Rotating Black String and Effective Teukolsky Equation in Braneworld

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In the Randall-Sundrum two-brane model (RS1), a Kerr black hole on the brane can be naturally identified with a section of rotating black string. To estimate Kaluza-Klein (KK) corrections on gravitational waves emitted by perturbed rotating black strings, we give the effective Teukolsky equation on the brane which is separable equation and hence numerically manageable. In this process, we derive the master equation for the electric part of the Weyl tensor $E_{\mu\nu}$ which would be also useful to discuss the transition from black strings to localized black holes triggered by Gregory-Laflamme instability.

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I. INTRODUCTION

The recent progress of superstring theory has provided a new picture of our universe, the so-called braneworld. The evidence of the extra dimensions in this scenario should be explored in the early stage of the universe or the black hole. In particular, gravitational waves are key probes because they can propagate into the bulk freely. Cosmology in this scenario has been investigated intensively. While, gravitational waves from black holes have been less studied so far. In this paper, we shall take a step toward this direction.

Here, we will concentrate on a two-brane model which is proposed by Randall and Sundrum as a simple and phenomenologically interesting model [1]. In this RS1 model, the large black hole on the brane is expected to be black string. Hence, it would be important to clarify how the gravitational waves generated in the perturbed bulk black string system and how the effects of the extra dimensions come into the observed signal of the gravitational waves. It is desired to have a basic formalism for analyzing gravitational waves generated by perturbed rotating black string.

It is well known in general relativity that the perturbation around Kerr black hole is elegantly treated in the Newman-Penrose formalism [2]. Indeed, Teukolsky derived a separable master equation for the gravitational waves in the Kerr black hole background [3]. The main purpose of this paper is to extend the Teukolsky formalism to the braneworld context and derive the effective Teukolsky equation.

The organization of this paper is as follows. In sec.II, we present the model and demonstrate the necessity of solving $E_{\mu\nu}$ in deriving the effective Teukolsky equation. In sec.III, a perturbed equation and the junction conditions for $E_{\mu\nu}$ are obtained. In sec.IV, we give the explicit solution for $E_{\mu\nu}$ using the gradient expansion method. Then, the effective Teukolsky equation is presented. The final section is devoted to the conclusion. In the appendix A, the formal solution for $E_{\mu\nu}$ is presented.

II. TEUKOLSKY EQUATION ON THE BRANE

Based on the Newman-Penrose (NP) null-tetrad formalism, in which the tetrad components of the curvature tensor are the fundamental variables, a master equation for the curvature perturbation was developed by Teukolsky for a Kerr black hole with source. The master equation is called the Teukolsky equation, and it is a wave equation for a null-tetrad component of the Weyl tensor $\Psi_0$ or $\Psi_4$, where $C_{pqrs}$ is the Weyl tensor and $\ell, n, m, \bar{m}$ are null basis in the NP formalism. All information about the gravitational radiation flux at infinity and at the event horizon can be extracted from $\Psi_0$ and $\Psi_4$. The Teukolsky equation is constructed by combining the Bianchi identity with the Einstein equations. The Riemann tensor in the Bianchi identity is written in terms of the Weyl tensor and the Ricci tensor. The Ricci tensor is replaced by the matter fields using the Einstein equations. In this way, the Bianchi identity becomes no longer identity and one can get a master equation in which the curvature tensor is the fundamental variable [4].

It is of interest to consider the four-dimensional effective Teukolsky equation in braneworld to investigate gravitational waves from perturbed rotating black strings. We consider an $S_1/Z_2$ orbifold space-time with the two branes as the fixed points. In this RS1 model, the two 3-branes are embedded in AdS$_5$ with the curvature radius $\ell$ and the brane tensions given by $\sigma_{\parallel} = 6/(\kappa^2 \ell)$ and $\sigma_{\perp} = -6/(\kappa^2 \ell)$. Our system is described by the action

$$S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \left( \mathcal{R} + \frac{12}{\ell^2} \right) - \sum_{i=\parallel,\perp} \int d^4x \sqrt{-g^{i\text{brane}}} \sigma_i + \sum_{i=\parallel,\perp} \int d^4x \sqrt{-g^{i\text{brane}}} E^{i\text{matter}}_{\text{matter}},$$

(1)
where $g_{\mu \nu}$, $R_{\mu \nu}$, $g_{\mu \nu}^{\text{brane}}$, and $\kappa^2$ are the 5-dimensional metric, the 5-dimensional scalar curvature, the induced metric on the $i$-brane, and the 5-dimensional gravitational constant, respectively.

Let $n$ be a unit normal vector field to branes. Using the extrinsic curvature $K_{\mu \nu} = -(1/2) \mathcal{L}_n g_{\mu \nu}$, 5-dimensional Einstein equations in the bulk become

$$\frac{1}{2} \frac{d}{d\tau} R + \frac{1}{2} K^2 - \frac{1}{2} K^{\alpha \beta} K_{\alpha \beta} = \frac{6}{\ell^2} , \quad (2)$$

$$K_{\mu \nu} - n_{\mu} n_{\nu} = 0 , \quad (3)$$

$$G_{\mu \nu} = -E_{\mu \nu} + \frac{3}{\ell^2} \delta_{\mu \nu} - K^{\alpha \nu} K_{\alpha \mu}$$

$$+ K K_{\mu \nu} \left( \frac{1}{2} \partial^\alpha (K_{\alpha \beta} - K^2) \right) , \quad (4)$$

where “the electric part” of the Weyl tensor

$$E_{\mu \nu} = \mathcal{L}_n K_{\mu \nu} + K_{\mu}^{\alpha} K_{\alpha \nu} + \frac{1}{\ell^2} g_{\mu \nu}$$

(5)

is defined. Here, (4) represents the 4-dimensional quantity and $\parallel$ is the covariant derivative with respect to the metric $g_{\mu \nu}(y, x^\mu)$. As the branes act as the singular sources, we also have the junction conditions

$$[K^\mu_{\nu} - \delta^\mu_{\nu} K] \bigg|_{\parallel} = \frac{\kappa^2}{2} \left( -\partial^\mu \partial^\nu + T^\mu_{\nu} \right) , \quad (6)$$

$$[K^\mu_{\nu} - \delta^\mu_{\nu} K] \bigg|_{\parallel} = -\frac{\kappa^2}{2} \left( -\delta^\mu_{\nu} + T^\mu_{\nu} \right) . \quad (7)$$

Since the Bianchi identity is independent of dimensions, what we need is the projected Einstein equations on the brane derived by Shiromizu, Maeda and Sasaki [3]. The first order perturbation of the projected Einstein equation is

$$G_{\mu \nu} = 8\pi G T_{\mu \nu} - \delta E_{\mu \nu} , \quad (8)$$

where $8\pi G = \kappa^2/\ell$. If we replace the Ricci curvature in the Bianchi identity to the matter fields and $E_{\mu \nu}$ using Eq. (8), then the projected Teukolsky equation on the brane is written in the following form,

$$[(\Delta + 3 \gamma - \gamma^* + 4 \mu + \mu^*)(D + 4 \epsilon - \rho) - (\delta^* - \tau^* + \beta^* + 3 \alpha + 4 \pi)(\delta - \tau + 4 \beta)] - 3 \Psi_2] \delta \Psi_4$$

$$= \frac{1}{2} (\Delta + 3 \gamma - \gamma^* + 4 \mu + \mu^*)$$

$$\times [(\delta^* - 2 \tau^* + 2 \alpha)(8\pi G T_{m* m*} - \delta E_{m* m*})$$

$$- (\Delta + 2 \gamma - 2 \gamma^* + \mu^*)(8\pi G T_{m* m*} - \delta E_{m* m*})]$$

$$+ \frac{1}{2} (\delta^* - \tau^* + \beta^* + 3 \alpha + 4 \pi)$$

$$\times [(\Delta + 2 \gamma + 2 \mu^*)(8\pi G T_{m* m*} - \delta E_{m* m*})$$

$$- (\delta^* - \tau^* + 2 \beta^* + 2 \alpha)(8\pi G T_{m* m*} - \delta E_{m* m*})] . \quad (9)$$

Here our notation follows that of [3]. We see the effects of a fifth dimension, $\delta E_{\mu \nu}$, is described as a source term in the projected Teukolsky equation. It should be stressed that the projected Teukolsky equation on the brane Eq. (9) is not a closed system yet. One must solve the gravitational field in the bulk to obtain $\delta E_{\mu \nu}$.

### III. Master Equation for $\delta E_{\mu \nu}$

To solve $\delta E_{\mu \nu}$, we must start with the 5-dimensional Bianchi identities. Using the Gauss equation and the Codacci equation, we obtain

$$\mathcal{L}_n B_{\mu \nu \lambda} + E_{\mu \nu}^{(\parallel)} - E_{\mu \lambda}^{(\parallel)}$$

$$+ K^{\alpha \nu} B_{\alpha \mu \lambda} + K^{\alpha \lambda} B_{\nu \alpha \mu} - K^{\alpha \nu} B_{\lambda \alpha \mu} = 0 \ , \quad (10)$$

$$B_{\mu [\nu \lambda \rho]} + K^{\sigma \rho} R_{\nu \lambda \sigma \rho} = 0 \ , \quad (11)$$

$$\mathcal{L}_n R_{\mu \nu \lambda \rho} + K^{\alpha \nu} R_{\alpha \lambda \rho} - K^{\alpha \lambda} R_{\nu \alpha \rho}$$

$$+ B_{\mu \lambda \nu \rho} - B_{\mu \rho \lambda \nu} = 0 \ , \quad (12)$$

$$\mathcal{L}_n R_{\nu [\lambda \rho \sigma]} = 0 \ , \quad (13)$$

where we have defined “the magnetic part” of the Weyl tensor

$$B_{\mu \nu \lambda} = K_{\mu \lambda \nu} - K_{\mu \nu \lambda} \ . \quad (14)$$

After combining (12) with (11) and putting the result into Eq. (12), we have

$$\mathcal{L}_n E_{\mu \nu} = B_{(\mu \nu) \rho} - \Sigma^{(4)}_{\mu \nu \alpha \beta}$$

$$\Sigma_{\mu \nu \alpha \beta} + \frac{1}{2} g_{\mu \nu} E_{\alpha \beta} \Sigma_{\alpha \beta}$$

$$- 2(\Sigma_{\mu \nu} E_{\alpha \nu} + \Sigma_{\nu \alpha} E_{\mu \nu})$$

$$+ \frac{1}{2} K E_{\mu \nu} + \frac{1}{2} g_{\mu \nu} \Sigma_{\alpha \beta} \Sigma_{\alpha \beta}$$

$$- 2 \Sigma_{\mu \nu} \Sigma_{\alpha \beta} \Sigma_{\beta \gamma} \Sigma_{\gamma \alpha}$$

$$+ \frac{1}{2} g_{\mu \nu} \Sigma_{\alpha \beta} \Sigma_{\beta \gamma} \Sigma_{\gamma \alpha} \ , \quad (15)$$

where we decomposed the extrinsic curvature into the traceless part and the trace part

$$K_{\mu \nu} = \Sigma_{\mu \nu} + \frac{1}{4} g_{\mu \nu} K \ . \quad (16)$$

Now we consider the perturbation of these equations. The background we consider is a Ricci flat string without source ($T_{\mu \nu} = 0$) whose metric is written as

$$ds^2 = dy^2 + e^{-2\Phi} g_{\mu \nu}(x^\mu) dx^\mu dx^\nu \ , \quad (17)$$

where $g_{\mu \nu}(x^\mu)$ is supposed to be the Ricci flat metric [4]. The variables $K_{\mu \nu}$, $E_{\mu \nu}$ and $B_{\mu \nu}$ for this background are given by

$$K_{\mu \nu} = \frac{1}{\ell} g_{\mu \nu} \ , \quad E_{\mu \nu} = B_{\mu \nu \lambda} = 0 \ . \quad (18)$$

Linearizing Eq. (15) around this background, we obtain

$$\delta E_{\mu \nu, y} = \delta B_{(\mu \nu) \rho}^{(4)} - \Sigma^{(4)}_{\mu \nu \alpha \beta} \delta \Sigma_{\alpha \beta} + \frac{1}{2} \delta E_{\mu \nu} \ . \quad (19)$$

Similarly, Eq. (12) reduces to

$$\delta B_{(\mu \nu) \alpha}^{(4)} = -\delta E_{\mu \nu | \lambda} + \delta E_{\mu \lambda | \nu} \ . \quad (20)$$
Using the following relation
\[ (\delta B_{(\mu\nu)\lambda})_{y} = (\delta B_{(\mu\nu)\lambda})_{y} - \frac{2}{\ell} \delta B_{(\mu\nu)\lambda}, \tag{21} \]
we can eliminate \( B_{(\mu\nu)\lambda} \) from Eqs. (19) and (20). Then, the equation of motion for \( \delta E_{\mu\nu} \) in the bulk is found as
\[ \left( \partial_{y}^{2} - \frac{4}{\ell} \partial_{y} + \frac{4}{\ell^{2}} \right) \delta E_{\mu\nu} = -e^{2\hat{\gamma}} L_{\mu\nu}^{\alpha\beta} \delta E_{\alpha\beta} \]
\[ = -e^{2\hat{\gamma}} \hat{L} \delta E_{\mu\nu}, \tag{22} \]
where \( \hat{L}_{\mu\nu}^{\alpha\beta} \) stands for the Lichnerowicz operator,
\[ \hat{L}_{\mu\nu}^{\alpha\beta} = \Box \delta_{\mu}^{\alpha} \delta_{\nu}^{\beta} + 2 R_{\mu}^{\alpha} \nu^{\beta}. \tag{23} \]
Here, the covariant derivative and the Riemann tensor are constructed from \( g_{\mu\nu}(x) \).

In order to deduce the junction conditions for \( \delta E_{\mu\nu} \) from Eqs. (19) and (20), we use Eq. (19) and
\[ \delta B_{\mu\nu\rho} = \delta K_{\mu\nu\rho} - \delta K_{\mu\rho\nu}. \tag{24} \]
As a result, the junction conditions on each branes become
\[ e^{2\hat{\gamma}} \left[ e^{-2\hat{\gamma}} \delta E_{\mu\nu}(y) \right]_{y=0} = -\frac{\kappa^{2}}{6} \Box [\mu_{\nu} - \frac{2}{\ell} \hat{L}_{\mu\nu}^{\alpha\beta} \left( \hat{T}_{\alpha\beta} - \frac{1}{3} g_{\alpha\beta} \hat{T} \right)], \tag{25} \]
\[ e^{2\hat{\gamma}} \left[ e^{-2\hat{\gamma}} \delta E_{\mu\nu}(y) \right]_{y=d} = \frac{\kappa^{2}}{6} \Box [\mu_{\nu} + \frac{2}{\ell} \hat{L}_{\mu\nu}^{\alpha\beta} \left( \hat{T}_{\alpha\beta} - \frac{1}{3} g_{\alpha\beta} \hat{T} \right)]. \tag{26} \]
Let \( \phi(x) \) and \( \bar{\phi}(x) \) be the scalar fields on each branes which satisfy
\[ \Box \phi = \kappa^{2} \frac{\Box}{6}, \tag{27} \]
\[ \Box \bar{\phi} = \kappa^{2} \frac{\Box}{6}, \tag{28} \]
respectively. In the Ricci flat space-time, the identity
\[ (\Box \phi)_{\mu\nu} = \hat{L}_{\mu\nu}^{\alpha\beta} \phi_{\alpha\beta} \tag{29} \]
holds. Thus, Eqs. (25) and (26) can be rewritten as
\[ e^{2\hat{\gamma}} \left[ e^{-2\hat{\gamma}} \delta E_{\mu\nu}(y) \right]_{y=0} = -\hat{L} \left( \phi_{\mu\nu} - g_{\mu\nu} \phi \right) - \frac{\kappa^{2}}{2} \hat{L} T_{\mu\nu} \equiv \hat{L} S_{\mu\nu}, \tag{30} \]
\[ e^{2\hat{\gamma}} \left[ e^{-2\hat{\gamma}} \delta E_{\mu\nu}(y) \right]_{y=d} = \hat{L} \left( \phi_{\mu\nu} - g_{\mu\nu} \phi \right) + \frac{\kappa^{2}}{2} \hat{L} T_{\mu\nu} \equiv -\hat{L} S_{\mu\nu}. \tag{31} \]
The scalar fields \( \phi \) and \( \bar{\phi} \) can be interpreted as the brane fluctuation modes. The formal solution for \( \delta E_{\mu\nu} \) using the Green function can be found in Appendix A.

IV. EFFECTIVE TEUKOLSKY EQUATION

A. Gradient Expansion Method

It is known that the Gregory-Laflamme instability occurs if the curvature length scale of the black hole \( L \) is less than the Compton wavelength of KK modes \( \sim \ell \exp(d/\ell) \). As we are interested in the stable rotating black string,
\[ \epsilon = \left( \frac{\ell}{L} \right)^{2} \ll 1 \tag{32} \]
is assumed. This means that the curvature on the brane can be neglected compared with the derivative with respect to \( y \). Our iteration scheme consists in writing the Weyl tensor \( E_{\mu\nu} \) in the order of \( \epsilon \). Hence, we will seek the Weyl tensor as a perturbative series
\[ \delta E_{\mu\nu}(y, x^{\mu}) = \delta E_{\mu\nu}^{(1)}(y, x^{\mu}) + \delta E_{\mu\nu}^{(2)}(y, x^{\mu}) + \delta E_{\mu\nu}^{(3)}(y, x^{\mu}) + \cdots. \tag{33} \]

1. First order

At first order, we can neglect the Lichnerowicz operator term. Then Eq. (22) become
\[ \left( \partial_{y}^{2} - \frac{4}{\ell} \partial_{y} + \frac{4}{\ell^{2}} \right) \delta E_{\mu\nu}^{(1)} = 0, \tag{34} \]
where the superscript \( (1) \) represents the order of the derivative expansion. This can be readily integrated into
\[ \delta E_{\mu\nu}^{(1)} = e^{2\hat{\gamma}} \left( C_{\mu\nu} \phi_{\nu}^{(1)} + \chi_{\mu\nu}^{(1)} \right), \tag{35} \]
where \( C_{\mu\nu} \) and \( \chi_{\mu\nu}^{(1)} \) are the constants of integration which depend only on \( x^{\mu} \) and satisfy the transverse \( (1) C_{\nu|\mu} = (1) \chi_{\nu|\mu} = 0 \) and traceless \( (1) C_{\mu\nu} = (1) \chi_{\mu\nu} = 0 \) constraints. The junction conditions on each branes at this order are
\[ e^{2\hat{\gamma}} \left[ e^{-2\hat{\gamma}} \delta E_{\mu\nu}^{(1)} \right]_{y=0,d} = 0. \tag{36} \]
Imposing this junction condition Eq. (36) on the solution (35), we see \( (1) C_{\mu\nu} = 0 \). Thus, we get the first order Weyl tensor
\[ (1) \delta E_{\mu\nu} = e^{2\hat{\gamma}} (1) \chi_{\mu\nu}(x), \tag{37} \]
where \( (1) \chi_{\mu\nu}(x) \) is arbitrary at this order. This should be determined from the next order analysis.
The next order solutions are obtained by taking into account the terms neglected at first order. At second order, Eq. (22) becomes
\[
\left( \partial_y^2 - \frac{4}{\ell} \partial_y + \frac{4}{\ell^2} \right) \delta E_{\mu\nu} = -\epsilon^2 \hat{L} \delta E_{\mu\nu} .
\]
(38)
Substituting the first order $E_{\alpha\beta}$ into the right hand side of Eq. (38), we obtain
\[
\delta E_{\mu\nu} \left|_{(2)} \right. = \epsilon^2 \hat{L} \delta E_{\mu\nu} = -\epsilon^2 \hat{L} \left( \mu\nu \right) \chi_{(1)} .
\]
(39)
where $C_{\mu\nu}$ and $\chi_{\mu\nu}$ are again the constants of integration at this order and satisfy the transverse and traceless constraint $\chi_{\mu\nu} = \chi_{\nu\mu} = 0$, etc.). The junction conditions at this order give
\[
\begin{align*}
\left. e^{-2\hat{L}} \delta E_{\mu\nu} \right|_{y=0} &= \hat{L} S_{\mu\nu} (1), \\
\left. e^{-2\hat{L}} \delta E_{\mu\nu} \right|_{y=d} &= -\Omega^2 \hat{L} S_{\mu\nu} (1).
\end{align*}
\]
(40)
Here $\Omega^2 = \exp \left[-\frac{2}{\ell} \hat{L} \right]$ is a conformal factor that relates the metric on the $\oplus$-brane to that on the $\ominus$-brane. Substituting Eq. (39) into the above junction conditions, we get
\[
\begin{align*}
\frac{1}{\ell} C_{\mu\nu} - \frac{1}{2} \hat{L} \chi_{\mu\nu} &= \hat{L} S_{\mu\nu} (1), \\
\frac{1}{\ell} C_{\mu\nu} - \frac{1}{2} \Omega^2 \hat{L} \chi_{\mu\nu} &= -\Omega^2 \hat{L} S_{\mu\nu} (1).
\end{align*}
\]
(41)
Eliminating $\chi_{\alpha\beta}$ from these equations, we obtain one of the constants of integration
\[
C_{\mu\nu} = -\frac{\ell}{2} \Omega^2 \left( \hat{L} S_{\mu\nu} (1) + \Omega^2 \hat{L} S_{\mu\nu} (1) \right). \tag{42}
\]
Similarly, eliminating $\chi_{\mu\nu}$ from Eqs. (42) and (43), we obtain the equation
\[
\hat{L} \chi_{\mu\nu} = \frac{2}{\ell} \Omega^2 \left( \hat{L} S_{\mu\nu} (1) + \Omega^2 \hat{L} S_{\mu\nu} (1) \right), \tag{44}
\]
which is easily integrated as
\[
\chi_{\mu\nu} = \frac{2}{\ell} \Omega^2 \left( \hat{L} S_{\mu\nu} (1) + \Omega^2 \hat{L} S_{\mu\nu} (1) \right). \tag{46}
\]
Comparing Eq. (46) with the analysis for the perturbations around the flat two-brane background, we see the above result corresponds to the zero mode contribution [3]. Hence, the KK corrections come from the second order corrections which are not yet determined.

3. Third order

In order to obtain KK corrections, we need to fix $\chi_{\mu\nu}$. For that purpose, we must proceed to third order analysis. At third order, we have
\[
\left( \partial_y^2 - \frac{4}{\ell} \partial_y + \frac{4}{\ell^2} \right) \delta E_{\mu\nu} = -\epsilon^2 \hat{L} \delta E_{\mu\nu} .
\]
(47)
The solution is
\[
\delta E_{\mu\nu} = \epsilon^2 \left( \mu\nu \right) \chi_{(1)} - \frac{\ell^2}{4} \epsilon^4 \hat{L} \chi_{\mu\nu} \left( 2 - \frac{\ell^2}{4} \hat{L} \right) \chi_{(2)} + \frac{\ell^6}{64} \epsilon^6 \hat{L}^2 \chi_{(3)} \left( 2 \right) . \tag{48}
\]
where $C_{\mu\nu}$ and $\chi_{\mu\nu}$ are the constants of integration at this order. Junction conditions yield
\[
\begin{align*}
\frac{\ell^3}{16} \hat{L} \chi_{\mu\nu} + \frac{\ell}{4} \hat{L} C_{\mu\nu} - \frac{\ell}{2} \hat{L} \chi_{\mu\nu} + \frac{1}{3} \hat{L} S_{\mu\nu} &= \hat{L} S_{\mu\nu}, \tag{49}
\frac{\ell^3}{16} \hat{L} \chi_{\mu\nu} - \frac{1}{2 \Omega^2} \left( d - \frac{\ell}{2} \right) \hat{L} C_{\mu\nu} + \frac{\ell^3}{16 \Omega^2} \hat{L} \chi_{\mu\nu}
\end{align*}
\]
and
\[
\begin{align*}
\hat{L} \chi_{\mu\nu} = \frac{1}{2} + \frac{d}{\ell} \frac{1}{\Omega^2} \left[ \hat{L} C_{\mu\nu} + \frac{\ell^2}{8} \left( 1 + \frac{1}{\Omega^2} \right) \hat{L} \chi_{\mu\nu} \\
+ \frac{2}{\ell} \left( 1 + \frac{1}{\Omega^2} \right) \left[ \hat{L} S_{\mu\nu} + \Omega^2 \hat{L} S_{\mu\nu} \right] \right]. \tag{51}
\end{align*}
\]
It is easy to obtain
\[
\begin{align*}
\chi_{\mu\nu} = \frac{1}{2} + \frac{d}{\ell} \frac{1}{\Omega^2} \left[ C_{\mu\nu} + \frac{\ell^2}{8} \left( 1 + \frac{1}{\Omega^2} \right) \hat{L} \chi_{\mu\nu} \\
+ \frac{2}{\ell} \left( 1 + \frac{1}{\Omega^2} \right) \left[ S_{\mu\nu} + \Omega^2 \hat{L} S_{\mu\nu} \right] \right]. \tag{53}
\end{align*}
\]
Thus, we have obtained KK corrections. In principle, we can continue this perturbative calculations to any order.
B. Effective Teukolsky Equation

Schematically, Teukolsky equation (9) takes the following form

\[
\hat{P}\delta\Psi_4 = \hat{Q} (8\pi GT_{nm}\cdot - \delta E_{nm}) + \cdots ,
\]

where \(\hat{P}\) and \(\hat{Q}\) are the operators in Eq.(9). What we needed is \(\delta E\) in the above equation. Now, we can write down \(\delta E\) on the brane up to the second order as

\[
\delta E_{\mu\nu}\bigg|_{y=0} = \frac{2}{\ell} \frac{\Omega^2}{1 - \Omega^2} \left[ \mathbf{S}_{\mu\nu} + \Omega^2 \mathbf{S}_{\mu\nu} \right] + \frac{1}{2} \frac{d}{\ell} \frac{1}{\Omega^2 - 1} \left[ \mathbf{L}\mathbf{S}_{\mu\nu} + \Omega^4 \mathbf{L}\mathbf{S}_{\mu\nu} \right] + \frac{\ell}{4} \mathbf{L}\mathbf{S}_{\mu\nu} + \Omega^2 \mathbf{L}\mathbf{S}_{\mu\nu} ,
\]

where

\[
\mathbf{S}_{\mu\nu} = -\phi_{\mu\nu} + g_{\mu\nu} \Box \phi - \frac{\kappa^2}{2} T_{\mu\nu} ,
\]

\[
\mathbf{S}_{\mu\nu} = -\phi_{\mu\nu} + g_{\mu\nu} \Box \phi - \frac{\kappa^2}{2} T_{\mu\nu} .
\]

Substituting this \(\delta E\) into (9), we get the effective Teukolsky equation on the brane. From Eq. (55), we see KK corrections give extra sources to Teukolsky equation. To obtain quantitative results, we must resort to numerical calculations. It should be stressed that the effective Teukolsky equation is separable like as the conventional Teukolsky equation. Therefore, it is suitable for numerical treatment.

Notice that our result in this section assume only Ricci flatness. If we do not care about separability, we can study the gravitational waves in the general Ricci flat background. Moreover, we can analyze other types of waves using \(\delta E\). As \(\delta E\) has 5 degrees of freedom which corresponds to the degrees of freedom of the bulk gravitational waves, one expect the scalar gravitational waves and vector gravitational waves. Without KK effects, no vector gravitational waves exist and the scalar gravitational waves can be described as the Brans-Dicke scalar waves. However, KK effects produce new effects which might be observable.

V. CONCLUSION

We formulated the perturbative formalism around the Ricci flat two-brane system. In particular, the master equation for \(\delta E\) is derived. The gradient expansion method is utilized to get a series solution. This gives the closed system of equations which we call the effective Teukolsky equations in the case of type D induced metric on the brane. This can be used for estimating the Kaluza-Klein corrections on the gravitational waves emitted from the perturbed rotating black string. Our effective Teukolsky equation is completely separable, hence the numerical scheme can be developed in a similar manner as was done in the case of 4-dimensional Teukolsky equation [10, 11].

It seems legitimate to regard a section of the black string as the black hole on the brane at low energy. However, if the gravitational radius of black string is smaller than the Compton wave length of the KK modes, the black string becomes unstable. Therefore, the localized black hole is expected to be realized in this case [12]. It would be interesting to investigate this transition analytically. In particular, the possibility of the classical evaporation is an interesting issue [13], because the resultant localized black hole is generically still large and hence AdS/CFT argument can be applicable.

It is possible to apply our master equation to the transition phenomena from black string to black hole. Interestingly, our master equation coincides with the 5-dimensional Einstein equations in the harmonic gauge. Hence, it must exhibit the Gregory-Laflamme instability. It would be interesting to see this instability from the brane point of view. We leave this interesting issue for the future work.

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APPENDIX A: FORMAL SOLUTION

To solve equation for \(\delta E\), we introduce the Green function

\[
\left[ \left( \partial^2_y - \frac{4}{\ell} \partial_y + \frac{4}{\ell^2} \right) \delta_{\mu}^{\alpha} \delta_{\nu}^{\beta} + e^{2\Phi} \mathbf{L}_{\mu\nu} \right] G_{\alpha\beta}^{\lambda\rho}(x, y; x', y') = \frac{-e^{4\Phi}}{\sqrt{-g}} \delta^4(x - x') \delta(y - y') \left( \delta_{\mu}^{(\alpha} \delta_{\nu}^{\beta)} - \frac{1}{4} g_{\mu\nu} g^{\lambda\rho} \right)
\]

with the boundary conditions

\[
\partial_y \left[ e^{-2\Phi} G_{\mu\nu}^{\alpha\beta} \right] \bigg|_{y=0,d} = 0 .
\]

Then, the formal solution is given by

\[
\delta E_{\mu\nu}(x, y) = \int d^4 x' \sqrt{-g} \left[ e^{-2\Phi} G_{\mu\nu}^{\lambda\rho} \partial_y \left( e^{-2\Phi} \delta E_{\lambda\rho} \right) \right] \bigg|_{y=d} \bigg|_{y=0} .
\]
Using junction conditions (30) and (31), we obtain

$$
\delta E_{\mu\nu}(x', y') = \int d^4x \sqrt{-g} \left[ G_{\mu\nu}(x, d; x', y') \Omega^4 \mathcal{L}_{\lambda\rho} + G_{\mu\nu}(x, 0; x', y') \mathcal{L}_{\lambda\rho} \right].
$$

(A4)

If we use the 4-dimensional Green function

$$
\hat{L}_{\mu\nu}^{\alpha\beta} - \lambda_n^2 \delta_{\mu\nu} \delta_{\alpha\beta} G(x, x'; \lambda_n^2) = -\delta(x - x'),
$$

(A5)

the Green function can be written as

$$
G_{\mu\nu}(x, y, x', y') = \sum_n \varphi_n(y) \varphi_n(y') G(x, x').
$$

(A6)

Here, mode function is given by

$$
\varphi_n(y) = N_n e^{2\pi y} \left[ J_0(\lambda_n e^y) - \frac{J_1(\lambda_n e^y)}{N_1(\lambda_n e^y)} N_0(\lambda_n e^y) \right],
$$

(A7)

where $N_n$ is a normalization constant which is determined by

$$
\int dy e^{-2\pi y} \varphi_n(y) \varphi_m(y) = \delta_{nm}.
$$

(A8)

The junction condition for the negative tension brane gives a KK-spectrum as

$$
J_1(\lambda_n e^y) - \frac{J_1(\lambda_n e^y)}{N_1(\lambda_n e^y)} N_0(\lambda_n e^y) = 0.
$$

(A9)

In the low energy regime $\lambda_n \ell \ll 1$, we get $J_1(\lambda_n e^{d/\ell}) = 0$. We see the KK-mass spectrum can be estimated as $\lambda_n \sim e^{-d/\ell}/\ell$. [14].

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