Dense 2-generator subsemigroups of $2 \times 2$ matrices

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Abstract

We show that the semigroup of real linear fractional transformations on a proper subinterval of the real line does not admit any 2-generator dense subsemigroups, and then we construct a 3-parameter family of examples of 3-generator dense subsemigroups. We also construct an explicit example of a 2-generator dense subsemigroup of $2 \times 2$ real matrices. In the complex case, we prove the existence of uncountably many 2-generator dense subsemigroups.

1 Introduction

Let $\mathcal{F}$ denote the set of real linear fractional transformations from $(0, \infty)$ into $(0, \infty)$ i.e., maps of the form

$$f(x) = \frac{ax + b}{cx + d}; \quad a, b, c, d \geq 0 \text{ and } ad - bc \neq 0.$$

The semigroup $\mathcal{F}$ (under the composition of functions) is isometric to the semigroup of $2 \times 2$ invertible matrices with nonnegative entries modulo (nonzero)
scalar multiplication. A sequence \( f_i \in \mathcal{F}, i \in \mathbb{N} \), is said to be convergent to \( f \in \mathcal{F} \) if for every \( x > 0 \), we have \( f_i(x) \to f(x) \) as \( i \to \infty \).

In [2], we found all pairs in \( \mathcal{F} \) that generate a topologically transitive semigroup on \((0, \infty)\) i.e., pairs \( f, g \in \mathcal{F} \) so that the orbit of (almost) every \( x > 0 \) under the action of the semigroup generated by \( f \) and \( g \) is dense in \((0, \infty)\). In this paper, we are interested in finding minimally generated dense subsemigroups of \( \mathcal{F} \). It turns out that we need at least three elements in \( \mathcal{F} \) to generate a dense subsemigroup of \( \mathcal{F} \), and we present a three-parameter family of examples of 3-generator dense subsemigroups of \( \mathcal{F} \) (see Theorem 4).

A subgroup of \( SL(2, \mathbb{C}) \), the group of complex \( 2 \times 2 \) matrices with determinant 1, is called elementary if the commutator of every two elements of infinite order in the subgroup has trace 2. Also, a subgroup of \( SL(2, \mathbb{C}) \) is called discrete if no sequence of distinct elements in the subgroup converges. Jørgensen [5] studied the non-elementary subgroups of \( SL(2, \mathbb{C}) \), and showed that a non-elementary subgroup is discrete if and only if each of its subgroups generated by two elements is discrete. In the real case, a subgroup of \( SL(2, \mathbb{R}) \) is discrete if and only if each subgroup generated by one element is discrete. Jørgensen also proved that every dense subgroup of \( SL(2, \mathbb{R}) \) has a dense subgroup generated by two elements. We asked the following question in [3] regarding the dense subsemigroups of \( n \times n \) matrices.

**Problem.** What is the least number of generators that can generate a dense subsemigroup of the set of \( n \times n \) matrices?

The semigroup generated by only one matrix can never be dense or even have a dense orbit (this can be seen by looking at the Jordan normal form of the matrix; see [6, 7]). In this paper, we will answer this question for \( n = 2 \): we construct an explicit example of a 2-generator dense subsemigroup of 2 \( \times \) 2 real matrices (see Example 11). In the complex case, we prove the existence of uncountably many 2-generator dense subsemigroups (see Example 13). We also show that the semigroup of \( 2 \times 2 \) matrices with nonnegative entries does not have any 2-generator dense subsemigroup, however it has 3-generator dense subsemigroups.

Although there is a lot of literature on dense subgroups in a variety of settings, dense subsemigroups, on the other hand, have rarely been studied. One might argue that it is more natural to seek dense subsemigroups instead of dense subgroups, since the semigroup structure is more natural than the
group structure. One hopes that many of the results on dense subgroups can be improved to existence results on dense subsemigroups. Below, we mention two examples.

X. Wang [8] has shown that every dense subgroup of the group of orientation preserving Möbius transformations on $S^n$ has a dense subgroup that is generated by at most $n$ elements, $n \geq 2$. A similar statement about the group $U(n, 1)$ was obtained by W. Cao [1]. In both settings, one could ask for minimally generated dense subsemigroups.

Here is how this paper is organized. In Section 2, we study dense subsemigroups of $F$. In Section 3, we will give a geometric description of the orbit of a given point in $(0, \infty)^2$ under the action of a pair of LFT’s. In Sections 4 and 5, we study dense subsemigroups of $2 \times 2$ matrices in real and complex cases.

2 Dense subsemigroups of $F$

In this section, we show that there are no 2-generator dense subsemigroups of $F$, and then we construct a 3-parameter family of examples of 3-generator dense subsemigroups. The proper interval of reals under consideration is $(0, \infty)$; however, using a conjugation by a linear fractional map, the results in this section are valid on any proper subinterval of reals. Given a pair of functions $f, g \in F$ and $x > 0$, the orbit of $x$ under the action of $\langle f, g \rangle$ (the semigroup generated by $f$ and $g$) is given by

$$\{f^{\alpha_1}g^{\beta_1}\ldots f^{\alpha_k}g^{\beta_k}(x) : \forall i \alpha_i, \beta_i \geq 0, k \geq 0\}.$$ 

The induced action of $f \in F$ on $(0, \infty)^2$ is defined by

$$f(x, y) = (f(x), f(y)).$$

We use the same character to denote $f \in F$ and its induced action on $(0, \infty)^2$. The conjugation $\theta : (0, \infty) \to (0, 1)$, defined by $\theta(x) = 1/(x + 1)$, gives a one-to-one correspondence between LFT’s from $[0, 1]$ to $[0, 1]$ and $F$. In particular if $\langle f, g \rangle$ is dense in $F$, then $\langle \hat{f}, \hat{g} \rangle$ is also dense in the set of LFT’s from $[0, 1]$ to $[0, 1]$, where $\hat{f} = \theta f \theta^{-1}$ and $\hat{g} = \theta g \theta^{-1}$.

We prove that there are no 2-generator dense subsemigroups in $F$ by contradiction. Suppose that $\langle f, g \rangle$ is dense in $F$. Then $\langle f, g \rangle$ must have dense orbits in $(0, \infty)$, and so, by the results in [2], one of the following occurs (up to order and a conjugation by a map of the form $ux^v$ with $u > 0$ and $v \in \{1, -1\}$).
I. i) $a, b \geq 1, c \geq 0, b > 1$ if $c = 0$, and

$$f(x) = \frac{x}{x + a}, \quad g(x) = bx + c.$$ 

ii) $a, b > 1, \ln a / \ln b$ is irrational, and

$$f(x) = \frac{x}{a}, \quad g(x) = bx.$$ 

II. i) $0 \leq c \leq 1, a > 0, b \geq 1, b > 1$ if $c = 0$, and

$$f(x) = \frac{a}{x + a}, \quad g(x) = bx + c.$$ 

ii) $a, b \geq 1$ and

$$f(x) = \frac{a}{x}, \quad g(x) = bx + 1.$$ 

III. $0 \leq c \leq 1, a > 0, b \geq 1, ab \leq 1$ if $c = 0$, and

$$f(x) = \frac{a}{x + a}, \quad g(x) = c + \frac{ab}{x}.$$ 

In case I, $f$ and $g$ are both increasing, and so the entire semigroup $\langle f, g \rangle$ is comprised of increasing maps. In particular, the induced action of $\langle f, g \rangle$ on $(0, \infty)^2$ preserves the regions $\{(x, y) : 0 \leq x \leq y\}$ and $\{(x, y) : x \geq y \geq 0\}$. It follows that in cases I, the action of $\langle f, g \rangle$ on $(0, \infty)^2$ has no dense orbits, and the semigroup $\langle f, g \rangle$ is not dense in $\mathcal{F}$. A similar argument eliminates case III.

Next, we eliminate the sub-case (ii) of case II, where $f(x) = a/x$ and $g(x) = bx + 1$. By conjugating these maps with $\theta(x) = 1/(x + 1)$, we get $\hat{f}(x) = (1 - x)/(ax - x + 1)$ and $\hat{g}(x) = x/(2x - bx + b)$ on $[0, 1]$. Since

$$\text{Im}(\hat{g}) = [0, 1/2], \quad \text{Im}(\hat{f} \hat{g}) = [1/(a + 1), 1],$$

and $(\hat{f})^2 = Id$ (the identity map), for every $(x, y) \in [0, 1]^2$ and $\hat{h} \in \langle \hat{f}, \hat{g} \rangle$ with $\hat{h} \neq \hat{f}, Id$, we have

$$|\hat{h}(x) - \hat{h}(y)| \leq \frac{a}{a + 1}.$$ 

Hence, the open set $\{(x, y) \in [0, 1]^2 : |x - y| > a/(a + 1)\}$ cannot contain more than one element of each orbit. However the orbit of the point $(0, 1)$
is dense under the action of LFT’s on $[0, 1]$, hence the orbit of $(0, 1)$ is dense under the action of any dense subsemigroup. Since $(f, g)$ has no dense orbits, we conclude that it is not dense in $\mathcal{F}$.

Next, suppose

$$f(x) = \frac{a}{x + a} \text{ and } g(x) = bx + c,$$

with $c \neq 0$. The conjugation by $\theta(x) = 1/(x + 1)$, gives the maps $\hat{f} = \theta f \theta^{-1}(x) = (ax - x + 1)/(2ax - x + 1)$ and $\hat{g} = \theta g \theta^{-1}(x) = x/(cx - bx + x + 1)$, and we have

$$\text{Im}(\hat{f}) = [1/2, 1] \text{ and } \text{Im}(\hat{g}) = [0, 1/(c + 1)].$$

It follows that for every $(x, y) \in [0, 1]^2$ and $\hat{h} \in \langle \hat{f}, \hat{g} \rangle$, we have

$$|\hat{h}(x) - \hat{h}(y)| \leq \max \left(\frac{1}{2}, \frac{1}{c + 1}\right).$$

If $c \neq 0$, then the open set $\{(x, y) \in [0, 1]^2 : |x - y| > \max(1/2, 1/(c + 1))\}$ cannot contain more than one element of each orbit i.e. the orbits are not dense, and so $(f, g)$ is not dense in $\mathcal{F}$ in this case either.

We study the remaining case of $f(x) = a/(x + a)$ and $g(x) = bx$ in more details below. By a conjugation $(x \to 1/x)$, we have the pair

$$R(x) = 1 + \frac{a}{x} \text{ and } S(x) = \frac{x}{b},$$

where $a > 0$ and $b > 1$.

Let $\Lambda$ be the semigroup of real linear fractional transformations generated by $R$ and $S$, and let $\bar{\Lambda}$ be the closure of $\Lambda$.

**Lemma 1.** For every $(\alpha_1, \ldots, \alpha_{k+1}) \in \mathbb{Z}^{k+1}$, $k \geq 0$, the map

$$f_\alpha(x) = \frac{b^{\alpha_{k+1}}x}{(b^{\alpha_1} \cdots + b^{\alpha_k})x/a + 1}$$

belongs to $\bar{\Lambda}$.

**Proof.** Proof is by induction on $k$. For $k = 0$, we need to show that for every $\alpha_1 \in \mathbb{Z}$, we have $b^{\alpha_1}x \in \bar{\Lambda}$. For positive integers $m$ and $n$, we have

$$S^m R^m R(x) = \frac{b^{-m}(a + x + ab^n x)}{x + a}.$$
Let \( l \) be a fixed integer, and set \( n = l + m \). Then as \( m \to \infty \), we have \( S^mRS^nR \to b'x/(1 + x/a) \), and so \( f_1(x) = b'x/(1 + x/a) \in \mathbb{A} \) for all \( l \in \mathbb{Z} \). Next, we let \( l \to \infty \), to get \( b^{a_1}x = \lim f_lS^l-a_1(x) \in \mathbb{A} \), which proves the basis of the induction.

Now suppose the assertion of the lemma is true for \( k \geq 0 \), and let \((\alpha_1, \ldots, \alpha_{k+2}) \in \mathbb{Z}^{k+2}\). By the inductive hypothesis, \( g(x) = b^{a_{k+1}}x/(Ax+1) \in \mathbb{A} \), where \( A = (b^{a_1} + \ldots + b^{a_k})/a \). For \( l = \alpha_{k+2} - \alpha_{k+1} \), it follows that

\[
f_lg(x) = \frac{b^{a_{k+2}}x}{(A + b^{a_{k+1}}/a)x + 1} \in \mathbb{A},
\]

and the inductive step is completed. \( \square \)

Given \( s \geq 0 \), there exists a sequence \( \{\alpha_i\}_{i=1}^{\infty} \) of integers so that \( sa = \sum_{i=1}^{\infty} b^{\alpha_i} \). It follows from Lemma 1 that

\[
T_s(x) = \frac{x}{sx + 1} = \lim_{k \to \infty} \frac{x}{(b^{a_1} + \ldots + b^{a_k})x/a + 1} \in \mathbb{A}. \quad (2.1)
\]

On the other hand,

\[
S^mRS^n(x) = \frac{1}{b^m} + \frac{a}{x} \to \frac{a}{x},
\]

as \( m \to \infty \). Hence, \( I(x) = a/x \in \mathbb{A} \).

**Lemma 2.** Let \( \alpha, \beta, \gamma \geq 0 \), so that \( 0 \leq \alpha - \beta \gamma \leq \min(1, \alpha^2) \). Then

\[
F(\alpha, \beta, \gamma)(x) = \frac{\alpha x + \beta}{\gamma x + 1} \in \mathbb{A}.
\]

**Proof.** It is sufficient to consider the case where \( \alpha, \beta, \gamma > 0 \) and \( 0 < \alpha - \beta \gamma \leq \min(1, \alpha^2) \). For \( u, v, w \geq 0 \), it follows from (2.1) that

\[
T_uIT_{v/a}IT_w(x) = \frac{(1 + vw)x + v}{(u + w + uwv)x + 1 + uv} \in \mathbb{A}.
\]

Now, given \( \alpha, \beta, \gamma \), we set

\[
u = \frac{-\sqrt{d} + 1}{\beta}, \quad v = \frac{\beta}{\sqrt{d}}, \quad w = \frac{-\sqrt{d} + \alpha}{\beta}, \quad \quad (2.2)
\]

where \( d = \alpha - \beta \gamma \). The conditions given in the Lemma guarantee that \( u, v, w \geq 0 \). These choices of \( u, v, w \) are made so that \( T_uIT_{v/a}IT_w = F(\alpha, \beta, \gamma) \), and the proof is completed. \( \square \)
For \( f(x) = (\alpha x + \beta)/(\gamma x + \delta) \in \mathcal{F} \) with \( \delta \neq 0 \), let

\[
\det(f) = \frac{1}{\delta^2}(\alpha \delta - \beta \gamma), \quad \sigma(f) = \frac{\alpha^2}{\delta^2}.
\]

Let \( \mathcal{F}^+ = \{ f \in \mathcal{F} : \det(f) \geq 0 \} \), and for \( k \in \mathbb{Z} \), let

\[
U_k = \{ f \in \mathcal{F}^+ : \det(f) \leq \min(b^k, b^{-k} \sigma(f)) \}.
\]

(2.3)

**Theorem 3.** \( \bar{\Lambda} \cap \mathcal{F}^+ = \bigcup_{k \in \mathbb{Z}} U_k \).

**Proof.** Since \( U_1 \subseteq \bar{\Lambda} \) by Lemma 2 and \( U_k = S^{-k}U_1 \), we have \( U = \bigcup_{k \in \mathbb{Z}} U_k \subseteq \Lambda \cap \mathcal{F}^+ \). Next, we show that \( \Lambda \cap \mathcal{F}^+ \subseteq U \). First we show that \( U \) is a semigroup under composition. To see this, let \( f \in U_k \) and \( g \in U_l \) for some \( k, l \in \mathbb{Z} \). Let \( f(x) = (\alpha x + \beta)/(\gamma x + 1) \) and \( g(x) = (ux + v)/(wx + 1) \). Then

\[
fg(x) = \frac{(\alpha u + \beta w)x + (\alpha v + \beta)}{(\gamma u + w)x + (\gamma v + 1)}.
\]

One verifies that

\[
0 \leq \det(fg) = \frac{(\alpha - \beta \gamma)(u - vw)}{(\gamma v + 1)^2} \leq \min\left(b^{k+l}, b^{-k-l} \left(\frac{\alpha u + \beta w}{\gamma v + 1}\right)^2\right),
\]

and so \( fg \in U \) i.e., \( U \) is a semigroup.

Next, a simple calculation shows that \( S^k(x) = F(b^{-k}, 0, 0) \in U_1 \) and \( RS^kR = F(b^k + 1/a, 1, 1/a) \in U_k \) for every \( k \geq 0 \). Now, every \( f \in \Lambda \cap \mathcal{F}^+ \) can be factored into terms of the form \( RS^kR \), and \( S^k \), and since \( U \) is a semigroup, it follows that \( \Lambda \cap \mathcal{F}^+ \subseteq U \). Since \( U \) is closed in \( \mathcal{F} \), we conclude that \( \Lambda \cap \mathcal{F}^+ \subseteq U \), and the proof is completed.

Theorem 3 implies that there are no 2-generator dense subsemigroups of LFT’s on \([0, \infty)\) (hence on any proper subinterval of reals), since \( U \) does not include every \( f \in \mathcal{F}^+ \). In the next theorem we show that there are 3-generator dense subsemigroups.

**Theorem 4.** Let \( a, c > 0 \) and \( b > 1 \) so that \( \ln c/\ln b \notin \mathbb{Q} \). Then the semigroup generated by \( 1 + a/x, x/b, \) and \( x/c \) is dense in the set of LFT’s on \([0, \infty)\).
Proof. Let $U$ be defined as in \cite{23}. Suppose that $\alpha, \beta, \gamma > 0$ so that $0 \leq \alpha - \beta \gamma$. Since $\ln c/\ln b \notin \mathbb{Q}$, it follows that there exist a sequence $\{k_i\}_{i=1}^{\infty}$ of integers and a sequence $\{l_i\}_{i=1}^{\infty}$ of positive integers so that $b^{k_i}c^{-l_i} \rightarrow \alpha$. Then, we have

$$\lim_{i \to \infty} \min(b^{k_i}c^{-l_i}, b^{-k_i}c^{l_i}\alpha^2) = \alpha,$$

and so for $i$ large enough, we have

$$\alpha - \beta \gamma \leq \min(b^{k_i}c^{-l_i}, b^{-k_i}c^{l_i}\alpha^2),$$

which in turn implies that

$$0 \leq c^l \alpha - (c^l\beta)\gamma \leq \min(b^{k_i}, b^{-k_i}(c^l\alpha)^2).$$

By Theorem \ref{thm:3}, we conclude that $F(c^l\alpha, c^l\beta, \gamma) \in \bar{\Lambda}$, and so $F(\alpha, \beta, \gamma) = c^{-1}F(c^l\alpha, c^l\beta, \gamma) \in \bar{\Lambda}$ as well. The case of $\beta = 0$ or $\gamma = 0$ follows by using a limiting process.

By composing $F(\alpha, \beta, \gamma)$ with $a/x$, we deduce that $F(u, v, w) \in \bar{\Lambda}$ for all $u, w \geq 0$ and $v > 0$. The case of $v = 0$ can be dealt with by using another limiting process. \hfill \Box

3 Orbit closures

In section 2, we showed that there are no 2-generator dense subsemigroups of $\mathcal{F}$. In this section, we study the induced action of the semigroup generated by $R(x) = 1 + a/x$ and $S(x) = x/b$ on $(0, \infty)^2$, and show that it has no dense orbits in $[0, \infty)^2$. On the other hand, the action of the conjugated maps $\hat{R} = \theta R \theta^{-1}$ and $\hat{S} = \theta S \theta^{-1}$, where $\theta = 1/(x + 1)$, on $[0,1]^2$ has dense orbits (where the only dense orbits are the orbits of $(x, y)$ with $x = 0$ or $y = 0$ except $(0,0)$ and $(1,1)$).

The next theorem describes the orbit closure of $(x, y) \in (0, \infty)^2$ under the action of $(R, S)$. It is more appropriate to give a geometric description of the orbit closures. Given a point $A = (x, y) \in (0, \infty)^2$, there exists a unique hyperbola tangential to the line $y = x$ at the origin that connects the origin to $A$. We denote this hyperbolic segment by $H(x, y)$. Also, we denote the infinite half-line in $(0, \infty)^2$ with slope 1 starting at $(x, y)$ by $L(x, y)$. Finally, let $\Omega(x, y)$ denote the closed region bounded by $H(x, y), L(x, y), H(a/x, a/y)$, and $L(a/x, a/y)$. If $x = y$, then this region degenerates to the half-line $y = x$, and so in this case we set $\Omega(x, x) = \{(t, t); t \geq 0\}$. In the sequel, $\bar{\Lambda}$ denotes
the closure of the semigroup generated by \( R(x) = 1 + a/x \) and \( S(x) = x/b \),
where \( a > 0 \) and \( b > 1 \). We begin with the following Lemma.

**Lemma 5.** Let \((x, y) \in (0, \infty)^2\). Then for every \((u, v) \in \Omega(x, y)\), there exists \( f \in \tilde{\Lambda} \) so that \( f(x, y) = (u, v) \).

**Proof.** Since \( \Omega(x, y) \) is invariant under \( I(x) = a/x \in \tilde{\Lambda} \), without loss of
generality, we assume that \( x \geq y \) and \( u \geq v \). If \( x = y \) or \( u = v \), the claim
follows from the fact that the orbits of \( \langle R, S \rangle \) on \([0, \infty)\) are all dense (and
that \((0, 0)\) belongs to every orbit closure). Thus, suppose that \( x > y \) and \( u > v \).
Since \((u, v) \in \Omega(x, y)\), we have
\[
   u \geq v \geq \max \left( u - x + y, \frac{uxy}{ux - uy + xy} \right). \tag{3.1}
\]

It follows from Lemma 2 (by setting \( \alpha = 1 \)) that maps of the form \( f(x) = \frac{x + \beta}{\gamma x + 1} \)
belong to \( \Lambda \), where \( \beta, \gamma \geq 0 \) and \( \beta \gamma < 1 \). We choose \( \beta \) and \( \gamma \)
so that \( f(x) = u \) and \( f(y) = v \). In fact, we need to have
\[
   \beta = \frac{xy(-u + v) + uv(x - y)}{ux - vy}, \quad \gamma = \frac{x - y - u + v}{ux - vy}.
\]
The conditions \( \beta, \gamma \geq 0 \) follow directly from (3.1). The condition \( \beta \gamma \leq 1 \) is
equivalent to
\[
   (ux - vy)^2 - (xy(-u + v) + uv(x - y))(x - y - u + v) \geq 0,
\]
which can be factorized as \((u - v)(v + x)(x - y)(u + y) > 0\), and the proof is
completed. \( \square \)

**Theorem 6.** Let \( R(x) = 1 + a/x \) and \( S(x) = x/b \), where \( a > 0 \) and \( b > 1 \).
Then for any \((x, y) \in (0, \infty)^2\), the orbit closure of \((x, y)\) under the action of
\( \langle R, S \rangle \) is given by
\[
   \bigcup_{k \in \mathbb{Z}} \Omega(b^k x, b^k y).
\]

**Proof.** Lemma 5 and the fact that \( b^k x \in \tilde{\Lambda} \), for all \( k \in \mathbb{Z} \), imply that the set
\( \Omega = \bigcup_{k \in \mathbb{Z}} \Omega(b^k x, b^k y) \) is included in the orbit closure of \((x, y)\). To show that
the orbit closure is included in \( \Omega \), it is sufficient to show that \( \Omega \) is invariant
under \( R \) and \( S \). The set \( \Omega \) is clearly invariant under \( S \). Moreover, we have
\( R(x) = 1 + a/x = M \circ I(x) \), where \( M(x) = x + 1 \) and \( I(x) = a/x \). Since \( \tilde{\Omega} \)
is invariant under both \( I \) and \( M \), we see that it is invariant under \( R \) as well,
and the proof is completed. \( \square \)
Theorem 6 shows that the orbits of $\langle R, S \rangle$ on $(0, \infty)$ are never dense. However, if the interval is finite, dense orbits exist. To see this, we use the conjugation $\theta(x) = 1/(x + 1)$ to move to the interval $[0, 1]$, and denote the conjugated maps by the hat notation.

**Proposition 7.** The orbit of $(x, y)$ under the action of the semigroup $\langle \hat{R}, \hat{S} \rangle$ is dense in $[0, 1]^2$ if and only if $(x, y)$ belongs to the perimeter of the square $[0, 1]^2$ except the vertices $(0, 0)$ and $(1, 1)$.

**Proof.** The claim that none of the orbits starting from an interior point are dense follows from Theorem 6. The orbits starting from $(0, 0)$ and $(1, 1)$ are clearly not dense. Since the point $(0, 1)$ belongs to the orbit of every point on the perimeter of $[0, 1]^2$ except $(0, 0)$ and $(1, 1)$, it is sufficient to prove that the orbit of $(0, 1)$ is dense. Let $\bar{O}$ denote the closure of the orbit of $(0, 1)$ in $[0, 1]^2$. Let $u \geq 0$ be arbitrary. By (2.1), after conjugation by $\theta$, we conclude that

$$\tilde{T}_u(x) = \frac{u(1-x) + x}{(u+1)(1-x) + x}$$

belongs to the closure of $\langle \hat{R}, \hat{S} \rangle$. It follows that $(f(0), f(1)) = (u/(u+1), 1) \in \bar{O}$, which implies that the segment $[0, 1] \times \{1\}$ is a subset of $\bar{O}$. By applying $\hat{R} = (1-x)/(2-2x + ax)$ to this segment, we obtain $[0, 1/2] \times \{0\} \subseteq \bar{O}$. By applying $\hat{S}$ to the segment repeatedly, we get $[0, 1] \times \{0\} \subseteq \bar{O}$. Now, for any $u \geq 0$, apply $T_u$ to $[0, 1] \times \{0\}$, we conclude that the segment $[u/(u+1), 1] \times \{u/(u+1)\}$ is in $\bar{O}$. It follows that $\Delta = \{(x, y) \in [0, 1]^2 : x \geq y\} \subseteq \bar{O}$. By applying $\hat{R}$ to $\Delta$, we get $\{(x, y) \in [0, 1]^2 : y \leq 1/2\} \subseteq \bar{O}$, and by applying $\hat{S}$ repeatedly to this latter set, we conclude that $[0, 1]^2 \subseteq \bar{O}$. □

4 Dense subsemigroups of $2 \times 2$ matrices

It immediately follows from Theorem 6 that there are no 2-generator dense subsemigroups of $\mathcal{F}$. This, in turn, implies that there are no 2-generator dense subsemigroups of the set of $2 \times 2$ matrices with nonnegative entries. In this section, we first show that there exist 3-generator subsemigroups of matrices with nonnegative entries. Recall that a real matrix is called unimodular if its determinant is $\pm 1$. 

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Lemma 8. Let $a, c > 0$ and $b > 1$ so that $\log c / \log b \notin \mathbb{Q}$. Then the semigroup generated by the matrices
\[
\begin{pmatrix}
1/c & 0 \\
0 & c
\end{pmatrix}, \begin{pmatrix}
1/b & 0 \\
0 & b
\end{pmatrix}, \begin{pmatrix}
1/a & a \\
1/a & 0
\end{pmatrix},
\]
is dense in the semigroup of unimodular real matrices with nonnegative entries.

Proof. Let us denote these three matrices by $A$, $B$, and $C$. Let $X = [y, z; s, t]$ be a $2 \times 2$ matrix with nonnegative entries and $\det(X) = \pm 1$. Without loss of generality, we can assume $\det(X) = 1$ and $t \neq 0$. By Theorem 4, for each $i \geq 1$, there exists a matrix $D_i = [\alpha_i, \beta_i; \gamma_i, \delta_i] \in \langle A, B, C \rangle$ so that
\[
\left| \frac{\alpha_i}{\delta_i} - \frac{y}{t} \right| + \left| \frac{\beta_i}{\delta_i} - \frac{z}{t} \right| + \left| \frac{\gamma_i}{\delta_i} - \frac{s}{t} \right| < \frac{1}{i}, \quad (4.1)
\]
It follows that
\[
\left| \frac{1}{\delta_i^2} - \frac{1}{t^2} \right| = \left| \frac{\alpha_i \delta_i - \beta_i \gamma_i}{\delta_i^2} - \frac{yt - zs}{t^2} \right| < \frac{\lambda}{i},
\]
where $\lambda$ depends only on $X$. And so $\delta_i \to t$ as $i \to \infty$, and consequently, by (4.1), we have $\alpha_i \to y$, $\beta_i \to z$, and $\gamma_i \to s$. In other words, $D_i \to X$ as $i \to \infty$, and the claim follows.

Corollary 9. Let $a > 0$ and $b > 1 > c > 0$ so that $\log c / \log b \notin \mathbb{Q}$. Then the semigroup generated by the matrices
\[
\begin{pmatrix}
c & 0 \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
1 & 0 \\
0 & b
\end{pmatrix}, \begin{pmatrix}
1 & a \\
1 & 0
\end{pmatrix}
\]
is dense in the set of $2 \times 2$ matrices with nonnegative entries.

Proof. Let $S$ denote the closure of the semigroup generated by these three matrices. We first show that $dI_{2 \times 2} \in S$ for every $d \geq 0$. Choose sequences of positive integers $k_i, l_i$ so that $b^{k_i}c^{l_i} \to d/a$. Then
\[
\begin{pmatrix}
0 & d \\
1 & 0
\end{pmatrix} = \lim_{i \to \infty} \begin{pmatrix}
c^{l_i} & b^{k_i}c^{l_i}a \\
1 & 0
\end{pmatrix} = \lim_{i \to \infty} \begin{pmatrix}
c^{l_i} & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & a \\
1 & 0
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & b^{k_i}
\end{pmatrix} \in S,
\]
and so $dI_{2 \times 2} = [0, d; 1, 0]^2 \in S$. Next, let $X$ be any $2 \times 2$ matrix with nonnegative entries and $\mu = \det(X) \neq 0$. Let $\hat{F} = F / \sqrt{\det(F)}$ for an
invertible matrix, and let \( \hat{S} = \{ \hat{F}, F \in S \} \). By Lemma 8, there exists \( D_i \in \hat{S} \) so that \( D_i \to \hat{X} \) as \( i \to \infty \). Choose \( d_i \) so that \( d_i D_i \in S \). Since we showed that \( (\sqrt{\mu/d_i})I_{2 \times 2} \in S \), we have

\[
X = \sqrt{\mu} \hat{X} = \lim_{i \to \infty} \frac{\sqrt{\mu}}{d_i} (d_i D_i) \in S,
\]

and so \( S \) contains every \( 2 \times 2 \) matrix with nonnegative entries. \( \square \)

The following corollary is an immediate consequence of Corollary 9.

**Corollary 10.** Let \( a > 0 \) and \( b > 1 > c > 0 \) so that \( \ln c / \ln b \notin \mathbb{Q} \). Then the semigroup generated by the matrices

\[
\begin{pmatrix} -c & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -b \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}
\]

is dense in the set of \( 2 \times 2 \) real matrices.

Now, we construct an explicit example of two \( 2 \times 2 \) matrices that generate a dense semigroup in the set of \( 2 \times 2 \) matrices in the real case:

**Example 11.** The semigroup of matrices generated by

\[
A = \begin{pmatrix} 1 & 1/2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -8/3 \end{pmatrix},
\]

is dense in the set of \( 2 \times 2 \) real matrices.

One verifies that \( ABA^3BA = [-2/9, 0; 0, 1] = C \), and so \( \langle A, B \rangle = \langle A, B, C \rangle \), which is dense in the set of \( 2 \times 2 \) real matrices by Corollary 10.

5 **The complex case**

In this section, we consider the set of \( 2 \times 2 \) complex matrices and prove a result analogous to Corollary 10 in the complex case. At the end of this section, we prove the existence of examples of 2-generator dense subsemigroups of \( 2 \times 2 \) complex matrices. In the sequel \( i = \sqrt{-1} \).

**Corollary 12.** Let \( a, b, c, u \in \mathbb{C} \) such that the following conditions hold:

i) \( a, u \neq 0 \),
ii) \( b = ri \) with \( r > 1 > |c| \), and

iii) the three numbers \( 1, \ln |c| / \ln |b|, \arg(c)/2\pi \) are rationally independent.

Then the semigroup generated by the matrices

\[
C = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \quad A = \begin{pmatrix} u & a \\ 1 & 0 \end{pmatrix}
\]

is dense in the set of \( 2 \times 2 \) complex matrices.

Proof. The argument presented in the proof of Lemma 1 works in the complex case for \( R(x) = u + a/x \) and \( S(x) = x/b \), as long as \( a, u \neq 0 \) and \( |b| > 1 \). For every complex number \( s \) there exists a sequence \( \{\alpha_j\}_{j=1}^{\infty} \) of integers so that

\[
sa = \sum_{j=1}^{k} b^{\alpha_j}.
\]

To see this, we note that every positive real number can be written as a series with terms of the form \( b^{4k} \), \( k \in \mathbb{Z} \), while every negative real number can be written as a series with the terms of the form \( b^{2k} \), \( k \in \mathbb{Z} \). Similarly, every purely imaginary number \( ti \) can be written as a series with terms of the form \( b^{4k+1} \), if \( t > 0 \), or terms of the form \( b^{4k+3} \) if \( t < 0 \). It then follows from (2.1) that \( T_s(x) = x/(sx + 1) \in \bar{\Lambda} \) for all \( s \in \mathbb{C} \). Now, the argument in Lemma 2 can be used to show that \( \left( \alpha x + \beta \right) / \left( \gamma x + 1 \right) \in \bar{\Lambda} \) for every \( \alpha, \beta, \gamma \in \mathbb{C} \) (since this time the equations (2.2) are always solvable if \( \beta, \alpha - \beta \gamma \neq 0 \). The cases where \( \beta = 0 \) or \( \alpha - \beta \gamma = 0 \) can be dealt with by taking limits).

So far we have shown that the semigroup generated by \( R(x) = u + a/x \) and \( S(x) = x/b \) is dense in the set of Möbius transformations (which is isometric to \( SL(2, \mathbb{C}) \)). The argument in Corollary 9 can be used to show that the semigroup generated by \( A, B, \) and \( C \) is dense, if we show that the set \( \langle b, c \rangle = \{b^m c^n : m, n \in \mathbb{N} \} \) is dense in \( \mathbb{C} \). Let \( z \) be an arbitrary nonzero complex number. It follows from condition (iii) and the multidimensional Kronecker’s approximation Theorem [4, §23.6] that for any \( \epsilon > 0 \) there exist positive integers \( m, n \) and an integer \( L \) so that

\[
\left| n \left( \frac{\arg(c)}{2\pi} \right) - \left( \frac{\arg(z)}{2\pi} \right) + L \right| < \epsilon, \quad (5.2)
\]

\[
\left| n \left( \frac{\ln |c|}{\ln |b|^4} \right) - \left( \frac{\ln |z|}{\ln |b|^4} \right) + m \right| < \epsilon. \quad (5.3)
\]

It follows from the inequalities (5.2) and (5.3) that \( |\ln |c^n b^{4m}| - \ln |z|| < \epsilon |b|^4 \) and \( |\arg(c^n b^{4m}) - \arg(z) + 2\pi L| < 2\pi \epsilon \). Since \( \epsilon \) was arbitrary, we conclude that \( \langle b, c \rangle \) is dense in \( \mathbb{C} \), and the proof is completed.

\[\square\]
We are now ready to prove the existence of examples of 2-generator dense subsemigroups of complex 2 \times 2 matrices. Recall that a set \( F \) is called \textit{cocountable} in \( E \) if \( E \setminus F \) is countable.

**Example 13.** For \( r > 3 \), let
\[
\begin{align*}
    b &= r i, \quad (5.4) \\
    u &= -\left(\frac{1}{2b}\right)^{1/5} \left( 8 + 2b + b^2 + 4\sqrt{4 + b^2} + b\sqrt{4 + b^2} \right)^{1/5}, \quad (5.5) \\
    a &= u^2(-2 - b + \sqrt{4 + b^2})/(2b), \quad (5.6) \\
    c &= \frac{1}{2} \left( 2 + b^2 + \sqrt{4 + b^2} - b - b\sqrt{4 + b^2} \right). \quad (5.7)
\end{align*}
\]

Then there exists a cocountable subset \( F \subseteq (3, \infty) \) so that, for every \( r \in F \), the semigroup generated by the matrices \( A = [u, a; 1, 0] \) and \( B = [1, 0; 0, b] \) is dense in the set of 2 \times 2 complex matrices.

**Proof.** We have selected \( a, b, c, \) and \( u \) so that \( ABA^3BA = C \) i.e., \( \langle A, B, C \rangle = \langle A, B \rangle \). Thus, we only need to verify the conditions of Corollary 12 for \( a, b, c, \) and \( u \). Clearly \( a, u \neq 0 \) and \( |b| = r > 1 \). By direct computation, we have
\[
|c|^2 = \frac{1}{2} \left( r^4 - 3r^2 + r\sqrt{r^2 - 4} - r^3\sqrt{r^2 - 4} \right) \in (0, 1),
\]
for all \( r > 3 \).

Now, \( f(r) = \arg(c)/2\pi \) and \( g(r) = \ln |c|/\ln |b| \) are both analytic functions of \( r \in (3, \infty) \). Let \( \mathcal{H} \) denote the set of \( r > 3 \) so that \( 1, f(r), g(r) \) are rationally dependent. We need to show that \( \mathcal{H} \) is a countable set. On the contrary, suppose that \( \mathcal{H} \) is uncountable. For each \( r \in \mathcal{H} \), there exists a triplet of integers \( (A(r), B(r), C(r)) \neq (0, 0, 0) \) so that
\[
A(r) + B(r)f(r) + C(r)g(r) = 0.
\]

The function \( r \mapsto (A(r), B(r), C(r)) \) maps the uncountable set \( \mathcal{H} \) to the countable set \( \mathbb{Z}^3 \setminus \{(0, 0, 0)\} \). It follows that there exist uncountably many values of \( r \) that are mapped to the same triplet \( (A, B, C) \neq (0, 0, 0) \), and so the equation
\[
H(r) = A + Bf(r) + Cg(r) = 0,
\]
has uncountably many solutions for \( r > 3 \). Since \( f \) and \( g \) are analytic functions of \( r \), it follows that \( H \) is an analytic function of \( r \), and so \( H(r) \equiv 0 \).
for all $r > 3$. On the other hand, as $r \to \infty$, one shows that $f(r) \to 1/4$ and $g(r) \to -1$, and so $A + B/4 - C = 0$. If $C \neq 0$, then $h(r) = (g(r) + 1)/(1/4 - f(r)) = B/C$, but it is easy to check that $h(r)$ is not constant (or alternatively check that $\lim_{r \to \infty} h(r) = 0$, which gives $B = 0$, and then because $g$ is not a constant function, it follows that $A = C = 0$). Hence $C = 0$, which in turn implies that $A = B = 0$, since $f$ is not a constant function either. This is a contradiction, and the proof is completed. □

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