Iwasawa nilpotency degree of non compact symmetric cosets in $\mathcal{N}$–extended Supergravity

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We analyze the polynomial part of the Iwasawa realization of the coset representative of non compact symmetric Riemannian spaces. We start by studying the role of Kostant’s principal $SU(2)_P$ subalgebra of simple Lie algebras, and how it determines the structure of the nilpotent subalgebras. This allows us to compute the maximal degree of the polynomials for all faithful representations of Lie algebras. In particular the metric coefficients are related to the scalar kinetic terms while the representation of electric and magnetic charges is related to the coupling of scalars to vector field strengths as they appear in the Lagrangian.

We consider symmetric scalar manifolds in $\mathcal{N}$–extended supergravity in various space-time dimensions, elucidating various relations with the underlying Jordan algebras and normed Hurwitz algebras. For magic supergravity theories, our results are consistent with the Tits-Satake projection of symmetric spaces and the nilpotency degree turns out to depend only on the space-time dimension of the theory.

These results should be helpful within a deeper investigation of the corresponding supergravity theory, e.g. in studying ultraviolet properties of maximal supergravity in various dimensions.
1. Introduction

In the present paper we analyze the polynomial part of the Iwasawa realization of the coset representative of non compact symmetric Riemannian spaces.

Such non compact forms of the Lie groups are relevant for supergravity [1, 2, 3, 4, 5, 6, 7, 8]. Here the scalar sector of the theory is described by a non linear sigma model based on a non compact symmetric space with negative curvature \( G/K \), where \( G \) is non compact and \( K \) is its maximal compact subgroup. In this framework \( G \) is the continuous group of \( U \)-duality transformations, called electric magnetic dualities in \( D = 4 \). Exceptions to this description are \( N = 1 \) and \( 2 \) theories, where the sigma model doesn’t need to be a symmetric space but it has a restricted holonomy (Hodge-Kähler for \( N = 1 \), special Kähler and Quaternionic for vector and hypermultiplets in \( N = 2 \)).

Moreover, non compact homogeneous spaces and their non compact duality group \( G \) appear in the description of the black hole orbits [9, 10, 11, 12] as well as duality invariant Bekenstein-Hawking entropy formula [13, 14, 15, 16] according to the attractor mechanism [17, 18, 19]. The study of the attractor mechanism in string theory has been pioneered in [20].

In the past few years there has been a strong development in 4-dimensional \( N = 8 \) and \( N = 4 \) supergravity theories (SUGRA) because of their unexpected remarkable ultraviolet behaviour. It has been discovered that \( d = 4 \) \( N = 8 \) SUGRA is finite in the ultraviolet up to four loops [21], while the pure \( N = 4 \) theory is finite up to 3 loops [22], which is not a priori guaranteed by supersymmetry, where a valid counterterm could in principle exist. \( N = 4 \) SUGRA, however, is already divergent at four loops [23], and when coupled with matter already at one loop [24, 25].

Perturbative finiteness of \( d = 4 \) \( N = 8 \) SUGRA is possible only if its \( E_{7(7)} \) symmetry is anomaly-free [26, 27, 28, 29]. Even when the symmetry is anomalous, as in pure \( N = 4 \) SUGRA, its Ward identities can restrict counterterms [30, 31, 32]. The latter can remain invariant under the Borel parabolic subgroup [32] of the original non compact duality group. At the quantum level the group
is expected to be broken to one of its discrete subgroups. An example of this phenomenon is the Dirac-Schwinger-Zwanziger quantization of black hole states.

The Iwasawa parametrization explicitly constructed at the group level in [33] provides a realization of the coset based on a nilpotent subalgebra. As the coset representative directly enters the Lagrangian of the corresponding supergravity theory, it naturally delivers a choice of the fields for which they appear polynomially in the Lagrangian, making the calculations significantly more manageable. This is relevant for the study of the ultraviolet properties of maximal supergravities in various dimensions.

Our analysis of the structure of the nilpotent subalgebra is based on the work of Kostant [34] who has introduced the concept of principal SU(2) subalgebra (principal triple). He has shown how such principal SU(2) characterizes the nilpotent subalgebras of maximal degree of the enveloping algebra in the adjoint representation. We generalize his results for the adjoint to all the faithful linear representations.

The principal triple is related to the appearance of W-algebras, which are relevant as symmetries for integrable Toda systems (for a review see e.g. [35]) and higher spins [36]. In particular, the W-algebra related to the principal SL(2) is finite and Abelian and the corresponding Toda system is called Abelian. In the framework of AdS higher spins, the choice of an embedding of SL(2) determines the asymptotical symmetry on the boundary, hence fixing the theory. It turns out [37] that only the W-algebras constructed using the principal embedding could admit a unitary representation for large values of the central charge.

We have been able to determine the degree of the polynomials in the nilpotent part of the Iwasawa decomposition for the faithful representations of all the non compact symmetric spaces. In particular we have applied our results to the symmetric scalar manifolds in Maxwell-Einstein theories of (super)gravity in various space-time dimensions for various signatures. We also determine the degree of the polynomials occurring in the biinvariant metric of the coset. We find that in the case of magic $\mathcal{N}=2$ supergravity the degree is independent of the particular representation appearing in the corresponding magic square, but depends only on the space-time dimension. On the other hand, in the case of theories associated to split normed algebras, like e.g. maximal supergravity, we discover an intriguing connection with the 1-form potential representations of the electric-magnetic (U-)duality group.

The plan of the paper is as follows.

In Sec. 2 we introduce the general setup, and define the Iwasawa nilpotency degree in a given representation of a Lie group. Then, in Secs. 2.1 and 2.2 the nilpotency degree is computed for the maximally non compact (split) form and for any other real form, respectively. Some examples are considered in Sec. 2.2.1. The degree of the polynomials occurring in the biinvariant metric of the coset is then computed in Sec. 3.

Sec. 4 presents three Tables summarizing the reasonings and the main results of this investigation. As an application, in Sec. 5 we consider the construction of the coset representative of non compact Riemannian symmetric manifolds, namely of symmetric scalar manifolds occurring in Maxwell-Einstein theories of gravity in various Lorentzian space-dimensions, possibly endowed with local supersymmetry. Our results are presented in a number of Tables. As respectively analyzed in Secs. 5.1 and 5.2, for magic supergravity theories our results are consistent with the Tits-Satake projection of symmetric spaces, whereas for theories related to split normed algebras (such as maximal supergravity), we find intriguing connections with the 1-form potential representations of the electric-magnetic (U-)duality group. In Sec. 5.3 we also comment on the relation between our results and the axionic U-duality generators related to five dimensions, exhibiting the universal degree of nilpotency 4.

Five Appendices conclude the paper. In Apps. A and B some basic facts on the Racah-Casimir polynomials of a Lie algebra $\mathfrak{g}$ and on the semispin groups are recalled. Then, in Apps. C, D and E...
the inverse Cartan matrices, the Dynkin diagrams and the Satake-type vectors of simple Lie algebras are reported.

2. **Iwasawa Nilpotency Degrees in a Given Representation**

Let us consider a simple Lie group $G$ of rank $l$. The corresponding root lattice $\Lambda_R$ is generated by $l$ simple roots $\alpha_1, \ldots, \alpha_l$, and it is a sub-lattice of the weight lattice $\Lambda_W$, which is the integer lattice generated by the fundamental weights $\mu^1, \ldots, \mu^l$. The simple roots define a real space $\mathbb{R}^l = \Lambda_R \otimes \mathbb{R}$, naturally endowed with a positive definite scalar product $(,)\), inherited from the Killing form of the Lie algebra $\mathfrak{g}$ of $G$. The fundamental weights are univocally related to the simple roots by

$$\langle \mu^i | \alpha_j \rangle := 2 \frac{\langle \mu^i | \alpha_j \rangle}{\langle \alpha_j | \alpha_j \rangle} = \delta^i_j. \quad (2.1)$$

The set of dominant weights is defined as the intersection between the weight lattice and the closure of the convex cone generated by the fundamental weights

$$\Lambda_W^+ := \{ m_1\mu^1 + \ldots + m_l\mu^l \mid m_i \in \mathbb{N} \cup \{0\} \}. \quad (2.2)$$

The dominant weights are in one-to-one correspondence with the irreducible representations (irreps.) of $\mathfrak{g}$, since each dominant weight is the maximal weight of an (unique up to isomorphisms) irrep., and all irreps. are of this kind. In particular, all irreps. can be obtained by $G$ -covariantly branching the tensor products of the representations associated to the fundamental weights (thus explaining the name fundamental). The representations associated to the fundamental weights $\mu_i$ are called fundamental representations $V(\mu_i)$ $(i = 1, \ldots, l)$; they are reported in App. D for all simple Lie groups.

Now, let us consider an irrep. $V(\mu_M)$ of $\mathfrak{g}$ associated to any maximal weight $\mu_M = m_1\mu^1 + \ldots + m_l\mu^l$. In such a representation, one can fix a basis of matrices $h_i$ for the (image under the representation map of a) Cartan subalgebra of $\mathfrak{g}$. To any simple root $\alpha_j$ $(j = 1, \ldots, l)$ it corresponds a matrix $\lambda_{\alpha_j} \in \text{End}(V(\mu_M))$, the so-called root matrix, which behaves as the raising operator by $\alpha_j$: if $v$ is a weight vector with weight $\mu$, then either $\lambda_{\alpha_j} v$ is zero or it is a weight vector with weight $\mu + \alpha_j$.

It is clear from this, that each root matrix is nilpotent of some order, and that, in particular, the linear span $\mathfrak{n}$ of positive root matrices (that are root matrices associated to positive roots) is made of nilpotent elements.

We are interested in determining the degree of nilpotency of the generic element in $\mathfrak{n}$ or in a suitable proper subspace, related to certain Iwasawa symmetric constructions, in any given representation. Let us first discuss the case of the symmetric spaces associated to the split real forms.

2.1. **Iwasawa Polynomials for the Split Real Form.** The maximally non compact (split) form of a simple Lie group $G$ (of rank $l$) is the unique real form having $l$ non compact Cartan generators, spanning the Cartan subalgebra $\mathfrak{c}$. Its maximal compact subgroup (mcs) $H$ has dimension $h = (\dim(G) - l)/2$, and it is the smallest maximal subgroup symmetrically embedded in $G$.

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1In general, $\Lambda_R$ is a proper sub-lattice of $\Lambda_W$, satisfying $\Lambda_W/\Lambda_R \simeq \mathbb{Z}$, where $\mathbb{Z}$ is the center of the covering group of $G$. For center-free groups, such as $G_2$, $F_4$, $E_6$, the two lattices $\Lambda_R$ and $\Lambda_W$ do coincide.

2Often, in the physical literature the name “fundamental representation” is reserved to the non-trivial, smallest (fundamental) irrep. For example, for the $\mathfrak{su}(N)$ algebras, all fundamental representations, and then all representations, can be obtained from the external powers of the smallest one: $V(\mu) = \wedge^k V(\mu^1)$. This can justify the physical notation. However this is not true for all cases. For example, for $\mathfrak{so}(2m - 1)$ all but one fundamental irreps. can be obtained from the smallest one, and all but two can be obtained for $\mathfrak{so}(2m)$. Indeed, the spinor representations must be constructed in an independent way, and in this sense are not “less fundamental” than the smallest one.
Let us consider the Iwasawa construction \[ \text{IRGS} \] of the non compact, irreducible, Riemannian, (globally) symmetric space\(^3\)
\[
M := \frac{G}{H},
\]
which has rank \( r = l \), and (real) dimension \((\dim(G) + l)/2\). It is here worth remarking that, even though in the split cases the rank \( r \) of the quotient is the same as the rank \( l \) of the group, we will use \( r \) in place of \( l \), wherever the rank of the manifold is relevant. This is in order to avoid confusion, and it will make the generalization beyond the split case clearer.

Let us work with any irreducible representation \( V(\mu_M) \). Let \( c_1, \ldots, c_r \) be a basis of non compact generators for the Cartan subalgebra \( \mathfrak{c} \), and let us fix any choice of positive roots with respect to \( \mathfrak{c} \). These are exactly \( h \), denoted by \( \alpha_a, a = 1, \ldots, h \), with \( \lambda_{\alpha_a} \) being the corresponding eigenmatrices (named root matrices in the treatment above). Now, the coset representative \( M \) of the symmetric space \( M(2.3) \) can be parameterized in terms of the parameters \( y_1, \ldots, y_r \) (coordinates of the maximal non-compact Cartan subalgebra \( \mathfrak{c} \)) and \( x_1, \ldots, x_h \) (coordinates pertaining to the \( h \) nilpotent eigenmatrices \( \lambda_{\alpha_a} \)'s) as follows\(^4\):
\[
M[y, \vec{x}] = \exp \left( \sum_{i=1}^{r} y^i c_i \right) \exp \left( \sum_{a=1}^{h} x^a \lambda_{\alpha_a} \right) = \left( \prod_{i=1}^{r} \exp \left( y^i c_i \right) \right) \exp \left( \sum_{a=1}^{h} x^a \lambda_{\alpha_a} \right).
\]
The generators \( c_i \) can be chosen so that \( \exp(\text{\(iy^i c_j\)}) \) is periodic. Note that the number \( I = h \) of Iwasawa (nilpotent) generators of the NISS \( M(2.3) \) generally satisfies \( \dim_{\mathbb{R}}(M) = r + I \) (cfr. Table 2), but in this very case also satisfies \( I = (\dim_{\mathbb{R}}(G) - l)/2 \). The first factor on the r.h.s. of \( (2.4) \) is the product of \( r \) exponentials of the \( y_i \)-coordinates, whereas, by virtue of the nilpotency properties of \( x^a \)'s, of which we want to determine the degree. In order to achieve this, let us consider the Taylor expansion:
\[
\exp \left( \sum_{a=1}^{h} x^a \lambda_{\alpha_a} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{a=1}^{h} x^a \lambda_{\alpha_a} \right)^n,
\]
in which the terms of degree \( n \) are given by
\[
M_n := \frac{1}{n!} \sum_{d_1 + \ldots + d_h = n, d_i \geq 0} \frac{(x^1)^{d_1}}{d_1!} \ldots \frac{(x^h)^{d_h}}{d_h!} \sum_{\sigma} \lambda_{\sigma_1} \ldots \lambda_{\sigma_n},
\]
where \( \sigma \) are the permutations of \( \{1 \times d_1, \ldots, h \times d_h\} \) \( \in \mathbb{N}^n \).

Our problem is to determine the largest \( n \) such that \( M_n \neq 0 \), whereas \( M_{n+1} = 0 \). The answer for the case of the adjoint representation can be found in \[33\], proposition 5.4:

Let \( \alpha_1, \ldots, \alpha_l \) be the simple roots. All other positive roots are expressed in terms of these as linear combinations with non-negative integer coefficients. In particular, there exists a longest root, whose coefficient are all maximal:
\[
\alpha_L = n_1 \alpha_1 + \ldots + n_l \alpha_l.
\]
Set
\[
q := n_1 + \ldots + n_l.
\]
Then, the proposition of Kostant states that the maximal nilpotency of the elements of \( \mathfrak{n} \) is \( 2q + 1 \), or equivalently, the maximal polynomial degree is \( 2q \). In particular, this is reached if all the coefficients of the simple roots are non vanishing.

\(^3\)For a review on irreducible, Riemannian, globally symmetric (IRGS) spaces in supergravity, see e.g. \[39\].

\(^4\)This (total) Iwasawa parametrization is usually named "standard" parametrization; it is used e.g. in \[10\].
Indeed, for all simple groups, $q$ is given by
\begin{equation}
q = j_l = C_G - 1,
\end{equation}
where $C_G$ is the Coxeter number of $G$, and $j_l$ is the maximal spin of the $\mathfrak{su}(2)_l$-irreps. into which the adjoint irrep. $\text{Adj}$ of $G$ branches under the maximal embedding
\begin{equation}
\text{Adj}(G) = \sum_{A=1}^{l} (2j_A + 1) = \sum_{A=1}^{l} S_{j_A}, \quad j_1 < \ldots < j_l.
\end{equation}

$S_{j_A}$ denotes the $\mathfrak{su}(2)_l$-irrep. of spin $j_A$ of the unique principal $SU(2)_l$ embedded in $G$ (cfr. App. [A]).

Let us now show how the Kostant proposition works and how it can be generalized to any other representation. A generic element $x \in \mathfrak{n}$ will take the form
\[ x = x^1 \lambda_1 + \ldots + x^h \lambda_h, \]
where $\lambda_i$, $i = 1, \ldots, h$ are all the positive root root matrices. With generic we mean that the coefficients corresponding to the simple roots are all non vanishing. Our aim is to compute the maximal power
\[ x^d \neq 0 \text{ but } x^{d+1} = 0. \]

Now, among all the weights there exists also a lowest weight $\mu_m$, with the property that $\mu_m - \alpha_i$ is not a weight if $\alpha_i$ is a positive root, and the corresponding eigenspace is one dimensional. Let $v_m$ be the corresponding eigenvector in the fixed basis. It follows immediately from the properties of the weight system that if $x^k v_m = 0$ then $x^{k+1} v = 0$ for any other weight vector $v$. Thus we can work only on $v_m$ to check nilpotency.

Now, the action of $x$ on $v_m$ is a combination of shifting of $v_m$ by positive roots (in the sense discussed above). So, we can look at the repeated multiplication by $x$ as a combination of paths in the set of weights, starting by $v_m$ and adding a positive root at each step (in all different possible ways). A path will stop when the addition of a further positive root will bring it out of the set of weights. This is equivalent to saying that all the paths will stop at the highest weight. Note that at a given step $x^k$, it may happen that one or more paths, but not all, stop. We call $k$ the length of the path. The corresponding term will be given by $v_M$ with coefficient a monomial of degree $k$ in the $x^i$. At the next step the corresponding term will be sent to zero, but $x^{k+1}$ will not vanish. It follows that the degree $d$ we are looking for corresponds to the step $d$ for which all the surviving paths (eventually only one) reach the highest weight. Thus, $d$ is the maximal possible length of a path from the lowest weight to the highest one.

Since a positive root is a combination of simple roots with integer non negative coefficients, any $^5v_M$ is the eigenvector in the basis corresponding to the maximal weight $\mu_M$. 

maximal path can be obtained by using only simple roots to realize each step. Thus, we immediately arrive to the following conclusion: setting

\[ \mu_M - \mu_m = \sum_{i=1}^{l} q_i \alpha_i, \]

where \( \alpha_i, i = 1, \ldots, l \) are the simple roots, and \( q_i \) are positive integer coefficients, the length of a maximal path is

\[ d = \sum_{i=1}^{l} q_i. \]

2.1.1. Computation of the Degree \( d \). We now state the following

**Proposition 1.** Let \( \Lambda \) a finite dimensional complex simple Lie algebra of rank \( l \), and \( \alpha_i, i = 1, \ldots, l \) a choice of simple roots. Let \( (\rho, V) \) the irreducible representation of \( \Lambda \) associated to the highest weight \( \mu_M \). Set

\[ \mu_M = \sum_{i=1}^{l} r_i \alpha_i. \]

Then

\[ s = \sum_{i=1}^{l} r_i \in \mathbb{Z}/2. \]

Moreover, assume \( x = \sum_{i=1}^{l} x^i \alpha_i \) to be generic. Then \( \exp x \) is a polynomial of degree \( d = 2s \) in the \( x^i \).

**Proof.** Let \( \mathfrak{c} \) the Cartan subalgebra. Since the simple roots form a basis of \( \mathfrak{c}^* \), the decomposition \( \mu_M = \sum_{i=1}^{l} r_i \alpha_i \) is always possible and unique, with \( r_i \) non negative rational coefficients. Let \( \mu_m \in \mathfrak{c}^* \) the minimal weight. We know that there exist positive integer coefficients \( q_i \) such that \( \mu_M - \mu_m = \sum_{i=1}^{l} q_i \alpha_i \).

Now, from the definition of \( \mu_m \), there exist non negative coefficients \( \tilde{r}_i, i = 1, \ldots, l \) such that

\[ \mu_m = -\sum_{i=1}^{l} \tilde{r}_i \alpha_i. \]

Inserting this into \( \mu_M = \sum_{i=1}^{l} q_i \alpha_i \), we see that \( r_i + \tilde{r}_i = q_i \) is an integer and

\[ d = \sum_{i=1}^{l} (r_i + \tilde{r}_i). \]

Since \( \{-\alpha_1, \ldots, -\alpha_l\} \) is a good candidate for a fundamental root system, we know that there exists an element \( w \) in the Weyl reflection group and a permutation \( \sigma \in \mathcal{P} \) such that \( -\alpha_i = w(\alpha_{\sigma(i)}) \). In particular \( \mu_m = w(\mu_M) \). Indeed \( w(\mu_M) \) is a weight. Assuming it is not the minimal weight, it would exist at least a simple root \( \alpha_i \) such that \( w(\mu_M) - \alpha_i = w(\mu_M + \alpha_{\sigma(i)}) \) is also a weight. Then, \( \mu_M + \alpha_{\sigma(i)} \) would be also a weight, thus contradicting the maximality of \( \mu_M \).

Applying this to \( \mu_m = -\sum_{i=1}^{l} \tilde{r}_i \alpha_i \) we get \( \tilde{r}_i = r_{\sigma(i)} \). Then

\[ \sum_{i=1}^{l} \tilde{r}_i = \sum_{i=1}^{l} r_{\sigma(i)} = \sum_{i=1}^{l} r_i. \]
which from (2.17) implies

\[ d = 2 \sum_{i=1}^{l} r_i. \]  

Note that if \( x \) was not generic, then some simple root would not enter in constructing the path and the degree of the polynomial would turn out to be lower. We are going to discuss this point in the next subsection. Note also that we have assumed for the algebra to be complex. However the proposition holds true also for the split form. Indeed, in the split case all the root matrices are in the real algebra and we do not need to complexify in order to proceed with the proof. Using the proposition and the results in [34], we see that \( s \) is the spin of the \( SL(2)_P \) principal subgroup of \( \Lambda \) in the given representation. In order to compute \( d \) for the given representation, it is sufficient to determine the degrees \( d^i = 2s^i \) associated to the fundamental weights \( \mu^i \). From the proof of the proposition it is clear that if \( \mu_M = \sum_{i} m_i \mu^i \) for non negative integers \( m_i \), then

\[ s = \sum_{i=1}^{l} m_i s^i. \]

Note that the degree \( d \) coincides with the level of \( \mu_M \), which is the number of steps necessary to reach \( \mu_M \) by acting on \( \mu_M \) with the negative of the simple roots. These have been computed for all fundamental representations and simple groups in Table 10 of [41]. It is however instructive and useful for the next application to see how they can be obtained.

Set

\[ \mu^i = \sum_{j=1}^{l} r^{ij} \alpha_j. \]  

By definition of fundamental weights

\[ \delta_k^i = \langle \mu^i | \alpha_k \rangle = \sum_{j=1}^{l} r^{ij} \langle \alpha_j | \alpha_k \rangle = \sum_{j=1}^{l} r^{ij} C_{jk}, \]  

where \( C = \{ C_{jk} \} \) is the Cartan matrix. Thus, we see that

\[ r^{ij} = C^{ij} \equiv (C^{-1})^{ij}, \]  

are given by the inverse of the Cartan matrix.

The principal spin associated to the representation \( V(\mu^i) \) is then

\[ s^i = \sum_{j=1}^{l} C^{ij}. \]  

A convenient way to express this result is as follows. Let \( \bar{s}_\Lambda \) be the column vector whose entries are the principal spins \( s^i \) of the fundamental representations of \( \Lambda \), and let \( \bar{\varepsilon} \) the column vector in \( \mathbb{R}^l \) with all entries equal to 1. Then

\[ \bar{s}_\Lambda = C_\Lambda^{-1} \bar{\varepsilon}, \]  

where we have specified the dependence of the Cartan matrix from \( \Lambda \). \( 2\bar{s}_\Lambda = R^\dagger_\Lambda \) is the transpose of the level vectors \( R_\Lambda \) given in Table 10 of [41].

\[ ^6\text{More precisely } s \text{ is the highest spin representation in the direct decomposition of } (\rho, V) \text{ under } SL(2)_P. \]
2.2. Iwasawa Polynomials for the other non compact, Real Forms. The construction we have developed up to now is valid for the Iwasawa decomposition associated to the split real form. For a generic real form not all the simple roots enter in the Iwasawa parametrization, so that the expected degree for the Iwasawa polynomials is lower than the one previously computed. We thus need to adapt our analysis to such general cases.

Thus, the roots can be separated into the \( \alpha \) roots of the simple roots, \(\alpha_i\), non-split, real (non compact) form of \( G \), where \( \alpha_i \) is in the complexification of \( g \), and \( h \) is the orthogonal complement (in the Lie algebra \( g \) of \( G \)) of the maximal compact subalgebra \( h \) of \( H = \text{mcs}(G) \); in other words, \( p \) denotes nothing but the generators of the coset \( G/H \) in the symmetric Cartan decomposition

\[
g = h \oplus p.
\]

Note that generally \( C_h \) is not the maximal Cartan subalgebra of \( h \).

Let us fix a basis \( k_1, \ldots, k_s \) for \( C_h \), and \( h_1, \ldots, h_r \) for \( C_p \) (\( s + r = l \)). Thus, the Cartan decomposition \((2.27)\) can be further refined as follows:

\[
g = h \oplus (C_p \oplus \tilde{p}),
\]

where \( \tilde{p} \) is the complement of \( C_p \) in \( p \). It is here worth recalling that, as the Cartan decomposition \((2.27)\) pertains to the maximal and symmetric embedding of \( h \) into \( g \), it holds that

\[
[h, h] \subseteq h, \quad [p, p] \subseteq h, \quad [h, p] \subseteq p.
\]

Let \( \mathfrak{t} \) be the Lie algebra of the normalizer \( K \) of \( C_p \), i.e. the largest Lie subalgebra of \( h \) such that \([\mathfrak{t}, C_p] = 0\). Then \((2.28)\) and \((2.29)\) imply that

\[
[\mathfrak{t}, C_p] = 0 \Rightarrow [\tilde{h}, C_p] \subseteq \tilde{p}, \quad [\tilde{p}, C_p] \subseteq \tilde{h},
\]

where \( \tilde{h} \) is the complement of \( \mathfrak{t} \) in \( h \).

It is here worth pointing out a convenient characterization of the roots of the simple, non compact non split, real Lie algebra \( g \). Since \( C_h \) is contained in \( h \) and commutes with \( C_p \), it is contained in \( \mathfrak{t} \) and is necessarily a Cartan subalgebra for it. Then, it follows that

\[
\text{rank}(\mathfrak{t}) = \text{dim}(C_h) = s.
\]

One can then represent all roots of \( g \) as the simultaneous eigenvalues of the operators \((a k_1, \ldots, a k_s; a h_1, \ldots, a h_r)\). By defining \( k := \text{dim} K \), the eigenvectors of the roots \( \alpha_{h,a} \) \((a = 1, \ldots, k-s)\) of \( \mathfrak{t} \) are in the complexification of \( \mathfrak{t} \) and thus in the kernel of \( a h_i \) \((i = 1, \ldots, r)\) : as a consequence, the last \( r \) components of the string of simultaneous eigenvalues of \((a k_1, \ldots, a k_s; a h_1, \ldots, a h_r)\) are zero. Actually, the roots \( \alpha_{h,a} \) are all the non-vanishing roots of \( g \) with this property: all the other ones are characterized by \((at least some)\) non-vanishing components among the last \( r \) ones in the string of simultaneous eigenvalues of \((a k_1, \ldots, a k_s; a h_1, \ldots, a h_r)\). We will call the corresponding roots \( \alpha_{p,b} \) \((b = 1, \ldots, 2(h - k))\). Indeed, these correspond to the non-vanishing roots of the \( h_i \), whose number, by \((2.30)\), is \( h - k \).

Thus, the roots can be separated into the \( \alpha_p \) and \( \alpha_h \) roots. In realizing the quotient only the first
ones survive, the last ones being projected to zero roots. As usual, one can divide the roots $\text{Rad}$ in positive ones $\text{Rad}^+$ and negative ones $\text{Rad}^-$, namely:

\[(2.32) \quad \text{Rad} = \text{Rad}^+ \oplus \text{Rad}^-.
\]

Correspondingly, this determines a decomposition of the root system associated to the Lie algebra $\mathfrak{p}$ of the symmetric coset $G/H$:

\[(2.33) \quad \text{Rad}_\mathfrak{p} = \text{Rad}_\mathfrak{p}^+ \oplus \text{Rad}_\mathfrak{p}^-.
\]

While in the split case $\text{Rad}_\mathfrak{p}$ is a reduced lattice system, in the non-split case this is generally not true anymore, and generically each root $\alpha$ is characterized by a multiplicity $m_\alpha \geq 1$; moreover, given a root $\alpha$, it can happen that $2\alpha$ or $\frac{1}{2}\alpha$ is also a root.

All the root systems $\Lambda_{G/H}$ associated to irreducible symmetric spaces $G/H$ have been classified by Araki [42]. The split forms are exactly the ones which satisfy $\Lambda_{G/H} = \text{R}_G$, where $\text{R}_G$ is the root system of $G$ itself. For a generic non compact real form of $G$, $\Lambda_{G/H}$ is not reduced, but it contains a maximal subsystem of roots (its reduction), which correspond to a root system of simple kind (but in general with non-trivial multiplicities). The complement of the reduction of a non-reduced $\Lambda_{G/H}$ possibly consists of further roots which are twice the reduced ones (they are dubbed double roots).

Now we can extend our results to the given real form. In this case the nilpotent elements are of the form $x = \sum_i x^i \lambda_i$, where the sum is extended over all root matrices corresponding to the positive roots of the lattice associated to the quotient symmetric manifold. Thus, not all the simple roots matrices will participate to the Iwasawa construction, but only the ones corresponding to roots of $\text{Lie}(G)$ that are not sent to zero in the projection on the quotient. As a consequence, in order to compute the degree of the corresponding Iwasawa polynomials, we have to project (2.12) on the quotient.\footnote{Note that a maximal weight is again maximal in the non reduced lattice after projection, but it is in general not unique.} If we call $\pi$ the projection, we get

\[(2.34) \quad \pi(\mu_M) - \pi(\mu_m) = \sum_{i=1}^r q_{ji} \pi(\alpha_{ji}),
\]

where $\alpha_{ji}$ are the roots that are not in the kernel of $\pi$. We can then proceed exactly as in the proof of the previous proposition to get for the degree of the Iwasawa polynomial

\[(2.35) \quad d = 2 \sum_{i=1}^r r_{ji}.
\]

Again, it is sufficient to compute only the spins associated to the fundamental representations, which as a consequence are

\[(2.36) \quad d' = \sum_{k=1}^r C^{ik} r_{jk}.
\]

Now, it is easy to determine which roots are projected into zero and which not. Indeed, to each real form, or, equivalently, to each non compact Irreducible Symmetric Space (NISS), it is associated a Satake diagram, indicating as black dots the simple roots in the kernel of $\pi$. A complete list of Satake diagrams can be found in [42]. Let us call $x$ of generic type in the Iwasawa nilpotent in the representation $(\rho, V)$ if

\[(2.37) \quad x = \sum x^i \lambda_i,
\]
where the sum is extended to all positive quotient roots, and the coefficients corresponding to simple roots are all non-vanishing. Thus, we have proved the following proposition:

**Proposition 2.** Let $\Lambda$ be a complex simple Lie algebra of rank $l$ and $\Lambda^T$ be its real form corresponding to a given symmetric space of type $T$ and rank $r$ (as listed in [43]). Consider the associated Satake diagram and let $\bar{e}^T$ the column vector in $\mathbb{R}^l$ with entry 1 if corresponding to a white dot and zero otherwise (Satake vector), and let $\{\bar{e}_i\}_{i=1}^l$ the canonical basis of $\mathbb{R}^l$. Let $x$ be of generic type in the corresponding Iwasawa nilpotent in the representation $V(\mu^i)$, $(i = 1, \ldots, l)$. Then

$$(2.38) \quad P(x) = \exp(x)$$

is a polynomial of degree $d = 2s_{\Lambda^T}$, where

$$(2.39) \quad d = 2s = \bar{e}_i \cdot C^{-1} \bar{e}^T.$$

It should be pointed out that the same construction can be extended to reducible representations and to semi-simple groups. In these cases the maximal polynomial degree will be determined by the maximal spin among all sub representations or among the simple factors, respectively.

A list of the inverse Cartan matrices can be found in appendix C, whereas all the Satake vectors are listed in appendix E.

2.2.1. Some Examples. In order to illustrate how the degree of nilpotency depends on the $G$-representation of the coset representative $M$, we consider the simplest case provided by the rank-1 symmetric space

$$(2.40) \quad \frac{G}{H} = \frac{SL(2, \mathbb{R})}{SO(2)},$$

which corresponds to setting $n = 1$ in the first row of Tables 1 and 3. The result $d = 1$ pertains to the coset representative in the fundamental irrep. 2 (spin $s = 1/2$) of $G_{nc} = SL(2, \mathbb{R})$, with a $2 \times 2$ coset representative $M$; this case is relevant both to four-dimensional $N = 4$ “pure” supergravity as well as to $N = 2$ supergravity minimally coupled to one Abelian vector multiplet (this latter theory can indeed be obtained from the former by a consistent truncation of graviphotons and gravitinos), and it has been recently reconsidered in Sec. 5 of [44].

On the other hand, the coset (2.40) can also be considered as the scalar manifold of the so-called $T^3$ model of $N = 2$, $D = 4$ Maxwell-Einstein supergravity, in which the unique Abelian vector multiplet is coupled non-minimally, but rather through a cubic holomorphic prepotential $F = T^3$, to the gravity multiplet. In this case, the relevant irrep. of $G_{nc} = SL(2, \mathbb{R})$ is the 4 (spin $s = 3/2$), and thus the coset representative $M$ is a $4 \times 4$ matrix. Since the irrep. 4 has weight $3\lambda_1$, from the reasoning above one obtains that the corresponding maximal degree is three times the fundamental one, and thus

$$(2.41) \quad d = 3$$

in 4 of $SL(2, \mathbb{R}) : d = 3$.

The result (2.41) on the degree of nilpotency $d + 1 = 4$ of the unique Iwasawa generator is consistent with the treatment recently given in [44]; indeed, for the $T^3$ model, which uplifts to five-dimensional minimal “pure” supergravity, the (partial) axionic Iwasawa construction exploited in [44] (in particular, cfr. Sec. 2 and App. B of [44]) is actually a total Iwasawa construction.

For any $sl(2, \mathbb{R})$-irrep. of weight $m\lambda_1$ (and thus spin $s = m/2$), the corresponding maximal Iwasawa degree concerning the IRGS (2.40) reads $d_S = 2s = m$, and thus the corresponding Iwasawa nilpotency degree is nothing but $m + 1$.

The negative constant scalar curvature of (2.40) for the $s = 1/2$ and $s = 3/2$ cases respectively is $R = -2$ and $R = -2/3$ (corresponding to $m = 1$ and $m = 3$), and they are the unique values for which this symmetric Kähler coset is a special manifold [5].
Considering IRGS relevant as scalar manifolds in supergravity theories (see e.g. [39] for a comprehensive review), one should consider only a few other non-fundamental irreps., such as the rank-3 antisymmetric skew-traceless $14'$ of $Sp(6,\mathbb{R})$.

Regarding the coset manifold $\frac{SL(2,\mathbb{R})}{SO(2)}$, another remark is in order. As it is discussed in the next Section 3, a further quantity that is interesting is the invariant metric tensor. Then it can be seen immediately that for $\frac{SU(1,1)}{U(1)}$ the degree $d_g$ of the metric vanishes: $d_g = 0$. This is a consequence of the fact that in this case the metric does not depend on the axions and hence on the nilpotent part of the Iwasawa decomposition. In other words, the metric of the upper half plane depends only on the imaginary part $\text{Im}(z)$ of the complex coordinate $z = \text{Re}(z) + i \text{Im}(z)$. For these results, compare the values $d_g = 0$ in the first row for $n = 1$ and in the third row of Table 3.

3. The Degree $d_g$ of Metric Polynomials

There is a second problem we are interested in. Let us consider a NISS $M$. It can be realized in a given representation $(\rho, V)$ by means of the Iwasawa parametrization, which, following our previous notations, can be formally written as

\begin{equation}
M = \exp(y) \exp(x),
\end{equation}

where $y = \sum_{i=1}^{r} y^i c_i$, where $c_i$, $i = 1, \ldots, r$ is a basis of $C_p$, and $x = \sum_{i} x^i \lambda_i$ is of generic type in the Iwasawa nilpotent $n$ in the representation $(\rho, V)$, apart from positive codimension submanifolds. Note that $C_p \oplus n$ is a (non simple) subalgebra so that the Lie algebra valued right-invariant one form

\begin{equation}
J_M := dM M^{-1}
\end{equation}

again lies in $C_p \oplus n$. From this, one obtains the invariant metric tensor $g_{ij}$ on $M$ in the standard way. In particular, $J_M$ will be polynomial in $x$ of a given degree $d$, so that the metric will be polynomial of degree $2d$. The aim of this section is to determine $d$.

To this end, let us first note that,

\begin{equation}
J_M = d \exp(y) \exp(-y) + Ad_{\exp(y)}(d \exp(x) \exp(-x)).
\end{equation}

Since $\tilde{J}(x) := d \exp(x) \exp(-x)$ lies in $n$, on which $Ad_{\exp(y)}$ acts diagonally, the polynomial part comes only from $\tilde{J}(x)$. Now

\begin{equation}
\tilde{J}(x) = \sum_{i} dx^i \int_{0}^{1} Ad_{\exp(vx)}(\lambda_i) dv,
\end{equation}

from which we can easily read the degree of $\tilde{J}(x)$. Indeed, for a fixed $i$, $Ad_{\exp(vx)} = \exp(vad_x)$ acts on $\lambda_i$ in the same way discussed in the previous section: $\lambda_i$ is the root matrix corresponding to the positive root $\alpha_i$; the action of $ad_x$ changes $\lambda_i$ to a combination of root matrices corresponding to roots of the form $\alpha_i + \alpha_j$, with $\alpha_j$ positive, and so on until reaching the highest root. Thus we can follow exactly the same reasoning as in the previous section: the main point is that it does not matter what representation we are starting from, as it is always the adjoint representation which is acting in this case. What is relevant is only the type of the NISS. For example, if we work with split form, then we have to work with the highest root, expressed in terms of the simple roots as

\begin{equation}
\alpha_H = \sum_{i=1}^{l} n^i \alpha_i.
\end{equation}

Since we construct the paths by starting from a positive root $\lambda_i$, the longest paths will be obtained by adding simple roots to a starting simple root. The maximal possible length of a path will then be $d = \sum_{i=1}^{l} n^i - 1 = C_G - 2$. 
If we work with a generic real form, then, as before, we will need to sum only over the coefficients \( n_{j_i} \) corresponding to the roots that are not in the kernel of the projector \( \pi \): \( d = \sum_{i=1}^{r} n_{j_i} \). We thus arrive at the following result:

**Proposition 3.** Let \( M = G/H \) a NISS of type \( T \) and \( g \) the corresponding positive definite biinvariant metric induced by the Killing form. Let \( x \) the coordinates in the nilpotent part of the standard Iwasawa parametrization of \( M \). Then \( g \) is a polynomial function of \( x \) of degree

\[
d_g = 2(s_{ad} - 1),
\]

where \( s_{ad} \) is the spin of the principal \( SL(2) \) in the adjoint representation. If \( \bar{n} = (n_1, \ldots, n_l) \) are the coordinates of the highest roots w.r.t. the simple roots, then

\[
d_g = 2(\bar{n} \cdot \bar{\varepsilon}^T - 1).
\]

Let us notice that, again, this result is easily generalized to the case of a reducible representation or to semisimple groups: the degree is the maximum degree among the irreducible or simple factors, respectively.

4. **Summary of Results**

The results of the above reasonings are collected in Tables 1, 2 and 3.

In the first column of Table 1, consistent with the notation e.g. in [42, 43], we indicate the label for the non compact, real form \( G_{nc} \) of \( G \), which in turn is given in the second column. \( H = mcs(G_{nc}) \) is given, along with possible discrete factors, in the third column. Then, following the classification of [42], the fourth column specifies the type of the root systems \( \Lambda_{G/H} \) associated to irreducible symmetric space \( G/H \). After [43], the fifth column provides the coefficients \( (n_1, \ldots, n_r) \) of the longest root \( \alpha_L \) (2.7). In the sixth column, the multiplicities for the roots and the double roots are given [42]: these are just a number for simply-laced root systems; on the other hand, in non-simply-laced systems they are given as a pair, in which the first and second entry give respectively the multiplicity of the long and short root.

In Table 2 we report the rank \( r \) of the NISS \( G_{nc}/mcs(G_{nc}) \), along with the number \( I \) of the corresponding nilpotent (i.e., Iwasawa) generators; the resulting (real) dimension dim of the NISS is nothing but \( I + r \), where \( r \) is the rank of \( G_{nc} \); finally, in the fifth column the character [38] \( \chi := nc - c \) of the coset is reported, where “nc” and “c” denote respectively the number of non compact resp. compact generators of the NISS itself.

In Table 3 we report the degrees of the Iwasawa polynomials associated to each NISS. In the first column we report \( G_{nc} \), in the second column the rank \( l \) of \( G_{nc} \), in column three we give the vector of degrees of Iwasawa polynomials associated to the fundamental representations, and in the last column the degree \( d_g \) of the polynomial part in the biinvariant metric \( g \).

Note that from the fundamental degrees reported in the second column of Table 3, the nilpotency degree of the Iwasawa polynomials can be obtained by simply adding one to each entry.
| T   | $G_{nc}$ | $H$            | $\lambda_{G/H}$ | $(n_1, \ldots, n_r)$ | $\tilde{m}_\lambda$, $\tilde{m}_{2\lambda}$ |
|-----|----------|----------------|------------------|-----------------------|-----------------------------------|
| AI  | $SL(n+1, \mathbb{R})$ | $SO(n+1)$      | $A_n (n \geq 1)$ | $(1, \ldots, 1)$      | $(1)$, $(0)$                      |
| AI  | $SU^*(2k)$ | $USp(2k)$      | $A_{k-1} (k > 1)$ | $(1, \ldots, 1)$      | $(4)$, $(0)$                      |
| AIa | $SU(p,q)$ | $S(U(p) \times U(q))$ | $B_{p}(1 < p < q)$ | $(2,2, \ldots, 2)$    | $(2(1,q-p), (0,1)$                |
| AIb | $SU(p,p)$ | $S(U(p) \times U(p))$ | $C_{p} (p > 1)$ | $(2,2, \ldots, 2,1)$ | $(1,2)$, $(0)$                    |
| AIV | $SU(1,n)$ | $S(U(1) \times U(n))$ | $A_{1}$ | $(2)$ | $(2n-2)$, $(1)$ |
| Bl  | $SO(n,n+1)$ | $SO(n) \times SO(n+1)$ | $B_{n} (n \geq 2)$ | $(1,2, \ldots, 2)$    | $(1,1), (0,0)$                    |
| Blb | $SO(p,q)$ | $SO(p) \times SO(q)$ | $B_{p}(1 < p < n)$ | $(1,2, \ldots, 2)$    | $(1,2(n-p)+1), (0,0)$             |
| BII | $SO(1,2n)$ | $SO(2n)$      | $A_{1}$ | $(1)$ | $(2n-1)$, $(0)$ |
| Cl  | $Sp(2n, \mathbb{R})$ | $U(n)$          | $C_{n} (n \geq 3)$ | $(2,2, \ldots, 2,1)$ | $(1,1), (0,0)$                    |
| CIIa| $USp(2p, 2q)$ | $USp(2p) \times USp(2q)$ | $B_{p}(1 \leq p \leq (n-1)/2)$ | $(2,2, \ldots, 2)$   | $(4n-8p)$, $(0,3)$                |
| CIIb| $USp(2k, 2k)$ | $USp(2k) \times USp(2k)$ | $C_{k}$ | $(2,2, \ldots, 2,1)$ | $(3,4), (0,0)$                    |
| DIa | $SO(r, n)$ | $SO(n) \times SO(n)$ | $D_{n} (n \geq 3)$ | $(1,2, \ldots, 2,1,1)$ | $(1)$, $(0)$                      |
| DIb | $SO(n-1, n+1)$ | $SO(n-1) \times SO(n+1)$ | $B_{n-1} (n > 2)$ | $(1,2, \ldots, 2)$    | $(1,2), (0,0)$                    |
| DIc | $SO(1,2n-1)$ | $SO(2n-1)$     | $A_{1}$ | $(1)$ | $(2n-2)$, $(0)$ |
| DIIa| $SO^*(4k+2)$ | $U(2k+1)$     | $B_{k} (k \geq 2)$ | $(2,2, \ldots, 2)$    | $(4,4), (0,1)$                    |
| DIIb| $SO^*(4k)$  | $U(2k)$        | $C_{k} (k \geq 2)$ | $(2,2, \ldots, 2,1)$ | $(1,4), (0,0)$                    |
| G   | $G_{2}(2)$ | $SO(4)/Z_2$   | $G_{2}$ | $(3,2)$ | $(1,1), (0,0)$ |
| FI  | $F_{4}(4)$ | $USp(6) \times USp(2)$ | $F_{4}$ | $(2,3,4,2)$ | $(1,1)$, $(0,0)$ |
| FII | $F_{4}(-20)$ | $SO(9)$       | $A_{1}$ | $(2)$ | $(8), (7)$ |
| EI  | $E_{6}(6)$ | $USp(8)/Z_2$  | $E_{6}$ | $(1,2,2,3,2,1)$ | $(1)$, $(0)$ |
| EIa | $E_{6}(2)$ | $(USp(2) \times SU(6))/Z_2$ | $F_{4}$ | $(2,3,4,2)$ | $(1,2), (0,0)$ |
| EIb | $E_{6}(-14)$ | $(U(1) \times SO(10))/Z_4$ | $B_{2}$ | $(2,2)$ | $(6,8), (0,1)$ |
| EIV | $E_{6}(-26)$ | $F_{4}$       | $A_{2}$ | $(1,1)$ | $(8), (0)$ |
| EV  | $E_{7}(7)$ | $SU(8)/Z_2$   | $E_{7}$ | $(2,2,3,4,3,2,1)$ | $(1)$, $(0)$ |
| EVI | $E_{7}(-5)$ | $(SU(2) \times SO(12))/Z_2$ | $F_{4}$ | $(2,3,4,2)$ | $(1,4), (0,0)$ |
| EVII| $E_{7}(-25)$ | $(U(1) \times E_{6})/Z_3$ | $C_{3}$ | $(2,2), (1)$ | $(1,8), (0,0)$ |
| EVIII| $E_{6}(8)$ | $SO(16)$ | $E_{6}$ | $(2,3,4,6,5,4,3,2)$ | $(1)$, $(0)$ |
| EIX | $E_{8}(-24)$ | $(SU(2) \times E_{7})/Z_2$ | $F_{4}$ | $(2,3,4,2)$ | $(1,8), (0,0)$ |

Table 1: In this Table, we list the main ingredients necessary to describe the Iwasawa construction of the non compact, irreducible, Riemannian, globally symmetric space $G_{nc}/H$. $S_{q}$ $(16)$ denotes the semispin group of type $D_{q}$ $(16)$ (see also appendix $B$). Note that, when listing the $nc$s $H$ in the second column, the universal coverings are considered, namely $SO(3) \equiv Spin(3) \simeq SU(2)$, $USp(4) \simeq SO(5) \equiv Spin(5)$, $USp(2) \simeq SU(2)$, $SO(6) \equiv Spin(6) \simeq SU(4)$, and $SO(4) \equiv Spin(4) \simeq SU(2) \times SU(2)$. Furthermore, local isomorphisms among non compact, real forms are used throughout (cfr. e.g. $(35)$). Notice that in $BI_{b}$, $p + q = 2n + 1$, in $CII_{a}$, $p + q = n$, and in $DI_{c}$, $p + q = 2n$. 


| $G_{nc}$         | $r$       | $I$       | $\dim(G_{nc}/\text{mcs}(G_{nc})) = I + r$ | $\chi$ |
|-----------------|-----------|-----------|------------------------------------------|--------|
| $SO(n+1, \mathbb{R})$ | $n$       | $(n+1)(n+2)/2 - n - 1$ | $(n+1)(n+2)/2 - 1$ | $n$ |
| $SU^{*}(2k)$   | $k - 1$   | $2k(k - 1)$ | $k(2k - 1) - 1$ | $-2k - 1$ |
| $SU(p,q)$      | $\min(p,q)$ | $2pq - \min(p,q)$ | $2pq$ | $-(p-q)^2 + 1$ |
| $SU(p,p)$      | $p$       | $2p^2 - p$  | $2p^2$ | $1$ |
| $SU(1,n)$      | $1$       | $2n - 1$   | $2n$ | $-n(n-2)$ |
| $SO(n,n+1)$    | $n$       | $n^2$      | $n(n+1)$ | $n$ |
| $SO(p,q)_{p+q=2n+1}$ | $\min(p,q)$ | $pq - \min(p,q)$ | $pq$ | $[(p+q) - (p-q)^2]/2$ |
| $SO(1,2n)$     | $1$       | $2n - 1$   | $2n$ | $n(3-2n)$ |
| $Sp(2n, \mathbb{R})$ | $n$       | $n^2$      | $n(n+1)$ | $n$ |
| $USp(2p,2q)$   | $\min(p,q)$ | $4pq - \min(p,q)$ | $4pq$ | $-(2p+2q) + (2p-2q)^2)/2$ |
| $USp(2k,2k)$   | $k$       | $k(4k-1)$  | $4k^2$ | $-2k$ |
| $SO(n,n)$      | $n$       | $n(n-1)$   | $n^2-1$ | $n-2$ |
| $SO(n-1,n+1)$  | $n-1$     | $n(n-1)$  | $n^2-1$ | $n-2$ |
| $SO(p,q)_{p+q=2n}$ | $\min(p,q)$ | $pq - \min(p,q)$ | $pq$ | $[(p+q) - (p-q)^2]/2$ |
| $SO(1,2n-1)$   | $1$       | $2(n-1)$  | $2n-1$ | $5n-2n^2-2$ |
| $SO^{*}(4k+2)$ | $2k+1$    | $(4k+2)(k-1/2)$ | $2k(2k+1)$ | $-2k-1$ |
| $SO^{*}(4k)$   | $2k$      | $4k(k-1)$  | $2k(2k-1)$ | $-2k$ |
| $G_{2}(2)$     | $2$       | $6$       | $8$ | $2$ |
| $F_{4}(4)$     | $4$       | $24$      | $28$ | $4$ |
| $F_{4}(-20)$   | $1$       | $15$      | $16$ | $-20$ |
| $E_{6}(6)$     | $6$       | $36$      | $42$ | $6$ |
| $E_{6}(2)$     | $4$       | $36$      | $40$ | $2$ |
| $E_{6}(-14)$   | $2$       | $30$      | $32$ | $-14$ |
| $E_{6}(-26)$   | $2$       | $24$      | $26$ | $-26$ |
| $E_{7}(7)$     | $7$       | $63$      | $70$ | $7$ |
| $E_{7}(-5)$    | $4$       | $60$      | $64$ | $-5$ |
| $E_{7}(-25)$   | $3$       | $51$      | $54$ | $-25$ |
| $E_{8}(8)$     | $8$       | $120$     | $128$ | $8$ |
| $E_{8}(-24)$   | $4$       | $108$     | $112$ | $-24$ |

Table 2: In the second column the rank $r$ of the symmetric coset $G_{nc}/\text{mcs}(G_{nc})$ is indicated, in the third column the number $I$ of Iwasawa (nilpotent) generators, in the fourth column the real dimension $\dim(G_{nc}/\text{mcs}(G_{nc}))$, and in the fifth column the character $\chi := nc - c$ of $G_{nc}$, where ”nc” and “c” denote respectively the number of non compact and of compact generators of $G_{nc}$ itself [17]. Note that for the maximally non compact (split) forms, $r = \chi = l$, where we recall that $l$ is the rank of $G$ itself; consequently, for the split forms $I$ is equal to the number of positive (or of negative) roots of $G$. Also, we notice the peculiarity of the minimally non compact real form of $E_6$, i.e. of $E_6(-26)$, for which $\chi = -\dim(G_{nc}/\text{mcs}(G_{nc}))$. 

of the corresponding polynomials. All computations are based on the inverse Cartan matrices and the nilpotency degree is given by $D_I$.

For the real forms there are no blackened nodes at all.

Table 3: In the second column the rank $l$ of the group $G$ is indicated, in the third column the vector of fundamental degrees, that are the degrees of the polynomials in the fundamental representations, accordingly.

Remark: Notice that for $SO^*(4k)$, following the convention in [42], we have assumed the even roots to be the non compact ones. However, from the symmetry of the Dynkin diagram, one sees that it is possible to obtain an isomorphic coset manifold by picking the last but one root instead of the last one as the non compact root. In this case the last two entries of the third column need to be switched accordingly.
5. Application to Supergravity

As an application of the set of general results for the degrees $d$ and $d_g$ of the Iwasawa polynomials resulting in the standard construction of the coset representative of non compact Riemannian symmetric manifolds, we consider the symmetric scalar manifolds of supergravity theories in various Lorentzian space-dimensions (for a comprehensive review, see e.g. [59]), namely $D = 3, 4, 5, 6$. We report the corresponding results in various Tables, and we address to the corresponding captions for further comments.

Summarizing, in Table 1 we consider the special Kähler vector multiplets’ scalar manifolds of $N = 2$ theories in $D = 4$, whereas in Table 5 we deal with the quaternionic Kähler scalar manifolds [58] (obtained through the so-called $c$-map [19]) of the theory dimensionally reduced (along a spacelike direction) down to $D = 3$ (after dualization) [6]. Moreover, the vector multiplets’ real special manifolds (obtained through the so-called $R$-map [50]) of the minimal theory obtained by oxidizing (i.e., uplifting) to $D = 5$ Lorentzian dimension are reported in Table 6. The scalar manifolds of $N > 2$–extended supergravity theories in $D = 4$ and $D = 5$ are considered in Tables 7 and 8 respectively, while the symmetric manifolds of $N > 4$–extended supergravity theories in $D = 3$ are given in Table 9.

Concerning $D = 6$, for brevity’s sake we will consider only minimal chiral theories. The scalar manifold of $(1, 0)$ chiral magic supergravity theories (based on $J^3_2 \sim \Gamma_{1,q+1}$, the Clifford algebra of $O(1, q + 1)$ [54]) is given by $SO(1,q+1) / SO(q+1)$ (with $q = 1, 2, 4, 8$ for $A = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively), with the relevant $G$-irrep. being the vector: $\mathbb{R}(SO(1,q+1)) = q + 2 = (1, 0, ..., 0)$ (for a discussion of $D = 6$ minimal chiral theories, see e.g. [52] and [53]). This formally matches the scalar manifold of the so-called non-Jordan symmetric sequence [54] (cfr. line 1 of Table 4), and thus $d = 2$ and $d_g = 0$, regardless of the value of $q$.

The general structures of (maximal and symmetric) group embeddings read as follows:

\[(5.1)\quad QConf \left( J^k_3 \right) \subset Conf \left( J^k_3 \right) \times SL(2, \mathbb{R}) ; \]
\[(5.2)\quad Conf \left( J^k_3 \right) \subset Str_0 \left( J^k_3 \right) \times SO(1, 1) ; \]
\[(5.3)\quad Str_0 \left( J^k_3 \right) \subset Str_0 \left( J^k_2 \right) \times SO(1, 1) , \]

where $QConf, Conf$ and $Str_0$ respectively denote the quasi-conformal, conformal and reduced structure symmetry groups of Jordan algebras (see e.g. [55]). In (5.1) $SL(2, \mathbb{R})$ is the so-called Ehlers group, related to the reduction of pure Einstein gravity to 3 dimensions, whereas in (5.2) and (5.3) $SO(1, 1)$ is the Kaluza-Klein compactification factor. Due to the properties of the Clifford algebra in certain dimensions, $Str_0 \left( J^k_2 \right)$, in some cases contains a commuting non-trivial compact factor $T ri(\mathbb{A}) / SO(\mathbb{A})$, where $T ri(\mathbb{A})$ and $SO(\mathbb{A})$ denote the triality group and norm-preserving group of the normed division algebra $\mathbb{A}$ (cfr. e.g. [56]). Note that

\[(5.4)\quad \frac{T ri(\mathbb{A})}{SO(\mathbb{A})} = \emptyset, SO(2), SO(3), \emptyset \text{ for } \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} , \]

and

\[(5.5)\quad Str_0 \left( J^k_2 \right) = SO(1, \dim_\mathbb{R} \mathbb{A} + 1) \times \frac{T ri(\mathbb{A})}{SO(\mathbb{A})} . \]

However, due to its compactness, this extra factor does not affect the structure of the (tensor multiplets’) real scalar manifold in $D = 6$.

By denoting with $G_{D,\mathbb{A}}$ the electric-magnetic $(U\text{-})$duality group of the $J^\mathbb{A}$-based magic supergravity in $D$ Lorentzian space-time dimensions, the supergravity interpretation of such symmetry groups is
the following:

\[ QConf \left( J^5_3 \right) = G_{3,\mathbb{A}}; \]  
\[ Conf \left( J^5_3 \right) = G_{4,\mathbb{A}}; \]  
\[ Str_0 \left( J^3_5 \right) = G_{5,\mathbb{A}}; \]  
\[ Str_0 \left( J^2_5 \right) = G_{6,\mathbb{A}}. \]

In the second row of Table 6, an extrapolation to \( n = 1 \), which is not reported in the Table, corresponds to the \( D = 5 \) uplift of the so-called \( ST^2 \) model of \( \mathcal{N} = 2, D = 4 \) Maxwell-Einstein supergravity. This is (the unique model) given by the dimensional reduction of “pure” \( (1,0) \) minimal chiral supergravity in \( D = 6 \). For this model the maximal degrees vanish: \( d = 0 \) and \( d_g = 0 \).

In the second row of Table 7, an extrapolation to \( n = 0 \), which is not reported in the Table, corresponds to “pure” \( \mathcal{N} = 4, D = 4 \) supergravity; the \( U \)-duality group is \( SL(2, \mathbb{R}) \times SU(4) \), and the relevant \( U \)-irrep. is \( (2,6) \). Indeed, the \( U(1) \) factor of the \( \mathcal{N} = 4 \) \( \mathcal{R} \)-symmetry \( U(4) \) is gauged by the complex (axio-dilatonic) scalar field of the gravity multiplet, whereas the \( SU(4) \) factor is not gauged, i.e. it is global. Concerning the non compact part of the \( U \)-duality, this theory is like the \( \mathcal{N} = 2 \) axio-dilatonic model (based on \( \mathbb{CP}^1 \)), describing the minimal coupling of 1 vector multiplet to the gravity multiplet \( [7] \); cfr. \( n = 1 \) in the first row of Table 4, but in the \( \mathcal{N} = 4 \) case the compact part is given by \( SO(6) \; vs. \; the \; U(1) \) factor in the \( \mathcal{N} = 2 \) model. It follows that the corresponding maximal Iwasawa degrees are \( d = 1 \) and \( d_g = 0 \).

In the first row of Table 8, an extrapolation to \( n = 0 \), which is not reported in the Table, corresponds to “pure” \( \mathcal{N} = 4, D = 5 \) supergravity: the \( U \)-duality group is \( SO(1,1) \times SO(5) \), where \( SO(5) \) is the (global) \( \mathcal{N} = 4 \) \( \mathcal{R} \)-symmetry. There is only one real scalar in the gravity multiplet, and the relevant \( U \)-rep. is \( (1,1) + (1,5) \) (cfr. e.g. \([63]\)). The maximal Iwasawa degrees are \( d = 0 \) and \( d_g = 0 \).

5.1. \( q \)-Independence of \( d \) and \( d_g \) for Magic Supergravities. From Tables 4, 5, 6 and the above observation implying \( d = 2 \) and \( d_g = 0 \) for \( D = 6 \), one can report the values of \( d \) and \( d_g \) for magic Maxwell-Einstein supergravities in \( D = 3, 4, 5, 6 \) Lorentzian space-time dimensions in Table 10.

Interestingly, for the magic Maxwell-Einstein supergravities \([4]\), the degrees \( d \) and \( d_g \) of the Iwasawa polynomial only depend on \( D \), and not on \( q \); this seems consistent with the fact that the corresponding homogeneous (symmetric) manifolds are in the same Tits-Satake universality class (see e.g. \([4]\) for a general treatment).

Note that the second, third and fourth rows of Table 10 are the same as the ones of Günaydin-Sierra-Townsend (GST) Magic Square \([4]\) \( L_3 (\mathbb{A}_s, \mathbb{B}) \) (for a comprehensive treatment of \( 4 \times 4 \) Magic Squares \( L_3 \)'s, cfr. \([65]\)). On the other hand, the relation between the first rows of Table 10 and of the GST Magic Square \( L_3 (\mathbb{A}_s, \mathbb{B}) \) can be expressed by observing that the maximal compact subgroup (mcs) of the symmetry groups occurring in the first row of Table 10 (namely, the \( D = 6 \) \( U \)-duality group \( G_{6,\mathbb{A}} = Str_0 \left( J^3_5 \right) \)) is a (maximal symmetric) subgroup of the mcs of \( Str_0 \left( J^2_5 \right) \), which is the group occurring in the first row of GST Magic Square \( L_3 (\mathbb{A}_s, \mathbb{B}) \):

\[ \text{mcs} \left( Str_0 \left( J^3_5 \right) \right) \subseteq_{\text{max}} \text{mcs} \left( Str_0 \left( J^2_5 \right) \right). \]

Such a relation can be trivially expressed by stating that the stabilizer of \( D = 6 \) (tensor multiplets') real scalar manifold is maximally (and symmetrically) embedded into the (vector multiplets') real special scalar manifold of the corresponding theory obtained by dimensional reduction (along a spacelike

---

\[ \text{Note that Table 10} \] considers all possible dimensions in which theories with 8 local supersymmetries can be defined, namely \( D = 3, 4, 5, 6 \).
direction) down to $D = 5$ (for some discussion on anomaly-freedom conditions of minimal chiral theories in $D = 6$, cfr. e.g. [52]).

5.2. $q$-Dependence of $d$ and $d_g$ for $J^{A_s}$-based Maxwell-Einstein (Super)Gravity Theories and $U$-Representations of 1-Form Potentials. For theories based on $J^{A_s}$ (where $A_s$ denotes the split form of the normed algebra $A_s$), the values of $d$ and $d_g$ are not $q$-independent, but rather they follow an interesting pattern. Essentially, the Iwasawa degree $d$ of the scalar manifold $G_{D,0_s}/mcs(G_{D,0_s})$ in $D$ dimensions (in the relevant $G_{D,0_s}$-irrep.) is related to the (real) dimension of the $G_{D+1,0_s}$-irrep. in which the 1-form (Abelian) potentials of maximal supergravity sit (cfr. e.g. [66][67]).

While the purely $D$-dependent values of $d$ and $d_g$ for $J^{A_s}$-based magic Maxwell-Einstein supergravities are consistent with the Tits-Satake projection for homogeneous (symmetric) manifolds [64], it is here worth observing that the non compact symmetry ($U$-duality) groups pertaining to $J^{A_s}$-based theories are the maximally non compact (namely, split) forms of Lie groups. As a consequence the corresponding homogeneous spaces are not necessarily in the same Tits-Satake universal class.

Tables 11-13 respectively illustrate the resulting pattern for $D = 3, 4, 5$ in some detail.

For the maximally supersymmetric theory based on $O_s$, the uplifts to $D > 5$ Lorentzian dimensions are known and well defined.

In $D = 6$, one can consider the following embedding in the maximal non-chiral $(2,2)$ supergravity:

\[(5.11) \quad SO(5,5) \supset s_{\text{max}}^s SL(5,\mathbb{R}) \times SO(1,1);\]
\[(5.12) \quad 16 = 10 - 1 + 5_3 + 1 - 5,\]
and the Iwasawa degree $d$ of the $D = 6$ scalar manifold $G_{6,0_s}/mcs(G_{6,0_s}) = SO(5,5)/(SO(5) \times SO(5))$ in the 16 is 5, namely the (real) dimension of the irrep. 10 of the $D = 7$ U-duality group $SL(5,\mathbb{R})$, occurring in (5.11)-(5.12) (in this case, $d_g = 12$).

In $D = 7$, the relevant embedding in $N = 4$ supergravity reads:

\[(5.13) \quad SL(5,\mathbb{R}) \supset s_{\text{max}}^s SL(3,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(1,1);\]
\[(5.14) \quad 10 = (3,2)_1 + (3',1)_4 + (1,1)_6,\]
and the Iwasawa degree $d$ of the $D = 7$ scalar manifold $G_{7,0_s}/mcs(G_{7,0_s}) = SL(5,\mathbb{R})/SO(5)$ in the 10 is 6, namely the (real) dimension of the irrep. $(3,2)$ of the $D = 8$ U-duality group $SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$ occurring in (5.13)-(5.14) (in this case, $d_g = 6$).

In $D = 8$, the relevant embedding in $N = 2$ supergravity is:

\[(5.15) \quad SL(3,\mathbb{R}) \times SL(2,\mathbb{R}) \supset s_{\text{max}}^s GL(2,\mathbb{R}) \times SO(1,1);\]
\[(5.16) \quad (3,2) = 2 \cdot (2_1 + 1_2),\]
and the Iwasawa degree $d$ of the $D = 8$ scalar manifold $G_{8,0_s}/mcs(G_{8,0_s}) = SL(3,\mathbb{R})/SO(3) \times SL(2,\mathbb{R})/SO(2)$ in the $(3,2)$ is 2, namely the (real) dimension of the irrep. 2 of the $D = 9$ U-duality group $GL(2,\mathbb{R})$ occurring in (5.15)-(5.16).

Finally, in $D = 9$ the relevant embedding in $N = 2$ supergravity is:

\[(5.17) \quad GL(2,\mathbb{R}) \supset s_{\text{max}}^s SO(1,1)_{IIA} \times SO(1,1);\]
\[(5.18) \quad 2 = 1_{\alpha} + 1_{-\alpha},\]
where the $SO(1,1)$-weight $\alpha$ in (5.18) depends on normalization, and the subscript “IIA” in the first $SO(1,1)$ in the r.h.s. of (5.17) characterizes it as the U-duality group of type IIA $D = 10$ supergravity, and distinguishes it from the second (Kaluzka-Klein) $SO(1,1)$. The Iwasawa degree $d$ of the $D = 9$ scalar manifold $G_{9,0_s}/mcs(G_{9,0_s}) = GL(2,\mathbb{R})/SO(2)$ in the 2 is 1, namely the (real) dimension of the $SO(1,1)_{IIA}$-singlet 1 occurring in (5.17)-(5.18).

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9We are grateful to Mario Trigiante for useful discussions on this point.
As mentioned above, the representation determining the Iwasawa degree $d$ of the scalar manifold $G_{D,0}/mcs(G_{D,0})$ in $D$ dimensions (in the relevant $G_{D,0}$-irrep) is the $G_{D+1,0}$-irrep. in which the 1-form (Abelian) potentials of maximal supergravity sit (cfr. e.g. [66, 67]). In this sense, one can understand why type IIB supergravity in $D = 10$ (along with its $SL(2,\mathbb{R})$ $U$-duality) has not been considered: it has no 1-form potentials. As discussed in [70, 71, 66], the relevant 1-form $U$-irrep. can be predicted by exploiting the infinite-dimensional (Kac-Moody extended) algebra $E_{11}$. It would be interesting, at least for maximal supergravity in $D = 3, ..., 9$, to understand the matching between the relevant Iwasawa degree $d$ of $(G_{D,0}/mcs(G_{D,0}))_{R_D}$ and the (real) dimension of the $G_{D+1,0}$-irrep. $R_{D+1}$ in terms of $E_{11}$; we leave this task for future further investigations.

5.3. Universal Nilpotency Degree of Axionic Iwasawa Generators. Considering symmetric cosets $G_4/H_4$ (2.3) which are scalar manifolds of Maxwell-Einstein theories of (super)gravity in $D = 4$ (Lorentzian) dimensions, it is here worth commenting on the relation between the (“standard”; cfr. Footnote 4) total Iwasawa construction (2.3):

\[
g_4 = h_4 \oplus a_4 \oplus n_4,
\]

exploited in the present paper, and the Iwasawa construction restricted to the axionic generators of the $D = 4$ $U$-duality group $G_4$, recently studied in [41]. Note that in (5.19) the sums are not direct, but they do respect the Lie algebra $g_4$: $h_4$ denotes the Lie algebra of $H_4 = mcs(G_4)$, while $a_4$ and $n_4$ respectively stand for the (maximal) Abelian (Cartan) non compact sub-algebra (whose dimension equals the rank $r$ of $G_4$; cfr. Table 2) and the maximal nilpotent set of nilpotent (namely, Iwasawa) generators, whose cardinality has been denoted by $I$ in Table 2.

As illustrative examples, we will consider two theories: the magic exceptional $\mathcal{N} = 2$ theory [4], based on $J^0_3$, and the maximal $\mathcal{N} = 8$ supergravity, based on $J^0_3$.

5.3.1. $\mathcal{N} = 2$, $D = 4$ Magic Exceptional ($J^0_3$). As yielded by (5.19), the relevant (maximal, symmetric) embedding pertaining to the Iwasawa decomposition of the $D = 4$ $U$-duality group $G_4 \equiv G_{4,0} = E_{7(-25)}$ reads

\[
\begin{align*}
E_{7(-25)} &\supset E_{6(-78)} \times U(1); \\
\begin{array}{c|c|c}
\text{dim}_{\mathbb{R}} (a_4) & 133 & \text{dim}_{\mathbb{R}} (n_4) \\
\hline
\text{dim}_{\mathbb{R}} (h_4) & 78_0 + 10 + 27_2 + \overline{27}_2,
\end{array}
\end{align*}
\]

where $\text{dim}_{\mathbb{R}} (a_4) = \text{rank} \left( \frac{E_{7(-25)}}{E_{6(-78)} \times U(1)} \right) = 3$, and $\text{dim}_{\mathbb{R}} (n_4) = I = 51$. The maximal degree of the polynomial in the Iwasawa decomposition of $\frac{E_{7(-25)}}{E_{6(-78)} \times U(1)}$ in the 56 of $E_{7(-25)}$ is $d = 9$.

On the other hand, the Iwasawa parametrization of the axionic generators of $E_{7(-25)}$, performed in [14], considers the maximal triangular subgroup of $E_{7(-25)}$, containing the $D = 5$ $U$-duality group $G_5 \equiv G_{5,0} = E_{6(-26)}$:

\[
\begin{align*}
E_{7(-25)} &\supset E_{6(-26)} \times SO(1,1)_{KK} \times_s T_{27}; \\
\begin{array}{c|c|c}
\text{dim}_{\mathbb{R}} (a_4) & 133 & \text{dim}_{\mathbb{R}} (n_4) \\
\hline
\text{dim}_{\mathbb{R}} (h_4) & 78_0 + 10 + 27_2 + \overline{27}_2,
\end{array}
\end{align*}
\]

where the “$KK$” subscript denotes the Kaluza-Klein nature of the commuting 1-dimensional factor, the subscript “$s$” indicates the semi-direct nature of the product, and $T_{27} \equiv \mathbb{R}^{27}$ corresponds to the 27 axionic Iwasawa generators, in one-to-one correspondence with the fifth component $A^{\mu}_5$ of the 27 Abelian vector potentials of the corresponding theory in $D = 5$; note that (5.22) is nothing but a different, non compact form of (5.20).

In this case, the maximal degree $d$ pertaining to the set of axionic generators $T_{27}$ of the rank-3 special Kähler symmetric coset $\frac{E_{7(-25)}}{E_{6(-78)} \times U(1)}$ is $d_{\text{axionic}} = 3$ [6, 14, 72, 73].
This is universal, namely it concerns the axionic generators of the global electric-magnetic isometries of all $D = 4$ theories which enjoy a $D = 5$ uplift, even in the case of non-symmetric nor homogeneous (vector multiplets’) scalar manifolds.

At the level of the coset manifolds, (5.22) and (5.23) can be reformulated as the isometry

$$\frac{E_{7(-25)}}{E_{6(-78)} \times U(1)} \sim \frac{E_{6(-26)}}{F_{4(-52)}} \times SO(1,1)_{KK} \times_s T_{27};$$ (5.24)

$$54 = 26 + 1 + 27,$$

(5.25)

where the numbers in the second line denote the real dimensions of the manifolds. For theories based on Frietenthal triple systems $\mathbf{F} (J_3)$ on rank-3 (simple or semi-simple) Jordan algebras $J_3$, (5.22) and (5.23) can be q-parametrized as follows:

$$G_{\mathbf{F}(J_3),4} \supset G_{J_3,5} \times SO(1,1)_{KK} \times_s T_{3q+3};$$ (5.26)

$$\text{Adj}(G_{\mathbf{F}(J_3),4}) \supset \text{Adj}(G_{J_3,5}) + 1_{1_0} + (3q+3)_{-2} + (3q+3)^I_{2},$$ (5.27)

where $G_{\mathbf{F}(J_3),4} = \text{Aut} (\mathbf{F} (J_3)) = \text{Conf} (J_3)$ and $G_{J_3,5} = \text{Str} (J_3)$ respectively denote the $D = 4$ and $D = 5$ U-duality groups; as mentioned above, for simple rank-3 Jordan algebras, $q := \dim_{\mathbb{R}} \mathbb{A}_q = \dim_{\mathbb{R}} \mathbb{A}_s$, whereas $q = (m + n - 4)/3$ for the class of semi-simple Jordan algebras $\mathbb{R} \oplus \Gamma_{m-1,n-1},$ where $\Gamma_{m-1,n-1}$ stands for the Clifford algebra of $O(m - 1, n - 1)$. It should be remarked that, since this is a $\mathcal{N} = 2, D = 4$ theory, $G_{J_3,5}$ is nothing but a non compact, real form of the $mcs (G_{\mathbf{F}(J_3),4}).$

Note that the “standard” total Iwasawa construction (2.1) (or, equivalently, (5.19)) takes into account of the maximal number $6q + 3$ of Iwasawa (nilpotent) generators, but with high maximal degree ($d = 9$ for all $J_3^A$-based magic theories in $D = 10; \text{cf. Sec. 5.1}$), while the Iwasawa construction exploits in [44] is restricted to the axionic generators of the $D = 4$ U-duality group $G_4$, but with lower (and universal) maximal degree ($d_{\text{axionic}} = 3$). At least in the theories related to rank-3 Jordan algebras, the $3q + 3$ axionic generators are related to the whole set of $6q + 3$ Iwasawa generators through some Wick’s rotation and linear combination: in fact, in the magic exceptional supergravity $E_{6(-78)} \times U(1)$ gets converted into $E_{6(-26)} \times SO(1,1)$.

5.3.2. $\mathcal{N} = 8, D = 4$ Maximal (J$^3_{G^2}$). The difference between the two aforementioned approaches to the Iwasawa construction is more striking in the maximal supergravity theory.

As yielded by (5.19), the relevant (maximal, symmetric) embedding pertaining to the Iwasawa decomposition of the $D = 4$ U-duality group $G_4 \equiv G_{4,0} = E_{7(7)}$ reads

$$E_{7(7)} \supset SU(8);$$ (5.28)

$$133_{G_4} = 63_\mathbb{h}_4 + 70_{a_4 \oplus n_4},$$ (5.29)

where $\dim_{\mathbb{R}} (a_4) = \text{rank} \left( \frac{E_{7(7)}}{SU(8)} \right) =: r = 7$, and $\dim_{\mathbb{R}} (n_4) =: I = 63$. Note that in this case the coset rank $r$ matches the group rank $l$ as well as the coset character $\chi$; this is a common feature of all theories based on split algebras $\mathbb{A}_s$ (resulting in split, i.e. maximally non compact, forms of the corresponding U-duality groups, as it is the case for $E_{7(7)}$). The maximal degree of the polynomial in the Iwasawa decomposition of $\frac{E_{7(7)}}{SU(8)}$ in the 56 of $E_{7(7)}$ is $d = 27$.

On the other hand, the Iwasawa parametrization of axionic generators of $E_{7(7)}$, performed in [44], considers the maximal triangular subgroup of $E_{7(7)}$, containing the $D = 5 U$-duality group $G_5 \equiv$
\( \mathcal{G}_{5,0} = E_{6(6)} : \)

\begin{align}
(5.30) \quad E_7(7) & \supset E_{6(6)} \times SO(1,1)_{KK} \times_s T_{27}; \\
(5.31) \quad \frac{133}{g_4} & = \frac{78_0}{g_5} + \frac{1_9}{so(1,1)_{KK}} + \frac{27_{-2} + 27'_{-2}}{T_{27}}.
\end{align}

Once again, note that \((5.30)\) is nothing but a non compact form of the maximal and symmetric embedding \((5.20)\) (different from the non compact form \((5.22)\)). In this case, the maximal degree pertaining to the set of axionic generators \(T_{27}\) of \(E_{7(7)} / SU(8)\) is \(d_{\text{axionic}} = 3\); it is the same as the one of \(E_{7(-25)} / E_{\text{al}(7)} \times U(1)\), because, as mentioned above, it is universal. In particular, at the level of cosets, \((5.30)\) and \((5.31)\) specify as

\begin{align}
(5.32) \quad \frac{E_7(7)}{SU(8)} & \sim \frac{E_{6(6)}}{USp(8)} \times SO(1,1)_{KK} \times_s T_{27}; \\
(5.33) \quad 70 & = 42 + 1 + 27.
\end{align}

It should be remarked that, since this is not an \(\mathcal{N} = 2, D = 4\) theory, \(G_{J,5}\) is not a non compact, real form of the mcs \((G_{F}(J,4))\).

In this case, the “standard” total Iwasawa construction \((2.4)\) (or, equivalently, \((5.19)\)) takes into account of the whole set of 63 Iwasawa (nilpotent) generators, but with high maximal degree \(d = 27\) (for the other \(\mathbb{H}\)- and \(\mathbb{C}\)-based theories, see Sec. \((5.2)\)), while the Iwasawa construction exploited in \((44)\) is restricted to the 27 axionic generators of the \(D = 4\) U-duality group \(E_7(7)\), but with lower (and universal) maximal degree \(d_{\text{axionic}} = 3\). As observed above, the 27 axionic generators are related to the whole set of 63 Iwasawa generators through some Wick’s rotation and linear combination.

5.3.3. The \(\mathcal{N} = 2, D = 4\) \(T^3\) Model. As already mentioned, among the theories with symmetric (vector multiplets) scalar manifolds, there is a unique case in which the two treatments under consideration coincide: it is the so-called \(T^3\) model of \(\mathcal{N} = 2, D = 4\) Maxwell-Einstein supergravity, in which the unique Abelian vector multiplet is coupled \textit{non-minimally}, but rather through a cubic holomorphic prepotential \(\mathcal{F} = T^3\), to the gravity multiplet.

In this case, \((5.19)\) trivially yields the (maximal, symmetric) embedding pertaining to the Iwasawa decomposition of the \(D = 4\) U-duality group \(G_4 = SL(2,\mathbb{R})\):

\begin{align}
(5.34) \quad SL(2,\mathbb{R}) & \supset U(1); \\
(5.35) \quad \frac{3}{g_4} & = \frac{1_0}{h_4} + \frac{1_2 + 1_{-2}}{a_4 + n_4},
\end{align}

where \(\text{dim}_{\mathbb{R}}(a_4) = \text{rank}\left(\frac{SL(2,\mathbb{R})}{U(1)}\right) = r = 1\), and \(\text{dim}_{\mathbb{R}}(n_4) = I = 1\). Note that also in this case the coset rank \(r\) matches the group rank \(l\) as well as the coset character \(\chi\), because \(SL(2,\mathbb{R})\) trivially is the split form of \(SU(2)\).

The maximal degree of the polynomial pertaining to the Iwasawa decomposition of \(\frac{SL(2,\mathbb{R})}{U(1)}\) in the the relevant irrep, \(4\) (spin \(s = 3/2\)) of \(SL(2,\mathbb{R})\) is \(d = 3\). Thus, it matches the universal result \(d_{\text{axionic}} = 3\), which does pertain to the Iwasawa parametrization of the unique axionic generator of \(SL(2,\mathbb{R})\) \((44)\), considering the maximal triangular subgroup of \(SL(2,\mathbb{R})\) itself:

\begin{align}
(5.36) \quad SL(2,\mathbb{R}) & \supset SO(1,1)_{KK} \times_s T_1; \\
(5.37) \quad \frac{3}{g_4} & = \frac{1_0}{so(1,1)_{KK}} + \frac{1_{-2} + 1_{2}}{T_1}.
\end{align}
Once again, note that (5.36) is nothing but a non compact form of the maximal and symmetric embedding (5.20). In this case the $D = 5$ U-duality group is empty ($G_5 = \emptyset$), because the $T^3$-model is the unique model which uplifts to “pure” minimal ($\mathcal{N} = 2$) $D = 5$ theory, in which only the gravity multiplet is present.

In particular, at the level of cosets, (5.36) and (5.37) respectively specify as

\[
\frac{SL(2,\mathbb{R})}{U(1)} \sim SO(1,1)_{KK} \times T_1; \\
2 = 1 + 1.
\]

The matching of the two approaches can be explained simply by observing that the unique generator of $\mathfrak{n}_4$ in (5.35) is $T_1$ occurring in (5.38).

| $\mathbb{CP}^n$ | $\mathbb{CP}^n = \frac{SU(1,n)}{SU(n) \times U(1)}$, $n \in \mathbb{N}$ | $\mathbb{CP}^n$ | $\mathbb{CP}^n = \frac{SU(1,n)}{SU(n) \times U(1)}$, $n \in \mathbb{N} \setminus \{0\}$ |
|-----------------|-------------------------------------------------|-----------------|-------------------------------------------------|
| $SU(1,1)$      | $U(1)$                                           | $SU(1,1)$      | $U(1)$                                           |
| $SU(2,1)$      | $SU(2,1)$                                       | $SU(2,1)$      | $SU(2,1)$                                       |

Table 4: $\mathcal{N} = 2$, $D = 4$ symmetric special Kähler vector multiplets’ scalar manifolds. If any, the related Euclidean rank-3 Jordan algebras are reported in brackets throughout (the notation of [39] is used). By defining $q := \dim_{\mathbb{A}} \mathbb{A} (= 1, 2, 4, 8$ for $A = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ respectively), the complex dimension of the $\mathcal{N} = 2, d = 4$ symmetric special Kähler manifolds based on $J_3^A$ is $3q + 3$ [57]. The irrep. $R$ of $G_{\mathfrak{uc}}$ relevant for supergravity is reported, along with the corresponding degrees $d$ and $d_g$. Note that $d = 9$ and $d_g = 8$ for every $q = 1, 2, 4, 8$. For the Dynkin notation of irreps., we adopt the conventions of [41] throughout. Notice that in the sixth row for both the representations $32$ and $32'$ of $SO^*(12)$ the degree is $9$. This is due to the fact that the highest weight corresponds to the non compact root (see Remark after Table 3).
Table 5: $\mathcal{N} = 4$, $D = 3$ symmetric quaternionic spaces, obtained through c-map [49] from the special Kähler manifolds of Table 4. Since the hypermultiplets are insensitive to dimensional reduction, these spaces can also be regarded as the hypermultiplets’ scalar manifolds in $D = 3, 4, 5, 6$ (for a discussion of anomaly-free conditions in minimal chiral $(1,0)$ theories in $D = 6$, see e.g. [52]). The relevant $G$ -irrep. is the adjoint. In general, starting from a special Kähler geometry with $\dim \mathbb{C} = n$, the c-map generates a quaternionic manifold with $\dim \mathbb{H} = n + 1$ [49]. Thus, the quaternionic dimensions of the spaces corresponding to $q = 1, 2, 4, 8$ is $3q + 4$ [49] [55]. The irrep. $\mathbf{R} = \text{Adj}(G_{nc})$ relevant for supergravity is reported, along with the corresponding degrees $d$ and $d_g$. Note that $d = 22$ and $d_g = 20$ for every $q = 1, 2, 4, 8$. The unique other class of symmetric quaternionic spaces is given by the non compact quaternionic projective spaces $\mathbb{HP}^n = Usp(2, 2n)/(Usp(2) \times Usp(2n))$ ($n \in \mathbb{N}$), which is not in the c-map image of any (symmetric) special Kähler manifold [49] (indeed, its $\Omega$-tensor - and thus, its corresponding quaternionic potential - identically vanishes [59]) for this class, still $\mathbf{R}(Usp(2, 2n)) = \text{Adj} = (2n + 3)(n + 1) = (2, 0, ..., 0)$, and $d = 4, d_g = 2$ if $n > 1, d = 2, d_g = 0$ if $n = 1$. 

| Quaternionic Symmetric Space | $\mathbf{R} = \text{Adj}(G)$ | $d$ | $d_g$ |
|-----------------------------|--------------------------|-----|-----|
| $SU(2n+1)/SU(n+1) \times SU(2) \times U(1)^*$ | $(n + 3)^2 - 1 = (1, 0, ..., 0, 1)$ | $8$ ($n > 1$) | $6$ ($n = 1$) |
| $SO(4n+2)/SO(n+2) \times SO(4)^*$ | $(n+2)(n+5)/2 = (0, 1, 0, ..., 0)$ | $14$ ($n > 2$) | $10$ ($n = 2, 1$) |
| $G_{2(2)}$/$O(3)$ | $14 = (10)$ | $10$ | $8$ |
| $F_{4(4)}/Usp(6) \times SU(2)$ | $(J^\mathbb{R}_3)$ | $52 = (10000)$ | $22$ | $20$ |
| $E_{6(2)}/SU(6) \times SU(2)$ | $(J^\mathbb{C}_3)$ | $78 = (000001)$ | $22$ | $20$ |
| $E_{7(-5)}/SO(12) \times SU(2)$ | $(J^{16}_3, \mathcal{N} = 4 \Leftrightarrow \mathcal{N} = 12)$ | $133 = (1000000)$ | $22$ | $20$ |
| $E_{6(-24)}/E_7 \times SU(2)$ | $(J^3_3)$ | $248 = (00000010)$ | $22$ | $20$ |
| Real Special Symmetric Space | R. | d | d_g |
|------------------------------|----|---|-----|
| $\frac{SO(1,n)}{SO(n)}$, $n \geq 2$ | $1 + n = (1,0,...,0)$ $(n > 2)$ $3 = (2)$ $(n = 2)$ | 2 | 0 |
| $SO(1,1) \otimes \frac{SO(1,n-1)}{SO(n-1)}$, $n \geq 2$ $(\mathbb{R} \oplus \Gamma_{n-1,1})$ | $(1,1) + (1,n) =$ $0(0) + (0)(1,0,...,0)$ $(n > 3)$ $(n = 3)$ $(n = 3)$ $(n = 2)$ | 2 $(n > 2)$ $0 (n = 2)$ | 0 |
| $\frac{SL(3,\mathbb{R})}{SO(3)} \left(J^R_3\right)$ | $6' = (02)$ | 4 | 2 |
| $\frac{SL(3,\mathbb{C})}{SU(3)} \left(J^C_3\right)$ | $9 = (3,\overline{3}) = (10)(01)$ | 4 | 2 |
| $\frac{SU^*(6)}{USp(6)} \left(J^R_3, N = 2 \Leftrightarrow \mathcal{N} = 6\right)$ | $15 = (01000)$ | 4 | 2 |
| $\frac{F_{6(-26)}}{E_7} \left(J^G_3\right)$ | $27 = (100000)$ | 4 | 2 |

Table 6: $\mathcal{N} = 2$, $D = 5$ symmetric real special vector multiplets’ scalar manifolds; they are nothing but the moduli spaces of non-BPS $Z \neq 0$ extremal black hole attractors of the corresponding theory in $D = 4$ [57]. They are obtained from the corresponding symmetric special Kähler manifolds of Table 4 through the so-called $R$-map, with the notable exception of the case $\frac{SO(1,n)}{SO(n)}$. From [60] the latter does not correspond to the $R$-map image of any symmetric space, as the minimally coupled $D = 4$ models based on $\mathbb{CP}^q$ cannot be uplifted to $D = 5$. If any, the related Euclidean rank-3 Jordan algebras are reported in brackets throughout. The real dimension of the manifolds based on $J^R_3$ is $3q + 2$ [57]. The irrep. $R$ of $G_{nc}$ relevant for supergravity is reported, along with the corresponding degrees $d$ and $d_g$. Note that $d = 4$ and $d_g = 2$ for every $q = 1, 2, 4, 8$. The coset $\frac{SO(1,n)}{SO(n)}$ is not related to a rank-3 Jordan algebra, and the corresponding $D = 4$ theory is based on an homogeneous non-symmetric manifold; for further details, see e.g. [51] (see also [61]). $\frac{SL(3,\mathbb{C})}{SU(3)}$ is the unique coset (at least among symmetric scalar manifolds in supergravity) belonging to the class of symmetric spaces $\frac{G_C}{G_R}$ [38], where $G_C$ is a complex (non compact ) (semi-)simple Lie group regarded as a real group, and $G_R$ is its compact, real form $(mcs \left(G_C\right) = G_R)$; in general, $\frac{G_C}{G_R}$ is a Riemann symmetric space with $\dim \mathbb{R}(\frac{G_C}{G_R}) = \dim \mathbb{R}(G_R)$, and $\text{rank}(\frac{G_C}{G_R}) = \text{rank}(G_R)$; this case deserves a separate analysis, which will be given elsewhere (we here anticipate the result relevant for the present treatment). Note that the $6'$ of $SL(3,\mathbb{R})$ is not fundamental; together with the spin $s = 3/2$ irrep. $4$ of $SL(2,\mathbb{R})$ in the so-called $D = 4$ $T^3$ model, this is the only case in which the relevant $G_{nc}$-repr. is not fundamental.
### Table 7: Scalar manifolds of $\mathcal{N} \geq 3$, $D = 4$ supergravity theories

| $\mathcal{N} = 3$ | $\mathcal{R}$ | $d$ | $d_g$ |
|--------------------|-----------------|-----|------|
| $\mathcal{N} = 4$ : $\frac{SU(3,n)}{SU(3) \times SU(n) \times U(1)}$, $n \in \mathbb{N}$ | $3 + n = (1,0,...,0)$ | $6$ ($n > 3$) | $10$ ($n > 3$) |
| | | $5$ ($n = 3$) | $8$ ($n = 3$) |
| | | $4$ ($n = 2$) | $4$ ($n = 2$) |
| | | $2$ ($n = 1$) | $2$ ($n = 1$) |

| $\mathcal{N} = 5$ | $\frac{SU(5)}{SU(5) \times U(1)}$ | $20$ $(00100)$ | $2$ | $2$ |

| $\mathcal{N} = 2 \leftrightarrow \mathcal{N} = 6$ | $\frac{SO^*(12)}{SU(6) \times U(1)}$ | $32$ $(000010)$ | $9$ | $8$ |

| $\mathcal{N} = 8$ | $\frac{F_4(7)}{SU(8)}$ | $56$ $(0000010)$ | $27$ | $32$ |

Table 7: Scalar manifolds of $\mathcal{N} \geq 3$, $D = 4$ supergravity theories. Note that the scalar manifolds of $\mathcal{N} = 6$ theory coincides with the one of $\mathcal{N} = 2$ magic supergravity based on $J_3^H$; in fact, these theories share the very same bosonic sector [62]. $M_{1,2} (\mathbb{O})$ is an exceptional Jordan triple system, generated by $2 \times 1$ vectors over $\mathbb{O}$ [4]. Note that, despite the different relevant irrep. $\mathcal{R}$ (6 vs. rank-3 self-dual real 20), the degree of the Iwasawa polynomial pertaining to $\mathcal{N} = 2$ supergravity minimally coupled to 5 vector multiplets (based on $\mathbb{CP}^5$) is the same as the one of $\mathcal{N} = 5$ theory, which is pure (no matter coupling allowed); indeed, both 6 and 20 are fundamental irreps. of $SU(1,5)$, and the string $(d_1, ... ,d_l)$ for $SU(1,n)$ is constant: $(d_1, ... ,d_l) = (2,1,1,1)$. Notice that in the fourth row for both the representations $32$ and $32'$ of $SO^*(12)$ the degree is 9. This is due to the fact that the highest weight corresponds to the non-compact root (see Remark after Table 6).

### Table 8: Scalar manifolds of $\mathcal{N} \geq 4$, $D = 5$ supergravity theories

| $\mathcal{N} = 4$ : $\frac{SO(1,1) \times SO(5,n-1)}{SO(n-1) \times SO(5)}$, $n \geq 2$ | $\mathcal{R}$ | $d$ | $d_g$ |
|-----------------------------------------------|-----------------|-----|------|
| $(1,1) + (1,n + 4) = (0) + (0) (1,0,...,0)$ | $10$ ($n > 5$) | $10$ ($n > 5$) |
| | $8$ ($n = 5$) | $12$ ($n = 5$) |
| | $6$ ($n = 4$) | $8$ ($n = 4$) |
| | $4$ ($n = 3$) | $4$ ($n = 3$) |
| | $2$ ($n = 2$) | $0$ ($n = 2$) |

| $\mathcal{N} = 2 \leftrightarrow \mathcal{N} = 6$ | $\frac{SU^*(6)}{U(6) \times SU(6)}$ | $15$ $(01000)$ | $4$ | $2$ |

| $\mathcal{N} = 8$ | $\frac{F_4(6)}{USp(8)}$ | $27$ $(100000)$ | $16$ | $20$ |

Table 8: Scalar manifolds of $\mathcal{N} \geq 4$, $D = 5$ supergravity theories. Note that the scalar manifold of the $\mathcal{N} = 6$ theory coincides with the one of $\mathcal{N} = 2$ magic supergravity based on $J_3^H$; in fact, these theories share the very same bosonic sector [62].
Table 9: Scalar manifolds of $\mathcal{N} \geq 5$, $D = 3$ supergravity theories. Note that the scalar manifold of the $\mathcal{N} = 12$ theory coincides with the one of $\mathcal{N} = 4$ magic supergravity based on $JH_3$; in fact, these theories share the very same bosonic sector [62]. Note that, as for the $\mathcal{N} = 4$ models with symmetric scalar manifold reported in Table 5, the relevant $G$-irrep. is the adjoint.
the theory based on split octonions 

Interestingly, the degree \(d\) of the corresponding Iwasawa polynomial and the metric degree \(d_g\) of \(G_{3,A_3}/mcs(G_{3,A_3})\) are related to the dimension of the real symplectic \(G_{4,A_3}\)-irrep \(\mathbf{R}_4\), in which the 2-form Abelian field strengths of the corresponding \(D = 4\) theory sit.
Table 12: Scalar manifolds of $D = 4$ Maxwell-Einstein gravity theories based on rank-3 simple Jordan algebras $J_{A_s}^3$ on split algebras $A_s = \mathbb{O}_s$ ($q = 8$), $\mathbb{H}_s$ ($q = 4$) and $\mathbb{C}_s$ ($q = 2$). The $D = 4$ and $D = 5$ U-duality groups are the conformal resp. reduced structure groups of $J_{A_s}^3 : G_{4,A_s} = \text{Conf} \left( J_{A_s}^3 \right)$, $G_{5,A_s} = \text{Str}_0 \left( J_{A_s}^3 \right)$. They occur in the third resp. second row of the symmetric Magic Square $L_3 (A_s, B_s)$ [68, 65]. In $D = 4$, only the theory based on split octonions $\mathbb{O}_s$ can be regarded as the scalar sector of a locally supersymmetric theory (namely with maximal supersymmetry in $D = 4 : \mathcal{N} = 8$), while the other theories are non-supersymmetric [69]. Interestingly, the degree $d$ of the corresponding Iwasawa polynomial is related to the dimension of the real symplectic $G_{5,A_s}$-irrep. $R_5$, in which the 2-form Abelian field strengths of the corresponding $D = 5$ theory sit. On the other hand, the metric degree $d_g$ of $G_{4,A_s}/mcs (G_{4,A_s})$ is nothing but $4q$. 

| $\mathcal{N} = 8 : E_7(7)/SU(8)$ | $E_7(7) \supset_s E_6(6) \times SO(1,1)$ | 27 | 32 |
| $\mathcal{N} = 0 : SO(6,6)/SO(6) \times SO(6)$ | $SO(6,6) \supset SL(6,\mathbb{R}) \times SO(1,1)$ | 15 | 16 |
| $\mathcal{N} = 0 : SL(6,\mathbb{R})/SO(6)$ | $SL(6,\mathbb{R}) \supset_s SL(3,\mathbb{R}) \times SO(1,1)$ | 9 | 8 |
Table 13: Scalar manifolds of $D = 5$ Maxwell-Einstein gravity theories based on rank-3 simple Jordan algebras $J^h_3$ on split algebras $A_h = \mathbb{O}_h, \mathbb{H}_h$ and $\mathbb{C}_h$. The $D = 5$ and $D = 6$ U-duality groups are the structure groups of $J^h_3$ resp. $J^h_2 \sim \Gamma_{q/2+1,q/2+1}$. $G_{5,A_h} = \text{Str}_0 \left( J^h_3 \right)$, and $G_{6,A_h} = \text{Str}_0 \left( J^h_2 \right) = SO(q/2+1,q/2+1) \times \text{Tri}(A_h) / SO(A_h)$. $G_{5,A_h}$ occur in the second row of the symmetric Magic Square $\mathcal{L}_3 (A_h, \mathbb{B}_h)$ [68, 65], whereas $G_{6,A_h}$ occur in the first row of Table III $\text{Tri}(A_h)$ and $SO(A_h)$ respectively denote the triality and norm-preserving groups of $A_h$, and the factor $\text{Tri}(A_h) / SO(A_h)$ is non-trivial only for $q = 4 \ (SL(2, \mathbb{R}))$ and for $q = 2 \ (SO(1,1))$. In $D = 5$, only the theory based on split octonions $\mathbb{O}_h$ can be regarded as the scalar sector of a locally supersymmetric theory (namely with maximal supersymmetry in $D = 5 : \mathcal{N} = 8$), while the other theories are non-supersymmetric. Interestingly, the degree $d$ of the corresponding Iwasawa polynomial is related to the dimension of the real symplectic $G_{6,A_h}$-irrep. $R_6$, in which the vector multiplets of the corresponding (anomaly-free) $D = 6$ theory sit.

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A systematic study of primitive invariant tensors of a compact, simple Lie algebra \(\mathfrak{g}\) has been the subject of a number of monographies and papers along the years; here, we will give a concise résumé, mainly based on \([74, 75, 76, 77]\), to which we address the reader for further elucidation and a list of references. 

The symmetric invariant tensors give rise to the so-called Racah-Casimir polynomials of \(\mathfrak{g}\); on the other hand, the skewsymmetric invariant tensors determine the non-trivial cocycles for the Lie algebra cohomology (see e.g. \([78]\)). By denoting the rank of \(\mathfrak{g}\) with \(l\), it is well known \([79]\) that there are \(l\) such invariant symmetric primitive polynomials of order \(\delta_A (A = 1, \ldots, l)\), which determine \(l\) independent primitive Racah-Casimir polynomials \(\{C_{\delta_A}\}_{A=1,\ldots,l}\) of the same order, as well as \(l\) skewsymmetric invariant primitive tensors \(\Omega^{(2\delta_A-1)}\) of order \(2\delta_A - 1\). The latter determine the non-trivial cocycles for the Lie algebra cohomology, their order being related to the topological properties of the associated compact group manifold \(G\) which, from the point of view of the real homology, behaves as products of \(l\) spheres \(S^{(2\delta_A-1)}\) \([80]\). Indeed, the Poincaré polynomial of \(G\) is of the form (see e.g. \([77]\))

\[
\begin{align}
 f_G(t) &= \prod_{A=1}^{l} \left(1 + t^{2\delta_A - 1}\right); \\
 \sum_{A=1}^{l} (2\delta_A - 1) &= \dim(G).
\end{align}
\]

Remarkably, the so-called principal \(SU(2)_P\) \([34]\) is (generally non-symmetrically) embedded in \(G\) such that \((2,1,1)\) holds, with \(\delta_A = j_A + 1\). All simple Lie groups admits an embedded principal \(SU(2)_P\), whose embedding is always maximal and non-symmetric. Exceptions are provided by \(SU(3)\), which embeds \(SO(3) \simeq SU(2)_P\) symmetrically, and by \(E_6\), which embeds \(SU(2)_P\) only next-to-maximally, i.e. through the 2-step chain of maximal embeddings:

\[
E_6 \supset \circ \, F_4 \supset SU(2)_P,
\]

where the subscript “s” in the first embedding denotes that it is symmetric.

The \(C_{\delta_A}\)'s are particular homogeneous polynomials in \(U(\mathfrak{g})\), the universal enveloping algebra of \(\mathfrak{g}\); indeed, they generate the center \(U(\mathfrak{g})^G\) of \(U(\mathfrak{g})\) itself. Remarkably, \(\{C_{\delta_A}\}_{A=1,\ldots,l}\) constitutes a complete, “minimal-degree”, finitely generating basis of the ring of invariant polynomials in \(\mathfrak{g}\) (see e.g. \([81]\)). Their degrees, i.e. the numbers \(\delta_A\), are known for all simple \(\mathfrak{g}\), and they are reported in Table 2; they can be computed by diagonalizing the Coxeter element, the product of simple Weyl reflections (see e.g. \([82]\)). The relation between \(\delta_A\)'s and the Cartan matrix of \(\mathfrak{g}\) has been determined in \([33]\), and recently reviewed in many cases in App. A of \([34]\). It should also be recalled that a neat derivation of the Betti numbers \(2\delta_A - 1 = 2j_A + 1\) of semisimple Lie groups is presented in \([85]\).

### Appendix B. Semispin Groups

Among the groups of type \(D_{2m}\) there are four interesting compact forms which we will shortly describe here \([86]\).

The first one is the spin group \(Spin(4m)\) that is a double covering of \(SO(4m)\). It is realized as usual as a multiplicative subgroup of the even Clifford algebra associated to \(\mathbb{R}^{4m}\) with the standard Euclidean product. If \(\{e_1, \ldots, e_{4m}\}\) is the canonical basis of \(\mathbb{R}^{4m}\) (naturally embedded into the Clifford algebra) then, setting \(e := e_1 \cdot \ldots \cdot e_{2m}\), one gets that \(\pm e\) generate the center of \(Spin(4m)\). Since \(e^2 = 1\) the center is thus a group \(\mathbb{Z}_2 \times \mathbb{Z}_2\)

\[
Z = \{1, e\} \times \{1, -e\} \cong \mathbb{Z}_1 \times \mathbb{Z}_2.
\]
Table 14: The orders $\delta_A$ of the Casimir invariant polynomials $\{C_{\delta_A}\}_{A=1,...,l}$, and the Coxeter number $C_G = \max \{\delta_A\}$ for each simple Lie algebra $\mathfrak{g}$. Recall $A_{n-1} = \mathfrak{su}(n)$, $B_n = \mathfrak{so}(2n + 1)$, $C_n = \mathfrak{sp}(2n)$, $D_n = \mathfrak{so}(2n)$. Simply-laced and non-simply-laced $\mathfrak{g}$'s are respectively listed on the left and right hand side.

| $\mathfrak{g}$ | $\delta_A$ | $C_G$ | $\mathfrak{g}$ | $\delta_A$ | $C_G$ |
|----------------|------------|-------|----------------|------------|-------|
| $A_{n-1}$      | $n$        | $n$   | $B_n$          | $2,4,\ldots,2n$ | $2n$  |
| $D_n$          | $2n - 2$   | $2n$  | $C_n$          | $2,4,\ldots,2n$ | $2n$  |
| $\mathfrak{e}_6$ | $12$       | $f_4$ | $\mathfrak{f}_4$ | $2,6,8,12$    | $12$  |
| $\mathfrak{e}_7$ | $18$       | $\mathfrak{g}_2$ | $2,6$ | $6$ |
| $\mathfrak{e}_8$ | $30$       | $\mathfrak{g}_2$ | $2,6$ | $6$ |

Table 15: Decomposition of the adjoint irrep. $(2.11)$ under the maximal embedding $(2.10)$ of $\mathfrak{su}(2)\mathfrak{f}$ into $\mathfrak{g}$ $[34]$.

Z contains a third $\mathbb{Z}_2$ subgroup, the diagonal subgroup $Z_3 = 1, -1$. Thus, one can construct three quotient groups $S(4m)_i := \text{Spin}(4m)/Z_i$. For $m = 2$ the triality provides an isomorphism among the three groups so that $S(8)_1 \cong SO(8)$. But for $m \geq 3$ there is no triality and only $S(4m)_1 \cong S(4m)_2$ are isomorphic. Thus we have two distinct quotients

$$(B.2) \quad SO(4m) \cong S(4m)_3, \quad S_4(4m) := S(4m)_1.$$  

The last one is called the semispin group.

Finally, one can consider the group $PSO(4m) = SO(4m)/\pm I_{4m}$, where $\pm I_{4m}$ is the image of the center of $\text{Spin}(4m)$ in the projection $\text{Spin}(4m) \to SO(4m)$.

**Appendix C. Inverse Cartan Matrices**

For sake of completeness, here we list the inverse Cartan matrices for all simple Lie groups.

$$C^{-1}_{A_n} = \frac{1}{n + 1} \begin{pmatrix} n & n - 1 & n - 2 & \ldots & 3 & 2 & 1 \\ n - 1 & 2(n - 1) & 2(n - 2) & \ldots & 3 \cdot 2 & 2 \cdot 2 & 2 \\ n - 2 & 2(n - 2) & 3(n - 2) & \ldots & 3 \cdot 3 & 3 \cdot 2 & 3 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 3 & 2 \cdot 3 & 3 \cdot 3 & \ldots & (n - 2) \cdot 3 & (n - 2) \cdot 2 & n - 2 \\ 2 & 2 \cdot 2 & 3 \cdot 2 & \ldots & (n - 2) \cdot 2 & (n - 1) \cdot 2 & n - 1 \\ 1 & 2 & 3 & \ldots & n - 2 & n - 1 & n \end{pmatrix}.$$
\( C_{B_n}^{-1} = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 1 & 1 \\
1 & 2 & 2 & \ldots & 2 & 2 & 2 \\
1 & 2 & 3 & \ldots & 3 & 3 & 3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & 2 & 3 & \ldots & n-2 & n-2 & n-2 \\
1 & 2 & 3 & \ldots & n-2 & n-1 & n-1 \\
\end{pmatrix} \),

(C.2)

\( C_{C_n}^{-1} = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 1 & 1/2 \\
1 & 2 & 2 & \ldots & 2 & 2 & 2/2 \\
1 & 2 & 3 & \ldots & 3 & 3 & 3/2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & 2 & 3 & \ldots & n-2 & n-2 & (n-2)/2 \\
1 & 2 & 3 & \ldots & n-2 & n-1 & (n-1)/2 \\
\end{pmatrix} \),

(C.3)

\( C_{D_n}^{-1} = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 1/2 & 1/2 \\
1 & 2 & 2 & \ldots & 2 & 2/2 & 2/2 \\
1 & 2 & 3 & \ldots & 3 & 3/2 & 3/2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1/2 & 2/2 & 3/2 & \ldots & (n-2)/2 & n/4 & (n-2)/4 \\
1/2 & 2/2 & 3/2 & \ldots & (n-2)/2 & n/4 & (n-2)/4 \\
\end{pmatrix} \),

(C.4)

\( C_{G_2}^{-1} = \begin{pmatrix} 2 & 3 \\
1 & 2 \\
\end{pmatrix} \),

(C.5)

\( C_{F_4}^{-1} = \begin{pmatrix} 2 & 3 & 4 & 2 \\
3 & 6 & 8 & 4 \\
2 & 4 & 6 & 3 \\
1 & 2 & 3 & 2 \\
\end{pmatrix} \),

(C.6)

\( C_{E_6}^{-1} = \begin{pmatrix} 4/3 & 5/3 & 2 & 4/3 & 2/3 & 1 \\
5/3 & 10/3 & 4 & 8/3 & 4/3 & 2 \\
2 & 4 & 6 & 4 & 2 & 3 \\
4/3 & 8/3 & 4 & 10/3 & 5/3 & 2 \\
2/3 & 4/3 & 2 & 5/3 & 4/3 & 1 \\
1 & 2 & 3 & 2 & 1 & 2 \\
\end{pmatrix} \),

(C.7)

\( C_{E_7}^{-1} = \begin{pmatrix} 2 & 3 & 4 & 3 & 2 & 1 & 2 \\
3 & 6 & 8 & 6 & 4 & 2 & 4 \\
4 & 8 & 12 & 9 & 6 & 3 & 6 \\
3 & 6 & 9 & 15/2 & 5 & 5/2 & 9/2 \\
2 & 4 & 6 & 5 & 4 & 2 & 3 \\
1 & 2 & 3 & 5/2 & 2 & 3/2 & 3/2 \\
2 & 4 & 6 & 9/2 & 3 & 3/2 & 7/2 \\
\end{pmatrix} \),

(C.8)
Here we show the Dynkin diagrams for all the simple groups, with the fundamental representations corresponding to the simple weights associated to the simple roots.

**A**$_n$:

**B**$_n$:

**C**$_n$:

**D**$_n$:
Appendix E. The Satake Type Vectors

Here we give a complete list of the vectors with entry 1 if corresponding to a white dot of the Satake diagram and zero otherwise. In the formulas $\bar{e}_i$, $i = 1, \ldots, n$ indicates the canonical basis of $\mathbb{R}^n$, where $n$ is the rank of the group $G_{nc}$ from which the NISS is realized, indicated in parenthesis. For the meaning of the indices $n$, $p$, $k$, refer to Table 3.

\begin{align*}
\bar{e}_{AI}(n) &= \sum_{i=1}^{n} \bar{e}_i, \\
\bar{e}_{AII}(2k-1) &= \sum_{i=1}^{k-1} \bar{e}_{2i}, \\
\bar{e}_{AIII}(2n-1) &= \sum_{i=1}^{2n-1} \bar{e}_i,
\end{align*}

(E.1) (E.2) (E.3)
(E.4) \[
\bar{\varepsilon}_{AII}(2p-1) = \sum_{i=1}^{p} (\bar{\varepsilon}_1 + \bar{\varepsilon}_{n-i}),
\]

(E.5) \[
\bar{\varepsilon}_{AIV}(n) = \bar{\varepsilon}_1 + \bar{\varepsilon}_n,
\]

(E.6) \[
\bar{\varepsilon}_{Bl_a(n)} = \sum_{i=1}^{n} \bar{\varepsilon}_i,
\]

(E.7) \[
\bar{\varepsilon}_{Bl_b(n)} = \sum_{i=1}^{p} \bar{\varepsilon}_i,
\]

(E.8) \[
\bar{\varepsilon}_{BI}(n) = \bar{\varepsilon}_1,
\]

(E.9) \[
\bar{\varepsilon}_{Cl}(n) = \sum_{i=1}^{n} \bar{\varepsilon}_i,
\]

(E.10) \[
\bar{\varepsilon}_{CII_a(2k)} = \sum_{i=1}^{k} \bar{\varepsilon}_{2i},
\]

(E.11) \[
\bar{\varepsilon}_{CII_b(2k)} = \sum_{i=1}^{p} \bar{\varepsilon}_{2i},
\]

(E.12) \[
\bar{\varepsilon}_{DI_a(n)} = \bar{\varepsilon}_{DI_b(n)} = \sum_{i=1}^{n} \bar{\varepsilon}_i,
\]

(E.13) \[
\bar{\varepsilon}_{DL_a(n)} = \sum_{i=1}^{p} \bar{\varepsilon}_i,
\]

(E.14) \[
\bar{\varepsilon}_{DII}(n) = \bar{\varepsilon}_1,
\]

(E.15) \[
\bar{\varepsilon}_{DIII_a(2k+1)} = \bar{\varepsilon}_{2k+1} + \sum_{i=1}^{k} \bar{\varepsilon}_{2i},
\]

(E.16) \[
\bar{\varepsilon}_{DIII_b(2k)} = \sum_{i=1}^{k} \bar{\varepsilon}_{2i},
\]

(E.17) \[
\bar{\varepsilon}_{G(2)} = \bar{\varepsilon}_1 + \bar{\varepsilon}_2,
\]

(E.18) \[
\bar{\varepsilon}_{FI(4)} = \bar{\varepsilon}_1 + \bar{\varepsilon}_2 + \bar{\varepsilon}_3 + \bar{\varepsilon}_4,
\]

(E.19) \[
\bar{\varepsilon}_{FII(4)} = \bar{\varepsilon}_4,
\]

(E.20) \[
\bar{\varepsilon}_{EI(6)} = \bar{\varepsilon}_{EII(6)} = \bar{\varepsilon}_1 + \bar{\varepsilon}_2 + \bar{\varepsilon}_3 + \bar{\varepsilon}_4 + \bar{\varepsilon}_5 + \bar{\varepsilon}_6,
\]

(E.21) \[
\bar{\varepsilon}_{EIII(6)} = \bar{\varepsilon}_1 + \bar{\varepsilon}_5 + \bar{\varepsilon}_6,
\]

(E.22) \[
\bar{\varepsilon}_{EIV(6)} = \bar{\varepsilon}_1 + \bar{\varepsilon}_5,
\]

(E.23) \[
\bar{\varepsilon}_{EV(7)} = \bar{\varepsilon}_1 + \bar{\varepsilon}_2 + \bar{\varepsilon}_3 + \bar{\varepsilon}_4 + \bar{\varepsilon}_5 + \bar{\varepsilon}_6 + \bar{\varepsilon}_7,
\]

(E.24) \[
\bar{\varepsilon}_{EVII(7)} = \bar{\varepsilon}_1 + \bar{\varepsilon}_2 + \bar{\varepsilon}_3 + \bar{\varepsilon}_5,
\]

(E.25) \[
\bar{\varepsilon}_{EVIII(8)} = \bar{\varepsilon}_1 + \bar{\varepsilon}_2 + \bar{\varepsilon}_3 + \bar{\varepsilon}_4 + \bar{\varepsilon}_5 + \bar{\varepsilon}_6 + \bar{\varepsilon}_7 + \bar{\varepsilon}_8,
\]

(E.26) \[
\bar{\varepsilon}_{EIX(8)} = \bar{\varepsilon}_1 + \bar{\varepsilon}_5 + \bar{\varepsilon}_6 + \bar{\varepsilon}_7.
\]
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