Article

Exact Solutions of the Nonlinear Modified Benjamin-Bona-Mahony Equation by an Analytical Method

Trad Alotaibi and Ali Althobaiti *

Department of Mathematics, Taif University, Taif 21944, Saudi Arabia; t.alotaibi@tu.edu.sa
* Correspondence: aa.althobaiti@tu.edu.sa

Abstract: The current manuscript investigates the exact solutions of the modified Benjamin-Bona-Mahony (BBM) equation. Due to its efficiency and simplicity, the modified auxiliary equation method is adopted to solve the problem under consideration. As a result, a variety of the exact wave solutions of the modified BBM equation are obtained. Furthermore, the findings of the current study remain strong since Jacobi function solutions generate hyperbolic function solutions and trigonometric function solutions, as liming cases of interest. Some of the obtained solutions are illustrated graphically using appropriate values for the parameters.

Keywords: the BBM equation; exact solution; the extended auxiliary equation method

1. Introduction

Nonlinear partial differential equations are widely used in science and engineering to model various nonlinear scenarios in real-life applications. For instance, these equations can be used in modeling fluid dynamical problems, wave propagation in corrugated media, the examination of earthquakes and seismic waves, and modeling optical fibers, etc. Despite being an old field of great attention, the field of fluid dynamics is still relevant. Various nonlinear dynamical models have been proposed for modeling the dynamics of surface water waves, such as the Benjamin-Bona-Mahony equation [1], Boussinesq equation [2,3], Korteweg–de-Vries equation and its modifications [4,5], Benney–Luke equation [6] and so on. More specifically, the model of interest in this manuscript is the modified Benjamin-Bona-Mahony (BBM) equation [7–10] and the references cited therein. In essence, the BBM equation is a nonlinear evolution equation that is used to model the dynamics of unidirectional water waves with small amplitude, in addition to being disturbed by dispersion and nonlinear effects [11]. Moreover, one could find various generalizations and modifications of this model, including its reduction to the Korteweg–de-Vries equation under a special assumption. However, the Korteweg–de-Vries equation is integrable via the inverse scattering approach, as against the current model of interest.

As highlighted above with regard to the unidirectional water waves with small amplitude, such propagation of waves is called a solitary wave [12–14]. Moreover, there exist several analytical and computational methods for finding the exact and approximate solitary wave solutions associated with diverse nonlinear evolution equations. In light of this, we mention some of the analytical methods in the literature, such as the Jacobi elliptic functions method [15], the exponential expansion method [16], the tanh-based expansion methods [17], the Kudryashov method [18], the G'/G-expansion method [19], the sub-equation method [20] and the modified exponential rational method [21]. For more on analytical methods, see [22–28]. Furthermore, we mention some of the well-known computational methods for approximate soliton solutions here, including the famous Adomian decomposition approach [29], the Laplace-homotopy perturbation method [30] and the homotopy analysis approach [31], among others.

The present paper is devoted to examining the modified BBM equation analytically. It is relevant to state here that four types of analytical solutions, including exponential [7],
periodic and hyperbolic [8–10] and rational [9,10] solutions have been obtained for the model in the literature. This study objectively intends to go beyond what is known in the open literature [7–10] by constructing more general exact solutions for the model via Jacobi functions. Thus, the modified auxiliary equation method [28] is adopted for the treatment in order to obtain these kinds of solutions. Moreover, the exact solutions are comprehensively analyzed further by determining their constraint conditions (if any). In addition, we aim to examine the obtained solutions graphically, as well as classifying them based on the features of the resultant profiles.

The current paper is organized as follows. Section 2 presents the outline of the methodology of concern. Section 3 applies the method presented in Section 2 on the governing modified BBM model. Section 4 is concerned with the discussion of the obtained results, while Section 5 gives some closing remarks as a concluding note.

2. The Modified Auxiliary Equation Method

In this section, we give the outline of the methodology of concern, the modified auxiliary equation method [28]. Thus, considering the partial differential equation below, we present the main steps of the method as follows:

\[ \chi(v, v_x, v_t, v_{xt}, v_{xx}, \ldots) = 0. \]  \hspace{1cm} (1)

**Step 1.** First, we use the transformation

\[ v(x, t) = V(\xi), \quad \xi = x - ct, \]  \hspace{1cm} (2)

where \( c \) is a constant to be determined. Then, upon employing the transformation given in Equation (2) in Equation (1), the following nonlinear ordinary differential equation is obtained:

\[ Q(V, V'', \ldots) = 0. \]  \hspace{1cm} (3)

**Step 2.** We assume that the solution of Equation (3) is given by

\[ V(\xi) = \sum_{i=-n}^{n} \eta_i \psi^i(\xi), \]  \hspace{1cm} (4)

where \( n \) is a positive integer, and \( \eta_i \) are arbitrary constants to be determined. Moreover, \( \psi(\xi) \) satisfies

\[ \psi'^2(\xi) = v_0 + v_1 \psi^2(\xi) + v_2 \psi^4(\xi), \]  \hspace{1cm} (5)

where \( v_0, v_1, v_2 \) are arbitrary constants. Moreover, Equation (5) admits the following solutions:

**Case 1.** If \( v_0 = 1, v_1 = -(1 + k^2), v_2 = k^2 \), then Equation (5) has a solution \( \psi(\xi) = \text{sn}(\xi, k) \), where \( \text{sn}(\xi, k) \) defines the Jacobi function and \( k \) denotes the elliptic modulus such that \( 0 < k < 1 \).

**Case 2.** If \( v_0 = 1 - k^2, v_1 = 2k^2 - 1, v_2 = -k^2 \), then Equation (5) has a solution \( \psi(\xi) = \text{cn}(\xi, k) \), and \( \text{cn}(\xi, k) \) defines the Jacobi function and \( k \) is as defined earlier.

**Case 3.** If \( v_0 = k^2 - 1, v_1 = 2 - k^2, v_2 = -1 \), then Equation (5) has a solution \( \psi(\xi) = \text{dn}(\xi, k) \), where \( \text{dn}(\xi, k) \) defines the Jacobi function.

**Case 4.** If \( v_0 = k^2, v_1 = -(1 + k^2), v_2 = 1 \), then Equation (5) has a solution \( \psi(\xi) = \text{ns}(\xi, k) \), where \( \text{ns}(\xi, k) \) defines the Jacobi function.

**Case 5.** If \( v_0 = 1 - k^2, v_1 = 2 - k^2, v_2 = 1 \), then Equation (5) has a solution \( \psi(\xi) = \text{cs}(\xi, k) \), where \( \text{cs}(\xi, k) \) defines the Jacobi function \( cs \).

**Case 6.** If \( v_0 = 1, v_1 = 2k^2 - 1, v_2 = k^2(k^2 - 1) \), then Equation (5) has a solution \( \psi(\xi) = \text{sd}(\xi, k) \), where \( \text{sd}(\xi, k) \) defines the Jacobi function \( sd \).

**Step 3.** The value of \( n \) in Equation (4) can be determined via the application of the homogeneous balancing principle [28].
Step 4. After substitution of Equation (4) together with Equation (5) into Equation (3) and setting all terms with the same power of $\psi(\xi)$ equal to zero, we find a set of overdetermined equations for $\eta_i$. Consequently, a solution of Equation (1) is obtained.

3. Analytical Solutions of the Modified BBM Equation

In this section, the modified auxiliary equation method will be employed to analytically treat the governing modified BBM equation. Therefore, we consider the modified BBM equation, which is given by [7–10]

$$u_t + u_x + u_{xxx} - \alpha u^2 u_x = 0, \quad (6)$$

where $\alpha$ is a nonzero constant. Hence, we shall use the transformation

$$u(x,t) = U(\xi), \quad \xi = x - ct, \quad (7)$$

where $c$ is a constant. Next, the governing model given in Equation (6) is written via Equation (7) in the following way:

$$(1 - c)U' - \alpha U^2 U' + U''' = 0. \quad (8)$$

Using the homogeneous balancing principle on Equation (8), we find $n = 1$. Thus, the solution of Equation (8) is written as

$$U(\xi) = \eta_0 + \eta_1 \psi(\xi) + \frac{\eta_{-1}}{\psi(\xi)}, \quad (9)$$

Then, after substituting Equation (9) with the use of Equation (5) into Equation (8) and thereafter setting the coefficients of $\psi(\xi)$ equal to zero, we obtain a system of algebraic equations. Solving this system for $\eta_0, \eta_1, \eta_{-1}$ and $c$, the following sets of solutions are obtained:

**Set 1.**

$$\eta_0 = 0, \eta_1 = 0, \eta_{-1} = \pm \frac{\sqrt{6}\nu_0}{\sqrt{\alpha}}, c = \nu_1 + 1. \quad (10)$$

On substituting these values into Equation (9), various exact solutions can be constructed as the following cases:

**Case 1.** If $\nu_0 = 1, \nu_1 = -(1 + k^2), \nu_2 = k^2$, then the modified BBM equation in Equation (6) has a solution in the form

$$u(\xi) = \pm \frac{\sqrt{6}}{\sqrt{\alpha} \sn(\xi, k)}, \quad (11)$$

The solution in Equation (11) further leads to

$$u(x,t) = \pm \frac{\sqrt{6} \coth(t + x)}{\sqrt{\alpha}}, \quad (12)$$

when $k \to 1$. Figure 1 represents the positive solutions given in Equations (11) and (12) when $\alpha = 1$. From this figure, it is can be observed that Figure 1a shows a singular partial periodicity, while Figure 1b shows a singular behavior.
Figure 1. The graphs (a, b) are the 3D plots of the positive solutions (11), (12), respectively. In (a), \( k = 0.8 \).

**Case 2.** If \( \nu_0 = 1 - k^2, \nu_1 = 2k^2 - 1, \nu_2 = -k^2 \), then Equation (6) admits the following solution:

\[
  u(\xi) = \pm \sqrt{6} \frac{\sqrt{1 - k^2}}{\sqrt{\alpha}} \text{cn}(\xi, k). \tag{13}
\]

**Case 3.** If \( \nu_0 = k^2 - 1, \nu_1 = 2 - k^2, \nu_2 = -1 \), then Equation (6) has a solution in the following form:

\[
  u(\xi) = \pm \sqrt{6} \frac{\sqrt{-1 + k^2}}{\sqrt{\alpha}} \text{dn}(\xi, k). \tag{14}
\]

**Case 4.** If \( \nu_0 = k^2, \nu_1 = -(1 + k^2), \nu_2 = 1 \), then Equation (6) satisfies the following solution:

\[
  u(\xi) = \pm \frac{\sqrt{6} k}{\sqrt{\alpha}} \text{ns}(\xi, k). \tag{15}
\]

Moreover, the above solution transforms to the following

\[
  u(x, t) = \pm \frac{\sqrt{6} \tanh(t + x)}{\sqrt{\alpha}}, \tag{16}
\]
when $k \to 1$. In addition, Figure 2 represents the positive solutions given in Equations (15) and (16) when $\alpha = 1$. From this figure, it can be observed that Figure 2a shows a periodicity, while Figure 2b shows a kink-type behavior. Moreover, the solution reported in Equation (16) is sometimes called a dark soliton solution.

![Figure 2](image)

**Figure 2.** The graphs (a, b) are the 3D plots of the positive solutions (15), (16), respectively. In (a), $k = 0.4$.

**Case 5.** If $\nu_0 = 1 - k^2, \nu_1 = 2 - k^2, \nu_2 = 1$, then Equation (6) has a solution

$$u(\xi) = \pm \frac{\sqrt{6} \sqrt{1 - k^2}}{\sqrt{\alpha} \cos(\xi, k)},$$

which reduces to the following

$$u(x, t) = \pm \frac{\sqrt{6} \tan(3t - x)}{\sqrt{\alpha}},$$

when $k \to 0$.

**Case 6.** If $\nu_0 = 1, \nu_1 = 2k^2 - 1, \nu_2 = k^2(k^2 - 1)$, then Equation (6) satisfies the following solution:

$$u(\xi) = \pm \frac{\sqrt{6}}{\sqrt{\alpha} \operatorname{sd}(\xi, k)}.$$
Moreover, the solution determined above transforms to
\[ u(x,t) = \pm \sqrt{6} \frac{\csc(2t - x)}{\sqrt{\alpha}}, \]  
(20)
as \( k \to 1 \).

**Set 2.**
\[ \eta_0 = 0, \eta_{-1} = 0, \eta_1 = \pm \sqrt{6} \frac{\sqrt{\nu_0}}{\sqrt{\alpha}}, c = \nu_1 + 1. \]  
(21)
Inserting these values into Equation (9), we find the following cases:

**Case 1.** If \( \nu_0 = 1, \nu_1 = -1 - k^2, \nu_2 = k^2 \), then Equation (6) admits the following solution:
\[ u(\xi) = \pm \sqrt{6} \frac{k \sn(\xi, k)}{\sqrt{\alpha}}. \]  
(22)

**Case 2.** If \( \nu_0 = k^2, \nu_1 = -1 + k^2, \nu_2 = 1 \), then Equation (6) satisfies the solution:
\[ u(\xi) = \pm \sqrt{6} \frac{\csc(\xi, k)}{\sqrt{\alpha}}. \]  
(23)

**Case 3.** If \( \nu_0 = 1 - k^2, \nu_1 = 2 - k^2, \nu_2 = 1 \), then Equation (6) has a solution
\[ u(\xi) = \pm \sqrt{6} \frac{\csc(\xi, k)}{\sqrt{\alpha}}. \]  
(24)

Moreover, when \( k \to 0 \), the above solution becomes
\[ u(x,t) = \pm \sqrt{6} \frac{\cot(3t - x)}{\sqrt{\alpha}}. \]  
(25)

Therefore, we give in Figure 3 the graphical representations of the obtained solutions (positive) in Equations (24) and (25) when \( \alpha = 1 \). The figure further shows singular periodicity in both (a) and (b). In addition, the solution reported in Equation (25) is sometimes called a singular soliton solution.

![Figure 3](image-url)
Figure 3. The graphs (a,b) are the 3D plots of the positive solutions (24), (25), respectively. In (a), $k = 0.8$.

Set 3.

\[
\eta_0 = 0, \eta_1 = -\sqrt{6}\sqrt{\nu_2}, \eta_{-1} = -\sqrt{6}\sqrt{\nu_0}, c = \nu_1 - 6\sqrt{\nu_0}\sqrt{\nu_2} + 1. \tag{26}
\]

Then, substituting these results into Equation (9), we obtain the following solution cases.

**Case 1.** If $\nu_0 = 1, \nu_1 = -(1 + k^2), \nu_2 = k^2$, then Equation (6) admits the solution

\[
u (\xi) = -\frac{\sqrt{6} \, k \, \text{sn}(\xi, k)}{\sqrt{\alpha}} - \frac{\sqrt{6}}{\sqrt{\alpha} \text{sn}(\xi, k)}. \tag{27}
\]

Figure 4 represents the solution determined in Equation (27) when $\alpha = 1$ and $k = 0.4$. The 3D plot in Figure 4 shows a singular solution. Further, the same solution expressed in Equation (27) reduces to the following expression

\[
u (x, t) = -\frac{\sqrt{6} \, \text{coth}(x + 7t)}{\sqrt{\alpha}} - \frac{\sqrt{6} \, \tanh(x + 7t)}{\sqrt{\alpha}}, \tag{28}
\]

when $k \to 1$.

Figure 4. The 3D plot of the solutions (27) when $k = 0.4$. 
Case 2. If \( \nu_0 = k^2, \nu_1 = -(1 + k^2), \nu_2 = 1 \), then Equation (6) has a solution
\[
    u(\xi) = -\sqrt{6} \, \text{ns}(\xi, k) \frac{k}{\sqrt{\alpha}} = -\sqrt{6} \, \text{ns}(\xi, k).
\] (29)

Case 3. If \( \nu_0 = 1 - k^2, \nu_1 = 2 - k^2, \nu_2 = 1 \), then the modified BBM equation given in Equation (6) has a solution in the form
\[
    u(\xi) = -\sqrt{6} \, \text{cs}(\xi, k) \frac{k}{\sqrt{\alpha}} = -\sqrt{6(1 - k^2)} \, \text{cs}(\xi, k).
\] (30)

Set 4.
\[
    \eta_0 = 0, \eta_1 = -\sqrt{6} \sqrt{v_2} \sqrt{\alpha}, \eta_{-1} = \sqrt{6} \sqrt{v_0} \sqrt{\alpha}, c = v_1 + 6 \sqrt{v_0} \sqrt{v_2} + 1.
\] (31)

By substituting these results into Equation (9), we find the following solution cases.

Case 1. If \( \nu_0 = 1, \nu_1 = -(1 + k^2), \nu_2 = k^2 \), then Equation (6) has a solution
\[
    u(\xi) = -\sqrt{6} \, \text{sn}(\xi, k) \frac{k}{\sqrt{\alpha}} + \sqrt{6} \, \text{sn}(\xi, k).
\] (32)

Moreover, when \( k \to 1 \), the above solution then becomes
\[
    u(x, t) = \sqrt{6} \tan(5t - x) \frac{1}{\sqrt{\alpha}} - \sqrt{6} \coth(5t - x) \frac{1}{\sqrt{\alpha}}.
\] (33)

Case 2. If \( \nu_0 = k^2, \nu_1 = -(1 + k^2), \nu_2 = 1 \), then Equation (6) has a solution
\[
    u(\xi) = -\sqrt{6} \, \text{ns}(\xi, k) \frac{k}{\sqrt{\alpha}} + \sqrt{6} \, \text{ns}(\xi, k).
\] (34)

Moreover, when \( k \to 1 \), the latter solution reduces to
\[
    u(x, t) = -\sqrt{6} \tan(5t - x) \frac{1}{\sqrt{\alpha}} + \sqrt{6} \coth(5t - x) \frac{1}{\sqrt{\alpha}}.
\] (35)

Case 3. If \( \nu_0 = 1 - k^2, \nu_1 = 2 - k^2, \nu_2 = 1 \), then the modified BBM equation earlier expressed in Equation (6) has a solution in the following form:
\[
    u(\xi) = -\sqrt{6} \, \text{cs}(\xi, k) \frac{k}{\sqrt{\alpha}} + \sqrt{6(1 - k^2)} \, \text{cs}(\xi, k).
\] (36)

Moreover, if \( k \to 0 \), the latter solution becomes
\[
    u(x, t) = \sqrt{6} \cot(9t - x) \frac{1}{\sqrt{\alpha}} - \sqrt{6} \tan(9t - x) \frac{1}{\sqrt{\alpha}}.
\] (37)

Set 5.
\[
    \eta_0 = 0, \eta_1 = \sqrt{6} \sqrt{v_2} \sqrt{\alpha}, \eta_{-1} = -\sqrt{6} \sqrt{v_0} \sqrt{\alpha}, c = v_1 + 6 \sqrt{v_0} \sqrt{v_2} + 1.
\] (38)

In the same manner, the following solution cases are obtained.
**Case 1.** If \( \nu_0 = 1, \nu_1 = -(1 + k^2), \nu_2 = k^2 \), then the modified BBM equation given Equation (6) admits the following solution:

\[
\begin{align*}
\frac{u(\xi)}{\sqrt{\alpha}} &= \frac{\sqrt{6} k \text{sn}(\xi, k)}{\sqrt{\alpha}} - \frac{\sqrt{6}}{\sqrt{\alpha} \text{sn}(\xi, k)}. \\
\text{(39)}
\end{align*}
\]

**Case 2.** If \( \nu_0 = k^2, \nu_1 = -(1 + k^2), \nu_2 = 1 \), then Equation (6) has a solution

\[
\begin{align*}
\frac{u(\xi)}{\sqrt{\alpha}} &= \frac{\sqrt{6} \text{ns}(\xi, k)}{\sqrt{\alpha}} - \frac{\sqrt{6} k}{\sqrt{\alpha} \text{ns}(\xi, k)}. \\
\text{(40)}
\end{align*}
\]

**Case 3.** If \( \nu_0 = 1 - k^2, \nu_1 = 2 - k^2, \nu_2 = 1 \), then the modified BBM equation given in Equation (6) has a solution in the form

\[
\begin{align*}
\frac{u(\xi)}{\sqrt{\alpha}} &= \frac{\sqrt{6} \text{cs}(\xi, k)}{\sqrt{\alpha}} - \frac{\sqrt{6} (1 - k^2)}{\sqrt{\alpha} \text{cs}(\xi, k)}. \\
\text{(41)}
\end{align*}
\]

Again, when \( k \to 0 \), the above solution becomes

\[
\begin{align*}
\frac{u(x, t)}{\sqrt{\alpha}} &= \frac{\sqrt{6} \tan(9t - x)}{\sqrt{\alpha}} - \frac{\sqrt{6} \cot(9t - x)}{\sqrt{\alpha}}. \\
\text{(42)}
\end{align*}
\]

**Set 6.**

\[
\eta_0 = 0, \eta_1 = \frac{\sqrt{6} \sqrt{v_2}}{\sqrt{\alpha}}, \eta_{-1} = \frac{\sqrt{6} \sqrt{v_1}}{\sqrt{\alpha}}, c = v_1 - 6 \sqrt{v_1} \sqrt{v_2} + 1. \\
\text{(43)}
\]

Furthermore, as in the preceding scenario, the following solution cases are acquired.

**Case 1.** If \( \nu_0 = 1, \nu_1 = -(1 + k^2), \nu_2 = k^2 \), then Equation (6) has a solution

\[
\begin{align*}
\frac{u(\xi)}{\sqrt{\alpha}} &= \frac{\sqrt{6} k \text{sn}(\xi, k)}{\sqrt{\alpha}} + \frac{\sqrt{6}}{\sqrt{\alpha} \text{sn}(\xi, k)}. \\
\text{(44)}
\end{align*}
\]

Further, the solution expressed in Equation (44) leads to

\[
\begin{align*}
\frac{u(x, t)}{\sqrt{\alpha}} &= \frac{\sqrt{6} \tanh(7t + x)}{\sqrt{\alpha}} + \frac{\sqrt{6} \coth(7t + x)}{\sqrt{\alpha}}, \\
\text{(45)}
\end{align*}
\]

when \( k \to 1 \).

**Case 2.** If \( \nu_0 = k^2, \nu_1 = -(1 + k^2), \nu_2 = 1 \), then Equation (6) has a solution

\[
\begin{align*}
\frac{u(\xi)}{\sqrt{\alpha}} &= \frac{\sqrt{6} \text{ns}(\xi, k)}{\sqrt{\alpha}} + \frac{\sqrt{6} k}{\sqrt{\alpha} \text{ns}(\xi, k)}. \\
\text{(46)}
\end{align*}
\]

**Case 3.** If \( \nu_0 = 1 - k^2, \nu_1 = 2 - k^2, \nu_2 = 1 \), then the modified BBM equation in Equation (6) has a solution in the form

\[
\begin{align*}
\frac{u(\xi)}{\sqrt{\alpha}} &= \frac{\sqrt{6} \text{cs}(\xi, k)}{\sqrt{\alpha}} + \frac{\sqrt{6} (1 - k^2)}{\sqrt{\alpha} \text{cs}(\xi, k)}, \\
\text{(47)}
\end{align*}
\]

such that, upon considering the limiting case of \( k \to 0 \), the latter solution becomes

\[
\begin{align*}
\frac{u(x, t)}{\sqrt{\alpha}} &= \frac{\sqrt{6} \tan(3t + x)}{\sqrt{\alpha}} + \frac{\sqrt{6} \cot(3t + x)}{\sqrt{\alpha}}. \\
\text{(48)}
\end{align*}
\]
4. Discussion of the Results
The current manuscript takes into consideration one of the most famous evolution equations that plays an important role in the dynamics of fluid flows: the modified BBM equation [7–10]. Moreover, the model is realized in various nonlinear physical processes. A reliable analytical integration scheme obtained by the named modified auxiliary equation method [28] has been used for the analytical examination. Importantly, this reliable method leads to six (6) different sets of exact solutions, in addition to the various solution cases associated with each solution set. Furthermore, diverse Jacobi functions have been obtained by the present reliable method, in addition to the provision of various periodic and hyperbolic function solutions, as limiting cases of interest. Finally, it is highly recommended to make use of the deployed method to study other forms of evolution equations, such as higher-order evolution equations involving different forms of nonlinearities.

5. Conclusions
In conclusion, the current study examines the modified BBM equation by constructing a variety of exact solutions. We have used the modified auxiliary equation method due to its documented applications in tackling diverse evolution equations. Comparing our results with the results in the literature, our solutions are found for the first time since they are given in terms of Jacobi functions. Moreover, certain solution cases of interest have been graphed, showing different forms of behavior, including periodicity, singular periodicity and kick-type, among others. Lastly, the present study may give more insights into the dynamicity of the modified BBM equation with regard to its response to spatial and temporal evolution, as shown graphically. As a future perspective, the current study can be extended to other forms of evolution equations featuring different forms of nonlinearities.

Author Contributions: Conceptualization, T.A.; methodology, A.A.; software, A.A.; validation, A.A.; formal analysis, T.A.; investigation, T.A. and A.A.; resources, T.A.; data curation, T.A.; writing—original draft preparation, T.A. and A.A.; writing—review and editing, T.A. and A.A.; visualization, A.A.; supervision, T.A. and A.A.; project administration, T.A. and A.A.; funding acquisition, T.A. and A.A. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by Taif University Researches Supporting Project number [1-442-60], Taif University, Taif, Saudi Arabia.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References
1. Benjamin, T.; Bona, J.; Mahony, J. Model equations for long waves in nonlinear dispersive systems. Philos. Trans. R. Soc. Lond. Ser. A 1972, 272, 47.
2. Rashid, S.; Kaabar, M.K.A.; Althobaiti, A.; Alqurashi, M.S. Constructing analytical estimates of the fuzzy fractional-order Boussinesq model and their application in oceanography. J. Ocean Eng. Sci. 2022. [CrossRef]
3. Ali Akbar, M.; Akinyemi, L.; Yao, S.-W.; Jhangeer, A.; Rezazadeh, H.; Khater, M.M.A.; Ahmad, H.; Inc, M. Soliton Solutions to the Boussinesq Equation Through Sine-Gordon Method and Kudryashov Method. Results Phys. 2021, 25, 104228. [CrossRef]
4. Che, H.; Yu-Lan, W. Numerical Solutions of Variable-Coefficient Fractional-in-Space KdV Equation with the Caputo Fractional Derivative. Fractal Fract. 2022, 6, 207. [CrossRef]
5. Ullah, H.; Fiza, M.; Khan, I.; Alshammari, N.; Hamadneh, N.N.; Islam, S. Modification of the Optimal Auxiliary Function Method for Solving Fractional Order KdV Equations. Fractal Fract. 2022, 6, 288. [CrossRef]
6. Khalid, K.A.; Nuruddeen, R.I. Analytical treatment for the conformable space-time fractional Benney-Luke equation via two reliable methods. Int. J. Phy. Res. 2017, 5, 109–114.
7. Khater, M.M.; Salama, S.A. Semi-analytical and numerical simulations of the modified Benjamin–Bona–Mahony model. J. Ocean. Eng. Sci. 2022, 7, 264–271. [CrossRef]
8. Khan, K.; Akbar, M.A.; Islam, S.M.R. Exact Solution for (1+1)-Dimensional Nonlinear Dispersive Modified Benjamin-Bona-Mahony Equation and Coupled Klein-Gordon Equations. Springer Plus 2014, 3, 724. [CrossRef]
9. Naher, H.; Abdullah, F.A. The modified Benjamin-Bona-Mahony equation via the extended generalized Riccati equation mapping method. *Appl. Math. Sci.* **2012**, *6*, 5495–5512.

10. Baskonus, H.M.; Bulut, H. Analytical studies on the (1+1)-dimensional nonlinear dispersive modified Benjamin-Bona-Mahony equation defined by seismic sea waves. *Waves Random Complex Media* **2015**, *25*, 576–586. [CrossRef]

11. Abbaspandy, S.; Shirzadi, A. The first integral method for modified Benjamin-Bona-Mahony equation. *Commun. Nonlinear Sci. Numer. Simul.* **2010**, *15*, 1759–1764. [CrossRef]

12. Hereman, W. *Shallow Water Waves and Solitary Waves, Encyclopedia of Complexity and Systems*; Springer: Berlin/Heidelberg, Germany, 2009.

13. Seadawy, A.R. Ionic acoustic solitary wave solutions of two-dimensional nonlinear Kadomtsev-Petviashvili Burgers equations in quantum plasma. *Math. Methods Appl. Sci.* **2017**, *40*, 1598–1607. [CrossRef]

14. Khalid, K.A.; Nurudddeen, R.I.; Adel, R.H. New exact solitary wave solutions for the extended (3+1)-dimensional Jimbo-Miwa equations. *Results Phys.* **2018**, *9*, 12–16.

15. Gepreel, K.A.; Nofal, T.A.; Althobaiti, A.A. The modified rational Jacobi elliptic functions method for nonlinear differential difference equations. *J. Appl. Math.* **2012**, *2012*, 427479. [CrossRef]

16. Islam, R.; Alam, M.N.; Hossain, A.K.M.K.S.; Roshid, H.O.; Akbar, M.A. Traveling wave solution of nonlinear evolution equation via exp(−φ(η))-expansion method. *Glob. Sci. Front. Res.* **2013**, *13*, 63–71.

17. Raslan, K.R.; Khalid, K.A.; Shallal, M.A. The modified extended tanh method with the Riccati equation for solving the space-time fractional EW and MEW equations. *Chaos Solitons Fractals* **2017**, *103*, 404–409. [CrossRef]

18. Rezazadeh, H.; Ullah, N.; Akinyemi, L.; Shah, A.; Mirhosseini-Alizamin, S.M.; Chu, Y.M.; Ahmad, H. Optical soliton solutions of the generalized non-autonomous nonlinear Schrödinger equations by the new Kudryashov’s method. *Results Phys.* **2021**, *24*, 104179. [CrossRef]

19. Kaewta, S.; Sirisubtawee, S.; Koonprasert, S.; Sungnul, S. Applications of the $G'/G^2$-Expansion Method for Solving Certain Nonlinear Conformable Evolution Equations. *Fractal Fract.* **2021**, *5*, 88. [CrossRef]

20. Gepreel, K.A.; Althobaiti, A.A. Exact solutions of nonlinear partial fractional differential equations using fractional sub-equations method. *Indian J. Phys.* **2014**, *88*, 293–300. [CrossRef]

21. Althobaiti, A.; Althobaiti, S.; El-Rashidy, K.; Seadawy, A.R. Exact solutions for the nonlinear extended KdV equation in a stratified shear flow using modified exponential rational method. *Results Phys.* **2021**, *29*, 104723. [CrossRef]

22. Rizvi, S.T.; Seadawy, A.R.; Akram, U.; Younis, M.; Althobaiti, A. Solitary wave solutions along with Painlevé analysis for the Ablowitz–Kaup–Newell–Segur water waves equation. *Mod. Phys. Lett. B* **2022**, *36*, 2150548. [CrossRef]

23. Rehman, H.U.; Seadawy, A.R.; Younis, M.; Rizvi, S.T.R.; Anwar, I.; Baber, M.Z.; Althobaiti, A. Weakly nonlinear electron-acoustic waves in the fluid ions propagated via a (3+1)-dimensional generalized Korteweg–de-Vries–Zakharov–Kuznetsov equation in plasma physics. *Results Phys.* **2022**, *33*, 105069. [CrossRef]

24. Nurudddeen, R.I.; Aboodh, K.S.; Khalid, K.A. Analytical investigation of soliton solutions to three quantum Zakharov-Kuznetsov equations. *Commun. Theor. Phys.* **2018**, *70*, 405–412. [CrossRef]

25. Seadawy, A.R.; Rizvi, S.T.; Ali, I.; Younis, M.; Ali, K.; Mahklouf, M.M.; Althobaiti, A. Conservation laws, optical molecules, modulation instability and Painlevé analysis for the Chen–Lee–Liu model. *Opt. Quantum Electron.* **2021**, *54*, 172. [CrossRef]

26. Akhineymi, L.; Rezazadeh, H.; Yao, S.W.; Akbar, M.A.; Khater, M.M.; Jhangeer, A.; Inc, M.; Ahmad, H. Nonlinear dispersion in parabolic law medium and its optical solitons. *Results Phys.* **2021**, *26*, 104411. [CrossRef]

27. Seadawy, A.R.; Lu, D. Bright and dark solitary wave soliton solutions for the generalized higher order nonlinear Schrödinger equation and its stability. *Results Phys.* **2017**, *7*, 43–48. [CrossRef]

28. Mahak N.; Akram G. The modified auxiliary equation method to investigate solutions of the perturbed nonlinear Schrödinger equation with Kerr law nonlinearity. *Optik* **2020**, *207*, 164467. [CrossRef]

29. Banaja, M.; Al Qarni, A.A.; Bakodah, H.O.; Biswas, A. Bright and dark solitons in cascaded system by improved Adomian decomposition scheme. *Optik* **2016**, *130*, 341–351. [CrossRef]

30. Shakhanda, R.; Goswami, P.; He, J.-H.; Althobaiti, A. An approximate solution of the time-fractional two-mode coupled Burgers equations. *Fract. Fract.* **2021**, *5*, 196. [CrossRef]

31. Alqudah, M.A.; Asrraf, R.; Rashid, S.; Singh, J.; Hammouch, Z.; Abdeljawad, T. Novel Numerical Investigations of Fuzzy Cauchy Reaction–Diffusion Models via Generalized Fuzzy Fractional Derivative Operators. *Fract. Fract.* **2021**, *5*, 151. [CrossRef]