ON ENERGY DISSIPATION THEORY AND NUMERICAL STABILITY FOR TIME-FRACTIONAL PHASE FIELD EQUATIONS *

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Abstract. For the time-fractional phase field models, the corresponding energy dissipation law has not been settled on both the continuous level and the discrete level. In this work, we shall address this open issue. More precisely, we prove for the first time that the time-fractional phase field models indeed admit an energy dissipation law of an integral type. In the discrete level, we propose a class of finite difference schemes that can inherit the theoretical energy stability. Our discussion covers the time-fractional gradient systems, including the time-fractional Allen-Cahn equation, the time-fractional Cahn-Hilliard equation, and the time-fractional molecular beam epitaxy models. Numerical examples are presented to confirm the theoretical results. Moreover, a numerical study of the coarsening rate of random initial states depending on the fractional parameter $\alpha$ reveals that there are several coarsening stages for both time-fractional Cahn-Hilliard equation and time-fractional molecular beam epitaxy model, while there exists a $-\alpha/3$ power law coarsening stage.

Key words. time-fractional gradient system, time-fractional phase field equations, Allen-Cahn equation, Cahn-Hilliard equation, energy dissipation law, energy stable scheme

AMS subject classifications. 65M12, 65M06, 35Q99, 74A50

1. Introduction. The phase-field method has been a powerful modeling and simulation tool in diverse research areas such as material sciences [4, 6, 48, 8, 21, 17], multi-phase flow [29, 25, 26, 31, 10, 20, 37, 27, 50, 5], biology and tumor growth [11, 22, 43, 34, 18, 51], to name a few. Most of the phase field formulations are based on a free energy function depending on an order parameter (the phase field) and a diffusive mechanism. Equations of the model are then obtained by using general relations of statistical physics. Phase field models are constructed from physical considerations, but contain a parameter or combination of parameters related to the interface width. Parameters of the model are then chosen by studying the limit of the model with this width going to zero, in such a way that one can identify this limit with the intended sharp interface model [15, 28, 45].

The well-known examples of phase field models include the Allen-Cahn (AC) equation [4], the Cahn-Hilliard (CH) equation [6], and the molecular beam epitaxy (MBE) models [8, 21]. A common feature of the above mentioned phase field models is that the associated free energy admits an energy dissipation law, or in other words, they are gradient flows.

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Taking the CH equation as an example, the associated governing equation yields

\[
\frac{\partial \phi}{\partial t} + \gamma (-\Delta) \left( -\varepsilon \Delta \phi + \frac{1}{4} F'(\phi) \right) = 0, \quad x \in \Omega \subset \mathbb{R}^d, \quad 0 < t \leq T.
\]  

(1.1)

where \( \varepsilon \) is an interface width parameter, \( \gamma \) is the mobility, and

\[
F(\phi) = \frac{1}{4}(1 - \phi^2)^2
\]

(1.2)

is of bistable type, admitting two local minima. For simplicity, we set \( \Omega = (0, 2\pi)^d \), and assume that \( \phi(\cdot, t) \) satisfies a periodic boundary condition. Other reasonable boundary conditions can also be considered, such as the Neumann boundary conditions and the dynamic contact line condition \([31, 37]\). The corresponding free energy functional of the CH equation is defined as

\[
E(\phi) := \int_{\Omega} \left( \frac{\varepsilon}{2} |\nabla \phi|^2 + \frac{1}{\varepsilon} F(\phi) \right) dx.
\]

(1.3)

There are other forms of energy definitions, but the one given above has a meaningful finite limit as \( \varepsilon \) goes to zero. The CH equation can be viewed as a gradient flow of the above energy with respect to the space \( H^{-1} \). It is well known that \( E(u) \) decreases in time and approaches a minimum:

\[
\frac{d}{dt} E(\phi) = - \int_{\Omega} \left| \nabla \left( -\varepsilon^2 \Delta \phi + F'(\phi) \right) \right|^2 dx \leq 0.
\]

(1.4)

Such an energy dissipation low plays an important role in developing stable numerical methods for dissipation systems due to its importance to the long time numerical simulation, see e.g., \([12, 13, 14, 44, 36, 49, 16, 46, 42]\) and references therein.

In recent years, fractional-type phase-field models have attracted more and more attentions \([1, 2, 3, 7, 30, 38, 23]\). For instance, consider a fractional type free energy \([38]\)

\[
E^\alpha(\phi) := \int_{\Omega} \left( \frac{\varepsilon^2}{2} |\nabla^\alpha \phi|^2 + F(\phi) \right) dx,
\]

(1.5)

where \( \nabla^\alpha \) is the fractional gradient \( \nabla^\alpha = \left( \frac{\partial^\alpha}{\partial x_1}, \ldots, \frac{\partial^\alpha}{\partial x_d} \right) \) and \( \{ \frac{\partial^\alpha}{\partial x_k} \}_k \) are fractional derivatives. One is interested in the following space-fractional CH equation

\[
\frac{\partial \phi}{\partial t} + (-\Delta)^\alpha \left( -\varepsilon^2 \Delta^\alpha \phi + F'(\phi) \right) = 0.
\]

(1.6)

It is obvious that for the modified energy functional (1.5), the energy dissipation law still holds, i.e.,

\[
\frac{d}{dt} E^\alpha(\phi) \leq 0.
\]

(1.7)

Another interesting approach is to keep the original free energy (1.3) unchanged, yet consider the associate gradient flow in the space \( H^{-\alpha} \), which yields another space-fractional CH equation \([1]\)

\[
\frac{\partial \phi}{\partial t} + (-\Delta)^\alpha \left( -\varepsilon^2 \Delta^\alpha \phi + F'(\phi) \right) = 0.
\]

(1.8)
It is also straightforward to show that the free energy admits an energy dissipation law. As reported in [1], the nature of the solution of the fractional CH with a general $\alpha > 0$ is qualitatively (and quantitatively) closer to the behavior of the classical CH equation (1.1) regardless of how close to zero the value of $\alpha$ is. An examination of the coarsening rates reveals that the asymptotic rate is rather insensitive to the value of $\alpha$ and, as a consequence, is close to the well-established rate observed for the classical CH equation.

The time-fractional phase field models have also been investigated recently. Consider the following time-fractional CH equation

$$\frac{\partial^\alpha}{\partial t^\alpha} \phi + \gamma (\Delta) \left( -\varepsilon \Delta \phi + \frac{1}{\varepsilon} F'(u) \right) = 0, \quad (1.9)$$

where $\alpha \in (0, 1)$, and $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo derivative defined as

$$\frac{\partial^\alpha}{\partial t^\alpha} \phi = C_0 D^\alpha \phi(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\phi'(s)}{(t-s)^\alpha} ds, \quad t > 0, \quad \alpha \in (0, 1). \quad (1.10)$$

It is shown in [23] by numerical examples that the free energy admits an energy dissipation law. However, a theoretical study (rigorous analysis) for the energy dissipation law is still open. This is the main motivation of the present work. We shall address this open issue in this work. In particular, our main contribution is three folds:

- In the continuous level, we shall prove the associated energy dissipation law for the time-fractional phase field models. Our discussion covers the time-fractional AC equation, the time-fractional CH equation, and the time-fractional MBE models. Extensions to general time-fractional gradient systems are straightforward. Moreover, a maximum principle property for the time-fractional AC equation is also presented.

- In the discrete level, we design a class of finite difference schemes that can inherit the discrete energy dissipation law. Numerical examples are presented to confirm the theoretical finding.

- Moreover we investigate the coarsening rate of the time-fractional phase-field models. Our numerical examples show that there are several coarsening stages in both the time-fractional CH equation and the time-fractional MBE equation, where a power-law coarsening rate stage with the powers depend on the time derivative order $\alpha$ exist.

The rest of the paper is organized as following. In Section 2, we shall present the energy dissipation law for time-fractional phase-field equations. In Section 3, a class of finite difference schemes will be proposed and we shall show that the proposed schemes can inherit the energy dissipation law in discrete level. Numerical examples will be presented in Section 4 to confirm our theoretical results and show dependence of the energy dissipation on the fractional order. Some conclusions will be given in Section 5.

2. Energy dissipation for time-fractional phase field equations. In this section, we will provide detail energy dissipation analysis for the three time-fractional models mentioned above. To this end, we first present the following lemma that plays an important role in our analysis.

**Lemma 2.1.** For any given $h \in L^2(0, T)$, $\alpha \in (0, 1)$, we have

$$\int_0^T \int_0^t \frac{h(s)h(t)}{(t-s)^\alpha} dsdt \geq 0. \quad (2.1)$$
Proof. First, let us introduce some notations and basic properties of time-fractional derivatives and integrals [19]. The Riemann-Liouville fractional integrals for $\alpha \in (0, 1)$ on finite interval $[0, T]$ are defined as:

$$
(I^\alpha_0 f)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad \text{for } t \geq 0,
$$

$$
(I^\alpha_T f)(t) := \frac{1}{\Gamma(\alpha)} \int_t^T \frac{f(s)}{(s-t)^{1-\alpha}} ds, \quad \text{for } t \leq T.
$$

The Liouville fractional integrals on the real axis $\mathbb{R}$ are defined as:

$$
(I^\alpha_+ f)(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t \frac{f(s)ds}{(t-s)^{1-\alpha}}, \quad (I^\alpha_- f)(t) := \frac{1}{\Gamma(\alpha)} \int_t^{\infty} \frac{f(s)ds}{(s-t)^{1-\alpha}}.
$$

They satisfy the following fractional integration by parts formula:

$$
\int_0^T (I^\alpha_0 f)(t)g(t)dt = \frac{1}{\Gamma(\alpha)} \int_0^T \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds g(t)dt
$$

$$
= \int_0^T f(s) \frac{1}{\Gamma(\alpha)} \int_s^T \frac{g(t)}{(t-s)^{1-\alpha}} dt ds = \int_0^T f(t)(I^\alpha_- g)(t)dt, \quad (2.2)
$$

$$
\int_{-\infty}^{\infty} (I^\alpha_+ f)(t)g(t)dt = \int_{-\infty}^{\infty} f(t)(I^\alpha_- g)(t)dt. \quad (2.3)
$$

For $\alpha > 0, \beta > 0, \alpha + \beta < 1$, $f \in L_1(\mathbb{R})$, it can be verified that the following semi-group properties for fractional integrals hold

$$
(I^\alpha_+ I^\beta_+ f)(t) = (I^\alpha+\beta_+ f)(t), \quad (I^\alpha_- I^\beta_- f)(t) = (I^\alpha+\beta_- f)(t). \quad (2.4)
$$

Next, let us define

$$
A_\alpha(f, g) := \frac{1}{\Gamma(1-\alpha)} \int_0^T \int_0^t \frac{f(s)g(t)}{(t-s)^\alpha} ds dt = \int_0^T (I^\alpha_0 f)(t)g(t)dt. \quad (2.5)
$$

The proof of the lemma is equivalent to demonstrate that $A_\alpha(h, h) \geq 0$. To this end, we first extend the function $h$ to $\mathbb{R}$ by

$$
\tilde{h}(t) = \begin{cases} 
    h(t), & t \in [0, T], \\
    0, & \text{otherwise}. 
\end{cases}
$$

Then we have

$$
A_\alpha(h, h) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{\infty} \int_{-\infty}^t \frac{\tilde{h}(s)\tilde{h}(t)}{|t-s|^\alpha} ds dt = \int_{-\infty}^{\infty} (I^{1-\alpha}_0 \tilde{h})(t)\tilde{h}(t)dt.
$$

By the semi-group property (2.4) and the fractional integration by parts, $A_\alpha$ can be rewritten as

$$
A_\alpha(h, h) = \int_{-\infty}^{\infty} (I^\beta_+ \tilde{h})(t)(I^\beta_- \tilde{h})(t)dt, \quad \beta = \frac{1-\alpha}{2}.
$$
Denote $\hat{h}$ the Fourier transform for any given $L^2$ function $h$, $\check{h}$ the conjugate of $h$. If both $f, g$ are $L^2$ function, then we have the Parseval’s formula (see, e.g., [32])
\[
\int_{-\infty}^{\infty} f(t)\overline{g(t)}\,dt = \int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi.
\]
The Fourier transform for the Liouville factional integrals are given by (see, e.g., [33, 19]):
\[
(FI_+^\alpha f)(\xi) = \frac{(\mathcal{F}f)(\xi)}{(-i\xi)^\alpha}, \quad (FI_-^\alpha f)(\xi) = \frac{(\mathcal{F}f)(\xi)}{(i\xi)^\alpha}, \quad \text{for } \alpha \in (0, 1), f \in L^1(\mathbb{R}),
\]
where $(\mp i\xi)^\alpha$ is defined as
\[
(\mp i\xi)^\alpha = |\xi|^\alpha e^{\mp\alpha\pi\operatorname{sgn}(\xi)/2}.
\]
For $h \in L^2([0, T])$, it is easy to see that $I_+^\alpha \check{h}, I_-^\alpha \check{h} \in L^2(\mathbb{R})$. Then by the Parseval’s formula and the convolution theorem we obtain
\[
A_\alpha(h, h) = \int_{-\infty}^{\infty} (I_+^\alpha \check{h})(t)(I_-^\alpha \check{h})(t)\,dt = \int_{-\infty}^{\infty} \frac{\check{h}(\xi)}{(-i\xi)^\alpha} \frac{\check{h}(\xi)}{(i\xi)^\beta} d\xi
\]
\[
= \int_{-\infty}^{\infty} \frac{|\check{h}(\xi)|^2}{|\xi|^{1-\alpha}} e^{i\beta \pi \operatorname{sgn}(\xi)} d\xi = \cos \left( \frac{1-\alpha}{2} \right) \int_{-\infty}^{\infty} \frac{|\check{h}(\xi)|^2}{|\xi|^{1-\alpha}} d\xi \geq 0,
\]
which gives
\[
A_\alpha(h, h) = \sin \left( \frac{\alpha \pi}{2} \right) \int_{-\infty}^{\infty} \frac{|\check{h}(\xi)|^2}{|\xi|^{1-\alpha}} d\xi \geq 0. \quad (2.6)
\]
This complete the proof. \square

By symmetry, we have
\[
A_\alpha(h, h) = \frac{1}{2} \frac{1}{\Gamma(1-\alpha)} \int_0^T \int_0^T \frac{h(s)h(t)}{|t-s|^{\alpha}} ds dt. \quad (2.7)
\]
Consequently, one easily get the following result from Lemma 2.1.

**Corollary 2.1.** For any given $h \in L^2(0, T)$, $\alpha \in (0, 1)$, we have
\[
\int_0^T \int_0^T \frac{h(s)h(t)}{|t-s|^{\alpha}} ds dt \geq 0. \quad (2.8)
\]

**2.1. The time-fractional Allen-Cahn equations.** Consider the following time-fractional AC equation:
\[
\frac{\partial^\alpha}{\partial t^\alpha} \phi = \gamma(\varepsilon \Delta \phi - \frac{1}{\varepsilon} f(\phi)), \quad \alpha \in (0, 1), \quad (x, t) \in \Omega \times [0, T],
\]
where $\phi$ is a function of $x$ and $t$, $\varepsilon$ is the thickness of the phase interface, $\gamma$ is a mobility constant, and $f(\phi)$ is a given nonlinear function corresponding to the derivative of the bulk energy density function $\mathcal{F}(\phi)$. The function $\mathcal{F}(\phi)$ is usually chosen as (1.2).
Note that $\alpha \in (0, 1)$ in (2.9) is the fractional order parameter. When $\alpha = 1$, the equation (2.9) is the standard AC equation, which satisfies a well-known energy dissipation property

$$\frac{d}{dt} E[\phi] = \frac{1}{\gamma} \left\| \frac{\partial}{\partial t} \phi \right\|^2,$$

or

$$E[\phi(T)] - E[\phi(0)] = \frac{1}{\gamma} \int_0^T \left\| \frac{\partial}{\partial t} \phi \right\|^2 dt,$$

where $E[\phi]$ is the system energy:

$$E[\phi] = \frac{\varepsilon}{2} \|\nabla \phi\|^2 + \frac{1}{\varepsilon} \langle F(\phi), 1 \rangle.$$

Here we use $\langle \cdot, \cdot \rangle$ to denote the $L^2(\Omega)$ inner product in spatial domain; $\| \cdot \|$ denotes the standard $L^2(\Omega)$ norm in spatial domain.

Our first question is whether the time-fractional AC equation satisfies an energy dissipation law similar to (2.10) or (2.11)? We shall give a positive answer in the following discussions.

**Theorem 2.1.** The time-fractional AC equation (2.9) satisfies a dissipation law

$$E[\phi(T)] - E[\phi(0)] = \frac{1}{\gamma} \int_0^T A_\alpha(\phi_t, \phi_t) dx dt \leq 0.$$

**Proof.** We first multiply both sides of (2.9) by $-\phi_t$ and then take integrations on both sides. This gives

$$- \int_0^T \int_\Omega \frac{\partial^\alpha \phi}{\partial t^\alpha} \phi_t dx dt = \gamma \int_0^T \int_\Omega (-\Delta \phi + f(\phi)) \phi_t dx dt = \gamma \int_0^T \frac{d}{dt} E[\phi] dt.$$

Consequently,

$$E[\phi(T)] - E[\phi(0)] = \frac{1}{\gamma \Gamma(1 - \alpha)} \int_\Omega \int_0^T \int_0^t \phi_t(x, s) \frac{d}{ds} \phi(x, t) ds dt dx.$$

Then the desired property (2.13) is a direct result of (2.14) and Lemma 2.1.

**Remark 2.1.** Note that Eq. (2.13) is an energy dissipation law of integral type, from which we have $E[\phi(T)] \leq E[\phi(0)]$, thus the solution is energy stable. However, Eq. (2.13), in general, does not lead to $\frac{d}{dt} E \leq 0$ or $\frac{d^\alpha}{dt^\alpha} E \leq 0$.

Besides the energy stability, the time-fractional AC equation also satisfies the following maximum principle.

**Theorem 2.2.** Suppose $\phi$ is a classical strong solution (i.e. $\Delta \phi \in C^0$ and $\phi_t \in C^0$) of time-fractional AC equation (2.9). If $|\phi(x, 0)| \leq 1$ for all $x \in \Omega$, then $|\phi(x, t)| \leq 1$ for all $x \in \Omega$ and $t > 0$.

**Proof.** Proof by contradiction. Suppose that $(x_0, t_0)$ is a inner point at which $\phi$ reach its maximum and $\phi(x_0, t_0) > 1$. Since $\phi$ is a classical solution, we have $\Delta \phi \leq 0$ and $f(\phi) = \phi^3 - \phi > 0$. Then the right hand side of Eq. (2.9) is negative. The remaining is to prove that the left hand side at $(x_0, t_0)$ can be negative. Define
\( \psi(x, t) = \phi(x, t) - \phi(x_0, t_0) \), then we have \( \frac{\partial^{\alpha} \phi}{\partial t^{\alpha}} = \frac{\partial^{\alpha} \psi}{\partial t^{\alpha}} \). By a direct calculation, we have

\[
\frac{\partial^{\alpha} \psi(x_0, t_0)}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \lim_{\delta \to 0^+} \int_0^{t_0-\delta} \frac{\psi(x_0, s)}{(t_0-s)^{\alpha+1}} ds
\]

In the last inequality, we used the fact \( \psi(x, t) \leq 0 \) and \( \psi(x_0, t_0) = 0 \). This result contradicts with the fact that the right hand side is negative. So, we have \( \phi(x, t) \leq 1 \) for all \( t > 0, x \in \Omega \).

A similar argument leads to \( \phi(x, t) \geq -1 \). The proof is completed. \( \square \)

### 2.2. The time-fractional Cahn-Hilliard equation

The analysis of the time-fractional AC equation can be extended to the time-fractional CH equation, which will be given in this subsection.

**Theorem 2.3.** The time-fractional CH equation

\[
\frac{\partial^{\alpha} \phi}{\partial t^{\alpha}} = \gamma A_{\phi}, \quad \mu = -\varepsilon \Delta \phi + \frac{1}{\varepsilon} f(\phi), \quad \alpha \in (0, 1),
\]

with periodic boundary condition or no-flux boundary condition

\[
\frac{\partial \mu}{\partial n} \bigg|_{\partial \Omega} = 0, \quad \frac{\partial \phi}{\partial n} \bigg|_{\partial \Omega} = 0
\]

satisfies the energy dissipation law

\[
E[\phi(T)] - E[\phi(0)] = -\frac{1}{\gamma} \int_{\Omega} A_{\phi}(\nabla \psi, \nabla \psi) dx \leq 0,
\]

where \( \psi = -\Delta^{-1} \phi_t \) is the solution of the equation

\[
-\Delta \psi = \phi_t
\]

with periodic or homogeneous Neumann boundary condition \( \partial_n \psi \big|_{\partial \Omega} = 0 \).

**Proof.** We first show that the time-fractional CH equation conserves the total mass. More precisely, if \( \phi \) is the solution of time-fractional CH equation (2.15) with periodic boundary condition or no-flux boundary condition (2.16), then

\[
\int_{\Omega} \phi(x, t) dx = \int_{\Omega} \phi(x, 0) dx, \quad \forall \ t \geq 0.
\]

To see this, integrating with respect to the spatial dimension for the first equation in (2.15) gives

\[
0 = \int_{\Omega} \frac{\partial^{\alpha} \phi}{\partial t^{\alpha}} dx = \frac{1}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{1}{(t-s)^{\alpha}} \frac{\partial \phi(x, s)}{\partial t} ds dx
\]

\[
= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^{\alpha}} \left( \int_{\Omega} \frac{\partial}{\partial t} \phi(x, s) dx \right) ds, \quad \forall \ t \geq 0.
\]
Then we obtain
\[ \frac{d}{dt} \int_{\Omega} \phi(x,t) dx = \int_{\Omega} \frac{\partial}{\partial t} \phi(x,t) = 0, \quad \text{for a.e. } t \geq 0. \]
Integrating the above equation, we obtain (2.19).

It follows from (2.19) that \( \int_{\Omega} \phi_t dx = 0 \). Therefore, \( \psi \) in (2.18) is well defined.
Pair the first equation of (2.15) with \( -\frac{1}{\gamma} \psi \), second equation with \( \phi_t \), and sum up the results, we have
\[ -\frac{1}{\gamma} \langle I^{1-\alpha} \nabla \psi, \nabla \psi \rangle = \frac{d}{dt} E. \]
Integrating both sides from 0 to \( T \), and by using Lemma 2.1, we obtain the energy dissipation law (2.17) for the time-fractional CH equation.

2.3. The time-fractional MBE Model. Now we consider the time-fractional MBE model:
\[ \frac{\partial^\alpha}{\partial t^\alpha} \phi = \gamma (-\varepsilon \Delta^2 \phi + \frac{1}{\varepsilon} \nabla \cdot f(\nabla \phi)), \quad \alpha \in (0,1), \quad (2.20) \]
where \( f(\nabla \phi) = (|\nabla \phi|^2 - 1)\nabla \phi \) in model with slope selection, \( f(\nabla \phi) = -\frac{\nabla \phi}{1 + |\nabla \phi|^2} \) in the model without slope selection.

Using similar arguments as in the last subsection, we can show that the time-fractional MBE models have the following energy dissipation property.

**Theorem 2.4.** Suppose the time-fractional MBE model satisfies periodic boundary condition or no-flux boundary condition
\[ \partial_n \Delta \phi|_{\partial \Omega} = \partial_n \phi|_{\partial \Omega} = 0. \quad (2.21) \]
Then the solution satisfies following energy dissipation law
\[ E_m[\phi(T)] - E_m[\phi(0)] \leq -\frac{1}{\gamma} \int_{\Omega} A_{\alpha}(\phi, \phi) dx \leq 0, \quad (2.22) \]
where
\[ E_m(\phi) = \frac{\varepsilon}{2} ||\Delta \phi||^2 + \frac{1}{\varepsilon} \langle F(\nabla \phi), 1 \rangle, \]
and
\[ F(v) = \begin{cases} 
\frac{1}{4} (|v|^2 - 1)^2, & \text{for model with slope selection}, \\
-\frac{1}{4} \ln (1 + |v|^2), & \text{for model without slope selection}.
\end{cases} \]

3. Energy stable finite difference schemes. In this section, we shall design first order energy stable finite difference schemes for the time-fractional phase field models (i.e., time-fractional AC equation, CH equation, and MBE model). To this end, we shall first review a commonly-used finite difference scheme.

One important application of fractional partial differential equations is the following time-fractional diffusion equation
\[ \frac{\partial^\alpha u}{\partial t^\alpha} = \Delta u + f(x,t), \quad \alpha \in (0,1), \quad (3.1) \]
which can be regard as the linearized version of time-fractional AC equation (2.9). For notation simplicity, we shall consider the one spatial dimension case, i.e., $\Delta u = \frac{\partial^2 u}{\partial x^2}$, with $x$ being the spatial variable. Let $\tau$ be the time stepsize, $t_k = k\tau$, $u^k(\cdot)$ be the numerical approximation of $u(\cdot, t_k)$. By applying the classical $L_1$ scheme (see, e.g., [24]) to the time-fractional derivative and treating other terms in an implicit way, one get the following scheme:

$$\sum_{j=0}^{k} b_j \frac{u^{k+1-j}(x) - u^{k-j}(x)}{\tau} = \frac{\partial^2 u^{k+1}(x)}{\partial x^2} + f(x, t_{k+1}),$$

(3.2)

where

$$b_j = \frac{1}{\Gamma(1-\alpha)} \int_{\tau}^{(j+1)\pi} 1 \cdot \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} [(j+1)^{1-\alpha} - j^{1-\alpha}], \; j \geq 0.$$

(3.3)

The derivation of the left hand side of (3.2) is given as below

$$\frac{\partial^u}{\partial \alpha}(x, t_{k+1}) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{k+1}} \frac{u_t(x, s)}{(t_{k+1} - s)^\alpha} ds$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{u(x, t_{j+1}) - u(x, t_j)}{\tau} \int_{t_j}^{t_{j+1}} \frac{ds}{(t_{k+1} - s)^\alpha} + r_{k+1}^{\alpha}$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} b_{k-j} \frac{u(x, t_{j+1}) - u(x, t_j)}{\tau} + r_{k+1}^{\alpha},$$

where the integer derivative $\frac{\partial^u}{\partial \alpha}u(x, t)$ in time interval $[t_j, t_{j+1}]$ is approximated with a first order Euler approach [39]. The finite difference scheme for the fractional differential operator is obtained by dropping the remainder $r_{k+1}^{\alpha}$, which is bounded by $c_n \tau^{2-\alpha}$ if the exact solution is sufficiently smooth [24].

Suppose the spatial discretization using Galerkin approach is accurate enough. Then the $H^1$ stability is available as the $L_1$ discretization of the fractional derivative satisfies the special property:

$$\sum_{j=0}^{k} b_j \frac{u^{k+1-j}(x) - u^{k-j}(x)}{\tau} = \frac{1}{\tau} \left[ b_0 u^{k+1} - \sum_{j=0}^{k-1} (b_j - b_{j+1}) u^{k-j} - b_k u^0 \right],$$

and

$$b_k > 0, \quad b_k - b_{k+1} > 0, \quad \sum_{j=0}^{k-1} (b_j - b_{j+1}) + b_k = b_0, \quad \forall \; k \geq 0.$$

(3.4)

By this property, if one pair the scheme (3.2) with $u^{k+1}$, then all the cross terms $w_j u^{k+1}, j = 0, \ldots, k$ can be bounded by square terms $(w_j)^2 + (u^{k+1})^2$. Hence the $H^1$ stability can be proved by a simple mathematical induction [24].
The time-fractional AC equation. We now consider the time-fractional AC equation (2.9). We adopt the numerical method described above for the linear part of the (2.9) and use a stabilization technique for the nonlinear bulk force. This leads to the following semidiscretized scheme for (2.9):

\[
\frac{1}{\tau} \sum_{j=0}^{k} b_j \left( \phi^{k+1-j}(x) - \phi^{k-j}(x) \right) = \varepsilon \Delta \phi^{k+1} - \frac{1}{\gamma} f(\phi^k) - \frac{S}{\gamma} (\phi^{k+1} - \phi^k), \quad k \geq 0,
\]

where \( S \) is a sufficiently large positive constant, \( \tau = T/n \) is the time-step, and \( b_j, j = 0, 1, \ldots \) are defined by (3.3). We first present the following Lemma.

**Lemma 3.1.** Let \( \alpha \in (0, 1) \) and \( b_j, j = 0, 1, \ldots \) are defined by (3.3). Then for any \( (u_1, \ldots, u_n)^T \in \mathbb{R}^n \), we have

\[
B := 2 \sum_{k=1}^{n} \sum_{j=1}^{k} b_{k-j} u_j u_k = \sum_{k=1}^{n} b_0 u_k^2 + \sum_{k=1}^{n} \sum_{j=1}^{k} b_{k-j} u_j u_k \geq \sum_{k=1}^{n} s_n u_k^2,
\]

where \( s_n = \left( \frac{n+1}{2} \right)^{-\alpha} \frac{1}{\Gamma(1-\alpha)} \tau^{1-\alpha} > 0 \).

**Proof.** We prove this by converting \( B \) into the form of Lemma 2.1. First, convert \( \{u_j, j = 1, \ldots, n\} \) into a piece-wise constant function on \([0, T]\) as

\[
u^n(t) = \begin{cases} u_{[t/\tau]+1}, & 0 \leq t < T, \\ 0, & \text{otherwise,} \end{cases}
\]

where \([t]\) stands for the integer part of real number \( t \). Obviously, \( u^n(t) \in L^2(0, T) \).

Then by Lemma 2.1, we have

\[
0 \leq \frac{2}{\tau} A_\alpha (u^n, u^n) = \frac{1}{\tau} \frac{1}{\Gamma(1-\alpha)} \int_0^T \int_0^T \frac{u^n(s)u^n(t)}{|t-s|^\alpha} ds dt
\]

\[
= \frac{1}{\tau} \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} \int_{(k-1)\tau}^{k\tau} \frac{u_k}{|t-s|^\alpha} ds dt
\]

\[
= \frac{1}{\tau} \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} u_k \sum_{j=1}^{n} \int_{(j-1)\tau}^{j\tau} \frac{1}{|t-s|^\alpha} ds dt = \sum_{k=1}^{n} \sum_{j=1}^{n} u_j u_k \tilde{b}_{k-j},
\]

where

\[
\tilde{b}_{k-j} = \frac{1}{\Gamma(1-\alpha)} \int_0^{(k+1)\tau} \int_0^{t} \frac{1}{|t-s|^\alpha} ds dt
\]

\[
= \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} \left( (k+1)^{2-\alpha} - 2k^{2-\alpha} + (k-1)^{2-\alpha} \right), \quad k \geq 1,
\]

\[
\tilde{b}_0 = \frac{2}{\Gamma(2-\alpha)} \int_0^{\tau} t^{1-\alpha} dt = \frac{2}{\Gamma(3-\alpha)} \tau^{1-\alpha}.
\]

It is easy to see that \( b_{k-j} \) is an approximation of \( \tilde{b}_{k-j} \) by evaluating the integration using a one side quadrature rule. To prove \( B \) is positive definite, we need to prove
that the difference between $B$ and $2A_n/\tau$ in the off-diagonal parts can be controlled by the difference in the diagonal part. We now verify this. For the diagonal term

$$2b_0 - \tilde{b}_0 = \frac{2\tau^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{2\tau^{1-\alpha}}{\Gamma(3-\alpha)} = 2b_0(1 - \frac{1}{2 - \alpha}) = \frac{2}{2 - \alpha} \frac{1}{\Gamma(1 - \alpha)} \tau^{1-\alpha} \geq 0.$$ 

For the off-diagonal term, denoting $F(x) = \frac{1}{2-\alpha}(x+1)^{2-\alpha} - \frac{1}{2-\alpha}x^{2-\alpha}$, we have

$$\tilde{b}_{|k|} - b_{|k|} = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \frac{(k+1)^{2-\alpha} - 2k^{2-\alpha} + (k-1)^{2-\alpha}}{2 - \alpha} - \left[ (k+1)^{1-\alpha} - k^{1-\alpha} \right]$$

$$= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} (F(k) - F(k-1) - F'(k))$$

$$= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \frac{1}{2} (1 - \alpha) \left( x^{-\alpha} - (x+1)^{-\alpha} \right), \quad \text{for some } x \in [k-1,k] \geq 0, \quad \forall \ k \geq 1.$$ 

Then,

$$\sum_{k=1}^{n} |\tilde{b}_{|k|} - b_{|k|}| = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \frac{(n+1)^{2-\alpha} - n^{2-\alpha} - 1}{2 - \alpha} - \left[ (n+1)^{1-\alpha} - 1 \right]$$

$$\leq \frac{\tau^{1-\alpha}}{\Gamma(1-\alpha)} \left( \frac{1}{2 - \alpha} - \frac{1}{2} (n+1)^{-\alpha} \right).$$

So the column sum of the off-diagonals are bounded by

$$c_0 := \frac{\tau^{1-\alpha}}{\Gamma(1-\alpha)} \left( \frac{2}{2 - \alpha} - \frac{(n+1)^{-\alpha}}{2} \right).$$

Hence $C = \{c_{k-j}\}_{k,j=1}^{n}$ is a symmetric $M$-matrix and positive definite, where $c_k = b_{|k|} - \tilde{b}_{|k|}$ for $k = \pm 1, \ldots, \pm n$. We have

$$B = \frac{2}{\tau} A_\alpha(u^n, u^n) + \sum_{k=1}^{n} \left[ 2b_0 - \tilde{b}_0 - c_0 \right] u_k^2 + \sum_{k=1}^{n} \sum_{j=1}^{n} c_{k-j} u_k u_j$$

$$\geq \frac{2}{\tau} A_\alpha(u^n, u^n) + \sum_{k=1}^{n} \left[ 2b_0 - \tilde{b}_0 - c_0 \right] u_k^2$$

$$= \frac{2}{\tau} A_\alpha(u^n, u^n) + s_n \sum_{k=1}^{n} u_k^2.$$ 

The proof is completed. □

We are now ready to give the following theorem, which shows that our numerical scheme is energy stable.

**Theorem 3.1.** The scheme (3.5) satisfies a discrete energy dissipation law as below

$$E[\phi^n] - E[\phi^0] \leq - \frac{s_0}{2\gamma A} \sum_{k=0}^{n-2} \| \delta_k \phi^{k+1} \|^2$$

$$- \sum_{k=0}^{n-1} \left\{ \frac{1}{\gamma^2} \| \nabla \delta_k \phi^{k+1} \|^2 + \left\langle \frac{S}{\gamma} - \frac{1}{2\gamma} F'(\phi^k), (\delta_k \phi^{k+1})^2 \right\rangle \right\}, \quad (3.6)$$
where $\delta_t \phi^{k+1} := \phi^{k+1} - \phi^k$. Define $L := \max f'(x)$. If $S \geq \gamma L/2\epsilon$, then the scheme is unconditional energy stable in the sense that

$$E[\phi^n] - E[\phi^0] \leq 0, \quad \forall \ n > 0.$$ 

**Proof.** Multiplying both sides of (3.5) by $\delta_t \phi^{k+1}$, and integrating in space, the resulting right-hand side is given by

$$\text{RHS} = -\frac{\epsilon}{2} \| \nabla \phi^{k+1} \|^2 - \frac{\epsilon}{2} \| \nabla \phi^k \|^2 + \frac{\epsilon}{2} \| \nabla \delta_t \phi^{k+1} \|^2 - \left( \frac{1}{\epsilon} f(\phi^k) \right) \delta_t \phi^{k+1} + \left( \frac{S}{\gamma} \right) (\delta_t \phi^{k+1})^2, 1 \right).$$

where $\xi^k(x)$ is between $\phi^k(x)$ and $\phi^{k+1}(x)$. On the other hand, the resulting left-hand side is given by

$$\text{LHS} = \frac{1}{\gamma t} \int_{\Omega} \sum_{j=0}^{k} b_j (\delta_t \phi^{k+1-j} \delta_t \phi^{k+1}) \, dx.$$ 

Summing up both sides for $k = 0, \ldots, n - 1$, we get

$$E[\phi^n] - E[\phi^0] = \frac{1}{\gamma t} \int_{\Omega} \sum_{j=0}^{k} \sum_{i=0}^{n-1} b_j (\delta_t \phi^{k+1-j} \delta_t \phi^{k+1}) \, dx$$

$$- \frac{1}{\gamma t} \sum_{k=0}^{n-1} \left\{ \frac{\epsilon}{2} \| \nabla \delta_t \phi^{k+1} \|^2 + \left( \frac{S}{\gamma} \right) (\delta_t \phi^{k+1})^2 \right\}.$$ 

We get the desired energy estimate (3.6) by using Lemma 3.1.

**Remark 3.1.** For the commonly used double-well potential $F$ defined in (1.2), $F''(\phi)$ does not have a upper bound. But it is a common practice to modify $F'(\phi)$ for $\phi$ larger than a given threshold to make $F''(\phi)$ have a upper bound [36, 9, 41, 42], such that $L$ is a finite constant and Theorem 3.1 makes sense. To get a smooth double-well potential with quadratic growth, one may introduce $\tilde{F}(\phi)$ as

$$\tilde{F}(\phi) = \begin{cases} \frac{1}{4} (\phi - 2)^2 + 6(\phi - 2) + \frac{9}{4}, & \phi > 2, \\ \frac{1}{4} (\phi^2 - 1)^2, & \phi \in [-2, 2], \\ \frac{1}{4} (\phi + 2)^2 + 6(\phi + 2) + \frac{9}{4}, & \phi < -2. \end{cases}$$

(3.7)

to replace $F'(\phi)$. This modification is acceptable since physically the order parameter should be in the region $[-1, 1]$. Mathematically, Theorem 2.2 says the time-fractional AC equation satisfies a maximum principle in the sense that if the initial value $|\phi(x, 0)| \leq 1$ for all $x \in \Omega$, then $|\phi(x, t)| \leq 1$ for all $x \in \Omega$, and $t > 0$. So by replacing $F$ with $\tilde{F}$, the exact solution would not changed if $\max_{x \in \Omega} |\phi(x, 0)| \leq 1$. Numerical experiments show that the scheme for the time-fractional AC equation works very well even no truncation is made to the usual double-well potential.

**Remark 3.2.** One can use convex-splitting [13, 14] methods instead of the stabilization method to get a unconditional stale scheme. Take the double well potential (1.2) as example, let

$$f_e(\phi) = \phi^3, \quad f_c(\phi) = \phi,$$ 

(3.8)
The corresponding convex splitting scheme for the time-fractional AC equation (2.9) is given as

\[
\frac{1}{\gamma} \sum_{j=0}^{k} b_j \frac{\phi^{k+1-j}(x) - \phi^{k-j}(x)}{\tau} = \varepsilon \Delta \phi^{k+1} - \frac{1}{\varepsilon} f_\iota(\phi^{k+1}) + \frac{1}{\varepsilon} f_\varepsilon(\phi^k), \quad k \geq 0, \quad (3.9)
\]

The scheme is unconditionally stable but at each time step one needs to solve a nonlinear equation.

**Remark 3.3.** It is known that the order of accuracy for the time fractional derivative is \(2 - \alpha\) if the underlying solution is sufficiently smooth. The semi-implicit treatment of the right hand side is first-order accurate. Consequently the entire scheme is first-order accurate. One can get high order schemes by using high order finite difference scheme for the time-fractional derivative and high order semi-implicit treatment for the right hand side as did in [44, 36, 40]. But the proof of the unconditional stability of second order linear stabilization schemes is much involved. Directly applying the stability analysis of the second order stabilization schemes proposed in [41, 42] will result in a stabilization constant that depends on the fractional order \(\alpha\).

**The time-fractional CH equation.** The above scheme can be easily extended to the time-fractional CH equation (2.15):

\[
\frac{1}{\gamma} \sum_{j=0}^{k} b_j \frac{\phi^{k+1-j}(x) - \phi^{k-j}(x)}{\tau} = \Delta \mu^{k+1}, \quad (3.10)
\]

\[
\mu^{n+1} = -\varepsilon \Delta \phi^{k+1} + \frac{1}{\varepsilon} f(\phi^k) + \frac{S}{\gamma} (\phi^{k+1} - \phi^k). \quad (3.11)
\]

Similar to the time-fractional AC equation case, one can prove the following energy dissipation property.

**Theorem 3.2.** The scheme (3.10)-(3.11) satisfies the following discrete energy dissipation law

\[
E[\phi^n] - E[\phi^0] \leq - \frac{S}{2\gamma} \sum_{j=0}^{n-1} \|\delta_t \phi^k\|^2_{L^1} \leq - \sum_{k=0}^{n-1} \left\{ \frac{\varepsilon}{2} \|\nabla \delta_t \phi^{k+1}\|^2 + \left( \frac{S}{\gamma} - \frac{1}{2\varepsilon} f'(\xi), (\delta_t \phi^{k+1})^2 \right) \right\}. \quad (3.12)
\]

If \(S \geq \frac{\gamma}{2\varepsilon} \max f'(x)\), then the scheme is unconditional energy stable in the sense that

\[
E[\phi^n] \leq E[\phi^0], \quad \forall \ n > 0.
\]

**Remark 3.4.** The CH equation does not satisfy a maximum principle, so usually one need to truncate \(F\) to have bounded \(F''\). The stabilization approach can be replaced with convex-splitting approach if one does not want to truncate \(F\), which will also lead to unconditional stable schemes. The main difference is, after time semi-discretization, one obtain a linear well-posed system in the stabilized approach, while a nonlinear system is obtained in the convex-splitting approach. The convex-splitting
scheme for the time-fractional CH equation with double well potential is given below:

\[
\frac{1}{\gamma} \sum_{j=0}^{k} b_j \frac{\phi^{k+1-j}(x) - \phi^{k-j}(x)}{\tau} = \Delta \mu^{k+1}, \quad (3.13)
\]

\[
\mu^{n+1} = -\varepsilon \Delta \phi^{k+1} + \frac{1}{\varepsilon} f_i(\phi^{k+1}) - \frac{1}{\varepsilon} f_e(\phi^k). \quad (3.14)
\]

**The time-fractional MBE equation.** The scheme and analysis also apply to the time-fractional MBE model (2.20). The corresponding scheme is

\[
\frac{1}{\gamma} \sum_{j=0}^{k} b_j \frac{\phi^{k+1-j}(x) - \phi^{k-j}(x)}{\tau} = -\varepsilon \Delta^2 \phi^{k+1} + \frac{1}{\varepsilon} \nabla \cdot f(\nabla \phi^k) + \frac{S}{\gamma} (\Delta \phi^{k+1} - \Delta \phi^k), \quad k \geq 0. \quad (3.15)
\]

Similarly, one can prove the following energy dissipation property.

**Theorem 3.3.** Suppose the time-fractional MBE model (2.20) with periodic boundary condition or no-flux boundary condition (2.21). Then the scheme (3.15) satisfies a discrete energy dissipation law as below

\[
E_m[\phi^n] - E_m[\phi^0] \leq -\frac{S_0}{2\gamma \tau} \sum_{j=0}^{n-1} \|\delta_t \phi^k\|^2 - \frac{\varepsilon}{2} \sum_{k=0}^{n-1} \|\nabla^2 \delta_t \phi^{k+1}\|^2 - \frac{S}{\gamma} \sum_{k=0}^{n-1} \left( \frac{1}{2\varepsilon} f'(\xi_k), (\nabla \delta_t \phi^{k+1})^2 \right). \quad (3.16)
\]

If \( S \geq \frac{\gamma}{2\varepsilon} \max f'(x) \), then the scheme is unconditional energy stable for any time step size.

Notice that for the time-fractional MBE model without slope selection, we have

\[ f'(v) = \frac{v^2 - 1}{(1 + |v|^2)^2}. \]

The derivative has a finite upper bound 1/8. The above scheme is unconditionally stable if one choose \( S = \frac{\gamma}{2\varepsilon} \). For the time-fractional MBE model with slope selection, one need to modify \( F \) to have bounded \( F'' \), since the quantity \( |\nabla \phi| \) is not uniformly bounded in the evolution, even though it is in the final equilibrium state. The stabilization approach can be replaced with convex-splitting approach if one does not want to truncate \( F \), which will also lead to unconditionally stable scheme. The corresponding convex-splitting scheme for the MBE model with slope selection is given:

\[
\frac{1}{\gamma} \sum_{j=0}^{k} b_j \frac{\phi^{k+1-j}(x) - \phi^{k-j}(x)}{\tau} = -\varepsilon \Delta^2 \phi^{k+1} + \frac{1}{\varepsilon} \nabla \cdot f_i(\nabla \phi^{k+1}) - \frac{1}{\varepsilon} \nabla \cdot f_e(\nabla \phi^k), \quad k \geq 0. \quad (3.17)
\]

**4. Numerical results.** In this section, we present numerically examples to verify our theoretical findings, and furthermore, we shall use our numerical schemes to study the coarsening rates of the time-fractional phase-field models. We shall solve phase-field problems in a 2-dimensional tensor product domain \( \Omega = [0,L_x] \times [0,L_y] \) with periodic boundary condition. A Fourier-Galerkin method is used for the spatial discretization.
Fig. 4.1. The snapshots of the solution of the time-fractional AC equation for $\alpha = 1, 0.5, 0.3$ (top, middle, bottom row, respectively).

Fig. 4.2. The energy dissipation of time-fractional AC equation with different $\alpha$.

4.1. Numerical results of the time-fractional AC equation. For the time-fractional AC equation, we take $L_x = L_y = 2$, $\varepsilon = 0.05$, $\gamma = 0.05$. We use stabilized scheme (3.5) with $S = 0.1$. We take $128 \times 128$ Fourier modes, and set the time step
as $\tau = 0.1$. The initial state is given as

$$\phi_0(x) = \tanh \left( \frac{1}{\sqrt{2} \varepsilon} \left( r - \frac{1}{4} \frac{1 + \cos(4\theta)}{16} \right) \right), \quad r = \sqrt{x^2 + y^2}, \quad \theta = \arctan \frac{y}{x}. \quad (4.1)$$

We investigate how the phase field evolve and energy dissipate for different fractional order. The phase field $\phi$ for $\alpha = 0.3, 0.5, 1$ at different time $t$ are given in Fig. 4.1, from which it is observed that as $\alpha$ decreases the relaxation time to the equilibrium get longer and longer.

The above assertion can be further verified by looking at the energy dissipation curves shown in Fig. 4.2. We can also observe from this figure that the dissipation of the time-fractional AC equation has a long tail effect.

![Fig. 4.3. The snapshots of the solution of the time-fractional CH equation for $\alpha = 1, 0.5, 0.3$ (top, middle and bottom row respectively).](image)

4.2. Numerical results of the time-fractional CH equation. For the time-fractional CH equation, we take $L_x = L_y = 2, \varepsilon = 0.05, \gamma = \varepsilon^2$. We take an uniformly random distribution field in $[-1, 1]$ as the initial state. We use stabilized scheme (3.10)-(3.11) with $S = 0.01$, and use $256 \times 256$ Fourier modes for spatial discretization. The time step is taken as $\tau = 0.001$. As proved in last section, the scheme is unconditional stable if one choose the stabilization constant $S$ big enough. However, to get very accurate numerical results, one need to use sufficiently small time steps.

We investigate how the phase field evolve and energy dissipate for different fractional order. The phase field $\phi$ for different time $t$ and $\alpha = 0.3, 0.5, 1$ are given in Fig 4.3, from which we see again that as $\alpha$ decreases the relaxation time to the equilibrium becomes longer.
The above assertion can be further verified by looking at the energy dissipation curves shown in Figure 4.4(a).

The overall energy dissipation process can be split into three stages. In the first stage, bulk force is the driving force, small scale phase separation forms. In the second stage, small structures interact with each other, and the energy dissipation rate seems to satisfy a power-law rule 

$$ E[\phi(t)] \approx C_\alpha t^{p_\alpha}. $$

By a data fitting, we found that 

$$ p_\alpha \approx -\alpha/3 $$

(cf. Fig. 4.4(b)), which is consistent to the $-1/3$ dissipation law of the classical CH equation. In the last stage the solution converges to a minimum.

4.3. Numerical results of the time-fractional MBE model. For the time-fractional MBE model with slope selection, we take $L_x = L_y = 2\pi, \varepsilon = 0.1, \gamma = \varepsilon$. We take a $[-0.001, 0.001]$ uniformly random distribution field as the initial state. We use stabilized scheme (3.15) with $S = 0.1$, and take $256 \times 256$ Fourier modes for spatial discretization. The time step is taken as $\tau = 0.001$.

We investigate how the phase field evolve and energy dissipate for different fractional order. The phase field $\phi$ for $\alpha = 0.4, 0.7, 1$ at several time $t$ are given in Fig 4.5. It demonstrates again that as $\alpha$ decreases the relaxation time to the equilibrium get longer.

By looking at the energy dissipation curves shown in Figure 4.6(a), we found that the overall energy dissipation process for the time-fractional MBE model can also be split into three stages. Similarly, in the second stage, the energy dissipation rate seems satisfying a power-law rule. The data fitting results in Fig. 4.6(b) show that the power again behaves like $p_\alpha \approx -\alpha/3$.

5. Discussion and conclusion. In this work, we have addressed an open issue on energy dissipation related to the time-fractional phase field equations. We prove in the continuous level that the time-fractional phase field equations admit an energy dissipation law of integral type. Moreover, in the discrete level, we propose a class of finite difference schemes that can inherit the discrete energy dissipation law. Our discussion covers time-fractional gradient systems, including the time-fractional AC equation, the time-fractional CH equation, and the time-fractional MBE models.
Fig. 4.5. The snapshots of the solution of the time-fractional MBE model with slope selection for $\alpha = 1$, $\alpha = 0.7$, $\alpha = 0.4$ (top, middle and bottom row respectively).

(a) The energy dissipation

(b) The power-law scaling

Fig. 4.6. The energy dissipation and power-law scaling of time-fractional MBE model with slope selection for different $\alpha$.

Numerical schemes and analysis proposed in this paper can be extended to high order methods and adaptive marching step size.

This seems to be the first effort in analyzing the time-fractional phase field models.
There are, however, several relevant issues need to be followed up.

- As discussed in Remark 2.1, we have presented the energy law \( E[\phi(T)] \leq E[\phi(0)] \). A related question is whether the following energy dissipation law holds:

\[
\frac{d}{dt} E \leq 0 \quad \text{or} \quad \frac{d^\alpha}{dt^{\alpha}} E \leq 0 ?
\]

- We have shown numerically that there exists a \(-\alpha/3\) power law coarsening stage, yet a rigorous proof is still needed.

- Another topic is to get rid of the artificial parameters used in the numerical schemes, i.e., \( S \). One possible refinement in this direction is to use the so-called energy quadratization (see, e.g., [48, 47]) or scalar auxiliary variable (SAV) approach (see, e.g., [35]), which leads to energy-stable schemes without introducing the artificial terms. Moreover, only first order schemes are presented in this work. Thus, designing high order energy stable schemes will be part of our future studies.

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