Electric–Magnetic Duality and the Dualized Standard Model

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Abstract

In these lectures I shall explain how a new-found nonabelian duality can be used to solve some outstanding questions in particle physics. The first lecture introduces the concept of electromagnetic duality and goes on to present its nonabelian generalization in terms of loop space variables. The second lecture discusses certain puzzles that remain with the Standard Model of particle physics, particularly aimed at nonexperts. The third lecture presents a solution to these problems in the form of the Dualized Standard Model, first proposed by Chan and the author, using nonabelian dual symmetry. The fundamental particles exist in three generations, and if this is a manifestation of dual colour symmetry, which by ’t Hooft’s theorem is necessarily broken, then we have a natural explanation of the generation puzzle, together with tested and testable consequences not only in particle physics, but also in astrophysics, nuclear and atomic physics.

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Introduction

In the first lecture I shall show you a new symmetry of gauge theory which results from certain topological and geometrical considerations. In the second lecture I shall discuss some theoretical problems in present-day particle physics which need to be solved and which we think this new dual symmetry may help to solve. Then in the last lecture I shall propose such a solution in the form of the Dualized Standard Model. What comes as a pleasant, but intriguing, surprise is that this model has implications not only in particle physics but also in astrophysics, nuclear and atomic physics as well.

Notations, conventions and dictionary

I shall use the following groups of terms synonymously:

- Maxwell theory, theory of electromagnetism, abelian theory;
- Yang–Mills theory, nonabelian (gauge) theory (however, nonabelian may take a truly mathematical meaning);
- Spacetime, Minkowski space.

I shall use the following notations unless otherwise stated:

- \( X = \) Minkowski space with signature + − − −
- \( \mu, \nu, \ldots = \) spacetime indices = 0,1,2,3
- \( i,j,\ldots = \) spatial indices or group indices
- repeated indices are summed
- \( G = \) gauge group = compact, connected Lie group
  (usually \( U(n), SU(n), O(n) \))

I shall make the following convenient assumptions: functions, manifolds, etc. are as well behaved as necessary; typically functions are continuous or smooth, manifolds are \( C^\infty \).

I shall use the units conventional in particles physics, in which \( h = 1, c = 1 \), the former being the reduced Planck’s constant and the latter the speed of light.
Further, since the audience is familiar with the language of fibre bundles, it may be useful to include the following dictionary:

- base space $\leftrightarrow$ spacetime
- structure group $\leftrightarrow$ gauge group
- principal bundle $\leftrightarrow$ gauge theory
- principal coordinate bundle $\leftrightarrow$ gauge theory in a particular gauge
- connection $\leftrightarrow$ gauge potential
- curvature $\leftrightarrow$ gauge field
- holonomy $\leftrightarrow$ phase factor
- bundle reduction $\leftrightarrow$ symmetry breaking
- section $\sigma: X \rightarrow E \leftrightarrow$ Higgs fields

1 Electric–magnetic duality

1.1 Gauge invariance

Consider an electrically charged particle in an electromagnetic field. The wavefunction of this particle is a complex-valued function $\psi(x)$ of $X$ (space-time). The phase of $\psi(x)$ is not a measurable quantity, since only $|\psi(x)|^2$ can be measured and has the meaning of the probability of finding the particle at $x$. Hence one is allowed to redefine the phase of $\psi(x)$ by an arbitrary (continuous) rotation independently at every spacetime point without altering the physics. This is the origin of gauge invariance or gauge symmetry. Yang and Mills generalized this phase freedom to an arbitrary element of a Lie group (originally they considered $SO(3)$).

In view of this arbitrariness, how can we compare the phases at neighbouring points in spacetime? In other words, how can we ‘parallelly propagate’ the phase? Well, we know we can if given a potential $A_\mu(x)$. This is the connection 1-form in the principal $G$-bundle, which is the exact geometric picture of a gauge theory. The potential is not a directly observable quantity because it transforms under a gauge transformation $S(x) \in G$ as:

\[ A_\mu(x) \mapsto S(x) A_\mu(x) S^{-1}(x) - \left( \frac{i}{g} \right) \partial_\mu S(x) S^{-1}(x). \]  \hspace{1cm} (1.1)

From the connection we can define the curvature 2-form, which is the gauge field, given in local coordinates by:

\[ F_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x) + ig[A_\mu(x), A_\nu(x)], \]  \hspace{1cm} (1.2)
where I have included the coupling constant $g$ whose numerical value deter-
mines the strength of the interaction. In abelian theory, the 6 components of
this skew rank-2 tensor give exactly the 3 components of each of the electric
and magnetic fields, so that in classical electromagnetism there is no need to
introduce the potential. However, the Bohm–Aharonov experiment dem-
onstrates that the potential is necessary to describe the motion of a quantum
particle (e.g. an electron) in an electromagnetic field. This really vindicates
the geometric description of gauge theory we have now.

In contrast to the abelian case the nonabelian field $F$ is not observable
as it is gauge covariant (that is, it is a tensorial 2-form) and not gauge in-
variant. In fact, Yang has proved, and that is what is also demonstrated
in the abelian case by the Bohm–Aharonov experiment, that it is the set of
variables comprising the holonomy of loops which describes a gauge theory
exactly. In other words, there is a 1–1 correspondence between such sets and
the sets of physical configurations of a gauge theory. Yang calls the holon-
omy the Dirac phase factor, which is usually written in a slightly misleading
fashion as:

$$\Phi(C) = P \exp ig \int_C A_\mu(x),$$

where the letter $P$ denotes path-ordering. We shall come back to these loop
variables later.

1.2 Sources and monopoles

Gauge invariance in the abelian case is Maxwell’s theory of light. To
include matter, we have to look at the charges and monopoles.

In building a physical theory, we must look among experimental facts to
collect our ingredients. The potential fixes only the Lie algebra. To select
out from among the locally isomorphic ones the correct Lie group we must
look at the particle spectrum, that is, what kind of and how many particles
exist or are postulated to exist.

Consider electrodynamics. Since we know that all charges are multiples
(in fact, just $\pm 1$ if we do not consider composite objects) of a fundamental
charge $e$, so that wavefunctions transform as

$$\psi \mapsto e^{\pm ieA} \psi,$$  \hspace{1cm} (1.4)

\footnote{This distinction is not quite accurate. In a fully quantized field theory, particles and
fields are synonymous. But it is a useful and geometrically meaningful distinction.}
we can parametrize the circle group $U(1)$ corresponding to the phase by $[0, 2\pi/e]$. In fact, \textit{charge quantization} is equivalent to having $U(1)$ as the gauge group of electromagnetism.

On the other hand, if we consider pure electromagnetism without charges, then the only relevant gauge transformations are those of $A_\mu$:

$$A_\mu \mapsto A_\mu + \partial_\mu \Lambda,$$

so that the group will just be the real line given by the scalar function $\Lambda(x)$.

Similarly for Yang–Mills theory, and for definiteness let us study an $\text{su}(2)$ theory. If it contains particles with a 2-component wave function $\psi = \{\psi_i, \ i = 1, 2\}$, then

$$\psi \mapsto S\psi, \quad S \in SU(2),$$

so that the effect of $S$ and $-S$ are not identical. In this case the gauge group is $SU(2)$. If, on the other hand, there are no charges so that the only gauge transformation one needs to consider is on the gauge potential $A_\mu(x)$:

$$A_\mu \mapsto S A_\mu S^{-1} - \frac{i}{g} \partial_\mu S S^{-1}.$$

Then the effects on $A_\mu$ of $S$ and $-S$ are identical, and these two elements should be identified, resulting in $SO(3)$ being the gauge group.

These considerations can also be cast in terms of representations. Charged particles in a Yang–Mills theory are in certain representations of the gauge group. What we are saying is the known result that the collection of all representations determines the group. In the above case, the gauge potential is in the 3-dimensional adjoint representation and the 2-component $\psi$ is in the 2-dimensional spinor representation. In the absence of the spinor representation, the group is $SO(3)$, but when spinors are present, the group must be $SU(2)$.

By the same arguments, the group generally denoted $SU(3) \times SU(2) \times U(1)$ describing the Standard Model of particle physics should really be quotiented out by a $\mathbb{Z}_6$ subgroup of its centre. However, if in future we either \textit{discover} or \textit{postulate} more particles, then the correct group will be different.

For the moment we wish to distinguish between two types of charged particles: sources and monopoles.

In a pure gauge theory, that is, one without matter, we have Yang–Mills equation:

$$D_\mu F^{\mu\nu} = 0,$$
where $D$ denotes the covariant derivative. *Electric sources* (or just *sources*) are those particles that give rise to a nonvanishing right hand side of the above equation:

$$D_\nu F^{\mu\nu} = -j^\mu, \quad j^\mu = g\bar{\psi}\gamma^\mu\psi,$$

(1.9)

where $j$ is called the current, and $\gamma^\mu$ is a Dirac gamma matrix, identifiable as a basis element of the Clifford algebra over spacetime.

*Magnetic monopoles* (or just *monopoles*), on the other hand, are topological in nature and are represented geometrically by nontrivial $G$-bundles. They are classified by elements of $\pi_1(G)$.

From the definition of magnetic charges by closed curves we can easily deduce the *Dirac quantization condition*. For instance, in the abelian case, the size of the circle is inversely proportional to $e$; hence we have, in appropriate units, for $e$ electric and $\tilde{e}$ magnetic:

$$e\tilde{e} = 2\pi.$$

(1.10)

Similarly for nonabelian charges:

$$g\tilde{g} = 4\pi,$$

(1.11)

the difference between the two cases being only a matter of conventional normalization.

However, in view of the electric–magnetic duality we shall study, the concepts of ‘electric’ and ‘magnetic’ are interchangeable depending on which description one uses. I hope to make this quite clear in the sequel.

Notice that I am using the terms ‘electric’ and ‘magnetic’ in a generalized sense (that is, not just for Maxwell theory), and this will be the case throughout this course.

### 1.3 Abelian dynamics and duality: the Wu–Yang criterion

Since we are in flat spacetime, the Hodge star, which is here more conveniently thought of as the duality operator, is defined by

$$^*F^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma},$$

(1.12)

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally skew symbol with the convention that $\epsilon^{0123} = -1$, and the sign is a consequence of the Minkowski signature.
It is well known that classical Maxwell theory is invariant under this duality operator. By this we mean that at any point in spacetime free of electric and magnetic charges we have the two dual symmetric Maxwell equations:

\[ \partial_\nu F^{\mu\nu} = 0 \quad [d \ast F = 0] \quad (1.13) \]
\[ \partial_\nu \ast F^{\mu\nu} = 0 \quad [d F = 0], \quad (1.14) \]

where I have put in square brackets the equivalent equations in the language of differential forms. Then by the Poincaré lemma we deduce immediately the existence of potentials \( A \) and \( \tilde{A} \) such that\(^2\)

\[ F_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x), \quad (1.15) \]
\[ \ast F_{\mu\nu}(x) = \partial_\nu \tilde{A}_\mu(x) - \partial_\mu \tilde{A}_\nu(x). \quad (1.16) \]

The two potentials transform independently under independent gauge transformations \( \Lambda \) and \( \tilde{\Lambda} \):

\[ A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x), \quad (1.17) \]
\[ \tilde{A}_\mu(x) \rightarrow \tilde{A}_\mu(x) + \partial_\mu \tilde{\Lambda}(x), \quad (1.18) \]

which means that the full symmetry of this theory is doubled to \( U(1) \times \tilde{U}(1) \), where the tilde on the second circle group indicates it is the symmetry of the dual potential \( \tilde{A} \). It is important to note that the physical degrees of freedom remain the same. This is clear because \( F \) and \( \ast F \) are related by an algebraic equation \((1.12)\). As a consequence the physical theory is the same: the doubled gauge symmetry is there all the time but just not so readily detected.

This dual symmetry means that what we call ‘electric’ or ‘magnetic’ is entirely a matter of choice.

In the presence of electric charges, the Maxwell equations appear usually as

\[ \partial_\nu F^{\mu\nu} = -j^\mu \quad (1.19) \]
\[ \partial_\nu \ast F^{\mu\nu} = 0. \quad (1.20) \]

The apparent asymmetry in these equations comes from the experimental fact that there is only one type of charges in nature which we choose to call

\(^2\)More precisely, \((1.16)\) follows from \((1.13)\) and \((1.15)\) from \((1.14)\).
electric’. But as we see, we could equally have thought of these as ‘magnetic’
charges and write instead
\[
\begin{align*}
\partial_\nu F^{\mu \nu} &= 0 \quad (1.21) \\
\partial_\nu *F^{\mu \nu} &= -j^\mu. \quad (1.22)
\end{align*}
\]
And if both types of charges existed in nature, then we would have the dual
symmetric pair:
\[
\begin{align*}
\partial_\nu F^{\mu \nu} &= -j^\mu \quad (1.23) \\
\partial_\nu *F^{\mu \nu} &= -\tilde{j}^\mu. \quad (1.24)
\end{align*}
\]
This duality goes in fact much deeper, as can be seen if we use the Wu–
Yang criterion to derive the Maxwell equations\(^3\).
Consider first pure electromagnetism. The free Maxwell action is:
\[
\mathcal{A}_F^0 = -\frac{1}{4} \int F_{\mu \nu} F^{\mu \nu}. \quad (1.25)
\]
The true variables of the theory as we said before are the \(A_\mu\), so in (1.25) we
should put in a constraint to say that \(F_{\mu \nu}\) is the curl of \(A_\mu\) (1.15). This can
be viewed as a topological constraint, because it is precisely equivalent to
(1.14). Using the method of Lagrange multipliers, we form the constricted
action
\[
\mathcal{A} = \mathcal{A}_F^0 + \int \lambda_\mu (\partial_\nu *F^{\mu \nu}), \quad (1.26)
\]
which we can now vary with respect to \(F_{\mu \nu}\), obtaining
\[
F^{\mu \nu} = 2\epsilon^{\mu \nu \rho \sigma} \partial_\rho \lambda_\sigma \quad (1.27)
\]
which implies (1.13). Moreover, the Lagrange multiplier \(\lambda\) is exactly the dual
potential \(\tilde{A}\).
This derivation is entirely dual symmetric, since we can equally well use
(1.13) as constraint for the action \(\mathcal{A}_F^0\), now considered as a functional of \(*F^{\mu \nu}\):
\[
\mathcal{A}_F^0 = \frac{1}{4} \int *F_{\mu \nu} *F^{\mu \nu}, \quad (1.28)
\]
and obtain (1.14) as the equation of motion.
\(^3\)What we present here is not the textbook derivation of Maxwell equations from an
action, but we consider this method to be much more intrinsic and geometric.
This method applies to the interaction of charges and fields as well. In this case we start with the free field plus free particle action:

$$\mathcal{A}^0 = \mathcal{A}_F^0 + \int \bar{\psi}(i\partial_\mu \gamma^\mu - m)\psi,$$  \hspace{1cm} (1.29)

where we assume the free particle $m$ to satisfy the Dirac equation. To fix ideas, let us consider this particle to be a magnetic monopole. Then the constraint we put in is (1.22):

$$\mathcal{A}' = \mathcal{A}^0 + \int \lambda_\mu (\partial_\nu *F_{\mu\nu} + \tilde{j}^\nu).$$ \hspace{1cm} (1.30)

Varying with respect to $F$ gives us (1.21), and varying with respect to $\bar{\psi}$ gives

$$(i\partial_\mu \gamma^\mu - m)\psi = -\tilde{e} \tilde{A}_\mu \gamma^\mu \psi.$$ \hspace{1cm} (1.31)

So the complete set of equations for a Dirac particle in an electromagnetic field is (1.21), (1.22) and (1.31) if considered as a magnetic monopole. The duals of these equations will describe the dynamics of an electric charge, say a positron, in an electromagnetic field.

We see from this that the Wu–Yang criterion actually gives us an intuitively clear picture of interactions. The assertion that there is a monopole at a certain spacetime point $x$ means that the gauge field on a 2-sphere surrounding $x$ has to have a certain topological configuration (e.g. giving a nontrivial bundle of a particular class), and if the monopole moves to another point, then the gauge field will have to rearrange itself so as to maintain the same topological configuration around the new point. There is thus naturally a coupling between the gauge field and the position of the monopole, or in physical language a topologically induced interaction between the field and the monopole.

As a side remark, I wish to point out that although the action $\mathcal{A}_F^0$ is not immediately identifiable as geometric in nature, the Wu–Yang criterion, by putting the topological constraint and the equation of motion on equal (or dual) footing, suggests that in fact it is geometric in a subtle not yet fully understood manner. Moreover, as pointed out, equation (1.27) says that the dual potential is given by the Lagrange multiplier of the constrained action.

### 1.4 Loop space variables

We would of course like to generalize this duality to the nonabelian Yang–Mills case. Although there is no difficulty in defining $*F^{\mu
u}$, which is again
given by (1.12), we immediately come to difficulties in the relation between field and potential, e.g. (1.2):

\[ F_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x) + ig[A_\mu(x), A_\nu(x)]. \]

First of all, despite appearances the Yang–Mills equation (1.8)

\[ D_\nu F^{\mu\nu} = 0 \]

and the Bianchi identity

\[ D_\nu *F^{\mu\nu} = 0 \quad (1.32) \]

are not dual-symmetric, because the correct dual of the Yang–Mills equation ought to be

\[ \tilde{D}_\nu *F^{\mu\nu} = 0, \quad (1.33) \]

where \( \tilde{D}_\nu \) is the covariant derivative corresponding to a dual potential. Secondly, the Yang–Mills equation, unlike its abelian counterpart (1.13), says nothing about whether the 2-form \( *F \) is closed or not. Nor is the relation (1.2) about exactness at all. In other words, Yang–Mills equation does not guarantee the existence of a dual potential, in contrast to the Maxwell case. In fact, Gu and Yang have constructed a counter-example. Because the true variables of a gauge theory are the potentials and not the fields, this means that Yang–Mills theory is not symmetric under the Hodge star operation (1.12).

Nevertheless, electric–magnetic duality is a very useful physical concept. So the natural step is to seek a more general duality transform (\( ^\sim \)) satisfying the following properties:

1. \( (\ ) \sim = \pm (\ ) \),
2. electric field \( F_{\mu\nu} \sim \rightarrow \) magnetic field \( \tilde{F}_{\mu\nu} \),
3. both \( A_\mu \) and \( \tilde{A}_\mu \) exist as potentials (away from charges),
4. magnetic charges are monopoles of \( A_\mu \), and electric charges are monopoles of \( \tilde{A}_\mu \),
5. \( ^\sim \) reduces to * in the abelian case.
One way to do so is to study the Wu–Yang criterion more closely. This reveals the concept of charges as topological constraints to be crucial even in the pure field case, as can be seen in the map below:

\[
\begin{array}{c|c}
A_\mu & \text{Defining constraint} \\
\text{exists as} & \partial_\mu F^{\mu\nu} = 0 \\
\text{potential for} & (dF = 0) \\
F^{\mu\nu} & \text{Poincaré} \\
(F = dA) & \text{Gauss}
\end{array}
\]

The point to stress is that, in the above abelian case, the condition for the absence of a topological charge (a monopole) exactly removes the redundancy of the variables \( F^{\mu\nu} \).

Now the nonabelian monopole charge was defined topologically as an element of \( \pi_1(G) \), and this definition also holds in the abelian case of \( U(1) \), with \( \pi_1(U(1)) = \mathbb{Z} \). So our first task is to write down a condition for the absence of a nonabelian monopole.

To fix ideas, let us consider the group \( SO(3) \), whose monopole charges are elements of \( \mathbb{Z}_2 \), which can be denoted by a sign: \( \pm \). The vacuum, charge +, that is, no monopole, is represented by a closed curve in the group manifold of even winding number, and the monopole charge − by a closed curve of odd winding number. It is more convenient\(^4\), however, to work in \( SU(2) \), which is the double cover of \( SO(3) \) and which has the topology of \( S^3 \). There the charge + is represented by a closed curve, and the charge − by a curve which winds an odd number of “half-times” round the sphere \( S^3 \). Since these charges are defined by closed curves, it is reasonable to try to write the constraint in terms of loop variable. This is what we shall now study. I must immediately say that our analysis is not as rigorous as we would like but at the moment that is the best we can do. Other treatments exist but they are not so adapted to the problem in hand.

\(^4\)This is because sometimes it is useful to identify the fundamental group of \( SO(3) \) with the centre of \( SU(2) \) and hence consider the monopole charge as an element of this centre.
Recall that we define the Dirac phase factor $\Phi(C)$ of a loop $C$ in (1.3), which can be rewritten as

$$\Phi[\xi] = P_s \exp ig \int_0^{2\pi} ds A_\mu(\xi(s))\dot{\xi}_\mu(s),$$

where we parametrize the loop $C$:

$$C : \{\xi^\mu(s) : s = 0 \rightarrow 2\pi, \xi(0) = \xi(2\pi) = \xi_0\},$$

and a dot denotes differentiation with respect to the parameter $s$. We thus regard loop variables in general as functionals of continuous piece-wise smooth functions $\xi$ of $s$. In this way, loop derivatives and loop integrals are just functional derivatives and functional integrals. This means that loop derivatives $\delta_\mu(s)$ are defined by a regularization procedure approximating delta functions with finite bump functions and then taking limits in a definite order. For integrals, we shall ignore, for want of something better, the question of infinite measure as usually done in physics.

Following Polyakov, we define the logarithmic loop derivative of $\Phi[\xi]$:

$$F_\mu[\xi|s] = \frac{i}{g} \Phi^{-1}[\xi] \delta_\mu(s) \Phi[\xi],$$

which acts as a kind of ‘connection’ in loop space since it tells us how the phase of $\Phi[\xi]$ changes from one loop to a neighbouring loop. We can represent it pictorially as in Figure 1. We can go a step further and define its ‘curvature’ in direct analogy with $F_{\mu\nu}(x)$:

$$G_{\mu\nu}[\xi|s] = \delta_\nu(s)F_\mu[\xi|s] - \delta_\mu(s)F_\nu[\xi|s] + ig[F_\mu[\xi|s], F_\nu[\xi|s]].$$

It can be shown that using the $F_\mu[\xi|s]$ we can rewrite the Yang–Mills action as

$$A_F^0 = -\frac{1}{4\pi \bar{N}} \int \delta \xi \int_0^{2\pi} ds \text{Tr}\{F_\mu[\xi|s]F^\mu[\xi|s]\} |\dot{\xi}(s)|^{-2},$$

where the normalization factor $\bar{N}$ is an infinite constant. However, the true variables of our theory are still the $A_\mu$. They represent 4 functions of a real variable, whereas the loop connections represent 4 functionals of the real function $\xi(s)$. Just as in the case of the $F_{\mu\nu}$, these $F_\mu[\xi|s]$ have to be constrained, but this time much more severely. Put in another way, we have
to find the constraint on $F_\mu[\xi|s]$ in order to recover $A_\mu$ to ensure that we are doing the same field theory as before.

It turns out that in pure Yang–Mills theory, the constraint that says there are no monopoles:

$$G_{\mu\nu}[\xi|s] = 0 \quad (1.39)$$

removes also the redundancy of the loop variables, exactly as in the abelian case. That this condition is necessary is easy to see. In the absence of a topological charge, that is, when the principal bundle is trivial, the potential $A_\mu$ is well-defined single-valued everywhere. Then the condition (1.39) follows directly from (1.34) and (1.36). The proof of the converse of this “extended Poincaré lemma” is fairly lengthy and will not be presented here. Granted this, we can now apply the Wu–Yang criterion to the action (1.38) and derive the Polyakov equation:

$$\delta_\mu(s) F^\mu[\xi|s] = 0, \quad (1.40)$$

which is the loop version of the Yang–Mills equation.

In the presence of a monopole charge $-$, the constraint (1.39) will have a nonzero right hand side:

$$G_{\mu\nu}[\xi|s] = -J_{\mu\nu}[\xi|s]. \quad (1.41)$$

The loop current $J_{\mu\nu}[\xi|s]$ has been written down explicitly, but it is a little complicated. However, its global form is much easier to understand. Recall
that $F^\mu[\xi[s]]$ can be thought of as a loop connection, for which we can form its ‘holonomy’. This is defined for a closed (spatial) surface $\Sigma$ (enclosing the monopole), parametrized by a family of closed curves $\xi_t(s)$, $t = 0 \rightarrow 2\pi$. The ‘holonomy’ $\Theta_\Sigma$ is then the total change in phase of $\Phi[\xi_t]$ as $t \rightarrow 2\pi$, and thus equals the charge $-I$.

It is instructive to examine how this result arises in detail in terms of the (patched) gauge potential (Figure 2). Without loss of generality, we shall choose the reference point $P_0 = \xi_0^\mu$ to be in the overlap region, say on the equator which corresponds to the loop $\xi_{t_e}$. Starting at $t = 0$, where $\Phi^{(N)}[\xi_0]$ is the identity, the phase factor $\Phi^{(N)}[\xi_t]$ traces out a continuous curve in $SU(2)$ until it reaches $t = t_e$. At $t = t_e$ one makes a patching transformation and goes over to $\Phi^{(S)}[\xi_t]$. From $t = t_e$ onwards, the phase factor $\Phi^{(S)}[\xi_t]$ again traces out a continuous curve until $t$ reaches $2\pi$, where it becomes again the identity and joins up with $\Phi^{(N)}[\xi_0]$. In order that the curve $\Gamma_\Sigma$ so traced out winds only half-way round $SU(2)$ while being a closed curved in $SO(3)$, as it should if $\Sigma$ contains a monopole, we must have

$$\Phi^{(N)}[\xi_{t_e}] = -\Phi^{(S)}[\xi_{t_e}],$$

(1.42)

which means for the holonomy

$$\Theta_\Sigma = (\Phi^{(S)}[\xi_{t_e}])^{-1} \Phi^{(N)}[\xi_{t_e}] = -I.$$ 

(1.43)
1.5 Nonabelian duality

To formulate an electric–magnetic duality which is applicable to non-abelian theory we find that we need to define yet another set of loop variables. Instead of the Dirac phase factor $\Phi[\xi]$ for a complete curve (1.34) we can define the parallel phase transport for part of a curve from $s_1$ to $s_2$:

$$
\Phi_\xi(s_2, s_1) = P_s \exp ig \int_{s_1}^{s_2} ds A_\mu(\xi(s)) \dot{\xi}^\mu(s).
$$

(1.44)

Then the new variables are defined as:

$$
E_\mu[\xi|s] = \Phi_\xi(s, 0) F_\mu[\xi|s] \Phi_\xi^{-1}(s, 0).
$$

(1.45)

These are not gauge invariant like $F_\mu[\xi|s]$ and may not be as useful in general but seem more convenient for dealing with duality. They can be pictorially represented as the bold curve in Figure 3 where the phase factors $\Phi_\xi(s, 0)$ have cancelled parts of the faint curve representing $F_\mu[\xi|s]$. In contrast to $F_\mu[\xi|s]$, therefore, $E_\mu[\xi|s]$ depends really only on a “segment” of the loop $\xi$ from $s_-$ to $s_+$. The rule for differentiation is to take this finite segment, introduce the delta function variation, and only then to take the limit as $s_- \to s_+$ in the prescribed manner.

In terms of these variables, equations (1.39) and (1.40) become:

$$
\delta_\nu(s) E_\mu[\xi|s] - \delta_\mu(s) E_\nu[\xi|s] = 0,
$$

(1.46)
and
\[ \delta^\mu(s)E_\mu[\xi|s] = 0. \] (1.47)

Using these new variables \( E_\mu[\xi|s] \) for the field, we now define their 'dual' \( \tilde{E}_\mu[\eta|t] \) as:
\[
\omega^{-1}(\eta(t)) \tilde{E}_\mu[\eta|t] \omega(\eta(t)) = -\frac{2}{N} \epsilon_{\mu\rho\sigma\tau} \eta^\nu(t) \int \delta \xi ds E^\rho[\xi|s] \dot{\xi}^\sigma(s) \dot{\xi}^{-2}(s) \delta(\xi(s) - \eta(t)), \] (1.48)
where \( \omega(x) \) is a (local) rotation matrix tranforming from the frame in which the orientation in internal symmetry space of the fields \( E_\mu[\xi|s] \) are measured to the frame in which the dual fields \( \tilde{E}_\nu[\eta|t] \) are measured. It can be shown that this dual transform satisfies all the 5 required conditions we listed before. Further, by differentiating (1.48) we can show that (1.47) is equivalent to
\[ \delta_\nu(t) \tilde{E}_\mu[\eta|t] - \delta_\mu(t) \tilde{E}_\nu[\eta|t] = 0, \] (1.49)
or that there are no monopoles in the dual field. Hence, by the ‘extended Poincaré Lemma’ mentioned before and via the dual of (1.39), we deduce that a dual potential \( \tilde{A}_\mu(x) \) will exist in this case.

We now have a situation which is the exact parallel of pure electromagnetism. The equations (1.46) and (1.49) are the equivalents of (1.13) and (1.14), and they each guarantee as before the existence of a potential, \( A_\mu(x) \) and \( \tilde{A}_\mu(x) \) respectively, again leading to a doubled gauge symmetry \( G \times \tilde{G} \). Similarly, the Wu–Yang criterion treats them dually: the use of one as constraint leads to the other as equation of motion. Here again as in the abelian case, the physical degrees of freedom remain the same.

The treatment of charges also follows the abelian case. To the free action
\[ \mathcal{A}^0 = -\frac{1}{4\pi N} \int \text{Tr}(E_\mu E^\mu) \dot{\xi}^{-2} + \int \bar{\psi}(i\partial_\mu \gamma^\mu - m)\psi \] (1.50)
we may impose the constraint
\[ \delta_\nu(s)E_\mu - \delta_\mu(s)E_\nu = -J_{\mu\nu}, \] (1.51)
obtaining (1.47) and the (dual) Dirac equation
\[ (i\partial_\mu \gamma^\mu - m)\psi = -\tilde{g} \tilde{A}_\mu \gamma^\mu \psi, \] (1.52)
where the dual potential $\tilde{A}_\mu$ is again given in terms of the Lagrange multipliers pertaining to the constraint \( [1.51] \).

The whole procedure can be dualized, just as in the abelian case. We thus recover full electric–magnetic duality for Yang–Mills theory.

In presenting the above duality we have skipped over many technical points and also neglected to clarify many ambiguities, because the derivations are long and not very transparent. It would of course be much nicer if there is a spacetime formulation of this nonabelian duality, but that is perhaps unlikely, particularly in view of some recent work by Bakaert and Cucu.

2 Some questions in present-day theoretical particle physics

I shall not of course even try to give you a comprehensive view of present-day particle theory. Instead I shall describe the background to some questions which we hope to answer using this new-found dual symmetry\(^5\). However, I must not give you the wrong impression that particle theory is full of problems. On the contrary, we now have a very good working model of particle physics. And it is precisely because the theory is so good that we can now ask detailed and profound questions.

2.1 The Standard Model of particle physics: a first look at the spectrum

The reason why gauge theories are so important is that not only electromagnetism is a gauge theory, but that we now believe that all particle interactions (that is, all except gravity) are gauge interactions. Moreover, all these interactions can now be amalgamated into one single gauge theory: the Standard Model (SM). This model knits together the two kinds of fundamental forces: strong and electroweak\(^6\).

The first thing to study is the particles themselves. The gauge structure provides a framework for classifying them. Within this framework the fundamental particles are:

---

\(^5\)In doing so I may have introduced some personal biases, but I shall endeavour not to do so.

\(^6\)The third is gravity, which we shall not discuss here.
Vector bosons (also known as gauge bosons): $\gamma$, $W^+, W^-, Z^0$, $g$

(photons; massive vector bosons; gluons)

Quarks: $t, b; c, s; u, d$

(top, bottom; charm, strange; up, down)

Leptons: $\tau, \nu_\tau; \mu, \nu_\mu; e, \nu_e$

(tauon, tau neutrino; muon, muon neutrino; electron, electron neutrino)

In a full quantum theory, these particles all have corresponding antiparticles.

The quarks and leptons are the charges we studied previously. Experimentally, all known fundamental fermions have spin $\frac{1}{2}$, and all known fundamental bosons have spin 1, although theory postulates the existence of certain scalars called Higgs particles (of which more later). The quarks are not experimentally observed, but are supposed to combine together to form most of the other particles observed in nature and in experiments.

The quarks interact strongly and electromagnetically, while the leptons have only electroweak interaction. The quarks are in the 3-dimensional fundamental representation of ‘colour’ $SU(3)$, while the electroweak group is usually written as $SU(2) \times U(1)$. An examination of all the charge assignments (as currently given) tells us that the correct gauge group of the SM is

$$SU(3) \times SU(2) \times U(1)/Z_6,$$

where $Z_6$ is a certain subgroup of the centre of the product group.

The first striking feature of the above classification is that the charges come in 3 copies, known as generations. For instance, there are the three electrically charged leptons $\tau, \mu, e$. Except for their very different masses:

$$m_\tau; m_\mu; m_e \approx 3000: 200: 1$$

they have the same SM quantum numbers and behave in extremely similar fashions. The 3 neutrinos $\nu_\tau, \nu_\mu, \nu_e$ also have similar interactions. The quarks also come in 3 generations: the $U$-quarks $(t, c, u)$ with electric charge $\frac{2}{3}$ and the $D$-quarks $(b, s, d)$ with electric charge $-\frac{1}{3}$.

The usual SM offers no explanation for the existence of 3 generations. Moreover, experiments tell us that most probably there are no more than 3 generations. This is one of the major puzzles of particle physics.

A fermion state, being an independent physical state, is orthogonal to any other fermion states. Hence the 3 $U$-quark states form an orthonormal triad in generation space (which is a 3-dimensional subspace of the total state space.
comprising all other quantum numbers). It is found experimentally that the orthonormal triad of the $D$-quark states are not exactly aligned with that of the $U$-quarks, and the transformation between the two triads is a unitary matrix known as the CKM matrix. The absolute values of the elements of this matrix are well measured, and the off-diagonal elements vary in magnitude from about 0.002 to 0.2. This phenomenon is known as ‘quark mixing’. The corresponding leptonic mixing matrix, known as the MNS matrix, is less well measured but found to have larger off-diagonal elements. They give rise to neutrino oscillations which we shall study further later.

Thus a third question that the SM does not answer is fermion mixing.

### 2.2 Symmetry breaking

The strong interaction and the electroweak interaction are both gauge theories, but they differ in one fundamental respect. While the strong interaction group $SU(3)$, called ‘colour symmetry’, is exact, the electroweak group is ‘broken’. The physical idea of symmetry breaking is that although the action is invariant under the action of the gauge group the vacuum, that is, the solution to the equation of motion corresponding to the lowest energy, is invariant under only a proper subgroup.

The gauge group of electroweak theory, when particle spectrum is taken into account properly, is the group $U(2)$, which is doubly-covered by $SU(2) \times U(1)$. Both have Lie algebra $su(2) \oplus u(1)$. Let $T_0$ be the generator of this $u(1)$, and $T_1, T_2, T_3$ be those of $su(2)$, where the notation for the $T_i$ is exactly the same as for ordinary spin, with $T_3$ is represented by the diagonal matrix

$$T_3 = -\frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

(2.2)

The ‘symmetry breaking’ is effected by introducing an extra term in the Yang–Mills action:

$$A_H = \int D_\mu \phi D^\mu \phi + V(\phi),$$

(2.3)

with a potential $V(\phi)$ given by

$$V(\phi) = -\frac{\mu^2}{2} |\phi|^2 - \frac{\lambda}{4} |\phi|^4 \quad (\lambda > 0)$$

(2.4)

where $\phi$ is an $SU(2)$ doublet of complex scalar fields, called Higgs fields:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}.$$

(2.5)
Here the covariant derivative $D_\mu$ will contain four gauge components: $W^1_\mu$, $W^2_\mu$, $W^3_\mu$ corresponding to the $\mathfrak{su}(2)$ part with coupling $g_2$, and $Y_\mu$ to the $\mathfrak{u}(1)$ part with coupling $g_1$.

If $\mu^2 > 0$, then the scalar field $\phi$ has mass $\mu$ and the vacuum (or ground state) corresponds to $\phi_0 = 0$. If $\mu^2 < 0$, we get the famous Mexican hat potential, and the vacuum (with $V(\phi)$ minimum) is given by

$$|\phi_0| = -\mu^2/\lambda = \eta \neq 0.$$  \hfill (2.6)

We now choose a gauge such that

$$\phi_0 = \eta \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \hfill (2.7)$$

In this way, the vacuum corresponds to a particular direction in the space of $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ and once this choice is made, the physics will no longer be invariant under the whole of the $U(2)$ group. In fact, since $\phi$ is a complex vector in $\mathbb{C}^2$, there will be a phase rotation left over after fixing a direction as above, and it is this ‘little group’ $U(1)$ that is identified as the abelian electromagnetic group we studied before. Geometrically the group $U(2)$ is a torus $S^3 \times S^1$, and the residual symmetry group is a ‘diagonal’ $U(1)$ of this torus, generated by the linear combination of $T_0$ and $T_3$ shown below.

For a quantum field theory, we look at quantum excitations around the vacuum $\phi_0$, giving rise to a new scalar field $\sigma$:

$$\phi(x) = \begin{pmatrix} 0 \\ \eta + \frac{\sigma(x)}{\sqrt{2}} \end{pmatrix}. \hfill (2.8)$$

If we now define the Weinberg angle

$$\sin \theta_W = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}, \hfill (2.9)$$

and fields

$$A_\mu = -\sin \theta_W W^3_\mu + \cos \theta_W Y_\mu, \hfill (2.10)$$

$$Z_\mu = \cos \theta_W W^3_\mu + \sin \theta_W Y_\mu, \hfill (2.11)$$

we can re-write the action $A^0_F+A_H$ in terms of the new fields $\sigma, W^1_\mu, W^2_\mu, Z_\mu, A_\mu$. By comparing each term with the Klein–Gordon lagrangian for a boson with mass $m$

$$- \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2$$ \hfill (2.12)
we can identify the massive fields $\sigma, W^1_\mu, W^2_\mu, Z_\mu$, while the field $A_\mu$ which remains massless we can identify as the electromagnetic field.

To describe the charges in electroweak theory we need to introduce further terms in the action

$$A_L = \int \bar{\psi} D_\mu \gamma^\mu \psi + \int \rho \bar{\psi}_L \phi \psi_R + h.c.$$  \hspace{1cm} (2.13)

As a first step we shall include only one lepton generation, that is, $e$ and $\nu_e$. In this case,

$$\psi_L = \left( \begin{array}{c} \nu_e \\ e \end{array} \right)_L, \quad \psi_R = e_R, \quad \psi = \psi_L + \psi_R,$$  \hspace{1cm} (2.14)

with

$$e_L = \frac{1}{2}(1 + \gamma_5)e, \quad e_R = \frac{1}{2}(1 - \gamma_5)e,$$  \hspace{1cm} (2.15)

and $\nu_e$ purely left-handed\footnote{In view of the recent positive experimental results on neutrino oscillations, this will need to be modified. See the next lecture.}. In (2.13) the second term is called the Yukawa term (with $\rho$ a constant), and “h.c.” means Hermitian conjugate. Again by comparing (2.13) with the Dirac lagrangian

$$\bar{\psi}(i\partial_\mu \gamma^\mu - m)\psi$$  \hspace{1cm} (2.16)

we can see that the electron acquires a mass through the Higgs field $\phi$ in the Yukawa term. The purely left-handed neutrino remains massless in this formulation.

Thus the main result of symmetry breaking is that some gauge fields and charges become massive. For many theorists, the one unsatisfactory aspect of this is that we do not have a good theoretical reason to introduce the Higgs fields in the first place.

### 2.3 The fermion mass matrices in SM

In a gauge theory under certain generally accepted assumptions the only way for particles to be massive is by the Higgs mechanism of symmetry breaking described above. By confronting the three component groups $SU(3)$, $SU(2)$, and $U(1)$ of the Standard Model with what is observed (or desired to be observed), we have the following situation as regards the mass. Of the gauge bosons, the 8 gluons of colour $SU(3)$ are massless, so is the particular
generator of $SU(2) \times U(1)$ corresponding to (2.10) identified as the photon. The 3 remaining gauge bosons are massive. Of the fermions (which are the charges), both the quarks and the charged leptons acquire mass through Yukawa terms involving the Higgs field. There is no theoretical reason to demand that the neutrinos are massless, and indeed they most probably have a small mass.

There are 3 generations to each of the 4 types of fermions: $U, D, L, N$ (for up-type quarks, down-type quarks, charged leptons and neutrinos respectively). Since the left-handed components of the $U$ and $D$ quarks, as well as those of the leptons $L$ and $N$, transform as doublets under $SU(2)$, it is not possible in general to find a common basis in generation space in which the 4 mass matrices are diagonal. Indeed, because of the observed mixing (CKM and MNS), we know that the mass matrices are not diagonal. Since mass is a measurable quantity, the problem is to extract the relevant values from these matrices.

The situation is made more complicated by the fact that quantum field theory as presently formulated can only yield measurable quantities by a perturbative calculation, and the only realistic way to do so is by summing Feynman diagrams. Even putting aside the question of ghost terms for a non-abelian gauge theory, we are immediately faced with two problems. Firstly each individual Feynman diagram usually contains divergent integrals. And even after regularizing these integrals one has to make sure that the perturbative series can be sensibly summed. These issues are dealt with under the heading of ‘renormalization’ and are definitely outside the scope of these lectures. What we need to know is that the renormalization procedure introduces a scale dependence on the physical quantities in the theory. A well-known example is the ‘running coupling constant’, which has a measurable effect.

The dependence on scale $t$ of any given quantity, such as the mass matrix, is explicitly known, via the relevant ‘renormalization group equation’. For example, for the quarks we have the first-order SM equations:

$$\frac{dU}{dt} = \frac{3}{32\pi^2}(UU^\dagger - DD^\dagger)U + (\Sigma_u - A_u)U, \quad (2.17)$$

$$\frac{dD}{dt} = \frac{3}{32\pi^2}(DD^\dagger - UU^\dagger)D + (\Sigma_d - A_d)D, \quad (2.18)$$

where the quantities $\Sigma$ and $A$ need not concern us here.

So for a mass matrix with both eigenvalues and eigenvectors depending on scale, it is not obvious how one can define the physical mass and the
physical state vector. In the next lecture we shall make a proposal for doing so in the Dualized Standard Model.

2.4 't Hooft’s theorem and duality

There is an unexplained lopsidedness about the SM, in that on the one hand we have an exact colour symmetry for the strong interaction where the charges (that is, quarks) are confined (that is, not observable in the free state), and on the other hand we have a broken gauge symmetry for the electroweak interaction where the charges (that is, leptons) are free.

In his famous study of the confinement problem 't Hooft introduces two loop quantities $A(C)$ and $B(C)$ which are operators in the Hilbert space of quantum states satisfying the commutation relation

$$A(C)B(C') = B(C')A(C) \exp(2\pi in/N), \quad (2.19)$$

for an $SU(N)$ gauge theory, where $n$ is the linking number between the two (spatial) loops $C$ and $C'$. The first operator is given explicitly by

$$A(C) = \text{tr} \Phi(C), \quad (2.20)$$

where $\Phi(C)$ is the Dirac phase factor (1.3). He describes these two quantities as:

- $A(C)$ measures the magnetic flux through $C$ and creates electric flux along $C$
- $B(C)$ measures the electric flux through $C$ and creates magnetic flux along $C$

So they play dual roles in the sense we have been considering in the first lecture. However, there was no “magnetic” potential available at the time, so that the definition of $B(C')$ was not explicit, only through the commutation relation above. But we have now in fact constructed the magnetic potential $\tilde{A}_\mu$, and using it to construct the dual operator $B(C) = \text{tr} \tilde{\Phi}(C)$ we can prove the commutation relation (2.19), so that we know that our duality is the same as 't Hooft’s. This also means that we can apply the following result to the duality we find.

't Hooft’s Theorem. *If the Wilson loop operator of an $SU(N)$ theory and its dual theory satisfy the commutation relation given above, then:*

$$SU(N) \text{ confined } \iff \tilde{SU}(N) \text{ broken}$$

$$SU(N) \text{ broken } \iff \tilde{SU}(N) \text{ confined}$$
Note that the second statement follows from the first, given that the operation of duality is its own inverse (up to sign).

The theorem does not hold for a $U(1)$ theory, where both $U(1)$ and $\tilde{U}(1)$ may exist in a Coulomb phase, that is, with long range potential ($\sim 1/r$).

The statement is phrased in terms of phase transition, and has profound implications. It has been a cornerstone for attempts to prove quark confinement ever since. We shall show in the next lecture how we use it to suggest a solution to the generation puzzle.

Coming back to the commutation relation, I wish to show you how to prove it in the abelian case, just to give you a taste of what is involved. The nonabelian case is too complicated to treat here.

In the abelian case, we do not need the trace, hence $A(C) = \Phi(C)$, $B(C') = \tilde{\Phi}(C')$, and the $\Phi$ are genuine exponentials. So if we can show the following relation for the exponents, we shall have proved the required commutation relation:

$$\left[ie \oint_C A_i dx^i, i\tilde{e} \oint_{C'} \tilde{A}_i dx^i\right] = 2\pi n i.$$

Using Stokes’ theorem the second integral

$$= -i\tilde{e} \int_{\Sigma_{C'}} \ast F_{ij} d\sigma^j = i\tilde{e} \int_{\Sigma_{C'}} E_i d\sigma^i,$$

where $\partial \Sigma_{C'} = C'$.

For simplicity, suppose the linking number $n = 1$. Then the loop $C$ will intersect $\Sigma_{C'}$ at some point $x_0$—if it intersects more than once, the other contributions will cancel in pairs, so we shall ignore them. So except for $x_0$, all points in $C$ are spatially separated from points on $\Sigma_{C'}$.

Using the canonical commutation relation for $A_i$ and $E_j$

$$[E_i(x), A_j(x')] = i\delta_{ij}\delta(x - x')$$

we get

$$\left[ie \oint_C A_i dx^i, i\tilde{e} \int_{\Sigma_{C'}} E_j d\sigma^j\right] = ie\tilde{e} = 2\pi i$$

by Dirac’s quantization condition.

Thus we have shown explicitly in the abelian case that our definition of duality coincides with ’t Hooft’s. The same is true in the nonabelian case.
3 The Dualized Standard Model (DSM)

3.1 Generation symmetry as dual colour

So far theory has made two predictions. First, duality tells us that if we start with an $SU(N)$ gauge theory we have a doubled gauge symmetry of $SU(N) \times \tilde{SU}(N)$. Secondly, 't Hooft’s theorem tells us that the symmetry $SU(N)$ is confined if and only if the symmetry $\tilde{SU}(N)$ is broken.

On the other hand, experiment gives us two pieces of information. First, there are three and only three generations of fermions, which are very similar except for their masses. Secondly, $SU(3)$ colour is confined.

It is therefore natural, at least to us, to put the two together. So the main assumption of the Dualized Standard Model (DSM) is that generation symmetry is dual colour. In doing so, not only do we explain the existence of exactly three generations, but also we dispense with the need to find an experimental niche for the dual colour symmetry which must exist by the theory.

As a reminder, here are again the four questions we noted when discussing the Standard Model, the first of which is now answered and the remainder of which we shall answer:

1. Why are there exactly three generations?
2. What is the origin of the Higgs fields?
3. Why are the fermion masses hierarchical?
4. Why do fermions mix in the patterns observed?

Since all fermions carry generation index, now identified as dual colour, they are dyons in the sense of having both ‘electric’ and ‘magnetic’ charges. We have already seen the ‘electric’ assignments in the Standard Model. Now a monopole of charge $n$ in SM has the following components: $\exp(2\pi in/3)$ of $SU(3)$, $(-1)^n$ of $SU(2)$, $n/3$ of $U(1)$. Putting these two together, we make the assignments indicated in Table 1, where bold numbers indicate the dimension of the representation. In the table, only the lightest generation is represented, as the two higher generations have identical charges.
Next we recall that, in the duality transform (1.48), $\omega(x)$ was originally conceived as the matrix relating the internal symmetry $G$-frame to the dual symmetry $\tilde{G}$-frame. The rows of $\omega$ therefore transform as the conjugate fundamental representation of the $G$-symmetry, i.e. as $\bar{3}$ of colour or $\bar{2}$ of weak isospin, while its columns transform as the fundamental representation of the dual $\tilde{G}$-symmetry, i.e. as $3$ of dual colour or $2$ of dual weak isospin. We want to relate these to the Higgs fields.

The idea of using frame vectors as dynamical variables is made familiar already in the theory of relativity where in the Palatini treatment or the Einstein-Cartan-Kibble-Sciama formalism the space-time frame vectors or vierbeins are used as dynamical variables. In gauge theory, frame vectors in internal symmetry space are not normally given a dynamical role, but it turns out that in the dualized framework they seem to acquire some dynamical properties, in being patched, for example, in the presence of monopoles. Moreover, they are space-time scalars belonging to the fundamental representations of the internal symmetry group, i.e. doublets in electroweak $SU(2)$ and triplets in dual colour $\tilde{SU}(3)$, and have finite lengths (as vev’s). They thus seem to have just the right properties to be Higgs fields, at least as borne out by the familiar example of the Salam-Weinberg breaking of the electroweak theory. Hence, one makes the second basic assumption in the DSM scheme, namely that these frame vectors are indeed the physical Higgs fields required for the spontaneously broken symmetries.

Having made these basic assumptions, let us now explore the consequences. First, making the frame vectors in internal symmetry space into

|        | $SU(3)$ | $SU(2)$ | $U(1)$ | $\tilde{SU}(3)$ | $\tilde{SU}(2)$ | $\tilde{U}(1)$ |
|--------|---------|---------|--------|-----------------|-----------------|----------------|
| $u_L$  | 3       | 2       | $1/3$  | 3               | 1               | $-2/3$        |
| $d_L$  | 3       | 2       | $1/3$  | 3               | 1               | $-2/3$        |
| $u_R$  | 3       | 1       | $4/3$  | 1               | 1               | 0             |
| $d_R$  | 3       | 1       | $-2/3$ | 1               | 1               | 0             |
| $\nu_L$ | 1     | 2       | $-1$   | 3               | 1               | $-2/3$        |
| $e_L$  | 1       | 2       | $-1$   | 3               | 1               | $-2/3$        |
| $\nu_R$ | 1     | 1       | 0      | 1               | 1               | 0             |
| $e_R$  | 1       | 2       | $-2$   | 1               | 1               | 0             |

Table 1: Charge assignments in DSM

### 3.2 Higgs fields as frame vectors

Next we recall that, in the duality transform (1.48), $\omega(x)$ was originally conceived as the matrix relating the internal symmetry $G$-frame to the dual symmetry $\tilde{G}$-frame. The rows of $\omega$ therefore transform as the conjugate fundamental representation of the $G$-symmetry, i.e. as $3$ of colour or $2$ of weak isospin, while its columns transform as the fundamental representation of the dual $\tilde{G}$-symmetry, i.e. as $3$ of dual colour or $2$ of dual weak isospin. We want to relate these to the Higgs fields.

The idea of using frame vectors as dynamical variables is made familiar already in the theory of relativity where in the Palatini treatment or the Einstein-Cartan-Kibble-Sciama formalism the space-time frame vectors or vierbeins are used as dynamical variables. In gauge theory, frame vectors in internal symmetry space are not normally given a dynamical role, but it turns out that in the dualized framework they seem to acquire some dynamical properties, in being patched, for example, in the presence of monopoles. Moreover, they are space-time scalars belonging to the fundamental representations of the internal symmetry group, i.e. doublets in electroweak $SU(2)$ and triplets in dual colour $\tilde{SU}(3)$, and have finite lengths (as vev’s). They thus seem to have just the right properties to be Higgs fields, at least as borne out by the familiar example of the Salam-Weinberg breaking of the electroweak theory. Hence, one makes the second basic assumption in the DSM scheme, namely that these frame vectors are indeed the physical Higgs fields required for the spontaneously broken symmetries.

Having made these basic assumptions, let us now explore the consequences. First, making the frame vectors in internal symmetry space into
dynamical variables and identifying them with Higgs fields mean that for dual colour $\tilde{SU}(3)$, we introduce 3 triplets of Higgs fields $\phi^{(a)}_a$, where $(a) = 1, 2, 3$ labels the 3 triplets and $a = 1, 2, 3$ their 3 dual colour components. Further, the 3 triplets having equal status, it seems reasonable to require that the action be symmetric under their permutations, although the vacuum need not be. An example of a Higgs potential which breaks both this permutation symmetry and also the $\tilde{SU}(3)$ gauge symmetry completely is as follows:

$$V[\phi] = -\mu \sum_{(a)} |\phi^{(a)}|^2 + \lambda \left\{ \sum_{(a)} |\phi^{(a)}|^2 \right\}^2 + \kappa \sum_{(a)\neq(b)} |\phi^{(a)}\phi^{(b)}|^2,$$  \hspace{2cm} (3.1)

a vacuum of which can be expressed without loss of generality in terms of the Higgs vacuum expectation values:

$$\phi^{(1)} = \zeta \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}, \quad \phi^{(2)} = \zeta \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}, \quad \phi^{(3)} = \zeta \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix},$$ \hspace{2cm} (3.2)

with

$$x^2 + y^2 + z^2 = 1,$$ \hspace{2cm} (3.3)

and

$$\zeta = \sqrt{\mu/2\lambda},$$ \hspace{2cm} (3.4)

$x, y, z$, and $\zeta$ being all real and positive. Indeed, this vacuum breaks not just the symmetry $\tilde{SU}(3)$ but the larger symmetry $\tilde{SU}(3) \times \tilde{U}(1)$ completely giving rise to 9 massive dual gauge bosons. And of the 18 real components in $\phi^{(a)}_a$, 9 are thus ‘eaten up’, leaving just 9 (dual colour) Higgs bosons.

### 3.3 The fermion mass matrices

Following the procedure outlined in the last lecture, we let the fermions acquire nonzero mass through additional terms in the Lagrangian. It turns out that, using frame vectors as Higgs fields as above, the $3 \times 3$ fermion mass matrix (one for each of the four types $T = U, D, L, N$) is of rank 1 and can thus be written in terms of one single normalized vector $v_T = (x, y, z)$ (without loss of generality we may assume $x \geq y \geq z$) as:

$$m = m_T \begin{pmatrix} x \\ y \\ z \end{pmatrix} (x \ y \ z) = m_T \begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix},$$ \hspace{2cm} (3.5)
where \( m_T \) is the only nonzero eigenvalue.

Evaluating the 1-loop Feynman diagrams in the standard way gives us the renormalization group equation for \( m \). We can show that the matrix \( m \) remains of factorized form under change of scale \( \mu \), so that all the physics content of \( m \) can be deduced from the running of the normalization factor \( m_T \) and the rotation of the unit vector \( v_T \). In particular, we have the following renormalization group equation for \( v_T \):

\[
\frac{d}{d(\ln \mu^2)} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{3}{64\pi^2} \rho^2 \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix},
\]

with

\[
\tilde{x} = \frac{x(x^2 - y^2)}{x^2 + y^2} + \frac{x(x^2 - z^2)}{x^2 + z^2}, \quad \text{cyclic},
\]

where \( \rho \) is constant, the same for all types \( T \), which gives the ‘speed’ at which the vector \( v_T \) runs. This means that all \( v_U, v_D, v_L, v_N \) lie on the same renormalization group equation trajectory on the unit sphere, Figure 4, a study of which will give us most of the results we want.

Recall from (3.5) that the mass matrix \( m \) has only one nonzero eigenvalue, but both it and the eigenvectors depend on the scale \( \mu \). The eigenvector corresponding to the single nonzero eigenvalue is \( v_T \), the radial vector of the trajectory on the unit sphere (Figure 4). Further, we can easily see that the renormalization group equation (3.6) has two fixed points: \( v = (1,0,0) \) at high (or infinite) energy, and \( v = (1/\sqrt{3},1/\sqrt{3},1/\sqrt{3}) \) at low (or zero) energy. These are important for what follows.

We raised the question in the last lecture about the ambiguity in defining physical masses and states when the mass matrix runs with scale. We now make the following proposal, a kind of working criterion which is applicable in DSM.

First we run \( m \) to a scale \( \mu \) such that \( \mu = m_T(\mu) \); this value we can reasonably call the mass of the highest generation. To fix ideas, let us concentrate on the \( U \) type quarks \( t, c, u \). The corresponding eigenvector \( v_t \) is then the state vector of the \( t \) quark. Having fixed this, we now know that the \( c \) and \( u \) lie in the 2-dimensional subspace \( V \) orthogonal to \( v_t \). As we go down in scale the eigenvector \( v_T \) rotates so that \( V \) is no longer the null eigenspace, and the projection of \( m \) onto \( V \) is a \( 2 \times 2 \) matrix, again of rank 1. We now repeat the procedure for this submatrix and determine both the mass and the state of the \( c \) quark (Figure 5). Once the \( c \) state is determined,
Figure 4: The trajectory traced out by $v_T$ as scale changes
we know the \( u \) state as well, as being the third vector of the orthonormal triad \((t, c, u)\). The \( u \) mass is similarly determined.

The above procedure can be repeated for the other three types of fermions\(^8\). The direction cosines of the two triads \((t, c, u)\) and \((b, s, d)\), in other words, the 9 inner products, will give us the CKM matrix, Figure 6. Similarly for the MNS mixing matrix of the leptons.

Our proposal has the following desirable properties:

- The 3 lepton state vectors (of the same type) are always orthogonal.
- The mixing matrix is always unitary.
- Mass hierarchy is automatic.

The last point comes about because the lower generation masses are obtained by the “leakage” from the single nonzero eigenvalue of \( m \) after the heaviest generation is fixed.

This now answers the remaining two of our four questions.

\(^8\)Neutrinos will need further special treatment. See next subsection.
Before we present the consequences of DSM, we need to make a small digression on neutrinos. Recall that in SM the neutrinos have only left-handed components, so that one cannot write a Yukawa term $\bar{\nu}_L \phi \nu_R$ as in (2.13). However, as we emphasized, there is no theoretical reason why right-handed neutrinos should not exist.

In fact, in view of the recent neutrino oscillation results from Superkamiokande, SNO and others, evidence is quite conclusive that neutrinos must have nonzero masses, although these are very small compared to the other fermions.

Neutrinos oscillate because the states $\nu_\tau$, $\nu_\mu$, $\nu_e$ in which they are produced, for example in $\beta$-decay where a neutron decays into a proton, an electron and a neutrino, are not mass eigenstates (that is, states which propagate according to the Dirac equation).

The most naive way of just introducing a right-handed component $\nu_R$ is not very satisfactory, because of their very small masses. One way to produce very small physical neutrino masses is via the ‘see-saw mechanism’. Because $\nu_R$ has no gauge charges whatsoever (in SM) there is an additional mass term, the Majorana mass term, one can write down. By postulating a

3.4 Neutrino masses

Figure 6: Two triads of state vectors for the quarks on the trajectory
large Majorana mass $B$ (which has no counterpart in the other fermions), the see-saw mechanism can produce a small $\nu$ mass when the following matrix is diagonalized:

$$
\begin{pmatrix}
0 & M \\
M & B \\
\end{pmatrix},
$$

where $M$ the Dirac mass (coming from the Yukawa term) need not be too different from that of the other fermions, and yet one can have a small eigenvalue $\sim M^2/B$.

Now DSM can quite naturally incorporate this feature and this is what has been done.

### 3.5 Consequences of DSM: masses and mixing

As the audience is mainly mathematical, I shall mention only very briefly the numerical results. We used the so-called Cabibbo angles $V_{us}$ and $V_{cd}$, which are the best measured experimentally among the mixing angles, as inputs. So for the quark CKM matrix:

$$
\begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
V_{cd} & V_{cs} & V_{cb} \\
V_{td} & V_{ts} & V_{tb} \\
\end{pmatrix}
$$

with experimental values

$$
\begin{pmatrix}
0.9742 - 0.9757 & 0.219 - 0.226 & 0.002 - 0.005 \\
0.219 - 0.225 & 0.9734 - 0.9749 & 0.037 - 0.043 \\
0.004 - 0.014 & 0.035 - 0.043 & 0.9990 - 0.9993 \\
\end{pmatrix} \quad (3.8)
$$

we obtain

$$
\begin{pmatrix}
0.9752 & 0.2215 & 0.0048 \\
0.2211 & 0.9744 & 0.0401 \\
0.0136 & 0.0381 & 0.9992 \\
\end{pmatrix} \quad (3.9)
$$

from our calculations. And for the lepton MNS matrix:

$$
\begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} \\
U_{\tau1} & U_{\tau2} & U_{\tau3} \\
\end{pmatrix}
$$
with experimental values

\[
\begin{pmatrix}
* & 0.4 - 0.7 & 0.0 - 0.15 \\
* & * & 0.56 - 0.83 \\
* & * & *
\end{pmatrix}
\] (3.10)

we obtain

\[
\begin{pmatrix}
0.97 & 0.24 & 0.07 \\
0.22 & 0.71 & 0.66 \\
0.11 & 0.66 & 0.74
\end{pmatrix}
\] (3.11)

from our calculations.

All except the so-called ‘solar neutrino angle’ $U_{e2}$ are well within experimental bounds. We shall now give a simple geometric reason for such good agreements, quite apart from the actual results of our numerical computation.

Since we have a curve on the unit sphere, we are reminded of the elementary classical differential geometry of a curve on a surface. Let $N$ be the normal to the surface at a given point, $T$ the tangent to the curve at that point, and $B = N \wedge T$. These 3 vectors form the Darboux triad, and the rotation matrix linking two neighbouring triads at $\Delta s$ apart are given by the Serret–Frenet–Darboux formulae\(^9\)
as

\[
\begin{pmatrix}
1 & -\kappa_g \Delta s & -\tau_g \Delta s \\
\kappa_g \Delta s & 1 & \kappa_n \Delta s \\
\tau_g \Delta s & -\kappa_n \Delta s & 1
\end{pmatrix},
\] (3.12)

where $\kappa_g$ is the geodesic curvature, $\kappa_n$ the normal curvature, and $\tau_g$ the geodesic torsion. Now for a unit sphere, $\kappa_n = 1$, $\tau_g = 0$, so that we have for the case in hand

\[
\begin{pmatrix}
1 & -\kappa_g \Delta s & 0 \\
\kappa_g \Delta s & 1 & \Delta s \\
0 & -\Delta s & 1
\end{pmatrix},
\] (3.13)

A comparison of (3.13) with (3.8) (or indeed (3.9)) and with (3.10) (or again (3.11)) will elicit the following remarkable features immediately:

1. The corner elements ($V_{ub}$, $V_{td}$) and respectively ($U_{e3}$, $U_{\tau1}$) are at most second order in the separation $\Delta s$, and hence vanishingly small.

\(^9\)These are similar to the well-known Serret–Frenet formulae for a space curve, with tangent, normal and binormal forming a moving triad.
2. The 4 other off-diagonal CKM elements are small compared with the
  diagonal elements, since they are of first order in the separation between
  the $t$ and the $b$ quarks, which is small as seen in Figure 4.

3. The elements $V_{cb}, V_{ts}$ for quarks are much smaller than their counter-
  parts $U_{\mu 3}, U_{\tau 2}$ for leptons, since they are to first order proportional to
  the separation, which is much smaller for quarks than for leptons as
  seen in Figure 4.

Incidentally, we also understand why it is more difficult to get the solar angle
$U_{\theta 2}$ correct, since this depends more on the details of the actual curve, namely,
its geodesic curvature. (The corresponding quark elements were inputs, as
already mentioned.)

What is quite amusing is that, although the $\tau$ and $\nu_3$ are quite far apart
on our trajectory (Figure 4), the relation (3.13) still seems to work, and
in fact if you take a piece of string and measure the ratio of the distance
between $\tau$ and $\nu_3$ and between $t$ and $b$, you will get roughly the correct
factor $U_{\mu 3}/V_{cb} \sim 20$. This must be the most inexpensive particle physics
experiment ever performed!

3.6 Renormalization fixed points

In determining the mixing angles the fermion masses necessarily come
into play, as explained in §3.3. However, I did not explicitly present the
mass results in the last subsection, because these are much more transparent
when examined from the point of view of the renormalization group equation
fixed points.

Recall that we need only concern ourselves with the single unit vector $v_T$.
As the scale $\mu$ decreases, this vector rotates from near the high energy fixed
point at $(1,0,0)$ to the low energy fixed point $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ tracing
out a trajectory (Figure 4). Figure 7 shows that part of the trajectory which
is near the high energy fixed point, and which is very nearly planar. Since
we are near a fixed point, rotation is slow in the sense that (i) the top quark
at 175 GeV is almost indistinguishable from the fixed point itself, and (ii)
after running down several decades of energy the rotation angle is still fairly
small. That the masses and mixing parameters come from a rotation angle
which remains small even when the energy range is large is very important for
the DSM results. This is because our renormalization group equation comes
from contributions from 1-loop diagrams only and if the relevant parameter
(here the rotation angle) were not small there would be no reason to expect
Figure 7: Trajectory near the high energy fixed point
1-loop calculations to be accurate over such a large range in scale. We see in Figure 7 that the DSM curve here goes smoothly through all the data points (mass and mixing). Notice that experimental estimates of quark masses are not without ambiguity, hence the large ‘error bars’ on the data points.

Since all three state vectors (in generation space) of the three generations are already fixed at the position of the second generation, Figure 7 tells us that they are accurately determined (for both quark types $U$ and $D$ and for charged leptons $L$), hence good agreement with experiment for the CKM mixing elements. To get the lowest generation masses $u, d, e$ one needs to run further down in energy, where the rotation angles become sizeable. While the values we get are still sensibly hierarchical (a necessary consequence of our scheme, §3.3) they are numerically inaccurate.

At the other extreme, the neutrinos have very small mass. The heaviest one $\nu_3$ is estimated to be $\sim 0.05$ eV, which is so near the low energy fixed point that its state vector is very well approximated by the fixed point itself $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, as confirmed by explicit calculations. Since we already know the charged lepton triad accurately, we can now deduce the ‘atmospheric’ angle $U_{\mu 3}$ and the ‘Chooz’ angle $U_{e3}$, the former near maximal and the latter very small, again as confirmed by calculations and exactly as data say. The ‘solar’ angle $U_{e2}$, however, depends on how the trajectory approaches the fixed point, that is, its tangent there, which we cannot obtain simply by extrapolation. For this angle, we do not get good agreement with experiment (see also the last subsection).

### 3.7 Further consequences of DSM

To summarize so far: with 3 parameters (fixed from data) we were able to calculate all the other quantities relevant to fermion masses and mixing. Of these, 9 are within experimental error, 1 nearly so, and only 2, the electron mass and the solar neutrino angle are outside experimental bounds.

With all but one piece (see below) of information in hand we can now look at DSM in three further broad areas of application.

First, the exchange of dual colour gauge bosons will induce new interactions, called ‘flavour-changing neutral current effects’. By assuming a lower bound of about 500 TeV for the mass of these dual colour bosons, we are able to satisfy experimental bounds for the following rather diverse interactions:

---

10 These were obtained about 3 years ago. We now believe that we may have a more reliable method of extrapolation which would give closer agreement with experiment, especially for the $u$ and $d$ quarks.
• rare hadron decays e.g. $K_L \rightarrow e^\pm \mu^\mp$;

• mass differences e.g. $K_L - K_S$;

• coherent muon-electron conversion on nuclei e.g. $\mu^- + Ti \rightarrow e^- + Ti$;

• muonium conversion e.g. $\mu^+e^- \rightarrow e^+\mu^-$;

• neutrinoless double beta decay e.g. $^{76}Ge \rightarrow ^{76}Se + 2e^-$.

Note that these involve not only particle physics but also nuclear and atomic physics.

Secondly, a rotating lepton mass matrix will mean that lepton quantum numbers may not be conserved, leading to phenomena of lepton flavour violation or ‘transmutation’. Using the parameters determined previously we have done calculations for the following interactions:

• decays e.g. $\Upsilon \rightarrow \mu^\pm\tau^\mp$;

• photo-transmutation e.g. $\gamma e^- \rightarrow \gamma\tau^-$;

• transmutational Bhabha e.g. $e^+e^- \rightarrow e^+\mu^-$;

all predicted cross-sections satisfying existing bounds.

Lastly, and rather surprisingly, DSM has something to say about the so-called ‘ultra high energy cosmic rays’ or ‘airshowers’. These are still an unsolved puzzle in astrophysics, and I shall describe them briefly, as they are something of an expected bonus for us.

Cosmic rays with energy $> 10^{20}$ eV pose a problem in astrophysics. Over the last 30 years about 12 such events have been observed, each producing some $10^{11}$ charged particles. If they are protons, they will lose their energy quickly by interacting with the cosmic microwave background: $p + \gamma_{2,7} \rightarrow \Delta + \pi$. Greisen, Zatsepin and Kuz’min (GZK) estimated that if these primary particles are indeed protons, they cannot therefore originate further than 50 Mpc away without losing their energy. However, there are no obvious proton sources of such high energy that near. Moreover, some possible pairs and triples have been observed, pointing back to the same source. But if they were protons, they would have been deflected by the inter-galactic magnetic field and one would not have been able to trace back their origin. So protons seem not to provide a solution. If these particles are neutrinos, on the other hand, they would not suffer from these constraints. But ordinary weakly
interacting neutrinos would not have a large enough cross-section with air nuclei to produce the many particles observed in each such event.

DSM offers a possible solution. At energies above the dual colour gluon mass neutrinos will have become strongly interacting, because they carry the generation index which is identified with dual colour. From the GZK bound we can deduce a lower bound for the mass of these bosons, which turns out to be around 500 TeV. That this lower bound coincides with the upper bound estimated from flavour-changing neutral currents above (more specifically, from $K_L - K_S$ mass difference) is a very pleasant surprise! Indeed, strongly interacting neutrinos in this scenario can actually solve all the above-mentioned problems:

- $\nu$ can escape strong electromagnetic field around any source, e.g. AGN such as MCG8-11-11;
- $\nu$ can survive a long journey through microwave background;
- near hadronic cross-section with air nuclei at high energy;
- pairs (or triplets) not deflected by inter-galactic electromagnetic field;
- highest energy event at $3 \times 10^{20}$ eV with no abundant lower energy events in same direction: $\nu$ interacts strongly only at high energy.

Indeed, for this last event from Fly’s Eye we estimated the height of the primary vertex and found it agrees substantially better with a neutrino-induced rather than a proton-induced shower, Figure 8.

More similar quantitative calculations within DSM can be done. With the planned Auger observatories, we hope that the question about the origin of airshowers will be settled by the new data in perhaps the next decade.

### 3.8 DSM: the future

We have travelled a long way. From the loop space formulation of gauge theory, by way of electric–magnetic duality and ’t Hooft’s theorem, we have arrived at actual numbers to confront with experiments and have so far done honourably indeed. But many things remain to be understood and done.

While the basis of DSM seems to have survived all experimental tests so far, we are sure that many details will need to be changed as we gain more understanding.

The following is but some of the ‘items on the agenda’ that come up immediately to our mind:
Figure 8: The positions of the maximum heights for varying $\theta$

- to understand the dual transform;
- to study further Higgs field as frame vectors;
- to see if there is a geometric origin to the Yukawa terms;
- to obtain a better picture of the middle-energy range;
- to understanding further the neutrinos.

No doubt more items will come up even before we start to tackle any of the above. We find this an exciting prospect.

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