Additive colorings of planar graphs

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Abstract

An additive coloring of a graph $G$ is an assignment of positive integers $\{1, 2, \ldots, k\}$ to the vertices of $G$ such that for every two adjacent vertices the sums of numbers assigned to their neighbors are different. The minimum number $k$ for which there exists an additive coloring of $G$ is denoted by $\eta(G)$. We prove that $\eta(G) \leq 468$ for every planar graph $G$. This improves a previous bound $\eta(G) \leq 5544$ due to Norin. The proof uses Combinatorial Nullstellensatz and coloring number of planar hypergraphs. We also demonstrate that $\eta(G) \leq 36$ for 3-colorable planar graphs, and $\eta(G) \leq 4$ for every planar graph of girth at least 13. In a group theoretic version of the problem we show that for each $r \geq 2$ there is an $r$-chromatic graph $G_r$ with no additive coloring by elements of any Abelian group of order $r$. 
1 Introduction

Let $G$ be a simple graph, and let $k$ be a positive integer. By a coloring of $G$ we mean any function $f$ from the set of vertices $V(G)$ to the set $\{1, 2, \ldots, k\}$. Given a coloring $f$, consider the induced function $S = S(f)$ on the set $V(G)$ defined by the formula

$$S(v) = \sum_{x \in N(v)} f(x),$$

where $N(v)$ denotes the set of neighbors of the vertex $v$ in $G$. The initial coloring $f$ is called an additive coloring of $G$ if $S(u) \neq S(v)$ for every pair of adjacent vertices $u$ and $v$. The minimum number $k$ for which there exists an additive coloring of $G$ is denoted by $\eta(G)$.

The notion of additive coloring was introduced in [4] as a vertex version of the 1-2-3-conjecture of Karoński, Łuczak, and Thomason [7]. In the original problem the numbers are assigned to the edges of a graph, and prospective color of a vertex $v$ is derived as the sum of numbers assigned to the edges incident to $v$. It is conjectured that for every connected graph (except $K_2$) one can produce a proper vertex coloring in this way using only three numbers— 1, 2, and 3. Currently best bound is 5, as proved by Kalkowski, Karoński, and Pfender [6].

In the related additive coloring problem no finite bound is possible since for cliques we have $\eta(K_n) = n$. We conjecture however, that perhaps $\eta(G) \leq \chi(G)$ for every graph $G$, where $\chi(G)$ denotes the usual chromatic number. This conjecture is widely open as it is not known whether $\eta(G)$ is bounded for bipartite graphs. In [4] we proved that $\eta(G) \leq 3$ for planar bipartite graphs, and also that $\eta(G) \leq 100280245065$ for general planar graphs. The later bound was improved to 5544 by Norin (personal communication). We present this proof in section 2 for completeness.

In this note we obtain a further improvement of this bound. Our main result asserts that $\eta(G) \leq 468$ for every planar graph $G$. The proof uses Combinatorial Nullstellensatz of Alon [1], and the coloring number of hypergraphs represented by planar bipartite graphs. For planar graphs of girth at least 13 we get a much better bound by 4, using a decomposition theorem from [3].
2 Coloring number of graphs and hypergraphs

We start with presenting an unpublished result of Norin. Recall that the coloring number \( \text{col}(G) \) of a graph \( G \) is the least integer \( k \) such that there exists a linear ordering of the vertices \( v_1, \ldots, v_n \) such that the number of backward neighbors of \( v_i \) (those contained in the set \( \{v_1, \ldots, v_{i-1}\} \)) is at most \( k-1 \), for every \( i = 1, 2, \ldots, n \). It is well known that \( \text{col}(G) \leq 6 \) for every planar graph \( G \).

**Theorem 1** (S. Norin) Let \( G \) be a graph with chromatic number \( \chi(G) = r \) and coloring number \( \text{col}(G) = k \). Let \( n_1, \ldots, n_r \) be \( r \) pairwise coprime integers, with \( n_i \geq k \) for all \( i = 1, 2, \ldots, k \). Then \( \eta(G) \leq n_1 \times \ldots \times n_r \). In particular, \( \eta(G) \leq 5544 \) for every planar graph \( G \) (by taking \( n_1 = 7, n_2 = 8, n_3 = 9, \) and \( n_4 = 11 \)).

**Proof.** Fix a proper coloring \( c \) of a graph \( G \) using colors \( \{1, 2, \ldots, r\} \). Also, fix a linear ordering of the vertices realizing \( \text{col}(G) = k \). Let \( n_1, \ldots, n_r \) be any positive integers such that \( \gcd(n_i, n_j) = 1 \) for every pair \( i \neq j \), with \( n_i \geq k \) for all \( i = 1, 2, \ldots, r \). Suppose now that each vertex \( v \) is assigned with certain weight \( n(v) \in \mathbb{Z}_{n_j} \), with \( j = c(v) \). Denote by \( S_i(v) \) the sum of weights of all the neighbors of \( v \) in color \( i \). More formally,

\[
S_i(v) = \sum_{x \in N(v) \cap c^{-1}(i)} n(x),
\]

where summation is in the group \( \mathbb{Z}_{n_j} \). Finally, let \( S(v) = (S_1(v), \ldots, S_r(v)) \).

Since no neighbor of \( v \) is colored with \( c(v) \), we have \( S_j(v) = 0 \) for \( j = c(v) \). Our aim is to modify weights \( n(v) \) greedily so that \( S_{c(v)}(u) \neq 0 \) for every backward neighbor \( u \) of \( v \). This will imply that \( S(u) \neq S(v) \) for every pair of adjacent vertices \( u \) and \( v \).

Suppose we have achieved this property for all vertices up to \( v_{i-1} \) by choosing appropriate weights \( n(v_1), \ldots, n(v_{i-1}) \). Now we have to find a weight for the vertex \( v_i \). Let \( j = c(v_i) \). For every backward neighbor \( u \) of \( v_i \) there is only one value of \( n(v_i) \) making \( S_j(u) = 0(\text{mod } n_j) \). Since there are at most \( k-1 \) backward neighbors of \( v_i \), there are only \( k-1 \) forbidden values for \( n(v_i) \). Since \( n_j > k-1 \), there is a free element of \( \mathbb{Z}_{n_j} \) for the weight \( n(v_i) \).

To get an additive coloring of graph \( G \) we assign to every vertex \( v \), an element \( f(v) = (f_1(v), \ldots, f_r(v)) \) of the group \( \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_r} \), defined by \( f_j(v) = n(v) \) if \( j = c(v) \), and \( f_j(v) = 0 \), otherwise. This completes the proof, as the group \( \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_r} \) is isomorphic to \( \mathbb{Z}_N \), where \( N = n_1 \times \ldots \times n_r \). □
The notion of coloring number can be generalized in a natural way for hypergraphs. Given a hypergraph \( H \) and a linear ordering of the vertices \( v_1, \ldots, v_n \), define the backward degree of vertex \( v_i \) as the number of different hyperedges of the form \( \{v_j\} \cup A \), with \( A \subseteq \{v_1, \ldots, v_{i-1}\} \) (we allow \( A \) to be empty). The coloring number \( \text{col}(H) \) of hypergraph \( H \) is the minimum \( k \) such that in some linear ordering of the vertices all backward degrees are at most \( k - 1 \). This definition differs slightly from the one given in [8], but it is appropriate for our purposes.

**Lemma 2** Let \( H \) be a hypergraph with \( \text{col}(H) = k \). Then there is a function \( f : V(H) \to \mathbb{Z}_k \) such that every hyperedge \( B \) satisfies

\[
\sum_{v \in B} f(v) \not\equiv 0 \pmod{k}.
\]

**Proof.** Start with a linear ordering of the vertices realizing \( \text{col}(H) \) and proceed greedily in that order. At each step there are at most \( k - 1 \) partial sums we have to account, and each of them is reset by exactly one value. Hence, there is always a good choice for the next value of \( f \).

Now we give an upper bound for the coloring number of hypergraphs arising from bipartite planar graphs.

**Lemma 3** Let \( G \) be a bipartite planar graph with bipartition classes \( X \) and \( Y \). Let \( H \) be a hypergraph on the set of vertices \( X \) whose incidence graph is \( G \). Then \( \text{col}(H) \leq 12 \). In particular, there exists a coloring \( f : X \to \mathbb{Z}_{12} \) satisfying condition:

\[
\sum_{x \in N(y)} f(x) \not\equiv 0 \pmod{12}
\]

for every non-isolated vertex \( y \in Y \).

**Proof.** We may assume that no two vertices in \( Y \) are twins (have exactly the same nonempty neighborhood), as multiple hyperedges do not count in backward degree. We shall prove that hypergraph \( H \) always contains a vertex of the usual degree at most 11. This is sufficient since a hypergraph \( H - x \) still does not contain multiple hyperedges, (therefore the incidence graph of \( H - x \) does not contain twins) and we may order the vertices of \( H \) by sequential deletion of such vertices.

Fix an embedding of \( G \) in the plane. Transform this embedding into a new plane graph \( P \) in the following way. For every vertex \( y \in Y \), draw a simple closed curve \( C(y) \) through
the neighbors of $y$ within $\varepsilon$-distance from the connecting edges, so that a simply connected region $F(y)$ arises with the following properties:

1. All neighbors of $y$ belong to $C(y)$.

2. All other points of the edges connecting $y$ to its neighbors (and $y$ itself) are in the interior of $F(y)$.

3. No other points of the embedding of $G$ are in $F(y)$.

Forget now about $y$'s and their edges inside regions $F(y)$. In this way we get a plane (pseudo)graph $P$ on the set of vertices $X$ whose faces can be properly 2-colored: color the faces $F(y)$ by black and all other faces by white. Notice that hyperedges of $H$ turned into black faces in $P$. Hence, $\text{deg}_H(v)$ is just the number of black faces incident to $v$.

We claim that there is always a vertex in $P$ incident to at most 11 black faces. First, shrink all loops and all 2-sided faces of $P$ to get a new pseudograph $Q$ whose faces have at least three vertices. Let $v$, $e$, and $f$ denote the number of vertices, edges, and faces in $Q$, respectively. So, we have $3f \leq 2e$, and by Euler’s formula we get $e \leq 3v - 6$. Hence, there must be a vertex $x$ of degree at most 5 in $Q$. Now, by the lack of twins in $G$, each edge incident to $x$ in $Q$ has multiplicity at most 4 in $P$. Also, there can be at most one loop at each vertex in $P$, by the same reason. Therefore degree of $x$ in $P$ is at most 22, and there are at most 11 black faces incident to $x$. The proof of the lemma is complete.

It is worth noticing that the above lemma is tight. To see this take the icosahedron on the vertex set $X$ and modify it in the following way: (1) subdivide each edge and each face of the icosahedron with one new vertex, (2) append a hanging edge to each vertex from $X$. The resulting graph is a twin-free planar bipartite graph in which every vertex in $X$ has degree 11.

3 Combinatorial Nullstellensatz

For the proof of our main result we will need a simple consequence of the celebrated Combinatorial Nullstellensatz of Alon. For the sake of completeness we provide also an elegant, simple proof due to Michalek [9].
Theorem 4 (Combinatorial Nullstellensatz) Let $\mathbb{F}$ be an arbitrary field, and let $P(x_1, \ldots, x_n)$ be a polynomial in the ring of polynomials $\mathbb{F}[x_1, \ldots, x_n]$. Suppose that there is a nonvanishing monomial $x_1^{k_1} \cdots x_n^{k_n}$ in $P$ such that $k_1 + \ldots + k_n = \deg(P)$. Then for every subsets $A_1, \ldots, A_n$ of the field $\mathbb{F}$, with $|A_i| \geq k_i + 1$, there are elements $a_i \in A_i$ such that $P(a_1, \ldots, a_n) \neq 0$.

Proof. We will proceed by induction on the degree of polynomial $P$. If $\deg(P) = 0$, then $P$ is a nonzero constant polynomial and the assertion holds trivially. Let $\deg(P) \geq 1$ and suppose the theorem is true for all polynomials of strictly smaller degree. Hence, for at least one $i \in \{1, \ldots, n\}$ we must have $k_i \geq 1$. Assume, for simplicity, that $k_1 \geq 1$, and let $a \in A_1$ be a fixed element. Using the algorithm of long division of polynomials, we may write

$$P = (x_1 - a)Q + R. \quad (*)$$

Indeed, we may treat $P$ as a polynomial in one variable $x_1$ with coefficients in the ring $\mathbb{F}[x_2, \ldots, x_n]$ and perform long division by the polynomial $(x_1 - a)$ to determine uniquely quotient $Q$ and remainder $R$. Since $\deg(x_1 - a) = 1$, the remainder $R$ must be a constant in $\mathbb{F}[x_2, \ldots, x_n]$, which means that it does not contain variable $x_1$. Hence, by the assumption on the nonvanishing monomial in $P$, the quotient $Q$ must have a nonvanishing monomial $x_2^{k_2-1}x_3^{k_3} \cdots x_n^{k_n}$ and $\deg(Q) = (k_1 - 1) + k_2 + \ldots + k_n$.

Suppose on the contrary that $P(x)$ vanishes on the set $A_1 \times \ldots \times A_n$. Take any element $x \in \{a\} \times A_2 \times \ldots \times A_n$ and substitute to equation $(*)$. Since $P(x) = 0$, we get that $R(x) = 0$. But $R$ does not contain variable $x_1$, so it follows that $R$ also vanishes on the whole set $A_1 \times \ldots \times A_n$. Take now any $x \in (A_1 \setminus \{a\}) \times A_2 \times \cdots \times A_n$ and substitute to equation $(*)$. Since $P(x) = 0$, $R(x) = 0$, and $(x_1 - a) \neq 0$, it follows that $Q(x) = 0$. This means that $Q$ vanishes on the whole set $(A_1 \setminus \{a\}) \times A_2 \times \cdots \times A_n$, which contradicts the inductive assumption.

The above theorem has many surprising applications in geometry, combinatorics, and number theory [1]. We used it in [4] to prove that every planar bipartite graph has an additive coloring from arbitrary lists of size at least three. Below we give a slight extension of this result, which will be useful later.

Theorem 5 Let $G$ be a bipartite graph whose edges can be oriented so that each vertex has indegree at most $k$. Suppose that each vertex $v$ is assigned with a list $L(v)$ of $k + 1$ real numbers. Then for every function $q : V(G) \to \mathbb{R}$ there is a coloring $f$ of the vertices such
that
\[ q(u) + \sum_{x \in N(u)} f(x) \neq q(v) + \sum_{x \in N(v)} f(x) \]
for every pair of adjacent vertices \( u \) and \( v \).

**Proof.** Let \( U = \{u_1, \ldots, u_m\} \) and \( V = \{v_1, \ldots, v_n\} \) be the bipartition classes of a graph \( G \). Let \( \{x_1, \ldots, x_m\} \) and \( \{y_1, \ldots, y_n\} \) be the variables assigned to the vertices of these classes, respectively. Denote by \( S(u) \) the sum of variables assigned to the neighbors of \( u \). Consider a polynomial \( P \) over the field of reals defined by
\[ P(x_1, \ldots, x_m, y_1, \ldots, y_n) = \prod_{u_i, v_j \in E(G)} (q(u_i) + S(u_i) - q(v_j) - S(v_j)). \]
We claim that \( P \) contains a nonvanishing monomial with exponents bounded by \( k \). Let \( \vec{G} \) be an orientation of \( G \) with indegrees bounded by \( k \). In every factor of \( P \) corresponding to edge \( u_i v_j \) choose one of the variables \( x_i \) or \( y_j \)–the one that corresponds to the vertex on which the arrow points. In this way we obtain monomial \( M = x_1^{k_1} \ldots x_m^{k_m} y_1^{l_1} \ldots y_n^{l_n} \) satisfying \( 0 \leq k_i, l_j \leq k \). Why is this monomial nonvanishing in \( P \)? It is because each variable occurs in factors of \( P \) with uniform sign (\( x_i \) with minus sign, \( y_j \) with plus sign). Hence, the sign of monomial \( M \) in \( P \) is uniquely determined by the sequence of exponents, and therefore its copies cannot cancel. Finally, to apply Combinatorial Nullstellensatz, notice that \( \deg(P) \) equals the number of edges in \( G \), which is the same as \( k_1 + \ldots + k_m + l_1 + \ldots + l_n \) since \( q(u_i) - q(v_j) \) are constants.

**Corollary 6** Every tree has an additive coloring from arbitrary lists of size two. Every bipartite planar graph has an additive coloring from arbitrary lists of size three.

**Proof.** Every tree has an orientation with at most one incoming edge to every vertex. Every bipartite planar graph has an orientation with indegrees bounded by two.

### 4 Main results

Let us start with a simpler case of planar 3-colorable graphs.

**Theorem 7** Every planar graph \( G \) with \( \chi(G) \leq 3 \) satisfies \( \eta(G) \leq 36 \).
Proof. Let \( V(G) = A \cup B \cup C \) be a partition of the vertex set of \( G \) into three independent sets. Let \( H \) be a subgraph of \( G \) on the set of vertices \( V(H) = A \cup B \cup C \) with the edge set

\[
E(H) = \{uv \in E(G) : u \in A \cup B \text{ and } v \in C\}.
\]

Clearly \( H \) is a bipartite graph. Hence, by Theorem 3, there is a function \( h : C \to \{1, 2, \ldots, 12\} \) such that the sum

\[
S_h(u) = \sum_{x \in N_H(u)} h(x)
\]

satisfies \( S_h(u) \neq 0(\text{mod } 12) \) for every vertex \( u \in A \cup B \) having at least one neighbor in \( C \). For other vertices the above sum is empty and we adopt \( S_h(u) = 0 \) by convention.

Consider now a bipartite subgraph \( F \) of \( G \) induced by the vertices \( A \cup B \). Assign to each vertex \( u \) in \( F \) the list \( L(u) = \{12, 24, 36\} \), and apply Theorem 5 with function \( q(u) = S_h(u) \). Let \( f \) be a coloring satisfying the assertion of Theorem 5. That is, \( f \) satisfies condition \( S_f(u) + S_h(u) \neq S_f(v) + S_h(v) \) for every edge \( uv \in E(F) \), where

\[
S_f(u) = \sum_{x \in N_P(u)} f(x).
\]

Finally, let \( g \) be a function defined on the whole set of vertices \( V(G) \) by joining \( f \) and \( h \):

\[
g(x) = \begin{cases} 
h(x) & \text{if } x \in C \\
f(x) & \text{if } x \in A \cup B \end{cases}
\]

We claim that \( g \) is an additive coloring of \( G \) over the set \( \{1, 2, \ldots, 36\} \). Let \( S(u) \) be the sum of \( g \)-labels over the whole neighborhood \( N(u) \), that is, \( S(u) = S_h(u) + S_f(u) \). Let \( uv \) be any edge in \( G \). If \( u \in A \cup B \) and \( v \in C \), then \( S_h(u) \neq 0(\text{mod } 12) \) and \( S_f(u) = 0(\text{mod } 12) \), thus \( S(u) \neq 0(\text{mod } 12) \). On the other hand, \( S_h(v) = S_f(v) = 0(\text{mod } 12) \), so \( S(v) = 0(\text{mod } 12) \). In the other case, if \( u \in A \) and \( v \in B \), condition \( S(u) \neq S(v) \) is guaranteed by construction of \( f \). This completes the proof.

The proof for 4-colorable planar graphs is similar in spirit, though a bit more technical.

Theorem 8 Every planar graph satisfies \( \eta(G) \leq 468 \).

Proof. Let \( V(G) = A \cup B \cup C \cup D \) be a partition of the vertex set of \( G \) into four independent sets. Let \( H_1 \) be a subgraph of \( G \) on the set of vertices \( (A \cup B) \cup C \) with the edge set

\[
E(H_1) = \{uv \in E(G) : u \in A \cup B \text{ and } v \in C\}.
\]
Clearly $H_1$ is a bipartite graph. Hence, by Theorem 3 there is a function $h_1 : C \to \mathbb{Z}_{12}$ such that the sum

$$S_{h_1}(u) = \sum_{x \in N_{h_1}(u)} h_1(x)$$

satisfies $S_{h_1}(u) \not\equiv 0 \pmod{12}$ for every vertex $u \in A \cup B$ with at least one neighbor in $C$. Now, let $H_2$ be a subgraph of $G$ on the set of vertices $(A \cup B \cup C) \cup D$ with the edge set

$$E(H_2) = \{uv \in E(G) : u \in A \cup B \cup C \text{ and } v \in D\}.$$ 

Clearly $H_2$ is a bipartite graph. Hence, by Theorem 3 there is a function $h_2 : D \to \mathbb{Z}_{13}$ such that the sum

$$S_{h_2}(u) = \sum_{x \in N_{h_2}(u)} h_2(x)$$

satisfies $S_{h_2}(u) \not\equiv 0 \pmod{13}$ for every vertex $u \in (A \cup B \cup C)$ having a neighbor in $D$.

Now, using functions $h_1$ and $h_2$, we define a new function $h : C \cup D \to \{1, 2, \ldots, 156\}$ as follows. First we extend $h_1$ and $h_2$ to the whole set $C \cup D$ by putting $h_1(x) = 0$ for $x \in D$ and $h_2(x) = 0$ for $x \in C$. Let $\sigma$ be a group isomorphism from $\mathbb{Z}_{12} \times \mathbb{Z}_{13}$ to $\mathbb{Z}_{156}$. For each $x \in C \cup D$ define $h(x)$ as the unique number in the range $\{1, 2, \ldots, 156\}$ satisfying congruence

$$h(x) \equiv \sigma((h_1(x), h_2(x))) \pmod{156}.$$ 

Let

$$S_h(u) = \sum_{x \in N(u) \cap (C \cup D)} h(x)$$

for every $u \in A \cup B$, where, as before, $S_h(u) = 0$ if $N(u) \cap (C \cup D) = \emptyset$. First we claim that $S_h(u) \not\equiv 0 \pmod{156}$ for every vertex $u \in A \cup B$ which has at least one neighbor in $C \cup D$. Indeed, since $\sigma$ is a group isomorphism we may write

$$S_h(u) = \sum_{x \in N(u) \cap (C \cup D)} h(x) = \sum_{x \in N(u) \cap (C \cup D)} \sigma((h_1(x), h_2(x))) = \sigma\left(\left(\sum_{x \in N(u) \cap C} h_1(x), \sum_{x \in N(u) \cap D} h_2(x)\right)\right) = \sigma((S_{h_1}(u), S_{h_2}(u))).$$

Hence, $S_h(u)$ cannot be zero in $\mathbb{Z}_{156}$, since at least one of the sums $S_{h_1}(u)$ or $S_{h_2}(u)$ is non-zero in its respective group. Notice also that $S_h(u) \not\equiv 0 \pmod{156}$ for every vertex $u \in C$ and having a neighbor in $D$, as in this case we have $S_h(u) = \sigma((0, S_{h_2}(u)))$ and $S_{h_2}(u) \not\equiv 0$ in $\mathbb{Z}_{13}$.
Consider now a bipartite subgraph $F$ of $G$ induced by the vertices $A \cup B$. Assign to each vertex $u$ in $F$ the list $L(u) = \{156, 312, 468\}$, and apply Theorem 5 with function $q(u) = S_h(u)$. Let $f$ be a coloring satisfying the assertion of Theorem 5. That is, $f$ satisfies condition $S_f(u) + S_h(u) \neq S_f(v) + S_h(v)$ for every edge $uv \in E(F)$, where

$$S_f(u) = \sum_{x \in N_F(u)} f(x).$$

Putting things together we define a function $g$ on the whole set of vertices $V(G)$ by joining $f$ and $h$:

$$g(x) = \begin{cases} h(x) & \text{if } x \in C \cup D \\ f(x) & \text{if } x \in A \cup B \end{cases}.$$

We claim that $g$ is an additive coloring of $G$ over the set $\{1, 2, \ldots, 468\}$. Let $S(u)$ be the sum of $g$-labels over the whole neighborhood $N(u)$, that is, $S(u) = S_h(u) + S_f(u)$. Let $uv$ be any edge in $G$. If $u \in A \cup B$ and $v \in C \cup D$, then $S_h(u) \neq 0 \pmod{156}$ while $S_f(u) = 0 \pmod{156}$, thus $S(u) \neq 0 \pmod{156}$. The other end of the edge satisfies $S_h(v) = S_f(v) = 0 \pmod{156}$, so $S(v) = 0 \pmod{156}$. If $u \in A$ and $v \in B$, condition $S(u) \neq S(v)$ is guaranteed by construction of $f$. We are left with the last case $u \in C$ and $v \in D$. Suppose on the contrary that $S(u) = S(v)$. Since $S_f(u) = S_f(v) = 0 \pmod{156}$, we get $S_h(u) = S_h(v)$ in $\mathbb{Z}_{156}$. But $S_h(u) = \sigma((0, S_{h_2}(u)))$ and $S_{h_2}(u) \neq 0$ in $\mathbb{Z}_{13}$, while $S_h(v) = \sigma((S_{h_1}(v), 0))$. This contradiction completes the proof.

A set of vertices $I$ in a graph $G$ is called two-independent if the distance between any two vertices of $I$ is at least three. In [3] it was proved that every planar graph of girth at least 13 has a vertex decomposition into two sets $I$ and $F$ such that $I$ is two-independent and $F$ induces a forest. Our last theorem follows easily from this result.

**Theorem 9** Every planar graph of girth at least 13 satisfies $\eta(G) \leq 4$.

**Proof.** Let $V(G) = I \cup F$, where $I$ is 2-independent and $F$ induces a forest. By Corollary 4 there is an additive coloring $f$ of the forest $F$ using labels $\{2, 4\}$. Extend this coloring to the whole graph $G$ by putting $f(i) = 1$ for each vertex $i \in I$. It is easy to see that $f$ is an additive coloring of $G$. ■
5 Finite abelian groups

The problem of additive coloring can be considered in a more general setting of Abelian (additive) groups. We may use elements of any such group $\Gamma$ as the labels of vertices and define the additive coloring the same way as before. Accordingly to our main conjecture, as well as to the methods we develop so far, one could expect that perhaps every graph has an additive coloring over some group whose order is equal to the chromatic number of the graph. We prove below that this is not true.

**Theorem 10** For every $r \geq 2$ there is a graph $G_r$ such that $\chi(G_r) = r$, and there is no additive coloring of $G_r$ over any finite Abelian group of order $r$. But there is an additive coloring of $G_r$ in $\mathbb{Z}_{r+1}$.

**Proof.** Let $P$ denote a path on five vertices $a, x, b, y, c$ (in that order). Consider a graph $H = H(r)$ obtained by blowing up each of the two vertices $x$ and $y$ to the clique $K_{r-1}$. Now, take $r$ copies of $H$, chose one vertex $v_i$ in any of the two cliques $K_{r-1}$ in each copy of $H$, and join all these vertices mutually to form a new clique $K_r$. We claim that in this way we constructed a graph $G_r$ satisfying the assertion of the theorem. It is not hard to see that $\chi(G_r) = r$. To prove the first part of the theorem, suppose that $\Gamma$ is any Abelian group of order $r$, and there is a coloring $f : V(G_r) \rightarrow \Gamma$ such that the sums $S(v)$ form a proper coloring of $G_r$. Notice that in any proper coloring of $H$ with $r$ colors, the vertices $a$, $b$, and $c$ must have the same color. Thus $s(a) = s(b) = s(c)$. Notice also that, by the definition of additive coloring we have $S(b) = S(a) + S(c)$, which implies that $S(a) = S(b) = S(c) = 0$ in every copy of $H$ in $G_r$. This implies in turn that $S(v) \neq 0$ for all other vertices of $G_r$. In particular, we get a proper coloring of the clique $K_r$ by non-zero elements of $\mathbb{Z}_r$, which is not possible.

For the second assertion we define explicitly an additive coloring $f : V(G_r) \rightarrow \mathbb{Z}_{r+1}$ as follows. Denote by $H_i$ the $i$th copy of the graph $H$ in $G_r$. Let $X_i$ and $Y_i$ denote the two cliques $K_{r-1}$ in $H_i$ obtained by blowing up the vertices $x$ and $y$, respectively. Also, let $a_i$, $b_i$, and $c_i$ be the respective copies of the end vertices and the middle vertex of the path $P$ in $H_i$. Finally, let $v_i$ denote the unique vertex of $H_i$ belonging to the clique $K_r$. We may assume that $v_i \in V(X_i)$. We have to distinguish two cases.

1. (**The number $r + 1$ is odd.**) Put $f(v_i) = f(b_i) = 0$ and $f(a_i) = f(c_i) = i$ for all $i = 1, 2, \ldots, r$. Then extend injectively the coloring using all labels from the set
Theorem 11 Let $A$ be a fixed Abelian group. The problem of deciding whether a given graph $G$ has an additive coloring over $A$ is NP-complete if $|A| \geq 3$, and polynomial for $A = \mathbb{Z}_2$. 

Proof. Let $|A| = k \geq 3$. For a given graph $G$, whose vertex set is $V(G) = \{v_1, \ldots, v_n\}$, consider a new graph $G'$ obtained by adding $n$ new vertices $\{v'_1, \ldots, v'_n\}$ and $n$ new edges $v_iv'_i$ for $i = 1, \ldots, n$. We prove that $G$ is $k$-colorable (in the usual sense) if and only if $G'$ is additively colorable over $A$. This will prove the first assertion of the theorem.
Obviously, if $G'$ has an additive coloring over $A$, then $G$ is $k$-colorable in the usual sense. For the other implication, assume that $G$ is $k$-colorable, and fix a proper coloring $c$ of $G$ using $A$ as the set of colors. Now fix a nonzero element $a \in A$ and define a new coloring $f$ of $G'$ in the following way:

1. If $c(v_i) = 0$, then $f(v_i) = a$.
2. If $c(v_i) \neq 0$, then $f(v_i) = 0$.
3. $f(v'_i) = c(v_i) - \sum_{x \in N_G(v_i)} f(v_i)$.

We claim that $f$ is a desired additive coloring of $G'$ over $A$. Indeed, the sum of colors around each vertex $v_i$ satisfies

$$S(v_i) = \sum_{x \in N_G(v_i)} f(v_i) + f(v'_i) = c(v_i),$$

so there are no conflicts in $G$. Also by definition of $f$ we have

$$S(v'_i) = f(v_i) \neq c(v_i) = S(v_i)$$

for each vertex $v'_i$. This prove the claim.

For the second assertion just notice that the problem reduces to recognizing if a given graph $G$ is bipartite, and then checking solvability of a system of linear equations of the form $Mx = y$ over $\mathbb{Z}_2$, where $M$ is the adjacency matrix of $G$, and $y$ is binary vector encoding a proper coloring of $G$. There are actually two possible such vectors for a connected bipartite graph $G$. This completes the proof. ■

6 Open problems

We conclude the paper with a short list of open questions concerning additive coloring of graphs.

**Conjecture 12** Every graph $G$ satisfies $\eta(G) \leq \chi(G)$. 


It is not known whether this is true for bipartite graphs. It is not even known if $\eta(G)$ is bounded for bipartite graphs. A heuristic argument is that the statement of the conjecture holds trivially if we extend the set of labels to real numbers. Indeed, any proper coloring of a $k$-colorable graph $G$ with a set of $k$ real numbers which is independent over rationals, gives an additive coloring of $G$. Another direction is to consider additive colorings in finite Abelian groups.

**Conjecture 13** Every graph $G$ has an additive coloring modulo $\chi(G) + 1$.

If true this is best possible, as we proved in section 5.

Our last problem arose as a vertex analog of the famous antimagic labeling conjecture of Ringel [5].

**Conjecture 14** Let $G$ be a simple graph on $n$ vertices in which no two vertices have the same neighborhood. Then there is a bijection $f : V(G) \rightarrow \{1, 2, \ldots, n\}$ such that

\[
\sum_{x \in N(u)} f(x) \neq \sum_{x \in N(v)} f(x)
\]

for any two distinct vertices $u$ and $v$.

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