Symmetry Factors of Feynman Diagrams for Scalar Fields

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Abstract

The symmetry factor of Feynman diagrams for real and complex scalar fields is presented. Being analysis of Wick expansion for Green functions, the mentioned factor is derived in a general form. The symmetry factor can be separated into two ones corresponding to that of connected and vacuum diagrams. The determination of symmetry factors for the vacuum diagrams is necessary as they play a role in the effective action and phase transitions in cosmology. In the complex scalar theory the diagrams different in topology may give the same contribution, hence inverse of the symmetry factor \((1/S)\) for total contribution is a summation of each similar ones \((1/S_i)\), i.e., \(1/S = \sum_i(1/S_i)\).

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1 Introduction

In quantum field theory, physical processes can be determined by the elements of S-matrix as describing by Feynman diagrams correspondingly. In calculating these
diagrams, one of the important issues is the determination of symmetry factors associated with them (see, for example, [1]). What is the symmetry factor? How to obtain it? To see these, let us take an expansion at $p$-order of the $n$-point correlation function in a real-scalar theory with interaction $L_{\text{int}} = (\lambda/4!)\phi^4$ as follows

$$
(1/p!)(1/4!)^p\langle 0|T[\phi(x_1)\phi(x_2)\ldots\phi(x_n)\phi^4(y_1)\phi^4(y_2)\ldots\phi^4(y_p)]|0\rangle,
$$

where a factor $(i\lambda)^p$ and integrations over $y_1, y_2, ..., y_p$ are omitted as they are well-known in Feynman rules. We would like to count the number of different contractions giving the same expression (corresponding to a Feynman diagram) [2]. This number is equal to $N/D$, the number of all possible contractions divided by the number of identical contractions. The overall constant of the diagram then becomes $S^{-1} \equiv (1/p!)(1/4!)^p N/D$. The $S$ is generally different from unit and actually called the symmetry factor of the diagram.

Let us look further. The numerator $N$ is a product of (a) $p!$ interchanges of vertices $y_1, y_2, ..., y_p$ and (b) $N_i$ self-contractions of vertex $y_i$ and placements of contractions into this vertex ($i = 1, 2, ..., p$). $N_i$ is equal to $4!$ if there is no self-contraction, $4!/2$—one self-contraction (single-bubble), $4!/8$—two self-contractions (double-bubble). Thus, $N = p!\prod_i N_i = [p!(4!)^p]/[2^s 8^d]$, where $s$ and $d$ are the numbers of vertices with single-bubble and vertices with double-bubble, respectively. Since a double-bubble contains two single-bubbles, the total number of single-bubbles is $\beta = s + 2d$. We can rewrite $N = [p!(4!)^p]/[2^\beta 2^d]$. In contrast, the determination of the denominator $D$ is not easy. Roundly, we evaluate it as follows: i) interchange of vertex-vertex contractions. If there are $n$ contractions, we have $n!$ interchanges, ii) interchange of vertices $y_1, ..., y_p$ giving identical contractions, i.e. an identical set of Feynman propagators. In this case there are $d!$ interchanges of $d$-type vertices $\times g'$ interchanges of the remaining vertices. The result is $D = g'/d!\prod_{n=2,3,...}(n!)^{\alpha_n}$, in which $\alpha_n$ is the vertex-pair number with $n$ contractions. The symmetry factor has the form

$$
S = g'/d!\prod_{n}(n!)^{\alpha_n} 2^d 2^\beta.
$$

The definition of $g'$ is not trivial [3]. Sometimes this causes a lot of problems. In the literatures, the symmetry factors in the real scalar field theory with connected diagrams are presented only, but not for the vacuum ones as well as those of the complex fields. It is to be noted that the vacuum diagrams have the applications in particle physics and cosmology such as in effective action and phase transition (see, for example, Refs [1,2]). A formula for calculating symmetry factors of such diagrams is necessarily provided. The aim of this work is to explicitly drive the above expression.
for symmetry factor in real scalar theory. We will achieve this by applying the Wick’s theorem, and by the way we will show that the vacuum diagrams are factorized [4] explicitly order by order of the perturbation theory. We also study its generalizations to complex scalar fields, and so forth. A list of symmetry factors corresponding to Feynman diagrams in the theories are also given.

This work is organized as follows: In the next section, Sec. 2 we provide some notations. In section 3 we formulate symmetry factor for the real scalar theory. In section 4 we generalize the case to the complex fields. The special features existing only in the complex theory are also given in this section. We summary our results and make conclusions in the last section, Sec. 5. The appendix is devoted to Feynman diagrams and respective symmetry factors in both the theories.

2 Notations

Let us first mention some ingredients in the $S$ matrix. The time-evolution operator is given in terms of action as [5]

$$U(t_1, t_2) = T \exp[iS_{int}(t_1, t_2, \hat{\phi})]$$

$$= N \left\{ \exp \left( \frac{1}{2} \frac{\delta}{\delta \phi_i} \Delta \frac{\delta}{\delta \phi_k} \right) \exp[iS_{int}(t_1, t_2, \phi)] \right\} |...|$$

(3)

where symbol $|...|$ denotes the fact that after differentiation classical fields $\phi_i$ are replaced by quantized ones $\hat{\phi}_i$. $T$ and $N$ stand for time-ordering and normal-ordering operators, respectively. The $S$-matrix is a limit of the time-evolution operator when $t_1 \to -\infty$ and $t_2 \to +\infty$. The c-number (Feynman propagator) function $\Delta(x, x')$ is defined as

$$\Delta(x, x') = T[\hat{\phi}(x)\hat{\phi}(x')] - N[\hat{\phi}(x)\hat{\phi}(x')].$$

(4)

For further presentation, the following formula is useful [5]

$$T \left\{ \prod_{i=1}^{n} F_i(\hat{\phi}) \right\} = N \left\{ \exp \left[ \frac{1}{2} \sum_{i} \frac{\delta}{\delta \phi_i} \Delta \frac{\delta}{\delta \phi_i} + \sum_{i<k} \frac{\delta}{\delta \phi_i} \Delta \frac{\delta}{\delta \phi_k} \right] \prod_{i=1}^{n} F_i(\phi_i) \right\} |...|$$

(5)

It is worth mentioning that the first term in r.h.s in (5) exists only for real field theory.

Let us recall that as presented in the introduction, every Feynman diagram has a symmetry factor. In Ref. [1, 6] this is given by

$$S = g^{2^d} \prod_{n=2,3,...} (n!)^{\alpha_n},$$

(6)
where $\alpha_n$ is the number of pairs of vertices connected by $n$ identical self-conjugate lines, $\beta$ is the number of lines connecting a vertex with itself, and $g$ is the number of permutations of vertices which leave the diagram unchanged with fixed external lines. It is worth noting that a factor $2^\beta$ connects with the factor $1/2$ at the first term in r.h.s in (5). Notice also that formula (6) works only for connected diagrams but not for the diagrams with vacuum ones. Let us derive the symmetry factor in the general case as sketched out in [2] for the real $\phi^4$ theory.

3 Symmetry factors in real scalar theory

Consider the model with interaction Lagrangian given by

$$L_{\text{int}}^r = \frac{\lambda}{4!} \phi^4$$

(7)

It is well known that there is a direct connection between the $S$-matrix elements and the Green functions. Hence we will deal with the Green functions defined as

$$G(x_1, x_2, ..., x_n) = \sum_{p=0}^{\infty} G^{(p)}(x_1, x_2, ..., x_n)$$

with the $p$–order expansion given by

$$G^{(p)}(x_1, x_2, ..., x_n) = \frac{i^p}{p!} \int_{-\infty}^{\infty} d^4 y_1 ... d^4 y_p \langle 0 | T [\phi(x_1) \cdots \phi(x_n) L_{\text{int}}^r(\phi(y_1)) \cdots L_{\text{int}}^r(\phi(y_p))] | 0 \rangle.$$ 

(8)

This is actually called the $p$–order Green function. The $G(x_1, ..., x_n)$ contains every $n$–point diagram in the theory including connected and disconnected ones.

Let us remind the reader that the four fields in $L_{\text{int}}^r(\phi(y))$ are equal times. Applying (5) for (7) we get

$$\phi^4(y) \sim T[\phi^4(y)] = N[\phi^4(y)] + 6N[\phi^2(y)] \hat{\Delta} + 3 \hat{\Delta} \hat{\Delta},$$

(9)

where $\hat{\Delta} \equiv \Delta(y, y)$ denotes the bubble diagram $\circ$. Let us denote three terms in (9) by $a$, $b$ and $c$:

$$a \equiv N[\phi^4(y)], \quad b \equiv N[\phi^2(y)] \hat{\Delta}, \quad c \equiv \hat{\Delta} \hat{\Delta}.$$ 

(10)

Then (9) is rewritten in the form

$$\phi^4 \sim T[\phi^4] = a + 6b + 3c.$$ 

(11)

The Green function (8) is invariant under permutations of the interaction Lagrangians. Hence, the production of these Lagrangians can be expanded as a summation of monomials of $a$'s, $b$'s and $c$'s so that all terms like $a^p b^q c^t$ with definite $p, q, t$
are equivalent under integration. The overall coefficients of the monomials in the expansion can be extracted by using the following multinomial formula

\[(x_1 + x_2 + \cdots + x_r)^p = \sum_{p_1, p_2, \ldots, p_r} \frac{p!}{p_1! p_2! \cdots p_r!} x_1^{p_1} \cdots x_r^{p_r}, \quad (12)\]

with \( p_1 + p_2 + p_3 + \cdots + p_r = p. \)

The equation (8) becomes

\[G^{(p)}(x_1, x_2, \ldots, x_n) = \frac{1}{p!} \left( \frac{i\lambda}{4!} \right)^p \sum_{p_1+p_2+p_3=p} \frac{p!}{p_1! p_2! p_3!} \int_{-\infty}^{\infty} d^4 y_1 \cdots d^4 y_p \times \langle 0|T[\phi(x_1)\ldots\phi(x_n) a^{p_1} b^{p_2} c^{p_3}]|0 \rangle, \quad (13)\]

where the variables under integrations are easily understood:

\[a^{p_1} b^{p_2} c^{p_3} = a(y_1) a(y_2) \ldots a(y_{p_1}) b(y_{p_1+1}) b(y_{p_1+2}) \ldots b(y_{p_1+p_2}) c(y_{p_1+p_2+1}) c(y_{p_1+p_2+2}) \cdots c(y_p). \]

For the further presentation, let us omit the summations and integrations and represent coefficients of \( b \) and \( c \) by

\[6 = \frac{4!}{2! 2!}, \quad 3 = \frac{4!}{2! 2! 2!}. \quad (14)\]

Thus the Green function can be rewritten in the form

\[G^{(n)}(x_1, x_2, \ldots, x_n) = (i\lambda)^p A B; \quad (15)\]

where

\[A \equiv \frac{(4!)^{p_2} (4!)^{p_3}}{(4!)^{p_1+p_2+p_3} (2!)^{p_2} (2!)^{p_2} (2!)^{p_3} (2!)^{p_3} p_1! p_2! p_3!}, \quad (16)\]

\[B \equiv \langle 0|T[\phi(x_1) \ldots \phi(x_n) a^{p_1} b^{p_2} c^{p_3}]|0 \rangle. \quad (17)\]

Note that, \( b \) associated with \( p_2 \), contains one bubble diagram, while \( c \) associated with \( p_3 \), contains two ones—double bubble \( \bigcirc \bigcirc \), hence if we call the \( \beta \) is the number of lines that connect a vertex with itself, then

\[\beta = p_2 + 2p_3. \quad (18)\]

Moreover, those bubbles can be factored out from the \( T \)-product in \( B \), i.e. the \( T \)-operator would not act on them:

\[B = \langle 0|T[\phi(x_1) \ldots \phi(x_n) (N(\phi^4))^{p_1} (N(\phi^2))^{p_2}]|0 \rangle \Delta^{p_2} \Delta^{2p_3}, \quad (19)\]
where the double bubbles as disconnected pieces are vacuum subdiagrams. Notice also that \(p_2, p_3\) just coincide with the notations \(s, d\) in the introduction, respectively.

Correspondingly, the coefficient \(A\) is interpreted as

\[
A = \left[ \frac{1}{(4!)^{p_1}(2!)^{p_2}p_1!p_2!} \right] \left[ \frac{1}{2^3(2!)^{p_3}p_3!} \right].
\]

(20)

In this formula, \(p_1!\) and \(p_2!\) are, respectively, the numbers of permutations of \(a\)'s and \(b\)'s vertices, similar to \(p!\) in (8). The \(4!\) (powered \(p_1\)) and \(2!\) (powered \(p_2\)) are symmetry factors (the number of permutations of identical interaction-fields) respectively associated with \(a\) and \(b\), similar to \(4!\) in (7). Totally, we get the factor of \(p_1!p_2!(4!)^{p_1}(2!)^{p_2}\) that is deduced as the first factor in (20). This factor would be reduced depending on the \(T\)-product expansion for \(B\). The second factor associated with the bubbles are constantly unchanged under \(T\)-product. It is obtained as follows: \(p_3!\) is the number of permutations of \(c\)'s vertices; \(2!\) (powered \(p_3\)) is the number of permutations of the two single-bubbles of any \(c\) vertex; the remainder \(\beta\) has been mentioned as above.

Next, to contract \(B\) under the \(T\)-product, let us call the reader’s attention to Eq. (4.45) in Ref. [2] as a guidance. The number of different contractions that give the same expression is production of: 1) \(p_1!p_2!\) interchanges of \(p_1\) \(a\)'s and \(p_2\) \(b\)'s vertices. 2) placement of contractions into a vertex, \(4!\) for \(a\) and \(2!\) for \(b\), thus \(4!)^{p_1}(2!)^{p_2}\) for \(p_1\) \(a\)'s and \(p_2\) \(b\)'s vertices. Note that there is no self-contraction for each vertex. 3) \(1/\Pi_{n=2,3...(n!)}^{\alpha_n}\) interchanges of vertex–vertex contractions, with \(n\)—the number of contractions and \(\alpha_n\)—the number of vertex pairs with \(n\) contractions. 4) If we call \(g'\) is interchange number of vertices \(a\)'s and \(b\)'s that does not topologically change the diagram, then the factor of \(1/g'\) should be multiplied to the result. In summary, the overall factor contributing to one diagram is

\[
\frac{p_1!p_2!(4!)^{p_1}(2!)^{p_2}}{g'\Pi_n(n!)^{\alpha_n}} A = \frac{1}{(g'p_3!2^3(2!)^{p_3}\Pi_n(n!)^{\alpha_n}}
\]

(21)

Thus the symmetry factor is given by

\[
S = g2^3(2!)^d \Pi_n(n!)^{\alpha_n},
\]

(22)

where \(d = p_3\), and \(g = g'p_3!\) has the same meaning as \(g'\). Let us notice that any vertex of \(a\)'s and \(b\)'s directly connected to the external points \(x_1, x_2, ..., x_n\) is not subject under the interchanges defining \(g'\). The examples in Ref. [3] and followings demonstrate this.

The resulting diagram is typically consisted of connected pieces (subdiagrams), a piece connected to \(x_1, x_2, ..., x_n\) and several pieces disconnected from all the external
points—vacuum bubbles, in which the double-bubble as mentioned is one of the cases. Label the connected piece by $V_c$ and the various possible disconnected pieces by:

$$V_k \in \quad , \quad \quad , \quad \quad , \quad \ldots$$

where $k = 1, 2, 3...$ Suppose that the diagram has $n_k$ pieces of the form $V_k$, for each $k$, in addition to $V_c$. Let the value of $g$ for the connected piece $V_c$ and disconnected pieces $V_k$ be $g_c$ and $g_k$, respectively. It is easy to obtain $g = \prod_l n_l!(g_l)^{n_l}$, where $l = c, k$ and $n_c = 1, n_1 = p_3$. Here $n_k!$ is the symmetry factor coming from interchanging the $n_k$ copies of $V_k$. We can therefore rewrite (22) as

$$S = \prod_l n_l!(S_l)^{n_l} = S_c \times \prod_k n_k!(S_k)^{n_k}, \quad (23)$$

where $S_l = S_c$, $S_k$ is the symmetry factor of $V_l$ having the same form as (22):

$$S_l = g_l 2^{\beta_l}(2!)^{d_l} \prod_n (n1)^{\alpha_n^l}, \quad (24)$$

where the parameters indexed by $l$ are those of $V_l$ satisfying $d_{l=1} = 1$, $d_{l\neq1} = 0$, $d = \sum_l n_l d_l$, $\beta = \sum_l n_l \beta_l$, $\alpha_n = \sum_l n_l \alpha_n^l$. It is worth mentioning that there is an additional factor of $2!$ associated with only double-bubble. This is in contradiction to the formula (6) as given in the literatures.

In calculating, let us recall that the symmetry factor of arbitrary diagrams is obtained from (22) or (23) while that of connected diagrams is given by (24). Since (22) and (24) have the same form, we can commonly use (22) for both the cases with the corresponding parameters being understood. The symmetry factors of some two-point connected diagrams are
We have also, for some vacuum bubbles,

- $S = 8$ ($g = 1, \beta = 2, d = 1, \alpha_n = 0$)
- $S = 16$ ($g = 2, \beta = 2, d = 0, \alpha_2 = 1$)
- $S = 48$ ($g = 2, \beta = 0, d = 0, \alpha_4 = 1$)
- $S = 24$ ($g = 2, \beta = 1, d = 0, \alpha_3 = 1$)

For general diagrams, we consider the following example:
$S = 2.2!(8)^2 = 256$, with the help of (23).
Alternatively, $S = 256\ (g = 2, \beta = 5, d = 2, \alpha_n = 0)$, from (22).

More examples of symmetry factors are given in the next sections. In the following if some parameter gets its trivial value such as $g = 1, \beta = 0$ or so on, the parameter will be not listed in parentheses. Next, we consider the case of complex scalar fields.

4 Symmetry factors in complex scalar theory

The interaction Lagrangian is given by

$$\mathcal{L}_{\text{int}}^c = \frac{\rho}{4}(\varphi^* \varphi)^2$$

(25)

Applying (5) one gets

$$(\varphi^* \varphi)^2 \sim T[(\varphi^* \varphi)^2] = N[(\varphi^* \varphi)^2] + 4N(\varphi^* \varphi)\hat{\Delta} + 2\hat{\Delta}\hat{\Delta},$$

(26)

where $\hat{\Delta}$, in this case, denotes the bubble diagram $\bigcirc$. As before, we call also the corresponding terms to $a, b, c$. The $p$-order Green function is

$$G^{(p)}(x_1, x_2, ..., x_n) = (i\rho)^p A_c \langle 0[T[\varphi(x_1)...\varphi^*(x_n)a^{p_1}b^{p_2}c^{p_3}]]0 \rangle,$$

(27)

where the integrations and summations are understood, and

$$A_c \equiv \frac{1}{4^{p_1}2^{p_3}p_1!p_2!p_3!}.$$  

(28)

It is to be noted that the unique case giving non-zero Green function when number of fields $\varphi$ in (27) equal their complex-conjugate ones $\varphi^*$.

Repeating previous analysis, we get contribution for one diagram

$$\frac{p_1!p_2!4^{p_1}}{g'\prod_n(n!)^{a_n}} A_c = \frac{1}{(g'p_3!)2^{p_3} \prod_n(n!)^{a_n}}.$$  

(29)
Hence, the symmetry factor in the theory under consideration is given by

\[ S = g^{2d} \prod_n (n!)^{\alpha_n}, \quad (30) \]

where \( d = p_3 \) is the number of double-bubbles, and \( g = g' p_3! \) is the number of interchanges of interacting-vertices leaving the diagram as well as its charged-scalar flows invariant. As before, the symmetry factor can be separated into several pieces corresponding to connected and vacuum diagrams, namely,

\[ S = S_c \times S_v. \quad (31) \]

The symmetry factor for these subdiagrams has the same form as \((31)\) in which \( d \) is nonzero only if it is associated with double-bubble.

It is emphasized that there is no factor \( 2^\beta \) in \((30)\). Note that \( n \) is the number of identical lines connecting two separated vertices with the same direction. The formula \((30)\) is just a generation of \((22)\) when discriminations among directions of scalar fields are included. We illustrate this by the following examples:

\[ (a) \ S = 1 \quad (b) \ S = 2 \ (\alpha_2 = 1) \quad (c) \ S = 8 \ (g = 2, \alpha_2 = 2) \]

In the diagram (a) the symmetry factor is 1 due to vanishing \( \beta \). In (b) we have only one set \( n = 2 \), while in (c) we have two sets with \( n = 2 \). Remember that in the real scalar theory, for the similar diagrams: \( n = 3 \) and \( n = 4 \), respectively. Many comparisons of symmetry factors of diagrams at three-order in the real and complex scalar theories are given in the appendix.

From Eq.\((31)\) it follows that the vacuum diagrams are factorized order by order of the perturbation theory. Thus the connected Green functions are defined, as in the literature, by the following formula

\[ <0|T[\varphi(x_1) \cdots \varphi(x_n)]|0> = \frac{\langle 0|T[\varphi(x_1) \cdots \varphi(x_n) \exp i \int d^4y L_{int}]|0\rangle}{\langle 0|\exp i \int d^4y L_{int}|0\rangle}, \quad (32) \]

where the denominator produces vacuum diagrams.

Next, let us discuss some special features of the complex theory. We consider two contributions with respective symmetry factors demonstrated as
It is easy to check that these contributions are the same. This is due to the fact that $\Delta(x, y) = \Delta(y, x)$ [7]. Hence, the contributions of this type can be determined by only one diagram with the symmetry factor given by

$$S^{-1} = S_1^{-1} + S_2^{-1}. \quad (33)$$

Thus $S = 24/5$.

It is to be noted that recently proposed hybrid inflationary scenario in which two scalar fields [8] $\phi$ and $\varphi$ with the coupling

$$\frac{\lambda}{2}(\phi^2\varphi^2) \quad (34)$$

It is easily to check out that our formula can be applied for such kind of interaction.

5 Conclusion

We have derived the symmetry factor for both real and complex scalar theories:

$$S = g2^\beta 2^d \prod_n (n!)^{\alpha_n}, \quad (35)$$

where $g$ is number of interchanges of vertices leaving the diagram topologically unchanged, $\beta$ number of lines connecting a vertex with itself ($\beta$ vanishes if the field is complex), $d$ number of double-bubbles, and $\alpha_n$ number of vertex-pairs connected by $n$-identical lines. Our result revises the usual formula for symmetry factor as given in the literature. The result in this paper can be easily generalized to the fields with higher spin.
We have also shown that in the complex scalar theory the diagrams different in topology may give the same contribution. The symmetry factor for contributions of such type is also obtained.

Our result explicitly shows that, as expected, the vacuum diagrams are factorized order by order of the perturbation theory.

Let us recall that the determination of symmetry factor is important because it is not only in part of modern quantum field theory, but also having applications in effective-potential calculations in extra-dimensional and cosmological theories.

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References

[1] T. P. Cheng and L. F. Li, *Gauge Theory of Elementary Particle Physics*, Clarendon Press (2004).
[2] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, Westview Press (1995).
[3] C. D. Palmer and M. E. Carrington, Can. J. Phys. 80, 847 (2002).
[4] L. H. Ryder, *Quantum field theory*, 2nd edition, Cambridge University Press, (1998).
[5] A. N. Vasiliev, *Functional methods in Quantum Field Theory and Statistics*, Leningrad University Press, (1976).
[6] Michio Kaku, *Quantum Field Theory, A Modern Introduction*, Oxford University Press (1993).
[7] W. Greiner and J. Reinhardt, *Field Quantization*, Springer (1996), p. 112.
[8] A. D. Linde, Phys. Lett. B 259, 38 (1991).
A Symmetry factors compared at three-order in the real and complex scalar theories

The left-side is the diagrams of the real scalar theory. The corresponding diagrams of the complex scalar theory are given on the right side.

\[ S = 3072 \ (g = 3!, \beta = 6, d = 3) \quad S = 48 \ (g = 3!, d = 3) \]

\[ S = 256 \ (g = 2, \beta = 5, d = 2) \quad S = 8 \ (g = 2, d = 2) \]

\[ S = 12 \ (\beta = 1, \alpha_3 = 1) \quad S = 2 \ (\alpha_2 = 1) \]
$S = 128 \ (g = 2, \beta = 4, d = 1, \alpha_2 = 1)$

$S = 4 \ (g = 2, d = 1)$

$S = 32 \ (\beta = 4, d = 1)$

$S = 2 \ (d = 1)$

$S = 32 \ (\beta = 3, d = 1, \alpha_2 = 1)$

$S = 2 \ (d = 1)$

$S = 32 \ (g = 2, \beta = 3, \alpha_2 = 1)$

$S = 2 \ (g = 2)$
\[ S = 48 \ (g = 3!, \beta = 3) \]

\[ S = 3 \ (g = 3) \]

\[ S = 8 \ (\beta = 3) \]

\[ S = 1 \]

\[ S = 32 \ (g = 2, \beta = 2, \alpha_2 = 2) \]

\[ S = 2 \ (g = 2) \]

\[ S = 8 \ (\beta = 2, \alpha_2 = 1) \]

\[ S = 1 \]
\[ S = 8 \ (g = 2, \beta = 2) \]

\[ S = 384 \ (g = 2, \beta = 2, d = 1, \alpha_4 = 1) \]

\[ S = 48 \ (\beta = 2, d = 1, \alpha_3 = 1) \]

\[ S = 24 \ (g = 2, \beta = 1, \alpha_3 = 1) \]

\[ S = 1 \]

\[ S = 16 \ (g = 2, d = 1, \alpha_2 = 2) \]

\[ S = 4 \ (d = 1, \alpha_2 = 1) \]

\[ S = 2 \ (\alpha_2 = 1) \]
\[ S = 8 \ (\beta = 1, \alpha_2 = 2) \]

\[ S = 1 \]

\[ S = 96 \ (g = 2, \beta = 1, \alpha_4 = 1) \]

\[ S = 8 \ (g = 2, \alpha_2 = 2) \]

\[ S = 4 \ (\beta = 1, \alpha_2 = 1) \]

\[ S = 2 \ (\alpha_2 = 1) \]

\[ S = 12 \ (g = 2, \alpha_3 = 1) \]

\[ S = 2 \ (\alpha_2 = 1) \]
\[ S = 48 \ (g = 3!, \ \alpha_2 = 3) \]

\[ S = 24 \ (g = 3, \ \alpha_2 = 3) \]

\[ S = 6 \ (g = 3!) \]

\[ S = 4 \ (\alpha_2 = 2) \]

\[ S = 1 \]

\[ S = 4 \ (\alpha_2 = 2) \]