Existence and non-existence of breather solutions in damped and driven nonlinear lattices

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We investigate the existence of spatially localised solutions, in the form of discrete breathers, in general damped and driven nonlinear lattice systems of coupled oscillators. Conditions for the exponential decay of the difference between the maximal and minimal amplitudes of the oscillators are provided which proves that initial non-uniform spatial patterns representing breathers attain exponentially fast a spatially uniform state preventing the formation and/or preservation of any breather solution at all. Strikingly our results are generic in the sense that they hold for arbitrary dimension of the system, any attractive interaction, coupling strength and on-site potential and general driving fields. Furthermore, our rigorous quantitative results establish conditions under which discrete breathers in general damped and driven nonlinear lattices can exist at all and open the way for further research on the emergent dynamical scenarios, in particular features of pattern formation, localisation and synchronisation, in coupled cell networks.

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Intrinsic localised modes (ILMs) or discrete breathers in nonlinear lattices have attracted significant interest recently, not least due to the important role they play in many physical realms where features of localisation in systems of coupled oscillators are involved (for a review see [1] and references therein), [2]-[4]. For conservative systems proofs of existence and (exponential) stability of breathers, as spatially localised and time-periodically varying solutions, were provided in [2] and [3] respectively. Analytical and numerical methods have been developed to continue breather solutions in conservative and dissipative systems starting from the anti-integrable limit [7],[8]. During recent years the existence of breathers has been verified in a number of experiments in various contexts including micro-mechanical cantilever arrays [9], arrays of coupled Josephson junctions [10], coupled optical wave guides [12], Bose-Einstein condensates in optical lattices [13], in coupled torsion pendula [14], electrical transmission lines [15], and granular crystals [16]. Regarding their creation mechanism in conservative systems, modulational instability (MI) provides the route to the formation of breathers originating from an initially spatially homogeneous state imposed to (weak) perturbations. To be precise, the MI of band edge plane waves triggers an inherent instability leading to the formation of a spatially localised state [17]. Departing from this often too ideal assumption of a conservative system requires, for more realistic models, the inclusion of dissipation. Accomplishing breather solutions in the presence of dissipation requests some compensating energy injection mechanism. As far as their persistence is concerned it is expected that breathers can be continued from a conservative system into a system augmented by weak dissipation and driving. Compared to their Hamiltonian (conservative) counterparts breathers in dissipative and driven lattice systems do not occur in families of localised solutions as they are provided by discrete sets of attractors for appropriate initial conditions contained in the corresponding basin(s) of attraction [18]-[25].

The aim of this work is to establish quantitative conditions in parameter space for the existence respectively non-existence of discrete breathers in general damped and driven anharmonic lattice systems. To this end we show that there exist parameter ranges such that for any launched localised state the difference between the maximal amplitude and the minimal amplitude of the oscillators decays exponentially fast. Consequently, a spatially uniform state is attained. Most importantly, this rules out the persistence of any non-uniform pattern such as breathers. Moreover, our results also identify parameter ranges for which no inherent instability that is able to trigger the formation of a localised pattern exists. Crucially, our rigorous quantitative results establish prerequisites for the existence of discrete breathers in general damped and driven nonlinear lattices beyond the validity of the continuation process starting from the anti-continuum limit [8],[18]-[21]. Our results are generic as they hold not only for any coupling strength but also for any on-site potential, any type of attractive interaction, any degree of attractive interaction, general driving fields and arbitrary dimension of the system.

We study the dynamics of general driven and damped nonlinear lattice systems of dimension $d$ given by the following system

$$
\ddot{q}_n = -U'(q_n) - \gamma \dot{q}_n + A(t) + B(t)q_n - \sum_{j \in N_r(n)} \kappa_j \left[ V'(q_{n+j} - q_n) + V'(q_n - q_{n-j}) \right],\]

with $n \in \mathbb{Z}^d$, $j, r \in \mathbb{N}^d$ and $N_r(n)$ is the set associated with the $r$ neighbours, $n+j$, of site $n$ with $1 \leq j \leq r$. The
variable \( q_n(t) \) is the amplitude of the oscillator at site \( n \) evolving in an anharmonic on-site potential \( U(q_n) \). The prime \( \prime \) stands for the derivative with respect to \( q_n \) and an overdot \( \dot{\cdot} \) represents the derivative with respect to time \( t \).

The on-site potential \( U \) is analytic and is assumed to have the following properties:

\[
U(0) = U'(0) = 0, \quad U''(0) > 0. \tag{2}
\]

In what follows we differentiate between soft on-site potentials and hard on-site potentials. For the former (latter) the oscillation frequency of an oscillator moving in the on-site potential \( U(q) \) decreases (increases) with increasing oscillation amplitude. A soft potential possesses at least one inflection point. If a soft potential possesses a single inflection point, denoted by \( q_i \), we suppose without loss of generality (w.l.o.g.) that \( q_i > 0 \). Then the following relations are valid

\[
U'(-\infty < q < 0) < 0, \quad U'(0 < q < q_i) > 0, \quad U''(q_i) = 0, \quad U''(-\infty < q < q_i) > 0. \tag{3}
\]

If \( U(q) \) possesses two inflection points denoted by \( q_{i,-} < 0 \) and \( q_{i,+} > 0 \) it holds that

\[
U'(q_{i,-} < q < 0) < 0, \quad U'(0 < q < q_{i,+}) > 0 \quad U''(q_{i,\pm}) = 0, \quad U''(q_{i,-} < q < q_{i,+}) > 0. \tag{4}
\]

We remark that \( U(q) \) can have more than two inflection points (an example is a periodic potential \( U(q) = -\cos(q) \)). However, in the frame of the current study we are only interested in motion between the inflection points adjacent to the minimum of \( U(q) \) at \( q = 0 \). Hence, in the forthcoming we suppose that for soft on-site potentials the motion at each lattice site \( n \) stays inbetween the inflection points, viz. \( q_{i,-} < g_n(t) < q_{i,+} \), where \( U(q) \) is convex.

Hard on-site potentials are, in addition to the assumptions in (2), characterised in their entire range of definition by

\[
U'(q < 0) < 0, \quad U'(q > 0) > 0, \quad U''(q) > 0. \tag{5}
\]

In contrast to soft potentials, since the hard potentials are by assumption convex in their entire range of definition no boundedness condition as for the motion in soft potentials is required.

Each oscillator interacts within the interaction radius \( r \) with its neighboring oscillators with (local) coupling strength \( \kappa_j > 0 \) (the interaction radius can range from next neighbour coupling to global coupling) via forces derived from an attractive interaction potential \( V(u) \) which is analytic and furthermore, is assumed to have the following features:

\[
V(0) = V'(0) = 0, \quad V''(0) \geq 0, \quad V''(u \neq 0) > 0. \tag{6}
\]

Thus \( V(u) \) is convex which is further characterised by \( V'(u > 0) > 0 \) and \( V'(u < 0) < 0 \). It is through the site-dependent coupling strength that heterogeneity enters the model. The interaction potential can be harmonic but also anharmonic such as for example in \( \beta \)-Fermi-Pasta-Ulam systems and Toda-type interactions.

The parameter \( \gamma > 0 \) regulates the strength of the damping. \( A(t) \) and \( B(t) \) are smooth functions representing general external time-dependent fields with

\[
\max_{t \in \mathbb{R}} A(t) = A_{\text{max}} < \infty, \quad \min_{t \in \mathbb{R}} A(t) = A_{\text{min}} > -\infty, \quad \max_{t \in \mathbb{R}} B(t) = B_{\text{max}} < \infty, \quad \min_{t \in \mathbb{R}} B(t) = B_{\text{min}} > -\infty.
\]

The \( A(t) \) and \( B(t) \) term in Eq. (1) is associated with direct and parametric driving respectively.

We investigate under which circumstances Eq. (1) possesses time-periodic and spatially localised solutions, viz. discrete breathers, \( q_n(t + T_b) = q_n(t) \), with period \( T_b = 2\pi/\omega_b \) where \( \omega_b \) denotes the breather frequency. We consider all possible standard breather solutions involving single-site breathers as well as multi-site breathers in the following referred to as single breathers and multibreathers. While for the former all the oscillators perform inevitably in-phase motion for the latter the oscillators perform in-phase and/or out-of-phase periodic motion with respect to a reference oscillator \( [18, 26] \). Multibreathers can also consist of arrays of single breathers, viz. the pattern is localised around more than a single site or a single group of sites. Note that as one-dimensional lattices are concerned, it is proven in \( [27, 28] \) that the only available stable multibreather solution are those with relative phase 0 (in-phase) and \( \pi \) (out-of-phase) between the lattice sites and phase-shift breathers do not exist. Hence our treatment of breathers is comprehensive.
In general, breathers, being supported by periodic closed orbits in phase space, are associated with periodic bounded motion of the oscillators inside their on-site potentials \( U(q) \). Periodic solutions require time-periodic external fields \( A(t) = A(t + T_A) \) and \( B(t + T_B) \) with appropriate periods \( T_A \) and \( T_B \).

We introduce the following quantities related to the extremal values of the coordinates:

\[
q_{\text{max}}(t) = \max_n q_n(t), \quad q_{\text{min}}(t) = \min_n q_n(t),
\]

and denote the difference between them by

\[
\Delta q(t) = q_{\text{max}}(t) - q_{\text{min}}(t) \geq 0.
\]

The difference between the associated velocities is denoted by \( \Delta \dot{q}(t) = \dot{q}_{\text{max}}(t) - \dot{q}_{\text{min}}(t) \). In general for breather solutions with period \( T_b \) it holds that \( \Delta q(t) = \Delta q(t + T_b) \) and \( \Delta \dot{q}(t) = \Delta \dot{q}(t + T_b) \). Non-uniform (uniform) states are characterised by non-vanishing (identically vanishing) \( \Delta q(t) \).

In the following we list the conditions satisfied by breather solutions described above:

We first discuss single breather solutions using a lattice site and the oscillators perform in-phase motion. Later we comment on multibreathers.

The difference \( \Delta q(t) \) involves inevitably the same two oscillators all the time. In fact, since for single breathers the pattern is spatially exponentially localised the two lattice sites involving \( q_{\text{max}}(t) \) and \( q_{\text{min}}(t) \) remain the same and only exchange their role after every change of sign of the periodically oscillating amplitudes. To be precise, the lattice sites \( n = \text{max} \) and \( n = \text{min} \) supporting the oscillators with \( q_{\text{max}} \) and \( q_{\text{min}} \) respectively during phases when \( q_n \geq 0 \) swap when the coordinates \( q_n \) become negative. To describe the behaviour of \( \Delta q(t) \) and \( \Delta \dot{q}(t) \) we express a period duration \( T_b = T_d + (T_b - T_d) \) as the sum of two stages of length \( T_d \) and \( T_b - T_d \) during which the coordinates \( q_n(t) \) possess opposite sign. We first consider stages of length \( T_d \) determined by \( kT_b \leq t \leq kT_b + T_d \) with \( k = 0, 1, \ldots \) during which the coordinates are either non-negative or non-positive depending on the initial conditions. (We recall that \( \Delta q \) is non-negative by definition.) At the beginning of each interval the values are w.l.o.g. given by \( \Delta q(kT_b) = \Delta q_0 = 0 \) and \( \Delta \dot{q}(kT_b) = \Delta \dot{q}_0 > 0 \). (We remark that in the following the temporal evolution is considered on such subintervals where \( \Delta q(t) \) is smooth.) Positive (negative) initial velocities \( \dot{q}_n(kT_b) > 0 \) \( \dot{q}_n(kT_b) < 0 \) with \( \dot{q}_{\text{max}}(kT_b) > \dot{q}_{\text{min}}(kT_b) > 0 \) \( \dot{q}_{\text{max}}(kT_b) < \dot{q}_{\text{min}}(kT_b) < 0 \) result in non-negative (non-positive) amplitudes \( q_n(t) \geq 0 \) \( q_n(t) \leq 0 \) during intervals \( kT_b \leq t \leq kT_b + T_d \). That is, all oscillators are at \( t = kT_b \) situated at the position \( q_n = 0 \), corresponding to the minimum position of the on-site potential, and \( \Delta \dot{q}(kT_b) \) and \( \Delta \dot{q}(kT_b) \) attains its maximum and minimum respectively. During \( kT_b \leq t \leq kT_b + T_d/2 \), the quantity \( \Delta \dot{q}(t) \) monotonically decreases resulting at \( t = kT_b + T_d/2 \) in \( \Delta \dot{q}(t) = 0 \) while the monotonically increasing quantity \( \Delta q(t) \) reaches its maximum. During \( kT_b + T_d/2 < t \leq kT_b + T_d \) both \( \Delta \dot{q}(t) \) and \( \Delta q(t) \) monotonically decrease attaining at the end of the interval \( kT_b + T_d \) their minima \( \Delta q(kT_b + T_d) = 0 \) and \( \Delta \dot{q}(kT_b + T_d) = -\Delta \dot{q}(kT_b) \).

For the subsequent stage of length \( T_b - T_d \), when the amplitudes \( q_n(t) \) have opposite sign compared to the previous interval, the motion of \( \Delta q(t) \) and \( \Delta \dot{q}(t) \) starts with the same values as at the beginning of the previous interval, viz. \( \Delta q_0 = 0 \) and \( \Delta \dot{q}_0 > 0 \) and the oscillator at the lattice site that previously supported \( q_{\text{max}} \) \( (\text{and} \ q_{\text{max}}) \) possesses now the minimal amplitude \( q_{\text{min}} \) \( (\text{and} \ q_{\text{min}}) \) and vice versa. However, \( \Delta q(t) \) and \( \Delta \dot{q}(t) \) resemble the behaviour of their counterparts during the previous interval.

As multibreathers are concerned the quantities \( \Delta q \) and \( \Delta \dot{q} \) exhibit qualitatively the same behaviour as for single breathers except that for phase differences \( \pi \) the oscillators with \( q_{\text{max}} > 0 \) and \( q_{\text{min}} < 0 \) possess opposite sign.

In order to establish conditions for the non-existence of breather solutions we consider the behaviour of \( \Delta q(t) \) and \( \Delta \dot{q}(t) \) w.l.o.g. on intervals

\[
I_k : kT_b + a \leq t \leq kT_b + T_d - a, \quad \text{with} \ k = 0, 1, 2, \ldots, \text{and} \ 0 < a < \frac{T_d}{2}, \tag{9}
\]

during which the amplitudes \( q_n(t) \) are for single breathers and multibreathers with phase difference 0 either exclusively non-negative or non-positive (see above) implying that the lattice site with \( q_{\text{max}} \) is fixed and so is the lattice site with \( q_{\text{min}} \). For multibreathers with phase difference \( \pi \) between the oscillators with \( q_{\text{max}} \) and \( q_{\text{min}} \) the same holds true regarding the fixed positions of the extremal coordinates except that \( q_{\text{max}} \) is always positive while \( q_{\text{min}} \) is always negative. (For multibreathers more than one lattice site may support \( q_{\text{max}} \) and/or \( q_{\text{min}} \).) Note that \( \Delta q(kT_b + a) = \Delta q(kT_b + T_d - a) > 0 \). For the forthcoming study it is appropriate to shift the time as \( \tilde{t} = t - a \) shifting the intervals \( I_k \) in (9) to

\[
\tilde{I}_k : kT_b \leq \tilde{t} \leq kT_b + T_d - 2a, \quad \text{with} \ k = 0, 1, 2, \ldots, \text{and} \ 0 < a < \frac{T_d}{2}, \tag{10}
\]

In what follows the tildes are omitted and at \( t = 0 \) the starting values \( \Delta q_0 \) and \( \Delta \dot{q}_0 \) are given by \( \Delta q(0) = \Delta q_0 > 0 \) and \( \Delta \dot{q}(0) = \Delta \dot{q}_0 > 0 \).
Δq(t) is smooth on the intervals \( I_k \). Furthermore, on each interval \( I_k \) it holds that \( Δq(t) \) is even with respect to \( t_k = kT_b + T_d/2 - a \) whereas \( Δ\dot{q}(t) \) is odd.

Exploiting the symmetry features and periodicity of \( Δq(t) \) and \( Δ\dot{q}(t) \) one obtains the following relations:

\[
\begin{align*}
Δq((k+1)T_b) &= Δq(kT_b + T_d - 2a) = Δq(kT_b), \\
Δ\dot{q}((k+1)T_b) &= -Δ\dot{q}(kT_b + T_d - 2a) = Ω(kT_b).
\end{align*}
\]

Crucially, the relations (11) and (12) constitute necessary conditions to be satisfied by breather solutions. Thus, for given values \( Δq(kT_b) \), \( Δ\dot{q}(kT_b) \) at the beginning of intervals \( I_k \) the solution \( Δq(kT_b + T_d - 2a) \), \( Δ\dot{q}(kT_b + T_d - 2a) \) at the end of intervals \( I_k \) can be utilised to derive a first recurrence (Poincaré) map \((Δq(jT_b), Δ\dot{q}(jT_b)) \) for which breathers constitute fixed points.

The time evolution of the difference variable \( Δq(t) \) is determined by the following equation

\[
\frac{d^2Δq}{dt^2} = -[U'(q_{max}) - U'(q_{min})] - \gamma(q_{max} - q_{min}) + B(t)(q_{max} - q_{min}) - \sum_{j∈N_r(n)} \{k_{max} [V'(q_{max} + j - q_{max}) + V'(q_{max} - q_{max} - j)] - k_{min} [V'(q_{min} + j - q_{min}) + V'(q_{min} - q_{min} - j)]\},
\]

with

\[
k_{max} = \max_{1≤j≤r} k_j; \quad k_{min} = \min_{1≤j≤r} k_j.
\]

Notice that the direct driving field \( A(t) \) has no impact on \( Δq(t) \). Regarding the maintenance of localisation the inequality \( Δq(t) ≥ 0 \) constitutes a necessary condition. Regarding the equal sign, for localised solutions, such as breathers, where the oscillators perform in-phase motion (and/or out-of-phase motion) \( Δq(t) \) is zero only at instants of time when the oscillators, whilst performing periodic motion inside their potential wells, attain simultaneously the given values \( ∆ \) for which breathers constitute fixed points.

For the forthcoming derivations of estimates we facilitate the following statement:

**Lemma:** For soft potentials \( U(q) \) with two inflection points \( q_{l,±} \) consider the interval

\[
I_s := [q_l, q_r] \text{ with } q_{l,−} < q_l, \text{ and } q_r < q_{l,+}.
\]

Then it holds that for any pair \( x, y ∈ I_s \) with \( x < y \)

\[
[U'(y) - U'(x)] > δ_s (y - x) > 0
\]

where the constant \( δ_s > 0 \) is given by

\[
δ_s = \min_{q ∈ I_s} U''(q) = \min [U''(q_l), U''(q_r)].
\]

For hard potentials consider the interval \( I_h := [x_l, x_r] \) with \( −∞ < x_l, x_r < ∞ \). Then it holds that for any pair \( x, y ∈ I_h \) with \( x < y \)

\[
[U'(y) - U'(x)] > δ_h (y - x) > 0
\]

and the constant \( δ_h > 0 \) is given by

\[
δ_h = \min_{q ∈ I_h} U''(q) = U''(0).
\]

**Proof:** Consider the expression

\[
F(x, y) = \frac{U'(y) - U'(x)}{y - x}.
\]
By assumptions (6) and (7) we have that on intervals $I_s$ and $I_h$ it holds that $U'(y) > U'(x)$ for $y > x$. Therefore the expression $F(x, y)$ is positive. Furthermore, by virtue of the mean value theorem there exist a point $z$ in $(x, y)$ such that
\[
\frac{U'(y) - U'(x)}{y - x} = U''(z) \geq \min_{q \in I_s, I_h} U''(q).
\] (21)

One has for soft potentials $\min_{q \in I_s} U''(q)) = \min[U''(q_i), U''(q_r)] > 0$, and therefore it holds that
\[
U'(y) - U'(x) \geq \min[U''(q_i), U''(q_r)](y - x) = \delta_s(y - x) > 0.
\] (22)

Similarly for hard potentials by the assumption (2) one has $\min_{q \in I_h} U''(q)) = U''(0) > 0$, so that
\[
U'(y) - U'(x) \geq U''(0)(y - x) = \delta_h(y - x) > 0.
\]
completing the proof.

Remark: To apply Lemma to the case of soft potentials with a single inflection point $q_i > 0$ one proceeds along the lines given above for the Lemma considering the interval $(-\infty, q_i]$ and $q_r < q_i$. The positive constant $\delta_s$ is given by $\delta = \min_{q \in I_s} U''(q)) = U''(q_r)$.

In the following we present conditions for which $\Delta q(t)$, associated with a breather solution satisfying the conditions listed above, exponentially decays which rules out the existence of breather solutions to Eq. (1).

**Theorem:** Let the relation $(\gamma/2)^2 > \omega^2_0 - B_{\text{max}} > 0$ be valid with $\omega^2_0 = \delta_s$ and $\omega^2_0 = \delta_h$ for soft and hard on-site potentials given in Eq. (17) and (19) respectively. Then it holds that Eq. (1) does not possesses breather solutions.

**Proof:** We prove the assertion by contradiction. That is we suppose that Eq. (1) exhibits breather solutions associated with periodic functions $\Delta q(t + T_h) = \Delta q(t)$ and $\Delta \dot{q}(t + T_h) = \Delta \dot{q}(t)$ satisfying the necessary conditions in (11) and (12). Using the conditions in (2) and (3) together with the Lemma enables us to bound the r.h.s. of Eq. (13) on each of the intervals $I_k$, $k = 0, 1, \ldots$, defined in (10), from above as follows:

\[
\frac{d^2\Delta q}{dt^2} = -[U'(q_{\text{max}}) - U'(q_{\text{min}})] - \gamma(q_{\text{max}} - q_{\text{min}}) + B(t)(q_{\text{max}} - q_{\text{min}})
- \sum_{j \in N, (n)} \kappa_{\text{max}} \left( V'(q_{\text{max}} + q_{\text{max}}) + V'(q_{\text{max}} - q_{\text{max}}) \right)
- \kappa_{\text{min}} \left( V'(q_{\text{min}} + q_{\text{min}}) + V'(q_{\text{min}} - q_{\text{min}}) \right)
\leq -\omega^2_0(q_{\text{max}} - q_{\text{min}}) - \gamma(q_{\text{max}} - q_{\text{min}}) + B_{\text{max}}(q_{\text{max}} - q_{\text{min}})
- \gamma \frac{d\Delta q}{dt}.
\]

Therefore, by the comparison principle for differential equations, $\Delta q(t)$ and $\Delta \dot{q}(t)$ are bounded from above by the solution of

\[
\frac{d^2\Delta q}{dt^2} = -\omega^2_0 - B_{\text{max}})\Delta q - \gamma \frac{d\Delta q}{dt}.
\] (23)

The solution to Eq. (23) with initial conditions $\Delta q_{0,k} = \Delta q(kT_h) > 0$, $\Delta \dot{q}_{0,k} = \Delta \dot{q}(kT_h) > 0$ is given for $(\gamma/2)^2 > \omega^2_0 - B_{\text{max}} > 0$ on each interval $I_k$ by

\[
\Delta q_k(t) = \text{exp} \left(-\frac{\gamma t}{2}\right) \left[ \frac{\Delta \dot{q}_{0,k} + \frac{7}{2} \Delta q_{0,k}}{W} \sinh(Wt)
+ \Delta q_{0,k} \cosh(Wt) \right],
\] (24)
\[ \Delta q_k(t) = \exp \left( -\frac{\gamma}{2} t \right) \left[ W \left( 1 - \left( \frac{\gamma}{2W} \right)^2 \right) \Delta q_{0,k} \right.
\]
\[ - \left. \frac{\gamma}{2W} \Delta q_{0,k} \right] \sinh(Wt) + \Delta q_{0,k} \cosh(Wt) \]

(25)

where the index \( k \) refers to the interval \( I_k \) and \( W = \sqrt{(\gamma/2)^2 - (\omega_0^2 - B_{max})} \). Due to the Eqs. \( 11,12 \), fulfilled by breather solutions, the following recursion relations are true

\[ \Delta q_{0,k+1} = \Delta q_k (kT_b + T_d - 2a) \]
\[ \Delta \dot{q}_{0,k+1} = -\Delta \dot{q}_k (kT_b + T_d - 2a) \]

(26)

(27)

with starting values \( \Delta q_{0,k=0} > 0 \) and \( \Delta \dot{q}_{0,k=0} > 0 \) (see above). Using the latter recursions and the notation \( Q_j = \Delta q(jT_b) \) and \( P_j = \Delta \dot{q}(jT_b) \) with \( j = 0,1, \ldots \) we cast the solution in form of a first recurrence (Poincaré map)

\[ \left( \begin{array}{c} Q_{j+1} \\ P_{j+1} \end{array} \right) = M \left( \begin{array}{c} Q_j \\ P_j \end{array} \right) \]

where the matrix \( M_j \) is given by

\[ M = E \left( \frac{(C + \frac{\gamma WS}{2W})}{W \left( \left( \frac{\gamma}{2W} \right)^2 - 1 \right) S} \left( \frac{1}{W} S - C \right) \right) \]

with entries

\[ E = \exp \left( -\frac{\gamma}{2} (T_d - 2a) \right) \]
\[ C = \cosh(W(T_d - 2a)) \]
\[ S = \sinh(W(T_d - 2a)). \]

(28)

(29)

(30)

For the determinant of \( M \) one obtains

\[ \det M = -E^2 \left( C^2 - S^2 \right) = -E^2. \]

(31)

As \( |\det M| < 1 \) the Poincaré map is contractive and for any initial condition \( \Delta q(0), \Delta \dot{q}(0) \) the quantities \( \Delta q(jT_b) \) and \( \Delta \dot{q}(jT_b) \) exponentially decay and fall eventually below their initial values \( \Delta q(0) = \Delta q_0 > 0 \) and \( \Delta \dot{q}(0) = \Delta \dot{q}_0 > 0 \) so that \( \Delta q(t) \) and \( \Delta \dot{q}(t) \) converge uniformly to zero which is in contradiction to the condition of periodic behaviour of non-vanishing \( \Delta q(t) = \Delta q(t + T_b) \) and \( \Delta \dot{q}(t) = \Delta \dot{q}(t + T_b) \) and the proof is complete.

\[ \square \]

Conclusively, our theorem provides conditions that rule out the existence and/or formation of breather solutions.

**Corollary:** Breather solutions to Eq. \( 11 \) can only exist for

\[ B_{max} > \omega_0^2. \]

(32)

Remarkably, the process of exponential decay takes place regardless of the amplitude of the external field \( A(t) \). Furthermore, exponential decay happens for any kind of attractive interaction potential \( V(u) \). As far as hard on-site potential \( U(q) \) is concerned, *only its curvature at the bottom, \( U''(0) \), plays a role* for the decay process and the larger is the curvature the faster is the exponential decay while increasing the amplitude of the parametric driving \( B_{max} \) has the opposite effect. Note that in order that the theorem applies the latter has to fulfill the constraint \( B_{max} < \omega_0^2 \). Importantly, the result holds for general driving fields. Interestingly, in our upper bound the decay rate turns out to be independent of the initial distribution of the amplitudes and velocities \( \{ q_0(0) \} \) and \( \{ \dot{q}_0(0) \} \). They influence the amplitude of the decay of \( \Delta q(t) \) and \( \Delta \dot{q}(t) \) though.

We stress that the hypothesis \( (\gamma/2)^2 > \omega_0^2 - B_{max} > 0 \) can be satisfied for arbitrarily small values of the damping strength \( \gamma \) as for given \( \omega_0^2 = U''(0) \) for hard on-site potentials \( (\omega_0^2 = \min[U''(q_l), U''(q_r)]) \) for soft on-site potentials) the amplitude of the parametric driving field \( B_{max} \) can be tuned to control the infimum of \( \gamma \) complying with the inequality. Hence, \( \gamma \) can be sufficiently small compared to a characteristic frequency of the system (which is e.g. given by oscillations near the bottom of a potential well with frequency determined by \( U''(0) \)) so that the system’s dynamics is kept away from the overdamped limit.
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FIG. 1: Time evolution of $\Delta q(t)$ exhibiting on average exponential decay in accordance with inequality (24) for the system given in Eq. (33). The values of the parameters are $\gamma = 0.35$, $A_0 = B_0 = 0.99$, $\Omega_A = \Omega_B = 2.0$, $\Theta_A = \Theta_B^0 = 0$, and $\kappa = 0.5$.

Finally we remark that it is certainly of interest to extend the present study to systems that are discrete not only in space but also in time utilising the methods outlined in [29].

For an illustration of the exponential decay of an initially localised solution we choose for the hard on-site potential

$$U(q) = \frac{1}{2}q^2 + \frac{1}{4}q^4.$$  

The interaction potential is harmonic and is given by

$$V(q_n - q_{n-1}) = \frac{1}{2}(q_n - q_{n-1})^2,$$

where the interaction radius is taken as $r = 1$ amounting to linear nearest-neighbour interaction and the coupling strength is uniform, i.e. $\kappa_n = \kappa$. As the external fields are concerned we consider time-periodically varying fields and set for the direct driving field $A(t)$

$$A(t) = A_0 \sin(\Omega_A t + \Theta_A^0),$$

with amplitude $A_0$, frequency $\Omega_A$ and phase $\Theta_A^0$. Similarly, for the parametric driving field $B(t)$ we choose

$$B(t) = B_0 \sin(\Omega_B t + \Theta_B^0),$$

with amplitude $B_0$, frequency $\Omega_B$ and phase $\Theta_B^0$.

The corresponding lattice system is given by

$$\ddot{q}_n = -q_n - q_n^3 + \kappa(q_{n+1} - 2q_n + q_{n-1}) + \beta \dot{q}_n + A_0 \sin(\Omega_A t + \Theta_A^0) + B_0 \sin(\Omega_B t + \Theta_B^0)q_n.  \tag{33}$$

In our simulation the system comprises $N = 100$ oscillators and periodic boundary conditions are imposed. We plot in Fig. 1 the temporal behaviour of $\Delta q(t)$ for the system (33) starting from a localised single hump solution peaked around the site $n = 50$ associated with initial conditions $q_n(0) = 1/cosh(n - 50)$ and $\dot{q}_n(0) = 0$. $\Delta q(t)$ exponentially decays on average which is in accordance with the inequality (24) bounding the amplitude of $\Delta q(t)$ from above. The corresponding spatio-temporal evolution of $q_n(t)$ is shown in Fig. 2 corroborating the exponential decay of a spatial pattern. Eventually the dynamics settles on a spatially uniform state and the oscillators perform identical motion,

$$q_n(t) = q(t) \text{ and } \dot{q}_n(t) = \dot{q}(t) \text{ for all } n,$$

entailing that the oscillators decouple. The oscillators undergo periodic motion on a limit cycle supporting periodic oscillations of the variables $q_n(t)$.

In conclusion, we have studied the persistence and formation of non-homogeneous patterns represented by breather solutions in general nonlinear damped and driven lattice systems. Sufficient conditions, in terms of the values of the parameters, have been provided which assure that no time-periodic non-uniform state can exist. To be precise, it has been proven that the difference between the maximal and minimal amplitudes of the lattice oscillators of a non-uniform time-periodic state decays exponentially fast. In this way we have proven that creation and/or preservation of time-periodic, spatially (localised) patterns is impossible. Notably our results are independent of the number of oscillators and hold for arbitrary dimension of the system. Conversely, rigorous quantitative conditions are identified under which
discrete breathers can exist in general driven and damped lattices at all. Furthermore, our generic results on the non-existence of time-periodic space-localised patterns and their formation in general nonlinear lattice systems open the way for further research on the emergent dynamical scenarios, in particular features of synchronisation, in coupled cell networks. Given that we have provided quantitative criteria in parameter space for the existence/nonexistence of discrete breathers the current work is also expected to stimulate further experimental studies of breathers in nonlinear damped and driven lattice systems.

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