The truncated matrix trigonometric moment problem with an open gap.

S.M. Zagorodnyuk

1 Introduction.

This paper is a continuation of our previous investigations on the truncated matrix trigonometric moment problem (briefly TMTMP) by the operator approach in [1],[2]. The truncated matrix trigonometric moment problem consists of finding a non-decreasing \(C_{N \times N}\)-valued function \(M(t) = (m_{k,l}(t))_{k,l=0}^{N-1}, t \in [0,2\pi], M(0) = 0\), which is left-continuous in \((0,2\pi]\), and such that

\[
\int_0^{2\pi} e^{int} dM(t) = S_n, \quad n = 0,1,\ldots,d,
\]

(1)

where \(\{S_n\}_{n=0}^d\) is a prescribed sequence of \((N \times N)\) complex matrices (moments). Here \(N \in \mathbb{N}\) and \(d \in \mathbb{Z}_+\) are fixed numbers. Set

\[
T_d = (S_{l-j})_{i,j=0}^{d} = \begin{pmatrix}
S_0 & S_{-1} & S_{-2} & \cdots & S_{-d} \\
S_1 & S_0 & S_{-1} & \cdots & S_{-d+1} \\
S_2 & S_1 & S_0 & \cdots & S_{-d+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
S_d & S_{d-1} & S_{d-2} & \cdots & S_0
\end{pmatrix},
\]

(2)

where

\[S_k := S_{-k}^*, \quad k = -d,-d+1,\ldots,-1,\]

and \(\{S_n\}_{n=0}^d\) are from [1]. It is well known that the following condition:

\[T_d \geq 0,\]

(3)

is necessary and sufficient for the solvability of the moment problem [1] (e.g. [3]). The solvable moment problem [1] is said to be determinate if it has a unique solution and indeterminate in the opposite case. We shall omit here an exposition on the history and recent results for the moment problem [1]. All that can be found in [1],[2].

Choose an arbitrary \(a \in \mathbb{N}\). Denote by \(S(\mathbb{D};\mathbb{C}_{a \times a})\) a set of all analytic in \(\mathbb{D}\), \(\mathbb{C}_{a \times a}\)-valued functions \(F_\zeta\), such that \(F_\zeta^*F_\zeta \leq I_a\), \(\forall \zeta \in \mathbb{D}\). In [2] we obtained a Nevanlinna-type parameterization for all solutions of the moment problem [1]:

1
Theorem 1 Let the moment problem (7), with \(d \in \mathbb{N}\), be given and condition (3), with \(T_d\) from (2), be satisfied. Suppose that the moment problem is indeterminate. All solutions of the moment problem (7) have the following form:

\[
M(t) = (m_{k,l}(t))_{k,l=0}^{N-1}, \quad \int_0^{2\pi} \frac{1}{1 - \zeta e^{it}} dM^T(t)
\]

\[
= \frac{1}{h_\zeta} A_\zeta - \frac{\zeta}{h_\zeta^2} B_\zeta F_\zeta \left( I_\delta + \frac{1}{h_\zeta} C_\zeta F_\zeta \right)^{-1} D_\zeta, \quad \zeta \in \mathbb{D}, \quad (4)
\]

where \(A_\zeta, B_\zeta, C_\zeta, D_\zeta\), are matrix polynomials defined by the given moments, with values in \(\mathbb{C}_{N \times N}, \mathbb{C}_{N \times \delta}, \mathbb{C}_{\delta \times \delta}, \mathbb{C}_{\delta \times N}\), respectively (\(0 \leq \delta \leq N\)). The scalar polynomial \(h_\zeta\) is also defined by the moments. Here \(F_\zeta \in S(\mathbb{D}; \mathbb{C}_{\delta \times \delta})\). Conversely, each function \(F_\zeta \in S(\mathbb{D}; \mathbb{C}_{\delta \times \delta})\) generates by relation (7) a solution of the moment problem (7). The correspondence between all functions from \(S(\mathbb{D}; \mathbb{C}_{\delta \times \delta})\) and all solutions of the moment problem (7) is one-to-one.

In this paper we shall study the moment problem (7) with an additional constraint posed on the matrix measure \(M_T(\delta), \delta \in \mathfrak{B}(\mathbb{T})\), generated by the function \(M(x)\) (see the precise definition of \(M_T(\delta)\) and other details below):

\[
M_T(\Delta) = 0, \quad (5)
\]

where \(\Delta\) is a given open subset of \(\mathbb{T}\) (called a gap). Here \(\mathbb{T}\) is viewed as a metric space with the metric \(r(z, w) = |z - w|\).

We present necessary and sufficient conditions for the solvability of the moment problem (7), (5). All solutions of the moment problem (7), (5) can be constructed by relation (7), where \(F_\zeta\) belongs to a certain subset of \(S(\mathbb{D}; \mathbb{C}_{\delta \times \delta})\).

Notations. As usual, we denote by \(\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+\), the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively; \(\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}\); \(\mathbb{D}_e = \{z \in \mathbb{C} : |z| > 1\}\); \(\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}\); \(\mathbb{T}_e = \{z \in \mathbb{C} : |z| \neq 1\}\). Let \(m, n \in \mathbb{N}\). The set of all complex matrices of size \((m \times n)\) we denote by \(\mathbb{C}_{m \times n}\). The set of all complex non-negative Hermitian matrices of size \((n \times n)\) we denote by \(\mathbb{C}^\geq_{n \times n}\). If \(M \in \mathbb{C}_{m \times n}\) then \(M^T\) denotes the transpose of \(M\), and \(M^*\) denotes the complex conjugate of \(M\). The identity matrix from \(\mathbb{C}_{n \times n}\) we denote by \(I_n\). By \(\mathfrak{B}(\mathbb{T})\) we denote a set of all Borel subsets of \(\mathbb{T}\).

If \(H\) is a Hilbert space then \((\cdot, \cdot)_H\) and \(\| \cdot \|_H\) mean the scalar product and the norm in \(H\), respectively. Indices may be omitted in obvious cases.
By $\mathbb{C}^N$ we denote the finite-dimensional Hilbert space of complex column vectors of size $N$ with the usual scalar product $(\vec{x}, \vec{y})_{\mathbb{C}^N} = \sum_{j=0}^{N-1} x_j \overline{y_j}$, for $\vec{x}, \vec{y} \in \mathbb{C}^N$, $\vec{x} = (x_0, x_1, \ldots, x_{N-1})^T$, $\vec{y} = (y_0, y_1, \ldots, y_{N-1})^T$, $x_j, y_j \in \mathbb{C}$.

For a linear operator $A$ in $H$, we denote by $D(A)$ its domain, by $R(A)$ its range, by $\text{Ker}\, A$ its null subspace (kernel), and $A^*$ means the adjoint operator if it exists. If $A$ is invertible then $A^{-1}$ means its inverse. $\overline{A}$ means the closure of the operator, if the operator is closable. If $A$ is bounded then $\|A\|$ denotes its norm. For a set $M \subseteq H$ we denote by $\overline{M}$ the closure of $M$ in the norm of $H$. For an arbitrary set of elements $\{x_n\}_{n \in I}$ in $H$, we denote by $\text{Lin}\{x_n\}_{n \in I}$ the set of all linear combinations of elements $x_n$, and $\text{span}\{x_n\}_{n \in I} := \text{Lin}\{x_n\}_{n \in I}$. Here $I$ is an arbitrary set of indices. By $E_H$ we denote the identity operator in $H$, i.e. $E_H x = x$, $x \in H$. In obvious cases we may omit the index $H$. If $H_1$ is a subspace of $H$, then $P_{H_1} = P^H_{H_1}$ is an operator of the orthogonal projection on $H_1$ in $H$.

By $S(D; N, N')$ we denote a class of all analytic in a domain $D \subseteq \mathbb{C}$ operator-valued functions $F(z)$, which values are linear non-expanding operators mapping the whole $N$ into $N'$, where $N$ and $N'$ are some Hilbert spaces.

For a closed isometric operator $V$ in a Hilbert space $H$ we denote: $M_\zeta(V) = (E_H - \zeta V)^{-1}$, $N_\zeta(V) = H \ominus M_\zeta(V)$, $\zeta \in \mathbb{C}$; $M_\infty(V) = R(V)$, $N_\infty(V) = H \ominus R(V)$.

### 2 The TMTMP with an open gap.

Let the moment problem (1), with $d \in \mathbb{N}$, be given and condition (3), with $T_d$ from (2), be satisfied. Let

$$T_d = (\gamma_{n,m})_{n,m=0}^{(d+1)N-1}, \quad S_k = (S_{k;l})_{s,l=0}^{N-1}, \quad -d \leq k \leq d,$$

where $\gamma_{n,m}, S_{k;s,l} \in \mathbb{C}$. Observe that

$$\gamma_{kN+s,rN+l} = S_{k-r;s,l}, \quad 0 \leq k, r \leq d, \quad 0 \leq s, l \leq N-1. \quad (6)$$

We repeat here some constructions from [1]. Consider a complex linear vector space $\mathfrak{H}$, which elements are row vectors $\vec{u} = (u_0, u_1, u_2, \ldots, u_{(d+1)N-1})$, with $u_n \in \mathbb{C}$, $0 \leq n \leq (d+1)N-1$. Addition and multiplication by a scalar are defined for vectors in a usual way. Set

$$\vec{e}_n = (\delta_{n,0}, \delta_{n,1}, \delta_{n,2}, \ldots, \delta_{n,(d+1)N-1}), \quad 0 \leq n \leq (d+1)N-1,$$
where $\delta_{n,r}$ is Kronecker's delta. In $\mathcal{H}$ we define a linear functional $B$ by the following relation:

$$B(\vec{u}, \vec{w}) = \sum_{n,r=0}^{(d+1)N-1} a_n b_r \gamma_{n,r},$$

where

$$\vec{u} = \sum_{n=0}^{(d+1)N-1} a_n \vec{\varepsilon}_n, \quad \vec{w} = \sum_{r=0}^{(d+1)N-1} b_r \vec{\varepsilon}_r, \quad a_n, b_r \in \mathbb{C}.$$

The space $\mathcal{H}$ with $B$ form a quasi-Hilbert space ([1]). By the usual procedure of introducing of the classes of equivalence (see, e.g. [1]), we put two elements $\vec{u}, \vec{w}$ from $\mathcal{H}$ to the same class of equivalence denoted by $[\vec{u}]$ or $[\vec{w}]$, if $B(\vec{u} - \vec{w}, \vec{u} - \vec{w}) = 0$. The space of classes of equivalence is a (finite-dimensional) Hilbert space. Everywhere in what follows it is denoted by $H$. Set

$$x_n := [\vec{\varepsilon}_n], \quad 0 \leq n \leq (d + 1)N - 1.$$

Then

$$(x_n, x_m)_H = \gamma_{n,m}, \quad 0 \leq n, m \leq (d + 1)N - 1, \quad (7)$$

and $\text{span}\{x_n\}_{n=0}^{(d+1)N-1} = \text{Lin}\{x_n\}_{n=0}^{(d+1)N-1} = H$. Set $L_N := \text{Lin}\{x_n\}_{n=0}^{N-1}$. Consider the following operator:

$$Ax = \sum_{k=0}^{dN-1} \alpha_k x_{k+N}, \quad x = \sum_{k=0}^{dN-1} \alpha_k x_k, \quad \alpha_k \in \mathbb{C}. \quad (8)$$

By [1, Theorem 1] all solutions of the moment problem (1) have the following form

$$M(t) = (m_{k,j}(t))_{k,j=0}^{N-1} \quad m_{k,j}(t) = (E_t x_k, x_j)_H, \quad (9)$$

where $E_t$ is a left-continuous spectral function of the isometric operator $A$. Conversely, each left-continuous spectral function of $A$ generates by (9) a solution of the moment problem (1). The correspondence between all left-continuous spectral functions of $A$ and all solutions of the moment problem (1), established by relation (9), is one-to-one.

By [1, Theorem 3] all solutions of the moment problem (1) have the following form

$$M(t) = (m_{k,j}(t))_{k,j=0}^{N-1}, \quad t \in [0, 2\pi], \quad (10)$$
where \( m_{k,j} \) are obtained from the following relation:

\[
\int_0^{2\pi} \frac{1}{1 - \zeta e^{it}} dm_{k,j}(t) = ([E_H - \zeta(A \oplus \Phi_\zeta)]^{-1} x_k, x_j)_H, \quad \zeta \in \mathbb{D}. \tag{11}
\]

Here \( \Phi_\zeta \in \mathcal{S}(D; N_0(A), N_\infty(A)) \). Conversely, each \( \Phi_\zeta \in \mathcal{S}(D; N_0(A), N_\infty(A)) \) generates by relations (10)-(11) a solution of the moment problem (1). The correspondence between all \( \Phi_\zeta \in \mathcal{S}(D; N_0(A), N_\infty(A)) \) and all solutions of the moment problem (1) is one-to-one.

Observe that the right-hand side of (11) may be written as \( (R_\zeta(A)x_k, x_j)_H \), where \( R_\zeta(A) \) is a generalized resolvent of the isometric operator \( A \). The correspondence between all generalized resolvents of \( A \) and all solutions of the moment problem is one-to-one, as well.

Consider an arbitrary solution \( M(x) \) of the moment problem (1). By the construction in [1, pp. 791-793], the corresponding spectral function \( E_t \) in (9) is generated by the left-continuous orthogonal resolution of unity \( \tilde{E}_t \) of a unitary operator \( \tilde{U}_0 \) in a Hilbert space \( H_1 \supseteq H \). Moreover, the following relation holds:

\[
\tilde{U}_0 = UU_0U^{-1},
\]

where \( U \) is a unitary transformation which maps \( L^2(M) \) onto \( H_1 \), and \( U_0 \) is the operator of multiplication by \( e^{it} \) in \( L^2(M) \).

Denote by \( \bar{E}(\delta) \), \( \delta \in \mathfrak{B}(\mathbb{T}) \), the orthogonal spectral measure of the unitary operator \( \tilde{U}_0 \). The spectral measure \( \bar{E}(\delta) \) and the resolution of the identity \( \bar{E}_t \) are related in the following way:

\[
\bar{E}_t = \bar{E}(\delta_t), \quad \delta_t = \{z = e^{i\tau} : 0 \leq \tau < t\}, \quad t \in [0, 2\pi).
\]

Therefore the spectral function \( E_t \) and the corresponding spectral measure \( E(\delta) \), \( \delta \in \mathfrak{B}(\mathbb{T}) \), satisfy the following relation:

\[
E_t = E(\delta_t), \quad t \in [0, 2\pi). \tag{12}
\]

Relation (12) may be rewritten in the following form:

\[
M(t) = (E(\delta_t)x_k, x_j)_H)_{k,j=0}^{N-1}, \quad t \in [0, 2\pi). \tag{13}
\]

Define the following \( \mathbb{C}_{N \times N}^{\geq} \)-valued measure on \( \mathfrak{B}(\mathbb{T}) \) (i.e. a \( \mathbb{C}_{N \times N}^{\geq} \)-valued function on \( \mathfrak{B}(\mathbb{T}) \) which is countably additive):

\[
M_T(\delta) = (E(\delta)x_k, x_j)_H)_{k,j=0}^{N-1}, \quad \delta \in \mathfrak{B}(\mathbb{T}). \tag{14}
\]
From this definition and relation (13) it follows that
\[ M(t) = M_{T}(δ_t), \quad t \in [0, 2\pi). \] (15)

Observe that
\[ \int_{\mathcal{T}} z^n dM_{T} = \int_{0}^{2\pi} e^{int} dM(t) = S_n, \quad n = 0, 1, \ldots, d. \] (16)

An arbitrary \( \mathbb{C}_{N \times N}^{>0} \)-valued measure \( \tilde{M}_{T}(δ), \delta \in \mathcal{B}(\mathbb{T}) \), satisfying the following relation:
\[ \tilde{M}_{T}(δ_t) = M(t), \quad t \in [0, 2\pi), \] (17)
coinsides with the matrix measure \( M_{T}(δ) \). In fact, we may consider the following functions:
\[ f_{k,j}(\delta; \alpha; \tilde{M}_{T}) = (\tilde{M}_{T}(δ)(\tilde{u}_k + \alpha \tilde{u}_j), \tilde{u}_k + \alpha \tilde{u}_j)_{\mathbb{C}^N} \geq 0, \]
where \( \alpha \in \mathbb{C}, \delta \in \mathcal{B}(\mathbb{T}), \tilde{u}_k = (δ_k,0,δ_k,1,\ldots,δ_k,N-1)^T, 0 \leq k, j \leq N-1 \). The scalar measures \( f_{k,j}(\delta; \alpha; M_{T}) \) and \( f_{k,j}(\delta; \alpha; \tilde{M}_{T}) \) coincide on \( δ_t, t \in [0, 2\pi) \).

Therefore they coincide on the minimal generated algebra \( Y \), which consists of all finite unions of disjoint sets of the form \( δ_{t_2,t_1} = \{ z = e^{it} : t_2 \leq t < t_1 \}, \quad t_1, t_2 \in [0, 2\pi) \). Since the Lebesgue continuation is unique, these scalar measures coincide. On the other hand, the entries of \( M_{T}(δ) \) and \( \tilde{M}_{T}(δ) \) are expressed via \( f_{k,j} \) by the polarization formula. Then \( \tilde{M}_{T}(δ) = M_{T}(δ) \).

During the investigation of the moment problem (1),(5), it is enough to assume that the corresponding moment problem (1) (with the same moments) is indeterminate. In fact, if the corresponding moment problem (1) has no solutions than the moment problem (1),(5) has no solutions, as well. If the corresponding moment problem (1) has a unique solution than this solution can be found explicitly, and then condition (6) may be verified directly.

**Proposition 1** Let the indeterminate moment problem (1) with \( d \in \mathbb{N} \) be given and the operator \( A \) in a Hilbert space \( H \) be constructed as in (8). Let \( \Delta \in \mathcal{B}(\mathbb{T}) \) be a fixed set. Let \( M(x), x \in [0,2\pi], \) be a solution of the moment problem (1), \( M_{T}(δ), \delta \in \mathcal{B}(\mathbb{T}), \) be the matrix measure which is defined by (14) with the corresponding spectral measure \( E(δ), \delta \in \mathcal{B}(\mathbb{T}) \). The following two conditions are equivalent:

(i) \( M_{T}(\Delta) = 0; \)
(ii) \(E(\Delta) = 0\).

**Proof.** (ii)⇒(i). It follows directly from the definition (14).

(i)⇒(ii). By (14) we may write:

\[
M^\top(\Delta) = ((E(\Delta)x_k, x_j)_H)^{N-1}_{k,j=0} = ((\tilde{E}(\Delta)x_k, x_j)_H)^{N-1}_{k,j=0} = 0. \tag{18}
\]

Choose arbitrary numbers \(l, m\): \(0 \leq l, m \leq dN + N - 1\). Let \(l = rN + k\), \(l = sN + j\), where \(0 \leq k, j \leq N - 1\), \(r, s \in \mathbb{Z}_+\). Then

\[
(\tilde{E}(\Delta)x_l, x_m)_H = (\tilde{E}(\Delta)\tilde{U}_0^r x_k, \tilde{U}_0^s x_j)_H = (\tilde{U}_0^{-s} \tilde{E}(\Delta)x_k, x_j)_H
\]

\[
= \int_T z^{r-s} d(\tilde{E}(\delta)\tilde{E}(\Delta)x_k, x_j)_H = \int_T z^{r-s} d(\tilde{E}(\delta \cap \Delta)x_k, x_j)_H = 0.
\]

Therefore \(E(\Delta) = 0\). \(\Box\)

Let \(z, w \in \mathbb{T}: z \neq w\). Let \(z = e^{it}, w = e^{iy}, 0 \leq t, y < 2\pi\). If \(t < y\), we denote

\[
l(z, w) = \{u = e^{i\tau} : t < \tau < y\}.
\]

If \(t > y\), we set \(l(z, w) = \mathbb{T} \setminus (l(w, z) \cup \{w\} \cup \{z\})\). Thus, \(l(z, w)\) is an open arc of \(\mathbb{T}\) with ends in \(z, w\).

Observe that an arbitrary open subset \(\Delta\) of \(\mathbb{T}, \Delta \neq \mathbb{T}\), is a finite or countable union of disjoint open arcs of \(\mathbb{T}\). In fact, suppose that \(\zeta_0 \in \mathbb{T}\): \(\zeta_0 \notin \Delta\). Then \(1 \notin \frac{1}{\zeta_0} \Delta = \{u = \frac{1}{\zeta_0} w, \ w \in \Delta\}\). Set

\[
\Omega := \left\{ x = \text{Arg} z, \ z \in \frac{1}{\zeta_0} \Delta \right\} \subseteq (0, 2\pi).
\]

The set \(\Omega\) is an open subset of \(\mathbb{R}\). Therefore it is a finite or countable union of disjoint open intervals \(l_j \subseteq (0, 2\pi)\). Then \(\frac{1}{\zeta_0} \Delta\) is a finite or countable union of disjoint open arcs of \(\mathbb{T}\). Consequently, \(\Delta\) has the same property.

The following proposition and theorem are simple consequences of Proposition 4.1 and Theorem 4.13 in [5] and the above proved property.

**Proposition 2** Let \(V\) be a closed isometric operator in a Hilbert space \(H\), and \(F(\delta), \delta \in \mathcal{B}(\mathbb{T}),\) be its spectral measure. Let \(\Delta\) be an open subset of \(\mathbb{T}\), \(\Delta \neq \mathbb{T}\). The following two conditions are equivalent:

\[
(i) \ F(\Delta) = 0;
\]

\[
(ii) \ The \ generalized \ resolvent \ R_\Delta(V), \ corresponding \ to \ the \ spectral \ measure \ F(\delta), \ admits \ analytic \ continuation \ on \ the \ set \ \mathbb{D} \cup \mathbb{D}_e \cup \overline{\Delta}, \ where \ \overline{\Delta} = \{z \in \mathbb{C} : \ z \in \Delta\}.
\]
Let $V$ be a closed isometric operator in a Hilbert space $H$, and $\zeta \in \mathbb{T}$. Suppose that $\zeta^{-1}$ is a point of the regular type of $V$. Consider the following operators (see [5, p. 270]):

$$W_\zeta P_{N_0}^H f = \zeta^{-1} P_{N_\infty}^H f, \quad f \in N_\zeta,$$

with the domain $D(W_\zeta) = P_{N_0}^H N_\zeta$;

$$S = S_\zeta = P_{N_0(V)}^H |_{N_\zeta(V)}, \quad Q = Q_\zeta = P_{N_\infty(V)}^H |_{N_\zeta(V)}.$$

Moreover (see [5, p. 271]), since $\zeta^{-1}$ is a point of the regular type of $V$, $S^{-1}$ exists and it is defined on the whole $N_0(V)$, $D(W_\zeta) = N_0(V)$, and

$$W_\zeta = \zeta^{-1} Q_\zeta S^{-1} \zeta.$$

**Theorem 2** Let $V$ be a closed isometric operator in a Hilbert space $H$, and $\Delta$ be an open subset of $\mathbb{T}$, $\Delta \neq \mathbb{T}$, such that

$$\zeta^{-1} \text{ is a point of the regular type of } V, \quad \forall \zeta \in \Delta,$$

and

$$P_{N_\infty(V)}^H M_\zeta(V) = M_\infty(V), \quad \forall \zeta \in \Delta.$$

Let $R_\zeta = R_\zeta(V)$ be an arbitrary generalized resolvent of $V$, and $F_\zeta \in \mathcal{S}(\mathbb{D}; N_0, N_\infty)$ corresponds to $R_\zeta(V)$ by Chumakin’s formula. The operator-valued function $R_\zeta(V)$ has an analytic continuation on a set $\mathbb{D} \cup \mathbb{D}_c \cup \Delta$ if and only if the following conditions hold:

1) $F_\zeta$ admits a continuation on a set $\mathbb{D} \cup \Delta$ and this continuation is continuous in the uniform operator topology;

2) The continued function $F_\zeta$ maps isometrically $N_0(V)$ on the whole $N_\infty(V)$, for all $\zeta \in \Delta$;

3) The operator $F_\zeta - W_\zeta$ is invertible for all $\zeta \in \Delta$, and

$$(F_\zeta - W_\zeta) N_0(V) = N_\infty(V), \quad \forall \zeta \in \Delta,$$

where $W_\zeta$ is from (21).

As it follows from Remark 4.14 in [5, p. 274], conditions (22), (23) are necessary for the existence of at least one generalized resolvent of $V$, which admits an analytic continuation on $\mathbb{T}_c \cup \Delta$. 

Proposition 3 Let \( V \) be a closed isometric operator in a finite-dimensional Hilbert space \( H \), and \( \Delta \) be an open subset of \( T \), \( \Delta \neq T \), such that condition (22) holds. Then condition (23) holds true.

Proof. By Corollary 4.7 in [5, p. 268] we may write:

\[
N_\infty(V) + M_\zeta(V) = H, \quad \forall \zeta \in \Delta.
\]

Applying \( P_H M_\infty(V) \) to the both sides of the latter equality we obtain relation (23). \( \square \)

By Proposition 3 and Theorem 2 we get the following result.

Theorem 3 Let \( V \) be a closed isometric operator in a finite-dimensional Hilbert space \( H \), and \( \Delta \) be an open subset of \( T \), \( \Delta \neq T \), such that condition (22) holds. Let \( R_{\zeta} = R_{\zeta}(V) \) be an arbitrary generalized resolvent of \( V \), and \( F_{\zeta} \in S(D; N_0, N_\infty) \) corresponds to \( R_{\zeta}(V) \) by Chumakin’s formula. The operator-valued function \( R_{\zeta}(V) \) has an analytic continuation on a set \( \mathbb{D} \cup \mathbb{D}_e \cup \Delta \) if and only if the following conditions hold:

1) \( F_{\zeta} \) admits a continuation on a set \( \mathbb{D} \cup \Delta \) and this continuation is continuous in the uniform operator topology;

2) The continued function \( F_{\zeta} \) maps isometrically \( N_0(V) \) into \( N_\infty(V) \), for all \( \zeta \in \Delta \);

3) The operator \( F_{\zeta} - W_{\zeta} \) is invertible for all \( \zeta \in \Delta \), where \( W_{\zeta} \) is from (21).

Let us return to the investigation of the moment problem (1). At first, we shall obtain some necessary conditions for the solvability of the moment problem (1), (5). We shall use the orthonormal bases constructed in [2].

Proposition 4 Let the indeterminate moment problem (7) with \( d \in \mathbb{N} \) be given and the operator \( A \) in a Hilbert space \( H \) be constructed as in (8). Let \( \Delta \) be an open subset of \( T \), \( \Delta \neq T \). Let \( \mathcal{A} = \{g_j(\zeta)\}_{j=0}^{\overline{\tau}-1}, \overline{\tau} \leq dN, \) be an orthonormal basis in \( M_\zeta(A) \), obtained by the Gram-Schmidt orthogonalization procedure from the following sequence:

\[
x_0 - \zeta x_N, x_1 - \zeta x_{N+1}, \ldots, x_{dN-1} - \zeta x_{dN+N-1}.
\]

Here \( \zeta \in \overline{\Delta} = \{z \in \mathbb{C} : \tau \in \Delta\} \). The case \( \overline{\tau} = 0 \) means that \( \mathcal{A} = \emptyset \), and \( M_\zeta(A) = \{0\} \). Then the following conditions are equivalent:

(a) \( \zeta \) is a point of the regular type of \( A \), \( \forall \zeta \in \Delta \).
(b) $M_{\zeta}(A) \neq \{0\}$, and the matrix $M_{E_{H-\zeta}A}$ is invertible, for all $\zeta \in \overline{\Delta}$.

Here we denote by $M_{E_{H-\zeta}A}$ the matrix of the operator $E_{H-\zeta}A$ with respect to the bases $\mathfrak{A}_2, \mathfrak{A}'_3$.

Conditions (a), (b) are necessary for the existence of a solution $M(x)$, $x \in [0, 2\pi]$, of the moment problem (1), such that $M_{\tau}(\Delta) = 0$. If conditions (a),(b) are satisfied then $\tau = \tau \geq 1$.

**Proof.** The implication (b) $\Rightarrow$ (a) is obvious. Conversely, suppose that $\Delta$ consists of points of the regular type of $A$. Choose an arbitrary $\zeta \in \overline{\Delta}$, and set $z_0 = \overline{\zeta} \in \Delta$. Then $(A - z_0E_H)^{-1}M_{\zeta}(A) = D(A)$. If $M_{\zeta}(A) = \{0\}$, we would get $D(A) = \{0\}$, $S_0 = 0$, and in this case the moment problem would be determinate ($M(x) \equiv 0$). Therefore $M_{\zeta}(A) \neq \{0\}$. The rest is obvious.

Suppose that there exists a solution $M(x)$, $x \in [0, 2\pi]$, of the moment problem (1), such that $M_{\tau}(\Delta) = 0$, where $M_{\tau}(\Delta)$, $\delta \in \mathfrak{B}(\mathbb{T})$, is the corresponding matrix measure. By Proposition 2 we get $E(\Delta) = 0$, where $E(\delta)$, $\delta \in \mathfrak{B}(\mathbb{T})$, is the corresponding spectral measure. By Proposition 3 this means that the corresponding generalized resolvent $R_{\zeta}(A)$ admits an analytic continuation on a set $T_\epsilon \cup \overline{\Delta}$. In this case, as it was noticed after Theorem 2 relation (22) holds for $\zeta \in \overline{\Delta}$. □

Consider the indeterminate moment problem (1), such as in Proposition 1 and suppose that condition (b) of Proposition 1 is satisfied. Set $\overline{L} := \text{Lin}\{g_0(\zeta), g_1(\zeta), ..., g_{\tau-1}(\zeta), x_0, x_1, ..., x_{N-1}\}, \zeta \in \overline{\Delta}$. Notice that $\overline{L} = H$. In fact, this follows from the following inclusion, which may be checked by the induction argument: $\{x_n\}_{n=0}^{kN+N-1} \subseteq \overline{L}, k = 0, 1, ..., d$.

Apply the Gram-Schmidt orthogonalization procedure to the following sequence:

$g_0(\zeta), g_1(\zeta), ..., g_{\tau-1}(\zeta), x_0, x_1, ..., x_{N-1}$.

Observe that the first $\tau$ elements are already orthonormal. During the orthogonalization of the rest $N$ elements we shall obtain an orthonormal set $\mathfrak{A}'_+ = \{g_j(\zeta)\}_{j=0}^{\delta-1}$. Observe that $\mathfrak{A}'_+$ is an orthonormal basis in $N_{\zeta}(A)$, $\zeta \in \overline{\Delta}$.

For the operator $A$ in the Hilbert space $H$ and an arbitrary $\zeta \in \overline{\Delta}$, we may construct the operators $S_{\zeta}, Q_{\zeta}$ from (20) with $V = A$. Let $M_{S_{\zeta}}$ ($M_{Q_{\zeta}}$) be the matrix of the operator $S_{\zeta}$ ($Q_{\zeta}$) with respect to the bases $\mathfrak{A}'_+$, $\mathfrak{A}_3$ (respectively, to the bases $\mathfrak{A}'_+, \mathfrak{A}'_3$):

$$M_{S_{\zeta}} = ((S_{\zeta}g_k(\zeta), u_j)_H)_{\tau \leq j \leq \tau + \delta - 1, 0 \leq k \leq \delta - 1}$$

$$= ((g_k(\zeta), u_j)_H)_{\tau \leq j \leq \tau + \delta - 1, 0 \leq k \leq \delta - 1}; \quad (25)$$
\[ M_{Q_\zeta} = \left( (Q_\zeta g_k(\zeta), v_j)_H \right)_{\tau \leq j \leq \tau + \delta - 1, \ 0 \leq k \leq \delta - 1}, \ \zeta \in \bar{\Delta}. \]  

(26)

Denote by \( \widetilde{W}_\zeta \) the matrix of the operator \( W_\zeta \) from (21) with respect to the bases \( \mathfrak{A}_3, \mathfrak{A}_3' \). Then

\[ \widetilde{W}_\zeta = \zeta^{-1}M_{Q_\zeta}M^{-1}_{\bar{S}_\zeta}, \ \zeta \in \bar{\Delta}. \]  

(27)

**Definition 1** Choose an arbitrary \( a \in \mathbb{N}, \Delta \subseteq \mathbb{T} \), and let \( Y_\zeta \) be an arbitrary \( C_{a \times a} \)-valued function, \( \zeta \in \bar{\Delta} \). By \( S(\mathbb{D}; C_{a \times a}; \Delta; Y) \) we denote a set of all functions \( G_\zeta \) from \( S(\mathbb{D}; C_{a \times a}) \) which satisfy the following conditions:

A) \( G_\zeta \) admits a continuation on \( \mathbb{D} \cup \bar{\Delta} \), and the continued function \( G_\zeta \) is continuous (i.e. each entry of \( G_\zeta \) is continuous);

B) \( G_\zeta^* G_\zeta = I_a \), for all \( \zeta \in \bar{\Delta} \);

C) The matrix \( G_\zeta - Y_\zeta \) is invertible for all \( \zeta \in \bar{\Delta} \).

We denote by \( S(\mathbb{D}; N_0(A), N_\infty(A); \Delta; W) \) a set of all functions \( \Phi_\zeta \) from \( S(\mathbb{D}; N_0(A), N_\infty(A)) \), which satisfy conditions 1)-3) of Theorem 3 with \( V = A \) and \( \bar{\Delta} \) instead of \( \Delta \).

Consider a transformation \( T \) which for an arbitrary function \( \Phi_\zeta \in S(\mathbb{D}; N_0(A), N_\infty(A)) \) put into correspondence the following \( C_{a \times a} \)-valued function \( F_\zeta \):

\[ F_\zeta = T\Phi = \left( (\Phi_\zeta u_k, v_j)_H \right)_{\tau \leq j, k \leq \tau + \delta - 1}, \ \zeta \in \bar{\mathbb{D}}. \]  

(28)

The transformation \( T \) is bijective, and it maps \( S(\mathbb{D}; N_0(A), N_\infty(A)) \) on the whole \( S(\mathbb{D}; C_{\delta \times \delta}) \). The following conditions: \( S(\mathbb{D}; N_0(A), N_\infty(A); \Delta; W) \neq \emptyset \), and \( S(\mathbb{D}; C_{\delta \times \delta}; \Delta; \bar{W}) \neq \emptyset \), are equivalent. If \( S(\mathbb{D}; C_{\delta \times \delta}; \Delta; \bar{W}) \neq \emptyset \), then

\[ TS(\mathbb{D}; N_0(A), N_\infty(A); \Delta; W) = S(\mathbb{D}; C_{\delta \times \delta}; \Delta; \bar{W}). \]  

(29)

All this can be checked directly by the definitions of the corresponding sets.

**Theorem 4** Let the indeterminate moment problem (7) with \( d \in \mathbb{N} \) be given and the operator \( A \) in a Hilbert space \( H \) be constructed as in (8). Let \( \Delta \) be an open subset of \( \mathbb{T} \), \( \Delta \neq \mathbb{T} \), and condition (b) of Proposition 4 be satisfied. The moment problem (7) has a solution \( M(x) \), \( x \in [0, 2\pi] \), such that \( M_\mathbb{T}(\Delta) = 0 \), if and only if \( S(\mathbb{D}; C_{\delta \times \delta}; \Delta; \bar{W}) \neq \emptyset \).

If \( S(\mathbb{D}; C_{\delta \times \delta}; \Delta; \bar{W}) \neq \emptyset \), then formula (4) establishes a one-to-one correspondence between all solutions \( M(x) \), \( x \in [0, 2\pi] \), of the moment problem (7), such that \( M_\mathbb{T}(\Delta) = 0 \), and all functions \( F_\zeta \in S(\mathbb{D}; C_{\delta \times \delta}; \Delta; \bar{W}) \).
Proof. By Proposition 4 we obtain that condition (22) holds for $V = A$ and with $\overline{\Delta}$ instead of $\Delta$.

Suppose that the moment problem (1) has a solution $M(x) \in [0, 2\pi]$, such that $M_T(\Delta) = 0$. By Proposition 4 we conclude that the corresponding generalized resolvent $R_\ast(A)$ admits an analytic continuation on $\mathbb{T}_e \cup \overline{\Delta}$. Let $\Phi_\zeta$ be the function from $\mathcal{S}(\mathbb{D}; N_0(A), N_\infty(A))$ which corresponds to $R_\ast(A)$ by Chumakin’s formula. By Theorem 5 we obtain that $\Phi_\zeta$ belongs to $\mathcal{S}(\mathbb{D}; N_0(A), N_\infty(A); \Delta; W)$. By (22) we get $\mathcal{S}(\mathbb{D}; \mathbb{C}_{\delta \times \delta}; \Delta; \overline{W}) \neq \emptyset$.

Conversely, suppose that $\mathcal{S}(\mathbb{D}; \mathbb{C}_{\delta \times \delta}; \Delta; \overline{W}) \neq \emptyset$. By (22) we can choose a function $\Phi_\zeta \in \mathcal{S}(\mathbb{D}; N_0(A), N_\infty(A); \Delta; W)$. Let $R_\ast(A)$ be the generalized resolvent of $A$ which corresponds to $\Phi_\zeta$ by Chumakin’s formula. By Theorem 5 we obtain that $R_\ast(A)$ admits an analytic continuation on $\mathbb{T}_e \cup \overline{\Delta}$. By Proposition 2 we conclude that $E(\Delta) = 0$. Finally, applying Proposition 4 we obtain that $M_T(\Delta) = 0$.

Consider the case $\mathcal{S}(\mathbb{D}; \mathbb{C}_{\delta \times \delta}; \Delta; \overline{W}) \neq \emptyset$.

Choose an arbitrary function $F_\zeta \in \mathcal{S}(\mathbb{D}; \mathbb{C}_{\delta \times \delta}; \Delta; \overline{W})$. Let $\Phi_\zeta \in \mathcal{S}(\mathbb{D}; N_0(A), N_\infty(A); \Delta; W)$ be such that $T\Phi_\zeta = F_\zeta$. Repeating the above arguments we conclude that for the corresponding solution $M(x) \in [0, 2\pi]$, it holds $M_T(\Delta) = 0$. By the construction of formula (4), for $F_\zeta$ it corresponds namely $M(x)$.

Conversely, choose an arbitrary solution $M(x) \in [0, 2\pi]$, of the moment problem (1) such that $M_T(\Delta) = 0$. Repeating the arguments at the beginning of the proof we obtain that $\Phi_\zeta$ belongs to $\mathcal{S}(\mathbb{D}; N_0(A), N_\infty(A); \Delta; W)$, where $\Phi_\zeta$ is related to the corresponding generalized resolvent by Chumakin’s formula. Then $F_\zeta = T\Phi_\zeta \in \mathcal{S}(\mathbb{D}; \mathbb{C}_{\delta \times \delta}; \Delta; \overline{W})$. Observe that $F_\zeta$ corresponds to $M(x)$ by formula (4).

The correspondence between solutions and functions from $\mathcal{S}(\mathbb{D}; \mathbb{C}_{\delta \times \delta}; \Delta; \overline{W})$ is one-to-one, as it follows from Theorem 1.

Example 2.1. Let $N = 3, d = 1$, $S_0 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $S_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

The moment problem (1) with moments $S_0, S_1$ was studied in [2]. We shall use orthonormal bases and other objects constructed therein. The following formula:

$$\int_0^{2\pi} \frac{1}{1 - \zeta e^{it}} dM_T(t) = \begin{pmatrix} \frac{1}{\zeta} & \frac{1}{\zeta} & 0 \\ \frac{1}{\zeta} & \frac{1}{\zeta} & 0 \\ 0 & 0 & 1 + \zeta^2 \frac{F_\zeta}{1 - \zeta^2 F_\zeta} \end{pmatrix}, \quad \zeta \in \mathbb{D},$$

establishes a one-to-one correspondence between all solutions $M(x), x \in \mathbb{D}$.
Let $\Delta = l(1,-1)$. Let us find solutions of the moment problem (1) which satisfy condition (5). Calculate the elements of the orthonormal bases $A^\zeta$ and $A^\zeta_+$:

$$g_0 = \frac{1}{\sqrt{2 - \zeta - \zeta^2}}(x_0 - \zeta x_3), \quad g_1 = \frac{1}{\sqrt{2}}(x_2 - \zeta x_5);$$

$$g_1 = \frac{1}{\sqrt{2}}(x_2 + \zeta x_5), \quad \zeta \in l(1,-1).$$

By (25), (26) we get:

$$\mathcal{M}_{Q^\zeta} = \frac{1}{\sqrt{2}}, \quad \mathcal{M}_{S^\zeta} = \frac{1}{\sqrt{2}} \zeta, \quad \zeta \in l(1,-1).$$

By (27) we get:

$$\tilde{W}_\zeta = \zeta^{-2}, \quad \zeta \in l(1,-1).$$

Observe that the function $F_\zeta \equiv 1$ belongs to the set $S(D; C_{1 \times 1}; l(1,-1); \zeta^{-2})$. By Theorem 4 the moment problem (1) with an additional constraint (5) is solvable. Formula (30) establishes a one-to-one correspondence between all solutions $M(x), x \in [0,2\pi]$, of the moment problem (1) with an additional constraint (5) and all functions $F_\zeta \in S(D; C_{1 \times 1}; l(1,-1); \zeta^{-2})$.

References

[1] Zagorodnyuk S. M. The truncated matrix trigonometric moment problem: the operator approach // Ukrainian Math. J. - 2011. - 63, no. 6. - P. 786–797.

[2] Zagorodnyuk S. M. Nevanlinna formula for the truncated matrix trigonometric moment problem // Ukrainian Math. J. - 2013. - 64, no. 8. - P. 1199–1214.

[3] Ando T., Truncated moment problems for operators // Acta Scientarum Math., (Szeged).- 1970.- 31, no. 4.- P.319–334.

[4] Berezanskii Ju. M. Expansions in Eigenfunctions of Selfadjoint Operators. - Amer. Math. Soc., Providence, RI, 1968. (Russian edition: Naukova Dumka, Kiev, 1965).
The truncated matrix trigonometric moment problem with an open gap.

S.M. Zagorodnyuk

This paper is a continuation of our previous investigations on the truncated matrix trigonometric moment problem in Ukrainian Math. J., 2011, 63, no. 6, 786-797, and Ukrainian Math. J., 2013, 64, no. 8, 1199-1214. In this paper we shall study the truncated matrix trigonometric moment problem with an additional constraint posed on the matrix measure $M_T(\delta)$, $\delta \in \mathcal{B}(\mathbb{T})$, generated by the sought function $M(x)$: $M_T(\Delta) = 0$, where $\Delta$ is a given open subset of $\mathbb{T}$ (called a gap). We present necessary and sufficient conditions for the solvability of the moment problem with a gap. All solutions of the moment problem with a gap can be constructed by a Nevanlinna-type formula.