The low–energy modes of a spin–imbalanced superfluid Fermi gas in the Bardeen-Cooper–Schrieffer (BCS) side are studied. The gas is assumed to be sufficiently dilute so that the pairing of atoms can be considered effective only in $s$–wave between fermions of different internal state. The order parameter at equilibrium is determined by the mean–field approximation, while the properties of the collective modes are calculated within a Gaussian approximation for the fluctuations of the order parameter. In particular we investigate the effects of asymmetry between the populations of the two different components and of temperature on the frequency and damping of collective modes. It is found that the temperature does not much affect the frequency and the damping of the modes, whereas an increase of the imbalance shifts the frequency toward lower values and enhances the damping sensitively. Besides the Bogoliubov–Anderson phonons, we observe modes at zero frequency for finite values of the wave–number. These modes indicate that an instability develops driving the system toward two separate phases, normal and superfluid.

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I. INTRODUCTION

Due to the addition of a further state parameter, the ratio between the populations of the two different internal states, imbalanced two–component Fermi gases can display a very rich phase diagram with respect to the symmetric case. For instance, besides the continuous phase–transition, we can observe at sufficiently low temperature a first–order transition between superfluid and normal phases, driven by the imbalance parameter. Moreover, the possibility of tuning the strength of interatomic interactions via Feshbach resonances allows to explore very different physical regimes. Therefore, the studies of such systems involve multifaceted aspects of many–body physics. These peculiar features have stimulated several theoretical works. By referring only to three–dimensional systems, see Refs. [1, 2] for a general and extended review on the subject and Refs. [3–7] for more recent studies of the phase structure of spin–imbalanced Fermi gases. In the meantime experimental investigations have been performed in order to assess the occurrence of different phases in trapped spin–imbalanced Fermi gases. More specifically, phase separations have been indicated by analyzing the density distributions of two–component Fermi gases [8–14].

As the imbalance parameter increases, the Fermi gas experiences different physical situations, which can give rise to qualitatively new effects with respect to the balanced case. In particular the system can exhibit [2] a first order phase transition and the onset of the gapless Sarma phase [15]. We should expect sizable effects of the degree of polarization on the dynamic behavior of the gas. In this paper we present a study on collective modes, having energy below the pair–breaking threshold, of a polarized Fermi gas in the Bardeen-Cooper-Schrieffer (BCS) regime. In particular, we are concerned with the combined effects of the asymmetry of populations and temperature. The parameters of the equilibrium state are determined within the saddle-point approximation to the functional integral representing the grand partition function of the system. For the collective modes we adopt a Gaussian approximation for the time–dependent fluctuations of the order parameter about its equilibrium value.

It is known that the saddle–point approximation or, equivalently, the mean–field approach to the description of the properties of a superfluid Fermi gas has only a qualitative validity when the unitary regime, corresponding to resonant two–body interactions, is approached. This limitation should be ascribed to the fact that the mean–field approximation does not include thermodynamic and quantum fluctuations [16]. The latter may have sizable effects in the region of the BCS–BEC (Bose–Einstein–Condensate) crossover or when the values of temperature are quite close to that of the normal–superfluid phase transition. There have been calculations [17, 18] taking into account the contribution of the fluctuations of the order parameter within a framework similar to that used here. However, the approach of Ref. [17] suffers from a severe drawback: quite generally, one can notice that the Goldstone theorem is violated when the saddle–point equation is just modified away from its mean–field expression [17]. In view of that, we prefer to follow the
mean-field approximation to calculate the equilibrium order parameter. Moreover, our calculations are limited to physical situations not very close to the unitary regime or to the critical temperature of the superfluid phase transition. Then, we may believe that our results are reliable at least at a qualitative level.

The mismatch of the Fermi surfaces of the different species might give rise to the inhomogeneous LOFF phase \[19\text{,}20\]. However, experiments have found no sign of such a phase in three-dimensional Fermi gases. So, its occurrence still remains an open problem \[8\text{–}12\]. Here, we consider only homogeneous phases.

II. FORMALISM

We consider a two-component Fermi gas, whose components are only distinguished by the internal quantum state, ”spin up” (+) and ”spin down” (−). We consider a sufficiently dilute gas so that we can assume that the interatomic interaction has a short range and mainly occurs in the s-wave channel. Because of the Pauli principle the interaction can be effective only between atoms of different species.

In the imaginary-time functional formalism an effective action for the pairing field can be obtained by using a Stratonovich-Hubbard transformation for the couples of fermion fields (see, e.g., Ref. \[21\]). Within this framework we derive the properties of the low-energy collective modes taking into account Gaussian fluctuations of the pairing field about its saddle point value. Here, we extend to the imbalanced case and to finite temperatures the formalism developed in Ref. \[17\] for the Gaussian fluctuations. Further formal details can also be found in Ref. \[22\].

For the sake of completeness and in order to introduce notations, we report the equation for the equilibrium pairing field \(\Delta_{0}\) evaluated within the saddle–point approximation to the effective action. A regularized expression in terms of the scattering amplitude \(a_{S}\) can be obtained with the substitution \[23\text{,}24\]

\[
\frac{1}{|\lambda|} \rightarrow - \frac{m}{4\pi a_{S}} + \int \frac{dk}{(2\pi)^{3}} \frac{1}{2\epsilon(k)},
\]

where \(\lambda < 0\) is the strength of a contact interaction between fermions and \(\epsilon(k) = k^{2}/(2m)\) is the fermion kinetic energy. It reads

\[
- \frac{m}{4\pi a_{S}} = \int \frac{dk}{(2\pi)^{3}} \left[ \frac{1}{4E(0)(k)} \right] \times \left( \tanh \left( \frac{\beta E_{+}(k)}{2} \right) + \tanh \left( \frac{\beta E_{-}(k)}{2} \right) - \frac{1}{2\epsilon(k)} \right),
\]

where \(\beta = 1/T\), \(E(0)(k) = \sqrt{\epsilon(k) - \mu}^{2} + \Delta_{0}^{2}\) with \(\mu = (\mu_{+} + \mu_{-})/2\) the average chemical potential, and \(E_{\pm}(k) = E(0)(k) \mp \eta\) are the quasiparticle energies with \(\eta = (\mu_{+} - \mu_{-})/2\) half the difference in chemical potentials (units such that \(\hbar = k_{B} = 1\) are used).

Adapting the formal calculations of Refs. \[17\text{,}22\] to the present case one can derive, for the propagator of the pairing-field fluctuations

\[
\sigma(q, \tau) = \Delta(q, \tau) - \Delta_{0},
\]

the 2 × 2 set of algebraic equations

\[
\hat{D}(\omega_{n}, q) = \frac{\lambda}{V} \hat{1} - \lambda \int \frac{dk}{(2\pi)^{3}} \hat{A}(\omega_{n}, q, k) \hat{D}(\omega_{n}, q),
\]

in the frequency representation, where \(\omega_{n}\) are the Matsubara frequencies and \(V\) is the normalization volume. The elements of the kernel \(\hat{A}(\omega_{n}, q, k)\) are explicitly given by

\[
A_{1,1}(\omega_{n}, q, k) = \left[ 1 - f(E_{\pm}(k)) - f(E'_{\pm}(k)) \right] \frac{u^{2}(k)u'^{2}(k)}{i\omega_{n} + E_{\pm}(k) + E'_{\pm}(k)} - \left[ 1 - f(E'_{\pm}(k)) - f(E_{\pm}(k)) \right] \frac{v^{2}(k)v'^{2}(k)}{i\omega_{n} - E'_{\pm}(k) - E_{\pm}(k)} \nonumber
\]

\[
- \left[ f(E'_{\pm}(k)) - f(E_{\pm}(k)) \right] \frac{v^{2}(k)u'^{2}(k)}{i\omega_{n} + E'_{\pm}(k) - E_{\pm}(k)} - \left[ f(E_{\pm}(k)) - f(E'_{\pm}(k)) \right] \frac{u^{2}(k)v'^{2}(k)}{i\omega_{n} + E_{\pm}(k) - E'_{\pm}(k)}
\]

(3)

and

\[
A_{1,2}(\omega_{n}, q, k) = - \frac{\Delta_{0}^{2}}{4E(0)(k)E'(0)(k)} \times \left[ 1 - f(E_{\pm}(k)) - f(E'_{\pm}(k)) \right] \frac{1}{i\omega_{n} + E_{\pm}(k) + E'_{\pm}(k)} \nonumber
\]

\[
- \left[ 1 - f(E'_{\pm}(k)) - f(E_{\pm}(k)) \right] \frac{1}{i\omega_{n} - E'_{\pm}(k) - E_{\pm}(k)} + \left[ f(E'_{\pm}(k)) - f(E_{\pm}(k)) \right] \frac{1}{i\omega_{n} + E'_{\pm}(k) - E_{\pm}(k)} + \left[ f(E_{\pm}(k)) - f(E'_{\pm}(k)) \right] \frac{1}{i\omega_{n} + E_{\pm}(k) - E'_{\pm}(k)}
\]

(4)

while the remaining matrix elements are determined by the symmetry relations

\[
A_{2,2}(\omega_{n}, q, k) = A_{1,1}(-\omega_{n}, -q, k),
\]

\[
A_{2,1}(\omega_{n}, q, k) = A_{1,2}(\omega_{n}, q, k).
\]

In the above equations \(f(E_{\pm}(k))\) and \(f(E'_{\pm}(k))\) are the quasiparticle Fermi distributions and the following shorthand notations are used:

\[
E_{\pm}(k), E'_{\pm}(k) = E(k) \mp \eta, E'(k) \mp \eta,
\]

\[
u(k), v(k), E(k) = u(q/2 + k), v(q/2 + k), E(q/2 + k)
\]
and

\[ u'(k), v'(k), E'(k) = u(q/2 - k), v(q/2 - k), E(q/2 - k), \]

where \( E(q/2 \pm k) = \sqrt{\xi^2(q/2 \pm k) + \Delta^2} \),

\[ u^2(q/2 \pm k) = \frac{1}{2} \left( 1 + \frac{\xi(q/2 \pm k)}{E(q/2 \pm k)} \right), \]

and

\[ v^2(q/2 \pm k) = \frac{1}{2} \left( 1 - \frac{\xi(q/2 \pm k)}{E(q/2 \pm k)} \right), \]

with \( \xi(q/2 \pm k) = (q/2 \pm k)^2/2m - \mu \).

The propagator \( \tilde{D}(\omega_n, q) \) represents the effective interaction between quasi–particles when the exchange of phonons is included. It is formally similar to the random phase approximation (RPA) to the medium effective interaction for normal Fermi systems [25]. The Green's functions of the pairing field \( \sigma(q, \tau) = \sigma^\dagger(q, \tau), \sigma(q, \tau) \) are obtained by subtracting from \( \tilde{D}(\omega_n, q) \) the matrix element, in the momentum representation, \( \lambda/V \) of the bare interaction, then by amputating the two external interaction vertices:

\[ \frac{V^2}{\lambda^2} \left( \tilde{D}(\omega_n, q) - \frac{\lambda}{\sqrt{V}} \right). \]

From Eqs. (2) we get an equivalent set of equations for the propagator of the pairing fluctuations, \( \tilde{\sigma}(q, \tau) \), per unit of volume,

\[ \tilde{D}(\omega_n, q) = -\frac{1}{|\lambda|} \left[ \frac{1}{|\lambda|} - \tilde{A}^T(\omega_n, q) \right]^{-1} \tilde{A}^T(\omega_n, q) \]

(5)

with

\[ \tilde{A}^T(\omega_n, q) = \int \frac{dk}{(2\pi)^3} \tilde{A}(\omega_n, q, k) \]

depending only on the magnitude of the wave–vector \( q \).

The poles of the Green's functions are given by the complex frequencies \( z = \omega(q) + i\Gamma(q) \) for which the determinant

\[ \det \left[ \frac{1}{|\lambda|} - \tilde{A}^T(z, q) \right] \]

vanishes. In the integrals for the diagonal elements \( A^T_{ij} \), the ultraviolet divergencies are removed by substituting the coupling constant \( \lambda \) with the scattering length as in the case of the gap equation (1).

For given values of the wave–number \( q \) the real part of the frequency represents the excitation energy of collective modes while the imaginary part gives their life–time. Then, we expect complex poles in the lower half–plane of \( z \), \( \Gamma(q) < 0 \), corresponding to damped oscillations. Since the Green's functions have a branch cut along the real axis \( \omega \), they should be analytically continued from the upper to the lower half–plane of \( z \) on another Riemann sheet [20, 21]. In our case this can be obtained by adding \( 2imA^T_{ij}(\omega + i\epsilon, q) \) to the advanced counterpart \( A^T_{ij}(\omega - i\epsilon, q) \), which is in turn analytic in the lower half–plane. In the Appendix the procedure is exemplified by explicit calculations in the limit of small wave–number \( q \). Complex frequencies, occurring even for low values of \( \omega \) and \( q \), emerge from the quasiparticle–quasihole singularities of the last two terms of Eqs. (3) and (4). We stress in passing that the factors \( [f(E_\pm(k) - f(E'_\pm(k)) \]

vanish only for \( T = 0 \). Then, an expansion at small values of \( q \) and \( \omega \) is not allowed in general [25]. Moreover, we observe that the onset, for sufficiently high imbalance \( |\eta| \geq |\Delta| \), of the gapless Sarma phase does not give rise to any critical behavior of the properties of collective modes, since in the denominators of Eqs. (3) and (4) the energies of the intermediate states do not depend on \( \eta \): \( E'_\pm(k) + E_\pm(k) = E'(k) + E(k) \) and \( E'_\pm(k) - E_\pm(k) = E'(k) - E(k) \).

III. RESULTS

In this paper we are concerned with the properties of low–energy collective modes, therefore we choose sufficiently low values of the wave–vector \( q/k_{F_\pm} \leq 0.1 \), where \( k_{F_\pm} \) is the Fermi momentum of the spin–up fermions, so that the corresponding energies are below the threshold for pair breaking.

To be specific, from now on we assume that the majority particles are in the intrinsic state "spin–up", \( n_+ \geq n_- \), where \( n_\pm \) are the densities of the two species of fermions. We note that for \( T \to 0 \) the polarization \( (n_+ - n_-) \) is vanishing necessarily [2, 29].
In order to explore a fairly large region of the phase diagram \((P,T)\) accessible by the system, with \(P = (n_+ - n_-)/(n_+ + n_-)\) the relative polarization, we use the value \(a_S k_F = -2\) for the interaction strength in actual calculations \([2,29]\). Furthermore, with this value of \(a_S\) the physical parameters of the system are not close to the area of the BCS–BEC cross–over, where fluctuations of the pairing field could be important and our approach can be rather questionable.

For given values of the temperature we have calculated the frequency and the life–time of collective modes as functions of the polarization, up to values of \(P\) near the line of the first–order phase transition, for \(T < T_{CP}\), or the second–order phase transition, for \(T < T_{CP}\). Here \(T_{CP}\) is the temperature of the tricritical point, where the normal state, the superfluid state and the instability region meet \([2,29]\). We have found that, besides a phonon–like pole, the propagator of the pairing field, \(\mathcal{D}(z,q)\), exhibits a pole at \(z \to 0\) for finite values of \(q\), when \(P\) approaches the border of the instability region \((T < T_{CP})\). In particular, the pole moves along the imaginary axis from the lower part to the upper part of the \(z\)–plane. The values of the couple \((T,P)\), for which the pole emerges, are in agreement with the instability curve obtained in Refs. \([2,29]\) within a thermodynamical approach. Finally, we note that for a given value of \(T\) the position of the pole in the \((P,T)\) plane moves to the right increasing \(q\), albeit slightly for the considered values of \(q\). Similar features have been found for the relevant response functions in investigations of first–order phase transitions within different contexts \([30,32]\).

The phase velocity \(c_S = \omega(q)/q\) and the damping rate \(\gamma_S = |\Gamma(q)|/q\) of the phonon–like mode are displayed in Figs. (1) and (2) respectively, as functions of the relative polarization for three values of the temperature: below, at and above the tricritical temperature \((T_{CP} = 0.109T_{F_+})\), with \(T_{F_+}\) the Fermi temperature of the majority particles. The curves end for values of \(P\) at the border of the instability region \((T = 0.08T_{F_+})\), at the tricritical point \((T = 0.109)\) and at the border of the second–order phase transition \((T = 0.12T_{F_+})\). Here we have used the value \(q = 0.01k_{F_+}\) for the wave–vector. However, explicit calculations show that for the considered values of \(q\) both the frequency and the life–time of the phonon–like excitations are linear functions of \(q\), with a good approximation.

One can see from Figs. (1) and (2) that the phase velocity as well as the damping rate show a sizable dependence on the polarization, whereas, in the interval of \(P\) where the collective modes subsist, they are slightly affected by the temperature. In particular, the growing with \(P\) of the damping rate \(\Gamma(q)/q\) can essentially be ascribed to the increase of the density of quasi–particle states

\[
4\pi k^2 \frac{\partial}{\partial k} \left[ f(E_+(k)) + f(E_-(k)) \right]
\]

with increasing the difference in chemical potentials, \(\mu_+ - \mu_-\). We note that the damping assumes appreciable values at relatively low temperature for sufficiently high values of \(P\). Whereas for a balanced gas it becomes significant only at temperatures close to that of the normal–superfluid transition \((T \sim 0.9T_C)\) \([33,34]\).

The spectral density of the pairing–field fluctuations can supply a more detailed insight of the low–energy excitation spectrum of the superfluid phase. This quantity corresponds to the imaginary part of the retarded propagator of the pairing field fluctuations \(\hat{\mathcal{D}}(q,\tau)\), and can be obtained by analytical continuation of the matrix element \(\mathcal{D}_{1,1}(\omega_n,q)\) to real frequencies \((i\omega_n \to \omega + i\epsilon)\)

\[
\rho(\omega,q) = -\frac{1}{\pi} \text{Im} \mathcal{D}_{1,1}(\omega,q).
\]

It is convenient to rewrite Eqs. (5) as

\[
\hat{\mathcal{D}}(\omega,q) = -\frac{1}{\lambda^2} \left( \frac{\hat{\lambda}}{|\lambda|} - A^T(\omega,q) \right)^{-1} - |\lambda| \hat{1}. \tag{7}
\]

Thus, the spectral density is given by

\[
\rho(\omega,q) = \frac{1}{\lambda^2} \frac{1}{\pi} \text{Im} \left[ \frac{\hat{\lambda}}{|\lambda|} - A^T(\omega,q) \right]^{-1}_{1,1}. \tag{8}
\]

and for the integrals we can exploit the regularization procedure used before, Eq. (1). The pre–factor \(1/\lambda^2\) in the above equation is a simple scale factor non affecting the relevant features of \(\rho(\omega,q)\). We can safely omit it and limit ourselves to examine the quantity \(\lambda^2 \rho(\omega,q)\). In any case the coupling constant \(\lambda\) could be determined consistently with the value of the scattering length \(a_S\). We note, however, that the relationship between \(\lambda\) and \(a_S\), used in the present work, is not suitable for this purpose, we should introduce a further parameter at least.

In Figs. (3) and (4) we show the calculated spectral density as a function of \(\omega\) for two values of the wave–vector, \(q = 0.01k_{F_+}\) and \(q = 0.1k_{F_+}\) respectively. The
FIG. 3: The modified spectral density of collective modes (see text) as function of the scaled frequency $\omega/\epsilon_{F+}$ for different values of temperature and relative polarization. Panel (a): $T = 0.08\epsilon_{F+}$, $P = 0$ (black line), $P = 0.1$ (red line) and $P = 0.19$ (green line). The inset in panel (a) displays a zoom on low–frequency region for $P = 0.19$. Panels (b) and (c) show the results for $T = 0.109\epsilon_{F+}$ and $T = 0.12\epsilon_{F+}$ respectively, with three values of polarization, $P = 0.0$ (black line), $P = 0.2$ (red line) and $P = 0.33$ (green line). The scaled wave–number is $q/k_{F+} = 0.01$ and the used value for $a_S k_{F+}$ is the same as in Fig.1.

FIG. 4: Same as Fig.3, but for $q/k_{F+} = 0.1$.

values of temperature are the same as in Figs. (1) and (2): below, at, and above the tricritical temperature. For each temperature three values of the relative polarization $P$ are considered, the last of which being in proximity of the curve of the first or second order phase–transition. The inset in the figures is a zoom on the region about $\omega = 0$, showing the occurrence of the instability pole. Furthermore, in the graphs with $T = 0.08T_{F+}$, we have chosen the highest value of $P$ taking into account that the instability border is shifted toward larger $P$ increasing
the value of \( q \).

We observe that in the two figures abscissae and widths of the peaks only differ by a scale factor, given by the ratio between the values of wave vector. This denotes a linear dependence on the wave vector of the complex frequency, as mentioned before. In addition, Figs. (3) and (4) show that the global properties of the spectral density depend mostly on polarization, whereas the temperature, in the considered range, has a weak impact. A remark is in order. For low values of the polarization an undamped hydrodynamic sound mode can survive until temperatures comparable to the critical one \[33, 34\]. Increasing the polarization the microscopic mechanism underlying the collective mode becomes different. The quasiparticle–quasihole states acquire a more and more important weight, yielding an anisotropic distortion of the equilibrium quasiparticle distribution, so that the requirement of local equilibrium for a hydrodynamic mechanism should be extended to include an interaction between quasi-particles.

We finish the survey on the properties of the spectral density giving a look at its composition in terms of amplitude and phase fluctuations, respectively. Their propagators are given by simple combinations of the matrix elements of \( \mathcal{D}(\omega, q) \):

\[
\mathcal{D}_\chi(\omega, q) = \frac{1}{4} \left[ \mathcal{D}_{1,1}(\omega, q) + \mathcal{D}_{2,2}(\omega, q) + \mathcal{D}_{1,2}(\omega, q) + \mathcal{D}_{2,1}(\omega, q) \right]
\]

and

\[
\mathcal{D}_\phi(\omega, q) = \frac{1}{4} \left[ \mathcal{D}_{1,1}(\omega, q) + \mathcal{D}_{2,2}(\omega, q) - \mathcal{D}_{1,2}(\omega, q) - \mathcal{D}_{2,1}(\omega, q) \right].
\]

Explicit calculations of the imaginary parts of these quantities show that, for both the considered values of \( q \), the phase and amplitude modes are almost decoupled: the phonon peak corresponds to a damped phase mode ( \( > 90\% \) ) in all the cases, whereas the instability pole ( \( T < T_{CP} \) ) corresponds to an unstable amplitude mode ( \( > 90\% \) ).

IV. SUMMARY AND CONCLUSIONS

In our approach to assess the occurrence of a superfluid phase and to determine the properties of low–energy excitations of an imbalanced two–component Fermi gas, only fermionic degrees of freedom come into play. For a sufficiently diluted gas only atoms, in different "spin" states and in the \( s \)-channel, can be considered paired.

So far, imbalanced Fermi gases have been studied at equilibrium mostly. A particular attention has been devoted to their properties in the unitarity regime. Here, we have looked into some aspects of the dynamics, which may play a role in the response of the gas to an external probe. We have focused our attention into the effects of the polarization of the gas on the low–energy excitation spectrum. The effects of the temperature have been taken into account as well. At finite temperatures, singularities, due quasiparticle excitations, emerge in the integrals. They give rise to damped collective modes, like the well–known Landau damping. In the case of a balanced Fermi gas, a strong increase of the width of the peak, corresponding to the Bogoliubov–Anderson mode, was observed for values of temperature close to the critical one \[33, 34\]. In our study we have found a similar behavior but with increasing the polarization of the gas, even at temperatures far from the critical one. Moreover, we have not observed any critical effect on the low–lying collective modes from the occurrence of the Sarma phase \[15\]. Actually, approaching the Sarma phase the density of quasiparticle states increases but continuously.

Our calculations show that, in addition to the pole related to the Bogoliubov–Anderson mode, the propagator of the pairing field exhibits a purely imaginary pole. The magnitude of the imaginary frequency is vanishing when the value of the polarization reaches the borders of the coexistence region of the superfluid and normal phases. This is connected to an instability situation, which develops toward the separation of the two phases. However we remark that, in the formalism used in the present paper, the normal phase, when the order parameter vanishes, reduces to an ideal gas. In order to attain a more realistic description of the phase separation, the formalism should be extended to include an interaction between quasi-particles.

Appendix

Appendix A: Analytic continuation

For excitation energies below the pair–breaking threshold the terms in Eqs. (3) and (4) giving rise to complex poles of the Green’s function of Eq. (5) are those which contain the factors \( f(E_\pm(k)) \) and \( f(E_{\pm}(k)) \). For small wave numbers \( q \), the factor \( q = q_{F_p} << \Delta_0 \) ), their contribution to \( A_{ij}^{T}(\omega_n, q) \) is the same for all the matrix elements. It is given by

\[
\delta A^T(\omega + i\epsilon, q) = -\int \frac{d\mathbf{k}}{(2\pi)^3} v_0^2(k) v_0^2(k) \left( \nabla_k f(E_\pm(k)) \cdot \mathbf{q} - \nabla_k f(E_{\pm}(k)) \cdot \mathbf{q} \right) \left( \frac{\omega + i\epsilon + 2q \cdot w}{\omega + i\epsilon - 2q \cdot w} \right),
\]

where the retarded case ( \( \omega_n \rightarrow \omega + i\epsilon \) ) is considered. In the above equation the notations \( w = \nabla_k E_{\pm}(k) \) and \( u_0(k), v_0(k) = u(k), v(k) \), with \( q = 0 \), are used ( see Eqs. (3) and (4) ).

The angular integration in Eq. (A.1) can be performed
analytically and gives
\[
\delta A^T(\omega + i\epsilon, q) = \int \frac{dk}{(2\pi)^2} \frac{k^2 u_2^2(k)v_1^2(k)}{e^{\beta E^{(0)}_k(k)} - 1} \times \left[\frac{e^{\beta E^{(0)}_k(k)}}{(1 + e^{\beta E^{(0)}_k(k)})^2} + \frac{e^{\beta E^{(0)}_k(k)}}{(1 + e^{\beta E^{(0)}_k(k)})^2}\right] \times \left[2 - \frac{\omega - 2qw}{2qw} \ln \left|\frac{\omega + 2qw + i\epsilon}{\omega - 2qw + i\epsilon}\right|\right].
\]

The analytic continuation to the lower half-plane of the complex frequency can be made by exploiting the property that in the complex plane the logarithm is a multi-valued function. The logarithms of the advanced counter-part, \(\delta A^T(\omega - i\epsilon, q)\), are evaluated on the next Riemann sheet:

\[
\ln(\omega - i\epsilon - 2qw) = \ln(|\omega - i\epsilon - 2qw|) + i(\arg(\omega - i\epsilon - 2qw) + 2\pi\theta(1 - s)),
\]

for \(s = \omega/2qw > 0\), or

\[
\ln(\omega - i\epsilon - 2qw) = \ln(|\omega - i\epsilon - 2qw|) + i(\arg(\omega - i\epsilon - 2qw) + 2\pi\theta(1 + s)),
\]

for \(s = \omega/2qw < 0\). This is equivalent to defining the analytic continuation of the retarded function as

\[
\delta A^{TR}(\omega - i\epsilon, q) = \delta A^T(\omega - i\epsilon, q) + 2i\text{Im}\delta A^T(\omega + i\epsilon, q).
\]

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