Linear theory of the structure of mechanisms

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Abstract. Structure analysis contains the definition of two structural characteristics, the one being the degree of abnormality in structure, which defines the balance of mobilities and structural bonds along the contour line of kinematic chain, and the other being the degree of irrationality, which defines balance of transverse mobilities and structural bonds; in combination they make structural equations of bonds and can be applied for structure analysis of plain and spatial mechanisms. Linear approach is also used to define the class of contour and whole mechanism because it shows the quantity of dimensions lacking for construction of the contour while binding the structural characteristics with the method of the following kinematic research: class of a contour is defined by the nonnegative value of the degree of its abnormality, while the class of a mechanism – by the algebraic sum of classes of the contours.

1. Tasks of the structure analysis
Structure analysis demands for a research of a structural layout of a mechanism for the purpose of assessment of its correspondence to two performance criteria: mobility – the capability of its elements to change their position in relationship to each other (positional relationship), and definiteness of motion (position) – unambiguity of positioning of the elements to the position of the input element. These criteria were originally stated by one of the founding fathers of the theory of mechanisms and machines Pafnuty Chebyshev together with the first structure formula in 1869 [1]. Structure (concept) layout with use of schematic symbols representing elements and flexible joints is presented without dimensions of the elements of positioning of axes of kinematic pairs, because during the stage of structure analysis one can’t distinguish between a structure layout of a plain and spatial mechanism. Due to the same reason such concepts like “plain formula” and “spatial formula” that were traditionally used for structure analysis of plain and spatial mechanisms become irrelevant [2,3].

A mechanism of a normal structure can be regarded as unconditionally operable if its layout meets two criteria of operability. The issue here is that the mechanisms that are really applied in the technics moved out of the framework of solely normal structures long ago and the technological development shows the necessity of use of mechanisms that have abnormal structure, for example, adaptive (differential) and indifferential mechanisms. In fact, mechanisms having abnormal structure and featuring special properties are the outstanding point for the development of technology; therefore the approach to structure analysis should be common and take into account their properties too. Leonid Assur, although his work [4] is devoted to the research and classification of groups of normal structure, pointed out the necessity of a research in the field of abnormal structures.

Another task of structure analysis is to define class of a mechanism, whereby it is known that the concept of a class binds the structure to kinematics and defines the method of a further kinematic
analysis. The literature shows different approaches to classification of mechanisms with different supportability of a classification feature, which, most often, has no physical sense [2, 3]. This drawback is mainly driven by the very group approach to the structural study that is aimed on the separation of structural layout into groups of normal structure that, as it is known, have kinematic definability; but the problem of finding the positions is in fact solved in a contour by contour but not in a group-by-group mode.

The solution for the aforementioned tasks is possible with use of a linear approach to the research of structure properties, at which structure bonds and mobilities are divided into axial and transversal in relation to the contour line of the kinematic chain.

2. Structure characteristics

The main dependencies that bind the structure parameters to the structure properties are structure formulas that serve for determination of the degree of abnormality and the degree of irrationality of a structure [5]. The degree of abnormality of a structure $S$ defines the balance of structural bonds and mobilities (possible motions) along the line of a contour generated by the axes of kinematic chain.

$$S = 3n - 2(p_5 + p_4 + p_3) - p_2 - p_1 - p_0,$$

where $n$ is the quantity of flexible elements;
$p_5$ - quantity of kinematic pairs of the fifth class (rotating and sliding);
$p_4$ - quantity of kinematic pairs of the fourth class (cylindrical);
$p_3$ - quantity of kinematic pairs of the third class (spherical);
$p_2$ - quantity of kinematic pairs of the second class (linear);
$p_1$ - quantity of kinematic pairs of the first class (pointed);
$p_0$ – quantity of input connections superposed on the input elements and arrowed.

Coefficients before the quantity of pairs signify that the unilateral (of the first and second class) produce only one connection (towards constriction/compression or extension) along the line of the contour, which is produced by a kinematic chain, and bilateral (lower pairs in Leonid Assur’s terms) pairs of the third, fourth and fifth class have two connections each (towards constriction/compression and extension). Formula (1) is the development of the known Chebyshev formula, which, as we know, took into account only rotating joints [1] and disregarded input connections. It is obvious that the theoretical justification would be also different; thus, for instance, the coefficient 3 before the quantity of flexible elements refers to the number of possible longitudinal motions relative to the three axes of reference within three-dimensional space, and not the quantity of plain mobilities.

The degree of irrationality of a structure $s$ defines a balance of structural bonds and mobilities around and across the line of the contour of an isolated kinematic chain. In this formula $3n$ equals the quantity of transversal possible motions subtracting transversal structural bonds.

$$s = 3n - 3p_5 - 2p_4 - p_3 - p_2.$$
tolerances for linear dimensions of the elements.

Correspondingly, the degree of irrationality equals zero: $s = 0$, if the structure is rational; if $s > 0$, there are transverse excessive mobilities relating to the contour line (along the axis of a kinematic pair); if $s < 0$, there are transverse excessive connections. Excessive connections arise in a closed contour and demand higher standards of accuracy of mutual arrangement of axes of the elements of kinematic pairs; therefore it is necessary to reduce their quantity as much as possible. The maximum quantity of excessive connections of such type in one closed contour comprised only of pairs of the fifth class (irrespective the quantity of elements) equals three; the decrease of class of kinematic pairs leads to the decrease of the quantity of excessive connections. Because the dimensions and relative position of elements is insignificant at the research of the structure, visualization of the linear model can be presented as a line for a kinematic chain with a random number of elements (see Figure 1, a); in this case contour line can be presented as a direct line that is rigidly fixed on both ends (see Figure 1, b). In this case the three excessive connections, that are mentioned afore, act on torsion $s_1$, flexion $s_2$ and shift $s_3$. Elimination of the three excessive connections makes the chain statically definable. On the other hand, the quantity of excessive connections equals the quantity of supplementary conditions that are necessary for static definability (equation of deformations) and kinematic mobility (requirement to positional relationship of axes of elements and pairs). Contour line of a non-looped chain is fixed on one end, therefore it contains no such conditions, i.e. has no transverse excessive connections.

![Figure 1. Linear model of a looped kinematic chain contour: a – kinematic chain line; b – structure connections types.](image)

The three aforementioned transverse excessive connections occur in every looped spatial or plain contour containing only fifth class pairs. Mobility of a plain contour is provided by alignment of axes of kinematic pairs, i.e. by kinematic (dimensional) conditions compensating the correspondent structural excessive connections.

The degree of abnormality according to the formula (1) on the Figure 1, b is represented by two connections: for stretching $S_1$ of the contour line and for compression $S_2$, whereby the chain is considered to be adaptive if $S>0$, and indifferent if $S<0$. Balance between mobilities and structural bonds is preserved in the chain with normal structure.

The formulas (1) and (2) at the member-by-member addition provide the so called Somov-Malyshev formula [2, 3], which is positioned as a spatial, but in fact provides a sum of degree of abnormality and degree of irrationality $W = S + s$, that in some cases can compensate each other and give a deficient idea of structural properties.

\[ W = 6n - 5p_5 - 4p_4 - 3p_3 - 2p_2 - p_1 - q. \]

In other words, this formula contains two unknowns $W$ and $q$, whereby instead of $p_0$ here we see $q$ – formally quantity of excessive connections, but actually this parameter should be an algebraic sum of excessive connections and redundant mobilities, both linear and transverse.

Quantity of input connections $p_0$ is an important structural parameter that influences the degree of abnormality (1) and, correspondingly, its structural properties. Four-bar linkage (see Figure 2, a) contains three flexible elements, four kinematic pairs of fifth class $A$, $B$, $C$ and $D$ and one input connection aligned with the element 1.

The degree of abnormality of the structure according to the formula (1) is as follows:

\[ S = 3n - 2p_5 - p_0 = 3 \cdot 3 - 2 \cdot 4 - 1 = 0 \]

i.e. the structure is normal, the quantity of structure connections equals the quantity of possible motions (mobilities).
The degree of irrationality of the four-bar linkage structure is as follows:

\[ s = 3n - 3p_5 = 3 \cdot 3 - 3 \cdot 4 = -3, \]

i.e. there are excessive connections within a looped contour produced by a kinematic chain. The connections are marked on the Figure 1, b as \( s_1, s_2, s_3 \). If there are no input connections in the four-bar linkage (Figure 2, b), its structure becomes adaptive along the contour line, and the same three excessive connections remain transversely to the contour line. One has to align two input connections (Figure 2, c) with the elements of a four-bar linkage structure to make it indifferent.

![Figure 2. Dependence of structural characteristics of a four-bar linkage on the quantity of input connections](image)

The same example proves the independence of structural characteristics: a change in the degree of abnormality doesn’t lead to a change in the degree of irrationality, and vice versa.

Non-looped chains have a rational structure. Besides the non-looped chains, gear-type units with helical engagement have rational structure. Gear-type three-link mechanism (see Figure 3) contains two flexible elements 1, 2, one of which is the input – gear 1, two are turning kinematic pairs of fifth class \( A \) and \( B \), as well as one unilateral pair \( C \), made up by gear and wheel teeth. If we regard this pair as linear (spur) it should be of the second class. Let us define structural characteristics:

\[ S = 3n - 2(p_5) - p_2 - p_0 = 3 \cdot 2 - 2 \cdot 2 - 1 - 1 = 0 \]
\[ s = 3n - 3p_5 - p_2 = 3 \cdot 2 - 3 \cdot 2 - 1 = -1 \]

i.e. the gear-type unit has normal structure, but there is one excessive connection transversely to the contour line. If this is a point (helical) gearing, i.e. if it builds up a kinematic pair of the first class, then \( s = 0 \) and the mechanism has rational structure.

![Figure 3. Gear-type three-link mechanism](image)

Finally, if the properties of adaptive mechanisms, containing linear excessive mobilities, were studied long ago and are known in the majority of ways (thus, for instance, in case of automotive differential), the structures with a linear excessive connection \( S =< 0 \) were named indifferent in the process of a research of the lever rounding mechanisms [6]. The use of a dialectic approach allows us to formulate their properties as opposed to the adaptive ones and from henceforth distribute their properties on all structures with linear excessive connections. Let us view this in the case of a crank-type sliding cross-head mechanism (Figure 4).
Figure 4. Approximate crank-type sliding cross-head mechanism of indifferent structure.

It contains a crank, tiller, slide and an arc rigidly bound to the tiller, drawn out of the guiding point M. We suppose that one of the arcs, shown on the figure, envelops the positions of a united line in its complex motion, whereby the line is parallel to the trajectory of the M point; in this case we may deem the arc and the line to be mutually enveloping elements of the kinematic pair of the first or second class. In this case the degree of abnormality (1) is

$$S = 3n - 2(p_5) - p_2 - p_0 = 3 \cdot 3 - 2 \cdot 4 - 1 - 1 = -1,$$

which refers to a linear excessive connection that is characteristic for an indifferent mechanism: if the aforementioned conditions are not met, the mechanism will lose its mobility. Shall we take into account all the three “arc-line” pairs shown on the figure there will be more excessive connections:

$$S = 3n - 2(p_5) - p_2 - p_0 = 3 \cdot 3 - 2 \cdot 4 - 3 - 1 = -3,$$

at that the mobility will remain, assuming that all structural excessive connections are compensated by the recurring kinematic conditions.

3. Mechanism class definition

The degree of abnormality of a structure and the degree of its irrationality are not only independent characteristics of the structure, but also universal characteristics, i.e. they are applicable for plain and for spatial mechanisms. Specifically, layouts of mechanisms can be imaged on every plane, including in axonometric projection, whereby the structural characteristics will differ and show the balance of structural bonds and mobilities specifically for this type of layout. To rectify the results, it is recommended to perform structure analysis on three levels of the depth of investigation: the first level consists of investigation of the projection on the plane, parallel to which the elements move, if the mechanism is plain, or on any other favorable plain, on which we can see positioning of the elements. The second level features the definition of structural characteristics of the layout, which is shown in axonometric projection, i.e. considering the recurrent connections (kinematic pairs). And the third level consists of a contour-by-contour structure analysis. The quantity of independent contours $c$ is defined according to the formula of Chaim Gochman:

$$c = \sum_{i=1}^{5} p_i - n$$  \hspace{1cm} (3)

where $p_i$ – is the quantity of kinematic pairs of all classes; and $n$ –in the quantity of flexible elements.

Independent contours are the contours in which there is at least one structure parameter, that isn’t part of other contours. For example, four-bar linkage on the Figure 2 contains three elements and four kinematic pairs of the fifth class; therefore the quantity of independent contours is as follows, according to the formula (3):
At that all structural parameters are calculated in this sole contour. Gear drive on the Figure 3 also contains one independent contour:
\[ c = \sum_{i=1}^{5} p_i - n = 4 - 3 = 1 \]
where \( \sum_{i=1}^{5} p_i = 3 \) – are two rotating and one linear kinematic pair;
\( n = 2 \) – are two flexible elements (gear wheels).

The layout on Figure 5 contains seven flexible elements and ten rotating kinematic pairs, because the quantity of independent contours equals three:
\[ c = \sum_{i=1}^{5} p_i - n = 10 - 7 = 3 \]

Let us assume contour \( K_1 \) to be the first. This contour contains an input element, three flexible elements and four rotating pairs. Its structural characteristics are:
\[ S = 3n - 2p_5 - p_0 = 3 \cdot 3 - 2 \cdot 4 - 1 = 0 \]
\[ s = 3n - 3p_5 = 3 \cdot 3 - 3 \cdot 4 = -3 \]

The obtained result shows that along the line of the second contour there is one excessive flexibility, and for construction of its position we lack one dimension, which means that the second contour is the first class contour. In the third contour \( K_3 \), comprising the remaining element and two rotating pairs, there is, vice versa, an excessive connection and, correspondingly, one odd dimension in the performance of a task on positions:
\[ S = 3n - 2p_5 = 3 \cdot 1 - 2 \cdot 2 = -1, \]
\[ s = 3n - 3p_5 = 3 \cdot 1 - 3 \cdot 2 = -3. \]

Because the negative value of the degree of abnormality of the contour speaks for a kinematic overdefiniteness. i.e. odd dimension, the performance of a task on positions has a solution for this particular contour, and it can be ranked as a zero class, as well as the first contour. As a general matter, algebraic sum of classes of the contours equals one, i.e. the mechanism belongs to the first class (even if the degree of its abnormality equals zero).

As for the degree of irrationality, because the three contours are constituted by pairs only of the fifth class, they all consist of three transversally-angular excessive connections each, irrespectively of the degree of abnormality.

Structural characteristics of the whole mechanism are algebraic sums of the correspondent structural characteristics of the contained contours:
\[ S = 3n - 2p_5 - p_0 = 3 \cdot 7 - 2 \cdot 10 - 1 = 0, \]

Figure 5. Structural layout with three independent contours.
\[ s = 3n - 3p_5 = 3 \cdot 7 - 3 \cdot 10 = -9, \]

which proves the normal structure of this layout.

Thus, let us agree to consider that non-negative value of the degree of abnormality of the contour to be the class of the contour, and algebraic sum of classes of the contours, being the parts of it – to be the class of a structural group and the whole mechanism. The class is, therefore, a connecting term between the structure and kinematics and shows how many dimensions (kinematic parameters) is lacking for construction of the position of this contour, group or mechanism, if we know all dimensions of the elements. According to the given definition, mechanisms on Figure 2, a, b and Figure 3 fall into zero class, and the layout on Figure 2, b and Figure 4 – first class. Correspondently, to construct the position of layouts of the mechanisms of zero class it is enough to know the dimensions of elements, and for the graphic construction of the position of mechanism of the first class we need to apply the method of false positions, or, in the analytic mode, simultaneous solution of the system of four equations of projections of contours 2 and 3.

Traditionally, the definition of a class is combined with the method of kinematic research, but we use quantity of sides of the looped contour, constituting structural group, as classification attribute. Insufficient justification of such criteria can be easily seen through the example of two driven structures with equal quantity of elements (4) and joints (6), one of which is conceptually regarded as the group of third class, the second – of the fourth class, which has no physical sense (see Figure 6). In fact this is one and the same group of normal structure, but the second group is the inversion of the first one, i.e. it is received by a replacement of a rack. Correspondently, the properties of the groups are identical and there should be one class. Let us illustrate this.

Figure 6. Inversion of the group of the first class.

Both layouts contain two independent contours each, but the second contour in the second layout is internal. The degree of abnormality of the first contour, containing three elements and four joints, is as follows:

\[ S = 3n - 2p_5 - p_0 = 3 \cdot 3 - 2 \cdot 4 - 0 = 1 \]

The degree of abnormality of the second contour, containing the remaining element and two joints:

\[ S = 3n - 2p_5 - p_0 = 3 \cdot 1 - 2 \cdot 2 - 0 = -1 \]

In general, the structure comes out to be normal, but the class of the first contour is first, and of the second – zero; therefore, both groups are of the first class and can be constructed by the method of false positions. In this particular case the class number has physical sense, because it is equal to the quantity of dimensions, necessary for solving of the task on positions for the purposes of the contour line.

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