Inductive Lusternik-Schnirelmann category in a model category

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Abstract
We introduce the notion of inductive category in a model category and prove that it agrees with the Ganea approach given by Doeraene. This notion also coincides with the topological one when we consider the category of (well-) pointed topological spaces.

Introduction

The Lusternik-Schnirelmann category $\text{cat}(X)$ of a space $X$, LS-category for short, is a homotopy numerical invariant which was introduced by the quoted authors in the early 30’s in their research on calculus of variations [12]. It has turn out to be an important invariant not only in algebraic topology but also in other important subjects in mathematics such as differential topology or dynamical systems. For an excellent introduction on LS-category theory we refer the reader to [2] and [11]. Unfortunately, although its definition is quite simple to establish, the LS-category of a space is hard to compute. Therefore since its beginnings there have been different attempts of giving alternative descriptions, approaches or reasonable bounds in a more algebraic way. There are four standard formulations of LS-category, at least equivalent for a large class of topological spaces:

1. The definition by coverings [5]: that it, the category $\text{cat}(X)$, of a space $X$ is the least $n$ (or infinite) for which there is a covering of $X$ by $n+1$ open subspaces, each of them contractible in $X$.

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2. The Whitehead characterization [16]: \( \text{cat}(X) \leq n \) if and only if the \( n+1 \) diagonal map can be factorized, up to homotopy, through the \( n \)-th fat wedge.

3. The Ganea characterization [6]: \( \text{cat}(X) \leq n \) if and only if the \( n \)-th Ganea fibration (fibre-cofibre construction) admits a homotopy section.

4. The inductive category [6], [7]: \( \text{indcat}(X) = 0 \) if and only if \( X \) is contractible; and \( \text{indcat}(X) \leq n \) if and only if there exists a cofibration \( A \to Y \to C \) such that \( \text{indcat}(Y) \leq n - 1 \) and \( C \) dominates \( X \).

The three latter notions are functorial and also the most usual and successful ones. These approaches have played an important role during the development of this invariant and we may assert that without them it would have been much more difficult to achieve many important accomplishments given in this subject.

At the same time, many direct-upper or lower-bounds of the LS-category have appeared such as the strong category, the weak category or the sigma-category. Yet, there is also an important technique for obtaining lower bounds. It consists of taking models for topological spaces in some algebraic category where a LS-category-type homotopy invariant is defined. Necessarily, this invariant must be established in an abstract homotopy setting-hopefully a model category in the sense of Quillen [13], [14]. Then, the algebraic LS-category of the model of \( X \) is a lower bound of the original LS-category of such space. We can mention the notorious work of Félix and Halperin [4] in rational homotopy, where they defined a numerical homotopy invariant, \( \text{cat}_0 \), in the category of commutative cochain algebras over \( \mathbb{Q} \) and proved that the LS-category of the rationalization of a space agrees with the \( \text{cat}_0 \) of its Sullivan model. Halperin and Lemaire [8] also defined important similar numerical invariants in certain full subcategories. Thus, different algebraic LS-category-type homotopy invariants have been appearing in several categories others than topological spaces.

In order to give a unified theory, basically a generalization of LS-category in all these categories, Doeraene [3] successfully introduced an intrinsic notion of LS-category of an object in a Quillen’s model category. Actually, he established what he called J-category, where cofibrations, fibrations and weak equivalences take part. This structure is determined by a certain set of axioms, which are sufficient to develop an abstract LS-category theory. Inspired by the topological case he gave two ’a priori’ different notions, analogous to the Ganea and Whitehead characterizations. A crucial point, the cube axiom, gives the expected equivalence between these notions. Later, a third equivalent notion, inspired
by the original topological LS-category notion by coverings, was established by Hess and Lemaire [10]. However, since now, it seems that nobody has noticed the lack of a notion of abstract inductive category in this framework, equivalent with the latter.

Our aim in this paper is to give this fourth equivalent notion in a model category, analogous to that of the topological inductive category. In fact, we prove that our abstract inductive category agrees with the Ganea approach in a J-category. It is also important to remark that in the case of topological spaces it coincides with the usual inductive category.

We have divided this article in two sections. In the first one we give some background about J-categories, necessary for the rest of the paper. Then, in section 2 we introduce the main notion of this paper, indcat, the inductive category, as well as the corresponding establishment indcat \equiv \text{cat}. For that goal we firstly set and give some properties of a certain notion of domination, a bit weaker than having a weak section, as introduced by Doeraene. Finally we display some examples where our theory may be of interest.

1 Preliminaries

This section is devoted to recall all the notions and results that will be used along this paper. We begin by giving the definition of a J-category.

A J-category is category \mathcal{C} together with a zero object 0 and three classes of morphisms called fibrations (\rightarrow), cofibrations (\hookrightarrow) and weak equivalences (\cong), satisfying the following set of axioms (J1)-(J5). Before stating such axioms some points should be clarified: A morphism which is both a fibration (resp. cofibration) and a weak equivalence is called trivial fibration (resp. trivial cofibration). An object \text{}B is called cofibrant model (resp. fibrant model) if every trivial fibration \text{}p : E \sim \rightarrow B admits a section (resp. if every trivial cofibration \text{}i : B \sim \rightarrow E admits a retraction).

1.1 The axioms of a J-category.

\begin{enumerate}
  \item [(J1)] Isomorphisms are trivial cofibrations and trivial fibrations. The composite of fibrations (resp. cofibrations) is a fibration (resp. a cofibration). Given \text{}f : X \rightarrow Y and \text{}g : Y \rightarrow Z morphisms, if any two of \text{}f, \text{}g, \text{}gf are weak equivalences then so is the third.
  \item [(J2)] For any fibration \text{}p : E \rightarrow B and morphism \text{}f : B' \rightarrow B the pull-back
exists in $\mathcal{C}$

$$
\begin{array}{ccc}
E' & \xrightarrow{\overline{\tau}} & E \\
\downarrow{\overline{\pi}} & & \downarrow{\pi} \\
B' & \xrightarrow{\overline{f}} & B \\
\end{array}
$$

and the base extension $\overline{\pi}$ is a fibration. Moreover, if $f$ is a weak equivalence then so is $\overline{\tau}$; and if $p$ is a weak equivalence then so is $\overline{\pi}$.

Dually, for any cofibration $i : A \hookrightarrow B$ and morphism $f : A \to A'$ the push-out of $i$ and $f$ exists, and the cobase extension $\overline{i}$ of $i$ is a cofibration. If $f$ is a weak equivalence then so is its cobase extension $\overline{f}$; and if $i$ is a weak equivalence then so is $\overline{i}$.

(J3) For any map $f : X \to Y$ there exist

(i) an $F$-factorization, that is, a factorization $f = p\tau$ where $\tau$ is a weak equivalence and $p$ is a fibration; and

(ii) a $C$-factorization, that is, a factorization $f = \sigma i$, where $i$ is a cofibration and $\sigma$ is a weak equivalence.

(J4) Given any object $X$ in $\mathcal{C}$, there exists a trivial fibration $F \xrightarrow{\sim} X$, where $F$ is a cofibrant model.

For the next axiom, we need the definitions of homotopy pull-back and homotopy push-out. A commutative square

$$
\begin{array}{ccc}
D & \xrightarrow{f'} & C \\
\downarrow{g'} & & \downarrow{g} \\
A & \xrightarrow{f} & B \\
\end{array}
$$

is said to be a homotopy pull-back if for some (equivalently any) $F$-factorization of $g$, the induced map from $D$ to the pull-back $E' = A \times_B E$ is a weak equivalence
We can also use an $F$-factorization of $f$ instead of $g$ (or both). The Eckmann-Hilton dual notion, taking a $C$-factorization and a push-out, is called homotopy push-out.

(J5) *The cube axiom.* Given any commutative cube where the bottom face is a homotopy push-out and the vertical faces are homotopy pull-backs then the top face is a homotopy push-out.

The fundamental construction which can be made in a $J$-category is that of the join of two objects over a third.

**Definition 1.1.** Given two morphisms $f : A \to B$ and $g : C \to B$ with the same target, we consider their join $A \ast_B C$ as follows. First consider any $F$-factorization of $g = pr$ and the pull-back of $f$ and $p$. Let $\tau$ and $\tau^*$ the base extensions of $f$ and $p$ respectively. Then take any $C$-factorization of $\tau^* = \sigma i$ and the push-out of $\tau$ and $i$. This push-out object is denoted by $A \ast_B C$ and called the join of $A$ and $C$ over $B$. The dotted induced map from $A \ast_B C$ to $B$ is called the join morphism of $f$ and $g$.

Two objects $X$ and $Y$ in $C$ are said to be *weakly equivalent* if there exists a finite chain of weak equivalences joining $X$ and $Y$

$$X \longrightarrow \cdots \longrightarrow Y$$

where the symbol $\bullet \longrightarrow \bullet$ means an arrow with either left or right orientation.

The object $A \ast_B C$ and the join map are well defined and symmetrical up to weak equivalence.

**Definition 1.2.** Let $f : A \to B$ and $g : C \to B$ be morphisms in $C$. We say that $f$ admits a weak lifting along $g$ if for some (equivalently for any) $F$-factorization
There exists a commutative diagram

\[ \begin{array}{ccc}
C & \xrightarrow{g} & E \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B
\end{array} \]

In the particular case \( f = \text{id}_B \) we say that \( g : C \to B \) admits a weak section.

Now we are ready to the definition of \( n \)-th-Ganea map as well as the category of a given object \( B \) in \( C \).

**Definition 1.3.** Let \( B \) an object in \( C \). The \( n \)-th Ganea object \( G_nB \), as well as the \( n \)-th Ganea map \( p_n^B : G_nB \to B \) (\( n \geq 0 \)) are constructed as follows: For \( n = 0 \) let \( p_0^B \) be the zero map \( 0 \to B \); if \( p_{n-1}^B \) is already constructed then \( p_n^B \) is defined as the join map of \( 0 \to B \) and \( p_{n-1}^B : G_{n-1} \to B \) (so \( G_nB = 0 \ast_B G_{n-1}B \)).

**Remark 1.4.** This construction is functorial. Given \( f : B \to B' \) any morphism in \( C \) we can construct a morphism \( G_n(f) : G_nB \to G_nB' \) such that \( p_n^{B'}G_n(f) = fp_n^B \).

**Definition 1.5.** \[3, 3.8\] We say that \( \text{cat}(B) \leq n \) if and only if the \( n \)-th Ganea map \( p_n : G_nB \to B \) admits a weak section. If no such \( n \) exists then \( \text{cat}(B) = \infty \).

Doeraene proved that this construction is invariant by weak equivalence, that is, if \( B \) and \( B' \) are weakly equivalent objects then \( \text{cat}(B) = \text{cat}(B') \).

We note that, actually, this definition holds in any pointed category (that is, with zero object) verifying (J1)-(J4) axioms, without the cube axiom (J5). The cube axiom is just needed to prove that a second definition, the Whitehead approach, is equivalent to this one \[3\]. Since we are just dealing with the Ganea approach, (J5) axiom will no be needed in this article. On the other hand, instead of (J3) and (J4) axioms we are interested in the following slightly stronger ones, given in any model category.

**M1** Given any commutative diagram of solid arrows

\[ \begin{array}{ccc}
A & \xrightarrow{i} & E \\
\downarrow & & \downarrow \\
X & \xrightarrow{p} & B
\end{array} \]

where \( i \) is a cofibration, \( p \) is a fibration and either \( i \) or \( p \) is a weak equivalence, then the dotted arrow exists making commutative the two triangles.
(M2) Any map \( f : X \to Y \), can be factored in two ways:

(i) \( f = p\tau \), where \( \tau \) is a trivial cofibration and \( p \) is a fibration (\( F \)-factorization).

(ii) \( f = \sigma i \), where \( i \) is a cofibration and \( \sigma \) is a trivial fibration (\( C \)-factorization).

From (M2) axiom, (J3) is obviously satisfied. Yet (J4) is also fulfilled; indeed for any object \( X \) we can consider the following factorization

\[
\begin{array}{ccc}
0 & \rightarrow & X \\
\downarrow & & \downarrow \\
QX & \rightarrow & RX
\end{array}
\]

obtaining, taking into account (M1) axiom, a cofibrant model.

Dually, we can also consider its fibrant model, obtained by the factorization of the zero map \( X \to 0 \) through a trivial cofibration \( i_X : X \sim RX \) followed by a fibration. We also note that given any map \( f : X \to Y \) it is possible to find a map \( Qf : QX \to QY \) (respectively \( Rf : RX \to RY \)) such that \( fp_X = p_Y Qf \) (respectively \( Rfi_X = i_Y f \)).

At the sight of these remarks, the framework in which we will be immersed throughout this paper could be a pointed proper model category. Nevertheless, the reader may also think that we are considering a pointed category \( C \) such that (J1),(J2), (M1) and (M2) axioms are satisfied.

## 2 The inductive category.

We begin by giving the notion of domination, which is weaker of that of having a weak section.

**Definition 2.1.** Given \( X,Y \) objects in \( C \) we will say that \( X \) dominates \( Y \) (denoted \( X \gg Y \)) if for some (equivalent any) cofibrant model \( QX \) of \( X \) and for some (equivalent any) fibrant model \( RY \) of \( Y \) there exists a morphism \( \alpha : QX \to RY \) such that \( i_Y : Y \sim RY \) admits a weak lifting along \( \alpha \).
Note that this definition agrees with the one in the case of topological spaces. The following properties will be useful for our notion of inductive category.

**Proposition 2.2.**

1. \( X \gg X \), for any object \( X \) in \( C \).
2. If \( X \) and \( Y \) are weakly equivalent then \( X \gg Z \) if and only if \( Y \gg Z \).
3. If \( X \xrightarrow{f} Y \) admits a weak section then \( X \gg Y \).
4. If \( X \gg Y \) then \( \text{cat } Y \leq \text{cat } X \).

**Proof:**

1. This item is easily proved taking the composite \( QX \xrightarrow{\sim} X \xrightarrow{\sim} RX \).
2. We can suppose, without losing generality, that there is a weak equivalence \( w : X \xrightarrow{\sim} Y \). Assume that \( X \gg Z \); then there is a morphism \( \alpha : QX \to RZ \) such that \( iZ \) admits a weak lifting along \( \alpha \).

We can take a cofibrant model of \( Y \) in such a way \( Qw : QX \xrightarrow{\sim} QY \) is a trivial cofibration. Since \( RZ \) is a fibrant object we can consider a morphism \( \lambda : QY \to RZ \) such that \( \lambda Qw = \alpha \). Taking into account that the existence of the weak lifting along \( \alpha \) is independent of the chosen factorization of \( \alpha \), we have that \( Y \gg Z \). We have just to take into account the following diagram:

\[
\begin{array}{cccccc}
QY & \xrightarrow{Qw} & QX \\
\downarrow & & \downarrow \sim \\
Z & \xrightarrow{\sim} & RX \\
\end{array}
\]

The converse is straightforward and left to the reader.

3. Consider a weak section for \( f \) a cofibrant model \( QX \) for \( X \) and a fibrant model \( RY \) for \( Y \). Consider also \( E \xrightarrow{h} E' \xrightarrow{g} RY \) an \( F \)-factorization of \( i_Y \circ q \). Then the result follows from
the following commutative diagram

\[
\begin{array}{ccc}
QX & \xrightarrow{hlp_X} & X \\
\downarrow & & \downarrow \\
E' & \xrightarrow{g} & RY \\
\downarrow & & \downarrow \\
Y & \xrightarrow{i} & RY \\
\end{array}
\]

\(hlp_X\)
\(g\)
\(i\)

iv) Suppose that \(\text{cat} X = n\) and denote by \(p : E \to RY\) and \(s : Y \to E\) as given in definition [2.1]. Since \(\text{cat} X = \text{cat} QX = \text{cat} E\), there exists \(\sigma\) a section of \(p^n : G_nE \to E\). Then we have a lifting

\[
\begin{array}{ccc}
Y & \xrightarrow{\sigma s} & G_n(RY) \\
\downarrow & & \downarrow \\
RY & \xrightarrow{id} & RY \\
\end{array}
\]

showing that \(\text{cat} Y = \text{cat} RY \leq n\).  

Now we recall the notion of weak push-out, given in [3].

**Definition 2.3.** [3, 2.3] Let \(f : A \to B, f' : A' \to B'\) and \(a : A' \to A\) be morphisms in \(\mathcal{C}\). We say that \(A' - A - B - B'\) forms a weak push-out if for some (equivalently, any) \(\mathcal{C}\)-factorization \(A' \xrightarrow{i} X \xrightarrow{\sigma} B'\) of \(f'\), one has a homotopy push-out \(A' - A - B\).

\[
\begin{array}{ccc}
A' & \xrightarrow{a} & A \\
\downarrow & & \downarrow \\
B' & \xleftarrow{\sim} & X \\
\downarrow & & \downarrow \\
B & \xleftarrow{\sigma} & X \\
\end{array}
\]

We say that \(f\) is the weak cobase extension of \(f'\) by \(a\).

In particular, if \(a : A' \to 0\) is the zero map we say that the map \(x\) (or just the object \(B\)) is the homotopy cofibre of \(f' : A' \to B'\).

Given \(f' : A' \to B'\) and \(a : A' \to A\) morphisms in \(\mathcal{C}\), the usual way to obtain their weak push-out (up to weak equivalence) is to consider a \(\mathcal{C}\)-factorization \(f' = \sigma i\) and then the push-out of \(i\) along \(a\). Considering the gluing lemma [1], this construction is well-defined and symmetrical up to weak equivalence (i.e. we may take a \(\mathcal{C}\)-factorization \(a = \sigma i\) and then form the push-out of \(i\) along \(f'\)).

Then, given \(f : A \to Y\) morphism in \(\mathcal{C}\), its cofibre sequence \(A \overset{i}{\to} Y \overset{p}{\to} C\)
might be obtained as the following push-out

\[
\begin{array}{c}
A \xrightarrow{f} Y \\
\downarrow k \quad \downarrow p \\
0 \xleftarrow{CA} C
\end{array}
\]

Now we are giving the main notion of this paper.

**Definition 2.4.** Let \(X\) be any object in \(C\). The inductive category of \(X\), \(\text{indcat } X\), is defined as follows: \(\text{indcat } X = 0\) if and only if the zero map \(0 \rightarrow X\) admits a weak section; for \(n \geq 1\), \(\text{indcat } X \leq n\) if and only if there exists a cofibre sequence

\[
A \xrightarrow{f} Y \xrightarrow{p} C
\]

such that \(\text{indcat } Y \leq n - 1\) and \(C \gg X\).

**Remark 2.5.** Considering proposition 2.2 it is straightforward to check that \(\text{indcat } X\) is well defined and it is invariant up to weak equivalence.

**Theorem 2.6.** Given \(X\) any object in \(C\) we have

\[\text{indcat } X = \text{cat } X\]

**Proof:**

By using inductive arguments and proposition 2.2 we can plainly see that \(\text{indcat } G_k(X) \leq k\), for all \(k \geq 0\). Therefore, and again by the same proposition 2.2 we have \(\text{indcat } X \leq \text{cat } X\).

For the inequality \(\text{cat } X \leq \text{indcat } X\) we proceed by induction on the integer \(\text{indcat } X = k\) and the object \(X\). The result is obvious for \(k = 0\). Now suppose that the statement is true for any object \(Z\) and \(k \leq n - 1\) and that \(\text{indcat } X = n\). Then, there is a cofibre sequence \(A \xrightarrow{f} Y \xrightarrow{p} C\) such that \(C \gg X\) and \(\text{cat } Y = \text{indcat } Y \leq n - 1\).

We explicitly give a construction, up to weak equivalence, of the \(n\)th Ganea
map of $C$ as explained in the next diagram:

\[
\begin{array}{c}
F_{n-1}(C) \xrightarrow{p_{n-1}^C} CA \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
G_{n-1}(C) \xrightarrow{p_{n-1}^C} C
\end{array}
\]

Observe that we can assume that $p_{n-1}^C : G_{n-1}C \to C$ is already a fibration. Then the homotopy fibre $F_{n-1}(C)$ of $p_{n-1}^C$ may be obtained, up to weak equivalence, as the pull-back of $p_{n-1}^C$ and $f : CA \to C$. Next we factor $p_{n-1}^C$, the base change of $p_{n-1}^C$, as a cofibration $\varepsilon$ followed by a trivial fibration $q$. Finally, take the push-out $G_n(C)$ of $\varepsilon$ and $f$ as well as the push-out map $p_n^C$.

Considering a section $s : Y \to G_{n-1}(Y)$ of $p_{n-1}^Y : G_{n-1}(Y) \to Y$ and the map $G_{n-1}(p) : G_{n-1}(Y) \to G_{n-1}(C)$ the Ganea construction of $p$ we can take the pull-back map

$$\lambda = (G_{n-1}(p)s, f, k) : A \to F_{n-1}(C).$$

By (M1) axiom take $q'$ any lift in the diagram

\[
\begin{array}{c}
A \xrightarrow{\varepsilon\lambda} CA \\
\downarrow k \downarrow \downarrow \downarrow \downarrow \\
CA \xrightarrow{f} CA
\end{array}
\]

Then one can straightforwardly check that the push-out map

$$\sigma = (\mu q', wG_{n-1}(p)s) : C \to G_n(C)$$

is a section of $p_n^C$. Following proposition 2.2.iv) we have the desired result.

**Remark 2.7.** We point out that all these notions and results have their dual in the sense of Eckmann-Hilton. In the obvious way the notion of inductive cocategory can be defined; and using the dual results we have that the inductive cocategory agrees with the Ganea approach of cocategory (by using cojoins).
2.1 Some examples

Now we review some examples of pointed categories in which the inductive category can be applied. Of course, any pointed proper model category may be taken.

(i) \( \text{Top}^w \), the category of well-pointed topological spaces and continuous maps which preserve the base point.

By a well-pointed space we mean a pointed space \( X \) in which the inclusion of the base point in \( X \) is a closed topological cofibration (that is, a closed map such that verifies the homotopy extension property). Considering

- Fibrations: the Hurewicz fibrations,
- Cofibrations: the closed cofibrations,
- Weak equivalences: the homotopy equivalences,

then \( \text{Top}^w \) together these classes of maps satisfies (J1), (J2), (M1) and (M2) axioms and all spaces are fibrant and cofibrant. In fact, Strøm proved that it verifies stronger conditions [15].

As it is well-known, T. Ganea [6], [7] proved that the inductive (co)category agrees with the Ganea approach of Lusternik-Schnirelmann (co)category in the context of topological spaces. Here we have displayed a different and slightly simpler proof of this fact.

(ii) The category \( S^\bullet \) of pointed simplicial sets and simplicial maps preserving base points, where

- Fibrations: Kan fibrations,
- Cofibrations: injective simplicial maps,
- Weak equivalences: maps whose geometric realizations are homotopy equivalences.

Then, \( S^\bullet \) is a proper model category [13] where all objects are cofibrant and the fibrant objects are the Kan complexes. We can consider Kan’s \( \text{Ex}^\infty \) functor which associates a Kan complex \( \text{Ex}^\infty L \) to any simplicial set \( L \) up to weak equivalence. This construction also involves a simplicial map \( \nu_L : L \rightarrow \text{Ex}^\infty L \), which is, actually, a fibrant model for \( L \). Then, given \( K, L \) pointed simplicial sets, \( K \) dominates \( L \) if there exists a simplicial
map \( \alpha : K \to \text{Ex}^\infty L \) such that \( \nu_L \) admits a weak section along \( \alpha \)

\[
\begin{array}{c}
\text{map} \quad K \\
\downarrow \alpha \\
\text{Ex}^\infty L
\end{array}
\]

(iii) The category \( \text{CDA}^* \) of augmented commutative cochain algebras over a field of characteristic zero. Considering

- Fibrations: Surjective maps.
- Cofibrations: KS-extensions.
- Weak equivalences: Quasi-isomorphism, that is, maps which induce isomorphisms in cohomology.

and the full subcategory \( \text{CDA}^{*\text{co}} \) of c-connected differential algebras. We can find in [9] and [11] that \( \text{CDA}^{*\text{co}} \) with the induced structure satisfies (J1), (J2), (M1) and (M2) axioms, where all objects are fibrant. The \( n \)th Ganea algebra of \( \Lambda X \), a minimal c-1-connected KS-complex, defined by Félix and Halperin [4] was interpreted by Doeraene in another way as a cojoin construction [3]. That is, the dual \( n \)th Ganea map \( \Lambda X \to G^n(\Lambda X) \) admits a weak retraction if and only if the projection \( \Lambda X \to (\Lambda X/\Lambda^{>n}X) \) admits a weak retraction. So in this case the Doeraene’s notion of cocat in \( \text{CDA}^{*\text{co}} \) agrees with the rational LS-category defined by Félix and Halperin. We have another equivalent definition in terms of inductive cocategory:

We say that indcocat \( \Lambda X \leq n \) if there is a fiber sequence (in \( \text{CDA}^{*\text{co}} \))

\[
F \to E \to B
\]

with indcocat \( E \leq n - 1 \) and there exists a map \( \Lambda V \to \Lambda X \) which admits a weak section, being \( \Lambda V \) a minimal model for \( F \).

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