Local geometrised Rankin-Selberg method for GL($n$)

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Abstract

Following Laumon [10], to a nonramified $\ell$-adic local system $E$ of rank $n$ on a curve $X$ one associates a complex of $\ell$-adic sheaves $\mathcal{K}_E$ on the moduli stack of rank $n$ vector bundles on $X$ with a section, which is cuspidal and satisfies Hecke property for $E$. This is a geometric counterpart of the well-known construction due to Shalika [17] and Piatetski-Shapiro [16]. We express the cohomology of the tensor product $\mathcal{K}_E \otimes \mathcal{K}_E$ in terms of cohomology of the symmetric powers of $X$. This may be considered as a geometric interpretation of the local part of the classical Rankin-Selberg method for GL($n$) in the framework of the geometric Langlands program.

0.1 General introduction

This is the first in a series of two papers, where we propose a geometric version of the classical Rankin-Selberg method for computation of the scalar product of two cuspidal automorphic forms on GL($n$) over a function field. This geometrization fits in the framework of the geometric Langlands program initiated by V. Drinfeld, A. Beilinson and G. Laumon.

Let $X$ be a smooth, projective, geometrically connected curve over $\mathbb{F}_q$. Let $\ell$ be a prime invertible in $\mathbb{F}_q$. According to the Langlands correspondence for GL($n$) over function fields (proved by L. Lafforgue), to any smooth geometrically irreducible $\mathbb{Q}_\ell$-sheaf $E$ of rank $n$ on $X$ is associated a (unique up to a multiple) cuspidal automorphic form $\varphi_E : \text{Bun}_n(\mathbb{F}_q) \rightarrow \overline{\mathbb{Q}}_\ell$, which is a Hecke eigenvector with respect to $E$. The function $\varphi_E$ is defined on the set $\text{Bun}_n(\mathbb{F}_q)$ of isomorphism classes of rank $n$ vector bundles on $X$.

The classical method of Rankin and Selberg for GL($n$) may be divided into two parts: local and global. The global result calculates for any integer $d$ the scalar product of two (appropriately normalized) automorphic forms

$$\sum_{L \in \text{Bun}_d(\mathbb{F}_q)} \frac{1}{\# \text{Aut} L} \varphi_{E_1}^*(L) \varphi_{E_2}(L),$$

where $\text{Bun}_d(\mathbb{F}_q)$ is the set of isomorphism classes of vector bundles $L$ on $X$ of rank $n$ and degree $d$, and $\# \text{Aut} L$ stands for the number of elements in $\text{Aut} L$. More precisely, this scalar product vanishes if and only if $E_1$ and $E_2$ are non isomorphic. In the case $E_1 \cong E_2 \rightarrow E$ the answer is expressed in terms of the action of the geometric Frobenius endomorphism on $H^1(X \otimes \overline{\mathbb{F}}_q, \mathcal{E}nd_E)$.

The computation of (1) is based on the equality of formal series

$$\sum_{d \geq 0} \sum_{(\Omega^{n-1} \rightarrow L) \in n \cdot \mathcal{M}_d(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(\Omega^{n-1} \hookrightarrow L)} \varphi_{E_1}(L) \varphi_{E_2}(L) t^d = L(E_1^* \otimes E_2, q^{-1} t)$$
Here \( n\mathcal{M}_d(\mathbb{F}_q) \) is the set of isomorphism classes of pairs \((\Omega^{n-1} \to L)\), where \( L \) is a vector bundle on \( X \) of rank \( n \) and degree \( d + n(n - 1)(g - 1) \), and \( \Omega \) is the canonical invertible sheaf on \( X \) \((\Omega^{n-1} \text{ is embedded in } L \text{ as a subsheaf, i.e., the quotient is allowed to have torsion}) \). We have denoted by \( L(E_1^* \otimes E_2, t) \) the \( L \)-function attached to the local system \( E_1^* \otimes E_2 \) on \( X \).

Recall that the existence of the automorphic form \( \varphi_E \) is a descent problem (cf. [1]). Using an explicit construction due to Shalika [17] and Piatetski-Shapiro [14], one associates to a smooth \( \mathbb{Q}_\ell \)-sheaf \( E \) of rank \( n \) on \( X \) a function \( \tilde{\varphi}_E : n\mathcal{M}_d(\mathbb{F}_q) \to \mathbb{Q}_\ell \), which is cuspidal and satisfies Hecke property with respect to \( E \). The Langlands conjecture predicts that when \( E \) is geometrically irreducible, \( \tilde{\varphi}_E \) is constant along the fibres of the projection \( n\mathcal{M}_d(\mathbb{F}_q) \to \text{Bun}_n^{d+n(n-1)(g-1)}(\mathbb{F}_q) \), that is, \( \tilde{\varphi}_E \) is the pull-back of a function \( \varphi_E \) on \( \text{Bun}_n(\mathbb{F}_q) \). So, \((2)\) is a statement independent of the Langlands conjecture. In fact, \((2)\) is of local nature; it is true for any local systems \( E_1 \) and \( E_2 \) of rank \( n \) on \( X \) after replacing \( \varphi_E \) by \( \tilde{\varphi}_E \).

Main result of this paper is a strengthened geometric version of the equality

\[
\sum_{(\Omega^{n-1} \to E) \in n\mathcal{M}_d(\mathbb{F}_q)} \frac{1}{\# \text{Aut}(\Omega^{n-1} \to L)} \tilde{\varphi}_{E_1^*}(L) \tilde{\varphi}_{E_2^*}(L) = q^{-d} \sum_{D \in X^{(d)}(\mathbb{F}_q)} \text{tr}(\text{Fr}, (E_1^* \otimes E_2)^{(d)}_D) \tag{3}
\]

of coefficients in \((2)\) for each \( d \geq 0 \). Here \( X^{(d)} \) is the \( d \)-th symmetric power of \( X \), \( (E_1^* \otimes E_2)^{(d)} \) is a constructable \( \mathbb{Q}_\ell \)-sheaf on \( X^{(d)} \) (cf. Sect. [1]), and \( \text{Fr} \) is the geometric Frobenius endomorphism.

Let \( n\mathcal{M}_d \) denote the moduli stack of pairs \((\Omega^{n-1} \to L)\), where \( L \) is a vector bundle of rank \( n \) and degree \( d + n(n - 1)(g - 1) \) on \( X \), and \( s \) is an inclusion of \( \mathcal{O}_X \)-modules. Following Drinfeld (\( n=2 \)) and Deligne (\( n=1 \), Laumon [11] has defined a complex of \( \mathbb{Q}_\ell \)-sheaves \( n\mathcal{K}'_E^d \) on \( n\mathcal{M}_d \), which is a geometric counterpart of \( \tilde{\varphi}_E \). The geometric Langlands conjecture predicts that when \( E \) is a smooth geometrically irreducible \( \mathbb{Q}_\ell \)-sheaf of rank \( n \) on \( X \), \( n\mathcal{K}'_E^d \) descends with respect to the projection \( n\mathcal{M}_d \to \text{Bun}_n \), where \( \text{Bun}_n \) is the moduli stack of rank \( n \) vector bundles on \( X \).

We establish for any smooth \( \mathbb{Q}_\ell \)-sheaves \( E_1, E_2 \) of rank \( n \) on \( X \) and any \( d \geq 0 \) a canonical isomorphism

\[
\text{R}\Gamma_c(n\mathcal{M}_d, n\mathcal{K}'_E^d \otimes n\mathcal{K}'_E^d) \simeq \text{R}\Gamma(X^{(d)}, (E_1^* \otimes E_2)^{(d)})(d)[2d],
\]

which is a geometric version of \((3)\). In fact, a more general statement is proved.

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### 0.2 Conventions and notation

0.2.1. Fix an algebraically closed ground field \( k \) of characteristic \( p > 0 \), a prime \( \ell \neq p \) and an algebraic closure \( \overline{\mathbb{Q}}_\ell \) of \( \mathbb{Q}_\ell \). All the schemes and stacks we use will be defined over \( k \). Throughout the paper, \( X \) will denote a fixed smooth projective connected curve of genus \( g \geq 1 \) (over \( k \)).

\(^1\)We normalize \( n\mathcal{K}'_E^d \) as in Remark [1] (cf. Sect. 2.1). This also gives a normalization of \( \tilde{\varphi}_E \) as the function ‘trace of Frobenius’ of \( n\mathcal{K}'_E^d \).
We will work with algebraic stacks in smooth topology and with (perverse) \( \mathbb{Q}_\ell \)-sheaves on them. If \( X \) is an algebraic stack locally of finite type then the notion of a (perverse) \( \mathbb{Q}_\ell \)-sheaf on \( X \) localizes in the smooth topology, and hence makes perfect sense. However, the corresponding derived category is problematic. We adopt the point of view that an appropriate formalism exists (it is partially established in [11]). Let \( f : X \to Y \) be a morphism of algebraic stacks. The functors \( f^*, f_*; f_! \) will be understood in the derived category sense.

We say that \( f \) is a generalized affine fibration of rank \( m \) in the following cases. First, if locally in smooth topology on \( Y \) there exists a homomorphism \( L \to L' \) of locally free coherent sheaves on \( Y \) and an \( L' \)-torsor \( Y' \to Y \) such that \( f \) is identified with \( Y'/L \to Y \), the quotient being taken in stack sense, and \( \text{rk } L' - \text{rk } L = m \). Second, if the map \( f \) can be written as the composition of generalized affine fibrations of first type of ranks \( m_1, \ldots, m_k \) with \( \sum m_i = m \). We essentially use the fact that for a generalized affine fibration \( f \) of rank \( m \) one has \( f_! \mathbb{Q}_\ell \cong \mathbb{Q}_\ell(-m)[−2m] \).

We fix a nontrivial additive character \( \psi : \mathbb{F}_p \to \mathbb{Q}_\ell^* \) and denote by \( L_\psi \) the Artin-Schreier sheaf on \( \mathbb{A}_k^1 \) associated to \( \psi \) (SGA4 1/2, [Sommes trig.], 1.7). Fix also a square root of \( p \) in \( \mathbb{Q}_\ell \) and define using it the sheaf \( \mathbb{Q}_\ell(\frac{1}{2}) \) over \( \text{Spec } \mathbb{F}_p \) and, hence, over \( \text{Spec } k \).

0.2.2. When we say that a stack \( Y \) classifies something, it should always be clear what an \( S \)-family of something is for any \( k \)-scheme \( S \), i.e., what is the groupoid \( \text{Hom}(S, Y) \) and what are the functors \( \text{Hom}(S_2, Y) \to \text{Hom}(S_1, Y) \) for each morphism \( S_1 \to S_2 \).

For example, if \( Y \) is the stack that classifies pairs \( M_1 \to M_2 \) with \( M_1 \) (resp. \( M_2 \)) being a coherent sheaf on \( X \) of generic rank \( i_1 \) and of degree \( d_1 \) (resp., of generic rank \( i_2 \) and of degree \( d_2 \)) then \( \text{Hom}(S, Y) \) is the groupoid whose objects are inclusions \( M_1 \to M_2 \) of coherent sheaves on \( S \times X \) that are \( S \)-flat and such that the quotient \( M_2/M_1 \) is also \( S \)-flat, and for any point \( s \in S \) the conditions on the generic rank and on the degree of \( M_i |_{s \times X} \) (i = 1, 2) hold. Morphisms from an object \( M_1 \to M_2 \) to an object \( M'_1 \to M'_2 \) are by definition the isomorphisms \( M_1 \to M'_1 \) and \( M_2 \to M'_2 \) making the natural diagram commutative.

We denote by \( \text{Sh}_0 \) the moduli stack of coherent sheaves on \( X \) of generic rank \( i \). This is an algebraic stack locally of finite type. Its connected components are numbered by \( d \in \mathbb{Z} \): the component \( \text{Sh}_0^d \) classifies coherent sheaves of rank \( i \) and of degree \( d \) on \( X \). The stack \( \text{Sh}_0^d \) is, in fact, of finite type.

By \( \text{Pic } X \subset \text{Sh}_1 \) we denote the open substack classifying invertible \( \mathcal{O}_X \)-modules. This is the Picard stack of \( X \). Its connected component \( \text{Pic}^d \) \( X \) classifies line bundles of degree \( d \) on \( X \).

Denote by \( \text{Sh}_0^d \subset \text{Sh}_0^d \) the open substack given by the property: for a scheme \( S \) an object \( F \) of \( \text{Hom}(S, \text{Sh}_0^d) \) lies in \( \text{Hom}(S, \text{Sh}_0^d) \) if the geometric fibre of \( F \) at any point of \( X \times S \) is of dimension at most \( n \). We write \( X^{(d)} \) for the \( d \)-th symmetric power of \( X \). By \( \text{div} : \text{Sh}_0^d \to X^{(d)} \) is denoted the morphism norm (cf. [8] 6). If \( D_1, \ldots, D_s \) are effective divisors on \( X \) then it sends the \( \mathcal{O}_X \)-module \( \mathcal{O}_{D_1} \oplus \ldots \oplus \mathcal{O}_{D_s} \) to \( D_1 + 2D_2 + \ldots + sD_s \).

0.2.3. Fix the maximal torus of diagonal matrices in \( \text{GL}(n) \) and the Borel subgroup of upper-triangular matrices. Then the set of weights of \( \text{GL}(n) \) is identified with \( \mathbb{Z}^n \). The fundamental weights are given by \( w_i = (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{Z}^n \), where \( 1 \) occurs \( i \) times \((i = 1, \ldots, n)\).

Define the following semigroups \( \Lambda^+_n \subset \Lambda_n \subset \Lambda_n^p \) consisting of weights. Let \( \Lambda_n = \mathbb{Z}^n \) and \( \Lambda_n^p = \{ \lambda \in \mathbb{Z}^n \mid \lambda_1 + \ldots + \lambda_i \geq 0 \text{ for all } i \} \). The superscript \( p \) should designate that \( \Lambda_n^p \) contains
the \( \mathbb{Z}_+ \)-span of positive roots. Set also \( \Lambda^+_\mu = \{ \lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0) \mid \lambda_i \in \mathbb{Z} \} \). Similarly, we let \( \Lambda^-_\mu = \{ \lambda = (0 \leq \lambda_1 \leq \ldots \leq \lambda_n) \mid \lambda_i \in \mathbb{Z} \} \).

For \( d \geq 0 \) we also introduce \( \Lambda_{n,d} \subseteq \Lambda_n \), \( \Lambda^-_{n,d} \subseteq \Lambda^-_n \) and so on, where the subscript \( d \) means that we impose the condition \( \sum \lambda_i = d \). The half sum of positive roots is denoted by \( \rho \).

For a weight \( \lambda \) of \( \text{GL}(n) \) we introduce the schemes \( X^+_\lambda, X^-_\lambda, X^\lambda \) and \( X^\lambda_\rho \) that should be thought of as the moduli schemes of \( \Lambda^+_\mu \) (resp., of \( \Lambda^-_\mu, \Lambda_\mu, \Lambda^\mu_\rho \)) -valued divisors on \( X \) of degree \( \lambda \). The precise definition is as follows.

Set \( X^\lambda_p = \prod_{i=1}^n X^{(\lambda_1+\ldots+\lambda_i)} \). A point of \( X^\lambda_p \) is a collection of (not necessarily effective) divisors \( (D_1, \ldots, D_n) \) on \( X \) with \( D_1 + \ldots + D_i \in X^{(\lambda_1+\ldots+\lambda_i)} \). Let \( X^\lambda \hookrightarrow X^\lambda_p \) be the closed subscheme given by \( D_i \geq 0 \) for all \( i \). Let \( X^\lambda_+ \) (resp., \( X^\lambda_- \)) be the closed subscheme of \( X^\lambda \) given by \( D_1 \geq \ldots \geq D_n \) (resp, \( D_1 \leq \ldots \leq D_n \)).

Given a closed point \( (D_i) \) of \( X^\lambda_p \) with \( D_i = \sum_x d_{i,x} x \), we associate to it a divisor on \( X \) with values in \( \Lambda^\mu_n \). The value of this divisor at \( x \) is the weight \( (d_{1,x}, \ldots, d_{n,x}) \). In the same way a closed point of \( X^\lambda \) (resp., \( X^\lambda_+, X^\lambda_- \)) can be viewed as a \( \Lambda_n \) (resp., \( \Lambda^+_n, \Lambda^-_n \)) -valued divisor on \( X \).

0.2.4. For \( \lambda \in \Lambda^+_{n,d} \) define the polynomial functor \( V^\lambda \) of a \( \mathbb{Q}_l \)-vector space \( V \) as follows. Let \( \lambda = (\lambda_1, \ldots, \lambda_{n'}, 0, \ldots, 0) \) with \( \lambda_{n'} > 0 \). Denote by \( U^\lambda \) the irreducible representation of \( S_n \) (over \( \mathbb{Q} \)) associated to \( \lambda \). So, for example, if \( \lambda = (d, 0, \ldots, 0) \) then \( U^\lambda \longrightarrow \mathbb{Q} \) is trivial, and if \( \lambda = (1, \ldots, 1) \) then \( U^\lambda \) is the signature representation. Set

\[
V^\lambda = (V^\otimes d \otimes \mathbb{Q} U^\lambda)^{S_d},
\]

where it is understood that \( S_d \) acts by permutations on \( V^\otimes d \) and diagonally on the tensor product. If \( m = \dim V < n' \) then \( V^\lambda = 0 \), otherwise \( V^\lambda \) is the irreducible representation of \( \text{GL}(V) \) of the highest weight \( (\lambda_1, \ldots, \lambda_{n'}, 0, \ldots, 0) \in \Lambda^+_{m,d} \).

1 Laumon’s perverse sheaf \( \mathcal{L}^d_E \)

Let \( E \) be a smooth \( \mathbb{Q}_l \)-sheaf on \( X \). Recall the definition of Laumon’s perverse sheaf \( \mathcal{L}^d_E \) on \( \text{Sh}_d \) associated to \( E \) (\( \square \)). Denote by \( \text{sym} : X^d \rightarrow X^{(d)} \) the natural map and consider the smooth \( \mathbb{Q}_l \)-sheaf \( E^{\square d} \) on \( X^d \). Notice that \( \text{sym}_i(E^{\square d})(d) \) is a perverse sheaf. Set

\[
E^{(d)} = (\text{sym}_i(E^{\square d}))^{S_d}
\]

Since \( E^{(d)} \) is a direct summand of \( \text{sym}_i(E^{\square d}) \), \( E^{(d)}(d) \) is also a perverse sheaf.

Denote by \( \mathcal{F}^{1^{\ldots,1}} \) (1 occurs \( d \) times) the stack of complete flags \( (F_1 \subset \ldots \subset F_d) \), where \( F_i \) is a coherent torsion sheaf on \( X \) of length \( i \). The morphism \( p : \mathcal{F}^{1^{\ldots,1}} \rightarrow \text{Sh}_0^d \) that sends \( (F_1 \subset \ldots \subset F_d) \) to \( F_d \) is representable and proper. The morphism \( q : \mathcal{F}^{1^{\ldots,1}} \rightarrow \text{Sh}_0^1 \times \ldots \times \text{Sh}_0^d \) that sends \( (F_1 \subset \ldots \subset F_d) \) to \( (F_1, F_2/F_1, \ldots, F_d/F_{d-1}) \) is a generalized affine fibration. This, in particular, implies that \( \mathcal{F}^{1^{\ldots,1}} \) is smooth.

Springer’s sheaf \( \text{Spr}^d_E \) on \( \text{Sh}_0^d \) is defined as

\[
\text{Spr}^d_E = p q^*(\text{div}^\times d)^*(E^{\square d})
\]
Since $p$ is small, $Sp^d_E$ is a perverse sheaf that coincides with the Goresky-MacPherson extension of its restriction to any nonempty open substack of $Sh^d_0$. It also carries a natural $S_r$-action (cf. Theorem 3.3.1 of [10]). Set
\[
L^d_E = \text{Hom}_{S_d}(\text{triv}, \sp^d_E),
\]
where triv denotes the trivial representation of the symmetric group $S_d$. Again, $L^d_E$ is a direct summand of $\sp^d_E$, so $L^d_E$ is perverse and coincides with the Goresky-MacPherson extension of its restriction to any nonempty open substack of $Sh^d_0$. We have a smooth morphism $X^{(d)} \to Sh^d_0$ that sends a divisor $D$ to $O_D$, and the pull-back of $L^d_E$ under this map is identified with $E^{(d)}$.

## 2 Main results

2.1 Fix $n > 0, d \geq 0$. Let $\Omega$ be the canonical invertible sheaf on $X$. Denote by $nQ_d$ the stack that classifies collections
\[
(0 = L_0 \subset L_1 \subset \ldots \subset L_n \subset L, \ (s_i)),
\]
where $L_n \subset L$ is a modification of rank $n$ vector bundles on $X$ with $\deg(L/L_n) = d$, $(L_i)$ is a complete fiber of subbundles on $L_n$, and $s_i : \Omega^{n-i} \to L_i/L_{i-1}$ is an isomorphism ($i = 1, \ldots, n$). We have a map $\mu : nQ_d \to \mathbb{A}^1_k$ which at the level of $k$-points sends the above collection to the sum of $n-1$ classes in
\[
k \mapsto \text{Ext}^1(\Omega^{n-i-1}, \Omega^{n-i}) \to \text{Ext}^1(L_{i+1}/L_i, L_i/L_{i-1})
\]
that correspond to the succesive extensions $0 \to L_i/L_{i-1} \to L_{i+1}/L_{i-1} \to L_{i+1}/L_i \to 0$.

Let $\beta : nQ_d \to \leq n Sh^d_0$ be the map that sends $[\mathbb{L}]$ to $L/L_n$. It is of finite type and smooth of relative dimension $b = b(n, d) = nd + (1-g) \sum_{i=1}^{n-2} i^2$. Therefore, $nQ_d$ is smooth and of finite type. So, if $E$ is a smooth $\mathbb{Q}_r$-sheaf on $X$ then on $nQ_d$ we have a perverse $\mathbb{Q}_r$-sheaf
\[
nE^{d,\psi}_E = \beta^*E^d_E \otimes \mu^*L_\psi[b](\frac{b}{2})
\]

Let $\pi_0 : nQ_d \to X^{(d)}$ be the map that sends $[\mathbb{L}]$ to the divisor $D \in X^{(d)}$ for which the inclusion of invertible sheaves $\land^nL_n \hookrightarrow \land^nL$ induces an isomorphism $\land^nL_n(D) \to \land^nL$. We also have a map $X^{(d)} \to \text{Pic}^d X$ that sends a divisor $D$ to $O_X(D)$.

Let $\mathcal{M}_d$ be the stack classifying pairs $(\Omega^{n-1} \hookrightarrow L)$, where $L$ is an $n$-bundle on $X$ with $\deg L - \deg((\Omega^{n-1}+(n-2)+\ldots+(n-n)) = d$.

The forgetful map $\zeta : nQ_d \to \mathcal{M}_d$ is representable, and the following diagram commutes:
\[
\begin{array}{ccc}
nQ_d & \xrightarrow{\pi_0} & X^{(d)} \\
\downarrow \zeta & & \downarrow \\
n\mathcal{M}_d & \xrightarrow{\theta} & \text{Pic}^d X,
\end{array}
\]
where \( \theta \) is the map that sends \( (\Omega^{n-1} \to L) \) to \( \det L \otimes \Omega^{(1-n)+(2-n)+\ldots+(n-n)} \). Denote by
\[
\pi : n \mathcal{Q}_{d} \times n \mathcal{M}_{d} n \mathcal{Q}_{d} \to X^{(d)} \times_{\text{Pic}^{d} X} X^{(d)}
\]
the morphism \( \pi_0 \times \pi_0 \). Since \( X^{(d)} \to \text{Pic}^{d} X \) is representable and separated, the diagonal map \( i : X^{(d)} \to X^{(d)} \times_{\text{Pic}^{d} X} X^{(d)} \) is a closed immersion. Our main result is the next theorem.

**Main Local Theorem.** For any smooth \( \mathbb{Q}_{\ell} \)-sheaves \( E, E' \) on \( X \) of ranks \( m, m' \) respectively with \( \min\{m, m'\} \leq n \) there exists a canonical isomorphism
\[
\pi_{(\mathcal{F}_{E,\psi} \boxtimes \mathcal{F}_{E',\psi^{-1}})} \cong i_{*}(E \otimes E')^{(d)}(d)[2d]
\]
in the derived category on \( X^{(d)} \times_{\text{Pic}^{d} X} X^{(d)} \).

**Remark 1.**

i) The stack \( n \mathcal{Q}_{d} \times n \mathcal{M}_{d} n \mathcal{Q}_{d} \) is of finite type, though \( n \mathcal{M}_{d} \) is not, so that \( \pi \) is of finite type but not representable.

ii) Define the complex \( n \mathcal{K}_{E}^{d} \) on \( n \mathcal{M}_{d} \) as \( n \mathcal{K}_{E}^{d} = \zeta(n \mathcal{F}_{E,\psi}^{d}) \). The geometric Langlands conjecture claims that if \( E \) is a smooth irreducible \( \mathbb{Q}_{\ell} \)-sheaf of rank \( n \) on \( X \) then for each \( d \geq 0 \) the complex \( n \mathcal{K}_{E}^{d} \) descends with respect to the projection \( n \mathcal{M}_{d} \to \text{Bun}_{n} \).

2.2 Actually we prove a more general statement. Recall that for \( \mathbb{Q}_{\ell} \)-vector spaces \( E, E' \) of dimensions \( m, m' \) respectively we have
\[
\text{Sym}^{d}(E \otimes E') = \bigoplus_{\lambda \in \Lambda_{r,d}^{+}} E^{\lambda} \otimes (E')^{\lambda},
\]
where \( r = \min\{m, m'\} \). To formulate the version of Main Local Theorem we actually prove, we globalize the above equality as follows.

For \( \lambda \in \Lambda_{r,d}^{+} \) and a smooth \( \mathbb{Q}_{\ell} \)-sheaf \( E \) on \( X \) we define a constructable \( \mathbb{Q}_{\ell} \)-sheaf \( E_{\lambda}^{+} \) on \( X_{\lambda}^{+} \) (cf. Sect. 3.1), which is a global analog of the corresponding polynomial functor. The fibre of \( E_{\lambda}^{+} \) at \( D = \sum \lambda_{x} \) \( x \) is the tensor product over closed points of \( X \)
\[
\otimes_{x \in X} (E_{x})^{\lambda_{x}},
\]
where \( E_{x} \) denotes the fibre of \( E \) at \( x \). For example, for \( \lambda = (d, 0, \ldots, 0) \) we have \( X_{\lambda}^{+} = X^{(d)} \) and \( E_{\lambda}^{+} = E^{(d)} \). Another example, for \( \lambda = \omega_{i} \) we obtain \( X_{\omega_{i}}^{+} = X \) and \( E_{\omega_{i}}^{+} = \omega_{i} E \).

Denote by \( \pi^{\lambda} : X_{\lambda}^{+} \to X^{(d)} \) the map that sends \( (D_{1} \geq \ldots \geq D_{n} \geq 0) \) \( \in X_{\lambda}^{+} \) to \( \sum D_{i} \).

**Lemma 1.** For any smooth \( \mathbb{Q}_{\ell} \)-sheaves \( E, E' \) on \( X \) of ranks \( m, m' \) resp. there is a canonical filtration
\[
0 = \leq 0(E \otimes E')^{(d)} \subset \leq 1(E \otimes E')^{(d)} \subset \ldots
\]
on \( (E \otimes E')^{(d)} \) by constructable subsheaves with the following property. First, if \( \min\{m, m'\} \leq n \) then \( \leq n(E \otimes E')^{(d)} = (E \otimes E')^{(d)} \). Secondly, there is a canonical refinement of this filtration such that
\[
\text{gr} \leq n(E \otimes E')^{(d)} \cong \bigoplus_{\lambda \in \Lambda_{r,d}^{+}} \pi^{\lambda}_{*}(E_{\lambda}^{+} \otimes E_{\lambda}^{+})
\]
for each \( n \).
Main Local Theorem. For any smooth $\mathbb{Q}_l$-sheaves $E, E'$ on $X$ there exists a canonical isomorphism

$$\pi_!([n]^{d}_E) \boxtimes [n]^{d}_{E'} \to \iota_*([n]^{E \otimes E'}_d)(d)[2d]$$

in the derived category on $X^{(d)} \times_{\text{Pic}^d} X^{(d)}$.

2.3 The proof consists of the following steps. Let us denote by $\mathcal{X}_d$ the stack classifying collections $(L, (t_i))$, where $L$ is a vector bundle on $X$ of rank $n$,

$$t_i : \Omega^{(n-1)+(n-2)+\ldots+(n-i)} \hookrightarrow \wedge^i L$$

is an inclusion of $\mathcal{O}_X$-modules $(i = 1, \ldots, n)$, and $\deg L - \deg(\Omega^{(n-1)+(n-2)+\ldots+(n-n)}) = d$.

Given an object of $n\mathcal{Q}_d$, we get the morphisms

$$t_i : \Omega^{(n-1)+\ldots+(n-i)} \hookrightarrow \wedge^i L_i \hookrightarrow \wedge^i L$$

This defines a map $\varphi : n\mathcal{Q}_d \to n\mathcal{X}_d$. Notice that $\zeta : n\mathcal{Q}_d \to n\mathcal{M}_d$ factors as $n\mathcal{Q}_d \xrightarrow{\varphi} n\mathcal{X}_d \to n\mathcal{M}_d$, where the second arrow is the forgetful map. Since $n\mathcal{X}_d \to n\mathcal{M}_d$ is representable and separated, the natural map $n\mathcal{Q}_d \times_{n\mathcal{X}_d} n\mathcal{Q}_d \to n\mathcal{Q}_d \times_{n\mathcal{M}_d} n\mathcal{Q}_d$ is a closed immersion. Let

$$\pi' : n\mathcal{Q}_d \times_{n\mathcal{X}_d} n\mathcal{Q}_d \to X^{(d)} \times_{\text{Pic}^d} X^{(d)}$$

be the restriction of $\pi$ to $n\mathcal{Q}_d \times_{n\mathcal{X}_d} n\mathcal{Q}_d$. The first step is to establish the following result.

Theorem A. For any smooth $\mathbb{Q}_l$-sheaves $E, E'$ on $X$ the natural map

$$\pi_!([n]^{d}_E) \boxtimes [n]^{d}_{E'} \to \pi_!([n]^{d}_E) \boxtimes [n]^{d}_{E'}$$

is an isomorphism.

Our proof of Theorem A will be based on Proposition [1], which is a corollary of the geometric Casselman-Shalika formula for GL($n$) (cf. [14, 15, 16]). We present it in Sect. 3 written independently of the rest of the paper.

The second step is as follows. Let $\tilde{\phi} : n\mathcal{X}_d \to X^{(d)}$ be the map that sends $(L, (t_i))$ to the divisor $D \in X^{(d)}$ such that $t_n$ induces an isomorphism

$$\Omega^{(n-1)+\ldots+(n-n)}(D) \cong \wedge^n L,$$

so that $\tilde{\phi} \circ \varphi = \pi_0$. We will write $f : n\mathcal{Q}_d \times_{n\mathcal{X}_d} n\mathcal{Q}_d \to X^{(d)}$ for the composition $n\mathcal{Q}_d \times_{n\mathcal{X}_d} n\mathcal{Q}_d \to n\mathcal{X}_d \xrightarrow{\tilde{\phi}} X^{(d)}$, where the first map is the natural projection. The morphism $f$ is of finite type but not representable. Since the diagram

$$\begin{array}{ccc}
n\mathcal{Q}_d \times_{n\mathcal{X}_d} n\mathcal{Q}_d & \xrightarrow{f} & n\mathcal{Q}_d \times_{n\mathcal{M}_d} n\mathcal{Q}_d \\
\downarrow & & \downarrow \pi \\
X^{(d)} & \xrightarrow{\iota} & X^{(d)} \times_{\text{Pic}^d} X^{(d)}
\end{array}$$

commutes, Main Local Theorem is just a combination of Theorem A with the following result.
Theorem B. For any smooth $\bar{\mathcal{Q}}_\ell$-sheaves $E, E'$ on $X$ there is a canonical isomorphism
\[
fi((nF_{E, \psi} \boxtimes nF'_{E', \psi^1}) \tilde{\to} \leq^n(E \otimes E')^{(d)}(d)[2d]
\] (5)

We present two different proofs of Theorem [3]. In the first proof, which occupies Sect. 6.1 through 6.5, we derive Theorem B from the following result.

Theorem C. Let $\leq^n \text{div} : \leq^n \text{Sh}_0^d \to X^{(d)}$ denote the restriction of $\text{div} : \text{Sh}_0^d \to X^{(d)}$. For any smooth $\bar{\mathcal{Q}}_\ell$-sheaves $E, E'$ on $X$ the complex $(\leq^n \text{div})_!(L_E^d \otimes L_{E'}^d)$ is placed in degrees $\leq -2d$, and for the highest cohomology sheaf of this complex we have canonically
\[
R^{-2d}(\leq^n \text{div})_!(L_E^d \otimes L_{E'}^d)(-d) \tilde{\to} \leq^n(E \otimes E')^{(d)}
\]

In Sect. 6.6 we present an alternative proof of Theorem [3]. The idea of this proof was communicated to the author by D. Gaitsgory. This proof requires the additional assumption: $\min\{\text{rk} E, \text{rk} E'\} \leq n$. The reader interested in the proof of Mail Local Theorem under this assumption may skip Sect. 6.1 through 6.5.

3 Around the geometric Casselman-Shalika formula for $\text{GL}(n)$

3.1 The purpose of Sect. 3 is to present Proposition 1, which is a corollary of the geometric Casselman-Shalika formulae for $\text{GL}(n)$ (cf. [14, 3, 13]). To formulate it we introduce some notation.

Fix $\lambda \in \Lambda^-_{n,d}$. Recall that $X^\lambda$ is the scheme of collections $(D_1, \ldots, D_n)$, where $D_i$ is an effective divisor on $X$ of degree $\lambda_i$ with $D_1 \leq \ldots \leq D_n$. Let
\[
i_\lambda : X^\lambda \to \leq^n \text{Sh}_0^d
\]
be the map that sends $(D_1, \ldots, D_n)$ to
\[
\Omega^{n-1}(D_1)/\Omega^{n-1} \oplus \Omega^{n-2}(D_2)/\Omega^{n-2} \oplus \ldots \oplus \mathcal{O}(D_n)/\mathcal{O}
\]
According to ([14], Theorem 3.3.8), if $E$ is a smooth $\bar{\mathcal{Q}}_\ell$-sheaf on $X$ then the complex $i_\lambda^* L_E^d$ is placed in degrees $\leq 2a(\lambda)$ with respect to the usual t-structure, where
\[
a(\lambda) \overset{\text{def}}{=} < \lambda, (n-1, n-2, \ldots, 0) >
\]
Moreover, if $m \in \mathbb{N}$ is such that $\lambda = (0, \ldots, 0, \lambda_{n-m+1}, \ldots, \lambda_n)$ with $\lambda_{n-m+1} > 0$ then $2a(\lambda)$-th cohomology sheaf of $i_\lambda^* L_E^d$ vanishes if and only if $\text{rk} E < m$.

For a weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ set $\lambda^t = (\lambda_n, \ldots, \lambda_1)$. Denote also by $t : X^\lambda \xrightarrow{\sim} X^{\lambda^t}$ the isomorphism that sends $(D_1, \ldots, D_n)$ to $(D_n, \ldots, D_1)$.

Definition 1. For any smooth $\bar{\mathcal{Q}}_\ell$-sheaf $E$ on $X$ define the sheaf $E^\lambda_+$ on $X^\lambda_+$ by
\[
E^\lambda_+ = \mathcal{H}^{2a(\lambda)}(i^* L_E^d)(a(\lambda))
\]
Define also the sheaf $E^{\lambda^t}_+$ on $X^{\lambda^t}_+$ by $E^{\lambda^t}_+ = t_+ E^\lambda_+$.
Let \( S^\lambda \to X^\lambda \) be the vector bundle whose fibre over \((D_1, \ldots, D_n)\) is the vector space of collections \((\sigma_1, \ldots, \sigma_{n-1})\), where

\[
\sigma_i \in \text{Hom}(\Omega^{n-i-1}, \Omega^{n-i}(D_i)/\Omega^{n-i})
\]

By \( \mu_S : S^\lambda \to \mathbb{A}^1 \) we will denote the map that at the level of \( k \)-points sends \((\sigma_1, \ldots, \sigma_{n-1})\) to the sum of \( n-1 \) classes in \( k \tilde{\to} \text{Ext}^1(\Omega^{n-i-1}, \Omega^{n-i}) \) corresponding to the pull-backs of

\[
0 \to \Omega^{n-i} \to \Omega^{n-i}(D_i) \to \Omega^{n-i}(D_i)/\Omega^{n-i} \to 0
\]

with respect to \( \sigma_i : \Omega^{n-i-1} \to \Omega^{n-i}(D_i)/\Omega^{n-i} \).

Let \( W^\lambda \) be the stack of collections: \((D_1, \ldots, D_n) \in X^\lambda \) and a flag \((F^1 \subset \ldots \subset F^n)\) of coherent torsion sheaves on \( X \) with trivializations

\[
F^i/F^{i-1} \tilde{\to} \Omega^{n-i}(D_i)/\Omega^{n-i}
\]

for \( i = 1, \ldots, n \). The projection \( \tau : W^\lambda \to X^\lambda \) is a generalized affine fibration of rank zero.

Let \( \kappa : W^\lambda \to S^\lambda \) be the morphism over \( X^\lambda \) defined as follows. Given an \( S \)-point of \( W^\lambda \), consider for \( i = 1, \ldots, n-1 \) the exact sequence

\[
0 \to \Omega^{n-i-1} \to \Omega^{n-i-1}(D_{i+1}) \to \Omega^{n-i-1}(D_{i+1})/\Omega^{n-i-1} \to 0
\]

(Here \( \Omega \) should be understood as the sheaf of relative differentials \( \Omega_{S \times X/S} \)). It induces a map

\[
\text{Hom}(\Omega^{n-i-1}, \Omega^{n-i}(D_i)/\Omega^{n-i}) \to \text{Ext}^1(\Omega^{n-i-1}(D_{i+1})/\Omega^{n-i-1}, \Omega^{n-i}(D_i)/\Omega^{n-i})
\]

which is an isomorphism of \( \mathcal{O}_{S \times X} \)-modules, because \( D_{i+1} \geq D_i \). The map \( \kappa \) sends this point of \( W^\lambda \) to \((\sigma_1, \ldots, \sigma_{n-1})\), where \( \sigma_i \) is the global section of \( \text{Hom}(\Omega^{n-i-1}, \Omega^{n-i}(D_i)/\Omega^{n-i}) \), whose image under \( \tilde{\text{Hom}} \) corresponds to the extension

\[
0 \to F^i/F^{i-1} \to F^{i+1}/F^{i-1} \to F^{i+1}/F^i \to 0
\]

Denote also by \( \beta_W : W^\lambda \to \leq^n \text{Sh}_d \) the morphism that sends \((F^1 \subset \ldots \subset F^n)\) to \( F^n \).

**Proposition 1.** For any smooth \( \mathbb{Q}_l \)-sheaf \( E \) on \( X \) there is a canonical isomorphism

\[
\pi (\beta_W^* \mathcal{L}_E \otimes \kappa^* \mu_S^* \mathcal{L}_\omega) \tilde{\to} E^\lambda
\]

(7)

The proof is given in Sect. 3.3.

3.2. Local lemma

For the convenience of the reader, we begin with a local counterpart of Proposition [4] working completely in a local setting.

Let \( \mathcal{O} \) be a complete local \( k \)-algebra with residue field \( k \), which is regular of dimension one
(that is, choosing a generator \( \omega \) of the maximal ideal \( m \subset \mathcal{O} \), one identifies \( \mathcal{O} \) with the ring \( k[[\omega]] \) of formal power series of one variable). Denote by \( K \) the field of fractions of \( \mathcal{O} \). Let \( \Omega \) be the
completed module of relative differentials of $\mathcal{O}$ over $k$ (so, $\Omega$ is a free $\mathcal{O}$-module generated by $d\omega$). For $i \geq 0$ we write $\Omega^i$ for the $i$-th tensor power of $\Omega$ (over $\mathcal{O}$). For an integer $m$ denote by $\Omega^i(m) \subset \Omega^i \otimes K$ the $\mathcal{O}$-submodule generated by $\omega^{-m}d\omega \otimes 1$.

Recall that we have fixed $\lambda \in \Lambda^-_{n,d}$. Consider the stack $W^\lambda_{\text{loc}}$ classifying collections: a flag of torsion sheaves $(F^1 \subset \ldots \subset F^n)$ over $\text{Spf} \mathcal{O}$ with trivializations

$$F^i/F^{i-1} \cong \Omega^{n-i}(\lambda_i)/\Omega^{n-i}$$

for $i = 1, \ldots, n$. (The subscript ‘loc’ will stand for local counterparts of certain stacks or morphisms). Clearly, $W^\lambda_{\text{loc}} \to \text{Spec} \ k$ is a generalized affine fibration of rank zero. We also have the scheme $S^\lambda_{\text{loc}}$ whose set of $k$-points is the set of $(\sigma_1, \ldots, \sigma_{n-1})$ with

$$\sigma_i \in \text{Hom}(\Omega^{n-i-1}, \Omega^{n-i}(\lambda_i)/\Omega^{n-i})$$

Besides, we have a map $(\mu_S)_{\text{loc}}: S^\lambda_{\text{loc}} \to \mathbb{A}^1$ that at the level of $k$-points sends $(\sigma_1, \ldots, \sigma_{n-1})$ to $\sum \text{res} \sigma_i$. One also defines a morphism $\kappa_{\text{loc}}: W^\lambda_{\text{loc}} \to S^\lambda_{\text{loc}}$ in the same way as $\kappa$.

Let $\mathcal{S}^\lambda_{n,d}(\mathcal{O})$ be the stack classifying coherent torsion sheaves $F$ on $\text{Spf} \mathcal{O}$ of length $d$ for which $\dim(\mathcal{F} \otimes \mathcal{O} k) \leq n$. It is stratified by locally closed substacks $\mathcal{S}^{\nu}(\mathcal{O})$ indexed by $\nu \in \Lambda^+_{n,d}$. The stratum $\mathcal{S}^{\nu}(\mathcal{O})$ classifies sheaves isomorphic to

$$\mathcal{O}/m^\nu_1 \oplus \ldots \oplus \mathcal{O}/m^\nu_n$$

Let $\mathcal{B}_\nu$ be the intersection cohomology sheaf associated to the constant sheaf on the stratum $\mathcal{S}^{\nu}(\mathcal{O})$. Let

$$\beta_{W,\text{loc}}: W^\lambda_{\text{loc}} \to \mathcal{S}^\lambda_{n,d}(\mathcal{O})$$

be the map sends $(F^1 \subset \ldots \subset F^n)$ to $F^n$.

Local version of Proposition \[ can be stated as follows.

**Lemma 2.** For any $\nu \in \Lambda^+_{n,d}$ we have canonically

$$\text{RG}^\ast_{\nu}(W^\lambda_{\text{loc}}, \beta_{W,\text{loc}}^\ast \mathcal{B}_\nu \otimes (\mu_S)_{\text{loc}}^\ast \mathcal{L}_\nu) \cong \begin{cases} 0, & \text{if } \nu' \neq \lambda \\ \mathbb{Q}_d[-d](a(\lambda)), & \text{if } \nu' = \lambda \end{cases}$$

**Proof** Denote by $\tilde{W}^\lambda_{\text{loc}}$ the stack of collections: a flag of torsion sheaves $(\tilde{F}^1 \subset \ldots \subset \tilde{F}^n)$ on $\text{Spf} \mathcal{O}$ with trivializations

$$\tilde{F}^i/\tilde{F}^{i-1} \cong \Omega^{n-i}/\Omega^{n-i}(-\lambda_{n-i+1})$$

for $i = 1, \ldots, n$. We have an isomorphism $W^\lambda_{\text{loc}} \cong \tilde{W}^\lambda_{\text{loc}}$ that sends $(F^1 \subset \ldots \subset F^n)$ to the flag $(\tilde{F}^1 \subset \ldots \subset \tilde{F}^n)$ with

$$\tilde{F}^i = \mathcal{E}_X^1(F^n/F^{n-i}, \Omega^{n-1})$$

for $i = 1, \ldots, n$. This duality allows to switch between dominant and anti-dominant weights of $\text{GL}(n)$.
Put \( \tilde{L}_i = \Omega^{n-1} \oplus \ldots \oplus \Omega^{n-i} \) for \( i = 1, \ldots, n \). Denote by \( \mathcal{G}r^{d,+}(\tilde{L}_n) \) the moduli scheme of \( \mathcal{O} \)-sublattices \( \mathcal{R} \subset L_n \) such that

\[
\dim(\tilde{L}_n/\mathcal{R}) = d
\]

Chosing a trivialization \( \tilde{L}_n \cong \mathcal{O}^n \), one identifies this scheme with the connected component \( \mathcal{G}r^{d,+} = \mathcal{G}r^{d,+}(\mathcal{O}^n) \) of the positive part of the affine grassmanian for \( \text{GL}(n) \).

We have a locally closed subscheme \( \tilde{S} \rightarrow \mathcal{G}r^{d,+}(\tilde{L}_n) \) whose set of \( k \)-points consists of \( \mathcal{R} \) with the following property. If \( \mathcal{R}_i = \mathcal{R} \cap L_i \) then the image of the inclusion \( \mathcal{R}_i/\mathcal{R}_{i-1} \hookrightarrow \Omega^{n-i} \) is \( \Omega^{n-i}(\lambda_{n-i+1}) \) for \( i = 1, \ldots, n \).

We have a map \( \tilde{\eta}_{\text{loc}} : \tilde{S} \rightarrow \mathcal{W}_\text{loc}^\lambda \) given by \( \tilde{F}_i = L_i/\mathcal{R}_i \) for \( i = 1, \ldots, n \). One checks that \( \tilde{\eta}_{\text{loc}} \) is an affine fibration of rank \( a(\lambda^i) \). We also have a smooth and surjective map

\[
\mathcal{G}r^{d,+}(\tilde{L}_n) \rightarrow \mathcal{S}_0^d(\mathcal{O})
\]

that sends \( \mathcal{R} \subset \tilde{L}_n \) to \( \tilde{L}_n/\mathcal{R} \), and we denote by \( \mathcal{G}r^\nu(\tilde{L}_n) \) the preimage of the stratum \( \text{Sh}_\nu(\mathcal{O}) \) under this map. The Goresky-MacPherson extension of \( \mathcal{Q}_\ell[2(\nu, \rho)](\langle \nu, \rho \rangle) \) from \( \mathcal{G}r^\nu(\tilde{L}_n) \) to its closure is a perverse sheaf denoted \( \mathcal{A}_\nu \) (cf. [14, 6, 15]). So, our assertion is nothing else but the geometric Casselman-Shalika formulae (cf. [2, 14, 15]):

\[
\mathcal{R}_\Gamma_c(\tilde{S}, \mathcal{A}_\nu \otimes \tilde{\eta}_{\text{loc}}^* \kappa_{\text{loc}}^i(\mu_S)^i_{\text{loc}} \mathcal{L}_\psi) \cong \left\{ \begin{array}{ll} 0, & \text{if } \nu^i \neq \lambda \\ \mathcal{Q}_\ell[-2(\nu, \rho)](\langle \nu, \rho \rangle), & \text{if } \nu^i = \lambda \end{array} \right.
\]

\[\square\]

We will need the above lemma in a bit different form. Put \( L_i = \Omega^{n-1}(\lambda_1) \oplus \ldots \oplus \Omega^{n-i}(\lambda_i) \) for \( i = 1, \ldots, n \). Let \( \mathcal{G}r^{d,+}(L_n) \) be the moduli scheme of \( \mathcal{O} \)-sublattices \( \mathcal{R} \subset L_n \) such that \( \dim(L_n/\mathcal{R}) = d \). As in the proof of Lemma 2, on \( \mathcal{G}r^{d,+}(L_n) \) we get a stratification by locally closed subschemes \( \mathcal{G}r^\nu(L_n) \) indexed by \( \nu \in \Lambda^+_n \) and the perverse sheaves \( \mathcal{A}_\nu \).

By \( S \subset \mathcal{G}r^{d,+}(L_n) \) we denote the locally closed subscheme whose set of \( k \)-points consists of sublattices \( \mathcal{R} \) with the following property. Let \( \mathcal{R}_i = \mathcal{R} \cap L_i \). The condition is that the image of the inclusion

\[
\mathcal{R}_i/\mathcal{R}_{i-1} \hookrightarrow L_i/L_{i-1} \rightarrow \Omega^{n-i}(\lambda_i)
\]

is the sublattice \( \Omega^{n-i} \subset \Omega^{n-i}(\lambda_i) \) for \( i = 1, \ldots, n \). We also have a map \( \eta_{\text{loc}} : S \rightarrow \mathcal{W}_\text{loc}^\lambda \) given by \( F_i = L_i/\mathcal{R}_i \) for \( i = 1, \ldots, n \). This is an affine fibration of rank \( a(\lambda) \).

Let \( \text{Spec } k \rightarrow S \) be the distinguished point that corresponds to \( \mathcal{R} = \Omega^{n-1} \oplus \ldots \oplus \mathcal{O} \). Put \( \mu_{\text{loc}} = (\mu_S)_{\text{loc}} \circ \kappa_{\text{loc}} \circ \eta_{\text{loc}} \). We will use Lemma 2 under the following form.

**Lemma 3.** For any \( \nu \in \Lambda^+_n \) we have canonically

\[
\mathcal{R}_\Gamma_c(S, \mathcal{A}_\nu \otimes \mu^*_{\text{loc}} \mathcal{L}_\psi) \cong \left\{ \begin{array}{ll} 0, & \text{if } \nu^i \neq \lambda \\ \mathcal{Q}_\ell[-2(\lambda, \rho)](\langle \lambda, \rho \rangle), & \text{if } \nu^i = \lambda \end{array} \right.
\]

Moreover, this isomorphism is obtained by applying the functor \( \mathcal{R}_\Gamma_c \) to the composition of the canonical maps

\[
\mathcal{A}_\nu \otimes \mu^*_{\text{loc}} \mathcal{L}_\psi \rightarrow i_* i^*(\mathcal{A}_\nu \otimes \mu^*_{\text{loc}} \mathcal{L}_\psi) \rightarrow \tau_{\geq 2(\lambda, \rho)}(i_* i^*(\mathcal{A}_\nu \otimes \mu^*_{\text{loc}} \mathcal{L}_\psi))
\]
Proof As is easy to see, if $\nu^t \neq \lambda$ then the fibre $i^*(A_\nu \otimes \mu^*_\text{loc} L_\psi)$ is placed in (usual) degrees strictly less then $2\langle \lambda, \rho \rangle$. For $\nu^t = \lambda$ this fibre equals

$$\mathbb{Q}_\ell[-2\langle \lambda, \rho \rangle](-\langle \lambda, \rho \rangle)$$

Since the closure of $Gr^\lambda(L_n)$ in $Gr^{d,+}(L_n)$ is the union of strata $Gr^\nu(L_n)$ with $\nu \leq \lambda^t$, our assertion follows from Lemma 2 combined with the geometric statement due to B.C.Ngo (cf. [15], Lemma 5.2, p.14):

i) $S \cap Gr^\nu(L_n) = \emptyset$ for $\nu < \lambda^t$

ii) $S \cap Gr^\nu(L_n)$ is the point $\text{Spec} k \hookrightarrow S$ for $\nu = \lambda^t$

□

Remark 2. The shift in degree and the twist is calculated using the following two formulas. For any weight $\nu$ of $\text{GL}(n)$ we have $a(\nu) - a(\nu^t) = 2\langle \nu, \rho \rangle$ and $2a(\nu) - 2\langle \nu, \rho \rangle = d(n - 1)$, where $d = \sum_{i} \nu_i$.

3.3 Proof of Proposition 3

We return to our notation in the global case (as in Sect. 3.1).

Step 1. Denote by $\mathcal{W}^\lambda$ the scheme of collections: $(D_1, \ldots, D_n) \in X^\lambda$, a digram

$$L_1 \subset \ldots \subset L_n$$

$$\cup \quad \cup$$

$$\mathcal{R}_1 \subset \ldots \subset \mathcal{R}_n,$$

where $L_i = \Omega^{n-1}(D_1) \oplus \ldots \oplus \Omega^{n-i}(D_i)$ for $i = 1, \ldots, n$, and $(\mathcal{R}_i)$ is a complete flag of vector subbundles on an n-bundle $\mathcal{R}_n$ such that the natural map

$$\mathcal{R}_i/\mathcal{R}_{i-1} \hookrightarrow L_i/L_{i-1} \sim \Omega^{n-i}(D_i)$$

induces an isomorphism $\mathcal{R}_i/\mathcal{R}_{i-1} \sim \Omega^{n-i}$. We have a map $\eta: \mathcal{W}^\lambda \to \mathcal{W}^\lambda$ over $X^\lambda$ given by

$$F^i = L_i/\mathcal{R}_i$$

for $i = 1, \ldots, n$. This is an affine fibration of rank $a(\lambda)$, so that $\eta_\mathbb{Q}_\ell \sim \mathbb{Q}_\ell (-a(\lambda))[-2a(\lambda)]$. Put $\tilde{\tau} = \tau \circ \eta$. We will replace the functor $\tau_!(\cdot)$ by

$$\tilde{\tau} \eta^*(\cdot)(a(\lambda))[2a(\lambda)]$$

The advantage is that $\tilde{\tau}$ is representable whence $\tau$ is not.

The morphism $\tilde{\tau}: \mathcal{W}^\lambda \to X^\lambda$ admits a canonical section $\xi: X^\lambda \to \mathcal{W}^\lambda$ defined by

$$\mathcal{R}_i = \Omega^{n-1} \oplus \ldots \oplus \Omega^{n-i}$$
for \(i = 1, \ldots, n\). Notice that \(\xi\) is a closed immersion. The following diagram commutes

\[
\begin{array}{ccc}
\mathcal{W}^\lambda & \xrightarrow{\eta} & \mathcal{W}^\lambda \\
\downarrow \beta_W & & \uparrow \xi \\
\leq^n \text{Sh}_0 & \xrightarrow{i^\lambda} & X^\lambda
\end{array}
\]

Besides, the composition

\[
X^\lambda \xrightarrow{\xi} \mathcal{W}^\lambda \xrightarrow{\eta} W^\lambda \xrightarrow{\kappa^* \mu_S^* \mathcal{L}_\psi} S^\lambda \mu_S^* \mathbb{A}^1
\]

is the zero map. Now applying the functor \(\tilde{n}\) to the canonical morphism

\[
\eta^* \left( \beta_W^* \mathcal{L}_E^d \otimes \kappa^* \mu_S^* \mathcal{L}_\psi \right) \to \xi^* \xi^* \left( \beta_W^* \mathcal{L}_E^d \otimes \kappa^* \mu_S^* \mathcal{L}_\psi \right)
\]

we get a map

\[
\tilde{n} \left( \beta_W^* \mathcal{L}_E^d \otimes \kappa^* \mu_S^* \mathcal{L}_\psi \right) \to i^*_\lambda \mathcal{L}_E^d(a(\lambda))[2a(\lambda)]
\]

Define (7) as the composition of (9) with the canonical map

\[
i^*_\lambda \mathcal{L}_E^d(a(\lambda))[2a(\lambda)] \to \mathcal{H}^{2a(\lambda)}(i^*_\lambda \mathcal{L}_E^d)(a(\lambda))
\]

Now we check that (7) is an isomorphism fibre by fibre.

**Step 2.** Fix a \(k\)-point \((D_1, \ldots, D_n)\) of \(X^\lambda\). Let \(D_i = \sum_x \lambda_{i,x} x\). So, the corresponding \(\Lambda_n^-\)-valued divisor on \(X\) associates to \(x \in X\) the anti-dominant weight

\[
\lambda_x = (\lambda_{1,x}, \ldots, \lambda_{n,x}) \in \Lambda_n^{-,d_x}
\]

with \(d_x = \sum_i \lambda_{i,x}\).

For \(i = 1, \ldots, n\) put \(L_i = \Omega^{n-1}(D_1) \oplus \ldots \oplus \Omega^{n-i}(D_i)\). For every closed point \(x \in X\) let

\[
((L_1)_x \subset \ldots \subset (L_n)_x)
\]

be the restriction of the flag \((L_1 \subset \ldots \subset L_n)\) to \(\text{Spec} \hat{O}_{X,x}\).

Let \(G_{r}^{d_x,+(L_n)_x}\) denote the moduli scheme of sublattices \(\mathcal{R} \subset (L_n)_x\) such that

\[
\dim((L_n)_x/\mathcal{R}) = d_x
\]

By \(S_x \subset G_{r}^{d_x,+(L_n)_x}\) we will denote the locally closed subscheme whose set of \(k\)-points consists of sublattices \(\mathcal{R} \subset (L_n)_x\) with the following property. Let \(\mathcal{R}_i = \mathcal{R} \cap (L_i)_x\). The condition is that the image of the natural inclusion

\[
\mathcal{R}_i/\mathcal{R}_{i-1} \hookrightarrow (L_i)_x/(L_{i-1})_x \xrightarrow{\sim} (\Omega^{n-i}(\lambda_{i,x} x))_x
\]

is the sublattice \(\Omega^{n-i}_x \subset (\Omega^{n-i}(\lambda_{i,x} x))_x\) for \(i = 1, \ldots, n\).
For every \( x \in X \) one defines the stack \( \mathcal{W}_x^\lambda \), the scheme \( S_x^\lambda \) with morphisms
\[
S_x \xrightarrow{\eta_x} \mathcal{W}_x^\lambda \xrightarrow{\kappa_x} S_x^{(\lambda(x))} \hookrightarrow \mathbb{A}^1,
\]
which are local counterparts of the corresponding stacks and morphisms for the weight \( \lambda(x) \). So, after the base change \(( D_1, \ldots, D_n ) : \text{Spec} \ k \to X^\lambda \), the diagram
\[
\mathcal{W}^\lambda \xrightarrow{\eta} \mathcal{W}^\lambda \xrightarrow{\kappa} \mathcal{S}^\lambda
\]
becomes
\[
\prod_{x \in X} S_x \to \prod_{x \in X} \mathcal{W}_x^\lambda \to \prod_{x \in X} S_x^\lambda,
\]
the morphisms being the product of morphisms \( \eta_x \) and \( \kappa_x \) respectively. The restriction of \( \mu_S : S^\lambda \to \mathbb{A}^1 \) to \( \prod_{x \in X} S_x^\lambda \) is the sum of morphisms \( (\mu_S)_x \).

For \( \nu \in \Lambda^+_{n,d} \), the perverse sheaf \( \mathcal{A}_\nu \) considered as a sheaf on \( \mathcal{G}^{d_1,+,d_2}((L_n)_x) \) will be denoted \( \mathcal{A}^\nu, x \) (cf. Sect. 3.2). From (\cite{5}, Proposition 3.1 and Lemma 4.2) it follows that the restriction of \( \eta^* \beta^* \mathcal{L}_E \) to \( \prod_{x \in X} S_x \) is identified with \( \bigoplus \mathcal{A}_\nu, x \) under the inclusion \( S_x \hookrightarrow \mathcal{G}^{d_1,+,d_2}((L_n)_x) \) (the sum being taken over \( \nu \in \Lambda^+_{n,d} \)).

Now combining Lemma 3 with (Theorem 3.3.8, \cite{10}), we get the desired assertion. This concludes the proof of Proposition 1.

4 Geometric Whittaker models for \( \text{GL}(n) \)

4.1 The stack \( n \mathcal{Y}_d \)

Consider the stack \( n \mathcal{X}_d \) defined in Sect. 2.3. We impose Plücker’s relations on a point \((L, (t_i))\) of \( n \mathcal{X}_d \), which mean that generically \((t_i)\) come from a complete flag of vector subbundles of \( L \). Our definition is justified by the following simple observation.

Let \( V \) be a vector space of dimension \( n \) (over any field). For \( n \geq k > i \geq 1 \) let \( \alpha_{k,i} : \wedge^k V \otimes \wedge^i V \to \wedge^{k+1} V \otimes \wedge^{i-1} V \) be the contraction map that sends \( u \otimes (v_1 \wedge v_2 \wedge \ldots \wedge v_i) \) to
\[
\sum_{j=1}^i (-1)^j (u \wedge v_j) \otimes (v_1 \wedge \ldots \wedge \hat{v}_j \wedge \ldots \wedge v_i)
\]

Lemma 4. Given nonzero elements \( t_i \in \wedge^i V \) for \( 1 \leq i \leq n \), the following are equivalent:
1) There exists a complete flag of vector subspaces \( 0 = V_0 \subset V_1 \subset \ldots \subset V_n = V \) such that \( t_i \in \wedge^i V_i \subset \wedge^i V \),
2) For \( n \geq k > i \geq 1 \) we have \( \alpha_{k,i}(t_k \otimes t_i) = 0 \).
Proof The statement is obvious in characteristic zero. Let us give an argument that holds in any characteristic. Write \( e_1 \wedge \ldots \wedge e_i \) for the image of \( e_1 \otimes \ldots \otimes e_i \) under \( V^{\otimes i} \to \wedge^i V \).

We construct by induction on \( k \) the elements \( e_1, \ldots, e_k \in V \) such that \( t_i = e_1 \wedge \ldots \wedge e_i \) for \( i = 1, \ldots, k \). Let \( e_1 = t_1 \), and assume that \( e_1, \ldots, e_{k-1} \) are already constructed.

To construct \( e_k \), we show by induction on \( i \) that \( t_k = e_1 \wedge \ldots \wedge e_i \wedge \omega_{k-i} \) for some \( \omega_{k-i} \in \wedge^{k-i} V \), and define \( e_k \) as \( \omega_1 \).

First, since \( \alpha_{k,1}(t_k \otimes t_1) = -t_k \wedge e_1 = 0 \), we get \( t_k = e_1 \wedge \omega_{k-1} \) for some \( \omega_{k-1} \in \wedge^{k-1} V \). Now assume that \( t_k = e_1 \wedge \ldots \wedge e_{i-1} \wedge \omega_{k-i+1} \) for some \( \omega_{k-i+1} \in \wedge^{k-i+1} V \) with \( i < k \). Then

\[
\alpha_{k,i}(t_k \otimes t_i) = \alpha_{k,i}(t_k \otimes (e_1 \wedge \ldots \wedge e_i)) = (-1)^i (t_k \wedge e_i) \otimes (e_1 \wedge \ldots \wedge e_{i-1}) = 0
\]

It follows that \( t_k \wedge e_i = 0 \). So, there exists \( \omega_{k-i} \in \wedge^{k-i} V \) such that \( t_k = e_1 \wedge \ldots \wedge e_i \wedge \omega_{k-i} \). We are done. \( \square \)

Now we define the closed substack \( n\mathcal{Y}_d \hookrightarrow n\mathcal{X}_d \) by the conditions \( \alpha_{k,i}(t_k \otimes t_i) = 0 \) for \( n \geq k > i \geq 1 \), where

\[
\alpha_{k,i} : \wedge^k L \otimes \wedge^i L \to \wedge^{k+1} L \otimes \wedge^{i-1} L
\]

are the contraction maps defined as above. Then the map \( \varphi \) factors through \( nQ_d \to n\mathcal{Y}_d \to n\mathcal{X}_d \).

We stratify \( n\mathcal{Y}_d \) by locally closed substacks \( \mathcal{V}^\lambda_p \subset n\mathcal{Y}_d \) numbered by \( \lambda \in \Lambda^p_{n,d} \). The stratum \( \mathcal{V}^\lambda_p \) is defined by the condition: the degree of the divisor of zeros of \( t_i : \Omega^{(n-1)+(n-i)} \to \wedge^i L \) equals \( \lambda_1 + \ldots + \lambda_i \) for \( i = 1, \ldots, n \). Recall that a point of \( X^\lambda_p \) is a collection of divisors \( (D_1, \ldots, D_n) \) on \( X \) with \( \deg(D_i) = \lambda_i \) and \( D_1 + \ldots + D_i \geq 0 \) for all \( i \). So, the stack \( \mathcal{V}^\lambda_p \) classifies collections:

\[
(0 = L'_0 \subset L'_1 \subset \ldots \subset L'_n = L, (s_i), (D_i) \in X^\lambda_p), \tag{10}
\]

where \( (L'_i) \) is a complete flag of subbundles on a rank \( n \) vector bundle \( L \) and

\[
s_i : \Omega^{(n-1)+(n-i)}(D_1 + \ldots + D_i) \to \wedge^i L'_i
\]

is an isomorphism.

Define the closed substack \( \mathcal{V}^\lambda \hookrightarrow \mathcal{V}^\lambda_p \) as \( \mathcal{V}^\lambda_p \times_{X^\lambda} X^\lambda \). So, if \( \lambda \notin \Lambda_n \) then \( \mathcal{V}^\lambda \) is empty. Notice that the projection \( \mathcal{V}^\lambda_p \times_{n\mathcal{Y}_d} nQ_d \to \mathcal{V}^\lambda_p \) factors through \( \mathcal{V}^\lambda \to \mathcal{V}^\lambda_p \), and for \( \lambda \in \Lambda_{n,d} \) the corresponding morphism \( \mathcal{V}^\lambda_p \times_{n\mathcal{Y}_d} nQ_d = \mathcal{V}^\lambda \times_{n\mathcal{Y}_d} nQ_d \to \mathcal{V}^\lambda \) is an affine fibration of rank \( a(\lambda) \).

4.2 The sheaves \( nP^d_{E,\psi} \)

**Definition 2.** For any smooth \( \mathcal{Q}_E \)-sheaf \( E \) on \( X \) put \( nP^d_{E,\psi} = \varphi!(nP^d_{E,\psi})\).

Clearly, the restriction of \( nP^d_{E,\psi} \) to a stratum \( \mathcal{V}^\lambda_p \) of \( n\mathcal{Y}_d \) vanishes outside the closed substack \( \mathcal{V}^\lambda \) of \( \mathcal{V}^\lambda_p \). For \( \lambda \in \Lambda_{n,d} \) denote by \( nP^\lambda_{E,\psi} \) the restriction of \( nP^d_{E,\psi} \) to \( \mathcal{V}^\lambda \).
Define the closed substack $V^\lambda \hookrightarrow \mathcal{V}^\lambda$ as $\mathcal{V}^\lambda \times_X \lambda$. Recall that the subscheme $X^\lambda$ of $X$ is given by the condition $0 \leq D_1 \leq \ldots \leq D_n$, where $(D_i) \in X^\lambda$. Let $\mu_\lambda : V^\lambda \rightarrow A^1$ be the map that at the level of $k$-points sends $[\mathbf{1}]$ to the sum of $n - 1$ classes in

$$k \sim \text{Ext}^1(\Omega^{n-1}(D_1), \Omega^{n-i}(D_i))$$

corresponding to the pull-backs of the successive extensions

$$0 \rightarrow L_i'/L_{i-1}' \rightarrow L_{i+1}'/L_i' \rightarrow \Omega^{n-i}(D_i) \rightarrow 0$$

with respect to the inclusion $\Omega^{n-i-1}(D_i) \rightarrow \Omega^{n-i-1}(D_{i+1})$.

**Proposition 2.** Let $E$ be a smooth $\overline{\mathbb{Q}}_l$-sheaf on $X$ of rank $m$ and $\lambda \in \Lambda_{n,d}$. Then

1) $nP^\lambda_{E,\psi}$ vanishes unless

$$\lambda_1 = \ldots = \lambda_{n-m} = 0$$

(\*)

2) Under the condition (\*) the complex $nP^\lambda_{E,\psi}$ is supported at $V^\lambda \hookrightarrow \mathcal{V}^\lambda$, its restriction to $V^\lambda$ is isomorphic to the tensor product of

$$\mu_\lambda^* L_\psi \left( \frac{b - 2a(\lambda)}{2} \right) [b - 2a(\lambda)]$$

with the inverse image of $E^\lambda_n$ under $V^\lambda \rightarrow X^\lambda$.

**Remark 3.** i) The sheaf $nP^d_{E,\psi}$ was also considered by Frenkel, Gaitsgory and Vilonen (\cite{6}, 4.3). They show that for any smooth $\overline{\mathbb{Q}}_l$-sheaf $E$ on $X$, $nP^d_{E,\psi}$ is a perverse sheaf and the Goresky-MacPherson extension of its restriction to any nonempty open substack of $nY_d$. Besides, the Verdier dual of $nP^d_{E,\psi}$ is canonically isomorphic to $nP^{d*,\psi-1}$ (4.6, 4.7, loc.cit.). We notice that the stratification of $nY_d$ used in (4.10, loc.cit.) is different from ours, so that our Proposition 2 is a strengthened version of (4.13, loc.cit.).

According to (\cite{6, 7}), in the case $\text{rk } E = n$ the perverse sheaf $nP^d_{E,\psi}$ can be thought of as a geometric counterpart of the Whittaker function canonically attached to $E$.

ii) For $m > 0$ let $mY_d \subset nY_d$ denote the open substack given by the conditions: the image of $t_i$ is a line subbundle in $\wedge^i L$ for $i = 1, \ldots, n - m$. In particular, $mY_d = nY_d$ for $m \geq n$. Then 1) of Proposition 2 claims that $nP^d_{E,\psi}$ is the extension by zero of its restriction to $mY_d \subset nY_d$.

iii) The relation between the sheaves $nP^d_{E,\psi}$ for different $n$ is as follows. Let $mQ_d$ be the preimage of $mY_d$ under $\varphi : nQ_d \rightarrow nY_d$. So, $mQ_d$ is the open substack of $nQ_d$ parametrizing collections of $\text{Ext}^i$ such that $L/L_{n-m}$ is locally free.

Denote by $m\mathcal{E}xt$ the stack of collections $(L_1 \subset \ldots \subset L_{n-m+1} \subset L, (s_i))$, where $L/L_{n-m}$ is a vector bundle on $X$ of rank $m$, and $s_i : \Omega^{n-i} \sim L_i/L_{i-1}$ is an isomorphism $(i = 1, \ldots, n-m+1)$. Let $\mu_n^* : m\mathcal{E}xt \rightarrow A^1$ be the composition $m\mathcal{E}xt \rightarrow n-m+1 \mathbb{Q}_0 \rightarrow A^1$, where the first arrow is the map that forgets $L$.
Let $m\mathcal{M}$ be the stack of pairs $(\Omega^{m-1} \hookrightarrow L)$, where $L$ is a vector bundle on $X$ of rank $m$.

Taking the quotient by $L_{n-m}$, we get a map $m\mathcal{E}xt \to m\mathcal{M}$, which is a generalized affine fibration.

For $1 \leq m \leq n$ there is a commutative diagram

\[
\begin{array}{ccc}
m_n Q_d & \xrightarrow{\sim} & m_n Q_d \times_{m\mathcal{M}} m_n \mathcal{E}xt \\
\downarrow & & \downarrow \varphi \times \text{id} \\
m_n Y_d & \xrightarrow{\sim} & m_n Y_d \times_{m\mathcal{M}} m_n \mathcal{E}xt; \\
\end{array}
\]

where the left vertical arrow is the restriction of $\varphi$. So, the restriction of $n\mathcal{P}_E$ to $m_n Y_d$ is isomorphic to

\[
m_n \mathcal{P}_E \boxtimes (\mu_n)^* L_\psi [b(m,d) - b(n,d)] \left( \frac{b(m,d) - b(n,d)}{2} \right)
\]

4.3 The support of $\mathcal{P}_E^\lambda$

In this subsection we prove the next lemma.

**Lemma 5.** The complex $n\mathcal{P}_E^\lambda$ vanishes outside the closed substack $\mathcal{V}^\lambda \hookrightarrow \mathcal{V}^\nu$.

This may be derived from the geometric Casselman-Shalika formulae, but we will give a direct proof. We start with the following sublemma. Given $\lambda \in \Lambda_n^-$ and $\nu \in \Lambda_n^+$, denote by $U^\nu_\lambda$ the stack of collections: $(D_1, \ldots, D_n) \in X^\lambda$, $(D'_1, \ldots, D'_n) \in X^\nu$, a diagram

\[
L'_1 \subset \ldots \subset L'_n \\
\cup \\
L_1 \subset \ldots \subset L_n,
\]

where $(L_i)$ (resp., $(L'_i)$) is a complete flag of vector subbundles on a rank $n$ vector bundle $L_n$ (resp., $L'_n$) on $X$ with trivializations

\[
L'_i/L_{i-1} \cong \Omega^{n-i}(D_i + D'_i)
\]

such that the image of the inclusion $L_i/L_{i-1} \hookrightarrow L'_i/L'_{i-1} \cong \Omega^{n-i}(D_i + D'_i)$ equals $\Omega^{n-i}(D_i)$ for $i = 1, \ldots, n$. Let

\[
\varphi^\nu_\lambda : U^\nu_\lambda \to (X^\lambda \times X^\nu) \times_{X^{\lambda+\nu}} \mathcal{V}^{\lambda+\nu}
\]

be the map that forgets the flag $(L_i)$. Here $X^\lambda \times X^\nu \to X^{\lambda+\nu}$ denotes the summation of divisors. The map $\varphi^\nu_\lambda$ is an affine fibration of rank $\alpha(\nu)$.

Let $X_{\lambda,\nu} \hookrightarrow X^\lambda \times X^\nu$ be the closed subscheme defined by

\[
D_i \geq D_{i-1} + D'_{i-1}
\]

for $i = 2, \ldots, n$. The composition $X_{\lambda,\nu} \hookrightarrow X^\lambda \times X^\nu \to X^{\lambda+\nu}$ factors through $X^{\lambda+\nu} \hookrightarrow X^{\lambda+\nu}$.

We also have a map $U^\nu_\lambda \to \mathcal{V}^\lambda$ that forgets $(L'_i)$ and $(D'_i)$. By abuse of notation, the composition of this map with $\mu_\lambda : \mathcal{V}^\lambda \to \mathbb{A}^1$ will also be denoted $\mu_\lambda$.
Sublemma 1. The complex $(\phi^*_\lambda)_{\mu^*_\lambda}L_\psi$ is supported at the closed substack $X_{\lambda,\nu} \times X_{\lambda+\nu} \psi^{\lambda+\nu}$ of $(X_{\lambda} \times X_{\nu}) \times X_{\lambda+\nu} \psi^{\lambda+\nu}$, and is isomorphic to the inverse image of

$$\mu^*_{\lambda+\nu}L_\psi[-2a(\nu)](-a(\nu))$$

from $\psi^{\lambda+\nu}$.

Proof. Let us decompose $\varphi^\nu_\lambda$ into two affine fibrations $\mathcal{U}_\lambda^\nu \to \mathcal{U}_\lambda^\nu \to (X_{\lambda} \times X_{\nu}) \times X_{\lambda+\nu} \psi^{\lambda+\nu}$ defined as follows. Let $\mathcal{U}_\lambda^\nu$ be the stack of collections:

- $(D_i) \in X_{\lambda}^x$, $(D_i') \in X_{\nu}^x$
- a complete flag of vector bundles $(L_1^x \subset \ldots \subset L_n^x)$ on $X$ with trivializations

$$L_i^x/L_{i-1}^x \otimes \Omega^{n-i}(D_i + D_i')$$

for $i = 1, \ldots, n$

- for $i = 1, \ldots, n - 1$ diagrams

$$0 \to L_i^x/L_{i-1}^x \cup L_{i+1}^x/L_i^x \cup L_i^x/L_{i-1}^x \to 0$$

where each row is an exact sequence of $O_X$-modules, and both left and right vertical arrows are compatible with (12).

Now define the morphism $\mathcal{U}_\lambda^\nu \to \mathcal{U}_\lambda^\nu$ by $F_i = L_{i+1}^x/L_i^x$ for $i = 1, \ldots, n - 1$, where $L_i$ are from diagram (11). One checks that this is an affine fibration of rank

$$(n - 2)\nu_1 + (n - 3)\nu_2 + \ldots + \nu_{n-2}$$

Define also $\varphi^\nu_\lambda$ as the map that forgets all $F_i$. This is an affine fibration of rank $\nu_1 + \ldots + \nu_{n-1}$.

Clearly, $\mu_\lambda : \mathcal{U}_\lambda^\nu \to \mathbb{A}^1$ is constant along the fibres of $\mathcal{U}_\lambda^\nu \to \mathcal{U}_\nu^\nu$. So, it suffices to prove the sublemma in the case $n = 2$.

In this case a fibre of $\varphi^\nu_\lambda$ is the affine space of maps $\xi : L_2/L_1 \to L_2'/L_1$ such that the diagram commutes

$$0 \to L_1/L_1 \to L_2'/L_1 \to L_2'/L_1 \to 0$$

where $i$ is the canonical inclusion compatible with trivializations. On this affine space we have a free and transitive action of $\text{Hom}(L_2/L_1, L_1'/L_1)$. The restriction of $\mu^*_{\lambda}L_\psi$ to this affine space is a sheaf that changes under the action of $\text{Hom}(L_2/L_1, L_1'/L_1)$ by a local system, say $\tilde{\mu}^*L_\psi$, where

$$\tilde{\mu} : \text{Hom}(L_2/L_1, L_1'/L_1) \to k$$

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is the following linear functional. It associates to \( s \in \text{Hom}(L_2/L_1, L_1'/L_1) \) the class of the pull-back of
\[
0 \to L_1 \to L_1' \to L_1'/L_1 \to 0
\] (13)
under the composition \( \mathcal{O}(D_1) \hookrightarrow \mathcal{O}(D_2) \xrightarrow{s} L_1'/L_1 \). The sequence (13) is just
\[
0 \to \Omega(D_1) \to \Omega(D_1 + D_1') \to \Omega(D_1 + D_1')/\Omega(D_1) \to 0
\]
So, \( \tilde{\mu} = 0 \) if and only if \( D_2 \geq D_1 + D_1' \). Besides, under this condition the pull-back of
\[
0 \to L_1' \to L_2' \to L_2'/L_1' \to 0
\]
under \( \mathcal{O}(D_1 + D_1') \hookrightarrow \mathcal{O}(D_2 + D_2') \xrightarrow{s} L_2'/L_1' \) is identified (after tensoring by \( \mathcal{O}(-D_1') \)) with the pull back of
\[
0 \to L_1 \to L_2 \to L_2'/L_1 \to 0
\]
under \( \mathcal{O}(D_1) \hookrightarrow \mathcal{O}(D_2) \xrightarrow{s} L_2'/L_1 \). Our assertion follows. \( \square \)

For \( m \geq 0 \) and \( \nu \in \Lambda_{m,d} \) denote by \( \mathcal{F}l^\nu \) the stack of flags \( (F^1 \subset \ldots \subset F^m) \), where \( F^i \) is a coherent torsion sheaf on \( X \) with \( \text{deg}(F^i/F^{i-1}) = \nu_i \) for \( i = 1, \ldots, m \). Let \( \text{div}^\nu : \mathcal{F}l^\nu \to X^\nu \) denote the composition
\[
\mathcal{F}l^\nu \to \text{Sh}^\nu_0 \times \ldots \times \text{Sh}^\nu_m \xrightarrow{\text{div} \times \ldots \times \text{div}} X^\nu
\]
Set also \( nQ^\nu = nQ_d \times_{\text{Sh}^d_{0}} \mathcal{F}l^\nu \), where \( \mathcal{F}l^\nu \to \text{Sh}^d_0 \) sends \( (F^1 \subset \ldots \subset F^m) \) to \( F^m \).

Denote by \( \frac{m}{n} J_d \) the set of \( n \times m \)-matrices \( e = (e^i_j) \) \( (1 \leq i \leq n, 1 \leq j \leq m) \) with \( e^i_j \in \mathbb{Z}_+ \), \( \sum_{i,j} e^i_j = d \). We have a map \( h : \frac{m}{n} J_d \to \Lambda_{n,d} \times \Lambda_{m,d} \) that sends \( e \) to \( (\lambda, \nu) \), where \( \lambda_i = \sum_j e^i_j \) and \( \nu_j = \sum_i e^i_j \). For \( e \in \frac{m}{n} J_d \) put \( Y^e = \prod_{i,j} X(e^i_j) \). So, \( Y^e \) classifies matrices of effective divisors \( (D^e_i) \) on \( X \) such that \( \text{deg}(D^e_i) = e^i_j \).

For every \( \lambda \in \Lambda_{n,d} \) the stack \( \mathcal{V}^{\lambda} \times_{\Lambda_{n,d}} nQ^\nu \) is stratified by locally closed substacks \( Q^\nu \hookrightarrow \mathcal{V}^{\lambda} \times_{\Lambda_{n,d}} nQ^\nu \) indexed by \( e \in \frac{m}{n} J_d \) such that \( h(e) = (\lambda, \nu) \). The stratum \( Q^\nu \) is the stack classifying collections:

- a diagram

\[
\begin{array}{cccccc}
L^m_1 & \subset & L^m_2 & \subset & \ldots & \subset & L^m_n \\
\cup & & \cup & & \cup & & \\
L^{m-1}_1 & \subset & L^{m-1}_2 & \subset & \ldots & \subset & L^{m-1}_n \\
\cup & & \cup & & \cup & & \\
\vdots & & \vdots & & \vdots & & \\
L^0_1 & \subset & L^0_2 & \subset & \ldots & \subset & L^0_n
\end{array}
\] (14)

where \( L^i_j \) is a vector bundle of rank \( i \) on \( X \), and all the maps are inclusions of \( \mathcal{O}_X \)-modules

- a matrix \( (D^e_i) \in Y^e \)
isomorphisms $L_i^m/L_{i-1}^m \iso \Omega^{n-i}(D_i^1 + \ldots + D_i^m)$ such that the image of the inclusion

$$L_i^j/L_{i-1}^j \hookrightarrow L_i^m/L_{i-1}^m \iso \Omega^{n-i}(D_i^1 + \ldots + D_i^m)$$

equals $\Omega^{n-i}(D_i^1 + \ldots + D_i^j)$ ($i = 1, \ldots, n; j = 0, \ldots, m$).

We have a natural map

$$\varphi^e : Q^e \to Y^e \times_{X^\lambda} \mathcal{V}^\lambda$$

that forgets all the rows in (14) except the top one. (Here $Y^e \to X^\lambda$ sends $(D_i^j)$ to $(D_i)$ with $D_i = \sum_j D_i^j$). The morphism $\varphi^e$ is an affine fibration of rank $a(\lambda)$.

Denote by $Y^e \hookrightarrow Y^e$ the closed subscheme given by the conditions:

1) for $i \leq n - j$ we have $D_i^j = 0$

2) for $1 \leq j \leq m - 1$ and $2 \leq i \leq n$ we have $D_i^1 + \ldots + D_i^j \geq D_i^{j+1} + \ldots + D_i^{j+1}$

The composition $Q^e \hookrightarrow nQ^e \to nQ_d \mu \to A^1$ will be denoted by $\mu_e$.

**Sublemma 2.** The complex $(\varphi^e)_! \mu_e^* L_\psi$ is supported at $Y^e \times_{X^\lambda} \mathcal{V}^\lambda \hookrightarrow Y^e \times_{X^\lambda} \mathcal{V}^\lambda$ and is isomorphic to the inverse image of

$$\mu_e^* L_\psi (-a(\lambda)) [-2a(\lambda)]$$

from $\mathcal{V}^\lambda$.

**Proof** Apply Sublemma 1 $m$ times forgetting successively the rows in diagram (14) starting from the lowest one and moving up. □

**Proof of Lemma 5**

Since $\mathcal{L}^d_E$ is a direct summand of the Springer sheaf $Spr^d_E$ (cf. Sect. 3), it suffices to show that the restriction of

$$\varphi_!(\beta^* Spr^d_E \otimes \mu^* L_\psi)$$

to $\mathcal{V}^\lambda$ vanishes outside $\mathcal{V}^\lambda$. Put $\nu = (1, \ldots, 1) \in \Lambda_{d,d}$. The composition $nQ^e \to nQ_d \mu \to A^1$ will also be denoted by $\mu$. By the projection formulae, we have to consider the direct image with respect to the projection

$$\mathcal{V}^\lambda \times_n \mathcal{Y}_d nQ^\nu \to \mathcal{V}^\lambda$$

(15)

of $pr_2^* \mu^* L_\psi$ tensored by some local system that comes from $X^\nu$. The stack $\mathcal{V}^\lambda \times_n \mathcal{Y}_d nQ^\nu$ is stratified by locally closed substacks $Q^e$ indexed by $e \in dJ_d$ such that $h(e) = (\lambda, \nu)$. The restriction of (15) to $Q^e$ can be decomposed as

$$Q^e \hookrightarrow Y^e \times_{X^\lambda} \mathcal{V}^\lambda \to X^\nu \times_{X^d(\nu)} \mathcal{V}^\lambda \to \mathcal{V}^\lambda$$

So, our assertion follows from Sublemma 2, because the composition $Y^e \hookrightarrow Y^e \to X^\lambda$ factors through $X^\lambda \hookrightarrow X^\lambda$. □
Remark 4. Using Sublemma 2, one may also check that for any \( \lambda \in \Lambda_{n,d} \) and any smooth \( \Q_{\ell} \)-sheaf \( E \) on \( X \), \( E^\lambda [\lambda_n] \) is a perverse sheaf on \( X^\lambda \).

4.4 Proof of Proposition 3

Recall that \( \mathcal{V}_\lambda \times_{n, \mathcal{V}_d} n \mathcal{Q}_d \) is the stack classifying collections: \( (D_1, \ldots, D_n) \in X^\lambda \), a diagram

\[
L'_1 \subset \ldots \subset L'_n
\]

\[
L_1 \subset \ldots \subset L_n,
\]

where \( (L'_i) \) (resp., \( L_i \)) is a complete flag of vector subbundles on a rank \( n \) vector bundle \( L'_n \) (resp., \( L_n \)) on \( X \) with trivializations

\[
L'_i/L_{i-1} \rightarrow \Omega^{n-i}(D_i)
\]

such that the image of the natural inclusion \( L_i/L_{i-1} \rightarrow L'_i/L_{i-1} \rightarrow \Omega^{n-i}(D_i) \) equals \( \Omega^{n-i} \) for \( i = 1, \ldots, n \). Denote by

\[
\eta : \mathcal{V}_\lambda \times_{n, \mathcal{V}_d} n \mathcal{Q}_d \rightarrow \mathcal{V}_\lambda \times_{X, \lambda} \mathcal{W}^\lambda
\]

the morphism over \( \mathcal{V}_\lambda \), whose composition with the projection \( \mathcal{V}_\lambda \times_{X, \lambda} \mathcal{W}^\lambda \rightarrow \mathcal{W}^\lambda \) sends \( \eta \) to the flag \( (L'_1/L_1 \subset L'_2/L_2 \subset \ldots \subset L'_n/L_n) \). One checks that \( \eta \) is a (representable) affine fibration of rank \( a(\lambda) \). Further, the composition

\[
\mathcal{V}_\lambda \times_{n, \mathcal{V}_d} n \mathcal{Q}_d \overset{\eta}{\rightarrow} \mathcal{V}_\lambda \times_{X, \lambda} \mathcal{W}^\lambda \overset{id \times \kappa}{\rightarrow} \mathcal{V}_\lambda \times_{X, \lambda} S^\lambda \overset{\mu \times \mu_S}{\rightarrow} \mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1
\]

coincides with the restriction of \( \mu : n \mathcal{Q}_d \rightarrow \mathbb{A}^1 \) to the substack \( \mathcal{V}_\lambda \times_{n, \mathcal{V}_d} n \mathcal{Q}_d \rightarrow n \mathcal{Q}_d \). So, our assertion follows from Proposition 3. \( \square \)

5 Proof of Theorem A

Recall the map \( \phi : n \mathcal{X}_d \rightarrow X^{(d)} \) (cf. Sect. 2.3). By abuse of notation, its restriction to \( n \mathcal{V}_d \rightarrow n \mathcal{X}_d \) is also denoted \( \phi \). We let

\[
\tilde{\pi} : n \mathcal{V}_d \times_{n, \mathcal{M}_d} n \mathcal{V}_d \rightarrow X^{(d)} \times_{\mathbb{P}^{d-1}_X} X^{(d)}
\]

be the morphism \( \phi \times \phi \). By \( \tilde{\pi}' \) we will denote the restriction of \( \tilde{\pi} \) to the diagonal \( n \mathcal{V}_d \rightarrow n \mathcal{V}_d \times_{n, \mathcal{M}_d} n \mathcal{V}_d \). Clearly, Theorem A is equivalent to the fact that the natural map

\[
\tilde{\pi}_!(n P^d_{E, \psi} \boxtimes n P^d_{E', \psi^{-1}}) \rightarrow \tilde{\pi}'_!(n P^d_{E, \psi} \boxtimes n P^d_{E', \psi^{-1}})
\]

is an isomorphism. For \( \lambda, \nu \in \Lambda_{n,d} \) we denote by \( \tilde{\pi}^{\lambda, \nu} \) the restriction of \( \tilde{\pi} \) to the substack

\[
\mathcal{V}_\lambda \times_{n, \mathcal{M}_d} \mathcal{V}_\nu \rightarrow n \mathcal{V}_d \times_{n, \mathcal{M}_d} n \mathcal{V}_d
\]

In the case \( \lambda = \nu \) we write \( \tilde{\pi}^{\lambda, \lambda} \) for the restriction of \( \tilde{\pi}^{\lambda, \lambda} \) to the diagonal \( \mathcal{V}_\lambda \rightarrow \mathcal{V}_\lambda \times_{n, \mathcal{M}_d} \mathcal{V}_\lambda \).

Using the stratification of \( n \mathcal{V}_d \times_{n, \mathcal{M}_d} n \mathcal{V}_d \) induced by both stratifications of the first and the second multiple (cf. Sect. 4.1), Theorem A is reduced to the following statement.
Proposition 3. For any \( \lambda, \nu \in \Lambda_{n,d} \) the direct image \( (\tilde{\pi}^{\lambda,\nu})_!(nP_{E,\psi}^\lambda \boxtimes nP_{E',\psi}^\nu) \) vanishes unless \( \lambda = \nu \). Under the condition \( \lambda = \nu \) the natural map

\[
(\tilde{\pi}^{\lambda,\lambda})_!(nP_{E,\psi}^\lambda \boxtimes nP_{E',\psi}^\nu) \rightarrow (\tilde{\pi}^{\lambda,\lambda})_!(nP_{E,\psi}^\lambda \boxtimes nP_{E',\psi}^\lambda)
\]

is an isomorphism.

Proof Put \( V_1 = V_{\lambda}^1 \times_{nM_d} V_{\nu}^1 \). The restriction of \( \tilde{\pi}^{\lambda,\nu} \) to \( V_1 \) can be decomposed as

\[
V_1 \xrightarrow{1^\pi} X_{\lambda}^1 \times_{\text{Pic}^d X} X_{\nu}^1 \rightarrow X^{(d)} \times_{\text{Pic}^d X} X^{(d)},
\]

where \( 1^\pi \) is the product of two projections \( V_{\lambda}^1 \rightarrow X_{\lambda}^1 \) and \( V_{\nu}^1 \rightarrow X_{\nu}^1 \). In the case \( \lambda = \nu \) we denote by \( \text{diag} : V_{\lambda}^1 \hookrightarrow V_1 \) the diagonal map.

By Proposition 3, our assertion is reduced to the next lemma.

Lemma 6. For any \( \lambda, \nu \in \Lambda_{n,d}^{-} \) the direct image \( 1^\pi!(\mu_{\lambda}^*L_\psi \boxtimes \mu_{\nu}^*L_{\psi^{-1}}) \) vanishes unless \( \lambda = \nu \). Under the condition \( \lambda = \nu \) the natural map

\[
1^\pi!(\mu_{\lambda}^*L_\psi \boxtimes \mu_{\lambda}^*L_{\psi^{-1}}) \rightarrow 1^\pi!(\text{diag})_*(\text{diag})^*(\mu_{\lambda}^*L_\psi \boxtimes \mu_{\lambda}^*L_{\psi^{-1}})
\]

is an isomorphism.

We need the next straightforward sublemma. Given a divisor \( D \) and a coherent sheaf \( M \) on \( X \) with a section \( O(D) \xrightarrow{s} M \), denote by \( \mathcal{E}xt_{M,D} \) the stack classifying extensions of \( \mathcal{O}_X \)-modules \( 0 \rightarrow \mathcal{O}(D) \rightarrow ? \rightarrow M \rightarrow 0 \), and by \( \mu_s : \mathcal{E}xt_{M,D} \rightarrow \mathbb{A}^1 \) the map that sends this extension to the class of its pull-back under \( s \).

Sublemma 3. If \( s \neq 0 \) then \( R\Gamma_c(\mathcal{E}xt_{M,D}, \mu_s^*\mathcal{L}_\psi) = 0 \).

Proof of the lemma The stack \( V_1 \) classifies collections: \( (D_1, \ldots, D_n) \in X_{\lambda}^1, (D'_1, \ldots, D'_n) \in X_{\nu}^1 \), two flags \( (L_1 \subset \ldots \subset L_n = L) \) and \( (L'_1 \subset \ldots \subset L'_n = L) \) of subbundles on a rank \( n \) vector bundle \( L \) on \( X \) with trivializations

\[
s_i : \Omega^{n-i}(D_i) \rightarrow L_i/L_{i-1} \text{ and } s'_i : \Omega^{n-i}(D'_i) \rightarrow L'_i/L'_{i-1}
\]

for \( i = 1, \ldots, n \) such that \( (L_1, s_1) \) and \( (L'_1, s'_1) \) coincide (in particular, we have \( D_1 = D'_1 \)).

Let \( V_i \hookrightarrow V_1 \) be the closed substack defined by the condition: the flags

\[
(L_1 \subset \ldots \subset L_i, (s_j)_{j=1,\ldots,i}) \text{ and } (L'_1 \subset \ldots \subset L'_i, (s'_j)_{j=1,\ldots,i})
\]

coincide. Let \( ^iL_\psi \) be the restriction of \( \mu_{\lambda}^*L_\psi \boxtimes \mu_{\nu}^*L_{\psi^{-1}} \) under \( V_i \hookrightarrow V_1 \). Let also

\[
{^i\pi} : V_i \rightarrow X_{\lambda}^1 \times_{\text{Pic}^d X} X_{\nu}^1
\]

be the restriction of \( 1^\pi \) to \( V_i \). Arguing by induction, we will show that for every \( i = 1, \ldots, n-1 \) the natural map

\[
({^i\pi})_!(^iL_\psi) \rightarrow ({}^{i+1}\pi)_!({^{i+1}}L_\psi)
\]

is an isomorphism.
is an isomorphism.

To do so, denote by $N_i$ the stack of collections: $(D_1, \ldots, D_n) \in X^\lambda$, $(D'_1, \ldots, D'_n) \in X'^\nu$, two flags $(M_{i+1} \subset \ldots \subset M_n = M)$ and $(M'_{i+1} \subset \ldots \subset M'_n = M)$ of subbundles on a rank $n - i$ vector bundle $M$ on $X$ with trivializations $s_j : \Omega^{n-j}(D_j) \tilde{\to} M_j/M_{j-1}$ and $s'_j : \Omega^{n-j}(D'_j) \tilde{\to} M'_j/M'_{j-1}$ for $j = i + 1, \ldots, n$. Let also $'N_i \hookrightarrow N_i$ be the closed substack defined by the condition:

$$(M_{i+1}, s_{i+1}) \text{ and } (M'_{i+1}, s'_{i+1})$$

coincide. Taking the quotient by $L_i = L'_i$, we get a morphism $\gamma : V_i \to N_i$, which is a generalized affine fibration. Further, we have a commutative diagram

$$
\begin{array}{ccc}
V_{i+1} & \hookrightarrow & V_i \\
\downarrow & & \downarrow \gamma \\
'N_i & \hookrightarrow & N_i \\
\end{array}
$$

where the square is cartesian. Applying Sublemma 3 for the section $s_{i+1} - s'_{i+1} : \Omega^{n-i-1}(D_i) \to M$, one checks that the complex $\gamma! (i \mathcal{L}_\psi)$ is supported at $'N_i$, and our assertion follows. $\square$

□(Proposition 3)

This concludes the proof of Theorem A.

6 Proof of Theorems B and C

6.1 Plan of the proof

Proposition 2 admits the following corollary.

**Corollary 1.** For any smooth $Q_\ell$-sheaves $E, E'$ on $X$ the complex

$$
n^S_{E,E'} \overset{\text{def}}{=} f_l(n^F_{E,\psi} \boxtimes n^F_{E',\psi-1})(-d)[-2d]
$$

is a sheaf on $X^{(d)}$ placed in degree zero. It has a canonical filtration by constructible subsheaves such that

$$
gr_n^S_{E,E'} = \oplus_{\lambda \in \Lambda^+_n,d} \pi^\lambda_+(E_+ \otimes E'_+^\lambda)
$$

For each $r \leq n$ there is a canonical inclusion $r^S_{E,E'} \subset n^S_{E,E'}$ compatible with filtrations.

**Proof** By the projection formulae, $n^S_{E,E'} \tilde{\to} \phi_l(n^\mathcal{P}_{E,\psi} \otimes n^\mathcal{P}_{E',\psi-1})(-d)[-2d]$. Calculate this direct image with respect to the stratification of $n\mathcal{Y}_d$ by locally closed substacks $\mathcal{Y}_p^\lambda$ indexed by $\lambda \in \Lambda_p^\lambda$ (cf. Sect. 4.1). Since the natural map $\mathcal{Y}_p^\lambda \to X^\lambda$ is a generalized affine fibration of rank $b - d - 2a(\lambda)$, our first assertion follows from Proposition 2.
Recall that we have open substacks \( r_nY_d \subset nY_d \) for \( r \leq n \) (cf. iii) of Remark 3 Sect. 4.2).

Let \( \leq^r \phi \) be the restriction of \( \phi \) to \( r_nY_d \). By loc.cit., \( \leq^r \phi((n\mathcal{P}_F \otimes n\mathcal{P}_E)(-d)[-2d] \to^r \mathcal{S}_E,E' \).

Our second assertion follows. \( \square \)

This reduces our proof of Theorem B to the following steps. For \( \nu \in \Lambda_{m,d}, \nu' \in \Lambda_{m',d} \) and \( c \overset{\text{def}}{=} (\nu, \nu') \) set \( V_c = X' \times_{X(d)} X'^\nu \). Recall our notation \( nQ^\nu = nQ_d \times \text{Sh}_0 \mathcal{F}_d \) (cf. Sect. 4.3). Let

\[
f_c : nQ^\nu \times_{nY_d} nQ'^\nu \to V_c
\]

denote the composition \( nQ^\nu \times_{nY_d} nQ'^\nu \to \mathcal{F}_d \times_{\text{Sh}_0 \mathcal{F}_d} \mathcal{F}_d' \to V_c \). The morphism \( f_c \) is of finite type. Let also \( nF_c \) be the restriction of \( f_c \) to the closed substack

\[
nQ^\nu \times_{nQ_d} nQ'^\nu \subset nQ^\nu \times_{nY_d} nQ'^\nu
\]

First step is as follows.

**Proposition 4.** The morphism \( f_c \) is of relative dimension \( \leq b - d \), and the natural map of the highest cohomology sheaves

\[
R^{2(b-d)}(f_c)_!(\mu^* \mathcal{L}_\psi \otimes \mu^* \mathcal{L}_{\psi-1}) \to R^{2(b-d)}(nF_c)_! \mathcal{Q}_\ell
\]

is an isomorphism.

Further, set \( W_c = \mathcal{F}_d \times_{\text{Sh}_0 \mathcal{F}_d} \mathcal{F}_d' \). Let \( \text{div}^c : W_c \to V_c \) be the map \( \text{div}^{\nu} \times \text{div}^{\nu'} \). Set \( \leq_n W_c = W_c \times_{\text{Sh}_0 \mathcal{F}_d} \leq_n \mathcal{F}_d \). Let also \( \leq_n \text{div}^c \) denote the restriction of \( \text{div}^c \) to \( \leq_n W_c \subset W_c \).

The morphism \( nF_c \) is decomposed as \( nQ^\nu \times_{nQ_d} nQ'^\nu \to \leq_n W_c \to \leq_n \text{div}^c \to V_c \), where \( \beta^c \) is the natural projection. Second step is the next lemma.

**Lemma 7.** i) \( \beta^c \) is smooth and surjective with connected fibres of dimension \( b \).

ii) \( \text{div}^c \) is of relative dimension \( \leq -d \), so that

\[
R^{2(b-d)}(nF_c)_! \mathcal{Q}_\ell(b - d) \to R^{-2d}(\leq_n \text{div}^c)_! \mathcal{Q}_\ell(-d)
\]

Now assume \( c = (\nu, \nu) \) with \( \nu = (1, \ldots, 1) \in \Lambda_{d,d} \). Let \( nS^c_{E,E'} \) denote the direct image under \( V_c \to X^{(d)} \) of the sheaf

\[
R^{2(b-d)}(f_c)_!(\mu^* \mathcal{L}_\psi \otimes \mu^* \mathcal{L}_{\psi-1})(b - d)
\]
tensored by the local system \( (E^{\otimes d}) \otimes (E'^{\otimes d}) \) on \( V_c \). The group \( S_d \times S_d \) acts naturally on \( nS^c_{E,E'} \). By Corolary 3 \( nS^d_{E,E'} \) is the sheaf of \( S_d \times S_d \)-invariants of \( nS^c_{E,E'} \).

Combining Lemma 7 and Proposition 4, we learn that the complex \( (\leq_n \text{div})_!(S_{pr^d_F} \otimes S_{pr^d_{E'}}) \) is placed in degrees \( \leq -2d \), and there is a canonical \( S_d \times S_d \)-equivariant isomorphism

\[
nS^c_{E,E'} \to R^{-2d}(\leq_n \text{div})_!(S_{pr^d_F} \otimes S_{pr^d_{E'}})(-d)
\]
Thus, Theorem [3] is reduced to Theorem [4].

6.2 The stack $\mathcal{Z}_d$

For $0 \leq i \leq n$ denote by $\mathcal{Q}_d$ the stack classifying collections

$$(0 = L_0 \subset L_1 \subset \ldots \subset L_i \subset F, (s_j)),$$

where $F \in \text{Sh}_i$, $(L_j)$ is a complete flag of vector subbundles on a rank $i$ vector bundle $L_i$, $\text{deg}(F/L_i) = d$, and $s_j : \Omega^{i-j} \to L_j/L_{j-1}$ is an isomorphism ($j = 1, \ldots, i$).

We have the open substack $\mathcal{Q}_d \subset \mathcal{Q}_d$ given by the condition: $F$ is locally free. We also have a map $\mathcal{Q}_d \to \text{Sh}_i$ that sends (17) to $F$. Define a substack

$$\mathcal{Z}_d \hookrightarrow \mathcal{Q}_d \times_{\text{Sh}_i} \mathcal{Q}_d$$

as follows. If $S$ is a scheme then an object

$$(F, (L_j, s_j), (L_j', s_j'))$$

of $\text{Hom}(S, \mathcal{Q}_d \times_{\text{Sh}_i} \mathcal{Q}_d)$ lies in $\text{Hom}(S, \mathcal{Z}_d)$ if the collections $(L_j, s_j)$ and $(L_j', s_j')$ coincide outside a closed subscheme of $S \times X$ finite over $S$.

Lemma 8. The map $\mathcal{Z}_d \hookrightarrow \mathcal{Q}_d \times_{\text{Sh}_i} \mathcal{Q}_d$ is a closed immersion. In particular, the stack $\mathcal{Z}_d$ is algebraic.

Proof. An object (18) of $\text{Hom}(S, \mathcal{Q}_d \times_{\text{Sh}_i} \mathcal{Q}_d)$ gives rise to a pair of sections

$$t_j : \Omega^{i-1} \to \ldots \to \Omega^{i-j} L_j \to \Lambda^j F$$

and

$$t_j' : \Omega^{i-1} \to \ldots \to \Omega^{i-j} L_j' \to \Lambda^j F$$

Clearly, (18) lies in $\text{Hom}(S, \mathcal{Z}_d)$ if and only if the support of $t_j - t_j'$ is a closed subscheme of $S \times X$ finite over $S$ (for all $j = 1, \ldots, i$).

Since $F$ is $S$-flat, $F$ (as well as its exterior powers) is locally free outside some closed subscheme of $S \times X$ finite over $S$. So, our assertion is a consequence of the following sublemma, communicated to the author by V. Drinfeld.

Sublemma 4. 1) Let $F$ be any coherent sheaf on $S \times X$, which is locally free outside a closed subscheme of $S \times X$ finite over $S$. Let $s$ be a global section of $F$. Consider the following subfunctor $Z$ of $S$ (on the category of $S$-schemes): a morphism $S' \to S$ belongs to $Z(S')$ if the pull-back of $s$ to $S' \times X$ vanishes outside a closed subscheme of $S' \times X$ finite over $S'$. Then the subfunctor $Z$ is closed.

2) Suppose in addition that $F$ is locally free. Then $S' \to S$ belongs to $Z(S')$ if and only if the pull-back of $s$ to $S' \times X$ vanishes.

25
We stratify and extend to a generalized affine fibration $n$ coincides, where $\det: \text{Sh}_n$ and $(18)$ of $i$ components). L coincide. Notice that taking the quotient by $i$ The map $\beta: nQ_d \to F/L_n$ whose restriction to $nZ_d$ coincides with $f$. We will see that $\tilde{f}$ is of relative dimension $\leq b - d$, but not of finite type (the stack $n\tilde{Z}_d$ even has infinitely many irreducible components).

For $k = 0, \ldots, i$ we have a closed substack $k\tilde{Z}_d \hookrightarrow \tilde{Z}_d$ given by the condition: for a point $(18)$ of $\tilde{Z}_d$ the flags

$$(0) = L_0 \subset \ldots \subset L_k, (s_j)_{j=1,\ldots,k}$$

and

$$(0) = L_0' \subset \ldots \subset L_k', (s_j')_{j=1,\ldots,k}$$

coincide. Notice that taking the quotient by $L_k = L_k'$, one gets a map $k\tilde{Z}_d \to i-k\tilde{Z}_d$, which is a generalized affine fibration. This observation will be a key point in the proof of Proposition $\square$

6.3 Dimensions counting

Proof of Lemma $\square$

i) The map $\beta^e$ is obtained by base change from the map $\beta: nQ_d \to \leq n\text{Sh}_d^d$, which is surjective and extends to a generalized affine fibration $nQ_d \to \text{Sh}_d^d$ that sends $(17)$ to $F/L_n$. ii) We stratify $W^c$ by locally closed substacks $U^e \hookrightarrow W^c$ indexed by $e \in m'J_d$ with $h(e) = (\nu, \nu')$. A point

$$(F^1 \subset \ldots \subset F^m = F, (F^1')' \subset \ldots \subset (F^{m'})' = F)$$

of $W^c$ lies in $U^e$ if

$$\text{deg}(F^j) = \sum_{k \leq i, t \leq j} e_k^j$$

for $1 \leq i \leq m, 1 \leq j \leq m'$. 26
where \( F^j_i = F^j \cap (F^j)' \). If \([13]\) is a point of \( U^e \) then for \( 1 \leq i \leq m, 1 \leq j \leq m' \) define \( \tilde{F}^j_i \in \text{Sh}_0 \) from the cocartesian square

\[
\begin{array}{ccc}
F^j_{i-1} & \rightarrow & \tilde{F}^j_i \\
\uparrow & & \uparrow \\
F^j_{i-1} & \rightarrow & \tilde{F}^j_{i-1}
\end{array}
\]

and put \( G^j_i = F^j_i / \tilde{F}^j_i \). Set also

\[ W^e = \prod_{i,j} \text{Sh}_0^{e_i} \]

The map \( U^e \rightarrow W^e \) that sends \([19]\) to the collection \((G^j_i)\) is a generalized affine fibration of rank zero. We have a map \( W^e \rightarrow Y^e \) that sends \( G^j_i \) to the collection \((\text{div} G^j_i)\), and define \( \text{div} : U^e \rightarrow Y^e \) as the composition \( U^e \rightarrow W^e \rightarrow Y^e \). Since for any \( i \geq 0 \) the morphism \( \text{div} : \text{Sh}_0 \rightarrow X^{(i)} \) is of relative dimension \( \leq -i \), our assertion follows. \( \square \)

Define the stack \( \tilde{\mathcal{Z}}^c \) by the cartesian square

\[
\begin{array}{ccc}
\tilde{i}\tilde{\mathcal{Z}}^c & \rightarrow & \tilde{i}\tilde{\mathcal{Z}}_d \\
\downarrow & & \downarrow \\
\mathcal{F}l^\nu \times_{X^\nu} \mathcal{F}l^\nu' & \rightarrow & \text{Sh}_d \times_{X^\nu} \text{Sh}_d,
\end{array}
\]

where the right vertical arrow sends \([18]\) to \( (F/L_i, F/L_i') \). Let \( \tilde{i}\mathcal{Z}^c \subset \tilde{i}\tilde{\mathcal{Z}}^c \) denote the preimage of \( \tilde{i}\mathcal{Z}_d \) under \( \tilde{i}\tilde{\mathcal{Z}}^c \rightarrow \tilde{i}\tilde{\mathcal{Z}}_d \). In particular, we have \( n\mathcal{Z}^c = n\mathcal{Q}^\nu \times_{n\mathcal{Y}_d} n\mathcal{Q}^\nu' \). Let

\[ \tilde{f}^c : n\tilde{\mathcal{Z}}^c \rightarrow V^c \]

denote the composition \( n\tilde{\mathcal{Z}}^c \rightarrow \mathcal{F}l^\nu \times_{X^\nu} \mathcal{F}l^\nu' \rightarrow \text{div}^\nu \times \text{div}^\nu' \rightarrow V^c \). The restriction of \( \tilde{f}^c \) to \( n\mathcal{Z}^c \) coincides with \( f^c \). Notice that \( \tilde{f}^c \) is locally of finite type, but not of finite type in general.

**Lemma 9.** The map \( \tilde{f}^c \) is of relative dimension \( \leq b - d \).

**Proof**

**Step 1.** The stack \( n\mathcal{Q}^\nu \times_{n\mathcal{Y}_d} n\mathcal{Q}^\nu' \) is stratified by locally closed substacks \( \mathcal{Q}^\nu \times_{\mathcal{Y}_d} \mathcal{Q}^\nu' \) indexed by pairs \( e \in mJ_d, e' \in m'J_d \) such that there exists \( \lambda \in \Lambda_{n,d} \) with \( h(e) = (\lambda, \nu), h(e') = (\lambda, \nu') \). (cf. Sect. 4.3).

The restriction of \( f^c : n\mathcal{Z}^c \rightarrow V^c \) to a stratum \( \mathcal{Q}^e \times_{\mathcal{Y}_d} \mathcal{Q}^{e'} \) is written as the composition

\[
\mathcal{Q}^e \times_{\mathcal{Y}_d} \mathcal{Q}^{e'} \xrightarrow{\varphi \times \varphi'} (Y^e \times_{X^\lambda} Y^{e'}) \times_{X^\lambda} Y^\lambda \rightarrow Y^e \times_{X^\lambda} Y^{e'} \rightarrow V^c
\]

Since \( \varphi : \mathcal{Q}^e \rightarrow Y^e \times_{X^\lambda} Y^\lambda \) is an affine fibration of rank \( a(\lambda) \), and \( Y^\lambda \rightarrow X^\lambda \) is a generalized affine fibration of rank \( b - d - 2a(\lambda) \), it follows that \( f^c \) is of relative dimension \( \leq b - d \).

**Step 2.** Stratify \( \text{Sh}_n \) by fixing the degree of the maximal torsion subsheaf of \( F \in \text{Sh}_n \). Consider the induced stratification of \( n\tilde{\mathcal{Z}}^c \). A stratum \( n\tilde{\mathcal{Z}}^c_k \subset n\tilde{\mathcal{Z}}^c \) classifies data: a point \([18]\) of \( n\tilde{\mathcal{Z}}_d \), two flags of subsheaves

\[
(L_n \subset L^1_n \subset \ldots \subset L^m_n = F)
\]

(20)

27
and

\[(L'_n \subset (L_n^1)' \subset \ldots \subset (L_n^{m'})' = F),\]

and an exact sequence \(0 \rightarrow F_0 \rightarrow F \rightarrow M \rightarrow 0\) of \(\mathcal{O}_X\)-modules, where \(F_0 \in \text{Sh}_0^k\) and \(M\) is a vector bundle on \(X\) of rank \(i\).

The preimages of flags \((20)\) and \((21)\) in \(F_0\) give rise to a point of \(\mathcal{F}l^\lambda \times_{\text{Sh}_0^k} \mathcal{F}l^{\lambda'}\) for some \(\lambda \in \Lambda_{m,k}, \lambda' \in \Lambda_{m',k}\). This yields a stratification of \(n \mathcal{Z}_k^c\) by locally closed substacks \(n \mathcal{Z}_k^c \hookrightarrow n \mathcal{Z}_k^c\) indexed by pairs \(\lambda \in \Lambda_{m,k}, \lambda' \in \Lambda_{m',k}\).

For an object of \(n \mathcal{Z}_{k,\lambda',\nu}\) the vector bundle \(M\) together with the images of the corresponding flags on \(F\) defines a point of \(n \mathcal{Q}^\nu_{\nu-\lambda} \times_{n \mathcal{Y}_{d-k}} n \mathcal{Q}^\nu_{\nu-\lambda'}\). The natural forgetful map

\[n \mathcal{Z}_k^c \hookrightarrow (\mathcal{F}l^\lambda \times_{\text{Sh}_0^k} \mathcal{F}l^{\lambda'}) \times (n \mathcal{Q}^\nu_{\nu-\lambda} \times_{n \mathcal{Y}_{d-k}} n \mathcal{Q}^\nu_{\nu-\lambda'})\]

is a generalized affine fibration of rank \(nk\). Recall that \(b = b(n, d)\) depends on \(n\) and \(d\) (cf. Sect. 2.2). By Step 1,

\[n \mathcal{Q}^\nu_{\nu-\lambda} \times_{n \mathcal{Y}_{d-k}} n \mathcal{Q}^\nu_{\nu-\lambda'} \rightarrow X^{\nu-\lambda} \times_{X^{(d-k)}} X^{\nu-\lambda'}\]

is of relative dimension \(\leq b(n, d - k) - (d - k) = b(n, d) - d - nk + k\). By ii) of Lemma \([\text{ii}]\),

\[\mathcal{F}l^\lambda \times_{\text{Sh}_0^k} \mathcal{F}l^{\lambda'} \rightarrow X^{\lambda} \times_{X^{(k)}} X^{\lambda'}\]

is of relative dimension \(\leq -k\). Our assertion follows. \(\square\)

6.4 Proof of Proposition \([\text{iii}]\)

Let \(k \mathcal{Z}_d\) be the preimage of \(i \mathcal{Z}_d\) under \(k \mathcal{Z}_d \hookrightarrow i \mathcal{Z}_d\). For \(k = 0, \ldots, i\) define the stacks \(k \mathcal{Z}_d \subset k \mathcal{Z}_c\) by the cartesian squares

\[
\begin{array}{ccc}
\mathcal{Z}_d & \rightarrow & \mathcal{Z}_c \\
\downarrow & & \downarrow \\
\mathcal{Z}_d & \rightarrow & \mathcal{Z}_c
\end{array}
\]

Denote by \(i \mathcal{L}_c\) the restriction of \(\mu^* \mathcal{L}_c \boxtimes \mu^* \mathcal{L}_{c-1}\) under the composition

\[i^* \mathcal{Z}_c \hookrightarrow n \mathcal{Z}_c \longrightarrow n \mathcal{Z}_d \longrightarrow n \mathcal{Q}_d \times_{n \mathcal{Y}_d} n \mathcal{Q}_d\]

Let also \(i f^c\) be the restriction of \(f^c\) to \(i^* \mathcal{Z}_c \hookrightarrow n \mathcal{Z}_c\). Arguing by induction, we will show that the natural map

\[R^2(b - d)(i f^c + 1)(i \mathcal{L}_c) \rightarrow R^2(b - d)(i + 1)(i + 1 \mathcal{L}_c)\]

is an isomorphism for \(i = 1, \ldots, n - 1\).

The map \(\mu : n \mathcal{Q}_d \rightarrow \mathbb{A}^1\) extends naturally to a morphism \(n \mathcal{Q}_d \rightarrow \mathbb{A}^1\) defined in the same way, it will also be denoted by \(\mu\). This allows to extend \(i \mathcal{L}_c\) to a local system \(i \mathcal{L}_d\) on \(i \mathcal{Z}_c\), where \(i \mathcal{L}_d\) is defined as the restriction of \(\mu^* \mathcal{L}_c \boxtimes \mu^* \mathcal{L}_{c-1}\) under the composition

\[i^* \mathcal{Z}_d \hookrightarrow n \mathcal{Z}_d \longrightarrow n \mathcal{Q}_d \times_{\text{Sh}_n} n \mathcal{Q}_d\]

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We have the diagram
\[
\begin{array}{c}
\frac{i+1}{n}Z^c \\ \downarrow \alpha \\
\frac{1}{n-i} \tilde{Z}^c \end{array} \rightarrow \begin{array}{c}
\frac{i}{n}Z^c \\ \downarrow \beta \\
n-i \tilde{Z}^c \end{array}
\]
\[
\delta \rightarrow \begin{array}{c}
i \tilde{Z}^c \\ \gamma \end{array}
\end{array}
\]
in which the square is cartesian, and \( \gamma \) is a generalized affine fibration of rank \( b(n, d) - b(n - i, d) \).

Since \( \delta \) is an open immersion,
\[
R^{\top} \beta(i \mathcal{L}_\psi) \rightarrow R^{\top} \gamma(i \tilde{\mathcal{L}}_\psi)
\]
is an isomorphism over the image of (the smooth map) \( \beta \). Applying Sublemma 1 for \( M = F/L_i; D = 0 \), and the section \( s_{i+1} - s'_{i+1} : \Omega^{n-i-1} \rightarrow M \), we learn that \( R^{\top} \gamma(i \tilde{\mathcal{L}}_\psi) \) is supported at \( 1/n-i \tilde{Z}^c \). Therefore,
\[
R^{\top} \beta(i \mathcal{L}_\psi) \rightarrow R^{\top} \alpha(i+1 \mathcal{L}_\psi)
\]
is an isomorphism. So, if \( c \) is decomposed as \( nZ^c \rightarrow n-i \tilde{Z}^c \rightarrow V^c \), where, by Lemma 9, the second map is of relative dimension \( \leq b(n-i, d) - d \).

Though \( n-i \tilde{Z}^c \rightarrow V^c \) is not of finite type in general, its restriction to the image of \( \beta \) is a morphism of finite type.

This concludes the proof of Proposition 4.

6.5 Proof of Theorem C

Recall that we have the map \( h : \frac{m'}{m} J_d \rightarrow \Lambda_m d \times \Lambda_{m', d} \), and for \( e \in \frac{m'}{m} J_d \) we write \( Y^e = \prod_{i,j} X^{(e_i)} \) (cf. Sect. 4.3). Let
\[
norm : \bigsqcup_{e \in h^{-1}(c)} Y^e \rightarrow V^c
\]
be the map that sends a matrix \( (D^j_i) \in Y^e \) to the collection \( ((D^i_i), (D^j_j)) \), where \( D^i_i = \sum_j D^j_i \) and \( D^i_j = \sum_i D^i_j \).

**Lemma 10.** i) The scheme \( V^c \) is of pure dimension \( d \), its irreducible components are numbered by the set \( h^{-1}(c) \). Namely, to \( e \in h^{-1}(c) \) there corresponds the component \( \norm(Y^e) \).

ii) \( \norm \) is the normalization of \( V^c \) (more precisely, \( \norm \) is a finite morphism, an isomorphism over an open dense subscheme of \( V^c \), and the scheme \( \bigsqcup_{e \in h^{-1}(c)} Y^e \) is smooth). So,
\[
\norm \mathcal{Q}[d] \rightarrow IC,
\]
where \( IC \) is the intersection cohomology sheaf on \( V^c \).

**Proof** Stratify \( V^c \) by locally closed subschemes \( eV^c \subset V^c \) indexed by \( e \in h^{-1}(c) \). First, define \( eV^c \) as the open subscheme of \( Y^e \) given by the condition:
\[
\text{if } i > k, l > j \text{ then } D^i_i \cap D^j_j = 0
\]
Then the composition \( e V^c \hookrightarrow Y^e \xrightarrow{\text{norm}} V^c \) is a locally closed immersion. As a subscheme of \( V^c \), \( e V^c \) is given by the condition:

\[
\text{for all } i, j \text{ we have } \deg((\sum_{k \leq i} D_k) \cap (\sum_{l \leq j} D'_l)) = \sum_{k \leq i, l \leq j} e_k^i
\]

For any \( e \in h^{-1}(c) \) the scheme \( e V^c \) is smooth, nonempty and irreducible of dimension \( d \). This concludes the proof. □

**Lemma 11.** There is a canonical isomorphism \( R^{-2d}(\text{div}^e)_! \mathbb{Q}_\ell(-d) \sim \text{norm}_* \mathbb{Q}_\ell \).

We need the following straightforward sublemma.

**Sublemma 5.** i) Let \( r : Y \rightarrow Y' \) be a separated morphism of schemes of finite type. Assume that the fibres of \( r \) are of dimension \( \leq d \). Let \( F \) be a smooth \( \mathbb{Q}_\ell \)-sheaf on \( Y \), \( U \subset Y \) be an open subscheme, and \( r_U \) be the restriction of \( r \) to \( U \). Then the natural map \( R^{2d}(r_U)_! F \rightarrow R^{2d}r_! F \) is injective.

ii) Let \( (U^j)_{j \in J} \) be a stratification of \( Y \) by locally closed subschemes that comes from a filtration of \( Y \) by closed subschemes. Let \( r^j \) be the restriction of \( r \) to \( U^j \). Then \( R^{2d}r_! F \) admits a filtration by subsheaves with successive quotients being \( R^{2d}r^j_! F \) \( (j \in J) \). □

**Proof of Lemma 11**

Recall that \( W^c \) is stratified by locally closed substacks \( U^e \hookrightarrow W^c \) indexed by \( e \in h^{-1}(c) \) and we have the maps \( \text{div}^e : U^e \rightarrow Y^e \) (cf. the proof of Lemma 7). The diagram commutes

\[
\begin{array}{ccc}
U^e & \hookrightarrow & W^c \\
\downarrow \text{div}^e & & \downarrow \text{div}^c \\
Y^e & \xrightarrow{\text{norm}} & V^c
\end{array}
\]

We have \( R^{-2d}(\text{div}^e)_! \mathbb{Q}_\ell(-d) \sim \mathbb{Q}_\ell \) canonically. Indeed, by Künneth formulae, this is reduced to the fact that for any \( i \geq 0 \) the fibres of \( \text{div} : \text{Sh}^i \rightarrow X^{(i)} \) are connected of dimension \( -i \).

By ii) of Sublemma 5 on \( R^{-2d}(\text{div}^e)_! \mathbb{Q}_\ell(-d) \) there is a filtration parametrized by the set \( h^{-1}(c) \) with successive quotients being \( (\text{norm}^e)_* \mathbb{Q}_\ell \). We claim that any filtration with these successive quotients degenerates canonically into a direct sum. Indeed,

i) the different successive quotients are supported on different irreducible components of \( V^c \), so our filtration degenerates into a direct sum over some open dense subscheme of \( V^c \);

ii) the sheaf \( (\text{norm}^e)_* \mathbb{Q}_\ell[d] \) is perverse, it is the Goresky-MacPherson extension of its restriction to any open dense subscheme of \( V^c \);

iii) the property “perverse and the Goresky-MacPherson extension of its restriction to a given open subscheme of \( V^c \)” is preserved for extensions.

□
Finally, assume \( c = (\nu, \nu) \) with \( \nu = (1, \ldots, 1) \in \Lambda_{d,d} \). Then the set \( h^{-1}(c) \) is in natural bijection with \( S_d \), and the map \( \text{norm} \) becomes

\[
\bigcup_{\sigma \in S_d} X^{\nu}_{\sigma} \to V^c,
\]

where \( X^{\nu}_{\sigma} = X^{\nu} \), and \( \text{norm} \) sends a point \( (x_1, \ldots, x_d) \in X^{\nu}_{\sigma} \) to \( ((x_1, \ldots, x_d), (x_{\sigma 1}, \ldots, x_{\sigma d})) \). The action of \( S_d \times S_d \) on \( V^c \) lifts naturally to an action on

\[
(E^{\boxtimes d} \boxtimes E'^{\boxtimes d}) \otimes \text{norm}_{*}\bar{Q}_{\ell},
\]

and it is easy to see that \( \text{pr}((E^{\boxtimes d} \boxtimes E'^{\boxtimes d}) \otimes \text{norm}_{*}\bar{Q}_{\ell})^{S_d \times S_d} \cong (E \otimes E')^{(d)} \) canonically, where \( \text{pr} : V^c \to X^{(d)} \) denotes the projection. On the other hand, by Lemma \([1]\),

\[
R^{-2d} \text{div}_{!}((Sp_E^d \otimes Sp_{E'}^d)^{(-d)}) \cong \text{pr}_{!}((E^{\boxtimes d} \boxtimes E'^{\boxtimes d}) \otimes \text{norm}_{*}\bar{Q}_{\ell})
\]

One checks that this isomorphism is \( S_d \times S_d \)-equivariant. Taking the invariants, one gets

\[
R^{-2d} \text{div}_{!}(L_{E}^{d} \otimes L_{E'}^{d})^{(-d)} \cong (E \otimes E')^{(d)}
\]

By i) of Sublemma \([2]\), the natural map \( R^{-2d}(\leq n \text{div}^c)!\bar{Q}_{\ell} \to R^{-2d}(\text{div}^c)!\bar{Q}_{\ell} \) is an inclusion. It follows that

\[
R^{-2d}(\leq n \text{div})!((Sp_E^d \otimes Sp_{E'}^d)^{(-d)}) \to R^{-2d} \text{div}_{!}(Sp_E^d \otimes Sp_{E'}^d)^{(-d)}
\]

is an inclusion. Taking the \( S_d \times S_{d'} \)-invariants in \([22]\), one gets an inclusion \( nS_{E,E'}^d \subset (E \otimes E')^{(d)} \), whose image is denoted \( \leq n(E \otimes E')^{(d)} \). Since \( n \) and \( d \) were arbitrary, Lemma \([1]\) follows now from Corollary \([4]\) and the proof of Theorem \([4]\) is completed.

So, Theorem \([4]\) and Main Local Theorem are also proved.

6.6 Second proof of Theorem \([3]\)

In this section we present an alternative proof of Theorem \([3]\) under the additional assumption:

\( \min\{\text{rk} E, \text{rk} E'\} \leq n \). The idea of this proof was suggested to the author by D. Gaitsgory.

Let \( n \text{ Mod}_d \) denote the stack classifying modifications \( (L \subset L') \) of rank \( n \) vector bundles on \( X \) with \( \deg(L'/L) = d \). Let \( q : n \text{ Mod}_d \to \text{Sh}_d \) be the map that sends \( (L \subset L') \) to \( L'/L \), and \( \text{supp} : n \text{ Mod}_d \to X^{(d)} \) denote \( \text{div} \circ q \). For \( d' \geq 0 \) let \( p_{Y} : n\mathcal{Y}_d \times_{\text{Bun}_n} n \text{ Mod}_d \to n\mathcal{Y}_{d+d'} \) be the map that sends \( ((t_{i}), L \subset L') \) to \( ((t'_{i}), L') \), where \( t'_{i} \) is the composition

\[
\Omega^{(n-1)+\ldots+(n-i)} \xrightarrow{t_{i}} \wedge^{i} L \hookrightarrow \wedge^{i} L'.
\]

The map \( p_{Y} \) is representable and proper. Let \( q_{Y} : n\mathcal{Y}_d \times_{\text{Bun}_n} n \text{ Mod}_d \to n\mathcal{Y}_d \) denote the projection. The map \( q_{Y} \) is smooth of relative dimension \( nd' \).

The key ingredient is the Hecke property of Whittaker sheaves \([1], 7.5\). It admits the following immediate corollary (the argument given in loc.cit. for \( \text{rk} E = n \) holds, in fact, for \( \text{rk} E \leq n \)).
Proposition 5. For any smooth $\mathcal{Q}_d$-sheaf $E$ on $X$ and any $d \geq 0$ there is a natural map

$$(q_Y \times \text{supp})!p_Y^*(n\mathcal{P}_{E,ψ}^{d+1}) \to n\mathcal{P}_{E,ψ}^d \boxtimes E(\frac{2-n}{2})[2-n],$$

which is an isomorphism if $\text{rk} E \leq n$. □

Let $\n\text{Mod}_d$ be the stack of flags $(L_0 \subset \ldots \subset L_d)$, where $(L_i \subset L_{i+1}) \in \n\text{Mod}_i$ for all $i$. Let $\text{supp} : \n\text{Mod}_d \to X^d$ be the map that sends $(L_0 \subset \ldots \subset L_d)$ to $(\text{div}(L_1/L_0), \ldots, \text{div}(L_d/L_{d-1}))$. Let $p : n\mathcal{Y}_d \times \text{Bun}_n \n\text{Mod}_d \to n\mathcal{Y}_d \times \text{Bun}_n \n\text{Mod}_d$ be the projection, and $\tilde{p}_Y : n\mathcal{Y}_d \times \text{Bun}_n \n\text{Mod}_d \to n\mathcal{Y}_d \times \text{Bun}_n \n\text{Mod}_d$ be the composition $p \circ \tilde{p}.

Corollary 2. For any smooth $\mathcal{Q}_d$-sheaf $E$ on $X$ and any $d, d' \geq 0$ there is a natural map

$$(q_Y \times \text{supp})!\tilde{p}_Y^*(n\mathcal{P}_{E,ψ}^{d+d'}) \to n\mathcal{P}_{E,ψ}^d \boxtimes E^{d+d'}(\frac{2d' - nd'}{2})[2d' - nd'],$$

which is an isomorphism if $\text{rk} E \leq n$.

Proof The map (23) is defined as follows. Let $p_\mathcal{Q} : \mathcal{Q}_d \times \text{Bun}_n \n\text{Mod}_d \to \mathcal{Q}_{d+d'}$ be the map that sends $(L_1 \subset \ldots \subset L_n \subset L \subset L')$ to $(L_1 \subset \ldots \subset L_n \subset L')$. Let $\tilde{p}_\mathcal{Q} : \mathcal{Q}_d \times \text{Bun}_n \n\text{Mod}_d \to \mathcal{Q}_{d+d'}$ denote the composition

$$\mathcal{Q}_d \times \text{Bun}_n \n\text{Mod}_d \to \mathcal{Q}_d \times \text{Bun}_n \n\text{Mod}_d \xrightarrow{p_\mathcal{Q}} \mathcal{Q}_{d+d'},$$

where the first arrow is the projection. Consider the commutative diagram

$$\begin{array}{ccc}
\mathcal{Q}_d \times \text{Bun}_n \n\text{Mod}_d & \xrightarrow{\tilde{p}_\mathcal{Q}} & \mathcal{Q}_{d+d'} \\
\downarrow \phi \times \text{id} & & \downarrow \phi \\
\mathcal{Y}_d \times \text{Bun}_n \n\text{Mod}_d & \xrightarrow{\tilde{p}_Y} & \mathcal{Y}_{d+d'}
\end{array}$$

Since $\mathcal{P}_{E,ψ}^{d+d'}$ is a direct summand of

$$(\tilde{p}_\mathcal{Q})!(n\mathcal{P}_{E,ψ}^d \boxtimes \text{supp}^* E^{d+d'})[nd'](\frac{nd'}{2}),$$

it follows that $n\mathcal{P}_{E,ψ}^{d+d'}$ is a direct summand of $(\tilde{p}_Y)^!(n\mathcal{P}_{E,ψ}^d \boxtimes \text{supp}^* E^{d+d'})[nd'](\frac{nd'}{2})$. This yields a morphism

$$\tilde{p}_Y^*(n\mathcal{P}_{E,ψ}^{d+d'}) \to n\mathcal{P}_{E,ψ}^d \boxtimes \text{supp}^* E^{d+d'}[nd'](\frac{nd'}{2})$$

Since $q_Y \times \text{supp}$ is smooth of relative dimension $d'(n-1)$, the desired map is obtained from the last one by the adjointness. To show that (23) is an isomorphism under the condition $\text{rk} E \leq n$, apply $d'$ times Proposition 3. □

Denote by $\n\text{rss} X^{d'} \subset X^{d'}$ the open subscheme that parametrizes pairwise different points $(x_1, \ldots, x_{d'}) \in X^{d'}$ (here ‘rss’ stands for ‘regular semisimple’). Let $\n\text{rss} \n\text{Mod}_{d'}$ be the preimage

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of $\tilde{\text{rss}} X^{d'}$ under $\tilde{\text{supp}}$. The symmetric group $S_{d'}$ acts on $\tilde{\text{rss}} \text{Mod}_{d'}$, and the restriction of $p_Y$ to $n \mathcal{Y}_d \times \text{Bun}_n \tilde{\text{rss}} \text{Mod}_{d'}$ is $S_{d'}$-invariant. So, the action of $S_{d'}$ on $n \mathcal{Y}_d \times \text{Bun}_n \tilde{\text{rss}} \text{Mod}_{d'}$ lifts to an action on

$$p_Y^* (n \mathcal{P}_{E,\psi}^{d+d'})$$

Since the restriction $\tilde{\text{rss}} \text{Mod}_{d'} \to \tilde{\text{rss}} X^{d'}$ of $\tilde{\text{supp}}$ is $S_{d'}$-equivariant, $S_{d'}$ acts on the complex

$$(q_Y \times \tilde{\text{supp}}) p_Y^* (n \mathcal{P}_{E,\psi}^{d+d'})$$

restricted to $n \mathcal{Y}_d \times \tilde{\text{rss}} X^{d'}$. On the other hand, $S_{d'}$ acts on $E^{\otimes d'}$ and, hence, on the right hand side of (23). Using the explicit description of the map (23) one easily proves the next lemma.

**Lemma 12.** For any smooth $\mathbb{Q}_E$-sheaf $E$ on $X$ the map (23) restricted to $n \mathcal{Y}_d \times \tilde{\text{rss}} X^{d'}$ is $S_{d'}$-equivariant. □

Recall that for any smooth $\mathbb{Q}_E$-sheaf $E$ on $X$ the Verdier dual of $\tilde{\text{rss}} \mathcal{P}_{E,\psi}^d$ is canonically isomorphic to $\tilde{\text{rss}} \mathcal{P}_{E',\psi}^d$ (see [3], 4.7). So, to prove Theorem 4 we must establish a canonical isomorphism

$$\phi^* \mathcal{R}\text{Hom}(n \mathcal{P}_{E,\psi}^d, n \mathcal{P}_{E',\psi}^d) \cong \mathcal{R}\text{Hom}(E, E')(d),$$

where $\phi : n \mathcal{Y}_d \to X^{(d)}$ is the map defined in Sect. 2.3. The statement of Theorem 4 being symmetric with respect to interchanging $E$ and $E'$, we assume $\text{rk} E \leq n$.

For any smooth $\mathbb{Q}_E$-sheaf $E$ on $X$ set $n \tilde{\mathcal{P}}_{E',\psi}^d = \mathcal{V} (\beta^* \mathcal{S} \mathcal{P}_{E}^\prime \otimes \mu^* \mathcal{L}_E)[b](\frac{1}{2})$. In other words, $n \tilde{\mathcal{P}}_{E,\psi}^d$ is a complex on $n \mathcal{Y}_d$ obtained by replacing in the definition of $n \mathcal{P}_{E,\psi}^d$ Laumon’s sheaf $\mathcal{L}_E^d$ by Springer’s sheaf $\mathcal{S} \mathcal{P}_{E}^\prime$. Theorem 5 follows now from the next statement.

**Proposition 6.** Let $E, E'$ be smooth $\mathbb{Q}_E$-sheaves on $X$ with $\text{rk} E \leq n$. Then there exists a canonical $S_d$-equivariant isomorphism

$$\phi^* \mathcal{R}\text{Hom}(n \mathcal{P}_{E,\psi}^d, n \tilde{\mathcal{P}}_{E',\psi}^d) \cong \text{sym}_* (\mathcal{R}\text{Hom}(E, E')^{\otimes d})$$

**Proof** The idea is that Proposition 5 is a tautological consequence of Corollary 2 obtained by applying the formalism of six functors. The equivariance property follows from Lemma 12. The precise argument is as follows.

Consider the commutative diagram

$$\begin{array}{ccc}
n \mathcal{Y}_0 \times \text{Bun}_n \text{Mod}_d & \xrightarrow{p_Y} & n \mathcal{Y}_d \\
\downarrow \psi & & \downarrow \phi \\
n \mathcal{Y}_0 \times X^{(d)} & \xrightarrow{q_Y \times \text{supp}} & X^{(d)}
\end{array}$$

Set for brevity $\Psi^0 = n \mathcal{P}_{E,\psi}^0$ (it does not depend on $E$, though does depend on $\psi$). By definition,

$$n \tilde{\mathcal{P}}_{E',\psi}^d \cong p_Y^* (q_Y^* \Psi^0 \otimes q^* \mathcal{S} \mathcal{P}_{E'}^\prime)[nd](\frac{nd}{2})$$

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Lemma 13. \(q_Y^* \Psi^0 \otimes q^* S^{pr^d}_{E'} \cong \text{RHom}(q^* S^{pr^d}_{E'}, q_Y^* \Psi^0)\) canonically and \(S_d\)-equivariantly.

**Proof** Using the fact that both \(q_Y\) and \(q_Y \circ p\) are smooth of relative dimension \(nd\), we get

\[
q_Y^* \Psi^0 \otimes q^* S^{pr^d}_{E'} \cong p_!(\text{supp}^* E'^d \otimes p^* q_Y^* \Psi^0) \cong p_* \text{RHom}(\text{supp}^* (E'^*)^d, p^* q_Y^* \Psi^0) \cong p_* \text{RHom}(\text{supp}^* (E'^*)^d, p^! q_Y^* \Psi^0[-2nd](-nd)) \cong \text{RHom}(p_! \text{supp}^* (E'^*)^d, q_Y^* \Psi^0) \quad \square
\]

Using the above lemma, we get

\[
\text{RHom}(p^* (\mathcal{P}_{E'; \psi}), \mathcal{P}_{E'; \psi}) \cong \text{RHom}(\mathcal{P}_Y^* (\mathcal{P}_{E'; \psi}), q_Y^* \Psi^0 \otimes q^* S^{pr^d}_{E'})[nd](\frac{nd}{2}) \cong \text{RHom}(\mathcal{P}_Y^* (\mathcal{P}_{E'; \psi}) \otimes q^* S^{pr^d}_{E'}, q_Y^* \Psi^0)[-nd](\frac{-nd}{2})
\]

Let \(j : \mathcal{Q}_0 \hookrightarrow \mathcal{Y}_0\) denote the natural open immersion. Since \(n \mathcal{Q}_0 \to \text{Spec } k\) is a generalized affine fibration, we have \(\Gamma(n \mathcal{Q}_0, \mathcal{O}_\mathcal{E}) \cong \mathcal{O}_X\). So, our assertion is reduced to the next lemma.

**Lemma 14.** There is a canonical \(S_d\)-equivariant isomorphism over \(\mathcal{Y}_0 \times X(d)\)

\[
(q_Y \times \text{supp})_* \text{RHom}(p_Y^* (\mathcal{P}_{E'; \psi}) \otimes q^* S^{pr^d}_{E'}, q_Y^* \Psi^0) \cong (j \times \text{id})_* (\mathcal{Q}_0 \boxtimes \text{sym}_0 \text{Hom}(E, E'^d)^d)[nd](\frac{nd}{2})
\]

**Proof** Let \(pr_Y : \mathcal{Y}_0 \times X^d \to \mathcal{Y}_0\) denote the projection. Using the commutative diagram

\[
\begin{array}{ccc}
n \mathcal{Y}_0 \times \text{Bun}_n \mathcal{Y}_0 \times \text{Mod}^d & \xrightarrow{\mathcal{P}} & n \mathcal{Y}_0 \times \text{Bun}_n \mathcal{Y}_0 \times \text{Mod}^d \\
\downarrow q_Y \times \text{supp} & & \downarrow q_Y \times \text{supp} \\
n \mathcal{Y}_0 \times X^d & \xleftarrow{\text{id} \times \text{sym}} & n \mathcal{Y}_0 \times X^d,
\end{array}
\]

we obtain

\[
(q_Y \times \text{supp})_* \text{RHom}(p_Y^* (\mathcal{P}_{E'; \psi}) \otimes q^* S^{pr^d}_{E'}, q_Y^* \Psi^0) \cong (q_Y \times \text{supp})_* p_* \text{RHom}(\text{supp}^* (E'^*)^d, q_Y^* \Psi^0) \cong (q_Y \times \text{supp})_* p_* \text{RHom}(p_Y^* (\mathcal{P}_{E'; \psi}) \otimes \text{supp}^* (E'^*)^d, p^! q_Y^* \Psi^0) \cong (\text{id} \times \text{sym})_* (q_Y \times \text{supp})_*(\text{supp}^* (E'^*)^d) \cong (\text{id} \times \text{sym})_* \text{RHom}((q_Y \times \text{supp})_!(\mathcal{P}_Y^* (\mathcal{P}_{E'; \psi}) \otimes \text{supp}^* (E'^*)^d), p_Y^* \Psi^0) \cong (\text{id} \times \text{sym})_* \text{RHom}(\Psi^0 \otimes (E \otimes E'^d)^d, p_Y^* \Psi^0)[nd - 2d](\frac{nd - 2d}{2})
\]

where the last isomorphism comes from Corolary 3. Since \(pr_Y\) is smooth of relative dimension \(d\), our assertion follows. \(\square\)

\(\square\)(Proposition 3)
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