UNIPOTENT BLOCKS AND WEIGHTED AFFINE WEA YL GROUPS

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INTRODUCTION

0.1. Let $G$ be a connected reductive algebraic group over $\mathbb{C}$. Let $un(G)$ be the set of unipotent conjugacy classes in $G$. Let $ls(G)$ be the set of all pairs $(c, \mathfrak{L})$ where $c \in un(G)$ and $\mathfrak{L}$ is an irreducible $\mathbb{C}$-local system on $c$, equivariant for the conjugation action of $G$. In [S76], Springer discovered a remarkable bijection between a certain subset $'ls(G)$ of $ls(G)$ and the set of irreducible representations of the Weyl group of $G$ (up to isomorphism). In [L84] I extended Springer result by defining the “generalized Springer correspondence” that is, a partition of $ls(G)$ into subsets (which could be called unipotent blocks) and a bijection, for each unipotent block $\beta$, between the set of objects in $\beta$ and the set $\text{Irr}(\mathcal{W}_\beta)$ of irreducible representations (up to isomorphism) of a certain Weyl group $\mathcal{W}_\beta$ associated to $\beta$. (The subset $'ls(G)$ is one of the unipotent blocks, namely the one containing $(\{1\}, \mathbb{C})$.) The arguments of [L84] were based on the use of perverse sheaves on $G$.

0.2. In the remainder of this introduction we assume that $G$ is almost simple, simply connected. Let $W$ be the affine Weyl group of type dual to that of $G$. In this paper we try to show that various information on unipotent elements of $G$ can be recovered from things which are more primitive than unipotent elements, namely from Weyl groups, their associated Hecke algebras and their representations. We want to recover such information from $W$ and its subgroups, without use of the geometry of $G$. Results of this type have been obtained in [L20] for the unipotent block $'ls(G)$ and this paper can be viewed as an attempt to extend [L20] to arbitrary unipotent blocks.

We now describe some earlier results in the same direction.

In [L79] it has been observed that there is a set defined purely in terms of $W$ which is an indexing set for $un(G)$, via the ordinary Springer correspondence. More precisely, this set describes $un(G)$ in terms of truncated induction from special representations of the various finite standard parabolic subgroups of $W$, see 6.3. The same process allows one to recover the order of the group of components of

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the centralizer of a unipotent element in the adjoint group of $G$ in terms of data attached to the various special representations in the previous sentence, see [L09]. In [L80] it was conjectured (and in [L89] proved) that the set of two-sided cells of $W$ is an indexing set for $\text{un}(G)$. These results suggest that it may be possible to recover other properties of unipotent elements of $G$ in terms of $W$.

0.3. Let $\Omega_W$ be the group of automorphisms of $W$ as a Coxeter group which induce an inner automorphism of $W$ modulo the subgroup of translations. In this paper, for any $\omega \in \Omega_W$, we define (without reference to $G$) a certain set $\mathcal{C}_\omega(W)$ of standard finite parabolic subgroups $W_J$ of $W$, stable under $\omega$, see §3. Moreover, we define a bijection between the set of unipotent blocks of $G$ and the set $\cup_{\omega \in \Omega_W} \mathcal{C}_\omega(W)$, see 6.5. (This bijection is closely related to the arithmetic/geometric correspondence [L95] in the study of unipotent representations of a simple $p$-adic group.)

One of the properties that we impose on $W_J \in \mathcal{C}_\omega(W)$ implies that a finite reductive group over $\mathbf{F}_q$ with Weyl group $W_J$ and with Frobenius acting on this Weyl group as $\omega$ should have a unipotent cuspidal representation (but we formulate this property without reference to finite reductive groups, see §2). This has the consequence that the subgroups $W_J$ which appear are rather few. In §5 we attach to each $W_J \in \mathcal{C}_\omega(W)$ an affine Weyl group $\mathcal{W}_J$ with a weight function $L_J : \mathcal{W}_J \to \mathbf{N}$. (This follows a procedure from [L90], [L95].) In this way to any unipotent block we have attached a weighted affine Weyl group. (We regard the group $\{1\}$ as a weighted affine Weyl group.) The weighted affine Weyl group $(\mathcal{W}_J, L_J)$ is different in general from the one defined in [L17]. In the remainder of this introduction we fix a unipotent block $\beta$ and we denote by $\omega, W_J, W_J, L_J, L_J$ the objects associated to $\beta$ as above.

In Theorem 6.10 we show that the quotient $\bar{W}_J$ of $\mathcal{W}_J$ by its group of translations (which is defined purely in terms of $W$) can be identified with the group $\mathcal{W}_\beta$ of 0.1. In §4 we define a function $c : \text{Irr}(\bar{W}_J) \to \mathbf{N}$ purely in terms of $W$. The values of $c$ are conjecturally related to dimensions of certain Springer fibres (see 6.11); this relation is unconditional for exceptional types. The function $c$ is used in 6.12(a) to express conjecturally the generalized Green functions [L86, 24.8] purely in terms of $W$; again this is unconditional for exceptional types. In 7.3 we state a conjecture which provides an indexing set for the set of two-sided cells in $\mathcal{W}_J$ relative to $L_J$; this is unconditional for exceptional types.

0.4. Notation. For a group $L$ we denote by $L_{\text{der}}$ the derived subgroup of $L$. For a finite group $\Gamma$ let $\text{Irr}(\Gamma)$ be the set of irreducible representations of $\Gamma$ over $\mathbf{C}$ (up to isomorphism).

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1. Weighted Weyl groups

1.1. Let $\mathcal{W}$ be a Coxeter group and let $\mathcal{S}$ be its set of simple reflections. Let $w \mapsto |w|$ be the length function on $\mathcal{W}$. We say that $\mathcal{W}$ is weighted if we are given a weight function $L : \mathcal{W} \to \mathbb{N}$ that is a function such that $L(ww') = L(w) + L(w')$ for any $w, w'$ in $\mathcal{W}$ such that $|ww'| = |w| + |w'|$. Let $Cell(\mathcal{W}, L)$ be the set of two sided cells of $(\mathcal{W}, L)$ in the sense of [L03].

Let $v \in \mathbb{C} - \{0\}$ be a non-root of 1 and let $\mathcal{H}_v$ be the Hecke algebra (over $\mathbb{C}$) associated to $\mathcal{W}, L, v$; thus $\mathcal{H}_v$ is the $\mathbb{C}$-vector space with basis $\{T_w \mid w \in \mathcal{W}\}$ with associative multiplication defined by the rules $T_w T_{w'} = T_{ww'}$ for any $w, w'$ in $\mathcal{W}$ such that $|ww'| = |w| + |w'|$ and $(T_\sigma + v^{-L(\sigma)})(T_\sigma - v^{L(\sigma)}) = 0$ for any $\sigma \in \mathcal{S}$.

1.2. In the remainder of this section we assume that $\mathcal{W}$ is a Weyl group with a given weight function $L : \mathcal{W} \to \mathbb{N}$. Let $\text{Irr}(\mathcal{H}_v)$ be the set of simple $\mathcal{H}_v$-modules (up to isomorphism). It is known that $\text{Irr}(\mathcal{W}), \text{Irr}(\mathcal{H}_v)$ are in canonical bijection. (We can assume that $\mathcal{W}$ is irreducible. If properties P1-P15 in [L03, §14] are assumed, then according to [L03], the algebras $\mathcal{H}_v, \mathbb{C}[\mathcal{W}]$ are canonically isomorphic and the result would follow. Now P1-P15 do hold when $\mathcal{W}$ is of type $E_7, E_8$ or $G_2$. Thus we can assume that $\mathcal{W}$ is of type other than $E_7, E_8$ or $G_2$. In this case, the result follows from the observation in [BC] according to which the various $E \in \text{Irr}(\mathcal{W})$ are characterized by their multiplicities in representations of $\mathcal{W}$ induced from the unit representation of various parabolic subgroups. Alternatively we can use [G11].)

We denote this bijection by $E \leftrightarrow E_v$. For $E \in \text{Irr}(\mathcal{W})$ we set

$$f_{E,L,v} = (\dim E)^{-1} \sum_{w \in \mathcal{W}} \text{tr}(T_w,E_v)\text{tr}(T_{w^{-1}},E_v) \in \mathbb{C}^*$$

and

$$D_{E,L,v} = f_{E,L,v}^{-1} \sum_{w \in \mathcal{W}} v^{2L(w)} \in \mathbb{C}.$$ 

From the known explicit formulas for $D_{E,L,v}$ we see that $D_{E,L,v}$ is a nonzero rational function in $v$. We define $a_{E,L}(E) \in \mathbb{N}$ by the requirement that $D_{E,L,v} v^{-2a_{E,L}(E)}$ is a rational function in $v$ whose value at $v = 0$ is $\neq 0$ and $\neq \infty$. In the case where $L = ||$ we write $a(E)$ instead of $a_{E,L}(E)$.

1.3. If $\mathcal{W}$ is of type $A_1$ and the value of $L_\mathcal{S}$ is $a \in \mathbb{Z}_{>0}$ then setting $q = v^{2a}$ we have $D_{E,L,v} = 1$ if $E = 1$ and $D_{E,L} = q$ if $E = \text{sgn}$; the corresponding values of $a_{E,L}$ are $0; a$. We now assume that $\mathcal{W}$ is of type $A_2$ and the values of $L|_\mathcal{S}$ are $a, a$ in $\mathbb{Z}_{>0}$. Setting $q = v^{2a}$ we write below the values of $D_{E,L,v}$ for various $E$ of dimension 1; 2; 1:

$$1; q^2 + q, q^3.$$ 

The corresponding values of $a_{E,L}$ are $0; a; 3a$. 


1.4. In this subsection we assume that \( \mathfrak{W} \) is of type \( B_2 \) and the values of \( \mathcal{L}|\triangledown \) are \( a, b \) in \( \mathbb{Z}_{>0} \). Setting \( q = v^{2a}, y = v^{2b} \) we write below the values of \( D_{E,\mathcal{L},v} \) for various \( E \) of dimension 1; 2; 1; 1:

\[
1; qy(q + 1)(y + 1)/(q + y); q^2(yq + 1)/(q + y); y^2(yq + 1)/(q + y); q^2y^2.
\]

The corresponding values of \( a_{E,\mathcal{L}} \) are:

\[
0; a + b - m; 2a - m; 2b - m; 2a + 2b
\]

where \( m = \min(a, b) \).

1.5. In this subsection we assume that \( \mathfrak{W} \) is of type \( G_2 \) and the values of \( \mathcal{L}|\triangledown \) are \( a, b \) in \( \mathbb{Z}_{>0} \). Setting \( q = v^{2a}, y = v^{2b}, \sqrt{qy} = v^{a+b} \) we write below the values of \( D_{E,\mathcal{L},v} \) for various \( E \) of dimension 1; 2; 1; 1:

\[
1; qy(q + 1)(y + 1)/(q + y); qy(q + 1)(y + 1)/(2q + \sqrt{qy} + y);
q^2(q^2y^2 + qy + 1)/(q^2 + qy + y^2); y^2(q^2y^2 + qy + 1)/(q^2 + qy + y^2);
q^3y^3.
\]

The corresponding values of \( a_{E,\mathcal{L}} \) are:

\[
0; a + b - m; a + b - m; 2a - 2m; 2b - 2m, 3a + 3b
\]

where \( m = \min(a, b) \).

1.6. In this subsection we assume that \( \mathfrak{W} \) is of type \( B_3 \) and the values of \( \mathcal{L}|\triangledown \) are \( a, b \) in \( \mathbb{Z}_{>0} \). Setting \( q = v^{2a}, y = v^{2b} \) we write below the values of \( D_{E,\mathcal{L},v} \) for various \( E \) of dimension 1; 3; 2; 1; 3; 3; 2; 3; 1; 1:

\[
1; qy(q^2 + q + 1)/(q + y); q^2(q + 1)(q^2y + 1)/(q + y);
q^2y^3(q + 1)(q^2y + 1)/(q + y); q^2y^3y^2(q + 1)(q^2y + 1)/(q + y);
q^6(qy + 1)(q^2y + 1)/(q^2 + y)(q + y)); q^6y^3.
\]

The corresponding values of \( a_{E,\mathcal{L}} \) are:

\[
0; a + b - m; 3a - m; 3b - m; a + 2b - m'; 3a + b - m'; 2a + 3b - m;
3a + 2b - m; 6a - m - m'; 6a + 3b
\]

where \( m = \min(a, b) \), \( m' = \min(2a, b) \).

1.7. In [L82] a definition of a partition of \( \operatorname{Irr}(\mathfrak{W}) \) into subsets called families was given. Repeating that definition but using \( a_{\mathcal{L}}(E) \) instead of \( a(E) \) for \( E \in \operatorname{Irr}(\mathfrak{W}) \) we obtain a partition of \( \operatorname{Irr}(\mathfrak{W}) \) into subsets called \( \mathcal{L} \)-families. (This definition appears in [L83, no.7].) Thus the families of [L82] are the same as the \(|-\)-families.
2. Sharp Weyl groups

2.1. Let \( \mathcal{W}, \mathcal{S}, || \) be as in 1.1. We assume that \( \mathcal{W} \) is a Weyl group. Let \( A_{\mathcal{W}} \) be the group of all automorphisms \( \gamma \) of \( \mathcal{W} \) preserving \( \mathcal{S} \) and such that whenever \( \sigma \neq \sigma' \) are in the same \( \gamma \)-orbit in \( \mathcal{S} \), the product \( \sigma \sigma' \) has order \( \geq 3 \). For \( \gamma \in A_{\mathcal{W}} \) let \( r(\gamma) \) be the number of \( \gamma \)-orbits on \( S \); let \( \text{ord}(\gamma) \) be the order of \( \gamma \). We shall also write \( \gamma \mathcal{W} \) instead of \( (\mathcal{W}, \gamma) \). (We sometimes write \( d\mathcal{W} \) instead of \( \gamma\mathcal{W} \) where \( d = \text{ord}(\gamma) \); when \( d = 1 \) we write \( \mathcal{W} \) instead of \( 1\mathcal{W} \).)

Let \( op \in A_{\mathcal{W}} \) be given by conjugation by the longest element of \( \mathcal{W} \).

For any \( E \in \text{Irr}(\mathcal{W}) \) and \( v \in C^* \) a non-root of 1, \( D_{E,||,v} \in C \) is defined as in 1.2 with \( \mathcal{L} = || \). It is known that \( D_{E,||,v} \) is a polynomial in \( v^2 \) with rational coefficients. Let \( z(E) \) be the largest integer \( \geq 0 \) such that \( D_{E,||,v}/(v^2 + 1)^{z(E)} \) is a polynomial in \( v^2 \). From the known formulas for \( D_{E,||,v} \) one can see that \( z(E) \leq r(op) \).

Assuming that \( \mathcal{W} \) is \( \{1\} \) or irreducible, we say that \( \mathcal{W} \) is sharp if

(a) there exists \( E_0 \in \text{Irr}(\mathcal{W}) \) such that \( z(E_0) = r(op) \)
and \( r(op)\text{ord}(op) \) is even. (The last condition is imposed to rule out a \( \mathcal{W} \) of type \( E_7 \).) Note that the family of \( \mathcal{W} \) containing \( E_0 \) in (a) is necessarily unique (when such \( E_0 \) exists); moreover \( E_0 \) itself is unique (when it exists) if it is assumed to be special.

Let \( \gamma \in A_{\mathcal{W}} \). Assuming that \( \mathcal{W} \) is \( \{1\} \) or irreducible, we say that \( \gamma \mathcal{W} \) is sharp if \( \mathcal{W} \) is sharp and \( \text{ord}(op\gamma) \) is odd. The last condition means that we have either \( \gamma = op \) or \( \text{ord}(\gamma) = 3 \) (hence \( \mathcal{W} \) is of type \( D_4 \)). For \( \gamma \mathcal{W} \) sharp we set \( a[\gamma \mathcal{W}] = a(E_0) \) where \( E_0 \) is as in (a).

2.2. Here is a complete list of the various sharp \( \gamma \mathcal{W} \) and the corresponding \( a[\gamma \mathcal{W}] \):

(i) \( \{1\} \), \( a[\gamma \mathcal{W}] = 0 \):
(ii) \( 2A_{(t^2 - 1)/2 - 1} \), \( t \in \{5, 7, 9, \ldots \} \), \( a[\gamma \mathcal{W}] = (t - 3)(t - 1)(t + 1)/48; \)
(iii) \( B_{(t^2 - 1)/4} \), \( t \in \{3, 5, 7, \ldots \} \), \( a[\gamma \mathcal{W}] = (t - 1)(t + 1)(2t - 3)/24; \)
(iv) \( D_{t^2/4} \), \( t \in \{4, 8, 12, \ldots \} \), \( a[\gamma \mathcal{W}] = (t - 2)t(2t + 1)/24; \)
(v) \( 2D_{t^2/4} \), \( t \in \{6, 10, 14, \ldots \} \), \( a[\gamma \mathcal{W}] = (t - 2)t(2t + 1)/24; \)
(vi) \( G_2, 3D_4, F_4, 2E_6, E_8, a[\gamma \mathcal{W}] = 1, 3, 4, 7, 16 \) respectively.

For a general \( \mathcal{W}, \gamma \) we say that \( \mathcal{W} \) is \( \gamma \)-irreducible if \( \mathcal{W} \) is a product of \( k \geq 1 \) irreducible Weyl groups \( \mathcal{W}_1, \ldots, \mathcal{W}_k \) and \( \gamma \) permutes \( \mathcal{W}_1, \ldots, \mathcal{W}_k \) cyclically. In this case we say that \( \gamma \mathcal{W} \) is sharp if \( \gamma^k \mathcal{W}_1 \) is sharp and we set \( a[\gamma \mathcal{W}] = k a[\gamma^{k-1} \mathcal{W}_1] \).

For a general \( \mathcal{W}, \gamma \), we have that \( \mathcal{W} \) is a product \( \mathcal{W}_1' \times \ldots \times \mathcal{W}_j' \) of Weyl groups such that each \( \mathcal{W}_j' \) is \( \gamma \)-stable and \( \gamma \)-irreducible. We say that \( \gamma \mathcal{W} \) is sharp if each \( \gamma \mathcal{W}_j' \) is sharp; we set \( a[\gamma \mathcal{W}] = \sum_j a[\gamma \mathcal{W}_j] \).

The objects in (i),(iii)-(v) can be viewed as vertices of a graph:

\[
\begin{align*}
1 \quad & - - - - B_{(3^2 - 1)/4} - - - - D_{4^2}/4 - - - - B_{(5^2 - 1)/4} - - - - \\
& - - - - 2D_{6^2/4} - - - - B_{(7^2 - 1)/4} - - - - D_{8^2}/4 - - - - B_{(9^2 - 1)/4} - - - - \\
(a) \quad & - - - - 2D_{10^2/4} - - - - B_{(11^2 - 1)/4} - - - - . . . .
\end{align*}
\]
We will attach to each vertex of this graph an index: the index of each of $B_{(t^2-1)/4}$, $D_{t^2/4}$, $^2D_{t^2/4}$ is $t$; the index of $\{1\}$ is 2.

From the objects in (i)-(v) we can form a second graph:

$$
\{1\} \times \{1\} \rightarrow -2A_{(5^2-1)/8-1} \rightarrow -\gamma(B_{(6/2)^2-1}/4 \times B_{((6/2)^2-1)/4}) \rightarrow
-2A_{(7^2-1)/8-1} \rightarrow -\gamma(D_{(8/2)^2/4} \times D_{(8/2)^2/4}) \rightarrow
-2A_{(9^2-1)/8-1} \rightarrow -\gamma(B_{((10/2)^2-1)/4} \times B_{((10/2)^2-1)/4}) \rightarrow
-2A_{(11^2-1)/8-1} \rightarrow -\gamma(D_{(12/2)^2/4} \times D_{(12/2)^2/4}) \rightarrow
\gamma(D_{(t/2)^2/4} \times D_{(t/2)^2/4}), ^2A_{(t^2-1)/8-1}, \gamma(B_{(t/2)^2-1}/4 \times B_{((t/2)^2-1)/4})
$$

Here $\gamma$ acts on $B_{((t/2)^2-1)/4} \times B_{((t/2)^2-1)/4}$ as an involution exchanging the two factors; it acts on $D_{(t/2)^2/4} \times D_{(t/2)^2/4}$ by permuting the two factors in such a way that $\gamma^2 = 1$ if $t \in \{8, 16, 24, \ldots\}$ and $\gamma^2 \neq 1$ if $t \in \{12, 20, 28, \ldots\}$. We will attach to each vertex of this graph an index: the index of each of

$$
\gamma(D_{(t/2)^2/4} \times D_{(t/2)^2/4}), ^2A_{(t^2-1)/8-1}, \gamma(B_{(t/2)^2-1}/4 \times B_{((t/2)^2-1)/4})
$$

is $t$; the index of $\{1\} \times \{1\}$ is 4.

3. Affine Weyl groups and the sets $C_\omega(W)$

3.1. In this section $W$ denotes an (irreducible) affine Weyl group. Let $\mathcal{T}$ be the set of all $w \in W$ such that the conjugacy class of $w$ is finite. (Such $w$ are said to be the translations of $W$.) Now $\mathcal{T}$ is a free abelian group of finite rank and of finite index in $W$. Let $w \mapsto |w|$ be the usual length function of $W$. Let $S$ be the set of simple reflections of $W$. Let $S^1$ be the set of all $\sigma \in S$ such that the sum of labels of edges of the Coxeter graph of $W$ which touch $\sigma$ is $\geq 3$. We have $\sharp(S^1) \leq 2$.

Let $\Omega_W$ be the (finite abelian) group of automorphisms of $W$ preserving $S$ whose restriction to $\mathcal{T}$ is given by conjugation by an element of $W$. If $S^1 \neq \emptyset$ let $\Omega_W'$ be the set of all $\omega \in \Omega_W$ such that $\omega$ restricted to $S^1$ is the identity map (this is a subgroup of $\Omega_W$); if $S^1 = \emptyset$ we set $\Omega_W' = \Omega_W$. We set $\Omega_W'' = \Omega_W - \Omega_W'$.

Let $\hat{W} = W/\mathcal{T}$ (a finite group). We show:

(a) If $\omega \in \Omega_W$ then $\omega : W \rightarrow W$ induces an inner automorphism of $\hat{W}$.

We can find $w \in W$ such that $\omega(\tau) = \text{Ad}(w)(\tau)$ for all $\tau \in \mathcal{T}$. Let $\zeta = \text{Ad}(w^{-1})w : W \rightarrow W$. We have $\zeta(\tau) = \tau$ for any $\tau \in \mathcal{T}$. Let $y \in W, \tau \in \mathcal{T}$. We have $y\tau y^{-1} \in \mathcal{T}$ hence $\zeta(y\tau y^{-1}) = y\tau y^{-1}$ that is $\zeta(y)\tau\zeta(y^{-1}) = y\tau y^{-1}$. Setting $y' = y^{-1}\zeta(y) \in W$ we have $y'\tau y'^{-1} = \tau$ for any $\tau \in W$. Now the action of $W/\mathcal{T}$ on $\mathcal{T}$ by conjugation is faithful hence $y' \in \mathcal{T}$. Thus $\zeta(y) \in y\mathcal{T}$ so that $w^{-1}\omega(y)w \in y\mathcal{T}$ and $\omega(y) \in wyw^{-1}\mathcal{T}$. This proves (a).

3.2. For any $J \subsetneq S$ let $W_J$ be the subgroup of $W$ generated by $J$ (a finite Weyl group).
Let $S_\ast$ be the set of all $\sigma \in S$ such that $W_{S - \{\sigma\}} \to \hat{W}$ (restriction of the obvious map $W \to \hat{W}$) is an isomorphism. We have $S_\ast \neq \emptyset$ and the obvious action of $\Omega_W$ on $S_\ast$ is simply transitive. If $J \subsetneq S$ let $W_J \to \hat{W}$ be the restriction of the obvious homomorphism $W \to \hat{W}$; this is an imbedding, so that $W_J$ can be viewed as a subgroup of $\hat{W}$. For any special representation $E \in \text{Irr}(W_J)$ there is a unique $E' \in \text{Irr}(\hat{W})$ such that $E'$ appears in $\text{Ind}_W^{\hat{W}}(E)$ and in the $a(E)$-th symmetric power of the conjugation representation of $\hat{W}$ on $C \otimes T$ (with $a(E)$ defined in terms of $W_J$); we set $E' = j_{W_J}^{\hat{W}}(E)$, see [L09, 1.3]. Let $\hat{S}(W)$ be the subset of $\text{Irr}(\hat{W})$ consisting of representations of the form $j_{W_J}^{\hat{W}}(E)$ for some $J \subsetneq S$ and some special $E \in \text{Irr}(W_J)$.

3.3. Let $\omega \in \Omega_W$. We define a set $\mathcal{C}_\omega(W)$ of Weyl subgroups $W_J$ of $W$ with $J \subsetneq S$. This set contains $\{1\}$. Now $\mathcal{C}_\omega(W) - \{1\}$ consists of the subgroups $W_J$ with $J \subsetneq S$, $J \neq \emptyset$ which satisfy the following requirements.

(i) $W_J$ is $\omega'$-stable for any $\omega' \in \Omega_W$.
(ii) $W_J$ is $\omega$-sharp.
(iii) $W_{S - J}$ is $\omega$-irreducible.
(iv) If $\sharp(S') = 2$ and $\Omega_W' = \emptyset$ then $W_J$ is the product of two vertices of the graph 2.2(a) which are joined by an edge.
(v) If $\omega \in \Omega_W'$, then $W_J$ is the product of two vertices of the graph 2.2(b) which are joined by an edge.

3.4. We now describe the set $\mathcal{C}_\omega(W)$ in each case. If $W$ is of affine type $A_{n-1}$, $n \geq 2$, then $\Omega_W$ is cyclic of order $n$; for any $\omega \in \Omega_W$, $\mathcal{C}_\omega(W)$ consists of a single element: $\{1\}$ (with $a[\{1\}] = 0$).

3.5. If $W$ is of affine type $E_6$ then $\Omega_W$ is cyclic of order 3. Let $\omega \in \Omega_W$. If $\omega = 1$ then $\mathcal{C}_\omega(W)$ consists of a single element: $\{1\}$ (with $a[\{1\}] = 0$). If $\omega \neq 1$ then $\mathcal{C}_\omega(W)$ consists of $\{1\}$ (with $a[\{1\}] = 0$) and of the subgroup $W_J$ of type $D_4$ (so that $^\omega W_J = 3D_4$ and $a[ ^\omega W_J] = 3$).

3.6. If $W$ is of affine type $E_7$ then $\Omega_W$ is cyclic of order 2. Let $\omega \in \Omega_W$. If $\omega = 1$ then $\mathcal{C}_\omega(W)$ consists of a single element: $\{1\}$ (with $a[\{1\}] = 0$). If $\omega \neq 1$ then $\mathcal{C}_\omega(W)$ consists of $\{1\}$ (with $a[\{1\}] = 0$) and of the subgroup $W_J$ of type $E_6$ (so that $^\omega W_J = 2E_6$ and $a[ ^\omega W_J] = 7$).

3.7. If $W$ is of affine type $E_8, F_4$ or $G_2$ and $\omega \in \Omega_W$ then $\omega = 1$ and $\mathcal{C}_\omega(W)$ consists of $\{1\}$ (with $a[\{1\}] = 0$) and of the subgroup $W_J$ of non-affine type $E_8, F_4$ or $G_2$ (respectively), with $a[W_J]$ equal to 16, 4, 1 respectively.

3.8. In the remainder of this section we assume that $W$ is of affine type $B_n (n \geq 3), C_n (n \geq 2)$ or $D_n (n \geq 4)$. Let $\omega \in \Omega_W$. Assume first that $\omega \in \Omega_W'$. Let $\mathcal{C}_\omega(W)$ be the set of all pairs $(t, s) \in \mathbb{N}^2$ such that

$t - s = \pm 1$ (type $B$), $t = s$ (type $C, D$),
$t = 0 \mod 4$ (type $B, D$ with $\omega = 1$), $t = 2 \mod 4$ (type $B, D$ with $\omega \neq 1$)
$t = 1 \mod 2$ (type $C$),
and for some \( r \in \mathbb{N} \) we have
\[
\begin{align*}
& t^2/4 + (s^2 - 1)/4 + r = n, \text{ (type } B), \\
& (t^2 - 1)/4 + (s^2 - 1)/4 + r = n, \text{ (type } C), \\
& t^2/4 + s^2/4 + r = n, \text{ (type } D). \\
\end{align*}
\]
that is, \( ts + 2r = 2n \) (type \( B, D \)), \( ts + 2r = 2n + 1 \) (type \( C \)).

In type \( B \) we define a bijection \( C_\omega(W) \xrightarrow{\sim} \mathcal{C}_\omega(W) \) by associating to \( W_j \in C_\omega(W) \) (assumed to be \( \neq \{1\} \)) the pair \((t, s)\) formed by the indexes \( t, s \) of the two vertices attached to \( W_j \) in 3.3(iv) and by associating to \( W_j = \{1\} \) the pair \((0, 1)\) (if \( \omega = 1 \)) or \((2, 1)\) (if \( \omega \neq 1 \)).

In type \( C, D \), we define a bijection \( C_\omega(W) \xrightarrow{\sim} \mathcal{C}_\omega(W) \) by associating to \( W_j \in C_\omega(W) \) (assumed to be \( \neq \{1\} \)) the pair \((t, s)\) (with \( t = s \)) where \( W_j \) is the product of a vertex of index \( t \) in 2.2(a) with itself and by associating to \( W_j = \{1\} \) the pair \((1, 1)\) (type \( C \)), \((0, 0)\) (type \( D \) with \( \omega = 1 \)), \((2, 2)\) (type \( D \) with \( \omega \neq 1 \)).

Assuming that \( W_j \) corresponds as above to \((t, s)\) we can compute \( a^{[\omega W_j]} \) in each case.
If \( s = t \in \{1, 3, 5, \ldots\} \) then
\[
a^{[\omega W_j]} = (t - 1)(t + 1)(2t - 3)/24 + (t - 1)(t + 1)(2t - 3)/24 \\
= (t - 1)(t + 1)(2t - 3)/12.
\]
If \( s = t \in \{0, 2, 4, \ldots\} \) then
\[
a^{[\omega W_j]} = (t - 2)t(2t + 1)/24 + (t - 2)t(2t + 1)/24 = (t - 2)t(2t + 1)/12.
\]
If \( t \in \{0, 2, 4, \ldots\}, s \in \{1, 3, 5, \ldots\} \), \( t - s = \pm 1 \) then
\[
a^{[\omega W_j]} = (t - 2)t(2t + 1)/24 + (s - 1)s(2s + 1)/24
\]
and this equals \((t - 1)t(t + 1)/6\) if \( s = t + 1 \) and \((t - 2)(t - 1)t/6\) if \( t = s + 1 \).

3.9. Assume next that \( \omega \in \Omega'_W \). Let \( \mathcal{C}_\omega(W) \) be the set of all pairs \((t, s)\) in \( \mathbb{N}^2 \) such that
\[
t - s = \pm 1, \\
t = 2 \mod 4 \text{ (type } C), \\
t = 0 \mod 8 \text{ (type } D \text{ with } \omega^2 = 1), \ t = 4 \mod 8 \text{ (type } D \text{ with } \omega^2 \neq 1),
\]
and for some \( r \in \mathbb{N} \) we have
\[
\begin{align*}
& ((t/2)^2 - 1)/2 + (s^2 - 1)/8 + 2r = n \text{ (type } C), \\
& (t/2)^2 + (s^2 - 1)/8 + 2r = n \text{ (type } D), \\
\end{align*}
\]
or equivalently \( ts/2 + 4r = 2n + 1 \) (type \( C \)), \( ts/2 + 4r = 2n \) (type \( D \)).

Note that \((2, 1)\) \( \in \mathcal{C}_\omega(W) \) in type \( C \) with \( n \) even and \((2, 3)\) \( \in \mathcal{C}_\omega(W) \) in type \( C \) with \( n \) odd. We define a bijection \( C_\omega(W) \xrightarrow{\sim} \mathcal{C}_\omega(W) \) by associating to \( W_j \in C_\omega(W) \) (assumed to be \( \neq \{1\} \)) the pair \((t, s)\) formed by the indexes of the two vertices attached to \( W_j \) in 3.3(v) and by associating to \( W_j = \{1\} \) the pair
(2, 1) (type C with n even), the pair (2, 3) (type C with n odd), the pair (0, 1) (type D with \( \omega^2 = 1 \)), the pair (4, 3) (type D with \( \omega^2 \neq 1 \)).

Assuming that \( W_J \) corresponds as above to \((t, s)\) we can compute \( a^{\omega W_J} \) in each case.

If \( t \in \{2, 6, 10, \ldots\} \) then

\[
a^{\omega W_J} = (s - 3)(s - 1)(s + 1)/48 + (t/2 - 1)(t/2 + 1)(t - 3)/24 +
(t/2 - 1)(t/2 + 1)(t - 3)/24
\]

which equals \((t - 2)(2t^2 - 5t - 6)/48\) if \( s = t - 1 \) and equals \((t - 2)(t + 2)(2t - 3)/48\) if \( s = t + 1 \).

If \( t \in \{0, 4, 8, 10, \ldots\} \) then

\[
a^{\omega W_J} = (s - 3)(s - 1)(s + 1)/48 + (t/2 - 2)(t/2 + 1)/24 + (t/2 - 2)(t/2)(t + 1)/24
\]

which equals \((t - 4)t(2t - 1)/48\) if \( s = t - 1 \) and equals \( t(2t^2 - 3t - 8)/48\) if \( s = t + 1 \).

**3.10.** Let \( \omega \in \Omega^W \). Let \( 'C_\omega(W) \) be the set of all pairs \((\delta, r)\) in \( \mathbb{N}^2 \) such that

- (type B) \( \delta + 2r = 2\sigma, \delta = 2 + 4 + 6 + \cdots + (2\sigma) \), where \( \sigma \in \mathbb{N} \) and \( \sigma = 0 \mod 4 \) or \( \sigma = 3 \mod 4 \) if \( \omega = 1 \); \( \sigma = 1 \mod 4 \) or \( \sigma = 2 \mod 4 \) if \( \omega \neq 1 \);
- (type C) \( \delta + 2r = 2n + 1, \delta = 1 + 3 + 5 + \cdots + (2\sigma - 1) \) where \( \sigma \in \mathbb{N} \) and \( \sigma = 1 \mod 2 \);
- (type D) \( \delta + 2r = 2n, \delta = 1 + 3 + 5 + \cdots + (2\sigma - 1) \) where \( \sigma \in \mathbb{N} \) and \( \sigma = 0 \mod 4 \) if \( \omega = 1, \sigma = 2 \mod 4 \) if \( \omega \neq 1 \).

We define a bijection

\[
'\mathcal{C}_\omega(W) \xrightarrow{\sim} '\mathcal{C}_\omega(W)
\]

by \((t, s) \mapsto (ts, r)\) where \( r \in \mathbb{N} \) is as in 3.8.

**3.11.** Let \( \Delta = \{t(t + 1)/2 \mid t \in \mathbb{N} \} \). Let \( \omega \in \Omega'_W \). Let \( 'C_\omega(W) \) be the set of all pairs \((\delta, r)\) in \( \Delta \times \mathbb{N} \) such that

- \( \delta + 4r = 2n + 1 \) (type C),
- \( \delta + 4r = 2n, \delta = 0 \mod 4 \) (type D, \( \omega^2 = 1 \)),
- \( \delta + 4r = 2n, \delta = 2 \mod 4 \) (type D, \( \omega^2 \neq 1 \)).

We define a bijection

\[
'\mathcal{C}_\omega(W) \xrightarrow{\sim} '\mathcal{C}_\omega(W)
\]

by \((t, s) \mapsto (ts/2, r)\) where \( r \in \mathbb{N} \) is as in 3.9.

**4. The function** \( c : \text{Irr}(\tilde{W}) \to \mathbb{N} \)

**4.1.** In this section \( \mathcal{W} \) denotes an irreducible affine Weyl group with a set \( \mathcal{S} \) of
simple reflections and with a given weight function \( \mathcal{L} : \mathcal{W} \to \mathbb{N} \). Let \( || \) be the
length function of \( \mathcal{W} \). Let \( \mathcal{T}_W \) be the group of translations of \( \mathcal{W} \) (see 3.1) and let
\( \tilde{\mathcal{W}} = \mathcal{W}/\mathcal{T}_W \) (a finite group). For any \( \mathcal{J} \subseteq \mathcal{S} \) we denote by \( \mathcal{W}_{\mathcal{J}} \) the subgroup of
\( \mathcal{W} \) generated by \( \mathcal{J} \) (a finite Weyl group). Let \( \tilde{\mathcal{W}}_{\mathcal{J}} \) be the image of \( \mathcal{W}_{\mathcal{J}} \) under the
obvious map \( W \to \hat{W} \). Note that the obvious map \( W_J \to \hat{W}_J \) is an isomorphism; we use this to identify \( W_J = \hat{W}_J \).

For any \( E \in \operatorname{Irr}(W) \) we define \( \Sigma(E) \) to be the set of all pairs \((J, E')\) where \( J \subset S, \#(J) = \#(S) - 1 \) and \( E' \in \operatorname{Irr}(W_J) \) is such that \( E' \) appears in the restriction of \( E \) to \( W_J = \hat{W}_J \). We set

\[
(c) \quad c_E = \max_{(J, E') \in \Sigma(E)} a_L(E') \in \mathbb{N},
\]

\[
\Sigma^*(E) = \{(J, E') \in \Sigma(E); a_L(E') = c_E\},
\]

where \( a_L(E') \) is defined as in 1.2 in terms of the Weyl group \( W_J \) with the weight function obtained by restricting \( L : W \to N \) to \( W_J \). We have \( \Sigma^*(E) \neq \emptyset \).

4.2. The function \( E \mapsto c_E \) has been computed explicitly in [L20] in the case where \( W \) is of exceptional type and \( L = \| \). We will now describe explicitly the map \( \operatorname{Irr}(W) \to N, E \mapsto c_E \) of 4.1 in several examples with \( L \neq \| \). Assume that either

(a) \( W \) is of affine type \( C_r, r \geq 2 \) and \( L|_S \) takes the values \( t, 1, 1, \ldots, 1, 1, s \) where \( t > 0, s > 0 \), or

(b) \( W \) is of affine type \( C_r, r \geq 2 \) and \( L|_S \) takes the values \( t, 2, 2, \ldots, 2, 2, s \) where \( t > 0, s > 0 \), or

(c) \( W \) is of affine type \( G_2 \) and \( L|_S \) has values \( 3, 3, 1 \).

In case (a),(b) we set \( u = \max(t, s) \). In each of the examples below we give a table with two rows whose columns are indexed by the various \( E \in \operatorname{Irr}(W) \). The first row represents the numbers \( c_E \). The second row represents the numbers \( a_L(E') \) for various \( E' \in \operatorname{Irr}(W_J) \) where \( J = S - \{s\} \) for some \( s \in S_\ast \) (see 3.2) which in case (a),(b) is chosen so that \( L(s) = u \). (We can identify \( W_J = \hat{W} \) hence \( E' \) can be identified with an \( E \in \hat{W}_J \).) Any entry \( e \) in the first row and column \( E \) is \( \geq \) than the entry \( e' \) in the second row and column \( E \). When \( e > e' \) we indicate some other \( J' \subsetneq S \) such that some \( E' \in \operatorname{Irr}(W_{J'}) \) appears in the restriction of \( E \) to \( W_{J'} \) and \( a_L(E') = e \). (We will specify such \( J' \) by specifying the type of \( W_{J'} \).)

Assume first (in case (a)) that \( r = 2 \) and \( t = s = u \geq 1 \). The table is

\[
0; 1; u; 2u; 2u + 2
0; 1; t; 2u - 1; 2u + 2
\]

with an additional \( J' \) with \( W_{J'} \) of type \( A_1 \times A_1 \), contributing \( 2u \) to the fourth column.

Assume next (in case (a)) that \( r = 2 \) and \( t = s \pm 1, u \geq 2 \). The table is

\[
0; 1; u; 2u - 1; 2u + 2
0; 1; u; 2u - 1; 2u + 2
\]

In this case there is no need for an additional \( J' \).
Assume next (in case (b)) that $r = 2$ and $t = s \pm 1$, $u \geq 2$. The table is

\[
\begin{align*}
0; u; 2; 2u - 1; 2u + 2 \\
0; u; 2; 2u - 2; 2u + 2
\end{align*}
\]

with an additional $J'$ with $W_{J'}$ of type $A_1 \times A_1$, contributing $2u - 1$ to the fourth column.

Assume next (in case (b)) that $r = 3$ and $t = s \pm 1$, $u \geq 2$. The table is

\[
\begin{align*}
0; u; 2; 3u - 3; 2u - 1; u + 2; 3u + 3; 2u + 4; 6; 3u + 12 \\
0; u; 2; 3u - 6; 2u - 2; u + 2; 3u + 2; 2u + 4; 6; 3u + 12
\end{align*}
\]

with additional $J'$ with $W_{J'}$ of type $A_1 \times B_2$, contributing $3u - 3$ to the fourth column, $2u - 1$ to the fifth column and $3u + 3$ to the seventh column.

Finally assume that we are in case (c). The table is

\[
\begin{align*}
0; 1; 3; 4; 9; 12 \\
0; 1; 3; 3; 7; 12
\end{align*}
\]

with an additional $J'$ with $W_{J'}$ of type $A_2$ contributing 9 to the fifth column and with an additional $J'$ with $W_{J'}$ of type $A_1 \times A_1$ contributing 4 to the fourth column.

4.3. We set

$$\nu(W, L) = \max_{s \in S^*} L(w_{0,s})$$

where $w_{0,s}$ is the element of maximal length of $W_{S - \{s\}}$.

4.4. Let $\sim$ be the equivalence relation on $\text{Irr}(\bar{W})$ generated by the relation $E_1 \sim E_2$ when $c_{E_1} = c_{E_2}$ and there exist $(J_1, E'_1) \in \mathcal{I}^*(E_1)$, $(J_2, E'_2) \in \mathcal{I}^*(E_2)$ such that $J_1 = J_2$ and $E'_1, E'_2$ are in the same $L$-family (see 1.7) of $\text{Irr}(W_{J_1}) = \text{Irr}(W_{J_2})$.

5. A weighted affine Weyl group

5.1. We preserve the setup of 3.1. For any $J \subseteq S$ let $w_J^0$ be the longest element of $W_J$. We now fix $\omega \in \Omega_W$, $J \subseteq S$ such that $W_J \in \mathcal{C}_\omega(W)$. Following [L90] (see also [L95]), to $W, \omega, J$ we associate a weighted affine Weyl group $(W_J, L_J)$. (A similar procedure was used for finite Weyl groups in [L78].) The proofs of various statements in this section can be extracted from [L95]. Let

$$W_J' = \{w \in W; wW_J = W_Jw, w \text{ has minimal length in } wW_J = W_Jw\};$$

this is a subgroup of $W$ stable under $\omega$; let $W_J = \{w \in W_J'; \omega(w) = w\}$ (another subgroup of $W$). Let $S_J$ be the set of all $\omega$-orbits on $S - J$. 

When \( \sharp(S_J) \geq 2 \) let \( T_J \) be the group of translations of the affine Weyl group \( W_J \) and let \( \tilde{W}_J = W_J/T_J \), a finite group. When \( \sharp(S_J) = 1 \) we have \( W_J = \{1\} \) and we set \( T_J = \{1\}, \tilde{W}_J = \{1\} \); in this case we regard \( W_J \) as a weighted affine Weyl group with weight function \( \mathcal{L}_J = 0 \).

We now assume that \( \sharp(S_J) \geq 2 \). If \( \theta \in S_J \) we set \( \tau_\theta = w_0^{J \cup \theta}w_0^J = w_0^J w_0^{J \cup \theta} \). (The last equality is a property of any \( W_J \in \mathcal{C}_\omega(W) \).) We have \( \tau_\theta \in W_J, \tau_\theta^2 = 1 \). Moreover \( W_J \) is a Coxeter group on the generators \( \{\tau_\theta; \theta \in S_J\} \) and with Coxeter relations \( (t_\theta t_{\theta'})^{m_{\theta, \theta'}} = 1 \) for any distinct \( \theta, \theta' \in S_J \) such that \( J \cup \theta \cup \theta' \neq S \), where

\[
m_{\theta, \theta'} = \frac{2(|w_0^{J \cup \theta} | - | w_0^J |)}{|w_0^{J \cup \theta'} | + | w_0^{J \cup \theta''} | - 2 | w_0^J |}.
\]

(When \( J \cup \theta \cup \theta' = S \) then \( \tau_\theta \tau_{\theta'} \) has infinite order.) Note that \( W_J \) is an (irreducible) affine Weyl group.

5.2. We define a weight function \( \mathcal{L}_J : W_J \to \mathbb{N} \). Let \( \theta \in S_J \). Let \( E_0 \) be the unique irreducible special representation of \( W_J \) such that \( z(E_0) = r(op) \) (notation of 2.1 with \( \mathcal{W} \) replaced by \( W_J \)). Let

\[
\mathcal{E} = \{ \tilde{E} \in \text{Irr}(W_{J \cup \theta}; \tilde{E} \text{ appears in } \text{Ind}_{W_J}^{W_{J \cup \theta}}(E_0)) \},
\]

\[
\mathcal{L}(\theta) = \max\{a(\tilde{E}); \tilde{E} \in \mathcal{E}\} - \min\{a(\tilde{E}); \tilde{E} \in \mathcal{E}\}.
\]

(Here \( a(\tilde{E}) \) is as in 1.2 with \( \mathcal{W} \) replaced by \( W_{J \cup \theta} \).) This defines the function \( \mathcal{L}_J : S_J \to \mathbb{Z}_{>0} \). This extends uniquely to a weight function \( \mathcal{W} \to \mathbb{N} \) denoted again by \( \mathcal{L}_J \) so that \( (W_J, \mathcal{L}_J) \) is a weighted affine Weyl group.

5.3. Let \( \omega \in \Omega_W, J \subseteq S \) be such that \( W_J \in \mathcal{C}_\omega(W) \). We now describe the pair \( (\mathcal{W}, \mathcal{L}) = (\mathcal{W}_J, \mathcal{L}_J) \) in each case. We will write \( S \) instead of \( S_J \).

If \( W \) is of affine type \( A_{n-1}, n \geq 2 \), let \( k = \text{ord}(\omega) \) (a divisor of \( n \)). We have \( W_J = \{1\} \). If \( k < n, W \) is of affine type \( A_{(n/k)-1} \) with \( \mathcal{L}|S \) being constant equal to \( k \). If \( k = n, \) we have \( W = \{1\} \).

Assume that \( W \) is of affine type \( E_6 \). If \( \omega = 1 \) then \( W_J = \{1\} \) and \( \mathcal{W} = W \) with \( \mathcal{L}|S \) being constant equal to 1. If \( \omega \neq 1 \) and \( W_J = \{1\} \) then \( \mathcal{W} \) is of affine type \( G_2 \) with the values of \( \mathcal{L}|S \) being 3, 3, 1. If \( \omega \neq 1 \) and \( W_J \) is of type \( D_4 \) then \( \mathcal{W} = \{1\} \).

Assume that \( W \) is of affine type \( E_7 \). If \( \omega = 1 \) then \( W_J = \{1\} \) and \( \mathcal{W} = W \) with \( \mathcal{L}|S \) being constant equal to 1. If \( \omega \neq 1 \) and \( W_J = \{1\} \) then \( \mathcal{W} \) is of affine type \( F_4 \) with the values of \( \mathcal{L}|S \) being 2, 2, 2, 1, 1. If \( \omega \neq 1 \) and \( W_J \) is of type \( E_6 \) then \( \mathcal{W} = \{1\} \).

Assume that \( W \) is of affine type \( E_8, F_4 \) or \( G_2 \). We have \( \omega = 1 \). If \( W_J = \{1\} \) then \( \mathcal{W} = W \) with \( \mathcal{L}|S \) being constant equal to 1. If \( W_J \) is of the non- affine type \( E_8, F_4 \) or \( G_2 \) (respectively) then \( \mathcal{W} = \{1\} \).

5.4. We now assume that \( W \) is of affine type \( B_{n} (n \geq 3), C_{n} (n \geq 2) \) or \( D_{n} (n \geq 4) \). Let \( (\delta, r) \in \mathcal{C}_\omega(W) \) be the image of \( W_J \) under the composition \( \mathcal{C}_\omega(W) \to \to \mathcal{C}_\omega(W) \to \mathcal{C}_\omega(W) \) (the first map as in 3.8, 3.9, the second map as in 3.10, 3.11).
If $r = 0$ then $\mathcal{W} = \{1\}$. If $r > 0$, $\delta > 0$, then $\mathcal{W}$ is of affine type $C_r$, with Coxeter graph

(a) $\boxed{t} = \boxed{1} - - \boxed{1} - - \cdots - - \boxed{1} - - \boxed{1} = \boxed{s}$

(b) $\boxed{t} = \boxed{2} - - \boxed{2} - - \cdots - - \boxed{2} - - \boxed{2} = \boxed{s}$

where the boxed entries are the values of $\mathcal{L}|_S$ and $t, s$ in $\mathbb{N}$ are defined by

$ts = \delta$, $t - s \in \{0, 1, -1\}$ (in (a) with $\omega \in \Omega'_W$),

$ts/2 = \delta$, $t - s \in \{1, -1\}$ (in (b) with $\omega \in \Omega''_W$).

(Here affine of type $C_1$ is taken to be the same as affine of type $A_1$.)

If $\delta = 0$ (hence $r > 0$), then either:

- $\mathcal{W}$ is of type $B_n$ or $D_n$, $\omega = 1$ and $\mathcal{W} = W$ with $\mathcal{L}|_S$ constant equal to 1, or
- $\mathcal{W}$ is of type $D_n$, $n = 2r \geq 6$, $\omega \in \Omega''_W$, $\omega^2 = 1$ and $\mathcal{W}$ is of affine type $B_r$, with the values of $\mathcal{L}|_S$ being 1, 2, 2, ..., 2.

5.5. We now list the various $(\mathcal{W}, \mathcal{L})$ which are associated to various $W, \omega, W_J \in \mathcal{C}_\omega(W)$.

$\mathcal{W} = \{1\}$ or $\mathcal{W}$ is an irreducible affine Weyl groups of type $E_6, E_7, E_8$ or $D_m, m \geq 4$ with $\mathcal{L}|_S$ constant equal to 1;

- $\mathcal{W}$ of affine type $A_{n-1}, n \geq 3$ with $\mathcal{L}|_S$ constant in $\mathbb{Z}_{>0}$;
- $\mathcal{W}$ of affine type $B_m, m \geq 3$ with the values of $\mathcal{L}|_S$ being 1, 1, ..., 1, 1 or 1, 2, 2, ..., 2;
- $\mathcal{W}$ of affine type $C_m, m \geq 1$ with the values of $\mathcal{L}|_S$ being $t, 1, 1, \ldots, 1, 1, s$ with $(t, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ such that $t - s \in \{0, 1, -1\}$;
- $\mathcal{W}$ of affine type $C_m, m \geq 2$ with the values of $\mathcal{L}|_S$ being $t, 2, 2, \ldots, 2, 2$ with $(t, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ such that $t - s \in \{1, -1\}$;
- $\mathcal{W}$ of affine type $G_2$ with the values of $\mathcal{L}|_S$ being 1, 1, 1 or 3, 3, 1;
- $\mathcal{W}$ of affine type $F_4$ with the values of $\mathcal{L}|_S$ being 1, 1, 1, 1, 1 or 2, 2, 2, 1, 1.

(Here affine of type $C_1$ is taken to be the same as affine of type $A_1$.)

5.6. Let $\omega \in \Omega_W$, $J \nsubseteq S$ be such that $W_J \in \mathcal{C}_\omega(W)$. Let $E \mapsto c_E$ be as in 4.1 with $\mathcal{W} = \mathcal{W}_J$. For $E, \tilde{E}$ in $\text{Irr}(\tilde{\mathcal{W}}_J)$ we write $E \leq \tilde{E}$ if $E = \tilde{E}$ or $c_E > c_{\tilde{E}}$. This is a partial order on $\text{Irr}(\tilde{\mathcal{W}}_J)$. For $E, \tilde{E}$ in $\text{Irr}(\tilde{\mathcal{W}}_J)$ we write $E \approx \tilde{E}$ if $c_E = c_{\tilde{E}}$. This is an equivalence relation on $\text{Irr}(\tilde{\mathcal{W}}_J)$.

Let $q$ be an indeterminate. Let $\mathcal{T}_J = Q(q) \otimes \mathcal{T}_J$. Now the obvious $\mathcal{W}_J$ action on $\mathcal{T}_J$ induces a linear action of $\mathcal{W}_J$ on $\mathcal{T}_J$.

For $E, \tilde{E}$ in $\text{Irr}(\tilde{\mathcal{W}}_J)$ we define

$$\Omega'_{E, \tilde{E}} = z(\tilde{\mathcal{W}})^{-1} \sum_{w \in \mathcal{W}_J} \text{tr}(w^{-1}, E)\text{tr}(w, \tilde{E}) \det(q - w, \mathcal{T}_J)^{-1} q^{-c_E - c_{\tilde{E}}} \in Q(q).$$

The following result can be deduced from Lemma 2.1 in [GM] (where it is attributed to the author).
Proposition 5.7. The system of equations

\[ \sum_{E', E \in \mathcal{W}_L; E' \leq E} P_{E', E} \Lambda'_{E', E} P_{E', E} = \Omega'_{E, E}, \]

\[ P_{E, E} = 1 \text{ for all } E, \]

\[ P_{E', E} = 0 \text{ if } E' \approx E, E \neq E', \]

\[ P_{E', E} = 0 \text{ if } E' \not\approx E, \]

\[ \Lambda'_{E', E} = 0 \text{ unless } E' \approx E \]

with unknowns

\[ P_{E', E} \in \mathbb{Q}(q), \Lambda'_{E', E} \in \mathbb{Q}(q), (E', E \in \text{Irr}(\mathcal{W}_L)) \]

has a unique solution.

6. The set \( G_\omega(G) \) and the bijection \( G_\omega(G) \overset{\sim}{\rightarrow} C_\omega(W) \)

6.1. Let \( le(G) \) be the set of subgroups of \( G \) (see 0.1) which are Levi subgroups of some parabolic subgroup of \( G \). Let \( ls^0(G) \) be the subset of \( ls(G) \) (see 0.1) consisting of the pairs \( (e, \mathfrak{L}) \) each of which is a unipotent block by itself. Let \( l\mathfrak{s}(G) \) be the set of unipotent blocks of \( G \). Note that \( ls^0(G) \) can be identified with a subset \( l\mathfrak{s}^0(G) \) of \( l\mathfrak{s}(G) \). Let \( Z_G \) be the group of components of the centre of \( G \) and let \( Z^*_G = \text{Hom}(Z_G, \mathbb{C}^*) \). We have a partition \( ls(G) = \bigsqcup_{\chi \in Z^*_G} ls_\chi(G) \) where \( ls_\chi(G) \) consists of all \( (e, \mathfrak{L}) \in ls(G) \) such that the natural action of \( \tilde{Z}_G \) on \( \mathfrak{L} \) is through \( \chi \). For any \( \chi \in Z^*_G \), \( ls_\chi(G) \) is a union of unipotent blocks. Hence we have a partition

\[ l\mathfrak{s}(G) = \bigsqcup_{\chi \in Z^*_G} l\mathfrak{s}_\chi(G) \]

where \( l\mathfrak{s}_\chi(G) \) is the set of unipotent blocks contained in \( ls_\chi(G) \). In [L84] it is shown that for any \( \chi \in Z^*_G \)

(a) the intersection \( ls^0(G) \cap ls_\chi(G) \) consists of at most one element.

Let \( \mathcal{G}'(G) \) be the set consisting of \( G \)-conjugacy classes of triples \( (L, e_1, \mathfrak{L}_1) \) with \( L \in le(G), (e_1, \mathfrak{L}_1) \in ls^0(L) \). In [L84] a bijection

(b) \( ls(G) \leftrightarrow \mathcal{G}'(G) \)

is established. To \( (L, e_1, \mathfrak{L}_1) \in \mathcal{G}'(G) \) we have associated in [L84, 4.4] a perverse sheaf \( \phi_K \) (up to shift) on \( G \) whose cohomology sheaves restricted to unipotent classes are direct sums of local systems in a fixed unipotent block. (This defines the unipotent blocks and the bijection (b).) If \( \chi \in Z^*_G \) and \( \beta \in l\mathfrak{s}_\chi(G) \), then the corresponding \( (L, e_1, \mathfrak{L}_1) \in \mathcal{G}'(G) \) is such that \( (e_1, \mathfrak{L}_1) \in ls_\chi_1(L) \) for some \( \chi_1 \in \mathbb{Z}_L^* \) which is uniquely determined by \( \chi \). (Note that \( \chi \) is the image of \( \chi_1 \) under the injective homomorphism \( \mathbb{Z}_L^* \rightarrow \mathbb{Z}_G^* \) induced by the obvious (surjective)
homomorphism $Z_G \to Z_L$.) Using (a) for $L$ we see that $(c_1, \mathcal{C}_1)$ is unique if it exists. Thus (b) gives rise to a bijection

(c) $\mathcal{L}_x(G) \leftrightarrow \mathcal{G}_x(G)$

where $\mathcal{G}_x(G)$ is the set of all $L \in le(G)$ (up to $G$-conjugacy) such that there exists $(c_1, \mathcal{C}_1) \in \mathcal{L}^{0}(L) \cap \mathcal{L}^{c}(L)$ where $\chi_1 \in Z_L^*$ maps to $\chi$ under $Z_L^* \to Z_G^*$. We set $\mathcal{G}(G) = \cup_{\chi \in Z_G^*} \mathcal{G}_x(G)$. Then (c) gives rise to a bijection

(d) $\mathcal{L}_x(G) \leftrightarrow \mathcal{G}(G)$.

In [L84] it is shown that the set of objects in a fixed unipotent block $\beta$ is in bijection with the set of irreducible representations of the normalizer $\mathcal{M}_\beta$ of $L$ modulo $L$ (with $L$ corresponding to $\beta$) and that $\mathcal{M}_\beta$ is naturally a Weyl group.

6.2. In the remainder of this section we assume that $G$ in 0.1 is almost simple, simply connected and that $W, S$ in 3.1 is the affine Weyl group associated with a simple algebraic group $H$ over $C$ of type dual to that of $G$. In particular, $T$ (see 3.1) can be identified with the group of one parameter subgroups of a maximal torus $T$ of $H$. For $h \in T$, the connected centralizer of $h$ in $H$ has Weyl group equal to a $W$-conjugate of $W_J$ for some $J \subsetneq S$. This gives a correspondence $h \leftrightarrow J$ between $T$ and $\{ J; J \subsetneq S \}$.

We can find an isomorphism $\iota : Z^*_G \sim \Omega_W$ such that the following holds: if $G = Spin_N(C)$, with $N \geq 5$ odd or $N \geq 10$ even, the subset $\Omega'_W$ of $\Omega_W$ (see 3.1) corresponds to the set of characters of $Z^*_G$ which do not factor through $SO_N(C)$. We shall identify $Z^*_G = \Omega_W$ via $\iota$.

The following result appears in [L79], [L89].

**Theorem 6.3.** Define $un(G) \to \text{Irr}(\tilde{W})$ by $c \mapsto E$ where $E$ is attached to $(c, C)$ under the Springer correspondence. This map defines a bijection $un(G) \sim \tilde{S}(\tilde{W})$ (notation of 3.2).

6.4. Let $\omega \in \Omega_W$. Let $L \in \mathcal{G}_\omega(G)$, let $(L, c_1, \mathcal{C}_1)$ be the corresponding object of $\mathcal{G}'_G$ (see 6.1) and let $\phi_IK$ be the associated complex of sheaves on $G$ (see 6.1); note that $\phi_IK[m]$ is a semisimple perverse sheaf for some $m$. Let $K_1$ be a simple perverse sheaf on $G$ such that $K_1$ is a direct summand of $\phi_IK[m]$. Then $K_1$ is a character sheaf on $G$. Let $C$ be the semisimple conjugacy class of $H$ attached to $K_1$ by the known classification of character sheaves. Note that $C$ is independent of the choice of $K_1$. There is a unique subset $J \subsetneq S$ such that for any $h \in T \cap C$ we have $h \leftrightarrow J$ (see 6.2).

**Theorem 6.5.** For $L \in \mathcal{G}_\omega(G)$ we define $J \subsetneq S$ as in 6.4. We have $W_J \in \mathcal{C}_\omega(W)$ and $L \mapsto W_J$ is a bijection

(a) $\mathcal{G}_\omega(G) \sim \mathcal{C}_\omega(W)$.

Note that the Theorem gives a parametrization of the set of unipotent blocks of $G$ which is purely in terms of $W$ and is thus independent of the geometry of $G$. The proof is given in the remainder of this section.
6.6. Let \( \omega \in \Omega_W \). We now describe explicitly (and independently of 6.4) a bijection

\[
(a) \quad \mathcal{G}_\omega(G) \rightarrow \mathcal{C}_\omega(W).
\]

The set \( \mathcal{G}_\omega(G) \) is computed in [L84]. Assume first that \( G = SL_n(C) \), \( n \geq 2 \). Note that \( Z^*_G = \Omega_W \) is a cyclic group of order \( n \). Let \( k = \text{ord}(\omega) \) (a divisor of \( n \)). Now \( \mathcal{G}_\omega(G) \) consists of a single \( L \in \text{le}(G) \) (up to conjugacy) such that \( L_{\text{der}} \cong SL_k(C) \times \ldots \times SL_k(C) \) (\( n/k \) copies). The bijection \((a)\) is the obvious bijection between sets with one element.

Assume that \( G \) is of type \( E_6 \). Then \( \Omega_W \) is cyclic of order 3. If \( \omega = 1 \) then \( \mathcal{G}_\omega(G) \) consists of a single object, a maximal torus \( L \). The bijection \((a)\) is the obvious bijection between sets with one element. If \( \omega \neq 1 \) then \( \mathcal{G}_\omega(G) \) consists of \( L = G \) and of \( L \in \text{le}(G) \) such that \( L_{\text{der}} = SL_3(C) \times SL_3(C) \) (up to conjugacy). Recall that \( \mathcal{C}_\omega(W) = \{\{1\}, W_J\} \) where \( W_J \) is of type \( D_4 \). We define \((a)\) by \( G \mapsto W_J, L \mapsto \{1\} \) where \( L \neq G \).

Assume that \( G \) is of type \( E_7 \). Then \( \Omega_W \) is cyclic of order 2. If \( \omega = 1 \) then \( \mathcal{G}_\omega(G) \) consists of a single object, a maximal torus \( L \). The bijection \((a)\) is the obvious bijection between sets with one element. If \( \omega \neq 1 \) then \( \mathcal{G}_\omega(G) \) consists of \( L = G \) and of \( L \in \text{le}(G) \) (up to conjugacy) such that \( L_{\text{der}} = SL_2(C) \times SL_2(C) \times SL_2(C) \) and \( L \) is contained in an \( L' \in \text{le}(G) \) with \( L'_{\text{der}} = SL_6(C) \) but \( L \) is not contained in an \( L'' \in \text{le}(G) \) with \( L''_{\text{der}} = SL_7(C) \). Recall that \( \mathcal{C}_\omega(W) = \{\{1\}, W_J\} \) where \( W_J \) is of type \( E_6 \). We define \((a)\) by \( G \mapsto W_J, L \mapsto \{1\} \) where \( L \neq G \).

Assume that \( G \) is of type \( E_8, F_4 \) or \( G_2 \). We have \( \omega = 1 \) and \( \mathcal{G}_\omega(G) \) consists of two objects, an \( L \) which is a maximal torus and \( G \). Recall that \( \mathcal{C}_\omega(W) = \{\{1\}, W_J\} \) where \( W_J \) is of the non-affine type \( E_8, F_4 \) or \( G_2 \) (respectively). We define \((a)\) by \( G \mapsto W_J, L \mapsto \{1\} \) where \( L \neq G \).

6.7. In this subsection we assume that \( G \) is \( Sp_{2n}(C), (n \geq 3) \), \( Spin_{2n+1}(C), (n \geq 2) \) or \( Spin_{2n}(C), (n \geq 4) \) so that \( W \) is of affine type \( B_n(n \geq 3), C_n(n \geq 2) \) or \( D_n(n \geq 4) \) (respectively).

From [L84] we see that \( \mathcal{G}_\omega(G) \) consists of all \( L \in \text{le}(G) \) (up to conjugacy) such that

\[
\begin{align*}
L_{\text{der}} & \cong Sp_4, \text{ for various } (\delta, r) \in '\mathcal{C}_\omega(W) \text{ (for } W \text{ of type } B) \\
L_{\text{der}} & \cong Spin_4, \text{ for various } (\delta, r) \in '\mathcal{C}_\omega(W) \text{ (for } W \text{ of type } C \text{ or } D, \omega \in \Omega_W) \\
L_{\text{der}} & \cong Spin_4 \times SL_2(C)^r, \text{ for various } (\delta, r) \in '\mathcal{C}_\omega(W) \text{ (for } W \text{ of type } C \text{ or } D, \\
& \omega \in \Omega_W).
\end{align*}
\]

Then \( L \mapsto (\delta, r) \) defines a bijection \( \mathcal{G}_\omega(G) \overset{\sim}{\rightarrow} '\mathcal{C}_\omega(W) \). Composing this with the inverses of the bijections \( '\mathcal{C}_\omega(W) \overset{\sim}{\rightarrow} '\mathcal{C}_\omega(W) \) (3.10, 3.11) and \( \mathcal{C}_\omega(W) \overset{\sim}{\rightarrow} '\mathcal{C}_\omega(W) \) (3.8, 3.9) we obtain the bijection 6.6(a) in our case.

This completes the definition of the bijection 6.6(a) in all cases. It can be verified that this associates to \( L \in \mathcal{G}_\omega(G) \) the same \( W_J \) as that defined in 6.5. Hence the map 6.5(a) is well defined and it is a bijection (the same as 6.6(a)).
6.8. Let \( \omega \in \Omega W, \beta \in \mathfrak{s}_{\omega}(G) \) with corresponding \( L \in \mathcal{G}_{\omega}(G) \). Under 6.5(a), \( L \) corresponds to \( W_J \in \mathcal{C}_{\omega}(W) \). Let \((W, \mathcal{L}, S, \mathcal{W}) = (W_J, \mathcal{L}_J, S_J, \mathcal{W}_J) \) be as in §5.

Let \((L, c_1, \mathcal{L}_1) \in \mathcal{G}'(G)\) be a triple corresponding to \( L \) as in 6.1. Let \( c_{\text{max}} \) be the unipotent class of \( G \) induced by \( c_1 \); let \( c_{\text{min}} \) be the unipotent class of \( G \) that contains \( c_1 \).

For any \( \mathfrak{c} \in \mathfrak{u}(G) \) we denote by \( b_{\mathfrak{c}} \) the dimension of the variety of Borel subgroups of \( G \) that contain a fixed element of \( \mathfrak{c} \). A case by case verification gives the following two results.

**Theorem 6.9.** We have

(a) \( b_{c_{\text{max}}} = a^{[\omega W_J]} \) (notation of 2.1).

(b) \( b_{c_{\text{min}}} - b_{c_{\text{max}}} = \nu(W, \mathcal{L}) \) (notation of 4.3).

**Theorem 6.10.** There exists a group isomorphism \( \mathfrak{W}_\beta \cong \mathcal{W} \) well defined up to composition with an inner automorphism of \( \mathcal{W} \) given by the action of an element in \( \Omega W \) (see 3.1). It carries the set of simple reflections of \( \mathfrak{W}_\beta \) into the image of \( S \) under \( \mathfrak{W} \rightarrow \mathcal{W} \).

**Conjecture 6.11.** Assume that \( W \neq \{1\} \). Let \((\mathfrak{c}, \mathcal{L}), (\mathfrak{c}', \mathcal{L}') \) be in \( \beta \) and let \( E, E' \) be in \( \text{Irr}(\mathfrak{W}_\beta) = \text{Irr}(\mathcal{W}) \) (this equality follows from 6.10). Assume that the generalized Springer correspondence [L84] associates \( E \) to \((\mathfrak{c}, \mathcal{L})\) and \( E' \) to \((\mathfrak{c}', \mathcal{L}')\). Then

(a) \( b_{\mathfrak{c}} - b_{c_{\text{max}}} = c_{E} \) where \( c_{E} \) is defined as in 4.1 in terms of \((W, \mathcal{L})\);

(b) we have \( E \sim E' \) if and only if \( \mathfrak{c} = \mathfrak{c}' \).

This holds in the case where \( G \) is of exceptional type. (In the case where \( G \) is of type \( E_8, F_4 \) or \( G_2 \) this follows from [L20].) This can be also verified in the cases where \((W, \mathcal{L})\) is as in the examples in 4.2.

6.12. For any \( i = (\mathfrak{c}, \mathcal{L}), i' = (\mathfrak{c}', \mathcal{L}') \) in \( \mathfrak{b} \) let \( \Omega_{i,i'} \in \mathbb{Q}(q) \) be as in [L86, 24.7] and let \( \Pi_{i',i} \in \mathbb{Q}(q) \) be as in [L86, 24.8]. Here \( q \) is an indeterminate. Let \( E, E' \) in \( \text{Irr}(\mathcal{W}) \) be corresponding to \( i, i' \) (respectively) under the generalized Springer correspondence. From the definitions we have \( \Omega_{i,i'} = f(q)\Omega_{E',E} \) where \( f(q) \in \mathbb{Q}(q) - \{0\} \) is independent of \( i, i' \). Assuming that 6.11(a) holds we see from 5.7 and [L86, 24.8] that

(a) \( \Pi_{i',i} = P_{E',E} \)

for any \( i, i' \) as above. In particular this holds in the case where \( G \) is of exceptional type. Note that \( \Pi_{i',i} \) measures the stalks of the intersection cohomology sheaf on the closure of \( \mathfrak{c} \) with coefficients in \( \mathcal{L} \) while \( P_{E',E} \) is determined purely in terms of \( W \).

7. Cells in the weighted affine Weyl group \( W_J \)

7.1. Let

(a) \( \zeta : \text{Cell}(W, ||) \cong \text{un}(G) \)

be the bijection defined in [L89].
7.2. We now fix \( \omega \in \Omega_W \) and \( \beta \in \mathfrak{l}_{\omega}(G) \) with corresponding \( L \in G_{\omega}(G) \). Under 6.5(a), \( L \) corresponds to \( J \subseteq S \) such that \( W_J \in C_{\omega}(W) \). Let \( W_J, S_J, L_J \) be as in \( \S 5 \). We assume that \( \sharp(S_J) \geq 2 \). Let \( \mathfrak{u}_\beta(G) \) be the set of all \( c \in \mathfrak{u}(G) \) such that \( (c, L) \in \beta \) for some \( L \). We define a map

(a) \( \mathfrak{A} : \text{Cell}(W_J, L_J) \to \mathfrak{u}_\beta(G) \)

assuming a conjecture in [L02, \S 25]. Let \( c \in \text{Cell}(W_J, L_J) \). Let \( E_0 \) be the unique irreducible special representation of \( W_J \) such that \( z(E_0) = r(op) \) (notation of 2.1 with \( \mathfrak{W} \) replaced by \( W_J \)) and let \( c_0 \in \text{Cell}(W_J, ||) \) be such that \( E_0 \) belongs to \( c_0 \). According to [L03, Conj.25.3] there is a well defined \( \tilde{c} \in \text{Cell}(W, ||) \) which contains \( yx \) for any \( y \in c_0, x \in c' \) (the product \( yx \) is taken in \( W \)). (We use the fact that, by results of [L84a], \( W_J \) satisfies the assumptions of [L03, 25.2].) We set \( \mathfrak{A}(c) = \zeta(\tilde{c}) \), with \( \zeta \) as in 7.1(a).

Conjecture 7.3. \( \mathfrak{A} \) is injective with image equal to \( \mathfrak{u}_\beta(G) \). Hence \( \mathfrak{A} \) defines a bijection \( \text{Cell}(W_J, L_J) \sim \to \mathfrak{u}_\beta(G) \).

This is a generalization of 7.1(a). Note that [L03, 25.2] holds if \( J = \emptyset \) in which case it states that any \( c \in \text{Cell}(W_J, L_J) \) is contained in a two-sided cell of \( (W, ||) \); this can be deduced from [L03, 10.14]. Using this one can define \( \mathfrak{A} \) unconditionally when \( W \) is of exceptional type and verify the conjecture in that case.

7.4. Let \( a : W_J \to \mathbb{N} \) be the \( a \)-function (see [L03]) of the weighted affine Weyl group \( (W_J, L_J) \). Let \( c \in \text{Cell}(W_J, L_J) \) and let \( c = \mathfrak{A}(c) \in \mathfrak{u}_\beta(G) \). We expect that

(a) for any \( w \in c \) we have \( a(w) = b_c - b_{\kappa_{\text{max}}} \).

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