COMPONENT GRAPHS OF VECTOR SPACES AND ZERO-DIVISOR GRAPHS OF ORDERED SETS

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ABSTRACT. In this paper, nonzero component graphs and nonzero component union graphs of finite dimensional vector space are studied using the zero-divisor graph of specially constructed 0-1-distributive lattice and the zero-divisor graph of rings. Further, we define an equivalence relation on nonzero component graphs and nonzero component union graphs to deduce that these graphs are the graph join of zero-divisor graphs of Boolean algebras and complete graphs. In the last section, we characterize the perfect and chordal nonzero component graphs and nonzero component union graphs.

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1. INTRODUCTION

The study of graphs associated with algebraic and ordered structures is an active and fruitful area of research. Algebraic structures mainly include groups, rings and vector spaces while ordered structures include posets, lattices and boolean algebras. There are many research articles on associating a graph with an algebraic structure (ordered structure) and investigating the algebraic (ordered) features of algebraic (ordered) structure using the associated graph, and vice versa. Cayley graphs of groups is the first example of this kind. This graph was introduced by Arthur Cayley in 1878. Few other graphs such as zero-divisor graphs of rings, comaximal ideal graphs of rings, nonzero component graphs and nonzero component union graphs of finite dimensional vector space are some examples of the graphs associated with algebraic structures, while zero-divisor graphs of posets and comparability graphs of posets are some of the graphs associated with ordered structures.

Beck [7] first introduced the concept of a zero-divisor graph \( \Gamma(R) \) of a commutative ring \( R \) with unity, where the vertex set is the set of elements of \( R \), and two vertices \( x \) and \( y \) are adjacent if \( xy = 0 \). Anderson and Livingston [6] modified this definition of zero-divisor graph \( \Gamma(R) \) of a ring \( R \) by considering the vertex set to be the set of all nonzero zero-divisors and the adjacency to be the same, that is, \( x \) and \( y \) are adjacent if \( xy = 0 \).

The zero-divisor graph \( G(P) \) of a poset \( P \) is defined and explored in a similar way. In [12], the concept of a poset’s zero-divisor graph is introduced, which is later modified in [17]. Assume that \( P \) is a poset with 0. Given any \( \emptyset \neq A \subseteq P \), the lower cone of \( A \) is given by \( A' = \{ b \in P \mid b \leq a \text{ for every } a \in A \} \). Define a zero-divisor of \( P \) to be any element of the set

\[
Z(P) = \left\{ a \in P \mid (\exists b \in P \setminus \{0\}) \{a,b\}^\ell = \{0\} \right\}.
\]
As in [17], the zero-divisor graph of $P$ is the graph $G(P)$ whose vertices are the elements of $Z(P) \setminus \{0\}$ such that two vertices $a$ and $b$ are adjacent if and only if $\{a, b\} \subseteq \{0\}$. If $Z(P) \neq \{0\}$, then clearly $G(P)$ has at least two vertices, and $G(P)$ is connected with diameter at most three [17 Proposition 2.1].

One good reason for studying the zero-divisor graphs of posets is that many other examples of graphs defined on algebraic structures, such as the noncyclic graph of a group, defined by Abdollahi and Hassanabadi [2], now perhaps better known as the complement of the enhanced power graph of the group, following [1]. So we have the possibility of proving results for many different types of algebraic structures by considering the zero-divisor graphs of posets. This paper is intended as an illustration of this principle.

Recently, Angsuman Das [9, 10] defined and studied the nonzero component graph (nonzero component union graph) of a finite dimensional vector space. Let $V$ be a vector space over field $F$ with $B = \{ v_1, \ldots, v_n \}$ as a basis and 0 as the null vector. Then any vector $a \in V$ can be uniquely expressed in linear combination of the form $a = a_1 v_1 + \cdots + a_n v_n$. We denote this representation as a basic representation of $a$ with respect to $\{ v_1, \ldots, v_n \}$. Define the skeleton of $a$ with respect to $B$, as

$S_B(a) = \{ v_i \mid a_i \neq 0, a = a_1 v_1 + \cdots + a_n v_n \}$.

Angsuman Das [9] defined the nonzero component graph $\mathcal{G}(V)$ with respect to $B$ as follows: The vertex set of graph $\mathcal{G}(V)$ is $V \setminus \{ 0 \}$ and for any $a, b \in V \setminus \{ 0 \}$, $a$ is adjacent to $b$ if and only if $a$ and $b$ share at least one $v_i$ with non-zero coefficient in their basic representation, that is, $a$ and $b$ are adjacent in $\mathcal{G}(V)$ if and only if $S_B(a) \cap S_B(b) \neq \emptyset$. It is easy to observe that the nonzero component graph $\mathcal{G}(V)$ with respect to a basis $B$ and the nonzero component graph $\mathcal{G}(V)'$ with respect to a basis $B'$ are isomorphic.

In another paper [10], Angsuman Das defined the nonzero component union graph $\mathcal{G}(V)$ with respect to $B$ as follows: The vertex set of graph $\mathcal{G}(V)$ is $V \setminus \{ 0 \}$ and for any $a, b \in V \setminus \{ 0 \}$, $a$ is adjacent to $b$ if and only if $S_B(a) \cup S_B(b) = B$. It is easy to observe that the nonzero component union graph $\mathcal{G}(V)$ and $\mathcal{G}(V)'$ with respect to basis $B$ and $B'$, respectively, are isomorphic. This graph is also known as the skeleton graph; see [3].

Thus, the nonzero component graph will mean the skeleton intersection graph and nonzero component union graph will mean the skeleton union graph. These names are justified by the adjacency of these two graphs.

Many results of nonzero component graphs and nonzero component union graphs of vector spaces are similar to as that of zero-divisor graphs of ordered sets. For example, both of these graphs are weakly perfect, both of these graphs are connected with diameter at most 2 etc. This motivated us to study these graphs from the perspective of the zero-divisor graph of an ordered sets and its join with a complete graph. By the join of two graphs $G$ and $H$, we mean a graph formed from disjoint copies of $G$ and $H$ by connecting each vertex of $G$ to each vertex of $H$. We denote the join of graphs $G$ and $H$ by $G \vee H$; see West [18].

In this paper, we prove the following main result.

**Theorem 1.1.** Let $V$ be a $n$-dimensional vector space over a field $F$ and let $t = |V|_{12 \ldots n} = (|F| - 1)^n$. Let $\mathcal{G}(V)$ and $\mathcal{G}(V)$ be the nonzero component graph and nonzero component union graph respectively.

1. $\mathcal{G}(V) = G'(L) \vee K_t$;
2. $\mathcal{G}(V) = G(L') \vee K_t$;

where $L$ is a lattice constructed from $V$, $L'$ is the dual lattice of $L$ and $G(L)$ is the zero-divisor graph of $L$.

In the last section, we apply results on chordal and perfect zero-divisor graphs of ordered sets to decide when the graphs $\mathcal{G}(V)$ and $\mathcal{G}(V)$ are perfect or chordal.
2. Preliminaries

We begin with the following necessary definitions and terminologies given in Devhare et al. [11] and Khandekar and Joshi [16].

**Definition 2.1** ([11] and [16]). Let $P$ be a poset. Given any $\emptyset \neq A \subseteq P$, the upper cone of $A$ is given by $A^a = \{ b \in P \mid b \geq a \text{ for every } a \in A \}$. If $a \in P$, then the sets $\{a\}^u$ and $\{a\}^f$ will be denoted by $a^u$ and $a^f$, respectively. By $A^u$, we mean $\{A^u\}$. Dually, we have the notion of $A^f$.

A poset $P$ with $0$ is called 0-distributive if $\{a, b\}^f = \{0\} = \{a, c\}^f$ implies $\{a, b, c\}^u = \{0\}$; see [14]. Note that if $\{b, c\}^u = \emptyset$, then $\{b, c\}^u = P$. A lattice $L$ with $0$ is said to be 0-distributive if $a \wedge b = 0$ and $a \wedge c = 0$ implies $a \wedge (b \vee c) = 0$. Hence it is clear that if a lattice $L$ is 0-distributive, then $L$, as a poset, is a 0-distributive poset. Dually, a lattice $L$ with $1$ is said to be 1-distributive if $a \lor b = 1$ and $a \lor c = 1$ implies $a \lor (b \wedge c) = 1$.

We set $\mathcal{D} = P \setminus Z(P)$. The elements $d \in \mathcal{D}$ are the dense elements of $P$.

Throughout, $P$ denotes a poset with $0$ and $q_i, i \in \{1, 2, \ldots, n\}$ are the atoms of $P$, where $n \geq 2$.

Afkhami et al. [3] partitioned the set $P \setminus \{0\}$ as follows.

Let $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, where $k > 0$. The notation $P_{i_1, i_2, \ldots, i_k}$ stands for the set

$$P_{i_1, i_2, \ldots, i_k} = \left\{ x \in P \mid x \in \left( \bigcap_{s=1}^{k} (q_{i_s})^u \right) \setminus \bigcup_{j=1}^{k} (q_{i_j})^u \right\}.$$  

Thus $P_{i_1, i_2, \ldots, i_k}$ is the set of elements $x$ of $P$ such that the atoms below $x$ are precisely $q_{i_1}, q_{i_2}, \ldots, q_{i_k}$.

In [4], the following observations are proved; these show that the sets just defined partition $P \setminus \{0\}$.

(1) If the index sets $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_{k'}\}$ of $P_{i_1, i_2, \ldots, i_k}$ and $P_{j_1, j_2, \ldots, j_{k'}}$, respectively, are distinct, that is, $\{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_{k'}\}$, then $(P_{i_1, i_2, \ldots, i_k}) \cap (P_{j_1, j_2, \ldots, j_{k'}}) = \emptyset$.

(2) $P \setminus \{0\} = \bigcup_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} P_{i_1, i_2, \ldots, i_k}$.

Define a relation $\approx$ on $P \setminus \{0\}$ as follows: $x \approx y$ if and only if $x, y \in P_{i_1, i_2, \ldots, i_k}$ for some part $P_{i_1, i_2, \ldots, i_k}$ of the partition just defined. Thus $x \approx y$ if $x$ and $y$ are above the same atoms of $P$.

The set of equivalence classes under $\approx$ of $P \setminus \{0\}$ will be denoted by

$$[P] = \{P_{i_1, i_2, \ldots, i_k} \mid \{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}, \text{ and } P_{i_1, i_2, \ldots, i_k} \neq \emptyset\}.$$  

Now, we set $[P] = [P] \cup P_0$, where $P_0 = \{0\}$. We define relation $\leq$ on $[P]$ as follows. $P_{i_1, i_2, \ldots, i_k} \leq P_{j_1, j_2, \ldots, j_m}$ if and only if $i_1 \leq j_1$, for some $a \in P_{i_1, i_2, \ldots, i_k}$ and for some $b \in P_{j_1, j_2, \ldots, j_m}$, where $P_{i_1, i_2, \ldots, i_k}, P_{j_1, j_2, \ldots, j_m} \in [P]$. It is not very difficult to prove that $( [P], \leq )$ is a poset. The least element of $([P], \leq )$ is $P_0$ and if $P$ has the greatest element $1$, then the greatest element of the poset $([P], \leq )$ is $P_{1, 2, \ldots, n}$.

The following statements (1)–(4) are essentially proved in [11] (see Lemma 4.2, Lemma 4.5). We write these statements in terms of $P_{i_1, i_2, \ldots, i_k}$ (see [10]). These properties will be used in the sequel.

**Lemma 2.2.** The following statements are true.

(1) If $q_1, q_2, \ldots, q_n$ are distinct atoms of $P$, then $[q_1], \ldots, [q_n]$ are distinct atoms of $[P]$. Note that $[q_i] = P_0$, for every $i \in \{1, \ldots, n\}$.

(2) If $a \leq b$ in $P$, then $[a] \leq [b]$ in $[P]$. Moreover, $P_{i_1, i_2, \ldots, i_k} \leq P_{j_1, j_2, \ldots, j_m}$ in $[P]$ if and only if $\{i_1, i_2, \ldots, i_k\} \subseteq \{j_1, j_2, \ldots, j_m\}$.

(3) $\{a, b\} = \{0\}$ in $P$ if and only if $\{[a], [b]\} = \{[0]\}$ in $[P]$. Note that the lower cones are taken in the respective posets. $P_{i_1, i_2, \ldots, i_k}$ and $P_{j_1, j_2, \ldots, j_m}$ are adjacent in $G([P])$ if and only if $\{i_1, i_2, \ldots, i_k\} \cap \{j_1, j_2, \ldots, j_m\} = \emptyset$. Further, $a \in V(G(P))$ if and only if $[a] \in V(G([P]))$. 

(4) Let \([a] \in V(G([P]))\). Then for any \(x, y \in [a], \{x, y\} \neq \emptyset\) in \(P\). Hence vertices of \([a]\) forms an independent set in \(G(P)\). Further, if \([(a), [b]] = ([0])\) in \([P]\), then for any \(x \in [a]\) and for any \(y \in [b], \{x, y\} = \emptyset\) in \(P\). In particular, \([a]\) and \([b]\) are adjacent in \(G([P])\) with \([a] = m, \) \([b] = n\), then the vertices of \([a]\) and \([b]\) forms an induced complete bipartite subgraph \(K_{m,n}\) of \(G(P)\). Moreover, for any \(x, y \in [a], \deg_{G(P)}(x) = \deg_{G(P)}(y)\).

3. INTERPLAY

With this preparation, we now give a relation between the skeleton intersection (union) graph of a finite dimensional vector space and the zero-divisor graph of a poset.

For this, let \(V\) be a finite dimensional vector space over a field \(\mathbb{F}\) with \(B = \{v_1, \ldots, v_n\}\) as a basis. Let \(1 \leq i_1 < i_2 \cdots < i_k \leq n\). The notation, \(V_I = \{0\}\) (null vector/zero vector) and \(V_{i_1, i_2 \cdots i_k}\) stands for the set

\[
V_{i_1, i_2 \cdots i_k} = \{a \in V \mid (a_j \neq 0) \Leftrightarrow (j \in \{i_1, \ldots, i_k\})\},
\]

where \(a = a_1 v_1 + \cdots + a_n v_n\). We also denote this set by \(V_I\), where \(I = \{i_1, i_2, \ldots, i_k\}\).

For any \(\{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}\), we have \(v_i + \cdots + v_i = w\) (say). Clearly, \(w \in V_{i_1, i_2 \cdots i_k}\). Thus \(V_{i_1, i_2 \cdots i_k} \neq \emptyset\) for every \(\{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}\). Also observe that \(V_{i_1, i_2 \cdots i_k} = ([\mathbb{F}] - 1)^k\) for all nonempty set \(\{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}\).

Since any vector \(a \in V\) can be uniquely expressed as \(a = a_1 v_1 + \cdots + a_n v_n\), we have \(V_{i_1 \cdots i_k} \cap V_{j_1 \cdots j_l} = \emptyset\) if \(\{i_1, \ldots, i_k\}\) and \(\{j_1, \ldots, j_l\}\) are distinct subsets of \(\{1, 2, \ldots, n\}\).

For any \(a \in V\), if \(a = 0\), then \(0 = a \in V_Q\); otherwise, if \(a \neq 0\), then \(a \in V_{i_1 \cdots i_k}\) for some nonempty subset \(\{i_1, \ldots, i_k\}\) of \(\{1, 2, \ldots, n\}\).

Thus, \(V = \bigcup_{I \subseteq \{1, \ldots, n\}} V_I\).

We define \([V] = \{V_I \mid I \subseteq \{1, \ldots, n\}\}\). Define a relation \(\leq\) on \([V]\) as: \(V_I \leq V_J\) if and only if \(I \subseteq J\). Clearly, \([V]\) is a poset. In fact, \([V]\) is a lattice under \(V_I \wedge V_J = V_{I \cap J}\) (meet) and \(V_I \vee V_J = V_{I \cup J}\) (join). Further, we observe that \([V]\) contains \(2^n\) elements and is a Boolean lattice isomorphic to \(P(X)\) for \(X = \{1, \ldots, n\}\). In fact, the map \(I \mapsto V_I\) for \(I \subseteq X\), is an isomorphism between \([V]\) and \(P(X)\).

Now, we construct a lattice \(L\) from \([V]\) such that the zero-divisor graph of \(L\) is related to the skeleton intersection graph and skeleton union graph of \(V\).

We replace \(V_I \in [V]\) by the chain of elements of \(V_I\) in \(L\) in some pre-determined well-order, where \(I \neq \emptyset\) and \(I \neq \{1, \ldots, n\}\). The elements \(V_\emptyset\) and \(V_{\{1, \ldots, n\}}\) of \([V]\) are replaced by 0 and 1 respectively, in \(L\).

We illustrate this construction with the following example.

Consider \(V\) as a 3-dimensional vector space over field \(\mathbb{F}\), where \(|\mathbb{F}| = 3\). Let \(B = \{v_1, v_2, v_3\}\) be a basis of \(V\) and let \(\mathbb{F} = \{\emptyset, \{1\}, \{1, 2\}\}\).

To derive the properties of \(L\), we need the following results which are straight forward. Hence proofs are omitted.

**Lemma 3.1.** Let \(L\) be a finite lattice and \(L'\) be a poset obtained from \(L\) by replacing an element of \(L\) by a bounded chain. Then the following statements hold:

1. The poset \(L'\) is a lattice.
2. If \(L\) is a 0-distributive (1-distributive) lattice, then \(L'\) is a 0-distributive (1-distributive) lattice.

Since \([V]\) is a Boolean lattice and every Boolean lattice is 0-distributive and 1-distributive, so is \(L\) by Lemma 3.1.

**Lemma 3.2.** The lattice \(L\) derived from \([V]\) is a 0-distributive as well as 1-distributive.
for some $t$ vertex set $V$ Theorem 3.3. preparation, we are ready to prove our two main theorems. Let $B$ Now, we make a very important observation. Recall that $L$ Note that $L$ denotes the dual of a lattice $L$ ...n) for any $x,y \in \mathbb{L}$ $(x \in V_I$ and $y \in V_J$ for some $I, J \subseteq \{1, \ldots, n\})$. 1) $x \land y = 0$ in $L$ if and only if $V_I \land V_J = V_{\emptyset}$ if and only if $I \cap J = \emptyset$. 2) $x \lor y = 1$ in $L$ if and only if $V_I \lor V_J = V_{\{1,\ldots,n\}}$ if and only if $I \cup J = \{1, \ldots, n\}$. Now, we make a very important observation. Recall that $B$ is a basis for $V$. For any $a \in V$, 1) If $a = 0$, then $S_B(a) = \emptyset$ and $a \in V_{\emptyset}$ 2) If $a \neq 0$, then $S_B(a) \subseteq B$. In particular, $S_B(a) = \{v_{i_1}, \ldots, v_{i_k}\} \subseteq B$ if and only if $a \in V_I$, where $I = \{i_1, \ldots, i_k\}$. Note that $L^0$ denotes the dual of a lattice $L$ (that is $a \leq b$ in $L$ if and only if $b \leq a$ in $L^0$), which is also lattice. If $L$ is a $0-1$-distributive lattice, then so is $L^0$. Let $G(\mathbb{L})$ be the zero-divisor graph of the lattice $\mathbb{L}$ and $G^c(\mathbb{L})$ be its graph complement. With this preparation, we are ready to prove our two main theorems. Theorem 3.3. $\mathbb{G}(V) = G^c(\mathbb{L}) \lor K_t$, where $t = |V_{12,\ldots,n}| = (|F| - 1)^n$. Proof. Note that $V(G(\mathbb{L})) = \mathbb{L} \setminus \{0, 1\}$, as $[\mathbb{V}]$ is Boolean. Further, $V(\mathbb{G}(V)) = V(G(\mathbb{L})) \cup V_{12,\ldots,n}$. For $u, v \in V \setminus \{0\} \cup V_{12,\ldots,n}$. The vertices $u$ and $v$ are adjacent in $G(\mathbb{L})$ if and only if $u \land v = 0$ in $\mathbb{L}$ if and only if $V_I \land V_J = V_{\emptyset}$, where $u \in V_I, v \in V_J$ if and only if $I \cap J = \emptyset$ and if only if $S_B(u) \cap S_B(v) = \emptyset$ if and only if $u$ and $v$ are not adjacent in $\mathbb{G}(V)$. This proves that the induced subgraph of $\mathbb{G}(V)$ on the vertex set $V \setminus \{0\} \cup V_{12,\ldots,n}$ is equal to $G^c(\mathbb{L})$. Let $w \in V_{12,\ldots,n}$ and $(0 \neq) u \in V$ such that $u \neq w$, then $S_B(w) = \{v_1, \ldots, v_n\}$ and $S_B(u) = \{v_{i_1}, \ldots, v_{i_k}\}$ for some $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ and $k \geq 1$. Clearly, $S_B(w) \cap S_B(u) \neq \emptyset$. This implies that any $w \in V_{12,\ldots,n}$ is adjacent to a nonzero $u \in V \setminus \{w\}$ in $\mathbb{G}(V)$. This proves that $\mathbb{G}(V) = G^c(\mathbb{L}) \lor K_t$, where $t = |V_{12,\ldots,n}| = (|F| - 1)^n$. Figure 1.
Theorem 3.4. \[ \text{UG}(V) = G(L^0) \cup K_t, \text{ where } t = |V_{12...n}| = (|F| - 1)^n. \]

Proof. Consider the induced subgraph of \( \text{UG}(V) \) on the set \( V \setminus \{0\} \cup V_{12...n} \). Note that the greatest element 1 of \( L \) is the zero element of \( L^0 \). Let \( u, v \in V \setminus \{0\} \cup V_{12...n} \). The vertices \( u \) and \( v \) are adjacent in \( G(L^0) \) if and only if \( u \land v = 1 \) in \( L \). Let \( u \lor v = 1 \) in \( L \) if and only if \( V_I \lor V_J = V_{12...n} \), where \( u \in V_I, v \in V_J \) if and only if \( I \cup J = \{1, 2, \ldots, n\} \) if and only if \( S_B(u) \cup S_B(v) = B \) if and only if \( u \) and \( v \) are adjacent in \( \text{UG}(V) \). This proves that the induced subgraph of \( \text{IG}(V) \) on the vertex set \( V \setminus \{0\} \cup V_{12...n} \) is equal to \( G(L^0) \).

Let \( w \in V_{12...n} \) and \( (0 \neq) u \in V \) such that \( u \neq w \). Then \( S_B(w) = \{v_1, \ldots, v_n\} \) and \( S_B(u) = \{v_1, \ldots, v_k\} \) for some \( \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\} \) and \( k \geq 1 \). Clearly, \( S_B(w) \cup S_B(u) = \{v_1, \ldots, v_n\} \). This implies that any \( w \in V_{12...n} \), is adjacent to any nonzero \( u \in V \setminus \{w\} \) in \( \text{UG}(V) \). This proves that \( \text{UG}(V) = G(L^0) \cup K_t \), where \( t = |V_{12...n}| = (|F| - 1)^n \). \( \square \)

From the above two results, it is clear that nonzero component union graph alias the skeleton union graph as well as nonzero component graph alias skeleton intersection graph can be studied using the zero-divisor graph of ordered sets; in particular, zero-divisor graph of 0-1 distributive lattices.

Let \( V \) be a \( n \)-dimensional vector space over a field \( F \) and let \( B = \{v_1, \ldots, v_n\} \) be a basis of \( V \). Then the map\[ a = a_1 v_1 + \cdots + a_n v_n \mapsto (a_1, \ldots, a_n) \]is a vector space isomorphism from \( V \) to \( F^n \), where \( F^n = F \times \cdots \times F \) (\( n \)-times). Now \( F^n \) is also a ring (the direct product of \( n \) copies of \( F \)). We prove that the skeleton union graph of \( V \) is the graph join of the complement of the ring-theoretic zero-divisor graph of \( F^n \) and a complete graph. In fact, we prove the following result.

Theorem 3.5. Let \( V \) be a \( n \)-dimensional vector space over a field \( F \) and let \( B = \{v_1, \ldots, v_n\} \) be a basis of \( V \). Then \( \text{IG}(V) \cong \Gamma^c(F^n) \cup K_t \), where \( t = |V_{12...n}| = (|F| - 1)^n \).

Proof. Let \( V \) be a \( n \)-dimensional vector space over a field \( F \) and let \( B = \{v_1, \ldots, v_n\} \) be a basis of \( V \). Consider the ring \( F^n \). The set of all nonzero zero-divisors of \( F^n \) is \( F^n \setminus \{(0, \ldots, 0) \cup U(F^n)\} \), where \( U(F^n) \) is the set of units of \( F^n \). Now an element \( a \in \text{U} \) satisfies \( a \in V_{12...n} \) if and only if \( (a_1, \ldots, a_n) \in U(F^n) \); for if \( a_i = 0 \) for all \( i \in \{1, \ldots, n\} \) then \( a_i^{-1} = a \) is also a ring (the direct product of \( n \) copies of \( F \)). We prove that skeleton intersection graph of \( V \) is the graph join of the complement of the ring-theoretic zero-divisor graph of \( F^n \) and a complete graph. In fact, we prove the following result.

From the proof of Theorem 3.3. any \( w \in V_{12...n} \) is adjacent to \( u \) for a nonzero \( u \in V \setminus \{w\} \) in \( \text{UG}(V) \). This proves that \( \text{IG}(V) \cong \Gamma^c(F^n) \cup K_t \), where \( t = |V_{12...n}| = (|F| - 1)^n \). \( \square \)

We will close this section by observing that the reduced graph of \( \text{IG}(V) \) (\( \text{UG}(V) \)) is the graph join of the zero-divisor graph of a Boolean ring (equivalently, Boolean lattice) and the complete graph. For this, we need the following results. It is well-known that the compressed lattice of a finite \( 0 \)-distributive lattice under the equivalence relation \( \approx \) (see Definition 2.1) is always a Boolean lattice; see [11] Lemma 4.6. In the next result, the finiteness of set of atoms is enough to prove that \( [L] \) is Boolean, if \( L \) is a \( 0 \)-distributive lattice.

Theorem 3.6 ([11] Theorem 2.15]). Let \( L \) be a \( 0 \)-distributive bounded lattice with finitely many atoms. Then \( [L] \) is a Boolean lattice.
Corollary 4.3.

Remark 4.2.

graph $G$

Strong Perfect Graph Theorem [8] asserts that a graph of every induced subgraph equals the order of the largest clique of that subgraph (clique number). The problem of finding induced cycle of length at least 4. A perfect graph is a graph in which the chromatic number is an equivalence relation on $V(G)$. The equivalence class of $v$ is the set \{ $u \in V(G) \mid u \equiv v$ \}, denoted by $[v]^{\equiv}$. Denote the set \{ $[v]^{\equiv} \mid v \in V(G)$ \} by $G_{\text{red}}$. Define $[u] - [v]$ is an edge in $E(G_{\text{red}})$ if and only if $u - v \in E(G)$, where $[u] \equiv [v]$.

**Remark 3.7.** It is easy to observe that, if $G^c(P)$ is the complement of the zero-divisor graph $G(P)$, then $(G^c(P))_{\text{red}} = G^c([P])$.

Bagheri et al. [5] considered the following relation on a graph $G$: $u \equiv v$ if and only if $N_G(u) = N_G(v)$. Clearly, $\equiv$ is an equivalence relation on $V(G)$. The equivalence class of $v$ is the set \{ $u \in V(G) \mid u \equiv v$ \}, denoted by $[v]^\equiv$. Denote the set \{ $[v]^\equiv \mid v \in V(G)$ \} by $V(G_{\text{red}})$. Define $[u]^\equiv - [v]^\equiv$ is an edge in $E(G_{\text{red}})$ if and only if $u - v \in E(G)$, where $[u]^\equiv \neq [v]^\equiv$. Let $[G]^\equiv$ be a simple graph whose vertex set is $V(G)$, and edge set is $E(G_{\text{red}})$.

**Remark 3.8.** It is easy to observe that, if $G(P)$ be the zero-divisor graph, then $[G(P)] = G([P])$.

Since $L$ is a 0–1-distributive lattice with $n$ atoms, then $[L]$ and $[L^\varnothing]$ both are Boolean lattices having $2^n$ elements. This implies that $G([L]) \cong G([L^\varnothing]) \cong \Gamma(\mathbb{Z}_2^n)$. In view of Theorem 3.3 and Remark 3.7 we have the following result.

**Corollary 3.9.**

1. $(\mathbb{L}(V))_{\text{red}} = \Gamma(\mathbb{Z}_2^n) \vee K_1$

2. $[\mathbb{L}(V)]^\equiv = \Gamma(\mathbb{Z}_2^n) \vee K_t$, where $t = |V_{12...n}| = (|F| - 1)^n$.

4. **Applications**

A chord of a cycle $C$ of graph $G$ is an edge that is not in $C$ but has both its end vertices in $C$. A graph $G$ is chordal if every cycle of length at least 4 has a chord, i.e., $G$ is chordal if and only if it does not contain induced cycle of length at least 4. A perfect graph is a graph in which the chromatic number of every induced subgraph equals the order of the largest clique of that subgraph (clique number). The Strong Perfect Graph Theorem [8] asserts that a graph $G$ is perfect if and only if neither $G$ nor $G^c$ contains an induced odd cycle of length at least 5.

**Theorem 4.1** ([10 Theorem 1.1]). Let $P$ be a finite poset such that $[P]$ is a Boolean lattice. Then

(A) $G(P)$ is chordal if and only if one of the following holds:

1. $P$ has exactly one atom;
2. $P$ has exactly two atoms with $|P_i| = 1$ for some $i \in \{1, 2\}$;
3. $P$ has exactly three atoms with $|P_i| = 1$ for all $i \in \{1, 2, 3\}$.

(B) $G^c(P)$ is chordal if and only if the number of atoms of $P$ is at most 3.

(C) $G(P)$ is perfect if and only if $P$ has at most 4 atoms.

**Remark 4.2.** It is easy to observe that the following conditions are equivalent:

1. $G + I_m$ is a chordal (resp. perfect) graph;
2. $G$ is a chordal (resp. perfect) graph;
3. $G \vee K_m$ is a chordal (resp. perfect) graph.

In view of Theorem 3.3 and Remark 4.2 we have the following result.

**Corollary 4.3.** Let $V$ be a finite dimensional vector space over finite field $F$. Then

1. The skeleton intersection graph $\mathbb{L}(V)$ is chordal if and only if $\dim(V) \leq 3.$
(2) The skeleton union graph $\text{UG}(V)$ is chordal if and only if either $\dim(V) = 1$ or $\dim(V) \in \{2, 3\}$ with $|F| = 2$.

(3) The skeleton intersection graph $\text{IG}(V)$ is perfect if and only if the skeleton union graph $\text{UG}(V)$ is perfect; this occurs if and only if $\dim(V) \leq 4$.

Many other properties of skeleton intersection and skeleton union graphs can be obtained from the zero-divisor graphs of posets. Such properties include being weakly perfect, Eulerian or Hamiltonian.

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