ON THE SYMMETRY OF SPATIALLY PERIODIC TWO-DIMENSIONAL WATER WAVES

FLORIAN KOGELBAUER
Faculty of Mathematics, University of Vienna
Oskar-Morgenstern-Platz 1
Vienna, A-1090, Austria

(Communicated by Adrian Constantin)

Abstract. We show that a spatially periodic solution to the irrotational two-dimensional gravity water wave problem, with the property that the horizontal velocity component at the flat bed is symmetric, while the acceleration at the flat bed is anti-symmetric with respect to a common axis of symmetry, necessarily constitutes a traveling wave. The proof makes use complex variables and structural properties of the governing equations for nonlinear water waves.

1. Introduction. Questions related to the symmetry of the wave profile as well as the underlying flow arise naturally in the study of spatially-periodic gravity water waves. For recent discussions of qualitative properties of solutions that guarantee symmetry of the wave, we refer to [6], [7], [11], [13], [14] and [15]. The sharpest results state that symmetry is ensured as soon as the wave profile is monotone between successive crests and troughs, under some relatively mild conditions on the underlying flows. In particular, monotonicity of the wave profile between successive crests and troughs for irrotational waves - the celebrated Stokes waves present this distinctive feature, see the discussion in [3] - guarantees symmetry of the wave profile. Symmetry properties of the underlying flow and the free surface are closely related to steady state solutions. In [8] the authors prove that a solution to the two-dimensional gravity water wave problem over a flat bed with the property that the free surface and the velocity field as well as the pressure admit at any instant of time an axis of symmetry necessarily defines a traveling wave. Since the symmetry assumptions in [8] require information about the velocity field and the pressure in the whole fluid domain, it is difficult to check them experimentally. We will present a reduced set of assumptions that guarantee symmetry in the fluid domain and that only involve measurements at the flat bed. Actually, we will show that a solution with the property that the horizontal velocity component at the flat bed is symmetric, while the acceleration at the flat bed is anti-symmetric with respect to a common axis of symmetry, necessarily constitutes a traveling wave.

Information about the flow at the flat bed is obtained via in situ measurements. For instance, pressure measurements at the flat bed can be used to recover the wave profile and to estimate the wave height, cf. [1], [2], [5] and [10].

2010 Mathematics Subject Classification. Primary: 35Q35; Secondary: 35B50.
Key words and phrases. Irrotational water waves, traveling wave.
The author is supported by the ERC grant NWFV 267116.
2. The governing equations. The governing equations for an ideal, incompressible, homogeneous fluid in two space dimensions under the influence of gravity consist of the equation of mass conservation

\[ u_x + v_y = 0, \quad (1) \]

coupled to Euler's equations

\[
\begin{align*}
    u_t + uu_x + vu_y &= -P_x, \\
    v_t + uv_x + vv_y &= -P_y - g,
\end{align*}
\]

where \((u,v) = (u(x,y,t), v(x,y,t))\) is the velocity field, \(P = P(x,y,t)\) is the pressure and \(g \approx 9.81 \text{ m/s}^2\) is the acceleration of gravity at the surface of the Earth.

In the study of two-dimensional water waves, we additionally require that there exists a free surface and that the motion of the fluid beneath the surface is independent from the motion of the air above, which is expressed in the kinematic boundary condition

\[ v = \eta_t + u \eta_x \quad \text{for} \quad y = \eta(x,t), \quad (3) \]

and the dynamic boundary condition

\[ P = P_{\text{atm}} \quad \text{on} \quad y = \eta(x,t), \quad (4) \]

where \(P_{\text{atm}}\) is the constant atmospheric pressure. We are concerned with waves propagating over a flat bed of mean depth \(d\). The fluid is allowed to slip at the flat bed, but cannot penetrate it:

\[ v = 0 \quad \text{on} \quad y = -d. \quad (5) \]

Additionally, we impose that the flow is irrotational, assuming that the vorticity is zero throughout the fluid:

\[ u_y = v_x. \quad (6) \]

The relations (1)-(6) constitute the governing equations for the irrotational water wave problem. For a detailed derivation and discussion, we refer to [4] and [9]. In the following, we will be dealing with periodic wave solutions with wavelength \(L\), therefore assuming that \(u, v, P\) and \(\eta\) are smooth and \(L\)-periodic in \(x\). The a priori unknown fluid domain at time \(t\) is therefore given by

\[ \Omega = \Omega(t) = \{(x,y) \in \mathbb{R}^2 : -L < x < L, -d < y < \eta(x,t)\}. \]

3. Main result. We will show that symmetry properties at the flat bed imply that the flow is that beneath a traveling wave.

**Theorem 3.1.** Let \((u,v,P,\eta)\) be a smooth \(x\)-periodic solution to the irrotational water wave problem (1)-(6), with the property that the horizontal velocity component \(u(x,-d,t)\) at the flat bed is symmetric about \(x = \lambda(t)\), while the acceleration \(\vec{a}(x,-d,t)\) at the flat bed is antisymmetric about \(x = \lambda(t)\) at any instant of time. Then the solution defines a traveling wave.

The first assumption reads as

\[ u(x,-d,t) = u(2\lambda(t) - x, -d, t) \quad (7) \]

at any instant of time. In order to compute the acceleration, let \(t \mapsto (X(t), Y(t))\) be the path of a fluid particle. The velocity relates to the flow field via \((\dot{X}(t), \dot{Y}(t)) = (u(X(t), Y(t), t), v(X(t), Y(t), t))\), while the acceleration is given by

\[ \dot{a} = (\dot{X}, \dot{Y}) = (u_t + uu_x + vu_y, v_t + uw_x + vv_y). \quad (8) \]
In view of (8), (7) and (5), the second assumption reads
\[ u_t(x, -d, t) = -u_t(2\lambda(t) - x, -d, t), \] (9)
since the function \( uu_x \) is anti-symmetric about \( x = \lambda(t) \).

**Remark 1.** Note that there is, in general, no a priori connection between the horizontal velocity component \( u \) and its time derivative \( u_t \). Taking the derivative in (7) with respect to \( t \), we get
\[ u_t(x, -d, t) = u_t(2\lambda(t) - x, -d, t) + 2\dot{\lambda}(t)u_x(2\lambda(t) - x, -d, t), \]
which does not immediately relate to the assumed symmetry of the velocity component \( u \) due to the unknown character of \( \dot{\lambda} \).

In the proof of Theorem (3.1) we will use the following result, proved in [8]:

**Lemma 3.2.** Only traveling wave solutions solution to the governing equations (1)-(6) have the property that
\begin{align*}
  u(x, y, t) & = u(2\lambda(t) - x, y, t), \\
  v(x, y, t) & = -v(2\lambda(t) - x, y, t), \\
  P(x, y, t) & = P(2\lambda(t) - x, y, t), \\
  \eta(x, t) & = \eta(2\lambda(t) - x, t)
\end{align*}
throughout the fluid domain.

The proof of Theorem (3.1) is organized as follows. First, we will show that the horizontal velocity component \( u \) is symmetric, while the vertical velocity component \( v \) is anti-symmetric in some sub-domain of \( \bar{\Omega} \subseteq \Omega \). Then we will show that a similar relation holds true for \( u_t \) and \( v_t \), while the pressure is symmetric in \( \bar{\Omega} \), which will permit us to infer by the dynamic boundary condition (4) that necessarily the free surface is symmetric. By Lemma (3.2), we then conclude that the flow defines a traveling wave solution.

**Proof.** Let us show that the velocity field \( (u, v) \) has a symmetry property in a sub-domain of \( \Omega \). The complex velocity field
\[ F(x, y, t) := u(x, y, t) - iv(x, y, t) \]
defines, by (1) and (6), at any instant of time a holomorphic function on the fluid domain \( \Omega \). Since \( F \) is real-valued at the flat bed in view of (5), we may apply the standard Schwartz reflection principle, cf. [12], to define
\[ F(z, t) = \begin{cases}
  F(z, t) & \text{if } z \in \Omega \\
  (F(\bar{z}, t))^* & \text{if } z \in \Omega^* 
\end{cases}, \] (10)
which is a holomorphic function on \( \Omega \cup \Omega^* \) at any instant of time. Here \( z = x + iy \) and \( z^* = x - iy \) denotes complex conjugation.

Now let \( R_\lambda : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (2\lambda - x, y) \) be the reflection with respect to the axis \( \{ x = \lambda \} \) and define \( \Omega_\lambda := R_\lambda(\Omega) \), the reflected fluid domain. Note that in general \( \Omega_\lambda \neq \Omega \), unless the free surface \( \eta \) is symmetric about \( \lambda \). We define the reflected complex velocity field as
\[ F^\lambda(z, t) := R_\lambda[(F(z, t))^*] = u(2\lambda(t) - x, y, t) + iv(2\lambda(t) - x, y, t), \]
which is a holomorphic function on \( \Omega_\lambda \) at any instant of time. Similar to (10), \( F^\lambda \) can be extended to a holomorphic function on \( \Omega \cup \Omega^*_\lambda \) at any instant of time. The function \( F(z, t) - F^\lambda(z, t) \) is holomorphic on the domain \( (\Omega \cup \Omega^*) \cap (\Omega_\lambda \cup \Omega^*_\lambda) \) and identically zero on the real axis by (5) and our first assumption (7). Therefore, by the Identity Theorem for holomorphic functions, cf.
it follows that $F(z, t) = F^\lambda(z, t)$ for all $z \in (\Omega \cup \Omega^*) \cap (\Omega_\lambda \cup \Omega^*_\lambda)$, which readily implies that
\[
\begin{align*}
u(x, y, t) &= -v(2\lambda(t) - x, y, t) \\
v(x, y, t) &= -\nu(2\lambda(t) - x, y, t)
\end{align*}
\]for all $(x, y) \in \Omega \cap \Omega_\lambda$ at any instant of time.

Following the same line of reasoning for $F_t(x, y, t) = u_t(x, y, t) - iv_t(x, y, t)$, which is holomorphic on $\Omega$, $F_t^\lambda(z, t) = R_\lambda((F_t(z, t))^*)$ and $F_t(z, t) + F_t^\lambda(z, t)$ by taking advantage of our second assumption (9), we deduce that
\[
\begin{align*}
u_t(x, y, t) &= -\nu_t(2\lambda(t) - x, y, t) \\
u_t(x, y, t) &= -\nu_t(2\lambda(t) - x, y, t)
\end{align*}
\]for all $(x, y) \in \Omega \cap \Omega_\lambda$ at any instant of time. To prove symmetry of the pressure on the domain $\Omega \cap \Omega_\lambda$, we deduce that by the symmetry conditions (13), (14), (11), (12) and the first Euler equation in (2) we have
\[
P_x(x, y, t) = -P_x(2\lambda(t) - x, y, t)
\]for all $(x, y) \in \Omega \cap \Omega_\lambda$ at any instant of time. Integrating (15) with respect to $x$ we obtain $P(x, y, t) = P(2\lambda(t) - x, y, t) + \alpha(y, t)$ for some function $\alpha$. Evaluating the expression at $x = \lambda$, it follows that $\alpha \equiv 0$ and that
\[
P(x, y, t) = P(2\lambda(t) - x, y, t)
\]for all $(x, y) \in \Omega \cap \Omega_\lambda$ at any instant of time.

So far, we have proved that the velocity field $(u, v)$ as well as the pressure $P$ satisfy a symmetry condition on the domain $\Omega \cap \Omega_\lambda$. In order to apply Lemma (3.2), we have to show that $\eta$ is symmetric with respect to $\lambda$ and hence $\Omega = \Omega_\lambda$. We observe that the pressure is a super-harmonic function in $\Omega$:
\[
\Delta P = -\partial_x(u_t + uu_x + vv_y) - \partial_y(v_t + uu_x + vv_y + g)
= -2(u_x^2 + u_y^2),
\]where we have used equations (2), (1) and (6). Assume by contradiction that the free surface is not symmetric about $\lambda$. Then there exists a boundary point $(x_0, \eta(x_0))$ that is mapped into the interior of $\Omega$ under the reflection $R_\lambda$, that is to say $(2\lambda(t) - x_0, \eta(x_0)) \in \Omega$. By the boundary condition (4) and the symmetry of the pressure (16), it follows that $P = P_{atm}$ at an interior point. Since $P_{atm} = \min_{\Omega} P$, cf. [4], this would contradict the super-harmonicity of $P$. Hence the wave profile is symmetric and we can now conclude that the flow defines a traveling wave.

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Received December 2015; revised January 2016.

E-mail address: florian.kogelbauer@univie.ac.at