GLOBAL SOLUTION TO THE RELATIVISTIC ENSKOG EQUATION
WITH NEAR-VACUUM DATA

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ABSTRACT. We give two hypotheses of the relativistic collision kernal and show the
existence and uniqueness of the global mild solution to the relativistic Enskog equation
with the initial data near the vacuum for a hard sphere gas.

1. INTRODUCTION

We will show the existence and uniqueness of the global mild solution to the relativistic
Enskog equation with near-vacuum initial data for a hard sphere gas. By using a similar
derivation as for the relativistic Boltzmann equation [4], the relativistic Enskog equation
can be obtained as follows (see [12]):

\[ \frac{\partial f}{\partial t} + \frac{\mathbf{p}}{p_0} \frac{\partial f}{\partial \mathbf{x}} = Q(f) \]  

(1.1)

where \( t \in [0, +\infty) \), \( \mathbf{x} \in \mathbb{R}^3 \), \( \mathbf{p} \in \mathbb{R}^3 \), \( p_0 = (1 + |\mathbf{p}|^2)^{1/2} \) and \( Q(f) = Q(f)(t, \mathbf{x}, \mathbf{p}) \) is the
relativistic Enskog collision operator which is expressed by the difference between the gain
and loss terms respectively defined by

\[ Q^+(f)(t, \mathbf{x}, \mathbf{p}) = \frac{a^2}{p_0} \int_{\mathbb{R}^3} \int_{S_+^2} d\omega F^+(\mathbf{f})(t, \mathbf{x}, \mathbf{p}') f(t, \mathbf{x} + a\omega, \mathbf{p}') B(g, \theta) \]  

(1.2)

\[ Q^-(f)(t, \mathbf{x}, \mathbf{p}) = \frac{a^2}{p_0} \int_{\mathbb{R}^3} \int_{S_+^2} d\omega F^-(\mathbf{f})(t, \mathbf{x}, \mathbf{p}) f(t, \mathbf{x} - a\omega, \mathbf{p}) B(g, \theta) \]  

(1.3)

where \( f = f(t, \mathbf{x}, \mathbf{p}) \) is a distribution function of a one-particle classical relativistic gas
without external forces, \( a \) is a diameter of hard sphere \( (a > 0) \) and \( F^\pm \) are two functionals
on a Banach space \( M \) defined in Section 2. More precisely speaking, \( F^\pm \) are high-density
collision frequencies which are physically defined by a geometrical factor \( Y \) that depends
on the space density \( \rho(t, \mathbf{x}) = \int_{\mathbb{R}^3} f(t, \mathbf{x}, \mathbf{p}) d^3\mathbf{p} \) at the time \( t \) and the point of contact
[e.g. \( F^+ = Y(\rho(t, \mathbf{x} + a\omega/2)) \) and \( F^- = Y(\rho(t, \mathbf{x} - a\omega/2)) \)]. The other different parts in
equations (1.2) and (1.3) are explained below.

\( \mathbf{p} \) and \( \mathbf{p}_1 \) are dimensionless momenta of two colliding particles immediately before
collision while \( \mathbf{p}' \) and \( \mathbf{p}'_1 \) are dimensionless momenta after collision; \( p_0 = (1 + |\mathbf{p}|^2)^{1/2} \)
and \( p_{10} = (1 + |\mathbf{p}_1|^2)^{1/2} \) are the dimensionless energy of two colliding particles
immediately before collision while \( p_0' = (1 + |\mathbf{p}'|^2)^{1/2} \) and \( p_{10}' = (1 + |\mathbf{p}'_1|^2)^{1/2} \) are the dimensionless
energy after collision. \( \mathbb{R}^3 \) is a three-dimensional Euclidean space and \( S_+^2 = \{ \omega \in S^2 : \omega(\mathbf{p}_1/p_{10} - \mathbf{p}/p_0) \geq 0 \} \) a subset of a unit sphere surface \( S^2 \). \( B(g, \theta) \) is given by
\( B(g, \theta) = g s^2 \sigma(g, \theta)/2 \), where \( \sigma(g, \theta) \) is a scattering cross section, \( s = |p_{10} + p_0|^2 - |\mathbf{p}_1 + \mathbf{p}|^2 \)
\( g = \sqrt{|\mathbf{p}_1|^2 - |p_{10} - p_0|^2}/2 \), \( \theta \) is the scattering angle defined in \([0, \pi]\) by \( \cos \theta = 1 - 2(p_0 - p_{10})(p_0 - p'_0) - (\mathbf{p} - \mathbf{p}_1)(\mathbf{p} - \mathbf{p}')/(4 - s) \). Obviously, \( s = 4 + 4g^2 \). \( \omega = (\cos \psi \sin \theta, \sin \psi \sin \theta, \cos \theta) \) varies on \( S_+^2 \), where \( 0 \leq \theta \leq \pi, 0 \leq \psi \leq 2\pi \). Put \( \mathbf{p}' = \mathbf{p} + q \omega \) and

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\( \mathbf{p}'_1 = \mathbf{p} - q \omega \). Then, by using a similar derivation as given by Glassey and Strauss \[11\] for the relativistic Boltzmann equation, we have

\[
q \equiv q(\mathbf{p}, \mathbf{p}_1, \omega) = \frac{2(p_0 + p_{10})p_0 p_{10}[\omega(\mathbf{p}_1/p_{10} - \mathbf{p}/p_0)]}{(p_0 + p_{10})^2 - [\omega(\mathbf{p} + \mathbf{p}_1)]^2},
\]

(1.4)

\[
g/\sqrt{s} = \frac{8(p_0 + p_{10})^2[\omega(\mathbf{p}_1/p_{10} - \mathbf{p}/p_0)]}{(p_0 + p_{10})^2 - [\omega(\mathbf{p} + \mathbf{p}_1)]^2}.
\]

(1.5)

Here the scattering angle \( \theta \) can be regarded as a function of the variables \( \mathbf{p}, \mathbf{p}_1 \) and \( \omega \), i.e., \( \theta \equiv \theta(\mathbf{p}, \mathbf{p}_1, \omega) \). \( \mathbf{p}' \) and \( \mathbf{p}'_1 \) are bounded for bounded pre-collisional momenta and lie on an ellipsoid when they are plotted in a plane, see detail explanation in \[1\].

The Moller velocity is defined as \( v_M = gs^{\frac{1}{2}}/(p_0 p_{10}) \), thus it can be found that \( v_M^2 = |\mathbf{p}/p_0 - \mathbf{p}_1/p_{10}|^2 - |\mathbf{p} \times \mathbf{p}_1/(p_0 p_{10})|^2 \) and that \( v_M \leq |\mathbf{p}/p_0 - \mathbf{p}_1/p_{10}| \).

As a comparison the relativistic Boltzmann equation is the relativistic Enskog equation with the factor \( a^2 F^\pm \) constant and the diameter \( a \) equal to zero in the density variables. Boltzmann’s equation provides a successful description for dilute gases and is no longer valid when the density of the gas increases. The Enskog equation proposed by Enskog \[8\] in 1922 is a modification of the Boltzmann equation to explain the dynamical behavior of the density profile of a moderately dense gas. It is thus a suitable idea that the relativistic Enskog equation is used to model a hard sphere relativistic gas.

There are many results about the relativistic Boltzmann equation, such as global existence proof of Dudyński & Ekiel-Jeżewska \[6,7\] and properties of the relativistic collision operator given by Glassey & Strauss \[10\], and background information of the classical Enskog equation may be found in \[2,8,14\]. An existence and uniqueness theorem has been given by Galeano et. al \[12\] for the global solution to the relativistic Enskog equation with data near the vacuum for a hard sphere gas. In their theorem (see Theorem 3.2 in \[12\]), a set \( M_R \) is defined by \( M_R = \{ f \in M : ||f|| \leq R \} \) with

\[
R^2 < \beta^4 |v|/(16 \pi^2 c L a)
\]

and an initial datum \( f_0 \) satisfies

\[
||f_0|| < R e^{\beta |x|^2}/2,
\]

where \( M \) is defined by

\[
M = \left\{ f \in C([-0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3) : \text{there exists } c > 0 \text{ such that} ||f(t, x, v)| \leq c e^{-\beta(\sqrt{1+|v|^2+|x+tv|^2})} \right\}
\]

with a norm

\[
||f|| = \sup_{t,x,v} \{ e^{\beta(\sqrt{1+|v|^2+|x+tv|^2})}|f(t, x, v)| \},
\]

c, L, a and \( \beta \) are positive constants, \( t \) is a time variable in \([0, \infty)\), \( x \) and \( v \) are space and momentum variables in \( \mathbb{R}^3 \) respectively. These assumptions imply that \( M_R \) is an empty set and that none of the initial datum \( f_0 \) occurs. Below let us prove this claim. We first prove that \( M_R \) is an empty set. Assume that \( M_R \) is not an empty set. Then there exists a function \( f \) in \( M_R \) that \( ||f|| \leq R \). Thus \( R \geq 0 \). By assumption (1.6), we know that

\[
R < \sqrt{\beta^4 |v|/(16 \pi^2 c L a)}.
\]

(1.8)

Since \( v \) is a momentum variables in \( \mathbb{R}^3 \), by setting \( v = 0 \), (1.8) shows that \( R < 0 \) as \( v = 0 \). This is in contradiction with \( R \geq 0 \). Hence \( M_R \) is an empty set. Next, we show that none of the initial datum \( f_0 \) occurs. Assume that there exists a function \( f_0 \) satisfying (1.7). It can be known from (1.7) that \( R > 0 \). By (1.6), (1.8) then follows. Since \( v \) is a momentum variables in \( \mathbb{R}^3 \), by letting \( v = 0 \), (1.8) shows that \( R < 0 \) as \( v = 0 \). This is in contradiction with \( R > 0 \). None of the initial datum \( f_0 \) hence occurs. If (1.6) and
\( (1.7) \) are replaced with \( R^2 \leq \beta^4|v|/(16\pi^2cLa) \) and \( \|f_0\| \leq Re^{\beta|x|^2}/2 \) respectively, then \( M_R = \{0\} \) and \( f_0 = 0 \). Thus one can deduce that a unique solution to the relativistic Enskog equation in their theorem is in fact zero. Hence this result is also trivial. Notice that \( v \) is a variable in \( \mathbb{R}^3 \) and that \( (1.6) \) holds for all \( v \in \mathbb{R}^3 \). If \( |v| \) in \( (1.6) \) is replaced with a positive constant \( v_0 \), that is, \( (1.6) \) is changed as \( R^2 < \beta^4v_0/(16\pi^2cLa) \), then the problem is non-trivial. Recently, global existence of mild solutions has been proved by Glassey \[9\] for the relativistic Boltzmann equation with near-vacuum data and many relevant papers of both classical and relativistic cases can be found in the reference. Now there is not yet this result for the relativistic Enskog equation. The aim of this paper is to extend this result into the case of the relativistic Enskog equation. In Section 2 two hypotheses of the relativistic collision kernel are given and a Banach space and its operators are constructed. Then an existence and uniqueness theorem of global mild solution to the relativistic Enskog equation with near-vacuum data is given in Section 3.

2. Hypotheses and Operators

Let us begin by assuming that there is a positive function \( m(x, p) \) such that a non-negative function \( B(g, \theta) = gs^4\sigma(g, \theta)/2 \) satisfies two hypotheses as follows:

\[
\frac{1}{p_0} \int_0^t d\tau \int_{\mathbb{R}^3} \frac{d^3p_1}{p_{10}} \int_{S^2_+} d\omega m(x + \tau p/p_0 - \tau p'/p_{10}, p') \times m(x + a\omega + \tau p/p_0 - \tau p_1'/p_{10}, p') B(g, \theta) \leq m(x, p)K, \tag{2.1}
\]

\[
\frac{1}{p_0} \int_0^t d\tau \int_{\mathbb{R}^3} \frac{d^3p_1}{p_{10}} \int_{S^2_+} d\omega m(x - a\omega + \tau p/p_0 - \tau p_1/p_{10}, p_1) B(g, \theta) \leq K. \tag{2.2}
\]

for any \( x \in \mathbb{R}^3, p \in \mathbb{R}^3, t \in \mathbb{R}^+ \) and some positive constant \( K \). It can be known from the recent work of Glassey \[9\] that there exist two such functions \( m(x, p) \) and \( B(g, \theta) \) satisfying \( (2.1) \) and \( (2.2) \). For example, as given by Glassey \[9\], we assume that

\[
m(x, p) = (1 + |x \times p|)^{-\left(1+\delta\right)/2}e^{-p_0}, \tag{2.3}\]

\[
\sigma \equiv \sigma(p, p_1, \omega) = |\omega(p_1 \times p)|\tilde{\sigma}(\omega)/|p_{10}g(1 + g^2)^{\delta+1/2}|, \tag{2.4}\]

for any fixed \( \delta \in (0, 1) \), where \( \tilde{\sigma}(\omega) \) is a non-negative, bounded and continuous function such that \( \int_{S^2_+} \tilde{\sigma}(\omega)/(1 + |z\omega|) d\omega \leq c_0|z|^{-1} \) for some positive constant \( c_0 \) and every non-zero element \( z \in \mathbb{R}^3 \). Thus a similar integral estimate to that developed by Glassey \[9\] leads to the fact that \( (2.1) \) and \( (2.2) \) hold if \( m(x, p) \) and \( B(g, \theta) \) are defined by \( (2.3) \) and \( (2.4) \) respectively. This indicates that our assumptions \( (2.1) \) and \( (2.2) \) are valid for the relativistic Enskog equation.

Then we can construct a subset \( M \) of a Banach space \( C([0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3) \), which has the property that every element \( f = f(t, x, p) \in M \) if and only if there exists a positive constant \( c \) such that \( f \) satisfies \( |f^\#(t, x, p)| \leq cm(x, p) \), where and below everywhere, \( f^\# \) is expressed as \( f^\#(t, x, p) = f(t, x + tp/p_0, p) \) for any measurable function \( f \) on \( (0, +\infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \). It follows that \( M \) is a Banach space when it has a norm \( \|f\| = \sup_{t, x, p} \{|f^\#(t, x, p)|m^{-1}(x, p)\} \). This space will be used below.

The relativistic Enskog equation \( (1.1) \) can be also written as

\[
\frac{d}{dt} f^\#(t, x, p) = Q(f)^\#(t, x, p),
\]
which leads to the following integral equation
\[
 f^\#(t, x, p) = f_0(x, p) + \int_0^t Q(f)^\#(\tau, x, p)d\tau. \tag{2.5}
\]
A function \( f(t, x, p) \) is called global mild solution to the Enskog equation \([11]\) if \( f(t, x, p) \) satisfies the above integral equation \((2.5)\) for almost every \((t, x, p) \in [0, +\infty) \times \mathbb{R}^3 \times \mathbb{R}^3\). The definition of the term “mild solution” also appears in the famous work of DiPerna and Lions \([5]\) where they show a global existence proof for the classical Boltzmann equation.

By \((1.2)\) and \((1.3)\), \( Q(f)^\#(t, x, p) \) can be rewritten as the difference between the gain and loss terms of two other forms
\[
 Q^+(f)^\#(t, x, p) = \frac{a^2}{p_0} \int_{\mathbb{R}^3} \frac{d^3P_1}{p_{10}} \int_{S^2_+} d\omega F^+(f) f^\#(t, x + tp/p_0 - tp'/p_{00}, p') \times f^\#(t, x + \omega + tp/p_0 - tp'/p_{10}, p_1) B(g, \theta), \tag{2.6}
\]
\[
 Q^-(f)^\#(t, x, p) = \frac{a^2}{p_0} \int_{\mathbb{R}^3} \frac{d^3P_1}{p_{10}} \int_{S^2_+} d\omega F^-(f) f^\#(t, x - \omega + tp/p_0 - tp_1/p_{10}, p_1) B(g, \theta). \tag{2.7}
\]

According to \((2.6)\) and \((2.7)\), we can finally build a Banach space \( \tilde{M} \) defined by \( \tilde{M} = \{ f^\# : f \in M \} \) with a norm \( \| f^\# \| = \| f \| \) and an operator \( J \) on \( \tilde{M} \) by
\[
 J(f)^\# = f_0(x, p) + \int_0^t Q(f)^\#(\tau, x, p)d\tau, \tag{2.8}
\]
since \( F^\pm \) can be in fact regarded as two functionals on \( \tilde{M} \).

### 3. Existence and Uniqueness

Let \( M_R \) be denoted by \( M_R = \{ f \in M : \| f \| \leq R \} \) for any \( R \in \mathbb{R}_+ \), where \( M \) is given in Section \( 2 \). We first have the following lemma:

**Lemma 3.1.** Suppose that the conditions \((2.7)\) and \((2.2)\) hold and that \( F^\pm \) are two functionals on \( M_R \) such that \( |F^+(f) - F^+(g)| \leq L(R)\| f - g \| \) for any \( f, g \in M_R \) where \( L(R) \) is a positive nondecreasing function on \( \mathbb{R}_+ \). Then
\[
 \int_0^t \| Q^+(f)^\#(\tau, x, p) \| d\tau \leq C(R)m(x, p)\| f \|^2,
\]
\[
 \int_0^t \| Q^-(f)^\#(\tau, x, p) \| d\tau \leq C(R)m(x, p)\| f \|^2
\]
for any \( f \in M_R \), where \( C(R) \) is a positive nondecreasing function on \( \mathbb{R}_+ \).

**Proof.** It can be first found from the assumption of the two functionals \( F^\pm \) that there exists a positive constant \( \tilde{L}(R) = L(R)R + |F^+(0)| + |F^-(0)| \) such that \( |F^+(f)| \leq \tilde{L}(R) \) for any \( f \in M_R \). It follows from \((2.6)\) and \((2.7)\) that
\[
 \int_0^t Q^+(f)^\#(\tau, x, p)d\tau \leq \frac{\tilde{L}(R)a^2}{p_0} \int_0^t d\tau \int_{\mathbb{R}^3} \frac{d^3P_1}{p_{10}} \int_{S^2_+} d\omega \| f \|^2 \times m(x + \tau p/p_0 - \tau p'/p_{00}, p_1)m(x + \omega + \tau p/p_0 - \tau p'_1/p_{10}, p_1) B(g, \theta), \tag{3.1}
\]
\[
 \int_0^t Q^-(f)^\#(\tau, x, p)d\tau \leq \frac{\tilde{L}(R)a^2}{p_0} \int_0^t d\tau \int_{\mathbb{R}^3} \frac{d^3P_1}{p_{10}} \int_{S^2_+} d\omega \| f \|^2 \times m(x, p)m(x - \omega + \tau p/p_0 - \tau p'_1/p_{10}, p_1) B(g, \theta). \tag{3.2}
\]
By (2.1) and (2.2), (3.1) and (3.2) give
\[\int_0^t Q^+(f)(\tau, x, p)d\tau \leq \tilde{L}(R)a^2 Km(x, p)||f||^2,\]
\[\int_0^t Q^-(f)(\tau, x, p)d\tau \leq \tilde{L}(R)a^2 Km(x, p)||f||^2.\]
Take $C(R) = \tilde{L}(R)a^2 K$. It follows obviously that Lemma 3.1 holds.

Then we can get the following theorem:

**Theorem 3.2.** Suppose that the conditions (2.1) and (2.2) hold and that $F^\pm$ are two functionals on $M_R$ such that $|F^\pm(f) - F^\pm(g)| \leq L(R)||f - g||$ for any $f, g \in M_R$ where $L(R)$ is a positive nondecreasing function on $R_+$. Then there exists a positive constant $R_0$ such that the relativistic Enskog equation (1.1) has a unique non-negative global mild solution $f = f(t, x, p) \in M_{R_0}$ through a non-negative initial data $f_0 = f_0(x, p)$ when sup$\{f_0(x, p)m^{-1}(x, p)\}$ is sufficiently small.

Theorem 3.2 shows that there exists a unique global mild solution to the relativistic Enskog equation (1.1) with the initial data near vacuum if a suitable assumption of the scattering kernel is given. Below is our proof of Theorem 3.2.

**Proof.** We first define a set $\tilde{M}_R$ by $\tilde{M}_R = \{f^\#: |||f^\#||| \leq R, f^\# \in \tilde{M}\}$, where $\tilde{M}$ is given in Section 2. By (2.8) and Lemma 3.1 we have
\[||J(f^\#)||m^{-1}(x, p) \leq ||f_0(x, p)||m^{-1}(x, p) + 2C(R)||f||^2 \leq R/2 + 2C(R)R^2\]
for any $f^\# \in \tilde{M}_R$ and $f_0$ with $||f_0|| \leq R/2$. Since $C(R)$ is a positive nondecreasing function on $R_+$, it follows that $|||J(f^\#)||| \leq R$ for sufficiently small $R$. Therefore $J$ is an operator from $\tilde{M}_R$ to itself for sufficiently small $R$. Similarly, it can be also found that $J$ is a contractive operator on $\tilde{M}_R$ for some suitably small $R$. Thus there exists a unique element $f^\# \in \tilde{M}_R$ such that $f^\# = J(f^\#)$, i.e., (2.5) holds. It then follows from the same argument as the one in [3] (or see [9, 13, 15]) that if $f_0(x, p) \geq 0$ then $f(t, x, p) \geq 0$. Hence the proof of Theorem 3.2 is finished.

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