Abstract. In this paper we use the description of free group factors as the von Neumann algebras of Berezin’s deformation of the upper half-plane, modulo PSL (2, \(\mathbb{Z}\)).

The derivative, in the deformation parameter, of the product in the corresponding algebras, is a positive Hochschild 2-cocycle, defined on a dense subalgebra. By analyzing the structure of the cocycle we prove that there is a generator \(\mathcal{L}\) for a quantum dynamical semigroup that implements the cocycle on a strongly dense subalgebra.

For \(x\) in the dense subalgebra, \(\mathcal{L}(x)\) is the (diffusion) operator

\[
\mathcal{L}(x) = \Lambda(x) - \frac{1}{2} \{T, x\},
\]

where \(\Lambda\) is the pointwise (Schur) multiplication operator with a symbol function related to the logarithm of the automorphic form \(\Delta\). The operator \(T\) is positive and affiliated with the algebra \(\mathcal{A}_t\) and \(T\) corresponds to \(\Lambda(1)\), in a sense to be made precise in the paper. After a suitable normalization, corresponding to a principal-value type method adapted for \(\text{II}_1\) factors, \(\Lambda\) becomes (completely) positive on a union of weakly dense subalgebras. Moreover, the 2-cyclic cohomology cocycle associated to the deformation may be expressed in terms of \(\Lambda\).

INTRODUCTION

In this paper we analyze the structure of the positive Hochschild cocycle that determines Berezin’s deformation [Be] of the upper half-plane \(\mathbb{H}\), modulo PSL (2, \(\mathbb{Z}\)).

As described in [Ra], the algebras \(\mathcal{A}_{t, t>1}\) in the deformation are \(\text{II}_1\) factors (free group factors, by [Dy] and [Ra], based on [Vo] whose elements are (reproducing) kernels \(k\) that are functions on \(\mathbb{H} \times \mathbb{H}\), analytic in the second variable and antianalytic in the first variable, diagonally PSL (2, \(\mathbb{Z}\))-invariant and subject to boundedness conditions (see [Ra]).

The product \(k \star_t l\) of two such kernels is the convolution product

\[
(k \star_t l)(\overline{z}, \xi) = c_t \int_{\mathbb{H}} k(\overline{z}, \eta) l(\overline{\eta}, \xi) |\overline{z}, \eta, \overline{\eta}, \xi|^t \, d\nu_t(\eta), \quad z, \xi \in \mathbb{H}.
\]
Here \([\overline{z}, \eta, \overline{\eta}, \xi]\) is the cross ratio \(\frac{(\overline{z} - \xi)(\overline{\eta} - \eta)}{(\overline{z} - \eta)(\overline{\eta} - \xi)}\), while \(d\nu_t\) is the measure on the upper half-plane \(\mathbb{H}\) defined by \(d\nu_t = (\text{Im}\, \eta)^{t-2} d\overline{\eta} d\eta\), and \(c_t\) is a constant.

For \(k, l\) in a weakly dense subalgebra \(\hat{A}_t\), that will be constructed later in the paper, the following Hochschild 2-cocycle is well defined:

\[ C_t(k, l) = \text{the derivative at } t, \text{ from above, of } s \to k \ast_s l. \]

Clearly

\[ C_t(k, l) = \frac{c'_t}{c_t} (k \ast_t l) + c_t \int_{\mathbb{H}} k(z, \eta) l(\overline{\eta}, \xi) [z, \eta, \overline{\eta}, \xi]^t \ln[z, \eta, \overline{\eta}, \xi] d\nu_0(\eta). \]

In what follows we will prove that \(C_t\) is always a completely positive Hochschild 2-cocycle (for example in the sense introduced in [CoCu]). More precisely, for all \(k_1, k_2, \ldots, k_N\) in \(\hat{A}_{t_0}\), \(l_1, l_2, \ldots, l_N\) in \(A_t\), we have that

\[ \sum_{i,j} \tau_{A_t}(l_i^* C_t(k_i^*, k_j) l_j) \leq 0. \]

This also holds true for more general, discrete, subgroups of \(\text{PSL}(2, \mathbb{R})\).

In the case of \(\text{PSL}(2, \mathbb{Z})\), it turns out that \(C_t(k, l)\) behaves like the corresponding cocycle obtained from the generator of a quantum dynamical semigroup: that is, there exists a (necessary completely diffusive, i.e., completely conditionally negative) \(L\) such that

\[ C_t(k, l) = L_t(k \ast_t l) - k \ast_t L_t(l) - L_t(k) \ast_t l. \]

It turns out that \(L\) is defined on a unital, dense subalgebra \(D_t\) of \(A_t\), and that \(L(k)\) belongs to the algebra of unbounded operators affiliated with \(A_t\). Moreover, by a restricting to a smaller, dense, but not unital subalgebra \(D^0_t\), the completely positive part of \(L\) will take values in the predual \(L^1(A_t)\).

The construction of \(L_t\) is done by using automorphic forms. Let \(\Delta\) be the unique (normalized) automorphic form for \(\text{PSL}(2, \mathbb{Z})\) in order 12. Then \(\Delta\) is not vanishing in \(\mathbb{H}\), so that the following expression

\[ \ln \varphi(\overline{z}, \xi) = \ln \left( \overline{\Delta(z)\Delta(\xi)[(\overline{z} - \xi)/(-2i)]^{12}} \right) = \ln \Delta(z) + \ln \Delta(\xi) + 12 \ln[(\overline{z} - \xi)/(-2i)], \quad z, \xi \in \mathbb{H}, \]

is well defined, and diagonally \(\Gamma\)-invariant, for a suitable choice of the logarithmic function.
Let $\Lambda$ be the multiplication operator on $A_t$, corresponding to pointwise (Schur) multiplication of a symbol $k$ by $\ln \varphi$. Then $\Lambda$ is defined on a weakly dense subalgebra $D_t$ of $A_t$. If \{a, b\} denotes the Jordan product $\{a, b\} = ab + ba$, then

$$\mathcal{L}(k) = \Lambda(k) - \frac{1}{2}\{T, k\}$$

where $T$ is related to $\Lambda(1)$ in a sense made explicit in §5. Moreover, by adding a suitable constant times the identity operator to the linear map $-\Lambda$, we get a completely positive map, defined on a weakly dense subalgebra.

By analogy with Sauvageot’s construction [Sau], the Hochschild 2-cocycle $C_t$ corresponds to a construction of a cotangent bundle associated with the deformation. Moreover, there is a “real and imaginary part” of $C_t$. Heuristically, this is analogous to the decomposition of $d$, the exterior derivative, on a Kähler manifold, into $\delta$ and $\overline{\delta}$ (we owe this analogy to A. Connes).

The construction of the “real part” of $C_t$ is done as follows. One considers the “Dirichlet form” $\mathcal{E}_t$ associated to $C_t$, which is defined as follows:

$$\mathcal{E}_t(k, l) = \tau_{A_t}(C_t(k, l)),$$

defined for $k, l$ in a weakly dense, unital subalgebra $\hat{A}_t$. Out of this one constructs the operator $Y_t$ defined by

$$\langle Y_t(k), l \rangle_{L^2(A_t)} = \mathcal{E}_t(k, l), \quad k, l \in \hat{A}_t.$$

The imaginary part of $C_t$ is rather defined as 2-cyclic cohomology cocycle. The formula for this cyclic [Ra, Ra2] cocycle is:

$$\Psi_t(k, l, m) = \tau_{A_t}([C_t(k, l) - (\nabla Y_t)(k, l)]m), \quad k, l, m \in \hat{A}_t,$$

with

$$(\nabla Y_t)(k, l) = Y_t(k, l) - kY_t(l) - Y_t(k)l.$$  

This is a construction similar to one used in [CoCu].

Let $\chi$ be the antisymmetric form defined on $D_t^0$, a weakly dense subalgebra of $A_t$, by the formula

$$\chi_t(k, l) = \frac{1}{2}[(\Lambda k, l) - (k, \Lambda(l))].$$

Then there is a nonzero constant $\beta$, depending on $t$, such that

$$\Psi_t(k, l, m) + \beta\tau_{A_t}(klm) = \chi_t(kl, m) - \chi_t(k, lm) + \chi_t(mk, l),$$

for $k, l, m$ in $D_t^0$. 
We will show in the paper that $L^2(\mathcal{A}_t)$ can be identified with the Bargmann-type Hilbert space of diagonally $\Gamma$-invariant functions on $\mathbb{H} \times \mathbb{H}$, that are square-summable on $F \times \mathbb{H}$, analytic in the second variable and antianalytic in the first variable. Here $F$ is a fundamental domain for $\text{PSL}(2, \mathbb{Z})$ in $\mathbb{H}$, and on $F \times \mathbb{H}$ we consider the invariant measure

$$d(z, \eta)^{2t} \, d\nu_0(z) \, d\nu_0(w) = \left( \frac{(\text{Im} \, z)^{1/2} (\text{Im} \, \eta)^{1/2}}{||(z - \eta)/(2i)||} \right)^{2t} \, d\nu_0(z) \, d\nu_0(w).$$

With this identification, the “real part” of $C_t$ is implemented (on $\hat{\mathcal{A}}_t$) by the analytic Toeplitz operator on $L^2(\mathcal{A}_t)$ (compression of multiplication) of symbol $\ln d$. The “imaginary part” of $C_t$ is implemented (on the smaller algebra $\mathcal{D}_t^0$) by the Toeplitz operator, on $L^2(\mathcal{A}_t)$, of symbol $\frac{1}{12} \ln \varphi$.

The expression that we have obtained for $C_t(k, l) = \mathcal{L}_t(k *_tl) - k *_t \mathcal{L}_t(l) - \mathcal{L}_t(k *_tl)$, $\mathcal{L}(k) = \Lambda(k) - \frac{1}{2} \{T, k\}$, is in concordance with known results in quantum dynamics: Recall that in Christensen and Evans [CE], by improving a result due to Lindblad [Li] and [GKS], it is proved that for every uniformly norm-continuous semigroup $(\Phi_t)_{t \geq 0}$ of completely positive maps on a von Neumann algebra $\mathcal{A}$, the generator $\mathcal{L} = \frac{d}{dt} \Phi_t$ has the following form:

$$\mathcal{L}(x) = \Psi(x) - \frac{1}{2} \{\Psi(1), x\} + i[H, x],$$

where $\Psi: \mathcal{A} \to \mathcal{A}$ is a completely positive map and $H$ is a bounded selfadjoint operator.

For a semigroup of completely positive maps that is only strongly uniformly continuous, the generator has a similar form, although $\mathcal{L}(x)$, for $x$ in $\mathcal{A}$, is defined as a quadratic form affiliated to the von Neumann algebra $\mathcal{A}$.

Conversely, given $\mathcal{L}$, a minimal semigroup may be constructed under certain conditions (see, e.g., [CF, Ho, MS, GS, Dav]), although the semigroup might not be conservative (i.e., unital) even if $\mathcal{L}(1) = 0$.

If $\mathcal{L}(x) = \Lambda(x) + (G^* x + xG)$, let $\hat{\Lambda}_t x = e^{-tG^*} x e^{-tG}$. Then in the case of $\mathcal{A} = B(H)$, the corresponding semigroup $\Phi_t$, satisfying the master equation

$$\frac{d}{dt} \langle \Phi_t(x), \xi, n \rangle = \langle \mathcal{L}(\Phi_t(x)) \xi, \eta \rangle$$

for $\xi, \eta$ in a dense domain, is constructed by the Dyson expansion [Ho]

$$\Phi_t(x) = \hat{\Lambda}_t(x) + \sum_{n \geq 0} \int \cdots \int \hat{\Lambda}_{t_1} \circ \Lambda \circ \hat{\Lambda}_{t_2 - t_1} \circ \cdots \circ \Lambda \circ \hat{\Lambda}_{t - t_n} \, dt_1 \, dt_2 \cdots dt_n,$$
which is proved to be convergent [CF, MS].

It is not clear if a minimal conservative semigroup exists for the quantum dynamical generator $L_t$ constructed in our paper. The quantum dynamical generators $L_t$ constructed in this paper have the following formal property:

Assume that there exists a family of completely positive maps $(\Phi_{s,t})_{s \geq t}$, with $\Phi_{s,t}: A_t \to A_s$ satisfying the following variant of the master equation:

\[
\frac{d}{ds}(\Psi_{s,t}(\Phi_{s,t}(X))) \bigg|_{s=s_0} = L_{s_0}(\Psi_{s_0,t}(\Phi_{s_0,t}(X))).
\]  

(0.1)

Then $\Phi_{s,t}$ would satisfy the Chapman-Kolmogorov condition:

$\Phi_{s,t} \Phi_{s,u} = \Phi_{s,v}; \quad s \geq t \geq v, \quad \Phi_{s,s} = \text{Id}.$

Moreover,

\[
\frac{d}{ds}(\Phi_{s,t}(X) \ast_s \Phi_{s,t}(Y)) \bigg|_{s=s_0}
\]

would be

$\mathcal{L}_{s_0}(\Phi_{s_0,t}(X), \Phi_{s_0,t}(Y)) + \mathcal{L}_{s_0}(\Phi_{s_0,t}(X)) \ast s_0 \Phi_{s_0,t}(Y) + \Phi_{s_0,t}(X) \ast s_0 \mathcal{L}_{s_0}(\Phi_{s_0,t}(Y)),$

which by the cocycle property would be

$\mathcal{L}_{s_0}(\Phi_{s_0,t}(X \ast s_0 Y)).$

Thus $\frac{d}{ds}(\Phi_{s,t}(X) \ast_s \Phi_{s,t}(Y)) = \frac{d}{ds}\Phi_{s,t}(X \ast t Y)$. If unicity (conservativity) holds, it would follow that $\Psi_{s,t}\Phi_{s,t}(X)$ would be a (unital) multiplicative map from $A_t$ into $A_s$.

At present we do not know if this conservativity condition of the minimal solution and the subsequent considerations hold true.

ACKNOWLEDGEMENT. This work was initiated while the author was visiting the Erwin Schroedinger Institute in Wien. This work was completed while the author was visiting IHP and IHES, to which the author is grateful for the excellent conditions and warm reception. The author acknowledges enlightening discussions with L. Beznea, P. Biane, A. Connes, P. Jorgensen, R. Nest, J.L. Sauvageot, and L. Zsido.

DEFINITIONS

We recall first some notions associated with Berezin’s deformation [Be] of the upper half-plane that were proved in [Ra] (see also [Ra1]), in the $\Gamma$-equivariant context.
We consider the Hilbert space $H_t = H^2(\mathbb{H}, d\nu_t)$, $t > 1$, of square-summable analytic functions on the upper half-plane $\mathbb{H}$, with respect to the measure $d\nu_t = (\text{Im} \ z)^{t-2} \ d\overline{z} \ dz$. $d\nu_0$ is the PSL$(2, \mathbb{R})$-invariant measure on $\mathbb{H}$. This space occurs as the Hilbert space for the series of projective unitary irreducible representations $\pi_t$ of PSL$(2, \mathbb{R})$ on $H_t$, $t > 1$ [Sal, Puk].

Recall that $\pi_t(g)$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in PSL$(2, \mathbb{R})$, are defined by means of left translation (using the Möbius action of PSL$(2, \mathbb{R})$ on $\mathbb{H}$) by the formula

$$(\pi_t(g)f)(z) = f(g^{-1}z)(cz + d)^{-t}, \quad z \in \mathbb{H}, \ f \in H_t.$$ 

Here the factor $(cz + d)^{-t}$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined by using a preselected branch of $\ln(cz + d)$ on $\mathbb{H}$, which is always possible [Sal]. If $t = n$ is an integer $\geq 2$, then $\pi_t$ is actually a representation of PSL$(2, \mathbb{R})$, in the discrete series.

Let $\Gamma$ be a discrete subgroup of finite covolume in PSL$(2, \mathbb{R})$ and consider the von Neumann algebra $\mathcal{A}_t = \{\pi_t(\Gamma)\}' \subseteq B(H_t)$ consisting of all operators that commute with $\pi_t(\Gamma)$. By generalizing a result of [AS], [Co], [Co1], [GHJ], it was proved in [Ra] that $\{\pi_t(\Gamma)\}''$ (the enveloping von Neumann algebra of the image of $\Gamma$ through $\pi_t$) is isomorphic to $\mathcal{L}(\Gamma, \sigma_t)$, which is the enveloping von Neumann algebra of the image of the left regular cocycle representation of $\Gamma$ into $B(l^2(\Gamma)))$. Thus $\mathcal{L}(\Gamma, \sigma_t)$ is a $\Pi_1$ factor. Here $\sigma_t$ is the cocycle coming from the projective unitary representation $\pi_t$.

Therefore, $H_t$, as a left Hilbert module over $\{\pi_t(\Gamma)\}'' \simeq \mathcal{L}(\Gamma, \sigma_t)$, has Murray–von Neumann dimension (see, e.g., [GHJ]) equal to $((t - 1)/\pi) \text{covol}(\Gamma)$ (this generalizes to projective unitary representations, by the formula in [AS, Co, GH]). The precise formula is

$$\dim_{\mathcal{L}(\Gamma, \sigma_t)} H_t = \dim_{\{\pi_t(\Gamma)\}''} H_t = \frac{t-1}{\pi} \text{covol}(\Gamma).$$

Hence the commutant $\mathcal{A}_t$ is isomorphic to $\mathcal{L}(\Gamma, \sigma_t)_{((t-1)/\pi) \text{covol}(\Gamma)}$. We use the convention to denote by $M_t$, for a type $\Pi_1$ factor $M$, the isomorphism class of $eMe$, with $e$ an idempotent of trace $t$. If $t > 1$, then one has to replace $M$ by $M \otimes M_N(\mathbb{C})$ (see [MvN]).

When $\Gamma = \text{PSL}(2, \mathbb{Z})$, the class of the cocycle $\sigma_t$ vanishes (although not in the bounded cohomology, see [BG]). Consequently, since in this case [GHJ]

$$\frac{t-1}{\pi} \text{covol}(\Gamma) = \frac{t-1}{12},$$


It follows that when $\Gamma = \text{PSL}(2, \mathbb{Z})$ we have

$$A_t \simeq L(\text{PSL}(2, \mathbb{Z}))(t-1)/12.$$ 

We want to analyze the algebras $A_t$ by means of Berezin’s deformation of $H$. Recall that the Hilbert space $H_t$ has reproducing vectors $e^t_z$, $z \in \mathbb{H}$, that are defined by the condition $\langle f, e^t_z \rangle = f(z)$, for all $f$ in $H$. The precise formula is

$$e^t_z(\xi) = \langle e^t_\xi, e^t_z \rangle = \frac{c_t}{[\overline{z} - \xi]/(-2i)]^t}, \quad \xi \in \mathbb{H}, \ c_t = \frac{t-1}{4\pi}.$$ 

Each operator $A$ in $B(H_t)$ then has a reproducing kernel $\hat{A}(\xi, \eta)$. To obtain the Berezin symbol, one normalizes so that the symbol of $A = \text{Id}$ is the identical function 1.

Thus the Berezin symbol of $A$ is a bivariable function on $\mathbb{H} \times \mathbb{H}$, antianalytic in the first variable, analytic in the second, and given by

$$\hat{A}(\xi, \eta) = \frac{\langle Ae^t_z, e^t_\xi \rangle}{\langle e^t_z, e^t_\xi \rangle}, \quad \xi, \eta \in \mathbb{H}.$$ 

We have that $\langle Ae^t_z, e^t_\xi \rangle$ is a reproducing kernel for $A \in B(H_t)$, and hence the formula for the symbol $\widehat{AB}$ of the composition of two operators $A, B$ in $B(H_t)$ is computed as

$$\widehat{AB}(\xi, \eta)\langle e^t_z, e^t_\xi \rangle\langle AB e^t_z, e^t_\xi \rangle = \langle e^t_z, e^t_\xi \rangle \int_\mathbb{H} \langle Ae^t_z, e^t_\eta \rangle \langle Be^t_\eta, e^t_\xi \rangle \, d\nu_t(\eta).$$

**Definition 0.1.** By making explicit the kernels involved in the product, one obtains the following formula: Let $\hat{A}(\xi, \eta) = k(\xi, \eta)$, $\hat{B}(\xi, \eta) = l(\eta, \xi)$, and let $(k *_t l)(\xi, \eta)$ be the symbol of $AB$ in $H_t$. Then

$$(k *_t l)(\xi, \eta) = c_t \int_\mathbb{H} (k(\xi, \eta)) (l(\eta, \xi)) [\overline{\xi}, \eta, \overline{\eta}, \xi]^t \, d\nu_0(\eta) \quad (0.2)$$

with $[\overline{\xi}, \eta, \overline{\eta}, \xi] = ((\overline{\xi} - \eta)(\overline{\overline{\eta}} - \eta))/((\overline{\xi} - \eta)(\overline{\overline{\eta}} - \xi))$.

Here one uses the choice of the branch of $\ln(\overline{\xi} - \xi) \in [-\pi, \pi]$ that appears in the definition of $e^t_z$ (see [Sal]).

The above definition can be extended, when the integrals are convergent, to an (associative) operation on the space of bivariable kernels, by the formula (0.2). One problem that remains open is to determine when a given bivariable function represents a bounded operator on $H_t$. 

Let \( d(\zeta, \eta) = \left(\frac{\text{Im} z}{\text{Im} \eta}\right)^{1/2} / \left|\frac{\zeta - \eta}{-2i}\right| \) for \( z, \eta \) in \( \mathbb{H} \). Then \( d(\zeta, \eta)^2 \) is the hyperbolic cosine of the hyperbolic distance between \( z, \eta \) in \( \mathbb{H} \). The following criterion was proven in [Ra].

**Criterion 0.2.** Let \( h \) be a bivariable function on \( \mathbb{H} \times \mathbb{H} \), antianalytic in the first variable, and analytic in the second variable. Consider the following norm: \( \|h\|_t^{\hat{}} \) is the maximum of the two quantities

\[
\sup_{z \in \mathbb{H}} \int |h(\zeta, \eta)| (d(\zeta, \eta))^t \, d\nu_0(\eta),
\]

\[
\sup_{\eta \in \mathbb{H}} \int |h(\zeta, \eta)| (d(\zeta, \eta))^t \, d\nu_0(\zeta).
\]

Then \( \|h\|_t^{\hat{}} \) is a norm on \( B(H_t) \), finer than the uniform norm, and the vector space of all elements in \( B(H_t) \) whose kernels have finite \( \| \cdot \|_t^{\hat{}} \) norm is an involutive, weakly dense, unital, normal subalgebra of \( B(H_t) \). We denote this algebra by \( \hat{B}(H_t) \).

In [Ra] we proved a much more precise statement about the algebra \( \hat{B}(H_t) \):

**Proposition 0.3 [Ra].** The algebra of symbols corresponding to \( \hat{B}(H_t) \) is closed under all the product operations \( *_s \), for \( s \geq t \). In particular \( \hat{B}(H_t) \) embeds continuously into \( \hat{B}(H_t) \) and its image is closed under the product in \( \hat{B}(H_t) \).

Since this statement will play an essential role in proving that the domains of some linear maps in our paper, are algebras, we’ll briefly recall the proof of this proposition:

Assume that \( k, l \) are kernels such that \( \|k\|_t^{\hat{}} \|l\|_t^{\hat{}} < \infty \). Consider the product of \( k, l \) in \( \mathcal{A}_s \). We are estimating

\[
\int |(k *_s l)(\zeta, \xi)| |d(\zeta, \xi)|^t \, d\nu_0(\xi).
\]

This should be uniformly bounded in \( z \).

The integrals are bounded by

\[
\int |k(\zeta, \eta)| |l(\eta, \xi)| |d(\zeta, \eta)|^t \, d\nu_0(\eta) \, d\nu_0(\xi).
\]

Since obviously

\[
|\zeta, \eta, \xi|^s = \left[\frac{d(\zeta, \eta)d(\eta, \xi)}{d(\zeta, \xi)}\right]^s,
\]

the integral is bounded by

\[
\int |k(\zeta, \eta)| |d(\zeta, \eta)|^t |l(\eta, \xi)| |d(\eta, \xi)|^t \cdot M(z, \eta, \xi) \, d\nu_0(\eta, \xi).
\]
If we can show that \( M(z, \eta, \xi) \) is a bounded function on \( \mathbb{H} \times \mathbb{H} \times \mathbb{H} \), then the last integral will be bounded by \( \|M\|_\infty \|k\|_t \|l\|_t \).

But it is easy to see that

\[
M(z, \eta, \xi) = \left| \frac{d(\overline{\tau}, \eta)d(\overline{\eta}, \xi)}{d(\overline{\tau}, \xi)} \right|^{s-t} = |\overline{\tau}, \eta, \xi|^{s-t}.
\]

This is a diagonally \( \text{PSL}(2, \mathbb{R}) \)-invariant function on \( \mathbb{H} \times \mathbb{H} \times \mathbb{H} \). Since \( d(z, \eta) \) is an intrinsic notion of the geometry on \( \mathbb{H} \) we can replace \( \mathbb{H} \) by \( \mathbb{D} \), the unit disk. Then the expression of \( d(z', \xi') \) becomes:

\[
(1 - |z'|^2)^{1/2}(1 - |\xi'|^2)^{1/2} \left| \frac{1}{1 - z'\xi'} \right|^{s-t} \left| \frac{1}{1 - z'\xi'} \right| \leq 2
\]

since \( t > 1 \). This completes the proof of Proposition 0.3. \( \square \)

In [Ra] we proved that there is a natural symbol map \( \Psi_{s,t}: B(H_t) \to B(H_s) \) defined as follows:

**Definition 0.4.** Let \( \Psi_{s,t}: B(H_t) \to B(H_s) \) be the map that assigns to every operator \( A \) in \( B(H_t) \) of Berezin symbol \( \hat{A}(\overline{\tau}, \eta) \), \( \overline{\tau}, \eta \in \mathbb{H} \), the operator \( \Psi_{s,t}(A) \) on \( B(H_s) \) whose Berezin symbol (as operator on \( H_s \)) coincides with the symbol of \( A \). Then \( \Psi_{s,t} \) is continuous on \( B(H_s) \).

A proof of this will be given in Section 1 and we will in fact prove even more, that is, that \( \Psi_{s,t} \) is a completely positive map.

Obviously one has

\[
\Psi_{s,t} \Psi_{s,v} = \Psi_{s,v} \quad \text{for} \quad s \geq t \geq v > 1,
\]

\[
\Psi_{s,s} = \text{Id} \quad \text{for} \quad s > 1.
\]

Assume \( k, l \) represent two symbols of bounded operators in \( B(H_t) \). Then the product \( k \ast_s l \) makes sense for all \( s \geq t \). The following definition of differentiation of the product structure then appears naturally. In this way we get a canonical Hochschild 2-cocycle associated with the deformation.
**Definition-Proposition 0.5** [Ra]. Fix $1 < t_0 < t$. Let $k, l$ be operators in $B(\widehat{H}_{t_0})$. Consider $k \ast_s l$ for $s \geq t$, and differentiate pointwise the symbol of this expression at $s = t$. Denote the corresponding kernel by $C_t(k, l) = k \ast'_t l$. Then $C_t(k, l)$ corresponds to a bounded operator in $B(H_t)$. Moreover, $C_t(k, l)$ has the following expression:

$$C_t(k, l) = \frac{d}{ds}(k \ast_s l) \bigg|_{s=t},$$

$$C_t(k, l)(\zeta, \xi) = \frac{c'_t}{c_t}(k \ast_s l)(\zeta, \xi)$$

$$+ c_t \int_{\mathbb{H}} k(\zeta, \eta)l(\overline{\eta}, \xi)[\zeta, \eta, \overline{\eta}, \xi]^t \ln[\zeta, \eta, \overline{\eta}, \xi] \, d\nu_0(\eta).$$

Moreover, by differentiation of the associativity property, it follows that $C_t(k, l)$ defines a Hochschild two-cocycle on the weakly dense subalgebra $\hat{B}(H_{t_0})$ (viewed as a subalgebra of $B(H_t)$ through the symbol map).

We now specialize this construction for operators $A \in A_t = \{\pi_t(\Gamma)\}'$, that is, operators that commute with the image of $\Gamma$ in $B(H_t)$. We have the following lemma, which was proved in [Ra].

**Lemma 0.6** [Ra]. Let $\Gamma$ be a discrete subgroup of finite covolume in $\text{PSL}(2, \mathbb{R})$. Assume $F$ is a fundamental domain of $\Gamma$ in $\mathbb{H}$ (of finite area $\nu_0(F)$ with respect to the $\text{PSL}(2, \mathbb{R})$-invariant measure $d\nu_0$ on $\mathbb{H}$).

Let $A_t = \{\pi_t(\Gamma)\}'$, which is a type II_1 factor with trace $\tau$. Then

1) Any operator $A$ in $A_t$ has a diagonally $\Gamma$-equivariant kernel $k = k_A(\zeta, \xi)$, $z, \xi \in \mathbb{H}$ (that is, $k(\zeta, \xi) = k(\overline{\zeta}, \gamma \xi)$, $\gamma \in \Gamma$, $z, \xi \in \mathbb{H}$).

2) The trace $\tau_A(k)$ is computed by

$$\frac{1}{\nu_0(F)} \int_F k(\zeta, z) \, d\nu_0(z).$$

3) More generally, let $P_t$ be the projection from $L^2(\mathbb{H}, d\nu_t)$ onto $H_t$. Let $f$ be a bounded measurable function on $\mathbb{H}$ that is $\Gamma$-equivariant and let $M_f$ be the multiplication operator on $L^2(\mathbb{H}, d\nu_t)$ by $f$. Let $T_f^t = P_t M_f P_t$ be the Toeplitz operator on $H_t$ with symbol $M_f$.

Then $T_f^t$ belongs to $A_t$ and

$$\tau(T_f^t A) = \frac{1}{\nu_0(F)} \int_F k_A(\zeta, z)f(z) \, d\nu_0(z).$$
4) $L^2(\mathcal{A}_t)$ is identified with the space of all bivariable functions $k$ on $\mathbb{H} \times \mathbb{H}$ that are analytic in the second variable, antianalytic in the first variable, and diagonally $\Gamma$-invariant. The norm of such an element $k$ is given by the formula

$$
\|k\|_{2,t} = \frac{1}{\text{area}(F)} \int \int_{F \times \mathbb{H}} |k(z, \eta)|^2 d(z, \eta)^{2t} d\nu_0(z) d\nu_0(\eta).
$$

We also note that the algebras $\widehat{B(H_t)}$, and the map $\Psi_{s,t}$, $s \geq t$, have obvious counterparts for $\mathcal{A}_t$. Obviously $\Psi_{s,t}$ maps $\mathcal{A}_t$ into $\mathcal{A}_s$ for $s \geq t$.

**Definition 0.7** [Ra]. Let $\mathcal{A}_t = \widehat{B(H_t)} \cap \mathcal{A}_t$. Then $\widehat{\mathcal{A}_t}$ is a weakly dense involutive, unital subalgebra of $\mathcal{A}_t$.

Moreover, $\widehat{\mathcal{A}_t}$ is closed under any of the operations $*_{s}$, for $s \geq t$. This means that $\Psi_{s,t}(k)\Psi_{s,t}(l) \in \Psi_{s,t}(\widehat{\mathcal{A}_t})$ for all $k, l$ in $\widehat{\mathcal{A}_t}$, $s \geq t$.

More generally, $\mathcal{A}_s$ is contained in $\widehat{\mathcal{A}_t}$ if $s < t - 2$, and $\widehat{\mathcal{A}_r}$ is weakly dense in $\mathcal{A}_t$ if $r \leq t$ (and hence $\widehat{\mathcal{A}_r}$ is weakly dense in $\mathcal{A}_t$ if $r \leq t$) [Ra, Proposition 4.6].

We also note that, as a consequence of the previous lemma, we can define for $1 < t_0 < t$

$$
C_t(k, l) = \left. \frac{d}{ds}(k *_{s} l) \right|_{s=t} \text{ for } k, l \text{ in } \widehat{\mathcal{A}_{t_0}}
$$

and we have the expression (0.2) of the kernel.

Another way to define $C_t(k, l)$ is to fix vectors $\xi, \eta$ in $H_t$ and to consider the derivative

$$
\frac{d}{ds} \langle (k *_{s} l)\xi, \eta \rangle_{H_t} = \langle C_t(k, l)\xi, \eta \rangle_{H_t}, \quad \xi, \eta \in H_t.
$$

For $k, l$ in $\widehat{\mathcal{A}_{t_0}}$, $t_0 < t$, this makes sense because $k *_{s} l$ is already the kernel of an operator in $\widehat{\mathcal{A}_{t_0}}$. 

§0. Outline of the paper

The paper is organized as follows:

In Section 1, we show, based on the facts proved in [Ra], that the symbol maps \( \Psi_{s,t}: \mathcal{A}_t \to \mathcal{A}_s \), for \( s \geq t \), are completely positive, unital and trace preserving. Consequently the derivative of the multiplication operation (keeping the symbols fixed) is a positive Hochschild 2-cocycle (see [CoCu]). In particular the trace of this Hochschild cocycle is a (noncommutative) Dirichlet form (see [Sau]).

In Section 2 we analyze positivity properties for families of symbols induced by intertwining operators. As in [GHJ], let \( S_{\Delta^\varepsilon} \) be the multiplication operator by \( \Delta^\varepsilon \), viewed as an operator from \( H_t \) into \( H_{t+12\varepsilon} \). Then \( S_{\Delta^\varepsilon} \) is an intertwiner between \( \pi_t|\Gamma \) and \( \pi_{t+12\varepsilon}|\Gamma \), with \( \Gamma = \text{PSL}(2,\mathbb{Z}) \). Here we use the following branch for \( \ln(cz+d) = \ln(j(\gamma, z)) \), which appears in the definition of \( \pi_t(\gamma), \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{PSL}(2,\mathbb{Z}) \). We define \( \ln(j(\gamma, z)) = \ln(\Delta(\gamma^{-1}z)) - \ln \Delta(z) \), which is possible since there is a canonical choice for \( \ln \Delta(z) \).

We use the fact that \( S_{\Delta^\varepsilon}S_{\Delta^\varepsilon}^* \) is a decreasing family of operators, converging to the identity as \( \varepsilon \to 0 \). Let

\[
\varphi(\overline{z}, \xi) = \overline{\Delta(z)\Delta(\xi)[(\overline{z} - \xi)/(-2i)]^{12}}.
\]

Then

\[
\ln \varphi(\overline{z}, \xi) = \ln \Delta(z) + \ln \Delta(\xi) + 12 \ln[(\overline{z} - \xi)/(-2i)],
\]

has the property that

\[
\left[ \left( -\frac{1}{12} \ln \varphi(\overline{z}_i, z_j) + \frac{c'_i}{c_t} \right) [(\overline{z}_i - z_j)/(-2i)]^{-t} \right]_{i,j}
\]

is a positive matrix for all \( z_1, z_2, \ldots, z_n \) in \( \mathbb{H} \) and for all \( t > 1 \).

In Section 3 we use the positivity proven in Section 2 to check that the operator of symbol multiplication by \( (\ln \varphi)(\varphi^\varepsilon + C_t) \) (for a suitable constant \( C_t \), depending only on \( t \)) is well defined on a weakly dense subalgebra of \( \mathcal{A}_t \). This operator gives a completely positive map on this subalgebra.

By a principal-value procedure, valid in a type \( \Pi_1 \) factor, we deduce that multiplication by \( (-\ln \varphi + c_{t,\mathcal{A}}) \) is a completely positive map \( \Lambda \), on a weakly dense unital subalgebra \( \mathcal{A} \) of \( \mathcal{A}_t \) (\( c_{t,\mathcal{A}} \) is a constant that only depends on \( t \) and \( \mathcal{A} \)). Multiplication by \( (\ln \varphi) \) maps \( \mathcal{A} \) into the operator affiliated with \( \mathcal{A}_t \).

In particular \( \Lambda(1) \) is affiliated with \( \mathcal{A}_t \). We obtain this result by checking that the kernels \( -((\varphi^\varepsilon - \text{Id})/\varepsilon) \) are decreasing as \( \varepsilon \downarrow \varepsilon_0, \varepsilon_0 > 0 \) (up to a small linear perturbation), to \( \varphi^{\varepsilon_0} \ln \varphi \), plus a suitable constant.
This is not surprising as \( \Lambda(1) = \ln \varphi(z, \xi) \) barely fails the summability criteria for \( L^1(A_t) \).

In Section 4, we analyze the derivatives \( X_t \) at \( t \) of the intertwining maps \( \theta_{s,t}: A_t \to A_s, s \geq t \), with \( \theta_{s,t}(k) = S_{\Delta(t-s)/12}kS^{*}_{\Delta(t-s)/12} \). The derivatives \( (X_t) \) are, up to a multiplicative constant, the operators defined in Section 3. The operator \( X_t \) is defined on a weakly dense unital subalgebra of \( A_t \).

We take the derivative of the identity satisfied by \( \theta_{s,t} \), which is

\[
\theta_{s,t}(k^*T_{t}^{\varphi(z-s/12)}(z)l) = \theta_{s,t}(k^*)\theta_{s,t}(l).
\]

This gives the identity

\[
X_t(k^*l) + k^*T_{t}^{\varphi(z-s/12)}(z) \varphi = C_t(k,l) + X_t(k^*l) + k^*X_tl,
\]

which holds on a weakly dense (nonunital) subalgebra.

Based on an estimate on the growth of the function \( |\ln \Delta(z)\Delta^\varepsilon(z)|, z \in \mathbb{H}, \) for fixed \( \varepsilon > 0 \), we prove in Section 5 that the positive, affiliated operators \( -\Lambda(1) \) and \( -T_{t}^{\varphi} \) are equal operators. We prove this by showing that there is an increasing family \( A_\varepsilon \) in \( A_t \) and dense domains \( D_0, D_1 \) (where \( D_0 \) is affiliated to \( A_t \)) such that \( \langle A_\varepsilon(z)\xi, \xi \rangle \to \langle -\Lambda(1)(z)\xi, \xi \rangle \) for \( z \) in \( D_0 \) and \( \langle A_\varepsilon(z)\xi, \xi \rangle \to \langle -T_{t}(z)\xi, \xi \rangle \) for \( \xi \) in \( D_1 \).

In Section 6 we analyze the cyclic cocycle associated with the deformation which is obtained from the positive Hochschild cocycle by discarding a trivial part.

The precise formula is

\[
\Psi_t(k, l, m) = \tau_{A_t}(C_t(k, l) - Y_t(kl) + (Y_t(k)l + k(Y_t(l))m),
\]

for \( k, l, m \) in a dense subalgebra, and

\[
\langle Y_t(k, l) \rangle = -\frac{1}{2} \tau_{A_t}(C_t(k, l^*)).
\]

We reprove a result in [Ra], that the cyclic cohomology cocycle

\[
\Psi(k, l, m) = \text{cst} \tau(klm)
\]

is implemented by

\[
\chi_t(k, l^*) = \langle X_tk, l \rangle - \langle k, X_tl \rangle
\]

for \( k, l \) in a dense subalgebra. Since the constant in the above formula is nonzero, this corresponds to nontriviality of \( \Psi_t \) on this dense subalgebra.
In Section 7 we analyze a dual form of the coboundary for $C_t(k,l)$, in which multiplication by $\varphi$ is rather replaced by the Toeplitz operator of multiplication by $\overline{\varphi}$ (compressed to $L^2(A_t)$). It turns out that the roles of $\Lambda(1)$ and $T_{\ln}^t \varphi$ are reversed in the functional equation satisfied by the coboundary.

In the appendix, giving up the complete positivity requirement and the algebra requirement on the domain of the corresponding maps, we find some more general coboundaries for $C_t$, which were hinted at in [Ra].
1. Complete positivity for the Hochschild 2-cocycle associated with the deformation

In this section we prove the positivity condition on the Hochschild 2-cocycle associated with Berezin’s deformation.

Denote for \( z, \eta \) in \( \mathbb{H} \) the expression
\[
d(\varpi, \eta) = \frac{(\text{Im} z)^{1/2}(\text{Im} \eta)^{1/2}}{[(\varpi - \eta)/(−2i)]^{1/2}}
\]
and recall that \( |d(\varpi, \eta)|^2 \) is the hyperbolic cosine of the hyperbolic distance between \( z, \eta \in \mathbb{H} \).

In [Ra] we introduced the following seminorm, defined for \( A \in B(H_s) \), given by the kernel \( k = k_A(\varpi, \xi), \varpi, \xi \in \mathbb{H} \):
\[
\|A\|_s = \|k\|_s = \max \left( \sup_{\varpi \in \mathbb{H}} \int |k(\varpi, \eta)| |d(\varpi, \eta)|^s \, d\nu_0(\eta), \sup_{\varpi \in \mathbb{H}} \int |k(\varpi, \eta)| |d(\varpi, \eta)|^s \, d\nu_0(\varpi) \right).
\]

The subspace of all elements \( A \) in \( B(H_s) \) (respectively \( A_s \)) such that \( \|A\|_s \) is finite is a closed, involutive Banach subalgebra of \( B(H_s) \) (respectively \( A_s \)) that we denote by \( \hat{B}(H_s) \) (respectively \( \hat{A}_s \)).

In [Ra] we proved that in fact \( \hat{A}_s \) (or \( \hat{B}(H_s) \)) is also closed under any of the products \(*_t\), for \( t \geq s \), and that there is a universal constant \( c_{s,t} \), depending on \( s, t \), such that
\[
\|k *_t l\|_s \leq c_{s,t} \|k\|_s \|l\|_s, \quad k, l \in \hat{A}_s.
\]

Also \( \hat{A}_s \) (or \( \hat{B}(H_s) \)) is weakly dense in \( A_s \) (respectively \( B(H_s) \)).

Let \( \Psi_{s,t}, s \geq t > 1 \), be the map that associates to any \( A \) in \( B(H_t) \) (respectively \( A_t \)) the corresponding element in \( B(H_s) \) (respectively \( A_s \)) having the same symbol (that is, \( \Psi_{s,t}(A) \in A_s \) has the same symbol as \( A \) in \( A_t \)). Then \( \Psi_{s,t} \) maps \( A_t \) continuously into \( A_s \).

In the next proposition we prove that \( \Psi_{s,t} \) is a completely positive map. This is based on the following positivity criterion proved in [Ra].

**Lemma 1.1 (Positivity criterion).** A kernel \( k(\varpi, \xi) \) defines a positive bounded operator in \( B(H_t) \), of norm less than 1, if and only if for all \( N \) in \( \mathbb{N} \) and for all \( z_1, z_2, \ldots, z_N \) in \( \mathbb{H} \) we have that the following matrix inequality holds:
\[
0 \leq \left[ \frac{k(\varpi_i, \varpi_j)}{[(\varpi_i - \varpi_j)/(−2i)]^t} \right]_{i,j=1}^{N} \leq \left[ \frac{1}{[(\varpi_i - \varpi_j)/(−2i)]^t} \right]_{i,j=1}^{N}.
\]
This criterion obviously holds at the level of matrices of elements in $M_p(\mathbb{C}) \otimes \mathcal{A}_t$.

**Lemma 1.2** (Matrix positivity criterion). If $[k_{p,q}]_{p,q=1}^N$ is a positive matrix of elements in $\mathcal{A}_t$ then for all $N$ in $\mathbb{N}$, all $z_1, z_2, \ldots, z_N$ in $\mathbb{H}$ the following matrix is positive definite:

$$
\begin{bmatrix}
\frac{k_{p,q}(z_i, z_j)}{[(z_i - z_j)/(2i)]^t} \\
\end{bmatrix}_{(i,p),(j,q) \in \{1,2,\ldots,P\} \times \{1,2,\ldots,N\}}.
$$

Conversely, if the entries $k_{p,q}$ represent an element in $\mathcal{A}_t$ and if the above matrix is positive, then $[k_{p,q}]_{p,q=1}^N$ is a positive matrix in $\mathcal{A}_t$.

**Proof.** Let $[k_{p,q}]_{p,q=1}^N$ be a matrix in $\mathcal{A}_t$. Then $k = [k_{p,q}]$ is positive if and only if for all vectors $\xi_1, \xi_2, \ldots, \xi_N$ in $H_t$ we have that

$$
\sum_{p,q} \langle k_{p,q} \xi_p, \xi_q \rangle_{H_t} \geq 0.
$$

Since $k_{p,q} = P_t k_{p,q} P_t$, where $P_t$ is the projection from $L^2(\mathbb{H}, \nu_t)$ onto $H_t$, it turns out that this is equivalent with the same statement which now must be valid for all $\xi_1, \xi_2, \ldots, \xi_N$ in $L^2(\mathbb{H}, \nu_t)$.

Thus we have that

$$
\sum_{p,q} \iint_{\mathbb{H}^2} \frac{k_{p,q}(z, w)}{[(z - w)/(2i)]^t} \bar{\xi}_p(z) \bar{\xi}_q(w) \, d\nu_t(z) \, d\nu_t(w) \geq 0
$$

for all $\xi_1, \xi_2, \ldots, \xi_N$ in $L^2(\mathbb{D}, \nu_t)$.

We let the vectors $\xi_p$ converge to the Dirac distributions, for all $p = 1, 2, \ldots, N$, $\sum_i \lambda_{ip} \delta_{zi} (\text{Im } z_i)^{-t/2}$, for all $p = 1, 2, \ldots, N$. By the above inequality we get

$$
\sum_{i,j,p,q} \frac{k_{p,q}(z_i, z_j)}{[(z_i - z_j)/(2i)]^t} \bar{\lambda}_{ip} \lambda_{jq} \geq 0
$$

for all choices of $\{\lambda_{ij}\}$ in $\mathbb{C}$. This corresponds exactly to the fact that the matrix

$$
\begin{bmatrix}
\frac{k_{p,q}(z_i, z_j)}{[(z_i - z_j)/(2i)]^t} \\
\end{bmatrix}_{(p,i),(q,j)}
$$

is positive. □

**Proposition 1.3.** The map $\Psi_{s,t} : \mathcal{A}_t \to \mathcal{A}_s$ which sends an element $A$ in $\mathcal{A}_t$ into the corresponding element in $\mathcal{A}_s$, having the same symbol, is unital and completely positive.
Proof. This is a consequence of the fact [ShS, Sal] that the matrix

\[
\begin{bmatrix}
1 \\
\left[ \frac{(\zeta_i - z_j)/(-2i)}{\varepsilon} \right]_{i,j}
\end{bmatrix}
\]

is a positive matrix for all \(\varepsilon\), all \(N\), all \(z_1, z_2, \ldots, z_n\) in \(\mathbb{H}\). Indeed

\[
\frac{1}{\left[ \frac{(\zeta - \xi)/(-2i)}{\varepsilon} \right]}
\]

(or \(1/(1 - \zeta \xi)^{\varepsilon}\)) is a reproducing kernel for a space of analytic functions, even if \(\varepsilon < 1/2\). □

We will now follow Lindblad’s [Li] argument to deduce that \(C_t(k, l)\) is a completely positive Hochschild 2-cocycle. We recall first the definition of the cocycle \(C_t\) associated with the deformation.

DEFINITION 1.4. Fix \(t > s_0 > 1\). Then the following formula defines a Hochschild 2-cocycle on \(\widehat{A}_{s_0}\).

\[
C_t(k, l)(\zeta, \xi) = \frac{d}{ds} \bigg|_{s=t} \langle k *_s l \rangle_{H_t}(\zeta, \xi)
\]

\[
= \frac{c_t}{c_t} (k *_t l)(\zeta, \xi) + c_t \int_{\mathbb{H}} k(\zeta, \eta) l(\eta, \xi) [\zeta, \eta, \eta, \xi]^t \ln[\zeta, \eta, \eta, \xi] d\nu_0(\eta).
\]

Indeed, it was proven in [Ra] that the above integral is absolutely convergent for \(k, l\) in \(\widehat{A}_{s_0}\), for any \(s_0 < t\).

The above definition may be thought of also in the following way. Fix vectors \(\xi, \eta\) in \(H_t\) and fix \(k, l\) in \(A_{s_0}\). Then \(k *_t l\), and \(k *_s l\) make sense for all \(s_0 < s < t\) and they represent bounded operators in \(A_t\). Thus the following derivative makes sense:

\[
\frac{d}{ds} \bigg|_{s=t} \langle k *_s l \xi, \eta \rangle_{H_t},
\]

and it turns out to be equal to

\[
\langle C_t(k, l) \xi, \eta \rangle_{H_t}.
\]

In the following lemma we use the positivity of \(\Psi_{s,t}\) to deduce the complete positivity of \(C_t\). We recall the following formal formula for \(C_t\) that was proved in [Ra1], [Ra2].
Lemma 1.5 [Ra]. Let $1 < t_0 < t$ and let $k, l, m$ belong to $\hat{A}_{t_0}$. Then the following holds:

$$\tau_{A_t}(C_t(k, l) *_t m) = \frac{d}{ds}\tau_{A_s}((k *_s l *_s m) - (k *_t l) *_s m)\bigg|_{s=t}^{s>t}.$$

In a more precise notation, the second term is

$$\frac{d}{ds}\tau_{A_s}([Ψ_{s,t}(k) *_s Ψ_{s,t}(l) - Ψ_{s,t}(k *_t l)] *_s Ψ_{s,t}(m))\bigg|_{s=t}^{s>t}.$$

The proof of the lemma is trivial, as long as one uses the absolute convergence of the integrals, which follows from the fact that the kernels belong to an algebra $\hat{A}_{t_0}$, for some $t_0 < t$.

The positivity property that we are proving for $C_t(k, l)$, is typical for coboundaries of the form $D(ab) - D(a)b - aD(b)$, where $D$ is the generator of dynamical semigroup. It is used by Sauvageot to construct the cotangent bimodule associated with a dynamical semigroup, and many of the properties in [Sau] can be transferred to $C_t$ with the same proof. Such positive (or negative) cocycles appear in the work of Connes and Cuntz (see also [CH]).

Proposition 1.6. Fix $1 < t_0 < t$, and for $k, l$ in $\hat{A}_{t_0}$, define

$$C_t(k, l) := \frac{d}{ds}(k *_s l)\bigg|_{s=t}^{s>t}.$$

Then for all $k_1, k_2, \ldots, k_N$ in $\hat{A}_{t_0}$, $l_1, l_2, \ldots, l_N$ in $A_t$, we have that

$$\sum_{i,j} \tau_{A_t}(l^*_i C_t(k^*_i, k^*_j) l_j) \geq 0.$$

This is the same as requiring the matrix $(C_t(k^*_i, k^*_j))_{i,j}$ to be negative in $M_N(A_t)$.

Proof. For $s \geq t$, let $f(s)$ be defined by the formula

$$f(s) = \tau\left(\sum_{i,j} (k^*_i *_s k_j - k^*_i *_t k_j) *_t (l^*_i *_t l_j)\right).$$

Using the $Ψ_{s,t}$ notation, this is

$$f(s) = \sum_{i,j} \tau((Ψ_{s,t}(k^*_i)Ψ_{s,t}(k_j) - Ψ_{s,t}(k^*_i k_j)) Ψ_{s,t}(l^*_i l_j)).$$
By the previous lemma, \( f'(t) \) is equal to \( \tau(C_t(k_i^*, j_l^*)l_j^*) \). In these terms, to prove the statement we must prove that \( f'(t) \geq 0 \). Clearly \( f(t) = 0 \).

By the generalized Cauchy–Schwarz–Stinespring inequality for completely positive maps, and since \( \Psi_{s,t} \) is unital, we get that the matrix
\[
D_{ij} = [\Psi_{s,t}(k_i^*)\Psi_{s,t}(j_l^*) - \Psi_{s,t}(k_i^*k_j^*)]
\]
is non-positive. Since \( Z_{ij} = \Psi_{s,t}(l_j^*i^*) \) is another positive matrix in \( M_n(A_s) \), we obtain that
\[
f(s) = \tau_{A_s \otimes M_n(\mathbb{C})}(DZ)
\]
is negative.

So \( f(s) \leq 0 \) for all \( s \geq t, f(0) = 0 \). Hence \( \frac{d}{ds} f(s) \big|_{s=t; s>t} \) is negative. \( \square \)

Appendix (to Section 1)

We want to emphasize the properties of the trace \( E_t(k, l) = -\tau(C_t(k, l)) \), \( k, l \in A_t \). Clearly \( E_t \) is a positive form on \( A_t \), and in fact it is obviously positive definite. Following [Sau], one can prove that \( E_t \) is a Dirichlet form. The following expression holds for \( E_t \).

**Lemma 1.7.** For \( 1 < t_0 < t, k, l \in \hat{A}_{t_0} \) we have that
\[
E_t(k, l) = \int \int_{F \times \mathbb{H}} k(\bar{z}, \eta) \overline{k(\bar{z}, \eta)} |d(\bar{z}, \eta)|^{2t} \ln |d(z, \eta)| \ d\nu_0(z, \eta),
\]
where \( F \) is a fundamental domain for \( \Gamma \) in \( \mathbb{H} \), and
\[
|d(\bar{z}, \eta)| = \left| \frac{\text{Im} z^{1/2} \text{Im} \eta^{1/2}}{|(\bar{z} - \eta)/(2i)|} \right|
\]
is the hyperbolic cosine of the hyperbolic distance between \( z \) and \( \eta \) in \( \mathbb{H} \).

Recall that \( L^2(A_t) \) is identified [Ra] with the Bargmann-type Hilbert space of functions \( k(\bar{z}, \eta) \) on \( \mathbb{H} \times \mathbb{H} \) that are antianalytic in the first variable, analytic in the second, diagonally \( \Gamma \)-invariant (that is, \( k(\gamma z, \gamma \eta) = k(\bar{z}, \eta), \gamma \in \Gamma, z, \eta \in \mathbb{H} \)), and square-summable:
\[
\|k\|_{L^2(A_t)}^2 = c_t \int \int_{F \times \mathbb{H}} |k(\bar{z}, \eta)|^2 |d(\bar{z}, \eta)|^{2t} \ d\nu_0(z, \eta).
\]

Let \( P_t \) be the projection from the Hilbert space of square-summable functions \( f \) on \( \mathbb{H} \times \mathbb{H} \) that are \( \Gamma \)-invariant and square-summable:
\[
c_t \int \int_{F \times \mathbb{H}} |f(\bar{z}, \eta)|^2 |d(\bar{z}, \eta)|^{2t} \ d\nu_0(\eta) \ d\nu_0(z) < \infty.
\]
The following proposition is easy to prove, but we won’t make any use of it in this paper.

**Proposition 1.8.** Let \( \phi \) be a bounded measurable \( \Gamma \)-invariant function on \( \mathbb{H} \times \mathbb{H} \). Let \( T_\phi \) be the Toeplitz operator of multiplication by \( \phi \) on the Hilbert space \( L^2(A_t) \), that is, \( T_\phi k = \mathcal{P}(\phi k), k \in L^2(A_t) \). Then \( T_\phi k = \mathcal{P}_t k \mathcal{P}_t \), where the last composition is in \( A_t \), by regarding \( k \) as an element affiliated to \( A_t \).

**Remark.** In this setting the positive form \( \mathcal{E}_t \) may be identified with the quadratic form on \( L^2(A_t) \) induced by the unbounded operator \( \mathcal{T}_{\ln d} \) where \( d = |d(z, \eta)| \) is defined as above.
§2. Derivatives of some one-parameter families of positive operators

In this section we consider some parametrized families of completely positive maps that are induced by automorphic forms (and fractional powers thereof). The automorphic forms are used as intertwining operators between the different representation spaces of \( \text{PSL}(2, \mathbb{Z}) \), consisting of analytic functions.

It was proved in [GHJ] that automorphic forms \( f \) for \( \text{PSL}(2, \mathbb{Z}) \) of weight \( k \), provide bounded multiplication operators \( S_f: H_t \to H_{t+k} \). The boundedness property comes exactly from the fact that one of the conditions for an automorphic form \( f \) of order \( k \) is

\[
\sup_{z \in \mathbb{H}} |f(z)|^2 \text{Im} z^k \leq M,
\]

which is exactly the condition that the operator of multiplication by \( f \) from \( H_t \) into \( H_{t+k} \) be norm bounded by \( M \).

Secondly, the automorphic forms have the (cocycle) \( \Gamma \)-invariance property as functions on \( \mathbb{H} \), that is,

\[
f(\gamma^{-1}z) = (cz+d)^{-k}f(z), \quad z \in \mathbb{H}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z}).
\]

Since \( \pi_t(\gamma) \), \( \pi_{t+k}(\gamma) \) act on the corresponding Hilbert space of analytic functions on \( \mathbb{H} \) by multiplication with the automorphic factor \((cz+d)^{-t}\), respectively \((cz+d)^{-t-k}\), this implies exactly that

\[
\pi_{t+k}(\gamma)S_f = S_f \pi_t(\gamma).
\]

Let \( f, g \) be automorphic forms of order \( k \). Let \( F \) be a fundamental domain for the group \( \text{PSL}(2, \mathbb{Z}) \) in \( \mathbb{H} \). It was proved in [GHJ] that the trace (in \( A_t \)) of \( S_f^*S_g \) is equal to the Petersson scalar product

\[
\frac{1}{\text{area } F} \langle f, g \rangle = \frac{1}{\text{area } F} \int_F f(z)\overline{g(z)}(\text{Im } z)^k \, d\nu_0(z).
\]  

In the next lemma we will prove that the symbol of \( S_f^*S_g \), as an operator on \( H_t \) belonging to \( \mathcal{A}_t \) (the commutant of \( \text{PSL}(2, \mathbb{Z}) \)) is (up to a normalization constant) \( f(z)g(\xi)[(\overline{z} - \xi)/(-2i)]^k \).

In particular this shows that the above formula (2.1) is explained by the trace formula \( \tau_{\mathcal{A}_t}(k) = (1/\text{area } F) \int_F k(\overline{z}, z) \, d\nu_0(z) \), applied to the operator \( k = S_f^*S_g \).

The role of the factor \([(\overline{z} - \xi)/(-2i)]^k \) is to make the function \( f(z)g(\xi)[(\overline{z} - \xi)/(-2i)]^k \) diagonally \( \text{PSL}(2, \mathbb{Z}) \)-invariant. It is easy to observe that \( S_f^*S_g \) is the Toeplitz operator on \( H_t \) with symbol \( f(z)g(z)(\text{Im } z)^k \). Note that, to form \( S_f^*S_g \), we
have the restriction $k < t - 1$, because $S_g^*$ has to map $H_t$ into a space $H_{t-k}$ that makes sense.

We observe that the symbol of $S_f k S_g^*$ for an operator $k$ on $H_{t+k}$ of symbol $k = k(\overline{z}, \xi)$ is

$$\frac{c_{t-p}}{c_t} f(z) g(\xi) [(\overline{z} - \xi)/(-2i)]^p k(\overline{z}, \xi),$$

if $f, g$ are automorphic forms of order $p$.

This also explains the occurrence of operators of multiplication with symbols $\Phi(\overline{z}, \xi)$ on the space $L^2(A_t)$, in this setting. In the terminology of the Appendix in the previous section those are the Toeplitz operators with analytic symbol $\Phi(\overline{z}, \xi)$, a diagonally PSL $(2, \mathbb{Z})$-invariant function. In the present setting, to get a bounded operator, we map $L^2(A_{t+k})$ into $L^2(A_t)$, by multiplying by $f(z) g(\xi) [(\overline{z} - \xi)/(-2i)]^k$.

In this section we will analyze the derivatives of a family of such operators.

Let $\Delta(z)$ be the unique automorphic form for PSL $(2, \mathbb{Z})$ in dimension 12 (this is the first order for which there is a nonzero space of automorphic forms).

We rescale this form by considering the normalized function $\Delta_1 = \Delta/c$, where the constant $c$ is chosen so that

$$\sup_{z \in \mathbb{H}} |\Delta_1(z)|^2 (\text{Im } z)^{12} \leq 1.$$

In the sequel we will omit the subscript 1 from $\Delta$. This gives that the norm $\|S\Delta\|$, as an operator from $H_t$ into $H_{t+12}$, is bounded by 1.

As $\Delta$ is a nonzero analytic function on the upper half-plane, one can choose an analytic branch for $\ln \Delta$. Consider the $\Gamma$-invariant function

$$\varphi(\overline{z}, \xi) = \ln (\overline{\Delta(z)}) + \ln \Delta(\xi) + 12 \ln [(\overline{z} - \xi)/(-2i)],$$

which we also write as

$$\varphi(\overline{z}, \xi) = \ln (\overline{\Delta(z)} \Delta(\xi) \cdot [(\overline{z} - \xi)/(-2i)]^{12}).$$

Defining $\pi_t(\gamma)$ for $\gamma$ in PSL $(2, \mathbb{Z})$ involves a choice of a branch for $\ln(cz + d)$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We define $\pi_t(\gamma)$, $\gamma \in \text{PSL } (2, \mathbb{Z})$, by using the factor $(cz + d)^{-t}$ corresponding to the following choice of the logarithm for $\ln(cz + d)$:

$$\ln \Delta(\gamma^{-1}z) - \ln \Delta(z) = \ln(cz + d),$$

$z \in \mathbb{H}$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in PSL $(2, \mathbb{Z})$. 

By making this choice for $\pi_t$ restricted to $\Gamma$, we do not change the algebra $A_t$, which is the commutant of $\{\pi_t(\Gamma)\}$, but we have the following.

With the above choice for $\ln(cz + d)$ and thus for $\pi_t(\gamma), \gamma \in \text{PSL}(2, \mathbb{Z})$, and for any $\varepsilon > 0$, we have that $S_{\Delta \varepsilon}$ is a bounded operator between $H_t$ and $H_{t+12\varepsilon}$, which intertwines $\pi_t$ and $\pi_{t+12\varepsilon}$ for all $t > 1, \varepsilon > 0$.

In the following lemma we make the symbol computation for operators of the form $S_f S_g^*$. Recall that $H_t, t > 1$ is the Hilbert space of analytic functions on $\mathbb{H}$ that are square-summable under $d\nu_t = (\text{Im } z)^{t-2} d\overline{z} dz$.

**Lemma 2.1.** Let $f, g$ be analytic functions on $\mathbb{H}$, $k$ a strictly positive integer, and $t > 1$. Assume that $M_f = \sup_z |f(z)|^2 \text{Im } z^k$, $M_g = \sup_z |g(z)|^2 \text{Im } z^k$ are finite quantities.

Let $S_f, S_g$ be the multiplication operators from $H_t$ into $H_{t+k}$ by the functions $f, g$. Then $S_f, S_g$ are bounded operators of norm at most $M_f, M_g$ respectively.

Moreover, the symbol of $S_f S_g^* \in B(H_t)$ is given by the formula

$$
\frac{c_{t-k}}{c_t} f(z) f(\xi) \frac{(\overline{z} - \xi)/(-2i)^k}{(\overline{z} - \xi)/(-2i)^k}.
$$

**Proof.** Before starting the proof we’ll make the following remark that should explain the role of the constant $c_t (= (t - 1)/4\pi)$ in this computation.

**Remark.** The quantity $c_t$ is a constant that appears due to the normalization in the definition of $H_t$, where we have chosen

$$
\|f\|^2_{H_t} = \int_{\mathbb{H}} |f(z)|^2 (\text{Im } z)^{t-2} d\overline{z} dz.
$$

Consequently the reproducing vectors $e^t_z$ (defined by $\langle f, e^t_z \rangle = f(z), f \in H_t, z \in \mathbb{H}$) are given by the following formula [Ba, Mi]:

$$
e^t_z(\xi) = \langle e^t_z, e^t_\xi \rangle = \frac{c_t}{(\overline{z} - \xi)/(-2i)^t}, \quad z, \xi \in \mathbb{H}.
$$

Consequently the normalized symbol of an operator $A$ in $B(H_t)$ is given by the formula $k_A(\overline{z}, \xi) = \langle Ae^t_z, e^t_\xi \rangle / \langle e^t_z, e^t_\xi \rangle, z, \xi \in \mathbb{H}$.

In the product formula we have that the symbol $k_{AB}(\overline{z}, \xi)$ of the product of two operators $A, B$ on $H_t$ with symbols $k_A, k_B$ is given by

$$
\langle e^t_z, e^t_\xi \rangle k_{AB}(\overline{z}, \eta) = \langle AB e^t_z e^t_\xi \rangle = \int_{\mathbb{H}} \langle Ae^t_z, e^t_\eta \rangle \langle Be^t_\xi, e^t_\eta \rangle d\nu_t(\eta).
$$
Thus
\[ k_{AB}(\zeta, \xi) = \langle e^t_z, e^t_{\xi} \rangle \int_{H} k_A(z, \eta) (e^t_z, e^t_\eta) k_B(\bar{\eta}, \xi) (e^t_\eta, e^t_\xi) \, d\nu_t(\eta) \]
\[ = \frac{[(\zeta - \xi)/(-2i)]^t}{c_t} \int_{H} k_A(z, \eta) \frac{c_t}{[(\zeta - \eta)/(-2i)]^t} k_B(\bar{\eta}, \xi) \frac{c_t}{[(\bar{\eta} - \xi)/(-2i)]^t} \, d\nu_t(\eta) \]
\[ = c_t \int_{H} k_A(z, \eta) k_B(\bar{\eta}, \xi) \, d\nu_0(\eta). \]

This accounts for the constant \( c_t \) that occurs in front of the product formula (otherwise if we proceed as in [Ba] and include the constant \( c_t \) in the measure \( d\nu_t \), the constant will still show up in the product formula).

In the proof of the lemma we use the following observation.

**Observation 2.2.** Let \( f, H_t, S_f \) be as in the statement of Lemma 2.1. Let \( e^t_z, e^{t+k}_z \) be the evaluation vectors at \( z \), in the spaces \( H_t \) and \( H_{t+k} \). Then
\[ S^*_f e^{t+k}_z = f(z) e^t_z, \quad z \in \mathbb{H}. \]

**Proof.** Indeed, since we will prove the boundedness of \( S_f \), we can check this by evaluating on a vector \( g \) in \( H_t \). We have
\[ \langle S^*_f e^{t+k}_z, g \rangle_{H_t} = \langle e^{t+k}_z, (S_f)g \rangle_{H_{t+k}} = \langle e^{t+k}_z, f g \rangle = \langle f g, e^{t+k}_z \rangle = f g(z). \]

On the other hand:
\[ \overline{\langle f(z), e^t_z, g \rangle_{H_t}} = \overline{f(z)} \langle e^t_z, g \rangle_{H_t} = \overline{f(z)} \langle g, e^t_z \rangle_{H_t} = \overline{f(z)} g(z). \]

This shows the equality of the two vectors. \( \square \)

We can now go on with the proof of Lemma 2.1. It is obvious that \( S_f, S_g \) are unbounded operators of norms \( M_f, M_g \). Indeed, for \( S_f \) we have that
\[ \|S_f g\|_{H^*}^2 = \int_{\mathbb{H}} |(S_f g)(z)| \, d\nu_{t+k}(z) \]
\[ = \int_{\mathbb{H}} |(f g)(z)|^2 \, d\nu_{t+k}(z) \]
\[ = \int_{\mathbb{H}} |f(z)|^2 |g(z)|^2 (\text{Im } z)^2 (\text{Im } z)^{t-2} \, d\zeta \, dz \]
\[ = \int_{\mathbb{H}} |g(z)|^2 (|f(z)|^2 (\text{Im } z)^k) \, d\nu_t(z) \]
\[ \leq M_f \int_{\mathbb{H}} |g(z)|^2 \, d\nu_t(z). \]
Hence \( \|S_f\| \leq M_f \).

To prove the second assertion, observe that the symbol \( k(z, \xi) \) of \( S_fS_g^* \), as an operator on \( H_t \), is given by the following formula:

\[
k(z, \xi) = \frac{\langle S_fS_g^*e_z^t, e_\xi^t \rangle_{H_t}}{\langle e_z^t, e_\xi^t \rangle_{H_t}}
\]

\[
= \frac{\langle S_g^*e_z^t, S_f^*e_\xi^t \rangle}{\langle e_z^t, e_\xi^t \rangle_{H_t}}
\]

\[
= \frac{g(z)\langle e_z^{t-k}, f(\xi)e_\xi^{t-k} \rangle_{H_{t-k}}}{\langle e_z^t, e_\xi^t \rangle_{H_t}}
\]

\[
= g(z)f(\xi)\frac{\langle e_z^{t-k}, e_\xi^{t-k} \rangle_{H_{t-k}}}{\langle e_z^t, e_\xi^t \rangle_{H_t}}
\]

\[
= g(z)f(\xi)\frac{c_{t-k}}{c_t} \frac{[(z - \xi)/(-2i)]^{t-k}}{[(z - \xi)/(-2i)]^t}
\]

\[
= \frac{c_{t-k}}{c_t} g(z)f(\xi)\frac{[(z - \xi)/(-2i)]^k}{k(z, \xi)}.
\]

This also works also for \( k \) not an integer (as \( \ln[(z - \xi)/(-2i)] \) is chosen once for all).

Let us finally note that the same arguments might be used to prove the following more general statement.

**Remark.** Let \( f, g \) be analytic functions as in the statement of the lemma, and let \( k \) be an operator in \( \mathcal{A}_t \). Then \( S_fkS_g^* \), which belongs to \( \mathcal{A}_{t+k} \) (if we think of \( S_f, S_g \) as bounded operators mapping \( H_t \) into \( H_{t+k} \)), has the following symbol:

\[
\frac{c_t}{c_{t+k}} f(\xi)g(z)[(z - \xi)/(-2i)]^k k(z, \xi).
\]

To show this, we have to evaluate

\[
\frac{\langle S_fkS_g^*e_z^{t+k}, e_\xi^{t+k} \rangle_{H_{t+k}}}{\langle e_z^{t+k}, e_\xi^{t+k} \rangle_{H_{t+k}}} = \frac{f(\xi)g(z)\langle ke_z^t, e_\xi^t \rangle_{H_t}}{\langle e_z^t, e_\xi^t \rangle_{H_t}} \cdot \frac{\langle e_z^t, e_\xi^t \rangle_{H_{t+k}}}{\langle e_z^{t+k}, e_\xi^{t+k} \rangle_{H_{t+k}}}
\]

\[
= \frac{c_t}{c_{t+k}} f(\xi)g(z)[(z - \xi)/(-2i)]^k k(z, \xi).
\]
In the next lemma we will deduce a positivity condition for kernels of operators that occur as generators of parametrized families $S_f S_g^*$, where $f, g$ are supposed to have a logarithm on $\mathbb{H}$.

**Lemma 2.3.** Assume that $f$ is a function as in Lemma 2.1. $f$ is analytic on $\mathbb{H}$ and we assume $M_f = \sup_{z \in \mathbb{H}} |f(z)|^2 (\text{Im} z)^k$ is less than 1.

Assume that $f$ is nonzero on $\mathbb{H}$, and choose a branch for $\ln f$ and hence for $f^\varepsilon$, $\varepsilon$ being strictly positive.

Let $\varphi(\overline{z}, \eta)$ be the function

$$\ln f(\overline{z}) + \ln f(\xi) + k \ln[(\overline{z} - \xi)/(2i)]$$

and use this as a choice for

$$\ln[f(z) f(\xi)[(\overline{z} - \xi)/(2i)]^k] = \varphi(\overline{z}, \xi).$$

Then for all $\varepsilon > 0$ the kernel

$$k_\varphi(\overline{z}, \eta) = k_{\varphi, t, \varepsilon}(\overline{z}, \xi) = \varphi(\overline{z}, \xi) \left[ \frac{c_t - k_{\varepsilon} c_t}{c_t} \ln \varphi - \frac{c_t'}{c_t} \right]$$

is nonpositive in the sense of $A_t$, that is, $k_{\varphi, t, \varepsilon}(\overline{z}_i, \overline{z}_j)$ is a nonpositive matrix for all choices of $N \in \mathbb{N}, z_1, z_2, \ldots, z_N \in \mathbb{H}$.

**Proof.** By the choice we just made it is clear that the norm of the operator $S_f^\varepsilon$ is always less than 1. We will also denote by $S_{f, t}^\varepsilon$ the corresponding operators, which act as a contraction from $H_t$ into $H_{t+k\varepsilon}$.

Consider the following operator-valued functions, with values in $H_t$:

$$f(\varepsilon) = S_{f, t}^{t-k\varepsilon} \left( S_{f, t}^{t-k\varepsilon} \right)^*.$$  

Obviously the symbol of $f(\varepsilon)$ is $(c_t - k_{\varepsilon} c_t) \varphi(\overline{z}, \xi)$, and moreover, $f(0) = 1$, $f(\varepsilon)$ is a decreasing map because for $0 \leq \varepsilon \leq \varepsilon'$, we have that

$$f(\varepsilon') = S_{f, t}^{t-k\varepsilon'} \left( S_{f, t}^{t-k\varepsilon'} \right)^* = S_{f, t}^{t-k\varepsilon} \left[ S_{f, t}^{t-k\varepsilon - \varepsilon' \varepsilon} \left( S_{f, t}^{t-k\varepsilon'} \right)^* \right] \left( S_{f, t}^{t-k\varepsilon} \right)^*$$

But the operator in the middle has norm less than 1, and hence we get that

$$f(\varepsilon') \leq S_{f, t}^{t-k\varepsilon} \left( S_{f, t}^{t-k\varepsilon} \right)^* = f(\varepsilon).$$

Fix $N$, and $z_1, z_2, \ldots, z_N$ in $\mathbb{H}$. Then (since the corresponding operators form a decreasing family)

$$g(\varepsilon) = \left[ \frac{f(\varepsilon)(\overline{z}_i, \overline{z}_j)}{[(\overline{z}_i - \overline{z}_j)/(2i)]^t} \right]_{i,j}$$

is a decreasing family of matrices, and $g(0) = \text{Id}$. Hence $g'(\varepsilon)$ must be a negative (nonpositive) matrix. Note that $f(\varepsilon)(\overline{z}_i, \overline{z}_j) = (c_t - k_{\varepsilon} c_t) \varphi(\overline{z}, \overline{z}_j)$. 

But \( g'(\varepsilon) \) has exactly the formula stated above, that is,

\[
g'(\varepsilon) = \frac{\varphi^e(z_i, z_j) \left[ \ln \varphi(z_i, z_j) \frac{c_t - k\varepsilon}{c_t} - k\frac{c'_t}{c_t} \right]}{[(z_i - z_j)/(-2i)]^t}
\]

This completes the proof. \( \square \)

By collecting the terms together, and since \( \frac{c'_t}{c_t} = \frac{1}{t-1} \), we obtain, for all \( \varepsilon \geq 0 \), the following

**Lemma 2.4.** With the notations from the previous lemma, for all \( \varepsilon \geq 0 \), the kernel

\[
k_{\varphi} = k_{\varphi, \varepsilon, t} = \varphi^e \left[ \ln \varphi - \frac{1}{t - 1 - k\varepsilon} \right]
\]

is nonpositive in \( A_t \). Precisely this means that for all choices of \( N \) in \( \mathbb{N} \) and \( z_1, z_2, \ldots, z_N \) in \( \mathbb{H} \) we have that

\[
k_{\varphi}(z_i, z_j) \frac{[(z_i - z_j)/(-2i)]^t}{[(z_i - z_j)/(-2i)]^t}
\]

is a nonpositive matrix.

The following observation will be used later in the proofs.

**Observation 2.5.** For any \( s > 1 \), the identity

\[
\tau_{A_s}(S_g S_g^*) = \frac{c_s}{c_{s+k}} \tau_{A_t}(S_g S_g^*),
\]

holds true for any automorphic form \( g \) of order \( k \).

**Proof.** We have that \((S_g S_g^*)\) is the Toeplitz operator (on \( H_s \)) with symbol

\[
|g(z)|^2 \text{ (Im } z)^k.
\]

Hence the trace \( \tau_{A_s}(S_g S_g^*) \) is

\[
\frac{1}{\text{area}(F)} \int_F |g|^2 \text{ (Im } z)^k \, dv_0(z).
\]

On the other hand the symbol of \((S_g S_g^*)\) (which is viewed here as an operator on \( H_{s+k} \)) is equal to

\[
(z, \xi) \rightarrow \frac{\langle S_g e_{z+k}^s, S_g^* e_{\xi+k}^s \rangle}{\langle e_{z+k}^s, e_{\xi+k}^s \rangle} = \frac{c_s}{c_{s+k}} g(z)g(\xi) [(\overline{z} - \xi)/(-2i)]^k,
\]

and hence the trace of the latter symbol is

\[
\frac{c_s}{c_{s+k}} \frac{1}{\text{area}(F)} \int_F |g(z)|^2 \text{ (Im } z)^k \, dv_0(z).
\]

\( \square \)
§3. Properties of the (unbounded) multiplication maps by 
\[ \ln|\Delta(z)\Delta(\xi)[(\zeta - \xi)/(2i)]^{12}| \] on different spaces of kernels

Let \( \varphi(\zeta, \xi) = \ln(\Delta(z)\Delta(\xi)[(\zeta - \xi)/(-2i)]^{12}) \). In this section we want to exploit the negativity properties of the kernels

\[ \varphi^{\varepsilon/12} \left( \frac{1}{12} \ln \varphi - \frac{1}{t - 1 - \varepsilon} \right). \]

By \( \odot \) we denote the operation of pointwise multiplication of symbols. It is the analogue of Schur multiplication on matrices or on a group algebra. When no confusion is possible we will omit the symbol \( \odot \) and just replace it by \( \cdot \).

We want to draw conclusions on the properties of the multiplication maps, defined on a suitable dense subspace of \( L^2(\mathcal{A}_t) \), by the formula

\[ \Lambda_{\varepsilon}(k) = k \odot \left( \varphi^{\varepsilon/12} \left[ \frac{1}{12} \ln \varphi - \frac{1}{t - 1 - \varepsilon} \right] \right). \]

For functions \( k(\zeta, \eta) \) on \( \mathbb{H} \times \mathbb{H} \) that are positive, but do not necessarily represent a positive operator, we will introduce the following definition.

**Definition 3.1.** A function \( k(\zeta, \eta) \) on \( \mathbb{H} \times \mathbb{H} \) that is analytic for \( \eta \) and anti-analytic for \( z \) will be called positive for \( \mathcal{A}_t \) if the following matrix

\[ \left[ \frac{k(\zeta_i, z_j)}{((\zeta_i - z_j)/(-2i))^{t}} \right]_{i,j=1}^{n} \]

is positive for all choices of \( N \in \mathbb{N} \) and \( z_1, z_2, \ldots, z_N \) in \( \mathbb{H} \).

The space of such kernels will be denoted by \( \mathcal{S}_t \).

The following remark is a trivial consequence of the fact that the Schur product of two positive matrices is positive, and a consequence of the description for positivity of kernels of operators in \( \mathcal{A}_t \) given in Section 1.

**Proposition 3.2.** For all numbers \( r, s > 1 \), the vector space \((\mathcal{A}_r)_+ \odot \mathcal{S}_s\) is contained in \( \mathcal{S}_{r+s} \) and \((\mathcal{A}_r)_+ \subseteq \mathcal{S}_r\).

**Proof.** Just observe that in fact \( \mathcal{S}_s \odot \mathcal{S}_r \) is contained in \( \mathcal{S}_{s+r} \). \( \square \)

The problem that we address in this section comes from the fact that the operator

\[ -\Lambda_{\varepsilon, r, s}(k) = k \odot \varphi^{\varepsilon/12} \left[ -\frac{1}{12} \ln \varphi + \frac{1}{r - 1 - \varepsilon} \right] \]

maps \( k \in (\mathcal{A}_s)_+ \) into \( \mathcal{S}_{r+s} \). Also \( \Lambda_{\varepsilon, r, s}(k)(\zeta, z) \) is integrable on \( F \), so it is tempting to infer that \( \Lambda_{\varepsilon, r, s}(k) \) belongs to \( L^1(\mathcal{A}_{r+s}) \). In fact, we conjecture that a kernel
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$k(\tau, \eta)$ in $\mathcal{S}_t$ that is also diagonally integrable on $F$ corresponds to an element in $L^1(\mathcal{A}_t)_t$. Since we are unable to prove the conjecture directly, we will use monotonicity properties for the derivatives of $\varphi^\varepsilon$.

If no constants were involved, we would simply say that $\varphi^\varepsilon(-\ln \varphi)$ is the increasing limit of the derivatives, since the second derivative would be negative. This doesn’t hold exactly, but the constants involved are small enough and have a negligible effect on the previous line of reasoning. This is done in the following lemma.

**Lemma 3.3.** Let $k$ be a positive kernel in $\mathcal{A}_s$. Fix $v > 1$ and let $\varepsilon > 0$ be small enough. Consider the following elements in $\mathcal{A}_{s+\varepsilon}$ defined by the kernels

$$\lambda_{\varepsilon,v,s}(k)(\tau, \xi) = \left[\frac{v - 1 - \varepsilon}{v - 1}\varphi^{\varepsilon/12}(\tau, \xi)\right] k(\tau, \xi).$$

Note that up to a multiplicative constant $\lambda_{\varepsilon,v,s}(k)$ is the kernel of $S_{\Delta^{\varepsilon/12}}kS^*_{\Delta^{\varepsilon/12}}$ in $\mathcal{A}_{s+\varepsilon}$. Let $\tilde{\lambda}_{\varepsilon,v,s}(k)$ be the image (through $\Psi_{v+2s,v+\varepsilon}$) of this kernel in $\mathcal{A}_{v+2s}$.

Then $\tilde{\lambda}_{\varepsilon,v,s}(k)$ is a decreasing family of positive kernels representing elements in $\mathcal{A}_{v+2s}$, and there exists a negative element $M(k) = M_{\varepsilon,v,s}(k)$ in $-L^1(\mathcal{A}_{v+2s})_+$ such that $M(k)$ is the derivative with respect to $\varepsilon$:

$$M(k) = \frac{d}{d\varepsilon} \tilde{\lambda}_{\varepsilon,v,s}(k).$$  \hspace{1cm} (3.1)

The derivative is computed in the strong convergence topology, on a dense domain $\mathcal{D} \subseteq H_{v+2s}$ affiliated with $\mathcal{A}_{v+2s}$.

The symbol of $M_{\varepsilon,v,s}(k)$ as an operator in $H_{v+2s}$ is equal to

$$\Lambda_{\varepsilon,v,s}(k)(\tau, \xi) = \frac{v - 1 - \varepsilon}{v - 1} k(\tau, \xi) \varphi^{\varepsilon/12} \left[\frac{1}{12} \ln \varphi - \frac{1}{v - 1 - \varepsilon}\right].$$

**Proof.** For simplicity of the proof we will use the notation $\varphi_1 = \varphi^{1/12}$. We prove first that the family $\lambda_{\varepsilon,v,s}(k)$ is a decreasing family in $\mathcal{A}_{v+s+\varepsilon}$ and hence in $\mathcal{A}_{v+2s}$.

Indeed, $((v - 1 - \varepsilon)/(v - 1)) \varphi^\varepsilon_1(\tau, \xi)$ is a decreasing family of operators in $\mathcal{A}_v$, and hence by Proposition 3.2 it follows that

$$\frac{v - 1 - \varepsilon}{v - 1} \varphi^\varepsilon_1(\tau, \xi) k(\tau, \xi)$$

is a decreasing family in $\mathcal{S}_{v+s}$, and hence in $\mathcal{S}_{v+s+\varepsilon}$. Since we know that these operators are already bounded in $\mathcal{A}_{v+\varepsilon}$, the first part of the statement follows immediately.
Denote by \( G(\varepsilon) = G(\varepsilon)(\overline{z}, \xi) \) the kernel represented by

\[
\frac{v - 1 - \varepsilon}{v - 1} \varphi_1^\varepsilon(\overline{z}, \xi) k(\overline{z}, \xi),
\]

which represents therefore a (decreasing) family in \( A_{v+s+\varepsilon} \) and hence in \( A_{v+2s} \). Fix \( \varepsilon_0 > 0 \) and let

\[
g_{\varepsilon}(\overline{z}, \xi) = \frac{G(\varepsilon)(\overline{z}, \xi) - G(\varepsilon_0)(\overline{z}, \xi)}{\varepsilon - \varepsilon_0}.
\]

Then \( g_{\varepsilon} \) is a negative (nonpositive) element in \( A_{v+2s} \). We want to find a formula for \( g_{\varepsilon'} - g_{\varepsilon} \). Obviously when \( \varepsilon \) converges to \( \varepsilon_0 \), the kernel \( g_{\varepsilon} \) converges (at least pointwise) to the kernel \( \Lambda_{\varepsilon_0,v,s}(k)(\overline{z}, \xi) \).

It is elementary calculus to find for \( H_{\varepsilon', \varepsilon}(\overline{z}, \xi) = g_{\varepsilon'}(\overline{z}, \xi) - g_{\varepsilon}(\overline{z}, \xi) \) the following pointwise expression:

\[
H_{\varepsilon, \varepsilon'} = (\varepsilon - \varepsilon') \int_0^1 \left( \int_0^1 tG''(\varepsilon(t, s)) \, ds \right) \, dt \tag{3.2}
\]

where \( \varepsilon(t, s) = s[(1 - t)\varepsilon_0 + t\varepsilon] + (1 - s)[(1 - t)\varepsilon_0 + t\varepsilon'] \) belongs to the interval determined by \( \varepsilon, \varepsilon', \varepsilon_0 \).

This formula holds at the level of kernels (that is, by evaluating both sides on any given points \( z, \xi \in \mathbb{H} \)).

On the other hand, because

\[
G(\varepsilon) = \frac{1}{v - 1}[(v - 1 - \varepsilon)\varphi_1^\varepsilon]k,
\]

we may compute immediately that

\[
G'(\varepsilon) = \frac{1}{v - 1}[(v - 1 - \varepsilon)\varphi_1^\varepsilon \ln \varphi_1 - \varphi_1^\varepsilon]k,
\]

\[
G''(\varepsilon) = \frac{1}{v - 1}[(v - 1 - \varepsilon) \ln^2 \varphi_1 - 2 \ln \varphi_1] \varphi_1^\varepsilon \cdot k.
\]

Furthermore we have the following expression for \( G''(\varepsilon) \):

\[
G''(\varepsilon) = \frac{v - 1 - \varepsilon}{v - 1} \varphi_1^\varepsilon \cdot k \left[ \ln^2 \varphi_1 - \frac{2}{v - 1 - \varepsilon} \ln \varphi_1 \right]
\]

\[
= \frac{v - 1 - \varepsilon}{v - 1} \left\{ k \left[ \varphi_1^{\varepsilon/2} \left( - \ln \varphi_1 + \frac{1}{v - 1 - \varepsilon} \right) \right]^2 - \frac{k \varphi_1^\varepsilon}{(v - 1 - \varepsilon)^2} \right\}.
\]

Thus we obtain further that

\[
G''(\varepsilon) = \frac{v - 1 - \varepsilon}{v - 1} k \left[ \varphi_1^{\varepsilon/2} \left( - \ln \varphi_1 + \frac{1}{v - 1 - \varepsilon} \right) \right]^2 - \frac{k \varphi_1^\varepsilon}{(v - 1 - \varepsilon)(v - 1)}.
\]
But because of the previous Proposition 3.2 and Lemma 2.4, we have that
\[
\left[ \varphi_1^{\varepsilon/2} \left( -\ln \varphi_1 + \frac{1}{v - \varepsilon} \right) \right]^2 = \left[ \varphi_1^{\varepsilon/2} \left( -\ln \varphi_1 + \frac{1}{v - \varepsilon/2 - 1 - \varepsilon/2} \right) \right]^2
\]
represents the square of an element
\[
\varphi_1^{\varepsilon/2} \left( -\ln \varphi_1 + \frac{1}{v - 1} \right)
\]
in \( S_{v-\varepsilon/2} \). The square of the above element consequently belongs to \( S_{2v-\varepsilon} \).

Hence
\[
\mathcal{R}(\varepsilon) = k \left[ \varphi_1^{\varepsilon/2} \left( -\ln \varphi_1 + \frac{1}{v - 1 - \varepsilon} \right) \right]^2,
\]
as a kernel, belongs to \( S_{s+2v-\varepsilon} \subseteq S_{s+2v} \).

In conclusion, we have just verified that
\[
G''(\varepsilon) = \mathcal{R}(\varepsilon) - \frac{k \varphi_1^\varepsilon}{(v-1)(v-1-\varepsilon)},
\]
where \( \mathcal{R}(\varepsilon) \) belongs to \( S_{2v+s} \).

Moreover, it is obvious that
\[
Q(\varepsilon) = \frac{k \varphi_1^\varepsilon}{(v-1)(v-1-\varepsilon)}
\]
is a bounded element in \( A_{2v+s} \), and that \( Q(\varepsilon) \) is consequently bounded by a constant \( C \), independent of all the variables \( v, s, \varepsilon \):
\[
Q(\varepsilon) \leq C \cdot \text{Id} \quad \text{in} \quad A_{2v+s},
\]
and hence
\[
Q(\varepsilon) \leq C \cdot \text{Id} \quad \text{in} \quad S_{2v+s}.
\]

We put this into the integral formula for
\[
H_{\varepsilon,\varepsilon'} = g(\varepsilon) - g'(\varepsilon') = \frac{G(\varepsilon) - G(\varepsilon_0)}{\varepsilon - \varepsilon_0} - \frac{G(\varepsilon') - G(\varepsilon_0)}{\varepsilon' - \varepsilon_0}
\]
to obtain that \( H_{\varepsilon,\varepsilon'} \) is of the form \((\varepsilon - \varepsilon') [\mathcal{R} - Q] \) where \( \mathcal{R} \) belongs to \( S_{2v+s} \) and \( Q \) belongs to \( (A_{v+2s})_+ \) and \( 0 \leq Q \leq C \cdot \text{Id} \).

But then \( \mathcal{R} \) belongs to \( A_{v+2s} \cap S_{v+2s} \), and hence \( \mathcal{R} \in (A_{v+2s})_+ \).

Thus, in \( A_{v+2s} \), we have (assuming \( \varepsilon - \varepsilon' \geq 0 \)) that
\[
H_{\varepsilon,\varepsilon'} \geq -(\varepsilon - \varepsilon')Q \geq -(\varepsilon - \varepsilon')C,
\]
so
\[ H_{\varepsilon, \varepsilon'} \geq - (\varepsilon - \varepsilon') C, \]
and therefore
\[ g_\varepsilon - g_\varepsilon' \geq C (\varepsilon' - \varepsilon). \]
Hence for \( \varepsilon \geq \varepsilon' \geq \varepsilon_0 \) we have that \( g_\varepsilon + C_\varepsilon \geq g_\varepsilon' + C_\varepsilon' \) in \( A_{v+2s} \), for a fixed positive constant \( C \).

Now recall that
\[ g(\varepsilon) = \frac{G(\varepsilon) - G(\varepsilon_0)}{\varepsilon - \varepsilon_0} \]
and that \( G(\varepsilon) \) itself was a decreasing family in \( A_{v+2s} \), so that \( g(\varepsilon) \) are negative elements in \( A_{v+2s} \).

Denote for simplicity \( h(\varepsilon) = - g(\varepsilon) \). Then what we just obtained is the following:

The operators \( h(\varepsilon) \) are positive elements in \( (A_{v+2s})_+ \). Moreover, \( h(\varepsilon) - C \varepsilon \leq h(\varepsilon') - C \varepsilon' \) if \( \varepsilon \geq \varepsilon' \), i.e., \( h(\varepsilon) - C \varepsilon \) is a decreasing family. By adding a big constant \( k \) to \( h(\varepsilon) \) we have that \( K + h(\varepsilon) - C \varepsilon \) is a decreasing family of positive elements in \( (A_{s+2v})_+ \).

Thus as \( \varepsilon \) decreases to \( \varepsilon_0 \) we have that \( K + h(\varepsilon) - C \varepsilon \) is an increasing family of positive operators in \( A_{v+s} \).

Moreover, the trace of \( h(\varepsilon) \) is equal to \( - \tau(g(\varepsilon)) \), which is
\[
- \int_F \left[ \frac{\varphi_1^\varepsilon(\overline{z}, z) - \varphi_1^{\varepsilon_0}(\overline{z}, z) \frac{\varepsilon - \varepsilon_0}{v-1}}{\varepsilon - \varepsilon_0} \right] k(\overline{z}, z) \, d\nu.
\]
This integral converges (in \( L^1(F \, d\nu_0) \)) to
\[
\int_F \left. \frac{d}{d\varepsilon} \frac{v-1-\varepsilon}{v-1} \varphi_1^\varepsilon(\overline{z}, z) \right|_{\varepsilon=\varepsilon_0} k(\overline{z}, z) \, d\nu_0(z)
\]
\[
= - \frac{v-1-\varepsilon_0}{v-1} \int_F \varphi_1^{\varepsilon_0}(\overline{z}, z) \left[ \ln \varphi_1 - \frac{1}{v-1-\varepsilon_0} \right] k(\overline{z}, z) \, d\nu_0(z),
\]
which is finite (the convergence is dominated here for example by \( C \varphi_1^{\varepsilon'_0} \), for some \( \varepsilon' \leq \varepsilon_0 \)).

Thus \( K + h(\varepsilon) - C(\varepsilon) \) are an increasing family in \( A_t \) (as \( \varepsilon \) decreases to \( \varepsilon_0 \)) and the supremum of the traces (in \( L^1(A_t) \)) is finite. By Lesbegue’s Dominated Convergence Theorem in \( L^1(A_t) \), the limit of \( K + h(\varepsilon) - C(\varepsilon) \) exists in \( L^1(A_t) \) and convergence is in the strong operator topology on a dense domain affiliated with \( A_t \). \( \square \)
**Corollary 3.4.** Let $\Lambda_{\varepsilon,v,s}(k)$ be the map, defined in the previous lemma, that associates to any positive $k$ in $A_s$ a positive element in $A_{v+2s}$, whose kernel is given by the formula:

$$\Lambda_{\varepsilon,v,s}(k)(\varpi,\xi) = \frac{v-1-\varepsilon}{v-1} k(\varpi,\xi) \phi^{\varepsilon/12} \left[ -\frac{1}{12} \ln \phi + \frac{1}{v-1-\varepsilon} \right].$$

Then $-\Lambda_{\varepsilon,v,s}$ is a completely positive map from $A_s$ into $L^1(A_{v+2s})$.

**Proof.** From the previous lemma we know that $\Lambda_{\varepsilon,v,s}(k)$ is well defined and belongs to $L^1(A_{v+2s})$. On the other hand $\Lambda_{\varepsilon,v,s}(k)$ is obtained by multiplication with a positive kernel in $S_v$, and hence (as in the proof of the complete positivity for $\Psi_{s,t}$) we obtain that $[\Lambda_{\varepsilon,v,s}(k_{pq})]_{p,q}$ is a positive in $M_N(\mathbb{C}) \otimes L^1(A_{v+2s})$, if $[k_{pq}]$ is a positive matrix in $M_N(\mathbb{C}) \otimes A_s$. □ □

**Corollary 3.5.** Let $\varepsilon_0 > 0$ and $t > 3+\varepsilon_0$. Let $\Lambda_\varepsilon$ be defined, on the space of all symbols $k$ representing operators in $\bigcup_{1<s<t-2-\varepsilon_0} A_s$, by the formula

$$\Lambda_{\varepsilon_0}(k) = \frac{d}{d\varepsilon}(k \otimes \phi^\varepsilon) \bigg|_{\varepsilon=\varepsilon_0}. $$

Note that the pointwise derivative of kernels is $(\phi^{\varepsilon_0} \ln \phi) \otimes k$.

Then $\Lambda_{\varepsilon_0}(k)$ belongs to $L^1(A_t)$, and moreover, the derivative is valid in the sense of the strong operator topology on a dense domain affiliated to $A_t$.

Fixing $1<s<t-2-\varepsilon_0$, there exists a sufficiently large constant $C$ (depending on $s,t,\varepsilon_0$) such that $-\Lambda_{\varepsilon_0} + Ck \otimes \phi^\varepsilon$ (and hence $-\Lambda_{\varepsilon_0} + C \cdot \text{Id}$) becomes a completely positive operator from $A_s$ into $A_t$.

**Proof.** Because of the condition $s<t-2-\varepsilon_0$, we can always find a constant $C$, by the previous lemma, such that the previous lemma applies to the operator $\Lambda_{\varepsilon_0} + Ck \otimes \phi^\varepsilon$. □

**Corollary 3.6.** Fix $t > 3$. For every $1<s<t-2$ and for every $k$ in $A_s$ there exists an (eventually unbounded) operator $\Lambda(k)$ (of symbol multiplication by $\ln \phi$) that is affiliated with $A_t$, and there exists a dense domain $\mathcal{D}$ in $H_t$ that is affiliated with $A_t$, such that the derivative

$$\frac{d}{d\varepsilon} \langle k \otimes \phi^\varepsilon, \xi, \eta \rangle_{H_t} \bigg|_{\varepsilon=0}$$

exists for all $\xi, \eta$ in $\mathcal{D}$ and is equal to

$$\langle \Lambda(k)\xi, \eta \rangle.$$
Moreover, there exists a constant $C$, depending only on $s, t$, such that for any positive matrix $[k_{p,q}]_{p,q=1}^P$ in $M_N(A_s)_+$, the operator matrix

$$-[(\Lambda + C \cdot \text{Id})(k_{p,q})]_{p,q=1}^N$$

represents a positive operator, affiliated with $A_t$.

**Remark.** The operator $k \odot \varphi^\varepsilon$ appearing in the previous statement is bounded. Indeed, modulo a multiplicative constant, $k \odot \varphi^\varepsilon$ is the symbol of $S_{\Delta^\varepsilon}kS_{\Delta^\varepsilon}^*$. If $k \in A_s$, then $S_{\Delta^\varepsilon}kS_{\Delta^\varepsilon}^*$ belongs to $A_{s+12\varepsilon}$, and since $s < t$, by choosing $\varepsilon$ small enough, we can assume that $S_{\Delta^\varepsilon}kS_{\Delta^\varepsilon}^*$ represents a bounded operator on $H_t$, and hence that the expression $\langle k \odot \varphi^\varepsilon \xi, \eta \rangle_{H_t}$ makes sense for all $\xi, \eta$ in $H_t$.

Before going to the proof of the statement of Corollary 3.6, we prove the following lemma (which will be used in the proof of Corollary 3.6) concerning the operator $k \odot \varphi^\varepsilon$ and the range of the operator $\Lambda_{\varepsilon,v,s}(k)$ defined in Lemma 3.3.

**Lemma 3.7.** With the notations from Lemma 3.3, let $k$ be an operator in $A_s$, $v > 1$, $\varepsilon > 0$. Let $M_{\varepsilon,v,s}(k)$ be the derivative (with respect to $\varepsilon$), which belongs to $L^1(A_{v+2s})$, of the decreasing family

$$\tilde{\lambda}_{\varepsilon,v,s}(k) = \frac{v - 1 - \varepsilon}{v - 1} \varphi^\varepsilon(\overline{z}, \xi)k(\overline{z}, \xi).$$

Then the range and init space of the unbounded operator $M_{\varepsilon,v,s}(k)$ are contained (and dense) in the closure of the range of $S_{\Delta^\varepsilon} \subseteq H_{2v+s}$ (more precisely in closure of the range of $S_{\Delta^\varepsilon}^{2v+s-\varepsilon}$).

**Proof.** Indeed, by what we have just proved, $M_{\varepsilon,v,s}(k)$ is the strong operator topology limit (on a dense domain affiliated with the von Neumann algebra), as $\varepsilon'$ decreases to $\varepsilon$, of the operators

$$\frac{G_{\varepsilon'}(\overline{z}, \xi) - G_{\varepsilon}(\overline{z}, \xi)}{\varepsilon' - \varepsilon}.$$ 

Recall that $G_{\varepsilon'}(\overline{z}, \xi)$ was the symbol (modulo a multiplicative constant) of the operator $S_{\Delta^\varepsilon}kS_{\Delta^\varepsilon}^*$.

Then by applying $(G_{\varepsilon'} - G_{\varepsilon})/(\varepsilon' - \varepsilon)$ to any vector $\xi$ in $H_t$, the outcome is already a vector in the closure of the range of $S_{\Delta^\varepsilon}$. This property is preserved in the limit. By selfadjointness the same is valid for the init space. \qed

We proceed now to the proof of Corollary 3.6.

**Proof of Corollary 3.6.** We start by constructing first the domain $D$. For $\varepsilon_0 > 0$ let $D_{\varepsilon_0} \subseteq H_t$ be the range of $(S_{\Delta^\varepsilon_0}^t)^*$, considered as an operator from $H_{t+12\varepsilon_0}$ into $H_t$. $D$ will be the increasing union (with respect to $\varepsilon_0$) of $D_{\varepsilon_0}$.
Let $B_{\varepsilon_0}$ be a right inverse, as an unbounded operator for the operator $S_{\Delta_0}$. Thus $B_{\varepsilon_0}$ acts from a domain dense in the closure of range $S_{\Delta_0}^t$ into $H_t$. It is clear that $B_{\varepsilon_0}$ is an intertwiner affiliated with the von Neumann algebras $A_t$ and $A_{t+12\varepsilon_0}$ (by von Neumann’s theory of unbounded operators, affiliated to a II\(_1\) factor [MvN]).

Thus, denoting by $P_{\varepsilon_0}$ the projection onto the closure of the range of $S_{\Delta_0}^t$ in $H_{t+12\varepsilon_0}$, the following properties hold true:

\[(S_{\Delta_0}^t)B_{\varepsilon_0} = P_{\varepsilon_0}.\]

By taking the adjoint, we obtain

\[B_{\Delta_0}^* (S_{\Delta_0}^t)^* = P_{\varepsilon_0}.\]

All compositions make sense in the algebra of unbounded operators affiliated with $A_t$ and $A_{t+12\varepsilon_0}$. On $H_{t+12\varepsilon_0}$, we let $M_{\varepsilon_0}(k)$ be the $L^1$ operator given by Corollary 3.4, whose symbol is

\[k \odot \varphi^{\varepsilon_0} \ln \varphi,\]

for $k$ in $A_s$.

We define $\Lambda_{\varepsilon_0}(k)$ by the following composition:

\[\Lambda_{\varepsilon_0}(k) = \frac{c_{t+12\varepsilon_0}}{c_t} B_{\varepsilon_0} M_{\varepsilon_0}(k) B_{\varepsilon_0}^*.\]

We want to prove that $\Lambda_{\varepsilon_0}$ does not depend on $\varepsilon_0$. Obviously (by [MvN]), the operator $\Lambda_{\varepsilon_0}(k)$ is affiliated with $A_t$.

Moreover, for $\xi, \eta$ in $D_{\varepsilon_0}$, which are thus of the form

\[\xi = S_{\Delta_0}^* \xi_1, \quad \eta = S_{\Delta_0}^* \eta_1,\]

for some $\xi_1, \eta_1$ in $H_{t+12\varepsilon_0}$, we have that

\[\langle \Lambda_{\varepsilon_0}(k) \xi, \eta \rangle = \langle \Lambda_{\varepsilon_0}(k) S_{\Delta_0}^* \xi_1, S_{\Delta_0}^* \eta_1 \rangle_{H_t}.\]

This is equal to

\[\frac{c_{t+12\varepsilon_0}}{c_t} \langle B_{\varepsilon_0} M_{\varepsilon_0}(k) B_{\varepsilon_0}^* S_{\Delta_0}^* \xi_1, S_{\Delta_0}^* \eta_1 \rangle_{H_t}\]

\[= \frac{c_{t+12\varepsilon_0}}{c_t} \langle P_{\varepsilon_0} M_{\varepsilon_0}(k) P_{\varepsilon_0} \xi_1, \eta_1 \rangle_{H_{t+12\varepsilon_0}}.\]

Because of Lemma 3.7, we know that this is further equal to

\[\frac{c_{t+12\varepsilon_0}}{c_t} \langle M_{\varepsilon_0}(k) \xi_1, \eta_1 \rangle.\]
We use the above chain of equalities to deduce that the definition of $\Lambda_{\varepsilon_0}(k)$ is independent of the choice of $\varepsilon_0$.

Indeed, assume we use another $\varepsilon'_0$, which we assume to be bigger than $\varepsilon_0$. Assume $\xi = S_{\Delta \varepsilon_0}^* \xi_2$. This is further equal to $S_{\Delta \varepsilon_0}^* S_{\Delta (\varepsilon'_0 - \varepsilon_0)}^* \xi_2$.

Then, by redoing the previous computations we arrive to the term

$$\frac{c_t + 12\varepsilon'_0}{c_t} \langle M_{\varepsilon'_0}(k)\xi_2, \eta_2 \rangle.$$

But on the other hand in this situation

$$\langle M_{\varepsilon_0}(k)\xi_1, \eta_1 \rangle = \frac{c_t + 12\varepsilon_0}{c_t} \langle M_{\varepsilon_0}(k)S_{\Delta (\varepsilon'_0 - \varepsilon_0)}^* \xi_2, S_{\Delta (\varepsilon'_0 - \varepsilon_0)}^* S_{\Delta \varepsilon_0}^* \eta_2 \rangle = \frac{c_t + 12\varepsilon_0}{c_t} \langle S_{\Delta (\varepsilon'_0 - \varepsilon_0)} M_{\varepsilon_0}(k)S_{\Delta (\varepsilon'_0 - \varepsilon_0)}^* \xi_2, \eta_2 \rangle.$$

To show independence of the choice of $\varepsilon_0$, we need consequently to prove that

$$\frac{c_t + 12\varepsilon_0}{c_t} \langle S_{\Delta (\varepsilon'_0 - \varepsilon_0)} M_{\varepsilon_0}(k)S_{\Delta (\varepsilon'_0 - \varepsilon_0)}^* \xi_2, \eta_2 \rangle$$

is equal to $\frac{c_t + 12\varepsilon_0}{c_t} \langle M_{\varepsilon'_0}(k)\xi_2, \eta_2 \rangle$.

Now all the operators are in $L^1$. Moreover, the symbol of

$$S_{\Delta (\varepsilon'_0 - \varepsilon_0)} M_{\varepsilon_0}(k)S_{\Delta (\varepsilon'_0 - \varepsilon_0)}^*$$

is

$$\frac{c_t + 12\varepsilon_0}{c_t} \varphi^\varepsilon_{\varepsilon_0 - \varepsilon_0}$$

times the symbol of $M_{\varepsilon_0}(k)$.

But the symbol of $M_{\varepsilon_0}(k)$ is $\varphi^{\varepsilon_0} \ln \varphi$ divided by $c_t + 12\varepsilon_0 / c_t$.

This shows independence of the choice of $\varepsilon_0$ (some care has to be taken when choosing $\xi_1, \eta_1, \xi_2, \eta_2$ given $\xi, \eta$). We always choose them in the init space of $S_{\Delta \varepsilon_0}^*$, respectively $S_{\Delta \varepsilon_0}^*$, $S_{\Delta \varepsilon'_0}^*$. By the von Neumann theorem we will be able to choose a common intersection domain for these operators.

Consequently, to check that the derivative of $\langle h \odot \varphi^\varepsilon \xi, \eta \rangle_{H_t}$ at $\varepsilon = 0$ is equal to the operator $\Delta k$ introduced in the statement of Corollary 3.6, we only have to check this for vectors $\xi, \eta$, that we assume to be of the form

$$\xi = S_{\Delta \varepsilon_0}^* \xi_1, \quad \eta = S_{\Delta \varepsilon_0}^* \eta_1.$$

Then, modulo a multiplicative constant, $\langle k \odot \varphi^\varepsilon \xi, \eta \rangle_{H_t}$ becomes

$$\langle k \odot \varphi^{\varepsilon + \varepsilon_0} \xi_1, \eta_1 \rangle_{H_{t + \varepsilon}}.$$
By a change of variables, the derivative at 0 of \( \langle k \odot \varphi^\varepsilon \xi, \eta \rangle_{H_t} \) becomes the derivative at \( \varepsilon_0 \) of the later expression: \( \langle k \odot \varphi^\varepsilon \xi_1, \eta_1 \rangle_{H_{t+\varepsilon}} \). Up to a multiplicative constant, this derivative exists and it is equal to \( \langle M(k)\xi_1, \eta_1 \rangle \), which is by definition \( \langle \Lambda(k)\xi_1, \eta_1 \rangle \).

Finally, observe that for any constant \( C \), \( \langle (\Lambda(k) + C)\xi_1, \eta_1 \rangle \) is equal to

\[ B_{\varepsilon_0}(M_{\varepsilon_0}(k) + C' S_{\Delta\varepsilon_0} k S_{\Delta\varepsilon_0}^* \)B_{\varepsilon_0}^* \]

for a constant \( C' \) obtained from \( C \) by multiplication by a normalization factor depending on \( t \) and \( \varepsilon_0 \).

Consequently, if \( [k_{p,q}]_{p,q=1}^N \) is a positive matrix in \( \mathcal{A}_s \), then by using the complete positivity result of Lemma 3.3, we infer that the matrix

\[ -[M_{\varepsilon_0}(k_{p,q}) + C' \varphi^\varepsilon_0 \odot k_{p,q}]_{p,q=1}^N \]

represents a positive operator in \( M_p(\mathbb{C}) \otimes (\mathcal{A}_{t+\varepsilon_0})^\ast \).

Since \( \varphi^\varepsilon_0 \odot k_{p,q} \) is \( S_{\Delta\varepsilon_0} k_{p,q} S_{\Delta\varepsilon_0}^* \), we get that \( -[\Lambda(k_{p,q})]_{p,q=1}^N \) is a positive matrix of operators affiliated to \( M_p(\mathbb{C}) \otimes \mathcal{A}_t \). \( \square \)

Remark. If want to deal with less general operators (paying the price of not including the identity operator in the domain of \( \Lambda \)), then we can take operators of the form \( S_{\Delta\varepsilon_0} k S_{\Delta\varepsilon_0}^* \) that belong to \( \mathcal{A}_s \), \( s < t - 2 \), \( s - \varepsilon_0 > 1 \), and then \( \Lambda(k) \) will be in \( L^1(\mathcal{A}_t) \), for such a kernel \( k \), directly from Lemma 3.3.
§4. Construction of an (unbounded) coboundary for the Hochschild cocycle in Berezin’s deformation

In this section we analyze the Hochschild 2-cocycle

\[ C_t(k, l) = \frac{d}{ds}(k \ast_s l) \bigg|_{s=t} \]

that arises in the Berezin deformation. We prove that the operator introduced in the previous section (§3) may be used to construct an operator \( L \) (defined on a dense subalgebra of \( A_t \)), taking values in the algebra of unbounded operators affiliated with \( A_t \). \( L \) will be defined on a dense subalgebra of \( A_t \).

The equation satisfied by \( L \) is

\[ C_t(A, B) = L_t(A \ast_t B) - A \ast_t L_t(B) - L_t(A) \ast_t B \]

and this will be fulfilled in the form sense (that is, by taking the scalar product with some vectors \( \xi, \eta \) in both sides).

The fact that \( L \) takes its values in the unbounded operators affiliated with \( A_t \) presents some inconvenience, but we recall that in the setting of type II_1 factors, by von Neumann theory [MvN], the algebra of unbounded (affiliated) operators is a well behaved algebra (with respect to composition, sum and the adjoint operations).

In fact we will prove that \( L \) comes with two summands

\[ L(k) = \Lambda(k) - \frac{1}{2}\{T, k\} \]

where \(-T\) is positive affiliated with \( A_t \) and \(-\Lambda\) a completely positive (unbounded) map. In the next section we prove that \( T \) is \( \Lambda(1) \).

For technical reasons (to have an algebra domain for \( L \)), we require that \( k \in \hat{A}_s, s < t - 2, \) since we know (by [Ra]) that the space of operators in \( A_t \) represented by such kernels is closed under taking the \( \ast_t \) multiplication (the multiplication in \( A_t \)).

The operator \( \Lambda \) will be (up to an additive multiple of the identity) multiplication of the symbol by \( \ln \varphi \). This operation is made more precise in Corollary 3.6.

If \( k \) is already of the form \( S_{\Delta \varepsilon_0} k S_{\Delta \varepsilon_0} \), for some \( k \) in \( A_{s - \varepsilon_0} \), \( s - \varepsilon_0 > 1, s < t - 2, \) then \( \Lambda(k) \) is an operator in \( L^1(A_t) \). In order to have the identity \( \text{Id} \) in the domain) we allow \( \Lambda \) to take its values in the operators affiliated with \( A_t \).

Consequently \( \Lambda(1) \) is just the positive operator, affiliated with \( A_t \), which corresponds to the symbol \( \ln \varphi = \ln(\Delta(z)\Delta(\xi)[(z - \xi)/(-2i)]^{12}) \) plus a suitable multiple of the identity.
To deduce the expression for $C_t(k, l)$ one could argue formally as follows:

\[
C_t(k, l)(z, \xi) = \frac{c_t'}{c_t} (k *_t l)(z, \xi)
\]

\[
+ c_t \int k(z, \eta)l(z, \xi)[z, \eta, \xi]^t \ln[z, \eta, \xi] \, d\nu_0(\eta).
\]  

(4.1)

At this point to get a $\Gamma$-invariant expression, we should decompose $\ln[z, \eta, \eta, \xi] = \ln[(z - \xi)(\xi - \eta))/((z - \eta)(\eta - \xi))]$ as a sum of $\Gamma$-invariant functions. The easier way to do that would be to write

\[
\ln \varphi[z, \eta, \eta, \xi] = \frac{1}{12} [\ln \varphi(z, \xi) + \ln \varphi(\eta, \eta) - \ln \varphi(\eta, \xi) - \ln \varphi(z, \eta)].
\]

If we use this expression back in (4.1) we would get four terms which are described as follows.

The term corresponding to $\ln \varphi(z, \xi)$ will come in front of the integral and give

\[
\frac{1}{12} \ln \varphi(z, \xi)(k *_t l)(z, \xi).
\]

The term corresponding to $\ln \varphi(z, \eta)$ would multiply $k(z, \eta)$ and would correspond formally to $\frac{1}{12}[(\ln \varphi)k] *_t l$.

The term corresponding to $\ln \varphi(\eta, \eta)$ would give the following integral:

\[
c_t \int k(z, \eta)(\ln \varphi(\eta, \eta))l(\eta, \xi)[z, \eta, \eta, \xi]^t \, d\nu_0(\eta).
\]

This is formally $\frac{1}{12} k *_t T_{\ln \varphi}^t *_t l$. If $\ln \varphi$ were a bounded function and $T_{\ln \varphi}^t$ the Toeplitz operator with this symbol, this expression would make perfect sense.

Putting this together we would get

\[
k *_t l = C_t(k, l) = \frac{c_t'}{c_t} k *_t l + \frac{1}{12} \ln \varphi(k *_t l) - \left[\left(\frac{1}{12} \ln \varphi\right) k\right] *_t l
\]

\[
- k *_t \left(\frac{1}{12} \ln \varphi\right) l + k *_t T_{(1/12)\ln \varphi}^t *_t l.
\]

This would give that $C_t(k, l)$ is implemented by the operator

\[
\mathcal{L}(k) = \left(\frac{1}{12} \ln \varphi - \frac{c_t'}{c_t}\right) k - \frac{1}{2} \{T_{(1/12)\ln \varphi}, k\},
\]

where by $\{a, b\}$ we denote the Jordan product $ab + ba$. 
This means that $C_t(k, l)$ is implemented by the operator $L(k)$, which resembles the canonical form of a generator of a dynamical semigroup: a positive map ($-\ln \varphi$ is a positive kernel, when adding a constant) minus a Jordan product.

To justify such a formula and the convergence of the integrals involved seems to be a difficult task, so we will follow a different but more rigorous approach, which consists in defining the operator $(-\ln \varphi)k$, as in the previous section, as a strong operator topology derivative.

To that end we introduce a family of completely positive maps that canonically connect the fibers of the deformation. These maps arise from automorphic forms, viewed (as in [GHJ]) as intertwining operators.

In the next lemma we give a precise meaning for the operator $T^t_{\ln \varphi}$, which is the (unbounded) Toeplitz operator acting on $H_t$ with symbol $\ln \varphi$.

**Lemma 4.1.** We define $T = T^t_{\ln \varphi}$, as a quadratic form, by

$$\langle T^t_{\ln \varphi} \xi, \xi \rangle_{H_t} = \int_{\mathbb{H}} (\ln \varphi) |\xi|^2 \, d\nu_t,$$

on the domain

$$\mathcal{D} = \left\{ \xi \in H_t \left| \int_{\mathbb{H}} (\ln \varphi) |\xi|^2 \, d\nu_t < \infty \right. \right\}.$$

Clearly, $\mathcal{D}$ is dense in $H_t$, as it contains $\mathcal{D}_0 = \bigcup_{\varepsilon > 0} \text{Range } S_{\Delta^\varepsilon}$, where $S_{\Delta^\varepsilon}$ is viewed as the operator of multiplication by $\Delta^\varepsilon$ from $H_{t-\varepsilon}$ into $H_t$.

Moreover, $T^t_{\ln \varphi}$ is the restriction to $H_t$ of the multiplication operator by $\ln \varphi$ on $L^2(\mathbb{H}, \nu_t)$. For $\xi, \eta$ in $\mathcal{D}_0$ we have that

$$\langle T^t_{\ln \varphi} \xi, \eta \rangle_{H_t} = \frac{d}{d\varepsilon} \langle T^t_{\varphi^\varepsilon} \xi, \eta \rangle_{H_t} \bigg|_{\varepsilon = 0}.$$

**Proof.** All that stated above is obvious: the last statement is justified because, if $S_{\Delta^0} : H_{t-12\varepsilon_0} \to H_t$, then $S_{\Delta^0}^* T^t_{\ln \varphi} S_{\Delta^0}$ is obviously equal to $T^t_{\ln \varphi^0 \varphi^0}$. \[ \square \]

In the next lemma we explain the role of automorphic forms as comparison operators between different algebras $\mathcal{A}_t$ (they are a sort of tool for making a differentiable field out of the algebras $\mathcal{A}_t$).

**Definition 4.2.** For $s \geq t$, let $\theta_{s,t} : \mathcal{A}_t \to \mathcal{A}_s$ be the completely positive map associating to $k$ in $\mathcal{A}_t$ the bounded operator in $\mathcal{A}_s$ defined as

$$\theta_{s,t}(k) = (S_{\Delta^{(s-t)/12}}) k (S_{\Delta^{(s-t)/12}})^*$$

Clearly the symbol of $\theta_{s,t}(k)$ is

$$\frac{c_t}{c_s} k(\overline{z}, \xi)(\varphi(\overline{z}, \xi))^{(s-t)/12}.$$
Also we have \( \theta_{s,t}(\theta_{t,v}(k)) = \theta_{s,v}(k) \) for all \( s \geq t \geq v \).

The following property is a trivial consequence of the definition of \( \theta_{s,t} \). It expresses the fact that \( \theta_{s,t} \) has an almost multiplicative structure, as follows.

**Lemma 4.3.** For \( s \geq t \) the following holds for all \( k, l \) in \( A_t \):

\[
\theta_{s,t} \left( k * T^t_{\phi(s-t)/12} * l \right) = \theta_{s,t}(k) * \theta_{s,t}(l).
\]

**Proof.** This is obvious since \( \theta_{s,t}(k) \theta_{s,t}(l) \) (with product in \( A_s \)) is equal to

\[ S_{\Delta(s-t)/12} k S^*_{\Delta(s-t)/12} S_{\Delta(s-t)/12} l S^*_{\Delta(s-t)/12}. \]

But an obvious formula shows \( S^*_{\Delta(s-t)/12} S_{\Delta(s-t)/12} \) is equal to \( T^t_{\phi(s-t)/12} \) (see, e.g., [Ra]).

We intend next to differentiate the above formula, in \( s \), while keeping \( t \) fixed. In order to do this we will need to differentiate \( \theta_{s,t}(k) \). One problem that arises is the fact that \( a \) priori \( \theta_{s,t}(k) \) belongs to \( A_s \) rather than \( A_t \). But if \( k \) belongs to some \( A_{t_0} \), with \( t_0 < t \), and \( s \) is sufficiently close to \( t \), then \( \theta_{s,t}(k) \) will be (up to a multiplicative constant) represented by the symbol of \( \theta_{s+t-t_0,t_0}(k) \). Since \( s \) was small, this defines (via \( \Psi_{t,t-t_0+s} \)) a bounded operator in \( A_t \). Thus for such \( k \) it makes sense to define \( \langle \theta_{s,t}(k) \xi, \eta \rangle_{H_t} \) for all vectors \( \xi, \eta \) in \( H_t \).

We differentiate this expression with respect to \( s \). The existence of the derivative, in the strong operator topology, was already shown in the previous section. We reformulate Corollary 3.6, in the new setting.

**Lemma 4.4.** Let \( t > 3 \) and let \( k \) belong to \( A_{s_0} \), where \( 1 < s_0 < t - 2 \). Then there exists a dense domain \( D_0 \) (eventually depending on \( k \)) that is affiliated with \( A_t \) such that the following expression:

\[
\langle X_t(k) \xi, \eta \rangle_{H_t} = \frac{d}{ds} \bigg|_{s=t} \langle \theta_{s,t}(k) \xi, \eta \rangle_{H_t}
\]

defines a linear operator \( X_t \) on \( D \), which is affiliated with \( A_t \) (and hence closable [Ha]).

Moreover, for a sufficiently large constant \( C \) (depending on \( s, t \)), \(-X_t + C \cdot \text{Id}\) becomes a completely positive map with values in the operators affiliated to \( A_t \).

Consider the (non-unital) subalgebra \( \tilde{A}_{s_0} \subseteq A_{s_0} \), which is also weakly dense, consisting of all operators \( k \) in \( A_{s_0} \) that are of the form \( S_{\Delta s_0} k_1 S^*_{\Delta s_0} \) (where \( S_{\Delta s_0} \) maps \( H_{s_0-12s_0} \) into \( H_s \)), where \( k_1 \) belongs to \( A_{s_0-12s_0} \) and \( s_0 - 12s_0 \) is assumed bigger than 1.
Then $X_t$ also maps $\tilde{A}_s$ into $L^1(A_t)$. For such a $k$ the limit in the definition of $X_t$ is in the strong operator topology on a dense, affiliated domain.

Before going into the proof we make the following remark (which is not required for the proof).

Remark. Since the kernel of the operator $\theta_{s,t}(k)$ (in $A_s$) is equal to

$$\frac{ct}{cs} \cdot k(\overline{z}, \xi)[\varphi(\overline{z}, \xi)]^{(s-t)/12},$$

it follows that $X_t(k)$ is associated (in a sense that doesn’t have to be made precise for the proof) to

$$\left(-\frac{ct}{c't} + \frac{1}{12} \ln \varphi \right) k,$$

which appeared in the formula in the introduction.

Proof of Lemma 4.4. Because of the form of the symbol we may use Corollary 3.6. □ □

The main result of our paper shows that, by accepting an unbounded coboundary, the Hochschild 2-cocycle appearing in Berezin’s deformation is trivial, and the coboundary (which is automatically dissipative) has a form very similar to the canonical expression of a generator of a quantum dynamical semigroup.

First we deduce a direct consequence out of the formula in Lemma 4.3.

PROPOSITION 4.5. Fix a number $t > 3$. Consider the algebra $\tilde{A}_t \subseteq A_t$ consisting of all $k \in A_s$ for some $s < t - 2$ that are of the form $S_{\Delta \varepsilon_0} k_1 S_{\Delta \varepsilon_0}^*$, for some $\varepsilon_0$ (such that $s - \varepsilon_0 > 1$) and $k_1 \in A_{s_0 - \varepsilon}$.

Let $X_t$ be the operator defined in the previous lemma. Then

$$\left. \frac{d}{ds} \right|_{s=t} \theta_{s,t} \left( k \ast_t T^t_{(s-t)/12} \ast_t l \right) = X_t(k \ast_t l) + k \ast_t T^t_{(1/12) \ln \varphi \ast_t l}, \tag{4.2}$$

$$\left. \frac{d}{ds} \right|_{s=t} [\theta_{s,t}(k) \ast_s \theta_{s,t}(l)] = X_t(k) \ast_t l + C_t(k, l) + k \ast_t X_t(l), \tag{4.3}$$

for all $k, l \in \tilde{A}_t$. Consequently the two terms on the right-hand side of (4.2) and (4.3) are equal, that is,

$$X_t(k \ast_t l) + k \ast_t T^t_{(1/12) \ln \varphi \ast_t l} = X_t(k) \ast_t l + C_t(k, l) + k \ast_t X_t(l).$$

Before proceeding to the proof of Proposition 4.5, we note that $\tilde{A}_t$ is indeed an algebra (see also the end of this section). Assume that $k, l$ are given, but that
they correspond to two different choices of \( s \), say \( s, s' \), with \( s' < s \). Because \( \Psi_{s,s'} \) maps \( A_{s'} \) into \( A_s \), we can assume \( s = s' \). Then when \( k = S_{\Delta s_0} k_1 S_{\Delta s_0}^* \), \( l = S_{\Delta s_0}^* l_1 S_{\Delta s_0} \), and say \( s_0 < s_0' \). Then we replace the expression of \( l \) as

\[
S_{\Delta s_0}^* \left[ S_{\Delta s_0'}^* \left( S_{\Delta s_0'} \right)^* \right] S_{\Delta s_0}^* ,
\]

and choose the new \( l_1 \) to be \( S_{\Delta s_0}^* l_1 \left( S_{\Delta s_0} \right)^* \).

**Proof of Proposition 4.5.** We will give separate proofs for each of the equalities (4.2), (4.3). Of course these are the product formula for derivatives, but the complicated nature of the operator functions obliges us to work on the nonunital algebra \( \tilde{A} \). This might be just a technical condition that perhaps could be dropped.

**Proof of equality (4.2).**

\[
\frac{d}{ds} \theta_{s,t} \left( k * T_{\varphi(s-t)/12}^t * l \right) = X_t(k * l) + k * T_{\varphi(s-t)/12}^t \ln \varphi * l.
\]

We start with the left-hand side: Denote \( P_s = k * T_{\varphi(s-t)/12}^t * l \), for fixed \( h, l \in \tilde{A} \).

We have to evaluate (against \( \langle \cdot, \xi \rangle \eta \), where \( \xi, \eta \) belong to a suitable dense domain \( D \) affiliated with \( A \)) the expression:

\[
\frac{\theta_{s,t}(P_s) - P_t}{s - t} = \theta_{s,t} \left( \frac{P_s - P_t}{s - t} \right) + \frac{\theta_{s,t}(P_t) - P_t}{s - t}.
\]

The second term converges when \( s \searrow t \), since \( P_t = k * l \), to \( \frac{d}{ds}|_{s=t} \theta_{s,t}(P_t) \), which is by definition \( X_t(k * l) \). Here we rely on the fact that \( \tilde{A} \) is an algebra, so that \( k * l \) belongs to the domain of \( X_t \).

For the limit of the expression \( \theta_{s,t}((P_s - P)/(s - t)) \), we note that

\[
\frac{P_s - P_t}{s - t} = k * T_{h_s,t}^t * l,
\]

with

\[
h_s,t = \frac{\varphi(s-t)/12 - \text{Id}}{s - t}.
\]

Assume now that \( k = S_{\Delta s_0} k_1 S_{\Delta s_0}^* \), \( l = S_{\Delta s_0} l_1 S_{\Delta s_0}^* \).

Then \( \theta_{s,t}(k * T_{h_s,t}^t * l) \) is equal to

\[
S_{\Delta(s-t)/12} \left( (S_{\Delta s_0} k_1 S_{\Delta s_0}^* ) * T_{h_s,t}^t \right) (S_{\Delta s_0} l_1 S_{\Delta s_0}^* ) S_{\Delta(s-t)/12}^*.
\]

This is easily seen to be equal to

\[
S_{\Delta((s-t)/12) + \varepsilon_0} \left[ k_1 * T_{\varphi_0}^t h_{s,t} * l_1 \right] S_{\Delta((s-t)/12) + \varepsilon_0}^*.
\]

\[
= \theta_{s,t - \varepsilon_0} \left( k_1 * T_{\varphi_0}^t h_{s,t} * l_1 \right).
\]
Denote \( \widetilde{P}_s = k_1 *_{t-\varepsilon_0} T^t_{\varphi^{\varepsilon_0} h_{s,t}} *_{t-\varepsilon_0} l_1. \)

As \( s \) decreases to \( t \), we have (as \( \varphi^{\varepsilon_0} \ln \varphi \) is bounded) that \( \widetilde{P}_s \) converges in the uniform operator topology, to \( k_1 *_{t-\varepsilon_0} T^t_{(1/12)\varphi^{\varepsilon_0} \ln \varphi} *_{t-\varepsilon_0} l_1 \). This is because \( \varphi^{\varepsilon_0} ((\varphi^{s-t}/12 - \text{Id})/(s-t)) \) converges uniformly to \( \frac{1}{12} \varphi^{\varepsilon_0} \ln \varphi \), since \( \varphi \) is a bounded function.

Also, if \( s \) is sufficiently close to \( t \), \( \theta_{s,t-\varepsilon}(k_1) \) defines for every \( k_1 \in \mathcal{A}_{t-\varepsilon_0} \) a bounded operator on \( \mathcal{A}_t \). Indeed, \( \theta_{s,t-\varepsilon_0}(k_1) \) has symbol (up to a multiplicative constant) equal to \( \varphi^{(s-t+\varepsilon_0)/12} k_1 \). This is well defined as the kernel of an operator in \( \mathcal{A}_t \), since \( k_1 \in \mathcal{A}_{t_0-\varepsilon_0} \).

Thus \( \theta_{s,t_0-\varepsilon} \) can be thought of as a completely positive map from \( \mathcal{A}_{t_0-\varepsilon} \) into \( \mathcal{A}_t \). Moreover, \( \theta_{s,t_0-\varepsilon}(1) \), which is \( S_{\Delta(s-t_0+\varepsilon)/12} S^*_{\Delta(s-t_0+\varepsilon)/12} \), is less than a constant \( C \) (not depending on \( s \)) times the identity.

Hence the linear maps \( \theta_{s,t_0-\varepsilon} \), acting from \( \mathcal{A}_{t_0-\varepsilon} \) into \( \mathcal{A}_t \), are uniformly bounded. Consequently, when evaluating

\[
\left| \left\langle \left( \theta_{s,t_0-\varepsilon}(\widetilde{P}_s) - \theta_{t,t_0-\varepsilon}(\widetilde{P}_t) \right) \xi, \eta \right\rangle \right| ,
\]

we can majorize by

\[
\left| \langle \theta_{s,t_0-\varepsilon}(\widetilde{P}_s - \widetilde{P}_t) \xi, \eta \rangle \right| + \left| \langle (\theta_{t,t_0-\varepsilon} - \theta_{s,t_0-\varepsilon}) (\widetilde{P}_t) \xi, \eta \rangle \right| .
\]

The first term goes to zero by uniform continuity of \( \theta_{s,t_0+\varepsilon} \) with respect to \( s \) (and since \( \| \widetilde{P}_s - \widetilde{P}_t \| \to 0 \)). The second goes to zero because of the pointwise strong-operator-topology continuity of the map \( s \to \theta_{s,t_0-\varepsilon} \).

Thus \( \theta_{s,t} ((P_s - P_t)/(s-t)) \) converges to \( \theta_{t,t_0-\varepsilon}(\widetilde{P}_t) \), which was

\[
S_{\Delta^{\varepsilon_0}} \left( k_1 *_{t-\varepsilon_0} T^t_{(1/12)\varphi^{\varepsilon_0} \ln \varphi} *_{t} l_1 \right) S^*_{\Delta^{\varepsilon_0}},
\]

which is equal to \( k *_{t} T^t_{\ln \varphi} *_{t} l \).

**Proof of equality (4.3).**

\[
\frac{d}{ds} \bigg|_{s=t} \theta_{s,t}(k) *_{s} \theta_{s,t}(l) = X_t k *_{t} l + C_t(k,l) + k *_{t} X_t.
\]

We verify this equality by evaluating it on \( \langle \cdot, \xi \rangle \eta, \xi, \eta \in \mathcal{D} \), where \( \mathcal{D} \) is a dense domain affiliated to \( \mathcal{A}_t \).

We write the expression

\[
\frac{\theta_{s,t}(k) *_{s} \theta_{s,t}(l) - k *_{t} l}{s-t}
\]
as
\[
\frac{\theta_{s,t}(k) \ast_s \theta_{s,t}(l) - \theta_{s,t}(k) \ast_t \theta_{s,t}(l)}{s - t} + \frac{\theta_{s,t}(k) \ast_t \theta_{s,t}(l) - k \ast_t l}{s - t}.
\]

We will analyze first the first summand and prove that
\[
\frac{\theta_{s,t}(k) \ast_s \theta_{s,t}(l) - \theta_{s,t}(k) \ast_t \theta_{s,t}(l)}{s - t}
\]
converges to \(C_t(k \ast_t l)\).

We use the symbols of \(k, l\), and then find that the symbols of \(\theta_{s,t}(k)\), \(\theta_{s,t}(l)\) are \(\phi(s - t)/12k\), \(\varphi(s - t)/12l\), up to multiplicative constants that we ignore here (because the argument has a qualitative nature).

Then the symbol of the expression in (4.4) is
\[
\int_H \left(k \varphi(s - t)/12 \right)(z, \eta) \left(l \varphi(s - t)/12 \right)(\overline{\eta}, \xi) \frac{[\overline{z}, \eta, \overline{\eta}, \xi]^s - [\overline{z}, \eta, \overline{\eta}, \xi]^t}{s - t} \nu_0(\eta).
\]

By the mean-value theorem, with \(\alpha_s(v) = sv + (1 - v)t\), the expression becomes
\[
\int_0^1 \int_H \left(k \varphi(s - t)/12 \right)(z, \eta) \left(l \varphi(s - t)/12 \right)(\overline{\eta}, \xi) [\overline{z}, \eta, \overline{\eta}, \xi]^\alpha_s(v) \ln[\overline{z}, \eta, \overline{\eta}, \xi] \nu_0(\eta) \, dv.
\]

Similarly (by ignoring the numerical factors due to the constants \(c_s\)) we have that
\[
C_t(\theta_{s,t}(k), \theta_{s,t}(l))
\]
contains the integral
\[
\int_H \left(k \varphi(s - t)/12 \right)(z, \eta) \left(l \varphi(s - t)/12 \right)(\overline{\eta}, \xi) [\overline{z}, \eta, \overline{\eta}, \xi]^t \ln[\overline{z}, \eta, \overline{\eta}, \xi] \nu_0(\eta).
\]

Taking the difference, we obtain the following integral:
\[
\int_H \left(k \varphi(s - t)/12 \right)(z, \eta) \left(l \varphi(s - t)/12 \right)(\overline{\eta}, \xi) 
\left(\frac{[\overline{z}, \eta, \overline{\eta}, \xi]^s - [\overline{z}, \eta, \overline{\eta}, \xi]^t}{s - t} - [\overline{z}, \eta, \overline{\eta}, \xi]^t \ln[\overline{z}, \eta, \overline{\eta}, \xi] \right) \nu_0(\eta).
\]

By Taylor expansion, this is \((s - t)\) times a term involving the integral
\[
\int_H \left(k \varphi(s - t)/12 \right)(z, \eta) \left(l \varphi(s - t)/12 \right)(\overline{\eta}, \xi) [\overline{z}, \eta, \overline{\eta}, \xi]^s' \left(\ln[\overline{z}, \eta, \overline{\eta}, \xi]\right)^2 \nu_0(\eta),
\]
where \(s'\) is the interval determined by \(s\) and \(t\).
We have to prove that the integral of the absolute value of the integrand in the above integral is bounded by a constant independent of the choices of \(s'\) (and \(s\)).

We write

\[
|\ln |\overline{z}, \eta, \overline{\eta}, \xi|| = \frac{d(|z, \eta|) d(\overline{\eta}, \xi)}{d(z, \xi)}.
\]

Then

\[
|\ln |\overline{z}, \eta, \overline{\eta}, \xi|| \leq |\ln |d(z, \eta)|| + |\ln |d(\overline{\eta}, \xi)|| + |\ln |d(z, \xi)||.
\]

Also we note that the logarithm in \(\ln |\overline{z}, \eta\rangle\) has bounded imaginary part, as the branches in

\[
\ln |(\overline{z} - \eta)/(-2i)|, \quad \ln |(\overline{\eta} - \xi)/(-2i)|, \quad \ln |(\overline{\xi} - \xi)/(-2i)|
\]

have imaginary part in the fixed segment \([0, 2\pi]\).

Thus the term that we have to evaluate will involve terms of the form

\[
\int_{\mathbb{H}} |k'(\overline{z}, \eta)| |l'(\overline{\eta}, \xi)| |d(\overline{z}, \eta)||^{s'} |d(\overline{\eta}, \xi)||^{s} \, d\nu_0(\eta),
\]

where \(k'(\overline{z}, \eta)\) could be \(k\phi^{(s-t)/12}(\overline{z}, \eta)\), eventually multiplied by a power (1 or 2) of \(\ln d(\overline{z}, \eta)\). A similar assumption holds for \(l\).

By the Cauchy–Schwarz inequality, this expression is bounded by products of:

\[
\left(\int_{\mathbb{H}} \left|k'(\overline{z}, \eta)\right|^2 |d(\overline{z}, \eta)|^{2s} \, d\nu_0(\eta)\right)^{1/2} \left(\int_{\mathbb{H}} \left|l'(\overline{\eta}, \xi)\right|^2 |d(\overline{\eta}, \xi)|^{2s} \, d\nu_0(\eta)\right)^{1/2}.
\] (4.9)

But such expressions are finite, because we know that \(k, l\) are in \(A_{t_0}\) for some fixed \(t_0 < t\), and hence in \(L^2(A_{t_0})\), and consequently the integral

\[
\int_{\mathbb{H}} \left|k(\overline{z}, \eta)\right|^2 |d(\overline{z}, \eta)|^{2t_0} \, d\nu_0(\eta)
\] (4.10)

is finite.

Moreover, \(\phi(\overline{z}, \xi) = \overline{\Delta(z)} \Delta(\xi)(|\overline{z} - \xi|)/(-2i)|^{12}\). This term is bounded in absolute value by \(|d(\overline{z}, \xi)|^{-12}\), so that \(\phi(\overline{z}, \xi)|^{(s-t)/12}\) is bounded by \(|d(\overline{z}, \xi)|^{s-t}\). Also \(|\ln |d(\overline{z}, \eta)|| d(\overline{z}, \eta)|^\varepsilon\) is bounded for any choice of \(\varepsilon\).

Thus, by choosing \(\varepsilon\) small enough, the finiteness of the integral in (4.10) implies the finiteness of the integral in (4.9).

Thus the integral in (4.7) tends to zero. This completes the proof that

\[
\frac{\theta_{s,t}(k)x_s\theta_{s,t}(l) - \theta_{s,t}(k)x_t\theta_{s,t}(l)}{s-t}
\]

converges to \(C_t(k, l)\).
The remaining term to be analyzed is
\[ \frac{d}{ds} \Bigg|_{s=t} \frac{\theta_{s,t}(k) \ast_t \theta_{s,t}(l) - k \ast_t l}{s-t}. \]
We have to show that the limit is \( (X_t(k)) \ast_t l + k \ast_t (X_t(l)) \) (evaluated on vectors \( \xi, \eta \) in a dense domain affiliated to \( A_t \)).

Fix the vector \( \xi, \eta \). Then we have to analyze the following sum:
\[ \langle \frac{\theta_{s,t}(l) - l}{s-t} \xi, \theta_{s,t}(k^* \eta) \rangle + \langle \frac{\theta_{s,t}(k) - k}{s-t} l \xi, \eta \rangle. \]
The second term obviously converges to \( \langle [X_t(k) \ast_t l] \xi, \eta \rangle \), and the first term is also convergent to \( \langle X_t(l) \xi k^*, \eta \rangle \), because \( ((\theta_{s,t}(l) - l)/(s-t)) \xi \), for \( \xi \) in a dense domain \( D \), converges in norm to \( X_t(l)\xi \). Indeed in Corollary 3.6 we proved that \( (\theta_{s,t}(l) - l)/(s-t) \) converges strongly to \( X_t \), on a dense domain topology, because the convergence (for \( l = S_{\Delta \epsilon_0} l_1(S_{\Delta \epsilon_0}^* l_1) \), \( l_1 \in A_{t_0-\epsilon_0} \)) comes by proving that the partial fractions \( -(\theta_{s,t}(l) - l)/(s-t) \) increase (modulo \( (s-t) \) times a constant) to \(-X_t l\).

This completes the proof. \( \square \)

We are now able to formulate our main result. We recall first the context of this result. The algebras \( A_t \) are the von Neumann algebras (type II\(_1\) factors) associated with Berezin’s deformation of \( \mathbb{H}/\text{PSL}(2,\mathbb{Z}) \). These algebras can be realized as subalgebras of \( B(H_t) \) where \( H_t \) is the Hilbert space \( H^2(\mathbb{H}, \text{Im} z^{t-2} \, d\overline{z} \, dz) \).

As such, every operator \( A \) in \( A_t \) (or \( B(H_t) \)) is given by a reproducing kernel: \( k_A \), which is a bivariable function on \( \mathbb{H} \), analytic in the second variable, antianalytic in the first and \( \text{PSL}(2,\mathbb{Z}) \)-invariant. The symbols are normalized so that the symbol of the identity is the constant function 1.

By using these symbols (that represent the deformation) we can define the \( \ast_t \) product of two symbols \( k, l \) by letting \( k \ast_t l \) be the product symbol in the algebra \( A_t \).

The Hochschild 2-cocycle associated with the deformation is defined by
\[ C_t(k, l) = \frac{d}{ds}(k \ast_s l) \bigg|_{s=t}. \]
The Hochschild cocycle condition is obtained by differentiation of the associativity identity.

The cocycle \( C_t \) is well defined on a weakly dense, unital subalgebra \( \hat{A}_t \) of \( A_t \). A sufficient condition that an element in \( A_t \), represented by a symbol \( k \), belongs to \( \hat{A}_t \), is that the quantity \( \|k\|_{\hat{A}_t} \), defined as the maximum of
\[ \sup_{z \in H} \int_{\mathbb{H}} |k(z, \eta)| \, |d(z, \eta)|^t \, d\nu_0(\eta), \]
and
\[ \sup_{\eta \in H} \int_{\mathbb{H}} |k(z, \eta)| |d(z, \eta)|^t \, d\nu_0(z), \]
be finite.

The algebra \( \hat{A}_t \) is the analogue of the Jolissaint algebra [Jo] for discrete groups.

We proved in Section 2 that the applications \( \Psi_{s,t} \) which map the operator \( A \) in \( A_t \) into the corresponding operator in \( A_s \) having the same symbol are completely positive.

This property proves that \( C_t \) is completely negative, that is, for all \( l_1, l_2, \ldots, l_N \) in \( A_t \), for all \( k_1, k_2, \ldots, k_N \) in \( \hat{A}_t \), we have that
\[ \sum l^*_i c(k^*_i, k_j) l_j \leq 0. \]

This property could be used to construct, as in [Sau], the cotangent bundle. In fact, here \( C_t \), or rather \( -C_t \), plays the role of \( \nabla L \), where \( L \) should be a generator of a quantum dynamical semigroup \( \Phi_t \) (thus \( L = \frac{d}{ds} \Phi_s \bigg|_{s=0} \)) and we have \( \nabla L(a, b) = L(a, b) - aL(b) - L(a)b \).

It is well known that \( \nabla L \) is completely negative [Li]. In our case, the role of the quantum dynamical semigroup is played by the completely positive maps \( \Psi_{s,t} \) that have the property \( \Psi_{s,t} \Psi_{t,v} = \Psi_{s,v}, \ s \geq t \geq v \). The generator \( L \) doesn’t make sense here, since \( \Psi_{s,t} \) takes its values in different algebras, depending on \( s \).

Instead we use the derivative of the multiplication operation, which formally is
\[ \frac{d}{ds} \Psi_{s,t}^{-1} (\Psi_{s,t}(k) *_s \Psi_{s,t}(l)) \bigg|_{s=t}, \]
as a substitute for \( \nabla L \).

All the above is valid for the general case of Berezin’s deformation of \( H/\Gamma \), where \( \Gamma \) is any discrete subgroup of \( \text{PSL}(2, \mathbb{R}) \) of finite covolume.

When specializing to \( \Gamma = \text{PSL}(2, \mathbb{Z}) \), we construct also the diffusive operator \( \mathcal{L} \), which plays the role of the generator of a dynamical semigroup.

In the next theorem we formulate our main result. We construct explicitly an operator \( \mathcal{L} \) such that
\[ \mathcal{L}(ab) - \mathcal{L}(a)b - a\mathcal{L}(b) = C_t(a, b). \]
We will show that \( \mathcal{L} \) is well defined on a weakly dense (non-unital) subalgebra \( \mathcal{D}_t^0 \) and the above relation holds for \( a, b \in \mathcal{D}_t^0 \) (which is obtained by considering suitable subalgebras of \( \hat{A}_s, s < t - 2 \)). Moreover, \( \mathcal{L} \) has an expression that is very similar to
the Lindblad [Li, CE, GSK, HP] form of the generator $L$ of a uniformly continuous semigroup. Recall that this expression is, in the uniformly continuous case,

$$L(x) = \Phi(x) - \frac{1}{2}\{\Phi(1), x\} + i[H, x],$$

where $\Phi$ is completely positive and $H$ is selfadjoint.

In our case (which is certainly not [Dav] corresponding to the uniformly continuous case) the generator $L(x)$ is defined rather as an unbounded operator (which is the approach taken in [Dav, CF, Ho, GS, MS]).

We prove that there exists a weakly dense, unital algebra $D_t$ containing $D_0^t$, and a linear map $\Lambda$ from $D_t$ into the operators affiliated with $A_t$, and a positive operator that is also affiliated to $A_t$, such that

$$L(x) = \Lambda(x) - \frac{1}{2}\{T, x\}.$$

Also $\Lambda$ maps $D_0^t$ into $L^1(A_t)$.

Moreover, $\Lambda$ has properties that are very similar to a completely positive map. We prove that there exists an increasing filtration $(B_{rt})_{1 < r < t - 2}$ of $D_t$, consisting of weakly dense subalgebras, such that, for a constant $C_0^t$ depending on $r$, $-[\Lambda + C_0^t \cdot \text{Id}]$ is a completely positive map on $B_{rt}$.

This means that when restricted to $B_{rt}$, $L$ has the form $L(x) = \Lambda'(x) - \frac{1}{2}\{T', x\}$, where $-\Lambda' = -[\Lambda + C_0^t \cdot \text{Id}]$ is a completely positive map and $T = T + C_0^t \cdot \text{Id}$.

**Theorem 4.6.** Let $A_t$, $t > 1$, with product operation $\ast$ be the von Neumann algebra (a type II$_1$ factor) associated with Berezin’s deformation of $\mathbb{H}/\text{PSL}(2, \mathbb{Z})$.

Let $C_t$ be the Hochschild 2-cocycle associated with the deformation

$$C_t(k, l) = \frac{d}{ds}k \ast_s l \bigg|_{s=t},$$

which is defined on the weakly dense subalgebra $\hat{A}_t$.

Then there exists a weakly dense (non-unital) subalgebra $D_0^t$ in $\hat{A}_t \subseteq A_t$ and $L_t$, a linear operator on $D_0^t$, with values in the algebra of operators affiliated with $A_t$, such that

$$C_t(k, l) = L_t(kl) - kL_t(l) - L_t(k)l, \quad k, l \in D_0^t.$$ 

Note that $-L_t$ is automatically completely dissipative.

Moreover, $L_t$ has the following expression. There exists a weakly dense, unital subalgebra $D_t$, such that $D_0^t \subseteq D_t \subseteq \hat{A}_t$, there exists $\Lambda_t$ defined on $D_t$ with values

$$L_t(x) = \Lambda_t(x) - \frac{1}{2}\{T, x\}.$$
in the operators affiliated to $\mathcal{A}_t$, and there exists $T$, a positive unbounded operator affiliated with $\mathcal{A}_t$ such that

$$\mathcal{L}_t(k) = \Lambda_t(k) - \frac{1}{2}\{T, k\}, \quad k \in \mathcal{D}_t^0.$$ 

Moreover, $\Lambda_t$ has the following complete positivity properties:

1) $\Lambda_t$ maps $\mathcal{D}_t^0$ into $L^1(\mathcal{A}_t)$;

2) There exists an increasing filtration of weakly dense, unital subalgebras $(\mathcal{B}_{s,t})_{1<s<t-2}$ of $\mathcal{D}_t$, with $\bigcup_s \mathcal{B}_{s,t} = \mathcal{D}_t$, and there exist constants $C_{s,t}$ such that $- [\lambda_t + C_{s,t} \cdot \text{Id}]$ is completely positive on $\mathcal{B}_{s,t}$.

Remark. At the level of symbols the operator $\Lambda_t$ has a very easy expression, namely $\Lambda_t(k)$ is the pointwise multiplication (the analogue of Schur multiplication) of $k$ with the $\Gamma$-equivariant symbol

$$\ln(\Delta(z)\Delta(\xi)[(z - \xi)/(2\pi i)]^{12}).$$

We identify, as in Section 1, $L^2(\mathcal{A}_t)$ with a Hilbert space of $\Gamma$-bivariable functions analytic in the first variable and antianalytic in the second.

Then $\Lambda$ corresponds to the (unbounded) analytic Toeplitz operator with symbol $\ln(\Delta(z)\Delta(\xi)[(z - \xi)/(2\pi i)]^{12})$.

**Proof of Theorem 4.6.** This was almost proved in Lemma 3.3 and Proposition 4.5, but we have to identify the ingredients. Here the algebra $\mathcal{D}_t^0$ is the union (with respect to $s, \varepsilon_0$)

$$\bigcup_{1<s-\varepsilon_0<s<t-2} S_{\Delta_{\varepsilon_0}} A_{s-\varepsilon_0} S_{\Delta^*_{\varepsilon_0}}^*.$$

It is obvious that $\mathcal{D}_t^0$ is an algebra (under the product on $\mathcal{A}_t$—the algebra $\mathcal{D}_t$ is the union $\bigcup_{1<s<t-2} \mathcal{A}_t$, viewed as an algebra of $\mathcal{A}_t$). The algebra $\mathcal{B}_{s,t}$ is the union (with respect to $\varepsilon$) of $\bigcup_{1<s-\varepsilon_0<s<t-2} S_{\Delta_{\varepsilon_0}} A_{s-\varepsilon_0} S_{\Delta^*_{\varepsilon_0}}^*$.

The operator $T$ is the Toeplitz operator with symbol $\frac{1}{12} \ln \varphi$, while $\Lambda_t$ is $X_t$, where $X_t$ was defined in Lemmas 3.3 and 4.4. In Proposition 4.5 we also proved that

$$C_t(a, b) = X_t(a *_t b) - X_t(a)_t b - a *_t X_t b + a *_t T_{(\ln \varphi)_{12}} *_t b, \quad \text{for all } a, b \in \mathcal{D}_t^0.$$ 

Clearly the term $a *_t T_{(\ln \varphi)_{12}} *_t b$ is a cohomologically trivial term, and hence $C_t(a, b)$ is implemented by $\mathcal{L}_t(a) = X_t(a) - \frac{1}{2}\{a, T_{(\ln \varphi)_{12}}\}$. Hence $C_t$ is implemented by $\mathcal{L}_t = \Lambda_t(a) - \frac{1}{2}\{a, T_{(\ln \varphi)_{12}}\}$. All the other properties for $\Lambda_t$ were proven in Section 3.
One also needs to show that the vector spaces $D_t = \bigcup_{s < t - 2} \hat{A}_s$ and

$$D_t^0 = \bigcup_{1 < s - \varepsilon_0 < s < t - 2} S_{\Delta \varepsilon_0} \hat{A}_{s-12\varepsilon_0} S_{\Delta \varepsilon_0}^*$$

are indeed algebras (in $A_t$). $D_t$ is obviously an algebra, since we proved (in [Ra]) that $\hat{A}_s$ is closed under $*_v$ for all $v \geq s$. Of course, if we take the product of different $\hat{A}_{s_1}$ and $\hat{A}_{s_2}$ we may embed them in $\hat{A}_{\max(s_1,s_2)}$.

To prove that $D_t^0$ is an algebra (in $A_t$) we will need to show first that we are reduced to proving that $S_{\Delta \varepsilon_0} \hat{A}_{s-12\varepsilon_0} S_{\Delta \varepsilon_0}^*$, for fixed $s$ and $\varepsilon_0$, is closed under the product $*_t$ in $A_t$.

Indeed if we do a product for different $s$, we may simply take the maximum of $s',s$. If we do a product corresponding to different $\varepsilon_0$’s, say $\varepsilon_0$ and $\varepsilon_1$, then we choose $\varepsilon_1$ to be the largest.

Then observe that for $k \in A_{s-12\varepsilon_1}$

$$S_{\Delta \varepsilon_1} k S_{\Delta \varepsilon_1}^* = S_{\Delta \varepsilon_0} (S_{\Delta \varepsilon_1 - \varepsilon_0} k S_{\Delta \varepsilon_1 - \varepsilon_0}) S_{\Delta \varepsilon_0}^*.$$

Now $S_{\Delta \varepsilon_1 - \varepsilon_0} k S_{\Delta \varepsilon_1 - \varepsilon_0}$ has symbol equal to (modulo a multiplicative constant) $\varphi_{\varepsilon_1 - \varepsilon_0}^e k$. Since $|\varphi| \leq d^{-12}$, it follows that $|\varphi|^e \leq d^{-12}$ and hence that $\varphi_{\varepsilon_1 - \varepsilon_0}^e k$ belongs to $\hat{A}_{s-12\varepsilon_1 + 12(\varepsilon_1 - \varepsilon_0)}$, which is $\hat{A}_{s-12\varepsilon_0}$.

Thus $S_{\Delta \varepsilon_1} \hat{A}_{s-12\varepsilon_1} S_{\Delta \varepsilon_1}^*$ is contained in $S_{\Delta \varepsilon_0} \hat{A}_{s-12\varepsilon_0} S_{\Delta \varepsilon_0}^*$.

Now we are reduced to showing that the product of two elements $S_{\Delta \varepsilon_0} k_1 S_{\Delta \varepsilon_0}^*$ and $S_{\Delta \varepsilon_0} l_1 S_{\Delta \varepsilon_0}^*$, $k_1, l_1 \in \hat{A}_{s-12\varepsilon_0}$, is again an element in $S_{\Delta \varepsilon_0} \hat{A}_{s-12\varepsilon_0} S_{\Delta \varepsilon_0}^*$.

But

$$(S_{\Delta \varepsilon_0} k_1 S_{\Delta \varepsilon_0}^*) _t (S_{\Delta \varepsilon_0} l_1 S_{\Delta \varepsilon_0}^*)$$

coincides with

$$S_{\Delta \varepsilon_0} [k_1 _t \hat{A}_{s-12\varepsilon_0} S_{\Delta \varepsilon_0}^* S_{\Delta \varepsilon_0} _t \hat{A}_{s-12\varepsilon_0} S_{\Delta \varepsilon_0}^*] S_{\Delta \varepsilon_0}.$$

Because $\hat{A}_{s-12\varepsilon_0}$ is closed under the product $*_t$, it is sufficient to show that

$$T^t_{\varphi_{\varepsilon_0}} = S_{\Delta \varepsilon_0}^* S_{\Delta \varepsilon_0}$$

belongs to $\hat{A}_{s-12\varepsilon_0}$. But this is a general fact contained in the following lemma.

**Lemma 4.7.** Assume $f$ is a bounded, measurable, $\Gamma$-equivariant function on $\mathbb{H}$. Let $T_f^t$ be the Toeplitz operator on $H_t$ with symbol $f$. Then $T_f^t$ belongs to $\hat{A}_t$. Moreover, $\|T_f^t\|_t \leq C\|f\|_\infty$, where $C$ is a constant depending on $t$.

**Proof.** Note the symbol of $T_f^t$ is given by the formula [Ra]

$$s_f(z, \xi) = \int_{\mathbb{H}} f(a)[z, a, \overline{a}, \xi]^t da_0(a), \quad z, \xi \in \mathbb{H}.$$
We have to check that the quantity

$$\sup_{z \in \mathbb{H}} \int_{\mathbb{H}} |S_f(z, \xi)| |d(z, \xi)|^t \, d\nu_0(\xi) \leq \|f\|_\infty$$

(and a similar one) is finite.

But the above integral is bounded by

$$\int \int_{\mathbb{H}^2} |f(a)| |[\overline{z}, a, \overline{\alpha}, \xi]|^t |d(z, \xi)|^t \, d\nu_0(a, \xi)$$

$$= \int \int_{\mathbb{H}^2} |f(a)| (d(\overline{z}, a))^t (d(\overline{\alpha}, \xi))^t \, d\nu_0(a, \xi)$$

$$= \int_{\mathbb{H}} f(a)(d(\overline{z}, a))^t \left( \int_{\mathbb{H}} d(\overline{\alpha}, \xi)^t \, d\nu_0(\xi) \right) \, d\nu_0(a).$$

But the inner integral is a constant $K_t$, depending just on $t$ and not on $z$. Thus we get

$$K_t \int_{\mathbb{H}} f(a)(d(\overline{z}, a))^t \, d\nu_0(a) \leq K_t^2 \|f\|_\infty.$$  \hfill \Box
§5. **Comparison of $T_{\ln \varphi}^t$ and $\Lambda(1)(\overline{z}, \xi) = \ln \varphi(\overline{z}, \xi) - (c'_t/c_t)$**

In this section we compare $\Lambda(1)$, which was constructed in Section 3, with $T_{\ln \varphi}^t$.

We recall that $\Lambda(1)$ is (up to an additive constant depending on the deformation parameter $t$)

$$(S_{\Delta^0})^{-1} \left( \frac{d}{d\varepsilon} S_{\Delta^\varepsilon} S_{\Delta^\varepsilon}^* \bigg|_{\varepsilon=\varepsilon_0} \right) (S_{\Delta^0}^*)^{-1}$$

where $S_{\Delta^\varepsilon}$ is acting on $H_{t+12\varepsilon_0}$, while $S_{\Delta^0}$ acts from $H_t$ into $H_{t+12\varepsilon_0}$. The inverse $(S_{\Delta^0})^{-1}$ is an unbounded operator with domain dense in the closure of the range of $S_{\Delta^0}$. We have explained in Section 4 that $\Lambda(1)$ corresponds, in a non-specified way, to the kernel $\ln \varphi(\overline{z}, \xi) - (c'_t/c_t)$.

Both $\Lambda(1)$ and $T_{\ln \varphi}^t\ln \varphi$ are positive and affiliated with $A_t$. Also recall, from Section 3, that the above definition for $\Lambda(1)$ translates into the fact that for $W = \bigcup_{\varepsilon_0} \text{Range } (S_{\Delta^0})^*$ we have that (up to a constant)

$$\Lambda(1) = \frac{d}{d\varepsilon} \left\langle S_{\Delta^\varepsilon} (S_{\Delta^\varepsilon}^*) w, w \right\rangle_{H_t} = \lim_{\varepsilon \searrow 0} \left\langle S_{\Delta^\varepsilon} S_{\Delta^\varepsilon}^* - \text{Id} \over \varepsilon \right\rangle w, w$$

Our main result proves that there exists a (possibly different) domain where $T_{\ln \varphi}^t$ is given by the same formula.

The main result is as follows:

**PROPOSITION 5.1.** There exists a densely defined $S_0 \subseteq H_t$, which is a core for $T_{\ln \varphi}^t$ (though not affiliated with $A_t$) such that the following holds true:

Let $G_\varepsilon$ be the bounded operator in $A_t$ given by $(1/\varepsilon) (S_{\Delta^\varepsilon} S_{\Delta^\varepsilon}^* - \text{Id})$. Clearly $G_\varepsilon$ has kernel $\hat{G}_\varepsilon(\overline{z}, \xi) = (1/\varepsilon) \left( (c_t/(c_t + \varepsilon)) \varphi(\overline{z}, \xi) - \text{Id} \right)$, and the kernels converge pointwise (as $\varepsilon$ tends to 0) to $\ln \varphi(\overline{z}, \xi) - (c'_t/c_t)$.

Then, for all $v_1, v_2$ in $S_0$, we have that

$$\langle T_{\ln \varphi} v_1, v_2 \rangle = \lim_{\varepsilon \searrow 0} \langle G_\varepsilon v_1, v_2 \rangle.$$

**Remark.** By comparison, the same holds true for $\Lambda(1)$: the only difference is that this happens on a different domain $W$ (in place of $S_0$) which is affiliated to $A_t$.

This will be proved in several steps, divided into the following lemmas.

**LEMMA 5.2.** Let

$$S = \left\{ \sum_{i=1}^N \frac{\lambda_i}{(z - \alpha_i)} e^{i\varepsilon_iz} \left| \text{Re } \alpha_i > 3, \varepsilon_i > 0, \lambda_i \in \mathbb{C}, N \in \mathbb{N} \right. \right\}.$$
Then $S$ is contained in all $H_t$, and dense in all $H_t$, $t > 1$.

Proof. Actually $\Re \alpha_i > 1$ would be sufficient for the convergence, but for later considerations we take $3$ instead of $1$. It is sufficient to consider a single term (so $N = 1$). We omit all the indices for $\alpha, a, \varepsilon$ and let $\lambda = 1$. We prove first that $f(z) = (1/(z - \overline{a})^\alpha) e^{i\varepsilon z}$ belongs to any $H_t$. Indeed we have

$$
\int_{\mathbb{H}} \left| \frac{1}{(z - \overline{a})^\alpha} e^{i\varepsilon z} \right|^2 \nu_t(z) = \int_{\mathbb{H}} \frac{1}{|z - \overline{a}|^{\Re \alpha}} e^{-(\Im z)\varepsilon} (\Im z)^{-2} d\overline{z} dz,
$$

which is obviously convergent as $\Re \alpha \geq 2$. \hfill \Box

In the next lemma, we enlarge that space $S$ to exhaust the range of all $S_{\Delta^\varepsilon}$.

Lemma 5.3. Let $S_{0,t} = \bigcup_{\varepsilon > 0} \Delta^\varepsilon S$, $t - \varepsilon > 1$. Then $S_{0,t}$ is dense in all $H_t$, $t > 1$.

Proof. We need only look at $S \subseteq H_{t-\varepsilon}$ and apply the operator $S_{\Delta^\varepsilon}$. \hfill \Box

Next we need a bound on $\Im (\ln \Delta(z))$. Recall that we are using a choice for $\ln \Delta(z)$ which comes from that fact that $\Delta(z)$ is nonzero in $\mathbb{H}$.

Lemma 5.4. Let $\ln \Delta(z)$ be the principal branch of the logarithm of the function $\Delta$. Then $|\Im \ln(\Delta(z))|$ is bounded by a constant times $(\Re z + (1/(\Im z)^2))$, as $\Im z \downarrow 0$.

Proof. We let $q = e^{2\pi i z}$ and use the following expansion for $\ln \Delta(z)$:

$$
\ln \Delta(z) = \frac{\pi i z}{12} + \sum_{n \geq 1} \ln (1 - q^n).
$$

When $r = |q| = |e^{2\pi i z}| = e^{-\pi y}$ tends to $1$ we have, with $q = re^{i\theta}$, $z = x + iy$, that

$$
\Im \ln \Delta(z) = \frac{\pi x}{12} + \sum_{n \geq 1} \arg((1 - r^n \cos n\theta) + ir^n \sin n\theta)
$$

$$
= \frac{\pi x}{12} + \sum_{n \geq 1} \tan^{-1} \left[ \frac{r^n \sin(n\theta)}{1 - r^n \cos n\theta} \right] .
$$

As $r \to 1$ this is dominated by

$$
\frac{\pi x}{12} + \sum_{n \geq 1} \frac{r^n \sin n\theta}{1 - r^n \cos n\theta},
$$

which in turn is dominated by

$$
\frac{\pi x}{12} + \sum_{n \geq 1} \frac{r^n}{1 - r^n}.
$$
This turns out to be
\[
\frac{\pi x}{12} + (r + r^2 + r^3 + \ldots) \\
+ (r^2 + r^4 + r^6 + r^8 + \ldots) \\
+ (r^3 + r^6 + r^9 + \ldots) \\
+ (r^4 + r^8 + \ldots) \\
+ (r^5 + r^{10} + \ldots) \\
+ \ldots
\]

and this is dominated by
\[
\frac{\pi x}{12} + \frac{c}{(1 - r)^2},
\]
for some constant \(c\).

Letting \(r = e^{-2\pi y}\), and using that \(\lim_{y \to 0} \left((1 - e^{-2\pi y})/y\right)\) is finite, it follows that
\[
|\text{Im} \ln \Delta(x)| \leq c \left(x + \frac{1}{y^2}\right) = c \left(\text{Re} \ z + \frac{1}{(\text{Im} \ z)^2}\right).
\]

\[\square\]

**Corollary 5.5.** For any \(\varepsilon > 0\), there exists a constant \(c_\varepsilon\) such that
\[
|\Delta^\varepsilon(z) \ln \Delta(z)| \leq c_\varepsilon \left(1 + \frac{1}{\text{Im} z}\right) \left(1 + \text{Re} \ z + \frac{1}{(\text{Im} z)^2}\right).
\]

**Proof.** We write
\[
|\Delta^\varepsilon(z) \ln(\Delta(z))| \leq |\Delta|^\varepsilon \ln |\Delta(z)| + |\Delta(z)|^\varepsilon |\text{Im} \ln \Delta(z)|.
\]

We note that \(|\Delta(z)|^2 \text{Im} z^{12}\) is a bounded function and hence
\[
|\Delta(z)| \leq \frac{c_1}{(\text{Im} z)^6}.
\]

Also, since \(|x^\varepsilon \ln x| \leq \text{const} \ (|x^{\varepsilon_1}, x^{\varepsilon_2}|)\) for \(x > 0\), where \(\varepsilon_1 > \varepsilon > \varepsilon_2\), we have that
\[
|\Delta(z)|^\varepsilon \ln |\Delta(z)| \leq \text{const} \ (|\Delta(z)|^{\varepsilon_1}, |\Delta(z)|^{\varepsilon_2})
\]
\[
\leq c \max \left(\frac{1}{(\text{Im} z)^{6\varepsilon_1}}, \frac{1}{(\text{Im} z)^{6\varepsilon_2}}\right) \leq c \left(1 + \frac{1}{(\text{Im} z)^6}\right).
\]
Similarly,

$$|\Delta(z)|^\varepsilon |\text{Im}(\ln(\Delta(z)))| \leq \frac{c}{(\text{Im} z)^6} \left[ x + \frac{1}{(\text{Im} z)^2} \right].$$

Putting the two inequalities together, we get

$$|\Delta^\varepsilon(z) \ln(\Delta(z))| \leq c \left( 1 + \frac{1}{(\text{Im} z)} + \frac{1}{(\text{Im} z)^6} \left[ x + \frac{1}{(\text{Im} z)^2} \right] \right),$$

which is thus smaller than

$$c \left( 1 + \frac{1}{(\text{Im} z)} \right) + x \left( 1 + \frac{1}{(\text{Im} z)} \right) + \left( 1 + \frac{1}{(\text{Im} z)} \right) \frac{1}{(\text{Im} z)^2}$$

$$= c \left( 1 + \frac{1}{(\text{Im} z)} \right) \left( 1 + \text{Re} z + \frac{1}{(\text{Im} z)^2} \right). \quad \Box$$

**Corollary 5.6.** Because $|\Delta(z)|$ has the order of growth of $|e^{2\pi i z}| = e^{-2\pi y}$, $y = \text{Im} z$, it follows, by first splitting $\Delta^\varepsilon(z) = \Delta^\varepsilon_1(z)\Delta^\varepsilon_2(z)$, that the growth of $|\Delta^\varepsilon(z) \ln \Delta(z)|$ will come from $1/\text{Im} z$ as $\text{Im} z \to 0$. Thus the above estimate can be improved to

$$|\Delta^\varepsilon(z) \ln \Delta(z)| \leq c \frac{\text{Re} z}{(\text{Im} z)^3} \left( e^{-\varepsilon_1 \text{Im} z} \right).$$

In the next lemma we establish the integral formula for $\langle T_{\ln \varphi}^t v, v \rangle$.

**Lemma 5.7.** Fix $t \geq 10$. For $v$ in $S_{0,t}$, the integral

$$\iint_{\mathbb{H}^2} \frac{\ln \varphi(\overline{z}, \xi)}{|z - \xi|^t} v(z) v(\overline{\xi}) \, d\nu_t(z, \xi)$$

is absolutely convergent and equal to

$$\int_{\mathbb{H}} \ln \varphi(\overline{z}, z) |v(z)|^2 \, d\nu_t(z) = \langle T_{\ln \varphi}^t v, v \rangle.$$  

**Proof.** We will make use of the fact that $v \in S_{0,t}$, so that

$$v(z) = \Delta^\varepsilon(z) v_1(z)$$

for some $\varepsilon > 0$ and for some $v_1 \in S$ (which is contained in $H_{t-\varepsilon}$).

We start by establishing the absolute convergence of the integral. The integral of the absolute value of the integrand is

$$\iint_{\mathbb{H}^2} \frac{\ln \varphi(\overline{z}, \xi) |\Delta^\varepsilon(z)|}{|z - \xi|^t} |v_1(z)| |\Delta^\varepsilon(\xi)||v_1(\xi)| \, d\nu_t(z, \xi).$$
We expand this into three terms, by using the expression

\[ \ln \varphi(\overline{\zeta}, \xi) = \ln \overline{\Delta(z)} + \ln \Delta(\xi) + 12 \ln(\overline{\zeta} - \xi), \quad \text{for } z, \xi \in \mathbb{H}. \]

We will analyse each term separately. Since the situations are similar we will do only the computation for the term involving \( |\ln \Delta(z)| \). The corresponding integral is

\[ \iint_{\mathbb{H}^2} \frac{|\ln \Delta(z)||\Delta^\varepsilon(z)|}{|\overline{z} - \xi|^t} |v_1(z)||\Delta^\varepsilon(\xi)||v_1(\xi)| \, d\nu_t(z, \xi). \quad (5.1) \]

Because \((\text{Im } z)^{t/2}(\text{Im } \xi)^{t/2}/|\overline{z} - \xi|^t\) is bounded above 1, the previous integral is in turn bounded by the integral

\[ \iint_{\mathbb{H}^2} |\ln \Delta(z)|\Delta^\varepsilon(z)||v_1(z)||\Delta^\varepsilon(\xi)||v_2(\xi)| (\text{Im } z)^{t/2-1}(\text{Im } \xi)^{t/2-1} \, dz \, d\overline{\zeta} \, d\xi \, d\xi. \]

We use the estimate from Corollary 5.6 to obtain that this integral is further bounded (up to a multiplicative constant \( c \)) by

\[ c \iint_{\mathbb{H}^2} \frac{\text{Re } z}{(\text{Im } z)^3} \, |v_1(z)||v_2(\xi)||e^{\varepsilon_1 \text{Im } z} e^{\varepsilon(\text{Im } \xi)} (\text{Im } z)^{t/2-2}(\text{Im } \xi)^{t/2} \, dz \, d\overline{\zeta} \, d\xi \, d\xi. \]

This comes to

\[ c \iint_{\mathbb{H}^2} (\text{Re } z) \, |v_1(z)||v_2(\xi)||e^{\varepsilon_1 \text{Im } z} e^{\varepsilon(\text{Im } \xi)} (\text{Im } z)^{t/2-5}(\text{Im } \xi)^{t/2} \, dz \, d\overline{\zeta} \, d\xi \, d\xi. \]

As long as \( t/2 - 5 \geq 0 \), the term \( e^{-\varepsilon_1(\text{Im } z)}(\text{Im } z)^{t/2-5} \) will be bounded by some \( e^{-\varepsilon'_1(\text{Im } z)} \).

Thus if \( t \geq 0 \), and with the price of replacing \( \varepsilon, \varepsilon_1 \) with some smaller ones, in order to kill growth of \((\text{Im } z)^{t/2-5} \) and \((\text{Im } \xi)^{t/2-2} \), we get a multiple of

\[ \iint_{\mathbb{H}^2} (\text{Re } z) \, |v_1(z)||v_2(\xi)||e^{\varepsilon_1 \text{Im } z} e^{-\varepsilon_2 \text{Im } \xi} \, dz \, d\overline{\zeta} \, d\xi \, d\xi. \]

But for \( z = x + iy \), \( |v_1(z)| \) involves powers of \( 1/x^3 \), which makes the integral absolutely convergent. Hence the integral in (5.1) is absolutely convergent.

In the next lemma we will prove that for \( v \in S_0 \), the integral

\[ c_t \iint_{\mathbb{H}^2} \frac{\ln[(\overline{\zeta} - \xi)/(-2i)]}{[(\overline{\zeta} - \xi)/(-2i)]^t} \, v_1(z)v_2(\xi) \, d\nu_t(\overline{\zeta}, \xi) \]
is absolutely convergent and equal to
\[
\int v_1(z)\overline{v_2(\xi)} \ln(\overline{z} - z) \, d\nu_t(z) + \frac{c_t'}{c_t}.
\]

We now complete the proof of Lemma 5.7:
\[
\int_{\mathbb{H}} \ln \varphi(\overline{z}, z) \, |v(z)|^2 \, d\nu_t(z) = \int \left[ \ln \Delta(z) + \ln \Delta(\bar{z}) + 12 \ln(\overline{z} - z) \right] |v(z)|^2 \, d\nu_t(z).
\]

We analyze each term separately.

We have
\[
\int \ln \Delta(z) v(z) v(\overline{z}) \, d\nu_t(z) = \int \ln \Delta(z) v(z) c_t \left( \int \frac{v(\xi)}{(z - \bar{\xi})^t} \, d\nu_t(\xi) \right) \, d\nu_t(z)
\]
\[
= c_t \int_{\mathbb{H}^2} \ln(\Delta(z)) v(z) v(\xi) \, d\nu_t(z, \xi) = c_t \int_{\mathbb{H}^2} \ln \Delta(\xi) v(\xi) v(z) \, d\nu_t(z, \xi).
\]

Similarly,
\[
\int \ln \Delta(\bar{z}) v(z) v(\overline{z}) \, d\nu_t(z) = c_t \int_{\mathbb{H}^2} \ln \Delta(\bar{z}) v(\xi) v(z) \, d\nu_t(z, \xi).
\]

We know that the integrals are absolutely convergent and that we may integrate in any order. Finally using the next lemma, we will have that
\[
\int_{\mathbb{H}} \ln(\text{Im} \, z) \, |v(z)|^2 \, d\nu_t(z) = c_t \int_{\mathbb{H}^2} \ln \frac{[\overline{z} - \xi]/(-2i)]^t}{[\overline{z} - \xi]/(-2i)]^t} v(z) v(\xi) \, d\nu_t(z, \xi) - \frac{c_t'}{c_t} \langle v, v \rangle_{H_t}.
\]

Putting this together we get that
\[
\langle T_{(1/12) \ln \varphi} v, v \rangle = c_t \int_{\mathbb{H}^2} \ln \frac{[\overline{z} - \xi]/(-2i)]^{12}}{[\overline{z} - \xi]/(-2i)]^t} v(z) v(\xi) \, d\nu_t(z, \xi) - \frac{c_t'}{c_t} \langle v, v \rangle_{H_t}.
\]

This completes the proof of Lemma 5.7. □

The following lemma was used above.

**Lemma 5.8.** For \( v \) in \( S_{0,t} \), we have that
\[
\int_{\mathbb{H}} \ln(\text{Im} \, z) \, |v(z)|^2 \, d\nu_t(z) = c_t \int_{\mathbb{H}^2} \ln \frac{[\overline{z} - \xi]/(-2i)]^t}{[\overline{z} - \xi]/(-2i)]^t} v(z) v(\xi) \, d\nu_t(z, \xi) - \frac{c_t'}{c_t} \langle v, v \rangle_{H_t}.
\]
Proof. Start with the identity
\[ v(\xi) = c_s \int_{\mathbb{H}} \frac{v(z)}{[\overline{z} - \xi]/(-2i)]^s} \, d\nu_t(z) = \langle v, c_t^t \rangle_{H_t}. \]

We differentiate this with respect to \( s \), at \( s = t \) (which is allowed because of the fast decay of the functions in \( S_{0,t} \)).

This gives us
\[ 0 = \frac{c_t'}{c_t} v(\xi) + c_t \int_{\mathbb{H}} \frac{v(z)[\ln(\operatorname{Im} z) - \ln([\overline{z} - \xi]/(-2i))]}{[(\overline{z} - \xi)/(-2i)]^t} \, d\nu_t(s). \]

Now we integrate over \( \mathbb{H} \), with respect to the measure \( v(\xi) \cdot d\nu_t(\xi) \). We get
\[ 0 = \frac{c_t'}{c_t} \| v \|_{H_t}^2 + c_t \iint_{\mathbb{H}^2} \frac{v(z)v(\xi)\ln \operatorname{Im} z}{[(\overline{z} - \xi)/(-2i)]^t} \, d\nu_t(z, \xi) \]
\[ - \iint_{\mathbb{H}^2} \frac{\ln([\overline{z} - \xi]/(-2i))v(z)v(\xi)}{[(\overline{z} - \xi)/(-2i)]^t} \, d\nu_t(z, \xi). \]

The second integral is
\[ \int_{\mathbb{H}} |v(z)|^2 \ln(\operatorname{Im} z) \, d\nu_t(z). \]

So we get the required identity.
This completes the proof of Lemma 5.7 and also the proof of Lemma 5.8. □

We now prove that the reproducing kernel
\[ \frac{1}{12} \ln \left( \Delta(z)\Delta(\xi)[(\overline{z} - \xi)/(-2i)]^{12} \right) - \frac{c_t'}{c_t} \]
is the derivative of \((c_t/(c_t+12(s-t))) S_{\Delta(s-t)/12}S^*_{\Delta(s-t)/12} \) on the space \( S_{0,t} \).

**Lemma 5.9.** For \( v_1, v_2 \in S_{0,t} \), we have that
\[ \iint_{\mathbb{H}^2} \frac{\ln \varphi(\overline{z}, \xi)v_1(z)v_2(\xi)}{[(\overline{z} - \xi)/(-2i)]^t} \, d\nu_t(z, \xi) \]
is the limit, when \( \varepsilon \searrow 0 \), of
\[ \iint_{\mathbb{H}^2} \frac{1/\varepsilon(\varphi(\overline{z}, \xi)^\varepsilon - \text{Id})}{[(\overline{z} - \xi)/(-2i)]^t} v_1(z)v_2(\xi) \, d\nu_t(z, \xi). \]
Proof. The convergence of the integrals involved in the limits was proved in Lemma 5.7. To check the value of the limit we will evaluate the difference. This is
\[
\int H^2 \left[ \frac{1/\varepsilon(\varphi(z,\xi)^\varepsilon - \text{Id}) - \ln \varphi(z,\xi)}{[\varphi(z,\xi)/(-2i)]^t} \right] v_1(z)\overline{v_2(\xi)} \, d\nu_t(z,\xi).
\]

We use the Taylor formula to express
\[
\int H^2 \left[ \frac{1/\varepsilon(\varphi(z,\xi)^\varepsilon - 1) - \ln \varphi(z,\xi)}{[\varphi(z,\xi)/(-2i)]^t} \right] v_1(z)\overline{v_2(\xi)} \, d\nu_t(z,\xi)
= \int H^2 \varepsilon \int_0^1 \frac{\varphi^{\varepsilon r}(z,\xi) \ln^2 \varphi(z,\xi)}{[\varphi(z,\xi)/(-2i)]^t} v_1(z)\overline{v_2(\xi)} \, dr \, d\nu_t(z,\xi).
\]

The same type of argument as in Lemma 5.7, because of the rapid decay of the vectors \(v_1, v_2\) in \(S_{0,t}\), proves that the integral
\[
\int_0^1 \int H^2 \left[ \frac{\varphi^{\varepsilon r}(z,\xi) \ln^2 \varphi(z,\xi)}{[\varphi(z,\xi)/(-2i)]^t} \right] v_1(z)\overline{v_2(\xi)} \, d\nu_t(z,\xi) \, dr
\]
is absolutely convergent with a bound independent of \(\varepsilon\). This completes the proof of Lemma 5.9.

To complete the proof of Proposition 5.1, it remains to check the fact that the operators \(G_\varepsilon = (S_{\Delta^\varepsilon}S_{\Delta^\varepsilon}^* - \text{Id})/\varepsilon\) are decreasing, after making a correction of the form \(-G_\varepsilon + \varepsilon K\), for a constant \(K\). This is done in the following lemma.

**Lemma 5.10.** Consider the bounded operators \(G_\varepsilon = (S_{\Delta^\varepsilon}S_{\Delta^\varepsilon}^* - \text{Id})/\varepsilon\), which are represented by the kernels
\[
\frac{1}{\varepsilon} \left[ \frac{c_t - 12\varepsilon}{c_t} \varphi(z,\xi)^\varepsilon - 1 \right].
\]

Then, there exists a constant \(K\) such that \(-G_\varepsilon + \varepsilon K\) is (as \(\varepsilon\) decreases to 0) an increasing family of positive operators in \(A_{2t+1}\).

**Proof.** Note that the kernel of \(S_{\Delta^\varepsilon}S_{\Delta^\varepsilon}^*\) is
\[
\frac{c_t - 12\varepsilon}{c_t} \left[ \overline{\Delta(z)\Delta(\xi)}[(\overline{z} - \xi)/(-2i)]^{12} \right]^\varepsilon\]
Hence the derivative is
\[
-12 \frac{c_t'}{c_t} + \ln \left[ \overline{\Delta(z)\Delta(\xi)}[(\overline{z} - \xi)/(-2i)]^{12} \right].
\]
Clearly $S_{\Delta^e} S_{\Delta^e}^*$ is a decreasing family. We will proceed as in Lemma 3.3. Let $s_\varepsilon = S(\varepsilon)$ be the kernel of $S_{\Delta^e} S_{\Delta^e}^*$ (as an operator on $H_t$). Then

$$s_\varepsilon(z, \xi) = \frac{c_t - 12\varepsilon}{c_t} (\varphi(z, \xi))^\varepsilon.$$

Let $G_\varepsilon = (s_\varepsilon - \text{Id})/\varepsilon$.

The first derivative of $s_\varepsilon$ (with respect to $\varepsilon$) is

$$-12\frac{c'_t}{c_t} \varphi^\varepsilon + \frac{c_t - 12\varepsilon}{c_t} \varphi^\varepsilon \ln \varphi.$$

The second derivative is

$$24\frac{c'_t}{c_t} \varphi^\varepsilon \ln \varphi - \frac{c_t - 12\varepsilon}{c_t} \varphi^\varepsilon (\ln \varphi)^2.$$

This is equal to (as $c'_t = 1$)

$$\frac{c_t - 12\varepsilon}{c_t} \varphi^\varepsilon \left[ (\ln \varphi)^2 - \frac{24 \ln \varphi}{c_t - 12\varepsilon} \right] = \frac{c_t - 12\varepsilon}{c_t} \varphi^\varepsilon \left[ (\ln \varphi) + \frac{12 \ln \varphi}{c_t - 12\varepsilon} \right]^2 - \frac{c_t - 12\varepsilon}{c_t} \varphi^\varepsilon \left( \frac{144}{(c_t - 12\varepsilon)^2} \right).$$

This is further equal to

$$\frac{c_t - 12\varepsilon}{c_t} \varphi^\varepsilon \left[ \ln \varphi + \frac{12 \ln \varphi}{c_t - 12\varepsilon} \right]^2 - \frac{144}{c_t (c_t - 12\varepsilon)} \varphi^\varepsilon.$$

For every $r > 1$, $S_{\Delta^e}^r (S_{\Delta^e}^r)^* = f(\varepsilon)$ is a decreasing family in $A_r$. By evaluating the kernel, which is

$$f(\varepsilon)(\overline{z}, \xi) = \frac{c_r - 12\varepsilon}{c_r} [\varphi(z, \xi)]^\varepsilon,$$

we get that $\frac{d}{d\varepsilon} f(\varepsilon)(\overline{z}, \xi)$ is a positive kernel for $A_r$. Since

$$c_{r - 12\varepsilon} = \frac{r - 12\varepsilon - 1}{\pi},$$

we obtain that

$$\frac{c_{r - 12\varepsilon}}{c_r} \varphi^\varepsilon \ln \varphi - 12\frac{c'_r}{c_r} \varphi^\varepsilon$$

represents a negative kernel for $A_r$.

We recall, from Section 3, that a kernel $k = k(\overline{z}, \xi)$ is positive for $A_r$ (even if $k$ does not necessary represent an operator in $A_r$) if

$$\left[ \frac{k(\overline{z}, z_j)}{(\overline{z_i} - z_j)^r} \right]_{i,j=1}^N$$

is a positive matrix for all choices of $z_1, z_2, \ldots, z_N$ in $H$, and for all $N$ in $\mathbb{N}$. 
We get that
\[
\varphi \left( \ln \varphi - \frac{1}{c r + 12\varepsilon} \right)
\]
represents a negative (nonpositive) kernel for \( A_r \).

Thus \( \varphi^{\varepsilon/2} \left[ \ln \varphi + \left( 12 / (r - 12 (\varepsilon/2) - (\varepsilon/2) - 1) \right) \right] \) is negative for \( A_{r+12\varepsilon} \), and hence the square
\[
\varphi^{\varepsilon} \left[ \ln \varphi + \frac{12 \varepsilon}{r - 12 \varepsilon - \varepsilon - 1} \right]^2 = \varphi^{\varepsilon} \left[ \ln \varphi - \frac{12 \varepsilon}{r - 12 \varepsilon - \varepsilon - 1} \right]^2
\]
is positive for \( A_{2r - 12\varepsilon} \).

Consequently, the kernels
\[
\frac{c_t - 12\varepsilon}{c_t} \varphi^{\varepsilon} \left[ \ln \varphi - \frac{12 \varepsilon}{t - 12\varepsilon - 1} \right]^2
\]
are positive for \( A_{2t + 13\varepsilon} \).

Now we note the trivial calculus formulae
\[
G_{\varepsilon} = \frac{S(\varepsilon) - \Id}{\varepsilon} = \int_0^1 S'(\varepsilon v) \, dv,
\]
\[
G_{\varepsilon'} = \frac{S(\varepsilon') - \Id}{\varepsilon'} = \int_0^1 S'(\varepsilon' v) \, dv.
\]

The above equalities hold pointwise, that is, when evaluating the corresponding kernels on points in \( \mathbb{H}^2 \). Hence
\[
G_{\varepsilon} - G_{\varepsilon'} = \int_0^1 (S'(\varepsilon v) - S'(\varepsilon' v)) \, dv
\]
\[
= \int_0^1 (\varepsilon v - \varepsilon' v) \int_0^1 S''(p(\varepsilon v) + (1 - p)\varepsilon' v) \, dp \, dv
\]
\[
= (\varepsilon - \varepsilon') \int_0^1 \int_0^1 v S''(\alpha(v, p)) \, dp \, dv,
\]
where \( \alpha(v, p) = p(\varepsilon v) + (1 - p)\varepsilon' v \leq \max(\varepsilon, \varepsilon') \).

We have proved that \( S''(\alpha(v, p)) \) is represented by a positive kernel \( R \), from which one has to subtract a quantity \( Q \) (which is precisely \( \frac{\text{const}}{c_{v-12\alpha(v,p)}} \varphi^{\alpha(v,p)} \)).

As such, by integration we obtain
\[
G_{\varepsilon} - G_{\varepsilon'} = (\varepsilon - \varepsilon') [R - Q],
\]
where $\mathcal{R}$ represents a positive kernel for $A_{2t-12 \min(\varepsilon, \varepsilon')}$. Moreover, $Q$ is a positive element in $A_{2t+12 \min(\varepsilon, \varepsilon')}$, and $Q$ is bounded by $c \cdot \text{Id}$, where $c$ is a universal constant.

Assume that $\varepsilon \geq \varepsilon'$; then in the sense of inequalities in $A_{2r+1}$ we have that

$$G_\varepsilon - G_{\varepsilon'} \geq (\varepsilon - \varepsilon')(Q).$$

Since $0 \leq Q \leq c \cdot \text{Id} \cdot \text{Id}$, we have that $0 \geq -Q \geq c \cdot \text{Id} \cdot \text{Id}$ (in $A_{2t+1}$). Consequently, in $A_{2t+1}$, we have that

$$G_\varepsilon - G_{\varepsilon'} \geq (\varepsilon - \varepsilon')(c).$$

Therefore, the following inequality holds in $A_{2t+1}$:

$$G_\varepsilon + \varepsilon c \geq G_{\varepsilon'} + \varepsilon' c.$$

If we take into account that $G_\varepsilon$ was negative and replace $G_\varepsilon$ by $H_\varepsilon = -G_\varepsilon$, then we get that in $A_{2t+1}$

$$(-G_\varepsilon) - \varepsilon c \leq (-G_{\varepsilon'}) - \varepsilon' c,$$

i.e., that if $\varepsilon \geq \varepsilon'$,

$$H_\varepsilon - \varepsilon c \leq H_{\varepsilon'} - \varepsilon c.$$

We have consequently proved that, in $A_{2t+1}$, the kernels

$$H_\varepsilon(\overline{z}, \xi) = -G_\varepsilon(\overline{z}, \xi) = -\frac{c_t - 12\varepsilon}{c_t} \varphi^\varepsilon + \text{Id}$$

are positive and they increase (when $\varepsilon$ decreases to zero, modulo an infinitesimal term) to $-(c_t'/c_t) + \ln \varphi(\overline{z}, \xi)$.

**Lemma 5.11.** Let $M \subseteq B(H)$ be a type II$_1$ factor and assume that $(H_n)_{n \in \mathbb{N}}$ is an increasing family of positive operators in $M$. Let

$$\mathcal{D}(X) = \{\xi \in H \mid \sup \langle H_n \xi, \xi \rangle < \infty\}$$

and assume that $\mathcal{D}(X)$ is weakly dense in $H$. Then $\mathcal{D}(X)$ is affiliated with $M$, and

$$\langle X\xi, \xi \rangle = \sup_n \langle H_n \xi, \xi \rangle, \xi \in \mathcal{D}(X),$$

defines an operator affiliated with $M$.

**Proof.** Clearly $\mathcal{D}(X) = \{\xi \mid \sup_n \|H_n^{1/2} \xi\| < \infty\}$ and as such $\mathcal{D}(X)$ is a subspace, because if $\|H_n^{1/2} \xi\| \leq A$, $\|H_n^{1/2} \eta\| \leq B$ for all $n$ then $\|H_n^{1/2} (\xi + \eta)\| \leq A + B$.

Moreover, $\mathcal{D}(X)$ is clearly invariant under $u' \in \mathcal{U}(M')$, and hence $\mathcal{D}(X)$ is affiliated to $M$. 

The quadratic linear form \( q_X(\xi) = \sup_n \langle H_n \xi, \xi \rangle \) is weakly lower semicontinuous, thus by [Co2], \( q_X \) defines a positive unbounded operator \( X \), affiliated with \( M \), with domain \( \mathcal{D}(X) \).

\[ \text{□ □} \]

**Corollary 5.12.** The following holds:

\[ T^t_{\ln \varphi} + \frac{c'_t}{c_t} \cdot \text{Id} = \Lambda (1). \]

**Proof.** Let \( H_\varepsilon = -G_\varepsilon + K \cdot \text{Id} + C_\varepsilon \), where \( G_\varepsilon \) are as in Lemma 5.10. Then by definition \( X = -\Lambda (1) + K \cdot \text{Id} \) coincides with the supremum of \( H_\varepsilon \) on \( S_0 = \bigcup S_{\Delta \varepsilon} \).

On \( S_0 \), which is a core for \( T^t_{\ln \varphi} + K \cdot \text{Id} \), the same holds for \( T^t_{\ln \varphi} + K \cdot \text{Id} \).

Thus \( T^t_{\ln \varphi}|_{S_0} \subseteq X \), hence \( \overline{T^t_{\ln \varphi}} \subseteq X \) and so \( T^t_{\ln \varphi} = X = \Lambda (1) \) by [MvN]. \( \text{□ □} \)
§6. The cyclic cocycle associated to the deformation

In [Ra] we introduced a cyclic cocycle $\Psi_t$, which lives on the algebra $\bigcup_{s<t} \hat{A}_s$, and we proved a certain form of nontriviality for this cocycle.

We recall first the definition of the cocycle $\Psi_t$ and then we will show the nontriviality of $\Psi_t$ by using a quadratic form deduced from the operator introduced in Corollary 3.6 and Lemma 4.4. The main result of this section will be the following:

**Theorem 6.1.** Let $t > 1$, let $\mathcal{B}_t = \bigcup_{s<t} \hat{A}_s$, which is a weakly dense subalgebra of $A_t$, and let $R_t$ be defined on $\mathcal{B}_t$ (with values in $\mathcal{B}_t$) by the formula

$$\langle R_t k, l \rangle_{L^2(A_t)} = -\frac{1}{2} \tau_{A_t}(\mathcal{C}_t(k, l^*)), \quad k, l \in \mathcal{B}_t$$

(that is, $R_t$ implements the Dirichlet form $\tau_t(\mathcal{C}_t(k, l^*))$).

Let $(\nabla R_t)(k, l) = R_t(k, l) - kR_t l - R_t(k)l$, which belongs to $\mathcal{B}_t$, if $k, l \in \mathcal{B}_t$, and let $\Psi_t$ be the cyclic cocycle associated with the deformation $[Ra]

$$\Psi_t(k, l, m) = \tau_{A_t}([\mathcal{C}_t(k, l) - (\nabla R_t)(k, l)]m), \quad k, l, m \in \mathcal{B}_t.$$ 

Let $\Lambda_0$ be the operator, on the weakly dense (non-unital) subalgebra $\mathcal{D}_t^0 \subseteq A_t$, introduced in Theorem 4.6, by requiring that $\Lambda_0(k)$ is the derivative at 0 of the operator represented in $A_t$ by the kernel $\varphi^e(\overline{z}, \xi)k(\overline{z}, \xi)$. Thus $\Lambda_0(k)(\overline{z}, \xi)$ is formally $k(\overline{z}, \xi) \ln \varphi(\overline{z}, \xi)$.

Let $\chi_t(k, l) = \langle \Lambda_0 k, l^* \rangle_{L^2(A_t)} - \langle k, \Lambda_0(l^*) \rangle_{L^2(A_t)}$ be the antisymmetric form associated with $\Lambda_0$. Then

$$\Psi_t(k, l, m) = \frac{c_t'}{c_t} \tau_{A_t}(klm) + \chi_t(k \ast_t l, m) + \chi_t(l \ast_t m, k) + \chi_t(m \ast_t k, l)$$

for $k, l, m \in \mathcal{D}_t^0$.

We will split the proof of this result into several steps: First we prove some properties about $\Lambda_0$ and its formal adjoint $\Lambda^+$. We start with the definition of $\Lambda^+$.

**Lemma 6.2.** Let $f$ be a bounded measurable function that is PSL $(2, \mathbb{Z})$-equivariant.

We define $\Lambda^+(T^t_f) = T^t_{f \ln \varphi}$. Then $\Lambda^+$ has the following properties:

1) Assume in addition that $f \ln \varphi(\overline{z}, z)$ is a bounded function. Then $(\Lambda_0 \mid \mathcal{D}_t)^* \subseteq \Lambda^+$ and

$$\tau_{A_t}(\Lambda_0(k)(T^t_f)^*) = \tau_{A_t}(k\Lambda^+(T^t_f)) = \frac{1}{\text{area } F} \int_F k(\overline{z}, z) f(z) \ln \varphi(\overline{z}, z) d\nu_0(z).$$
2) For $k, l$ in $D_t^0$, we have that
\[ \tau(k\Lambda^+(T_f^t)l) = \tau(\Lambda_0(l \ast k)T_f^t). \]

**Proof.** The proof of this proposition is obvious, since the integrals are absolutely summable. For part 2 we remark that $k\Lambda^+(T_f^t)l$ has symbol
\[ c_t \int k(z, \eta)[f(\eta, \eta) \ln \phi(\eta, \eta)]\phi(\eta, \xi) d\nu_t(\eta). \]

Hence by summability, the trace is
\[ \frac{c_t}{\text{area} F} \int_{F \times \mathbb{H}} k(z, \eta)l(\eta, z) f(\eta, \eta) \ln \phi(\eta, \eta) d\nu_t(z, \eta), \]
which is exactly $\tau(\Lambda(l \ast k)T_f^t \ln \varphi)$. This completes the proof. \(\square\)

Recall that in Section 2 we introduced the densely defined operator $T_{\ln d}$ on $L^2(\mathcal{A}_t)$, given by the formula
\[ \langle T_{\ln d} k, l \rangle = \frac{c_t}{\text{area} F} \int_{F \times \mathbb{H}} k(z, \eta)l(\eta, z) f(\eta, \eta) \ln \phi(\eta, \eta) d\nu_t(z, \eta), \]
which is well defined for $k, l$ in algebra $\hat{\mathcal{B}}_t$. We note that $T_{\ln d}$ acts like a Toeplitz operator on $L^2(\mathcal{A}_t)$, with symbol $\ln d$. In the next lemma we establish the relation between the operator $T_{\ln d}$ and the operator $R_t$.

**Lemma 6.3.** The operator $R_t$, defined by the property
\[ \langle R_t k, l \rangle = -\frac{1}{2} \tau(C_t(k, l)), \]
has the following simple expression in terms of $T_{\ln d}$:
\[ R_t = -\frac{1}{2} c_t' - \langle T_{\ln d} k, l \rangle, \quad k, l \in \hat{\mathcal{B}}_t. \]

**Proof.** Indeed, we have that, for $k, l \in \mathcal{B}_t \subseteq \hat{\mathcal{A}}_t$,
\[ C_t(k, l) = \frac{c_t'}{c_t} (k \ast_t l) + c_t \int_{\mathbb{H}} k(z, \eta)l(\eta, \xi) z, \eta, \eta, \xi \ln [z, \eta, \eta, \xi] d\nu_t(\eta). \]
If we make $\xi = z$ in the above expression and then integrate over $F$ to get the trace of $C_t(k, l)$, we get

$$
\tau(C_t(k, l)) = \frac{c'_t}{c_t} \tau(k \ast_t l) + \frac{c_t}{\text{area } F} \int_{F \times \mathbb{H}} k(\overline{z}, \eta)l(\overline{z}, \eta) |d(\overline{z}, \eta)|^{2t} \ln d^2 d\nu_t(\eta)
$$

$$
= \frac{c'_t}{c_t} \tau(k \ast_t l) + 2 \langle T \ln \varphi, k, l \rangle.
$$

This completes the proof. \qed

In the next proposition we prove a relation between $\Lambda_0 + \Lambda^+$ and the other terms (remark that $\Lambda^+$ is not necessarily the adjoint of $\Lambda_0$, but rather we define $\Lambda^+(T^t_f) = T^t_{f \ln \varphi}$ whenever possible).

**Proposition 6.4.** For all $k, l \in \mathcal{D}^0_t$ we have

$$
\langle \Lambda_0 k, l^* \rangle + \langle k, \Lambda_0 l^* \rangle = \tau(k T^t_{f \ln \varphi} l) + \tau(l T^t_{f \ln \varphi} k) - 2 \langle T \ln \varphi, k, l \rangle. \tag{6.1}
$$

Consequently if we define “Re $\Lambda_0$” (formwise) by the relation

$$
\langle (\text{Re } \Lambda_0) k, l \rangle = \frac{1}{2} \left( \langle \Lambda_0 k, l \rangle + \langle k, \Lambda_0 l \rangle \right),
$$

then

$$
\langle (\text{Re } \Lambda_0) k, l^* \rangle = \frac{1}{2} \tau(k T l + l T k) + \langle R_t k, l^* \rangle + \frac{1}{2} \frac{c'_t}{c_t} \langle k, l^* \rangle_{L^2(A_t)} \tag{6.2}
$$

**Proof.** We prove first the relation (6.1). For $k, l \in \mathcal{D}^0_t$, we have that

$$
\langle \Lambda_0 k, l^* \rangle_{L^2(A_t)} + \langle k, \Lambda_0 l^* \rangle_{L^2(A_t)}
$$

is equal to

$$
\frac{c_t}{\text{area } F} \int_{F \times \mathbb{H}} \ln \varphi(\overline{z}, \eta) + \ln \varphi(\overline{\eta}, z) \, k(\overline{z}, \eta)l(\overline{\eta}, z) d(\overline{z}, \eta)^{2t} d\nu_0(z, \eta).
$$

Since

$$
\ln \varphi(\overline{z}, \eta) + \ln \varphi(\overline{\eta}, z) = \ln \varphi(\overline{z}, z) + \ln \varphi(\overline{\eta}, \eta) - \ln[d(\overline{z}, \eta)]^2,
$$

we get the relation (6.1).

Dividing by 2 we get

$$
\frac{1}{2} \left[ \langle \Lambda_0 k, l^* \rangle + \langle k, \Lambda_0 l^* \rangle \right] = \frac{1}{2} \left[ \tau(k T^t_{f \ln \varphi} l) + \tau(l T^t_{f \ln \varphi} k) \right] - \langle T \ln \varphi, k, l^* \rangle_{L^2(A_t)}.
$$
The definition of $R_t$ and the previous lemma complete the proof. □

Recall that in §4, we proved that for all $k, l$ in $\mathcal{D}_t^0$,

$$C_t(k, l) = \frac{c'_t}{c_t} \text{Id} + kT_{\ln \varphi}^t l + \Lambda_0(kl) - \Lambda_0(k)l - k\Lambda_0(l). \quad (6.3)$$

We want to use (6.3) to find an expression for

$$C_t(k, l) - (\Delta R_t)(k, l)$$

by taking the trace of the product of $m \in \mathcal{D}_t^0$ with the previous expression.

**Notation.** We denote $T = T_{\ln \varphi}^t$ and let $\langle \text{Sym}_{\varphi} k, l \rangle = \frac{1}{2}[\tau(kTl^*) + \tau(l^*Tk)]$, for $k, l \in \mathcal{D}_t^0$. Hence

$$\tau_{A_t}(\text{Sym}_{\varphi}(k)l) = \frac{1}{2}[\tau(kTl) + \tau(lTk)].$$

In this terminology the relation in Proposition 6.4 becomes

$$\langle (\text{Re } \Lambda_0)k, l^* \rangle = \langle \text{Sym}_{\varphi} k, l \rangle + \langle R_t k, l^* \rangle + \frac{1}{2} \frac{c'_t}{c_t} \langle k, l^* \rangle.$$

Note that in the relation above, the scalar product refers to the scalar product on $L^2(\mathcal{A}_t)$. Moreover, the following relations hold true:

$$\tau(\text{Sym}_{\varphi}(kl)m) = \frac{1}{2}[\tau(kTlm) + \tau(mTkl)], \quad (6.4)$$

$$\tau((\text{Sym}_{\varphi} k)lm) = \tau(\text{Sym}_{\varphi}(k)(lm)) = \frac{1}{2}[\tau(kTlm) + \tau(lmTk)], \quad (6.5)$$

$$\tau(k(\text{Sym}_{\varphi}(l))m) = \tau(\text{Sym}_{\varphi}(mk)l) = \frac{1}{2}[\tau(lTmk) + \tau(mTk)]. \quad (6.6)$$

**Lemma 6.5.** For all $k, l, m$ in $\mathcal{D}_t^0$ we have that

$$E = \text{Sym}_{\varphi}(kl) - (\text{Sym}_{\varphi} k)l - k(\text{Sym}_{\varphi} l) + kT^t l = 0.$$

To check this, one has to verify that $\tau(Em) = 0$ for all $m$ in $\mathcal{D}_t^0$.

**Proof.** We have to check that the expression

$$\tau(kTlm) + \tau(mTk) - \tau(kTlm) - \tau(lTk) - \tau(lTmk) - \tau(mTk) + 2\tau(kTlm)$$
vanishes. But
\[ \tau(mTkl) = \tau(lmTk), \]
\[ \tau(klTm) = \tau(lTmk). \]

After cancelling the above terms we are left to check that
\[ -\tau(kTlm) - \tau(mkTl) + 2\tau(kTlm) \]
is equal to zero, which is obvious since \( k, l, m \in D_t^0 \). This completes the proof. \( \square \)

We now decompose \( \tau(\Lambda_0(k)l) \) in the following way:
\[ \tau(\Lambda_0(k), l^*) = \langle \text{(Re } \Lambda_0(k)), l \rangle + i\langle \text{(Im } \Lambda_0(k)), l \rangle, \]
where
\[ \langle \text{(Im } \Lambda_0(k)), l^* \rangle = (1/2i)[\langle \Lambda_0(k), l^* \rangle - \langle k, \Lambda_0(l^*) \rangle]. \quad (6.7) \]

We now can proceed to the proof of Theorem 6.1.

Proof of Theorem 6.1. We have
\[ C_t(k, l) = \frac{c_t'}{c_t} kl + kTl + \Lambda_0(kl) - \Lambda_0(k)l - k\Lambda_0(l). \]

Hence by taking the scalar product with an \( m \) in \( A_t \), that is, computing \( \tau(C_t(k, l)m) \), we obtain
\[ \tau(C_t(k, l)m) = \frac{c_t'}{c_t} \tau(klm) + \tau(kTlm) + \tau(\Lambda_0(kl)m) - \tau(\Lambda_0(k)lm) - \tau(\Lambda_0(k)mk) \]
\[ = \frac{c_t'}{c_t} \tau(klm) + \tau([kTl]m) + \langle \text{Re } \Lambda_0(kl), m^* \rangle \]
\[ - \langle \text{Re } \Lambda_0(k), (lm)^* \rangle - \langle \text{Re } \Lambda_0(l), (mk)^* \rangle \]
\[ + i\langle \text{Im } \Lambda_0(kl), m^* \rangle - i\langle \text{Im } \Lambda_0(k), (lm)^* \rangle - i\langle \text{Im } \Lambda_0(l), (mk)^* \rangle. \]

By using the relation
\[ \langle \text{Re } \Lambda_0(k), l^* \rangle = \langle R_t k, l^* \rangle + \frac{1}{2c_t} \langle k, l^* \rangle + \langle \text{Sym}_\varphi k, l^* \rangle, \]
we obtain that \( \tau(C_t(k, l)m) \) is equal to
\[ \tau([(\Delta R_t)(k, l)]m) \]
plus the following terms:

\[ \langle \text{Sym}_\varphi (kl), m^* \rangle - \tau (\text{Sym}_\varphi (k)lm) - \tau (\text{Sym}_\varphi (l)mk), \quad (6.8) \]

plus the terms

\[ \frac{c'_t}{c_t} \tau (klm) + \left( \frac{1}{2} \frac{c'_t}{c_t} - \frac{1}{2} \frac{c'_t}{c_t} - \frac{1}{2} \frac{c'_t}{c_t} \right) \tau (klm), \quad (6.9) \]

plus the terms

\[ i\langle \text{Im}\Lambda_0 kl, m^* \rangle - i\langle \text{Im}\Lambda_0 (k), (lm)^* \rangle - i\langle \text{Im}\Lambda_0 (l), (mk)^* \rangle. \quad (6.10) \]

The terms in (6.8) add up to zero, as it was proved in Lemma 6.5. The terms in (6.9) add up to \( \frac{1}{2} \left( \frac{c'_t}{c_t} \right) \tau (klm) \).

Since \( \chi_t(k, l) = \frac{1}{2} \left( \langle \Lambda_0 k, l^* \rangle - \langle k, \Lambda_0 l^* \rangle \right) = i\langle \text{Im}\Lambda_0 (k), l^* \rangle \), we obtain, by adding the terms from (6.8), (6.9), (6.10), that

\[ \tau (C_t (k, l) m) = \tau ([\Delta R_t] (k, l)] m) + \frac{1}{2} \frac{c'_t}{c_t} \tau (klm) + \chi_t (kl, m) - \chi_t (k, lm) - \chi_t (l, mk). \]

Thus

\[ \Psi_t (k, l, m) = \frac{1}{2} \frac{c'_t}{c_t} + \chi_t (kl, m) - \chi_t (k, lm) - \chi_t (l, mk). \]

\[ \square \]

**Lemma 6.6.** Let \( t > 1 \). Assume that \( k, l \) are such that \( k = k_1 \ast_t k_2 \), \( l = l_1 \ast_t l_2 \), \( k_i, l_i \in D_t^0 \). Then

\[ \tau (k_2 \mathcal{L}_t (l) k_1) + \tau (l_2 \mathcal{L}_t (k) l_1) + \tau (C_t (k, l)) = -\frac{c'_t}{c_t} \tau (k \ast_t l). \]

**Proof.** Recall that

\[ \mathcal{L}_t = \left( \Lambda_0 - \frac{c'_t}{c_t} \cdot \text{Id} \right) - \frac{1}{2} \{ T, \cdot \}. \quad (6.11) \]

Also \( \tau (C_t (k, l)) = (c'_t/c_t) \tau (k \ast_t l) + 2 \langle T_{\ln d} k, l \rangle \). Also

\[ \tau (\Lambda_0 (k) \cdot l) + \tau (k \Lambda_0 (l)) = \tau (kTl) + \tau (lTk) - 2 \langle T_{\ln d} k, l \rangle. \quad (6.12) \]

Moreover,

\[ \tau (kTl) = \tau (\Lambda_0 (l \ast_t k)) = \int_F \ln \varphi \cdot (l \ast_t k) (\bar{z}, z) \, d\nu_0 (z). \quad (6.13) \]
Hence, by (6.11),

\[ \tau(k_2 L(t) k_1) = \tau(k_2 \Lambda_0 (l) k_1) - \frac{c'_t}{c_t} \tau(kl) - \frac{1}{2} \tau(k_2 Tk_1) - \frac{1}{2} \tau(k_2 lTk_1) \]

and

\[ \tau(l_2 L(t) (k) l_1) = \tau(l_2 \Lambda_0 (k) l_1) - \frac{1}{2} (-\tau(l_2 Tkl_1) - \tau(l_2 kTl_1)) - \frac{c'_t}{c_t} \tau(kl) . \]

So by (6.11), (6.12),

\[
\begin{align*}
\tau(k_2 L(t) (l) k_1) + \tau(l_2 L(t) (k) l_1) + \tau(C_t (k, l)) & = \tau(\Lambda_0 (k) l) + \tau(\Lambda_0 (l) k) - 2 \frac{c'_t}{c_t} \tau(kl) - \tau(kTl) - \tau(lTk) + \tau(C_t (k, l)) \\
& = \tau(kTl) + \tau(lTk) - 2 \frac{c'_t}{c_t} \tau(kl) - \tau(kTl) - \tau(lTk) \\
& - 2 \langle T ln d k, l \rangle + 2 \langle T ln d k, l \rangle + \frac{c'_t}{c_t} \tau(kl) \\
& = - \frac{c'_t}{c_t} \tau(kl) .
\end{align*}
\]

\[\square\]
\section{A dual solution; closability of $\Lambda$}

In this section we analyze the Hilbert-space dual of the operator $\Lambda(k), k \in D_0^t$, introduced in Section 4. This is achieved by analyzing the derivative of the one-parameter family of completely positive maps $\chi_{s,t}: \mathcal{A}_t \to \mathcal{A}_s, 1 < s \leq t$, defined as follows:

$$\chi_{s,t}(k) = S_{\Delta(s-t)/12} k (S_{\Delta(s-t)/12})^*, \quad k \in \mathcal{A}_t.$$  

Recall that the Hilbert space $L^2(\mathcal{A}_t)$ is naturally identified with the Hilbert space of all kernels $k = k(\tau, \xi)$ on $\mathbb{H} \times \mathbb{H}$ that are diagonally $\Gamma$-equivariant, $\Gamma = \text{PSL}(2, \mathbb{Z})$. The kernels are also required to be square-summable with respect to the measure $[d(z, \xi)]^{2t} \, d\nu_0(z) \, d\nu_0(\xi)$ on $F \times \mathbb{H}$. (Recall that $F$ is a fundamental domain for $\Gamma$ in $\mathbb{H}$.)

Consider the Hilbert space $L_t$ of all measurable functions on $\mathbb{H} \times F$ that are square-summable with respect to the measure $d^{2t} \, d\nu_0 \times d\nu_0$. This space is obviously identified with a space of $\Gamma$-invariant (diagonally) functions on $\mathbb{H} \times \mathbb{H}$, square-summable over $F \times \mathbb{H}$.

We let $\mathcal{P}$ be the orthogonal projection from $L^t$ into $L^2(\mathcal{A}_t)$. Let $\Phi$ be a measurable (diagonally) $\Gamma$-equivariant function on $\mathbb{H} \times \mathbb{H}$. With the above identification let $M_\Phi$ be the (eventually unbounded) operator on $L_t$, defined by multiplication with $\Phi$ on $L_t$. Correspondingly, there is a Toeplitz operator $\mathcal{T}_\Phi = \mathcal{P}M_\Phi\mathcal{P}$, densely defined on $L^2(\mathcal{A}_t)$.

For example, the map $\Lambda$, constructed in Section 4, is $\mathcal{T}_\Phi$ with $\Phi = \ln \varphi$. In Section 6, Lemma 6.3, we have proved that the operator $R_t$ defined by

$$\langle R_t k, l \rangle_{L^2(\mathcal{A}_t)} = -\frac{1}{2} \frac{d}{ds} \tau_{\mathcal{A}_s}(k \ast_s l^*),$$

defined for $k, l$ in an algebra, is exactly $-\mathcal{T}_{\ln d} - \frac{1}{2} \left( c_t/^c_t \right) \text{Id}$.

Let also $P_t$ be the orthogonal projection from $L^2(\mathbb{H}, d\nu_t)$ onto $H_t$. Recall that the formula for $\pi_t$ has a trivial extension to a projective unitary representation, $\tilde{\pi}_t$ (given by the same formula as $\pi_t$), on functions on $L^2(\mathbb{H}, d\nu_t)$. Moreover, $P_t = P_t \tilde{\pi}_t = \tilde{\pi}_t P_t = P_t \pi_t P_t$. Let $\tilde{\mathcal{A}}_t \subseteq B(L^2(H, \nu_t))$ be the commutant of $\tilde{\pi}_t(\Gamma)$. By [A], this is a type II$_\infty$ factor, such that $L^2(\tilde{\mathcal{A}}_t)$ is canonically identified with $L_t$. Consequently, at least, for $k$ in $L^2(\tilde{\mathcal{A}}_t) \cap \tilde{\mathcal{A}}_t$, it makes sense to consider $\mathcal{P}(k) = P_t k P_t$.

\textbf{Lemma 7.1.} Let $\mathcal{P}_t$ be the orthogonal projection from $L_t$ (identified with $L^2(\tilde{\mathcal{A}}_t)$) into $L^2(\mathcal{A}_t)$.

Then $\mathcal{P}_t(k)$ is given by the formula $P_t k P_t$, which is well defined for $k \in L^2(\tilde{\mathcal{A}}_t) \cap \tilde{\mathcal{A}}_t$ and then extended by continuity. For such a $k$, the kernel of $(\mathcal{P}_t k)(\tau, \xi), \quad \tau, \xi \in \mathbb{H}.$
\(z, \xi \text{ in } \mathbb{H}, \) is given by the formula

\[
(\mathcal{P}_t)(\varpi, \xi) = c_t^2 [(\varpi - \xi)/(2i)]^t \int_{\mathbb{H}^2} \frac{k(\eta_1, \eta_2)}{(\varpi - \eta_1)^t (\eta_1 - \eta_2)^t (\eta_2 - \xi)^t} \, d\nu_t(\eta_1) \, d\nu_t(\eta_2).
\]

**Proof.** One can check immediately that the map \(\mathcal{P}(k) = P_t k P_t, \) for \(k \) in \(L^2(\tilde{A}_t), \) defines an orthogonal projection on \(L^2(\mathcal{A}_t), \)

The formula for \(\mathcal{P}(k)\) follows by writing down the corresponding kernels, and it holds as long as \(k\) is in \(L^2(\tilde{A}_t).\) \(\square\)

In the next lemma we will prove that the Toeplitz operators of symbols \(\ln \varphi\) and \(\ln \varphi\) have dense domain (in \(L^2(\mathcal{A}_t)\)) and that they are adjoint to each other.

**Lemma 7.2.** Let \(\varphi(\varpi, \xi) = \frac{1}{2} \ln [\Delta(z) \Delta(\xi)] [(\varpi - \xi)/(2i)]^{12}\) (as in Section 4). Let \(\tilde{D}_t^0\) be the union of \(S_{\Delta \varepsilon} \tilde{A}_{t-3-\varepsilon} S_{\Delta \varepsilon}^*\) with respect to \(\varepsilon > 0, t - 3 - \varepsilon > 1.\)

Then \(\text{Dom}(\mathcal{M}_\varphi) \cap H_t\) contains the weakly dense subalgebra \(\tilde{D}_t^0.\) Consequently \(\tilde{D}_t^0\) is contained in the domain of \(\mathcal{P} \mathcal{M}_\varphi \mathcal{P}, \) which is the Toeplitz operator \(T_\varphi.\)

Before beginning the proof of the lemma, we note the following consequence:

**Corollary 7.3** The operator \(\Lambda\) introduced in §4 (restricted to \(\tilde{D}_t^0\)) coincides with \(\mathcal{M}_{(1/12) \ln \varphi} k,\) acting on the same domain. Moreover, the operators \(T_{\ln \varphi}\) and \(T_{\ln \varphi}\) are densely defined and \(T_{\ln \varphi} \subseteq (T_{\ln \varphi})^*, \) \(T_{\ln \varphi} \subseteq (T_{\ln \varphi})^*.\) Consequently, these operators are closable in \(L^2(\mathcal{A}_t).\)

**Proof.** The fact that \(\Lambda(k)\) is equal to \(\mathcal{M}_{(1/12) \ln \varphi} k\) for \(k\) in \(\tilde{D}_t^0\) is a consequence of the fact that \(\mathcal{M}_\varphi(k)\) is the \(L^2\)-valued derivative at 0 of the differentiable, \(L_t\)-valued function \(\varepsilon \to \mathcal{M}_\varphi(k).\) This is based on the arguments in the proof (below) of Lemma 7.2. Hence, \(T_{\ln \varphi}(k)\) is the derivative at 0 of the differentiable, \(L^2(\mathcal{A}_t)\)-valued function \(\varepsilon \to T_{\varphi \varepsilon}.\) \(\square\)

**Proof of Lemma 7.2.** We have to check that for \(k\) in \(\tilde{D}_t^0\) having the expression \(k = S_{\Delta \varepsilon} k_1 S_{\Delta \varepsilon}^*,\) with \(k_1 \in \tilde{A}_{t-2-\varepsilon}\) (so that up to a constant \(k(\varpi, \xi) = \varphi^\varepsilon k_1(\varpi, \xi),\) \(z, \xi \in \mathbb{H}),\) the integral

\[
\iint_{F \times \mathbb{H}} |\ln \varphi(z, \xi)|^2 \left| \Delta(z) \Delta(\xi)(\varpi - \xi)^{12} \right|^\varepsilon |k_1(z, \xi)| \, d^2t \, d\nu_0(z)
\]

is (absolutely) convergent.

Since \(k_1\) belongs to \(\tilde{A}_{t-2-\varepsilon}\) we may free up a small power \(\alpha\) of \(d,\) so that the integral

\[
\iint_{F \times \mathbb{H}} |k_1(\varpi, \xi)| \, d^{2t-4-\alpha} \, d\nu_0(z, \xi)
\]
is still convergent. We proved in § 5 that for any \( \varepsilon < \varepsilon' \), there exists a positive constant \( C_{\varepsilon, \varepsilon'} \), such that

\[
|\ln \Delta(z) \Delta^\varepsilon(z)| \leq C_{\varepsilon, \varepsilon'} \frac{\text{Re } z}{(\text{Im } z)^2} e^{-\varepsilon' \text{Im } z}.
\]

When evaluating the integral in (7.1), we will have to find an estimate for each of the terms that arise by writing

\[
\ln \varphi(z, \xi) = \ln \Delta(z) + \ln \Delta(\xi) + 12 \ln(\overline{z} - \xi).
\]

After taking the square, we see that it remains to prove that the integrals containing the following quadratic terms are finite:

\[
|\ln \Delta(z)|^2 |\Delta(z)|^\varepsilon d^\alpha,
\]

\[
|\ln \Delta(\xi)|^2 |\Delta(\xi)|^\varepsilon d^\alpha,
\]

\[
|\ln[(\overline{z} - \xi)/(2i)]|^2 |\Delta^\varepsilon(\xi)|.
\]

We analyze, for example, the term involving \( |\ln \Delta(z)|^2 \). By using Corollary 5.6, we note that the integral is consequently bounded by

\[
\iint_{F \times \mathbb{H}} \frac{(\text{Re } z)^2}{(\text{Im } z)^6} e^{-\varepsilon' \text{Im } z} e^{-\varepsilon \text{Im } \xi} |\overline{z} - \xi|^{12\varepsilon} \cdot (d(z, \xi))^{2t} |k_1(\overline{z}, \xi)|^2 d\nu_0(z).
\]

We write \((d(z, \xi))^{2t} = (d(z, \xi))^{2(t-3)} \cdot d(z, \xi)^6\) to get that the above integral is bounded by

\[
\iint_{F \times \mathbb{H}} e^{-\varepsilon' \text{Im } z} e^{-\varepsilon \text{Im } \xi} \frac{(\text{Im } \xi)^6}{|\overline{z} - \xi|^{12-12\varepsilon}} \cdot (d(z, \xi))^{2(t-3)} |k_1(z, \xi)|^2 d\nu_0(z, \xi).
\]

Because of the term \( e^{-\varepsilon \text{Im } \xi} \), by eventually multiplying with a constant, we can neglect the term \( (\text{Im } \xi)^6 \).

Thus we are led to analyze the following integral:

\[
\iint_{F \times \mathbb{H}} (\text{Re } z)^2 e^{-\varepsilon' \text{Im } z} e^{-\varepsilon \text{Im } \xi} \frac{1}{|\overline{z} - \xi|^{12-12\varepsilon}} (d(z, \xi))^{2(t-3)} |k_1(z, \xi)|^2 d\nu_0(z).
\]

Because \((z, \xi) \in F \times \mathbb{H}\), it follows that there is a constant \( C \) such that \(|\overline{z} - \xi| \geq C\), for \( z, \xi \in F \times \mathbb{H}\). Also \((\text{Re } z)^2/|\overline{z} - \xi|^2\) is bounded from above on this region.
Thus the above integral is bounded by a constant times
\[
\int F \times \mathbb{H} \int (d(z, \xi))^{2(t-3)} |k_1(\overline{z}, \xi)|^2 \, d\nu_0(z, \xi),
\]
which is finite if \( k_1 \in A_{t-3-\varepsilon} \).

The terms with \( |\ln[(\overline{z} - \xi)/(2i)]| \) are solved by absorbing \( |\ln[(\overline{z} - \xi)/(2i)]| \) into some power of \( |z - \xi| \).

Clearly \( \mathcal{PM}_{\ln \varphi} \) has the same domain as \( \mathcal{PM}_{\ln \varphi} \). This is precisely the vector space of all \( k \in L^2(A_t) \) such that \( |k(\ln \varphi)|^2 = |k(\ln \varphi)|^2 \) is summable on \( F \times \mathbb{H} \), with respect to the measure \( dt \times d\nu_0 \times d\nu_0 \). This completes the proof. □ □

We introduce the following definition which will be used in the dual solution for the cohomology problem, corresponding to \( C_t \).

**Definition 7.4.** Let \( \chi_{s,t} : A_t \to A_s \) be defined by the formula
\[
\chi_{s,t}k = S_{\Delta(t-s)/12}^* k S_{\Delta(t-s)/12},
\]
for \( k \) in \( ac_t \). Here \( s \leq t \).

In the next proposition we analyze the relation between the derivative of \( \chi_{s,t} \) at \( s = t \), \( s \nrightarrow t \), with the derivative of \( \theta_{s',t} \), at \( s' = t \), \( (s' \searrow t) \), introduced in Section 4.

**Definition 7.5.** For \( t > 1 \), we let \( \mathcal{D}_t^+ \) be the algebra consisting of all \( k \) in \( A_s \) that, for some \( s < t \), are of the form \( S_{\Delta s}^* k_1 S_{\Delta s} \), for some \( \varepsilon > 0 \) such that \( s + \varepsilon < t \) and \( k_1 \in A_{s+\varepsilon} \).

Clearly \( \mathcal{D}_t^+ \) is a weakly dense, unital subalgebra of \( A_t \).

**Lemma 7.6.** Fix \( t > 1 \). Assume that \( k \) in \( \mathcal{D}_t^+ \) has the expression \( k = S_{\Delta s}^* k_1 S_{\Delta s} \), \( k_1 \in A_s, \varepsilon > 0, \varepsilon + s < t \). Then
\[
k = T_{\varphi}^\varepsilon(k_1) T_{\Delta s}^\varepsilon(\Delta \varepsilon(\overline{z} - \xi)/(2i))^{12\varepsilon}(k_1).
\]

**Remark.** Note that by putting the variables \( \overline{z}, \xi \), we indicated that \( k_1 \) is multiplied by a function, that contrary to \( k_1 \), is antianalytic in the second variable and analytic in the first. Thus \( T_{\varphi}^\varepsilon \) corresponds to a Toeplitz operator with an “antianalytic” symbol.

**Proof of Lemma 7.6.** Let \( T(k) = S_{\Delta s}^* k S_{\Delta s} \). The statement follows immediately from the fact that the adjoint of \( T \), as a map on \( L^2(A_t) \), is \( l \to S_{\Delta s} l S_{\Delta s}^* \). □
In the next proposition we clarify the relation between the operator $Y_t k$ defined as $\frac{d}{ds} \chi_{s,t}(k)\bigg|_{s=t; \ s \neq t}$ and the operator $X_t$ introduced in Section 4.

First we recall that the "real part" associated with the deformation is given by the Dirichlet form $\mathcal{E}_s(k, l) = \frac{d}{ds} \tau_{A_s}(k * s l) = \tau_{A_s} c_s(k, l)$.

**Definition 7.7.** Recall (from Section 6) that the real part of the cocycle $C_t$ is the operator $R_t$ given by the formula

$$\langle R_t k, l^* \rangle = -\frac{1}{2} \cdot \tau_{A_s}(k * s l)\bigg|_{s=t; \ s \neq t} = -\frac{1}{2} \mathcal{E}_s(k, l).$$

This holds for all $k, l$ in $\bigcup_{r<t} L^2(A_r)$, where $L^2(A_r)$ is identified with a vector subspace of $L^2(A_t)$ via the symbol map $\Psi_{t,r}$.

Moreover, in Section 6 we proved that $R_t$ has the following expression:

$$R_t = \mathcal{T}_{\ln} - \frac{1}{2} \frac{c'_t}{c_t} \text{Id}.$$ 

In the next proposition we construct the dual object for the generator used in Section 4.

**Proposition 7.8.** For any $k$ in $D_t^+ \subseteq A_t$, the limit

$$Y_t(k) = \frac{d}{ds} \Psi_{t,s}(\chi_{s,t} k)\bigg|_{s=t; \ s \neq t}$$

exists in $L^2(A_t)$. Moreover, we have that

$$Y_t = - (\mathcal{T}_{(1/12) \ln \varphi})^* - \frac{c'_t}{c_t} \text{Id} + 2R_t.$$ 

The adjoint $(\mathcal{T}_{(1/12) \ln \varphi})^*$ is obtained by first restricting $\mathcal{T}_{(1/12) \ln \varphi}$ to $\tilde{D}_t^0$ and then taking the adjoint.

**Proof.** Indeed, $\chi_{s,t}(k)$ may be identified with the Toeplitz operator (on $L^2(A_s)$) with symbol

$$\varphi^{(t-s)/12} = \overline{\Delta^{(t-s)/12}(z) \Delta^{(t-s)/12}(\xi)((z-\xi)/(2i))^{(t-s)}}.$$ 

Thus $\chi_{s,t}(k)$ is (modulo a multiplicative constant)

$$\mathcal{P}_s \left[ \mathcal{M}_{\varphi^{(t-s)/12}} k \right],$$
and hence $\Psi t, s \chi s, t (k)$ is

$$\Psi t, s \mathcal{P} s \left[ M_{\varphi (t-s)/12} (k) \right]$$

The derivative at $s = t$ consequently involves two components.

One component is the derivative $\frac{d}{ds} \Psi t, s (\mathcal{P} s k) |_{s \neq t}$, which gives the summand corresponding to $R_t$, i.e., $-(c_l/c_t) + 2R_t$.

The other component is $\frac{d}{ds} \mathcal{P} t (M_{\varphi (t-s)/12} (k))$, which gives the multiplication-by-$\varphi$ part. Indeed, recall that $k$ belongs to $\mathcal{D}^+ t \subseteq \mathcal{A}_t$, and hence $k$ is of the form $S^*_{\Delta \varepsilon} k_1 S_{\Delta \varepsilon}$ for some $\varepsilon_0 > 0$ such that $s + \varepsilon_0 < t$ and $k_1 \in \mathcal{A}_{s+\varepsilon_0}$. But then

$$\mathcal{P} t (M_{\varphi (t-s)/12} (k)) = \mathcal{P} t (M_{\varphi (t-s)/12} (\mathcal{P} s (M_{\varphi \varepsilon_0} (k_1))))$$

Since $\overline{\varphi}$ plays the role of an antianalytic symbol, it follows that this is further equal to

$$\mathcal{P} t \left( M_{\varphi [(t-s)/12+\varepsilon_0]} (k_1) \right).$$

The derivative (in the $s$ variable) of $s \to \varphi [(t-s)/12+\varepsilon_0]$ at $s = t$ exists, by the method in Lemma 7.2 in $L_t$, and it is equal to

$$-\frac{1}{12} \ln \varphi \cdot \overline{\varphi \varepsilon_0} \cdot k_1.$$

Thus, in the Hilbert space $L^2 (\mathcal{A}_t)$, we have that

$$\frac{d}{ds} \mathcal{P} t (M_{\varphi (t-s)/12} (k)) = -\mathcal{P} t (M_{(1/12)} (\ln \varphi) \overline{\varphi \varepsilon_0} (k_1))$$

$$= -\mathcal{P} t \left( M_{(1/12)} \ln \varphi (\mathcal{P} t + 12\varepsilon_0 \varphi \varepsilon_0) (k_1) \right)$$

$$= -\mathcal{P} t \left( M_{(1/12)} \ln \varphi (k) \right).$$

This completes the proof. □

We use the above arguments to prove that also the operator $Y_t = \frac{d}{ds} \chi s, t$ implements a coboundary for $\mathcal{C}_t$.

**Lemma 7.9.** For $k, l$ in $\mathcal{D}^+_t$, we have that $\frac{d}{ds} \chi s, t (k \ast t \varphi^{t-s} (\overline{\varphi}, \xi) \ast t l) |_{s \neq t}$ is equal to $Y_t (k \ast t l) - k \ast t \Lambda (1) \ast t l$.

**Proof.** Since $k, l$ are in $\mathcal{D}^+_t$, there exists $\varepsilon_0 > 0$ and there are $k_1, l_1 \in \mathcal{A}_{t+12\varepsilon_0}$ such that $k = S^*_{\Delta \varepsilon} k_1 S_{\Delta \varepsilon}$, $l = S^*_{\Delta \varepsilon} l_1 S_{\Delta \varepsilon}$. This gives that

$$k \ast t \Lambda (1) \ast t l = S^*_{\Delta \varepsilon} [k_1 \ast t+\varepsilon \varphi \ln \varphi \ast t+\varepsilon l_1] S_{\Delta \varepsilon},$$

where by $\varphi \ln \varphi$ we understand the unbounded operator defined in §3, corresponding to

$$\varphi^2 (\overline{\varphi}, \xi) \ln \varphi (\overline{\varphi}, \xi).$$
As in the proof of Proposition 4.5, when computing this derivative, we have a trivial summand plus a more complicated summand, corresponding to the symbol

\[
\lim_{s \searrow t} \Psi_{t,s} \left[ \mathcal{P}_s \varphi^{t-s} \left[ k \ast_t \frac{\varphi^{t-s} - \text{Id}}{t-s} \ast_t l \right] \right].
\]

Because of the assumptions, the inside term

\[
k \ast_t \frac{\varphi^{t-s} - \text{Id}}{t-s} \ast_t l
\]

is equal to

\[
S_{\Delta \varepsilon_0} \left[ k_1 \ast_{t+\varepsilon_0} \varphi^{t-s} - \text{Id} \ast_{t+\varepsilon_0} l \right] S^*_{\Delta \varepsilon_0}
\]

\[
= \mathcal{P}_t \left( \varphi^{\varepsilon_0} \left( k_1 \ast_{t+\varepsilon_0} \varphi^{t-s} - \text{Id} \ast_{t+\varepsilon_0} l \right) \right).
\]  

(7.2)

But the methods in the proof of the density of the domain of \( \mathcal{M}_{\ln \varphi} \) may also be used to prove that

\[
\varphi^{\varepsilon_0} \left( \frac{\varphi^{t-s} - \text{Id}}{t-s} \right)
\]

converges, as \( s \searrow t \), in \( L^2(\mathcal{A}_{t+12\varepsilon_0}) \) to \( -\varphi^{\varepsilon_0} \ln \varphi \).

Since \( \Psi_{t,s} \mathcal{P}_s \) converges strongly to the identity, and the norm of \( \Psi_{t,s} \mathcal{P}_s \) as an operator from \( L^2(\hat{\mathcal{A}}_s) \) into \( L^2(\mathcal{A}_t) \) is bounded by 1, it follows that the expression in (7.2) converges to

\[
\mathcal{P}_t(\varphi^{\varepsilon_0}[k_1 \ast_{t+\varepsilon_0} \varphi^{\varepsilon_0}(-\ln \varphi) \ast_{t+\varepsilon_0} l]),
\]

which is \( k \ast_t (-\Lambda(1)) \ast_t l \).

Similarly we have the following lemma.

**Lemma 7.10.** For \( k, l \) in \( \mathcal{D}_t^+ \) we have that

\[
\frac{d}{ds} \left[ \chi_{t,s}(k) \ast_s \chi_{t,s}(l) \right]_{s \searrow t} = Y_t(k) \ast_t l + \mathcal{C}_t(k, l) + k \ast_t Y_t(l).
\]

**Proof.** Again this derivative has three summands. The first summand is

\[
\lim_{s \searrow t} \frac{\chi_{t,s}(k) \ast_s \chi_{t,s}(l) - \chi_{t,s}(k) \ast_t \chi_{t,s}(l)}{t-s}.
\]

The same type of argument as in Proposition 4.5 gives that this is \( \mathcal{C}_t(k, l) \).
From the remaining two summands, the only one that is complicated is
\[
\chi_{s,t}(k) \ast_t \frac{\chi_{s,t}(l) - l}{t - s}.
\]

Because for \(l\) in \(D^+_t\) we have that \((\chi_{s,t}(l) - l) / (t - s)\) converges in \(L^2(\mathcal{A}_t)\) to \(Y_t l\), and since \(\chi_{s,t}(k)\) is bounded in \(L^2(\mathcal{A}_t)\) as \(s \nearrow t\), it follows that this term converges too, to \(k \ast_t Y_t l\).

The remaining term trivially converges to \(Y_t k \ast_t l\).

This completes the proof of the lemma. \(\square\)

As a corollary we obtain the following result.

**Proposition 7.11.** Let \(D^+_t\) be as in Definition 7.5. Assume \(t > 3\); then for all \(k, l\) in \(D^+_t\), we have that
\[
Y_t(k \ast_t l) - Y_t k \ast_t l - k \ast_t Y_t l - k \ast_t \frac{1}{12} \ln \varphi \ast_t l = C_t(k, l).
\]

Here by \(\frac{1}{12} \varphi(\tau, \xi)\) we understand \(\Lambda(1) = \mathcal{M}_{\ln \varphi}(1)\), the operator constructed in Corollary 3.6 and in Lemma 4.4.

**Proof.** Indeed the identity
\[
\chi_{s,t}(k \ast_t \varphi^{t-s} \ast_t l) = \chi_{s,t}(k) \ast_s \chi_{s,t}(l)
\]

is obvious, valid for all \(k, l\) in \(D^+_t\), \(s \leq t\).

By differentiation, and using the two previous lemmas, we get our result. \(\square\)

**Remark.** Recall that in Proposition 6.4 we proved that if \(\Lambda_0(k) = \mathcal{M}_{\ln \varphi} k\) (for \(k\) in \(D^+_0\)), then, denoting \(S = \text{Sym} \varphi\), we have
\[
\langle \Lambda_0(k), l \rangle + \langle k, \Lambda_0(l) \rangle = 2 \langle Sk, l \rangle + 2 \langle R_t k, l \rangle + \frac{c'_t}{c_t} \langle k, l \rangle.
\]

If \(k\) belonged to \(\tilde{D}^+_0\) (which is the domain of \(\Lambda_0\)) and also to the domain of \(Y_t\), which is \(D^+_t\), then the above relation could be rewritten as
\[
\Lambda_0 + \Lambda_0^* = 2S + 2R_t + \frac{c'_t}{c_t}. \tag{7.3}
\]

Recall that \(S = \text{Sym} \varphi\) is the operator defined by
\[
\langle Sk, l \rangle = \tau_{\mathcal{A}_t}(k T l^*) + \tau_{\mathcal{A}_t}(l^* T k).
\]
But on the intersection of the domains we have (from Proposition 7.8) that
\[ Y_t = -\Lambda_0^* + 2R_t - \frac{c_t'}{c_t} \text{Id}. \] (7.4)

Consequently
\[ \Lambda_0^* = -Y_t + 2R_t - \frac{c_t'}{c_t} \text{Id}. \]

Thus, by (7.3), for \( k \) in \( \tilde{D}_t^0 \cap D_t^+ \) we get that
\[ \Lambda_0 = 2S + Y_t + \frac{c_t'}{c_t} \text{Id}, \]
and hence that
\[ X_t = \Lambda_0 - \frac{c_t'}{c_t} \text{Id} = 2S + Y_t + \frac{c_t'}{c_t} \text{Id}, \] (7.5)

where equality holds on \( \tilde{D}_t^0 \cap D_t^+ \).

Now we compare the way \( X_t, Y_t \) implement a coboundary for \( C_t(k,l) \). Recall the notation \( \langle \nabla \Phi \rangle (k,l) = \Phi(k,l) - k\Phi(l) - \Phi(k)l \).

Thus we have proved that
\[ \nabla X_t(k,l) = C_t(k,l) - kT_t^l \text{ln} \phi_k, \quad k,l \text{ in } \tilde{D}_t^0, \] (7.6)
\[ \nabla Y_t(k,l) = C_t(k,l) - k\Lambda(1)l, \quad k,l \text{ in } D_t^+. \] (7.7)

Now if \( k,l \) were in \( \tilde{D}_t^0 \cap D_t^+ \), it would follow, by substituting (7.5) into (7.6), that
\[ 2\nabla S_t(k,l) + \nabla Y_t(k,l) - \frac{c_t'}{c_t} k^* l = C_t(k,l) - kT_t^l \text{ln} \phi_k. \] (7.8)

By using (7.7) in (7.8) we get
\[ 2(\nabla S_t)(k,l) - k\Lambda(1)l - \frac{c_t'}{c_t} (k^* l) = -kT_t^l \text{ln} \phi_k, \]
and thus for \( k,l \) in \( \tilde{D}_t^0 \cap D_t^+ \) we would get that
\[ k \left[ T_t^l \text{ln} \phi - \left( \Lambda(1) + \frac{c_t'}{c_t} \right) l \right] = 2\nabla S_t(k,l) \] (7.9)

for all \( k,l \) in \( \tilde{D}_t^0 \cap D_t^+ \). But recall that
\[ \langle S_t(k), l^* \rangle = \tau(kTl^* + l^*Tk). \]

This corresponds, at least formally, to the fact that \( S_t k = kT + Tk \) and hence \( (\nabla S_t)(k,l) \) is \( 2kT_t^l \text{ln} \phi_k \).

Thus (7.9) would imply directly that
\[ T_t^l \text{ln} \phi = \Lambda(1) + \frac{c_t'}{c_t} \]
if \( \tilde{D}_t^0 \cap D_t^+ \) is nonzero.
Appendix

A more general coboundary for $C_t$

In this appendix, we want to construct a more general solution for a coboundary (which is necessarily unbounded, see [Ra]) for $C_t$. This will be constructed out of a measurable function $g$ that has the same $\Gamma$-invariance properties as $\ln \Delta(z)$. By this construction we will lose the complete positivity properties of the solution.

Recall that $L_t$ consists of all kernels $k$ or $H \times H$, that are diagonally $\Gamma$-invariant and square summable on $F \times H$, with respect to the measure $dt \times d\nu_0 \times d\nu_0$. Also recall that the elements in $L_t$ are canonically identified with operators in the $\Pi_{\infty}$ factor of all operators that commute with $\tilde{\pi}_t(\Gamma)$, acting on $L^2(H, d\nu_t)$.

**Proposition A1.** Let $g$ be a measurable function $H$ such that the bivariable function $\theta$ in $H \times H$ defined by

$$\theta(z, \xi) = \overline{g(z)} + g(\xi) + \ln(|z - \xi|/(2i))$$

is $\Gamma$-invariant. (It is this point which makes the problem solvable, by this method, only for $\text{PSL}(2, \mathbb{Z})$.)

Let $D_g, \tilde{D}_g$ consist of all $k$ in $L^2(A_t)$ (respectively $L_t$) such that $k \cdot \theta$ still belongs to $L_t$. Let $M_\theta$ be the (unbounded) operator with domain $\tilde{D}_g$, of multiplication by $\theta$. Let $T_\theta = P_t M_\theta|_{L^2(A_t)}$ and let $T_\theta^t$ be the Toeplitz operator with symbol $\theta(z, z) = \text{Re} g(z) + \ln(z - \overline{z})$.

Let $k, l$ be in $D_g$ such that $k, l$ also belong to the domain of $C_t(k, l)$. Then

$$M_\theta(k *_t l) - M_\theta k *_t l - k *_t M_\theta l + M_\theta(k, l) = C_t(k, l),$$

where $M_\theta(k, l)$ is a bimodule map, equal to $k *_t T_\theta^t *_t l$, if $T_\theta^t$ exists.

Consequently, by taking $P_t$ on the left- and right-hand sides, the same will hold true for $T_\theta = P_t M_\theta|_{L^2(A_t)}$.

**Proof.** Indeed $C_t(k, l)$ is given by the kernel

$$C_t(k, l)(\overline{z}, \xi) = \frac{c_t^l}{c_t}(k *_t l)(\overline{z}, \xi)$$

$$+ c_t \int_{H} k(z, \eta)l(\overline{z}, \xi)[\overline{z}, \eta, \overline{\eta}, \xi]^t \ln[\overline{z}, \eta, \overline{\eta}, \xi] d\nu_0(\eta).$$

On the other hand

$$M_\theta(k *_t l) - M_\theta k *_t l - k *_t M_\theta l$$
has the kernel
\[ c_t \int_{\mathbb{H}} k(\overline{z}, \eta)l(\overline{\eta}, \xi)\overline{[\overline{\xi}, \eta, \xi]^t}(\theta(z, \xi) - \theta(z, \eta) - \theta(\eta, \xi)) d\nu_0(\eta). \]

Since \( \theta(z, \xi) - \theta(z, \eta) - \theta(\eta, \xi) \) is equal to \( \theta(\eta, \eta) \), it follows that
\[ C_t(k, l) - [M_\theta(k *_t l) - M_\theta k *_t l - k *_* M_\theta l] \]
is given by the kernel
\[ \frac{[(\overline{z} - \xi)/(-2i)]^t}{[(\overline{z} - \eta)/(-2i)]^t[(\overline{\eta} - \xi)/(-2i)]^t} \left( \theta(\eta, \eta) + \frac{c_t'}{c_t} \right) d\nu_0(\eta), \]
which indeed corresponds to \( T_\theta^t + (c_t'/c_t) \cdot \text{Id} \), as long as we can make sense of the unbounded Toeplitz operator \( T_\theta^t \).

A dual version could be obtained if we consider
\[ \langle 2R_t k, l^\ast \rangle_{L^2(A_t)} = -\frac{d}{ds} \tau_{A_s}(k *_s l^\ast) \bigg|_{s=t}, \]
which is in other terms
\[ 2R_t = -\frac{c_t'}{c_t} - 2T_{\ln d}. \]

One can check immediately that
\[ [C_t(k, l) - (\nabla 2R_t)(k, l)](\overline{z}, \xi) \]

\[ = \frac{c_t'}{c_t} \tau(k *_t l) + c_t \int_{\mathbb{H}} k(\overline{z}, \eta)l(\overline{\eta}, \xi) \ln[\overline{z}, \eta, \overline{\eta}, \xi][\overline{\xi}, \eta, \overline{\eta}, \xi]^t d\nu_0(\eta) \]
\[ + \int_{\mathbb{H}} k(\overline{z}, \eta)l(\overline{\eta}, \xi)[-\ln(d(z, \xi))^2 + \ln(d(z, \eta))^2 + \ln(d(\eta, \xi))^2] d\nu_0(\eta) \]
\[ - \left( -\frac{c_t'}{c_t} + \frac{c_t'}{c_t} + \frac{c_t'}{c_t} \right) \tau(k *_t l) \]
\[ = c_t \int_{\mathbb{H}} k(\overline{z}, \eta)l(\overline{\eta}, \xi)[\overline{\xi}, \eta, \overline{\eta}, \xi]^t \]
\[ \cdot \{ \ln([\overline{z}, \eta, \overline{\eta}, \xi]) - 2 \ln d(z, \xi) + 2 \ln d(z, \eta) + 2 \ln d(\eta, \xi) \} d\nu_0(\eta) \]
\[ = -c_t \int_{\mathbb{H}} k(\overline{z}, \eta)l(\overline{\eta}, \xi)[\overline{\xi}, \eta, \overline{\eta}, \xi] \cdot \ln[\overline{\xi}, \eta, \overline{\eta}, \xi] d\nu_0(\eta). \]

Then consider \( g \) such that
\[ \theta(z, \xi) = g(z) + \overline{g(\xi)} + \ln(z - \overline{\xi}) \]
is $\Gamma$-invariant.

The same argument as above gives that for $k, l$ in $\mathcal{D}(M_\theta)$ s.t. $k \ast_t l \in \mathcal{D}(M_\theta)$ and $k, l$ in $\text{Dom}(R_t)$, $k \ast_t l$ in $\text{Dom}(R_t)$ we have that

$$M_\theta(k \ast_t l) - M_\theta k \ast_t l - k \ast_t M_\theta l + k \ast_t T_\theta \ast_t l$$

is equal to $C_t(k, l) - 2\nabla R_t(k, l)$.

Finally, remark that we have proved that for $k, l$ in $\mathcal{D}_t^+$, which is the vector space of all $k$ that are of the form $P_t(k_1 \phi^\varepsilon)$, the expression

$$M_{\phi^\varepsilon}(k \ast_t l) - M_{\phi^\varepsilon} k \ast_t l - k \ast_t M_{\phi^\varepsilon} l + k \ast_t \Lambda(1) \ast_t l - C_t(k, l)$$

is orthogonal to $P_t(L_t^t)$ (in other words, if we apply $P_t$ to the left and right we get 0).

If we could extend the above relation to all $k, l$ in $\text{Dom}(M_{\phi^\varepsilon}) \cap \text{Dom} R_t$ such that $k \ast_t l$ belongs to the same domain, then the above relation, by the considerations at the end of §7, would imply that $\Lambda(1) - (c_t'/c_t) \cdot \text{Id}$ coincides (on an affiliated domain) with $T_{\ln \phi}$.

Note that this corresponds formally to the fact that

$$T_{\ln \phi}^t = T_{\ln(\eta - \eta)}^t + T_{\ln \Delta}^t + T_{\ln \Delta}^t.$$

On the other hand $T_{\ln(\eta - \eta)}^t = P_{\ln(\eta - \eta)} - (c_t'/c_t) \cdot \text{Id}$, while $T_{\ln \Delta}^t$ is clearly, on its domain, $\mathcal{P}_{\ln \Delta}$ (and similarly for $T_{\ln \Delta}^t$).

If the domains had nonzero intersection, one could directly conclude that

$$\mathcal{S}_{\ln \Delta}^t + \mathcal{S}_{\ln \Delta}^t + \mathcal{P}_{\ln(\eta - \eta)} = \Lambda(1).$$
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