IMPROVED EVALUATION OF THE NNLO QCD EFFECTS IN THE TAU DECAY, $e^+e^-$ ANNIHILATION INTO HADRONS AND DEEP-INELASTIC SUM RULES

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Abstract
A systematic method is proposed for analyzing the renormalization scheme uncertainties in the next-next-to-leading order QCD predictions, based on a condition which eliminates schemes that give rise to large cancellations in the expression for the characteristic scheme invariant combination of the expansion coefficients. Using this method it is shown that the QCD corrections to the tau lepton decay are rather stable with respect to change of the scheme, provided that an improved formula is used, which involves numerical evaluation of the contour integral in the complex energy plane with the Adler function under the integral. Optimized predictions for the tau decay corrections are given. It is shown that also in the case of the of QCD corrections to $e^+e^-$ annihilation into hadrons the conventional expansion has sizable scheme dependence, even at large energies. However, a considerable improvement is obtained when the QCD corrections are expressed as a contour integral, with the Adler function under the integral, resumming in this way the large $\pi^2$ contributions. In the case of the corrections to the Bjorken sum rule for polarized structure functions it is found that for $n_f = 4$ they are insensitive to change of the scheme. However, the $n_f = 3$ expression is found to be strongly scheme dependent at lower energies.

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Perturbative predictions in finite order of perturbation expansion depend on the choice of the renormalization scheme (RS). In the next-to-next-to-leading order (NNLO) approximation, when mass effects are neglected, the freedom of choice of RS may be characterized by two independent continuous parameters. The differences between predictions in various schemes are formally of higher order in the coupling constant, but numerically they are significant for phenomenology.

There are two things that one should do with the RS dependence of perturbative approximants. First, using various heuristic arguments of physical or technical character, one should make a careful choice of the scheme, obtaining an “optimized” perturbative prediction. Several propositions have been discussed in the literature [1], including a very interesting prescription based on the Principle of Minimal Sensitivity (PMS) [2]. Secondly, one should investigate how strongly predictions change when one moves away from the preferred RS. By calculating the variation in the predictions over a set of \textit{a priori} acceptable schemes one obtains an estimate of the reliability of the “optimized” prediction. A systematic method for obtaining such estimates has been recently presented in [3]. This method is based on a condition which eliminates from the analysis the schemes that give rise to unnaturally large expansion coefficients in the expansion for the physical quantity or the beta-function.

Let us consider a NNLO expression for a physical quantity $\delta$, depending on one energy variable $P$, in the massless approximation:

$$
\delta(P^2) = a(P^2)[1 + r_1 a(P^2) + r_2 a^2(P^2)],
$$

(1)

where $a(\mu^2) = g^2(\mu^2)/(4\pi^2)$ denotes the coupling constant that satisfies the NNLO renormalization group equation:

$$
\mu \frac{da}{d\mu} = -b a^2 (1 + c_1 a + c_2 a^2),
$$

(2)

The expansion coefficients $r_i$ and $c_2$ depend on the choice of the renormalization scheme — the relevant formulas have been collected for example in [4] — but there exists a combination of these coefficients which is independent of the scheme:

$$
\rho_2 = c_2 + r_2 - c_1 r_1 - r_1^2.
$$

(3)

As was discussed in [1, 3], this combination provides a natural RS independent characterization of the magnitude of the NNLO correction. The RS invariant $\rho_2$ may be used to distinguish between “good” and “bad” schemes. Indeed, one usually identifies unnatural schemes with ones that introduce large expansion coefficients in an artificial way. However, the combination (3) must stay the same even for very bad schemes, which may be achieved only by the presence of large cancellations between various terms in the expression for $\rho_2$ [4]. It is then useful to introduce a function:

$$
\sigma_2(r_1, r_2, c_2) = |c_2| + |r_2| + c_1|r_1| + r_1^2.
$$

(4)

This function measures the degree of cancellation in $\rho_2$, and gives a quantitative meaning to the notion of “naturalness” of any chosen RS. It is clear that a large cancellation in $\rho_2$ would imply that the value of $\sigma_2$ would be large compared to $|\rho_2|$. (Obviously $\sigma_2 \geq |\rho_2|$.) If we have any preference for using some renormalization scheme, we should also include in the analysis predictions obtained in schemes which have the same, or smaller, value of $\sigma_2$. For example, for the PMS scheme we have:

$$
r_1^{PMS} = O(a^{PMS}), \quad c_2^{PMS} = 1.5 \rho_2 + O(a^{PMS}),
$$

(5)
which implies \( \sigma_2(PMS) \approx 2|\rho_2| \). More generally, we may write the condition on the acceptable schemes in the form:

\[
\sigma_2(r_1, r_2, c_2) \leq l |\rho_2|.
\] (6)

The constant \( l \) in the condition (6) controls the degree of cancellations that we want to allow in the expression for \( \rho_2 \). The value \( l = 2 \) in (6) is the minimal value of \( l \) for which the PMS scheme falls into the “allowed” region of scheme parameters. The proposition of (4, 3) is to calculate the variation of the predictions for \( \delta \) over the set of schemes satisfying the condition (6), and use this variation as a quantitative estimate of reliability of the perturbative predictions. As was discussed in (4, 3), it is convenient to use \( r_1 \) and \( c_2 \) to parametrize the freedom of choice of the RS in the NNLO approximants.

It should be stressed that no claim is made that in this way the actual theoretical error of the calculation is obtained, i.e. there is of course no theorem that guarantees that the difference between the NNLO prediction and the true result would lie within the obtained “error bound.” However, the region of scheme parameters satisfying the condition (6) with \( l = 2 \) appears to be the minimal set of schemes that has to be taken into account — consequently, if strong scheme dependence is obtained for these schemes, it is an unambiguous sign that the perturbation series is not reliable. Also, evaluating by our method the variation in predictions for several quantities with the same value of \( l \) in the condition (6) we obtain a very good estimate of relative reliability of the predictions. The obtained estimates may then be used for example to assign different weights to different observables in the global fit of \( \Lambda_{\overline{MS}} \).

Let us begin with the QCD correction \( \delta_\tau \) to the tau lepton decay into hadrons. The QCD prediction for \( \delta_\tau \) may be expressed as a contour integral in the complex energy plane (4):

\[
\delta_\tau = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left( 1 + 2e^{i\theta} - 2e^{3i\theta} - e^{4i\theta} \right) \left[ \delta_{\Pi}(-\sigma) \right]_{|\sigma = -m_\tau^2 e^{i\theta}} ,
\] (7)

where \( \delta_{\Pi}(-\sigma) \) is the so called Adler function, which has the form (4). The contour integral has been initially evaluated by expanding \( a(-\sigma) \) in terms of \( a(m_\tau^2) \). This led to the frequently used expansion for \( \delta_\tau \) in terms of \( a(m_\tau^2) \), which has the form (4), with \( \rho_2 = -5.476 \) (4, 7). The region of scheme parameters satisfying the condition (6) with \( l = 2 \) is approximately given by \( r_1 \in (-1.54, 2.11) \) and \( c_2 \in (-8.21, 2.74) \). Applying to this expansion the method outlined above we find very strong RS dependence. This confirms observations made in (4). However, as was pointed out in (4, 7), one may evaluate \( \delta_\tau \) in an improved way, using under the contour integral the renormalization group improved expression for \( \delta_{\Pi} \), and calculating the contour integral numerically. In this way one resums large \( \pi^2 \) corrections which would otherwise appear in higher orders. Let us note that for \( \delta_{\Pi} \) one has \( \rho_2^{\Pi} = 5.238 \), which has the same magnitude, but an opposite sign compared to value obtained for the “naive” expansion. A complete analysis of the RS dependence of the improved expression for \( \delta_\tau \) has been described in (11) (4 contains some discussion of scale dependence of the improved expression.) The QCD predictions obtained in the improved evaluation appear to be quite stable with respect to change of RS, despite the low energy scale of the process. The problem of finding the PMS predictions for the improved expression has been considered. This is nontrivial since for the improved approximant we cannot use the set of algebraic PMS equations given in (2). It was found that the location of the critical point closest to the \( l = 2 \) region of allowed scheme parameters is well approximated by \( r_1 = 0 \) and \( c_2 = 1.5 \rho_2^{\Pi} \), for most values of \( m_\tau/\Lambda_{\overline{MS}} \). Therefore predictions in this scheme have been taken as preferred predictions for the phenomenological analysis. Assuming \( (R_\tau)_{exp} = 3.591 \pm 0.036 \) as an averaged experimental value (see discussion of the experimental results in the Appendix of (11)) it was found that \( \Lambda_{\overline{MS}}^{(3)} = \)
376(\text{opt})^{+15}_{-14}(\text{th},l=2) \pm 29(\text{exp}) \text{ MeV}. \text{ For the } l = 3 \text{ region the variation is } \pm^{26}_{21} \text{MeV. This corresponds to: } \alpha_{\overline{MS}}^{\text{MS}}(m_{c}^{2}) = 0.332^{+0.008}_{-0.007}(\text{th},l=2) \pm 0.015(\text{exp}). \text{ For the } l = 3 \text{ region we obtain variation of } \pm^{0.014}_{0.010}. \text{ Extrapolating to } m_{Z}^{2} \text{ we find } \alpha_{\overline{MS}}^{\text{MS}}(m_{Z}^{2}) = 0.1190^{+0.0009}_{-0.0008}(\text{th},l=2) \pm 0.0017(\text{exp}). \text{ (For the } l = 3 \text{ region we find a considerable variation of the predictions for } Q \text{ equal to approximately for } r_{1} \in (-1.65, 1.0) \text{ and } c_{2} \in (-2.8, 8.3). \text{ Changing the scheme parameters in this region we find a considerable variation of the predictions for } Q^{2}/\Lambda_{\overline{MS}}^{2} \text{ below approximately } 6^{3}, \text{ as is shown in Fig.2. This indicates that perturbative predictions in this region are not reliable, even though we may still obtain PMS predictions. Unfortunately, in the case of the Bjorken sum rule we cannot expect that the RS dependence would be reduced in a way similar to tau decay or } e^{+}e^{-} \text{ annihilation. Other ways of improving the QCD perturbation expansion in this case have to be investigated. It should be noted however that the RS dependence ambiguities are still comparable in magnitude to the uncertainties related to the present experimental accuracy of } R_{\tau}. \text{ In the case of the QCD correction } \delta_{e^{+}e^{-}} \text{ to the } e^{+}e^{-} \text{ annihilation into hadrons, for } n_{f} = 5, \text{ we have } \rho_{2}^{+}e^{-} = -15.055 \text{ }[3, 4] \text{ (this does not include a very small singlet correction, which is added separately). \text{ The region of scheme parameters satisfying the condition } (3) \text{ with } l = 2 \text{ is approximately given by } r_{1} \in (-4.8, 4.2) \text{ and } c_{2} \in (-22.6, 7.5), \text{ and for } l = 3 \text{ it is approximately } r_{1} \in (-5.5, 4.9) \text{ and } c_{2} \in (-30.2, 15.1). \text{ Changing the scheme parameters in the } l = 2 \text{ region we find for } s/\Lambda_{\overline{MS}}^{2} = 75^{2} \text{ the variation in the predictions from 0.0491 to 0.0537, and for the } l = 3 \text{ region from 0.0454 to 0.0539. We see that even though the considered energy is high and the perturbation series should be reliable, the scheme dependence of the predictions is surprisingly large. This is a consequence of the fact that } \delta_{e^{+}e^{-}} \text{ has very large NNLO correction, which is reflected by the magnitude of the RS invariant. However, closer examination shows that major part of the NNLO correction comes from the term proportional to } \pi^{2}, \text{ which appears in the process of analytic continuation from spacelike to timelike momenta. The large contributions from the } \pi^{2} \text{ terms may be avoided — similarly as in the case of the tau decay — by expressing } \delta_{e^{+}e^{-}} \text{ as a contour integral in the complex energy plane, with the renormalization group improved expression for the Adler function under the integral [11]:}
\begin{equation}
\delta_{e^{+}e^{-}}(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \delta_{\Pi}(-\sigma)|_{\sigma = -s e^{i\theta}}. \tag{8}
\end{equation}
\text{ For the Adler function for } n_{f} = 5 \text{ we have } \rho_{2}^{\Pi} = -2.969. \text{ The improved expression for } \delta_{e^{+}e^{-}} \text{ appears to be much more stable with respect to change of RS as compared to the conventional expression [12]. Also, the } l = 2, 3 \text{ regions of the scheme parameters are much smaller than in the case of conventional expansion, because the RS invariant is much smaller. In Fig.1 we show the contour plot of } \delta_{e^{+}e^{-}}, \text{ as a function of scheme parameters, for } s/\Lambda_{\overline{MS}}^{2} = 75^{2}. \text{ We see that the variation of predictions over the } l = 2 \text{ region of scheme parameters is in fact negligible from the point of view of phenomenological applications.} \text{ For } n_{f} = 3, 4 \text{ the same effect of reduced RS dependence in the improved formula is found. Incidentally, for } n_{f} = 5, \text{ the optimized prediction obtained from the contour integral is close to the prediction obtained with the conventional expansion in the } \overline{MS} \text{ scheme and the PMS prediction in the conventional expansion. This is not the case for other number of flavors.} \text{ In the case of the QCD correction to the Bjorken sum rule for the polarized structure functions [13] it was found that the expression for } n_{f} = 4, \text{ with } \rho_{2} = 1.330 \text{ is very stable with respect to the change of the RS. The situation with } n_{f} = 3 \text{ predictions is quite different. For } n_{f} = 3 \text{ we have } \rho_{2} = 5.476, \text{ so that the } l = 2 \text{ region of scheme parameters extends approximately for } r_{1} \in (-1.65, 1.0) \text{ and } c_{2} \in (-2.8, 8.3). \text{ Changing the scheme parameters in this region we find a considerable variation of the predictions for } Q^{2}/\Lambda_{\overline{MS}}^{2} \text{ below approximately } 6^{3}, \text{ as is shown in Fig.2. This indicates that perturbative predictions in this region are not reliable, even though we may still obtain PMS predictions. Unfortunately, in the case of the Bjorken sum rule we cannot expect that the RS dependence would be reduced in a way similar to tau decay or } e^{+}e^{-} \text{ annihilation. Other ways of improving the QCD perturbation expansion in this case have to be investigated. It should be noted that the QCD correction to the Gross-}
Llewellyn-Smith sum rule has almost identical expansion as in the case of the Bjorken sum rule, so that our remarks apply also to that case.

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I apologize to all authors whose related work has not been mentioned here. A complete bibliography for the discussed topics greatly exceeds 50 papers.

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Figure Captions

Fig.1. Contour plot of $\delta^{e^+e^-}$ for $s/\Lambda_{\overline{MS}}^2 = 75^2$, as a function of the scheme parameters, obtained by evaluating the contour integral numerically. The regions of allowed scheme parameters for $l = 2, 3, 6, 10$ are also indicated.

Fig.2. QCD correction to the Bjorken sum rule for $n_f = 3$, as a function of $\sqrt{Q^2}/\Lambda_{\overline{MS}}$, for different values of $r_1$ and $c_2$ belonging to the $l = 2$ allowed region: a)(-1.65,0), b)(0.5,5.27), c)(-1.0,5.27), d)(-1.65,5.27). The dashed curve indicates prediction in the $\overline{MS}$ scheme.
