Regressions with Fractional $d = 1/2$ and Weakly Nonstationary Processes

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Abstract

Despite major theoretical advances, important classes of fractional models (e.g. ARFIMA models) have not yet been fully characterised in terms of asymptotic theory. In particular, no limit theory is available for general additive functionals of fractionally integrated processes of order $d = 1/2$. Such processes cannot be handled by existing asymptotic results for either stationary or nonstationary processes, i.e. by existing LLNs or FCLTs. Their asymptotics must therefore be derived by novel arguments, such as are developed in this paper. In the course of doing so, we develop new limit theory for a broader class of linear processes lying on the boundary between stationarity and nonstationarity – what we term weakly nonstationary processes. This includes, as leading examples, both fractional processes with $d = 1/2$, and arrays of autoregressive processes with roots drifting slowly towards unity. We apply our new results to the asymptotics of both parametric and kernel regression estimators.

Keywords: Fractional process, half unit root, kernel regression, mildly integrated process, weakly nonstationary process
1 Introduction

Autoregressive Fractionally Integrated Moving Average (ARFIMA) models, and their vector generalisations, are one of the most important classes of stochastic processes in statistics. Yet the limit theory for these processes still remains to be fully characterised. In particular, no asymptotic theory is currently available for additive functionals of fractionally integrated processes of order \( d \) (hereafter ‘I(\(d\))’) when \( d = 1/2 \). This value is of particular significance as it demarcates the boundary between stationarity and nonstationarity for these processes.

Existing results suggest the following dichotomy in the limit theory for fractional processes. Firstly, in the stationary region (\(|d| < 1/2\)), laws of large numbers (LLNs) for dependent processes apply, yielding limiting results of the form

\[
\frac{1}{n} \sum_{t=1}^{n} f(x_t) \overset{p}{\to} E f(x_t)
\]

for suitably integrable \( f \) (see e.g. Taqqu, 1975; Giraitis, 1985; Giraitis and Surgailis, 1985). On the other hand, when \( d \in (1/2, 3/2) \), the asymptotics are driven by functional central limit theorems (FCLTs) of the form

\[
n^{1/2-d}x_{[nr]} \overset{d}{\to} W_d(r),
\]

where \( W_d \) denotes a fractional Brownian motion (fBM) of order \( d \) (see e.g. Taqqu, 1975, Phillips, 1987a; Chan and Wei, 1988; Marinucci and Robinson, 2000). Together with the continuous mapping theorem, or more general results on the convergence of integral functionals, such FCLTs yield

\[
\frac{1}{n} \sum_{t=1}^{n} f(n^{1/2-d}x_t) \overset{d}{\to} \int_{0}^{1} f(W_d(r))dr
\]

These differences in the limit theory are not only of theoretical interest, but also practically relevant to problems of estimation and inference in regression models with fractionally integrated regressors. For example, consider the linear regression

\[
y_t = \beta x_{t-1} + u_t,
\]

where \( u_t \) is a martingale difference sequence (e.g. with respect to the natural filtration for \((x_t, u_t)\)). Then for \(|d| < 1/2\), a LLN together with a martingale central limit theorem (CLT) yields

\[
n^{1/2}(\hat{\beta}_{LS} - \beta) \overset{d}{\to} N \left(0, E \left(u_t^2 / E \left(x_t^2 \right) \right) \right),
\]
i.e. the least squares (LS) estimator \( \hat{\beta} \) is \( n^{-1/2} \)-consistent for \( \beta \) and asymptotically normal. On the other hand for \( d \in (1/2, 3/2) \), the weak convergence of stochastic integrals yields

\[
n^d(\hat{\beta}_{LS} - \beta) \xrightarrow{d} \left[ \int_0^1 W_d(r)^2 dr \right]^{-1} \int_0^1 W_d(r) dB_u(r),
\]

where \( B_u(r) \) is a Brownian motion (see e.g. Phillips, 1995; Robinson and Hualde, 2003). In this case, the LS estimator enjoys a faster rate of convergence, but its limiting distribution is such that conventional inferential procedures – i.e. those based on a normal approximation to the \( t \)-statistic – are no longer valid.

This paper develops new limit theory for the case where \( d = 1/2 \), showing that there is in fact a trichotomy in the asymptotics of fractional processes. Although \( I(1/2) \) processes are nonstationary – thus ruling out the application of a LLN in the manner of (1) – FCLTs do not apply to these processes either. For example, in the case of the ‘type II’ fractional process: while the finite dimensional distributions of \( X_n(r) := Var(x_n)^{-1/2}x_{nr} \) do converge, their limit is a nonseparable Gaussian ‘white noise’ process \( G \), which has the property that \( G(r) \) and \( G(s) \) are independent for every \( r \neq s \). Since the sample paths of \( G \) are not a.s. bounded, this convergence cannot be strengthened to weak convergence with respect to the uniform or Skorokhod topologies; nor does \( \{X_n\} \) satisfy even the much weaker regularity conditions sufficient for the convergence of integral functionals in the form of (2) (see Gikhman and Skorokhod, 1969, p. 485, Theorem 1). Despite this, we shall prove that for \( I(1/2) \) type II processes

\[
\frac{1}{n} \sum_{t=1}^{n} f(\beta_n^{-1} x_t) \xrightarrow{p} \int_{\mathbb{R}} f(x) \varphi_1(x) dx
\]  

(3)

where \( \beta_n^2 = Var(x_n) \), and \( \varphi_{\sigma^2} \) denotes a normal density with mean zero and variance \( \sigma^2 \). The counterpart of (3) for type I processes is

\[
\frac{1}{n} \sum_{t=1}^{n} f(\beta_n^{-1} x_t) \xrightarrow{d} \int_{\mathbb{R}} f(x + X^-) \varphi_{1/2}(x) dx
\]

(4)

\(^1\)For the distinction between ‘type I’ and ‘type II’ fractional processes, due to Marinucci and Robinson (1999), see Section 2.1.
where $X^− \sim N[0,1/2]$. Results of the kind (3) and (4) for general nonlinear transformations are new to the literature – the limit theory for $I(1/2)$ processes previously developed by Shimotsu and Phillips (2005) and Hualde and Robinson (2011) being limited to specific functionals that arise in the context of memory estimation.

En route to (3) and (4), we develop a general framework that is able to handle the asymptotics of a much broader class of processes (and arrays thereof) that share two of the key characteristics of $I(1/2)$ processes: (a) they are sufficiently nonstationary to resist the application of existing LLNs; and (b) their dependence is sufficiently weak that their finite-dimensional distributions converge to those of a nonseparable Gaussian process. We term these weakly nonstationary processes (WNPs). We develop a set of general high-level conditions that express (a) and (b) in more precise mathematical terms, and which allow us to establish the asymptotics of

$$
\frac{1}{n} \sum_{t=1}^{n} f(\beta_n^{-1} x_t) \quad \text{and} \quad \frac{\beta_n}{nh_n} \sum_{t=1}^{n} K \left( \frac{x_t - x}{h_n} \right)
$$

where $\beta^2_n = Var(x_n)$, $f$ is locally integrable, $K$ is integrable, and $h_n$ denotes a bandwidth sequence. We also provide an accompanying set of lower-level sufficient conditions that may be more directly verified for linear processes.

Beyond the ARFIMA class, another important class of processes to which our low-level conditions apply are the mildly integrated (MI) processes considered by Giraitis and Phillips (2006) and Phillips and Magdalinos (2007, 2009). These are closely related to the Nearly Integrated (NI) processes developed by Chan and Wei (1987) and Phillips (1987), and more recently extended by Buchmann and Chan (2007). Both MI and NI processes may be defined in terms of an array as

$$
x_t(n) = (1 - \kappa_n^{-1})x_{t-1}(n) + v_t, \quad x_0(n) = 0,
$$

where $v_t$ is a stationary process and $\kappa_n > 0$ with $\kappa_n \to \infty$, so that the autoregressive coefficient becomes increasingly proximate to unity as $n$ grows. Both NI and MI processes thus describe wide sense autoregressive processes, which have a root in the vicinity of unity. They have accordingly been used to investigate the behaviour of various inferential procedures under local departures from unit roots (e.g. Mikusheva, 2007, and Duffy, 2017), and in the construction of robust inferential procedures (e.g. Magdalinos and Phillips, 2009; Kostakis, Magdalinos and Stamatogiannis, 2014).
The crucial difference between NI and MI processes concerns the assumed growth rate of the sequence $\kappa_n$. NI processes are defined by $\kappa_n/n \to c \neq 0$, with the consequence that $n^{-1/2}x_{[nr]}$ converges weakly to an Ornstein-Uhlenbeck process, leading to convergence results akin to (2). MI processes have $\kappa_n/n \to 0$, which tilts $x_t(n)$ closer to stationarity. As a consequence, FCLTs no longer apply, and it is shown in this paper that limits analogous to (3) hold instead. Again, these results are new to the literature. Existing results due to Giraitis and Phillips (2006) and Phillips and Magdalinos (2007) consider only quadratic transformations of MI processes driven by short memory linear processes errors. Duffy (2017) provides limit theory for bounded and Lipschitz continuous kernel functionals of MI processes driven by short memory errors. Our results extend these papers both by allowing for much more general nonlinear transformations, and for MI processes that are driven by long memory errors – an extension of MI processes analogous to that of Buchmann and Chan (2007) for NI processes.

Our general results on the asymptotics of the functionals in (5) appear in Sections 2 and 3. We then apply these to deduce the large-sample behaviour of both parametric and nonparametric estimators in regression models involving WNPs – or non-linear transformations thereof – as regressors (Section 4). In conjunction with suitable martingale central limit theorems (Hall and Heyde, 1980; and Wang, 2014), we show that both the ordinary least squares (OLS) and Nadaraya-Watson (NW) kernel regression estimators have limit distributions that are mixed normal when the regressor is $I(1/2)$ type-I, and normal when the regressor is $I(1/2)$ type-II or MI. Therefore, in regressions with WNPs conventional inferential procedures remain valid and none of the modifications familiar from inference with nonstationary processes (e.g. Phillips, 1995; Robinson and Hualde, 2003) are required. Proofs of the main results of the paper appear in Section 5; proofs of auxiliary lemmas are given in the Supplementary Material.

**Notation.** For two deterministic sequences $\{a_n\}$ and $\{b_n\}$, $a_n \sim b_n$ denotes $\lim_{n \to \infty} a_n/b_n = 1$. For two random variables $X$ and $Y$, $X \sim Y$ and $X \overset{d}{=} Y$ denote distributional equality between $X$ and $Y$. For a real number $x$, $\lfloor x \rfloor$ denotes its integer part. $1\{A\}$ denotes the indicator function for the set $A$. $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_+^*$ are the extended, the positive, and strictly positive real numbers respectively. A $k \times k$ diagonal matrix with diagonal elements $a_1, \ldots, a_k$ is written $\text{diag}(a_1, \ldots, a_k)$. $:=$ denotes definitional equality. $\sigma(X)$ denotes the
sigma algebra generated by random variable $X$. All limits are taken as $n \to \infty$ unless otherwise indicated.

2 Additive functionals of standardised WNPAs

2.1 $I(1/2)$ and MI processes

We first consider the asymptotics of additive functionals of standardised WNPAs: that is, quantities of the form

$$
\frac{1}{n} \sum_{t=1}^{n} f(\beta_n^{-1} x_t)
$$

(7)

where $f$ is locally integrable, and $\beta_n^2 = Var(x_n)$.

As noted in the introduction, two classes of processes with which we are particularly concerned are the $I(1/2)$ and MI processes. To define these more formally, let $\{\xi_t\}_{t \in \mathbb{Z}}$ denote an i.i.d. sequence with mean zero and unit variance (further properties of which will be specified by Assumption INN in Section 2.3 below), and define the linear processes

$$
x_t = \sum_{j=0}^{t-1} \phi_j v_{t-j} \quad \text{where} \quad v_t = \sum_{i=0}^{\infty} c_i \xi_{t-i}
$$

(8)

for some coefficient sequence $\{c_i\}_{i \in \mathbb{Z}}$.

The definition of a ‘fractionally integrated process’ used this paper closely follows that of Marinucci and Robinson (1999; see also Robinson and Hualde, 2003, p. 1728-1730). These authors classify non-stationary fractional processes as type I or type II according to the ‘type’ of the fractional Brownian motion (fBM) to which their finite dimensional distributions converge (upon standardisation). Although these authors consider processes with a long memory component that is specified ‘parametrically’ – via the expansion of an autoregressive lag polynomial $(1 - L)^d$ (see also Remark 2.4(a) below) – their classification extends straightforwardly to the case where this is instead formulated ‘semi-parametrically’, in terms of the decay rate of the coefficients $\{\phi_j\}$. Thus we shall say that for $d \in (1/2, 1)$, $x_t$ is an $I(d)$ process of

- type I: if $\phi_j = 1 \forall j$, $c_s \sim \ell(s)s^{d-2}$ and $\sum_{s=0}^{\infty} c_s = 0$; and
• type II: if \( \phi_j \sim \ell(j) j^{d-1} \), \( \sum_{s=0}^{\infty} |c_s| < \infty \), and \( \sum_{s=0}^{\infty} c_s \neq 0 \);

where \( \ell \) is slowly varying at infinity (henceforth, ‘SV’) in the sense of Bingham, Goldie and Teugels (1987, p. 6). Fractional processes of this kind have been widely studied (for \( d \neq 1/2 \)): see e.g. Jeganathan (1999, 2004, 2008) for the type I case (and also Taqqu, 1975; Astrauskas, 1983; Kasahara and Maejima, 1988); and Robinson and Hualde (2003), Marmol and Velasco (2004), Phillips and Shimotsu (2005), Hualde and Robinson (2011) for the type II case. The preceding definitions extend naturally to the case where \( d = 1/2 \), and give the definitions of \( I(1/2) \) processes (of each type) used throughout this paper – though the standardised processes will not converge to an fBM of either type in the case. Note also that if \( \phi_n \to 0 \) sufficiently rapidly, then these processes may in fact be stationary (see Remark 2.4(b) below).

MI processes, which were defined in (6) above, can also be encompassed within the framework of (8) if we allow \( \phi_j \) to depend on \( n \) as per

\[ \phi_j = \phi_j(n) = (1 - \kappa_n^{-1})^j \]

where \( \kappa_n > 0 \), \( \kappa_n \to \infty \) and \( \kappa_n/n \to 0 \). Previous work on these processes has assumed that \( v_t \) is short memory in the sense that \( \sum_{s=0}^{\infty} |c_s| < \infty \) (see e.g. Giraitis and Phillips, 2006; Magdalinos and Phillips, 2007). Our results apply to this case, but we shall also allow \( v_t \) to have long memory in the sense that \( c_s \sim s^{-m} \) for \( m \in (1/2, 1) \), thereby extending this previous work much in the manner of Buchmann and Chan’s (2007) extension of earlier work on NI processes.

Both \( I(1/2) \) and MI processes may thus be regarded as instances of (arrays of) linear processes formed from the underlying i.i.d. sequence \( \{\xi_t\} \), which we denote generally as

\[ x_t(n) = \sum_{k=0}^{\infty} a_{k,t}(n) \xi_{t-k} \tag{9} \]

where (suppressing the dependence of these quantities on \( n \) for the sake a readability, as we shall do freely below)

\[ a_{k,t} = \sum_{j=0}^{(t-1) \land k} \phi_j c_{k-j} = \begin{cases} \sum_{j=0}^{k} \phi_j c_{k-j} =: a_k & \text{if } 0 \leq k \leq t - 1, \\ \sum_{j=0}^{t-1} \phi_j c_{k-j} =: a_{k,t} & \text{if } k \geq t. \end{cases} \tag{10} \]
Note, in particular, that \(a_{k,t}\) does not depend on \(t\) for \(1 \leq k \leq t - 1\), and we accordingly denote these coefficients by simply \(a_k\). For \(k \geq t\), the notation \(a_{-k,t}\) reminds us that these coefficients refer to innovations dated \(t \leq 0\).

### 2.2 Asymptotics under high-level conditions

To allow our limit theory to cover a broad class of processes, which encompass the examples of \(I(1/2)\) and MI processes as a special cases, this section provides high-level conditions (Assumption HL) under which the limit of (7) may be obtained. We subsequently provide sufficient conditions for these to be satisfied by general linear processes of the form (9)–(10) (Assumption LP in Section 2.3 below). These ‘linear process conditions’ are then, in turn, verified for \(I(1/2)\) and MI processes under certain low-level technical conditions (Assumption LL in Section 2.4 below).

**Assumption HL (high-level conditions)**

**HL0** Let \(\{X_t(n)\}_{t=1}^n\), \(n \in \mathbb{N}\) be a random array and \(\{\mathcal{F}_t\}_{t=-\infty}^\infty\) a filtration such that \(X_t(n)\) is \(\mathcal{F}_t\)-measurable for all \(t\) and \(n\). Let \(\{\beta_n\}\) denote a non-negative sequence with \(\beta_n \to \infty\).

**HL1** \(X_t(n) = X_t(n)^+ + X_t(n)^- + R_t(n)\), where \(X_t(n)^-\) is \(\mathcal{F}_0\)-measurable, and \(\sup_{1 \leq t \leq n} \mathbb{P}\{\beta_n^{-1}|R_t(n)| > \epsilon\} \to 0\) for every \(\epsilon > 0\).

**HL2** There are random variables \(X^+\) and \(X^-\), where \(X^+\) has bounded density \(\Phi_{X^+}\) such that: for every \(\delta \in (0,1)\) and \(\{t_n\}\) with \([n\delta]\) \(\leq t_n \leq n\)

(a) \(\beta_{n}^{-1}X_{t_n}(n)^+ \overset{d}{\to} X^+\), conditionally on \(\mathcal{F}_0\) in the sense that for all bounded and continuous \(h : \mathbb{R} \to \mathbb{R}\)

\[
\mathbb{E}\left[h\left(\beta_{n}^{-1}X_{t_n}(n)^+\right) \mid \mathcal{F}_0\right] \overset{p}{\to} \int_{\mathbb{R}} h(x) \Phi_{X^+}(x)dx; \text{ and}
\]

(b) \(\beta_{n}^{-1}X_{t_n}(n)^- \overset{d}{\to} X^-\), and \(\beta_{n}^{-1}[X_n(n) - X_{t_n}(n)^-] \overset{p}{\to} 0\).

**HL3** \(\beta_{-t}X_t(n)\) has Lebesgue density \(\mathcal{D}_{n,t}(x)\) such that for some \(n_0 \geq t_0 \geq 1\),

\[
\sup_{n \geq n_0, t_0 \leq t \leq n} \sup_x \mathcal{D}_{n,t}(x) < \infty
\]
For every bounded and Lipschitz continuous $g : \mathbb{R} \to \mathbb{R}$

$$\frac{1}{n} \sum_{t=1}^{n} g \left( \beta_n^{-1} X_t (n) \right) = \frac{1}{n} \sum_{t=1}^{n} E \left[ g \left( \beta_n^{-1} X_t (n) \right) \mid F_0 \right] + o_p(1).$$

For some $\lambda \in (0, \infty)$ and some $n_0 \geq 1$

(a) $\sup_{n \geq n_0, 1 \leq t \leq n} E \left| \beta_n^{-1} X_t (n) \right|^\lambda < \infty$; or
(b) $\sup_{n \geq n_0, 1 \leq t \leq n} E \exp \left( \lambda \left| \beta_n^{-1} X_t (n) \right| \right) < \infty$.

For some $n_0 \geq t_0$, $\sup_{n \geq n_0} \frac{\beta_n}{n} \sum_{t=t_0}^{n} \beta_t^{-1} < \infty$, where $t_0$ is as in HL3.

Remark 2.1. (a) When $X_t(n) = x_t(n)$, the linear process array in (9), condition HL1 is trivially satisfied by splitting $x_t(n)$ into two weighted sums as per

$$x_t(n) = \sum_{k=0}^{t-1} a_k(n) \xi_{t-k} + \sum_{k=t}^{\infty} a_k^{-}(n) \xi_{t-k} =: x_t^+(n) + x_t^-(n) \quad (11)$$

(b) HL2 expresses one of the key properties of a WNP: that its finite dimensional distributions should converge (upon standardisation), albeit not to those of a separable process. Its requirements are best illustrated by the $I(1/2)$ type I process with $\ell(x) = 1$. In this case, the application of a CLT for weighted sums yields that for every $r, s \in (0, 1]$,

$$\beta_n^{-1}(x_{[nr]}^+, x_{[ns]}^+) \overset{d}{\to} (\eta_r, \eta_s) \quad (12)$$

where $\beta_n^2 = Var(x_n) \sim C \cdot \ln n$, and $\eta_r$ and $\eta_s$ are independent $N[0, 1/2]$ random variables. We thus have the marginal convergence of each coordinate of $\beta_n^{-1}x_{[nr]}^+$ to identical distributional limits, as per HL2(a) – and if $\xi_t$ is i.i.d., the required conditional convergence holds trivially. In that case, (12) holds jointly with (and independently of)

$$\beta_n^{-1}(x_{[nr]}^-, x_{[ns]}^-) \overset{d}{\to} (\eta^-, \eta^-) \quad (13)$$

where $\eta^- \sim N[0, 1/2]$. Note, in particular, the degeneracy in the joint distribution of the limit in (13), consistent with the second part of HL2(b).

(c) HL3 is useful for establishing $L_1$-approximations for functionals of WNPs: that is, they allow convergence results proved under the requirement
that \( f \) in (7) is bounded and continuous to be extended to a much broader class of integrable functions. High-level conditions similar to HL3 have been employed for similar purposes in many previous works, e.g. Akonom (1993), Jeganathan (2004, 2008), Pötscher (2004), Gao, King, Lu and Tjøstheim (2009), Wang and Phillips (2009a,b; 2012) among others.

(d) Together with HL2, HL4 is the principal condition that expresses the requirement that a WNP should not be too strongly dependent. It would fail both for \( I(d) \) processes with \( d > 1/2 \), and for NI processes – and indeed for any process for which \( \beta_{n-1} X_{[nt]}(n) \) converges weakly to a process with continuous sample paths.

Under the preceding high-level conditions, we have the following result, the proof of which appears in Section 5.

**Theorem 2.1** Suppose \( f : \mathbb{R} \to \mathbb{R} \) is locally Lebesgue integrable, and HL0–HL4, HL6 hold. Further suppose \( f \) is bounded or that the following hold:

(i) There is a \( Y \subseteq \mathbb{R} \) such that \( \int_{\mathbb{R}} |f(x+y)| \Phi_X(x) dx < \infty \) for all \( y \in Y \), and \( P(X^- \in Y) = 1 \);

(ii) \( n^{-1} \sum_{t=1}^{n_0-1} f(\beta_{n-1} X_t(n)) = o_p(1) \), for \( t_0 \) as in HL3;

(iii) For \( \lambda' \in (0, \lambda) \), where \( \lambda \) is as in HL5, either:

(a) \( |f(x)| = O(|x|^\lambda') \), as \( |x| \to \infty \) and HL5(a) holds; or

(b) \( |f(x)| = O(\exp(\lambda' |x|)) \), as \( |x| \to \infty \) and HL5(b) holds.

Then as \( n \to \infty \),

\[
\frac{1}{n} \sum_{t=1}^{n} f(\beta_{n-1} X_t(n)) \overset{d}{\to} \int_{\mathbb{R}} f(x+X^-) \Phi_X(x) dx. \tag{14}
\]

Further, if \( \tilde{f} : \mathbb{R} \to \overline{\mathbb{R}} \) is such that \( \tilde{f}(x) = f(x) \) almost everywhere, then (14) holds with \( f \) replaced by \( \tilde{f} \).

**Remark 2.2.** (a) If \( f \) is bounded, conditions (i)–(iii) hold trivially, and HL5 is unnecessary for (14). If \( f \) is additionally Lipschitz, then HL3 and HL6 may also be dispensed with.

(b) Condition (ii) of Theorem 2.1 is a technical requirement that has also been employed in other studies that develop limit theory for functionals of nonstationary processes, e.g. Jeganathan (2004) and Pötscher (2004).
2.3 Sufficient conditions for linear processes

We next provide conditions under which linear process arrays of the form (9) satisfy the preceding high-level conditions. Although we do not require that these processes be generated according to the model (8), we shall make one simplifying assumption consistent with that model: that the coefficients $a_{k,t}$ should not depend on $t$ for $1 \leq k \leq t - 1$. Our conditions thus envisage an array of the form

$$x_t(n) = \sum_{k=0}^{t-1} a_k(n) \xi_{t-k} + \sum_{k=t}^{\infty} a_{k,t}(n) \xi_{t-k} =: x_t^+(n) + x_t^-(n).$$

Our first assumption concerns the innovation sequence $\{\xi_t\}$.

Assumption INN (innovations).

(i) $\xi_t$ is i.i.d. with $E\xi_1 = 0$ and $Var(\xi_1) = \sigma_\xi^2 < \infty$.

(ii) $\xi_1$ has an absolutely continuous distribution, and a characteristic function $\psi_\xi(\lambda)$ that satisfies $\int_\mathbb{R} |\psi_\xi(\lambda)|^\theta d\lambda < \infty$, for some $\theta \in \mathbb{N}$.

To state our conditions on the coefficients on the linear process in (15), we first define

$$\beta_{n,t}^2 := Var(x_t(n)) = \sigma_\xi^2 \sum_{k=0}^{t} a_k(n)^2 + \sigma_\xi^2 \sum_{k=0}^{t} a_{k,t}(n)^2 =: (\beta_{n,t}^+)^2 + (\beta_{n,t}^-)^2$$

and set $\beta_n := \beta_{n,n}$, for $n \in \mathbb{N}$ and $t \in \{1, \ldots, n\}$. The following conditions will always be applied in conjunction with Assumption INN, and are stated in terms of the $\theta$ and the i.i.d. sequence $\{\xi_t\}$ appearing in that assumption.

Assumption LP (linear process).

LP1 $x_t(n)$, $a_k(n)$, and $a_{-k,t}(n)$ are as in (15).

LP2 (a) $E|\xi_1|^\lambda < \infty$ for some $\lambda \in [2, \infty)$ or (b) $\xi_1$ has a finite moment generating function (m.g.f.) in a neighbourhood of zero.

LP3 $\beta_{n,t} < \infty$ for all $n, t \geq 1$. Further, there are sequences $q_t, \overline{q}_t \in \mathbb{N}$ where $\overline{q}_t - q_t \to \infty$, $\overline{q}_t \leq t - 1$ as $t \to \infty$, and a $\delta \in (0, 1]$ such that for some $n_0 \geq t_0 \in \mathbb{N}$ and all $\rho > 0$:
(a) $\inf_{n \geq n_0, t_0 \leq t \leq n} \beta_t^{-2} \sum_{k=0}^{\bar{q}_t} \theta \beta_t a_k(n)^2 =: C_1 > 0$;
(b) $\sup_{n \geq n_0, t_0 \leq t \leq n} \beta_t^{-\delta} \sup_{\bar{q}_t \leq k \leq \bar{q}_n} |a_k(n)| \vee 1 =: C_2 < \infty$;
(c) $\sup_{n \geq n_0, t_0 \leq t \leq n} \exp \left\{ -\rho \beta_t^{2(1-\delta)} \right\} \beta_t \sup_{\bar{q}_t \leq k \leq \bar{q}_n+\theta} |a_k(n)|^{-1} < \infty$.

**LP4** For some $q \geq \theta$, each $n \in \mathbb{N}$ and $1 \leq t \leq n$ there are $\{k_{n,t,l}\}_{l=1}^q \in \mathbb{Z}_+$ such that for all $t_0 \in \mathbb{N}$

$$\liminf_{n \to \infty} \inf_{1 \leq l \leq q, 1 \leq t \leq t_0} a_{k_{n,t,l}}(n) > 0.$$ 

**LP5** $\lim_{n \to \infty} n^{-1/2} \beta_n^{-1} \sum_{k=0}^{\bar{q}_n} |a_k(n)| = 0$.

**LP6** For any $\delta \in (0, 1)$ and $\{t_n\}$ such that $[n\delta] \leq t_n \leq n$:

(a) $\beta_n^{-1} [\beta_{t_n,n}, \beta_{t_n,n}] \to [\sigma_+, \sigma_-]$ with $\sigma_+ > 0$ and $\sigma_- \geq 0$;
(b) $\beta_n^{-1} \left( \sup_{0 \leq k \leq n} |a_k(n)| + \sup_{1 \leq t \leq n, k \geq t} |a_k^-(n)| \right) \to 0$;
(c) $\beta_n^{-2} \sum_{l=-\infty}^{0} \left[ a_{t_n-n,l}^+(n) - a_{t_n-l,n}^-(n) \right]^2 \to 0$.

**LP7** $\sup_{n \geq n_0, 1 \leq t \leq n} \beta_n^{-2} \beta_{n,t} < \infty$ for some $n_0 \geq 1$.

**Remark 2.3.** The principal relationships between the preceding conditions, and the high-level conditions (Assumption HL) as they would be applied to $X_t(n) = x_t(n)$ may be summarised as follows; these are also stated more formally as Proposition 2.1 below.

(a) LP2 and LP7 imply that $\beta_n^{-1} x_t(n)$ has finite moments of a sufficient order (as per HL5). For $\beta_n^{-1} x_t(n)$ to have finite $\lambda$-moments, it is sufficient that $\xi_t$ also have finite $\lambda$-moments; $\beta_n^{-1} x_t(n)$ will have finite exponential moments if $\xi_t$ has an m.g.f. that is finite in a neighbourhood of zero.

(b) LP3 and LP4 ensure that $\beta_t^{-1} x_t(n)$ has a uniformly bounded density, as required by HL3. Under Assumption INN, a weighted sum involving at least $\theta$ of the innovations $\xi_t$ will have an integrable characteristic function. $\beta_t^{-1} x_t^2(n) = \sum_{k=0}^{t-1} \beta_t^{-1} a_k(n) \xi_{t-k}$ will thus have a density bounded uniformly over $n$ and $t$, provided that the $L_1$ norm of its characteristic function can be uniformly bounded. This in turn requires that: (i) the variance of $\beta_t^{-1} x_t^2(n)$ can be bounded away from zero; and (ii) it is never dominated by less than $\theta$ of the innovations that contribute to it. Both are ensured by LP3, at least for $n$ and $t$ sufficiently large. LP4 entails that for every $t$, a sufficient number
of coefficients \( \{a_{k,t}(n)\} \) are bounded away from zero; together with \( \text{LP3} \) it is sufficient for \( \text{HL3} \) to hold with \( t_0 = 1 \).

(c) \( \text{LP5} \) can be understood as a kind of weak dependence condition, which is used solely to verify \( \text{HL4} \).

(d) \( \text{LP6} \) permits a central limit theorem for weighted sums to be applied to each of \( \beta_n^{-1} x_t^+(n) \) and \( \beta_n^{-1} x_t^-(n) \), as required by \( \text{HL2} \). \( \text{LP6(a)} \) determines the limiting variance of each of these two terms, while \( \text{LP6(b)} \) is a negligibility requirement on the linear process coefficients, akin to a Lindeberg condition. Finally, \( \text{LP6(c)} \) implies the second part of \( \text{HL2(b)} \).

**Proposition 2.1 (LP ⇒ HL)** Suppose Assumptions \( \text{LP1} \) and \( \text{INN} \) hold. Then for \( \mathcal{F}_t := \sigma(\{\xi_r\}_{r \leq t}) \), \( X_t^+ := x_t^+(n) \), and \( X_t^- := x_t^-(n) \):

(i) \( \text{LP6} \Rightarrow \text{HL2} \) with \( [X^+, X^-] \sim N[0, \text{diag}(\sigma_+^2, \sigma_-^2)] \).

(ii) (a) \( \text{LP3} \Rightarrow \text{HL3} \);

(b) \( \text{LP3} \) and \( \text{LP4} \) ⇒ \( \text{HL3} \) with \( t_0 = 1 \).

(iii) \( \text{LP5} \Rightarrow \text{HL4} \).

(iv) (a) \( \text{LP2(a)} \) and \( \text{LP7} \Rightarrow \text{HL5(a)} \) with \( \lambda \geq 2 \);

(b) \( \text{LP2(b)}, \text{LP6(b)} \) and \( \text{LP7} \Rightarrow \text{HL5(b)} \).

### 2.4 Verification for leading examples

We turn finally to the application of our ‘intermediate-level’ results to \( I(1/2) \) and MI processes. That is, we now specialise to the case of a linear process array of the form

\[
x_t(n) = \sum_{j=0}^{t-1} \phi_j(n) v_{t-j}, \quad \text{where} \quad v_t = \sum_{i=0}^{\infty} c_i \xi_{t-i}, \quad (16)
\]

and \( \phi_j(n) \) and \( c_i \) are as specified by the ‘low level conditions’ given immediately below. These conditions further elaborate the definitions of \( I(1/2) \) and MI processes that were introduced in Section 2.1 and impose certain technical requirements that facilitate the verification of Assumption LP.
**Assumption LL (low-level conditions).** \(x_t(n)\) is generated by (16). The function \(\ell\) is SV such that \(L(n) := \int_1^n elder(2x/x)dx < \infty\) for all \(n \in \mathbb{N}\), \(L(n) \to \infty\) as \(n \to \infty\), and either:

\[\begin{align*}
\text{LL1} & \quad \phi_j \sim \ell(j) j^{-1/2}, \; \phi_0 \neq 0, \; \sum_{s=0}^{\infty} |c_s| < \infty, \; \sum_{s=0}^{\infty} c_s \neq 0, \; \text{and either} \\
(a) & \lim sup_{j \to \infty} j |c_j| < \infty, \; \text{or} \\
(b) & \lim_{n \to \infty} \ell(n)^{-1} \sum_{j=|n\delta|} j^{1/2} |c_j| = 0 \text{ for some } 0 < \delta < 1; \; \text{or}
\end{align*}\]

Or: for \(\rho_n = 1 - \kappa_n^{-1}\) - where \(\kappa_n > 0, \kappa_n = n^\alpha \ell_s(n)\) for \(\ell_s\) SV and \(\alpha \in [0, 1)\), \(\kappa_n \to \infty\) and \(\sup_{n \geq 1, 1 \leq t \leq n} \sup_{t} \kappa_t / \kappa_n < \infty\) - either:

\[\begin{align*}
\text{LL2} & \quad \phi_j = 1 \forall j \geq 0, \; c_s \sim \ell(s) s^{-3/2}, \; \text{and} \; \sum_{s=0}^{\infty} c_s = 0.
\end{align*}\]

Remark 2.4 (a) LL1 and LL2 respectively imply that \(x_t(n)\) is an I(1/2) process of types II and I, according to the definitions of these processes given in Section 2.1 above. LL1 also specifies some additional regularity conditions ((a) and (b)) that are used principally to ensure \(x_t(n)\) satisfies condition LP3 (implying that \(\beta_t^{-1} x_t(n)\) has a uniformly bounded density). For verifying the other requirements of Assumption LP, these conditions can be dispensed with. (Similar conditions have appeared elsewhere in the literature, e.g. Jeganathan, 2008 and Hualde and Robinson, 2011.)

These regularity conditions are sufficiently weak to ensure that parametric ARFIMA models with \(d = 1/2\) are covered by LL1 and LL2, which may be demonstrated as follows.

**Parametric type II:** Consider the ARFIMA(1/2) type II model

\[ (1 - L)^{1/2} x_t = v_t 1 \{ t > 1 \}, \quad \text{where} \quad A(L) v_t = B(L) \xi_t, \quad (17) \]

where \(A\) and \(B\) denote finite-order polynomials in the lag operator \(L\). In this case, it is possible to write \(x_t = \sum_{j=0}^{m-1} \phi_j v_t - j\), where \(\{ \phi_j \}_{j \geq 0}\) are the coefficients in the power series expansion of \((1 - L)^{-1/2}\); and so \(\phi_0 = 1\) and \(\phi_j \sim C \cdot j^{-1/2}\) for some \(C \in (0, \infty)\) (see e.g. p. 673 in Johansen and Nielsen, 2012). Further, if all the roots of \(A\) lie outside the unit circle, \(v_t = A(L)^{-1} B(L) \xi_t\) is a linear process with geometrically decaying coefficients (e.g. Theorem 3.1.1 in Brockwell and Davis, 1991). Thus LL1(a) and LL1(b) are easily satisfied. Since \(\ell(x) = 1\), we have trivially that \(L(n) \to \infty\).
Parametric type I: Under LL2,

\[ x_t = x_{t-1} + v_t, \quad x_0 = 0 \]  

(18)

i.e. a random walk driven by a fractional error process \( v_t \), integrated of order \( \delta = -1/2 \). Under LL2, we may specify that \( v_t = \sum_{j=0}^{\infty} \pi_j \varepsilon_{t-j}, \) where \( \varepsilon_t = \sum_{j=0}^{\infty} \theta_j \xi_{t-j}, \) \( |\theta_j| = O(j^{-q}) \) for \( q \geq 3/2 \), and \( \sum_{j=0}^{\infty} \theta_j \neq 0 \). This encompasses the case where \( \varepsilon_t \) is a stationary ARMA process. Further, suppose that \( \pi_j \) are the coefficients in the series expansion of \((1-x)^{1/2}\) (i.e. \((I-L)^{-1/2}v_t = \varepsilon_t\)). Then \( \pi_j \sim C \cdot j^{-3/2} \) and \( \sum_{j=0}^{\infty} \pi_j = 0 \). In this case we can write \( v_t = \sum_{i=0}^{\infty} c_i \xi_{t-i} \) with \( c_i = \sum_{j=0}^{\infty} \theta_j \pi_{i-j} \). To see that \( \{c_i\} \) satisfies LL2, note that: (a) \( c_i \sim \pi_i \sum_{j=0}^{\infty} \theta_j \) (see Lemma 7.1(ii) in Appendix B); and (b) \( \sum_{i=0}^{\infty} c_i = 0 \) by Mertens’ Theorem for Cauchy products (see Thm 8.46 in Apostol, 1981).

(b) Since only the values taken by \( \ell(x) \) for sufficiently large \( x \in \mathbb{N} \) are relevant to LL1 and LL2, we may require \( L(n) := \int_1^n |\ell^2(x)/x|dx < \infty \) for all \( n \in \mathbb{N} \) without loss of generality.\footnote{This follows the fact that a SV function is locally bounded on \([x_0, \infty)\) for any \( x_0 \) sufficiently large: see Lemma 1.3.2 in Bingham et al. (1987).}

It may be shown that \( Var(x_n) \sim C \cdot L(n) \) under LL1 and LL2, where \( L(n) \) is itself an SV function. The divergence or convergence of \( L(n) \) effectively demarcates the boundary between the WNPs considered in this paper, and stationary linear processes. Either is possible: e.g. \( \ell(n) = 1 \) gives \( L(n) \sim \ln n \) and \( \ell(n) = (\ln n)^{-1/2} \) gives \( L(n) \sim \ln \ln n \), whereas \( \ell(n) = (\ln n)^{-1} \) gives a bounded \( L(n) \). When \( L(n) \) diverges, \( L(n)^{-1/2}x_n \) will obey a CLT (under Assumption INN) – and \( x_n \) will thus satisfy Assumption HL2 above. But when \( L(n) \) is bounded, no CLT applies: and indeed in this case we have under LL1 that \( \sum_{j=0}^{\infty} \theta_j^2 < \infty \), so that \( x_t \) is stationary.

(c) The requirements (in LL3 and LL4) that \( \kappa_n \) be regularly varying and \( \sup_{n \geq 1, 1 \leq t \leq n} \kappa_t/\kappa_n < \infty \) are helpful in unifying notation and simplifying some derivations – most particularly in the context of verifying Assumption HL3. They can be dispensed with at the expense of a considerably more involved exposition.

Proposition 2.2 next shows that under Assumption LL and Assumption INN, the WNPs specified by (16) satisfy Assumption LP.
Proposition 2.2 (LL ⇒ LP) Suppose that Assumption INN and Assumption LL hold. Then LP3, LP5, LP6 and LP7 hold. If \( \theta = 1 \) then LP4 also holds.

Our main low-level result is essentially a direct consequence of Theorem 2.1 and Propositions 2.1-2.2. Recall that \( \varphi_{\sigma^2}(x) \) denotes the \( N[0,\sigma^2] \) density, and \( \beta_n^2 = \text{Var}(x_n(n)) \).

Theorem 2.2 Suppose \( f : \mathbb{R} \to \mathbb{R} \) is locally Lebesgue integrable, and Assumptions INN and LL hold. Further suppose:

(i) Either: (a) for each \( t' \in \mathbb{N} \) fixed, \( n^{-1} \sum_{t=1}^{t'} f (\beta_n^{-1} x_t(n)) = o_p(1) \); or
(b) Assumption INN holds with \( \theta = 1 \).

(ii) For some \( \lambda' \in (0, \infty) \), as \( |x| \to \infty \) either
(a) \( |f(x)| = O(|x|^\lambda') \), and \( E|\xi_1|^{2\lambda} < \infty \) for some \( \lambda > \lambda' \); or
(b) \( |f(x)| = O(\exp (\lambda' |x|)) \), and \( \xi_1 \) has a finite m.g.f. in a neighborhood of zero.

Then
\[
\frac{1}{n} \sum_{t=1}^{n} f (\beta_n^{-1} x_t(n)) \overset{d}{\to} \begin{cases} 
\int_{\mathbb{R}} f(x) \varphi_1(x) dx & \text{under LL(1,3,4)} \\
\int_{\mathbb{R}} f(x + X^-) \varphi_{1/2}(x) dx & \text{under LL2} 
\end{cases} \tag{19}
\]
where \( X^- \sim N[0,1/2] \). Further, suppose that \( \tilde{f} : \mathbb{R} \to \mathbb{R} \) is such that \( \tilde{f}(x) = f(x) \) almost everywhere. Then (19) holds with \( f \) replaced by \( \tilde{f} \).

Remark 2.5 (a) Theorem 2.2 bridges existing asymptotic results for \( I(d) \) processes of order \( |d| < 1/2 \) and \( d \in (1/2,3/2) \). There are similarities and differences between (19) and the limit theory that applies in these two cases. Firstly, there is some analogy with the LLN results that hold when \( |d| < 1/2 \). As in that case, the limit term in (19) is determined by an expectation, but with the difference that the expectation in (19) is with respect to a limiting distribution, rather than the invariant distribution of a strictly stationary process. Secondly, the limiting density (\( \varphi_1 \) or \( \varphi_{1/2} \)) is obtained via application of a CLT, and in this respect the limit theory is analogous to that for \( d \in (1/2,3/2) \), which involves the application of FCLTs. In that case, the weak
limits of additive functionals are stochastic, being a functional of an fBM as in (2) above. This nondegeneracy of the limit carries over to the I(1/2) type I process (LL2), as evinced by the presence of \( X^{-} \sim N[0, 1/2] \) on the r.h.s. of (19). (The type II process can be regarded as a truncated type I process, and the additional variability of the latter appears to make its behaviour closer to that of an I(d) processes with \( d > 1/2 \)).

(b) Theorem 2.2 generalises the limit theory of Giraitis and Phillips (2006) and Phillips and Magdalinos (2007) for MI processes to general nonlinear functionals. Those two papers consider quadratic functions (i.e. \( f(x) = x^2 \)) of MI processes driven by short memory linear processes errors. Using a direct approach they show, in particular, that \( \frac{1}{b_n} \sum_{t=1}^{n} x_t \to^d 1 \). Theorem 2.2 provides a more general characterisation of the aforementioned limit term; indeed the result in (19) specialises to

\[
\int_{\mathbb{R}} f(x) \phi_1 dx = \int_{\mathbb{R}} x^2 \phi_1 dx = 1
\]

in this case.

(c) (19) is formulated in terms of the standardised process \( \beta_n^{-1} x_t(n) \). The growth rate of \( \beta_n \) is of interest insofar as it affects the convergence rate of parametric and nonparametric regression estimators, as will be discussed in the next section. It may be shown (see Lemma LVAR in Appendix A1) that \( \beta_n \sim \gamma_n^2 \nu_n^2 \), where

\[
[\gamma_n^2, \nu_n^2] := \begin{cases} 
[L(n), \sigma_s^2 (\sum_{s=0}^\infty c_s)^2] & \text{under LL1} \\
[L(n), 8\sigma_s^2] & \text{under LL2} \\
[\kappa_n, \sigma_s^2 (\sum_{s=0}^\infty c_s)^2 / 2] & \text{under LL3} \\
[\kappa_n^{3-2m}, \Gamma (2 - 2m) B (1 - m, 2m - 1)] & \text{under LL4}
\end{cases}
\]

(\( \Gamma, B \) are Gamma and Beta functions respectively). (19) may accordingly be restated as

\[
\frac{1}{n} \sum_{t=1}^{n} f(\gamma_n^{-1} x_t(n)) \to^d \begin{cases} 
\int_{\mathbb{R}} f(x) \phi_{\nu_n^2} (x) dx & \text{under LL(1,3,4)} \\
\int_{\mathbb{R}} f(x + \nu_n X^{-}) \phi_{\nu_n^2/2} (x) dx & \text{under LL2}
\end{cases}
\]

(see also Lemma CLT in Appendix A1).

(d) The existence of moments of \( \{\xi_j\} \) under Assumption INN, ensures that with appropriate scaling \( x_{[nr]}(n) \) satisfies a CLT. As a result the limiting distributions that appear in (19) are normal. Some preliminary work of the
authors’ shows that Theorem 2.2 can be extended to the case where \( \{\xi_j\} \) is in the domain of attraction of an \( \alpha \)-stable law with parameter \( 0 < \alpha \leq 2 \) (e.g. Astrauskas, 1983; Kasahara and Maejima, 1988; Jeganathan, 2004), in which case other stable distributions (rather than the normal distribution) will appear in the limit. We leave extensions of this kind for future work.

(e) Under the requirement that \( \lim_{n \to \infty} n^{-1} f (\beta_n^{-1} z_n) = 0 \) for any finite convergent sequence \( \{z_n\} \), condition (i.a) of Theorem 2.2 is satisfied. For example, suppose Assumption INN and LL3 or LL4 hold. Then

\[
x_t(n) = \sum_{j=0}^{t-1} \rho_j(n) v_{t-j} \xrightarrow{a.s.} \sum_{j=0}^{t-1} v_{t-j}
\]

where the r.h.s. is a.s. finite. Hence condition (i.a) holds under the stated requirement on \( f \).

3 Kernel functionals of WNPs

We now turn to the limiting behaviour of kernel functionals of the form

\[
\frac{\beta_n}{h_n n} \sum_{t=1}^{n} K \left( \frac{x_t(n) - x}{h_n} \right),
\]

(21)

where \( x \in \mathbb{R}, h_n \) is a bandwidth sequence, and \( K \) is an integrable kernel function satisfying

**Assumption K (kernel).** \( K : \mathbb{R} \to \mathbb{R} \) is such that \( K \) and \( K^2 \) are Lebesgue integrable.

Whereas in (7) the nonlinear transformation \( f \) is applied to the standardised process \( \beta_n^{-1} x_t(n) \), in (21) \( K \) is applied to the unstandardised process \( x_t(n) \). This leads to a different limit theory, which is partly reflected in the different normalisations of the sums in (7) and (21). Nonetheless, the results of the preceding section turn out to be highly relevant for the asymptotics of (21). To explain why this is the case, we return to the setting of Assumption HL above, which we recall refers to a general triangular array \( \{X_t(n)\} \) that is not required to conform to a specific time series model. We shall augment this assumption by the following additional smoothness conditions on the density of the increments of \( X_t(n) \). To state these, let

\[
\Omega_n(\eta) := \left\{ \{s, t\} \in \mathbb{N}^2 : |\eta m| \leq s \leq |1 - \eta| n, |\eta m| + s \leq t \leq n \right\},
\]
for $\eta \in (0, 1)$.

**HL7** Let $X_0(n) = 0$ and $t > s \geq 0$. Conditionally on $F_s$, $\beta_{t-s}^{-1}(X_t(n) - X_s(n))$ has density $D_{t,s,n}(x)$ such that for some $n_0, t_0 \geq 1$

$$\sup_{n \geq n_0, 0 \leq s \leq t, t-s \geq t_0} \sup_x D_{t,s,n}(x) < \infty.$$

**HL8** For all $q_0, q_1 > 0$

$$\lim_{\eta \downarrow 0} \limsup_{n \to \infty} \sup_{(s,t) \in \Omega_0(\eta)} \sup_{|x| \leq q_0 q_1} |D_{t,s,n}(x) - D_{t,s,n}(0)| = 0$$

**HL9** For $t_0$ as in **HL7**:

(a) $\lim_{\eta \to 0} \lim_{n \to \infty} \frac{\beta_n}{n} \sum_{t=t_0}^{\lfloor n(1-\eta) \rfloor} \beta_t^{-1} = 0$;

(b) $\lim_{\eta \to 0} \lim_{n \to \infty} \frac{\beta_n}{n} \sum_{t=t_0}^{\lfloor n(1-\eta) \rfloor} \beta_t^{-1} = 0$;

(c) $\lim_{\eta \to 0} \lim_{n \to \infty} \frac{\beta_n}{n} \sup_{0 \leq s \leq (1-\eta) n} \sum_{t=s+t_0}^{s+\lfloor n(1-\eta) \rfloor} \beta_t^{-1} = 0$

(d) $\limsup_{n \to \infty} \frac{\beta_n}{n} \sup_{0 \leq s \leq n-1} \sum_{t=s+t_0}^{s+\lfloor n(1-\eta) \rfloor} \beta_t^{-1} < \infty$;

(e) For each $\eta \in (0, 1)$, there exist $l_0 > 0$ and $l_1 \in (0, 1)$ such that $\liminf_{n \to \infty} \beta_n^{-1} \inf_{(s,t) \in \Omega_0(\eta)} \beta_{t-s}^{-1} \geq \eta^{l_1}/l_0$.

Conditions **HL7** and **HL9** may be regarded as strengthened versions of **HL3** and **HL6**, and are closely related to Assumptions 2.3 in Wang and Phillips (2009a). Indeed, under these conditions, an $L_1$-approximation argument developed by those authors and Jeganathan (2004) yields that, for $t_0$ as in **HL7**,

$$\frac{\beta_n}{h_n} \sum_{t=t_0}^{n} K\left(\frac{X_t(n) - x}{h_n}\right) = \frac{1}{n} \sum_{t=t_0}^{n} \varphi_{e^2}(\beta_n^{-1} X_t(n)) \int_{\mathbb{R}} K(u)du + o_p(1)$$

as $n \to \infty$ and then $\varepsilon \to 0$ (see **Lemma 5.3** below); recall that $\varphi_{e^2}$ denotes the density of a normal random variable with mean zero and variance $\sigma^2$.

The leading order term on the r.h.s. clearly has the same form as the l.h.s. of (19) and is thus is amenable to a direct application of **Theorem 2.1**—which under **HL0–HL4** and **HL6** entails

$$\frac{1}{n} \sum_{t=1}^{n} \varphi_{e^2}(\beta_n^{-1} X_t(n)) \overset{d}{\to} \int \varphi_{e^2}(x + X^-) \Phi_{X^+}(x)dx, \quad \text{as } n \to \infty$$

$$\overset{p}{\to} \Phi_{X^+}(-X^-), \quad \text{as } \varepsilon \to 0.$$  (22)

by dominated convergence. We thus have the following high-level result, which is the counterpart of **Theorem 2.1** for kernel functionals.

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Theorem 3.1 Suppose that, in addition to Assumptions K, HL0–HL2, HL4 and HL6–HL9, the following hold:

(i) \( \{h_n\} \) is a positive sequence with \( \beta^{-1}n h_n + \beta_n(nh_n)^{-1} \to 0 \); and

(ii) for each \( x \in \mathbb{R} \) and \( t_0 \) as in HL7, \( \beta_n h_n \sum_{t=1}^{t_0-1} K \left( \frac{X_t(n)-x}{h_n} \right) = o_p(1) \).

Then
\[
\frac{\beta_n}{h_n} \sum_{t=1}^{n} K \left( \frac{X_t(n)-x}{h_n} \right) \xrightarrow{d} \Phi_X^* (-X^-) \int_{\mathbb{R}} K(u)du.
\]

To apply the preceding to the processes covered by Assumption LL above, it is simply a matter of verifying that these also satisfy HL7–HL9 – the other requirements of Assumption HL having already been verified by Propositions 2.1 and 2.2. This leads to the following counterpart of Theorem 2.2.

Theorem 3.2 Suppose that, in addition to Assumption K:

(i) \( x_t(n) \) is given by (16), and satisfies Assumptions INN and LL;

(ii) \( \{h_n\} \) satisfies condition (i) of Theorem 3.1 and

(iii) for each \( x \in \mathbb{R} \) and \( t' \in \mathbb{N} \), \( \beta_n h_n \sum_{t=1}^{t'} K \left( \frac{x_t(n)-x}{h_n} \right) = o_p(1) \).

Then
\[
\frac{\beta_n}{h_n} \sum_{t=1}^{n} K \left( \frac{x_t(n)-x}{h_n} \right) \xrightarrow{d} \int_{\mathbb{R}} K(u)du \cdot \begin{cases} \varphi_1(0) & \text{under } LL(1,3,4) \\ \varphi_{1/2}(X^-) & \text{under } LL2 \end{cases}
\]

where \( X^- \sim N[0,1/2] \).

Remark 3.1 (a) Theorem 3.2 fills the gap in existing asymptotic theory for kernel functionals of linear processes. A general theory for stationary linear processes, including \( I(d) \) processes with \( 0 \leq d < 1/2 \), is given in Wu and Mielniczuk (2002). Supposing that \( \int_{\mathbb{R}} K = 1 \) for simplicity, under their conditions kernel functionals converge to the invariant density of the stationary linear process. Jeganathan (2004, 2008) provides limit theorems
for kernel functionals of $I(d)$ processes with $1/2 < d < 3/2$. In that case, kernel functionals converge to the local time of a fractional Brownian motion – so their limit is an occupation density rather than the invariant density of some strictly stationary process. The limiting behaviour of kernel functionals of $I(1/2)$ processes is intermediate between these two cases. These converge to the density of a random variable, rather than to an occupation density, but the density corresponds to a limiting random variate, rather than the invariant density of a stationary process.

(b) Under Assumption INN, Assumption LL and some additional requirements on $x_t(n)$, similar to LP4 (see also Wang and Phillips 2009a), it is possible to show that HL7 and HL9 hold with $t_0 = 1$. In this case, condition (iii) of Theorem 3.2 is redundant. We do not pursue extensions of this kind here, in order to keep the paper to a manageable length.

(c) Theorem 3.2 nests a similar result provided by Duffy (2017) for bounded kernel functionals of MI processes driven by short memory errors (Assumption LL3), which unlike Assumption K requires $K$ to be bounded and Lipschitz continuous.

4 Application to regression estimators

The limit theory developed in the preceding sections is fundamental to the asymptotics of both parametric and nonparametric least squares estimators, in models involving WNPs as regressors. To illustrate some of the potential applications of our results, in this section we consider: (a) parametric estimation of $\beta \in \mathbb{R}$ in the model

$$y_t = \beta g(x_{t-1}(n)) + u_t$$

where $g$ is a known nonlinear transformation, using the least squares (LS) estimator

$$\hat{\beta} = \frac{\sum_{t=2}^{n} g(x_{t-1}(n))y_t}{\sum_{t=2}^{n} g^2(x_{t-1}(n))}$$

(Section 4.1); and (b) nonparametric estimation of the unknown function $g$ in the model

$$y_t = g(x_{t-1}(n)) + u_t$$

by the kernel regression (Nadaraya–Watson; NW) estimator

$$\hat{g}(x) = \frac{\sum_{t=2}^{n} K[(x_{t-1}(n) - x)/h_n]y_t}{\sum_{t=2}^{n} K[(x_{t-1}(n) - x)/h_n]}$$
In both (23) and (25) \( u_t \) is assumed to be m.d.s. with respect to the \( \{F_t\} \), the natural filtration for \( \{(u_t, \xi_t)\} \), and \( x_t(n) \) an \( I(1/2) \) or MI process (as per Assumption LL). \(^3\)

Since the regressor in (23) and (25) is predetermined (i.e. \( F_{t-1} \)-measurable) relative to the error \( u_t \), both are instances of so-called ‘predictive’ or ‘reduced form’ regression models. Correlation between \( x_t(n) \) and \( u_t \) is not precluded here: such correlation does not affect the consistency of either the LS or NW estimators – and if \( x_t(n) \) is stationary, both estimators will be asymptotically normal. If \( x_t(n) \) is nonstationary – being e.g. \( I(d) \) with \( 1/2 < d < 3/2 \) or near integrated – then the NW estimator is mixed normal, and standard methods of inference remain valid (see Wang and Phillips, 2009a, 2012). However, the results of Park and Phillips (1999, 2001) show that the parametric LS estimator of \( \beta \) has a non-standard limiting distribution, unless either \( g \) is itself integrable, or \( x_t(n) \) and \( u_t \) satisfy a very restrictive ‘long-run orthogonality’ condition. This necessitates either non-standard inferential procedures, or the use of suitably modified estimators that enjoy mixed normal limiting distributions, such as are developed by Phillips (1995) and Robinson and Hualde (2003).

When \( x_t(n) \) is a WNP, Theorems 4.1 and 4.2 below respectively establish that the large-sample behaviour of both the LS and NW estimators is similar to the case where \( x_t(n) \) is stationary. Both estimators are either asymptotically normal or mixed normal, ensuring the validity of conventional methods of inference (e.g. \( t \)-statistics will be asymptotically standard normal). These results in turn follow from the application of the results of the preceding sections and suitable martingale central limit theorems to quantities of the form

\[
\left( \sum_{t=2}^{n} G_{n,t-1}^j \right)^{-1} \sum_{t=2}^{n} G_{n,t-1} u_t
\]

upon a suitable rescaling, where: (i) for the LS estimator, \( j = 2, G_{n,t} = H_g[\beta_n^{-1} x_t(n)] \) for \( H_g \) as in Definition AHF below and \( \beta_n = Var(x_n(n)) \); (ii) for the NW estimator, \( j = 1 \) and \( G_{n,t} = K[(x_t(n) - x)/h_n] \). Under the assumption that \( E[u_t^2 \mid F_{t-1}] = \sigma_u^2 \) a.s., the limiting behaviour of both the

\(^3\)The model in (23) is linear in the parameters. It would also be possible to extend our results to cover the estimation of models in which \( g \) is non-linearly parametrised, along the lines of Park and Phillips (2001) and Chan and Wang (2015).
denominator in (26) and the conditional variance of the martingale in the numerator,
\[ \sum_{t=2}^{n} E\{G_{n,t-1}u_t \mid \mathcal{F}_{t-1}\} = \sigma_u^2 \sum_{t=2}^{n} G_{n,t-1}^2, \tag{27} \]
can be deduced directly from Theorems 2.2 and 3.2. In cases where, upon rescaling, (27) converges to a deterministic limit (as under LL1, LL3 and LL4), Theorem 3.2 in Hall and Heyde (1980) establishes the asymptotic normality of (26); whereas when (27) converges in distribution to a random limit (as under LL2), it is necessary to invoke an extension of that result due to Wang (2014).

### 4.1 Parametric regression

The connection between the LS estimator in (24), which involves nonlinear (and generally non-integrable) transformations of the unstandardised regressor \( x_t(n) \), and the limit theory developed in Section 4.1 above, where these regressors appear standardised by \( \beta_n = \text{Var}(x_n(n)) \), may be established via an argument due to Park and Phillips (1999, 2001). For that purpose, we need the following

**Definition AHF (asymptotically homogeneous function).** Let \( \{x_t(n)\} \) denote a random array and \( \beta_n^2 = \text{Var}(x_n(n)) \). A function \( g : \mathbb{R} \to \mathbb{R} \) is *asymptotically homogeneous* for \( x_t(n) \), if for each \( \lambda > 0 \), \( g \) admits the decomposition
\[ g(x) = \kappa_g(\lambda)H_g(x/\lambda) + R_g(x, \lambda), \]
where \( \kappa_g : \mathbb{R}_+^* \to \mathbb{R}_+^* \), \( H_g : \mathbb{R} \to \mathbb{R} \), \( R_g : \mathbb{R} \times \mathbb{R}_+^* \to \mathbb{R} \)
and
\[ R_{g,n}^2 := \frac{1}{\kappa_g(\beta_n)n} \sum_{t=1}^{n} E|R_g(x_t(n), \beta_n)|^2 = o(1). \tag{28} \]

AHFs encompass a wide range of commonly used regression functions, such as polynomial functions, cumulative distribution functions (with \( H_g(u) = 1\{u > 0\} \)), and logarithmic functions (with \( H_g(u) = 1 \)); see Park and Phillips (1999, 2001) for some further examples. Such a condition as
\[ \lim_{\lambda \to \infty} \kappa_g(\lambda)^{-1} \sup_x |R_g(x, \lambda)| = 0 \]
is sufficient, but not necessary, for (28) to hold.

The relevance of AHFs for LS estimators can be seen from the fact that if \( g \) is an AHF with limit homogeneous component \( H_g \), such that \( H^2_g \) and \( \{x_t(n)\} \) satisfies both requirements of Theorem 2.2, then

\[
\frac{1}{\kappa^2(\beta_n)n} \sum_{t=1}^{n} g^2(x_t(n)) = \frac{1}{n} \sum_{t=1}^{n} H^2_g(\beta_n^{-1} x_t(n)) + o_p(1)
\]

\[
d \rightarrow \begin{cases} 
\int_{\mathbb{R}} H^2_g(x) \varphi_1(x) dx & \text{under } LL_1(1,3,4) \\
\int_{\mathbb{R}} H^2_g(x+X^-) \varphi_{1/2}(x) dx & \text{under } LL_2
\end{cases}
\]

where \( X^- \sim N[0, 1/2] \), by (28) and the Cauchy-Schwarz inequality, and then Theorem 2.2. Reasoning along these lines thus yields our main result on parametric LS estimators. To state it, let \( MN[0, \zeta^2] \) denote a mixed normal distribution with mixing variate \( \zeta^2 \) (i.e. which has characteristic function \( \mu \mapsto \mathbb{E} e^{-\zeta^2 \mu^2/2} \)).

**Theorem 4.1** Let \( \{y_t\}_{t=1}^{n} \) be generated by (23), \( F_t := \sigma(\{\xi_s, u_s\}_{s \leq t}) \), and suppose that:

(i) \( x_t(n) \) satisfies (16) and Assumptions INN and LL;

(ii) \( \{u_t, F_t\}_{t \geq 1} \) is a martingale difference sequence such that \( \mathbb{E}[u_t^2 | F_{t-1}] = \sigma_u^2 \) a.s. for some constant \( \sigma_u^2 \in \mathbb{R}_+^* \);

(iii) \( \sup_{1 \leq t \leq n} \mathbb{E}[u_t^2 1\{|u_t| \geq A_n\} | F_{t-1}] = o_p(1) \), for all deterministic \( 0 < A_n \to \infty \); and

(iv) \( g(x) \) is AHF for the array \( \{x_t(n)\} \), with limit homogeneous component \( H_g \) such that \( H^2_g \) satisfies the requirements of Theorem 2.2.

Then \( \kappa_g(\beta_n) \sqrt{n}(\hat{\beta}_LS - \beta) \xrightarrow{d} MN[0, \zeta^2] \), where

\[
\zeta^2 = \sigma_u^2 \cdot \begin{cases} 
\left[\int_{\mathbb{R}} H^2_g(x) \varphi_1(x) dx\right]^{-1} & \text{under } LL_1(1,3,4), \\
\left[\int_{\mathbb{R}} H^2_g(x+X^-) \varphi_{1/2}(x) dx\right]^{-1} & \text{under } LL_2
\end{cases}
\]

where \( X^- \sim N[0, 1/2] \).
Remark 4.1 (a) For I(1/2) type II and MI processes $\hat{\beta}_n$ is asymptotically normal; for I(1/2) type I processes it is merely mixed normal, because of the limiting variate $X^\prime$. In either case, the associated $t$ statistic will be asymptotically standard normal.

(b) In a linear regression model, i.e. $g(x) = x$, we may take $H_g(u) = u$ and $\kappa_g(\lambda) = \lambda$, and it follows that the LS estimator has convergence rate $\beta_n\sqrt{n}$, which is faster than the $\sqrt{n}$-convergence rate that obtains when the regressor is stationary. For I(1/2) processes the gain in convergence rate is given by the slowly varying factor $L(n)$ (see Remark 2.4(b)).

(c) If $g(x) = F(x)$ for some cumulative distribution function $F$, then $H_g = 1\{x > 0\}$ as noted, and so

$$\int_{\mathbb{R}} H_g^2(x) \varphi_1(x) dx = \int_{\mathbb{R}} 1\{x > 0\} \varphi_1(x) dx = 1/2.$$ 

If $g(x) = \ln|x|$, then $H_g = 1$ and so in all cases the r.h.s. of (30) is $N[0, \sigma_u^2]$.

4.2 Kernel regression

We conclude with the asymptotics of the NW estimator. As noted above, fundamental limit theorems for kernel functionals of nonstationary fractional processes with $d > 1/2$ can be found in Jeganathan (2004, 2008; see also Borodin and Ibragimov, 1995 and the references therein for some earlier results). The following theorem is a direct consequence of Theorem 3.2 and certain martingale central limit theorems, and is complementary to the recent work of Wang and Phillips (2009a,b; 2012) who develop estimation and testing procedures in the context of nonparametric regression with fractional $d > 1/2$ processes.

Theorem 4.2 Let $\{y_t\}_{t=1}^n$ be generated by (25) and suppose that:

(i) conditions (i)–(iii) of Theorem 4.1 hold;

(ii) $K$ and $K^2$ satisfy Assumption $K$, with $\int_{\mathbb{R}} K(u) du = 1$, and $h_n + \beta_n/nh_n \to 0$;

(iii) for each $t' \in \mathbb{N}$ and $x \in \mathbb{R}$, $\frac{\beta}{nh_n} \sum_{t=1}^{t'} K([x_t(n) - x]/h_n) = o_p(1)$; and

(iv) there is a real function $\overline{g}(x, z)$ such that $|g(x + hz) - g(x)| \leq h^\mu \overline{g}(x, z)$ for some $\mu \in (0, 1]$ for all $h$ sufficiently small, and $\int_{\mathbb{R}} \overline{g}(x, z) |K(z)| dz < \infty$ for every $x \in \mathbb{R}$. 

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Then \( \hat{g}(x) \xrightarrow{p} g(x) \). If in addition \( nh_n^{1+2\mu}/\beta_n \to 0 \), then
\[
\sqrt{nh_n/\beta_n} (\hat{g}(x) - g(x)) \xrightarrow{d} MN \left[ 0, \frac{\sigma^2_u}{\varsigma^2 \int_{\mathbb{R}} K^2(u)du} \right],
\]
where \( \varsigma^2 = \varphi_1(0) \) under \( LL1, LL3 \) and \( LL4 \), and \( \varsigma^2 = \varphi_{1/2}(X^-) \) for \( X^- \sim N[0,1/2] \) under \( LL2 \).

**Remark 4.2 (a)** Condition (iv) of Theorem [4.2] is a smoothness requirement on the regression function \( g \) that is also utilised by Wang and Phillips (2009a,b).

**Remark 4.2 (b)** Notice that because \( \beta_n \to \infty \), the convergence rate is slower than that for stationary processes. In particular, for \( I(1/2) \) processes the reduction in the convergence speed is given by a slowly varying factor.

## 5 Proofs of main results

This Section provides proofs of our main theorems under high level conditions (i.e. Theorems 2.1 and 3.1). (The proofs of the remaining results are given in Appendices A, B and C.) Throughout \( C \in (0, \infty) \) denotes a generic constant which may take different values at each appearance. We first state three technical lemmas.

**Lemma 5.1** Suppose \( f : \mathbb{R} \to \mathbb{R} \) is locally integrable, and let \( \varepsilon, \eta > 0 \). Then there is a Lipschitz continuous \( f_{\varepsilon,\eta} \) such that \( \int_{|x| \leq \eta} |f(x) - f_{\varepsilon,\eta}(x)| \, dx < \varepsilon \) and \( f_{\varepsilon,\eta}(x) = 0 \) for \( |x| > \eta \).

**Lemma 5.2** Let \( X_n \) and \( Y_n \) be real valued random sequences on some probability space \( (\Omega, \mathcal{A}, P) \) and \( \mathcal{F} \subset \mathcal{A} \) a \( \sigma \)-field, for which

(i) \( X_n \xrightarrow{d} X \sim F_X \), conditionally on \( \mathcal{F} \), in the sense that \( \mathbb{E} (h(X_n) \mid \mathcal{F}) \xrightarrow{P} \int_{\mathbb{R}} h(x)dF_X(x) \) for all \( h : \mathbb{R} \to \mathbb{R} \) bounded and continuous; and

(ii) \( Y_n \xrightarrow{d} Y \), where \( Y_n \) is \( \mathcal{F} \)-measurable for each \( n \).

Then for all \( g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) bounded and Lipschitz continuous,
\[
\mathbb{E} (g(X_n,Y_n) \mid \mathcal{F}) \xrightarrow{d} \int_{\mathbb{R}} g(x,Y)dF_X(x).
\]
Lemma 5.3  Suppose that:

(i) HL0, and HL7-HL9 hold;
(ii) K satisfies Assumption K; and
(iii) \{h_n\} is such that \(\beta_n^{-1} h_n + \beta_n (nh_n)^{-1} \to 0\).

Then for all \(x \in \mathbb{R}\),

\[
\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} E \left| \frac{1}{h_n n} \sum_{t=0}^{n} K \left( \frac{X_t(n) - x}{h_n} \right) - \frac{1}{n} \sum_{t=0}^{n} \varphi_{x^2} \left( \beta_n^{-1} X_t(n) \right) \right| = 0. \tag{31}
\]

The proof of Lemma 5.1 follows from Theorem 2.26 in Folland (1999), while the proof Lemma 5.2 is given in Appendix A3. Finally, Lemma 5.3 follows from Lemma 7 in Jeganathan (2004) and arguments similar to those used in Wang and Phillips (2009a, pp. 725-728).

Proof of Theorem 2.1. Let \(\varepsilon > 0\). By Lemma 5.1 for each \(\varepsilon > 0\) there is a Lipschitz continuous \(f_\varepsilon(x)\) such that \(\int_{|x| < \varepsilon^{-1}} |f(x) - f_\varepsilon(x)|\,dx < \varepsilon\) and \(f_\varepsilon(x) = 0\) for \(|x| > \varepsilon^{-1}\). We shall prove that:

\[
\frac{1}{n} \sum_{t=1}^{n} f_\varepsilon \left( \beta_n^{-1} X_t(n) \right) = \frac{1}{n} \sum_{t=1}^{n} f_\varepsilon \left( \beta_n^{-1} X_t(n) \right) + o_p(1), \tag{32}
\]

as \(n \to \infty\) then \(\varepsilon \to 0\); that for each \(\varepsilon > 0\)

\[
\frac{1}{n} \sum_{t=1}^{n} f_\varepsilon \left( \beta_n^{-1} X_t(n) \right) \overset{d}{\to} \int_{\mathbb{R}} f_\varepsilon(x + X^-) \Phi_X(x)\,dx, \tag{33}
\]

as \(n \to \infty\); and that

\[
\int_{\mathbb{R}} f_\varepsilon(x + X^-) \Phi_X(x)\,dx \overset{a.s.}{\to} \int_{\mathbb{R}} f(x + X^-) \Phi_X(x)\,dx. \tag{34}
\]

as \(\varepsilon \to 0\), where the integral on the r.h.s. exists a.s. due to condition (i) of Theorem 2.1. In view of (32)-(34), (14) then follows from Theorem 4.2 in Billingsley (1968). The final part of the theorem (for \(\tilde{f} : \mathbb{R} \to \mathbb{R}\) with \(\tilde{f} = f\) a.e.) in turn follows straightforwardly from each \(X_t(n)\) having a density (as per HL3).
Proof of (32). We prove (32) under condition (iii.a) of Theorem 2.1. The proof under (iii.b) is identical and therefore omitted. By condition (iii.a) we may choose \( \varepsilon > 0 \) sufficiently small that \( |f(x)| \leq C|x|^{\lambda'} \) for all \( |x| \geq \varepsilon^{-1} \). Decompose

\[
f(x) = f(x) 1\{|x| \leq \varepsilon^{-1}\} + f(x) 1\{|x| > \varepsilon^{-1}\} =: f_{1,\varepsilon}(x) + f_{2,\varepsilon}(x),
\]

where \( \int_{|x| \leq \varepsilon^{-1}} |f_{1,\varepsilon}(x) - f_\varepsilon(x)| \, dx < \varepsilon \), and \( |f_{2,\varepsilon}(x)| \leq C|x|^{\lambda'} 1\{|x| > \varepsilon^{-1}\} \). Letting \( \tilde{X}_t(n) := \beta_n^{-1}X_t(n) \), in view of condition (ii) of Theorem 2.1, (32) will follow once we have shown that

\[
\frac{1}{n} \sum_{t=t_0}^{n} E|f_{1,\varepsilon}(\tilde{X}_t(n)) - f_\varepsilon(\tilde{X}_t(n))| + \frac{1}{n} \sum_{t=1}^{n} E|f_{2,\varepsilon}(\tilde{X}_t(n))| \to 0,
\]

as \( n \to \infty \) and then \( \varepsilon \to 0 \), for \( t_0 \) as in condition (ii).

To that end, note that by HL5(a) and condition (iii.a) of Theorem 2.1 there is an \( n_0 \geq 1 \) and a \( \lambda > \lambda' \) such that \( \sup_{n \geq n_0, 1 \leq t \leq n} E|\tilde{X}_t(n)|^{\lambda'} < \infty \). Hence \( |\tilde{X}_t(n)|^{\lambda'} \) is uniformly integrable, and so for \( n \geq n_0 \)

\[
\frac{1}{n} \sum_{t=1}^{n} E|f_{2,\varepsilon}(\tilde{X}_t(n))| \leq \sup_{n \geq n_0, 1 \leq t \leq n} E|\tilde{X}_t(n)|^{\lambda'} 1\{|\tilde{X}_t(n)| > \varepsilon^{-1}\} \to 0,
\]

as \( \varepsilon \to 0 \). This gives the required negligibility of the second l.h.s. term in (35). For the first l.h.s. term, we note that by HL3 that

\[
\frac{1}{n} \sum_{t=t_0}^{n} E|f_{1,\varepsilon}(\tilde{X}_t(n)) - f_\varepsilon(\tilde{X}_t(n))| \leq \frac{1}{n} \sum_{t=t_0}^{n} \int_{\mathbb{R}} \left| f_{1,\varepsilon}\left(\frac{\beta_n x}{\beta_n}\right) - f_\varepsilon\left(\frac{\beta_n x}{\beta_n}\right) \right| D_{n,t}(x) \, dx \leq \sup_{n \geq n_0, t_0 \leq t \leq n} \int_{\mathbb{R}} |\beta_n f_{1,\varepsilon}(x) - f_\varepsilon(x)| \, dx \cdot \frac{\beta_n}{n} \sum_{t=t_0}^{n} \beta_t^{-1} \leq C \int_{|x| \leq \varepsilon^{-1}} |f_{1,\varepsilon}(x) - f_\varepsilon(x)| \, dx \leq C \varepsilon \to 0
\]

as \( \varepsilon \to 0 \), where the second inequality holds by HL6, for \( n \) sufficiently large.
Proof of (**33**). Let \( E_0(\cdot) := E(\cdot \mid \mathcal{F}_0) \), and \( \delta \in (0, 1) \). By HL4 and the boundedness of \( f_\varepsilon \), we have
\[
\frac{1}{n} \sum_{t=1}^{n} f_\varepsilon(X_t(n)) = \frac{1}{n} \sum_{t=\lceil n\delta \rceil +1}^{n} E_0[f_\varepsilon(X_t(n))] + o_p(1) \tag{36}
\]
as \( n \to \infty \) and then \( \delta \to 0 \). Now let \( \tilde{X}_t^+(n) := \beta_n^{-1}X_t^+(n) \) and \( \tilde{X}_t^-(n) := \beta_n^{-1}X_t^-(n) \). Since \( f_\varepsilon \) is bounded and Lipschitz, it follows from HL1 that
\[
\frac{1}{n} \sum_{t=\lceil n\delta \rceil +1}^{n} E \left| E_0[f_\varepsilon(\tilde{X}_t(n))] - E_0[f_\varepsilon(\tilde{X}_t^+(n) + \tilde{X}_t^-(n))] \right|
\leq C \sup_{1 \leq t \leq n} E(\beta_n^{-1}|R_t(n)| \land 1) \to 0 \tag{37}
\]
as \( n \to \infty \) for each \( \delta \in (0, 1) \); and from HL2(b) that
\[
\frac{1}{n} \sum_{t=\lceil n\delta \rceil +1}^{n} E \left| E_0[f_\varepsilon(\tilde{X}_t^+(n) + \tilde{X}_t^-(n))] - E_0[f_\varepsilon(\tilde{X}_t^+(n) + \tilde{X}_t^-(n))] \right|
\leq C \sup_{\lceil n\delta \rceil +1 \leq t \leq n} E(|\tilde{X}_t^-(n) - \tilde{X}_t^-(n)| \land 1)
= CE(|\tilde{X}_t^-(n) - \tilde{X}_t^-(n)| \land 1) \to 0 \tag{38}
\]
as \( n \to \infty \) for each \( \delta \in (0, 1) \), where \( \ell_n \in \{\lceil n\delta \rceil + 1, \ldots, n\} \) may always be chosen such that the final equality holds. Finally, by HL2(a), Theorem 2.1(ii) in Billingsley (1968) and Lemma 5.2 we have as
\[
\frac{1}{n} \sum_{t=\lceil n\delta \rceil +1}^{n} E_0 [f_\varepsilon(\beta_n^{-1}[X_t^+(n) + X_t^-(n)])]
\to (1 - \delta) \int_{\mathbb{R}} f_\varepsilon(x + X^-) \Phi_{X^+}(x)dx \tag{39}
\]
as \( n \to \infty \), for each \( \delta \in (0, 1) \). Hence (**33**) follows from (**36**)–(**39**) and Theorem 4.2 in Billingsley (1968).

Proof of (**34**). Let \( y \in \mathcal{Y} \) for \( \mathcal{Y} \) as in condition (i) of Theorem 2.1.
Noting \( f_\varepsilon(x) = 0 \) for \(|x| > \varepsilon\), and \( \Phi_{X^+} \) is bounded under \( \text{HL2} \), we have

\[
\int |f_\varepsilon(x + y) - f(x + y)|\Phi_{X^+}(x)dx \\
\leq \sup_{u} \Phi_{X^+}(u) \int_{|x| \leq \varepsilon^{-1}} |f_\varepsilon(x) - f(x)|dx + \int_{|x| > \varepsilon^{-1}} f(x)\Phi_{X^+}(x - y)dx \\
= C\varepsilon + o(1)
\]

as \( \varepsilon \to 0 \), where the negligibility of the second r.h.s. term follows by condition (i) of \textbf{Theorem 2.1}. Noting that \( P\{X^- \in \mathcal{Y}\} = 1 \) completes the proof. ■

**Proof of Theorem 3.1** By \textbf{Lemma 5.3} and condition (ii) of the theorem,

\[
\frac{\beta_n}{h_n} \frac{1}{n} \sum_{t=1}^{n} K \left( \frac{x_t(n) - x}{h_n} \right) = \frac{1}{n} \sum_{t=1}^{n} \varphi_{\varepsilon^2}(\beta_n^{-1} X_t(n)) + o_p(1)
\]

as \( n \to \infty \) and then \( \varepsilon \to 0 \). Noting that \( \text{HL7} \) implies \( \text{HL3} \), we have by \textbf{Theorem 2.1} that

\[
\frac{1}{n} \sum_{t=1}^{n} \varphi_{\varepsilon^2}(\beta_n^{-1} X_t(n)) \xrightarrow{d} \int_{\mathbb{R}} \varphi_{\varepsilon^2}(x + X^-)\Phi_{X^+}(x)dx,
\]

as \( n \to \infty \). Finally, as argued in (22) above,

\[
\int_{\mathbb{R}} \varphi_{\varepsilon^2}(x + X^-)\Phi_{X^+}(x)dx \to \Phi_{X^+}(-X^-)
\]

as \( \varepsilon \to 0 \). The result then follows by Theorem 4.2 in Billingsley (1968). ■

### 6 Appendix A:

**Appendix A** provides proofs for the rest of results of Section 2 (see \textbf{Appendix A2}), and a proof for \textbf{Lemma 5.2} (see \textbf{Appendix A3}). **Appendix A1** states some useful lemmas required for the aforementioned proofs. The proofs of these lemmas are given in **Appendix B**. Recall that throughout \( C \in (0, \infty) \) denotes a generic constant which may take different values at each appearance.

**Appendix A1a (technical lemmas)**
Lemma 6.1 Let $\xi \sim (0, \sigma^2)$ and $i = \sqrt{-1}$. There is some $0 < \rho_* < \infty$ such that
\[ |E(e^{i\xi \lambda})| \leq e^{-\rho_*(\lambda^2 \wedge 1)}. \]

We introduce some notation that is utilised in the subsequent lemma. Set
\[\{\eta_t, q_t, \eta\} := \{t, \lfloor L(t) \rfloor, \sqrt{\frac{t}{\ell(t)}}\}, \text{ under LL1-LL2}\]
\[\{\kappa_t, \lfloor \kappa_t / 2 \rfloor, 1/\sqrt{2}\}, \text{ under LL3-LL4}\]
\[
\delta_t := \begin{cases} 1, & \text{under LL1-LL3} \\ \kappa_t^{-1-m}, & \text{under LL4} \end{cases}
\]

Lemma 6.2 Suppose that Assumption LL holds. There are $n_0, n_1 > 0$ and constants $0 < D_1 \leq D_2 < \infty$ such that for all $n \geq n_0$ and $n_1 \leq t \leq n$ the following hold:

(i) $D_1 \leq \delta_t^{-1} \inf_{q_t+\theta \leq k \leq \eta_t} \eta_k |a_k(n)| \leq \delta_t^{-1} \sup_{0 \leq k \leq \eta_t} \eta_k |a_k(n)| \leq D_2$, for all $\theta \in \mathbb{N}$.

(ii) $\delta_n^{-1} \sup_{0 \leq k \leq n} \eta_k |a_k(n)| \leq D_2$.

(iii) $\lim_{n \to \infty} \gamma_n^{-1} \sup_{1 \leq t \leq n, k \geq t} |a_{k,t}(n)| = 0$.

Remark A1. Note that Lemma 6.2(ii,iii) implies $\lim_{n \to \infty} \gamma_n^{-1} \sup_{1 \leq t \leq n, k \geq 0} |a_{k,t}(n)| = 0$.

Lemma 6.3 Suppose that $\varphi_j \in \mathbb{R}$ for all $j \in \mathbb{N}$, $\varsigma$ is SV (in the sense of Bingham et al., 1987; p. 6) and $x_0 \in \mathbb{N}$ is such that $\varsigma(x)$ is locally bounded for all $x \geq x_0$. The following hold:

(i) Suppose that $\varphi_j \sim j^l \varsigma(j)$, where $l > -1$. Then for all $0 \leq s < r < \infty$ as $n \to \infty$:
\[ \frac{1}{n^{1+l\varsigma(n)}} \sum_{j=\lfloor ns \rfloor+1}^{\lfloor nr \rfloor} \varphi_j \to \int_s^r x^l dx. \]

(ii) Suppose that $\varphi_j \sim j^l \varsigma(j)$, where $l < -1$. Then as $n \to \infty$:
\[ \frac{1}{n^{1+l\varsigma(n)}} \sum_{j=n}^{\infty} \varphi_j \to \int_1^\infty x^l dx. \]
(iii) Suppose that \( S(x) := \int_{x_0}^{x} \xi(u) \, du \) is such that \( S(x) \to \infty \) as \( x \to \infty \).

Then as \( n \to \infty \)
\[
\sum_{j=x_0}^{n} \frac{\xi(j)}{j} \sim S(n).
\]

**Lemma 6.4** Suppose that \( \rho_n = 1 + \frac{c}{\kappa_n} \) with \( c < 0 \) and \( q \in \mathbb{N} \).

(i) For \( n \) large
\[
\begin{align*}
& (a) \quad \left| \rho_n^{qj} - \exp \left\{ qj \frac{c}{\kappa_n} \right\} \right| \leq 2 \exp \left\{ \frac{c}{2\kappa_n} qj \right\} qj \left( \frac{c}{\kappa_n} \right)^2, \\
& (b) \quad \sup_{1 \leq j \leq \tau_n} \left| \rho_n^{qj} - \exp \left\{ qj \frac{c}{\kappa_n} \right\} \right| = O \left( \frac{\tau_n}{\kappa_n^2} \right).
\end{align*}
\]

(ii) For \( \kappa_n/\lambda_n \to 0 \) and all \( 0 \leq \delta < |c| \) we have
\[
\kappa_n e^{q\frac{\lambda_n}{\kappa_n}} \left| \rho_n^{q\lambda_n} - \exp \left\{ \frac{c q \lambda_n}{\kappa_n} \right\} \right| = o(1).
\]

(iii) For \( k_n > \tau_n \geq 0 \), we have
\[
\sum_{t=1}^{k_n} \rho_n^{q_t} - \sum_{t=1}^{\tau_n} \rho_n^{q_t} = O \left( \kappa_n \rho_n^{q \tau_n} \right).
\]

If in addition \( \tau_n^{-1} \kappa_n \ln (\kappa_n) \to 0 \), then \( \kappa_n \rho_n^{q \tau_n} = o(1) \).

**Appendix A1b (intermediate lemmas)**

**Lemma UBD** (uniformly bounded density) Suppose that **Assumption INN, LP1, and LP3** with \( \beta_n \to \infty \) as \( n \to \infty \).

(i) The following hold:

(a) \( \beta_t^{-1} x_t \sim (n) \) possesses density \( D_{i,n}(x) \) such that for some \( n_0, t_0 \in \mathbb{N} \),
\[
\sup_{n \geq n_0, t_0} \sup_{\leq x} D_{i,n}(x) < \infty;
\]

(b) If in addition **LP4** holds, then the above holds with \( t_0 = 1 \).

(ii) Let \( i = \sqrt{-1} \) and \( \psi_{i,n}(\lambda) := E \exp (i \lambda x_t \sim (n) / \beta_t) \). The following hold:
For some $n_0, t_0 \in \mathbb{N}$,
\[
\lim_{A \to \infty} \sup_{n \geq n_0, t_0 \leq t \leq n} \int_{|\lambda| \geq A} |\psi_{t,n}(\lambda)| \, d\lambda = 0;
\]

(b) If in addition LP4 holds, then the above holds with $t_0 = 1$.

**Remark A.2 (a)** Note that the results of parts (ii.a) and (ii.b) imply those of (i.a) and (i.b) respectively. Recall that uniform integrability (of the characteristic function of $x_t(n)/\beta_t$) implies integrability, which in turn implies uniform bounded density e.g. Feller (1971, p. 516). **Lemma UBD(i)** is utilised for the proof of Theorem 2.1 whilst its stronger version (i.e. **Lemma UBD(ii)**) is employed for the proof of Theorem 3.2.

(b) It follows easily from the arguments used for the proof of **Lemma UBD** that parts (i.a) and (i.b) also apply to $\beta_t^{-1}x_t^+(n)$.

**Lemma LVAR** (limit variance) Suppose that $x_t(n)$ is given by (1.1), and Assumption INN(i) and Assumption LL hold. Let $t_n \in [\lfloor rn \rfloor, n]$, where $0 < r \leq 1$ and $\gamma_n, \mathcal{V}_\infty$ as in (2.1). The following hold:

Under LL1 or LL3 or LL4 as $n \to \infty$
\[
\text{Var} \left( \gamma_n^{-1}x_{t_n}(n) \right) = \text{Var} \left( \gamma_n^{-1}x_{t_n}^+(n) \right) + o(1) \to \mathcal{V}_\infty;
\]

Under LL2 as $n \to \infty$
\[
\text{Var} \left( \gamma_n^{-1}x_{t_n}^+(n) \right), \text{Var} \left( \gamma_n^{-1}x_{t_n}^-(n) \right) \to \mathcal{V}_\infty/2.
\]

Further, let $0 < s \leq r \leq 1$. Then as $n \to \infty$
\[
\text{Var} \left( \gamma_n^{-1} \left( x_{\lfloor rn \rfloor}^-(n) - x_{\lfloor sn \rfloor}^-(n) \right) \right) \to 0.
\]

**Lemma CLT** (central limit theorem). Suppose that $x_t(n)$ is given by (1.1), and Assumption INN(i) and Assumption LL hold. Let $t_n \in [\lfloor rn \rfloor, n]$, where $0 < r \leq 1$ and $\gamma_n, \mathcal{V}_\infty$ as in (2.1). Then as $n \to \infty$
\[
\gamma_n^{-1} \left[ x_{t_n}^+(n), x_{t_n}^-(n) \right] \overset{d}{\to} N(0, \Sigma),
\]
where $\Sigma = \text{diag} (\mathcal{V}_\infty/2, \mathcal{V}_\infty/2)$ under LL2, and $\Sigma = \text{diag} (\mathcal{V}_\infty, 0)$ otherwise.
Remark A.3 The proofs of Lemma IVAR and Lemma CLT do not utilise the restrictions on coefficients \( \{c_j\} \) given by conditions LL1(a,b).

Appendix A2

Proof of Proposition 2.1 (i) For \( \{\xi_k^*\} \overset{d}{=} \{\xi_k\} \) and \( \{\xi_k^*\} \perp \{\xi_k\} \) using (15) write

\[
\beta_n^{-1} \left[ x_{t_n}^+ (n), x_{t_n}^- (n) \right] \overset{d}{=} \beta_n^{-1} \left[ \sum_{k=0}^{t_n-1} a_k (n) \xi_k, \sum_{k=t_n}^\infty a_{k,t_n}^- (n) \xi_k^* \right].
\]

Next, note that due to LP6(b) as \( n \to \infty \)

\[
\beta_n^{-1} \left[ \sup_{0 \leq k \leq t_n-1} |a_k (n)| + \sup_{k \geq t_n} |a_{k,t_n}^- (n)| \right] = o(1).
\]

In view of the above and LP6(a), the weak limit result follows immediately from Lemma 2.1 of Abadir, Distaso, Giraitis and Koul (2014). Further, for the second part of HL2(b) note that due to condition LP6(c)

\[
\beta_n^{-2} \mathbb{E} \left[ \sum_{k=t_n}^\infty a_{k,t_n}^- (n) \xi_{t_n-k} - \sum_{k=n}^\infty a_{k,n}^- (n) \xi_{n-k} \right]^2 = \beta_n^{-2} \mathbb{E} \left[ \sum_{l=-\infty}^0 a_{t_n-l,t_n}^- (n) \xi_l - \sum_{l=-\infty}^0 a_{n-l,n}^- (n) \xi_l \right]^2
\]

\[
l = t_n - k \to k = t_n - l
\]

\[
= \beta_n^{-2} \sigma_k^2 \sum_{l=-\infty}^0 \left[ a_{t_n-l,t_n}^- (n) - a_{n-l,n}^- (n) \right]^2 = o(1).
\]

(ii) This part follows from Lemma UBD (Appendix A1b, above).

(iii) The proof uses arguments similar to those used in Wu and Mielniczuk (2002, p. 1443) and Duffy (2017, Lemma C.3)). Set \( \mathbf{E}_t (.) = \mathbb{E} (.) \mid \mathcal{F}_t \), \( t \in \mathbb{Z} \). Set \( \mathbf{E}_t (.) = \mathbb{E} (.) \mid \mathcal{F}_t \), \( t \in \mathbb{Z} \).

Recall that

\[
x_t (n) = \sum_{k=0}^{t-1} a_k (n) \xi_{t-k} + x_t^- (n).
\]

Write

\[
g (x_t (n)) - \mathbf{E}_0 g (x_t (n)) = \sum_{s=0}^{t-1} \left\{ \mathbf{E}_{t-s} g (x_t (n)) - \mathbf{E}_{t-s-1} g (x_t (n)) \right\}.
\]

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Let \( g_{t,n} := g (x_t(n)/\beta_n) \). In view of the above and the fact that \( \{E_t g_{t,n}, \mathcal{F}_r\}_{r \in \mathbb{Z}} \) is a martingale we get

\[
I_n := \frac{1}{n} E \left| \sum_{t=1}^{n} g_{t,n} - \sum_{t=1}^{n} E_0 g_{t,n} \right|
\]

\[
= \frac{1}{n} E \left| \sum_{s=0}^{t-1} \sum_{t=1}^{n} \{E_{t-s} g_{t,n} - E_{t-s-1} g_{t,n} \} \right| \leq \frac{1}{n} E \left| \sum_{s=0}^{n} \sum_{t=s+1}^{n} \{E_{t-s} g_{t,n} - E_{t-s-1} g_{t,n} \} \right|^2
\]

Next, write

\[
x_t(n) = \sum_{k=0}^{s-1} a_k(n) \xi_{t-k} + a_s(n) \xi_{t-s} + \sum_{k=s+1}^{t-1} a_k(n) \xi_{t-k} + x_t^{-}(n)
\]

\[
= \sum_{k=0}^{s-1} a_k(n) \xi_{t-k} + a_s(n) \xi^* + \sum_{k=s+1}^{t-1} a_k(n) \xi_{t-k} + x_t^{-}(n) =: x_{t,s}^*,
\]

where \( \xi^* \overset{d}{=} \xi_1 \) and \( \xi^* \perp \{\xi_j\} \). Therefore, \( \{E_{t-s} g (x_t(n)/\beta_n), E_{t-s-1} g (x_t(n)/\beta_n)\} \equiv \{E_{t-s} g (x_t(n)/\beta_n), E_{t-s} g (x_{t,s}^*(n)/\beta_n)\} \) which implies that

\[
I_n \leq \frac{1}{n} \sum_{s=0}^{n} \left\{ \sum_{t=s+1}^{n} E \left| E_{t-s} g_{t,n} - E_{t-s-1} g_{t,n} \right|^2 \right\}^{1/2}
\]

\[
= \frac{1}{n} \sum_{s=0}^{n} \left\{ \sum_{t=s+1}^{n} E \left| E_{t-s} g_{t,n} - E_{t-s} g (x_{t,s}^*(n)/\beta_n) \right|^2 \right\}^{1/2}
\]
\[
\leq \frac{1}{n} \sum_{s=0}^{n} \left\{ \sum_{t=s+1}^{n} \mathbb{E} \left| g_{t,n} - g \left( x_{t,s}^* (n) / \beta_n \right) \right|^2 \right\}^{1/2}
\]
\[
= \frac{1}{n} \sum_{s=0}^{n} \left\{ \sum_{t=s+1}^{n} \mathbb{E} \left| g \left( x_t (n) / \beta_n \right) - g \left( x_{t,s}^* / \beta_n \right) \right|^2 \right\}^{1/2}
\]
\[
\leq C \sqrt{\mathbb{E} [\xi_1 - \xi^*]^2} \sum_{s=0}^{n} \left\{ \frac{\beta_n^{-2}}{n} \sum_{s=0}^{n} |a_s (n)|^2 \right\}^{1/2} = \frac{C}{\beta_n \sqrt{n}} \sum_{s=0}^{n} |a_s (n)| = o(1),
\]
where we have used the Lipschitz continuity of \( g(.) \) and \( \text{LP5} \).

(iv) Recall \( \beta_{n,t}^2 = \text{Var} (x_t (n)) \). For part (a) without loss of generality suppose that \( \lambda \geq 2 \). By Theorem 2 of Whittle (1960) and \( \text{LP1} \) we have for \( n \geq n_0 \)

\[
\sup_{1 \leq t \leq n} \mathbb{E} \left| \frac{x_t (n)}{\beta_n} \right|^{\lambda} \leq C_\lambda \mathbb{E} \left[ |\xi_1|^\lambda \right] \sup_{1 \leq t \leq n} \left( \beta_n^{-2} \sum_{k=0}^{\infty} a_{k,t}^2 \right)^{\lambda/2}
\]
\[
= C_\lambda \mathbb{E} [\xi_1]^\lambda \sup_{1 \leq t \leq n} \left( \frac{\beta_n^{-2} \text{Var} (x_t (n))}{\mathbb{E} [\xi_1^2]} \right)^{\lambda/2} \leq \frac{C_\lambda \mathbb{E} [\xi_1]^\lambda}{\mathbb{E} [\xi_1^2]^{1/\lambda}} \left( \sup_{n_0 \leq n, 1 \leq t \leq n} \beta_n^{-2} \beta_{n,t}^{-2} \right)^{\lambda/2} < \infty,
\]
where \( 0 < C_\lambda < \infty \), and the last inequality follows from \( \text{LP2(a)} \) and \( \text{LP7} \).

Next, we show part (b). For \( \lambda \in (0, \infty) \), consider

\[
\mathbb{E} e^{\beta_n^{-1} |x_t (n)|} = \mathbb{E} e^{\beta_n^{-1} \lambda x_t (n)} 1 \{x_t (n) \geq 0\} + \mathbb{E} e^{-\beta_n^{-1} \lambda x_t (n)} 1 \{x_t (n) < 0\}
\]
\[
\leq \mathbb{E} e^{\beta_n^{-1} \lambda x_t (n)} + \mathbb{E} e^{-\beta_n^{-1} \lambda x_t (n)}.
\]
We shall prove the result for the first term on the r.h.s. above. The proof for the second term is identical. Note that \( \text{LP2(b)} \) implies that \( \xi_t \) is sub-exponential i.e. there are \( \epsilon_2, b_\epsilon > 0 \) such that \( \mathbb{E} \exp (\mu \xi_t) \leq \exp (\mu^2 \epsilon_2 / 2) \) for all \( |\mu| < b_\epsilon^{-1} \). Next, for each \( \lambda \in (0, \infty) \) there is some \( n_0 \geq 1 \) such that \( \sup_{n_0 \leq n, 1 \leq t \leq n} \lambda \beta_n^{-1} |a_{k,t} (n)| < b_\epsilon^{-1} \) due to \( \text{LP6(b)} \). To see this note that from \( \text{LP5} \) and \( \text{LP6(b)} \) we get as \( n \to \infty \)

\[
\beta_n^{-1} \sup_{1 \leq t \leq n} \sup_{k \geq 0} |a_{k,t} (n)| \leq \beta_n^{-1} \left[ \sup_{0 \leq k \leq n} |a_k (n)| + \sup_{1 \leq t \leq n, k \geq t} |a_{k,t}^{-} (n)| \right] = o(1).
\]
In view of the above for \( n \) large and Fatou’s lemma we have

\[
E e^{\beta_n^{-1} \lambda x_t(n)} = E \exp \left( \beta_n^{-1} \lambda \sum_{k=0}^{\infty} a_{k,t}(n) \xi_{t-k} \right) \leq \liminf_{M \to \infty} \prod_{k=0}^{M} E \exp \left( \beta_n^{-1} \lambda a_{k,t}(n) \xi_{t-k} \right)
\]

\[
\leq \liminf_{M \to \infty} \prod_{k=0}^{M} \exp \left( \beta_n^{-2} \lambda^2 a_{k,t}(n)^2 \xi^2/2 \right) = \exp \left( \beta_n^{-2} \lambda^2 \sum_{k=0}^{\infty} a_{k,t}(n)^2 \xi^2/2 \right)
\]

\[
= \exp \left( \beta_n^{-2} \lambda^2 \text{Var} \left( x_t(n) \right) \xi^2/2\sigma^2 \right) \leq \exp \left( \frac{\xi^2 \lambda^2}{2\sigma^2} \sup_{n \geq n_0, 1 \leq t \leq n} \beta_n^{-2} \beta_n^{-2} \right) < \infty,
\]

where the last inequality follows from \( \text{LP7} \).

\[\square\]

**Proof of Proposition 2.2** Without loss of generality set \( E \xi_1^2 = 1 \).

**Proof of LP3.** Part (a): Let \( \gamma_n \) and \( \mathcal{V}_\infty \) as in (20). By **Lemma LVAR** as \( t \to \infty \)

\[
\gamma_t^{-2} \beta_t^2 \to \mathcal{V}_\infty.
\]

In view of this for \( n \geq t \) and as \( t \to \infty \) we get

\[
\beta_t^{-2} \sum_{k=q_t+\theta}^{\eta_t} a_k(n)^2 = \mathcal{V}_\infty^{-1} \gamma_t^{-2} \sum_{k=q_t+\theta}^{\eta_t} a_k(n)^2 + o(1).
\]

Suppose that (40) holds. By **Lemma 6.2(i)** as \( n, t \to \infty \) we have

\[
\gamma_t^{-2} \sum_{k=q_t+\theta}^{\eta_t} a_k(n)^2 = \gamma_t^{-2} \delta_t^2 \sum_{k=q_t+\theta}^{\eta_t} \eta_k \delta_t^{-2} a_k(n) / \eta_k^2
\]

\[
\geq D_1^2 \gamma_t^{-2} \delta_t^2 \sum_{k=q_t+\theta}^{\eta_t} \eta_k^{-2} = (1) D_1^2 \left( 1 + o(1) \right),
\]

and the result follows from the above if we show \( = (1) \) which what we set out to do next. First, note that under **Assumption LL1-LL2** by **Lemma 6.3(iii)** as \( t \to \infty \)

\[
\frac{\delta_t^2}{\gamma_t^2} \sum_{k=q_t+\theta}^{\eta_t} \eta_k^{-2} = L(t)^{-1} \sum_{k=\lfloor L(t) \rfloor + \theta}^{t} \frac{\ell(k)^2}{k} = 1 + o(1).
\]

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Next, under Assumption LL3-LL4 as $t \to \infty$

\[
\frac{\delta^2}{\gamma_t^2} \sum_{k=\underline{q}_t+\theta}^{\bar{q}_t} \eta_{k-2} = 2\delta^2 \left( \frac{\eta_t - \underline{q}_t - \theta + 1}{\gamma_t^2} \right) = 2\delta^2 \left( \frac{\lfloor \kappa_t \rfloor - \lfloor \kappa_t/2 \rfloor - \theta + 1}{\gamma_t^2} \right) = 1 + o(1),
\]

where we have used the fact that $\kappa_t \delta^2_t = \gamma_t^2$.

Part (b): Set $\delta = (1 - m)/(3/2 - m)$ under Assumption LL4, and $\delta = 0$ otherwise. Note that in this case $\delta_t$ introduced in (40) becomes $\delta_t = \gamma_t^\delta$ (recall that $\gamma_n = \kappa_n^{3/2-m}$ under Assumption LL4). In view of this, by (41) and Lemma 6.2(i) there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $n_0 \leq t \leq n$ we have

\[
\beta_t^{-\delta} \sup_{0 \leq k \leq \bar{q}_t} |a_k(n)| \leq C \gamma_t^{-\delta} \sup_{0 \leq k \leq \bar{q}_t} |a_k(n)| \leq C \cdot D_2.
\]

Part (c): Let $\rho > 0$. By (41) we have $C/2 \leq \beta_t / \gamma_t \leq C$ for some $0 < C < \infty$ and $t$ large. In view of this and Lemma 6.2(i), as $t \to \infty$

\[
\exp \left\{ -\rho \beta_t^{2(1-\delta)} \right\} \beta_t \sup_{q \leq k \leq \bar{q}_t+\theta} |a_k(n)|^{-1} \leq C \exp \left\{ -\rho (C \gamma_t/2)^{2(1-\delta)} \right\} \gamma_t \sup_{q \leq k \leq \bar{q}_t+\theta} |a_k(n)|^{-1}
\]

\[
= C \exp \left\{ -\rho (C \gamma_t/2)^{2(1-\delta)} \right\} \gamma_t \sup_{q \leq k \leq \bar{q}_t+\theta} \left| \gamma_t^{-\delta} a_k(n) \right|^{-1}
\]

\[
= C \exp \left\{ -\rho (C \gamma_t/2)^{2(1-\delta)} \right\} \gamma_t \sup_{q \leq k \leq \bar{q}_t+\theta} \left| \delta_t^{-1} \eta_k \eta^{-\delta} a_k(n) \right|^{-1}
\]

\[
\leq D_1^{-1} C \exp \left\{ -\rho (C \gamma_t/2)^{2(1-\delta)} \right\} \gamma_t \sup_{q \leq k \leq \bar{q}_t+\theta} \eta_k := T_t.
\]

Now recall that by (21) and (40), under Assumption LL1-LL2, $\gamma_t = \sqrt{L(t)}$, $q_t = \lfloor L(t) \rfloor$, $\delta = 0$ and $\eta_t = \sqrt{t}/\ell(t)$. Therefore, it can be easily seen that as $t \to \infty$

\[
T_t \leq D_1^{-1} C \exp \left\{ -\rho C^2 L(t)/4 \right\} \sqrt{L(t)} (L(t) + \theta) = o(1).
\]

Further, note that under Assumption LL3-LL4 $\eta_t = \sqrt{1/2}$. In view of this and the fact that $\gamma_t^{1-\delta} \to \infty$, we get as $t \to \infty$,

\[
T_t \leq \sqrt{2} D_1^{-1} C \exp \left\{ -\rho (C \gamma_t/2)^{2(1-\delta)} \right\} \gamma_t^{1-\delta} = o(1).
\]
Proof of LP4 ($\theta = 1$) For a sequence $\{\pi_j\}_{j=0}^{\infty}$ define $b_t(\{\pi_j\}, k) := \sum_{j=0}^{t+k} \pi_j c_{k-j}$. Hence, under Assumption LL we can also write $x_t(n)$ as

$$x_t(n) = \sum_{k=0}^{\infty} b_t(\{\phi_j(n)\}, k) \xi_{t-k}, \quad \phi_j(n) = \begin{cases} \phi_j, \text{under } LL1-LL2 \\ \rho_{t_j}^i, \text{under } LL3-LL4 \end{cases}$$

First, suppose that LL1-LL2 hold. Then for each $t \geq 1$ there is some $k^*_t \geq 0$ such that $b_t(\{\phi_j\}, k^*_t) \neq 0$. Note that under our assumptions $\phi_0 \neq 0$. To see that the aforementioned statement is true, suppose that there some $t \geq 1$ such that $b_t(\{\phi_j\}, k) = 0$ for all $k$ i.e.

$$\begin{align*}
&b_t(\{\phi_j\}, 0) = \phi_0 c_0 \\
&b_t(\{\phi_j\}, 1) = \phi_0 c_1 + \phi_1 c_0 \\
&b_t(\{\phi_j\}, 2) = \phi_0 c_2 + \phi_1 c_1 + \phi_2 c_0 \\
&\vdots \quad \vdots
&b_t(\{\phi_j\}, t-1) = \phi_0 c_{t-1} + \ldots + \phi_{t-2} c_1 + \phi_{t-1} c_0 \\
&b_t(\{\phi_j\}, t) = \phi_0 c_t + \ldots + \phi_{t-1} c_1 + \phi_t c_0 \\
&b_t(\{\phi_j\}, t+1) = \phi_0 c_{t+1} + \phi_1 c_t + \ldots + \phi_{t-2} c_2 + \phi_{t-1} c_1 \\
&b_t(\{\phi_j\}, t+2) = \phi_0 c_{t+2} + \phi_1 c_{t+1} + \ldots + \phi_{t-1} c_3 + \phi_t c_2 \\
&\vdots \quad \vdots
\end{align*}$$

This implies that $c_j = 0$ for all $j \geq 0$, which contradicts Assumptions LL1-LL2 i.e. the facts that either $\sum_{j=0}^{\infty} c_j \neq 0$ or $c_j \sim \ell(j)j^{-3/2}$. Hence, indeed for each $t \geq 1$ there is some $k^*_t \geq 0$ such that $b_t(\{\phi_j\}, k^*_t) \neq 0$. Therefore, for any $t_0 \in \mathbb{N}$.

$$\inf_{1 \leq t \leq t_0} |b_t(\{\phi_j\}, k^*_t)| > 0.$$  

Next, suppose that LL3-LL4 hold. Using the same argument given above we can show that for each $t \geq 1$ there is some $k^*_t \geq 0$ such that $b_t(\{1\}, k^*_t) \neq 0$. Then

$$\lim \inf_{n \to \infty} \inf_{1 \leq t \leq t_0} |b_t(\{\rho_n^i\}, k^*_t)| = \inf_{1 \leq t \leq t_0} |b_t(\{1\}, k^*_t)| > 0,$$

as required.

Proof of LP5 Set $R_n := \frac{1}{\beta_n \sqrt{n}} \sum_{k=0}^{n} |a_k(n)|$. Under Assumption LL1-LL2 (recall in this case $\eta_k = \sqrt{k}/\ell(k)$, $\delta_n = 1$) by Lemma 6.2(ii), Lemma 39.
and Lemma 6.3(i) we get as \( n \to \infty \)

\[
R_n \leq \frac{\delta_n D_2}{\beta_n \sqrt{n}} \sum_{k=0}^{n} \eta_k^{-1} = \frac{D_2}{\beta_n \sqrt{n}} \sum_{k=1}^{n} \ell(k)k^{-1/2} + o(1)
\]

\[
\leq \frac{C}{\sqrt{L(n)} \sqrt{n}} \sum_{k=1}^{n} \ell(k)k^{-1/2} = \frac{C \ell(n)}{\sqrt{L(n)}} \left( \int_{0}^{1} x^{-1/2} dx + o(1) \right) = o(1),
\]

where the last approximation is due to Proposition 1.5.9a in Bingham et al. (1987). Next,

\[
R_n \leq \frac{1}{\beta_n \sqrt{n}} \sum_{k=0}^{n} \sum_{j=0}^{k} |\rho_n^j c_{k-j}|
\]

\[
= \frac{1}{\beta_n \sqrt{n}} \sum_{j=0}^{n} |\rho_n^j| \sum_{k=j}^{n} |c_{k-j}| = \frac{1}{\beta_n \sqrt{n}} \sum_{j=0}^{n} |\rho_n^j| \sum_{l=0}^{n-j} |c_l|
\]

\[
[l = k - j]
\]

\[
\leq (1) \frac{C\kappa_n}{\beta_n \sqrt{n}} \sum_{l=0}^{\infty} |c_l| = (2) \frac{C\kappa_n}{\gamma_n \sqrt{n}} = (3) \frac{C\sqrt{\kappa_n}}{\sqrt{n}} = o(1),
\]

where \( \leq (1) \) is due to Lemma 6.4(iii), \( = (2) \) due to Lemma 6.4(iii), and \( = (3) \) due to LL3. Similarly, under Assumption LL4 using Lemma LVAR, Lemma 6.3(i) and Lemma 6.4(iii) we get

\[
R_n \leq \frac{C}{\beta_n \sqrt{n}} \sum_{k=0}^{n} |\rho_n^k| \sum_{k=0}^{\infty} |c_k| \leq \frac{C\kappa_n}{\beta_n \sqrt{n}} \sum_{k=0}^{n} |c_q| = \frac{C\kappa_n^{m-1/2}}{\sqrt{n}} \left( \sum_{q=1}^{n} q^{-m} + o(1) \right)
\]

\[
= \frac{C\kappa_n^{m-1/2} n^{-1-m}}{\sqrt{n}} \left( \int_{0}^{1} x^{-m} dx + o(1) \right) = O \left( (\kappa_n/n)^{m-1/2} \right) = o(1).
\]

Proof of LP6 Part (a) It is a direct consequence of Lemma LVAR. Part (b) is a direct consequence of Lemma 6.2(ii,iii) and Lemma LVAR. For part (c) note that by Lemma LVAR under LL2

\[
\beta_n^{-2} Var \left[ x_{-t_n}(n) - x_{-n}(n) \right] \to 0.
\]

Further, under LL1, LL3 and LL4 again by Lemma LVAR we have

\[
\beta_n^{-2} \left[ Var \left( x_{-t_n}(n) \right) + Var \left( x_{-n}(n) \right) \right] \to 0.
\]
In view of the above the result follows from the fact that

\[ \beta_n^{-2} \sum_{t=-\infty}^{0} \left[ a_{t_1} - a_{t_2} (n) \right]^2 = \beta_n^{-2} \text{Var} \left[ x_{t_1} (n) - x_{t_2} (n) \right] \]

\[ \leq \beta_n^{-2} \left\{ \text{Var} \left[ x_{t_1} (n) \right] + \text{Var} \left[ x_{t_2} (n) \right] + 2 \sqrt{\text{Var} \left[ x_{t_1} (n) \right] \text{Var} \left[ x_{t_2} (n) \right]} \right\} . \]

Proof of LP7 First note that under Assumption LL and INN, it can be easily checked that \( \text{Var} \left( x_t (n) \right) \) is a finite sequence i.e. \( \text{Var} \left( x_t (n) \right) \in \mathbb{R}_+ \) for all \( 1 \leq t \leq n, n \in \mathbb{N} \).

Suppose that LL1 or LL2 holds. Without loss of generality we assume that \( \gamma_n^2 > 0 \). Otherwise we can set \( \gamma_n^2 = 1 \lor L(n) \) which is strictly positive and (eventually) increasing. By Lemma LVAR, we have as \( n \to \infty \)

\[ \beta_n^{-2} \sup_{1 \leq t \leq n} \text{Var} \left( x_t \right) \leq \beta_n^{-2} \sup_{1 \leq t \leq n} \gamma_n^2 \sup_{1 \leq t \leq n} \gamma_t^{-2} \text{Var} \left( x_t \right) \]

\[ = \beta_n^{-2} \gamma_n^2 \sup_{1 \leq t \leq n} \gamma_t^{-2} \text{Var} \left( x_t \right) \leq C \beta_n^{-2} \gamma_n^2 = C \gamma_n^{-1} + o(1), \]

where \( \gamma_n > 0 \), and the last inequality above follows from the fact that \( \gamma_t^{-2} \text{Var} \left( x_t \right) \) is a finite convergent sequence (Lemma LVAR).

Next, suppose that LL3 or LL4 holds. Note that in this case \( x_t (n) \) is an array. We have \( \text{Var} \left( x_t (n) \right) = \text{Var} \left( x_t^+ (n) \right) + \text{Var} \left( x_t^- (n) \right) \). For \( 1 \leq t \leq n \), the first term

\[ \text{Var} \left( x_t^+ (n) \right) = \sum_{k=0}^{t-1} \left( \sum_{j=0}^{k} \rho_n^j c_{k-j} \right)^2 \leq \sum_{k=0}^{n-1} \left( \sum_{j=0}^{k} \rho_n^j c_{k-j} \right)^2 = \text{Var} \left( x_n^+ (n) \right) . \]

In view of the above and Lemma LVAR we have as \( n \to \infty \)

\[ \beta_n^{-2} \sup_{1 \leq t \leq n} \text{Var} \left( x_t^+ (n) \right) \leq \beta_n^{-2} \text{Var} \left( x_n^+ (n) \right) = 1 + o(1). \]

Hence, \( \beta_n^{-2} \sup_{1 \leq t \leq n} \text{Var} \left( x_t^+ (n) \right) \) is bounded for \( n \) large. Next, we show that \( \beta_n^{-2} \sup_{1 \leq t \leq n} \text{Var} \left( x_t^- (n) \right) \) is bounded for \( n \) large enough. Consider

\[ \sup_{1 \leq t \leq n} \text{Var} \left( x_t^- (n) \right) = \sup_{1 \leq t \leq n} \sum_{k=t}^{n} \left( \sum_{j=0}^{t-1} \rho_n^j c_{k-j} \right)^2 \leq \sup_{1 \leq t \leq n} \sum_{k=t}^{n} \left( \sum_{j=0}^{k} \rho_n^j c_{k-j} \right)^2 . \]
\[
\leq \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k \wedge n} |\rho_n^j c_{k-j}| \right)^2 = \sum_{k=0}^{n-1} \left( \sum_{j=0}^{k} |\rho_n^j c_{k-j}| \right)^2 + \sum_{k=n}^{\infty} \left( \sum_{j=0}^{n} |\rho_n^j c_{k-j}| \right)^2
\]

\[
=: C_n + D_n.
\]

Notice that \(C_n\) and \(D_n\) resemble \(\text{Var}(x^+ n)\) and \(\text{Var}(x^- n)\) respectively. Therefore, using the same arguments as those used for the proof of Lemma LVAR, we can show that \(\gamma_n^{-2} D_n = o(1)\). Further,

\[
\gamma_n^{-2} C_n \rightarrow \begin{cases} 
\left( \sum_{s=0}^{\infty} |c_s| \right)^2 / 2, & \text{under LL3} \\
\mathcal{V}_\infty, & \text{under LL4}
\end{cases}
\]

where \(\mathcal{V}_\infty\) is as in (20). The requisite result follows from the above and the fact that \(\beta_n^{-2} \gamma_n^{-2} \rightarrow \mathcal{V}_\infty^{-1}\) where \(\mathcal{V}_\infty > 0\). \(\blacksquare\)

**Proof of Theorem 2.2** We show that the assumptions of Theorem 2.1 hold true, under the assumptions of Theorem 2.2 with \(X_t(n) = x_t(n), t \geq 1\). We start with the result for \(f\).

(i) HL0 & HL1: Let \(F_t = \sigma(\{\xi_{\tau_t}\}_{\tau_t \leq t}), t \in \mathbb{Z}\), and for \(t \geq 1\), \([X_t^+(n), X_t^-(n)] = [x_t^+(n), x_t^-(n)]\) where \(x_t^+(n), x_t^-(n)\) are given in (15). It can be easily seen that in this case HL0 and HL1 hold due Assumption INN and Assumption LL.

(ii) HL2: By Lemma LVAR, LP6(a) holds with \([\sigma^2_+, \sigma^-] = [1, 0]\) under LL(1,3,4) and \([\sigma^2_+, \sigma^2_-] = [1/2, 1/2]\) otherwise. In view of this, HL2 by Propositions 2.1 and 2.2.

(iii) HL3-HL4: These conditions are direct consequences of Propositions 2.1 and 2.2.

(iv) HL6: In view of Assumption LL, Lemma LVAR (see also eq. (20)) implies that \(\beta_n \sim n^a \zeta(n)\) where \(0 \leq a < 1\) and \(\zeta(x) \text{ SV in some neighborhood of infinity. Therefore, for some } t_0 \geq 1\) and as \(n \rightarrow \infty\) Lemma 6.3(i) yields

\[
\frac{\beta_n}{n} \sum_{t=t_0}^{n} \beta_t^{-1} = \int_0^1 x^{-a} dx + o(1),
\]

and therefore HL6 holds.

We have established HL0-HL4 and HL6 under the assumptions of the current Theorem. Hence, by Theorem 2.1 (14) holds for \(f\) bounded. For \(f\) unbounded we also need validating conditions (i)-(iii) of Theorem 2.1 which is what we set out to do next. First, notice that HL2 holds with
\([X^+, X^-] \sim N \left[0, \text{diag}(\sigma_+^2, \sigma_-^2)\right]\), where \(\sigma_+^2, \sigma_-^2\) are as in part (ii) above. Hence, \(E \int_R |f(x + X^-)| \Phi_{X+}(x) dx = \int_R |f(x)| \varphi_1(x) dx\) and the latter is finite due to condition (ii) of Theorem 2.2. This in turn implies that condition (i) of Theorem 2.1 holds. Next, condition (ii) of Theorem 2.1 holds due to condition (i) of the current Theorem. Finally, condition (iii) of Theorem 2.1 holds by condition (ii) of the current Theorem, and Propositions 2.1.2.2.

Next, we show the result for \(\tilde{f}\). Note that under our assumptions, for all \(n\) large enough and each \(1 \leq t \leq n\), there is at least one coefficient \(a_{k,t}(n)\) in \(x_t(n) = \sum_{k=0}^{\infty} a_{k,t}(n) \xi_{t-k}\), that is bounded away from zero, as shown in the proof Proposition 2.2. In view of this and the fact that each \(\{\xi_t\}\) has a density (Assumption INN), it follows that \(\{x_t(n)\}_{t=1}^{\infty}\) also has a density for \(n\) large (e.g. Lukacs, 1970; Theorem 3.3.2). Hence, for \(n\) large \(\frac{1}{n} \sum_{t=1}^{n} \{f(\beta_n^{-1} x_t(n)) - \tilde{f}(\beta_n^{-1} x_t(n))\} = 0\) a.s., and the result follows.

**Proof of Theorem 3.2** We check that the assumptions of Theorem 3.1 hold true, under the conditions of Theorem 3.2 with \(X_t(n) = x_t(n), t \geq 1\).

(i) **HL0-HL2, HL4 and HL6** can be established using identical arguments as those used in the proof of Theorem 2.2.

(ii) Next, we show **HL7**. Note that from (15) we can write

\[
x_t^+(n) = \sum_{k=0}^{t-1} a_k(n) \xi_{t-k} = \sum_{l=1}^{t} a_{t-l}(n) \xi_l.
\]

Hence, for \(t > s\)

\[
x_t(n) - x_s(n) = \sum_{l=1}^{t} a_{t-l}(n) \xi_l - \sum_{l=1}^{s} a_{s-l}(n) \xi_l + x_t^-(n) - x_s^-(n)
\]

\[
= \sum_{l=s+1}^{t} a_{t-l}(n) \xi_l + \sum_{l=1}^{s} [a_{t-l}(n) - a_{s-l}(n)] \xi_l + x_t^-(n) - x_s^-(n) =: \sum_{l=s+1}^{t} a_{t-l}(n) \xi_l + \mathcal{X}_{t,s}(n)
\]

The first term on the r.h.s. in the last line above is

\[
\sum_{l=s+1}^{t} a_{t-l}(n) \xi_l = \sum_{k=0}^{t-s-1} a_k(n) \xi_{t-k} \overset{d}{=} x_{t-s}^+(n).
\]

\([l = t - k \rightarrow k = t - l]\)
It follows from Lemma UBD(ia) (see also Remark A.2) that \( \beta_i^{-1} x_i^+ (n) \) possesses a density, \( D_{t,n}^+ (x) \) say, such that \( \sup_{n \geq n_0, t_0 \leq t \leq n} \sup_x D_{t,n}^+ (x) < \infty \), for some \( n_0 \geq t_0 \geq 1 \). In view of the above, conditionally on \( \mathcal{F}_s \), \( \beta_i^{-1} [x_t (n) - x_s (n)] \) has a density \( D_{t,s,n} (x) \) such that \( D_{t,s,n} (x) = D_{t-s,n} (x - \mathcal{X}_{t,s} (n)) \) and

\[
\sup_{n \geq n_0, 0 \leq s \leq t \leq n, t-s \geq t_0} \sup_x D_{t-s,n} (x - \mathcal{X}_{t,s} (n)) \leq \sup_{n \geq n_0, t_0 \leq t \leq n} \sup_x D_{t,n}^+ (x) < \infty.
\]

(iii) HL8 follows from Lemma CLT, Lemma UBD(ii) and arguments similar to those used in Wang and Phillips (2009a), p. 729-731.

(vi) Next, we show HL9. HL9(a,b) follows from similar arguments as those used for establishing HL6 in the proof of Theorem 2.2. For HL9(c) recall that \( \beta_n \sim n^\alpha \zeta (n) \) where \( 0 \leq \alpha < 1 \) and \( \zeta (x) \) SV. Again using similar arguments as those used in the proof of Theorem 2.2 we get as \( n \to \infty \) first and then as \( \eta \downarrow 0 \)

\[
\frac{\beta_n}{n} \sup_{0 \leq s \leq (1-\eta)n} \sum_{t=s+t_0}^{s+\lfloor \eta n \rfloor} \beta_{t-s}^{-1} = \frac{\beta_n}{n} \sup_{0 \leq s \leq (1-\eta)n} \sum_{t=s+t_0}^{s+\lfloor \eta n \rfloor} \beta_{t-s}^{-1} \to \int_0^\eta x^{-\alpha} dx + o(1) \equiv o(1).
\]

HL9(d), follows using similar arguments as those used for HL9(c). We finally show HL9(e). Let \( 0 < \eta < 1 \). As \( n \to \infty \)

\[
\sup_{(t,s) \in \Omega_n (\eta)} \frac{\beta_{t-s} - (t-s)^\alpha}{n^\alpha} = (1) \sup_{(t,s) \in \Omega_n (\eta)} \frac{(t-s)^\alpha \zeta (t-s) - (t-s)^\alpha}{n^\alpha} + o(1)
\]

\[
\leq \sup_{\frac{\lfloor \eta n \rfloor}{n} \leq x \leq 1} \left| \frac{(nx)^\alpha \zeta (nx)}{n^\alpha \zeta (n)} - x^\alpha \right| \leq \sup_{\eta/2 \leq x \leq 1} \left| \frac{(nx)^\alpha \zeta (nx)}{n^\alpha \zeta (n)} - x^\alpha \right|
\]

\[
= \sup_{\eta/2 \leq x \leq 1} x^\alpha \left| \frac{\zeta (nx)}{\zeta (n)} - 1 \right| \leq \sup_{\eta/2 \leq x \leq 1} \left| \frac{\zeta (nx)}{\zeta (n)} - 1 \right| = (2) o(1),
\]

where = (1) follows from arguments similar to those used for the proof of eq. (L3) and = (2) is due to Theorem 1.2.1 in Bingham et al. (1987). Hence, as \( n \to \infty \)

\[
\beta_n^{-1} \inf_{(t,s) \in \Omega_n (\eta)} \left| \beta_{t-s} \right| = \inf_{(t,s) \in \Omega_n (\eta)} (t-s)^\alpha / n^\alpha + o(1),
\]

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where \( \inf_{(t,s) \in \Omega_n(\eta)} (t - s)^\alpha / n^\alpha \geq \lfloor m \eta \rfloor^\alpha / n^\alpha \geq \eta^\alpha / 2 \), and the result follows.

\[ \text{Appendix A3} \]

**Proof of Lemma 5.2** By Theorems 6.3 and 6.4 in Kallenberg (2001), there is a probability kernel \( \nu_n \) from \((\Omega, \mathcal{F})\) to \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) such that
\[
\mathbb{E}(g(X_n, Y_n) \mid \mathcal{F})^{a.s.} = \int_{\mathbb{R}} g(x, Y_n) d\nu_n(x) =: h_n(Y_n).
\]

Now using the above and condition (i), for each \( y \in \mathbb{R} \) fixed we have
\[
h_n(y) \overset{a.s.}{=} \mathbb{E}(g(X_n, y) \mid \mathcal{F}) \overset{p}{\to} \int_{\mathbb{R}} g(x, y) dF_X(x) =: h(y).
\]

Moreover, by the Lipschitz continuity of \( g \)
\[
|h_n(y) - h_n(y')| \leq \int_{\mathbb{R}} |g(x, y) - g(x, y')| d\nu_n(x) \leq C |y - y'|.
\]

Hence, \( \{h_n(y)\} \) is stochastically equicontinuous on \( \mathbb{R} \), whence if follows that \( h_n(y) \overset{p}{\to} h(y) \) uniformly on every compact subset of \( \mathbb{R} \) (see for example Theorem 1 and Lemma 1 in Andrews, 1992).

Finally, fix \( \varepsilon > 0 \) and choose \( M_\varepsilon \) such that \( \limsup_{n \to \infty} \mathbb{P}(|Y_n| > M_\varepsilon) < \varepsilon \), which is possible since \( Y_n \overset{d}{\to} Y \). Then as \( n \to \infty \)
\[
\mathbb{P}(|h_n(Y_n) - h(Y_n)| > \varepsilon) \leq \mathbb{P}(\{|h_n(Y_n) - h(Y_n)| > \varepsilon\} \cap \{|Y_n| \leq M_\varepsilon\})
+ \mathbb{P}(|Y_n| > M_\varepsilon) \leq \mathbb{P}\left(\sup_{|y| \leq M_\varepsilon} |h_n(y) - h(y)| > \varepsilon\right) + \varepsilon \to \varepsilon,
\]

by the uniform convergence in probability of \( h_n \) on compacta. In view of the above,
\[
\mathbb{E}(g(X_n, Y_n) \mid \mathcal{F})^{a.s.} = h_n(Y_n) = h(Y_n) + o_p(1) \overset{d}{\to} \int_{\mathbb{R}} g(x, Y) dF_X(x),
\]

where the last approximation is due to condition (ii) and the fact that \( h(y) \) is continuous in \( y \).
7 Appendix B

Appendix B1 states some technical lemmas that are subsequently utilised for the proofs of the preliminary lemmas of Appendix A1. The proofs of the preliminary lemmas of Appendix A, are given in Appendix B2a whilst Appendix B2b contains proofs for the technical lemmas of Appendix A1. The proofs of the of the technical lemmas of Appendix B1 are given in Appendix B3.

Appendix B1 (technical lemmas)

Lemma 7.1 The following hold.

(i) Suppose that $\phi_j \sim \ell(j) j^{-1/2}$, where $\ell(x)$ SV, $\sum_{j=0}^{\infty} |c_j| < \infty$, and one of the following holds:

(a) $\lim sup_{j \to \infty} j |c_j| < \infty$; or

(b) $\lim_{s \to \infty} \ell^{-1}(s) \sum_{j=\lfloor s \delta \rfloor}^{\infty} j^{1/2} |c_j| = 0$ for some $0 < \delta < 1$.

Then as $s \to \infty$

$$\sum_{j=0}^{s} \phi_j c_{s-j} = \phi_s \sum_{j=0}^{\infty} c_j + o\left(\ell(s)/s^{1/2}\right).$$

(ii) Suppose that $\pi_j \sim \ell_\pi(j) j^{-q_1}$, $\theta_j \sim \ell_\theta(j) j^{-q_2}$ for some $q_1, q_2 \geq 1$, $\sum_{j=0}^{\infty} (|\pi_j| + |\theta_j|) < \infty$, and $\ell_\pi(x), \ell_\theta(x)$ are SV. Then as $s \to \infty$

$$\sum_{j=0}^{s} \pi_j \theta_{s-j} = \pi_s \sum_{j=0}^{\infty} \theta_j + o\left(\pi_s\right) + \theta_s \sum_{j=0}^{\infty} \pi_j + o(\theta_s).$$

If in addition $\sum_{j=0}^{\infty} \pi_j = 0$ and $\theta_j = O(\pi_j)$, then as $s \to \infty$

$$\sum_{j=0}^{s} \pi_j \theta_{s-j} = \pi_s \sum_{j=0}^{\infty} \theta_j + o\left(\pi_s\right).$$

Remark. Lemma 7.1(i) implies that as $s \to \infty$

$$\frac{s^{1/2}}{\ell(s)} \sum_{j=0}^{s} \phi_j c_{s-j} = \sum_{j=0}^{\infty} c_j + o(1).$$

Therefore, if $\sum_{j=0}^{\infty} c_j \neq 0$, $(\sqrt{s}/\ell(s)) \sum_{j=0}^{s} \phi_j c_{s-j}$, is bounded away from zero for $s$ large.
Lemma 7.2 The following hold.

(i) Under Assumption LL4

(a) \[ \sup_{n \geq t, 0 \leq k \leq \kappa t} \left| \frac{1}{\kappa_n} a_k(n) - \int_0^{k/\kappa_n} e^{- \left( \frac{k}{\kappa_n} - \frac{m}{\kappa_n} \right)x} x^{-m} dx \right| = o(1), \text{ as } t \to \infty; \]

(b) \[ \sup_{0 \leq k \leq n} \left| \frac{1}{\kappa_n} a_k(n) - \int_0^{k/\kappa_n} e^{- \left( \frac{k}{\kappa_n} - \frac{m}{\kappa_n} \right)x} x^{-m} dx \right| = o(1), \text{ as } n \to \infty; \]

(c) \[ \sup_{1 \leq t \leq n, k \geq t} \left| \frac{1}{\kappa_n} a_k(n) - \int_0^{k/\kappa_n} e^{- \left( \frac{k}{\kappa_n} - \frac{m}{\kappa_n} \right)x} x^{-m} dx \right| = o(1), \text{ as } n \to \infty. \]

(ii) Under Assumption LL4, there are \( n_0, n_1 \in \mathbb{N} \) such that

\[ \inf_{n \geq n_0, n_1 \leq t \leq n_0} \left| \int_0^{k/\kappa_n} e^{- \left( \frac{k}{\kappa_n} - \frac{m}{\kappa_n} \right)x} x^{-m} dx > 0 \right. \]

Lemma 7.3 For \( 1/2 < m < 1 \) we have the representation

\[ \int_0^{\infty} e^{2y} \left( \int_0^y e^{-x} x^{-m} dx \right)^2 dy = \Gamma (2 - 2m) B (1 - m, 2m - 1), \]

where \( \Gamma(.) \) and \( B(.,.) \) are Gamma and Beta functions respectively.

Appendix B2a (proofs of preliminary lemmas of Appendix A)

Proof of Lemma UBD. We only prove part (i). Part (ii) follows easily from minor modifications of the arguments used below.

(i.a) We shall prove that the characteristic function of \( x_t(n)/\beta_t \) has a uniformly bounded \( L_1 \)-norm i.e. \( \sup_{n \geq n_0, t_0 \leq t \leq n} \int_\lambda |\psi_{t,n}(\lambda)| d\lambda < \infty \) for some \( n_0 \geq t_0 \geq 1 \). This implies the conclusion of part (i.a). Consider

\[ \int_{\mathbb{R}} \left| \mathbb{E} \left( e^{i\lambda x_t(n)/\beta_t} \right) \right| d\lambda \leq \int_{\mathbb{R}} \left| \mathbb{E} \left( e^{i\lambda x_t^+(n)/\beta_t} \right) \right| d\lambda = \int_{\mathbb{R}} \left| \mathbb{E} \left( e^{i\lambda \left( \sum_{k=0}^{t-1} a_k(n)\xi_{t-k} \right)/\beta_t} \right) \right| d\lambda \]

\[ = \left( \int_{|\lambda| \leq \beta_t^{-1/2} c^{-1}_t} + \int_{|\lambda| \geq \beta_t^{-1/2} c^{-1}_t} \right) \prod_{k=0}^{t-1} \psi_{t,\beta_{t}^{-1/2} a_k(n)} \left( \frac{\lambda}{\beta_t^{-1/2} a_k(n)} \right) \right) d\lambda =: I_{1,n}(t) + I_{2,n}(t), \]
where $\delta$ and $C_2$ are given in Assumption LP3(b). By Lemma 6.1 and in view of LP3(b) for $n$ and $t$ large enough the first term is

$$I_{1,n}(t) \leq \int_{|\lambda| \leq \beta_t^{-1}C_2^{-1}} \prod_{k=\underline{q}}^{\overline{q}} \left| \psi_\xi \left( \frac{\lambda}{\beta_t} a_k(n) \right) \right| d\lambda$$

$$\leq \int_{|\lambda| \leq \beta_t^{-1}C_2^{-1}} \prod_{k=\underline{q}}^{\overline{q}} \exp \left( -\rho_* \frac{\lambda^2}{\beta_t^2} a_k^2(n) \right) d\lambda = \int_{|\lambda| \leq \beta_t^{-1}C_2^{-1}} \exp \left( -\rho_* \frac{\lambda^2}{\beta_t^2} \sum_{k=\underline{q}}^{\overline{q}} a_k^2(n) \right) d\lambda$$

$$\mu = \left( \beta_t^{-2} \sum_{k=\underline{q}}^{\overline{q}} a_k^2(n) \right)^{1/2}$$

$$\leq \left( \inf_{n \geq n_0, t_0 \leq t \leq n} \beta_t^{-2} \sum_{k=\underline{q}}^{\overline{q}} a_k^2(n) \right)^{-1/2} \int_{\mathbb{R}} \exp \left( -\rho_* \mu^2 \right) d\mu < \infty,$$

where the last inequality follows from LP3(a). Hence, $\sup_{n \geq n_0, t_0 \leq t \leq n} I_{1,n}(t) < \infty$ for some $n_0 \geq t_0 \in \mathbb{N}$.

Next, we show that $I_{2,n}(t)$ is bounded. For $n \geq t$ and $t$ large enough we get

$$I_{2,n}(t) \leq \int_{|\lambda| \geq \beta_t^{-1}C_2^{-1}} \prod_{k=\underline{q}}^{\overline{q}} \left| \psi_\xi \left( \frac{\lambda}{\beta_t} a_k(n) \right) \right| d\lambda$$

$$= \int_{|\lambda| \geq \beta_t^{-1}C_2^{-1}} \prod_{k=\underline{q}}^{\overline{q} + \theta - 1} \left| \psi_\xi \left( \frac{\lambda}{\beta_t} a_k(n) \right) \right| \prod_{k=\overline{q} + \theta}^{\overline{q}} \left| \psi_\xi \left( \frac{\lambda}{\beta_t} a_k(n) \right) \right| d\lambda,$$

with $\theta$ as in Assumption INN. For $|\lambda| \geq \beta_t^{-1}C_2^{-1}$ and $q_t + \theta \leq k \leq \overline{q}_t$ we get

$$\left| \psi_\xi \left( \frac{\lambda}{\beta_t} a_k(n) \right) \right| \leq (1) \exp \left\{ -\rho_* \left( \frac{\lambda}{\beta_t} a_k(n) \right)^2 \wedge 1 \right\}$$

$$\leq \exp \left\{ -\rho_* \left( \frac{\beta_t^{-1}C_2^{-1} a_k(n)}{\beta_t} \right)^2 \wedge 1 \right\} \leq \exp \left\{ -\rho_* \left( \frac{C_2^{-1} a_k(n)}{\beta_t^\delta} \right)^2 \right\},$$

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where $0 < \rho_* < \infty$, and $\leq (1)$ is due to Lemma 6.1. It follows from the above and LP3(a) that

$$
\sup_{|\lambda| \geq \beta_t^1 - \delta C_2^{-1}} \prod_{k=q_t}^q |\psi_{\xi}(\lambda/\beta_t(n))| \leq \prod_{k=q_t}^q \exp \left\{-\rho_* \left(\frac{C_2^{-1}}{\beta_t} \lambda k(n) \right)^2 \right\}
$$

$$
= \exp \left\{-\frac{\rho_* \sum_{k=q_t}^q \lambda^2 k(n)}{\beta_t^2 C_2^2} \right\}
= \exp \left\{-\frac{\rho_* \beta_t^{-2} \sum_{k=q_t}^q \lambda^2 k(n)}{\beta_t^2 (\delta - 1) C_2^2} \right\}
\leq \exp \left\{-\rho_* C_2^2 \beta_t^2 (\delta - 1)/C_2^2 \right\} =: Q_t,
$$

where the last inequality follows from LP3(a). Note that $Q_t \to 0$ as $t \to \infty$, because $\beta_t \to \infty$. Hence, for $n \geq t$ and $t$ large we get

$$
I_{2,n}(t) \leq Q_t \int_{|\lambda| \geq \beta_t^1 - \delta C_2^{-1}} \prod_{k=q_t}^q |\psi_{\xi}(\lambda/\beta_t(n))| d\lambda \leq (2) Q_t \int_{|\lambda| \geq \beta_t^1 - \delta C_2^{-1}} \prod_{k=q_t}^q |\psi_{\xi}(\lambda/\beta_t(n))| d\lambda \leq Q_t \beta_t \prod_{k=q_t}^q |\lambda k(n)|^{-1} \int_{R} |\psi_{\xi}(\lambda)|^{\theta} d\lambda \leq (3) \sup_{n \geq q_t, t \leq t_0} Q_t \beta_t \sup_{q_t \leq k \leq q_t + \theta - 1} |a_k(n)|^{-1} \int_{R} |\psi_{\xi}(\lambda)|^{\theta} d\lambda < \infty,
$$

where $\leq (2)$ follows from Lemma 7 in Jeganathan (2008), $\leq (3)$ is due to Assumption INN and LP3(c), and the result follows.

(i.b) For the second part let $1 \leq t \leq t_0 < \infty$. We have for $n$ large

$$
\int_{R} \left| E(e^{i\lambda x_t(n)/\beta_t}) \right| d\lambda \leq \int_{R} \prod_{l=1}^q \left| E(e^{i\lambda x_{n,l}(n)/\beta_t}) \right| d\lambda
$$
\[
\leq \prod_{l=1}^{q} \left\{ \int_{\mathbb{R}} \psi_{\xi} \left( \frac{\lambda}{\beta_t} a_{k_{n,t,l},t}(n) \right) \bigg| d\lambda \right\}^{1/q} = \prod_{l=1}^{q} \left\{ \int_{\mathbb{R}} \frac{\beta_t}{a_{k_{n,t,l},t}(n)} \bigg| \psi_{\xi} (\lambda) \bigg| d\lambda \right\}^{1/q} \\
\leq \sup_{1 \leq t \leq t_0} \beta_t \left( \inf_{1 \leq t \leq q, 1 \leq t \leq t_0} a_{k_{n,t,l},t}(n) \right) \int_{\mathbb{R}} \bigg| \psi_{\xi} (\lambda) \bigg| d\lambda < \infty,
\]

where the last inequality is due to Assumption INN, LP4 and the fact that \( \beta_t \) is a finite sequence (LP3). \( \blacksquare \)

**Proof of Lemma LVAR.** For convenience set \( \sigma_{\xi}^2 = 1 \). Throughout we also use the convention \( \sum_{j=b}^{a} u_j = 0 \) for \( a < b \). Further, under LL1-LL2 without loss of generality set \( t_n = n \). The result for the general case follows immediately from the arguments used below and the fact that \( L(t_n)/L(n) \to 1 \) (see Lemma 6.3(iii)).

(i) **LL1**: We prove the result under LL1, without utilising conditions (a) or (b) i.e. the only restrictions imposed on \( \{c_j\} \) in this proof are \( \sum_{j=0}^{\infty} |c_j| < \infty \) and \( \sum_{j=0}^{\infty} c_j \neq 0 \). Recall that \( \ell(x) \) is a SV on \([\ell, \infty)\), \( \ell > 0 \). For convenience and without loss of generality extend the domain of definition by setting \( \ell(x) = 1 \) for all integer \( x \leq \ell \). Further, given that \( \phi_j \) can be approximated by \( \ell(j)j^{-1/2} \) for \( j \) large, there is no loss of generality if we assume that \( 0 < \ell(j) < \infty \) for all \( j \). Note that all SV functions are eventually non zero and finite (e.g. Bingham et al. 1987, Lemma 1.3.2). These conventions will simplify the derivations below. Write \( x_t = x_t(n) \), and consider

\[
\gamma_n^{-2} Var(x_n^+) = \gamma_n^{-2} \sum_{s=0}^{n-1} \left( \sum_{l=0}^{s} \phi_{l} c_{s-l} \right)^2 = \gamma_n^{-2} \sum_{s=0}^{n-1} \left( \sum_{l=0}^{s} \phi_{s-l} c_{l} \right)^2 \\
= \gamma_n^{-2} \sum_{s=0}^{n-1} \sum_{l=0}^{s} \phi_{l}^2 c_{s-l}^2 + 2\gamma_n^{-2} \sum_{s=0}^{n-1} \sum_{l=1}^{s} \sum_{k=0}^{s-l} \phi_{s-l} \phi_{s-k} c_{l} c_{k} =: A_n + 2B_n.
\]

We shall show that as \( n \to \infty \)

\[
B_n = \gamma_n^{-2} \sum_{s=1}^{n-1} \ell(s)^2 s^{-1} \sum_{k=0}^{n} c_k c_l + o(1).
\]

Similarly, it can be shown that

\[
A_n = \gamma_n^{-2} \sum_{s=1}^{n-1} \ell(s)^2 s^{-1} \sum_{k=0}^{n} c_k^2 + o(1).
\]
Further, noting that $\gamma_n^2 = L(n)$, **Lemma 6.3(iii)** gives $\gamma_n^{-2} \sum_{s=1}^{n-1} \ell(s)^2 s^{-1} = 1 + o(1)$. In view of the above, we get as $n \to \infty$

$$\gamma_n^{-2} \text{Var}(x_n^+) = \sum_{k=0}^{\infty} c_k^2 + 2 \sum_{k=0}^{\infty} \sum_{l=k+1}^{\infty} c_k c_l + o(1) = \left( \sum_{k=0}^{\infty} c_k \right)^2.$$ 

Subsequently we shall show that

$$\gamma_n^{-2} \text{Var}(x_n^-) = o(1).$$

First, we set out to show (42). Write

$$B_n = \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} \sum_{s=l}^{n-1} \phi_{s-l} \phi_{s-k} c_l c_k = \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} \left( \sum_{s=l}^{n-1} \frac{1}{L(n)+1} + \sum_{s=l}^{n-1} \phi_{s-l} \phi_{s-k} c_l c_k \right).$$

Note that as $n \to \infty$

$$\gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} \sum_{s=l}^{n-1} \phi_{s-l} \phi_{s-k} c_l c_k \leq \left( \sup_{s \geq 0} |\phi_s| \right)^2 \sqrt{L(n)} \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} |c_l c_k| \leq C \left( \sum_{l=1}^{\infty} |c_l| \right)^2 \sqrt{L(n)} \gamma_n^{-2} = O \left( L(n)^{-1/2} \right) = o(1).$$

Set $\lambda(k) := \ell(k) k^{-1/2}$. Hence, as $n \to \infty$

$$B_n = \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} \sum_{s=l}^{n-1} \phi_{s-l} \phi_{s-k} c_l c_k + o(1) = \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} \sum_{s=l}^{n-1} c_l c_k \lambda(s-l) \lambda(s-k) + o(1) =: B'_n + o(1), \quad (43)$$

where the approximation in the last line above will be demonstrated in detail later. Set

$$G_n := \sup_{\sqrt{L(n)} \leq x \leq n} \sup_{1 \leq y < \infty} \sum_{j=1}^{2} \left| \frac{\lambda(xy)^j}{\lambda(x)^j} - y^{-j/2} \right|, \quad (44)$$

$$G_{s,k,l} := \left| \left( \frac{s-k}{s-l} \right)^{-1/2} - 1 \right| + \left| \left( \frac{s}{s-l} \right)^{-1} \right|. \quad 51$$
Note that for $j = \{1, 2\}$, $\lambda(x)^j$ is regularly varying with index $-j/2$, and $L(n) \to \infty$ as $n \to \infty$. Therefore, by Theorem 1.5.2. in Bingham et al. (1987) we have $\lim_{n \to \infty} G_n = 0$. In view of this, as $n \to \infty$

$$ |B_n' - B_n''| := \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} \sum_{s=0}^{n} |c_l c_k \left[ \lambda(s - l) \lambda(s - k) - \lambda(s)^2 \right]| $$

$$ \leq \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} \sum_{s=0}^{n} |c_l c_k| \left\{ \left| \lambda(s - l) \lambda(s - k) - \lambda(s) \right|^2 \right\} $$

$$ = \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} |c_l c_k| \sum_{s=0}^{n} \lambda(s - l)^2 \left\{ \left| \frac{\lambda(s - k)}{\lambda(s - l)} - 1 \right| + \left| \frac{\lambda(s)^2}{\lambda(s - l)^2} - 1 \right| \right\} $$

$$ \leq \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} |c_l c_k| \sum_{s=0}^{n} \lambda(s - l)^2 \left\{ \left| \frac{\lambda(s - k)}{\lambda(s - l)} - \left( \frac{s - k}{s - l} \right)^{-1/2} \right| + \left| \frac{\lambda(s)^2}{\lambda(s - l)^2} - \left( \frac{s}{s - l} \right)^{-1} \right| \right\} $$

$$ \leq \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} |c_l c_k| \sum_{s=0}^{n} \lambda(s - l)^2 \left[ G_n + G_{s,k,l} \right] =: R_{1,n} + R_{2,n}, $$

where we have used the fact that for $j = \{1, 2\}$

$$ \sup_{1 \leq l \leq n} \sup_{0 \leq k \leq l-1} \sup_{s \leq s \leq n} \left| \frac{\lambda(s - k)^j}{\lambda(s - l)^j} - \left( \frac{s - k}{s - l} \right)^{-j/2} \right| \leq \sup_{1 \leq l \leq n} \sup_{0 \leq k \leq l-1} \sup_{s \leq s \leq n} \left| \frac{\lambda((s - l)^j)}{\lambda(s - l)^j} \right| $$

$$ \leq \sup_{1 \leq l \leq n} \sup_{0 \leq k \leq l-1} \sup_{s \leq s \leq n} \left| \frac{\lambda(x)^j}{\lambda(x)^j} - y^{-j/2} \right|. $$

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The first term

\[ R_{1,n} \leq G_n \gamma_n^{-2} \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} |c_l c_k| \sum_{q=1+\sqrt{L(n)}}^{\infty} \lambda(q)^2 \]

\[ \leq G_n \gamma_n^{-2} \left( \sum_{l=1}^{\infty} |c_l| \right)^2 \sum_{q=1}^{n} \lambda(q)^2 = G_n \left( \sum_{l=1}^{\infty} |c_l| \right)^2 (1 + o(1)) = o(1), \]

where the first approximation shown above is due to Lemma 6.3(iii). The second term

\[ R_{2,n} = \gamma_n^{-2} \left( \sum_{l=1}^{\sqrt{L(n)}} + \sum_{l=1+\sqrt{L(n)}}^{n-1} \right) \sum_{k=0}^{l-1} |c_l c_k| \sum_{q=1+\sqrt{L(n)}}^{\infty} \lambda(q)^2 \left[ \left( \frac{q}{q + (l-k)} \right)^{1/2} - 1 \right] \]

\[ + \left| \frac{q}{q + l} - 1 \right| : = R'_{2,n} + R''_{2,n}. \]

Note that

\[ \left| \left( \frac{q}{q - (l-k)} \right)^{1/2} - 1 \right| + \left| \frac{q}{q + l} - 1 \right| \leq \left( \frac{l-k}{q + (l-k)} \right)^{1/2} + \frac{l}{q + l}. \]

Hence,

\[ R'_{2,n} \leq \gamma_n^{-2} \sum_{l=1}^{\sqrt{L(n)}} \sum_{k=0}^{l-1} |c_l c_k| \sum_{q=1+\sqrt{L(n)}}^{\infty} \lambda(q)^2 \left[ \frac{\sqrt{L(n)} \sqrt{L(n)}}{q^{1/2}} + \frac{\sqrt{L(n)}}{q} \right] \]

\[ \leq C \gamma_n^{-2} \sqrt{L(n)} \left( \sum_{l=1}^{\infty} |c_l| \right)^2 \sum_{q=1}^{\infty} \lambda(q)^2 q^{-1/2} = O \left( L(n)^{-1/2} \right) = o(1), \]

where we have used the fact that \( \gamma_n^2 = L(n) \). Further,

\[ R''_{2,n} \leq 2 \gamma_n^{-2} \sum_{l=1+\sqrt{L(n)}}^{n-1} \sum_{k=0}^{l-1} |c_l c_k| \sum_{q=1+\sqrt{L(n)}}^{\infty} \lambda(q)^2 \leq 2 \sum_{l=\sqrt{L(n)}}^{\infty} |c_l| \sum_{k=0}^{\infty} |c_k| (1 + o(1)) = o(1). \]

Therefore, as \( n \to \infty \) we get \( |B'_n - B''_n| = o(1) \).
where (1) follows from Lemma 6.3(iii), and (2) from Kronecker’s lemma e.g. Hall and Heyde (1980), p. 31 (recall that $L(n) = \gamma_n^2$). In view of the above, and using again Lemma 6.3(iii), we have as $n \to \infty$

$$B_n = B''_n + o(1) = \sum_{l=0}^{n} \sum_{k=0}^{l-1} c_l c_k + o(1) = \sum_{l=k+1}^{n} c_k c_l,$$

which yields (42).

Next, we show (43). Note that given $\varepsilon > 0$ and some integer $N_{\varepsilon}$ we have

$$\left| \frac{\phi_{s-l} \phi_{s-k}}{\lambda(s-l) \lambda(s-k)} - 1 \right| < \varepsilon$$

for $s-l, s-k \geq N_{\varepsilon}$. Hence, as $n \to \infty$ first and then as $\varepsilon \downarrow 0$

$$|B_n - B'_n| \leq \gamma_n^{-2} \sum_{l=1}^{n} \sum_{k=0}^{l-1} \sum_{s=l+1+N_{\varepsilon}}^{s} \left| \frac{\phi_{s-l} \phi_{s-k}}{\lambda(s-l) \lambda(s-k)} - 1 \right| \lambda(s-l) \lambda(s-k) |c_l c_k|$$

$$\leq \varepsilon \gamma_n^{-2} \sum_{l=1}^{n} \sum_{k=0}^{l-1} \sum_{s=l+1}^{s} \lambda(s-l) \lambda(s-k) |c_l c_k| \leq \varepsilon C \varepsilon \downarrow 0$$
where the last inequality follows from the fact that
\[
\gamma_n^{-2} \sum_{l=1}^{n} \sum_{k=0}^{l-1} \sum_{s=l+1}^{n} \lambda(s-l) \lambda(s-k) |c_l c_k|
\]
sembles \(B'_n\) which is convergent.

Finally, consider
\[
\gamma_n^{-2} \text{Var} (x^-_n) = \sum_{s=n}^{\infty} \left( \sum_{j=0}^{n/2-1} \phi_j c_{s-j} \right)^2 \leq \sum_{s=n}^{\infty} \left( \sum_{j=0}^{n/2-1} |\phi_j c_{s-j}| \right)^2 =: E_n + 2F_n.
\]

Recall that \(\lambda(x) = \ell(x)x^{-1/2}\). The first term
\[
E_n = \gamma_n^{-2} \sum_{s=n}^{\infty} \left( \sum_{j=0}^{\lfloor n/2 \rfloor} \phi_j^2 c_{s-j}^2 \right) =: E'_n + E''_n.
\]

Next, as \(n \to \infty\)
\[
E'_n = \gamma_n^{-2} \sum_{j=0}^{\lfloor n/2 \rfloor} \phi_j^2 \sum_{s=n}^{\infty} c_{s-j}^2 = \gamma_n^{-2} \sum_{j=0}^{\lfloor n/2 \rfloor} \phi_j^2 \sum_{l=0}^{\lfloor n/2 \rfloor} \phi_j^2 \sum_{s=0}^{\infty} c_{s-l+j}^2 = o(1).
\]

\([l = s - j]\)

The second term as \(n \to \infty\)
\[
E''_n = \gamma_n^{-2} \sum_{j=0}^{\lfloor n/2 \rfloor} \phi_j^2 \sum_{s=n}^{\infty} c_{s-j}^2 \leq \gamma_n^{-2} \sum_{j=0}^{\lfloor n/2 \rfloor} \phi_j^2 \sum_{s=0}^{\infty} c_s^2 \leq \gamma_n^{-2} \sum_{j=0}^{n/2} \phi_j^2 \sum_{s=0}^{\infty} c_s^2 = o(1).
\]

\(= C \gamma_n^{-2} \sum_{j=0}^{\lfloor n/2 \rfloor} \phi_j^2 = C \gamma_n^{-2} \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{\ell(j)^2}{j} + o(1) = C \gamma_n^{-2} \left[ \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{\ell(j)^2}{j} - \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{\ell(j)^2}{j} \right] = C \left[ 1 + o(1) - \frac{\gamma_n^{2}}{\gamma_n^{2}} (1 + o(1)) \right] = o(1),\)
where the first approximation shown above follows from similar arguments as those used for the proof of (43), whilst the second approximation is due to Lemma 6.3(iii) and the fact that \( \gamma_n^2 = L(n) \). The final term is

\[
F_n = \gamma_n^{-2} \sum_{s=n}^{n-1} \sum_{j=0}^{n} \sum_{j'=j+1}^{n} |\phi_j \phi_{j'} c_{s-j} c_{s-j'}| \leq \gamma_n^{-2} \sum_{j=\sqrt{L(n)+1}}^{n-1} \sum_{j'=j+1}^{n} \sum_{s=n}^{\infty} |\phi_j \phi_{j'} c_{s-j} c_{s-j'}|
\]

\[
+ \gamma_n^{-2} \sqrt{L(n)} \left( \sup_{j \in \mathbb{N}} |\phi_j| \sum_{s=0}^{\infty} |c_s| \right)^2
\]

\[
= \gamma_n^{-2} \sum_{j=\sqrt{L(n)+1}}^{n-1} \sum_{j'=j+1}^{n} \lambda(j) \lambda(j') \sum_{s=n}^{\infty} |c_{s-j} c_{s-j'}| + o(1) + O \left( L(n)^{-1/2} \right), \ (45)
\]

where the last approximation will be demonstrated in detail later. Set \( G_n \) as in (44) and note that

\[
\sup_{\sqrt{L(n)} \leq j \leq n} \sup_{j' \leq n} \left| \frac{\lambda \left( j' \frac{j'}{j} \right)}{\lambda(j)} - \left( \frac{j'}{j} \right)^{-1/2} \right| = \sup_{\sqrt{L(n)} \leq x \leq n} \sup_{1 \leq y \leq \infty} \left| \frac{\lambda(xy)}{\lambda(x)} - y^{-1/2} \right| \leq G_n.
\]

In view of this consider

\[
\left| \gamma_n^{-2} \sum_{j=\sqrt{L(n)+1}}^{n-1} \sum_{j'=j+1}^{n} \lambda(j) \lambda(j') \sum_{s=n}^{\infty} |c_{s-j} c_{s-j'}| - F_n \right| :=
\]

\[
\gamma_n^{-2} \left| \sum_{j=\sqrt{L(n)+1}}^{n-1} \sum_{j'=j+1}^{n} \left[ \lambda(j) \lambda(j') - \lambda(j)^2 \right] \sum_{s=n}^{\infty} |c_{s-j} c_{s-j'}| \right|
\]

\[
\leq \gamma_n^{-2} \left| \sum_{j=\sqrt{L(n)+1}}^{n-1} \sum_{j'=j+1}^{n} \lambda(j)^2 \frac{\lambda(j')}{\lambda(j)} - 1 \right| \sum_{s=n}^{\infty} |c_{s-j} c_{s-j'}| 
\]

\[
\leq \gamma_n^{-2} \sum_{j=\sqrt{L(n)+1}}^{n-1} \lambda(j)^2 \sum_{s=n}^{\infty} \sum_{j'=j+1}^{n} |c_{s-j} c_{s-j'}| \left[ G_n + \left( \frac{j'}{j} \right)^{-1/2} - 1 \right]
\]

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\[ F'_n \leq \gamma_n^{-2} \sum_{j=1}^{n} \lambda(j) \sum_{s=n}^{\infty} |c_{s-j+s'}| \]

\[ = C \gamma_n^{-2} \left( \sum_{j=1}^{[n/2]} + \sum_{j=[n/2]+1}^{n} \right) \lambda(j) \sum_{l=n-j}^{\infty} |c_l| \leq C \gamma_n^{-2} \left[ \sum_{j=1}^{[n/2]} \sum_{l=\infty}^{\infty} |c_l| + \sum_{j=1}^{n} \lambda(j)^2 \right] \]

\[ = C \left[ (1 + o(1)) \sum_{l=[n/2]}^{\infty} |c_l| + 1 + o(1) - \gamma_n^{2} \frac{[n/2]}{\gamma_n^{2}} (1 + o(1)) \right] = o(1), \]

where the last line above is due to Lemma 6.3(iii). Finally, we show that the approximation in \((45)\) holds. As before, note that \(|\varphi_j \varphi_{j'}/\lambda(j) \lambda(j') - 1| < \varepsilon\) for \(j, j' \geq N_\varepsilon\). Hence, for \(n\) large enough such that \(\sqrt{L(n)} > N_\varepsilon\) we have

\[ F''_n := \gamma_n^{-2} \sum_{j=1+\sqrt{L(n)}}^{n} \left[ \varphi_j \varphi_{j'} - \lambda(j) \lambda(j') \right] \sum_{s=n}^{\infty} |c_{s-j+s'}| \]

\[ \leq \varepsilon \gamma_n^{-2} \sum_{j=1+\sqrt{L(n)}}^{n} \lambda(j) \lambda(j') \sum_{s=n}^{\infty} |c_{s-j+s'}| \]

Noting that the term above resembles that in \((45)\), which in convergent, we get as \(n \to \infty\) first and then as \(\varepsilon \downarrow 0\), \(F''_n = o(1)\).
(ii) LL2: Without loss of generality set $\ell(j) = 1$ (i.e. $c_j \sim j^{-3/2}$). The result for the general case follows from Lemma 6.3(ii) and arguments similar to those used in the previous part. In view of the fact that $\phi_j = 1$, write

$$x_t = \sum_{s=0}^{\infty} \sum_{j=0}^{(t-1)\land s} \phi_j c_{s-j} \xi_{t-s} = \sum_{s=0}^{t-1} \sum_{j=0}^{s} c_{s-j} \xi_{t-s} + \sum_{s=t}^{\infty} \sum_{j=0}^{\infty} c_{s-j} \xi_{t-s} =: x_t^+ + x_t^-.$$  

Using the fact that $\sum_{j=0}^{\infty} c_j = 0$, the first term

$$\gamma_n^{-2} Var(x_n^+) = \gamma_n^{-2} \sum_{s=0}^{n-1} \left( \sum_{j=0}^{s} c_{s-j} \right)^2 = \gamma_n^{-2} \sum_{s=0}^{n-1} \left( \sum_{r=s+1}^{\infty} c_r \right)^2$$

where the approximation above can be established using arguments similar to those used in part (i), eq. (43). Hence, in view of the above Euler summation (Lemma 6.3(ii)-(iii)) gives

$$\gamma_n^{-2} Var(x_n^+) = \gamma_n^{-2} \sum_{s=0}^{n-1} \left( \sum_{r=s+1}^{\infty} c_r \right)^2$$

Similarly we get as $n \to \infty$,

$$\gamma_n^{-2} Var(x_n^-) = \gamma_n^{-2} \sum_{s=n}^{\infty} \left( \sum_{j=0}^{s-1} c_{s-j} \right)^2 = \gamma_n^{-2} \sum_{s=n}^{\infty} \left( \sum_{j=s+1-n}^{s} c_l \right)^2 = \gamma_n^{-2} \sum_{s=n+1}^{\infty} \left( \sum_{j=s-n}^{s} l^{-3/2} \right)^2 + o(1)$$

Finally, we have

$$\gamma_n^{-2} Var(x_n) = \gamma_n^{-2} \sum_{s=n}^{\infty} \left( \sum_{j=0}^{s-1} c_{s-j} \right)^2 = \gamma_n^{-2} \sum_{s=n}^{\infty} \left( \sum_{j=s+1-n}^{s} c_l \right)^2 = \gamma_n^{-2} \sum_{s=n+1}^{\infty} \left( \sum_{j=s-n}^{s} l^{-3/2} \right)^2 + o(1)$$

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\[
= \frac{4\gamma_n^{-2}}{n} \sum_{s=n+1}^{\infty} \left( \left[ x^{-1/2} \right] \right) \left( \left[ \frac{s}{n} \right] \right) - \left( \left[ \frac{r}{n} \right] \right) \left( \left[ \frac{s}{n} \right] \right)^{-1/2} \right)^2
= 4\gamma_n^{-2} \int_{1+\frac{1}{n}}^{\infty} (x^{-1/2} - (x - 1)^{-1/2})^2 dx + o(1) = 4\gamma_n^{-2} \int_{1/n}^{\infty} ((u + 1)^{-1/2} - u^{-1/2})^2 du
\]

\[
= 4\gamma_n^{-2} \left( \int_{1/n}^{1} + \int_{1}^{\infty} \right) ((u + 1)^{-1/2} - u^{-1/2})^2 du = 4 (A_n + B_n).
\]

Using the mean value theorem the second term
\[
\gamma_n^{-2} B_n = \gamma_n^{-2} \int_{1}^{\infty} ((u + 1)^{-1/2} - u^{-1/2})^2 du \leq \gamma_n^{-2} \int_{1}^{\infty} u^{-3} du = o(1).
\]

The first term
\[
A_n = \gamma_n^{-2} \int_{1/n}^{1} ((u + 1)^{-1} + u^{-1} - 2 (u + 1)^{-1/2} u^{-1/2}) du
\]
\[
= 1 + o(1) - 2\gamma_n^{-2} \int_{1/n}^{1} (u + 1)^{-1/2} u^{-1/2} du = 1 + o(1),
\]
where we have used the fact that
\[
\gamma_n^{-2} \int_{1/n}^{1} (u + 1)^{-1/2} u^{-1/2} du \leq \gamma_n^{-2} \int_{0}^{1} u^{-1/2} du = o(1).
\]

Proof of the increments result: Write
\[
x_t^- = \sum_{k=t}^{t-1} \sum_{j=0}^{t-1} c_{k-j} \xi_{t-k} = \sum_{i=-\infty}^{0} \sum_{j=0}^{t-1} c_{t-i-j} \xi_{i} = \sum_{i=-\infty}^{0} \sum_{l=1}^{t} c_{l-i} \xi_{i}
\]

\[
[x_{nr}^- - x_{ns}^-] = \sum_{i=-\infty}^{0} \sum_{l=1}^{|nr|} c_{l-i} \xi_{i} - \sum_{i=-\infty}^{0} \sum_{l=1}^{|ns|} c_{l-i} \xi_{i}
\]

\[
\gamma_n^{-2} E \left( [x_{nr}^- - x_{ns}^-] \right)^2 = \gamma_n^{-2} E \left( \sum_{i=-\infty}^{0} \sum_{l=1}^{|nr|} c_{l-i} \xi_{i} - \sum_{i=-\infty}^{0} \sum_{l=1}^{|ns|} c_{l-i} \xi_{i} \right)^2
\]
\[ \gamma_n^{-2} \sum_{i=0}^{[nr]} c_{i-i} = \gamma_n^{-2} \sum_{i=0}^{[nr]} \left( \sum_{j=0}^{[ns]} (l-i)^{-3/2} \right)^2 \]

where the approximation shown above follows from similar arguments as those used in the proof of \( \gamma_n^2 \). Further, using arguments similar to those used in part (ii) above we also get \( \int_s^\infty \left[ (r-s+u)^{-1/2} - u^{-1/2} \right]^2 du < \infty \). In view of this, Euler summation yields

\[ \gamma_n^{-2} \mathbb{E} \left( x_{[nr]} - x_{[ns]} \right)^2 = \gamma_n^{-2} \sum_{i=0}^{[nr]} \left( \int_{[ns]}^{[ns]+1} \left( x - \frac{i}{n} \right)^{-3/2} dx \right) + o(1) = 4 \gamma_n^{-2} \sum_{i=0}^{[nr]} \left( \left( \frac{nr}{n} + \frac{i}{n} \right)^{-3/2} - \left( \frac{ns}{n} + \frac{i}{n} \right)^{-3/2} \right)^2 \]

\[ = 4 \gamma_n^{-2} \int_0^\infty \left[ \left( \frac{nr}{n} + x \right)^{-1/2} - \left( \frac{ns}{n} + x \right)^{-1/2} \right]^2 dx + o(1) \]

\[ = 4 \gamma_n^{-2} \int_{[ns]+1}^{\infty} \left[ \left( \frac{nr}{n} - [ns] - 1 + u \right)^{-1/2} - u^{-1/2} \right]^2 du \]

\[ \leq 4 \gamma_n^{-2} \int_s^\infty \left[ (r-s+u)^{-1/2} - u^{-1/2} \right]^2 du + o(1) = O \left( \gamma_n^{-2} \right). \]

(iii) Consider

\[ \gamma_n^{-2} \text{Var} \left( x_{[n]}^+ (n) \right) = \gamma_n^{-2} \sum_{s=0}^{t_n-1} \left( \sum_{j=0}^{s} \rho_n^{l_j} c_{s-j} \right)^2 = \gamma_n^{-2} \sum_{s=0}^{t_n} \left( \sum_{l=0}^{s} \rho_n^{s-l} c_l \right)^2 + o(1) \]

\[ = \gamma_n^{-2} \sum_{s=0}^{t_n-1} \sum_{l=0}^{s} \rho_n^{2(s-l)} c_l^2 + 2 \gamma_n^{-2} \sum_{s=0}^{t_n-1} \sum_{l=1}^{s-1} \sum_{k=0}^{l-1} \rho_n^{s-l} c_l \rho_n^{s-k} c_k =: A_n + 2B_n. \]
We shall show that as $n \to \infty$

$$B_n = \frac{1}{2} \sum_{l=1}^{\infty} c_l \sum_{k=0}^{l-1} c_k + o(1) \quad (46)$$

Further, it can be shown that

$$A_n = \frac{1}{2} \sum_{j=0}^{\infty} c_j^2 + o(1). \quad (47)$$

The result follows immediately from (46) and (47). We only consider in detail $B_n$ which is more complicated than $A_n$. Recall that $\gamma_n^2 = \kappa_n$ and write

$$B_n = \sum_{l=1}^{t_n-1} c_l \sum_{k=0}^{l-1} c_k \rho_n^{l-k} \rho_n^{2(s-l)} = \sum_{l=1}^{t_n-1} c_l \sum_{k=0}^{l-1} c_k \rho_n^{l-k} \rho_n^{2q} \quad [q = s-l]$$

$$= \frac{\gamma_n^{-2}}{1 - \rho_n^2} \sum_{l=1}^{t_n} c_l \sum_{k=0}^{l-1} c_k \rho_n^{l-k} (1 - \rho_n^{2(t_n-l)}) + o(1)$$

$$= \frac{1}{2} \left( \sum_{l=1}^{t_n} c_l \sum_{k=0}^{l-1} c_k \rho_n^{l-k} - \sum_{l=1}^{t_n} c_l \sum_{k=0}^{l-1} c_k \rho_n^{l-k} \rho_n^{2(t_n-l)} \right) =: \frac{1}{2} (B'_n - B''_n)$$

Let $m_n > 0$ integer valued sequence such that $\frac{1}{m_n} + \frac{m_n}{\kappa_n} \to 0$. Consider first

$$B'_n = \sum_{l=1}^{t_n} c_l \sum_{k=0}^{l-1} c_k \rho_n^{l-k} = \left( \sum_{l=1}^{m_n} + \sum_{l=m_n+1}^{t_n} \right) c_l \sum_{k=0}^{l-1} c_k \rho_n^{l-k} =: C'_n + C''_n.$$  

In view of Lemma 6.4(i)

$$C'_n = \sum_{l=1}^{m_n} c_l \sum_{k=0}^{l-1} c_k e^{-\frac{l-k}{m_n}} + O \left( \frac{m_n}{\kappa_n^2} \right) = \sum_{l=1}^{\infty} c_l \sum_{k=0}^{l-1} c_k + o(1),$$

where the last approximation follows from the fact that $\sup_{1 \leq l \leq m_n, 0 \leq k \leq l-1} |e^{-\frac{l-k}{m_n}} - 1| \leq \frac{m_n}{\kappa_n}$. Next, using the fact that $|\rho_n| \leq 1$ for $n$ large we get

$$|C''_n| \leq \sum_{l=m_n+1}^{\infty} |c_l| \sum_{k=0}^{\infty} |c_k| = o(1).$$
Further, for $n$ large enough

$$|B''_n| \leq \left( \sum_{l=1}^{m_n} + \sum_{l=m_n+1}^{n} \right) \sum_{k=0}^{l-1} \left| c_l c_k \rho_{n_{l-k}}^{2(t_n-l)} \right| \leq \sum_{l=1}^{m_n} \left| c_l \right| \sum_{k=0}^{l-1} \left| c_k \right| \left| \rho_n \right|^{2(t_n-l)}$$

$$+ \sum_{l=m_n+1}^{\infty} \left| c_l \right| \sum_{k=0}^{\infty} \left| c_k \right| \leq \left( \sum_{k=0}^{\infty} \left| c_k \right| \right)^2 \left| \rho_n \right|^{2(t_n-m_n)} + o(1) = o(1),$$

where the last approximation is due to **Lemma 6.4(ii)**, and hence (40) holds.

Finally, consider

$$\gamma_n^{-2} \text{Var} \left( x_{t_n}^{-} (n) \right) \leq \gamma_n^{-2} \sum_{s=t_n}^{\infty} \left( \sum_{j=0}^{t_n} \left| \rho_n^j c_{s-j} \right| \right)^2$$

$$= \gamma_n^{-2} \left( \sum_{j=0}^{t_n} \sum_{s=t_n}^{\infty} \rho_n^{2j} c_{s-j}^2 + 2 \sum_{s=t_n}^{\infty} \sum_{j=0}^{t_n-1} \sum_{j'=j+1}^{t_n} \left| \rho_n^j c_{s-j} \rho_n^{j'} c_{s-j'} \right| \right) =: D_n + 2E_n$$

Consider

$$D_n \leq \gamma_n^{-2} \sum_{j=0}^{t_n} \rho_n^{2j} \sum_{l=t_n-j}^{\infty} c_l^2 = \gamma_n^{-2} \left( \sum_{j=0}^{\lfloor t_n/2 \rfloor} + \sum_{j=\lfloor t_n/2 \rfloor + 1}^{t_n} \right) \rho_n^{2j} \sum_{l=t_n-j}^{\infty} c_l^2 =: D'_n + D''_n.$$ 

**Lemma 6.4(iii)** is utilised in the subsequent approximations. First in view of the fact that $\gamma_n^2 = \kappa_n$, the first term

$$D'_n \leq \gamma_n^{-2} \sum_{j=0}^{\lfloor t_n/2 \rfloor} \rho_n^{2j} \sum_{l=t_n-j}^{\infty} c_l^2 \leq C \sum_{l=\lfloor t_n/2 \rfloor}^{\infty} c_l^2 = o(1).$$

Next, for some $0 < \delta < 1$ **Lemma 6.4(ii)-(iii)** yields

$$D''_n \leq \gamma_n^{-2} \sum_{j=\lfloor t_n/2 \rfloor + 1}^{t_n} \rho_n^{2j} \sum_{l=0}^{\infty} c_l^2 = O \left( e^{-2(\lfloor t_n/2 \rfloor + 1) \kappa_n} \right) + o \left( \kappa_n^{-1} e^{-2\delta \lfloor t_n/2 \rfloor \kappa_n} \right) = o(1).$$

Finally, for $n$ large we have

$$|E_n| = \gamma_n^{-2} \sum_{j=0}^{t_n-1} \sum_{j'=j+1}^{t_n} \rho_n^{2j} \sum_{s=t_n}^{\infty} |c_{s-j} c_{s-j'}| = \gamma_n^{-2} \sum_{j=0}^{t_n-1} \sum_{s=t_n}^{\infty} \rho_n^{2j} \sum_{l=s-t_n}^{\infty} |c_l| \sum_{s=2j}^{s-j-1} |c_l|$$

$$[l = s-j]$$

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and therefore \( \kappa \). Subsequently, we show that

\[
\int_0^1 e^{-2y} \left( \int_0^y e^{x} x^{-m} dx \right)^2 dy = \Gamma(2-2m) B(1-m, 2m-1) < \infty,
\]

where \( B(x,y) \) and \( \Gamma(x) \) are Beta and Gamma function respectively.

We start with the proof of (48). Notice that

\[
Var \left( x_{i_n}^+ (n) \right) - Var \left( x_{\bar{\varepsilon}_n}^+ (n) \right) = \left[ Var \left( x_{i_n}^+ (n) \right) - Var \left( x_{\bar{\delta}_n}^+ (n) \right) \right] I \{ \delta_n < t_n \}.
\]
Now for $\delta_n < t_n$,

$$\gamma_n^{-2} \left[ \text{Var} \left( x_{t_n}^+ (n) \right) - \text{Var} \left( x_{\epsilon n}^+ (n) \right) \right] = \gamma_n^{-2} \sum_{s=\delta_n}^{t_n-1} \left( \sum_{j=0}^{s} \rho_n^j c_{s-j} \right)^2$$

$$\leq \gamma_n^{-2} \sum_{s=\delta_n}^{t_n} \sum_{j=0}^{s} \rho_n^{2j} c_{s-j}^2 + 2 \gamma_n^{-2} \sum_{s=\delta_n}^{t_n} \sum_{j=0}^{s-1} \sum_{k=0}^{j+1} \rho_n^j c_{s-j} \rho_n^k c_{s-k} =: A_n + 2B_n.$$  

Let $\bar{\delta}_n := \lfloor \delta_n/2 \rfloor$. As $n \to \infty$, the second term

$$\gamma_n^2 B_n = \sum_{j=0}^{t_n} \sum_{s=\delta_n \vee (j+1)}^{t_n} \sum_{k=j+1}^{s} \left| \rho_n^j \rho_n^k c_{s-j} c_{s-k} \right| = \sum_{j=0}^{t_n-1} \sum_{k=j+1}^{t_n} \sum_{s=\delta_n \vee k}^{t_n} \rho_n^j \rho_n^k c_{s-j} c_{s-k}$$

$$\leq \sum_{j=0}^{t_n} \sum_{k=j+1}^{t_n} \sum_{s=\delta_n \vee k}^{t_n} \rho_n^j \rho_n^k c_{s-j} c_{s-k} = \left( \sum_{j=0}^{\delta_n} \sum_{k=j+1}^{\delta_n \vee k} + \sum_{j=0}^{t_n-1} \sum_{k=j+1}^{t_n} \sum_{s=\delta_n \vee k}^{t_n} \rho_n^j \rho_n^k c_{s-j} c_{s-k} \right) + \sum_{j=\delta_n+1}^{t_n} \sum_{k=j+1}^{t_n} \sum_{s=\delta_n \vee k}^{t_n} \rho_n^j \rho_n^k c_{s-j} c_{s-k}.$$  

The first term

$$\gamma_n^2 B'_n = \sum_{j=0}^{\delta_n} \sum_{k=j+1}^{\delta_n \vee k} \sum_{s=\delta_n \vee k}^{t_n} \rho_n^j \rho_n^k c_{s-j} c_{s-k} = \sum_{j=0}^{\delta_n} \sum_{k=j+1}^{\delta_n} \sum_{l=\delta_n \vee k - k}^{t_n-k} \rho_n^j \rho_n^k c_{l} c_{l+k}$$

$$\leq \sum_{j=0}^{\delta_n} \sum_{k=j+1}^{\delta_n} \sum_{l=\delta_n \vee k - k}^{\infty} \rho_n^j \rho_n^k c_{l}^2 \leq \sum_{j=0}^{\delta_n} \sum_{k=j+1}^{\delta_n} \sum_{l=\delta_n - \delta_n}^{\infty} \rho_n^j \rho_n^k c_{l}^2 \leq \sum_{j=0}^{\delta_n} \rho_n^j \sum_{k=j}^{\delta_n} \rho_n^k \sum_{l=\delta_n}^{\infty} c_{l}^2.$$
Hence, using Lemmas 6.3(iii) and 6.4(iii) we get

\[ B_n' \leq C \frac{\kappa_n}{\gamma_n^2} \sum_{l=\delta_n}^{\infty} c_l^2 \leq C \frac{\kappa_n^2}{\gamma_n^2} \sum_{l=\delta_n}^{\infty} c_l^2 \]

\[ = C \frac{\kappa_n^2}{\gamma_n^2} \sum_{l=\delta_n}^{\infty} l^{-2m} + o(1) = C \frac{\kappa_n^2}{\gamma_n^2} \frac{\delta_n^{1-2m}}{\delta_n} \sum_{l=\delta_n}^{\infty} \left( \frac{l}{\delta_n} \right)^{-2m} \]

\[ = C \frac{\kappa_n^2}{\gamma_n^2} \delta_n^{1-2m} \left[ \int_1^\infty x^{-2m} dx + o(1) \right] = o(1), \]

where the last approximation follows from the fact that

\[ \delta_n^{1-2m} \rho_n^2 \gamma_n^2 \sim \delta_n^{1-2m} \rho_n^2 \gamma_n^2 \sim \delta_n^{1-2m} = o(1). \]

Similarly, as \( n \to \infty \)

\[ B_n'' = \gamma_n^{-2} \sum_{j=0}^{\delta_n} \sum_{k=\delta_n+1}^{\infty} \sum_{s=\delta_n}^{\infty} \rho_{n,j}^k c_{s-j} c_{s-k} = \gamma_n^{-2} \sum_{j=0}^{\delta_n} \sum_{k=\delta_n+1}^{\infty} \sum_{s=\delta_n}^{\infty} \rho_{n,j}^k \left| c_{s-j} c_{s-k} \right| \]

\[ \leq \sum_{l=0}^{\infty} c_l^2 \gamma_n^{-2} \sum_{j=0}^{\delta_n} \sum_{k=\delta_n+1}^{\infty} \rho_{n,j}^k \leq C \frac{\kappa_n}{\gamma_n^2} \sum_{j=0}^{\delta_n} \rho_{n,j}^k \leq C \frac{\kappa_n}{\gamma_n^2} \sum_{j=0}^{\infty} \rho_{n,j}^k = C \frac{\kappa_n}{\gamma_n^2} \rho_n^2 = C \frac{\kappa_n}{\gamma_n^2} \rho_n^2 = o(1), \]

where the last approximation above follows from Lemma 6.3(iii) and the fact that \( \delta_n^{-1} \kappa_n \ln(\kappa_n) \to 0 \) (which imply \( \kappa_n \rho_n^2 = o(1) \)). Similarly,

\[ |B_n'''| = \gamma_n^{-2} \sum_{j=\delta_n+1}^{\infty} \sum_{k=j+1}^{\infty} \sum_{s=\delta_n}^{\infty} \rho_{n,j}^{k} c_{s-j} c_{s-k} \leq \gamma_n^{-2} \sum_{j=\delta_n+1}^{\infty} \sum_{k=j+1}^{\infty} \rho_{n,j}^{k} \sum_{s=\delta_n}^{\infty} c_s^2 \]

\[ \leq C \frac{\kappa_n}{\gamma_n^2} \sum_{j=\delta_n}^{\infty} \rho_{n,j}^{2j} = C \frac{\kappa_n^2}{\gamma_n^2} \rho_n^{2\delta_n} = C \left( \frac{\kappa_n \rho_n^3}{\gamma_n^2} \right)^2 = o(\gamma_n^{-2}). \]

Finally, using similar arguments we can also show that \( \gamma_n^{-2} A_n = o(1) \). Therefore, (19) holds. Next, we show (19). Consider

\[ Var \left( x_{\hat{c}_n}^\infty (n) \right) - Q_{\hat{c}_n} (t_n) = \left[ Var \left( x_{\hat{c}_n}^\infty (n) \right) - Q_{\hat{c}_n} (t_n) \right] I \{ \delta_n < t_n \}. \]
Recall that \( Q_\kappa (t_n) := \sum_{s=t_n}^{\infty} \left( \sum_{j=0}^{\delta_n-1} \rho_n^j c_{s-j} \right)^2 \) and therefore \( \Var (x_{\kappa n}^- (n)) = Q_{\kappa n} (t_n). \) For \( \delta_n < t_n, \) write

\[
\gamma_n^{-2} \Var (x_{\kappa n}^- (n)) = \gamma_n^{-2} \sum_{s=t_n}^{\infty} \left( \sum_{j=0}^{\delta_n-1} \rho_n^j \right)^2 \sum_{j=\delta_n}^{t_n-1} \rho_n^{2j} c_{s-j}^2 \\
+ \frac{\gamma_n^{-2}}{2} \left( \sum_{j=0}^{\delta_n-1} \rho_n^j \sum_{k=\delta_n}^{t_n-1} \rho_n^k c_{s-j-k} \right) \sum_{s=t_n}^{\infty} \rho_n^j c_{s-j} \rho_n^k c_{s-k} \\
\]

Hence,

\[
\gamma_n^{-2} \Var (x_{\kappa n}^- (n)) - \gamma_n^{-2} Q_\kappa (t_n) = \gamma_n^{-2} \sum_{s=t_n}^{\infty} \sum_{j=\delta_n}^{t_n-1} \rho_n^{2j} c_{s-j}^2 \\
+ 2\gamma_n^{-2} \left( \sum_{j=0}^{\delta_n-1} \rho_n^j \sum_{k=\delta_n}^{t_n-1} \rho_n^k c_{s-j-k} \right) \sum_{s=t_n}^{\infty} \rho_n^j c_{s-j} \rho_n^k c_{s-k} := C_n + 2D_n.
\]

As \( n \to \infty, \) the second term above

\[
|D_n| \leq \gamma_n^{-2} \left( \sum_{j=0}^{\delta_n} \rho_n^j \sum_{k=\delta_n}^{t_n-1} \rho_n^k c_{s-j-k} \right) \sum_{s=t_n}^{\infty} \rho_n^j c_{s-j} \rho_n^k c_{s-k} \\
\leq \sum_{s=0}^{\infty} c_s \gamma_n^{-2} \left[ \sum_{j=0}^{\delta_n} \rho_n^j \sum_{k=\delta_n}^{t_n-1} \rho_n^k + \sum_{j=\delta_n}^{t_n} \rho_n^j \sum_{k=\delta_n}^{t_n-1} \rho_n^k \right] \\
\leq C_{\delta_n} \left[ \rho_n^\delta c_{s-j} \sum_{j=0}^{\delta_n} \rho_n^j + \sum_{j=\delta_n}^{t_n} \rho_n^j \sum_{k=\delta_n}^{t_n-1} \rho_n^k \right] \leq C_{\delta_n} \left[ \kappa_n \rho_n^\delta c_{s-j} \rho_n^2 \right] = o(1),
\]

where we have used Lemma 6.3(iii) and the fact that for \( n \) large, \( \kappa_n \rho_n^\delta \leq \kappa_n \rho_n^2 = o(1). \) Similarly, we can show that \( C_n = o(1). \) Therefore, (49) holds.

Next, we show (50). Write

\[
\gamma_n^{-2} \left| \Var (x_{\kappa n}^- (n)) - \sum_{s=0}^{t_n-1} \left( \sum_{j=0}^{s} e^{-\frac{s}{\kappa_n} c_{s-j}} \right)^2 \right| = \gamma_n^{-2} \left| \sum_{s=0}^{t_n-1} \left( \sum_{j=0}^{s} \rho_n^j \right) \left( \sum_{j=0}^{s} e^{-\frac{s}{\kappa_n} c_{s-j}} \right)^2 \right| \\
= \gamma_n^{-2} \left| \sum_{s=0}^{t_n-1} \left( \sum_{j=0}^{s} \rho_n^j c_{s-j} - \sum_{j=0}^{s} e^{-\frac{s}{\kappa_n} c_{s-j}} \right) \left( \sum_{j=0}^{s} \rho_n^j c_{s-j} + \sum_{j=0}^{s} e^{-\frac{s}{\kappa_n} c_{s-j}} \right) \right|
\]

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= γ_n^2 \left| \sum_{s=0}^{\varepsilon_n - 1} \left( \sum_{j=0}^{s} (\rho_n^j - e^{-\frac{j}{\kappa_n}}) c_{s-j} \right) \left( \sum_{j=0}^{s} \rho_n^j c_{s-j} + \sum_{j=0}^{s} e^{-\frac{j}{\kappa_n}} c_{s-j} \right) \right|

\leq γ_n^2 \varepsilon_n \left( \sum_{j=0}^{\varepsilon_n} \left| \rho_n^j - e^{-\frac{j}{\kappa_n}} \right| c_{s-j} \right) \left( \sum_{j=0}^{\varepsilon_n} \left| \rho_n^j - e^{-\frac{j}{\kappa_n}} \right| c_{s-j} \right) + 2 \sum_{j=0}^{\varepsilon_n} \left| e^{-\frac{j}{\kappa_n}} c_{s-j} \right|

= γ_n^2 \varepsilon_n \left( \sum_{j=0}^{\varepsilon_n} \left| \rho_n^j - e^{-\frac{j}{\kappa_n}} \right| c_{s-j} \right)^2 + 2γ_n^2 \varepsilon_n \left( \sum_{j=0}^{\varepsilon_n} \left| \rho_n^j - e^{-\frac{j}{\kappa_n}} \right| c_{s-j} \right) \left( \sum_{j=0}^{\varepsilon_n} \left| e^{-\frac{j}{\kappa_n}} c_{s-j} \right| \right)

=: E_n + 2F_n.

By Lemma 6.4(i), Euler summation and using similar arguments to those used for the proof of (13) in part (i), the first term is

\[ E_n \leq \left( \frac{\varepsilon_n}{\kappa_n^2} \right)^2 \gamma_n^{-2} \varepsilon_n \sum_{l=0}^{\varepsilon_n} \left| c_l \right|^2 = \left( \frac{\varepsilon_n}{\kappa_n} \right)^2 \gamma_n^{-3/2m} \varepsilon_n \sum_{l=0}^{\varepsilon_n} \left( \frac{1}{\varepsilon_n} \sum_{l=0}^{\varepsilon_n} \left( \frac{l}{\varepsilon_n} \right)^{-m} \right)^2 + o(1) \]

\[ = \frac{\varepsilon_n^{5-2m}}{\kappa_n^{7-2m}} \left[ \int_{0}^{1} \left( \int_{0}^{\gamma_n} x^{-m} dx \right)^2 dy + o(1) \right] = O \left( \frac{\varepsilon_n^{5-2m}}{\kappa_n^{7-2m}} \right) = (1) o(1), \quad (52) \]

where = (1) follows from the definition of \( \varepsilon_n \). Similarly,

\[ F_n \leq \frac{\varepsilon_n}{\kappa_n^2} \gamma_n^{-2} \varepsilon_n \sum_{l=0}^{\varepsilon_n} \left| c_{s-j} \right| \sum_{j=0}^{s} \left( e^{-\frac{j}{\kappa_n}} c_{s-j} \right) \]

\[ \leq \sqrt{\left( \frac{\varepsilon_n}{\kappa_n^2} \right)^2 \gamma_n^{-2} \varepsilon_n \sum_{l=0}^{\varepsilon_n} \left| c_l \right|^2} \gamma_n^{-2} \varepsilon_n \sum_{s=0}^{\varepsilon_n} \left( \sum_{j=0}^{s} \left| e^{-\frac{j}{\kappa_n}} c_{s-j} \right| \right)^2 \leq \sqrt{O \left( \frac{\varepsilon_n^{5-2m}}{\kappa_n^{7-2m}} \right)} = o(1), \]

where we have used (52) the fact that as \( n \to \infty \)

\[ \gamma_n^{-2} \varepsilon_n \left( \sum_{j=0}^{s} \left| e^{-\frac{j}{\kappa_n}} c_{s-j} \right| \right)^2 \to \int_{y=0}^{\infty} e^{-2y} \left( \int_{x=0}^{y} e^x x^{-m} dx \right)^2 dy, \]

which is justified by the arguments that lead to (53) below. In view of the above Euler summation gives as \( n \to \infty \)
\[ \gamma_n^{-2} \text{Var} \left( x^+_{\varepsilon_n} (n) \right) = \gamma_n^{-2} \sum_{s=0}^{\varepsilon_n-1} \left( \sum_{j=0}^{s} e^{-\frac{1}{\kappa_n} c_{s-j}} \right)^2 + o(1) = \gamma_n^{-2} \sum_{s=0}^{\varepsilon_n} e^{-\frac{1}{\kappa_n}} \left( \sum_{k=0}^{s} e^{\frac{k}{\kappa_n}} c_k \right)^2 + o(1) \]

\[ [k = s - j \rightarrow j = s - k] \]

\[ = \kappa_n^{-\varepsilon_n} \left( \frac{1}{\kappa_n} \sum_{k=1}^{\varepsilon_n} \left( \frac{k}{\kappa_n} \right)^{-m} \right)^2 + o(1) = \int_0^{\varepsilon_n/\kappa_n} e^{-2y} \left( \int_{1/\kappa_n}^{y} e^x x^{-m} dx \right)^2 dy + o(1) \]

\[ = \int_0^{\infty} e^{-2y} \left( \int_{1/\kappa_n}^{\infty} e^x x^{-m} dx \right)^2 dy + o(1), \quad (53) \]

and this shows (50). Next, we show (51). Using (49) write

\[ \gamma_n^{-2} \text{Var} \left( x^-_{t_n} (n) \right) - \gamma_n^{-2} \sum_{s=t_n}^{\infty} \left( \sum_{j=0}^{s} e^{-\frac{1}{\kappa_n} c_{s-j}} \right)^2 = \gamma_n^{-2} Q_{\varepsilon_n} (t_n) - \gamma_n^{-2} \sum_{s=t_n}^{\infty} \left( \sum_{j=0}^{s} e^{-\frac{1}{\kappa_n} c_{s-j}} \right)^2 + o(1) \]

\[ = \gamma_n^{-2} \sum_{s=t_n}^{\infty} \left[ \sum_{j=0}^{\varepsilon_n-1} \rho^j_{s-j} - \sum_{j=0}^{\varepsilon_n-1} e^{-\frac{1}{\kappa_n} c_{s-j}} \right] \left[ \sum_{j=0}^{\varepsilon_n-1} \rho^j_{s-j} \right] + \sum_{j=0}^{\varepsilon_n-1} e^{-\frac{1}{\kappa_n} c_{s-j}} \]

\[ = \gamma_n^{-2} \sum_{s=t_n}^{\infty} \left[ \sum_{j=0}^{\varepsilon_n-1} \left( \rho^j_{s-j} - e^{-\frac{1}{\kappa_n} c_{s-j}} \right) \right] \left[ \sum_{j=0}^{\varepsilon_n-1} \left( \rho^j_{s-j} - e^{-\frac{1}{\kappa_n} c_{s-j}} \right) + 2 \sum_{j=0}^{\varepsilon_n-1} e^{-\frac{1}{\kappa_n} c_{s-j}} \right] \]

\[ = \gamma_n^{-2} \sum_{s=t_n}^{\infty} \sum_{j=0}^{\varepsilon_n-1} \left( \rho^j_{s-j} - e^{-\frac{1}{\kappa_n} c_{s-j}} \right)^2 + 2 \gamma_n^{-2} \sum_{s=t_n}^{\infty} \left[ \sum_{j=0}^{\varepsilon_n-1} \left( \rho^j_{s-j} - e^{-\frac{1}{\kappa_n} c_{s-j}} \right) \right] \sum_{j=0}^{\varepsilon_n-1} e^{-\frac{1}{\kappa_n} c_{s-j}} \]

\[ := G_n + 2H_n. \]

Using again Lemma 6.4(i) and Euler summation we have

\[ G_n \leq \gamma_n^{-2} \sum_{s=t_n}^{\infty} \left[ \sum_{j=0}^{\varepsilon_n-1} \left( \rho^j_{s-j} - e^{-\frac{1}{\kappa_n} c_{s-j}} \right) \right]^2 \leq C \left( \frac{\varepsilon_n}{\kappa_n^2} \right)^2 \gamma_n^{-2} \sum_{s=t_n}^{\infty} \left( \sum_{j=0}^{\varepsilon_n-1} c_{s-j} \right)^2 \]

\[ \leq C \left( \frac{\varepsilon_n}{\kappa_n^2} \right)^2 \gamma_n^{-2} \sum_{s=t_n}^{\infty} \left( \sum_{l=s-\varepsilon_n}^{s} \left| c_l \right| \right)^2 = C \frac{\varepsilon_n^{-5} - 2m}{\kappa_n^{-2m}} \sum_{s=t_n}^{\infty} \left( \frac{1}{\varepsilon_n} \right)^{s-\varepsilon_n} \left( \frac{l}{\varepsilon_n} \right)^{-m} + o(1) \]

\[ [l = s - j] \]
where we have used the facts that
\[ \varepsilon = C\varepsilon \]

Similarly,
\[ \sum \epsilon_n \text{Var} \left( x_n^{-} \right) \]
\[ \leq \sqrt{O \left( \frac{\varepsilon_n^{5-2m}}{(k_n^{-})^{7-2m}} \right)} = o(1), \]

where we have used the facts that \( \varepsilon_n^{5-2m} (k_n^{-})^{7-2m} = o(1) \) and
\[ \gamma_n^{-2} \sum_{s=t_n}^{\infty} \left( \sum_{j=0}^{\epsilon_n} e^{-\frac{j}{\varepsilon_n} c_{s-j}} \right)^2 = \int_{y=t_n/\kappa_n}^{\infty} e^{-2y} \left( \int_{x=0}^{y} e^{x} x^{-m} dx \right)^2 dy + o(1) = o(1), \]

which is justified by the arguments that lead to (54) below. In view of the above, Euler summation gives as \( n \to \infty \)
\[ \gamma_n^{-2} \text{Var} \left( x_n^{-} \right) = \gamma_n^{-2} Q_{x_n} (t_n) + o(1) = \gamma_n^{-2} \sum_{s=t_n}^{\infty} \left( \sum_{j=0}^{\epsilon_n} e^{-\frac{j}{\varepsilon_n} c_{s-j}} \right)^2 + o(1) \]
\[ [k = s - j \to j = s - k] \]
\[ = \gamma_n^{-2} \sum_{s=t_n}^{\infty} \left( \sum_{k=s-\epsilon_n}^{s} e^{-\frac{s-k}{\varepsilon_n} c_k} \right)^2 = \kappa_n^{-(3-2m)} \sum_{s=t_n}^{\infty} e^{-2\frac{s}{\varepsilon_n}} \left( \sum_{k=s-\epsilon_n}^{s} e^{\frac{k}{\varepsilon_n} c_k} \right)^2 + o(1) \]
\[ = \frac{1}{\kappa_n} \sum_{s=t_n}^{\infty} e^{-2\frac{s}{\varepsilon_n}} \left( \frac{1}{\kappa_n} \sum_{k=s-\epsilon_n}^{s} e^{\frac{k}{\varepsilon_n} (\frac{k}{\kappa_n})^{-m}} \right)^2 \]
\[ = \int_{y=t_n/\kappa_n}^{\infty} e^{-2y} \left( \int_{x=0}^{y} e^{x} x^{-m} dx \right)^2 dy + o(1) \]
\[
\leq \int_{y=t_{n}/\kappa_{n}}^{\infty} e^{-2y} \left( \int_{x=0}^{y} e^{x-m} dx \right)^{2} dy = o(1), \tag{54}
\]
where the last approximation shown above is due to dominated convergence and the fact that \( t_{n}/\kappa_{n} \to \infty \) (recall that \( \int_{y=0}^{\infty} e^{-2y} \left( \int_{x=0}^{y} e^{x-m} dx \right)^{2} dy < \infty \), by Lemma 7.3). ■

**Proof of Lemma CLT.** The result follows from Lemma LVAR and arguments similar to those used in the proof of Proposition 2.1(i). ■

Appendix B2b (proofs of technical lemmas of Appendix A)

**Proof of Lemma 6.1.** Fix, \( \rho \in (0, 1/2) \). Then there is some 0 < \( \delta_{\rho} < 1 \) and some \( \vartheta_{\rho} > 0 \) (e.g. Feller, 1971, p. 516) such that

\[
|\psi_{k}(\lambda)| \leq \begin{cases} 
 e^{-\rho \lambda^2}, & |\lambda| \leq \delta_{\rho} \\
 e^{-\vartheta_{\rho}}, & |\lambda| > \delta_{\rho} 
\end{cases}.
\tag{55}
\]

Set \( \rho_{*} := \rho \wedge \vartheta_{\rho} \). It follows from (55) that

\[
|\psi_{k}(\lambda)| \leq e^{-\rho_{*} \lambda^2} \left\{ |\lambda| \leq \delta_{\rho} \right\} + e^{-\varrho_{*} \lambda^2} \left\{ |\lambda| > \delta_{\rho} \right\} \\
= e^{-\rho_{*} \lambda^2} \{ |\lambda| \leq \delta_{\rho} \} + e^{-\varrho_{*} \lambda^2} \{ |\lambda| > \delta_{\rho} \} \\
\leq e^{-\rho_{*} \lambda^2} \{ |\lambda| \leq 1 \} + e^{-\varrho_{*} \lambda^2} \{ |\lambda| > 1 \} = e^{-\rho_{*} (\lambda^2 \wedge 1)},
\]
as required. ■

**Proof of Lemma 6.2 Part (i) & (ii):** Without loss of generality set \( \theta = 0 \). It can be easily seen that the arguments below hold true for any \( \theta \geq 0 \). First, suppose that Assumption LL1 holds. By Lemma 7.1 as \( k \to \infty \)

\[
\delta_{k}^{-1} \eta_{k} |a_{k}(n)| = \sqrt{\frac{k}{\ell(k)}} \left| \sum_{j=0}^{k} \phi_{j} c_{k-j} \right| = \left| \sum_{j=0}^{\infty} c_{j} \right| + o(1).
\]

In view of this and the fact that \( \sum_{j=0}^{\infty} |c_{j}| < \infty \) and \( \left| \sum_{j=0}^{\infty} c_{j} \right| > 0 \), the result for part (i) and (ii) follows immediately.
Under Assumption LL2 using Lemma 6.3(ii), we get as $k \to \infty$
\[
\delta_t^{-1} \eta_k a_k (n) = \frac{k^{1/2}}{\ell(k)} \sum_{j=0}^{k} c_j = - \frac{k^{1/2}}{\ell(k)} \sum_{j=k+1}^{\infty} c_j = - \frac{k^{1/2}}{\ell(k)} \sum_{j=k+1}^{\infty} \ell(j) j^{-3/2} + o(1)
\]
\[
= - \int_{1}^{\infty} x^{-3/2} dx + o(1) = - \frac{1}{1 - 3/2} [x^{-1/2}]_{1}^{\infty} + o(1) = -2,
\]
and the result follows. Notice that the approximation in the first line above is justified as follows. Fix $\varepsilon > 0$. Then for $k$ large enough $\frac{c_i}{(\ell(j) j^{-3/2} - 1)} < \varepsilon$. Hence, as $k \to \infty$ first and then as $\varepsilon \to 0$
\[
\frac{k^{1/2}}{\ell(k)} \sum_{j=k+1}^{\infty} |c_j - \ell(j) j^{-3/2}| = \frac{k^{1/2}}{\ell(k)} \sum_{j=k+1}^{\infty} \left| \frac{c_j}{\ell(j) j^{-3/2}} - 1 \right| \ell(j) j^{-3/2}
\]
\[
= \varepsilon \frac{k^{1/2}}{\ell(k)} \sum_{j=k+1}^{\infty} \ell(j) j^{-3/2} \to \varepsilon \int_{1}^{\infty} x^{-3/2} dx + o(1) \xrightarrow{\varepsilon \to 0} o(1),
\]
as required.

Suppose that Assumption LL3 holds and without loss of generality set $\eta_k = 1$. Choose $n_0$ large enough such that $0 < \rho_n < 1$ for $n \geq n_0$. Hence, $0 < \rho_n^{k-j} \leq 1$ for all $0 \leq k - j$ and $n \geq n_0$. We get
\[
\delta_t^{-1} \sup_{0 \leq k} \eta_k |a_k (n)| = \sup_{0 \leq k} \left| \sum_{j=0}^{k} \rho_n^{k-j} c_j \right| \leq \sup_{0 \leq k} \left| \sum_{j=0}^{k} \rho_n^{k-j} \right| \left| c_j \right| \leq \sum_{j=0}^{\infty} \left| c_j \right| < \infty,
\]
and this proves the upper bound in both part (i) and (ii).

For the lower bound note again that $0 < \rho_n < 1$ for $n$ large enough. Therefore as $n \to \infty$
\[
\Delta_n := \sup_{n \geq t \geq 1, 0 \leq k \leq \kappa_f} \delta_t^{-1} \eta_k \left| a_k (n) - \rho_n^{k} \sum_{j=0}^{k} c_j \right| = \sup_{t \geq 1, 0 \leq k \leq \kappa_f} \left| \sum_{j=0}^{k} \rho_n^{k-j} c_j - \rho_n^{k} \sum_{j=0}^{k} c_j \right|
\]
\[
\leq \sup_{n \geq t \geq 1, 0 \leq k \leq \kappa_f} \left| \rho_n^{k} \right| \left| \sum_{j=0}^{k} \rho_n^{-j} c_j - \sum_{j=0}^{k} c_j \right|
\]
\[
\leq \sup_{n \geq t \geq 1} \sum_{j=0}^{\kappa_f} \left| \rho_n^{-j} - 1 \right| \left| c_j \right| \leq \sum_{j=0}^{\infty} \left| \rho_n^{-j} - 1 \right| 1 \left\{ j \leq \kappa_f \kappa_n \right\} \left| c_j \right| = o(1), \tag{56}
\]
where $\infty > \kappa_* := \sup_{1 \leq t \leq n} \kappa_t / \kappa_n$ and the last approximation is due to dominated convergence. To see this note that for $n$ large enough, $|\rho_n^{-j} - 1| \{j \leq \kappa_n\}$ is a finite. Further, it is bounded because as $n \to \infty$

$$|\rho_n^{-j} - 1| \{j \leq \kappa_n\} \leq |\rho_n^{-\kappa_n} + 1| = |e + 1| + o(1).$$

Therefore, noting that $\lim_{n \to \infty} \rho_n^{-j} = 1$ for all $j$ fixed, dominated convergence yields $\Delta_n \to 0$. Next, recall that $\sum_{j=0}^{\infty} c_j \neq 0$. Hence, there is $k_1 > 0$ such for all $k \geq k_1$, $|\sum_{j=0}^{k} c_j| \geq |\sum_{j=0}^{\infty} c_j| / 2 > 0$. Choose $n_1$ large enough such that $|\kappa_t / 2| \geq k_1$ for $t \geq n_1$. Hence, in view of the uniform convergence result of (56) as $n \to \infty$ we have

$$\inf_{n_1 \leq t \leq n, |\kappa_t / 2| \leq |\kappa_t|} \delta_t^{-1} |\eta_k a_k(n)| = \inf_{n_1 \leq t \leq n, |\kappa_t / 2| \leq |\kappa_t|} |\rho_n^{-k} | \sum_{j=0}^{k} c_j | + o(1)$$

$$\geq |\rho_n^{\kappa_n}| \inf_{k \leq k_1} \sum_{j=0}^{k} c_j \geq |\rho_n^{\kappa_n}| \sum_{j=0}^{\infty} c_j / 2 = e^{-\kappa_n} \sum_{j=0}^{\infty} c_j / 2 + o(1),$$

and the result follows from the fact that the limit term above is strictly positive.

Finally, suppose that Assumption LL4 holds. As before without loss of generality, set $\eta_k = 1$. We start with the proof for the upper bound of part (i). Again note that $0 < \rho_n < 1$ for $n$ large. Hence, for $n \geq t$ as we get as $t \to \infty$

$$\delta_t^{-1} \sup_{0 \leq k \leq \gamma_t} \eta_k |a_k(n)| = \kappa_t^{-(1-m)} \sup_{0 \leq k \leq |\kappa_t|} \sum_{j=0}^{k} \rho_n^{-j} c_j \leq \kappa_t^{-(1-m)} \sup_{0 \leq k \leq |\kappa_t / 2|} \sum_{j=0}^{k} \rho_n^{-j} |c_j|

\leq \kappa_t^{-(1-m)} \sum_{j=0}^{k} |c_j| = \kappa_t^{-(1-m)} \sum_{j=1}^{\gamma_t} \left( \frac{1}{\kappa_t} \right)^{-m} + o(1) = \int_0^1 x^{-m} dx + o(1),$$

where the last approximation above follows from Lemma 6.3(i). For the upper bound of part (ii) we have from Lemma 7.2

$$\delta_n^{-1} \sup_{0 \leq k \leq n} \eta_k |a_k(n)| = \kappa_n^{-(1-m)} \sup_{0 \leq k \leq n} \sum_{j=0}^{k} \rho_n^{-j} c_j = \kappa_n^{-(1-m)} \sup_{0 \leq k \leq n} \int_0^{k / \kappa_n} e^{-\left( \frac{\rho_n^{-j}}{k / \kappa_n} \right)} x^{-m} dx + o(1)$$

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\[
\leq \kappa_n^{-(1-m)} \left( \sup_{0 \leq k \leq \kappa_n} + \sup_{\kappa_n \leq k \leq n} \right) \int_0^{k/\kappa_n} e^{-(\frac{x}{\kappa_n}-x)} \, dx
\]
\[
\leq \kappa_n^{-(1-m)} \left[ \int_0^1 ex^{-m} \, dx + \kappa_n^{-(1-m)} \sup_{k \geq \kappa_n} \int_1^{k/\kappa_n} e^{-(\frac{k}{\kappa_n}-x)} x^{-m} \, dx \right]
\]
\[
= \kappa_n^{-(1-m)} \left[ C + \sup_{k \geq \kappa_n} \left[ e^{-\left(\frac{k}{\kappa_n}-x\right)} \right]^{k/\kappa_n} \right] = O\left( \kappa_n^{-(1-m)} \right) = o(1). \tag{57}
\]

For the lower bound note that by Lemma 7.2 for \( n \geq t \) and as \( t \to \infty \) we have

\[
\inf_{\frac{k}{\kappa_n} \leq k \leq \kappa_n} \delta_{t}^{-1} \eta_k |a_k(n)| = \inf_{\frac{k}{\kappa_n} \leq k \leq \kappa_n} \left| \frac{1}{\kappa_t^{1-m}} \sum_{j=0}^{k} \rho_n^{k-j} c_j \right|
\]
\[
= \inf_{\frac{k}{\kappa_n} \leq k \leq \kappa_n} \int_0^{k/\kappa_n} e^{-\left(\frac{k}{\kappa_n}-x\right)} x^{-m} \, dx + o(1) \geq C > 0.
\]

**Part (iii):** Under Assumption LL1-LL3 we have as \( n \to \infty \)

\[
\gamma_n^{-1} \sup_{1 \leq t \leq n, k \geq t} |a_{k,t}^{-}(n)| \leq \gamma_n^{-1} \sup_{1 \leq t \leq n, k \geq t} \sum_{j=0}^{t} |\phi_j(n) c_{k-j}| \leq \gamma_n^{-1} C \sum_{j=0}^{\infty} |c_j| = o(1).
\]

Under Assumption LL4 using Lemma 7.2 we get

\[
\gamma_n^{-1} \sup_{1 \leq t \leq n, k \geq t} |a_{k,t}^{-}(n)| = \gamma_n^{-1} \sup_{1 \leq t \leq n, k \geq t} \left| \sum_{j=0}^{t} \phi_j(n) c_{k-j} \right|
\]
\[
= \kappa_n^{-1/2} \left( \sup_{1 \leq t \leq n, k \geq t} \int_{(k-t)/\kappa_n}^{k/\kappa_n} e^{-\left(\frac{k}{\kappa_n}-x\right)} x^{-m} \, dx + o(1) \right)
\]
\[
\leq \kappa_n^{-1/2} \left( \sup_{1 \leq k \leq \kappa_n} + \sup_{k \geq \kappa_n} \right) \int_0^{k/\kappa_n} e^{-\left(\frac{k}{\kappa_n}-x\right)} x^{-m} \, dx = O(\kappa_n^{-1/2}) = o(1),
\]

where the last equality follows from the same argument that leads to (57) above.

**Proof of Lemma 6.3** Note that by Lemma 1.3.2. In Bingham et al. (1987) there is some \( x_0 > 0 \) such that \( \varsigma(x) \) is locally bounded for all \( x \geq x_0 \). We start with the proof of part (i).
(i) Without loss of generality set \( s = 0 \). The result for \( s > 0 \) follows immediately from the arguments shown below. Note that \(|\varphi_j / j^l \zeta(j) - 1| < \varepsilon\) for all \( j \geq N_\varepsilon \) and some \( N_\varepsilon \geq x_0 \). Further, for all \( j \) large enough \( (j \geq N_\varepsilon \text{ say}) \) \( \zeta(j) \) is strictly positive and finite. Hence, as \( n \to \infty \) first, and then as \( \varepsilon \to 0 \)

\[
\left| \frac{1}{n^{1+l} \zeta(n)} \sum_{j=1}^{[nr]} \varphi_j - \frac{1}{n^{1+l} \zeta(n)} \sum_{j=1+\delta}^{[nr]} j^l \zeta(j) \right| \\
\leq \frac{1}{n^{1+l} \zeta(n)} \sum_{j=1+\delta}^{[nr]} |\varphi_j - j^l \zeta(j)| + \frac{\max_{1 \leq j \leq N_\varepsilon} |\varphi_j|}{n^{1+l} \zeta(n)} \to 0 = \frac{1}{n^{1+l} \zeta(n)} \sum_{j=1+\delta}^{[nr]} |\varphi_j - j^l \zeta(j)| + o(1)
\]

\[
\leq \frac{\varepsilon}{n^{1+l} \zeta(n)} \sum_{j=1+\delta}^{[nr]} j^l \zeta(j) = \varepsilon \left( \int_0^r x^l dx + o(1) \right) = o(1),
\]

where \( =_{(1)} \) is demonstrated by (58) below. Fix \( \delta \) such that \( r > \delta > 0 \) and consider

\[
\frac{1}{n^{1+l} \zeta(n)} \sum_{j=1+\delta}^{[nr]} j^l \zeta(j) = \frac{1}{n^{1+l} \zeta(n)} \sum_{j=\lceil n\delta \rceil +1}^{[nr]} j^l \zeta(j) + \frac{1}{n^{1+l} \zeta(n)} \sum_{j=1+\delta}^{\lceil n\delta \rceil} j^l \zeta(j) =: S_{n,\delta} + T_{n,\delta}.
\]

First, consider

\[
\left| S_{n,\delta} - \frac{1}{n^{1+l}} \sum_{j=\lceil n\delta \rceil +1}^{[nr]} j^l \right| \leq \frac{1}{n} \sum_{j=\lceil n\delta \rceil +1}^{[nr]} \left( \frac{j}{n} \right)^l \left| \frac{j \zeta(j)}{\zeta(n)} - 1 \right|.
\]

Now notice that by the uniform convergence Theorem for regularly varying functions (e.g. Theorem 1.5.2. in Bingham et al., 1987) as \( n \to \infty \)

\[
\sup_{[n\delta]+1 \leq j \leq n} \left| \frac{j \zeta(j)}{\zeta(n)} - 1 \right| = \sup_{n \delta+1 \leq x \leq 1} \left| \frac{\zeta(x)}{\zeta(n)} - 1 \right| \leq \sup_{\delta \leq x \leq 1} \left| \frac{\zeta(x)}{\zeta(n)} - 1 \right| = o(1).
\]

In view of this Euler summation gives as \( n \to \infty \) first, and then as \( \delta \to 0 \)

\[
S_{n,\delta} = \int_\delta^r x^l dx + o(1) = \int_0^r x^l dx + o(1). \tag{58}
\]

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Hence, it suffices showing that $T_{n, \delta}$ is negligible. Set $0 < \eta < 1 + l$. By Potter’s Theorem (e.g. Theorem 1.5.6 in Bingham et al., 1987) there is a positive integer $X_{\eta}$ such that for all $n \geq j \geq X_{\eta}$

$$\zeta(j)/\zeta(n) \leq (n/j)^{\eta}.$$ 

Therefore, as $n \to \infty$ first, and then as $\delta \to 0$

$$|T_{n, \delta}| \leq \frac{1}{n^{1 + l_{\zeta}(n)}} \sum_{j=1+\bar{N}_{\varepsilon}}^{[n\delta]} j^{l} |\zeta(j)| = \frac{1}{n^{1 + l_{\zeta}(n)}} \left( \sum_{j=1+\bar{N}_{\varepsilon}}^{X_{\eta}} + \sum_{j=X_{\eta}+1}^{[n\delta]} \right) j^{l} |\zeta(j)|$$

$$= \frac{\max_{j \in N : 1 + \bar{N}_{\varepsilon} \leq j \leq X_{\eta}} |\zeta(j)|}{n^{1 + l_{\zeta}(n)}} + \frac{2}{n} \sum_{j=X_{\eta}+1}^{[n\delta]} \left( \frac{j}{n} \right)^{l-\eta} n \to \infty o(1) + 2 \int_{0}^{\delta} x^{l-\eta} dx \delta \to 0 o(1).$$

(ii) Using arguments similar those used above we have

$$\frac{1}{n^{1 + l_{\zeta}(n)}} \sum_{j=n}^{\infty} \varphi_{j} = \frac{1}{n^{1 + l_{\zeta}(n)}} \sum_{j=n}^{\infty} j^{l} \zeta(j) + o(1).$$

Next, let $R_{p}(x) := x^{p} \zeta(x)$ (i.e. regularly varying with index $p$). For $l < -1$, set $l_{1} = l - l_{2}$ with $l_{2} = (1 + l)/2$. Notice that $l_{2} < 0$ and $l_{1} < -1$. Therefore, as $n \to \infty$

$$\left| \frac{1}{n^{1 + l_{\zeta}(n)}} \sum_{j=n}^{\infty} j^{l} \zeta(j) - \frac{1}{n} \sum_{j=n}^{\infty} \left( \frac{j}{n} \right)^{l_{1}} \right| \leq \frac{1}{n} \sum_{j=n}^{\infty} \left| \frac{j^{l_{2} \zeta(j)}}{n^{l_{2} \zeta(n)}} - \left( \frac{j}{n} \right)^{l_{2}} \right| \left( \frac{j}{n} \right)^{l_{1}}$$

$$= \lim_{M \to \infty} \frac{1}{n} \sum_{j=n}^{M} \left| \frac{R_{l_{2}}(j)}{R_{l_{2}}(n)} - \left( \frac{j}{n} \right)^{l_{2}} \right| \left( \frac{j}{n} \right)^{l_{1}} \leq \sup_{1 \leq y < \infty} \left| \frac{R_{l_{2}}(yn)}{R_{l_{2}}(n)} - y^{l_{2}} \right| \lim_{M \to \infty} \frac{1}{n} \sum_{j=n}^{M} \left( \frac{j}{n} \right)^{l_{1}}$$

$$= \sup_{1 \leq y < \infty} \left| \frac{R_{l_{2}}(yn)}{R_{l_{2}}(n)} - y^{l_{2}} \right| \frac{1}{n} \sum_{j=n}^{\infty} \left( \frac{j}{n} \right)^{l_{1}} = (1) \sup_{1 \leq y < \infty} \left| \frac{R_{l_{2}}(yn)}{R_{l_{2}}(n)} - y^{l_{2}} \right| \left( \int_{1}^{\infty} x^{l_{1}} dx + o(1) \right) = (2) o(1),$$

where $(1)$ is due to Euler summation and $(2)$ due to Theorem 1.5.2. in Bingham et al. (1987).

(iii) Set $s(x) := \zeta(x)/x$. Recall that for all $x \geq x_{0}$, $s(x)$ is locally bounded. Due to the measurability and positiveness of $s(x)$ (c.f. p. 6 in Bingham et al., 1987), the integral

$$S(x) = \int_{x_{0}}^{x} \frac{s(u)}{u} du,$$
is defined in the Lebesgue sense and is finite due to the local boundedness of \( \zeta(x)/x \) on finite intervals far enough to the right. Further, \( S(x) \) is SV due to Theorem 1.5.9a in Bingham et al. (1987).

Define \( \underline{s}(x) := \inf_{x_0 \leq u \leq x} s(u) \). Clearly, \( \underline{s}(x) \) is decreasing. Further, by Bingham et al. (1987), Theorem 1.5.3.

\[ \underline{s}(x) \sim s(x) \]

and \( \underline{s}(x) \) is regularly varying of index \( -1 \). The latter implies that \( \underline{s}(x) \) is of the form \( \zeta(x)/x \), for some SV function \( \zeta(x) \) (e.g. Theorem 1.4.1. in Bingham et al., 1987). We first show that

\[ S(n) := \int_{x_0}^{n} \underline{s}(x)dx \sim S(n) \tag{59} \]

and then that

\[ \sum_{j=x_0}^{n} \frac{s(j)}{j} \sim \underline{S}(n), \tag{60} \]

which are sufficient for the requisite result. We start with the proof of (59).

First note that

\[ S(n)^{-1} \left| \int_{x_0}^{n} \underline{s}(u)du - \int_{x_0}^{n} s(u)du \right| = (1) S(n)^{-1} \left| \int_{x_0}^{n} \underline{s}(u)du - \int_{x_0}^{n} s(u)du \right| + o(1) \]

\[ \leq S(n)^{-1} \int_{x_0}^{n} \left| \frac{s(u)}{s(u)} - 1 \right| s(u)du \leq \sup_{S(n) \leq u \leq n} \left| \frac{s(u)}{s(u)} - 1 \right| S(n)^{-1} \int_{x_0}^{n} s(u)du \]

\[ = (2) o(1) [1 + o(1)], \]

where \( = (1) \) follows from the fact that \( S(n)^{-1} [S(S(n)) + \underline{S}(S(n))] = o(1) \) (recall that \( S \) and \( \underline{S} \) are SV) and \( = (2) \) is due to \( \underline{s}(x) \sim s(x) \).

For the proof of (60) note that

\[ \left| S(n)^{-1} \sum_{j=x_0}^{n} \frac{s(j)}{j} - 1 \right| \leq S(n)^{-1} \sum_{j=x_0}^{n} \left| \frac{s(j)}{j} - \frac{s(j)}{j} \right| + S(n)^{-1} \sum_{j=x_0}^{n} \frac{s(j)}{j} - 1 \]

\[ = S(n)^{-1} \sum_{j=x_0}^{n} \left| \frac{s(j)}{s(j)} - 1 \right| + S(n)^{-1} \sum_{j=x_0}^{n} \frac{s(j)}{j} - 1 \]

\[ = T_{1,n} + T_{2,n} \]

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First, we show that the second term $T_{2,n} = o(1)$. For a function $f$ denote its total variation on the interval $[a, b]$ as $V(f)_a^b$. Using integration by parts the second term is

$$T_{2,n} = S(n)^{-1} \left| \sum_{j=x_0}^{x} \frac{S(j)}{j} \right| = S(n)^{-1} \left| \sum_{j=x_0}^{x} \frac{S(j)}{j} - \int_{x_0}^{x} \frac{S(x)}{x} \, dx \right| + o(1)$$

$$= S(n)^{-1} \left| \int_{x_0}^{x} \frac{S(x)}{x} \, dx \right| + o(1) = \left| S(n)^{-1} \int_{x_0}^{x} (x - |x|) \, d\left( \frac{S(x)}{x} \right) \right|$$

$$\leq S(n)^{-1} V\left( \frac{S(x)}{x} \right)_{x_0}$$

$$= (3) S(n)^{-1} \left( \frac{S(x_0)}{x_0} - \frac{S(n)}{n} \right) = o(1) (1 + o(1)) = o(1),$$

where $= (3)$ follows from the fact that $S(x)/x$ is decreasing. Next, fix $\varepsilon > 0$ and choose an integer $N_\varepsilon \geq x_0$ such that $\left| \frac{S(j)}{j} - 1 \right| < \varepsilon$ for $j > N_\varepsilon$. Then as $n \to \infty$ first and then as $\varepsilon \to 0$

$$T_{1,n} \leq o(1) + \varepsilon S(n)^{-1} \sum_{j=N_\varepsilon}^{x} \frac{S(j)}{j} = o(1) + \varepsilon (1 + o(1)) \to 0,$$

where $= (4)$ is due to the fact that $T_{2,n} = o(1)$.

**Proof of Lemma 6.4** (i) Choose $n$ large enough such that $|c|/\kappa_n < 1/2$. Further, let $\bar{x}$ be a mean value such that $0 \geq \bar{x} \geq c/\kappa_n > -1/2$, for $n$ large. Hence, by the mean value for $n$ large enough we get

$$\rho_n^{qj} = \exp \{ qj \ln \rho_n \} = \exp \left\{ qj \ln \left( 1 + \frac{c}{\kappa_n} \right) \right\} = \exp \left\{ qj \left[ \frac{c}{\kappa_n} - \frac{1}{2! (1 + \bar{x})^2} \left( \frac{c}{\kappa_n} \right)^2 \right] \right\}.$$

In view of the above and the fact that $|e^x - 1| \leq |x|$ for all $x \leq 0$ we get for $n$ large enough

$$\left| \rho_n^{qj} - \exp \left\{ qj \frac{c}{\kappa_n} \right\} \right| = \left| \exp \left\{ qj \frac{c}{\kappa_n} - qj \frac{1}{2! (1 + \bar{x})^2} \left( \frac{c}{\kappa_n} \right)^2 \right\} - \exp \left\{ qj \frac{c}{\kappa_n} \right\} \right|$$
\[\exp \left\{ q \frac{c}{\kappa_n} \right\} \exp \left\{ -\frac{jq}{2! (1 + \bar{x})^2} \left( \frac{c}{\kappa_n} \right)^2 \right\} - 1 \leq \exp \left\{ q j \frac{c}{\kappa_n} \right\} \frac{jq}{2! (1 + \bar{x})^2} \left( \frac{c}{\kappa_n} \right)^2 \leq \exp \left\{ q j \frac{c}{\kappa_n} \right\} 2jq \left( \frac{c}{\kappa_n} \right)^2.\]

This shows part (a). Part (b) is an immediate consequence of the above.

(ii) It follows from part (i) that for \( j = \lambda_n \) and \( n \) large

\[
\kappa_n e^{q^2 \frac{\lambda_n}{\kappa_n}} \left| \rho_n^{q \lambda_n} - \exp \left\{ \frac{q \lambda_n c}{\kappa_n} \right\} \right| \leq \kappa_n e^{q^2 \frac{\lambda_n}{\kappa_n}} 2 \exp \left\{ \frac{q \lambda_n c}{\kappa_n} \right\} \left( \frac{c}{\kappa_n} \right)^2 \leq 2 \exp \left\{ q \left( c + \delta \right) \right\} = 2 \exp \left\{ q \left( c + \delta \right) + o(1) \right\} = o(1).
\]

(iii) Let \( \tau_n \leq k_n \). As \( n \to \infty \)

\[
\sum_{t=1}^{k_n} \rho_n^{q \tau_t} - \sum_{t=1}^{\tau_n} \rho_n^{q \tau_t} = \rho_n \frac{1 - \rho_n^{q \kappa_n}}{1 - \rho_n} - \rho_n \frac{1 - \rho_n^{q \tau_n}}{1 - \rho_n} = \rho_n \frac{1 - \rho_n}{1 - \rho_n} \left( \rho_n^{q \tau_n} - \rho_n^{q \kappa_n} \right) = o \left( \kappa_n \rho_n^{q \tau_n} \right),
\]

where the last approximation is due to the binomial theorem i.e. \( \frac{\rho_n}{1 - \rho_n} = \rho_n \left( -\frac{1}{\rho_n} \right)^{q \tau_n} = O(\kappa_n), \) and the fact that \( \rho_n^{q \tau_n} \geq \rho_n^{q \kappa_n} \) for \( n \) large (recall that \( \tau_n \leq k_n \)).

Now suppose that \( \kappa_n^{-1} \kappa_n \ln(\kappa_n) \to 0 \). By part (ii) for all \(|c| > \delta > 0 \)

\[
\rho_n^{q \tau_n} = e^{cq \frac{\tau_n}{\kappa_n}} + o \left( \kappa_n^{-1} e^{-\delta q \frac{\tau_n}{\kappa_n}} \right),
\]

Hence, as \( n \to \infty \)

\[
\kappa_n \rho_n^{q \tau_n} = \kappa_n \left[ e^{cq \frac{\tau_n}{\kappa_n}} + o \left( \kappa_n^{-1} e^{-\delta q \frac{\tau_n}{\kappa_n}} \right) \right] = \kappa_n e^{cq \frac{\tau_n}{\kappa_n}} + o \left( e^{-\delta q \frac{\tau_n}{\kappa_n}} \right).
\]
\[ = e^{a_n (cq + \ln(e_n) a_n)} + o(1) = e^{a_n (cq + o(1))} = o(1), \]
as required. ■

Appendix B3 (proofs of technical lemmas of Appendix B)

Proof of Lemma \textbf{7.1} \hspace{1em} (i) Set \( \lambda (s) := \ell (s) s^{-1/2} \). Note that for all \( \varepsilon > 0 \), there some \( N_{\varepsilon} \) such that \( |\phi_{s-l}/\lambda (s-l) - 1| < \varepsilon \) when \( s-l > N_{\varepsilon} \) and \( \sup_{l>N_{\varepsilon}} \lambda (l) < \varepsilon \). We have

\[
\sum_{j=0}^{s} \phi_{j} c_{s-j} = (1) \sum_{l=0}^{s-1} \phi_{s-l} c_{l} + o (\lambda (s)) = (2) \sum_{l=0}^{s-1-N_{\varepsilon}} \lambda (s-l) c_{l} + o (\lambda (s)), \quad (61)
\]

where \((1)\) follows from condition (a) or (b), whilst \((2)\) is demonstrated in detail later. Note that under condition (a), \((1)\) follows immediately. Under (b) as \( s \to \infty \) large

\[
|c_{s}| \leq \sum_{j=s}^{\infty} |c_{j}| = s^{-1/2} \sum_{j=s}^{\infty} j^{1/2} |c_{j}| = \lambda (s) \ell (s) s^{-1} \sum_{j=s}^{\infty} j^{1/2} |c_{j}| = o (\lambda (s)). \quad (62)
\]

Next, set \( m_{s} := |s \delta| \) where \( 0 < \delta < 1 \). Consider

\[
\sum_{l=0}^{s-1-N_{\varepsilon}} \lambda (s-l) c_{l} = \left( \sum_{l=0}^{m_{s}} + \sum_{l=m_{s}+1}^{s-1-N_{\varepsilon}} \right) \lambda (s-l) c_{l} := T_{1}(s) + T_{2}(s).
\]

The first term as \( s \to \infty \)

\[
\left| T_{1}(s) - \lambda (s) \sum_{l=0}^{m_{s}} c_{l} \right| \leq \sum_{l=0}^{m_{s}} |\lambda (s) - \lambda (s-l)| |c_{l}|
\]

\[
\leq \sum_{l=0}^{m_{s}} \left| \frac{\lambda (s-l)}{\lambda (s)} - 1 \right| \lambda (s) |c_{l}| \leq \sum_{l=0}^{m_{s}} \left| \frac{\lambda (s-l)}{\lambda (s)} - \left( \frac{s-l}{s} \right)^{-1/2} \right|
\]

\[
+ \left| \left( \frac{s-l}{s} \right)^{-1/2} - 1 \right| \lambda (s) |c_{l}|
\]

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\begin{align*}
\leq_{\text{(1)}} \sup_{1-\delta \leq x \leq 1} \frac{\lambda(sx)}{\lambda(s)} - x^{-1/2} & \left| \lambda(s) \sum_{l=0}^{\infty} |c_l| + \lambda(s) \sum_{l=0}^{m_s} \frac{l^{1/2}}{(s-l)^{1/2}} |c_l| \right| \\
=_{\text{(2)}} o(\lambda(s)) + \lambda(s) \frac{1}{(s-m_s)^{1/2}} \sum_{l=0}^{m_s} l^{1/2} |c_l| & =_{\text{(3)}} o(\lambda(s)),
\end{align*}

where \(\leq_{\text{(1)}}\) follows from the fact that
\[
\left( \frac{s}{s-l} \right)^{1/2} - 1 = \frac{s^{1/2} - (s-l)^{1/2}}{(s-l)^{1/2}} \leq \frac{l^{1/2}}{(s-l)^{1/2}},
\]

\(=_{\text{(2)}}\) from the uniform convergence theorem for regularly varying functions (e.g. Bingham et al. 1987, Theorem 1.5.2.), and \(=_{\text{(3)}}\) from Kronecker’s lemma (e.g. p. 31 in Hall and Heyde, 1980).

Next, using condition (i.a) and Lemma 6.3(i) as \(s \to \infty\) and then as \(\delta \to 1\) we have
\[
|T_2(s)| \leq \sum_{l=m_s}^{s-1} \lambda(s-l) l^{-1/2} |c_l| \\
\leq Cm_s^{-1} \sum_{l=m_s}^{s-1} \lambda(s-l) = Cm_s^{-1} \sum_{l=1}^{s-m_s+1} \lambda(l) \\
= C\ell(s)\sqrt{s} \left( \int_{0}^{1-\delta} x^{-1/2} dx + o(1) \right) = C\frac{\lambda(s)}{\delta} \int_{0}^{1-\delta} x^{-1/2} dx (1 + o(1)) = o(\lambda(s)).
\]

Next, we consider \(T_2(s)\) under condition (ib). Using the same arguments that lead to \(=_{\text{(2)}}\) we get as \(s \to \infty\)
\[
|T_2(s)| \leq \sup_{l>N_s} \lambda(l) \sum_{l=m_s}^{s-1-N_s} |c_l| \leq C\frac{\lambda(s)}{\delta^{1/2}} \sum_{l=m_s}^{\infty} l^{1/2} |c_l| = o(\lambda(s)).
\]

Finally, we show that the second approximation in \(=\) holds. Consider
\[
R_s := \sum_{l=0}^{s-1} \phi_{s-l} c_l - \sum_{l=0}^{s-1-N_s} \lambda(s-l) c_l \\
= \sum_{l=0}^{s-1-N_s} [\phi_{s-l} - \lambda(s-l)] c_l + \sum_{l=s-N_s}^{s-1} \phi_{s-l} c_l =: R_{1,s} + R_{2,s}.
\]
The first term

\[ R_{1,s} \leq \sum_{l=0}^{s-1-N_{\varepsilon}} \left| \frac{\phi_{s-l}}{\lambda(s-l)} - 1 \right| \lambda(s-l) |c_l| \leq \varepsilon \sum_{l=0}^{s-1-N_{\varepsilon}} \lambda(s-l) |c_l|. \]

It follows from similar arguments to those used above that \( \sum_{l=0}^{s-1-N_{\varepsilon}} \lambda(s-l) |c_l| = O(\lambda(s)) \). Hence, as \( s \to \infty \) first, and then as \( \varepsilon \downarrow 0 \) we have \( \lambda(s)^{-1} R_{1,s} = o(1) \). Further, it can be easily seen (e.g. from (62)) that

\[ R_{2,s} \leq \sum_{l=s-N_{\varepsilon}}^{s-1} |\phi_{s-l} c_l| \leq \sup_{j \in \mathbb{N}} |\phi_j| \sum_{l=s-N_{\varepsilon}}^{s-1} |c_l| = o(\lambda(s)). \]

(ii) Without loss of generality assume that \( \pi(j) = \ell_{\pi}(j) j^{-q_{1}} \) and \( \theta(j) = \ell_{\theta}(j) j^{-q_{2}} \). Using similar arguments as in those in used in part (i) we get

\[ \sum_{l=0}^{s} \pi_{s-l} \theta_l = \sum_{l=0}^{s-1} \pi_{s-l} \theta_l + o(1), \]

As before, set \( m_{s} := |s\delta| \) where \( 0 < \delta < 1 \). Then

\[ \sum_{l=0}^{s-1} \pi_{s-l} \theta_l = \left( \sum_{l=0}^{m_{s}} + \sum_{l=m_{s}+1}^{s-1} \right) \pi_{s-l} \theta_l := T_{1}(s) + T_{2}(s). \]

Again, using similar arguments as those used in the part (i) as \( s \to \infty \) the first term

\[ T_{1}(s) = \sum_{l=0}^{m_{s}} \pi_{s-l} |\theta_l| \leq \sum_{l=0}^{m_{s}} \left| \pi(s-l) \theta_l \right| \leq \sum_{l=0}^{m_{s}} \left| \frac{\pi(s-l)}{\pi(s)} - \left( \frac{s-l}{s} \right)^{-q_{1}} \right| \left| \pi(s-l) \theta_l \right| \]

\[ \leq \sup_{1-\delta \leq x \leq 1} \left| \frac{\pi(s-x)}{\pi(s)} - x^{-q_{1}} \right| \sum_{l=0}^{\infty} \left| \theta_l \right| + \sup_{1-\delta \leq x \leq 1} \left| \frac{\pi(s-x)}{\pi(s)} - x^{-q_{1}} \right| \sum_{l=0}^{m_{s}} \left( \frac{s^{q_{1}} - (s-l)^{q_{1}}}{(s-l)^{q_{1}}} \right) |\theta_l| \]

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\[
\leq (3) \ o(\pi(s)) + q_1(s) \frac{q_1^{q_1-1}}{(s-m_s)^{q_1}} \sum_{l=0}^{m_s} l |\theta_l| = (4) \ o(\pi(s)),
\]
where \( \leq (3) \) follows from the mean value theorem for
\[
|(s-l)^{q_1} - s^q_1| = |q_1(s-l)^{q_1-1}| \leq q_1 s^{q_1-1} l
\]
with \( 0 \leq l \leq m_s \) being a mean value, and \( = (4) \) from Kronecker’s lemma. Next, noting that \( T_2(s) = \sum_{l=1}^{s-m_s-1} \pi(l) \theta(s-l) \), and using the same arguments as those used above we get as \( s \to \infty \)
\[
\left| T_2(s) - \theta(s) \sum_{l=1}^{s-m_s} \pi(l) \right| \leq |\theta(s)| \sum_{l=1}^{s-m_s} |\theta(s-l) - \theta(s)| |\pi(l)|
\]
\[
\leq |\theta(s)| \sum_{l=1}^{s-m_s} \left| \frac{\theta(s-l)}{\theta(s)} \right| \left( \frac{s-l}{s} \right)^{-q_2} \left( \frac{s-l}{s} \right)^{-q_2} \left| \pi(l) \right|
\]
\[
\leq |\theta(s)| \sup_{0 \leq \delta \leq 1} \left| \frac{\theta(s)}{\theta(s)} \right| \left( \frac{s}{s} \right)^{-q_2} \left( \frac{s}{s} \right)^{-q_2} \left| \pi(l) \right|
\]
\[
\leq o(\theta(s)) + |\theta(s)| \frac{q_2 s^{q_2-1}}{m_s^{q_2}} \sum_{l=1}^{s-m_s} l |\pi(l)|
\]
\[
= o(\theta(s)) + |\theta(s)| \left( \delta^{-q_2} q_2 + o(1) \right) s^{-1} \sum_{l=1}^{s-m_s} l |\pi(l)| = o(\theta(s)) ,
\]
as required for the first part of (ii). The second part of (ii) follows easily from the above. ■

Proof of Lemma 7.2. (i) We start with part (a). Recall that under LL4 \( \alpha_k(n) = \sum_{j=0}^{k} \rho_n^{k-j} c_j \), for \( 0 \leq k \leq t-1 \). Next, for \( t \leq n \), as \( t \to \infty \)

Lemma 6.4(ia) and Euler summation yield
\[
\kappa_t^{-(1-m)} \sup_{0 \leq k \leq t} \left| \sum_{j=0}^{k} \rho_n^{k-j} c_j - \sum_{j=0}^{k} \exp \left\{ \frac{-k-j}{\kappa_n} \right\} c_j \right| \leq \kappa_t^{-(1-m)} \sup_{0 \leq k \leq n} \left| \sum_{l=0}^{k} \rho_n^{l} \exp \left\{ \frac{-l}{\kappa_n} \right\} c_{k-l} \right|
\]
\[
[ l = k - j \leftrightarrow j = k - l ]
\]
\[
\leq C \kappa_t^{-(1-m)} \sup_{0 \leq k \leq n} \left| \sum_{l=0}^{k} \exp \left\{ \frac{-l}{2\kappa_n} \right\} \frac{l}{\kappa_n} c_{k-l} \right| \leq C \sup_{j \geq 0} |c_j| \kappa_t^{-(1-m)} \sum_{l=0}^{n} \exp \left\{ \frac{-l}{2\kappa_n} \right\} \frac{l}{\kappa_n^2},
\]
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\[ \lim_{n \to \infty} C_{k_t}^{-(1-m)} \left[ \int_0^{n} e^{-x/2} dx + o(1) \right] \leq C_{k_t}^{-(1-m)} \int_0^{\infty} e^{-x/2} dx \]

Next, as \( t \to \infty \) we get

\[
\sup_{1 \leq k \leq \kappa_t} \left| \kappa_t^{-(1-m)} \sum_{j=0}^{k} e^{-\frac{k-j}{\kappa_t}} c_j - \int_0^{k/\kappa_t} e^{-\frac{(k-j)x}{\kappa_t}} x^{-m} dx \right|
\]

\leq \sup_{1 \leq k \leq \kappa_t} \left| \frac{1}{k} \sum_{j=1}^{k} e^{-\frac{k-j}{\kappa_t}} \left( \frac{j}{k} \right)^{-m} - \int_0^{k/\kappa_t} e^{-\frac{(k-j)x}{\kappa_t}} x^{-m} dx \right|

+ \sup_{1 \leq k \leq \kappa_t} \left| \frac{1}{k^{1-m}} \sum_{j=1}^{k} e^{-\frac{k-j}{\kappa_t}} c_j - \frac{1}{k^{1-m}} \sum_{j=1}^{k} e^{-\frac{k-j}{\kappa_t}} \left( \frac{j}{k} \right)^{-m} \right| + o(1) = A_{n,t} + B_{n,t}

Recall that for a function \( g(x) \), \( V \left( g(x) \right)^k \) is its total variation on the interval \([a,b]\). Set \( \tilde{c} := -\kappa_t/\kappa_n \), \( g(x) := e^{-\tilde{c}x} x^{-m} \) and \( g(x)' := \partial g(x)/\partial x \). Using standard arguments, the first term is

\[
A_{n,t} \leq \kappa_t^{-1} \sup_{1 \leq k \leq n} V \left( e^{-k \tilde{c}} g(x) \right)^{k/\kappa_t} = \kappa_t^{-1} \sup_{1 \leq k \leq n} e^{-k \tilde{c}} \int_{1/\kappa_t}^{k/\kappa_t} |g(x)|' dx,
\]

where \( g(x)' \) is the derivative of \( g \). Note that

\[
g(x)' := -\tilde{c}e^{-\tilde{c}x} x^{-m} - me^{-\tilde{c}x} x^{-m-1} = e^{-\tilde{c}x} x^{-m-1} \left[ -\tilde{c}x - m \right]
\]

Hence,

\[
|g(x)|' = \begin{cases} 
  g(x)', & x \geq -m/\tilde{c} \\
  -g(x)', & 0 < x < -m/\tilde{c}
\end{cases}
\]

Consider the term

\[
\int_{1/\kappa_t}^{k/\kappa_t} |g(x)|' dx = - \int_{1/\kappa_t}^{-m/\tilde{c}} g(x)' dx + \int_{-m/\tilde{c}}^{k/\kappa_t} g(x)' dx = - \left[ g(x) \right]_{1/\kappa_t}^{-m/\tilde{c}} + \left[ g(x) \right]_{-m/\tilde{c}}^{k/\kappa_t}
\]

For \( 1 \leq k \leq n \) and using the fact that \( \sup_{a \geq 1, 1 \leq t \leq n} \kappa_t/\kappa_n < \infty \) we have \( t, n \) large

\[
V \left( e^{-k \tilde{c}} g(x) \right)^{k/\kappa_t} \leq e^{-k \tilde{c}} \left[ 2g(-m/\tilde{c}) + g(1/\kappa_t) + g(k/\kappa_t) \right]
\]

\[
= e^{-k \tilde{c}} \left( \frac{-m}{-m} \right)^m + e^{-k \tilde{c}} \left( \frac{k}{\kappa_t} \right)^m + e^{-k \tilde{c}} \left( \frac{1}{\kappa_t} \right)^m
\]

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\[
\begin{align*}
&= e^{-\frac{1}{k_n}} \left[ 2e^m \left( \frac{\kappa_t}{m \kappa_n} \right)^m + e^{\frac{\epsilon t}{m^\ast n}} \kappa_t^m + e^{\frac{\epsilon t}{m^\ast n}} (\kappa_t/k)^m \right] \\
&= e^{-\frac{1}{k_n}} \left[ 2e^m \left( \frac{\kappa_t}{m \kappa_n} \right)^m + e^{\frac{\epsilon t}{m^\ast n}} \kappa_t^m + e^{\frac{\epsilon t}{m^\ast n}} (\kappa_t/k)^m \right] \\
&\leq C (\kappa_t/\kappa_n)^m + C \kappa_t^m + (\kappa_t/k)^m \leq C \kappa_t^m.
\end{align*}
\]

Hence, for \( n \geq t \) as \( t \to \infty \)

\[ A_{n,t} \leq C \kappa_t^{m-1} = o(1). \]

For the second term note that for each \( \epsilon > 0 \) there \( N_\epsilon > 0 \) such that \(|c_j/j^m - 1| < \epsilon \) for \( j > N_\epsilon \). Hence, for \( n \geq t \) as \( t \to \infty \) first and then as \( \epsilon \to 0 \)

\[
B_{n,t} \leq \kappa_t^{(1-m)} \sup_{1 \leq k \leq \kappa} \sum_{j=1}^k e^{-\frac{k-1}{k_n}} |c_j - j^{-m}| \leq \kappa_t^{(1-m)} \left( \sup_{N_\epsilon < k \leq \kappa} \sum_{j=N_\epsilon+1}^k e^{-\frac{k-1}{k_n}} |c_j - j^{-m}| + CN_\epsilon \right)
\]

\[
= \kappa_t^{(1-m)} \sup_{N_\epsilon < k \leq \kappa} \sum_{j=1}^k e^{-\frac{k-1}{k_n}} \left| \frac{c_j}{j^{-m}} - 1 \right| j^{-m} + o(1) \leq \epsilon \kappa_t^{(1-m)} \sup_{1 \leq k \leq n} \sum_{j=1}^k e^{-\frac{k-1}{k_n}} j^{-m}
\]

\[
\leq \epsilon \left[ A_{n,t} + \sup_{1 \leq k \leq \kappa} \int_0^{k/k_n} e^{-\frac{k-1}{k_n}} x^{-m} dx \right] \leq \epsilon C \epsilon \to 0,
\]

where the last inequality is justified as follows. First, note that from before \( A_{n,t} = o(1) \) for \( n \geq t \to \infty \). Further, using again the fact that \( \sup_{n \geq 1, 1 \leq k \leq n} \kappa_t/\kappa_n < \infty \), we get

\[
\sup_{1 \leq k \leq \kappa} \int_0^{k/k_n} e^{-\frac{k-1}{k_n}} x^{-m} dx \leq e^{\frac{2}{m^\ast n}} \int_0^1 x^{-m} dx \leq C \int_0^1 x^{-m} dx < \infty,
\]

and the result follows.

For part (b) first, note that by Lemma 6.4{(ia)} as \( n \to \infty \)

\[
\kappa_n^{-(1-m)} \sup_{0 \leq k \leq n} \left| \sum_{j=0}^k \rho_n^{k-j} c_j - \sum_{j=0}^k \exp \left\{ -\frac{k-j}{\kappa_n} \right\} c_j \right| \leq C \kappa_n^{-(1-m)} \left( \int_0^{\infty} e^{-x/2} dx + o(1) \right) = o(1).
\]

In view of this the result follows from similar arguments as those used for the proof of part (a).
We shall prove the first equality. Consider
\[ n \sum_{t=0}^{t-1} \rho_{t} c_{k-j} \{ t \leq k \} = \sum_{t=k-t+1}^{k} \rho_{t} c_{k-1} \{ t \leq k \}. \]
Using arguments similar to those used above we have as \( n \to \infty \)
\[
\sup_{1 \leq t \leq n, k \geq t} \left| \frac{1}{\rho_{n}^{1-m}} a_{k,t}^{-1}(n) \right| - \int_{(k-t)/\rho_{n}}^{k/\rho_{n}} e^{-(\frac{k}{\rho_{n}} - x)} x^{-m} dx \leq \kappa_{n}^{-1} \sup_{1 \leq t \leq n, k \geq t} \nu \left( e^{\frac{k}{\rho_{n}} q(x)} \right)^{k/\rho_{n}}
\]
with \( q(x) := e^{x} x^{-m} \). As before, for \( 1 \leq t \leq n \) and \( k > t \) we have as \( n \to \infty \)
\[
\kappa_{n}^{-1} \nu \left( e^{\frac{k}{\rho_{n}} q(x)} \right)^{k/\rho_{n}} = \kappa_{n}^{-1} e^{\frac{k}{\rho_{n}}} \int_{(k-t)/\rho_{n}}^{k/\rho_{n}} |q(x)| dx = \kappa_{n}^{-1} e^{\frac{k}{\rho_{n}}} \left( \int_{m}^{m} - \int_{(k-t)/\rho_{n}}^{k/\rho_{n}} q(x) dx \right)
\]
\[
= \kappa_{n}^{-1} e^{\frac{k}{\rho_{n}}} \left[ [q(x)]_{m}^{k/\rho_{n}} - [q(x)]_{(k-t)/\rho_{n}}^{m} \right]
\]
\[
= \kappa_{n}^{-1} e^{\frac{k}{\rho_{n}}} \left[ e^{\frac{k}{\rho_{n}}} \left( \frac{k_{n}}{k} \right)^{m} - 2e^{m} \left( \frac{1}{m} \right)^{m} + e^{\frac{k}{\rho_{n}}} \left( \frac{k_{n}}{k-t} \right)^{m} \right]
\]
\[
\leq \kappa_{n}^{-1} \left[ \left( \frac{k_{n}}{k} \right)^{m} + e^{\frac{k}{\rho_{n}}} \left( \frac{k_{n}}{k-t} \right)^{m} + 2e^{\frac{k}{\rho_{n}}} e^{m} \left( \frac{1}{m} \right)^{m} \right]
\]
\[
\leq 2k_{n}^{-1} + 2e^{m} \left( \frac{1}{m} \right)^{m} \kappa_{n}^{-1} = o(1).
\]

(ii) First, note that \( [k_{t}/2] \leq k \leq k_{t} \). Then for \( n \geq t \) we get as \( t \to \infty \)
\[
\int_{0}^{k_{t}/\rho_{n}} e^{-(\frac{k}{\rho_{n}} - \frac{k_{t}}{\rho_{n}} x)} x^{-m} dx \geq \int_{0}^{[k_{t}/2]/\rho_{n}} e^{-(\frac{k}{\rho_{n}} - \frac{k_{t}}{\rho_{n}} x)} x^{-m} dx \geq \int_{0}^{[k_{t}/2]/\rho_{n}} e^{-\frac{k}{\rho_{n}} x} dx
\]
\[
= e^{-\frac{k}{\rho_{n}}} \int_{0}^{1/2} dx + o(1) \geq \frac{1}{2} e^{-\frac{\rho_{n}}{k_{t}}} \geq \frac{1}{2} \exp \left( - \sup_{n \geq 1, 1 \leq t \leq \kappa} \frac{k_{t}}{\rho_{n}} \right) > 0,
\]
as required. \( \blacksquare \)

**Proof of Lemma 7.3.** The second equality above follows directly from the definition of Gamma and Beta function (note that \( B(x, y) = \int_{0}^{\infty} t^{x-1} (1 + l)^{-x-y} dl \)).
We shall prove the first equality. Consider
\[
= \int_{y=0}^{\infty} \exp \{ 2cy \} \left( \int_{x=0}^{y} \exp \{ -cx \} x^{-m} dx \right)^{2} dy
\]
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\[
\begin{align*}
&= \int_{y=0}^{\infty} \int_{x=0}^{y} \int_{x_*=0}^{y} \exp \{ c (2y - x - x_*) \} x^{-m} x_*^{-m} dx dx_* dy \\
&= \int_{x=0}^{\infty} \int_{x_*=0}^{\infty} \int_{y=x_*/x_*}^{\infty} \exp \{ c (2y - x - x_*) \} x^{-m} x_*^{-m} dx dx_* dy \\
&= \int_{x=0}^{\infty} \int_{x_*=0}^{\infty} \int_{y=x_*/x_*}^{\infty} y \cdot e^{-c(x+x_*)} x^{-m} x_*^{-m} dy dx_* \\
&= \frac{1}{2c} \int_{x=0}^{\infty} \int_{x_*=0}^{\infty} \int_{y=x_*/x_*}^{\infty} e^{2cy} y \cdot e^{-c(x+x_*)} x^{-m} x_*^{-m} dy dx_* \\
&= \frac{1}{2c} \int_{x=0}^{\infty} \int_{x_*=0}^{\infty} \left[ e^{2cy} y \cdot e^{-c(x+x_*)} x^{-m} x_*^{-m} \right] dy dx_* \\
&= -\frac{1}{2c} \int_{x=0}^{\infty} \left[ \left( \int_{x_*=x}^{\infty} e^{2cy} y \cdot e^{-c(x+x_*)} x^{-m} x_*^{-m} dx_* \right) \right] dx \\
&= -\frac{1}{2c} \left[ \int_{x=0}^{\infty} e^{cx} x^{-m} x_*^{-m} dx_* \right] dx \\
&= -\frac{1}{2c} \left[ \int_{x=0}^{\infty} e^{cx} x^{-m} x_*^{-m} dx_* + \int_{x_*=x}^{\infty} e^{cx} x^{-m} x_*^{-m} dx_* \right] dx \\
&= -\frac{1}{2c} \left[ \int_{x=0}^{\infty} e^{cx} x^{-m} x_*^{-m} dx_* \right] dx \\
&= -\frac{1}{2c} \left[ \int_{x=0}^{\infty} e^{cx} x^{-m} x_*^{-m} dx_* \right] dx + \int_{x_*=x}^{\infty} e^{cx} x^{-m} x_*^{-m} dx_* = T_1 + T_2
\end{align*}
\]

The first term

\[
T_1 = -\frac{1}{2c} \int_{y=0}^{\infty} e^{cy} y^{-m} \int_{x=0}^{y} e^{-cx} x^{-m} dx dy \\
\quad [x = y - z \rightarrow z = y - x]
\]

\[
= -\frac{1}{2c} \int_{y=0}^{\infty} e^{cy} y^{-m} \int_{z=y}^{0} e^{-c(y-z)} (y-z)^{-m} dz dy = -\frac{1}{2c} \int_{y=0}^{\infty} y^{-m} \int_{z=0}^{y} e^{cz} (y-z)^{-m} dz dy \\
\quad [q = y - z \rightarrow y = q + z]
\]

\[
= -\frac{1}{2c} \int_{z=0}^{\infty} e^{cz} \int_{q=0}^{\infty} (q+z)^{-m} q^{-m} dz dq = -\frac{1}{2c} \int_{z=0}^{\infty} e^{cz} \int_{q=0}^{\infty} z^{-2m} (\frac{q}{z} + 1)^{-m} (\frac{q}{z})^{-m} dz dq
\]
\[
\begin{align*}
[ u = \frac{q}{z} \rightarrow zdu &= dq] \\
&= -\frac{1}{2c} \int_{z=0}^{\infty} e^{cz} z^{1-2m} \int_{u=0}^{\infty} (u + 1)^{-m} u^{-m} du dz \\
&= -\frac{1}{2c} (-c)^{2m-1} \int_{z=0}^{\infty} e^{cz} (-cz)^{1-2m} dz \int_{u=0}^{\infty} (u + 1)^{-m} u^{-m} du \\
&= -\frac{1}{2c} (-c)^{2m-1} \frac{1}{-c} \int_{v=0}^{\infty} e^{-v} v^{1-2m} dv \int_{u=0}^{\infty} (u + 1)^{-m} u^{-m} du \\
&= \frac{1}{2} (-c)^{2m-3} \int_{v=0}^{\infty} e^{-v} v^{1-2m} dv \int_{u=0}^{\infty} (u + 1)^{-m} u^{-m} du \\
\end{align*}
\]

The second term
\[
T_2 = -\frac{1}{2c} \int_{y=0}^{\infty} e^{-cy} y^{-m} \int_{x=y}^{\infty} e^{cx} x^{-m} dx dy \\
[x = z - y \rightarrow x = z + y] \\
= -\frac{1}{2c} \int_{y=0}^{\infty} e^{-cy} y^{-m} \int_{z=0}^{\infty} e^{cz} (z+y)^{-m} dz dy \\
= -\frac{1}{2c} \int_{y=0}^{\infty} \int_{z=0}^{\infty} e^{cz} (z+y)^{-m} dz dy \\
= -\frac{1}{2c} \int_{y=0}^{\infty} \int_{z=0}^{\infty} e^{cz} (z+y)^{-m} dy dz \\
= -\frac{1}{2c} \int_{z=0}^{\infty} e^{cz} z^{-2m} \int_{y=0}^{\infty} (z^{-1} y)^{-m} (1 + z^{-1} y)^{-m} dy dz \\
= -\frac{1}{2c} \int_{z=0}^{\infty} e^{cz} z^{-1-2m} \int_{l=0}^{\infty} l^{-m} (1 + l)^{-m} dl \\
= -\frac{1}{2c} (-c)^{2m} \int_{z=0}^{\infty} e^{cz} (-cz)^{-1-2m} dz \left( \int_{l=0}^{\infty} l^{-m} (1 + l)^{-m} dl \right) \\
= -\frac{1}{2c} (-c)^{2m-1} \int_{z=0}^{\infty} e^{cz} (-cz)^{-1-2m} dz \left( \int_{l=0}^{\infty} l^{-m} (1 + l)^{-m} dl \right) \\
= -\frac{1}{2c} (-c)^{2m-1} \left( -\frac{1}{c} \right) \int_{s=0}^{\infty} e^{-s} s^{-1-2m} ds \left( \int_{l=0}^{\infty} l^{-m} (1 + l)^{-m} dl \right)
\]
\[ T_1 + T_2 = \frac{1}{(-c)^{3-2m}} \Gamma (2 - 2m) \left( \int_{l=0}^{\infty} l^{-m} (1 + l)^{-m} \, dl \right) \]

Therefore,

\[ \frac{1}{(-c)^{3-2m}} \Gamma (2 - 2m) B(1 - m, 2m - 1) \]

Note that

\[ B(x, y) = \int_{l=0}^{\infty} l^{x-1} (1 + l)^{-x-y} \, dl \]

Hence,

\[ \int_{l=0}^{\infty} l^{-m} (1 + l)^{-m} \, dl = \int_{l=0}^{\infty} l^{(1-m)-1} (1 + l)^{-(1-m)+(1-m)-m} \, dl \]

\[ = \int_{l=0}^{\infty} l^{(1-m)-1} (1 + l)^{-(1-m)-(2m-1)} \, dl = B(1 - m, 2m - 1). \]

\[ \blacksquare \]

8 Appendix C

Appendix C provides proofs for the Theorems of Section 3.

**Proof of Theorem 4.1** The LS is

\[ \kappa_g(\beta_n) \sqrt{n} \left( \hat{\beta} - \beta \right) = \frac{1}{\sqrt{n}} \sum_{t=2}^{n} H_g \left( \beta_n^{-1} x_{t-1}(n) \right) u_t + \frac{1}{\kappa_g(\beta_n) \sqrt{n}} \sum_{t=2}^{n} R_g \left( x_{t-1}(n), \beta_n \right) u_t \]

\[ = \frac{1}{\sqrt{n}} \sum_{t=2}^{n} H_g \left( \beta_n^{-1} x_{t-1}(n) \right) u_t + \frac{1}{\kappa_g(\beta_n) \sqrt{n}} \sum_{t=2}^{n} R_g \left( x_{t-1}(n), \beta_n \right) u_t + o_p(1), \]

where we have used the fact that (see \[28\])

\[ \mathbb{E} \left( \frac{1}{\kappa_g(\beta_n) \sqrt{n}} \sum_{t=2}^{n} R_g \left( x_{t-1}(n), \beta_n \right) u_t \right)^2 = \frac{\sigma_u^2 \kappa_g(\beta_n)^2 n}{\kappa_g(\beta_n)^2 n} \sum_{t=2}^{n} \mathbb{E} R_g \left( x_{t-1}(n), \beta_n \right)^2 = o(1). \]
Next, using Theorem 2.2 we get as $n \to \infty$

$$\frac{1}{n} \sum_{t=2}^{n} H_g \left( \beta_n^{-1} x_{t-1}(n) \right)^2 \overset{d}{\to} \int_{\mathbb{R}} H_g \left( x + X^{-} \right)^2 \varphi_{\alpha^2} \, dx =: V_{H_2^2}.$$ 

Therefore, by Wang (2014, Theorem 3.2 and Remark 5) it suffices showing that

$$\sup_{1 \leq t \leq n} \left| H_g \left( \beta_n^{-1} x_t(n) \right) \right| \sqrt{n} = o_p(1), \quad (64)$$

for obtaining the joint limit

$$\left[ \frac{1}{n} \sum_{t=2}^{n} H_g \left( \beta_n^{-1} x_{t-1}(n) \right)^2, \frac{1}{\sqrt{n}} \sum_{t=2}^{n} H_g \left( \beta_n^{-1} x_{t-1}(n) \right) u_t \right] \overset{d}{\to} \left[ V_{H_2^2}, MN \left( 0, \sigma_u^2 V_{H_2^2} \right) \right],$$

which in view of (63) is sufficient for the requisite result. Note that the mixing variate $X^-$ is non degenerate only under LL2 (recall that under LL1, LL3 and LL4 $X^- = 0$), in which case $X^-$ is the distribution limit of some functional of $\{ \xi_0, \xi_{-1}, \ldots \}$ that is $\mathcal{F}_0$-measurable (see Lemma CLT). Therefore, it can be easily seen that eq. (2.3) in Wang (2014) is unnecessary (see p. 523-525 in Wang, 2014). Finally, note that the following Lindeberg condition is sufficient for (64) (e.g. eq. (3.5) in Hall and Heyde, 1980): For all $\eta > 0$, as $n \to \infty$

$$\frac{1}{n} \sum_{t=1}^{n} H_g \left( \beta_n^{-1} x_t(n) \right)^2 \left\{ \left| H_g \left( \beta_n^{-1} x_t(n) \right) \right| > \eta \sqrt{n} \right\} = o_p(1). \quad (65)$$

We set out to prove (65) next. By condition (iia) of Theorem 2.2 for large $x$, $H_g \left( x \right)^2 \leq C \left| x \right|^\lambda'$ for some $\lambda' > 0$. Further, under the conditions of Theorem 4.1 Assumptions HL0-HL5(a) and HL6 hold with $X_t(n) = x_t(n)$ (as demonstrated by Propositions 2.1 and 2.2). In particular, $\sup_{n \geq n_0, 1 \leq t \leq n} \mathbb{E} \left| \beta_n^{-1} x_t(n) \right|^\lambda < \infty$, for some $\lambda > \lambda'$ (HL5(a)). Hence, $\left| \beta_n^{-1} x_t(n) \right|^\lambda$ is uniformly integrable. Further, $\beta_t^{-1} x_t(n)$ possesses density $\mathcal{D}_{t,n}(x)$ such that $\sup_{n \geq n_0, t_0 \leq t \leq n} \mathcal{D}_{t,n}(x) < \infty$, for some $n_0 \geq t_0 \geq 1$ (HL3). Without loss of generality suppose that $t_0 = 1$. Next, note that due to condition (vi), $H_g \left( x \right)^2$ is locally integrable. In view of the above, for all $\eta > 0$ and for $\varepsilon > 0$ large enough we have as $n \to \infty$

$$\frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left[ H_g \left( \beta_n^{-1} x_t(n) \right)^2 \left\{ \left| H_g \left( \beta_n^{-1} x_t(n) \right) \right| > \eta \sqrt{n} \right\} \right]$$
\[
\leq \frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left[ H_g \left( \beta_n^{-1} x_t(n) \right)^2 \mathbb{I} \left\{ \left| \beta_n^{-1} x_t(n) \right| < \varepsilon \right\} \mathbb{I} \left\{ \left| H_g \left( \beta_n^{-1} x_t(n) \right) \right| > \eta \sqrt{n} \right\} \right] + C \frac{1}{n} \sum_{t=1}^{n} \mathbb{E} \left[ \left| \beta_n^{-1} x_t(n) \right|^{\lambda'} \mathbb{I} \left\{ \left| \beta_n^{-1} x_t(n) \right|^{\lambda'/2} > \eta \sqrt{n} \right\} \mathbb{I} \left\{ \left| \beta_n^{-1} x_t(n) \right| > \varepsilon \right\} \right]
\]

\[
\leq \frac{1}{n} \sum_{t=1}^{n} \int x H_g \left( \beta_t \beta_n \right)^2 \mathbb{I} \left\{ \left| \beta_t \beta_n \right| < \varepsilon \right\} \mathbb{I} \left\{ \left| H_g \right| > \eta \sqrt{n} \right\} D_{t,n}(x) dx + C \sup_{1 \leq t \leq n} \mathbb{E} \left[ \left| \beta_n^{-1} x_t(n) \right|^{\lambda'} \mathbb{I} \left\{ \left| \beta_n^{-1} x_t(n) \right|^{\lambda'} > \eta^2 n \right\} \right]
\]

\[
\leq (1) C \frac{\beta_n h_n}{n} \sum_{t=1}^{n} \beta_t^{-1} \int_{|x|<\varepsilon} H_g(x)^2 \mathbb{I} \left\{ \left| H_g \right| > \eta \sqrt{n} \right\} dx + o(1)
\]

\[
\leq (2) C \int_{|x|<\varepsilon} H_g(x)^2 \mathbb{I} \left\{ \left| H_g \right| > \eta \sqrt{n} \right\} dx + o(1) = o(1),
\]

where the approximation in \( \leq (1) \) is due to the fact \( \left| \beta_n^{-1} x_t(n) \right|^{\lambda'} \) is uniformly integrable and the approximation \( \leq (2) \) is due to HL6 and dominated convergence (recall that by condition (vi), \( f(x) = H_g(x)^2 \) a.e. where \( f(x) \in \mathbb{R} \) and locally integrable). □

**Proof of Theorem 4.2** Part (a) follows from Theorem 3.2 and arguments similar to those used in Wang and Phillips (2009a,b). For part (b) note that by Theorem 4.2 we have as \( n \to \infty \)

\[
\frac{\beta_n}{h_n} \sum_{t=1}^{n} K \left( \frac{x_t(n) - x}{h_n} \right)^2 \xrightarrow{\text{d}} \varphi_{\sigma^2_+} (X^-) \int_{\mathbb{R}} K(x)^2 dx \equiv V_{K^2}, \quad (66)
\]

\[
\frac{\beta_n}{h_n} \sum_{t=1}^{n} K \left( \frac{x_t(n) - x}{h_n} \right) \xrightarrow{\text{d}} \varphi_{\sigma^2_+} (X^-) \int_{\mathbb{R}} K(x) dx \equiv V_K. \quad (67)
\]

In fact it can be seen from the arguments used in the proof of Theorem 2.1 that (66)-(67) hold jointly. Next, the NW estimator is

\[
\sqrt{\frac{h_n}{\beta_n}} (\hat{g}(x) - g(x)) = \sqrt{\frac{\beta_n}{h_n}} \sum_{t=1}^{n} K \left( \frac{x_{t-1}(n) - x}{h_n} \right) u_t + O_p \left( \frac{nh_n^{1+2\mu}}{\beta_n} \right), \quad (68)
\]

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where the approximation above follows from condition (iv) and arguments similar as those used by Wang and Phillips (2009a,b). By (66) and Wang (2014, Theorem 3.2 and Remark 5) showing that
\[
\sqrt{n} \sup_{1 \leq t \leq n} \left| K \left( \frac{x_{t-1}(n) - x}{h_n} \right) \right| = o_p(1),
\]
(69)
is sufficient for the joint limit
\[
\sqrt{n} \sum_{t=1}^{n} K \left( \frac{x_t(n) - x}{h_n} \right)^2
\]
(70)
Next, we show (69) by establishing a Lindeberg condition as in the previous proof. Note that under our assumptions, HL3 and HL6 hold with \( X_t(n) = x_t(n) \). Without loss of generality suppose that HL3 holds with \( t_0 = 1 \). Then for all \( \eta > 0 \), as \( n \to \infty \) we have
\[
\frac{\beta_n}{h_n} \sum_{t=1}^{n} \mathbb{E} \left[ K \left( \frac{x_t(n) - x}{h_n} \right)^2 1 \left\{ K \left( \frac{x_t(n) - x}{h_n} \right) > \eta \sqrt{\frac{h_n}{\beta_n}} \right\} \right]
\]
\[
\leq \left( \eta \sqrt{\frac{h_n}{\beta_n}} \right)^{-2} \frac{\beta_n}{h_n} \sum_{t=1}^{n} \mathbb{E} \left[ K \left( \frac{x_t(n) - x}{h_n} \right)^4 \right]
\]
\[
= \left( \eta \sqrt{\frac{h_n}{\beta_n}} \right)^{-4} \frac{\beta_n}{h_n} \sum_{t=1}^{n} \int_{\mathbb{R}} \left[ K \left( \frac{\beta_t y - x}{h_n} \right) \right]^4 \mathcal{D}_{t,n}(y) dy
\]
\[
\leq (1) \ C \left( \eta \sqrt{\frac{h_n}{\beta_n}} \right)^{-4} \frac{\beta_n}{n} \sum_{t=1}^{n} \beta_t^{-1} \int_{\mathbb{R}} K(y)^4 dy \leq (2) \ C \left( \eta \sqrt{\frac{h_n}{\beta_n}} \right)^{-4} \to 0,
\]
where \( \leq (1) \) follows from HL3 and \( \leq (2) \) from HL6 and the fact that \( \int_{\mathbb{R}} K(y)^4 dy < \infty \) (see condition (ii)). Now using (66)-(68) and (70) we get
\[
\sqrt{\frac{h_n}{\beta_n}} (\hat{g}(x) - g(x)) = \frac{\beta_n}{h_n} \sum_{t=1}^{n} K \left( \frac{x_{t-1}(n) - x}{h_n} \right)^2 \sqrt{\frac{\beta_n}{h_n}} \sum_{t=1}^{n} K \left( \frac{x_{t-1}(n) - x}{h_n} \right) u_t
\]
\[
+ o_p(1)
\]
\[
\int_{\mathbb{R}} K(x)^2 \, dx \frac{\sqrt{\frac{\beta_n}{h_n^2}} \sum_{t=1}^{n} K \left( \frac{x_{t-1}(n) - x}{h_n} \right) u_t}{\frac{\beta_n}{h_n^2} \sum_{t=1}^{n} K \left( \frac{x_{t-1}(n) - x}{h_n} \right) u_t} + o_p(1) \xrightarrow{d} MN \left( 0, \frac{\sigma_u^2 \int_{\mathbb{R}} K(x)^2 \, dx}{\varphi^{4}(X^-) \left[ \int_{\mathbb{R}} K(x) \, dx \right]^2} \right),
\]

as required. ■

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