Abstract. We derive a free boson representation of the Yangian double $DY_k(sl_N)$ with arbitrary level $k$ using the observation that there is a correspondence between the $q$-affine algebra and Yangian double associated with the same Cartan matrix. Vertex operator and screening currents are not obtained in the same way.
1. Introduction

$q$-algebra and Yangian were proposed by Drinfeld as generalizations of classical Lie algebras with nontrivial Hopf algebra structures [7, 8, 10]. Following the Faddeev-Reshetikhin-Takhtajan formalism [11], both kinds of algebras can be considered as associative algebras defined through the Yang-Baxter relation (i.e. RLL-relations) with the structure constants determined by the solutions of the quantum Yang-Baxter equation (QYBE). $q$-affine algebra [10] and Yangian double [17, 18, 19] with center are respectively affine extensions of $q$-algebra and Yangian. $q$-affine algebra corresponds to the trigonometric solution of QYBE and Yangian corresponds to the rational one—if one considers the Reshetikhin-Semenov-Tian-Shansky realization [26] which is the affine analog of Faddeev-Reshetikhin-Takhtajan formalism—and they both were proved to have important applications in certain physical problems, especially in describing the dynamical symmetries and calculating the correlation functions and/or form factors of some two-dimensional exactly solvable lattice statistical model and $(1+1)$-dimensional completely integrable quantum field theories [4, 23, 28]. In such applications, the infinite-dimensional representations of $q$-affine algebra and Yangian double are often required, especially the representations with higher ($k > 1$) level.

In practice, realization of complicated algebra in terms of a relatively simple one is proved to be quite effective and useful. In this aspect, the Heisenberg algebra (or free boson) representation has become a common method for obtaining representations of ($q$-)affine algebras. For examples, the free boson representations of $U_q(\widehat{sl_2})$ with an arbitrary level have been obtained in Refs. [27, 24, 23, 20, 1]. Free boson representation of $U_q(\widehat{sl_N})$ with level 1 was constructed in [14]. Free boson representations of $U_q(\widehat{sl_3})$ and $U_q(\widehat{sl_N})$ with arbitrary level were constructed in [2] and [3] respectively. For the Yangian doubles, the free field representation of $DY_h(\widehat{sl_2})$ with level $k$ was constructed in [22]. The level 1 free boson representation of $DY_h(\widehat{sl_N})$ was given in [15]. But free field representations for Yangian doubles of higher rank and with arbitrary level are still unknown.

In this paper we shall address the problem of free field representation of the Yangian double $DY_h(\widehat{sl_N})$ with arbitrary level $k$. For this purpose we largely rely on the result of [3] on free field representation of $U_q(\widehat{sl_N})$ with arbitrary level and observe that there is a simple correspondence between the $q$-affine algebra $U_q(\widehat{sl_N})$ and the Yangian double $DY_h(\widehat{sl_N})$. This correspondence makes our derivation of free field representation for $DY_h(\widehat{sl_N})$ greatly simplified. However, we have been unable to obtain the vertex operators and screening currents for $DY_h(\widehat{sl_N})$ following the same spirit.

2. $q$-AFFINE AND YANGIAN DOUBLE CORRESPONDENCE

In this section we first establish the correspondence between the $q$-affine algebra $U_q(\widehat{sl_N})$ and the Yangian double $DY_h(\widehat{sl_N})$. For this and the subsequent purposes we use the Drinfeld current realizations for both algebras. Other realizations such as the Reshetikhin-Semenov-Tian-Shansky realization of them can be found in [26] and [15] (which are actually the
quotient algebras of $U_q(gl_N)$ and $DY_2(gl_N)$ respectively with respect to a Heisenberg subalgebra), and the equivalence (algebra isomorphism) to Drinfeld realizations were given in [8] and [15] respectively. We remark that the Ding-Frenkel isomorphism only provides an algebra isomorphism but not a Hopf algebra isomorphism for $q$-affine algebras at least in the $gl_2$ case [8]. Whether this is also the case for Yangian doubles is an interesting open problem.

2.1. Drinfeld currents realization of $U_q(sl_N)$. $U_q(sl_N)$ is an associative algebra generated by the Drinfeld generators $E_n^{\pm,i} (n \in \mathbb{Z})$, $H_n^i (n \in \mathbb{Z})$ ($i = 1, 2, ..., N - 1$) and the center $\gamma$. Let

$$K_i = \exp \left( (q - q^{-1}) \frac{1}{2} H_i^0 \right),$$

then we can write the Drinfeld currents in the form of formal power series of the complex parameter $z$ with coefficients given by the above generators,

$$H^i(z) = \sum_{n \in \mathbb{Z}} H_n^i z^{-n-1}, \quad E_{n}^{\pm,i}(z) = \sum_{n \in \mathbb{Z}} E_{n}^{\pm,i} z^{-n-1},$$

$$\psi_\pm^i(z) = \sum_{n \in \mathbb{Z}} \psi_\pm^i z^n = K_i^{\pm 1} \exp \left( \pm (q - q^{-1}) \sum_{n>0} H_n^i z^{-n} \right).$$

The generating relations for $U_q(sl_N)$ in terms of these currents can be written as follows [3],

$$[\psi_\pm^i(z), \psi_\pm^j(w)] = 0,$$  \hspace{1cm} (1)

$$(z - q^{a_{ij}} \gamma^{-1} w)(z - q^{-a_{ij}} \gamma w)\psi_+^i(z)\psi_+^j(w) = (z - q^{a_{ij}} \gamma w)(z - q^{-a_{ij}} \gamma^{-1} w)\psi_+^i(z)\psi_+^j(z),$$ \hspace{1cm} (2)

$$(z - q^{\pm e_{ij}} \gamma^{\mp} \frac{1}{2} w)\psi_+^i(z)E_{\pm}^{i,j}(w) = (q^{\pm e_{ij}} \gamma z - \gamma^{\mp} w)E_{\pm}^{i,j}(w)\psi_+^i(z),$$ \hspace{1cm} (3)

$$(z - q^{\pm e_{ij}} \gamma^{\mp} \frac{1}{2} w)E_{\pm}^{i,j}(z)\psi_+^i(w) = (q^{\pm e_{ij}} \gamma z - \gamma^{\mp} w)\psi_+^i(w)E_{\pm}^{i,j}(z),$$ \hspace{1cm} (4)

$$[E_{\pm}^{i,j}(z), E_{-}^{i,j}(w)] = \frac{\delta_{ij}}{(q - q^{-1}) z w} \left( \delta(z^{-1} w \gamma)\psi_+^i(\gamma^{\frac{1}{2}} w) - \delta(z^{-1} w \gamma^{-1})\psi_+^i(\gamma^{-\frac{1}{2}} w) \right),$$ \hspace{1cm} (5)

$$(z - q^{\pm e_{ij}} \gamma)E_{\pm}^{i,j}(z)E_{\pm}^{i,j}(w) = (q^{\pm e_{ij}} \gamma z - w)E_{\pm}^{i,j}(w)E_{\pm}^{i,j}(z),$$ \hspace{1cm} (6)

$$E_{\pm}^{i,j}(z)E_{\pm}^{i,j}(w) = E_{\pm}^{i,j}(w)E_{\pm}^{i,j}(z) \text{ for } a_{ij} = 0,$$ \hspace{1cm} (7)

$$E_{\pm}^{i,j}(z)E_{\pm}^{i,j}(z_1)E_{\pm}^{i,j}(z_2)E_{\pm}^{i,j}(w) - (q + q^{-1})E_{\pm}^{i,j}(z_1)E_{\pm}^{i,j}(w)E_{\pm}^{i,j}(z_2)$$

$$+ E_{\pm}^{i,j}(w)E_{\pm}^{i,j}(z_1)E_{\pm}^{i,j}(z_2) + \text{(replacement: } z_1 \leftrightarrow z_2) = 0 \text{ for } a_{ij} = -1,$$ \hspace{1cm} (8)

where $a_{ij}$ are elements of the Cartan matrix of the type $A_{N-1}$ and

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n.$$
In this paper we only consider $q$-affine algebra and Yangian double as associative algebras and do not care about the Hopf algebra aspect.

2.2. The Yangian double $DY_h(sl_N)$. As an associative algebra, the Yangian double $DY_h(sl_N)$ is generated by the Drinfeld generators $\{h_{il}, e_{il}^\pm | i = 1, 2, ..., N-1; \ l \in \mathbb{Z}_{\geq 0}\}$ and the center $c$. In terms of the formal power series (Drinfeld currents)

\[ H_i^+(u) = 1 + \hbar \sum_{l \geq 0} h_{il} u^{-l-1}, \quad H_i^-(u) = 1 - \hbar \sum_{l < 0} h_{il} u^{-l-1}, \]

\[ E_i^\pm(u) = \sum_{l \in \mathbb{Z}} e_i^\pm u^{-l-1} \]

we can write the generating relations for $DY_h(sl_N)$ as follows [15],

\[
\begin{align*}
[H_i^+(u), H_j^+(v)] &= 0, \\
(u_+ - v_+ + B_{ij} h)(u_+ - v_+ - B_{ij} h)H_i^+(u)H_j^+(v) &= (u_+ - v_+ - B_{ij} h)(u_+ - v_+ - B_{ij} h)H_i^+(u)H_j^+(v), \\
(u_+ - v_+ \pm B_{ij} h)H_i^+(u)E_j^+(v) &= (u_+ - v_+ \pm B_{ij} h)E_j^+(v)H_i^+(u), \\
(u_+ - v_+ \pm B_{ij} h)E_j^+(v) &= (u_+ - v_+ \pm B_{ij} h)E_j^+(v), \\
(u - v_\mp B_{ij} h)E_i^+(u)E_j^+(v) &= (u - v_\mp B_{ij} h)E_i^+(u)E_j^+(v),
\end{align*}
\]

\[
\begin{align*}
[E_i^+(u), E_j^-(v)] &= \frac{1}{\hbar} \delta_{ij} (\delta(u_+ - v_+)H_i^+(v_+) - \delta(u_+ - v_-)H_i^-(v_-)), \\
E_i^+(u_1)E_i^+(u_2)E_j^+(v) &= 2E_i^+(u_1)E_j^+(v)E_i^+(u_2) \\
+ E_j^+(v)E_i^+(u_1)E_i^+(u_2) + \text{(replacement: } u_1 \leftrightarrow u_2) = 0 \text{ for } |i - j| = 1, \\
E_i^+(u)E_j^+(v) &= E_j^+(v)E_i^+(u) \text{ for } |i - j| > 1,
\end{align*}
\]

where

\[ u_{\pm} = u \pm \frac{1}{4} \hbar c \]

and

\[ B_{ij} = \frac{1}{2} a_{ij}. \]

2.3. $q$-affine-Yangian double correspondence. Our central goal is to establish a free boson representation of the Yangian double $DY_h(sl_N)$. For this we would like to use the known results [3] for the $q$-affine algebra $U_q(\widehat{sl_N})$ by establishing a correspondence principle between these two algebras. Such a correspondence principle has been expected for some time and was “quite mysterious” as stated in Ref. [15].

For the present authors, however, such a correspondence is rather obvious by making use of the Drinfeld current realizations for both $U_q(\widehat{sl_N})$ and $DY_h(sl_N)$. For other realizations no such an obvious observation could be obtained. We give the following
Observation 1. \((q\text{-affine-Yangian double correspondence})\). The following gives a simple correspondence between \(U_q(\hat{sl}_N)\) and \(DY_\hbar(sl_N)\) as associative algebras

\[
\begin{align*}
q &\to e^{\frac{\hbar}{2}}, \quad \gamma \to e^{\frac{u}{2}}, \\
z &\to e^{u}, \\
\psi^i_\pm(z) &\to H^\pm_i(u), \\
z E^\pm,i(z) &\to E^\pm_i(u)
\end{align*}
\]

in the limit \(\hbar \to 0, u \to 0\) up to the linear approximation in \(\hbar\) and \(u\).

We remark that the above observation only gives a rule for obtaining equations (9-16) from (1-8) and does not imply any more fundamental Hopf algebraic or algebraic relations.

3. Free boson representation of \(DY_\hbar(sl_N)\) with arbitrary level

In this section we shall consider our central problem—the establishment of a free boson representation of \(DY_\hbar(sl_N)\) with arbitrary level. For \(N = 2\) this problem has already been solved in Ref. [22]. For generic \(N\), the desired expressions are rather complicated and our construction depend largely on the observation 1 and the result of [3]. One crucial difference of our construction from the one in [3] is that, in our case, the Yangian double \(DY_\hbar(sl_N)\) should be realized through ordinary Heisenberg algebras (i.e. without deformation), whereas in Ref. [3], \(U_q(sl_N)\) was realized via a set of \(q\)-deformed Heisenberg algebras. Therefore our observation 1 has to be used in somewhat a nontrivial way (for example, the vertex operators and screening currents cannot be obtained using our correspondence principles).

3.1. Free bosons and Fock space. We introduce the following set of \(N^2 - 1\) Heisenberg algebras with generators \(a^i_n\) (1 \(\leq i \leq N - 1\)), \(b^i_j\) and \(c^i_{ij}\) (1 \(\leq i < j \leq N\)) with \(n \in \mathbb{Z} - \{0\}\) and \(p_{a^i}, q_{a^i}\) (1 \(\leq i \leq N - 1\)), \(p_{b^i}, q_{b^i}, p_{c^i}, q_{c^i}\) (1 \(\leq i < j \leq N\)),

\[
\begin{align*}
[a^i_n, a^j_m] &= (k + g)B_{ij}n\delta_{n+m,0}, \\
[b^i_j, b^k_l] &= -\eta^{i,j}B_{ij}\delta_{n+m,0}, \\
c^i_{j,m} &= n\delta^{i,j} \delta^{i,j} \delta_{n+m,0}. \\
p_{a^i}[0] &= p_{a^i} = 0, \\
q_{a^i}[0] &= q_{a^i} = 0.
\end{align*}
\]

where \(g = N\) is the dual Coxeter number for the Cartan matrix of type \(A_N - 1\).

The Fock space corresponding to the above Heisenberg algebras can be specified as follows. Let \(|0\rangle\) be the vacuum state defined by

\[
\begin{align*}
a^i_n|0\rangle &= b^i_n|0\rangle = c^i_n|0\rangle = 0 \quad (n > 0), \\
p_{a^i}|0\rangle &= p_{b^i}|0\rangle = p_{c^i}|0\rangle = 0.
\end{align*}
\]

Define
\[ |l_a, l_b, l_c\rangle = \exp \left( \sum_{i,j=1}^{N-1} \sum_{n>0} \frac{1}{k + g} (B^{-1})^{ij} a_i^+ a_j - \sum_{1 \leq i<j \leq N} l_{bij} q_{bij} + \sum_{1 \leq i<j \leq N} l_{cij} q_{cij} \right) |0\rangle, \]

it can be shown that the following equations hold,

\[ a_i^+ |l_a, l_b, l_c\rangle = b_i^+ |l_a, l_b, l_c\rangle = c_i^+ |l_a, l_b, l_c\rangle = 0 \quad (n > 0), \]

\[ p_a^+ |l_a, l_b, l_c\rangle = l_a^+ |l_a, l_b, l_c\rangle, \]

\[ p_{bij}^+ |l_a, l_b, l_c\rangle = l_{bij}^+ |l_a, l_b, l_c\rangle, \]

\[ p_{cij}^+ |l_a, l_b, l_c\rangle = l_{cij}^+ |l_a, l_b, l_c\rangle. \]

The Fock space \( \mathcal{F}(l_a, l_b, l_c) \) is then generated by the actions of the negative modes of \( a^i, b^{ij}, c^{ij} \). We shall see later that this Fock space actually forms a (Wakimoto-like \([30, 12]\)) module for the Yangian double \( \text{DY}_k(sl_N) \) with level \( k \).

For \( X = a^i, b^{ij}, c^{ij} \), let us now define

\[
X(u; A, B) = \sum_{n>0} \frac{X_n}{n} (u + Ah)^n - \sum_{n>0} \frac{X_n}{n} (u + Bh)^{-n} + \log(u + Bh) p_X + q_X, \\
X_+(u; B) = -\sum_{n>0} \frac{X_n}{n} (u + Bh)^{-n} + \log(u + Bh) p_X, \\
X_-(u; A) = \sum_{n>0} \frac{X_n}{n} (u + Ah)^n + q_X, \\
X(u; A) = X(u; A, A), \quad X(u) = X(u, 0).
\]

Then we have

\[ : \exp(X(u; A, B)) := \exp(X_-(u; A)) \exp(X_+(u; B)). \]

Following the standard quantum field theory we have

\[
X^\alpha(u; A, B) X^\beta(v; C, D) = \langle X^\alpha(u; A, B) X^\beta(v; C, D) \rangle + \langle X^\alpha(u; A, B) X^\beta(v; C, D) \rangle, \tag{17}
\]

where \(^1\)

\[
\langle a^i(u; A, B) a^j(v; C, D) \rangle = (k + g) B_{ij} \log(u - v + (B - C)h), \\
\langle b^{ij}(u; A, B) b^{ij'}(v; C, D) \rangle = -\delta^{ij} \delta^{ij'} \log(u - v + (B - C)h), \\
\langle c^{ij}(u; A, B) c^{ij'}(v; C, D) \rangle = \delta^{ij} \delta^{ij'} \log(u - v + (B - C)h),
\]

and all other contractions vanish. From eq.\(^{(17)}\) it is easy to calculate that

\(^1\)Here and below, all OPE relations should be understood to hold in the analytic continuation sense.
Moreover, we have the following relations,

\[ : \exp (X^\alpha(u; A, B)) : = \exp (X^\beta(v; C, D)) : \]

\[
= \exp \left( (X^\alpha(u; A, B)X^\beta(v; C, D)) \right) \exp (X^\alpha(u; A, B)) \exp (X^\beta(v; C, D)) : . \quad (18)
\]

It should be noticed that equations (17) and (18) hold unchanged if we change everywhere \( X^\alpha(u; A, B) \rightarrow X^\alpha(u; B) \) and \( X^\beta(v; C, D) \rightarrow X^\beta(v; C) \).

For later use let us introduce some more definitions. For \( X = b^ij \), \( c^ij \), define

\[
\dot{X}_\pm(u) = \mp \left( X_\pm(u; -\frac{1}{2}) - X_\pm(u; \frac{1}{2}) \right).
\]

For the bosonic fields \( a^i(u; A, B) \), define

\[
\dot{a}_+^i(u) = a_+^i(u; 0) - a_+^i(u; k + g),
\]

\[
\dot{a}_-^i(u) = \frac{1}{k + g} \sum_{j,l=1}^{N-1} (B^{-1})^{ij} \left( a_-^i(u; B_{ij}) - a_-^i(u; -B_{ij}) \right). \quad (20)
\]

It is easy to obtain the following operator product expansion (OPE) relations,

\[
\exp (\dot{a}_+^i(u)) \exp (\dot{a}_-^i(v))
\]

\[
= \frac{(u-v+B_{ij}h)(u-v+(k+g+B_{ij})h)}{(u-v+B_{ij}h)(u-v+(k+g-B_{ij})h)} \exp (\dot{a}_-^i(v)) \exp (\dot{a}_+^i(u)), \quad (21)
\]

\[
\exp (\dot{b}_+^{ij}(u)) \exp (\dot{b}_-^{ij}(v))
\]

\[
= \frac{(u-v)^2}{(u-v-h)(u-v+h)} \delta^{ij} \exp (\dot{b}_-^{ij}(v)) \exp (\dot{b}_+^{ij}(u)),
\]

\[
\exp (\dot{c}_+^{ij}(u)) \exp (\dot{c}_-^{ij}(v))
\]

\[
= \frac{(u-v-h)(u-v+h)}{(u-v)^2} \delta^{ij} \exp (\dot{c}_-^{ij}(v)) \exp (\dot{c}_+^{ij}(u)).
\]

Moreover, we have the following relations,

\[
\exp (\dot{b}_+^{ij}(u)) : \exp (\dot{b}_-^{ij}(v)) : 
\]

\[
= \frac{(u-v+B_{ij}h)}{u-v+(k+g+B_{ij})h} \delta^{ij} \exp (\dot{b}_-^{ij}(v)) \exp (\dot{b}_+^{ij}(u)),
\]

\[
\exp (\dot{c}_+^{ij}(u)) : \exp (\dot{c}_-^{ij}(v)) : 
\]

\[
= \frac{(u-v+B_{ij}h)}{u-v+(k+g-B_{ij})h} \delta^{ij} \exp (\dot{c}_-^{ij}(v)) \exp (\dot{c}_+^{ij}(u)).
\]
To specify the correspondence of our notations and that of Ref. [3] for \( q \)-bosons, we give the second observation.

**Observation 2.** The expressions \( \hat{a}_i^\pm(u), \hat{b}_{ij}^\pm(u) \) and \( \hat{c}_{ij}^\pm(u) \) correspond to the fields \( a_i^\pm(q^\pm k \pm g \pm z), b_{ij}^\pm(z) \) and \( c_{ij}^\pm(z) \) of Ref. [3] respectively.

**Remark 1.** Notice that in Ref. [3], the explicit expressions for \( a_i^\pm(q^\pm k \pm g \pm z) \) are symmetric with respect to \( + \leftrightarrow - \), but this is not the case for \( \hat{a}_i^\pm(u) \). The partial reason for this difference is that, for \( q \)-affine algebras, the Drinfeld currents \( \psi_i^\pm(z) \) are defined in a symmetric way in \( H_i^\pm \), whilst for Yangian doubles the currents \( H_i^\pm \) are defined asymmetrically.

In the next subsection, we shall see that, despite the difference stated in Remark 1, the above observations are rather useful to guess the bosonic expressions for the Drinfeld currents of \( DY_\hbar(sl_N) \).

### 3.2. Free boson representation of \( DY_\hbar(sl_N) \) with level \( k \)

Let us define

\[
H_i^\pm(u) =: \exp \left\{ \sum_{l=1}^{i-1} \hat{b}_{l+1}^i(u \pm \frac{1}{2}(l+1) \hbar) - \sum_{l=1}^{i-1} \hat{b}_{l}^{i+1}(u \pm \frac{1}{2}(l+1) \hbar) \right. \\
+ \hat{a}_i^\pm(u \pm \frac{1}{4} k \hbar) \\
+ \sum_{l=i+1}^{N} \hat{b}_{l}^i(u \pm \frac{1}{2} k \hbar) - \sum_{l=i+2}^{N} \hat{b}_{l}^{i+1}(u \pm \frac{1}{2} k \hbar) \right\} , \tag{22}
\]

\[
E_i^\pm(u) = -\frac{1}{\hbar} \sum_{m=1}^{i-1} : \exp \left\{ (b + c)^m i(u + \frac{1}{2} (m-1) \hbar) \right. \\
\times \left[ \exp \left( \hat{b}^{m,i+1}_+(u + \frac{1}{2} (m-1) \hbar) - (b + c)^{m,i+1}_+(u + \frac{1}{2} m \hbar) \right) \\
- \exp \left( \hat{b}^{m,i+1}_-(u + \frac{1}{2} (m-1) \hbar) - (b + c)^{m,i+1}_-(u + \frac{1}{2} (m-2) \hbar) \right) \right] \\
\times \exp \left\{ \sum_{l=i+1}^{m-1} \left[ \hat{b}^{l,i+1}_+(u + \frac{1}{2} (l-1) \hbar) - \hat{b}^{l}_+(u + \frac{1}{2} l \hbar) \right] \right\} : , \tag{23}
\]
\[ E^+_i(u) = \frac{1}{\hbar} \left\{ \sum_{m=1}^{i-1} : \exp \left( (b+c)^{m,i+1}(u - \frac{1}{2}(k + m)\hbar) \right) \right\} \]
\[ \times \left[ \exp \left( -\hat{b}^m_i(u - \frac{1}{2}(k + m)\hbar) \right) - (b+c)^{m,i}(u - \frac{1}{2}(k + m - 1)\hbar) \right) + \hat{a}^+_i(u) + \sum_{l=i+1}^{N} \hat{b}^{(1)}_+(u - \frac{1}{2}(k + l)\hbar) - \sum_{l=i+2}^{N} \hat{b}^{(1),l}_+(u - \frac{1}{2}(k + l - 1)\hbar) \right] + : \exp \left( (b+c)^{i+1}(u - \frac{1}{2}(k + i)\hbar) \right)
\times \exp \left( \hat{a}^+_i(u) + \sum_{l=i+1}^{N} \hat{b}^{(1)}_+(u - \frac{1}{2}(k + l)\hbar) - \sum_{l=i+2}^{N} \hat{b}^{(1),l}_+(u - \frac{1}{2}(k + l - 1)\hbar) \right)
- \sum_{m=i+2}^{N} : \exp \left( (b+c)^{m,i+1}(u - \frac{1}{2}(k + m - 1)\hbar) \right) \times \left[ \exp \left( \hat{b}^{(1),m}_+(u + \frac{1}{2}(k + m - 1)\hbar) \right) - (b+c)^{i+1,m}(u + \frac{1}{2}(k + m)\hbar) \right)
- \exp \left( \hat{b}^{(1),m}_-(u + \frac{1}{2}(k + m - 1)\hbar) \right) \right]) \times \exp \left( \sum_{l=m}^{N} \left[ \hat{b}^{(1)}_+(u + \frac{1}{2}(k + l)\hbar) - \hat{b}^{(1),l}_+(u + \frac{1}{2}(k + l - 1)\hbar) \right] \right). \quad (24) \]

The following proposition is the main result of this paper:

**Proposition 1.** The fields \( H^+_i(u) \), \( E^+_i(u) \) defined in equations (22), (23) and (24) are well-defined on the Fock space \( \mathcal{F}(l_u, l_b, l_c) \) and satisfy equations (24) with \( c = k \) and

\[ E^+_i(u)E^+_j(v) \simeq E^+_j(v)E^+_i(u) \sim \text{reg.} \quad \text{for} \ B_{ij} = 0, \]
\[ (u - v \mp B_{ij}\hbar)E^+_i(u)E^+_j(v) \simeq (u - v \pm B_{ij}\hbar)E^+_j(v)E^+_i(u) \sim \text{reg.} \quad \text{for} \ B_{ij} \neq 0, \]
\[ E^+_i(u)E^+_j(v) - E^+_j(v)E^+_i(u) \sim \text{reg.} + \frac{1}{\hbar} \left( \delta(u_+ - v_+)H^+_i(v_+) - \delta(u_+ - v_-)H^+_i(v_-) \right), \]

where \( \text{reg.} \) means some regular expressions and \( \simeq \) and \( \sim \) imply “equals up to” such expressions.
Proof: The proposition follow by straightforward but tedious calculations. Actually the calculations are step by step analogous to that of Ref. [3] for q-affine case. So we omit all such calculations and only refer to [3] and remind the readers of our correspondence rules (Observations 1 and 2).

Remark 2. In proving Proposition 1, only the OPE relation (21) for \( \hat{a}_+ \) is used and the exact expressions for the fields \( \hat{a}_\pm \) are not important. Actually, there are infinite many choices for \( \hat{a}_\pm \) which satisfy the relation (21). For example, the following is another example which differs from the original definitions (19,20),

\[
\hat{a}_+^i(u) = \frac{1}{k + g} \sum_{j,i=1}^{N-1} (B^{-1})^{ij} a_+^j(u; \frac{k + g}{2} - B_{ij}) - a_+^j(u; \frac{k + g}{2} + B_{ij}),
\]

\[
\hat{a}_-^i(u) = a_-^i(u; -\frac{k + g}{2}) - a_+^i(u; \frac{k + g}{2}).
\]

However, no matter which choice we use, we cannot make the definition of \( \hat{a}_+^i(u) \) and \( \hat{a}_-^i(u) \) symmetric, i.e. no violation of Remark 1 could occur.

Remark 3. While \( N = 2 \), equations (22,24) become

\[
H^+(u) = \exp \left( \hat{b}_+(u + \frac{1}{4} \hbar) + \hat{b}_+(u + \frac{1}{2} \hbar) + \hat{a}_+(u - \frac{1}{4} \hbar) \right) : \hspace{1cm}
\]

\[
H^-(u) = \exp \left( \hat{b}_-(u - \frac{1}{4} \hbar) + \hat{b}_-(u - \frac{1}{2} \hbar) + \hat{a}_-(u + \frac{1}{4} \hbar) \right) : \hspace{1cm}
\]

\[
E^+(u) = -\frac{1}{\hbar} \left[ \exp \left( \hat{b}_+(u) - (b + c)(u + \frac{\hbar}{2}) \right) - \exp \left( \hat{b}_-(u) - (b + c)(u - \frac{\hbar}{2}) \right) \right]
\]

\[
E^-(u) = \frac{1}{\hbar} \left[ \exp \left( (b + c)(u + \frac{1}{2} \hbar) \right) \exp \left( \hat{a}_+(u) + \hat{b}_+(u + \frac{1}{2} \hbar) \right) - \exp \left( (b + c)(u - \frac{1}{2} \hbar) \right) \exp \left( \hat{a}_-(u) + \hat{b}_-(u - \frac{1}{2} \hbar) \right) \right]
\]

which is different from the result of Ref. [22] for \( DY_h(sl_2)_k \). The reason for this difference is that, first, the Yangian double \( DY_h(sl_2)_k \) of Ref. [22] is realized in an asymmetric way which differs from the symmetric one which we are using by a shift of parameter [19]; Second, as we have remarked in Remarks 1 and 3, for the same realization of Yangian double, there still exist infinite many choices for the bosonization formulas. Therefore the difference of our result (27,28) from that of Ref. [22] is reasonable.

4. Problems in obtaining Vertex operators and screening currents

After successfully obtained the bosonization formulas for the Yangian double \( DY_h(sl_N) \) using the correspondence rules (Observations 1 and 2), one naturally expects that the Vertex operators and screening currents for \( DY_h(sl_N) \) could also be obtained in the same way. In this section we briefly give why this is difficult.
Following Observations \([\underline{1}]\) and \([\underline{2}]\) and Ref. \([^3\underline{}\)]), we expect that the screening currents for \(DY_h(sl_N)\) might be written in the following form,

\[
S^i(u) = \exp \left( X^i[a](u) \right) : \bar{S}^i(u),
\]

where \(\bar{S}^i(u)\) is nothing but \(E_{N-i}^j(u)\) with the replacement \(\hat{b}_\pm^j \rightarrow \pm b^{N+1-j,N+1-1}, (b+c)^j \rightarrow (b+c)^N+1-j,N+1-1, \) and \(X^i[a](u)\) is some field depending only on \(a^i\) but not on \(b^j\) and \(c^j\).

In the \(q\)-affine case, \(X^i[a](u)\) is just the field \(- (\frac{1}{k+1} a^i) (z; \frac{k+g}{2})\). At present, we expect that \(\exp \left( X^i[a](u) \right) : \) have the following OPE relations with \(\exp \left( \hat{a}_+^i(u) \right) \) and \(\exp \left( \hat{a}_-^i(u) \right) \) (see equations (C.17), (C.18) of Ref. \([^3\underline{}\)]).

\[
\exp(\hat{a}_+^i(u)) : \exp(\hat{a}_-^i(v)) : = \frac{u - v + (\frac{k+g}{2} - B_{ij})h}{u - v + (\frac{k+g}{2} + B_{ij})h} : \exp(\hat{a}_-^i(v)) : \exp(\hat{a}_+^i(u)).
\]

\(\text{(31)}\)

\[
\exp(\hat{a}_-^i(u)) : \exp(\hat{a}_+^i(v)) : = \frac{u - v + (\frac{k+g}{2} + B_{ij})h}{u - v + (\frac{k+g}{2} - B_{ij})h} : \exp(\hat{a}_-^i(v)) : \exp(\hat{a}_+^i(u)).
\]

\(\text{(32)}\)

Notice that in the rational factors of equations (\[31\]) and (\[32\]), the \(\frac{k+g}{2}\) appear with the same sign in both the numerators and the denominators. This fact makes it difficult to obtain an explicit expression for \(X^i[a](u)\). For examples, if we adopt the definitions \([\underline{14}]\) and \([\underline{15}]\) of \(\hat{a}_\pm^i(u)\), then the “positive frequency part” of \(X^i[a](u)\) can be easily seen to be equal to \(\hat{a}_-(u + \frac{k+g}{2})\), but the “negative frequency part could not be written in a simple form (and it is not known whether it is possible to write down such an expression). If we adopt the definitions \([\underline{24}]\) and \([\underline{25}]\) instead of \([\underline{13}]\) and \([\underline{14}]\), then the negative frequency part of \(X^i[a](u)\) can be obtained easily but the positive frequency part is unknown. That is why we could not obtain a simple analogy of screening currents for Yangian double and \(q\)-affine algebras. Due to similar reasons the vertex operators for \(DY_h(sl_N)\) is also not obtained from that of \(U_q(sl_N)\).

5. Discussions

In this paper we established the free boson representation of the Yangian double \(DY_h(sl_N)\) with arbitrary level \(k\). Our construction is based on the crucial correspondence Observations \([\underline{1}]\) and \([\underline{2}]\). Such representations of the Yangian double \(DY_h(sl_N)\) are expected to be useful in calculating the correlation functions of various quantum integrable systems in \((1+1)\)-spacetime dimensions, e.g. the spin Calogero-Sutherland model \([\underline{28}]\), quantum nonlinear Schrodinger equation \([\underline{21}]\) and some field theoretic models such as Thirring model, Gross-Neveu model with \(U(N)\) gauge symmetries etc. Our representation of \(DY_h(sl_N)\) may also be used to analyze the behavior of Yangian double at the critical level \(k = -g\), a very fascinating area of great interest of study \([\underline{13}]\).
Besides what have been solved in this paper, the unsolved problem of the construction of vertex operators and screening currents are also of great interest. Especially if we know these quantities we could have been able to calculate the cohomology of the action of our bosonization formulas on the Fock spaces $\mathcal{F}(l_a, l_b, l_c)$. We hope these problems could be solved in the future.
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