Decomposition of stochastic flows with automorphism of subbundles component

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Abstract

We show that given a $G$-structure $P$ on a differentiable manifold $M$, if the group $G(M)$ of automorphisms of $P$ is large enough, then there exists the quotient of stochastic flows $\phi_t$ by $G(M)$, in the sense that $\phi_t = \xi_t \circ \rho_t$ where $\xi_t \in G(M)$, the remainder $\rho_t$ has derivative which is vertical, transversal to the fibres of $P$. This geometrical context generalises previous results where $M$ is a Riemannian manifold and $\phi_t$ is decomposed with an isometric component, see [12] and [15], which in our context corresponds to the particular case of an $SO(n)$-structure on $M$.

Key words: decomposition of stochastic flows, automorphisms of $G$-structures, infinitesimal automorphisms, stochastic exponential, symplectic flows.

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1 Introduction

Let $M$ be a connected differentiable manifold, with $\text{Diff}(M)$ the group of smooth diffeomorphisms of $M$. Consider the following Stratonovich stochastic differential equation (sde) on $M$:

$$dx_t = \sum_{i=0}^{k} X_i(x_t) \circ dW^i_t$$

with initial condition $x_0 \in M$, where $X_0, X_1, \ldots, X_k$ are smooth vector fields on $M$, $(W^0_t) = t$, and $(W^i_t, \ldots, W^k_t)$ is a Brownian motion in $\mathbb{R}^k$, defined over an appropriate filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$. We shall denote by $\varphi_t : \Omega \times M \to M$ the stochastic flow associated to the diffusion generated by this equation, which we shall assume to exist for all $t \geq 0$, e.g. assuming that the derivatives of the vector fields of the sde are bounded.

In this article we study decompositions of the stochastic flow $\varphi_t$ such that one of the components in the decomposition is a diffusion in the group of automorphisms of

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a $G$-structure on $M$, i.e. a subbundle of the principal bundle of linear frames in each tangent space of $M$, with $G$ as the structure group. Our main result shows that given a $G$-structure $P$ on a differential manifold $M$, if the group of automorphisms $G(M)$ is large enough, then there exists the quotient of the stochastic flow $\varphi_t$ by $G(M)$, in the sense that, for a fixed initial conditions $x_0 \in M$ and a frame $u_0 \in P$, $\varphi_t = \xi_t \circ \rho_t$ where $\xi_t \in G(M)$, the remainder $\rho_t(x_0) = x_0$ for all $t \geq 0$ and $\rho_t$ has linearization which is vertical, transversal to the fibre of $P$ at $u_0$, precisely $\rho_t(u_0) = u_0 \cdot q_t$, where $q_t$ is a process which lives in the exponential of a complementary subspace of the Lie algebra of $G$. We explore geometrically interesting examples where these complementary spaces are Lie subalgebras of $\text{gl}(n, \mathbb{R})$.

This geometrical $G$-structure context generalises previous results of Liao [12] and Ruffino [15] where they decompose $\varphi_t$ with an isometric component, which in our context corresponds to the particular case of an $SO(n)$-structure on $M$. In those articles the decomposition (quotient) has been used to calculate the Lyapunov exponents and the matrix of rotation, respectively.

In the following section we shall recall the definitions and basic properties of $G$-structures on a differentiable manifold $M$. The main results are presented in Section 3. In Section 4 we present a sequence of examples and applications. The reader will notice that the approach used here can also be used, with few adaptations, to proof the same results for control and nonautonomous flows. The fact that we deal with stochastic equations will guarantee that if the original system of equation (1) has nondegenerate (Equation (3)), then the support of $\xi_t$ will be the connected component of the identity of $G(M)$. We remark also that the component $\xi_t$ illustrates, for $r = 1$ the stochastic calculus of order $r$ for diffusions in $M$ which are automorphisms of $G$-structures of contact order $r$, cf. Akiyama [2].

2 Geometric set up

In this section we introduce the geometrical objects involved in the general technique of decomposition of flows given a group of automorphisms of a subbundle of a principal bundle, which most of interesting case are $G$-structure. We refer mainly to the classical Kobayashi [9] or Kobayashi andNomizu [10].

Let $M$ be a smooth connected $n$-dimensional manifold $M$. We shall denote by $GL(M)$ the principal bundle of linear frames in each tangent space of $M$, i.e. the set of linear isomorphisms $u : \mathbb{R}^n \to T_x M$ for all $x \in M$ with the natural projection $\pi : GL(M) \to M$ where $Gl(n, \mathbb{R})$ is the structural group.

We shall consider a complete connection $\Gamma$ in $GL(M)$, i.e. such that any segment of $\Gamma$-geodesic in $M$ defined for parameter $t$ in an interval can be extended to all $t \in \mathbb{R}$. The corresponding $\text{gl}(n, \mathbb{R})$-value 1-form connection will be denoted by $\omega$, where $\text{gl}(n, \mathbb{R})$ denotes the Lie algebra of $Gl(n, \mathbb{R})$. For any vector $v \in \mathbb{R}^n$, there exists a standard horizontal vector field $B(v)$ in $GL(M)$ given by the following: at a frame $u$, $B(v)$ is the unique horizontal vector in $T_u GL(M)$ such that $d_u \pi (B) = u(v)$. Alternatively, if $\theta$ is the canonical form $\theta_u : T_u GL(M) \to \mathbb{R}^n$ given by $\theta_u(Y) = u^{-1}d\pi(Y)$, then $\theta(B(v)) = v$. Let $H_1, ..., H_n$ be the standard horizontal vector fields
generated by the canonical basis \( e_1, \ldots, e_n \) of \( \mathbb{R}^n \) and let \( \{ E_i^j \} \) be the fundamental vector fields corresponding to the basis \( \{ E_i^j \} \) of the Lie algebra \( \mathfrak{gl}(n, \mathbb{R}) \), where the fundamental vector fields in \( \text{GL}(M) \) is given, for each \( A \) in the Lie algebra \( \mathfrak{gl}(n, \mathbb{R}) \), by
\[
A^*(u) = \frac{d}{dt}[u \exp(tA)]|_{t=0}.
\]

These \( n^2 + n \) vector fields \( \{(H_k)_u, (E_i^j)_u\} \) form a basis of \( T_u \text{GL}(M) \) for every \( u \in \text{GL}(M) \) with \( \theta(H_k) = e_k \) and \( \omega(E_i^j) = E_i^j \).

A diffeomorphism \( \varphi : M \to M \) induces naturally an automorphism \( \varphi_* \) of the bundle \( \text{GL}(M) \); which maps a frame \( u = (X_1, \ldots, X_n) \) at a certain point \( x \in M \) into the frame \( \varphi_*(u) = (\varphi_*X_1, \ldots, \varphi_*X_n) \) at \( \varphi(x) \in M \). The canonical form \( \theta \) is invariant by the pull-back \( \varphi^*\theta = \theta \). In particular, we say that a diffeomorphism \( \varphi : M \to M \) is an affine transformation if the derivative \( \varphi_* : TM \to TM \) maps horizontal curves into horizontal curves. Equivalently, \( \varphi \) is affine if \( \varphi_* \) preserves the horizontal subspaces established by the connection \( \Gamma \), or yet, if it preserves the connection form \( \varphi^*\omega = \omega \).

Let \( X \) be a smooth vector field on \( M \), and let \( \eta_t \) be its associated one-parameter flow. The natural lift \( \delta X \) of \( X \), denoted by \( \delta X \) (\( u \)), is a vector field in \( \mathfrak{X}(\text{GL}(M)) \), the Lie algebra of vector fields in the frame bundle, given by:
\[
\delta X(u) = \frac{d}{dt}\eta_t(u)|_{t=0}.
\]

The lift \( \delta X \) can also be characterised by the following properties simultaneously:
\begin{enumerate}
  \item \( R_{a*}\delta X(u) = \delta X(ua) \) for every \( a \in G \);
  \item \( L_{\delta X} \theta = 0 \) (Lie derivative);
  \item \( d\pi(\delta X(u)) = X(\pi(u)) \), for every \( u \in \text{GL}(M) \).
\end{enumerate}

Given a complete linear connection \( \Gamma \) and its connection form \( \omega \) on \( M \), a vector field \( X \) on \( M \) is an infinitesimal affine transformation if the associated flow \( \eta_t \) are affine transformations for all \( t \in \mathbb{R} \). Affine infinitesimal transformations, denoted by \( a(M) \) is a Lie algebra isomorphic to the subalgebra of \( \omega \)-preserving elements of \( \mathfrak{X}(\text{GL}(M)) \):
\[
a(M) = \{ \delta X \in \mathfrak{X}(\text{GL}(M)) : L_{\delta X} \omega = 0 \}.
\]

The Lie algebra \( a(M) \) has dimension at most \( n^2 + n \), and for any \( u \in \text{GL}(M) \) the linear mapping \( a(M) \to T_u \text{GL}(M) \), given by \( X \mapsto \delta X(u) \) is injective. When \( \dim a(M) = n^2 + n \) then \( \Gamma \) is flat. See e.g. Kobayashi and Nomizu [10] Chap. 3, Thm 2.3 among others.

Fundamental vector fields \( A^*(u) \), with \( A \in \mathfrak{gl}(n, \mathbb{R}) \) generate the vertical subspace \( T_u^{v} \text{GL}(M) \) of \( T_u \text{GL}(M) \). For \( Y \in T_x M \), let \( \nabla_X Y = \nabla X(Y) \) be the covariant derivative defined by the Riemannian connection. The vertical component of the canonical lift is given precisely by \( (\delta X)^v(u) = \nabla X(u) \). Hence, there is a unique matrix \( [\tilde{X}(u)] \in \mathfrak{gl}(n, \mathbb{R}) \) which, acting on the right, equals the covariant derivative of \( \nabla X \) acting on the left: i.e. \( \nabla X(u) = u[\tilde{X}(u)] \). We refer to Cordero et al. [4].
2.1 \(G\)-structures

A \(G\)-structure on \(M\) is a reduction of the frame bundle \(GL(M)\) to a subbundle \(P\), with structure Lie group \(G \subset GL(n, \mathbb{R})\). Given a closed subgroup \(G\), the existence of a \(G\)-structure depends intrinsically on the topology of \(M\): More precisely, there exists a one to one correspondence between \(G\)-structures and cross sections of the associated fibre bundle \(GL(M)/G\), see [10, Chap. I, Prop. 5.6]. A list of interesting examples includes the following: There exist \(GL(n, \mathbb{R})^+\)-structures if and only if \(M\) is orientable; there exist \(Sl(n, \mathbb{R})\)-structures if and only if \(M\) has a volume form; paracompactness guarantees the existence of \(O(n, \mathbb{R})\)-structures via Riemannian metrics; finally note that a manifold \(M\) is parallelizable if and only if there exist a \(\{1\}\)-structure. For these properties and further example, we refer to [9], [10], the classical Sternberg [16], among others.

Given a \((k, l)\)-tensor \(K\) over the Euclidean space \(\mathbb{R}^n\), let \(G\) be the group of linear transformations in \(\mathbb{R}^n\) which is \(K\)-invariant, i.e. for \((v_1, \ldots, v_k, f_1, \ldots, f_l) \in (\mathbb{R}^n)^k \times (\mathbb{R}^{n*})^l\), we have that \(g \in G\) if

\[ K(v_1, \ldots, v_k, f_1, \ldots, f_l) = K(gv_1, \ldots, gv_k, g^*f_1, \ldots, g^*f_l). \]

We say that a corresponding \(G\)-structure over \(M\) is induced by the tensor \(K\). A such \(G\)-structure extends naturally the tensor \(K\) to a tensor field \(k\) on \(M\) defined by: for each \(x \in M\), given a linear isomorphism \(u : \mathbb{R}^n \to T_xM \in P\), for \(w_1, \ldots, w_k \in T_xM\) and \(z_1, \ldots, z_l \in T_xM^*\), we assign

\[ k_x(w_1, \ldots, w_k, z_1, \ldots, z_l) = K(u^{-1}w_1, \ldots, u^{-1}w_k, u^{-1*}z_1, \ldots, u^{-1*}z_l). \]

The invariance of \(K\) by \(G\) guarantees that the definition above is independent of the choice of \(u\).

Let \(\varphi : M \to M\) be a smooth diffeomorphism, given a \(G\)-structure \(P\), if \(\varphi_*\) maps \(P\) into itself, we call \(\varphi\) an automorphism of the \(G\)-structure \(P\). A vector field \(X\) on \(M\) is called an infinitesimal automorphism of a \(G\)-structure \(P\) if it generates a local 1-parameter group of automorphisms of \(P\). A vector field \(X\) on \(M\) is an infinitesimal automorphism if and only if \(L_Xk = 0\), where \(k\) is the tensor field associated to the \(G\)-structure \(P\).

To illustrate, we recall that \(GL(n, \mathbb{R})^+\)-structures has the group of automorphisms given by diffeomorphisms which preserve orientation, and the infinitesimal automorphisms are differentiable vector fields. For \(Sl(n, \mathbb{R})\)-structures, the group of automorphisms is given by diffeomorphisms which preserve volume, and the infinitesimal automorphisms are vector fields with vanishing divergent. For \(O(n, \mathbb{R})\)-structures, the group of automorphisms is given by isometries and the infinitesimal automorphisms are Killing vector fields. For a \(\{1\}\)-structure, the group of automorphisms degenerates to identity and the infinitesimal automorphisms degenerates to the zero vector field.
3 Main Results

We shall consider a differentiable manifold $M$ which admits a $G$-structure $P$ induced by a tensor field $k$, where $G$ is a subgroup of $Gl(n, \mathbb{R})$. Let $\pi_0 : P \to M$ be the restriction of $\pi : GL(M) \to M$. We shall fix an initial condition $x_0 \in M$ and an initial frame $u_0 \in P$ in the tangent space at $x_0$, i.e. $\pi_0(u_0) = x_0$.

We assume that $M$ is endowed with a complete connection $\nabla$. Let $g(M) = \{X \in a(M) : L_X k = 0\}$, since the Lie derivative $L_{[X,Y]} = [L_X, L_Y]$ (see e.g. [10] Prop 3.4 p.32), we have that $g(M)$ is a Lie algebra of affine vector fields which preserve the tensor $k$. Given a vector field $X$ in $g(M)$, the associated flow $\eta_t$ is a family of automorphisms of $P$. Hence $\eta_t(u_0) \in P$ consequently, $\delta X(u_0) \in T_{u_0}P$. We shall denote by $\gamma$ the restriction of the canonical lift $\delta$ to vector fields in $g(M)$:

\[ \gamma : g(M) \to T_{u_0}P \]
\[ X \mapsto \delta X(u_0). \]  

It is injective since it is the restriction of the linear and injective mapping $a(M) \to T_{u_0}GL(M)$, $X \mapsto \delta X(u_0)$.

The finite dimensional Lie group generated by $g(M)$ will be denoted by $G(M)$. So, if $\xi_t$ is a process in $G(M)$ then $\xi_t$ is an affine transformation which preserves the tensor field $k$.

We shall use freely the Lie group terminology to deal with flows and vector fields, e.g. given $\xi \in G(M)$ and $Y \in g(M)$, $\xi Y$ and $Y \xi$ are the tangent vectors at $T_t G(M)$ obtained by left and right translations of $Y$ respectively. Clearly, $\xi \mapsto \xi Y$ are left invariant vector field on $G(M)$. Analogously for the right translation $Y \xi$.

Given initial conditions $x_0 \in M$ and $u_0 \in \pi^{-1}(x_0) \subset P$ we shall fix a projection $p : T_{u_0}GL(M) \to T_{u_0}P$ such that $p(HT_{u_0}GL(M)) = HT_{u_0}P$ and $p(VT_{u_0}GL(M)) = VT_{u_0}P$. By the linear dependence of the fundamental vector fields with the elements in the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$, such a projection $p$ is equivalent, and will be identified with the same notation, of a projection in the corresponding Lie algebras $p : \mathfrak{gl}(n, \mathbb{R}) \to \mathfrak{p}$, where $\mathfrak{p}$ is the Lie algebra of $G$.

We shall assume the following hypotheses:

**H1** (Existence of a $G$-structure) The manifold $M$ admits a $G$-structure $P$. Given initial conditions $x_0 \in M$ and $u_0 \in \pi^{-1}(x_0) \subset P$, we shall fix a projection $p : T_{u_0}GL(M) \to T_{u_0}P$ such that $p(HT_{u_0}GL(M)) = HT_{u_0}P$ and $p(VT_{u_0}GL(M)) = VT_{u_0}P$.

**H2** (The Lie algebra $g(M)$ is big enough) The projections $p[\delta Ad(\xi)X_j(u_0)]$ are in the image $Im(\gamma)$, where $\gamma$ is given in equation (2), for all vector fields $X_j$, $j = 0, 1, ..., k$ of the sde (1) and for any automorphism $\xi$ in $G(M)$.

The geometric and dynamical meaning of different choices of projection $p$ in (H1) above will be clear in the corollaries and examples after the main result (Theorem 3.1) below. This theorem generalise to automorphisms of $G$-structures the factorizations presented in Liao [12], Ruffino [15], Colonius and Ruffino [6] for stochastic and control flows.
Theorem 3.1 (Decompositions in automorphisms of G-structure) Assume the conditions (H1) and (H2) above. Then the stochastic flow \( \varphi_t \) of equation (1) decomposes as \( \varphi_t = \xi_t \circ \rho_t \), where \( \xi_t \) is a diffusion in the group \( G(M) \) of automorphisms of \( P \), the remainder \( \rho_t \) is a process in \( \text{Diff}(M) \) such that \( \rho_t(x_0) = x_0 \) and \( \rho_{t*}(u_0) = u_0 \cdot q_t \), where \( q_t \) is a process in \( \langle \exp(ker \ p) \rangle \), the Lie subgroup generated by \( \exp(ker \ p) \).

Proof:

Let \( \xi_t \) be the solution of the following sde in the group \( G(M) \) with initial condition \( \xi_0 = \text{id} \):

\[
d\xi_t = \sum_{j=0}^{k} (L_{\xi_t})_* \left[ \text{Ad}(\xi_t^{-1})X_j \right]^p \circ dW_t^j,
\]

where \( \left[ \text{Ad}(\xi_t^{-1})X_j \right]^p \) is the unique vector field in \( g(M) \) such that \( \delta \left[ \text{Ad}(\xi_t^{-1})X_j \right]^p(u_0) = p(\delta \text{Ad}(\xi_t^{-1})X_j(u_0)) \). Note that the horizontal components of these two vectors coincide, i.e. in \( T_{x_0}M \) we have that \( \left[ \text{Ad}(\xi_t^{-1})X_j \right]^p(x_0) = \text{Ad}(\xi_t^{-1})X_j(x_0) \). The process \( \xi_t \) is an stochastic exponential in the sense of Hakim-Dowek and Lépingle [8] or Catuogno and Ruffino [5]. Since the process is the stochastic exponential of a process in the Lie algebra \( g(M) \), then \( \xi_t \) is a diffusion in the group \( G(M) \) of automorphisms of \( P \).

For the remainder \( \rho_t = \xi_t^{-1} \varphi_t \), by the Itô formula in the group, as in Kunita [11, pp. 208-209] and the fact that

\[
d\xi_t^{-1} = -\sum_{j=0}^{k} \left[ \text{Ad}(\xi_t^{-1})X_j \right]^p \xi_t^{-1} \circ dW_t^j,
\]

we have that

\[
d\rho_t = \sum_{j=0}^{k} \left\{ \text{Ad}(\xi_t^{-1})X_j - \left[ \text{Ad}(\xi_t^{-1})X_j \right]^p \right\}(\rho_t) \circ dW_t^j.
\]

Since at \( x_0 \), the vector field of the equation above vanishes, the derivative process \( \rho_{t*}u_0 \) starting at \( u_0 \), has no horizontal component and satisfies the linear sde in \( T_{u_0}GL(M) \):

\[
d\rho_{t*} = \sum_{j=0}^{k} \delta\{\xi_{t*}^{-1}X_j\} - [\xi_{t*}^{-1}(X_j)]^p \rho_{t*} \circ dW_t^j.
\]

The derivative \( \rho_{t*} \) at \( u_0 \) in the fibre \( \pi^{-1}(x_0) \) acts on the left as a linear transformation which has no horizontal component neither vertical component along the action of \( p \). Hence it has a vertical component in \( T_{u_0}GL(M) \) along a process which is in the Lie group generated by the kernel of the projection \( p \), i.e. \( \rho_{t*}u_0 = u_0 \cdot q_t \), where \( q_t \) is in \( \langle \exp(ker \ p) \rangle \subseteq GL(n, \mathbb{R})(n) \).

In other words, the theorem above says that there exists a right quotient of a stochastic flow in the differentiable manifold \( M \) with respect to the group of automorphisms of a \( G \)-structure if the adjoint by automorphisms of each vector field
involved in the SDE is equal to a certain infinitesimal transformations at $x_0$ and both have the same canonical lift to $u_0$. The “remainder” of the quotient is the process $\rho_t$.

If the Lie algebra $\mathfrak{p}$ has a complementary Lie subalgebra $\mathfrak{q}$, i.e. $\mathfrak{gl}(n, \mathbb{R}) = \mathfrak{p} \oplus \mathfrak{q}$, there is a natural choice for the projection $p$ in hypothesis (H1). In this case, the derivative of the remainder $\rho_t$ becomes uniquely characterised at the initial condition.

**Corollary 3.2** If the Lie algebra $\mathfrak{p}$ of the Lie group $G$ associated to the $G$-structure $P$ has a complementary Lie algebra $\mathfrak{q}$, then there exists a decomposition of the stochastic flow $\varphi_t$ of equation (11) as $\varphi_t = \xi_t \circ \rho_t$, where $\xi_t$ is a diffusion in the group $G(M)$ of automorphisms of $P$, $\rho_t$ is a process in $\text{Diff}(M)$ such that $\rho_t(x_0) = x_0$ and $\rho_{ts}(u_0) = u_0 \cdot q_t$, where $q_t$ is a process in the Lie group $\langle \exp \mathfrak{q} \rangle$ generated by the Lie algebra $\mathfrak{q}$.

The processes $q_t$ which satisfies the property above is unique.

**Proof:** Just take the projection $p : \mathfrak{gl}(n, \mathbb{R}) \rightarrow \mathfrak{p}$ of hypothesis (H1) along the subspace $\mathfrak{q}$ and apply Theorem 3.1. For the uniqueness of the process $\eta_t$ in the group $\langle \exp \mathfrak{q} \rangle$, let $\xi_t \circ \rho_t$ and $\tilde{\xi}_t \circ \tilde{\rho}_t$ be two distinct decomposition of $\varphi_t$ with $\tilde{\rho}_t(u_0) = u_0 \tilde{q}_t$ and $\rho_{ts}(u_0) = u_0 q_t$. We have that $\xi_t^{-1} \circ \tilde{\xi}_t$ is an automorphism of the $G$-structure $P$ which fix the point $x_0$ for all $t \in \mathbb{R}$. Hence, the derivative $(\xi_t^{-1} \circ \tilde{\xi}_t)_s(u_0)$ is a vertical translation in the fibre $\pi^{-1}(x_0) \subset P$. On the other hand, since the action of $\rho_s$ is equivariant:

$$(\xi_t^{-1} \circ \tilde{\xi}_t)_s(u_0) = (\rho_t \tilde{\rho}_t^{-1})_s(u_0) = u_0 q_t q_t^{-1}.$$ 

The unique element which is a vertical translation of $u_0$ in $P$ and has the form above with $q_t q_t^{-1} \in \langle \exp \mathfrak{q} \rangle$ is $u_0$ itself, it follows that $\tilde{q}_t = q_t$.

The uniqueness stated above with local properties does not imply that the decomposition $\varphi_t = \xi_t \circ \rho_t$ is unique with these properties if the group of diffeomorphisms of $M$ which preserves the tensor associated with the $G$-structure is large enough. One may find a, say deterministic curve of tensor preserving diffeomorphisms $\eta_t$, with $\eta_0 = \text{Id}$ and which restricted to an open neighbourhood of $x_0$ is the identity for all $t \geq 0$. In this case $\tilde{\xi}_t = \xi_t \eta_t$ and $\tilde{\rho}_t = \eta_t^{-1} \rho_t$ also satisfies the local conditions stated in the corollary, although in this case $\xi_t$ may no longer be an affine transformation. For example, in the infinite dimensional group of diffeomorphisms which preserve volume we do not have uniqueness. In the isometry group the uniqueness holds as stated in [12] and [15].

The diffusion of $G$-automorphism $\xi_t$ in the main theorem induces a Markov process in $M$ which is not necessarily time homogeneous, see e.g. Liao [14]. The next result shows an alternative decomposition $\varphi_t = \xi_t \rho_t$ where $\xi_t$ is a flow (time homogeneous) in $M$ itself, instead of in $G(M)$:
Theorem 3.3 Under conditions (H1) and (H2) above, the stochastic flow $\varphi_t$ of equation (1) factorizes as $\varphi_t = \xi_t \circ \rho_t$, where $\xi_t$ is an stochastic flow in $M$ which preserves the $G$-structure. The remainder component $\rho_t$ is a process in $\text{Diff}(M)$ such that the vector fields of its sde vanish at $\xi_t^{-1}(x_0)$.

Proof: Define $\xi_t$ as the solution of the following right invariant stochastic equation.

$$d\xi_t = \sum_{j=0}^{k} [X_j]^p \xi_t \circ dW^j_t,$$

where, as in the proof of the theorem, $[X_j]^p$ is the unique vector field in $g(M)$ such that $\delta[X_j]^p(u_0) = p(\delta X_j(u_0))$. Again, note that the horizontal components of these two vectors coincide, i.e. in $T_{x_0}M$ we have that $[X_j]^p(x_0) = X_j(x_0)$.

For the remainder $\rho_t = \xi_t^{-1} \varphi_t$, again by the Itô formula we have that

$$d\rho_t = \sum_{j=0}^{k} \text{Ad}(\xi_t^{-1})(X_j - [X_j]^p) \rho_t \circ dW^j_t.$$

Last property of the statement follows directly from this equation.

Remark: It might be possible to obtain, locally, analogous results of this section with a linear connection by using local affine transformations. This approach would demand to reconstruct locally the theory of (global) affine transformations (as in e.g. Kobayashi and Nomizu [10, pp.234-235]). To prevent the reader from further geometrical technicalities, here we have assumed completeness of the connection.

Corollary 3.4 With the same hypotheses of Theorem 3.1 we have the following left remainder decomposition $\varphi_t = \tilde{\rho}_t \circ \xi_t$ where $\xi_t$ is a diffusion in the group $G(M)$ of automorphisms of the $G$-structure, the remainder component $\tilde{\rho}_t$ is a process in $\text{Diff}(M)$ such that $\tilde{\rho}_t(\xi_t(x_0)) = \xi_t(x_0)$.

Proof: Take the diffusion $\xi_t$ in Theorem 3.1, i.e. equation (3). From the theorem we have that $\varphi_t(x_0) = \xi_t(x_0)$. Hence, for any time $0 \geq t$ the left hand side remainder considered here $\tilde{\rho}_t = \varphi_t \xi_t^{-1}$, is a random diffeomorphism which fixes $\xi_t(x_0)$ a.s., moreover it satisfies the non-autonomous equation.

$$d\tilde{\rho}_t = \sum_{j=0}^{k} \{X_j - \text{Ad}(\varphi_t)[\text{Ad}(\xi_t^{-1})X_j]^p\}(\rho_t) \circ dW^j_t.$$
4 Examples

We recall initially two interesting particular cases of decomposition in the literature where the hypothesis of Corollary 3.2 holds due to global geometrical properties of the manifold. In the first case, the $G$-structure preserves the metric tensor, i.e. $g(M)$ is the algebra of Killing vector fields:

**Theorem 4.1** ([12], [15]) *If $M$ is a simply connected Riemannian manifold with constant curvature, then every stochastic flow of the Stratonovich equation (1) satisfies the hypotheses of Corollary 3.2, hence, for each initial condition $x_0 \in M$ and $u_0$ an orthonormal frame in $T_{x_0} M$, the flow $\varphi_t$ admits the decomposition:

$$\varphi_t = \xi_t \circ \rho_t,$$

where $\xi_t$ are isometries of $M$, $\rho_t(x_0) = x_0$ and the linearization $\rho_t^*(u_0) = u_0 q_t$, where $q_t$ is a process in the group of upper triangular matrices.*

**Proof:** In this case the dimension of $g(M)$ is maximal $n(n+1)/2$, (see e.g. [9]) and $\mathfrak{gl}(n, \mathbb{R}) = \mathfrak{s}\mathfrak{o}(n, \mathbb{R}) \oplus \mathfrak{q}$, with $\mathfrak{q}$ the Lie algebra of upper triangular matrices.

This decomposition is used to study in the same context the radial and angular asymptotic behaviour: The Lyapunov exponents comes from the remainder $\rho_t$, [12]; and the matrix of rotation comes from the isometries $\xi_t$, [15].

The second particular case refers to decompositions with affine transformation component, i.e. $\xi_t$ is a diffusion in the group of diffeomorphisms which preserves the connection.

**Theorem 4.2** ([15]) *In Euclidean spaces, given the stochastic flow of the Stratonovich equation (7) for each initial condition $x_0 \in M$ and $u_0$ a frame in $T_{x_0} M$, then

$$\varphi_t = \xi_t \circ \rho_t,$$

where $\xi_t$ are affine transformations, $\rho_t(x_0) = x_0$ and the linearization $\rho_t^* = Id$.*

**Proof:** It is a direct consequence of Corollary 3.2. Again, the dimension of $g(M)$ is maximal $n(n+1)$, (see e.g. [10]) and $\mathfrak{p} = \mathfrak{gl}(n, \mathbb{R})$, hence it degenerates the derivative of the remainder $\rho_t$.

The theorem above holds in a more general geometrical context: essentially the group of affine transformations has to be large enough (depending also on the sde considered). Again, for sufficiently large groups of affine transformations of compact manifolds, Theorem 4.2 implies Theorem 4.1, since in this case, affine transformations are isometries (see e.g. Kobayashi [9 Cor.2.4]).
4.1 Decomposition of symplectic flows

Let \((M, \omega)\) be a symplectic manifold, where \(\omega\) here denotes a closed nondegenerate 2-form in \(M\). In this example we consider a stochastic symplectic flow \(\varphi_t\) in \(M\) associated with the sde

\[dx_t = \sum_{i=0}^k X_i(x_t) \circ dW_t^i\]  \hspace{1cm} (9)

where \(X_0, X_1, \ldots X_k\) are smooth symplectic vector fields on \(M\). The flow \(\varphi_t\) is a symplectic transformation of \(M\), i.e., \(\varphi_t^* \omega = \omega\) almost surely (see e.g. Kunita [11]).

Consider the following decomposition of the symplectic Lie algebra \(\mathfrak{sp}(n, \mathbb{R}) = \mathfrak{p} \oplus \mathfrak{q}\) where

\[p = \left\{ \begin{pmatrix} A & -S \\ S & A \end{pmatrix} : A^t = -A \text{ and } S^t = S \right\}.
\]

And

\[q = \left\{ \begin{pmatrix} \Delta & S \\ 0 & -\Delta^t \end{pmatrix} : S^t = S \text{ and } \Delta \text{ is upper triangular} \right\}.
\]

Given an element \(A \in \mathfrak{sp}(n, \mathbb{R})\), we write \(A = A_p + A_q\) with \(A_p \in \mathfrak{p}\) and \(A_q \in \mathfrak{q}\). Note that the Lie algebra \(\mathfrak{p}\) is a Lie subalgebra of skew-symmetric matrices. Therefore, the subgroup generated \(\langle \exp \mathfrak{p} \rangle \subset \text{Sp}(2n, \mathbb{R}) \cap \text{SO}(2n, \mathbb{R})\). We assume that \(M\) admit a \(\langle \exp \mathfrak{p} \rangle\)-structure \(P\), with \(P\) been a differentiable subbundle of the symplectic \(\text{Sp}(n)\)-structure of \(\text{GL}(M)\).

We consider the projection \(p : \mathfrak{sp}(n, \mathbb{R}) \to \mathfrak{p}\) along the subspace \(\mathfrak{q}\). We have a direct consequence of Corollary 3.2:

**Corollary 4.3** The symplectic stochastic flow \(\varphi_t\) has a decomposition \(\varphi_t = \xi_t \circ \rho_t\), where \(\xi_t\) is a diffusion in the group of isometries, \(\rho_t(x_0) = x_0\) and \(\rho_t^*(u_0) = u_0 q_t\), for some process \(q_t\) in the subgroup \(\exp(\mathfrak{q})\) of the symplectic group.

This decomposition with the choice of subalgebras \(\mathfrak{p}\) and \(\mathfrak{q}\) as above is not the same decomposition with isometric component of Theorem 4.1. To illustrate, consider this simple example:

\[\dot{x}_t = u_1(t)A(x_t) + u_2(t)B(x_t)\]

with non commutative

\[A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}\]

and the bounded measurable control functions given by the indicator functions of intervals \(u_1(t) = 1_{[0,1]}(t), u_2(t) = 1_{[1,\infty]}(t)\). The initial conditions are \(x_0\) the origin and \(u_0\) the canonical basis of \(\mathbb{R}^{2n}\).
The solution of the control equation above is the symplectic diffeomorphism $\varphi_t$, which has the following decomposition: For $0 \leq t \leq 1$

$$\varphi_t = Id \circ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \begin{pmatrix} 1 & 0 \\ 0 & -t & 1 \end{pmatrix} \end{pmatrix}$$

(10)

and for $t \geq 1$,

$$\varphi_t = \begin{pmatrix} \text{Rot}(t-1) & 0 \\ 0 & \text{Rot}(t-1) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \end{pmatrix},$$

(11)

where the 2-dimensional rotation

$$\text{Rot}(s) := \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix},$$

for $s \in \mathbb{R}$. By commutativity of the initial frame $u_0$ with each component, one sees that the first matrices in the product of equations (10) and (11) are the $\xi_t$ component of the decomposition of Corollary 4.3 and the second matrices are the remainder $\rho_t$. This decomposition differs from the decomposition of Theorem 4.1 since in the example here the derivative of the remainder in $u_0$ is not described by the right action of an upper triangular matrix.

### 4.2 Volume preserving component

In this example we give detailed calculation for the decomposition with a volume preserving component. Consider initially that our sde of equation (1) is in the Euclidean space $\mathbb{R}^n$. We shall consider the canonical volume tensor $v = dx_1 \wedge \ldots dx_n$ and the corresponding infinite dimensional group of volume preserving diffeomorphisms. The theory of the previous section applies here to find a decomposition of the solution flow $\varphi_t$ as $\xi_t \circ \rho_t$, where the component $\xi_t$ is in the intersection of affine transformations and the volume preserving group of diffeomorphisms. In our terminology, it corresponds precisely to work with the $\text{Sl}(n, \mathbb{R})$-structure, which here trivializes as $P = \mathbb{R}^n \times \text{Sl}(n, \mathbb{R})$.

Given an initial condition $x_0$ and an initial frame $u_0 \in P$, we consider the following basis for the tangent space $T_{u_0}P$:

$$\{Y_1, \ldots, Y_n, Y_{n+1}, \ldots, Y_{(n^2 + n - 1)}\},$$

where, for $j = 1, \ldots, n$, the horizontal elements are $Y_j = e_j$, the canonical basis and for $j = (n + 1), \ldots, (n^2 + n - 1)$, $Y_j = A_j$, with $(A_j)$ a basis of $\text{sl}(n, \mathbb{R})$.

For $A \in \mathfrak{gl}(n, \mathbb{R})$ there is a unique decomposition $p \oplus q$ given by

$$A = (A - \frac{\text{tr}(A)}{n} Id) + \frac{\text{tr}(A)}{n} Id = A_p + A_q$$
where \( p = \mathfrak{sl}(n, \mathbb{R}) \) and \( q \simeq \mathbb{R} \) are Lie algebras of matrices. Hence Corollary 3.2 applies and we get:

**Corollary 4.4** The stochastic flow \( \varphi_t \) has a unique decomposition \( \varphi_t = \xi_t \circ \rho_t \), where \( \xi_t \) is a diffusion process in the intersection of affine transformations and the group of volume preserving diffeomorphisms, \( \rho_t(x_0) = x_0 \) and \( \rho_t(u_0) = q_t u_0 \), for some process \( q_t \in \mathbb{R} \). The process \( q_t \) carries the information of the sum of the Lyapunov exponents of the flow \( \varphi_t \):

\[
\lim_{t \to \infty} \frac{1}{t} \log q_t = \sum_{i=1}^{n} \lambda_i
\]

where \( \lambda_i \) are the \( n \) Lyapunov exponents (with possible repetition).

**Proof:** The first part follows directly from Corollary 3.2 and the commutativity of the elements of \( q \). The second statement follows because

\[
\sum_{i=1}^{n} \lambda_i = \lim_{t \to \infty} \frac{1}{t} \log \| \varphi_t \|,
\]

and \( q_t = \| \varphi_t \| \). See e.g. Arnold [1], Baxendale [3], Elworthy [7], Liao [13] among others. □

More generally, for a Riemannian manifold \( M \) that admits an \( Sl(n, \mathbb{R}) \)-structure \( P \), we have similarly the following decomposition of the vertical component of the natural lift of \( X \):

\[
\nabla X = (\nabla X - \frac{\text{div} X}{n} \text{Id}) + \frac{\text{div} X}{n} \text{Id},
\]

So, the same decomposition and Lyapunov property of Corollary 4.4 holds.

### 4.3 Cascade Decompositions

In the special geometrical conditions where there exists a sequence of subbundles \( P^1 \subset P^2 \subset \cdots \subset P^n \subset GL(M) \) which are all \( G \)-structure, repeating conveniently the decomposition technique described in the previous section allows a cascade (geometrical filtration) of decompositions. We denote, as before, by \( G^i(M) \) and \( g^i(M) \) the subgroup of automorphisms of \( P^i \) and the Lie algebra of infinitesimal automorphisms, respectively for \( i = 1, \ldots, n \). The Lie algebra of infinitesimal automorphisms is a flag of subalgebras of vector fields

\[
g^1(M) \subset \cdots \subset g^{n-1}(M) \subset g^n(M)
\]

where \( g^i(M) = \{ X \in a(M) : L_X k^i = 0 \} \), \( i = 1, \ldots, n \), for some tensor field \( k^i \).

We shall assume that, for \( i = 1, 2, \ldots, n \):

**\( \text{C1} \)** There exists a sequence of \( G \)-structure \( P^1 \subset P^2 \subset \cdots \subset P^n \subset GL(M) \) such that for an initial condition \( x_0 \in M \) and \( u_0 \in P^1 \), there is a sequence of projections \( p_i : T_{u_0}P^{i+1} \rightarrow T_{u_0}P^i \), with \( p_i(HT_{u_0}P^{i+1}) = HT_{u_0}P^i \) and \( p_i(VT_{u_0}P^{i+1}) = VT_{u_0}P^i \), where \( P^{n+1} = GL(M) \).
The projections \(p_n \circ p_{n-1} \circ \cdots \circ p_i \circ \delta \text{Ad}(\xi)X_j(u_0)\) are in the image \(\text{Im}(\delta g^i(M))\), for all vector fields \(X_j\), \(j = 0, 1, \ldots, k\) of the sde [1] and for any automorphism \(\xi\) of \(P^i\).

**Corollary 4.5 (Cascade decomposition)** Under conditions (C1) and (C2) above, we have the following decomposition of the stochastic flow:

\[
\varphi_t = \xi_1^i \circ \xi_2^i \circ \cdots \xi_n^i \circ \rho_t,
\]

where for \(1 \leq i \leq n\), \(\xi_i^i \in G^k(M)\), the partial compositions \((\xi_i^1 \circ \cdots \xi_i^n)\) are diffusions in \(G^i(M)\), \((\xi_i^1 \circ \cdots \xi_i^n)\) are elements of the form:

\[
\rho_t(x_0) = u_0q_t^{(i)},
\]

where \(q_t^{(i)}\) is a process in the Lie subgroup \(\exp(\ker p_n \circ p_{n-1} \circ \cdots p_i) \subset \text{GL}(n, \mathbb{R})\).

**Proof:** For each subbundle \(P^n\) of \(\text{GL}(M)\), consider the projection \(p_n \circ p_{n-1} \circ \cdots p_i : T_u \text{GL}(M) \to T_u P^i\). Condition (C2) implies hypothesis (H2), hence, by Theorem 3.1 there exists a decomposition \(\varphi_t = \xi_t^{(i)} \rho_t^{(i)}\) such that \(\xi_t^{(i)}\) is a diffusion in \(G^i(M)\) and \(\rho_t^{(i)} \circ u_0 = u_0q_t^{(i)}\). The result follows by taking \(\xi_t^{(i)} = \xi_t^{(i)}\) and by induction

\[
\xi_t^i = (\xi_t^{(i-1)})^{-1} \circ \xi_t^{(i)}
\]

for \(1 \leq i \leq n\) and \(\rho_t = \rho_{t}^{(n)}\).

A direct example of cascade decomposition occurs if we put together the results of Theorems 4.1, 4.2 and 4.4. If the geometry of the manifold is simple enough (large groups of isometries and affine transformations) then the flow decomposes as

\[
\varphi_t = \xi_1^i \circ \xi_2^i \circ \xi_n^i \circ \rho_t,
\]

where \(\xi_1^i\) is a diffusion in the group of isometries of \(M\), \(\xi_1^i \circ \xi_2^i\) is a diffusion in the group of volume preserving transformations of \(M\), \(\xi_1^i \circ \xi_2^i \circ \xi_3^i\) is a diffusion in the group of affine transformations of \(M\) and \(\rho_t(x_0) = x_0\) with linearization at \(x_0\) given by \(\rho_t = Id\).

Another natural class of examples of cascade decomposition which satisfies hypotheses (C1) and (C2) is a stochastic flow \(\varphi_t\) in \(\mathbb{R}^n\) where the Lie subalgebras \(p_i\) are elements of the form: \(p_1 = \mathfrak{so}(n, \mathbb{R})\) and the sequence is a flag in the subalgebras of the form

\[
p_i = \mathfrak{so}(n, \mathbb{R}) \oplus \begin{pmatrix} B_{k_1} & 0 \\ 0 & \ddots \\ 0 & 0 \end{pmatrix}
\]

where \(B_{k_j}\) are \(k_j \times k_j\) upper triangular matrices with \(\sum_{j=1}^s k_j = n\). One finds many choices of a chain of subalgebras \(p_1 = \mathfrak{so}(n, \mathbb{R}) \subset p_2 \subset \cdots p_n = \mathfrak{gl}(n, \mathbb{R})\) with natural projections as stated in hypothesis (C2).
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