Halphen’s transform and middle convolution

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Abstract
We show that the Halphen transform of a Lamé equation can be written as the symmetric square of the Lamé equation followed by an Euler transform. We use this to compute a list of Lamé equations with (non-) arithmetic Fuchsian monodromy group. It contains all those Lamé equations where the quaternion algebra $A$ over $k$ associated to the arithmetic Fuchsian group is a quaternion algebra $A$ over $\mathbb{Q}$.

1 Introduction

Besides the Gauss hypergeometric differential equation, perhaps the Lamé equations are the most well studied second order differential equations

$$p(x)y'' + \frac{1}{2}p'(x)y' + q(x)y = 0,$$

$$p(x) = 4\prod_{i=1}^{3}(x-e_i) = 4x^3 - g_2x - g_3, \quad q(x) = -(n(n+1)x - H)$$

Of special interest are those Lamé equations with finite monodromy group, having therefore algebraic solutions, studied by Baldassarri, Beukers and van der Waall, Chudnovsky and Chudnovsky, Dwork and many others (cf. [2], [4], [8] just to mention some papers). Lamé equations also occur in the context of Grothendieck’s $p$-curvature conjecture (cf. [8, p. 15]). This conjecture says that if the $p$-curvature of a differential equation is zero modulo $p$ for almost all primes $p$ then its monodromy group is finite. More generally, it is conjectured that if the $p$-curvature is globally nilpotent then the differential equation is geometric (also called coming from geometry, Picard-Fuchs) (s. [1, Chap. II §1]).

These conjectures are proven by Chudnovsky and Chudnovsky in the Lamé case for $n$ being an integer. In this case the monodromy group is a dihedral group or reducible (s. [8 Thm. 2.1]). Moreover they showed that, for a given exponent scheme, there is only a finite number of Lamé operators that are globally nilpotent (s. [8 Thm. 2.3]).

One also knows that if the monodromy group of a Lamé equation is an arithmetic Fuchsian group of signature $(1,e)$ then it comes from geometry. This gives examples of geometric second order differential equation with 4 singularities (s. [8]). In [17] Krammer determined one such example and showed that it is not a (weak) pullback of a hypergeometric differential equation contradicting a conjecture of Dwork that any globally nilpotent second order differential equation on $\mathbb{P}^1/\bar{\mathbb{Q}}$ has either algebraic solutions, or is a weak pullback of a Gauss hypergeometric differential equation (cf. [17, Section 11]).

But there is also the Halphen transform that changes the Lamé equation into another second order differential equation, again a Heun equation. This was used in [8] for $n =$

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Thus we obtain the equations arise from the classification of families of elliptic curves having 4 singular fibres. Here we have considered the case when the monodromy group of the above Heun equations is geometric. Going the converse way in Section 3, starting with special geometric Heun equations, that were jointly with H. Movasati computed in [18], listed in Table 3.1:

| Nr. in [18] | \( p(x) y'' + \frac{1}{2} p'(x) y' + \left(\alpha x + \frac{1}{2} H\right)y = 0 \) | \( p(x) \) |
|------------|-------------------------------------------------|----------------|
| 1          | \( p(x) y'' + p'(x) y' + (x + 1/3)y = 0 \)        | \( x(x^2 + x + 1/3) \) |
| 2          | \( p(x) y'' + p'(x) y' + xy = 0 \)                | \( x(x - 1)(x + 1) \) |
| 3          | \( p(x) y'' + p'(x) y' + (x - 1/4)y = 0 \)        | \( x(x - 1)(x + 1/8) \) |
| 4          | \( p(x) y'' + p'(x) y' + (x + 3)y = 0 \)          | \( x(x^2 + 11x - 1) \) |
| 5          | \( p(x) y'' + p'(x) y' + (35/36x - 3/2)y = 0 \)   | \( x(x^2 - 14/3x + 9) \) |
| 6          | \( \lambda p(x) y'' + p'(x) y' + ((35/36)x - 9/8)y = 0 \) | \( x(x - 1)(x - 81/32) \) |
| 7          | \( p(x) y'' + p'(x) y' + ((35/36)x - 45/2)y = 0 \) | \( x(x - 1)(x - 81/32) \) |
| 8          | \( p(x) y'' + p'(x) y' + ((35/36)x - 20/27)y = 0 \) | \( x(x - 1)(x - 32/27) \) |
| 9          | \( p(x) y'' + p'(x) y' + (15/16x + 1/8)y = 0 \)   | \( x(x^2 + 13/32x + 1/8) \) |
| 10         | \( p(x) y'' + p'(x) y' + (15/16x - 3/2)y = 0 \)   | \( x(x - 1)(x - 4) \) |
| 11         | \( p(x) y'' + p'(x) y' + (15/16x - 12)y = 0 \)    | \( x(x - 1)(x - 128/3) \) |
| 12         | \( p(x) y'' + p'(x) y' + (8/9x)y = 0 \)           | \( x(x - 1)(x + 1) \) |
| 13         | \( p(x) y'' + p'(x) y' + (8/9x - 8/27)y = 0 \)    | \( x(x - 1)(x - 2/7) \) |

Thus we obtain the Table 3.2 of geometric Lamé equations with (non)-arithmetic Fuchsian monodromy group:

| Nr. | \( p(x) y'' + \frac{1}{2} p'(x) y' - (n(n + 1)x - H)y = 0 \) | \( p(x) = \) |
|-----|-------------------------------------------------|----------------|
| 1   | \( p(x) y'' + \frac{1}{2} p'(x) y' + (1/4x + 1/12)y = 0 \) | \( 4x(x^2 + x + 1/3) \) |
| 2   | \( p(x) y'' + \frac{1}{2} p'(x) y' + (1/4)xy = 0 \)       | \( 4x(x - 1)(x + 1) \) |
| 3   | \( p(x) y'' + \frac{1}{2} p'(x) y' + (1/4x - 1/32)y = 0 \) | \( 4x(x - 1)(x + 1/8) \) |
| 4   | \( p(x) y'' + \frac{1}{2} p'(x) y' + (1/4x + 1/4)y = 0 \)  | \( 4x(x^2 + 11x - 1) \) |
| 5   | \( p(x) y'' + \frac{1}{2} p'(x) y' + (2/9x - 1/3)y = 0 \)  | \( 4x(x^2 - 14/3x + 9) \) |
| 6   | \( p(x) y'' + \frac{1}{2} p'(x) y' + ((2/9)x - 31/128)y = 0 \) | \( 4x(x - 1)(x - 81/32) \) |
| 7   | \( p(x) y'' + \frac{1}{2} p'(x) y' + ((2/9)x - 2)y = 0 \)  | \( 4x(x - 1)(x - 81) \) |
| 8   | \( p(x) y'' + \frac{1}{2} p'(x) y' + ((2/9)x - 7/36)y = 0 \) | \( 4x(x - 1)(x - 32/27) \) |
| 9   | \( p(x) y'' + \frac{1}{2} p'(x) y' + (3/16x + 3/128)y = 0 \) | \( 4x(x^2 + 13/32x + 1/8) \) |
| 10  | \( p(x) y'' + \frac{1}{2} p'(x) y' + (3/16x - 1/4)y = 0 \)  | \( 4x(x - 1)(x - 4) \) |
| 11  | \( p(x) y'' + \frac{1}{2} p'(x) y' + (3/16x - 13/12)y = 0 \) | \( 4x(x - 1)(x - 128/3) \) |
| 12  | \( p(x) y'' + \frac{1}{2} p'(x) y' + (5/36x)y = 0 \)       | \( 4x(x - 1)(x + 1) \) |
| 13  | \( p(x) y'' + \frac{1}{2} p'(x) y' + (5/36x - 1/36)y = 0 \) | \( 4x(x - 1)(x - 2/7) \) |

Here we have considered the case when the monodromy group of the above Heun equations is contained in SL₂(ℤ) and has at least 3 unipotent monodromy group generators. These equations arise from the classification of families of elliptic curves having 4 singular fibres in [13]. Thus being rational pull-backs of (geometric) Gauss hypergeometric differential equations they are geometric.
In general the Euler transform does not commute with pullbacks and in general destroys properties of the monodromy group like being arithmetic or even being discrete (s. e.g. [9]). Hence most of the Lamé equations obtained in the above way are not pull-backs of a Gauss hypergeometric differential equation.

In the literature we found only Lamé equations with arithmetic Fuchsian monodromy group (s. [17] and [8]) or geometric Heun equations with nondiscrete monodromy group (s. [9]) being no pullbacks of hypergeometric differential equations, providing counter examples to the above mentioned conjecture of Dwork. (Recently also differential equations with 5 singularities and non-arithmetic Fuchsian monodromy group were computed in [6].)

The Lamé equations with arithmetic Fuchsian groups appear all in principle in [8] but not all are listed there. In addition they are obtained via numerical solutions of the uniformization problem of punctured tori. However via our approach we think that Dwork’s conjecture can be modified in the following way: Any globally nilpotent second order differential equation on \( \mathbb{P}^1/\mathbb{Q} \) is related to a Gauss hypergeometric differential equation via geometric operations (cf. also Beukers’ way relating Krammer’s example to a Lauricella hypergeometric function of type FD and so saving Dwork’s conjecture).

In Section 4 we determine the monodromy group generators of the Heun equations we started with and show how to obtain the corresponding monodromy group generators of the Lamé equations.

2 The Halphen transform of a Lamé equation

We recall some properties of the Halphen transform taken from [19, chap IX] and [7, p. 60-62].

The Lamé equation can be written in the algebraic form

\[
p(x)y'' + \frac{1}{2}p'(x)y' + q(x)y = 0,
\]

\[
p(x) = 4\prod_{i=1}^{3}(x - e_i) = 4x^3 - g_2x - g_3. \quad q(x) = -(n(n+1)x - H)
\]

with Riemann scheme

\[
\left(\begin{array}{cccc}
e_1 & e_2 & e_3 & \infty \\
0 & 0 & 0 & -\frac{n}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{n+1}{2}
\end{array}\right)
\]

or in the elliptic form

\[
y'' - (n(n+1)p(u) - H)y = 0,
\]

where \( p(u) \) denotes the Weierstrass \( p(u) \)-function. The Halphen transform is obtained by putting

\[
u = 2v, \quad y = p'(v)^{-n}z.
\]

Then \( z \) satisfies

\[
z'' - 2n\frac{p''(v)}{p'(v)}z' + 4(n(2n-1)p(v) + H)z.
\]
Putting \( x = p(v) \) we get again the algebraic form
\[
p(x) y'' + \left( \frac{1}{2} - n \right) p'(x) y' + 4(n(2n - 1)x + H)y = 0
\]
with Riemann scheme
\[
\begin{pmatrix}
e_1 & e_2 & e_3 & \infty \\
0 & 0 & 0 & -2n \\
n + \frac{1}{2} & n + \frac{1}{2} & n + \frac{1}{2} & \frac{1}{2} - n
\end{pmatrix}.
\]
We will show that the Halphen transform of a Lamé equation can be written as the symmetric square of the Lamé equation followed by an Euler transform.

Thus we recall the

**Remark. 2.1.** ([7, p. 59]) The symmetric square of a second order differential equation
\[
y'' + q_1(x)y' + q_2(x)y = 0
\]
can be written as
\[
y''' + 3q_1(x)y'' + (q_1'(x) + 4q_2(x) + 2q_1(x)^2)y' + 2(q_2'(x) + 2q_1(x)q_2(x))y = 0.
\]
(I.e. all products of solutions of (1) satisfy (2)). Moreover in the Lamé case we have
\[
p(x) = 4(x - e_1)(x - e_2)(x - e_3),
\]
and therefore the symmetric square of a Lamé equation is
\[
(3) \quad p(x)y''' + \frac{3}{2} p'(x)y'' + \left( \frac{p''(x)}{2} - 4(n(n + 1)x - H) \right)y' - 2n(n + 1)y = 0.
\]

Using the formula for Euler integral in [16, Chap. 3.3, 3.4] we get the following

**Lemma. 2.2.** Let \( f \) be a solution of (1). Then the Euler integral \( \int_\gamma f(t)(x - t)^{-1-\mu}dt \) over a Pochhammer double loop \( \gamma \) satisfies
\[
p(x) y''' + \left( \frac{3}{2} + \mu \right) p'(x)y'' + r_1(x)y' + r_0(x)y = 0,
\]
where \( p(x) = 4(x - e_1)(x - e_2)(x - e_3), \)
\[
r_1(x) = 4(6x + 2(- \sum e_i))(\frac{\mu(\mu - 1)}{2} + \frac{3}{2}\mu + \frac{1}{2}) - 4(n(n + 1)x - H),
\]
\[
r_0(x) = 4\mu^3 + 6\mu^2 + 2\mu - 4\mu(n(n + 1)) - 2n(n + 1) = 2(2\mu + 1)(\mu - n)(\mu + n + 1).
\]
Thus if we choose \( \mu \) such that \( r_0(x) = 0 \) then we get again a second order differential equation:

**Corollary. 2.3.** Let \( p(x) = 4(x - e_1)(x - e_2)(x - e_3) \). Then we get the following special cases in the above lemma.
a) If \( \mu = -\frac{1}{2} \) then
\[
p(x)y'' + p'(x)y' + (4H - \sum_{i=1}^{3} e_i - (2n + 3)(2n - 1)x)y = 0.
\]
b) If \( \mu = n \) then
\[
p(x)y'' + (\frac{3}{2} + n)p'(x)y' + (4H - (n + 1)^2 \sum_{i=1}^{3} e_i + 4(n + 1)(2n + 3)x)y = 0.
\]
c) If \( \mu = -n - 1 \) then
\[
p(x)y'' + (\frac{1}{2} - n)p'(x)y' + (4(H - n^2 \sum_{i=1}^{3} e_i) + 4n(2n - 1)x)y = 0.
\]

The case c) gives the Halphen transform, since we had there assumed \( \sum_{i=1}^{3} e_i = 0 \).

Since convolution and tensor products are geometric operations (cf. [1, Chap. II §1]) we get the following

**Corollary. 2.4.** The Halphen transform preserves geometric differential equations.

**Remark. 2.5.** Applying the Euler integral and factoring out trivial subspaces is exactly the middle convolution operation (s. [11]).

We apply Corollary 2.3 a) to the following example studied by Krammer in [17], which is considered as a counterexample to a conjecture of Dwork, being not a (weak) pull-back of a Gauss hypergeometric differential equation: (It also appears in [8, p. 23])

**Example 2.6.** The geometric Lamé equation with arithmetic monodromy group of signature (1,3)
\[
p(x)y'' + \frac{1}{2} p'(x)y' + \left(\frac{2}{9}x - 2\right)y, \quad p(x) = 4x(x - 1)(x - 81)
\]
becomes after applying Corollary 2.3 a) the Heun equation
\[
p(x)y'' + p'(x)y' + \left(\frac{35}{9}x - 90\right)y = 0
\]
with unipotent local monodromy at 0, 1, and 81. We will see in the next section (Table 3.1, row 9) that this differential equation is a rational pullback of a hypergeometric differential equation.

### 3 Examples

In this section we go the converse way of Corollary 2.3 to obtain a list of Lamé equations with (arithmetic) Fuchsian monodromy group. We start with a list of 13 second order differential equations having 4 regular singularities (Heun equations). All these arise from rational pull-backs of the Gauss hypergeometric function \( _2F_1(\frac{1}{12}, \frac{1}{12}, \frac{5}{4}, x) \). In addition their monodromy group is contained in \( \text{SL}_2(\mathbb{Z}) \) and they posses at least 3 unipotent monodromy group generators. Note that some of the 17 differential equations we obtained in [18] from Herfurtner’s list, where Herfurtner has classified families of elliptic curves with 4 singular fibres, coincide. Thus only 13 remain.
Table 3.1. Heun equations having at least 3 unipotent monodromy group generators and monodromy group in $SL_2(\mathbb{Z})$ being pullbacks of Gauss hypergeometric differential equations, taken from [18]:

| Nr. | Herfurtner’s notation in [18] | $p(x)y'' + p'(x)y' + (\alpha \beta x + \tilde{H})y = 0$ | $p(x)$ |
|-----|-------------------------------|-------------------------------------------------|--------|
| 1   | $I_1I_1I_9$                  | $p(x)y'' + p'(x)y' + (x + 1/3)y = 0$          | $x(x^2 + x + 1/3)$ |
| 2   | $I_1I_1I_5$                  | $p(x)y'' + p'(x)y' + xy = 0$                   | $x(x-1)(x+1)$ |
| 3   | $I_1I_2I_3$                  | $p(x)y'' + p'(x)y' + (x - 1/4)y = 0$          | $x(x-1)(x+1/8)$ |
| 4   | $I_1I_1I_5I_5$              | $p(x)y'' + p'(x)y' + (x + 3)y = 0$            | $x(x^2 + 11x - 1)$ |
| 7   | $I_1I_1I_5I_5$              | $p(x)y'' + p'(x)y' + (35/36x - 3/2)y = 0$    | $x(x^2 - 14/3x + 9)$ |
| 8   | $I_1I_1I_5I_5$              | $p(x)y'' + p'(x)y' + ((35/36)x - 9/8)y = 0$ | $x(x-1)(x - 81/32)$ |
| 9   | $I_1I_1I_5I_5$              | $p(x)y'' + p'(x)y' + ((35/36)x - 45/2)y = 0$ | $x(x-1)(x - 81)$ |
| 10  | $I_2I_3I_5I_5$              | $p(x)y'' + p'(x)y' + ((35/36)x - 20/27)y = 0$, | $x(x-1)(x - 32/27)$ |
| 11  | $I_1I_1I_5I_5$              | $p(x)y'' + p'(x)y' + (15/56x + 1/8)y = 0$   | $x(x^2 + 13/32x + 1/8)$ |
| 12  | $I_1I_1I_5I_5I_5$           | $p(x)y'' + p'(x)y' + (15/56x - 3/2)y = 0$    | $x(x-1)(x - 4)$ |
| 13  | $I_1I_1I_5I_5I_5$           | $p(x)y'' + p'(x)y' + (15/56x - 12)y = 0$,    | $x(x-1)(x - 128/3)$ |
| 15  | $I_1I_1I_5I_5$              | $p(x)y'' + p'(x)y' + (8/9x)y = 0$,            | $x(x-1)(x + 1)$ |
| 16  | $I_1I_1I_5I_5$              | $p(x)y'' + p'(x)y' + (8/9x - 8/27)y = 0$,    | $x(x-1)(x - 2/27)$ |

Herfurtner’s list is indexed by the local monodromy in $SL_2(\mathbb{Z})$, where $I_k$ denotes the unipotent class in $SL_2(\mathbb{Z})$ with entry $k$ in the upper diagonal and $II, III$ and $IV$ classes of elliptic elements of order 6, 4 and 3 resp.

This list gives rise to the following list of Lamé equations via Corollary [23] and the relations between the coefficients of the Heun equation and the Lamé equation given there:

$-n(n + 1) = \alpha \beta - 3/4, \quad H = \tilde{H} + \sum_{i=1}^{3} (c_i)/4$

Thus we get the following table of geometric Lamé equations:

Table 3.2.

| Nr. | [8] p.23 | $p(x)y'' + \frac{1}{7}p'(x)y' - (n(n+1)x - H)y = 0$ | $p(x)$ |
|-----|----------|-------------------------------------------------|--------|
| 1   | $p(x)y'' + \frac{1}{7}p'(x)y' + (1/4x + 1/12)y = 0$ | $4x(x^2 + x + 1/3)$ |
| 2   | $p(x)y'' + \frac{1}{7}p'(x)y' + (1/4x + 1/4)y = 0$ | $4x(x - 1)(x + 1)$ |
| 3   | $p(x)y'' + \frac{1}{7}p'(x)y' + (1/4x - 3/2) = 0$ | $4x(x - 1)(x + 1/8)$ |
| 4   | $p(x)y'' + \frac{1}{7}p'(x)y' + (1/4x + 1/4)y = 0$ | $4x(x^2 + 11x - 1)$ |
| 7   | $p(x)y'' + \frac{1}{7}p'(x)y' + (2/9x - 1/3) = 0$ | $4x(x^2 - 14/3x + 9)$ |
| 8   | $(1,3); (2)$ | $p(x)y'' + \frac{1}{7}p'(x)y' + (2/9)x - 31/128y = 0$ | $4x(x-1)(x - 81/32)$ |
| 9   | $(1,3); (4)$ | $p(x)y'' + \frac{1}{7}p'(x)y' + (2/9)x - 2y = 0$, | $4x(x-1)(x - 81)$ |
| 10  | $p(x)y'' + \frac{1}{7}p'(x)y' + (2/9)x - 7/36y = 0$, | $4x(x-1)(x - 32/27)$ |
| 11  | $p(x)y'' + \frac{1}{7}p'(x)y' + (3/16x + 3/128)y = 0$ | $4x(x^2 + 13/32x + 1/8)$ |
| 12  | $(1,2); (2)$ | $p(x)y'' + \frac{1}{7}p'(x)y' + (3/16x - 1/4)y = 0$, | $4x(x-1)(x - 4)$ |
| 13  | $(1,2); (3)$ | $p(x)y'' + \frac{1}{7}p'(x)y' + (3/16x - 13/12)y = 0$, | $4x(x-1)(x - 128/3)$ |
| 15  | $p(x)y'' + \frac{1}{7}p'(x)y' + (5/36x)y = 0$, | $4x(x-1)(x + 1)$ |
| 16  | $p(x)y'' + \frac{1}{7}p'(x)y' + (5/36x - 1/36)y = 0$, | $4x(x-1)(x - 2/27)$ |
The entry $(1, e); (x)$ in the second column refers to the Lamé equation $(x)$ with arithmetic monodromy group of signature $(1, e)$ given in [8, p. 23].

**Remark. 3.3.** (i) It is mentioned in [8, Section 3] that for all 73 (s. [22]) arithmetic Fuchsian subgroups $\Gamma$ of signature $(1, e)$

$$\Gamma = \langle \alpha, \beta, \gamma | \alpha \beta \alpha^{-1} \beta^{-1} \gamma = -1_2, \gamma^e = -1_2 \rangle,$$

where $\alpha$ and $\beta$ are hyperbolic elements of $\text{SL}_2(\mathbb{R})$, there exists a corresponding Lamé equation, defined over $\overline{\mathbb{Q}}$. Using numerical solutions of the (inverse) uniformization problem for the punctured tori they were computed. But only some of these we listed there. Our list of Lamé equations contains all those where the quaternion algebra $A$ over $k$ associated to the arithmetic Fuchsian group is a quaternion algebra $A$ over $\mathbb{Q}$ (s. [22]), i.e. Nr. 1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13.

(ii) Note that Nr. 15 and 16 do not have an arithmetic monodromy group, since these examples do not appear in the classification of Takeuchi (s. [22]). However since

$$|\text{tr}(A_1A_2A_3)| = 1 = 2\cos(\pi/3),$$

the monodromy group is Fuchsian with signature $(0; 2, 2, 3, 0)$, s. [20, Thm 1].

**Proof.** It follows from the following section that the monodromy group of the Lamé equation is contained in $\text{SL}_2(\mathbb{R})$. Thus the comparison with the Fricke parameters in [22] and Lemma 4.9 (using Remark 4.11) yields the claim. \hfill \Box

**Corollary. 3.4.** Nr. 15 and 16 provide examples of second order differential equations with 4 singularities and nonarithmetic Fuchsian monodromy group coming from geometry. Further the monodromy group is not commensurable to a triangle group. This follows from the well known fact that the trace field $\mathbb{Q}(\text{tr}(\Gamma^2))$ generated be the traces of the squares of the elements of a finitely generated non-elementary subgroup $\Gamma$ of $\text{SL}_2(\mathbb{C})$ is an invariant of the commensurability class.

Lamé equations with unipotent monodromy at infinity were already studied by Chudnovsky and Chudnovsky in [8] via Halphen transform and symmetric squares. There it was also mentioned that the Heun cases 1, 2, 3, 4 are pull-backs of hypergeometric differential equations. Using computer aided computations the following conjecture was stated:

**Conjecture 3.5** (Chudnovsky and Chudnovsky). Lamé equations with $n = -\frac{1}{2}$ defined over $\mathbb{Q}$ are not globally nilpotent except for the 4 classes listed as 1, 2, 3, 4 above.

Note that Beukers also studied Heun equations with 4 unipotent monodromy group generators in [3] and Lamé equations with unipotent monodromy in [5]. Next we list Heun equations with 3 unipotent monodromy group generators via rational Belyi functions (i.e. rational function which are only ramified at 0, 1 and $\infty$) that do not appear in [18]. In these cases the monodromy group is a subgroup of a nonarithmetic triangle group. The conditions for right choice of Belyi-functions $j(x)$ and the hypergeometric differential equation follow from [18, Sec. Belyi functions]. Since the computation of the Heun equations is analogous to the one in [18] we skip it. We only list $j(x)$ and the corresponding hypergeometric function that yield the Heun equation.
Lemma. 3.6. Let \( j(x) = \frac{j_1(x)}{j_2(x)} \) and \( _2F_1(a, b, c, x) \) be as in the list below:

\[
j(x) \quad \text{ramification data} \quad _2F_1(a, b, c, x) \quad \text{Riemann scheme}
\]

\[
i) \quad \frac{(x^2 - 10x + 5)^2x}{(5x^2 - 10x + 1)^x} \quad (2, 2, 1), (5), (2, 2, 1) \quad _2F_1(\frac{13}{20}, \frac{3}{20}, 1, x) \quad \begin{pmatrix} 0 & 1 & \infty \end{pmatrix}
\]

\[
ii) \quad \frac{64(3x - 1)^2}{272x^4(3x^2 + 285x + 40)} \quad (5), (2, 2, 1), (3, 1, 1) \quad _2F_1(\frac{3}{20}, \frac{3}{20}, 1, x) \quad \begin{pmatrix} 0 & 1 & \infty \end{pmatrix}
\]

\[
iii) \quad \frac{(x + 80)^3x^2(25x - 48)}{64(3x - 1)^4} \quad (3, 2, 1), (2, 2, 2), (5, 1) \quad _2F_1(\frac{7}{20}, \frac{3}{20}, 1, x) \quad \begin{pmatrix} 0 & 1 \end{pmatrix} \end{pmatrix}
\]

\[
iv) \quad \frac{-x^4(25x^2 + 44x + 20)}{256(x + 1)^2} \quad (4, 1, 1), (2, 2, 2), (5, 1) \quad _2F_1(\frac{7}{20}, \frac{3}{20}, 1, x) \quad \begin{pmatrix} 0 & 1 \end{pmatrix} \end{pmatrix}
\]

\[
v) \quad \frac{-(x - 1)^2(81x^2 + 14x + 1)}{256x^3} \quad (2, 1, 1), (4), (3, 1) \quad _2F_1(\frac{13}{20}, \frac{5}{20}, 1, x) \quad \begin{pmatrix} 0 & 1 \end{pmatrix} \end{pmatrix}
\]

Then the function \( j_2^{-a}(x) \) \( _2F_1(a, b, c, j(x)) \) is a solution of

\[
p(x)g'' + p'(x)g' + (a/3x - q)y = 0 \quad p(x)
\]

\[
i) \quad p(x)g'' + p'(x)g' + (15/16x - 9/16)y = 0 \quad x(x^2 - 2x + 1/5)
\]

\[
ii) \quad p(x)g'' + p'(x)g' + (15/16x - 9/4)y = 0 \quad x(x^2 - 57/8x + 72/5)
\]

\[
iii) \quad p(x)g'' + p'(x)g' + (99/100x + 45/4)y = 0 \quad x(x - 1)(x + 125/3)
\]

\[
iv) \quad p(x)g'' + p'(x)g' + (99/100x + 3/4)y = 0 \quad x(x^2 + 11/5x + 5/4)
\]

\[
v) \quad p(x)g'' + p'(x)g' + (35/36x + 5/8)y = 0 \quad x(x^2 + 176/81x + 96/81)
\]

Corollary. 3.7. Via Corollary 2.3 we get the following Lamé equations:

\[
p(x)g'' + \frac{1}{p(x)}p'(x)y' + (\alpha/3x - q)y = 0 \quad p(x)
\]

\[
i) \quad p(x)g'' + \frac{1}{p(x)}p'(x)y' + (3/16x - 1/16)y = 0 \quad 4x(x^2 - 2x + 1/5)
\]

\[
ii) \quad p(x)g'' + \frac{1}{p(x)}p'(x)y' + (3/16x - 15/32)y = 0 \quad 4x(x^2 - 57/8x + 72/5)
\]

\[
iii) \quad p(x)g'' + \frac{1}{p(x)}p'(x)y' + (6/25x + 13/12)y = 0 \quad 4x(x - 1)(x + 125/3)
\]

\[
iv) \quad p(x)g'' + \frac{1}{p(x)}p'(x)y' + (6/25x + 1/5)y = 0 \quad 4x(x^2 + 11/5x + 5/4)
\]

\[
v) \quad p(x)g'' + \frac{1}{p(x)}p'(x)y' + (2/9x + 53/648)y = 0 \quad 4x(x^2 + 176/81x + 96/81)
\]

Note that example iii) appears (after the Möbius transformation \( \phi(x) = (-125/3 - 1)x + 1 \)) in [5, p.23/24] and has an arithmetic Fuchsian monodromy group with signature \((1, 5)\), while the corresponding Heun equation has no arithmetic monodromy group (s. [21]).

Proof. This follows from Corollary 2.3 as in the computation of Table 3.2.

Also geometric Heun equations with 4 equal exponent differences and monodromy group contained in SL_2(\mathbb{R}) yield after the inverse Halphen transform geometric Lamé equations: Those Heun equations can be computed for example as rational pullbacks of Gauss hypergeometric differential equations with local projective monodromy order \((2, 3, 7), (2, 3, 8), (2, 3, 9), (2, 3, 10)\) or \((3, 3, 5)\). In this cases the Heun equation has local projective monodromy orders \((3, 3, 3)\). We demonstrate this via the following example:
**Example 3.8.** If one uses the Belyi function \(j(x) = \frac{(x-1)^3(x^2+3x+6)}{10x^2-15x+6}\) with ramification data \((3)(1)(1), (5), (3)(1)(1)\) as pullback for \(2F_1\left(\frac{1}{15}, \frac{2}{5}, \frac{2}{3}, x\right)\) one obtains after a Möbius transformation the Heun equation

\[
p(x)y'' + \frac{2}{3}p'(x)y' + \left(\frac{2}{9}x - 44/243\right)y = 0, \quad p(x) = x(x - 1)(x - 32/27).
\]

The corresponding monodromy group is a subgroup of finite index of the arithmetic triangle group corresponding to \(2F_1\left(\frac{1}{15}, \frac{2}{5}, \frac{2}{3}, x\right)\). Applying the inverse Halphen transformation we get using Corollary 2.3(c) the Lamé equation

\[
p(x)y'' + \frac{1}{2}p'(x)y' - (n(n+1)x-H)y = 0, \quad p(x) = 4x(x-1)(x-32/27), \quad n = \frac{-1}{6}, H = \frac{13}{108}.
\]

### 4 Monodromy

Here we determine the monodromy group generators of the Heun equations in Table 3.1. We will state some remarks concerning the change of the monodromy group under the symmetric square and the Euler-integral with \(\mu = -\frac{1}{2}\). Hence we can determine the monodromy group generators of the corresponding Lamé equations.

Via the Riemann-Hilbert correspondence a second order Fuchsian differential equation with 4 singularities is uniquely determined by its monodromy group generators:

**Definition. 4.1.** We call a tuple \(A = (A_1, \ldots, A_4), A_i \in \text{SL}_2(\mathbb{C})\), a tuple of monodromy group generators in \(\text{SL}_2(\mathbb{C})\) if

\[
A_1A_2A_3A_4 = \text{id}_2.
\]

It is well known that the monodromy group representation is uniquely determined by the Fricke parameters:

**Remark. 4.2.** [12] p. 365-366] Let \(A\) be a tuple of monodromy group generators in \(\text{SL}_2(\mathbb{C})\) and

\[
a_1 = \text{tr}(A_1), \quad a_2 = \text{tr}(A_2), \quad a_3 = \text{tr}(A_3), \quad a_4 = \text{tr}(A_4)
\]

and

\[
x = \text{tr}(A_1A_2), \quad y = \text{tr}(A_2A_3), \quad z = \text{tr}(A_1A_3).
\]

Then the parameters \((a_1, a_2, a_3, a_4, x, y, z)\) satisfy the Fricke relation

\[
\sum_{i=1}^{4} a_i^2 + \prod_{i=1}^{4} a_i + x^2 + y^2 + z^2 + xyz - (a_1a_2 + a_3a_4)x - (a_1a_4 + a_2a_3)y - (a_1a_3 + a_2a_4)z = 4.
\]

A nice well known application is the following

**Corollary. 4.3.** Let the monodromy group act irreducibly. Then it leaves a hermitian form invariant if and only if all Fricke parameters are real numbers.

If the form is positive definite then the group is contained in \(\text{SU}_2(\mathbb{R})\) and if it is indefinite then the group is contained in \(\text{SL}_2(\mathbb{R})\).

**Corollary. 4.4.** Let the monodromy group of a Lamé equation act irreducibly and leave an indefinite hermitian form invariant. Then the Fricke parameters \((x, y, z)\) are of absolute value \(\geq 2\).
Proof. The group generated by $A_1$ and $A_2$ is an irreducible dihedral group. If the product would be an elliptic element then this subgroup would leave a positive definite form invariant or $A_1A_2 = \pm \text{Id}_2$. The latter case implies that the monodromy group is a dihedral group. Thus the claim follows.

It is also well known that the braid group $B_2 = \langle \beta_1, \beta_2 \rangle$ acts on monodromy group generators the following way: (This describes the deformation of the paths $\gamma_i$ in the fundamental group)

$$\pi(\mathbb{P}^1 \setminus \{t_1, t_2, t_3, \infty\}, x_0) = \langle \gamma_1, \ldots, \gamma_4 \mid \gamma_1 \cdots \gamma_4 = 1 \rangle$$

if we switch the singularities.

Remark. 4.5. The braid group $B_2 = \langle \beta_1, \beta_2 \rangle$ acts on $A$ via

$$\beta_1(A) = (A_2, A_2^{-1}A_1A_2, A_3, A_4) \quad \beta_2(A) = (A_1, A_3, A_3^{-1}A_2A_3, A_4)$$

$$\beta_1 : (a_1, a_2, a_3, a_4, x, y, z) \mapsto (a_2, a_1, a_3, a_4, x, \tilde{z}, y), \quad \tilde{z} = a_1a_3 + a_2a_4 - z - xy$$

$$\beta_2 : (a_1, a_2, a_3, a_4, x, y, z) \mapsto (a_1, a_3, a_2, a_4, z, y, \tilde{x}), \quad \tilde{y} = a_1a_2 + a_3a_4 - x - yz$$

We consider the case where we have local unipotent monodromy at least at 3 singularities.

Corollary. 4.6. Let $A_1, \ldots, A_3$ be unipotent elements and

$$x = n_1 + 2, \quad y = n_2 + 2, \quad z = n_3 + 2.$$ 

Then

$$(\sum n_i + 2 - a_4)^2 + \prod n_i = 0.$$ 

Moreover, for special values of $a_4$ we get the following relations:

(i) If $A_4$ is also unipotent, then $a_i = 2, i = 1, \ldots, 4$, and the Fricke relations simplify to

$$(x - 4)^2 + (y - 4)^2 + (z - 4)^2 = 20 - xyz.$$ 

If we put

$$x = n_1N + 2, \quad y = n_2N + 2, \quad z = n_3N + 2, \quad N = \gcd(n_1, n_2, n_3) \in \mathbb{N}$$

we obtain

$$(n_1 + n_2 + n_3)^2 + n_1n_2n_3N = 0.$$ 

Further the second solution $n_i'$ of the quadratic equation for $n_i$ is obtained via the corresponding braid group action and we get $n_in_i' = (n_j + n_k)^2$.

(ii) If $A_4$ is an elements of order 4 then $a_4 = 0$ and

$$(\sum n_i + 2)^2 + \prod n_i = 0.$$ 

(iii) If $A_4$ is an elements of order 6 then $a_4 = 1$ and

$$(\sum n_i + 1)^2 + \prod n_i = 0.$$
(iv) If $A_4$ is an elements of order 3 then $a_4 = -1$ and
\[
(\sum n_i + 3)^2 + \prod n_i = 0.
\]

**Proof.** This follows using the above remark and direct computations with the Fricke relations.
\[
x^2 + y^2 + z^2 + xyz - 4 = (\sum n_i + 4)^2 + \prod n_i
\]
\[
\sum_{i=1}^{4} a_i^2 + \prod_{i=1}^{4} a_i - (a_1 a_2 + a_3 a_4)x - (a_1 a_4 + a_2 a_3)y - (a_1 a_3 + a_2 a_4)z = (2 + a_4)^2 - 2(2 + a_4)(\sum n_i + 4).
\]
Since $a_1 = a_2 = a_3 = 2$ the claim is readily to check. □

**Corollary. 4.7.** Let $A$ be a tuple of monodromy group generators in $\text{SL}(\mathbb{C})$ with unipotent elements $A_1, \ldots, A_3$. If one of the Fricke parameters $x = n_1 + 2, y = n_2 + 2, z = n_3 + 2$ is greater than 2 then the minimal triples in the braid group orbit are of the form
\[
(x, y, z) = (n_1 + 2, 2, -n_1 + a_4)
\]
or
\[
(x, y, z) = (n_1 + 2, n_1 + 2, -2), \quad a_4 = -2.
\]

The corresponding monodromy group generators are
\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & -1 + \frac{a_4 - 2}{n_1} \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
-n_1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & -\frac{a_4 - 2}{n_1} \\
-n_1 & a_4 - 1
\end{pmatrix}, \quad n_1 \neq 0.
\]
\[
\begin{pmatrix}
1 & b \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & c \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
-1 & -4 \\
1 & 3
\end{pmatrix},
\begin{pmatrix}
-1 & n_1 + 4 \\
0 & -1
\end{pmatrix}.
\]

**Proof.** By Corollary 4.6 we have
\[
(\sum n_i + a)^2 + \prod n_i = 0, \quad 0 \leq 2 - a_4 = a \leq 4.
\]
Let $(n_1, n_2, n_3)$ be a minimal solution in the braid group orbit. Then
\[
|n_2 + n_3 + a| \geq n_1 \geq n_2 \geq 0, \quad 0 \leq -n_3 \leq |n_1 + n_2 + a|.
\]

Case i) Let $n_1 + n_3 + a \geq 0$. If $n_3 + a \geq 0$ then
\[
(n_1 + n_2)^2 \leq (n_1 + n_2 + n_3 + a)^2 = -n_1 n_2 n_3 \leq n_1 n_2 a \leq 4 n_1 n_2.
\]
This implies
\[
n_1 = n_2, \quad a = 4 = -n_3.
\]
If $n_3 + a < 0$ then
\[
n_1 \leq |n_2 + n_3 + a| \leq |n_3 + a| \leq n_1.
\]
Hence $n_1 + n_3 + a = 0$, $n_2 = 0$.

Case ii) Let $n_1 + n_3 + a < 0$. Then
\[
n_2^2 \geq ((n_1 + n_3 + a) + n_2)^2 = -n_1 n_2 n_3,
\]
since $-n_3 \leq (n_1 + n_2 + a)$. Thus $n_2 = 0$. □
Lemma. 4.8. Let $x, y, z$ be negative integers satisfy the relation in corollary 4.6. Then there exist in the braid group orbit the following minimal solutions (with respect to $|x| + |y| + |z|$):

| case | $a_4$ | $(n_1, n_2, n_3)$ |
|------|------|------------------|
| i)   | $a_4 = 2$ | $N = 5$ ($-1, -4, -5$) |
|      |       | $N = 6$ ($-1, -2, -3$) |
|      |       | $N = 8$ ($-1, -1, -2$) |
|      |       | $N = 9$ ($-1, -1, -1$) |
| ii)  | $a_4 = 0$ | $(-5, -12, -15)$ ($-6 - 8, -12$) ($-7, -7, -9$) |
| iii) | $a_4 = 1$ | $(-5, -16, -20)$ ($-6, -10, -15$) ($-7, -8, -14$) |
|      |       | $(-8, -8, -9)$ |
| iv)  | $a_4 = -1$ | $(-5, -8, -10)$ ($-6, -6, -9$) |

Proof. Case i): Let $a_2 = 2$ and $n_3 \leq n_2 \leq n_1 < 0$ be a minimal solution. Then we get a second solution $(n_3', n_2, n_1)$ with $n_3' \leq n_2 \leq n_1 < 0$ by minimality, where $n_3n_3' = (n_1 + n_2)^2$. Thus

$$-n_3 \leq -(n_1 + n_2)$$

and

$$4n_3^2 \leq (n_1 + n_2 + n_3)^2 = -Nn_1n_2n_3.$$ 

On the other hand we get the inequality

$$9n_3^2 \geq -Nn_1n_2n_3 \iff \frac{-n_3}{n_1n_2} \geq \frac{N}{9} (\iff 0 \geq n_1 \geq \frac{-9}{N}).$$

This gives

$$\frac{N}{4} \geq \frac{-n_3}{n_1n_2} \geq \frac{-n_1 - n_2}{n_1n_2} = \frac{-1}{n_1} + \frac{-1}{n_2}.$$ 

On the other hand we have

$$4(n_1 + n_2)^2 \geq (n_1 + n_2 + n_3)^2 = -Nn_1n_2n_3.$$ 

Hence

$$4(-\frac{1}{n_1} + \frac{-1}{n_2})^2 \geq \frac{-Nn_3}{n_1n_2} \geq \frac{N^2}{9}$$

and we obtain

$$\frac{N}{4} \geq \left( \frac{-1}{n_1} + \frac{-1}{n_2} \right) \geq \frac{N}{6}.$$ 

First we consider solutions with $n_1 = -1$: Then $n_1 = -1$ and $n_3 \in \{n_2, n_2 - 1\}$. If the tuple is of the form

i) $(-1, n_2, n_2)$ then

$$(-1 - n_2 - n_2)^2 = Nn_2^2 \Rightarrow n_2 = -1.$$ 

Thus $N = 9$ and $(-1, n_2, n_2) = (-1, -1, -1)$. 

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ii) \((-1, n_2, n_2 - 1)\) then
\[
(-1 + n_2 + n_2 - 1)^2 = Nn_2(n_2 - 1) \Rightarrow (N - 4)n_2 = -4.
\]
Hence \(N > 4\) and \(n_2 \mid 4\). This gives the cases \(N = 8\) and \((-1, n_2, n_2 - 1) = (-1, -1, -2)\) or \(N = 6\) and \((-1, n_2, n_2 - 1) = (-1, -2, -3)\) or \(N = 5\) and \((-1, n_2, n_2 - 1) = (-1, -4, -5)\).

All other solutions start with at least \(n_1 = -2\). Hence \(N \leq 6\).

Next we consider solutions with \(n_1 = -2\): Then \(n_1 = -1\) and \(n_3 \in \{n_2, n_2 - 1, n_2 - 2\}\). If the tuple is of the form

i) \((-2, n_2, n_2)\) then
\[
(-2 - n_2 - n_2)^2 = 2Nn_2^2 \Rightarrow n_2 \mid 2.
\]
But \(n_2 = 2\) yields a contradiction.

ii) \((-2, n_2, n_2 - 1)\) we get
\[
(-2 + n_2 + n_2 - 1)^2 = Nn_2(n_2 - 1) \Rightarrow n_2 - 1 \mid 1, \quad n_2 \mid 3
\]
a contradiction.

iii) \((-2, n_2, n_2 - 2)\) then
\[
(-2 + n_2 + n_2 - 2)^2 = Nn_2(n_2 - 2) \Rightarrow 2(n_2 - 2) = Nn_2.
\]
Thus \(n_2 \mid 4\) and \(N - 2 \mid 4\). This yields the cases
\[
N = 4, \quad (-2, -2, -4),
\]
\[
N = 3, \quad (-2, -4, -6).
\]

For the remaining cases we only have to check \(N \leq 4\).

For \(N = 4\) we get the solution \((-3, -3, -6)\). For \(N = 3\) we get \((-3, -3, -3)\) and for \(N = 2\) we get \((-3, -6, -9)\) or \((-4, -4, -8)\). But in these cases \(gcd(n_1, n_2, n_3) > 1\).

Finally assume that \(1 = gcd(n_1, n_2, n_3) = N\). Then
\[
(n_1 + n_2 + n_3)^2 = -n_1n_2n_3
\]
implies that \(gcd(n_i, n_j) = 1\) for \(i \neq j\). Hence \(-n_i\) is a square. Thus the equation reduced modulo 3 has no solutions.

For the other cases the proof is analogous.

For case i) see also Gutzwiller [14].

Now we can easily determine the corresponding tuple of monodromy group generators:
Lemma 4.9. We list the corresponding tuples of monodromy group generators for the minimal tuples in Lemma 4.8. Note that the monodromy group generators for \( N = 3 \) and \( N = 9 \) are conjugate in \( \text{SL}_2(\mathbb{Q}) \) by a diagonal matrix. The same holds for \( N = 4 \) and \( N = 8 \).

\[
\begin{array}{cccc}
N = 3, & (1 & 0), & (-2 & 3), & (-5 & 12), & (1 & 3) \\
N = 4, & (1 & -2), & (5 & -2), & (1 & 0), & (-3 & -4) \\
N = 5, & (1 & 0), & (1 & 0), & (-9 & 20), & (11 & -25) \\
N = 6, & (1 & 0), & (-5 & 2), & (-5 & 3), & (11 & 1) \\
N = 8, & (1 & 1), & (16 & -3), & (8 & 1), & (8 & 5) \\
N = 9, & (1 & 0), & (-2 & 1), & (-9 & 4), & (1 & 1) \\
\end{array}
\]

\[
\begin{array}{cccc}
(n_1, n_2, n_3) & A_1 & A_2 & A_3 & A_4 \\
(-6, -6, -9) & (1 & 6), & (1 & 0), & (-2 & 9), & (-5 & 21) \\
(-5, -8, -10) & (1 & 5), & (3 & 4), & (1 & 0), & (-1 & 1) \\
(-5, -16, -20) & (1 & 5), & (2 & 1), & (-3 & 4), & (-4 & 7) \\
(-6, -10, -15) & (1 & 5), & (1 & 1), & (-3 & 4), & (-1 & 3) \\
(-7, -8, -14) & (1 & 7), & (1 & 1), & (3 & 4), & (3 & 13) \\
(-8, -8, -9) & (1 & 7), & (1 & 1), & (2 & 9), & (-1 & 4) \\
(-5, -12, -15) & (1 & 5), & (1 & 1), & (3 & 4), & (-1 & 6) \\
(-6, -8, -12) & (1 & 6), & (1 & 1), & (3 & 7), & (1 & 3) \\
(-7, -7, -9) & (1 & 7), & (1 & 1), & (1 & 4), & (-1 & 5) \\
\end{array}
\]

Corollary 4.10. These tuple correspond exactly (up to the braid group action) to the differential equations in Table 1.

From this list one can also determine explicitly the monodromy group generators for the corresponding Lamé equations via the following

Lemma 4.11. Let \( A \) be a tuple of monodromy group generators in \( \text{GL}_2(\mathbb{C}) \) with \( A_1, A_2, A_3 \) being involutions. Taking the symmetric square and applying the middle convolution \( MC_{-1} \) (s. [10]) we obtain a tuple of monodromy group generators \( B \) in \( \text{SL}_2(\mathbb{C}) \) with unipotent elements \( B_1, B_2, B_3 \).
The transformation of the Fricke parameters is

$$(0, 0, 0, a_4, x, y, z) \mapsto (2, 2, 2, -a_4^2 - 2, -(x^2 - 2), -(y^2 - 2), -(z^2 - 2)) .$$

Moreover if $A$ preserves a hermitian form then the monodromy group generated by $B$ is contained in $\text{SL}_2(\mathbb{R})$. Further

$$-(x^2 - 2), -(y^2 - 2), -(z^2 - 2) \leq -2 \text{ or } -2 \leq -(x^2 - 2), -(y^2 - 2), -(z^2 - 2) \leq 2 .$$

**Proof.** The symmetric square of $A$ gives a tuple $C$ in $\text{SL}_3(\mathbb{C})$ where $C_1, C_2$ and $C_3$ are reflections. If $A_4$ is semi-simple with eigenvalues $\alpha, -\alpha^{-1}$ then $C_4$ has eigenvalues $\alpha^2, \alpha^{-2}, -1$.

By [11, Thm 2.4 i)] we get

$$MC_{-1}(C) = B \in \text{GL}_m(\mathbb{C})^4, \quad m = \sum_{i=1}^{3} \text{rk}(C_i - 1) + \text{rk}(-C_4 - 1) - 3 = 2$$

and the change of the eigenvalues under the middle convolution [11, Lemma 2.6] gives a tuple of monodromy group generators $B$ in $\text{SL}_2(\mathbb{C})$ with unipotent elements $B_1, B_2, B_3$ and $B_4$ being semi-simple with eigenvalues $-\alpha^2, -\alpha^{-2}$. The eigenvalues of $C_iC_j$ are $\gamma_{ij}, \gamma_{ij}^{-1}, 1$ where $\text{tr}(C_1C_2) = x^2 - 1, \text{tr}(C_2C_3) = y^2 - 1$ and $\text{tr}(C_1C_3) = z^2 - 1$. Since the convolution commutes also with coalescing (s. [10, Lemma 5.6]) the eigenvalues of $B_iB_j$ are $-\gamma_{ij}, -\gamma_{ij}^{-1}$.

Hence

$$\text{tr}(B_iB_j) = -\text{tr}(C_iC_j) + 1$$

and the claim follows.

If $A_4$ is not semi-simple then its Jordanform is $iJ(2)$, where $J(2)$ denotes a Jordanblock of length 2. Then $C_4$ has Jordanform $-J(3)$. By [11, Thm 2.4 i)] we get

$$MC_{-1}(C) = B \in \text{GL}_2(\mathbb{C})^4$$

with 4 unipotent elements $B_i$. The claim follows in this case as above.

Since the symmetric square and the convolution with $-1$ preserve a hermitian form by [11, Thm 2.4 v),vi)] $B$ leaves an indefinite form invariant if $A$ preserves a hermitian form.

By Corollary 4.4 the claim concerning the values of the Fricke parameters follows.

Note that we do not have to assume that the corresponding differential equations are Heun equations (having no apparent singularity).

Next we show that the map $A \to B$ is surjective. However if $B$ leaves a hermitian form invariant this doesn’t imply that this also true for $A$.

**Corollary. 4.12.** Let $B$ be a tuple of monodromy group generators in $\text{SL}_2(\mathbb{R})$ with unipotent elements $B_1, B_2$ and $B_3$. Then there exists a tuple of monodromy group generators $A$ in $\text{GL}_2(\mathbb{C})$ with reflections $A_1, A_2$ and $A_3$ as in Lemma 4.11.

Further the Fricke parameters $x, y, z$ w.r.t. $B$ are either all $\leq -2$ or all of the form $\lambda + \bar{\lambda}, \lambda \bar{\lambda} = 1$ if and only if the monodromy group generators $A$ leave a hermitian form invariant.

Moreover if $B_4$ is also unipotent then the Fricke parameters $x, y, z$ are all $\leq -2$ if and only if the monodromy group generators $A$ leave a hermitian form invariant.
Proof. Let the Fricke parameters $x, y$ and $z$ in $\mathbb{R}$. We consider first the case where $B_4$ is a semi-simple element with eigenvalues $(\alpha_4, \alpha_4^{-1})$. If we apply the middle convolution $MC_{-1}$ to the tuple $B$ of monodromy group generators we get a tuple $C = (C_1, \ldots, C_4)$, $C_i \in \text{GO}_3(\mathbb{C})$, where $C_1, \ldots, C_3$, are reflections and $C_4$ has eigenvalues $(-1, -\alpha_4, -\alpha_4^{-1})$. This follows from \cite{11} Thm 2.4 i)] since

$$MC_{-1}(B) = C \in \text{GL}_m(\mathbb{C})^4, \quad m = \sum_{i=1}^{3} \text{rk}(B_i - 1) + \text{rk}(-B_4 - 1) - 2 = 3$$

and the change of the eigenvalues under the middle convolution \cite{11} Lemma 2.6] and the fact that the convolution with $-1$ preserves a hermitian form but changes a symplectic form to an orthogonal one [\cite{11} Thm 2.4 v),vi)]. Since the convolution commutes also with coalescing (s. \cite{10} Lemma 5.6]) the eigenvalues of $C_i C_j$ are the negative eigenvalues of $B_i B_j$ and 1. Since $S^2(\text{SL}_2(\mathbb{C})) \cong \text{SO}_3(\mathbb{C})$, we find a tuple $A$ of monodromy group generator in $\text{SL}_2(\mathbb{C})$. But $\text{trace}(B_i B_j) > 2$ would imply pure imaginary eigenvalues of absolute value greater than 1 of the element $A_i A_j$ in $\text{SL}_2(\mathbb{C})$. Hence in such a case there is no invariant form. If $0 < |\text{trace}(B_i B_j)| < 2$ then the group generated by $A$ has to leave a positive definite form invariant and an indefinite form if $|\text{trace}(B_i B_j)| > 2$ by Corollary 4.4.

If $-B_4$ is unipotent, then $A_1, \ldots, A_4$ are reflection in $\text{GL}_2(\mathbb{C})$ and the generated group is therefore a dihedral group. Hence the claim follows.

Finally if $B_4$ is also unipotent then the Jordan form of $-A_4$ is a Jordan block of length 3. Hence the group generated by $C$ contains a unipotent element. Thus the hermitian form has to be indefinite. Hence the claim follows again by Corollary 4.4.

We conclude with the following

**Question 4.13.** Let $A$ be a tuple of monodromy group generators with unipotent elements $A_1, \ldots, A_3$. Let further the monodromy group be an irreducible subgroup of $\text{SL}_2(\mathbb{R})$. If one of the Fricke parameters $x, y, z$ is greater than 2 as in Corollary 4.7 then the corresponding differential equation is via geometric operations related to a differential equation without hermitian form by the above lemma. Thus the following question arises: **Is it true that the monodromy group of a geometric differential equation leaves a hermitian form invariant?**

In this case one could rule out all examples arising from Corollary 4.7 as being geometric. It is known that monodromy group of a Picard-Fuchs equation leaves a hermitian form invariant (s.\cite{13}). However this doesn’t imply that this also holds for the absolute irreducible components. At least we do not see this.

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