Metallic Nanosphere in a Magnetic Field: an Exact Solution

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We consider the electron gas moving on the surface of a sphere in a uniform magnetic field. An exact solution of the problem is found in terms of oblate spheroidal functions, depending on the parameter \( p = \Phi/\Phi_0 \), the number of flux quanta piercing the sphere. The regimes of weak and strong fields are discussed, the Green’s functions are found for both limiting cases in the closed form. In the weak fields the magnetic susceptibility reveals a set of jumps at half-integer \( p \). The strong field regime is characterized by the formation of Landau levels and localization of the electron states near the poles of the sphere defined by a direction of the field. The effects of coherence within the sphere are lost when its radius exceeds the mean-free path.

The electronic properties of cylindrical and spherical carbon macromolecules\(^1\) have attracted much of theoretical interest last years. A large part of these studies is devoted to the band structure calculations\(^2\) and to the effects of topology for the transport and mechanical properties of these nanostructures. The main interest to the topological aspects is connected with the carbon nanotubes, where one can investigate the effective model, based on the band calculations and incorporating the particular geometry of the object\(^3\).

At the same time, the problem of the topology appears to be a general one and may be studied independently from the physics of carbon materials. The recent advances in technology\(^4\) let one think of a wider class of the spherical nanostructures, e.g. the spheres coated by metal films\(^5\), whose properties may differ from those of planar objects.

In this paper we study the gas of electrons moving within a thin spherical layer in an applied magnetic field. We find the exact solution of this problem in terms of oblate spheroidal functions. This physical application of the theory of spheroidal functions is demonstrated apparently for the first time\(^6\). We show the jumps in the susceptibility of the system in “weak” fields and the localization of the electronic states in “strong” fields. The last effect could be experimentally investigated for the hemispherical tips of nanotubes\(^7\). At the intermediate fields the sophisticated structure of the functions makes the analytical treatment impossible and the numerical methods should be used for the analysis of observable quantities.

\[ \mathcal{H} = \frac{1}{2m_e} (-i \nabla + eA)^2 + U(r), \]  
where \( m_e \) is the (effective) mass of an electron, \( -e \) its charge; we have set \( \hbar = c = 1 \) and omitted the trivial term connected with the spin of the particle. We assume that the total number of particles \( N \) (with one projection of spin) is fixed and defines the value of the chemical potential \( \mu \) and the areal density \( \nu = N/(4\pi r_0^2) \). The confining potential \( U(r) = 0 \) at \( r_0 < r < r_0 + \delta r \) and \( U(r) \to \infty \) otherwise. We focus our attention on the limit \( \delta r \ll r_0 \), when the variables of the problem are separated. The radial component \( R(r) \) of the wave function is a solution of the Shrödinger equation with the quantum well potential. Henceforth we ignore the radial component and put \( r = r_0 \) in the remaining angular part of the Hamiltonian \( H_{\Omega} \); it can be done if \( \mu \) lies below the first excited level of \( R(r) \), which in turn means \( \delta r \ll \nu^{-1/2} \). We choose the direction of the field \( \mathbf{B} \) along the \( \hat{z} \)–axis and as a north pole of the sphere \(( \theta = 0)\) and look for the eigenfunctions to the Shrödinger equation \( H_{\Omega} \Psi = E \Psi \) in the form \( \Psi(\theta, \phi) = S(\theta)e^{im\phi} \). Defining the dimensionless energy \( \varepsilon = 2m_e r_0^2 E \) and setting \( \eta = \cos \theta \), we write

\[ \frac{\partial}{\partial \eta} (1 - \eta^2) \frac{\partial S}{\partial \eta} + \left[ \varepsilon - 2m \rho - \frac{m^2}{1 - \eta^2} - p^2 (1 - \eta^2) \right] S = 0. \]  

We introduced here the important dimensionless parameter

\[ p = eBr_0^2/2 = \pi Br_0^2/\Phi_0 = r_0^2/(2l^2) = m_e r_0^2 \omega_c/2, \]  
with the magnetic flux quantum \( \Phi_0 = 2\pi/e = 2 \cdot 10^{-15} \) T·m\(^2\), the magnetic length \( l_\ast = (eB)^{-1/2} \), the cyclotron frequency \( \omega_c = eB/m_e \). Note that for a sphere of radius \( r_0 = 10 \) nm one has \( p = 1 \) at the field \( B \approx 6 \) T.

The equation (2) is known as the spheroidal differential equation and was extensively studied previously\(^8\). The solutions to it are given by the oblate (angular) spheroidal functions \( S_{lm}(p, \eta) \) with the corresponding eigenvalues \( \varepsilon_{lm}(p) \).

I. A SETTING-UP OF THE PROBLEM

Let us consider the electrons moving on a surface of the sphere of radius \( r_0 \). In the presence of the uniform magnetic field \( \mathbf{B} \) we choose the gauge of the vector potential as a vector product \( \mathbf{A} = \frac{1}{2} (\mathbf{B} \times \mathbf{r}) \). The Hamiltonian of the system is given by

\[ \mathcal{H} = \frac{1}{2m_e} (-i \nabla + eA)^2 + U(r), \]  
where \( m_e \) is the (effective) mass of an electron, \( -e \) its charge; we have set \( \hbar = c = 1 \) and omitted the trivial term connected with the spin of the particle. We assume that the total number of particles \( N \) (with one projection of spin) is fixed and defines the value of the chemical potential \( \mu \) and the areal density \( \nu = N/(4\pi r_0^2) \). The confining potential \( U(r) = 0 \) at \( r_0 < r < r_0 + \delta r \) and \( U(r) \to \infty \) otherwise. We focus our attention on the limit \( \delta r \ll r_0 \), when the variables of the problem are separated. The radial component \( R(r) \) of the wave function is a solution of the Shrödinger equation with the quantum well potential. Henceforth we ignore the radial component and put \( r = r_0 \) in the remaining angular part of the Hamiltonian \( H_{\Omega} \); it can be done if \( \mu \) lies below the first excited level of \( R(r) \), which in turn means \( \delta r \ll \nu^{-1/2} \). We choose the direction of the field \( \mathbf{B} \) along the \( \hat{z} \)–axis and as a north pole of the sphere \(( \theta = 0)\) and look for the eigenfunctions to the Shrödinger equation \( H_{\Omega} \Psi = E \Psi \) in the form \( \Psi(\theta, \phi) = S(\theta)e^{im\phi} \). Defining the dimensionless energy \( \varepsilon = 2m_e r_0^2 E \) and setting \( \eta = \cos \theta \), we write

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The equation (2) is known as the spheroidal differential equation and was extensively studied previously\(^8\). The solutions to it are given by the oblate (angular) spheroidal functions \( S_{lm}(p, \eta) \) with the corresponding eigenvalues \( \varepsilon_{lm}(p) \).
It is known that spheroidal functions belong to the simplest class of special functions which are not essentially hypergeometric ones. For the spheroidal functions there are no recurrence relations, generating function representations etc., which are characteristic for the classical special functions. The spectrum \( \varepsilon_{lm}(p) \) is found as the eigenvalues of the infinite matrices. For different sets of functions orthogonal on the interval \((-1, 1)\), these matrices are reduced to the tridiagonal form and could be further analyzed within the chain fraction formalism or by explicit (numerical) diagonalization. Perhaps, the mostly known quantum-mechanical application of the spheroidal functions is the problem of an electron in two-center Coulomb potential \( (\mathbf{H}_2^+ \text{ molecule}) \).

II. THE WEAK FIELD, SUSCEPTIBILITY

Let us first concentrate on the case of the weak field. In the absence of the field, the spectrum is that of a free rotator model and the solutions to \( (5) \) are the associated Legendre polynomials \( \bar{P}_l^m(\eta) \).

\[
\varepsilon_{lm}(0) = l(l+1),
\]

\[
S_{lm}(0, \eta) = \sqrt{\frac{2l+1}{4\pi r_0^2}} \frac{\Gamma(l+1)}{\Gamma(l+|m|)!} P_l^m(\eta).
\]

According to \( (6) \) the wave functions are the spherical harmonics \( \Psi_{lm}(\theta, \phi) = r_0^{-1} Y_{lm}(\theta, \phi) \) and are normalized on the surface of the sphere, \( r_0 \int |\Psi|^2 \sin \theta \, d\theta \, d\phi = 1 \). This normalization facilitates the comparison of our results with the case of planar geometry.

For \( p \neq 0 \) one develops the perturbation theory around the initial wave functions \( (6) \) as long as \( p^2 \leq 4 \bar{P}^2 \). The energies and wave functions in this case are given by series in \( p^2 \):

\[
\varepsilon_{lm}(p) = l(l+1+2pm + \frac{p^2}{2}) + O \left[ \frac{p^2}{l} \right], 
\]

\[
S_{lm}(p, \eta) \propto \bar{P}_l^m(\eta) + \frac{m^2}{l(l+1)} O \left[ \frac{p^2}{l} \right] 
\]

To clarify the criterion \( p^2/l \lesssim 1 \) we consider the case when the field ceases to be small for electrons near the Fermi level \( l \approx r_0 \sqrt{4\pi \nu} \), mostly contributing to the physical properties. This case corresponds to the following relation

\[
B r_0^3 \sim \left[ \frac{\hbar c}{e} \right]^2 e^2 \sqrt{\nu} \sim 137^2 \omega_d.
\]

Hence the field cannot be treated as a perturbation, if the energy of the magnetic field in the volume of the sphere is \( 10^4 \) times larger than the characteristic plasma frequency. For densities \( \nu \sim 10^{14} \text{ cm}^{-2} \) and \( r_0 = 10 \text{ nm} \) it corresponds to the fields greater than 40 T.

For the weak field regime it is interesting to observe the following property of the spectrum \( (6) \). For simplicity we consider the situation when the \( L \)th unperturbed level is completely filled and \((L+1)\)th level is empty. Linearizing the spectrum we have \( \varepsilon_{lm} \sim 2L(l-L-1/2+pm/L) \). It is clear that at \( p > 1/2 \) the state \((l = L+1, m = -L-1)\) is energetically more favorable than the state \((l = L, m = L)\), with the resulting change in the occupation of the levels.

This phenomenon is accompanied by the jumps in the static susceptibility. Consider the free energy \( F = -T \sum_{l,m} \ln(\varepsilon - E_{lm})/T + 1 \). Then, apart from the Pauli spin contribution, the magnetic (differential) susceptibility \( \chi = \partial M/\partial B = -\partial^2 F/\partial B^2 \) is given by

\[
\chi = -\frac{\mu_B^2 m_e r_0^2}{2} \sum_{lm} \left[ \frac{\partial^2 \varepsilon_{lm}}{\partial p^2} n_F(\varepsilon_{lm}) + \left[ \frac{\partial \varepsilon_{lm}}{\partial p} \right]^2 n_F'(\varepsilon_{lm}) \right]
\]

with the Bohr magneton \( \mu_B = e/2m_e \), the Fermi function \( n_F(\varepsilon) \) and \( n_F'(\varepsilon) \) its derivative. The first term here gives the diamagnetic contribution \( \chi^D \sim -2/3 \mu_B^2 m_e r_0^2 N \) and the second term is the paramagnetic one. At low temperatures \( \omega_c \lesssim T \ll \mu \) we let \( n_F'(E_{lm}) \approx -\delta(\varepsilon_{lm} - \mu) \) and change \( \sum_{l,m} \approx \int_{-1}^1 dm \). Performing the integration we get

\[
\chi = \frac{N^2(\mu_B^2/2\mu)}{2/3 + p^{-3} \int_{-1}^1 (l - L - 1/2)^2}
\]

where the summation is restricted by \( |l - L - 1/2| < p \). From \( (7) \) we see that \( \chi \) exhibits the jumps at \( p = 1/2, 2/3, 2/5, \ldots \) and \( \chi \to 0 \) in the formal limit \( p \to \infty \). This behavior is shown in the Fig. \( (7) \) together with the quantity \( M/B \). At the other fillings, i.e. at other values of \( \mu \), the jumps of \( \chi \) take place at other values of \( p \) and the qualitative picture remains true. The amplitude of the jumps is \( N \) times larger than the Pauli spin susceptibility \( \sim N \mu_B^2/\mu \); this coherent effect vanishes if the coherence on the sphere is lost due to the finite quasiparticle lifetime (see below).

III. THE WEAK FIELD, GREEN’S FUNCTION

Let us discuss the properties of the electron Green’s function. For the two points on the sphere \( \mathbf{r} \leftrightarrow (\theta, 0) \) and \( \mathbf{r}' \leftrightarrow (\theta', \phi) \) we define

\[
G(\mathbf{r}, \mathbf{r}', \omega) = \sum_{lm} \frac{\Psi_{lm}(\theta', \phi) \Psi_{lm}(\theta, 0)}{\omega - E_{lm}}.
\]

with \( E_{lm} = (2m_e r_0^2)^{-1} \varepsilon_{lm} \).

In the absence of the magnetic field \( \Psi_{lm}(\theta, \phi) = r_0^{-1} Y_{lm}(\theta, \phi) \) and one can find (see below) an exact representation of \( G \) through the Legendre function

\[
G(\omega)_{B=0} \equiv G^0(\omega) = -\frac{m_e}{2 \cos \pi a} P_{-1/2+n}(\cos \Omega),
\]

where we introduced \( a = \sqrt{2m_e r_0^2(\mu + \omega) + 1/4} \) and the distance \( \Omega \) on the sphere :
\[ \cos \Omega = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos \phi \]

We immediately see from (4) the logarithmic singularity in the one-point correlator (\( \Omega = 0 \)) as it should be. It is instructive to find out how the usual expressions for the planar geometry are recovered for the large radius of the sphere, \( r_0 \to \infty \); in this case \( a \sim r_0 \), while \( \Omega \to |r - r'|/r_0 \equiv r/r_0 \). In the limit \( a \gg 1 \) and for \( a \sin \Omega \gtrsim 1 \) one has in the main order of \( a^{-1} \)

\[ G^0(\omega) \simeq -\frac{m_e}{\sqrt{2\pi a \sin \Omega}} \frac{\cos(a\pi - a\Omega - \pi/4)}{\cos \pi a}, \quad (10) \]

The existence of two oscillating exponents \( \pm i(a\pi - a\Omega - \pi/4) \) corresponds to quantum coherence of two waves. One propagates along the shortest way between two points and another wave goes along the longest way, turning round the sphere. In the theory of metals one considers \( |\omega| \ll \mu = k_F^2/(2m_e) \), while the finite quasiparticle lifetime can be modulated by ascribing the imaginary part to \( \omega \). We write in this sense \( \sqrt{2m_c}\omega = k_F + i/l_{mf} \) with the mean free path \( l_{mf} \). When \( r_0 \gtrsim l_{mf} \), we see that the coherence breaks and the only surviving exponent in (4) has the form

\[ G^0_{\text{damped}}(\omega) \simeq -\frac{m_e}{\sqrt{2\pi k_Fr}} \exp \left[ i k_F r + \frac{\pi}{4} - \frac{r}{l_{mf}} \right], \quad (11) \]

which expression coincides with the usual findings [3].

It is possible to obtain the closed form of the Green’s function in the weak field regime. We sketch the corresponding derivation below.

First we neglect the \( p^2/l \) terms in the wave function (8) and write \( Y_{lm}(\theta, \phi)Y_{lm}(\theta, 0) = (4\pi)^{-1}(2l + 1)e^{-i\beta p}P_l^m(\cos \theta)P^m_{l+1}(\cos \theta) \). Using (8) and dropping the terms containing \( p^2/a \sim p^2/l \), we have

\[ \frac{2l + 1}{\omega - \epsilon_{lm}} \simeq \left[ \frac{a + pm}{a} - l - \frac{1}{2} \right] - \left[ \frac{a + pm}{a} + l + \frac{1}{2} \right]^{-1} \]

Now we represent the appearing fraction as the Taylor series \( (a + pm/a - z)^{-1} = \exp \left( \frac{pm}{a} \frac{d}{dz} \right) (a - z)^{-1} \) and substitute \( m \) by \( \frac{\omega}{i\beta} \) in the last exponent. After that we can sum over \( m \) in (8) and represent the remaining sum over \( l \) as the contour integral, to obtain

\[ G(\omega) = \exp \left[ \frac{p}{a} \frac{\partial^2}{\partial \alpha \partial \phi} \right] \int \frac{d\nu}{2\pi i} \frac{m_e}{2\pi \nu} \frac{P_{l\nu}(-\cos \Omega)}{a - \nu - 1/2} \]

\[ = \exp \left[ \frac{p}{a} \frac{\partial^2}{\partial \alpha \partial \phi} \right] G^0(\omega), \quad (12) \]

with \( G^0(\omega) \) given by (8). When \( a \gg 1 \) the expression (12) could be simplified at the above condition \( a \sin \Omega \gtrsim 1 \). In this case we use (10) and observe that, upon differentiating over \( \phi \), the main contribution of order of \( a \) stems from the numerator \( \exp(\pm i\alpha \Omega) \) of (11). Then one uses the identity \( \exp \left[ \pm i\alpha \frac{\partial}{\partial \alpha} \right] = \frac{1}{a} \exp \left[ \pm i\frac{\partial}{\partial \alpha} \right] a \) and finds

\[ G(\omega) \simeq -\frac{m_e}{\sqrt{2\pi a \sin \Omega}} \exp \left[ \frac{i\alpha}{2} \left( \frac{\pi}{4} - \frac{\pi}{a} \right) \right] \frac{e^{i\alpha(\pi - \Omega) - i\pi/4}}{\cos \pi(a + p\beta)} + \frac{e^{-i\alpha(\pi - \Omega) + i\pi/4}}{\cos \pi(a - p\beta)} \]

(13)

with \( \beta = \sin \theta \sin \theta' \sin \phi/\sin \Omega \). As before, in the presence of damping \( r_0 > l_{mf} \), the coherence on the sphere breaks and the only surviving term in (13) has the form

\[ G_{\text{damped}}(\omega) \simeq \exp(\mp i\Omega' r)/2l_{mf}^2 G^0_{\text{damped}}(\omega), \quad (14) \]

in accordance with previous results (see e.g. [3]).

IV. THE STRONG FIELD REGIME

Let us now turn to the case of strong fields, \( p \to \infty \). We define the integer number \( n \geq 0 \) as follows: \( 2n = l - |m| \) for even \( l - |m| \) and \( 2n + 1 = l - |m| \) for odd \( l - |m| \). The value of \( n \) has a simple meaning, it corresponds to the number of zeroes of wave function \( S_{lm}(p, \cos \theta) \) within the interval \( \theta \in (0, \pi/2) \), by analogy with the known property of \( P_n^m(\cos \theta) \). The spectrum of (8) is then given by a series [4]

\[ \epsilon_{lm}(p) = 4p [n + (m + |m| + 1)/2] - (s^2 - m^2 + 1)/2 + O(s^3/p), \quad (15) \]

with \( s = 2n + |m| + 1 \); one has \( s = l + 1 \) for even (odd) values of \( l - m \). The eigenfunctions are given by

\[ S_{lm}^\pm (p, \eta) = \tilde{S}_{lm}(p, \eta) \pm \tilde{S}_{lm}(p, -\eta), \quad (16) \]

where plus (minus) sign corresponds to even (odd) values of \( l - m \). The functions \( \tilde{S}_{lm}(p, \eta) \) are found as a series in the Laguerre polynomials \( L_n(x) \); in the main order of \( p^{-1} \) they can be written as

\[ \tilde{S}_{lm}(p, \eta) \simeq c(1 - \eta)^{|m|/2} e^{-p(1-\eta)} L_n^{|m|}(2p(1 - \eta)), \quad (17) \]

\[ c = \left[ \frac{2^{|m|}p^{|m|+1}n!}{2\pi r_0^2(n + |m|)!} \right]^{1/2} \]

We see that in the main order of \( p \) the Landau quantization takes place, i.e. the spectrum (15) is that of quantum oscillator with the cyclotron frequency being the energy quantum [8]. The convergence and hence the applicability of the series (15) is given by the condition \( p \gtrsim s \sim l \). In its turn, it means [9] that all \( n \) zeroes of the approximate eigenfunction \( L_n^{|m|}(2p(1 - \eta)) \) in (17) lie in the northern hemisphere, \( \eta > 0 \), as they should.

We notice the following important property of the spectrum (15). For given non-positive \( m \) the values of \( \epsilon_{lm} \) corresponding to \( l = 2n + |m| \) and \( l = 2n + |m| + 1 \) coincide. This property and the form of the wave function (8) can be understood as follows. At \( p \to \infty \) the field-induced potential \( p^2 \sin^2 \theta \) in (8) localizes the particles.
near the poles of the sphere. This form of two-well potential leads to the discussed degeneracy of energy levels, while the total wave function is given by a symmetrization \[ \sum_{l} P_{l}^{m}(p, \eta) \] of the wave functions \( P_{l}^{m}(p, \eta) \) related to each of the wells. The possibility of quantum tunneling between the wells lifts the degeneracy and produces the exponentially small splitting \( \sim e^{-2p} \) between the states \( S_{l+}^{m}(p, \eta) \) and \( S_{l-}^{m}(p, \eta) \).

It would be tempting to obtain the Green’s function at \( p \to \infty \) in the closed form. It can be done at the following simplifications. First, we neglect the exponentially small splitting and notice that \( S^{+}(\eta)S^{+}(\eta') + S^{-}(\eta)S^{-}(\eta') = 2[S(\eta)S(\eta') + \tilde{S}(\eta)\tilde{S}(\eta')] \), i.e. the correlations within one hemisphere only survive. Now we leave only the leading terms \( |16| \) of the wave functions and \( O(p) \) terms in energies \( |13| \). The necessity to restrict the summation in \( |3| \) by the terms with \( s \lesssim p \) could be modelled by inclusion of the cutoff factor \( e^{-s\delta} \) with \( \delta \sim p^{-1} \) into \( |3| \). We put \( \mu = 0 \) for simplicity of writing and raise the denominator into the exponent, \( (\omega - E_{nlm})^{-1} = i\delta \int dt e^{i(E_{nlm}-\omega)t} \). Next we perform the summation over \( n \) with the use of bilinear generating function for the Laguerre polynomials \( |4| \):

\[
\sum_{n=0}^{\infty} \frac{n!}{(n + |m|)!} L_{n}^{|m|}(x) L_{n}^{|m|}(y) z^{n} = \frac{(xy)^{|m|/2}}{1 - z} \exp \left(-z \frac{x + y}{1 - z} \right) I_{|m|} \left( 2 \sqrt{xyz} \right).
\]

with the modified Bessel function \( I_{n}(w) \); in our case \( x = 2p(1 - \eta), y = 2p(1 - \eta') \) and \( z = e^{\iota \omega_{c}t} \). Making use of the property \( I_{|m|}(iw) = i^{m} J_{m}(w) \) and rescaling \( t \omega_{c}/2 \to t \) we arrive at the intermediate formula for \( \eta, \eta' > 0 \):

\[
G(\omega) \simeq -\frac{m_{e}}{4\pi} \int_{0}^{\infty} \frac{dt}{\sin(t + i\delta)} e^{-2i\omega_{c}t} e^{-2it\omega/\omega_{c}} \frac{1}{1 - z} \exp \left(-z \frac{x + y}{1 - z} \right) I_{|m|} \left( 2 \sqrt{xyz} \right) \times \sum_{m=-\infty}^{\infty} e^{i(t - \phi + \pi/2)m} J_{m} \left( \frac{\sqrt{xy}}{\sin(t + i\delta)} \right).
\]

The summation over \( m \) is now easily done \( \sum_{m} e^{im\phi} J_{m}(w) = e^{i\omega_{c}t} \) and we obtain the Green’s function as a sum of two terms, referring to two hemispheres. The “northern” term, corresponding to the above expression, is given by

\[
G^{n}(\omega) \simeq -\frac{m_{e}}{4\pi} e^{i\omega_{c}t} \int_{0}^{i\delta} \frac{dt}{\sin(t)} e^{-2i(t - \delta)\omega/\omega_{c}} \frac{1}{1 - z} \exp \left(-z \frac{x + y}{1 - z} \right).
\]

with the analogue of the distance on the sphere \( \rho = 2p[2\eta - \eta'] - 2\sqrt{(1 - \eta)(1 - \eta')(\cos(\phi + i\delta))} \) and that of the vector product \( v = 2p\sqrt{(1 - \eta)(1 - \eta')(\cos(\phi + i\delta))} \).

The form of the “southern” term is obtained by changing \( \eta \to -\eta \) and \( \eta' \to -\eta' \) in these expressions. The last integral is reduced for vanishing \( \delta \) to the hypergeometric function \( \Psi(a, b; z) \) and we obtain the “northern” term in the form :

\[
G^{n}(\omega) \simeq -\frac{m_{e}}{4\pi} e^{i\omega_{c}t - 2i\omega/\omega_{c}} \int_{0}^{\infty} \frac{dt}{\sin(t)} e^{-2i(t - \delta)\omega/\omega_{c}} \frac{1}{1 - z} \exp \left(-z \frac{x + y}{1 - z} \right) \Psi \left[ \frac{1}{2} - \frac{\omega}{\omega_{c}}, 1; \rho \right],
\]

\[
= -\frac{m_{e}}{4\pi} e^{i\omega_{c}t - 2i\omega/\omega_{c}} \sum_{n=0}^{\infty} \frac{L_{n}(\rho)}{\omega_{c} - n - 1/2} \tag{19}
\]

The last equation is similar to previous findings \( |13| \).

Let us discuss the applicability of \( |19| \). Our derivation was straightforward until \( |13| \), while the last step demanded \( \delta \sim 1/p \to 0 \). The incomplete restructuring of the spectrum into the Landau level scheme is absent in the usual planar geometry, wherein one would put \( \delta = 0 \) and the expression \( |13| \) would be exact. In the spherical case we cannot treat the higher levels with \( l \gtrsim p \) in an analytical way, it is mimicked in \( |13| \) by the appearance of the exponential cutoff at \( \omega \gtrsim p\omega_{c} \).

A subtler issue in justifying \( |19| \) is the shift of the last integration in \( |13| \) to the interval \( (-\pi/2, \pi/2) \). The integration over the remaining segments \( (\pm\pi/2, \pm\pi/2 + i\delta) \) yields the factor \( \cos(\pi\omega/\omega_{c}) \), thus the contribution of these segments to the Green’s function has no poles in \( \omega \). This contribution could be combined with the smooth (real) part of \( G(\omega) \) stemming from the consideration of the highest energy levels.

As a result, one may conclude that the expression \( |19| \) correctly reproduces the basic properties of the Green’s function for \( \omega < p\omega_{c} \).

The finite value of the cutoff parameter \( \delta \) in \( |13| \) becomes important for the one-point correlation function, i.e. at \( \eta = \eta' \) and \( \phi = 0 \). The residues of the Green’s function define the local density of states (LDOS) by the relation \( N(r) = \int d\omega n_{p}(\omega) [G(\mathbf{r}, \omega - i0) - G(\mathbf{r}, \omega + i0)]/(2\pi i) \). If we assume that all lowest Landau levels are filled by the electrons, then it follows from \( |19| \) that the LDOS is given by \( N(r) = 2pe^{2i\omega/\omega_{c}}/(2\pi\sqrt{2}) \). In this case the finite \( \delta \sim 1/p \) provides a smooth variation of LDOS of the form

\[
N(\mathbf{r}) \propto \exp[-O(\sin^{2}\theta)],
\]

which variation is absent in the usual planar geometry. This result, however, may depend on the approximations made and requires further numerical investigation.

It should be stressed that at the intermediate fields, at \( p > 1 \) and \( p \gtrsim l \lesssim p^{2} \), the spectrum and wave functions are not principally reduced to closed expressions of hypergeometric type. The numerical methods are indispensable here. We calculated the evolution with \( p \) of the energy levels with \( l = 0, \ldots, 5 \) by diagonalizing the 200 \( \times \) 200 triadagonal matrices in the basis of \( P_{l}^{m}(\eta) \). The results are shown in the Fig. \( |4| \) where we made a following convention. From the general property of Eq. \( |2| \)
it follows that $\varepsilon_{l-m}(p) = \varepsilon_{l,m}(-p)$ with formally negative $p$. For better readability of the graph, it is possible then to show all the data, plotting only half of them with one sign of $m$ but in the formally extended region of $p$. One can see in the Fig. 2 that the degeneracy at $p = 0$ is eventually changed by the Landau levels formation at $p = 10$. In the intermediate region $p \sim 3$ the absence of any structure in the levels’ scheme is noted. The mesh of lines at the intermediate fields mimicks the chaotic behavior, although this is not so. Both in the quantum problem considered here and in its classical counterpart one has two variables ($\theta, \phi$) and two integrals of motion, the energy and the projection of the angular momentum onto the field direction. To obtain chaos, it suffices to break the rotation symmetry, $\phi \rightarrow \phi + \delta \phi$. The latter problem is, however, beyond the scope of this study.

In conclusion, we demonstrate the exact solution of the electron gas on the sphere in the magnetic field. In the limits of weak and strong fields this solution is reduced to the hypergeometric functions and the observables quantities are found in the closed form. In the case of intermediate fields, the solution is not essentially hypergeometric function and the observables require further numerical analysis.

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FIG. 1. The dependence of the static susceptibility $\chi$ on the number $p$ of magnetic flux quanta piercing the sphere.
FIG. 2. The dependence of the energy levels $\varepsilon_{lm}$ on the number $p$ of magnetic flux quanta piercing the sphere. For convenience of presentation, we plotted the evolution of $\varepsilon_{lm}(p)$ with $m \leq 0$ on the rhs of the plot, at $p > 0$. The evolution of $\varepsilon_{lm}(p)$ with $m \geq 0$ is depicted on the lhs of this plot for the formally negative $p$, see the text for additional explanations.