AN OPTIMAL TWO PARAMETER BOUNDS FOR THE IDENTRIC MEAN

OMRAN KOUBA

Abstract. In this note we obtain sharp bounds for the identric mean in terms of a two parameter family of means. Our results generalize and extend recent bounds due to Y. M. Chu & al. (2011), and to M.-K. Wang & al. (2012).

1. Introduction

Given two distinct positive real numbers \(a\) and \(b\), we recall that the arithmetic mean \(A(a, b)\), the geometric mean \(G(a, b)\), the harmonic mean \(H(a, b)\), and the identric mean \(I(a, b)\), are respectively defined by

\[
A(a, b) = \frac{a + b}{2}, \quad G(a, b) = \sqrt{ab}, \quad H(a, b) = \frac{2ab}{a + b}, \quad I(a, b) = \frac{1}{e} \left( \frac{a^a}{b^b} \right)^{1/(a-b)}.
\]

Inequalities relating means in two arguments have attracted and continue to attract the attention of mathematicians. Many recent papers were concerned in comparing these means.

For instance, H. Alzer and S. Qui considered in [1] the following inequality relating the identric, geometric and arithmetic means:

\[
\alpha A(a, b) + (1 - \alpha) G(a, b) < I(a, b) < \beta A(a, b) + (1 - \beta) G(a, b),
\]

they proved that it holds, for every distinct positive numbers \(a\) and \(b\), if and only if \(\alpha \leq 2/3\) and \(\beta \geq 2/e\).

This was later complemented by T. Trif [6] who proved that, for \(p \geq 2\) and every distinct positive numbers \(a\) and \(b\), we have

\[
\alpha A^p(a, b) + (1 - \alpha) G^p(a, b) < I^p(a, b) < \beta A^p(a, b) + (1 - \beta) G^p(a, b),
\]

if and only if \(\alpha \leq (2/e)^p\) and \(\beta \geq 2/3\).

In another direction we proved in [3] that the inequality

\[
I^p(a, b) < \frac{2}{3} A^p(a, b) + \frac{1}{3} G^p(a, b)
\]

holds true for every distinct positive numbers \(a\) and \(b\), if and only if \(p \geq \ln \left( \frac{3}{2} \right) / \ln \left( \frac{5}{3} \right) \approx 1.3214\), and that the reverse inequality holds true for every distinct positive numbers \(a\) and \(b\), if and only if \(p \leq 6/5 = 1.2\).

In this paper we consider the two parameter family of means \(Q_{t,s}(a, b)\), defined for \(s \geq 1\) and \(t \in [0, 1/2]\), by

\[
Q_{t,s}(a, b) = G^s(ta + (1-t)b, tb + (1-t)a)A^{1-s}(a, b). \tag{1.1}
\]
Similar means were previously considered by several authors. For instance
\[ Q_{t,2}(a, b) = H(ta + (1 - t)b, tb + (1 - t)a) \]
was considered in by Y.-M. Chu, M.-K. Wang and Z.-K. Wang in [2] where it was compared to the identric mean. The same authors compared also
\[ Q_{t,1}(a, b) = G(ta + (1 - t)b, tb + (1 - t)a) \]
to the identric mean in their recent work [7].

We will see later that, for distinct positive real numbers \(a\) and \(b\), the function \(t \mapsto Q_{t,s}(a, b)\) is continuous and increasing. Moreover, for \(s \geq 1\) and every distinct positive numbers \(a\) and \(b\), we have
\[ Q_{0,s}(a, b) \leq Q_{0,1}(a, b) = G(a, b) < I(a, b) < A(a, b) = Q_{1/2,s}(a, b). \]
Therefore, it is natural to consider, for \(s \geq 1\), the sets
\[
\mathcal{L}_s = \{ t \in [0, 1/2] : \text{for all positive } a, b \text{ with } a \neq b, Q_{t,s}(a, b) < I(a, b) \}, \\
\mathcal{U}_s = \{ t \in [0, 1/2] : \text{for all positive } a, b \text{ with } a \neq b, I(a, b) < Q_{t,s}(a, b) \}.
\]

Using the fact that \(t \mapsto Q_{t,s}(a, b)\) is increasing, we see that \(\mathcal{L}_s\) and \(\mathcal{U}_s\) are intervals.

In this work, (see Theorem 3.1), we will determine in terms of \(s \geq 1\), the values \(p_s \in (0, 1/2)\) and \(q_s \in (0, 1/2)\) such that \(\mathcal{L}_s = [0, p_s]\) and \(\mathcal{U}_s = [q_s, 1/2]\). These results extend those of Y.-M. Chu \& al. [2] and M.-K. Wang \& al. [7], with simpler and unified proofs.

\section{Preliminaries}

The following lemmas pave the way to the main theorem. In the next Lemma 2.1 we study a family of functions, using simple methods from classical analysis.

\textbf{Lemma 2.1.} For \(s \geq 1\) and \(u \in [0, 1]\), we consider the real function \(f_{u,s}\) defined on \([0, 1)\) by
\[
f_{u,s}(x) = 1 - \frac{1}{2x} \ln \left( \frac{1 + x}{1 - x} \right) - \frac{1}{2} \ln(1 - x^2) + \frac{s}{2} \ln(1 - ux^2). \tag{2.1}
\]
(a) The necessary and sufficient condition to have \(f_{u,s}(x) > 0\) for \(x \in (0, 1)\), is that \(3su \leq 1\).
(b) The necessary and sufficient condition to have \(f_{u,s}(x) < 0\) for \(x \in (0, 1)\), is that \(u + (2/e)^{2/s} \geq 1\).

\textit{Proof.} We consider only the case \(u \in (0, 1]\), since \(f_{0,s}\) is independent of \(s\) and positive on \((0, 1)\). It is straightforward to see that \(f'_{u,s}(x) = h_{u,s}(x)/x^2\) where
\[
h_{u,s}(x) = -x + \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right) - \frac{sux^3}{1 - ux^2}
\]
and that
\[
h'_{u,s}(x) = \frac{x^2}{(1 - x^2)(1 - ux^2)^2} T_{u,s}(x^2)
\]
where \(T_{u,s}\) is the trinomial defined by
\[
T_{u,s}(X) = (1 - s)u^2 X^2 - (2 - 3s - su)u X + (1 - 3su).
\]
Noting that \( T_{u,s}(1) = (1-u)^2 \geq 0 \) and \( T_{u,s}(0) = 1 - 3su \), we see that we have two cases:

- First, \( T_{u,s}(0) \geq 0 \), or equivalently \( 3su \leq 1 \). Again, we distinguish two cases:
  
  - If \( s = 1 \), then clearly the zero of \( T_{u,1} \) does not belong to \((0, 1)\) and \( T_{u,s} \) has a positive sign on \((0, 1)\).
  
  - If \( s > 1 \), then the coefficient of \( X^2 \) in \( T_{u,s} \) is negative, and the fact that both \( T_{u,s}(0) \) and \( T_{u,s}(1) \) are nonnegative, implies that \( z_0 \leq 0 < 1 \leq z_1 \) where \( z_0 \) and \( z_1 \) are the zeros of \( T_{u,s} \). Hence, \( T_{u,s} \) has also a positive sign on \((0, 1)\) in this case.

It follows that in this case \( h_{u,s} \) is increasing on \([0, 1)\). But \( h_{u,s}(0) = 0 \), so \( h_{u,s} \) is positive on \((0, 1)\). This implies that \( f_{u,s} \) is increasing on \((0, 1)\). Finally, the fact that \( \lim_{x \to 0^+} f_{u,s}(x) = 0 \) implies that \( f_{u,s}(x) > 0 \) for every \( x \in (0, 1) \) in this case.

- Second, \( T_{u,s}(0) < 0 \), or equivalently \( 3su > 1 \). This means that \( T_{u,s} \) has a unique zero \( z_0 \) in the interval \((0, 1)\), (because \( \text{deg}(T_{u,s}) \leq 2 \)).

  - If \( u = 1 \), then \( z_0 = 1 \) and \( h_{1,s} \) is decreasing on \([0, 1]\). But \( h_{1,s}(0) = 0 \), so \( h_{1,s} \) is negative on \((0, 1)\). This implies that \( f_{1,s} \) is decreasing on \((0, 1)\). Finally, we have \( \lim_{x \to 0^+} f_{1,s}(x) = 0 \) and consequently \( f_{1,s}(x) < 0 \) for every \( x \in (0, 1) \).

  - If \( u < 1 \), then \( z_0 \in (0, 1) \). So \( h_{u,s} \) is decreasing on \([0, z_0]\) and increasing on \([z_0, 1]\). But \( h_{u,s}(0) = 0 \) so \( h_{u,s}(z_0) < 0 \). On the other hand \( \lim_{x \to 1^-} h_{u,s}(x) = +\infty \). So there exists a unique real number \( y_0 \in (z_0, 1) \) such that \( h_{u,s}(y_0) = 0 \). Thus \( h_{u,s}(x) < 0 \) for \( x \in (0, y_0) \) and \( h_{u,s}(x) > 0 \) for \( x \in (y_0, 1) \). This implies that \( f_{u,s} \) is decreasing on \((0, y_0)\) and increasing on \((y_0, 1)\). Finally, we have \( \lim_{x \to 0^+} f_{u,s}(x) = 0 \) and \( \lim_{x \to 1^-} f_{u,s}(x) = \ln(e(1-u)s/2) \).

This shows that the necessary and sufficient condition for \( f_{u,s} \) to be negative on \((0, 1)\) is that \( u = 1 \) or \( u < 1 \) and \( \ln(e(1-u)s/2) \leq 0 \) which is equivalent to the condition \( 1 \leq u + (2/e)^{2/s} \).

This achieves the proof of Lemma 2.1 \( \square \)

Next we introduce the set \( \mathcal{D} \) defined as follows:

\[
\mathcal{D} = \{(a, b) \in \mathbb{R}^2 : a > b > 0 \}.
\]

It is sufficient to consider couples \((a, b)\) from \( \mathcal{D} \), since the considered means are symmetric functions of their arguments. The next Lemma 2.2 explains why the family of functions studied in Lemma 2.1 is important to our study.

**Lemma 2.2.** Consider \((a, b) \in \mathcal{D}\) and let \( v = \frac{a-b}{a+b} \).

(a) For \( s \geq 1 \) and \( t \in [0, 1/2] \), we have

\[
\ln \left( \frac{Q_{t,s}(a,b)}{A(a,b)} \right) = \frac{s}{2} \ln \left( 1 - (1 - 2t)^2 v^2 \right).
\]

(b) Also, for the identric mean we have

\[
\ln \left( \frac{I(a,b)}{A(a,b)} \right) = -1 + \frac{1}{2} \ln(1 - v^2) + \frac{1}{2v} \ln \left( \frac{1+v}{1-v} \right).
\]
Proof. Indeed, (a) follows from the simple fact that
\[ G(ta + (1 - t)b, tb + (1 - t)a) = A(a, b) \sqrt{1 - (1 - 2t)^2 \left(\frac{a - b}{a + b}\right)^2} \]
To see (b) we note that
\[ I(a, b) \leq A(a, b) = 1 + e^{-2a/b} \left(\frac{a}{a+b}\right)^{\frac{a-b}{2(a+b)}} \left(1 - \frac{a-b}{a+b}\right)^{\frac{a+b}{2(a+b)}} \]
\[ = e \left(1 + \frac{a-b}{a+b}\right)^{\frac{a+b}{2(a+b)}} \left(1 - \frac{a-b}{a+b}\right)^{\frac{a+b}{2(a+b)}} \]
\[ = e \left(1 + v\right)^{\frac{1}{2} + \frac{a+b}{2(a+b)}} \left(1 - v\right)^{\frac{1}{2} - \frac{a+b}{2(a+b)}} = 1 \]
This concludes the proof of Lemma 2.1.

Remark 2.1. In particular, it follows from Lemma 2.1 (a), that the function \( t \mapsto Q_{t,s}(a, b) \) is continuous and increasing as announced in the introduction.

Remark 2.2. Combining (a) and (b) from Lemma 2.1 we see immediately that if \( f_{a,s} \) is the function defined in Lemma 2.1 then, for every \((a, b) \in D\) we have
\[ \ln \left(\frac{Q_{t,s}(a, b)}{I(a, b)}\right) = f_{(1-2t)^2,s} \left(\frac{a-b}{a+b}\right), \]
and this explains the importance of the family of functions studied in Lemma 2.1 to our study.

3. The Main Theorem

Theorem 3.1. Let \( s \) be a real number such that \( s \geq 1 \), and define the sets
\[ \mathcal{L}_s = \{ t \in [0, 1/2] : \forall (a, b) \in D, \ Q_{t,s}(a, b) < I(a, b) \}, \]
\[ \mathcal{U}_s = \{ t \in [0, 1/2] : \forall (a, b) \in D, \ I(a, b) < Q_{t,s}(a, b) \}. \]
Then
\[ \mathcal{L}_s = \left[0, \frac{1}{2} - \frac{1}{2} \sqrt{1 - \left(\frac{2}{e}\right)^{2/s}}\right] \quad \text{and} \quad \mathcal{U}_s = \left[\frac{1}{2} - \frac{1}{2\sqrt{3s}}, \frac{1}{2}\right]. \]

Proof. First note that
\[ \left\{ \frac{a-b}{a+b} : (a, b) \in D \right\} = (0, 1). \]
So, using Remark 2.2 we see that \( t \in \mathcal{L}_s \) if and only if \( f_{(1-2t)^2,s}(x) < 0 \) for every \( x \in (0, 1) \). Using Lemma 2.1 we see that this is equivalent to \((1 - 2t)^2 + (2/e)^{2/s} \geq 1 \) or \((1 - \sqrt{1 - (2/e)^{2/s}})/2 \geq t \). This proves that
\[ \mathcal{L}_s = \left[0, \frac{1}{2} - \frac{1}{2} \sqrt{1 - \left(\frac{2}{e}\right)^{2/s}}\right]. \]
Similarly using Remark 2.2 we see that \( t \in \mathcal{U}_s \) if and only if \( f_{(1-2t),s}(x) > 0 \) for every \( x \in (0,1) \). Using Lemma 2.1 again we see that this is equivalent to \( 3s(1-2t)^2 \leq 1 \) or \( (1-1/\sqrt{3s})/2 \leq t \). This proves that
\[
\mathcal{U}_s = \left[ \frac{1}{2} - \frac{1}{2\sqrt{3s}}, \frac{1}{2} \right].
\]
The proof of Theorem 3.1 is complete. \( \square \)

The following two corollaries correspond to the particular cases \( s = 2 \) and \( s = 1 \). They give the bounds obtained in [2] and [7].

**Corollary 3.2** (see [2]). The necessary and sufficient condition on \( p, q \) from \([0,1/2]\) to have
\[
H(pa + (1-p)b, pb + (1-p)a) < I(a,b) < H(qa + (1-q)b, qb + (1-q)a)
\]
for every distinct positive numbers \( a \) and \( b \), is that
\[
p \leq 1 - \frac{\sqrt{1-2/e}}{2} \quad \text{and} \quad q \geq \frac{6 - \sqrt{6}}{12}.
\]

**Corollary 3.3** (see [7]). The necessary and sufficient condition on \( p, q \) from \([0,1/2]\) to have
\[
G(pa + (1-p)b, pb + (1-p)a) < I(a,b) < G(qa + (1-q)b, qb + (1-q)a)
\]
for every distinct positive numbers \( a \) and \( b \), is that
\[
p \leq 1 - \frac{\sqrt{1-4/e^2}}{2} \quad \text{and} \quad q \geq \frac{3 - \sqrt{3}}{6}.
\]

In the next corollary, the lower bound is an inequality due to H.-J. Seiffert [5], and can be also found in [4]. While the upper bound is new and to be compared with the results of J. Sándor and T. Trif in [4].

**Corollary 3.4.** For every positive numbers \( a \) and \( b \), we have
\[
\exp \left( \frac{1}{6} \left( \frac{a - b}{a + b} \right)^2 \right) \leq \frac{A(a,b)}{I(a,b)} \leq \exp \left( \frac{\ln e}{2} \left( \frac{a - b}{a + b} \right)^2 \right)
\]

**Proof.** Indeed, for \( s \geq 1 \) let
\[
p_s = \frac{1}{2} - \frac{1}{2} \sqrt{1 - \left( \frac{2}{e} \right)^{2/s}} \quad \text{and} \quad q_s = \frac{1}{2} - \frac{1}{2\sqrt{3s}}.
\]

Using Theorem 3.1 for every \( (a,b) \in \mathcal{D} \), we have
\[
Q_{p_s,s}(a,b) < I(a,b) < Q_{q_s,s}(a,b).
\]
This can be written as follows
\[
\frac{A(a,b)}{Q_{p_s,s}(a,b)} < \frac{A(a,b)}{I(a,b)} < \frac{A(a,b)}{Q_{q_s,s}(a,b)}.
\]
and using Lemma 2.2, it is equivalent to
\[
\left(1 - \frac{v^2}{3s}\right)^{-s/2} < \frac{A(a, b)}{I(a, b)} < \left(1 - \left(1 - \left(\frac{2}{e}\right)^{2/s}\right)v^2\right)^{-s/2}
\]
where \(v = (a - b)/(a + b)\). Now letting \(s\) tend to \(+\infty\) we obtain
\[
e^{v^2/6} \leq \frac{A(a, b)}{I(a, b)} \leq e^{(\ln e)(v^2)},
\]
which is the conclusion of Corollary 3.4.

In fact, because of the “limit argument” in the proof of Corollary 3.4, we lost the strict inequalities for distinct positive real arguments. But, studying the family of functions \((g_t)_{t \in (0, +\infty)}\) defined by
\[
g_t(x) = 1 - \frac{1}{2x} \ln \left(\frac{1 + x}{1 - x}\right) - \frac{1}{2} \ln(1 - x^2) - tx^2,
\]
using similar arguments to those used in Lemma 2.1, we can prove the following exact version of Corollary 3.4, which extends the results of Seiffert [5] and those of Sándor and Trif [4].

**Theorem 3.5.** The necessary and sufficient condition on \(p, q\) from \((0, +\infty)\) to have
\[
\forall (a, b) \in \mathcal{D}, \quad \exp \left(p \left(\frac{a - b}{a + b}\right)^2\right) < \frac{A(a, b)}{I(a, b)} < \exp \left(q \left(\frac{a - b}{a + b}\right)^2\right)
\]
is that \(p \leq \frac{1}{6}\) and \(q \geq \ln(\frac{\ln e}{2})\).

**References**

[1] H. ALZER and S.-L. QIU, Inequalities for means in two variables, *Arch. Math.*, (Basel), 80 (2003), 201–215.

[2] Y.-M. CHU, M.-K. WANG, and Z.-K. WANG, A Sharp Double Inequality between Harmonic and Identric Means, *Abstract and Applied Analysis*, vol. 2011, Article ID 657935, (2011), 7 pages.

[3] O. KOUBA, New bounds for the identric mean of two arguments, *J. Inequal. Pure and Appl. Math.*, 9(3) (2008) Art.71. [ONLINE : Available at http://jipam.vu.edu.au/article.php?sid=1008].

[4] J. SÁNDOR and T. TRIF, Some new inequalities for means of two arguments, *Int. J. Math. Math. Sci.*, 25 (2001), 525–532.

[5] H.-J. SEIFFERT, Ungleichungen für elementare Mittelwerte [Inequalities for elementary means], *Arch. Math.* (Basel) 64 no. 2 (1995), 129–131 (German).

[6] T. TRIF, Note on certain inequalities for means in two variables, *J. Inequal. Pure and Appl. Math.*, 6(2) (2005), Art.43. [ONLINE : Available at http://jipam.vu.edu.au/article.php?sid=512].

[7] M.-K. WANG, Z.-K. WANG, AND Y.-M. CHU, An optimal double inequality between geometric and identric means, *Applied Mathematics Letters*, Vol. 25, Issue 3, (2012), 471–475.

Department of Mathematics, Higher Institute for Applied Sciences and Technology, P.O. Box 31983, Damascus, Syria.

E-mail address: omran_kouba@hiast.edu.sy