We introduce a framework for the construction of completely positive maps for subsystems of indistinguishable fermionic particles. In this scenario, the initial global state is always correlated, and it is not possible to tell system and environment apart. Nonetheless, a reduced map in the operator sum representation is possible for some sets of states where the only non-classical correlation present is exchange.

PACS numbers: 03.67.Mn, 03.65.Aa

I. INTRODUCTION

The characterization of the dynamics of a system that may be correlated with other systems has been subject of investigation in several areas, varying from quantum information processing to condensed matter physics [1][2]. A closed system evolves unitarily according to the Schrödinger equation. On the other hand, the dynamics of a subsystem is not necessarily unitary, and the theory of open quantum systems provides the mathematical framework to treat it. In this context, we speak of system and environment, and say that the system, which is just a part of the whole, is open. If system and environment start in a uncorrelated global state (factorable), then the dynamics is guaranteed to be completely positive (CP). However, if the system is initially correlated with the environment, the map associated with the dynamics of the system may not be completely positive or, as we will see, is valid only for a subset of the state space. In recent years, more attention has been given to the construction of reduced dynamical maps with different initial conditions [3][8], mainly motivated by discussions between Pechukas and Aliciki [9][11]. Pechukas introduced the idea of ‘assignment map’ (Φ), which characterizes initial system-environment states (ΦρS = ρS'E) for open quantum systems, and showed that imposing three ‘natural’ conditions, namely: (linearity) Φ preserves mixtures; (consistency) it is consistent, in the sense that ρS = T(E(ΦρS)); (positivity) and ΦρS is positive for all positive ρS; this implies the initial state of the system and environment is factorable (ΦρS = ρS ⊗ ρE). To deal with the problem of characterizing reduced dynamics of initial correlated systems, Pechukas [9][11] suggested to giving up positivity. On the other hand, Aliciki [10] argued to either giving up consistency or linearity. In the end, the conclusion is that, one way or the other, the domain of validity of the assignment map must be restricted. Afterwards, Stelmachovic et al. [3] studied the influence of initial correlations between system and environment in the dynamics of the system, making clear that taking into account such correlations is paramount to the correct description of the evolution. They showed an instructive example with two qubits (one for the system, one for the environment), evolving under a C-NOT gate: both a maximally entangled state and a maximally mixed global state have the same one-qubit local maximally mixed states, but the evolution is radically different. In a comment to [3], Salgado et al. [12] showed for two qubits that, whatever the initial correlations, the system dynamics has the Kraus representation form, and is consequently completely positive, whenever the global dynamics is locally unitary. This was then proved for bipartite global systems of arbitrary dimension by Hayashi et al. [13]. Later on many authors worked out sets of classically [3][6] or quantum [7][8] correlated initial global states that guarantee complete positivity of the reduced dynamics. The subject has recently regained impetus, with many interesting discussions [5][14][15].

In this work we are interested in the construction of the reduced dynamical map in the case of systems of N indistinguishable particles, in particular fermions, which are always correlated, and for which an usual tensor product structure between ‘system’ and ‘environment’ is absent. The subtle notion of quantum correlations of indistinguishable particles has been investigated by many authors, with introduction of seminal ideas, as entanglement of modes [19], or entanglement of particles [20][29]. Our own group has scrutinized the concept of entanglement of particles [23][24], and made interesting applications [27]. More recently, the concept of ‘quantumness of correlations’ of indistinguishable particles was explored by Iemini et al. [28], and Debarba et al. [29]. It is well established that the exchange correlations generated by mere antisymmetrization of the state, due to indistinguishability of their fermions, does not result in entanglement or, more generally, in quantumness [28][29]. To the best of our knowledge, the role of initial exchange correlations in the reduced dynamics is still unexplored.

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We propose a framework to construct completely positive maps representing the dynamics of a single particle reduced state.

This paper is organized as follows. In Sec.II we briefly discuss particle correlation in the antisymmetric subspace. In Sec.III we identify a class of initial global states that give rise to completely positive reduced dynamics. In Sec.IV we illustrate the formalism with an example of two fermions under a quadratic Hamiltonian. Conclusions are presented in Sec.V.

II. CORRELATIONS IN THE ANTISYMMETRIC SUBSPACE

Composed distinguishable quantum systems are described by density operators over a composition of Hilbert spaces of individual subsystems, by means of the tensor product:

\[ \rho_{1 \ldots N} : \mathcal{H}_N \to \mathcal{H}_N, \]

where \( \mathcal{H}_N = \mathcal{H}_{1}^{L_1} \otimes \cdots \otimes \mathcal{H}_{N}^{L_N}, \) \( N \) is the number of subsystems, \( L_i \) is the dimension of \( i \)th subsystem, and \( \rho_{1 \ldots N} \in \mathcal{D}(\mathcal{H}_1^{L_1} \otimes \cdots \otimes \mathcal{H}_N^{L_N}), \) with \( \mathcal{D} \) the set of density operators (positive semidefinite and trace-one operators). In these systems, the tensor product structure between the subsystems plays an important role to the characterization of correlations as entanglement \[ 35 \] and quantumness \[ 33, 34 \]. However, the state space of \( N \) indistinguishable fermions is described by the antisymmetrized composed Hilbert space (Fig. 1):

\[ \mathcal{F}_N^F = \mathcal{A}(\mathcal{H}_1^L \otimes \cdots \otimes \mathcal{H}_N^L), \]

where \( N \) is the number of fermions and \( L \) is the number of accessible modes. Note that this space does not support a tensor product structure and have a more suitable description in the second quantization formalism. Therefore a basis in this subspace can be constructed out of fermionic operators \( \{a_k\}_{k=1}^L \), satisfying the usual anticommutation relations:

\[ \{a_l, a_k^\dagger\} = \delta_{k,l}, \quad \{a_k, a_l\} = \{a_k^\dagger, a_l^\dagger\} = 0. \]

where \( a_k \) and \( a_k^\dagger \) are annihilation and creation operators for the \( k \)’th mode, respectively. A single particle orthogonal basis is formed by the set of states \( \{a_k^\dagger|0\}\}_{k=1}^L \), where \( |0\rangle \) represents the vacuum.

As mentioned in the Introduction, the correlation of indistinguishable particles, mostly entanglement, was study by many groups \[ 20, 26 \], giving rise to many definitions that agree with each other in the fermionic case, in the sense that the set of unentangled states can be written as a convex sum of Slater determinants. More generally, with studies in quantumness \[ 25, 29 \], we can define states where the only non-classical correlation present is exchange, which leads to the following definition:

**Definition 1.** A fermionic state \( \omega \in \mathcal{D}(\mathcal{F}_N^F) \) has no quantumness of correlation if it can be decomposed as a convex combination of orthogonal Slater determinants, namely,

\[ \omega = \sum_{\vec{k}} p(\vec{k}) |a_{\vec{k}}^\dagger|0\rangle \langle 0| a_{\vec{k}}^\dagger, \]

where \( \vec{k} = (k_1, \ldots, k_N) \) is an \( N \)-tuple denoting the modes occupied by the fermions, with \( k_i = 1, \ldots, L \). \( p(\vec{k}) \) is a probability distributions and \( a_{\vec{k}}^\dagger|0\rangle \equiv a_{k_1}^\dagger \cdots a_{k_N}^\dagger |0\rangle \).

As we are interested in exploring the role of initial exchange correlations in the reduced dynamics of fermionic systems, we will choose the initial global fermionic state in the set with no quantumness, according to Definition 1.

III. DYNAMICAL MAPS FOR REDUCED STATES OF FERMIONIC SYSTEMS

In this section we introduce the formalism to describe the dynamics of a single fermion in a closed system of \( N \) fermions. More precisely, given a system of \( N \) indistinguishable fermions in the state \( \rho(0) \), evolving under the unitary \( U_t \), which preserves the total number of particles, we want to obtain the dynamical map \( \Phi_t \), which

![FIG. 1: Pictorial view of Hilbert space with (a) tensor product structure (\( \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \)), and (b) antisymmetric space without tensor product structure, where the particle states overlap. A partial trace over a subsystem in the antisymmetric space has information about the whole system, since the particles are indistinguishable.](image)
evolves the one-particle reduced state ρr = TrN−1(ρ(0)), see Fig. 2. Since the fermionic states are restricted to the antisymmetric sector of the Hilbert space, it is not possible to start with initial states in the tensor product form. As discussed in the Introduction, one way to deal with the problem of obtaining completely positive maps, characterizing the dynamics of states initially correlated with an external system, is to restrict the domain of the map. Using the fact that the Kraus representation assures completely positivity \[1,2\], we will show that for some sets of initial states with no quantumness of correlations, we can construct completely positive maps for the reduced state.

The construction of the single-fermion dynamical map, in the simplest scenario of a closed system of two fermions initially in a pure state, \(\rho(0) = |\psi\rangle\langle\psi|\), gives us a good grasp on the general features of the formalism, and includes all the technical problems of the general case. The generalisation to \(N\) fermions mixed states is straightforward and performed in Appendix B.

Let us consider a set of states in the antisymmetric space of 2 fermions and \(L+1\) modes, that can be written in a given basis of Slater determinants as:

\[
S^\mu_{2,pure} = \{a^\dagger_i a^\dagger_k |0\rangle \langle 0| a_k a_\mu \}_{k=0}^L,
\]

where \(\mu\) is a fixed mode. Note that \(\mu\) labels a reference mode, and different values of \(\mu\) lead to distinct sets.

Let us calculate the one-particle reduced state by tracing out one fermion from Eq.(5). Assuming that \(\{f_k\}_{k=0}^L\) is an orthonormal basis of fermionic creation operators for the space of a single fermion \(F_{L+1}^1\), thus \(f_k^\dagger = \sum_k V_k a^\dagger_k\), \(V\) is a unitary matrix of dimension \(L+1\). The partial trace over one particle is given by \(\rho_r = \frac{1}{2} \sum_{k=0}^L f_k \rho f_k^\dagger\). The explicit calculation of the matrix element \((\rho_r)_{i,j}\) goes as follows:

\[
(\rho_r)_{i,j} = \langle 0 | f_j^\dagger \left(\frac{1}{2} \sum_{k=0}^L f_k \rho f_k^\dagger\right) f_i^\dagger |0\rangle \\
= \frac{1}{2} \sum_{k=0}^L \langle 0 | f_k^\dagger f_i f_k \rho f_k^\dagger |0\rangle \\
= \frac{1}{2} Tr_1 (f_i^\dagger f_j \rho),
\]

where we used the fermionic anti-commutation relations and the cyclicity of the trace. Now we can write the set of single-fermion reduced states of Eq.(5):

\[
S^\mu_{r(2)} = Tr_1 (S^\mu_{2,pure}) \\
= \left\{ \frac{1}{2} a^\dagger_k |0\rangle \langle 0| a_k + \frac{1}{2} a^\dagger_\mu |0\rangle \langle 0| a_\mu \right\}_k^L,
\]

with \(\mu\) a fixed mode. Assuming the dynamics of \(\rho(0) \in S^\mu_{2,pure}\) is given by the unitary \(U_t\), we can define a CP map \(\Phi_t\) for the dynamics of the single-fermion reduced state \(\rho_r(t) = S^\mu_{r(2)}\), i.e., a CP map \(\Phi_t : S^\mu_{2,pure} \mapsto F_{L+1}^1\) as follows:

**Definition 2.** A dynamical map \(\Phi_t^\mu\) for the single-fermion reduced state \(\rho_r(t) \in S^\mu_{r(2)}\), of a 2-fermion pure state initially with no quantumness of correlations, \(\rho(0) \in S^\mu_{2,pure}\), evolving under the global unitary \(U_t\), has the operator sum representation \(\Phi_t^\mu[\rho_r] = \sum_{j=0}^L K_j^\mu \rho_r K_j^\mu\), with the Kraus operators

\[
K_i^\mu = f_i U_t a^\dagger_\mu.
\]

**Proof.** If the 2-fermion state evolves according to \(\rho(t) = U_t \rho(0) U_t^\dagger\), the reduced density matrix is:

\[
\rho_r(t) = Tr_1 (U_t a^\dagger_\mu a_\mu U_t^\dagger |0\rangle \langle 0| a_k a_\mu U_t^\dagger) \\
= \sum_{l=0}^L f_l U_t a^\dagger_\mu \left( \frac{1}{2} a^\dagger_k |0\rangle \langle 0| a_k \right) a_\mu U_t^\dagger f_l^\dagger,
\]

where in the last equation we used the definition of fermionic partial trace (Eq.(3)) and the anti-commutation relations. Using the fact that we cannot create more than one fermion in the same mode (Pauli exclusion principle), we can add a second null term in Eq.(3), in order to recover the reduced state in the form of Eq.(9).

\[
\rho_r(t) = \sum_{l=0}^L f_l U_t a^\dagger_\mu \left( \frac{1}{2} a^\dagger_k |0\rangle \langle 0| a_k \right) a_\mu U_t^\dagger f_l^\dagger \\
+ \sum_{l=0}^L f_l U_t a^\dagger_\mu \left( \frac{1}{2} a^\dagger_\mu |0\rangle \langle 0| a_\mu \right) a_\mu U_t^\dagger f_l^\dagger,
\]

which can be written as,

\[
\rho_r(t) = \sum_{l=0}^L f_l U_t a^\dagger_\mu \rho_r(0) a_\mu U_t^\dagger f_l^\dagger \\
= \sum_{l=0}^L K_i^\mu \rho_r(0) K_l^\mu,
\]
with $K^\mu_i = f_i U_i a^\mu_i$.

Due to the restriction of the map domain (Eq. 7), the relation between Kraus operators and trace preservation can be written as,

$$\sum_{i} K^\mu_i K^{\mu}_i = \text{diag} (\lambda_0, \lambda_1, \ldots, \lambda_L),$$

with $\lambda_i \neq \lambda_\mu = 2$ and $\lambda_\mu = 0$, since

$$\text{Tr} (\rho_t (t)) = \text{Tr} [\text{diag} (\lambda_0, \lambda_1, \ldots, \lambda_L) \rho_t (0)] = 1,$$

where $\rho_t (0) \in \mathcal{S}^{\text{pure}} (\mathcal{E})$. This can be checked by computing the matrix elements of $\sum_{i} K^\mu_i K^{\mu}_i$, in the basis $\{a^\mu_k (0)\}_{k=0}^L$, namely:

$$\sum_{i=0}^L (K^\mu_i K^{\mu}_i)_{i,j} = \langle 0| a_i \sum_{l=0}^L K^\mu_l \left( \sum_{k=0}^L a^\mu_k |0\rangle \langle a_k | \right) K^\mu_i a^\mu_j |0\rangle$$

$$= \langle 0| a_i a_\mu U_t^\dagger \left( \sum_{k,l} f^\mu_{k,l} a^\mu_k |0\rangle \langle a_k f_l | \right)$$

$$\times U_\mu a^\mu_j |0\rangle,$$

(14)

where we used in the first line the identity $\sum_{k} a^\mu_k |0\rangle \langle a_k | = \mathbb{I}_{\mathcal{J}^{2L+1}}$.

Since $\{a_i\}$ and $\{f_i\}$ are both orthonormal bases, there exists a unitary $V$, of dimension $L + 1$, which performs the single particle transformation $f^\mu_i = \sum_{m} V_{m} a^\mu_{m}$, we can simplify the term

$$\left( \sum_{k,l} f^\mu_{k,l} a^\mu_k |0\rangle \langle a_k f_l | \right)$$

$$= \left( \sum_{k,l,m,n} V_{m,l} V^*_n a^\mu_m a^\mu_{k,l} |0\rangle \langle a_k a_n | \right)$$

$$= \left( \sum_{k,m} a^\mu_m a^\mu_k |0\rangle \langle a_k a_m | \right) = 2 \mathbb{I}_{\mathcal{J}^{2L+1}},$$

(15)

therefore, we have:

$$\sum_{i=0}^L (K^\mu_i K^{\mu}_i)_{i,j} = 2 \langle 0| a_i a_\mu a^\mu_j |0\rangle$$

$$= \begin{cases} 2, & \text{if } i = j, i \neq \mu, j \neq \mu, \\ 0, & \text{otherwise} \end{cases}.$$  

(16)

As mentioned before, fixing different values of the reference mode $\mu$, generates distinct maps $\Phi^\mu_t$ with domain $\mathcal{S}^{\mu,\text{pure}} (\mathcal{E})$. Now let us compare these distinct maps. We know that given two sets $\mathcal{S}^{\mu,\text{pure}} (\mathcal{E})$ and $\mathcal{S}^{\nu,\text{pure}} (\mathcal{E})$, with fixed modes $\mu$ and $\nu$, there exists a unitary $V \in \mathcal{U}(\mathcal{J}^{2L+1})$ such that $a^\mu_k a^\mu_j |0\rangle = V a^\nu_k a^\nu_j |0\rangle$. Therefore, any pair of maps $\Phi^\mu_t$ and $\Phi^\nu_t$ have the Kraus operators $\{K^\mu_i = f_i U_i a^\mu_i\}_j$ and $\{E^\nu_j = f_j U_j V a^\nu_j\}_j$, respectively. We can compute an upper bound to the norm difference of the (Choi-Jamiolkowski) dynamical matrices $D_{\Phi^\mu}$ and $D_{\Phi^\nu}$, associated with the maps, which is proved in Appendix B.B.1:

$$\|D_{\Phi^\mu} - D_{\Phi^\nu}\|_1 \leq d^2 L^2 \sup_{a^\mu_k |0\rangle \langle a^\nu_n | \in \mathcal{J}^{2L+1}} \| \left( a^\mu_k |0\rangle \langle a^\nu_n | - V^T a^\nu_n |0\rangle \langle a^\nu_n | V^* \right) \|_1,$$

(17)

where $d$ is the dimension of $\mathcal{J}^{2L+1}$. It is illustrative to compare this bound with its counterpart in the case of distinguishable particles, where we have initially uncorrelated system $S$ and environment $E$ forming a closed global system, whose dynamics is described by a unitary $U_{S:E}$. Assuming two dynamical maps, $\Phi_t$ and $\Lambda_t$, constructed from different initial states of the environment, we have the two sets of Kraus operators $\{K_a = \langle a | U_{S:E} | 0\rangle\}_a$ and $\{E_a = \langle a | U_{S:E} (I_S \otimes V_E) | 0\rangle\}_a$, respectively. Then the following inequality, which is proved in Appendix B.B.2 holds:

$$\|D_{\Phi} - D_{\Lambda}\|_1 \leq d_S^2 \| |0\rangle \langle 0 | - V_E |0\rangle \langle 0 | V_E^* \|_1,$$

(18)

where $d_S$ is the dimension of the Hilbert space of the system $S$. It is important to emphasize that the two frameworks are completely different. A tensor product structure between system and environment is absent in our context of indistinguishable fermions. Another remark is that the two maps in the distinguishable particles case have the same domain, which in general is not true in the case of indistinguishable fermions.

IV. EXAMPLES OF ONE-PARTICLE DYNAMICAL MAPS OF INDISTINGUISHABLE FERMIONS

In this section we illustrate our formalism, deriving the Kraus operators for the dynamics of one-fermion reduced state of two distinct two-particle Hamiltonians. To simplify the discussion, we assume initial pure global state, such that the Kraus operators $\{K^\mu_i = f_i U_i a^\mu_i\}$ have domain given by Eq. (7).

A. Non-interacting Hamiltonian

Our first example, consisting of a non-interacting Hamiltonian, shows the consistency of our formalism. As no correlation can be created, and the initial global Hamiltonian, shows the consistency of our formalism. The Hamiltonian can be written in terms of fermionic operators as $H = \sum_{i,j} M_{i,j} a^\dagger_i a_j$, and has the
following diagonal form: \( H = \sum_k \lambda_k b_k^\dagger b_k \), where
\[
b_k^\dagger = \sum_i V_{k,i} a_i^\dagger, \tag{19}\n\]
\[
a_j^\dagger = \sum_k V_k^{*} b_k^\dagger, \tag{20}\n\]
\( \lambda_k \) are the single particle energy excitations and \( V \) is the unitary that diagonalizes \( M \). The dynamical evolution is given by the unitary \( U_t = \exp(-i \sum_k \lambda_k b_k^\dagger b_k) \). Now, we form the Kraus operators using Eq. [3], with the choice \( \{j_k^\dagger\}_k = \{b_k\}_k = 0 \), namely: \( K^\mu_j = b_t U_t a_\mu \). The matrix elements of the Kraus operator are explicitly:
\[
(K^\mu_j)_{m,n} = (0|b_m b_t U_t a_\mu b^n_1|0) = (0|b_m b_t U_t \left( \sum_k V^*_{k,m} b_k^\dagger \right) b^n_1|0) = \sum_k V^*_{k,m} e^{-i(\lambda_k + \lambda_\mu)} (\delta_{l,k} \delta_{m,n} - \delta_{m,k} \delta_{l,n}), \tag{21}\n\]
thus
\[
K^\mu_j = \sum_m e^{-i(\lambda + \lambda_\mu)} (V^*_{m,k} b_m^\dagger|0\rangle\langle 0|b_m - V^*_{m,k} b_m^\dagger|0\rangle\langle 0|b_t). \tag{22}\n\]
The map acts on its domain (Eq. [7]) as the unitary \( U_t \):
\[
\rho_t(t) = \frac{1}{2} \sum_{m,n} \left( V^*_{m,k} V_{n,k} + V^*_{m,k} V_{n,k} \right) e^{-i(\lambda_k - \lambda_\mu)} b_m^\dagger b_n|0\rangle\langle 0|b_t
\]
\[
= \rho_t(0) U_t^\dagger. \tag{23}\n\]

\section*{B. Four Level Interacting System}

Consider two spin-1/2 fermions, in a lattice of two sites, whose dynamics is given by the following Hamiltonian:
\[
H = -\sum_{\sigma=\uparrow,\downarrow} \left( a_{\uparrow}^\dagger \sigma a_{\uparrow} \sigma + h.c. \right) + \sum_{j=1}^{2} n_j n_j + v_1 n_1 n_2, \tag{24}\n\]
where \( a_{\uparrow,\downarrow}^\dagger \) and \( a_{\uparrow,\downarrow} \) are creation and annihilation operators, respectively, of a fermion at site \( j \) with spin \( \sigma \), \( a_{\uparrow,\downarrow} = a_{\uparrow}^\dagger a_{\downarrow} \) and \( n_j = n_{\uparrow,\downarrow} + n_{\downarrow,\uparrow} \) are the number operators. The first term of the Hamiltonian characterizes hopping (tunnelling) between sites, while the second and third terms characterize the on-site and intersite interactions, parametrized by \( u \) and \( v \), respectively. In the basis \( a_{\uparrow,\downarrow}^\dagger|0\rangle \in \mathcal{F}_2^2 \), where \( \vec{k} = (k_1, k_2) \) has six possible configurations,
\[
\vec{k} \in \{(1,\uparrow), (1,\downarrow), (1,\uparrow,\downarrow), (1,\downarrow,\uparrow), (1\downarrow,\uparrow), (2,\uparrow,\downarrow)\}, \tag{25}\n\]
we obtain the following matrix representation for the Hamiltonian:
\[
H = \begin{pmatrix}
  u & 0 & -1 & 1 & 0 & 0 \\
  0 & v & 0 & 0 & 0 & 0 \\
 -1 & 0 & v & 0 & 0 & -1 \\
 0 & 0 & 0 & v & 1 & 0 \\
 0 & 0 & 1 & 0 & v & 0 \\
 0 & -1 & 0 & 0 & 0 & u \\
\end{pmatrix}. \tag{26}\n\]

Now we form the Kraus operators \( K^\mu_j = a_j U_t a_\mu \), with the choice \( \{f_j^\dagger\}_k = \{a_k\}_k = 0 \). If the unitary \( V \) diagonalizes the Hamiltonian, \( D = VHV^\dagger \), we can write \( U_t \) as:
\[
U_t = \sum \left( e^{-iD_t} \right)_{k,k'} \left( V_{l,k,l,k'} \right)^{\dagger} \left( a_\mu \right)^{\dagger} \left( 0 \right) \langle 0 | a_\mu. \tag{27}\n\]

According to Eq. [6] we have:
\[
K^\mu_j = a_j U_t a_\mu = \sum \left( e^{-iD_t} \right)_{k,k'} \left( V_{l,k,l,k'} \right)^{\dagger} \left( a_\mu \right)^{\dagger} \left( 0 \right) \langle 0 | a_\mu. \tag{28}\n\]

Using the anti-commutation relations, the last line of Eq. [26] reduces to:
\[
\left( \delta_{j,k} a_{k_1}^\dagger a_{k_2}^\dagger \right) \left( 0 \right) \langle 0 | a_{k_2}^\dagger a_{k_1} \langle 0 | a_\mu, \tag{29}\n\]

and finally,
\[
K^\mu_j = \sum \left( e^{-iD_t} \right)_{k,k'} \left( V_{l,k,l,k'} \right)^{\dagger} \left( a_\mu \right)^{\dagger} \left( 0 \right) \langle 0 | a_{k_1} a_{k_2}. \tag{30}\n\]

The unitary \( V \) can now be written explicitly as:
\[
V = \begin{pmatrix}
  -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 \\
 a(u,v) & 0 & b(u,v) & -b(u,v) & 0 & a(u,v) \\
 b(u,v) & 0 & -a(u,v) & a(u,v) & 0 & b(u,v) \\
\end{pmatrix}, \tag{31}\n\]

while the explicit form of \( D \) is:
\[
D = \text{diag} \left( u, v, v, v, \frac{1}{2} \left( u + v - \sqrt{\Delta(u,v)^2 + 16} \right), \frac{1}{2} \left( u + v + \sqrt{\Delta(u,v)^2 + 16} \right) \right), \tag{32}\n\]

with \( \Delta(u,v) = v - u \),
\[
a(u,v) = \frac{\Delta(u,v) + \sqrt{\Delta(u,v)^2 + 16}}{2 \left[ \Delta(u,v) + \sqrt{\Delta(u,v)^2 + 16} \right] + 16}. \tag{33}\n\]
and
\[ b(u, v) = \frac{4}{\sqrt{2 \left( \Delta(u, v) + \sqrt{\Delta(u, v)^2 + 16} \right)^2 + 16}}. \]

V. CONCLUSION

In systems of indistinguishable fermions, antisymmetrization eliminates the notion of separability, and the very concept of correlation, which is an important ingredient in obtaining CP maps for open systems, becomes subtle. We showed that it is possible to write a CP map for a single fermion, which is part of a system on \( N \) indistinguishable particles, for sets of initial global states with no quantumness of correlation. We also illustrated our formalism with examples of CP maps corresponding to a non-interacting and an interacting Hamiltonian of two fermions. The extension of our formalism to subsystems with more than one indistinguishable particle, and for the case of bosons presents no difficulty. As many properties of many-body Hamiltonians can be inferred from the single particle reduced state, an interesting investigation would be if any computational gain can be obtained by the employment of the formalism developed in this article.

ACKNOWLEDGMENTS

We acknowledge financial support by the Brazilian agencies INCT-IQ (National Institute of Science and Technology for Quantum Information), FAPEMIG, and CNPq.

Appendix A: Dynamical Map for Single-Fermion Reduced State - General Case with Initial Mixed States

A.1. System of Two Fermions

Consider a set of mixed quantum states in the antisymmetric space of \( L + 1 \) modes and two fermions, \( \rho(0) \in \mathcal{F}_2^{L+1} \), written in a basis of Slater determinants:
\[ S_2^p = \left\{ \rho(0) = \sum_{\mu \in \Sigma, k \in \Gamma} p(\mu) q(k) a_\mu^\dagger a_k^\dagger |0\rangle a_\mu a_k \mid p \text{ fixed} \right\}, \tag{A1} \]
with both \( \Sigma \) and \( \Gamma \) finite, and disjoint, \( \Sigma \cap \Gamma = \emptyset \). Let \(|\Sigma| = d, |\Gamma| = L - d\), and \( Z_{L+1} = \{0, 1, \ldots, L\} \). We took the \( d \) elements of \( \Sigma \) from \( Z_{L+1} \), and the set \( \Gamma \) as
\[ Z_d \setminus \Sigma. \]
Tracing out one fermion from \( S_2^p \), we obtain the single-fermion reduced states, \( \{\rho_r(0)\} \):
\[ S_{r(2)}^p = \left\{ \rho_r(0) = \frac{1}{2} \sum_{k \in \Gamma} q(k) a_k^\dagger |0\rangle a_k + \frac{1}{2} \sum_{\mu \in \Sigma} p(\mu) a_\mu^\dagger |0\rangle a_\mu \mid \text{p fixed} \right\}. \tag{A2} \]

Definition 3. A CP map \( \Phi_\rho^p \), describing the dynamics of the single particle reduced state \( \rho_r(0) \in S_{r(2)}^p \), can be written in Kraus representation as:
\[ \Phi_\rho^p[\rho_r(0)] = \sum_{j=0}^L \sum_{\mu \in \Sigma} K_{j,\mu}^p \rho_r(0) K_{j,\mu}^p \dagger, \tag{A3} \]
with the Kraus operators:
\[ K_{i,\mu}^p = f_i U_i a_\mu^\dagger \sqrt{p(\mu)} \prod_{m \in \Sigma} \left( 1 - a_m^\dagger a_m \right), \tag{A4} \]
\[ \text{Proof.} \] The one-particle reduced dynamics can be expressed as \( \rho_r(t) = Tr_1(U_t \rho(0) U_t^\dagger) \):
\[ \rho_r(t) = \frac{1}{2} \sum_{k=0}^L f_k U_k \left( \sum_{\mu \in \Sigma, k \in \Gamma} p(\mu) q(k) a_\mu^\dagger a_k^\dagger |0\rangle \langle 0| a_\mu a_k \right) U_k^\dagger f_k^\dagger \]
\[ = \sum_{i=0}^L \sum_{\mu \in \Sigma} \sqrt{p(\mu)} f_i U_i a_\mu^\dagger \left( \frac{1}{2} \sum_{k \in \Gamma} q(k) a_k^\dagger |0\rangle \langle 0| a_k \right) \times \sqrt{p(\mu)} a_\mu U_i^\dagger f_i^\dagger. \tag{A5} \]
Defining an operator \( \prod_{m \in \Sigma} (1 - a_m^\dagger a_m) \) that annihilates fermions in \( \Sigma \), and leaves states unchanged otherwise, we can write
\[ \rho_r(t) = \frac{1}{2} \sum_{k=0}^L f_k U_k a_\mu^\dagger \prod_{m \in \Sigma} \left( 1 - a_m^\dagger a_m \right) \times \left( \frac{1}{2} \sum_{k \in \Gamma} q(k) a_k^\dagger |0\rangle \langle 0| a_k \right) \prod_{m \in \Sigma} \left( 1 - a_m^\dagger a_m \right) a_\mu U_i^\dagger f_i^\dagger \sqrt{p(\mu)}. \tag{A6} \]
Note that
\[ \prod_{m \in \Sigma} (1 - a_m^\dagger a_m) \left( \frac{1}{2} \sum_{j \in \Sigma} p(j) a_j^\dagger |0\rangle \langle 0| a_j \right) = 0. \tag{A7} \]
Adding Eq. (A7) to Eq. (A6), Definition 3 is proven:

\[ \rho_r(t) = \sum_{l=0}^{L} \sum_{\mu \in \Sigma} f_l U_l a_{\mu}^\dagger \sqrt{p(\mu)} \prod_{m \in \Sigma} (1 - a_{m}^\dagger a_m) \times \left( \frac{1}{2} \sum_{k \in \Gamma} q(k) a_k^{\dagger} \langle 0 | 0 \rangle a_k + \frac{1}{2} \sum_{j \in \Sigma} p(j) a_j^{\dagger} \langle 0 | 0 \rangle a_j \right) \]

\[ \times \prod_{m \in \Sigma} \left(1 - a_{m}^\dagger a_m\right) \sqrt{p(\mu)} a_m U_l^\dagger f_l^\dagger \]

\[ = \sum_{l=0}^{L} \sum_{\mu \in \Sigma} K_{l,\mu}^p \rho_r(0) K_{l,\mu}^{p \dagger} \] \hspace{1cm} (A8)

\[ \square \]

### A.2. System of N-Fermions

Consider a set of states \( \rho \in \mathcal{F}_{N+1}^{L+1} \), with no quantum-

\[ \mathcal{S}_N^p = \begin{cases} \rho(0) = \sum_{\mu \in \Sigma} \sum_{k \in \Gamma} p(\mu_1, \cdots, \mu_{N-1}) q(k) \\
\times a_{\mu} k \langle 0 | 0 \rangle a_k, \text{ } p \text{ fixed} \end{cases} \] \hspace{1cm} (A9)

where \( \mu = (\mu_1, \cdots, \mu_{N-1}) \), \( \Sigma = (\Sigma_1, \cdots, \Sigma_{N-1}) \) are \( N-1 \)-tuples, and \( p(\mu) \), \( q(k) \) are probability distributions. The sets \( \Sigma_j \) and \( \Gamma \) are finite, and disjoint \( \Sigma_j \cap \Gamma = \emptyset \) \( \forall j \). With \( |\Sigma| = \bigcup_{i=1}^{N-1} \Sigma_i = d \), \( |\Gamma| = L - d \), and \( \mathcal{Z}_{L+1} = \{0, 1, \cdots, L\} \), we took the \( d \) elements of \( \bigcup_{i=1}^{L-N-1} \Sigma_i \) from \( \mathcal{Z}_{L+1} \), and the set \( \Gamma \) as \( \mathcal{Z}_d \setminus \bigcup_{i=1}^{N-1} \Sigma_i \). Note that \( d \) is the number of accessible modes for \( N-1 \) fermions, thus \( d \geq N - 1 \).

Tracing \( N-1 \) fermions out from (A9), we obtain the set of single-fermion reduced states \( \rho_r(0) \):

\[ \mathcal{S}_{r(N)}^p = \begin{cases} \rho_r(0) = \frac{1}{N} \sum_{k \in \Gamma} q(k) a_k^{\dagger} \langle 0 | 0 \rangle a_k + \sum_{j=1}^{L-N-1} \sum_{\mu \in \Sigma_j} p_j(\mu) a_{\mu}^{\dagger} \langle 0 | 0 \rangle a_{\mu}, \text{ } p_j \text{ fixed} \end{cases} \] \hspace{1cm} (A10)

where \( p_j(\mu) = \sum_{\mu \setminus \mu_j} p(\mu_1, \cdots, \mu_{N-1}) \) is the marginal distribution.

**Definition 4.** A CP map \( \Phi_r^p \) describing the dynamics of the single particle reduced state \( \rho_r(0) \in \mathcal{S}_{r(N)}^p \), can be written in Kraus representation as:

\[ \Phi_r^p[\rho_r(0)] = \sum_{l, \mu} K_{l,\mu}^p \rho_r(0) K_{l,\mu}^{p \dagger}, \] \hspace{1cm} (A11)

with the Kraus operators:

\[ K_{l,\mu}^p = \sqrt{p(\mu_1, \cdots, \mu_{N-1}) \times f_l U_l a_{\mu}^\dagger \prod_{m \in \Sigma_i} (1 - a_{m}^\dagger a_m) .} \] \hspace{1cm} (A12)

The proof of Definition 4 is *mutatis mutandis* the same performed for Definition 3.

### Appendix B: Norm Bound

#### B.1. Fermionic System

**Theorem 1.** Consider two maps \( \Phi \) and \( \Lambda \), with Kraus operators \( K_j = f_j U_{a_{\mu}} \), and \( E_j = f_j UV_{a_{\mu}} \), respectively. Then the following inequality holds:

\[ \| D_\Phi - D_\Lambda \|_1 \leq \sup_{a_{\mu}^\dagger |0 \rangle \langle 0 | a_{\nu}^\dagger \in \mathcal{F}_{L+1}^{2L}} \| \left( a_{\mu}^\dagger |0 \rangle \langle 0 | a_{\nu}^\dagger - V^T a_{\mu}^\dagger |0 \rangle \langle 0 | V^* \right) \|_1, \] \hspace{1cm} (B1)

where \( d \) is the dimension of \( \mathcal{F}_{L+1}^{2L} \), \( (k_1, k_2) \) is a 2-tuple indicating the modes occupied by a pair of fermions, with \( k_1 = 0, \cdots, L \), and \( V \) is a unitary operator, \( V : \mathcal{F}_{L+1}^{2L} \rightarrow \mathcal{F}_{L+1}^{2L} \).

**Proof.** Writing the dynamical matrix of a map \( \Phi \) in terms of the Kraus operators \( \{K_j\} \):

\[ D_\Phi = \sum_j \text{vec}(K_j) \text{vec}(K_j)^\dagger, \] \hspace{1cm} (B2)

where the vec operation is defined by \( \text{vec}(|x\rangle \langle y|) = |x\rangle \otimes |y\rangle \), we obtain:

\[ \| D_\Phi - D_\Lambda \|_1 = \| \sum_j \left( \text{vec}(K_j) \text{vec}(K_j)^\dagger - \text{vec}(E_j) \text{vec}(E_j)^\dagger \right) \|_1 \]

\[ = \| \sum_j \left( \text{vec}(f_j U_{a_{\mu}}) \text{vec}(f_j U_{a_{\mu}})^\dagger - \text{vec}(a_{\mu} UV_{a_{\nu}}) \text{vec}(a_{\nu} UV_{a_{\mu}})^\dagger \right) \|_1. \] \hspace{1cm} (B3)

Using the following identity for matrices:

\[ \text{vec}(ABC) = (A \otimes C^T) \text{vec}(B), \] \hspace{1cm} (B4)

we have,

\[ \| D_\Phi - D_\Lambda \|_1 = \| \sum_j \left( f_j \otimes a_{\mu}^\dagger \text{vec}(U) \text{vec}(U)^\dagger a_{\mu}^\dagger - f_j \otimes a_{\mu}^\dagger \text{vec}(V^T \text{vec}(U))^\dagger f_j \otimes V^* a_{\mu}^\dagger \right) \|_1. \] \hspace{1cm} (B5)
With the unitary operator $U$ written as,

$$U = \sum_{\vec{k},\vec{\ell}} u_{\vec{k},\vec{\ell}} a_{\vec{k}}^\dagger |0\rangle \langle a_{\vec{\ell}}, \quad (B6)$$

where $\vec{k} = (k_1, k_2)$, Eq. (B5) becomes:

$$\|D_\Phi - D_\Lambda\|_1 = \| \sum_{\vec{k},\vec{\ell},\vec{l},\vec{r}} u_{\vec{k},\vec{l}}^* u_{\vec{k},\vec{l}} \left[ \left( f_{\vec{k}} a_{\vec{k}}^\dagger |0\rangle \otimes a_{\vec{\ell}}^\dagger |0\rangle \right) \times \left( |0\rangle f_{\vec{r}}^\dagger \otimes |0\rangle V^* a_{\vec{l}}^\dagger |0\rangle \right) \right] \|_1 + \| \sum_{\vec{k},\vec{\ell},\vec{l},\vec{r}} u_{\vec{k},\vec{l}}^* u_{\vec{k},\vec{l}} \left[ \left( f_{\vec{k}} a_{\vec{k}}^\dagger |0\rangle \otimes a_{\vec{\ell}}^\dagger |0\rangle \right) \times \left( |0\rangle f_{\vec{r}}^\dagger \otimes |0\rangle V^* a_{\vec{l}}^\dagger |0\rangle \right) \right] \|_1. \quad (B7)$$

Using some norm properties, as triangle inequality ($\|X + Y\| \leq \|X\| + \|Y\|$), positive scalability ($\|\alpha X\| = |\alpha| \|X\|, \alpha \in \mathbb{C}$), and tensor product ($\|X_1 \otimes X_2\| = \|X_1\| \|X_2\|$) and the definition of fermionic partial trace of one particle, we can write:

$$\|D_\Phi - D_\Lambda\|_1 \leq \sum_{\vec{k},\vec{l}} \left| u_{\vec{k},\vec{l}}^* u_{\vec{k},\vec{l}} \right| \left\| Tr_1 \left( a_{\vec{k}}^\dagger |0\rangle \langle a_{\vec{l}} \right) \right\|_1 \times \left\| a_{\vec{\ell}}^* \left( a_{\vec{k}}^\dagger |0\rangle \langle a_{\vec{\ell}} - V^T a_{\vec{k}}^\dagger |0\rangle \langle a_{\vec{\ell}} V^* a_{\vec{k}}^\dagger \right) a_{\vec{l}}^T \right\|_1. \quad (B8)$$

As the trace norm is non-increasing under partial trace ($\|Tr_{X_1}(X)\|_1 \leq \|X\|_1$), we also have $\|X\|_1 = \|X^T\|_1 = \|X \|_1$. Therefore:

$$\|D_\Phi - D_\Lambda\|_1 \leq \sum_{\vec{k},\vec{l}} \left| u_{\vec{k},\vec{l}}^* u_{\vec{k},\vec{l}} \right| \left\| a_{\vec{k}}^\dagger |0\rangle \langle a_{\vec{l}} \right\|_1 \times \left\| a_{\vec{\ell}}^* \left( a_{\vec{k}}^\dagger |0\rangle \langle a_{\vec{\ell}} - V^T a_{\vec{k}}^\dagger |0\rangle \langle a_{\vec{\ell}} V^* a_{\vec{k}}^\dagger \right) a_{\vec{l}}^T \right\|_1. \quad (B9)$$

As $\|a_{\vec{k}}^\dagger |0\rangle \langle a_{\vec{l}}\|_1 = Tr\sqrt{a_{\vec{k}}^\dagger a_{\vec{l}}} = 1$, and $\|a_{\vec{\ell}}\|_1 = Tr\sqrt{a_{\vec{\ell}}} = L$ is the number of states $\{a_{\vec{k}}^\dagger |0\rangle\}$ with occupied mode $\mu$:

$$\|D_\Phi - D_\Lambda\|_1 \leq L^2 \sum_{\vec{k},\vec{l}} \left| u_{\vec{k},\vec{l}}^* u_{\vec{k},\vec{l}} \right| \left\| a_{\vec{k}}^\dagger |0\rangle \langle a_{\vec{l}} - V^T a_{\vec{k}}^\dagger |0\rangle \langle a_{\vec{l}} V^* \right\|_1. \quad (B10)$$

From the definition of unitary operators we have, $\sum_k u_{k\ell}^* u_{k\ell} = \delta_{i,j}$, therefore:

$$\|D_\Phi - D_\Lambda\|_1 \leq L^2 \sum_{\vec{k},\vec{l}} \left| a_{\vec{k}}^\dagger |0\rangle \langle a_{\vec{l}} - V^T a_{\vec{k}}^\dagger |0\rangle \langle a_{\vec{l}} V^* \right\|_1. \quad (B12)$$

Finally:

$$\|D_\Phi - D_\Lambda\|_1 \leq d^2 L^2 \sup_{a_{\vec{k}}^\dagger |0\rangle \langle a_{\vec{l}} \in \mathcal{X}_{\vec{k}}^+} \left\| a_{\vec{k}}^\dagger |0\rangle \langle a_{\vec{l}} - V^T a_{\vec{k}}^\dagger |0\rangle \langle a_{\vec{l}} V^* \right\|_1. \quad (B13)$$

B.2. System of Distinguishable Particles

Theorem 2. Assume two maps $\Phi$ and $\Lambda$, with Kraus operators $\{K_a = \langle a|U_{SE}|0\rangle\}_a$ and $\{E_a = \langle a|U_{SE}(|I_S \otimes V_E)|0\rangle\}_a$, respectively. Then the following inequality holds:

$$\|D_\Phi - D_\Lambda\|_1 \leq d_S^2 \|0\rangle \langle 0| - V_E|0\rangle \langle 0| V_E^\dagger \|_1, \quad (B14)$$

where $d_S$ is the dimension of the Hilbert space of the system $S$.

Proof. Writing the dynamical matrix of a map $\Phi$ in the Choi representation:

$$D_\Phi = \sum_{i,j=1}^{d_S} \Phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|, \quad (B15)$$

we obtain:

$$\|D_\Phi - D_\Lambda\|_1 = \frac{1}{d_S^2} \sum_{i,j=1}^{d_S} \Phi(|i\rangle\langle j|) \otimes |i\rangle\langle j| - \sum_{i,j=1}^{d_S} \Lambda(|i\rangle\langle j|) \otimes |i\rangle\langle j| \|_1$$

$$= \left\| \sum_{i,j=1}^{d_S} \sum_a K_a |i\rangle K_a^\dagger - E_a |i\rangle E_a^\dagger \right\|_1 \otimes |i\rangle\langle j| \|_1$$

$$\leq \sum_{i,j=1}^{d_S} \left\| \sum_a K_a |i\rangle K_a^\dagger - E_a |i\rangle E_a^\dagger \right\|_1 \otimes |i\rangle\langle j| \|_1. \quad (B16)$$

Thus, by the definition of Kraus operators above:

$$K_a |i\rangle K_a^\dagger - E_a |i\rangle E_a^\dagger =$$

$$= \langle a| \left\{ U_{SE}|i\rangle \langle j| \otimes (0\rangle \langle 0|_E - V|0\rangle \langle 0|_E V^\dagger) U_{SE}^\dagger \right\} |a\rangle E. \quad (B17)$$
substituting in Eq. \( \text{[B16]} \), and using \( \| X \otimes Y \| = \| X \| \| Y \| \):

\[
\| D_\Phi - D_\Lambda \|_1 \leq d_2^2 \| \left\{ \sum_a \left( a \left| U_{SE}^{-1} | i \rangle \otimes \langle 0 \rangle \langle 0 | - V | 0 \rangle \langle 0 | V^\dagger \right| U_{SE}^{-1} \right) a \right\} \|_1
\]

\[
= d_2^2 \| \left\{ \left| U_{SE}^{-1} | j \rangle \otimes \langle 0 \rangle \langle 0 | - V | 0 \rangle \langle 0 | V^\dagger \right| U_{SE}^{-1} \right) \|_1,
\]

where we used that \( \sum_a \langle a | X | a \rangle = \text{Tr}_F(Y) \). Finally, as trace distance is invariant under unitary operations, the statement is proved:

\[
\| D_\Phi - D_\Lambda \|_1 \leq d_2^2 \| (\langle 0 | - V | 0 \rangle | V^\dagger) \|_1.
\]

**B19**