log-Coulomb Gases in the Projective Line of a $p$-Field

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Abstract—This article extends recent results on log-Coulomb gases in a $p$-field $K$ (i.e., a nonarchimedean local field) to those in its projective line $\mathbb{P}^1(K)$, where the latter is endowed with the $\text{PGL}_2$-invariant Borel probability measure and spherical metric. Our first main result is an explicit combinatorial formula for the canonical partition function of log-Coulomb gases in $\mathbb{P}^1(K)$ with arbitrary charge values. Our second main result is called the "$(q+1)$th Power Law", which relates the grand canonical partition functions for one-component gases in $\mathbb{P}^1(K)$ (where all particles have charge 1) to those in the open and closed unit balls of $K$ in a simple way. The final result is a quadratic recurrence for the canonical partition functions for one-component gases in both unit balls of $K$ and in $\mathbb{P}^1(K)$. In addition to efficient computation of the canonical partition functions, the recurrence provides their "$q \to 1$" limits and "$q \mapsto q^{-1}$" functional equations.

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1. INTRODUCTION

A Coulomb gas in dimension $d \geq 1$ is a system of charged particles in $\mathbb{R}^d$ with pairwise interactions given by a Coulomb potential (or Coulomb kernel),

$$G_d(x, x') := \begin{cases} -\log \|x - x'\| & \text{if } d = 1, 2, \\ \|x - x'\|^{2-d} & \text{if } d > 2, \end{cases}$$

for distinct $x, x' \in \mathbb{R}^d$,

where $\|\cdot\|$ is the usual norm on $\mathbb{R}^d$. For $d \geq 2$, $G_d$ arises as a fundamental solution to the $d$-dimensional Poisson equation and is thus a natural generalization of the familiar $d = 3$ case in classical electromagnetism. To emphasize the logarithms, $G_1$ and $G_2$ are called log-Coulomb potentials and the respective gases in $\mathbb{R}$ and $\mathbb{R}^2$ are called log-Coulomb gases. The latter appear in numerous areas of physics and mathematics including potential theory [4], the study of Ginzburg–Landau vortices, the Fractional Quantum Hall Effect, and Fekete points [17], and somewhat surprisingly, random matrix theory [13]. A detailed overview of these and other applications of log-Coulomb gases can be found in [6, 17], and the references therein.

Mirroring the $d \geq 2$ situation above, [16] showed that the $p$-adic analogue of the Poisson equation has a fundamental solution of essentially the same form as the $\mathbb{R}^d$ one, thus establishing an analogous notion of Coulomb gas in $\mathbb{Q}_p^d$. Indeed, the $p$-adic Coulomb potential is defined just as in (1.1), with $\mathbb{R}^d$ and $\|\cdot\|$ simply replaced by $\mathbb{Q}_p^d$ and a respective natural norm $\|\cdot\|_p$. As was recently shown in [23], the $p$-adic Coulomb potential has much in common with its $\mathbb{R}^d$ counterpart, including $\Gamma$-convergence of Hamiltonians and the existence of Frostman equilibrium measures (as discussed for $\mathbb{R}^d$ in [17]).

The present article continues the recent statistical-mechanical investigation of $p$-adic log-Coulomb gases appearing in [21, 24], and [19], adapted to the projective setting described in [7]. We will give
several formulas for the canonical partition functions and grand canonical partition functions for log-Coulomb gases in the projective line \( \mathbb{P}^1(K) \), where \( K \) is an arbitrary \( p \)-field (of which \( \mathbb{Q}_p \) is an example). As seen in \[24\] and \[21\], a distinct advantage of working with \( K \) (instead of \( \mathbb{R} \)) is a seamless generalization to multi-component (i.e. mixed-charge) gases. This will be demonstrated again in our main formula for canonical partitions of log-Coulomb gases in \( \mathbb{P}^1(K) \) with arbitrary charge values (Theorem 2.1 in §2.1). Returning to the one-component case and taking inspiration from the so-called \( q \)th Power Law in \[19\], we give a similar relationship for log-Coulomb gas in \( \mathbb{P}^1(K) \) called the \((q+1)\)th Power Law (Theorem 2.4 in §2.2). Finally, we will use both Power Laws to provide a quadratic recurrence that allows for more efficient computation of one-component canonical partition functions, while also revealing a \( q \mapsto q^{-1} \) symmetry and \( q \mapsto 1 \) limiting behavior inspired by \[20\] and \[1\] (Theorem 2.6 in §2.3).

To properly contextualize the three theorems mentioned above, we devote the next section to an account of the relevant statistical-mechanical terminology, notation, and literature concerning log-Coulomb gases in \( \mathbb{R} \). Afterward, we give a brief review of the basic properties of \( p \)-fields, their projective lines, and some combinatorial notions from \[21\] that will be used to set up our results.

1.1. log-Coulomb Gases and Partition Functions

Like ideal gases, paramagnets, Ising models, and other many-particle systems, a Coulomb gas can be understood as a statistical model, which is abstractly defined in \[12\] as a “state space” endowed with a collection of probability measures. The two models we will use for Coulomb gases are the canonical ensemble and grand canonical ensemble, whose state spaces are comprised of all possible particle configurations (or “microstates”). In the canonical ensemble, the number of particles is assumed to be fixed and the collection of probability measures is parametrized by the inverse temperature \( \beta \) (a convenient auxiliary for temperature). In the grand canonical ensemble, the number of particles is allowed to vary and the collection of probability measures is parametrized both by \( \beta \) and by the fugacity \( f \) (a convenient auxiliary for chemical potential). Each model provides a description of the macroscopic properties of the gas in terms of those parameters, and this description is made explicit by an associated partition function. Examples of canonical and grand canonical ensembles with closed-form partition functions are standard in most statistical mechanics texts (for instance, \[10\]), though such formulas are difficult to find in general.

The following description of the Coulomb gas canonical ensemble is adapted from \[6, 17\], and \[18\]. Consider a system comprised of \( N \geq 2 \) particles with locations \( x_1, \ldots, x_N \in \mathbb{R}^d \) and respective constant charge values \( q_1, \ldots, q_N \in \mathbb{R} \). We assume the particles are labeled, so that whenever \( i \neq j \), the \( i \)th and \( j \)th particles are always distinguishable (even if \( q_i = q_j \)). This ensures that the distinct configurations of the particles correspond uniquely to the tuples \( (x_1, \ldots, x_N) \in (\mathbb{R}^d)^N \), and thus we call \((\mathbb{R}^d)^N\) the state space and call each tuple a microstate. The collection of probability measures on \((\mathbb{R}^d)^N\) are called Gibbs measures and are defined as follows.

- The total energy (or Hamiltonian) function \( H : (\mathbb{R}^d)^N \to (-\infty, +\infty) \) defined by
  \[
  H(x_1, \ldots, x_N) := E(x_1, \ldots, x_N) + V(x_1, \ldots, x_N),
  \]
  where the first term is the electrostatic (or interaction) energy
  \[
  E(x_1, \ldots, x_N) := \begin{cases}
  \sum_{1 \leq i < j \leq N} q_i q_j G_d(x_i, x_j) & \text{if } x_i \neq x_j \text{ for all } i < j, \\
  +\infty & \text{otherwise},
  \end{cases}
  \]
  and \( V : (\mathbb{R}^d)^N \to (-\infty, +\infty) \) is a localizing (or confining) potential, suitably chosen to ensure that \( H(x_1, \ldots, x_N) \to +\infty \) “fast enough” as any of \( \|x_1\|, \ldots, \|x_N\| \) increases to +\( \infty \).

- We postulate that the gas is in thermal equilibrium with a heat reservoir at absolute temperature \( T > 0 \). The latter is an idealized “ambient” system with which the gas can freely exchange energy while leaving \( T \) fixed. The inverse temperature parameter \( \beta := \frac{1}{k_B T} \) serves as a convenient auxiliary for \( T \), where \( k_B > 0 \) is a Boltzmann constant chosen so that \( \beta H(x_1, \ldots, x_N) \) is dimensionless. In particular, thermal equilibrium between the gas and heat reservoir ensures that \( \beta \) is constant with respect to \( H(x_1, \ldots, x_N) \).
The canonical partition function $Z_N$ is formally defined for $\beta > 0$ by the integral

$$Z_N(\beta) := \int_{(\mathbb{R}^d)^N} e^{-\beta H(x_1, \ldots, x_N)} \, dx_1 \ldots dx_N.$$  

(1.3)

Note that the domain of $Z_N$, i.e. the set of positive $\beta$ for which the integral converges, depends on the particular choices of $q_1, \ldots, q_N$, and $V$ in the definition of $H$. The familiar reader will recognize that we have excluded a factor of $1/N!$ (or a similar combinatorial factor) from the definition of $Z_N$, as particles were explicitly assumed to be distinguishable.

For each $\beta$ in the domain of $Z_N$, the associated Gibbs measure $\mathbb{P}_\beta$ is defined by

$$d\mathbb{P}_\beta(x_1, \ldots, x_N) := \frac{1}{Z_N(\beta)} e^{-\beta H(x_1, \ldots, x_N)} \, dx_1 \ldots dx_N.$$  

(1.4)

This measure makes precise the intuitive notion that the gas “prefers” low energy microstates, and that this preference grows stronger as $T$ decreases. Indeed, the density appearing in $d\mathbb{P}_\beta$ decays exponentially as $H(x_1, \ldots, x_N)$ increases, with decay rate inversely proportional to $T$.

With the collection of Gibbs measures in hand (one $\mathbb{P}_\beta$ for each $\beta$ in the domain of $Z_N$), one takes expectations against them to find macroscopic quantities as functions of $\beta$ (and hence of $T$). In fact, some can be computed directly from $Z_N(\beta)$, such as the dimensionless free energy $-\log Z_N(\beta)$, mean energy $-\partial/\partial \beta \log Z_N(\beta)$, and energy fluctuation (variance) $\partial^2/\partial \beta^2 \log Z_N(\beta)$, provided those derivatives exist (see Section 4.6 in [10]). However, in light of current literature on Coulomb gases, finding a closed formula and explicit domain for $Z_N$ depends quite delicately on the chosen charge values $q_1, \ldots, q_N$, even when $d = 1$ and $V$ is relatively simple.

Among the simplest examples of $Z_N$ is that for log-Coulomb gas in $\mathbb{R}$ confined to an “infinite potential well” in $[0, 1]$. In this case the Hamiltonian has a confining potential of the form

$$V(x_1, \ldots, x_N) = \begin{cases} 0 & \text{if } x_i \in [0, 1] \text{ for all } i, \\ \infty & \text{otherwise}, \end{cases}$$

and thus, with the convention $e^{-\beta(+\infty)} = 0$ for $\beta > 0$, (1.3) takes the form

$$Z_N(\beta) = \int_{[0,1]^N} \prod_{1 \leq i < j \leq N} |x_i - x_j|^{q_i q_j} \beta \, dx_1 \ldots dx_N.$$  

(1.6)

Alternatively, this $Z_N$ can be understood as the canonical partition function for a canonical ensemble whose state space is simply $[0, 1]^N$ and whose Hamiltonian $H : [0, 1]^N \to (-\infty, +\infty]$ is simply $H(x_1, \ldots, x_N) = E(x_1, \ldots, x_N)$ without a confining potential. The one-component version of (1.6) (where $q_1 = \cdots = q_N = 1$) is a special case of Selberg’s integral, which is discussed in great detail in [8]. In particular, its domain extends to all complex $\beta$ with $\Re(\beta) > -2/N$, and for such $\beta$ it admits the closed form

$$Z_N(\beta)|_{q_1=\cdots=q_N=1} = \prod_{k=1}^{N} \frac{\Gamma^2(1 + (k - 1)\beta/2) \cdot \Gamma(1 + k\beta/2)}{\Gamma(2 + (N - 1 + k - 1)\beta/2) \cdot \Gamma(1 + \beta/2)}.$$  

(1.7)

A closely related example concerns log-Coulomb gas in $\mathbb{R}$ subject to a “harmonic potential well”. In this case the Hamiltonian has a confining potential of the form $V(x_1, \ldots, x_N) = \frac{1}{2}k_BT(x_1^2 + \cdots + x_N^2)$, and thus (1.3) takes the form

$$Z_N(\beta) = \int_{\mathbb{R}^N} \prod_{1 \leq i < j \leq N} |x_i - x_j|^{q_i q_j} e^{-\frac{1}{2}(x_1^2 + \cdots + x_N^2)} \, dx_1 \ldots dx_N.$$  

(1.8)

The one-component version of (1.8) is known as Mehta’s integral, which also converges for complex $\beta$ with $\Re(\beta) > -2/N$ and takes the closed form

$$Z_N(\beta)|_{q_1=\cdots=q_N=1} = (2\pi)^{N/2} \prod_{k=1}^{N} \frac{\Gamma(1 + k\beta/2)}{\Gamma(1 + \beta/2)}.$$  

(1.9)
The evident similarity between (1.7) and (1.9) is an artifact of Bombieri's proof of the latter, which expresses Mehta's integral as a limit of Selberg integrals [8]. Prior to Bombieri's general proof, Mehta and Dyson had established (1.9) for $\beta = 1, 2, 4$, while showing that the respective Gibbs measures are the probability laws for the eigenvalues of the $N \times N$ gaussian orthogonal ($\beta = 1$), unitary ($\beta = 2$), and symplectic ($\beta = 4$) matrix ensembles [13]. Another variant of (1.6) and (1.8) is the canonical partition for the circular ensemble, namely

$$Z_N(\beta) = \int_{T_N} \prod_{1 \leq i < j \leq N} |z_i - z_j|^q \, d\mu(z_1) \ldots d\mu(z_N),$$

(1.10)

where $T$ is the unit circle in $\mathbb{C}$ and $\mu$ is the multiplicative Haar measure given by $d\mu(z) = \frac{dz}{2\pi}$. One can also use Selberg integrals to deduce a closed-form for the one-component version of (1.10), namely

$$Z_N(\beta) \bigg|_{q_1=\ldots=q_N=1} = (2\pi)^N \frac{\Gamma(1+N\beta/2)}{\Gamma(N(1+\beta/2))},$$

(1.11)

which is also valid for all complex $\beta$ with $\text{Re}(\beta) > -2/N$. In fact, for $\beta = 1, 2, 4$, the respective Gibbs measures are the probability laws for the eigenvalues of the $N \times N$ uniform orthogonal ($\beta = 1$), unitary ($\beta = 2$), and symplectic ($\beta = 4$) matrix ensembles [8].

Closed forms for (1.8) and (1.10) have also been found in a handful of multi-component cases (where $q_1, \ldots, q_N$ are not all equal). For instance, given arbitrary $q_1, \ldots, q_N \in \mathbb{Z}_{>0}$, [9] presents a method for explicit evaluation of (1.10) at $\beta = 2$ and [18] presents a similar method for evaluating (1.8) at $\beta = 1, 2, 4$, in terms of Pfaffians and Berezin integrals. Similar tools were used in [14] for a detailed analysis in the \{q_1, \ldots, q_N\} = \{1, 2\} and $\beta = 1$ case. However, the strategies presented in [9, 18], and [14] do not generalize beyond those special choices of $\beta$ and $q_i$ in an obvious way.

For the analogues of (1.6), (1.8), and (1.10) over a $p$-field $K$ (such as $\mathbb{Q}_p$), it turns out that the general multi-component versions are only slightly more difficult to evaluate than their one-component versions [21], and this will also be the case for the $\mathbb{P}^1(K)$ analogue. With these comparisons in mind, we define the following “generalized canonical partition function” in order to treat arbitrary choices of $q_1, \ldots, q_N \in \mathbb{R}$ and various state spaces in a unified way.

**Definition 1.1.** Suppose $X$ is a compact metric space with metric $d$, suppose $X$ has a finite positive Borel measure $\mu$, and suppose $N \geq 2$ is an integer. Write $\mu^N$ for the product measure on $X^N$ and denote tuples $(s_{ij})_{1 \leq i < j \leq N} \in \mathbb{C}^{N(N-1)/2}$ by $s$. For $s \in \mathbb{C}^{N(N-1)/2}$ we formally define

$$Z_N(X, s) := \int_{X^N} \prod_{1 \leq i < j \leq N} d(x_i, x_j)^{s_{ij}} \, d\mu^N,$$

and for $\beta \in \mathbb{C}$ we formally define

$$Z_N(X, \beta) := Z_N(X, s)\bigg|_{s=\ldots=\beta} = \int_{X^N} \prod_{1 \leq i < j \leq N} d(x_i, x_j)^{\beta} \, d\mu^N.$$  

Depending on $(X, d, \mu)$, $s$, and $\beta$, the integrals $Z_N(X, s)$ and $Z_N(X, \beta)$ may or may not converge.

**Remark 1.2.** Given $q_1, \ldots, q_N \in \mathbb{R}$, note that Definition 1.1 generalizes two of the examples above. Indeed, taking $X = [0, 1]$ with the usual metric $d$ and Lebesgue measure $\mu$, we can restate (1.6) as

$$Z_N(\beta) = Z_N([0, 1], s)\bigg|_{s=(q_1, \ldots, q_N)}$$

(1.12)

or simply $Z_N(\beta) = Z_N([0, 1], \beta)$ if $q_1 = \ldots = q_N = 1$. Similarly, taking $X = \mathbb{T}$ with the chordal metric and $\mu$ the multiplicative Haar measure ($d\mu(z) = \frac{dz}{2\pi}$ as above), we can restate (1.10) as

$$Z_N(\beta) = Z_N(\mathbb{T}, s)\bigg|_{s=(q_1, \ldots, q_N)}$$

(1.13)

or simply $Z_N(\beta) = Z_N(\mathbb{T}, \beta)$ if $q_1 = \ldots = q_N = 1$. 

Like its closed form, the domain of the integral $\mathcal{Z}_N(X, s)$ (the subset of $\mathbb{C}^{N(N-1)/2}$ on which it converges) is generally difficult to describe explicitly. For example, when $X = \mathbb{Z}_p$ with the $p$-adic (ultra)metric $d(x, x') = |x - x'|_p$ and additive Haar measure $\mu$, the familiar reader may recognize the integral $\mathcal{Z}_N(\mathbb{Z}_p, s)$ as a multivariate Igusa zeta function attached to the collection of polynomials \( \{ x_i - x_j : 1 \leq i < j \leq N \} \subset \mathbb{Z}_p[x_1, \ldots, x_N] \). For general collections \( \{ f_1, \ldots, f_k \} \subset \mathbb{Z}_p[x_1, \ldots, x_N] \), [11] used resolution of singularities to describe the domains of such zeta functions as convex open polytopes in $\mathbb{C}^k$ (i.e. intersections of open half-spaces). Even for our relatively simple collection \( \{ x_i - x_j : 1 \leq i < j \leq N \} \), the number of half-spaces used to describe the associated polytope in $\mathbb{C}^{N(N-1)/2}$ grows exponentially with $N$ (see Proposition 1.8 below).

Whatever the full domain of $\mathcal{Z}_N(X, s)$ may be, the properties of $(X, d, \mu)$ ensure that it at least contains the closed polytope \( \{ s \in \mathbb{C}^{N(N-1)/2} : \Re(s_{ij}) \geq 0 \text{ for all } i < j \} \). Indeed, compactness of $X$ and continuity of $d$ imply $\diam(X) = \sup\{ d(x, x') : x, x' \in X \}$ is finite, so taking $s$ with $\Re(s_{ij}) \geq 0$ for all $i < j$ leads to a bound that is uniform for all $(x_1, \ldots, x_N) \in X^N$:

$$\prod_{1 \leq i < j \leq N} d(x_i, x_j)^{s_{ij}} = \prod_{1 \leq i < j \leq N} d(x_i, x_j)^{\Re(s_{ij})} \leq \diam(X)^{\sum_{1 \leq i < j \leq N} \Re(s_{ij}) \cdot \mu(X)^N}.$$ 

Thus, absolute convergence of $\mathcal{Z}_N(X, s)$ for such $s$ follows from finiteness of $\mu^N(X^N) = (\mu(X))^N$. For later use, we state this fact and some immediate consequences as a lemma.

**Lemma 1.3.** Let $(X, d, \mu)$ be as in Definition 1.1 and let $\diam(X) = \sup\{ d(x, x') : x, x' \in X \}$. If $N \geq 2$ and $s \in \mathbb{C}^{N(N-1)/2}$ satisfies $\Re(s_{ij}) \geq 0$ for all $i < j$, then the integral $\mathcal{Z}_N(X, s)$ converges absolutely and satisfies

$$|\mathcal{Z}_N(X, s)| \leq \diam(X)^{\sum_{1 \leq i < j \leq N} \Re(s_{ij}) \cdot (\mu(X))^N}.$$ 

In particular, if $N \geq 2$ and $\beta \in \mathbb{C}$ satisfies $\Re(\beta) \geq 0$, then the integral $\mathcal{Z}_N(X, \beta)$ converges absolutely and satisfies

$$|\mathcal{Z}_N(X, \beta)| \leq \diam(X)^{\frac{N(N-1)}{2} \beta} \cdot (\mu(X))^N.$$ 

We now fix a space $(X, d, \mu)$ as in Definition 1.1 and give a brief description of the grand canonical ensemble for a log-Coulomb gas in $X$, adapted from [19] and Section 4.9 of [10]. In this model we assume all particles are indistinguishable (i.e. labels are ignored) with all charge values equal to 1. We continue to assume the gas is in thermal equilibrium with a heat reservoir, so that its energy may vary while the inverse temperature $\beta$ remains constant. Additionally, we now assume the gas can exchange particles with the reservoir with chemical potential $\eta$, a real parameter that measures the amount of energy gained/lost by the gas when it gains a particle. Explicitly, in the instance that the gas has $N \geq 2$ particles located at $x_1, \ldots, x_N \in X$, its energy is given by a modified Hamiltonian

$$-\eta N + H(x_1, \ldots, x_N) = \begin{cases} -\eta N - \sum_{1 \leq i < j \leq N} \log d(x_i, x_j) & \text{if } x_i \neq x_j \text{ for all } i < j, \\ +\infty & \text{otherwise}. \end{cases}$$

Since the particles are indistinguishable, the order of $x_1, \ldots, x_N$ is irrelevant and the associated microstate can be represented by any permutation of the tuple $(x_1, \ldots, x_N) \in X^N$. Therefore, writing $S_N$ for the symmetric group on $\{1, \ldots, N\}$, we now define a microstate as the $S_N$-orbit of $(x_1, \ldots, x_N)$, namely \( \{(x_{\sigma(1)}, \ldots, x_{\sigma(N)}) : \sigma \in S_N\} \), not just the tuple itself. In particular, a microstate is a subset of $X^N$ with at most $\#S_N = N!$ elements, though it still has a well-defined energy because $-\eta N + H(x_{\sigma(1)}, \ldots, x_{\sigma(N)}) = -\eta N + H(x_1, \ldots, x_N)$ for all $\sigma \in S_N$. To set up the appropriate analogue of (1.3), we adopt the convention $e^{-\beta(\pm \infty)} = 0$ for $\beta > 0$ and note the following equivalence of conditions for a tuple $(x_1, \ldots, x_N) \in X^N$:

$$e^{-\beta(-\eta N + H(x_1, \ldots, x_N))} > 0 \text{ for all } \beta > 0 \iff -\eta N + H(x_1, \ldots, x_N) < +\infty$$

$$\iff x_1, \ldots, x_N \text{ are distinct}$$
With this in mind, the “sum” of the above exponential over all \( N \)-particle microstates is given by

\[
\frac{1}{N!} \int_{X^N} e^{-\beta(-\eta N + H(x_1, \ldots, x_N))} \, d\mu^N = Z_N(X, \beta) \frac{(e^{\beta \eta})^N}{N!}, \tag{1.14}
\]

with \( Z_N(X, \beta) \) as in Definition 1.1. Since the integrand is constant on each microstate and positive only for those with finite energy, the factor \( 1/N! \) corrects for “\( N! \)-fold over-counting”, i.e. the fact that a finite-energy microstate is represented by precisely \( N! \) elements of \( X^N \). To make (1.14) accommodate the possibility that \( N = 0 \) or \( N = 1 \), we replace \( H(x_1, \ldots, x_N) \) by 0 (since there is no particle-particle interaction in this case) and accordingly replace \( X^N, \mu^N, \) and \( Z_N \) as follows. There is a unique microstate with \( N = 0 \) (the empty gas), so we understand \( X^0 \) as the singleton containing it and let \( \mu^0 \) be the measure on \( X^0 \) satisfying \( \mu^0(X^0) = 1 \). When \( N = 1 \), the space of 1-particle microstates is simply \( X^N = X \) with \( \mu^N = \mu \). Thus, with the conventions \( Z_0(X, \beta) = 1 \) and \( Z_1(X, \beta) = \mu(X) \) for all \( \beta > 0 \), the form and interpretation of (1.14) are valid for all \( N \geq 0 \).

Finally, the \textit{grand canonical partition function} is the “sum” over the full state space, defined formally for \( \beta > 0 \) and \( \eta \in \mathbb{R} \) by

\[
\sum_{N=0}^{\infty} \frac{1}{N!} \int_{X^N} e^{-\beta(-\eta N + H(x_1, \ldots, x_N))} \, d\mu^N = \sum_{N=0}^{\infty} Z_N(X, \beta) \frac{(e^{\beta \eta})^N}{N!}. \tag{1.15}
\]

Following [19], we introduce the \textit{fugacity} parameter \( f \) in place of \( e^{\beta \eta} \) above, regard \( \beta \) and \( f \) abstractly as independent complex numbers, and define a generalized version of (1.15).

**Definition 1.4.** Let \( (X, d, \mu) \) be as in Definition 1.1 and define \( Z_0(X, \beta) := 1 \) and \( Z_1(X, \beta) := \mu(X) \) for all \( \beta \in \mathbb{C} \). For complex \( \beta \) and \( f \), formally define

\[
Z(f, X, \beta) := \sum_{N=0}^{\infty} Z_N(X, \beta) \frac{f^N}{N!}.
\]

If \( (X, d, \mu) \) happens to satisfy \( \text{diam}(X) \leq 1 \) and \( \beta \) is a complex number with \( \text{Re}(\beta) \geq 0 \), then Lemma 1.3, \( Z_0(X, \beta) = 1 \), and \( Z_1(X, \beta) = \mu(X) \) together imply \( |Z_N(X, \beta)| \leq (\mu(X))^N \) for all \( N \geq 0 \). This observation leads immediately to the following lemma.

**Lemma 1.5.** Let \( (X, d, \mu) \) be as in Definition 1.1 with \( \text{diam}(X) = \sup\{d(x, x') : x, x' \in X\} \leq 1 \), and fix a complex number \( \beta \) with \( \text{Re}(\beta) \geq 0 \). The power series defining \( Z(f, X, \beta) \) converges for all \( f \in \mathbb{C} \), and therefore \( Z(f, X, \beta) \) is an entire function of \( f \in \mathbb{C} \). In particular, \( \partial/\partial f \) can be applied to \( Z(f, X, \beta) \) term-by-term any number of times, and \( Z_N(X, \beta) \) can be recovered from \( Z(f, X, \beta) \) via

\[
Z_N(X, \beta) = \left. \frac{\partial^N}{\partial f^N} Z(f, X, \beta) \right|_{f=0}.
\]

for every \( N \geq 0 \).

In addition to the canonical partition functions for one-component log-Coulomb gases in \( X \), a formula for \( Z(f, X, \beta) \) allows for direct computation of several important quantities. For instance, at a fixed inverse temperature \( \beta > 0 \), Lemma 1.5 implies that \( Z(f, X, \beta) \) restricts to a smooth (and strictly positive) function of \( f > 0 \). Then \( \log Z(f, X, \beta) \) is a smooth function of \( f > 0 \) and the expected number of particles in the gas is given by

\[
\frac{1}{Z(f, X, \beta)} \sum_{N=0}^{\infty} N \cdot Z_N(X, \beta) \frac{f^N}{N!} = \frac{1}{Z(f, X, \beta)} f \frac{\partial}{\partial f} Z(f, X, \beta) = f \frac{\partial}{\partial f} \log Z(f, X, \beta).
\]

Evaluating at \( f = e^{\beta \eta} \) recovers the \( \eta \)-dependent definition of expected particle number in Section 4.9 of [10], namely \( \langle N \rangle = \frac{1}{\beta} \frac{\partial}{\partial \eta} \log Z(e^{\beta \eta}, X, \beta) \).
1.2. **p-Fields, Projective Lines, and Splitting Chains**

The following account of well-known properties of \( p \)-fields is taken from Chapter 1 of [22]. For the remainder of this paper, we fix a \( p \)-field \( K \), write \( | \cdot | \) for its canonical absolute value, write \( d \) for the associated metric (i.e., \( d(x, y) := |x - y| \)), and define

\[
R := \{ x \in K : |x| \leq 1 \} \quad \text{and} \quad P := \{ x \in K : |x| < 1 \}.
\]

The closed unit ball \( R \) is the maximal compact subring of \( K \), the open unit ball \( P \) is the unique maximal ideal in \( R \), and the group of units is \( R^\times = R \setminus P = \{ x \in K : |x| = 1 \} \). The residue field \( R/P \) is isomorphic to \( \mathbb{F}_q \) for some prime power \( q \), and there is a canonical isomorphism of the cyclic group \((R/P)^\times\) onto the group of \((q - 1)\)th roots of unity in \( K \). Fixing a primitive such root \( \xi \in K \) and sending \( P \to 0 \) extends the isomorphism \((R/P)^\times \to \{ 1, \xi, \ldots, \xi^{q - 2} \}\) to a bijection \( R/P \to \{ 0, 1, \xi, \ldots, \xi^{q - 2} \} \) with inverse \( y \mapsto y + P \). Therefore \( \{ 0, 1, \xi, \ldots, \xi^{q - 2} \} \) is a complete set of representatives for the cosets of \( P \subset R \), i.e.

\[
R = P \cup (1 + P) \cup (\xi + P) \cup \cdots \cup (\xi^{q - 2} + P).
\]  

(1.16)

Fix a uniformizer \( \pi \in K \) (any element satisfying \( P = \pi R \)) and let \( \mu \) be the unique additive Haar measure on \( K \) satisfying \( \mu(R) = 1 \). The open balls in \( K \) are precisely the sets of the form \( y + \pi^n R \) with \( y \in K \) and \( n \in \mathbb{Z} \), and every such ball is compact with measure \( \mu(y + \pi^n R) = |\pi|^n = q^{-n} \). In particular, each of the \( q \) cosets of \( P \) in (1.16) is a compact open ball with measure \( q^{-1} \), and two elements \( x, y \in R \) satisfy \( |x - y| = 1 \) if and only if they belong to different cosets.

We henceforth reserve the symbols \( K, | \cdot |, d, R, P, q, \xi, \pi, \mu \) for the items just described, and we distinguish \( | \cdot | \) from the standard absolute value on \( \mathbb{C} \) by writing \( | \cdot |_C \) for the latter. For any \( v \in \mathbb{Z} \) and \( y \in K \), note that Definition 1.1 applies to the compact space \( X = y + \pi^n R \) endowed with \( d \) and \( \mu \). More explicitly, for \( N \geq 2 \) and \( s = (s_{ij})_{1 \leq i < j \leq N} \in \mathbb{C}^{N(N - 1)/2} \), we have the (formally defined) integral

\[
\mathbb{Z}_N(y + \pi^n R, s) = \int_{(y + \pi^n R)^N} \prod_{1 \leq i < j \leq N} |x_i - x_j|^{s_{ij}} \, d\mu^N.
\]

Whenever \( \mathbb{Z}_N(y + \pi^n R, s) \) converges, its value is independent of \( y \) because \( \mu \) is translation invariant and the integrand is invariant under the diagonal translation \( (x_1, \ldots, x_N) \mapsto (y + x_1, \ldots, y + x_N) \). In particular, when \( v = 0 \) or \( v = 1 \), \( \mathbb{Z}_N(y + \pi^n R, s) \) is respectively equal to \( \mathbb{Z}_N(R, s) \) or \( \mathbb{Z}_N(P, s) \).

We now recall the definition of the projective line of \( K \) and several of its well-known properties. We will closely follow the setup in [7], though much of it can also be found in [5] and Chapter 1 of [15]. The **projective line** of \( K \) is the quotient space \( \mathbb{P}^1(K) = (K^\times \setminus \{ 0 \}) / \sim \), where \( (x_0, x_1) \sim (y_0, y_1) \) if and only if \( y_0 = \lambda x_0 \) and \( y_1 = \lambda x_1 \) for some \( \lambda \in K^\times \). Thus we may understand \( \mathbb{P}^1(K) \) concretely as the set of symbols \( [x_0 : x_1] \) with \( (x_0, x_1) \in K^2 \setminus \{ (0, 0) \} \), subject to the relation \( [x_0 : \lambda x_1] = [x_0 : x_1] \) for all \( \lambda \in K^\times \) and endowed with the topology induced by the quotient \( (x_0, x_1) \mapsto [x_0 : x_1] \). The projective line is compact and metrizable by the **spherical metric** \( \delta : \mathbb{P}^1(K) \times \mathbb{P}^1(K) \to \{ 0 \} \cup \{ q^{-v} : v \in \mathbb{Z}_{\geq 0} \} \), which is defined via

\[
\delta([x_0 : x_1], [y_0 : y_1]) := \frac{|x_0 y_1 - x_1 y_0|}{\max\{|x_0|, |x_1|\} \cdot \max\{|y_0|, |y_1|\}}.
\]

(1.17)

In particular, every open set in \( \mathbb{P}^1(K) \) is a union of balls of the form

\[
B_\phi([x_0 : x_1]) := \{ [y_0 : y_1] \in \mathbb{P}^1(K) : \delta([x_0 : x_1], [y_0 : y_1]) \leq q^{-v} \}
\]

(1.18)

with \( [x_0 : x_1] \in \mathbb{P}^1(K) \) and \( v \in \mathbb{Z}_{\geq 0} \), and every such ball is open and compact. The projective linear group \( \mathrm{PGL}_2(R) \) is the quotient of \( \mathrm{GL}_2(R) = \{ A \in M_2(R) : \det(A) \in R^\times \} \) by its center, namely \( Z = \{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in R^\times \} \cong R^\times \). It is straightforward to verify that the rule

\[
\phi[x_0 : x_1] := [ax_0 + bx_1 : cx_0 + dx_1],
\]

where \( \phi \in \mathrm{PGL}_2(R) \) and \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R) \) is any representative of \( \phi \), gives a well-defined transitive action of \( \mathrm{PGL}_2(R) \) on \( \mathbb{P}^1(K) \).

---

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Lemma 1.6 (PGL$_2$($R$)-invariance [7]). The spherical metric satisfies
\[ \delta(\phi[x_0 : x_1], \phi[y_0 : y_1]) = \delta([x_0 : x_1], [y_0 : y_1]) \]
for all $\phi \in$ PGL$_2$($R$) and all $[x_0 : x_1], [y_0 : y_1] \in \mathbb{P}^1(K)$. There is also a unique Borel probability measure $\nu$ on $\mathbb{P}^1(K)$ satisfying
\[ \nu(\phi(M)) = \nu(M) \]
for all $\phi \in$ PGL$_2$($R$) and all Borel subsets $M \subset \mathbb{P}^1(K)$.

In particular, for each $v \in \mathbb{Z}_{\geq 0}$ the relation $\phi(B_v[x_0 : x_1]) = B_v(\phi[x_0 : x_1])$ defines a transitive PGL$_2$($R$) action on the set of balls of radius $q^{-v}$, and thus $\nu(B_v[x_0 : x_1])$ depends only on $v$. With the metric $\delta$ and measure $\nu$ in hand, we may now apply Definition 1.1 to $X = \mathbb{P}^1(K)$. Explicitly, for $N \geq 2$ and $s \in \mathbb{C}^{N(N-1)/2}$, we have the (formally defined) integral
\[ Z_N(\mathbb{P}^1(K), s) = \int_{(\mathbb{P}^1(K))^N} \prod_{1 \leq i < j \leq N} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}} \, dv^N. \]

A significant portion of the present paper is dedicated to determining a closed formula and explicit domain for $Z_N(\mathbb{P}^1(K), s)$. Just like those for $Z_N(R, s)$, it will turn out that the domain and formula for $Z_N(\mathbb{P}^1(K), s)$ can be stated neatly in terms of the following combinatorial objects from [21].

**Definition 1.7 (Splitting chains).** A splitting chain of order $N \geq 2$ and length $L \geq 1$ is a tuple $\mathfrak{h} = (\mathfrak{h}_0, \ldots, \mathfrak{h}_L)$ of partitions of $[N] = \{1, \ldots, N\}$ satisfying the proper refinement condition
\[ \{[N] \} = \mathfrak{h}_0 > \mathfrak{h}_1 > \mathfrak{h}_2 > \cdots > \mathfrak{h}_L = \{\{1\}, \ldots, \{N\}\}. \]

(a) Each non-singleton part $\lambda \in \mathfrak{h}_0 \cup \mathfrak{h}_1 \cup \cdots \cup \mathfrak{h}_L$ is called a branch of $\mathfrak{h}$. We write $\mathcal{B}(\mathfrak{h})$ for the set of all branches of $\mathfrak{h}$, i.e.,
\[ \mathcal{B}(\mathfrak{h}) := (\mathfrak{h}_0 \cup \cdots \cup \mathfrak{h}_{L-1}) \setminus \mathfrak{h}_L. \]

(b) Each $\lambda \in \mathcal{B}(\mathfrak{h})$ must appear in a final partition $\mathfrak{h}_L$ before refining into two or more parts in $\mathfrak{h}_{L+1}$, so we define its depth $\ell_{\mathfrak{h}}(\lambda) \in \{0, 1, \ldots, L-1\}$ and degree $\deg_{\mathfrak{h}}(\lambda) \in \{2, 3, \ldots, N\}$ by
\[ \ell_{\mathfrak{h}}(\lambda) := \max\{\ell : \lambda \in \mathfrak{h}_\ell\} \text{ and } \deg_{\mathfrak{h}}(\lambda) := \#\{\lambda' : \lambda' \in \mathfrak{h}_{\ell_{\mathfrak{h}}(\lambda)+1} : \lambda' \subset \lambda\}. \]

(c) We say that $\mathfrak{h}$ is reduced if each $\lambda \in \mathcal{B}(\mathfrak{h})$ satisfies $\lambda \in \mathfrak{h}_\ell \iff \ell = \ell_{\mathfrak{h}}(\lambda)$.

Write $\mathcal{R}_N$ for the set of reduced splitting chains of order $N$ and define
\[ \Omega_N := \bigcap_{\lambda \subseteq [N]} \{ s : \Re(e_\lambda(s)) > 0 \} \quad \text{where} \quad e_\lambda(s) := \#\lambda - 1 + \sum_{i < j} s_{ij}. \]

Proposition 3.15 and Theorem 2.6(c) in [21] imply the following proposition, which shall later be generalized slightly in order to prove the main results of this paper.

**Proposition 1.8.** Suppose $N \geq 2$. The integral $Z_N(R, s)$ converges absolutely if and only if $s \in \Omega_N$, and in this case its value is given by the finite sum
\[ Z_N(R, s) = \frac{q^{\sum_{1 \leq i < j \leq N} s_{ij}}}{\prod_{\mathfrak{h} \in \mathcal{R}_N} \prod_{\lambda \in \mathcal{B}(\mathfrak{h})} \frac{(q-1)\deg_{\mathfrak{h}}(\lambda)-1}{q^{e_\lambda(s)}-1}}. \]

Here $(q-1)_n$ stands for the degree $n$ falling factorial $(z)_n = z(z-1) \cdots (z-n+1) \in \mathbb{Z}[z]$ evaluated at the integer $z = q-1$. 
The following example demonstrates Proposition 1.8 for $R = \mathbb{Z}_p$ and $N = 3$.

**Example 1.9.** The integral

$$Z_3(\mathbb{Z}_p, s) = \int_{\mathbb{Z}_p^3} |x_1 - x_2|^{s_{12}} |x_1 - x_3|^{s_{13}} |x_2 - x_3|^{s_{23}} \, d\mu^3$$

converges absolutely if and only if the complex tuple $s = (s_{12}, s_{13}, s_{23})$ belongs to the open polytope $\Omega_3 \subset \mathbb{C}^3$, which by part (c) of Definition 1.7 is the intersection of the four open half-spaces

$$\{s : \text{Re}(e_{(1,2)}(s)) > 0\} = \{s : \text{Re}(s_{12}) > -1\},$$

$$\{s : \text{Re}(e_{(1,3)}(s)) > 0\} = \{s : \text{Re}(s_{13}) > -1\},$$

$$\{s : \text{Re}(e_{(2,3)}(s)) > 0\} = \{s : \text{Re}(s_{23}) > -1\},$$

$$\{s : \text{Re}(e_{(1,2,3)}(s)) > 0\} = \{s : \text{Re}(s_{12} + s_{13} + s_{23}) > -2\}.$$

For $s \in \Omega_3$, we have that $Z_3(\mathbb{Z}_p, s)$ is equal to $p^{s_{12} + s_{13} + s_{23}}$ times a sum over the reduced splitting chains $\mathfrak{h} \in \mathcal{R}_3$, of which there are four. We tabulate them below, along with the associated branch sets and summand values:

| $\mathfrak{h}$ | $\mathcal{B}(\mathfrak{h})$ | $\prod_{\lambda \in \mathcal{B}(\mathfrak{h})} \frac{(p - 1)_{\text{deg}(\lambda) - 1}}{p^{\mathcal{M}(\lambda)} - 1}$ |
|----------------|----------------------------|-----------------------------------------------|
| $\{1, 2, 3\}$, $\{1, 2\} \{3\}$, $\{1\} \{2\} \{3\}$ | $\{1, 2, 3\}$, $\{1, 2\}$ | $\frac{(p - 1)_{2-1}}{p^{2 + s_{12} + s_{13} + s_{23}} - 1} \cdot \frac{(p - 1)_{2-1}}{p^{1 + s_{12}} - 1}$ |
| $\{1, 2, 3\}$, $\{1, 3\} \{2\}$, $\{1\} \{2\} \{3\}$ | $\{1, 2, 3\}$, $\{1, 3\}$ | $\frac{(p - 1)_{2-1}}{p^{2 + s_{12} + s_{13} + s_{23}} - 1} \cdot \frac{(p - 1)_{2-1}}{p^{1 + s_{13}} - 1}$ |
| $\{1, 2, 3\}$, $\{1\} \{2, 3\}$, $\{1\} \{2\} \{3\}$ | $\{1, 2, 3\}$, $\{2, 3\}$ | $\frac{(p - 1)_{2-1}}{p^{2 + s_{12} + s_{13} + s_{23}} - 1} \cdot \frac{(p - 1)_{2-1}}{p^{1 + s_{23}} - 1}$ |
| $\{1, 2, 3\}$, $\{1\} \{2\} \{3\}$ | $\{1, 2, 3\}$ | $\frac{(p - 1)_{3-1}}{p^{2 + s_{12} + s_{13} + s_{23}} - 1}$ |

Finally, summing the rightmost column, simplifying falling factorials, and pulling out some common factors gives

$$Z_3(\mathbb{Z}_p, s) = \frac{p^{s_{12} + s_{13} + s_{23}}}{p^{2 + s_{12} + s_{13} + s_{23}} - 1} \cdot \left( (p - 1)(p - 2) + (p - 1)^2 \left[ \frac{1}{p^{1 + s_{12}} - 1} + \frac{1}{p^{1 + s_{13}} - 1} + \frac{1}{p^{1 + s_{23}} - 1} \right] \right).$$

This formula readily implies explicit formulas for the canonical partition functions of 3-particle log-Coulomb gases in $\mathbb{Z}_p$, with any desired choices of $q_1, q_2, q_3 \in \mathbb{R}$. Here are three examples:

- If $q_1 = 1$, $q_2 = 2$, and $q_3 = 3$, the canonical partition function $Z_3(\beta)$ is obtained by evaluating $Z_3(\mathbb{Z}_p, s)$ at $(s_{12}, s_{13}, s_{23}) = (q_1q_2\beta, q_1q_3\beta, q_2q_3\beta) = (2\beta, 3\beta, 6\beta)$, which belongs to $\Omega_3$. 

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if and only if \(\text{Re}(\beta) > -1/6\). Therefore the partition function converges precisely when 
\(\text{Re}(\beta) > -1/6\), and for these \(\beta\) it satisfies

\[
Z_3(\beta) = \int_{\mathbb{Z}_p^3} |x_1 - x_2|^2 \beta |x_1 - x_3|^2 \beta |x_2 - x_3|^3 \beta \, d\mu^3
\]

\[
= \frac{p^{11\beta}}{p^{2+11\beta} - 1} \cdot \left( (p - 1)(p - 2) + (p - 1)^2 \left[ \frac{1}{p^{1+2\beta} - 1} + \frac{1}{p^{1+3\beta} - 1} + \frac{1}{p^{1+6\beta} - 1} \right] \right).
\]

Through this formula, \(Z_3\) extends to a function that is meromorphic on \(\mathbb{C}\) with poles belonging to the set

\[
-\frac{1}{2} + \frac{\pi i}{\log(p)} \mathbb{Z} \cup -\frac{1}{3} + \frac{2\pi i}{3\log(p)} \mathbb{Z} \cup -\frac{2}{11} + \frac{2\pi i}{11\log(p)} \mathbb{Z} \cup -\frac{1}{6} + \frac{\pi i}{3\log(p)} \mathbb{Z}.
\]

- If \(q_1 = -1, q_2 = 2, \) and \(q_3 = 2\), the canonical partition function \(Z_3(\beta)\) is obtained by evaluating \(Z_3(\mathbb{Z}_p, s)\) at \((s_{12}, s_{13}, s_{23}) = (q_1 q_2 \beta, q_1 q_3 \beta, q_2 q_3 \beta) = (-2\beta, -2\beta, 4\beta)\), which belongs to \(\Omega_3\) if and only if \(-1/4 < \text{Re}(\beta) < 1/2\). Therefore the partition function converges precisely when \(-1/4 < \text{Re}(\beta) < 1/2\), and for these \(\beta\) it satisfies

\[
Z_3(\beta) = \int_{\mathbb{Z}_p^3} |x_1 - x_2|^{2\beta} |x_1 - x_3|^{2\beta} |x_2 - x_3|^{3\beta} \, d\mu^3
\]

\[
= \frac{1}{p^{2\beta} - 1} \cdot \left( (p - 1)(p - 2) + (p - 1)^2 \left[ \frac{2}{p^{1-2\beta} - 1} + \frac{2}{p^{1+3\beta} - 1} \right] \right).
\]

Through this formula, \(Z_3\) extends to a function that is meromorphic on \(\mathbb{C}\) with poles belonging to the set

\[
-\frac{1}{4} + \frac{\pi i}{2\log(p)} \mathbb{Z} \cup \frac{1}{2} + \frac{\pi i}{\log(p)} \mathbb{Z}.
\]

- If \(q_1 = q_2 = q_3 = 1\), the canonical partition function \(Z_3(\beta)\) (i.e. \(Z_3(\mathbb{Z}_p, \beta)\) in Definition 1.1) is obtained by evaluating \(Z_3(\mathbb{Z}_p, s)\) at \((s_{12}, s_{13}, s_{23}) = (\beta, \beta, \beta)\), which belongs to \(\Omega_3\) if and only if \(\text{Re}(\beta) > -2/3\). Therefore the partition function converges precisely when \(\text{Re}(\beta) > -2/3\), and for these \(\beta\) it satisfies

\[
Z_3(\beta) = \int_{\mathbb{Z}_p^3} |x_1 - x_2|^{\beta} |x_1 - x_3|^{\beta} |x_2 - x_3|^{\beta} \, d\mu^3
\]

\[
= \frac{p^{3\beta}}{p^{2+3\beta} - 1} \cdot \left( (p - 1)(p - 2) + \frac{3(p - 1)^2}{p^{1+\beta} - 1} \right).
\]

Through this formula, \(Z_3\) extends to a function that is meromorphic on \(\mathbb{C}\) with poles belonging to the set

\[
-1 + \frac{2\pi i}{\log(p)} \mathbb{Z} \cup -\frac{2}{3} + \frac{2\pi i}{3\log(p)} \mathbb{Z}.
\]

2. STATEMENT OF RESULTS

2.1. The Projective Analogue

Our first main result is the following \(\mathbb{P}^1(K)\)-analogue of Proposition 1.8:

**Theorem 2.1.** Suppose \(N \geq 2\). The integral \(Z_N(\mathbb{P}^1(K), s)\) converges absolutely if and only if \(s \in \Omega_N\), and in this case its value is given by the finite sum

\[
Z_N(\mathbb{P}^1(K), s) = \frac{1}{(q + 1)^{N-1}} \sum_{n \in \mathbb{N}_N} \frac{q^{N + \sum_{i<s} 1 - \deg_n([N])}}{q^{1 - \deg_n([N])}} \prod_{\lambda \in \mathbb{B}(n)} \frac{(q - 1)^{\deg_n([N])}}{q^{\deg_n([N])} - 1}.
\]

The summand for each \(n \in \mathbb{R}_N\) is defined for all prime powers \(q\), as the denominator \(q + 1 - \deg_n([N])\) is cancelled by the factor \((q - 1)^{\deg_n([N])} - 1\) inside the product over \(\lambda \in \mathbb{B}(n)\).
Since we recently described $\Omega_3$, tabulated $\Re_3$, and wrote a formula for $Z_3(\mathbb{Z}_p, s)$ in Example 1.9, we can demonstrate Theorem 2.1 for $K = \mathbb{Q}_p$ and $N = 3$ with only a few simple modifications.

**Example 2.2.** The integral $Z_3(\mathbb{P}^1(\mathbb{Q}_p), s) =$

$$
\int_{(\mathbb{P}^1(\mathbb{Q}_p))^3} \delta([x_{1,0}: x_{1,1}], [x_{2,0}: x_{2,1}])^{s_{12}} \delta([x_{1,0}: x_{1,1}], [x_{3,0}: x_{3,1}])^{s_{13}} \delta([x_{2,0}: x_{2,1}], [x_{3,0}: x_{3,1}])^{s_{23}} \, dv^3
$$

converges absolutely if and only if the complex tuple $s = (s_{12}, s_{13}, s_{23})$ satisfies $\Re(s_{12}) > -1$, $\Re(s_{13}) > -1$, $\Re(s_{23}) > -1$, and $\Re(s_{12} + s_{13} + s_{23}) > -2$. For such $s$ it evaluates to

$$
Z_3(\mathbb{P}^1(\mathbb{Q}_p), s) = \frac{1}{(p+1)^2} : \frac{1}{p^{1+s_{12}+s_{13}+s_{23}} - 1} \left( (p-1)(p^{3+s_{12}+s_{13}+s_{23}} - 2) + (p-1)(p^{3+s_{12}+s_{13}+s_{23}} - 1) \left[ \frac{1}{p^{1+s_{12}} - 1} + \frac{1}{p^{1+s_{13}} - 1} + \frac{1}{p^{1+s_{23}} - 1} \right] \right).
$$

The evident similarities between Proposition 1.8 and Theorem 2.1 follow from explicit relationships between the metrics and measures on $K$ and those on $\mathbb{P}^1(K)$. These relationships also play a role in the forthcoming results, so they are worth recalling now. Note that $[x_0 : x_1] \neq [1 : 0]$ if and only if $x_1 \neq 0$, in which case $x = x_0/x_1$ is the unique element of $K$ satisfying $[x : 1] = [x_0 : x_1]$. Therefore the rule $\iota(x) := [x : 1]$ defines a bijection $\iota : K \to \mathbb{P}^1(K) \setminus \{[1 : 0]\}$ and relates the metric structures of $K$ and $\mathbb{P}^1(K)$ in a simple way: Given $x, y \in K$, (1.17) implies $\delta(\iota(x), [1 : 0]) = (\max\{1, |x|\})^{-1}$ and

$$
\delta(\iota(x), \iota(y)) = \begin{cases} |x - y| & \text{if } x, y \in R, \\ 1 & \text{if } x \in R \text{ and } y \notin R, \text{ or if } x \notin R \text{ and } y \in R, \\ |1/x - 1/y| & \text{if } x, y \notin R. \end{cases} \tag{2.1}
$$

Recall that the *normalized valuation* is the surjective continuous homomorphism $v_K : K^\times \to \mathbb{Z}$ defined by $v_K(x) := -\log_q |x|$ for $x \in K^\times$, then extended to $0 \in K$ via $v_K(0) := \infty$. Among many other uses, $v_K$ allows the strong triangle equality (i.e. $|x + y| = \max\{|x|, |y|\}$ whenever $|x| \neq |y|$) to be rewritten as $v_K(x + y) = \min\{v_K(x), v_K(y)\}$ whenever $v_K(x) \neq v_K(y)$. Using this fact, the definition of $B_v[x_0 : x_1]$ in (1.18), and the formula in (2.1), one easily verifies that for any $y \in K$ and $v \in \mathbb{Z}_{>0}$ we have

$$
\iota(y + \pi^v R) = \begin{cases} B_v(\iota(y)) & \text{if } y \in R, \\ B_v(\pi^{-v} \iota(y)) & \text{if } y \notin R. \end{cases} \tag{2.2}
$$

In other words, $\iota$ sends the open ball of radius $r \in (0, 1)$ centered at $y \in K$ onto the open ball of radius $r/\max\{1, |y|^2\}$ centered at $\iota(y) \in \mathbb{P}^1(K) \setminus \{[1 : 0]\}$. In particular, $\iota : K \to \mathbb{P}^1(K) \setminus \{[1 : 0]\}$ is a homeomorphism that restricts to an isometry on $R$ and a contraction on $K \setminus R$.

The map $\iota$ also relates the measures on $K$ and $\mathbb{P}^1(K)$ in a simple way: Given $v > 0$ and a complete set of representatives $y_1, \ldots, y_{q^v} \in R$ for the cosets of $\pi^v R \subset R$, applying (2.2) to the partition $R = (y_1 + \pi^v R) \sqcup \cdots \sqcup (y_{q^v} + \pi^v R)$ yields

$$
\iota(R) = B_v[y_1 : 1] \sqcup \cdots \sqcup B_v[y_{q^v} : 1].
$$

Therefore $PGL_2(R)$-invariance of $\nu$ (Lemma 1.6) implies $\nu(\iota(R)) = q^v \cdot \nu(B_v[0 : 1])$. On the other hand,

$$
\iota(K \setminus R) = \iota(\{x : |x| \geq q\}) = \{\iota(x) : \delta(\iota(x), [1 : 0]) \leq q^{-1}\} = B_1[1 : 0] \setminus \{[1 : 0]\}
$$

implies $\iota(R) \sqcup B_1[1 : 0] = \iota(R) \sqcup \iota(K \setminus R) \sqcup \{[1 : 0]\} = \mathbb{P}^1(K)$ and hence

$$
\nu(\iota(R)) + \nu(B_1[1 : 0]) = \nu(\mathbb{P}^1(K)) = 1.
$$
Since \( \nu(\iota(R)) = q \cdot \nu(B_1[1 : 0]) \), it follows that \( \nu(\iota(R)) = q/(q + 1) \), and hence every ball \( B_v[x_0 : x_1] \subset \mathbb{P}^1(K) \) with \( v > 0 \) has measure \( q^{-v} \cdot q/(q + 1) \). Combining this with (2.2), one concludes that the measure \( \nu \) on \( \mathbb{P}^1(K) \setminus \{[1 : 0] \} \) pulls back along \( \iota \) to an explicit measure on \( K \):

\[
\nu(\iota(M)) = \frac{q}{q + 1} \int_M (\max\{1, |x|^2\})^{-1} \, d\mu \quad \text{for any Borel subset } M \subset K. \tag{2.3}
\]

Finally, (1.16) and (2.2) give a nice refinement of \( \mathbb{P}^1(K) = \iota(R) \cup B_1[1 : 0] \) in terms of the \((q - 1)\)th roots of unity in \( K \), which should be understood as the projective analogue of (1.16):

\[
\mathbb{P}^1(K) = B_1[0 : 1] \cup B_1[1 : 1] \cup B_1[\xi : 1] \cup \cdots \cup B_1[\xi^{q-2} : 1] \cup B_1[1 : 0]. \tag{2.4}
\]

Indeed, all \( q + 1 \) of the parts in the partition are balls with measure \( 1/(q + 1) \) and radius \( q^{-1} \), and two points \([x_0 : x_1], [y_0 : y_1] \in \mathbb{P}^1(K) \) satisfy \( \delta([x_0 : x_1], [y_0 : y_1]) = 1 \) if and only if \([x_0 : x_1] \) and \([y_0 : y_1] \) belong to different parts. Note that \( \iota \) sends \( R^\times \) onto the “equator” \( \iota(R^\times) \), i.e. the set of points in \( \mathbb{P}^1(K) \) with \( \delta \)-distance 1 from both the “south pole” \([0 : 1]\) and the “north pole” \([1 : 0]\).

### 2.2. Relationships Between Grand Canonical Partition Functions

In this section we establish relationships between the grand canonical partition functions for log-Coulomb gases in \( R, P \), and \( \mathbb{P}^1(K) \). More precisely, we will find three simple equations that relate the functions \( Z(f, R, \beta) \), \( Z(f, P, \beta) \), and \( Z(f, \mathbb{P}^1(K), \beta) \) as defined in Definition 1.4. The definitions of \( R \) and \( P \) readily imply \( \text{diam}(R) = 1 \) and \( \text{diam}(P) = q^{-1} < 1 \), and one easily deduces \( \text{diam}(\mathbb{P}^1(K)) = 1 \) from (2.1). Thus, for any fixed \( \beta > 0 \), Lemma 1.5 implies that the series defining \( Z(f, R, \beta) \), \( Z(f, P, \beta) \), and \( Z(f, \mathbb{P}^1(K), \beta) \) are absolutely convergent for all \( f \in \mathbb{C} \). Sinclair recently found an elegant relationship between the first two, which is closely related to the partition of \( R \) in (1.16):

**Proposition 2.3** (The \( q \)th Power Law [19]). For all \( \beta > 0 \) and \( f \in \mathbb{C} \), we have

\[
Z(f, R, \beta) = (Z(f, P, \beta))^q.
\]

Roughly speaking, the \( q \)th Power Law states that a log-Coulomb gas in \( R \) exchanging energy and particles with a heat reservoir “factors” into \( q \) identical sub-gases (one in each coset of \( P \)) that exchange energy and particles with the reservoir. Since the series

\[
Z(f, R, \beta) = \sum_{N=0}^{\infty} Z_N(R, \beta) \frac{f^N}{N!} \quad \text{and} \quad Z(f, P, \beta) = \sum_{N=0}^{\infty} Z_N(P, \beta) \frac{f^N}{N!},
\]

converge (absolutely) for all \( f \in \mathbb{C} \), the series equation \( Z(f, R, \beta) = (Z(f, P, \beta))^q \) is equivalent to the coefficient equalities

\[
\frac{Z_N(R, \beta)}{N!} = \sum_{N_0 + \cdots + N_q = N} \prod_{k=0}^{q-1} \frac{Z_{N_k}(P, \beta)}{N_k!} \quad \text{for all } N \geq 0. \tag{2.5}
\]

The \( \beta = 1 \) case of (2.5) is given in [2], in which the positive number \( Z_N(R, 1)/N! \) is recognized as the probability that a random monic polynomial in \( R[x_1, \ldots, x_N] \) splits completely in \( R \). The more general \( \beta > 0 \) case given in [19] makes explicit use of the partition of \( R \) into cosets of \( P \) (as in (1.16)). In §3, we will use the analogous partition of \( \mathbb{P}^1(K) \) into \( q + 1 \) balls (recall (2.4)) to show that

\[
\frac{Z_N(\mathbb{P}^1(K), \beta)}{N!} = \sum_{N_0 + \cdots + N_q = N} \prod_{k=0}^{q} \left( \frac{q}{q + 1} \right)^{N_k} \frac{Z_{N_k}(P, \beta)}{N_k!} \quad \text{for all } \beta > 0 \text{ and } N \geq 0. \tag{2.6}
\]

As we saw for the \( q \)th Power Law, since the series \( Z(f, \mathbb{P}^1(K), \beta) \) also converges (absolutely) for all \( f \in \mathbb{C} \), the coefficient identities (2.6) are equivalent to our second main result:
Theorem 2.4 (The \((q + 1)\)th Power Law). For all \(\beta > 0\) and \(f \in \mathbb{C}\), we have
\[
Z(f, \mathbb{P}^1(K), \beta) = (Z(f, \mathbb{P}^1(R), \beta))^{q+1}.
\]

Like the \(q\)th Power Law, the \((q + 1)\)th Power Law roughly states that a log-Coulomb gas in \(\mathbb{P}^1(K)\) exchanging energy and particles with a heat reservoir “factors” into \(q + 1\) identical sub-gases in the balls \(B_1[0 : 1], B_1[1 : 1], B_1[\xi : 1], \ldots, B_1[\xi^{q-2} : 1], B_1[1 : 0]\) (each isometrically homeomorphic to \(P\)), with fugacity \(\frac{qf}{q+1}\). The \(q\)th Power Law allows the \((q + 1)\)th Power Law to be written more crudely as
\[
Z(f, \mathbb{P}^1(K), \beta) = Z(\frac{qf}{q+1}, R, \beta) \cdot Z(\frac{qf}{q+1}, P, \beta),
\] (2.7)

which is to say that the gas in \(\mathbb{P}^1(K)\) “factors” into two sub-gases: one in \(\iota(R)\) and one in \(B_1[1 : 0]\) (which are respectively isometrically homeomorphic to \(R\) and \(P\)), both with fugacity \(\frac{qf}{q+1}\).

2.3. Functional Equations and a Quadratic Recurrence

Although Proposition 1.8 and Theorem 2.1 provide explicit formulas for \(Z_N(R, \beta)\) and \(Z_N(\mathbb{P}^1(K), \beta)\), they are not efficient for computation because they require a complete list of reduced splitting chains of order \(N\). For a practical alternative, we take advantage of both Power Laws and the following ideas from \([2]\) and \([19]\).

Suppose \(\beta > 0\), and recall that the \(q\)th Power Law \(Z(f, R, \beta) = (Z(f, P, \beta))^q\) is an equality of absolutely convergent power series, valid for all \(f \in \mathbb{C}\). Thus we can apply \(Z(f, P, \beta) \cdot \frac{\partial}{\partial f}\) to the equation \(Z(f, R, \beta) = (Z(f, P, \beta))^q\) to get
\[
Z(f, P, \beta) \cdot \frac{\partial}{\partial f} Z(f, R, \beta) = q \cdot Z(f, R, \beta) \cdot \frac{\partial}{\partial f} Z(f, P, \beta),
\]
apply the derivatives term-by-term (by Lemma 1.5), then expand each side as power series in \(f\). The two resulting series must have the same coefficients because they both converge and agree for all \(f \in \mathbb{C}\). Explicitly, equating the coefficients of \(f^{N-1}\) gives
\[
\sum_{k=1}^{N} \frac{Z_{N-k}(R, \beta) Z_k(R, \beta)}{(N-k)! (k-1)!} = q \cdot \sum_{k=1}^{N} \frac{Z_{N-k}(R, \beta) Z_k(P, \beta)}{(N-k)! (k-1)!} \quad \text{for all } N \geq 1.
\] (2.8)

This equation is merely a restatement of \(\mu(R) = q \cdot \mu(P)\) when \(N = 1\), so suppose \(N \geq 2\). The identities \(Z_j(P, \beta) = q^{-j-\left(\frac{j}{2}\right)\beta} Z_j(R, \beta)\) follow easily from Definition 1.1 and the homothetic change of variables \(R^j \rightarrow (\pi R)^j = P^j\), allowing (2.8) to be rewritten completely in terms of \(R\). After doing so, gathering the two \(k = N\) summands on one side of the equation and gathering all of the \(k < N\) summands on the other side gives
\[
(1 - q^{1-N-(\frac{N}{2})\beta}) \frac{Z_N(R, \beta)}{(N-1)!} = \sum_{k=1}^{N-1} \frac{(1 - q^{1-N-(\frac{N}{2})\beta}) - q^{-(N-k)-(\frac{N}{2})\beta}}{(N-k)! (k-1)!} \frac{Z_{N-k}(R, \beta) Z_k(R, \beta)}{(N-k)! (k-1)!} \quad \text{for all } N \geq 1.
\] (2.9)

Since \(\beta > 0\), the factor \(1 - q^{1-N-(\frac{N}{2})\beta}\) is nonzero and can be divided out of both sides, and a careful re-distribution of powers of \(q\) provides the identities
\[
\frac{q^{1-k-(\frac{k}{2})\beta} - q^{-(N-k)-(\frac{N}{2})\beta}}{1 - q^{1-N-(\frac{N}{2})\beta}} = \frac{\sinh \left( \frac{\log(q)}{2} \left[ \left( N + \left(\frac{N}{2}\right)\beta \right) \left( 1 - \frac{2k}{N} \right) + 1 \right] \right)}{\sinh \left( \frac{\log(q)}{2} \left[ \left( N + \left(\frac{N}{2}\right)\beta \right) - 1 \right] \right)} \cdot \frac{q^{\frac{n}{2}(\frac{N}{2})\beta}}{\frac{q^{\frac{n}{2}(\frac{N}{2})\beta}}{q^{\frac{k}{2}(\frac{N}{2})\beta} \cdot q^{\frac{k}{2}(\frac{N}{2})\beta}}}
\]
for \(1 \leq k \leq N - 1\). Finally, applying these identities and a minor adjustment to the factorials, (2.9) becomes the recurrence
\[
\frac{Z_N(R, \beta)}{N! q^{\frac{N}{2}(\frac{N}{2})\beta}} = \sum_{k=1}^{N-1} \frac{\sinh \left( \frac{\log(q)}{2} \left[ \left( N + \left(\frac{N}{2}\right)\beta \right) \left( 1 - \frac{2k}{N} \right) + 1 \right] \right)}{\sinh \left( \frac{\log(q)}{2} \left[ \left( N + \left(\frac{N}{2}\right)\beta \right) - 1 \right] \right)} \cdot \frac{Z_{N-k}(R, \beta)}{(N-k)! q^{\frac{k}{2}(\frac{N}{2})\beta}} \cdot \frac{Z_k(R, \beta)}{k! q^{\frac{k}{2}(\frac{N}{2})\beta}}.
\]
The lefthand side is identically 1 if \( N = 0 \) or \( N = 1 \), so induction confirms that it is polynomial in ratios of hyperbolic sines for all \( N \geq 0 \). In particular, its dependence on \( q \) is carried only by the factor \( \log(q) \) appearing inside all of the hyperbolic sines, which motivates the following lemma.

**Lemma 2.5 (The Quadratic Recurrence).** Set \( F_0(t, \beta) = F_1(t, \beta) = 1 \) for all \( \beta \in \mathbb{C} \) and all \( t \in \mathbb{R} \). For \( N \geq 2 \), \( \Re(\beta) > -2/N \), and \( t \in \mathbb{R} \), define \( F_N(t, \beta) \) by the recurrence

\[
F_N(t, \beta) := \begin{cases} 
\sum_{k=1}^{N-1} k \cdot \frac{\sinh \left( \frac{1}{2} \left( N + \frac{N}{2} \beta \right) \left( 1 - \frac{2k}{N} \right) \right)}{\sinh \left( \frac{1}{2} \left( N + \left( \frac{N}{2} \beta \right) \right) \right)} \cdot F_{N-k}(t, \beta) \cdot F_k(t, \beta) & \text{if } t \neq 0, \\
\sum_{k=1}^{N-1} k \cdot \frac{\left( N + \frac{N}{2} \beta \right) \left( 1 - \frac{2k}{N} \right) + 1}{\left( N + \left( \frac{N}{2} \beta \right) \right) - 1} \cdot F_{N-k}(0, \beta) \cdot F_k(0, \beta) & \text{if } t = 0.
\end{cases}
\]

(a) For fixed \( N \geq 2 \) and fixed \( t \), the function \( \beta \mapsto F_N(t, \beta) \) is holomorphic for \( \Re(\beta) > -2/N \).

(b) For fixed \( N \geq 2 \) and fixed \( \beta \), the function \( t \mapsto F_N(t, \beta) \) is smooth and even on \( \mathbb{R} \).

To establish part (a), recall that if \( a, b, c, d \in \mathbb{C} \) are constants such that \( c \beta + d \in \mathbb{C} \setminus \pi i \mathbb{Z} \) whenever \( \Re(\beta) > -2/N \), then \( \beta \mapsto (a \beta + b)/(c \beta + d) \) and \( \beta \mapsto \sinh(a \beta + b)/\sinh(c \beta + d) \) are holomorphic for \( \Re(\beta) > -2/N \). The claim then follows by induction on \( N \) and the fact that the set of holomorphic functions (on \( \Re(\beta) > -2/N \)) is closed under addition and multiplication. To establish part (b), recall that if \( z, w \in \mathbb{C} \) are constants with \( \Re(z) > 0 \) and \( \Re(w) > 0 \), then the function \( h : \mathbb{R} \to \mathbb{C} \) defined by

\[
h(t) = \begin{cases} 
\sinh(tz)/\sinh(tw) & \text{if } t \neq 0, \\
z/w & \text{if } t = 0,
\end{cases}
\]

is smooth and even. The claim then follows by induction on \( N \) and the fact that smooth even functions on \( \mathbb{R} \) are also closed under addition and multiplication.

An interesting and immediate consequence of The Quadratic Recurrence and the preceding discussion is the formula

\[
Z_N(R, \beta) = N! q^{-\frac{1}{2}(\beta)} F_N(\log(q), \beta) \quad \text{for } N \geq 0 \text{ and } \beta > 0.
\]

The equation is obviously true for all \( \beta \in \mathbb{C} \) when \( N = 0 \) or \( N = 1 \), so suppose \( N \geq 2 \). Both sides are holomorphic for \( \Re(\beta) > -2/N \) (recall Proposition 1.8), so their agreement for \( \beta > 0 \) implies that they actually agree for \( \Re(\beta) > -2/N \). Thus, Lemma 2.5 provides a computationally efficient alternative to the “one-component case” of Proposition 1.8 (when \( s_{ij} = \beta \) for all \( i < j \)) and extends \( Z_N(R, \beta) \) to a smooth function of \( q \in (0, \infty) \). Moreover, it clarifies the following \( q \mapsto q^{-1} \) symmetry:

\[
Z_N(R, \beta)|_{q \rightarrow q^{-1}} = N! q^{-\frac{1}{2}(\beta)} F_N(\log(q^{-1}), \beta) = N! q^{-\frac{1}{2}(\beta)} F_N(\log(q), \beta) = q^{-\frac{1}{2}(\beta)} Z_N(R, \beta).
\]

The Quadratic Recurrence serves the projective analogue as well. For \( \beta > 0 \), expanding both sides of (2.7) into power series in \( f \) (which both converge for all \( f \in \mathbb{C} \)) yields the coefficient equations

\[
\frac{Z_N([1]^K, \beta)}{N!} = \sum_{k=0}^{N} \frac{q}{q+1}^N \frac{Z_{N-k}(R, \beta) Z_k(P, \beta)}{(N-k)! k!} \quad \text{for all } N \geq 0.
\]

The equation is obviously true for all \( \beta \in \mathbb{C} \) when \( N = 0 \) or \( N = 1 \), so suppose \( N \geq 2 \). Both sides are holomorphic for \( \Re(\beta) > -2/N \) by Proposition 1.8 and Theorem 2.1, and hence (2.10) is valid for \( \Re(\beta) > -2/N \). The identities \( Z_j(P, \beta) = q^{-j-\frac{1}{2}(\beta)} Z_j(R, \beta) \) and \( Z_j(R, \beta) = j! q^{j-\frac{1}{2}(\beta)} F_j(\log(q), \beta) \) allow the \( k \)th summand to be rewritten as

\[
\left( \frac{q}{q+1} \right)^N \frac{Z_{N-k}(R, \beta) Z_k(P, \beta)}{(N-k)! k!} = \frac{q^{\frac{1}{2}(N+j)(\beta)(1-\frac{2j}{N})}}{2 \cosh \left( \frac{\log(q)}{2} \right)^N} \cdot F_{N-k}(\log(q), \beta) \cdot F_k(\log(q), \beta).
\]
so adding two copies of the sum in (2.10) together, pairing the kth term of the first copy with the 
(N − k)th term of the second copy, and dividing by 2 gives

\[
\frac{Z_N(\mathbb{P}^1(K), \beta)}{N!} = \sum_{k=0}^{N} \frac{\cosh\left(\frac{\log(q)}{2} \left( N + \binom{N}{2} \beta \right) \left( 1 - \frac{2k}{N} \right) \right)}{(2 \cosh\left(\frac{\log(q)}{2}\right))^N} \cdot F_{N-k}(\log(q), \beta) \cdot F_k(\log(q), \beta).
\]

Through this formula, which is valid for \( \text{Re}(\beta) > -2/N \), we see that \( Z_N(\mathbb{P}^1(K), \beta) \) also extends to a 
smooth function of \( q \in (0, \infty) \) and is invariant under the involution \( q \mapsto q^{-1} \). We conclude this section 
by summarizing these observations:

**Theorem 2.6 (Efficient Formulas and Functional Equations).** Suppose \( N \geq 2 \) and \( \text{Re}(\beta) > -2/N \), 
and define \( (F_k(t, \beta))_{k=0}^N \) as in Lemma 2.5. The values of the one-component canonical partition 
functions for \( R \) and \( \mathbb{P}^1(K) \) are given by the respective formulas

\[
Z_N(R, \beta) = N! q^{\frac{1}{2}(N+1)\beta} F_N(\log(q), \beta) \quad \text{and} \quad Z_N(\mathbb{P}^1(K), \beta) = N! \sum_{k=0}^{N} \frac{\cosh\left(\frac{\log(q)}{2} \left( N + \binom{N}{2} \beta \right) \left( 1 - \frac{2k}{N} \right) \right)}{(2 \cosh\left(\frac{\log(q)}{2}\right))^N} \cdot F_{N-k}(\log(q), \beta) \cdot F_k(\log(q), \beta),
\]

which extend \( Z_N(R, \beta) \) and \( Z_N(\mathbb{P}^1(K), \beta) \) to smooth functions of \( q \in (0, \infty) \) satisfying

\[
Z_N(R, \beta)|_{q \mapsto q^{-1}} = q^{-\frac{1}{2}(N-1)\beta} Z_N(R, \beta) \quad \text{and} \quad Z_N(\mathbb{P}^1(K), \beta)|_{q \mapsto q^{-1}} = Z_N(\mathbb{P}^1(K), \beta).
\]

It should be noted here that the \( q \mapsto q^{-1} \) functional equation at left is a special case of the one proved 
in [3], and that both functional equations closely resemble the ones in [20].

**3. PROOFS OF THE MAIN RESULTS**

This section will establish the proofs of Theorems 2.1 and 2.4. The common step in both is a 
decomposition of \( (\mathbb{P}^1(K))^N \) into \( (q+1)^N \) cells that are isometrically homeomorphic to \( P^N \). This, 
combined with the metric and measure properties in §1.1.2, forms the key relationship between the 
canonical partition functions for \( X = \mathbb{P}^1(K) \) and \( X = P \). We will prove this relationship first, then 
conclude the proofs of Theorems 2.1 and 2.4 in their own subsections.

**3.1. Decomposing the Integral over \( (\mathbb{P}^1(K))^N \)**

We begin with an integer \( N \geq 2 \) that shall remain fixed for the rest of this section, reserve the symbol 
\( s \) for a complex tuple \( (s_{ij})_{1 \leq i < j \leq N} \), and fix the following notation to better organize the forthcoming 
arguments:

**Notation 3.1.** Let \( I \) be a subset of \( [N] = \{1, \ldots, N\} \).

- For any set \( X \) we write \( X^I \) for the product \( \prod_{i \in I} X = \{x_I = (x_i)_{i \in I} : x_i \in X\} \) and assume \( X^I \) has the product topology if \( X \) is a topological space.

- We write \( \mu^I \) for the product Haar measure on \( K^I \) satisfying \( \mu^I(R^I) = 1 \), and we make this 
consistent for \( I = \emptyset \) by giving the singleton space \( K^\emptyset = R^\emptyset = \{0\} \) measure 1. We also write 
\( \nu^I \) for the product measure on \( (\mathbb{P}^1(K))^I \), with the same convention for \( I = \emptyset \).
• For a compact subset $X \subset K$ we set $Z_{\emptyset}(X, s) := 1$ and

$$Z_I(X, s) := \int_X I \prod_{i \neq j \in I} |x_i - x_j|^{s_{ij}} \, d\mu^I$$

if $I \neq \emptyset$.

Note that $Z_I(X, s)$ is constant with respect to the entry $s_{ij}$ if $i \in [N] \setminus I$ or $j \in [N] \setminus I$, and it coincides with $Z_N(X, s)$ (as in Definition 1.1) if $I = [N]$.

• Using the constant $q = \#(R/P)$, we write $(I_0, \ldots, I_q) \vdash [N]$ for an ordered partition of $[N]$ into at most $q + 1$ parts. That is, $(I_0, \ldots, I_q) \vdash [N]$ means $I_0, \ldots, I_q$ are $q + 1$ disjoint ordered subsets of $[N]$ with union equal to $[N]$, where some $I_k$ may be empty.

In addition to the above, it will be useful to consider $I$-analogues of splitting chains:

**Definition 3.2.** Suppose $I \subset [N]$. An $I$-splitting chain of length $L \geq 0$ is a tuple $\mathfrak{h} = (\mathfrak{h}_0, \ldots, \mathfrak{h}_L)$ of partitions of $I$ satisfying the proper refinement condition

$$\{I\} = \mathfrak{h}_0 > \mathfrak{h}_1 > \mathfrak{h}_2 > \cdots > \mathfrak{h}_L = \{\{i\} : i \in I\}.$$  

If $\#I \geq 2$, we define $B(\mathfrak{h})$, $\ell_{\mathfrak{h}}(\lambda)$, and $\operatorname{deg}_{\mathfrak{h}}(\lambda) \in \{2, 3, \ldots, \#I\}$ just as in Definition 1.7. Otherwise $B(\mathfrak{h})$ will be treated as the empty set and there is no need to define $\ell_\mathfrak{h}$ or $\operatorname{deg}_\mathfrak{h}$. Finally, we call an $I$-splitting chain $\mathfrak{h}$ reduced if each $\lambda \in B(\mathfrak{h})$ satisfies $\lambda \in \mathfrak{h}_1 \iff \ell = \ell_{\mathfrak{h}}(\lambda)$, write $\mathcal{R}_I$ for the set of reduced $I$-splitting chains, and define

$$\Omega_I := \bigcap_{\lambda \in I \setminus \#I > 1} \{s : \operatorname{Re}(e_\lambda(s)) > 0\} \quad \text{where} \quad e_\lambda(s) := \#\lambda - 1 + \sum_{i < j} s_{ij}.$$  

Note that $\mathcal{R}_{\emptyset} = \emptyset$ because $I = \emptyset$ has no partitions, $\Omega_{\emptyset} = \mathbb{C}^{N(N - 1)/2}$ because $\Omega_{\emptyset}$ is an intersection of subsets of $\mathbb{C}^{N(N - 1)/2}$ taken over an empty index set, and $e_\emptyset(s) = -1$ for a similar reason. For each singleton $\{i\}$, the set $\mathcal{R}_{\{i\}}$ is comprised of a single splitting chain of length zero, $\Omega_{\{i\}} = \mathbb{C}^{N(N - 1)/2}$ for the same reason as the $I = \emptyset$ case, and similarly $e_{\{i\}}(s) = 0$. At the other extreme, taking $I = [N]$ in Definition 3.2 recovers Definition 1.7 and gives $\Omega_I = \Omega_N$.

**Proposition 3.3.** For any $v \in \mathbb{Z}$ and any nonempty subset $I \subset [N]$, the integral $Z_I(\pi^v R, s)$ converges absolutely if and only if $s \in \Omega_I$, and in this case

$$Z_I(\pi^v R, s) = \frac{1}{q^{(v-1)(e_\mathfrak{h}(s)+1)+\#I}} \sum_{\mathfrak{h} \in \mathcal{R}_I} \prod_{\lambda \in B(\mathfrak{h})} \frac{(q - 1)_{\operatorname{deg}_{\mathfrak{h}}(\lambda)-1}}{q^{e_\lambda(s)} - 1}. $$

In particular, we recover Proposition 1.8 by taking $I = [N]$ and $v = 0$.

**Proof.** First suppose $I$ is a singleton, so that the product inside the integral $Z_I(\pi^v R, s)$ is empty and hence

$$Z_I(\pi^v R, s) = \int_{(\pi^v R)^I} d\mu^I = \int_{\pi^v R} d\mu = q^{-v}. $$

This integral is constant, and hence absolutely convergent, for all $s \in \mathbb{C}^{N(N - 1)/2} = \Omega_I$. On the other hand, $\mathcal{R}_I$ consists of a single $I$-splitting chain, namely the one-tuple $\mathfrak{h} = \{\{I\}\}$. Then $B(\mathfrak{h}) = \emptyset$ and $e_I(s) = 0$ imply

$$\frac{1}{q^{(v-1)(e_\mathfrak{h}(s)+1)+\#I}} \sum_{\mathfrak{h} \in \mathcal{R}_I} \prod_{\lambda \in B(\mathfrak{h})} \frac{(q - 1)_{\operatorname{deg}_{\mathfrak{h}}(\lambda)-1}}{q^{e_\lambda(s)} - 1} = \frac{1}{q^{(v-1)(e_\emptyset(s)+1)+1}} \prod_{\lambda \in \emptyset} \frac{(q - 1)_{\operatorname{deg}_{\emptyset}(\lambda)-1}}{q^{e_\lambda(s)} - 1} = q^{-v}.$$
as well, so the claim holds for any singleton subset \( I \subset [N] \). Now suppose \( I \) is not a singleton. By relabeling \( I \) we may assume \( I = [n] \) where \( 2 \leq n \leq N \). By Proposition 3.15 and Lemma 3.16(c) in [21], the integral

\[
Z_I(R, s) = \mathcal{Z}_n(R, s) = \int_{R^n} \prod_{1 \leq i < j \leq n} |x_i - x_j|^{s_{ij}} \, d\mu^n
\]

(where \( \mu^n \) is the \( n \)-fold product measure on \( R^n \)) converges absolutely if and only if \( s \) belongs to the intersection

\[
\bigcap_{\Phi \in \mathcal{R}_n} \bigcap_{\lambda \in \mathcal{B}(\Phi)} \{ s : \text{Re} (\epsilon\lambda(s)) > 0 \},
\]

and for such \( s \) we have

\[
\mathcal{Z}_n(R, s) = q^{\epsilon\lambda(s)+1-n} \sum_{\Phi \in \mathcal{R}_n} \prod_{\lambda \in \mathcal{B}(\Phi)} \frac{(q - 1)^{\deg\lambda - 1}}{q^{\deg\lambda(s)} - 1}.
\]

Changing variables in the integral \( Z_n(R, s) \) by the homothety \( R^n \rightarrow (\pi^n R)^n \) gives

\[
\mathcal{Z}_N(\pi^n R, s) = q^{-\nu(\epsilon|N|(s)+1)} \cdot \mathcal{Z}_n(R, s) = \frac{1}{q^{(\nu-1)(\epsilon|N|(s)+1)+n}} \sum_{\Phi \in \mathcal{R}_n} \prod_{\lambda \in \mathcal{B}(\Phi)} \frac{(q - 1)^{\deg\lambda - 1}}{q^{\deg\lambda(s)} - 1},
\]

and the first equality implies that the domain of absolute convergence for \( \mathcal{Z}_n(\pi^n R, s) \) is also the intersection appearing in (3.1). But every subset \( \lambda \subset [n] \) with \( \# \lambda > 1 \) appears as a branch in at least one reduced splitting chain of order \( n \), so the intersection in (3.1) is precisely \( \Omega([n]) \). Therefore the claim holds for \( I = [n] \), and hence for any non-singleton subset \( I \subset [N] \).

Theorem 3.4. Suppose \( N \geq 2 \). The integral \( \mathcal{Z}_N(\mathbb{P}^1(K), s) \) converges absolutely if and only if \( s \in \Omega_N \), and in this case its value is

\[
\mathcal{Z}_N(\mathbb{P}^1(K), s) = \left( \frac{q}{q + 1} \right)^N \sum_{(I_0, \ldots, I_q) \in [N]} \prod_{k=0}^q \mathcal{Z}_{I_k}(P, s).
\]

Proof. The partition of \( \mathbb{P}^1(K) \) in (2.4) can be rewritten in the form

\[
\mathbb{P}^1(K) = \bigcup_{k=0}^q \phi_k(B_1[0 : 1]),
\]

where \( \phi_k \in PGL_2(R) \) is the element represented by \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) if \( k = 0 \), \( \begin{pmatrix} 1 & k^{-1} \\ 0 & 1 \end{pmatrix} \) if \( 0 < k < q \), or \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) if \( k = q \). This leads to a partition of the \( N \)-fold product,

\[
(\mathbb{P}^1(K))^N = \bigcup_{(I_0, \ldots, I_q) \in [N]} C(I_0, \ldots, I_q),
\]

where each part is a “cell” of the form

\[
C(I_0, \ldots, I_q) := \{ ([x_0 : x_1], \ldots, [x_N : x_{N+1}] ) \in (\mathbb{P}^1(K))^N : [x_i : x_i] \in \phi_k(B_1[0 : 1]) \iff i \in I_k \}
= \prod_{k=0}^q (\phi_k(B_1[0 : 1]))^I_k.
\]
Accordingly, the integral $\int_{\mathbb{P}^1(K)} \mathbb{Z}_N(P, s) \, d\nu^N$, (3.2) summed over all $(I_0, \ldots, I_q) \leadsto [N]$. Since every cell $C(I_0, \ldots, I_q)$ has positive measure, the integral $\mathbb{Z}_N(P, s)$ converges absolutely if and only if the integral in (3.2) converges absolutely for all $(I_0, \ldots, I_q) \leadsto [N]$. Fix one $(I_0, \ldots, I_q)$ for the moment. By (2.4) and the definition of the $\phi_k$'s above, note that the entries of each tuple $([x_{i,0} : x_{i,1}], \ldots, [x_{N,0} : x_{N,1}] \in C(I_0, \ldots, I_q)$ satisfy $\delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}}$ if and only if $i$ and $j$ belong to different parts of $(I_0, \ldots, I_q)$. Therefore the integrand in (3.2) factors as

$$
\prod_{1 \leq i < j \leq N} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}} = \prod_{k=0}^q \prod_{i,j \in I_k} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}},
$$

and the measure on $C(I_0, \ldots, I_q)$ factors in a similar way, namely $\prod_{k=0}^q d\nu^I_k$ where $\nu^I_k$ is the product measure on $(\mathbb{P}^1(K))^I_k$. Now Fubini’s Theorem for positive functions and $PGL_2(R)$-invariance give

$$
\int_{\mathbb{P}^1(K)} \prod_{1 \leq i < j \leq N} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}} \, d\nu^N
$$

$$
= \prod_{k=0}^q \int_{(B_1[0;1])^{I_k}} \prod_{i,j \in I_k} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}} \, d\nu^I_k
$$

$$
= \prod_{k=0}^q \int_{(B_1[0;1])^{I_k}} \prod_{i,j \in I_k} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}} \, d\nu^I_k,
$$

so the integral in (3.2) converges absolutely if and only if all $q + 1$ of the integrals of the form

$$
\int_{(B_1[0;1])^{I_k}} \prod_{i,j \in I_k} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}} \, d\nu^I_k
$$

(3.3) converge absolutely. The change of variables $P^I_k \to (B_1[0;1])^{I_k}$ given by $\iota : P \to B_1[0;1]$ in each coordinate, along with (2.1), (2.2), and (2.3), allows the integral in (3.3) to be rewritten as $\frac{q}{q+1}^{\#I_k} \mathbb{Z}_{I_k}(P, s)$, and thus Proposition 3.3 implies that it converges absolutely if and only if $s \in \Omega_{I_k}$. Therefore the integral over $C(I_0, \ldots, I_q)$ in (3.2) converges absolutely if and only if $s \in \Omega_{I_0} \cap \cdots \cap \Omega_{I_q}$, and in this case Fubini’s Theorem for absolutely integrable functions, $PGL_2(R)$-invariance, and the change of variables above allow it to be rewritten as

$$
\int_{C(I_0, \ldots, I_q)} \prod_{1 \leq i < j \leq N} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}} \, d\nu^N
$$

$$
= \prod_{k=0}^q \int_{(B_1[0;1])^{I_k}} \prod_{i,j \in I_k} \delta([x_{i,0} : x_{i,1}], [x_{j,0} : x_{j,1}])^{s_{ij}} \, d\nu^I_k
$$

$$
= \prod_{k=0}^q \left( \frac{q}{q+1} \right)^{\#I_k} \mathbb{Z}_{I_k}(P, s)
$$

$$
= \left( \frac{q}{q+1} \right)^N \prod_{k=0}^q \mathbb{Z}_{I_k}(P, s).
$$
Finally, since $Z_N(\mathbb{P}^1(K), s)$ is the sum of these integrals over all $(I_0, \ldots, I_q) \vdash [N]$, it converges absolutely if and only if
\[
s \in \bigcap_{(I_1, \ldots, I_q) \vdash [N]} (\Omega_{I_1} \cap \cdots \cap \Omega_{I_q}) = \bigcap_{I \subset [N]} \Omega_I.
\]
The last equality of intersections holds because each subset $I \subset [N]$ with $\#I > 1$ appears as a part in at least one of the ordered partitions $(I_1, \ldots, I_q) \vdash [N]$, and because none of the parts with $\#I_k \leq 1$ affect the intersection (because $\Omega_{I_k} = \mathbb{C}^{N(N-1)/2}$ for such $I_k$). The intersection of $\Omega_I$ over all $I \subset [N]$ with $\#I > 1$ is clearly equal to $\Omega_{[N]} = \Omega_N$ by Definition 3.2, so the proof is complete. 

3.2. Finishing the Proof of Theorem 2.1

Theorem 3.4 established that the integral $Z_N(\mathbb{P}^1(K), s)$ converges absolutely if and only if $s \in \Omega_N$, and for such $s$ it gave
\[
Z_N(\mathbb{P}^1(K), s) = \left(\frac{q}{q+1}\right)^N \sum_{(I_0, \ldots, I_q) \vdash [N]} \prod_{k=0}^q Z_{I_k}(P, s).
\]
(3.4)

It remains to show that the righthand sum can be converted into the sum over $n \in \mathbb{R}_N$ proposed in Theorem 2.1.

Proof of Theorem 2.1. We begin by breaking the terms of the sum in (3.4) into two main groups. The simpler group is indexed by those $(I_0, \ldots, I_q)$ with $I_j = [N]$ for some $j$ and $I_k = \emptyset$ for all $k \neq j$, in which case $Z_{I_k}(P, s) = Z_N(P, s)$ and $Z_{I_k}(P, s) = 1$ for all $k \neq j$. Therefore each of the group’s $q + 1$ terms (one for each $j \in \{0, \ldots, q\}$) contributes the quantity $\prod_{k=0}^q Z_{I_k}(P, s) = Z_N(P, s)$ to the sum in (3.4) for a total contribution of
\[
(q + 1)Z_N(P, s) = \frac{q + 1}{q^N} \sum_{n \in \mathbb{R}_N} \prod_{\lambda \in \mathcal{B}(n)} \frac{(q - 1)^{\deg_{\lambda}(\lambda) - 1}}{q^{\deg_{\lambda}(\lambda)} - 1}
\]
by the $v = 1$ and $I = [N]$ case of Proposition 3.3. The other group of terms is indexed by the ordered partitions $(I_0, \ldots, I_q) \vdash [N]$ satisfying $I_0, \ldots, I_q \subset [N]$. To deal with them carefully, we fix one such $(I_0, \ldots, I_q)$ and for the moment, note that the number $d$ of nonempty parts $I_k$ must be at least 2. Thus we have indices $k_1, \ldots, k_d \in \{0, \ldots, q\}$ with $I_{k_j} \neq \emptyset$, and for every $k \in \{0, \ldots, q\} \setminus \{k_1, \ldots, k_d\}$ we have $I_k = \emptyset$ and hence $Z_{I_k}(P, s) = 1$. For the nonempty sets $I_{k_j}$, Proposition 3.3 expands $Z_{I_{k_j}}(P, s)$ as a sum over $\mathbb{R}_{I_{k_j}}$ (whose elements shall be denoted $\mathfrak{n}_{k_j}$ instead of $n$) and hence
\[
\prod_{k=0}^q Z_{I_k}(P, s) = \prod_{j=1}^d \frac{1}{q^{\#I_{k_j}}} \sum_{\mathfrak{n}_{k_j} \in \mathbb{R}_{I_{k_j}}} \prod_{\lambda \in \mathcal{B}(\mathfrak{n}_{k_j})} \frac{(q - 1)^{\deg_{\lambda}(\lambda) - 1}}{q^{\deg_{\lambda}(\lambda)} - 1}
\]
\[
= \frac{1}{q^N} \sum_{(\mathfrak{n}_1, \ldots, \mathfrak{n}_d) \in \mathbb{R}_{I_{k_1}} \times \cdots \times \mathbb{R}_{I_{k_d}}} \prod_{j=1}^d \prod_{\lambda \in \mathcal{B}(\mathfrak{n}_{k_j})} \frac{(q - 1)^{\deg_{\lambda}(\lambda) - 1}}{q^{\deg_{\lambda}(\lambda)} - 1}
\]
\[
= \frac{1}{q^N} \sum_{(\mathfrak{n}_1, \ldots, \mathfrak{n}_d) \in \mathbb{R}_{I_{k_1}} \times \cdots \times \mathbb{R}_{I_{k_d}}} \prod_{\lambda \in \mathcal{B}(\mathfrak{n}_{k_1} \cup \cdots \cup \mathfrak{n}_{k_d})} \frac{(q - 1)^{\deg_{\lambda}(\lambda) - 1}}{q^{\deg_{\lambda}(\lambda)} - 1}.
\]

We now make use of a simple correspondence between the tuples $(\mathfrak{n}_1, \ldots, \mathfrak{n}_d) \in \mathbb{R}_{I_{k_1}} \times \cdots \times \mathbb{R}_{I_{k_d}}$ and the reduced splitting chains $\mathfrak{n} = (\mathfrak{n}_0, \mathfrak{n}_1, \ldots, \mathfrak{n}_d) \in \mathbb{R}_N$ satisfying $\mathfrak{n}_1 = \{I_{k_1}, \ldots, I_{k_d}\}$. To establish it, note that each $\mathfrak{n} \in \mathbb{R}_N$ corresponds uniquely to its branch set $\mathcal{B}(\mathfrak{n})$ (Lemma 2.5(b) of [21]), which generalizes in an obvious way to reduced $I$-splitting chains (for any nonempty $I \subset [N]$). Now if $\mathfrak{n} =
\((\mathfrak{n}_0, \mathfrak{n}_1, \ldots, \mathfrak{n}_L) \in \mathcal{R}_N\) satisfies \(\mathfrak{n}_1 = \{I_{k_1}, \ldots, I_{k_d}\}\), the corresponding branch set \(\mathcal{B}(\mathfrak{n})\) decomposes as

\[\mathcal{B}(\mathfrak{n}) = \{[N]\} \cup \bigsqcup_{j=1}^{d} \{\lambda \in \mathcal{B}(\mathfrak{n}) : \lambda \subseteq I_{k_j}\}.\]

Each of the sets \(\{\lambda \in \mathcal{B}(\mathfrak{n}) : \lambda \subseteq I_{k_j}\}\) is the branch set \(\mathcal{B}(\mathfrak{n}_j)\) for a unique \(\mathfrak{n}_j \in \mathcal{R}_{I_{k_j}}\), so in this sense \(\mathfrak{n}\) “breaks” into a unique tuple \((\mathfrak{n}_1, \ldots, \mathfrak{n}_d) \in \mathcal{R}_{I_{k_1}} \times \cdots \times \mathcal{R}_{I_{k_d}}\). On the other hand, any tuple \((\mathfrak{n}_1, \ldots, \mathfrak{n}_d) \in \mathcal{R}_{I_{k_1}} \times \cdots \times \mathcal{R}_{I_{k_d}}\) can be “assembled” as follows. Since \(\{I_{k_1}, \ldots, I_{k_d}\}\) is a partition of \([N]\), taking the union of the \(d\) branch sets \(\mathcal{B}(\mathfrak{n}_1), \ldots, \mathcal{B}(\mathfrak{n}_d)\) and the singleton \([\{N]\}\) forms the branch set \(\mathcal{B}(\mathfrak{n})\) for a unique \(\mathfrak{n} \in \mathcal{R}_N\). It is clear that “breaking” and “assembling” are inverses, giving a correspondence \(\mathfrak{n} \in \mathcal{R}_N : \mathfrak{n}_1 = \{I_{k_1}, \ldots, I_{k_d}\} \leftrightarrow \mathcal{R}_{I_{k_1}} \times \cdots \times \mathcal{R}_{I_{k_d}}\) under which each identification \(\mathfrak{n} \leftrightarrow (\mathfrak{n}_1, \ldots, \mathfrak{n}_d)\) amounts to a branch set equation, i.e.,

\[\mathcal{B}(\mathfrak{n}) \setminus \{[N]\} = \mathcal{B}(\mathfrak{n}_1) \sqcup \cdots \sqcup \mathcal{B}(\mathfrak{n}_d).\]

In particular, each \(\lambda \in \mathcal{B}(\mathfrak{n}) \setminus \{[N]\}\) is contained in exactly one \(\mathcal{B}(\mathfrak{n}_j)\), and \(\deg_\mathfrak{n}(\lambda) = \deg_{\mathfrak{n}_j}(\lambda)\) by Definition 1.7 in this case. These facts allow the sum over \(\mathcal{R}_{I_1} \times \cdots \times \mathcal{R}_{I_d}\) above to be rewritten as a sum over all \(\mathfrak{n} \in \mathcal{R}_N\) with \(\mathfrak{n}_1 = \{I_{k_1}, \ldots, I_{k_d}\}\), and each product over \(\lambda \in \mathcal{B}(\mathfrak{n}_1) \sqcup \cdots \sqcup \mathcal{B}(\mathfrak{n}_d)\) inside it is simply a product over \(\lambda \in \mathcal{B}(\mathfrak{n}) \setminus \{[N]\}\). We conclude that an ordered partition \((I_0, \ldots, I_q) \vdash [N]\) with \(I_0, \ldots, I_q \subseteq [N]\) contributes the quantity

\[\prod_{k=0}^{q} \frac{1}{q^{N_k}} \sum_{\mathfrak{n}_i \in \mathcal{R}_{I_{k_i}}} \prod_{\lambda \in \mathcal{B}(\mathfrak{n}) \setminus \{[N]\}} \frac{(q - 1)^{\deg_\mathfrak{n}(\lambda) - 1}}{q^{\deg_\lambda(s) - 1}} \quad (3.6)\]

to the sum in (3.4), where \((I_{k_1}, \ldots, I_{k_d})\) is the (unordered) subset of nonempty parts in that particular ordered partition. We must now sum the contribution in (3.6) over all possible \((I_0, \ldots, I_q) \vdash [N]\) with \(I_0, \ldots, I_q \subseteq [N]\). Given a partition \(\{\lambda_1, \ldots, \lambda_d\} \vdash [N]\) with \(d \geq 2\), note that there are precisely \((q + 1)_d = (q + 1) \cdot (q + 1) \cdot \cdots \cdot (q + 1)\) ordered partitions \((I_0, \ldots, I_q) \vdash [N]\) such that \(\{I_{k_1}, \ldots, I_{k_d}\} = \{\lambda_1, \ldots, \lambda_d\}\). Therefore summing (3.6) over all \((I_0, \ldots, I_q) \vdash [N]\) with \(I_0, \ldots, I_q \subseteq [N]\) gives

\[\sum_{(I_0, \ldots, I_q) \vdash \{[N]\}} \prod_{k=0}^{q} \frac{1}{q^{N_k}} \sum_{\mathfrak{n}_i \in \mathcal{R}_{I_{k_i}}} \prod_{\lambda \in \mathcal{B}(\mathfrak{n}) \setminus \{[N]\}} \frac{(q - 1)^{\deg_\mathfrak{n}(\lambda) - 1}}{q^{\deg_\lambda(s) - 1}} = \frac{q + 1}{q^N} \sum_{\lambda_1, \ldots, \lambda_d \vdash [N]} \prod_{\lambda \in \mathcal{B}(\mathfrak{n}) \setminus \{[N]\}} \frac{(q - 1)^{\deg_\mathfrak{n}(\lambda) - 1}}{q^{\deg_\lambda(s) - 1}}\]

Given a partition \(\{\lambda_1, \ldots, \lambda_d\} \vdash [N]\), those splitting chains \(\mathfrak{n} \in \mathcal{R}_N\) with \(\mathfrak{n}_1 = \{\lambda_1, \ldots, \lambda_d\}\) all have \(\deg_\mathfrak{n}([N]) = \#\mathfrak{n}_1 = d\) by Definition 1.7. Moreover, no \(\mathfrak{n} \in \mathcal{R}_N\) is missed or repeated in the sum of sums above, so it can be rewritten as

\[\sum_{(I_0, \ldots, I_q) \vdash \{[N]\}} \prod_{k=0}^{q} \frac{1}{q^{N_k}} \sum_{\mathfrak{n}_i \in \mathcal{R}_{I_{k_i}}} \prod_{\lambda \in \mathcal{B}(\mathfrak{n}) \setminus \{[N]\}} \frac{(q - 1)^{\deg_\mathfrak{n}(\lambda) - 1}}{q^{\deg_\lambda(s) - 1}}\]

\[= \frac{q + 1}{q^N} \sum_{\mathfrak{n} \in \mathcal{R}_N} \left(\frac{(q - 1)^{\deg_\mathfrak{n}([N]) - 1}}{q^{\deg_\mathfrak{n}([N]) - 1}}\right) \prod_{\lambda \in \mathcal{B}(\mathfrak{n}) \setminus \{[N]\}} \frac{(q - 1)^{\deg_\mathfrak{n}(\lambda) - 1}}{q^{\deg_\lambda(s) - 1}}\]

\[= \frac{q + 1}{q^N} \sum_{\mathfrak{n} \in \mathcal{R}_N} \left(\frac{q^{N + \sum_{i<j} s_{ij} - q}}{q^{1 - \deg_\mathfrak{n}([N])}}\right) \prod_{\lambda \in \mathcal{B}(\mathfrak{n}) \setminus \{[N]\}} \frac{(q - 1)^{\deg_\mathfrak{n}(\lambda) - 1}}{q^{\deg_\lambda(s) - 1}}.\]
Note that the summand for each $h \in \mathcal{R}_N$ is still defined for any prime power $q$ since the denominators $(q - 1)^{\text{deg}_h([N])} - 1$ and $q + 1 - \text{deg}_h([N])$ (which vanish when $q = \text{deg}_h([N]) - 1$) are cancelled by the numerator $(q - 1)^{\text{deg}_h([N])} - 1$ appearing in the product over $\lambda \in \mathcal{B}(h)$. Finally, we evaluate the righthand side of (3.4) by combining the sum directly above with that in (3.5) and multiplying through by $(\frac{q}{q+1})^N$. This yields the desired formula for $Z_N(\mathbb{P}^1(K), s)$:

\[ Z_N(\mathbb{P}^1(K), s) = \frac{1}{(q + 1)^N - 1} \sum_{h \in \mathcal{R}_N} \left( 1 + \frac{q^{N + \sum_{i < j} s_{ij} - q}}{q + 1 - \text{deg}_h([N])} \right) \prod_{\lambda \in \mathcal{B}(h)} \frac{(q - 1)^{\text{deg}_h([N])} - 1}{q^{\lambda(s)} - 1} \]

\[ = \frac{1}{(q + 1)^N - 1} \sum_{h \in \mathcal{R}_N} q^{N + \sum_{i < j} s_{ij} - q} \prod_{\lambda \in \mathcal{B}(h)} \frac{(q - 1)^{\text{deg}_h([N])} - 1}{q^{\lambda(s)} - 1} \]

\[ \square \]

3.3. Finishing the Proof of Theorem 2.4

Our final task is to prove the $(q + 1)$th Power Law, which we noted in §2.2 is equivalent to the equations in (2.6). That is, it remains to prove

\[ \frac{Z_N(\mathbb{P}^1(K), \beta)}{N!} = \sum_{N_0 + \cdots + N_q = N} \prod_{k=0}^{q} \left( \frac{q}{q + 1} \right)^{N_k} Z_{N_k}(P, \beta) = \frac{1}{N!} N_{N_0! \cdots N_q!} \]

For all $\beta > 0$ and $N \geq 0$.

Proof. Fix $N \geq 0$ and $\beta > 0$, and fix $s$ via $s_{ij} = \beta$ for all $i < j$, so that $Z_N(\mathbb{P}^1(K), s) = Z_N(\mathbb{P}^1(K), \beta)$ and $Z_I(P, s) = Z_{\#I}(P, \beta)$ for any subset $I \subset [N]$. The formula in §3.4 relates these functions of $\beta$ via

\[ Z_N(\mathbb{P}^1(K), \beta) = Z_N(\mathbb{P}^1(K), s) \]

\[ = \left( \frac{q}{q + 1} \right)^N \sum_{(I_0, \ldots, I_q) \subseteq [N]} \prod_{k=0}^{q} Z_{\#I_k}(P, s) \]

\[ = \sum_{(I_0, \ldots, I_q) \subseteq [N]} \prod_{k=0}^{q} \left( \frac{q}{q + 1} \right)^{\#I_k} Z_{\#I_k}(P, \beta). \]

For each choice of $q + 1$ ordered integers $N_0, \ldots, N_q \geq 0$ satisfying $N_0 + \cdots + N_q = N$, there are precisely

\[ \binom{N}{N_0, \ldots, N_q} = \frac{N!}{N_0! \cdots N_q!} \]

ordered partitions $(I_0, \ldots, I_q) \subseteq [N]$ satisfying $\#I_0 = N_0, \ldots, \#I_q = N_q$. Finally, grouping ordered partitions according to all possible ordered integer choices establishes the desired equation:

\[ Z_N(\mathbb{P}^1(K), s) = \frac{1}{N!} \cdot \sum_{N_0 + \cdots + N_q = N} \prod_{k=0}^{q} \left( \frac{q}{q + 1} \right)^{N_k} Z_{N_k}(P, \beta) \]

\[ = \frac{1}{N!} \cdot \sum_{N_0 + \cdots + N_q = N} \binom{N}{N_0, \ldots, N_q} \prod_{k=0}^{q} \left( \frac{q}{q + 1} \right)^{N_k} Z_{N_k}(P, \beta) \]

\[ = \sum_{N_0 + \cdots + N_q = N} \prod_{k=0}^{q} \left( \frac{q}{q + 1} \right)^{N_k} Z_{N_k}(P, \beta). \]

\[ \square \]
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