A KINEMATIC FORMULA FOR
THE TOTAL ABSOLUTE CURVATURE

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Abstract. Let $A$ and $B$ be compact PL-subspaces of some euclidean space $E^n$. We show that for these a kinematic formula for the total absolute curvature holds in analogy to the classical one.

Introduction

Let $E^n$ be an euclidean space and $E^n \supset A, B$ be compact tame sets. The notion “tame” will be explained from case to case. The classical kinematic formula says:

$$\int_{G} \chi(A \cap gB) \, dG = \sum_{k=0}^{n} c_k V_k(A)V_{n-k}(B).$$

Here $\chi$ is the Euler characteristic, the $c_k$ are universal constants:

$$c_k = c(n,k) := \binom{n}{k}^{-1} \omega_k \omega_{n-k} / \omega_n,$$

where $\omega_i$ is the volume of the $i$-dimensional unit ball and $G = O(n) \ltimes \mathbb{R}^n$ is the group of all euclidean motions of $\mathbb{R}^n$, endowed with the product of Haar measure and Lebesgue measure. The $V_k$, $k = 0, \ldots, n$ are functionals for tame sets, which are known under different names: Minkowski–functional, cross sectional measure (Quermaß), generalized volumes, Lipschitz–Killing invariant, ... One of the possible definitions is:

$$V_k(A) = \int_{\text{Graff}(n,n-k)} \chi(A \cap E) \, dE,$$

where $\text{Graff}(n,n-k)$ is the affine Grassmannian of $(n-k)$-planes in $E^n$, provided with a $G$-invariant measure. In Section 2 for PL-spaces (piecewise linear spaces) we will consider a more intrinsic definition for the $V_k$ and similarly for absolute curvature measures (compare Notation 2.3 and Corollary 3.8).

Now, concerning the notion “tame” it is more reasonable to consider tame classes of sets. So, tame sets are members of a tame class.

The kinematic formula is known for the following tame classes:

- Convex sets (Blaschke, Hadwinger and many others [K-R]),
- Manifolds with and without boundary (Chern [C], Santalo [S]).

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PL-sets (Wintgen, Cheeger–Müller–Schrader [C-M-S]),
Sets with positive reach (Rataj, Zähle [R-Z], [Z]),
Definable sets with respect to an o-minimal system [vdD], in particular sub-analytic sets (Fu [Fu2], Bröcker–Kuppe [Br-K]).

For more details and generalizations see also [Sch-W], [H-Sch].

Let $\dim(A) = m$. One has $V_0 = \chi(A)$, $V_m(A) = \text{vol}_m(A)$, $V_{m-1}(A) = \frac{1}{2} \text{vol}_{m-1}(\partial A)$ if $A$ is a manifold with boundary, $V_k(A) = 0$ for $k > m$.

We are going to show a kinematic formula for the absolute curvature, to be defined below, at least for PL-sets. On the way, we find a new proof for the usual kinematic formula in the PL-case, which does not use approximation by manifolds and reduction to Chern’s result as in [C-M-S].

However, kinematic formulas for the total absolute curvature are not known for other settings not even for manifolds except for special situations in dimension 2, see [G-R-S-T] and for linear kinematic formulas [Ba], [R-Z] (compare also Corollary 3.8). The reason for this is, that the total absolute curvature is very sensitive concerning approximation by triangulations.

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1. CURVATURE MEASURES

Let $E^n \supset X$ be a compact manifold, possibly with boundary. Let

$$N(X) := \{ (x, a) \mid x \in X, a \in S^{n-1}, a \perp T_x(X) \}$$

be the unit normal bundle. We denote by

$$\gamma : N(X) \to S^{n-1}, \quad (x, a) \mapsto a$$

the Gauß map.

The absolute curvature $\tau(X)$ is defined by

$$\tau(X) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \# \gamma^{-1}(a) \, da .$$

Similarly, one has the Gauß curvature, for which we choose an orientation on $N(X)$. Then

$$\sigma(X) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \sum_{(x, a) \in N(X)} \text{sign}(\gamma(x, a)) \, da .$$

One has

$$\sigma(X) = \chi(X) \quad \text{Gauß–Bonnet},$$

$$\tau(X) \geq \sum_{k=0}^n b_k(X) \quad \text{Chern–Lashof} .$$

Here $b_0, b_1, b_2, \ldots, b_n$ are the Betti numbers. On the way, we will prove these formulas in a more general setting. Note that the quantities under the integrals are finite up to a set of measure 0 in $S^{n-1}$. 
One may localize $\sigma$ and $\tau$, thus getting curvature measures. Also this we will do in a more general setting. Let $X \subset E^n$, where $X$ is compact and belongs to a tame class. Let $x \in X$. For $r > 0$ the homeomorphy type of $B(x, r) \cap X$ does not depend on $r$ if $r$ is sufficiently small.

**Definition 1.1.** Let $x \in X$, $r > 0$ and $0 < \delta \ll r$. Also let $a \in S^{n-1}$. Then

$$C(X, x, a) = \{ y \in X \cap B(x, r) | -\delta \leq \langle y-x, a \rangle \leq \delta \}$$

is called the cone of $X$ at $x$ in direction $a$, and

$$L(X, x, a) = \{ y \in X \cap B(x, r) | \langle y-x, a \rangle = \delta \}$$

is called the link of $X$ at $x$ in direction $a$.

Again, up to homeomorphy, the pair of spaces $(C(X, x, a)/L(X, x, a))$ is independent of the choices of $r$ and $\delta$.

**Definition 1.2.**

a) $\sigma(X, x, a) := 1 - \chi(L(X, x, a))$ is called the index of $X$ at $x$ in direction $a$.

b) $\tau(X, x, a) := |b(L(X, x, a)) - 1|$ is called the absolute index of $X$ at $x$ in direction $a$. Here $b$ is the sum of all Betti numbers, that is:

$$b(L(X, x, a)) = b_0(L(X, x, a)) + \cdots + b_n(L(X, x, a)).$$

**Remark 1.3** (compare [Kü]).

a) $\sigma(X, x, a) = \sum_{k=0}^{n} (-1)^k b_k(C(X, x, a)/L(X, x, a))$.

b) $\tau(X, x, a) = \sum_{k=0}^{n} b_k(C(X, x, a)/L(X, x, a))$.

This can be seen by regarding the long exact homology sequence together with the fact that $C(X, x, a)$ is contractible.

Let $a \in S^{n-1}$ such that $\varphi(y) := -\langle a, y \rangle$ is a Morse function [G-M, Chap. 2.1]. Suppose that $x$ is a critical point for $\varphi$ and that there is no other critical point $x'$ with $\varphi(x') = \varphi(x) =: \alpha$. Then $\varphi^{-1}(\alpha + \delta)$ is homeomorphic to the space which one gets by attaching $C(X, x, a)$ at $\varphi^{-1}(\alpha - \delta)$ along $L(X, x, a)$ [G-M, 3.5.4].

If there are several critical points, where $\varphi$ takes the same value, then the result is similar. One has just to attach the different cones simultaneously. Anyway, there are only finitely many critical points for $\varphi$. Thus we get:

**Proposition 1.4.** In the situation above:

a) $\chi(X) = \sum_{x \text{ critical for } \varphi} \sigma(X, x, a)$,

b) $\sum_{k=0}^{n} b_k(X) \leq \sum_{x \text{ critical for } \varphi} \tau(X, x, a)$. 
Proof. a) follows directly from the additivity of $\chi$ regarding that $\chi(C(X, x, a)) = 1$ for every critical point $x$ for $\varphi$, b) follows from the calculations in [Mi, §5]. □

Note that again, up to a set of measure 0 in $S^{n-1}$, $\varphi(y) = -\langle a, y \rangle$ is a Morse function for $a \in S^{n-1}$.

Now we are able to define curvature measures:

**Definition 1.5.** Let $X \supset U$ be a Borel set.

a) $\sigma(U) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \sum_{x \in U} \sigma(X, x, a) \, da$.

b) $\tau(U) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \sum_{x \in U} \tau(X, x, a) \, da$.

So we get by Proposition 1.4:

**Proposition 1.6.** Let $E^n \supset X$ be compact and belong to a tame class.

a) $\sigma(X) = \chi(X)$ Gauß–Bonnet,

b) $\tau(X) \geq \sum_{k=0}^{n} b_k(X)$ Chern–Lashof.

**Remark 1.7.** Let $E^n \supset X$ be a manifold $a \in S^{n-1}$ such that $\varphi : y \mapsto \langle a, y \rangle$ is a Morse function on $X$ with critical point $x$ of index $\lambda$ (compare [Mi, § 3]). Then $L(X, x, a)$ is homotopically equivalent to $S^{\lambda-1}$ or $\emptyset$. Hence

$$\sigma(X, x, a) = 1 - \chi(L(X, x, a)) = (-1)^{\lambda},$$

$$\tau(X, x, a) = \left| \sum_{k=0}^{n} b_k(L(X, x, a)) - 1 \right| = 1.$$  

So our curvature measures coincide with the usual ones.

**Remark 1.8.** J. Fu [Fu1] studied a large class of “tame” sets for which a generalized Morse theory exists (independent of [G-M]) and he defined corresponding indices. This work is closely related to [R-Z].

2. PL-spaces

We have seen that we have similar situations for the curvatures $\sigma$ and $\tau$ respectively. Let us consider this in more generality:

**Notation 2.1.** Let $T$ be a tame class. Consider all isotopy classes of pairs $(X/Y)$, $X, Y \in T$, both compact.

A curvature map $\varphi : \{(X/Y)\} \rightarrow \mathbb{R}$ assigns to each class $(X/Y)$ a real number $\varphi(X/Y)$ such that
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(i) \( g((X/Y) + (X'/Y')) = g(X/Y) + g(X'/Y') \),
where + on the left hand side means disjoint union.

(ii) \( g((X/Y) \times (X'/Y')) = g(X/Y) \cdot g(X'/Y') \).
Note that \( (X/Y) \times (X'/Y') = (X \times X' / X \times Y' \cup X' \times Y) \).

(iii) \( g(x, x) = 0, \ g(x, \emptyset) = 1 \) for a singleton \( X = x \).

Example 2.2. Let \( b_k(X/Y) \) be the \( k \)th Betti number of \( (X/Y) \). The tameness of the class \( \mathcal{T} \) guarantees that the Betti numbers exist in all cases we need.

a) \( g(X/Y) = \sum_{k=0}^{\infty} (-1)^k b_k(X/Y) \),

b) \( g(X/Y) = \sum_{k=0}^{\infty} b_k(X/Y) \),

c) \( g(X/Y) = b_0(X/Y) \).

This follows from the Künneth formula [Ma, XI, § 6].

Now let \( E^n \supset X, X \in \mathcal{T}, x \in X \) and \( S \) be the \((n - 1)\)-sphere in \( E^n \). According to Definition 1.1 we set, for \( a \in S \),

\[ g(X, x, a) := g(C(X, x, a)/L(X, x, a)) \]

and also

\[ g(X) := \frac{1}{|S|} \int_S \sum_{x \in X} g(X, x, a) \, da . \]

The tameness guarantees that, up to a set of measure 0 in \( S \), the sum is finite.

More generally, we get a curvature measure \( \sigma(X, U) \) assigning to every open set \( U \subset X \) the value

\[ \frac{1}{|S|} \int_S \sum_{x \in U} g(X, x, a) \, da . \]

The cases a) and b) in Example 2.2 lead to the curvature measures \( \sigma \) and \( \tau \) respectively which we considered in Section 1.

From now on let \( \mathcal{T} = \text{PL} \), the class of piecewise linear spaces.

Notation 2.3. Let \( E^n \supset X \in \text{PL}, X \) compact. For \( 0 \leq k \leq n \) we denote by \( X_k \) the \( k \)-skeleton of \( X \).

The curvature measure on \( X \) is a Dirac measure, concentrated at the vertices of \( X \). So

\[ g(X) = \sum_{x \in X_0} g(x) . \]

Let \( F \in X_k \). We denote by \( \text{Aff}(F) \) the affine hull of \( F \). Let \( \text{Int}(F) \) be the set of interior points of \( F \). Also, for \( x \in \text{Int}(F) \) let \( \text{Aff}(F) \perp \) be the affine orthogonal complement of \( \text{Aff}(F) \) such that \( x \in \text{Aff}(F) \perp \).
Let \( a \in S^{n-1} \). We set

\[
g(X, F, x, a) := g(X \cap \text{Aff}(F)^\perp, x, a) .
\]

This is obviously independent of \( x \), so we set

\[
g(X, F, a) := g(X, F, x, a) ,
\]

and correspondingly

\[
g(X, F) := \left| S^{n-k-1} \right|^{-1} \int_{S^{n-k-1}} g(X, F, x, a) \, da .
\]

Finally, we set

\[
W_k(F) := |F| g(X, F) ,
\]

where \( |F| \) is the \( k \)-volume of \( F \), and

\[
W_k(X) := \sum_{F \in X_k} W_k(F) .
\]

For \( g = \sigma \) one has \( W_k(X) = V_k(X) \) (see the Introduction).

**Remark 2.4.** Let \( x \in X, a \in S^{n-1} \) and \( H = \{ y \in E \mid \langle y - x, a \rangle = 0 \} \).

Assume that the following condition holds:

(\( * \)) \hspace{1cm} \( H \) does not contain any face \( F \) of \( X \) unless \( F = \{ x \} \).

Then the following pairs of spaces are isotopic:

\begin{enumerate}[(a)]
  \item \( (C(X, x, a) / L(X, x, a)) \)
  \item \( (C'(X, x, a) / L'(X, x, a)) := (X \cap B^n(x, r) / X \cap S^{n-1}(x, r) \cap \{ y \in C \mid \langle a, y \rangle \geq 0 \}) \),
    where \( r > 0 \) is so small that \( B^n(x, r) \cap X_0 = \emptyset \) or \( \{ x \} \).
\end{enumerate}

Note that for \( x \in X \) condition (\( * \)) holds for all \( y \in S^{n-1} \) up to a set of measure 0.

Now let \( X_1 \) and \( X_2 \) be two compact PL-spaces in \( E^n \), and let \( G \) be the group of all euclidean motions. For \( g \in G \), and \( X \cap gX_2 \), we have to look at the vertices and their curvature measures. Such a vertex appears if \( F_1 \) is a \( k \)-face of \( X_1 \), \( F_2 \) an \((n-k)\)-face of \( F_2 \), \( x \in F_1 \), \( z \in F_2 \) and \( g(z) = x \). Since the curvature measures are \( G \)-invariant, we may consider the following situation:

\( F_1 \) is a \( k \)-face of \( X_1 \), \( F_2 \) an \((n-k)\)-face of \( X_2 \), \( \text{Int}(F_1) \cap \text{Int}(F_2) = \{ x \} \), say, \( x = 0 \).

Let \( E_i \) be the linear hull of \( F_i \), \( i = 1, 2 \), and assume, at first, that \( E^n = E_1 \perp E_2 \).

Then \( x \) is a vertex of \( E_1 \cap X_2 \) and \( E_2 \cap X_1 \). In this situation we have the fundamental

**Proposition 2.5.** Let \( a \in S^{n-1} \) such that (\( * \)) holds for \( a \) with respect to \( X_1 \cap X_2 \) at 0. Moreover, let \( E_1 \perp E_2 \). Let \( a = a_1 + a_2, a_i \in E_i \) and \( \bar{a}_i = |a_i|^{-1}a_i \).

(Note that by condition (\( * \)) we have \( a_i \neq 0 \) for \( i = 1, 2 \).)

Then

\[
(C / L) \simeq (C_1 / L_1) \times (C_2 / L_2) ,
\]

where \( C_i / L_i \) denotes the class of \( \bar{a}_i \) modulo \( L_i \) and \( a_i \) modulo \( L_i \).
where
\[ C = C(X_1 \times X_2, 0, a), \quad L = L(X_1 \cap X_2, 0, a), \]
\[ C_i = C(X_j \cap E_i, 0, a_i), \quad L_i = L(X_j \cap E_i, 0, a_i), \]
for \( i = 1, 2 \) and \( j = (i \mod 2) + 1 \).

**Proof.** Inside a ball \( B(0, r) \), \( r \) sufficiently small, we have \( X_i \cap B(0, r) = (X_i \cap E_j) \times E_i \). Therefore every \( x \in X_1 \cap X_2 \cap B(0, r) \) corresponds to a pair \((x_1, x_2)\) with \( x_i \in X_i \cap E_j \).

Now \((*)\) holds also for \( X_i \cap E_j = X_i \cap E_j^\perp \) with respect to \( a_i \), \( i = 1, 2 \). For if we had a face \( \{0\} \neq F_i \subset a_i^\perp \cap B(0, r) \), then \( F_i \times \{0\} \) would be a face in \( X_1 \cap X_2 \cap B(0, r) \cap a^\perp \). So we may prove the claim for pairs according to type b) in Remark 2.4.

Let
\[ C' = X_1 \cap X_2 \cap B(0, r), \quad L' = X_1 \cap X_2 \cap S(0, r) \cap \{ y \in X_1 \cap X_2 | \langle a, y \rangle \geq 0 \}, \]
\[ C_i' = E_i \cap X_j \cap B(0, r/2), \quad L_i' = E_i \cap X_j \cap S(0, r/2) \cap \{ y \in E_i \cap X_j | \langle a_i, y \rangle \geq 0 \}. \]

The addition
\[ \alpha : E_1 \times E_2 \to E, \quad (x_1, x_2) \mapsto x_1 + x_2 \]
restricts to a homeomorphism \( \alpha : C_i' \times C_j' \to K \subset C' \), where \( K \) contains a neighbourhood of 0 in \( C' \). We call \( \text{Fr}(K) := \{ \text{endpoints of halflines in } K \} \) the *frontier* of \( K \). Here
\[ \alpha \left( (C_i' \cap S(0, r/2)) \times (C_j' \cap S(0, r/2)) \right) = \text{Fr}(K). \]

Now
\[ \alpha \left( L_i' \times C_j' \cup (L_i' \times C_j') \right) = M := \{ y \in \text{Fr}(K) | \langle a, y \rangle \geq 0 \}. \]

On the other hand, the natural retraction
\[ \text{ret} : B(0, r) \setminus \{0\} \to S(0, r) \]
extends to a homeomorphism \((K/M) \to (C'/L')\) \( \square \)

**Remark 2.6.** Proposition 2.5 remains true if \( E_1 \) and \( E_2 \) are not orthogonal. To see this, introduce a new scalar product \( \langle , \rangle' \) by setting \( \langle , \rangle' = \langle , \rangle \) on \( E_i, i = 1, 2 \) and \( \langle e_1, e_2 \rangle' = 0 \) for \( e_i \in E_i \). However, then one can no longer identify \( E_i \) and \( E_j^\perp \) with respect to \( \langle , \rangle \). This will play a role in Propositions 3.2 and 3.3.

3. The kinematic formula

We keep the situation of Section 2. So \( E = E^n \) is an euclidean space, \( E = E_1 \perp E_2 \), \( \dim E_1 = k \), \( \dim E_2 = n - k \). Let \( S, S_1, S_2 \) be spheres of dimension \( n - 1, k - 1 \), \( n - k - 1 \) in \( E, E_1, E_2 \) respectively. In \( S \) we have subspheres \( T_i := S \cap E_i, i = 1, 2 \). Let \( S' := S \setminus (T_1 \cup T_2) \). Then \( S' \) is dense in \( S \). For \( s \in S' \) let \( s = s_1 + s_2 \) with \( s_i \in E_i \).

**Remark 3.1.** Let \( PS^1 := \{ (t_1, t_2) \in S^1 | t_1 > 0, t_2 > 0 \} \). Then
\[ f : S' \to PS^1 \times S_1 \times S_2, \quad s \mapsto (||s_1||, ||s_2||, s_1, s_2) \]
is a diffeomorphism, where \( s_i = ||s_i||^{-1}s_i \). The inverse is
\[ f^{-1} : PS^1 \times S_1 \times S_2 \to S', \quad ((t_1, t_2), u_1, u_2) \mapsto t_1u_1 + t_2u_2. \]
One has for the volume element
\[ dS' = t_1^{k-1}t_2^{n-k-1}dt_1 dt_2 \, du_1 du_2 \, . \]

We come back to the situation of Proposition 2.5: \( E \supset X_1, X_2 \) are compact PL-spaces. \( F_i \) is a face in \( X_i \) and \( E_i = \text{Aff}(F_i) \) for \( i = 1, 2 \). Moreover, \( \{0\} = \text{Int}(F_1) \cap \text{Int}(F_2) \).

**Proposition 3.2.** Let \( E_1 \perp E_2 \). Then
\[ \varrho(X_1 \cap X_2, 0) = \varrho(X_1 \cap E_2, 0) \varrho(X_2 \cap E_1, 0) = \varrho(X_1, F_1) \varrho(X_2, F_2) \, . \]

**Proof.**
\[
\begin{align*}
\varrho(X_1 \cap X_2, 0) &= |S|^{-1} \int_S \varrho(X_1 \cap X_2, 0, s) \, ds \\
&= |S|^{-1} \int_{PS^1 \times S_1 \times S_2} t_1^{k-1}t_2^{n-k-1} \varrho(X_1 \cap E_2, 0, u_2) \varrho(X_2 \cap E_1, 0, u_1) d(t_1, t_2) \, du_1 \, du_2 \\
&= |S|^{-1} \int_{PS^1} t_1^{k-1}t_2^{n-k-1} \int_{S_1 \times S_2} \varrho(X_1 \cap E_2, 0, u_2) \varrho(X_2 \cap E_1, 0, u_1) \, du_1 \, du_2 \, d(t_1, t_2) \\
&= \varrho(X_1 \cap E_2, 0) \varrho(X_2 \cap E_1, 0) |S|^{-1} \int_{PS^1} t_1^{k-1}t_2^{n-k-1} |S_1| |S_2| \, d(t_1, t_2) \\
&= |S|^{-1} |S| \varrho(X_1 \cap E_2, 0) \varrho(X_2 \cap E_1, 0) \, .
\end{align*}
\]

The preceding proposition is no longer true if \( E_1 \) and \( E_2 \) are not perpendicular (but still complementary), since the measure on \( S_i \), which one gets from projecting \( E_j^\perp \to E_i \), is different from the canonical measure.

So we have to change the measures \( du_1 \) and \( du_2 \) by \( \mu_1(s_1) \, ds_1 \) and \( \mu_2(s_2) \, ds_2 \) respectively, where \( \mu_i \) is a smooth function on \( S_i \) for which
\[ (1) \quad \int_{S_i} \mu_i(s_i) \, ds_i = 1, \quad i = 1, 2. \]

We get as before
\[
\begin{align*}
\varrho(X_1 \cap X_2, 0) &= |S|^{-1} \int_S \varrho(X_1 \cap X_2, 0, s) \, ds \\
&= |S|^{-1} \int_{PS^1 \times S_1 \times S_2} t_1^{k-1}t_2^{n-k-1} \varrho(X_1 \cap E_2, 0, s_2) \varrho(X_2 \cap E_1, 0, s_1) \cdot d(t_1, t_2) \mu_1(s_1) \, ds_1 \mu_2(s_2) \, ds_2 \\
&= |S|^{-1} \int_{PS^1} t_1^{k-1}t_2^{n-k-1} \int_{S_1 \times S_2} \varrho(X_1 \cap E_2, 0, s_2) \varrho(X_2 \cap E_1, 0, s_1) \cdot \mu_1(s_1) \, ds_1 \mu_2(s_2) \, ds_2 \, d(t_1, t_2) \, .
\end{align*}
\]
Nothing can be said about this. Therefore we take averages. Let \( G_i = O(E_i) \) for \( i = 1, 2 \).

**Proposition 3.3.**

\[
\int_{G_1 \times G_2} \varrho(g_2 X_1 \cap g_1 X_2) \, dg_1 \, dg_2 = \varrho(X_1 \cap E_1^+, 0) \cdot \varrho(X_2 \cap E_2^+, 0)
\]

\[
= \varrho(X_1, F_1) \cdot \varrho(X_2, F_2)
\]

**Proof.** By the preceding formula (2) we have

\[
\int_{G_1 \times G_2} \varrho(g_2 X_1 \cap g_1 X_2) \, dg_1 \, dg_2
\]

\[
= |S|^{-1} \int_{P^{S^1}} \int_{t^{k-1} t^{n-k-1}} \int_{S_1 \times S_2} \varrho(g_2 X_1 \cap E_1^+, 0, s_2) \varrho(g_1 X_2 \cap E_2^+, 0, s_1) \cdot \mu_1(s_1) \, ds_1 \mu_2(s_2) \, ds_2 \, d(t_1, t_2) \, dg_1 \, dg_2
\]

\[
= |S|^{-1} \int_{P^{S^1}} \int_{t^{k-1} t^{n-k-1}} \int_{S_1 \times S_2 \times G_1 \times G_2} \varrho(g_2 X_1 \cap E_1^+, 0, s_2) \varrho(g_1 X_2 \cap E_2^+, 0, s_1) \cdot \mu_1(g_1(s_1)) \mu_2(g_2(s_2)) \, ds_1 \, ds_2 \, d(t_1, t_2)
\]

after substituting \( s_i \mapsto g_i(s_i) \) and reversing the ordering of integrations

\[
= |S|^{-1} \int_{P^{S^1}} \int_{t^{k-1} t^{n-k-1}} \int_{S_1 \times S_2} \varrho(g_j X_i \cap E_i^+, 0, s_j) = \varrho(X_i \cap E_i^+, 0, s_j)
\]

\[
= |S|^{-1} \int_{P^{S^1}} \int_{t^{k-1} t^{n-k-1}} \int_{S_1 \times S_2} \varrho(X_1 \cap E_1^+, 0, s_2) \varrho(X_2 \cap E_2^+, 0, s_1) \, ds_1 \, ds_2 \, d(t_1, t_2)
\]

\[
= \varrho(X_1 \cap E_1^+, 0) \cdot \varrho(X_2 \cap E_2^+, 0).
\]

\( \square \)

**Notation 3.4.** Let \( E_k \) and \( E^{n-k} \) be complementary subspaces of \( E^n \). Let \( e_1, \ldots, e_k \) be an orthonormal basis of \( E_k \) and \( e_{k+1}, \ldots, e_n \) be an orthogonal basis of \( E^{n-k} \).

Then we denote

\[
|E_k : E^{n-k}| := |\det(e_1, \ldots, e_n)|.
\]

**Remark 3.5.** Let \( F_1 \) be a \( k \)-face in \( X_1 \), \( F_2 \) an \( (n-k) \)-face in \( X_2 \) such that \( \text{Aff}(F_i) = E_i + a_i \), \( 0 \in E_i \) for \( i = 1, 2 \). Then

\[
\int_{F_1 \cap F_2 + x \neq \emptyset} dx = |F_1| |F_2| |E_1 : E_2|.
\]

Recall that \( G \) is the group of all euclidean motions of \( E \) with canonical measure \( dg \).

From Proposition 3.3 and Remark 3.5 we get
Proposition 3.6.

\[ \int_G g(x_1 \cap g x_2, x) \, dg = |F_1| |F_2| c(n, k) g(x_1, F_1) g(x_2, F_2) \]

\[ = c(n, k) W_k(F_1) W_{n-k}(F_2), \]

where

\[ c(n, k) = \int |E_1 : g E_2| \, dg. \]

Note that \( |E_1 : g E_2| \) only depends on the coset \( O(E) / O(E_1) \times O(E_2) \).

One may compute \( c(n, k) \) directly or as in the proof below. Now, for the computation of \( \int_G g(x_1 \cap g x_2) \, dg \) we just gather all occurrences where a \( k \)-face of \( X_1 \) intersects an \( (n-k) \)-face of \( X_2 \) and apply Proposition 3.6.

Theorem 3.7 (Kinematic formula). Let \( E = \mathbb{R}^n \) be an euclidean vector space, \( G = O(E) \times E \) the group of all euclidean motions endowed with the product of canonical Haar measure and Lebesgue measure and let \( E \supseteq X_1, X_2 \) be compact PL-spaces. Then

\[ \int_G g(x_1 \cap g x_2) \, dg = \sum_{k=0}^{n} \binom{n}{k}^{-1} \frac{\omega_k \omega_{n-k}}{\omega_n} W_k(X_1) W_{n-k}(X_2). \]

Proof. It remains to compute the universal constants \( c_k \). For this consider the unit cube \( Q_k \) of dimension \( k \). Then

\[ \int_G g(x_1 \cap g(r Q_i)) \, dg = \sum_{i=n-k}^{n} c_i W_i(X_1) W_{n-i}(r Q_k) \]

\[ = \sum_{i=n-k}^{n} c_i W_i(X_1) \cdot r^{n-i} W_{n-i}(Q_k). \]

Dividing by \( r^k \) and letting \( r \to \infty \) we get

\[ (3) \]

\[ c_{n-k} W_{n-k}(X_1) = \int_G g(x_1 \cap g E^k) \, dg. \]

Next, interchanging \( X_1 \) and \( X_2 \), we observe that \( c_k = c_{n-k} \) for \( k = 0, \ldots, n \).

Let \( B_k \) be the unit ball in \( \mathbb{R}^k \), \( \omega_k = \text{vol}_k(B_k) \). We may choose \( g = \sigma \).

After approximating \( B_n \) by a convex polytope, we may compute

\[ p(r) := \int_G g(B_n \cap g(r B_n)) \, dg \]

in two ways. Directly we get

\[ p(r) = \omega_n (1 + r)^n = \omega_n \sum_{k=0}^{n} \binom{n}{k} r^k, \]
and by (3) we get

\[ p(r) = \sum_{k=0}^{n} c_k W_{n-k}(B_n) W_k(B_n) r^k \]

\[ = \sum_{k=0}^{n} c_k^{-1} \int_G \varrho(B_n \cap gE^k) \, dg \int_G \varrho(B_n \cap gE^{n-k}) \, dg \, r^k \]

\[ = \sum_{k=0}^{n} c_k^{-1} \omega_k \omega_{n-k} r^k . \]

Comparing homogenous parts we get

\[ c_k = \binom{n}{k}^{-1} \omega_k \omega_{n-k} . \]

\[ \square \]

**Corollary 3.8** (Linear kinematic formula).

Let \( E^n \supset X \) be a compact PL-space. Then

\[ W_k(X) = \binom{n}{k} \frac{\omega_n}{\omega_k \omega_{n-k}} \int_G \varrho(X \cap gE^{n-k}) \, dg = \int \varrho(X \cap E) \, dE . \]

For \( \varrho = \tau \) this holds also for manifolds [Ba] and even for sets of positive reach [R-Z], but as we mentioned before, very little is known for the absolute curvature in more general situations.

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