On the complexity of isometric immersions of hyperbolic spaces in any codimension

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Abstract. Although the Nash theorem solves the isometric embedding problem, matters are inherently more involved if one is further seeking an embedding that is well-behaved from the standpoint of submanifold geometry. More generally, consider a Lipschitz map $F : M^m \to \mathbb{R}^n$, where $M^m$ is a Hadamard manifold whose curvature lies between negative constants. The main result of this paper is that $F$ must perform a substantial compression: For every $r > 0$ and integer $k \geq 2$ there exist $k$ geodesic balls of radius $r$ in $M^m$ that are arbitrarily far from each other, but whose images under $F$ are bunched together arbitrarily close in the Hausdorff sense of $\mathbb{R}^n$. In particular, every isometric embedding $\mathbb{H}^m \to \mathbb{R}^n$ of hyperbolic space must have a complex asymptotic behavior, regardless of how high the codimension is. Hence, there is no truly simple way to realize $\mathbb{H}^m$ isometrically inside any Euclidean space.

1 Introduction.

The Nash embedding theorem ([12], [15]), to the effect that any Riemannian manifold $(M^m, g)$ can be isometrically embedded as a bounded subset of some high dimensional Euclidean space, represents a landmark in Riemannian geometry. Questions about the smoothness of the isometric immersion, connections with topology and partial differential equations, as well as the smallest dimension of the receiving space, have also attracted considerable attention over the years ([6], [7], [8]).

Exploring the surrounding landscape further, one comes across the natural idea of establishing the existence of an isometric immersion that, from the perspective of global submanifold geometry, is as well-behaved as the intrinsic geometry allows.

For instance, if $(M^m, g)$ is non-compact but complete, one might aim for the existence of a proper isometric immersion (or embedding) $F : M^m \to \mathbb{R}^n$, for some $n > m$. A more refined problem, albeit vaguely stated, would be to produce a proper isometric embedding whose behavior at infinity is as tame as possible.

A suitable testing ground for these ideas is $\mathbb{H}^m$, the complete simply-connected space of constant sectional curvature $-1$. For reasons that were not clear, it has been rather difficult to realize the Nash theorem in the case of $\mathbb{H}^m$, namely to produce explicit isometric embeddings of $\mathbb{H}^m$ into some Euclidean space.

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One such result is due to Henke and Nettekoven [10], where $\mathbb{H}^m$ is properly isometrically embedded in $\mathbb{R}^{6m-6}$ as a smooth complete graph over an $m$-dimensional subspace (see also [1], [9]). How can the existence of such a proper isometric embedding be reconciled with the fact that hyperbolic space $\mathbb{H}^m$ is much larger at infinity than any Euclidean space? What is the role of the codimension?

The main finding of this paper is that, in order to accommodate the different orders of growth at infinity, any isometric embedding $F : \mathbb{H}^m \to \mathbb{R}^n$ - regardless of regularity, dimension or codimension - must exhibit a high degree of asymptotic complexity, which is expressed in a precise quantitative fashion.

In fact, as we shall see below, our arguments can be implemented in the broader context of Lipschitz maps $M^m \to \mathbb{R}^n$, where $M^m$ is a Hadamard manifold with curvature bounded away from zero.

Before stating the main result, we explain its geometric meaning in an informal way:

(†) For every $r > 0$ and integer $k \geq 2$ there exist $k$ geodesic balls of radius $r$ in $\mathbb{H}^m$ that are arbitrarily far apart, but whose images under $F$ are arbitrarily Hausdorff-close in $\mathbb{R}^n$.

When $F$ is the lift to $\mathbb{H}^m$ of an isometric immersion of a compact hyperbolic manifold and $r$ is large enough, the images of the balls actually coincide with $F(\mathbb{H}^m)$. If $F$ is a proper isometric embedding, as in [10], and $r$, $k$ are arbitrary but fixed, one obtains from (†) the following “dynamical” picture, that helps in the visualization of the embedding:

(‡†) There is a sequence of configurations of $k$ balls of radius $r$ in $\mathbb{H}^m$, the distance between any two balls in a configuration going to infinity, such that the isometric images under $F$ of the $k$ balls form a sequence of “stacks” in $\mathbb{R}^n$, each one with $k$ “layers”, with the property that the stacks tend to infinity in $\mathbb{R}^n$ while their thickness tends to zero.

Hence, despite the fact that $\mathbb{H}^m$ is a simple space, any isometric embedding $\mathbb{H}^m \hookrightarrow \mathbb{R}^n$, proper or not, must be rather complex, as the submanifold in $\mathbb{R}^n$ fails to stabilize at infinity.

The phenomenon described above helps to explain why isometric embeddings of hyperbolic spaces into Euclidean spaces are so hard to produce explicitly, since any candidate for an isometric embedding must exhibit a priori a specific complex asymptotic behavior.

There are simple models of hyperbolic geometry that retain some features of Euclidean geometry, for instance the classical models of Poincaré, Lobachevsky, Minkowski-Lorentz and Cayley-Klein. On the other hand, our results reveal that, despite Nash’s theorem, there is no truly simple way to actually realize hyperbolic spaces isometrically inside any Euclidean space, no matter how high the codimension is allowed to be.

We observe that matters are much simpler when the roles of the spaces are reversed. For instance, horospheres provide well-behaved examples of isometric embeddings $\mathbb{R}^n \hookrightarrow \mathbb{H}^m$, $m = n + 1$.

As mentioned before, our main result holds for maps that are more general than isometric immersions $\mathbb{H}^m \to \mathbb{R}^n$. Its formal statement reads as follows:
Theorem 1.1. Let \( m \geq 2, n \geq 1 \) be integers, \( M^m \) a Hadamard manifold whose curvature is bounded above by a negative constant, and \( F : M^m \rightarrow \mathbb{R}^n \) a Lipschitz map. Then, for every \( r > 0, \epsilon \in (0,1) \) and integer \( k \geq 2 \), there are points \( p_1, \ldots, p_k \in M^m \) for which the geodesic balls \( B(p_i, r) \) satisfy, for all distinct \( i, j \in \{1, \ldots, k\} \):

i) The Riemannian distance between \( B(p_i, r) \) and \( B(p_j, r) \) is at least \( \epsilon^{-1} \).

ii) The Euclidean distance between \( F(B(p_i, r)) \) and \( F(B(p_j, r)) \) is at most \( \epsilon \).

If the curvature of \( M^m \) lies between negative constants, then i) and iii) below hold:

iii) The Hausdorff distance between \( F(B(p_i, r)) \) and \( F(B(p_j, r)) \) is at most \( \epsilon \).

The proof of Theorem 1.1 to be given in the next section, is based on a careful study of the interplay between the asymptotic growth of some specially defined packings of geodesic balls in the strongly curved Hadamard manifold \( M^m \), and the massive compression they must undergo under the action of a Lipschitz map that takes values in some Euclidean space.

The conceptual remarks below are meant to shed some light on Theorem 1.1 vis-a-vis the nature of the known examples of isometric immersions of the known examples of isometric immersions of flat Euclidean spaces, for instance the totally geodesic ones, are strongly proper. But observe that the graph \( S \) of the
example above is flat, proper, but not strongly proper. Isometric immersions of hyperbolic spaces into $\mathbb{R}^n$, on the other hand, behave quite differently. Although [10] provides examples of proper isometric embeddings $\mathbb{H}^m \hookrightarrow \mathbb{R}^{6m-6}$, Corollary 1.2 implies:

**Corollary 1.3.** There are examples of proper isometric immersions $F : \mathbb{H}^m \to \mathbb{R}^n$, but no such $F$ can be strongly proper.

Needless to say, the full force of Theorem 1.1 provides much more information about isometric immersions $\mathbb{H}^m \to \mathbb{R}^n$ than Corollary 1.3.

For the sake of completeness, we mention the well-known problem that $\mathbb{H}^m$ cannot be $C^2$ isometrically immersed in $\mathbb{R}^{2m-1}$, although this conjecture is not the focus of the present work (indeed, our results are valid in arbitrary codimension). For background on this problem, as well as related works, see [4], [5], [11], [14], [16] - [20].

We would like to stress that since Theorem 1.1 holds for Lipschitz functions, it is conceivable that it may be of use in other problems in geometric analysis, besides isometric immersions.

Given the somewhat general nature of our arguments, we expect Theorem 1.1 to admit formulations in other settings as well, provided that there is a notion of hyperbolicity that can be played against the idea of polynomial growth.

In conclusion, despite the fact that the Nash theorem solves the isometric embedding problem, matters are inherently more involved from the standpoint of the geometry of submanifolds. Indeed, as it will be seen in this paper, in some cases there are global obstructions at work that preclude the existence of isometric embeddings with tame asymptotic behavior, regardless of the codimension.

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## 2 Lipschitz maps and asymptotic densities.

This section contains the proof of Theorem 1.1 presented after some preparatory material. Given a complete non-compact $m$-dimensional Riemannian manifold $M^m$, $R \in (0, \infty)$, $C \in (0, R)$, and $p \in M^m$, denote by $\#(p, C, R; M^m)$ the maximum number of disjoint metric balls of radius $C$ that are contained in the open ball $B(p, R)$.

**Lemma 2.1.** Let $M^m$ be a Hadamard manifold with curvature bounded away from zero. Then, for all $p \in M^m$ and $C > 0$, $\#(p, C, R; M^m)$ grows exponentially as $R \to \infty$.

In the special case $M^m = \mathbb{H}^m$, a non-computational proof of this result can be given using the fact that there are compact hyperbolic manifolds $P^m$ with an arbitrarily large injectivity radius, together with the exponential growth of the fundamental group of $P^m$. Since we were unable to locate a reference for Lemma 2.1 in the case of variable curvature, a detailed proof will be provided.

The following standard result ([2], [3]) will be used in Lemmas 2.1 and 2.4.
Lemma 2.2. Let $M^n$ and $\tilde{M}^n$ be Riemannian manifolds and suppose that $\tilde{K}_\sigma(\sigma) \geq K_\sigma(\sigma)$, for all $x \in \tilde{M}$, $\tilde{\sigma} \in T_\sigma \tilde{M}$, $\sigma \in T_\theta M$. Let $p \in M$, $\tilde{p} \in \tilde{M}$ and fix a linear isometry $\iota : T_p M \to T_{\tilde{p}} \tilde{M}$. Let $r > 0$ such that $\exp_p|_{B_r(0)}$ is a diffeomorphism and $\exp_p|_{\tilde{B}_r(0)}$ is non-singular. Let $c : [0, a] \to \exp_p(B_r(0)) \subset M$ be smooth and define $\tilde{c} : [0, a] \to \exp_{\tilde{p}}(\tilde{B}_r(0)) \subset \tilde{M}$ by $\tilde{c}(s) = \exp_{\tilde{p}} \circ \iota \circ \exp_p^{-1}(c(s))$, $s \in [0, a]$. Then $l(c) \geq l(\tilde{c})$.

In order to prove Lemma 2.1 we may assume, without loss of generality, that the sectional curvature $K$ of $M^n$ is at most $-1$. As before, denote by $\mathbb{H}^m$ the $m$-dimensional hyperbolic space and fix $p \in M^n$, $\tilde{p} \in \mathbb{H}^m$. Since $\exp_p : T_p M \to M$, $\exp_{\tilde{p}} : T_{\tilde{p}} \mathbb{H}^m \to \mathbb{H}^m$ are diffeomorphisms, for any fixed linear isometry $\iota : T_p M \to T_{\tilde{p}} \mathbb{H}^m$ the map $\phi := \exp_{\tilde{p}} \circ \iota \circ \exp_p^{-1} : M^n \to \mathbb{H}^m$ is also a diffeomorphism. Moreover, by Lemma 2.2,

$$d_M(x, y) \geq d_{\mathbb{H}^m}(\phi(x), \phi(y)), \quad x, y \in M^n. \quad (2.1)$$

Our strategy will be to identify a suitable two-dimensional surface with the property that the maximum number of disjoint balls of radius $C$ that are contained in $B(p, R)$, and whose centers lie in the said surface, already grows exponentially as $R \to \infty$.

Given $R > 2C$, let $\alpha \in (0, \pi/2)$ be such that

$$\sin \alpha = \frac{\sinh C}{\sinh (R - C)}. \quad (2.2)$$

Let $k$ be largest positive integer that satisfies $k\alpha \leq \pi - \alpha$, so that

$$k > \frac{\pi - \alpha}{\alpha} - 1. \quad (2.3)$$

Let $u, w \in T_p M$ be orthogonal unit vectors and, for each integer $0 \leq j \leq k$, set

$$v_j = \cos(2ja)u + \sin(2ja)w.$$

We claim that, for all $0 \leq i < j \leq k$, the smallest angle $\angle(v_i, v_j)$ between $v_i$ and $v_j$ is at least $2\alpha$. In fact, taking the inner product of $v_i$ and $v_j$ we obtain

$$\cos \angle(v_i, v_j) = \cos(2ja - 2ia) = \cos (2\pi - (2ja - 2ia)).$$

If $2ja - 2ia \leq \pi$, then $\angle(v_i, v_j) = (j - i)2\alpha \geq 2\alpha$. On the other hand, if $2ja - 2ia > \pi$ we have

$$\angle(v_i, v_j) = 2\pi - (j - i)2\alpha \geq 2\pi - 2k\alpha.$$

That $\angle(v_i, v_j) \geq 2\alpha$ is valid also in this case is an immediate consequence of the above inequality and our choice of $k$.

For $0 \leq j \leq k$, consider the geodesic in $M$ given by $\gamma_j(t) = \exp_p(tv_j)$, and let $p_j = \gamma_j(R - C)$.

Lemma 2.3. $d_M(p_i, p_j) \geq 2C$ for all distinct $i, j$ in $\{0, 1, \ldots, k\}$. 

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To prove Lemma 2.3, for each $j$ such that $0 \leq j \leq k$ consider the geodesic in $\mathbb{H}^m$ defined by $\tilde{\gamma}_j(t) = \exp_{\tilde{p}}(t\iota(v_j))$, and let $\tilde{p}_j = \tilde{\gamma}_j(R - C)$. Since $\phi(p_j) = \tilde{p}_j$, it follows from (2.1) that

$$d_M(p_i, p_j) \geq d_{\mathbb{H}^m}(\tilde{p}_i, \tilde{p}_j), \quad i, j \in \{0, 1, ..., k\}, \quad i \neq j. \quad (2.4)$$

We can assume that $\angle(v_i, v_j)$ is strictly less than $\pi$, otherwise $\tilde{\gamma}_i$ and $\tilde{\gamma}_j$ would be opposite geodesics, and so

$$d_{\mathbb{H}^m}(\tilde{p}_i, \tilde{p}_j) = 2(R - C) > 2C.$$

At this point in the proof we need to invoke a classical formula in hyperbolic trigonometry, the so-called hyperbolic law of sines [13, p. 432]. This formula states that if a triangle in the hyperbolic plane has sides of lengths $a, b, c$, and the corresponding opposite angles have measures $\lambda, \mu, \nu$, then

$$\frac{\sinh a}{\sin \lambda} = \frac{\sinh b}{\sin \mu} = \frac{\sinh c}{\sin \nu}.$$

Let $\tilde{q}$ be the midpoint of the segment $\tilde{p}_i\tilde{p}_j$, so that the triangle $\tilde{p}\tilde{q}\tilde{p}_i$ has a right angle at $\tilde{q}$ and an angle of measure $\frac{1}{2}\angle(v_i, v_j)$ at $\tilde{p}$. Applying the hyperbolic law of sines to the triangle $\tilde{p}\tilde{q}\tilde{p}_i$, one obtains

$$\sinh \left( d_{\mathbb{H}^m}(\tilde{q}, \tilde{p}_i) \right) = \sinh(R - C) \sin \left( \frac{\angle(v_i, v_j)}{2} \right). \quad (2.5)$$

Since, by the previous claim, $\angle(v_i, v_j) \geq 2\alpha$, it follows from (2.2) and (2.5) that

$$\sinh \left( d_{\mathbb{H}^m}(\tilde{q}, \tilde{p}_i) \right) \geq \sinh(R - C) \sin \alpha = \sinh C. \quad (2.6)$$

Lemma 2.3 now follows from (2.4) and (2.6):

$$d_M(p_i, p_j) \geq d_{\mathbb{H}^m}(\tilde{p}_i, \tilde{p}_j) = 2d_{\mathbb{H}^m}(\tilde{q}, \tilde{p}_i) \geq 2C. \quad (2.7)$$

We now resume the proof of Lemma 2.1. As $d_M(p_i, p_j) = R - C$, it follows from the triangle inequality that $B(p_j, C) \subset B(p, R)$ for $0 \leq j \leq k$. Moreover, by Lemma 2.3 $B(p_i, C) \cap B(p_j, C) = \emptyset$ for all distinct $i, j \in \{0, 1, ..., k\}$. From (2.2) and (2.3), we then obtain

$$\#(p, C, R; M^m) \geq k + 1 > \frac{\pi - \alpha}{\alpha} \geq \frac{\pi - \alpha}{\alpha} = \frac{\sin \alpha}{\alpha} \frac{\pi - \alpha}{\sin \alpha} \geq \frac{1}{\alpha} \frac{\pi - \alpha}{\sinh C} \sinh(R - C) > \frac{1}{2} \frac{\pi}{\alpha} \frac{\pi}{\sinh C} \sinh(R - C). \quad (2.8)$$

Since, by (2.2), $\alpha \to 0$ as $R \to \infty$, it follows from (2.8) that $\#(p, C, R; M^m)$ grows exponentially with $R$, for every $C > 0$ fixed. This concludes the proof of Lemma 2.1. □
Lemma 2.4. Let $M^m$ be a Hadamard manifold whose sectional curvature $K$ is bounded from below. Then, for all $\rho > 0$ and $\delta > 0$ there exist a positive integer $l = l(\rho, \delta)$ and maps $\sigma_1, ..., \sigma_l : M \to M$ such that, for all $p \in M$,

i) $\sigma_j(p) \in B(p, \rho)$, \hspace{0.5cm} 1 \leq j \leq l,

ii) $B(p, \rho) \subset \bigcup_{j=1}^l B(\sigma_j(p), \delta)$.

To prove the lemma, set $b = \inf_M K > -\infty$ and let $\tilde{M}$ be the complete simply-connected $m$-dimensional Riemannian manifold with constant sectional curvature $b$. Fix $\tilde{q} \in \tilde{M}$ and let $\tilde{q}_1, ..., \tilde{q}_l \in B(\tilde{q}, \rho)$ be such that

$$B(\tilde{q}, \rho) \subset \bigcup_{j=1}^l B(\tilde{q}_j, \delta).$$  \hspace{1cm} (2.9)

For $q \in M$ fixed, consider orthonormal bases $\{\tilde{v}_1, ..., \tilde{v}_m\}$ and $\{v_1, ..., v_m\}$ of $T_{\tilde{q}}\tilde{M}$ and $T_qM$, respectively. For each $p \in M$, $p \neq q$, let $\{V_1(p), ..., V_m(p)\}$ be the (orthonormal) basis of $T_pM$ obtained by the parallel transport of $v_1, ..., v_m$ along the (unique) geodesic joining $q$ to $p$. Consider also the linear isometry $\iota_p : T_{\tilde{q}}\tilde{M} \to T_pM$ satisfying $\iota_p(\tilde{v}_i) = V_i(p)$, and define a diffeomorphism $\phi_p : \tilde{M} \to M$ by $\phi_p = \exp_p \circ \iota_p \circ \exp^{-1}_{\tilde{q}}$.

For all $p \in M$ and $j \in \{1, ..., l\}$, set $\sigma_j(p) = \phi_p(\tilde{q}_j) \in M$. Since $\phi_p(B(\tilde{q}, \rho)) = B(p, \rho)$, we have $\sigma_j(p) \in B(p, \rho)$ whenever $1 \leq j \leq l$. Given $x \in B(p, \rho)$, we obtain from (2.9) that $\phi_p^{-1}(x) \in B(\tilde{q}_j, \delta)$ for some $j$, $1 \leq j \leq l$. Applying Lemma 2.2 with the roles of $M$ and $\tilde{M}$ interchanged, one has

$$d_{\tilde{M}}(\phi_p^{-1}(x), \phi_p^{-1}(y)) \geq d_{\tilde{q}}(\tilde{q}_j, \tilde{q}_j),$$

and so $d_M(x, \sigma_j(p)) \leq d_{\tilde{M}}(\phi_p^{-1}(x), \tilde{q}_j) < \delta$, which establishes ii). \hspace{1cm} $\Box$

With these preliminaries out of the way, we are ready to begin the proof of Theorem 1.1. To this end, fix $p_0 \in M^m$ and let $C$ be a positive number to be specified later. For each $R > C$, take a collection $\hat{C}_{R,C}$ of disjoint balls of radius $C$ inside the ball $B(p_0, R) \subset M^m$ such that $|\hat{C}_{R,C}| = \#(p_0, C, R; M^m)$.

According to Lemma 2.1, the cardinality $|\hat{C}_{R,C}|$ grows exponentially as $R \to \infty$. In particular, one has

$$\lim_{R \to \infty} \frac{|\hat{C}_{R,C}|}{R^a} = \infty. \hspace{1cm} (2.10)$$

Consider, for each $R > C$, a subfamily $\mathcal{C}_{R,C}$ of $\hat{C}_{R,C}$ satisfying:

a) If $|\mathcal{C}_{R,C}| > 1$ and $p, q$ are centers of distinct balls in $\mathcal{C}_{R,C}$, then $||F(p) - F(q)|| \geq \frac{1}{n}$.

b) $\mathcal{C}_{R,C}$ is maximal with respect to property a).
Writing $D(q,t)$ for the Euclidean ball in $\mathbb{R}^n$ of radius $t$ and center $q$, we observe that if $|\mathcal{C}_{R,C}| > 1$ and $B(q_i, C), B(q_j, C)$ are distinct balls in $\mathcal{C}_{R,C}$, then

$$D \left( F(q_i), \frac{1}{3C} \right) \cap D \left( F(q_j), \frac{1}{3C} \right) = \emptyset. \quad (2.11)$$

Indeed, if (2.11) were to fail, the triangle inequality would imply $||F(q_i) - F(q_j)|| < \frac{2}{3C}$, contradicting a) above.

Denote by $L$ the Lipschitz constant of $F$. Since

$$||F(q_i) - F(p_0)|| \leq Ld(q_i, p_0) < LR$$

we have, for all $x \in D\left(F(q_i), 1/3C\right)$,

$$||x - F(p_0)|| \leq ||x - F(q_i)|| + ||F(q_i) - F(p_0)|| < \frac{1}{3C} + LR.$$

As a consequence,

$$\bigcup_{q_i} D \left( F(q_i), \frac{1}{3C} \right) \subset D \left( F(p_0), LR + \frac{1}{3C} \right), \quad (2.12)$$

where $q_i$ runs over the centers of all balls in $\mathcal{C}_{R,C}$.

An individual ball $D(q,t)$ in $\mathbb{R}^n$ has volume $c(n)t^n$, the explicit value of the constant $c(n)$ being unimportant for our current purposes. The volume of each ball $D\left(F(q_i), 1/3C\right)$ is a fixed constant, say $\lambda_0$. There are $|\mathcal{C}_{R,C}|$ such balls in $\mathbb{R}^n$ and, as observed in (2.11), they are pairwise disjoint. Thus, the volume of the union in (2.12) is $\lambda_0|\mathcal{C}_{R,C}|$ and, furthermore,

$$\lambda_0|\mathcal{C}_{R,C}| \leq c(n)(LR + 1/3C)^n.$$

In particular,

$$\limsup_{R \to \infty} \frac{|\mathcal{C}_{R,C}|}{R^n} < \infty. \quad (2.13)$$

It follows from (2.10) and (2.13) that, for all sufficiently large $R$, say $R > R_0$, the inclusion $\mathcal{C}_{R,C} \subset \hat{\mathcal{C}}_{R,C}$ is proper.

Consider any ball $B(p,C)$ from $\hat{\mathcal{C}}_{R,C} - \mathcal{C}_{R,C}$. The family $\{B(p,C)\} \cup \mathcal{C}_{R,C}$ consists of disjoint balls of radius $C$ and so, by a) and the maximality of $\mathcal{C}_{R,C}$ that was stipulated in b), one can select a (not necessarily unique) ball $B(q,C)$ in $\mathcal{C}_{R,C}$ such that

$$||F(p) - F(q)|| < \frac{1}{C}. \quad (2.14)$$

Any such assignment $B(p,C) \leadsto B(q,C)$ gives rise to a map

$$\Theta_{R,C} : \hat{\mathcal{C}}_{R,C} - \mathcal{C}_{R,C} \to \mathcal{C}_{R,C}, \quad R > R_0.$$
We claim that when $R$ tends to infinity, the cardinality of some fiber of $\Theta_{R,C}$ becomes larger than any specified integer $j$.

Indeed, if not,
\[
|\hat{C}_{R,C}| = |C_{R,C}| + |\hat{C}_{R,C} - C_{R,C}|
\leq |C_{R,C}| + j|\Theta_{R,C}(\hat{C}_{R,C} - C_{R,C})|
\leq (1 + j)|C_{R,C}|
\]
contradicting (2.10) and (2.13).

Hence, by (2.14) and the previous assertion about the fibers of $\Theta_{R,C}$, there are points $q, p_1, \ldots, p_k \in M^m$ such that
\[
\min_{1 \leq i,j \leq k, i \neq j} d(p_i, p_j) \geq 2C \quad \text{and} \quad \max_{1 \leq i \leq k} ||F(p_i) - F(q)|| < \frac{1}{C}.
\]
In particular,
\[
\max_{1 \leq i,j \leq k} ||F(p_i) - F(p_j)|| < \frac{2}{C}.
\]
Choosing $C = 2(2r\epsilon + 1)/\epsilon$, we obtain (ii) in the statement of Theorem 1.1. To see that (i) is also valid with this choice of $C$, observe that, for all $x \in B(p_i, r)$ and $y \in B(p_j, r)$,
\[
d(x, y) \geq d(p_i, p_j) - 2r \geq 2C - 2r > \frac{2r\epsilon + 1}{\epsilon} - 2r = \frac{1}{\epsilon}.
\]

We now move on to the second half of the theorem, and assume that the sectional curvature of $M^m$ is bounded from above and below by negative constants.

By Lemma 2.4, with $\rho = r$ and $\delta = \frac{\epsilon}{2L}$, there exist a positive integer $l$ and maps $\sigma_1, \ldots, \sigma_l : M \to M$ such that, for all $p \in M$,
\[
\sigma_j(p) \in B(p, r), \quad 1 \leq j \leq l,
\]
\[
B(p, r) \subset \bigcup_{j=1}^{l} B(\sigma_j(p), \epsilon/2L).
\]

In order to control the Hausdorff distance, we introduce the following augmentation of the map $F$:
\[
\hat{F} : M^m \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n = \mathbb{R}^{nl}, \quad \hat{F}(p) = (F(\sigma_1(p)), \ldots, F(\sigma_l(p))).
\]

Fix $p_0 \in M^m$, and let $C$ be a positive number to be specified later. As before, for each $R > C$ denote by $\hat{C}_{R,C}$ a collection of disjoint balls of radius $C$ that are contained in the ball $B(p_0, R) \subset M^m$ and satisfy $|\hat{C}_{R,C}| = \#(p_0, C, R; M^m)$. 9
By Lemma 2.1, one has
\[
\lim_{R \to \infty} \frac{\tilde{C}_{R,C}}{R^{nl}} = \infty. \tag{2.18}
\]

Consider, for each \( R > C \), a subfamily \( \tilde{C}_{R,C} \) of \( \tilde{C}_{R,C} \) satisfying:

a) If \( |\tilde{C}_{R,C}| > 1 \) and \( p, q \) are centers of distinct balls in \( \tilde{C}_{R,C} \), then \( ||\hat{F}(p) - \hat{F}(q)|| \geq \frac{1}{C} \).

b) \( \tilde{C}_{R,C} \) is maximal with respect to property a).

From this point on, we write \( D(q, t) \) for the ball in \( R^{nl} \) of radius \( t \) and center \( q \). If \( |\tilde{C}_{R,C}| > 1 \) and \( B(q_i, C), B(q_j, C) \) are distinct balls in \( \tilde{C}_{R,C} \) then, by a) above,
\[
D\left( \hat{F}(q_i), \frac{1}{3C} \right) \cap D\left( \hat{F}(q_j), \frac{1}{3C} \right) = \emptyset. \tag{2.19}
\]

From
\[
||F(\sigma_s(p)) - F(\sigma_s(p_0))|| \leq Ld(\sigma_s(p), \sigma_s(p_0)) \leq L[d(\sigma_s(p), p) + d(p, p_0) + d(p_0, \sigma_s(p_0))]
\]
and (2.15), one obtains
\[
||F(\sigma_s(p)) - F(\sigma_s(p_0))|| \leq L(d(p, p_0) + 2r),
\]
\[
||\hat{F}(p) - \hat{F}(p_0)|| \leq \sum_{s=1}^{l} ||F(\sigma_s(p)) - F(\sigma_s(p_0))||^2 < lL^2(d(p, p_0) + 2r)^2.
\]

Then, for all \( x \in D(\hat{F}(q_i), 1/3C) \) we have
\[
||x - \hat{F}(p_0)|| \leq ||x - \hat{F}(q_i)|| + ||\hat{F}(q_i) - \hat{F}(p_0)|| < \frac{1}{3C} + \sqrt{lL}(R + 2r).
\]

As a consequence,
\[
\bigcup_{q_i} D\left( \hat{F}(q_i), \frac{1}{3C} \right) \subset D\left( \hat{F}(p_0), \sqrt{lL}(R + 2r) + \frac{1}{3C} \right), \tag{2.20}
\]
where \( q_i \) runs over the centers of all balls in \( \tilde{C}_{R,C} \).

Denoting by \( \lambda \) the volume of each ball \( D(\hat{F}(q_i), 1/3C) \), it follows from (2.19) and (2.20) that
\[
\lambda|\tilde{C}_{R,C}| \leq c(nl)(\sqrt{lL}(R + 2r) + 1/3C)^{nl},
\]
where \( c(nl) \) is the volume of the unit ball in \( R^{nl} \). In particular,
\[
\limsup_{R \to \infty} \frac{|\tilde{C}_{R,C}|}{R^{nl}} < \infty. \tag{2.21}
\]
It follows from (2.18) and (2.21) that, for all sufficiently large $R$, say $R > R_0$, the inclusion $\tilde{C}_{R,C} \subset \hat{C}_{R,C}$ is proper. Hence, by a) and the maximality of $\tilde{C}_{R,C}$ that was stipulated in b), for any ball $B(p, C)$ from $\hat{C}_{R,C} - \tilde{C}_{R,C}$ one can select a (not necessarily unique) ball $B(q, C)$ in $\tilde{C}_{R,C}$ such that
\[
||\hat{F}(p) - \hat{F}(q)|| < \frac{1}{C}. \tag{2.22}
\]
Any such assignment $B(p, C) \sim B(q, C)$ gives rise to a map
\[
\tilde{\Theta}_{R,C} : \hat{C}_{R,C} - \tilde{C}_{R,C} \to \tilde{C}_{R,C}, \quad R > R_0.
\]
If all fibers of $\tilde{\Theta}_{R,C}$ had size at most a fixed integer $j$, an argument similar to the discussion following (2.14) would show that $|\hat{C}_{R,C}| \leq (1 + j)\tilde{C}_{R,C}$, contradicting (2.18) and (2.21).

Hence, by (2.22) there are points $q, p_1, \ldots, p_k \in M$ for which
\[
\min_{1 \leq i, j \leq k, i \neq j} d(p_i, p_j) \geq 2C \quad \text{and} \quad \max_{1 \leq i \leq k} ||\hat{F}(p_i) - \hat{F}(q)|| < \frac{1}{C}.
\]
In particular,
\[
\max_{1 \leq i, j \leq k} ||\hat{F}(p_i) - \hat{F}(p_j)|| < \frac{2}{C}. \tag{2.23}
\]
For $x \in B(p_i, r)$ and $y \in B(p_j, r)$, one has
\[
d(x, y) \geq d(p_i, p_j) - 2r \geq 2C - 2r,
\]
and so, under the hypothesis that the curvature lies between negative constants, Theorem 1.1 i) follows by choosing any $C$ satisfying
\[
C \geq \frac{2r\epsilon + 1}{2\epsilon}.
\]
Given $x \in B(p_i, r)$, (2.16) implies that there is $s \in \{1, \ldots, l\}$ with $d(x, \sigma_s(p_i)) < \epsilon/2L$, so that, by the Lipschitz condition,
\[
||F(x) - F(\sigma_s(p_i))|| < \frac{\epsilon}{2}.
\]
Since, by (2.17) and (2.23),
\[
||F(\sigma_s(p_i)) - F(\sigma_s(p_j))|| \leq ||\hat{F}(p_i) - \hat{F}(p_j)|| < \frac{2}{C},
\]
we have
\[
||F(x) - F(\sigma_s(p_j))|| < \frac{\epsilon}{2} + \frac{2}{C}. \tag{2.24}
\]
Finally, choosing
\[
C \geq \max \left\{ \frac{2r\epsilon + 1}{2\epsilon}, \frac{4}{\epsilon} \right\}
\]
one sees from (2.24) that $F(x)$ lies in the $\epsilon$-neighborhood of the set $F(B(p_j, r))$. As $x \in B(p_i, r)$ is arbitrary, $F(B(p_i, r))$ is contained in the $\epsilon$-neighborhood of $F(B(p_j, r))$. Theorem 1.1 iii) now follows by reversing the roles of $i$ and $j$ in the argument above.
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