PRESENTATIONS FOR THE COHOMOLOGY RINGS OF TREE Braid Groups

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Abstract. If \( \Gamma \) is a finite graph and \( n \) is a natural number, then \( U^n_G \), the unlabelled configuration space of \( n \) points on \( \Gamma \), is the space of all \( n \)-element subsets of \( \Gamma \). The fundamental group of \( U^n_G \) is the \( n \)-strand braid group of \( \Gamma \), denoted \( B_n \).

We build on earlier work to compute presentations of the integral cohomology rings \( H^*(U^n_T; \mathbb{Z}) \), where \( T \) is any tree and \( n \) is arbitrary. The results suggest that \( H^*(U^n_T; \mathbb{Z}) \) is the exterior face ring of a simplicial complex.

1. Introduction

Fix a finite graph \( \Gamma \) and a natural number \( n \). The labelled configuration space of \( n \) points on \( \Gamma \), denoted \( C^n_G \), is the space

\[
\left( \prod_{i=1}^{n} \Gamma \right) - \Delta,
\]

where \( \Delta \) is the collection of ordered \( n \)-tuples \((x_1, \ldots, x_n)\) such that \( x_i = x_j \) for some \( i \neq j \). The unlabelled configuration space, denoted \( U^n_G \), is the quotient of \( C^n_G \) by the action of the symmetric group \( S_n \), where \( S_n \) permutes the factors. The fundamental groups of these spaces are, respectively, the pure braid group \( PB_n \) and the braid group \( B_n \) of \( n \) strands on the graph \( \Gamma \).

This paper continues the study of the braid groups \( B_n \) begun in [6]. Here we completely compute the cohomology ring \( H^*(U^n_T; \mathbb{Z}) \), where \( T \) is a tree. The spaces \( U^n_G \) are aspherical for any \( n \) and any \( \Gamma \) [9], so, as an immediate result, we also compute the cohomology ring \( H^*(B_n; \mathbb{Z}) \) of any tree braid group, i.e., any braid group \( B_n \), where \( T \) is a tree.

In [5], Lucas Sabalka and I used partial information about the cohomology rings \( H^*(B_n; \mathbb{Z}/2\mathbb{Z}) \) of tree braid groups in order to characterize those groups \( B_n \) that are right-angled Artin groups. The crucial idea in computing the ring structure is to introduce a “cup product complex” \( \overline{UD}^n_T \), which is obtained from a certain cubical complex \( UD^n_T \) (described in Subsection 2.1) by identifying all opposite faces of all cubes \( C \) in \( UD^n_T \). The main argument of [5]: i) showed that \( \overline{UD}^n_T \) is a subcomplex of a high-dimensional torus; ii) showed that the map \( g^* : H^*(\overline{UD}^n_T; \mathbb{Z}/2\mathbb{Z}) \to H^*(UD^n_T; \mathbb{Z}/2\mathbb{Z}) \) induced by the quotient map \( q : UD^n_T \to \overline{UD}^n_T \) is surjective, and iii) explicitly described \( q^* \). As a result, it was possible to compute \( a \cup b \) for many elements \( H^*(UD^n_T; \mathbb{Z}/2\mathbb{Z}) \).

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Here we achieve two improvements. First, we work with integral coefficients. Second, we give a complete set of generators for the ideal $\text{Ker} q^*$, where $q^*$ is the map on cohomology induced by the quotient map $q : U^D^n \to \hat{U}^D^n$. The argument, in outline, goes like this. The quotient map $q : U^D^n \to \hat{U}^D^n$ induces a map $q^* : C^*(\hat{U}^D^n; \mathbb{Z}) \to C^*(U^D^n; \mathbb{Z})$ on cochains. All boundary maps in $C_*(\hat{U}^D^n; \mathbb{Z})$ are 0, so cohomology classes in $H^*(\hat{U}^D^n; \mathbb{Z})$ are essentially the same as cochains in $C^*(\hat{U}^D^n; \mathbb{Z})$. The images of the standard basis for $C^*(\hat{U}^D^n; \mathbb{Z})$ under the map $q^*$ are thus cellular cocycles, which we call \textit{standard cocycles}. The standard cocycles can be easily visualized using what we call \textit{cloud pictures}. A certain type of standard cocycle is called \textit{critical}; these form a complete set of representatives for a basis of $H^*(U^D^n; \mathbb{Z})$. We describe an algorithm which takes a standard cocycle $\phi$ as input and gives a sum $\Sigma$ of critical cocycles as output, where $\Sigma$ and $\phi$ are cohomologous. Moreover, the difference $\Sigma - \phi$ is a sum of coboundaries from a certain family, which we describe quite explicitly using another type of picture. This easily gives the presentation promised by the title.

The paper concludes with the computation of several examples, which suggest that the cohomology rings $H^*(U^D^n; \mathbb{Z})$ are all exterior face rings (see Subsection 4.3 for a definition).

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2. General Background on Configuration Spaces of Graphs

2.1. Configuration spaces of graphs and graph braid groups. Let $\Gamma$ be a finite graph, and fix a natural number $n$. The \textit{labelled configuration space} of $n$ points on $\Gamma$, denoted $C^n\Gamma$, is the space

$$\left(\prod^n \Gamma\right) - \Delta,$$

where $\Delta$ is the set of all points $(x_1, \ldots, x_n) \in \prod^n \Gamma$ such that $x_i = x_j$ for some $i \neq j$. The \textit{unlabelled configuration space} of $n$ points on $\Gamma$, denoted $UC^n\Gamma$, is the quotient of the labelled configuration space by the action of the symmetric group $S_n$, where the action permutes the factors. The $n$-\textit{strand braid group} of $\Gamma$, denoted $B_n\Gamma$, is the fundamental group of the unlabelled configuration space of $n$ points on $\Gamma$.

It will be convenient to use a space somewhat smaller than $UC^n\Gamma$ in our applications. Let $\Delta'$ denote the union of those open cells of $\prod^n \Gamma$ whose closures intersect $\Delta$. Let $D^n\Gamma$ denote the space $(\prod^n \Gamma) - \Delta'$. Note that $D^n\Gamma$ inherits a CW complex structure from the Cartesian product, and that an open cell in $D^n\Gamma$ has the form $c_1 \times \cdots \times c_n$, where each $c_i$ is either a vertex or the interior of an edge, and the closures of the $c_i$ are mutually disjoint. Let $UD^n\Gamma$ denote the quotient of $D^n\Gamma$ by the action of the symmetric group $S_n$, which permutes the coordinates. An open cell in $UD^n\Gamma$ has the form $\{c_1, \ldots, c_n\}$, where each $c_i$ is either a vertex or the interior of an edge, and the closures of any two of the $c_i$ are disjoint. The set notation is used to indicate that order does not matter.

In many circumstances, $C^n\Gamma$ (respectively, $UC^n\Gamma$) is homotopy equivalent to $D^n\Gamma$ (respectively, $UD^n\Gamma$). Specifically:
Theorem 2.1. [1] For any \( n > 1 \) and any graph \( \Gamma \) with at least \( n \) vertices, the labelled (unlabelled) configuration space of \( n \) points on \( \Gamma \) strongly deformation retracts onto \( D^n \Gamma \) (\( UD^n \Gamma \)) if

1. each path between distinct vertices of degree not equal to 2 passes through at least \( n - 1 \) edges; and
2. each path from a vertex to itself which is not null-homotopic in \( \Gamma \) passes through at least \( n + 1 \) edges.

A graph \( \Gamma \) satisfying the conditions of this theorem for a given \( n \) is called sufficiently subdivided for this \( n \). It is clear that every graph is homeomorphic to a graph that is sufficiently subdivided for \( n \), no matter what \( n \) may be.

We will sometimes call a vertex of degree 3 or more essential.

Throughout the rest of the paper, we work exclusively with the space \( UD^n \Gamma \), where \( \Gamma \) is sufficiently subdivided for \( n \). Also, from now on, “edge” and “cell” will mean “closed edge” and “closed cell”, respectively.

2.2. An ordering on vertices of \( \Gamma \) and a classification of cells in \( UD^n \Gamma \).

Choose a maximal tree \( T \) in \( \Gamma \). Pick a vertex \( * \) of valence 1 in \( T \) to be the root of \( T \). Choose an embedding of the tree \( T \) into the plane. We define an order on the vertices of \( T \) (and, thus, on vertices of \( \Gamma \)) as follows. Begin at the basepoint \( * \) and walk along the tree, following the leftmost branch at any given intersection, and consecutively number the vertices in the order in which they are first encountered. (When you reach a vertex of degree one, turn around.) The vertex adjacent to \( * \) is assigned the number 1. See Figure 1. Note that this numbering depends only on the choice of \( * \) and the embedding of the tree.

Let \( \iota(e) \) and \( \tau(e) \) denote the endpoints of a given edge \( e \) of \( \Gamma \). We orient each edge to go from \( \iota(e) \) to \( \tau(e) \), and so that \( \iota(e) < \tau(e) \). (Thus, if \( e \subseteq T \), the
geodesic segment \( [\tau(e), *] \) in \( T \) must pass through \( \epsilon(e) \). Note that this convention for orientation is the opposite of that in [4] and [6].

We use the order on the vertices to classify the cells \( c \) of \( UD^n \Gamma \) as collapsible, critical, or redundant. These terms come from discrete Morse theory. Our arguments will use this theory only indirectly, so we omit a general introduction, which can be found in [8]. Strictly speaking, our cells are collapsible, critical, or redundant relative to a discrete gradient vector field \( W \) on \( UD^n T \), which depends only on the choices of embedding of \( T \) in the plane and basepoint *. A thorough account of the discrete gradient vector field \( W \) on \( UD^n T \) can be found in [6].

We require several definitions. Edges outside of \( T \) are called deleted edges. If \( c = \{c_1, \ldots, c_{n-1}, v\} \) and \( e \) is the unique edge in \( T \) such that \( \tau(e) = v \), then \( v \) is blocked with respect to \( c \) if \( v = * \) or \( \{c_1, \ldots, c_{n-1}, e\} \) is not a cell of \( UD^n \Gamma \), i.e., if \( c_i \cap e \neq \emptyset \) for some \( i \in \{1, \ldots, n - 1\} \). Otherwise, \( v \) is unblocked. If \( c = \{c_1, \ldots, c_{n-1}, e\} \), the edge \( e \) is \textit{not order-respecting} (with respect to \( c \)) if

\[
\begin{align*}
(1) & \text{ there is a vertex } v \text{ in } c \text{ such that} \\
& (a) \text{ } v \text{ is adjacent to } \iota(e), \text{ and} \\
& (b) \text{ } \iota(e) < v < \tau(e), \text{ or} \\
(2) & \text{ } e \text{ is a deleted edge.}
\end{align*}
\]

Otherwise, the edge \( e \) is \textit{order-respecting} with respect to \( c \).

It will often be useful to have another definition. If \( v \) is a vertex in the tree \( T \), we say that two vertices \( v_1 \) and \( v_2 \) lie in the same direction from \( v \) if the geodesics \( [v, v_1], [v, v_2] \subseteq T \) coincide in some neighborhood of \( v \). It follows that there are \( n \) directions from a vertex of degree \( n \) in \( T \). We number these directions 0, 1, 2, \ldots, \( n - 1 \), beginning with the direction represented by \( [v, *] \), which is numbered 0, and proceeding in clockwise order. We will sometimes write \( g(v_1, v_2) \) (where \( v_1 \neq v_2 \)) to refer to the direction from \( v_1 \) to \( v_2 \).

Suppose that we are given a cell \( c = \{c_1, \ldots, c_n\} \) in \( UD^n \Gamma \). Assign each cell \( c_i \) in \( c \) a number as follows. A vertex of \( c \) is given the number from the above traversal of \( T \). An edge \( e \) of \( c \) is given the number for \( \tau(e) \). Arrange the cells of \( c \) in a sequence \( S \), from the least- to the greatest-numbered. (Here the basepoint * is numbered 0.)

**Definition 2.2.** We classify each cell \( c \) of \( UD^n \Gamma \) as follows:

\[
\begin{align*}
(1) & \text{ If an unblocked vertex occurs in } S \text{ before all of the order-respecting edges} \\
& \text{ in } c \text{ (if any), then } c \text{ is redundant.} \\
(2) & \text{ If an order-respecting edge occurs before any unblocked vertex, then } c \text{ is} \\
& \text{ collapsible.} \\
(3) & \text{ If all vertices of } c \text{ are blocked and no edge of } c \text{ is order-respecting, then } c \text{ is} \\
& \text{ critical.}
\end{align*}
\]

(Of course, this definition is ultimately a theorem. See Theorem 3.6 of [6].)

**Example 2.3.** We give a short example to illustrate the previous definitions. Consider the ‘Y’-graph, which is homeomorphic to the capital letter \( Y \). We consider the discretized configuration space \( UD^3 Y \) for \( Y \) sufficiently subdivided. Figure 2 depicts three different closed 1-cells in \( UD^3 Y \).

In a), the vertices (large dots in the figure) are numbered 0 and 4, and the edge (represented by a thick line segment) is numbered 5. The vertex 0 is blocked. The vertex 4 is unblocked, so the cell in a) is redundant. Note that the edge is order-respecting.
Figure 2. These are three different 1-cells in $U^3D^3Y$. The cells are redundant, critical, and collapsible (respectively).

In b), the vertices are numbered 0 and 3, and the edge is again numbered 5. The vertex 0 is again blocked, and so is the vertex 3. The edge $e$, numbered 5, is not order-respecting, since the vertex numbered 3 lies between the initial and terminal vertices of $e$ (numbered 2 and 5, respectively), and vertex 3 is adjacent to vertex 2. It follows from this that the 1-cell in b) is critical.

In c), the vertices are numbered 0 and 5, and the edge is numbered 3. The vertex 0 is blocked. The edge $e$ is order-respecting, since $\iota(e)$ is numbered 2, and $\tau(e)$ is numbered 3, and thus it is impossible to satisfy condition (1b) in the above definition of order-respecting edges. (This particular edge must be order-respecting in any cell $c$ containing it.) It follows that the cell in c) is collapsible. Note that the vertex numbered 5 is blocked.

2.3. On the homology of tree braid groups. Discrete Morse theory (which is implicit in the previous subsection) allows one to compute the homology groups of a complex $X$ from a simplified chain complex $(M_i(X), \partial_i)$, called the Morse complex. A basis for the $i$th chain group $M_i(X)$ is given by the collection of critical $i$-cells (that is, critical relative to a discrete gradient vector field $W$ – see [8]). The boundary maps $\partial_i$ have natural definitions, but we won’t need them here.

In [4], we gave a computation of the integral homology groups of the spaces $U^D^nT$ in the form of the following theorem.

**Theorem 2.4.** [4] Let $(M_i(U^D^nT), \partial_i)$ be the Morse complex relative to the discrete gradient vector field $W$ (as defined in [6]). Each map $\partial_i$ is 0. In particular, the $i$th integral homology group of $U^D^nT$ is free abelian of rank $c(i)$, where $c(i)$ is the number of critical $i$-cells in $U^D^nT$.

In particular, we can simply identify $H_i(U^D^nT)$ with the free abelian group generated by critical $i$-cells, and we do this freely from now on. In some cases, we will nevertheless need to have a description of homology representatives in the standard cellular chain complex $(C_i(U^D^nT), \partial_i)$. The following Proposition will be sufficient.

**Proposition 2.5.** [7] Let $X$ be a finite regular CW complex endowed with a discrete gradient vector field $W$. The isomorphism between $H_i(M(X))$ and $H_i(C(X))$ is
Lemma 2.6. Subsection 2.2, we get the following lemma.

Lemma 2.7. Suppose that \( \mathcal{C} \) is a chain complex for \( X \) with \( n \) elements. In a general cubical complex \( X \) each cube \( c \) must be given an orientation by specifying an identification \( \phi : [0,1]^k \to c \). The boundary \( \partial(c) \) in the cellular chain complex for \( Y \) may then be computed from the above formula, using suitable identifications, as determined by \( \phi \). With the conventions adopted at the end of Subsection 2.2, we get the following lemma.

2.4. The cellular boundary maps in the cubical complexes \( U^m T \). Consider the standard \( n \)-cube \( [0,1]^n \). The standard orientation for this standard \( n \)-cube is a certain choice of orthogonal frame, namely the ordered \( n \)-tuple \( (v_1, \ldots, v_n) \), where \( v_i \) is the \( i \)-th dimensional vector which is 0 in every coordinate except for the \( i \)-th one, where it is 1.

The boundary map \( \partial : C_n([0,1]^n) \to C_{n-1}([0,1]^n) \) can be described as follows [2]:

\[
\partial([0,1]^n) = \sum_{i=1}^{n} (-1)^n (A_i - B_i),
\]

where \( A_i \) is the \((n-1)\)-dimensional cube \([0,1] \times \cdots \times \{0\} \times \{1\} \) in which the factor \( \{0\} \) occurs in the \( i \)-th place. The \((n-1)\)-dimensional cube \( B_i \) is the same, except that the \( i \)-th factor is \( \{1\} \) instead of \( \{0\} \). All of the cubes \( A_i, B_i \) are given the standard orientations \((v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)\).

In a general cubical complex \( X \) each cube \( c \) must be given an orientation by specifying an identification \( \phi : [0,1]^k \to c \). The boundary \( \partial(c) \) in the cellular chain complex for \( X \) may then be computed from the above formula, using suitable identifications, as determined by \( \phi \). With the conventions adopted at the end of Subsection 2.2, we get the following lemma.

Lemma 2.6. Let \( c = \{e_1, e_2, \ldots, e_k, v_{k+1}, \ldots, v_n\} \) be a \( k \)-cell in \( U^m T \), where \( \phi(e_1) < \phi(e_2) < \phi(e_3) < \ldots < \phi(e_k) \), and \( T \) is a tree. Suppose that all cells of \( U^m T \) are given their standard orientations (as in (2.2)).

\[
\partial(c) = \sum_{i=1}^{k} (-1)^{i} (c_{\phi(e_i)} - c_{\phi(e_i)}),
\]

where \( c_{\phi(e_i)} \) is the \((k-1)\)-cube obtained from \( c \) by replacing the edge \( e_i \) with its initial vertex \( \phi(e_i) \). The \((k-1)\)-cube \( c_{\phi(e_i)} \) is defined analogously, and both are given their standard orientations.

Proof. This is a matter of checking definitions and the consistency of the orientations. \( \square \)

The following lemma, an immediate consequence of the previous one, is true because of our very careful choice of orientations on the cubes of \( U^m T \).

Lemma 2.7. Suppose that \( c = \{e_1, \ldots, e_{i-1}, e_i, e_{i+1}, \ldots, e_k, v_{k+1}, \ldots, v_n\} \), \( c' = \{e_1, \ldots, e_{i-1}, e'_i, e_{i+1}, \ldots, e_k, v_{k+1}, \ldots, v_n\} \), \( \phi(e_1) < \phi(e_2) < \phi(e_3) < \ldots < \phi(e_k) \), and \( \phi(e_i) = \phi(e'_i) \). The cell \( c_{\phi(e_i)} = c'_{\phi(e'_i)} \) occurs in both of the sums \( \partial(c) \) and \( \partial(c') \).
and with the same signs. The cells \( c_\tau(e_i), c'_\tau(e'_i) \) occur in the sums \( \partial(c) \) and \( \partial(c') \) (respectively), and with the same signs.

\[ \square \]

3. Background on the Cohomology Rings of Tree Braid Groups

3.1. The cup product complex \( \hat{U}D^nT \). In \([5]\), we introduced a complex \( \hat{U}D^nT \) as an aid to computing cup products in \( H^*(\hat{U}D^nT) \). Here we summarize the definition of this complex and collect a few useful facts. We emphasize the standing assumption that \( T \) is a tree, since certain of the following propositions (3.1, for instance) aren’t necessarily true for general graphs.

If \( c = \{c_1, c_2, \ldots, c_n\} \) and \( c' = \{c'_1, c'_2, \ldots, c'_n\} \) are cells of \( \hat{U}D^nT \), write \( c \sim c' \) if:

1. \( c \) and \( c' \) have the same collections of edges, and
2. if \( \{e_1, \ldots, e_k\} \) and \( \{e'_1, \ldots, e'_k\} \) are the edges in \( c \) and \( c' \) (respectively) then each component of \( T - (e_1 \cup e_2 \cup \ldots \cup e_k) = T - (e'_1 \cup e'_2 \cup \ldots \cup e'_k) \) contains the same number of vertices from \( c \) as from \( c' \).

It is obvious that \( \sim \) is an equivalence relation. We let \( [c] \) denote the equivalence class of \( c \).

There is a natural partial order \( \leq \) defined on equivalence classes. Write \( [c] \leq [c'] \) if there exist representatives \( c \in [c], c' \in [c'] \) (without loss of generality) such that \( c \) is a face of \( c' \) in the complex \( \hat{U}D^nT \). In other words, \( [c] \leq [c'] \) if there is a collection \( \{e_{i_1}', e_{i_2}', \ldots, e_{i_k}'\} \subseteq c' \) such that \( c \) may be obtained from \( c' \) by replacing the edges \( e_{i_1}', e_{i_2}', \ldots, e_{i_k}' \) in \( c' \) with some combination of their initial and terminal vertices.

Proposition 3.1. \([5]\) The relation \( \leq \) is a well-defined partial order with the property that any collection \( \{[c_1], \ldots, [c_m]\} \) having an upper bound also has a least upper bound.

\[ \square \]

Proposition 3.2. \([5]\) If \( \hat{c} \) is a \( j \)-cell in \( \hat{U}D^nT \), then there is a unique collection \( \{[c_1], \ldots, [c_j]\} \) of equivalence classes of \( 1 \)-cells such that \( [\hat{c}] \) is the least upper bound of \( \{[c_1], \ldots, [c_j]\} \).

Indeed, \( \{[c_1], \ldots, [c_j]\} = \{[c'] \mid [c'] \leq [\hat{c}]; \, \dim(c') = 1\} \).

\[ \square \]

These propositions establish a natural one-to-one correspondence between equivalence classes of \( j \)-cells and \( j \)-element collections of equivalence classes of \( 1 \)-cells having an upper bound.

For each equivalence class \( [c] \) of \( 1 \)-cells, introduce a copy of \( S^1 \), denoted \( S^1_{[c]} \), where \( S^1_{[c]} \) is given the CW structure having a single open \( 1 \)-cell \( e^1_{[c]} \) and a single \( 0 \)-cell. We choose an orientation for each \( 1 \)-cell \( e^1_{[c]} \) (or, in other words, an equivalence class of characteristic map \( h_{[c]} : [0, 1] \to e^1_{[c]} \)). The complex \( \hat{U}D^nT \) is the subcomplex of \( \prod_{[c]} S^1_{[c]} \) obtained by throwing out all open cells of the form \( e^1_{[c_1]} \times \ldots \times e^1_{[c_k]} \) where the collection \( \{[c_1], \ldots, [c_k]\} \) has no upper bound. If the collection \( \{[c_1], \ldots, [c_k]\} \) (where \( c_1, \ldots, c_k \) are \( 1 \)-cells) has a least upper bound \( [c] \), then we label the open cell \( e^1_{[c_1]} \times \ldots \times e^1_{[c_k]} \) by \([c]\). Each cell of \( \hat{U}D^nT \) is given the natural product orientation. (Note that this orientation depends upon an ordering of the factors. We will order the factors so that \( [c_i] \) occurs before \([c_j]\) if \( \iota(e_i) < \iota(e_j) \), where \( e_i \) is the unique edge occurring in \( c_i \), and \( e_j \) is the unique edge occurring in \( c_j \). If \( \iota(e_i) = \iota(e_j) \), then the ordering may be chosen arbitrarily.\) Propositions 3.1 and 3.2 show that every equivalence class \([c] \) of cells occurs as the label of some cell in \( \hat{U}D^nT \), and that each such label occurs uniquely.
The complex $\hat{U^nD^nT}$ can also be described as a quotient of $U^nD^nT$. We define a quotient map $q : U^nD^nT \to \hat{U^nD^nT}$ as follows. The map $q$ sends the $j$-cell $c = \{e_1, e_2, \ldots, e_j, v_{j+1}, \ldots, v_n\}$ in $U^nD^nT$, where $i(e_1) < i(e_2) < \ldots < i(e_j)$, to the cell labelled $[c]$ in $\hat{U^nD^nT}$. Let $f : [0, 1]^j \to c$ be the standard characteristic map for $c$ (as described at the end of Section 2). For each cell $[c]$ in $\hat{U^nD^nT}$, we choose an orientation $h : [0, 1]^j \to [c]$ satisfying $h(t_1, \ldots, t_j) = (h_{[c]}(t_1), \ldots, h_{[c]}(t_j))$, where $\{[c_1], \ldots, [c_j]\}$ is the unique collection of equivalence classes of 1-cells having $[c]$ as its least upper bound, and $[c_i]$ is the unique equivalence class of this collection satisfying $e_i \in c_i$. We define the map $q : U^nD^nT \to \hat{U^nD^nT}$ cell by cell to be the unique map satisfying $q \circ f = h$. The argument from 4.3 in [5] shows that this assignment does indeed induce a quotient map $q : U^nD^nT \to \hat{U^nD^nT}$, and that each open cell $c$ in the domain is mapped homeomorphically to the open cell $[c]$. Our current choice of orientation also shows that positive cells are mapped to positive cells.

We would now like to describe the cohomology rings $H^*(\hat{U^nD^nT}; \mathbb{Z})$. For this we need a definition. If $K$ is a finite simplicial complex, then the exterior face ring of $K$ over $\mathbb{Z}$, denoted $\Lambda[K]$, is defined by

$$\Lambda[K] \cong \Lambda[v_1, \ldots, v_m]/I,$$

where $\Lambda[v_1, v_2, \ldots, v_m]$ denotes the ordinary integral exterior ring [10] generated by the collection of vertices of $K$, and $I$ is the ideal generated by the monomials $v_{i_1}v_{i_2}\ldots v_{i_n}$ such that $\{v_{i_1}, \ldots, v_{i_n}\}$ is not a simplex of $K$.

It is now straightforward to describe the integral cohomology ring of $\hat{U^nD^nT}$. For any equivalence class $[c]$ of $j$-cells, let $\hat{\phi}_{\{c\}}$ denote the $j$-cocycle satisfying $\hat{\phi}_{\{c\}}([c]) = 1$ and $\hat{\phi}_{\{c\}}([c']) = 0$ for all $[c'] \neq [c]$. We note that there is no real distinction between cocycles and cohomology classes (since all boundary maps in the cellular chain complex for $\hat{U^nD^nT}$ are 0), so we also let $\hat{\phi}_{\{c\}}$ denote a cohomology class whenever it is convenient to do so.

**Proposition 3.3.** [5] The integral cohomology ring $H^*(\hat{U^nD^nT})$ is isomorphic to $\Lambda[K]$, where $K$ is a simplicial complex on the vertex set $V$ consisting of all equivalence classes of 1-cells in $U^nD^nT$. The simplices of $K$ consist of all sets $\{[c_{i_1}], \ldots, [c_{i_m}]\} \subseteq V$ having an upper bound.

The isomorphism $\Phi : H^*(\hat{U^nD^nT}) \to \Lambda[K]$ sends a $j$-dimensional cohomology class $\hat{\phi}_{\{c\}}$ to $[c_{i_1}]\{c_{i_2}\} \ldots [c_{i_j}]$, where $\{[c_{i_1}], \ldots, [c_{i_j}]\}$ is the unique collection having $[c]$ as its least upper bound.

Here the ordering of the factors in the product $[c_{i_1}]\{c_{i_2}\} \ldots [c_{i_j}]$ is such that $i(e_1) < i(e_2) < \ldots < i(e_j)$, where $e_i$ is the unique edge in $c_i$, for $i = 1, \ldots, j$.

**3.2. Standard cocycles on $U^nD^nT$.** We have the following description of the map $q^* : H^*(\hat{U^nD^nT}; \mathbb{Z}) \to H^*(U^nD^nT; \mathbb{Z})$ on cohomology.

**Proposition 3.4.** The map $q^*$ sends $\hat{\phi}_{\{c\}}$ to $\phi_{\{c\}} \in C^*(U^nD^nT; \mathbb{Z})$ on the level of cellular cocycles, where

$$
\begin{align*}
\phi_{\{c\}}(\hat{c}) &= 1 \quad \text{if } \hat{c} \sim c \\
\phi_{\{c\}}(\hat{c}) &= 0 \quad \text{otherwise}.
\end{align*}
$$

**Proof.** This follows from the description of the map $q : U^nD^nT \to \hat{U^nD^nT}$, and from our choice of orientation as in the previous subsection. \qed
We call the cocycles $\phi_{[c]}$ standard. A standard cocycle $\phi_{[c]}$ is called critical if the equivalence class $[c]$ contains at least one critical cell.

**Lemma 3.5.** [5] If $[c]$ contains at least one critical cell, then it contains exactly one, and every other cell in $[c]$ is redundant. 

**Corollary 3.6.** The collection of critical cocycles represent a basis for $H^*(U^D^n T)$. In particular, $q^*$ is surjective.

**Proof.** Let $\phi_{[c]}$ be a critical $i$-cocycle, and suppose without loss of generality that $c$ is a critical $i$-cell. We evaluate $\phi_{[c]}$ on cycle representatives for a basis of $H_i(U^D^n T)$. Recall that one such collection of cycles $\{c_1, \ldots, c_j\}$ corresponds to the set $\{c_1, c_2, c_3, \ldots, c_j\}$ of all critical $i$-cells, where a given cycle $c_k$ has the form $c_k+$ (collapsible cells).

Consider $\phi_{[c]}(c_k)$. The support of $\phi_{[c]}$ contains no collapsible cells, so $\phi_{[c]}(c_k) = \phi_{[c]}(c_k)$. Since the support of $\phi_{[c]}$ contains exactly one critical cell (namely $c$), it follows that $\phi_{[c]}(c_k) = 1$ if $c = c_k$, and $\phi_{[c]}(c_k) = 0$ otherwise.

It follows from this that the cohomology classes $\langle \phi_{[c_1]}, \ldots, \langle \phi_{[c_j]} \rangle$ map to the dual basis $\hat{c}_1, \ldots, \hat{c}_j$ in $\text{Hom}(H_n(U^D^n T), \mathbb{Z})$ via the universal coefficient isomorphism $H^*(U^D^n T) \to \text{Hom}(H_1(U^D^n T), \mathbb{Z})$. 

4. A Presentation for $H^*(B_n T)$ as a Ring

4.1. Pictures of standard cocycles and other necessary definitions. Let $\phi_{[c]}$ be a standard cocycle on $U^D^n T$. If $c = \{v_1, \ldots, v_j, e_{j+1}, \ldots, e_n\}$, then by definition $\phi_{[c]}$ is supported on all cells $\hat{c} = \{\hat{v}_1, \ldots, \hat{v}_j, \hat{e}_{j+1}, \ldots, \hat{e}_n\}$ such that: i) $\{e_{j+1}, \ldots, e_n\} = \{\hat{e}_{j+1}, \ldots, \hat{e}_n\}$ and ii) $C \cap \{v_1, \ldots, v_j\} = C \cap \{\hat{v}_1, \ldots, \hat{v}_j\}$ for every connected component $C$ of $T - (e_{j+1} \cup \ldots \cup e_n) = T - (\hat{e}_{j+1} \cup \ldots \cup \hat{e}_n)$. Thus, a standard $i$-cocycle is completely determined by: i) a collection of connected components $\text{Comp}(T - (e_1 \cup \ldots \cup e_i))$ for a given collection $E = \{e_1, \ldots, e_i\}$ of disjoint closed edges, and ii) an assignment $f : \text{Comp}(T - (e_1 \cup \ldots \cup e_i)) \to \mathbb{Z}^+ \cup \{0\}$ such that $\sum c f(c) = n - i$. We call a pair $(\text{Comp}(T - (e_1 \cup \ldots \cup e_i)), f)$ satisfying i) and ii) a cloud picture of a standard cocycle, since we will often indicate such a pair using pictures as in the following example. (Note: in practice, we will usually think of the elements of $\text{Comp}(T - (e_1 \cup \ldots \cup e_i))$ as the 0-skeletons of the connected components described above in i.)

**Example 4.1.** Let $T$ be as in Figure 1, and consider pairs $(\text{Comp}(T - e_{19}), f_1), (\text{Comp}(T - e_{16}), f_2)$ (where $e_i$ is the unique edge of $T$ satisfying $\tau(e_i) = v_i$, for $i = 16, 19$), and $f_1 : \{C_0, C_1, C_2\} \to \mathbb{Z}^+ \cup \{0\}$, $f_2 : \{C_0', C_1', C_2'\} \to \mathbb{Z}^+ \cup \{0\}$ satisfying:

\[
\begin{align*}
f_1(C_0) &= 1 & f_2(C_0') &= 2 \\
f_1(C_1) &= 2 & f_2(C_1') &= 1 \\
f_1(C_2) &= 0 & f_2(C_2') &= 0.
\end{align*}
\]

(Here $C_i$ is the connected component of $T - e_{19}$ lying in the direction $i$ from $\iota(e_{19}) = v_9$; similarly $C_i'$ lies in direction $i$ from $\iota(e_{16})$.)

We represent each of these pairs as pictures (see Figure 3). Each picture can be interpreted in two ways. First, we can see each picture as a description of a given equivalence class of, in this case, 1-cells. The picture on the left, for instance, represents the equivalence class of the 1-cell $c = \{e_{19}, *, v_{10}, v_{11}\}$, where $*$ is the
basepoint, and $v_{10}$ and $v_{11}$ are the vertices numbered 10 and 11 in Figure 1. The elements of the indicated equivalence class are 1-cells $c'$ such that: i) the unique (in this case) 1-cell of $c'$ is $e_{19}$, and ii) exactly two vertices of $c'$ lie inside the cloud at the bottom left (which consists of the set $\{v_{10}, \ldots, v_{18}\}$), iii) exactly one vertex of $c'$ lies inside the cloud at the bottom right (which consists of the vertices $\{*, v_1, \ldots, v_8\}$).

We leave it as an easy exercise to describe the equivalence class symbolized by the picture on the right in Figure 3. Second, we can see each picture as a representative of the unique standard cocycle supported on the given equivalence class.

**Figure 3.** These cloud pictures are determined by the pairs $(\text{Comp}(T - e_{19}), f_1)$ and $(\text{Comp}(T - e_{16}), f_2)$, respectively.

Now Propositions 3.3 and 3.4 imply that the product of the above two cocycles (cohomology classes), in the given order, is as in Figure 4. The main point to check is that the cloud picture in Figure 4 is the least upper bound of the two pictures in Figure 3. This can be done very easily. Simply choose a representative $c$ of the equivalence class $[c]$ determined by the picture in Figure 4, and find two faces $c_1$, $c_2$ of $c$ such that $[c_1]$ and $[c_2]$ have the pictures in Figure 3. One possibility is to let $c = \{e_{19}, e_{16}, *, v_{13}\}$, $c_1 = \{e_{19}, v_{12}, *, v_{13}\}$, and $c_2 = \{v_9, e_{16}, *, v_{13}\}$.

**Figure 4.** This is the product of the two cocycles from Figure 3.

We introduce some definitions which will be useful when we work with standard cocycles. First, we refer to the elements $C$ of $\text{Comp}(T - (e_1 \cup \ldots \cup e_j))$ as clouds of the given standard cocycle. If $C$ is a cloud picture and $e$ is an edge of $C$, we say that the cloud picture $C'$ is obtained from $C$ by breaking the edge $e$ and combining
clouds if \( C' \) is the result of: i) removing the edge \( e \) from the defining list for \( C \), and ii) replacing the clouds incident with \( e \) by a single cloud (which also contains the edge \( e \)). The resulting cloud’s number is 1 greater than the sum of the numberings of the old clouds. For instance, the cloud picture on the left in Figure 3 is obtained from the cloud picture in Figure 4 by breaking the edge \( e_{16} \) and combining clouds.

\[ \square \]

We now assign an ordered pair of integers to each standard cocycle. We need several definitions. If \( c = \{e_1, \ldots, e_l, v_{l+1}, \ldots, v_n\} \) is an \( i \)-cell of \( U D^n T \), then we call the vertices \( \iota(e_1), \ldots, \iota(e_l) \) special vertices of \( c \). The \( c \)-depth \( \text{dep}_c(v_k) \) of a vertex \( v_k \in c \) is the number of special vertices lying on the geodesic segment \([*, v_k]\) in \( T \). The total vertex depth \( \text{dep}(c) \) of \( c \) is \( \sum_{v \in c} \text{dep}_c(v) \).

If \( c \) is an arbitrary cell of \( U D^n T \), let \( r(c) \) be a choice of cell in \([c]\) with the property that all of its vertices are blocked. An edge \( e \in c \) is called bad if it is order-respecting in \( r(c) \). Note that “badness” of an edge is well-defined on equivalence classes \([c]\), in the sense that if \( c \sim c' \) and \( e \in c, c' \) then \( e \) is bad in \( c \) if and only if \( e \) is bad in \( c' \). (On the other hand, easy examples show that there exist \( c, c', c \sim c' \) for which it is possible to choose \( r(c) \) and \( r(c') \) such that \( r(c) \neq r(c') \).) The number of bad edges in \( c \) is denoted \( \text{bad}(c) \).

We define the rank of \( c \), denoted \( \text{rank}(c) \), to be the pair \((-\text{dep}(c), \text{bad}(c))\). The set of ranks is lexicographically ordered, i.e., \( \text{rank}(c) < \text{rank}(c') \) if \( \text{dep}(c) > \text{dep}(c') \) or \( \text{dep}(c) = \text{dep}(c') \) and \( \text{bad}(c) < \text{bad}(c') \). It is clear moreover that there can be no infinite descending chains. Note that the functions \( \text{dep} \) and \( \text{bad} \) are well-defined on equivalence classes, so we can define the rank \( \text{rank}([c]) \) of an equivalence class to be equal to the rank of \( c \). Similarly, the \( \text{rank} \) of a standard cocycle is the rank of any cell in its support.

4.2. A complete list of relators. Let \( C = (\text{Comp}(T - (e_1 \cup \ldots \cup e_i)), f) \) be a cloud picture of a standard \( i \)-cocycle; let \( E = \{e_1, \ldots, e_l\} \), and \( \text{Comp}(T - (e_1 \cup \ldots \cup e_i)) = \{C_1, \ldots, C_k\} \). Let \( v' \) be a special vertex of one (equivalently, any) of the cells \( e \) in the support of the cocycle determined by \( C \). Let \( e' \) be the unique edge of \( c \) having \( v' \) as its initial endpoint. We define an \((i - 1)\)-dimensional cochain \( R_{C, v'} = (P_{C, v'}, f_{C, v'}) \) in terms of a modified “cloud notation”. The set \( P_{C, v'} \) partitions the vertices of \( T - (\cup_{e \in E - \{v'\}} e) \); each member of \( P_{C, v'} \) is precisely the same as a member of \( \text{Comp}(T - (e_1 \cup \ldots \cup e_i)) \), with two exceptions: i) the component \( C_\tau \in \text{Comp}(T - (e_1 \cup \ldots \cup e_i)) \) which is adjacent to \( \tau(e') \) is replaced by the set \( C'_\tau = C_\tau \cup \{\tau(e')\} \) in \( P_{C, v'} \), and ii) the component \( C_\iota \in \text{Comp}(T - (e_1 \cup \ldots \cup e_i)) \) which is adjacent to \( \iota(e') = v' \) and lies in the \( 0 \)th direction from \( v' \) is replaced by the set \( C'_\iota = C_\iota \cup \{\iota(e')\} \). The function \( f_{C, v'} \): i) agrees with \( f \) on \( P_{C, v'} \cap \text{Comp}(T - (e_1 \cup \ldots \cup e_i)) \), ii) \( f_{C, v'}(C'_\iota) = f(C_\iota) \), and iii) \( f_{C, v'}(C'_\tau) = f(C_\tau) + 1 \).

This new “cloud notation” describes the support of a cochain \( R_{C, v'} \). It is perhaps best to give the definition of this cochain with help from an example (Example 4.2 below).

Note that our definition of \( R_{C, v'} \) implicitly assumes that the degree of the vertex \( \tau(e') \) is less than or equal to 2. This is by far the most important case. We will treat the case in which the degree of \( \tau(e') \) is at least 3 after Theorem 4.5.

We now describe such a cochain \( R_{C, v'} \) and its coboundary \( \delta(R_{C, v'}) \) in an explicit case.

Example 4.2. Consider the standard cocycle on the left in Figure 5. If we check
the right in “cloud notation”. Thus a positively oriented cell $c$ of $U^\alpha T$ lies in the support of $R_{C,v_9}$ (and $R_{C,v_9}(c) = 1$) if and only if: i) the only edge of $c$ is $e_{16}$; ii) $c$ contains exactly one of the cells from the set $\{v_{13}, v_{14}, v_{15}\}$; iii) $c$ contains exactly one of the cells from $\{*, v_1, \ldots, v_9\}$, and iv) $c$ contains exactly one of the cells from $\{v_{19}, \ldots, v_{27}\}$.

Let us consider the support of the (2-dimensional) coboundary $\delta(R_{C,v_9})$. Suppose that $c' \in \text{supp}(\delta(R_{C,v_9}))$. Let $c'_1, c'_2 \in c'$ be the two 1-cells in $c' \in \text{supp}(\delta(R_{C,v_9}))$. The coboundary $\delta(R_{C,v_9})$ evaluated at $c'$ is the following sum for a suitable choice of sign:

$$\delta(R_{C,v_9})(c') = \pm \left( R_{C,v_9}(c'_{\tau(e_1)}) - R_{C,v_9}(c'_{\iota(e_1)}) \right) \mp \left( R_{C,v_9}(c'_{\tau(e_2)}) - R_{C,v_9}(c'_{\iota(e_2)}) \right),$$

where, for instance, $c'_{\tau(e_1)}$ is the face of $c'$ obtained by replacing $e_1' \in c'$ with $\tau(e_1')$. The cells $c'_{\tau(e_1)}, c'_{\iota(e_1)}$ have analogous definitions.

It follows easily from this that one of $e_{14}'$, $e_2'$ is $e_{16}$ (since $c' \in \text{supp}(\delta(R_{C,v_9}))$; we assume $e_1' = e_{16}$, without loss of generality. The cells $c'_{\tau(e_1)}, c'_{\iota(e_1)}$ are thus annihilated by $R_{C,v_9}$. We have

$$\delta(R_{C,v_9})(c') = \mp \left( R_{C,v_9}(c'_{\tau(e_2)}) - R_{C,v_9}(c'_{\iota(e_2)}) \right).$$

It follows directly from this that precisely one of $c'_{\tau(e'_2)}, c'_{\iota(e'_2)}$ lies in the support of $R_{C,v_9}$. It then follows that $c'_2$ must connect a vertex in one “cloud” to a vertex in another. Thus $c'_2$ is either $e_{10}$ or $e_{19}$. (Note also that if $c'_2$ connects two clouds as above, then it follows easily that at most one face of $c'$ lies in the support of $R_{C,v_9}$.)

The support of $\delta(R_{C,v_9})$ can now be determined directly. If $c' \in \text{supp}(\delta(R_{C,v_9}))$, then $c' = \{e_{16}, e_{10}, v', v''\}$ or $c' = \{e_{16}, e_{19}, v', v''\}$. The above reasoning also shows that the entire contribution to the value of $\delta(R_{C,v_9})$ comes from a single face $c''$ of $c'$ lying in the support of $R_{C,v_9}$. There are four cases:

1. $c' = \{e_{16}, e_{10}, v', v''\}$ and $c'' = \{e_{16}, \iota(e_{10}), v', v''\};$
2. $c' = \{e_{16}, e_{10}, v', v''\}$ and $c'' = \{e_{16}, \tau(e_{10}), v', v''\};$
3. $c' = \{e_{16}, e_{19}, v', v''\}$ and $c'' = \{e_{16}, \iota(e_{19}), v', v''\};$
4. $c' = \{e_{16}, e_{19}, v', v''\}$ and $c'' = \{e_{16}, \tau(e_{19}), v', v''\}.$
Case (1) can be realized if \( v' \in \{v_{19}, \ldots, v_{27}\} \) and \( v'' \in \{v_{13}, v_{14}, v_{15}\} \). Case (2) is clearly impossible, since \( \tau(e_{10}) \) is in a cloud which is labelled 0, and thus \( c'' \) fails to be in \( \text{supp}(R_{C,v}) \). Case (3) can be realized if \( v' \in \{v_{20}, \ldots, v_{27}\} \) and \( v'' \in \{v_{13}, v_{14}, v_{15}\} \). Case (4) can be realized if \( v' \in \{*, v_1, \ldots, v_8\} \) and \( v'' \in \{v_{13}, v_{14}, v_{15}\} \).

Reviewing the considerations in (1)-(4) above, we see that \( \text{supp}_\delta(R_{C,v}) \) is the union of the (disjoint) supports of three standard cocycles; their cloud pictures are Figure 6, written (from left to right) in the order (4), (1), (3). The sum in Figure 6 is equal to 0 on cohomology since it is a coboundary. Note that the standard cocycle on the left is \( \phi \).

Now \( \text{rank}(\phi) = (-2, 1) \); the ranks of the standard cocycles on the right are both \((-3, 1)\). We therefore have succeeded in representing the cohomology class of \( \phi \) using a sum of cocycles of rank smaller than \( \text{rank}(\phi) \).

We now consider the features from the previous example which hold more generally. Let \( C \) be the cloud picture of a standard cocycle \( \phi \) (of any dimension), let \( v \) be a special vertex of \( \phi \), and let \( e \) be the (unique) edge of \( \phi \) such that \( \iota(e) = v \). A cell \( c' \) in the support of \( \delta(R_{C,v}) \) has the following properties:

1. The edge set \( E(c') \) contains the edge set \( E(R_{C,v}) \). The one remaining edge \( e' \in E(c') \) must connect distinct clouds of \( R_{C,v} \). Thus, there are (a priori, at least) \( d(v) - 1 \) possibilities for \( e' \).
2. Exactly one face of \( c' \) lies in the support of \( R_{C,v} \), and this face can be obtained by replacing the edge \( e' \) with either its initial or its terminal endpoint. We therefore have the formula
   \[
   \delta(R_{C,v})(c') = R_{C,v}(\tau(e')) - R_{C,v}(\iota(e')),
   \]
   where, for example, \( c'_{\tau(e')} \) is the face of \( c' \) obtained by replacing the edge \( e' \) with \( \tau(e') \) in the definition of \( c' \). This equality holds only up to a choice of multiplication of the right side of the equality by \( \pm 1 \), but this choice doesn’t matter, since we are only interested in equalities on the level of cohomology, and the left side of the equation is, of course, 0 on cohomology. We ignore this choice of sign from now on.

3. We can divide the cells \( c' \) into two groups, according to whether \( c'_{\tau(e')} \) or \( c'_{\iota(e')} \) lies in the support of \( R_{C,v} \). By Lemma 2.7, we can write
   \[
   \delta(R_{C,v}) = \Sigma_{C,v,\tau} - \Sigma_{C,v,\iota}.
   \]
   Here
   \[
   \Sigma_{C,v,\tau} = \chi\left(\{c' \mid c'_{\tau(e')} \in \text{supp} R_{C,v}\}\right)
   \]
and \(\chi\) denotes a characteristic function. The function \(\Sigma_{\mathcal{C},v,\tau}\) is defined similarly.

(4) We now examine the functions \(\Sigma_{\mathcal{C},v,\tau}\) and \(\Sigma_{\mathcal{C},v,\iota}\). We can partition the support of \(\Sigma_{\mathcal{C},v,\tau}\) into \(d(v) - 1\) pieces, according to the location of the edge \(e'\), which can point in any of \(d(v) - 1\) directions. (In practice, some elements of the indicated partition will be empty, as in Case (2) from Example 4.2, but this causes no problems. It will simply mean that some of the standard cocycles in the sums to be defined below are 0.) For \(i = 1, \ldots, d(v) - 1\), we let \(e'_i\) denote the edge with initial vertex \(v\) which lies in the direction \(i\) from \(v\). For \(i = 0, 1, \ldots, d(v) - 1\), we let \(C_i\) denote the element of the partition \(\mathcal{P}_{\mathcal{C},v}\) which lies in the \(i\)th direction from \(v\). (Recall that \(R_{\mathcal{C},v} = (\mathcal{P}_{\mathcal{C},v}, f_{\mathcal{C},v})\).)

A routine check shows that

\[
\Sigma_{\mathcal{C},v,\tau} = \sum_{i=1}^{d(v)-1} \Psi_{\mathcal{C},v,\tau,i}
\]

where \(\Psi_{\mathcal{C},v,\tau,i}\) is the standard cocycle supported on all cells \(c\) such that: i) \(E(c) = E(R_{\mathcal{C},v}) \cup \{e'_i\}\), and ii) exactly \(f_{\mathcal{C},v}(C_j)\) vertices of \(c\) lie in \(C_j\) for \(j \neq i\); exactly \(f_{\mathcal{C},v}(C_i) - 1\) vertices of \(c\) lie in \(C_i\).

Similarly,

\[
\Sigma_{\mathcal{C},v,\iota} = \sum_{i=1}^{d(v)-1} \Psi_{\mathcal{C},v,\iota,i}
\]

where \(\Psi_{\mathcal{C},v,\iota,i}\) is the standard cocycle supported on all cells \(c\) such that: i) \(E(c) = E(R_{\mathcal{C},v}) \cup \{e'_i\}\), and ii) exactly \(f_{\mathcal{C},v}(C_i)\) vertices of \(c\) lie in \(C_i\) for \(i > 0\), and \(f_{\mathcal{C},v}(C_0) - 1\) vertices of \(c\) lie in \(C_0\).

Since the descriptions of \(\Sigma_{\mathcal{C},v,\tau}\) and \(\Sigma_{\mathcal{C},v,\iota}\) from (4) are somewhat unpleasant notationally, we give a computation of \(\delta(R_{\mathcal{C},v}) = \Sigma_{\mathcal{C},v,\tau} - \Sigma_{\mathcal{C},v,\iota}\) in the example below.

**Example 4.3.** Let \(R_{\mathcal{C},v}\) denote the cochain determined by the cloud representative in Figure 7 a). We have abandoned the cloud notation, omitted the numbering,

\[
\begin{align*}
a) & \quad \begin{array}{c}
\begin{array}{c}
2
\end{array} \\
3
\end{array} \\
& \quad \begin{array}{c}
\begin{array}{c}
1
\end{array} \\
2
\end{array} \\
& \quad \begin{array}{c}
\begin{array}{c}
*\end{array} \\
1
\end{array}
\end{align*}

b) & \quad \begin{array}{c}
\begin{array}{c}
2
\end{array} \\
1
\end{array} \\
& \quad \begin{array}{c}
\begin{array}{c}
1
\end{array} \\
2
\end{array} \\
& \quad \begin{array}{c}
\begin{array}{c}
*\end{array} \\
0
\end{array}
\end{align*}

\begin{align*}
c) & \quad \begin{array}{c}
\begin{array}{c}
2
\end{array} \\
1
\end{array} \\
& \quad \begin{array}{c}
\begin{array}{c}
1
\end{array} \\
3
\end{array} \\
& \quad \begin{array}{c}
\begin{array}{c}
*\end{array} \\
0
\end{array}
\end{align*}
\]

**Figure 7.** a) shows a picture representative of the cochain \(R_{\mathcal{C},v}\); b) is the sum \(\Sigma_{\mathcal{C},v,\tau}\), and c) is \(\Sigma_{\mathcal{C},v,\iota}\).

and even avoided specifying the subdivision of the tree \(T\). (A sufficient subdivision
of $T$ would have 7 edges along each of the radial arms of the tree. In a), the integer labels on the arms of the tree represent the number of vertices which lie inside the cloud containing the given arm. For this purpose, we regard the central vertex as belonging to the cloud which also contains the basepoint. The notation in b) and c) should be unambiguous.

The two constituent sums of the coboundary $\delta(R_{e,v})$ are depicted in Figure 7, parts b) and c). The terms in b) are $\Sigma_{C,v,\tau}$; the terms in c) are $\Sigma_{C,v,\tau}$. □

Note that the ranks of the cocycles in Figure 7, parts b) and c), are $(-5, 1), (-5, 0), (-5, 0), (-6, 1), (-6, 0), (-6, 0)$, and $(-6, 0)$, respectively (read from left to right). Thus there is a unique standard cocycle in the sum $\Sigma_{C,v,\tau} - \Sigma_{C,v,\tau}$ having maximal rank (namely the first cocycle, in this case). This is in fact a general feature of all coboundaries $\delta(R_{e,v})$ with $C$ and $v$ arbitrary.

**Proposition 4.4.** Let $\Psi$ be a standard cocycle with cloud diagram $C$. Let $e$ be a bad edge of $\Psi$ and let $v$ be the initial vertex of $e$.

The coboundary $\delta(R_{e,v})$ can be expressed as a sum $\sum \Psi_i$ of standard cocycles, where

1. the cocycle $\Psi$ occurs exactly once in $\sum \Psi_i$, and
2. every other term in $\sum \Psi_i$ (if any) is a standard cocycle of rank strictly less than $\text{rank}(\Psi)$.

**Proof.** Suppose that $e$ lies in direction $k$ from $v$, where $k \in \{1, \ldots, d(v) - 1\}$, or $k = 1$ if $d(v) = 1$. Let $C_j$, for $j = 0, 1, \ldots, d(v) - 1$, be the cloud of $R_{e,v}$ lying in the $j$th direction from $v$. It follows from the fact that $e$ is bad that $\phi_{C,v}(C_j) = 0$ for all $0 < j < k$; by the definition of $R_{e,v}$, $f_{e,v}(C_k) = 1 + f_{e}(C_k)$. In particular, $f_{e,v}(C_k) > 0$.

Now we consider the sum $\Sigma_{C,v,\tau}$:

$$\Sigma_{C,v,\tau} = \sum_{i=1}^{d(v)-1} \Psi_{C,v,\tau,i}$$

$$= \Psi_{C,v,\tau,k} + \sum_{i=k+1}^{d(v)-1} \Psi_{C,v,\tau,i}.$$

The cocycle $\Psi_{C,v,\tau,k}$ is equal to $\Psi$ by definition. Each term of the remaining sum is either identically 0 or the edge $e'_i$ $(i > k)$ is no longer bad. Since the total vertex depth of each term in $\Sigma_{C,v,\tau}$ is identical, and $\text{bad}(\Psi_{C,v,\tau,i}) < \text{bad}(\Psi_{C,v,\tau,k})$ for $i > k$, it follows that $\text{rank}(\Psi_{C,v,\tau,k}) > \text{rank}(\Psi_{C,v,\tau,i})$, for $i > k$.

In the sum

$$\Sigma_{C,v,i} = \sum_{i=1}^{d(v)-1} \Psi_{C,v,i},$$

each cocycle $\Psi_{C,v,i}$ satisfies $\text{dep}(\Psi_{C,v,i}) = \text{dep}(\Psi) + 1$, so $\text{rank}(\Psi_{C,v,i}) < \text{rank}(\Psi)$ for all $i$. □

Fix a cochain $R_{e,v}$, where $C$ is a cloud picture and $v$ is a special vertex of a bad edge of $C$. The discussion before Proposition 4.4 shows that $\delta(R_{e,v})$ can be written uniquely as a sum of standard cocycles. (Indeed, the non-zero standard cocycles make up a linearly independent set, and even have pairwise disjoint supports.) We let $\delta(R_{e,v})$ denote the unique cellular cochain in $C^*(DU^{n}T; \mathbb{Z})$ which maps onto
δ(R_{C,v}) under the function on cochains induced by the canonical quotient map $q : UD^nT \to \overline{UD^nT}$. Since every cochain in $C^*(\overline{UD^nT}; \mathbb{Z})$ is a cocycle, we can safely identify $\delta(R_{C,v})$ with the element of cohomology it represents, and we shall do so freely and without further notice.

**Theorem 4.5.** The kernel of the map $q^* : H^*(\overline{UD^nT}; \mathbb{Z}) \to H^*(UD^nT; \mathbb{Z})$ is generated by the cohomology classes $\delta(R_{C,v})$, where $C$ is a cloud picture with at least one bad edge $e$ and $v$ is the initial vertex of $e$.

In particular,

$$H^*(UD^nT; \mathbb{Z}) \cong A[K]/I,$$

where $I$ is the ideal generated by the $\delta(R_{C,v})$ (as above), and $K$ is the simplicial complex described in Proposition 3.3.

**Proof.** We work with standard cocycles in $C^*(UD^nT)$, which are in perfect one-to-one correspondence with the canonical basis in $C^*(\overline{UD^nT})$ ($= H^*(\overline{UD^nT})$).

Define a strict partial order $<$ on standard cocycles as follows. Let $\phi$ be a standard cocycle, let $C$ be its cloud picture, suppose that $C$ has at least one bad edge $e$, and let $v = \iota(e)$. If $\phi_1$ is a standard cocycle we write $\phi_1 < \phi$ if $\text{supp} \phi_1 \subseteq \text{supp} \delta(R_{C,v})$ and $\phi_1 \neq \phi$. (Thus, for instance, the left-most standard cocycle $\phi$ in Figure 6 is strictly greater than the other cocycles in Figure 6.) Let $<$ also denote the transitive closure of $<$. That $<$ is a strict partial order follows easily from the fact that $\text{rank}(\phi_1) < \text{rank}(\phi)$ by Proposition 4.4.

Consider a minimal standard cocycle $\phi$ under the partial order $<$. By the definition of $<$, this implies either: i) that $\phi$ has no bad edges, or ii) any sum $\delta(R_{C,v})$ (where $C$ is the cloud picture of $\phi$ and $v$ is the initial vertex of a bad edge) satisfies $\phi = \delta(R_{C,v})$. In case i), $\phi$ is a critical cocycle, and in case ii) $\phi$ is cohomologous to 0.

Let $\sum_i \phi_i$ be a finite sum of standard cocycles. We let the rank of $\sum_i \phi_i$ be the maximum of the ranks of the non-critical standard cocycles occurring in $\sum_i \phi_i$. The rank of a sum of critical cocycles can simply be defined as $(-\infty, -\infty)$. Suppose, for the sake of simplicity in notation, that there is a unique non-critical standard cocycle $\phi_k$ occurring in the sum such that $\text{rank}(\phi_k) > \text{rank}(\phi_i)$ for any non-critical standard cocycle $\phi_i$ in $\sum_i \phi_i$ such that $i \neq k$. The cloud picture $C_k$ for $\phi_k$ has at least one bad edge $e$, and we let $\iota(e) = v$. We have the equality

$$\sum_i \phi_i = \left( \sum_{i \neq k} \phi_i \right) + (\phi_k - \delta(R_{C_k,v}))$$

on the level of cohomology. Note that, by Proposition 4.4, the cocycle on the right side is of smaller rank.

In the general case, there are several non-critical standard cocycles $\phi_i$ (rather than a unique one) satisfying $\text{rank}(\phi_i) = \text{rank}(\phi_{i_2}) = \ldots = \text{rank}(\phi_{i_k}) = \text{rank}(\sum_i \phi_i)$. In this case, the same procedure should be performed for each such $\phi_i$ simultaneously.

It follows that a given sum $\Sigma$ of standard cocycles, not all of which are critical, can be rewritten as $\Sigma'$, where $\text{rank}(\Sigma') < \text{rank}(\Sigma)$. The collection of all possible ranks is finite, so this rewriting procedure must terminate, and the result will be a (possibly empty) sum of critical cocycles. During this process, we apply only relations (coboundaries) of the form $\delta(R_{C,v})$ as in the statement of the theorem.
If the sum $\Sigma$ is cohomologous to 0, then the rewriting procedure expresses $\Sigma$ as a sum of coboundaries $\delta(R_{C,v})$, as required.

Finally, we mention how to modify the above rewriting procedure to cover the case in which the degree of $\tau(e)$ is greater than 2, for some defining edges $e$ of the standard cocycle $\phi$. Fix such an edge $e$. Consider the cochain $R'$, defined using "cloud notation", and satisfying: i) one of the clouds of $R'$ consists of the vertex $\tau(e)$ alone, and this cloud is assigned the number 1; ii) each of the clouds incident with the edge $e$ is the same as before, except for the cloud $C_0$ lying in direction 0 from $\tau(e)$, which gains the vertex $\iota(e)$. These clouds are all numbered as they were in $\phi$.

We can again express the coboundary $\delta(R')$ as a sum $\Sigma_{\tau} - \Sigma_\iota$. We leave it as an exercise to show that $\Sigma_{\tau}$ consists simply of the standard cocycle $\phi$ itself, and that $\Sigma_\iota$ is a sum of standard cocycles in which the edge $e$ has been replaced by various edges $e'$ (one for each term of the sum) satisfying $\iota(e') = \tau(e)$. This in particular implies that $\tau(e')$ has degree less than 3 in every case. (Here we assume that no two vertices of degree greater than two are adjacent. This can be guaranteed by subdividing $T$.)

Now note that, after applying this procedure repeatedly, we can represent the cohomology class of $\phi$ by a sum of standard cocycles, none of which have edges $e$ such that $d(\tau(e)) > 2$. At this point, we can apply the procedure of Theorem 4.5 without change, and ultimately express the cohomology class of $\phi$ in terms of critical cocycles. Note that the procedure of that Theorem will never change the special vertices of a standard cocycle, so it is never necessary to apply the special procedure described here again.

4.3. A Refinement of Ghrist’s Conjecture. The main theorem and (especially) the examples to come in Section 5 suggest a possible general description of the cohomology rings $H^*(UD^n T; \mathbb{Z})$.

**Conjecture 4.6.** Fix a finite tree $T$ and a positive integer $n$. Let $K$ be the simplicial complex associated to the complex $\hat{UD}^n T$, as in Proposition 3.3.

$$H^*(UD^n T; \mathbb{Z}) \cong \Lambda[K]/I',$$

where $I'$ is the ideal generated by all monomials $[c_1] \ldots [c_k]$, where the least upper bound $[c]$ of $\{[c_1], \ldots, [c_k]\}$ contains no critical cell, where we regard $[c]$ as an equivalence class of cells in $UD^n T$.

In particular, each $H^*(UD^n T; \mathbb{Z})$ is the exterior face ring of a simplicial complex (as implicitly described above).

Note that $I'$ is in fact quite different from the ideal $I$ in the statement of Theorem 4.5. The conjecture is motivated by the fact that

$$\Lambda[K]/I \cong \Lambda[K]/I'$$

in all of the existing computations. These computations become very long even for small trees $T$ and integers $n$, however, so the conjecture is ultimately based on only a few examples.

Robert Ghrist [9] asked if all braid groups of planar graphs are right-angled Artin groups. (Strictly speaking, he conjectured that all pure braid groups of planar graphs are right-angled Artin groups. Here we follow our practice from [5] and refer to either version of the conjecture as Ghrist’s conjecture.) A right-angled
Artin group is a group $G$ defined by a presentation of the form \( \langle x_1, x_2, \ldots, x_n \mid [x_i, x_j] \in \mathcal{I} \rangle \), where $\mathcal{I} \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ is an arbitrary indexing set. Equivalently, a right-angled Artin group $G^r$ may be defined by a simplicial graph $\Gamma$. The generators of $G^r$ are vertices of $\Gamma$, and the generators $v_i, v_j$ commute if they are adjacent in $\Gamma$. The class of right-angled Artin groups thus includes, at opposite extremes, free groups (where $\mathcal{I} = \emptyset$ or $\Gamma$ is a discrete set) and free abelian groups (where $\mathcal{I} = \{1, \ldots, n\} \times \{1, \ldots, n\}$ or $\Gamma$ is a complete graph).

There are well-known classifying spaces for right-angled Artin groups [3]. Fix a right-angled Artin group $G^r$. Begin with a copy of $S^1$, denoted $S^1_1$, having a single 0-cell and a single 1-cell, for each generator $v_i$. Form the product $\prod_{i=1}^{n} S^1_i$. Each open $j$-cell $e$ in the product corresponds naturally to a collection of $j$ generators, one for each 1-cell which contributes a factor to the $j$-cell $e$. We throw out an open $j$-cell if and only if the corresponding set of generators fails to form a copy of the complete graph on $j$ vertices in $\Gamma$. The resulting complex $K_{1}$ is a $K(G^r, 1)$ complex. It is easy to describe the cohomology ring of $K_{1}$ (and, thus, the cohomology ring of $G^r$) using this complex (see [10]): $H^*(K_{1}; \mathbb{Z})$ is the exterior face ring of the largest simplicial complex $K$ having $\Gamma$ as its 1-skeleton. A simplicial complex that is determined by its 1-skeleton in this sense is called flag.

It follows that Conjecture 4.6 is a weakened version of Ghrist’s conjecture. Indeed, the latter conjecture (which was disproved in [5]) would have implied (as above) that $H^*(U D^n \Gamma; \mathbb{Z})$ is the exterior face ring of a flag complex. A version of Conjecture 4.6 is also open for planar graphs, i.e., it is conceivable that the cohomology ring of $U D^n \Gamma$ is an exterior face algebra for $\Gamma$ planar and $n$ arbitrary, although the current positive evidence is not strong. We note finally that the cohomology rings of non-planar graph braid groups are not necessarily exterior face rings, since $U D^2 K_5$ and $U D^2 K_{3,3}$ are both homotopy equivalent to non-orientable surfaces [1].

5. Examples

We conclude with some sample calculations of cohomology rings $H^*(U D^n T; \mathbb{Z})$. It will be helpful to have a few lemmas for simplifying the calculations.

**Lemma 5.1.** Let $\phi$ be a standard cocycle, let $e$ be a bad edge of $\phi$, and let $v = i(e)$. For $i = 0, 1, \ldots, d(v) - 1$, let $C_i$ denote the cloud adjacent to $v$ in the direction $i$. Suppose that the edge $e$ points in direction $k > 0$ from $v$. If $f(C_i) = 0$ for all $i \neq k$, then $\phi$ is cohomologous to 0.

**Proof.** The description of $\delta(R_{C,v})$ implies that $\phi$ is the only standard cocycle occurring in the sum $\sum_{C,v} - \sum_{C,v}$. \( \Box \)

**Lemma 5.2.** Let $\phi, \psi$ be standard cocycles on $U D^n T$. If $[\phi] \cup [\psi] \neq 0$ (where the brackets denote cohomology classes), then for any cloud $C$ of $\phi$, $f_{\phi}(C) \geq \sum_{C \subseteq C'} f_{\psi}(C') + |\{e \in E(\psi) \mid e \subseteq C\}|,$

where the sum on the right runs over all clouds $C'$ of $\psi$ that are contained in $C$.

**Proof.** If $[\phi] \cup [\psi] \neq 0$, then there is some standard cocycle $\nu$ such that each of the standard cocycles $\phi, \psi$ may be obtained from $\nu$ by breaking certain collections
of edges in the cloud picture for \( \nu \) and combining clouds. The fact that the cup product \([\phi] \cup [\psi]\) is non-zero also implies that \(E(\phi) \cap E(\psi) = \emptyset\).

The cloud picture for \( \phi \) is obtained from \( \nu \) by breaking each edge \( e \in E(\psi) \subseteq E(\nu) \) and combining all of \( \nu \) incident with these edges. The description of \( \nu \) implies that each cloud of \( \nu \) is contained in one of the clouds of \( \phi \). If we fix a cloud \( C \) of \( \phi \) and break all edges \( e \in E(\psi) \subseteq E(\nu) \), then after combining all of the clouds \( C' \) of \( \nu \) that are contained in \( C \) there are precisely

\[
f_\phi(C) = \left[ \sum_{C' \subseteq C} f_\nu(C') \right] + \{ e \in E(\psi) \mid e \subseteq C \},
\]

vertices in the cloud \( C \), where the sum on the right runs over all clouds \( C' \) of \( \nu \) that are contained in \( C \).

We consider only the clouds \( C' \) of \( \nu \) that are also clouds of \( \psi \). Omitting the others (if any) from the above equality, we get

\[
f_\phi(C) \geq \left[ \sum_{C' \subseteq C} f_\nu(C') \right] + \{ e \in E(\psi) \mid e \subseteq C \},
\]

where the sum runs over all clouds \( C' \subseteq C \) which are clouds of both \( \psi \) and \( \nu \).

We claim that \( f_\nu(C') = f_\nu(C') \) for all such clouds \( C' \); the lemma clearly follows from this claim. The statement that \( C' \) is a cloud of \( \nu \) and \( \psi \) implies that \( C' \) touches only edges of \( \psi \). Now the required equality follows from the fact that \( \psi \) is obtained from \( \nu \) by breaking edges of \( \phi \) and combining the clouds; this procedure doesn’t change the number of vertices in \( C' \) by our assumptions, thus proving the claim.

\[\square\]

**Lemma 5.3.** Let \( \phi \) be a critical 1-cocycle defined on \( UD^n T \); and let \([\phi]\) denote its cohomology class. If \([\phi] \cup [\theta] \neq 0\) for some other cohomology class \([\theta] \in H^1(UD^n T)\) \((j \geq 1)\), then there is some cloud \( C \) of \( \phi \) such that:

1. \( C \) contains at least one essential vertex of \( T \), and
2. \( f_\phi(C) \geq 2 \).

**Proof.** Fix a critical 1-cocycle \( \phi \). If there is a cohomology class \([\theta]\) as in the statement of the Lemma, then there is some critical 1-cocycle \( \theta' \) such that \([\phi] \cup [\theta'] \neq 0\), since \( H^*(UD^n T) \) is generated by \( H^1(UD^n T) \), and a basis for \( H^1(UD^n T) \) is determined by the cohomology classes of critical 1-cocycles, by Corollary 3.6.

Since \([\phi] \cup [\theta'] \neq 0\), there is some \([\psi] \in H^2(UD^n T) - \{0\} \) where \( \psi \) is a standard 2-cocycle (which need not be critical) and the equivalence class \([\psi]\) is the least upper bound of \( \{[\phi], [\theta']\} \). Let \( e \) be the edge of \( \psi \) which is not an edge of \( \phi \). Now \([\psi] \neq 0\), so, by Lemma 5.1, at least one of the clouds \( C_1 \) incident with \( e \) satisfies \( f_\psi(C_1) > 0 \). It follows that \( \phi \), which is the result of breaking the edge \( e \) in \( \psi \) and combining clouds, satisfies \( f_\phi(C) \geq 2 \), where \( C \) is the unique cloud of \( \phi \) containing \( e \). This proves property (2). We have that \( e \) is the edge of \( \theta' \), and since this is a critical cocycle, \( i(e) \) must be essential. It follows that \( C \) also satisfies (1). \[\square\]

**Example 5.4.** We consider \( H^*(UD^4 T) \), where \( T \) is the tree depicted Figure 8, with the given choices of embedding and basepoint \(*\).

A critical 1-cocycle \( \phi \) on \( UD^4 T \) is a standard cocycle satisfying two conditions:

i) the (unique) edge \( e \) of \( \phi \) is one of \( e_A, e_B, e_C \), where \( e_X \) is the unique edge such
that \( \iota(e_X) = X \) and \( e_X \) points in direction 2 from \( X \); ii) the cloud \( C \) in direction 1 from \( \iota(e) \) satisfies \( f(C) > 0 \). Theorem 4.5 implies that the critical 1-cocycles determine a distinguished basis for \( H^1(U^D T) \).

Fix a particular critical 1-cocycle \( \phi \) and let \( [\phi] \) denote its cohomology class. If \( [\phi] \) cups non-trivially with some other member \( [\psi] \) of the distinguished basis (where \( \psi \) is a critical 1-cocycle), then it follows from Lemma 5.3 that some cloud \( C \) of \( \phi \) contains at least one essential vertex and satisfies \( f_\phi(C) \geq 2 \). It follows that there are exactly four critical 1-cocycles \( \phi \) having non-trivial cup products; these cocycles label the vertices in Figure 9.

The edges are to be labelled by (2-cocycle representatives of) the cup products of their endpoints. The precise details of this labelling are an exercise. The cup product of any other pair of elements of the standard basis for \( H^1(U^D T) \) is 0. It follows from Proposition 4.1 of [4] that \( H^1(U^D T) \cong \mathbb{Z}^{18} \) and \( H^2(U^D T) \cong \mathbb{Z}^{3} \) as abelian groups.

It now follows that \( H^*(U^D T; \mathbb{Z}) \cong \Lambda(K) \), where \( K \) is the simplicial complex consisting of a line segment that is divided into 3 pieces (as in Figure 9) and 14 isolated vertices. □

**Example 5.5.** Consider now the cohomology ring \( H^*(U^D H) \), where \( H \) is the graph that is homeomorphic to the letter “H”. Proposition 4.1 from [4] implies that \( H^1(U^D H; \mathbb{Z}) \cong \mathbb{Z}^{20} \), \( H^2(U^D H; \mathbb{Z}) \cong \mathbb{Z}^{5} \), and \( H^n(U^D H; \mathbb{Z}) \cong 0 \) for \( n > 2 \). We let \( A \) denote the left-hand essential vertex and \( B \) denote the right-hand essential vertex. Our basepoint \( * \) is the vertex of degree 1 at the bottom left.
The critical 1-cocycles $\phi$ of $UD^5H$ (which represent a distinguished basis for $H^1(UD^5H; \mathbb{Z})$) are the standard 1-cocycles determined by the following two properties: i) the (unique) edge $e$ of $\phi$ satisfies $\iota(e) = A$ or $\iota(e) = B$ and points in direction 2 from $\iota(e)$; ii) the cloud $C$ in direction 1 from $\iota(e)$ satisfies $f_{\phi}(C) > 0$.

If $[\phi]$ cups non-trivially with some cohomology class $[\psi]$ (where $\phi$, $\psi$ are both critical 1-cocycles), then, by Lemma 5.3, there must be some cloud $C'$ of $\phi$ which contains some essential vertex and such that $f_{\phi}(C') \geq 2$. This leaves only six elements of the distinguished basis for $H^1(UD^5H)$ which have non-trivial cup products; cocycle representatives for these cohomology classes label the vertices in Figure 10.

The edges should be labelled by (cocycle representatives of) the corresponding cup products. Note that the edge labels represent a basis for $H^2(UD^5H)$. We conclude that $H^*(UD^5H) \cong \Lambda(K)$, where $K$ is the simplicial complex consisting of 14 isolated vertices and the letter “H” (as cellulated in Figure 10).

**Example 5.6.** Let $T$ be the tree depicted in Figure 11. It is a consequence of Theorem 3.7 from [4] that $H^0(UD^4T) \cong 0$, $H^1(UD^4T) \cong \mathbb{Z}^{50}$, $H^2(UD^4T) \cong \mathbb{Z}^{18}$, and $H^n(UD^4T) \cong 0$ for $n > 2$.

We compute the cohomology ring $H^*(UD^4T; \mathbb{Z})$. We will need to consider a large number of cocycles in our argument, so it will be useful to introduce new notation. We describe a critical 1-cocycle $\phi$ by a 4- (or 5-) tuple as follows.

1. The first entry is $\iota(e)$, where $e$ is the unique edge of the cocycle $\phi$. By the description of critical cocycles, it follows that $\iota(e) \in \{A, B, C, D, E\}$.
Figure 11. Here is the tree $T$ that is under consideration in Example 5.6.

(2) The last 3 (or 4) entries are the numbers of cells lying in, respectively, the directions 0, 1, 2, and (possibly) 3 from $\iota(e)$. We therefore have a 4-tuple if $\iota(e) \in \{A, C, D, E\}$; 5-tuple if $\iota(e) \in \{B\}$. Note that this cell count includes the edge $e$.

(3) Finally, we indicate the location of the edge $e$ by placing a bar over the $(i + 2)$nd entry of the tuple, if the direction from $\iota(e)$ to $\tau(e)$ is $i$.

For instance, if we use this notation to describe the six critical cocycles which label the vertices in Figure 10, then, reading from the top left, we get

$$(B, 2, 1, \bar{2}); (A, 0, 1, \bar{4}); (B, 2, 2, \bar{1}); (A, 1, 1, \bar{3}); (B, 3, 1, \bar{1}); (A, 0, 2, \bar{3}).$$

We use a somewhat modified notation in Figure 12. For instance, the 4-tuple $(A, 0, 1, 3)$ is replaced with a circled “A”; the second, third, and fourth entries of the 4-tuple are written in clockwise order around the interior of the circle, beginning from below the “A”. The “3” is circled to indicate the location of the edge.

A vertex represents the cohomology class determined by its label. Two vertices are connected by an edge if they represent 1-cocycles having a common upper bound. In almost every case, the upper bound in question is a critical cocycle. There are exactly 7 edges representing non-critical cocycles (these edges are dotted in the picture; the numbering of the edges is keyed to the enumerated points below).

We consider each case in turn. Let, for instance, $(A, 0, 1, 3) - (E, 2, 1, \bar{1})$ denote the edge running between the indicated vertices.

(1) $(B, 0, 0, 2, 2) - (D, 2, 1, \bar{1}), (B, 0, 2, 0, 2) - (C, 2, 1, \bar{1}), (B, 0, 2, 2, 0) - (C, 2, 1, 1)$ Each of these edges represents a standard 2-cocycle which satisfies the hypotheses of Lemma 5.1. Accordingly, all of the cup products in question are 0.

(2) $(B, 1, 2, \bar{1}, 0) - (C, 2, 1, \bar{1})$ The standard 2-cocycle represented by the given edge has the form in Figure 13.

The bad edge $e$ of the standard cocycle $\phi$ with cloud picture $C$ depicted in Figure 13 a) satisfies $\iota(e) = B$. Figure 13 b) depicts the cochain $R_{C,B}$, where the central vertex of degree 4 belongs to the same cloud as the basepoint $\ast$. In Figure 13 c), we see $\delta(R_{C,B})$. If we apply the rewrite procedure from Theorem 4.5 to the final 3 terms in c) (and leave the first term alone), we arrive at the sum of standard cocycles in d). (The precise details in this last step are an exercise.) The sum $\Sigma$ in d) is clearly a sum of coboundaries – this is a direct consequence of the way we computed it – so $\Sigma = 0$ on cohomology. It follows directly that the cup product of the cohomology
Figure 12. Here is part of the simplicial complex $K$ which determines the cohomology ring $H^*(UD^4T)$. The seven dotted edges can be eliminated by a change of basis in $H^*(UD^4T)$. 
Figure 13. a) This standard cocycle $\phi$ is the result of cupping $(B, 1, 2, 1, 0)$ and $(C, 2, 1, 1)$. b) This is the cochain $R_{C, B}$. c) This is the coboundary $\delta(R_{C, B})$. d) This is the result of applying the procedure from Theorem 4.5 to the final three terms of the sum from c).

classes $(B, 1, 2, 1, 0)$ and $(C, 2, 1, 1)$ is represented by $\phi - \Sigma$, and $\phi - \Sigma$ is a sum of critical cocycles.

We would like to choose a basis for $H^*(UD^4T; \mathbb{Z})$ that will give the simplest possible formulas for the cup product. To achieve this, consider the result of breaking the left-hand edge in each of the cloud pictures of d). We arrive at the sum $\Sigma' = (B, 1, 2, 1, 0) + (B, 3, 1, 0) - (B, 2, 1, 1)$ of critical cocycles. It is straightforward to check that the cup product of $[\Sigma']$ with any other cohomology class of dimension at least 1 is 0. Indeed, by Lemma 5.2, it suffices to check that the cup product of $[\Sigma']$ with $[(C, 2, 1, 1)]$ is 0, but the product in question is represented by the sum $\Sigma$ in Figure 13 d), which is 0 on cohomology. We replace the class $[(B, 1, 2, 1, 0)]$ with $[(B, 1, 2, 1, 0) + (B, 0, 3, 1, 0) - (B, 0, 2, 1, 1)]$ in our standard basis.

(3) $(B, 1, 0, 2, 1) - (D, 2, 1, 1)$ This edge is labelled by the following cloud picture:

This case follows the pattern of 2); we apply the rewrite procedure from Theorem 4.5 to express the above cocycle $\phi$ as a sum of critical cocycles $\Sigma''$, i.e.,

$$\phi = \Sigma'',$$

where the equality is valid on the level of cohomology. The sum $\phi - \Sigma''$ is sum of standard cocycles, each of which has a given fixed edge $e$ such that $\iota(e) = D$. If we break this edge in each standard cocycle of the sum $\phi - \Sigma''$ and combine clouds, then we arrive at the sum $\hat{\Sigma} = (B, 1, 0, 2, 1) +$
This is the standard cocycle $\phi$ representing the cup product of $(B,1,0,2,\bar{1})$ with $(D,2,1,\bar{1})$.

$$(B,0,1,2,\bar{1}) + (B,0,0,3,\bar{1}).$$

This represents a cohomology class which, as in 2), has no non-zero cup products. We replace $[(B,1,0,2,\bar{1})]$ with $[\Sigma]$ in the standard basis.

(4) $(B,1,2,0,\bar{1}) - (C,2,1,\bar{1})$ This case again follows the pattern of 2) and 3).

We replace the cohomology class $[(B,1,2,0,1)]$ in the standard basis with $[(B,1,2,0,1) + (B,0,3,0,\bar{1}) + (B,2,1,\bar{1})]$.

(5) $(B,0,2,\bar{1},1) - (C,2,1,\bar{1})$ Replace $[(B,0,2,\bar{1},1)]$ with $[(B,0,2,\bar{1},1) + (B,0,2,1,\bar{1})]$.

Items 1)-5) show that we can arrange, after a change of basis, for the seven dashed edges in Figure 12 to be deleted. It is routine to check that the 2-dimensional cohomology classes which should label the edges in the remaining graph form a basis for $H^2(UD^4T)$. The cup product ring is thus the exterior face ring of the graph consisting of a copy of $K_4$ with three additional edges attached at each vertex, along with 34 isolated vertices.

\[\square\]

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