The radial defocusing energy-supercritical cubic nonlinear wave equation in $\mathbb{R}^{1+5}$

Aynur Bulut

Institute for Advanced Study, Einstein Dr., Princeton, NJ 08540, USA

E-mail: abulut@math.ias.edu

Received 24 June 2013, revised 26 June 2013
Accepted for publication 12 March 2014
Published 22 July 2014

Recommended by R de la Llave

Abstract
In this work, we establish a frequency localized version of the classical Morawetz inequality adapted to almost periodic solutions of the defocusing cubic nonlinear wave equation in dimension $d = 5$. As an application, we conclude that radial solutions to this equation that remain bounded in the critical homogeneous Sobolev space exist globally in time and scatter.

Keywords: global well-posedness, radial nonlinear wave equation, energy-supercritical, frequency localized Morawetz inequality

Mathematics Subject Classification: 35L71, 35B44, 35L15, 35P25

1. Introduction

The goal of this work is to establish a frequency-localized version of the Morawetz inequality adapted to a class of almost periodic solutions of the defocusing cubic nonlinear wave equation in dimension $d = 5$.

\[ (NLW) \begin{cases} u_{tt} - \Delta u + |u|^2 u = 0, \\ (u(0), u_t(0)) = (u_0, u_1) \in H^s_x(\mathbb{R}^5) \times H^{s-1}_x(\mathbb{R}^5), \end{cases} \]

where the critical regularity is given by $s_c = \frac{3}{2}$, $u$ maps $I \times \mathbb{R}$ to $\mathbb{R}$ and $0 \in I \subset \mathbb{R}$ is a time interval. Note that if the domain is taken as $\mathbb{R}^d$, $d \geq 1$, the critical regularity becomes $\frac{d}{2} - 1$.

We will use the following notion of solution:

Definition 1.1. A function $u : I \times \mathbb{R}^5 \to \mathbb{R}$ with $0 \in I \subset \mathbb{R}$ is a solution to (NLW) if $(u, u_t)$ belongs to $C(I; H^{3/2}_x \times H^{1/2}_x) \cap L^6_{t,x}(K \times \mathbb{R}^5)$ for every $K \subset I$ compact and $u$ satisfies the Duhamel formula

\[ u(t) = \cos(t|\nabla|)u_0 + \frac{\sin(t|\nabla|)}{|\nabla|}u_1 + \int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|}F(u(t')) \, dt' \]

for every $t \in I$. 

0951-7715/14/081859+19$33.00 © 2014 IOP Publishing Ltd & London Mathematical Society Printed in the UK
The class of almost periodic solutions has recently arisen as a fundamental object in the study of global well-posedness and asymptotic properties of solutions for nonlinear Schrödinger and wave equations. We recall the definition of this class.

**Definition 1.2.** A solution $u$ to (NLW) with time interval $I$ is said to be almost periodic modulo symmetries if $(u, u_t) \in L^\infty(I; \dot{H}^{3/2}_x \times \dot{H}^{1/2}_x)$ and there exist functions $N : I \to \mathbb{R}^+, x : I \to \mathbb{R}^3$ and $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \in I$ and $\eta > 0$,

$$\int_{|x-x(t)| \geq C(\eta)/N(t)} ||\nabla||^{3/2} u(t, x) ||^2 + ||\nabla||^{1/2} u_t(t, x) ||^2 \, dx \leq \eta,$$

and

$$\int_{|\xi| \geq C(\eta)/N(t)} |\xi|^3 |\hat{u}(t, \xi)|^2 + |\xi| |\hat{u}_t(t, \xi)|^2 \, d\xi \leq \eta.$$ 

The main result which we establish in this work then takes the following form.

**Theorem 1.3 (Frequency localized Morawetz estimate).** If $u : I \times \mathbb{R}^3 \to \mathbb{R}$ is an almost periodic solution to (NLW) on $I^* = \cup J_k \subset \mathbb{R}$ with $N(t) = N_k > 1$ on each $J_k$ and $(u, u_t) \in L^\infty(I ; \dot{H}^{3/2}_x \times \dot{H}^{1/2}_x)$, then for any $\eta > 0$ there exists $N_0 = N_0(\eta) > 0$ such that for all $N \leq N_0$ one has

$$\int_{I_0} \int_{\mathbb{R}^3} \frac{|u| \geq N(t, x) |^4}{|x|} \, dx \, dt \leq \eta C(u) (N^{-1} + |I_0|)$$

on any compact interval $I_0 = \cup J_k$.

As an application of theorem 1.3, we give a short proof that a priori bounds in the critical homogeneous Sobolev space lead to global well-posedness and scattering for solutions to (NLW) in the radial case (the non-radial case has been studied for dimensions $d \geq 6$ in [2] and for dimension $d = 5$ in [3], using substantially different techniques).

To frame our discussion, we briefly recall the scaling properties of (NLW). If we define

$$u_\lambda(t, x) := \lambda u(\lambda t, \lambda x)$$

then the map $u \mapsto u_\lambda$ carries the set of solutions of (NLW) to itself. Moreover, this map preserves the $\dot{H}^{3/2}_x \times \dot{H}^{1/2}_x$ norm of the initial data, and therefore the space $\dot{H}^{3/2}_x \times \dot{H}^{1/2}_x$ is referred to as the critical space with respect to the scaling. We also recall that solutions to (NLW) conserve the energy,

$$E(u(t), u_t(t)) = \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u(t, x)|^2 + \frac{1}{2} |u_t(t, x)|^2 + \frac{1}{4} |u(t, x)|^4 \, dx,$$

which is finite for solutions to (NLW) on $\mathbb{R}^d$ when $s_c = 1$. In view of this, we call the $d$-dimensional version of (NLW) energy-supercritical when $s_c > 1$, that is $d \geq 6$.

In [2], global well-posedness for (NLW) on higher-dimensional domains $\mathbb{R}^d$, $d \geq 6$, is established by making use of the concentration compactness approach introduced by Kenig and Merle [11, 12], reducing the question to an analysis of three specific blow-up scenarios as in [20, 21]. The key part of this analysis is to show that in each of these scenarios solutions have finite energy, for which a major tool is the double Duhamel technique [6, 17, 19]. This is the source of the restriction to dimensions $d \geq 6$.

In the five-dimensional setting of this work, this analysis becomes much more subtle. In [3], which establishes the global-well-posedness result for the non-radial problem, a delicate spatial localization is invoked in order to recover a version of the double Duhamel technique. On the other hand, in the radial setting, a short and self-contained proof can be recovered by appealing to theorem 1.3.
Theorem 1.4. Suppose \( u : I \times \mathbb{R}^3 \to \mathbb{R} \) is a solution to (NLW) with radial initial data, maximal interval of existence \( I \subseteq \mathbb{R} \), and satisfying the a priori bound
\[
(u, u_t) \in L^\infty_t(I; H^{3/2}_x \times H^{1/2}_x).
\]
Then \( u \) is global and
\[
\|u\|_{L^5_t(L^{\infty})} \leq C
\]
for some constant \( C = C(\|u\|_{L^\infty_t(I; H^{3/2}_x \times H^{1/2}_x)}) \). Furthermore, \( u \) scatters both forward and backward in time.

As in [2, 3], our proof of theorem 1.4 is a proof by contradiction following the concentration compactness approach. Equipped with the frequency localized Morawetz estimate of energy in ruling out the blow-up scenarios.

In addition to the works [2, 3] described above treating the non-radial case, the global well-posedness result has been established in [13] in the case of dimension \( d = 3 \) with the energy-supercritical nonlinearity \( u^p u, p > 4 \) for radially symmetric initial data; see also [14]. Moreover, the problem has been treated for general (possibly non-radial) initial data in three dimensions with \( p \) even [20], and for the radial case in higher dimensions with a range of \( p \) dependent on the dimension [21]. In [21], the restriction on \( p \) corresponding to five spatial dimensions is \( \frac{4}{3} < p < 2 \), excluding the cubic case treated in this work.

We now outline our strategy for proving theorem 1.4. We first recall the following result due to Kenig and Merle [13], which shows that the failure of theorem 1.4 gives the existence of a minimal counterexample which belongs to the class of almost periodic solutions.

Theorem 1.5 (Theorem 1.5). Suppose that theorem 1.4 failed. Then there exists a radial solution \( u : I \times \mathbb{R}^3 \to \mathbb{R} \) to (NLW) with maximal interval of existence \( I \),
\[
(u, u_t) \in L^\infty_t(I; H^{3/2}_x \times H^{1/2}_x)
\]
such that \( u \) is a minimal blow-up solution in the following sense: for any solution \( v \) with maximal interval of existence \( J \) such that \( \|v\|_{L^5_t(J \times \mathbb{R}^3)} = \infty \), we have
\[
\sup_{t \in I} \|u(t), u_t(t)\|_{H^{3/2}_x \times H^{1/2}_x} \leq \sup_{t \in J} \|v(t), v_t(t)\|_{H^{3/2}_x \times H^{1/2}_x}.
\]
Moreover, \( u \) is almost periodic modulo symmetries.

We remark that the proof of theorem 1.5 appearing in [13] is presented in the setting of periodic modulo symmetries.

Theorem 1.6. Suppose that theorem 1.4 failed. Then there exists a radial solution \( u : I \times \mathbb{R}^3 \to \mathbb{R} \) to (NLW) with maximal interval of existence \( I \) such that \( u \) is almost periodic modulo symmetries, \( (u, u_t) \in L^\infty_t(I; H^{3/2}_x \times H^{1/2}_x) \), \( \|u\|_{L^5_t(I \times \mathbb{R}^3)} = \infty \), and there exists \( \delta > 0 \) and a family of disjoint intervals \( \{J_k\}_{k \geq 1} \) with \( I^+ = \bigcup J_k \),
\[
N(t) = \widetilde{N}_k \geq 1 \text{ for } t \in J_k, \quad \text{and} \quad |J_k| = \delta \widetilde{N}_k^{-1}.
\]
Moreover, either
\[
|I^+| < \infty \quad \text{or} \quad |I^+| = \infty.
\]
This theorem is proved by applying a rescaling argument to the function obtained in theorem 1.5 to find another almost periodic solution with $N(t) \geq 1$ for $t \in I^+$ (see theorem 1.3 in [12]). One then observes that the function $N(t)$ obeys $N(s) \sim u N(t)$ for $|s - t| \leq \delta N(t)^{-1}$ and $\delta$ suitably chosen, as a consequence of the scaling symmetry and local theory for (NLW). This property is proved in the NLS setting in [16, corollary 3.6] (see also [25]); however, the arguments apply equally to (NLW). After a suitable modification of $N(t)$ and $C(t)$, the desired result is obtained.

In theorem 1.6 we divide the solutions of (NLW) into two classes depending on the control granted by the frequency localized Morawetz estimate, theorem 1.3. This is inspired by recent works in the mass and energy critical NLS settings [7, 26]. In the present context, this corresponds to distinguishing the cases $|I^+| < \infty$ and $|I^+| = \infty$; we also note that this distinction is also present in [12].

We next give a quick remark concerning the decay of norms of the Littlewood–Paley projections of $u$.

**Remark 1.7.** Suppose that $u$ is as in theorem 1.6. The property $\inf_{t \in I^+} N(t) = \inf_k \tilde{N}_k \geq 1$ along with the definition of almost periodicity implies

$$\lim_{N \to 0} [\|u_N\|_{L^\infty ([t^*, T^*]; H^2_x)} + \|\partial_t u_N\|_{L^\infty ([t^*, T^*]; H^1_x)}] = 0.$$ 

The proof of theorem 1.4 is therefore reduced to the task of showing that solutions satisfying the properties given in theorem 1.6 cannot occur. This is accomplished in section 5.

We now conclude this section by giving an outline of the rest of the paper. In section 2, we recall some preliminaries and establish our notation. Section 3 is then devoted to the proof of a frequency localized version of the Strichartz inequality which will be essential to obtain the frequency localized Morawetz estimate. This estimate is then proved in section 4. Section 5 is then devoted to the proof of theorem 1.4.

**2. Preliminaries**

In this section, we introduce the notation and some standard estimates that we use throughout the paper. We write $L^q_t L^r_x$ to indicate the space–time norm

$$\|u\|_{L^q_t L^r_x} = \left( \int_\mathbb{R} \|u(t)\|_{L^r_x}^q \, dt \right)^{1/r}$$

with the standard convention when $q$ or $r$ is equal to infinity. If $q = r$, we shorten the notation and write $L^q_x$.

We write $X \lesssim Y$ to mean that there exists a constant $C > 0$ such that $X \leq CY$, while $X \preceq Y$ indicates that the constant $C = C(u)$ may depend on $u$. We use the symbol $\nabla$ for the derivative operator in only the space variables.

Throughout the exposition, we define the Fourier transform on $\mathbb{R}^5$ by

$$\hat{f}(\xi) = (2\pi)^{-5/2} \int_{\mathbb{R}^5} e^{-ix\cdot\xi} f(x) \, dx.$$ 

We also denote the homogeneous Sobolev spaces by $H^s_x(\mathbb{R}^5)$, $s \in \mathbb{R}$, equipped with the norm $\|f\|_{H^s_x} = \|\nabla^s f\|_{L^2_x}$ where the fractional differentiation operator is given by $\nabla^s f(\xi) = |\xi|^s \hat{f}(\xi)$.

For $s \geq 0$, we say that a pair of exponents $(q, r)$ is $H^s_x$-wave admissible if $q, r \geq 2$, $r < \infty$ satisfy

$$\frac{1}{q} + \frac{2}{r} \leq 1 \quad \text{and} \quad \frac{1}{q} + \frac{5}{r} = \frac{5}{2} - s.$$
We then define the following Strichartz norms. For each \( I \subset \mathbb{R} \) and \( s \geq 0 \), we set
\[
\|u\|_{S_s(I)} = \sup_{(q,r) \text{ wave admissible}} \|u\|_{L^q_t L^r_x(I \times \mathbb{R}^d)},
\]
\[
\|u\|_{N_s(I)} = \inf_{(q,r) \text{ wave admissible}} \|u\|_{L^q_t L^r_x(I \times \mathbb{R}^d)}.
\]

Suppose \( u : I \times \mathbb{R}^5 \rightarrow \mathbb{R} \) with time interval \( 0 \in I \subset \mathbb{R} \) is a solution to the nonlinear wave equation
\[
\begin{cases}
  u_{tt} - \Delta u + F = 0 \\
  (u, u_t)|_{t=0} = (u_0, u_1) \in \dot{H}^s(\mathbb{R}^5) \times \dot{H}^{s-1}(\mathbb{R}^5), \quad \mu \in \mathbb{R}.
\end{cases}
\]

Then for all \( s, \tilde{s}, \mu \in \mathbb{R} \) we have the inhomogeneous Strichartz estimates \([8, 10],\)
\[
\|\nabla^l u\|_{S_{s-l}(I)} + \|\nabla^{l-1} u_t\|_{S_{s-l}(I)} \lesssim \|(u_0, u_1)\|_{\dot{H}^s \times \dot{H}^{s-1}} + \|\nabla^l F\|_{N_{s-l}(I)}.
\]
whenever \( \mu \geq 0, \mu - s \geq 0 \) and \( 1 + \tilde{s} - \mu \geq 0. \)

Here, we note that our assumption \( u \in L^\infty_s(K \times \mathbb{R}^5) \) on the solution in definition 1.1, combined with the local theory and the Strichartz estimates, implies
\[
\|\nabla^l u\|_{S_{s-l}(K)} < \infty
\]
for \( s \in [0, \frac{5}{2}] \) and \( K \subset I \) compact.

Turning to the notion of almost periodicity, we recall that this property is equivalent to the following condition: there exist functions \( N : I \rightarrow \mathbb{R}^+ \) and \( x : I \rightarrow \mathbb{R}^2 \) such that the set
\[
K = \left\{ \left( \frac{1}{N(t)} u \left( t, x(t) + \frac{x}{N(t)} \right), \frac{1}{N(t)} u_t \left( t, x(t) + \frac{x}{N(t)} \right) \right) : t \in I \right\},
\]
has compact closure in \( \dot{H}^{3/2}_x(\mathbb{R}^5) \times \dot{H}^{1/2}_x(\mathbb{R}^5) \). In particular, if \( u \) is almost periodic, then for every \( \eta > 0 \) there exists \( C(\eta) > 0 \) such that for all \( t \in I \),
\[
\int_{|x-x(t)| \geq C(\eta)/N(t)} |\nabla u(t,x)|^{5/2} \, dx + \int_{|x-x(t)| \geq C(\eta)/N(t)} |u_t(t,x)|^{5/2} \, dx \leq \eta.
\]

Moreover, for every nonzero almost periodic solution \( u \) to (NLW) there exists \( C(u) > 0 \) such that for every compact \( K \subset I \)
\[
\frac{1}{C(u)} \int_K N(t) \, dt \leq \|u\|_{L^6_s(K \times L^2_x)}^6 \leq C(u) \left( 1 + \int_K N(t) \, dt \right),
\]
(2.3)

and the corresponding estimates for the norms \( \|\nabla^l u\|_{L^6_s(K \times L^2_x)} \) and \( \|u\|_{L^6_s(K \times L^{20/3}_x)} \). The above bounds are consequences of almost periodicity and the Strichartz estimates (2.1). For completeness, we sketch the argument for the upper bounds on the stated norms (without loss of generality, we consider the norm \( \|u\|_{L^6_s(K \times L^2_x)} \)): fixing \( \epsilon > 0 \), partition the interval \( K \) into subintervals \( K_j \) on which \( \int_{K_j} N(t) \, dt < \epsilon \) and use the Strichartz estimate to control \( \|u\|_{L^6_s(K_j \times L^2_x)} \) by a suitable power of itself and a small error term. The argument concludes using a standard bootstrap argument and summing the contributions from each \( K_j \). For more details, we refer to the analogous estimates in the NLS setting (see, for instance, [18, lemma 5.21] and [26, lemma 1.7]).

We next recall some basic facts from the Littlewood–Paley theory that will be used frequently in the following. Let \( \phi(\xi) \) be a real valued radially symmetric bump function
supported in the ball \( \{ \xi \in \mathbb{R}^d : |\xi| \leq \frac{M}{N} \} \) which equals 1 on the ball \( \{ \xi \in \mathbb{R}^d : |\xi| \leq 1 \} \). For any dyadic number \( N = 2^k, k \in \mathbb{Z} \), we define the following Littlewood–Paley operators:

\[
P_{\leq N} \tilde{f}(\xi) = \phi(\xi/N) \hat{f}(\xi),
\]

\[
P_{> N} \tilde{f}(\xi) = (1 - \phi(\xi/N)) \hat{f}(\xi),
\]

\[
P_{N} \tilde{f}(\xi) = (\phi(\xi/N) - \phi(2\xi/N)) \hat{f}(\xi).
\]

Similarly, we define \( P_{= N} \) and \( P_{\geq N} \) with

\[
P_{< N} = P_{\leq N} - P_{N}, \quad P_{\geq N} = P_{> N} + P_{N},
\]

and also

\[
P_{M < N} := P_{\leq N} - P_{\leq M} = \sum_{M < N_i \leq N} P_{N_i}
\]

whenever \( M \leq N \).

These operators commute with one another and with derivative operators. Moreover, they are bounded on \( L^p \) for \( 1 \leq p \leq \infty \) and obey the following Bernstein inequalities:

\[
\| |\nabla|^s P_{\leq N} f \|_{L^p} \lesssim N^s \| P_{\leq N} f \|_{L^p},
\]

\[
\| P_{> N} f \|_{L^p} \lesssim N^{-s} \| P_{\leq N} |\nabla|^s f \|_{L^p},
\]

\[
\| |\nabla|^{1+s} P_{N} f \|_{L^p} \sim N^{2s} \| P_{N} f \|_{L^p},
\]

with \( s \geq 0 \) and \( 1 \leq p \leq \infty \).

**3. Frequency localized Strichartz estimate**

We now obtain a frequency localized version of the Strichartz estimates that we will use as a main ingredient in proving the frequency localized Morawetz estimate in section 4. The proof of this result is inspired by recent progress in the mass and energy critical nonlinear Schrödinger equation [7, 26].

**Theorem 3.1 (Frequency localized Strichartz estimate).** Suppose that \( u \) is an almost periodic solution to (NLW) with maximal interval of existence \( I, (u, u_t) \in L^\infty_t(I; \dot{H}^{1/2}_x \times \dot{H}^{1/2}_x) \), and such that there exist disjoint intervals \( \{ J_k \}_{k \geq 1} \) with \( I^* = \cup J_k \) and for every \( k, N(t) = \widetilde{N}_k \in [1, \infty) \) on \( J_k \) and \( |J_k| = \delta \widetilde{N}_k^{-1} \).

Then there exists \( C = C(u) > 0 \) such that for all dyadic \( N \) and compact intervals \( I_0 = \cup J_k \subset I^* \) we have

\[
\| |\nabla|^{1/2} u_{\leq N} \|_{L^3(I_0; L^6)} \leq C(u) (1 + (N|I_0|)^{1/2}).
\]

(3.1)

Moreover, for every \( \eta > 0 \) there exists \( N_0 > 0 \) such that for \( N < N_0 \) we have

\[
\| |\nabla|^{1/2} u_{\leq N} \|_{L^3(I_0; L^6)} \leq C(u) \eta (1 + (N|I_0|)^{1/2}).
\]

(3.2)

Before we proceed with the proof of the theorem, we record the following related estimates, derived by interpolating (3.1) and (3.2) with the a priori bound on \( L^\infty_t(I; \dot{H}^{3/2}_x \times \dot{H}^{1/2}_x) \).

**Corollary 3.2.** Let \( u \) be as in theorem 3.1. Then there exists \( C(u) > 0 \) such that

- for each dyadic \( N > 0 \) and compact interval \( I_0 = \cup J_k \subset I^* \) we have

\[
\| u_{> N} \|_{L^2(I_0; L^{\infty} \cap L^{20\eta})} \leq C(u) N^{-1/2} (1 + N|I_0|)^{1/3},
\]

\[
\| u_{> N} \|_{L^2(I_0; L^{20\eta})} \leq C(u) N^{-1} (1 + N|I_0|)^{1/4},
\]
and for each \( \eta > 0 \) there exists \( N_0 > 0 \) such that for \( N < N_0 \) we have
\[
\| \nabla u_{\leq N} \|_{L^1_t(L^6_x)} \leq C(u) \eta (1 + N|I_0|)^{1/3},
\]
\[
\| \nabla u_{\leq N} \|_{L^1_t(L^{20/7}_x)} \leq C(u) \eta (1 + N|I_0|)^{1/4}.
\]

**Proof of theorem 3.1.** We begin by showing (3.1). Let \( I_0 \subset I^* \) be given as stated and observe that the bound (2.4) implies that (3.1) holds with \( C_1(u) \) for all
\[
N \geq \frac{\int_{I_0} N(t) \, dt}{|I_0|}.
\]

For general dyadic numbers \( N \), we proceed by induction. Fix
\[
C(u) = \max \{ C_1(u), 1 \}
\]
to be determined, and suppose that (3.1) holds for all \( N \) larger than some \( N_0 \). Our goal is to show that (3.1) holds for \( N = N_1 := N_0/2 \) (with \( C(u) \) unchanged). Towards this end, we apply the Strichartz inequality to obtain
\[
\| \nabla \|^{3/4} u_{\leq N_1} \|_{L^1_t(L^6_x)} \leq \inf_{t \in I_0} \| (u_{\leq N_1}(t), \partial_t u_{\leq N_1}(t)) \|_{H^{1/2}_x \times H^{1/2}_x} + \| \nabla [^{5/4} P_{\leq N_1}[u^3]] \|_{L^1_t(L^{5/4}_x)}.
\]
(3.3)

In the rest of the proof, all space–time norms will be over the set \( I_0 \times \mathbb{R}^3 \), unless otherwise indicated.

To estimate the nonlinear term in (3.3), we fix \( 0 < \eta_0 \leq \frac{1}{4} \) (to be determined later in the argument) and use the almost periodicity of \( u \) to choose \( c_0 = c_0(\eta_0) \) such that
\[
\| \nabla \|^{3/4} u_{\leq N_1(\eta_0)} \|_{L^1_t(L^6_x)} \leq \eta_0.
\]
(3.4)

Then, writing
\[
u(t) = u_{\leq N_1(\eta_0)}(t) + u_{> N_1(\eta_0)}(t)
\]
and using the identity
\[
(u_{> N_1(\eta_0)}(t) + u_{\leq N_1(\eta_0)}(t))^3 = u_{> N_1(\eta_0)}(t)^3 + 3u_{> N_1(\eta_0)}(t) u_{\leq N_1(\eta_0)}(t) u(t) + u_{\leq N_1(\eta_0)}(t)^3,
\]
we obtain
\[
\| \nabla \|^{3/4} u_{\leq N_1(\eta_0)} \|_{L^1_t(L^6_x)} \leq \inf_{t \in I_0} \| (u_{\leq N_1}(t), \partial_t u_{\leq N_1}(t)) \|_{H^{1/2}_x \times H^{1/2}_x} + \| \nabla [^{5/4} P_{\leq N_1}[u_{> N_1(\eta_0)}^3]] \|_{L^1_t(L^{5/4}_x)} + \| \nabla [^{5/4} P_{\leq N_1}[u_{> N_1(\eta_0)} u_{\leq N_1(\eta_0)} u(t)]] \|_{L^1_t(L^{5/4}_x)} + \| \nabla [^{5/4} P_{\leq N_1}[u_{\leq N_1(\eta_0)}^3]] \|_{L^1_t(L^{5/4}_x)}.
\]
(3.5)

Furthermore, we bound the last term by a multiple of
\[
\| \nabla [^{5/4} P_{\leq N_1}[u_{\leq N_1(\eta_0)} u_{\leq c_0 N(\eta_0)}^2]] \|_{L^1_t(L^{5/4}_x)} + \| \nabla [^{5/4} P_{\leq N_1}[u_{\leq c_0(\eta_0)}^3]] \|_{L^1_t(L^{5/4}_x)} + \| \nabla [^{5/4} P_{\leq N_1}[u_{\leq c_0(\eta_0)}^2]] \|_{L^1_t(L^{5/4}_x)},
\]
where we have set \( P_{\leq} = P_{\leq N_1(\eta_0)} \) and used the decomposition
\[
P_{\leq} u(t) = P_{\leq} u_{\leq c_0 N(\eta_0)}(t) + P_{> c_0 N(\eta_0)}(t),
\]
and where \( c_0 \) is chosen in (3.4).
Thus, it suffices to bound (3.5) through (3.9). Before estimating each of these terms, we will need the following estimate, which is obtained via Hölder’s inequality in time and interpolation: for each dyadic $M > 0$,

$$\|\nabla^{5/4} u \|_{L^6_t L^{20/7}_x} \leq (M|I_0|)^{1/4} \|\nabla u \|_{L^4_t L^{20/7}_x}$$

$$\lesssim (M|I_0|)^{1/2} + \|\nabla u \|_{L^{20/7}_x}^2$$

$$\lesssim (M|I_0|)^{1/2} + \|\nabla^{3/4} u \|_{L^{20/7}_x} \|\nabla^{5/4} u \|_{L^6_t L^{20/7}_x}$$

$$\lesssim (M|I_0|)^{1/2} + \|\nabla^{3/4} u \|_{L^4_t L^{20/7}_x}$$

(3.10)

where in obtaining the last inequality, we have used the \textit{a priori} bound $(u, u_t) \in L^\infty_t (\dot{H}^{3/2}_x \times \dot{H}^{3/2}_x)$.

With this bound in hand, we are now ready to estimate the above terms. For (3.5), we note that an application of Bernstein’s inequality gives

$$\left(3.5\right) \lesssim N_1^{5/4} \|u^3 \|_{L^\infty_t L^{13}_x}$$

$$\lesssim N_1^{5/4} \sum_{M > N_1/2} \|u_M \|_{L^6_t L^{20/7}_x} \|u \|_{L^\infty_t L^{20/7}_x}$$

$$\lesssim_u N_1^{5/4} \sum_{M > N_1/2} M^{-5/4} \|\nabla^{3/4} u \|_{L^4_t L^{20/7}_x}$$

$$\lesssim \eta_0^{3/4} C_2(u) C(u)(N_1|I_0|)^{1/2} + \eta_0^{5/4} C_2(u) C(u),$$

where to obtain the last line we have used (3.10) followed by the induction hypothesis. We may use the same argument to estimate (3.6), obtaining

$$\left(3.6\right) \lesssim \eta_0^{3/4} C_2(u) C(u)(N_1|I_0|)^{1/2} + \eta_0^{5/4} C_2(u) C(u).$$

On the other hand, to estimate (3.7), we apply the fractional product rule [5, 15] to obtain

$$\left(3.7\right) \lesssim \|\nabla^{5/4} u \|_{L^6_t L^{20/7}_x} \|\left[ P_{\leq N_1/20} u \right] \|_{L^\infty_t L^{13}_x}$$

$$+ \|u \|_{L^6_t L^{20/7}_x} \|\nabla^{5/4} [ P_{\leq c_0 N_1} u ] \|_{L^\infty_t L^{20/13}_x}$$

$$\lesssim_u \eta_0^{2} \left[ \|\nabla^{5/4} u \|_{L^6_t L^{20/7}_x} + \|\nabla^{3/4} u \|_{L^4_t L^{20/7}_x} \right],$$

where to obtain the second inequality we have used (3.4) to estimate the first term and the Sobolev embedding, fractional product rule, and (3.4) to estimate the second term. Then, using (3.10) and the induction hypothesis once again, we obtain

$$\left(3.7\right) \lesssim \eta_0^{3/4} C_3(u) (\eta_0^{-1/2} C(u)(N_1|I_0|)^{1/2} + C(u)).$$

We now turn our attention to the two remaining terms. In what follows, we will use the notation $v(t)$ to refer to either of the functions $P_{\leq N_1/20} u \in (I_0(t))$ and $P_{\leq c_0 N_1} u \in (I_0(t))$. In particular, using Bernstein’s inequalities combined with the fractional product rule, we obtain the preliminary bound

$$\|\nabla^{5/4} P_{\leq N_1} \left[ u \|_{L^6_t L^{20/7}_x} \right] \|_{L^6_t (L^{4}_x)}$$

$$\lesssim N_1^{1/2} \|\nabla^{3/4} [ u \|_{L^6_t L^{20/7}_x} ] \|_{L^6_t L^{20/13}_x}$$

$$+ N_1^{1/2} \|u \|_{L^6_t L^{20/7}_x} \|u \|_{L^6_t L^{20/7}_x} \|\nabla^{3/4} P_{\leq c_0 N_1} u \|_{L^6_t L^{20/7}_x}.$$  

(3.11)
We then use the fractional product rule again combined with the \textit{a priori} bound \((u, u_t) \in L_t^\infty(I; \dot{H}^{3/2} \times \dot{H}^{1/2})\) to bound the factor
\[
\|\nabla\|^{1/4}u_{\leq N_t/m}\|_{L_x^\infty L_{t,x}^{20/3}} \lesssim \|\nabla\|^{3/4}u_{\leq N_t/m}\|_{L_x^\infty L_{t,x}^{20/3}}\|v\|_{L_x^\infty L_t^1} + \|u_{\leq N_t/m}\|_{L_x^\infty L_t^1}\|\nabla\|^{3/4}v\|_{L_x^\infty L_{t,x}^{20/3}} \lesssim_u 1.
\] (3.12)

Invoking this bound in (3.11), we obtain
\[
\max\{(3.8), (3.9)\} \lesssim \left( \sum_{J_k \subset I_t} \|\nabla\|^{5/4}P_{\leq N_t} \left[u_{\leq N_t/m}vP_{\leq u_{\geq c_0}N(t)}\right]\right)^{1/2} N_t^{1/2} \left( \sum_{J_k \subset I_t} \|u_{> c_0 N_k}\|_{L_x^2 J_k}^2 + \|\nabla\|^{3/4}u_{> c_0 N_k}\|_{L_x^3 J_k}^{12} \right)^{1/2} \]
\begin{equation}
\lesssim_u \left( \sum_{J_k \subset I_t} \frac{1}{c_0 N_k} \left[ \|\nabla\|^{1/2}u_{> c_0 N_k}\|_{L_x^2 J_k}^2 + \|\nabla\|^{3/4}u_{> c_0 N_k}\|_{L_x^3 J_k}^{12} \right] \right)^{1/2} \lesssim \frac{C_4(u)}{c_0^{1/2}} (N_t|I_0|)^{1/2}. \tag{3.13}
\end{equation}

where in the second term of (3.11) we use the bound (2.3) in the form
\[
\|u\|_{L_x^1(I_t; L_t^1)} \leq C(u)(1 + \delta) \lesssim_u 1. \tag{3.14}
\]

Moreover, using Bernstein’s inequalities and the bounds \(\|\nabla\|^{1/2}u\|_{L_x^2(J_k; L_t^1)} \lesssim_u 1\) (which follows from (2.4) and the Sobolev embedding) and \(\|\nabla\|^{5/4}u\|_{L_x^3(J_k; L_t^{2})} \lesssim_u 1\) (as remarked in the discussion following (2.4), this bound is obtained via an argument identical to that used to prove (2.4)),
\[
(3.13) \lesssim_u \left( \sum_{J_k \subset I_t} \frac{1}{c_0 N_k} \left[ \|\nabla\|^{1/2}u_{> c_0 N_k}\|_{L_x^2 J_k}^2 + \|\nabla\|^{3/4}u_{> c_0 N_k}\|_{L_x^3 J_k}^{12} \right] \right)^{1/2} \lesssim \frac{C_4(u)}{c_0^{1/2}} (N_t|I_0|)^{1/2}. \]

Combining the estimates of (3.5) through (3.9), we obtain
\[
\|\nabla\|^{3/4}u_{\leq N_t}\|_{L_x^2(I_t; L_t^1)} \leq \left[ C_0 \inf_{t \in I_t} \|\nabla\|^{1/2}u_{\leq N_t}(t), \|\nabla\|^{3/4} u_{\leq N_t}(t)\|_{\dot{H}^{1/2} \times \dot{H}^{1/2}} \right] + 2C^{3/4}C_2(u)C(u)(N_t|I_0|)^{1/2} + \eta_0^{1/2} \tag{3.15}
\]
\[+ \frac{C_4(u)}{c_0^{1/2}} (N_t|I_0|)^{1/2} + \eta_0^{-1/2}(N_t|I_0|)^{1/2} + \frac{C_4(u)}{c_0^{1/2}} (N_t|I_0|)^{1/2}. \]

We now choose \(\eta_0\) sufficiently small (depending on \(C_2(u)\) and \(C_4(u)\)) to ensure that
\[
\|\nabla\|^{3/4}u_{\leq N_t}\|_{L_x^2(I_t; L_t^1)} \leq \left[ C_0 \inf_{t \in I_t} \|\nabla\|^{1/2}u_{\leq N_t}(t), \|\nabla\|^{3/4} u_{\leq N_t}(t)\|_{\dot{H}^{1/2} \times \dot{H}^{1/2}} + \frac{2C(u)}{3}(N_t|I_0|)^{1/2} \right] \]
\[+ \frac{2C(u)}{3} \frac{C_4(u)}{c_0^{1/2}} (N_t|I_0|)^{1/2}. \]

We now choose \(C(u)\) large enough so that
\[
C(u) > \max\left\{ \frac{3C_4(u)}{c_0(\eta_0)^{1/2}}, \frac{3C_0}{c_0(\eta_0)^{1/2}} \right\}.
\]

1867
With such a choice of \( C(u) \) we obtain
\[
\| |\nabla|^{3/4} u \|_{L^6} \leq C(u)(1 + (N_1|I_0|)^{1/2}),
\]  
(3.16)
completing the induction.

We now turn to (3.2). Let \( \eta > 0 \) be given and fix \( N_0 = N_0(\eta) > 0 \) to be determined later in the argument. Let \( N \leq N_0 \) be given and recall that (3.1) is satisfied for all \( N > 0 \). As a consequence, (3.15) is satisfied for any \( \eta_0 \in (0, \frac{1}{2}] \) with \( N_1 \) replaced by \( N \). More precisely, after setting
\[
f(N) = \| (u_{\leq N}, \partial_t u_{\leq N}) \|_{L^6(\mathbb{H}^{3/2}_N \times H^{1/2}_N)} + \sup_{J_k \subset I} \| u_{\leq N} \|_{L^6(J_k; L^4_2)},
\]
we have
\[
\| |\nabla|^{3/4} u \|_{L^6} \lesssim_{\eta} f(N) + \eta_0^{3/4} ((N|I_0|)^{1/2} + \eta_0^{1/2})
\]
\[
+ \eta_0^2 \eta (N|I_0|)^{1/2} + \frac{f(N)/\eta_0}{e^{3/2}} - (N|I_0|)^{1/2}
\]
(3.17)
for any \( \eta_0 \in (0, \frac{1}{2}] \), where we have replaced \( C_2(u) \) in (3.15) by \( f(N/\eta_0) \) in view of (3.11) and (3.12). More precisely, replacing the constant on the right-hand side of (3.12) by a multiple of \( f(N/\eta_0) \) and using the inequality \( \| u_{\leq N/\eta_0} \|_{L^6(J_k; L^4_2)} \leq f(N/\eta_0) \) gives the claim. We next show that \( f(N) \to 0 \) as \( N \to 0 \). Indeed, invoking the Strichartz inequality (2.1) and using the decomposition \( u = u_{\leq N/\eta} + u_{N/\eta} \), we obtain
\[
f(N) \lesssim \| (u_{\leq N}, \partial_t u_{\leq N}) \|_{L^6(\mathbb{H}^{3/2}_N \times H^{1/2}_N)} + \sup_{J_k \subset I} \| |\nabla|^{5/4} P_{\leq N}[u^3] \|_{L^2(J_k; L^4_2)}
\]
\[
\lesssim \| (u_{\leq N}, \partial_t u_{\leq N}) \|_{L^6(\mathbb{H}^{3/2}_N \times H^{1/2}_N)} + \sup_{J_k \subset I} \left[ \| |\nabla|^{5/4} P_{\leq N}[u^3] \|_{L^2(J_k; L^4_2)} + \| |\nabla|^{5/4} P_{\leq N}[u_1^3] \|_{L^2(J_k; L^4_2)} \right]
\]
(3.18)
\[
\lesssim \| (u_{\leq N}, \partial_t u_{\leq N}) \|_{L^6(\mathbb{H}^{3/2}_N \times H^{1/2}_N)} + \sup_{J_k \subset I} \left[ N^{5/4} \| u_{N/\eta} \|_{L^6(J_k; L^{12/5})} \| u \|_{L^2(J_k; L^4_2)} \right]
\]
(3.19)
for any \( N > 0 \), where we have used the Bernstein inequalities followed by the Hölder inequality for the second and third terms of (3.18) and the fractional product rule for the fourth term of (3.18). We then bound the second and third terms in (3.19) using the Bernstein inequalities followed by (3.14) along with the bounds \( \| |\nabla|^{5/4} u \|_{L^6(J_k; L^{12/5})} \lesssim_{u} 1 \) and \( \| u \|_{L^4(J_k; L^{20/7})} \lesssim_{u} 1 \) (see the discussion following (2.4)) to obtain, for \( N < 1 \),
\[
f(N) \lesssim_{u} \| (u_{N/\eta}, \partial_t u_{N/\eta}) \|_{L^6(\mathbb{H}^{3/2}_N \times H^{1/2}_N)} + N^{5/8},
\]
which tends to 0 as \( N \to 0 \) as a consequence of remark 1.7. With this limit in hand, we choose \( \eta_0 \) small enough to ensure \( \eta_0^{3/4} \leq \eta \) and \( N_0 \) small enough to guarantee that \( N < N_0 \) implies \( f(N) < \eta \) and \( f(N/\eta_0) < \eta \eta_0 (\eta_0)^{1/2} \). The inequality (3.17) then gives
\[
\| |\nabla|^{3/4} u \|_{L^6} \lesssim_{\eta} \eta (1 + (N|I_0|)^{1/2})
\]
as desired. 
\qed

Proof of corollary 3.2. We note that interpolation gives
\[
\| u_{N} \|_{L^3_t L^6_x} \lesssim \| u_{N} \|_{L^6_t L^4_x} \| u_{N} \|_{L^2_t L^4_x}
\]
(3.20)
and
\[
\|u_{>N}\|_{L^4_t L^8_{x}} \lesssim \|\nabla|^{5/4}u_{>N}\|_{L^\infty_t L^{20/9}_{x}} \|\nabla|^{-5/4}u_{>N}\|_{L^2_t L^4_{x}}. \tag{3.21}
\]
The Sobolev inequality followed by the boundedness of the Littlewood–Paley projection then yields
\[
\|\nabla|^{5/4}u_{>N}\|_{L^\infty_t L^{20/9}_{x}} \lesssim \|(u, ut)\|_{L^\infty_t (H^{1/2} \times H^{1/2})}.
\]
On the other hand, the Bernstein inequalities along with theorem 3.1 give the bounds
\[
\|u_{>N}\|_{L^2_t L^4_{x}} \lesssim \sum_{M>N} M^{-3/4} \|\nabla|^{3/4}u_M\|_{L^2_t L^4_{x}}
\lesssim \sum_{M>N} M^{-3/4} \|\nabla|^{3/4}u_{\leq 2M}\|_{L^2_t L^4_{x}}
\lesssim_a \sum_{M>N} M^{-3/4} (1 + (M|I_0|)^{1/2})
\lesssim_a N^{-3/4} (1 + N|I_0|)^{1/2}
\]
and (by an identical argument)
\[
\|\nabla|^{-5/4}u_{>N}\|_{L^2_t L^4_{x}} \lesssim_a N^{-2} (1 + N|I_0|)^{1/2}.
\]
Thus, we obtain
\[
(3.20) \lesssim_a N^{-1/2} (1 + N|I_0|)^{1/3}, \quad (3.21) \lesssim_a N^{-1} (1 + N|I_0|)^{1/4}
\]
as desired.

The bounds on \(\|\nabla u_{\leq N}\|_{L^2_t L^4_{x}}\) and \(\|\nabla u_{\leq N}\|_{L^\infty_t L^{20/9}_{x}}\) are obtained by interpolating (3.2) with the \textit{a priori} bound \((u, ut) \in L^\infty_t (H^{1/2} \times H^{1/2})\).

\section{4. Proof of theorem 1.3: frequency-localized Morawetz estimate}

In this section, we establish the frequency-localized Morawetz estimate. The proof of this result is inspired by recent analysis in the energy critical NLS setting \([26]\). We begin by deriving a form of the classical Morawetz estimate with forcing term \(\mathcal{N}\); for the classical form, see \([23, 24]\). To obtain this, when \(u\) is a solution to \(u_t - \Delta u + N = 0\), we set
\[
M(t) = \int_{\mathbb{R}^3} -a_j(x)u_j(t, x)u_j(t, x) - \frac{1}{2} a_{jj}(x)u(t, x)u_t(t, x) \, dx,
\]
where \(a : \mathbb{R}^3 \rightarrow \mathbb{R}\), subscripts indicate partial derivatives, and we have used the summation convention. A brief calculation then yields the identity
\[
\frac{dM}{dt}(t) = \int_{\mathbb{R}^3} a_{jk}(x)u_j(t, x)u_k(t, x) + \frac{1}{2} a_{jj}(x)\mathcal{N}, u \big|_j - \frac{1}{4} a_{jkk}(x)u(t, x)^2 \, dx,
\]
with \(\{f, g\} := f\nabla g - g\nabla f\), where the subscript on \(\{\mathcal{N}, u\}\) denotes the \(j\)th component. Taking \(a(x) = |x|\), integrating in time, and using the fundamental theorem of calculus, we then have
\[
\int_{J} \int_{\mathbb{R}^3} \left( \delta_{jk} \frac{x_{jk}}{|x|^3} \right) u_j(t,x)u_k(t,x) + \frac{8}{|x|^3} u(t,x)^2 \, dx \, dt \lesssim \sup_{t \in I} |M(t)| \quad (4.1)
\]
for every \(I \subset \mathbb{R}\). Moreover, the triangle inequality followed by the Cauchy–Schwartz and Hardy inequalities gives
\[
|M(t)| \lesssim \|u_t\|_{L^\infty_t L^2_{x}} \|\nabla u\|_{L^\infty_t L^2_{x}} \quad (4.2)
\]
for all \(I \in I\). Combining (4.1) with (4.2), observing that the first term on the left-hand side of (4.1) is non-negative and invoking an approximation argument, we obtain the following.
Lemma 4.1 (Morawetz estimate). Suppose \( u : I \times \mathbb{R}^5 \to \mathbb{R} \) solves \( u_{tt} - \Delta u + \mathcal{N} = 0 \). Then,

\[
\int_I \int_{\mathbb{R}^5} \frac{x \cdot \{N(t, x), u(t, x)\}}{|x|} \, dx \, dt \lesssim \|u_t\|_{L^p_t L^q_x} \|\nabla u\|_{L^r_t L^s_x}. \tag{4.3}
\]

We also recall the following Hardy-type bound, which will be used to estimate the error terms resulting from the frequency localization (see, for instance, [1, 9] and the references cited therein).

Proposition 4.2 (Hardy-type bound). Fix \( 1 < p < \infty \), and \( 0 \leq \alpha < 5 \). Then there exists \( C = C(\alpha, p) > 0 \) such that for every \( g \in S(\mathbb{R}^5) \),

\[
\| |x|^{-\alpha/p} g(x) \|_{L^q_t(\mathbb{R}^5)} \lesssim C(\alpha, p) \| \nabla |x|^{\alpha/p} g(x) \|_{L^r_t(\mathbb{R}^5)}. \tag{4.4}
\]

In particular, we prove the following:

Proof of theorem 1.3. Fix a compact time interval \( I_0 = \bigcup J_k \subset I^* \). In what follows, all space–time norms will be taken over \( I_0 \times \mathbb{R}^5 \), unless otherwise indicated. Let \( \eta > 0 \) be given, and fix \( N_0 > 0 \) to be determined later in the argument. Let \( N \leq N_0 \) be given. We begin by observing that the Morawetz estimate (4.3) applied to \( u_{\geq N} \) yields

\[
\int_{I_0} \int_{\mathbb{R}^5} \frac{x \cdot \{P_{\geq N}[u(t, x)^3], u_{\geq N}(t, x)\}}{|x|} \, dx \, dt \lesssim \|\partial_t u_{\geq N}\|_{L^p_t L^q_x} \|\nabla u_{\geq N}\|_{L^r_t L^s_x}. \tag{4.5}
\]

Note that by remark 1.7, we may choose \( N_1 > 0 \) so that \( N \leq N_1 \) implies

\[
\|u_{\leq N}, \partial_t u_{\leq N}\|_{L^p_t(\mathbb{R}^5 \times \mathcal{H}^{5/2})} < \eta^{1/2}.
\]

Now, by choosing \( N_0 \) small enough so that \( N_0 < \eta N_1 \), we may estimate the right-hand side of (4.5) by

\[
\begin{align*}
\left( \|\partial_t u_{\leq N}, \partial_t u_{\geq N}\|_{L^p_t L^q_x} + \|\partial_t u_{\geq N}\|_{L^p_t L^q_x} \right) \cdot \left( \|\nabla u_{\leq N}, \partial_t u_{\leq N}\|_{L^p_t L^q_x} + \|\nabla u_{\geq N}, \partial_t u_{\geq N}\|_{L^p_t L^q_x} \right)
\lesssim (N_1 - \frac{1}{2}) \|u_{\geq N}\|_{L^p_t \mathcal{H}^{5/2}}^2 + N_1^{-1/2} \|\nabla u_{\geq N}\|_{L^p_t \mathcal{H}^{5/2}}^2.
\end{align*}
\]

We now estimate the left-hand side of (4.5). For this, we use the identity

\[
\{P_{\geq N}[u(t)^3], u_{\geq N}(t)\} = \{u(t)^3, u(t)\} - \{u_{<N}(t)^3, u_{<N}(t)\}
- \{u(t)^3 - u_{<N}(t)^3, u_{<N}(t)\}
- \{P_{<N}[u(t)^3], u_{\geq N}(t)\}
\]

to obtain

\[
\int_{I_0} \int_{\mathbb{R}^5} \frac{x \cdot \{P_{\geq N}[u(t, x)^3], u_{\geq N}(t, x)\}}{|x|} \, dx \, dt
= \int_{I_0} \int_{\mathbb{R}^5} \frac{x \cdot \{u(t, x)^3, u(t, x)\}}{|x|} \, dx \, dt
- \int_{I_0} \int_{\mathbb{R}^5} \frac{x \cdot \{u_{<N}(t, x)^3, u_{<N}(t, x)\}}{|x|} \, dx \, dt
- \int_{I_0} \int_{\mathbb{R}^5} \frac{x \cdot \{P_{<N}[u(t, x)^3], u_{\geq N}(t, x)\}}{|x|} \, dx \, dt. \tag{4.7}
\]

1870
A simple calculation then shows \( \{ f^3, f \} = -\frac{1}{2} \nabla [f^4] \), so that integrating the first two terms in (4.7) by parts gives

\[
\int_{I_0} \int_{\mathbb{R}^3} \frac{x \cdot \{ P_{>N} [u(t, x)^3], u_{\geq N} (t, x) \}}{|x|} \, dx \, dt
\]

\[
= \int_{I_0} \int_{\mathbb{R}^3} \frac{2(|u(t, x)|^4 - |u_{<N} (t, x)|^4)}{|x|} \, dx \, dt
\]

\[
- \int_{I_0} \int_{\mathbb{R}^3} x \cdot \{ u(t, x)^3 - u_{<N} (t, x)^3, u_{<N} (t, x) \} \, dx \, dt
\]

\[
- \int_{I_0} \int_{\mathbb{R}^3} x \cdot \{ P_{<N} [u(t, x)^3], u_{\geq N} (t, x) \} \, dx \, dt.
\]  

On the other hand, applying the decomposition 

\[
u = u_{<N} + u_{\geq N}
\]  

gives

\[
\int_{I_0} \int_{\mathbb{R}^3} |u_{\geq N} (t, x)|^4 \, dx \, dt \lesssim \int_{I_0} \int_{\mathbb{R}^3} |u(t, x)|^4 - |u_{<N} (t, x)|^4 \, dx \, dt
\]

\[
+ \sum_{i=1}^3 \int_{I_0} \int_{\mathbb{R}^3} |u_{<N} (t, x)|^4 |u_{\geq N} (t, x)|^i \, dx \, dt.
\]

In view of (4.5) and (4.8), we therefore obtain the bound

\[
\int_{I_0} \int_{\mathbb{R}^3} |u_{\geq N} (t, x)|^4 \, dx \, dt \lesssim \eta N^{-1} + \sum_{i=1}^3 (I)_i + (II) + (III),
\]

where we have set

\[
(I)_i = \int_{I_0} \int_{\mathbb{R}^3} \frac{|u_{<N} (t, x)|^{4-i} |u_{\geq N} (t, x)|^i}{|x|} \, dx \, dt, \quad i = 1, \ldots, 3,
\]

\[
(II) = \left| \int_{I_0} \int_{\mathbb{R}^3} x \cdot \{ u(t, x)^3 - u_{<N} (t, x)^3, u_{<N} (t, x) \} \, dx \, dt \right|, \quad \text{and}
\]

\[
(III) = \left| \int_{I_0} \int_{\mathbb{R}^3} x \cdot \{ P_{<N} [u(t, x)^3], u_{\geq N} (t, x) \} \, dx \, dt \right|.
\]

We estimate each of these terms individually. For \( (I)_i \), we use the Hölder inequality with the Hardy-type bound (4.4), along with the Sobolev embedding and corollary 3.2 (after choosing \( N_0 \) sufficiently small) to obtain the bounds

\[
(I)_i \lesssim \| u_{\geq N} \|_{L_t^1 L_x^{20/7}} \left\| \frac{u_{<N}}{|x|} \right\|_{L_t^{3} L_x^{10/3}}^3
\]

\[
\lesssim \| u_{\geq N} \|_{L_t^1 L_x^{20/7}} \| \nabla \|^{1/3} u_{\geq N} \|_{L_t^2 L_x^6}^3
\]

\[
\lesssim \| u_{\geq N} \|_{L_t^1 L_x^{20/7}} \| \nabla u_{<N} \|_{L_t^2 L_x^{20/7}}^3
\]

\[
\lesssim \eta^3 (N^{-1} + |I_0|).
\]

For the term \( (I)_2 \), we write

\[
(I)_2 \lesssim \int_{I_0} \int_{\mathbb{R}^3} \frac{|u_{<N} (t, x)| |u_{\geq N} (t, x)|}{|x|} \left( |u_{<N} (t, x)|^2 + |u_{\geq N} (t, x)|^2 \right) \, dx \, dt
\]

\[
\lesssim (I)_1 + (I)_3,
\]

\[
1871
\]
while for the term \((I)_3\), we note that for each \(\epsilon > 0\),

\[
(II) \lesssim \int_{I_0} \int_{R^3} \frac{|u_{<N}(t, x)|}{|x|} \frac{|u_{\geq N}(t, x)|^3}{|x|} \, dx \, dt + \int_{I_0} \int_{R^3} \frac{|u_{<N}(t, x)|}{|x|} \frac{|u_{\geq N}(t, x)|^3}{|x|} \, dx \, dt \
\leq \epsilon \int_{I_0} \int_{R^3} \frac{|u_{\geq N}(t, x)|^4}{|x|} \, dx \, dt + \frac{1}{\epsilon^2} (I)_{1}.
\]

We now estimate term \((II)\). Using the identity

\[
\{ u^3 - u_{<N}^3 \} = 2(u^3 - u_{<N}^3) \nabla u_{<N} - \nabla ((u^3 - u_{<N}^3) u_{<N}),
\]

we apply the triangle inequality and integrate by parts in the second term of the resulting integral to obtain

\[
(II) \lesssim \int_{I_0} \int_{R^3} \frac{1}{|x|} \frac{3}{7} \sum_{i=1}^{3} \int_{I_0} \left| u_{\geq N}(t, x) \right|^3 \left| \nabla u_{<N}(t, x) \right| \, dx \, dt + \int_{I_0} \int_{R^3} \frac{1}{|x|} \frac{3}{7} \sum_{i=1}^{3} (I)_{i}.
\]

We now use the Hölder inequality, Sobolev embedding and corollary 3.2 to estimate the first term,

\[
\| u_{\geq N} \|_{L^2_{7,1}} \| \nabla u_{<N} \|_{L^1_{2,1}} \lesssim \| u_{\geq N} \|_{L^2_{7,1}} \| u_{<N} \|_{L^6_{7,1}} \| \nabla u_{<N} \|_{L^6_{2,7}} \lesssim \eta N^{-1} (1 + N|I_0|),
\]

the second term,

\[
\| u_{\geq N} \|_{L^2_{7,1}} \| \nabla u_{<N} \|_{L^1_{2,1}} \lesssim \| u_{\geq N} \|_{L^2_{7,1}} \| u_{<N} \|_{L^6_{7,1}} \| \nabla u_{<N} \|_{L^6_{2,7}} \lesssim \eta N^{-1} (1 + N|I_0|),
\]

and the third term,

\[
\| u_{\geq N} \|_{L^2_{7,1}} \| \nabla u_{<N} \|_{L^1_{2,1}} \lesssim \| u_{\geq N} \|_{L^2_{7,1}} \| u_{<N} \|_{L^6_{7,1}} \| \nabla u_{<N} \|_{L^6_{2,7}} \lesssim \eta N^{-1} (1 + N|I_0|).
\]

Combining these estimates then gives

\[
(II) \lesssim \eta N^{-1} (1 + N|I_0|) + \sum_{i=1}^{3} (I)_{i}.
\]

To continue, we estimate the remaining term, \((III)\). In a similar manner as above, we use the identity

\[
\{ P_{<N}[u(t)^3], u_{\geq N}(t) \} = \nabla (P_{<N}[u(t)^3] u_{\geq N}(t)) - 2u_{\geq N}(t) \nabla P_{<N}[u(t)^3]
\]

1872
and integrate by parts in the first term of the resulting integral to obtain

\[ (III) \lesssim \int_0^T \int_{\mathbb{R}^2} \frac{|P_{<N}[u(t, x)]^3|u_{\geq N}(t, x)|}{|x|} \, dx \, dt + \int_0^T \int_{\mathbb{R}^2} |u_{\geq N}(t, x)\nabla P_{<N}[u(t, x)]^3| \, dx \, dt \]

\[ \lesssim \sum_{i=0}^2 \left\| P_{<N}[u_i^2 u_{\geq N}]^3 u_{\geq N} \right\|_{L_t^1 L_x^\infty} \| u_{\geq N} \nabla P_{<N}[u_{\geq N}^3] \|_{L_t^3 L_x^\infty}. \]

We estimate the terms containing the gradient and remark that the other terms may then be bounded through the use of the Hardy-type inequality (4.4). In particular, we apply the Hölder, Bernstein and Sobolev inequalities along with corollary 3.2 to obtain, for the first term (using the bound from (4.6)),

\[ \left\| u_{\geq N} \nabla P_{<N}[u_{\geq N}^3] \right\|_{L_t^1 L_x^\infty} \lesssim \left\| u_{\geq N} \right\|_{L_t^\infty L_x^{13/4}} \left\| \nabla P_{<N}[u_{\geq N}^3] \right\|_{L_t^1 L_x^2} \lesssim \eta N^{-1}(1 + N |I_0|) \]

for the second term,

\[ \left\| u_{\geq N} \nabla P_{<N}[u_{\geq N}^2 u_{\geq N}] \right\|_{L_t^1 L_x^\infty} \lesssim \left\| u_{\geq N} \right\|_{L_t^\infty L_x^2} \left\| \nabla P_{<N}[u_{\geq N}^2 u_{\geq N}] \right\|_{L_t^1 L_x^2} \lesssim \eta N^{-1}(1 + N |I_0|) \]

for the third term,

\[ \left\| u_{\geq N} \nabla P_{<N}[u_{\geq N}^2 u_{\geq N}] \right\|_{L_t^1 L_x^\infty} \lesssim \left\| u_{\geq N} \right\|_{L_t^{13/7} L_x^{20/7}} \left\| \nabla P_{<N}[u_{\geq N}^2 u_{\geq N}] \right\|_{L_t^1 L_x^{13/13}} \lesssim \eta N^{-1}(1 + N |I_0|) \]

and for the fourth term,

\[ \left\| u_{\geq N} \nabla P_{<N}[u_{\geq N}^3] \right\|_{L_t^1 L_x^\infty} \lesssim \left\| u_{\geq N} \right\|_{L_t^{20/7} L_x^{20/7}} \left\| \nabla P_{<N}[u_{\geq N}^3] \right\|_{L_t^1 L_x^{20/13}} \lesssim \eta N^{-1}(1 + N |I_0|) \]

Combining these estimates, we obtain

\[ \int_0^T \int_{\mathbb{R}^2} \frac{|u_{\geq N}(t, x)|^4}{|x|} \, dx \, dt \lesssim \eta (N^{-1} + |I_0|) + C_\epsilon (I_1) + \epsilon \int_0^T \int_{\mathbb{R}^2} \frac{|u_{\geq N}(t, x)|^4}{|x|} \, dx \, dt. \]

Choosing \( \epsilon \) sufficiently small, we obtain

\[ \int_0^T \int_{\mathbb{R}^2} \frac{|u_{\geq N}(t, x)|^4}{|x|} \, dx \, dt \lesssim \eta (N^{-1} + |I_0|) \]

as desired. \( \square \)
5. Proof of theorem 1.4

As described in the introduction, in order to prove theorem 1.4 it suffices to rule out the possibility of solutions satisfying the properties given in theorem 1.6.

To handle the case $|I^*| < \infty$, treated in section 5.1, we show that the solution at time $t$ must be supported in space inside a ball centred at the origin with radius shrinking to 0 as $t$ approaches the blow-up time. This is then shown to be incompatible with the conservation of energy.

On the other hand, to handle the case $|I^*| = \infty$, which we treat in section 5.2, we observe that for given $\eta > 0$ theorem 1.3 implies the bound
\[
\int_{I_0} \int_{\mathbb{R}^3} \frac{|u_N(t, x)|^4}{|x|} \, dx \, dt \lesssim \eta (N^{-1} + |I_0|)
\]
for $N$ sufficiently small and all $I_0 \subset I^*$ compact. We then obtain a bound from below on the left-hand side of this inequality by a multiple of $|I_0|$ (up to a small error term). Choosing $\eta$ sufficiently small then gives the desired contradiction.

5.1. Finite time blow-up solution

In this section, we rule out the existence of finite time blow-up solutions satisfying the properties stated in theorem 1.6. Arguing as in [2, 13, 20, 21], this is accomplished by showing that such solutions must have zero energy, which in the defocusing case implies that the solution must be identically zero, contradicting its blow-up.

In particular, we have the following theorem.

**Theorem 5.1.** Suppose that $u$ is an almost periodic solution to (NLW) with maximal interval of existence $I$, satisfying the properties given in theorem 1.6. Then the case $|I^*| < \infty$ cannot occur.

**Proof of theorem 1.3.** Let $u$ be given as stated and suppose to the contrary that $|I^*| < \infty$. By the time reversal and scaling symmetries we may assume that $\sup I = 1$.

We first show that
\[
\sup u(t, \cdot), \quad \sup u_t(t, \cdot) \subset B(0, 1-t), \quad 0 < t < 1.
\]
Indeed, the almost periodicity of $u$ in the form of (2.2) gives that for all $\epsilon > 0$ there exists $R = R(\epsilon) > 0$ such that for every $0 < s < 1$ we have
\[
\int_{|x| \geq \frac{s}{N(s) + s - t}} |\nabla u(s, x)|^{5/2} + |u_t(s, x)|^{5/2} \, dx < \epsilon.
\]
An invocation of the finite speed of propagation (see, for instance, [2, proposition 5.1]) then gives
\[
\int_{|x| \geq \frac{s}{N(s) + s - t}} |\nabla u(t, x)|^{5/2} + |u_t(t, x)|^{5/2} \, dx \leq \epsilon
\]
whenever $0 < t < s < 1$, yielding
\[
\limsup_{s \to 1} \int_{|x| \geq \frac{s}{N(s) + s - t}} |\nabla u(t, x)|^{5/2} + |u_t(t, x)|^{5/2} \, dx \leq \epsilon
\]
for $t \in (0, 1)$. On the other hand, recalling $N(t) \to \infty$ as $t \to 1$ (a consequence of the local theory and the almost periodicity), for all $t \in (0, 1)$ and $\eta > 0$ we have
\[
\{ x : |x| \geq 1 - t + \eta \} \subset \left\{ x : |x| \geq \frac{R}{N(s)} + s - t \right\}.
\]
when \( s = s(t, \eta) \) is sufficiently close to 1. Combining this inclusion with (5.2) and letting \( \eta \) and \( \epsilon \) tend to zero, we obtain

\[
\int_{|x| \geq 1 - t} |\nabla u(t, x)|^{5/2} + |u_t(t, x)|^{5/2} \, dx = 0,
\]

which in turn yields that \((u(t, \cdot)\) is constant a.e. on \( \{x : |x| \geq 1 - t\} \) as well as \( \text{supp} \, u_t(t, \cdot) \subseteq B(0, 1 - t) \). To bound the support of \( u \), we note that \( u \) belongs to \( L^\infty_t L^4_x \), which gives (5.1).

To continue, by (5.1), we write the energy by

\[
E(u, u_t) = \int_{|x| \leq 1 - t} \frac{1}{2} |\nabla u(t, x)|^2 + \frac{1}{2} |u_t(t, x)|^2 + \frac{1}{4} |u(t, x)|^4 \, dx \\
\lesssim (1 - t) \left[ \|\nabla u(t)\|^2_{L^5_x} + \|u_t(t)\|^2_{L^5_x} + \|u(t)\|^4_{L^5_x} \right] \\
\lesssim u 1 - t
\]

where we have used the \textit{a priori} bound \((u, u_t) \in L^\infty_t (H^{3/2}_x \times H^{5/2-1}_x)\). Letting \( t \to 1 \) and using the conservation of energy, we obtain \( u \equiv 0 \), contradicting its blow-up. \( \square \)

5.2. Infinite time blow-up solution

In this section, we consider the second class of solutions identified in theorem 1.6, almost periodic solutions to (NLW) which blow up in infinite time. By making use of a frequency localized variant of the concentration of potential energy along with the frequency localized Morawetz estimate obtained in section 4, we obtain a bound on the length of the maximal interval of existence, contradicting the assumption of infinite time blow-up. When combined with the results of the previous section, this completes the proof of theorem 1.4.

In particular, we prove the following theorem.

**Theorem 5.2.** There is no solution \( u \) to (NLW) satisfying the properties of theorem 1.6 with \(|I^*| = \infty\).

**Proof of theorem 1.3.** Suppose to the contrary that such a solution \( u \) existed. We begin by showing that there exists \( C > 0 \) and \( N_0 > 0 \) such that for all \( N \leq N_0 \) and every \( k \geq 1 \),

\[
\int_J \int_{|x| \leq C/\tilde{N}_k} |u(t, x)|^4 \, dx \, dt \geq u \tilde{N}_k^{-2}.
\]  

(5.3)

To show this claim, we recall that [21, lemma 2.6] gives the existence of \( C \) such that for every \( k \geq 1 \),

\[
\int_J \int_{|x| \leq C/\tilde{N}_k} |u(t, x)|^4 \, dx \, dt \geq u \tilde{N}_k^{-2}.
\]

An application of Minkowski's inequality then gives

\[
\left( \int_J \int_{|x| \leq C/\tilde{N}_k} |u(t, x)|^4 \, dx \, dt \right)^{1/4} \\
\geq \left( \int_J \int_{|x| \leq C/\tilde{N}_k} |u(t, x)|^4 \, dx \, dt \right)^{1/4} - \left( \int_J \int_{|x| \leq C/\tilde{N}_k} |u(t, x)|^4 \, dx \, dt \right)^{1/4} \\
\geq u \tilde{N}_k^{-2} - \left( \int_J \int_{|x| \leq C/\tilde{N}_k} |u(t, x)|^4 \, dx \, dt \right)^{1/4}.
\]

(5.4)
On the other hand, fixing $\eta_1 > 0$ and applying Hölder’s inequality along with remark 1.7, we obtain that for $N$ sufficiently small
\begin{align*}
\int_{|x| \leq C/N_k} |u_{\leq N/2}(t, x)|^4 \, dx \lesssim \tilde{N}_k^{-1} \left( \int_{|x| \leq C/N_k} |u_{\leq N/2}(t, x)|^5 \, dx \right)^{4/5} \\
\lesssim \eta \tilde{N}_k^{-1} \|u_{\leq N/2}\|_{L^6}^4 \lesssim \eta_1^4/N_k.
\end{align*}
This implies the bound
\begin{equation}
\int_{|x| \leq C/N_k} |u_{\leq N/2}(t, x)|^4 \, dx \lesssim \frac{n_1^4}{N_k^2},
\end{equation}
so that, after choosing $\eta_1$ sufficiently small and substituting this bound into (5.4), we obtain (5.3).

We now fix $\eta > 0$ to be determined later in the argument and recall that theorem 1.3 implies the existence of $N_1 \in (0, N_0)$ such that for all $N \leq N_1$ and $I_0 = \cup J_k \subset I$ compact,
\begin{equation}
\int_{I_0} \int_{\mathbb{R}^n} |u_{\geq N}(t, x)|^4 \, dx \, dt \lesssim \eta (N^{-1} + |I_0|). \tag{5.5}
\end{equation}
Combining (5.5) with (5.3) then gives
\begin{align*}
\eta(N^{-1} + |I_0|) \geq \eta \sum_{J_k \subset I_0} \int_{|x| \leq C/N_k} |u_{\geq N}(t, x)|^4 \, dx \, dt \\
\geq \eta \sum_{J_k \subset I_0} \tilde{N}_k \int_{|x| \leq C/N_k} |u_{\geq N}(t, x)|^4 \, dx \, dt \\
\geq \eta \sum_{J_k \subset I_0} \tilde{N}_k^{-1} \\
\geq \eta |I_0| \tag{5.6}
\end{align*}
for all $N \leq N_1$, where we have used the assumption of radial initial data to ensure that $x(t) \equiv 0$ in the definition of almost periodicity (in the general case, the factor $1/|x|$ above is replaced by $1/|x - x(t)|$). Choosing $\eta$ sufficiently small (depending on the constant in (5.6)), we obtain the bound
\begin{equation}
|I_0| \lesssim \eta N^{-1}
\end{equation}
for all $N \leq N_1$ and all $I_0$. Fixing $N$ and letting $I_0$ tend to $I^*$ then gives the desired contradiction. \square

**Acknowledgments**

The author would like to thank M Visan as well as N Pavlović and W Beckner for discussions concerning the content of this paper. The author also wishes to thank M Visan for careful reading and comments on the manuscript.

This material is based upon work supported by the National Science Foundation under agreement Nos DMS-0635607 and DMS-0808042. Any opinions, finding and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.
References

[1] Beckner W 2008 Pitt’s inequality with sharp convolution estimates Proc. Am. Math. Soc. 136 1871–85
[2] Bulut A 2010 Global well-posedness and scattering for the defocusing energy-supercritical cubic nonlinear wave equation J. Funct. Anal. 263 1609–60
[3] Bulut A 2011 The defocusing energy-supercritical cubic nonlinear wave equation in dimension five Trans. Am. Math. Soc. at press (arXiv:1112.0629)
[4] Bulut A, Czubak M, Li D, Pavlovic N and Zhang X 2009 Stability and unconditional uniqueness of solutions for energy critical wave equations in high dimensions Commun. Partial Diff. Eqns 38 575–607
[5] Christ F M and Weinstein M 1991 I. Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation J. Funct. Anal. 100 87–109
[6] Colliander J, Keel M, Staffilani G, Takaoka H and Tao T 2008 Global well-posedness, scattering and blow-up for the energy-critical nonlinear Schrödinger equation in ℝ3 Ann. Math. 167 767–865
[7] Dodson B 2009 Global well-posedness and scattering for the defocusing, L2-critical, nonlinear Schrödinger equation when d \geq 3 J. Am. Math. Soc. 25 429–63
[8] Ginibre J and Velo G 1995 Generalized Strichartz inequalities for the wave equation J. Funct. Anal. 133 50–68
[9] Herbst I W 1977 Spectral theory of the operator (p2 + m2)1/2 – Ze2/r Commun. Math. Phys. 53 285–94
[10] Keel M and Tao T 1998 Endpoint Strichartz estimates Am. J. Math. 120 955–80
[11] Kenig C and Merle F 2006 Global well-posedness, scattering and blow-up for the energy critical, focusing, non-linear Schrödinger equation in the radial case Invent. Math. 166 645–75
[12] Kenig C and Merle F 2008 Global well-posedness, scattering and blow-up for the energy critical focusing non-linear wave equation Acta Math. 201 147–212
[13] Kenig C and Merle F 2008 Nondispersive radial solutions to energy supercritical non-linear wave equations, with applications Am. J. Math. 133 1029–65
[14] Kenig C and Merle F 2011 Radial solutions to energy supercritical wave equations in all odd dimensions Discrete Contin. Dyn. Syst. 31 1365–81
[15] Kenig C and Ponce G and Vega L 1993 Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle Commun. Pure Appl. Math. 46 527–620
[16] Killip R, Tao T and Visan M 2009 The cubic nonlinear Schrödinger equation in two dimensions with radial data J. Eur. Math. Soc. (JEMS) 11 1203–58
[17] Killip R and Visan M 2010 The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher Am. J. Math. 132 361–424
[18] Killip R and Visan M 2009 Nonlinear Schrödinger equations at critical regularity. Evolution equations (Clay Mathematics Institute Lecture Notes vol 17) (Providence, RI: American Mathematical Society) pp 325–437
[19] Killip R and Visan M 2010 Energy-supercritical NLS: critical H1-bounds imply scattering Commun. Partial Diff. Eqns 35 945–87
[20] Killip R and Visan M 2011 The defocusing energy-supercritical nonlinear wave equation in three space dimensions Trans. Am. Math. Soc. 363 3893–934
[21] Killip R and Visan M 2011 The radial defocusing energy-supercritical nonlinear wave equation in all space dimensions Proc. Am. Math. Soc. 139 1805–17
[22] Killip R and Visan M 2012 Global well-posedness and scattering for the defocusing quintic NLS in three dimensions Anal. PDE 5 855–85
[23] Morawetz C 1975 Notes on time decay and scattering for some hyperbolic problems Regional Conf. Series in Applied Mathematics No 19 (Philadelphia, PA: SIAM)
[24] Morawetz C and Strauss W 1972 Decay and scattering of solutions of a nonlinear relativistic wave equation Commun. Pure Appl. Math. 25 1–31
[25] Tao T, Visan M and Zhang X 2008 Minimal-mass blowup solutions of the mass-critical NLS Forum Math. 20 881–919
[26] Visan M 2011 Global well-posedness and scattering for the defocusing cubic NLS in four dimensions Int. Math. Res. Not. 2012 1037–67