On weighted Hardy inequality with two-dimensional rectangular operator – extension of the E. Sawyer theorem

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Abstract: A characterization is obtained for those pairs of weights \( v \) and \( w \) on \( \mathbb{R}_+^2 \), for which the two-dimensional rectangular integration operator is bounded from a weighted Lebesgue space \( L^p_w(\mathbb{R}_+^2) \) to \( L^q_w(\mathbb{R}_+^2) \) for \( 1 < p \neq q < \infty \), which is an essential complement to E. Sawyer’s result \cite{14} given for \( 1 < p \leq q < \infty \). Besides, we declare that the E. Sawyer theorem is actual if \( p = q \) only, for \( p < q \) the criterion is less complicated. The case \( q < p \) is new.

Key words: Rectangular integration operator; Hardy inequality; weighted Lebesgue space.

MSC: 26D10, 47G10

1 Introduction

Let \( n \in \mathbb{N} \). For Lebesgue measurable functions \( f(y_1, \ldots, y_n) \) on \( \mathbb{R}_+^n := (0, \infty)^n \) the \( n \)-dimensional rectangular integration operator \( I_n \) is given by the formula

\[
I_n f(x_1, \ldots, x_n) := \int_0^{x_1} \cdots \int_0^{x_n} f(y_1, \ldots, y_n) \, dy_1 \cdots dy_n \quad (x_1, \ldots, x_n > 0).
\]

The dual transformation \( I_n^* \) has the form

\[
I_n^* f(x_1, \ldots, x_n) := \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f(y_1, \ldots, y_n) \, dy_1 \cdots dy_n \quad (x_1, \ldots, x_n > 0).
\]

Let \( 1 < p, q < \infty \) and \( v, w \geq 0 \) be weight functions on \( \mathbb{R}_+^n \). Consider Hardy’s inequality

\[
\left( \int_{\mathbb{R}_+^n} (I_n f)^q w \right)^{\frac{1}{q}} \leq C_n \left( \int_{\mathbb{R}_+^n} f^p v \right)^{\frac{1}{p}} \quad (f \geq 0) \tag{1}
\]

on the cone of non-negative functions in weighted Lebesgue space \( L^p_w(\mathbb{R}_+^n) \). The constant \( C_n > 0 \) in (1) is assumed to be the least possible and independent of \( f \). For a fixed weight \( v \) and a parameter \( p > 1 \) the space \( L^p_w(\mathbb{R}_+^n) \) consists of all measurable on \( \mathbb{R}_+^n \) functions \( f \) such that \( \int_{\mathbb{R}_+^n} |f|^p v < \infty \).

The problem of characterizing the inequality (1) is well known and has been considered by many authors (see \cite{14} and references therein). The one-dimensional case of this inequality has been completely studied (see \cite{5, 7, 13}). However, for \( n > 1 \) difficulties arise, preventing characterizing (1) without additional restrictions on \( v \) and \( w \). Nevertheless, E. Sawyer’s result is well known for arbitrary \( v, w \) in the case \( 1 < p \leq q < \infty \).

To formulate it we denote \( p' := p/(p-1) \) and \( \sigma := v^{1-p'} \).

**Theorem 1.1.** \cite{14} Theorem 1A] Let \( n = 2 \) and \( 1 < p \leq q < \infty \). The inequality (1) holds for all measurable non-negative functions \( f \) on \( \mathbb{R}_+^2 \) if and only if

\[
A_1 := A_1(p, q) := \sup_{(t_1, t_2) \in \mathbb{R}_+^2} \left[ I^*_2 w(t_1, t_2) \right]^{\frac{1}{2}} \left[ I^*_2 \sigma(t_1, t_2) \right]^{\frac{1}{2}} \frac{1}{p} < \infty, \tag{2}
\]

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\begin{align}
A_2 & := A_2(p, q) := \sup_{(t_1, t_2) \in \mathbb{R}^2_+} \left( \int_0^{t_1} \int_0^{t_2} (I_2 \sigma(t_1, t_2))^{q} w \, dt_1 \right)^{\frac{1}{q}} \left[ I_2 \sigma(t_1, t_2) \right]^{-\frac{1}{p'}} < \infty, \\
A_3 & := A_3(p, q) := \sup_{(t_1, t_2) \in \mathbb{R}^2_+} \left( \int_0^{\infty} \int_0^{\infty} (I_2^* \sigma \sigma)^{p'} \, dt_1 \right)^{\frac{1}{p'}} \left[ I_2^* \sigma(t_1, t_2) \right]^{-\frac{1}{p}} < \infty,
\end{align}
and \( C_2 \approx A_1 + A_2 + A_3 \) with equivalence constants depending on parameters \( p \) and \( q \).

Note that in one–dimensional case the analogs of the conditions \((2)–(4)\) are equivalent to each other \([2]\). For \( n = 2 \) this, generally speaking, is not true. Moreover, as shown in \([14], \S \ 4\) for \( p = q = 2 \), no two of the conditions \((2)–(4)\) guarantee \((1)\). However, the construction of the second counterexample in \([14], \S \ 4\) is unexpandable to the case \( p < q \).

The purpose of this paper is to obtain new criteria for the fulfillment of Hardy’s inequality \((1)\) for \( n = 2 \) and \( 1 < p \neq q < \infty \). The solution to this problem is contained in Theorem \( 2.1 \) (see \( \S \ 2 \)). In \( \S \ 3 \) an alternative sufficient condition is found for \( v \) and \( w \), when \((1)\) is true for \( n = 2 \) and \( 1 < q < p < \infty \). Recall that the criterion for \((1)\) when \( n = 2 \) and \( 1 < p \leq q < \infty \), established in \([14]\), is that the sum of three independent functionals is bounded (see Theorem \( 1.1 \)). It is proven in Theorem \( 2.1 \) that for \( 1 < p \neq q < \infty \) the inequality \((1)\) is characterized by only one functional.

Analogs of Theorem \( 2.1 \) are also valid for the dual operator \( I_2^* \) and mixed Hardy operators (see \([14], \text{Remark 1}\) for details).

In \( \S \ 4 \), for completeness, we present known results about the operator \( I_n \) for arbitrary \( n \), provided that at least one of the two weight functions in \((1)\) is factorizable, that is, can be represented as a product of \( n \) one–dimensional functions.

Since \( A_1 \leq C_2 \), we may and shall assume that \( I_2 \sigma(x, y) < \infty \) and \( I_2^* w(x, y) < \infty \) for any \( (x, y) \in \mathbb{B}_R^2 \). In particular, \( \sigma, w \in L^1_{\text{loc}}(\mathbb{B}_R^2) \).

Throughout the work, the notation of the form \( \Phi \lesssim \Psi \) means that the relation \( \Phi \leq c \Psi \) holds with some constant \( c > 0 \), independent of \( \Phi \) and \( \Psi \). We write \( \Phi \approx \Psi \) in the case of \( \Phi \lesssim \Psi \lesssim \Phi \). The symbols \( \mathbb{Z} \) and \( \mathbb{N} \) are used for denoting the sets of integers and natural numbers, respectively. The characteristic function of the subset \( E \subset \mathbb{R}_+^n \) is denoted by \( \chi_E \). Symbols := and :=: are used to define new values.

## 2 Main result

Denote
\[
\alpha(p, q) := \frac{p^2(q - 1)}{q - p}, \quad p < q;
\]
\[
\beta(p, q) := \frac{2^{q+1}}{2^q - 1} \cdot \begin{cases} 
2^{\frac{q+1}{2}}, & \frac{r}{p} \geq 1, \\
1, & \frac{r}{p} < 1,
\end{cases} \quad q < p,
\]
where \( 1/r := 1/q - 1/p; A := A_1 \).

\[
B := B_1 := B_1(p, q) := \left( \int_{\mathbb{B}_R^2} d_y \left[ I_2 \sigma(x, y) \right]^{\frac{q}{p}} d_x \left[ - I_2^* w(x, y) \right]^{\frac{q}{p}} \right)^{\frac{1}{q}} = \left( \int_{\mathbb{B}_R^2} \left[ I_2 \sigma(x, y) \right]^{\frac{q}{p}} d_x d_y \left[ I_2^* w(x, y) \right]^{\frac{q}{p}} \right)^{\frac{1}{q}},
\]
where the last two equalities follow by integration by parts; also
\[
B_2 := B_2(p, q) := \left( \int_{\mathbb{R}^d_+} [I_2 \sigma(x, y)]^{-\frac{2}{p'}} d_x d_y \left( \int_0^x \int_0^y (I_2 \sigma)^q w \right)^{\frac{1}{p'}} \right),
\]
\[
B_3 := B_3(p, q) := \left( \int_{\mathbb{R}^d_+} [I_2^* w(x, y)]^{-\frac{2}{p'}} d_x d_y \left( \int_0^\infty \int_y^\infty (I_2^* w)^{q'} \sigma \right)^{\frac{1}{q'}} \right).
\]
Notice that
\[
\lim_{q \to p} B_i(p, q) = A_i(p, p), \quad i = 1, 2, 3.
\]
Let us recall the result we need in what follows from the work [3].

**Proposition 2.1.** [3, Proposition 2.1] Let \(0 < \gamma < \infty\) and let \(\{a_k\}_{k \in \mathbb{Z}}, \{\rho_k\}_{k \in \mathbb{Z}}, \{\tau_k\}_{k \in \mathbb{Z}}\) be non-negative sequences.

(a) If \(\rho := \inf_{k \in \mathbb{Z}} \rho_{k+1}/\rho_k > 1\) then
\[
\sum_{k \in \mathbb{Z}} \left( \sum_{m \geq k} a_m \right)^\gamma \rho_k \leq \sum_{m \in \mathbb{Z}} a_m^{\gamma} \rho_m \gamma \frac{\rho_{\gamma}}{\rho^{\gamma - 1} \rho^{\gamma - 1}}, \quad 0 < \gamma \leq 1, \quad \gamma > 1.
\]

(b) If \(\tau := \sup_{k \in \mathbb{Z}} \tau_{k+1}/\tau_k < 1\) then
\[
\sum_{k \in \mathbb{Z}} \left( \sum_{m \leq k} a_m \right)^\gamma \tau_k \leq \sum_{m \in \mathbb{Z}} a_m^{\gamma} \tau_m \gamma \frac{\tau_{\gamma}}{(\tau^{\gamma - 1} (1 - \tau^{\gamma - 1}))}, \quad 0 < \gamma \leq 1, \quad \gamma > 1.
\]

We start with some auxiliary technical statements.

**Lemma 2.1.** Let \(0 \leq a < b < c < d < \infty\) and \(0 \leq c < d < \infty\). If \(1 < p < q < \infty\) then
\[
V_{(a, b) \times (c, d)} := \int_a^b \int_c^d w(x, y) \left( \int_a^x \int_c^y \sigma \right)^q dy dx \leq \alpha(p, q) \left( \int_a^b \int_c^d \sigma \right)^q A^q.
\]
For \(1 < q < p < \infty\) the following inequality holds:
\[
V_{(a, b) \times (c, d)} \leq \beta(p, q) \left( \int_a^b \int_c^d \sigma \right)^q \times \left[ \int_a^b \int_c^d \chi_{\text{supp } w(x, y)} dy \left( I_2 \sigma(x, y) \right)^\frac{q}{p'} dx \left( \int_0^\infty \int_y^\infty \left( I_2^* w(x, y) \right)^{q'} \sigma \right)^{\frac{1}{q'}} \right]^{\frac{1}{q'}}.
\]
**Proof.** Assume \(1 < p < q < \infty\) and write
\[
V_{(a, b) \times (c, d)} = \int_a^b \int_c^d \left( \int_a^x \int_c^y \sigma \right)^q dy \left[ \int_y^d w(x, t) dt \right] dx
\]
\[
= q \int_a^b \int_c^d \left( \int_a^x \int_c^y \sigma \right)^q \left( \int_a^x \sigma(s, y) ds \right) \left( \int_y^d w(x, t) dt \right) dy dx
\]
\[
= q \int_c^d \int_a^b \left( \int_a^x \int_c^y \sigma \right)^q \left( \int_a^x \sigma(s, y) ds \right) dx \left[ \int_x^d \int_y^d w \right] dy
\]
\[
= q \int_a^b \int_c^d \left( \int_a^x \int_c^y \sigma \right)^q \left( \int_a^x \sigma(s, y) ds \right) \left( \int_c^y \sigma(x, t) dt \right)
\]
\[
+ q \left( \int_a^x \int_c^y \sigma \right)^q \left( \int_a^x \sigma(x, t) dt \right) \left( \int_c^y \sigma(x, t) dt \right) dx dy.
\]
Then
\[
V_{(a,b) \times (c,d)} \leq q A^q \int_a^b \int_c^d \left\{ (q - 1) \left( \int_a^x \int_c^y \sigma \right)^{\frac{q}{p} - 2} \left( \int_a^x \sigma(s, y) \, ds \right) \left( \int_c^y \sigma(x, t) \, dt \right) \right. \\
\left. + \left( \int_a^x \int_c^y \sigma \right)^{\frac{q}{p} - 1} \sigma(x, y) \right) \, dx \, dy.
\]

The assertion of the lemma for the case \( p < q \) follows from the chain of inequalities:
\[
q \int_a^b \int_c^d \left\{ (q - 1) \left( \int_a^x \int_c^y \sigma \right)^{\frac{q}{p} - 2} \left( \int_a^x \sigma(s, y) \, ds \right) \left( \int_c^y \sigma(x, t) \, dt \right) \\
\left. + \left( \int_a^x \int_c^y \sigma \right)^{\frac{q}{p} - 1} \sigma(x, y) \right) \right\} \, dx \, dy \]
\[
= p \int_a^b \int_c^d \left\{ q \left( \frac{q}{p} - 1 \right) \left( \int_a^x \int_c^y \sigma \right)^{\frac{q}{p} - 2} \left( \int_a^x \sigma(s, y) \, ds \right) \left( \int_c^y \sigma(x, t) \, dt \right) \\
\left. + \frac{q}{p} \left( \int_a^x \int_c^y \sigma \right)^{\frac{q}{p} - 1} \sigma(x, y) \right) \right\} \, dx \, dy \]
\[
\leq p \int_a^b \int_c^d \left\{ \frac{q}{p} \left( \frac{q}{p} - 1 \right) \left( \int_a^x \int_c^y \sigma \right)^{\frac{q}{p} - 2} \left( \int_a^x \sigma(s, y) \, ds \right) \left( \int_c^y \sigma(x, t) \, dt \right) \\
\left. + \frac{q}{p} \left( \int_a^x \int_c^y \sigma \right)^{\frac{q}{p} - 1} \sigma(x, y) \right) \right\} \, dx \, dy \]
\[
= \left[ p + \frac{pq(p - 1)}{q - p} \right] \int_a^b \int_c^d \left\{ \frac{q}{p} \left( \frac{q}{p} - 1 \right) \left( \int_a^x \int_c^y \sigma \right)^{\frac{q}{p} - 2} \left( \int_a^x \sigma(s, y) \, ds \right) \left( \int_c^y \sigma(x, t) \, dt \right) \\
\left. + \frac{q}{p} \left( \int_a^x \int_c^y \sigma \right)^{\frac{q}{p} - 1} \sigma(x, y) \right) \right\} \, dx \, dy
\]
\[
= \alpha(p, q) \int_a^b \int_c^d \left\{ \frac{q}{p} \left( \frac{q}{p} - 1 \right) \left( \int_a^x \int_c^y \sigma \right)^{\frac{q}{p} - 2} \left( \int_a^x \sigma(s, y) \, ds \right) \left( \int_c^y \sigma(x, t) \, dt \right) \\
\left. + \frac{q}{p} \left( \int_a^x \int_c^y \sigma \right)^{\frac{q}{p} - 1} \sigma(x, y) \right) \right\} \, dx \, dy = \alpha(p, q) \left( \int_a^b \int_c^d \sigma \right)^{\frac{q}{p}}.
\]

Now suppose that \( q < p \). By analogy with the proof of [14 Theorem 1A] we define the domains
\[
\omega_k := \left\{ (x, y) \in (a, b) \times (c, d) : \int_a^x \int_c^y \sigma > 2^k \right\}, \quad -\infty < k \leq K_{\sigma}.
\]
The restriction \( K_{\sigma} < \infty \) follows from the condition [14 (1.6)], which is necessary for any relations between \( p \) and \( q \). Then
\[
V_{(a,b) \times (c,d)} = \sum_{k \leq K_{\sigma}} \int_{\omega_k \setminus \omega_{k+1}} w(x, y) \left( \int_a^x \int_c^y \sigma \right)^q \, dy \, dx \]
\[
\leq 2^q \sum_{k \leq K_{\sigma}} 2^{kq} \left| \omega_k \setminus \omega_{k+1} \right| w \leq 2^q \sum_{k \leq K_{\sigma}} 2^{kq} \left| \omega_k \right| w.
\]
Using Hölder’s inequality with exponents $r/q$, it follows from Proposition 2.1(a) with analogously, \[ \left| \omega_k \right|_w = \left( \int_{\alpha_k}^{b} \int_{b}^{d} w \chi_{\omega_k} = \left( \int_{\alpha_k}^{b} d_y \left[ - \left( \int_{\alpha_k}^{b} \int_{y}^{d} w \chi_{\omega_k} \right) \right] \right) ^{\frac{r}{q}} \right) . \]

Since \[ \left[ - \left( \int_{x}^{b} \int_{y}^{d} w \chi_{\omega_k} \right) ^{r/q} \right]_{x} = 0 \] out of the set $\omega_k \cap \text{supp} \; w$ for each fixed $y \geq \beta_k$ and, analogously, \[ \left[ - \left( \int_{x}^{b} \int_{y}^{d} w \chi_{\omega_k} \right) ^{r/q} \right]_{y} = 0 \] outside $\omega_k \cap \text{supp} \; w$ for all $x \geq \alpha_k$, then

\[ \left| \omega_k \right|_w = \left( \int_{\omega_k} \chi_{\text{supp} \; w} (x, y) d_x d_y \left( \int_{x}^{b} \int_{y}^{d} w \right) ^{\frac{r}{q}} \right) ^{\frac{q}{r}} \]

Due to the choice of $\omega_k$, \[ 2^{2 \alpha} \left| \omega_k \right|_w = 2^{2 \alpha} \left( \int_{\omega_k} \chi_{\text{supp} \; w} (x, y) d_x d_y \left( \int_{x}^{b} \int_{y}^{d} w \right) ^{\frac{r}{q}} \right) ^{\frac{q}{r}} \]

It follows from Proposition 2.1(a) with $\rho = 2$ and $\gamma = q/r < 1$ that

\[ \sum_{k \leq K_\sigma} 2^{2 \alpha} \left( \int_{\omega_k} \chi_{\text{supp} \; w} (x, y) d_x d_y \left( \int_{x}^{b} \int_{y}^{d} w \right) ^{\frac{r}{q}} \right) ^{\frac{q}{r}} \]

\[ = \sum_{k \leq K_\sigma} 2^{2 \alpha} \left( \int_{\omega_k \cap \omega_{m+1}} \chi_{\text{supp} \; w} (x, y) d_x d_y \left( \int_{x}^{b} \int_{y}^{d} w \right) ^{\frac{r}{q}} \right) ^{\frac{q}{r}} \]

\[ \leq \frac{2^{2 \alpha}}{2^{2 \beta}} \beta (p, q) \sum_{k \leq K_\sigma} 2^{2 \alpha} \left( \int_{\omega_k \cap \omega_{m+1}} \chi_{\text{supp} \; w} (x, y) d_x d_y \left( \int_{x}^{b} \int_{y}^{d} w \right) ^{\frac{r}{q}} \right) ^{\frac{q}{r}} \]

Using Hölder’s inequality with exponents $r/q$ and $p/q$, we obtain

\[ \sum_{k \leq K_\sigma} 2^{2 \alpha} \left( \int_{\omega_k \cap \omega_{m+1}} \chi_{\text{supp} \; w} (x, y) d_x d_y \left( \int_{x}^{b} \int_{y}^{d} w \right) ^{\frac{r}{q}} \right) ^{\frac{q}{r}} \]

\[ \leq 2^{2 \alpha} 2^{2 \alpha} \sum_{k \leq K_\sigma} \left( \int_{\omega_k \cap \omega_{m+1}} \chi_{\text{supp} \; w} (x, y) d_x d_y \left( \int_{x}^{b} \int_{y}^{d} w \right) ^{\frac{r}{q}} \right) ^{\frac{q}{r}} \]
Since \( r/q > 1 \) and \( r/p' > 1 \), then integrating by parts over the variable \( y \) yields

\[
\int_a^b \int_c^d \chi_{\text{supp}w}(x, y) \left( \int_x^y \int_y^\sigma \tilde{y} \, dy \, dx \right)^{\frac{q}{q'}} d_x \left( \int_x^y w \right)^{\frac{r}{r'}} d_y \int_y^\sigma \tilde{y} \, d_x \left( \int_x^y w \right)^{\frac{r}{r'}} d_y \leq \int_a^b \int_c^d \chi_{\text{supp}w}(x, y) d_y \left[ I_2 \sigma(x, y) \right]^{\frac{r}{r'}} d_x \left( -[I_2^* w(x, y)]^{\frac{r}{r'}} \right). \]

A similar statement holds with the (inner) integral of \( w \).

**Lemma 2.2.** Let \( 0 \leq a < b < \infty \) and \( 0 \leq c < d < \infty \). If \( 1 < p < q < \infty \) then

\[
W_{(a,b) \times (c,d)} := \int_a^b \int_c^d \sigma(x, y) \left( \int_x^y d_y \right)^{\frac{q}{q'}} d_x \leq \alpha(q', p') \left( \int_a^b \int_c^d w \right)^{\frac{q'}{q'} A'}. \]

In the case \( 1 < q < p < \infty \)

\[
W_{(a,b) \times (c,d)} \leq \beta(q', p') \left( \int_a^b \int_c^d w \right)^{\frac{q'}{q'}} \times \left[ \int_a^b \int_c^d \chi_{\text{supp} \sigma(x, y)} d_y \left[ I_2 \sigma(x, y) \right]^{\frac{r}{r'}} d_x \left( -[I_2^* w(x, y)]^{\frac{r}{r'}} \right) \right]^{\frac{r}{r'}}. \]

Introduce notations: \( \alpha := \alpha(p, q), \beta := \beta(p, q), \alpha' := \alpha(q', p'), \beta' := \beta(q', p') \),

\[
C_{\alpha, \alpha'} := 3^q \left[ \left( \frac{2}{3} \right)^q \max \left\{ \alpha, 2q(q')^{\frac{q}{q'}} \right\} \left( \frac{2^p - 1}{2^{p-1} - 1} \right)^{\frac{r}{r'}} + 3^{\frac{q}{q'}} \left( \frac{3^{q-1} - 1}{3^{q-1} - 1} \right)^{\frac{r}{r'}} \right],
\]

\[
C_{\beta, \beta'} := 3^q \left[ \left( \frac{2}{3} \right)^q \max \left\{ \beta, 2q(p')^{q-1} \left( \frac{q}{p'} \right)^\frac{q}{q'} \right\} \left( \frac{2^{p-1} - 1}{2^{p-1} - 1} \right)^{\frac{r}{r'}} + 3^{\frac{q}{q'}} \left( \frac{3^{q-1} - 1}{3^{q-1} - 1} \right)^{\frac{r}{r'}} \right].
\]

The main result of the work is the following statement.

**Theorem 2.1.** Let \( 1 < p \neq q < \infty \). If \( p < q \) then the inequality

\[
\left( \int_{\mathbb{R}_+^2} (I_2 f)^q \right)^{\frac{1}{q}} \leq C_2 \left( \int_{\mathbb{R}_+^2} f_{pv} \right)^{\frac{r}{r'}} \quad (f \geq 0)
\]

holds if and only if \( A < \infty \). Besides,

\[ A \leq C_2 \leq C_{\alpha, \alpha'} A. \]

In the case \( q < p \) the inequality \((3)\) is true if and only if \( B < \infty \). Moreover,

\[ 2^{-\frac{q}{p}} \left( \frac{q}{p} \right)^{\frac{q}{q'}} \left( \frac{p}{r'} \right)^{\frac{r}{r'}} B \leq C_2 \leq C_{\beta, \beta'} B. \]

**Proof.** (Sufficiency) Similarly to how it was done in E. Sawyer’s paper [14] for the case \( 1 < p \leq q < \infty \), we show that the conditions of the theorem are sufficient, limiting ourselves to proving the inequality \((3)\) on the subclass \( M \subset L^{p}_v(\mathbb{R}_+^2) \) of all functions
f \geq 0 \text{ bounded on } \mathbb{R}_+^2 \text{ with compact supports contained in the set } \{I_2 \sigma > 0\}. \text{ Then the inequality } \{5\} \text{ for arbitrary } 0 \leq f \in L_{k}^0(\mathbb{R}_+^2) \text{ follows by the standard arguments.}

Suppose \( A < \infty \) for \( p < q \) (or \( B < \infty \) in the case of \( q < p \)) and fix \( f \in M. \) By analogy with the proof of [14, Theorem 1A], we define the domains

\[
\Omega_k := \{I_2 f > 3^k\}, \quad k \in \mathbb{Z}.
\]

Then, by our assumptions on \( f \), there exists \( K \in \mathbb{Z} \) such that \( \Omega_k \neq \emptyset \) for \( k \leq K, \) \( \Omega_k = \emptyset \) for \( k > K, \) \( \bigcup_{k \in \mathbb{Z}} \Omega_k = \mathbb{R}_+^2 \) and

\[
3^k < I_2 f(x, y) \leq 3^{k+1}, \quad k \leq K, \quad (x, y) \in (\Omega_k \setminus \Omega_{k+1}).
\]

We can write down that

\[
\int_{\mathbb{R}_+^2} (I_2 f)^q w = \sum_{k \leq K-2} \int_{\Omega_{k+2} \setminus \Omega_{k+3}} (I_2 f)^q w \leq 3^{3q} \sum_{k \leq K-2} 3^{kq} |\Omega_{k+2} \setminus \Omega_{k+3}|_w,
\]

where \( |\Omega_{k+2} \setminus \Omega_{k+3}|_w := \int_{\Omega_{k+2} \setminus \Omega_{k+3}} w \) and \( \Omega_K \setminus \Omega_{K+1} = \Omega_K, \) since \( \Omega_{K+1} \) is empty.

Next, as in the proof of [14, Theorem 1A], we introduce rectangles. For this, we fix \( k \) such that \( \Omega_{k+1} \neq \emptyset, \) and choose points \((x_j^k, y_j^{k-1})\), \( 1 \leq j \leq N = N_k, \) lying on the boundary \( \partial \Omega_k \) in such a way to have \((x_j^k, y_j^{k-1})\) belonging to \( \partial \Omega_{k+1} \) for \( 2 \leq j \leq N \) and \( \Omega_{k+1} \subset \bigcup_{j=1}^{N} S_j^k, \) where \( S_j^k \) is a rectangle of the form \((x_j^k, \infty) \times (y_j^k, \infty)\). We also define rectangles \( S_j^k = (x_j^k, x_{j+1}^k) \times (y_j^k, y_{j+1}^k) \) for \( 1 \leq j \leq N \) and \( R_j^k = (0, x_j^k) \times (0, y_j^k), \)

\( \tilde{R}_j^k = (x_j^k, x_{j+1}^k) \times (y_j^k, y_{j+1}^k) \) and \( T_j^k = (x_j^k, \infty) \times (y_j^k, \infty) \) for \( 1 \leq j \leq N - 1. \) Put \( y_0^k = x_{N+1}^k = \infty (\text{see Figure 1}). \)

Now we choose the sets \( E_j^k \subset T_j^k \) so that \( E_j^k \cap E_i^k = \emptyset \) for \( j \neq i \) and \( \bigcup_j E_j^k = (\Omega_{k+2} \setminus \Omega_{k+3}) \cap \left( \bigcup_j T_j^k \right). \) Since \( \Omega_{k+2} \setminus \Omega_{k+3} \subset \Omega_{k+1} \subset \left( \bigcup_j T_j^k \right) \cup \left( \bigcup_j S_j^k \right), \) then

\[
3^{-3q} \int_{\mathbb{R}_+^2} (I_2 f)^q w \leq \sum_{k,j} 3^{kq} |E_j^k|_w + \sum_{k,j} 3^{kq} |\tilde{S}_j^k \cap (\Omega_{k+2} - \Omega_{k+3})|_w =: I + II.
\]

Fig. 1

\[\begin{align*}
\text{Fig. 1} \quad \Omega_k \quad \Omega_{k+1} \\
S_j^k \\
\tilde{R}_j^k \\
\tilde{T}_j^k \\
x_1^k \quad x_2^k \quad x_3^k
\end{align*}\]
To estimate II we denote $D^k_j := \mathcal{S}^k_j \setminus \Omega_{k+3}$ and turn to the reasoning of E. Sawyer on page 6 in [14], from which it follows that

$$I_2(\chi_{D^k_j} f)(x, y) > 3^k \text{ if } (x, y) \in \mathcal{S}^k_j \cap (\Omega_{k+2} \setminus \Omega_{k+3}).$$

Further, according to [14] p. 6,

$$|\mathcal{S}^k_j \cap (\Omega_{k+2} \setminus \Omega_{k+3})| \leq 3^{-k} \int_{\mathcal{S}^k_j \cap (\Omega_{k+2} \setminus \Omega_{k+3})} I_2(\chi_{D^k_j} f)(x, y) \, dx \, dy$$

$$\leq 3^{-k} \int_{D^k_j} \left( \int_{x}^{y} \int_{y}^{f} w(x, y) \, dx \, dy \right)$$

$$= 3^{-k} \int_{D^k_j} f(s, t) \left( \int_{s}^{\infty} \int_{t}^{\infty} w(\chi_{D^k_j}) \, ds \, dt \right)$$

$$\leq 3^{-k} \left( \int_{D^k_j} f^p \right)^{\frac{1}{p}} \left( \int_{D^k_j} \sigma(s, t) \left( \int_{s}^{\infty} \int_{t}^{\infty} w(\chi_{D^k_j}) \right)^{\frac{q}{p}} \, ds \, dt \right)^{\frac{1}{q}}. \quad (8)$$

By applying Lemma 2.2 to $(a, b) \times (c, d) = \mathcal{S}^k_j$, we obtain for $p < q$ that

$$\mathcal{W}_{\mathcal{S}^k_j} = \int_{D^k_j} \sigma(s, t) \left( \int_{s}^{\infty} \int_{t}^{\infty} w(\chi_{D^k_j}) \right)^{\frac{q}{p}} \, ds \, dt \leq \alpha' \mathcal{W}^p \mathcal{S}^k_j \mathcal{W}^q \quad (9)$$

and in the case $q < p$

$$\mathcal{W}_{\mathcal{S}^k_j} \leq \beta' \mathcal{S}^k_j \mathcal{W}^q \left( \int_{D^k_j} d_y [I_2 \sigma(x, y)] \mathcal{W}^q \, dx \left( -[I_2 w(x, y)] \mathcal{W}^q \right) \right)^{\frac{p}{q}}.$$

For $q < p$, from this and Hölder’s inequality with $q$ and $q'$,

$$(\beta')^{-\frac{q}{p}} \cdot II \leq \sum_{k, j} 3^{k(q-1)} \left( \int_{D^k_j} f^p \right)^{\frac{1}{p}} \left( \int_{D^k_j} d_y [I_2 \sigma(x, y)] \mathcal{W}^q \, dx \left( -[I_2 w(x, y)] \mathcal{W}^q \right) \right)^{\frac{q}{p}} \mathcal{S}^k_j \mathcal{W}^q \left( \sum_{k, j} \left( \int_{D^k_j} f^p \right)^{\frac{q}{p}} \left( \int_{D^k_j} d_y [I_2 \sigma(x, y)] \mathcal{W}^q \, dx \left( -[I_2 w(x, y)] \mathcal{W}^q \right) \right)^{\frac{q}{p}} \right)^{\frac{p}{q}} \mathcal{S}^k_j \mathcal{W}^q.$$

On the strength of [14] (2.6)

$$\sum_{j=1}^{N_k} \chi_{s^k_j} \leq 3^{-k} \chi_{\Omega_k} I_2 f \text{ for all } k.$$ 

Then

$$\sum_{k, j} 3^{kq} \mathcal{S}^k_j \mathcal{W} \sum_{k} 3^{kq} \chi_{s^k_j} \mathcal{W} = \sum_{k} 3^{kq} \int_{R^2_+} \chi_{s^k_j} \mathcal{W} \leq \sum_{k} 3^{k(q-1)} \int_{R^2_+} \chi_{\Omega_k} (I_2 f) \mathcal{W}$$

$$= \sum_{k} 3^{k(q-1)} \int_{R^2_+} \chi_{\Omega_k \setminus \Omega_{k+1}} (I_2 f) \mathcal{W} = \sum_{m} 3^{m(q-1)} \int_{R^2_+} \chi_{\Omega_m \setminus \Omega_{m+1}} (I_2 f) \mathcal{W} \sum_{m \geq k} 3^{(k-m)(q-1)}$$

$$\leq 3^{q-1} \sum_{m} 3^{m(q-1)} \int_{R^2_+} \chi_{\Omega_m \setminus \Omega_{m+1}} (I_2 f) \mathcal{W}$$

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and, therefore,
\[ \sum_{k,j} 3^{kq} |S^k_j|_w \leq \frac{3^{q-1}}{3^{q-1} - 1} \sum_{m} \int_{\Omega_m \setminus \Omega_{m+1}} (I_2f)^q w = \frac{3^{q-1}}{3^{q-1} - 1} \int_{\mathbb{R}^2_+} (I_2f)^q w. \]

Further, Hölder’s inequality with \( p/q, r/q \) and the estimate \( \sum_{k,j} \chi_{D^k_j} \leq \sum_k \chi_{\Omega_k \setminus \Omega_{k+1}} \leq 3 \) entail
\[ \sum_{k,j} \left( \int_{D^k_j} f^{p^*} v \right)^{\frac{q}{pq}} \left( \int_{D^k_j} d_y [I_2 \sigma(x,y)] \right)^{\frac{p}{pq}} \left( \int_{D^k_j} [I_2 w(x,y)] \right)^{\frac{q}{q}} \leq \left( \sum_{k,j} \int_{D^k_j} f^{p^*} v \right)^{\frac{q}{pq}} \left( \sum_{k,j} \int_{D^k_j} d_y [I_2 \sigma(x,y)] \right)^{\frac{p}{pq}} \left( \int_{D^k_j} [I_2 w(x,y)] \right)^{\frac{q}{q}} \leq 3 \left( \int_{\mathbb{R}^2_+} f^{p^*} v \right)^{\frac{q}{pq}} \left( \int_{\mathbb{R}^2_+} d_y [I_2 \sigma(x,y)] \right)^{\frac{p}{pq}} \left( \int_{\mathbb{R}^2_+} [I_2 w(x,y)] \right)^{\frac{q}{q}}. \]

Thus, for \( q < p \),
\[ II \leq 3 (\beta')^{\frac{q}{p}} B \left( \frac{3^{q-1}}{3^{q-1} - 1} \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^2_+} f^{p^*} v \right)^{\frac{q}{pq}} \left( \int_{\mathbb{R}^2_+} (I_2f)^q w \right)^{\frac{1}{q}}. \tag{10} \]

In the case \( p < q \) a similar estimate of the form
\[ II \leq 3 (\alpha')^{\frac{q}{p}} A \left( \frac{3^{q-1}}{3^{q-1} - 1} \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^2_+} f^{p^*} v \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^2_+} (I_2f)^q w \right)^{\frac{1}{q}}. \tag{11} \]

follows from \([3], [9]\) and the reasoning on pages 6–7 in \([14]\).

To estimate \( I \) in \((7)\), in full accordance with the proof of \([14] \) Theorem 1A, pp. 8–9], we put \( g \sigma := f \) and write:
\[ 3^q I = \sum_{k,j} 3^{(k+1)q} |E^k_j|_w = \sum_{k,j} |E^k_j|_w \left( \int_{R^k_j} f \right)^q = \sum_{k,j} |E^k_j|_w \|R^k_j\|_{\sigma}^{q} \left( \frac{1}{|R^k_j|_{\sigma}} \int_{R^k_j} g \sigma \right)^q. \tag{12} \]

For an integer \( l \), by \( \Gamma_l \) we denote the set of pairs \((k,j)\) such that \( |E^k_j|_w > 0 \) and
\[ 2^l < \frac{1}{|R^k_j|_{\sigma}} \int_{R^k_j} g \sigma \leq 2^{l+1}, \quad (k,j) \in \Gamma_l \]
and observe that \( \Gamma_l \cap \Gamma_{l'} = \emptyset, l' \neq l'' \).

For fixed \( l \) the family \( \{U^l_{i}\}_{i=1}^{i(l)} \) consists of maximal rectangles from the collection \( \{R^k_j\}_{(k,j) \in \Gamma_l} \), that is, each \( R^k_j \) with \((k,j) \in \Gamma_l \) is contained in some \( U^l_{i} \) (or coincides with it). In \([14]\) p. 8 it is shown that \( \tilde{U}^l_{i} \) are disjoint for fixed \( l \), where we denote \( \tilde{U}^l_{i} = \tilde{R}^l_{i} \) if \( U^l_{i} = R^k_j \).

Let \( \chi^l_{i} \) be the characteristic function of the union of the sets \( E^k_j \) over all \((k,j) \in \Gamma_l \) such that \( R^k_j \subset U^l_{i} \). Further, following \([14] \) (2.13)], we arrive to
\[ \sum_{(k,j) \in \Gamma_l} |E^k_j|_w |R^k_j|_{\sigma}^{q} = \sum_{i=1}^{i(l)} \sum_{R^k_j \subset U^l_{i}} \int_{E^k_j} w [I_2(\chi_{U^l_{i}})(x^k_j, y^k_j)]^{q} \leq \sum_{i=1}^{i(l)} \int_{\mathbb{R}^2_+} \chi^l_{i} w [I_2(\chi_{U^l_{i}})]^{q}. \tag{13} \]
By analogy with \([14, (2.8)]\), let us first show the validity of the estimate
\[
\int_{\mathbb{R}_+^n} \chi_i^l w [I_2(\chi_i^l \sigma)]^q \leq \max \left\{ \beta, 2q(p')^{q-1} \left( \frac{q}{r} \right) \right\} (B_i^l)^q |U_i^l|_\sigma^\frac{q}{p}
\]
for \(U_i^l = (0, a) \times (0, b)\) in the case \(q < p\), where
\[
(B_i^l)^r = \int_{\mathbb{R}_+^n} \chi_i^l(x, y) dy [I_2\sigma(x, y)]^\frac{1}{p'} \int_{\mathbb{R}_+^n} \left( -[I_2^l w(x, y)]^\frac{1}{q} \right) \cdot
\]
On \((0, a) \times (0, b) = U_i^l\), in view of Lemma 2.1,
\[
\mathbf{V}_{U_i^l} = \int_{U_i^l} \chi_i^l w (I_2\sigma)^q \leq \beta \left( \int_{U_i^l} \chi_i^l(x, y) dy [I_2\sigma(x, y)]^\frac{1}{p'} \int_{\mathbb{R}_+^n} \left( -[I_2^l w(x, y)]^\frac{1}{q} \right) \right) \cdot
\]
On the rectangle \((a, \infty) \times (b, \infty)\) we obtain the estimate:
\[
\int_{(a, \infty) \times (b, \infty)} \chi_i^l w |U_i^l|_\sigma^\frac{q}{p} = \left( \int_{(a, \infty) \times (b, \infty)} \chi_i^l(x, y) dx dy [I_2^l w \chi_i^l(x, y)]^\frac{1}{q} \right) \cdot
\]
whence by integration by parts
\[
\int_{\mathbb{R}_+^n} \chi_i^l(x, y) [I_2\sigma(x, y)]^\frac{1}{p'} \int_{\mathbb{R}_+^n} \left( -[I_2^l w \chi_i^l(x, y)]^\frac{1}{q} \right) \cdot
\]
In the first of the two mixed cases — \((0, a) \times (b, \infty)\) and \((a, \infty) \times (0, b)\) — we obtain, using the criteria for the fulfillment of the one-dimensional weighted Hardy inequality for \(f^p(x) = \int_0^b \sigma(x, y) dy\) (see \([11, \text{§ 1.3.2}]\)):
\[
\int_{(a, \infty) \times (b, \infty)} \chi_i^l(x, y) w(x, y) \left( \int_0^x \int_0^b \sigma(s, t) dt \right) \cdot
\]

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\[ \leq (p')^{q-1} \left( \frac{q}{r} \right)^{\frac{q}{p}} \left( \int_0^a \int_b^c \chi_L^* (s,t) \left[ I_2 \sigma(s,t) \right] \frac{d_s d_t}{t} \left( \int_s^t \int_t^c \chi_L^* w \right)^{\frac{q}{r}} \right)^{\frac{q}{r}} | U|^\frac{q}{r}_\sigma \]

\[ \leq q(p')^{q-1} \left( \frac{q}{r} \right)^{\frac{q}{p}} \left( \int_{R^*} \chi_L^* (s,t) \left[ I_2 \sigma(s,t) \right] \frac{d_s d_t}{t} \left( \int_s^t \int_t^c \chi_L^* w \right)^{\frac{q}{r}} \right)^{\frac{q}{r}} | U|^\frac{q}{r}_\sigma \]

\[ \leq q(p')^{q-1} \left( \frac{q}{r} \right)^{\frac{q}{p}} \left( \int_{R^*} \chi_L^* (s,t) d_t \left[ I_2 \sigma(s,t) \right] \frac{d_s}{t} \left( - \left[ I^*_w (s,t) \right]^{\frac{q}{r}} \right) \right)^{\frac{q}{r}} | U|^\frac{q}{r}_\sigma, \quad (15) \]

The second mixed case is estimated in a similar way. So, (14) is proven. Continuing (13), we obtain, using (14) (2.11)] and H"older's inequality with $r/q, p/q$:

\[ \sum_{(k,j) \in \Gamma_1} | E_j^k |_{\sigma} R^k_{\sigma} \leq \max \left\{ \beta, 2q(p')^{q-1} \left( \frac{q}{r} \right)^{\frac{q}{p}} \right\} \sum_i (B_i^r) \left( \frac{2^{-l} \int_{U_{[g \in 2^{l-3}]}} g \sigma}{g} \right)^{\frac{q}{p}} \]

\[ \leq \max \left\{ \beta, 2q(p')^{q-1} \left( \frac{q}{r} \right)^{\frac{q}{p}} \right\} \left( \sum_i (B_i^r) \right)^{\frac{q}{p}} \left( \sum_i 2^{-l} \int_{U_{[g \in 2^{l-3}]}} g \sigma \right)^{\frac{q}{p}} \]

\[ \leq \max \left\{ \beta, 2q(p')^{q-1} \left( \frac{q}{r} \right)^{\frac{q}{p}} \right\} 2^{-lq/p} (B_i) q \left( \int_{g \in 2^{l-3}} g \sigma \right)^{\frac{q}{p}}. \]

The last estimate is valid with

\[ (B_i) := \int_{U_{[u(k,j) \in \Gamma_1} E_j^k (x,y) d_y \left[ I_2 \sigma(x,y) \left[ \chi_{\geq 1} \right] \frac{d_x}{t} \left( - \left[ I^*_w (x,y) \right]^{\frac{q}{r}} \right) \right] \sum_{l \in (p-1)} 2^{l(p-1)} \chi_{\geq 2^{l-3}} \leq \frac{2^{q-1}}{2^{p-1}} \frac{q}{r} \quad (16) \]

due to the fact that for fixed $l$ the rectangles $U^l_i$ do not intersect (see [14, p. 8]). Combining it with (12), we obtain, taking into account the relation

\[ \sum_l 2^{l(p-1)} \chi_{\geq 2^{l-3}} \leq \frac{2^{q-1}}{2^{p-1}} \frac{q}{r} \quad (16) \]

\[ H"older's \ inequality \ with \ r/q \ and \ p/q \ and \ the \ fact \ that \ all \ E_j^k \ are \ disjoint: \]

\[ I \leq \left( \frac{2}{3} \right)^q \sum_l 2^{lq} \sum_{(k,j) \in \Gamma_1} | E_j^k |_{\sigma} R^k_{\sigma} \]

\[ \leq \left( \frac{2}{3} \right)^q \max \left\{ \beta, 2q(p')^{q-1} \left( \frac{q}{r} \right)^{\frac{q}{p}} \right\} \sum_l 2^{lq} (B_i) q \left( \int_{g \in 2^{l-3}} g \sigma \right)^{\frac{q}{p}} \]

\[ \leq \left( \frac{2}{3} \right)^q \max \left\{ \beta, 2q(p')^{q-1} \left( \frac{q}{r} \right)^{\frac{q}{p}} \right\} \left( \sum_l (B_i) \right)^{\frac{q}{p}} \left( \int_{g \in 2^{l-3}} g \sigma \right)^{\frac{q}{p}} \]

\[ \leq \left( \frac{2}{3} \right)^q \max \left\{ \beta, 2q(p')^{q-1} \left( \frac{q}{r} \right)^{\frac{q}{p}} \right\} 2^{q-1} \frac{q}{r} B^q \left( \frac{2^{q-1}}{2^{q-1}} \frac{q}{r} \right)^{\frac{q}{p}} \]

Combining (16) with (10) we arrive at the required upper bound for $q < p$.

For $p < q$, the term $I$ is estimated identically to the case $p \leq q$ in (14) p. 9, i.e.

\[ I \leq \left( \frac{2}{3} \right)^q \max \left\{ \alpha, 2q(p')^{q-1} \right\} \left( \frac{2^{q-1}}{2^{p-1}} \frac{q}{r} \right)^{\frac{q}{p}} A^q \left( \int_{R^*} f^p \right)^{\frac{q}{p}}, \quad (17) \]
relying on an analog of the inequality (14) of the form

\[ \int_{\mathbb{R}^2_+} \chi_1^i w I_2^i (\chi_{U^i})^q \leq \max \{ \alpha, 2q(2q')^{\frac{1}{2}} \} A^q \left| U^i_{\sigma} \right|^\frac{q}{q'} \text{ for } U^i = (0, a) \times (0, b). \]

Note that in this case, unlike (2.8), to perform the estimate on the rectangle \((0, a) \times (0, b) = U^i\) one should apply the statement of Lemma 2.1, from which it follows that

\[ V_{U^i} \leq \alpha \left| U^i_{\sigma} \right|^\frac{q}{q'} A^q. \]

The final upper estimate

\[ \int_{\mathbb{R}^2_+} (f_2 f)^q w \leq C \left( \int_{\mathbb{R}^2_+} f P_v \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2_+} (f_2 f)^q w \right)^{\frac{1}{2}} + C^q \left( \int_{\mathbb{R}^2_+} f P_v \right)^{\frac{q}{q'}} \]

follows from (7) combined with (11) and (17) for \(p < q\) (or (10) and (16) if \(q < p\)) with 

\(C = A \cdot C_{\alpha, \alpha'}\) in case \(p < q\) and \(C = B \cdot C_{\beta, \beta'}\) for \(q < p\).

(Necessity) The validity of \(A \leq C_2\) follows by substituting \(f = \chi_{(0, a) \times (0, b)}\) into the initial inequality (6). To establish \(B \leq C_2\) in the case \(q < p\), we apply the test function

\[ f(s, y) = \sigma(s, y) \left[ \int_s^\infty \left[ I_2 \sigma(x, y) \right] \hat{\psi} \left[ I_2^* w(x, y) \right] \hat{\psi} \left( \int_y^\infty w(x, t) \, dt \right) \, dx \right]^{\frac{q}{q'}} = : \sigma(s, y) J(s, y) \]

into (13). Then

\[ \int_{\mathbb{R}^2_+} f P_v = \int_{\mathbb{R}^2_+} \sigma(s, y) [J(s, y)]^p \, ds \, dy \]

\[ = \int_{\mathbb{R}^2_+} \left[ I_2 \sigma(x, y) \right] \hat{\psi} \left[ I_2^* w(x, y) \right] \hat{\psi} \left( \int_y^\infty w(x, t) \, dt \right) \left( \int_0^x \sigma(s, y) \, ds \right) \, dx \, dy \]

\[ = \frac{p^q}{r^2} \int_{\mathbb{R}^2_+} d_y \left[ I_2 \sigma(x, y) \right] \hat{\psi} \left[ I_2^* w(x, y) \right] \hat{\psi} \left( \int_y^\infty w(x, t) \, dt \right) \left( \int_0^x \sigma(s, y) \, ds \right) \, dx \]

\[ = : \frac{q}{r} [J_1(s, y)]^p + \frac{q}{q'} [J_2(s, y)]^p. \quad (18) \]

To estimate the left-hand side of the inequality (6), we write

\[ [J(s, y)]^p = \frac{q}{r} \left[ I_2 \sigma(s, y) \right] \hat{\psi} \left[ I_2^* w(s, y) \right] \hat{\psi} \]

\[ + \frac{q}{q'} \int_s^\infty \left[ I_2 \sigma(x, y) \right] \hat{\psi}^{-1} \left[ I_2^* w(x, y) \right] \hat{\psi} \left( \int_x^\infty \sigma(x, t) \, dt \right) \, dx \]

\[ = : \frac{q}{r} [J_1(s, y)]^p + \frac{q}{q'} [J_2(s, y)]^p. \quad (19) \]

Then, for our chosen \(f\),

\[ F(u, z) := \int_0^u \int_0^z f = \int_0^u \int_0^z \sigma(s, y) J(s, y) \, dy \, ds \]

\[ \geq 2^{\frac{1}{q'}} \left( \frac{q}{r} \right)^{\frac{1}{2}} \int_0^u \int_0^z \sigma(s, y) J_1(s, y) \, dy \, ds + \left( \frac{q}{q'} \right)^{\frac{1}{2}} \int_0^u \int_0^z \sigma(s, y) J_2(s, y) \, dy \, ds \]

\[ = : 2^{\frac{1}{q'}} (F_1 + F_2). \quad (12) \]
To estimate $F_2$, we observe that
\[
\left(\frac{q'}{q}\right)^\frac{1}{\gamma} F_2 = \int_0^u \int_0^z \sigma(s, y) J_2(s, y) dy ds \\
\geq \left[ I^*_2 w(u, z) \right] \frac{\sigma}{\gamma'} \int_0^u \int_0^z \sigma(s, y) \left[ \int_s^u [ I_2 \sigma(x, y) ] \frac{1}{\sigma_x} \left( \int_0^y \sigma(x, t) dt \right) dx \right] \frac{1}{\gamma'} dy ds.
\]
Since
\[
\int_s^u [ I_2 \sigma(x, y) ] \frac{1}{\sigma_x} \left( \int_0^y \sigma(x, t) dt \right) dx \leq \frac{q'}{r'} \left[ I_2 \sigma(u, y) \right] \frac{1}{\gamma'},
\]
then
\[
\int_0^u \int_0^z \sigma(s, y) \left[ \int_s^u [ I_2 \sigma(x, y) ] \frac{1}{\sigma_x} \left( \int_0^y \sigma(x, t) dt \right) dx \right] \frac{1}{\gamma'} dy ds \\
\geq \left( \frac{q'}{r'} \right)^\frac{1}{\gamma'} \int_0^u \int_0^z \sigma(s, y) [ I_2 \sigma(u, y) ] \frac{1}{\gamma'} \int_s^u [ I_2 \sigma(x, y) ] \frac{1}{\sigma_x} \left( \int_0^y \sigma(x, t) dt \right) dx dy ds \\
\geq \left( \frac{q'}{r'} \right)^\frac{1}{\gamma'} [ I_2 \sigma(u, z) ] \frac{1}{\gamma'} \int_0^u \int_0^z \sigma(s, y) \left[ [ I_2 \sigma(x, y) ] \frac{1}{\sigma_x} \left( \int_0^y \sigma(x, t) dt \right) dx \right] dy dx \\
= \left( \frac{q'}{r'} \right)^\frac{1}{\gamma'} [ I_2 \sigma(u, z) ] \frac{1}{\gamma'} \int_0^u \int_0^z \left[ [ I_2 \sigma(x, y) ] \frac{1}{\sigma_x} \left( \int_0^y \sigma(x, t) dt \right) dx \right] dy dx.
\]
and, therefore,
\[
F_2 \geq \left( \frac{q}{q'} \right)^\frac{1}{\gamma} \left( \frac{r}{q' q} \right)^\frac{1}{\gamma'} [ I_2 \sigma(u, z) ] \frac{1}{\gamma'} \left[ [ I_2 \sigma(u, z) ] \frac{1}{\gamma'} \int_0^u \int_0^z [ I_2 \sigma(x, y) ] \frac{1}{\sigma_x} \left( \int_0^y \sigma(x, t) dt \right) dx dy ds \\
\times \int_0^u \int_0^z [ I_2 \sigma(x, y) ] \frac{1}{\sigma_x} \left( \int_0^y \sigma(x, t) dt \right) dx dy \\
= : \left( \frac{q}{q'} \right)^\frac{1}{\gamma} \left( \frac{r}{q' q} \right)^\frac{1}{\gamma'} [ I_2 \sigma(u, z) ] \frac{1}{\gamma'} \left[ [ I_2 \sigma(u, z) ] \frac{1}{\gamma'} \int_0^u \int_0^z [ I_2 \sigma(x, y) ] \frac{1}{\sigma_x} \left( \int_0^y \sigma(x, t) dt \right) dx dy ds.
\]

For $F_1$ we obtain:
\[
F_1 = \left( \frac{q}{q'} \right)^\frac{1}{\gamma} \int_0^u \int_0^z \sigma(s, y) [ I_2 \sigma(s, y) ] \frac{1}{\gamma'} [ I_2 \sigma(u, z) ] \frac{1}{\gamma'} \int_0^u \int_0^z [ I_2 \sigma(x, y) ] \frac{1}{\sigma_x} \left( \int_0^y \sigma(x, t) dt \right) dx dy ds \\
\geq \left( \frac{q}{q'} \right)^\frac{1}{\gamma} [ I_2 \sigma(u, z) ] [ I_2 \sigma(u, z) ] \frac{1}{\gamma'} \int_0^u \int_0^z \sigma(s, y) [ I_2 \sigma(s, y) ] \frac{1}{\gamma'} \int_0^u \int_0^z [ I_2 \sigma(x, y) ] \frac{1}{\sigma_x} \left( \int_0^y \sigma(x, t) dt \right) dx dy ds \\
= : \left( \frac{q}{q'} \right)^\frac{1}{\gamma} [ I_2 \sigma(u, z) ] [ I_2 \sigma(u, z) ] \frac{1}{\gamma'} \int_0^u \int_0^z [ I_2 \sigma(x, y) ] \frac{1}{\sigma_x} \left( \int_0^y \sigma(x, t) dt \right) dx dy ds.
\]

It holds that
\[
F(u, z) \geq 2^{-\frac{1}{\gamma}} \left( \frac{q}{q'} \right)^\frac{1}{\gamma} [ I_2 \sigma(u, z) ] [ I_2 \sigma(u, z) ] \frac{1}{\gamma'} \left[ [ I_2 \sigma(u, z) ] [ I_2 \sigma(u, z) ] \frac{1}{\gamma'} \int_0^u \int_0^z [ I_2 \sigma(x, y) ] \frac{1}{\sigma_x} \left( \int_0^y \sigma(x, t) dt \right) dx dy ds \\
+ \frac{r}{q'} \int_0^u \int_0^z \sigma(s, y) \left[ [ I_2 \sigma(x, y) ] \frac{1}{\sigma_x} \left( \int_0^y \sigma(x, t) dt \right) dx \right] dy dx \\
= \left( \frac{q'}{r} \right)^\frac{1}{\gamma} \int_0^u \int_0^z \sigma(s, y) \left[ [ I_2 \sigma(x, y) ] \frac{1}{\sigma_x} \left( \int_0^y \sigma(x, t) dt \right) dx \right] dy dx \\
- \frac{q'}{r} \int_0^u \int_0^z \sigma(s, y) \left[ [ I_2 \sigma(x, y) ] \frac{1}{\sigma_x} \left( \int_0^y \sigma(x, t) dt \right) dx \right] dy dx.
\]

Integrating by parts we find:
\[
[ I_2 \sigma(u, z) ] \frac{1}{\gamma} \left( \int_0^y \sigma(x, t) dt \right) dx dy \\
= \left( \frac{q'}{r} \right)^\frac{1}{\gamma} \int_0^u \int_0^z \sigma(s, y) \left[ [ I_2 \sigma(x, y) ] \frac{1}{\sigma_x} \left( \int_0^y \sigma(x, t) dt \right) dx \right] dy dx \\
- \frac{q'}{r} \int_0^u \int_0^z \sigma(s, y) \left[ [ I_2 \sigma(x, y) ] \frac{1}{\sigma_x} \left( \int_0^y \sigma(x, t) dt \right) dx \right] dy dx.
\]

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Hence,

\[ F(u, z) \geq 2^{-\frac{1}{q'}} \left( \frac{q}{r} \right)^{\frac{1}{p'}} \frac{p'}{r} [I_2 \sigma(u, z)] \frac{q}{\sigma} [I_2^* w(u, z)] \frac{q}{\sigma}. \]  

(21)

We write making use of (19):

\[
\int_{\mathbb{R}_+^2} (I_2 f)^q w = \int_{\mathbb{R}_+^2} f(x, y) \left( \int_x^\infty \int_y^\infty w(u, z) [F(u, z)]^{q-1} dz \right) dxdy \\
\geq 2^{-\frac{1}{q'}} \frac{p'}{r} \int_{\mathbb{R}_+^2} \sigma(x, y) \left( \int_x^\infty \int_y^\infty w F^{q-1} \right) \left\{ \left( \frac{q}{r} \right)^{\frac{1}{p'}} [I_2 \sigma(x, y)] \frac{q}{\sigma} [I_2^* w(x, y)] \frac{q}{\sigma} \right\} dxdy \\
+ \left( \frac{q}{q'} \right) \int_x^\infty \left[ [I_2 \sigma(s, y)] \frac{q}{\sigma}^{-1} [I_2^* w(s, y)] \frac{q}{\sigma} \left( \int_0^y \sigma(s, t) \frac{dt}{ds} \right) \right] dxdy \\
= : 2^{-\frac{1}{q'}} (G_1 + G_2).
\]

(22)

\[ G_1 = \left( \frac{q}{q'} \right) \int_{\mathbb{R}_+^2} \sigma(x, y) [I_2 \sigma(x, y)] \frac{q}{\sigma}^{-1} [I_2^* w(x, y)] \frac{q}{\sigma} \left( \int_0^y \sigma(s, t) \frac{dt}{ds} \right) dxdy \]

(23)

It is true for \( G_2 \):

\[
\left( \frac{q}{q'} \right)^{\frac{1}{p'}} G_2 = \int_{\mathbb{R}_+^2} \sigma(x, y) \left[ \int_x^\infty [I_2 \sigma(s, y)] \frac{q}{\sigma}^{-1} [I_2^* w(s, y)] \frac{q}{\sigma} \left( \int_0^y \sigma(s, t) \frac{dt}{ds} \right) \right] dxdy \\
\times \left( \int_x^\infty \int_y^\infty w(u, z) [F(u, z)]^{q-1} dz \right) dxdy \\
= \int_{\mathbb{R}_+^2} \int_0^u \sigma(x, y) \left[ \int_x^\infty [I_2 \sigma(s, y)] \frac{q}{\sigma}^{-1} [I_2^* w(s, y)] \frac{q}{\sigma} \left( \int_0^y \sigma(s, t) \frac{dt}{ds} \right) \right] dxdy \\
\times \left( \int_y^\infty w(u, z) [F(u, z)]^{q-1} dz \right) dudy \\
\geq \int_{\mathbb{R}_+^2} \int_0^u \sigma(x, y) \left[ \int_x^u [I_2 \sigma(s, y)] \frac{q}{\sigma}^{-1} [I_2^* w(s, y)] \frac{q}{\sigma} \left( \int_0^y \sigma(s, t) \frac{dt}{ds} \right) \right] dxdy \\
\times \left( \int_y^\infty w(u, z) [F(u, z)]^{q-1} dz \right) dudy \\
\geq \int_{\mathbb{R}_+^2} [I_2^* w(u, y)] \frac{q}{\sigma} \int_0^u \sigma(x, y) \left[ \int_x^u [I_2 \sigma(s, y)] \frac{q}{\sigma}^{-1} \left( \int_0^y \sigma(s, t) \frac{dt}{ds} \right) \right] dxdy \\
\times \left( \int_y^\infty w(u, z) [F(u, z)]^{q-1} dz \right) dudy \\
\geq \left( \frac{r}{q'} \right)^{\frac{1}{p'}} \int_{\mathbb{R}_+^2} [I_2 \sigma(u, y)] \frac{1}{\sigma} [I_2^* w(u, y)] \frac{1}{\sigma} \\
\times \left( \int_0^u \sigma(x, y) \left[ \int_x^u [I_2 \sigma(s, y)] \frac{1}{\sigma}^{-1} \left( \int_0^y \sigma(s, t) \frac{dt}{ds} \right) \right] dxdy \right)
\]
\[ \times \left( \int_y^\infty w(u,z)[F(u,z)]^{q-1} \, dz \right) \, du dy \]
\[ \geq \left( \frac{q'}{q} \right)^{\frac{r}{q'}} \int_{R^2_+} [I_2 \sigma(u,y)]^{-\frac{q'}{r}} [I_2^* w(u,y)]^{\frac{r}{q'}} \left( \int_y^\infty w(u,z) \, dz \right) [F(u,y)]^{q-1} \]
\[ \times \left[ \int_0^u [I_2 \sigma(s,y)]^{\frac{r}{q'}} \left( \int_0^s \sigma(x,y) \, dx \right) \left( \int_0^y \sigma(s,t) \, dt \right) \right] \, du dy. \]

Integrating by parts we find
\[
\int_0^u [I_2 \sigma(s,y)]^{\frac{r}{q'}} \left( \int_0^s \sigma(x,y) \, dx \right) \left( \int_0^y \sigma(s,t) \, dt \right) \, ds
= \frac{q'}{r} \int_0^u [I_2 \sigma(s,y)]^{\frac{r}{q'}} \left( \int_0^s \sigma(x,y) \, dx \right) \, ds - \frac{q'}{r} \int_0^u [I_2 \sigma(s,y)]^{\frac{r}{q'}} \sigma(s,y) \, ds.
\]

Hence, continuing the reasoning, we obtain for \( G_2 \) using (21):
\[
\left( \frac{q'}{q} \right)^{\frac{r}{q'}} G_2 \geq 2^{-\frac{q'}{r}} \left( \frac{q'}{q} \right)^{\frac{r}{q'}} \left( \frac{q'}{r} \right)^{\frac{r}{q'}} \left( \frac{q'}{r} \right)^{\frac{r}{q'}} \int_{R^2_+} [I_2^* w(u,y)]^{\frac{r}{q'}} \left( \int_y^\infty w(u,z) \, dz \right) \]
\[ \times \left[ [I_2 \sigma(u,y)]^{\frac{r}{q'}} \int_0^u \sigma(x,y) \, dx - \int_0^u [I_2 \sigma(s,y)]^{\frac{r}{q'}} \sigma(s,y) \, ds \right] \, du dy. \quad (24) \]

Since
\[
\int_{R^2_+} [I_2^* w(u,y)]^{\frac{r}{q'}} \left( \int_y^\infty w(u,z) \, dz \right) \left[ \int_0^u [I_2 \sigma(s,y)]^{\frac{r}{q'}} \sigma(s,y) \, ds \right] \, du dy
= \frac{q'}{r} \int_{R^2_+} [I_2^* w(u,y)]^{\frac{r}{q'}} [I_2 \sigma(u,y)]^{\frac{r}{q'}} \sigma(u,y) \, du dy
\]
then from (23) we obtain, applying (23) and (24),
\[
2^{\frac{r}{q'}} \int_{R^2_+} (I_2 f)^q w \geq \left( \frac{q'}{q} \right)^{\frac{r}{q'}} \left( \frac{p'}{r} \right)^{q-1} \int_{R^2_+} \sigma(x,y) [I_2 \sigma(x,y)]^{\frac{r}{q'}} [I_2^* w(x,y)]^{\frac{r}{q'}} \, dxdy
+ \left( \frac{q'}{q} \right)^{\frac{r}{q'}} \left( \frac{p'}{r} \right)^{q-1} \int_{R^2_+} [I_2^* w(u,y)]^{\frac{r}{q'}} \left( \int_y^\infty w(u,z) \, dz \right) [I_2 \sigma(u,y)]^{\frac{r}{q'}} \left( \int_0^u \sigma(x,y) \, dx \right) \, du dy
- \left( \frac{q'}{q} \right)^{\frac{r}{q'}} \left( \frac{p'}{r} \right)^{q-1} \frac{q'}{r} \int_{R^2_+} \sigma(x,y) [I_2 \sigma(x,y)]^{\frac{r}{q'}} [I_2^* w(x,y)]^{\frac{r}{q'}} \, dxdy
= \left( \frac{q'}{q} \right)^{\frac{r}{q'}} \left( \frac{p'}{r} \right)^{q-1} \frac{q'}{p} \int_{R^2_+} \sigma(x,y) [I_2 \sigma(x,y)]^{\frac{r}{q'}} [I_2^* w(x,y)]^{\frac{r}{q'}} \, dxdy
+ \left( \frac{q'}{q} \right)^{\frac{r}{q'}} \left( \frac{p'}{r} \right)^{q-1} \frac{q'}{r} \int_{R^2_+} d_u \left( - [I_2^* w(u,y)]^{\frac{r}{q'}} \right) \, du [I_2 \sigma(u,y)]^{\frac{r}{q'}} \, dxdy
\geq \left( \frac{q'}{q} \right)^{\frac{r}{q'}} \left( \frac{p'}{r} \right)^{q-1} \left( \frac{q'}{r} \right)^{q-1} B^r.
\]

In view of (13), the required lower bound for \( C_2 \) in the case \( q < p \) is proven. \( \square \)

Recall that in the case \( p \leq q \) the best constant \( C_2 \) of the two–dimensional inequality (13) is equivalent to \( \sum_{i=1}^3 A_i \) (see Theorem 11). However, by virtue of the statements of Lemmas 2.1 and 2.2 for \( p < q \) the following inequalities take place:
\[
A_1 \leq C_2 \leq C_{1,1} \left[ A_1 + A_2 + A_3 \right] \leq C_{1,1} \left[ 1 + \alpha(p,q)^{\frac{r}{q'}} + \alpha(q',p')^{\frac{r}{q'}} \right] A_1. \quad (25)
\]
Moreover,
\[ \lim_{p \uparrow q} [\alpha(p, q) + \alpha(q', p')] = \infty. \]

Thus, the last estimate in (25) and the upper bound in the main theorem have blow-up for \( p \uparrow q \).

Estimates similar to (25) hold also in the case \( q < p \) if conditions \( r/p \geq 1 \) and \( r/q' \geq 1 \) are simultaneously satisfied, namely,
\[
\left(\frac{q}{r}\right)^{\frac{1}{p'}} \left(\frac{p'}{2r}\right)^{\frac{1}{q'}} B_1 \leq C_2 \leq C_{1,1} \left[ B_1 + B_2 + B_3 \right] \leq C_{1,1} \left[ 1 + \beta(p, q) + \beta(q', p') \right] B_1,
\]
(26)

where
\[ \beta(p, q) = \frac{2^{1/q+1}}{(2r-q)/r - 1)^{1/r} (2q/r - 1)^{1/p}}. \]

Observe that
\[ \lim_{q \uparrow p} [\beta(p, q) + \beta(q', p')] = \infty. \]

In the rest cases, the following inequalities take place for \( q < p \):
\[
\left(\frac{q}{r}\right)^{\frac{1}{p'}} \left(\frac{p'}{2r}\right)^{\frac{1}{q'}} B_1 \leq C_2 \leq \begin{cases} 
C_{1,\beta'} \left[ B_1 + B_2 \right] \leq C_{1,\beta'} \left[ 1 + \beta(p, q) \right] B_1, & \frac{r}{p} \geq 1 & \text{and} & \frac{r}{q} < 1, \\
C_{\beta,1} \left[ B_1 + B_3 \right] \leq C_{\beta,1} \left[ 1 + \beta(q', p') \right] B_1, & \frac{r}{p} < 1 & \text{and} & \frac{r}{q} \geq 1, \\
C_{\beta,\beta'} B_1, & \frac{r}{p} < 1 & \text{and} & \frac{r}{q} < 1.
\end{cases}
\]
(27)

On the strength of the restrictions on the parameters \( p \) and \( q \), all coefficients in (27) are finite. In the first zone \( r \to \infty \) only if \( p, q \to \infty \); similarly, in the second zone \( r \to \infty \) only if \( p, q \to 1 \); and in the third zone \( r \) cannot approach \( \infty \). In addition, \( C_{1,1} \) in (20) does not diverge for \( q \uparrow p \), and, therefore, the second inequality gives an upper bound in Sawyer’s theorem for \( p = q \), since \( \lim B_i = A_i, i = 1, 2, 3 \) (see (23)).

The upper estimates in (26)–(27) can be proven similarly to the upper bound for \( C_2 \) in the case \( q < p \) in the main theorem. The only difference is that for \( r/p \geq 1 \), instead of Lemma 2.1 one should use the inequality
\[
\text{V}_{(a,b) \times (c,d)} \leq \left[ I_2 \sigma(b, d) \right]^{\frac{1}{p'}} \left[ \int_a^b \int_c^d \frac{\chi_{\text{supp } w(x, y)}}{[I_2 \sigma(x, y)]^{1/p}} \, dx \, dy \left( \int_0^\infty \int_0^\infty (I_2 \sigma)^q w \right)^{\frac{1}{q'}} \right]^\frac{1}{p'}.
\]

Similarly, for \( r/q' \geq 1 \), instead of Lemma 2.2, the following estimate should be applied:
\[
\text{W}_{(a,b) \times (c,d)} \leq \left[ I_2^* w(a, c) \right]^{\frac{1}{p'}} \left[ \int_a^b \int_c^d \frac{\chi_{\text{supp } w(x, y)}}{[I_2^* w(x, y)]^{1/p'}} \, dx \, dy \left( \int_0^\infty \int_0^\infty (I_2^* w)^{q'} \sigma \right)^{\frac{1}{q'}} \right]^\frac{1}{p'}.
\]

To establish \( B_2 \leq \beta(p, q) B_1 \) we split \( \mathbb{R}_2^+ \) into domains \( \omega_k \) (as in Lemma 2.1). Then
\[
\int_{\mathbb{R}_2^+} [I_2 \sigma(x, y)]^{-\frac{1}{q'}} \, dx \, dy \left( \int_0^x \int_0^y (I_2 \sigma)^q w \right)^{\frac{1}{q'}} = \sum_{k \leq K} \int_{\omega_k \setminus \omega_{k+1}} [I_2 \sigma(x, y)]^{-\frac{1}{q'}} \, dx \, dy \left( \int_0^x \int_0^y (I_2 \sigma)^q w \right)^{\frac{1}{q'}}.
\]
From Proposition 2.1(b) with $\tau = \frac{q}{\gamma}$ and $\gamma = r/q$, we have:

$$\sum_{k \leq K_{\sigma}} 2^{-kr/p} \int_{\mathbb{R}^2_+ \setminus \omega_{k+1}} d_x d_y \left( \int_0^x \int_0^y (I_2\sigma)^q w \right)^{\frac{r}{q}} \leq \sum_{k \leq K_{\sigma}} 2^{-kr/p} \int_{\mathbb{R}^2_+ \setminus \omega_{k+1}} d_x d_y \left( \int_0^x \int_0^y (I_2\sigma)^q w \right)^{\frac{r}{q}}.$$

Since

$$\int_{\mathbb{R}^2_+ \setminus \omega_{k+1}} d_x d_y \left( \int_0^x \int_0^y (I_2\sigma)^q w \right)^{\frac{r}{q}} = \int_{\mathbb{R}^2_+ \setminus \omega_{k+1}} \chi_{\mathbb{R}^2_+ \setminus \omega_{k+1}}(x, y) \ d_x d_y \left( \int_0^x \int_0^y (I_2\sigma)^q w \right)^{\frac{r}{q}} = \left( \int_{\mathbb{R}^2_+ \setminus \omega_{k+1}} (I_2\sigma)^q w \right)^{\frac{r}{q}},$$

then we have

$$\sum_{k \leq K_{\sigma}} 2^{-kr/p} \int_{\mathbb{R}^2_+ \setminus \omega_{k+1}} d_x d_y \left( \int_0^x \int_0^y (I_2\sigma)^q w \right)^{\frac{r}{q}} = \sum_{k \leq K_{\sigma}} 2^{-kr/p} \left( \int_{\mathbb{R}^2_+ \setminus \omega_{k+1}} (I_2\sigma)^q w \right)^{\frac{r}{q}}.$$  

From Proposition 2.1(b) with $\tau = \frac{q}{\gamma}$ and $\gamma = r/q$, we get:

$$\sum_{k \leq K_{\sigma}} 2^{-kr/p} \left( \int_{\mathbb{R}^2_+ \setminus \omega_{k+1}} (I_2\sigma)^q w \right)^{\frac{r}{q}} \leq \sum_{k \leq K_{\sigma}} 2^{-kr/p} \left( \sum_{m \leq k} \int_{\omega_m \setminus \omega_{m+1}} (I_2\sigma)^q w \right)^{\frac{r}{q}} \leq \frac{2r/p}{(2(r-q)/p - 1)(2q/r - 1)^{r/p}} \sum_{k \leq K_{\sigma}} 2^{-kr/p} \left( \int_{\omega_k \setminus \omega_{k+1}} (I_2\sigma)^q w \right)^{\frac{r}{q}} \leq \frac{2^{r/p+r}}{(2(r-q)/p - 1)(2q/r - 1)^{r/p}} \sum_{k \leq K_{\sigma}} 2^{kr/p'} \left( \int_{\omega_k \setminus \omega_{k+1}} w \right)^{\frac{r}{q}}.$$  

By analogy with the proof of Lemma 2.1, we can write

$$|\omega_k \setminus \omega_{k+1}|^{\frac{r}{q}} w \leq |\omega_k|^{\frac{r}{q}} w = \int_{\omega_k \setminus \omega_{k+1}} d_x d_y [I^*_2 (\chi_{\omega_k} w)(x, y)]^{\frac{r}{q}} = \int_{\omega_k} d_x d_y [I^*_2 w(x, y)]^{\frac{r}{q}}.$$  

Hence (see Proposition 2.1(a)),

$$\sum_{k \leq K_{\sigma}} 2^{kr/p'} \left( \int_{\omega_k \setminus \omega_{k+1}} w \right)^{\frac{r}{q}} \leq \sum_{k \leq K_{\sigma}} 2^{kr/p'} \int_{\omega_k} d_x d_y [I^*_2 w(x, y)]^{\frac{r}{q}} \leq \sum_{k \leq K_{\sigma}} 2^k \int_{\omega_k} [I_2\sigma(x, y)]^{\frac{r}{q'}} d_x d_y [I^*_2 w(x, y)]^{\frac{r}{q}} \leq 2 \sum_{k \leq K_{\sigma}} \int_{\omega_k \setminus \omega_{k+1}} [I_2\sigma(x, y)]^{\frac{r}{q'}} d_x d_y [I^*_2 w(x, y)]^{\frac{r}{q}} = B_1.$$  

Similarly, one can show that $B_3 \leq \beta(q', p')B_1$. Thus, (26) and (27) are valid.
3 Sufficient condition

The one–dimensional analog of the condition (2) is the boundedness of the Muckenhoupt constant \( B \), of the condition (3) — the boundedness of the Tomaselli functional \( B \) in definition (11), and the analogs of the constants \( B_1 \), \( B_2 \) are the Maz’ya–Rozin [7, § 1.3.2] and Persson–Stepanov [10, Theorem 3] functionals, respectively. The constants have been generalized to the scales of equivalent conditions in [11] (see also [2] for the case of the Persson–Stepanov functional). The constants have been generalized to the scales of equivalent conditions in [11] (see also [2] for the case \( p \leq q \)). In the following theorem we find a sufficient condition for the inequality (3) to hold, having the form (28), where \( B_v \) is a two–dimensional analog of the constant \( B_{\text{MR}}(1/r) \) from [11] in the one–dimensional case.

**Theorem 3.1.** Let \( 1 < q < p < \infty \). The inequality (6) holds if

\[
B_v := \left( \int_{\mathbb{R}_+^2} \sigma(u, z) \left( \int_u^\infty \left( \int_z^\infty (I_2\sigma)^{q-1} w \right)^{\frac{r}{q}} du \right)^{\frac{1}{r}} \right)^{\frac{1}{q}} < \infty,
\]

where \( C_2 \lesssim B_v \).

**Proof.** We apply Sawyer’s scheme of partitioning \( \mathbb{R}_+^2 \) into rectangles from the proof of the sufficiency in Theorem 2.1. Compared to Figure 1, Figure 2 below has a rectangle \( Q^k_j = (0, x_j^k) \times (0, y_j^k) \) added.

Denote \( \widetilde{E}_j^k := E_j^k \cup (\tilde{S}_j^k \cap (\Omega_{k+2} - \Omega_{k+3})) \). Then (see [11])

\[
\int_{\mathbb{R}_+^2} (I_2 f)^q w \approx \sum_{k,j} 3^{kq} |\widetilde{E}_j^k|_w.
\]

Put \( g\sigma := f \) and write

\[
\sum_{k,j} 3^{kq}|\widetilde{E}_j^k|_w = \sum_{k,j} |\widetilde{E}_j^k|_w \left( \int_{Q_j^k} f \right)^q = \sum_{k,j} |\widetilde{E}_j^k|_w |Q_j^k|^q \left( \frac{1}{|Q_j^k|_\sigma} \int_{Q_j^k} g\sigma \right)^q.
\]
For an integer \( l \) by \( \Gamma_l \) we denote the set of pairs \((k, j)\) such that \(|\tilde{E}^k_j|_w > 0\) and
\[
2^l < \frac{1}{|Q^k_j|_\sigma} \int_{Q^k_j} g \sigma \leq 2^{l+1}, \quad (k, j) \in \Gamma_l.
\]

By analogy with how it was done in the proof of \cite{14, Theorem 1A}, we show that
\[
2^{l-1} < \frac{1}{|Q^k_j|_\sigma} \int_{Q^k_j} g \sigma \chi_{\{g > 2^{l-1}\}}, \quad \text{for all } j, k.
\]

Indeed, this follows from the fact that
\[
2^l < \frac{1}{|Q^k_j|_\sigma} \int_{Q^k_j} g \sigma = \frac{1}{|Q^k_j|_\sigma} \left[ \int_{Q^k_j \cap \{g > 2^{l-1}\}} g \sigma + \int_{Q^k_j \cap \{g \leq 2^{l-1}\}} g \sigma \right] \leq \frac{1}{|Q^k_j|_\sigma} \int_{Q^k_j \cap \{g > 2^{l-1}\}} g \sigma + 2^{l-1}.
\]

Further, we write for fixed \( l \):
\[
\sum_{(k, j) \in \Gamma_l} |\tilde{E}^k_j|_w |Q^k_j|_\sigma \overset{\text{(31)}}{\leq} 2^{-l} \sum_{(k, j) \in \Gamma_l} |\tilde{E}^k_j|_w |Q^k_j|_\sigma \int_{Q^k_j} g \sigma \chi_{\{g > 2^{l-1}\}}
\]
\[
\overset{\text{Combining the last estimate and (30), we obtain}}{=} 2^{-l} \sum_{(k, j) \in \Gamma_l} \int_{\tilde{E}^k_j} w(x, y) \left[I_2 \sigma(x, y)\right]^{q-1} \left( \int_0^x \int_0^y g \sigma \chi_{\{g > 2^{l-1}\}} \right) dx dy.
\]

Since \( 2^{l_0-1} < g(s, t) \leq 2^{l_0} \) almost everywhere for fixed \((s, t)\) then \( g(s, t) > 2^{l-1} \) for \( l \leq l_0 \) and, therefore,
\[
\sum_{l \leq l_0} 2^{(q-1)} \chi_{\{g > 2^{l-1}\}} = \sum_{l \leq l_0} 2^{(q-1)} = 2^{l_0(q-1)} \sum_{l \leq l_0} 2^{(l-l_0)(q-1)} \approx 2^{l_0(q-1)}.
\]

From this and Hölder’s inequalities with exponents \( p/q \) and \( r/q \), we find that
\[
\sum_{k, j} 3^{kq} |\tilde{E}^k_j|_w \overset{\text{by}}{\leq} \sum_{k, j} \int_{E^k_j} w(x, y) \left[I_2 \sigma(x, y)\right]^{q-1} \left( \int_0^x \int_0^y g^q(s, t) \sigma(s, t) ds dt \right) dx dy
\]
\[
= \int_{\mathbb{R}^2_+} w(x, y) \left[I_2 \sigma(x, y)\right]^{q-1} \left( \int_0^x \int_0^y g^q(s, t) \sigma(s, t) ds dt \right) dx dy
\]
\[
= \int_{\mathbb{R}^2_+} g^q(s, t) \sigma(s, t) \left( \int_{t}^{\infty} \int_{s}^{\infty} w(x, y) \left[I_2 \sigma(x, y)\right]^{q-1} dx dy \right) ds dt
\]
\[
\leq \left( \int_{\mathbb{R}_+^2} g^p \sigma \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}_+^2} \sigma(s,t) \left( \int_t^\infty \int_s^\infty (I_2 \sigma)^{q-1} w \right)^{\frac{1}{q}} ds \, dt \right)^{\frac{1}{p}}
= B_v^p \left( \int_{\mathbb{R}_+^2} g^p \sigma \right)^{\frac{1}{p}},
\]
(31)
since the sets \( E_j \) are disjoint and \( g^p \sigma = f^p \nu \). The estimates \( \text{(29)} \) and \( \text{(31)} \) imply the validity of \( \text{(28)} \) for all \( f \) from the subclass \( M \).

There is also a dual statement of the last theorem with the functional

\[
B_w := \left( \int_{\mathbb{R}_+^2} w(u,z) \left( \int_0^u \int_0^z (I_2^w \sigma)^{p'-1} \right)^{\frac{1}{p'}} du \, dz \right)^{\frac{1}{p}}
\]
instead of \( B_v \). The proof of this fact is similar and can be carried out through the operator \( I_2^w \).

If the weights \( v \) and \( w \) are factorizable, then the condition \( B_v < \infty \) (or \( B_w < \infty \)) is necessary and sufficient for the \( \text{(II)} \) to be true in the case of \( 1 < q < p < \infty \), moreover \( C_2 \approx B_v \approx B_w \).

4 Multidimensional case with factorizable weights

It was established by A. Wedestig in [16] (see also [17]) for the case \( n = 2 \) that if the weight function \( v \) in \( \text{(I)} \) is factorizable, that is, \( v(x_1, x_2) = v_1(x_1)v_2(x_2) \), then it is possible to characterize the inequality \( \text{(I)} \) by only one functional for all \( 1 < p \leq q < \infty \).

**Theorem 4.1.** [17] Theorem 1.1 Let \( n = 2 \), \( 1 < p \leq q < \infty \), \( s_1, s_2 \in (1, p) \) and \( v(x_1, x_2) = v_1(x_1)v_2(x_2) \). Then the inequality \( \text{(I)} \) holds for all \( f \geq 0 \) if and only if

\[
A_W(s_1, s_2) := \sup_{(t_1, t_2) \in \mathbb{R}_+^2} \left[ I_1 \sigma_1(t_1) \right]^\frac{1}{p} \left[ I_1 \sigma_2(t_2) \right]^\frac{1}{p} \\
\times \left( \int_t^\infty \int_t^\infty \left( I_1 \sigma_1 \right)^{\frac{q(p-s_1)}{p}} \left( I_1 \sigma_2 \right)^{\frac{q(p-s_2)}{p}} w \right)^{\frac{1}{q}} < \infty,
\]

where \( \sigma_i := v_i^{1-p'} \), \( i = 1, 2 \). Moreover, \( C_2 \approx A_W(s_1, s_2) \) with equivalence constants dependent on parameters \( p, q \) and \( s_1, s_2 \) only.

The result of this theorem can be generalized to \( n > 2 \).

A number of statements similar to [17] Theorem 1.1 were obtained in [12] under the condition that weight functions \( v \) or \( w \) satisfy

\[
v(y_1, \ldots, y_n) = v_1(y_1) \ldots v_n(y_n)
\]
(32)
or

\[
w(x_1, \ldots, x_n) = w_1(x_1) \ldots w_n(x_n).
\]
(33)

**Theorem 4.2.** [12] Theorems 2.1, 2.2 Let \( 1 < p \leq q < \infty \) and the weight function \( v \) satisfy the condition \( \text{(II)} \). Then the inequality \( \text{(I)} \) holds for all \( f \geq 0 \)

(i) if and only if \( A_{M_n} < \infty \), where

\[
A_{M_n} := \sup_{(t_1, \ldots, t_n) \in \mathbb{R}_+^n} \left[ I_1^w w(t_1, \ldots, t_n) \right]^\frac{1}{q} \left[ I_1 \sigma_1(t_1) \right]^\frac{1}{p} \ldots \left[ I_1 \sigma_n(t_n) \right]^\frac{1}{p};
\]

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(ii) if and only if \( A_{T_n} < \infty \), where

\[
A_{T_n} = \sup_{(t_1, \ldots, t_n) \in \mathbb{R}_+^n} \left[ I_1 \sigma_1(t_1) \right]^{-\frac{1}{q}} \cdots \left[ I_1 \sigma_n(t_n) \right]^{-\frac{1}{q}} \left( \int_0^{t_1} \cdots \int_0^{t_n} (I_1 \sigma_1)^{q} \cdots (I_1 \sigma_n)^{q} w \right)^{\frac{1}{q}}.
\]

Besides, \( C_n \approx A_{M_n} \approx A_{T_n} \) with equivalence constants depending on \( p, q \) and \( n \).

**Theorem 4.3.** [12] Theorems 2.4, 2.5 Let \( 1 < p \leq q < \infty \) and the weight \( w \) satisfy the condition (33). Then the inequality (1) is true

(i) if and only if \( A_{M_n}^* < \infty \), where with \( \sigma := v^{1-p'} \)

\[
A_{M_n}^* := \sup_{(t_1, \ldots, t_n) \in \mathbb{R}_+^n} \left[ I_1^* \sigma_1(t_1) \right]^{\frac{1}{p'}} \cdots \left[ I_1^* \sigma_n(t_n) \right]^{\frac{1}{p'}} \left( \int_0^{t_1} \cdots \int_0^{t_n} (I_1^* \sigma_1)^{p'} \cdots (I_1^* \sigma_n)^{p'} \right)^{\frac{1}{p'}}.
\]

(ii) if and only if \( A_{T_n}^* < \infty \), where

\[
A_{T_n}^* = \sup_{(t_1, \ldots, t_n) \in \mathbb{R}_+^n} \left[ I_1^* w_1(t_1) \right]^{-\frac{1}{p}} \cdots \left[ I_1^* w_n(t_n) \right]^{-\frac{1}{p}} \left( \int_0^{t_1} \cdots \int_0^{t_n} (I_1^* w_1)^{p'} \cdots (I_1^* w_n)^{p'} \right)^{\frac{1}{p'}}.
\]

Besides, \( C_n \approx A_{M_n}^* \approx A_{T_n}^* \) with equivalence constants depending on \( p, q \) and \( n \).

**Theorem 4.4.** [12] Theorems 3.1, 3.2 Let \( 1 < q < p < \infty \). Suppose that the weight function \( v \) in (1) satisfies the condition (32) and \( I_1 \sigma_1(\infty) = \ldots = I_1 \sigma_n(\infty) = \infty \). Then (1) is valid for all \( f \geq 0 \) on \( \mathbb{R}_+^n \) with \( C_n < \infty \) independent of functions \( f \)

(i) if and only if \( B_{MR_n} < \infty \), where

\[
B_{MR_n} := \left( \int_{\mathbb{R}_+^n} \left[ I_1^* w(t_1, \ldots, t_n) \right]^{\frac{1}{p}} \left[ I_1 \sigma_1(t_1) \right]^{\frac{1}{q}} \cdots \left[ I_1 \sigma_n(t_n) \right]^{\frac{1}{q}} \right)^{\frac{1}{q}} dt_1 \cdots dt_n;
\]

(ii) if and only if \( B_{PS_n} < \infty \), where

\[
B_{PS_n} := \left( \int_{\mathbb{R}_+^n} \left( \int_0^{t_1} \cdots \int_0^{t_n} [I_1 \sigma_1(t_1)]^q \cdots [I_1 \sigma_n(t_n)]^q w(x_1, \ldots, x_n) dx_1 \cdots dx_n \right)^{\frac{1}{q}} \right)^{\frac{1}{p}} dt_1 \cdots dt_n.
\]

Moreover, \( C_n \approx B_{MR_n} \approx B_{PS_n} \) with equivalence constants dependent on \( p, q \) and \( n \).

**Theorem 4.5.** [12] Theorems 3.3, 3.4 Let \( 1 < q < p < \infty \). Assume that \( w \) in (1) satisfies (33) and \( I_1^* w_1(0) = \ldots = I_1^* w_n(0) = \infty \). Then (1) is valid for all \( f \geq 0 \) on \( \mathbb{R}_+^n \) with \( C_n < \infty \) independent of functions \( f \)

(i) if and only if \( B_{MR_n}^* < \infty \), where

\[
B_{MR_n}^* := \left( \int_{\mathbb{R}_+^n} \left[ I_n \sigma(t_1, \ldots, t_n) \right]^{\frac{1}{p'}} \left[ I_1^* w_1(t_1) \right]^{\frac{1}{q'}} \cdots \left[ I_1^* w_n(t_n) \right]^{\frac{1}{q'}} \right)^{\frac{1}{q'}} dt_1 \cdots dt_n;
\]

(ii) if and only if \( B_{PS_n}^* < \infty \), where

\[
B_{PS_n}^* := \left( \int_{\mathbb{R}_+^n} \left( \int_0^{t_1} \cdots \int_0^{t_n} (I_1^* w_1)^{p'} \cdots (I_1^* w_n)^{p'} \right)^{\frac{1}{p'}} \cdots \left[ I_1 \sigma_n(t_n) \right]^{\frac{1}{q'}} \right)^{\frac{1}{p'}} dt_1 \cdots dt_n.
\]

Moreover, \( C_n \approx B_{MR_n}^* \approx B_{PS_n}^* \) with equivalence constants dependent on \( p, q \) and \( n \).
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