VARIATIONAL METHODS FOR BREACHER SOLUTIONS OF NONLINEAR WAVE EQUATIONS

RAINER MANDEL AND DOMINIC SCHEIDER

Abstract. We construct infinitely many real-valued, time-periodic breather solutions of the nonlinear wave equation
\[ \partial_{tt}U - \Delta U = Q(x)|U|^{p-2}U \quad \text{on } \mathbb{T} \times \mathbb{R}^N \]
with suitable \( N \geq 2, \ p > 2 \) and localized nonnegative \( Q \). These solutions are obtained from critical points of a dual functional and they are weakly localized in space. Our abstract framework allows to find similar existence results for the nonlinear Klein-Gordon equation and biharmonic wave equations.

1. Introduction

Breathers are real-valued, time-periodic and spatially localized solutions of nonlinear equations describing the propagation of waves on \( \mathbb{R}^N \times \mathbb{R} \) where \( N \in \mathbb{N} \). The existence of breather solutions appears to be a rare phenomenon and up to now, most work in this area is related to the discussion of explicit examples such as the famous sine-Gordon breather for the \((1+1)\)-sine-Gordon equation [1]. A number of (in-)stability results for such explicit breathers [2–5] is available. Nonexistence results can be found in [10, 23, 29]. The construction of non-explicit breather solutions is a very difficult task. In papers by Hirsch, Reichel [19, Theorem 1.3] and Blank, Chirilus-Bruckner, Lescarret, Schneider [6] this was achieved for nonlinear wave equations of the form
\[ s(x)\partial_{tt}u - u_{xx} + q(x)u = f(x, u) \quad (x, t \in \mathbb{R}) \]
following two completely different approaches. The methods from [6] come from spatial dynamics and rely on center manifold reductions. For one very specific choice of periodic step functions \( s, q \) (multiples of each other) and the nonlinearity \( f(x, u) = u^3 \), the authors prove the existence of \( L^\infty \)-small periodic breather solutions that are exponentially localized in space. The particular choice for \( s \) and \( q \) is motivated by the underlying spectral theory of periodic Hill operators, also called Floquet theory. Using variational methods instead, Hirsch and Reichel [19] proved the existence of (spatially) square integrable breather solutions under appropriate assumptions on the nonlinearity. The latter include power-type nonlinearities \( f(x, u) = |u|^{p-1}u \) with \( 1 < p < p^* \) for some \( p^* \) depending on the choice of \( s \) and \( q \). Again, the potentials \( s, q \) are of very special form in order to ensure suitable spectral properties. More precisely, it is required that for all \( k \in \mathbb{Z} \) the spectrum of the linear operator associated with the \( k \)-th mode does not contain 0 in a uniform sense. This makes it possible to have

Date: January 13, 2021.

Key words and phrases. Wave equation, Breather, Dual variational methods, Helmholtz equation.
a strong localization in space. All of these results concern the case of one spatial dimension $N = 1$. The Bethe-Sommerfeld Conjecture about the number of gaps of periodic Schrödinger operators (see [24, Section 6.1.3]) suggests that the above approach can hardly be generalized to higher space dimensions that we discuss next.

In the case $N \geq 2$ we are aware of very few results. The first deals with a semilinear curl-curl wave equation in $\mathbb{R}^3 \times \mathbb{R}$ where $-u_{xx}$ is replaced by $\nabla \times \nabla \times u$ in (1) and $u$ is a three-dimensional vector field on $\mathbb{R}^3$. Using that this part in the equation actually vanishes for gradient fields, Plum and Reichel [30] succeeded in proving the existence of exponentially localized breather solutions via ODE methods for suitable radially symmetric coefficient functions $s, q$ and power-type nonlinearities $f$. As far as we know, this is the only result dealing with strongly localized breathers in higher dimensions, i.e., $U(t, \cdot) \in L^2(\mathbb{R}^N)$ for almost all $t \in \mathbb{R}$. Recently, the second author suggested a new construction of (even in time) breathers [32] for the cubic Klein-Gordon equation that we will refer to as weakly localized in space. Those satisfy $U(\cdot, t) \in L^q(\mathbb{R}^N)$ for almost all $t \in \mathbb{R}$ for some $q > 2$ and we believe that in general $U(t, \cdot) \notin L^2(\mathbb{R}^N)$ holds due to rather small decay rates at infinity, presumably $U(t, x) \sim |x|^{(1-N)/2}$ as $|x| \to \infty$. That approach relies on the $L^p$-theory for Helmholtz equations on $\mathbb{R}^N$ and bifurcation techniques allow to prove the existence of infinitely many branches consisting of polychromatic radially symmetric breather solutions that emanate from a nontrivial stationary solution of the problem. The solutions are of the form

$$U(t, x) = \sum_{k \in I_s} e^{ikt} u_k(x), \quad I_s \subseteq \mathbb{Z}$$

with radially symmetric Fourier modes $u_k = \overline{u}_{-k}$ infinitely many of which are non-zero. Imposing radial symmetry is of course a significant restriction.

In this paper we propose a dual variational approach for the construction of weakly localized breather solutions of abstract nonlinear wave equations without any symmetry assumptions on the coefficients. This approach goes back to Brézis, Coron, Nirenberg [7] and Rabinowitz [31] in the context of classical nonlinear wave equations in bounded domains. Dual variational methods have the advantage that they overcome the strong indefiniteness of the corresponding functional in the classical variational formulation. They apply in the case $N \geq 2$ and do not rely on specific “nonstandard” choices for the elliptic part of the wave operator that seem to be needed for the construction of strongly localized breathers via classical variational methods when $N = 1$, see [19]. The dual variational method is typically implemented in function spaces different from $L^2$ or $H^1$ so that the constructed solutions do not necessarily give rise to strongly localized breathers. In fact, we shall prove the existence of weakly localized breathers only. It is entirely unclear whether strongly localized breathers of nonlinear wave equations exist in the case $N \geq 2$.

We prove the existence of weakly localized breathers for nonlinear wave equations of the form

$$\partial_t U + \mathcal{L} U = Q(x) |U|^{p-2} U \quad \text{on } \mathbb{R} \times \mathbb{R}^N.$$
Since we want (3) to hold in a distributional sense that we will make precise below, we only need to assume that there is some \( m \in \mathbb{N} \) and a suitable \( q \in [p, \infty] \) such that \( \mathcal{L} : W^{m, \infty}(K) \to L^q(\mathbb{R}^N) \) is a bounded linear map for all compact \( K \subset \mathbb{R}^N \), see assumption (A1) below. Here and in the following, \( q' \) denotes the Hölder conjugate exponent of \( q \) defined by \( \frac{1}{q} + \frac{1}{q'} = 1 \). As a model case one may have in mind \( \mathcal{L} = -\Delta \). In contrast to [32] we can deal with nonradial \( Q \) and obtain the existence of an unbounded sequence of breathers. In order to avoid “bad” modes, we look for breather solutions that enjoy additional symmetry properties with respect to time. To include such symmetries in our analysis we introduce a parameter \( s \in \{1, \ldots, 5\} \) that stands for

\[
\begin{align*}
(s = 1) & \text{ no additional symmetry,} \\
(s = 2) & \text{ } U(t, x) = U(-t, x), \text{ i.e., } U \text{ is even in time,} \\
(s = 3) & \text{ } U(t, x) = -U(-t, x), \text{ i.e., } U \text{ is odd in time,} \\
(s = 4) & \text{ } U(t + \pi, x) = U(t, x), \text{ i.e., } U \text{ is } \pi\text{-periodic,} \\
(s = 5) & \text{ } U(t + \pi, x) = -U(t, x), \text{ i.e., } U \text{ is } \pi\text{-antiperiodic.}
\end{align*}
\]

For those symmetries the relevant modes \( k \in \mathcal{I}_s \subset \mathbb{Z} \) in the corresponding Fourier expansions come from the sets

\[
\mathcal{I}_1 := \mathbb{Z}, \quad \mathcal{I}_2 := \mathbb{Z}, \quad \mathcal{I}_3 := \mathbb{Z} \setminus \{0\}, \quad \mathcal{I}_4 := 2\mathbb{Z}, \quad \mathcal{I}_5 := 2\mathbb{Z} + 1.
\]

Accordingly, we look for functions \( u_k = \overline{u_k} \) \( (k \in \mathcal{I}_s) \) in order to ensure that the solution \( U \) given by (2) is real-valued. In the case \( s = 2 \) resp. \( s = 3 \) observe that the symmetry assumption even requires \( u_k \) resp. \( iu_k \) to be real-valued. Regarding \( s = 4 \) let us mention that general periods \( T > 0 \) can be discussed, as we will explain in Section [2.5]. We will assume that the following conditions are satisfied for \( k \in \mathcal{I}_s \):

(A1) There are bounded symmetric operators \( \mathcal{R}_k : L^q(\mathbb{R}^N) \to L^q(\mathbb{R}^N) \) for some \( q \in [p, \infty] \) that satisfy \( \mathcal{R}_k = \mathcal{R}_{-k}, \| \mathcal{R}_k \| \leq C(k^2 + 1)^{-\frac{\alpha}{2}} \) for some \( \alpha > 1 - \frac{2}{p} \) as well as

\[
\int_{\mathbb{R}^N} \mathcal{R}_k f \cdot (\mathcal{L} - k^2) \phi \, dx = \int_{\mathbb{R}^N} f \phi \, dx \quad \text{for all } f \in L^q(\mathbb{R}^N), \phi \in C_c^\infty(\mathbb{R}^N)
\]

where, for some \( m \in \mathbb{N} \), \( \mathcal{L} : W^{m, \infty}(K) \to L^q(\mathbb{R}^N) \) is a bounded linear map for all compact \( K \subset \mathbb{R}^N \).

(A2) \( Q \in L^q/(q-p)(\mathbb{R}^N), Q \geq 0, Q \not\equiv 0 \) and the linear operators \( v \mapsto \mathcal{R}_k^Q[v] := Q^{1/p} \mathcal{R}_k[Q^{1/p}v] \) are compact from \( L^p(\mathbb{R}^N) \) to \( L^p(\mathbb{R}^N) \).

(A3) There are \( \omega_k \in L^p(\mathbb{R}^N) \) with \( \int_{\mathbb{R}^N} \omega_k \mathcal{R}_k^Q[\omega_k] \, dx > 0 \).

Our convention is that functions belonging to \( L^r(\mathbb{R}^N), 1 \leq r \leq \infty \), are real-valued and \( \mathcal{R}_k \) is extended to complex-valued functions by linearity, i.e., \( \mathcal{R}_k(f + ig) := \mathcal{R}_k f + i\mathcal{R}_k g \) for \( f, g \in L^q(\mathbb{R}^N) \).

We briefly comment on these assumptions. The operators \( \mathcal{R}_k \) from (A1) can be interpreted as distributional right inverses of \( \mathcal{L} - k^2 \) that may even exist when classical inverses are not available. This is for instance the case in our model example \( \mathcal{L} = -\Delta \) as we will show in
we require these functions to solve equation (3) in the following sense: notably for the verification of the Palais-Smale condition for the dual functional

The growth bound on the norms ensures the convergence of Fourier series in topologies that are suitable for our analysis. Assumption (A2) is needed for our dual variational approach, notably for the verification of the Palais-Smale condition for the dual functional. In contrast, (A2) is often violated in the important special case $Q \equiv 1$ so that our approach does not apply in this case. We refer to Section 2.5 for further observations about this case and other more general nonlinearities. Finally, (A3) is a technical assumption that holds in many applications. It is for instance satisfied if $Q$ is positive and for all $k \in \mathcal{I}$ there are test functions $\phi_k \in C_c^\infty(\mathbb{R}^N)$ such that the inequality $\int_{\mathbb{R}^N} \phi_k (\mathcal{L} - k^2) \phi_k \, dx > 0$ holds. Indeed, choosing $\omega_k := \omega_{k,\delta}^\phi$ for sufficiently small $\delta > 0$ where $\omega_{k,\delta}^\phi := Q^{-1/p}1_{Q>\delta}(\mathcal{L} - k^2)\phi_k \in L^{p'}(\mathbb{R}^N)$ we obtain

$$\lim_{\delta \to 0^+} \int_{\mathbb{R}^N} \omega_{k,\delta}^{\phi} \mathcal{R}_k \left[ \omega_{k,\delta}^{\phi} \right] \, dx = \lim_{\delta \to 0^+} \int_{\mathbb{R}^N} (1_{Q>\delta}(\mathcal{L} - k^2)\phi_k) \mathcal{R}_k [1_{Q>\delta}(\mathcal{L} - k^2)\phi_k] \, dx$$

$$= \int_{\mathbb{R}^N} ((\mathcal{L} - k^2)\phi_k) \mathcal{R}_k [(\mathcal{L} - k^2)\phi_k] \, dx$$

$$= \int_{\mathbb{R}^N} \phi_k (\mathcal{L} - k^2) \phi_k \, dx > 0.$$ 

Hence, (A3) holds for positive $Q$ provided that $\mathcal{L}$ is a uniformly elliptic differential operator on $\mathbb{R}^N$, say of the form $\sum_{|\alpha| \leq 2m} a_\alpha(x) \partial^\alpha$ with locally integrable coefficients $a_\alpha$ and $\sum_{|\alpha|=2m} a_\alpha(x)(i\xi)^\alpha \geq c|\xi|^{2m}$ for all $x, \xi \in \mathbb{R}^N$ and some $c > 0$. Indeed, in that case one may choose $\phi_k = \phi(t_k \cdot)$ for some test function $\phi \in C_c^\infty(\mathbb{R}^N)$ and $t_k > 0$ sufficiently large, see the proof of Corollary 4 for the details in the case $\mathcal{L} = (-\Delta)^2$. In Section 2 we will see a number of settings where all our assumptions hold.

Assuming (A1)-(A3) for all modes $k \in \mathcal{I}$ we are going to prove the existence of breather solutions $U$ of (3) in the Banach space $L^q(\mathbb{R}^N, L^p(\mathbb{T}))$ consisting of all elements of $L^q(\mathbb{R}^N, L^p(\mathbb{T}))$ having the symmetry indexed by $s$. The norm on these spaces is given by

$$\|W\|_{L^q(\mathbb{R}^N, L^p(\mathbb{T}))} := \left\| \left( \|W(\cdot,x)\|_{L^p(\mathbb{T})} \right)^q \right\|_q = \left( \int_{\mathbb{R}^N} \left( \int_{\mathbb{T}} |W(t,x)|^p \, dt \right)^{q/p} \, dx \right)^{1/q},$$

where $\| \cdot \|_q = \| \cdot \|_{L^q(\mathbb{R}^N)}$ is the standard norm in $L^q(\mathbb{R}^N)$ and $\mathbb{T} \simeq [0,2\pi]$ stands for the torus.

More precisely, we will speak of $2\pi$-periodic real-valued distributional breather solutions since we require these functions to solve equation (3) in the following sense:

$$\int_{\mathbb{T} \times \mathbb{R}^N} U (\partial_{tt} - \mathcal{L}) \Phi \, dt \, dx = \int_{\mathbb{T} \times \mathbb{R}^N} Q(x) |U|^{p-2}U \, \Phi \, dt \, dx \quad \forall \Phi \in C_c^\infty(\mathbb{R}^N, C_c^\infty(\mathbb{T})).$$
Here, \( \Phi \in C_c^\infty(\mathbb{R}^N, C^\infty(\mathbb{T})) \) means that there is a compact subset \( K \subseteq \mathbb{R}^N \) such that \( \Phi : \mathbb{T} \times \mathbb{R}^N \to \mathbb{R} \) is smooth, 2\( \pi \)-periodic in time and the support of \( \Phi(t, \cdot) \) is contained in \( K \) for all \( t \in \mathbb{T} \). Our main result is the following.

**Theorem 1.** Assume \( N \in \mathbb{N}, 2 < p < \infty \) and (A1)-(A3) for all \( k \in \mathcal{I}_s \) where \( s \in \{1, \ldots, 5\} \). Then the nonlinear wave equation \((3)\) admits an unbounded sequence of 2\( \pi \)-periodical distributional real-valued breather solutions \( U_j \in L^p(\mathbb{R}^N, L^p_\mathcal{T}) \), \( j \in \mathbb{N}_0 \), in the sense of \((5)\).

We add that our breather solutions are either constant or polychromatic. The latter means that at least two Fourier modes \( u_k, u_l \) with \( k \neq l, k, l \in \mathbb{N}_0 \) of the solution are non-zero. Indeed, plugging the ansatz \( U(t, x) = \cos(kt)u_k(x) \) or \( U(t, x) = \sin(kt)u_k(x) \) and nontrivial functions \( u_k \) into \((3)\) one infers \( k = 0 \), so \( U \) is necessarily constant in time. In our applications below this will be avoided by choosing \( s \in \{3, 5\} \) because of \( 0 \notin \mathcal{I}_s \). In the case \( p \in \{3, 4, \ldots\} \) we can conclude as in \cite{22} Theorem 1 (iii)] that nonconstant in time breathers have in fact infinitely many nontrivial modes, which we believe to be the typical situation. Concerning the regularity of our breather solutions, we point out that, proceeding as in the proof of Proposition \( 2 \) below, it is possible to prove (local) Sobolev-regularity of solutions provided that estimates of the form

\[
\|R_k f\|_{W^{m,r}(K)} \leq C(k^2 + 1)^{\frac{\hat{\alpha}}{2}}(\|f\|_{s_1} + \|f\|_{s_2})
\]

hold for suitable \( K \subseteq \mathbb{R}^N \), \( m \in \mathbb{N}, s_1, s_2, r \in [1, \infty] \) and \( \hat{\alpha} > 0 \) such that the corresponding Fourier series converge. However, we could not verify such an estimate for large enough \( \hat{\alpha} > 0 \) in our applications, so we have to leave the regularity issue as an open problem. We expect the Propositions \( 2 \) below, the regularity of our breather solutions, we point out that, proceeding as in the proof of Proposition \( 2 \) below, it is possible to prove (local) Sobolev-regularity of solutions provided that estimates of the form

\[
\|R_k f\|_{W^{m,r}(K)} \leq C(k^2 + 1)^{\frac{\hat{\alpha}}{2}}(\|f\|_{s_1} + \|f\|_{s_2})
\]

hold for suitable \( K \subseteq \mathbb{R}^N \), \( m \in \mathbb{N}, s_1, s_2, r \in [1, \infty] \) and \( \hat{\alpha} > 0 \) such that the corresponding Fourier series converge. However, we could not verify such an estimate for large enough \( \hat{\alpha} > 0 \) in our applications, so we have to leave the regularity issue as an open problem. We expect that in the case \( \mathcal{L} = -\Delta \) the spatial decay of our breather solutions is \( U(t, x) \sim |x|^{(1-N)/2} \) and hence similar to the spatial decay of monochromatic complex-valued solutions of the form \( e^{ikt}u(x) \) that are modeled by a single Helmholtz equation instead of infinitely many coupled ones. Similarly, we expect that the breathers oscillate in the sense that they have infinitely many nodal domains. Given that the physical model requires for real-valued solutions, our results indicate that complex-valued waves of the form \( e^{ikt}u(x) \) provide a reasonable simplified model for breather solutions.

We outline how this paper is organized. In Section \( 2 \) we show how Theorem \( 1 \) applies in concrete situations. In particular, we prove the existence of infinitely many breathers of nonlinear wave equations and Klein-Gordon equations on \( \mathbb{R}^N \). Moreover, we indicate further possible generalizations of our approach. In Section \( 3 \) we motivate our variational approach and present the proof of Theorem \( 1 \) relying on the technical results contained in the Propositions \( 2 \). The proofs of the latter are presented in Section \( 4 \).

### 2. Applications and Examples

#### 2.1. Breather Solutions for the Wave Equation on \( \mathbb{R}^N \)

We show that Theorem \( 1 \) applies to classical nonlinear wave equations on \( \mathbb{R}^N \) with power-type nonlinearities

\[
\partial_t U - \Delta U = Q(x)|U|^{p-2}U \quad \text{on } \mathbb{T} \times \mathbb{R}^N.
\]

To verify (A1)-(A3) we need distributional right inverses for operators of the form \(-\Delta - k^2 \) for \( k \in \mathcal{I}_s \) and suitable \( s \in \{1, \ldots, 5\} \). From \cite{22}, Theorem 2.3, \cite{18}, Theorem 6] \( (N \geq 3) \)
and [13] Theorem 2.1] (N = 2) we infer that the operators

\[ R_k f := \lim_{\varepsilon \to 0^+} \text{Re} \left[ F^{-1} \left( \frac{F f}{|.|^2 + k^2 - i\varepsilon} \right) \right] \]

are suitable for that purpose. Here, \( F \) denotes the Fourier transform in \( \mathbb{R}^N \). For the asymptotics with respect to \( k \) in the estimate \( (7) \) we refer to [22] Theorem 2.3] (N = 3) and inequality (8) in [15] (N = 2).

**Lemma 1** (Kenig, Ruiz, Sogge). Let \( N \in \mathbb{N}, N \geq 3 \) and assume \( \frac{2(N+1)}{N-1} \leq r \leq \frac{2N}{N-2} \). For every \( k \in \mathbb{Z} \setminus \{0\} \) the operator \( R_k : L^r(\mathbb{R}^N) \to L^r(\mathbb{R}^N) \) is a bounded and symmetric distributional right inverse of \(-\Delta - k^2\) satisfying

\[ \|R_k f\|_r \leq C |k|^{-2 + \frac{N}{2} - \frac{N}{r}} \|f\|_r, \]

for some \( C > 0 \). In the case \( N = 2 \) the same holds for \( 6 \leq r < \infty \).

Notice that Lemma [1] can also be seen as a special case of Lemma [3] below and thus of [9, Theorem 4], so we do not provide a proof here. We stress that this result does not provide a distributional right inverse for \(-\Delta\), which is why we have to consider symmetries that exclude the zero mode. This and [1] motivate the choice \( s \in \{3,5\} \), so that we obtain the existence of infinitely many odd-in-time 2\( \pi \)-periodic breathers and infinitely many \( \pi \)-antiperiodic breathers.

**Corollary 1** (The Wave Equation). Assume \( N \in \mathbb{N}, N \geq 2 \) and \( Q \in L^{\frac{2}{p}}(\mathbb{R}^N), Q \geq 0, Q \neq 0 \) where \( p, q \) satisfy

\[ 2 < p < \frac{2(N+1)}{N-1}, \quad \frac{2(N+1)}{N-1} < q < \frac{2Np}{(N-1)p-2}. \]

Then, for \( s \in \{3,5\} \), the nonlinear wave equation \( (\mathcal{E}) \) admits an unbounded sequence of 2\( \pi \)-periodic real-valued distributional breather solutions \( U_j \in L^q(\mathbb{R}^N; L_p^p(\mathbb{T})) \), \( j \in \mathbb{N}_0 \).

**Proof of Corollary** [1]. We verify the assumptions (A1) - (A3) for \( \mathcal{L} = -\Delta \). As indicated above, the choice \( s \in \{3,5\} \) implies \( I_s \subset \mathbb{Z} \setminus \{0\} \) so that \( k \in I_s \) implies \( |k| \geq 1 \). In particular, the previous lemma applies and yields real-valued, bounded, symmetric linear operators \( R_k \) that are distributional right inverses of \(-\Delta - k^2\) and satisfy

\[ \|R_k f\|_q \leq C(k^2 + 1)^{-\alpha/2} \|f\|_q \quad (k \in I_s) \]

for \( \alpha = 2 - \frac{N}{q} + \frac{N}{q} \). Here we have used that our assumptions imply \( \frac{2(N+1)}{N-1} \leq q < \frac{2N}{N-2} \). From \( q < \frac{2Np}{(N-1)p-2} \) we moreover infer \( \alpha > 1 - \frac{2}{p} \). So assumption (A1) holds. The compactness of the Birman-Schwinger operator \( R_k^Q \) is proved as in [14] Lemma 4.1] (N \geq 3) resp. [13, Section 3] (N = 2) or [17] Lemma 3.1]. (We will provide more details in the proof of Corollary 3 below.) Taking \( \omega_k := \nu_0(k\cdot) \) for the function \( \nu_0 \) from [14] Lemma 4.2 (ii) we find that (A3) holds as well. Hence, Theorem [1] yields the existence of an unbounded sequence of distributional breathers in \( L^q(\mathbb{R}^N; L_p^p(\mathbb{T})) \).

□
2.2. Breather Solutions for the Klein-Gordon Equation. We study the nonlinear Klein-Gordon equation
\( \partial_{tt} U - \Delta U + m^2 U = Q(x)|U|^{p-2} U \) on \( T \times \mathbb{R}^N \).

Much like for the wave equation, we deduce from Theorem 1 the following Corollary 2 (The Klein-Gordon Equation). Assume \( N \in \mathbb{N}, N \geq 2, m > 0 \) and \( Q \in L_ \frac{1}{q-1}(\mathbb{R}^N), Q > 0 \) where \( p,q \) satisfy
\[
2 < p < \frac{2(N+1)}{N-1}, \quad \frac{2(N+1)}{N-1} < q < \frac{2Np}{(N-1)p-2}.
\]

Then the nonlinear Klein-Gordon equation \( (8) \) admits an unbounded sequence of \( 2\pi \)-periodic real-valued breather solutions \( U_j \in L_\frac{1}{q'}(\mathbb{R}^N; L^p(T)), j \in \mathbb{N}_0 \). Here, \( s \) can be chosen as follows:
(i) if \( m \notin \mathbb{N} \), then \( s \in \{1, \ldots, 5\} \),
(ii) if \( m \in 2N-1 \), then \( s = 4 \) (\( \pi \)-periodic breathers),
(iii) if \( m \in 2N \), then \( s = 5 \) (\( \pi \)-antiperiodic breathers).

Since the proof is very much the same as for the wave equation, we omit it. Let us remark that our choice of \( s \) again ensures that we avoid the modes \( k \in \mathcal{I}_s \) with
\[
m^2 - k^2 = 0.
\]

In the study of the operators \( -\Delta + m^2 - k^2 \), there may now occur a finite number of operators \( (k \in \mathcal{I}_s \) with \( k^2 < m^2 \)) with classical \( L^2 \)-inverses given by a convolution with positive exponentially decaying kernels, see (2.21) in [22]. The mapping properties of these well-understood Bessel potential operators are in fact much better than the ones mentioned in Lemma 1 because all \( r \in [2, \frac{2N}{N-2}] \) resp. \( r \in [2, \infty) \) are allowed in (7) if \( N \geq 3 \) resp. \( N = 2 \). Moreover, \( -\Delta + m^2 - k^2 \) is uniformly elliptic so that the arguments presented in the Introduction imply the validity of (A2) and (A3) under the assumption \( Q > 0 \).

2.3. Breather Solutions for Fractional and Biharmonic Wave Equations. We consider the problem
\( \partial_{tt} U + (-\Delta)\gamma U = Q(x)|U|^{p-2} U \) on \( T \times \mathbb{R}^N \)
for general \( \gamma > \frac{N}{N+1} \). As in the case of the classical wave equation one finds distributional right inverses of \( (-\Delta)\gamma - k^2 \) with the aid of the Limiting Absorption Principle that allows to make sense of the limits
\[
\mathcal{R}_k^\gamma f := \lim_{\varepsilon \to 0^+} \operatorname{Re} \left[ \mathcal{F}^{-1} \left( \frac{\mathcal{F} f}{|\cdot|^{2\gamma} - k^2 - i\varepsilon} \right) \right].
\]

This follows from a result by Huang, Yao, Zheng [21, Corollary 3.2].

Lemma 2 (Huang, Yao, Zheng). Let \( N \in \mathbb{N}, N \geq 3 \) and assume \( \frac{2(N+1)}{N-1} \leq r < \frac{2N}{(N-2)}, \gamma > \frac{N}{N+1} \). For every \( k \in \mathbb{Z} \setminus \{0\} \) the operator \( \mathcal{R}_k^\gamma : L^r(\mathbb{R}^N) \to L^r(\mathbb{R}^N) \) is a bounded and symmetric distributional right inverse of \( (-\Delta)\gamma - k^2 \) satisfying
\[
\|\mathcal{R}_k^\gamma f\|_r \leq |k|^{-2\gamma} \frac{N-\gamma}{\pi r} \|f\|_r.
\]
The previous lemma provides the existence of distributional right inverses for all $\gamma > \frac{N}{N+1}$. Notice that this restriction on $\gamma$ is needed to ensure $\frac{2(N+1)}{N-1} < \frac{2N}{(N-2\gamma)^+}$. As in the case of the Laplacian we expect a similar result to hold in the two-dimensional case $N = 2$. In the following result we apply Lemma 2 in order to prove the existence of breathers to fractional nonlinear wave equations just as in the case $\gamma = 1$ discussed in Corollary 1. We stress that this includes the case $\gamma = 2$ of biharmonic nonlinear wave equations.

**Corollary 3 (Fractional Wave Equations).** Assume $N \in \mathbb{N}, N \geq 3, \gamma > \frac{N}{N+1}$ and $Q \in L^{\frac{4}{\gamma'}}(\mathbb{R}^N), Q > 0$ where $p, q$ satisfy

$$2 < p < \frac{2\gamma(N+1)}{(2-\gamma)N-\gamma}_+,$$

$$2(N+1) \frac{N-1}{N-1} \frac{2Np}{((N-\gamma)p-2\gamma)_+}.$$

Then, for $s \in \{3, 5\}$, the nonlinear fractional wave equation $(\mathcal{N})$ admits an unbounded sequence of $2\pi$-periodic real-valued distributional breather solutions $U_j \in L^s(\mathbb{R}^N, L^q_T(\mathbb{T}))$, $j \in \mathbb{N}_0$.

**Proof of Corollary 3.** As in the proof of Corollary 1 the previous lemma yields assumption (A1) for $\alpha = 2 - \frac{N}{\gamma'} + \frac{N}{\gamma q'}$. For the verification of (A3) we follow [26, Lemma 3.1]. We define $\omega^\delta_k := Q^{-1/p}\tilde{\omega}_k 1_{\Omega \geq \delta} \in L^p(\mathbb{R}^N)$ for $\delta > 0$ and the support of $\mathcal{F}(\tilde{\omega}_k)$ is contained in $\{\xi \in \mathbb{R}^N : |\xi|^{2\gamma} > k^2\}$. Then the above definition for $\mathcal{R}_k^\gamma$ implies

$$\lim_{\delta \to 0^+} \int_{\mathbb{R}^N} \omega^\delta_k Q^{1/p} \mathcal{R}_k^\gamma(Q^{1/p} \omega^\delta_k) \, dx = \int_{\mathbb{R}^N} \tilde{\omega}_k \mathcal{R}_k^\gamma(\tilde{\omega}_k) \, dx$$

$$= \lim_{\varepsilon \to 0} \text{Re} \left( \int_{\mathbb{R}^N} \frac{|\mathcal{F}(\tilde{\omega}_k)|^2}{|\xi|^{2\gamma} - k^2 - i\varepsilon} \, d\xi \right)$$

$$= \int_{\mathbb{R}^N} \frac{|\mathcal{F}(\tilde{\omega}_k)|^2}{|\xi|^{2\gamma} - k^2} \, d\xi > 0.$$

So choosing $\delta > 0$ sufficiently small and $\omega_k = \omega^\delta_k \in L^p(\mathbb{R}^N)$ yields (A3).

The verification of (A2) is standard for classical Schrödinger operators of second order. In order to see that in the fractional case nothing really changes, we repeat the main arguments here. In view of

$$\|Q^{1/p} \mathcal{R}_k^\gamma(Q^{1/p} v)\|_p \leq \|Q\|^{1/p}_{\frac{4}{\gamma'}} \|\mathcal{R}_k^\gamma(Q^{1/p} v)\|_q$$

it suffices to prove that $v \mapsto \mathcal{R}_k^\gamma[\Gamma v]$ is compact from $L^p(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ where $\Gamma := Q^{1/p}$. We may without loss of generality assume that $\Gamma$ is bounded with compact support. Indeed, choosing $\Gamma_n \to \Gamma$ in $L^{\frac{4}{\gamma'}}(\mathbb{R}^N)$ with $\Gamma_n$ bounded and compact support, we find

$$\|\mathcal{R}_k^\gamma[\Gamma v] - \mathcal{R}_k^\gamma[\Gamma_n v]\|_q \leq \|\mathcal{R}_k^\gamma[(\Gamma - \Gamma_n)v]\|_q \leq \|\mathcal{R}_k^\gamma\|_{\infty} \|\Gamma - \Gamma_n\|_{\frac{4}{\gamma'}} \|v\|_{p'}.$$

Having proved that $v \mapsto \mathcal{R}_k^\gamma[\Gamma_n v]$ is compact for each $n \in \mathbb{N}$ we can thus conclude that $v \mapsto \mathcal{R}_k^\gamma[\Gamma v]$ is compact as the limit of compact operators with respect to the uniform operator topology. So it remains to prove the compactness of $v \mapsto \mathcal{R}_k^\gamma[\Gamma v]$ from $L^p(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ assuming that $\Gamma$ is bounded with compact support.
Let $B \subset \mathbb{R}^N$ be any bounded ball. The compactness of $v \mapsto \chi_B \mathcal{R}^\gamma_k[\Gamma v]$ follows from the fractional Rellich-Kondrachov Theorem, see [11] Corollary 7.2. By the same argument as above, it remains to show $\|\chi_{\mathbb{R}^N \setminus B} \mathcal{R}^\gamma_k[\Gamma v]\|_{p' \to q} \to 0$ as $B \nearrow \mathbb{R}^N$. To verify In order to apply Theorem 1, we have to check the conditions (A1),(A2),(A3). As-

Proof. of 2

\[ \mathcal{R}^\gamma_k f = G^\gamma_k \ast f, \quad \text{where } G^\gamma_k(z) := \lim_{\varepsilon \to 0^+} \text{Re} \left[ \mathcal{F}^{-1} \left( \frac{1}{1 - |\Gamma_{\varepsilon} - k^2 - i\varepsilon|^\gamma} \right)(z) \right]. \]

The formulas (3.8), (3.8') in [21] Corollary 3.2] show that the kernel function satisfies $|G^\gamma_k(z)| \leq C_k |z|^{-\frac{N}{2}}$ if $|z| \geq 1$ for some $C_k > 0$. Hence, for $M := \text{supp}(\Gamma)$ and $x \in \mathbb{R}^N$ such that $\text{dist}(x, M) \geq 1$ we have

\[ |\mathcal{R}^\gamma_k[\Gamma v](x)| \leq C_k \int_M |x - y|^{-\frac{N}{2}} |\Gamma(y)||v(y)| \, dy \leq \tilde{C}_k |x|^{-\frac{N}{2}} \|\Gamma\|_p \|v\|_{p'}. \]

This yields for large enough balls $B$

\[ \|\chi_{\mathbb{R}^N \setminus B} \mathcal{R}^\gamma_k[\Gamma v]\|_q \leq C \|\Gamma\|_p \|v\|_{p'} \left( \int_{\mathbb{R}^N \setminus B} |x|^\frac{(N-1)}{2} \, dx \right)^\frac{1}{q} \]

and the conclusion follows due to $q > \frac{2(N+1)}{N-1} > \frac{2N}{N-1}$. \hfill \Box

2.4. Breather Solutions for the perturbed Wave Equation. We consider

(10) \[ \partial_t U - \Delta U + V(x)U = Q(x)|U|^{p-2}U \quad \text{on } \mathbb{T} \times \mathbb{R}^N \]

where now $V$ is a short-range potential. In [9] Theorem 4] the following generalization of Lemma 1 is proved.

Lemma 3 (Cossetti, Mandel). Let $N \in \mathbb{N}, N \geq 3$ and assume $V \in L^{\frac{N}{2}}(\mathbb{R}^N) + L^{\frac{N+1}{2}}(\mathbb{R}^N)$ and $\frac{2(N+1)}{N-1} \leq r \leq \frac{2N}{N-2}$. For every $k \in \mathbb{Z} \setminus \{0\}$ there is a bounded and symmetric distributional right inverse $\mathcal{R}_k : L^r(\mathbb{R}^N) \to L^r(\mathbb{R}^N)$ of $-\Delta + V(x) - k^2$ satisfying

\[ \|\mathcal{R}_k f\|_r \leq |k|^{-2+\frac{N}{r}} \|f\|_{r'}. \]

Corollary 4. Assume $N \in \mathbb{N}, N \geq 2$, $V$ as in Lemma 3 and $Q \in L^{\frac{N}{2r}}(\mathbb{R}^N), Q > 0$ where $p, q$ satisfy

\[ 2 < p < \frac{2(N+1)}{N-1}, \quad \frac{2(N+1)}{N-1} < q < \frac{2Np}{(N-1)p-2}. \]

Then, for $s \in \{3, 5\}$, the perturbed nonlinear wave equation (10) has an unbounded sequence of $2\pi$-periodic real-valued distributional breather solutions $U_j \in L^s(\mathbb{R}^N; L^2_0(\mathbb{T})), j \in \mathbb{N}_0$.

Proof. In order to apply Theorem 1, we have to check the conditions (A1),(A2),(A3). Assumption (A1) follows from Lemma 3 and (A2) is a special case of [9] Proposition 8. To verify (A3) we proceed as outlined in the Introduction by choosing a test function $\phi_k \in C_c^\infty(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} \phi_k(-\Delta + V(x) - k^2) \phi_k \, dx > 0$. To this end we write $V = V_1 + V_2$ with $V_1 \in L^{N/2}(\mathbb{R}^N), V_2 \in L^{(N+1)/2}(\mathbb{R}^N)$. Replacing $V_1, V_2$ by $V_1 1_{|V| > R}$ respectively $V_1 1_{|V| \leq R} + V_2$
for large enough $R > 0$ if necessary we can assume that $\|V_1\|_{N/2} \leq \delta$ some given $\delta > 0$. From Hölder’s inequality and the Interpolation inequality

$$\|\psi\|^2_{\frac{2(N+1)}{N}} \leq \|\psi\|^2_{\frac{2N+1}{N+1}} \|\psi\|_2^{\frac{2}{N+1}} \leq \frac{\delta}{1 + \|V_2\|_{\frac{N+1}{N}}} \|\psi\|^2_{\frac{2N}{N+1}} + C_\delta \|\psi\|^2_2$$

for all $\psi \in C^\infty_c(\mathbb{R}^N)$

for some $C_\delta > 0$ depending on $\delta, V_2$ we therefore get

$$\int_{\mathbb{R}^N} \phi_k(-\Delta + V(x) - k^2)\phi_k \, dx$$

$$= \int_{\mathbb{R}^N} |\nabla \phi_k|^2 + (V(x) - k^2)|\phi_k|^2 \, dx$$

$$\geq ||\nabla \phi_k||_2^2 - \left(\|V_1\|_2 \|\phi_k\|_{\frac{2N}{N+1}}^2 + \|V_2\|_{\frac{N+1}{N}} \|\phi_k\|_2^{\frac{2(N+1)}{N}}\right) - k^2 ||\phi_k||_2^2$$

$$\geq \|\nabla \phi_k\|_2^2 - 2\delta ||\phi_k||_{\frac{2N}{N+1}}^2 - (C_\delta + k^2) ||\phi_k||_2^2$$

$$\geq (1 - 2\delta C_\delta^2) \|\nabla \phi_k\|_2^2 - (C_\delta + k^2) ||\phi_k||_2^2.$$

Here, $C_\delta > 0$ comes from Sobolev’s Embedding Theorem. Choosing $0 < \delta < \frac{1}{2C_\delta^2}$ and $\phi_k = \phi^\ast(t_k \cdot)$ for some fixed nontrivial $\phi^\ast \in C^\infty_c(\mathbb{R}^N)$ and large enough $t_k > 0$ we get the result. □

2.5. Generalizations. Before going on with the proof of our main result we indicate further generalizations of our method.

- **(General periods)** In our main result we presented the theory for $2\pi$-periodic breathers. Clearly, by rescaling, there is an analogous theory for $T$-periodic breathers for any given $T > 0$. Analytically this does not change much, but explicit criteria in applications need to be adapted. For instance, in Corollary [2] dealing with the Klein-Gordon equation the conditions on the mass $m$ in (i),(ii),(iii) need to be replaced by the corresponding conditions on $\frac{m}{2\pi}$.

- **(Negative $Q$)** In (A2) we assume nonnegative $Q$ since this is required by the classical dual variational approach that we implement in our paper. In the context of Helmholtz-type problems it is possible to deal with $Q \leq 0$ as well. Indeed, following [26], Section 3 we can slightly modify our functional $J$ from (11) below to cover this case. In this way one obtains the existence of infinitely many breathers in that case. In the case of an elliptic operator $\mathcal{L}$, say $\mathcal{L} = -\Delta$, the corresponding solutions are not stationary and hence polychromatic provided that $Q < 0$ (no matter what $s \in \{1, \ldots, 5\}$ is). As explained right after Theorem [1] this indeed follows once we know that constant in time solutions do not exist, and that is true because the maximum principle implies that solutions of $-\Delta U = Q|U|^{p-2}U$ are necessarily trivial due to $Q < 0$. Sign-changing $Q$ can be treated in the context of Helmholtz equations [28] and it might be that these techniques can be adapted to construct breather solutions.

- **(Non-Euclidean Settings)** Resolvent estimates of type $L^p - L^q$ also hold in the hyperbolic space, see [8, Theorem 2.3] and [20, Theorem 1.2]. The decay rate of the
operator norms of the corresponding distributional right inverses with respect to $k$, however, is not known as far as we can see. We expect that our method applies once a bound as in (A1) is proved.

- **(General evolutions)** The wave operator $\partial_{tt} + \mathcal{L}$ can be replaced by $P(-i\partial_t) + \mathcal{L}$ where $P : \mathbb{R} \to \mathbb{R}$ is a polynomial. In that case, distributional right inverses for $\mathcal{L} + P(k)$ are needed instead of $\mathcal{L} - k^2$ and the definition of $\mathcal{I}_s$ and Proposition 2 (i) have to be adapted in order to ensure the compatibility of the operator $\mathcal{R}$ (see the following section) with the imposed symmetries. The results for odd respectively even polynomials $P$ will be different here.

- **(Systems)** One may ask whether breathers also exist for coupled nonlinear wave equations. Following our approach, this leads to infinite systems of coupled nonlinear Helmholtz systems. We believe that some ideas from the paper [27] about $2 \times 2$-Nonlinear Helmholtz Systems can be used.

- **(General nonlinearities)** Our paper deals with power-type nonlinearities, but the dual variational technique is actually more flexible. To apply this method to a general nonlinearity $f(x, u)$ (replacing $Q(x)|u|^{p-2}u$ in (3)), one has to require the invertibility of $z \mapsto f(x, z)$ for almost all $x \in \mathbb{R}^N$. This is guaranteed by imposing a monotonicity assumption with respect to $z$. Moreover, this inverse needs to give rise to a dual functional having the Mountain Pass Geometry on an appropriate Banach space. In some cases, if the nonlinearity does not behave like a pure power, Orlicz spaces can be used, see for instance [12]. Being interested in an unbounded sequence of breather solutions, one may further impose that the nonlinearity is odd with respect to the second entry. We have to admit that the particularly important case $f(x, u) = |u|^{p-2}u$ is not covered by any of our examples. The reason is that assumption (A2) may not hold, for instance due to the translation invariance of $\mathcal{R}_k$. Here, more sophisticated variational methods such as the Concentration-Compactness Principle could prove to be useful in order to overcome the lack of compactness. An idea for a fixed point approach aiming at the construction of small breather solutions for general nonlinearities or $f(x, u) = |u|^{p-2}u$ can be found in [25].

### 3. Proof of Theorem 1

To motivate our variational approach we introduce the formal Fourier series expansion

$$ U(t, x) = \sum_{k \in \mathcal{I}_s} e^{ikt} u_k(x) \quad \text{with Fourier modes } u_k(x) := [U]_k(x) := \frac{1}{2\pi} \int_T e^{-ikt} U(t, x) \, dt. $$

Recall that $\mathcal{I}_s$ collects the frequencies that are needed for building up breather solutions $U$ with the symmetry indexed by $s \in \{1, \ldots, 5\}$ as in the Introduction. Plugging this ansatz into (3) we are lead to the infinite system of equations

$$ (\mathcal{L} - k^2) u_k = [Q|U|^{p-2}U]_k = Q^{1/p}[Q^{1/p'}|U|^{p-2}U]_k \quad (k \in \mathcal{I}_s). $$
We introduce the dual variable $V := Q^{1/p'}|U|^{p-2}U$ with formal Fourier series expansion
\[ V(t, x) = \sum_{k \in I_s} e^{ikt} v_k(x) \quad \text{with} \quad v_k(x) = [V]_k(x) = \frac{1}{2\pi} \int_T e^{-ikt} V(t, x) \, dt. \]

To find solutions of the above-mentioned infinite system, it is sufficient to solve the following equations:
\[
Q^{1/p} u_k = \mathcal{R}^Q_k \left[ (Q^{1/p'} |U|^{p-2} U)_k \right] \quad \text{for all} \ k \in I_s
\]
\[
\Rightarrow \quad Q^{1/p} U(t, \cdot) = \sum_{k \in I_s} e^{ikt} \mathcal{R}^Q_k \left[ (Q^{1/p'} |U|^{p-2} U)_k \right]
\]
\[
\Rightarrow \quad |V(t, \cdot)|^{p-2} V(t, \cdot) = \sum_{k \in I_s} e^{ikt} \mathcal{R}^Q_k [v_k]
\]
\[
\Rightarrow \quad |V|^{p-2} V = \mathcal{R}[V]
\]

where the operator $\mathcal{R}$ is defined via
\[
\mathcal{R}[V](t, x) := \sum_{k \in I_s} e^{ikt} \mathcal{R}^Q_k [v_k](x) \quad \text{with} \quad v_k(x) = [V]_k(x) = \frac{1}{2\pi} \int_T e^{-ikt} V(t, x) \, dt.
\]

Similarly, we can derive the formula $U = (Q^{-1/p} \mathcal{R})[V]$ where
\[
(Q^{-1/p} \mathcal{R})[V] := \sum_{k \in I_s} e^{ikt} \mathcal{R}_k [Q^{1/p} v_k].
\]

Notice that this operator makes sense under our hypothesis $Q \geq 0$ and $Q^{1/p}(Q^{-1/p} \mathcal{R})[V] = \mathcal{R}[V]$. Since $\mathcal{R}$ will turn out to be symmetric, the above equation for $V$ has a variational structure. It is the Euler-Lagrange equation of the functional
\[
J(V) := \frac{1}{p'} \int_{\mathbb{T} \times \mathbb{R}^N} |V|^{p'} \, d(t, x) - \frac{1}{2} \int_{\mathbb{T} \times \mathbb{R}^N} V \mathcal{R}[V] \, d(t, x)
\]
so that we are lead to prove the existence of critical points. This motivates the following discussion of the functional $J$ and finishes the nonrigorous introductory part of this section.

We start our rigorous analysis of the functional by proving that $J$ is well-defined and continuously differentiable on the Banach space $X_s^{p'}$ where, from now on, $X_s^r := L^r(\mathbb{R}^N, L^r_s(\mathbb{T}))$ for $r \in (1, \infty)$ and $s \in \{1, \ldots, 5\}$. These spaces were introduced at the beginning of the paper. We will need the Hausdorff-Young inequality for Fourier series that we recall for the convenience of the reader.

**Proposition 1** (Hausdorff-Young). Let $p \in [2, \infty]$. Then there is a $C > 0$ such that
\[
\|f\|_{L^p(\mathbb{T})} \leq C \|\hat{f}\|_{\ell^{p'}(\mathbb{Z})}
\]
\[
\|g\|_{L^p(\mathbb{T})} \leq C \|g\|_{L^{p'}(\mathbb{T})}
\]
whenever $\hat{f} \in \ell^{p'}(\mathbb{Z})$ and $g \in L^{p'}(\mathbb{T})$. Here, $\hat{g}(k) := \frac{1}{2\pi} \int_T e^{-ikt} g(t) \, dt$ for $k \in \mathbb{Z}$.

The proofs of the following propositions are postponed to Section 4.
Proposition 2. Assume \((A1),(A2)\).

(i) The operator \( \mathcal{R} : X_{s}^{p'} \to X_{s}^{p} \) is well-defined, continuous, symmetric and compact.
(ii) The operator \( Q^{-1/p} \mathcal{R} : X_{s}^{p'} \to L^q(\mathbb{R}^N, L^p_s(\mathbb{T})) \) is well-defined and continuous.

Using part (i) of the previous proposition we show that \( J \) satisfies the assumptions of the Symmetric Mountain Pass Theorem. This allows to conclude that there is an unbounded sequence of critical points. Those will provide the breather solutions \( U \) after inverting the formal passage to the dual variables from the beginning of this section.

Proposition 3. Assume \((A1),(A2),(A3)\). Then the functional \( J : X_{s}^{p'} \to \mathbb{R} \) as in (11) is even, continuously differentiable and has the Mountain Pass Geometry:

(i) \( J(0) = 0 \) and there are \( r, \delta > 0 \) with \( J(V) \geq \delta \) for all \( V \in X_{s}^{p'} \) with \( \|V\|_p = r \).
(ii) There is an increasing sequence of linear subspaces \( \mathcal{W}^{(m)} \subseteq X_{s}^{p'} \) of dimension \( m \) and radii \( R_m > r \) such that \( J(V) < 0 \) for all \( V \in \mathcal{W}^{(m)} \) with \( \|V\|_p > R_m \).
(iii) \( J \) satisfies the Palais-Smale condition.

The last of our preparatory results shows that each critical point of \( J \) indeed provides a \( 2\pi \)-periodic real-valued distributional breather solution as claimed in Theorem 1. Here we use part (ii) of Proposition 2.

Proposition 4. Assume \((A1),(A2)\) and let \( V \in X_{s}^{p'} \) be a nontrivial critical point of \( J \). Then the function \( U := (Q^{-1/p} \mathcal{R})[V] \in L^q(\mathbb{R}^N, L^p_s(\mathbb{T})) \) is a nontrivial \( 2\pi \)-periodic real-valued distributional breather solution of the nonlinear wave equation (3) in the sense of (5).

We summarize the arguments mentioned above to prove our main result.

Proof of Theorem 1.
By Proposition 3 the functional \( J \) satisfies all assumptions of the Symmetric Mountain Pass Theorem [16, Corollary 7.23]. So there is an unbounded sequence of critical values of \( J \). Since \( J \) maps bounded sets to bounded sets, we thus get an unbounded sequence of critical points \( (V_j)_{j \in \mathbb{N}_0} \). By Proposition 4 the substitution \( U_j := (Q^{-1/p} \mathcal{R})[V_j] \) yields the asserted infinite sequence of \( 2\pi \)-periodic breather solutions of the nonlinear wave equation (3) in \( L^q(\mathbb{R}^N, L^p_s(\mathbb{T})) \). This sequence is unbounded because Hölder’s inequality implies

\[
\|Q|^{1/p} \|U_j\|_q = \|Q^{1/p} \frac{1}{q-1} \|U_j\|_q \geq \|Q^{1/p} U_j\|_p = \|V_j\|_p^{p'-1} \not\to \infty \quad (j \to \infty) .
\]

Here, in the last equality we used \( Q^{1/p} U_j = Q^{1/p} (Q^{-1/p} \mathcal{R})[V_j] = \mathcal{R}[V_j] = |V_j|^{p'-2} V_j \) where the last equality follows from \( J'(V_j) = 0 \). \( \square \)
4. Proofs of auxiliary results

Proof of Proposition 2.

Step 1: Proof of (i) – Well-definedness and continuity.

We show that the series in the definition of $\mathcal{R}[V]$ converges in $L^p(\mathbb{R}^N, L^p(T))$ and that $\mathcal{R}$ preserves the time-symmetry, i.e., $\mathcal{R}(X^p) \subset X^p$. To prove the first point we use the estimate

$$||R^Q_k w||_p = ||Q^{1/p} R_k (Q^{1/p} w)||_p$$

$$\leq ||Q^{1/p}||_{\frac{p}{q-p}} ||R_k(Q^{1/p} w)||_q$$

$$\leq ||Q^{1/p}||_{\frac{p}{q-p}} C(k^2 + 1)^{-\alpha/2} ||Q^{1/p} w||_{q'}$$

Next we prove an estimate for sums over finitely many modes $k \in J_s$ that will imply the well-definedness of $\mathcal{R}$ after some Cauchy sequence argument. So let $J_s \subset I_s$ be any finite subset. Then:

$$\left\| \sum_{k \in J_s} e^{ik \cdot \cdot} R^Q_k [v_k] \right\|_{L^p(\mathbb{R}^N, L^p(T))} = \left\| \sum_{k \in J_s} e^{ik \cdot \cdot} R^Q_k [v_k] \right\|_{L^p(T)}$$

$$= \left\| \left( \sum_{k \in J_s} |R^Q_k [v_k]|^{p'} \right)^{1/p'} \right\|_p$$

$$\leq \left( \sum_{k \in J_s} \left\| R^Q_k [v_k] \right\|_{p-1}^{p'} \right)^{1/p'}$$

$$\leq C_1 \left( \sum_{k \in J_s} (k^2 + 1)^{-\frac{\alpha'}{2}} \left\| v_k \right\|_{p'}^{p'} \right)^{1/p'}$$

$$= C_1 \left( \int_{\mathbb{R}^N} \sum_{k \in J_s} (k^2 + 1)^{-\frac{\alpha'}{2}} |v_k(x)|^{p'} dx \right)^{1/p'}$$
\[
\begin{align*}
&\frac{p}{p+2} \leq C_1 \left( \int_{\mathbb{R}^N} \left( \sum_{k \in \mathcal{J}_s} (k^2 + 1)^{-\frac{\alpha p}{2(p-2)}} \right)^{\frac{p}{p-2}} \left( \sum_{k \in \mathcal{J}_s} |v_k(x)|^p \right)^{\frac{p}{p}} \, dx \right)^{1/p'} \\
&\leq C_1 \left( \sum_{k \in \mathcal{J}_s} (k^2 + 1)^{-\frac{\alpha p}{2(p-2)}} \right)^{\frac{p}{p-2}} \left\| \|V(\cdot, x)\|_{L^p(\mathbb{T})} \right\|_{p'} \\
&= C_1 \left( \sum_{k \in \mathcal{J}_s} (k^2 + 1)^{-\frac{\alpha p}{2(p-2)}} \right)^{\frac{p}{p-2}} \left\| V \right\|_{L^p(\mathbb{R}^N, L^{p'}(\mathbb{T}))}.
\end{align*}
\]

So we have proved
\[
(15) \quad \left\| \sum_{k \in \mathcal{J}_s} e^{ik \cdot x} \mathcal{R}_k^Q [v_k]_{L^p(\mathbb{R}^N, L^{p'}(\mathbb{T}))} \right\| \leq C_1 \left( \sum_{k \in \mathcal{J}_s} (k^2 + 1)^{-\frac{\alpha p}{2(p-2)}} \right)^{\frac{p}{p-2}} \left\| V \right\|_{L^p(\mathbb{R}^N, L^{p'}(\mathbb{T}))}.
\]

In view of (15) and \(\frac{\alpha p}{p-2} > 1\), which holds by assumption (A1), a Cauchy sequence argument proves that the corresponding infinite sum converges. So \(\mathcal{R}[V] \in L^p(\mathbb{R}^N, L^{p'}(\mathbb{T}))\) is well-defined and
\[
\left\| \mathcal{R}[V] \right\|_{L^p(\mathbb{R}^N, L^{p'}(\mathbb{T}))} \leq C_1 \left( \sum_{k \in \mathcal{J}_s} (k^2 + 1)^{-\frac{\alpha p}{2(p-2)}} \right)^{\frac{p}{p-2}} \left\| V \right\|_{L^p(\mathbb{R}^N, L^{p'}(\mathbb{T}))}.
\]

Notice that \(\mathcal{R}\) maps real-valued functions to real-valued functions because so do the operators \(\mathcal{R}_k^Q\). So it remains to prove the time-symmetry preserving property of \(\mathcal{R}\), i.e., \(\mathcal{R}(X^p_s) \subset X^p_s\).

Recall that elements \(V \in X^p_s\) are real-valued by assumption and thus satisfy \(v_k = v_{-k}\) for all \(k \in \mathbb{Z}\) and all \(s = 1, \ldots, 5\).

- For \(s = 1\) there is nothing to prove.
- For \(s = 2\) any \(V \in X^p_s\) is even in time. Equivalently, all Fourier modes \(v_k = v_{-k}\) are real-valued. This is true also for \(\mathcal{R}V\) because \(\mathcal{R}_k = \mathcal{R}_{-k}\) by (A1) implies
  \[
  [\mathcal{R}V]_k = \mathcal{R}_k^Q [v_k] = \mathcal{R}_{-k}^Q [v_{-k}] = [\mathcal{R}V]_{-k},
  \]
  \[
  [\mathcal{R}V]_k = \mathcal{R}_k^Q [v_k] = \mathcal{R}_k^Q [v_{-k}] = \mathcal{R}_k^Q [v_{-k}] = [\mathcal{R}V]_k.
  \]
- For \(s = 3\) any \(V \in X^p_s\) is odd in time, i.e., the zero mode \(v_0 = 0\) vanishes and the other Fourier modes \(v_k = -v_{-k}\) are purely imaginary. Again, this is true also for \(\mathcal{R}V\) because the zero mode does not occur in \(\mathcal{I}_s\) and
  \[
  [\mathcal{R}V]_k = \mathcal{R}_k^Q [v_k] = \mathcal{R}_{-k}^Q [-v_{-k}] = -[\mathcal{R}V]_{-k},
  \]
  \[
  [\mathcal{R}V]_k = \mathcal{R}_k^Q [v_k] = \mathcal{R}_k^Q [-v_k] = -[\mathcal{R}V]_k.
  \]
- For \(s = 4\) any \(V \in X^p_s\) is \(\pi\)-periodic, i.e., the modes with odd \(k\) vanish. Since \(\mathcal{I}_4 = 2\mathbb{Z}\) this is true as well for \(\mathcal{R}V\).
For $s = 5$ any $V \in X^p_s$ is $\pi$-antiperiodic, i.e., the modes with even $k$ vanish. Since $\mathcal{I}_s = 2\mathbb{Z} + 1$ this is true as well for $\mathcal{R}V$.

**Step 2:** Proof of (i) – Symmetry and compactness.

The symmetry of $\mathcal{R}$, which means
\[
\int_{\mathbb{T} \times \mathbb{R}^N} \mathcal{R}[V](t, x)W(t, x) \, dt \, dx = \int_{\mathbb{T} \times \mathbb{R}^N} V(t, x)\mathcal{R}[W](t, x) \, dt \, dx
\]
for all $V, W \in X^p_s$, follows from the continuity of $\mathcal{R} : X^p_s \to X^p_s$ and the symmetry of $\mathcal{R}^Q_k$. We now turn to the proof of compactness. Assume that $(V^{(n)})_n$ is a bounded sequence in $X^p_s$ with $\|V^{(n)}\|_{L^{p'}} \leq C_V$ for all $n \in \mathbb{N}$. We aim to show that a subsequence of $(\mathcal{R}[V^{(n)}])_n$ converges in $X^p_s$. Here, for almost all $t \in \mathbb{T}$ and $n \in \mathbb{N}$,
\[
\mathcal{R}[V^{(n)}](t, \cdot) = \sum_{k \in \mathcal{I}_s} e^{ikt} \mathcal{R}^Q_k [v^{(n)}_k] \quad \text{in } L^p(\mathbb{R}^N).
\]
From Hölder’s inequality we infer
\[
\|v^{(n)}_k\|_{L^{p'}} = \left\| \frac{1}{2\pi} \int_{\mathbb{T}} e^{ikt} V^{(n)}(t, \cdot) \, dt \right\|_{L^{p'}} \leq (2\pi)^{-\frac{1}{p'}} \|V^{(n)}\|_{L^{p'}(\mathbb{T})} \leq C_V.
\]
So all sequences $(v^{(n)}_k)_n$ are bounded in $L^{p'}(\mathbb{R}^N)$. The compactness of $\mathcal{R}^Q_k$ from (A2) combined with a standard diagonal sequence argument provides $y_k \in L^p(\mathbb{R}^N)$, $k \in \mathcal{I}_s$, and a subsequence $(v^{(n_j)}_k)_j$ with
\[
\forall k \in \mathcal{I}_s \quad \mathcal{R}^Q_k [v^{(n_j)}_k] \to y_k \quad \text{in } L^p(\mathbb{R}^N) \text{ as } j \to \infty.
\]
We claim that this implies
\[
\mathcal{R}[V^{(n_j)}] \to \sum_{k \in \mathcal{I}_s} e^{ik} \cdot y_k \quad \text{in } X^p_s \text{ as } j \to \infty.
\]
Before verifying (17) we check that the term on the right indeed belongs to $L^p(\mathbb{R}^N, L^p(\mathbb{T}))$. Indeed, for all finite $\mathcal{J}_s \subset \mathcal{I}_s$ we have
\[
\left\| \sum_{k \in \mathcal{J}_s} e^{ik} \cdot y_k \right\|_{L^p(\mathbb{R}^N, L^p(\mathbb{T}))} = \lim_{j \to \infty} \left\| \sum_{k \in \mathcal{J}_s} e^{ik} \cdot \mathcal{R}^Q_k [v^{(n_j)}_k] \right\|_{L^p(\mathbb{R}^N, L^p(\mathbb{T}))}
\]
\[
\leq C_1 \left( \sum_{k \in \mathcal{J}_s} (k^2 + 1)^{-\frac{\alpha p}{2(p-2)}} \right)^{\frac{p-2}{p}} \lim_{j \to \infty} \left\| V^{(n_j)} \right\|_{L^{p'}(\mathbb{R}^N, L^{p'}(\mathbb{T}))}
\]
\[
\leq C_1 C_V \left( \sum_{k \in \mathcal{J}_s} (k^2 + 1)^{-\frac{\alpha p}{2(p-2)}} \right)^{\frac{p-2}{p}} < \infty.
\]
As above, this and \( \alpha > 1 - \frac{2}{p} \) implies \( \sum_{k \in \mathcal{I}_s} e^{ik \cdot y} \in L^p(\mathbb{R}^N, L^p(\mathbb{T})) \). Next we verify (17). So let \( \varepsilon > 0 \). From the previous statement we obtain some finite subset \( \mathcal{J}_s \subset \mathcal{I}_s \) such that for all \( j \in \mathbb{N} \)

\[
\left\| \sum_{k \in \mathcal{I}_s \setminus \mathcal{J}_s} e^{ik \cdot y} \right\|_{L^p(\mathbb{R}^N, L^p(\mathbb{T}))} \leq \frac{\varepsilon}{4},
\]

\[
\left\| \sum_{k \in \mathcal{J}_s} e^{ik \cdot \mathcal{R}_k^Q [v_k^{(n_j)}]} \right\|_{L^p(\mathbb{R}^N, L^p(\mathbb{T}))} \leq C_R \left( \sum_{k \in \mathcal{I}_s \setminus \mathcal{J}_s} (k^2 + 1)^{-\frac{\alpha p}{2(\alpha - 1)}} \right)^{\frac{p-2}{p}} \left\| V^{(n_j)} \right\|_{L^{p'}(\mathbb{R}^N, L^{p'}(\mathbb{T}))} \leq C_R C_V \left( \sum_{k \in \mathcal{I}_s \setminus \mathcal{J}_s} (k^2 + 1)^{-\frac{\alpha p}{2(\alpha - 1)}} \right)^{\frac{p-2}{p}} < \frac{\varepsilon}{4}.
\]

So (16) and again the Hausdorff-Young inequality yield some \( j_0 = j_0(\varepsilon) \in \mathbb{N} \) such that for all \( j \geq j_0 \) the following holds.

\[
\left\| \sum_{k \in \mathcal{J}_s} e^{ik \cdot \mathcal{R}_k^Q [v_k^{(n_j)}]} - \sum_{k \in \mathcal{I}_s} e^{ik \cdot y} \right\|_{L^p(\mathbb{R}^N, L^p(\mathbb{T}))} \leq \left( \sum_{k \in \mathcal{J}_s} \left\| \mathcal{R}_k^Q [v_k^{(n_j)}] - y_k \right\|_{p'} \right)^{\frac{1}{p'}} < \frac{\varepsilon}{2}.
\]

Combining all these estimates, we infer for \( j \geq j_0 \)

\[
\left\| \sum_{k \in \mathcal{J}_s} e^{ik \cdot \mathcal{R}_k^Q [v_k^{(n_j)}]} - \sum_{k \in \mathcal{I}_s} e^{ik \cdot y} \right\|_{L^p(\mathbb{R}^N, L^p(\mathbb{T}))} < \varepsilon,
\]

which finishes the proof of (17) because \( X_s^p \) is closed in \( L^p(\mathbb{R}^N, L^p(\mathbb{T})) \).

**Step 3:** Proof of (ii).

We essentially repeat the estimate from the first step where the exponent \( q \) replaces \( p \) in the spatial Lebesgue norm in order to take the missing factor \( Q^{1/p} \) into account. Given that \( q \geq p' \) and that our summation is over finitely many indices \( k \in \mathcal{J}_s \) only, we obtain

\[
\left\| \sum_{k \in \mathcal{J}_s} e^{ik \cdot \mathcal{R}_k^Q [Q^{1/p} v_k]} \right\|_{L^p(\mathbb{T})} \leq C \left( \sum_{k \in \mathcal{J}_s} \left\| \mathcal{R}_k [Q^{1/p} v_k] \right\|_{p'} \right)^{1/p'} \leq C \left( \sum_{k \in \mathcal{J}_s} (k^2 + 1)^{-\frac{p'}{2}} \left\| Q^{1/p} v_k \right\|_{p'} \right)^{1/p'} \leq C \left( \sum_{k \in \mathcal{J}_s} (k^2 + 1)^{-\frac{p'}{2}} \left\| Q^{1/p} \right\|_{p'} \left\| v_k \right\|_{p'} \right)^{1/p'} \leq C \left( \sum_{k \in \mathcal{J}_s} (k^2 + 1)^{-\frac{p'}{2}} \left\| Q^{1/p} \right\|_{p'} \left\| v_k \right\|_{p'} \right)^{1/p'}.
\]
\[ \leq C \|Q\|^{1/p}_{q-p} \left( \sum_{k \in J_s} (k^2 + 1)^{-\frac{ap^2}{2p-2}} \|v_k\|_{p'} \right)^{1/p'} \]

\[ \leq \text{step 1} \]

\[ \leq C \|Q\|^{1/p}_{q-p} \left( \sum_{k \in J_s} (k^2 + 1)^{-\frac{ap}{2(p-1)}} \right)^{\frac{p-2}{p}} \|V\|_{L^p'(\mathbb{R}^N,L^p'(\mathbb{T}))}. \]

Since the sum on the right is bounded independently of \( J_s \subset I_s \), we get the result. \( \square \)

**Proof of Proposition 3.**

We prove that the functional

\[ J : X_{p'}^p \rightarrow \mathbb{R}, \quad J(V) := \frac{1}{p'} \int_{T \times \mathbb{R}^N} |V|^p' \, d(t, x) - \frac{1}{2} \int_{T \times \mathbb{R}^N} V \mathcal{R}[V] \, d(t, x) \]

satisfies the assumptions of the Symmetric Mountain Pass Theorem. It is straightforward to deduce from Proposition 2 and \( X_{p'}^p \subset L^p'(\mathbb{R}^N,L^p'(\mathbb{T})) \) that \( J \) is well-defined, even and of class \( C^1 \).

(i) Assuming \( \|V\|_{L^p'(\mathbb{R}^N,L^p'(\mathbb{T}))} = r \), we estimate using \( C_R := \|\mathcal{R}\|_{p' \rightarrow p} < \infty \) and get

\[ J(V) = \frac{1}{p'} \int_{T \times \mathbb{R}^N} |V|^p' \, d(t, x) - \frac{1}{2} \int_{T \times \mathbb{R}^N} V \mathcal{R}[V] \, d(t, x) \geq \left( \frac{1}{p'} - \frac{C_R}{2} r^{2-p'} \right). \]

Hence, the claim (i) holds for \( r = (C_Rp')^{-1/(2-p')} \) and \( \delta = r^{p'}/2p' > 0 \).

(ii) According to (A3) we find \( \omega_k \in L^p'(\mathbb{R}^N) \) such that w.l.o.g.

\[ \int_{\mathbb{R}^N} \omega_k \mathcal{R}_k^Q[\omega_k] \, dx = \frac{2}{\pi} \quad \text{for all } k \in I_s. \]

With that, we choose for positive \( k \in I_s \)

\[ V_k(t, x) := w_k(x)T_k(t) := \begin{cases} w_k(x) \cos(kt) & \text{if } s \in \{1, 2, 4\}, \\ w_k(x) \sin(kt) & \text{if } s \in \{3, 5\}. \end{cases} \]

This choice guarantees \( V_k \in X_{p'}^p \) for all \( k \in I_s \). Moreover, for positive \( k, k' \in I_s \)

\[ \int_{T \times \mathbb{R}^N} V_k \mathcal{R}[V_k] \, d(t, x) = \int_T T_{k'}(t)T_k(t) \, dt \cdot \int_{\mathbb{R}^N} \omega_{k'} \mathcal{R}_k^Q[\omega_k] \, dx \begin{cases} = 0 & \text{if } k \neq k', \\ = 2 & \text{if } k = k'. \end{cases} \]
So the $V_k$ are linearly independent. Indeed, $\sum_{k \in J_s} c_k V_k = 0$ for some $c_k \in \mathbb{R}$ and finite subset $J_s \subset \mathcal{I}_s$ implies $c_{k'} = 0$ for all $k' \in J_s$, because of
\[
0 = \int_{\mathbb{T} \times \mathbb{R}^N} V_{k'} \mathcal{R} \left[ \sum_{k \in J_s} c_k V_k \right] \, dt, \quad x = \sum_{k \in J_s} c_k \int_{\mathbb{T} \times \mathbb{R}^N} V_{k'} \mathcal{R}[V_k] \, dt, \quad = 2c_{k'}.
\]
Choosing nested subsets $\mathcal{I}_s \subset \mathcal{I}_s$ with $j$ positive elements and $\mathcal{M}_j := \text{span}\{V_k : k \in \mathcal{I}_j\}$ we thus get $\dim \mathcal{M}_j = j$. For any fixed $j \in \mathbb{N}$, equivalence of norms provides a constant $c_j > 1$ with
\[
\frac{1}{c_j} \left( \sum_{k \in \mathcal{I}_j} \beta_k^2 \right)^{1/2} \leq \left\| \sum_{k \in \mathcal{I}_j} \beta_k V_k \right\|_{L^{p'}(\mathbb{R}^N, L^p(\mathbb{T}))} \lesssim c_j \left( \sum_{k \in \mathcal{I}_j} \beta_k^2 \right)^{1/2} \quad \text{whenever } \beta_k \in \mathbb{R}, k \in \mathcal{I}_j.
\]
For $R > r$ and some arbitrary element $V = \sum_{k \in \mathcal{I}_j} \beta_k V_k \in \mathcal{M}_j$ with $\|V\|_{L^{p'}(\mathbb{R}^N, L^p(\mathbb{T}))} = R$, we obtain the estimate
\[
J(V) = \frac{1}{p'} \int_{\mathbb{T} \times \mathbb{R}^N} |V|^{p'} \, dt, \quad x = \frac{1}{2} \int_{\mathbb{T} \times \mathbb{R}^N} V \mathcal{R}[V] \, dt, \quad \mathcal{R}[V_k] \, dt, \quad = 2 \cdot R' - \sum_{k \in \mathcal{I}_j} \beta_k^2 \quad \text{and} \quad \leq \frac{1}{p'} \cdot R' - \frac{1}{c_j} \cdot R^2.
\]
Since $p' < 2$, we thus conclude for $R_j := \max \left\{ r, \left( \frac{c_j^2}{p'} \right)^{1/(2-p')} \right\}$ that $J(V) < 0$ whenever $V \in \mathcal{M}_j$ with $\|V\|_{L^{p'}(\mathbb{R}^N, L^p(\mathbb{T}))} > R_j$.

(iii) Take any Palais-Smale sequence $(V_n)_n$ for $J$, that is, $V_n \in X_{s}^{p'}$ with
\[
J'(V_n) \to 0 \quad \text{in} \quad \left( X_{s}^{p'} \right)', = X_{s}^{p}, \quad J(V_n) \to c
\]
where $c > 0$ denotes the Mountain Pass level. We claim that the sequence $(V_n)_n$ is bounded. Indeed, assuming otherwise, the identity
\[
J'(V_n)[V_n] - 2J(V_n) = \left( 1 - \frac{2}{p'} \right) \int_{\mathbb{T} \times \mathbb{R}^N} |V_n|^{p'} \, dt, \quad x \to \infty
\]
leads in the limit $n \to \infty$ to the contradictory statement
\[
0 = \limsup_{n \to \infty} \frac{J'(V_n)[V_n] - 2J(V_n)}{\|V_n\|_{L^{p'}(\mathbb{R}^N, L^p(\mathbb{T}))}} = \limsup_{n \to \infty} \left( 1 - \frac{2}{p'} \right) \|V_n\|_{L^{p'}(\mathbb{R}^N, L^p(\mathbb{T}))} = -\infty.
\]
Hence, we may assume w.l.o.g. that $V_n \rightharpoonup V$ weakly in $X_{s}^{p'}$ for some $V \in X_{s}^{p'}$. Due to the compactness of $\mathcal{R}$ (see Proposition 2), this implies $\mathcal{R}[V_n] \to \mathcal{R}[V]$ strongly in $L_{s}^{p}(\mathbb{T} \times \mathbb{R}^N)$.
Hence, we obtain
\[ \int_{T \times \mathbb{R}^N} V_n \mathcal{R}[V_n] \, d(t, x) \to \int_{T \times \mathbb{R}^N} V \mathcal{R}[V] \, d(t, x). \]

As in the proof of [13, Lemma 5.2] we conclude \( V_n \to V \) in the strong sense. Indeed, weak convergence implies
\[ \|V\|_{L^{p'}(\mathbb{T} \times \mathbb{R}^N)} \leq \liminf_{n \to \infty} \|V_n\|_{L^{p'}(\mathbb{T} \times \mathbb{R}^N)} \]
and the convexity of \( t \mapsto |t|^{p'} \) yields, as \( n \to \infty, \)
\[ \frac{1}{p'} \left\| V \right\|_{L^{p'}(\mathbb{T} \times \mathbb{R}^N)} - \frac{1}{p'} \left\| V_n \right\|_{L^{p'}(\mathbb{T} \times \mathbb{R}^N)} \leq \int_{T \times \mathbb{R}^N} |V_n|^{p'-2} V_n (V - V_n) \, d(t, x) \]
\[ = J'(V_n)[V - V_n] + \int_{\mathbb{R}^N} V_n \mathcal{R}[V - V_n] \, d(t, x) = o(1), \]
whence
\[ \limsup_{n \to \infty} \|V_n\|_{L^{p'}(\mathbb{T} \times \mathbb{R}^N)} \leq \|V\|_{L^{p'}(\mathbb{T} \times \mathbb{R}^N)}. \]

We conclude that the sequence of norms converges to the norm of the weak limit \( V \). By uniform convexity of \( L^{p'}(\mathbb{T} \times \mathbb{R}^N) \), this implies \( V_n \to V \) as \( n \to \infty \) and the statement is proved. \( \square \)

**Proof of Proposition 4.**
We consider a nontrivial critical point \( V \in L^{p'}_s(\mathbb{T} \times \mathbb{R}^N) \) of the functional \( J \). Since \( \mathcal{R} \) is symmetric by Proposition 2 (i), the Euler-Lagrange equation reads
\[ (18) \quad |V|^{p'-2} V = \mathcal{R}[V] \quad \text{in } X^{p'}_s = L^{p'}(\mathbb{R}^N, L^p_s(\mathbb{T})). \]

From Proposition 2 (ii) and (18) we infer \( U := (Q^{-1/p} \mathcal{R})[V] \in L^q(\mathbb{R}^N, L^p_1(\mathbb{T})) \). We will use \( Q^{1/p} U = (Q^{-1/p} \mathcal{R})[V] = \mathcal{R}[V] = |V|^{p'-2} V \) and thus
\[ Q|U|^{p-2} U = Q^{1/p} \cdot |Q^{1/p} U|^{p-2} Q^{1/p} U = Q^{1/p} V. \]

Using these facts, we have to verify
\[ \int_{T \times \mathbb{R}^N} Q|U|^{p-2} U \Phi \, d(t, x) = \int_{T \times \mathbb{R}^N} U (\partial_{tt} + \mathcal{L})\Phi \, d(t, x) \]
for all \( C^\infty_c(\mathbb{R}^N, C^\infty(\mathbb{T})) \). To this end we proceed step by step.

We first verify the above identity for real-valued test functions of the form
\[ \Phi(t, x) := \sum_{k \in \mathcal{J}_s} e^{-ikt} \phi_k(x), \quad \mathcal{J}_s \subset \mathcal{I}_s \text{ finite}, \phi_k \in C^\infty_c(\mathbb{R}^N) (k \in \mathcal{J}_s) \text{ such that } \Phi \in X^{p'}_s. \]

To see this we use the Euler-Lagrange equation and the definition of \( U \) from above.
\[ \int_{T \times \mathbb{R}^N} \sum_{k \in \mathcal{J}_s} e^{-ikt} \phi_k(x) \Phi(t, x) \, d(t, x) \]
\[= \int_{T \times \mathbb{R}^N} Q(x)^{1/p} V(t,x) \Phi(t,x) \, \mathrm{d}(t,x)\]

\[= \sum_{k \in J_s} \int_{\mathbb{R}^N} Q(x)^{1/p} \phi_k(x) \left[ \int_T e^{-ikt} V(t,x) \, \mathrm{d}t \right] \, \mathrm{d}x\]

\[= 2\pi \sum_{k \in J_s} \int_{\mathbb{R}^N} Q(x)^{1/p} \phi_k(x) \, \mathrm{d}x\]

\[= 2\pi \sum_{k \in J_s} \mathcal{R}[Q^{1/p} \phi_k](x) (\mathcal{L} - k^2) \phi_k(x) \, \mathrm{d}x\]

\[= \int_{\mathbb{R}^N} \sum_{k \in I_s} \mathcal{R}[Q^{1/p} \phi_k](x) \left( \mathcal{L} - k^2 \left[ \int_T e^{ikt} \Phi(t,\cdot) \, \mathrm{d}t \right] \right) (x) \, \mathrm{d}x\]

\[= \int_{\mathbb{R}^N} \sum_{k \in I_s} \mathcal{R}[Q^{1/p} \phi_k](x) \left[ \int_T e^{ikt} \mathcal{L} \Phi(t,\cdot) \, \mathrm{d}t + \partial_t \left[ e^{ikt} \Phi(t,x) \right] \right] \, \mathrm{d}x.\]

We now integrate by parts. Since \(\Phi(\cdot, x)\) is periodic, the boundary terms in

\[\int_0^{2\pi} \Phi \partial_u \left[ e^{ikt} \right] \, \mathrm{d}t = \left[ ik \Phi \cdot e^{ikt} + \left( \partial_t \Phi \right) \cdot e^{ikt} \right]_0^{2\pi} + \int_0^{2\pi} e^{ikt} \partial_u \Phi \, \mathrm{d}t\]

vanish for a.e. \(x \in \mathbb{R}^N\). So we get

\[\int_{T \times \mathbb{R}^N} Q(t,x)|U(t,x)|^{p-2} U(t,x) \Phi(t,x) \, \mathrm{d}(t,x)\]

\[= \int_{\mathbb{R}^N} \sum_{k \in I_s} \mathcal{R}[Q^{1/p} \phi_k](x) \left[ \int_T e^{ikt} (\partial_u + \mathcal{L}) \Phi \, \mathrm{d}t \right] \, \mathrm{d}x\]

\[= \int_{\mathbb{R}^N} \sum_{k \in I_s} \int_T e^{ikt} \mathcal{R}[Q^{1/p} \phi_k](x) (\partial_u + \mathcal{L}) \Phi \, \mathrm{d}t \, \mathrm{d}x\]

\[= \int_{T \times \mathbb{R}^N} \sum_{k \in I_s} e^{ikt} \mathcal{R}[Q^{1/p} \phi_k](x) (\partial_u + \mathcal{L}) \Phi(t,x) \, \mathrm{d}t \, \mathrm{d}x\]

\[= \int_{T \times \mathbb{R}^N} (Q^{-1/p} \mathcal{R}[V][\partial_u + \mathcal{L}] \Phi(t,x) \, \mathrm{d}t \, \mathrm{d}x\]

\[= \int_{T \times \mathbb{R}^N} U (\partial_u + \mathcal{L}) \Phi(t,x) \, \mathrm{d}t \, \mathrm{d}x.\]
Next we extend this identity to more general test functions. We claim that the above identity even holds for
\[ \Phi(t, x) := \sum_{k \in \mathcal{J}} e^{-ikt} \phi_k(x) \quad \mathcal{J} \subset \mathbb{Z} \text{ finite}, \quad \phi_k = \overline{\phi_{-k}} \in C_c^\infty(\mathbb{R}^N) \quad (k \in \mathcal{J}). \]

Indeed, given that \( \mathcal{J} \) is finite, we have \( \Phi \in L^p(\mathbb{R}^N, L^p(\mathbb{T})) \). Moreover, since the functions \( U(\cdot, x) \) and \( Q|U(\cdot, x)|^{p-2}U(\cdot, x) \) have the symmetry indexed by \( s \) for almost all \( x \in \mathbb{R}^N \), the time-symmetry requirement for the test function is not a true restriction. In fact, it imposes extra assumptions on the \( \phi_k \) only for \( s \in \{2, 3\} \), see the explanations near (3), but those restrictions are not necessary since integration of sin against cos-functions over the interval \( [0, 2\pi] \) gives 0. Moreover, \( U(\cdot, x) \) and \( Q|U(\cdot, x)|^{p-2}U(\cdot, x) \) are \( L^2(\mathbb{T}) \)-orthogonal to the modes \( e^{-ikt} \) with \( k \in \mathbb{Z} \setminus \mathcal{I}_s \). So the nonlinear wave-type equation actually holds in the distributional sense for test functions \( \Phi \) as above.

It remains to pass to the limit \( \mathcal{J} \nearrow \mathbb{Z} \) because of
\[ \Phi(t, x) = \sum_{k \in \mathbb{Z}} e^{-ikt} \phi_k(x), \quad \phi_k(x) := \frac{1}{2\pi} \int_{\mathbb{T}} e^{ikt} \Phi(x, t) \, dt \quad \text{where} \quad \Phi \in C_c^\infty(\mathbb{R}^N, C_c^\infty(\mathbb{T})). \]

To see that this passage is possible, we choose a compact set \( K \subset \mathbb{R}^N \) such that \( \Phi(\cdot, t)s \) and hence all the \( \phi_k \) have support contained in \( K \). Then
\begin{align}
\sum_{k \in \mathbb{Z}} \left| \int_{T \times K} U(\partial_t + \mathcal{L}) (e^{-ikt} \phi_k(x)) \, d(t, x) \right| & \leq \sum_{k \in \mathbb{Z}} \int_{T \times K} |U||(\mathcal{L} - k^2)\phi_k| \, d(t, x) \\
& \leq \sum_{k \in \mathbb{Z}} \|U\|_{L^q(K, L^p(\mathbb{T}))} \|(\mathcal{L} - k^2)\phi_k\|_{L^{q'}(K, L^{p'}(\mathbb{T}))} \\
& \leq \sum_{k \in \mathbb{Z}} \|U\|_{L^q(\mathbb{R}^N, L^p(\mathbb{T}))} (2\pi)^{1/q'} \left( \|\mathcal{L} \phi_k\|_{L^{q'}(K)} + k^2 \|\phi_k\|_{L^{q'}(K)} \right) \\
& \leq \|U\|_{L^q(\mathbb{R}^N, L^p(\mathbb{T}))} (2\pi)^{1/q'} \sum_{k \in \mathbb{Z}} \left( \|\phi_k\|_{W^{m,\infty}(K)} + k^2 |k|^{2}\|\phi_k\|_{L^{\infty}(K)} \right)
\end{align}

Since \( \Phi \) is smooth and periodic with respect to \( t \), we get from integration by parts
\[ \|\phi_k\|_{W^{m,\infty}(K)} \leq 2(k^2 + 1)^{-1}(\|\Phi\|_{W^{m,\infty}(T \times K)} + \|\partial_t \Phi\|_{W^{m,\infty}(T \times K)}), \]
\[ \|\phi_k\|_{L^{\infty}(K)} \leq 2(k^2 + 1)^{-1}(\|\Phi\|_{W^{m,\infty}(T \times K)} + \|\partial_{ttt} \Phi\|_{W^{m,\infty}(T \times K)}), \]
for all \( k \in \mathbb{Z} \). As a consequence, this above series converges. This shows that we can pass to the limit \( \mathcal{J} \nearrow \mathbb{Z} \) on the right hand side of the above distributional formulation of the wave equation. A similar estimate for the left hand side gives that the nonlinear wave equation is satisfied in the sense of (3), which finishes the proof.
\[ \square \]
VARIATIONAL METHODS FOR BREATHER SOLUTIONS OF NONLINEAR WAVE EQUATIONS

ACKNOWLEDGEMENTS

The authors thank Wolfgang Reichel (KIT) for several discussions leading to an improvement of the manuscript. Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

REFERENCES

[1] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur. Method for solving the sine-Gordon equation. Phys. Rev. Lett., 30:1262–1264, 1973.

[2] M. A. Alejo. Nonlinear stability of Gardner breathers. J. Differential Equations, 264(2):1192–1230, 2018.

[3] M. A. Alejo, L. Fanelli, and C. Muñoz. The Akhmediev breather is unstable. São Paulo J. Math. Sci., 13(2):391–401, 2019.

[4] M. A. Alejo and C. Muñoz. Nonlinear stability of MKdV breathers. Comm. Math. Phys., 324(1):233–262, 2013.

[5] M. A. Alejo, C. Muñoz, and J. M. Palacios. On the variational structure of breather solutions I: Sine-Gordon equation. J. Math. Anal. Appl., 453(2):1111–1138, 2017.

[6] C. Blank, M. Chirilus-Bruckner, V. Lescarret, and G. Schneider. Breather solutions in periodic media. Comm. Math. Phys., 302(3):815–841, 2011.

[7] H. Brézis, J.-M. Coron, and L. Nirenberg. Free vibrations for a nonlinear wave equation and a theorem of P. Rabinowitz. Communications on Pure and Applied Mathematics, 33 (5): 667–684, 1980.

[8] J.-B. Casteras and R. Mandel. On Helmholtz Equations and Counterexamples to Strichartz Estimates in Hyperbolic Space. International Mathematics Research Notices, 01 2020.

[9] L. Cossetti and R. Mandel. A limiting absorption principle for Helmholtz systems and time-harmonic isotropic Maxwell equations, 2020. arXiv:2009.05087.

[10] J. Denzler. Nonpersistence of breather families for the perturbed sine Gordon equation. Comm. Math. Phys., 158(2):397–430, 1993.

[11] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker’s guide to the fractional Sobolev spaces. Bull. Sci. Math., 136(5):521–573, 2012.

[12] G. Évêquoz. A dual approach in Orlicz spaces for the nonlinear Helmholtz equation. Z. Angew. Math. Phys., 66(6):2995–3015, 2015.

[13] G. Évêquoz. Existence and asymptotic behavior of standing waves of the nonlinear Helmholtz equation in the plane. Analysis, 37 (2): 55–68, 2019.

[14] G. Évêquoz and T. Weth. Dual variational methods and nonvanishing for the nonlinear Helmholtz equation. Advances in Mathematics, 280 : 690–728, 2015.

[15] R. L. Frank. Eigenvalue bounds for Schrödinger operators with complex potentials. Bull. Lond. Math. Soc., 43(4):745–750, 2011.

[16] N. Ghoussoub. Duality and perturbation methods in critical point theory. Cambridge tracts in mathematics. Cambridge Univ. Press, Cambridge, 1993.

[17] M. Goldberg and W. Schlag. A limiting absorption principle for the three-dimensional Schrödinger equation with \( L^p \) potentials. Int. Math. Res. Not., (75):4049–4071, 2004.

[18] S. Gutiérrez. Non trivial \( L^q \) solutions to the Ginzburg-Landau equation. Mathematische Annalen, 328 (1): 1–25, 2004.

[19] A. Hirsch and W. Reichel. Real-valued, time-periodic localized weak solutions for a semilinear wave equation with periodic potentials. Nonlinearity, 32 (4): 1408–1439, 2019.

[20] S. Huang and C. D. Sogge. Concerning \( L^p \) resolvent estimates for simply connected manifolds of constant curvature. J. Funct. Anal., 267(12):4635–4666, 2014.

[21] S. Huang, X. Yao, and Q. Zheng. Remarks on \( L^p \)-limiting absorption principle of Schrödinger operators and applications to spectral multiplier theorems. Forum Math., 30(1):43–55, 2018.
[22] C. E. Kenig, A. Ruiz, and C. D. Sogge. Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators. *Duke Math. J.*, 55(2): 329–347, 1987.

[23] M. Kowalczyk, Y. Martel, and C. Muñoz. Nonexistence of small, odd breathers for a class of nonlinear wave equations. *Lett. Math. Phys.*, 107(5):921–931, 2017.

[24] P. Kuchment. An overview of periodic elliptic operators. *Bull. Amer. Math. Soc. (N.S.)*, 53(3):343–414, 2016.

[25] R. Mandel. Uncountably many solutions for nonlinear Helmholtz and curl-curl equations. *Adv. Nonlinear Stud.*, 19(3):569–593, 2019.

[26] R. Mandel, E. Montefusco, and B. Pellacci. Oscillating solutions for nonlinear Helmholtz equations. *Z. Angew. Math. Phys.*, 68(6):Paper No. 121, 19, 2017.

[27] R. Mandel and D. Scheider. Dual variational methods for a nonlinear Helmholtz system. *NoDEA Nonlinear Differential Equations Appl.*, 25(2):Paper No. 13, 26, 2018.

[28] R. Mandel, D. Scheider, and T. Yesil. Dual variational methods for an indefinite nonlinear helmholtz equation, 2020. arXiv:2011.07808.

[29] C. Muñoz and G. Ponce. Breathers and the dynamics of solutions in KdV type equations. *Comm. Math. Phys.*, 367(2):581–598, 2019.

[30] M. Plum and W. Reichel. A breather construction for a semilinear curl-curl wave equation with radially symmetric coefficients. *J. Elliptic Parabol. Equ.*, 2(1-2):371–387, 2016.

[31] P. H. Rabinowitz. Free vibrations for a semilinear wave equation. *Communications on Pure and Applied Mathematics*, 31 (1): 31–68, 1978.

[32] D. Scheider. Breather solutions of the cubic Klein-Gordon equation. *Nonlinearity*, 33(12):7140–7166, 2020.

R. Mandel
Karlsruhe Institute of Technology
Institute for Analysis
Englerstrasse 2
D-76131 Karlsruhe, Germany
Email address: Rainer.Mandel@kit.edu

D. Scheider
Karlsruhe Institute of Technology
Institute for Analysis
Englerstrasse 2
D-76131 Karlsruhe, Germany
Email address: dominic.scheider@kit.edu