HEAT CONDUCTION WITH MEMORY: A SINGULAR KERNEL PROBLEM

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Dedicated to Mauro Fabrizio on his 70th Birthday.

Abstract. The existence and uniqueness of solution to an integro-differential problem arising in heat conduction with memory is here considered. Specifically, a singular kernel problem is analyzed in the case of a multi-dimensional rigid heat conductor. The choice to investigate a singular kernel material is suggested by applications to model a wider variety of materials and, in particular, new materials whose heat flux relaxation function may be superiorly unbounded at the initial time $t = 0$. The present study represents a generalization to higher dimensions of a previous one concerning a 1-dimensional problem in the framework of linear viscoelasticity with memory. Specifically, an existence theorem is here proved when initial homogeneous data are assumed. Indeed, the choice of homogeneous data is needed to obtain the a priori estimate in Section 2 on which the subsequent results, are based.

1. Introduction. The problem here addressed to is within the framework of rigid heat conduction with memory. The model takes its origin in Cattaneo’s celebrated paper [11], based on previous results by Coleman [15] and Gurtin and Pipkin [25]. Further investigations on the physical model concerning thermodynamics of materials with memory are due to Coleman [15], McCarthy [28], Coleman and Dill [16], Giorgi and Gentili [21], Gurtin [24] followed by Fabrizio, Gentili and Reynolds [19]. Specifically, on the basis of the Gurtin - Pipkin model [25], the thermodynamic theory developed by Fabrizio, Gentili and Reynolds [19] is adopted. Accordingly, the thermodynamic state of the material is determined when the temperature $u$, its history $u^t$:

\[ u^t(x, \tau) := u(x, t - \tau), \]  

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are given together with the temperature gradient and the integrated history of the temperature gradient

\[ g(x, \tau) := \nabla u(x, \tau), \quad \bar{g}(x, \tau) := \int_{t-\tau}^{t} g(x, s) \, ds. \]  

(2)

The internal energy \( e \), then, is assumed to be:

\[ e(x, t) = \alpha(x)u(x, t), \]  

(3)

where \( x \in \Omega \subset \mathbb{R}^3 \) denotes the position within the heat conductor, \( t \in [0, +\infty) \), the time variable; while the specific heat, \( \alpha(x) \), in general, depends on the position within the heat conductor. Hence, the energy \( e \) (3), depends linearly on the relative temperature \( u := \theta - \theta_0 \) which is the difference between the absolute temperature \( \theta \) and a fixed reference temperature \( \theta_0 \), i.e. the temperature of the surrounding environment assumed not to be affected by the thermodynamical status of the conductor, according to [16].

Here, again as stated in [19], the Clausius-Duhem inequality, which representing the second law of thermodynamics, can be written as

\[ \frac{\partial \eta}{\partial t} \geq \frac{1}{\theta} \frac{\partial e}{\partial \theta} + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta^2}, \]  

(4)

wherein, respectively, \( \eta \) denotes the entropy, \( \theta \) the absolute temperature, \( \mathbf{g} \) its gradient, \( e \) the energy, and \( \mathbf{q} \) the heat flux. Furthermore, all these quantities, which depend on the space and time variables, are subject to suitable constitutive assumptions. Within the linear theory, Fabrizio, Gentili and Reynolds [19] introduce the free energy \( \psi = e - \theta \eta \) and, then, \( \zeta := \theta_0(e - \theta_0 \eta) \), therein termed free pseudo energy, since it enjoys most of the properties which characterize a canonical free energy even if it is not dimensionally homogenous to an energy. Hence, as shown in [19] in the framework of linear approximation, the inequality (4) can be expressed via

\[ \frac{\partial \zeta}{\partial t} \leq u \frac{\partial e}{\partial t} - \mathbf{q} \cdot \mathbf{g}. \]  

(5)

Then, constitutive equations, which relate the internal energy \( e \) and the free energy \( \zeta \), are of the form

\[ e = \hat{e}(\theta(x, t), \bar{g}(x, t)) \]  

(6)

\[ \zeta = \hat{\zeta}(\theta(x, t), \bar{g}(x, t)). \]  

(7)

As it is well known, while the internal energy \( e \), in the linear case, is represented by (3), there are many expressions of the free energy \( \zeta \) since there is a whole class of functionals \( \hat{\zeta} \) which satisfies the inequality (5). An up to date study on free energies in heat conductors with memory is due to Amendola, Fabrizio and Golden [2].

Here, for sake of simplicity, the specific heat is assumed to be independent on the position within the heat conductor, that is, \( \alpha(x) = \alpha_0 \), \( \alpha_0 \in \mathbb{R}^+ \). So that, the internal energy (3) reads

\[ e(x, t) = \alpha_0 u(x, t). \]  

(8)

Let

\[ \nabla u^t(x, \tau) := \nabla u(x, t - \tau) \in \mathbb{R}^3 \]  

(9)

denote the history of the temperature-gradient, then, the heat flux \( \mathbf{q} \in \mathbb{R}^3 \) is given by

\[ \mathbf{q}(x, t) = -\int_0^{\infty} k(x, \tau) \nabla u(x, t - \tau) \, d\tau \]  

(10)
where the heat flux relaxation function \([19]\) is represented by \(k(\mathbf{x}, \tau)\), which, when the heat conductor is homogeneous and isotropic, depends only on the time variable. This is the case here considered. Accordingly, the heat flux relaxation function is given by \(k(t)\); it is assumed to satisfy the following requirements
\[
k(t) > 0, \quad \dot{k}(t) \leq 0, \quad \ddot{k}(t) \geq 0, \quad t \in (0, \infty).
\]
and
\[
k \in L^1(0, T) \cap C^2(0, T) \quad \forall T \in \mathbb{R}^+.
\]
Note that, as stated in \([19]\), conditions \((11)\) imposed on \(k\), together with \(\alpha_0 > 0\), guarantee that \(\zeta_G\) represents the free energy, termed Graffi free energy \([2]\):
\[
\zeta_G (u, \tilde{g}^t) := \frac{\alpha_0}{2} \left[ u(\mathbf{x}, t) \right]^2 - \frac{1}{2} \int_0^\infty k' (\tau) \left( \tilde{g}^t(\mathbf{x}, \tau) \right)^2 d\tau.
\]
Such a free energy satisfies Clausius-Duhem inequality \((5)\). In addition, usually, the further condition \(\dot{k} \in L^1(0, T), \quad \forall T \in \mathbb{R}^+\) is imposed; then, \(k_0 := k(0)\) is finite and represents the initial value of the heat flux relaxation function, termed initial heat flux relaxation coefficient.

When \(r(\mathbf{x}, t)\) denotes the source term and the positive constant \(\alpha_0 = 1\), according to \([19]\), the energy balance equation
\[
\frac{\partial e}{\partial t} = -\nabla \cdot \mathbf{q}(\mathbf{x}, t) + r(\mathbf{x}, t)
\]
allows to obtain the following evolution equation which models the temperature evolution within a rigid heat conductor with memory
\[
u_t = -\nabla \cdot \mathbf{q}(\mathbf{x}, t) + r(\mathbf{x}, t) .
\]
The latter, on use of the expression \((10)\) of the heat flux, reads:
\[
u_t = \int_0^\infty k(\tau) \Delta u (\mathbf{x}, t - \tau) d\tau + r(\mathbf{x}, t) ,
\]
which, more conveniently, can be written under the following form
\[
u_t = \int_0^t k(\tau) \Delta u (\mathbf{x}, t - \tau) d\tau + f(\mathbf{x}, t) ,
\]
i.e.
\[
u_t = \int_0^t k(t - \tau) \Delta u (\mathbf{x}, t) d\tau + f(\mathbf{x}, t) ,
\]
where the term \(f(\mathbf{x}, t)\) denotes an external source term which also includes the history of the material and \(\mathbf{x}\) denotes the position within a smooth compact domain \(\Omega \subset \mathbb{R}^3\). The source term \(f\) is assumed to be regular enough to be amenable to integration w.r. to time and is \(L^2\) - integrable w.r. to the space variable. Equation \((17)\) is subject to the initial and boundary conditions
\[
u_0 \equiv u(\cdot, 0) = 0, \quad \text{in } \Omega \subset \mathbb{R}^3 ,
\]
\[
u = 0, \quad \text{on } \Sigma = \partial \Omega \times (0, T).
\]
Regular evolution problems in this framework are considered in \([9, 10]\) where, in turn, existence and uniqueness and asymptotic behaviour are studied pointing out the relevance of the adopted choice of the form of the free energy. Let us, now, introduce \(K\), termed integrated relaxation function, via
\[
K(\xi) := \int_0^\xi k(\tau) d\tau , \quad K(0) = 0.
\]
On use of $K$, the integro-differential problem (17-19) can be re-written under the following equivalent integral form

$$
\begin{align*}
  u(x, t) &= \int_0^t K(t - \tau) \Delta u(x, \tau) d\tau + \int_0^t f(x, \tau) d\tau.
\end{align*}
$$

(21)

Notably, the choice to adopt (21) as the model equation to study the rigid heat conduction problem allows to impose weaker requirements on the relaxation function $k$ aiming to model a wider class of materials. Here, on the basis of the analytical analogies between the models which describe the displacement within a viscoelastic body, on one side, and the temperature evolution in a rigid heat conductor, on the other side, see for instance [7, 8], a new problem is studied in this second framework. Specifically, the novelty consists in that the functional requirements (12) $k$ is assumed to fulfill are compatible with both $k(0)$ finite as well as with $\lim_{t \to 0^+} k(t) = +\infty$; indeed, in the integral formulation (21) of the problem, only the integrated relaxation function $K(t)$ which, by definition, is equal to zero when evaluated at $t = 0$, appears, while, the kernel of the integro-differential problem under investigation exhibits a singularity at $t = 0$. The importance of such a case is testified by various interesting results [12, 30, 22, 26, 27, 32, 33, 37]; most of them concerned about problems arising in viscoelasticity studied in abstract form. In particular, [12, 30, 22, 33, 37] are concerned about the asymptotic behaviour of solutions. On the other hand, the relevance of new materials, such as the so-called smart materials is clear from the many Conferences as well as books, such as [29, 31, 20] devoted to the subject. Furthermore, here, we extend to a 3-dimensional rigid heat conductor with memory an existence and uniqueness result recently obtained in the case of a 1-dimensional viscoelasticity problem [5]. The present investigation is part of a wide research project concerning the behavior of materials with memory. Thus, both thermodynamical as well as mechanical properties of material with memory are of interest. In particular, viscoelastic solids are studied in [3, 4, 5]. The magneto-elastic problem is investigated in previous works [13, 14, 34] while results concerning viscoelasticity problems are comprised in [35, 36, 7].

In Section 2 the general strategy of our approach to the problem is introduced. Specifically, the problem under investigation is, as a first step, connected to a second order integro-differential problem whose kernel is singular at $t = 0$. Then, on use of the free energy functional, an a priori estimate is obtained. On the basis of the proved a priori estimate, in the subsequent Section 3, the existence of weak solutions admitted by the problem under investigation, is shown to follow. Specifically, the proof that our problem admits a weak solution is based on the existence of the solution of suitably constructed approximated problems. Then, the convergence of these solutions to the solution limit is shown. Section 4 is devoted to establish uniqueness of the obtained weak solution. In the last Section 5 some perspectives and open problems are mentioned.

2. An a priori estimate to an approximated heat conduction evolution problem. In this Section the singular integro-differential evolution problem is considered. It, on one hand, is related to the integral problem (21) which exhibits no singularity, and, on the other one, can be approximated on introduction of a small parameter $\varepsilon$. Specifically, given any $0 < \varepsilon \ll 1$, a regular integro-differential problem, denoted by $P^\varepsilon$, which approximates the singular one is introduced. Notably, the regular problem $P^\varepsilon$ admits a unique solution. Furthermore, it admits the integral problem (21) as a limit for $\varepsilon \to 0$, as shown in this Section.
Furthermore, here, an estimate which allows to prove the needed convergence of the approximated solutions as $\varepsilon$ goes to zero, is established. The result relies on a technical result, represented by the following Lemma 2.1. The latter plays a key role in proving the subsequent Theorem. Remarkably, this result is connected to the definition of free energy in linear thermodynamics with memory (see, for example, [1] and references therein).

The temperature evolution within a rigid heat conductor with memory is modeled by the evolution equation (17). It, letting $s := t - \tau$, can be written as follows

$$u_t = \int_0^t k(t - s) \Delta u(x, s) \, ds + f(x, t),$$

wherein the kernel may be singular at the origin. Hence, the time shifted heat flux relaxation function is defined by

$$k^\varepsilon(t - \tau) := k(\varepsilon + t - \tau),$$

and the approximated problems, obtained from (17) after the change of variable $\tau := t - s$, are introduced via

$$u_t^\varepsilon = \int_0^t k(\varepsilon + t - \tau) \Delta u^\varepsilon(x, \tau) \, d\tau + f(x, t),$$

wherein the unknown is denoted by $u^\varepsilon(x, t)$. Then, the following formulation of the problem is chosen since it is amenable to known existence and uniqueness results as shown in the following.

Hence, consider the linear integro-differential equation obtained on partial derivation, with respect to the time variable $t$, of (24):

$$u_{tt}^\varepsilon = k(\varepsilon)\Delta u^\varepsilon + \int_0^t \dot{k}(\varepsilon + t - \tau)\Delta u^\varepsilon(\tau) d\tau + f_t,$$

together with the associated initial and boundary conditions

$$u^\varepsilon|_{t=0} = 0, \quad u_t^\varepsilon|_{t=0} = f(x, 0), \quad u^\varepsilon|_{\partial \Omega \times (0,T)} = 0, \quad t < T. \tag{26}$$

Existence and uniqueness of solution to the linear problem (25) – (26) are stated in [17], [18].

First of all, equation (25) is re-written in a more convenient form: that is, let $s := t - \tau$ in (25) which, then, reads

$$u_{tt}^\varepsilon = k(\varepsilon)\Delta u^\varepsilon + \int_0^t \dot{k}(s + \varepsilon)\Delta u^\varepsilon(t - s) \, ds + f_t. \tag{27}$$

Then, the following Lemma can be stated.

**Lemma 2.1.** Let $u^\varepsilon$ be the unique solution to the problem (25) – (26), then it follows

$$\frac{1}{2} \frac{d}{dt} \int_\Omega k(t + \varepsilon) |\nabla u^\varepsilon|^2 \, dx - \frac{1}{2} \frac{d}{dt} \int_\Omega \dot{k}(s + \varepsilon) |\nabla u^\varepsilon(t) - \nabla u^\varepsilon(t - s)|^2 \, dx +$$

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |u_t^\varepsilon|^2 \, dx = \int_\Omega f_t \, u_t^\varepsilon \, dx + \frac{1}{2} \int_\Omega \dot{k}(t + \varepsilon) |\nabla u^\varepsilon|^2 \, dx +$$

$$- \frac{1}{2} \int_0^t ds \int_\Omega \dot{k}(s + \varepsilon) |\nabla u^\varepsilon(t) - \nabla u^\varepsilon(t - s)|^2 \, dx \tag{28}$$
Equation (25), wherein all the superscripts $\varepsilon$ are omitted for notation simplicity, can be written in the following equivalent form

$$u_{tt} - k(t + \varepsilon)\Delta u + \int_0^t k(s + \varepsilon) [\Delta u(t) - \Delta u(t - s)] \, ds = f_t$$

(29)

then, multiplication of equation (29) by $u_t$ and integration over $\Omega$, give

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |u_t|^2 \, dx + \int_\Omega k(t + \varepsilon) \nabla u \cdot \nabla u_t \, dx +$$

$$+ \int_\Omega u_t(t) \, dx \int_0^t k(s + \varepsilon) [\Delta u(t) - \Delta u(t - s)] \, ds = \int_\Omega f_t u_t \, dx$$

(30)

that is

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |u_t|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_\Omega k(t + \varepsilon) |\nabla u|^2 \, dx - \frac{1}{2} \int_\Omega \dot{k}(t + \varepsilon) |\nabla u|^2 \, dx -$$

$$- \int_\Omega \int_0^t \dot{k}(s + \varepsilon) \nabla u_t \cdot [\nabla u(t) - \nabla u(t - s)] \, dx \, ds = \int_\Omega f_t u_t \, dx.$$ 

(31)

Now, observe that

$$- \int_\Omega \int_0^t \dot{k}(s + \varepsilon) \nabla u_t \cdot [\nabla u(t) - \nabla u(t - s)] \, dx \, ds$$

$$= - \frac{1}{2} \frac{d}{dt} \int_0^t ds \int_\Omega \dot{k}(s + \varepsilon) |\nabla u(t) - \nabla u(t - s)|^2 \, dx +$$

$$+ \frac{1}{2} \int_\Omega \dot{k}(t + \varepsilon) |\nabla u(t) - \nabla u(0)|^2 \, dx +$$

$$- \int_\Omega \int_0^t \dot{k}(s + \varepsilon) \nabla u_t(t - s) \cdot [\nabla u(t) - \nabla u(t - s)] \, dx \, ds =$$

(32)

$$= - \frac{1}{2} \frac{d}{dt} \int_0^t ds \int_\Omega \dot{k}(s + \varepsilon) |\nabla u(t) - \nabla u(t - s)|^2 \, dx +$$

$$+ \frac{1}{2} \int_\Omega \dot{k}(t + \varepsilon) |\nabla u(t) - \nabla u(0)|^2 \, dx +$$

$$+ \int_\Omega \int_0^t \dot{k}(s + \varepsilon) \frac{d}{ds} |\nabla u(t - s) \cdot [\nabla u(t) - \nabla u(t - s)] \, dx \, ds =$$

$$= - \frac{1}{2} \frac{d}{dt} \int_0^t ds \int_\Omega \dot{k}(s + \varepsilon) |\nabla u(t) - \nabla u(t - s)|^2 \, dx +$$

$$+ \frac{1}{2} \int_\Omega \dot{k}(t + \varepsilon) |\nabla u(t) - \nabla u(0)|^2 \, dx -$$

$$- \frac{1}{2} \int_\Omega \int_0^t \dot{k}(s + \varepsilon) \frac{d}{ds} |\nabla u(t) - \nabla u(t - s)|^2 \, dx \, ds =$$
\[ \frac{1}{2} \int_{\Omega} k(t + \varepsilon) |\nabla u|^2 \, dx + \frac{1}{2} \int_{\Omega} |u_t|^2 \, dx \leq \int_0^t \int_\Omega f_t \, u_t \, dx \, ds + \int_\Omega f(x, 0) |\nabla u(0)|^2 \, dx + \frac{1}{2} \int_{\Omega} |f(x, 0)|^2 \, dx; \]

the latter, in the case of the homogeneous initial conditions, here considered, reads:

\[ \frac{1}{2} \int_{\Omega} k(t + \varepsilon) |\nabla u|^2 \, dx + \frac{1}{2} \int_{\Omega} |u_t|^2 \, dx \leq \int_0^t \int_\Omega f_t \, u_t \, dx \, ds + \frac{1}{2} \int_{\Omega} |f(x, 0)|^2 \, dx. \]

which allows to write

\[ \frac{1}{2} \int_{\Omega} k(t + \varepsilon) |\nabla u|^2 \, dx + \frac{1}{2} \int_{\Omega} |u_t|^2 \, dx - \int_0^t \int_{\Omega} |u_t|^2 \, dx \, ds \leq C(f) \]

and, Gronwall’s lemma implies

\[ \frac{1}{2} \int_{\Omega} k(t + \varepsilon) |\nabla u|^2 \, dx + \frac{1}{2} \int_{\Omega} |u_t|^2 \, dx \leq e^{T} C(f) \]

Moreover, since \( k(t + \varepsilon) \geq k(T + 1) \), it holds

\[ \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\Omega} |u_t|^2 \, dx \leq \gamma e^{T} C(f) \]

wherein \( \gamma = \max\{k(T + 1)^{-1}, 1\} \). This last estimate is required later on.

3. **Integral problem: Solution existence.** Here the approximated problem, let it be denoted as

\[ P_\varepsilon : \]

\[ u_\varepsilon^0 = 0, \quad u_\varepsilon^t|_{t=0} = f(x, 0), \quad u_\varepsilon|_{\partial \Omega \times (0, T)} = 0, \quad t < T, \]

given by (25) − (26) is considered. The aim of this Section is to prove that the solution \( u_\varepsilon \) of the singular problem under investigation exists. This result is achieved via the introduction of the regular problem \( P_\varepsilon \), whose solution, denoted as \( u_\varepsilon \), exists converges and, in addition, in the limit \( \varepsilon \to 0 \) converges to the solution of the singular problem. Note that the estimate (37) implies there exists a subsequence \( \{\varepsilon_h\}, h \in \mathbb{N} \) such that there exists a convergent subsequence of solutions \( \{u^{\varepsilon_h}\} \)

\[ u^{\varepsilon_h} \rightharpoonup u \text{ weakly in } H^1(0, T, H_0^1(\Omega)) \text{ as } \varepsilon_h \to 0; \]

\[ u^{\varepsilon_h} \to u \text{ strongly in } L^2(\Omega \times (0, T)) \text{ as } \varepsilon_h \to 0; \]
hence
\[ \exists \, u(t) = \lim_{\varepsilon_h \to 0} u^{\varepsilon_h}(t) \text{ in } L^2(\Omega \times (0, T)), \] (40)
where \( u^{\varepsilon_h} \) is solution to the problem (25) – (26). This convergence result allows to prove the following.

**Theorem 3.1.** The limit function \( u \) represents a weak solution to the integral problem (21).

**Proof.** Let \( u^{\varepsilon_h} \) be a solution of (25) – (26) and, hence, solution to
\[ P^{\varepsilon_h} : \ u^{\varepsilon_h}(t) = \int_0^t K^{\varepsilon_h}(t-\tau)\Delta u^{\varepsilon_h}(\tau)\, d\tau + \int_0^t f(\tau)\, d\tau, \] (41)
where
\[ K^{\varepsilon_h}(\xi) := \int_0^\xi k(\varepsilon_h + \tau)\, d\tau. \] (42)
Consider the weak formulation of the integral problems \( P^{\varepsilon_h} \) expressed in (41). Accordingly, let us introduce the test functions \( \varphi \), which depend on both the time and space variables and satisfy homogeneous boundary conditions at the smooth boundary \( \partial \Omega \) of \( \Omega \subset \mathbb{R}^3 \).
\[ \varphi \in C^\infty(\Omega \times (0, T)) , \ s.t. \ \varphi|_{\partial \Omega} = 0 \ \forall \ t \in (0, T) , \] (43)
then, multiply (41) by \( \varphi \) and integrate over \( \Omega \times (0, T) \). It follows
\[ \int_0^T \int_\Omega u^{\varepsilon_h}(t)\varphi dx\, dt + \int_0^T \int_\Omega \varphi \left\{ \int_0^t K^{\varepsilon_h}(t-\tau)\Delta u^{\varepsilon_h}(\tau)\, d\tau + \int_0^t f(\tau)\, d\tau \right\} dx\, dt. \] (44)
Then, via the Lemmas in the following, the existence of the solution admitted by the singular problem is shown to be given by \( u = \lim_{\varepsilon_h \to 0} u^{\varepsilon_h} \).

First of all, observe that the term
\[ \int_0^T \int_\Omega \varphi \left\{ \int_0^t f(\tau)\, d\tau \right\} dx\, dt \] (45)
refers to the history of the material with memory, assumed to be regular and, hence, the boundedness of \( \Omega \times (0, T) \), implies that the integral over \( \Omega \times (0, T) \), in (45), is also bounded. In addition, (45) does not depend on \( \varepsilon_h \) and, hence, it is unchanged in the limit \( \varepsilon_h \to 0 \). Consequently, only the quantity
\[ \int_0^T \int_\Omega \varphi \, d\Omega \int_0^t K^{\varepsilon_h}(t-\tau)\Delta u^{\varepsilon_h}(\tau)\, d\tau \] (46)
needs to be considered. First of all, note that it can be written in a more convenient form. Recalling that the test functions \( \varphi \) satisfy the homogeneous b.c. (43), then, via integration by parts with respect to the space variable, twice, it follows:
\[ \int_0^T \int_\Omega \varphi \, d\Omega \int_0^t K^{\varepsilon_h}(t-\tau)\Delta u^{\varepsilon_h}(\tau)\, d\tau = \int_0^T \int_\Omega \Delta \varphi \, d\Omega \int_0^t K^{\varepsilon_h}(t-\tau)u^{\varepsilon_h}(\tau)\, d\tau. \] (47)
Since
\[ \int_0^t K^{\varepsilon_h}(t-\tau)u^{\varepsilon_h}(\tau)\, d\tau = \int_0^t K^{\varepsilon_h}(s)u^{\varepsilon_h}(t-s)\, ds, \] (48)
adding and subtracting $K(s)$, the r.h.s. of (47) can be re-written the following equivalent form:

$$
\int_0^T \int_\Omega \Delta \varphi \int_0^t K^{\varepsilon_h}(t - \tau)u^{\varepsilon_h}(\tau)d\tau dx dt = 0
$$

Hence, the proof of the Theorem is implied by (38–39) together with the following Lemma.

**Lemma 3.2.** Given the integral problem (41–43), then

$$
\lim_{\varepsilon_h \to 0} \int_0^T \int_\Omega \Delta \varphi \int_0^t [K^{\varepsilon_h}(s) - K(s)] u^{\varepsilon_h}(t - s) d\sigma dx dt = 0
$$

**Proof.** Note that, $\forall (x,t) \in \Omega \times (0,T)$

$$
|u| \leq C|\Omega|, |\Delta \varphi| \leq M,
$$

furthermore

$$
|K^{\varepsilon_h}(s) - K(s)| = |K(\varepsilon_h + s) - K(s)| = \int_s^{\varepsilon_h + s} k(\tau)d\tau
$$

hence, since $k \in L^1(0,T)$, Lebesgue’s Theorem implies the limit convergence.

4. **Integral problem: Solution uniqueness.** In this Section the solution uniqueness is considered referring to the solution $u$ whose existence and regularity are considered in the previous Section and, hence, the following Theorem can be stated.

**Theorem 4.1.** Let $u$ be a solution admitted by (21) then $u$, whose regularity is specified in Theorem 3.1, is unique.

**Proof.** Assume (21) admits two different solutions, say $v$ and $\tilde{v}$, then the linearity implies that also any linear combination of them is again a solution to (21). Thus, consider $w := v - \tilde{v}$; it turns out to solve

$$
\int_0^t K(t - \tau)\Delta w(\tau)d\tau
$$

subject to homogeneous initial and boundary conditions, by definition. In particular, $w(x,t)|_{\partial \Omega} = 0$, $\forall t \in (0,T)$. Let us now consider the weak solution

$$
\int_0^T \int_\Omega w(t)\psi(x,t)dx dt = \int_0^T \int_\Omega \psi(x,t) \int_0^t K(t - \tau)\Delta w(\tau)d\tau dx dt,
$$

that is

$$
\int_0^T \int_\Omega w(t)\psi(x,t)dx dt = \int_0^T \int_\Omega \Delta \psi(x,t) \int_0^t K(t - \tau)wd\tau dx dt.
$$
Let, now, consider the eigenfunctions, which assume value 0 on the smooth boundary $\partial \Omega$, of the Laplace operator $\Delta$, here denoted as $\nu_n(x)$. Then, on introduction of $w(x, t)$ of the form

$$w(x, t) = \sum_{n=1}^{\infty} \beta_n(t) \nu_n(x) \ , \quad (56)$$

and of the test function $\psi(x, t)$

$$\psi(x, t) = \varphi(t) \nu_m(x) \ , \quad m \in \mathbb{N} \ ; \quad (57)$$

substitution of both (56–57), recalling the orthogonality of the eigenfunctions $\nu_n(x)$, gives

$$\int_0^T \varphi(t) \left[ \beta_m(t) - \int_0^t K(t-\tau)m^2 \beta_m(t)d\tau \right] dt = 0. \quad (58)$$

The arbitrariness of the test function $\psi(x, t)$ implies:

$$\beta_m(t) = \int_0^t K(t-\tau)m^2 \beta_m(t)d\tau \quad (59)$$

and, hence,

$$|\beta_m(t)| \leq K(T)m^2 \int_0^t |\beta_m(\tau)|d\tau \quad (60)$$

so that Gronwall’s Lemma implies $\beta_m(t) = 0$, $\forall m \in \mathbb{N}$. \hfill \Box

5. Concluding remarks. The results here obtained refer to a wide research project concerning the mechanical behavior of materials. Indeed, on the basis of the well known analogy between the analytical properties of the models in linear rigid heat conduction with memory and linear viscoelasticity, see for instance [7, 8], this result represents a generalization to a 3-dimensional rigid heat conduction problem of the result of existence and uniqueness of the solution admitted by a 1-dimensional singular viscoelasticity problem the authors prove in [5].

To achieve the existence result, here, initial homogeneous data are considered. Indeed, homogeneous data are needed to obtain, in Section 2, the a priori estimate on which the subsequent results are based. Thus, the applicability of the presented results is not too wide, however, this is the counter part of the unboundedness of the heat flux relaxation function $k$ at the origin. Further, and more general, cases are currently under investigation with special concern to Neumann boundary conditions, on one side, and to a singular magneto-viscoelasticity problem [6], on the other. As a final remark, note that the estimate here presented are related to the free energy functional. Connections with various different forms of free energy are currently under investigation.

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REFERENCES

[1] G. Amendola and S. Carillo, Thermal work and minimum free energy in a heat conductor with memory, Quart. J. of Mech. and Appl. Math., 57 (2004), 429–446.
[2] G. Amendola, M. Fabrizio and J. M. Golden, Thermodynamics of Materials with Memory. Theory and Applications, Springer, New York, 2012.
[3] S. Carillo, V. Valente and G. Vergara Caffarelli, A result of existence and uniqueness for an integro-differential system in magneto-viscoelasticity, Applicable Analysis: An International Journal, 1563-504X, First published on 19 August 2010, 90 (2011), 1791–1802.
[4] S. Carillo, V. Valente and G. Vergara Caffarelli, An existence theorem for the magnetic-viscoelastic problem, Discrete and Continuous Dynamical Systems Series S., 5 (2012), 435–447.
[5] S. Carillo, V. Valente and G. Vergara Caffarelli, A linear viscoelasticity problem with a singular memory kernel: an existence and uniqueness result, Differential and Integral Equations, 26 (2013), 1115–1125.
[6] S. Carillo, M. Chipot, V. Valente and G. Vergara Caffarelli, in preparation, (2014).
[7] S. Carillo, Some remarks on materials with memory: heat conduction and viscoelasticity, Journal of Nonlinear Mathematical Physics Supplement 1, 12 (2005), 163–178.
[8] S. Carillo, Evolution problems in materials with fading memory, Mathematical (Catania), 62 (2007), 93–105.
[9] S. Carillo, An evolution problem in materials with fading memory: Solution’s existence and uniqueness, Complex Variables and Elliptic Equations An International Journal, 56 (2011), 481–492.
[10] S. Carillo, Materials with mMemory: Free energies & solutions’ exponential decay, Commun. Pure Appl. Anal., 9 (2010), 1235–1248.
[11] C. Cattaneo, Sulla conduzione del calore, Atti Sem. Mat. Fis. Università Modena, 3 (1948), 83–101.
[12] V. V. Chepyzhov, E. Mainini and V. Pata, Stability of abstract linear semigroups arising from heat conduction with memory, Asymptotic Analysis, 50 (2006), 209–291.
[13] M. Chipot, I. Shafrir, V. Valente and G. Vergara Caffarelli, A nonlocal problem arising in the study of magneto-elastic interactions, Boll. UMI Serie IX, 1 (2008), 197–222.
[14] M. Chipot, I. Shafrir, V. Valente and G. Vergara Caffarelli, On a hyperbolic-parabolic system arising in magneto-elasticity, J. Math. Anal. Appl., 352 (2009), 120–131.
[15] B. D. Coleman, Thermodynamics of materials with memory, Arch. Rat. Mech. Anal., 17 (1964), 1–46.
[16] B. D. Coleman and E. H. Dill, On thermodynamics and stability of materials with memory, Arch. Rat. Mech. Anal., 51 (1973), 1–53.
[17] C. M. Dafermos, An abstract Volterra equation with applications to linear viscoelasticity, J. Diff. Equations, 7 (1970), 554–569.
[18] C. M. Dafermos, Asymptotic stability in viscoelasticity, Arch. Rat. Mech. Anal., 37 (1970), 297–308.
[19] M. Fabrizio, G. Gentili and D. W. Reynolds, On rigid heat conductors with memory, Int. J. Eng. Sci., 36 (1998), 765–782.
[20] M. Fabrizio, B. Lazzeri and A. Morro, Mathematical Models and Methods for Smart Materials, Series on Advances in Mathematics for Applied Sciences, World Scientific Publishing Co., Inc., River Edge, NJ, 62, 2002.
[21] C. Giorgi and G. Gentili, Thermodynamic properties and stability for the heat flux equation with linear memory, Quart. Appl. Math., 51 (1993), 343–62.
[22] C. Giorgi and V. Pata, Asymptotic behavior of a nonlinear hyperbolic heat equation with memory, Nonlinear Differential Equations and Applications, 8 (2001), 157–171.
[23] M. Grasselli and A. Lorenzi, Abstract nonlinear Volterra integro-differential equations with nonsmooth kernels, Atti. Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., 2 (1991), 43–53.
[24] M. E. Gurtin, Modern Continuum Thermodynamics, Mechanics Today, 1 (1972), 168–213.
[25] M. E. Gurtin and A. C. Pipkin, A general theory of heat conduction with finite wave speeds, Arch. Rat. Mech. Anal., 31 (1968), 113–126.
[26] J. Janno and L. von Woltersdorf, Identification of weakly singular memory kernels in viscoelasticity, ZAMM Z. Angew. Math. Mech., 78 (1998), 391–403.
[27] J. Janno and L. von Wollersdorf, Identification of weakly singular memory kernels in heat conduction, Z. Angew. Math. Mech., 77 (1997), 243–257.
[28] M. McCarthy, Constitutive equations for thermomechanical materials with memory, Int. J. Eng. Sci., 8 (1970), 467–474.
[29] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity. An Introduction to Mathematical Models, Imperial College Press, London, 2010.
[30] E. Mainini and G. Mola, Exponential and polynomial decay for first order linear Volterra evolution equations, Quart. Appl. Math., 67 (2009), 93–111.
[31] B. Miara, G. Stavroulakis and V. Valente, Topics on Mathematics for Smart Systems, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.
[32] R. K. Miller and A. Feldstein, Smoothness of solutions of Volterra integral equations with weakly singular kernels, SIAM J. Math. Anal., 2 (1971), 242–258.
[33] N.-E. Tatar, Exponential decay for a viscoelastic problem with a singular kernel, Zeitschrift für Angewandte Mathematik und Physik, 60 (2009), 640–650.
[34] V. Valente and G. Vergara Caffarelli, On the dynamics of magneto-elastic interactions: Existence of solutions and limit behavior, Asymptotic Analysis, 51 (2007), 319–333.
[35] G. Vergara Caffarelli, Dissipativity and uniqueness for the one-dimensional dynamical problem of linear viscoelasticity, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Mat. Natur., 82 (1990), 483–488.
[36] G. Vergara Caffarelli, Dissipativity and existence for the one-dimensional dynamical problem of linear viscoelasticity, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Mat. Natur., 82 (1988), 489–496.
[37] S. T. Wu, Exponential decay for a nonlinear viscoelastic equation with singular kernels, Acta Mathematica Scientia, 32 (2012), 2237–2246.

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