ON A POSITIVITY PRESERVATION PROPERTY FOR
SCHRÖDINGER OPERATORS ON RIEMANNIAN MANIFOLDS

OGNJEN MILATOVIC

Abstract. We study a positivity preservation property for Schrödinger operators with singular potential on geodesically complete Riemannian manifolds with non-negative Ricci curvature. We apply this property to the question of self-adjointness of the maximal realization of the corresponding operator.

1. Introduction

In his landmark paper [Ka1], Kato proved a powerful distributional inequality, today known as Kato’s inequality, which has since found numerous applications in self-adjointness (and m-accretivity) problems in $L^2(\mathbb{R}^n)$ for Schrödinger operators with a singular potential $V$. In this context, it is desirable that the “negative part” of $V$, that is max$(-V, 0)$, satisfy the positivity preservation property described below.

Positivity Preservation Property (PPP). Let $F \in L^1_{\text{loc}}(\mathbb{R}^n)$ be a non-negative function. Then, there exists $\lambda_0 \geq 0$ so that if $\lambda > \lambda_0$, $u \in L^2(\mathbb{R}^n)$, $Fu \in L^1_{\text{loc}}(\mathbb{R}^n)$, and
\[-\Delta u + \lambda u - Fu \geq 0, \quad \text{in distributional sense,}\]
then $u \geq 0$.

Brézis and Kato [BK] showed that (PPP) holds for (non-negative) functions $F \in L^\infty(\mathbb{R}^n) + L^p(\mathbb{R}^n)$ with $p = \frac{n}{2}$ for $n \geq 3$, $p > 1$ for $n = 2$, and $p = 1$ for $n = 1$, together with the assumption $F \in L^{n/2+\epsilon}(\mathbb{R}^n)$, $\epsilon > 0$, in dimensions $n = 3$ and $n = 4$. The proof of (PPP) in [BK] was based on elliptic equation theory and Sobolev space techniques.

Subsequently, using stochastic analysis techniques, Devinatz [De] showed that (PPP) holds for (non-negative) functions $F \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfying the property
\[
\lim_{\alpha \to \infty} \left( \sup_{x \in \mathbb{R}^n} \frac{1}{4\pi^{n/2}} \int_{\mathbb{R}^n} \frac{F(x-y)}{|y|^{n-2}} \left( \int_{0}^{\infty} \tau^{n/2-2} e^{-\tau} d\tau \right) dy \right) < 1.
\]

We should note that the results of [De] include those of Jensen [J]. As an application of (PPP), the papers [BK, De, J] studied the self-adjointness problem of the corresponding Schrödinger operator.

In the context of a Riemannian manifold $M$, a simpler variant of (PPP) with $F \equiv 0$, which we label as (PPP-0), was considered in Proposition B.3 of [BMS], where it was shown that (PPP-0) holds under $C^\infty$-bounded geometry assumption.

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on $M$, that is, $M$ has a positive injectivity radius and all Levi–Civita derivatives of the curvature tensor of $M$ are bounded. The main point here is that the corresponding proof of \cite{BMS} depends on the existence of a sequence of smooth compactly supported functions $\chi_k$ with the following properties:

(C1) $0 \leq \chi_k(x) \leq 1$, $x \in M$, $k = 1, 2, \ldots$;
(C2) for every compact set $K \subset M$, there exists $k_0$ such that $\chi_k = 1$ on $K$, for $k \geq k_0$;
(C3) $\sup_{x \in M} |d\chi_k(x)| \to 0$ as $k \to \infty$.
(C4) $\sup_{x \in M} |\Delta \chi_k(x)| \to 0$ as $k \to \infty$.

While the existence of a sequence $\chi_k$ satisfying (C1), (C2), and (C3) on an arbitrary geodesically complete Riemannian manifold is well known (see \cite{K}), a sequence satisfying all four properties has not yet been constructed (to our knowledge) in such a general context.

Very recently, G"uneysu \cite{G4} has improved (PPP-0) result considerably. In particular, in the context of a geodesically complete Riemannian manifold with non-negative Ricci curvature, the author of \cite{G4} has constructed a sequence $\chi_k$ satisfying (C1)–(C4) and proved (PPP-0). We should also note that the paper \cite{G4} contains, among other things, a study of (PPP-0) in the setting of $L^p$ spaces with $p \in [1, \infty]$.

Let us point out that under $C^\infty$-bounded geometry assumptions on $M$, an earlier study \cite{Mi} showed that (PPP) holds for (non-negative) functions $F$ belonging the Kato class (see Section 3.1 below) and satisfying the following additional assumption: $F \in L^p_{\text{loc}}(M)$ with $p = n/2 + \epsilon$, with some arbitrarily small $\epsilon > 0$, for the case $2 \leq n \leq 4$; $p = n/2$ for the case $n \geq 5$. We note that the paper \cite{Mi} used the latter assumption for elliptic equation and Sobolev space arguments. Based on recent developments in path-integral representations for semi-groups of Schrödinger operators with singular potential on Riemannian manifolds and the construction of cut-off functions satisfying (C1)–(C4) above, as seen in G"uneysu’s works \cite{G1, G2, G3, G4}, we will study (PPP) for a class functions $F$ that shares some properties with \cite{L1} and includes, in particular, Kato class. In this regard, within the class of non-negative Ricci curvature, our results include those in \cite{Mi}. In particular, we eliminate the assumption $F \in L^p_{\text{loc}}(M)$ with $p$ as described above. Finally, as an application of the corresponding (PPP), we give sufficient conditions for the self-adjointness of the “maximal” realization of the Schrödinger operator with electric potential whose negative part satisfies the same assumptions as $F$ in (PPP).

For reviews of results concerning the question of self-adjointness of Schrödinger operators in $L^2(\mathbb{R}^n)$ and $L^2(M)$, see, for instance, \cite{CFKS} and \cite{BMS}. For more recent studies, see the papers \cite{Ba} BGP \cite{GK} \cite{G4} \cite{GP}.

Finally, we remark that it might be possible to obtain a variant of (PPP) for perturbations of Dirichlet forms by measures. For the background on Dirichlet forms and their perturbations by measures, see, for instance, the book \cite{FOT}, papers \cite{KT1} \cite{KT2} \cite{SV}, and references therein.

2. Results

2.1. Notations. Let $(M, g)$ be a connected smooth Riemannian $n$-manifold without boundary. Throughout the paper, by $\Delta$ we denote the corresponding negative Laplace–Beltrami operator on $M$, by $d\mu$ the volume measure of $M$, by $C^\infty(M)$ the
space of complex-valued smooth functions on \( M \), by \( C_c^\infty(M) \) the space of complex-valued smoothly compactly supported functions on \( M \), by \( \Omega^1(M) \) the space of smooth 1-forms on \( M \), by \( L^2(M) \) the space of square integrable complex-valued functions on \( M \), and by \( \langle \cdot, \cdot \rangle \) the usual inner product on \( L^2(M) \). Additionally, \( p(t, x, y) \) denotes the heat kernel of \( M \) as in Theorem 7.13 in [Gr]. We should emphasize that in this paper \( p(t, x, y) \) corresponds to \( e^{-t(-\Delta/2)} \), \( t \geq 0 \), instead of \( e^{-t(-\Delta)} \).

### 2.2. Positivity Preservation Property

We are ready to formulate sufficient conditions for the positivity preservation property introduced in Section 1.

**Theorem 2.1.** Assume that \( M \) is a geodesically complete connected Riemannian manifold with non-negative Ricci curvature. Let \( F: M \to [0, \infty) \) be a measurable function satisfying the following property: there exists \( t_0 > 0 \) such that

\[
\sup_{x \in M} \left( \int_0^{t_0} \int_{M} p(s, x, y) F(y) \, d\mu(y) \, ds \right) < 1.
\]

Then, there exists \( \lambda_\ast \geq 0 \) such that if \( \lambda > \lambda_\ast \) and \( u \in L^2(M) \) and \( Fu \in L^1_{\text{loc}}(M) \) and \( u \) satisfies the distributional inequality

\[
(-\Delta/2 - F + \lambda)u \geq 0,
\]

then \( u \geq 0 \) a.e. on \( M \).

**Remark 2.2.** If \( F \) belongs to Kato class, then (2.1) is satisfied; see Section 3.1 below.

### 2.3. Hermitian Vector Bundles and Bochner Laplacian

We will formulate our self-adjointness result for Schrödinger operators acting on Hermitian vector bundles over \( M \). Before doing so, we explain some additional notations. Let \( E \to M \) be a smooth Hermitian vector bundle over \( M \) with underlying Hermitian structure \( \langle \cdot, \cdot \rangle_x \) and the corresponding norms \( | \cdot |_x \) on fibers \( E_x \). Smooth sections of \( E \) will be denoted by \( C^\infty(E) \) and compactly supported smooth sections by \( C^\infty_c(E) \). With \( d\mu \) as in Section 2.1 for all \( 1 \leq p < \infty \) we obtain the \( L^p(E) \) spaces of sections \( L^p(E) \) with norms \( \| \cdot \|_p \). The space of essentially bounded sections of \( E \) will be denoted by \( L^\infty(E) \) with the corresponding norm \( \| \cdot \|_\infty \). The notation \( (\cdot, \cdot)_{L^2(E)} \) or just \( \langle \cdot, \cdot \rangle \), when there is no danger of confusion, stands for the usual inner product in \( L^2(E) \).

Let \( \nabla \) be a Hermitian connection on \( E \) and let \( \nabla^* \) be its formal adjoint with respect to \( (\cdot, \cdot)_{L^2(E)} \). In what follows, we will consider the so-called Bochner Laplacian operator \( \nabla^* \nabla: C^\infty(E) \to C^\infty(E) \). For example, if we take \( \nabla = d \), where \( d: C^\infty(M) \to \Omega^1(M) \) is the standard differential, then \( d^* d: C^\infty(M) \to C^\infty(M) \) is just the (non-negative) Laplace–Beltrami operator \( -\Delta \).

We are interested in the Schrödinger-type differential expression

\[
L_V = \nabla^* \nabla/2 + V,
\]

where \( V \) is a measurable section of \( \text{End} E \) such that \( V(x): E_x \to E_x \) is a self-adjoint operator for almost every \( x \in M \).

For every \( x \in M \) we have the following canonical decomposition:

\[
V(x) = V^+(x) - V^-(x),
\]

where

\[
V^+(x) := P_+(x)V(x) \quad \text{and} \quad V^-(x) := -P_-(x)V(x),
\]

Here, \( P_+(x) := \chi_{[0, \infty)}(V(x)) \) and \( P_-(x) := \chi_{(-\infty, 0]}(V(x)) \), and \( \chi_G \) denotes the characteristic function of the set \( G \).
Lemma 3.1. If \( f \) is a measurable function on \( M \), then \( \sigma_{\max}(f) \) is a (real-valued) measurable function on \( M \).

2.4. Self-adjoint Realization of \( H_V \). Assume that \( V \in L^1_{\text{loc}}(E) \) and \( \sigma_{\max}(V^-) \in L^1_{\text{loc}}(M) \). We define \( S \) as an operator in \( L^2(E) \) by \( Su = L_Vu \) with the following domain \( \text{Dom}(S) \):

\[
\{ u \in L^2(E) : V^+ u \in L^1_{\text{loc}}(E), \sigma_{\max}(V^-)u \in L^1_{\text{loc}}(E), \text{ and } L_Vu \in L^2(E) \}
\]

Here, the expression \( L_Vu \) is understood in distributional sense.

Theorem 2.3. Assume that \( M \) is a geodesically complete connected Riemannian manifold with non-negative Ricci curvature. Assume that \( V^+ \in L^1_{\text{loc}}(E) \) and \( \sigma_{\max}(V^-) \) satisfies the property (2.1). Then \( S \) is a self-adjoint operator.

Remark 2.4. If \( \sigma_{\max}(V^-) \) satisfies the property (2.1), then \( \sigma_{\max}(V^-) \in L^1_{\text{loc}}(M) \); see Lemma 3.2 below.

Remark 2.5. For the operator \( H_V = -\Delta + V \) acting on scalar functions, the conditions \( V^+ u \in L^1_{\text{loc}}(E) \) and \( \sigma_{\max}(V^-)u \in L^1_{\text{loc}}(E) \) are equivalent to \( Vu \in L^1_{\text{loc}}(M) \). In this case, (2.4) describes the “maximal” realization of \( H_V \) in the sense of [Ka2].

3. Proof of Theorem 2.1

We first recall two definitions from [KT2].

3.1. Dynkin and Kato Classes. Let \( p(t,x,y) \) be as in Section 2.1. We say that a measurable function \( f : M \to \mathbb{R} \) belongs to Dynkin class relative to \( p(t,x,y) \) and write \( f \in S^0_D \) if \( |f| \) satisfies (2.1). We say that a measurable function \( f : M \to \mathbb{R} \) belongs to Kato class relative to \( p(t,x,y) \) and write \( f \in S^0_K \) if

\[
\lim_{t \to 0^+} \sup_{x \in M} \int_0^t \int_M p(s,x,y)|f(y)| \, d\mu(y) \, ds = 0.
\]

Clearly, we have the inclusion \( S^0_K \subset S^0_D \). For \( \alpha > 0 \), set

\[
r_\alpha(x,y) := \int_0^{\infty} e^{-\alpha t}p(t,x,y) \, dt.
\]

For \( \alpha > 0 \) and \( f \in S^0_D \) define

\[
c_\alpha(f) := \sup_{x \in M} \int_M r_\alpha(x,y)|f(y)| \, d\mu(y).
\]

By Lemma 3.2 in [KT2], we have \( c_\alpha(f) < \infty \) for all \( \alpha > 0 \). We now set

\[
c(f) := \inf_{\alpha > 0} c_\alpha(f).
\]

Lemma 3.1. If \( f \in S^0_D \) then \( c(f) < 1 \).

Proof. By Lemma 3.1 in [KT1] (or Proposition 2.7(a) in [G2]), for any measurable function \( f : M \to \mathbb{R} \) and all \( \alpha, t > 0 \) we have

\[
(1 - e^{-\alpha t}) \sup_{x \in M} \int_M r_\alpha(x,y)|f(y)| \, d\mu(y) \leq \sup_{x \in M} \int_0^t \int_M p(s,x,y)|f(y)| \, d\mu(y) \, ds.
\]
Since \( f \in S^0_D \), there exists \( t = t_0 > 0 \) such that the right hand side is less than 1. Consequently, we get
\[
c_\alpha(f) < \frac{1}{1 - e^{-\alpha t_0}},
\]
for all \( \alpha > 0 \), and from here \( c(f) < 1 \) follows easily. \( \square \)

The following lemma follows from Proposition 2.7(b) in [G2]:

**Lemma 3.2.** If \( f \in S^0_D \) then \( f \in L^1_{\text{loc}}(M) \).

**Remark 3.3.** The proof of Proposition 2.7(b) in [G2] uses strict positivity of \( p(t, x, y) \), which requires connectedness of \( M \).

**Remark 3.4.** In the sequel, for any \( x \in M \), the symbol \( \mathbb{P}^x \) stands for the law of a Brownian motion \( X_t \) on \( M \) starting at \( x \), and \( \mathbb{E}^x \) denotes the expected value corresponding to \( \mathbb{P}^x \). Our hypothesis on \( M \) ensure that \( M \) is stochastically complete (see [G1]); hence, the lifetime of \( X_t \) is \( \zeta = \infty \). We should emphasize that in this paper \( \mathbb{P}^x \) is \(-\Delta/2\) diffusion, as opposed to \(-\Delta \) diffusion.

**Remark 3.5.** We should note that the geodesic completeness and non-negative Ricci curvature assumptions are not used until Lemma 3.10 below. Also, in the absence of stochastic completeness, path-integral formulas below can be rewritten by taking into account the lifetime \( \zeta \) of \( X_t \).

**Lemma 3.6.** If \( 0 \leq f \in S^0_D \) then there exist constants \( \beta > 0 \) and \( \gamma > 0 \) such that
\[
\sup_{x \in M} \mathbb{E}^x \left[ e^{\int_0^t f(X_s) \, ds} \right] \leq \beta e^{\gamma t},
\]
for all \( t > 0 \).

**Proof.** First note that we can write
\[
\int_0^t \int_M p(s, x, y) f(y) \, d\mu(y) \, ds = \mathbb{E}^x \left[ \int_0^t f(X_s) \, ds \right].
\]
By the definition of the class \( S^0_D \), there exists \( t^* > 0 \) such that
\[
\nu_t := \sup_{x \in M} \mathbb{E}^x \left[ \int_0^t f(X_s) \, ds \right] < 1,
\]
for all \( 0 < t \leq t^* \). By Khasminskii’s Lemma (see Lemma 3.37 in [LHB]) we have
\[
\sup_{x \in M} \mathbb{E}^x \left[ e^{\int_0^t f(X_s) \, ds} \right] \leq \frac{1}{1 - \nu_t},
\]
for all \( 0 < t \leq t^* \). From here on we may repeat the proof of Lemma 3.38 of [LHB] to conclude that
\[
\sup_{x \in M} \mathbb{E}^x \left[ e^{\int_0^t f(X_s) \, ds} \right] \leq \left( \frac{1}{1 - \nu_{t^*}} \right)^{\lfloor t/t^* \rfloor + 1},
\]
for all \( t > 0 \), where \( \lfloor a \rfloor := \max\{k \in \mathbb{Z} : k \leq a\} \).

Setting \( \beta = \frac{1}{1 - \nu_{t^*}} \) and \( \gamma = \frac{1}{t^*} \log \left( \frac{1}{1 - \nu_{t^*}} \right) \), we obtain (3.3). \( \square \)
3.2. Quadratic Forms. In what follows, all quadratic forms are considered in the space $L^2(M)$. Let $w \in L^2_{loc}(M)$. Set $w^+ := \max(w, 0)$ and $w^- := \max(-w, 0)$, so that $w = w^+ - w^-$. Define

$$Q_0(u) := \frac{1}{2} \int_M |du|^2 d\mu,$$

with the domain $D(Q_0) = \{u \in L^2(M) : Q_0(u) < \infty\}$. The form $Q_0$ is non-negative, densely defined (since $C^\infty_c(M) \subset D(Q_0)$), and closed. Define $Q_{w^+}(u) := (w^+ u, u)$ with the domain $D(Q_{w^+}) = \{u \in L^2(M) : w^+ |u|^2 \in L^1(M)\}$. The form $Q_{w^+}$ is non-negative, densely defined (since $C^\infty_c(M) \subset D(Q_{w^+})$), and closed (by Theorem VI.1.11 in [Ka3] and Example VI.1.15 in [Ka3]). Finally, we define $Q_{w^-}(u) := -(w^- u, u)$ with the domain $D(Q_{w^-}) = \{u \in L^2(M) : w^- |u|^2 \in L^1(M)\}$. The form $Q_{w^-}$ is symmetric and densely defined.

Lemma 3.7. Assume that $w^- \in S^0_D$. Then there exist $a \in [0, 1)$ and $b \geq 0$ such that

$$|Q_{w^-}(u)| \leq a|Q_0(u)| + b|u|^2, \quad \text{for all } u \in D(Q_0).$$

Proof. Let $c_\alpha(w^-)$ be as in (3.2). We have already observed that $w^- \in S^0_D$ implies $c_\alpha(w^-) < \infty$ for all $\alpha > 0$. By Theorem 3.1 in [SV] we have

$$(w^- u, u) \leq \frac{c_\alpha(w^-)}{2} \int_M |du|^2 d\mu + \alpha c_\alpha(w^-) |u|^2,$$

for all $u \in D(Q_0)$ and all $\alpha > 0$. By Lemma 3.1 we have $c(w^-) < 1$. Hence, there exists $\alpha^*$ such that $c_{\alpha^*}(w^-) < 1$, which shows (3.4). \qed

By Lemma 3.7 above and Theorem VI.1.33 in [Ka3], the form $Q_{0,w} := (Q_0 + Q_{w^+}) + Q_{w^-}$ is densely defined, closed and semi-bounded from below with $D(Q_w) = D(Q_0) \cap D(Q_{w^+}) \subset D(Q_{w^-})$. Let $H(w)$ denote the semi-bounded from below self-adjoint operator in $L^2(M)$ associated to $Q_{0,w}$ by Theorem VI.2.1 of [Ka3].

3.3. Semigroup Associated to $H(-w^-)$. As seen from the proof of Lemma 3.7 for $w^- \in S^0_D$, there exists $\alpha_*$ such that $c_{\alpha_*}(w^-) < 1$, and the form $Q_{0,-w^-} := Q_0 + Q_{w^-}$ is semi-bounded from below by $-\alpha_* c_{\alpha_*}(w^-)$. Let $H(-w^-)$ be the corresponding self-adjoint (semi-bounded from below) operator and let $U_{2,-w^-}(t) := e^{-tH(-w^-)}, \ t \geq 0$, be the corresponding $C_0$-semigroup in $L^2(M)$. The following Lemma was proven in Theorem 3.3 of [SV].

Lemma 3.8. Assume that $w^- \in S^0_D$. Then, the operators $U_{2,-w^-}(t)$ act as $C_0$-semigroups in $L^p(M)$, for all $p \in [1, \infty)$, and we label those semigroups as $U_{p,-w^-}(t)$. Moreover, there exist $C \geq 0$ and $\omega \in \mathbb{R}$ (depending only on $\alpha_*$ and $c_{\alpha_*}(w^-)$) such that

$$\|U_{p,-w^-}(t)\|_{L^p \rightarrow L^p} \leq Ce^{\omega t},$$

for all $p \in [1, \infty)$ and $t \geq 0$.

3.4. Path Integral Representation of $U_{2,-w^-}(t)$. Let $X_t$ be as in Remark 3.4. For $w^- \in S^0_D$ we have the Feynman–Kac formula

$$U_{2,-w^-}(t)g(x) = \mathbb{E}^x \left[ e^{\int_0^t w^-(X_s) \, ds} g(X_t) \right],$$

where $\mathbb{E}^x$ denotes the expectation with respect to the increment process $\{X_t \mid t \geq 0\}$ started at $x$. The Feynman–Kac formula provides a probabilistic representation of the semigroup $U_{2,-w^-}(t)$ for $w^- \in S^0_D$. This representation is particularly useful in the study of stochastic processes and Feynman–Kac formulas.
for all \( g \in L^2(M) \), all \( t \geq 0 \), and a.e. \( x \in M \). In the Kato-case \( w^- \in S^0_\text{K} \), the formula (3.6) was proven in Theorem 2.9 of [G3]. The same proof works for \( w^- \in S^0_D \) thanks to (3.4) and the the following property: \( w^- \in S^0_D \) implies

\[
\mathbb{P}^x[w^-(X_*) \in L^1_{\text{loc}}(0, \infty)] = 1, \quad \text{a.e. } x \in M.
\]

For the latter property see the proof of Lemma 2.4(b) in [G3], which works without change for the class \( S^0_\text{D} \) instead of \( S^0_\text{K} \).

**Lemma 3.9.** If \( w^- \in S^0_D \) then for all \( g \in L^2(M) \cap L^\infty(M) \) and all \( t \geq 0 \) we have

\[
\|U_{2,-w^-}(t)g\|_\infty \leq \beta e^{\gamma t}\|g\|_\infty,
\]

where \( \beta > 0 \) and \( \gamma > 0 \) are some constants.

**Proof.** The lemma follows by combining (3.7) and (3.3). \( \square \)

**3.5. Cut-off Functions.** The following lemma was proven in Theorem 2.2 of [G4].

**Lemma 3.10.** Assume that \( M \) is a geodesically complete Riemannian manifold with non-negative Ricci curvature. Then there exists a sequence of functions \( \chi_k \in C^\infty_c(M) \) satisfying the properties (C1)–(C4) from Section 1.

**3.6. Sobolev Space.** Let \( \tilde{H}^2(M) \) denote the space of measurable functions \( u: M \to \mathbb{C} \) such that

\[
\|u\|_{\tilde{H}^2} := \|u\| + \|du\| + \|\Delta u\| < \infty,
\]

where \( \|du\| \) denotes the norm in \( L^2(\Lambda^1T^*M) \).

**Lemma 3.11.** Assume that \( M \) is a geodesically complete Riemannian manifold with non-negative Ricci curvature. Let \( 0 \leq u \in \tilde{H}^2(M) \). Then there exists a sequence of functions \( 0 \leq u_k \in C^\infty_c(M) \) such that \( \|u_k - u\|_{\tilde{H}^2} \to 0 \), as \( k \to \infty \).

**Proof.** In this proof, \( (\tilde{H}^2(M))^+ \) and \( (C^\infty_c(M))^+ \) denote the sets of non-negative elements of \( \tilde{H}^2(M) \) and \( C^\infty_c(M) \) respectively. Let \( u \in (\tilde{H}^2(M))^+ \) and let \( \chi_k \) be the sequence of cut-off functions as in Lemma 3.10. We will first show that the set of compactly supported elements of \( (\tilde{H}^2(M))^+ \) is dense in \( (\tilde{H}^2(M))^+ \). To do this, first note that

\[
d(\chi_k u) = u d\chi_k + \chi_k du
\]

and

\[
\Delta(\chi_k u) = \chi_k(\Delta u) + 2(d\chi_k, du) + u(\Delta \chi_k).
\]

If we denote the Riemannian metric of \( M \) by \( r = (r_{jk}) \), the notation \( \langle \kappa, \psi \rangle \) in (3.8) for 1-forms \( \kappa = \kappa_j dx^j \) and \( \psi = \psi_k dx^k \) means

\[
\langle \kappa, \psi \rangle := r^{jk} \kappa_j \psi_k,
\]

where \( (r^{jk}) \) is the inverse matrix to \( (r_{jk}) \), and the standard Einstein summation convention is understood. Now the property \( \|\chi_k u - u\|_{\tilde{H}^2} \to 0 \), as \( k \to \infty \), easily follows from (3.7), (3.8), and (C1)–(C4). This shows that the set of compactly supported elements of \( (\tilde{H}^2(M))^+ \) is dense in \( (\tilde{H}^2(M))^+ \). It remains to show that \( (C^\infty_c(M))^+ \) is dense in the set of compactly supported elements of \( (\tilde{H}^2(M))^+ \). To see this, we start with a compactly supported element \( u \in (\tilde{H}^2(M))^+ \). Since the support of \( u \) is compact, using a partition of unity, we may assume that \( u \) is supported in a coordinate chart \( (G, \phi) \) of \( M \) such that \( \phi(G) = K_1 \), where \( K_1 \) is
an open ball of radius 1 in \(\mathbb{R}^n\). Applying the Friederichs mollification procedure to \(u \circ \phi^{-1}\), we obtain a sequence of non-negative smooth functions \(v_j\) with support in \(K_j\) converging to \(u \circ \phi^{-1}\) with respect to \(\| \cdot \|_{W^{2,2}}\), as \(j \to \infty\), where \(\| \cdot \|_{W^{2,p}}\) stands for the usual Sobolev norm in \(\mathbb{R}^n\), with \(k\) indicating the highest derivative and \(p\) the corresponding \(L^p\)-space. Then \(v_j \circ \phi\) converges to \(u\) in the norm \(\| \cdot \|_{\tilde{H}^2}\), as \(j \to \infty\).

With the above preparations, the proof of Theorem 2.1 proceeds as that of (PPP) in [De].

**Proof of Theorem 2.1.** Let \(F\) be as in hypotheses of the Theorem. Define \(F_k := \min(F, k)\), \(k \in \mathbb{Z}_+\), and consider the semigroup \(U_{2,-F_k}(t)\) as in Section 3.3. Denote the generator of this semigroup by \(H(-F_k)\). As \(F_k \in L^\infty(M)\) and \(M\) is geodesically complete, it is well known that \((-\Delta/2 - F_k)|_{C^\infty(M)}\) is essentially self-adjoint and its (self-adjoint) closure is \((-\Delta/2 - F_k)|_{C^\infty(M)}\). Let \(F_k\) and \(F\) coincide with \(H(-F_k)\), which, in turn, coincides with the operator sum \(H(0) - F_k\), where \(F_k\) stands for the corresponding multiplication operator by the function \(F_k\).

Noting \(-F_k \geq -F\) and using the representation \(3.9\) together with \(3.10\) we have

\[
\|U_{2,-F_k}(t)\|_{L^2 \to L^2} \leq \|U_{2,-F}(t)\|_{L^2 \to L^2} \leq Ce^{\omega t},
\]

where \(U_{2,-F}(t)\) is the semigroup corresponding to \(H(-F)\) as in Section 3.3.

\(\lambda_* := \max\{\omega, \gamma, \alpha_* c_{\alpha_*}(F)\}\), where \(\gamma\) is as in Lemma 3.10 and \(\alpha_* c_{\alpha_*}(F)\) is as in Section 3.3. For \(\lambda > \lambda_*\), the (linear) operator \((\lambda + H(-F_k))^{-1} : L^2(M) \to L^2(M)\) is bounded. Let \(g \in L^2(M) \cap L^\infty(M)\) and \(g \geq 0\). For \(k \in \mathbb{Z}_+\), define

\[
(3.10) \quad u_k := (\lambda + H(-F_k))^{-1}g.
\]

Using the representation

\[
(3.11) \quad (\lambda + H(-F_k))^{-1}g = \int_0^\infty e^{-\lambda t}U_{2,-F_k}(t)g \, dt,
\]

the estimate \(3.9\) and the inequality \(\lambda > \lambda_*\), we obtain

\[
(3.12) \quad \|u_k\| \leq C \int_0^\infty e^{-(\lambda - \omega)t} \|g\| \, dt \leq C_1 \|g\|.
\]

for all \(k \in \mathbb{Z}_+\), with some constant \(C_1 \geq 0\).

Note that \(u_k \geq 0\) by \(3.11\), \(3.10\) and the assumption \(g \geq 0\). By Lemma 3.10 we have

\[
0 \leq u_k(x) = \int_0^\infty e^{-\lambda t}U_{2,-F_k}(t)g \, dt \\
\leq \int_0^\infty e^{-\lambda t}\|U_{2,-F_k}(t)g\|_\infty \, dt \\
\leq \beta \int_0^\infty e^{-(\lambda - \gamma)t} \|g\|_\infty \, dt \leq C_2 \|g\|_\infty,
\]

where \(C_2 \geq 0\) is a constant, and in the last inequality we used \(\lambda > \lambda_* \geq \gamma\).
By the definition of $u_k$ we have
\[(\lambda + H(-F_k))u_k = g.\]
Taking the inner product in $L^2(M)$ with $u_k$, using the fact that $H(0)$ is the operator associated to the form $Q_0$, and recalling the inequality $-F_k \geq -F$, we obtain
\[
(g, u_k) = ((\lambda + H(0) - F_k)u_k, u_k) = \lambda \|u_k\|^2 + \frac{1}{2} \int_M |du_k|^2 d\mu - (F_k u_k, u_k)
\]
\[
\geq \lambda \|u_k\|^2 + \frac{1}{2} \int_M |du_k|^2 d\mu - (F u_k, u_k),
\]
which, upon combining with (3.4) and rearranging, leads to
\[
(g, u_k) \geq \frac{1-a}{2} \int_M |du_k|^2 d\mu + (\lambda - b)\|u_k\|^2.
\]
From the last inequality we get
\[
\frac{1-a}{2} \int_M |du_k|^2 d\mu \leq (g, u_k) + (b-\lambda)\|u_k\|^2
\]
(3.14)
\[
\leq |b-\lambda|(C_1^2\|g\|^2 + C_1\|g\|^2),
\]
where in the last estimate we used Cauchy–Schwarz inequality and (3.12).

Let $0 \leq \psi \in C_c^\infty(M)$, let $u$ be as in the hypothesis of the theorem, and let $0 \leq g \in L^2(M) \cap L^\infty(M)$ and $u_k$ be as in (3.10). We have the following equality:
\[
(\psi u, g) = (\psi u, (-\Delta/2 + \lambda - F_k)u_k).
\]
Using (3.14) and the property
\[
0 \leq u_k \in \text{Dom}(H(-F_k)) = \{v \in L^2(M) : \Delta v \in L^2(M)\},
\]
we have $0 \leq u_k \in \tilde{H}^2(M)$. Thus, by Lemma 3.11 without loss of generality, we may assume that $0 \leq u_k \in C_c^\infty(M)$ in (3.15), which we will do from now on.

Using (3.7) and (3.8) we have
\[
(\psi u, (-\Delta/2 + \lambda - F_k)u_k) = ((-\Delta/2)(\psi u), u_k) + \lambda(u, \psi u_k)
\]
\[
+ ((F - F_k)\psi u, u_k) - (F\psi u, u_k)
\]
\[
= ((-\Delta/2 + \lambda - F)\psi u, u_k) + ((-\Delta/2)\psi u, u_k)
\]
\[
- (du, (d\psi)u_k) + ((F - F_k)\psi u, u_k)
\]
\[
geq ((-\Delta/2)\psi u, u_k) - (du, (d\psi)u_k) + ((F - F_k)\psi u, u_k),
\]
where in the last inequality we used $0 \leq \psi u_k \in C_c^\infty(M)$ and the assumption (2.2).

Using the fact that $-\Delta = d^*d$ we have
\[
(du, (d\psi)u_k) = (u, d^*(d\psi)u_k)) = (u, (d^*d\psi)u_k) - (ud\psi, du_k)
\]
(3.16)
\[
= ((-\Delta \psi)u, u_k) - (ud\psi, du_k),
\]
which upon combining with the preceding estimate and (3.15) leads to
\[
(\psi u, g) \geq ((F - F_k)\psi u, u_k) + ((-\Delta/2)\psi u, u_k) + (ud\psi, du_k).
\]
(3.17)
Let us replace $0 \leq \psi \in C_c^\infty(M)$ by a sequence $\chi_m$ of cut-off functions from Lemma 3.10. Using (3.12), (3.13), and the properties of $\chi_m$, it is easy to see that the last two terms on the right hand side of (3.17) converge to 0 as $m \to \infty$. 
4. Quadratic Forms in Vector-Bundle Setting.

4.1. Proof of Theorem

Let $E$, $\nabla$, and $V$ be as in hypotheses of Theorem. We begin by describing $L^2(E)$ analogues of quadratic forms from Section 3.2.

4.1.1. Quadratic Forms in Vector-Bundle Setting. Define

$$Q_{\nabla,0}(u) := \frac{1}{2} \int_M |\nabla u|^2 \, d\mu$$

with the domain $D(Q_{\nabla,0}) = \{ u \in L^2(E) : \nabla u \in L^2(T^*M \otimes E) \}$. Note that $Q_{\nabla,0}$ is non-negative, densely defined and closed. Next we define $Q_{V+}(u) = (V^+u, u)$ with the domain $D(Q_{V+}) = \{ u \in L^2(E) : (V^+u, u) \in L^1(M) \}$. The form $Q_{V+}$ is non-negative, densely defined and closed (by Theorem VI.1.11 in [Ka3] and Example VI.1.15 in [Ka3]). Finally, we define $Q_{V-}(u) = -(V^-u, u)$ with the domain $D(Q_{V-}) = \{ u \in L^2(E) : (V^-u, u) \in L^1(M) \}$. The form $Q_{V-}$ is densely defined and symmetric.

**Lemma 4.1.** Let $\sigma_{\text{max}}(V^-)$ be as in hypotheses of Theorem. Then there exist $a \in [0, 1)$ and $b \geq 0$ such that

$$\int_M \langle V^-u, u \rangle \, d\mu \leq (a/2)\|\nabla u\|^2_{L^2(T^*M \otimes E)} + b\|u\|^2_{L^2(E)}, \quad (4.1)$$

for all $u \in D(Q_{\nabla,0})$.

**Proof.** Let $u \in D(Q_{\nabla,0})$ and let $Q_0$ be as in Section 3.2. By Corollary 2.5 in [C2] we have $|u| \in D(Q_0)$, and

$$\|d|u||^2_{L^2(A^1T^*M)} \leq \|\nabla u\|^2_{L^2(T^*M \otimes E)}. \quad (4.2)$$

Using (3.4) and (4.2) we obtain

$$\int_M \langle V^-u, u \rangle \, d\mu \leq \int_M \sigma_{\text{max}}(V^-)|u|^2 \, d\mu \leq (a/2)\|d|u||^2_{L^2(A^1T^*M)} + b\|u\|^2_{L^2(M)} \leq (a/2)\|\nabla u\|^2_{L^2(T^*M \otimes E)} + b\|u\|^2_{L^2(E)}, \quad \text{for all } u \in D(Q_{\nabla,0}),$$

where $a$ and $b$ are as in $4.1$.

As a consequence of Lemma 4.1, analogously as in Section 3.2 the form $Q_{\nabla, V} := Q_0 + Q_{V+} + Q_{V-}$ is densely defined, closed and semi-bounded from below with

$$D(Q_{\nabla, V}) = D(Q_{\nabla,0}) \cap D(Q_{V+}) \subset D(Q_{V-}).$$

Let $H(V)$ denote the semi-bounded from below self-adjoint operator in $L^2(E)$ associated to $Q_{\nabla, V}$. 

We now consider the term $((F - F_k)\chi_m u, u_k)$. For a fixed $m \in \mathbb{Z}_+$, using the property (3.13) we have

$$(F - F_k)\chi_m u_k \to 0, \quad \text{a.e. } x \in M, \quad \text{as } k \to \infty.$$
4.2. Description of $H_V(V)$. By Lemma 3.2 we have $\sigma_{\max}(V^-) \in L^1_{\text{loc}}(M)$, which together with (4.1) and geodesic completeness of $M$, means that the hypothesis of Theorem 1.2 in [Mi] are satisfied. The latter theorem gives the following description of $H_V(V)$:

$$\text{Dom}(H_V(V)) = \{u \in D(Q_V); \langle V^+ u, u \rangle \in L^1(M) \text{ and } ((\nabla^* \nabla/2)u + Vu) \in L^2(E)\}.$$ 

and $H_V(V)u = (\nabla^* \nabla/2)u + Vu$, for all $u \in \text{Dom}(H_V(V))$.

In the proof of Theorem 2.3 we will use Kato’s inequality for Bochner Laplacian, whose proof is given in Theorem 5.7 of [BMS].

**Lemma 4.2.** Let $M$ be a connected Riemannian manifold (not necessarily geodesically complete). Let $E$ be a Hermitian vector bundle over $M$, and let $\nabla$ be a Hermitian connection on $E$. Assume that $w \in L^1_{\text{loc}}(E)$ and $\nabla^* \nabla w \in L^1_{\text{loc}}(E)$. Then

$$-\Delta |w| \leq \text{Re} \langle \nabla^* \nabla w, \text{sign } w \rangle_{E_x},$$ 

where

$$\text{sign } w(x) = \begin{cases} \frac{w(x)}{|w(x)|} & \text{if } w(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 4.3.** The original version of Kato’s inequality was proven in [Ka1].

**Proof of Theorem 2.3** Note that for all $u \in \text{Dom}(H_V(V))$ we have $\langle V^+ u, u \rangle \in L^1(M)$ and $\langle V^- u, u \rangle \in L^1(M)$, where the latter inclusion follows by (4.1). Thus, as observed in (4.3) of [Mi], the mentioned two inclusions and hypotheses on $V$ imply $V^+ u \in L^1_{\text{loc}}(E)$ and $\sigma_{\max}(V^-) u \in L^1_{\text{loc}}(E)$. Now we just compare the descriptions of $H_V(V)$ and $S$ to conclude that the operator relation $H_V(V) \subset S$ holds.

It remains to prove that $\text{Dom}(S) \subset \text{Dom}(H_V(V))$. Let $u \in \text{Dom}(S)$. Let $\lambda_*$ be as in Theorem 2.1. Since $H_V(V)$ is a semi-bounded from below (self-adjoint) operator, we can select $\lambda > \lambda_*$ large enough so that $H_V(V) + \lambda$ is a positive self-adjoint operator. With this selection of $\lambda$, the operator $(H_V(V) + \lambda)^{-1}: L^2(E) \to L^2(E)$ is bounded. Since $u \in \text{Dom}(S)$, we may define $v := (H_V(V) + \lambda)^{-1}(S + \lambda)u$

and write

$$(H_V(V) + \lambda)v = (S + \lambda)u.$$ 

Since $H_V(V) \subset S$, we can rewrite the last equality as

$$(S + \lambda)w = 0,$$

where $w := u - v$.

Since $w \in \text{Dom}(S)$, we have $V^+ w \in L^1_{\text{loc}}(E)$ and $\sigma_{\max}(V^-) w \in L^1_{\text{loc}}(E)$. Furthermore, from (4.4) we get

$$(\nabla^* \nabla/2)w = -Vw - \lambda w \in L^1_{\text{loc}}(E).$$

By Lemma 4.2 we have

$$-\langle \Delta/2 |w| \leq \text{Re} \langle (\nabla^* \nabla/2)w, \text{sign } w \rangle_{E_x} = \text{Re} \langle -(V + \lambda)w, \text{sign } w \rangle_{E_x}$$

$$\leq (\sigma_{\max}(V^-) - \lambda)|w|,$$

which leads to

$$-\Delta/2 - \sigma_{\max}(V^-) + \lambda)|w| \leq 0.$$
Since $\sigma_{\text{max}}(V^-)|w| \in L_{\text{loc}}^1(M)$, we may use Theorem 2.1 with $F = \sigma_{\text{max}}(V^-)$ to conclude $|w| \leq 0$ a.e. on $M$. This shows that $w = 0$ a.e. on $M$, i.e. $u = v$ a.e. on $M$; therefore, $u \in \text{Dom}(H_V(V))$. □

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Department of Mathematics and Statistics, University of North Florida, Jacksonville, FL 32224, USA

E-mail address: omilatov@unf.edu