Semi-semiclassical theory of quantum quenches in one dimensional systems

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We develop a hybrid semiclassical method to study the dynamics of one dimensional systems in and out of equilibrium. Our method handles internal degrees of freedom completely quantum mechanically, and accounts efficiently for entanglement entropy generation by these. In non-equilibrium situations, we can follow time evolution up to timescales at which local thermalization occurs. As an application, we investigate the quench dynamics and phase fluctuations of a pair of tunnel coupled one dimensional Bose condensates described by the sine–Gordon model.

Fundamental questions concerning the coherent time evolution, relaxation and thermalization of isolated quantum systems have been brought into the focus of attention by recent progress in experimental techniques [1–4]. Experiments on cold atomic gases allow us to engineer a broad range of lattice and continuum Hamiltonians in a controlled fashion, and to monitor the coherent time evolution of these systems through measuring multi-point correlation functions [5], accessing the quantum state via site-resolved quantum microscopy [6], and even measuring the entanglement properties of the system [7]. These experiments call for the development of new analytical and numerical methods that are able to describe non-equilibrium dynamics in closed quantum systems.

The most efficient numerical methods are so far the matrix product state (MPS) based time dependent Density Matrix Renormalization Group (t-DMRG) [8, 9] and Time Evolving Block Decimation (TEBD) [10] methods, both applicable for one dimensional lattice models. In most cases, they are both limited by finite size effects, and seem to work only for short times preceding thermalization. Moreover, there is no single efficient method to simulate non-equilibrium dynamics in continuum systems, relevant for ongoing matter wave interferometry [11], although Hamiltonian truncation methods [12] may offer a promising direction.

A successful analytical method to describe the dynamics of a broad range of gapped lattice and continuum systems in and out of equilibrium is the so-called semiclassical (SC) approach, originally developed to study the dynamics of finite temperature systems in equilibrium [13–15] and later successfully applied to quantum quenches [16–19], i.e. to non-equilibrium situations in which the system evolves unitarily starting from some prepared initial state. The SC method is applicable in gapped systems whenever the quasiparticles’ Compton wavelength (or the thermal wavelength) is much shorter than their average separation, $d = \rho^{-1}$, with $\rho$ the quasiparticles density. If the energy of the initial state is not too high, it acts as a weak source of almost pointlike quasiparticles, following classical trajectories (see Fig. 1.a). Quantum mechanical effects become relevant only when they collide. In the standard SC treatment, quasiparticles are furthermore assumed to move slowly and thus collisions are described by a universal purely reflective scattering matrix [20]. This universal SC (uSC) approach permits the derivation of precious analytical results [14, 15, 18, 19], but also suffers from a few artifacts [15, 19]; certain correlation functions and expectation values fail to decay and internal degrees of freedom remain just locally entangled.

Here we propose a novel method we dub ‘semi-
semiclassical” (sSC), where we hybridize the standard SC approach with TEBD: we compute the time evolution of the internal degrees of freedom completely quantumechanically using TEBD, while treating the orbital motion semiclassically. Our sSC method captures efficiently thermalization and entropy production, can be applied to any gapped system (integrable or not), and is able to reach time scales much beyond conventional TDMRG and TEBD. As a proof of principle, we use our method to describe quantum quenches in the sine-Gordon model, relevant for two coupled 1D quasi-condensates (see Fig. 1b), and studied intensely theoretically [21–24] and in matter wave interference experiments using nano-fabricated atom chips [3, 25].

Model.— The sine-Gordon model is defined as

$$H = \frac{\hbar c}{2} \int dx \left[ \frac{2\pi}{K} \Pi(x)^2 + \frac{K}{2\pi} \left[ \partial_x \varphi(x) \right]^2 \right] - \int dx 2\Delta_0 \cos[\varphi(x)], \quad (1)$$

with $\Pi(x)$ denoting the field conjugate to $\varphi(x)$, and $c$ the speed of sound. In case of coupled condensates, $\varphi(x) = \varphi_1(x) - \varphi_2(x)$ represents the relative phase of the condensates, $K$ stands for the Luttinger parameter of each condensate, and the cosine term accounts for Josephson tunneling between them. For locally interacting bosons $K \geq 1$, with $K \gg 1$ corresponding to weakly interacting bosons. The cosine perturbation is relevant for $K > 1/4$, and for $K < 1/2$ there are only kink excitations in the system, solitons and antisolitons. In the classical field theory, these kinks interpolate between neighboring minima of the cosine potential and have topological charges $\sigma = \pm \sigma_0$ corresponding to changes $\varphi \to \varphi \pm 2\pi$, respectively. For $K > 1/2$ their bound states, the so-called breathers are also present. However, having no topological charge, breathers are supposed to be irrelevant for the phase correlations studied here, and shall therefore be neglected in what follows.

We illustrate our method on the out of equilibrium time evolution of the coupled 1D condensates after changing the potential barrier that separates them. For small changes in $\Delta_0$, the gas of quasi-particles (kinks) created in the quench will be dilute and the SC method can be applied. Furthermore, by momentum conservation, a homogeneous but spatially localized perturbation gives rise to pairs of kinks flying away from each other with the same velocity and opposite topological charge. Thus the post-quench state will be populated by independent soliton-antisoliton pairs created with a velocity distribution $f(v)$. The precise form of $f(v)$ depends on details of the quench protocol, but turns out to be unimportant in the present calculation. The initial state considered here resembles to the ones appearing in previous studies of split condensates [26, 27] as well as other systems [23, 28–33], where the initial state was taken to be a coherent superposition of uncorrelated zero momentum quasiparticle pairs.

Within our sSC approach, however, phase coherence of the pairs does not play any role.

The semi-semiclassical (sSC) method.— In the SC approach, the quantum mechanical average is replaced by an ensemble average over initial positions and velocities of soliton-antisoliton pairs. The corresponding semiclassical configurations consist of space-time diagrams with pairs of straight lines (kink trajectories) originating from the $t = 0$ axis, distributed independently from each other, uniformly in space with random slopes corresponding to the distribution $f(v)$ (see Fig. 1a), and an initial distribution of paired charges.

Since quantum mechanical effects become relevant only at collisions, we can approximately factorize the many body wave function into an orbital and a charge part of the form $|\Psi(x, \sigma, t)\rangle \approx |\psi_{\text{orb}}(x, t)\rangle \otimes |\chi(\sigma, t)\rangle$, where

$$|\Psi_{\text{orb}}(x, t)\rangle = S \int d^N x \prod_{j=1}^N \delta(x_j - x_j^0 - v_j t)|x_1, \ldots, x_N\rangle,$$

$$|\chi(\sigma, t)\rangle = \sum_{\sigma_1, \sigma_2, \ldots} A_{\sigma_1, \sigma_2, \ldots} |\sigma_1, \sigma_2, \ldots \sigma_N\rangle. \quad (2a)$$

Here $x_j^0$ denotes the initial position of the $j$th kink of velocity $v_j$ and topological charge $\sigma_j$, and $S$ stands for symmetrization.

In the original uSC approach charges are treated classically: quasi-particles scatter as impenetrable billiard balls while preserving their charges. This follows from the perfectly reflective universal low energy two-body S-matrix, $S^{\sigma_1, \sigma_2} = (-1)^{\delta_{\sigma_1, \sigma_2}} \delta_{\sigma_1, \sigma_2}$ (notice the labeling convention), describing the scattering of quasi-particles with vanishing momenta. Using this perfectly reflective S-matrix allows one to obtain a number of exact results for thermal gases and near adiabatic quantum quenches [14, 15, 18, 19]. However, as soon as the quench is not slow enough, faster quasi-particles are inevitably created. Collisions involving these faster particles are not captured by the universal scattering matrix, and the true velocity dependent scattering matrix must be used (cf. Fig. 1a). The sine-Gordon model is integrable and this two-particle S-matrix is exactly known [34, 35]. The matrix elements $S_{\sigma_1, \sigma_2}^{\pm} = S_R(v_1, v_2)$ and $S_{\sigma_1, \sigma_2}^{\mp} = S_T(v_1, v_2)$, in particular, describe reflection and transmission, respectively. They satisfy $|S_T|^2 + |S_R|^2 = 1$, and for small velocities transmission vanishes as $|S_T(v_1, v_2)|^2 \propto (v_1 - v_2)^2$.

We can incorporate the non-trivial S-matrix in two different ways. In the first method, charges are still treated classically, but at each collision either a perfect transmission or perfect reflection takes place with probabilities given by the modulus square of the S-matrix elements. This method neglects interference effects but can be implemented as a classical Monte–Carlo simulation. A more refined approach is to treat the charge part of the wave function in Eq. (2b) in a fully quantum mechanical manner with an MPS-based method. As the quench protocol
generates neutral kink pairs with equal probability amplitude, the initial wave function is a dimerized state, i.e. a product of Bell pairs:

$$|\chi(\sigma, t = 0)\rangle = \prod_{j=1}^{N/2} \left( |+\rangle_{2j-1} |-\rangle_{2j} + |-\rangle_{2j-1} |+\rangle_{2j} \right) \sqrt{2} .$$ (3)

The charge wave function evolves only through collisions and it is frozen between collisions. If line \( j \) and \( j + 1 \) intersect at time \( \ell_k \), the change of the wave function is

$$|\chi(\sigma, \ell_k, +)\rangle = \hat{S}_{j,j+1}(\ell_k) |\chi(\sigma, \ell_k, -)\rangle ,$$ (4)

where \( \hat{S}_{j,j+1}(\ell_k) \) acts nontrivially only on charges \( j \) and \( j + 1 \) via the 2-body \( S \)-matrix evaluated at the velocities of the colliding quasiparticles. In this way we mapped the dynamics of the charges to that of an effective quantum spin chain. The time evolution operator of this spin chain depends on the underlying semiclassical trajectories and is given as a product of local unitary two-body operators, efficiently treated by an MPS-based algorithm.

**Phase distribution.**— The space resolved relative phase \( \phi(x) \) between the condensates can be measured directly via interferometry experiments [3, 5, 25, 36]. We first use the classical Monte Carlo approach to determine the full phase distribution after the quench, a quantity also analyzed in the experiment of Ref. [5]. Semiclassically, each kink is a domain wall that separates two domains with a phase difference of \( \pm 2\pi \). Domains separated by \( s \) kinks have a phase difference \( \Delta \phi = 2\pi \sum_{i=1}^{s} \phi_i \). The phase in a given domain being constant, its distribution function is a sum of weighted Dirac-delta peaks located at integer multiples of \( 2\pi \),

$$P(\phi, t) = \sum_{n \in \mathbb{Z}} \epsilon_{2\pi n}(t) \delta(\phi - 2\pi n) .$$ (5)

In real experiments, these delta peaks get broadened by quantum and thermal fluctuations. To take this into account, we estimated the phase fluctuations \( \langle \chi(x) \chi^\dagger(x) \rangle \) around the minima of the cosine potential for experimentally relevant parameter values (see [35]), and broadened the delta-functions accordingly.

The resulting phase distribution \( P(\phi, t) \) is shown in Fig. 2.a at different instances for a typical small quench. For the momentum distribution of quasiparticles, \( f(p) \), we used the ansatz \( f(p) \propto p^2 \exp(-p^2/p_0^2) \), [37] motivated by overlap expressions in the transverse field Ising model [38]. Immediately after the quench, there is a single central peak at \( \phi = 0 \) and \( c_0(0) = 1 \), but side peaks emerge as the system evolves in time. The two side peaks at \( \phi = \pm 2\pi \) emerge already at the uSC level: initially, domains of phase \( \phi = \pm 2\pi \) grow ballistically between separating soliton-antisoliton pairs (see Fig. 1.c), therefore \( c_0 \) decreases while \( c_{2\pi,2\pi} \) increase linearly in time. Within the uSC approach, the weight of these side peaks saturates at values \( c_{\pm 2\pi} \rightarrow 1/4 \) once the spatial location (coordinate) of the kinks becomes randomized, while the central peak levels off at \( c_0 \rightarrow 1/2 \). We observe this ‘pre-thermalized’ behavior by sSC at intermediate times in simulations with small quasiparticle velocities |\langle \sigma|\rangle/c \approx 0.05. However, after a fast relaxation, collisions start to dominate beyond the collision time, \( t > 0 \), \( c_0 \rightarrow c_0 = \rho^f \int_0^\infty dv \rho_v(\phi) \), and phase diffusion takes place due to transmissive collisions. The corresponding phase diffusion constant \( D_\phi \) is an increasing function of the ratio \( |\langle \sigma|\rangle/c \) and vanishes in the universal limit \( |\langle \sigma|\rangle/c \rightarrow 0 \), where the \( S \) matrix becomes purely reflective. Phase diffusion gives rise to domains of phases \( \phi = \pm 4\pi, \pm 6\pi, \ldots \), as reflected by the emergence of further side peaks and an increasing number of non-zero coefficients in Eq. (5) (see Fig. 2.b), all predicted to vanish as \( c_{2\pi n} \sim 1/\sqrt{t} \) for very long times, a clear fingerprint of phase diffusion.

**Entanglement entropy.**— Our method is able to follow the propagation and growth of entanglement in the charge sector. The most widely used entanglement measure is the entanglement entropy of a subsystem \( A \), defined as the von Neumann entropy of its reduced density matrix: \( S = -\text{Tr}_A[\rho_A \log \rho_A] \) where \( \rho_A = \text{Tr}_A|\Psi(x, \sigma, t)\rangle \langle \Psi(x, \sigma, t)| \). Here we focus on the simple case when \( A \) is half of the total system.

The initial charge wave function \( |\chi\rangle \) has alternating bond entanglement entropies of \( \log 2 \) and 0. At \( t = 0^+ \), the spatial extension of the entangled bonds is zero, so with probability 1 the cut between the two halves of the system falls between entangled pairs resulting in zero entanglement entropy. Within the uSC approximation, at \( t > 0 \), pairs with one member on the left and another member on the right entangle the two halves of the system. Since the uSC \( S \)-matrix is fully reflective, each kink remains maximally entangled with its original partner, but entanglement remains local. At very long
times, the cut between the two halves of the system either falls between pairs or cuts a pair in two with probability 1/2. Consequently, the entanglement entropy saturates at a value $S_{\text{sat}} = \log 2/2$. The full time evolution of the entropy is simple to calculate within the uSC approach which yields an exponential relaxation to $S_{\text{sat}}$. $S_{\text{uSC}}(t) = (1 - e^{-t/\tau}) \log 2/2$, (solid black line in Fig. 3). The saturation of $S_{\text{uSC}}(t)$ is clearly an artefact of the uSC method. In reality, the topological charges of remote quasiparticles become entangled with time due to transmissive scattering processes. The sSC approach captures the corresponding entropy production: the entanglement entropy does not saturate after the initial transient, but rather grows linearly in time without bound in an infinite system, $S(t) = \alpha t$, with a growth rate $\alpha \propto \langle |v| \rangle$.

Correlation functions and thermalization.— The sSC method is also suitable for computing the out-of-equilibrium evolution of correlation functions. We consider first the expectation values $G_\alpha(t) = \langle e^{i\alpha \varphi(x,t)} \rangle = \langle e^{i\alpha \varphi(x,t)} e^{-i\alpha \varphi(0,0)} \rangle$. The function $G_\alpha(t)$ is essentially the coherence factor, directly accessible through matter wave interferometry [3]. The standard uSC approach has been recently used to compute $G_\alpha(t)$ [19], which was found to decay exponentially to a value $\cos^2(\pi \alpha)$, apart from a prefactor incorporating vacuum fluctuations, now set to one (see Fig. 4.a). This surprising behavior is related to the fact that in the uSC approach the phase is pinned to the values $\varphi = 0, \pm 2\pi$. In contrast, within the sSC approach, the phase diffuses with time and $G_\alpha(t \to \infty)$ no longer remains finite; after a fast transient described approximately by uSC, $G_\alpha(t)$ is found to decay exponentially to zero with a rate depending on $\langle |v| \rangle / c$.

The equal time two-point correlation functions $C_\alpha(x-x';t) = \langle e^{i\alpha \varphi(x,t)} e^{-i\alpha \varphi(x',t)} \rangle$ are also accessible experimentally [39], and the integrated quantity, $\int dx \int d\rho C_\alpha(x-x';t)$, is directly related to the contrast of the interference fringes. Fig. 4.b shows the time evolution of $C_\alpha(x,t)$ for $\alpha = 1/2$. Here again, the uSC approach yields a quick relaxation to a state, where the phase is pinned and, accordingly, long-ranged correlations persist, $C_\alpha(|x|,t \to \infty) \sim \cos^4(\pi \alpha)$.

In sharp contrast to traditional uSC, within the sSC we find that $C_\alpha(x-x',\infty) = \exp(-2 \sin^2(\pi \alpha) |x-x'|)$ which coincides with the thermal equilibrium value computed within the standard finite temperature SC approach [20]. The sSC method thus captures apparently local thermalization and pre-thermalization. We should emphasize here that sSC incorporates phase relaxation at the fully quantum-mechanical level, and the final state we obtain is not completely thermal: it should rather be viewed as a pre-thermalized state, where local correlations and expectation values of vertex operators look thermal, but the velocity distribution of the quasiparticles remains non-thermal.

Thermalization thus takes place in several steps in a split condensate. For small values of $\langle |v| \rangle / c$, first a quick relaxation takes place to a first pre-thermalized state with pinned phase and a non-thermal quasiparticle velocity distribution, but with kink positions randomized. Next, the coherent evolution of the charge wave function gives rise to phase (quantum) diffusion. At this second pre-thermal stage, phase correlation functions look thermal, though the velocity distribution of quasiparticles is still non-thermal and the charge sector is fully coherent.
To reach a truly thermal state with thermal quasiparticle velocity distribution a further relaxation step and coupling to some external environment such as the symmetrical phase mode in the coupled condensate experiment are ultimately needed.

Conclusions.— The versatile semi-semiclassical method developed here has a broad range of applicability. It is suitable for studying the dynamics of non-integrable lattice systems, spin chains as well as continuum 1D systems. The differences between these translate into differences in the nature of quasiparticles and their 2-body scattering matrix, which makes our method ideal for identifying universal aspects of the dynamics. The semiclassical aspect allows for heuristic interpretations and relatively long simulation times, while the quantum mechanical description of the scattering of quasiparticles allows us to go beyond the limitations of the standard semiclassical approach and opens the way to study the propagation of entanglement. Possible connections with the recent work [40] would be interesting to analyze. Although here we focused on the non-equilibrium time evolution, the method can also be used to investigate dynamical correlation functions at finite temperature beyond the uSC approximation.

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Semi-semiclassical theory of quantum quenches in one-dimensional systems

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S-I. THE HAMILTONIAN AND SINE–GORDON DESCRIPTION OF COUPLED QUASICONDENSATES

The Hamiltonian of two coupled 1D condensates is given by [1]

\[ H = \sum_{j=1,2} \int dx \left\{ \frac{\hbar^2}{2m} \partial_x \psi_j(x) \partial_x \psi_j(x) + \frac{g}{2} \psi_j(x) \psi_j(x) \psi_j(x) \psi_j(x) + [V(x) - \mu] \psi_j(x) \psi_j(x) \right\} - \hbar J \int dx \left[ \psi_1^\dagger(x) \psi_2(x) + \psi_2^\dagger(x) \psi_1(x) \right]. \] (S-1)

where \( \psi_1(x), \psi_2(x) \) are the bosonic fields in the two condensates, \( V(x) \) is the longitudinal potential which we neglect in the following, \( J \) is the tunnel coupling between the condensates, and

\[ g = \frac{2\hbar^2 a_s}{m l_\perp^2} \left( 1 - 1.0325 \frac{a_s}{\xi} \right)^{-1} \] (S-2)

is the strength of the atom-atom interaction within each condensate. Here \( l_\perp = \sqrt{\hbar/(m \omega_\perp)} \) with \( \omega_\perp \) being the frequency of the radial confining potential and \( a_s \) denotes the s-wave scattering length of the atoms. The strength of interaction is often parameterized by the dimensionless combination

\[ \gamma = \frac{mg}{\hbar^2 n}. \] (S-3)

In the absence of coupling, the condensates can be described within the bosonization framework by the Hamiltonians

\[ H_j = \frac{\hbar c}{2} \int dx \left\{ \frac{\pi}{K} \Pi_j(x)^2 + \frac{K}{\pi} \left[ \partial_x \psi_j(x) \right]^2 \right\}, \] (S-4)

where \( [\psi_j(x), \Pi_j(x')] = i\delta(x - x') \). The speed of sound, \( c \), and the Luttinger parameter \( K \) can be computed from the exact Bethe Ansatz solution of the model. The asymptotic expansions for small and large couplings are

\[ K \approx \frac{\pi}{\sqrt{\gamma}} \left( 1 - \frac{\sqrt{\gamma}}{2\pi} \right)^{-1/2} \approx \hbar \pi \sqrt{\frac{n}{mg}}, \quad c \approx \sqrt{\frac{mg}{m}} \quad \text{for } \gamma \lesssim 10, \] (S-5a)

\[ K \approx (1 + 4/\gamma), \quad c \approx \hbar \pi n/m \quad \text{for } \gamma \gg 1. \] (S-5b)

Due to Galilean invariance, \( cK = \hbar \pi n/m \) holds for all \( \gamma \). The density fluctuations are suppressed at wavelengths smaller than the healing length which is also used as a short distance cutoff. For small \( \gamma \) it is

\[ \xi_h = 1/(n\sqrt{\gamma}) = \hbar / \sqrt{m g n} \approx \hbar / m c, \] (S-6)

while at strong coupling \( \xi_h \approx 1/n \). The coupling between the condensates is captured by the term \( 2\Delta \cos(\varphi_1 - \varphi_2) \), where \( \Delta \approx \hbar J n \) for weak coupling, but it can be renormalized at strong interactions. It is convenient to introduce the total and relative phase \( \varphi_{\pm} = \varphi_1 \pm \varphi_2 \) and \( \Pi_{\pm} = (\Pi_1 \pm \Pi_2)/2 \) in terms of which the two Hamiltonians decouple. For the phase difference we obtain the sine–Gordon Hamiltonian [1]

\[ H_- = \frac{\hbar c}{2} \int dx \left[ \frac{2\pi}{K} \Pi_-(x)^2 + \frac{K}{2\pi} \left[ \partial_x \varphi_-(x) \right]^2 \right] - \int dx 2\Delta \cos(\varphi_-(x)). \] (S-7)
S-II. FLUCTUATIONS OF THE PHASE

We are interested in the fluctuations of $\phi^-$ around the minima of the cosine potential. We will calculate this in the harmonic approximation, i.e. by expanding the cosine up to quadratic order which yields

$$H_\pm \approx \frac{hc}{2} \int dx \left[ \frac{2\pi}{K} \Pi_\pm(x)^2 + \frac{K}{2\pi} [\partial_x \phi_-(x)]^2 + \frac{2\Delta}{hc} \phi_-(x)^2 \right],$$

(S-8)

a free massive boson theory with mass gap

$$m_0 = \sqrt{\frac{4\Delta \hbar \pi}{c^2 K}} = \sqrt{\frac{4\Delta m}{\pi c^2}}.$$  

(S-9)

The mode expansion of the fields are

$$\phi(x) = \sqrt{\frac{\pi c}{K}} \int \frac{dp}{2\pi} \frac{1}{\sqrt{\omega(p)}} \left[ b_pe^{ipx/h} + b_p^* e^{-ipx/h} \right],$$

(S-10)

$$\Pi(x) = -i \frac{e}{2\hbar} \sqrt{\frac{K}{\pi c}} \int \frac{dp}{2\pi} \sqrt{\omega(p)} \left[ b_pe^{ipx/h} - b_p^* e^{-ipx/h} \right]$$

(S-11)

with $\omega(p) = \sqrt{p^2 + m_0^2 c^2}$ and $[b_p, b_p^*] = 2\pi \delta(p - p')$.

The finite temperature equal time correlation function of the phase is

$$\langle \phi(x) \phi(x') \rangle_T = \frac{\pi c}{K} \int \frac{dp}{2\pi} \frac{1}{\omega(p)} e^{ip(x-x')/\hbar} \coth(\beta \omega(p)/2),$$

(S-12)

where we used $\langle b_p b_{p'} \rangle = 2\pi \delta(p - p') f_T(p)$ with $f_T(p) = 1/(e^{\omega(p)/(k_B T)} - 1)$ the thermal Bose–Einstein distribution and $\beta = 1/(k_B T)$.

A. Quantum fluctuations ($T = 0$)

At zero temperature the integral can be evaluated in closed form,

$$\langle \phi(x) \phi(x') \rangle_{T=0} = \frac{\pi c}{K} \int \frac{dp}{2\pi} \frac{1}{\omega(p)} e^{ip(x-x')/\hbar} = \frac{1}{K} K_0[m_0|x - x'|/\hbar] = \frac{1}{K} K_0 \left( \frac{\Delta x}{l_\Delta} \right),$$

(S-13)

where $\Delta x = |x - x'|$, $K_0(z)$ is the modified Bessel function of the second kind and

$$l_\Delta \equiv \frac{\hbar}{m_0 c} = \sqrt{\frac{\hbar^2 c}{4m\Delta}}$$

(S-14)

is the Compton wavelength associated with the mass $m_0$ which physically corresponds to the “healing length of the relative phase”. For small separation

$$\langle \phi(x) \phi(x') \rangle_{T=0} = -\frac{1}{K} \left\{ \log \left( \frac{\Delta x}{2l_\Delta} \right) \left[ 1 + \left( \frac{\Delta x}{2l_\Delta} \right)^2 \right] + \gamma_E \right\} + \mathcal{O}\left( \frac{\Delta x}{l_\Delta} \right)^2,$$

(S-15)

so it is logarithmically divergent ($\gamma_E$ is the Euler–Mascheroni constant). By introducing a minimum distance $\alpha$ we obtain

$$\langle \phi^2 \rangle_{T=0} \approx -\frac{1}{K} \left[ \log \left( \frac{\alpha}{2l_\Delta} \right) + \gamma_E \right].$$

(S-16)

For e.g. $l_\Delta/\alpha \approx 10$, $\langle \phi^2 \rangle_{T=0} \approx 2.4/K$. As $K > 1$, this means that the quantum fluctuation of the phase satisfies $\Delta \phi \lesssim 1.5$ and remains small compared to $2\pi$, so the approximation of the cosine by a quadratic potential is justified.
B. Thermal fluctuations (low temperature)

Let us now compute the correlation function focusing on the low temperature limit, i.e. when $k_B T \ll m_0 c^2$. The thermal contribution is given by the expression

$$\langle \varphi(x)\varphi(x') \rangle_{\text{therm}} = \frac{\pi c}{K} \int \frac{dp}{2\pi} \frac{1}{\omega(p)} e^{ip(x-x')/\hbar} (\coth(\beta \omega(p)/2) - 1),$$  \hspace{1cm} (S-17)

which is perfectly well-behaved for large $p$. Expanding the hyperbolic cotangent in powers of $e^{-\beta \omega(p)}$, changing the momentum integration variable to relativistic rapidity, and shifting the integration contour we arrive at

$$\langle \varphi(x)\varphi(x') \rangle_{\text{therm}} = \frac{2}{K} \sum_{n=1}^{\infty} K_0 \left( \sqrt{(n\pi q/2K)^2 + (\Delta x/l_\Delta)^2} \right),$$  \hspace{1cm} (S-18)

where

$$q = \frac{\lambda_T}{l_\Delta} = \frac{2K m_0 c^2}{\pi \kappa_B T}$$ \hspace{1cm} (S-19)

is the ratio of the thermal phase coherence length $\lambda_T = 2\hbar^2 n/(mk_B T)$ and $l_\Delta$. For the fluctuations of $\varphi$ this implies

$$\langle \varphi^2 \rangle_T = \langle \varphi^2 \rangle_{T=0} + \langle \varphi^2 \rangle_{\text{therm}} \approx \frac{1}{K} \left[ - \log \left( \frac{\alpha}{2l_\Delta} \right) - \gamma_E + 2 \sum_{n=1}^{\infty} K_0 \left( \frac{n\pi q}{2K} \right) \right].$$  \hspace{1cm} (S-20)

In the low temperature limit $\pi q/(2K) \gg 1$ so we can expand the Bessel functions:

$$K_0 \left( \frac{n\pi q}{2K} \right) = \sqrt{\frac{K}{nq}} e^{-n\pi q/(2K)} \left( 1 - \frac{K}{4nq} + O \left( \frac{K}{nq} \right)^2 \right).$$  \hspace{1cm} (S-21)

Each term in the sum in Eq. (S-20) is exponentially suppressed with respect to the previous one, so we can truncate the series at the first term, leading to

$$\langle \varphi^2 \rangle_T \approx \frac{1}{K} \left[ - \log \left( \frac{\alpha}{2l_\Delta} \right) - \gamma_E + 2 \sqrt{\frac{K}{q}} e^{-m_0 c^2/(k_B T)} \right].$$  \hspace{1cm} (S-22)

The calculation is consistent if $\langle \varphi^2 \rangle_T \ll 1$ so the approximation of the cosine by a quadratic potential is justified.

S-III. S-MATRIX OF THE SINE–GORDON MODEL

The 2-particle S-matrix describing the scattering of two kinks is exactly known [2]. The energy and momentum of the incoming and outgoing particles are conveniently parameterized in terms of the relativistic rapidity as $E = mc^2 \cosh \theta$, $p = mc \sinh \theta$. The 2-particle S-matrix is given by a four by four matrix in the basis $|++, +-, -+, --\rangle$:

$$S = \begin{pmatrix} S & ST & SR & S_T \\ ST & SR & S_T & S \\ SR & S_T & S & SR \\ S_T & S & SR & ST \end{pmatrix},$$  \hspace{1cm} (S-23)

where due to relativistic invariance all entries depend only on the relative rapidity $\theta = \theta_1 - \theta_2$ of the two incoming kinks. Here $ST$ is the amplitude of transmission and $SR$ is the amplitude of reflection. The term $S$ accounts for the phase picked up by the wave function upon scattering of two kinks of the same charge:

$$S(\theta) = -\exp \left\{ -i \int dt \frac{\sinh(t(\pi-\xi))}{t \sinh^{1/2} \frac{t}{2} \cosh^{1/2} \frac{t}{2} \sin(\theta)} \right\},$$  \hspace{1cm} (S-24)

where

$$\xi = \frac{\pi}{4K} - 1.$$  \hspace{1cm} (S-25)
The transmission and reflection factors are given by

\[ S_T(\theta) = \frac{\sinh \frac{\pi \theta}{\xi}}{\sinh \frac{\pi (i\pi - \theta)}{\xi}} S(\theta), \]
\[ S_R(\theta) = i \frac{\sin \frac{\pi \theta}{\xi}}{\sinh \frac{\pi (i\pi - \theta)}{\xi}} S(\theta). \]

and they satisfy \(|S_T|^2 + |S_R|^2 = 1\). For small \(\theta\) they behave as \(|S_T|^2 \propto \theta^2\), \(|S_R|^2 \propto 1 - \theta^2\), thus at small rapidities the scattering of kinks of opposite charges is almost purely reflective.

**S-IV. DETAILS OF THE TEBD ALGORITHM**

In our numerical calculations a kink configuration can be visualized graphically by a superposition of pairs of straight lines that describe propagation of the soliton-antisoliton pairs. At \(t = 0\) they start to move ballistically in opposite directions with equal velocities (see Fig. S-1(a)). In this representation the collision of two kinks corresponds to the intersection of two lines. The quantum effects become relevant only at collision times and they do not affect the orbital motion of the quasiparticles, which allows us to factorize the wave function as in Eq. (2) in the main body of the paper. We construct such a kink configuration and index all the lines accordingly, such that at any time \(t\), we know the position \(x\) of each kink. Then we look for the intersections of the lines and index them too, each intersection \(I\) being characterized by a space-time coordinate \((x_I, t_I)\). These coordinates are then ordered chronologically. In this way, we completely characterize the orbital motion of the kinks, which corresponds to the construction or the orbital part of the wave function \(|\Psi_{orb}(x, t)\rangle\) in Eq. (2b).

To propagate the solution for the charge part of the wave function, we map the dynamics of the charges to an effective quantum spin chain model. To do that, we reindex the lines after each intersection as in Fig. S-1(b). In this way, the quasiparticle trajectories have zig-zagged shapes and the collisions between quasiparticles always happen between neighboring trajectories, allowing for the use of a TEBD algorithm [3, 4]. The time evolution of the charge part of the wave function, \(|\chi(\sigma, t)\rangle\), is represented in Fig. S-1(c). The \(t = 0\) wave function \(|\chi(\sigma, t = 0)\rangle\) given in Eq. (3) is first organized as a MPS state which is then evolved in time. The full time evolution operator \(U(t)\) is the chronological product of unitary two-body S-matrix operators that scatter pairs of quasiparticles only at the intersection times \(t_I\).

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