On Signed Domination of Grid Graph

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Abstract

Let $G(V, E)$ be a finite connected simple graph with vertex set $V(G)$. A function $f: V(G) \rightarrow \{-1, 1\}$ is a signed dominating function if for every vertex $v \in V(G)$, the sum of closed neighborhood weights of $v$ is greater or equal to 1. The signed domination number $\gamma_s(G)$ of $G$ is the minimum weight of a signed dominating function on $G$. In this paper, we calculate the signed domination numbers of the Cartesian product of two paths $P_m$ and $P_n$ for $m = 6, 7$ and arbitrary $n$.

Keywords

Grid Graph, Cartesian Product, Signed Dominating Function, Signed Domination Number

1. Introduction

Let $G$ be a finite simple connected graph with vertex set $V(G)$. The neighborhood of $v$, denoted $N(v)$, is set $\{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood of $v$, denoted $N[v]$, is set $N(v) \cup \{v\}$. The function $f$ is a signed dominating function if for every vertex $v \in V$, the closed neighborhood of $v$ contains more vertices with function value 1 than with $-1$. The weight of $f$ is the sum of the values of $f$ at every vertex of $G$. The signed domination number of $G$, $\gamma_s(G)$, is the minimum weight of a signed dominating function on $G$.

In [1] [2] [3] [4], Dunbar et al. introduced this concept, in [5] Haas and Wexler had found the signed domination number of $P_2 \times P_n$ and $P_2 \times C_n$. In [6] Hosseini gave a lower and upper bound for the signed domination number for any graph. In [7] Hassan, Al Hassan and Mostafa had found the signed domination number of $P_m \times P_n$ for $m = 3, 4, 5$ and arbitrary $n$.

We consider when we represent the $P_m \times P_n$ graph. The weight of the black circle is 1, and the white circles refer to the graph vertices which weight $-1$.

Let $f$ be a signed dominating function of the $P_m \times P_n$ and $A = \{v \in V : f(v) = 1\}$,
2. Main Results

In this paper we will show tow theorem to find the signed domination number of Cartesian product of \( P_m \times P_n \).

**Theorem 2.1.** For \( n \geq 1 \) then

\[
\gamma_s(P_m \times P_n) = \begin{cases} 
2n; & \text{If } n \equiv 1 \pmod{5}, \\
2n + 2; & \text{If } n \equiv 2 \pmod{5}, \\
2n + 4; & \text{If } n \equiv 0, 3, 4 \pmod{5}.
\end{cases}
\]

Proof:

Let \( f \) be a signed dominating function of \( (P_m \times P_n) \), then for any \( j \) were \( 2 \leq j \leq n - 3 \), then \( \sum_{k=j-1}^{j+2} |B_k| \leq 8 \). We discuss the following cases:

Case a. \( |B| = 4 \):

we notice that the first and last columns can’t include four of the \( B \) set vertices, but in the case \( 2 \leq j \leq n - 3 \) and \( |B| = 4 \), then the vertices \((1, j), (3, j), (4, j), (6, j) \in B \), and all of the \( j - 1 \text{th}, j + 1 \text{th} \) column’s vertices don’t contain any one of the \( B \) set vertices, so the \((1, j + 2), (6, j + 2) \) vertices, then the \( j + 2 \text{th} \) column includes three of the \( B \) set vertices at most (Figure 1).

Case b. \( |B| = 3 \):

We discuss the following cases:

b-1. If \((1, j), (3, j), (4, j) \in B \) then both of the \( j - 1 \text{th}, j + 1 \text{th} \) columns include at most one of the \( B \) set vertices, then the \( j + 2 \text{th} \) column includes at most three of the \( B \) set vertices.

b-2. If \((1, j), (3, j), (5, j) \in B \) then the \( j - 1 \text{th} \) and \( j + 1 \text{th} \) columns include at most two of the \( B \) set vertices, and the \( j + 1 \text{th} \) column includes three of the \( B \) set vertices.

b-3. If \((1, j), (3, j), (6, j) \in B \) then both of the \( j - 1 \text{th}, j + 1 \text{th} \) columns include at most one of the \( B \) set vertices. And the \( j + 2 \text{th} \) column includes two of the \( B \) set vertices.

b-4. If \((1, j), (4, j), (5, j) \in B \) then only one of the \( j - 1 \text{th}, j + 1 \text{th} \) columns include at most one of the \( B \) set vertices, so \((1, j + 2) \in A \), then the \( j + 2 \text{th} \) column includes at most three of the \( B \) set vertices.

![Figure 1. Case a.](image)
b-5. If \((1, j), (4, j), (6, j) \in B\) then both of the \(j−1^\text{th}, j+1^\text{th}\) columns include at most one of the \(B\) set vertices. Also \((1, j+2), (4, j+2)\) and \((6, j+2) \in A\) then only two of the \(j+2^\text{th}\) vertices belong to \(B\) set.

b-6. If \((2, j), (3, j), (6, j) \in B\) then only one of the \(j−1^\text{th}, j+1^\text{th}\) column’s vertices belong to the \(B\) set vertices, then the \(j+2^\text{th}\) column include at most four of the \(B\) set vertices (Figure 2).

Case c. \(|B| = 2\):

We discuss the following cases:

c-1. If \((1, j), (3, j) \in B\) then all of the \(j−1^\text{th}, j+1^\text{th}, j+2^\text{th}\) columns include at most two of the \(B\) set vertices (Figure 3).

c-2. If \((1, j), (4, j) \in B\) and the \(j−1^\text{th}\) column include two of the \(B\) set vertices then the \(j+1^\text{th}\) column include at most one of the \(B\) set vertices, so the \(j+2^\text{th}\) column include at most three vertices (Figure 4).

c-3. If \((1, j), (5, j) \in B\) or \((1, j), (6, j) \in B\), then all of the \(j−1^\text{th}, j+1^\text{th}, j+2^\text{th}\) columns include at most two of the \(B\) set vertices (Figure 5).

c-4. If \((2, j), (3, j) \in B\) then if the \(j−1^\text{th}\) column includes two of the \(B\) set vertices, then the \(j+1^\text{th}\) column includes at most one of the \(B\) set vertices, so the \(j+2^\text{th}\) column includes at most three vertices (Figure 6).

Figure 2. Case b.

Figure 3. Case c-1.

Figure 4. Case c-2.
c-5. If \((2, j), (4, j) \in B\) then the \(j - 1^{\text{th}}\) column includes at most three of the \(B\) set vertices, it is \((2, j - 1), (4, j - 1), (6, j - 1) \in B\), so the \(j + 1^{\text{th}}\) column includes one of the \(B\) set vertices, also the \(j + 2^{\text{th}}\) column includes three of the \(B\) set vertices and both of the \(j - 2^{\text{th}}, j + 3^{\text{th}}\) columns don’t include any one of the \(B\) set vertices, so the \(j + 4^{\text{th}}\) column includes four of the \(B\) set vertices and the \(j - 3^{\text{th}}\) column includes three of the \(B\) set vertices. then the eight columns include sixteen of the \(B\) set vertices. In other cases stay 
\[ \sum_{k=-1}^{j+2} |B_k| \leq 8 \] (Figure 7).

c-6. If \((2, j), (5, j) \in B\) then all of the \(j - 1^{\text{th}}, j + 1^{\text{th}}, j + 2^{\text{th}}\) columns include at most two of the \(B\) set vertices (Figure 8).

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**Figure 5.** Case c-3.

**Figure 6.** Case c-4.

**Figure 7.** Case c-5.

**Figure 8.** Case c-6.
c-7. If \((3, j), (4, j) \in B\) then all of the \(j-1\text{th}, j+1\text{th}, j+2\text{th}\) columns include at most two of the \(B\) set vertices (Figure 9).

Case d. \(|B| = 1\):
We discuss the following cases:
\(d-1.\) If \((1, j) \in B\) or \((3, j) \in B\) or \((4, j) \in B\) or \((6, j) \in B\) then the \(j-1\text{th}\) column includes at most three of the \(B\) set vertices also both of the \(j+1\text{th}, j+2\text{th}\) columns include at most two of the \(B\) set vertices (Figure 10).

\(d-2.\) If \((2, j) \in B\) or \((5, j) \in B\) then both of the \(j-1\text{th}, j+1\text{th}\) columns includes at most three of the \(B\) set vertices, and the \(j+2\text{th}\) column includes at most one of the \(B\) set vertices (Figure 11).

From the previous cases we conclude \(\gamma_s(P_6 \times P_n) \geq 2n\).

To find the upper bound of the signed domination number of \((P_6 \times P_n)\) graph, let’s define (Figure 12).

\[
B = \{(1, 1 + 5i), (6, 1 + 5i) : 0 \leq i \leq \left\lfloor \frac{n-1}{5} \right\rfloor \}
\cup \{(3, 2 + 5i), (4, 2 + 5i) : 0 \leq i \leq \left\lfloor \frac{n+2}{5} \right\rfloor \}
\cup \{(2, 3 + 5i), (5, 3 + 5i) : 0 \leq i \leq \left\lfloor \frac{n+3}{5} \right\rfloor \}
\cup \{(2, 4 + 5i), (5, 4 + 5i) : 0 \leq i \leq \left\lfloor \frac{n+4}{5} \right\rfloor \}
\cup \{(3, 5 + 5i), (4, 5 + 5i) : 0 \leq i \leq \left\lfloor \frac{n+5}{5} \right\rfloor \}
\]

Figure 9. Case c-7.

Figure 10. Case d-1.

Figure 11. Case d-2.
Case $n \equiv 1 \pmod{5}$.

If $B$ is the previously defined set and represents the vertices have the weight $-1$, then every one of the $P_6 \times P_n$ vertices achieves the signed dominating function, and $|B| \geq 2n$, then: $\gamma_s(P_6 \times P_n) \leq 6n - 2(2n) = 2n$. Consequently:

$$\gamma_s(P_6 \times P_n) = 2n : n = 1 \pmod{5} \quad \text{(Figure 13)}.$$  

Case $n \equiv 2 \pmod{5}$.

In this case, we delete one of the two vertices $(3, n)$ or $(4, n)$ from the previously defined set $B$ vertices, then the signed domination number will increase by 2 than the signed domination number in case of $n \equiv 1 \pmod{5}$, and $f$ remains a signed dominating function of the graph. Consequently:

$$\gamma_s(P_6 \times P_n) = 2n + 2 : n = 2 \pmod{5} \quad \text{(Figure 14)}.$$  

Case $n \equiv 0, 3, 4 \pmod{5}$.

In this case we delete the $B$ set vertices in the last column, then the signed domination number will increase by 4 than signed domination number in case of $n \equiv 1 \pmod{5}$. And $f$ remains a signed dominating function of the graph.

Consequently: $\gamma(P_6 \times P_n) = 2n + 4 : n = 0, 3, 4 \pmod{5} \quad \text{(Figure 15)}.$

Lemma 2.1.

Let $f$ be a signed domination function of $(P_7 \times P_n)$, and $B$ the graph vertices set which having the weight $-1$, Then for any $j$ were $1 \leq j \leq n - 1$, then $\sum_{k=1}^{j} |B_k| \leq 5$.

Except the following cases:

$(3, j), (5, j) \in B$, $(1, j), (3, j), (5, j) \in B$, $(2, j), (4, j), (6, j) \in B$, $(3, j), (5, j), (7, j) \in B$, $(1, j), (3, j), (5, j), (7, j) \in B$.

Then $\sum_{k=1}^{j} |B_k| \leq 6$ and in this case $|B_{n-2}| + |B_{n-1}| \leq 5$.

Proof:

For any $j$ were $1 \leq j \leq n$ then $|B| \leq 4$.

Case a. $|B| = 4$:

The $j + 1$th column includes at most one of the $B$ set vertices, except case $(1, j), (3, j), (5, j), (7, j) \in B$, then the $j + 1$th column includes two of the $B$ set vertices (Figure 16).

Case b. $|B| = 3$:

The $j + 1$th column includes at most two vertices except in the following cases:

$(1, j), (3, j), (5, j) \in B$, $(2, j), (4, j), (6, j) \in B$, $(3, j), (5, j), (7, j) \in B$. Then $|B_{n-1}| = 3$ (Figure 17).

Case c. $|B| = 2$:

The $j + 1$th column includes at most three vertices, except in case $(3, j), (5, j) \in B$, then the $j + 1$th column includes four of the $B$ set vertices (Figure 18).
Figure 13. Case \( n \equiv 1 \pmod{5} \).

Figure 14. Case \( n \equiv 2 \pmod{5} \).

Figure 15. Case a.

Figure 16. Case a.

Figure 17. Case b.
Figure 18. Case c.

In case $|B| = 1$ or $|B| = 0$ it’s proofed easily because $|B_{j+1}| \leq 4$.

Lemma 2.2.

Let $f$ be a signed domination function of $(P_7 \times P_n)$ and $B$ the graph vertices set which having the weight $-1$, then $|B_1| + |B_2| + |B_3| \leq 6$. Except for a case $(2, 3)$, $(3, 3)$, $(6, 3) \in B$. Then $|B_1| + |B_2| + |B_3| \leq 7$. In this case $|B_4| = 1$.

Proof:

Case a. $|B_2| = 3$:

If $(1, 3), (3, 3), (5, 3) \in B$ or $(2, 3), (4, 3), (6, 3) \in B$ then the second column include three vertices of the $B$ set vertices, and the first column doesn’t include any one of the $B$ set vertices (Figure 19).

Case b. $|B_2| = 2$:

If $(1, 3), (3, 3), (7, 3) \in B$ or $(1, 3), (4, 3), (5, 3) \in B$ or $(1, 3), (4, 3), (6, 3) \in B$, then the second column include two vertices of the $B$ set vertices, and the first column doesn’t include any one of the $B$ set vertices.

If $(1, 3), (3, 3), (4, 3) \in B$ or $(5, 3), (6, 3) \in B$, then the second column include two vertices of the $B$ set vertices, and the first column include one of the $B$ set vertices.

If $(2, 3), (3, 3), (6, 3) \in B$, then the second column include two vertices of the $B$ set vertices, and the first column include two vertices of the $B$ set vertices. In this case the fourth column at most include one of the $B$ set vertices (Figure 20).

Case b. $|B_2| = 1$:

If $(1, 3), (4, 3), (7, 3) \in B$, then the second column include one of the $B$ set vertices, and the first column include one of the $B$ set vertices (Figure 21).

Remark 2.1. $|B_{n-2}| + |B_{n-1}| + |B_n| \leq 6$. Except for a case $(2, n-2), (3, n-2), (6, n-2) \in B$. Then $|B_{n-2}| + |B_{n-1}| + |B_n| \leq 7$. In this case $|B_{n-3}| = 1$, and prove as in the lemma (2.2.)

Theorem 2.2. Let $n$ be a positive integer

If $n \equiv 0, 2 \pmod{5}$, then $\gamma_s(P_7 \times P_n) = \frac{11n}{5} + 6$;

If $n \equiv 1, 3 \pmod{5}$, then $\gamma_s(P_7 \times P_n) = \frac{11n}{5} + 7$;
If $n \equiv 4 \pmod{5}$, then $\gamma_s(P_7 \times P_n) = \frac{11n}{5} + 8$.

Proof:
Case $n \equiv 0 \pmod{5}$.
Let $f$ be a signed domination function of the $P_7 \times P_n$. And $B$ the graph vertices set which having the weight $-1$. Then for any $j$ were $1 \leq j \leq n - 3$ then

$\sum_{i=j-1}^{j+3} |B_i| \leq 12$.

Case a. $|B| = 4$:
Then we discuss the following cases:

a-1. If $(2, j), (3, j), (5, j), (6, j) \in B$ then both of the $j - 1^{th}, j + 1^{th}$ columns don’t include any one of the B set vertices, so $|B_{j-1}| + |B_j| + |B_{j+1}| \leq 4$. And according to lemma1 then $|B_{j+2}| + |B_{j+3}| \leq 6$.

a-2. If $(1, j), (3, j), (4, j), (6, j) \in B$ or $(1, j), (3, j), (4, j), (7, j) \in B$ or $(1, j), (3, j), (5, j), (6, j) \in B$. Then one of the $j - 1^{th}$ or $j + 1^{th}$ column includes one of the B set vertices, as $|B_{j-1}| + |B_{j+1}| \leq 6$.

a-3. If $(1, j), (3, j), (5, j), (7, j) \in B$ then both of the $j - 1^{th}, j + 1^{th}$ columns include two of the B set vertices, as $|B_{j-2}| + |B_{j+2}| \leq 6$ (Figure 22).

Case b. $|B| = 3$:
We discuss the following cases:
b-1. If \((1, j), (4, j), (7, j) \in B\) then at most one of the \(j - 1^{th}\) columns vertices and also at most one of the \(j + 1^{th}\) vertices belongs to the \(B\) set vertices. Then the number of the vertices from the \(B\) set in the five successive columns remains less or equal to 12 (Figure 23).

b-2. If \((1, j), (3, j), (4, j) \in B\) or \((1, j), (4, j), (5, j) \in B\) or \((1, j), (5, j), (6, j) \in B\) or \((2, j), (3, j), (5, j) \in B\) or \((2, j), (3, j), (6, j) \in B\) or \((2, j), (4, j), (5, j) \in B\) then at most two of the \(j - 1^{th}\) columns vertices and also at most one of the \(j + 1^{th}\) vertices belongs to the \(B\) set vertices. Then the number of the vertices from the \(B\) set in the five successive columns remains less or equal to 12 (Figure 24).

b-3. If \((2, j), (4, j), (6, j) \in B\) then at most one of the two vertices \((2, j - 1), (2, j + 1)\) and one of the two vertices \((4, j - 1), (4, j + 1)\), And one of the two vertices \((6, j - 1), (6, j + 1)\) may be of the \(B\) set vertices. Then the number of the vertices from the \(B\) set in the five successive columns remains less or equal to 12 (Figure 25).

b-4. If \((1, j), (3, j), (5, j) \in B\) then the \(j - 1^{th}\) column includes at most three of the \(B\) set vertices. In case \(|B_{j-1}| = 3\). Then \((3, j - 1), (5, j - 1), (7, j - 1) \in B\) so \((6, j + 1), (6, j + 2) \in B\). Thus it remains in the \(j + 2^{nd}\) column three successive vertices include at most two of the \(B\) set vertices, so the \(j + 3^{rd}\) column includes at most two of the \(B\) set vertices (Figure 26).

b-5. If \((1, j), (3, j), (7, j) \in B\) then both of the \(j - 1^{th}, j + 1^{th}\) columns include at most two of the \(B\) set vertices.

b-5-1. If \((3, j - 1), (5, j - 1) \in B\) then \((4, j + 1), (5, j + 1) \in B\) and \((2, j + 2), (6, j + 2) \in B\) then three of the \(j + 3^{rd}\) column vertices belongs to the \(B\) set vertices.

b-5-2. If \((4, j - 1), (5, j - 1) \in B\) then \((3, j + 1), (5, j + 1) \in B\) and \((2, j + 2), (5, j + 2) \in B\) or \((2, j + 2), (6, j + 2) \in B\), then at most three of the \(j + 3^{rd}\) column vertices belong to the \(B\) set vertices (Figure 27).

b-6. If \((1, j), (3, j), (6, j) \in B\) then the \(j - 1^{th}\) column includes at most two of the \(B\) set vertices, in this case the \(j + 1^{th}\) column includes at most two of the \(B\) set vertices, and the \(j + 2^{nd}\) column includes at most three vertices and the \(j + 3^{rd}\) column includes at most two vertices of the \(B\) set vertices (Figure 28).

Case c. \(|B| = 2\):

c-1. If \((1, j), (4, j) \in B\) or \((1, j), (7, j) \in B\) then both of the \(j - 1^{th}, j + 1^{th}\) columns include at most two of the \(B\) set vertices, then the \(j - 1^{th}, j^{th}, j + 1^{th}\) columns
Figure 23. Case b-1.

Figure 24. Case b-2.

Figure 25. Case b-3.

Figure 26. Case b-4.

Figure 27. Case b-5.
include at most six of the $B$ set vertices, as any two columns include at most six vertices (Figure 29).

c-2. If $(1, j), (3, j) \in B$ then the $j − 1$th column includes at most three vertices, because one of the two vertices $(3, j − 1) \in B$ or $(4, j − 1) \in B$ and either the two vertices $(5, j − 1)$ and $(6, j − 1)$ or $(5, j − 1)$ and $(7, j − 1)$ belong to the $B$ set vertices.

c-2-1. If $(3, j − 1) \in B$ the $j + 1$th column includes at most three of the $B$ set vertices, in this case the $j + 2$th column includes at most one of the $B$ set vertices, and the $j + 3$th column includes at most three vertices. Or the $j + 2$th column includes two of the $B$ set vertices and the $j + 3$th column includes at most three vertices.

Figure 28. Case b-6.

c-2-2. If $(4, j − 1) \in B$ then the $j + 1$th column includes at most three of the $B$ set vertices, in this case $(3, j + 1), (5, j + 1), (6, j + 1) \in B$ and $(2, j + 2) \in B$, so $(2, j + 3), (4, j + 3), (5, j + 3), (7, j + 3) \in B$, then the $j − 2$th column includes at most one of the $B$ set vertices, then $\sum_{i=j-3}^{j+2} |B_i| \leq 12$. Also the $j + 4$th column doesn’t include any one of the $B$ set vertices, so $\sum_{i=j-1}^{j+4} |B_i| \leq 12$. And according to lemma 2-1 note $|B_{j+5}| + |B_{j+6}| \leq 6$, so $|B_{j+7}| \leq 6$. Then every ten successive columns include at most twenty four of the $B$ set vertices (Figure 30).

c-3. If $(1, j), (5, j) \in B$ then the $j − 1$th column includes at most three of the $B$ set vertices, so the $j + 1$th and $j + 2$th columns includes at most two of the $B$ set vertices, and the $j + 3$th column includes at most three vertices (Figure 31).

c-4. If $(1, j), (6, j) \in B$ then the $j − 1$th column includes at most three vertices, in this case the $j + 1$th column includes at most two of the $B$ set vertices, also the $j + 2$th column includes three of the $B$ set vertices, and the $j + 3$th column includes at most two vertices (Figure 32).

c-5. If $(2, j), (3, j) \in B$ then the $j − 1$th column includes at most three of the $B$ set vertices, then the $j + 1$th column includes two of the $B$ set vertices which are $(5, j + 1), (6, j + 1)$, also $(1, j + 2), (3, j + 2), (4, j + 2) \in B$, and the $j + 3$th column includes only one of the $B$ set vertices (Figure 33).

c-6. If $(2, j), (4, j) \in B$ then the $j − 1$th column includes at most three of the $B$ set vertices, so the $j + 1$th column includes at most two of the $B$ set vertices, in this case the $j + 2$th column includes at most three of the $B$ set vertices, and the $j + 3$th column includes at most two vertices (Figure 34).

c-7. If $(2, j), (5, j) \in B$ then the $j − 1$th column includes at most three of the $B$ set vertices, and the $j + 1$th column includes two of the $B$ set vertices, then the $j + 2$th, $j + 3$th columns include at most five of the $B$ set vertices (Figure 35).
c-8. If \((2, j), (6, j) \in B\) then both of the \(j−1\)th, \(j+1\)th columns include at most three of the \(B\) set vertices, so the \(j+2\)th column includes at most one of the \(B\) set vertices, and the \(j+3\)th column includes at most three vertices (Figure 36).
c-9. If \((3, j), (4, j)\) \(\in B\) then the \(j - 1\)th column includes at most three of the \(B\) set vertices, then the \(j + 1\)th column includes at most three of the \(B\) set vertices, then the \(j + 2\)th column includes only one of the \(B\) set vertices, and the \(j + 3\)th column includes at most three vertices (Figure 37).

c-10. If \((3, j), (5, j)\) \(\in B\) then the \(j - 1\)th column includes at most four of the \(B\) set vertices, so the \(j + 1\)th column includes at most two vertices, then the \(j + 2\)th column includes at most three of the \(B\) set vertices, and the \(j + 3\)th column includes at most one vertex (Figure 38).

Case d. \(|B| = 1|:

In this case the \(j + 1\)th, \(j + 2\)th columns include at most five of the \(B\) set vertices, so if the \(j + 3\)th, \(j + 4\)th columns include six of the \(B\) set vertices, then the number of the vertices in the five columns is less or equal to 12 (Figure 39).

We note from all the previous cases \(|B| \leq \frac{12n}{5}\). Then \(\gamma_s(P_7 \times P_n) \geq 7n - 2\left(\frac{12n}{5}\right) = \frac{11n}{5}\).

To find the upper bound of the signed domination number of \((P_7 \times P_n)\) graph, let’s define (Figure 40).
If $B$ the graph vertices set which having the weight $-1$, then every one of the $P_3 \times P_n$ graph vertices achieves the signed domination function and $|B| \geq \left\lceil \frac{12n}{5} \right\rceil$. 

\begin{align*}
B = & \left\{(4,5j),(6,5j) : 0 \leq j \leq \left\lceil \frac{n}{5} \right\rceil \right\} \\
\cup & \left\{(2,5j+1),(4,5j+1),(6,5j+1) : 0 \leq j \leq \left\lceil \frac{n-1}{5} \right\rceil \right\} \\
\cup & \left\{(2,5j+2),(5,5j+2),(7,5j+2) : 0 \leq j \leq \left\lceil \frac{n-2}{5} \right\rceil \right\} \\
\cup & \left\{(3,5j+3),(5,5j+3) : 0 \leq j \leq \left\lceil \frac{n-3}{5} \right\rceil \right\} \\
\cup & \left\{(1,5j+4),(3,5j+4),(6,5j+4) : 0 \leq j \leq \left\lceil \frac{n-4}{5} \right\rceil \right\}.
\end{align*}
According to lemma 2-2 we deleted the vertex (4, 1) from the previously defined set $B$ vertices in all cases, then $\gamma_s(P_r \times P_s) \geq \left\lceil \frac{11n}{5} \right\rceil + 2$.

Case $n \equiv 0, 2 \pmod{5}$.

According to lemma 2-2, then in case $n \equiv 0 \pmod{5}$, we delete the vertices (3, $n$), (6, $n$), so in case $n \equiv 2 \pmod{5}$, we delete the vertex (4, $n$). Then the signed domination number will increase by 4.

Consequently:

$$\gamma_s(P_r \times P_s) = \left\lceil \frac{11n}{5} \right\rceil + 2 + 4 = \left\lceil \frac{11n}{5} \right\rceil + 6 : n \equiv 0, 2 \pmod{5}.$$

(Figure 41).

Case $n \equiv 1, 3 \pmod{5}$.

When we add one column on case $n \equiv 0 \pmod{5}$, note that the number of vertices will increase by 7, and the number of set $B$ vertices will increase by 2, in this case

$$\gamma_s(P_r \times P_s) = \left\lceil \frac{11(n-1)}{5} \right\rceil + 2 + 7 = \left\lceil \frac{11n}{5} \right\rceil + 7 : n \equiv 1 \pmod{5}.$$

When we add three columns on case $n \equiv 0 \pmod{5}$, note that the number of vertices will increase by 21, and the number of set $B$ vertices will increase by 5, in this case

$$\gamma_s(P_r \times P_s) = \left\lceil \frac{11(n-3)}{5} \right\rceil + 2 + 21 - 2 \times 5 = \left\lceil \frac{11n}{5} \right\rceil + 7 : n \equiv 3 \pmod{5}.$$

Consequently: $\gamma_s(P_r \times P_s) \geq \left\lceil \frac{11n}{5} \right\rceil + 7 : n \equiv 1, 3 \pmod{5}$. (Figure 42)

Case $n \equiv 4 \pmod{5}$.

When we add four column on case $n \equiv 0 \pmod{5}$, note that the number of vertices will increase by 28, and the number of set $B$ vertices will increase by 9, in this case (Figure 43)

$$\gamma_s(P_r \times P_s) = \left\lceil \frac{11(n-4)}{5} \right\rceil + 2 + 28 - 2 \times 7 = \left\lceil \frac{11n}{5} \right\rceil + 8 : n \equiv 4 \pmod{5}.$$

Figure 41. Case $n \equiv 0, 2 \pmod{5}$.

Figure 42. Case $n \equiv 1, 3 \pmod{5}$. 
3. Conclusion

In this paper, we studied the signed domination numbers of the Cartesian product of two paths $P_m$ and $P_n$ for $m = 6, 7$ and arbitrary $n$. We will work to find the signed domination numbers of the Cartesian product of two paths $P_m$ and $P_n$ for arbitraries $m$ and $n$, and special graphs.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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