), then $\mu(-1/2 + iy)$ belongs to $L^2(-\infty, +\infty)$ (statement (ii) of Theorem 3), and expansion (18) converges in the sense of the $L^2$–norm; the coefficients $c_n$ are then obtained by the use of the orthonormality condition (17). □

The coefficients $c_n$ can be evaluated as follows.

**Theorem 4.** The following equality holds true:

$$c_n = 2\sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mu_k P_n \left[ -i \left( k + \frac{1}{2} \right) \right], \quad (20)$$

where $P_n(\cdot)$ are the Pollaczek polynomials.

Proof. Integral (19) can be evaluated by means of the method of contour integration. Set in formula (19): $-1/2 + iy = z$, and accordingly $y = -i(z + 1/2)$. Then, performing an integration along the contour $\gamma$ shown in Fig. 1A, and taking into account the asymptotic behavior of the gamma function, we obtain

$$\int_{-\infty}^{+\infty} \mu(z) \Gamma(-z) P_n \left[ -i \left( z + \frac{1}{2} \right) \right] \, dz = \int_{C^+} \mu(z) \Gamma(-z) P_n \left[ -i \left( z + \frac{1}{2} \right) \right] \, dz, \quad (21)$$

where $C^+$ is a path which encircles the real positive semi–axis of the $z$–plane in counterclockwise sense (see Fig. 1B). Then, using the theorem of residues we get

$$-\frac{i}{\sqrt{\pi}} \int_{C^+} \mu(z) \Gamma(-z) P_n \left[ -i \left( z + \frac{1}{2} \right) \right] \, dz = 2\sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mu_k P_n \left[ -i \left( k + \frac{1}{2} \right) \right]. \quad (22)$$
3. Watson resummation of a class of $\mathcal{H}$–transformed power series

We prove the following theorem.

**Theorem 5.** Expansion (8), where the terms $\{\mu_n\}_0^\infty$ form a Hausdorff sequence satisfying condition (11) with $p \geq 2 + \epsilon$ ($\epsilon > 0$), can be rewritten in the following form:

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mu_n x^n = \frac{\sqrt{2}}{x+1} \sum_{n=0}^{\infty} u_n i^n \left( \frac{x-1}{x+1} \right)^n \quad (x > 0),$$

(23)

where

$$u_n = \sqrt{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mu_k P_n \left[ -i \left( k + \frac{1}{2} \right) \right],$$

(24)

$P_n(\cdot)$ being the Pollaczek polynomials. The convergence of expansion (23) is uniform on any compact subset of the real positive axis.

**Proof.** We start by rewriting expansion (8) in the following form:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mu_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mu_n e^{n\alpha} \quad (\alpha = \ln x).$$

(25)

Next, using once again the theorem of residues, we rewrite the sum (25) as the following integral:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mu_n e^{n\alpha} = \frac{1}{2\pi i} \int_{C^+} \Gamma(-z) \mu(z) e^{\alpha z} \, dz,$$

(26)

where the path $C^+$ encircles the real positive semi–axis of the $z$–plane (see Fig. 1B). Equality (26) holds true since:

(i) $\mu(z)$ is the Carlsonian interpolation of the moments $\{\mu_n\}$;

(ii) for $z = n$ ($n = 0, 1, 2, \ldots$), one has $\mu(n) = \mu_n$, and the function $\Gamma(-z) = \Gamma(-n)$ is singular and has simple poles with residues $(-1)^n/n!$.

We can now close the contour $C^+$ as shown in Fig. 1A. We have, by exploiting the asymptotic behavior of the gamma function

$$\oint C^+ \Gamma(-z) \mu(z) e^{\alpha z} \, dz = 0.$$

(27)

From (27) and using the Stirling formula for the gamma function,

$$\int_{C^+} \Gamma(-z) \mu(z) e^{\alpha z} \, dz = \int_{-\frac{1}{2}+i\infty}^{\frac{1}{2}+i\infty} \Gamma(-z) \mu(z) e^{\alpha z} \, dz$$

$$= i \int_{-\infty}^{+\infty} \Gamma \left( \frac{1}{2} - iy \right) \mu \left( iy - \frac{1}{2} \right) e^{\alpha iy - 1/2} \, dy.$$

(28)

In the latter integral we use (15) and (18). Then from (26) and (28) we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mu_n e^{n\alpha} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Gamma \left( \frac{1}{2} - iy \right) \mu \left( iy - \frac{1}{2} \right) e^{\alpha iy - 1/2} \, dy$$

$$= \frac{e^{-\alpha/2}}{2\pi\sqrt{\pi}} \int_{-\infty}^{+\infty} \sum_{n=0}^{\infty} c_n \Gamma \left( \frac{1}{2} - iy \right) \Gamma \left( \frac{1}{2} + iy \right) P_n(y) e^{iy} \, dy.$$

(29)
Using the formula
\[
\Gamma \left( \frac{1}{2} - iy \right) \Gamma \left( \frac{1}{2} + iy \right) = \frac{\pi}{\cosh(\pi y)},
\]
from (29) it follows that
\[
\int_{-\infty}^{+\infty} \sum_{n=0}^{\infty} c_n P_n(y) \cosh(\pi y) e^{i\alpha y} dy = 2 \pi i \mathcal{F}^{-1} \left\{ \sum_{n=0}^{\infty} c_n \frac{P_n(y)}{\cosh(\pi y)} \right\},
\]
where \( \mathcal{F} \) denotes the Fourier integral operator. Interchanging integration and summation, we have, from formulae (29)–(31):
\[
e^{-\alpha/2} \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} c_n \int_{-\infty}^{+\infty} P_n(y) \cosh(\pi y) e^{i\alpha y} dy = e^{-\alpha/2} \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} c_n \left( -\frac{d}{d\alpha} \right) \left[ \frac{1}{\cosh(\alpha/2)} \right]^{n}
\]
Substituting \( \alpha = \ln x \) in (32) yields
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mu_n x^n = \frac{\sqrt{2}}{x+1} \sum_{n=0}^{\infty} u_n i^n \left( \frac{x-1}{x+1} \right)^n (x > 0),
\]
where
\[
u_n = \frac{\sqrt{2}}{\sqrt{x+1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mu_k P_n \left[ -i \left( k + \frac{1}{2} \right) \right].
\]
It remains to prove that the series at the r.h.s. of formula (33) converges uniformly on any compact subset of the positive real axis. Using the Schwarz inequality,
\[
\left| \sum_{n=0}^{\infty} u_n i^n \left( \frac{x-1}{x+1} \right)^n \right| \leq \left( \sum_{n=0}^{\infty} |u_n|^2 \right)^{1/2} \left( \sum_{n=0}^{\infty} \left| \frac{x-1}{x+1} \right|^{2n} \right)^{1/2}.
\]
The sum \( \sum_{n=0}^{\infty} |u_n|^2 \) can be shown to be finite by using the Parseval equality on expansion (18). The sum \( \sum_{n=0}^{\infty} \left| \frac{x-1}{x+1} \right|^{2n} \) can be easily reduced to the series \( \sum_{n=0}^{\infty} y^n \), \( y = (\frac{x-1}{x+1})^2 \), which is uniformly convergent on any compact set \( y \leq y_0 < 1 \).

4. Truncation of the resummed expansion

We hereafter assume that only a finite number of Hausdorff moments \( \mu_k \) (see formula (34)) are given, and, furthermore, we suppose that they can also be affected by noise, being typically round-off numerical errors. Accordingly, they will be denoted by \( \mu_k^{(\eta)} \), \( \eta \) denoting the order of magnitude of the numerical noise. Precisely, we state: \( \left| \mu_k - \mu_k^{(\eta)} \right| \leq \eta \) (\( k = 0, 1, 2, \ldots, k_0; \eta > 0 \); \( k_0 + 1 \) is the number of Hausdorff moments which are supposed to be known. Next, we introduce the following finite sums:
\[
u_n^{(\eta,k_0)} = \sqrt{2} \sum_{k=0}^{k_0} \frac{(-1)^k}{k!} \mu_k^{(\eta)} P_n \left[ -i \left( k + \frac{1}{2} \right) \right].
\]
With obvious notation we write: \( u_n^{(0, \infty)} = u_n \). Then, the following two auxiliary lemmas can be proved.

**Lemma 1.** The following statements hold true:

(i) \[ \sum_{n=0}^{\infty} |u_n^{(0, \infty)}|^2 = \frac{1}{\pi} \int_{-\infty}^{+\infty} \left| \mu \left( -\frac{1}{2} + iy \right) \right|^2 dy = C \quad (C = \text{const}); \]

(ii) \[ \sum_{n=0}^{\infty} |u_n^{(\eta, k_0)}|^2 = +\infty; \]

(iii) \[ \lim_{k_0 \to \infty} u_n^{(\eta, k_0)} = u_n^{(0, \infty)} = u_n \quad (n = 0, 1, 2, \ldots); \]

(iv) If \( m_0(\eta, k_0) \) is defined as

\[ m_0(\eta, k_0) = \max \left\{ m \in \mathbb{N} : \sum_{n=0}^{m} |u_n^{(\eta, k_0)}|^2 \leq C \right\}, \]

then

\[ \lim_{k_0 \to \infty} m_0(\eta, k_0) = +\infty; \]

(v) The sum

\[ M_{m}^{(\eta, k_0)} = \sum_{n=0}^{m} |u_n^{(\eta, k_0)}|^2 \quad (m \in \mathbb{N}), \]

satisfies the following properties:

(a) It increases for increasing values of \( m \);

(b) the following relationship holds true:

\[ M_{m}^{(\eta, k_0)} \geq |u_n^{(\eta, k_0)}|^2 m \sim +\infty \frac{1}{(k_0)^2} (2m)^{2k_0} \quad (k_0 \text{ fixed}). \]

**Proof.** The proof is given, with minor modifications, in Ref. [4]. \( \square \)

**Lemma 2.** The following equality holds true:

\[ \lim_{k_0 \to \infty} \sum_{n=0}^{m_0(\eta, k_0)} \left| u_n^{(\eta, k_0)} - u_n^{(0, \infty)} \right|^2 = 0. \]

**Proof.** See Ref. [4]. \( \square \)

Next, we introduce the following notations:

\[ g(x) \equiv g^{(0, \infty)} = \sum_{n=0}^{\infty} (-1)^n \mu_n x^n = \frac{\sqrt{2}}{x + 1} \sum_{n=0}^{\infty} u_n^{(0, \infty)} i^n \left( \frac{x - 1}{x + 1} \right)^n, \]

\[ g^{(\eta, k_0)} = \frac{\sqrt{2}}{x + 1} \sum_{n=0}^{m_0(\eta, k_0)} u_n^{(\eta, k_0)} i^n \left( \frac{x - 1}{x + 1} \right)^n, \]

\[ u_n^{(\eta, k_0)} = \sum_{k=0}^{k_0} \sqrt{2} \frac{(-1)^k}{k!} J_k(\eta) P_n \left[ -i \left( k + \frac{1}{2} \right) \right]. \]
Then the following is true.

**Theorem 6.** The following equality holds:
\[
\lim_{k_0 \to \infty} \lim_{\eta \to 0} \left| g^{(\eta, k_0)} - g^{(0, \infty)} \right| = 0. \quad (48)
\]

**Proof.** We have
\[
\lim_{\eta \to 0} \left| g^{(\eta, k_0)} - g^{(0, \infty)} \right| = \left| \frac{\sqrt{2}}{x + 1} \right| \sum_{n=0}^{\infty} u_n^{(\eta, k_0)} \left( \frac{x - 1}{x + 1} \right)^n \left( \frac{x - 1}{x + 1} \right)^n - \sum_{n=0}^{\infty} u_n^{(0, \infty)} \left( \frac{x - 1}{x + 1} \right)^n
\]
\[
\leq \left| \frac{\sqrt{2}}{x + 1} \right| \sum_{n=0}^{\infty} \left| u_n^{(\eta, k_0)} - u_n^{(0, \infty)} \right| \left( \frac{x - 1}{x + 1} \right)^n \leq \left( \sum_{n=0}^{\infty} \left| u_n^{(\eta, k_0)} - u_n^{(0, \infty)} \right|^2 \right)^{1/2} \left( \sum_{n=0}^{\infty} \left| \frac{x - 1}{x + 1} \right|^{2n} \right)^{1/2}.
\]

Now, using the Schwarz inequality,
\[
\left( \sum_{n=0}^{\infty} \left| u_n^{(\eta, k_0)} - u_n^{(0, \infty)} \right| \right)^2 \leq \left( \sum_{n=0}^{\infty} \left| u_n^{(\eta, k_0)} \right|^2 \right) \left( \sum_{n=0}^{\infty} \left| \frac{x - 1}{x + 1} \right|^{2n} \right)^{1/2}.
\]

Since
\[
m_0(\eta, k_0) \xrightarrow{k_0 \to \infty, \eta \to 0} +\infty, \quad \sum_{n=0}^{\infty} \left| u_n^{(\eta, k_0)} \right|^2 < +\infty,
\]
and
\[
\sum_{n=0}^{\infty} \left| \frac{x - 1}{x + 1} \right|^{2n} < +\infty \quad \text{for } x > 0
\]
(see Lemma 11), the r.h.s. of formula (48) tends to zero as \( k_0 \to +\infty, \eta \to 0 \). Now,
\[
\left| \sum_{n=0}^{m_0(\eta, k_0)} \left( u_n^{(\eta, k_0)} - u_n^{(0, \infty)} \right) \left( \frac{x - 1}{x + 1} \right)^n \right| \leq \left| \sum_{n=0}^{m_0(\eta, k_0)} \left| u_n^{(\eta, k_0)} - u_n^{(0, \infty)} \right| \right| \left| \frac{x - 1}{x + 1} \right|^n
\]
\[
\leq \left( \sum_{n=0}^{m_0(\eta, k_0)} \left| u_n^{(\eta, k_0)} - u_n^{(0, \infty)} \right|^2 \right)^{1/2} \left( \sum_{n=0}^{m_0(\eta, k_0)} \left| \frac{x - 1}{x + 1} \right|^{2n} \right)^{1/2}.
\]

From Lemma 2 it follows that
\[
\lim_{k_0 \to \infty} \lim_{\eta \to 0} \sum_{n=0}^{m_0(\eta, k_0)} \left| u_n^{(\eta, k_0)} - u_n^{(0, \infty)} \right|^2 = 0,
\]
(52)
while \( \lim_{k_0 \to \infty} \sum_{n=0}^{m_0} (x-1)/(x+1)^{2n} \) is finite for \( x > 0 \).

5. Connection with the Hausdorff Moment Problem

The classical Hausdorff moment problem can be formulated as follows \([1],[14]\):

**Problem.** Given a sequence of real numbers \( \{\mu_n\}_{0}^{\infty} \), find a function \( u(t) \) such that

\[
\mu_n = \int_{0}^{1} t^n u(t) \, dt \quad (n = 0, 1, 2, \ldots).
\]  

This problem is ill-posed in the sense of Hadamard \([8]\): Suppose, for instance, that we are looking for a solution in the space \( X = L^2[0,1] \), and assume that a solution in this space exists and is unique, but it does not depend continuously on the data. Further, in practical cases only a finite number of moments \( \{\mu_n\}_{0}^{N} \) are known. We must then look for a solution in a finite-dimensional subspace \( X_{N+1} \) of \( X \). Therefore, any function which is orthogonal to \( X_{N+1} \) cannot be recovered: the solution is not unique. From the numerical point of view, we are led to the inversion of matrices which are severely ill-conditioned. We shall return on these questions later. For now we assume that a countable set of noiseless moments \( \{\mu_n\}_{0}^{\infty} \) are given, and prove the following theorem.

**Theorem 7.** Suppose that the real sequence \( \{\mu_n\}_{0}^{\infty} \) of Hausdorff moments satisfy condition (17) with \( p \geq 2 + \epsilon \) (\( \epsilon > 0 \)). Then the function \( u(t) \) can be represented by the following expansion, which converges in the \( L^2 \)–norm:

\[
u(t) = \sum_{n=0}^{\infty} u_n \Phi_n(t), \quad u_n = \sqrt{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mu_k \Psi_n \left[-i \left(k + \frac{1}{2}\right)\right],
\]

where

\[
\Phi_n(t) = i^n \sqrt{2} e^{-t} L_n(2t),
\]

and \( \Phi_n(\cdot) \) and \( L_n(\cdot) \) are the Pollaczek and the Laguerre polynomials, respectively.

**Proof.** In formula (18) set \( z = -1/2 + iy \). Recalling that the support of the function \( f(s) = e^{-s/2} u(e^{-s}) \) belongs to \( \mathbb{R}^+ \),

\[
\mu \left( \frac{1}{2} + iy \right) = \int_{-\infty}^{\infty} e^{-isy} e^{-s/2} u(e^{-s}) \, ds = \mathcal{F} \left\{ e^{-s/2} u(e^{-s}) \right\},
\]

where \( \mathcal{F} \) denotes the Fourier transform operator. Let us now return to the expansion (18) and to formula (15), which gives the expression of the functions \( \Psi_n(y) \). In particular, in the integral representation of the Euler gamma function \( \Gamma(1/2 + iy) \):

i.e., \( \Gamma(1/2 + iy) = \int_{0}^{\infty} e^{-t} e^{iy-1/2} \, dt \), we set \( t = e^{-s} \):

\[
\Gamma \left( \frac{1}{2} + iy \right) = \int_{-\infty}^{\infty} e^{-e^{-s}} e^{-s/2} e^{-isy} \, ds = \mathcal{F} \left\{ e^{-e^{-s} e^{-s/2}} \right\}.
\]

Since the function \( e^{-e^{-s} e^{-s/2}} \) belongs to the Schwartz space \( S_\infty \),

\[
\mathcal{F}^{-1} \left\{ \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{1}{2} + iy \right) P_n(y) \right\} = \mathcal{F}^{-1} \left\{ \Psi_n(y) \right\} = P_n \left( -i \frac{d}{ds} \left[ \frac{1}{\sqrt{\pi}} e^{-e^{-s} e^{-s/2}} \right] \right),
\]

\( 58 \).
therefore, from equality (56), and recalling once again expansion (18), we obtain
\[ e^{-s/2}u(e^{-s}) = \sum_{n=0}^{\infty} c_n P_n \left(-i \frac{d}{ds}\left[\frac{1}{\sqrt{\pi}} e^{-e^{-s/2}}\right]\right). \] (59)

Reverting to the variable \(t = e^{-s}\), we have:
\[ u(t) = \sum_{n=0}^{\infty} c_n \frac{1}{\sqrt{\pi t}} \left\{ P_n \left(i t \frac{d}{dt}\left[\sqrt{t} e^{-t}\right]\right)\right\} = \sum_{n=0}^{\infty} u_n \Phi_n(t), \] (60)
where \(u_n = c_n / \sqrt{2\pi}\), and the functions \(\Phi_n(t)\) are given by
\[ \Phi_n(t) = \sqrt{2} \frac{1}{\sqrt{t}} P_n \left(i t \frac{d}{dt}\left[\sqrt{t} e^{-t}\right]\right) = \sqrt{2} i^n e^{-t} L_n(2t), \] (61)
where the \(L_n(\cdot)\) are the Laguerre polynomials. Note that the functions \(\Phi_n(t)\) form a complete basis in \(L^2(0, \infty)\), and that the convergence of expansion (60) is in the sense of the \(L^2\)-norm [18]. □

We now analyze the truncation of expansion (60). We introduce the approximation
\[ u^{(\eta, k_0)}(t) = \sum_{n=0}^{m_0(\eta, k_0)} u_n^{(\eta, k_0)} \Phi_n(t), \] (62)
where the coefficients \(u_n^{(\eta, k_0)}\) are given by formula (36), and \(m_0(\eta, k_0)\) is defined by formula (40).

**Theorem 8.**
\[ \lim_{k_0 \to \infty} \lim_{\eta \to 0} \|u^{(\eta, k_0)} - u\|_{L^2[0, +\infty)} = 0. \] (63)

**Proof.**
\[ \|u^{(\eta, k_0)} - u\|_{L^2[0, +\infty)}^2 = \sum_{n=m_0(\eta, k_0)}^{\infty} |u_n|^2 + \sum_{n=0}^{m_0(\eta, k_0)} |u_n - u_n^{(\eta, k_0)}|^2. \] (64)

The statement of the theorem follows from Lemmas 1 and 2. □

As already remarked, the Hausdorff moment problem, formulated as above, is severely ill-posed. In principle one could use regularization procedures [5, 12], among which the Tikhonov’s or Tikhonov–based methods are the most popular [7, 17]. Each of these procedures consists in restricting the class of admissible solutions to a compact subspace of the solution space (for instance, a subspace of \(X = L^2[0, 1]\)), by introducing suitable bounds on the solutions. However, some problems remain, and in particular the determination of the so–called regularization parameter, whose optimal choice requires a precise knowledge of the majorizations on the solutions and on the noise affecting the data.

The method presented above does not make use of any a–priori knowledge on the solution and on the data. In several cases the truncation given by formula (62) can be easily determined by the properties of the truncation number \(m_0(\eta, k_0)\), illustrated by Lemma 1 and, in particular, by the statements (iv) and (v) (for numerical examples, see the next section). Then, the statement of Theorem 8 guarantees the convergence of approximation (62) to the solution, as the number of data increases to infinity, and the noise tends to zero (see (63)). If the number...
of data is too small, or the noise is too large, or, finally, if the function \( u \) to be determined is irregular (i.e., presents discontinuities of various types), the sum \( M_m^{(\eta,k_0)} \) (see (43) in Lemma 1) can present no plateau (see next section), and the method cannot be used. However, this negative result still provides information: the continuity which could at best be restored with classical regularization procedures remains extremely weak.

Returning to the Hausdorff–transformed power series of type \( \mathcal{S} \), it is worth remarking on the following fact: the method of resummation which we have presented is affected by the same type of ill–posedness illustrated above in connection with the solution of the Hausdorff moment problem. More precisely, we face the ill–posedness connected to the reconstruction of the function \( \mu(iy - 1/2) \) from the sequence of Hausdorff moments \( \{\mu_k\}_{k_0} \). We can thus advance the following critical remark: the method of resummation presented transforms the type of pathology affecting expansion \( \mathcal{S} \), i.e., slow convergence, into another type of pathology, i.e., ill-posedness. However, the ill-posedness of the problem, cured by the truncation procedure presented above, has, at least in the case of regular Hausdorff transformation, much milder effects than the slow convergence pathology on the actual goal to be achieved: the numerical evaluation of functions of type \( \mathcal{S} \). In other words, the regularization of the ill-posedness cures the drawbacks of the slow–convergence.

6. Numerical analysis

Return to formulae (23), (24), and set \( x = 1/r \), \((r \in [1, +\infty))\). Then

\[
g \left( \frac{1}{r} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mu_n \left( \frac{1}{c} \right)^n \left( \frac{c}{r} \right)^n = \frac{\sqrt{2}}{c/r + 1} \sum_{n=0}^{\infty} u_n^{(c)} i^n \left( \frac{c/r - 1}{c/r + 1} \right)^n \quad (c \geq 1),
\]

(65)

where

\[
u_n^{(c)} = \sqrt{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mu_k \left( \frac{1}{c} \right)^k P_n \left[ -i \left( k + \frac{1}{2} \right) \right].
\]

(66)

Next, we take for \( c \) the value \( r \in [1, +\infty) \); then the rightmost expansion in (65) and formula (66) reproduce once again the original series \( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mu_n (1/r)^n \), since \( P_0(\cdot) = 1 \). The same type of result can be obtained when the number of moments \( \mu_k \) is finite. We can thus conclude that the expansion that we have proposed can always be reduced to the standard one in the interval \((0, 1] \), where the original series converges rapidly. We shall now show that the resummed expansion converges much more rapidly than the original one for \( x \gg 1 \). First we discuss in detail how to manage this new expansion numerically, and, in particular, how the truncation number \( m_0(\eta,k_0) \) can be determined.

The problem of evaluating \( m_0(\eta,k_0) \) is intimately related to the price that must be paid for coping with the ill–posedness of the analytic continuation involved in the reconstruction of the function \( \mu(iy - 1/2) \) from the sequence of Hausdorff moments \( \{\mu_k\}_{k_0} \). As remarked in Section 5, most regularization procedures generally require a priori bounds on the solution and on the data. Then one is led to introduce in the regularizing algorithm a truncation, or a filtering, which depends on the a priori information on the solution that one is supposed to have. Our procedure does not require any prior knowledge, and the truncation number \( m_0(\eta,k_0) \) (see Section 4) can be determined by analyzing the sum \( M_m^{(\eta,k_0)} \) versus \( m \) (see 42).
From statement (v) of Lemma 1 and from formula (41) it follows that, if $k_0$ is sufficiently large and $\eta$ is sufficiently small, then $M_{m}^{(n,k_0)}$ presents a plateau, and, after that, it starts growing as a power of $(2k_0)$ (see (43) and Figs. 2B, 3C and 4A). The extension of the plateau varies, and increases as the number of moments $\mu_k$ becomes larger (see Fig. 2B). Analogous results are obtained even in the case $\eta \neq 0$ (see Fig. 4A). A simple algorithm for the automatic derivation of $m_0$ can be implemented along the following lines. We start by observing that the knowledge of the asymptotic behavior of $M_{m}^{(n,k_0)}$, for large $m$, allows us to restrict the range of $m_0$ by defining an upper limit $m_\alpha$ ($m_0 < m_\alpha$), which represents the value of $m$ where approximately the asymptotic behavior sets in; in practice, $m_\alpha$ is set as the value of $m$ where $M_{m}^{(n,k_0)}$ starts being close enough to its asymptotic behavior.

The candidate plateaux are then located by selecting the extended intervals of $m < m_\alpha$ where the modulus of the first numerical derivative of $M_{m}^{(n,k_0)}$ is sufficiently small. Finally, $m_0$ is chosen as the largest value of $m$ belonging to the interval which is closest, but inferior, to $m_\alpha$. It should be noticed that the choice of $m_0$ within the plateau is not critical for the accuracy of the final result. For the sake of completeness, it should also be mentioned that the erratic behavior of the noise can produce very short plateaux located between the true value of $m_0$ and before $M_{m}^{(n,k_0)}$ starts following its asymptotic behavior (i.e., for $m = m_\alpha$). In this case our procedure could fail to recover the correct value of $m_0$; this drawback has been solved heuristically by simply rejecting plateaux shorter than a given threshold length.

We test our method by comparing different evaluations of the confluent hypergeometric function $\Phi(1, \ell + 1; -x)$ (see formula (9)). The latter can be directly evaluated as the Laplace transform of the following function:

$$u(t) = \begin{cases} 
\ell(1-t)^{(\ell-1)}, & t \in [0, 1], \\
0, & t \in (1, +\infty).
\end{cases}$$

(67)

On the other hand, one can evaluate $\Phi(1, \ell + 1; -x)$ by means of the standard expansion at the r.h.s. of formula (9) (truncated at a certain $n = n_0$), and finally these results can be compared with those obtained by the Watson resummation method (in particular, see (16)). The results are illustrated in Figs. 2A.

In Fig. 2A the plots of $\Phi(1, 2; -x)$, computed by using the standard formula (9) with different values of $n_0$, are shown. It is evident how the deviation from the true function rapidly explodes as $x$ increases. Moreover, even using more moments $\mu_n$, i.e., increasing $n_0$, the situation does not get better significantly. Figure 2B shows the sum $M_{m}^{(n,k_0)}$ versus $m$ for various values of $k_0$ ($k_0 = 5, 10, 15, 20$). It can be seen the presence of the plateaux, whose length increases as the number of moments $\mu_k$ used in the computation, i.e., $k_0$, increases (see (11) in Lemma 1). This figure shows how the truncation number $m_0(0, k_0)$ can be determined. The comparison among the true $\Phi(1, 2; -x)$, computed analytically as the Laplace transform of the function $u(t)$ in (67) (solid line), the evaluation obtained by the truncated standard expansion (9) (dashed line), and truncated resummed expansion (10) (filled dots), is shown. In this case, $k_0 = n_0 = 20$. From the inspection of $M_{m}^{(0,20)}$ in Fig. 2B, it is recovered that the plateau approximately ranges from $m = 8$ through $m = 38$. The computation shown in Fig. 2C was made with $m_0(0, 20) = 30$. It is evident how the accuracy of the computation increases considerably when the
Figure 2: Computation of the confluent hypergeometric function $\Phi(1, \ell + 1; -x)$ with $\ell = 1$. (A) Computations of $\Phi(1, 2; -x)$ by means of expansion (9) truncated at $n = n_0$, for different values of $n_0$. The number near each plot indicates the value of $n_0$ used for the computation. The solid line represents the actual function $\Phi(1, 2; -x)$. (B) $M^{(0,k_0)}_m$ versus $m$ for $k_0 = 5, 10, 15, 20$ (see (42)). $(k_0 + 1)$ is the number of moments $\mu_k$ used for the computation. (C) $\Phi(1, 2; -x)$ computed by means of the resummed expansion (46) (dots) by using 21 moments $\mu_k$ (i.e., $k_0 = 20$). $m_0(0, k_0) = 30$ has been set according to the analysis of the function $M^{(0,20)}_m$ in (B). The solid line represents the actual function $\Phi(1, 2; -x)$. The dashed line shows the computation made by using expansion (9) truncated at $n_0 = 20$. (D) Plot of the relative error of reconstruction $|\Phi_{\text{appr}} - \Phi_{\text{true}}|/\Phi_{\text{true}}$ in the range $x \in [1 : 100]$; $n_0 = k_0 = 20$. The solid line shows the relative error computed by using expansion (9). The dashed lines indicate the relative error of the computations made with the resummed expansion (10), with $m_0(0, 20) = 6, 30, 45$.

The resummed expansion is used. This fact is made even more clear in Fig. 2D, where the relative error of computation $|\Delta \Phi/\Phi| = |(\Phi_{\text{appr}} - \Phi_{\text{true}})/\Phi_{\text{true}}|$ over the range $x \in [1 : 100]$ is shown. The error made by using the standard expansion (solid line) diverges, whereas the error made by using the resummed expansion (dashed
error rapidly grows. In order to show the interplay between
small, whereas when
varying in the range \([0 : 50]\). As long as
computations do not change significantly from that at
For any other value of
it is shown the relative error for different values of
, evidentiated in Fig. 3A, where the root mean square error of the computation of
within the plateau of
is displayed even for the cases \(m_0 \) does not lie within the plateau of \(M_m^{(0,k_0)}\) (see also Fig. 3B). In Fig. 3C the sums \(M_m^{(0,20)}\) have been plotted for various values of \(\ell\) (\(\ell = 1, 2, 3, 4, 5, 8, 15, 20\)), while in Fig. 3D the corresponding computations of \(\Phi(1, \ell + 1; -x)\) (\(\ell = 5, 8, 15, 20\)), made by means of the resummed expansion 16 with \(k_0 = 20\) and \(m_0(0, 20) = 10\) are compared with the true functions. From the analysis of \(M_m^{(0,20)}\) in Fig. 3C it can be seen that the value \(m_0(0, 20) = 10\) lies outside the plateau resulting for \(\ell = 15\) and \(\ell = 20\); correspondingly, in Fig. 3D the computation of \(\Phi(1, \ell + 1; -x)\) with \(\ell = 15\) and \(\ell = 20\) clearly deviate from the corresponding true functions. Finally, Fig. 4 illustrates the analysis in the case of noisy moments \(\mu_k^{(\eta)}\). To obtain the \(\mu_k^{(\eta)}\), the moments \(\mu_k\) have been noised by adding white noise uniformly distributed in the interval \([-\eta, \eta]\). Figure 4A shows the sum \(M_m^{(\eta,20)}\) for various values of the noise parameter \(\eta\) \((\eta = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}; \ell = 1)\). These plots show how the plateau gets shorter as the noise level increases. In Fig. 4B some examples of computation of \(\Phi(1, 2; -x)\) by using the standard expansion for various values of \(\eta\) are shown, while in Fig. 4C the computations made by using the resummed expansion are given. This panel shows that a significant deviation from the actual function arises only in the case of quite noisy moments \((\eta \sim 10^{-1})\), and remains quite acceptable for small levels of noise. The difference of accuracy achieved by using the two types of expansion are made evident in Fig. 4D, where the root mean square error, in the range \(x \in [0, 10]\), is shown as a function of \(\eta\). The dots indicate the error referred to the standard expansion, while the crosses indicate that referred to the resummed expansion. Over a wide range of noise level, the error made by using the resummed expansion remains orders of magnitude smaller than that made by using the standard expansion.

APPENDIX

The main properties of the Pollaczek polynomials \(P_n^{(1/2)}(y)\) are briefly summarized:

1. In terms of the hypergeometric series 2,
   \[
P_n^{(1/2)}(y) = i^n \, _2F_1 \left(-n, \frac{1}{2} + iy, 1; 2\right).
   \] (A.1)

2. They satisfy the following recurrence relation 2:
   \[
   (n + 1)P_{n+1}^{(1/2)}(y) - 2yP_n^{(1/2)}(y) + nP_{n-1}^{(1/2)}(y) = 0,
   \] (A.2a)
   \[
P_{-1}^{(1/2)}(y) = 0, \quad P_0^{(1/2)}(y) = 1.
   \] (A.2b)
Figure 3: (A) Root mean square error of the computation made with the resummed expansion (46) with respect to the actual function $\Phi(1, 2; -x)$ in the range $x \in [0 : 20]$, versus $m_0(0, k_0); k_0 = 20$. (B) Relative error of computation of $\Phi(1, 2; -x)$ by using the resummed expansion (46) in the range $x \in [1 : 1000]$. $m_0 = 10$ is kept fixed, while $k_0$ varies: $k_0 = 0, 4, 8, 10, 14$. (C) Computation of the confluent hypergeometric function $\Phi(1, \ell + 1; -x)$ for different values of $\ell$. In this panel the plots of $M^{(0,20)}_m$ (i.e., $k_0 = 20$) versus $m$ are shown. Each plot has been computed by using the moments $\mu_k = (\ell + k)^{-1}$ with $\ell = 1, 2, 3, 4, 5, 8, 15, 20$. (D) Plots of $\Phi(1, \ell + 1; -x)$ computed by using the resummed expansion (46) (dots), for various values of $\ell$: $\ell = 5, 8, 15, 20; k_0 = 20$ and $m_0(0, 20) = 10$ have been kept fixed. The lines represent the actual $\Phi(1, \ell + 1; -x)$.

(3) The generating function is given by:

$$\sum_{n=0}^{\infty} z^n P_{n}^{(1/2)}(y) = (1 - i z)^{(iy - 1)/2} (1 + i z)^{(iy - 1)/2} \quad (|z| < 1).$$

(A.3)
Figure 4: Computation of $\Phi(1,2;−x)$ by using noisy moments $\mu_k^{(\eta)}$. The moments $\mu_k$ have been noised by adding white noise uniformly distributed in the interval $[−\eta, \eta]$. (A) $M_m^{(\eta,20)}$ versus $m$ computed for $\eta$ ranging from $\eta = 10^{-1}$ through $\eta = 10^{-6}$ with step $10^{-1}$. The rightmost solid line indicates the noiseless $M_m^{(0,20)}$. (B) Comparison between the actual function $\Phi(1,2;−x)$ (solid line) and the computations made by using expansion (9), with $\eta = 10^{-2}, 10^{-3}, 10^{-4}$. The dashed line labelled by "0" has been computed by using the noiseless moments $\mu_k$. The number of moments used is $(n_0 + 1) = 21$. (C) Comparison between the actual function $\Phi(1,2;−x)$ (solid line) and the computations made by using the resummed expansion (46); $k_0 = 20$. The filled dots indicate the computation made with $\eta = 10^{-3}$ and $m_0(10^{-3}, 20) = 5$ (see also panel (A)). The dashed lines represent the computations made with $\eta = 10^{-2}, m_0(10^{-2}, 20) = 4$, and $\eta = 10^{-1}, m_0(10^{-1}, 20) = 2$. (D) Root mean square error of the computation of $\Phi(1,2;−x)$ versus $\eta$ in the range $x \in [0 : 10]; k_0 = n_0 = 20$. The dots indicate the error made by using expansion (9), while the crosses represent the error made by using expansion (46). The truncation numbers are: $m_0(10^{-6}, 20) = 30$, $m_0(10^{-5}, 20) = 15$, $m_0(10^{-4}, 20) = 10$, $m_0(10^{-3}, 20) = 5$, $m_0(10^{-2}, 20) = 4$, $m_0(10^{-1}, 20) = 2$.

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