The law of the iterated logarithm for two-dimensional stochastic Navier–Stokes equations

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Abstract. We implement the Azencott method to prove the moderate deviation principle for the two-dimensional incompressible stochastic Navier–Stokes equations in a bounded domain. The Strassen’s compact law of iterated logarithm is then achieved as an application.

1. Introduction

The classical Azencott method, first introduced in [3,37], is applied here to prove the moderate deviation principle for the two-dimensional incompressible stochastic Navier–Stokes equations (SNSE) in incompressible flow given by

\[
\frac{\partial u^\varepsilon(t)}{\partial t} + (u^\varepsilon(t) \cdot \nabla) u^\varepsilon(t) + \nabla p(t) = f(t) + \Delta u^\varepsilon(t) + \sqrt{\varepsilon} \sigma(t, u^\varepsilon(t)) \frac{dW(t)}{dt},
\]

\[
(\nabla \cdot u^\varepsilon)(t, y) = 0, \quad y \in D, \quad t > 0, \quad u^\varepsilon(t, y) = 0, \quad y \in \partial D, \quad t \geq 0,
\]

\[
u^\varepsilon(0, y) = u_0(y), \quad y \in D,
\]

where \( p(t, y) \) and \( u^\varepsilon(t, y) \in \mathbb{R}^2 \) are the pressure and velocity fields, respectively. A deterministic external force \( f(t, y) \in \mathbb{R}^2 \) is assumed to be given, along with a noise coefficient \( \sigma(t, u^\varepsilon(t)) \) of a Q-Wiener process \( W(\cdot) \), with properties provided later in Sect. 2.

The area of large deviations has proven to be useful in many fields such as in statistical mechanics, finance, queueing theory and communications to determine the probability of a quantity under study to exceed a given threshold. For example in finance, large deviations has been used in [24] to find the probability of the total loss becoming greater than the portfolio default benchmark. The probability of the waiting time in a queue exceeding a prescribed time is also studied in [30] by means of large deviations. For other similar results see [16,22,29,31]. Another closely related area of study is moderate deviations, for which one proves large deviations for the

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centered process multiplied by a rate of convergence slower than the rate used for large deviations.

The majority of work on large deviations regarding the SNSE has been established based on the weak convergence approach introduced by [8, 9]. In [15] this technique was applied to obtain the large deviation principle for a general class of stochastic PDEs (SPDEs) for which two-dimensional incompressible SNSE was a special case. For two-dimensional viscous, incompressible SNSE, this approach was taken in [41] where the noise term tends to zero and in [5] where the viscosity is set to vanish. As for moderate deviations, considering the same general class of SPDEs introduced in [15] to achieve large deviations, authors in [46] proved the moderate deviation principle for this class with multiplicative noise and in unbounded domains. For moderate deviations in a bounded domain, the weak convergence method was applied for two-dimensional viscous, incompressible SNSE with multiplicative noise in [44] and with Lévy noise in [19].

Our purpose here is to revisit a classical method for large deviations, called the Azencott method in order to bring another perspective and emphasize the applications it offers. In the literature, after the introduction of the weak convergence approach, many authors have preferred this technique over the Azencott method to avoid the technical time discretized exponential estimates. It is important to note that although the two methods are different in nature, they both essentially aim to prove two main conditions to achieve the large or moderate deviations principle. The first condition is to verify that the map, \( \Phi : h \to X_h \) is continuous, where \( X_h(t) \) is the unique solution to the skeleton PDE equation. This ensures that the rate function is a good rate function. The second condition forms the stochastic controlled equation by adding the controlled term to the original equation and proves the convergence in distribution of this equation to the skeleton equation as the noise term is set to vanish. The Freidlin–Wentzell inequality is derived in the Azencott method for the second condition and plays a major role in proving the Strassen’s compact law of the iterated logarithm (LIL).

For stochastic Navier–Stokes equations, large or moderate deviation principle by the Azencott method is new and to the best of our knowledge, this is the first result on an LIL type estimate for this equation and it aims to add more insight to the asymptotic behavior of these equations.

After the observation first made by [18] (see Lemma 1.4.3), many authors have proved the Strassen’s compact LIL as a consequence of their large or moderate deviation results (see for instance [1, 4, 13, 20, 36]). Similarly, we use our result on moderate deviations established by the Azencott method to achieve the Strassen’s compact LIL. For a detailed introduction to LIL and its different types in the literature such as classical LIL, Chover’s and Chung’s LIL we recommend [6].

LIL offers a rate of convergence of the stochastic equation to its deterministic counterpart as the term representing the noise in the system is set to go to zero. It has been applied for example in [26–28] in queueing theory to measure this rate for
the number of customers at time $t$, the workload, busy time of the server and the total number of customer departures. In the case of heavy traffic, Minkevičius [32], Minkevičius and Sakalauskas [33] and Minkevičius et al. [34] prove the LIL for the length and sojourn time process of the queue, respectively. For applications in finance we refer the reader to [23,47] for LIL on the risk process.

We begin in Sect. 2 with some background on the LIL and provide statements of the main results along with notations needed for the rest of the paper. Section 3 is devoted to proving the moderate deviation principle by the Azencott method and afterwards the compact LIL is established in Sect. 4.

2. Preliminaries and main results

In this section, we provide the notations and background needed for the paper. Let $D \subset \mathbb{R}^2$ be a bounded open domain with smooth boundary $\partial D$. For convenience, we will denote $u^\varepsilon(t) \in \mathbb{R}^2$ as $u^\varepsilon(t)$ where it is understood that our setting is in two dimensions. We next introduce the standard spaces and notations for the deterministic Navier–Stokes equation (cf. [43]). Let

$$V := \left\{ u \in H^1_0(D; \mathbb{R}^2) : \nabla \cdot u = 0 \right\},$$

$$H := \left\{ u \in L^2(D; \mathbb{R}^2) : \nabla \cdot u = 0 \right\},$$

with norms

$$\|u\|_V = \left( \int_D |\nabla u|^2 \, dx \right)^{1/2}, \quad \text{and} \quad \|u\|_H = \left( \int_D |u|^2 \, dx \right)^{1/2},$$

respectively. Note that spaces $H$ and $V$ are the closures of the divergence free smooth compactly supported functions in $\|\cdot\|_{L^2}$ and $\|\cdot\|_{H^1}$, respectively. Here we will follow the conventional notation in denoting the norm in $H$ as $\|\cdot\|$ and the norm in $V$ as $\|\cdot\|$. For space $H$, divergence, $\nabla \cdot u = 0$, is understood in distributional sense and $u \cdot \hat{n} \rvert_{\partial\Omega} = 0$ is well-defined (see Theorems 1.4 and 1.6 of [43]). Both $H$ and $V$ are Hilbert spaces, and in particular, $V$ may be equipped with the inner product

$$(u, v)_V = \sum_{i,j=1}^2 \int_D \partial_i u_j \partial_i v_j \, dx.$$

Letting $H'$ and $V'$ denote the dual spaces of $H$ and $V$, respectively, we identify $H$ with $H'$ by the Riesz representation theorem to obtain, $V \leftrightarrow H \equiv H' \leftrightarrow V'$. where the embeddings are dense and compact. Furthermore, using the Helmholtz-Leray projection, $P_H : L^2(D; \mathbb{R}^2) \rightarrow H$, we define

$$Au := -P_H \Delta u, \quad \forall u \in H^2(D; \mathbb{R}^2) \cap V,$$

$$B(u, v) := P_H ((u \cdot \nabla) v), \quad \forall u, v \in D(B) \subset V \times V,$$
where $A$ is a positive-definite, self-adjoint operator referred to as the Stokes operator. Operators $A$ and $B$ may be defined explicitly as follows:

$$ (Au, v) := \sum_{i,j=1}^{2} \int_{D} \partial_{i} u_{j} \partial_{i} v_{j} \, dx, $$

(2)

$$ (Bu, w) := \sum_{i,j=1}^{2} \int_{D} u_{i} \partial_{i} v_{j} \, w_{j} \, dx =: b(u, v, w). $$

(3)

One may observe that $b(u, v, w) = -b(u, w, v)$ leading to $b(u, v, v) = 0$. For estimates derived in the rest of the paper we have the following inequalities derived in [41,44]:

$$ |b(u, v, w)| \leq 2 \|u\|^{1/2} \cdot |u|^{1/2} \cdot \|v\|^{1/2} \cdot |v|^{1/2} \cdot \|w\|, $$

(4)

$$ |b(u, u, v)| \leq \frac{1}{2} \|u\|^{2} + c \|v\|^{4}_{L_{4}} \cdot |u|^{2}, $$

(5)

$$ |(B(u) - B(v), u - v)| \leq \frac{1}{2} \|u - v\|^{2} + c |u - v|^{2} \|v\|^{4}_{L_{4}}, $$

(6)

where $B(u) := B(u, u)$. Projecting system (1) onto the divergence free vector-fields by $P_{H}$ and still denoting $P_{H} f(t)$ as $f(t)$, we obtain the preferred abstract version of SNSE as follows:

$$ du^{\varepsilon}(t) + Au^{\varepsilon}(t) \, dt + B(u^{\varepsilon}(t)) \, dt = f(t) \, dt + \sqrt{\varepsilon} \sigma(t, u^{\varepsilon}(t)) \, dW(t), $$

(7)

in a probability space, $(\Omega, \mathcal{F}, P)$, where $W(\cdot)$ is an $H$ valued $\{\mathcal{F}_{t}\}_{t \geq 0}$-adapted Q-Wiener process, that may be written as

$$ W(t) := \sum_{j=1}^{\infty} \sqrt{\lambda_{j}} e_{j} \beta_{j}(t), $$

for an infinite sequence of independent, standard one dimensional $\{\mathcal{F}_{t}\}_{t \geq 0}$ Brownian motions and a complete orthonormal system $\{e_{j}\}_{j=1}^{\infty}$ in $H$ satisfying $Q e_{j} = \lambda_{j} e_{j}$, where $\lambda_{j}$ is the $j$th eigenvalue of the covariance operator $Q$. Furthermore, we define the Hilbert space, $H_{0} := Q^{1/2} H$, with inner product,

$$ (u, v)_{0} = \left( Q^{-1/2} u, Q^{-1/2} v \right), $$

for all $u, v \in H_{0}$, where the embedding of $H_{0}$ in $H$ is Hilbert–Schmidt. We let $L_{Q}(H_{0} : H)$ be the space of linear operators $S : H_{0} \mapsto H$ such that $S Q^{1/2}$ is a Hilbert–Schmidt operator from $H$ to $H$, with norm, $\|S\|_{L_{Q}} := \sqrt{\text{tr}(S QS^{*})}$. For more background on the Navier–Stokes equations in the deterministic setting we recommend [39,40,43]. We now state the assumptions required for our results.

Assumption (H1): (i) $|\nabla \times f| \leq K$, where $K$ is a positive constant and $f \in L^{4}(0, T; \mathcal{V}')$ i.e. there exists a positive constant, $K_{0}$, such that

$$ \int_{0}^{T} \|f(s)\|^{4}_{\mathcal{V}'} \, ds < K_{0}, $$
(ii) \( \sigma : [0, T] \times V \to L_Q(H_0; H) \) is bounded, satisfies the linear growth condition and is Lipschitz continuous. That is, for all \( u, v \in V \) and all \( t \in [0, T] \):

\[
\|\sigma(t, u)\|_{L_Q} \leq K_1, \quad \|\sigma(t, u)\|_{L_Q}^2 \leq K_2 (1 + \|u\|^2),
\]

and \( \|\sigma(t, u) - \sigma(t, v)\|_{L_Q} \leq K_3 \|u - v\| \). \( \tag{8} \)

Assumption (H2): suppose Assumption (H1) holds and additionally suppose

\[
\|\text{curl } \sigma(t, u)\|_{L_Q}^2 \leq \tilde{K}_0 + \tilde{K}_1 \|u\|^2,
\]

for \( u \in D(A) \). For our results the following estimates achieved in \([41,44]\) are frequently used.

**Lemma 1.** (Proposition 2.3 in \([41]\) and Proposition 3.1 in \([44]\)) If \( u^\varepsilon(t) \) is the solution of SNSE (7), then under Assumption (H1), for any \( \varepsilon < \frac{1}{2K_0} \wedge \frac{1}{4K_0} \wedge \frac{1}{2K_2} \),

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |u^\varepsilon(t)|^2 + \int_0^T \|u^\varepsilon(s)\|^2 ds \right) \leq K_4, \tag{10}
\]

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |u^\varepsilon(t)|^4 + \int_0^T |u^\varepsilon(s)|^2 \|u^\varepsilon(s)\|^2 ds \right) \leq K_5, \tag{11}
\]

\[
\sup_{0 \leq t \leq T} |u^0(t)|^2 + \int_0^T \|u^0(s)\|^2 ds \leq K_6, \tag{12}
\]

\[
\int_0^T \|u^0(s)\|_{L^4}^4 ds \leq \sup_{0 \leq t \leq T} |u^0(t)|^2 \int_0^T \|u^0(s)\|^2 ds \leq K_7, \tag{13}
\]

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |u^\varepsilon(t) - u^0(t)|^2 + \int_0^T \|u^\varepsilon(s) - u^0(s)\|^2 ds \right) \leq \varepsilon K_8, \tag{14}
\]

where each constant above depends on \( T \) and \( K_0 \).

We say that a family \( \{u^\varepsilon(\cdot)\}_{\varepsilon > 0} \) satisfies the moderate deviation principle, if the family \( \{v^\varepsilon(\cdot)\}_{\varepsilon > 0} \) defined as \( v^\varepsilon(t) := (a(\varepsilon)/\sqrt{\varepsilon})(u^\varepsilon(t) - u^0(t)) \) obeys the large deviation principle where conditions on \( a(\varepsilon) \) are \( a(\varepsilon) > 0 \) and \( a(\varepsilon)/\sqrt{\varepsilon} \to \infty \) as \( \varepsilon \) tends to zero. This ensures that the rate of decay of moderate deviation given by \( a(\varepsilon) \) is at a slower speed than the rate of decay for large deviation given by \( \sqrt{\varepsilon} \). Let \( \mathcal{H}_0 \) be the space consisting of absolutely continuous functions, \( h : [0, T] \to H_0 \) such that \( \int_0^T |h(s)|_0^2 ds < \infty \) and denote \( S_N := \{h \in \mathcal{H}_0 : \int_0^T |h(s)|_0^2 ds \leq N\} \). Then for every \( h \in S_N \), the controlled PDEs for \( v^\varepsilon(t) \), also referred to as the skeleton equation, is given by

\[
\frac{dX^h(t)}{dt} + AX^h(t)dt = -B(X^h(t), u^0(t))dt \tag{15}
\]

\[
-\mathbb{B}(u^0(t), X^h(t))dt + \mathbb{B}(u^0(t), h(t))dt + \mathbb{B}(u^0(t), X^h(t))dt + \mathbb{B}(u^0(t), h(t))dt,
\]
where $X^h(0) = 0$ and for which there exists a unique solution, denoted as $\Gamma^0(\int_0^T h(s)ds)$, in $\mathcal{C}([0, T]; H) \cap L^2(0, T; V)$ (cf. [44]). We denote

$$\varepsilon_0 := \min \left\{ 1, \frac{1}{2K_0^2}, \frac{1}{4K_0}, \frac{1}{2K_2} \right\}. \quad (16)$$

**Theorem 1.** Family $\{u^\varepsilon(\cdot)\}_{\varepsilon \in (0, \varepsilon_0)}$ satisfies the moderate deviation principle in $\mathcal{C}([0, T]; H) \cap L^2(0, T; V)$ with speed $a(\varepsilon)^2$ and the rate function

$$I(v) = \frac{1}{2} \inf_{\infty} \int_0^T |h(s)|^2 ds, \quad \text{for } v = \Gamma^0(\int_0^T h(s)ds), \ h \in \mathcal{H}_0, \ \text{otherwise}. \quad (17)$$

For the above theorem, we first prove the result for $a(\varepsilon) = 1/\sqrt{2 \log \log \frac{1}{\varepsilon}}$, which satisfies the required conditions on $a(\varepsilon)$ and gives the process

$$Z^\varepsilon(t) := \frac{1}{\sqrt{2 \varepsilon \log \log \frac{1}{\varepsilon}}} (u^\varepsilon(t) - u^0(t)), \quad (18)$$

namely,

$$Z^\varepsilon(t) = - \int_0^t A Z^\varepsilon(s)ds - \int_0^t B \left( Z^\varepsilon(s), u^\varepsilon(s) \right) ds - \int_0^t B \left( u^0(s), Z^\varepsilon(s) \right) ds + \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \int_0^t \tilde{\sigma}(s, Z^\varepsilon(s))dW(s), \quad (19)$$

with

$$\tilde{\sigma}(t, Z^\varepsilon(t)) := \sigma \left( t, \sqrt{2 \varepsilon \log \log \frac{1}{\varepsilon}} Z^\varepsilon(t) + u^0(t) \right).$$

Note that based on assumptions in (8) and (9), the following conditions hold:

$$\|\tilde{\sigma}(t, Z^\varepsilon(t))\|_{L^2}^2 \leq K_9 \left( 1 + 4\varepsilon \log \log \frac{1}{\varepsilon} \|Z^\varepsilon(t)\|^2 + 2\|u^0(t)\|^2 \right), \quad (20)$$

$$\|\tilde{\sigma}(t, Z^1(t)) - \tilde{\sigma}(t, Z^2(t))\|_{L^2} \leq K_{10} \sqrt{2 \varepsilon \log \log \frac{1}{\varepsilon}} \|Z^1(t) - Z^2(t)\|, \quad (21)$$

$$\|\text{curl } \tilde{\sigma}(t, Z^\varepsilon(t))\|_{L^2}^2 \leq \tilde{K}_2 + \tilde{K}_3 \sqrt{2 \varepsilon \log \log \frac{1}{\varepsilon}} (\|Z^\varepsilon(t)\| + \|u^0(t)\|). \quad (22)$$

The following is the definition of Strassen’s Compact LIL and the result achieved in this paper. For more information and similar results on this type of LIL we recommend [4, 15, 21, 45].

**Definition 1.** A class of functions, $\mathcal{F}$ satisfies Strassen’s Compact LIL with respect to an i.i.d. sequence of random variables, $\{X_j\}_{j \geq 1}$ if there exists a compact set $J$ in $\ell_\infty(\mathcal{F})$ such that $\{X_j\}_{j \geq 1}$ is a.s. relatively compact and its limit set is precisely $J$.

**Theorem 2.** Family $\{Z^\varepsilon(\cdot)\}_{\varepsilon \in (0, \varepsilon_0)}$ is relatively compact in $\mathcal{C}([0, T]; H) \cap L^2(0, T; V)$ and its set of limit points is exactly

$$L := \{ g \in \mathcal{C}([0, T]; H) \cap L^2(0, T; V) : I(g) \leq \frac{N}{2} \}, \text{where } I(g) \text{ is given by (17).}$$
3. Moderate deviations

We prove the moderate deviation principle for \( \{u^ε(\cdot)\}_{ε \in (0, ε₀)} \) by establishing the large deviation principle for \( \{Z^ε(\cdot)\}_{ε \in (0, ε₀)} \). The Azencott method implemented here may be described as follows. Consider two families of random variables \( \{Y^ε_1\}\) taking values in Polish spaces \( E_1, E_2 \), respectively with the corresponding metrics \( d_1, d_2 \). Suppose \( \{Y^ε_1\}\) satisfies the large deviation principle with rate function \( \widetilde{I}(g) \) where \( g \in E_1 \). Let \( a > 0 \) be fixed and \( Φ : \{I < δ\} \to E_2 \). For any \( R > 0, ρ > 0 \), suppose there exists \( ε > 0 \) such that for all \( 0 < ε ≤ \bar{ε} \), the following inequality:

\[
P \left( d_2(Y^ε_2, Φ(f)) ≥ ρ, d_1(Y^ε_1, f) < η \right) ≤ \exp \left( -\frac{R}{ε^2} \right),
\]

referred to as the Freidlin–Wentzell inequality, holds for any \( g \in E_1 \) with \( \widetilde{I}(g) ≤ a \). In addition, suppose the map \( Φ(\cdot) \) is continuous with respect to the topology of \( E_1 \) when restricted to compact sets \( \{I ≤ a\}_{a > 0} \) for every positive constant \( a \). Then \( \{Y^ε_1\} \) also satisfies the large deviation principle with rate function, \( I(h) := \inf\{\widetilde{I}(g) : Φ(g) = h\} \) for any \( h \in E_2 \). In the setting of the stochastic PDEs, the map \( Φ(\cdot) \) is the unique solution of the skeleton equation, which in our case is given by (15). We let \( Y^ε_1 := (1/√2 \log \log 1/ε) W \), which by an extension of the result of the Schilder’s theorem to Q-Wiener process, is known to satisfy the large deviation principle with a good rate function (for a proof see Theorem 3.1 in Chapter 6 of [2]). Thus, it is sufficient to prove that \( h \mapsto X^h \) is continuous and achieve inequality (23). For examples of large deviations results using this technique for SPDEs we refer the reader to [7, 12, 14, 35].

We begin by establishing the continuity of the map \( h \mapsto X^h(\cdot) \). For simplicity of notation, we denote \( L^∞(0, T; H) \) as \( L^∞_{H_0} \) and for the rest of the article we let the norm in \( C([0, T]; H) \cap L^2(0, T; V) \) be denoted as \( E(T) \), namely,

\[
\|u(t)\|_{E(T)} := \left( \sup_{0 ≤ t ≤ T} |u(t)|^2 + \int_0^T \|u(s)\|^2 ds \right)^{1/2}.
\]

**Lemma 2.** For every \( h ∈ H_0 \) and \( a > 0 \), the map \( h \mapsto X^h \) is continuous in \( C([0, T]; H) \cap L^2(0, T; V) \) with respect to the uniform convergence topology when restricted to the level set \( \{I ≤ a\}_{a > 0} \).

**Proof.** Let \( a > 0 \) and \( h, k ∈ H_0 \) such that \( |h|_0 ∨ |k|_0 ≤ a \), then we have using \( b(u, v, v) = 0 \):

\[
\begin{align*}
|X^h(t) − X^k(t)|^2 &+ 2 \int_0^t \|X^h(s) − X^k(s)\|^2 ds \\
&= -2 \int_0^t \left( B(X^h(s) − X^k(s), u^0(s)), X^h(s) − X^k(s) \right) ds \\
&+ 2 \int_0^t \left( σ(s, u^0(s))(h(s) − k(s)), X^h(s) − X^k(s) \right) ds.
\end{align*}
\]
By inequality (5) and Young’s inequality along with Assumption (H1) we obtain

\[
|X^h(t) - X^k(t)|^2 + 2 \int_0^t \|X^h(s) - X^k(s)\|^2 ds \\
\leq \int_0^t \|X^h(s) - X^k(s)\|^2 ds + 2 \int_0^t \|u^0(s)\|_{L^4} |X^h(s) - X^k(s)|^2 ds \\
+ 2\|h(s) - k(s)\|^2_{L^\infty} + \frac{1}{2} \int_0^t \|\sigma(s, u^0(s))\|^2_{L_0} |X^h(s) - X^k(s)|^2 ds \\
\leq \int_0^t \left( 2c\|u^0(s)\|^4_{L^4} + \frac{K_2}{2} + \frac{K_2}{2}\|u^0(s)\|^2 \right) |X^h(s) - X^k(s)|^2 ds \\
+ \int_0^t \|X^h(s) - X^k(s)\|^2 ds + 2\|h(s) - k(s)\|^2_{L^\infty},
\]

which by Gronwall’s inequality yield

\[
|X^h(t) - X^k(t)|^2 \\
\leq 2\|h(s) - k(s)\|^2_{L^\infty} \exp \left( \int_0^T 2c\|u^0(s)\|^4_{L^4} + \frac{K_2}{2} + \frac{K_2}{2}\|u^0(s)\|^2 ds \right).
\]

Noting inequalities (12) and (13) and using (25) in the bound of (24), we achieve the continuity of the map \( h \mapsto X^h \). \( \square \)

Now we focus on obtaining the Freidlin–Wentzell inequality (23), which for our model is, for any \( R > 0 \) and \( \rho > 0 \), there exists \( \eta > 0 \) and \( \bar{\varepsilon} > 0 \) such that for all \( 0 < \varepsilon \leq \bar{\varepsilon} \):

\[
P \left( \|Z^\varepsilon(t) - X^h(t)\|_{\mathcal{E}(T)} \geq \rho, \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} W - h \right)_{L^\infty} < \eta \right) \\
\leq \exp \left( -2R \log \log \frac{1}{\varepsilon} \right),
\]

where as noted earlier, \( \{(1/\sqrt{2 \log \log \frac{1}{\varepsilon}}) W\}_{\varepsilon > 0} \) satisfies the large deviation principle. By Girsanov’s transformation theorem, inequality (26) is implied by the following inequality (see [12,35,38] for more details):

\[
P \left( \|\tilde{Z}^\varepsilon(t) - X^h(t)\|_{\mathcal{E}(T)} \geq \rho, \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} W \right)_{L^\infty} < \eta \right) \\
\leq \exp \left( -2R \log \log \frac{1}{\varepsilon} \right),
\]

with
\[ \tilde{Z}^\varepsilon(t) = -\int_0^t A\tilde{Z}^\varepsilon(s)ds - \int_0^t B(u^\varepsilon(s), \tilde{Z}^\varepsilon(s))ds - \int_0^t B(\tilde{Z}^\varepsilon(s), u^0(s))ds \\
+ \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \int_0^t \tilde{\sigma}(s, \tilde{Z}^\varepsilon(s))dW(s) + \int_0^t \tilde{\sigma}(s, \tilde{Z}^\varepsilon(s))h(s)ds. \]

Now we apply a time discretization on \( \tilde{Z}^\varepsilon(t) \) by letting \( \Delta_1^n := [t^n_0, t^n_{j+1}) \), where, \( n \in \mathbb{N}\{0\} \), \( j = 0, 1, 2, \ldots, 2^n - 1 \) and \( t_j^n = (Tj)/2^n \). To obtain (27), it is sufficient to prove that there exists \( \beta > 0, n_0 \in \mathbb{N}\{0\} \) and \( \varepsilon > 0 \) such that for all \( n \geq n_0, \varepsilon \in (0, \varepsilon) \):

\[ P (\|\tilde{Z}^\varepsilon(t) - \tilde{Z}^\varepsilon(t_j^n)\|_{\mathcal{E}(T)} > \beta) \leq \exp \left( -2R \log \log \frac{1}{\varepsilon} \right), \tag{28} \]

and

\[ P \left( \|\tilde{Z}^\varepsilon(t) - X^h(t)\|_{\mathcal{E}(T)} \geq \rho, \left\| \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} W \right\|_{L^\infty_H} < \eta, \right. \]

\[ \|\tilde{Z}^\varepsilon(t) - \tilde{Z}^\varepsilon(t_j^n)\|_{\mathcal{E}(T)} \leq \beta \right) \leq \exp \left( -2R \log \log \frac{1}{\varepsilon} \right). \tag{29} \]

For this purpose, the following lemmas are proved and applied. For better presentation, their proofs are given in the Appendix. Recall \( \varepsilon_0 \) defined in (16).

**Lemma 3.** For any \( \varepsilon \in (0, \varepsilon_0) \) and \( p \geq 1 \),

\[ \mathbb{E} \sup_{0 \leq s \leq t} |\tilde{Z}^\varepsilon(s)|^{2p} + p\mathbb{E} \int_0^T |\tilde{Z}^\varepsilon(s)|^{2(p-1)} \|\tilde{Z}^\varepsilon(s)\|^2 ds \leq \tilde{M}_p(T), \tag{30} \]

where \( \tilde{M}_p(T) \) is a positive constant independent of \( \varepsilon \).

With the same set of techniques used to prove the above estimates, we may derive the following bounds, (31) and (32), for the proof of which we refer the reader to Lemma 4.2 in [25] and Proposition 4.4 in [44], respectively.

**Lemma 4.** For any \( p \geq 1 \) if \( \varepsilon < \frac{2}{1+2p} \),

\[ \mathbb{E} \sup_{0 \leq s \leq T} |u^\varepsilon(s)|^{2p} + \mathbb{E} \int_0^T |u^\varepsilon(s)|^{2(p-1)} \|u^\varepsilon(s)\|^2 ds \leq \tilde{N}_p(\varepsilon, T, u^\varepsilon(0)), \tag{31} \]

and for any \( N > 0 \) and \( h \in \mathcal{H}_0 \),

\[ \sup_{h \in \mathcal{S}_N} \left( \sup_{0 \leq t \leq T} |X^h(t)|^2 + \int_0^T \|X^h(s)\|^2 ds \right) \leq \tilde{N}(T, N). \tag{32} \]
In addition, the proposition below is used, which is established in [5] for 2D stochastic Navier Stokes equation having viscosity, \( v > 0 \), given as

\[
du_h^v(t) = -\left( vAu_h^v(t) + B(u_h^v(t), u_h^v(t)) \right) dt + \tilde{\sigma}_v(t, u_h^v(t))h(t)dt + \sqrt{v}\sigma_v(t, u_h^v(t))dW(t),
\]

\( u_h^v(0) = \eta. \)

**Proposition 1.** (Proposition 2.2 in [5]) Let \( p \geq 2 \), \( \mathbb{E}\|\eta\|^{2p} < \infty \) and suppose Assumption (H2) holds. Then given \( M > 0 \), there exists a positive constant \( C_2(p, M) \) such that for \( v \in (0, v_0] \) and \( h \in S_M \),

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \|u_h^v(t)\|^{2p} + v \int_0^T |Au_h^v(s)|^2 ds \right) \leq vC_2(p, M)(1 + \mathbb{E}\|\eta\|^{2p}). \tag{33}
\]

The main idea in their proof is to apply the curl operator to the solution and then use the Itô’s formula and a stochastic Gronwall inequality offered by Lemma A.1 of [15]. With the condition, \( |\nabla \times f| \leq K \) given in Assumption (H1), we may achieve the same inequality for \( u^\varepsilon(t) \) by setting \( \tilde{\sigma}(t, u^\varepsilon(t)) = 0 \). In addition, we may obtain this inequality for \( \tilde{Z}^\varepsilon(t) \) by the same proof by noting the following equality,

\[
(\text{curl } B(u, v), \text{curl } v) = 0,
\]

for \( u, v \in H^{2.2} \) given as equation (6.7) in [5]. Thus, we have

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|^{2p} + \int_0^T |Au^\varepsilon(s)|^2 ds \right) \leq \varepsilon\tilde{C}_1(p, M), \tag{34}
\]

and

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \|\tilde{Z}^\varepsilon(t)\|^{2p} + \int_0^T |A\tilde{Z}^\varepsilon(s)|^2 ds \right) \leq \frac{1}{2 \log \log \frac{1}{\varepsilon}}\tilde{C}_2(p, M). \tag{35}
\]

Moreover, similar to the proof of Theorem 3.1 of [5], the following may be established:

\[
\sup_{0 \leq t \leq T} \|u^0(t)\|^{2p} \leq C(M)(\|u^0(0)\|^{2p}) := \tilde{C}_3(2p). \tag{36}
\]

\[
\sup_{0 \leq t \leq T} \|X^h(t)\|^{2p} \leq C(M)(\|X^h(0)\|^{2p}) := \tilde{C}_4(2p). \tag{37}
\]

**Lemma 5.** For every positive constant \( R > 0 \) and \( \beta > 0 \) there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \) and \( \varepsilon \in (0, \varepsilon_0) \),

\[
P \left( \|\tilde{Z}^\varepsilon(t) - \tilde{Z}^\varepsilon(t_i^\varepsilon)\|_{\mathbb{E}(T)} > \beta \right) \leq \exp \left( -2R \log \log \frac{1}{\varepsilon} \right). \tag{38}
\]
Proof. Using the time discretization introduced earlier, we apply Itô’s formula to find
for \( t \in [t^n_i, t^n_{i+1}) \), \( s^n_i \in [0, t^n_i] \), and stopping time,
\[ \tau_N := \inf \{ t : \| \tilde{Z}^e(t) - \tilde{Z}^e(t^n_i) \|^2_{E(t)} > N \} , \]
where \( N \in \mathbb{N} \),
\[
| \tilde{Z}^e(t \land \tau_N) - \tilde{Z}^e(t^n_i \land \tau_N) |^2 + 2 \int_{t^n_i \land \tau_N}^{t \land \tau_N} \| \tilde{Z}^e(s) - \tilde{Z}^e(s^n_i) \|^2 ds
\]
\[
= -2 \int_{t^n_i \land \tau_N}^{t \land \tau_N} (A \tilde{Z}^e(s^n_i), \tilde{Z}^e(s) - \tilde{Z}^e(s^n_i)) ds
\]
\[
- 2 \int_{t^n_i \land \tau_N}^{t \land \tau_N} (B(u^e(s), \tilde{Z}^e(s)), \tilde{Z}^e(s) - \tilde{Z}^e(s^n_i)) ds
\]
\[
- 2 \int_{t^n_i \land \tau_N}^{t \land \tau_N} (B(\tilde{Z}^e(s), u^0(s)), \tilde{Z}^e(s) - \tilde{Z}^e(s^n_i)) ds
\]
\[
+ \frac{2}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \int_{t^n_i \land \tau_N}^{t \land \tau_N} (\tilde{\sigma}(s, \tilde{Z}^e(s))dW(s), \tilde{Z}^e(s) - \tilde{Z}^e(s^n_i)) ds
\]
\[
+ 2 \int_{t^n_i \land \tau_N}^{t \land \tau_N} (\tilde{\sigma}(s, \tilde{Z}^e(s))h(s), \tilde{Z}^e(s) - \tilde{Z}^e(s^n_i)) ds
\]
\[
+ \frac{1}{2 \log \log \frac{1}{\varepsilon}} \int_{t^n_i \land \tau_N}^{t \land \tau_N} \| \tilde{\sigma}(s, \tilde{Z}^e(s)) \|^2_{L^2} ds
\]
\[
= I_0(t^n_i \land \tau_N, t \land \tau_N) + I_1(t^n_i \land \tau_N, t \land \tau_N) + I_2(t^n_i \land \tau_N, t \land \tau_N)
\]
\[
+ I_3(t^n_i \land \tau_N, t \land \tau_N) + I_4(t^n_i \land \tau_N, t \land \tau_N) + I_5(t^n_i \land \tau_N, t \land \tau_N).
\]

Similar to the bounds in the Appendix derived for the proof of Lemma 3, we will take the supremum up to time \( t \land \tau_N \) and then expectation and determine the bounds for \( \mathbb{E}|I_j(t^n_i \land \tau_N, t \land \tau_N)| \) for \( j = 0, 1, \ldots, 5 \), \( j \neq 3 \), for which (35) is used. For \( I_0(t^n_i \land \tau_N, t \land \tau_N), \) by applying the Young’s inequality, we obtain
\[
\mathbb{E}|I_0(t^n_i \land \tau_N, t \land \tau_N)| \leq 2 \mathbb{E} \int_{t^n_i \land \tau_N}^{t \land \tau_N} (A \tilde{Z}^e(s^n_i), \tilde{Z}^e(s)) ds + 2 \mathbb{E} \int_{t^n_i \land \tau_N}^{t \land \tau_N} \| \tilde{Z}^e(s^n_i) \|^2 ds
\]
\[
\leq 2 \mathbb{E} \int_{t^n_i \land \tau_N}^{t \land \tau_N} \| \tilde{Z}^e(s^n_i) \| \| \tilde{Z}^e(s) \| ds + 2 \mathbb{E} \int_{t^n_i \land \tau_N}^{t \land \tau_N} \| \tilde{Z}^e(s^n_i) \|^2 ds
\]
\[
\leq 4 \mathbb{E} \int_{t^n_i \land \tau_N}^{t \land \tau_N} \| \tilde{Z}^e(s^n_i) \|^2 ds + \frac{1}{2} \mathbb{E} \int_{t^n_i \land \tau_N}^{t \land \tau_N} \| \tilde{Z}^e(s) \|^2 ds
\]
\[
\leq \frac{9 \tilde{C}_2(1, M)}{4 \log \log \frac{1}{\varepsilon}} | t \land \tau_N - t^n_i \land \tau_N |.
\]

Note that by Cauchy–Schwarz inequality
\[
\mathbb{E} \int_0^t \| u^e(s) \| \| u^e(s) \| ds \leq \int_0^t \left( \mathbb{E} \| u^e(s) \|^2 \right)^{1/2} \left( \mathbb{E} | u^e(s) |^2 \right)^{1/2} ds \]
\[
\leq \left( \int_0^t \mathbb{E} \| u^e(s) \|^2 ds \right)^{1/2} \left( \int_0^t \mathbb{E} | u^e(s) |^2 ds \right)^{1/2} \leq \sqrt{\varepsilon \tilde{C}_1(1, M) K_4 t}.
\]
Applying the above estimate along with (4) and Young’s inequality, lead to
\[ \mathbb{E}|I_1(t_i^n \wedge \tau_N, t \wedge \tau_N)| = 2 \left| \mathbb{E} \int_{t_i^n \wedge \tau_N}^{t \wedge \tau_N} \left( B(u^\varepsilon(s), \tilde{Z}^\varepsilon(s)), \tilde{Z}^\varepsilon(s_i^n) \right) \, ds \right| \]
\[ \leq \frac{1}{2} \mathbb{E} \int_{t_i^n \wedge \tau_N}^{t \wedge \tau_N} \|u^\varepsilon(s)\| |u^\varepsilon(s)| \, ds + 8\mathbb{E} \int_{t_i^n \wedge \tau_N}^{t \wedge \tau_N} \|\tilde{Z}^\varepsilon(s)| \|\tilde{Z}^\varepsilon(s)| \|\tilde{Z}^\varepsilon(s_i^n)|^2 \, ds \]
\[ \leq \frac{1}{2} \mathbb{E} \int_{t_i^n \wedge \tau_N}^{t \wedge \tau_N} \|u^\varepsilon(s)\| |u^\varepsilon(s)| \, ds + 2\mathbb{E} \int_{t_i^n \wedge \tau_N}^{t \wedge \tau_N} \|\tilde{Z}^\varepsilon(s)|^2 \|\tilde{Z}^\varepsilon(s)|^2 \, ds \]
\[ + 8\mathbb{E} \int_{t_i^n \wedge \tau_N}^{t \wedge \tau_N} \|\tilde{Z}^\varepsilon(s_i^n)|^4 \, ds \]
\[ \leq \left( \frac{\sqrt{\varepsilon C_1(1, M) K_4}}{2} + \frac{\tilde{C}_2(1, M) \tilde{M}_1(T) + \frac{4 \tilde{C}_2(2, M)}{\log \log \frac{1}{\varepsilon}}}{t \wedge \tau_N - t_i^n \wedge \tau_N} \right) |t \wedge \tau_N - t_i^n \wedge \tau_N|. \]

Furthermore, by (4) and (5) we have
\[ \mathbb{E}|I_2(t_i^n \wedge \tau_N, t \wedge \tau_N)| \]
\[ \leq 2 \mathbb{E} \int_{t_i^n \wedge \tau_N}^{t \wedge \tau_N} b(\tilde{Z}^\varepsilon(s), \tilde{Z}^\varepsilon(s), u^0(s)) \, ds + \mathbb{E} \int_{t_i^n \wedge \tau_N}^{t \wedge \tau_N} b(\tilde{Z}^\varepsilon(s), u^0(s), \tilde{Z}^\varepsilon(s_i^n)) \, ds \]
\[ \leq 2 \mathbb{E} \int_{t_i^n \wedge \tau_N}^{t \wedge \tau_N} \|\tilde{Z}^\varepsilon(s)|^2 \, ds + 2\mathbb{E} \int_{t_i^n \wedge \tau_N}^{t \wedge \tau_N} c \|u^0(s)| \|u^0(s)| \|\tilde{Z}^\varepsilon(s)|^2 \, ds \]
\[ + 2\mathbb{E} \int_{t_i^n \wedge \tau_N}^{t \wedge \tau_N} \|\tilde{Z}^\varepsilon(s)| \|\tilde{Z}^\varepsilon(s)| \, ds + \frac{1}{2} \int_{t_i^n \wedge \tau_N}^{t \wedge \tau_N} \|u^0(s)| \|u^0(s)| \, ds \]
\[ + 2\mathbb{E} \int_{t_i^n \wedge \tau_N}^{t \wedge \tau_N} \|\tilde{Z}^\varepsilon(s_i^n)|^4 \, ds \]
\[ \leq \left( \frac{\tilde{C}_2(1, M)}{2 \log \log \frac{1}{\varepsilon}} + 2 K_9 \tilde{M}_1(T) \tilde{C}_3(2) + 2 \sqrt{\frac{\tilde{C}_2(1, M) \tilde{M}_1(T)}{\log \log \frac{1}{\varepsilon}}} + \frac{K_9 \tilde{C}_3(1)}{2} + \frac{\tilde{C}_2(2, M)}{\log \log \frac{1}{\varepsilon}} \right) \times |t \wedge \tau_N - t_i^n \wedge \tau_N|. \]

The Cauchy–Schwarz, Hölder’s and Young’s inequalities may again be invoked to obtain
\[ \mathbb{E}|I_4(t_i^n \wedge \tau_N, t \wedge \tau_N)| \leq 2 \mathbb{E} \int_{t_i^n \wedge \tau_N}^{t \wedge \tau_N} \|\tilde{\sigma}(s, \tilde{Z}^\varepsilon(s))\|_{L_Q} |h(s)|_0 |\tilde{Z}^\varepsilon(s) - \tilde{Z}^\varepsilon(s_i^n)| \, ds \]
\[ \leq 2 \sqrt{N} \left( \mathbb{E} \int_{t_i^n \wedge \tau_N}^{t \wedge \tau_N} \|\tilde{\sigma}(s, \tilde{Z}^\varepsilon(s))\|_{L_Q} |\tilde{Z}^\varepsilon(s) - \tilde{Z}^\varepsilon(s_i^n)| \right)^{1/2} \]
\[ \leq 2 \sqrt{N} \left( \mathbb{E} \|\tilde{\sigma}(s, \tilde{Z}^\varepsilon(s))\|_{L_Q}^2 \mathbb{E}|\tilde{Z}^\varepsilon(s) - \tilde{Z}^\varepsilon(s_i^n)|^2 \right)^{1/2} \]
\[ \leq 2 \sqrt{NK_9} \left( 2 \tilde{M}_1(T) (1 + 2\varepsilon \tilde{C}_2(1, M) + 2 \tilde{C}_3(2)) \right)^{1/2} |t \wedge \tau_N - t_i^n \wedge \tau_N|^{1/2}. \]
Similarly, inequality (20) implies
\[
\mathbb{E}|I_5(t^n_i, t \wedge \tau_N)| \leq \frac{K_9}{2 \log \log \frac{1}{\varepsilon}} \left( 1 + 2\varepsilon \tilde{C}_2(1, M) + 2\tilde{C}_3(2) \right) |t \wedge \tau_N - t^n_i \wedge \tau_N|.
\]
(39)

Since \( |t \wedge \tau_N - t^n_i \wedge \tau_N| \leq T 2^{-n} \), then there exists \( n_0 \in \mathbb{N} \), such that for all \( n \geq n_0 \) and \( j = 0, 1, \ldots, 5 \) except \( j = 3 \),
\[
P \left( I_j(t^n_i \wedge \tau_N, t \wedge \tau_N) > \frac{\beta^2}{6} \right) \leq \exp \left( -2R \log \log \frac{1}{\varepsilon} \right),
\]
by Chebyshev’s inequality, for any given \( R > 0 \) and \( 0 < \varepsilon < \varepsilon_0 \). As for \( j = 3 \):
\[
P \left( I_3(t^n_i \wedge \tau_N, t \wedge \tau_N) > \frac{\beta^2}{6} \right)
\leq P \left( \exp \left( \int_{t^n_i \wedge \tau_N}^{t \wedge \tau_N} \left( \alpha(s, \tilde{Z}(s))dW(s), \tilde{Z}(s) - \tilde{Z}(s^n_i) \right)^2 \right) \right)
> \exp \left( \frac{\beta^4}{72 \log \log \frac{1}{\varepsilon}} \right).
\]

Inspired by the proof of Theorem 3.2 in [11] we write
\[
\mathbb{E} \exp \left( \int_{t^n_i \wedge \tau_N}^{t \wedge \tau_N} \left( \alpha(s, \tilde{Z}(s))dW(s), \tilde{Z}(s) - \tilde{Z}(s^n_i) \right)^2 \right)
\leq \mathbb{E} \lim_{m \to \infty} \sum_{k=0}^{m} \frac{1}{k!} \left( \int_{t^n_i \wedge \tau_N}^{t \wedge \tau_N} \left( \alpha(s, \tilde{Z}(s))dW(s), \tilde{Z}(s) - \tilde{Z}(s^n_i) \right)^2 \right)^2,
\]
(40)
and since by Burkholder–Davis–Gundy inequality,
\[
\mathbb{E} \left( \int_{t^n_i \wedge \tau_N}^{t \wedge \tau_N} \left( \alpha(s, \tilde{Z}(s))dW(s), \tilde{Z}(s) - \tilde{Z}(s^n_i) \right)^2 \right)^2
\leq \mathbb{E} \left( \int_{t^n_i \wedge \tau_N}^{t \wedge \tau_N} \| \alpha(s, \tilde{Z}(s)) \|^2_{L_Q} |\tilde{Z}(s) - \tilde{Z}(s^n_i)|^2 ds \right)^{k}
\leq \frac{1}{2^k} \mathbb{E} \left( \int_{t^n_i \wedge \tau_N}^{t \wedge \tau_N} \| \alpha(s, \tilde{Z}(s)) \|^4_{L_Q} ds \right)^k + \frac{1}{2^k} \mathbb{E} \left( \int_{t^n_i \wedge \tau_N}^{t \wedge \tau_N} |\tilde{Z}(s) - \tilde{Z}(s^n_i)|^4 ds \right)^k
\leq \left( \frac{K_9}{2^k} \left( 1 + 16\varepsilon^2 \tilde{C}_2(2, M) + 4\tilde{C}_3(4) + \tilde{M}_2(T)^k \right) \right) |t \wedge \tau_N - t^n_i \wedge \tau_N|^k,
\]
we have by Chebyshev’s inequality along with an application of Monotone convergence theorem on (40),
\[
P \left( I_3(t^n_i \wedge \tau_N, t \wedge \tau_N) > \frac{\beta^2}{6} \right) \leq \exp \left( -2R \log \log \frac{1}{\varepsilon} \right),
\]
for sufficiently large \( n \in \mathbb{N} \), and noting that the above estimate holds for any \( \beta > 0 \), inequality (38) is obtained. □

Next we aim to derive the required exponential bound for the second inequality given by (29).

**Lemma 6.** For any \( R > 0 \) and \( \rho > 0 \), there exists \( \eta > 0 \), \( \beta > 0 \), and \( n_0 \in \mathbb{N}\setminus\{0\} \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) and \( n \geq n_0 \),

\[
P\left( \| \tilde{Z}^\varepsilon(t) - X^h(t) \|_{E(T)} \geq \rho, \left\| \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} W \right\|_{L^\infty(T)} < \eta \right) \leq \exp \left( -2R \log \log \frac{1}{\varepsilon} \right).
\]

(41)

**Proof.** Observe that under condition,

\[
\| \tilde{Z}^\varepsilon(t) - \tilde{Z}^\varepsilon(t^n_i) \|_{E(T)} \leq \beta,
\]

\[
\rho \leq \| \tilde{Z}^\varepsilon(t) - X^h(t) \|_{E(T)} \leq \beta + \| X^h(t) - \tilde{Z}^\varepsilon(t^n_i) \|_{E(T)}.
\]

Noting that Lemma 5 holds for any \( \beta > 0 \), then for any \( \rho > 0 \), we may choose \( \beta > 0 \) such that \( \xi := \rho - \beta > 0 \) to obtain

\[
\| X^h(t) - \tilde{Z}^\varepsilon(t^n_i) \|_{E(T)} \geq \xi.
\]

For \( t \in \Delta^n_i \) we have

\[
X^h(t) - \tilde{Z}^\varepsilon(t^n_i) = X^h(0) - \int_0^{t^n_i} \left( AX^h(s) - A\tilde{Z}^\varepsilon(s) \right) ds - \int_0^{t^n_i} AX^h(s) ds
\]

\[
- \int_0^{t^n_i} B \left( X^h(s) - \tilde{Z}^\varepsilon(s), u^0(s) \right) ds - \int_0^{t^n_i} B(X^h(s), u^0(s)) ds
\]

\[
- \int_0^{t^n_i} B(u^0(s), X^h(s)) ds + \int_0^{t^n_i} B(u^\varepsilon(s), \tilde{Z}^\varepsilon(s)) ds + \int_0^{t} \sigma(s, u^0(s)) h(s) ds
\]

\[
- \int_0^{t^n_i} \tilde{\sigma}(s, \tilde{Z}^\varepsilon(s)) h(s) ds - \frac{1}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \int_0^{t^n_i} \tilde{\sigma}(s, \tilde{Z}^\varepsilon(s)) dW(s).
\]

Applying similar estimates as in the proof of Lemma 5 with bound in (37) we arrive at

\[
\mathbb{E} \sup_{0 \leq s \leq t} |X^h(s) - \tilde{Z}^\varepsilon(s^n_i)|^2 + \mathbb{E} \int_0^T \| X^h(s) - \tilde{Z}^\varepsilon(s^n_i) \|^2 ds
\]

\[
\leq |X^h(0)|^2 + c_1 t^n_i + c_2 |t - t^n_i| + c_3 t
\]

\[
+ 2 \sqrt{2 \log \log \frac{1}{\varepsilon}} \mathbb{E} \left| \int_0^{t^n_i} \left( \tilde{\sigma}(s, \tilde{Z}^\varepsilon(s)) dW, X^h(s) - \tilde{Z}^\varepsilon(s^n_i) \right) \right|,
\]
where $c_1, c_2, c_3$ are constants that depend on $T, \tilde{C}_2(1, M)/(\log\log(1/\varepsilon)), \tilde{C}_3(2)$ and $\tilde{C}_4(2)$. Thus, noting that $t \in \Delta^n_t$, and following the same reasoning as in the proof of Lemma 5 we achieve (41).

It may be observed that the above results can be generalized to achieve the moderate deviations for $\{u^\varepsilon(\cdot)\}_{\varepsilon > 0}$ by the Azencott method with any $a(\varepsilon)$ satisfying the required conditions, $a(\varepsilon) > 0$ and $a(\varepsilon)/\sqrt{\varepsilon} \to \infty$ as $\varepsilon$ tends to zero. Here we focused on the moderate deviation principle for the special case of $a(\varepsilon) = 1/\sqrt{2\log\log(1/\varepsilon)}$ to be able to achieve the LIL in the next section.

4. Strassen’s compact LIL

We begin by showing the relative compactness property of the process $\{Z^\varepsilon(\cdot)\}_{\varepsilon \in (0, \varepsilon_0)}$ in space $\mathcal{C}([0, T]; H) \cap L^2(0, T; V)$ as required by Theorem 2. To this end, the following result proved in [25] is applied, where the statement of their lemma is modified to match our setting. We make the remark that since the global in time well-posedness of solutions is known for our process, the convergence offered by the theorem holds for any time $t \in [0, T]$ and the use of stopping times is not necessary.

**Theorem 3.** (Lemma 5.1 in [25]) Let $B_1$ and $B_2$ be Banach spaces with norms, $\| \cdot \|_1$ and $\| \cdot \|_2$, respectively such that $B_2 \subset B_1$ is a continuous embedding. Suppose $\{X^\varepsilon\}_{\varepsilon > 0}$ is a family of $B_2$-valued stochastic process defined on $\mathcal{E}(T) := \mathcal{C}([0, T]; B_1) \cap L^2(0, T; B_2)$ a.s. If for some $M > 1$ and $T > 0$,

$$\lim_{\varepsilon_1 \to 0} \sup_{\varepsilon_1 \geq \varepsilon_2} \mathbb{E}\|X^{\varepsilon_1} - X^{\varepsilon_2}\|_{\mathcal{E}(T)} = 0,$$  \hspace{1cm} (42)

$$\lim_{S \to 0} \sup_{\varepsilon > 0} P\left(\|X^\varepsilon\|_{\mathcal{E}(T \wedge \bar{S})} > \|X^\varepsilon(0)\|_1 + M - 1\right) = 0,$$  \hspace{1cm} (43)

then for some subsequence, $\{X^{\varepsilon_\ell}\}_{\varepsilon_\ell > 0}$ and process $X \in \mathcal{E}(T)$, the convergence, $\|X^{\varepsilon_\ell} - X\|_{\mathcal{E}(T)} \to 0$ holds a.s. as $\varepsilon_\ell \to 0$.

We proceed by verifying conditions (42) and (43) for our model. Observing that $Z^\varepsilon(t)$ has the same terms as $\tilde{Z}^\varepsilon(t)$ with an addition of the term, $\int_0^t \tilde{\sigma}(s, \tilde{Z}^\varepsilon(s))h(s)ds$, we deduce estimates (30) and (35) for $Z^\varepsilon(t)$ and for simplicity keep the same notation for the bounds.

Let $V^\varepsilon(t) = Z^{\varepsilon_1}(t) - Z^{\varepsilon_2}(t)$, then applying the Itô formula, taking the supremum over time up to $t \wedge \tau_M$, where $\tau_M := \inf\{t : \|V^\varepsilon(t)\|_{\mathcal{E}(T)}^2 \geq M\}$ for some $M > 0$, and afterwards taking the expectation gives

$$\mathbb{E}\sup_{0 \leq s \leq t \wedge \tau_M} |V^\varepsilon(s)|^2 + 2 \int_0^{t \wedge \tau_M} \mathbb{E}\|V^\varepsilon(s)\|^2ds \leq 2\mathbb{E}\int_0^{t \wedge \tau_M} \left(-\left(\langle B\left(Z^{\varepsilon_1}(s), u^{\varepsilon_1}(s)\right), V^\varepsilon(s)\rangle + \langle B\left(Z^{\varepsilon_2}(s), u^{\varepsilon_2}(s)\right), V^\varepsilon(s)\rangle\right)dsight.$$
\[
+ 2E \sup_{0 \leq t \leq T \wedge \tau_M} \int_0^s \left( \frac{\tilde{\sigma}(\ell, Z^{\varepsilon_1}(\ell))}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} - \frac{\tilde{\sigma}(\ell, Z^{\varepsilon_2}(\ell))}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \right) dW(\ell), V^{\varepsilon}(s) \right) \right. \\
+ \left. \mathbb{E} \int_0^{T \wedge \tau_M} \left( \frac{1}{2 \log \log \frac{1}{\varepsilon_1}} \|\tilde{\sigma}(s, Z^{\varepsilon_1}(s))\|^2_{L_Q} + \frac{1}{2 \log \log \frac{1}{\varepsilon_2}} \|\tilde{\sigma}(s, Z^{\varepsilon_2}(s))\|^2_{L_Q} \right) ds, \right]
\]
where for each term on the right hand side, estimates may be made along the same lines as those in the proof of Lemma 3 to obtain
\[
\lim_{\varepsilon_1 \to 0, \varepsilon_1 \geq \varepsilon_2} \mathbb{E} \left( \sup_{s \in [0, T \wedge \tau_M]} |Z^{\varepsilon_1}(s) - Z^{\varepsilon_2}(s)|^2 + \mathbb{E} \int_0^{T \wedge \tau_N} \|Z^{\varepsilon_1}(s) - Z^{\varepsilon_2}(s)\|^2 ds \right) = 0,
\]
which is equivalent to (42) after setting \(M\) tend to infinity. As for (43), applying Itô’s formula then taking the supremum over \(t \in [0, T \wedge S]\) yields
\[
\sup_{t \in [0, T \wedge S]} |Z^{\varepsilon}(t)|^2 + \int_0^{T \wedge S} \|Z^{\varepsilon}(s)\|^2 ds \leq 2c \int_0^{T \wedge S} |Z^{\varepsilon}(s)|^2 \|u^{\varepsilon}(s)\|^4_{L_4} ds \\
+ \frac{2}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \sup_{s \in [0, T \wedge S]} \int_0^s (\tilde{\sigma}(\ell, Z^{\varepsilon}(\ell)) dW(\ell), Z^{\varepsilon}(\ell)) \\
+ \frac{1}{2 \log \log \frac{1}{\varepsilon}} \int_0^{T \wedge S} \|\tilde{\sigma}(s, Z^{\varepsilon}(s))\|^2_{L_Q} ds,
\]
where inequality (5) was applied. Hence, we obtain
\[
P \left( \|Z^{\varepsilon}(t)\|^2_{E(T \wedge S)} > M - 1 \right) \\
\leq P \left( 2c \int_0^{T \wedge S} |Z^{\varepsilon}(s)|^2 \|u^{\varepsilon}(s)\|^4_{L_4} ds > \frac{(M - 1)}{3} \right) \\
+ P \left( \frac{2}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \sup_{s \in [0, T \wedge S]} \int_0^s (\tilde{\sigma}(\ell, Z^{\varepsilon}(\ell)) dW(\ell), Z^{\varepsilon}(\ell)) > \frac{(M - 1)}{3} \right) \\
+ P \left( \frac{1}{2 \log \log \frac{1}{\varepsilon}} \int_0^{T \wedge S} \|\tilde{\sigma}(s, Z^{\varepsilon}(s))\|^2_{L_Q} ds > \frac{(M - 1)}{3} \right). \]
We may apply Doob’s and Chebyshev inequalities for the second and remaining two probabilities, respectively, to arrive at
\[
P \left( \|Z^{\varepsilon}(t)\|^2_{E(T \wedge S)} > M - 1 \right) \leq \frac{6c}{(M - 1)} \mathbb{E} \left( \int_0^{T \wedge S} |Z^{\varepsilon}(s)|^2 \|u^{\varepsilon}(s)\|^4_{L_4} ds \right) \\
+ \frac{k(M, K_9)}{\log \log \frac{1}{\varepsilon}} \mathbb{E} \int_0^{T \wedge S} \left( 1 + 4\varepsilon \log \log \frac{1}{\varepsilon} |Z^{\varepsilon}(s)|^2 + 2\|u^{\varepsilon}(s)\|^2 \right) |Z^{\varepsilon}(s)|^2 ds.
\]
\[
\begin{align*}
&+ \frac{3K_9}{2(M-1) \log \log \frac{1}{\varepsilon}} \mathbb{E} \int_0^{T \land S} \left( 1 + 4\varepsilon \log \log \frac{1}{\varepsilon} \|Z^\varepsilon(s)\|^2 + 2\|u_0^\varepsilon(s)\|^2 \right) ds \\
\leq & (T \land S) \left( \frac{6c \tilde{M}(T)K_6 \tilde{C}_3(2)}{M-1} \right) \\
&+ (T \land S) k(M, K_9) \left( \sqrt{M_2(T)(1+4\tilde{C}_3(4))} \frac{\log \log \frac{1}{\varepsilon}}{\log \log \frac{1}{\varepsilon}} + \sqrt{8\varepsilon^2 \tilde{M}(T)\tilde{C}_2(2, M)} \right) \\
&+ (T \land S) \left( 1 + 2\varepsilon \tilde{C}_2(1, M) + 2\tilde{C}_3(2) \right) \frac{3K_9}{2(M-1) \log \log \frac{1}{\varepsilon}}.
\end{align*}
\]

Thus, by taking the supremum on \( \varepsilon \in (0, \varepsilon_0) \) and afterwards letting \( S \) tend to zero we achieve condition (43) in our setting.

Next to verify that the set \( L \) given in Theorem 2 is the limit set, we prove by the following lemma that each element of set \( L \) is a limit point of \( \{ Z^\varepsilon(\cdot) \}_{\varepsilon \in (0, \varepsilon_0)} \). For this lemma, we let \( c > 1 \) and consider the process depending on \( \frac{1}{c^j} \) for \( j \geq 1 \), instead of \( \varepsilon > 0 \) for better presentation.

**Lemma 7.** For any \( c > 1 \) and \( g(t) \in L \), there exists \( j_0 > \frac{1}{\log c} \log \frac{1}{\varepsilon_0} \), such that

\[
P \left( \left\| \frac{1}{c^j} Z^\varepsilon(t) - g(t) \right\|_{\tilde{E}(T)}^2 \leq \varepsilon \right) = 1,
\]

for all \( j \geq j_0 \) and \( \varepsilon > 0 \).

**Proof.** For a constant \( \eta > 0 \), let

\[
F_j := \left\{ \left\| \frac{1}{c^j} Z^\varepsilon(t) - g(t) \right\|_{\tilde{E}(T)}^2 \leq \varepsilon \right\}, \quad G_j := \left\{ \left\| \frac{1}{\sqrt{2 \log \log c^j}} W - h \right\|_{L^\infty_H} \leq \eta \right\},
\]

where \( g(t) \) is any element in the set \( L \) and \( h \in \mathcal{H}_0 \) such that \( g(t) = X^h(t) \) and

\[
\frac{1}{2} \int_0^t \left( h(s) \right)^2 ds \leq \frac{N}{2}.
\]

Since the Strassen’s compact LIL is known for Brownian paths (see [42]), we have

\[
P \left( \limsup_{j \to \infty} \left\| \frac{1}{\sqrt{2 \log \log c^j}} W - h \right\|_{L^\infty_H} > \eta \right) = 0, \quad (44)
\]

Also note that by the Freidlin–Wentzell inequality (26) we have for any \( R > 1 \),

\[
P(F_j^c \cap G_j) \leq \exp(-2R \log \log c^j) \leq \frac{C}{j^{2R}},
\]

which by the Borel–Cantelli lemma yields

\[
P \left( \limsup_{j \to \infty} F_j^c \cap G_j \right) = 0. \quad (45)
\]
Now by (44),
\[ 1 = P \left( \limsup_{j \to \infty} G_j \right) \leq P \left( \limsup_{j \to \infty} G_j \cap F_j \right) + P \left( \limsup_{j \to \infty} G_j \cap F_j^c \right) \]
\[ \leq P \left( \limsup_{j \to \infty} F_j \right), \]
giving \( P(\limsup_{j \to \infty} F_j) = 1 \) to complete the proof. \( \Box \)

Inspired by [4, 10, 20] to ensure that set \( L \) is the only limit set of \( \{ Z^\varepsilon(\cdot) \}_{\varepsilon \in (0, \varepsilon_0)} \), we let \( \overline{L} := \{ g : \| g(t) - L \|_{\mathcal{E}(T)}^2 \geq \varepsilon \} \), which implies by the definition of set \( L \) that \( I(g) > N/2 + \delta \) for some \( \delta > 0 \). By the moderate deviations result of the previous section, we have for the closed set, \( \overline{L} \),
\[ \limsup_{j \to \infty} \frac{1}{2 \log \log c^j} \log P \left( Z^{\frac{1}{c^j}} \in \overline{L} \right) \leq -I(g) < -\left( \frac{N}{2} + \delta \right), \]
which leads to
\[ P(\frac{1}{c^j} \in \overline{L}) \leq \exp \left( -2 \left( \frac{N}{2} + \delta \right) \log \log c^j \right) \leq \frac{k}{j^{N+2\delta}}, \]
giving
\[ P \left( \limsup_{j \to \infty} \left\| Z^{\frac{1}{c^j}}(t) - L \right\|_{\mathcal{E}(T)}^2 \geq \varepsilon \right) = 0, \]
by Borel–Cantelli lemma. Thus, we achieve the limit of \( \| Z^{\frac{1}{c^j}}(t) - L \|_{\mathcal{E}(T)} \) to zero a.s.

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Appendix

Proof of Lemma 3. Letting $\tau_N \coloneqq \inf \{ t > 0 : \sup_{0 \leq t \leq T} |\tilde{Z}^\varepsilon(t)|^2 + \int_0^t \|\tilde{Z}^\varepsilon(s)\|^2 ds > N \}$ we apply the Itô’s formula, then take the supremum over time up to $t \wedge \tau_N$ and afterwards expectation to obtain

$$
\mathbb{E} \sup_{0 \leq s \leq t \wedge \tau_N} |\tilde{Z}^\varepsilon(s)|^2 + 2\mathbb{E} \int_0^{t \wedge \tau_N} \|\tilde{Z}^\varepsilon(s)\|^2 ds
= -2\mathbb{E} \int_0^{t \wedge \tau_N} \left( B \left( \tilde{Z}^\varepsilon(s), u^0(s) \right), \tilde{Z}^\varepsilon(s) \right) ds
+ 2\mathbb{E} \int_0^{t \wedge \tau_N} (\tilde{\sigma}(s, \tilde{Z}^\varepsilon(s)) h(s), \tilde{Z}^\varepsilon(s)) ds
+ \frac{2}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau_N} \int_0^s (\tilde{\sigma}(\ell, \tilde{Z}^\varepsilon(\ell)) dW(\ell), \tilde{Z}^\varepsilon(\ell))
+ \frac{1}{2 \log \log \frac{1}{\varepsilon}} \mathbb{E} \int_0^{t \wedge \tau_N} \|\tilde{\sigma}(s, \tilde{Z}^\varepsilon(s))\|_{L^2} ds
= I_1(t \wedge \tau_N) + I_2(t \wedge \tau_N) + \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau_N} I_3(s) + I_4(t \wedge \tau_N).
$$

We proceed to estimate the above terms. Using inequality (5) we have

$$
I_1(t \wedge \tau_N) = 2\mathbb{E} \int_0^{t \wedge \tau_N} b(\tilde{Z}^\varepsilon(s), \tilde{Z}^\varepsilon(s), u^0(s)) ds
\leq \int_0^{t \wedge \tau_N} \mathbb{E}\|\tilde{Z}^\varepsilon(s)\|^2 ds + 2c \int_0^{t \wedge \tau_N} \mathbb{E} \sup_{0 \leq \ell \leq s} |\tilde{Z}^\varepsilon(\ell)|^2 \|u^0(s)\|_{L^4}^4 ds.
$$

(46)

By (35) along with Cauchy–Schwarz and Young’s inequalities we obtain

$$
I_2(t \wedge \tau_N) \leq \frac{1}{2} \mathbb{E} \int_0^{t \wedge \tau_N} \|\tilde{\sigma}(s, \tilde{Z}^\varepsilon(s))\|^2_{L^2} ds + 2\mathbb{E} \int_0^{t \wedge \tau_N} |h(s)|_0^2 \|\tilde{Z}^\varepsilon(s)\|^2 ds
\leq \frac{K_9}{2} (T + 2K_6) + 2\varepsilon K_9 \left( \log \log \frac{1}{\varepsilon} \right) \mathbb{E} \int_0^{t \wedge \tau_N} \|\tilde{Z}^\varepsilon(s)\|^2 ds
+ 2\mathbb{E} \int_0^{t \wedge \tau_N} \sup_{0 \leq \ell \leq s} |\tilde{Z}^\varepsilon(\ell)|^2 |h(s)|_0^2 ds
\leq \frac{K_9}{2} (T + 2K_6) + \varepsilon K_9 \tilde{C}_2(1, M) T + 2\mathbb{E} \int_0^{t \wedge \tau_N} \sup_{0 \leq \ell \leq s} |\tilde{Z}^\varepsilon(\ell)|^2 |h(s)|_0^2 ds,
$$

where inequality (20) was also applied. Thanks to the Burkholder–Davis–Gundy inequality and inequality (12) we obtain
\[\mathbb{E} \sup_{0 \leq s \leq t \wedge \tau_N} I_3(s) \leq \frac{6}{\sqrt{2 \log \log \frac{1}{\epsilon}}} \mathbb{E} \left( \int_0^{t \wedge \tau_N} \| \tilde{\alpha}(s, \tilde{Z}^\epsilon(s)) \|_{L_2}^2 \| \tilde{Z}^\epsilon(s) \|^2 ds \right)^{1/2} \]

\[\leq \frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau_N} |\tilde{Z}^\epsilon(s)|^2 + \frac{9K_9}{\log \log \frac{1}{\epsilon}} \int_0^{t \wedge \tau_N} \left( 1 + 4\epsilon \log \log \frac{1}{\epsilon} \mathbb{E} \| \tilde{Z}^\epsilon(s) \|^2 + 2\|u^0(s)\|^2 \right) ds \]

\[\leq \frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau_N} |\tilde{Z}^\epsilon(s)|^2 + \frac{9K_9}{\log \log \frac{1}{\epsilon}} (T + 2K_6) + \frac{36\epsilon K_9 T}{\log \log \frac{1}{\epsilon}} \tilde{C}_2(1, M),\]

In addition, since \(\epsilon < 1\),

\[I_4(t \wedge \tau_N) \leq \frac{K_9}{2 \log \log \frac{1}{\epsilon}} (T + 2K_6) + \frac{\epsilon K_9 T}{\log \log \frac{1}{\epsilon}} \tilde{C}_2(1, M) \leq \frac{K(T, K_6, K_9, M)}{\log \log \frac{1}{\epsilon}}.\]

Hence, we arrive at

\[\frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau_N} |\tilde{Z}^\epsilon(s)|^2 + \int_0^{t \wedge \tau_N} \mathbb{E} \| \tilde{Z}^\epsilon(s) \|^2 ds \]

\[\leq K(T, \epsilon) + \int_0^{t \wedge \tau_N} \mathbb{E} \sup_{0 \leq t \leq s} |\tilde{Z}^\epsilon(t)|^2 \left( 2c \|u^0(s)\|_{L_4}^4 + 2|h(s)|_0^2 \right) ds.\]

Furthermore, observe that \(\epsilon < 1\) implies \(1/(\log \log(1/\epsilon)) < 1\) giving

\[K(T, \epsilon) = \frac{K_1(T)}{\log \log \frac{1}{\epsilon}} + \frac{K_9}{2} (T + 2K_6) + \epsilon K_9 \tilde{C}_2(1, M) T \leq M_1(T),\]

where \(M_1(T)\) does not depend on \(\epsilon\). Now an application of Gronwall’s inequality yields

\[\mathbb{E} \sup_{0 \leq s \leq t \wedge \tau_N} |\tilde{Z}^\epsilon(s)|^2 + \int_0^{t \wedge \tau_N} \mathbb{E} \| \tilde{Z}^\epsilon(s) \|^2 ds \]

\[\leq M_1(T) \exp \left( \int_0^{t \wedge \tau_N} \left( 2c \|u^0(s)\|_{L_4}^4 + 2|h(s)|_0^2 \right) ds \right) \leq \tilde{M}_1(T),\]

in which we have noted that \(h \in \mathcal{H}_0\) and used inequality (13). Now we let \(N\) tend to infinity.

For \(p > 1\), we let \(\tilde{\tau}_N := \inf \{ t > 0 : \sup_{0 \leq s \leq t} |\tilde{Z}^\epsilon(s)|^{2p} + 2p \int_0^t |\tilde{Z}^\epsilon(s)|^{2(p-1)} ds > N \}\), then we apply the Itô’s formula first to \(|\tilde{Z}^\epsilon(s)|^2\) and then to the map \(x \mapsto x^p\) as follows:

\[d|\tilde{Z}^\epsilon(t)|^{2p} = p \left(|\tilde{Z}^\epsilon(t)|^2\right)^{p-1} \frac{d|\tilde{Z}^\epsilon(t)|^2}{2} \frac{d|\tilde{Z}^\epsilon(t)|^2}{2} + \frac{1}{2} p(p-1)|\tilde{Z}^\epsilon(t)|^{2(p-2)} d(|\tilde{Z}^\epsilon(t)|^2).\]

More precisely,
\[
|\hat{Z}^\varepsilon(t \wedge \tau_N)|^2p + 2p \int_0^{t \wedge \tau_N} |\hat{Z}^\varepsilon(s)|^{2(p-1)}|\hat{Z}^\varepsilon(s)|^2ds
\]
\[
= -2p \int_0^{t \wedge \tau_N} |\hat{Z}^\varepsilon(s)|^{2(p-1)} \left(B(\hat{Z}^\varepsilon(s), u^0(s)), \tilde{Z}^\varepsilon(s)\right) ds
\]
\[
+ \frac{2p}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \int_0^{t \wedge \tau_N} |\tilde{Z}^\varepsilon(s)|^{2(p-1)} \left(\tilde{\sigma}(s, \tilde{Z}^\varepsilon(s))dW(s), \tilde{Z}^\varepsilon(s)\right) ds
\]
\[
+ 2p \int_0^{t \wedge \tau_N} |\tilde{Z}^\varepsilon(s)|^{2(p-1)} (\tilde{\sigma}(s, \tilde{Z}^\varepsilon(s)))h(s), \tilde{Z}^\varepsilon(s)\) ds
\]
\[
+ \frac{p}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \left|\tilde{Z}^\varepsilon(s)\right|^2 \left(\tilde{\sigma}(s, \tilde{Z}^\varepsilon(s)), \tilde{Z}^\varepsilon(s)\right)\frac{2}{|\tilde{Z}^\varepsilon(s)|^{2(p-2)}} ds
\]
\[
= I_1(t \wedge \tau_N) + I_2(t \wedge \tau_N) + I_3(t \wedge \tau_N) + I_4(t \wedge \tau_N) + I_5(t \wedge \tau_N).
\]

After taking the supremum on time up to \(t \wedge \tau_N\), and then expectation, we estimate each term as follows:

\[
\mathbb{E}I_1(t \wedge \tau_N) = 2p \mathbb{E} \int_0^{t \wedge \tau_N} |\tilde{Z}^\varepsilon(s)|^{2(p-1)} b(\tilde{Z}^\varepsilon(s), \tilde{Z}^\varepsilon(s), u^0(s)) ds
\]
\[
\leq p \mathbb{E} \left|\tilde{Z}^\varepsilon(s)\right|^{2(p-1)} \left|\tilde{Z}^\varepsilon(s)\right|^2 ds
\]
\[
+ 2p \mathbb{E} \left|\sup_{0 \leq \ell \leq s} \left|\tilde{Z}^\varepsilon(s)\right|^{2p} \left|u^0(s)\right|^4 \right| ds.
\]

Note that by Young’s inequality,

\[
\mathbb{E} \int_0^{t \wedge \tau_N} |\tilde{Z}^\varepsilon(s)|^{2(p-1)} \left|\tilde{\sigma}(s, \tilde{Z}^\varepsilon(s))\right|^2_{L^q} ds
\]
\[
\leq \frac{p-1}{p} \mathbb{E} \int_0^{t \wedge \tau_N} |\tilde{Z}^\varepsilon(s)|^2 ds + \frac{1}{p} \mathbb{E} \int_0^{t \wedge \tau_N} \left|\tilde{\sigma}(s, \tilde{Z}^\varepsilon(s))\right|^2_{L^q} ds
\]
\[
\leq \frac{p-1}{p} \mathbb{E} \int_0^{t \wedge \tau_N} |\tilde{Z}^\varepsilon(s)|^2 ds
\]
\[
+ \frac{K_0}{p} \mathbb{E} \int_0^{t \wedge \tau_N} \left(1 + 4\varepsilon \log \log \frac{1}{\varepsilon} \left|\tilde{Z}^\varepsilon(s)\right|^2 + 2 \left|u^0(s)\right|^2\right)^p ds
\]
\[
\leq k(p) \mathbb{E} \int_0^{t \wedge \tau_N} |\tilde{Z}^\varepsilon(s)|^2 ds + k(p) T (1 + (2\varepsilon)^p \tilde{C}_2(p, M) + 2p \tilde{C}_3(2p)),
\]
\[(47)\]

Now with the help of Burkholder–Davis–Gundy inequality and estimate (47) above we obtain
Moreover, applying the Cauchy–Schwarz and Young’s inequalities lead to

\[
\mathbb{E} \sup_{0 \leq s \leq t \wedge \tilde{\tau}_N} I_2(s) \leq \frac{6p}{\sqrt{2 \log \log \frac{1}{\varepsilon}}} \mathbb{E} \left( \int_0^{t \wedge \tilde{\tau}_N} \left| \tilde{Z}^\varepsilon(s) \right|^4 (p-1) \left\| \tilde{\sigma}(s, \tilde{Z}^\varepsilon(s)) \right\|_{L^Q}^2 \left| \tilde{Z}^\varepsilon(s) \right|^2 ds \right)^{1/2}
\]

\[
\leq \frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t \wedge \tilde{\tau}_N} \left| \tilde{Z}^\varepsilon(s) \right|^{2p} + \frac{9p^2}{\log \log \frac{1}{\varepsilon}} \mathbb{E} \int_0^{t \wedge \tilde{\tau}_N} \left| \tilde{Z}^\varepsilon(s) \right|^{2(p-1)} \left\| \tilde{\sigma}(s, \tilde{Z}^\varepsilon(s)) \right\|_{L^Q}^2 ds
\]

\[
\leq \frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t \wedge \tilde{\tau}_N} \left| \tilde{Z}^\varepsilon(s) \right|^{2p} + \frac{9k(p)}{\log \log \frac{1}{\varepsilon}} \mathbb{E} \int_0^{t \wedge \tilde{\tau}_N} \left| \tilde{Z}^\varepsilon(s) \right|^{2p} ds
\]

\[
+ \frac{9k(p)T}{\log \log \frac{1}{\varepsilon}} \left( (2\varepsilon)^p \tilde{C}_2(p, M) + 2^p \tilde{C}_3(2p) \right).
\]

The same reasoning implies

\[
\mathbb{E} I_3(t \wedge \tilde{\tau}_N) \leq 2p \mathbb{E} \int_0^{t \wedge \tilde{\tau}_N} \left| \tilde{Z}^\varepsilon(s) \right|^{2(p-1)} \left\| \tilde{\sigma}(s, \tilde{Z}^\varepsilon(s)) \right\|_{L^Q} \left| h(s) \right|_0 \left| \tilde{Z}^\varepsilon(s) \right| ds
\]

\[
\leq 2p^2 \mathbb{E} \int_0^{t \wedge \tilde{\tau}_N} \left| \tilde{Z}^\varepsilon(s) \right|^{2(p-1)} \left\| \tilde{\sigma}(s, \tilde{Z}^\varepsilon(s)) \right\|_{L^Q}^2 ds
\]

\[
+ \frac{1}{2} \mathbb{E} \int_0^{t \wedge \tilde{\tau}_N} \left| h(s) \right|_0^2 \left| \tilde{Z}^\varepsilon(s) \right|^{2p} ds
\]

\[
\leq 2k(p) \mathbb{E} \int_0^{t \wedge \tilde{\tau}_N} \left| \tilde{Z}^\varepsilon(s) \right|^{2p} ds + 2k(p)T (1 + (2\varepsilon)^p \tilde{C}_2(p, M) + 2^p \tilde{C}_3(2p))
\]

\[
+ \frac{1}{2} \mathbb{E} \int_0^{t \wedge \tilde{\tau}_N} \left| h(s) \right|_0^2 \left| \tilde{Z}^\varepsilon(s) \right|^{2p} ds.
\]

The same reasoning implies

\[
\mathbb{E} I_4(t \wedge \tilde{\tau}_N) + \mathbb{E} I_5(t \wedge \tilde{\tau}_N) \leq \frac{k(p)}{\log \log \frac{1}{\varepsilon}} \mathbb{E} \int_0^{t \wedge \tilde{\tau}_N} \left| \tilde{Z}^\varepsilon(s) \right|^{2p} ds
\]

\[
+ \frac{Tk(p)}{\log \log \frac{1}{\varepsilon}} \left( (2\varepsilon)^p \tilde{C}_2(p, M) + 2^p \tilde{C}_3(2p) \right),
\]

where we have used

\[
\mathbb{E} I_5(t \wedge \tilde{\tau}_N) \leq \frac{2p(p-1)}{\log \log \frac{1}{\varepsilon}} \mathbb{E} \int_0^{t \wedge \tilde{\tau}_N} \left\| \tilde{\sigma}(s, \tilde{Z}^\varepsilon(s)) \right\|_{L^Q}^2 \left| \tilde{Z}^\varepsilon(s) \right|^{2(p-1)} ds.
\]

Thus, we obtain
\[\frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t \wedge \tilde{\tau}} |\tilde{Z}^\varepsilon(s)|^{2p} + p \mathbb{E} \int_0^{t \wedge \tilde{\tau}} |\tilde{Z}^\varepsilon(s)|^{2(p-1)} \|	ilde{Z}^\varepsilon(s)\|^2 ds \leq TK(p, M) \left(1 + \varepsilon^p + \frac{1 + \varepsilon^p}{\log \log \frac{1}{\varepsilon}}\right) \]

\[+ \int_0^{t \wedge \tilde{\tau}} \mathbb{E} \sup_{0 \leq \ell \leq s} |\tilde{Z}^\varepsilon(\ell)|^{2p} \left(3k(p) + \frac{10k(p)}{\log \log \frac{1}{\varepsilon}} + \frac{1}{2} |h(s)|_0^2 + 2pc \|u_0(s)\|_{L^4}^4\right) ds.\]

Noting,

\[TK(p, M) \leq T k(p) := M_p(T),\]

we have by Gronwall’s inequality

\[\mathbb{E} \sup_{0 \leq s \leq t \wedge \tilde{\tau}} |\tilde{Z}^\varepsilon(s)|^{2p} + p \mathbb{E} \int_0^{t \wedge \tilde{\tau}} |\tilde{Z}^\varepsilon(s)|^{2(p-1)} \|	ilde{Z}^\varepsilon(s)\|^2 ds \leq M_p(T) \exp \left(3k(p) + \frac{10k(p)}{\log \log \frac{1}{\varepsilon}}\right) T + \int_0^T \frac{1}{2} |h(s)|_0^2 + 2pc \|u_0(s)\|_{L^4}^4 ds\]

\[=: \tilde{M}_p(T),\]

where \(\tilde{M}_p(T)\) is a positive constant using the assumption that \(h \in \mathcal{H}_0\) and applying inequality (13). Now letting \(N\) to go to infinity we obtain the result.

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