GEOMETRIC $K$-HOMOLOGY WITH COEFFICIENTS II: 
THE ANALYTIC THEORY AND ISOMORPHISM

ROBIN J. DEELEY

ABSTRACT. We discuss the analytic aspects of the geometric model for $K$-homology with coefficients in $\mathbb{Z}/k\mathbb{Z}$ constructed in [11]. In particular, using results of Rosenberg and Schochet, we construct a map from this geometric model to its analytic counterpart. Moreover, we show that this map is an isomorphism in the case of a finite CW-complex. The relationship between this map and the Freed-Melrose index theorem is also discussed. Many of these results are analogous to those of Baum and Douglas in the case of spin$^c$ manifolds, geometric K-homology, and Atiyah-Singer index theorem.

1. Introduction

This is the second in a pair of papers whose topic is the construction of a geometric model for $K$-homology with coefficients using spin$^c$ $\mathbb{Z}/k\mathbb{Z}$-manifold theory. Despite this, we have tried to make our treatment as self-contained as possible. In the first paper, [11], the cycles and relations for this model were described. In addition, it was shown that the model fits into the correct Bockstein sequence for the coefficient group $\mathbb{Z}/k\mathbb{Z}$. The goal of this paper is the construction (based on results in [21] and [22]) of an analytic model for $K$-homology with coefficients in $\mathbb{Z}/k\mathbb{Z}$ and the construction of a map (defined at the level of cycles) from the geometric model in [11] to this analytic model. The main result is that this map is an isomorphism for finite CW-complexes (see Theorem 71).

The reader should note the similarity with a number of constructions due to Baum and Douglas (c.f., [3], [4], etc). However, a number of our constructions involve noncommutative $C^*$-algebras (rather than the $C^*$-algebra of continuous function on a manifold in the case considered by Baum and Douglas). These algebras are constructed (based on [21]) using groupoids, but we have endeavoured to make them accessible to the reader unfamiliar with groupoid $C^*$-algebras. We also use some KK-theory but the amount is quite limited. Thus, prerequisites are limited to an understanding of the Baum-Douglas model for (geometric) K-homology and the Fredholm module picture of (analytic) K-homology (due to Kasparov, [17]).

The first section summarizes required results from the literature. In particular, we discuss geometric and analytic K-homology and summarize results contained in the first paper [11]. The reader is directed to [7] and [14] for more on K-homology and to [11] (and the references therein) for more on $\mathbb{Z}/k\mathbb{Z}$-manifolds, the Freed-Melrose index theorem and the construction of geometric K-homology with coefficients in $\mathbb{Z}/k\mathbb{Z}$. We note that $\mathbb{Z}/k\mathbb{Z}$-manifolds were first introduced by Sullivan (see [18], [23], [24]).

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The second section contains the main results of the paper. Namely, the construction of the map from geometric K-homology with coefficients in $\mathbb{Z}/k\mathbb{Z}$ to analytic K-homology with coefficients in $\mathbb{Z}/k\mathbb{Z}$ and the proof that this map is an isomorphism in the case of a finite CW-complex. To put these results in context, we now review the analogous results in K-homology (c.f., [7]). The reader may recall (also see Definition 23 below) that a geometric cycle in K-homology is given by a triple, $(M, E, f)$, where $M$ is compact $\text{spin}^c$ manifold, $E$ is a vector bundle, and $f$ is a continuous map from $M$ to $X$; $(X$ is the space whose K-homology we are modeling). An analytic cycle is given by a Fredholm module over a $C^*$-algebra, $C(M)$; (Based on results in [22], we define analytic K-homology with coefficients in $\mathbb{Z}/k\mathbb{Z}$ to be $K^*(C(X) \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z}))$.) The map (which is analogous to the map $\mu$ above) from geometric $\mathbb{Z}/k\mathbb{Z}$-cycles to analytic $\mathbb{Z}/k\mathbb{Z}$-cycles is defined in three steps:

1. To the $\text{spin}^c$ manifold in a geometric cycle, $(M, E, f)$, we associated a $C^*$-algebra, $C(M)$;
2. We use the $\text{spin}^c$-structure and the vector bundle, $E$, to produce a Fredholm module (denoted $\{D_E\}$) in the K-homology of $C(M)$;
3. Finally, the continuous map induces a map from the K-homology of $C(M)$ to the K-homology of $C(X)$ (denoted by $f_*$), which we apply to $\{D_E\}$ to get a class in the K-homology of $C(X)$.

The map from geometric cycles to analytic cycles is then defined to be

$$\mu: (M, E, f) \mapsto f_*(\{D_E\})$$

In [7], this map is shown to be an isomorphism in the case when $X$ is a finite CW-complex.

The results in the case of $\mathbb{Z}/k\mathbb{Z}$-coefficients can be summarized as follows. We recall (see Definitions 34 and 42 below) that a geometric $\mathbb{Z}/k\mathbb{Z}$-cycle is a triple, $((Q, P), (E, F), f)$, where $(Q, P)$ is a compact $\text{spin}^c$ $\mathbb{Z}/k\mathbb{Z}$-manifold, $(E, F)$ is a $\mathbb{Z}/k\mathbb{Z}$-vector bundle, and $f$ is a continuous map from $(Q, P)$ to $X$ (the space whose K-homology with coefficients in $\mathbb{Z}/k\mathbb{Z}$ we are modeling). On the other hand, an analytic $\mathbb{Z}/k\mathbb{Z}$-cycle is a Fredholm module over the $C^*$-algebra $C(X) \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z})$, where $C^*(pt; \mathbb{Z}/k\mathbb{Z})$ is the mapping cone of the inclusion of $\mathbb{C}$ into the $k$ by $k$ matrices. (Based on results in [22], we define analytic K-homology with coefficients in $\mathbb{Z}/k\mathbb{Z}$ to be $K^*(C(X) \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z}))$.) The map (which is analogous to the map $\mu$ above) from geometric $\mathbb{Z}/k\mathbb{Z}$-cycles to analytic $\mathbb{Z}/k\mathbb{Z}$-cycles is defined in three steps:

1. Using Rosenberg’s construction (see [21]), we associated a $C^*$-algebra to the $\mathbb{Z}/k\mathbb{Z}$-manifold, $(Q, P)$ (denoted by $C^*(Q, P; \mathbb{Z}/k\mathbb{Z})$);
2. Again, following Rosenberg, we use the $\text{spin}^c$-structure and the $\mathbb{Z}/k\mathbb{Z}$-vector bundle, $(E, F)$, to produce a Fredholm module (denoted $\{D_{(E, F)}\}$) in the K-homology of $C(Q, P; \mathbb{Z}/k\mathbb{Z})$;
3. Finally, the continuous map, $f$, induces a $*$-homomorphism from $C(X) \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z})$ to $C^*(Q, P; \mathbb{Z}/k\mathbb{Z})$ and hence a map from the K-homology of $C^*(Q, P; \mathbb{Z}/k\mathbb{Z})$ to the K-homology of $C(X) \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z})$. We denote this map by $\tilde{f}^*$ and then apply it to $\{D_{(E, F)}\}$ to get a class in the K-homology of $C(X) \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z})$.

To summarize, the map from the geometric theory to the analytic theory is defined via

$$\Phi: ((Q, P), (E, F), f) \mapsto \tilde{f}^*(\{D_{(E, F)}\})$$
The proof that this map is well-defined is not trivial. The most involved part is the bordism relation (see Theorem 70). To show that $\Phi$ is an isomorphism for finite CW-complexes, we use the Bockstein sequences for both the geometric and analytic models, the Five Lemma, and the fact that $\mu$ is an isomorphism for finite CW-complexes (c.f., Theorem 6.2 of [7]).

In the final section of the paper, we discuss the relationship between our results and index theory. In particular, we discuss the fact that it follows from our construction that the Freed-Melrose index theorem for $\text{spin}^c$ $\mathbb{Z}/k\mathbb{Z}$-manifolds can be conceptualized as a specific case of the isomorphism from geometric to analytic $K$-homology with coefficient in $\mathbb{Z}/k\mathbb{Z}$. This is analogous to Baum and Douglas’ conceptualization of the Atiyah-Singer index theorem as a specific case of the isomorphism between geometric and analytic $K$-homology. We also discuss index pairings in this section.

2. Preliminaries

We follow Chapters 8 to 11 of Higson and Roe’s book [14], the paper of Baum, Higson, and Schick [7], and [11] to introduce models for $K$-homology and to summarize the main results we need in our constructions.

2.1. Analytic $K$-homology via Kasparov cycles. Throughout, $A$ will denote a separable $C^*$-algebra. We begin by recalling a few basic definitions.

Definition 1. A Fredholm module over $A$ is a triple, $(\mathcal{H}, \rho, F)$, where

1. $\mathcal{H}$ is a separable Hilbert space;
2. $\rho: A \to B(\mathcal{H})$ is a representation;
3. $F \in B(\mathcal{H})$ such that each of $(F^2 - I)\rho(a)$, $(F^* - F)\rho(a)$ and $[F, \rho(a)]$ are compact.

Definition 2. Let $p$ be a nonnegative integer. Then a $p$-graded Fredholm module is a Fredholm module, $(\mathcal{H}, \rho, F)$, with the following additional structure:

1. $\mathcal{H}$ is a graded Hilbert space;
2. $\rho(a)$ is an even operator for each $a \in A$;
3. $F$ is an odd operator;
4. $\varepsilon_1, \ldots, \varepsilon_p$ are odd operators on $\mathcal{H}$ such that

\[ \varepsilon_j = -\varepsilon_j^*, \quad \varepsilon_j^2 = -1, \quad \varepsilon_i \varepsilon_j + \varepsilon_j \varepsilon_i = 0 \quad (i \neq j) \]

Moreover, we have that, for each $i = 1, \ldots, p$ and $a \in A$,

\[ [\rho(a), \varepsilon_i] = 0 \quad \text{and} \quad F \varepsilon_i + \varepsilon_i F = 0 \]

Remark 3. We will refer to (ungraded) Fredholm modules as $(-1)$-graded so as to include these objects in the framework of Definition 2. We refer to the operators, $\{\varepsilon_i\}_{i=1}^p$, as multigrading operators. The zero Fredholm module is obtained by taking the zero Hilbert space, zero representation, and zero operator.

Definition 4. Let $(\mathcal{H}, \rho, F)$ be a Fredholm module over $A$ and let $U: \mathcal{H} \to \mathcal{H}'$ be a unitary operator. Then $(\mathcal{H}', U\rho U^*, UFU^*)$ is also a Fredholm module over $A$. We say that such a module is unitarily equivalent to $(\mathcal{H}, \rho, F)$. If the Fredholm module is $p$-graded, then the unitary $U$ must preserve gradings (in particular, $\mathcal{H}'$ must be graded). We note that, in this case, the unitary also produces the multigrading operators, $\varepsilon_j' := U\varepsilon_j U^*$ (for $j = 1, \ldots p$), required by Definition 2.
Definition 5. Let \((\mathcal{H}, \rho, F_t)\) be a family of Fredholm modules parameterized by \(t \in [0, 1]\). If the function \(t \mapsto F_t\) is norm continuous, then we say that this family defines an operator homotopy between \((\mathcal{H}, \rho, F_0)\) and \((\mathcal{H}, \rho, F_1)\) and say that these Fredholm modules are operator homotopic. In the \(p\)-graded case, one must take a family of \(p\)-graded Fredholm modules.

Definition 6. The direct sum of two \(p\)-graded Fredholm modules, \((\mathcal{H}, \rho, F)\) and \((\mathcal{H}', \rho', F')\), is defined to be \((\mathcal{H} \oplus \mathcal{H}', \rho \oplus \rho', F \oplus F')\). We note that in the \(p\)-graded case, the multigrading operators are taken to be the direct sum of the multigrading operators of \((\mathcal{H}, \rho, F)\) and \((\mathcal{H}', \rho', F')\) (i.e., we take the multigrading operators given by \(\varepsilon_i \oplus \varepsilon_i'\), where \(\{\varepsilon_i\}_{i=1}^p\) and \(\{\varepsilon_i'\}_{i=1}^p\) are the multigrading operators associated to \((\mathcal{H}, \rho, F)\) and \((\mathcal{H}', \rho', F')\) respectively).

Definition 7. We define the Kasparov \(K\)-homology groups \(K^{-p}(A)\) to be the abelian group defined by generators and relations as follows. As generators, we take unitary equivalence classes of \(p\)-graded Fredholm modules defined by generators and relations as follows. As generators, we take the \(p\)-graded case, one must take a family of \(p\)-graded Fredholm modules.

Definition 8. Let \(X\) be a second countable, locally compact Hausdorff spaces. We will denote these groups by \(K^{\text{even}}(A)\) and \(K^{\text{odd}}(A)\). We will also use the notation \(K^*(A)\) for these two groups. However, it should be clear from the context whether we are referring to \(K^{-p}(A)\) (\(p = -1, 0, \ldots\)) or \(K^{\text{even/odd}}(A)\).

We now apply the theory we have developed to the case of commutative \(C^*\)-algebras (i.e., locally compact Hausdorff space). If \(X\) is a locally compact Hausdorff space, then the assumption that \(A = C(X)\) is separable is equivalent to \(X\) being second countable.

Definition 9. Let \(X\) be a second countable, locally compact Hausdorff spaces. We define \(K_p(X) := K^{-p}(C_0(X))\).

Theorem 9. (c.f. Theorem 2.11 of [14])
The functor \(K_{\ast}(\cdot)\) is a generalized homology theory. Moreover, on the category of finite CW-complexes, this homology theory is equal to \(K\)-homology. (We defined \(K\)-homology to be the generalized homology theory obtained from \(K\)-theory using duality (c.f. [4]).)

2.2. Dirac operators. We now discuss Dirac operators on \(\text{spin}^c\)-manifolds following the treatment in [7] and [14].

Definition 10. Let \(M\) be a smooth manifold and \(V\) be a smooth, Euclidean vector bundle over it. A \(p\)-graded Dirac structure on \(V\) is a smooth, \(\mathbb{Z}/2\)-graded, Hermitian vector bundle, \(S\), over \(M\) along with the following additional structure:

1. An \(\mathbb{R}\)-linear vector bundle morphism

\[ V \to \text{End}(S) \]

which maps each \(v \in V_x\) to a skew-adjoint, odd endomorphism. Moreover, if we denote the action of \(v\) on \(u \in S_x\) by \(u \mapsto v \cdot u\), then

\[ v \cdot (v \cdot u) = -\|v\|^2 u \]
A family of odd endomorphisms, \( \varepsilon_1, \ldots, \varepsilon_p \), of \( S \) such that the following relations hold:

\[
\varepsilon_j = -\varepsilon_j^*, \quad \varepsilon_j^2 = -1, \quad \varepsilon_i \varepsilon_j + \varepsilon_j \varepsilon_i = 0 \quad (i \neq j)
\]

Moreover, for any \( x \in M \) and any \( v \in V_x \), each \( \varepsilon_j \) commutes with the endomorphism associated to \( v \) (i.e., commutes with \( u \mapsto v \cdot u \)).

We will often refer to \( S \), defined as above, as a Dirac bundle (with respect to \( M \) and \( V \)). If we do not make reference to a vector bundle, then the reader should assume that \( V \) is the tangent bundle of \( M \). That is, a Dirac structure on \( M \) is defined to be a Dirac structure on the pair \( M \) and \( V = TM \).

**Example 11.** The prototypical example (c.f. Section 4 of [7]), is the case when \( M \) is a \( \text{spin}^c \)-manifold and \( V = TM \) (\( S \) in this case is the spinor-bundle).

**Example 12.** Let \( M \) be a smooth manifold, \( E \) and \( V \) be vector bundles over \( M \) and \( S \) be a \( p \)-graded Dirac bundle for \( M \) and \( V \). We denote the bundle morphism in Item 1 of Definition 10 by \( c : V \to \text{End}(S) \) and the family of odd endomorphisms in Item 2 of Definition 10 by \( \{ \varepsilon_j \}_{j=1}^p \). Then \( S \otimes E \) can be given a \( p \)-graded Dirac bundle structure by taking

\[
V \to \text{End}(S \otimes E)
\]
defined by \( v \mapsto c(v) \otimes 1 \) as the bundle morphism required in Item 1 of Definition 10 and using operators: \( \tilde{\varepsilon}_j = \varepsilon_j \otimes 1 \) for Item 2.

**Definition 13.** Let \( M \) be a Riemannian manifold with a \( p \)-graded Dirac bundle structure, \( S \), on its tangent bundle. We shall call an odd, symmetric, order one linear partial differential operator, \( D \), acting on the compactly supported smooth sections of \( S \), a Dirac operator if it commutes with the operators, \( \varepsilon_j \) (in Definition 10), and

\[
[D, f] \cdot u = \text{grad}(f) \cdot u
\]
for each \( f \in C^\infty(M) \) and section \( u \) of \( S \).

**Theorem 14.** (c.f., Theorem 10.6.5 of [14])

Let

1. \( M \) be a compact Riemannian manifold;
2. \( D \) be a Dirac operator associated to a \( p \)-graded Dirac bundle \( S \) on \( TM \);
3. \( \mathcal{H} = L^2(M, S) \) the Hilbert space of square-integrable sections of \( S \) (see Chapter 10 of [14]);
4. \( \rho \) be the representation of \( C(M) \) on \( \mathcal{B}(\mathcal{H}) \) via pointwise multiplication;
5. \( F = D(t + D^2)^{-\frac{1}{2}} \).

Then the triple \( (\mathcal{H}, \rho, F) \) is a \( p \)-graded Fredholm module for the \( C^* \)-algebra, \( C(M) \). We note that the multi-grading operators required by Definition 2 are the family of skew-adjoint endomorphisms given in Definition 10.

**Definition 15.** Using the notation of the previous theorem, we denote the class obtained from this construction by \( [D] \in K^{-p}(C(M)) \). There is a more involved construction for open manifolds without boundary (see Chapters 10 and 11 of [14] for details). In particular, when \( M \) is an open \( \text{spin}^c \)-manifold we can form a class in the \( K \)-homology of \( C_0(M) \) from a Dirac operator. We again denote this by \( [D] \in K^{-p}(C_0(M)) \).
Example 16. If $M$ is a spin$^c$ manifold, then it has a canonical Dirac bundle associated to its tangent bundle, namely its spinor bundle (denoted $S_{TM}$). Moreover, if $E$ is a complex vector bundle over $M$, we can form the Dirac bundle, $S_{TM} \otimes E$, using the process described in Example 12. If $D$ is a Dirac operator on $S_{TM}$ and $E$ has a Hermitian metric, then we can form a Dirac operator, $D_E$, for $S_{TM} \otimes E$ (c.f. [14]). We will refer to $D_E$ as the Dirac operator twisted by $E$.

We discuss manifolds with boundary since we will need this for the development of spin$^c$ $\mathbb{Z}/k\mathbb{Z}$-manifolds (see Definitions 34 and 39).

Definition 17. Let $S$ be a $p$-graded Dirac bundle on a Riemannian manifold, $\bar{M}$, with boundary $\partial \bar{M}$ and interior $M$. We let $e_1$ denote the outward pointing unit normal vector field on $\partial \bar{M}$, and define a map acting on the sections of $S$ restricted to $\partial M$ via

$$X : u \mapsto (-1)^{\partial u} e_1 \cdot \varepsilon_1 \cdot u$$

where $\partial u$ denotes the degree of the section $u$. This mapping defines an automorphism, which is even, self-adjoint, and $X^2 = I$. Moreover, $X$ commutes with both the multiplication operators $u \mapsto Y \cdot u$ (here $Y$ is tangent vector orthogonal to $e_1$) and each of the operators $\varepsilon_i$ (for $i \neq 1$).

We then define a $(p-1)$-graded Dirac bundle on $\partial M$ by taking $S_{\partial M}$ to be the eigenbundle of $X$ associated to the eigenvalue +1 and the multigrading operators $\varepsilon_2, \ldots, \varepsilon_p$. We refer to the Dirac bundle, $S_{\partial M}$, as the boundary of $S$.

Definition 18. Let $X$ be a locally compact Hausdorff space and $U$ an open subspace of $X$. If $i$ denotes the inclusion of $U$ into $X$, then $i$ induces an inclusion of $C_0(U)$ into $C_0(X)$, which in turn induces a map on $K$-homology. We denote the induced map by

$$i^! : K_p(X) \to K_p(U)$$

and call it the wrong-way map on $K$-homology.

The proof of the next theorem can be found in Chapters 10 and 11 of [14]. In particular, the reader is directed to Propositions 11.2.12 and 11.2.15 of [14].

Theorem 19. (c.f., Theorem 3.5 of [17])

To each Dirac operator, $D$, on a $p$-graded Dirac bundle over a smooth manifold without boundary, $M$, the associated class $[D] \in K_p(M)$ (see Definition 15) has the following properties:

1. $[D]$ depends only on the Dirac bundle (not on the particular choice of operator $D$).
2. Let $U$ be an open submanifold of $M$ and $D_U$ be the Dirac operator on $U$ given by the restriction of Dirac operator, $D_M$, on $M$. Then $[D_U]$ maps to $[D_M]$ under the wrong way map on $K$-homology (see Definition 18) induced from inclusion $U \hookrightarrow M$.
3. Suppose $\bar{M}$ is the interior of a Riemannian manifold, $\bar{M}$, with boundary $\partial \bar{M}$. Also let $S$ be a $p$-graded Dirac bundle on $\bar{M}$ with associated Dirac operator $D$. We denote by $D_{\partial \bar{M}}$ a Dirac operator associated to $S_{\partial \bar{M}}$ (see the construction in Definition 17). Then the boundary map in Kasparov $K$-homology,

$$\partial : K_p(M) \to K_{p-1}(\partial \bar{M})$$
satisfies
\[ \partial [D_M] = [D_{\partial M}] \]

We will need to compare the \( K \)-homology classes of Dirac operators on \( p \)-dimensional manifolds with those on \( (p + 2k) \)-dimensional manifolds when considering vector bundle modification. To this end, we consider the relationship between the Dirac operator on a fiber bundle with the Dirac operator on its base. Let \( M \) be a compact (or possibly just closed) Riemannian manifold and \( P \) a principal bundle over \( M \) with structure group a compact Lie group, which we denote by \( G \). Moreover, assume that \( N \) is a closed Riemannian manifold with an action of \( G \) given by isometries. We then form \( Z := P \times_G N \) and have the following exact sequence
\[ 0 \to V \to TZ \to \pi^*(TM) \to 0 \]
where \( \pi \) denotes the projection map \( Z \to M \) and \( V \) is the vertical tangent bundle defined by \( V := P \times_G TN \). Choosing a splitting, we obtain a Riemannian metric on \( TZ \) (i.e., the metric induced from the isomorphism \( TZ \cong V \oplus \pi^*(TM) \)).

We now construct a Dirac operator on \( Z \) from the ones on \( N \) and \( M \). We begin with the Dirac bundles. Let \( S_M \) be a \( p \)-graded Dirac bundle on \( M \) and \( S_N \) be a 0-graded Dirac bundle on \( N \). Moreover, assume that the action of \( G \) on \( N \) respects the bundle \( S_N \). We then define
\[
S_V := P \times_G S_N \\
S_Z := S_V \otimes \pi^*(S_M)
\]
We note that \( \otimes \) denotes the graded tensor product. The bundle, \( S_Z \), is a \( p \)-graded Dirac bundle on \( Z \). We define the action (required by Definition 10) as follows. Elements of the form \( v \oplus w \in V \oplus \pi^*(TM) \) act via \( v \otimes 1 + 1 \otimes w \) on \( S_V \otimes \pi^*(S_M) \). We then associate a Dirac operator to this bundle and denote it by \( D_Z \).

Now, using Theorem 14 (in fact, its generalization discussed in Definition 15), we have \( [D_Z] \in K_p(Z) \). Moreover, by Theorem 19, this class only depends on the bundle \( S_N \). Using the projection map, \( \pi : Z \to M \), we obtain
\[ \pi_*([D_Z]) \in K_p(M) \]
We now relate this class to the class associated to the bundle \( S_M \) (denoted \( [D_M] \in K_p(M) \)).

**Theorem 20.** (c.f., Proposition 3.6 of [7])
We use the notation of the previous paragraphs. We assume that there exists a \( G \)-equivariant Dirac operator for \( S_N \) (denoted \( D_N \)) whose kernel is the trivial representation of \( G \). Moreover, we assume that this eigenspace is spanned by an even section of \( S_N \). Then
\[ \pi_*([D_Z]) = [D_M] \in K_p(M) \]

**Remark 21.** The previous theorem is related to the “multiplicative property” of the index defined by Atiyah and Singer (see p. 504 of [2]).

**Remark 22.** We note that Proposition 3.11 of [7] implies that we may take \( N = S^{2n} \) (oriented as the boundary of the unit ball) and \( G = SO(2n) \) in the statement of Theorem 20.
2.3. Geometric K-homology via Baum-Douglas theory. We review the geometric model for $K$-homology developed by Baum and Douglas. As mentioned above, we have followed [7] for this development. In particular, the reader is direct to Section 5 of this paper for more details.

**Definition 23.** Let $X$ be a locally compact Hausdorff space. A Baum-Douglas cycle (i.e., $K$-cycle) over $X$ is a triple, $(M, E, f)$, where

1. $M$ is a smooth, compact manifold with a fixed $spin^c$-structure (hencefore referred to as a compact $spin^c$-manifold);
2. $E$ is a smooth, Hermitian vector bundle over $M$;
3. $f : M \to X$ is a continuous map.

**Definition 24.** Given two Baum-Douglas cycles, $(M_1, E_1, f_1)$ and $(M_2, E_2, f_2)$, we define their disjoint union to be the cycle:

$$(M_1 \cup M_2, E_1 \cup E_2, f_1 \cup f_2)$$

**Remark 25.** In light of this definition, the reader should note that we do not assume that the manifold in a Baum-Douglas cycle is connected. However, we will say that $(M, E, f)$ has even (resp. odd) dimension if each of the connected components of $M$ has even (resp. odd) dimension.

**Definition 26.** Given a Baum-Douglas cycle, $(M, E, f)$, we define its opposite to be the cycle

$$(-M, E, f)$$

where $-M$ denote the $spin^c$-manifold with the opposite $spin^c$-structure (see Definition 4.8 in [7]).

**Definition 27.** A Baum-Douglas cycle with boundary is a triple, $(\bar{M}, \bar{E}, \bar{f})$, where $\bar{M}$ is a $spin^c$-manifold with boundary and $\bar{E}$ and $\bar{f}$ are as in Definition 23 (with $M$ replaced by $\bar{M}$).

**Definition 28.** We let $\sim$ denote the equivalence relation (on Baum-Douglas cycles) generated by the following:

1. **Direct Sum/Disjoint Union:** Let $(M_1, E_1, f_1)$ and $(M_2, E_2, f_2)$ be two Baum-Douglas cycles with the same manifold and map. Then we define

$$(M_1, E_1, f_1) \cup (M_2, E_2, f_2) \sim (M_1 \oplus E_2, f_1 \cup f_2)$$

2. **Bordism:** Let $(M, E, f)$ and $(M', E', f')$ be two Baum-Douglas cycles. Then

$$(M, E, f) \sim_{\text{bord}} (M', E', f')$$

if there exists cycle with boundary, $(\bar{M}, \bar{E}, \bar{f})$, such that

$$(\partial \bar{M}, \bar{E}|_{\partial \bar{M}}, \bar{f}|_{\partial \bar{M}}) = (M, E, f) \cup (-M', E', f')$$

where we note that $-M'$ denotes taking the opposite $spin^c$-structure (see Definition 26).

3. **Vector Bundle Modification:** Let $(M, E, f)$ be a Baum-Douglas cycle and $W$ a $spin^c$-vector bundle over $M$ with even dimensional fibers. We construct a new Baum-Douglas cycle as follows. Let $M'$ be the unit sphere bundle of the direct sum of $V$ with the trivial rank one line bundle. In particular,

$$M' = B(V) \cup_{S(V)} B(V)$$
where $B(V)$ is the unit ball bundle of $V$ and $S(V)$ is the sphere bundle of $V$.

Next, we construct a vector bundle as follows. To begin, using the theory of $\frac{1}{2}$-spin representations (c.f., p. 18 of [19]), one can construct bundles $H_+$ and $H_-$ over $B(V)$. Moreover, we also have that

$$H_+|_{S(V)} \cong H_-|_{S(V)}$$

Thus, using the clutching construction (c.f., Proposition 7.1 of [15]), we can form a vector bundle, $H$, over $M'$. Finally, if we let $\pi$ denote the bundle projection $M' \to M$, then we have Baum-Douglas cycle:

$$(M', H \otimes \pi^*(E), f \circ \pi)$$

We then define

$$(M, E, f) \sim (M', H \otimes \pi^*(E), f \circ \pi)$$

**Definition 29.** We let $K^{geo}_0(X) := \{\text{even Baum-Douglas-cycles}\}/\sim$

$K^{geo}_1(X) := \{\text{odd Baum-Douglas-cycles}\}/\sim$

**Proposition 30.** $K^{geo}_*(X)$ is an abelian group with addition defined by disjoint union, inverse defined by taking the opposite cycle, and identity given by the empty cycle.

**Definition 31.** We define a map from $K^{geo}_*(X)$ to $K^{ana}_*(X)$. To do so, let $(M, E, f)$ be a Baum-Douglas cycle and let $[D_E]$ denote the analytic cycle in $K^{ana}_*(M)$ constructed via Definition 15. Also, let $f_*$ denote the map induced on K-homology by $f$. We then define

$$\mu : K^{geo}_*(X) \to K^{ana}_*(X)$$

$$(M, E, f) \mapsto f_*([D_E])$$

**Remark 32.** It is not completely clear that the map, $\mu$, is well-defined; we direct the reader to Theorem 6.1 of [7] for a proof of this fact.

**Theorem 33.** (c.f., Theorem 6.2 of [7]) Let $X$ be a finite CW-complex and

$$\mu : K^{geo}(X) \to K^{ana}(X)$$

be defined as in Definition 31. Then $\mu$ is an isomorphism.

2.4. Geometric $\mathbb{Z}/k\mathbb{Z}$-cycles. We now discuss the main results of [11]. We begin by recalling some definitions from [13].

**Definition 34.** Let $Q$ be an oriented, smooth compact manifold with boundary. We assume that the boundary of $Q$, $\partial Q$, decomposes into $k$ disjoint manifolds, $(\partial Q)_1, \ldots, (\partial Q)_k$. A $\mathbb{Z}/k\mathbb{Z}$-structure on $Q$ is an oriented manifold $P$, a disjoint collar neighbourhood, $V_i$ for each $(\partial Q)_i$, and orientation preserving diffeomorphisms $\gamma_i : V_i \to (0, 1] \times P$. A $\mathbb{Z}/k\mathbb{Z}$-manifold is a $Q$ with fixed $\mathbb{Z}/k\mathbb{Z}$-structure. We denote this by $(Q, P, \gamma_i)$. We sometimes drop the maps from this notation and denote a $\mathbb{Z}/k\mathbb{Z}$-manifold by $(Q, P)$.
Remark 35. From the data, \((Q, P, \gamma_i)\), we can create a singular space. To do so, we note that the diffeomorphisms, \(\{\gamma_i\}^k_{i=1}\), induce a diffeomorphism between \(\partial Q\) and \(P \times \mathbb{Z}/k\mathbb{Z}\). The singular space is then created by collapsing each \(\{x\} \times \mathbb{Z}/k\mathbb{Z} \in P \times \mathbb{Z}/k\mathbb{Z}\) to a point.

Example 36. We consider the manifold with boundary, denoted by \(Q\), given in Figure 1 and take \(P = S^1\). Then one can easily see that \((Q, P)\) has the structure of a \(\mathbb{Z}/3\)-manifold.

Example 37. Any compact oriented manifold without boundary is a \(\mathbb{Z}/k\mathbb{Z}\)-manifold for any \(k\). One takes \(P = \emptyset\) and notes that \((M, \emptyset)\) has the structure required by Definition 34.

Remark 38. Using the process described in Remark 35, we can think of a \(\mathbb{Z}/k\mathbb{Z}\)-manifold as a singular space. Then for any point in this singular space, there is a neighbourhood that is either diffeomorphic to a neighbourhood in \(\mathbb{R}^n\) or is of the form shown in Figure 2. The number of “sheets of paper” is equal to \(k\).

Definition 39. Let \((Q, P, \gamma_i)\) be a \(\mathbb{Z}/k\mathbb{Z}\)-manifold.
1. A Riemannian metric on \((Q, P, \gamma_i)\) is a choice of Riemannian metrics on the manifolds \(Q\) and \(P\) with the additional property that the \(\gamma_i\) are isometries with respect to these metrics. The reader should note that we take the product metric on \((0, 1] \times P\).
2. A \(\mathbb{Z}/k\mathbb{Z}\)-fiber bundle over \((Q, P, \gamma_i)\) is a triple \((E, F, \theta_i)\) where \(E\) and \(F\) are fiber bundles over \(Q\) and \(P\) respectively and, for each \(i\), \(\theta_i\) is a lift of \(\gamma_i\) which is an isomorphism between \(E|_{V_i}\) and \(\pi^*(F)\) where \(\pi^*(F)\) is the pullback to \((0, 1] \times P\). In particular, a \(\mathbb{Z}/k\mathbb{Z}\)-vector bundle is a triple \((E, F, \theta_i)\) where the \(E\) and \(F\) are vector bundles.
3. A spin\(^c\)-structure on an oriented, even dimensional, orthogonal \(\mathbb{Z}/k\mathbb{Z}\)-bundle \((V_Q, V_P)\) is a Hermitian \(\mathbb{Z}/k\mathbb{Z}\)-vector bundle, \((S_Q, S_P)\), equipped with a Clifford action of \(V_Q\) as skew-adjoint endomorphisms of \(S_Q\) that is fiberwise irreducible. On each collaring neighborhood, \(V_i\), the Clifford action is the pullback of the product Clifford action on \((0, 1] \times P\); the Clifford action on \(P\) is given by a representation of \(V_P\) as skew-adjoint endomorphisms of \(S_P\) that is fiberwise irreducible.
4. A \(\mathbb{Z}/k\mathbb{Z}\)-manifold is spin\(^c\) if there exists a spin\(^c\)-structure on its tangent bundle. We assume a given spin\(^c\) \(\mathbb{Z}/k\mathbb{Z}\)-manifold comes with a fixed spin\(^c\) structure. The opposite of a spin\(^c\) \(\mathbb{Z}/k\mathbb{Z}\)-manifold, \((Q, P)\), is denoted by \(-(Q, P)\).
We will also need to consider $\mathbb{Z}/k\mathbb{Z}$-manifolds with boundary.

**Definition 40.** Let $\bar{Q}$ be an $n$-dimensional, oriented, smooth, compact manifold with boundary. In addition, assume we are given $k$ disjoint, oriented embeddings of an $(n-1)$-dimensional, oriented, smooth, compact manifold with boundary, $\bar{P}$, into $\partial \bar{Q}$. Using the same notation as Definition 34, we denote this as a triple, $(\bar{Q}, \bar{P}, \gamma_i)$ (or just $(\bar{Q}, \bar{P})$), where $\{\gamma_i\}_{i=1}^k$ denote the $k$ disjoint oriented embeddings. We refer to such a triple as a $\mathbb{Z}/k\mathbb{Z}$-manifold with boundary. The boundary of such an object is defined to be $\partial \bar{Q} - \operatorname{int}(k\bar{P})$ where $k\bar{P}$ denotes the $k$ copies of $\bar{P}$ in $\partial \bar{Q}$. We note that the boundary has a natural $\mathbb{Z}/k\mathbb{Z}$-manifold structure induced by identifying the $k$ copies of the boundary of $\bar{P}$ (see Remark 41 for more on the boundary).

**Remark 41.** If a $\mathbb{Z}/k\mathbb{Z}$-manifold $(Q, P)$ is the boundary of the $\mathbb{Z}/k\mathbb{Z}$-manifold with boundary, $(\bar{Q}, \bar{P})$, then

\begin{align*}
\partial \bar{Q} &= Q \cup_{\partial Q} (k\bar{P}) \\
\partial \bar{P} &= P
\end{align*}

Figure 3 illustrates the decomposition of $\partial \bar{Q}$. 
Figure 3. Decomposition of $\partial \check{Q}$

**Definition 42.** Let $X$ be a compact space. A $\mathbb{Z}/k\mathbb{Z}$-cycle over $X$ is a triple, $((Q, P), (E, F), f)$, where $(Q, P)$ is a spin$^c$ $\mathbb{Z}/k\mathbb{Z}$-manifold, $(E, F)$ is a $\mathbb{Z}/k\mathbb{Z}$-vector bundle and $f$ is a continuous map from $(Q, P)$ to $X$.

**Remark 43.** We note that the continuous map from $(E, F)$ to $X$ must respect the $\mathbb{Z}/k\mathbb{Z}$-structure. If the compact space $(X$ in Definition 42) is clear from the context, then we will refer to $\mathbb{Z}/k\mathbb{Z}$-cycles, rather than $\mathbb{Z}/k\mathbb{Z}$-cycles over $X$.

**Definition 44.** Given a $\mathbb{Z}/k\mathbb{Z}$-cycle, $((Q, P), (E, F), f)$, we will denote its opposite by $(-(Q, P), (E, F), f)$ where $-(Q, P)$ is the spin$^c$ $\mathbb{Z}/k\mathbb{Z}$-manifold with the opposite spin$^c$ structure.

We now define operations on $\mathbb{Z}/k\mathbb{Z}$-cycles. The reader should note the similarity with the operations on the cycles which defined the Baum-Douglas model.

**Definition 45.** Let $((Q, P), (E, F), f)$ and $((\hat{Q}, \hat{P}), (\hat{E}, \hat{F}), \hat{f})$ be $\mathbb{Z}/k\mathbb{Z}$-cycles. Then the disjoint union of these cycles is given by the cycle $((Q \cup \check{Q}, P \cup \check{P}), (E \cup \check{E}, F \cup \check{F}), \check{f} \cup f)$

**Definition 46.** We say a $\mathbb{Z}/k\mathbb{Z}$-cycle, $((Q, P), (E, F), f)$, is a boundary if there exist

1. A smooth compact spin$^c$ $\mathbb{Z}/k\mathbb{Z}$-manifold with boundary, $(\check{Q}, \check{P})$,
2. A smooth Hermitian $\mathbb{Z}/k\mathbb{Z}$-vector bundle $(V, W)$ over $(\check{Q}, \check{P})$,
3. A continuous map $\Phi : (\check{Q}, \check{P}) \to X$,

such that $(Q, P)$ is the $\mathbb{Z}/k\mathbb{Z}$-boundary of $(\check{Q}, \check{P})$, $(E, F) = (V, W)|_{\partial(\check{Q}, \check{P})}$, and $f = \Phi|_{\partial(Q, P)}$. We say that $((Q, P), (E, F), f)$ is bordant to $((\hat{Q}, \hat{P}), (\hat{E}, \hat{F}), \hat{f})$ if $((Q, P), (E, F), f) \cup (-((\hat{Q}, \hat{P}), (\hat{E}, \hat{F}), \hat{f})$ is a boundary.
Definition 47. We now define vector bundle modification for \( \mathbb{Z}/k\mathbb{Z} \)-cycles. Let \( ((Q, P), (E, F), f) \) be a \( \mathbb{Z}/k\mathbb{Z} \)-cycle and \( (W, V) \) be an even-dimensional spin\( ^c \) \( \mathbb{Z}/k\mathbb{Z} \)-vector bundle over \( (Q, P) \). We note that \( (Q, E, f) \) is a Baum-Douglas cycle with boundary and \( (P, F, f|_P) \) is a Baum-Douglas cycle. As such we can define the \( \mathbb{Z}/k\mathbb{Z} \)-vector bundle modification of \( ((Q, P), (E, F), f) \) by \( (W, V) \) to be the Baum-Douglas vector bundle modification of the cycles \( (Q, E, f) \) and \( (P, F, f|_P) \) by \( W \) and \( V \) respectively. The compatibility required by the definition of a \( \mathbb{Z}/k\mathbb{Z} \)-vector bundle ensures that the result of such a modification forms a \( \mathbb{Z}/k\mathbb{Z} \)-cycle.

Definition 48. We define \( K(X; \mathbb{Z}/k\mathbb{Z}) \) to be the set of equivalence classes of \( \mathbb{Z}/k\mathbb{Z} \)-cycles where the equivalence relation is given by the following:

1. If \( \epsilon_1 = ((Q, P), (E_1, F_1), \phi) \) and \( \epsilon_2 = ((Q, P), (E_2, F_2), \phi) \) are \( \mathbb{Z}/k\mathbb{Z} \)-cycles, then
   \[ \epsilon_1 \cup \epsilon_2 \sim ((Q, P), (E_1 \oplus E_2, F_1 \oplus F_2), \phi) \]
2. Bordant \( \mathbb{Z}/k\mathbb{Z} \)-cycles are defined to be equivalent;
3. If \( \epsilon \) is a \( \mathbb{Z}/k\mathbb{Z} \)-cycle and \( (W, V) \) is an even-dimensional spin\( ^c \) \( \mathbb{Z}/k\mathbb{Z} \)-vector bundle over it, then \( \epsilon \) is defined to be equivalent to its vector bundle modification by \( (W, V) \).

Proposition 49. The set \( K(X; \mathbb{Z}/k\mathbb{Z}) \) is a group with the operation of disjoint union. In particular, the identity element is given by the class of the trivial cycle and the inverse of a cycle is given by its opposite cycle (see Definition 47). We then denote by \( K_0(X; \mathbb{Z}/k\mathbb{Z}) \) (resp. \( K_1(X; \mathbb{Z}/k\mathbb{Z}) \)) the subgroup of \( K(X; \mathbb{Z}/k\mathbb{Z}) \) containing \( \mathbb{Z}/k\mathbb{Z} \)-cycles where \( Q \) is even (resp. odd) dimensional.

The Bockstein sequence for the model takes the following form (see Theorem 80 [11] for details).

Theorem 50. The following sequence is exact.

\[
\begin{array}{ccc}
K_0(X) & \xrightarrow{k} & K_0(X) \\
\uparrow & & \downarrow \\
K_1(X; \mathbb{Z}/k\mathbb{Z}) & \xleftarrow{r} & K_1(X) \\
\end{array}
\]

where the maps are
1. \( k : K_*(X) \to K_*(X) \) is given by multiplication by \( k \);
2. \( r : K_*(X) \to K_*(X; \mathbb{Z}/k\mathbb{Z}) \) takes a cycle \( (M, E, f) \) to \( ((M, \emptyset), (E, \emptyset), f) \);
3. \( \delta : K_*(X; \mathbb{Z}/k\mathbb{Z}) \to K_{*+1}(X) \) maps the cycle \( ((Q, P), (E, F), f) \) to \( (P, F, f) \).

3. Main Results

In the previous sections, we have summarized the construction of two models for \( K \)-homology. Since each of these models define the same homology theory, one would like to define an isomorphism between them at the level of cycles. A natural map between cycles in geometric \( K \)-homology (i.e., Baum-Douglas cycles) and cycles in analytic \( K \)-homology (i.e., Fredholm modules) was developed in the original works of Baum and Douglas [3, 4, 5, etc.]. In [7], this map was shown to be an isomorphism for finite CW-complexes. This map was discussed (albeit briefly) in Section 2.3 (in particular, see Theorem 33).
In this section, we deal with this isomorphism for $K$-homology with coefficients in $\mathbb{Z}/k\mathbb{Z}$. In [22], Schochet defines an analytic model for $K$-homology with coefficients. We use results of Rosenberg (see [21]) to link Schochet’s analytic cycles to $\mathbb{Z}/k\mathbb{Z}$-manifold theory. Here, if we let $X$ be a compact Hausdorff space, then we defined analytic $K$-homology with coefficients in $\mathbb{Z}/k\mathbb{Z}$ of $X$ to be $K^\ast(C(X) \otimes C^\ast(pt; \mathbb{Z}/k\mathbb{Z}))$, where $C^\ast(pt; \mathbb{Z}/k\mathbb{Z})$ is a certain $C^\ast$-algebra (see Definition 51).

Next, we construct a map from the geometric model developed in [11] (also see Section 2.4 above) to this analytic model. To do so, we introduce (and generalize) the construction of a groupoid $C^\ast$-algebra from a $\mathbb{Z}/k\mathbb{Z}$-manifold developed in [21]. We then define the required map from geometric cycles to analytic cycles and prove (under the condition that $X$ is a finite CW-complex) that it is an isomorphism.

The construction of $C^\ast$-algebras from $\mathbb{Z}/k\mathbb{Z}$-manifolds introduced in [21] uses the theory of groupoid $C^\ast$-algebras. The theory of groupoid $C^\ast$-algebras is developed in great detail in [20]. We will not need the full power of this theory and the reader unfamiliar with it could possibly take Equation 6 as the definition of the $C^\ast$-algebra associated to a $\mathbb{Z}/k\mathbb{Z}$-manifold.

3.1. Analytic $K$-homology with coefficients in $\mathbb{Z}/k\mathbb{Z}$. We construct an analytic model for $K$-homology with coefficients in $\mathbb{Z}/k\mathbb{Z}$ which is compatible with index theory on $\mathbb{Z}/k\mathbb{Z}$-manifolds. Our choice is based on results in [21] and [22]. To begin, we introduce the $C^\ast$-algebra which will play the same role in $\mathbb{Z}/k\mathbb{Z}$-index theory as the point (i.e., the $C^\ast$-algebra, $\mathbb{C}$) plays in the case of the Atiyah-Singer index theorem.

**Definition 51.** We let $C^\ast(pt; \mathbb{Z}/k\mathbb{Z})$ denote the mapping cone of the inclusion of $\mathbb{C}$ into the $k$ by $k$ matrices, $M_k$. That is, we let

\begin{equation}
C^\ast(pt; \mathbb{Z}/k\mathbb{Z}) := \{ f \in C_0([0, \infty), M_k) | f(0) \text{ is a multiple of } I_k \}
\end{equation}

Basic properties of mapping cones imply that we have the following exact sequence of $C^\ast$-algebras:

$$0 \to C_0((0, \infty), M_k) \to C^\ast(pt; \mathbb{Z}/k\mathbb{Z}) \to \mathbb{C} \to 0$$

By considering the associated six-term exact sequence in $K$-homology, we have the following result.

**Lemma 52.** Let $C^\ast(pt; \mathbb{Z}/k\mathbb{Z})$ be defined as in Equation 3. Then

$$K^0(C^\ast(pt; \mathbb{Z}/k\mathbb{Z})) \cong \mathbb{Z}/k\mathbb{Z}$$

$$K^1(C^\ast(pt; \mathbb{Z}/k\mathbb{Z})) \cong 0$$

Based on the previous lemma and Sections 5 and 6 of [22], we have the following definition for analytic $K$-homology with coefficients in $\mathbb{Z}/k\mathbb{Z}$.

**Definition 53.** Let $X$ be a compact Hausdorff space. Then, we let $K_p^\ast \text{ana}(X; \mathbb{Z}/k\mathbb{Z}) = K^{-p}(C(X) \otimes C^\ast(pt, \mathbb{Z}/k\mathbb{Z}))$.

**Remark 54.** The Bockstein sequence for $K_p^\ast \text{ana}(X; \mathbb{Z}/k\mathbb{Z})$ is given by the six-term exact sequence associated to short exact sequence of $C^\ast$-algebras:

$$0 \to C(X) \otimes C_0((0, \infty), M_k) \to C(X) \otimes C^\ast(pt; \mathbb{Z}/k\mathbb{Z}) \to C(X) \to 0$$

We note that we must also use Bott periodicity to put this six-term exact sequence into the form:
3.2. **Rosenberg’s Groupoid $C^*$-algebra.** We generalize Rosenberg’s construction in [21] (also see [16]). Rosenberg’s construction is discussed in Example 59 below. Moreover, Figure 4 should be helpful to the reader during both the discussion of this example and the general construction.

We will work in a general framework, but Examples 59 and 61 are our main concern. The setup is the following. Let $N$ be a smooth manifold. We note that $N$ may have boundary and is not necessarily compact. Moreover, suppose that

1. $N = M^Q \cup_\Sigma M^P$ where $M^Q$ and $M^P$ are manifolds with boundary and $\Sigma$ is a manifold (possibly with boundary);
2. Moreover, we assume that $\Sigma \subseteq \partial M^Q$ and $\Sigma \subseteq \partial M^P$;
3. $M^P = k \cdot R$ for some manifold $R$ and $\Sigma = k \cdot \Sigma_R$ for some manifold $\Sigma_R$;

An important case of this setup was considered by Rosenberg in [21]. It is the following. Let $(Q, P)$ be a $\mathbb{Z}/k\mathbb{Z}$-manifold. Then, using the notation above, we take

4. $M^Q = Q$, $M^P = \partial Q \times [0, \infty)$, $\Sigma = \partial Q$, $R = P \times [0, \infty)$, $\Sigma_R = P$

We will discuss this case in more detail in Example 59 below.

Returning to the general setup, we construct a groupoid via an equivalence relation on $N$. We note that Figure 4 illustrates the important special case (i.e., Equation 4). The relation is defined by

1. If $n \in M^Q$, then $n$ is equivalent only to itself. We note that this includes points in $\partial Q$ and hence points in $\Sigma$.
2. If $n, n' \in M^P - \Sigma$, then $n \sim n'$ if and only if
$$\pi(n) = \pi(n') \in R$$
where $\pi$ denotes the trivial covering map $M^P \rightarrow R$.

**Definition 55.** Using the notation and constructions in the previous paragraphs, we let $\mathcal{G} \subset N \times N$ denote the groupoid associated to this equivalence relation and let $C^*(\mathcal{G})$ denote the associated groupoid $C^*$-algebra.

**Remark 56.** Before the statement of the next theorem, we make a remark about notation. In the setup above, we have not assumed any compactness properties; hence we will work with $C_0$-functions. In particular, if $W$ is a manifold with boundary then $C_0(W)$ denotes continuous functions which vanish at $\infty$ but which take nonzero values on the boundary of $W$. We will use $C_0(\text{int}(W))$ to denote the continuous functions that vanish at $\infty$ and on the boundary. We also note that $M_k$ denotes the $k$ by $k$ matrices and if $M$ is a space, then $C_0(M, M_k)$ denote the continuous functions from $M$ to $M_k$ which vanish at $\infty$. 

$$
\begin{align*}
K^a_0(X) & \longrightarrow K^a_0(X) \longrightarrow K^a_0(X; \mathbb{Z}/k\mathbb{Z}) \\
\uparrow & \\
K^a_1(X; \mathbb{Z}/k\mathbb{Z}) & \leftarrow K^a_1(X) \leftarrow K^a_1(X)
\end{align*}
$$
Figure 4. The equivalence relation on the \( \mathbb{Z}/3 \)-manifold from Example [36].

**Proposition 57.** Let \( C^*(\mathcal{G}) \) be the \( C^* \)-algebra from Definition [55]. Then it is isomorphic to

\[
\{(f, g) \in C_0(M^Q) \oplus C_0(R, M_k) \mid g|_{\Sigma_R} \text{ is diagonal and } f|_{\Sigma} = g|_{\Sigma_R}\}
\]

We note that the statement \( f|_{\Sigma} = g|_{\Sigma_R} \) is more correctly written as

\[
\alpha(f|_{\Sigma}) = g|_{\Sigma_R}
\]

where \( \alpha : C_0(\Sigma) \cong \oplus_{i=1}^k C_0(\Sigma_R) \to M_k(C_0(\Sigma_R)) \) is the diagonal inclusion.

**Proof.** We only sketch the ideas of the proof, leaving the details for the interested reader. To begin, we review some notation. Recall (see the paragraphs preceding Definition [55]) that \( N = M^Q \cup_{\Sigma} M^P \) where \( M^P = k \cdot R \) and \( \mathcal{G} \) denotes the equivalence relation (defined above) on \( N \). We will denote an element of \( \mathcal{G} \), \( n_1 \sim n_2 \), as \( (n_1, n_2) \in N \times N \). In addition, if \( p \in R \), then we let \( p_1, \ldots, p_k \) denote the preimages of \( p \) under the (trivial) covering map \( M^P \to R \).

Let \( h \in C_c(\mathcal{G}) \) (i.e., a continuous function with compact support on \( \mathcal{G} \)) and define a map:

\[
h \mapsto (f_h, g_h) \in C_0(M^Q) \oplus C_0(R, M_k)
\]

as follows. For \( q \in M^Q \),

\[
f_h(q) := h(q, q)
\]

and, for \( p \in R - \Sigma_R \), we define

\[
g_h(p) := [h(p_i, p_j)]_{i=1}^k_{j=1} \ldots_{j=1}^k
\]

where we have used the definition of \( \{p_i\}_{i=1}^k \) discussed in the first paragraph of the proof. Finally, for \( \tilde{p} \in \Sigma_R \), we define \( g_h(\tilde{p}) \) to be the diagonal \( k \times k \) matrix with entries along the diagonal given by

\[
h(p_1, p_1), \ldots, h(p_k, p_k)
\]
It is now left to the reader to show that this map is well-defined and extends to an isomorphism from \( C^*(G) \) to the \( C^* \)-algebra in the statement of the proposition (i.e., Equation 57).

\[ \square \]

**Corollary 58.** The \( C^* \)-algebra from Definition 55 (i.e., \( C^*(G) \)) fits into the following exact sequence:

\[ 0 \to C_0(R - \Sigma_R) \otimes M_k \to C^*(G) \to C_0(M^\mathbb{Q}) \to 0 \]

**Example 59.** The prototypical example of a groupoid of the form discussed in Example 59. We will denote \( C^*(G) \) by \( \mathcal{G} \). The content of Proposition 55 in this case takes the form:

\[ C^*(Q,P;\mathbb{Z}/k\mathbb{Z}) \cong \{ (f,g) \in C(Q) \oplus C_0(P \times [0,\infty), M_k) \mid \]

\[ g|_{\partial Q \times \{0\}} \text{ is diagonal and } f|_{\partial Q} = g|_{P \times \{0\}} \}

We recall that the statement \( f|_{\partial Q} = g|_{P \times \{0\}} \) is more correctly written as

\[ \alpha(f|_{\partial Q}) = g|_{P \times \{0\}} \]

where \( \alpha : C(\partial Q) \cong \oplus_{i=1}^k C(P) \to M_k(C(P)) \) is the diagonal inclusion.

In addition, we have the following exact sequence (see Equation 54 or [21]).

\[ 0 \to C_0(\mathbb{R}) \otimes C(P) \otimes M_k \to C^*(Q,P;\mathbb{Z}/k\mathbb{Z}) \to C(Q) \to 0 \]

**Remark 60.** If we are given a \( \mathbb{Z}/k\mathbb{Z} \)-manifold of the form \((M,\emptyset)\) (where \( M \) is a compact manifold), then we let

\[ C^*(M,\emptyset;\mathbb{Z}/k\mathbb{Z}) := C(M) \otimes C^*(pt;\mathbb{Z}/k\mathbb{Z}) \]

**Example 61.** In this example, we consider the case of a \( \mathbb{Z}/k\mathbb{Z} \)-manifold with boundary. We will form two \( C^* \)-algebra; they are analogous to \( C_0(\text{int}(W)) \) and \( C(W) \) in the case of a compact manifold with boundary \( W \). To fix notation, we let \((\bar{Q},\bar{P})\) be a \( \mathbb{Z}/k\mathbb{Z} \)-manifold with boundary, \( (Q,P) \).

We begin with the \( C^* \)-algebra which is analogous to \( C_0(\text{int}(W)) \). In the notation of our basic setup (see the discussion preceding Definition 55), we let

\[ M^Q = \bar{Q} - Q, M^P = \partial \bar{Q} - Q \times [0,\infty), \Sigma = \partial Q - \bar{Q}, R = \text{int}(\bar{P}) \times [0,\infty), \Sigma_R = \text{int}(\bar{P}) \]

and form the associated groupoid \( C^* \)-algebra, which we denote by \( C_0(\hat{Q},\hat{P};\mathbb{Z}/k\mathbb{Z}) \).

Proposition 57 implies that

\[ C_0(\hat{Q},\hat{P},\hat{\pi}) = \{ (f,g) \in C(\hat{Q}) \oplus C_0(\text{int}(\hat{P}) \times [0,\infty), M_k) \mid \]

\[ f|_Q = 0, g|_{\text{int}(\hat{P}) \times \{0\}} \text{ is diagonal and } f|_{\partial Q - Q} = g|_{\text{int}(\hat{P}) \times \{0\}} \}\]

We note that we have used the identification discussed in Proposition 57.

Next, we discuss the \( C^* \)-algebra which is analogous to \( C(W) \) in the case of a manifold with boundary. Again, using the setup discussed before Definition 55 we let

\[ M^Q = \bar{Q} - \text{int}(Q), M^P = \partial \bar{Q} - \text{int}(Q) \times [0,\infty), \Sigma = \partial \bar{Q} - \text{int}(Q), R = \bar{P} \times [0,\infty), \Sigma_R = \bar{P} \]
In this case, we denote the associated groupoid $C^*$-algebra by $C^*(\tilde{Q}, \tilde{P}; \mathbb{Z}/k\mathbb{Z})$.

Proposition 57 implies that
\[
C^*(\tilde{Q}, \tilde{P}, \tilde{\pi}) = \{(f, g) \in C(\tilde{Q}) \oplus C_0(\tilde{P} \times [0, \infty), M_k) \mid g|_{\tilde{P} \times \{0\}} \text{ is diagonal and } f|_{\tilde{\partial}Q - \text{int}(\tilde{Q})} = g|_{\tilde{P} \times \{0\}}\}
\]

We note that we have again used the identification discussed in Proposition 57.

Moreover, we have the following exact sequence:
\[
0 \to C_0(\tilde{Q}, \tilde{P}; \mathbb{Z}/k\mathbb{Z}) \to C^*(\tilde{Q}, \tilde{P}; \mathbb{Z}/k\mathbb{Z}) \to C^*(Q, P; \mathbb{Z}/k\mathbb{Z}) \to 0
\]

This is the $\mathbb{Z}/k\mathbb{Z}$-version of the exact sequence:
\[
0 \to C_0(\text{int}(\tilde{W})) \to C(W) \to C(\partial W) \to 0
\]
in the case of a manifold with boundary $\tilde{W}$.

We now consider natural classes in the $K$-homology of these groupoid $C^*$-algebras. We have followed [21] for this development. The setup is as follows. Let $(Q, P)$ be a spin$^c$ $\mathbb{Z}/k\mathbb{Z}$-manifold with $\dim(Q) = n$ and $D$ be the Dirac operator on it (possibly twisted by a $\mathbb{Z}/k\mathbb{Z}$-vector bundle). Let $S_Q$ denote the Dirac bundle to which $D$ is associated. Then $S_Q$ extends to a Dirac bundle on $N$ (which we denote by $S_N$) and $D$ also extends to all of $N$. We denote the extension of the operator, $D$, to $N$ by $D_N$. By Theorem 14 and Definition 15 we can form $[D_N] \in K^{-n}(C_0(N))$. Moreover, this class is equivariant with respect to the groupoid associated to $(Q, P)$; hence $D_N$ defines a class in $K^{-n}(C^*(Q, P; \mathbb{Z}/k\mathbb{Z}))$. To simplify notation, we will denote the class produced from this construction by $[D] \in K^{-n}(C^*(Q, P; \mathbb{Z}/k\mathbb{Z}))$.

Moreover, the same construction applies verbatim to produce a class $[D] \in K^{-n}(C_0^*(Q, P; \mathbb{Z}/k\mathbb{Z}))$ where $(Q, P)$ is a spin$^c$ $\mathbb{Z}/k\mathbb{Z}$-manifold with boundary and $D$ is again the Dirac operator (possibly twisted by a $\mathbb{Z}/k\mathbb{Z}$-vector bundle). The next proposition summarizes basic properties of the $K$-homology classes produced by this construction.

**Proposition 62.** Let $(Q, P)$ be a spin$^c$ $\mathbb{Z}/k\mathbb{Z}$-manifold with $\dim(Q) = n$ and $D_{(Q,P)}$ denote the Dirac operator on $(Q, P)$ (possibly twisted by a $\mathbb{Z}/k\mathbb{Z}$-vector bundle). Then the associated class $[D_{(Q,P)}] \in K^{-n}(C^*(Q, P; \mathbb{Z}/k\mathbb{Z}))$ has the following properties:

1. Let $(W, V)$ be a spin$^c$ $\mathbb{Z}/k\mathbb{Z}$-vector bundle over $(Q, P)$ with the dimension of the fibers equal to $2k$, and let $(E, F)$ denote a vector bundle over $(Q, P)$. Moreover, we assume that $[D_{(Q,P)}]$ is the Dirac operator of $(Q, P)$ twisted $(E, F)$. We denote the spin$^c$ $\mathbb{Z}/k\mathbb{Z}$-manifold produced by the vector bundle modification of $(Q, P)$ by $(W, V)$ by $(Q^W, P^V)$. Let $\pi$ denote the projection $(Q^W, P^V) \to (Q, P)$ and $\tilde{\pi}$ denote the induced map from $C^*(Q^W, P^V; \mathbb{Z}/k\mathbb{Z})$ to $C^*(Q^W, P^V; \mathbb{Z}/k\mathbb{Z})$. Finally, let $[D_{(Q^W, P^V)}] \in K^{-p-2k}(C^*(Q^W, P^V; \mathbb{Z}/k\mathbb{Z}))$ denote the classes associated to the Dirac operator (now twisted by vector bundle $(H_Q \otimes \pi^*(E), H_P \otimes \pi^*(F))$; see Definitions 28 and 47). Then
\[
\tilde{\pi}^*[D_{(Q^W, P^V)}] = [D_{(Q,P)}] \in K^{-n}(C^*(Q, P; \mathbb{Z}/k\mathbb{Z}))
\]

2. Suppose that $(Q, P)$ is the boundary of the $\mathbb{Z}/k\mathbb{Z}$-manifold, $(\tilde{Q}, \tilde{P})$. If we let
\[
\partial : K^{-n-1}(C_0^*(\tilde{Q}, \tilde{P}; \mathbb{Z}/k\mathbb{Z})) \to K^{-n}(C^*(Q, P; \mathbb{Z}/k\mathbb{Z}))
\]
be the boundary map of the six-term exact sequence associated to the exact sequence

\[ 0 \to C^*_0(Q, \tilde{P}; \mathbb{Z}/k\mathbb{Z}) \to C^*(\tilde{Q}, \tilde{P}; \mathbb{Z}/k\mathbb{Z}) \to C^*(Q, P; \mathbb{Z}/k\mathbb{Z}) \to 0 \]

then

\[ \partial[D_{\tilde{Q}, \tilde{P}}] = [D_{(Q, P)}] \]

**Proof.** In both cases, the result follows from the corresponding result in the commutative case and the fact that maps involved respect the groupoid used in the construction of $C^*$-algebras associated to $\mathbb{Z}/k\mathbb{Z}$-manifolds. We leave the details to the reader; the general idea is as follows.

For item 1) in the statement of the proposition, we use Theorem 20 and Remark 22. These results imply that $[D_N] = \pi^*[D_{N,W}] \in K^{−p}(C_0(N))$

where $\hat{W}$ is the extension of the vector bundle $W$ to the manifold $N = Q \cup_{\partial Q} \partial Q \times [0, \infty)$. Moreover, the required result follows from the fact that the classes (i.e., $[D_N]$ and $[D_{N,W}]$) and the map $\pi$ respect the groupoid used to define $C^*(Q, P; \mathbb{Z}/k\mathbb{Z})$.

The second item in the statement follows in a similar way; the required result from the commutative case being item 3) of Theorem 19. □

One of the advantages of Rosenberg’s definition is that we have formed a class in $K^{−p}(C^*(Q, P; \mathbb{Z}/k\mathbb{Z}))$. To define the index, we need only produce a “collapse to point” map. While to define a map between analytic and geometric $K$-homology with coefficients, we will need to pull-back the Dirac class from $K^{−p}(C^*(Q, P; \mathbb{Z}/k\mathbb{Z}))$ to $K_p^{an}(X; \mathbb{Z}/k\mathbb{Z})$.

We now consider the case of the former (i.e., the index map). The reader should recall that, following Rosenberg, we defined the $C^*$-algebra, $C^*(pt; \mathbb{Z}/k\mathbb{Z})$, to play the role of a point. In particular, Rosenberg replaces the collapse to point map in classical index theory with the $*$-homomorphism, $c$, defined by

\[ c : C^*(pt; \mathbb{Z}/k\mathbb{Z}) \to C^*(Q, P; \mathbb{Z}/k\mathbb{Z}) \]

\[ h \mapsto h(0)1_Q \oplus (1_P \otimes h) \]

where we have used Equation 8 and the fact that

\[ C^*(pt; \mathbb{Z}/k\mathbb{Z}) \to \{ f \in C_0([0, \infty), M_k) | f(0) \text{ diagonal} \} \]

**Definition 63.** Using the notation of the previous paragraph, we define the analytic index of $D$ to be the image of $[D]$ under the map on $K$-homology induced by the map which collapses $Q$ to a point. That is, the analytic index of $D$ is defined to be $c^*([D])$ where $c : C^*(pt; \mathbb{Z}/k\mathbb{Z}) \to C^*(Q, P; \mathbb{Z}/k\mathbb{Z})$ was defined above in Equation 8.

### 3.3. Map between geometric and analytic $K$-homology with coefficients.

We now define the map between cycles which will lead to an isomorphism between analytic and geometric $K$-homology with coefficients in $\mathbb{Z}/k\mathbb{Z}$. Given a $\mathbb{Z}/k\mathbb{Z}$-cycle, we need to make sense of pulling back the class of the Dirac operator in $K^*(C^*(Q, P; \mathbb{Z}/k\mathbb{Z}))$ to a class in $K_p^{an}(X; \mathbb{Z}/k\mathbb{Z})$. To do so, we need to show that, if $f : (Q, P) \to X$ is a continuous map, then it induces a $*$-homomorphism from $C(X) \otimes C^*(pt, \mathbb{Z}/k\mathbb{Z})$ to $C^*(Q, P; \mathbb{Z}/k\mathbb{Z})$. We denote by $f^*$ the map induced on $K$-homology by this $*$-homomorphism and then define the map between geometric and analytic $K$-homology with coefficients in $\mathbb{Z}/k\mathbb{Z}$.
Definition 64. Given a continuous map, \( f : (Q, P) \to X \), we define a \( * \)-homomorphism via

\[
\tilde{f} : C(X) \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z}) \to C^*(Q, P; \mathbb{Z}/k\mathbb{Z})
\]

where we have denoted by \( f_Q^* \) and \( f_P^* \) the \( * \)-homomorphism induced by \( f|_Q \) and \( f|_P \). Note that we have also used Equation 6 and the fact that \( C^*(pt; \mathbb{Z}/k\mathbb{Z}) \to \{ g \in C_0([0, \infty), M_k) | g(0) \text{ diagonal } \} \)

Remark 65. In the case when \( X = pt \), the \( * \)-homomorphism from Definition 64, \( C \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z}) \to C^*(Q, P; \mathbb{Z}/k\mathbb{Z}) \), is the same as the one defined in Equation 8. (One must first identify \( C \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z}) \) with \( C^*(pt; \mathbb{Z}/k\mathbb{Z}) \)).

Definition 66. Let \( X \) be a compact Hausdorff space. Let \( \Phi \) be the map between geometric \( \mathbb{Z}/k\mathbb{Z} \)-cycles and analytic \( \mathbb{Z}/k\mathbb{Z} \)-cycles defined by

\[
((Q, P), (E, F), f) \mapsto \tilde{f}^*([D_{(E,F)}])
\]

where \([D_{(E,F)}] \in K^*(C^*(Q, P; \mathbb{Z}/k\mathbb{Z}))\) is the class of the Dirac operator twisted by \((E, F)\) and \( \tilde{f}^* \) is the map on \( K \)-homology induced from \( f \) (see Definition 64).

Our first goal is to show that the map, \( \Phi \), is well-defined. We must show that the class in analytic \( K \)-homology is invariant under the relations on the geometric \( \mathbb{Z}/k\mathbb{Z} \)-cycles. The proof for the disjoint union operation is trivial.

The case of vector bundle modification follows from Item 1 in Proposition 62.

Corollary 67. Let \( X \) be a compact Hausdorff space, \((Q, P), (E, F), f\) a \( \mathbb{Z}/k\mathbb{Z} \)-cycle over \( X \), and \((W, V)\) a \( \text{spin}^c \mathbb{Z}/k\mathbb{Z} \)-vector bundle over \((Q, P)\) with even dimensional fiber. Then, using the same notation as the previous theorem, we have that

\[
\tilde{f}^*([D_{(E,F)}]) = (\tilde{f} \circ \tilde{\pi})^*([D_{(E,W,F,V)}])
\]

Proof. Based on Definition 63 we have \( * \)-homomorphisms

\[
\tilde{f}_{Q,P} : C(X) \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z}) \to C^*(Q, P; \mathbb{Z}/k\mathbb{Z})
\]

\[
\tilde{f}_{Q^W,P^V} : C(X) \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z}) \to C^*(Q^W, P^V; \mathbb{Z}/k\mathbb{Z})
\]

Moreover, based on the definition of vector bundle modification, we have that

\[
\tilde{\pi} \circ \tilde{f}_{Q,P} = \tilde{f}_{Q^W,P^V}
\]

Now, Item 1) of Proposition 62 implies

\[
\tilde{f}_{Q^W,P^V}([D_{Q^W,P^V}]) = (\tilde{\pi} \circ \tilde{f}_{Q,P})^*([D_{Q^W,P^V}])
\]

\[
= \tilde{f}_{Q,P}^*\tilde{\pi}^*([D_{Q^W,P^V}])
\]

\[
= \tilde{f}_{Q,P}^*([D_{Q,P}])
\]

This completes the proof that \( \Phi \) respects vector bundle modification. The case of \( \mathbb{Z}/k\mathbb{Z} \)-bordism is less clear; we must prove that the \( \tilde{f}^*([D_{(Q,P)}]) \) vanishes in \( K^*_{\text{ana}}(X; \mathbb{Z}/k\mathbb{Z}) \) if the cycle \((Q, P), (E, F), f\) is a \( \mathbb{Z}/k\mathbb{Z} \)-boundary (in the sense of Definition 66).
To begin, we recall the proof in the classical case (c.f., [6], [7], or Exercise 11.8.10 in [14]). Let $\bar{M}$ be a compact spin$^c$-manifold with boundary and denote its boundary by $\partial M$ and its interior by $M$. We have the following exact sequence of $C^*$-algebras:

$$(9) \quad 0 \to C_0(M) \to C(\bar{M}) \to C(\partial M) \to 0$$

Applying the analytic K-homology functor, we get the following six-term exact sequence.

$$\begin{array}{cccc}
K^{even}(C(\partial M)) & \longrightarrow & K^{even}(C(\bar{M})) & \longrightarrow & K^{even}(C_0(M)) \\
\uparrow & & & & \downarrow \\
K^{odd}(C_0(M)) & \longleftarrow & K^{odd}(C(\bar{M})) & \longleftarrow & K^{odd}(C(\partial M))
\end{array}$$

Moreover, assume we are given a continuous function, $f : \bar{M} \to X$, where $X$ is a compact Hausdorff space. Let $\tilde{f} : C(X) \to C(\bar{M})$ and $\tilde{f}|_{\partial M} : C(X) \to C(\partial M)$ denote the $*$-homomorphisms induced from the continuous functions $f$ and $f|_{\partial M}$. Moreover, we denote the $*$-homomorphism, $C(\bar{M}) \to C(\partial M)$, induced from the inclusion of $\partial M$ into $\bar{M}$ by $r$. It follows that $\tilde{r} \circ \tilde{f} = \tilde{f}|_{\partial M}$.

We now consider $(\tilde{f}|_{\partial M})^*([D_{\partial M}])$. We have that $\partial[D_M] = [D_{\partial M}]$ and hence

$$r^*([D_{\partial M}]) = (r^* \circ \partial)[D_M] = 0$$

is zero by exactness. Based on the fact that the map induced from $f$ factors through the map $r^* : K^*(C(\partial M)) \to K^*(C(\bar{M}))$, it follows that

$$(\tilde{f}|_{\partial M})^*([D_{\partial M}]) = (r \circ \tilde{f})^*([D_{\partial M}]) = \tilde{f}^* (r^*([D_{\partial M}])) = 0$$

This proves the required cobordism invariance.

We now use the groupoid $C^*$-algebras constructed in Examples 59 and 61 to generalize the classical construction to give a proof of the cobordism invariance (see Theorem 70 below).

We recall that given, $(\bar{Q}, \bar{P})$, a $\mathbb{Z}/k\mathbb{Z}$-manifold with boundary, $(Q, P)$, we have constructed two $C^*$-algebras (see Example 61 for details):

$$
\begin{align*}
C^*_0(\bar{Q}, \bar{P}, \bar{\pi}) &= \{ (f, g) \in C(\bar{Q}) \oplus C_0(\bar{P} \times [0, \infty), M_k) \\
&\quad | f|_{\bar{Q}} = 0 , g|_{\bar{P} \times \{0\}} \text{ is diagonal and } f|_{\partial \bar{Q} - \text{int}(Q)} = g|_{\bar{P} \times \{0\}} \} \\
C^*(\bar{Q}, \bar{P}, \bar{\pi}) &= \{ (f, g) \in C(\bar{Q}) \oplus C_0(\bar{P} \times [0, \infty), M_k) \\
&\quad | g|_{\bar{P} \times \{0\}} \text{ is diagonal and } f|_{\partial \bar{Q} - \text{int}(Q)} = g|_{\bar{P} \times \{0\}} \} 
\end{align*}
$$

We note that we must identify the diagonal of $C(\bar{P}, M_k)$ with the direct sum of $k$-copies of $C(\bar{P}) \cong C(\partial \bar{Q} - \text{int}(Q))$. We also have the following exact sequences involving the above two $C^*$-algebras and the $C^*$-algebra of Rosenberg (see Example 59):

$$(10) \quad 0 \to C^*_0(\bar{Q}, \bar{P}, \bar{\pi}) \to C^*(\bar{Q}, \bar{P}, \bar{\pi}) \to C^*(Q, P, \pi) \to 0$$

where the $*$-homomorphism,

$$r : C^*(\bar{Q}, \bar{P}, \bar{\pi}) \to C^*(Q, P, \pi)$$

is given by restriction and the $*$-homomorphism,

$$i : C^*_0(\bar{Q}, \bar{P}, \bar{\pi}) \to C^*(\bar{Q}, \bar{P}, \bar{\pi})$$

is given by inclusion.
This exact sequence is the $\mathbb{Z}/k\mathbb{Z}$-version of the exact sequence

$$0 \to C_0(M) \to C(M) \to C(\partial M) \to 0$$

In particular, we have the following.

**Lemma 68.** Let $(W, Z, \tilde{\pi})$ be a $\mathbb{Z}/k\mathbb{Z}$-manifold with boundary, $(Q, P, \pi)$ and $f : (W, Z, \tilde{\pi}) \to X$ be a continuous function into a compact Hausdorff space, $X$. Then we have $*$-homomorphisms,

$$\tilde{f} : C(X) \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z}) \to C^*(\tilde{Q}, \tilde{P}, \tilde{\pi})$$

Moreover, using the notation of the previous paragraph, $r \circ \tilde{f} = \tilde{f}|_{(Q,P,\pi)}$.

**Proof.** The homomorphisms are defined in a similar way to Definition [64]. In fact, the second homomorphism in the statement of the lemma is defined there. To be clear, the first homomorphism is defined by

$$\tilde{f} : C(X) \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z}) \to C^*(\tilde{Q}, \tilde{P}, \tilde{\pi})$$

$$\quad h \otimes r \mapsto (h \circ f) \oplus ((h \circ f) \circ r)$$

Since the homomorphism between $r : C^*(\tilde{Q}, \tilde{P}, \tilde{\pi}) \to C^*(Q, P, \pi)$ is also given by restriction, we have the result. \hfill \Box

**Corollary 69.** Let $X$ be a compact Hausdorff space and $((Q,P),(E,F),f)$ be a $\mathbb{Z}/k\mathbb{Z}$-cycle over $X$, which is the boundary of $((Q,P),(E,F),f)$. Then the map $\tilde{f}^*$ (see Definition [64]) factors through the map induced from the inclusion of the boundary. In other words, we have the following commutative diagram:

$$\begin{array}{ccc}
K^*(C^*(Q,P,\pi)) & \xrightarrow{r} & K^*(C^*(\tilde{Q},\tilde{P},\tilde{\pi})) \\
\downarrow{\tilde{f}^*} & & \downarrow{(\tilde{f})^*} \\
K^*(C(X) \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z})) & & \end{array}$$

**Theorem 70.** If $((Q,P),(E,F),f)$ is a $\mathbb{Z}/k\mathbb{Z}$-cycle which is a boundary, then $\tilde{f}^*([D_{(Q,P),(E,F)}]) = 0$ in $K^*_a(C(X); \mathbb{Z}/k\mathbb{Z}) = K^*(C(X) \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z}))$.

**Proof.** Denote by $((\tilde{Q},\tilde{P}), (\tilde{E},\tilde{F}), \tilde{f})$ the cycle which has boundary, $((Q,P),(E,F),f)$. We have the following exact sequence relating the $C^*$-algebras associated to these objects:

$$0 \to C^*_0(\tilde{Q},\tilde{P}) \xrightarrow{\partial} C^*(\tilde{Q},\tilde{P}) \xrightarrow{r} C^*(Q,P) \to 0$$

Applying the analytic $K$-homology functor, we get a long exact sequence.

$$\to K^*(C^*_0(\tilde{Q},\tilde{P})) \xrightarrow{\partial} K^{*+1}(C^*(Q,P)) \xrightarrow{r} K^{*+1}(C^*(\tilde{Q},\tilde{P})) \to K^{*+1}(C^*_0(\tilde{Q},\tilde{P})) \to$$

Based on item 2) of Proposition [62] and exactness, we have that

$$0 = (r^* \circ \partial)([D_{(Q,P),(E,F)}]) = r^*([D_{(Q,P),(E,F)}])$$

Moreover, using Corollary [69] we have that

$$\tilde{f}^*([D_{(Q,P),(E,F)}]) = \tilde{F}^*(r^*([D_{(Q,P),(E,F)}])) = 0$$

\hfill \Box
3.4. Isomorphism. We have now completed the proof that the map $\Phi$ is well-defined. In the case when $X$ is a finite CW-complex we have more; namely:

**Theorem 71.** Let $X$ be a finite CW-complex. Then the map

$$\Phi : K^\text{geo}_*(X; \mathbb{Z}/k\mathbb{Z}) \to K^\text{ana}_*(X; \mathbb{Z}/k\mathbb{Z})$$

constructed in Definition 66 is an isomorphism.

**Proof.** To begin, we prove that the Bockstein sequences of the analytic and geometric models fit into the following commutative diagram (we show only the $K_0$-part of the relevant commutative diagram):

$$
\begin{array}{ccccccc}
\longrightarrow & K^\text{geo}_0(X) & \longrightarrow & K^\text{geo}_0(X) & \longrightarrow & K^\text{geo}_0(X; \mathbb{Z}/k\mathbb{Z}) & \longrightarrow \\
\downarrow & \uparrow \mu & & \downarrow & \uparrow \mu & & \downarrow & \Phi \\
\longrightarrow & K^\text{ana}_0(X) & \longrightarrow & K^\text{ana}_0(X) & \longrightarrow & K^\text{ana}_0(X; \mathbb{Z}/k\mathbb{Z}) & \longrightarrow \\
\end{array}
$$

where we recall that $\mu$ denotes the map defined in Theorem 33. Commutativity follows from the following three facts:

1. $\mu$ is a group homomorphism and hence commutes with multiplication by $k$;
2. The $C^*$-algebra associated to a $\mathbb{Z}/k\mathbb{Z}$-manifold of the form $(M, \emptyset)$ is $C(M) \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z})$ and the $*$-homomorphism from $C(X) \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z})$ to $C(M) \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z})$ is $f \otimes \text{id}$ (see Remark 60 for more details);
3. The commutative diagram:

$$
\begin{array}{ccc}
C(X) \otimes C_0(0, \infty) \otimes M_k & \longrightarrow & C(X) \otimes C(pt; \mathbb{Z}/k\mathbb{Z}) \\
\downarrow f \mu \otimes \text{id} \otimes \text{id} & & \downarrow f \\
C(P) \otimes C_0(0, \infty) \otimes M_k & \longrightarrow & C^*(Q, \mathbb{Z}/k\mathbb{Z})
\end{array}
$$

and the fact that $[D_{(Q, P)}]$ is mapped to $[D_P]$ under the map:

$$K^*(C^*(Q, P; \mathbb{Z}/k\mathbb{Z})) \to K^*(C(P) \otimes C_0(0, \infty) \otimes M_k) \cong K^{*-1}(C(P))$$

The result then follows using the exactness of the Bockstein sequences, the fact that the map between geometric and analytic $K$-homology is an isomorphism, and the Five Lemma. □

**Remark 72.** If we proved that $K^\text{geo}_*(X; \mathbb{Z}/k\mathbb{Z})$ is isomorphic to $K^\text{ana}_*(X; \mathbb{Z}/k\mathbb{Z})$ via $\Phi$ directly (i.e., without using the Bockstein exact sequence), then we could prove the exactness of the Bockstein sequence for $K_*(X; \mathbb{Z}/k\mathbb{Z})$ using the corresponding result for $K^\text{ana}_*(X; \mathbb{Z}/k\mathbb{Z})$. This process would be analogous to the method used in [7]. There, the isomorphism between geometric and analytic $K$-homology is used to prove that Baum and Douglas’ geometric model for $K$-homology has a long exact sequence.

4. CONNECTION WITH THE Freed-Melrose INDEX THEOREM

The isomorphism from geometric $K$-homology (i.e., the Baum-Douglas model) to analytic $K$-homology (i.e., Kasparov model via Fredholm modules) leads to the Atiyah-Singer index theorem by considering the case of a point. That is, from the commutativity of the following diagram:
In the case of geometric $K$-homology with coefficients in $\mathbb{Z}/k\mathbb{Z}$, we have an analogous diagram; namely,

\[ K^\text{geo}_0(pt) \xrightarrow{\mu} K^\text{ana}_0(pt) \]

\[ \downarrow \text{ind}_{\text{top}} \quad \downarrow \text{ind}_{\text{ana}} \]

\[ \mathbb{Z} \]

\[ K^\text{geo}_0(pt; \mathbb{Z}/k\mathbb{Z}) \xrightarrow{\Phi} K^\text{ana}_0(pt; \mathbb{Z}/k\mathbb{Z}) \]

\[ \downarrow \text{ind}_{\text{top}}^F \quad \downarrow \text{ind}_{\text{ana}}^F \]

\[ \mathbb{Z}/k\mathbb{Z} \]

In this case, the commutativity of the diagram is the statement of the Freed-Melrose index theorem (c.f., [21]).

We can also consider pairings. For example, the pairing

\[ K^0(X) \times K^0(X; \mathbb{Z}/k\mathbb{Z}) \rightarrow K^0(pt; \mathbb{Z}/k\mathbb{Z}) \]

is given (analytically) by the Kasparov product:

(11) \[ KK^0(C, C(X)) \times KK^0(C(X) \otimes C^*(pt; \mathbb{Z}/k\mathbb{Z}), \mathbb{C}) \rightarrow KK^0(C^*(pt; \mathbb{Z}/k\mathbb{Z}), \mathbb{C}) \]

While, on the geometric side, we have the following. Let $V$ be a vector bundle over $X$ and $((Q,P), (E,F), f)$ be a geometric $\mathbb{Z}/k\mathbb{Z}$-cycle. Then we have pairing given by:

\[ \text{ind}_{\text{top}}^F(P^{(Q,P)}_{(E \otimes f^*(V), F \otimes (f|_P)^*(V))}) \]

This pairing naturally extends to K-theory and, moreover, is equal to the Kasparov pairing above (i.e., Equation [11]). This follows from the associativity of the Kasparov product and the fact that the pairing between a $\mathbb{Z}/k\mathbb{Z}$-vector bundle and the Dirac operator on $(Q, P)$ is given by twisting the operator by the bundle. Finally, using the orginal formulation of the Freed-Melrose index theorem (c.f., Corollary 5.4 in [12]), we have that this pairing is also equal to the Fredholm index of the operator $D^Q_{(E \otimes f^*(V))}$ with the Atiyah-Patodi-Singer boundary conditions (see [11]).

We have a similar pairing between the groups $K^1(X)$ and $K^1_1(X; \mathbb{Z}/k\mathbb{Z})$. However, the reader may recall that the pairing between $K^1(X)$ and $K^1_1(X)$ is given by the index of a Toeplitz operator (c.f., the introduction of [10] for details). Thus one is led to ask if there is an analogous index theorem for $\mathbb{Z}/k\mathbb{Z}$-manifolds. To the author’s knowledge, this is unknown. However, in [10], an index for Toeplitz operators on odd-dimensional manifolds with boundary is developed. Thus it is natural to ask if, in the case of a $\mathbb{Z}/k\mathbb{Z}$-manifold, the mod $k$ reduction of this index is a topological invariant and, moreover, if it is equal to the pairing between $K^1(X)$ and $K^1_1(X; \mathbb{Z}/k\mathbb{Z})$. Both these questions represent future work.
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Email address: rjdeeley@uni-math.gwdg.de
MATHEMATHISCHES INSTITUT, GEORG-AUGUST UNIVERSITÄT, BUSSENSTRASSE 3-5, 37073 GÖTTINGEN, GERMANY