BICONSERVATIVE HYPERSURFACES WITH CONSTANT SCALAR CURVATURE IN SPACE FORMS

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Abstract. Biconservative hypersurfaces are hypersurfaces which have conservative stress-energy tensor with respect to the bienergy, containing all minimal and constant mean curvature hypersurfaces. The purpose of this paper is to study biconservative hypersurfaces $M^n$ with constant scalar curvature in a space form $N^{n+1}(c)$. We prove that every biconservative hypersurface with constant scalar curvature in $N^4(c)$ has constant mean curvature. Moreover, we prove that any biconservative hypersurface with constant scalar curvature in $N^n(c)$ is either an open part of a certain rotational hypersurface or a constant mean curvature hypersurface. These solve an open problem proposed recently by D. Fetcu and C. Oniciuc for $n \leq 4$.

1. Introduction

In 1983, Eells and Lemaire [7] introduced the biharmonic map to classify maps between two Riemannian manifolds $(M^n, g)$ and $(N^m, \tilde{g})$. A biharmonic map $\phi : (M^n, g) \rightarrow (N^m, \tilde{g})$ is defined as a critical point of the bienergy functional $E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 dv_g$ where $\tau(\phi)$ is the tension field associated to $\phi$. By computing the first variational formulae, a biharmonic map $\phi$ is characterized by the vanishing of the associated bitension field:

$$\tau_2(\phi) := -\Delta \tau(\phi) - \text{trace} R^N(d\phi, \tau(\phi))d\phi = 0.$$ (1.1)

A submanifold $M^n$ of $N^m$ is called a biharmonic if the isometric immersion defining the biharmonic submanifold is a biharmonic map. The study of biharmonic submanifolds has attracted great attentions since then in geometry. Many important research results have been carried out to investigate the existence and classification problems of biharmonic submanifolds in some model spaces. More details and an overview about the historic development concerning biharmonic submanifolds can be found in [10, 24] and the references therein.

During the study of biharmonic submanifolds, Caddeo et al. [1] introduced the concept of biconservative immersions from the principle of a stress-energy tensor for the bienergy. An isometric immersion $\phi : (M^n, g) \rightarrow (N^m, \tilde{g})$ is said to be biconservative if its associated divergence of the stress-bienergy tensor $S_2$ is zero. Note that the biconservative immersion $\phi$ is given by the vanishing of the tangent part of the bitension field (c.f. [1, 23]). Hence, biharmonic submanifolds are automatically biconservative submanifolds.

In [1], Caddeo et al. made a first contribution on the classification problem of biconservative surfaces in 3-dimensional Riemannian space forms $N^3(c)$. Soon after, the first author [11] found a new biconservative surface in the hyperbolic 3-space $\mathbb{H}^3$ and provided a complete explicit classification of biconservative surfaces in 3-dimensional Riemannian space forms $N^3(c)$. Later, Montaldo, Oniciuc and Ratto [17] studied biconservative surfaces in an arbitrary Riemannian manifold; in particular, they found a remarkable property that the Hopf differential associated to...
a biconservative surface in an arbitrary Riemannian manifold is holomorphic if and only if the surface has constant mean curvature. The local parametric equations of biconservative hypersurfaces in 4-dimensional space forms were obtained in [15,19]. Furthermore, the global and uniqueness properties of biconservative surfaces (or hypersurfaces) have been investigated in a series of papers [8,9,18–20].

On the other hand, the study of minimal hypersurfaces of constant scalar curvature \( R \) in the unit sphere \( S^{n+1} \) has received great attentions in geometry. In 1960’s, Simons [25], Lawson [22] and Chern-do Carmo-Kobayashi [2] pioneered the study of minimal hypersurfaces with constant scalar curvature in the unit sphere. In particular, S. S. Chern proposed a famous conjecture: for a closed minimal hypersurface with constant scalar curvature \( R \) in the unit sphere \( S^{n+1} \), the set of \( R \) should be discrete. Later, Verstraelen et al [26] reformulated a stronger version of Chern’s Conjecture as following: any closed minimal hypersurface in the unit sphere \( S^{n+1} \) with constant scalar curvature is isoparametric. In 1993, Peng-Terng [21] and Chang [4] solved Chern’s Conjecture for \( n = 3 \). Furthermore, Chang [5] proved that any closed hypersurface in the unit sphere \( S^4 \) with constant mean curvature (CMC) and constant scalar curvature is isoparametric. Cheng and Wan [6] showed that any complete hypersurface with constant mean curvature and constant scalar curvature in a space form \( N^4(c) \) is isoparametric. For \( n \geq 4 \), Chern’s conjecture remains open. Some important progress had been made recently, see for examples [27,28] and references therein.

Since the class of biconservative hypersurfaces is a generalization of minimal (or CMC) hypersurfaces, it is very interesting to investigate whether Chern’s conjecture holds for biconservative hypersurfaces instead of using minimality (or CMC) for hypersurfaces in space forms. In their recent survey [10], Fetcu and Oniciuc proposed a challenging problem on biconservative hypersurfaces as following:

**Fetcu-Oniciuc’s Problem:** Classify all biconservative hypersurfaces with constant scalar curvature in a space form \( N^{n+1}(c) \).

For a non-CMC biconservative surface in \( N^3(c) \), a relation between the Gauss curvature \( K \) and mean curvature \( H \) holds that \( K = -3H^2 + c \) (c.f. [11], [18]). Assuming that the Gauss curvature is constant, it implies that the mean curvature \( H \) is also constant. Therefore, any biconservative surface with constant Gauss curvature (or constant scalar curvature) in \( N^3(c) \) is CMC. Hence the problem of the case of \( n = 2 \) was known.

In this paper, we firstly solve Fetcu-Oniciuc’s problem for \( n = 3 \):

**Theorem 1.1.** Any biconservative hypersurface with constant scalar curvature in \( N^4(c) \) has constant mean curvature.

As applications of Theorem 1.1, we use Chang’s result [5] to obtain a global result for biconservative hypersurfaces:

**Corollary 1.2.** Any closed biconservative hypersurface with constant scalar curvature in sphere \( S^4 \) is isoparametric.

Even more, using Cheng and Wan’s result [6], we have

**Corollary 1.3.** Any complete biconservative hypersurface with constant scalar curvature in a space form \( N^4(c) \) is isoparametric.

Secondly, we investigate Fetcu-Oniciuc’s problem for \( n = 4 \). Interestingly, we found a different class of non-CMC biconservative hypersurfaces with constant scalar curvature in \( N^5(c) \). Precisely, we prove
Theorem 1.4. Any biconservative hypersurface with constant scalar curvature in $N^5(c)$ is either CMC or contained in a certain non-CMC rotational hypersurface, where the rotational hypersurface has two distinct principal curvatures with $-\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ and the scalar curvature $R$ of this rotational hypersurface is $12c$.

The main idea to prove Theorem 1.4 (also Theorems 1.1) is based on the approach developed in [13] for settling Chen’s biharmonic conjecture. By converting the differential equations related to principal curvature functions into a system of algebraic differential equations, the solution can be determined completely. Furthermore, making use of the elimination method we get a polynomial function equation and derive a contradiction. Combining these with using a result of do Carmo and Dajczer [3], we prove Theorem 1.4. The current approach seems to be quite useful, see also Guan-Li-Vrancken’s work [14].

Remark 1. For a non-CMC rotational hypersurface in $N^5(c)$, if its principal curvatures satisfying a linear relation

$$-\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4,$$

then the rotational hypersurface must be biconservative (see details in Section 5).

Remark 2. Li and Wei showed in [16] that there are no compact embedded rotational hypersurfaces with constant scalar curvature $n(n-1)$ of $S^{n+1}$ other than the Riemannian product $S^{n-1}\left(\sqrt{\frac{n-2}{n}}\right) \times S^1\left(\sqrt{\frac{2}{n}}\right)$ and round geodesic spheres. Then we remark that for $c = 1$, the rotational hypersurface in Theorem 1.4 is not compact embedded.

The paper is organized as follows. In Section 2, we recall some background on the theory of hypersurfaces and derive a useful lemma (Lemma 2.2) for biconservative hypersurfaces with constant scalar curvature. In Section 3, we study the case of biconservative hypersurfaces in 4-dimensional space forms. We establish three key lemmas (Lemmas 3.1-3.3) to prove Theorem 1.1. In Section 4, we consider the case of biconservative hypersurfaces in a 5-dimensional space form and give a proof of Theorem 1.4. In Section 5, We give details to verify Remark 1.

2. Preliminaries

Let $N^{n+1}(c)$ be an $(n+1)$-dimensional Riemannian space form with constant sectional curvature $c$. For an isometric immersion $\phi : M^n \rightarrow N^{n+1}(c)$, we denote by $\nabla$ the Levi-Civita connection of $M^n$ and $\tilde{\nabla}$ the Levi-Civita connection of $N^{n+1}(c)$. For any $X, Y, Z \in C(TM)$, the Gauss and Codazzi equations are given by

$$R(X, Y)Z = c\langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle AY, Z \rangle AX - \langle AX, Z \rangle AY,$$

$$\nabla_X AY = \nabla_Y AX - A(\nabla_X Y).$$

Note that here $A$ is the shape operator satisfying

$$(\nabla_X A)Y = \nabla_X (AY) - A(\nabla_X Y),$$

and the curvature tensor of $M^n$ is defined to be

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$
Using Codazzi equation (2.2), it yields
e_{i} \in \mathbb{P}

Let \( A \) be the shape operator of the hypersurface \( M^{n} \), then the function \( A \) is given by

\[
A(A^{\top}) = \lambda_{1}, \cdots, \lambda_{n},
\]

where \( \lambda_{1}, \cdots, \lambda_{n} \) are the principal curvatures of \( M^{n} \). The definition of the mean curvature means \( \sum_{i=1}^{n} \lambda_{i} = nh \), and hence

\[
\lambda_{2} + \cdots + \lambda_{n} = -3\lambda_{1}.
\]

The squared length of the second fundamental form, denoted by \( S \), is defined as

\[
S = \text{trace} A^{2} = \sum_{i=2}^{n} \lambda_{1}^{2} + \lambda_{i}^{2}.
\]

By Gauss equation (2.1), it is easy to check that the scalar curvature \( R \) of \( M^{n} \) is given by

\[
R = n(n-1)c + 4\lambda_{1}^{2} - S.
\]

Combining (2.9) with (2.10) gives

\[
\sum_{i=2}^{n} \lambda_{1}^{2} = n(n-1)c + 3\lambda_{1}^{2} - R.
\]

Since \( e_{1} \) is parallel to \( \text{grad} H \), it follows that \( e_{i}(H) \neq 0 \) and \( e_{i}(H) = 0 \) for \( 2 \leq i \leq n \), and hence

\[
e_{1}(\lambda_{1}) \neq 0, \quad e_{i}(\lambda_{1}) = 0, \quad 2 \leq i \leq n.
\]

Set \( \nabla_{e_{i}} e_{j} = \sum_{k=1}^{n} \omega_{ij}^{k} e_{k} \) (\( 1 \leq i, j \leq n \)), where \( \omega_{ij}^{k} \) is the connection coefficients of \( M^{n} \). By the compatibility conditions \( \nabla_{e_{k}}(e_{i}, e_{i}) = 0 \) and \( \nabla_{e_{k}}(e_{i}, e_{j}) = 0 \) (\( i \neq j \)), we have

\[
\omega_{ki}^{i} = 0, \quad \omega_{ki}^{j} + \omega_{kj}^{i} = 0, \quad i \neq j.
\]

Using Codazzi equation (2.2), it yields

\[
e_{i}(\lambda_{j}) = (\lambda_{i} - \lambda_{j}) \omega_{ij}^{i},
\]

(2.14)

\[
(\lambda_{i} - \lambda_{j}) \omega_{ki}^{j} = (\lambda_{k} - \lambda_{j}) \omega_{ik}^{j}.
\]

(2.15)
Lemma 2.1. Let $T_n$ to

Proof. Put $\omega_i$ By multiplying respectively, using (2.8) we have

Similarly to the Lemma 3.2 in [12], it follows from Gauss equation that

Denote $f_k = \sum_{i=2}^n (\omega_{ii}^1)^k$, $k \in \mathbb{N}^*$. For the sake of simplicity, we write $\lambda = \lambda_1$, $T = f_1$, $T' = e_1(T)$, $T'' = e_1 e_1(T)$, $T''' = e_1 e_1 e_1(T)$ and $T'''' = e_1 e_1 e_1 e_1(T)$.

By modifying an argument in [13], we have

Lemma 2.1. Let $M^n$ be a proper biconservative hypersurface with constant scalar curvature in forms $5$. Then the functions $f_1, \cdots, f_5$ can be written as

\[
\begin{align*}
\frac{dT}{\lambda} &= \lambda T, \\
f_1 &= T, \\
f_2 &= T' + 3\lambda^2 - (n - 1)c, \\
f_3 &= \frac{1}{2} T'' - (\lambda^2 + c)T + 6\lambda\lambda', \\
f_4 &= \frac{1}{6} T''' - \frac{4}{3}(\lambda^2 + c)T' - \frac{7}{6}\lambda\lambda'T + 2\lambda^2 \\
&\quad + \frac{7}{2}\lambda\lambda'' + \frac{n^2 - 10}{2}c\lambda^2 - \frac{1}{2}R\lambda^2 + (n - 1)c^2, \\
f_5 &= \frac{1}{24} T'''' - \frac{5}{6}(\lambda^2 + c)T'' - \frac{35}{24}\lambda\lambda'T' \\
&\quad - \frac{1}{24}(11\lambda\lambda'' + 7\lambda^2 - 24\lambda^3 - 48\lambda c\lambda^2 - 24\lambda c^2)T' \\
&\quad + \frac{11}{8}\lambda\lambda''' + \frac{15}{8}\lambda'\lambda'' - 2\lambda^3\lambda' - \frac{134 - 5n^2}{12}c\lambda\lambda' - \frac{5}{12}\lambda\lambda'R.
\end{align*}
\]

Proof. Put $g_1 = \sum_{i=2}^n \lambda_i \omega_{ii}^1$. Taking the sum of $i$ from 2 to $n$ in (2.17) and (2.16) respectively, using (2.8) we have

\[
\begin{align*}
f_2 &= 3\lambda^2 + e_1(f_1) - (n - 1)c = T' + 3\lambda^2 - (n - 1)c, \\
g_1 &= \lambda T - 3e_1(\lambda) = \lambda T - 3\lambda'.
\end{align*}
\]

By multiplying $\omega_{ii}^1$ on both sides of equation (2.17), taking the sum of $i$ and using (2.19)–(2.20) gives

\[
\begin{align*}
f_3 &= \frac{1}{2} e_1(f_2) - \lambda g_1 - cf_1 \\
&= \frac{1}{2} T'' - (\lambda^2 + c)T + 6\lambda\lambda'.
\end{align*}
\]

Put $g_2 = \sum_{i=2}^n \lambda_i (\omega_{ii}^1)^2$. Differentiating $g_1 = \sum_{i=2}^n \lambda_i \omega_{ii}^1$ along $e_1$, using (2.16)–(2.17), it follows that

\[
g_2 = \frac{1}{2} \left\{ e_1(g_1) - \lambda \sum_{i=2}^n \lambda_i^2 + \lambda f_2 + 3c\lambda \right\}.
\]

In view of (2.19)–(2.20) and (2.11), the above equation becomes

\[
g_2 = \lambda T' + \frac{1}{2} \lambda T - \frac{3}{2} \lambda' - \frac{1}{2}c\lambda^2 - \frac{1}{2} R.
\]

By multiplying $(\omega_{ii}^1)^2$ on both sides of equation (2.17), taking the sum of $i$ from 2 to $n$ gives

\[
f_4 = \frac{1}{3} e_1(f_3) - \lambda g_2 - cf_2.
\]

Using (2.19), (2.21) and (2.23), $f_4$ can be written as

\[
f_4 = \frac{1}{6} T''' - \frac{4}{3}(\lambda^2 + c)T' - \frac{7}{6}\lambda\lambda'T + 2\lambda^2 \\
&\quad + \frac{7}{2}\lambda\lambda'' + \frac{n^2 - 10}{2}c\lambda^2 - \frac{1}{2}R\lambda^2 + (n - 1)c^2.
\]
Put $g_3 = \sum_{i=2}^{n} \lambda_i^2 \omega_{ii}^1$. Multiplying equation (2.16) by $\lambda_i$, using (2.11) and (2.20) yields

$$g_3 = \frac{1}{2} e_1 (\sum_{i=2}^{n} \lambda_i^2) + \lambda g_1 = \lambda^2 T.$$  \hspace{1cm} (2.25)

Put $g_4 = \sum_{i=2}^{n} \lambda_i (\omega_{ii}^1)^3$. Differentiating $g_2 = \sum_{i=2}^{n} \lambda_i (\omega_{ii}^1)^2$ with respect to $e_1$ and using (2.16)–(2.17), we have

$$g_4 = \frac{1}{3} (e_1 (g_2) + \lambda f_3 - 2 \lambda g_3 - 2 c g_1),$$  \hspace{1cm} (2.26)

which together with (2.20), (2.21), (2.23) and (2.25) yields

$$g_4 = \frac{1}{2} \lambda T'' + \frac{1}{2} \lambda' T'' + (\frac{1}{6} \lambda'' - \lambda^3 - c \lambda) T$$
$$- \frac{1}{2} \lambda''' + (2 \lambda^2 + \frac{1}{6} R) \lambda' + \frac{1}{6} (16 - n^2) c \lambda'.$$  \hspace{1cm} (2.27)

By multiplying $(\omega_{ii}^1)^3$ on both sides of equation (2.17) and taking the sum of $i$, we have

$$f_5 = \frac{1}{4} e_1 (f_4) - \lambda g_4 - c f_3.$$  \hspace{1cm} (3.1)

It follows from (2.21), (2.24) and (2.26) that

$$f_5 = \frac{1}{24} T''' - \frac{5}{6} (\lambda^2 + c) T'' - \frac{35}{24} \lambda \lambda' T'$$
$$- \frac{1}{24} (11 \lambda \lambda'' + 7 \lambda'^2 - 24 \lambda^4 - 48 c \lambda^2 - 24 c^2) T$$
$$+ \frac{11}{8} \lambda \lambda''' + \frac{15}{8} \lambda' \lambda'' - 2 \lambda^3 \lambda' - \frac{134 - 5 n^2}{12} c \lambda' - \frac{5}{12} \lambda' R.$$  \hspace{1cm} (3.2)

3. Biconservative hypersurfaces in $N^4(c)$

In this section, we consider biconservative hypersurfaces with constant scalar curvature in $N^4(c)$. In order to prove the Theorem 1.1 we need the following lemmas. Firstly, an easy computation shows that

**Lemma 3.1.** Let $f_k = (\omega_{22}^1)^k + (\omega_{33}^1)^k$. Then the functions $f_1, \cdots, f_4$ satisfy the following two algebraic equations:

$$f_1^3 - 3 f_1 f_2 + 2 f_3 = 0,$$  \hspace{1cm} (3.1)
$$f_1^4 - 2 f_1^2 f_2 - f_2^2 + 2 f_4 = 0.$$  \hspace{1cm} (3.2)

**Proof.** From the definition of $f_k$, one has

$$f_3 = (\omega_{22}^1 + \omega_{33}^1) \{ (\omega_{22}^1)^2 - \omega_{22}^1 \omega_{33}^1 + (\omega_{33}^1)^2 \}$$
$$= f_1 \{ f_2 - \frac{1}{2} (f_1^2 - f_2) \}$$
$$= \frac{1}{2} f_1 (3 f_2 - f_1^2)$$  \hspace{1cm} (3.3)
and
\[
\begin{align*}
f_4 &= (\omega_{22}^1)^4 + (\omega_{33}^1)^4 + 2(\omega_{22}^1)^2(\omega_{33}^1)^2 - 2(\omega_{22}^1)^2(\omega_{33}^1)^2 \\
&= (\omega_{22}^1 + \omega_{33}^1)^2 - 2(\omega_{22}^1)^2(\omega_{33}^1)^2 \\
&= f_2^2 - \frac{1}{2}(f_1^2 - f_2)^2 \\
&= \frac{1}{2}(f_2^2 + 2f_1^2 - f_2 - f_2^4).
\end{align*}
\]
(3.4)

Therefore, (3.3) and (3.4) give (3.1) and (3.2) respectively.

By Lemma 3.1 and Lemma 3.2, we can get the following lemma.

Lemma 3.2. Let \( \phi : M^3 \to N^4(c) \) be an orientable proper biconservative hypersurface with constant scalar curvature. Then we have \( e_i(T) = 0 \) for \( i = 2, 3 \).

Proof. Assume that \( T \neq 0 \). Substituting \( f_1, f_2, f_3, f_4 \) of (2.18) into (3.1) and (3.2), we obtain two equations
\[
\begin{align*}
T'' - 3TT' + T^3 + (4c - 11\lambda^2)T + 12\lambda\lambda' &= 0, \quad (3.5) \\
\frac{1}{3}T''' - T'' + \left( \frac{4}{3}c - \frac{26}{3}\lambda^2 - 2T^2 \right)T' + T^4 + (4c - 6\lambda^2)T^2 \\
&- \frac{7}{3}\lambda\lambda' + 9\lambda^4 - \lambda^2R + 11c\lambda^2 + 7\lambda\lambda'' + 4\lambda^2 &= 0.
\end{align*}
\]

Differentiating (3.5) with respect to \( e_1 \) and using (3.5)-(3.6), we can eliminate \( T''' \), \( T'' \) and get
\[
5\lambda T' - 5\lambda T^2 + 7\lambda' T + 9\lambda^3 + \lambda R - 11c\lambda - 3\lambda'' = 0. \quad (3.7)
\]
Note that \( \lambda \neq 0 \). Differentiating (3.7) with respect to \( e_1 \) and using (3.5), one has
\[
(5\lambda T + 12\lambda')T' - 5\lambda T^3 - 5\lambda T^2 + 7\lambda'' T + 55\lambda^3 T \\
- 20\lambda Tc + R\lambda' - 33\lambda^2\lambda' - 11c\lambda' - 3\lambda'' = 0.
\]

Suppose that \( 5\lambda T + 12\lambda' = 0 \), then \( T = -\frac{12\lambda'}{5\lambda} \), in this case the lemma follows immediately. Assume that \( 5\lambda T + 12\lambda' \neq 0 \). Combining (3.7) with (3.8) gives
\[
p_1 T + p_2 = 0, \quad (3.9)
\]
where
\[
p_1 = 50\lambda\lambda'' - 84\lambda R^2 + 230\lambda^4 - 5R\lambda^2 - 45c\lambda^2, \\
p_2 = -15\lambda\lambda'' + 36\lambda' \lambda'' - 273\lambda^3 \lambda' - 7R\lambda\lambda' + 77c\lambda\lambda'.
\]

**Case A:** \( p_1 \neq 0 \). It follows that \( T = -p_2/p_1 \), and the lemma follows.

**Case B:** \( p_1 = 0 \). It follows from (3.9) that \( p_2 = 0 \). By applying \( p_1 = 0 \) and \( p_2 = 0 \), we can eliminate \( \lambda'' \), \( \lambda' \) and get
\[
-252\lambda^2 - 60\lambda^4 + 2485R\lambda^2 - 12635c\lambda^2 = 0. \quad (3.10)
\]
Differentiating (3.10) along \( e_1 \) and using \( p_1 = 0 \) gives the following expression:
\[
-25980\lambda^4 - 61495R\lambda^2 + 321545c\lambda^2 + 10584\lambda R^2 = 0. \quad (3.11)
\]
Finally, combining (3.10) with (3.11) gives a non-trivial quadratic equation of \( \lambda \) with constant coefficients as follows:
\[
228\lambda^2 - 343R + 1673c = 0. \quad (3.12)
\]
Hence, \( \lambda \) must be a constant, which is impossible.

By a similar argument as Lemma 3.4 in [13], we have
Lemma 3.3. Let $\phi : M^3 \to N^4(c)$ be an orientable proper biconservative hypersurface. Then $e_i(\lambda_j) = 0$ for $2 \leq i, j \leq 3$.

We are now in a position to prove Theorem 1.1.

The proof of Theorem 1.1:

We need only to consider two cases:

Case 1. Suppose that $M^3$ has two distinct principal curvatures. Then one has $\lambda \neq \lambda_2 = \lambda_3$. Due to (2.8) we have $\lambda_2 = \lambda_3 = -\frac{3}{2} \lambda$, which together with (2.11) implies that $\lambda^2 = 4c - \frac{2}{3} R =$constant. Therefore, we conclude that the mean curvature $H = -\frac{2}{3} \lambda$ is also a constant.

Case 2. Suppose that $M^3$ has three distinct principal curvatures. Combining (2.8) with (2.11), we may eliminate $\lambda_2$ and get

$$6\lambda^2 + 6\lambda_2 + 2\lambda_2^2 - 6c + R = 0. \tag{3.13}$$

Differentiating (3.13) with respect to $e_1$ leads to

$$(6\lambda + 3\lambda_2)e_1(\lambda) + (3\lambda + 2\lambda_2)e_1(\lambda_2) = 0. \tag{3.14}$$

Similarly, we have

$$(6\lambda + 3\lambda_3)e_1(\lambda) + (3\lambda + 2\lambda_3)e_1(\lambda_3) = 0. \tag{3.15}$$

Taking into account (2.8) we know $3\lambda + 2\lambda_2 = \lambda_2 - \lambda_3 \neq 0$ and $3\lambda + 2\lambda_3 = \lambda_3 - \lambda_2 \neq 0$. Suppose that $6\lambda + 3\lambda_2 = 0$. Then (3.14) implies $e_1(\lambda_2) = 0$. Hence $e_1(\lambda) = -\frac{1}{2}e_1(\lambda_2) = 0$, which shows that $\lambda$ must be a constant. This contradicts to our assumption. Thus, we get $2\lambda + \lambda_2 \neq 0$. In the same manner we have that $2\lambda + \lambda_3 \neq 0$. Differentiating (3.14) along $e_1$ yields

$$e_1e_1(\lambda_2) = -\frac{3(2\lambda + \lambda_2)}{3\lambda + 2\lambda_2}e_1e_1(\lambda) - \frac{6(3\lambda^2 + 3\lambda_2 + \lambda_2^2)}{(3\lambda + 2\lambda_2)^3}(e_1(\lambda))^2. \tag{3.16}$$

Combining (2.16) with (2.17) shows that

$$e_1\left(\frac{e_1(\lambda_2)}{\lambda_2 - \lambda}\right) = \left(\frac{e_1(\lambda_2)}{\lambda_2 - \lambda}\right)^2 + \lambda\lambda_2 + c,$$

which further reduces to

$$e_1\frac{e_1(\lambda_2)}{\lambda_2 - \lambda} + e_1(\lambda_2)e_1(\lambda)\left(\frac{e_1(\lambda_2)}{\lambda_2 - \lambda}\right)^2 - 2\left(\frac{e_1(\lambda_2)}{\lambda_2 - \lambda}\right)^2 - \lambda\lambda_2 - c = 0. \tag{3.17}$$

By applying (3.14) and (3.16) to (3.17), we may eliminate $e_1e_1(\lambda_2), e_1(\lambda_2)$, and finally obtain

$$(\lambda\lambda_2 + c)(\lambda - \lambda_2)^2(3\lambda + 2\lambda_2)^3$$

$$+ 9(28\lambda^3 + 51\lambda^2\lambda_2 + 30\lambda\lambda_2^2 + 6\lambda_2^3)(e_1(\lambda))^2$$

$$+ 3(\lambda_2 - \lambda)(2\lambda + \lambda_2)(3\lambda + 2\lambda_2)^2e_1e_1(\lambda) = 0. \tag{3.18}$$

Similarly, one has

$$(\lambda\lambda_3 + c)(\lambda - \lambda_3)^2(3\lambda + 2\lambda_3)^3$$

$$+ 9(28\lambda^3 + 51\lambda^2\lambda_3 + 30\lambda\lambda_3^2 + 6\lambda_3^3)(e_1(\lambda))^2$$

$$+ 3(\lambda_3 - \lambda)(2\lambda + \lambda_3)(3\lambda + 2\lambda_3)^2e_1e_1(\lambda) = 0. \tag{3.19}$$

We claim that

$$\omega^1_{22}\omega^1_{33} = -\lambda_2\lambda_3 - c. \tag{3.20}$$

In fact, using the Gauss equation (2.1) gives

$$\langle R(e_2, e_3)e_2, e_3 \rangle = -\lambda_2\lambda_3 - c.$$
From the definition \((\ref{def:curvature})\) of curvature tensor \(R\), one can deduce that
\[
\langle R(e_2, e_3) e_2, e_3 \rangle = \langle \nabla_{e_2} \nabla_{e_3} e_2 - \nabla_{e_3} \nabla_{e_2} e_2 - \nabla_{[e_2, e_3]} e_2, e_3 \rangle
\]
\[
= \langle \nabla_{e_2} (\omega_{22} e_k) - \nabla_{e_3} (\omega_{22} e_k) - \nabla_{(e_2 e_3 - e_3 e_2)} e_2, e_3 \rangle
\]
\[
= e_2 (\omega_{22}^3) + \omega_{22}^3 \omega_{2k}^3 - e_3 (\omega_{22}^3) - \omega_{22}^3 \omega_{3k}^3 - \omega_{22}^3 \omega_{k2}^3 + \omega_{32}^3 \omega_{2k}^3
\]
\[
= -e_2 (\omega_{23}^3) + \omega_{23}^3 \omega_{21}^3 - e_3 (\omega_{22}^3) + \omega_{22}^1 \omega_{33}^1
\]
\[
- (\omega_{23}^3 \omega_{12}^1 - \omega_{32}^3 \omega_{22}^3) + (\omega_{32}^3 \omega_{12}^1 + \omega_{23}^3 \omega_{33}^3).
\]

In view of Lemma \((\ref{lem:codazzi})\) and the Codazzi equation \((\ref{eq:codazzi})\), we see at once that \(\omega_{22}^3 = \omega_{33}^2 = 0\). In addition, it follows from \((3.15)\) in \((\ref{ref})\) that \(\omega_{23}^1 = \omega_{32}^1 = 0\). Therefore, we obtain that
\[
\langle R(e_2, e_3) e_2, e_3 \rangle = \omega_{22}^3 \omega_{33}^1.
\]

Consequently, our claim follows.

Putting \(\omega_{ii}^1 = \frac{e_1(\lambda_1)}{\lambda_1 - \lambda} \) into \((\ref{eq:omega1})\) gives rise to
\[
\frac{e_1(\lambda_2)}{\lambda_2 - \lambda} \cdot \frac{e_1(\lambda_3)}{\lambda_3 - \lambda} = -\lambda_2 \lambda_3 - c, \tag{3.21}
\]
which together with \((\ref{eq:omega1})\)–\((\ref{eq:omega3})\) leads to
\[
\left(\frac{e_1(\lambda)}{\lambda_2 - \lambda}\right)^2 = \frac{\lambda_2 \lambda_3 (\lambda_2 - \lambda)(\lambda_3 - \lambda)(3\lambda + 2\lambda_2)(3\lambda + 2\lambda_3)}{9(2\lambda + \lambda_2)(2\lambda + \lambda_3)}, \tag{3.22}
\]
After eliminating \(e_1 e_1(\lambda)\) between \((\ref{eq:omega1})\) and \((\ref{eq:omega2})\), using \((\ref{eq:omega3})\) and \(\lambda_3 = -3\lambda - \lambda_2\), we get
\[
3\lambda_2^6 + 27\lambda_2^2 \lambda_3^3 + (99\lambda^2 - 4c)\lambda_2^3 + (189\lambda^3 - 24c\lambda)\lambda_2^2
\]
\[
+ (196\lambda^4 - 55c\lambda^2)\lambda_2^2 + (102\lambda^5 - 57c\lambda^3)\lambda_2 + 24\lambda^6 - 20c\lambda^4 = 0. \tag{3.23}
\]
Repeating the division algorithm for polynomial to \((\ref{eq:omega1})\) and \((\ref{eq:omega1})\), we could eliminate \(\lambda_2\) and obtain a non-trivial polynomial equation of \(\lambda\) with constant coefficients as follows:
\[
24\lambda^6 + (176c - 28R)\lambda^4 + (196cR - 18R^2 - 528c^2)\lambda^2
\]
\[
- 3R^3 + 46cR^2 - 228cR + 360c^3 = 0. \tag{3.24}
\]
Therefore, we derive that \(\lambda\) must be a constant, which contradicts to our assumption.

We complete the proof of Theorem \((\ref{thm:biconservative})\).

### 4. Biconservative hypersurfaces in \(N^5(c)\)

In this section, we will restrict ourselves to biconservative hypersurfaces with constant scalar curvature in \(N^5(c)\). To prove Theorem \((\ref{thm:biconservative})\) we need the following Lemma derived in \((\ref{ref})\).

\textbf{Lemma 4.1.} \((\ref{ref})\) For \(f_k = (\omega_{ii})^k + (\omega_{ii}^1)^k + (\omega_{ii}^1)^k\), the functions \(f_1, \cdots, f_5\) satisfy:

\[
f_4^1 - 6f_1^2 f_2 + 3f_2^3 + 8f_1 f_3 - 6f_4 = 0, \tag{4.1}
\]
\[
f_5^1 - 5f_1^2 f_2 + 5f_2^3 f_3 + 5f_2 f_3 - 6f_5 = 0. \tag{4.2}
\]

By applying Lemma \((\ref{lem:omega1})\) and \((\ref{lem:omega2})\) we can prove the following

\textbf{Lemma 4.2.} Let \(\phi : M^4 \rightarrow N^5(c)\) be an orientable proper biconservative hypersurface with constant scalar curvature. Then \(e_i(T) = 0\) for \(i = 2, 3, 4\).
Proof. Substituting (2.18) into (4.1) and (4.2) yields

\[- T''' + 4TT'' + 3T'^2 - (6T^2 - 26\lambda^2 + 10c)T' + T^4 - (26\lambda^2 - 10c)T^2 \quad (4.3)\]
\[+ 55\lambda\lambda' T - 21\lambda\lambda'' - 12\lambda'^2 + 27\lambda^4 + (3R - 72c)\lambda^2 + 9c^2 = 0,\]
\[- T'''' + (10T' + 10T^2 + 50\lambda^2 - 10c)T'' - (20T^3 + 20\lambda^2 T + 20cT) \quad (4.4)\]
\[- 155\lambda\lambda'^3 T'' + 47^5 + (40c - 80\lambda^2)T^3 + 120\lambda\lambda'T^2 + (11\lambda\lambda'' + 7\lambda'^2 - 84\lambda^4 - 48c\lambda^2 + 36\lambda^2)T - 33\lambda\lambda'' - 45\lambda\lambda'' + (408\lambda^2 + 10R - 252c)\lambda\lambda' = 0.\]

Using the above two equations, we can eliminate $T''''$, $T'''$ and obtain a second-order differential equation with respect to $T$ as follows:

\[- 6\lambda T'' + (18\lambda T - 12\lambda')T' - 6\lambda T^3 + 12\lambda T^2 + (48\lambda^3 - 10\lambda'')\quad (4.5)\]
\[+ 3R\lambda - 60c\lambda)T + 3\lambda'' + (27c - R - 75\lambda^2)\lambda' = 0.\]

By (4.3) and (4.5), it would allow us to eliminate $T'''$ and $T''$. Then we get the following first-order differential equation with respect to $T$:

\[a_1 T' - a_1 T^2 + a_2 T + a_3 = 0, \quad (4.6)\]

where

\[a_1 = 36\lambda^2 - 22\lambda'' - 108\lambda^4 + 3R\lambda^2,\]
\[a_2 = 30\lambda\lambda'' - 13\lambda\lambda''' + 93c\lambda\lambda' - 255\lambda^3\lambda' - 5R\lambda\lambda',\]
\[a_3 = 3\lambda\lambda'' - 9\lambda\lambda''' + (5\lambda^2 + 27c - R)\lambda\lambda'' + (147\lambda^2 + 3R - 81c)\lambda^2 - 162\lambda^6 + (432c - 18R)\lambda^4 - 54\lambda^2 c^2.\]

Consider the following cases:

**Case 1.** $a_1 = 0$. The equation (4.6) reduces to $a_2 T + a_3 = 0$.

**Case 1.1.** $a_2 \neq 0$. We have $T = -a_3/a_2$, which implies that $e_i(T) = 0$.

**Case 1.2.** $a_2 = 0$. Then $a_1 = 0$ and $a_2 = 0$ lead to the following two equations:

\[36\lambda^2 - 22\lambda'' - 108\lambda^4 + 3R\lambda^2 = 0, \quad (4.7)\]
\[30\lambda\lambda'' - 13\lambda\lambda''' + 93c\lambda\lambda' - 255\lambda^3\lambda' - 5R\lambda\lambda' = 0. \quad (4.8)\]

By (4.7)–(4.8), eliminating $\lambda'''$, $\lambda''$ and $\lambda'$, we finally obtain a non-trivial polynomial equation of $\lambda$:

\[96\lambda^2 - 121R + 1302c = 0.\]

Noting that both $R$ and $c$ are constant, we know that $\lambda$ is also constant. This is a contradiction.

**Case 2.** $a_1 \neq 0$. From (4.5)–(4.6) we can eliminate the terms of $T'''$, $T''$ step by step and get

\[b_1 T + b_2 = 0, \quad (4.9)\]
where
\[
b_1 = 10368\lambda^4 \chi''' - 288\lambda^2 \chi'' R + 2112\lambda \chi''' - 3456\lambda^2 \chi'''
- 107136\lambda^3 \chi'' + 972\lambda \chi''' R + 26784\lambda^4 \chi'' R - 270864\lambda^4 \chi'' c
+ 12540\lambda^4 \chi''' + 288360\lambda^6 \chi'' - 9108\lambda^4 \chi'' R - 270864\lambda^4 \chi'' c
- 3336\lambda^3 \chi'' R + 330516\lambda^2 \chi'' - 558\lambda^2 \chi'' R - 9108\lambda^2 \chi'' R c
- 7128\lambda^2 \chi'' R - 3840\lambda^2 \chi'' R - 37752\lambda^2 \chi'' R - 2100\lambda \chi'' R
+ 396\lambda^3 \chi c - 8800\lambda^3 c - 467910\lambda^3 \chi^2 - 42084\lambda^3 \chi^2 R
+ 665604\lambda^3 \chi^2 c - 244944\lambda^4 + 354\lambda^2 \chi R^2 - 4248\lambda \chi R^2 c
- 40230\lambda \chi R^2 c + 45486\lambda^5 - 4860\lambda^7 R - 419904\lambda^7 c - 1188\lambda^5 R^2
+ 31104\lambda \chi R^2 - 34992\lambda^5 c^2 + 27\lambda^3 R^3 - 540\lambda^3 \chi R^2 c + 972\lambda^3 \chi R^2 c,
\]
\[
b_2 = 648\lambda^2 \chi''' + 54\lambda^2 \chi'' R - 1944\lambda \chi''' - 396\lambda^2 \chi'''
+ 12366\lambda^3 \chi''' R - 18\lambda \chi''' R - 1674\lambda \chi''' R c + 630\lambda \chi''' R
- 1440\lambda \chi''' R + 32076\lambda^6 \chi''' - 4158\lambda^4 \chi'' R + 73224\lambda^4 \chi'' R c
+ 24066\lambda^3 \chi'' R - 6876\lambda^2 \chi'' R - 9\lambda \chi'' R^2 + 486\lambda^2 \chi'' R c
- 11340\lambda^2 \chi'' R - 636\lambda \chi'' R + 2106\lambda \chi'' R c + 792\lambda \chi'' R
- 6156\lambda \chi'' R c - 1890\lambda \chi'' R c + 2640\lambda \chi'' R c - 253098\lambda^5 \chi''
+ 31554\lambda \chi'' R - 285876\lambda^3 \chi'' R c - 133248\lambda^2 \chi'' R c + 89208\lambda \chi'' R c
+ 192\lambda \chi'' R^2 - 4626\lambda \chi'' R c + 39366\lambda \chi'' R c^2 - 400\lambda \chi'' R^2 c
+ 10800\lambda \chi'' R c - 702756\lambda^8 \chi + 10368\lambda^6 \chi R + 506412\lambda^6 \chi R c
+ 528174\lambda^4 \chi R^2 + 1863\lambda^4 \chi R^2 c + 36612\lambda^4 \chi R c - 323676\lambda^4 \chi R^2 c
- 13572\lambda^2 \chi R^3 + 38988\lambda^2 \chi R^3 c - 9\lambda \chi R R^2 + 243\lambda R^2 R c
- 3564\lambda^2 \chi R R^2 c + 30132\lambda^2 \chi R c^2 + 29808\lambda^5 - 126\lambda R^2 R^2
+ 1728\lambda \chi R R^3 - 1458\lambda R^2 c.
\]

If \(b_1 = 0\) and \(b_2 = 0\), then we may eliminate the terms of \(\chi'''', \chi''', \chi'', \chi', \lambda', \lambda''\), \lambda'' item by item, and finally obtain a non-trivial polynomial equation concerning \(\lambda\) with constant coefficients. This shows that \(\lambda\) is a constant and contradicts to our assumption. Hence the lemma follows. \(\Box\)

According to Lemma 3.4 in [13], the similar proof remains valid for the case that the number of the principal curvatures is 2, 3 or 4. Summarizing, we have the following lemma.

Lemma 4.3. Let \(\phi : M^4 \rightarrow N^5(c)\) be an orientable proper biconservative hypersurface. Then \(e_i(\lambda_j) = 0\) for \(2 \leq i, j \leq 4\).

Using Lemma 4.3, the following results hold.

Lemma 4.4. (c.f. [12]) If \(\lambda_2, \lambda_3\) and \(\lambda_4\) are different from each other, then we have
\[
\omega_{23}^4(\lambda_3 - \lambda_4) = \omega_{32}^4(\lambda_2 - \lambda_4) = \omega_{32}^4(\lambda_3 - \lambda_2),
\]
\[
\omega_{23}^4 + \omega_{34}^2 \omega_{13}^2 + \omega_{43}^2 \omega_{14}^2 = 0,
\]
\[
\omega_{23}^4(\omega_{33}^1 - \omega_{14}^1) = \omega_{32}^4(\omega_{22}^1 - \omega_{14}^1) = \omega_{13}^2(\omega_{33}^1 - \omega_{22}^1).
\]
Lemma 4.5. (c.f. \cite{12}) Under the same hypotheses of Lemma 4.4, one has

\begin{align}
\omega_{22} \omega_{33} - 2 \omega_{23} \omega_{32} &= -\lambda_2 \lambda_3 - c, \\
\omega_{22} \omega_{44} - 2 \omega_{24} \omega_{42} &= -\lambda_2 \lambda_4 - c, \\
\omega_{33} \omega_{44} - 2 \omega_{34} \omega_{43} &= -\lambda_3 \lambda_4 - c.
\end{align}

The proof of Theorem 1.4:

For \( n = 4 \), combining (2.11) with (2.8) gives

\[ \lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4 = \frac{1}{2} \left\{ \left( \sum_{i=2}^{4} \lambda_i \right)^2 - \sum_{i=2}^{4} \lambda_i^2 \right\} = \frac{1}{2} R + 3 \lambda^2 - 6c. \]  

(4.16)

Using arithmetic geometric mean inequality, it follows that

\[ 3 \sum_{i=2}^{4} \lambda_i^2 \geq \left( \sum_{i=2}^{4} \lambda_i \right)^2, \]

which together with (2.8) and (2.11) gives

\[ 12c - R \geq 0, \]  

(4.17)

where the equality holds if and only if \( \lambda_2 = \lambda_3 = \lambda_4 \).

Based on discussing the number of distinct principal curvatures, in what follows we need to consider three cases.

**Case 1.** Assume that \( M^4 \) is a biconservative hypersurface with two distinct principal curvatures in \( \mathbb{N}_5(c) \). Then \( \lambda = -2H \) and \( \lambda_2 = \lambda_3 = \lambda_4 = 2H \). In this case, the scalar curvature \( R \) satisfies \( R = 12c \). Applying a well-known result of do Carmo and Dajczer (Theorem 4.2 of \cite{3}), one see immediately that \( M^4 \) is contained in a rotational hypersurface in \( \mathbb{N}_5(c) \).

**Case 2.** Suppose that \( M^4 \) is a proper biconservative hypersurface with three distinct principal curvatures and proper biconservative. We may assume that \( \mu := \lambda_2 = \lambda_3 \neq \lambda_4 \). According to (2.8) and (2.11), we have

\begin{align}
2 \mu + \lambda_4 &= -3 \lambda, \\
2 \mu^2 + \lambda_4^2 &= 12c + 3 \lambda^2 - R.
\end{align}

(4.18)

(4.19)

Eliminating \( \lambda_4 \) from (4.18) and (4.19), one has

\[ (\lambda + \mu)^2 = 2c - \frac{1}{6} R. \]  

(4.20)

Since the scalar curvature \( R \) is constant, we have \( e_1(\mu) = -e_1(\lambda) \). It follows from (2.16) and (2.17) that

\begin{align}
e_1 \left( \frac{e(\mu)}{\mu - \lambda} \right) &= \left( \frac{e(\mu)}{\mu - \lambda} \right)^2 + \lambda \mu + c, \\
e_1 \left( \frac{e(\lambda)}{\lambda_4 - \lambda} \right) &= \left( \frac{e(\lambda)}{\lambda - \lambda} \right)^2 + \lambda \lambda_4 + c.
\end{align}

(4.21)

(4.22)

Using \( e(\mu) = -e(\lambda) \) in (4.21), we check at once that

\[ (\lambda - \mu)e_1 e_1(\lambda) - 3(e_1(\lambda))^2 = (\lambda \mu + c)(\lambda - \mu)^2. \]  

(4.23)

Eliminating \( \lambda_4 \) between (4.18) and (4.22) and using the fact \( e(\mu) = -e(\lambda) \) again, it follows immediately that

\[ (4 \lambda + 2 \mu)e_1 e_1(\lambda) - 3(e_1(\lambda))^2 = -(3 \lambda^2 + 2 \lambda \mu - c)(4 \lambda + 2 \mu)^2. \]  

(4.24)
Noting that $\lambda$, $\mu$ and $\lambda_4$ are entirely different from each other, we conclude from (4.18) that $\lambda - \mu$ and $4\lambda + 2\mu$ can not vanish in same neighborhood. Then, we can eliminate the terms of $e_i (\lambda)$ and obtain
\[
3(\lambda + \mu)(e_i (\lambda))^2 = -(4\lambda + 2\mu)(\lambda - \mu)(4\lambda^2 + \lambda\mu - c). \tag{4.25}
\]

Observing that $M^4$ has three distinct principal curvatures, we deduce that $\lambda + \mu \neq 0$. Hence,
\[
(e_1 (\lambda))^2 = -\frac{1}{3} (4\lambda + 2\mu)(\lambda - \mu)(4\lambda^2 + \lambda\mu - c). \tag{4.26}
\]

Similarly to (3.20), it follows from Gauss equation that
\[
\omega_{11}^1 = \omega_{33}^1 = -\mu_4 - c, \tag{4.27}
\]
which together with (2.16), (4.18) and $e_1 (\mu) = -e_1 (\lambda)$ indicates that
\[
(e_1 (\lambda))^2 = (4\lambda + 2\mu)(\lambda - \mu)(2\mu^2 + 3\lambda\mu - c). \tag{4.28}
\]

After eliminating the terms of $(e_1 (\lambda))^2$ between (4.26) and (4.28), one has
\[
2\lambda^2 + 5\lambda\mu + 3\mu^2 - 2c = 0. \tag{4.29}
\]

By (1.29) and (1.20), we can eliminate $\mu^2$, $\mu$ and get a quadratic equation of $\lambda$ as follows:
\[
2(R - 12c)\lambda^2 + 3R^2 - 48Rc + 192c^2 = 0. \tag{4.30}
\]

This clearly implies that $\lambda$ is a constant, which is a contradiction.

**Case 3.** Consider $M^4$ as a proper biconservative hypersurface with four distinct principal curvatures in $N^5 (c)$. We will drive a contradiction again. The proof will be divided into the following two subcases.

**Case 3.1.** $\omega_{43}^1 \neq 0$, $\omega_{42}^2 \neq 0$, and $\omega_{43}^1 \neq 0$. According to Lemma 4.3 one has $e_i (\omega_{ii}^1) = 0$ and $e_i (\omega_{di}) = 0$ for $i = 2, 3, 4$. Note that (4.10) and (4.12) reduce to
\[
\frac{\omega_{33}^1 - \omega_{14}^1}{\lambda_3 - \lambda_4} = \frac{\omega_{33}^1 - \omega_{22}^1}{\lambda_3 - \lambda_2} = \frac{\omega_{14}^1 - \omega_{22}^1}{\lambda_1 - \lambda_2} := \alpha,
\]
where $\alpha$ is a smooth function satisfying $e_1 (\alpha) = 0$ for $i = 2, 3, 4$. Hence there exists another smooth function $\beta$ satisfying $e_i (\beta) = 0$ such that
\[
\omega_{ii}^1 = \alpha \lambda_i + \beta, \quad i = 2, 3, 4. \tag{4.31}
\]

Differentiating with respect to $e_1$ on both sides of equation (4.31), using (2.16) and (2.17) we get
\[
e_1 (\alpha) = \lambda (\alpha^2 + 1) + \alpha \beta, \tag{4.32}
\]
\[
e_1 (\beta) = \lambda \alpha \beta + \beta^2 + c. \tag{4.33}
\]

Taking a sum on $i$ in (4.31) and using (2.8), one has
\[
\sum_{i=2}^{4} \omega_{ii}^1 = -3\alpha \lambda + 3\beta. \tag{4.34}
\]

Taking into account (4.11) and (4.13)-(4.15) lead to
\[
\omega_{22}^1 \omega_{33}^1 + \omega_{22}^1 \omega_{14}^1 + \omega_{33}^1 \omega_{14}^1 = -\lambda_2 \lambda_3 - \lambda_2 \lambda_4 - \lambda_3 \lambda_4 - 3c,
\]
which combining with (4.31) further reduces to
\[
(1 + \alpha^2) (\lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4) + 2\alpha (\lambda_2 + \lambda_3 + \lambda_4) + 3\beta^2 + 3c = 0. \tag{4.35}
\]

By substituting (2.8) and (4.16) into (4.35), it follows that
\[
(1 + \alpha^2) (\frac{1}{2} R + 3\lambda^2 - 6c) - 6\alpha \beta + 3\beta^2 + 3c = 0
\]
and hence
\[ 6\beta^2 - 12\lambda\alpha\beta + (6\lambda^2 + R - 12c)\alpha^2 + 6\lambda^2 + R - 6c = 0. \] \tag{4.36}

Moreover, differentiating (2.8) with respect to \( e_1 \) and using (2.16), we get
\[ -3e_1(\lambda) = \sum_{i=2}^{4} (\lambda_i - \lambda)\omega_{ii}. \] (4.37)

Using (4.31), (2.8) and (2.11), the above equation becomes
\[ e_1(\lambda) = -\frac{1}{3} \sum_{i=2}^{4} (\lambda_i - \lambda)(\alpha\lambda_i + \beta) \]
\[ = -\frac{1}{3} \sum_{i=2}^{4} \{ \alpha\lambda_i^2 + (\beta - \lambda\alpha)\lambda_i - \lambda\beta \} \]
\[ = -\frac{1}{3} \{ \alpha(12c + 3\lambda^2 - R) - 3\lambda(\beta - \lambda\alpha) - 3\lambda\beta \} \]
\[ = \left( \frac{1}{3}R - 4c - 2\lambda^2 \right) \alpha + 2\lambda\beta. \]

Differentiating (4.36) with respect to \( e_1 \) and using (4.32)–(4.33) one has
\[ 6\beta^3 - 18\lambda\alpha\beta^2 + (18\lambda^2 + 12c - R)\alpha^2\beta + 6(\lambda^2 + c)\beta \]
\[ + (3R - 6\lambda^2 - 36c)\alpha^3 + (3R - 6\lambda^2 - 42c)\lambda\alpha = 0. \] \tag{4.38}

Differentiating (4.38) with respect to \( e_1 \) and using (4.32)–(4.33) and (4.37), then
\[ 18\beta^4 - 72\alpha\beta^3\lambda + (108\lambda^2 - 9R + 108c)\alpha^2\beta^2 \]
\[ + (12\lambda^2 + 24c)\beta^2 + (24R - 72\lambda^2 - 288c)\alpha^3\beta \lambda \]
\[ + (11R - 24\lambda^2 - 180c)\alpha\beta\lambda + (18\lambda^2 - 3R + 36c)\lambda^2\alpha^4 \]
\[ + (R^2 - 24Rc + 144c^2)\alpha^4 + (12\lambda^2 + 24c)\alpha^2\lambda^2 \]
\[ + (R^2 - 27Rc + 180c^2)\alpha^2 + (3R - 6\lambda^2 - 36c)\lambda^2 + 6c^2 = 0. \]

Combining (4.36) and (4.38), we eliminate the terms of \( \beta^3, \beta^2 \) and derive
\[ (R - 12c)\{ (2\alpha^2 + 1)\beta - 4\alpha^3\lambda - 4\alpha\lambda \} = 0. \] \tag{4.40}

From (4.36) and (4.40) we can continue to eliminate \( \beta^2, \beta \) and obtain
\[ 6(\alpha^2 + 1)(4\alpha^4 + 12\alpha^2 + 1)\lambda^2 \]
\[ + (4\alpha^6 + 8\alpha^4 + 5\alpha^2 + 1)R \]
\[ - (48\alpha^6 + 72\alpha^4 + 36\alpha^2 + 6)c = 0. \] \tag{4.41}

On the other hand, we can eliminate the terms of \( \beta^4, \beta^3, \beta^2 \) by (4.36) and (4.39), then we have
\[ (R - 12c)\{ (12\alpha^2 + 22)\alpha\beta\lambda + (12\lambda^2 + 6R - 72c)\alpha^4 \]
\[ + (26\lambda^2 + 7R - 68c)\alpha^2 + (14\lambda^2 + R - 8c) \} = 0. \] \tag{4.42}

Eliminating \( \beta \) by (4.40) and (4.42), we get
\[ 2(\alpha^2 + 1)(36\alpha^4 + 64\alpha^2 + 7)\lambda^2 \]
\[ + (12\alpha^6 + 20\alpha^4 + 9\alpha^2 + 1)R \]
\[ - (144\alpha^6 + 208\alpha^4 + 84\alpha^2 + 8)c = 0. \] \tag{4.43}
Similarly, eliminating $\lambda^2$ between (4.41) and (4.43), we finally obtain

$$(32R - 504c)\alpha^6 + (20R - 420c)\alpha^4 - (14R - 30c)\alpha^2 - (2R - 9c) = 0. \quad (4.44)$$

We conclude from (4.44) that $\alpha$ must be a constant, which together with (4.41) implies that $\lambda$ is a constant. This is a contradiction.

**Case 3.2.** $\omega_{23}^4 = \omega_{32}^4 = \omega_{13}^4 = 0$. In this case, it follows from (4.13), (4.14), (4.15) that

$$\omega_{22}^1 = -\lambda_2 \lambda_3 - c, \quad \omega_{12}^1 = -\lambda_2 \lambda_4 - c, \quad \omega_{33}^1 = -\lambda_3 \lambda_4 - c. \quad (4.45)$$

Taking the sum of (4.45), (4.46) and (4.47) gives

$$\omega_{22}^1 + \omega_{22}^1 + \omega_{33}^1 = -3c - \lambda_2 \lambda_3 - \lambda_2 \lambda_4 - \lambda_3 \lambda_4. \quad (4.48)$$

Substituting (4.13) into (4.48) gives

$$\omega_{22}^1 + \omega_{22}^1 + \omega_{33}^1 = 3c - 3\lambda^2 - \frac{1}{2}R. \quad (4.49)$$

From the expressions of $f_1$ and $f_2$, we have

$$\omega_{22}^1 + \omega_{22}^1 + \omega_{33}^1 = \frac{1}{2}(f_1^2 - f_2) = \frac{1}{2}(T^2 - T^\prime - 3\lambda^2 + 3c). \quad (4.50)$$

Combining (4.49) with (4.50) gives

$$T^2 - T^\prime + 3\lambda^2 - 3c + R = 0. \quad (4.51)$$

Using (4.45)–(4.47) again, it follows that

$$(\omega_{22}^1 + \omega_{33}^1 + \omega_{44}^1)^2 + (\lambda_2 \lambda_3 + c)(\lambda_2 \lambda_4 + c)(\lambda_3 \lambda_4 + c) = 0. \quad (4.52)$$

Let $K = \lambda_2 \lambda_3 \lambda_4$. By (2.8) and (4.13), we obtain

$$\lambda_2 \lambda_3 + c)(\lambda_2 \lambda_4 + c)(\lambda_3 \lambda_4 + c) \quad (4.53)$$

$$= (\lambda_2 \lambda_3 \lambda_4)^2 + c^2(\lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_2 \lambda_4) + c\lambda_2 \lambda_3 \lambda_4(\lambda_2 + \lambda_3 + \lambda_4) + c^3$$

$$= K^2 + c^2 \left( \frac{1}{2}R + 3\lambda^2 - 6c \right) - 3c\lambda K + c^3$$

$$= K^2 - 3c\lambda K + c^2\lambda^2 - 5c^3 + \frac{1}{2}c^2 R. \quad (4.54)$$

An easy computation shows that

$$f_1^2 - f_3 = 3\{ (\omega_{22}^1)^2 + (\omega_{14}^1)^2 + (\omega_{44}^1)^2 + (\omega_{22}^1 + \omega_{14}^1)^2 \}
+ 6\omega_{22}^1 \omega_{33}^1 \omega_{44}^1
= 3 \sum_{i=2}^4 (\omega_{ii}^1)^2 (f_1 - \omega_{ii}^1) + 6\omega_{22}^1 \omega_{33}^1 \omega_{44}^1
= 3f_1 f_2 - 3f_3 + 6\omega_{22}^1 \omega_{33}^1 \omega_{44}^1,$$

which together with the expressions of $f_1$, $f_2$ and $f_3$ implies that

$$\omega_{22}^1 \omega_{33}^1 \omega_{44}^1 = \frac{1}{6}(f_1^2 - 3f_1 f_2 + 2f_3) \quad (4.54)$$

$$= \frac{1}{6}(T^3 - 11\lambda^2 T + 7c R + T^\prime - 3TT^\prime + 12\lambda R).$$
Substituting (4.53) and (4.54) back to (4.52) yields to
\[ (T^3 - 11\lambda^2 T + 7c T + T'' - 3TT' + 12\lambda \lambda')^2 + 36\left(K^2 - 3c\lambda K + 3c^2\lambda^2 - 5c^3 + \frac{1}{2}c^2 R\right) = 0. \] (4.55)

Using (2.8) and (2.9) again, we find
\[ \lambda_3 \lambda_4 = \frac{1}{2} \left\{ (\lambda_3 + \lambda_4)^2 - (\lambda_3^2 + \lambda_4^2) \right\} \] (4.56)
\[ = \frac{1}{2} \left\{ (-3\lambda - \lambda_2)^2 - (12c + 3\lambda^2 - R - \lambda_2^2) \right\} \]
\[ = 3\lambda^2 + 3\lambda \lambda_2 + \lambda_2^2 - 6c + \frac{1}{2}R. \]
Similarly, we can see that
\[ \lambda_2 \lambda_4 = 3\lambda^2 + 3\lambda \lambda_3 + \lambda_3^2 - 6c + \frac{1}{2}R, \] (4.57)
\[ \lambda_2 \lambda_3 = 3\lambda^2 + 3\lambda \lambda_4 + \lambda_4^2 - 6c + \frac{1}{2}R. \] (4.58)

Applying (4.56) - (4.58) yields
\[ \omega_{22}^3 \lambda_3 \lambda_4 + \omega_{33}^1 \lambda_2 \lambda_4 + \omega_{44}^1 \lambda_2 \lambda_3 = (3\lambda^2 - 6c + \frac{1}{2}R)(\omega_{22}^1 + \omega_{33}^1 + \omega_{44}^1) + 3\lambda(\lambda_2 \omega_{22}^1 + \lambda_3 \omega_{33}^1 + \lambda_4 \omega_{44}^1) + (\lambda_2^2 \omega_{22}^1 + \lambda_3^2 \omega_{33}^1 + \lambda_3 \omega_{44}^1) = (3\lambda^2 - 6c + \frac{1}{2}R)f_1 + 3\lambda g_1 + g_3. \] (4.59)

which together with the expression of \( f_1, g_1 \) and \( g_3 \) leads to
\[ K' := e_1(K) = e_1(\lambda_2)\lambda_3 \lambda_4 + e_1(\lambda_3)\lambda_2 \lambda_4 + e_1(\lambda_4)\lambda_2 \lambda_3 = \omega_{22}^1(\lambda_2 - \lambda)\lambda_3 \lambda_4 + \omega_{33}^1(\lambda_3 - \lambda)\lambda_2 \lambda_4 + \omega_{44}^1(\lambda_4 - \lambda)\lambda_2 \lambda_3 = (\lambda_2^2 \omega_{22}^1 + \lambda_3^2 \omega_{33}^1 + \lambda_3 \omega_{44}^1) - \lambda(\omega_{22}^1 \lambda_3 \lambda_4 + \omega_{33}^1 \lambda_2 \lambda_4 + \omega_{44}^1 \lambda_2 \lambda_3) = Kf_1 - \lambda(3\lambda^2 - 6c + \frac{1}{2}R)f_1 - 3\lambda g_1 - \lambda g_3 = KT - 7\lambda^3 T + 6c\lambda T - \frac{1}{2}\lambda RT + 9\lambda^2 \lambda'. \]

Differentiating (4.53) with respect to \( e_1 \) and using (4.60) give rise to
\[ 36K^2 T + (162\lambda T\alpha + 324\lambda^2 \alpha' - 18RT\lambda - 252T\lambda^3 - 54\lambda'c)K + 27RT \lambda^2 c + 3T^5 T' - 3T^4 T'' - 22T^4 \lambda \lambda' - 12T^3 T'^2 - 44T^3 T' \lambda^2 + 28T^3 T c + T^3 T'' + 12T^3 \lambda \alpha' + 12T^3 \lambda^2 + 12T^2 T' T'' + 102T^2 T' \lambda \lambda' + 33T^2 T'' \lambda^2 - 21T^2 T'' c + 242T^2 \lambda^3 \lambda' - 154T^2 \lambda \lambda' c + 9TT'^3 + 66TT'^2 \lambda^2 - 42TT'^2 e - 3TT' T'' + 121TT' \lambda^4 - 154TT' \lambda^2 c - 36TT' \lambda \lambda' - 36TT' \lambda'^2 + 49TT' \lambda' c - 58TT'' \lambda' \lambda' - 11TT'' \lambda'^2 + 7TT'' e + 378T \lambda^3 c - 132T \lambda^3 \lambda'' - 396T \lambda^3 \lambda'^2 - 324T \lambda^3 \lambda' e + 84T \lambda'^2 c - 3T'^2 T'' - 36T'^2 \lambda' \lambda' - 11T' T'' \lambda'^2 + 7T' T'' c - 132T' \lambda^3 \lambda' + 84T' \lambda' c + T'' T'' + 12T'' \lambda \lambda' + 12T'' \lambda^2 + 12T'' \lambda' - 486\lambda^3 \lambda' c + 144\lambda^2 \lambda' \lambda'' + 144\lambda \lambda'^2 + 108\lambda \lambda' c = 0. \]
Biconservative hypersurfaces with constant scalar curvature in forms

Eliminating the terms of $K^2$ from (4.55) and (4.61) shows that
d_1K + d_2 = 0, \quad (4.63)
where

d_1 = 18RT\lambda + 252T\lambda^3 - 270T\lambda c - 324\lambda^2\lambda' + 54\lambda',
d_2 = -27RT\lambda^2c + 18RTc^2 + T^7 - 9T^5T' - 22T^5\lambda^2 + 14T^5c
+ 57^4T'' + 46T^4\lambda\lambda' + 21T^3T'^2 + 110T^3T\lambda^2 - 70T^3T'c
- T^3T''' + 121T^3\lambda^4 - 154T^3\lambda^2c - 12T^3\lambda c - 12T^3\lambda'^2
+ 49T^3c^2 - 18T^2T'T'' - 174T^2T'\lambda\lambda' - 55T^2T''\lambda^2 + 35T^2T''c
- 506T^2\lambda^3\lambda' + 322T^2\lambda\lambda'c - 9TT^{63} - 66TT'^2\lambda^2 + 42TT'^2c
+ 3TT'T''' - 121TT\lambda^4 + 154TT'\lambda^2c + 36TT'\lambda c - 36TT'\lambda'^2
- 49TT'c^2 + 4TT'^2 + 82TT''\lambda\lambda' + 11TT''\lambda^2 - 7TT''c
- 378T^3\lambda^2c + 132T^3\lambda\lambda' + 540T^3\lambda^2\lambda' + 432T^3\lambda^2\lambda'^2 - 84T^3\lambda\lambda'^2c
- 84T^3\lambda'^2c - 180Tc^3 + 3T^2T'' + 36TT'\lambda c + 11TT''\lambda^2 - 7TT''c
+ 132T^3\lambda^3\lambda' - 84T'^2\lambda\lambda'c - T''T''' - 12T''\lambda\lambda' - 12T''\lambda'^2
- 12T''\lambda c + 486\lambda^3\lambda'c - 144\lambda^2\lambda'^2 - 144\lambda\lambda'^3 - 108\lambda\lambda'^2.

When $d_1 = 0$, it follows immediately that $d_2 = 0$. As in the proof of Lemma 4.2 we can certainly eliminate $T'''$, $T''$, $T'$ and $T$ gradually, and obtain a polynomial equation concerning $\lambda$ and its derivatives. In addition, similar arguments applied to (4.51) and $d_1 = 0$ gives another polynomial equation concerning $\lambda$ and its derivatives. From these two equations, we may eliminate all the derivatives of $\lambda$, and get a non-trivial polynomial equation of $\lambda$ with constant coefficients. Therefore, the mean curvature $H$ must be constant, a contradiction.

Now we assume that $d_1 \neq 0$. By use of (4.63), we may eliminate $K$ in (4.55) and have

$$(T^3 - 11\lambda^2T + 7cT + T'' - 3TT' + 12\lambda\lambda')^2 + 36\left(\frac{d_2}{d_1}\right)^2 + 3c\lambda\left\frac{d_2}{d_1}\right + 3c^2\lambda^2 - 5c^3 + \frac{1}{2}c^2R = 0. \quad (4.64)$$

Using an analysis similar to the above, we can eliminate $T''''$, $T'''$, $T'$ and $T$ between (4.51) and (4.64), and deduce a polynomial equation concerning $\lambda$ and its derivatives. In the proof of Lemma 4.2 we find that if $a_1 = 0$, then (4.6) shows that $a_2T + a_3 = 0$, which together with (4.51) can deduce another polynomial equation concerning $\lambda$ and its derivatives after eliminating the terms of $T'$ and $T$. The rest of the proof runs as before. If $a_1 \neq 0$, the three equations (4.6), (4.51) and (4.64) can be handled in the same way. We complete the proof of Theorem 1.4.

5. A FINAL REMARK

According to Theorem 1.4, any non-CMC biconservative hypersurface with constant scalar curvature in the 5-dimensional space form $N^5(c)$ is a certain rotational hypersurface with two principal curvatures verifying

$$-\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4. \quad (5.1)$$

A natural question is whether the converse is also true. In the following, we show that any non-CMC rotational hypersurface verifying (5.1) must be biconservative.
Let us recall the explicit parametric equation of the rotational hypersurface in $S^5$ from [3]. Consider the profile curve $\gamma$ by

$$\gamma(s) = (h_1(s), 0, 0, h_2(s), h_3(s))$$

for some smooth function $h_i \ (i = 1, 2, 3)$. Then the parametrization of the rotational hypersurface in $S^5$ can be written as

$$f(s, t_1, t_2, t_3) = (h_1(s)\varphi_1, h_1(s)\varphi_2, h_1(s)\varphi_3, h_2(s), h_3(s)),$$

where $\varphi(t_1, t_2, t_3) = (\varphi_1, \varphi_2, \varphi_3)$ is an orthogonal parametrization of the unit sphere. Hence $\varphi_i = \varphi_i(t_1, t_2, t_3)$ and $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 1$. Since the profile curve $\gamma$ belongs to $S^5$ and the parameter $s$ can be chosen as its arc length, it follows that

$$h_1^2 + h_2^2 + h_3^2 = 1 \text{ and } h_1'^2 + h_2'^2 + h_3'^2 = 1.$$ 

It is straightforward to compute that

$$\frac{\partial f}{\partial s} = (h_1'\varphi_1, h_1'\varphi_2, h_1'\varphi_3, h_2', h_3'),$$

$$\frac{\partial f}{\partial t_i} = (h_1 \frac{\partial \varphi_1}{\partial t_i}, h_1 \frac{\partial \varphi_2}{\partial t_i}, h_1 \frac{\partial \varphi_3}{\partial t_i}, 0, 0), \quad i = 2, 3, 4.$$ 

Choosing a frame $\{\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t_1}, \frac{\partial f}{\partial t_2}, \frac{\partial f}{\partial t_3}\}$, do Carmo-Dajczer [3] further showed that the principal curvatures $\lambda_i$ of $M^4$ are given by

$$\lambda_1 = \frac{h_1'' + h_1}{\sqrt{1 - h_1'^2 - h_1'^2}},$$

$$\lambda_i = -\frac{\sqrt{1 - h_i^2 - h_i'^2}}{h_i}, \quad i = 2, 3, 4.$$ 

Notice that the principal curvatures $\lambda_i$ are functions depending only on the variable $s$.

Since the principal curvatures satisfy (5.1) on the rotational hypersurfaces in Theorem 1.4 from (5.2) and (5.3) we get a second order ODE that

$$h_1(h_1'' + h_1) = 1 - h_1^2 - h_1'^2$$

and the mean curvature $H$ is given by

$$H = -\frac{h_1'' + h_1}{2\sqrt{1 - h_1'^2 - h_1'^2}}.$$ 

It is straightforward to check that $\frac{\partial f}{\partial s}$ is a principal direction and the corresponding principal curvature is $\lambda_1 (-2H)$. Hence we conclude that this rotation hypersurface $M^4$ satisfies (2.6) and must be biconservative. Similar arguments can be applied to the cases for the rotational hypersurfaces with (5.1) in $N^5(c)$ for $c \leq 0$.

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