Crossover scaling in the Domany-Kinzel cellular automaton

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Abstract. We consider numerically the crossover scaling behavior from the directed percolation universality class to the compact directed percolation universality class within the one-dimensional Domany-Kinzel cellular automaton. Our results are compared to those of a recently performed field theoretical approach. In particular, the value of the crossover exponent $\phi = 2$ is confirmed.

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1. Introduction

We consider the crossover scaling behavior between two non-equilibrium universality classes. Both universality classes describe second order phase transitions into an absorbing state where the dynamics of the model is trapped for ever. The first class is the famous directed percolation (DP) universality class (see [1, 2, 3, 4] for recent reviews). The second class is the so-called compact directed percolation (CDP) universality class which is also known as the voter universality class within the mathematical literature. Whereas the one-dimensional CDP behavior is well understood due to a mapping to random walks, the stochastic process of DP is still analytically unsolved. Numerical investigations of this crossover trace back to [5]. Here, we numerically examine the crossover between both universality classes within the one-dimensional Domany-Kinzel (DK) cellular automaton [6]. This model is well suited for the corresponding crossover investigation since its phase diagram exhibits a line of DP-like transitions, terminating in a CDP endpoint. In particular, we consider the temporal evolution of the order parameter. The behavior of this quantity reflects the crossover and will be useful to determine the crossover exponent. Our results confirm those of a recently performed field theoretical approach [7]. In particular, the value of the crossover exponent, which is expected to be exact, is confirmed.

2. Directed Percolation

The stochastic process of directed percolation (DP) can be considered as a paradigm of non-equilibrium phase transitions into an absorbing state (see [1, 2, 3, 4] for recent reviews). The process of directed percolation can be represented by the Langevin equation [8] (in the Ito sense, see e.g. [9])

$$\lambda^{-1} \partial_t \rho_a(r,t) = \tau \rho_a(r,t) - u \rho_a(r,t)^2 + \nabla^2 \rho_a(r,t) + \eta(r,t)$$  (1)
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which describes the order parameter $\rho_a(r, t)$, i.e., the density of active sites, on a mesoscopic scale. A positive order parameter occurs above the transition point ($\tau = 0$) whereas the absorbing state $\rho_a = 0$ is approached for negative $\tau$, below the transition point. Furthermore, $\eta$ denotes the noise which accounts for fluctuations of the particle density $\rho_a(r, t)$. According to the central limit theorem, $\eta(r, t)$ is a Gaussian random variable with zero mean and whose correlator is given by

$$\langle \eta(r, t) \eta(r', t') \rangle = \lambda^{-1} \Gamma \rho_a(r, t) \delta(r - r') \delta(t - t'). \quad (2)$$

Notice that the noise ensures that the system is trapped in the absorbing state $\rho_a(r, t) = 0$. Furthermore, higher order terms such as $\rho_a(r, t)^3, \rho_a(r, t)^4, \ldots$ (or $\nabla^4 \rho_a(r, t), \nabla^6 \rho_a(r, t), \ldots$) are irrelevant under renormalization group transformations as long as $u > 0$. Negative values of $u$ give rise to a first order phase transition whereas $u = 0$ is associated with a tricritical point (see [10, 11, 7] for field theoretical as well as [12, 13] for recently performed numerical investigations of tricritical directed percolation).

As usual for second order phase transitions the system obeys certain scaling laws close to criticality (see [2] or [4] for complete discussions of the critical scaling behavior of absorbing phase transitions). For example, the steady state order parameter scales (up to higher orders) as

$$\rho_a \propto \tau^\beta \quad (3)$$

for $\tau > 0$. Additionally to the steady state scaling behavior, the dynamical scaling is also expressed in terms of power laws

$$\rho_a(t) \propto t^{-\alpha}, \quad P_a(t) \propto t^{-\delta}. \quad (4)$$

The first law describes the order parameter decay at criticality starting from a fully occupied lattice. The second power law describes the temporal evolution of the survival probability $P_a$ of an initially isolated single seed of activity. This probability is related to the probability that a given site belongs to a percolating cluster

$$P_{perc} \propto \tau^{\beta'}, \quad (5)$$

for $\tau > 0$. Often this quantity is considered as the order parameter of the percolation transition, additionally to the steady state density $\rho_a$. Above the upper critical dimension $d_{c, DP} = 4$, the exponents equal their mean field values, e.g. $\beta = 1$ and $\beta' = 1$.

Below the upper critical dimension renormalization group techniques have to be applied to determine the critical exponents. In that case path integral formulations are more adequate than the Langevin equation approach (see e.g. [14]). Furthermore, they provide a deeper understanding of the stochastic process. Stationary correlation functions as well as response functions can be determined by calculating path integrals with weight $\exp \left( -J \right)$, where the dynamic functional $\mathcal{J}$ describes the considered stochastic process. Up to higher irrelevant orders the dynamic functional associated with directed percolation is given by [8, 15, 16, 17]

$$\mathcal{J}[\hat{\rho}_a, \rho_a] = \lambda \int d^d r dt \hat{\rho}_a \left[ \lambda^{-1} \partial_t \rho_a - (\tau + \nabla^2) \rho_a - \left( \frac{\Gamma}{2} \hat{\rho}_a - u \rho_a \right) \rho_a \right] \quad (6)$$

where $\hat{\rho}_a(r, t)$ denotes the response field conjugated to the Langevin noise field $\eta$. The functional $\mathcal{J}$ is invariant under the duality transformation (so-called rapidity reversal in Reggeon field theory)

$$\mu^{-1} \hat{\rho}_a(r, t) \leftrightarrow -\mu \rho_a(r, -t) \quad (7)$$
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with the redundant variable $\mu^2 = 2u/\Gamma$. As usual, the duality transformation defines a dual stochastic process that might differ from the original one \cite{19}. At criticality, the average density of particles of the dual process $\rho_{\text{dual}}^a$ equals asymptotically the survival probability \cite{7} via

$$P_a(t) \sim \mu^2 \rho_{\text{dual}}^a(t).$$

Here, $P_a(t)$ denotes again the probability that a cluster generated by a single seed is still active after $t$ time steps. On the other hand, $\rho_{\text{dual}}^a(t)$ describes the particle decay of the dual process started from a fully occupied lattice. The self-duality of directed percolation implies $\rho_a(t) = \rho_{\text{dual}}^a(t)$ and therefore \cite{20,7}

$$P_a(t) \sim \mu^2 \rho_a(t).$$

The asymptotic equivalence ensures that both quantities have the same exponents \cite{20}, including

$$\alpha = \delta.$$ \hspace{1cm} (10)

Taking the scaling laws $\beta = \nu_\parallel \alpha$ and $\beta' = \nu_\parallel \delta$ into account ($\nu_\parallel$ denotes the exponent of the temporal correlations) equation (10) implies the identity

$$\beta = \beta'$$ \hspace{1cm} (11)

for the universality class of directed percolation.

It is worth mentioning that equation (9) follows from the rapidity reversal symmetry of the dynamical functional $J$, i.e., it is a specific property of the directed percolation universality class. Thus, compared to general absorbing phase transitions, the number of independent critical exponents for directed percolation is reduced. Furthermore, the self-duality is expressed within the field theoretical treatment of the minimal Langevin equation of directed percolation. Hence, the important rapidity reversal is usually reflected on a coarse grained level only and holds often only asymptotically. In other words, it is often masked on a microscopic level, i.e. it does not necessarily represent a physical symmetry of microscopic models \cite{5}. Nevertheless, the robustness and the ubiquity of the DP universality class is expressed by the conjecture of Janssen and Grassberger \cite{8,21}: short-range interacting systems, exhibiting a continuous phase transition into an absorbing state, belong to the DP universality class, if they are characterized by a one-component order parameter. Different universal scaling behavior is expected to occur in the presence of additional symmetries or relevant disorder effects.

3. Domany-Kinzel automaton and Compact Directed Percolation

Several lattice models are known which exhibit an absorbing phase transition belonging to the directed percolation universality class. Famous examples are the contact process \cite{22}, the pair-contact process \cite{23}, the Ziff-Gulari-Barshad model \cite{24}, and the threshold transfer process \cite{25} (see e.g. \cite{26} for a detailed scaling analysis of various DP-like lattice models). Another well known 1 + 1-dimensional stochastic cellular automaton exhibiting directed percolation scaling behavior is the Domany-Kinzel (DK) automaton \cite{6}. It is defined on a diagonal square lattice where the lattice sites are either empty ($n = 0$) or occupied by a particle ($n = 1$). At discrete time steps a parallel update procedure is performed according to the following rules (see figure 1). A site at time $t$ is occupied with probability $p_2$ ($p_1$) if both (only one) backward sites (at time $t - 1$) are occupied. Otherwise the site remains empty.
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Notice that the Domany-Kinzel automaton depends on the two parameters $p_1$ and $p_2$. This dependence results in a non-trivial phase diagram which is shown in figure 2. Some particular processes are included in the Domany-Kinzel automaton. For example, the process of bond directed percolation corresponds to the choice $p_2 = p_1(2 - p_1)$ whereas site directed percolation is obtained for $p_1 = p_2$ [6]. Furthermore, for $p_2 = 0$ the Domany-Kinzel automaton equals to the cellular automata rule "18" of Wolfram’s classification scheme [27, 28].

Instructive insights into the behavior of the system can be gained already from a simple mean field treatment (see e.g. [29, 30, 31, 32]). Within a one-site approximation the above presented reaction scheme leads to the differential equation for the density of active (occupied) sites (see [4] for a detailed consideration)

$$\partial_t \rho_a(p_1, p_2) = 2(2p_1 - 1)\rho_a - 2(2p_1 - p_2)\rho_a^2.$$  \hspace{1cm} (12)

Focusing on the steady state behavior ($\partial_t \rho_a = 0$) we find that the order parameter is given by

$$\rho_a(p_1, p_2) = 0 \quad \vee \quad \rho_a(p_1, p_2) = \frac{2p_1 - 1}{2p_1 - p_2}.$$  \hspace{1cm} (13)

The first solution is stable for $p_1 < 1/2$ and corresponds to the inactive phase. The active phase is described by the second solution which is stable for $p_1 > 1/2$. The phase diagram of the Domany-Kinzel automaton exhibits two phases separated by the critical line $p_{1,c} = 1/2$ (see figure 2). Along this critical line the order parameter vanishes linearly ($\beta_{\rho_a} = 1$) in the active phase

$$\rho_a(p_1, p_2) = \frac{1}{1 - p_2} \delta p + O(\delta p^2)$$  \hspace{1cm} (14)

with $\delta p = (p_1 - p_c)/p_c$. Thus, the critical exponent of the order parameter equals the mean field value of directed percolation as along as $p_2 < 1$, i.e., all critical points on the transition line (except of the termination point $p_2 = 1$) belong to the universality class of directed percolation. The last statement is also valid for the 1 + 1-dimensional Domany-Kinzel automaton although the transition line as well as the values of the critical exponents differ from the mean field approximations (see figure 2).

Clearly, the above derived mean field results are not valid for $p_2 = 1$. In that case no particle annihilation takes place within a domain of occupied sites. Thus,
creation and annihilation processes are bounded to the domain walls where empty and occupied sites are adjacent. This corresponds to the particle-hole symmetry

\[ n \leftrightarrow 1 - n \]  

which is also reflected in the mean field differential equation

\[ \partial_t \rho_a(p_1, p_2 = 1) = 2 \left( 2p_1 - 1 \right) \rho_a \left( 1 - \rho_a \right) = 2 \delta p \rho_a \left( 1 - \rho_a \right). \]  

To be precise, equation (16) is invariant under the transformation \( \rho_a \leftrightarrow 1 - \rho_a \) and \( \delta p \leftrightarrow -\delta p \). Beside the empty lattice, the fully occupied lattice is now another absorbing state. The behavior of the domain wall boundaries can be mapped to a simple random walk and the domains of particles grow on average for \( \delta p > 0 \) whereas they shrink for \( \delta p < 0 \). Thus the steady state density \( \rho_a \) is zero below \( p_c = 1/2 \) and \( \rho_a = 1 \) above \( p_c \). At the critical value the order parameter \( \rho_a \) exhibits a jump.

The associated critical exponent \( \beta_{\text{CDP}} = 0 \) differs in all dimensions from the directed percolation values \( \beta_{\text{DP}} \). Since domain branching does not take place the dynamical process for \( p_2 = 1 \) is often termed compact directed percolation (CDP). The critical behavior equals that of the 1 + 1-dimensional voter model and it is analytically tractable due to the mapping to random walks. Exact results are derived for the critical exponents \( \beta = 0, \beta' = 1, \nu_\parallel = 2, \nu_\perp = 1 \), as well as for certain finite-size scaling functions. In particular, the domain growth from a single seed exhibits a one-to-one correspondence to a pair of annihilating random walkers. That correspondence allows the calculation of the percolation probability

\[ P_{\text{perc}}(p_1) = \begin{cases} 0 & \text{if } p_1 < 1/2, \\ \frac{2p_1 - 1}{p_1} & \text{if } p_1 > 1/2, \end{cases} \]  

yielding the percolation exponent \( \beta'_{\text{CDP}} = 1 \). In contrast to directed percolation the universality class of compact directed percolation is characterized by the inequality

\[ \beta_{\text{CDP}} \neq \beta'_{\text{CDP}}. \]  

The number of independent critical exponents is therefore greater than for directed percolation. In summary, the universality class of compact directed percolation is characterized by a continuously vanishing percolation order parameter \( P_{\text{perc}} \) and algebraically diverging correlations lengths, indicating a second order phase transition. But due to the misleading discontinuous behavior of the steady state order parameter \( \rho_a \) the phase transition was often considered as first order.

Within a field theoretical approach, the CDP process can be described by the Langevin equation

\[ \lambda^{-1} \partial_t \rho_a = \tau \rho_a (1 - \rho_a) + \nabla^2 \rho_a + \eta. \]  

In order to ensure that the fully occupied and empty lattice is absorbing the noise correlator obeys

\[ \langle \eta(r, t) \eta(r', t') \rangle = \lambda^{-1} \Gamma \rho_a (r, t) [1 - \rho_a (r, t)] \delta(r - r') \delta(t - t'). \]  

Simple dimensional counting shows that the noise is irrelevant for \( d > 2 \), i.e., the value of the upper critical dimension is \( d_{\ast, \text{CDP}} = 2 \). Associated to this Langevin equation is the response functional

\[ \mathcal{J} [\hat{\rho}_a, \rho_a] = \lambda \int d^4 r d t \hat{\rho}_a \left[ \lambda^{-1} \partial_t \rho_a - \tau \rho_a (1 - \rho_a) - \nabla^2 \rho_a - \frac{\Gamma}{2} \hat{\rho}_a \rho_a (1 - \rho_a) \right]. \]
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Figure 2. The phase diagram of the $1+1$-dimensional Domany-Kinzel automaton. The bold circles indicate the numerically determined transition points (the solid line is just to guide the eye). The dashed line corresponds to the simplest mean field approximation (see text). In both cases, all transition points along the critical lines belong to the universality class of directed percolation except of the termination point ($p_1 = 1/2$, $p_2 = 1$).

The functional is invariant under the transformation
\begin{align*}
\rho_a(r, t) &\leftrightarrow 1 - \rho_a(r, -t), \\
\tilde{\rho}_a(r, t) &\leftrightarrow -\tilde{\rho}_a(r, -t), \\
\tau &\leftrightarrow -\tau ,
\end{align*}
which reflects the particle-hole symmetry and differs obviously from the rapidity-reversal symmetry of directed percolation.

In the following we consider the crossover scaling from CDP to DP. It is instructive to express the mean field equation (12) in terms of the distances to the critical line $\tau = 2\delta p$ and to the termination point $\kappa = 2(1 - p_2)$
\begin{equation}
\partial_t \rho_a(p_1, p_2) = \tau \rho_a(1 - \rho_a) - \kappa \rho_a^2. \tag{23}
\end{equation}

From the CDP point of view ($\kappa = 0$), the particle-hole symmetry is broken for any positive $\kappa$, i.e., for any vertical distance from the termination point in the phase diagram. The crossover from the CDP to the DP behavior can be described in terms of scaling forms. For example, the steady state order parameter obeys for any positive value of $\lambda$
\begin{equation}
\rho_a(\tau, \kappa) \sim \lambda^{-\beta_{\text{CDP}}} \tilde{r}(\tau \lambda, \kappa \lambda^\phi) \tag{24}
\end{equation}
with $\beta_{\text{CDP}} = 0$ and where $\phi$ denotes the crossover exponent. Setting $\lambda = \kappa^{-1/\phi}$ we obtain $\rho_a(\tau, \kappa) = \tilde{r}(\tau \kappa^{-1/\phi}, 1)$. Comparing this result with the steady state solution (see equation (13))
\begin{equation}
\rho_a(\tau, \kappa) = \frac{\tau/\kappa}{1 + \tau/\kappa} \tag{25}
\end{equation}
we can identify the crossover exponent as well as the crossover scaling function
\begin{equation}
\phi_{\text{MF}} = 1, \quad \tilde{r}_{\text{MF}}(x, 1) = \frac{x}{1 + x} . \tag{26}
\end{equation}
Studying the crossover beyond the mean field level, we have to add to the Langevin equation an appropriate term which breaks the particle hole-symmetry. For example, within the above presented mean field approximation the particle-hole symmetry is broken by \(-\kappa\rho_a^2\) (see equation 23). Alternatively, a term of linear order \(\rho_a\) acts in the same way. The latter case is examined in ref. [7] by Janssen. Performing a field theoretical treatment, Janssen derived the crossover exponent

\[
\phi = \begin{cases} 
\frac{2}{d} & \text{if } d < d_c = 2 \\
1 & \text{if } d \geq d_c = 2.
\end{cases}
\]

(27)

Remarkably, these field theoretical results are expected to be exact since the involved diagrams can be summed exactly (see [7] for a detailed discussion). In the following we consider the 1 + 1-dimensional Domany-Kinzel automaton in the vicinity of the termination point and confirm numerically the result \(\phi_{d=1} = 2\).

4. Numerical Analysis of the Crossover Behavior

At the beginning of our analysis, we determine several critical points in order to obtain an estimate of the transition line. For this purpose, we consider the dynamical properties of the system along various lines \(p_2 = kp_1\) for \(k\) ranging from 0.1 up to 1.99 (see table 1). For each value, we vary \(p_1\) and determine the critical point. Therefor, we use a standard spreading procedure, i.e., we consider the spreading of an initial seed and measure the survival probability \(P_a(t)\). At the critical point \(p_{1,c}\), the survival probability obeys a power-law behavior with the critical exponent \(\delta = \alpha = 0.159464(6)\), see equation (4). Slightly above or below the critical value, the survival probability exhibits deviations from a pure power-law, resulting in a significant (left or right) curvature in a double-logarithmic plot (not shown). In this way we get a sufficient estimate of the transition line. The corresponding data are shown in figure 2 and are listed in table 1.

As well known from scaling theory, the crossover exponent \(\phi\) determines, additionally to the crossover scaling behavior, the shape of the transition line [34]. In our case, the phase boundary is expected to obey asymptotically

\[
\tau_c \propto \kappa^{1/\phi}.
\]

(28)

Here, the crossover parameter \(\kappa\) equals again the distance to the termination point along the \(p_2\)-direction, i.e., \(\kappa \propto (1 - p_2)\). On the other hand, \(\tau_c\) is given by

\[
\tau_c \propto p_{1,c}(\kappa) - p_{1,c}(\kappa = 0) = p_{1,c}(p_2) - \frac{1}{2}.
\]

(29)

Thus, the transition line behaves as

\[
p_{1,c}(\kappa) = p_{1,c}(p_2) = \frac{1}{2} + \text{const} (1 - p_2)^{1/\phi}.
\]

(30)

Therefore, we plot in figure 3 the crossover parameter \(1 - p_2\) as a function of \(p_{1,c} - 1/2\) in a double logarithmic plot. Approaching the termination point, the transition line agrees well with equation (28) and confirms the field theoretical result \(\phi = 2/d\) [7], i.e., \(\phi = 2\) for \(d = 1\). But as can be seen, equation (28) describes only the asymptotic behavior close to the termination point, i.e., for \(\kappa \ll 0.1\). This reflects the notorious difficulties to determine the crossover exponent \(\phi\) from the phase boundary since it is in general not clear whether the asymptotic scaling regime is actually reached.
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As already mentioned above, all critical points along the phase transition line (except of the termination point) belong to the universality class of directed percolation. Thus the scaling behavior at all determined critical points has to obey the rapidity reversal and equation (30) has to be fulfilled. In order to check its validity we have measured first the order parameter decay (started from a fully occupied lattice) and second the survival probability (of an initially single seed) at the critical point. Examples are shown in figure 4 for \( p_2 = p_1(2-p_1) \) (bond directed percolation) as well as for \( p_2 = p_1/2 \). Bond directed percolation is a particular case since the rapidity reversal is microscopically obeyed (see e.g. [2]). Therefore, equation (9) holds for all \( t \) with \( \mu = 1 \), i.e.,

\[
P_a(t) = \rho_a(t).
\]

This can be seen in figure 4, where the survival probability \( P_a(t) \) and the order parameter \( \rho_a(t) \) collapse onto the same function.

Except of this particular case, the order parameter decay equals the survival probability behavior only asymptotically. Furthermore, the variable \( \mu \) is non-trivial, i.e., \( \mu \neq 1 \). Both effects can be seen in figure 4 for \( p_2 = p_1/2 \). After a certain transient \( P_a(t) \) and \( \rho_a(t) \) are characterized by the same power-law decay, i.e., \( \alpha = \delta \). Choosing the correct value of \( \mu \) (here \( \mu \approx 1.4 \)) the tails of both functions collapse onto the same curve. Worth mentioning, the transient regime increases by approaching the termination point. But more important, the factor \( \mu \) varies along the transition line. Moving along the critical line into the direction of the CDP termination point, \( \mu \) decreases. Approaching the termination point, \( \mu \) tends to zero signalling the violation of equation (9). In other words, \( \mu \rightarrow 0 \) reflects the breakdown of the rapidity reversal caused by the change of the universality class from DP to CDP. In figure 5, \( \mu^2 \) is plotted as a function of the distance from the termination point along the transition line. Notice that \( \mu^2 \) vanishes linearly as a function of \( p_1 \). In that way, \( \mu \) can be used to parameterize the critical line of the one-dimensional DK automaton alternatively to the crossover parameter \( \kappa \). But in the below presented scaling analysis, we prefer to use \( \kappa \) since the accuracy of the determination of \( \kappa \), i.e., of the critical line, is significantly higher than that of \( \mu \).

Figure 3. The transition points (bold circles) of the 1 + 1-dimensional Domany-Kinzel automaton according to equation (30). The dashed line indicates the analytically expected behavior with the crossover exponent \( \phi = 2 \).
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Various quantities, for example \( \rho_a(t) \) as well as \( P_a(t) \), reflect the crossover and are therefore well suited for our analysis. But in contrast to the survival probability \( (\delta_{\text{CDP}} > 0) \) the order parameter decay \( \rho_a(t) \) has the advantage that the rescaling affects only one axis since \( \beta_{\text{CDP}} = \alpha_{\text{CDP}} = 0 \). Thus, for the sake of simplicity, we focus our attention to the scaling behavior of the order parameter decay for critical points along the transition line. In order to describe the temporal evolution we incorporate

Figure 4. The order parameter decay \( \rho_a(t) \) (started from a fully occupied lattice) and the survival probability \( P_a(t) \) (started from a single seed) for two distinct critical points of the Domany-Kinzel automaton. For bond-DP the dual symmetry is fulfilled microscopically leading to \( \rho_a(t) = P_a(t) \). In general, this equivalence is only asymptotically obeyed i.e., \( \rho_a(t) \sim \mu^2 P_a(t) \). Here, the curves for \( p_2 = p_1/2 \) are shown.

Table 1. Estimates of critical points of the 1+1-dimensional Domany-Kinzel automaton. Bond directed percolation \( (p_2 = p_1(2 - p_1)) \) is denoted by bDP. Site directed percolation corresponds to \( k = 1 \). The data of [37] are obtained by series expansions. In 1+1-dimension, this method yields data of significantly higher accuracy than usual Monte Carlo methods.

| \( k \) | \( p_{1,c} \) | \( p_{2,c} = kp_{1,c} \) | \( \mu \) |
|---|---|---|---|
| 0 | 0.811(1) | 0 | 1.5747(34) |
| 0.1 | 0.8015(4) | 0.08015 | 1.4876(32) |
| 0.25 | 0.7894(3) | 0.19745 | 1.3957(31) |
| 0.5 | 0.7668(2) | 0.3834 | 1.3957(31) |
| 0.7 | 0.74515(7) | 0.521605 | 1.3195(30) |
| 0.8 | 0.73300(10) | 0.5864 | 1.2809(34) |
| 1.0 | 0.70548515(20) [37] | 0.70548515 | 1.1902(30) |
| 1.2 | 0.67316(11) | 0.807792 | 1.1046(26) |
| bDP | 0.644700185(5) [37] | 0.837362052 | 1 |
| 1.45 | 0.62585(9) | 0.9074825 | 0.93737(20) |
| 1.6 | 0.594305(15) | 0.950888 | 0.82863(22) |
| 1.67 | 0.57870(8) | 0.966429 | 0.76165(26) |
| 1.8 | 0.54865(7) | 0.98757 | 0.60868(24) |
| 1.9 | 0.52469(6) | 0.996011 | 0.44501(30) |
| 1.95 | 0.5124250(15) | 0.99922875 | 0.31626(38) |
| 1.99 | 0.5024969(15) | 0.99996903 | 0.14060(44) |

Now we consider the crossover scaling behavior using dynamical simulations. Various quantities, for example \( \rho_a(t) \) as well as \( P_a(t) \), reflect the crossover and are therefore well suited for our analysis. But in contrast to the survival probability \( (\delta_{\text{CDP}} > 0) \) the order parameter decay \( \rho_a(t) \) has the advantage that the rescaling affects only one axis since \( \beta_{\text{CDP}} = \alpha_{\text{CDP}} = 0 \). Thus, for the sake of simplicity, we focus our attention to the scaling behavior of the order parameter decay for critical points along the transition line. In order to describe the temporal evolution we incorporate
an additional scaling field to the steady state scaling form
\[ \rho_s(\tau, t, \kappa) \sim \lambda^{-\beta_{CDP}} \tilde{r}(\tau \lambda, t \lambda^{-\nu_{1,CDP}}, \kappa \lambda^\phi). \]  

Along the transition line ($\tau = 0$), the order parameter has to obey the scaling form
\[ \rho_s(0, t, \kappa) \sim \tilde{r}(0, 1, \kappa t^{\phi/2}) \]  

where we made use of the analytical results $\beta_{CDP} = 0$, $\nu_{1,CDP} = 2$ and where we have set $t \lambda^{-\nu_{1,CDP}} = 1$.

The inset of figure 6 displays the order parameter decay close to the termination point ($k = 1.8, 1.9, 1.95, 1.99$ see Table 1). Starting from the fully occupied lattice the order parameter remains nearly constant ($\alpha_{CDP} = 0$) before a crossover to the directed percolation power-law behavior takes place ($\alpha_{DP} \approx 0.159$). The first transient regime increases by approaching the termination point. Using the field theoretical result $\phi = 2$ these different curves have to collapse onto a single curve by plotting $\rho_s$ as a function of $\kappa t$. The obtained data collapse (see figure 6) confirms the field theoretical result.

5. Summary

We have investigated the crossover scaling behavior from directed percolation to compact directed percolation within the Domany-Kinzel automaton. Our results confirm the field theoretical results, in particular the crossover exponent $\phi = 2/d$. Furthermore, we have analysed the rapidity reversal symmetry along the transition line. To our knowledge, the important field theoretical parameter $\mu$ is numerically determined for the first time. Approaching the termination point this parameter vanishes, reflecting the breakdown of the rapidity reversal.

For the sake of completeness we mention that the crossover from DP to CDP along the transition line is also reflected in the morphology of the spreading clusters. As pointed out in [2], the varying morphology corresponds to the change of short-range
correlations, leaving the long-range correlations (i.e., the scaling behavior) unchanged unless the termination point is reached.

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