HOPF ALGEBRA EXTENSION OF A ZAMOLOCHIKOV ALGEBRA AND ITS DOUBLE

JINTAI DING

ABSTRACT. The particles with a scattering matrix \( R(x) \) are defined as operators \( \Phi_i(z) \) satisfying the relation
\[
R_{i,j}(x_1/x_2)\Phi_i(x_1)\Phi_j(x_2) = \Phi_i(x_2)\Phi_j(x_1)
\]
The algebra generated by those operators is called a Zamolochikov algebra. We construct a new Hopf algebra by adding half of the FRTS construction of a quantum affine algebra with this \( R(x) \). Then we double it to obtain a new Hopf algebra such that the full FRTS construction of a quantum affine algebra is a Hopf subalgebra inside. Drinfeld realization of quantum affine algebras is included as an example. This is a further generalization of the constructions in [DI].

1. Introduction.

In physics, the particles with a scattering matrix \( R(x) \) in \( \text{End}(V) \otimes \text{End}(V) \) are defined with the operators \( \Phi_i(x) \) index by a linear independent basis of \( V \) such that
\[
R_{i,j}(x_1/x_2)\Phi_i(x_1)\Phi_j(x_2) = \Phi_i(x_2)\Phi_j(x_1)
\]
where \( V \) is a vector space, \( x \) is a parameter in \( \mathbb{C} \). This naturally gives an algebra with these current generators \( \Phi_i(x) \), which we will call a Zamolochikov algebra. However this algebra is not given a Hopf algebra structure. We construct a Hopf algebra on this algebra by adding structure with the ideas coming from the structures of the affine quantum groups.

The definition of quantum groups discovered by Drinfeld and Jimbo is presented as a deformation of the simple Lie algebra by the basic generators and the relations based on the data coming from the corresponding Cartan matrix. The extension of the realization of the affine Kac-Moody algebra \( \hat{g} \) associated to a simple Lie algebra \( g \) as a central extension of the corresponding loop algebra \( g \otimes \mathbb{C}[t, t^{-1}] \) [G] has two different approaches. The first approach was given by Faddeev, Reshetikhin and Takhtajan [FRT], who obtained a realization of the quantum loop algebra \( U_q(g \otimes [t, t^{-1}]) \) via a canonical solution of the Yang-Baxter equation depending on a parameter \( z \in \mathbb{C} \). This approach was completed by Reshetikhin and Semenov-Tian-Shansky [RS] by incorporating the central extension in the previous realization. We call this approach FRTS construction. The second approach was given by Drinfeld, who [Dr2] gave a realization of the quantum affine algebra \( U_q(\hat{g}) \) and its special degeneration called the
Yangian. As an algebra, this realization is equivalent to the FRTS construction [DF] through certain Gauss decomposition for the case of $U_q(\hat{\mathfrak{gl}}(n))$. Although we cannot extend the conventional comultiplication to the current operators of Drinfeld to derive a closed comultiplication formula, Drinfeld also gave the Hopf algebra structure for such a formulation [DF], which [DM] [DI] we used to study vertex operators and zeros and poles of the quantum current operators. In the Drinfeld realization of quantum affine algebras, the structure constants are certain rational functions $g_{ij}(z)$. In [DI], we generalize this type of Hopf algebras by substituting $g_{ij}(z)$ by other functions that satisfy the functional property of $g_{ij}(z)$.

In this paper, we will use the idea of FRTS construction to define a Hopf algebra generated by a current operator valued matrix on $V$, $L(x)$, such that

$$R(x_1/x_2)L_1(x_1)L_2(x_2) = L_2(x_2)L_1(x_1)R(x_1/x_2),$$

where $L_1(x) = L(x) \otimes 1$ and $L_2(x) = 1 \otimes L(x)$; and the commutation relation between the particles and this new operator matrix $L(x)$ is presented as

$$\Phi(x_1)_1L(x_2)_2 = R(x_1/x_2q^{c1/2})^{-1}L(x_2)_2\Phi(x_1)_1.$$

This relation can be interpreted as that $\Phi(x)$ is an intertwiner for the algebra generated by $L(x)$ [FR]. With this we can define a comultiplication on the algebra generated by $L(x)$ and $\Phi(x)$, where the comultiplication for $L(z)$ comes from FRTS construction and the comultiplication of $\Phi(x)$ is defined as

$$\Delta(\Phi(z)) = \Phi(x) \otimes 1 + L(xq^{c1/2}) \otimes \Phi(zq^{c1}),$$

which is a generalization of Drinfeld construction. Then combining the idea of FRTS construction and Drinfeld realization, we give a double for such a construction, where the FRTS construction is a Hopf subalgebra and Drinfeld realization is a special case of our realization with certain diagonal $R(x)$. This paper is basically a result of the combination of the two approaches, the FRTS construction and the Drinfeld realization.

This paper contains two sections. In the first section, we define the Zamolochikov algebra and present its Hopf algebra extension. The second section is to describe the double of such a construction and the related examples.

2. Hopf algebra extension of a Zamolochikov algebra

Let $V$ be the vector space $\mathbb{C}^n$. Let $x$ be a parameter in $\mathbb{C}$. A function valued R-matrix $R(x)$ is an function valued operator in $\text{End}(V) \otimes \text{End}(V)$, which satisfies the so-called Yang-Baxter equation:

$$R_{12}(z)R_{13}(z/w)R_{23}(w) = R_{23}(w)R_{13}(z/w)R_{12}(z),$$
where \( R_{12}(x) = f_{ij}(x) \sum_{ij} a_i \otimes b_j \otimes 1 = R(x) \otimes 1, R_{13}(x) = f_{ij}(x) \sum_{ij} a_i \otimes 1 \otimes b_j, R_{23}(x) = f_{ij}(x) \sum_{ij} 1 \otimes a_i \otimes b_j = 1 \otimes R, \) and \( R(x) = f_{ij}(x) \sum_{ij} a_i \otimes b_j. \) We also require that \( R(x) \) satisfies the unitary condition

\[
R_{21}(z)^{-1} = R(z^{-1}),
\]

where \( R_{21}(z) = f_{ij}(x) \sum_{ij} b_j \otimes a_i. \)

**Definition 2.1.** The associative algebra \( P[R(x)] \) is an algebra generated by operators \( \Phi_i(x) \) indexed by a linear independent basis \( e_i \) of \( V. \) Let \( \Phi(x) = \Sigma \Phi_i(x) \otimes e_i. \) The commutation relations are presented as:

\[
R(x_1/x_2)\Phi(x_1)_1\Phi(x_2)_2 = \Phi(x_2)_2\Phi(x_1)_1,
\]

where \( \Phi(x_1)_1\Phi(x_2)_2 = \Sigma \Phi_i(x_1)\Phi_j(x_2)e_i \otimes e_j \) and \( \Phi(x_2)_2\Phi(x_1)_1 = \Sigma \Phi_j(x_2)\Phi_i(x_1)e_i \otimes e_j \)

As explained in the introduction, this system is used in the description particles \( (\Phi_i(x)) \) in physics with the scattering matrix \( R(x) \) and in some other context. This relation is also satisfied by the vertex operators for quantum affine algebras \([FR][DFJMN], \) etc and this type of system also appeared in describing the elliptic type of algebras \([FO] \). However, they are all described as algebras not Hopf algebras. Following the idea in \([DI, B], \) we would like to extend this algebra with additional current operators coming from the FRTS construction to give a Hopf algebra structure to such a system.

**Definition 2.2.** The algebra \( EP[R(x)] \) is an associative algebra generated by \( \Phi_i(x) \) indexed by a linear independent basis \( e_i \) of \( V, l_{ij}(x) \) indexed by the linear independent basis \( e_{ij} \) of \( \text{End}(V) \) and a central element \( k. \) Let \( \Phi(x) = \Phi_i(x) \otimes e_i \) and the operator valued matrix \( L(x) = \Sigma l_{ij}(x) \otimes e_{ij}, \) such that \( L(x) \) is invertible. They satisfies the commutation relations:

\[
R(x_1/x_2)\Phi(x_1)_1\Phi(x_2)_2 = \Phi(x_2)_2\Phi(x_1)_1,
\]

\[
\Phi(x_1)_1L(x_2)_2 = R(q^{i/2}x_1/x_2)^{-1}L(x_2)_2\Phi_i(x_1)_1,
\]

\[
R(x_1/x_2)L(x_1)_1L(x_2)_2 = L(x_2)_2L(x_1)_1R(x_1/x_2).
\]

Here \( \Phi(x_1)_1L(x_2)_2 = \Sigma \Phi_i(x_1)l_{kl}(x_2)e_i \otimes e_{kl}, \) \( L(x_2)_2\Phi_i(x_1)_1 = \Sigma l_{kl}(x_2)\Phi_i(x_1)e_i \otimes e_{kl}, \) \( L(x_1)_1L(x_2)_2 = PL(x_1)_2L(x_2)_1P = \Sigma l_{ij}(x_1)L_i(x_2)e_{ij} \otimes e_{kl}, \) and \( P \) is the permutation operator.

**Theorem 2.1.** The algebra \( EP[R(x)] \) has a Hopf algebra structure, which are given by the following formulae.
**Coproduct** $\Delta$

(0) $\Delta(q^c) = q^c \otimes q^c$,

(1) $\Delta(\Phi_i(z)) = \Phi_i(z) \otimes 1 + \sum L_{ij}(zq^c) \otimes \Phi_j(zq^{c^i})$,

(2) $\Delta(L_{ij}(z)) = \sum L_{ik}(zq^{-}\tilde{\Phi}) \otimes L_{kj}(zq^{\tilde{\Phi}}),$

where $c_1 = c \otimes 1$ and $c_2 = 1 \otimes c$.

**Counit** $\varepsilon$

$\varepsilon(q^c) = 1 \quad \varepsilon(L_{ij}(z)) = \delta_{ij},$

$\varepsilon(\Phi_i^+(z)) = 0.$

**Antipode** $\alpha$

(0) $\alpha(q^c) = q^{-c},$

(1) $\alpha(\Phi_i(z)) = \Sigma - (L(zq^{-}\tilde{\Phi})^{-1})_{ij}\Phi_j(zq^{-c}),$

(2) $\alpha(L(z)) = (L(z))^{-1}.$

We will use the notation to denote the comultiplication.

$\Delta \Phi(x_1) = \Phi(x_1) \tilde{\otimes} 1 + L(x_1q^{\tilde{\Phi}}) \tilde{\otimes} \Phi(x_1q^{c_1});$

$\Delta(L(x_2)) = (L(x_2q^{-\tilde{\Phi}}) \tilde{\otimes} L(x_2q^{\tilde{\Phi}})).$

Proof. For the comultiplication above we have that

$\Delta \Phi(x_1) \Delta L(x_2) =$

$(\Phi(x_1) \tilde{\otimes} 1 + L(x_1q^{\tilde{\Phi}}) \tilde{\otimes} \Phi(x_1q^{c_1}))_1(L(x_2q^{-\tilde{\Phi}}) \tilde{\otimes} L(x_2q^{\tilde{\Phi}}))_2 =$

$R(x_1/x_2q^{\tilde{\Phi}+\tilde{\Phi}})^{-1}(L(x_2q^{-\tilde{\Phi}}) \tilde{\otimes} L(x_2q^{\tilde{\Phi}}))_2(\Phi(z) \tilde{\otimes} 1)_1 +$

$R(x_2/x_1q^{-\tilde{\Phi}})^{-1}(L(x_2q^{-\tilde{\Phi}}) \tilde{\otimes} L(x_2q^{\tilde{\Phi}}))_1(L(x_1q^{\tilde{\Phi}}) \tilde{\otimes} \Phi(x_1q^{c_1}))_1.$

$R(x_1/x_2)\Delta \Phi(x_1) \Delta \Phi(x_2) =$

$(\Phi(x_1) \tilde{\otimes} 1 + L(x_1q^{\tilde{\Phi}}) \tilde{\otimes} \Phi(x_1q^{c_1}))_1(\Phi(x_2) \tilde{\otimes} 1 + L(x_2q^{\tilde{\Phi}}) \tilde{\otimes} \Phi(x_2q^{c_1}))_2 =$

$(\Phi(x_2) \tilde{\otimes} 1)_2(\Phi(x_1) \tilde{\otimes} 1)_1 + (L(x_2q^{\tilde{\Phi}}) \tilde{\otimes} \Phi(x_2q^{c_1}))_2(\Phi(x_1) \tilde{\otimes} 1)_1 +$

$(L(x_1q^{\tilde{\Phi}}) \tilde{\otimes} \Phi(x_1q^{c_1}))_1(L(x_2q^{\tilde{\Phi}}) \tilde{\otimes} \Phi(x_2q^{c_1}))_2 + R_{21}(x_2/x_1)^{-1}L(x_1q^{\tilde{\Phi}}) \tilde{\otimes} \Phi(x_1q^{c_1}))_1(\Phi(x_2) \tilde{\otimes} 1)_2 =$

$\Delta \Phi(x_2) \Phi(x_1).$

This construction of comultiplication follows partially the idea of constructing comultiplications for the quantum Lie algebra $[B]$, where the cases without the parameter $x$ are given. With our construction, we can extend the Hopf algebra structures to the special Zamolochikov algebra $Z_{n,k}(\xi, \tau)$, which is defined as the algebra generated $\Phi(z)$ with an Belavin elliptic R-matrix $R(z)[FO]$. We expect that the new Hopf
algebra structure should be very useful in the study of the representation theory of the elliptic Zamolochikov algebras and hopefully even the related Sklyanin elliptic algebras.

3. The Double of $EP[R(x)]$

In this section, we will present a double of the algebra $EP[R(x)]$ following the Drinfeld realization of the quantum affine algebra $U_q(\mathfrak{sl}(2))$.

**Definition 3.1.** The algebra $DEP[R(x)]$ is an associative algebra generated by $\Phi_i(x)$ indexed by a linear independent basis $e_i$ of $V$, $l_{ij}(x)$ and $l_{ij}^*(x)$ indexed by the linear independent basis $e_i$ of $V^*$, the dual space of $V$, and a central element $k$. Let $\Phi(x) = \Phi_i(x) \otimes e_i$, $\Phi^*(x) = \Phi_i^*(x) \otimes e_i^*$ the operator valued matrix $L(x) = \sum l_{ij}(x) \otimes e_{ij}, L^*(x) = \sum l_{ij}^*(x) \otimes e_{ij}$, such that $L(x)$ and $L^*(x)$ are invertible. They satisfy the commutation relations:

$$R(x_1/x_2)\Phi(x_1)_1\Phi(x_2)_2 = \Phi(x_2)_2\Phi(x_1)_1,$$

$$\Phi(x_1)_1L(x_2)_2 = R(q^{c/2}x_1/x_2)^{-1}L(x_2)_2\Phi(x_1)_1,$$

$$R(x_1/x_2)L(x_1)_1L(x_2)_2 = L(x_2)_2L(x_1)_1R(x_1/x_2).$$

$$R(x_1/x_2)L^*(x_1)_1L^*(x_2)_2 = L^*(x_2)_2L^*(x_1)_1R(x_1/x_2),$$

$$R(x_1/x_2q^{-c})L(x_1)_1L^*(x_2)_2 = L^*(x_2)_2L^*(x_1)_1R(x_1/x_2q^{-c}),$$

$$\Phi^*(x_2)_2\Phi^*(x_1)_1 = \Phi^*(x_1)_1\Phi^*(x_2)_2R_{21}(x_2/x_1),$$

$$L^*(x_2)_2\Phi^*(x_1)_1 = \Phi^*(x_1)_1L^*(x_2)_2R_{21}(q^{-c/2}x_2/x_1),$$

$$\Phi(x_1)_1\Phi^*(x_2)_2^* - \Phi^*(x_2)_2\Phi(x_1)_1 = 1/(q - q^{-1})(L^*(wq^{c/2})\delta(z/wq^{-c}) - \delta(z/wq^c)L^*(zq^{c/2}),$$

$$L(x_1)_1\Phi^*(x_2)_2R_{21}(q^{-c/2}x_2/x_1) = \Phi^*(x_2)_2L(x_1)_1,$$

$$R(q^{c/2}x_1/x_2)L^*(x_1)_1\Phi(x_2)_2 = \Phi(x_2)_2L^*(x_1)_1.$$  

Here $\Phi(x_1)_1L(x_2)_2 = \sum \Phi_i(x_1)_1L_{kl}(x_2)e_i \otimes e_{kl}$, $L(x_2)_2\Phi(x_1)_1 = \sum L_{kl}(x_2)\Phi_i(x_1)_1e_i \otimes e_{kl}$, $L(x_1)_1L(x_2)_2 = PL(x_2)_2L(x_1)_1P = \sum L_{ij}(x_1)L_i(x_2)e_{ij} \otimes e_{kl}$, and the others are defined in the same way. $P$ is the permutation operator. $\delta(z)$ is the distribution with the support at 1.

**Theorem 3.1.** $DEP[R(x)]$ has an Hopf algebra structure. The comultiplication $\Delta$, the counit $\varepsilon$ and the antipode $\alpha$ are given by the following formulas.
Coproduct \( \Delta \)

\[
(0) \quad \Delta(q^c) = q^c \otimes q^c, \\
(1) \quad \Delta(\Phi_i(z)) = \Phi_i(z) \otimes 1 + \sum L_{ij}(zq^2) \otimes \Phi_j(zq^{2i}), \\
(2) \quad \Delta(L_{ij}(z)) = \sum L_{ik}(zq^{-\varphi}) \otimes L_{kj}(zq^{2\varphi}), \\
(3) \quad \Delta(\Phi^*_i(z)) = 1 \otimes \Phi^*_i(z) + \sum \Phi^*_j(zq^{c^2}) \otimes L^*_ij(zq^{2\varphi}), \\
(2) \quad \Delta(L^*_ij(z)) = \sum L^*_ik(zq^{2\varphi}) \otimes L^*_kj(zq^{-\varphi}).
\]

where \( c_1 = c \otimes 1 \) and \( c_2 = 1 \otimes c. \)

Counit \( \varepsilon \)

\[
\varepsilon(q^c) = 1 \quad \varepsilon(L(z)) = \varepsilon(L^*(z)) = I, \\
\varepsilon(\Phi(z)) = 0 = \varepsilon(\Phi^*(z)).
\]

Antipode \( a \)

\[
(0) \quad a(q^c) = q^{-c}, \\
(1) \quad a(\Phi(z)) = -L(zq^{-\varphi})^{-1}\Phi(zq^{-c}), \\
(2) \quad a(\Phi^*(z)) = -\Phi^*(zq^{-c})L^*(zq^{-\varphi})^{-1}, \\
(3) \quad a(L(z)) = L(z)^{-1}, \\
(4) \quad a(L^*(z)) = L^*(z)^{-1}.
\]

Proof.

\[
\Delta\Phi^*(z)_1\Delta\Phi(w)_2 R_{21}(w/z) = \\
(1 \otimes \Phi^*(z) + \Phi^*(zq^{c^2}) \otimes L^*(zq^{2\varphi}))1(1 \otimes \Phi^*(w) + \Phi^*(wq^{c^2}) \otimes L^*(wq^{2\varphi}))2 R_{21}(x_1/x_2) = \\
(1 \otimes \Phi^*(w)_2(1 \otimes \Phi^*(z)_1 + (\Phi^*(wq^{c^2}) \otimes L^*(wq^{2\varphi}))_2(1 \otimes \Phi^*_1(z)_1) + \\
(\Phi^*(zq^{c^2}) \otimes L^*(zq^{2\varphi}))_1(1 \otimes \Phi^*(w))_2 R_{21}(w/z)^{-1} + (1 \otimes \Phi^*(w) + \Phi^*(wq^{c^2}) \otimes L^*(wq^{2\varphi}))_2 \times \\
(1 \otimes \Phi^*(z) + \Phi^*(zq^{c^2}) \otimes L^*(zq^{2\varphi}))_1 + (\Phi^*(wq^{c^2}) \otimes L^*(wq^{2\varphi}))_2(\Phi^*(zq^{c^2}) \otimes L^*(zq^{2\varphi}))_1.
\]

\[
\Delta\Phi^*(z)_1\Delta L^*(w)_2 R_{21}(q^{-(c_1+c_2)/2}w/z) = \\
(1 \otimes \Phi^*(z) + \Phi^*(zq^{c^2}) \otimes L^*(zq^{2\varphi}))_1(L^*(wq^{2\varphi}) \otimes L^*(wq^{-\varphi}))_2 R_{21}(q^{-(c_1+c_2)/2}w/z) = \\
\Delta L^*(w)_2 \Delta\Phi^*(z)_1.
\]

\[
\Delta L(z)_1\Delta\Phi^*(w)_2 R_{21}(q^{-(c_1+c_2)/2}w/z) = \\
(L(zq^{-\varphi}) \otimes L(zq^{\varphi}))_1(1 \otimes \Phi^*(w) + \Phi^*(wq^{c^2}) \otimes L^*(wq^{2\varphi}))_2 R_{21}(q^{-(c_1+c_2)/2}w/z) = \\
(1 \otimes \Phi^*(w)_2(L(zq^{-\varphi}) \otimes L(zq^{\varphi}))_1 + (L(zq^{-\varphi}) \otimes L(zq^{\varphi}))_1(\Phi^*(wq^{c^2}) \otimes L^*(wq^{2\varphi}))_2 R_{21}(q^{-(c_1+c_2)/2}w/z) = \\
(1 \otimes \Phi^*(w)_2(L(zq^{-\varphi}) \otimes L(zq^{\varphi}))_1 + (\Phi^*(wq^{c^2}) \otimes L^*(wq^{2\varphi}))_2(L(zq^{-\varphi}) \otimes L(zq^{\varphi}))_1 =
\]

6
\[
\Delta(\Phi^*(w))_2 \Delta(L(x_1))_1,
\]
\[
R(q^{(c_1 + c_2)/2}z/w) \Delta L^*(z)_1 \Delta \Phi(w)_2 = \]
\[
R(q^{(c_1 + c_2)/2}z/w)(L^*(zq^2) \otimes L^*(zq^{-2}))_1(\Phi(w) \otimes 1 + L(wq^2) \otimes \Phi(wq^c))_2 = \]
\[
(\Phi(w) \otimes 1)_2(L^*(zq^2) \otimes L^*(zq^{-2}))_1 + R_{21}(q^{-(c_1 + c_2)/2}w/z)(L^*(zq^2) \otimes L^*(zq^{-2}))_1L(wq^2) \otimes \Phi(wq^c))_2 = \]
\[
\Delta \Phi(w)_2 \Delta L^*(z)_1.
\]
\[
\Delta \Phi(z)_1 \Delta \Phi(w)_2 - \Delta \Phi^*(w)_2 \Delta \Phi(z)_1 = \]
\[
(\Phi(z) \otimes 1 + L(zq^2) \otimes \Phi(zq^c))_1(1 \otimes \Phi^*(w) + \Phi^*(wq^c) \otimes L^*(wq^2))_2 - \]
\[
(1 \otimes \Phi^*(w) + \Phi^*(wq^c) \otimes L^*(wq^2))_2(\Phi(z) \otimes 1 + L(zq^2) \otimes \Phi(zq^c))_1 = \]
\[
0 + 1/(q - q^{-1})(L^*(wq^{c_1 + c_2}z) \otimes L^*(wq^2)) - \delta(z/wq^{-c_1 - c_2})L(zq^{c_1/2}) \otimes L^*(wq^2) + \]
\[
L(zq^2) \otimes (1/(q - q^{-1})(L^*(wq^2) \delta(z/wq^{c_1 + c_2})) - \delta(z/wq^{-c_1 + c_2})L(zq^{c_1 + c_2/2}) + \]
\[
(L(zq^2) \otimes \Phi(zq^c))_1(\Phi^*(wq^c) \otimes L^*(wq^2))_2 - (\Phi^*(wq^c) \otimes L^*(wq^2))_2(L(zq^2) \otimes \Phi(zq^c))_1.
\]
Because
\[
(L(zq^2) \otimes 1)_1(\Phi^*(wq^c))_2 \otimes 1)_2R_{21}(q^{-(c_1 + c_2)}w/z) = (\Phi^*(wq^c))_2 \otimes 1)_2(L(zq^2) \otimes 1)_1,
\]
and
\[
(R_{21}(q^{-(c_1 + c_2)}w/z))^{-1}(1 \otimes \Phi(zq^c))_1L^*(wq^2))_2 = L^*(wq^2))_2(1 \otimes \Phi(zq^c))_1,
\]
we have that
\[
\Delta \Phi(z)_1 \Delta \Phi(w)_2^* - \Delta \Phi^*(w)_2 \Delta \Phi(z)_1 = \]
\[
1/(q - 1^{-1})(\Delta L(z/wq^{(c_1 + c_2)/2}) \delta(z/wq^{-(c_1 + c_2)}) - \delta(z/wq^c)\Delta L^*(zq^{(c_1 + c_2)/2}).
\]
In all the setting above, we assume \(l_{ij}(z), l_{ij}^*(z), \Phi_1(z)\) and \(\Phi_1^*(z)\) are functional operators, namely the operator depending the variable \(z\). On the other hand, we can assume that \(z\) is a formal variable and \(l_{ij}(z) = \Sigma_{n \in \mathbb{Z}} l_{ij}(n)z^{-n}, l_{ij}^*(z) = \Sigma_{n \in \mathbb{Z}} l_{ij}^*(n)z^{-n}, \Phi_1(z) = \Sigma_{n \in \mathbb{Z}} \Phi_1(n)z^{-n}, \Phi_1^*(z) = \Sigma_{n \in \mathbb{Z}} \Phi_1^*(n)z^{-n}\). We can define an algebra \(DZP[R(x)]\).

Let \(R'(z) = R(z)f(z)\), where \(f(z)\) is the common divisor of all the functions \(F(z)\), such that \(F(z)R(z)\) has no poles.

7
Definition 3.2. The algebra $DZP[R(x)]$ is an associative algebra generated by $\Phi_i(x)$ indexed by a linear independent basis $e_i$ of $V$, $l_{ij}(x)$ and $l_{ij}^*(x)$ indexed by the linear independent basis $e_i^*$ of $V^*$, the dual space of $V$, and a central element $k$. Let $\Phi(x) = \Phi_i(x) \otimes e_i$, $\Phi^*(x) = \Phi_i^*(x) \otimes e_i^*$ the operator valued matrix $L(x) = \sum l_{ij}(x) \otimes e_{ij}, L^*(x) = \sum l_{ij}^*(x) \otimes e_{ij}$, such that $L(x)$ and $L^*(x)$ are invertible. They satisfies the commutation

$$R'(x_1/x_2)\Phi(x_1)\Phi(x_2) = f(x_1/x_2)\Phi(x_2)\Phi(x_1),$$

$$\Phi(x_1)L(x_2) = R(q^{1/2}x_1/x_2)^{-1}L(x_2)\Phi(x_1),$$

$$R(x_1/x_2)L(x_1)L(x_2) = L(x_2)L(x_1)R(x_1/x_2),$$

$$R(x_1/x_2)L^*(x_1)L^*(x_2) = L^*(x_2)L^*(x_1)R(x_1/x_2),$$

$$f(x_1/x_2)\Phi^*(x_2)\Phi^*(x_1) = \Phi^*(x_1)\Phi^*(x_2)R_{21}(x_2/x_1),$$

$$L^*(x_2)\Phi^*(x_1) = \Phi^*(x_1)L^*(x_2)R_{21}(q^{-1/2}x_2/x_1),$$

$$\Phi(x_1)\Phi(x_2)^2 - \Phi^*(x_2)\Phi(x_1) = 1/(q-q^{-1})(L^*(wq^{1/2})\delta(z/wq^{-1}) - \delta(z/wq)\delta(z/q^{1/2}),$$

$$L(x_1)\Phi^*(x_2)R_{21}(1/2q^{1/2}x_2/x_1) = \Phi^*(x_2)R_{21}(x_1).$$

Here $\Phi(x_1)L(x_2) = \sum \Phi_i(x_1)L_{kl}(x_2)e_i \otimes e_{kl}$, $L(x_2)\Phi(x_1) = \sum L_{kl}(x_2)\Phi_i(x_1)e_i \otimes e_{kl}$, $L(x_1)L(x_2) = PL(x_1)L(x_2)P = \sum L_{ij}(x_1)L_{i}(x_2)e_{ij} \otimes e_{kl}$, and the others are defined in the same way as above. $P$ is the permutation operator. $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$. The operator $R(z)$ and $R_{21}(z)$ are expanded in appropriate directions.

If the poles of the matrix of $R(z)$ are beyond a finite disc around zero, we can always impose the condition that $l_{kl}(n) = 0 = l_{kl}^*(-n)$, $l_{ij}(0) = l_{ij}^*(0)$, for $n < 0$, $i < j$. Then the condition of the invertibility is equivalent to requires that $l_{ii}(0)$ and $l_{ii}^*(0)$ are invertible.

Example 3.1. Let $v$ be one dimensional, and $R(z) = z - wq^2/zq^2 - w$. Let $l_{11}(n) = 0 = l_{11}^*(-n)$, for $n < 0$. Then the algebra $DZP[R(x)]$ is the quantum affine algebra $U_q(\hat{sl}(2))$. If we choose $R(z)$ to be other functions with the property $R(z) = (R(z^{-1}))^{-1}$, then it is an algebra defined in [DI] as an generalization of the the quantum affine algebra $U_q(\hat{sl}(2))$.

Example 3.2. Let $v = C^n$ and $R(z) = \sum (z - wq^2)/(zq^2 - w)e_{ii} \otimes e_{ii} + \sum (z - wq^{-1})/(zq^{-1} - w) (e_{ii} \otimes e_{i+1,i+1,i+1} + e_{i+1,i+1} \otimes e_{ii})$. Let $l_{kl}(n) = 0 = l_{kl}^*(-n) = l_{ij}(0) = l_{ij}^*(0)$, for $n < 0$, $i < j$. Then the algebra $DZP[R(x)]$ is an algebra, whose quotient (modular the cubic relations) is $U_q(\hat{sl}(n))$. If we substitute $z - wq^2/zq^2 - w$ and $(z - wq^{-1})/(zq^{-1} - w)$ by other functions, it will be the generalization of $U_q(\hat{sl}(n))$ without the cubic relations [DI].
Example 3.3. Let $v = C^n$ and $R(z)$ be the projection of the universal R-matrix $\mathfrak{R}$ of $U_q(\hat{\mathfrak{sl}}(n))$. Let $l_{ik}(n) = 0 = l_{ki}^*(-n) = l_{ij}(0) = l_{ji}^*(0)$, for $n < 0$, $i < j$. The operator $L(z)$ can be identified with the operator $(id \otimes \pi_V)R_21(zq^{c/2})$ and $L^*(z)$ with the operator $(id \otimes \pi_V)\mathfrak{R}^{-1}(z^{-1}q^{-c/2})$. The subalgebra generated by $L(z)$ and $L^*(z)$ is isomorphic to $U_q(\hat{\mathfrak{sl}}(n))$. It can see that the algebra $DZP[R(x)]$ should be isomorphic to $U_q(\hat{\mathfrak{sl}}(n+1))$, because when $q$ goes to 1, this algebra degenerate into $\hat{\mathfrak{sl}}(n+1)$.

From the definition, we can see both the subalgebra generated by $\Phi(z)$, $L(z)$ and $L^*(z)$ and the subalgebra generated by $\Phi^*(z)$, $L(z)$ and $L^*(z)$ are the Hopf algebras. If we take $R(z)$ to be the projection of the universal R-matrix $\mathfrak{R}$ of $U_q(\hat{\mathfrak{sl}}(n))$ on certain linear spaces, we will derive unconventional Hopf algebras from those subalgebras.

It is clear our new algebras can be viewed as a simple generalization of the Drinfeld realization of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}(2))$, where the function $g(z)$ is substitute by a matrix $R(z)$, the operators are substituted by the vector valued operators and the relations looks the same. However such a generalization is highly non-trivial in the sense that all the Hopf algebra structures are preserved, in the other words, those new algebras are Hopf algebras, whose comultiplication, counit and antipode symbolically are the same. These new Hopf algebras should be very useful in various applications in mathematics and physics, for example, the study of the representation theory of the elliptic Zamolochikov algebras.

Acknowledgment. We would like to thank M. Jimbo and B. Feigin for useful discussions. This project is supported by the grant Reward research (A) 08740020 from the Ministry of Education of Japan.

References

[B] D. Bernard Quantum Lie algebra and differential calculus on quantum groups, Preprint, SPht-90-119

[DFJMN] Diagonalization of the XXZ Hamiltonian by vertex operators, CMP, 151, 1993, 89-153

[DF] J. Ding, I. B. Frenkel Isomorphism of two realizations of quantum affine algebra $U_q(\hat{\mathfrak{sl}}(n))$, CMP, 156, 1993, 277-300 Physics

[DI] J. Ding, K. Iohara Generalization and deformation of the quantum affine algebras, RIMS-1090

[DM] J.Ding and T. Miwa Zeros and poles of quantum current operators and the quantum integrable condition, RIMS-1092.

[Dr1] V. G. Drinfeld Hopf algebra and the quantum Yang-Baxter Equation, Dokl. Akad. Nauk. SSSR, 283, 1985, 1060-1064

[Dr2] V. G. Drinfeld New realization of Yangian and quantum affine algebra, Soviet Math. Doklady, 36, 1988, 212-216

[FRT] L. D. Faddeev, N. Yu, Reshetikhin, L. A. Takhtajan Quantization of Lie groups and Lie algebras, Yang-Baxter equation in Integrable Systems, (Advanced Series in Mathematical Physics 10) World Scientific, 1989, 299-309.

[FO] B. Feigin, V. Odesski Sklyanin Elliptic algebras Funkts. Anal. Prilozhen., 23, 3, 1989, 45-54
B. Feigin, V. Odesski Vector bundles on Elliptic curve and Sklyanin algebras RIMS-1032, q-alg/9509021

E. Frenkel, B. Feigin Quantum W-algebra and elliptic algebras RIMS-1027, q-alg/9508009.

I. B. Frenkel, N. Yu, Reshetikhin Quantum affine algebras and holonomic difference equations, CMP, 146, 1992, 1-60

H. Garland The arithmetic theory of loop groups, Publ. Math. IHES 52, 1980, 5-136

M. Jimbo A q-difference analogue of U(g) and Yang-Baxter equation, Lett. Math. Phys. 10, 1985, 63-69

N.Yu. Reshetikhin, M.A. Semenov-Tian-Shansky Central Extensions of Quantum Current Groups, LMP, 19, 1990

E. K. Sklyanin On some algebraic structures related to the Yang-Baxter equation Funkts. Anal. Prilozhen, 16, No. 4, 1982, 22-34

JINTAI DING, RIMS, KYOTO UNIVERSITY