Universality of the least singular value and singular vector delocalisation for Lévy non-symmetric random matrices

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Abstract
In this paper we consider $N \times N$ matrices $D_N$ with i.i.d. entries all following an $\alpha$–stable law divided by $N^{1/\alpha}$. We prove that the least singular value of $D_N$, multiplied by $N$, tends to the same law as in the Gaussian case, for almost all $\alpha \in (0, 2)$. This is proven by considering the symmetrization of the matrix $D_N$ and using a version of the three step strategy, a well known strategy in the random matrix theory literature. In order to apply the three step strategy, we also prove an isotropic local law for the symmetrization of matrices after slightly perturbing them by a Gaussian matrix with a similar structure. The isotropic local law is proven for a general class of matrices that satisfy some regularity assumption. We also prove the complete delocalization for the left and right singular vectors of $D_N$ at small energy, i.e., for energies at a small interval around 0.

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1 Introduction and main results

1.1 Introduction

The asymptotic behavior of the spectrum of random matrices has been a crucial topic of studies since Wigner’s semicircle law, first proven in [1]. The study of the asymptotic spectral behavior of Wishart matrices was the next important result, firstly investigated in [2], although Wishart matrices actually preceded Wigner matrices.

The Wishart matrices, and more generally the covariance matrices, play a significant role in various scientific fields. See for example [3] and [4] for applications in statistics, [5] for application in economics and [6] for application in population genetics. Several spectral properties of these matrices have been investigated. We focus on the case that the entries of the matrix are identically distributed, independent random variables (i.i.d.). For those matrices some significant results concern the limit of the largest eigenvalue, the asymptotic behavior of the correlation functions and the asymptotic bulk and edge behavior. For example, see [7] or the lecture notes concerning the singular values [8]. These results are proven for matrices whose entries have finite variance.

Besides those results, an important problem in random matrix theory is the asymptotic behavior of the least eigenvalue of covariance matrices, when the matrices’ dimensions are equal. Note that the inverse of the least singular value of a matrix is equal to the operator norm of its inverse, so an estimate of the least singular value gives control to the probability that the inverse has large norm and also gives control to its condition number. To name an illustrative application, this estimate of the least singular value for various random matrix models, plays an important role in the analysis of the performance of algorithms, see [9]. In the case that the entries of the matrices are normally distributed, the limiting distribution has been described in Theorem 4.2 of [10], by directly computing the density of the smallest singular value multiplied by $N$. In the general i.i.d. case, under the assumption of finite moments of sufficiently large order, the least singular value is proven to tend to the same law as the least singular value of a Gaussian random matrix, in Theorem 1.3 of [11]. This phenomenon, the same asymptotic distribution for the least singular value of a matrix as in the Gaussian case, will be called universality of the least singular value for the matrix. Lastly, in the most recent papers [12] and [13] the authors proved that universality of the least singular values holds for more general classes of matrices.

The above results have been focused on the finite variance cases. In the case of infinite second moment, and more specifically in the case of stable entries, there are not so many results concerning the behavior of
the spectrum of covariance matrices. There are some results, mostly concerning the limit of the E.S.D. of such matrices \([14],[15]\) and the limit of the largest eigenvalues \([16],[17]\). Moreover there are also some generalizations, which concern the limit of the largest eigenvalue of heavy tailed autocovariance matrices in \([18]\) and covariance matrices with heavy tailed \(m\)-dependent entries in \([19]\). Despite that, progress has been made concerning the symmetric matrices with heavy tailed entries. In \([20]\), the authors found the limit of the empirical spectral distribution of such matrices. Next, in \([21]\) and \([22]\) the authors proved some version of local law and examined the localization and delocalization of the eigenvectors in each of these cases. Moreover, in \([15]\) and \([23]\), the authors gave a better understanding of the limiting distribution of the empirical spectral distribution by proving the convergence of resolvent of the matrix to the root of a Poisson weighted infinite tree in some operator space. Recently, in \([24]\) and \([25]\), the authors showed complete delocalization of the eigenvectors whose eigenvalues belong in some interval around 0, GOE statistics for the correlation function and described the precise limit of the eigenvectors respectively.

In this paper we prove universality for the least singular value of random matrices with i.i.d. \(\alpha\)-stable entries. The methods we use also imply the complete singular vector delocalization for such matrices at small energies. We prove these results using a version of the three step strategy, a strategy developed in the last decade, which is suitable in order to obtain universality results for random matrix models, see \([26]\).

The basic inspiration for this paper is Theorem 2.5 in \([24]\), which proves universality of the correlation functions for symmetric Lévy random matrices at small energies. Both the intermediate local law, Theorem 3.1 and the theorem concerning the comparison of the entries of the resolvent, Theorem 6.4, are similar to Theorem 3.5 and Theorem 3.15 of \([24]\) respectively, adjusted to our set of matrices. For the intermediate local law we also use a lot of results from \([21]\) and \([22]\).

Results and methods from \([27]\) and \([12]\) had significant influence to this paper as well. In particular the isotropic local law in Sections 5 is an analogue of Theorem 2.1 in \([27]\), proven for a different class of matrices. Moreover universality for the least singular value of random matrices after perturbing them by a Brownian motion Matrix can be found in Theorem 3.2 of \([12]\). So several results from Sections 4 and 6 are based or influenced by results of \([12]\).

### 1.2 Main results

Fix a parameter \(\alpha \in (0, 2)\). A random variable \(Z\) is called \((0, \alpha)\) \(\alpha\)-stable law if

\[
\mathbb{E}(e^{itZ}) = \exp(-\alpha^\alpha |t|^{\alpha}), \text{ for all } t \in \mathbb{R}.
\]

**Definition 1.1.** Set

\[
\sigma := \left(\frac{\pi}{2 \sin(\frac{\pi}{2} \theta(\alpha))}\right)^{1/\alpha} > 0,
\]

and let \(J\) be a symmetric random variable with finite variance and let \(Z\) be a \((0, \alpha)\) \(\alpha\)-stable random variable, independent from \(J\). Then, define the matrix \(D_N(\alpha) = \{d_{ij}\}_{1 \leq i \leq \sqrt{N}}\) to be random matrix with i.i.d. entries, all having the same law as \(N^{-1/\alpha}(J + Z)\). In what follows, we may omit explicitly indicating the dependence of the matrices \(D_N\) on the parameters \(\alpha\) and \(N\), and use the notation \(D\).

Lastly, fix parameters \(C_1, C_2\) such that

\[
\frac{C_1}{N^{\alpha} + 1} \leq \mathbb{P}(|d_{ij}| \geq t) \leq \frac{C_2}{N^{\alpha} + 1}.
\]

Such parameters exist due to the tail properties of the stable distribution. See \([28]\), Property 1.2.8.

The parameter \(\alpha\) is chosen in \([1.2]\) like so, in order to keep our notation consistent with previous works such as \([24],[25],[22]\) and \([21]\). This parameter can be altered by a rescaling. Moreover, denote \(\rho_{sc}\) the probability density function of the semicircle law, i.e.,

\[
\text{GOE denotes the Gaussian Orthogonal Ensemble, i.e., symmetric matrix with independent entries (up to symmetry) where the non-diagonal entries have law } N(0, 1) \text{ and the diagonal } N(0, 2).
\]
\[ \rho_{ac}(x) = \mathbf{1}\{ |x| \leq 2 \} \frac{1}{2\pi} \sqrt{4 - x^2}. \]

Furthermore, set
\[ \xi := \frac{P_a(0)}{\rho_{ac}(0)}, \quad \text{(1.4)} \]
where \( \rho_a \) is the density of the limiting distribution of the empirical measure of the singular values of \( D \) and their negative ones and is described in Proposition 2.15.

In what follows we will use the standard Big \( O \) notation. Specifically given two functions \( f, g \), we will say \( f = O(g) \) if and only if there exists a constant \( C > 0 \) independent of any other parameter such that
\[ \limsup_{x \to \infty} \frac{|f(x)|}{g(x)} = C < \infty, \quad \text{(1.5)} \]
where the constant \( C > 0 \) will be independent of any other parameter. If the constant \( C \) depends on some parameter(s) \( c \) defined earlier, we will write \( f = O_c(g) \). Moreover if the constant \( C = 0 \) then we will write \( f = o(g) \).

Our main result shows that the least singular values of \( D_N \) are universal as \( N \) tends to infinity. The analogous result for matrices with finite variance entries was proven in \[11\]. We also prove that the left and right singular vectors of \( D_N \) are completely delocalized for small energies, in the following sense.

**Theorem 1.2.** There exists a countable set \( \mathcal{A} \), subset of \( (0, 2) \), with no accumulation points in \( (0, 2) \) such that the following holds. Let \( \{D_N(a)\}_{N \in \mathbb{N}} \) be sequences of matrices, where \( D_N(a) \in \mathbb{R}^{N \times N} \) with i.i.d. entries all following \( N^{-1/4}(Z + J) \), where \( Z, J \) as in Definition 1.1. Then for every \( a \in (0, 2) \setminus \mathcal{A} \):

1. Let \( s_1(D_N(a)) \) denote the least singular value of \( D_N(a) \). Then, there exists \( c > 0 \) such that for all \( r \geq 0 \)
\[ \mathbb{P}\left( N \xi s_1(D_N(a)) \leq r \right) = 1 - \exp\left(-\frac{r^2}{2} - r\right) + O_r(N^{-c}). \quad \text{(1.6)} \]

2. For each \( \delta > 0 \) and \( D > 0 \) there exist constants \( C = C(a, \delta, D) > 0 \) and \( c = c(a) \) such that:
\[ \mathbb{P}\left( \max u : u \in \mathcal{B}_N \right) > N^{d - \frac{1}{2}} \leq CN^{-D}. \quad \text{(1.7)} \]

where \( \mathcal{B}_N \) is the set of eigenvectors of \( D_ND_N^T \) or \( D_N^2 \), normalized with the Euclidean norm, whose corresponding eigenvalues belong to the set \([-c, c]\).

The proof of Theorem 1.2 can be found in Subsection 6.3.

**Remark 1.3.** The set \( \mathcal{A} \) for which Theorem 1.2 cannot be applied is conjectured to be empty. Its presence is due to some \( a \)-dependent fixed point equations in \[22\], which we use and can be inverted only if \( a \notin \mathcal{A} \).

Moreover, we can generalize the proof of Theorem 1.2 to the joint distribution of the bottom \( k \) singular values in the following sense.

**Theorem 1.4.** Fix a positive integer \( k \). Let \( \mathcal{A} \subseteq (0, 2) \) be the countable set of Theorem 1.2. Then define, as in Definition 1.1 \( \{D_N\}_{N \in \mathbb{N}} \) with i.i.d. entries all following \( N^{-1/4}(Z + J) \), where \( Z \) is \((0, a)\)-stable for \( a \in (0, 2) \setminus \mathcal{A} \). Also let \( \{L_N\}_N \) be a sequence of \( N \times N \) i.i.d. matrices, with entries following the same law as a centered normal random variable with variance \( \frac{1}{N} \).

Also for any matrix \( A \) define
\[ \Lambda_k(A) := (s_1(A), \ldots, s_k(A)), \]
where \( \{s_i(A)\}_{i \in [N]} \) are the singular values of \( A \) arranged in increasing order. Also denote \( 1_k = (1, \cdots, 1) \)

\( \text{for all } E \in \mathbb{R}^k \)
\[ \Omega(E) := \{ x \in \mathbb{R}^k : x_i \leq E_i \text{ for all } i \in [k] \}. \]

Then there exists \( c > 0 \) such that for all \( E \in \mathbb{R}^k \)
\[ \mathbb{P}(\Lambda_k(L_N) \in \Omega(E - N^{-c}1_k)) - O_E(N^{-c}) \leq \mathbb{P}(\Lambda_k(\xi D_N) \in \Omega(E)) \leq \mathbb{P}(\Lambda_k(L_N) \in \Omega(E + N^{-c}1_k)) + O_E(N^{-c}). \quad \text{(1.8)} \]
The proof of Theorem \ref{thm:main} is similar to that of Theorem \ref{thm:main} and therefore is omitted. Note that the universal limiting distribution of $\Lambda_N(L_N)$ is explicitly given in \cite{MM2008}.

Moreover, by the way that we will prove Theorem \ref{thm:main} we can prove a similar result for the gap probability in the symmetric case. The proof of the following corollary will again be omitted due to its similarity to the proof of Theorem \ref{thm:main}.

**Corollary 1.5.** Let $M_N$ be an $N \times N$ symmetric matrix with i.i.d. entries (with respect to symmetry) and let all entries follow the same law as $N^{-1/\alpha}(Z+J)$, where $Z, J$ are defined in Definition \ref{def:1} for $a \in (0, 2) \setminus \mathcal{A}$. Here $\mathcal{A}$ is the set of Theorem \ref{thm:main}. Also let $W_N$ be a GOE matrix ($N \times N$ symmetric, with i.i.d. centered Gaussian entries, with variance $N^{-1}$). Arrange the eigenvalues of $M_N$ and $W_N$ in increasing order. Then there exists $\delta > 0$ such that for any $r > 0$,

$$\mathbb{P}\left( \# \left\{ i \in [N] : N\lambda_i(M_N) \in \left( -\frac{r}{2}, \frac{r}{2} \right) \right\} = 0 \right) - \mathbb{P}\left( \# \left\{ i \in [N] : N\lambda_i(W_N) \in \left( -\frac{r}{2}, \frac{r}{2} \right) \right\} = 0 \right) \leq O(e^{-\delta}).$$

(1.9)

For the Gaussian case, the limiting distribution of the gap probability is given in Theorem 3.12 of \cite{MM2008}.

**Remark 1.6.** Note that by Theorem \ref{thm:main} the least singular value of a random matrix with i.i.d. entries, all following an $\alpha$–stable distribution, are of order $O(N^{-\frac{1}{\alpha}})$ for $a \in (0, 2) \setminus \mathcal{A}$. So for $a \in (0, 1) \cap \mathcal{A}$ the least singular value, without normalization, tends to $\infty$, which is different from the finite variance case.

## 2 Preliminaries and sketch of the proof

### 2.1 Preliminaries

In this subsection we present some necessary definitions and lemmas.

Firstly fix parameters $a, b, \rho, \nu$ such that

$$a \in (0, 2), \quad \nu = \frac{1}{a} - b > 0, \quad 0 < \rho < \nu, \quad \frac{1}{4a} < \nu < \frac{1}{4 - 2a}, \quad a\rho < (2 - a)\nu.$$ (2.1)

Note that given $a \in (0, 2)$, such parameters will exists. Moreover $\nu > 0$ is the level on which we will truncate the matrix $D_N$ in (2.2). This truncation is crucial to our analysis as is explained later in Subsection 2.2. The rest of the restrictions for the parameters in (2.1), will be explained later in the choice of $\epsilon_0$ in (6.19), in the proof of Theorem 5.4.

Next we give some preliminaries definitions and lemmas.

**Definition 2.1.** For each $a \in (0, \infty)$ and $u \in \mathbb{C}^N$ we will use the notation

$$\|u\|_a = \left( \sum_{i=1}^N |u_i|^a \right)^{1/a}.$$

Moreover if $N = 1$ and $a = 2$, we will use the notation $|u|$ for the Euclidean norm.

**Definition 2.2.** Fix an $N \times N$ matrix $Y$. Then the empirical spectral distribution of $Y$ is the measure

$$\mu_Y := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i(Y)},$$

where $\delta_x$ is the Dirac measure for $x \in \mathbb{R}$ and $\{\lambda_i(Y)\}_{i \in [N]}$ are the eigenvalues of $Y$. We will also use the notation $\lambda_{\text{max}}(Y)$ for the largest eigenvalue of $Y$.

**Definition 2.3.** Let $M$ be an $N \times N$ real matrix. Then the $2N \times 2N$ matrix

$$\begin{bmatrix} 0 & M^T \\ M & 0 \end{bmatrix}$$

is called the symmetrization of $M$. 

We also define the matrices $A$ whose symmetrization is $X$.

Then for each $z$, the elements of $X$ (in the non-diagonal blocks) are called the $b$-removals of a deformed $(0, \sigma)$ $\alpha$-stable law. We also define the matrices $A_N := H_N - X_N$, the matrix $E_N$ whose symmetrization is $A_N$ and the matrix $K_N$ whose symmetrization is $X_N$, i.e.,

$$X_N = \begin{bmatrix} 0 & K_N^T \\ K_N & 0 \end{bmatrix}, \quad A_N = \begin{bmatrix} 0 & E_N^T \\ E_N & 0 \end{bmatrix}$$

Furthermore, define the matrix $L_N$ to be an $N \times N$ matrix with i.i.d. entries all following the law of a normal, centered random variable with variance $\frac{1}{N}$, and its symmetrization $W_N$. In what follows, we may omit the dependence of the matrices defined here on $N$, for notational convenience.

**Remark 2.5.** Note that the eigenvalues of $H$ are exactly the singular values of $D$ and their respective negative ones since

$$\det(j \cdot I_{2N} - H) = \det(j^2 \cdot I_N - D^T D).$$

Moreover, note that if we prove delocalization for the eigenvectors of $H$ in the sense of the second part of Theorem 1.2, then we will have an understanding over the delocalization of the left and right singular vectors of $D$, because of the following remark.

**Remark 2.6.** If $J_1, J_2$ are the matrices with columns the normalized left and right singular vectors of $D$, which by the singular value decomposition gives us that $J_1 DJ_2 = \text{diag}(s_1, s_2, \ldots, s_N)$, then one can compute that the matrix

$$\frac{1}{\sqrt{2}} \begin{bmatrix} J_2^T & -J_1 \\ J_1 & J_2 \end{bmatrix}$$

has columns the normalized eigenvectors of $H$.

So in what follows, we will focus on proving delocalization for the eigenvectors and universality of the least positive eigenvalue for $H$.

We will use the notation $\text{Im}(z)$ for the imaginary part of any $z \in \mathbb{C}$ and $\mathbb{C}^+ := \{z \in \mathbb{C} : \text{Im}(z) > 0\}$.

Furthermore we need the following definitions.

**Definition 2.7.** Let $M$ be an $N \times N$ matrix. The matrix $Y = (M - zI)^{-1}$ for $z \in \mathbb{C}^+$ is called the resolvent of $M$ at $z$.

**Definition 2.8.** (Stieltjes transform) Let $M$ be an $N \times N$ matrix and let $\mu_M$ be its empirical spectral distribution. Then for each $z \in \mathbb{C}^+$, we define its Stieltjes transform as the normalized trace of its resolvent, i.e.,

$$m_N^z(x) := \frac{1}{N} \text{tr}(M - zI)^{-1}.$$ 

In what follows, we might omit the dependence on the dimension of the Stieltjes transform or on the matrix, when it is clear to which matrix we refer.

**Definition 2.9.** In what follows we will use the following notation.

$$t := N \text{Var}(E_{1,1}).$$

Moreover, in Corollary 2.11 we prove that $t \to 0$ as $N \to \infty$.

In the next Lemma we give an estimate for the entries of $A$. 

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**Definition 2.4.** Let $H_N$ be the symmetrization of $D_N$, i.e.,

$$H_N = \begin{bmatrix} 0 & D_N^T \\ D_N & 0 \end{bmatrix}.$$
Lemma 2.10 (24), Lemma 4.1. Let $R \geq N^{-1/\alpha}$ and $p > a$. Then there exist a small constant $c = c(a, p, C_1)$ and a large constant $C = C(a, p, C_2)$ such that

$$cN^{-1} R^{\alpha - a} \leq \mathbb{E}[D_{1,1}] \leq CR^{-1}R^{\alpha - a}.$$ 

Here $D_{1,1}$ is the $(1, 1)$-entry of $D_N$. Here $C_1, C_2$ are the parameters from (1.3).

A direct application of the previous result for $R = N^{-\nu}$ and $p = 2$ implies the following.

Corollary 2.11. The entries of $E_N$ satisfy the following

$$cN^{(\alpha - 2)} \leq \text{Var}(E_{1,1}) \leq CN^{\alpha(\alpha - 2)}.$$ 

Remark 2.12. Note that the convergence of the E.S.D. of a sequence of random matrices, implies that the typical scale of an eigenvalue is $\frac{1}{N}$ (at least in the bulk of the spectrum) of the limiting distribution of the E.S.D.

Definition 2.13. Let $F(u)$ be a family of events indexed by some parameter(s) $u$. We will say that $F(u)$ holds with overwhelming probability, if for any $D > 0$ there exists an $N(D, u)$ such that for all $N \geq N(D, u)$

$$\mathbb{P}(F(u)) \geq 1 - N^{-D}.$$ 

uniformly in $u$.

Next we present a measure, for which in Theorem 3.14 we will prove that it is the limiting distribution of the E.S.D. of $X_N$.

Definition 2.14. Let $M_N$ be a sequence of symmetric $N \times N$ matrices with i.i.d. entries (up to symmetry) and for each $N \in \mathbb{N}$ let all the entries follow the same law $N^{-1/\alpha}(Z + J)$, where $Z, J$ are defined in Definition 1.1. In what follows for any $z \in \mathbb{C}^+$, we will use the notation

$$m_\nu(z) := \frac{1}{N} \lim_{N \to \infty} \text{tr}(M_N - zI)^{-1}. \quad (2.4)$$

So $m_\nu$ is the Stieltjes transform of the limiting distribution of the E.S.D. of the sequence of matrices $M_N$, see Theorem 1.4 of [20]. The properties of $m_\nu$ are described next in Proposition 2.15.

Proposition 2.15. The Stieltjes transform $m_\nu(z)$ of the limiting distribution of the E.S.D. of the matrices $M_N$ satisfies the following equation

$$m_\nu(z) = i \psi_{\alpha, z}(y(z)),$$

where

$$\phi_{\alpha, z}(x) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z - 1} e^{itx} e^{-\Gamma(1 - \frac{2}{\alpha}) \frac{x^2}{2}} dt, \quad (2.5)$$

$$\psi_{\alpha, z}(x) = \int_0^\infty e^{itx} e^{-\Gamma(1 - \frac{2}{\alpha}) \frac{x^2}{2}} dt, \quad (2.6)$$

$$y(z) = \phi_{\alpha, z}(y(z)). \quad (2.7)$$

where (2.7) is proven to have a unique solution on $\mathbb{C}^+$. Moreover the limiting probability density function $\rho_\alpha$ is bounded, absolutely continuous, analytic except at a possible finite set and with density at 0 given by

$$\rho_\alpha(0) = \frac{1}{\pi} \frac{1}{\Gamma(1 + \frac{2}{\alpha})} \left( \frac{\Gamma(1 - \frac{2}{\alpha})}{\Gamma(1 + \frac{2}{\alpha})} \right)^{1/\alpha}.$$ 

These results are proven in Proposition 1.1 of [14] and Theorem 1.6 of [23].

Remark 2.16. Later in Theorem 3.14, we will prove that the Stieltjes transform of $X_N$ also converges to $m_\nu$. So we will refer to the measure whose Stieltjes transform is $m_\nu$, as the limiting measure of the E.S.D. of $X_N$. 


2.2 Sketch of the proof

Now we are ready to present a sketch of the proof. At this point we will try to avoid as much technicalities as possible. In order to prove universality, meaning the same asymptotic distribution for the least singular value of $D_N$ as in the Gaussian case, we are going to follow the three step strategy, a well known strategy in random matrix theory literature. Some of the most fundamental results concerning this method can be found in [26] and in [30], which focus on proving universality of the correlation function for symmetric matrices. The key idea is that after a slight perturbation of a random matrix by a Brownian Motion matrix, the resulting matrix should behave as a Gaussian one, given that the initial matrix satisfies some mild assumption concerning its Stieltjes transform. This idea is exploited in the study of the evolution of the eigenvalues and the eigenvectors via stochastic differential equations. This method was crucial to the proof of the Wigner-Dyson-Mehta conjecture, see for example [26]. The three step strategy has also been used in establishing universality of the least singular value for random matrices, see for example [12], [13]. Specifically in our case:

- **First step:** We investigate the asymptotic spectral behavior of $X$ at an "intermediate" scale. At this step we prove that the matrix $X$ satisfies the necessary conditions, which insure that after a slight perturbation by a Brownian motion matrix universality will hold. This is done in Section 3. Note that by definition, $X$ contains the "big" elements of $H$. So the first step involves proving two estimates. One comparison of the Stieltjes transform of the E.S.D. of $X$ with the Stieltjes transform of its limiting measure, and one bound for resolvent entries of $X$. Set $m_X(z)$ the Stieltjes transform of $X$ and $R_{ij}(z)$ the resolvent of $X$ at $z$. In particular we wish to show that the following events

$$|m_{ij}(z) - m_X(z)| = o(1), \quad (2.8)$$

$$\max_{j \in [2N]} |R_{ij}(z)| = O\left(\log^C(N)\right), \text{ for some } C > 0. \quad (2.9)$$

hold with overwhelming probability for any $z : \text{Im}(z) \geq N^{-\frac{1}{3}}$ for any small enough $\delta > 0$ and $\text{Re}(z)$ in some $N$–independent interval. These results are called intermediate because the natural scale would be $\text{Im}(z) \geq N^{-1+\delta}$, as is explained Remark 2.12.

- **Second step:** We consider the perturbed matrix $X + \sqrt{t}W$, where $W$ is the symmetrization of a full centered Gaussian matrix with i.i.d. entries with variance $\frac{1}{N}$, and $t$ is chosen so that the variances of the entries of $\sqrt{t}W$ and of $A$ match. It can be computed that $t \sim N^{\alpha(2)}$.

The level of the intermediate scale local law in the previous step, is justified in this part of the proof. In order to apply universality Theorems for the matrices after slightly perturbing by Brownian motion matrices, see for example Theorem 3.2 of [12], we wish the variances of the $\sqrt{t}W$ to be above the intermediate scale of the local law. Since $N^{\delta - \frac{1}{4}} = o(t)$, for small enough $\delta > 0$, this is implied.

Roughly, what we need to prove at this step is that the desired properties, delocalization of the eigenvectors and universality of the least singular value, hold for the matrix $X + \sqrt{t}W$.

So for the first part of the second step, we prove universality of the least positive eigenvalue for $X + \sqrt{t}W$. This is based on the regularity of the Stietljes transform of $X$, proven in the previous step, and some results from [12]. More precisely at the first part of the second step we prove that

$$\lim_{N \to \infty} \mathbb{P}\left(N\sqrt{t}h_{ij}(X + \sqrt{t}W) \geq r\right) = \lim_{N \to \infty} \mathbb{P}\left(N\sqrt{t}h_{ij}(W) \geq r\right), \text{ for any } r \in \mathbb{R}^+. \quad (2.10)$$

This result is proven in Section 4.

In most of the universality-type theorems the fact that the entries have finite variances play a significant role, see for example Lemma 15.4 in [26]. In [12] the authors showed universality of the least singular value for sparse random matrix models. Firstly, they prove universality of the least singular value for the sparse models after slightly perturbing them by a Brownian motion matrix and then they remove the Brownian motion matrix. This is done by using results which
take advantage of the fact that the entries have finite variance, for example see Lemma 5.14 in [12]. Since our model does not have entries with finite variance, we will need to compare the matrices $X + \sqrt{t}W$ and $X + A$ with a different method. Essential to that method is the fact that the resolvent entries of $X + \sqrt{t}W$ do not grow very fast. Specifically set $T_{i,j}(z)$ to be the resolvent of $X + \sqrt{t}W$ at $z$. In particular in Section 5 we prove that for any small $\delta > 0$ the $\delta$--dependent events

$$\sup_{i,j}|T_{i,j}(z)| \leq N^\delta$$

(2.11)

hold with overwhelming probability, and for all $z : \text{Im}(z) \geq N^{\epsilon-1}$ for any small $\epsilon > 0$, very close to the natural scale in Remark 2.12. It is known that bounds as the one in (2.11) imply the complete eigenvector delocalization for the matrix $X + \sqrt{t}W$.

In order to establish (2.11), we prove something better. A universal result which compares the entries of the resolvent of any matrix, which satisfies some mild regularity assumption, Assumption 5.1 with the additive free convolution of the matrix with the semicircle law. Thus, the largest part of Section 5 is mostly independent for the rest of the paper.

**Third step:** We first compare the resolvent of $X + A$ and $X + \sqrt{t}W$. During the second step we have proven the desired properties, eigenvector delocalization and universality of the least eigenvalue for the matrix $X + \sqrt{t}W$, so we need to find a way to quantify the transition from the matrix $X + \sqrt{t}W$ to $X + A$ in order to prove the same properties for $H$. This is done by introducing the matrices

$$H^\gamma := X + \sqrt{t}(1 - \gamma^2)^{1/2}W + \gamma A, \text{ for all } \gamma \in [0, 1].$$

We manage to prove that the resolvent entries of $H^\gamma$ are asymptotically close for all $\gamma \in [0, 1]$, in Theorem 6.4. Similarly we study the continuity properties for $\gamma \in [0, 1]$ of the functions

$$q \left( \frac{N}{\pi} \int_{\phi}^{\gamma} \text{Im}(m_y(E + i\eta))dE \right),$$

(2.12)

where $m_y$ is the Stieltjes transform of the matrix $H^\gamma$ and $\eta$ is of order $N^{-\delta-1}$, below the natural scale. Eventually in (3.35) we prove that the functions defined in (2.12) are asymptotically close for any $\gamma \in [0, 1]$.

Next we introduce the functions

$$\iota_N(Y, r) := \# \{ i \in [N] : \iota_l(Y) \in (-r, r) \},$$

where $Y$ is a symmetric $N \times N$ matrix, $[\iota_l]_{l \in [N]}$ are the eigenvalues of $Y$ and $r$ is any positive number. So, it suffices to prove that there exists $c > 0$ such that for any $r > 0$

$$\left| \mathbb{P} \left[ \iota_N \left( X + \sqrt{t}G, \frac{r}{N} \right) = 0 \right] - \mathbb{P} \left[ \iota_N \left( X + A, \frac{r}{N} \right) = 0 \right] \right| \leq O(N^{-c}).$$

(2.13)

In order to prove the latter, we approximate the quantities $\mathbb{P}(\iota_N(H^\gamma, \frac{r}{N}) = 0)$ by appropriately choosing functions of the form (2.12). This is done in Lemma (6.13). After combining the results above, we conclude the proof in Subsection 6.3

### 3 Intermediate local law for $X$

Consider the matrices $H_N$ and $X_N$ as they are defined in Definition 2.4. In this section we are going to establish the local law (Theorem 5.14) for the $b$-removals of the matrix $H$, i.e., the matrix $X$. What we mean by local law is convergence of the Stieltjes transform of $X$ to its asymptotic limit, for complex numbers $z$ that depend on the dimension $N$ in some sense.

We will also use the notation

$$R(z) = (X - (E + i\eta)I)^{-1}.$$

(3.1)
for $z = E + i\eta$. In what follows we might abbreviate the dependence from the parameter $z$.

A precise formulation of this result is the following. There exists $C = C(a, b, \delta)$ such that

$$
\mathbb{P} \left( \sup_{E \in \mathbb{C}} \sup_{i \eta \leq N^{\delta/4}} \left| m_a(E + i\eta) - m_b(E + i\eta) \right| \geq \frac{1}{N^{\delta/8}} \right) \leq \exp \left( -\frac{(\log(N))^2}{C} \right).$$  \hspace{1cm} (3.2)

where the properties of $m_a(z)$ are described in Proposition 2.15.

We also prove that for all $z$ for which the local law holds, the diagonal entries of the resolvent of $X$ are almost bounded. More specifically for any large enough $N \in \mathbb{N}$ it is true that,

$$
\mathbb{P} \left( \sup_{E \in \mathbb{C}} \sup_{i \eta \leq N^{\delta/4}} \max |R_{ij}| > C \log^2(N) \right) \leq C \exp \left( -\frac{(\log(N))^2}{C} \right).$$  \hspace{1cm} (3.3)

In order to establish those results we will need to analyze the resolvent of $X$, in order for us to compare it with $m_a$. The main influence for this step is Theorem 3.5 of [23], where an intermediate local law is proven for symmetric heavy tailed random matrices. The main difference of the proof of the intermediate local law for our set of matrices from the symmetric case is that, by construction, only half of the minors of the resolvent will participate in the sum of the reductive formula from Schur complement formula. This difference is not crucial since we also prove that each of the diagonal entries of the resolvent of the matrix is identically distributed. Moreover, by Corollary 3.2 of [22], the sum of half of the diagonal entries of the resolvent is concentrated around its mean, like the sum of all its diagonal entries. The rest of the proof remains almost the same, but we will include most of the proofs for completeness of the paper.

Firstly we need to give a more detailed description of the limiting distribution.

### 3.1 Preliminaries for the intermediate local law

In [20] the authors proved that the E.S.D. of symmetric matrices with heavy tailed entries, converge in distribution to a deterministic measure and they described it. Next in [14], the authors described the limit of the sample covariance matrices. Next the authors in [22] and [21] proved local laws for symmetric heavy tailed matrices at an intermediate scale larger than $N^\delta$. Lastly in [24], the authors proved a local law at the intermediate scale $N^{\delta/4}$. All the previously mentioned results, are based on solving a fixed point equation. In the most recent results, these fixed equations are solved more generally, in a metric space which we are going to present in this subsection.

Next we present the metric space in which we will work with in order to prove an intermediate local law for $X$. The results we present here can also be found in [22].

**Definition 3.1.** For any $u, v \in \mathbb{C}$ define the following "inner product"

$$(u|v) := u \Re(v) + \bar{u} \Im(v) = \Re(u)(\Re(v) + \Im(v)) + \imath \Im(u)(\Re(v) - \Im(v)).$$

One may compute the following

$$(u|1) = u, \quad (-\imath u|e^{\imath \theta/4}) = \Im(u) \sqrt{2}, \quad |(u|v)| \leq 2|u||v|. \hspace{1cm} (3.4)$$

**Definition 3.2.** Set $\mathbb{K} = \mathbb{C}^+ \cap \{ z \in \mathbb{C} : \Re(z) > 0 \}$ and $\mathbb{K}^+ = \mathbb{K}$. Let $H_{u}$ be the space of the $C^1$, $g : \mathbb{K}^+ \to \mathbb{C}$ such that $g(ju) = j^r g(u)$ for each $j > 0$. Set also $S^1_{\mathbb{K}} = S^1 \cap \mathbb{K}^+$ where $S^1$ is the unit sphere on $\mathbb{C}$ with respect to the Euclidean norm. Following equation (10) of [22], define for each $r \in [0, 1]$ a norm on $H_r$

$$|g|_r = \sup_{u \in S^1_{\mathbb{K}}} \sqrt{f(\ell(u))^r \partial_1 g(u)^2 + |(\ell(u))^r \partial_2 g(u)|^2}.$$

Here,

$$\partial_1 g(x + iy) = \frac{dg(x + iy)}{dx}.$$
and likewise
\[ \partial_2 g(x + iy) = \frac{dg(x + iy)}{dy}. \]
Next, define the spaces \( H_{w,r} \) the completion of \( H_w \) with respect to the \( |g|_r \) norm. Further define \( H_w^\delta \subseteq H_{w,r} \) to be the set \( \{ g \in H_{w,r} : \inf_{u \in S^1} |\operatorname{Re}(g(u))| > \delta \} \). Also define the set
\[ H_{w,0}^\delta = \bigcup_{\delta > 0} H_{w,r}^\delta. \]
Further, abbreviate \( H_{w}^\delta := H_{w,0}^\delta. \)

**Remark 3.3.** For any \( g \in H_r \), by construction it is true that
\[ |g|_\infty \leq |g|_r. \]

Next, we present some lemmas concerning the metric spaces we presented and the fixed point equation we wish to solve.

**Lemma 3.4** \([22]. \) Let \( r \in (0,1) \) and \( u \in S^1 \) and \( x_1, x_2 \in \mathbb{K}^+ \) and let \( \eta \in (0,1) \) such that \( |x_1|, |x_2| \leq \eta^{-1} \). Set \( F_k(u) = (x_k|u|^2) \) for \( k \in \{1,2 \} \). Then there exists a constant \( C(r) \) such that for any \( s \in (0,r) \) we have that
\[ |F_k|_{1-r,s} \leq C|x_k|^s, \quad |F_k - F_2|_{1-r,s} \leq C\eta^{-s}(|x_1 - x_2|^s + \eta^s|x_1 - x_2|^s). \]  
(3.5)
Furthermore, if we further assume that \( \operatorname{Re}(x_1), \operatorname{Re}(x_2) \geq t \) and set \( G_k(u) = (x_k^t|u|^t) \), then there exists a constant \( C = C(t) \) such that
\[ |G_k - G_2|_{1-r,s} \leq C\eta^{2t-1}|x_1 - x_2|. \]  
(3.6)

**Definition 3.5.** Following Section 3.2 of \([22]\), for any numbers \( h \in \tilde{K}, u \in S^1 \) and \( g \in H_{a/2} \) define the functions,
\[ F_{h,g}(u) = \int_0^{\pi/2} \int_0^{\infty} \int_0^{\infty} \left[ \exp(-r^{a/2}g(e^{i\theta}) - (rh|e^{i\theta}^2)) - \exp(-r^{a/2}g(e^{i\theta} + uy) - (urh|u) - (rh|e^{i\theta}^2))r^{a/2-1}\right. \]
\[ \cdot y^{a/2-1} dyd\theta \]
(3.7)
and
\[ Y_f(u) = Y_{x,f}(u) = c_a F_{-x,f}(\tilde{u}), \]
where
\[ c_a = \frac{a}{2\pi a/2 \Gamma(a/2)} . \]

**Lemma 3.6** \([22], \) If \( f \in H_{a/2,r} \) then \( F_{0,g} \in H_{a/2,r} \). Also if \( f \in H_{a/2,r} \) and \( \operatorname{Re}(h) > 0 \) then \( F_{g,h} \in H_{a/2,r} \).

Next for any \( f \in H_{a/2} \) and \( p > 0 \) define the functions,
\[ r_{p,x}(f) = \frac{a_{1-p}}{\Gamma(p/2)} \int_0^{\infty} \int_0^{\pi/2} y^{-p} \exp((uyz|e^{i\theta}) - y^{a/2}f(e^{i\theta})) \sin(2\theta)^p y^{p-1} d\theta dy, \]
\[ s_{p,x}(x) = \frac{1}{\Gamma(p)} \int_0^{\infty} y^{p-1} \exp(-iyz - xy^{a/2}/y) dy. \]

**Lemma 3.7** \([22], \) There exists a countable subset \( \mathcal{A} \subseteq (0,2) \) with no accumulation points such for any \( r \in (0,1) \) and \( a \in (0,2) \setminus \mathcal{A} \), there exists a constant \( c = c(a, r) \) with the property that:
There exists a unique function \( \Omega_0 \in H_{a/2} \) such that \( \Omega_0 = Y_{0,\Omega_0}. \) Additionally for \( \operatorname{Im}(z) > 0 \) and \( |z| \leq c \), there exists a unique function \( f_0 = \Omega_x \in H_{a/2} \) that solves \( f = Y_{x,f} \) with \( |f - \Omega_0| \leq c. \) Moreover the function satisfies \( \Omega_x(e^{i\theta/4}) \geq c \) and for any \( p > 0 \) there exists a constant \( C = C(a, p) \) such that \( |r_{p,x}(\Omega_x)| \leq C. \)

**Lemma 3.8.** \([22], \) Proposition 3.4) Adopt the notation of the previous lemma. After decreasing \( c \) if necessary, there exists a constant \( C > 0 \) such that the following holds.
If \( \operatorname{Im}(z) > 0, |z| \leq c \) and \( |f - \Omega_x| \leq c, \) then
\[ |f - \Omega_x| \leq C|f - Y_{x,f}|_r. \]
Theorem 3.13. Let $\alpha, \eta > 0$. There exists a constant $C = C(\alpha, p, r) > 0$ such that, for any $g \in \mathcal{H}_{a/2,r}$ and $h \in \mathbb{K}$, we have that
\[
|F_2(g)| \leq C(\Re \eta)^{-\alpha/2} + C|g_i(\Re \eta))^{-\alpha/2},
\]
\[
|r_{p,x}(g)| \leq C(\Re \eta)^{-p}, \quad |s_{p,x}(g)| \leq C(\Re \eta)^{-p}.
\]

Lemma 3.9. (Lemma 4.1) Let $\alpha, \eta > 0$. There exists a constant $C = C(\alpha, p, r) > 0$ such that, for any $f, g \in \mathcal{H}_{a/2,r}$, and $z \in \mathbb{C}$
\[
|Y_f(z) - Y_g(z)| \leq C|f - g|_{\infty} + |f - g|_{\infty}(|f|_{r} + |g|_{r}).
\]
Furthermore, for any $p > 0$ there exists a constant $C' = C'(\alpha, a, r, p)$ such that for any $f, g \in \mathcal{H}_{a/2,r}$ and for any $z \in \mathbb{C}$ and $x, y \in \mathbb{K}$ with $\Re(x), \Re(y) \geq a$ we have that
\[
|r_{p,x}(f) - r_{p,x}(g)| \leq C'|f - g|_{\infty}, \quad |s_{p,x}(x) - s_{p,x}(y)| \leq C'|x - y|.
\]

The reason to present all the tools in this subsection is explained in the following Remark.

Remark 3.11. Due to Lemma 4.4. of [22], $\Omega_{1,x}(\xi_3(1))$, which is defined in Lemma 3.7, is exactly the limiting Stieltjes transform in Proposition 2.15.

3.2 Statement of the intermediate local law

In this subsection, we state the local law for the matrix $X$ and state a stronger theorem which will imply the local law.

Before we present the theorem, we give some definitions. Recall the notation from Subsection 3.1.

Definition 3.12. Define the following quantities
\[
u_x(u) := \Gamma \left(1 - \frac{\alpha}{2}\right)(-iR_{a/2})^{a/2}. \quad y_x := \mathbb{E}(\nu_x(u)),
\]
\[I_p := \mathbb{E}((-iR_{a/2})^p), \quad J_p := \mathbb{E}((|R_{a/2}|^p).
\]
where we have omitted the dependence from the dimension $N$ in the notation we used.

In what follows, keep in mind the definition of the functions $r_{p,x}$ and $s_{p,x}$ in Subsection 3.1. Next, we present the theorem which will imply the intermediate local law proved at Subsection 3.7.

Theorem 3.13. Let $\alpha \in (0, 2), b \in (0, \frac{1}{2}), s \in (0, \frac{a}{2}), p > 0, \varepsilon \in (0, 1]$ and $N \in \mathbb{N}$. Set $\delta = (\frac{1}{a} - b)(2 - a)/10$. Suppose $z = E + i\eta \in \mathbb{C}$ with $E, \eta \in \mathbb{R}$. Assume that:
\[
z = E + i\eta, \quad |\varepsilon| \leq \frac{1}{\varepsilon}, \quad \eta \geq N^{\varepsilon - \alpha/4}, \quad \mathbb{E}(\Im(R_{a/2})) \geq \varepsilon, \quad \mathbb{E}|R_{a/2}|^2 \leq \varepsilon^{-1}, \quad \text{for all } i \in \{2N\}.
\]

Then, there exists a constant $C = C(\alpha, \varepsilon, b, s, p) > 0$ such that
\[
|y_x - Y_{\varepsilon}|_{1-\alpha/2} \leq C \log^2(N) \left(\frac{1}{(\eta^2 N)^{a/8}} + \frac{1}{N^9} + \frac{1}{N^{s} \eta^{a/2}}\right),
\]
\[
|y_x - Y_{\varepsilon}|_{1} \leq C \log^2(N) \left(\frac{1}{(\eta^2 N)^{a/8}} + \frac{1}{N^9} + \frac{1}{N^{s} \eta^{a/2}}\right).
\]
\[
|y_x - Y_{\varepsilon}|_{1} \leq C \log^2(N) \left(\frac{1}{(\eta^2 N)^{a/8}} + \frac{1}{N^9} + \frac{1}{N^{s} \eta^{a/2}}\right).
\]
\[
|J_p - r_{p,x}(y_x)| \leq C \log^2(N) \left(\frac{1}{(\eta^2 N)^{a/8}} + \frac{1}{N^9} + \frac{1}{N^{s} \eta^{a/2}}\right).
\]
Moreover
\[
\inf_{u \in \mathcal{S}_i} \Re(\nu_x(u)) > \frac{1}{C}.
\]
and
\[
\mathbb{P}\left(\max_{u \in \{2N\}} |R_{a/2}| > C \log^2(N) \right) \leq C \exp\left(-\frac{(\log(N))^2}{C}\right).
\]
**Theorem 3.14** (Local law). There exists a countable set \( \mathcal{A} \subseteq (0, 2) \) with no accumulation points in \((0, 2) \) such that for each \( a \in (0, 2) \setminus \mathcal{A} \) the following holds. Fix \( b \in (0, \frac{1}{2}) \), \( \delta = (\frac{1}{a} - b)(2 - a) / 10 \) and \( \delta \in (0, \min(b, \frac{1}{2})) \). Then there exists a constant \( C = C(a, b, \delta, p) > 0 \) such that

\[
\mathbb{P}\left( \sup_{x \in \mathbb{R}^2} \left| m_N(z) - i s_1 \gamma_2(\gamma_2(1)) \right| > \frac{1}{N^{a/8}} \right) \leq C \exp\left( \frac{-\log^2(N)}{C} \right).
\]

Furthermore,

\[
\sup_{x \in \mathbb{S}_2} |\gamma_x(\eta) - \Omega_x(\eta)| \leq \frac{C}{N^{a/8}}
\]

and

\[
\mathbb{P}\left( \sup_{x \in \mathbb{S}_2} \max_{j \in [2N]} |R_{y,j} - C \log^2(N)| \leq C \exp\left( \frac{-\log(N)^2}{C} \right)
\]

Where \( D_{c, \delta} = \{E + i\eta : E \in (-\frac{1}{2}, -\frac{1}{2}), \frac{1}{\eta} \geq N^{\delta^{-1}/2} \} \), \( m_N(z) \) is the Stieltjes transform of \( X \) and \( \Omega_x(\eta) \) is defined in Lemma 3.7.

**Proof** Of Theorem 3.14 given Theorem 3.13. The proof is similar to the proof of Theorem 7.6 given Theorem 7.8 and Lemmas 3.7 3.8 (there called Lemma 7.2 and Lemma 7.3) in [24], so it will be omitted. \( \square \)

### 3.3 General results concerning the resolvent and the eigenvalues of a matrix

Firstly we present a well-known result that compares the eigenvalues of a matrix with the eigenvalues of its minors.

**Lemma 3.15** (Weyl’s inequality). Let \( R, M, Q \in \mathbb{R}^{N^2} \) some symmetric matrices such that

\[
M = R + Q.
\]

Let \( \mu_i, \rho_i, q_i \) be the eigenvalues of \( M, R, Q \) respectively arranged in decreasing order. Then

\[
q_j + \rho_k \leq \mu_i \leq q_r + \rho_s
\]

for any indices such that \( j + k - n \geq i \geq r + s - 1 \).

In the rest of this subsections we present some general results concerning the resolvent of a matrix. Most of them are known results, but we include them because they will be useful in the proof of Theorem 3.14.

**Lemma 3.16.** Let \( M_1, M_2 \) be two invertible, \( N \times N \) matrices then the following identity is true

\[
M_1^{-1} - M_2^{-1} = M_1^{-1}(M_2 - M_1)M_2^{-1}.
\]

Moreover if \( Y = (M_1 - zI)^{-1} \) such that \( z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \text{Im}(z) > 0\} \) then

\[
|Y_{ij}| \leq \frac{1}{\text{Im}(z)}, i,j \in [N].
\]

**Proof.** The identity (3.18) follows trivially by a right multiplication on both sides by the element \( M_1 \) and a left multiplication on both sides by the element \( M_2 \). Moreover (3.19) follows trivially from the spectral theorem. \( \square \)

**Definition 3.17.** In what follows in this section we will use the following notation. Consider \( M \) to be any \( N \times N \) matrix. Let \( J \subseteq [N] \). We will use the notation \((M^{(i)} - zI)^{-1}\) for the resolvent of the matrix \( M^{(i)} \), where \( M^{(i)} \) is the matrix \( M \) with the \( i \)-th row and column being replaced by zero vectors, for each \( i \in J \).

**Lemma 3.18.** Let \( M \) be an \( N \times N \) matrix and \( z \in \mathbb{C}^+ \). Then we have the following complements formula

\[
\frac{1}{(M - zI)_{i,i}^{-1}} = M_{i,i} - z - \sum_{k \notin [N] \setminus [i]} M_{i,k}(M^{(i)} - zI)^{-1}M_{k,i}.
\]

Next, we present the Ward identity. That is, for each \( J \subseteq [N] \) and \( j \in [N] \setminus J \) it is true that

\[
\sum_{k \in [N] \setminus J} |(M^{(j)} - zI)^{-1}M_{j,k}|^2 = \frac{\text{Im}((M^{(j)} - zI)^{-1})}{\text{Im}(z)}.
\]
Proof. The estimates [3.20] and [3.21] can be found in (8.8) and (8.3) in [26], respectively. □

**Lemma 3.19** ([22].Lemma 5.5). Let $M$ be an $N \times N$ matrix. For any $r \in (0, 1]$, $z \in \mathbb{C}^+$, $\eta = \text{Im}(z)$ and $i \in [N]$ we have the following deterministic bound

$$
\frac{1}{N} \sum_{i=1}^{N} |(M - zI)^{-1}_{ii} - (M^{(i)} - zI)^{-1}_{ii}|^r \leq \frac{4}{(N\eta)^r}.
$$

**Corollary 3.20.** ([24].Cor 5.7) Let $M$ be an $N \times N$ matrix. For any $r \in [1, 2]$, $z \in \mathbb{C}^+$, $\eta = \text{Im}(z)$ and $i \in [N]$ we have the deterministic estimate,

$$
\frac{1}{N} \sum_{i=1}^{N} |(M - zI)^{-1}_{ii} - (M^{(i)} - zI)^{-1}_{ii}|^r \leq \frac{4}{(N\eta)^r} \leq \frac{8}{N\eta^r}.
$$

### 3.4 Concentration results for the resolvent of a matrix

In this subsection, we present various identities and inequalities concerning the resolvent and the eigenvalues of a matrix.

Next we present some concentration inequalities.

**Lemma 3.21.** Let $N$ be an even positive integer and let $A = (a_{ij})_{i,j \in [N]}$ such that the rows $A_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ are mutually independent for each $i \in [N]$. Let $B = (A - zI)^{-1}$ and $z = E + i\eta$ where $\eta > 0$. Then for any Lipchitz function $f$ with Lipchitz norm $L_f$ and any $x > 0$, we have that,

$$
P \left[ \frac{2}{N} \sum_{i=1}^{N} f(B_{ii}) - \frac{2}{N} \sum_{i=1}^{N} f(B_{ii}) \right] \geq x \right] \leq 2 \exp \left( - \frac{N\eta^2 x^2}{8L_f^2} \right),
$$

and

$$
P \left[ \frac{2}{N} \sum_{i=1}^{N} f(B_{ii} + \frac{\eta}{2}, E + \frac{\eta}{2}) - \frac{2}{N} \sum_{i=1}^{N} f(B_{ii}) \right] \geq x \right] \leq 2 \exp \left( - \frac{N\eta^2 x^2}{8L_f^2} \right).
$$

**Proof.** The proof is similar to the respective proof for the Stieltjes transform in Lemma C.4 of [21]. We will sketch the proof for the first $N2^{-1}$ diagonal entries. The proof for the remaining $N2^{-1}$ entries is similar. More precisely, for any two deterministic Hermitian matrices $C$ and $B$, let $R(C)$ and $R(B)$ be their resolvents at $z$. Then it is proven in equation (91) of Lemma C.4 of [21] that:

$$
\frac{1}{N} \sum_{k=1}^{N/2} R_{k,k}(B) - R_{k,k}(C) \leq \frac{1}{N} \sum_{k=1}^{N} |R_{k,k}(C) - R_{k,k}(B)| \leq \text{rank}(C - B)2(\text{Im}(z)N)^{-1}.
$$

So if one considers the function

$$
F([x_i]_{i=1}^{N}) = \frac{1}{N} \sum_{k=1}^{N/2} f(R_{k,k}(X)), \quad [x_i]_{i=1}^{N} : x_i \in \mathbb{C}^{i-1} \times \mathbb{R},
$$

where $X$ is a Hermitian matrix with the $i$-th row of $X$ being $x_i$. Note that it suffices to describe the entries of the $i$-th row until the $i$-th column since the remaining elements will be filled by the properties of the Hermitian matrices. So if we consider two elements $X, X' \in \cup_{i=1}^{N} \mathbb{C}^{i-1} \times \mathbb{R}$ with only the $i$-th vector of $X$ and $X'$ different, then one has:

$$
|F(X) - F(X')| \leq \text{rank}(X - X')2(\text{Im}(z)N)^{-1} \leq 4(\text{Im}(z)N)^{-1}.
$$

since by construction, one has that $\text{rank}(X - X') \leq 2$. Now the desired inequality comes from Azuma-Hoeffding inequality, see Lemma 1.2 in [31]. □

**Corollary 3.22.** One can apply the previous Lemma to get the following concentration results. Fix an $N \times N$ symmetric random matrix $Y$ with i.i.d. entries (up to symmetry), where $N$ is an even integer, with resolvent
matrix $B = (Y - zI)^{-1}$ for $z = E + i\eta$. Then the following bounds are true:

\[
P \left( \frac{2}{N} \sum_{k=1}^{N/2} \left| B_{k,k} - \mathbb{E} B_{k,k} \right| \geq \frac{4 \log(N)}{(N\eta^2)^{1/2}} \right) \leq 2 \exp \left( -\frac{(\log(N))^2}{2} \right). \tag{3.25}
\]

Moreover for any $a \in (0, 2)$ there exists a constant $C = C(a)$ such that,

\[
P \left( \frac{2}{N} \sum_{k=1}^{N/2} \left| (-iB_{k,k})^{a/2} - \mathbb{E} (-iB_{k,k})^{a/2} \right| \geq x \right) \leq 2 \exp \left( -\frac{N(\eta^{a/2} x)^{2/a}}{C} \right). \tag{3.26}
\]

The same results hold for the remaining $N/2$ diagonal entries of $R$.

**Proof.** The first two inequalities are true by direct application of Lemma 3.21 for the functions $f(x) = x$ and $f(x) = \text{Im}(x)$ respectively.

For the third inequality let $c > 0$ and fix $\phi_c : \mathbb{C} \rightarrow \mathbb{R}^+$, such that

\[
\phi_c(z) = \begin{cases} 0 & |z| \leq c. \\ \frac{1}{x}(|z| - c) & |z| \in (c, 2c). \\ 1 & |z| \geq 2c. \end{cases} \tag{3.27}
\]

Note that the function $\phi_c$ is Lipschitz with Lipschitz constant bounded by $\frac{1}{x}$. Then define the function

\[
f_c(z) = (iz)^{a/2} \phi_c(z). \tag{3.28}
\]

Since $|1 - \phi_c(z)|(-iz)^{a/2} \leq (2c)^{a/2}$ for all $z \in \mathbb{C}^+$, it is clear that $|(-iz)^{a/2}| \leq f_c(z) + (2c)^{a/2}$. Moreover note that the function $f_c(z)$ is Lipschitz with constant bounded by $2c^{a-1}$.

So for any $x \geq 0$ fix $c$ such that $(2c)^{a/2} = x/4$. Then

\[
P \left( \frac{2}{N} \sum_{k=1}^{N/2} (-iB_{k,k})^{a/2} - \mathbb{E} (-iB_{k,k})^{a/2} \right) \geq x \right) \leq 2 \exp \left( -\frac{N(\eta^{a/2} x)^{2/a}}{C} \right).
\]

The following result is an analogue of Lemma 5.3 in \cite{22} for concentration of only half of the resolvent diagonal entries. The proof is analogous.

**Lemma 3.23.** Let $N$ be an even and positive integer, $A = \{a_{ij}\}_{1 \leq i,j \leq N}$ a symmetric matrix with independent entries (up to symmetry). Fix $u \in S^1$, $\alpha \in (0, 2)$ and $s \in (0, \frac{\alpha}{2})$. Moreover define the resolvent matrix $B = (A - z_0 I)^{-1}$ for $z_0 = E + i\eta \in \mathbb{C}^+$.

Then if we denote $f_u : \mathbb{C} \rightarrow \mathbb{C}$ such that $f_u(z) = (iz)^{a/2}$, there exists constant $C = C(\alpha) > 0$ such that

\[
P \left( \frac{2}{N} \sum_{i=1}^{\frac{N}{2}} \left| f_u(B_{i,i}) - \mathbb{E} f_u(B_{i,i}) \right| \geq x \right) \leq C(\eta^{a/2} x)^{-1/s} \exp \left( -\frac{N(\eta^{a/2} x)^{2/a}}{C} \right).
\]

A similar estimate is true for the concentration of the second half of the diagonal entries of the resolvent.

**Proof.** By definition of the norms in Definition 3.22 we need to bound the following quantities

\[
P \left( \sup_{u \in S^1} \left| \frac{2}{N} \sum_{k=1}^{\frac{N}{2}} f_u(B_{k,k}) - \mathbb{E} \frac{2}{N} \sum_{k=1}^{\frac{N}{2}} f_u(B_{k,k}) \right| \geq x \right) \quad \text{for any } x > 0, \tag{3.31}
\]

\[
P \left( \sup_{u \in S^1} \left| (u)^{1+2s/a} \sum_{k=1}^{\frac{N}{2}} f_u(B_{k,k}) - \mathbb{E} \frac{2}{N} \sum_{k=1}^{\frac{N}{2}} f_u(B_{k,k}) \right| \geq x \right) \quad \text{for any } x > 0. \tag{3.32}
\]
Fix $u \in S^1_r$ and $c > 0$. Then, similarly to the proof of Lemma 3.21 in Corollary 3.22, we can construct a function $\phi_c : C \to \mathbb{R}^r$, which is $\frac{1}{c}$-Lipschitz function and for which it is true that if we decompose $f_u$ in the following sense,

$$f_u(z) = \phi_c f_u(z) + (1 - \phi_c) f_u(z) = f_{1,u}(z) + f_{2,u},$$

then $f_{2,u}(z)$ is bounded by $(2c)^{1-\frac{r}{2}+s}$ and $f_{1,u}(z)$ is Lipschitz with constant bounded by $c' c^{r-\frac{r}{2}}$, for some other absolute constant $c'$. So for any $x > 0$, if one fixes $c$ such that $(2c)^{1-\frac{r}{2}+s} = \frac{x}{2}$ it is implied that

$$P \left( \left| \frac{2}{N} \sum_{k=1}^{\frac{N}{2}} f_u(B_{k,k}) - \mathbb{E} \left[ \frac{2}{N} \sum_{k=1}^{\frac{N}{2}} f_u(B_{k,k}) \right] \right| \geq x \right) \leq P \left( \left| \frac{2}{N} \sum_{k=1}^{\frac{N}{2}} f_{1,u}(B_{k,k}) - \mathbb{E} \left[ \frac{2}{N} \sum_{k=1}^{\frac{N}{2}} f_{1,u}(B_{k,k}) \right] \right| \geq \frac{x}{2} \right).$$

So by a direct application of Lemma 3.21 for the function $f_{1,u}$ one can conclude that

$$P \left( \left| \frac{2}{N} \sum_{k=1}^{\frac{N}{2}} f_{1,u}(B_{k,k}) - \mathbb{E} \left[ \frac{2}{N} \sum_{k=1}^{\frac{N}{2}} f_{1,u}(B_{k,k}) \right] \right| \geq x \right) \leq \exp \left( -\frac{N(x\eta)^{2}}{C} \right).$$

for some constant $C = C(a)$. Moreover due to the deterministic bounds in (3.14) and (3.19), we can restrict to the case that $x \leq 4\eta^{-\frac{r}{2}}$. Furthermore, by (4.6) in [22] for any $c \in (0, 1)$, any $c$-net of the sphere $S^1_r$ has cardinality at most $\frac{x}{\eta}$. Set $F^r$ one $c$-net of the sphere. So for any $x \in (0, 4\eta^{-\frac{r}{2}})$, fix $c$ such that $(2c\eta^{-1})^{1-\frac{r}{2}+s} = \frac{x}{4}$. Thus by (3.35), we conclude that

$$P \left( \sup_{u \in F^r} \left| \frac{2}{N} \sum_{k=1}^{\frac{N}{2}} f_u(B_{k,k}) - \mathbb{E} \left[ \frac{2}{N} \sum_{k=1}^{\frac{N}{2}} f_u(B_{k,k}) \right] \right| \geq x \right) \leq P \left( \sup_{u \in F^r} \left| \frac{2}{N} \sum_{k=1}^{\frac{N}{2}} f_{1,u}(B_{k,k}) - \mathbb{E} \left[ \frac{2}{N} \sum_{k=1}^{\frac{N}{2}} f_{1,u}(B_{k,k}) \right] \right| \geq \frac{x}{2} \right).$$

So we get (3.31) after using the union bound and (3.35) to bound (3.36).

It remains to prove (3.32). For this note that

$$\frac{2}{N} \sum_{k=1}^{\frac{N}{2}} f_{1,u}(B_{k,k}) = \mathbb{E} \left[ \frac{2}{N} \sum_{k=1}^{\frac{N}{2}} f_{1,u}(B_{k,k}) \right] + \frac{1}{4} \left( 1 - \frac{a}{2} + s \right) (B_{k,k}|u) \frac{x}{4} - (B_{k,k}|u)/(B_{k,k}|u).$$

So we can treat the function $g_u(z) = (iz|u)(iz|j)$ analogously $f_u(z)$ in (3.33). In particular we have the following decomposition

$$g_u(z) = \phi_c g_u(z) + (1 - \phi_c) g_u(z) = g_{1,u}(z) + g_{2,u},$$

where $g_{1,u}$ is Lipschitz with constant bounded by $c_0 c^{r-\frac{r}{2}}/|u|l$ and $g_{2,u}$ is bounded by $c_0 c^{1+\frac{r}{2}}/|u|l$, for some absolute constant $c_0$. So for any $x > 0$ let be a number such that $c_0 c^{1+\frac{r}{2}} = x/4$, we get that

$$P \left( (|u|)^{1-\frac{r}{2}+s} \left( \frac{2}{N} \sum_{k=1}^{\frac{N}{2}} f_{1,u}(B_{k,k}) - \mathbb{E} \left[ \frac{2}{N} \sum_{k=1}^{\frac{N}{2}} f_{1,u}(B_{k,k}) \right] \right) \geq x \right) \leq P \left( (|u|)^{1-\frac{r}{2}+s} \left( \frac{2}{N} \sum_{k=1}^{\frac{N}{2}} g_{1,u}(B_{k,k}) - \mathbb{E} \left[ \frac{2}{N} \sum_{k=1}^{\frac{N}{2}} g_{1,u}(B_{k,k}) \right] \right) \geq x/2 \right).$$

By a direct application of Lemma 3.21 we can bound (3.40). The last part of the proof is completed by a $c$-net argument, completely analogously to (3.36). □

**Lemma 3.24** [22], Lemma 5.4. Let $(y_1, y_2, \ldots, y_N)$ be a Gaussian random vector whose covariance matrix is the Id. Fix $a \in (0, 2), s \in (0, a/2)$. Moreover, let $|h_k| \in (C^+)^N$ such that $|h_k| \leq \eta^{-1}$, for some $\eta > 0$. Then for each $j \in \mathbb{N}$ define the following quantities

$$f_j(u) = \langle h_j | u \rangle^{a/2} |y_j|^a, \quad g_j(u) = \langle h_j | u \rangle^{a/2} \mathbb{E} |y_j|^a.$$

Then there exists a constant $C = C(a)$ such that

$$P \left( \left| \frac{1}{N} \sum_{j=1}^{N} f_j - g_{j+N} \right| \right)_{1-a/2+s} \geq x \right) \leq C(\eta^{a/2}x)^{-1/\sigma} \exp \left( -\frac{N(\eta^{a/2}x)^{2/\sigma}}{C} \right).$$

**Remark 3.25.** Due to the deterministic bound (3.19), we can apply Lemma 3.24 for any number of the diagonal entries of the resolvent of a matrix.
3.5 Gaussian and stable random variables

In this subsection we present several results concerning Gaussian random variables and their interaction with the quantities we study.

**Lemma 3.26.** (Lemma 6.4) Let \(N \in \mathbb{N}\) and \(x\) be a \(b\)-removal of a \((0, a)\)-stable distribution, as is defined in Definition 6.4. Then let \(\hat{X}\) be an \(N\)-dimensional vector with independent entries all with law \(N^{-1/a}x\). Then for any \(u \in \mathbb{R}\) and for \(A\) a non-negative symmetric matrix and \(Y\) an \(N\)-dimensional centered Gaussian vector with covariance matrix \(\text{Id}\) it is true that,

\[
\mathbb{E} \left[ \exp \left( -\frac{u^2}{2} \langle AX, \hat{X} \rangle \right) \right] = \mathbb{E} \exp \left( -\frac{|u|^a \|A^{1/2}Y\|_a^a}{N} \right) \exp \left( O(u^2 N^{(2-a)/(b-1/a)-1} \log(N) \text{tr}(A)) \right) + N \exp \left( -\frac{\log^2(N)}{2} \right). \tag{3.41}
\]

**Lemma 3.27.** (Lemma 6.5) Let \(N\) be a positive integer and let \(r, d\) be positive real numbers such that \(0 < r < 2 < d \leq 4\). Denote \(w = (w_1, w_2, \cdots, w_N)\) to be a centered \(N\)-dimensional Gaussian random variable with covariance matrix \(U_{ij} = \mathbb{E}(w_i w_j)\) for \(i, j \in [N]\). Denote \(V_j = \mathbb{E}(w_j^d)\) for each \(j \in [N]\) and define

\[
U = \frac{1}{N^2} \sum_{i,j \in [N]} U_{ij}^2, \quad V = \frac{1}{N} \sum_{j=1}^N V_j, \quad X = \sum_{i=1}^N V_j^{d/2}, \quad p = \frac{d-r}{d-2}, \quad q = \frac{d-r}{2-r}.
\]

Then if \(V > 100 \log^{10}(N) U^{1/2}\) there exists a constant \(C = C(a, r)\):

\[
\mathbb{P} \left( \frac{|u_r|^p}{N} < \frac{V^p}{C(X(\log(N))^8)^p/q} \right) \leq C \exp \left( -\frac{(\log(N))^2}{2} \right).
\]

3.6 Bounds for the resolvent of \(X\).

Recall the notation \(R\) for the resolvent of \(X\) and let \(X^{(i)}\) is the matrix \(X\) with its \(i\)-th row and column replaced by 0 vector, as in Definition 3.17.

In what follows we will use the following notation \(R^{(i)} = (X^{(i)} - z I)^{-1}\) and

\[
S_i(z) = \sum_{j \in [2N] \setminus i} X_{ij}^2 R_{ij}^{(i)}(z) \quad \text{and} \quad T_i(z) = X_{ii} - U_i(z) \quad \text{where} \quad U_i(z) = \sum_{j \in [2N]} \sum_{k \in [2N] \setminus i} X_{ik} R_{jk}^{(i)}(z) X_{kj}, \quad i \in [2N]. \tag{3.43}
\]

For notational convenience, we will omit the dependence of \(S_i(z), T_i(z)\) and \(U_i(z)\) from \(z\) and \(N\), the dimension of the matrix. By the resolvent equality in Lemma 3.18, one has that

\[
R_{i,i} = \frac{1}{T_i - z - S_i}. \tag{3.44}
\]

Moreover for each \(i \in [2N]\), one has that \(\text{Im}(R^{(i)})\) is positive definite, since it is symmetric and by the spectral theorem its eigenvalues are

\[
\eta \left( j \lambda_j(X^{(i)}) + E \right)^2 + \eta^2 > 0, \quad j \in [2N], \tag{3.45}
\]

where \(j \lambda_j(X^{(i)})\) are the eigenvalues of \(X^{(i)}\). So it is true that

\[
\text{Im}(S_i) \geq 0 \quad \text{and} \quad \text{Im}(S_i - T_i) \geq 0. \tag{3.46}
\]

In addition, the diagonal entries of the resolvent \(R_{i,i}\) are identically distributed. This is proven in the following Lemma.

**Lemma 3.28.** The random variables \(R_{i,i}\), for each \(i \in [2N]\), are identically distributed.

**Proof.** Note that due to Schur’s complement formula it is true that for any \(N \times N\) matrices \(A, B, C, D\), if \(A, D\) are invertible then

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & * \\ * & * \end{bmatrix} \begin{bmatrix} D - CA^{-1}B \end{bmatrix}. \]
So if one sets $A = D = -zI$, $C = K$ and $B = K^T$ it is true that $R_{il} = \frac{z(K^T K - z^2 I)}{u}$ for $i \in [N]$ and $R_{il} = \frac{z(K^T z - z^2 I)}{u}$ for $i \in [2N] \setminus [N]$. Thus we can conclude that for each $i \in [N]$ the diagonal term $R_{ii}$ has the same law as $R_{i+N+i,N}$. Moreover, for $i,j \in [N]$ or $i,j \in [2N] \setminus [N]$ it is easy to see that the matrix $X$ retains its law after the permutation of $i$–th column and row to the $j$–th. All these imply that the diagonal terms $R_{il}$ have the same law for each $i \in [2N]$.

Note that the Lemma above would not be true if the dimensions of the matrix, whose symmetrization is $X$, were not equal.

Moreover since the matrix $X$ has 0 at its diagonal blocks, one may compute that

$$S_i = \sum_{j=1}^{N} X_{i,N+j}^2 R_{N+j,N+j}^{(0)}, \quad T_i = -\sum_{j,k \in [2N] \setminus \{i\}} X_{i,j} X_{k,i} R_{j,k}^{(1)}. \quad (3.47)$$

Keep in mind that we want to prove Theorem 3.13, so in what follows in this section we will operate under the assumption that $\null^{(3.9)}$ holds.

The following is the analogue of Proposition 7.9 in [24], adjusted to our set of matrices.

**Proposition 3.29.** For each $i \in [2N]$ there exists a constant $C = C(a, \epsilon, b) > 1$ such that

$$\Pr \left( \operatorname{Im}(S_i) < \frac{1}{C \log(N)^C} \right) \leq C \exp \left( -\frac{(\log(N))^2}{C} \right).$$

**Proof.** We will prove the estimate for $S_1$, since $R_{ii}$ are identically distributed for $i \in [2N]$ due to Lemma 3.28. Set the event:

$$\mathcal{E} = \left\{ \left\| \frac{1}{N} \sum_{j=1}^{N} R_{N+j,N+j}^{(1)} \right\|_F \leq \frac{8 \log(N)}{(N\eta)^{1/2}} + \frac{16}{N\eta} \right\}.$$

By Corollary 3.22 and Lemma 3.20 one has that $\Pr(\mathcal{E}^c) \leq 2 \exp(-\log(N)^2)$.

Observe that $\operatorname{Im}(S_1) = \langle A \tilde{X}, \tilde{X} \rangle$, where $A$ is an $N$–dimensional diagonal matrix with entries $A_{ij} = \operatorname{Im}(R_{N+j,N+j}^{(1)})$ and $\tilde{X}$ is an $N$–dimensional vector with entries $\tilde{X}_j = X_{i,N+j}$. So we can apply Markov inequality for $u = (\log(N)^2/2)(2 \log(2))^{1/2}$ to get that:

$$\Pr(\operatorname{Im}(S_1) < 1(\mathcal{E}) \log(N)^{-4/\alpha}) \leq 2E(1(\mathcal{E}) \exp \left( -\frac{u^2}{2} \langle A \tilde{X}, \tilde{X} \rangle \right).$$

Next, we can apply Lemma 3.26 and after bounding $\text{tr}(A)$ by $C = C'(a, \epsilon, b, \epsilon)$, which we can do since we work on the set $\mathcal{E}$ and since it is true that $E \operatorname{Im}(R_{1,1}) \leq (E|R_{1,1}|^2)^{1/2} \leq \epsilon^{1/4}$ due to our assumption that $\null^{(3.9)}$ in [3.9] hold. We conclude that

$$\Pr \left( \left\| A^{1/2} Y \right\|_F^2 / N \leq \frac{1}{C N} \right) \leq C \exp \left( -\log^2(\log(N)) \right) + C \exp \left( \frac{-(\log(N))^2}{C} \right),$$

where $Y$ is a Gaussian vector with covariance matrix the identical, as mentioned in Lemma 3.26. Thus it remains to prove a lower bound for

$$\frac{\left\| A^{1/2} Y \right\|_F^2}{N} \geq \frac{1}{N} \sum_{j=1}^{N} \left| \operatorname{Im}(R_{N+i,j}^{(1)}) \right|^{1/2} \|Y_j\|^2. \quad (3.48)$$

Note that for $s \in (0, \frac{2}{3})$ and by Remark 3.3

$$\left| \frac{1}{N} \sum_{j=1}^{N} \left| \operatorname{Im}(R_{N+i,j}^{(1)}) \right|^{1/2} \|Y_j\|^2 \right| \leq \sup_{u \in \mathbb{S}^2} \frac{1}{N} \sum_{j=1}^{N} \left( \left| -u R_{N+i,j}^{(1)} \right|^{1/2} \|Y_j\|^2 \right) \leq \frac{1}{N} \sum_{j=1}^{N} \left| -u R_{N+i,j}^{(1)} \right|^{1/2} \|Y_j\|^2 \quad (3.49)$$

$$\leq \frac{1}{N} \sum_{j=1}^{N} \left( \left| -u R_{N+i,j}^{(1)} \right|^{1/2} \|Y_j\|^2 \right) \leq \frac{1}{N} \sum_{j=1}^{N} \left( \left| -u R_{N+i,j}^{(1)} \right|^{1/2} \|Y_j\|^2 \right) \quad (3.50)$$

$$\leq \left( 1 - \frac{1}{2} \left( \frac{1}{2} \left( \frac{1}{2} \right)^{1/2} \|Y_j\|^2 \right) \right) \quad (3.51)$$
So we can apply Lemma 3.24 for
\[ x = C \frac{\log^4 N}{N^{a/4} \eta^{a/2}}. \]
and \( s = \frac{4}{3} \) to get that the inequality
\[
\left| \frac{1}{N} \sum_{j=1}^{N} \text{Im}(R_{N+j,N+j}^{(1)})^{a/2} |y_j|^a - \frac{1}{N} \sum_{j=1}^{N} \text{Im}(R_{N+j,N+j}^{(1)})^{a/2} \text{E}|y_j|^a \right| \leq x
\] (3.52)
holds with probability at least \( 1 - C \exp(-\frac{\log^2(N)}{C}) \). Thus, it is sufficient to give a lower bound to
\[
\frac{1}{N} \sum_{j=1}^{N} \text{Im}(R_{N+j,N+j}^{(1)})^{a/2} \text{E}|y_j|^a.
\] (3.53)
in order to obtain a lower bound for the quantity in (3.48).

Next we apply again Lemma 3.20 for \( r = \frac{2}{3} \), and since for any \( u_1, u_2 \in \mathbb{R}^+ \) and \( r \in (0, 1) \) it is true that
\[ |u_1^r - u_2^r| \leq |u_1 - u_2|^r, \]
we obtain that
\[
\text{E}|y_1|^a \frac{1}{N} \sum_{j=1}^{N} \left| \text{Im}(R_{N+j,N+j}^{(1)})^{a/2} - \text{Im}(R_{N+j,N+j}^{(1)})^{a/2} \right| \leq \frac{4}{(\eta N)^{a/2}} |y_1|^a.
\]
So we have concluded that the event that
\[
\frac{\|A^{1/2} Y\|_a^a}{N} \geq O\left( \frac{1}{(\eta N)^{a/2}} \right) + O\left( \frac{1}{(\eta N^{1/2})^{a/2}} \right) + C' \frac{1}{N} \sum_{j=1}^{N} \text{Im}(R_{N+j})^{a/2}.
\]
holds with probability at least \( 1 - C \exp\left(-\frac{\log^2(N)}{C} \right) \). Restricting again on the set \( E \) and using the concentration inequality 3.25 and our hypothesis 3.9 one can conclude that there exists \( C = C(a, \epsilon, b) \) such that
\[
\mathbb{P}\left( \frac{\|A^{1/2} Y\|_a^a}{CN} \leq \epsilon \right) \leq \exp\left(-\frac{-\log^2(N)}{C} \right).
\]
which finishes the proof. \( \square \)

The following is the analogue of Proposition 7.10 in [24], adjusted to our set of matrices.

**Proposition 3.30.** For each \( i \in [2N] \) there exists a constant \( C = C(a, \epsilon, b) > 1 \) such that
\[
\mathbb{P}\left( \text{Im}(S_i - T_i) < \frac{1}{C(\log(N))} \right) \leq C \exp\left(-\frac{(\log(N))^2}{C} \right).
\] (3.54)
Moreover
\[
\mathbb{P}\left( \max_{j \in [2N]} |R_{j,i}| > C \log^{1/2}(N) \right) \leq C \exp\left(-\frac{(\log(N))^2}{C} \right).
\] (3.55)

**Proof.** By construction, one can prove that for \( A = [\text{Im}(R_{i,j}^{(1)})]_{i,j \in [2N],[N]} \) and \( X = \{X_{i,N+j}\}_{i \in [N]} \) it is true that,
\[ \text{Im}(S_i - T_i) = \langle A \hat{X}, \hat{X} \rangle. \]
So after applying Lemma 3.26 like in Proposition 3.29 one has that
\[
\mathbb{P}\left( \text{Im}(S_i - T_i) < \frac{1}{\log^{4/5}(N)} \right) \leq C \mathbb{E}\exp\left(-\frac{C \log^2(N) \|A^{1/2} Y\|_a^a}{N} \right) + C \exp\left(-\frac{-\log^2(N)}{C} \right),
\]
where \( Y \) is again a centered \( N \)-dimensional Gaussian random variable with covariance matrix equal to the identical.

Next, we want to apply Lemma 3.27 in order to establish a lower bound for \( \frac{1}{N} \|A^{1/2} Y\|_a^a \). Following the notation of Lemma 3.27 set
\[ w_i = (A^{1/2} Y)_i, \quad V_j = \text{Im}(R_{j,j}^{(1)}), \quad U_{j,k} = \text{Im}(R_{j,k}^{(1)}(z)). \]
\[ X' = \frac{1}{N} \sum_{t=1}^{N} Y_{a,t}^{e/2, N_{t+1}} \quad U = \frac{1}{N^2} \sum_{j \in [2N]} U_{kj} \quad r = a, \ d = 2 + e. \] (3.56)

So one may apply Lemma [3.16] and Lemma [3.18] to get that

\[ U \leq \frac{4}{N^2} \sum_{j \in [2N]} U_{kj}^2 \leq \frac{4}{N^2} \sum_{j \in [2N]} |\text{Im}(R_{1j})|^2 \leq \frac{4}{N^2} \sum_{j=1}^{2N} |\text{Im}(R_{1j})| \leq \frac{4}{N^2}. \] (3.57)

Next, we can approximate \( V = \frac{1}{N} \sum_{j=1}^{N} V_{N_{t+1}} \) by \( \frac{1}{N} \sum_{j=1}^{N} \text{Im}(R_{N_{t+1}j}) \) due to the deterministic bound in Lemma [3.20] and then approximate \( \frac{1}{N} \sum_{j=1}^{N} \text{Im}(R_{N_{t+1}j}) \) by \( \frac{1}{2} \sum_{j=1}^{N} \text{Im}(R_{1j}) \) due to Corollary [3.16] on an event which holds with probability at least \( 1 - 2 \exp \left(-\frac{\log^4(N)}{8}\right) \). The approximation procedure described above is identical to the similar approximation described in Proposition [3.29]. So after taking into account the Hypothesis [3.9], we have that

\[ \mathbb{E} \text{Im}(R_{1j}) \geq \left( \mathbb{E} |\text{Im}(R_{1j})|^{a/2} \right)^{2/a} \geq \epsilon^{2/a}. \]

Thus, it is implied that

\[ \mathbb{P} \left( \frac{|V|}{C} < 1 \right) < C \exp \left( -\frac{\log^2 N}{C} \right). \] (3.58)

So after combining (3.57) and (3.58), we get that for sufficient large \( N \) it is true that

\[ \mathbb{P} \left( |V| \leq 100 \log^{10}(N) U^{1/2} \right) \leq C \exp \left( -\frac{\log^2(N)}{C} \right). \]

Next we need to bound \( X' \) from (3.56). Note that again we can apply Lemma [3.20] to get that

\[ \left| \frac{1}{N} \sum_{j=1}^{N} \text{Im}(R_{j+N_{t+1}}) - X' \right| \leq \frac{1}{N} \sum_{j=1}^{N} |\text{Im}(R_{j+N_{t+1}})^{a/2} - \text{Im}(R_{1j})^{a/2}| \leq \frac{1}{N} \sum_{j=1}^{2N} |R_{1j} - R_{1j}^{(1)}|^{a/2} \leq \frac{4}{(N \eta)^{a/2}}. \] (3.59)

Moreover, since the function \( f(y) = 1 \) \( ||\text{Im}(y)|| \leq \eta ||\text{Im}(y)||^{a/2} + 1 \) \( ||\text{Im}(y)|| \geq \eta ||\text{Im}(y)||^{a/2} \) is Lipschitz with Lipschitz-constant \( L = a \eta^{-1/2} \), we can apply Lemma [3.21] for \( x = N^{-1/2} \eta^{1/2} \log(N) \) to get that

\[ \mathbb{P} \left( \left| \frac{1}{N} \sum_{j=1}^{N} |\text{Im}(R_{N_{t+1}j})|^{1/2} \right| - \frac{1}{N} \mathbb{E} |\text{Im}(R_{N_{t+1}j})|^{1/2} \right| \geq \frac{\log(N)}{N^{1/2} \eta^{1/2}} \right) \leq 2 \exp \left( -\frac{\log^2(N)}{8 \alpha^2} \right). \] (3.60)

So after combining (3.60), (3.59) with (3.9) and specifically with the fact that \( \mathbb{E} |\text{Re}(R_{1j})|^{a/2} \leq \mathbb{E} |\text{Re}(R_{1j})|^{a/2} \leq \frac{1}{e^{a/2}} \), we get that

\[ \mathbb{P} (|X'| > C) \leq C \exp \left( -\frac{\log^2(N)}{C} \right). \] (3.61)

for sufficient large universal constant \( C \). So the bounding for \( \frac{1}{N} ||A_{1/2} Y||^2 \) comes from a direct application of Lemma [3.27] with the bounding for \( V \) and \( X' \) proven in (3.57) and (3.61).

Note that (3.55) is a corollary of (3.54) and (3.44).

The following is the analogue of Proposition 5.9 in [24], adjusted to our set of matrices.

**Lemma 3.31.** There exists some constant \( C = C(a) \) such that for any \( x \geq 1 \) and for any \( i \in [2N] \), it is true that

\[ \mathbb{P} \left( |T_i| \geq \frac{C x^{1/2}}{(N \eta^{1/2})^{1/2}} \right) \leq \frac{C x^{1/2}}{x^{1/2}} \] (3.62)
We will bound each of the terms inside the sum in (3.68) individually. Firstly

Then

Thus combining (3.70), (3.69) and (3.66) we get that for some absolute constant $s$

For the first term on the right-hand-side of (3.65) note that by Markov’s inequality, the independence of $\{X_{i,j}\}_{j \in [2N]}$ and $R^{(1)}$ and the symmetry of the random variables $X_{i,j}$

We will bound each of the terms inside the sum in (3.68) individually. Firstly

The last inequality in (3.69) can be found in the proof of Proposition 5.9 [24]. Moreover, due to (3.21) and (3.19) one has that

Thus combining (3.70), (3.69) and (3.66) we get that for some absolute constant $C = C(\alpha)$ it is true that

Setting $s = x^{1/2}$, we get (3.65). 

### 3.7 Proof of Theorem 3.13

In order to prove Theorem 3.13, we wish to replace the entries of $X$ by $\alpha$–stable entries in several quantities, for example in quantities defined in (3.47), in order to use the properties of the $\alpha$–stable distribution.

Firstly consider the following

**Definition 3.32.** Define the following quantities:

$$\omega_x(u)^{(i)} = \Gamma \left(1 - \frac{\alpha}{2}\right) (|iz - \ell S_{\ell}| u)^{\alpha/2}, \quad \omega_x(u) = \mathbb{E} \omega_x(u)^{(i)}.$$  

$$G_{\ell} = \sum_{j \neq \ell \in N} Z_{j,\ell} R_{j,\ell}^{(i)}, \quad \Psi_x(u) = \Gamma \left(1 - \frac{\alpha}{2}\right) (|iz - \ell G_{\ell}| u)^{\alpha/2}, \quad \psi_x(u) = \mathbb{E} \Psi_x(u).$$

Here $Z_{j,\ell}$ are i.i.d. random variables from the definition of the matrix $D_N$, all with law $N^{-1/\alpha} Z$ where $Z$ is a $(0, \sigma)$ $\alpha$–stable random variable as in Definition 1.1.
Lemma 3.33. For any $p > 0$ there exists a constant $C = C(\alpha, \epsilon, b, s, p)$ such that
\[
\left| \mathbb{E}|R_{di}|^p - |(-S_i - z)^{-1}|^p \right| \leq \frac{C \log^C(N)}{(N\eta^2)^{\alpha/8}}. \tag{3.72}
\]
\[
\left| \mathbb{E}((-\overline{R}_{di})^p - \mathbb{E}([-iz - iS_i])^p \right| \leq \frac{C \log^C(N)}{(N\eta^2)^{\alpha/8}}. \tag{3.73}
\]
\[
|\gamma_z - \tilde{\omega}_z|_1-1^{-1} \leq \frac{C \log^C(N)}{(N\eta^2)^{\alpha/8}}. \tag{3.74}
\]

Proof. Let $C_1, C_2, C_3$ the constants from Propositions 3.29, 3.30 and Lemma 3.31 respectively and set $C = \max\{C_1, C_2, C_3\}$. Moreover let $E_1, E_2$ the events whose probability we bound in Proposition 3.29 and 3.30 respectively and set $E = E_1 \cup E_2$.

Note that due to our assumptions in 3.9, 3.19 and 3.46 it is true that
\[
\frac{1}{\text{Im}(S_i - T_i + z)} \leq N^{1/2}, \quad \frac{1}{\text{Im}(S_i + z)} \leq N^{1/2}. \tag{3.75}
\]
Furthermore by (5.5) in [24] one has that for any $u > 0$
\[
\left| |R_{di}|^p - |(-S_i - z)^{-1}|^p \right| \leq 1 \left( |T_i| < u \right) (p - 1)u \left( \frac{1}{\text{Im}(S_i - T_i + z)} \right)^{p+1} + \frac{1}{\text{Im}(S_i + z)} \right)^{p+1} \tag{3.76}
\]
\[
+ 1 \left( |T_i| \geq u \right) \left( \frac{1}{\text{Im}(S_i - T_i + z)} \right)^{p} + \frac{1}{\text{Im}(S_i + z)} \right)^{p} \tag{3.77}
\]
So one by Propositions 3.29 and 3.30 one has that
\[
\mathbb{E} 1(E^c) \left| |R_{di}|^p - |(-S_i - z)^{-1}|^p \right| \leq 2u(p - 1)C^{p+1} \log^{p+1} N + 2\mathbb{P}(|T_i| \geq u) C^p \log^C(N) \tag{3.78}
\]
\[
\mathbb{E} 1(E) \left| |R_{di}|^p - |(-S_i - z)^{-1}|^p \right| \leq 2uN^{(p+1)/2} \exp \left( \frac{-\log^2 N}{C} \right) \tag{3.79}
\]
So after setting $u = (N\eta^2)^{-1/4}$ and applying Lemma 3.31 we get 3.72.

The proof of 3.73 is analogous and therefore it is omitted.

For the proof of 3.74 note that

- By 3.5 applied for $x_1 = (iT_i - iS_i - iz)^{-1}, x_2 = (-iS_i - iz)^{-1}$ and for $r = z$ and $\eta = (2C \log^{2C} N)^{-1}$, we get that there exists a constant $C' = C(\alpha) > 0$ such that for any $u > 0$ it is true that

\[
1(E^c) \left| |T_i| < u \right) |e - \omega_{z^{1-1/2}}| \leq C' \left( 2C \log^{2C} N \right)^{\frac{1}{2}}. \tag{3.80}
\]

- Moreover again by 3.5 for the same $x_1, x_2$ and $r$ as before and for $\eta = N^{-1/2}$ there exists a constant $C' = C(\alpha)$ such that

\[
1(E) |e - \omega_{z^{1-1/2}}| \leq 2C' \left( 2C \log^{2C} N \right)^{\frac{1}{2}}. \tag{3.81}
\]

Note that by definition $E \omega_z = \tilde{\omega}_z$ and $E \xi_z = y_z$. So after summing 3.83, 3.81 and 3.85, taking expectation and applying Propositions 3.29 and 3.30 and Lemma 3.31 for $x = (N\eta^2)^{1/4}$, we get 3.74. □
3.7.1 Fixed point equation

In this subsection we establish the asymptotic fixed point equation. Firstly, we show that the quantities in Definition 3.32 are approximately equal to the respective quantities of the Stieltjes transform, i.e., the quantities defined in Definition 3.12. The latter is proven in the following proposition.

Proposition 3.34. It is true that for any \( p \in \mathbb{N} \),
\[
\left| \mathbb{E}[R_{i,j}]^p - \mathbb{E}([-z - G_i]^p] \right| \leq \frac{C \log^C(N)}{(N \eta^2)^{a/8}} + \frac{\log^C(N)}{N^{4\theta}},
\]
and
\[
\left| \mathbb{E}([-iR_{i,j}]^p - \mathbb{E}([-iz - iG_i]^p] \right| \leq \frac{\log^C(N)}{(N \eta^2)^{a/8}} + \frac{\log^C(N)}{N^{4\theta}}.
\]

Proof. We first present two facts.

- One can show that there exists a constant \( C = C(\alpha) > 0 \) such that
  \[
  \mathbb{P}(|S_1 - G_1| \geq N^{-4\theta}) \leq C(1 + \mathbb{E}(R_{1,1}))N^{-4\theta},
  \]
  similarly to the proof of Lemma 6.8 in [24]. As a result, by Assumption 3.9 we have that
  \[
  \mathbb{P}(|S_1 - G_1| \geq N^{-4\theta}) \leq CN^{-4\theta},
  \]
  for some constant \( C = C(\alpha, \epsilon) \).

- For each \( i \in [2N] \) there exists a constant \( C = C(\alpha, \epsilon, b) > 1 \) such that
  \[
  \mathbb{P}\left( \text{Im}(G_i) < \frac{1}{C \log(N) a} \right) \leq C \exp\left( -\frac{(\log(N))^2}{C} \right).
  \]

The proof of (3.90) is completely analogous to the proof of Proposition 3.29 after replacing the usage of Lemma 3.26 with Lemma B.1 in [21]. Therefore it is omitted.

Moreover note that due to Lemma 3.33 it is sufficient to prove that for any \( p \in \mathbb{N} \),
\[
\left| \mathbb{E}|z - S_1|^p - \mathbb{E}|(-z - G_1)^p] \right| \leq \frac{C \log^C(N)}{N^{4\theta}}.
\]
(3.91)
\[
\left| \mathbb{E}|(-iz - iS)|^p - \mathbb{E}|(-iz - iG_i)|^p] \right| \leq \frac{C \log^C(N)}{N^{4\theta}}.
\]
(3.92)
\[
|\tilde{\alpha}_z - \psi_z|_{1-\alpha/2+s} \leq \frac{C \log^C(N)}{N^{4\theta}}.
\]
(3.93)

Given (3.89) and (3.90), the proof of (3.91), is completely analogous to the proof of Lemma 3.33 therefore it is omitted.

Moreover, we have the following results which will be used in order to establish the limiting fixed point equation. The following Lemma will be the basis for the approximation of the fixed point equation.

Lemma 3.35. Recall Definition 3.5. It is true that,
\[
\Psi_z(u) = \mathbb{E}_D(Y_z(u)),
\]
where \( Y_z \) is as in Definition 3.5 \( D = \{y_j\}_{j=1}^N \) is an \( N \)-dimensional Gaussian random variable independent from any other quantity with covariance matrix being the identical, \( \mathbb{E}_D \) denotes the expectation with respect to the random variable \( D \) and
\[
\zeta(u) = \frac{1}{N} \sum_{j=1}^N \left( -\frac{|R_{i,j}^1\mid_{N+1,N+1}}{N} \right)^{\alpha/2} \frac{|y_j|^\alpha}{\mathbb{E}|y_j|^\alpha}.
\]

Also,
\[
\mathbb{E}(-iz - iG_1)^p = \mathbb{E}_D s_p(z(1)), \quad \mathbb{E}|z - G_1|^p = \mathbb{E}_D r_p(z(1)).
\]
Proof. This Lemma is a corollary of [22 Corollary 5.8] □

So in Proposition [3.34] we manage to approximate the quantities involving $G_1$, such as $y_x(u)$, by the analogous quantities involving $R_{ij}$, such as $y_x(u)$. In order to establish the asymptotic fixed point equation, we will need to approximate the function $\zeta(u)$ mentioned in Lemma [3.35] by $y_x(u)$ and then take advantage of [3.34]. This approximation is done via the following Lemma.

**Lemma 3.36.** There exists a constant $C = C(\alpha, \epsilon, s) > 1$ such that
\[
P \left( |\zeta - y_x|_{1-a/2+s} > \frac{C \log^C(N)}{N^{a/2} \eta^{a/2}} \right) \leq C \exp \left( -\frac{\log^2(N)}{C} \right).
\] (3.95)

**Proof.** Firstly note that $\zeta$ is close to $E_D \zeta$ with high probability due to Lemma [3.24] for appropriate $x$, i.e.,
\[
P \left( |\zeta - E_D \zeta|_{1-a/2+s} \geq \frac{\log^y(N)}{N^{a/2} \eta^{a/2}} \right) \leq C \exp \left( -\frac{\log^2 N}{C} \right).
\] (3.96)

for some constant $C = C(\alpha)$. Next, note that by Lemma [3.23] applied for the matrix $X^{(1)}$ and for appropriate $x$ one has that
\[
P \left( |E_{X^{(1)}} E_D \zeta - E_D \zeta|_{1-a/2+s} \geq \frac{\log^y(N)}{N^{a/2} \eta^{a/2}} \right) \leq C \exp \left( -\frac{\log^2 N}{C} \right).
\] (3.97)

for some appropriately chosen constant $C = C(\alpha)$. Here $E_{X^{(1)}}$ denotes the mean value with respect to the law of the matrix $X^{(1)}$. Next by Lemma [3.4] one has that
\[
\sum_{i=1}^{N} \frac{1}{N} |R_{N+N+1,i} - R_{N+N+1,i}^{(1)}|_{1-a/2+s} \leq C \eta^{-a/2} \frac{1}{N} \sum_{i=1}^{N} \left( |R_{N+N+1,i} - R_{N+N+1,i}^{(1)}|^{a/2} + \eta^{s} |R_{N+N+1,i} - R_{N+N+1,i}^{(1)}|^2 \right).
\] (3.98)

So after applying Lemma [3.20] and since $R_{ij}$ are identical distributed, one has the deterministic bound
\[
|E_{X^{(1)}} E_D \zeta - y_x| \leq C \left( \frac{1}{N} \eta^{s} N^{a/2} + \frac{1}{N} \right).
\] (3.99)

So after combining (3.96), (3.97) and (3.99), we get the desired inequality. □

Next, we give some more approximating results.

**Corollary 3.37.** There exists a constant $C = C(\alpha, \epsilon, s) > 0$ such that
\[
|y_x|_{1-a/2+s} < C, \quad \inf_{u \in S_+^1} \text{Re}(y_x(u)) > \frac{1}{C},
\] (3.100)

\[
P \left( \inf_{u \in S_+^1} \zeta(u) < \frac{1}{C} \right) < C \exp \left( -\frac{\log^2(N)}{C} \right).
\] (3.101)

**Proof.** By [3.95], the estimate in [3.101] is a consequence of [3.100].

For [3.95] note that due to the first estimate in (3.5) one has that there exists a constant $C = C(s)$ such that
\[
\left| (-iR_{ij}u)^{a/2} \right|_{1-s/2} \leq C |R_{ij}|^{a/2}.
\] (3.102)

By integrating (3.102) and by the definition of $y_x$ in Definition [3.12] one has that,
\[
|y_x|_{s/2} \leq C \Gamma \left( 1 - \frac{a}{2} \right) \mathbb{E} |R_{ij}|^{a/2} \leq C \Gamma \left( 1 - \frac{a}{2} \right) \left( \mathbb{E} |R_{ij}|^{a/2} \right)^{a/4} \leq C \left( 1 - \frac{a}{2} \right).
\] (3.103)

Where in the first inequality in (3.103) we used (3.102), in the second we used Holder’s inequality and in the third we used our Assumption [3.9]. So the first estimate in (3.100) is proven.

For the second estimate in (3.100) one has that for any $u \in S_+^1$
\[
\text{Re} y_x(u) = \Gamma \left( 1 - \frac{a}{2} \right) \mathbb{E} \text{Re} (iR_{ij}u)^{a/2} \geq \Gamma \left( 1 - \frac{a}{2} \right) \mathbb{E} \left( \text{Re} (iR_{ij}u)^{a/2} \right)
\] (3.104)

\[
\geq \Gamma \left( 1 - \frac{a}{2} \right) \mathbb{E} \text{Im} R_{ij}^{a/2} \geq \Gamma \left( 1 - \frac{a}{2} \right) \epsilon.
\] (3.105)
where in the first inequality in (3.104) we used the fact that \( \text{Re} \, c' \geq (\text{Re} \, c)' \) for any \( c \in \mathbb{K}^+ \) and \( r \in (0, 1) \), see the proof of Lemma 7.18 in [24]. In the second inequality we used the fact that \( \text{Re}(c)u \geq \text{Re}(c) \) for any \( c \in \mathbb{K}^+ \) and \( u \in S^1 \), and in the third we used our Assumption 3.9. Thus the second estimate in (3.100) is proven. □

Before presenting the proof of Theorem 3.13 we need a last approximation result.

**Lemma 3.38.** There exists a constant \( C = C(a, \varepsilon, s) \) such that

\[
P\left( |\psi_z - Y_{\kappa_1}|_{1/ -\delta + s} > \frac{C \log^2(N)}{N^{s/2} \eta^{a/2}} \right) < C \exp \left( -\frac{\log^2(N)}{C} \right). \tag{3.106}
\]

**Proof.** The strategy of the proof is firstly to approximate \( Y_{\kappa_1} \) by \( Y_{\kappa} \) and then use Lemma 3.35. 

- For the approximation of \( Y_{\kappa_1} \) and \( Y_{\kappa} \): Let \( C_1, C_2 \) be the constants mentioned in Lemma 3.36 and Corollary 3.37. Set \( C = 2 \max\{C_1, C_2\} \). Moreover define the following sets

\[
E_1 = \left\{ |\zeta - 2|_{1/ -\delta + s} > \frac{C \log^2(N)}{N^{s/2} \eta^{a/2}} \right\} \tag{3.107}
\]

\[
E_2 = \left\{ \inf_{u \in S^1} \text{Re} \, \zeta(u) < \frac{1}{C} \right\}. \tag{3.108}
\]

By Lemma 3.36 and Corollary 3.37 one has that

\[
P(E_1 \cup E_2) \leq C \exp \left( -\frac{\log^2 N}{C} \right). \tag{3.109}
\]

Set \( F \) the complement event of \( E_1 \cup E_2 \). So

\[
1(F) |Y_{\kappa} - Y_{\kappa_1}|_{1/ -\delta + s} \leq 1(F) C_1 |\zeta - 2|_{1/ -\delta + s} \left( 1 + |\zeta|_{1/ -\delta + s} + |\zeta|_{1/ -\delta + s} \right) \leq 1(F) \frac{C \log^2(N)}{N^{s/2} \eta^{a/2}} \left( 1 + \frac{2}{C} \right) \tag{3.110}
\]

where in the first inequality of 3.110 we used Lemma 3.10 and Remark 3.3 (\( C_1 \) is the constant mentioned in Lemma 3.10 and the fact that \( \gamma_2, \zeta \in H^{1/C}_{1/ -\delta + s} \) by Corollary 3.37 and the definition of the set \( F \). For the second inequality we used again the definition of \( F \), Corollary 3.37 and Lemma 3.36.

Now working on the event \( E_1 \cup E_2 \) we get that by Lemma 3.3 and Corollary 3.37 we there exists a constant \( C' > 0 \) such that

\[
1(E_1 \cup E_2) |Y_{\kappa_1}|_{1/ -\delta + s} \leq C' \eta^{-\delta} (1 + C) 1(E_1 \cup E_2) \tag{3.111}
\]

- Note that similarly to the proof of 3.75 one can prove that

\[
\left| \frac{1}{G_1 + z} \right| \leq \frac{1}{\eta} \tag{3.112}
\]

Thus, we can apply 3.5 to get that there exists a constant \( C = C(a) \) such that

\[
|\psi_z|_{1/ -\delta + s} \leq C \eta^{-a/2} \tag{3.113}
\]

So by Lemma 3.35 one has that

\[
|\psi_z - Y_{\kappa_1}|_{1/ -\delta + s} \leq E(1(F) |Y_{\kappa} - Y_{\kappa_1}|_{1/ -\delta + s} + E(1(E_1 \cup E_2) |\psi_z|_{1/ -\delta + s} + E(1(E_1 \cup E_2) |Y_{\kappa_1}|_{1/ -\delta + s} \tag{3.114}
\]

Now (3.106) is proven by combining 3.109, 3.113, 3.111, 3.114 and 3.110. □

Next, the proof of the main theorem of this subsection is presented.
Proof of Theorem 4.1. Note that, (3.10) is a consequence of (3.8) and (3.106). Additionally, (3.14) is already proven in (3.55). Lastly, note that (3.13) is a consequence of (3.100). So all that remains is to establish (3.11) and (3.12) in order to complete the proof. We will prove only (3.12). The proof of (3.11) is similar and will be omitted.

To that end, define the sets $E_1$ and $E_2$ as in (3.109) and $F$ the complement event of $E_1 \cup E_2$. So by the first estimate in (3.8) and Remark 3.3, one has that

$$1(F)|r_{p,x}(\xi) - r_{p,x}(\eta)| \leq 1(F)C|\eta - \xi|$$

for some constant $C'$. By the definition of the event $F$ and Lemma 3.35, we get the bound in (3.12) on the event $F$.

On the event $E_1 \cup E_2$ we can use the deterministic bound in Lemma 3.9 to get that

$$1(E_1 \cup E_2)|r_{p,x}(\xi) - r_{p,x}(\eta)| \leq 2C''\eta^{-p}1(E_1 \cup E_2)$$

for some other constant $C''$. Now the bound in (3.12) on the event $1(E_1 \cup E_2)$ is a consequence of (3.109) and (3.116). \qed

4 Universality for the least singular value after short time

At this section, universality of the least eigenvalue for the matrices $X + \sqrt{t}W$ is proven. More precisely:

**Theorem 4.1.** Let $L_N$ be an $N \times N$ matrix with i.i.d. entries all following the Gaussian distribution with mean 0 and variance $\frac{1}{N}$, independent from $H_N$. Then denote $W$ be the symmetrization of $L_N$. Let $\tilde{W}$ be an independent copy of $W$. Moreover, for every matrix $Y$ denote $\tilde{h}_N(Y)$ to be the smallest positive eigenvalue of $Y$. Then, for all $\alpha \in (0, 2)$ for which local law, Theorem 3.14 holds there exists $\varepsilon(\alpha) > 0$ such that for all $r > 0$

$$|\mathbb{P}(N\tilde{h}_N(X + \sqrt{sW}) \geq r) - \mathbb{P}(N\tilde{h}_N(W) \geq r)| \leq \frac{1}{N^{\varepsilon(\alpha)}}.\quad (4.1)$$

for all $s \in (N^{2\delta-\frac{1}{2}}, N^{-2\delta})$. Note that $\xi$ is the constant defined in (1.4).

The proof of Theorem 4.1 can be found in paragraph 4.

In order to begin the proof we need the following definition.

**Definition 4.2.** For an $N \times N$ matrix $J$ with eigenvalues $\{\lambda_i(J)\}_{i \in [N]}$ we define the free additive convolution of $J$, with $s$ times the semicircle law, to be the probability measure with Stieltjes transform $m_{s,fc}$, such that

$$m_{s,fc}(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i(J) - z - sm_{s,fc}(z)}.$$

It can be proven that the equation above has a unique solution. Moreover we denote by $\rho_{s,fc}(E)$ the density of the free convolution given by $\rho_{s,fc}(E) = \frac{1}{\pi} \lim_{\epsilon \to 0} \text{Im}(m_{s,fc}(E + i\epsilon))$.

**Remark 4.3.** For $z \in D_{c_{\alpha},\delta}$, the set for which the local law holds in Theorem 3.14 and $s \in (N^{2\delta-\frac{1}{2}}, N^{-2\delta})$, one has that $|m_{s,fc}(z) - m_{N,s}(z)| \leq \frac{1}{N\eta}$ with overwhelming probability, as is proven in Theorem 4.5 of [12]. Here $m_{s,fc}$ is the Stieltjes transform of the free additive convolution of $X$ with $s$ times the semicircle law and $m_{N,s}$ is the Stieltjes transform of the E.S.D of the matrix $X + \sqrt{sW}$, where $W$ is the symmetrization of a matrix with i.i.d. entries all following the Gaussian distribution with mean 0 and variance $\frac{1}{N}$. Moreover the following stability result is true, due to Lemma 4.1 of [12].

$$c \leq \text{Im}(m_{s,fc}(z)) \leq C.\quad (4.2)$$

In order to establish Theorem 4.1, we wish to apply Theorem 3.2 in [12] but we need to take into account Remark 7.6 of [13]. So firstly we state the following.

**Lemma 4.4.** Fix $s \in (N^{2\delta-\frac{1}{2}}, N^{-2\delta})$ for appropriate small $\delta$. Then

$$|\rho_s(x) - \rho_{s,fc}(x)| \leq N^{-\frac{\delta}{4}},$$

for $x \in (-\frac{1}{c_{\alpha}}, \frac{1}{c_{\alpha}})$ and $\alpha, \delta$ are parameters satisfying the assumptions of Theorem 3.14 and $C_{\alpha}$ the constant mentioned in the statement of Theorem 3.14.
Proof. The proof of the lemma is due to the local law Theorem 3.14 and similar to the proof of [33], Lemma 3.4, so it is omitted. □

Proof of Theorem 4.1. Firstly we apply Theorem 3.2 of [12] to the sequence of matrices \( \rho_{\text{uc}}(0)X_N \). Note that due to Theorem 3.14 the matrix \( X \) satisfies the assumptions of Theorem 3.2 for \( g = N^{\delta - \frac{1}{2}} \) and \( G = N^{-\delta} \) with overwhelming probability, for any small enough \( \delta > 0 \). So for all \( s_0, s_1 \in (N^{2\delta - \frac{1}{2}}, N^{-\delta}) \) such that \( s_0 = \frac{N^{\delta_0}}{N} \), \( s_1 = \frac{N^{\delta_1}}{N} \) with \( \delta_0 < \delta_1 < \frac{1}{2} \), there exists \( \delta' > 0 \) and a coupling of \( \tilde{h}_N(X + \sqrt{s_1 + s_0}W) \) and

\[
\left| \frac{\rho_{\text{uc}}(0)}{\rho_{\text{uc}, h}(0)} \tilde{h}_N(X + \sqrt{s_1 + s_0}W) - \tilde{h}_N(W + \sqrt{s_1 + s_0}W') \right| \leq \frac{1}{N^{4.1}},
\]

where \( W, W' \) are independent copies of \( W \). Moreover, by the properties of the Gaussian law, one has that \( W + \sqrt{s_1 + s_0}W' \) has the same law as \( \sqrt{1 + s_1 + s_0}W'' \), where \( W'' \) is again an independent copy of \( W \). But by Slutsky’s theorem one has that \( \lim_{N \to \infty} \tilde{h}_N(\sqrt{1 + s_1 + s_0}W'') \approx \lim_{N \to \infty} \tilde{h}_N(W) \). So one has that for each \( r > 0 \),

\[
\left| \mathbb{P} \left( N\tilde{h}_N(X + \sqrt{s_1 + s_0}W) \geq r \right) - \mathbb{P} \left( N\tilde{h}_N(W) \geq r \right) \right| \leq \frac{1}{N^{4.1}},
\]

where we have violated the notation in 4.4 by keeping the same constant \( \delta \). Next, since Remark 4.3 and Lemma 4.4 are true, one has that

\[
\left| \mathbb{P} \left( N\tilde{h}_N(X + \sqrt{s_1 + s_0}W) \geq r \right) - \mathbb{P} \left( N\tilde{h}_N(W) \geq r \right) \right| \leq \frac{1}{N^{4.1}}.
\]

Moreover for \( s_1, s_2 \in (N^{2\delta - \frac{1}{2}}, N^{-\delta}) \), such that \( s_1 < s_2 \), one can apply Weyl’s inequality, Lemma 3.15 to get that

\[
\tilde{h}_N(X + \sqrt{s_1}W) - \tilde{h}_N(X + \sqrt{s_2}W) \geq (s_1 - s_2)\tilde{h}_{\min}(W) \geq 0.
\]

The first inequality of (4.6) comes from the bottom of Weyl’s inequality, for the \( \frac{N}{2} + 1 \)–th eigenvalues of \( X + \sqrt{s_1}W \) and \( X + \sqrt{s_2}W \) when the eigenvalues are arranged in decreasing order. Note that in the notation we normally use, we have arranged the eigenvalues in decreasing order with respect to their absolute values.

The second inequality comes from the fact that \( \tilde{h}_{\min}(W) \) is the negative of the maximum singular value of \( L \). So (4.5) implies that if \( s_1 \leq s_2 \) then

\[
\tilde{h}_N(X + \sqrt{s_1}W) \geq \tilde{h}_N(X + \sqrt{s_2}W).
\]

Finally, fix \( s \in (N^{2\delta - \frac{1}{2}}, N^{-\delta}) \) and \( s_1 = \frac{N^{\delta_1}}{N}, s_2 = \frac{N^{\delta_2}}{N} \) parameters such that

\[
s_1 - \frac{N^{\delta_2}}{N} > N^{2\delta - \frac{1}{2}}, \quad s_1 < s, \quad s_2 - \frac{N^{\delta_2}}{N} \geq s \quad \text{and} \quad s_2 < N^{-\delta}.
\]

So by construction, one has that \( \tilde{h}_N(X + \sqrt{s_1}W) \) and \( \tilde{h}_N(X + \sqrt{s_2}W) \) are both universal in the sense of (4.5) and \( s_1 < s < s_2 \). So by (4.7),

\[
N\tilde{h}_N(X + \sqrt{s_2}W) \leq N\tilde{h}_N(X + \sqrt{s_2}W) \leq N\tilde{h}_N(X + \sqrt{s_1}W),
\]

which implies Theorem 4.1. □

Corollary 4.5. The least singular value of \( X + \sqrt{t}W \) is universal in the sense of Theorem 4.1, where \( t \) is defined in Definition 2.9.

Proof. We just need to show that \( t \) belongs to the interval \( (N^{2\delta - \frac{1}{2}}, N^{-\delta}) \), for any small enough \( \delta > 0 \), and then apply Theorem 4.1. Note that the latter claim is true due to the way \( \nu \) is chosen in (2.1), i.e.,

\[
0 < \nu(2 - a) < \frac{1}{2},
\]

and since \( t \) is of order \( N^{-\nu(2-a)} \). □
5 Isotropic local law for the perturbed matrices at the optimal scale

At this point we have proven, in Theorem 3.14, that some kind of regularity holds for the matrix $X$. Specifically we have proven that with high probability, the Stieltjes transform of $X$ converges to its deterministic limit, and its diagonal entries of its resolvent are logarithmically bounded, for complex numbers with imaginary parts of order just above $N^{-\frac{1}{4}}$. So, at this section we "justify" the reason why we have split the matrix $H$ into its "big" and "small" elements, i.e., the matrices $X$ and $A$, in Definition 2.4. More precisely, we proved that given the regularity properties of $X$ and after perturbing it by a Gaussian component, then the matrix becomes even more regular in some sense. Thus, what will remain to investigate is whether the "small" elements of $H$ preserve this regularity, which will be proven in the next section.

Specifically, at this section we show that for any small $\delta > 0$, the event $\delta$–dependent events

$$\left\{ \sup_{V_{\alpha,s}} \sup_{t} |T_{\alpha}(z)| \leq N^\delta \right\}, \quad (5.1)$$

hold with overwhelming probability. Here $D_{\alpha,s} = \{ E + in : E \in \left( \frac{1}{2C_1}, \frac{1}{2C_2} \right), \eta \in \left[ N^{5-1}, \frac{1}{4C_1} \right] \}$ and $C_\alpha$ is the constant mentioned in Theorem 3.13. This is stated in Corollary 5.15.

In order to prove the latter, we will show a general result which can be used for a general class of matrices. So except from Corollary 5.15 the rest of this section is independent from the rest of the paper. The general result we prove in Theorem 5.6 is an approximation of the resolvent of the symmetrization of a slightly perturbed by a Gaussian component matrix, which initially satisfies some regularity assumptions, Assumption 5.1. This resolvent is approximated by a quantity which involves the free additive convolution of the initial matrix with the semicircle law and the eigenvectors of the initial matrix. This approximation is achieved at any direction on the sphere, so it is called isotropic local law.

The isotropic local law is an analogue of Theorem 2.1 in [27] for our set of matrices, i.e., matrices perturbed by Gaussian factors with $0$ at the diagonal blocks. In [27] an isotropic local law is proven for matrices after perturbing them by a symmetric Brownian motion matrix.

This kind of results demands precise computations for the resolvent entries. In our case the "target" matrix, with which we compare the resolvent, is a diagonal matrix which lives in $M_N(M_2(\mathbb{C}))$. This increases the complexity of the calculations from the symmetric case where the "target matrix" is diagonal, but eventually this increase is not that significant.

5.1 Terminology

Firstly we introduce the terminology of [27].

For any $N$–dependent random variables $Y_1, Y_2$ we denote

1. $Y_1 \leq Y_2$ if there exists a universal constant $C > 0$ such that $|Y_1| \leq CY_2$.
2. $Y_1 \preceq_k Y_2$ if there exists a constant $C_k$ (which depends on some $k$) such that $|Y_1| \leq C_k Y_2$.
3. $Y_1 \ll Y_2$ if there is a positive constant $c$ such that $Y_1N^c \leq Y_2$.

5.2 Statement of the main result of this section

Assumption 5.1. Let $V$ be a deterministic $N \times N$ matrix. Denote $\tilde{V}$ the symmetrization of $V$ and $m_\tilde{V}$ the Stieltjes transform of $\tilde{V}$. Assume that there exists a large constant $\alpha > 1$ such that

1. $\| \tilde{V} \|_{sp} \leq N^\alpha$.
2. $\alpha^{-1} \leq \text{Im}(m_\tilde{V}(z)) \leq \alpha$, for all $z \in \{ E + in : E \in (E_0 - r, E_0 + r), h_s \leq \eta \leq 1 \}$, for some $N$–dependent constants $r, h_s$, such that: $\frac{1}{N} \ll h_s \ll r \leq 1$.

Moreover, fix $c > 0$ some arbitrary small constant and set

$$\psi = \frac{N^c}{N}.$$ (5.2)
Theorem 5.6. The parameter \( W \) with \( V \) and \( \lambda \) in Theorem 3.14 and since for fixed large \( D > 0 \), one can compute by \((1.3)\) that any given entry of \( X \) has magnitude greater than \( N^{2D} \) with probability less than \( CN^{-2D} \), which implies that

\[
P\left( \| X \|_{\text{op}} \geq N^{-2D} \right) \leq CN^{-2D}.
\]

Remark 5.3. Let \( V \) be a deterministic \( N \times N \) matrix. Due to the singular value decomposition of \( V \), there exist \( J_1, J_2 \) two orthogonal \( N \times N \) matrices, such that \( \Sigma = J_2 V J_1 \), where \( \Sigma \) is a diagonal matrix with diagonal entries the singular values of \( V \). Then denote \( \tilde{V} \) the symmetrization of \( V \) and set

\[
U = \begin{bmatrix} J_1^T & 0 \\ 0 & J_2 \end{bmatrix}.
\]

Then it is true that

\[
U \tilde{V} U^T = \begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix}.
\]

Moreover, note that \( U \) is orthogonal.

Definition 5.4. Suppose \( V \) is a deterministic matrix which satisfies the Assumption \([5.1]\) for some \( N \)-dependent constants \( h_*, r \). Then for any \( k \in (0, 1) \), define the set

\[
\mathbb{D}_k = \left\{ z = E + i\eta : E \in (E_0 - (1 - k)r, E_0 + (1 - k)r), \frac{\psi^4}{N} \leq \eta \leq 1 - kr \right\}.
\]

The parameter \( \psi \) is defined in \([5.2]\).

Definition 5.5. Recall the definition of the the Stieltjes transform of the Empirical spectral distribution of \( \tilde{V} \) with \( s \)-times the semicircle law in Definition \([4.2] \). We will use the following notation

\[
m_{s,k}(z) = \frac{1}{N} \sum_{i \in [N]} g_i(s, z), \quad \text{with} \quad g_i(s, z) = \frac{1}{\hat{\lambda}_i - z - sm_{s,k}(z)}
\]

and \( \hat{\lambda}_i \) are the eigenvalues of \( \tilde{V} \) arranged in increasing order so that \( \hat{\lambda}_i = -\hat{\lambda}_{-i} \).

Theorem 5.6. Let \( V \) be a deterministic matrix that satisfies the Assumptions \([5.1]\). Denote the matrix

\[
G(z, s) = (\tilde{V} + \sqrt{s} W - z I)^{-1}.
\]

Here \( W \) is the symmetrization of a matrix with \( i.i.d. \) entries, all following the Gaussian law with mean 0 and variance \( \frac{1}{N} \). Moreover, fix \( U \) to be the orthogonal matrix constructed in Remark \([5.3]\) for \( V \). Moreover, fix \( k \in (0, 1) \), \( s : h_* \ll s \ll r \) and \( q \in \mathbb{R}^N : \| q \|_2 = 1 \). Then it is true that,

\[
\left| \langle q, G(z, s) q \rangle - \sum_{i=-N}^{N} \frac{1}{2} (g_i + g_{-i})(s, z) (u_i, q)^2 - \sum_{i=1}^{N} (g_i - g_{-i})(s, z) (u_i, q)(u_{i+N}, q) \right| \leq \frac{\psi^2}{\sqrt{N}} \text{Im} \left( \sum_{i=1}^{N} ((u_i, q)^2 + (u_{i+N}, q)^2)(g_i(s, z) + g_{-i}(s, z)) \right),
\]

with overwhelming probability, uniformly for all \( z \in \mathbb{D}_k \). Here \( u_i \) denote the columns of \( U \).

Set \( C_j \), for \( j \in \{1, 2\} \), to be the \( N \times N \) diagonal matrices with their \( i \)-th diagonal element equal to \( g_i + (-1)^{i+1} g_{-i} \). Fix the \( 2N \times 2N \) matrix

\[
C = \begin{bmatrix} C_1 & C_2 \\ C_2 & C_1 \end{bmatrix}.
\]

In general, what Theorem \([5.6]\) states is that the matrix \( G(z, s) \) can be well approximated by \( UCU' \), since

\[
\langle q, UCU' q \rangle = \sum_{i=-N}^{N} \frac{1}{2} (g_i + g_{-i})(s, z) (u_i, q)^2 + \sum_{i=1}^{N} (g_i - g_{-i})(s, z)(u_i, q)(u_{i+N}, q).
\]

Moreover we can reduce the proof of Theorem \([5.6]\) to the diagonal case.
**Theorem 5.7.** Fix $V = \text{diag}(v_1, \cdots, v_N)$ a diagonal matrix which satisfies Assumption 5.1. Moreover set $W$ to be the symmetrization of a matrix $L$ with i.i.d. entries, all following the Gaussian law with 0 mean and $\frac{1}{N}$ variance. Define the resolvent $G(z, s) = (\tilde{V} + \sqrt{s}W - zI)^{-1}$. Fix $k \in (0, 1)$, $h, < s < r$ and $q : \|q\|_2 = 1$. Then

$$
\langle q, G(s, z)q \rangle - \frac{1}{2} \sum_{i=N}^{N} (g_i + g_{-i})(s, z)q_i^2 - \sum_{i=1}^{N} (g_i - g_{-i})(s, z)q_iq_{i+N} \\
\leq \frac{\eta^2}{\sqrt{N}} \min \left( \sum_{i=1}^{N} (q_i^2 + q_{i+N}^2)(g_i(s, z) + g_{-i}(s, z)) \right),
$$

holds with overwhelming probability uniformly for all $z \in \mathbb{D}_k$.

The proof of Theorem 5.7 can be found in paragraph 5.3.

**Proof of Theorem 5.6 assuming Theorem 5.7** Let $V$ be a general deterministic matrix with singular value decomposition $\Sigma = J_2 V J_1$ where $J_1$ and $J_2$ are orthogonal matrices. Define $U$ as in Remark 5.3. Then

$$
U(\tilde{V} + \sqrt{s}W)U^T = \begin{bmatrix} 0 & \Sigma + J^T \sqrt{s}L J \\ \Sigma + J_2 \sqrt{s}L J_1 & 0 \end{bmatrix}.
$$

But $L$ is invariant under orthogonal transformation, so $J_2 L J_1$ has the same law as $L$. This implies that $U(\tilde{V} + \sqrt{s}W)U^T$ has the same law as $U\tilde{V}U^T + \sqrt{s}W$. Next, by the properties of the inner product, one has that

$$
\langle q, (\tilde{V} + \sqrt{s}W - zI)^{-1}q \rangle = \langle q, U(\tilde{V}U^T + UWU^T - zI)^{-1}U^T q \rangle = \langle U^T q, (U\tilde{V}U^T + UWU^T - zI)^{-1}U^T q \rangle.
$$

By a similar computation for $\langle q, UC U^T q \rangle$, one reduces the problem in bounding

$$
\left| \langle q, U(\tilde{V}U^T + \sqrt{s}W - zI)^{-1}q \rangle - \sum_{i=1}^{N} q_i^2 g_i(s, z) + \sum_{i=N}^{[N]} q_i q_{i+N}(g_i - g_{-i})(s, z) \right|,
$$

which is true by a direct application of Theorem 5.7.

So it suffices to prove Theorem 5.7 i.e., to consider $V$ to be diagonal. Moreover we have the following identities.

**Remark 5.8.** Let $V$ be a deterministic diagonal matrix which satisfies Assumptions 5.1. Then consider the following matrix $F = \{F_{ij}\}_{i,j \in [N]}$, where

$$
F_{ij} := \begin{bmatrix} 0 & [V + \sqrt{s}L]_{ij} \\ [V + \sqrt{s}L]_{ij} & 0 \end{bmatrix}, \text{ for all } i, j \in [N].
$$

Note that there exists a unitary matrix $S$, the product of permutation matrices, such that if we set $F = S^T (V + \sqrt{s}W) S$ then $G(s, z) = S (F - zI)^{-1} S^T$ and

$$(F - zI)^{-1}_{ij} = \begin{bmatrix} G_{ij} & G_{i+Nj} \\ G_{ij+N} & G_{i+Nj+N} \end{bmatrix},$$

where $G_{ij}$ are the entries of $G(s, z)$.

It is more convenient to work with the matrix $F$ and its resolvent as it can be thought as a full symmetric matrix in $M_N(M_2(\mathbb{C}))$, instead of a symmetric matrix with 0 at the diagonal blocks in $M_{2N}(\mathbb{C})$.

5.3 **Proof of Theorem 5.7**

In this subsection we will prove Theorem 5.7. First, we present some results from [12], necessary for the proof.
Proposition 5.9 ([12], Theorem 4.5). Fix $s$ as in Theorem 5.7, the parameter $\psi$ defined in (5.2) and $k \in (0, 1)$. Then it is true that,

$$|m_s(z) - m_{s, k}(z)| \leq \frac{\psi}{N^2}$$

holds with overwhelming probability uniformly for all $z \in \mathbb{D}_k$. Here $m_s(z)$ is the Stieltjes transform of $\tilde{V} + \sqrt{s}W$.

Lemma 5.10. Fix $s$ and $k$ as in Theorem 5.7 Then uniformly for all $z \in \mathbb{D}_k$, there exists a constant $C > 1$ such that:

$$C^{-1} \leq |m_{s, k}(z)| \leq C,$$

$$|m_{s, k}(z)| \leq \frac{1}{N} \sum_{i=1}^{N} |g_i| + |g_{-i}| \leq C \log(N).$$

Proof. These estimates can be found in [12] Lemma 4.1 and Lemma 4.12.

Moreover the following estimates hold:

Lemma 5.11. Fix $T \subseteq [2N]$ such that $|T| \leq \log(N)$, which consists of pairs of indeces $\{k, k + N\}$ for $k \in [N]$. Moreover set $(H + \sqrt{s}G)^T$ the sub matrix of $H + \sqrt{s}G$ with the $i$-th columns and row removed for all $i \in T$ and $G^T(s, z) = \left((H + \sqrt{s}G)^T - z1_{2N-N}\right)^{-1}$. Then the following estimates hold with overwhelming probability.

$$\left|G_{ii}^T - \frac{1}{2} (g_i + g_{-i})\right| \leq \frac{(|g_i| + |g_{-i}|)^2}{\sqrt{N\eta}} \text{ for all } i \in [2N] \setminus T,$$

$$\left|G_{iN+i}^T - \frac{1}{2} (g_i - g_{-i})\right| \leq \frac{(|g_i| + |g_{-i}|)^2}{\sqrt{N\eta}} \text{ for all } i \in [2N] \setminus T,$$

$$\left|G_{ij}^T\right| \leq \frac{\min(|g_i|, |g_{-i}|, |g| + |g_{-i}|)}{\sqrt{N\eta}} \leq \frac{(|g_i| + |g_{-i}|)(|g| + |g_{-i}|)}{\sqrt{N\eta}} \text{ for all } i, j \in [2N] \setminus T.$$

Proof. The first two estimates are proven by the Schur Complement formula and the bounds (4.69) and (4.89) from [12]. The last bound is given in [12], equation (4.70) and (4.79).

Next, we present a bound for the diagonal and the anti-diagonal entries of $G(s, z)$.

Lemma 5.12. Adopt the notation of Theorem 5.7. Then it is true that with overwhelming probability

$$\left|\langle q, G(s, z)q \rangle - \frac{1}{2} \sum_{i=-N}^{N} q_i^2 (g_i(s, z) + g_{-i}(s, z)) - \sum_{i=1}^{N} q_i q_{i+N}(g_i - g_{-i})\right| \leq \frac{\psi^2}{\sqrt{N\eta}} \text{ Im} \left(\sum_{i=-N}^{N} q_i^2 (g_i(s, z) + g_{-i}(s, z))\right)$$

$$+ \frac{\psi^2}{\sqrt{N\eta}} \text{ Im} \left(\sum_{i=1}^{N} (g_i + g_{-i})(q_i^2 + q_{-i}^2)\right) + \left|\sum_{i,j=1}^{N} G_{ij} q_i q_j\right|,$$

Proof. One has that,

$$\langle q, G(s, z)q \rangle = \sum_{i=1}^{2N} q_i^2 G_{ii} + 2 \sum_{i=1}^{N} G_{i+N+i} q_i q_{i+N} + \sum_{i,j=1}^{N} q_i q_j G_{ij}.$$

So for the first part on the right side of the equality in (5.15), one can apply (5.10) to get that,

$$\sum_{i=-N}^{N} q_{i+N}^2 \left|G_{ii} - \frac{1}{2} (g_i + g_{-i})\right| \leq \frac{\psi}{\sqrt{N\eta}} \sum_{i=-N}^{N} q_{i+N}^2 (|g_i| + |g_{-i}|)^2 \leq \frac{2\psi}{\sqrt{N\eta}} \sum_{i=-N}^{N} q_{i+N}^2 + \frac{2\psi}{\sqrt{N\eta}} \sum_{i=-N}^{N} q_{i+N} |g_i|^2.$$

Next, we can apply Proposition 2.8 from [27] to get that with overwhelming probability,

$$\sum_{i=-N}^{N} q_{i+N}^2 \left|G_{ii} - \frac{1}{2} (g_i + g_{-i})\right| \leq \frac{2\psi}{\sqrt{N\eta}} \text{ Im} \left(\sum_{i=-N}^{N} (g_i + g_{-i}) q_{i+N}^2\right).$$
Similarly, by (5.11) one has that with overwhelming probability,
\[
\sum_{i=1}^N |q_i q_{i+N}| G_{i,N+1} - \frac{1}{2}(g_i - g_{i+1}) \leq \frac{2\psi}{\sqrt{N_1}} \text{Im} \left( \sum_{i=1}^N (g_i + g_{i+1})q_i^2 + q_{i+N}^2 \right). 
\tag{5.16}
\]
\[
\sum_{i=1}^N |q_i q_{i+N}| G_{N+i+1} - \frac{1}{2}(g_i - g_{i+1}) \leq \frac{2\psi}{\sqrt{N_1}} \text{Im} \left( \sum_{i=1}^N (g_i + g_{i+1})q_i^2 + q_{i+N}^2 \right). 
\tag{5.17}
\]

So in order to prove Theorem 5.7 it suffices to prove that
\[
\mathbb{E} Z^{2k} \leq k Y^{2k} \quad \text{for all } k \in \mathbb{N},
\tag{5.18}
\]
where,
\[
Z := \left| \sum_{i \neq j \mod N} q_i q_j G_{ij} \right| \quad \text{and} \quad Y = \frac{\log N}{\sqrt{N_1}} \text{Im} \left( \sum_{i=1}^N (g_i + g_{i+1})q_i^2 + q_{i+N}^2 \right).
\tag{5.19}
\]

By (5.18), one can obtain Theorem 5.7 by Markov’s inequality, which will imply
\[
\left| \sum_{i \neq j \mod N} q_i q_j G_{ij} \right| \leq \frac{\psi^2}{\sqrt{N_1}} \text{Im} \left( \sum_{i=1}^N (g_i + g_{i+1})q_i^2 + q_{i+N}^2 \right)
\tag{5.20}
\]
with overwhelming probability. More precisely for any \( D > 0 \) if we fix \( k : c k \geq D \) and sufficient large \( N \) such that \( N^{-k-D} \geq C_k \). Here \( c \) is the constant in the definition of \( \psi \) in (5.2) and \( C_k \) is implied in (5.18). Thus, one can apply Markov’s inequality in order to get that:
\[
\mathbb{P} \left( Z \geq \frac{\psi}{\log(N)} Y \right) = \mathbb{P} \left( Z^{2k} \geq \left( \frac{\psi}{\log(N)} Y \right)^{2k} \right) \leq \frac{C_k \log 2^k(N)}{N^{2k}} \leq \frac{C_k}{N^k} \leq \frac{1}{N^D}.
\]

Next, we give an analysis for the moments of \( Z \). Firstly, note that
\[
\mathbb{E} |Z|^{2k} = \sum_{B} q_B q_B q_B \cdots q_B \mathbb{E} X_{b_1,b_2} X_{b_3,b_4} \cdots X_{b_{2k-1},b_{2k}},
\tag{5.21}
\]
where the sum is taken over all \( B \subseteq [2N]^{4k} \) such that \( b_{2i-1} \neq b_{2i} \mod N \) and \( X_{b_{2i-1},b_{2i}} = G_{b_{2i-1},b_{2i}} \) for \( i \in [k] \) and \( X_{b_{2i-1},b_{2i}} = \tilde{G}_{b_{2i-1},b_{2i}} \) and \( i \in [2k] \setminus [k] \). Furthermore, we can continue the analysis of the sum such that,
\[
\mathbb{E} |Z|^{2k} = \sum_{B} \sum_{b_i \in [b_i,b_i+N]} q_B q_B q_B \cdots q_B \mathbb{E} X_{b_1,b_2} X_{b_3,b_4} \cdots X_{b_{2k-1},b_{2k}}.
\tag{5.22}
\]
Now the sum is considered, firstly over all \( B \subseteq [i \mod N]^{4k} \) with the restriction that \( B_{2i-1} \neq B_{2i} \) and then over the possible \( b_i = k \) such that \( k \in [N] \) or \( k \in B_i \) or \( b_i = k + N \) for \( k \in [N] \) and \( k \in B_i \). Next, for every summand in (5.22) set \( T = \cup_{b_i \in [b_i,b_i+N]} [b_i,b_i+N] \). Moreover set the diagonal block matrices
\[
[M_{1-2}]_{\text{block}} = \begin{bmatrix} m_{a_1,a_1} & 0 \\ 0 & m_{a_2,a_2} \end{bmatrix}
\]
and
\[
[\bar{M}]_{\text{block}} = \begin{bmatrix} m(T) & 0 \\ 0 & m^{(T)} \end{bmatrix},
\]
where \( m(S)(z) \) is the trace of the resolvent of \((V + \sqrt{5}W)^{(S)}\) divided by \( 2N \). Here \( S \) is any subset of \([2N]\) and \((V + \sqrt{5}W)^{(S)}\) is the minor of \((V + \sqrt{5}W)\) with rows and columns not included in \( S \). Moreover set
\[
\Phi_{1-2} = \begin{bmatrix} -z & \bar{c}_1 \\ \bar{c}_1 & -z \end{bmatrix}
\]
and
\[

W'_{1-2} = \begin{bmatrix} 0 & \bar{c}_2 \\ \bar{c}_2 & 0 \end{bmatrix}.
\]
Adopting the notation of Proposition [5.8] and since \( G(s, z) = S(F - zI)^{-1}S^T \), one can apply Schur complements formula to get that
\[
(F - zI)^{-1}_{i\in\mathcal{T}} = (\Phi_{i\in\mathcal{T}} + \sqrt{s}W_{i\in\mathcal{T}} - s(W')_{i\in\mathcal{T}})_{i\in\mathcal{T}}(S^T G(s, z) S)'_{i\in\mathcal{T}} (S^T G(s, z) S)'_{i\in\mathcal{T}}^{-1} (S^T G(s, z) S)'_{i\in\mathcal{T}} = (D - E^1 - E^2 - E^3)^{-1},
\]
(5.23)
where
\[
D = \Phi_{i\in\mathcal{T}} - sM^{(T)}_{i\in\mathcal{T}}, \quad E^1 = s(M^T - M^T_{i\in\mathcal{T}}), \quad E^2 = -\sqrt{s}W', \quad E^3 = s(W')_{i\in\mathcal{T}}(S^T G(s, z) S)'_{i\in\mathcal{T}} (S^T G(s, z) S)'_{i\in\mathcal{T}} - M^T.
\]
(5.25)
(5.26)

Next, we wish to estimate the operator norm of the matrix \( ED^{-1} \). We will show that,
\[
|ED^{-1}|_{\text{op}} \leq C_k \frac{\psi}{\sqrt{N\eta}}
\]
with overwhelming probability. Here \( E = \sum_{i=1}^{3} E^i \).

More precisely, firstly note that \( D \) is a \( 2N \times 2N \) dimensional matrix with 0 at the all non-diagonal \( 2 \times 2 \) blocks and with diagonal blocks equal to \( D_{ii} = \begin{bmatrix} -z - m_{s,k}(z) & j_i \\ j_i & -z - m_{s,k}(z) \end{bmatrix} \).

So the inverse of \( D \) will preserve the same structure. Thus, we can compute that:
\[
D_{ii}^{-1} = \frac{1}{2} \begin{bmatrix} g_i + g_{-i} & g_i - g_{-i} \\ g_i - g_{-i} & g_i + g_{-i} \end{bmatrix}.
\]

Moreover since \( \text{Im}(z + sm_{s,k}(z)) \geq (s + \eta) \), we get that \( |g_i| \leq \frac{1}{s+\eta} \), for all \( i : |i| \in [N] \). All these imply that all the entries of \( D^{-1} \) are bounded by \( \frac{1}{s+\eta} \) up to some universal constant. So it is implied that
\[
|D^{-1}|_{\text{op}} \leq k \frac{1}{s+\eta}.
\]
(5.28)

Next, similarly to the proof of (2.16) in [27] one can prove that
\[
|E|_{\text{op}} \leq k (s + \eta) \frac{\psi}{\sqrt{N\eta}}.
\]
(5.29)

So after combining (5.28) and (5.29), we get (5.27). Set \( \mathcal{A} \) to be the event where (5.27) holds with overwhelming probability. Then it is true that for appropriately large \( N \)
\[
P(\mathcal{A}^c) \leq N^{-(4a+6)k}.
\]
(5.30)

where \( \alpha \) is given in the Assumptions [5.1] Next by Taylor’s expansion on the event \( \mathcal{A} \) one has
\[
(F - zI)^{-1}_{i\in\mathcal{T}} = (D - E)^{-1} = \sum_{l=0}^{f-1} D^{-1}(ED^{-1})^l + (D - E)^{-1}(ED)^{-f},
\]
(5.31)
where \( f \) can be chosen to be arbitrary large. We choose \( f = \lceil \frac{3k(a + 1)}{c} \rceil \), where \( c \) is mentioned in the definition of \( \psi \) in [5.2] and \( a \) is mentioned in the Assumption [5.1]. Moreover since all the non diagonal \( 2 \times 2 \) blocks of \( D^{-1} \) are 0, we can ignore the case of \( l = 0 \) in (5.31), since we are interested in the elements of \( G_{i\in\mathcal{T}} \), such that \( b_i \neq b_{i+1} \mod N \). Moreover set \( X_{b_i, b_{i+1}} = (D^{-1}(ED^{-1})^l)_{b_i, b_{i+1}} \) and \( X_{b_i, b_{i+1}}^{\infty} = ((D - E)^{-1}(ED)^{-l})_{b_i, b_{i+1}} \).

So in order to prove (5.18), firstly we need to bound \( Y \) from below. Note that similarly to [27] (2.13) one has
\[
\text{Im} \left( \sum_{i=-N}^{N} q_i^2 N \eta_i \right) = \sum_{i=-N}^{N} \eta s \text{Im}(m_{s,k}(z)) \frac{q_i^2}{|\eta_i(0) - z - m_{s,k}(z)|^2} \geq \frac{\eta}{N^2 a},
\]
(5.32)
due to the fact that $z \in \mathbb{D}_k$. Assumption 5.1 and 5.9. So it is easily implied that

$$ Y \geq \frac{\eta \psi \log(N)}{N^{2a} \sqrt{N\eta}}. \quad (5.33) $$

Returning to the analysis of equation 5.22, one has that

$$ \sum_{b} \sum_{b_i=[b_i,k]} q_{b_1} q_{b_2} \cdots q_{b_k} \mathbb{E} X_{b_1,b_2} X_{b_3,b_4} \cdots X_{b_{k-1},b_k} \mathbb{1}(\mathcal{A}) + O(N^{2k} \eta^{-2k}) \mathbb{P}(\mathcal{A}^c). \quad (5.34) $$

where the second part on the right hand side of the equation comes from the fact that $X_{b_i,b_{i+1}}$ are uniformly bounded by $\eta^{-1}$ and the fact that $|q_i| \leq N^{1/2}$ since $|q_2| = 1$. So one has that

$$ \eta^{-2k} \mathbb{P}(\mathcal{A}^c) \left| \sum_{i\neq j} q_i q_j \right|^{2k} = \eta^{-2k} \mathbb{P}(\mathcal{A}^c) \left( \sum_{i=1}^{2N} |q_i| \sum_{j \neq i} |q_j| \right)^{2k} \leq \eta^{-2k} \mathbb{P}(\mathcal{A}^c) \left( \sum_{i=1}^{2N} N^{1/2} |q_i| \right)^{2k} \leq N^{2k} \eta^{-2k} \mathbb{P}(\mathcal{A}^c). \quad (5.35) $$

Next by (5.30), (5.33) and the fact that $z \in \mathbb{D}_k$ one has that

$$ N^{2k} \eta^{-2k} \mathbb{P}(\mathcal{A}^c) \leq N^{-2k/2} \eta^{-2k} \leq \frac{\eta}{N^{2k}} \leq Y^{2k}. $$

So, we have proven that we can constrain the event that $\mathcal{A}$ holds. Returning again to the analysis of the sum in the form 5.21 one has that

$$ \sum_{b} q_{b_1} q_{b_2} \cdots q_{b_k} \mathbb{E} X_{b_1,b_2} X_{b_3,b_4} \cdots X_{b_{k-1},b_k} \mathbb{1}(\mathcal{A}) = \sum_{b} q_{b_1} q_{b_2} \cdots q_{b_k} \mathbb{E} \mathbb{1}(\mathcal{A}) \prod_{i=1}^{k-1} \sum_{b_i} X_{b_{i+1},b_i}^{(i)} \quad (5.36) $$

$$ + \sum_{b} q_{b_1} q_{b_2} \cdots q_{b_k} \mathbb{E} \mathbb{1}(\mathcal{A}) X_{b_{2t-1},b_{2t}}^{(2t-1)} \prod_{j \leq t} X_{b_{2j-1},b_{2j}} \prod_{j \geq t+1} \left( X_{b_{2j-1},b_{2j}} - X_{b_{2j-1},b_{2j}}^{(2j)} \right). \quad (5.37) $$

We will show that the second part of the right hand side of the equation is negligible on the event $\mathcal{A}$. Note that since $\|\{(D-E)^{-1}\}_{op} \leq k 1 \over \eta$ and since 5.27 holds in $\mathcal{A}$ we get that

$$ |X_{b_{2j-1},b_{2j}}^{(2j)}| \leq k 1 \over \eta \left( \psi \over \sqrt{N\eta} \right)^f. $$

All these, imply that

$$ \left| \sum_{b} q_{b_1} q_{b_2} \cdots q_{b_k} \mathbb{E} \mathbb{1}(\mathcal{A}) X_{b_{2t-1},b_{2t}}^{(2t-1)} \prod_{j \leq t} X_{b_{2j-1},b_{2j}} \prod_{j \geq t+1} \left( X_{b_{2j-1},b_{2j}} - X_{b_{2j-1},b_{2j}}^{(2j)} \right) \right| \leq k N^{2k} \eta \left( \psi \over \sqrt{N\eta} \right)^f. \quad (5.38) $$

The $N^{2k}$ factor in 5.38, comes from bounding the quantity $|\sum_{i \neq j} q_i q_j|$. By the way $f$ is chosen, we get that

$$ \frac{N^{2k}}{\eta} \left( \psi \over \sqrt{N\eta} \right)^f \leq k \left( \eta \over N^{2a} \right)^{2k} \left( \psi \over \sqrt{N\eta} \right)^{2k} \leq k Y^{2k}. $$

So, the remaining quantity in the sum we need to bound is

$$ \sum_{1 \leq b_1 < b_2 < \cdots < b_{k-1}} \sum_{b} q_{b_1} q_{b_2} \cdots q_{b_k} \mathbb{E} \mathbb{1}(\mathcal{A}) \prod_{i=1}^{k-1} X_{b_{i+1},b_i}^{(i)}. $$

Moreover due to Cauchy–Schwarz inequality one can show that

$$ \mathbb{E} \mathbb{1}(\mathcal{A}) \prod_{i=1}^{k-1} X_{b_{i+1},b_i}^{(i)} \leq \mathbb{E} \prod_{i=1}^{k-1} X_{b_{i+1},b_i}^{(i)} + \mathbb{E}(\mathcal{A}^c) \mathbb{E} \prod_{i=1}^{k-1} X_{b_{i+1},b_i}^{(i)} \leq k Y^{2k}. $$

So, we will work with the right hand side of the last inequality, meaning we won't focus anymore on the event $\mathcal{A}$. Moreover we will focus on the first summand of the right hand side of the inequality, since the second one can be treated analogously.
Next, we can transform the previously mentioned quantity in a more appropriate form. Firstly, note for each \( i \in [k] \):

\[
X^{(i)}_{b_{2i-1}, b_{2i}} = \sum_{a \in \mathbb{N}} (D^{-1}a)^{(i)} a_1^{(i)} a_2^{(i)}
\]

and similarly for \( i \in [2k] \setminus [k] \):

\[
X^{(i)}_{b_{2i-1}, b_{2i}} = \sum_{a \in \mathbb{N}} (D^{-1}a)^{(i)} a_1^{(i)} a_2^{(i)},
\]

where the sum is taken over all \( a \subseteq \mathbb{T}^{l+1} \), i.e., all \( l \)-tuples with the restriction that \( a_1^{(i)} = b_{2i} \) and \( a_1^{(i)} = b_{2i+1} \). So, since this is true for all \( i \in [2k] \) one can show:

\[
\prod_{i=1}^{2k} X^{(i)}_{b_{2i-1}, b_{2i}} = \sum_a \prod_{i=1}^{2k} (D^{-1}a)^{\prime} a_1^{(i)} a_2^{(i)},
\]

where the sum is taken over all \( a = (a_1^1, a_2^2, \ldots, a_k^k) \) and for \( i \in [2k] \setminus [k] \), the \( \prime \) denotes the conjugate.

Next set

\[
E_{[a]}^{(i)} = \begin{bmatrix}
E_{[a_1^1]}^{(i)} & E_{[a_1^1] + N [a_1^1]} \\
E_{[a_1^1] + N [a_1^1]} & E_{[a_1^1] + N [a_1^1]} + N
\end{bmatrix},
\]

where \([a_1^1] \) is the least positive integer which is equal to \( a_1^1 \mod (N) \). Moreover set

\[
x(a_1^1) = 1 [a_1^1 = [a_1^1] + N] + 1.
\]

Furthermore, since \( D^{-1} \) consists of zero at the non diagonal \( 2 \times 2 \) blocks, one has that for \( j \neq 1 \)

\[
(ED^{-1})_{a_j^{(i)}, a_k^{(i)}} = \left( E_{[a_j]}^{(i)} \right)_{a_j^{(i)}, a_k^{(i)}} = \left( E_{[a_j]}^{(i)} \right)_{a_j^{(i)}, a_k^{(i)}} + \left( E_{[a_j]}^{(i)} \right)_{a_j^{(i)}, a_k^{(i)}} + \left( E_{[a_j]}^{(i)} \right)_{a_j^{(i)}, a_k^{(i)}}
\]

and similarly for \( j = 1 \)

\[
(D^{-1}ED^{-1})_{a_j^{(i)}, a_k^{(i)}} = \left( D^{-1}E \right)_{a_j^{(i)}, a_k^{(i)}} \left( D^{-1}E \right)_{a_j^{(i)}, a_k^{(i)}} \left( D^{-1}E \right)_{a_j^{(i)}, a_k^{(i)}}
\]

So it is implied that

\[
\prod_{i=1}^{2k} X^{(i)}_{b_{2i-1}, b_{2i}} \geq \sum_{a \in \mathbb{N}} \sum_{e} \prod_{i=1}^{2k} \left( D^{-1}a \right)^{(i)} E_{[a_j]}^{(i)}, c_i^{(i)} c_j^{(i)}
\]

\[
\prod_{j=1}^{2k} \left( D^{-1}a \right)^{(i)} E_{[a_j]}^{(i)}, c_i^{(i)} c_j^{(i)}
\]

Here \( e \) is any subset of \([1, 2][\Lambda_0^m, \Lambda_0^{l+1}] \). So since the entries of \( D^{-1} \) are deterministic and for all \( a_j^{(i)} \) the entries of \( D^{-1}[a_j^{(i)}, a_k^{(i)}] \) are bounded by \( |g_i^{(a_j^{(i)})}| + |g_i^{(a_k^{(i)})}| \), it is true that

\[
\prod_{i=1}^{2k} X^{(i)}_{b_{2i-1}, b_{2i}} \leq \prod_{a \in \mathbb{N}} \sum_{j \in \Lambda_0} \left| g_i^{(a_j^{(i)})} \right| \sum_{e \in \mathbb{N}} \prod_{i=1}^{2k} \left( E_{[a_j]}^{(i)} \right)^{(i)}, c_i^{(i)}
\]

Next we will show an important inequality, necessary to estimate the expectation of the products in the previous equations.

**Lemma 5.13.** It is true that for each array \((a_j^{(i)})\) with entries in \( \mathbb{T} \),

\[
\sum_{e} \prod_{i=1}^{2k} \left( E_{[a_j]}^{(i)} \right)^{(i)}, c_i^{(i)} \prod_{j=1}^{2k} \left( E_{[a_j]}^{(i)} \right)^{(i)}, c_j^{(i)}
\]

\[
\leq \frac{(\psi \log(N))^{\frac{1}{2}}}{{(N^2)^{k/2}}} X \left( \left[ a_1^{(1)} \right], \left[ a_1^{(2)} \right], \left[ a_1^{(3)} \right] \cdots \left[ a_1^{(l+1)} \right], \left[ a_2^{(1)} \right], \left[ a_2^{(2)} \right] \cdots \left[ a_2^{(l+1)} \right] \cdots \left[ a_2^{(2k)} \right] \right).
\]

where \( X(\cdot) \) is the indicator function that indicates if every element in the array appears an even number of times.
Proof. Note that by the definition of the matrices, one has that

$$E_{\{a_1^{(i)}\} \cup \{c_1^{(i)}\}} = E_{\{a_1^{(i)}\} \cup \{c_1^{(i)}\}}^1 + E_{\{a_1^{(i)}\} \cup \{c_1^{(i)}\}}^2 + E_{\{a_1^{(i)}\} \cup \{c_1^{(i)}\}}^3.$$  

Moreover, after conditioning on the matrix $W_{\mathbb{T},T}$, the matrices $E^1, E^2, E^3$ are independent since $E^1$ is dependent only on $(F - 2\mathbb{Z})^{(T)}$ and is diagonal and deterministic, $E^2$ depends only on $W_{\mathbb{T},T}$ and $E^3$ depends on $G^T$ and $W_{[2N],T}$. We will use the notation $E_T$ for the conditional expected value. So in order to prove Lemma 5.13 it suffices to show that

$$\sum_{e} E_T \left| \sum_{i=1}^{2k} \left( E_{\{a_1^{(i)}\} \cup \{c_1^{(i)}\}} \right)^{\prime} \prod_{j \neq 1} \left( E_{\{a_1^{(i)}\} \cup \{c_1^{(i)}\}} \right)^{\prime} \right|_{x(a_1^{(i)}), c_1^{(i)}}, \leq k \left( \frac{s}{N\eta} \right)^{\sum_k} X \left( \left( \{a_j^{(i)}\} \right)_{i \in [2k], j \in [l]} \right) \leq X \left( \left( \{a_j^{(i)}\} \right)_{i \in [2k], j \in [l]} \right) \left( \frac{\psi \log(N)}{(N\eta)^{\Sigma_{1/2}}} \right), \tag{5.45}$$

$$\leq X \left( \left( \{a_j^{(i)}\} \right)_{i \in [2k], j \in [l]} \right) \left( \frac{\psi \log(N)}{(N\eta)^{\Sigma_{1/2}}} \right), \tag{5.46}$$

$$\leq X \left( \left( \{a_j^{(i)}\} \right)_{i \in [2k], j \in [l]} \right) \left( \frac{\psi \log(N)}{(N\eta)^{\Sigma_{1/2}}} \right), \tag{5.47}$$

This is true, since if we assume 5.45, 5.46, 5.47 hold for any array then

$$\sum_{e} E_T \left| \sum_{i=1}^{2k} \left( E_{\{a_1^{(i)}\} \cup \{c_1^{(i)}\}} \right)^{\prime} \prod_{j \neq 1} \left( E_{\{a_1^{(i)}\} \cup \{c_1^{(i)}\}} \right)^{\prime} \right|_{x(a_1^{(i)}), c_1^{(i)}}, \leq k \left( \frac{s}{N\eta} \right)^{\sum_k} X \left( \left( \{a_j^{(i)}\} \right)_{i \in [2k], j \in [l]} \right) \leq X \left( \left( \{a_j^{(i)}\} \right)_{i \in [2k], j \in [l]} \right) \left( \frac{\psi \log(N)}{(N\eta)^{\Sigma_{1/2}}} \right), \tag{5.48}$$

$$\leq k \sum_{p_i, p_2, p_3} \left| \sum_{e} \prod_{i \in [3]} E_T \left[ \{a_1^{(i)}\} \cup \{c_1^{(i)}\} \right] \left( E_{\{a_1^{(i)}\} \cup \{c_1^{(i)}\}} \right)^{\prime} \prod_{j \neq 1} \left( E_{\{a_1^{(i)}\} \cup \{c_1^{(i)}\}} \right)^{\prime} \right|_{x(a_1^{(i)}), c_1^{(i)}}, \tag{5.49}$$

$$\leq k \sum_{p_i, p_2, p_3} \sum_{i \in [3]} X \left( \left( \{a_j^{(i)}\} \right) \right) \leq X \left( \left( \{a_j^{(i)}\} \right) \right) \left( \frac{s + \psi \log(N)}{(N\eta)^{\Sigma_{k}}} \right), \tag{5.50}$$

$$\leq k \sum_{p_i, p_2, p_3} \left| \sum_{i \in [3]} \psi \left( \frac{\log(N)}{(N\eta)^{\Sigma_{k}}} \right) \right|, \tag{5.51}$$

where the sum is taken over all 3-partitions $P^1, P^2, P^3$ of the set $\{i, j : i \in [2k], j \in [l] + 1\}$ such that if $a_1^{(i)} \in P^u$ then $a_1^{(j)} \in P^u$ for each $j \in [l] \cap (2N + 1)$, $i \in [2k]$ and $y \in [3]$. So the number of these partitions depends only on $k$ which implies the last inequality.

For the first inequality 5.45 note that the matrix $E^1$ is diagonal and its diagonal entries are bounded by $s/\eta$ due to Theorem 4.5 of [12] and the interlacing properties of the minors of the eigenvalues. So it is implied that $\left| E_{\{a_1^{(i)}\} \cup \{c_1^{(i)}\}} \right|_{op} \leq 1 \{a_1^{(i)} = a_1^{(i+1)}\} s/\eta$. So

$$\sum_{e} E_T \left| \sum_{i=1}^{2k} \left( E_{\{a_1^{(i)}\} \cup \{c_1^{(i)}\}} \right)^{\prime} \prod_{j \neq 1} \left( E_{\{a_1^{(i)}\} \cup \{c_1^{(i)}\}} \right)^{\prime} \right|_{x(a_1^{(i)}), c_1^{(i)}}, \leq \sum_{e} E_T \left| \sum_{i=1}^{2k} E_{\{a_1^{(i)}\} \cup \{c_1^{(i)}\}} \prod_{j \neq 1} E_{\{a_1^{(i)}\} \cup \{c_1^{(i)}\}} \right|_{op}, \tag{5.52}$$

$$\leq \sum_{i \in [2k]} \left| \sum_{j \in [l+1]} \left( \frac{\psi s}{N\eta} \right)^{\Sigma_k} X \left( \left( \{a_j^{(i)}\} \right)_{i \in [2k], j \in [l+1]} \right). \tag{5.53}$$
For the second inequality (5.46) by the way \(E^2\) was defined, one can compute that
\[
\left| \sum_{\mathbf{c}} \mathbb{E} \sum_{i=1}^{2k} \left( \mathbf{E}^2_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}_{\pm}\}} \right)_{\mathbf{c} \cdot \mathbf{c}, \mathbf{c}, \mathbf{c}, \mathbf{c}} \prod_{j=1}^{2} \left( \mathbf{E}^2_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{x}(\mathbf{x}^{(0)}_{\pm})} \right| \quad (5.54)
\]
\[
= \left| \sum_{\mathbf{c} \in (1,2)} \mathbb{E} \sum_{i=1}^{2k} \left( \mathbf{E}^2_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}_{\pm}\}} \right)_{\mathbf{c} \cdot \mathbf{c}, \mathbf{c}, \mathbf{c}, \mathbf{c}} \prod_{j=1}^{2} \left( \mathbf{E}^2_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{x}(\mathbf{x}^{(0)}_{\pm})} \right| \quad (5.55)
\]
where \([\alpha]_2\) the least positive integer such that it is equal to \(\alpha \mod 2\). Note that \(\left( \mathbf{E}^2_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}_{\pm}\}} \right)_{\mathbf{c} \cdot \mathbf{c}, \mathbf{c}, \mathbf{c}, \mathbf{c}} = -\sqrt{\mathbb{E} \omega_{\alpha_{ij}^{(0)}},\alpha_{ij}^{(0)}_{\pm}}\). Moreover, since the non zero entries of \(W''\) are independent, symmetric, normal random variables with variance \(\frac{1}{N}\), the product is non-zero only if every pair \((\alpha_{ij}^{(0)}),\{\alpha_{ij}^{(0)}_{\pm}\}\) in the product appears an even number of times. All these imply that
\[
\sum_{\mathbf{c} \in (1,2)} \mathbb{E} \sum_{i=1}^{2k} \left( \mathbf{E}^2_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}_{\pm}\}} \right)_{\mathbf{c} \cdot \mathbf{c}, \mathbf{c}, \mathbf{c}, \mathbf{c}} \prod_{j=1}^{2} \left( \mathbf{E}^2_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{x}(\mathbf{x}^{(0)}_{\pm})} \leq_k \sum_{\mathbf{c} \in (1,2)} \left( \frac{S}{N} \right)^{k/2} \mathbb{E}(\mathbb{E}^2_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}_{\pm}\}})\),
\]
which implies (5.46). The constant which is implied in the last inequality can be chosen to be \(2k \prod_{i=1}^{k} \mathbb{E}(\mathbb{E}^2_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}_{\pm}\}})\), which is a large constant depending only on \(k\) since \(\sqrt{\mathbb{E} \omega_{\alpha_{ij}^{(0)}},\alpha_{ij}^{(0)}_{\pm}} \sim N(0, 1)\).

For the third inequality, (5.47), one can show that
\[
\sum_{\mathbf{c} \in (1,2)} \mathbb{E} \sum_{i=1}^{2k} \left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{c} \cdot \mathbf{c}, \mathbf{c}, \mathbf{c}, \mathbf{c}} \prod_{j=1}^{2} \left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{x}(\mathbf{x}^{(0)}_{\pm})} \quad (5.56)
\]
\[
= \sum_{\mathbf{c} \in (1,2)} \mathbb{E} \left[ \sum_{i=1}^{2k} \left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{c} \cdot \mathbf{c}, \mathbf{c}, \mathbf{c}, \mathbf{c}} + 1 \left( a_1^{(0)} = a_2^{(0)} \right) \left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{x}(\mathbf{x}^{(0)})} \right] \quad (5.57)
\]
\[
\cdot \left( \prod_{j=1}^{2} \left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{x}(\mathbf{x}^{(0)}_{\pm})} + 1 \left( a_1^{(0)} = a_2^{(0)} \right) \left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{x}(\mathbf{x}^{(0)}_{\pm})} \right) \quad (5.58)
\]
\[
= \sum_{\mathbf{c} \in (1,2)} \mathbb{E} \sum_{i=1}^{2k} \left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{c} \cdot \mathbf{c}, \mathbf{c}, \mathbf{c}, \mathbf{c}} \prod_{j=1}^{2} \left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{x}(\mathbf{x}^{(0)}_{\pm})} \quad (5.59)
\]
where
\[
\left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{c} \cdot \mathbf{c}, \mathbf{c}, \mathbf{c}, \mathbf{c}} = \begin{cases} \left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{c} \cdot \mathbf{c}, \mathbf{c}, \mathbf{c}, \mathbf{c}} \cdot \frac{1}{N} \mathbf{1} \left( I \neq J \right) or I \neq m \end{cases}
\]

So by construction one can compute that
\[
\left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{c} \cdot \mathbf{c}, \mathbf{c}, \mathbf{c}, \mathbf{c}} = \left( \frac{S}{N} \right)^{k/2} \sum_{\mathbf{c} \in (1,2)} \mathbb{E} \left[ \sum_{i=1}^{2k} \left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{c} \cdot \mathbf{c}, \mathbf{c}, \mathbf{c}, \mathbf{c}} \prod_{j=1}^{2} \left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{x}(\mathbf{x}^{(0)}_{\pm})} \right] \quad (5.60)
\]

As a result
\[
\sum_{\mathbf{c} \in (1,2)} \mathbb{E} \sum_{i=1}^{2k} \left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{c} \cdot \mathbf{c}, \mathbf{c}, \mathbf{c}, \mathbf{c}} \prod_{j=1}^{2} \left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{x}(\mathbf{x}^{(0)}_{\pm})} = \quad (5.61)
\]
\[
\left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{c} \cdot \mathbf{c}, \mathbf{c}, \mathbf{c}, \mathbf{c}} = \quad (5.62)
\]
\[
\sum_{\mathbf{c} \in (1,2)} \mathbb{E} \sum_{i=1}^{2k} \left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{c} \cdot \mathbf{c}, \mathbf{c}, \mathbf{c}, \mathbf{c}} \prod_{j=1}^{2} \left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{x}(\mathbf{x}^{(0)}_{\pm})} = \quad (5.63)
\]
\[
\left( \mathbf{E}^3_{\{\alpha_{ij}^{(0)}\},\{\alpha_{ij}^{(0)}\}} \right)_{\mathbf{c} \cdot \mathbf{c}, \mathbf{c}, \mathbf{c}, \mathbf{c}} = \quad (5.64)
\]
Next set $\mathcal{G}$ to be the graph with vertices $\{[\beta^1_1], [\beta^1_2], \ldots, [\beta^{2k}_{2l}]\}$ and with edges $([\beta^i_{l-1}], [\beta^i_l])$. Set $\rho(\mathcal{G})$ the indicator function that every vertex of $\mathcal{G}$ is adjacent to at least two edges. $v = ([\beta^1_1], [\beta^1_2], \ldots, [\beta^{2k}_{2l}])$. $v_r\in[v]$ the non-repeating vertices of $\mathcal{G}$, $d_r$ the multiplicity of $v_r$, and $o$ the number of self loops in $\mathcal{G}$. So, by (5.11), (5.10) and (5.12) one has that with overwhelming probability

$$
\rho(\mathcal{G}) \leq k \frac{\psi \Sigma_{\mathcal{G}}^{1/2}}{N^2} \prod_{r \in [v]} (|g_r|^k/2 + |g_r-p|^k/2).
$$

Thus one can show similarly to the proof of (2.40) in [27] that the following holds with overwhelming probability

$$
\rho(\mathcal{G}) \sum_{\mathcal{G} \in \mathcal{E}} \prod_{i=1}^{2k} \left( G_{[\beta^i],[\beta^i_1]} \right) \prod_{j=1}^{2k} \left( G_{[\beta^j],[\beta^j_1]} \right) \sum_{a} \sum_{c} \left| \sum_{i=1}^{2k} \left( G_{[\beta^i],[\beta^i_1]} \right) \prod_{j=1}^{2k} \left( G_{[\beta^j],[\beta^j_1]} \right) \sum_{a} \sum_{c} \left| \sum_{i=1}^{2k} \left( G_{[\beta^i],[\beta^i_1]} \right) \prod_{j=1}^{2k} \left( G_{[\beta^j],[\beta^j_1]} \right) \phi_{[\beta^i],[\beta^i_1]} \phi_{[\beta^j],[\beta^j_1]} \right| \leq k \frac{(\psi \log(N))^\Sigma_{\mathcal{G}}^{1/2}}{N^2}.
$$

Next we need to bound the quantity

$$
\sum_{i=1}^{2k} \left( W_{[\beta^i],[\beta^i_1]} \right) \left| \phi_{[\beta^i],[\beta^i_1]} \phi_{[\beta^j],[\beta^j_1]} \right| \prod_{j=1}^{2k} \left( W_{[\beta^j],[\beta^j_1]} \right) \left| \phi_{[\beta^j],[\beta^j_1]} \phi_{[\beta^l],[\beta^l_1]} \right| \leq k \frac{(\psi \log(N))^\Sigma_{\mathcal{G}}^{1/2}}{N^2}.
$$

Note, that in order for the product to be different than 0, every pair $([\alpha^i], [\beta^i])$ must appear an even number of times. Moreover in order for the product to be different than 0, for each $i, j$, the number of consecutive pairs $([\alpha^m], [\beta^m])$ and $([\alpha^{m+1}_r], [\beta^{m+1}_r])$ for $m \in [2k]$ and $r \in [l]$ such that exactly one of them is equal to $[\alpha^i], [\beta^j]$ must be two. Furthermore, for each $i, j$, if such pairs do not exist, then the number of consecutive pairs which are both equal to $[\alpha^i], [\beta^j]$ must be at least 2, or else the product would be 0. The latter is true since either the square of a centered Gaussian random variable minus its variance would appear, either the product of two independent centered Gaussian random variables would appear.

So it is implied that in order for the product above to not be zero, it is demanded that $\rho(\mathcal{G}) = 1$ and $X([\alpha^i]) = 1$. Here $\mathcal{G}$ is the graph which is associated with $\beta^i$. So by a trivial bounding in the moments of Gaussian random variables one can show that

$$
\left| \sum_{i=1}^{2k} \left( W_{[\beta^i],[\beta^i_1]} \right) \left| \phi_{[\beta^i],[\beta^i_1]} \phi_{[\beta^j],[\beta^j_1]} \right| \prod_{j=1}^{2k} \left( W_{[\beta^j],[\beta^j_1]} \right) \left| \phi_{[\beta^j],[\beta^j_1]} \phi_{[\beta^l],[\beta^l_1]} \right| \leq k \frac{(\psi \log(N))^\Sigma_{\mathcal{G}}^{1/2}}{N^2}.
$$

Thus, the proof of the lemma is complete after combining (5.70) and (5.67).

We are now ready to present the proof of Theorem 5.7.

**Proof of Theorem 5.7** Note that Lemma 5.13 holds for every sequence of indexes. In our case though, by construction, every term in $[\alpha^i]$ appears a non-zero even number of times since they appear consecutive times for $j \neq 1, l + 1$. So one has that $X([\alpha^i], [\alpha^j], [\alpha^k]) = X([\alpha^i], [\alpha^j], [\alpha^k]) = X([\mathcal{B}])$. So by a direct application of Lemma 5.13

$$
\sum_{a} \sum_{c} \left| \sum_{i=1}^{2k} \left( D_{[\alpha^i], [\alpha^i_1]} \right) \left| \phi_{[\alpha^i], [\alpha^i_1]} \phi_{[\alpha^j], [\alpha^j_1]} \right| \prod_{j=1}^{2k} \left( D_{[\alpha^j], [\alpha^j_1]} \right) \left| \phi_{[\alpha^j], [\alpha^j_1]} \phi_{[\alpha^l], [\alpha^l_1]} \right| \right| \leq k \frac{(\psi \log(N))^\Sigma_{\mathcal{G}}^{1/2}}{(\psi \log(N))^\Sigma_{\mathcal{G}}^{1/2}} X([\mathcal{B}]).
$$

(5.74)
As a result,

$$\sum_{1 \leq s, t \leq 2k} \sum_{b_i, b_j \in \{0, 1, \ldots, N\}} |q_{b_i}||q_{b_j}| \cdots |q_{b_{2k}}| \left| \prod_{i=1}^{2k} X_{b_{2i-1}, b_{2i}} \right| \leq k$$

(5.75)

$$\sum_{1 \leq s, t \leq 2k} \sum_{b_i, b_j \in \{0, 1, \ldots, N\}} \frac{(\psi \log(N))^\sum_l (s + \eta)^\sum_{l/2} X(B)}{(N^2)^{\sum_l 1/2}} \sum_{b_i, b_j \in \{0, 1, \ldots, N\}} |q_{b_i}||q_{b_j}| \cdots |q_{b_{2k}}| \sum_{a} \prod_{i,j} \left( |g_{a_{ij}}| + |g_{a_{ij}}| \right) \leq k$$

(5.76)

$$\sum_{1 \leq s, t \leq 2k} \sum_{b_i, b_j \in \{0, 1, \ldots, N\}} \frac{(\psi \log(N))^\sum_l (s + \eta)^\sum_{l/2} X(B)}{(N^2)^{\sum_l 1/2}} \sum_{b_i, b_j \in \{0, 1, \ldots, N\}} |q_{b_i}||q_{b_j}| \cdots |q_{b_{2k}}| \sum_{a} \prod_{i,j} \left( |g_{a_{ij}}| + |g_{a_{ij}}| \right) \leq k$$

(5.77)

where the sum now is considered over all $A \subseteq B^{l+2k}$ with the restriction that $[a_i^t] = B_{2i-1}$ and $[a_i^{t+1}] = B_{2i}$.

Moreover note that the array $[a_i^t]$ defines a partition on the set $\{(i, j) : i \in [2k], j \in [l, i+1]\}$ such that $(i, j)$ belongs to the same block of the partition with $(i', j')$ if and only if $a_i^t = a_i^{t'}$. Furthermore, denote $n = |B|$, $d_i$ the number of times the $i-th$ element of $B$, which we denote with $y_i$, appears without repetition and $r_i$ such that $r_i + d_i$ is the number of times the $i-th$ element of $B$ appears in $A$. Note that since we are interested in the sequences that $X(B) = 1$, it is implied that $d_i$ are all even. So it is true that

$$\sum d_i = 2k \quad 2k + \sum r_i = \sum l.$$  

Moreover, notice that each induced partition mentioned before, uniquely determines the quantities $d_i$, $l_i$, and each block of the partition has at least two elements, since $X(B) = 1$. So we can modify the sum, into first summing over all partitions $P$ and then over all $A$-possible choices in the partition. Note that $B$ is completely described by the set $A$. So one has that

$$\sum_{1 \leq s, t \leq 2k} \sum_{b_i, b_j \in \{0, 1, \ldots, N\}} \frac{(\psi \log(N))^\sum_l (s + \eta)^\sum_{l/2} X(B)}{(N^2)^{\sum_l 1/2}} \sum_{P} \sum_{A \subseteq \text{partition}} X\left([a_i^t]_{i \in [2k], j \in [l, i+1]}\right) \leq k$$

(5.78)

$$\cdot \sum_{1 \leq s, t \leq 2k} \sum_{b_i, b_j \in \{0, 1, \ldots, N\}} \frac{(\psi \log(N))^\sum_l (s + \eta)^\sum_{l/2} X(B)}{(N^2)^{\sum_l 1/2}} \sum_{P} \sum_{A \subseteq \text{partition}} X\left([a_i^t]_{i \in [2k], j \in [l, i+1]}\right) \leq k$$

(5.79)

$$\sum_{1 \leq s, t \leq 2k} \sum_{b_i, b_j \in \{0, 1, \ldots, N\}} \frac{(\psi \log(N))^\sum_l (s + \eta)^\sum_{l/2} X(B)}{(N^2)^{\sum_l 1/2}} \sum_{P} \sum_{A \subseteq \text{partition}} X\left([a_i^t]_{i \in [2k], j \in [l, i+1]}\right) \leq k$$

(5.80)

$$\sum_{1 \leq s, t \leq 2k} \sum_{b_i, b_j \in \{0, 1, \ldots, N\}} \frac{(\psi \log(N))^\sum_l (s + \eta)^\sum_{l/2} X(B)}{(N^2)^{\sum_l 1/2}} \sum_{P} \sum_{A \subseteq \text{partition}} X\left([a_i^t]_{i \in [2k], j \in [l, i+1]}\right) \leq k$$

(5.81)

where in the last inequality we used the fact that $\psi \log(N)(\sqrt{N^2})^{-1} \leq 1$, the fact that $\sum l_i \geq 2k$, and the fact that both the number of partitions and the number of possible $l_i$ are bounded by constants depending only on $k$. For the second to last inequality, we used Proposition 2.18-inequality (2.38) in [27], the facts that $d_i \geq 2$ and that $\sum d_i = 2k$.

This finishes the proof of Theorem [5.7] \hfill \square

### 5.4 Bounding the perturbed matrices at the optimal scale

At this subsection we are going to essentially bound the entries of the resolvent $G(s, z)$ at the optimal scale $\text{Im}(z) = N^{-1}$, for all matrices $\tilde{V}$ that are initially bounded by an $N$-dependent parameter. Next, we will apply this result to the matrix $X$, which is initially bounded due to [3.17] with high probability. Thus we will prove that the matrix $X$, after slightly perturbing it, has essentially bounded resolvent entries at the optimal scale $N^{-1}$, for any small enough, positive $\delta$.

**Proposition 5.14.** Let $\tilde{V}$ be an $N \times N$ matrix and consider $\tilde{V}$ the symmetrization of $D$. Suppose that $\tilde{V}$ satisfies Assumption [5.] for some parameters $h, r$ at energy level $E_0 = 0$ and that there exists an $N$-dependent parameter $B \in (0, \frac{1}{h})$ such that $\max |(\tilde{V} - zI)_{ij}| \leq B$. Then for any $\delta > 0$ and $s : N^6 h_s \leq s \leq r N^{-\delta}$, it is true
that for any $D > 1$ there exists $C = C(\delta, D)$ such that
\[ \mathbb{P} \left( \sup_{D, ij} \sup_{s, z} |G(s, z)| \geq BN^2 \right) \leq CN^{-D}, \]
where $G(s, z) = (\tilde{V} + \sqrt{W} - z)^{-1}$, $W$ is the symmetrization of an i.i.d. Gaussian matrix with centered entries and common variance $\frac{1}{N}$ and $D = \{E + i\eta : E \in (-\frac{1}{2}, \frac{1}{2}), \eta \in [N^{\delta - 1}, 1 - \frac{1}{2}] \}$.

**Proof.** By a direct application of Theorem 5.6 for $q_k = 1 \{ i = k \}$ for any $k \in [2N]$ (without loss of generality suppose $k \in [N]$), one has that with overwhelming probability uniformly on $D$ it is true that,
\[ |G_{k, k}(s, z)| \leq \sum_{i = -N}^{N} ((|g_i| + |g_{-i}|)(u_{i+1}(0), q_k))^2 + \sum_{i = 1}^{2N} ((|g_i| + |g_{-i}|)(u_i(0), q_k))(\langle u_{i+1}(0), q_k \rangle) \]
\[ + \frac{N^{3/2}}{\sqrt{N\eta}} \left| \sum_{i = 1}^{N} (g_i + g_{-i})(\langle u_i, q_k \rangle + \langle u_{i+1}, q_k \rangle) \right|. \]

Note that by definition the $k$–th element of each of the columns/rows of $U$ is 0 for all the columns/rows with index larger than $N$. Moreover by definition $N^2 \leq N\eta$. So it is implied that the above bound becomes
\[ |G_{k, k}(s, z)| \leq \sum_{i = 1}^{N} ((|g_i| + |g_{-i}|)|u_{i+k}|. \]

Furthermore, due to Schur’s complement formula, one can prove, as in Lemma 3.28, that
\[ G_{k, k} = z\tilde{G}_{k, k}(z^2), \]
where $\tilde{G}$ is the resolvent of the matrix $D^T D$. Moreover, one may compute that
\[ D^T D = \left( (U)_{i \in [N], j \in [N]} \right)^T \Sigma^2 (U)_{i \in [N], j \in [N]}, \]
where $\Sigma$ is the diagonal matrix with the singular values of $D$. So it is true that
\[ \left| \sum_{i = 1}^{N} u_{k, i}^2 \frac{1}{\lambda_i^2 - z^2} \right| = \left| \frac{1}{z} \tilde{G}_{k, k}(z) \right| \leq \frac{B}{|z|}, \]
with overwhelming probability uniformly on $z \in \{ z = E + i\eta : E \in (-r, r), h, \eta \leq \eta \leq 1 \}$. Thus if we consider the sets $\mathcal{A}_m(0) = (2^{-m}h, 2^{-m}h) \cup (-2^{-m}h, -2^{-m}h)$ and set $z = i\eta$, it is true that
\[ \max_{j \in [N]} \left| \sum_{i \in \mathcal{A}_m} u_{k, i}^2 \right| \leq \min[B\eta 2^{4m}, 1]. \]

where the bounding by 1 is true due to the fact that the eigenvectors are considered normalized. After this observation the proof continues in a completely analogous way to the Proof of Proposition 3.9 in [24] and so it is omitted.

\[ \Box \]

**Corollary 5.15.** Adopt the notation of Section 3. Let $\mathcal{A}$ be the set mentioned in Theorem 3.13. For all $a \in (0, 2) \setminus \mathcal{A}$ consider the matrix $X + \sqrt{W}$, where $t = t(N)$ is defined in Definition 2.9. Set $\{ T_i \}_{i \in [2N]}(z)$ the resolvent of $X + \sqrt{W}$ at $z$. Then it is true that for any $D > 0$ and $\delta > 0$, there exists a constant $C' = C'(a, v, \rho, \delta, D)$ such that
\[ \mathbb{P} \left( \sup_{D, \lambda \in \mathcal{A}} \left( \sup_{D, ij} |T_i(z)| \right) \geq N^\delta \right) \leq C' N^{-D}, \]
where $D_{C, \delta} = \{ E + i\eta : E \in (-\frac{1}{2}, \frac{1}{2}), \eta \in [N^{\delta - 1}, 1 - \frac{1}{2}] \}$ and $C_\alpha$ is the constant mentioned in Theorem 3.13.

**Proof.** Due to 3.17, 3.15 and since $t$ belongs to the desired interval $(N^{2\delta - 4}, N^{-2\delta})$, as is mentioned in the proof of Corollary 4.5 the proof of Corollary 5.15 is just an application of Proposition 5.14 to our set of matrices.

\[ \Box \]

**Remark 5.16.** Note that bounding the entries of the resolvent of $X + \sqrt{W}$ as we did in Corollary 5.15 at scale $N^{\delta - 1}$, implies the complete eigenvector delocalization in the sense of Theorem 1.2. The proof of the latter claim is well-known and can be found in the proof of Theorem 6.3 in [33].
6 Establishing universality of the least singular value and eigenvector delocalization

Thus far, we have proven both universality of the least singular value, Corollary 4.5, and complete eigenvector de-localization, Remark 5.16; for the matrix $X + \sqrt{t}W$ in the sense of Theorem 1.2. What we need to prove next, is that the transition from $X + \sqrt{t}W$ to $X + A$ is smooth enough to preserve both the eigenvector delocalization and universality of the least singular value. A first step to that direction is Theorem 6.4 whose proof is more or less the same as its symmetric counterpart in [24]. Furthermore what we manage, is to extend Theorem 3.15 of [24] to its "integrating analogue" in Proposition 6.7 which is not very difficult given Theorem 6.4. Proposition 6.7 is the milestone for the comparison of the least positive eigenvalues of $X + A$ and $X + \sqrt{t}W$.

Firstly, we will use a convenient decomposition of the elements of $H$ in order to express the dependence of the "small" and the "large" entries of $H$ with Bernoulli random variables.

**Definition 6.1.** Define the following random variables for $i, j \in [2N] : |i - j| \geq N$

$$\psi_{i,j} = \mathbb{P}\left(|h_{i,j}| \geq N^{-\rho}\right), \quad x_{i,j} = \frac{\mathbb{P}\left(|h_{i,j}| \in (N^{-\nu}, N^{-\rho})\right)}{\mathbb{P}\left(|h_{i,j}| \leq N^{-\rho}\right)},$$

and

$$\mathbb{P}[a_{i,j} \in I] = \frac{\mathbb{P}(h_{i,j} \in I \cap (-N^{-\nu}, N^{-\nu}))}{\mathbb{P}(|h_{i,j}| < N^{-\nu})}, \quad \mathbb{P}(c_{i,j} \in I) = \frac{\mathbb{P}(h_{i,j} \in (-\infty, N^{-\rho}) \cup (N^{\rho}, \infty) \cap I)}{\mathbb{P}(|h_{i,j}| \geq N^{-\rho})},$$

$$\mathbb{P}(b_{i,j} \in I) = \frac{\mathbb{P}(h_{i,j} \in (N^{-\nu}, N^{-\rho}) \cap h_{i,j} \in I)}{\mathbb{P}(h_{i,j} \in [N^{-\nu}, N^{-\rho}])},$$

for any interval $I$, subset of $\mathbb{R}$.

Moreover we define each bunch of

$$(6.1) [x_{i,j}]_{i,j \in 2N: |i-j| \geq N}, [\psi_{i,j}]_{i,j \in 2N: |i-j| \geq N}, [a_{i,j}]_{i,j \in 2N: |i-j| \geq N},$$

$$(6.2) [b_{i,j}]_{i,j \in 2N: |i-j| \geq N}, [c_{i,j}]_{i,j \in 2N: |i-j| \geq N}$$

to be independent up to symmetry and independent amongst them for different indexes $i, j$.

**Definition 6.2.** Define the following matrices

$$A_{i,j} = \begin{cases} (1 - \psi_{i,j})(1 - x_{i,j})a_{i,j}, & i,j : |i-j| \geq N \\ 0, & \text{otherwise} \end{cases}$$

$$(6.3)$$

$$B_{i,j} = \begin{cases} (1 - \psi_{i,j})x_{i,j}b_{i,j}, & i,j : |i-j| \geq N \\ 0, & \text{otherwise} \end{cases}$$

$$(6.4)$$

$$C_{i,j} = \begin{cases} \psi_{i,j}c_{i,j}, & i,j : |i-j| \geq N \\ 0, & \text{otherwise} \end{cases}$$

$$(6.5)$$

$$\Psi_{i,j} = \begin{cases} \psi_{i,j}, & i,j : |i-j| \geq N \\ 0, & \text{otherwise} \end{cases}$$

$$(6.6)$$

Note that by definition $H = A + B + C$ and $X = B + C$.

Next we define the way to quantify the transition from $X + \sqrt{t}W$ to $X + A$.

**Definition 6.3.** Define the matrices

$$H^v = \gamma A + \sqrt{t}(1 - \gamma^2)^{1/2}W + X, \quad \gamma \in [0, 1]$$

and $G^v(z) = (H^v - zI)^{-1}$.
6.1 Green function Comparison

Next we present a comparison theorem for the resolvent entries of $H^v$.

**Theorem 6.4.** Let $a, b, \rho, v$ be constants that satisfy (2.1). Additionally suppose that $a \in (0, 2) \setminus \mathcal{A}$ as in Theorem 3.14. Moreover let $F : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$
\sup_{|x| \leq 2^n} |F^{(x)}(x)| \leq N^{C_0}, \quad \sup_{|x| \leq 2^n} |F^{(x)}(x)| \leq N^{C_0},
$$

(6.7)

for some absolute constant $C_0 > 0$, for some integer $n = n(a, b, \rho, v, C_0)$ sufficiently large and any $\epsilon > 0$ and $\mu \in [n]$. Furthermore fix $z = E + in\gamma$ for $E \in \mathbb{R}$ and $\gamma \geq N^{-2}$. Moreover for any matrix $\Psi$ denote $\mathbb{E}_\Psi$ the conditional expectation with respect to $\Psi$. Set

$$
\Xi(z) = \sup_{y \in [0, 1]} \max_{|n|} \mathbb{E}_\Psi |F^{(n)} \text{Im}(G^v_{ij}(z))|,
$$

(6.8)

$$
\Omega_0(z, \epsilon) = \begin{cases} 
\sup_{|y| \leq n} |G^v_{ij}(z)| \leq N^\epsilon, & G_0(z, \epsilon) = 1 - \mathbb{E}_\Psi (\Omega_0(z, \epsilon)). 
\end{cases}
$$

(6.9)

Then there exist $\epsilon = \epsilon(a, b, \rho, v)$ and $\omega = \omega(a, b, \rho, v)$ such that for any matrix $\Psi$ with at most $N^{l+q_\rho+\epsilon}$ non-zero entries, there exists a constant $C = C(a, v, \rho)$ so that

$$
\sup_{y \in [0, 1]} \left| \mathbb{E}_\Psi F \left( \text{Im}(G^v_{ij}(z)) \right) - \mathbb{E}_\Psi F \left( \text{Im}(G^0_{ij}(z)) \right) \right| \leq CN^{-\omega} (\Xi(z) + 1) + CQ_0(z, \epsilon)N^{C+C_0}, \quad \text{for all } i, j \in [2N].
$$

(6.10)

A similar bound to (6.10) can be proven if one replaces $\text{Im}(G^v_{ij}(z))$ and $\text{Im}(G^0_{ij}(z))$ with $\text{Re}(G^v_{ij}(z))$ and $\text{Re}(G^0_{ij}(z))$ respectively.

**Proof.** The proof is similar to the proof of Theorem 3.15 in [24]. Next we give a short description of the main ideas behind the proof. We will do so only for the imaginary parts $\text{Im}(G^v_{ij}(z))$. The proof for the real parts $\text{Re}(G^v_{ij}(z))$ is completely analogous.

Fix $z \in \mathbb{C}$ and $F : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the hypothesis of Theorem 6.4.

Firstly note that since $G^v = G^v(H^v - zI)G^v$, it is true that

$$
\frac{d}{dy} G^v = \frac{d}{dy} G^v(H^v - zI)G^v + G^v(H^v - zI)\frac{d}{dy} G^v + G^v \frac{d}{dy} (H^v - zI)G^v.
$$

So it is implied that

$$
-\frac{d}{dy} G^v = G^v \frac{d}{dy} (H^v) G^v,
$$

(6.11)

where the derivative $\frac{d}{dy}$ is considered in every entry. So by (6.11) and Leibniz integral rule, it is true that

$$
\left| \frac{d}{dy} \mathbb{E}_\Psi G^v_{ij} \right| = \sum_{p, q, r, \nu, \rho \in [2N]} \mathbb{E}_\Psi G^v_{ij} \left( A_{p, q} - \frac{\gamma^{1/2} \mu_{p, q}}{(1 - \gamma^2)^{1/2}} \right) G^v_{ij}. 
$$

Thus, in order to prove (6.10) it is sufficient to show that there exists a constant $C = C(a, v, \rho) > 0$ such that

$$
\sum_{p, q, r, \nu, \rho \in [2N]} \mathbb{E}_\Psi \text{Im}(G^v_{ij} G^v_{ij}) \left( A_{p, q} - \frac{\gamma^{1/2} \mu_{p, q}}{(1 - \gamma^2)^{1/2}} \right) F'(\text{Im}(G^v_{ij})) \leq \frac{C}{(1 - \gamma^2)^{1/2}} (N^{-\omega}(\Xi(z) + 1) + Q_0 N^{C+C_0})
$$

(6.12)

and then integrate over any interval of the form $(0, y')$ with $y' \in (0, 1)$. The proof of (6.12) is completely analogous to the proof of Proposition 4.4 in [24]. So we will give a sketch of the proof. Firstly fix $p, q, \in [2N] : \left| p - q \right| \geq N$ and set the matrices

$$
D_{a, b} = \begin{cases} 
H^v_{a, b} & (i, f) \notin ((p, q), (q, p)) \\ X_{p, q} & \text{else}
\end{cases}
$$

(6.13)

$$
E_{a, b} = \begin{cases} 
H^v_{a, b} & (a, b) \notin ((p, q), (q, p)) \\ C_{p, q} & \text{else}
\end{cases}
$$

(6.14)
Moreover set
\[ \Gamma = H^\nu - D, \quad \Lambda = D - E \]  
\[ R = (D - zI)^{-1}, \quad U = (E - zI)^{-1}. \]  
(6.15)  
(6.16)
So by Lemma 3.16 and as we have mentioned in the proof of Theorem 5.7, one can apply Taylor’s Theorem for matrices to get that
\[ G' - R = -R \Gamma R + (R \Gamma)^2 R - (R \Gamma)^3 G'. \]  
Moreover, by a Taylor expansion for the function \( F' \), it is true that for some \( \zeta_0 \in [\text{Im}(G'_{ij})]\), \( \text{Im}(R_{ij}) \) and \( \zeta = \text{Im}(R_{ij}) - \text{Im}(G'_{ij}) \).
\[ F'(\text{Im}(G'_{ij})) = F'(\text{Im}(R_{ij})) + \mathcal{O}(3)(\text{Im}(R_{ij})) + \frac{3^2}{2} F'(\text{Im}(R_{ij})) + \frac{3^3}{6} F^{(4)}(\zeta_0). \]  
(6.18)
where we have denoted \( F^{(l)}(x) = \frac{d^l}{dx^l} F(x) \) for all \( l \in \mathbb{N} \).
So by combining (6.17) and (6.19), one can notice that each of the \((p,q)\)-summand in (6.12) can be viewed as a sum of finite number of monomials of \( A_{p,q} \) and \( t^{1/2} u_{p,q} \) with coefficients depending on the matrices \( R \) and \( G' \). These monomials can be categorized into the following cases:

1. The product of even degree of terms, i.e., \( \prod_{k=1}^n \varphi_{k} \) such that \( \sum \kappa_r \) is even and \( \varphi_{k} \) is equal either to \( (R \Gamma)^k R \), either equal to \( \text{Im}((R \Gamma)^k R) \), either to \( \text{Re}((R \Gamma)^k R) \) for some \( s \in \mathbb{N} \) and \( \kappa_r \in \{0, 1, 2\} \). Then for any \( m \in \{0, 1, 2\} \) it is true that
\[ E \varphi F'(m)(\text{Im}(R_{ij})) \left( A_{p,q} - \frac{\varphi_{k} t^{1/2} u_{p,q}}{(1 - y^2)^{1/2}} \right) \leq N^{-2} \frac{C}{(1 - y^2)^{1/2}} (N^{-\alpha}(\Xi + 1) + \Theta_0 N^{1/2}). \]  
(6.19)

2. The terms that contain \( F^{(4)}(\zeta_0) \) can be bounded by a Taylor expansion similarly to Lemma 4.7 in [24]. More precisely one can show that
\[ E \varphi \text{Im}(G'_{ij}) \left( A_{p,q} - \frac{\varphi_{k} t^{1/2} u_{p,q}}{(1 - y^2)^{1/2}} \right)^3 \frac{3^2 F(3)}{2} \leq N^{-2} \frac{C}{(1 - y^2)^{1/2}} (N^{-\alpha}(\Xi + 1) + \Theta_0 N^{1/2}). \]  
(6.20)

3. Analogously to the previous bound, one can prove that for the \( s \)-products of \( \varphi_{k} \), when \( s \in \{1, 2, 3, 4\}, k_r \in \{0, 1, 2, 3\} \) and \( \sum k_r \geq 3 \), it holds that for any \( m \in \{0, 1, 2\} \),
\[ E \varphi F'(m)(\text{Im}(R_{ij})) \left( A_{p,q} - \frac{\varphi_{k} t^{1/2} u_{p,q}}{(1 - y^2)^{1/2}} \right)^s \leq N^{-2} \frac{C}{(1 - y^2)^{1/2}} (N^{-\alpha}(\Xi + 1) + \Theta_0 N^{1/2}). \]  
(6.21)

4. The remaining terms are the monomials of 2- degree. So it can be proven that,
\[ E \varphi(\text{Im}(R \Gamma R)) F'(\text{Im}(R_{ij})) \left( A_{p,q} - \frac{\varphi_{k} t^{1/2} u_{p,q}}{(1 - y^2)^{1/2}} \right) \leq N^{-2} \frac{C}{(1 - y^2)^{1/2}} (N^{-\alpha}(\Xi + 1) + \Theta_0 N^{1/2}). \]  
(6.22)
The proof of these inequalities is a consequence of further comparison between the entries of the matrices $R$ and $U$, similar to the one which was done for the matrices $G'$ and $R$ before.

So after summing over all possible $(p,q)$ and taking into account that $t \sim N^{(a-2)\nu}$ and that there are at most $N^{1+\alpha\nu+c}$ non-zero entries of $\Psi$ with overwhelming probability, see the proof of Corollary 5.6 one has that \[(6.12)\] holds, which finishes the proof. □

In what follows, set $C_n$ the constant mentioned in Theorem 3.14 So due to Theorem 6.3 one can prove the following.

**Proposition 6.5.** Let $a, b, v, \rho$ as in (2.1). Moreover fix $c > 0$ arbitrary small. Then for each $\delta > 0$ and $D > 0$ there exists a constant $C = C(a, \rho, v, b)$ such that

$$
P \left( \sup_{p \in [0,1]} \sup_{\epsilon \in \left[ -\frac{1}{2a}, \frac{1}{2a} \right]} \sup_{\eta \in N^{\delta-1}} \max_{\gamma} |G^\gamma_{ij}(E + \eta)| \geq N^\delta \right) \leq CN^{-D}. \quad (6.26)$$

The constant $C_n$ in (6.26) is the constant mentioned in Theorem 3.14.

*Proof.* The proof is based on Theorem 6.4 and is similar to the proof of Proposition 3.17 in [24], so we will just describe the key ideas behind the proof. The proof is done in steps. Set $p = \left[ \frac{D+30}{\delta} \right]$ and consider the function $F_{2p}(x) = |x|^{2p} + 1$. Note that $F_{2p}$ satisfies the hypothesis of Theorem 6.4. Moreover by Corollary 5.15 there exists a constant $C' = C'(a, b, v, \rho)$ such that

$$
P \left( \sup_{p \in [0,1]} \sup_{\epsilon \in \left[ -\frac{1}{2a}, \frac{1}{2a} \right]} \sup_{\eta \in N^{\delta-1}} \max_{\gamma} |G^\gamma_{ij}(E + \eta)| \right) \leq C'N^{-D}.$$

Fix $\epsilon_0$ and $\omega$, the constants from the application of Theorem 6.4 for the function $F_p$. Moreover define the quantities

$$\mathcal{B}(\delta, \eta) = P \left( \sup_{p \in [0,1]} \max_{\gamma} |G^\gamma_{ij}(E + \eta)| \geq N^\delta \right),$$

for $E \in \left[ -\frac{1}{2a}, \frac{1}{2a} \right]$ and $\eta \geq N^{\delta-1}$. Set $s = \frac{c}{4}$. Then one can show that there exists a constant $A = A(\delta, D)$ such that,

$$\mathcal{B}(\delta, \eta) \leq AN^A \mathcal{B}(\frac{\epsilon_0}{2}, N^s \eta) + AN^{-D}, \quad (6.27)$$

which can be proven by (i) integrating over $\Psi$ in the conclusion of Theorem 6.4 for $F_p$ after using (6.28), (ii) Corollary 5.15 (iii) Markov’s Inequality applied for $\gamma \in N^{-20} \cap (0, 1)$ and (iv) the deterministic estimates in the end of the proof of Lemma 4.3 in [24].

Thus in order to conclude, one can use induction over all $k \in \left[ -1, \left[ \frac{1-s}{4} \right] \right]$ to show that

$$\mathcal{B}(\frac{\epsilon_0}{2}, N^{-k\eta}) \leq AN^{-D}$$

and then extend to all $E \in \left[ -\frac{1}{2a}, \frac{1}{2a} \right]$ and $\eta \geq N^{\delta-1}$ by deterministic estimates of the form $|G^\gamma(z) - G^\gamma(z')| \leq N^\delta |z - z'|$ for an appropriately chosen grid. □

**Corollary 6.6.** Fix $F : \mathbb{R} \to \mathbb{R}$ such that it satisfies the assumption of Theorem 6.4 and $E \in \left[ \frac{1}{2a}, \frac{1}{2a} \right]$ and $\eta \geq N^{\delta-1}$, for an arbitrary small $c > 0$. Then there exists a constant $C = c(a, b, v, \rho, C_0)$ and a large constant $C = C(a, b, v, \rho)$ such that

$$\sup_{p \in [0,1]} \left| \mathbb{E}F(\Im(G^\gamma_{ij}(z))) - \mathbb{E}F(\Im(G^\gamma_{ij}(z))) \right| \leq CN^{-c}, \text{ for all } i,j \in [2N].$$

*Proof.* Firstly note that due to Chernoff bound there exists a constant $C'$ such that

$$\mathbb{P} \left( \{ (i,j) : H_{ij} \in [N^{-\rho}, \infty] \not\in \left( \frac{N^{1+\rho} C}{C'}, \frac{CN^{1+\rho}}{C'} \right) \} \right) \leq C' \exp \left( \frac{-N}{C'} \right). \quad (6.28)$$
Set $\Omega = \{ (i,j) : |H_{ij}| \in [N^{-p}, \infty) \} \in \left( \frac{N^{1+q_p}}{N}, C'N^{1+q_p} \right)$. Moreover by the deterministic estimate $|G_{ij}^\gamma| \leq \eta^{-1} \leq N$ and the hypothesis for $F$ one has that $|F(\text{Im}(G_{ij}^\gamma))| \leq N^{C_0}$ and hence,

$$\mathbb{E}F(\text{Im}(G_{ij}^\gamma(z))) - F(\text{Im}(G_{ij}^\gamma(z))) \leq \mathbb{E} \mathbf{1}(\Omega) F(\text{Im}(G_{ij}^\gamma(z))) - F(\text{Im}(G_{ij}^\gamma(z))) + N^{C_0}C \exp \left( \frac{-N}{C} \right).$$

Note that on the set $\Omega$ we can apply Theorem 6.4. Moreover by Proposition 6.5 one has that $Q_0(z, \epsilon) \leq C N^{-D}$ for any $D > 0$ and similarly show that

$$\Xi \leq N^{C_0}Q_0(z) + CN^{C_0}.\epsilon.$$ 

So the proof is complete after choosing an appropriately large $D > 0$.

Next, we extend the comparison result in such way that we can use in order to approximate the gap probability.

**Proposition 6.7.** Fix parameters $a, b, \rho, \nu$ as in [2.1]. Let $q : \mathbb{R} \to \mathbb{R}$ a $C^\infty$ function with all its derivatives bounded by an absolute constant $M$ greater than 1. Then for any $\eta \geq N^{-2}$ and any positive sequence $r(N)$ such that $\lim r(N) = r > 0$ there exist constants $\omega = \omega(a, \rho, \nu, b, r), \epsilon = \epsilon(a, \rho, \nu, b, r)$ and $C = C(a, \rho, \nu, b, r)$ such that

$$\sup_{y \in [0, 1]} \left| \mathbb{E}q\left( \int_{-\frac{\omega}{\eta}}^{\omega} \sum_{i=1}^{2N} \text{Im} G_{ii}^\gamma(y + i\eta)dy \right) - \mathbb{E}q\left( \int_{-\frac{\epsilon}{\eta}}^{\epsilon} \sum_{i=1}^{2N} \text{Im} G_{ii}^\gamma(y + i\eta)dy \right) \right| \leq C \left( MN^{-\omega} + MN^C Q(\epsilon, \eta) \right). \tag{6.29}$$

where

$$Q(\epsilon, \eta) = \mathbb{P} \left( \sup_{E \in [-\eta, \eta]} \max_{ij} |G_{ij}^\gamma(E + i\eta)| \geq N^\epsilon \right).$$

Moreover if we suppose that $\eta \geq N^{c-1}$, for arbitrary small $\xi > 0$, then there exists a constant $c = c(a, \rho, \nu, b)$ such that,

$$\sup_{y \in [0, 1]} \left| \mathbb{E}q\left( \int_{-\frac{\omega}{\eta}}^{\omega} \sum_{i=1}^{2N} \text{Im} G_{ii}^\gamma(y + i\eta)dy \right) - \mathbb{E}q\left( \int_{-\frac{\xi}{\eta}}^{\xi} \sum_{i=1}^{2N} \text{Im} G_{ii}^\gamma(y + i\eta)dy \right) \right| \leq CN^{-c}. \tag{6.30}$$

**Proof.** For simplicity we will assume $r$ is a constant. The proof of (6.29) is similar to the proof of Theorem 6.4. Next we highlight the differences.

Note that similarly to the proof of Corollary 6.6 it is sufficient to prove that for any matrix $\Psi$ with at most $N^{1+q_p}$ non-zero entries it is true that

$$\sup_{y \in [0, 1]} \left| \mathbb{E}\Psi q\left( \int_{-\frac{\omega}{\eta}}^{\omega} \sum_{i=1}^{2N} \text{Im} G_{ii}^\gamma(y + i\eta)dy \right) - \mathbb{E}\Psi q\left( \int_{-\frac{\xi}{\eta}}^{\xi} \sum_{i=1}^{2N} \text{Im} G_{ii}^\gamma(y + i\eta)dy \right) \right| \leq C \left( MN^{-\omega} + MN^C Q(\epsilon) \right).$$

Furthermore one can compute the derivative of the previous quantity with respect to $y$, as in Theorem 6.4. Thus by Leibniz integral rule, Fubini Theorem and (6.11) it is true that

$$\left| \frac{d}{dy} \mathbb{E}\Psi q\left( \int_{-\frac{\omega}{\eta}}^{\omega} \sum_{i=1}^{2N} \text{Im} G_{ii}^\gamma(y + i\eta)dy \right) \right| \leq \int_{-\frac{\omega}{\eta}}^{\omega} \sum_{i=1}^{2N} \sum_{p,q \in [2N]} |q_{p,q}|^{1/2} \left| \text{Im}(G_{ii}^\gamma G_{ii}^\nu) \right| \left| A_{p,q} - \frac{y^{1/2}u_{p,q}}{(1 - y^2)^{1/2}} \right| dy.$$

As a result it is sufficient to prove that for any $y \in \left( -\frac{\tau}{\eta}, \frac{\tau}{\eta} \right)$,

$$\sum_{p,q \in [2N]} |q_{p,q}|^{1/2} \left| \text{Im}(G_{ii}^\gamma G_{ii}^\nu) \right| \left| A_{p,q} - \frac{y^{1/2}u_{p,q}}{(1 - y^2)^{1/2}} \right| \leq \frac{C}{(1 - y^2)^{1/2}} M N^{-\omega} + Q(\epsilon, \eta) N^{C_0}. \tag{6.31}$$
But the proof of (6.31) is similar to the proof of (6.12), with the main difference located in the Taylor expansion which now instead of being applied as in (6.18), it will be applied for the quantities $\int_{y}^{r} \sum_{i=1}^{2N} \text{Im} G^r_i(y + i\eta) dy$ and $\int_{y}^{r} \sum_{i=1}^{2N} \text{Im} R_i(y + i\eta) dy$ for fixed $p, q$. But eventually, this does not affect the proof since each $(p,q)$-summand can again be expressed into monomials of $A_{p,q}$, $w_{p,q}$, which do not depend on the parameters $\eta$ and $y$, and since the quantity $G_0(\varepsilon, \eta)$ is replaced by $Q(\varepsilon, \eta)$ in the bound.

Moreover, if we assume $\eta \geq N^{-1}$, then $Q(\varepsilon, \eta)$ is smaller than $N^{-D}$ for any $D$ and for sufficient large $N$. Thus similarly to the proof of (6.6), one can prove (6.30) by (6.29).

\[ \square \]

Moreover, we wish to prove that the righthand side of (6.29) tends to $0$ as $N$ tends to infinity for $\eta = O(\frac{1}{N^{1/2}})$, below the natural scale. This is achieved via the following lemma.

**Lemma 6.8** (Lemma 2.1). Let $Y$ be an $N \times N$ matrix. Set the following quantity

$$\Gamma(Y, E + i\eta) = \max \{ 1, \max_{ij} |(Y - (E + i\eta)I_j^{-1})| \}. $$

Then for any $M \geq 1$ and $\eta > 0$ the following deterministic inequality holds

$$\Gamma(Y, E + i\frac{\eta}{M}) \leq M \Gamma(Y, E + i\eta).$$

(6.33)

**Corollary 6.9.** Fix $\varepsilon$ and $\delta$ arbitrary small positive numbers. Set $\eta_1 = N^{-\varepsilon/2}$. Then by (6.33) and (6.26) one has that for any $D > 0$ and for sufficient large $N$, it is true that

$$\mathbb{P} \left( \sup_{\eta \in [0,1]} \sup_{(y, E) \in [i\pi + \frac{1}{2}, i\pi - \frac{1}{2}]} \max_{ij} |G^0_{ij}(Y + i\eta_1)| \geq N^D \right) \leq C N^{-D}.$$

(6.34)

So in the setting of Proposition 6.7, it is implied that there exist two positive constants $C = C(a, b, v, \rho, r)$, $c = c(a, b, \rho, v)$ such that

$$\sup_{\eta \in [0,1]} \mathbb{E} \left( \left| \frac{1}{\sqrt{N}} \sum_{i=1}^{2N} \text{Im} G^0_{ii}(y + i\eta_1) dy - \mathbb{E} \left[ \frac{1}{\sqrt{N}} \sum_{i=1}^{2N} \text{Im} G^0_{ii}(y + i\eta_1) dy \right] \right| \right) \leq C N^{-c}.$$

(6.35)

### 6.2 Approximation of the gap probability

The goal of this subsection is to approximate the gap probability, i.e., the probability that there are no eigenvalues in an interval, by $C^\infty$ functions of the Stieltjes transform as in Proposition 6.7. In order to prove the latter, we use similar tools as in Section 5 of [24]. First, define the following quantities for any $r > 0$, $\gamma > 0$ and $\eta \geq 0$.

$$X_{r}(t) = 1 \{ t \in (-x, x) \}, \text{ for all } x \in \mathbb{R}$$

$$\delta_{r}(x) = \frac{1}{r} \text{Im} \left( \frac{1}{x - it} \right) = \frac{1}{r \pi} \frac{\eta}{x^2 + \eta^2}, \text{ for all } x \in \mathbb{R}$$

(6.36)

$$\text{Tr} \times \times \delta_{r}(H') = \frac{1}{r} \int_{-\frac{1}{r \pi}}^{\frac{1}{r \pi}} \sum_{i=1}^{2N} \text{Im} G^0_{ii}(x + it) dx = \frac{1}{r} \int_{-\frac{1}{r \pi}}^{\frac{1}{r \pi}} \sum_{i=1}^{2N} \frac{\eta}{(\eta_i(H') - x)^2 + \eta^2} dx.$$

Moreover for any $N \times N$ matrix $Y$ with eigenvalues denoted by $\lambda_i(Y)$ and for any $E_1, E_2, E \in \mathbb{R}$ such that $E_1 \leq E_2$ and $E > 0$, we denote

$$i_0(Y, E_1, E_2) = \# \{ i \in [N] : \lambda_i(Y) \in (E_1, E_2) \},$$

$$i_0(Y, E) = \# \{ i \in [N] : \lambda_i(Y) \in (-E, E) \}.$$  

(6.37)

Moreover set $[\lambda_i']_{i \in [2N]}$ the eigenvalues of $H'$ arranged in increasing order.

**Lemma 6.10.** For any $\gamma \in [0, 1]$ and $I \subseteq \left( -\frac{1}{r \pi}, \frac{1}{r \pi} \right)$ such that $|I| = N^{\gamma/2 - 1}$, it is true that

$$\left| \left\{ i \in [2N] : \lambda_i' \in I \right\} \right| \leq 2|I|^{1+\gamma/2}$$

with overwhelming probability.
\textbf{Proof.} For the convenience of notation, suppose that
\[ I = (E - \eta, E + \eta) \]

Moreover set the event
\[ \Omega_\eta = \left\{ \sup_{y \in [0, 1]} \sup_{E \in \Omega} \max_{ij} |G_{ij}(E + i\eta)| \geq N^{c/2} \right\}. \tag{6.38} \]

By (6.35), \( \Omega_\eta^c \) holds with overwhelming probability. Then
\[ 1 \left( \frac{\Omega_\eta^c}{2N} \right) \geq 1 \left( \frac{\Omega_\eta^c}{2N} \right) \sum_{i=1}^{2N} \text{Im}(G_{ii}(E + i\eta)) \geq 1 \left( \frac{\Omega_\eta^c}{2N} \right) \sum_{i \in I} \frac{\eta}{(j_i - E)^2 + \eta^2} \tag{6.39} \]
\[ \geq 1 \left( \frac{\Omega_\eta^c}{2N|I|} \right) \sum_{i \in I} \left\{ \frac{\eta}{(j_i - E)^2 + \eta^2} \right\} \tag{6.40} \]

Next fix \( \epsilon > 0 \) arbitrary small and \( r \in \mathbb{R} \). Set
\[ \eta_1 = N^{-1-3\epsilon}, \quad l = N^{-1-3\epsilon}, \quad l_i = lN^{2\epsilon}. \]

\textbf{Lemma 6.11.} For any \( \gamma \in [0, 1] \), it is true that there exists an absolute constant \( C \) such that with overwhelming probability
\[ \left| k_N \left( H^\gamma, \frac{r}{N} \right) - \text{Tr} \ X_r * \partial_{\eta_i}(H^\gamma) \right| \leq C \left( N^{-2\epsilon} + 2k_N \left( H^\gamma, -\frac{r}{N} + l_\gamma, \eta_\gamma \right) + k_N \left( H^\gamma, -\frac{r}{N} - l_\gamma, -\frac{r}{N} \right) \right). \]

\textbf{Proof.} Firstly note that by elementary computation as in (6.10) of [35] one has that,
\[ |X_{\frac{r}{N}}(x) - X_r * \partial_{\eta_i}(x)| \leq C\eta_1 \left( \frac{2r}{Nd_1(x)d_2(x)} + \frac{X_{\frac{r}{N}}(x)}{d_1(x) + d_2(x)} \right), \tag{6.41} \]
where \( C \) is some absolute constant, \( d_1(x) = \left| \frac{r}{N} + x \right| + \eta_1 \) and \( d_2 = \left| \frac{r}{N} - x \right| + \eta_1 \). Moreover note that the right hand side of (6.41) is always bounded by an absolute constant and is \( O(\eta_1/l) \) if \( \min d_i \geq l \). Thus by Lemma 6.10 one has that with overwhelming probability
\[ \left| k_N \left( H^\gamma, \frac{r}{N} \right) - \text{Tr} \ X_r * \partial_{\eta_i}(H^\gamma) \right| \leq C \left( \text{Tr} (f_1(H^\gamma) + f_2(H^\gamma)) + \eta_1 \right) \left( H^\gamma, -\frac{r}{N} + l \right) + k_N \left( H^\gamma, -\frac{r}{N} - l \right) \tag{6.42} \]
\[ + C \left( H^\gamma, -\frac{r}{N} - l, 0 \right) + k_N \left( H^\gamma, -\frac{r}{N} - l \right) \tag{6.43} \]
where
\[ f_1(x) = 1 \left\{ x \leq -E - l \right\}, \quad f_2(x) = 1 \left\{ x \geq E + l \right\}. \]

So in order to complete the proof we need to show that the first term on the right side of the inequality is of order \( N^{-2\epsilon} \). Note that due to Lemma 6.10 and the fact that the length of the interval \( \left( -\frac{r}{N} - l, \frac{r}{N} + l \right) \) is smaller than \( N^{\epsilon-1} \) for any \( \zeta > 0 \) one has that
\[ \frac{\eta_1}{l} k_N \left( H^\gamma, -\frac{r}{N} - l, -\frac{r}{N} \right) \leq N^{-2\epsilon} \text{ with overwhelming probability.} \]

Moreover after splitting the interval \( \left( -\frac{1}{3C}, -\frac{1}{3C} - l \right) \) into intervals with length \( O(N^{-1}) \), like in [12] (5.61) and since
\[ f_1(x) \leq \frac{\eta_1}{E - x} \left\{ x \in \left( -\frac{1}{3C}, -\frac{r}{N} - l \right) \right\} + \frac{\eta_1}{E - x} \left\{ x \in \left( -\infty, -\frac{1}{3C} \right) \right\}, \]
one can show that \( \text{Tr} f_1(H^\gamma) \leq N^{-2\epsilon} \). Similar bound can be proven for \( \text{Tr} f_2(H^\gamma) \). \( \square \)

\textbf{Lemma 6.12.} For any \( \gamma \in [0, 1] \) there exists an absolute constant \( C \) such that
\[ \text{Tr} X_r * \partial_{\eta_i}(H^\gamma) = CN^{-\epsilon} \leq k_N \left( H^\gamma, \frac{r}{N} \right) \leq \text{Tr} X_r * \partial_{\eta_i}(H^\gamma) + CN^{-\epsilon}. \tag{6.44} \]
\textbf{Proof.} We will prove the second inequality of (6.44). The proof of the first inequality is similar.

First note that by definition, one has that for any \(y_1 \geq y_2 > 0\),
\[
l_{2N}(H^r, y_2) \leq l_{2N}(H^r, y_1)
\]
(6.45)

\[
\text{Tr } X_{y_2} \ast \partial_{\eta_l}(H^r) \leq \text{Tr } X_{y_1} \ast \partial_{\eta_l}(H^r)
\]
(6.46)

So we get that with overwhelming probability
\[
l_{2N}\left(H^r, \frac{r}{N}\right) \leq \frac{1}{l_1} \int_{\frac{r}{N}}^{\frac{r}{N} + 1} l_{2N}\left(H^r, \frac{r}{N} + y\right)dy
\]
(6.47)

\[
\leq \frac{1}{l_1} \left( \int_{\frac{r}{N}}^{\frac{r}{N} + 1} \text{Tr } X_{\eta_l} \ast \partial_{\eta_l}(H^r) + C\left(N^{-2\epsilon} + l_{2N}(H^r, y - l, -y + l) + l_{2N}(H^r, -y - l, -y + l)\right)dy \right)
\]
(6.48)

\[
\leq \text{Tr } X_r \ast \partial_{\eta_l+1}(H^r) + CN^{-\epsilon}
\]
(6.49)

In the first inequality of (6.47) we used (6.45), in the second we used Lemma 6.11 and in the third we used Lemma 6.13.

\[
\text{In (6.52), we used the Markov inequality for the random variable }
\]
\[
\tilde{q}(x) = 0 \text{ for } x \in \left(-\infty, \frac{-2}{9}\right) \cup \left(\frac{2}{9}, \infty\right)
\]
\[
\tilde{q}(x) = 1 \text{ for } x \in \left(\frac{-2}{9}, \frac{2}{9}\right)
\]
\[
\tilde{q}(x) \text{ is decreasing on } \left(\frac{2}{9}, \frac{2}{9}\right).
\]

In the following Lemma we prove the approximation of the gap probability of \(H^r \) by function of the form appearing in 6.35.

\textbf{Lemma 6.13.} For any \(y \in [0, 1] \) and \(D > 0 \) it is true that
\[
\mathbb{E}\tilde{q}\left(\text{Tr } X_r \ast \partial_{\eta_l+1}(H^r)\right) - N^{-D} \leq \mathbb{P}\left(l_{2N}\left(H^r, \frac{r}{N}\right) = 0\right) \leq \mathbb{E}\tilde{q}\left(\text{Tr } X_r \ast \partial_{\eta_l-1}(H^r)\right) + N^{-D}.
\]
(6.50)

\textbf{Proof.} By Lemma 6.12 and for large enough \(N\), it is true that if \(l_{2N}(H^r, \frac{r}{N}) = 0\) then \(\text{Tr } X_r \ast \partial_{\eta_l-1}(H^r) \leq \frac{1}{9}\) with overwhelming probability. This implies that for any large \(D > 0\) and for \(N\) sufficiently large one has that,
\[
\mathbb{P}\left(l_{2N}\left(H^r, \frac{r}{N}\right) = 0\right) \leq \mathbb{P}\left(\text{Tr } X_r \ast \partial_{\eta_l-1}(H^r) \leq \frac{1}{9}\right) + N^{-D} \leq \mathbb{P}\left(\text{Tr } X_r \ast \partial_{\eta_l-1}(H^r) \leq \frac{2}{9}\right) + N^{-D}
\]
(6.51)

\[
= \mathbb{P}\left[\tilde{q}\left(\text{Tr } X_r \ast \partial_{\eta_l-1}(H^r)\right) \geq 1\right] + N^{-D} \leq \mathbb{E}\tilde{q}\left(\text{Tr } X_r \ast \partial_{\eta_l-1}(H^r)\right) + N^{-D}
\]
(6.52)

In 6.52, we used the Markov inequality for the random variable \(\tilde{q}\left(\text{Tr } X_r \ast \partial_{\eta_l-1}(H^r)\right)\). So we have proven the second inequality of 6.50.

For the first, note that again by Lemma 6.12 with overwhelming probability it is true that if \(\tilde{q}\left(\text{Tr } X_r \ast \partial_{\eta_l+1}(H^r)\right) \leq \frac{3}{5}\) then \(l_{2N}(H^r, \frac{r}{N}) \leq CN^{-\epsilon} + \frac{2}{5}\). Thus.
\[
\mathbb{E}\tilde{q}\left(\text{Tr } X_r \ast \partial_{\eta_l+1}(H^r)\right) \leq \mathbb{P}\left[\tilde{q}\left(\text{Tr } X_r \ast \partial_{\eta_l+1}(H^r)\right) \leq \frac{2}{9}\right]
\]
(6.53)

\[
\leq \mathbb{P}\left[l_{2N}\left(H^r, \frac{r}{N}\right) \leq CN^{-\epsilon} + \frac{2}{9}\right] + N^{-D} = \mathbb{P}\left[l_{2N}\left(H^r, \frac{r}{N}\right) = 0\right] + N^{-D}.
\]
(6.54)

\section{6.3 Proof of Theorem 1.2}

At this subsection we prove Theorem 1.2. Fix \(r \in (0, \infty) \) and \(\epsilon > 0\) small enough. Set \(\eta_l = N^{-1-\epsilon}\) and \(l = N^{-1-9\epsilon}\). Furthermore \(r \in (0, \infty)\). Let \(\tilde{q}(x)\) denote the function defined before Lemma 6.13.
• For the first part of Theorem 1.2 note that due to [6.35] and Lemma 6.13 one has that there exist constants $C = C(r) > 0$ and $c > 0$, such that for large enough $D > 0$ it is true that

$$
\mathbb{E} \bar{q}\left(\text{Tr} X_r * \partial_{\eta^{-1}}(H^0)\right) - N^{-D} - CN^{-c} \leq \mathbb{E} \bar{q}\left(\text{Tr} X_r * \partial_{\eta^{-1}}(H^1)\right) - N^{-D}.
$$

(6.55)

$$
\leq \mathbb{P}\left(\zeta_N \left(\frac{H^1, r}{N}\right) = 0\right) \leq \mathbb{E} \bar{q}\left(\text{Tr} X_r * \partial_{\eta^{-1}}(H^1)\right) + N^{-D} \leq \mathbb{E} \bar{q}\left(\text{Tr} X_r * \partial_{\eta^{-1}}(H^0)\right) + CN^{-c} + N^{-D}.
$$

(6.56)

Next note that by the definition of the symmetrization of a matrix, the gap probability is actually the tail distribution of the smallest singular value, i.e.,

$$
\mathbb{P}\left(\zeta_N \left(\frac{H^1, r}{N}\right) = 0\right) = \mathbb{P}\left(\mathbf{s}_1(D_N) \geq \frac{r}{N}\right).
$$

Moreover note that the limiting distribution of the least singular value of a Gaussian matrix is $1 - \exp(-r^2/2 - r)$ as mentioned in Theorem 1.3. of [11]. Let $L_N$ be a matrix with i.i.d. entries all following the Gaussian law with mean 0 and variance $N^{-1}$. Set $s_1(L_N)$ the least singular value of $L_N$. Let $W_N$ be the symmetrization of $L_N$. As before one can notice that

$$
\mathbb{P}\left(\zeta_N \left(\frac{E_N, r}{N}\right) = 0\right) = \mathbb{P}\left(s_1(D_N) \geq \frac{r}{N}\right).
$$

So after another application of Lemma 6.13 for the matrix $H^0$ and Corollary 4.5 for $r' = r\xi^{-1}$, where $\xi$ is defined in [1.4], one has that there exists a small constant $\hat{c} > 0$ and a large constant such that

$$
\mathbb{P}\left(Ns_1(L_N) \geq r - N^{-c}\right) - CN^{-\hat{c}} \leq \mathbb{P}\left(\xi s_1(D_N) \geq r\right) \leq \mathbb{P}\left(Ns_1(L_N) \geq r + N^{-c}\right) + CN^{-\hat{c}},
$$

which implies universality of the least singular value for $D_N$ multiplied by $N\xi$.

• For the proof of the second part, it is well-known that bounding the entries of the resolvent implies the complete eigenvector delocalization. So by [6.26], one can prove the complete eigenvector delocalization as in Theorem 6.3 in [33].

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