Solution of the Adiabatic Limit Problem. QED Without Infinites

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Abstract

In this work we give positive solution to the adiabatic limit problem in causal perturbative QED, as well as give a contribution to the solution of the convergence problem for the perturbative series in QED, by using white noise construction of free fields. The method is general enough to be applicable to more general causal perturbative QFT, such as Standard Model with the Higgs field.

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1 Introduction

This work is concerned with the causal perturbative approach to Quantum Field Theories (QFT), initiated by Stückelberg, Bogoliubov and Shirkov [6], and developed mainly by Epstein, Glaser [14], Blanchard, Seneor, Duch [4] [8], Dücht, Krahe and Scharf and Fredenhagen [9]-[12], [13].

In causal perturbative approach to QFT the infra-red-divergence (IR) problem is clearly separated from the ultra-violet-divergence (UV) problem by using a space-time function \( x \mapsto g(x) \) as coupling “constant”. The UV-problem is essentially solved within this approach, [14], – the origin of infinite counter terms of the renormalization scheme is well understood by now, i. e. using the counter terms (renormalization) is equivalent to the causal perturbative construction of the perturbative series due to Bogoliubov-Epstein-Glaser (scalar massive field), developed further for QED, and other physical theories with non abelian gauge mainly by Dücht, Krahe and Scharf, [9]-[12], where no infinite counter terms appear but instead one uses recurrence rules for the construction of the chronological product of fields regarded as operator-valued distributions. The renormalization scheme is now incorporated into the following recurrence rules for the chronological product \([14],[9]-[12],[13],[46]):

1) causality,
2) symmetricity,  
3) unitarity,  
4) Translational covariance (Lorentz covariance is not used),  
5) Ward identities – quantum version of gauge invariance (e. g. in case of QED),  
6) preservation of the Steinmann scaling degree,  

part of the remaining freedom may be reduced by imposing the natural field equations for the interacting field (which is always possible for the standard gauge fields) and the rest of the remaining freedom is pertinent to the Stückelberg-Petermann renormalization group. All the recurrence rules should be regarded as important physical laws which incorporate the whole content of the standard pragmatic approach including the renormalization scheme. Causality implies locality for perturbatively constructed (using the Epstein-Glaser method \[14\]) algebras of localized fields \( \mathcal{F}(\mathcal{O}) \) regarded as “smeared out” operator-valued distributions, where \( g \) is constant (equal to the electric charge in case of QED) within the open space-time region \( \mathcal{O} \) – the only step where the UV-problem shows up and is solved by the use of Epstein-Glaser method. The IR-problem is solved only partially, i. e. nets \( \mathcal{O} \mapsto \mathcal{F}(\mathcal{O}) \) of algebras \( \mathcal{F}(\mathcal{O}) \) of local (unbounded) operator localized fields have likewise been constructed perturbatively \[13\], but in the sense of formal power series only.

The most important and still open problems are the following.

(a) The problem of existence of the adiabatic limit (\( g \mapsto \text{constant function over the whole space-time} \)) in each order separately. This is the IR-problem or the Adiabatic Limit Problem.

(b) The convergence of the formal perturbative series for interacting fields (with \( g = 1 \)).

In this work we give a positive solution to the Adiabatic Limit Problem for QED, i.e. the problem (a), and give a contribution to the problem (b) for QED. The method is based solely on substitution into the casual perturbative series the free fields of the theory which are constructed with the help of white noise calculus. The whole causal perturbative method of Bogliubov-Epstein-Glaser remains unchanged. The whole point in constructing the free fields within the white noise set up lies in the fact that it allows us to treat them equivalently as integral kernel operators with vector-valued kernels in the sense of Obata \[38\], and opens us to the effective theory of such operators worked out by the Japanese School of Hida. Using the calculus of such operators we show that the class of integral kernel operators represented (or representing) free fields allows the operations of differentiation (similarly as Schwartz distributions) integration, point-wise Wick product, integration of Wick product integral kernel operators (including spatial integration), convolution of Wick product integral
kernel operators with tempered distributions, and splitting into advanced and retarded parts of integral kernel operators with causal supports. Thus all operations needed for the causal perturbation series have a well defined mathematical meaning if understood as operations performed upon integral kernel operators in the sense of Obata. Therefore the free fields, understood as integral kernel operators with vector-valued kernels in the sense of Obata, can be inserted into the formulas for the higher order contributions to the interacting fields. After the insertion we obtain each order term contribution to interacting fields in a form of finite sums of well defined integral kernel operators with vector-valued kernels, similarly as for the free fields themselves or for the Wick products of free fields.

But the most essential point is that these formulas do not lose their rigorous mathematical meaning even if we put in them the intensity-of-interaction function $g$ equal 1 everywhere over the whole space-time. The contributions still preserve their meaning of integral kernel operators with vector valued kernels, which belong to the same general class of integral kernel operators as the Wick products of free fields. We therefore arrive at the positive solution of the Problem (a) in QED. But at the same time we obtain the interacting fields in the form of Fock expansions into integral kernel operators with vector-valued kernels in the sense of [38], with precise estimate of the convergence, which allows us to give a computationally effective criteria for the convergence of the perturbative series, i.e non-trivial contribution to the solution of the Problem (b).

The method is general enough to be capable of application to other QFT with non abelian gauge.

In this manner we obtain causal perturbative QED in which there are no infra-red nor ultra-violet divergences and get insight into problems which were beyond the reach of the conventional approach involved into renormalization. In particular we hope that have given a step forward on the way in giving a rigorous construction of a non-trivial (and realistic) quantum interacting field. Some of the prominent analyst place this problem also among the most important unsolved problems in the contemporary analysis, compare [50].

Presented work is the separated part of the work [59] which is focused on the solution of the “Adiabatic Limit Problem”. The whole work [59] also contains exploration of the problems which were beyond the conventional method: 1) analysis of the structure of infra-red states, 2) relationship between the interacting fields and the classical gravitational field. Because both of them require a considerable amount of harmonic analysis on $SL(2, \mathbb{C})$ or a non-trivial extension of the harmonic analysis on $T^4 \otimes SL(2, \mathbb{C})$ over to Krein-isometric representations in the Krein-Hilbert space, we have decided to separate off the part devoted to the “Adiabatic Limit Problem”, not immediately involved into the representation theory.

From the purely mathematical point of view the present work may be considered as an immediate extension of the works: [26], [38], [39], of Hida and his school, on the so called integral kernel operators and Fock expansions into integral kernel operators.
The following Subsection of Introduction gives a more detailed formulation of our result.

1.1 Adiabatic Limit Problem and its solution. Short account

We keep the causal method of Stückelbeg-Bogoliubov-Epstein-Glaser unchanged, with the only proviso: we insert into the formulas the free fields of the theory which are constructed with the help of white noise Hida operators – construction of free fields which goes back to Berezin and later improved by the Japanese school of Hida. This allows us to interpret the free fields as integral kernel operators with vector-valued distribution kernels in the sense of Obata. The rest part of the work is reduced to application of the white noise calculus of integral kernel operators, which essentially is reduced to the proof that the operations involved in the causal perturbative construction of the higher order contributions are well defined when applied to the integral kernel operators defined by free fields. The main difficulty lies in the white noise construction of the free fields, namely the free Dirac and electromagnetic fields $\psi, A$, as finite sums

$$\psi = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0}), \quad A = \Xi_{0,1}(\kappa'_{0,1}) + \Xi_{1,0}(\kappa'_{1,0}) \in \mathcal{L}( (E) \otimes \mathcal{E}, (E)^*)$$

(of two) well defined integral kernel operators, in the sense of Obata \[38\], with vector valued distributional kernels $\kappa, \kappa'$ which belong respectively to

$$\mathcal{L}(E, \mathcal{E}^*),$$

Here $E$ is the respective nuclear space of restrictions of the Fourier transforms $\tilde{\varphi}$ of all space-time test functions $\varphi \in \mathcal{E}$ to the respective orbit $\mathcal{O}$ in the momentum space determining the representation of the $T_4 \otimes SL(2, \mathbb{C})$ acting in the single particle Hilbert space of the respective field, $\psi$ or $A$. $\mathcal{L}(E, \mathcal{E}^*)$ denotes the space of all linear continuous operators $E \to \mathcal{E}^*$, i.e. $\mathcal{E}^*$-valued distributions over the corresponding orbit $\mathcal{O}$ in the momentum space (recall that $\mathcal{O}$ is equal to the positive energy sheet of the hyperboloid $p \cdot p = m^2$ in the momentum space in case of field of mass $m$). We endow $\mathcal{L}(E, \mathcal{E}^*)$ with the natural topology of uniform convergence on bounded sets. $(E), (E)^*$ is the nuclear Hida subspace of the Fock space of the corresponding free field, and its strong dual space.

Moreover in order to construct the useful commutative algebra of operators to which the perturbative expansion can naturally be applied, we need a construction of the free fields, $\psi, A$, with as explicit representation of the Poincaré group in their Fock spaces as possible. Unfortunately no construction of these two most important fields in the whole of QFT, namely $\psi$ and $A$, based on the theory of representations of $T_4 \otimes SL(2, \mathbb{C})$, has been achieved, which is a well known fact, compare \[22\], p. 48, \[33\], \[34\]. This is because this problem cannot be solved within the ordinary unitary representations of the $T_4 \otimes SL(2, \mathbb{C})$ group. We have been forced to extend the Mackey theory of induced representations over to a more general class of representations in order to solve this unsolved problem, compare Section 12 of \[59\] for this extension. But this is not the whole
problem, because we additionally need a white noise constructions of these two free fields $\psi$ and $A$. This construction is essentially worked out for the simplest massive free scalar field by mathematicians [27], and its generalization to other massive fields (if the group theoretical aspect is ignored) presents no essential difficulties. But concerning the mass less fields, such e. g. as $A$, the white noise construction is far not so obvious and in fact (as to the author’s knowledge) has not been done before. This is because the white noise construction of the mass less fields requires the modification of the space-time test space $\mathcal{E}$ which cannot be equal $S(\mathbb{R}^4;\mathbb{C}^4)$ but instead it has to be equal to the space $E = S^0(\mathbb{R}^4;\mathbb{C}^4)$.

Namely $\varphi \in S^0(\mathbb{R}^4;\mathbb{C}^4)$ if and only if its Fourier transform $\tilde{\varphi} \in S^0(\mathbb{R}^4;\mathbb{C}^4)$, and $S^0(\mathbb{R}^4;\mathbb{C}^4)$ is the subspace of $S(\mathbb{R}^4;\mathbb{C}^4)$ of all those functions which have all derivatives vanishing at zero. Correspondingly we have the nuclear algebra $E$ of all restrictions of Fourier transforms to the corresponding orbit $\mathcal{O}$ (positive energy sheet of the cone) of the elements of the test space $\mathcal{E} = S^0(\mathbb{R}^4;\mathbb{C}^4)$, equal to $E = S^0(\mathbb{R}^3;\mathbb{C}^4)$ (of $\mathbb{C}^4$-valued functions in case of the field $A$, but for the $r$-component mass less fields we will have $\mathbb{C}^r$-valued functions here). This is related to the singularity of the cone orbit $\mathcal{O}$ at the apex – the orbit pertinent to the representation associated with mass less fields, i.e. the positive sheet of the cone in the momentum space (note that each sheet of the massive hyperboloid $\mathcal{O}_{m,0,0} = \{p \cdot p = m^2\}$ in the momentum space is everywhere smooth only for the massive orbit of the point $\vec{p} = (m, 0, 0)$ with $m \neq 0$, the zero mass orbits of $\vec{p} = (1, 0, 0, 1)$ or $(-1, 0, 0, 1)$, i.e. the positive and negative energy sheets of the cone are singular at the apex). The need for the modification of the space-time test space $\mathcal{E}$, when passing to mass less fields, may seem unexpected for those readers which compare it with the construction of mass less fields in the sense of Wightman, which allows the ordinary Schwartz test space also for the mass less fields. We nonetheless choose the white noise construction of free fields as much more adequate mathematical interpretation of the (free) quantum field. Among other things the white noise construction provides a much deeper insight into the Wick product construction of free fields at the same space-time point, which moreover fits well with the needs of the causal perturbative approach. “Wick product” construction due to Wighman and Gårding (although also rigorous) is not very much useful for the realistic causal perturbative QFT, such as QED. Again that the Wightman-Gårding “Wick product” is not useful in practical computations such as the causal perturbative approach, or in construction of conserved currents corresponding to the Noether theorem (which in fact is the basis for the Canonical Quantization Postulate) has been recognized by Segal [49], a prominent analyst who devoted much part of his research to the mathematical analysis of the Wick product construction.

Thus we give here white noise construction of the free field $\psi$ with the explicit construction of the representation of $T_4 \otimes SL(2, \mathbb{C})$, compare Sections 2, 2.2, 2.3. The white noise construction of the electromagnetic potential field $A$ in the Gupta-Bleuler gauge with explicit construction of the Krein-isometric representation of $T_4 \otimes SL(2, \mathbb{C})$ acting in the Krein-Fock space of the free field $A$ is given in Sections 4 and 5 of [59]. As to the author’s knowledge it has not been done before. In Subsections 2.4 and 2.10 there are summarized some of
the results obtained in [59] concerning the free field $A$ which are used in this work.

In fact the white noise construction of the free fields is not a new idea and goes back to Berezin. Subsequently it was developed mainly by Hida and his school.

The fact that the test space $S^{00}(\mathbb{R}^4; \mathbb{C}^4)$ contains no non-zero elements with compact support does not destroy splitting of causal homogeneous distributions into retarded and advanced parts, because the pairing functions of mass less fields, such as $A$, are homogeneous distributions. The test space $S^{00}(\mathbb{R}^4; \mathbb{C}^4)$ is flexible enough to contain non-zero element for each conic-type set, supported on this set. This allows splitting of causal homogeneous distributions (Subsection 5.7 or [59]).

Having given the free fields, $\psi$ and $A$, constructed as (finite sums of) integral kernel operators with vector-valued kernels, we show that the operations of differentiation, Wick product at the same space-time point, integration of the Wick product and its convolution with tempered distribution are well defined within the class of integral kernel operators to which the free fields and Wick product belongs (Subsection 2.7). In particular the formulas for each $n$-th order contributions, with the intensity of the interaction function $g = 1$, are equal to finite sums

$$\psi^{(n)}_{int}(g = 1, x) = \sum_{l,m} \Xi_{l,m}(\kappa_{l,m}(x)),$$

$$A^{(n)}_{int}(g = 1, x) = \sum_{l,m} \Xi_{l,m}(\kappa'_{l,m}(x)),$$

of integral kernel operators (similarly we have for $\Xi_{l,m}(\kappa'_{l,m}(x))$)

$$\Xi_{l,m}(\kappa_{l,m}(x)) = \sum_{s_1, \ldots, s_{l+m} \in \mathbb{R}^3(l+m)} \int \kappa_{l,m}(s_1, p_1, \ldots, s_{l+m}, p_{l+m}; x) a_{s_1}(p_1)^+ \cdots a_{s_{l+m}}(p_{l+m}) d^3 p_1 \cdots d^3 p_{l+m},$$

where $a_s(p)^+, a_s(p)$ are the creation and annihilation operators, constructed here as Hida operators in the tensor product of the Fock spaces of the free fields $\psi, A$, in the normal order, with the first $l$ factors equal to the creation operators and the last $m$ equal to the annihilation operators. Here

$$\kappa_{l,m} \in \mathcal{L}(E^{\otimes(l+m)}, \mathcal{E}_1'), \quad \mathcal{E}_1 = S(\mathbb{R}^4; \mathbb{C}^4)$$

$$\kappa'_{l,m} \in \mathcal{L}(E^{\otimes(l+m)}, \mathcal{E}_2'), \quad \mathcal{E}_2 = S^{00}(\mathbb{R}^4; \mathbb{C}^4)$$

with each factor $E$ in the tensor product $E^{\otimes(l+m)}$ equal

$$E = S(\mathbb{R}^3; \mathbb{C}^4) \text{ or } E = S^{00}(\mathbb{R}^3; \mathbb{C}^4).$$

Each of the operators $\Xi_{l,m}(\kappa_{l,m}(x)), \Xi_{l,m}(\kappa'_{l,m}(x))$ determines a well defined
integral kernel operator

$$\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}((E) \otimes \mathcal{E}_1, (E)^*) \cong \mathcal{L}(\mathcal{E}_1, \mathcal{L}((E), (E)^*)),$$

$$\Xi_{l,m}(\kappa'_{l,m}) \in \mathcal{L}((E) \otimes \mathcal{E}_2, (E)^*) \cong \mathcal{L}(\mathcal{E}_2, \mathcal{L}((E), (E)^*))$$

with vector-valued distribution kernel $\kappa_{l,m}$, respectively, $\kappa'_{l,m}$, in the sense of Obata [38], where $(E)$ is the nuclear Hida subspace in the tensor product of the Fock spaces of the fields $\psi$ and $A$. The integral kernel operators $\Xi_{l,m}(\kappa_{l,m}(x))$, $\Xi_{l,m}(\kappa'_{l,m}(x))$ are uniquely determined by the condition

$$\langle \langle \Xi_{l,m}(\kappa_{l,m})(\Phi \otimes \phi), \Psi \rangle \rangle = \langle \kappa_{l,m}(\eta_{\psi, \psi}), \phi \rangle, \quad \Phi, \Psi \in (E), \phi \in \mathcal{E}_1,$n

$$\langle \langle \Xi_{l,m}(\kappa'_{l,m})(\Phi \otimes \phi), \Psi \rangle \rangle = \langle \kappa'_{l,m}(\eta_{\psi, \psi}), \phi \rangle, \quad \Phi, \Psi \in (E), \phi \in \mathcal{E}_2,$n

where

$$\eta_{\psi, \psi}(s_1, p_1, \ldots, s_{l+m}, p_{l+m}) = \langle \langle a_{s_1}(p_1)^+ \cdots a_{s_{l+m}}(p_{l+m}) \Phi, \Psi \rangle \rangle.$$

Note that

$$\eta_{\psi, \psi} \in E^{\otimes (l+m)}, \quad \Phi, \Psi \in (E),$$

with the canonical pairing $\langle\langle \cdot, \cdot \rangle\rangle$ on $(E)^* \times (E)$. These results are contained as a particular case of Theorem 5 of Subsection 2.7 of Section 3.

Moreover the interacting fields, in the adiabatic limit $g = 1$, can be understood as Fock expansions

$$\psi_{\text{int}}(g = 1) = \sum_{l,m} \Xi_{l,m}(\kappa_{l,m}),$$

$$A_{\text{int}}(g = 1) = \sum_{l,m} \Xi_{l,m}(\kappa'_{l,m}),$$

into integral kernel operators in the sense of [38] with all terms $\Xi_{l,m}(\kappa_{l,m}(x))$, $\Xi_{l,m}(\kappa'_{l,m}(x))$ equal to integral kernel operators with vector-valued kernels, and all belonging to the class indicated above. Even more, most of the terms $\Xi_{l,m}(\kappa_{l,m}(x))$, $\Xi_{l,m}(\kappa'_{l,m}(x))$ behave even much more “smoothly” (although it is not necessary for the theory to work) and

$$\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}((E) \otimes \mathcal{E}_1, (E)) \cong \mathcal{L}(\mathcal{E}_1, \mathcal{L}((E), (E))),$$

$$\Xi_{l,m}(\kappa'_{l,m}) \in \mathcal{L}((E) \otimes \mathcal{E}_2, (E)) \cong \mathcal{L}(\mathcal{E}_2, \mathcal{L}((E), (E))).$$

In particular the first order contribution $A^{(1)}_{\text{int}}(g = 1)$, given by

$$A^{(1)}_{\text{int}}(g = 1, x) = -\frac{e}{4\pi} \int d^3 x_1 \frac{1}{|x_1 - x|} : \overline{\psi}^{\gamma\mu} \psi : (x_0 - |x_1 - x|, x_1),$$

to the interacting potential field, belongs to

$$\mathcal{L}((E) \otimes \mathcal{E}_2, (E)) \cong \mathcal{L}(\mathcal{E}_2, \mathcal{L}((E), (E))).$$
2 White noise construction of the Dirac and electromagnetic potential fields as integral kernel operators. Fundamental operations performed upon integral kernel operators

Here we present the white noise construction of the free quantized Dirac field within the white noise set-up of Hida, Obata and Saitô [20], [39], and which is a rigorous realization of the field along the lines suggested (partially heuristically) by Berezin [3]. This construction can be regarded as a far reaching extension of the definition due to Wightman [60] of the (free) field, and enters into the analysis of the distributional (generalized) states. We should emphasize here that the definition of Wightman is operationally and computationally much weaker. In general the two definitions are not equivalent. The main advantage we gain when constructing free fields within the white noise formalism is that we can give a rigorous meaning to the (free) quantum field of the so called integral kernel operator with vector-valued distributional kernel (in the sense [38] or [39], Chap. 6.3), which would be impossible within Wightman set-up. This allows to give the meaning of integral kernel operators (with vector-valued kernels) to the (generalized) operators under the formula (17.1) in [6], p. 154, or equivalently to the (generalized) operators (43) of [14], Sect. 4, p. 229. In particular when constructing free fields according to Berezin-Hida we obtain Theorem 0 of [14] as a corollary to theorems 2.2 and 2.6 of [20] and Thm. 3.13 of [38] with the domain $\mathcal{D}_0$ replaced with the so called Hida test space of white noise functionals. Moreover using the Berezin-Hida construction of free fields we gain a rigorous formulation and proof of the so called “Wick theorem”, as stated in [6], Chap. III. It should be emphasized that Wightman’s definition of the (free) field [60], does not provide sufficient computational basis for any rigorous formulation and proof of the “Wick theorem” for free fields as stated in [6], Chap. III. Note also that the (free) field constructed within the white noise calculus is well defined at space-time point as a generalized operator transforming the so called Hida space into its strong dual.

One should note that although the definition of the “Wick product” of Wightman and Gårding [61] based on the Wightman’s definition [60] of the field, is mathematically rigorous, it suffers at several crucial points from being computationally ineffective in computations which are important from the physical point of view:

1) The space-time averaging limits in Wightman and Gårding’s [61] definition of the “Wick product” are by no means canonical and involve a considerable amount of arbitrariness.

2) Although Wightman and Gårding [61] are able to construct their own “Wick products” which, after smearing out over space-time domains becomes well defined densely defined unbounded operators, it would be difficult to investigate the closability questions for these operators, their
eventual self-adjointness, as well as averaging over space-like (equal-time) surfaces, within the method of Wightman and Gårding. But the equal-time averagings are involved through conserved currents when we consider Noether theorem for free fields – fundamental from the more conventional, and used by physicists, approach to commutation rules and the more traditional proof of the Pauli theorem for free fields (compare [6]).

3) Wightman and Gårding definition of the “Wick product” [61] is not a sufficient basis for the strict formulation and proof of the “Wick theorem” as stated in [6], Chap. III, so fundamental for the causal approach to QFT which avoids ultraviolet divergences. Note in particular that Theorem 0 of [14] is formulated and proved on the basis of partially heuristic (but solid) arguments of the more traditional approach presented in [6], Chap. III, which uses the free fields at specified space-time points in the intermediate stage, and which are not merely symbolic in their character (contrary to what we encounter in the Wightman-Gårding’s approach). White noise construction of free fields on the other hand do provide a sufficient basis for the rigorous formulation and proof of “Wick theorem” for free fields of [6], Chap. III.

4) But most of all when constructing free fields using the white noise formalism, as integral kernel operators with vector-valued kernels, we are able to give a rigorous meaning to each order term contribution to interacting fields in QED (within the causal perturbative approach), of an integral kernel operator with vector-valued distribution kernel (in the sense [38]), which defines a well defined operator valued distribution on the space-time test space – a continuous map from the space-time test space to the linear space of continuous linear operators on the Hida space into its dual (with the standard topology of uniform convergence on bounded sets). Each such contribution can be averaged in the states of the Hida subspace and defines a scalar distribution as a functional of space-time test function. The crucial point is that these contributions do not loose this rigorous sense even for the “coupling space-time function $g$” put everywhere equal to unity, which allows to avoid both: ultraviolet and infra-red infinities in the perturbative (causal) approach to QED. For a detailed proof of this assertion and analysis of the all higher order contributions to the Dirac and electromagnetic potential interacting fields, compare Subsection 2.7 Sect. 5. In particular we can reach in this way a positive solution to the existence problem for the adiabatic limit in QED using a method which is applicable to interactions and fields of more general character, e.g. to the Standard Model.

For these reasons we regard the white noise construction of (free) fields of Berezin-Hida as integral kernel operators (with vector-valued distributional kernels) as more adequate mathematical interpretation of the (free) quantum field than the one proposed by Wightman [60].
In this Section we present white noise Berezin-Hida construction of the free Dirac field as an integral kernel operator with vector-valued distributional kernel in the sense of Obata [38]. In the work [59] we give the white noise construction of the free electromagnetic potential field \( A \), which again may be interpreted as integral kernel operator with vector-valued distributional kernel in the sense of Obata [38], compare Subsections 2.9 and 2.10 where some of the results of [59] concerning the free field \( A \) are summarized (in fact these Subsections are borrowed from [59]).

We present the construction of the Dirac field \( \psi \) in several steps, keeping the presentation as general as possible, in order to make it to serve as an introduction to the construction of (free) local fields within the white noise formalism.

Firstly, we give definition of the Hilbert space which is subject to second quantization functor, and then in the remaining four steps quantize it. The steps are realized in the following Subsections: 2.1, 2.2, 2.3, 2.4, 2.6. Subsection 2.6 is the longest, but it contains an introduction to the papers [26], [38] on integral kernel operators with scalar-valued and respectively vector-valued distributional kernels in fermi and bose Fock spaces (note that [26], [38] give detailed analysis for the bose case), which is of use in the remaining part of the whole work, and which is not so much pertinent to the specific Dirac field \( \psi \), but which is important for general local fields constructed within the white noise calculus. In particular we are using the cited theorems of [26], [38] on integral kernel operators in the proof of Bogoliubov-Shirkov Hypothesis (equivalently the classic Pauli theorem) for the Dirac field \( \psi \) (Subsection 2.8) and for the electromagnetic potential field (Subsection 5.9 of [59]); and finally in the analysis of contributions to interacting fields in QED (Subsection 2.7).

Subsection 2.7 is devoted to the proof that the contributions to interacting fields in causal perturbative spinor QED are well defined integral kernel operators with vector-valued kernels in the sense of Obata [38] whenever we are using in the causal construction of interacting fields the free fields which themselves are well defined integral kernel operators in the sense of Obata. Nonetheless Subsection 2.7 is of more general character not pertinent to the special case of spinor QED. It is devoted to the fundamental operations performed upon the free fields, understood as integral kernel operators with vector-valued kernels, which serve as fundamental computational rules in construction of the theory, in particular in construction of the perturbative series for interacting fields such as: Wick product of free fields, derivation and integration operations. These operations have general character and can be extended over other causal perturbative QFT.

We add two additional Subsections 2.5 and 2.8. Subsection 2.5 gives a motivation for using white noise calculus and for using the construction of fields due to Berezin-Hida, as integral kernel operators with vector-valued kernels. The Subsection 2.8 contains comparison with the standard realization of the free Dirac field and is devoted to the Bogoliubov-Shirkov Postulate (first Noether theorem for free fields and the classic Pauli theorem on spin-statistics relation).

In this Section \( m > 0 \) has the constant value equal to the electron mass.
2.1 Definition of the Hilbert space $\mathcal{H}$ which is then subject to the second quantization functor $\Gamma$

This is the Hilbert space $\mathcal{H}$ of bispinor solutions $\phi$ (regular function-like distributions on the Schwartz space $S(\mathbb{R}^4;\mathbb{C}^4)$ of testing bispinors transforming according to the law (27) of Subsection 2.1 of [59]) of the Dirac equation

$$(i\gamma^\mu \partial_\mu)\phi = m\phi,$$

with the inner product

$$(\tilde{\phi}, \tilde{\phi}') = \int_{x^\alpha = \text{const.}} \left( \phi(x), \phi'(x) \right) \Theta(p) \, d^4x,$$ (1)

and transformation law (27) of Subsection 2.1 of [59], compare e.g. [46] or [6]. This means that the Fourier transform $\tilde{\phi}$ of the bispinor $\phi \in \mathcal{H}$ (regular distribution) is concentrated on the disjoint sum of the positive and negative energy orbits $\mathcal{O}_{m,0,0,0} \sqcup \mathcal{O}_{-m,0,0,0}$ and $\tilde{\phi}$ cannot be regarded as ordinary function on the full range of $p \in \mathbb{R}^4$ of the momentum space. Nonetheless $\tilde{\phi}$ is a well defined (singular, i.e. non-function-like) distribution in the Schwartz space

$S(\mathbb{R}^4;\mathbb{C}^4) = S(\mathbb{R}^4;\mathbb{C}) \oplus S(\mathbb{R}^4;\mathbb{C}) \oplus S(\mathbb{R}^4;\mathbb{C}) \oplus S(\mathbb{R}^4;\mathbb{C})$

of bispinors on $\mathbb{R}^4$ (transforming according to (24) and (25), Subsect. 2.1 of [59]). It defines an ordinary bispinor-function $p \mapsto \tilde{\phi}(p)$ on the disjoint sum $\mathcal{O}_{m,0,0,0} \sqcup \mathcal{O}_{-m,0,0,0}$ of the positive and resp. negative energy orbits, which we denote likewise by the symbol $\tilde{\phi}$ (although it makes sense as a function only on the disjoint sum of the respective orbits and not on the whole $\mathbb{R}^4$ space), and which is square integrable with respect to the inner product (compare (28), Subsect. 2.1 of [59]) induced by the above inner product (1) in $\mathcal{H}$. Namely for $\phi \in \mathcal{H}$, the action of the Fourier transform $\tilde{\phi}$ on $\tilde{f} \in S(\mathbb{R}^4;\mathbb{C})$ is by definition equal to the integration of the product of the mentioned function $p \mapsto \tilde{\phi}(p)$ by the restriction of $\tilde{f}$ to the disjoint sum $\mathcal{O}_{m,0,0,0} \sqcup \mathcal{O}_{-m,0,0,0}$ along $\mathcal{O}_{m,0,0,0} \sqcup \mathcal{O}_{-m,0,0,0}$ with respect to the invariant measure on $\mathcal{O}_{m,0,0,0} \sqcup \mathcal{O}_{-m,0,0,0} \subset \mathbb{R}^4$ induced by the invariant measure $d^4p$ on $\mathbb{R}^4$. Thus, by definition of the singular distribution $\delta(P = 0)$, where $P$ is a smooth function on $\mathbb{R}^4$ such that $\text{grad} P \neq 0$ on the surface $P = 0$ (compare [17], Chap. III), we have

$$\int \phi(x) f(x) \, d^4x = \langle \tilde{\phi}, \tilde{f} \rangle = \int \tilde{\phi}(p) \tilde{f}(p) \, d^4p$$

$$= \int \delta(p \cdot p - m^2) \tilde{\phi}(p) \tilde{f}(p) \, d^4p$$

$$= \int \delta(p \cdot p - m^2) \Theta(p_0) \tilde{\phi}(p) \tilde{f}(p) \, d^4p + \int \delta(p \cdot p - m^2) \Theta(-p_0) \tilde{\phi}(p) \tilde{f}(p) \, d^4p$$

$$= \int_{\mathcal{O}_{m,0,0,0}} \tilde{\phi}(p) \tilde{f}_{\mathcal{O}_{m,0,0,0}}(p) \, d\mu_{m,0}(p) + \int_{\mathcal{O}_{-m,0,0,0}} \tilde{\phi}(p) \tilde{f}_{\mathcal{O}_{-m,0,0,0}}(p) \, d\mu_{-m,0}(p).$$
From now on we agree to denote the ordinary bispinor function $\tilde{\phi}$ on the disjoint sum $\mathcal{O}_{m,0,0,0} \sqcup \mathcal{O}_{-m,0,0,0}$ (equal to the distributional Fourier support of the distribution $\tilde{\phi}$) by the same symbol $\phi$ as the distributional Fourier transform $\tilde{\phi}$ of $\phi \in \mathcal{H}$ (although $\tilde{\phi}$ makes sense as the ordinary function only on the support of the distribution $\phi$, which as a “function” is intentionally equal zero outside the support, which makes a precise sense when $\phi$ is regarded as distribution defined as above).

In short for $\phi \in \mathcal{H}$ we can write

$$\phi(x) = \int_{\mathcal{O}_{m,0,0,0}} \phi(p) e^{-ip \cdot x} d\mu_{m,0}(p) + \int_{\mathcal{O}_{-m,0,0,0}} \phi(p) e^{-ip \cdot x} d\mu_{-m,0}(p);$$

or

$$\phi(x) = \int_{\mathcal{O}_{m,0,0,0}} \phi(p) e^{-ip \cdot x} d\mu_{m,0}(p) + \int_{\mathcal{O}_{-m,0,0,0}} \phi(p) e^{-ip \cdot x} d\mu_{-m,0}(p)$$

$$= \int_{\mathbb{R}^3} \phi(p_0(\vec{p})) e^{-(i|p_0(\vec{p})|t - i\vec{p} \cdot \vec{x})} \frac{d^3\vec{p}}{2|p_0(\vec{p})|} - \int_{\mathbb{R}^3} \phi(-\vec{p}, -|p_0(\vec{p})|) e^{i|p_0(\vec{p})|t - i\vec{p} \cdot \vec{x}} \frac{d^3\vec{p}}{2|p_0(\vec{p})|},$$

$$p_0(\vec{p}) = \pm \sqrt{\vec{p} \cdot \vec{p} + m^2}. \quad \text{(2)}$$

Here of course $p = (p_0(\vec{p}), \vec{p}) = (\sqrt{\vec{p} \cdot \vec{p} + m^2}, \vec{p})$ on $\mathcal{O}_{m,0,0,0}$ and $p = (p_0(\vec{p}), \vec{p}) = (-\sqrt{\vec{p} \cdot \vec{p} + m^2}, \vec{p})$ on $\mathcal{O}_{-m,0,0,0}$.

In particular for the solution $\phi \in \mathcal{H}$ whose Fourier transform $\tilde{\phi}$ is concentrated on the positive energy orbit $\mathcal{O}_{m,0,0,0}$ we have

$$\phi(x) = \phi(x, t) = \int_{\mathcal{O}_{m,0,0,0}} \phi(p) e^{-ip \cdot x} d\mu_{m,0}(p)$$

$$= \int_{\mathbb{R}^3} \phi(p_0(\vec{p})) e^{-(i|p_0(\vec{p})|t - i\vec{p} \cdot \vec{x})} \frac{d^3\vec{p}}{2|p_0(\vec{p})|}, \quad p_0(\vec{p}) = \sqrt{\vec{p} \cdot \vec{p} + m^2}.$$  

Similarly we have for the solution $\phi \in \mathcal{H}$ whose Fourier transform is concentrated on the negative energy orbit $\mathcal{O}_{-m,0,0,0}$:

$$\phi(x) = \phi(x, t) = \int_{\mathcal{O}_{-m,0,0,0}} \phi(p) e^{-ip \cdot x} d\mu_{-m,0}(p)$$

$$= \int_{\mathbb{R}^3} \phi(-\vec{p}, -|p_0(\vec{p})|) e^{i|p_0(\vec{p})|t - i\vec{p} \cdot \vec{x}} \frac{d^3\vec{p}}{2|p_0(\vec{p})|}, \quad p_0(\vec{p}) = -\sqrt{\vec{p} \cdot \vec{p} + m^2}.$$  

We have the following equality for the solutions $\phi, \phi' \in \mathcal{H}$ whose Fourier
transforms \( \tilde{\phi}, \tilde{\phi}' \) are concentrated on the positive energy orbit \( \mathcal{O}_{m,0,0} \):

\[
\int_{x^0 = t = \text{const.}} \left( \phi(\vec{x}, t), \phi'(\vec{x}, t) \right) \cdot d^3x = \int_{\mathcal{O}_{m,0,0}} \left( \tilde{\phi}(p), \tilde{\phi}'(p) \right) \cdot \frac{d\mu_{m,0}(p)}{2p_0} = \int_{\mathbb{R}^3} \left( \tilde{\phi}(\vec{p}, p_0(\vec{p})), \tilde{\phi}'(\vec{p}', p_0(\vec{p}')) \right) \cdot \frac{d^3\vec{p}}{2p_0(\vec{p})}, \quad p_0(\vec{p}) = \sqrt{\vec{p} \cdot \vec{p} + m^2}.
\]

Similarly we have for the solutions \( \phi, \phi' \in \mathcal{H} \) whose Fourier transforms \( \tilde{\phi}, \tilde{\phi}' \) are concentrated on the negative energy orbit \( \mathcal{O}_{-m,0,0} \):

\[
\int_{x^0 = t = \text{const.}} \left( \phi(\vec{x}, t), \phi'(\vec{x}, t) \right) \cdot d^3x = -\int_{\mathbb{R}^3} \left( \tilde{\phi}(\vec{p}, p_0(\vec{p})), \tilde{\phi}'(\vec{p}', p_0(\vec{p}')) \right) \cdot \frac{d^3\vec{p}}{2p_0(\vec{p})}, \quad p_0(\vec{p}) = -\sqrt{\vec{p} \cdot \vec{p} + m^2}.
\]

Note that the last expression is equal to \textit{minus} the inner product (33) of Subsection 2.1 of \([59]\) of the (Fourier transforms of) bispinors \( \phi, \phi' \) on the Hilbert space of Fourier transforms of bispinors, concentrated on \( \mathcal{O}_{-m,0,0} \) (up to the irrelevant constant factor \( m > 0 \)), introduced in Subsection 2.1 of \([59]\).

Consider now the induced representation

\[
U^{(m,0,0,0) L^{1/2}}
\]

of \( T_k \circ SL(2, \mathbb{C}) \), concentrated on the orbit \( \mathcal{O}_{(m,0,0,0)} \). Now we apply the isometric map \( V^\oplus \) to the space of this representation followed by the Fourier transform (20) (of Introduction to Sect. 2 of \([59]\) with the orbit \( \mathcal{O}_{\bar{p}} = \mathcal{O}_{(m,0,0,0)} \)), where \( V^\oplus \) is the map defined in Example 1 (Subsection 2.1 of \([59]\)). Let us denote the composed map just by \( \bar{V}^\oplus \). The image of \( \bar{V}^\oplus \) lies in \( \mathcal{H} \). Indeed because of eq. (28) of Subsection 2.1 of \([59]\) it is even isometric.

Similarly consider the representation

\[
U^{(-m,0,0,0) L^{1/2}}
\]

of \( T_k \circ SL(2, \mathbb{C}) \), concentrated on the orbit \( \mathcal{O}_{(-m,0,0,0)} \). To the space of this representation we apply the map \( \bar{V}^\oplus \) equal to \( V^\oplus \) followed by the Fourier transform (20) (Introduction to Section 2 of \([59]\) with the orbit \( \mathcal{O}_{\bar{p}} = \mathcal{O}_{(-m,0,0,0)} \)), where \( V^\oplus \) is the map defined in Example 1, Subsection 2.1 of \([59]\). Its image
likewise lies in $\mathcal{H}$ and by the same (28) of Subsection 2.1 of [59] – which is also valid for $V \oplus$ – it is isometric too. Now the image $\mathcal{H}_{m,0}^{\oplus}$ of the representation space of the representation $\tilde{V} \oplus$ lies in the positive eigenspace subspace $E_+ \mathcal{H}$ of the essentially self adjoint Dirac hamiltonian operator $H = -i\gamma^0\gamma^k \partial_k + m\gamma^0 = -i\alpha^k \partial_k + m\gamma^0$ acting on $\mathcal{H}$, where $E_+$ is the spectral projection corresponding to all positive spectral values of $H$. Similarly the image $\mathcal{H}_{m,0}^{\ominus}$ of the space of the representation $\tilde{V}$ under the map $\tilde{\varphi}(p) = \int \varphi(x) e^{ipx} d^4x$

of any element of $\mathcal{H}$ is concentrated on the set theoretical sum $\mathcal{O}_{(m,0,0,0)} \cup \mathcal{O}_{(-m,0,0,0)}$ of the orbits $\mathcal{O}_{(m,0,0,0)}$ and $\mathcal{O}_{(-m,0,0,0)}$, isometrically into $\mathcal{H}$.

On the other hand the only eigenvalues of the matrix $\gamma^0$ are 1 and -1, so it follows from the theorem of Section 10.1, Part II, Chapter II of [16] (compare also [19]-[21]), that the ordinary Fourier transform

$$U_{(m,0,0,0)} L^{1/2} \oplus U_{(-m,0,0,0)} L^{1/2},$$

concentrated on the sum theoretic set $\mathcal{O}_{(m,0,0,0)} \cup \mathcal{O}_{(-m,0,0,0)}$ of the orbits $\mathcal{O}_{(m,0,0,0)}$ and $\mathcal{O}_{(-m,0,0,0)}$. Thus the operator $V \oplus \tilde{V} \ominus$ regarded as operator on the space of the representation $\tilde{V} \ominus$ is onto $\mathcal{H}$, and therefore it is unitary, so that

$$E_+ \mathcal{H} = \mathcal{H}_{m,0}^{\ominus} \quad \text{and} \quad E_- \mathcal{H} = \mathcal{H}_{-m,0}^{\ominus}.$$ 

Therefore in the Hilbert space $\mathcal{H} = \mathcal{H}_{m,0}^{\ominus} \oplus \mathcal{H}_{-m,0}^{\ominus}$ there acts the unitary\footnote{\textsuperscript{1}Please, note also that the representation $V \oplus U_{(m,0,0,0)} L^{1/2} (V\oplus)^{-1} \oplus V \ominus U_{(-m,0,0,0)} L^{1/2} (V\ominus)^{-1}$, concentrated on $\mathcal{O}_{m,0,0,0} \cup \mathcal{O}_{-m,0,0,0}$ is unitary, similarly as the representation $V \ominus (U_{(m,0,0,0)} L^{1/2} \oplus U_{(m,0,0,0)} L^{1/2}) (V\ominus)^{-1}$ (compare Example 1, Subsection 2.1 of [59]) concentrated on $\mathcal{O}_{m,0,0,0}$.} representation

$$\tilde{V} \oplus U_{(m,0,0,0)} L^{1/2} (V \ominus)^{-1} \oplus \tilde{V} \ominus U_{(-m,0,0,0)} L^{1/2} (V \ominus)^{-1}$$

concentrated on $\mathcal{O}_{(m,0,0,0)} \cup \mathcal{O}_{(-m,0,0,0)}$, with

$$\tilde{V} \oplus U_{(m,0,0,0)} L^{1/2} (V \ominus)^{-1}$$

(7)
acting on $H_{m,0}^\oplus$ and with

$$\widetilde{V} \otimes U_{(-m,0,0,0)} L^{1/2} (\widetilde{V})^{-1}$$

acting on $H_{-m,0}^\oplus$.

To the Hilbert space $\mathcal{H}$ treated as if it was the single particle space we apply the fermionic functor of second quantization $\Gamma$, and obtain the standard absorption and emission operators. Next we split them (i.e. we consider their restrictions resp. to $H_{m,0}^\oplus$ or $H_{-m,0}^\ominus$) according to the splitting $\mathcal{H} = H_{m,0}^\oplus \oplus H_{-m,0}^\ominus = E_+ \mathcal{H} \oplus E_- \mathcal{H}$ of the space $\mathcal{H}$, compare e.g. [46]. We observe then that the absorption and emission operators restricted to $H_{m,0}^\oplus$ compose a fermionic free field and similarly the restrictions of the absorption and emission operators restricted to $H_{-m,0}^\ominus$ and that the the two sets of operators commute and are independent in consequence of the orthogonality of the subspaces $H_{m,0}^\oplus$ and $H_{-m,0}^\ominus$ (e.g. [46]). That is we have two independent fermionic quantizations: the functor $\Gamma$ applied to $H_{m,0}^\oplus$ and the functor $\Gamma$ applied to $H_{-m,0}^\ominus$ with the tensor product of the two independent sets of annihilation and creation operators acting in the tensor product of fermionic Fock spaces $\Gamma(H_{m,0}^\oplus) \otimes \Gamma(H_{-m,0}^\ominus) = \Gamma((H_{m,0}^\oplus \oplus H_{-m,0}^\ominus)_L^{1/2} (\widetilde{V} \otimes U_{(-m,0,0,0)} L^{1/2} (\widetilde{V})^{-1})).$

In order to repair the energy sign of the free Dirac field on $\Gamma(H_{m,0}^\oplus) \otimes \Gamma(H_{-m,0}^\ominus)$ we interchange the absorption and emission operators in $\Gamma(H_{-m,0}^\ominus)$. In this manner we obtain the following construction which may be described in the following four steps.

### 2.2 Application of the Segal second quantization functor to the subspace $H_{m,0}^\oplus$

To the subspace $H_{m,0}^\oplus$ we apply the Segal’s functor $\Gamma$ of fermionic quantization and obtain the fermionic Fock space

$$\mathcal{H}_F^\oplus = \Gamma(H_{m,0}^\oplus) = \mathbb{C} \oplus H_{m,0}^\oplus \oplus (H_{m,0}^\oplus) \hat{\otimes}^2 \oplus (H_{m,0}^\oplus) \hat{\otimes}^3 \oplus \ldots;$$

with the unitary representation

$$\Gamma(\widetilde{V} \otimes U_{m,0,0,0} L^{1/2} (\widetilde{V})^{-1}) = \bigoplus_{n=0,1,2\ldots} \left(\widetilde{V} \otimes U_{m,0,0,0} L^{1/2} (\widetilde{V})^{-1}\right)^\hat{\otimes}^n,$$

where in the formulas $(\cdot)^\hat{\otimes}^n$ stands for $n$-fold anti-symmetrized tensor product, and $(\cdot)^\otimes_n$ with $n = 0$ applied to the representation gives the trivial representation on $\mathbb{C}$ with each representor acting on $\mathbb{C}$ as multiplication by 1.

In this and in the following Sections, we will encounter essentially two types of topological vector spaces and operators acting upon them: 1) Hilbert spaces and 2) nuclear spaces (the Schwartz $\mathcal{S}(\mathbb{R}^n)$ space of test functions on $\mathbb{R}^n$ is an example of a nuclear space). Correspondingly we will use respectively 1) the Hilbert space tensor product $\otimes$ (if applied to Hilbert spaces, elements of Hilbert
spaces and operators upon them) and respectively projective tensor product \( \otimes \) (if applied to nuclear spaces, their elements and operators acting upon them); for definition, and properties of these standard constructions we refer e.g. to \( [30] \), \( [58] \), \( [45] \).

The linear spaces we encounter (Hilbert spaces and nuclear spaces) will be always over \( \mathbb{R} \) or over \( \mathbb{C} \), but whenever they are over \( \mathbb{C} \) they will be equal to complexifications of real (Hilbert or nuclear) spaces with naturally defined complex conjugation \( (\cdot)^* \) in them.

Note that by Riesz representation theorem for such Hilbert spaces \( \mathcal{H}' \) we have natural identification of linear continuous functionals on \( \mathcal{H}' \) with the elements of the adjoint Hilbert space \( \mathcal{H}'^* \), which in fact becomes an isomorphism of Hilbert spaces if we appropriately introduce the multiplication by a number and the inner product into the space of linear functionals on \( \mathcal{H}' \). Recall that the adjoint space \( \mathcal{H}'^* \) have the same set of elements as \( \mathcal{H}' \), but with scalar multiplication by a number \( \alpha \in \mathbb{C} \) and inner product defined by

\[
\alpha u \text{ in } \mathcal{H}'^* = \overline{\alpha} u \text{ in } \mathcal{H}',
\]

\[
(u, v) \text{ in } \mathcal{H}'^* = (v, u) \text{ in } \mathcal{H}'.
\]

With such a Hilbert space structure on \( \mathcal{H}'^* \) the map \( \mathcal{H}' \ni u \mapsto \overline{u} \in \mathcal{H}'^* \) defines a canonical linear isomorphism. In the sequel we will regard the dual space \( \mathcal{H}'^* \) as the adjoint space \( \mathcal{H}'^* \) with elements the same as elements of \( \mathcal{H}' \) (Riesz isomorphism).

For operators on Hilbert spaces we are using the standard notation for the ordinary adjoint operation with the superscript \( * \), with the exception of the annihilation operators, denoting the operators which are adjoint to them with the superscript \( + \) instead \( * \) (which is customary in physical literature). If working with operators \( A \) transforming (continuously) one nuclear space into another \( E_1 \to E_2 \), we use the superscript \( * \) to denote the linear dual (transposed) operator \( A'^*: E_2^* \to E_1^* \), transforming continuously the strong dual space \( E_2^* \) into the strong dual space \( E_1^* \), for definition and general properties of transposition we again refer to \( [58] \). For operator \( A \) transforming (continuously) nuclear space into nuclear space we denote by \( A^+ \) the operator \( (\cdot)^* \circ A^* \circ (\cdot)^* \), i.e. the linear dual of \( A \) composed with complex conjugation (say Hermitean adjoint = linear transposition + complex conjugation).

In the standard way we obtain the map from \( \mathcal{H}_{m,0}^\oplus \ni \tilde{\phi} \) to the families \( a_{\oplus}(\tilde{\phi}), a_{\oplus}^+(\tilde{\phi}) = a_{\oplus}(\overline{\tilde{\phi}})^+ \) of ordinary annihilation and creation operators in the
fermionic Fock space $\Gamma(\mathcal{H}_{m,0}^\oplus)$ fulfilling the canonical anticommutation relations:

\[
\{a_\oplus(\phi), a_\oplus(\phi')^+\} = \left(\phi, \phi'\right)_{\mathcal{H}_{m,0}^\oplus} = \int_{x^0=t=\text{const.}} \left(\phi(\vec{x}, t), \phi'(\vec{x}, t)\right)_c^3 x \text{d}^3x
\]

\[
= \int_{\mathcal{O}_{m,0,0}} \left(\phi(p), \phi'(p)\right)_{c^4} \frac{\text{d}\mu_{m,0}(p)}{2p_0}
\]

\[
= \int_{\mathbb{R}^3} \left(\phi(\vec{p}, p_0(\vec{p})), \phi'(\vec{p}, p_0(\vec{p}))\right)_{c^4} \frac{\text{d}^3\vec{p}}{(2p_0(\vec{p}))^2},
\]

\[
p_0(\vec{p}) = \sqrt{\vec{p} \cdot \vec{p} + m^2}.
\]

Here and in the rest part of this Section we identify the Hilbert space $\mathcal{H}_{m,0}^\oplus = E_+\mathcal{H}$ of positive energy distributional solutions $\phi$ of the Dirac equation with the ordinary functions $\tilde{\phi}$ on the orbit $\mathcal{O}_{m,0,0,0}$ which they induce on the orbit in the manner described above. Correspondingly we identify the Hilbert space $\mathcal{H}$ of distributional solutions $\phi$ of Dirac equation with the ordinary functions $\tilde{\phi}$ on the disjoint sum of orbits $\mathcal{O}_{m,0,0,0} \sqcup \mathcal{O}_{-m,0,0,0}$ ($= \text{supp} \tilde{\phi}$ of $\tilde{\phi}$ regarded as distribution). Similarly we identify the Hilbert space $\mathcal{H}_{-m,0} = E_-\mathcal{H}$ of negative energy distributional solutions $\phi$ of Dirac equation with the corresponding ordinary functions $\tilde{\phi}$ on $\mathcal{O}_{m,0,0,0} \sqcup \mathcal{O}_{-m,0,0,0}$ having the support on $\mathcal{O}_{-m,0,0,0}$.

In the later stage of the construction of the free Dirac field we will need a unitary involutive (and thus self-adjoint) operator $\mathcal{I}$, which we call parity number operator, canonically related to the Fock space construction. In order to indicate the relation of the parity number operator $\mathcal{I}$ to the corresponding Fock space $\Gamma(\mathcal{H}_{m,0}^\oplus)$, we use the subscript $\oplus$: $\mathcal{I}_\oplus$.

In order to define $\mathcal{I}_\oplus$ recall that every element $\Phi \in \Gamma(\mathcal{H}_{m,0}^\oplus)$ may be uniquely represented as the sum

\[
\Phi = \sum_{n \geq 0} \Phi_n
\]

over all $n = 0, 1, 2, \ldots$ of the orthogonal components $\Phi_n \in (\mathcal{H}_n)^{\oplus n}$ – the so called $n$-particle states, with

\[
\|\Phi\|^2 = \sum_{n \geq 0} \|\Phi_n\|^2 < +\infty.
\]
following state
\[ \text{In}_{\oplus} \Phi = \sum_{n \geq 0} (-1)^n \Phi_n. \]

It is evident that In\(_{\oplus}\) is unitary and involutive (thus self-adjoint)
\[ \text{In}_{\oplus}^2 = 1, \quad \text{In}_{\oplus}^* = \text{In}_{\oplus} \]
and that In\(_{\oplus}\) anti-commutes with the annihilation (and creation) operators:
\[ a_{\oplus}(\tilde{\phi}) \text{In}_{\oplus} = -\text{In}_{\oplus} a_{\oplus}(\tilde{\phi}). \]

Note that the unitary involution In on general Fock space, and in particular In\(_{\oplus}\), commutes with any (bounded or even unbounded) operator \(B\) which transforms the closed subspaces of fixed particle number into themselves (in case \(B\) is unbounded we assume \(\text{Dom} B\) to be a linear subspace or still more generally with \(\text{Dom} B\) to be closed under operation of multiplication by \(-1\)). In particular In (or In\(_{\oplus}\)) commutes with any operator of the form
\[ B = \Gamma(A) = \sum_{n=0}^{\infty} A^{\otimes n}, \]
namely:
\[ [\Gamma(A), \text{In}_{\oplus}] = 0 \quad \text{on} \quad \text{Dom} \Gamma(A), \]
irrespectively if \(A\) is bounded or not, but with linear \(\text{Dom} A\) and \(\text{Dom} \Gamma(A)\).

This in particular means that the operator In\(_{\oplus}\) commutes:
\[ \left[ \Gamma \left( \tilde{V}^{\oplus} U_{(m,0,0,0)} L^{1/2} (\tilde{V}^{\oplus})^{-1} \right), \quad \text{In}_{\oplus} \right] = 0 \]
with the representation of \(T_4 \otimes \text{SL}(2, \mathbb{C})\) acting in the Fock space \(\Gamma(H_{m,0}^{\oplus})\).

**REMARK 1.** Note that in literature, e.g. [7], there is frequently used the following construction of annihilation and creation operators, in a general Fock space (here we concentrate on the fermionic Fock space) \(\Gamma(H')\). For each \(u \in H'\) of the single particle space \(H'\) we define the operators \(a(u), a^{\dagger}(u) = a(u)^{\dagger}\) which by definition act on general element
\[ \Phi = \sum_{n \geq 0} \Phi_n, \quad \Phi_n \in H'^{\otimes n} \]
with
\[ \|\Phi\|^2 = \sum_{n \geq 0} \|\Phi_n\|^2 < +\infty, \]
of the Fock space $\Gamma(\mathcal{H}')$, in the following manner

1. $a(u)(\Phi = \Phi_0) = 0$,
2. $a(u)\Phi = \sum_{n \geq 0} n^{1/2} \pi_1 \Phi_n$,
3. $a(u)^+ \Phi = \sum_{n \geq 0} (n + 1)^{1/2} u \circ \Phi_n$.

Here $\otimes$ and $\otimes_1$ denote respectively the anti-symmetrized $n$-fold tensor product and the anti-symmetrized 1-contraction, uniquely determined by the formulae

$$v_1 \otimes \cdots \otimes v_n = (n!)^{-1} \sum_{\pi} \text{sign}(\pi) v_{\pi(1)} \otimes \cdots \otimes v_{\pi(n)}, \quad v_i \in \mathcal{H},$$

$$u \otimes_1 v_1 \otimes \cdots \otimes v_n = (n!)^{-1} \sum_{\pi} \text{sign}(\pi) \langle u, v_{\pi(1)} \rangle v_{\pi(2)} \otimes \cdots \otimes v_{\pi(n)}, \quad u \in \mathcal{H}^*, v_i \in \mathcal{H},$$

with the sums ranging over all permutations $\pi$ of the natural numbers $1, \ldots, n$, and with the evaluation $\langle u, v_{\pi(1)} \rangle$ of $u$, understood as a linear functional $\mathcal{H}^*$, on $v_{\pi(1)} \in \mathcal{H}$ equal

$$\langle u, v_{\pi(1)} \rangle = (u, v_{\pi(n)})$$

to the inner product of the elements $u, v_{\pi(n)} \in \mathcal{H}$. Note that in all the relevant physical situations the single particle Hilbert spaces and the corresponding Fock spaces have natural real structure and are equal to complexifications of real Hilbert spaces with naturally defined complex conjugations $\overline{\cdot}$ in them. Recall also that the map $\mathcal{H}' \ni u \mapsto \overline{u}$ defines a linear isomorphism of the Hilbert space $\mathcal{H}'$ into the adjoint Hilbert space $\overline{\mathcal{H}'}$, which in turn can be identified with the Hilbert space of linear functionals on $\mathcal{H}'$, by the Riesz representation theorem.

However we will interchangeably be using another, unitarily equivalent, realization of the annihilation and creation operators in the Fock space, which is more frequently used by mathematicians (and fits well with that used e.g. in [20], [38], [39], [27] for bosons, when adopting their results to the fermion case), because we will refer to the works [20], [38], [39], in the following part of our work. Let us call it the modified realization of annihilation-creation operators in the Fock space. This realization used by mathematicians is more natural for the interpretation of the creation and annihilation operators as derivations (or graded derivations in case of fermi Fock space) on a nuclear (skew-commutative, or say Grassmann, in case of fermi Fock space) algebra of Hida test functions on an (infinite-dimensional) strong dual space to a nuclear space.

In order to define it we first slightly modify the norm (12) of a general element (11) and put for its square instead

$$\|\Phi\|^2_0 = \sum_{n \geq 0} n! \|\Phi_n\|^2.$$
Then we define the annihilation and creation operators through their action on
general such element \( \Phi \) given by the following formulae

1) \( a(u)(\Phi = \Phi_0) = 0 \),

2) \( a(u)\Phi = \sum_{n \geq 0} n u \tilde{\otimes}_1 \Phi_n \),

3) \( a(u)^+ \Phi = \sum_{n \geq 0} u \tilde{\otimes}_1 \Phi_n \).

The unitary operator:

\[
U\left( \sum_{n \geq 0} \Phi_n \right) = \sum_{n \geq 0} (n!)^{-1/2} \Phi_n, \quad U^{-1}\left( \sum_{n \geq 0} \Phi_n \right) = \sum_{n \geq 0} (n!)^{1/2} \Phi_n,
\]

with the convention that \( 0! = 1 \), gives the unitary equivalence between the two
realizations of the annihilation and creation operators in the Fock spaces, as well
as of the representations of \( T_4 \circ SL(2, \mathbb{C}) \) in the corresponding Fock spaces.

2.3 Application of the Segal second quantization functor
to the space \( \mathcal{H}_{-m,0}^{\text{c}} \) of spinors conjugated to the spinors of the subspace \( \mathcal{H}_{-m,0}^{\ominus} \)

In the next step we apply the functor \( \Gamma \) of fermionic second quantization to the
subspace \( \mathcal{H}_{-m,0}^{\ominus} \) and obtain the fermionic Fock space

\[
\Gamma(\mathcal{H}_{-m,0}^{\ominus}) = \mathbb{C} \oplus \mathcal{H}_{-m,0}^{\ominus} \oplus (\mathcal{H}_{-m,0}^{\ominus})^\otimes 2 \oplus (\mathcal{H}_{-m,0}^{\ominus})^\otimes 3 \oplus \ldots;
\]

but the above mentioned interchange of the emission and absorption operators in \( \Gamma(\mathcal{H}_{-m,0}^{\ominus}) \) results in replacing the single particle Hilbert space \( \mathcal{H}_{-m,0}^{\ominus} = E_+ \mathcal{H} \)
with a conjugated one \( \mathcal{H}_{-m,0}^{\text{c}} \) and in replacing of the representation \( \mathfrak{S} \) acting
in \( \mathcal{H}_{-m,0}^{\ominus} \) with another conjugated representation acting in the Hilbert space
\( \mathcal{H}_{-m,0}^{\text{c}} \).

This procedure is the well known basis for the solution of the “negative
energy states problem” in relativistic quantum field theory, therefore we only
sketch briefly the general lines, presenting only the final results in case of the free
quantum Dirac field respecting the Dirac equation. Namely the solution is based
on the observation that the negative energy solutions lying in \( \mathcal{H}_{-m,0}^{\ominus} = E_+ \mathcal{H} \)
(classically the negative energy solutions of the equation which is to be fulfilled
by the quantized field, here of the Dirac equation \( D \phi = m \phi \) (30) of Subsection
2.1 of [59]), should not be interpreted as negative energy solutions of the original
equation (here Dirac equation), but rather as a kind of conjugation of positive
energy solutions of a conjugation of the original (here Dirac) equation, with the
conjugation depending on the actual kind of field. In particular for the scalar (complex) field fulfilling the Klein-Gordon equation the conjugation coincides with the ordinary complex conjugation (but only accidentally).

For (free) Dirac field respecting Dirac equation the conjugation is slightly more complicated and the conjugated equation does not coincide with the original Dirac equation. In the more general higher spin local fields the conjugation is similar as for the Dirac equation, and is easy to guess with its general definition being naturally determined by the general construction of the single particle Hilbert space of the field (with local transformation law).

Namely in general case of globally hyperbolic space-time and a free field, say $\phi$, on it we can extract the essential points of the construction of the free field on the flat Minkowski manifold, although the particular computations would be much less easy to handle. In any case the space-time manifold with its globally hyperbolic causal structure (given by a Lorenzian metric) is crucial, together with the type of field $\phi$ with its local transformation rule fixing the associated type of bundle with $\phi$ ranging over its sections, and respecting a hyperbolic differential equation $D\phi = m\phi$. If a preferable and natural assumptions of analytic type are put on the pseudo-riemannian space-time manifold (compare e.g. \cite{57}, \cite{2}) then the Lorenzian metric induces a Krein structure in the space of sections $\phi$ (compare the formulas (37), (38) of Subsect. 2.3 of \cite{59} in the special case of flat Minkowski space-time and the Dirac bispinors $\phi$ on it with the transformation law (39) of Subsect. 2.1 of \cite{59}). We expect the corresponding differential operator $D$ to be not merely Krein-self-adjoint, but moreover that it allows a Krein-orthogonal spectral decomposition similar to that obtained in Subsect. 2.3 of \cite{59} for the ordinary Dirac operator $D$ (in particular it is of spectral-type). This assumption is nontrivial, as in the Krein space Krein-self-adjoint operator in general does not allow any spectral decomposition of the type obtained in Subsect. 2.3 of \cite{59} for $D$ (compare e.g. the classic Dunford-Schwartz analysis of the type of generalized spectral decompositions of non-normal operators). In particular the method of extension of the construction of a free field on more general space-times proposed here have a rather restricted domain of validity, and is confined to situations with rather very special kind of corresponding hyperbolic differential operators $D$ allowing “regular” Krein-orthogonal spectral decompositions. Of course in general the spectral Krein-orthonormal decomposition of $D$ may contain a discrete component, or even consist of purely discrete part, depending on topology of the space-time manifold.

Next we consider the generalized eigenspace, which we agreed to denote by $\mathcal{H}$, of the Krein-self-adjoint operator $D$, corresponding to the eigenvalue $m$, and which consists of all distributional solutions $\phi$ of the equation $D\phi = m\phi$. The closed subspaces of generalized eigenspaces corresponding to the generalized eigenvalues of $D$ inherit nondegenerate Krein-space structure from the initial Krein space of sections $\phi$ in which $D$ acts. The restriction of the Krein-self-adjoint operator $D$ to this subspace $\mathcal{H}$ is not only Krein-self-adjoint but likewise self-adjoint with respect to the inherited Krein space and Hilbert space structures on $\mathcal{H}$, with well defined direct sum structure $\mathcal{H} = E_+ \mathcal{H} \oplus E_- \mathcal{H}$ with closed subspaces $E_\pm \mathcal{H}$ which are orthogonal and Krein orthogonal and with
nondegenerate Krein space structure. Moreover the operator $D$ is of spectral-type and admits generalized spectral Krein-orthogonal decomposition in the sense of Gelfand-Mackey, explicitly computed in Subsections 2.1-2.3 of [59], with each generalized eigenspace which inherits nondegenerate Hilbert space and Krein space structure. This is far not the case for general Krein-selfadjoint operator, compare [5]. In particular the space $\mathcal{H}$ of generalized eigenvectors of $D$ corresponding to the generalized real eigenvalue $m > 0$ (say mass) inherits nondegenerate and natural Krein space structure, in particular Hilbert space structure. We expect that the space-time manifold, especially its causal structure, allows to pick up the natural discrete operation of time-orientation-reversing in terms of an involutive unitary operator (say the sign $(\mathcal{H}) = \mathcal{H}|\mathcal{H}|-1$ of the Hamiltonian operator $H$ in $\mathcal{H}$) with the property that the change of time orientation transformation acts through sign $(\mathcal{H})$ as an involutive unitary which exchanges positive energy subspace $E_+\mathcal{H}$ with the negative energy subspace $E_-\mathcal{H}$ of $\mathcal{H}$. In case of globally hyperbolic and highly symmetric spacetimes with time symmetry (e.g. Einstein Static Universe) this plan is within our grasp. In particular the harmonic analysis of [40]-[42] is sufficiently effective on the Einstein Universe to allow e.g. construction of QED on it together with the proof of its convergence, compare [55]. In general the conjugation corresponding to the division of “positive” and “negative energy” solution subspaces $E_+\mathcal{H}$ and $E_-\mathcal{H}$ of the space of distributional solutions of $D\phi = m\phi$ is easy to guess and is strongly suggested by the geometric context. Construction of the involutive unitary which corresponds to the division into “positive” and “negative energy” solution subspaces is more tricky when time symmetry is lacking at the space-time geometry level, and reflects the conformal (causal) structure of space-time in the operator-spectral format. In fact construction of this division involves spectral decomposition of non-normal, Krein-self-adjoint operator $D$, and as we know there are no general theorems which would assure existence of such decompositions nor its sufficiently regular behaviour. This is the essential source of difficulty in achieving the honest division into “positive” and “negative” frequency modes. Once a generalized spectral Krein-orthogonal decomposition of $D$, similar to that presented in Subsections 2.1-2.3 of [59] is successful, the involutive unitary and the corresponding conjugation can be easily guessed. This is the case e.g. for the Einstein Universe, compare [40]-[42]. It can be achieved by explicit expansion of the general solution of the Dirac equation $D\phi = m\phi$ into “Einstein spinor modes” (as called by Segal and Zhou) and explicit division of the modes into positive and negative frequency parts. This is a good example to study the relationship of the conformal structure and the corresponding involutive unitary operator. Still more interesting case we obtain for de Sitter spacetime lacking time symmetry, but with the sufficiently reach harmonic analysis to study quantum fields on it. At least one example (of scalar quantum field on the three dimensional de Sitter spacetime), which comes naturally, we will encounter when studying infrared fields in Section 7 of [59]. The generalized regular Krein-isometric decomposition of $D$ (with finite but arbitrary high dimension of the fibre of the fibre bundle of sections of the corresponding Clifford module), providing the corresponding Krein-orthogonal
decomposition of the initial Krein space acted on by $D$, serves as the generalization of the Fourier transform $V_F$ of Subsections 2.1-2.8 of [59] in case of less symmetric globally hyperbolic spacetimes.

After this general remark concerning construction of free fields on more general space-time manifolds, let us back to the construction of the free Dirac field on the flat Minkowski space-time, or more precisely, to the conjugation, which accompany the division $\mathcal{H} = E_+ \mathcal{H} \oplus E_- \mathcal{H}$ into positive and negative energy solutions of the ordinary Dirac equation $D\phi = m\phi$ constructed as above.

As remarked earlier, the negative energy solutions $\phi$ should be interpreted as conjugations of positive energy solutions $\phi^c$ of the conjugated Dirac equation\(^2\). The representation space of the conjugated representation is defined as the Hilbert space $\mathcal{H}_{c,-m,0}$ of conjugated bispinors

$$\tilde{\phi}^c(p) = \tilde{\phi}(-p)^+ = (\tilde{\phi}(-p))^T$$

with $\tilde{\phi} = V_0 \tilde{\psi}_{-m,0}$ ranging over the Hilbert space $\mathcal{H}_{c,-m,0}$ of bispinors concentrated on the orbit $\mathcal{O}_{-m,0,0,0}$ (i.e. with $\tilde{\psi}_{-m,0}$ ranging over the Hilbert space of the representation $U(-m,0,0,0)^L_{1/2}$ concentrated on $\mathcal{O}_{-m,0,0,0}$, compare Example 1, Subsection 2.1 of [59]). Here $(\cdot)^T$ stands for transposition operation and

$$(\gamma^\mu)^c = (\gamma^\mu)^T = \gamma^{\mu+}.$$\(^3\)

In the space-time coordinates, i.e. after Fourier transformation, the formula for conjugation is equivalent to

$$\phi^c(x) = \phi(x)^+ = (\phi(x))^T.$$\(^4\)

On the Hilbert space $\mathcal{H}_{c,-m,0}$ of conjugated bispinors there is defined the (con-\(^2\)In the standard notation used by physicist the conjugated spinor $\phi^c$ is written as $\phi^+ = \tilde{\phi}^T$, which we have already reserved for the operator conjugation of operators in the Fock space. The complex conjugation followed by transposition we agree to denote in this section by using the $+$ superscript interchangibly with the conjugation superscript $c$, which is customary in physical literature concerning Dirac bispinors and Dirac equation.

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jugated) inner product

\[(\phi^c, \phi')_c = ((\bar{\phi})^c, (\bar{\phi}')^c)_c = (\phi', \phi)\]

\[= \int_{x^0 = t = \text{const.}} (\phi'(x, t), \phi(x, t)) c^4 dx\]

\[= \int_{\mathbb{R}^3} (\bar{\phi}'(-\vec{p}, -|p_0(\vec{p})|), \bar{\phi}(-\vec{p}, -|p_0(\vec{p})|)) c^4 \frac{d^3\vec{p}}{(2p_0)^2}\]

\[= -\int_{\mathcal{O}^{-m,0,0,0}} (\bar{\phi}'(p), \bar{\phi}(p)) c^4 \frac{d\mu_{m,0}(p)}{2|p_0|} = (\bar{\phi}', \bar{\phi})_{\mathcal{H}^{c\ominus}_{m,0}}, p_0(\vec{p}) = -\sqrt{\vec{p} \cdot \vec{p} + m^2}.

where \((\cdot, \cdot)\) is the inner product (11) in the Hilbert space \(H^{c\ominus}_{m,0} \subset H\) of distributional solutions (whose Fourier transforms are concentrated on \(\mathcal{O}^{-m,0,0,0}\)) of Dirac equation defined above, which induces, through Fourier transform, the inner product \((\cdot, \cdot)\) on their Fourier transforms. In the Hilbert space \(H^{c\ominus}_{-m,0}\) there are defined the operations of multiplication by a number \(\alpha \in \mathbb{C}\) and addition by the respective ordinary operations in \(H^{c\ominus}_{-m,0}\), in the following manner

\[\alpha \cdot (\bar{\phi})^c = (\bar{\alpha})^c, \quad (\bar{\phi})^c + (\bar{\phi}')^c = (\bar{\phi} + \bar{\phi}')^c, \quad \bar{\phi}, \bar{\phi}' \in \mathcal{H}^{c\ominus}_{-m,0}.

From the formula (13) one easily see that the Fourier transforms of the conjugated bispinors are concentrated on the positive energy orbit \(\mathcal{O}^{m,0,0,0}\) in the momentum space, and thus they are positive energy solutions of the conjugated Dirac equation (13).

Then on the conjugated Hilbert space \(H^{c\ominus}_{m,0}\) (of conjugated bispinors concentrated on the positive energy orbit \(\mathcal{O}^{-m,0,0,0}\)) there acts naturally the representation

\[\{\widetilde{V} \ominus U_{(-m,0,0,0)} L^{1/2} (\widetilde{V})^{-1}\}^c\]

conjugated to

\[\widetilde{V} \ominus U_{(-m,0,0,0)} L^{1/2} (\widetilde{V})^{-1}\]

with the general definition of conjugation

\[U^c(\bar{\phi})^c = (U \bar{\phi})^c.

Because the spin corresponding to the conjugated representation (14) is likewise 1/2 and the orbit is equal \(\mathcal{O}_{m,0,0,0}\), then one can guess that (15) is likewise equivalent to (3), by Mackey’s classification. Indeed one can construct explicit equivalence similarly as \(V^\ominus U\) in Example 1 (Subsection 2.1 of [59]) with additional transpositions and complex conjugations in this construction.
Thus to the space $H_{c,-m,0}$ we apply the Segal’s functor $\Gamma$ of fermionic quantization and obtain the fermionic Fock space

$$H_{c,-m,0} = \Gamma(H_{c,-m,0}) = C \oplus H_{c,-m,0} \oplus (H_{c,-m,0})^{\otimes 2} \oplus (H_{c,-m,0})^{\otimes 3} \oplus \ldots ;$$

with the unitary representation

$$\Gamma\left(\{\tilde{V} \otimes U_{(-m,0,0,0)} L^{1/2} (V \otimes)^{-1}\}^c\right) = \bigoplus_{n=0,1,2\ldots} \left(\{\tilde{V} \otimes U_{(-m,0,0,0)} L^{1/2} (V \otimes)^{-1}\}^c\right)^{\otimes n}.$$

The conjugation $(\tilde{\phi})^c$ of the bispinor function concentrated on $\sigma_{-m,0,0,0}$ will be sometimes denoted by $\tilde{\phi}^c$ in order to simplify notation. We construct in the standard manner the map

$$H_{c,-m,0} \ni \tilde{\phi}^c \rightarrow a_{\otimes} (\tilde{\phi}^c), \quad a_{\otimes}^\dagger (\tilde{\phi}^c) = a_{\otimes} (\tilde{\phi}^c)^\dagger$$

from $H_{c,-m,0}$ to the families of (ordinary operators, not distributions) of annihilation and creation operators acting in the fermionic Fock space $\Gamma(H_{c,-m,0})$, fulfilling the canonical anticommutation relations:

$$\{a_{\otimes} (\tilde{\phi}^c), \quad a_{\otimes} (\tilde{\phi}^c)^\dagger\} = (\tilde{\phi}^c, \tilde{\phi}^c)^c_{H_{c,-m,0}}$$

$$= (\tilde{\phi}^c, \tilde{\phi}^c)^c_{H_{c,-m,0}}$$

$$= (\tilde{\phi}^c, \tilde{\phi}^c)^c_{H_{c,-m,0}}$$

$$= - \int_{\sigma_{-m,0,0,0}} (\tilde{\phi}^c (p), \tilde{\phi}^c (p))_{H_{c,-m,0}} \frac{d\mu_{m,0} (p)}{2|p_0|}$$

$$= \int_{\mathbb{R}^3} (\tilde{\phi}^c (-\vec{p}, -|p_0(\vec{p})|), \tilde{\phi}^c (-\vec{p}, -|p_0(\vec{p})|))_{c^4} \frac{d^3 \vec{p}}{(2|p_0(\vec{p})|)^2},$$

$$p_0(\vec{p}) = -\sqrt{\vec{p} \cdot \vec{p} + m^2}.$$

In particular the representation of the group $T_4 \otimes SL(2, \mathbb{C})$ which acts in the Fock space $H_{\otimes}^c$ is equal

$$\Gamma\left(\{\tilde{V} \otimes U_{(-m,0,0,0)} L^{1/2} (V \otimes)^{-1}\}^c\right).$$

Of course on the Fock space $H_{\otimes}^c = \Gamma(H_{c,-m,0})$ we have the corresponding parity number (unitary and involutive) operator $In_{\otimes}$ fulfilling

$$In_{\otimes}^2 = 1, \quad In_{\otimes}^* = In_{\otimes},$$

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and such that $\ln{\mathcal{O}}$ anticommutes with the annihilation (and creation) operators:

$$\left\{ a_{\mathcal{O}} (\bar{\phi} |_{m,0,0,0}^c), \ln{\mathcal{O}} \right\} = 0.$$ 

Of course the operator $\ln{\mathcal{O}}$ commutes:

$$[\Gamma \left( \{\bar{\mathcal{V}} \ominus U(\bar{-m},0,0,0)\mathcal{L}^{1/2} \mathcal{V} \ominus 1 \}^c \right), \ln{\mathcal{O}}] = 0$$

with the representation of $T_4 \otimes \mathcal{SL}(2, \mathbb{C})$ acting in the Fock space $\Gamma(\mathcal{H}_{m,0}^{\mathbb{C}})$ and with any operator of the form $\Gamma(A)$ (bounded or unbounded with linear $\text{Dom} \, \Gamma(A)$ in $\Gamma(\mathcal{H}_{m,0}^{\mathbb{C}})$).

### 2.4 The Fock-Hilbert space $\mathcal{H}_F$ of the free Dirac field $\psi$

The Hilbert space $\mathcal{H}_F$ of the free Dirac field is defined as the application of the fermion second quantization functor $\Gamma$ to the “single particle” Hilbert space $\mathcal{H}_{m,0}^{\mathbb{C}} = \mathcal{H}_{m,0}^\oplus \mathcal{H}_{m,0}^{\ominus \mathbb{C}}$, an orthogonal sum of the Hilbert spaces $\mathcal{H}_{m,0}^\oplus$ and $\mathcal{H}_{m,0}^{\ominus \mathbb{C}}$.

Therefore, by the known property of the functor $\Gamma$, it is equal to the tensor product

$$\mathcal{H}_F = \mathcal{H}_{m,0}^{\mathbb{C}} \otimes \mathcal{H}_{m,0}^{\mathbb{C}} = \Gamma(\mathcal{H}_{m,0}^\oplus \otimes \Gamma(\mathcal{H}_{m,0}^{\ominus \mathbb{C}}) = \Gamma(\mathcal{H}_{m,0}^\oplus \otimes \mathcal{H}_{m,0}^{\ominus \mathbb{C}})$$

of the fermion Fock spaces $\mathcal{H}_{m,0}^{\mathbb{C}} = \Gamma(\mathcal{H}_{m,0}^\oplus)$ and $\mathcal{H}_{m,0}^{\mathbb{C}} = \Gamma(\mathcal{H}_{m,0}^{\ominus \mathbb{C}})$ with the representation

$$\left[ \bigoplus_{n=0,1,2,\ldots} \left( \mathcal{V} \ominus \mathcal{O} U(m,0,0,0) \mathcal{L}^{1/2} \mathcal{V} \ominus 1 \right)^n \right] \otimes \left[ \bigoplus_{n=0,1,2,\ldots} \left( \{\mathcal{V} \ominus \mathcal{O} U(-m,0,0,0) \mathcal{L}^{1/2} \mathcal{V} \ominus 1 \}^c \right)^n \right]$$

of the group $T_4 \otimes \mathcal{SL}(2, \mathbb{C})$ acting in the Hilbert space $\mathcal{H}_F$.

Now observe that

$$\{ \mathcal{V} \ominus \mathcal{O} U(\bar{-m},0,0,0) \mathcal{L}^{1/2} (\mathcal{V} \ominus 1) \}^c = (\mathcal{V} \ominus 1)^{-1} \{ \mathcal{V} \ominus \mathcal{O} U(\bar{m},0,0,0) \mathcal{L}^{1/2} \}^c (\mathcal{V} \ominus 1)^+.$$ 

Because by Mackey’s construction of induced representation it follows that

$$\{ \mathcal{V} \ominus \mathcal{O} U(\bar{m},0,0,0) \mathcal{L}^{1/2} \}^c = S^{-1} \{ \mathcal{V} \ominus \mathcal{O} U(\bar{m},0,0,0) \mathcal{L}^{1/2} \} S$$

with some (involutive) unitary operator $S$, we have

$$\{ \mathcal{V} \ominus \mathcal{O} U(\bar{-m},0,0,0) \mathcal{L}^{1/2} (\mathcal{V} \ominus 1) \}^c = U_0^{-1} \{ \mathcal{V} \ominus \mathcal{O} U(\bar{m},0,0,0) \mathcal{L}^{1/2} \} U_0, \quad U_0 = S (\mathcal{V} \ominus 1)^+.$$ 

Thus the joint spectrum of the translation generators of the representation acting in the Hilbert space $\mathcal{H}_F$ of the free Dirac field thus constructed is concentrated on the positive energy cone $C_+$, i.e. it is a positive energy field.
Into the Fock-Hilbert space $\mathcal{H}_F$ of the free Dirac field we again introduce in the standard manner the families

$$\mathcal{H}^{\oplus}_{m,0} \oplus \mathcal{H}^{\ominus}_{m,0} \ni \overline{\phi}_1 \oplus \overline{\phi}_2 \rightarrow a'(\overline{\phi}_1 \oplus \overline{\phi}_2), \quad a'^+(\overline{\phi}_1 \oplus \overline{\phi}_2) = a'(\overline{\phi}_1 \oplus \overline{\phi}_2)^+, \quad (15)$$

fulfilling canonical anticommutation relations

$$\left\{ a'(\overline{\phi}_1 \oplus \overline{\phi}_2), a'(\overline{\phi}'_1 \oplus \overline{\phi}'_2) \right\} = \left( \overline{\phi}_1 \oplus \overline{\phi}_2, \overline{\phi}'_1 \oplus \overline{\phi}'_2 \right)_{\mathcal{H}^{\oplus}_{m,0} \oplus \mathcal{H}^{\ominus}_{m,0}} = \left( \overline{\phi}_1, \overline{\phi}'_1 \right)_{\mathcal{H}^{\oplus}_{m,0}} + \left( \overline{\phi}_2, \overline{\phi}'_2 \right)_{\mathcal{H}^{\ominus}_{m,0}}, \quad (16)$$

where $(\cdot, \cdot)_{\mathcal{H}}$ stands for the inner product on the Hilbert space $\mathcal{H}$. Here $\overline{\phi}_1, \overline{\phi}'_1 \in \mathcal{H}^{\oplus}_{m,0}$ and $\overline{\phi}_2, \overline{\phi}'_2 \in \mathcal{H}^{\ominus}_{m,0}$.

It follows that

$$a'(\overline{\phi}_1 \oplus 0) = a_\oplus(\overline{\phi}_1) \oplus \text{In}_\ominus, \quad \overline{\phi}_1 \in \mathcal{H}^{\oplus}_{m,0}, \quad (17)$$

$$a'(0 \oplus \overline{\phi}_2) = 1 \otimes a_\ominus(\overline{\phi}_2), \quad \overline{\phi}_2 \in \mathcal{H}^{\ominus}_{m,0} \quad (18)$$

and

$$a'(\overline{\phi}_1 \oplus \overline{\phi}_2) = a_\oplus(\overline{\phi}_1) \oplus \text{In}_\ominus + 1 \otimes a_\ominus(\overline{\phi}_2). \quad (19)$$

Note that the equality $\Gamma(\mathcal{H}_1 \oplus \mathcal{H}_2) = \Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}_2)$ expresses in fact existence of a canonical unitary isomorphism respecting the relevant Fock structure with particular importance of the canonical nature of the identification (a mere existence of a unitary map, here in the context of separable Hilbert spaces, is trivial and would tell us nothing as there is plenty of such maps devoid of any relevance). The point is that the identification makes the following equality to hold

$$a(u \oplus v) = a_1(u) \otimes \text{In}_2 + 1 \otimes a_2(v),$$

for the corresponding annihilation and creation operators: $a(u \oplus v), a(u \oplus v)^+$ acting in $\Gamma(\mathcal{H}_1 \oplus \mathcal{H}_2)$, $a_1(u), a_1(u)^+$ acting in $\Gamma(\mathcal{H}_1)$ and $a_2(v), a_2(v)^+$ in $\Gamma(\mathcal{H}_2)$. Recall that $\text{In}_2$ is the involutive unitary (and self-adjoint) parity number operator in Fock space $\Gamma(\mathcal{H}_2)$. In fact in case of the fermionic Fock spaces we have two canonical choices for the identification of the spaces $\Gamma(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and $\Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}_2)$. The second identification makes the following equality to hold

$$a(u \oplus v) = a_2(u) \otimes 1 + \text{In}_1 \otimes a_2(v)$$

with the parity number involution $\text{In}_1$ of the Fock space $\Gamma(\mathcal{H}_1)$. Thus in particular we can use the other canonical identification, where instead of (17), (18), (19) we had

$$a'(\overline{\phi}_1 \oplus 0) = a_\oplus(\overline{\phi}_1) \otimes 1, \quad \overline{\phi}_1 \in \mathcal{H}^{\oplus}_{m,0},$$

$$a'(0 \oplus \overline{\phi}_2) = 1 \otimes a_\ominus(\overline{\phi}_2), \quad \overline{\phi}_2 \in \mathcal{H}^{\ominus}_{m,0},$$

$$a'(\overline{\phi}_1 \oplus \overline{\phi}_2) = a_\oplus(\overline{\phi}_1) \oplus 1 \otimes a_\ominus(\overline{\phi}_2).$$

In case of the boson Fock spaces we have essentially one canonical identification of the Fock spaces $\Gamma(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and $\Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}_2)$ which makes the following equality to hold

$$a(u \oplus v) = a_1(u) \otimes 1 + \text{In} \otimes a_2(v).$$

Therefore during the construction of a field with integer spin, which is not essentially neutral (with antiparticles), when the fermionic functor $\Gamma$ is replaced with bosinic and the anti-commutation relations are replaced with commutation relations, the involutive unitary and selfadjoint operators $\text{In}_\oplus$ and $\text{In}_\ominus$ are replaced here with the unital operator $1$. 

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Here $\text{In}_\odot$ is the parity number (involutive and self-adjoint unitary) operator in the Fock space $\Gamma(H^\oplus_{m,0})$ anticommuting with $a_\odot(\tilde{\phi}_1)$. The operators $a_\odot(\tilde{\phi}_1)$ act on $\Gamma(H^\oplus_{m,0})$ and $a_\odot(\tilde{\phi}_2)$. $\text{In}_\odot$ act on $\Gamma(H^\oplus_{-m,0})$.

In order to simplify notation the operators (17) and (18) understood as operators in the total Fock space

$$H_F = H^\oplus_F \otimes H^\ominus_F \quad \text{(115)} = \Gamma(H^\oplus_{m,0}) \otimes \Gamma(H^\oplus_{-m,0}) = \Gamma(H^\oplus_{m,0} \oplus H^\ominus_{-m,0})$$

of the free Dirac field will likewise be denoted by $a_\oplus(\tilde{\phi}_1)$ and $a_\ominus(\tilde{\phi}_2)$, where $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are understood as elements $\tilde{\phi}_1 \oplus 0$ and $0 \oplus \tilde{\phi}_2$ of the Hilbert space $H^\oplus_{m,0} \oplus H^\ominus_{-m,0}$ respectively, especially when the context suggest with what Fock space we are working.

Note in particular that for the operators (17) and (18), understood as operators on $H_F$ and denoted simply by $a_\oplus(\tilde{\phi}_1)$ and $a_\ominus(\tilde{\phi}_2)$, we have the following canonical anticommutation relations (which follow from (116))

$$\begin{align*}
\{a_\oplus(\tilde{\phi}_1), a_\ominus(\tilde{\phi}_1)^+\} &= (\tilde{\phi}_1, \tilde{\phi}_1^\dagger)_{H^\oplus_{m,0}}, \\
\{a_\ominus(\tilde{\phi}_2), a_\ominus(\tilde{\phi}_2)^+\} &= (\tilde{\phi}_2, \tilde{\phi}_2^\dagger)_{H^\ominus_{-m,0}}, \\
\{a_\oplus(\tilde{\phi}_1), a_\ominus(\tilde{\phi}_2)\} &= \{a_\ominus(\tilde{\phi}_1), a_\ominus(\tilde{\phi}_2)^+\} = 0,
\end{align*}$$

(20)

where again $\tilde{\phi}_1$, $\tilde{\phi}_2$ and $\tilde{\phi}_1$ are understood respectively as elements $\tilde{\phi}_1 \oplus 0$, $\tilde{\phi}_1 \oplus 0$ and $0 \oplus \tilde{\phi}_2$, $0 \oplus \tilde{\phi}_2$ of the Hilbert space $H^\oplus_{m,0} \oplus H^\ominus_{-m,0}$.

The functor $\Gamma$ allows us to have a clear insight into the structure of the representation of $T_4 \otimes SL(2, \mathbb{C})$ acting in $H_F$, as by construction it behaves functorially under the application of $\Gamma$, applied separately to $H^\oplus_{m,0}$ and $H^\ominus_{-m,0}$, and preserves the structure $H_F = \Gamma(H^\oplus_{m,0}) \otimes \Gamma(H^\ominus_{-m,0})$ because both $H^\oplus_{m,0}$ and $H^\ominus_{-m,0}$ are invariant for the representation of $T_4 \otimes SL(2, \mathbb{C})$ in the single particle Hilbert space $\mathcal{H} = H^\oplus_{m,0} \oplus H^\ominus_{-m,0}$. In particular by the general properties of $\Gamma$ the representation of $T_4 \otimes SL(2, \mathbb{C})$ acting in $H_F$ is naturally equivalent to the representation

$$\Gamma\left(U(m,0,0) L^{1/2} \right) \otimes \Gamma\left(U(m,0,0) L^{1/2} \right) = \left[ \bigoplus_{n=0,1,2,...} \left(U(m,0,0) L^{1/2} \right) \right] \otimes \left[ \bigoplus_{n=0,1,2,...} \left(U(m,0,0) L^{1/2} \right) \right],$$

with the equivalence given by the unitary operator $\Gamma(V^\otimes) \otimes \Gamma(S(V^\otimes)^\dagger)$.

Recall also the simple functorial property of $\Gamma$: for any group representations $U_1$ and $U_2$, $\Gamma(U_1 \otimes U_2)$ is naturally equivalent to $\Gamma(U_1) \otimes \Gamma(U_2)$. Thus the Hilbert space $H_F$ is naturally equivalent to the ordinary (in the mathematical sense) Fock space with the representation of $T_4 \otimes SL(2, \mathbb{C})$ in the single particle Hilbert space $\mathcal{H} = H^\oplus_{m,0} \oplus H^\ominus_{-m,0}$ equivalent to $U(m,0,0) L^{1/2} \otimes U(m,0,0) L^{1/2}$. 

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2.5 Quantum Dirac free field $\psi$ as a Wightman operator-valued distribution. Motivation for white noise construction.

In order to construct quantum Dirac field, $\psi$, we need a more subtle structure than just the Fock space, as the quantum field is something which could be called suggestively “operator-valued distribution”, and which in turn is motivated by the classic analysis of measurement of quantum fields due to Bohr and Rosenfeld. In fact the precise mathematical interpretation is in fact still on the way. Intentionally (direction initiated by Wightman) quantum field, say $\psi$, is regarded as a map $f \mapsto \psi(f)$ with $\psi(f)$, intentionally equal

$$\int \psi(x)f(x)\,d^4x = \sum_\alpha \int \psi^\alpha(x)f^\alpha(x)\,d^4x, \quad (21)$$

which maps continuously a specified test space (here the Schwartz’s space $\mathcal{S}(\mathbb{R}^4; \mathbb{C}^4)$ of bispinors $f$ on the space-time) into a specified class of (in general unbounded) operators $L(D)$ on a dense domain $D$ of the Hilbert space, i.e. of the Fock space $\mathcal{H}^\oplus_F = \Gamma(\mathcal{H}^\oplus_{m,0}) \otimes \Gamma(\mathcal{H}^{\otimes c}_{m,0}) = \Gamma(\mathcal{H}^\oplus_{m,0} \oplus \mathcal{H}^{\otimes c}_{m,0})$ in case of the field $\psi$ in question, with a specified sequentially complete topology on $L(D)$ respecting the nuclear theorem and a nuclear topology on the test space, compare [60] and [62] for a more detailed treatment. This should be regarded as the first step toward the precise mathematical interpretation of the notion of quantum field introduced by the founders of QED, and in fact this is one possible approach, most popular among mathematical physicists working within the “axiomatic approach to QFT”. There is also another possible approach, initiated by Berezin [3] and developed by mathematicians [26], [38], [39]. Although Wightman’s definition of the quantum (free) field does not fit well with the causal approach to QFT, we give a general remark on it before passing to the Berezin-Hida white noise construction – more adequate here.

In the Wightman’s construction of (free) quantum field the integral expression (21), and especially the quantum field $\psi(x)$ at a specified space-time point, has only symbolic character, lacking any immediate meaning even when considering free field(s), such as $\psi$. This is just like the symbol $\psi(x)$ for a symbolic evaluation at $x$ of a “function” which symbolizes (when – again symbolically – integrated with a test function $f$) the value at $f$ of a proper distribution – singular generalized function. In particular when considering a free field $\psi$, the value $\psi(f)$ for a space-time test (say bispinor function $f \in \mathcal{S}(\mathbb{R}^4; \mathbb{C}^4)$) is obtained through the creation and annihilation operators evaluated at the Fourier transform $\tilde{f}$ restricted to the orbit $\mathcal{O}$ pertinent to the representation defining the field(s) $\psi$ (in case of presence of antiparticles the representation is not irreducible and evaluation of the creation operator, acting over the Fock space over the single particle Hilbert space of conjugated solutions is involved, and even in general one has to consider many orbits in presence of more complicated fields or several field(s)). The expression (21) is given a meaning whenever ap-
plied to the vectors of the allowed domain $D$, only very indirectly, utilizing the quantity $\psi(f)$, $f \in S(\mathbb{R}^4; \mathbb{C}^4)$, which must be defined as the primary datum, together with the appropriate domain $D$, compare [60], §3-3. For the free Dirac field $\psi$, the expression $\psi(f)$, $f \in S(\mathbb{R}^4; \mathbb{C}^4)$, is defined through the creation $a_\oplus((P^\oplus \tilde{f})_\sigma)^+$ and annihilation $a_\ominus(P^\ominus \tilde{f}|_\sigma)$ operators:

$$\psi(f) = a_\ominus(P^\ominus \tilde{f}|_{e_{m,0,0,0}}) + a_\oplus((P^\oplus \tilde{f}|_{e_{m,0,0,0}})^+) \tag{22}$$

evaluated respectively at $P^\oplus \tilde{f}|_\sigma$ and $(P^\ominus \tilde{f}|_{e_{m,0,0,0}})^\circ$. Here $\tilde{f}$ is the ordinary Fourier transform of spacetime bispinor $f$, and $\tilde{f}|_{e_{m,0,0,0}}$, $\tilde{f}|_{e_{m,0,0,0}}$ the respective restrictions of $\tilde{f}$ to the orbits $\mathcal{O}_{m,0,0,0}, \mathcal{O}_{-m,0,0,0}$:

$$\tilde{f}|_{e_{m,0,0,0}}(p_0, p) = \tilde{f}(\sqrt{|p|^2 + m^2}, p), \quad \tilde{f}|_{e_{m,0,0,0}}(p_0, p) = \tilde{f}(\sqrt{|p|^2 + m^2}, p).$$

Here $P^\oplus$ is the projection operator acting on bispinors $\tilde{f}|_{e_{m,0,0,0}}$ concentrated on $\mathcal{O}_{m,0,0,0}$ and projecting on the Hilbert space $\mathcal{H}_{m,0}^\oplus$, defined in Subsection 2.1 of [59]. $P^\ominus$ is the projection operator which projects bispinors $\tilde{f}|_{e_{m,0,0,0}}$ concentrated on $\mathcal{O}_{-m,0,0,0}$ on the Hilbert space $\mathcal{H}_{m,0}^\ominus$, and defined in Subsection 2.1 of [59], so that

$$P^\oplus \tilde{f}|_{e_{m,0,0,0}}(p) \stackrel{def}{=} P^\oplus(p) \tilde{f}(p), \quad p = (\sqrt{|p|^2 + m^2}, p) \in \mathcal{O}_{m,0,0,0},$$

$$P^\ominus \tilde{f}|_{e_{m,0,0,0}}(p) \stackrel{def}{=} P^\ominus(p) \tilde{f}(p), \quad p = (\sqrt{|p|^2 + m^2}, p) \in \mathcal{O}_{-m,0,0,0}.$$  
Finally $(\cdot)^\circ$ stands for the conjugation defined in Subsection 2.3. By construction $P^\oplus \tilde{f}|_\sigma$ and $(P^\ominus \tilde{f}|_{e_{m,0,0,0}})^\circ$ belong respectively to $\mathcal{H}_{m,0}^\oplus$ and $\mathcal{H}_{-m,0}^\ominus$ whenever $f \in S(\mathbb{R}^4; \mathbb{C}^4)$, and thus belong to the single particle Hilbert space $\mathcal{H}_{m,0}^\oplus \oplus \mathcal{H}_{-m,0}^\ominus$, so that the expressions $a_\ominus((P^\ominus \tilde{f}|_{e_{m,0,0,0}})^\circ)$ and $a_\oplus(P^\oplus \tilde{f}|_\sigma)$ make sense. Moreover both operators $P^\oplus, P^\ominus$ of multiplication by the projectors $P^\oplus(p), p \in \mathcal{O}_{m,0,0,0}$ and respectively $P^\ominus(p), p \in \mathcal{O}_{-m,0,0,0}$, commute by construction with the Fourier transformed Dirac operator of point-wise multiplication by the matrix $p_0 \gamma^0 - p_k \gamma^k$ (summation with respect to $k = 1, 2, 3$) on the Hilbert spaces $\mathcal{H}_{m,0}^\oplus$ and $\mathcal{H}_{-m,0}^\ominus$ of bispinors $\tilde{f}|_{e_{m,0,0,0}}$ and respectively $\tilde{f}|_{e_{-m,0,0,0}}$ concentrated respectively on $\mathcal{O}_{m,0,0,0}$ and $\mathcal{O}_{-m,0,0,0}$, so that

$$\psi((i \gamma^\mu \partial_\mu - m \mathbf{1})f) = 0, \quad f \in S(\mathbb{R}^4; \mathbb{C}^4),$$

and the field $\psi$ fulfills the free Dirac equation as expected, because the algebraic relation

$$[p_0 \gamma^0 - p_k \gamma^k - m \mathbf{1}]P^\oplus \tilde{f}|_{e_{m,0,0,0}}(p) = 0, \quad p = (p_0, p) \in \mathcal{O}_{m,0,0,0},$$

$$[p_0 \gamma^0 - p_k \gamma^k - m \mathbf{1}]P^\ominus \tilde{f}|_{e_{-m,0,0,0}}(p) = 0, \quad p = (p_0, p) \in \mathcal{O}_{-m,0,0,0}. \tag{23}$$

of orbits with a finite range of possible mass parameters and the corresponding field which is called in this case a generalized free field. We describe the case of the quantum Dirac field in details below.
holds on the Hilbert spaces $\mathcal{H}_m^{\oplus,0}$ and $\mathcal{H}_m^{\ominus,0}$ of bispinors $\tilde{f}^1_{\epsilon_{m,0,0,0}}$ and respectively $\tilde{f}^1_{\epsilon_{-m,0,0,0}}$, concentrated on $\mathcal{O}_{m,0,0,0}$ and respectively on $\mathcal{O}_{-m,0,0,0}$, compare Subsection 2.1 of [59]. Indeed that $\psi$ fulfills the homogeneous Dirac equation, can also be immediately seen by noting that the Fourier transformed operator defining homogeneous Dirac equation is equal to point-wise multiplication by the matrix

$$[p_0 \gamma^0 - p_k \gamma^k - m 1_4] = [\not p - m]$$

and that the projection operators $P^\oplus, P^\ominus$, commuting with it, are equal to operators of multiplication by the projection matrices

$$P^\oplus(p) = \frac{1}{2m} [\not p + m], \quad p \in \mathcal{O}_{m,0,0,0},$$

$$P^\ominus(p) = \frac{1}{2m} [\not p + m], \quad p \in \mathcal{O}_{-m,0,0,0},$$

compare Appendix 4 formula (143). From this and from the fact that

$$[\not p + m][\not p - m] = [\not p - m][\not p + m] = [p \cdot p - m^2] 1_4 = 0, \quad p \in \mathcal{O}_{m,0,0,0},$$

$$[\not p + m][\not p - m] = [\not p - m][\not p + m] = [p \cdot p - m^2] 1_4 = 0, \quad p \in \mathcal{O}_{-m,0,0,0},$$

the commutativity of $[p_0 \gamma^0 - p_k \gamma^k - m 1_4]$ with $P^\oplus(p)$ on $\mathcal{O}_{m,0,0,0}$ and with $P^\ominus(p)$ on $\mathcal{O}_{-m,0,0,0}$, as well as the relations (23) are easily seen to hold, so that our assertion follows.

Note that in the formula (22) we have used the simplified notation for the operator (17) and for the operator adjoint to (18). For the operator $a_\oplus(P^\oplus \tilde{f}^1_{\epsilon_{m,0,0,0}})$ in the formula (22) the reader should read

$$a'(P^\oplus \tilde{f}^1_{\epsilon_{m,0,0,0}} \oplus 0) = a_\oplus(P^\oplus \tilde{f}^1_{\epsilon_{m,0,0,0}}) \otimes \text{In}_\ominus$$

(24)

and for the operator $a_\ominus\left((P^\ominus \tilde{f}^1_{\epsilon_{-m,0,0,0}})^c\right)^+$ in (22) the reader should read

$$a'(0 \oplus (P^\ominus \tilde{f}^1_{\epsilon_{-m,0,0,0}})^c)^+ = 1 \otimes a_\ominus\left((P^\ominus \tilde{f}^1_{\epsilon_{-m,0,0,0}})^c\right)^+.$$  

(25)

On the left hand sides of the last two formulas we have the standard annihilation and creation operators $a'(u \oplus v), a'(u \oplus v)^+$ acting on the Fock space

$$\mathcal{H}_F = \Gamma(\mathcal{H}_{m,0}^{\oplus} \oplus \mathcal{H}_{-m,0}^{\ominus}) = \Gamma(\mathcal{H}_{m,0}^{\oplus}) \otimes \Gamma(\mathcal{H}_{-m,0}^{\ominus})$$

of the free Dirac field introduced in Subsection 2.3. On the right hand sides of the last two formulas we have the annihilation and creation operators $a_\oplus(P^\oplus \tilde{f}^1_{\epsilon_{m,0,0,0}})$ and $a_\ominus\left((P^\ominus \tilde{f}^1_{\epsilon_{-m,0,0,0}})^c\right)^+$ acting respectively in the Fock spaces $\Gamma(\mathcal{H}_{m,0}^{\oplus})$ and $\Gamma(\mathcal{H}_{-m,0}^{\ominus})$, and defined respectively in Subsections 2.2 and 2.3. For definition of
the unitary involutive (and thus self-adjoint) operator \( \mathrm{In}_{\ominus} \) we refer to Subsections 2.2 and 2.3.

Thus the formula (22) should properly be written as

\[
\psi(f) = a'(P^\oplus \tilde{f}|_{\mathcal{O}_m,0,0,0} \oplus 0) + a'(0 \oplus (P^\ominus \tilde{f}|_{\mathcal{O}_{-m,0,0,0}})^c)^+. \tag{26}
\]

In fact \( \psi(f) \) is antilinear in \( f \), but the additional complex conjugation will make it linear operator-valued distribution. We have not placed this conjugation explicitly in order to simplify notation.

It should be stressed however that the structure \( \mathcal{H}_F = \mathcal{H}_F^\oplus \otimes \mathcal{H}_F^\ominus = \Gamma(\mathcal{H}_m^\oplus) \otimes \Gamma(\mathcal{H}_{-m}^\ominus) \) of the Hilbert space of the free quantum Dirac field \( \psi \), as well as the tensor product form of the operators (24) and (25) in (26) does not mean that the quantum Dirac field may be treated as sum of two independent fields of electrons and positrons. Indeed the quantized Dirac field, equal to the linear combination (26) of operators \( \psi(f) \), cannot be treated as sum of field operators respectively in \( \Gamma(\mathcal{H}_m^\oplus) \) and \( \Gamma(\mathcal{H}_{-m}^\ominus) \) simply because the arguments

\[
P^\oplus \tilde{f}|_{\mathcal{O}_m,0,0,0} \quad \text{and} \quad (P^\ominus \tilde{f}|_{\mathcal{O}_{-m,0,0,0}})^c
\]

in the operators (24) and (25) entering the formula (26) for \( \psi(f) \) are not independent. Indeed by choosing a function \( f \) from the test space \( \mathcal{S}(\mathbb{R}^4;\mathbb{C}^4) \) we predetermine the restrictions

\[
\tilde{f}|_{\mathcal{O}_m,0,0,0} \quad \text{and} \quad \tilde{f}|_{\mathcal{O}_{-m,0,0,0}}
\]

of its Fourier transform to the orbits \( \mathcal{O}_m,0,0,0 \) and \( \mathcal{O}_{-m,0,0,0} \), which cannot be varied independently one from another. This dependence, imposed on

\[
\tilde{f}_1 = \tilde{f}|_{\mathcal{O}_m,0,0,0} \quad \text{and} \quad \tilde{f}_2 = \tilde{f}|_{\mathcal{O}_{-m,0,0,0}}
\]

by the fact that they come from restrictions to the orbits of the Fourier transform of one and the same \( f \), cannot be realized by any natural relation put on the two \( a \text{ priori} \) independent fields of electrons and positrons, and realized through (24) and (25) with two independent arguments \( f \), respectively, in (24) and (25).

The domain \( \mathcal{D} \) of the field \( \psi \), due to the interpretation initiated by Wightman, is not determined uniquely but in any case contains at least the domain \( \mathcal{D}_0 \) which arises by the action of polynomials expressions in

\[
\psi(f_1), \psi(f_2), \ldots, \quad f_i \in \mathcal{S}(\mathbb{R}^4;\mathbb{C}^4)
\]

on the vacuum \( |0\rangle = \Psi_0 \). However we know that the domain must be considerably larger if \( L(\mathcal{D}) \) is supposed to satisfy kernel theorem in accordance to the

---

5The operator \( \mathrm{In}_{\ominus} \) is replaced with the unital operator in case of integer spin (non-neutral) field.

6Both treated as tensor product operators on \( \Gamma(\mathcal{H}_m^\oplus) \otimes \Gamma(\mathcal{H}_{-m}^\ominus) \), the first having the second factor trivial and equal to the fundamental unitary involution \( \mathrm{In}_{\ominus} \) and vice versa for the second, with the first factor trivial and equal to the unit operator.
result of [62]. In particular it must contain the domain called \( D_1 \) in [60], p. 107, but it is even not clear for the free field determined by an irreducible representation corresponding to a single orbit that \( L(D_1) \) satisfies the theorem on kernel as stated in [62]. We only know, by the result of [62], that such domain \( D \) exists on which \( L(D) \) satisfies the theorem on kernel (with the “strong topology” on \( L(D) \)), and contains the domain called \( D_1 \) in [60], p. 107.

More generally for any \( f \in S(\mathbb{R}^{4k}) = S(\mathbb{R}^4)^{\otimes k} \) and for any system of free fields \( \psi_1, \ldots, \psi_k \) one can give a meaning of a well defined vector in the dense domain \( D \) of the Fock space of the total system to the expression of the form

\[
\Psi = \int d^4x_1 \ldots d^4x_k f(x_1, \ldots, x_k) \psi_1(x_1) \ldots \psi_k(x_k) \Psi_0, \tag{27}
\]

and then for any field \( \psi \) of the considered system of free fields and for any \( \Psi \) of the form (27) one can give a meaning by a limit process to the expression

\[
\psi(f)\Psi \tag{28}
\]

thus giving a meaning to \( \psi_1(x_1) \ldots \psi_k(x_k) \) of an operator-valued distribution over the test space \( S(\mathbb{R}^4)^{\otimes k} \) on the domain containing all vectors of the form (27), compare [60], §3-3. This is achieved by noting first that

\[
(\Psi_0, \psi_1(f_1) \ldots \psi_k(f_k) \Psi_0)
\]

is a well defined and separately continuous multilinear functional of the arguments \( f_i \) in the nuclear topology on the Schwartz space \( S(\mathbb{R}^4) \). Thus by the ordinary Schwartz kernel theorem it follows that there exists a unique distribution \( \mathcal{W}(x_1, \ldots, x_k) \) such that

\[
\int \mathcal{W}(x_1, \ldots, x_k) f_1(x_1)f_2(x_2) \ldots f_k(x_k) d^4x_1 \ldots d^4x_k = (\Psi_0, \psi_1(f_1) \ldots \psi_k(f_k) \Psi_0)
\]

for any \( f_i \in S(\mathbb{R}^4) \). Using this fact (as in [60], p. 107) we next show that the states

\[
\Psi_J = \sum_{j=1}^{J} \psi_1(f_{1j}) \ldots \psi_k(f_{kj}) \Psi_0
\]

converge in norm of the Fock space whenever the functions

\[
f_J(x_1, \ldots, x_k) = \sum_{j=1}^{J} f_{1j}(x_1)f_{2j}(x_2) \ldots f_{kj}(x_k)
\]

converge to \( f \) in \( S(\mathbb{R}^4)^k = S(\mathbb{R}^{4k}) \). The limit of \( \Psi_J \) is defined as the vector \( \Psi \) giving the meaning to the expression (27). The value (28) is defined as the limit of \( \psi(f)\Psi_J \), and gives a well defined “operator-valued” distribution by the pre-closed character of the operators \( \psi(f) \) on the domains \( D_0 \subset D_1 \), compare [62].
In Wightman approach it is the formula (22) which gives the meaning to the symbolic expression (21) when applied to the elements of the domain $\mathcal{D}$.

For a given free field (or a system of free fields $\psi_1, \psi_2, \ldots, \psi_k$) one can give, within the mentioned Wightman approach, a meaning to the expression

$$\partial^{\alpha_1} \psi_1 \ldots \partial^{\alpha_k} \psi_k : f(x) : = \int : \partial^{\alpha_1} \psi_1(x) \ldots \partial^{\alpha_k} \psi_k(x) : f(x) \, d^4x \quad (29)$$

as a limit, giving an operator-valued distribution $[61]$. However here for definition of the “Wick product” due to $[61]$ and using Wightman’s definition of the field the limit process involved here is devoid of any natural choice, as the “Wick product field” of Wightman and Gårding is obtained from an operator-valued distribution in several spacetime variables, and then as a limit we obtain operator valued distribution in just one space-time variable. Such definition involves a considerable amount of unnatural and rather arbitrary choices in selecting a (class of) limit(s) of passing from test function spaces in just one space-time variable to the test space in several space-time variables, compare $[61]$ for one possible choice of the limit process.

Unfortunately the method of $[61]$ is not efficient (for boson, and particularly for mass less fields) in the investigation of the closability of the operator (29) or its eventual self-adjointness nor for the proof of the “Wick theorem” $[6]$, Chap. III, useful in the causal perturbative approach to QED. Similarly the space-time averaging as presented in $[61]$ is not applicable to the averaging over space-like Cauchy hypersurfaces of their “Wick product fields”, necessary in construction of the conserved currents appearing in the Noether theorem for free fields. In particular the Quantization Postulate for free fields as formulated in $[6]$, Chap. 2, §9.4, cannot be simply treated with Wightman-Gårding method, and for

$$\begin{align*}
: \partial^{\alpha_1} \psi_1(x) \partial^{\beta} \psi(x) := \lim_{x_1, x_2 \to x} \left[ \partial^{\alpha_1} \psi_1(x_1) \partial^{\beta} \psi(x_2) \Psi_0 \right], \\
: \partial^{\alpha} \psi(x) \partial^{\beta} \psi(x) \partial^{\gamma} \psi(x) := \lim_{x_1, x_2, x_3 \to x} \left[ \partial^{\alpha} \psi_1(x_1) \partial^{\beta} \psi(x_2) \partial^{\gamma} \psi(x_3) \Psi_0 \right],
\end{align*}$$

and is again only heuristic, and strictly speaking is meaningless as a definition of operator-valued distribution, as it involves limit process of passing from test space of one space-time variable to test space of several space-time variables, which is not specified there. The reader which would like to know the concrete choice of the possible limit process involved there which is meant by the authors will have to consult the paper $[61]$. 

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7For the opposite direction, i.e. for passing from distribution of one variable to distribution of several variables, we would have the natural choice given by the map defined by the restriction to the diagonal, which is continuous between the test spaces. Reverse direction is by no means natural nor unique. The reader should also note that the “definition” of the Wick product in $[60]$, §3-2, p. 104, which merely says:

$$\begin{align*}
\partial^{\alpha_1} \psi_1(x_1) \partial^{\beta} \psi(x_2) & = \lim_{x_1, x_2 \to x} \left[ \partial^{\alpha_1} \psi_1(x_1) \partial^{\beta} \psi(x_2) \Psi_0 \right], \\
\partial^{\alpha} \psi(x_1) \partial^{\beta} \psi(x_2) \partial^{\gamma} \psi(x_3) & = \lim_{x_1, x_2, x_3 \to x} \left[ \partial^{\alpha} \psi_1(x_1) \partial^{\beta} \psi(x_2) \partial^{\gamma} \psi(x_3) \Psi_0 \right],
\end{align*}$$

and so on . . .
zero mass fields this Postulate seems to be intractable with Wightman-Gårding method.

This is somewhat unsatisfactory because the causal method, which is successful in avoiding ultraviolet infinities (also avoiding infrared infinities for the adiabatically switched off interaction at infinity), expresses the interacting fields in terms of time ordered products of Wick polynomials of free fields, and is substantially based on the “Wick theorem” for free fields as stated in [6], Chap. III. Essentially this “theorem” allows to treat the (generalized) operators of the type (compare Theorem 0 in [14])

\[ \int \kappa(x_1, \ldots, x_k) : \partial^{a_1} \psi_1(x_1) \ldots \partial^{a_k} \psi_k(x_k) : \, d^4x_1 \ldots d^4x_k, \]  

with numerical, “translationally invariant” \((\kappa(x_1 + a, \ldots, x_k + a) = \kappa(x_1, \ldots, x_k))\), distributions \(\kappa \in \mathcal{S}(\mathbb{R}^{4k})^* = (\mathcal{S}(\mathbb{R}^4)^*)^\otimes k\) which, when integrated with test functions \(f \in \mathcal{S}(\mathbb{R}^{4k}) = \mathcal{S}(\mathbb{R}^4)^\otimes k\), define an operator valued distribution

\[ f \rightarrow \int f(x_1, \ldots, x_k) \kappa(x_1, \ldots, x_k) : \partial^{a_1} \psi_1(x_1) \ldots \partial^{a_k} \psi_k(x_k) : \, d^4x_1 \ldots d^4x_k. \]  

It is therefore not satisfactory that already at the free field level the “Wick theorem” in the form needed for the causal perturbative approach is not clearly related to the free field defined according to Wightman [60].

In spite of this inconvenience, “Wick theorem” of [6], Chap III, provides partially heuristic (but honest) basis for construction of “operator-valued distributions” of the type \(\kappa\), compare Theorem 0 of [14]. This turned up to be effective in the realization of the causal approach program of Stückelberg-Bogoliubov. As realized later by Epstein and Glaser [14] the causal approach of Stückelberg-Bogoliubov provides a perturbative method which avoids ultraviolet infinities (and also infrared but with the unphysical adiabatically switched off interaction at infinity which, especially in case of QED, needs a further analysis of the behaviour of the theory when the physical interaction is restored, say by adiabatical switching on the interaction at infinity). The essential improvement of the causal method of Stückelberg-Bogoliubov added by Epstein and Glaser is the carefull splitting of the operator-valued distributions of the type \(\kappa\) with causally supported distribution kernels \(\kappa\) into the retarded and advanced parts – a task which we encounter in the causal construction of the perturbative series. Epstein and Glaser [14] reduce this task to the splitting of the numerical causally supported distribution kernels \(\kappa\) into the retarded and advanced part. In fact this reduction of the splitting of operator-valued distribution to the splitting of the numerical distribution kernels \(\kappa\) does not proceed

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8The mentioned weaknesses of Wightman-Gårding definition of the “Wick product” have also been noted by I. E. Segal, compare e.g. [49], [50].

9In fact we are interested here in distributions \(\kappa\) which arise as tensor products of the pairings of the corresponding free fields \(\partial^{a_i} \psi_i\) and when the interaction does not contain derivatives we may confine attention in (30) to the case where derivatives are absent, i.e. with all the multiindices \(a_i = 0\). In particular all such distributions have the mentioned invariance property.
by any rigorous proof, but again seems to be a reliable assumption, which can automatically be proved at the same level of rigour as the “Wick theorem” for free fields of [6], Chap III.

Now in case of the first and higher order contributions to the interacting field (in the scalar : $\phi^4$ : massive theory) Epstein and Glaser [15] were able to prove that on a dense domain $\mathcal{D}$ containing $\mathcal{D}_0$ the contributions (taken separately) converge in norm of the Fock space when evaluated on the states of $\mathcal{D}$, provided the intensity-of-interaction-function $g$ converges suitably to a constant function (i.e. for the adiabatically switched on interaction). This suggests that the higher order contributions (taken separately) to the interacting field may represent an operator-valued distribution in Wightman sense, at least for massive scalar : $\phi^4$ : theory, with the interaction restored at infinity.

But because for QED similar convergence has so far been not successful (for the adiabatic limit of restoring the interaction at infinity), and because there are even evident counterexamples for the existence of a domain containing $\mathcal{D}_0$ on which such a limit could exist, some physicists come to the conclusion that the causal perturbative method cannot provide any sensible contributions to the interacting fields in QED.

But we claim that such conclusion would be premature. This is because the quantum field as defined by Wightman is not the one which is satisfactory from the physical point of view, in particular it does not provide sufficient basis for the “Wick theorem” for free fields needed for the causal perturbative method or even for the Noether theorem for free fields. The fact that the contributions to the interacting field in the massive scalar : $\phi^4$ : theory compose a Wightman field is from the physical point of view completely irrelevant and in fact accidental. Similarly the fact that the contributions to the interacting fields in QED do not form Wightman fields (which can be rather safely assumed) is completely irrelevant from the physical point of view.

A serious physical problem would arise if we had the following situation summarized by the following two hypotheses:

1) Assume we are using a “knew” mathematically rigorous construction of the (free) field in the causal perturbative method, which would be satisfactory in giving a solid basis for a rigorous formulation and proof of the “Wick theorem” for free fields, in giving a strict mathematical meaning to the field at specified space-time point (of course it cannot be ordinary operator in the Fock or generally Hilbert space of the field), which moreover allows to treat rigorously expressions like (30).

2) Assume that the higher order contributions to interacting fields cannot be interpreted as fields in this “knew” satisfactory sense, when we put the intensity-of-interaction-function $g$ equal everywhere to one (i.e. with the interaction restored at infinity).

If we had this situation we would be in a serious trouble, but fortunately we are not.
In this context we should recall the classic work of Berezin [3] who pointed out that there exists a natural construction of quantum free field(s), which gives a meaning to the field $\psi(x)$ at each specified space-time point $x$. Although $\psi(x)$ is not an ordinary operator in the Fock space, nonetheless it has a meaning as a generalized operator mapping continuously a dense nuclear subspace $(E)$ of the Fock space $\mathcal{H}_F = \Gamma(\mathcal{H}_{m,0} \oplus \mathcal{H}_{-m,0})$ into its strong dual $(E)^*$. The point is that the nuclear space $(E)$ is uniquely determined by the space-time geometry and by the transformation rule of the field, leaving no “hand-made” manipulations. Later on Hida, Obata and Saito [26], [39] converted Berezin’s ideas [3] into a very elegant construction of free fields in terms of white noise formalism. But perhaps the most important fact is that, when using the Berezin-Hida white noise construction of free fields, the expressions (21), (30) and the expression

$$\int f(x_1, \ldots, x_k) \kappa(x_1, \ldots, x_k) : \partial^{\alpha_1} \psi_1(x_1) \cdots \partial^{\alpha_k} \psi_k(x_k) : d^4x_1 \cdots d^4x_k$$

in (31) all become particular examples of a wide class of integral kernel operators, transforming continuously the Hida nuclear test space $(E)$ into its strong dual $(E)^*$. We denote the linear space of all operators transforming continuously $(E)$ into $(E)^*$ (resp. into $(E)$) by $\mathcal{L}((E), (E)^*)$ (resp. by $\mathcal{L}((E), (E))$) and endow with the topology of uniform convergence on bounded sets. Theory of such operators is computationally very effective, compare [26], [39], [38]. In particular there exists a theory of Fock expansions of operators from $\mathcal{L}((E), (E)^*)$ into series of integral kernel operators of the type (31), symbol calculus for such operators, as well as effective criteria put on the numerical distribution kernels $\kappa$ under which the corresponding integral kernel operator (30) from $\mathcal{L}((E), (E)^*)$ belongs to $\mathcal{L}((E), (E))$, i.e. transforms continuously the Hida test space $(E)$ into itself, and thus represents a densely defined ordinary operator in the Fock space. In particular as a corollary from the general theory of integral kernel operators we obtain a theorem that the map (31) is continuous from the nuclear test space to the space $\mathcal{L}((E), (E))$ endowed with the topology of uniform convergence on bounded sets, for massive free fields $\psi_k$ and the same theorem holds for the map $f \mapsto \psi(f)$, with $\psi(f)$ defined by (21), and now (21) becomes to be a well defined operator-valued distribution. If among the fields $\psi_k$ there are mass less fields, then still (31) is a well defined integral kernel operator and can be averaged in the states of the Hida subspace and each such average defines a well defined scalar distribution (as a function of $f$), compare Subsection 2.6.

It thus follows that the Berezin-Hida white noise construction of free fields fulfills the requirement put on the “knew” construction of the free field of the above stated Assumption 1). When we use this construction for free fields and put into the causal perturbative series for interacting fields, then each order contribution to interacting fields with the intensity-of-interaction-function $g$ equal 1, becomes a well defined integral kernel operator

$$\int \kappa(x_1, \ldots, x_k, x) : \partial^{\alpha_1} \psi_1(x_1) \cdots \partial^{\alpha_k} \psi_k(x_k) : d^4x_1 \cdots d^4x_k,$$
(or a finite sum of such), which can be understood as integral kernel operator

\[ \int \kappa'(x_1, \ldots, x_k) : \partial^{\alpha_1} \psi_1(x_1) \cdots \partial^{\alpha_k} \psi_k(x_k) : d^4 x_1 \ldots d^4 x_k, \]

but with the vector-valued distributional kernel \( \kappa' \) with values in the strong dual to the space of space-time test function space (tempered distributions), and moreover, the map

\[ f \mapsto \int f(x) \kappa(x_1, \ldots, x_k, x) : \partial^{\alpha_1} \psi_1(x_1) \cdots \partial^{\alpha_k} \psi_k(x_k) : d^4 x_1 \ldots d^4 x_k d^4 x, \]

is continuous from the nuclear space-time test function space to the space \( \mathcal{L}(\mathcal{E}, \mathcal{E}^\ast) \), which can be averaged in the states of the Hida subspace, and each such average defines a scalar distribution as a functional of the test function \( f \). A proof of this assertion for contributions of each order to the interacting Dirac field and electromagnetic potential field in QED can be found in Subsection 2.7, compare also [3]. Thus fortunately the above Assumption 2) is false.

Thus improving the method of Stückelberg and Bogoliubov, corrected by the careful splitting of Epstein and Glaser, still further by using the Berezin-Hida construction of free fields, understood as integral kernel operators with vector-valued kernels, we obtain well defined contributions to the interacting fields, as integral kernel operators with vector-valued kernels, which are well defined operator-valued distributions continuously mapping the test space into the space \( \mathcal{L}(\mathcal{E}, \mathcal{E}^\ast) \) endowed with the topology of uniform convergence on bounded sets, and defined as integral kernel operators with vector valued kernels. They can be averaged in the states of the Hida subspace and each such average defines a scalar distribution as a functional of the test function. Moreover we open up in this way the perturbative series to the general mathematical theory of Fock expansions for operators in \( \mathcal{L}(\mathcal{E}, \mathcal{E}^\ast) \) into integral kernel operators, [39], [38], [37], here defined by the integral kernel operators corresponding to the contributions of each individual order. Thus not only divergences at each order separately are not encountered, but we also acquire a new mathematical tool for the investigation of the convergence of the perturbative series for interacting fields. Therefore we obtain in this manner a perturbation method which, from the start to the end, uses well defined mathematical objects without encountering any ultraviolet nor infrared divergences; but moreover we can subject the convergence of the perturbative series for interacting fields to computationally effective criteria.

In fact the integral kernels \( \kappa \) in [39] which we are interested in are of special form because their Fourier transforms \( \tilde{\kappa} \) are concentrated on the Cartesian product of the orbits corresponding to the respective free fields, and can be regarded as distributions on the tensor products of nuclear spaces of restrictions of the Fourier transforms of test function spaces to the corresponding orbits. Denoting the nuclear spaces of restrictions of the Fourier transforms of the test functions to the corresponding orbits, respectively by \( E_1, E_2, \ldots \) (depending on the number of free fields in the system) we can restrict attention to the integral

\[ 39 \]
kernel operators in the momentum picture which are of the form

\[
\Xi_{i,m}(\kappa) = \int \kappa(k_1, \ldots, k_i, p_1, \ldots, p_m) \times \\
	imes a_1(k_1)^+ \cdots a_l(k_l)^+ a_1(p_1) \cdots a_m(p_m) d^3k_1 \cdots d^3k_l d^3p_1 \cdots d^3p_m, \quad (32)
\]

with kernels \(\kappa\) as numerical distributions, i.e. belonging to

\[
E_1^* \otimes \cdots \otimes E_l^* \otimes \cdots \otimes E_m = \mathcal{L}(E_1 \otimes \cdots \otimes E_l \otimes E_1 \otimes \cdots \otimes E_m, \mathbb{C})
\]

(when considering the so called \(n\)-point distributions in the expansion of the scattering matrix or when computing (67) of Subsection 2.9 of [59] or with kernels \(\kappa\) as vector-valued distributions, i.e. belonging to

\[
\mathcal{L}(E_1 \otimes \cdots \otimes E_l \otimes E_1 \otimes \cdots \otimes E_m, \mathcal{E}^*)
\]

when considering contributions to interacting fields. For reasons we have explained in [59] (compare also the following Sections) we have to consider two different kinds of nuclear spaces \(\mathcal{E}\) of space-time test \(\mathbb{C}\)-valued functions, correspondingly to the zero mass fields and to the massive fields (or correspondingly to the orbit \(\mathcal{O}_{1,0,0,1}\) which is given by one sheet of the light cone in momentum space or to the orbit \(\mathcal{O}_{m,0,0,0}\) which is given by one sheet of the two-sheeted hyperboloid of fixed mass in the momentum space). In the first massive case the nuclear space \(\mathcal{E}\) of space-time test \(\mathbb{C}\)-valued functions run over the ordinary Schwartz space \(\mathcal{S}(\mathbb{R}^4; \mathbb{C})\), in the second case \(\mathcal{E}\) is equal to the closed subspace \(\mathcal{S}^0(\mathbb{R}^4; \mathbb{C})\) of \(\mathcal{S}(\mathbb{R}^4; \mathbb{C})\) of all those functions whose Fourier transforms vanish at zero together with all their derivatives. Now in each case \(E_i\) is equal either to the nuclear space of restrictions of the Fourier transforms of elements of \(\mathcal{E}\) (equal either \(\mathcal{S}(\mathbb{R}^4; \mathbb{C})\) or \(\mathcal{S}^0(\mathbb{R}^4; \mathbb{C})\)) to the corresponding orbit \(\mathcal{O}_{m,0,0,0}\) or \(\mathcal{O}_{1,0,0,1}\). Denoting the nuclear space of Fourier transforms of the elements of \(\mathcal{S}^0(\mathbb{R}^n; \mathbb{C})\) by \(\mathcal{S}^0(\mathbb{R}^n; \mathbb{C})\), we see that \(E_i\) is equal respectively \(\mathcal{S}(\mathbb{R}^3; \mathbb{C})\) or \(\mathcal{S}^0(\mathbb{R}^3; \mathbb{C})\), correspondingly to the corresponding orbit \(\mathcal{O}\). The operators \(a_i(k_i)^+, a_i(p_i)\) in (32) compose canonocal pairs of commuting or anticommuting generalized operators at the specified points \(k_i\) and \(p_i\) of in the cartesian coordinates on the corresponding orbit \(\mathcal{O}_i\) in the momentum space, constructed within the white noise setup.

### 2.6 Quantum Dirac free field \(\psi\) as an integral kernel operator with vector-valued distributional kernel within the white noise construction of Berezin-Hida-Obata

In constructing the quantum free Dirac field \(\psi\) according to Berezin-Hida, we proceed in sense in a totally opposite direction in comparison to Wightman. Namely Wightman restricts the arguments \(u \oplus v \in \mathcal{H}' = \mathcal{H}_{m,0}^{\oplus} \oplus \mathcal{H}_{-m,0}^{\oplus}\) of the operators \(a'(u \oplus v), a'(u \oplus v)^+\) in (26) to the nuclear subspace \(E \cong \mathcal{S}(\mathbb{R}^3; \mathbb{C}^4)\) of all those \(u \oplus v\) for which \(u\) are equal to

\[
u = P^\oplus f|_{\mathcal{E}_{-m,0,0,0}}, \quad f \in \mathcal{S}(\mathbb{R}^4; \mathbb{C}^4)
\]
By construction the creation operators \( a_a \) cause the inclusion of \((E, H)\) operators continuously transforming the nuclear dense space \((E, H)\) with a nuclear (Hida) dense subspace \((E, H)\). The second quantized level with the corresponding Gelfand triple we should do it in such a manner which allows lifting of this construction to the second quantized level with the corresponding Gelfand triple

\[
\begin{align*}
E & \subset \mathcal{H}' \subset E^* \\
\mathcal{H}^\oplus_{m,0} \oplus \mathcal{H}^\otimes_{m,0}
\end{align*}
\]

In the following steps he keeps the arguments \( u \oplus v \) of the annihilation and creation operators \( a'(u \oplus v), a'(u \oplus v)^+ \) within the nuclear space \( E \), and with the domain \( D \) of the operators \( a'(u \oplus v), a'(u \oplus v)^+ \) which is not uniquely nor naturally determined.

According to Berezin-Hida we choose quite an opposite direction: we extend the domain of the arguments \( u \oplus v \) of the creation and annihilation operators \( a'(u \oplus v), a'(u \oplus v)^+ \) to include also generalized states (elements of the strong dual \( E^* \approx \mathcal{S}(R^3; C^4)^* \) – tempered distributions) \( u \oplus v \), like the plane wave solutions. This is exactly what is needed (and used but at the formal level) in the (formal) proof of the so called “Wick theorem” for free fields, presented in [4], Chap. III. By utilizing the rigorous construction of the Hida operators \( a'(u \oplus v), a'(u \oplus v)^+ \) we convert this formal proof into a rigorous one.

This is achieved in the following manner. First we introduce the nuclear space \( E \) as above, which composes with the single particle Hilbert space \( \mathcal{H}' = \mathcal{H}^\oplus_{m,0} \oplus \mathcal{H}^\otimes_{m,0} \), a Gelfand triple

\[
E \subset \mathcal{H}' \subset E^* \\
\mathcal{H}^\oplus_{m,0} \oplus \mathcal{H}^\otimes_{m,0}
\]

with a nuclear (Hida) dense subspace \((E)\) in the Fock space \( \Gamma(E') = \Gamma(\mathcal{H}^\oplus_{m,0} \oplus \mathcal{H}^\otimes_{m,0}) \). For each \( u \oplus v \in E^* \) the annihilation operators \( a'(u \oplus v) \) become operators continuously transforming the nuclear dense space \((E)\) into itself. Because the inclusion of \((E)\) into the strong dual \((E)^*\) is continuous, the operators \( a'(u \oplus v) \) can be naturally regarded as continuous operators \((E) \rightarrow (E)^*\). By construction the creation operators \( a'(u \oplus v)^+, u \oplus v \in E^* \), are equal \( \cdot \circ a'(u \oplus v)^* \circ (\cdot) \), i.e., to the linear duals \( a'(u \oplus v)^* \) of the annihilation operators \( a'(u \oplus v) \) composed with complex conjugation, and thus transform continuously the strong dual \((E)^*\) into itself, and can be naturally regarded as continuous operators \((E) = (E)^* \rightarrow (E)^* \) (because \((E)\) is reflexive). For \( u \oplus v \in E \) the operators \( a'(u \oplus v), a'(u \oplus v)^+ \) become operators transforming continuously the nuclear dense space \((E)\) into itself and thus belong to \( \mathcal{L}((E), (E)) \). Moreover the maps

\[
\begin{align*}
E \ni u \oplus v & \mapsto a'(u \oplus v) \in \mathcal{L}((E), (E)), \\
E \ni u \oplus v & \mapsto a'(u \oplus v)^+ \in \mathcal{L}((E), (E)),
\end{align*}
\]

41
are continuous when $L^2((E),(E))$ – the linear space of linear continuous operators from $(E)$ into $(E)$ – is given the natural nuclear topology of uniform convergence on bounded sets.

Therefore it is important to have the Gelfand triple $E \subset H' \subset E^*$ in the form which allows its lifting to the Fock space and the construction of the Hida test space $(E)$ composing the Gelfand triple $(E) \subset \Gamma(H') \subset (E)^*$. This is in particular the case when we have the nuclear space $E \subset H'$ in the standard form, [39]. Namely let $(\mathcal{O},d\mu_O)$ be a topological space $\mathcal{O}$ with a Baire (or Borel) measure $d\mu_O$. Then we assume that $H'$ is naturally unitarily equivalent to the Hilbert space of $C$-valued measurable (equivalence classes modulo equality almost everywhere) and square summable functions $L^2(\mathcal{O},d\mu_O)$. Next we assume that $E \subset H'$ is naturally unitarily equivalent, with the same unitary equivalence $U$ which also defines an isomorphism of $E$ with the standard countably Hilbert nuclear space $S_A(\mathcal{O};\mathbb{C}) \subset L^2(\mathcal{O},d\mu_O;\mathbb{C})$, composing a Gelfand triple

$$S_A(\mathcal{O};\mathbb{C}) \subset L^2(\mathcal{O},d\mu_O;\mathbb{C}) \subset S_A(\mathcal{O};\mathbb{C})^*,$$

and fulfilling the Kubo-Takenaka conditions. For standard construction of a nuclear space $S_A(\mathcal{O};\mathbb{C}) \subset L^2(\mathcal{O},d\mu_O;\mathbb{C})$ as arising from a standard (self-adjoint with nuclear or Hilbert Schmidt $A^{-1}$) operator $A$ on $L^2(\mathcal{O},d\mu_O;\mathbb{C})$, fulfilling Kubo-Takenaka conditions, compare [39], or Subsection 5.1 of [59].

In this situation we have the natural lifting of the Gelfand triple over to the Fock space:

$$(S_A(\mathcal{O};\mathbb{C})) \subset \Gamma(L^2(\mathcal{O},d\mu_O;\mathbb{C})) \subset (S_A(\mathcal{O};\mathbb{C}))^*,$$

constructed from the standard operator $\Gamma(A)$ in $\Gamma(L^2(\mathcal{O},d\mu_O;\mathbb{C}))$. That the operator $\Gamma(A)$ will be standard whenever $A$ is, also for the fermionic functor $\Gamma$ and under the same assumptions for $A$ as in the boson case, can be proved in exactly the same way as in [39], Lemma 3.1.2, for the bosonic case (the proof is even simpler in fermi case because the occupation numbers assume only the values 0 or 1 in this case).

Eventually we have the initial standard Gelfand triple in the single particle Hilbert space $H'$ given in the standard form only up to a unitary isomorphism:

$$S_A(\mathcal{O};\mathbb{C}) \subset L^2(\mathcal{O};\mathbb{C}) \subset S_A(\mathcal{O};\mathbb{C})^*,$$

$$E \subset H' \subset E^*,$$

with the vertical arrows indicating the unitary operator (and its inverse) $U : H' \to L^2(\mathcal{O};\mathbb{C})$ whose restriction to $E$ defines an isomorphism $U : E \to S_A(\mathcal{O};\mathbb{C})$ of nuclear spaces and whose linear transposition $U^*$ defines isomorphism $S_A(\mathcal{O};\mathbb{C})^* \to E^*$. The nuclear space $E \subset H'$ then corresponds to the standard operator $U^{-1}AU$ on $H'$, and can be be constructed from it (compare [39] or Subsection 5.1 of [59]).
The last Gelfand triples can be lifted to the corresponding Fock spaces together with the corresponding isomorphisms determined by the unitary operator $\Gamma(U)$: its restriction to $(E) \subset \Gamma(L^2(\mathcal{O}; \mathbb{C}))$ transforming continuously $(E) \to (S_A(\mathcal{O}; \mathbb{C}))$, or linear transposition of this restriction, defining the isomorphism $(E)^* \to (S_A(\mathcal{O}; \mathbb{C}))^*$:

$$
\begin{align*}
(S_A(\mathcal{O}; \mathbb{C})) & \subset \Gamma(L^2(\mathcal{O}; \mathbb{C})) \subset (S_A(\mathcal{O}; \mathbb{C}))^* \\
\downarrow & \quad \downarrow \\
(E) & \subset \Gamma(\mathcal{H}') \subset (E)^*.
\end{align*}
$$

In this case we have the following relations for the annihilation (and correspondingly creation) operators

$$
\begin{align*}
(\mathcal{U}(U)^+ a(U^{-1}(u \oplus v)) \mathcal{U}(U) &= a'(u \oplus v), \\
\mathcal{U}(U)^+ a(U^{-1}(u \oplus v))^+ \mathcal{U}(U) &= a'(u \oplus v)^+, \\
u \oplus v & \in E^*.
\end{align*}
$$

Here the Hida operators $a'(u \oplus v), a'(u \oplus v)^+$ coincide with the ordinary annihilation and creation operators $a'(u \oplus v), a'(u \oplus v)^+$ (defined in Subsection \ref{subsec:hida}) on the Hida subspace $(E) \subset \Gamma(\mathcal{H}') \subset (E)^*$ of the Fock space $\Gamma(\mathcal{H}') = \Gamma(\mathcal{H}^0_{m,0} \oplus \mathcal{H}^\infty_{m,0})$, whenever $u \oplus v \in E \subset \mathcal{H}' = \mathcal{H}^0_{m,0} \oplus \mathcal{H}^\infty_{m,0} \subset E^*$. Similarly $a(w), a(w)^+$ coincide with the standard annihilation and creation operators on the Hida subspace $(S_A(\mathcal{O}; \mathbb{C}))$ of the Fock space $\Gamma(L^2(\mathcal{O}; \mathbb{C}))$, whenever $w \in S_A(\mathcal{O}; \mathbb{C}) \subset L^2(\mathcal{O}; \mathbb{C}) \subset S_A(\mathcal{O}; \mathbb{C})^*$. In this case we can restrict the creation and annihilation operators $a'(u \oplus v), a'(u \oplus v)^+$ to the Hida subspace $(E)$ and regard them as elements of $\mathcal{L}((E), (E))$ (and respectively $a(w), a(w)^+ \in \mathcal{L}((S_A(\mathcal{O}; \mathbb{C})), (S_A(\mathcal{O}; \mathbb{C}))^*))$ and similarly restrict the linear dual composed with complex conjugation $\Gamma(U)^+ = \overline{\cdot} \circ \Gamma(U)^* \circ \overline{\cdot} : (S_A(\mathcal{O}; \mathbb{C}))^* \to (E)^*$ to the subspace $(E)$, where it coincides with the ordinary inverse $\Gamma(U)^{-1}$ of the unitary operator $\Gamma(U)$, and with the inverse $U^{-1} = \overline{\cdot} \circ U^{*-1} \circ \overline{\cdot}$ of the linear dual $U^* : S_A(\mathcal{O}; \mathbb{C})^* \to E^*$ to $U$ composed with conjugations degenerating to $U^{-1} = U$ on the subspace $E \subset E^*$. In this particular case the general formula \ref{eq:general FORMULA} degenerates to

$$
\begin{align*}
\Gamma(U)^{-1} a(U(u \oplus v)) \Gamma(U) &= a'(u \oplus v), \\
\Gamma(U)^{-1} a(U(u \oplus v))^+ \Gamma(U) &= a'(u \oplus v)^+, \\
u \oplus v & \in E \subset E^*.
\end{align*}
$$

But the formula \ref{eq:general FORMULA} is valid generally for the operators $a'(u \oplus v), a'(u \oplus v)^+ \in \mathcal{L}((E), (E)^*),$ $a(w), a(w)^+ \in \mathcal{L}((S_A(\mathcal{O}; \mathbb{C})), (S_A(\mathcal{O}; \mathbb{C}))^*),
understood in the sense of Hida with $u \oplus v \in E^*$, or respectively \( w \in \mathcal{S}_A(\mathcal{O}; \mathbb{C})^* \), and with $\Gamma(U)$ understood as a continuous isomorphism

\[
(\mathcal{E}) \rightarrow (\mathcal{S}_A(\mathcal{O}; \mathbb{C}))
\]

of nuclear spaces in the first formula of (34) and with $\Gamma(U)^+ = (\cdot) \circ \Gamma(U)^* \circ (\cdot)$ as its continuous dual isomorphism

\[
(\mathcal{S}_A(\mathcal{O}; \mathbb{C}))^* \rightarrow (\mathcal{E})^*
\]

composed with complex conjugation in (34). Below we give generalized operators $a(\cdot)$, $a'(\cdot)$ (and respectively $a(\cdot)$, $a(\cdot)^+$), due to Hida, which make sense also for $u \oplus v$ (respectively $w$), lying in the space dual to $\mathcal{E}$, respectively dual to $\mathcal{S}_A(\mathcal{O}; \mathbb{C})$.

In order to simplify notation we agree to write the last isomorphisms (34) (and their particular case (35)) induced by $U$ simply identifying the corresponding operators, namely

\[
a(U^{\pm 1}(u \oplus v)) = a'(u \oplus v), \quad a(U^{\pm 1}(u \oplus v))^+ = a'(u \oplus v)^+, \quad u \oplus v \in E^*,
\]

\[
a(U(u \oplus v)) = a'(u \oplus v), \quad a(U(u \oplus v))^+ = a'(u \oplus v)^+, \quad u \oplus v \in \mathcal{E} \subset \mathcal{E}^*,
\]

(36)

as operators transforming continuously Hida spaces into their strong duals (in the first case) or as operators transforming continuously Hida spaces into Hida spaces (in the second case).

Note that in our case the initial Gelfand triple $\mathcal{E} \subset \mathcal{H}^{\otimes}_{m,0} \oplus \mathcal{H}^{\otimes}_{-m,0} \subset E^*$ over the single particle Hilbert space $\mathcal{H}' = \mathcal{H}^{\otimes}_{m,0} \oplus \mathcal{H}^{\otimes}_{-m,0}$ does not have the standard form, because the single particle Hilbert space $\mathcal{H}'$ does not have the form $L^2(\mathcal{E}', d\mu_\mathcal{E}; \mathbb{C})$. Indeed note that the Hilbert space

\[
L^2(\mathbb{R}^3, d^3p/(2p_0(p))^2; \mathbb{C}^4) = \oplus_1^4 L^2(\mathbb{R}^3, d^3p/(2p_0(p))^2; \mathbb{C})
\]

\[
= L^2(\mathbb{R}^3 \sqcup \mathbb{R}^3 \sqcup \mathbb{R}^3 \sqcup \mathbb{R}^3, d^3p/(2p_0(p))^2; \mathbb{C})
\]

does have the required form $L^2(\mathcal{E}, d\mu_\mathcal{E}; \mathbb{C})$, with

\[
\mathcal{E}' = \mathbb{R}^3 \sqcup \mathbb{R}^3 \sqcup \mathbb{R}^3 \sqcup \mathbb{R}^3
\]

equal to the disjoint sum of four copies of $\mathbb{R}^3$ and the direct sum measure $d\mu_\mathcal{E}$ coinciding with $\frac{d^3p}{2p_0(p)}$ on each copy $\mathbb{R}^3$. But recall that although in our case the values $\vec{\phi}(p)$ of the bispinors $\vec{\phi} \in \mathcal{H}^{\otimes}_{m,0}$ concentrated on the positive energy orbit $\mathcal{E}_{m,0,0,0}$ range over $\mathbb{C}^4$, nonetheless $\mathcal{H}^{\otimes}_{m,0}$ does not have the standard form

\[
L^2(\mathbb{R}^3, d^3p/(2p_0(p))^2; \mathbb{C}^4),
\]

because for each fixed $p$ the vectors $\vec{\phi}(p_0(p))$, with $\vec{\phi}$ ranging over $\mathcal{H}^{\otimes}_{m,0}$, do not span $\mathbb{C}^4$, but are equal to the image $\text{Im} P^{\otimes}(p, p_0(p)) \neq \mathbb{C}^4$, for $p =$
(p, p_0(p)) \in \mathcal{O}_{m,0,0,0}$, because rank $P^\oplus(p, p_0(p)) = 2 \neq 4$ (compare Subsection 2.1 of [59], where the projection operator $P^\oplus$ of point-wise multiplication by $P^\oplus(p)$, $p \in \mathcal{O}_{m,0,0,0}$, acting on bispinors concentrated on the orbit $\mathcal{O}_{m,0,0,0}$ is defined).

Similarly $H_{m,0}^{\otimes_c}$ does not have the standard form

$$L^2(\mathbb{R}^3, d^3p/(2p_0(p))^2; \mathbb{C}^4)$$

in spite of the fact that the conjugations $\tilde{\phi}^c \in H_{-m,0}^{\otimes_c}$ of the bispinors $\tilde{\phi} \in H_{-m,0}^{\otimes_c}$ concentrated on the negative energy orbit $\mathcal{O}_{-m,0,0,0}$ take their values in $\mathbb{C}^4$, because $\{\tilde{\phi}(p, p_0(p)), \tilde{\phi} \in H_{-m,0}^{\otimes_c}\} = \text{Im} P^\oplus(p, p_0(p)) \neq \mathbb{C}^4$ with rank $P^\oplus(p, p_0(p)) = 2 \neq 4$, for $p = (p, p_0(p)) \in \mathcal{O}_{-m,0,0,0}$.

But there exists a natural unitary isomorphism $U$ (in fact a class of such natural $U$)

$$U : \mathcal{H} = H_{m,0}^{\otimes_c} \oplus H_{-m,0}^{\otimes_c} \longrightarrow L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4)$$

between the single particle Hilbert space $\mathcal{H}'$ and the Hilbert space

$$L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4) = \oplus L^2(\mathbb{R}^3, d^3p; \mathbb{C}) = L^2(\mathbb{R}^3 \sqcup \mathbb{R}^3 \sqcup \mathbb{R}^3 \sqcup \mathbb{R}^3, d^3p; \mathbb{C}),$$

which moreover restricts to an isomorphism between the nuclear spaces of Schwartz bispinors in $E \subset \mathcal{H}'$ and Schwartz functions in $S(\mathbb{R}^3; \mathbb{C}^4) = S_A(\mathbb{R}^3; \mathbb{C}^4) \subset L^2(\mathbb{R}^3, d^3p; \mathbb{C})$.

Indeed for $\tilde{\phi} \in H_{m,0}^{\otimes_c}$, $\tilde{\phi}^c \in H_{-m,0}^{\otimes_c}$ we put

$$U(\tilde{\phi} \oplus (\tilde{\phi}^c)^c) \overset{df}{=} (\tilde{\phi})_1 + (\tilde{\phi})_2 + (\tilde{\phi})_3 + (\tilde{\phi})_4 \in \oplus_4^4 L^2(\mathbb{R}^3; \mathbb{C}) = L^2(\mathbb{R}^3; \mathbb{C}^4), \quad (37)$$

where

$$(\tilde{\phi})_1(p) = (\tilde{\phi})_1(p) \overset{df}{=} \frac{1}{2p_0(p)} u_1(p) \tilde{\phi}(p_0(p), p), \quad p_0(p) = \sqrt{|p|^2 + m^2},$$

$$(\tilde{\phi})_2(p) = (\tilde{\phi})_2(p) \overset{df}{=} \frac{1}{2p_0(p)} u_2(p) \tilde{\phi}(p_0(p), p), \quad p_0(p) = \sqrt{|p|^2 + m^2},$$

and

$$(\tilde{\phi})_3(p) = (\tilde{\phi})_3(p) \overset{df}{=} \frac{1}{2p_0(p)} v_1(p) \tilde{\phi}(-p_0(p), -p), \quad p_0(p) = -\sqrt{|p|^2 + m^2},$$

and

$$(\tilde{\phi})_4(p) = (\tilde{\phi})_4(p) \overset{df}{=} \frac{1}{2p_0(p)} v_1(p) \tilde{\phi}(-p_0(p), -p) \tilde{\phi}^c(|p_0(p)|, p).$$

$$p_0(p) = -\sqrt{|p|^2 + m^2},$$

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Here \( u_s(p), v_s(-p), s = 1, 2 \), are the Fourier transforms of the complete system of solutions of the Dirac equation, given by the formula (33) of Appendix 4 in the so-called chiral representation of Dirac gamma matrices (which we have used in Subsection 2.1 of [59]); or by the formula (141) of Appendix 4 in the so-called standard representation of the Dirac gamma matrices. It follows that for any \( (\tilde{\phi})_1 = (\tilde{\phi})_{1+}, (\tilde{\phi})_2 = (\tilde{\phi})_{2+}, (\tilde{\phi})_3 = (\tilde{\phi})_{1-}, (\tilde{\phi})_4 = (\tilde{\phi})_{2-} \in L^2(\mathbb{R}^3; \mathbb{C}) \) we have

\[
U^{-1} \left( (\tilde{\phi})_{1+} \oplus (\tilde{\phi})_{2+} \oplus (\tilde{\phi})_{1-} \oplus (\tilde{\phi})_{2-} \right) \overset{df}{=} \tilde{\phi} \oplus (\tilde{\phi})^c \in \mathcal{H}_{m,0}^0 \oplus \mathcal{H}_{-m,0}^c. \tag{38}
\]

where

\[
\tilde{\phi}(p_0(p), p) \overset{df}{=} \sum_{s=1,2} 2p_0(p) (\tilde{\phi})_{s+}(p) u_s(p), \quad p_0(p) = \sqrt{|p|^2 + m^2}
\]

and

\[
((\tilde{\phi})^c(\phi_0(p), p))^T = \tilde{\phi}'(-|p_0(p)|, -p) \overset{df}{=} \sum_{s=1,2} 2p_0(p) (\tilde{\phi})_{s-}(p) v_s(p),
\]

\[
p_0(p) = -\sqrt{|p|^2 + m^2}.
\]

That \( U^{-1} \) is indeed equal to the inverse of the operator \( U \) follows immediately from the relations (136) for \( \phi \in \mathcal{H}_{m,0}^0 \) and from the relations (137) for \( \tilde{\phi}^c \in \mathcal{H}_{-m,0}^c \) of Appendix 4. That \( U^{-1} \) is isometric follows immediately from the orthonormality relations (134) for \( u_s(p), v_s(p), s = 1, 2 \). That \( U \) is isometric follows immediately from the relations (136) for \( \phi \in \mathcal{H}_{m,0}^0 \) and from the relations (137) for \( \tilde{\phi}^c \in \mathcal{H}_{-m,0}^c \) of Appendix 4. That \( U \) transforms isomorphically the indicated nuclear spaces follows from the fact that the components of \( u_s(p), v_s(p), s = 1, 2 \), are all multipliers of the Schwartz algebra \( \mathcal{S}(\mathbb{R}^3; \mathbb{C}) \).

Note here that there are more than just one canonical choice of the solutions \( u_s(p), v_s(-p), s = 1, 2 \), with smooth components belonging to the algebra of multipliers or even convolutors of \( \mathcal{S}(\mathbb{R}^3; \mathbb{C}) \). Indeed having given one choice \( u_s(p), v_s(-p), s = 1, 2 \), we can apply the unitary operator to \( u_s(p), v_s(-p), s = 1, 2 \), of multiplication by a unitary matrix with components smoothly depending on \( p \) and belonging to the algebra of multipliers of \( \mathcal{S}(\mathbb{R}^3; \mathbb{C}) \), and which rotates the initial \( u_s(p), v_s(-p), s = 1, 2 \), within the 2-dimensional images respectively of \( P^c(p_0(p), p) \) or \( P^c(-p_0(p), p) \). We obtain in this way various isomorphisms \( U \) and the corresponding unitary equivalent realizations of the Dirac field.
Recall, please, that the nuclear Schwartz space $S(\mathbb{R}^3; \mathbb{C}^4)$ can be obtained as a standard countably Hilbert nuclear space

$$S(\mathbb{R}^3; \mathbb{C}^4) = S_A(\mathbb{R}^3; \mathbb{C}^4) \subset L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4) = \oplus_1^4 L^2(\mathbb{R}^3, d^3p; \mathbb{C})$$

with the standard operator $A$ on

$$L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4) = \oplus_1^4 L^2(\mathbb{R}^3, d^3p; \mathbb{C})$$

equal to the direct sum

$$A = \oplus H_{(3)}$$

of four copies of the three dimensional oscillator Hamiltonian operator

$$H_{(3)} = -\Delta_p + p \cdot p + 1$$
on

$L^2(\mathbb{R}^3, d^3p; \mathbb{C})$,

compare e.g. [23], Appendix 9 of [59], or [56].

Summing up we will construct the Gelfand triples

$$L^2(\mathbb{R}^3 \sqcup \mathbb{R}^3 \sqcup \mathbb{R}^3 \sqcup \mathbb{R}^3, d^3p; \mathbb{C})$$

related by vertical isomorphisms induced by the unitary operator $U$:

$$U : \mathcal{H}' = \mathcal{H}_{m,0}^\oplus + \mathcal{H}_{m,0}^{\oplus c} \rightarrow \oplus L^2(\mathbb{R}^3; \mathbb{C})$$

with restriction to the nuclear space $E$ mapping isomorphically

$$E \rightarrow S_A(\mathbb{R}^3; \mathbb{C}^4) = \mathcal{S}(\mathbb{R}^3; \mathbb{C}^4)$$

with $A$ defined by [39]. The first triple has the standard form, and can be lifted with the help of $\Gamma(A)$. Thus we may define in the standard form the Hida operators $a(w), a(w)^+$ in the Fock space $\Gamma(\oplus L^2(\mathbb{R}^3; \mathbb{C}))$. The corresponding Hida operators $a'(u \oplus v), a'(u \oplus v)^+$ in the Fock space $\Gamma(\mathcal{H}')$ of the free Dirac field need not be separately constructed, and can be expressed with the help of the standard Hida operators $a(w), a(w)^+$ in the Fock space $\Gamma(\oplus L^2(\mathbb{R}^3; \mathbb{C}))$, by utilizing the isomorphism induced by $U$. Namely Hida operators $a'(u \oplus v), a'(u \oplus v)^+$ can be expressed by the Hida operators $a(w), a(w)^+$ as in the formula [36], namely:

$$a(U^{-1}(u \oplus v)) = a'(u \oplus v), \ a(U^{-1}(u \oplus v))^+ = a'(u \oplus v)^+, \ u \oplus v \in E^*,$$
$$a(U(u \oplus v)) = a'(u \oplus v), \ a(U(u \oplus v))^+ = a'(u \oplus v)^+, \ u \oplus v \in E \subset E^*.$$
The plan of the rest part of this Subsection is the following. First, we give the white noise construction of the Hida operators $a(w), a(w)^+$ obtained by lifting to the Fock space of the first (standard) Gelfand triple in \([40]\). In the next step we utilize the natural unitary isomorphism $U$ given by \([37]\), which induces the isomorphism of the Gelfand triples in \([40]\). Namely, using the unitary isomorphism $U$ and the Hida operators $a(w), a(w)^+$ corresponding to the lifting of the first triple in \([40]\) we compute the Hida operators $a'(u \oplus v), a'(u \ominus v)^+$ in the Fock space $\Gamma(H')$ (which enter into the Dirac field \([29]\)), using the formula \([36]\).

Let us concentrate now on the first (standard) of the Gelfand triples in \([40]\) and its lifting to the Fock space $\Gamma(\oplus L^2(\mathbb{R}^3; \mathbb{C}))$, together with the Hida definition of the Hida operators $a(w), a(w)^+, w \in S_A(\mathbb{R}^3)^* = S(\mathbb{R}^3)^*$. We only recall definition and some basic facts, referring e.g. to \([39]\), \([27]\), \([38]\), \([53]\), for more information.

We are using here the modified realization of annihilation-creation operators in the Fock space, defined in the Remark 1 of Subsection 2.2. It fits well with that used by Hida, Obata, Saitô, \([26]\), \([39]\), \([38]\), for boson case, when adopting the results of \([26]\), \([39]\), \([38]\), concerning integral kernel operators, to fermion case.

**REMARK 2.** It should be emphasized here that the results of \([26]\), \([39]\), \([38]\), concerning the so called integral kernel operators and their Fock expansions, can be proved without any essential changes also for the fermion case after \([26]\), \([39]\), \([38]\). Note that these theorems (e.g. Lemma 2.2, Thm. 2.2, Thm. 2.6. of \([26]\), or Thm. 3.13 of \([38]\)) could have been formulated and proved as well for the so called general Fock space

$$\Gamma_{\text{general}}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^\otimes n$$

without symmetrizing or antisymmetrizing the tensor products. In particular symmetrization (antisymmetrization) plays no fundamental role in the proof of these theorems, which are based on the norm estimations of the $m$-contractions $\otimes_m, \otimes^m$. Their eventual symmetrizations $\tilde{\otimes}_m, \tilde{\otimes}^m$ (or antisymmetrizations), which arise in the latter stage when restricting attention to the boson (or fermion) case, has nothing to do with these estimations and allows to state the analogous results for boson as well as for the fermion case.

Although differences between the fermi and boson case which arise nothing to do with the analysis of integral kernel operators (in which we are mostly interested), we should mention here some of them. The fundamental difference is that the algebra structure of the nuclear Hida test space, determined by the tensor product, is not commutative but skew commutative, due to the antisymmetry of the tensors in the fermi Fock space, and cannot be naturally realized as a nuclear function space on the strong dual $\mathcal{E}^*$ with multiplication defined by pointwise multiplication (because such multiplication is always commutative). In connection with this we have no natural isomorphism of the Fermi Fock space to
the space of square integrable functions on $E^*$ with the Gaussian measure on $E^*$ (no Wiener-Itô-Segal decomposition based on commutative infinite-dimensional measure space is possible). Of course a mere existence of a unitary map between the fermi Fock space and an $L^2$ space over a Gaussian measure space is trivial, but there are plenty of such maps devoid of any relevance. Naturality of the Wiener-Itô-Segal decomposition for the bose case is crucial. In order to keep a natural nature, e.g. preserving the algebra structure of the Hida test space (now skew commutative), in extending Wiener-Itô-Segal decomposition to the fermi case, a non-commutative extension of abstract integration is needed, and has been provided by Segal (note however that Segal [52] is not using a non-commutative extension of ordinary measure – but of a weak distribution on a Hilbert space). Because these questions concerning non commutative character of the multiplicative structure of the Hida test space in case of fermi case are not immediately related to the calculus of Fock expansions of integral kernel operators, developed in [26], [39], [38], we do not enter these questions in our work. In particular we do not exploit in any substantial manner the fact that Hida annihilation operators can be interpreted as graded derivations on the $\mathbb{Z}_2$ graded skew commutative nuclear algebra of Hida test functionals. The only practical consequence of this fact we feel in computations concerning integral kernel operators is that we confine ourselves to skew-symmetric kernels (in variables corresponding to fermi Hida creation-annihilation operators) in order to keep one-to-one correspondence between the kernels and corresponding operators.

But there is a relevant tool for computations which must be treated in slightly different manner in the two cases – bose and fermi case. Namely the symbol calculus, initiated by Berezin [3] and developed mainly by Obata [37], [38], must be realized in a slightly different manner for fermi case in comparison with the bose case. It order to adopt the symbol calculus of Obata to the fermi case it is convenient first to divide the fermi fock space $\Gamma(\mathcal{H}')$ into the subspaces $\Gamma_+(\mathcal{H}')$ of even elements
\[
\Phi = \sum_{n=0}^{\infty} \Phi_n,
\]
(with even $n$ in this decomposition), and $\Gamma_-(\mathcal{H}')$ of odd elements $\Phi$ (with $n$ odd in this decomposition). Similarly we do for the nuclear spaces $(E) = (E)^+ \oplus (E)^-$, $(E)^* = (E)^+_* \oplus (E)^-_*$. Next we note that for $\xi \in E^\otimes m$ (and generally $\xi \in E^\otimes m$ with even $m$) the exponential map
\[
\xi \mapsto \Phi \xi = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \xi^\otimes n \in (E)^+
\]
is well defined and continuous. Using this exponential map we utilize the Obata symbol for even operators, i.e. transforming $(E)^+ \rightarrow (E)^+_+$ and $(E)^- \rightarrow (E)^-_+$. The odd operators, i.e. transforming $(E)^+ \rightarrow (E)^*_-$ and $(E)^- \rightarrow (E)^-_+$ are reduced to even by multiplication by one Hida (creation, respectively annihilation) operator. Finally we note that any continuous operator $(E) \rightarrow (E)^*$ is naturally a direct sum of an even and an odd operator; compare [53].
Let $| \cdot |_0, (\cdot , \cdot )_0$ denote the standard $L^2$ norm and inner product on 
\[
L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4) = \bigoplus_1^4 L^2(\mathbb{R}^3, d^3p; \mathbb{C})
\]
and by the same symbol $| \cdot |_0$, after [26] and [39], we denote the Hilbert space norm on the Hilbert space tensor product
\[
L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4)^{\otimes n},
\]
as well as its restriction to the antisymmetrized tensor product
\[
L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4)^{\tilde{\otimes} n}.
\]
Recall that 
\[
| f |_k = |(A^{\otimes n})^k f |_0 \quad f \in \text{Dom} (A^{\otimes n})^k \subset L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4)^{\otimes n}
\]
(in particular well defined for $f \in S_A(\mathbb{R}^3, \mathbb{C}^4)^{\tilde{\otimes} n}$).

Let $\| \cdot \|_0, ((\cdot , \cdot ))_0$ denote the Hilbert space norm and the corresponding inner product on Fock space defined by the formula (convention used by [26], [38], compare Remark 1 of Subsection 2.2)
\[
\| \Phi \|_0^2 = \sum_{n=0}^\infty n! |\Phi_n|_0^2
\]
for $\Phi$ with decomposition
\[
\Phi = \sum_{n=0}^\infty \Phi_n, \text{ with } \Phi_n \in L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4)^{\tilde{\otimes} n}.
\]
Recall that by definition
\[
\| \Phi \|_k = \| \Gamma(A)^k \Phi \|_0 \text{ and } |\Phi_n|_k = |(A^{\otimes n})^k \Phi_n|_0
\]
for $\Phi \in \Gamma(L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4))$ and $\Phi_n \in L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4)^{\tilde{\otimes} n}$.

It follows in particular that the general element 
\[
\Phi = \sum_{n=0}^\infty \Phi_n, \text{ with } \| \Phi \|_0^2 = \sum_{n=0}^\infty n! |\Phi_n|_0^2 < \infty, \quad (41)
\]
of the Fock space $\Gamma(L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4))$ belongs to the Hida test space $(S_A(\mathbb{R}^3; \mathbb{C}^4)) \subset \Gamma(L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4))$ if $\Phi_n \in S_A(\mathbb{R}^3; \mathbb{C}^4)^{\tilde{\otimes} n}$ for all $n = 0, 1, 2, \ldots$ and
\[
\sum_{n=0}^\infty n! |\Phi_n|_k < \infty \text{ for all } k \geq 0.
\]
In this case
\[
\| \Phi \|_k^2 = \sum_{n=0}^\infty n! |\Phi_n|_k < \infty \text{ for all } k \geq 0. \quad (42)
\]
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Note that the norms 
\[ \| \Phi \|_k = \| \Gamma(A)^k \Phi \|_0 \] with \( \Phi \in (S_A(\mathbb{R}^3; \mathbb{C}^4)) \)
are well defined on the Hida space \((S_A(\mathbb{R}^3; \mathbb{C}^4)) \subset \Gamma(L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4))\) also for \( k \) equal to any negative integer. Completion of \((S_A(\mathbb{R}^3; \mathbb{C}^4))\) with respect to the Hilbertian norm
\[ \| \cdot \|_{-k} = \| \Gamma(A)^{-k} \cdot \|_0 \] with fixed \( k \in \mathbb{N} \)
is equal to a Hilbert space, which we denote
\[ (S_A(\mathbb{R}^3; \mathbb{C}^4))_{-k}, \quad (43) \]
and which is also equal do the completion of \( \text{Dom} \Gamma(A)^{-k} \) (equal to the whole Fock space \( \text{Dom} \Gamma(A)^{-k} = \Gamma(L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4)) \) for \( k = 0, 1, 2, \ldots \)) with respect to the norm \( \| \cdot \|_{-k} \). The Hilbert space \((43)\) is for each \( k \geq 0 \) canonically isomorphic, including the case \( k = 0 \), (Riesz isomorphism) to the Hilbert space dual of the Hilbert space
\[ (S_A(\mathbb{R}^3; \mathbb{C}^4))_{k}, \quad (44) \]
compare [39]. Recall that the Hilbert space \((44)\) is equal to the completion of the domain \( \text{Dom} \Gamma(A)^k \) with respect to the norm \( \| \cdot \|_k \). The Hilbert spaces \((44)\) compose an inductive system, [18], [39], with natural continuous inclusions
\[ (S_A(\mathbb{R}^3; \mathbb{C}^4))_{-0} \subset (S_A(\mathbb{R}^3; \mathbb{C}^4))_{-1} \subset (S_A(\mathbb{R}^3; \mathbb{C}^4))_{-2} \subset \ldots \subset (S_A(\mathbb{R}^3; \mathbb{C}^4))^* \]
which is dual to the projective system
\[ (S_A(\mathbb{R}^3; \mathbb{C}^4))_{0} \subset (S_A(\mathbb{R}^3; \mathbb{C}^4))_{1} \subset (S_A(\mathbb{R}^3; \mathbb{C}^4))_{2} \subset \ldots \subset (S_A(\mathbb{R}^3; \mathbb{C}^4))^* \]
\[ (45) \]
defining the Hida space \((S_A(\mathbb{R}^3; \mathbb{C}^4))\). The two systems \((46)\) and \((45)\) can be joined into single system of Hilbert spaces with comparable and compatible norms, by using the natural isomorphism of the dual to the adjoint space
\[ \Gamma(L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4))^* \cong \Gamma(L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4)) = (S_A(\mathbb{R}^3; \mathbb{C}^4))_{-0} \]
to the Hilbert space

$$\Gamma(L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4)) = (\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4))_0$$

(Riesz isomorphism, compare [15, 39]), and noting that the elements of the Hilbert space $H$ and its adjoint space $\overline{H}$ are the same:

$$\begin{align*}
(s_A(\mathbb{R}^3; \mathbb{C}^4))_0 & \subset \ldots \subset (s_A(\mathbb{R}^3; \mathbb{C}^4))_2 \subset (s_A(\mathbb{R}^3; \mathbb{C}^4))_1 \subset (s_A(\mathbb{R}^3; \mathbb{C}^4))_0 = \\
\Gamma(L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4)) & \subset (s_A(\mathbb{R}^3; \mathbb{C}^4))_0 \subset \ldots \subset (s_A(\mathbb{R}^3; \mathbb{C}^4))_2 \subset (s_A(\mathbb{R}^3; \mathbb{C}^4))_1 \subset (s_A(\mathbb{R}^3; \mathbb{C}^4))_0.
\end{align*}$$

The strong dual $(\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4))^*$ of the Hida space $(\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4))$ is equal to the inductive limit of the system (45). Recall that the Hida space $(\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4))$ itself is equal to the projective limit of the system (10), compare [39].

Similarly as for the elements of Hida (or Fock) space, likewise each element $\Phi \in (\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4))^*$ of the strong dual to the Hida space has a unique decomposition

$$\Phi = \sum_{n=0}^{\infty} \Phi_n, \quad \text{with } \Phi_n \in (\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4))^*.$$

(47)

In this case there exists a natural $k$ such that

$$\|\Phi\|_{-k}^2 = \sum_{n=0}^{\infty} n! |\Phi_n|_{-k}^2 < \infty.$$

Note that we have natural real and complex structure on the spaces we encounter here with well defined complex conjugation (7). In particular, if we denote the dual pairings on $\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)^* \times \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)$ and on $(\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4))^* \times (\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4))$ by $\langle \cdot, \cdot \rangle$ and respectively by $\langle \langle \cdot, \cdot \rangle \rangle$ then we have

$$\langle \xi, \eta \rangle = \langle \xi, \eta \rangle_0, \quad \text{for } \xi \in \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4) \subset \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)^*, \eta \in \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4),$$

$$\langle \langle \Psi, \Phi \rangle \rangle = \langle \langle \Psi, \Phi \rangle \rangle_0, \quad \text{for } \Psi \in (\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4))^*, \Phi \in (\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)).$$

Now we are ready to define the Hida operators $a(w), a(w)^+$, $w \in \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)^*$ in the Fock space $\Gamma(L^2(\mathbb{R}^3, d^3p; \mathbb{C}^4))$ corresponding to the first (standard) Gelfand triple in (45).

Namely for each $w \in \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)^*$, and each general element (11) of the Hida space we define Hida annihilation operator $a(w)$ which by definition acts on the element $\Phi$ given by (11) according to the following formula

1) $a(w)(\Phi = \Phi_0) = 0$,

2) $a(w)\Phi = \sum_{n \geq 0} n w \otimes_1 \Phi_n$. 

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Now we define the Hida creation operator \( a(w)^+ \), \( w \in S_A(\mathbb{R}^3; \mathbb{C}^4)^* \), transforming the strong dual \( (S_A(\mathbb{R}^3; \mathbb{C}^4))^* \) of the Hida space into itself. Namely let \( w \in S_A(\mathbb{R}^3; \mathbb{C}^4)^* \) and let \( \Phi \) be any general element \( \{17\} \) of the strong dual \( (S_A(\mathbb{R}^3; \mathbb{C}^4))^* \). The action of the Hida creation operator \( a(w)^+ \), \( w \in S_A(\mathbb{R}^3; \mathbb{C}^4)^* \), on such \( \Phi \) is by definition equal

\[
a(w)^+ \Phi = \sum_{n \geq 0} w \circ \Phi_n.
\]

Here as well as in the definition of the Hida annihilation operator the tensor product \( \otimes \) and its 1-contraction \( \otimes_1 \) (antisymmetrized \( \hat{\otimes} \), \( \hat{\otimes}_1 \)) is equal to the projective tensor product over the respective nuclear spaces:

\[
S_A(\mathbb{R}^3; \mathbb{C}^4)^*, S_A(\mathbb{R}^3; \mathbb{C}^4)^{\otimes n}, S_A(\mathbb{R}^3; \mathbb{C}^4)^{\hat{\otimes} n},
\]

\[
(S_A(\mathbb{R}^3; \mathbb{C}^4)^{\otimes n})^*, (S_A(\mathbb{R}^3; \mathbb{C}^4)^{\hat{\otimes} n})^*,
\]

In this case (of nuclear spaces) tensor product is essentially unique with the projective tensor product coinciding with the equicontinuous tensor product. Recall that

\[
v_1 \otimes \cdots \otimes v_n = (n!)^{-1} \sum_\pi \text{sign}(\pi) v_{\pi(1)} \otimes \cdots \otimes v_{\pi(n)},
\]

with \( v_i \) in the respective space, and that the antisymmetrized 1-contraction \( \hat{\otimes}_1 \) is uniquely determined by the formula

\[
u_1 \hat{\otimes} \cdots \hat{\otimes} v_n = (n!)^{-1} \sum_\pi \text{sign}(\pi) \langle u, v_{\pi(1)} \rangle v_{\pi(2)} \otimes \cdots \otimes v_{\pi(n)},
\]

with the sums ranging over all permutations \( \pi \) of the natural numbers \( 1, \ldots, n \), and with the evaluation \( \langle u, v_{\pi(1)} \rangle \) of \( u \) on \( v_{\pi(1)} \), which restricts to

\[
\langle u, v_{\pi(1)} \rangle = \langle \pi, v_{\pi(n)} \rangle_0 \text{ whenever } u \in S_A(\mathbb{R}^3; \mathbb{C}^4) \subset S_A(\mathbb{R}^3; \mathbb{C}^4)^*,
\]

It follows that \( a(w), w \in S_A(\mathbb{R}^3; \mathbb{C}^4)^* \), transforms continuously the Hida space into the Hida space

\[
a(w) : (S_A(\mathbb{R}^3; \mathbb{C}^4)^* \to (S_A(\mathbb{R}^3; \mathbb{C}^4)^*),
\]

for a proof compare e.g. \( \{39\} \), \( \{53\} \). By compositing it with the natural continuous inclusion \( (S_A(\mathbb{R}^3; \mathbb{C}^4))^* \subset (S_A(\mathbb{R}^3; \mathbb{C}^4)^*)^* \), we can also regard the Hida annihilation operator \( a(w), w \in S_A(\mathbb{R}^3; \mathbb{C}^4)^* \), as a continuous operator

\[
a(w) : (S_A(\mathbb{R}^3; \mathbb{C}^4)) \to (S_A(\mathbb{R}^3; \mathbb{C}^4))^*.
\]

It follows by general property of transposition, \( \{58\} \), that \( a(w)^*, w \in S_A(\mathbb{R}^3; \mathbb{C}^4)^* \), maps continuously the strong dual of the Hida space into itself

\[
a(w)^* : (S_A(\mathbb{R}^3; \mathbb{C}^4))^* \to (S_A(\mathbb{R}^3; \mathbb{C}^4))^*.
\]
By composing it with the dual

\[(\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)) \cong (\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4))^{**} \subset (\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4))^*\]

of the natural inclusion \((\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)) \subset (\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4))^*\), we can regard the Hida creation operator \(a(w)^*\), \(w \in \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)^*\), as a continuous operator

\[a(w)^* : (\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)) \rightarrow (\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4))^*\]

It turns out that

\[a(w)^+ = (\overline{\cdot}) \circ a(w)^* \circ (\overline{\cdot}), \ w \in \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)^*,\]

for \(a(w)^*, a(w)^+\) understood as maps of the strong dual of the Hida space into itself (or resp. as maps transforming the Hida space into its strong dual); compare \([39], [53]\).

**REMARK.** Note that in fact the definition of the Hida operator used by mathematicians is slightly different in comparison to ours with the additional complex conjugation

mathematicians’ \(a(w) = \overline{\text{ours } a(w)}\).

In particular ours \(a(w)\) is anti-linear in \(w\), which is the convention accepted in physical literature. This is the conjugation \(A^+ = (\overline{\cdot}) \circ A^* \circ (\overline{\cdot})\) equal to the linear transpose composed with complex conjugations, which connects the Hida generalized annihilation \(a(w)\) and creation operators \(a(w)^+\), due to the convention which we have accepted, and which is used by physicists. In the convention accepted by mathematicians it is the ordinary linear transpose which connects the generalized Hida annihilation \(a(w)\) and creation operators \(a(w)^*\).

In the mathematical literature the fact that the Hida annihilation operator \(a(w)\) is a \((\mathbb{Z}_2\text{-graded in fermi case})\) derivation on the Hida nuclear algebra (with the multiplication defined by the antisymmetrized tensor product \(\hat{\otimes}\)) is reflected by the following notation introduced by Hida:

\[D_w \overset{df}{=} a(w), \ w \in \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)^* = \mathcal{S}(\mathbb{R}^3; \mathbb{C}^4)^*\]

(here the convention used by mathematicians is better because their

\[D_w \overset{df}{=} a(w)\]

is linear in \(w\), and in bose case when the Hida space is realized as commutative algebra of functions on \(\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)^*\), the Hida annihilation operator \(a(w)\) is indeed equal to the Gâteaux derivation in the direction of \(w\) and not in direction \(\overline{w}\).

Recall that \(\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4) = \mathcal{S}(\mathbb{R}^3; \mathbb{C}^4) = \bigoplus_1^4 \mathcal{S}(\mathbb{R}^3; \mathbb{C})\) we regard as the nuclear space of complex valued functions \(f\) on four disjoint copies of \(\mathbb{R}^3\) whose restrictions \(f_s\) to each \(s\)-th copy coincide with the Schwartz functions in \(\mathcal{S}_{H(3)}(\mathbb{R}^3; \mathbb{C}) = \)
$\mathcal{S}(\mathbb{R}^3; \mathbb{C})$. In particular for each value of the discrete index $s \in \{1, 2, 3, 4\}$, corresponding to each copy, and for each point $p \in \mathbb{R}^3$, we have well defined Dirac delta-functional $\delta_{s,p} \in \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)^* = \mathcal{S}(\mathbb{R}^3; \mathbb{C}^4)^*$ defined by

$$\delta_{s,p}(f) = f_s(p),$$

i.e. the evaluation of the restriction of $f$ to the $s$-th copy of $\mathbb{R}^3$ at the point $p$ of that copy. Simply speaking $\delta_{s,p}$ is the evaluation functional at fixed point $(s, p)$ of the disjoint sum $\mathbb{R}^3 \sqcup \mathbb{R}^3 \sqcup \mathbb{R}^3 \sqcup \mathbb{R}^3$.

The generalized Hida annihilation and creation operators $a(w), a(w)^+$ evaluated at $w = \delta_{s,p}$ equal to the Dirac delta functionals $\delta_{s,p}$ have special importance, and have special notation in mathematical literature

$$\partial_{s,p} \overset{df}{=} D_{s,p} \overset{df}{=} a(\delta_{s,p}), \quad \partial_{s,p}^+ = D_{s,p}^+ = a(\delta_{s,p})^+$$

reflecting the derivation-like character of these generalized Hida operators, and are called Hida’s differential operators. But we have also widely used notation for operators in physical literature, with whom the Hida differential operators should be identified. Namely generalized Hida operators should be identified with the operators frequently written by physicists in the following manner

$$a_s(p) \overset{df}{=} D_{s,p} \overset{df}{=} \partial_{s,p} \overset{df}{=} a(\delta_{s,p}),$$

$$a_s(p)^+ \overset{df}{=} D_{s,p}^+ \overset{df}{=} \partial_{s,p}^+ \overset{df}{=} a(\delta_{s,p})^+.$$  

More precisely the operators $a_s(p), a_s(p)^+$ for $s = 1, 2$ should be identified with the operators $b_s(p), b_s(p)^+$ for $s = 1, -1$ of the book [46], p. 82 (or with the operators $b_s^-(p), b_s^+(p)$, $s = 1, 2$, of the book [4], p. 123). The operators $a_s(p), a_s(p)^+$ for $s = 3, 4$ should respectively be identified with the operators $d_s(p), d_s(p)^+$ for $s = 1, -1$, of the book [46], p. 82 (or respectively with the operators $d_s^-(p), d_s^+(p)$, $s = 1, 2$, of the book [4], p. 123).

Note that because the Dirac delta functional $\delta_{s,p}$ is real $\delta_{s,p} = \delta_{s,p}$ (i.e. commutes with complex conjugation), then

$$a(\delta_{s,p})^+ = \partial_{s,p}^+ = a(\delta_{s,p})^* = \partial_{s,p}^*,$$

so that for Hida’s differential operators the linear adjunction $\partial_{s,p}^*$ coincides with the Hermitean adjunction $\partial_{s,p}^*$. We may thus summarize the notation used here with that used by other authors in the following table
Now we remind some basic results of the calculus of integral kernel operators constructed mainly by Hida, Obata, and Saitô, which we will use here and in the following Sections (especially in Section 3).

Before doing it we make a general remark concerning norm estimations of the left \( \otimes_l \) and right \( \otimes_r \) antisymmetrized (or symmetrized) \( l \)-contractions (compare \[39\])

\[
|f \otimes_l \! g|_k, |\hat{F} \otimes_l \! \hat{g}|_{-k}, |F \otimes_l \! g|_{-k}, \quad \hat{F} \in \left( S_A(\mathbb{R}^3; \mathbb{C}^4) \otimes (l+m) \right)^*, f, \hat{g} \in S_A(\mathbb{R}^3; \mathbb{C}^4) \otimes (l+n).
\]

Namely passing from estimations for the norms

\[
|f \otimes^l \! g|_k, |F \otimes^l \! g|_{-k}, \quad \text{for } F \in \left( S_A(\mathbb{R}^3; \mathbb{C}^4) \otimes (l+m) \right)^*, f, g \in S_A(\mathbb{R}^3; \mathbb{C}^4) \otimes (l+n),
\]

with non antisymmetrized (or non symmetrized \( F, f \) and \( g \)), summarized in Prop. 3.4.3, Lemma 3.4.4, 3.4.5, to estimations with symmetrized or antisymmetrized \( \hat{F}, \hat{f} \) and \( \hat{g} \) we note that we have

\[
F \otimes \! g = F \otimes^l \! g = \pm F \otimes_l \! g = \pm F \otimes_l \! g,
\]

for \( F \in \left( S_A(\mathbb{R}^3; \mathbb{C}^4) \otimes (l+m) \right)^*, g \in S_A(\mathbb{R}^3; \mathbb{C}^4) \otimes (l+n), \)

and

\[
|\hat{f}|_k \leq |\! f|_k, \quad f \in S_A(\mathbb{R}^3; \mathbb{C}^4) \otimes^n, k \in \mathbb{Z}.
\]

in each case: for symmetrization as well as for antisymmetrization \( \hat{(\cdot)} \). This allows to restate the estimations for non symmetrized/antisymmetrized \( F, f \) and \( g \) (summarized in Prop. 3.4.3, Lemma 3.4.4, 3.4.5) in the form of propositions analogous to Prop. 3.4.7, 3.4.8, 3.4.9 in \[39\] for the contractions of antisymmetrized \( \hat{F}, \hat{G}, \hat{g}, \hat{f} \) on exactly the same footing as for symmetrized \( \hat{F}, \hat{G}, \hat{g}, \hat{f} \) (as we have already mentioned in Remark 2). In particular theorems concerning integral kernel operators and Fock expansions, in both cases 1) of scalar-valued kernels \[26, 37\], and 2) of vector-valued kernels \[38\], can be stated and proved exactly as in \[26, 37, 38\] also for the fermi case. The only difference which arises in fermi case (compared to the bose case) comes from additional factor

| \( a_{s=1}(p) \) | \( a_{s=2}(p) \) | \( a_{s=3}(p) \) | \( a_{s=4}(p) \) |
|---|---|---|---|
| \( \delta_{s=1,p} \) | \( \delta_{s=2,p} \) | \( \delta_{s=3,p} \) | \( \delta_{s=4,p} \) |
| \( b_{s=1}(p) \) | \( b_{s=1}(p) \) | \( b_{s=1}(p) \) | \( b_{s=1}(p) \) |
| \( \hat{a}_{s=1}(p) \) | \( \hat{a}_{s=2}(p) \) | \( \hat{a}_{s=3}(p) \) | \( \hat{a}_{s=4}(p) \) |

Hida-Obata \[39\], Scharf \[46\], Bogoliubov-Shirkov \[10\]
(-1) depending on the degree of the involved tensors. In particular we should note that for nonsymmetrized \( F \in \left( S_A(\mathbb{R}^3; \mathbb{C}^4)^{\otimes k} \right)^* \), \( G \in \left( S_A(\mathbb{R}^3; \mathbb{C}^4)^{\otimes l} \right)^* \), and \( h \in S_A(\mathbb{R}^3; \mathbb{C}^4)^{\otimes (k+l+m)} \), we have
\[
F \otimes_k (G \otimes_l h) = (G \otimes F) \otimes_{k+l} h \text{ in this order!}
\]
and thus by antisymmetrization \((\cdot)\) we get
\[
\hat{F} \otimes_k \hat{(G \otimes_l \hat{h})} = (\hat{G} \otimes \hat{F}) \otimes_{k+l} \hat{h} = (-1)^{(\text{deg} \hat{F})(\text{deg} \hat{G})} (\hat{F} \otimes \hat{G}) \otimes_{k+l} \hat{h},
\]
\[
\text{deg} \hat{F} \overset{df}{=} k, \text{deg} \hat{G} \overset{df}{=} l;
\]
(instead of Proposition 3.4.8 of [39] with symmetrization \((\cdot)\) in bose case, where the factor \((-1)^{(\text{deg} \hat{F})(\text{deg} \hat{G})}\) degenerates to 1).

Similarly we have for \( F \in \left( S_A(\mathbb{R}^3; \mathbb{C}^4)^{\otimes l} \right)^* \), \( G \in \left( S_A(\mathbb{R}^3; \mathbb{C}^4)^{\otimes m} \right)^* \), and \( f \in S_A(\mathbb{R}^3; \mathbb{C}^4)^{\otimes (l+n)} \)
\[
\langle F \otimes_l f, G \otimes_m g \rangle = \langle F \otimes G, f \otimes^n g \rangle.
\]
Again passing to the subspaces of antisymmetrized tensors we obtain
\[
(\hat{F} \otimes_l \hat{f}, \hat{G} \otimes_m \hat{g}) = (\hat{F} \otimes \hat{G}, \hat{f} \otimes^n \hat{g}) = (-1)^{m(\text{deg} \hat{f})} \langle \hat{F} \otimes \hat{G}, \hat{f} \otimes^n \hat{g} \rangle,
\]
(instead of Prop. 3.4.9 in [39] with symmetrization \((\cdot)\) for bose case). The replacements of symmetrization \((\cdot)\) with antisymmetrization \((\cdot)\) (with the appropriate factors \(-1\)) in the analysis of integral kernel operators in [39], are rather obvious, thus we leave the detailed inspection to the reader as an exercise. We mention only some particular cases in explicit form.

In particular we have the following analogue of Thm 4.1.7 of [39].

**THEOREM 1.** Let \( \Phi \in \left( S_A(\mathbb{R}^3; \mathbb{C}^4) \right)^* \) be any element of the Hida space, and let
\[
\Phi = \sum_{n=0}^{\infty} \Phi_n, \quad \Phi_n \in S_A(\mathbb{R}^3; \mathbb{C}^4)^{\otimes n}
\]
be its decomposition (thus fulfilling (43)). Then for
\[
y_1, \ldots, y_m \in S_A(\mathbb{R}^3; \mathbb{C}^4)^*
\]
we have
\[
D_{y_1} \cdots D_{y_m} \Phi = \sum_{n=0}^{\infty} (-1)^{m-1} \frac{(n+m)!}{n!} (y_1 \otimes \cdots \otimes y_m) \otimes_m \Phi_{m+n}.
\]
Moreover, for any \( k \geq 0, \ q > 0 \) and \( \Phi \in \left( S_A(\mathbb{R}^3; \mathbb{C}^4) \right) \) we have
\[
\| D_{y_1} \cdots D_{y_m} \Phi \| \leq \rho^{-q/2} m^{m/2} \left( \frac{\rho^{-q}}{2q e m \rho} \right)^{m/2} \| y_1 \|^{-(k+q)} \cdots \| y_m \|^{-(k+q)} \| \Phi \|_{k+q}.
\]

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Moreover, for any \(k > \eta\) where in fermi case, or symmetric in bose case) continuous operator \(L\) responding to \(\kappa\), the following (analogue of Lemma 4.3.1 in [39] or Lemma 2.1 in [26]):

**THEOREM 2.** Namely (compare Thm. 4.3.2 in [39] or Thm. 2.2. of [26]) for the summand dimensional oscillator hamiltonian operator and taking the sum as the direct sum, we achieve by eventually adding the unit operator to the ordinary 3-

**LEMMA 1.** For any elements \(\Phi, \Psi \in (S_{\mathcal{A}}(\mathbb{R}^3; \mathbb{C}^4))\) of the Hida space we put \((s_i, t_i) \in \{1, \ldots, 4\}, k_i, p_i \in \mathbb{R}^3\)

\[
\eta_{k, \Psi}(s_1, k_1, \ldots, s_l, k_l, t_1, p_1, \ldots, t_m, p_m) = \left\langle \left(\partial_{s_1, k_1}^* \cdots \partial_{s_l, k_l}^* \partial_{t_1, p_1}^* \cdots \partial_{t_m, p_m}^* \Phi, \Psi\right) \right\rangle,
\]

then for any \(k > 0\) we have

\[
|\eta_{k, \Psi}| \leq \rho^{-k} (l^m)^{1/2} \left(\frac{\rho^{-k}}{2k\hbar\rho}\right)^{(l+m)/2} \|\Phi\|_k \|\Psi\|_k.
\]

In particular, \(\eta_{k, \Psi} \in S_{\mathcal{A}}(\mathbb{R}^3; \mathbb{C}^4)^{\otimes (l+m)}\).

This allows analysis of an important class of integral kernel operators \(\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(S_{\mathcal{A}}(\mathbb{R}^3; \mathbb{C}^4)^{\otimes (l+m)})^*\), corresponding to \(\kappa_{l,m} \in (S_{\mathcal{A}}(\mathbb{R}^3; \mathbb{C}^4)^{\otimes (l+m)})^* = S(\mathbb{R}^3; \mathbb{C}^4)^{\otimes (l+m)}\), and written

\[
\Xi_{l,m}(\kappa_{l,m}) = \sum_{s_1, s_2, \ldots, t_m = 1}^{4} \int_{\mathbb{R}^3} \kappa_{l,m}(s_1, k_1, \ldots, s_l, k_l, t_1, p_1, \ldots, t_m, p_m) \times
\]

\[\times \partial_{s_1, k_1}^* \cdots \partial_{s_l, k_l}^* \partial_{t_1, p_1}^* \cdots \partial_{t_m, p_m}^* d^3k_1 \cdots d^3k_l d^3p_1 \cdots d^3p_m. (48)\]

**THEOREM 2.** Namely (compare Thm. 4.3.2 in [39] or Thm. 2.2. of [26]) for any \(\kappa_{l,m} \in (S_{\mathcal{A}}(\mathbb{R}^3; \mathbb{C}^4)^{\otimes (l+m)})^* = S(\mathbb{R}^3; \mathbb{C}^4)^{\otimes (l+m)}\) there exists (uniquely corresponding to \(\kappa_{l,m}\) if \(\kappa_{l,m}\) is antisymmetric: \(\kappa_{l,m} \in (S(\mathbb{R}^3; \mathbb{C}^4)^{\otimes (l+m)}\) in fermi case, or symmetric in bose case) continuous operator \(\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(S_{\mathcal{A}}(\mathbb{R}^3; \mathbb{C}^4)^{\otimes (l+m)})^*\), written as in (48), such that

\[
\left\langle \left(\Xi_{l,m}(\kappa_{l,m}) \Phi, \Psi\right) \right\rangle = \left\langle \kappa_{l,m}, \eta_{k, \Phi}\right\rangle, \quad \Phi, \Psi \in S_{\mathcal{A}}(\mathbb{R}^3; \mathbb{C}^4),
\]

where

\[
\eta_{k, \Psi}(s_1, k_1, \ldots, s_l, k_l, t_1, p_1, \ldots, t_m, p_m) = \left\langle \left(\partial_{s_1, k_1}^* \cdots \partial_{s_l, k_l}^* \partial_{t_1, p_1}^* \cdots \partial_{t_m, p_m}^* \Phi, \Psi\right) \right\rangle.
\]

Moreover, for any \(k > 0\) with \(|\kappa_{l,m}|_{-k} < \infty\) it holds

\[
\|\Xi_{l,m}(\kappa_{l,m}) \Phi\|_{-k} \leq \rho^{-k} (l^m)^{1/2} \left(\frac{\rho^{-k}}{2k\hbar\rho}\right)^{(l+m)/2} \|\kappa_{l,m}\|_{-k} \|\Phi\|_k.
\]

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We have the following important theorem (Thm. 4.3.9 of [39], Thm. 2.6 of [26]) which provides necessary and sufficient condition for the integral kernel operator (48) to be continuous not merely as an operator on the Hida space into its strong dual, but likewise as operator transforming continuously the Hida space into itself (thus becoming ordinary densely defined operator in the Fock space):

**THEOREM 3.** Let \( \kappa_{l,m} \in (\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)^\otimes(l+m))^* \). Then

\[
\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}\left( (\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)), (\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)) \right)
\]

if and only if \( \kappa_{l,m} \in \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)^\otimes l \otimes (\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)^\otimes m)^* \). In that case, for any \( k \in \mathbb{Z}, q > 0 \) with \( \alpha + \beta \leq 2q \), it holds

\[
\|\Xi_{l,m}(\kappa_{l,m})\Phi\|_k \leq \rho^{-q/2}(l^m m^l)^{1/2} \left( \frac{\rho^{-\alpha/2}}{-\alpha \ln \rho} \right)^{1/2} \left( \frac{\rho^{-\beta/2}}{-\beta \ln \rho} \right)^{m/2} |\kappa_{l,m}|_{l,m,k,-(k+q)} \|\Phi\|_{k+q},
\]

for all \( \Phi \in (\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4))^* \).

Here for \( f \in (\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)^\otimes(l+m))^* \) we have defined after [39], Chap. 3.4

\[
|f|_{l,m,k,q} \overset{df}{=} \left( \sum_{i,j} |\langle f, e(i) \otimes e(j) \rangle|^2 |e(i)|_2^2 |e(j)|_2^2 \right)^{1/2}, \quad k, q \in \mathbb{R}.
\]

Recall that here we have used (after [39]) the multiindex notation

\[
e(i) = e_{i_1} \otimes \cdots \otimes e_{i_l}, \quad i = (i_1, \ldots, i_l),
\]

\[
e(j) = e_{j_1} \otimes \cdots \otimes e_{j_m}, \quad j = (j_1, \ldots, j_m),
\]

with \( \{e_j\}_{j=0}^\infty \) being the complete orthonormal system in

\[
L^2(\mathbb{R}^3; \mathbb{C}^4) = L^2(\mathbb{R}^3 \sqcup \mathbb{R}^3 \sqcup \mathbb{R}^3 \sqcup \mathbb{R}^3; \mathbb{C})
\]

of eigenvectors of the operator \( A \) defined by [39]: \( Ae_j = \lambda_j e_j \), which belong to the nuclear Schwartz space

\[
e_j \in \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4) = \mathcal{S}_A(\mathbb{R}^3 \sqcup \mathbb{R}^3 \sqcup \mathbb{R}^3 \sqcup \mathbb{R}^3; \mathbb{C}).
\]

In our case

\[
\mathbb{R}^3 \sqcup \mathbb{R}^3 \sqcup \mathbb{R}^3 \sqcup \mathbb{R}^3 \ni (s, \mathbf{p}) \mapsto e_j(s, \mathbf{p}) = e_j(\mathbf{p}), \quad s \in \{1, 2, 3, 4\},
\]

where \( \{e_j\}_{j=0}^\infty \) is the system of products \( e_j = h_{n_j} h_{m_j} h_{l_j}, \lambda_j = \mu_{n_j} + \mu_{m_j} + \mu_{l_j} + 1 \) of Hermite functions – composing the complete orthonormal system of eigenfunctions of the hamiltonian operator \( H_{(3)} \) in \( L^2(\mathbb{R}^3; \mathbb{C}) \) of the three dimensional.
oscillator (here $\mu_i$ is the eigenvalue corresponding to the Hermite function $h_i$ of the one dimensional oscillator Hamiltonian $H_1$). When considering the white noise construction of zero mass fields we will likewise encounter another family of nuclear spaces $S_A(\mathbb{R}^3, \mathbb{C}^4) = S^0(\mathbb{R}^3, \mathbb{C}^4)$, or $S_A(\mathbb{R}^3, \mathbb{C}^n) = S^0(\mathbb{R}^3, \mathbb{C}^n)$ with another standard operator $A = \oplus A^{(3)}$ on $L^2(\mathbb{R}^3, \mathbb{C}^4)$, or on $L^2(\mathbb{R}^3, \mathbb{C}^n)$, with $A^{(3)} \neq H(3)$.

In particular we have the following Corollary (the fermi analogue of Prop. 4.3.10 of [39]).

**COROLLARY 1.** For $y \in S_A(\mathbb{R}^3, \mathbb{C}^4)^*$ it holds that

$$D_y = \Xi_{0,1}(y) = \sum_{s=1}^{4} \int_{\mathbb{R}^3} y(s, p) \partial_s p \, d^3p, \quad D^+_y = \Xi_{1,0}(y) = \sum_{s=1}^{4} \int_{\mathbb{R}^3} y(s, p) \partial^*_s p \, d^3p.$$ 

In particular, 

$$\partial_s p = \Xi_{0,1}(\delta_s p), \quad \partial^*_s p = \Xi_{1,0}(\delta_s p).$$

For $y \in S_A(\mathbb{R}^3, \mathbb{C}^4) \subset S_A(\mathbb{R}^3, \mathbb{C}^4)^*$

$$\Xi_{0,1}(y), \Xi_{1,0}(y) \in \mathcal{L}\left((S_A(\mathbb{R}^3, \mathbb{C}^4)), (S_A(\mathbb{R}^3, \mathbb{C}^4))\right)$$

and the linear maps

$$S_A(\mathbb{R}^3, \mathbb{C}^4) \ni y \mapsto \Xi_{0,1}(y) = D_y \in \mathcal{L}\left((S_A(\mathbb{R}^3, \mathbb{C}^4)), (S_A(\mathbb{R}^3, \mathbb{C}^4))\right)$$

$$S_A(\mathbb{R}^3, \mathbb{C}^4) \ni y \mapsto \Xi_{1,0}(y) = D^+_y \in \mathcal{L}\left((S_A(\mathbb{R}^3, \mathbb{C}^4)), (S_A(\mathbb{R}^3, \mathbb{C}^4))\right)$$

are continuous.

Moreover, for $y_1, \ldots, y_m \in S_A(\mathbb{R}^3, \mathbb{C}^4)^*$ it holds

$$D_{y_1} \cdots D_{y_m} = \Xi_{0,m}(y_1 \otimes \cdots \otimes y_m)$$

$$= \Xi_{0,m}(y_1 \otimes \cdots \otimes y_m)$$

$$= \sum_{s_1, \ldots, s_m=1}^{4} \int_{\mathbb{R}^3} y_1(s_1, p_1) \cdots y_1(s_m, p_m) \partial_{s_1} p_1 \cdots \partial_{s_m} p_m \, d^3p_1 \cdots d^3p_m$$

$$= (m!)^{-1} \sum_{\pi \in \mathfrak{S}_m} \text{sign } \pi \sum_{s_1, \ldots, s_m=1}^{4} \int_{\mathbb{R}^3} y_1(s_{\pi(1)}, p_{\pi(1)}) \cdots y_m(s_{\pi(m)}, p_{\pi(m)}) \times \partial_{s_1} p_1 \cdots \partial_{s_m} p_m \, d^3p_1 \cdots d^3p_m,$$

where $\pi$ runs over the set $\mathfrak{S}_m$ of all permutations of the numbers $1, 2, \ldots, m$.

Note that because for $y, y' \in S_A(\mathbb{R}^3, \mathbb{C}^4)^*$, $\xi, \xi' \in S_A(\mathbb{R}^3, \mathbb{C}^4)$ all the opera-
\[ D_y = \Xi_{0,1}(y) = \sum_{s=1}^{4} \int y(s, \mathbf{p}) \partial_s \mathbf{p} d^3 \mathbf{p}, \quad \text{and} \quad D_\xi^+ = \Xi_{1,0}(\xi) = \sum_{s=1}^{4} \int \xi(s, \mathbf{p}) \partial_s^* \mathbf{p} d^3 \mathbf{p}, \]
\[ D_{y'} = \Xi_{0,1}(y') = \sum_{s=1}^{4} \int y'(s, \mathbf{p}) \partial_s \mathbf{p} d^3 \mathbf{p}, \quad \text{and} \quad D_{\xi'}^+ = \Xi_{1,0}(\xi') = \sum_{s=1}^{4} \int \xi'(s, \mathbf{p}) \partial_s^* \mathbf{p} d^3 \mathbf{p}. \]

They are frequently written in the form (which should be understood properly in a rigorous sense explained below)
\[
\{ \Xi_{0,1}(y), \Xi_{1,0}(\xi) \} = \langle y, \xi \rangle I, \quad \{ \Xi_{0,1}(y), \Xi_{0,1}(y') \} = \{ \Xi_{1,0}(\xi), \Xi_{1,0}(\xi') \} = 0,
\]

or
\[
\{ D_y, D_\xi^+ \} = \{ y, \xi \} I, \quad \{ D_y, D_{y'} \} = \{ D_\xi^+, D_{\xi'} \} = 0.
\]

They are frequently written in the form (which should be understood properly in a rigorous sense explained below)
\[
\{ \partial_{s, \mathbf{p}}, \partial_{s', \mathbf{p}'} \} = \delta_{s,s'}(\mathbf{p}', \mathbf{p}'), \quad \{ \partial_{s, \mathbf{p}}, \partial_{s', \mathbf{p}'} \} = \{ \partial_{s, \mathbf{p}}, \partial_{s', \mathbf{p}'} \} = 0,
\]

or using the notation of physicists
\[
\{ a_s(\mathbf{p}), a_{s'}(\mathbf{p}')^+ \} = \delta_{ss'}(\mathbf{p} - \mathbf{p}'), \quad \{ a_s(\mathbf{p}), a_{s'}(\mathbf{p}') \} = \{ a_s(\mathbf{p}), a_{s'}(\mathbf{p}')^+ \} = 0,
\]

or (like in [46], p. 82)
\[
\begin{align*}
\{ b_s(\mathbf{p}), b_{s'}(\mathbf{p}')^+ \} &= \delta_{ss'}(\mathbf{p} - \mathbf{p}'), \quad \{ b_s(\mathbf{p}), b_{s'}(\mathbf{p}') \} = \{ b_s(\mathbf{p}), b_{s'}(\mathbf{p}')^+ \} = 0, \\
\{ d_s(\mathbf{p}), d_{s'}(\mathbf{p}')^+ \} &= \delta_{ss'}(\mathbf{p} - \mathbf{p}'), \quad \{ d_s(\mathbf{p}), d_{s'}(\mathbf{p}') \} = \{ d_s(\mathbf{p}), d_{s'}(\mathbf{p}')^+ \} = 0, \\
\{ b_s(\mathbf{p}), d_{s'}(\mathbf{p}')^+ \} &= 0, \quad s, s' = 1, -1,
\end{align*}
\]

with the obvious identifications
\[
D_y = a(y) = a(y|_{s=1} + y|_{s=2} + y|_{s=3} + y|_{s=4}) = b(y|_{s=1} + y|_{s=2} + 0 + 0) + d(0 + 0 + y|_{s=3} + y|_{s=4})
\]
\[
= \sum_{s=1}^{4} \int y(s, \mathbf{p}) a_s(\mathbf{p}) d^3 \mathbf{p},
\]

\[
b(y|_{s=1} + y|_{s=2} + 0 + 0) = \sum_{s=1}^{2} \int y(s, \mathbf{p}) a_s(\mathbf{p}) d^3 \mathbf{p} = \sum_{s=1}^{2} \int y(s, \mathbf{p}) b_{-2s+3}(\mathbf{p}) d^3 \mathbf{p},
\]

\[
d(0 + 0 + y|_{s=3} + y|_{s=4}) = \sum_{s=3}^{4} \int y(s, \mathbf{p}) a_s(\mathbf{p}) d^3 \mathbf{p} = \sum_{s=3}^{4} \int y(s, \mathbf{p}) d_{-2s+7}(\mathbf{p}) d^3 \mathbf{p}
\]
for

\[ y \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^*. \]

The relations (50) or equivalently (51) should be interpreted properly. Namely the first set of relations (49) in the particular case \( y, \xi \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) \) reduces to

\[ \{ \Xi_{0,1}(y), \Xi_{1,0}(\xi) \} = (\overline{y}, \xi)_0 \mathbf{1} \]

with the inner product \((\cdot, \cdot)_0\) on \( L^2(\mathbb{R}^3; \mathbb{C}^4) \). Using the continuity of the inner product \((\cdot, \cdot)_0\) in the nuclear topology of \( \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) \subset L^2(\mathbb{R}^3; \mathbb{C}^4) \) (compare [18], Ch. I.4.2) and nuclearity of \( \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) \), it follows that the bilinear map \( y \times \xi \mapsto (\overline{y}, \xi)_0 \mathbf{1} \) defines an operator-valued distribution:

\[
\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) \otimes \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) \ni \zeta \mapsto \Xi_{0,0}(\zeta)
\]

where \( \tau \in (\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) \otimes \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4))^* \) is defined by

\[
\langle \tau, y \otimes \xi \rangle = (\overline{y}, \xi)_0 = (y, \xi), \quad y, \xi \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4),
\]

therefore we have

\[
\Xi_{0,0}(y \otimes \xi) = \{ \Xi_{0,1}(y), \Xi_{1,0}(\xi) \}
\]

\[
= \sum_{s,s'} \int_{\mathbb{R}^3 \times \mathbb{R}^3} y \otimes \xi(s', \mathbf{p}', s, \mathbf{p}) \, \delta_{s,s'} \, \delta(\mathbf{p} - \mathbf{p}') \, 1 \, d^3p d^3p' \]

\[
= \sum_{s,s'} \int_{\mathbb{R}^3 \times \mathbb{R}^3} y(s', \mathbf{p}') \, \xi(s, \mathbf{p}) \, \delta_{s,s'} \, \delta(\mathbf{p} - \mathbf{p}') \, 1 \, d^3p d^3p',
\]

and

\[
\{ \partial_{s, \mathbf{p}}, \partial^*_{s', \mathbf{p}'} \} = \delta_{s,s'} \, \delta(\mathbf{p} - \mathbf{p}') \mathbf{1}.
\]

Note here that within the white noise construction of Hida the operators \( \partial_{s, \mathbf{p}}, \partial^*_{s, \mathbf{p}} \) are well defined at each point \((s, \mathbf{p}) \in \mathbb{R}^3 \subset \mathbb{R}^3 \subset \mathbb{R}^3 \subset \mathbb{R}^3 \), and there is no need for treating them as operator-valued distributions when using the calculus for integral kernel operators.

The exceptional situations, which involve more factors \( \partial_{s, \mathbf{p}}, \partial^*_{s, \mathbf{p}} \), are well described in non “normal” order, in which we are forced to treat them as distributions are however easily and naturally grasped within the white noise calculus. The first such situation where we need to use distributional interpretation we encounter when trying to give proper meaning to (50) or equivalently (51) which formally involve both

\[
\partial_{s', \mathbf{p}'} \partial_{s, \mathbf{p}} \text{ and } \partial_{s, \mathbf{p}} \partial^*_{s', \mathbf{p}'};
\]

with more than just one factor of the type \( \partial_{s, \mathbf{p}}, \partial^*_{s, \mathbf{p}} \) containing both \( \partial_{s, \mathbf{p}} \) and the adjoint operator \( \partial^*_{s, \mathbf{p}} \). Note that the first of the expressions (that in the
“normal” order) in (52) is meaningful as a continuous operator transforming the Hida space into its dual. But the second expression in (52) is meaningless as a generalized operator on the Hida space (or its dual). Nonetheless both expressions in (52) are well defined as operator-valued distributions. Indeed the corresponding maps

\[ \chi \times \xi \mapsto \Xi_{1,0}(\xi) \circ \Xi_{0,1}(\chi), \quad \chi \times \xi \mapsto \Xi_{0,1}(\chi) \circ \Xi_{1,0}(\xi) \]

are bilinear and separately continuous as maps

\[ \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) \times \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) \longrightarrow \mathcal{L}\left( (\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)), (\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)) \right). \]

Therefore by nuclearity of \( \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) \) and \( \mathcal{L}\left( (\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)), (\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)) \right) \) there exist the corresponding operator-valued distributions, written

\[ \chi \otimes \xi \mapsto \sum_{s,s'=1}^4 \int_{\mathbb{R}^3} \chi \otimes \xi(s',p',s,p) \partial_{s',p'} \partial_{s,p} d^3p' d^3p = \Xi_{1,1}(\chi \otimes \xi) = \Xi_{1,0}(\chi) \circ \Xi_{1,0}(\xi), \] (53)

and

\[ \chi \otimes \xi \mapsto \sum_{s,s'=1}^4 \int_{\mathbb{R}^3} \chi \otimes \xi(s',p',s,p) \partial_{s,p} \partial_{s',p'} d^3p' d^3p = \Xi_{0,1}(\chi) \circ \Xi_{0,1}(\xi), \] (54)

continuous as maps

\[ \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^{\otimes 2} \longrightarrow \mathcal{L}\left( (\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)), (\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)) \right). \]

Here in the formula (54) the “distributional integral kernel”, say operator-valued distribution \( \partial_{s,p} \partial_{s',p'} \), has only formal meaning, and cannot be interpreted as any actual generalized operator on the Hida space. But the integral in the formula (53) represents an integral kernel operator so that the equalities in the formula (53) is actually a theorem which can immediately be checked by application of definition of Hida operators. But likewise the operator \( \Xi_{0,1}(\chi) \circ \Xi_{1,0}(\xi) \) in the formula (54), transforming continuously the Hida space into itself, can be expressed as a (here finite) sum of integral kernel operators. This follows from the general theorem, [37] Thm. 6.1 or [39], Thm 4.5.1 (which can as well be proved for fermi case without any essential changes in the proof of [37], [39]). However our case is so simple that the corresponding decomposition of the operator \( \Xi_{0,1}(\chi) \circ \Xi_{1,0}(\xi) \) into the sum of integral kernel operators can be
proven to be equal
\[ \chi \otimes \xi \mapsto \Xi_{0,1}(\chi) \circ \Xi_{1,0}(\xi) \]
\[ = -\Xi_{1,1}(\chi \otimes \xi) + \Xi_{0,0}(\chi \otimes \xi) \]
\[- \sum_{s,s'=1}^{4} \int_{R^3} \chi \otimes \xi(s',p',s,p) \partial_{s',p'}^{*} \partial_{s,p} d^3p' d^3p + (\chi,\xi)_{0,1} \]
\[ = - \sum_{s,s'=1}^{4} \int_{R^3} \chi \otimes \xi(s',p',s,p) \partial_{s',p'}^{*} \partial_{s,p} d^3p' d^3p \]
\[ + \sum_{s,s'=1}^{4} \int_{R^3} \chi \otimes \xi(s',p',s,p) \{ \partial_{s,p}^{*}, \partial_{s',p'}^{*} \} d^3p' d^3p, \quad (55) \]

using the definition of Hida operators and the relations (49).

The operator-valued distribution (55) is called the normal order form distribution:
\[ \partial_{s,p}^{*}, \partial_{s',p'}^{*} : \text{pairing} \]
\[ \partial_{s,p}^{*}, \partial_{s',p'}^{*} = \partial_{s,p} \partial_{s',p'} + \{ \partial_{s,p}^{*}, \partial_{s',p'}^{*} \} \]

Similarly we have for decomposition of the operator-valued distributions involving more factors
\[ \cdots \partial_{s_i,p_i}, \cdots \partial_{s_j,p_j}, \cdots \quad (56) \]
of the type \( \partial_{s,p}^{*}, \partial_{s',p'}^{*} \), not necessary normally ordered, into sum of components with “normally” ordered Hida’s differential operators, and similarly as in the “Wick theorem” in [4], Chap. III. Note that although reduction of such distributions into “normal form” follows from the general theorem for decompositions of the corresponding operators
\[ \cdots \circ \Xi_{0,1}(\chi_i) \circ \cdots \circ \Xi_{1,0}(\xi_j) \circ \cdots \quad (57) \]
transforming continuously the Hida space into itself into sums of integral kernel operators ([37] Thm. 6.1 or [39], Thm 4.5.1), the simple operator (57) can be decomposed by induction, using the definition of Hida operators and the relations (49). We may also compute decompositions of more involved distributions then (56) which contain “normally ordered” factors \( \partial_{s,p}^{*}, \partial_{s',p'}^{*} \) with both \( \partial_{s,p}^{*} \) and \( \partial_{s',p'}^{*} \) evaluated at the same point \((s,p)\), as well defined distributions:
\[ \cdots \partial_{s_i,p_i}, \cdots \partial_{s_j,p_j}^{*}, \partial_{s_j,p_j}, \cdots \quad (58) \]
with the corresponding operators
\[ \cdots \circ \Xi_{0,1}(\chi_i) \circ \cdots \circ \Xi_{1,1}(\xi_j \otimes 1) \circ \cdots \quad (59) \]
transforming continuously the Hida space into itself. Here \( \tau \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) \otimes \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^* \) is uniquely determined by the formula
\[ \langle \tau, y \otimes \xi \rangle = \langle y, \xi \rangle = (\mathcal{F},\xi)_{0,1}, \quad y, \xi \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4). \]
By Theorem 3 the operator $\Xi_{1, 1}((\xi_j \otimes 1)\tau)$, with $\xi_j \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)$, belongs to
\[ \mathcal{L} \left( (\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)) \right), \]
and the map
\[ \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) \ni \xi_j \mapsto -\Xi_{1, 1}((\xi_j \otimes 1)\tau) \in \mathcal{L} \left( (\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)) \right), \]
is continuous, similarly as for the remaining integral kernel operators $\Xi_{0, 1}(\chi_i), \ldots$ in (59), so that indeed (59) determines a well defined distribution transforming continuously
\[ \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^{\otimes n} \longrightarrow \mathcal{L} \left( (\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)) \right). \]

By the general theorem ([37] Thm. 6.1 or [39], Thm 4.5.1) the operator (59) can be uniquely decomposed into (here finite) sum of integral kernel operators, thus providing the decomposition of the distribution (58) into sum of components, each in the “normal order”. We do not enter here into the investigation of the “Wick theorem” for distributions expressed as simple monomials in the Hida differential operators. In fact the “Wick theorem” of [6], Chap. III, involves the free field operators and not merely the (simpler) operators $a(\delta_s, p) = \partial_s p = a_s(p)$, $a(\delta_s, p)^* = \partial_s^{p'} = a_s(p')^+$. It is true that Wick theorem for free field operators may be immediately reduced to the Wick theorem for the corresponding $\partial_s p = a_s(p), \partial_s^{p'} = a_s(p')^+$ by utilizing the corresponding unitary isomorphisms $U$ (relating the standard Gelfand triples over the corresponding $L^2(\mathbb{R}^3; \mathbb{C}^n)$ with that over the single particle Hilbert spaces), in our case of Dirac field the isomorphism $U$ relating the Gelfand triples [10], which serves to construct the field out of the standard Hida operators through the formula (36). However starting with “Wick theorem” for the standard Hida differential operators wouid not be the correct succession for doing things, because we are interested in very special kind of distributions to be decomposed, which arise as polynomials of free fields containing concrete form of (Wick ordered) interacting term (or terms). Therefore we should first construct explicitly the free fields in thems of Hida differential operators (as special kinds of integral kernel operators, with vector-valued kernels), and then prove “Wick theorem” for polynomials of free fields containing the Wick ordered polynomials as interaction terms.

Here we have only taken the opportunity to emphasize the proper mathematical basis for the “Wick theorem for free fields” as stated in [6], Chap. III, which becomes a particular case of general theorem, [37] Thm. 6.1 or [39], Thm 4.5.1 (extended on generalized operators in the tensor product of several Fock – bose and fermi – spaces) on decomposition of operators transforming continuously the Hida space into itself into a series of integral kernel operators.

Summing up the discussion of the relations (51) or equivalently (51) and of the “Wick theorem for Hida differential operators”, we should emphasize that
the first standard Gelfand triple in (40), we can now utilize the unitary isomorphism Dirac field as Hida generalized operator, using transforming continuously (50) or (51) should be understood as equalities of operator valued distributions, transforming continuously

$$\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^{\otimes 2} \to \mathcal{L}\left( (\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)), (\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)) \right).$$

Now having given the Hida operators $a(\delta_{s,p}) = \partial_{s,p} = a_s(p), \partial_{s',p'} = a_{s'}(p')^+, a(w), a(w)^*, w \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^* \text{ corresponding to the Fock lifting } \Gamma \text{ of the first standard Gelfand triple in } (40), \text{ we can now utilize the unitary isomorphism } U, \text{ given by } (37), \text{ relating the triples in } (40), \text{ and then construct the free Dirac field as Hida generalized operator, using } a(\delta_{s,p}) = \partial_{s,p} = a_s(p), a(\delta_{s,p})^+ = \partial_{s',p'} = a_{s'}(p')^+, a(w), a(w)^*, w \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^* \text{ and the formula } (36):

$$\psi(\phi) = a'(P^\oplus \phi|_{\mathcal{E}_{m,0,0,0}} \oplus 0) + a'(0 \oplus (P^\ominus \phi|_{\mathcal{E}_{-m,0,0,0}})^c)^+$$
$$= a\left(U\left(P^\oplus \phi|_{\mathcal{E}_{m,0,0,0}} \oplus 0\right)\right) + a\left(U\left(0 \oplus (P^\ominus \phi|_{\mathcal{E}_{-m,0,0,0}})^c\right)\right),$$

for

$$0 \oplus (P^\ominus \phi|_{\mathcal{E}_{-m,0,0,0}})^c,$$
and $P^\oplus \phi|_{\mathcal{E}_{m,0,0,0}} \oplus 0 \in E, \phi \in \mathcal{E} = \mathcal{S}(\mathbb{R}^4; \mathbb{C}^4) = \mathcal{S}_\mathcal{H}_{H(4)}(\mathbb{R}^3, \mathbb{C}^4)).$

But the (free) Dirac field $\psi$ (and in general quantum free field) is naturally an integral kernel operator with well defined kernel equal to integral kernel operator

$$\psi^n(x) = \sum_{s=1}^{4} \int_{\mathbb{R}^3} \kappa_{0,1}(s, p; a, x) \partial_{s,p} d^3p + \sum_{s=1}^{4} \int_{\mathbb{R}^3} \kappa_{1,0}(s, p; a, x) \partial_{s,p}^* d^3p$$
$$= \Xi_{0,1}(\kappa_{0,1}(a, x)) + \Xi_{1,0}(\kappa_{1,0}(a, x)),$$

with vector-valued distributional kernels $\kappa_{lm}(a, x)$ representing distributions

$$\kappa_{lm} \in \mathcal{L}\left(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^{\otimes (l+m)}, \mathcal{L}\left(\mathcal{E}, \mathbb{C}\right)\right) \cong \mathcal{L}\left(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^{\otimes (l+m)}, \mathcal{E}^*\right)$$
$$\cong (\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^{\otimes (l+m)})^* \otimes \mathcal{E}^* \cong \mathcal{L}\left(\mathcal{E}^*\right) \cong \mathcal{L}\left(\mathcal{E}, \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^*\right).$$

in the sense of Obata [38]. In fact we have used the standard nuclear space $\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)$ instead of the isomorphic nuclear space $E$, because we have discarded the isomorphism $\Gamma(U)$ in (44) or in (45), and realize the Hida operators $a'$ in the Fock lifting of the standard Gelfand triple in (40). We will find such $\mathcal{L}\left(\mathcal{E}, \mathbb{C}\right) \cong \mathcal{E}^*$-valued distribution kernels $\kappa_{0,1}, \kappa_{1,0} \in \mathcal{L}\left(\mathcal{E}, \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^*\right)$ \cong
\( \mathcal{L}( \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}(\mathcal{E}, \mathbb{C})) \) that

\[
\psi(\phi) = a'(P^{\phi} \phi|_{\mathcal{E}_{m,0,0,0}} \oplus 0) + a'(0 \oplus (P^{\phi} \phi|_{\mathcal{E}_{m,0,0,0}})\cdot) + a\left(U(P^{\phi} \phi|_{\mathcal{E}_{m,0,0,0}} \oplus 0) + a\left(0 \oplus (P^{\phi} \phi|_{\mathcal{E}_{m,0,0,0}})\cdot)\right)\right) + \\
= \sum_{s=1}^{4} \int_{\mathbb{R}^3} \kappa_{0,1}(\phi)(s, p) \partial_s p \, d^3 p + \sum_{s=1}^{4} \int_{\mathbb{R}^3} \kappa_{1,0}(\phi)(s, p) \partial_s^* p \, d^3 p
\]

\[
= \Xi_{0,1}(\kappa_{0,1}(\phi)) + \Xi_{1,0}(\kappa_{1,0}(\phi)), \quad \phi \in \mathcal{E} = \mathcal{S}(\mathbb{R}^4; \mathbb{C}^4).
\]

Here \( \kappa_{0,1}, \kappa_{1,0} \in \mathcal{L}(\mathcal{E}, \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^*) \cong \mathcal{L}(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}(\mathcal{E}, \mathbb{C})) \) are vector valued distributions represented with the following distribution kernels

\[
\kappa_{0,1}(s, p; a, x) = \begin{cases} 
\frac{1}{2|p_0(p)|} u_{s}^a(p)e^{-ip\cdot x} & \text{with } p = (|p_0(p)|, p) \in \mathcal{E}_{m,0,0,0} \quad \text{if } s = 1, 2 \\
0 & \text{if } s = 3, 4
\end{cases}
\]

(61)

\[
\kappa_{1,0}(s, p; a, x) = \begin{cases} 
\frac{1}{2|p_0(p)|} v_{s-2}^a(p)e^{ip\cdot x} & \text{with } p = (|p_0(p)|, p) \in \mathcal{E}_{m,0,0,0} \quad \text{if } s = 1, 2 \\
0 & \text{if } s = 3, 4
\end{cases}
\]

(62)

Here \( \kappa_{0,1}(\phi), \kappa_{1,0}(\phi) \) denote the kernels representing distributions in \( \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^* \) which are defined in the standard manner

\[
\kappa_{0,1}(\phi)(s, p) = \sum_{a=1}^{4} \int_{\mathbb{R}^3} \kappa_{0,1}(s, p; a, x) \phi^a(x) \, d^4 x
\]

and analogously for \( \kappa_{1,0}(\phi), \) and such that

\[
\kappa_{0,1} : \mathcal{E} \ni \phi \mapsto \kappa_{0,1}(\phi) \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^*, \\
\kappa_{1,0} : \mathcal{E} \ni \phi \mapsto \kappa_{1,0}(\phi) \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^*
\]

belong to \( \mathcal{L}(\mathcal{E}, \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^*) \cong \mathcal{L}(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}(\mathcal{E}, \mathbb{C})). \) We should emphasize here that in case of free fields the the vector-valued distributions \( \kappa_{0,1}, \kappa_{1,0} \) are regular function like distributions with distribution kernels \( \kappa_{0,1}(s, p; a, x), \kappa_{0,1}(s, p; a, x) \)
equal to ordinary functions, determining functions

\[
\begin{align*}
(a, x) &\mapsto \kappa_{0,1;s,p}(a, x) \overset{\text{df}}{=} \kappa_{0,1}(s, p; a, x) \quad \in \mathcal{O}_M \subset \mathcal{E}^*, \quad (s, p) \in \sqcup \mathbb{R}^3,
(a, x) &\mapsto \kappa_{1,0;s,p}(a, x) \overset{\text{df}}{=} \kappa_{1,0}(s, p; a, x) \quad \in \mathcal{O}_M \subset \mathcal{E}^*, \quad (s, p) \in \sqcup \mathbb{R}^3,
(s, p) &\mapsto \kappa_{0,1;0,a,x}(s, p) \overset{\text{df}}{=} \kappa_{0,1}(s, p; a, x) \quad \in \mathcal{O}_{M,A} \subset \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^*,
(s, p) &\mapsto \kappa_{1,0;0,a,x}(s, p) \overset{\text{df}}{=} \kappa_{1,0}(s, p; a, x) \quad \in \mathcal{O}_{M,A} \subset \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^*,
\end{align*}
\]

which belong respectively to the function algebra of multipliers \(\mathcal{O}_M\) of the nuclear algebra \(\mathcal{E} = \mathcal{S}(\mathbb{R}^4; \mathbb{C}^4) = \mathcal{S}_{\mathcal{H}(4)}(\mathbb{R}^3, \mathbb{C}^4)\) (in the first two cases), and respectively to the algebra of multipliers \(\mathcal{O}_{M,A}\) of the nuclear algebra \(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) = \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)\) (in the last two cases). These statements can be understood in the sense that for each fixed value of the respective discrete index, \(a\) or \(s\), the functions \(x \mapsto \kappa_{i,m}(s, p; a, x)\) or \(p \mapsto \kappa_{0,1}(s, p; a, x)\), belong respectively to the algebra of multipliers of \(\mathcal{S}(\mathbb{R}^4; \mathbb{C}) = \mathcal{S}_{\mathcal{H}(4)}(\mathbb{R}^3, \mathbb{C})\) or convolutors of \(\mathcal{S}_{\mathcal{H}(4)}(\mathbb{R}^3, \mathbb{C})\). But according to our general prescription, we should also note that \(\mathcal{E} = \mathcal{S}(\mathbb{R}^4; \mathbb{C}^4) = \mathcal{S}_{\mathcal{H}(4)}(\mathbb{R}^3, \mathbb{C}^4) = \mathcal{S}_{\mathcal{H}(4)}(\mathbb{R}^3, \mathbb{C})\) can be treated as nuclear algebra of \(\mathbb{C}\)-valued functions on the disjoint sum \(\sqcup \mathbb{R}^4\) of four disjoint copies of \(\mathbb{R}^4\), with the natural point-wise multiplication rule of any two such functions. So that the algebra \(\mathcal{O}_M\) of multipliers is well defined and coincides with all those functions whose restrictions to each copy \(\mathbb{R}^4\) belongs to the algebra of multipliers of \(\mathcal{S}(\mathbb{R}^4; \mathbb{C}) = \mathcal{S}_{\mathcal{H}(4)}(\mathbb{R}^3, \mathbb{C})\). The algebra of convolutors \(\mathcal{O}_C\) of \(\mathcal{E}\), is also well defined with the ordinary Fourier transform exchanging the convolution and point-wise multiplication if we define action of translation \(T_b, b \in \mathbb{R}^4\) on \((a, x) \in \sqcup \mathbb{R}^4\) as equal \(T_b(a, x) = (a, x + b)\). Similarly the algebras \(\mathcal{O}_{M,A}(\mathbb{R}^3; \mathbb{C}^4), \mathcal{O}_{M,A}(\mathbb{R}^3; \mathbb{C}^4)\), of multipliers and convolutors of \(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) = \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4) = \mathcal{S}(\sqcup \mathbb{R}^3, \mathbb{C})\) are well defined, where the last is the algebra of all such functions on \(\sqcup \mathbb{R}^3\) with restrictions to each copy \(\mathbb{R}^3\) belonging to \(\mathcal{S}(\mathbb{R}^3, \mathbb{C}) = \mathcal{S}_{\mathcal{H}(3)}(\mathbb{R}^3, \mathbb{C})\).

Note in particular that the integrals in the pairings

\[
\langle \kappa_{0,1}(\phi), \zeta \rangle = \sum_{s=1}^{4} \int_{\mathbb{R}^4 \times \mathbb{R}^3} \kappa_{0,1}(\phi)(s, p) \, \zeta(s, p) \, d^3p
\]

\[
= \sum_{s=1}^{4} \sum_{a=1}^{4} \int_{\mathbb{R}^3} \kappa_{0,1}(\phi)(s, p; a, x) \, \zeta(s, p) \, d^4x \, d^3p, \quad \zeta \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4), \phi \in \mathcal{E},
\]

are not merely symbolic but actual well defined Lebesgue integrals.\(^{10}\)

We have the following

\(^{10}\)Here for the case of the Dirac field. But we have analogous situation for other fields
**Lemma 2.** Let \( \phi \in \mathcal{D} = S(\mathbb{R}^4; \mathbb{C}^4) \) and \( \kappa_{0,1}, \kappa_{1,0} \) be the vector-valued distributions ([81]) and respectively ([82]). Then

\[
\kappa_{0,1}(\tilde{\phi})(s, p) = (P^s\tilde{\phi}|_{\mathcal{L}_{m,0,0,0}})_s(p), \quad s = 1, 2,
\]

\[
\kappa_{0,1}(\tilde{\phi})(s, p) = 0, \quad s = 3, 4,
\]

\[
\kappa_{1,0}(\tilde{\phi})(s, p) = 0, \quad s = 1, 2,
\]

\[
\kappa_{1,0}(\tilde{\phi})(s, p) = (P^s\tilde{\phi}|_{\mathcal{L}_{-m,0,0,0}})_s(p), \quad s = 3, 4,
\]

where \((P^s\tilde{\phi}|_{\mathcal{L}_{m,0,0,0}})_s\) stands for the \(s\)-th component of

\[
U\left(P^s\tilde{\phi}|_{\mathcal{L}_{m,0,0,0}} \oplus 0\right), \quad \text{for } s = 1, 2
\]

or respectively \((P^s\tilde{\phi}|_{\mathcal{L}_{-m,0,0,0}})_s\) stands for the \(s\)-th component of

\[
U\left(0 \oplus (P^s\tilde{\phi}|_{\mathcal{L}_{-m,0,0,0}})^c\right), \quad \text{for } s = 3, 4
\]

in the image of the unitary isomorphism ([87]).

We have by definition for \(s = 1, 2\)

\[
\kappa_{0,1}(\tilde{\phi})(s, p) = \sum_{a=1}^{4} \frac{u_a^s(p)}{2p_0(p)} \int_{\mathbb{R}^4} \phi^a(x)e^{ipx}d^4x = \sum_{a=1}^{4} \frac{u_a^s(p)}{2p_0(p)} \tilde{\phi}^a(p_0(p), p)
\]

\[
= \sum_{a=1}^{4} \frac{u_a^s(p)}{2p_0(p)} \tilde{\phi}^a(p_0(p), p) = \frac{1}{p_0(p)} u_a(p)^+ \tilde{\phi}(p_0(p), p) \]

\[
= \frac{1}{2p_0(p)} u_a(p)^+ (P^s\tilde{\phi}(p_0(p), p)) = (P^s\tilde{\phi}|_{\mathcal{L}_{m,0,0,0}})_s(p), \quad \text{for } s = 1, 2.
\]

Here the first four equalities follow by definition, the fifth equality follows from the property ([11]) (compare Appendix [4] of \(u_a(p)\), and recall that the last term \((P^s\tilde{\phi}|_{\mathcal{L}_{m,0,0,0}})_s\) is equal to the complex conjugation of the \(s\)-th direct summand in

\[
U\left(P^s\tilde{\phi}|_{\mathcal{L}_{m,0,0,0}} \oplus 0\right), \quad \text{for } s = 1, 2
\]

by definition ([87]) of the unitary isomorphism \(U\).

With the standard Hilbert space \(L^2(\mathbb{R}^3; \mathbb{C}^4)\) and the standard operator \(A\) in ([80]) possibly replaced with corresponding standard \(L^2(\mathbb{R}^3; \mathbb{C}^n)\) and \(A = \oplus H_{(3)}\) or \(= \oplus A_{(3)}\). In this case \(S_{A=\oplus H_{(3)}}(\mathbb{R}^3; \mathbb{C}^n) = S(\mathbb{R}^3; \mathbb{C}^n)\) or \(S_{A=\oplus A_{(3)}}(\mathbb{R}^3; \mathbb{C}^n) = S^0(\mathbb{R}^3; \mathbb{C}^n), \delta = S_{A=\oplus H_{(4)}}(\mathbb{R}^4; \mathbb{C}^n) = S(\mathbb{R}^4; \mathbb{C}^n)\) or \(\delta = S_{A=\oplus A_{(4)}}(\mathbb{R}^4; \mathbb{C}^n) = S^0(\mathbb{R}^4; \mathbb{C}^n)\) (compare Section 5 of ([89])) and with the corresponding unitary isomorphism \(U\) joining the corresponding spectral triples analogous to ([24]). In this case the summation with respect to the indices \(s, a\) runs over \(\{1, 2, \ldots, n\}\).
Here the equalities follow by definition, except the fifth equality, which follows
from Lemma 2 and from (36) it follows
\[ \kappa_{s-2}(p) = \sum_{a=1}^{2} \phi_{s-2}(x)e^{-ipx}d^4x = \sum_{a=1}^{2} \phi_{s-2}(x)e^{-ip|\nu(p)|, -p} \]

The rest part:
\[ \kappa_{0,1}(\phi)(s, p) = 0, \quad s = 3, 4, \]
\[ \kappa_{1,0}(\phi)(s, p) = 0, \quad s = 1, 2, \]
of our Lemma follows immediately from definition (61) and respectively (62) of the distributions \( \kappa_{0,1}, \kappa_{1,0} \).

From Lemma 2 and from (63) it follows

**LEMMA 3.** Let \( \kappa_{0,1} \) and \( \kappa_{1,0} \) be the vector-valued distributions (61) and respectively (62). Then the equality (64) holds true:

\[ \psi(\phi) = a'(P^\phi|e_{m,0,0,0}) + 0 + a'(0 \oplus (P^\phi|e_{m,0,0,0})^c)^+ \]

\[ = a\left( U(P^\phi|e_{m,0,0,0}) + 0 \right) + a\left( U(0 \oplus (P^\phi|e_{m,0,0,0})^c)^+ \right) \]

\[ = \sum_{s=1}^{4} \int \kappa_{0,1}(\phi)(s, p) \partial_s p d^3p + \sum_{s=1}^{4} \int \kappa_{1,0}(\phi)(s, p) \partial_s p d^3p \]

\[ = \Xi_{0,1}(\phi) + \Xi_{1,0}(\phi), \quad \phi \in \mathcal{E} = S(\mathbb{R}^4; \mathbb{C}^4). \]

Indeed, we have

\[ \sum_{s=1}^{4} \int \kappa_{0,1}(\phi)(s, p) \partial_s p d^3p = \sum_{s=1}^{2} \int (P^\phi|e_{m,0,0,0})_s(p) \partial_s p d^3p \]

\[ = a\left( (P^\phi|e_{m,0,0,0})_1 \oplus (P^\phi|e_{m,0,0,0})_2 \oplus 0 \right) = a\left( U(P^\phi|e_{m,0,0,0}) + 0 \right) \]

\[ = a'(P^\phi|e_{m,0,0,0} + 0). \]
Here the first three equalities follow from Lemma 2 and Corollary 1; the last equality follows from (36).

Similarly we have

\[
\sum_{s=1}^{4} \int \kappa_{1,0}(\bar{\sigma})(s, \mathbf{p}) \partial_{s,p}^* \mathbf{d}^3 \mathbf{p} = \sum_{s=3}^{4} \int \left( P^\equiv \tilde{\phi}_{e_{m,0,0,0}} \right)_s(p) \partial_{s,p}^* \mathbf{d}^3 \mathbf{p} = a \left( 0 \oplus 0 \oplus \left( P^\equiv \tilde{\phi}_{e_{m,0,0,0}} \right)_4 \right) = a \left( U \left( 0 \oplus \left( P^\equiv \tilde{\phi}_{e_{m,0,0,0}} \right)^{\mathcal{C}} \right) \right) = a' \left( 0 \oplus \left( P^\equiv \tilde{\phi}_{e_{m,0,0,0}} \right)^{\mathcal{C}} \right).
\]

Here the first three equalities follow from Lemma 2 and Corollary 1; the last equality follows from (36).

Let \( \mathcal{O}_C = \mathcal{O}_C(\mathbb{R}^4; \mathbb{C}^4) \) be the predual of of the Schwartz algebra of convolutors \( \mathcal{O}_C' = \mathcal{O}_C(\mathbb{R}^4; \mathbb{C}^4) \), which means that each component of each element of \( \mathcal{O}_C \) belongs to the Horváth predual \( \mathcal{O}_C(\mathbb{R}^4; \mathbb{C}) \) of the ordinary Schwartz convolution algebra \( \mathcal{O}_C(\mathbb{R}^4; \mathbb{C}) \). For detailed construction and definition of \( \mathcal{O}_C(\mathbb{R}^4; \mathbb{C}) \) and \( \mathcal{O}_C(\mathbb{R}^4; \mathbb{C}) \), compare [47], [28] or [30], or finally compare the summary of their properties presented in Appendix 5.

The following Lemma holds true (and we have in general analogous Lemma for a local field understood as a sum of integral kernel operators with vector-valued kernels)

**LEMMA 4.** For the \( \mathcal{L}(\mathcal{E}, \mathbb{C}) \)-valued (or \( \mathcal{E}^* \)-valued) distributions \( \kappa_{0,1}, \kappa_{1,0} \), given by (61) and (62), in the equality (60) defining the Dirac \( \psi \) field we have

\[
\left( (a, x) \mapsto \sum_s \int \kappa_{0,1}(s, \mathbf{p}; a, x) \xi(s, \mathbf{p}) \mathbf{d}^3 \mathbf{p} \right) \in \mathcal{O}_C \subset \mathcal{O}_M \subset \mathcal{E}^*, \quad \xi \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4),
\]

\[
\left( (a, x) \mapsto \sum_s \int \kappa_{1,0}(s, \mathbf{p}; a, x) \xi(s, \mathbf{p}) \mathbf{d}^3 \mathbf{p} \right) \in \mathcal{O}_C \subset \mathcal{O}_M \subset \mathcal{E}^*, \quad \xi \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4),
\]

\[
\left( (s, \mathbf{p}) \mapsto \sum_a \int \kappa_{0,1}(s, \mathbf{p}; a, x) \phi^a(x) \mathbf{d}^4 x \right) \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4), \quad \phi \in \mathcal{E},
\]

\[
\left( (s, \mathbf{p}) \mapsto \sum_a \int \kappa_{1,0}(s, \mathbf{p}; a, x) \phi^a(x) \mathbf{d}^4 x \right) \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4), \quad \phi \in \mathcal{E}.
\]

Moreover the maps

\[
\kappa_{0,1} : \mathcal{E} \ni \phi \mapsto \kappa_{0,1}(\phi) \in \mathcal{S}_A(\mathbb{R}^3, \mathcal{C}^4),
\]

\[
\kappa_{1,0} : \mathcal{E} \ni \phi \mapsto \kappa_{1,0}(\phi) \in \mathcal{S}_A(\mathbb{R}^3, \mathcal{C}^4)
\]

are continuous (for \( \kappa_{0,1}, \kappa_{1,0} \) understood as maps in

\[
\mathcal{L}(\mathcal{E}, \left( \mathcal{S}_A(\mathbb{R}^3, \mathcal{C}^4) \right)^*) \cong \mathcal{L}(\mathcal{S}_A(\mathbb{R}^3, \mathcal{C}^4), \mathcal{L}(\mathcal{E}, \mathcal{C})),
\]

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and, equivalently, the maps \( \xi \mapsto \kappa_{0,1}(\xi), \xi \mapsto \kappa_{1,0}(\xi) \) can be extended to continuous maps

\[
\begin{align*}
\kappa_{0,1} : & \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)^* \ni \xi \mapsto \kappa_{0,1}(\xi) \in \mathcal{E}^*, \\
\kappa_{1,0} : & \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)^* \ni \xi \mapsto \kappa_{1,0}(\xi) \in \mathcal{E}^*,
\end{align*}
\]

(for \( \kappa_{0,1}, \kappa_{1,0} \) understood as maps \( \mathcal{L}(\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{E}(\mathcal{C})) \cong \mathcal{L}(\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{E}^*) \)). Therefore not only \( \kappa_{0,1}, \kappa_{1,0} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{E}(\mathcal{C})) \), but both \( \kappa_{0,1}, \kappa_{1,0} \) can be (uniquely) extended to elements of

\[
\mathcal{L}(\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)^*, \mathcal{E}(\mathcal{C})) \cong \mathcal{L}(\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)^*, \mathcal{E}^*) \cong \mathcal{L}(\mathcal{E}, \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)).
\]

That for each \( \xi \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4) \) the functions \( \kappa_{0,1}(\xi), \kappa_{1,0}(\xi) \) given by (here \( x = (x_0, \mathbf{x}) \))

\[
\begin{align*}
(a, x) & \mapsto \sum_{s=1}^{4} \int_{\mathbb{R}^3} \kappa_{0,1}(s, \mathbf{p}; a, x) \xi(s, \mathbf{p}) \, d^3\mathbf{p} = \sum_{s=1}^{2} \int_{\mathbb{R}^3} \frac{v_4^s(\mathbf{p})}{2p_0(\mathbf{p})} \xi(s, \mathbf{p}) e^{-ip_0(\mathbf{p})x_0+i\mathbf{p} \cdot \mathbf{x}} \, d^3\mathbf{p}, \\
(a, x) & \mapsto \sum_{s=1}^{4} \int_{\mathbb{R}^3} \kappa_{1,0}(s, \mathbf{p}; a, x) \xi(s, \mathbf{p}) \, d^3\mathbf{p} = \sum_{s=3}^{4} \int_{\mathbb{R}^3} \frac{v_4^{s-2}(\mathbf{p})}{2p_0(\mathbf{p})} \xi(s, \mathbf{p}) e^{ip_0(\mathbf{p})x_0-i\mathbf{p} \cdot \mathbf{x}} \, d^3\mathbf{p},
\end{align*}
\]

belong to \( \mathcal{O}_C \subset \mathcal{O}_M \subset \mathcal{E}^* \) is immediate. Indeed, that they are smooth is obvious, similarly as it is obvious the existence of such a natural \( N \) (it is sufficient to take here \( N = 0 \)) that for each multiindex \( \alpha \in \mathbb{N}^4 \) the functions

\[
(a, x) \mapsto (1 + |x|^2)^{-N}|D_{x_0}^\alpha \kappa_{0,1}(\xi)(a, x)|, \quad (a, x) \mapsto (1 + |x|^2)^{-N}|D_{x_0}^\alpha \kappa_{1,0}(\xi)(a, x)|
\]

are bounded. Here \( D_{x_0}^\alpha \kappa_{l,m}(\xi) \) denotes the ordinary derivative of the function \( \kappa_{l,m}(\xi) \) of \( |\alpha| = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 \) order with respect to space-time variables \( x = (x_0, x_1, x_2, x_3) \), and here \( |x|^2 = (x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2 \). The first statement of the Lemma equivalently means that if we fix the value of the discrete index \( a \) in the above functions

\[
(a, x) \mapsto \kappa_{0,1}(\xi)(a, x), \quad (a, x) \mapsto \kappa_{1,0}(\xi)(a, x),
\]

then we obtain functions which belong to the algebra of convolutors of the algebra

\[
\mathcal{S}(\mathbb{R}^4; \mathbb{C}) = \mathcal{S}_{H_{\alpha}}(\mathbb{R}^4; \mathbb{C}).
\]

of \( \mathbb{C} \)-valued functions.

Consider now the functions (in both formulas below the variable \( p = (|p_0(\mathbf{p})|, \mathbf{p}) \))
is restricted to the positive energy orbit $\mathcal{O}_{m,0,0,0}$

\[
(s, p) \mapsto \kappa_{0,1}(\phi)(s, p) = \frac{4}{2|p_0(p)|} \int_{\mathbb{R}^3} \phi^a(x) e^{-ixp} \, d^4x
\]

\[
= \sum_{a=1}^{4} \frac{u^a(p)}{2|p_0(p)|} \phi^a|_{\sigma_{m,0,0,0}}(-p),
\]

\[
(s, p) \mapsto \kappa_{1,0}(\phi)(s, p) = \frac{4}{2|p_0(p)|} \int_{\mathbb{R}^3} \phi^a(x) e^{ixp} \, d^4x
\]

\[
= \sum_{a=1}^{4} \frac{v^a(p)}{2|p_0(p)|} \phi^a|_{\sigma_{m,0,0,0}}(p),
\]

with $\phi \in \mathcal{S}(\mathbb{R}^4; \mathbb{C}^4)$. That both functions $\kappa_{0,1}(\phi), \kappa_{1,0}(\phi)$ depend continuously on $\phi$ as maps

$\mathcal{E} = \mathcal{S}(\mathbb{R}^4; \mathbb{C}^4) \longrightarrow \mathcal{S}_{A}(\mathbb{R}^3, \mathbb{C}^4) = \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$

follows from: 1) continuity of the Fourier transform as a map on the Schwartz space, as well as 2) from the continuity of the restriction to the orbits $\mathcal{O}_{m,0,0,0}$ and $\mathcal{O}_{-m,0,0,0}$ (with $m \neq 0$) regarded as a map from $\mathcal{S}(\mathbb{R}^4; \mathbb{C})$ into $\mathcal{S}(\mathbb{R}^3; \mathbb{C})$, and finally 3) from the fact that the functions $p \mapsto \frac{u^a(p)}{2|p_0(p)|}$ and $p \mapsto \frac{v^a(p)}{2|p_0(p)|}$ are multipliers of the Schwartz algebra $\mathcal{S}(\mathbb{R}^3; \mathbb{C})$, compare Appendix 4 and Appendix 5.

**REMARK.** Note here that the continuity of the maps

$\kappa_{0,1}: \mathcal{E} \ni \phi \mapsto \kappa_{0,1}(\phi) \in \mathcal{S}_{A}(\mathbb{R}^3, \mathbb{C}^4)$,

$\kappa_{1,0}: \mathcal{E} \ni \phi \mapsto \kappa_{1,0}(\phi) \in \mathcal{S}_{A}(\mathbb{R}^3, \mathbb{C}^4)$

is based on the continuity of the restriction to the orbits $\mathcal{O}_{m,0,0,0}$ and $\mathcal{O}_{-m,0,0,0}$, regarded as a map $\mathcal{E} = \mathcal{S}(\mathbb{R}^4; \mathbb{C}) \rightarrow \mathcal{S}(\mathbb{R}^3; \mathbb{C})$ between the ordinary Schwartz spaces. This continuity breaks down for the orbit equal to the light cone $\mathcal{O}_{1,0,0,1}$, because of the singularity at the apex. Therefore the space-time test space

$\mathcal{E} = \mathcal{S}_{A(1)}(\mathbb{R}^4; \mathbb{C}^n) = \mathcal{S}^{00}(\mathbb{R}^4; \mathbb{C}^n) \neq \mathcal{S}(\mathbb{R}^4; \mathbb{C}^n)$

cannot be equal $\mathcal{S}(\mathbb{R}^4; \mathbb{C}^n)$ and the standard operator $A \neq \oplus H_{(3)}$ with

$\mathcal{S}_{A}(\mathbb{R}^3, \mathbb{C}^n) = \mathcal{S}_{A(2)}(\mathbb{R}^3; \mathbb{C}^n) = \mathcal{S}^{0}(\mathbb{R}^3, \mathbb{C}^n) \neq \mathcal{S}(\mathbb{R}^3; \mathbb{C}^n)$,

for fields based on representations pertinent to the light cone orbit $\mathcal{O}_{1,0,0,1}$, if the continuity of the said maps $\phi \mapsto \kappa_{0,1}(\phi), \phi \mapsto \kappa_{1,0}(\phi)$ is to be preserved. But the said continuity of the map $\phi \mapsto \kappa_{1,0}(\phi)$ is necessary and sufficient (as we will soon see, compare Corollary 2) for the field $\psi = \Xi_{0,1}(\kappa_{1,0}) + \Xi_{1,0}(\kappa_{1,0})$ to be continuous

$\phi \mapsto \Xi_{0,1}(\kappa_{1,0}(\phi)) + \Xi_{1,0}(\kappa_{1,0}(\phi))$.
as a map in
\[ \mathcal{L} \left( \mathcal{E}, \mathcal{L}((E), (E)) \right), \]
i.e. necessary and sufficient condition for \( \psi = \Xi_{0,1}(\kappa_{1,0}) + \Xi_{1,0}(\kappa_{1,0}) \) to be a well defined operator valued distribution. Therefore the space-time test function space \( \mathcal{E} \) for zero mass fields must be modified and cannot coincide with the ordinary Schwartz space. This is at least the case for zero mass fields constructed as above as integral kernel operators with vector-valued kernels in the sense of Obata [38], [39], Chap. 6.3. Obata provided detailed analysis of are equal to integral kernel operators (48) with scalar valued kernel with vector-valued kernel we should give here general theorems for which

\[ \Xi_{l,m}(\kappa_{l,m}(a, x)) \]
\[ = \sum_{s_1, \ldots, s_l; t_1, \ldots, t_m=1}^{4} \int_{(\mathbb{R}^3)^{l+m}} \kappa_{l,m}(s_1, k_1, \ldots, s_l, k_t, t_1, \ldots, t_m, p_m; a, x) \times \]
\[ \partial_{s_1, k_1} \cdots \partial_{s_l, k_l} \partial_{t_1, p_1} \cdots \partial_{t_m, p_m} d^3k_1 \cdots d^3k_l d^3p_1 \cdots d^3p_m, \]
for which

\[ \Xi_{l,m}(\kappa_{l,m}(\phi)) \]
\[ = \sum_{s_1, \ldots, s_l; t_1, \ldots, t_m=1}^{4} \int_{(\mathbb{R}^3)^{l+m}} \kappa_{l,m}(\phi)(s_1, k_1, \ldots, s_l, k_t, t_1, \ldots, t_m, p_m) \times \]
\[ \partial_{s_1, k_1} \cdots \partial_{s_l, k_l} \partial_{t_1, p_1} \cdots \partial_{t_m, p_m} d^3k_1 \cdots d^3k_l d^3p_1 \cdots d^3p_m, \]
are equal to integral kernel operators [38] with scalar valued kernels \( \kappa_{l,m}(\phi) \in \left( \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^{\otimes (l+m)} \right)^* \), and with

\[ \kappa_{l,m} \in \mathcal{L}(\mathcal{E}, \left( \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^{\otimes (l+m)} \right)^*) \equiv \mathcal{L}(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^{\otimes (l+m)}, \mathcal{L}(\mathcal{E}, \mathcal{C})) \]
\[ = \mathcal{L}(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^{\otimes (l+m)}, \mathcal{E}^*), \]
worked out by Obata [38], [39], Chap. 6.3. Obata provided detailed analysis of the bose case, but in a manner easily adopted to the fermi case, and moreover he
analyzed slightly more general case of integral kernel operators with $\mathcal{L}(\mathcal{E}, \mathcal{E}^*)$-valued distributions

$$\kappa_{l,m} \in \mathcal{L}(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) \otimes (l+m), \mathcal{L}(\mathcal{E}, \mathcal{E}^*)).$$  

We only need to analyse the special case of $\mathcal{L}(\mathcal{E}, \mathbb{C}) \cong \mathcal{E}^*$-valued distribution kernels

$$\kappa_{l,m} \in \mathcal{L}(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) \otimes (l+m), \mathcal{L}(\mathcal{E}, \mathbb{C})) \cong \mathcal{L}(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) \otimes (l+m), \mathcal{E}^*).$$  

In fact in realistic QFT, such as QED, we have several free fields, coupled with lagrangian equal to a Wick polynomial of free fields (we have in view the causal perturbative approach). Therefore we need to consider a generalization of the Gelfand triples to the case of integral kernel operators in tensor product of, say $N$, (fermi and/or bose) Fock spaces $\mathcal{F}^0$ over the corresponding single particle Hilbert spaces $\mathcal{H}_i$, the corresponding standard Gelfand triples

$$L^2(\sqcup \mathbb{R}^3, d^3p; \mathbb{C})$$

$$\mathcal{S}_{A_i}(\mathbb{R}^3, \mathbb{C}^{r_i}) \subset \sqcup \mathcal{H}_i \subset \mathcal{S}_{A_i}(\mathbb{R}^3, \mathbb{C}^{r_i})^*$$

$$\downarrow \quad \downarrow$$

$$E_i \subset \mathcal{H}_i \subset \downarrow \quad \downarrow$$

$$E_i^*$$

(the analogues of (40)) with the corresponding unitary isomorphisms $U_i$ (analogues of the isomorphism $U$ joining the Gelfand triples (44)). We only need to analyse the special case of $\mathcal{L}(\mathcal{E}, \mathbb{C}) \cong \mathcal{E}^*$-valued distribution kernels

$$\kappa_{l,m} \in \mathcal{L}(\mathcal{S}_{A_{n_1}}(\mathbb{R}^3, \mathbb{C}^{r_1}) \otimes \cdots \otimes \mathcal{S}_{A_{n_m}}(\mathbb{R}^3, \mathbb{C}^{r_m}) \otimes \cdots \otimes \mathcal{S}_{A_{n_{l+m}}}(\mathbb{R}^3, \mathbb{C}^{r_{l+m}}), \mathcal{L}(\mathcal{E}, \mathbb{C})).$$

(64)

Here

$$\mathcal{E} = \mathcal{S}_{B}(\sqcup \mathbb{R}^W; \mathbb{C}) = \mathcal{S}_{B_{p_1}}(\mathbb{R}^4; \mathbb{C}^{q_1}) \otimes \cdots \otimes \mathcal{S}_{B_{p_M}}(\mathbb{R}^4; \mathbb{C}^{q_M})$$

$$\subset L^2(\sqcup \mathbb{R}^W; \mathbb{C}) = L^2(\mathbb{R}^4, \mathbb{C}^{q_1}) \otimes \cdots \otimes L^2(\mathbb{R}^4, \mathbb{C}^{q_M}),$$

with

$$B = B_{p_1} \otimes \cdots \otimes B_{p_M}, \quad p_k \in \{1, 2\},$$

on

$$L^2(\sqcup \mathbb{R}^W; \mathbb{C}) = L^2(\mathbb{R}^4, \mathbb{C}^{q_1}) \otimes \cdots \otimes L^2(\mathbb{R}^4, \mathbb{C}^{q_M}),$$

$$W = 4M, \quad q_k, M = 1, 2, \ldots$$

$$\sqcup \mathbb{R}^W = q_1q_2\cdots q_M$$

disjoint copies of $\mathbb{R}^W$

Moreover we have only two possibilities for $A_i, B_i, i = 1, 2$, on each respective $L^2(\mathbb{R}^3, \mathbb{C}^{r_i}), L^2(\mathbb{R}^4, \mathbb{C}^{q_i})$:

$$\mathcal{S}_{A_n}(\mathbb{R}^3; \mathbb{C}^{r_i}) = \mathcal{S}_{\oplus H_{\mathbb{R}}}(\mathbb{R}^3; \mathbb{C}^{r_i}) = \mathcal{S}(\mathbb{R}^3; \mathbb{C}^{r_i}),$$

or

$$\mathcal{S}_{B_n}(\mathbb{R}^4; \mathbb{C}^{q_i}) = \mathcal{S}_{\oplus A_{\mathbb{R}}}(\mathbb{R}^4; \mathbb{C}^{q_i}) = \mathcal{S}(\mathbb{R}^4; \mathbb{C}^{q_i}),$$

or

$$\mathcal{S}_{B_n}(\mathbb{R}^4; \mathbb{C}^{q_i}) = \mathcal{S}_{\oplus A_{\mathbb{R}}}(\mathbb{R}^4; \mathbb{C}^{q_i}) = \mathcal{S}(\mathbb{R}^4; \mathbb{C}^{q_i}).$$
Here we have the nuclear spaces $S^{00}(\mathbb{R}^4;\mathbb{C}^n)$, $S^0(\mathbb{R}^3;\mathbb{C}^n)$, and the standard operators $A^{(n)}$ in $L^2(\mathbb{R}^n,\mathbb{C})$, constructed in Subsections 5.2-5.5 and 5.8 of [59]. $H_{(4)}$ is the Hamiltonian operator on $L^2(\mathbb{R}^4;\mathbb{C})$ of the 4-dimensional oscillator, compare Appendix 9 of [59]. Here $\tilde{\cdot}$ = $\mathcal{F}(\cdot)$ stands for the Fourier transform image. Note that

$$S_{\mathbb{E}A^{(4)}}(\mathbb{R}^4;\mathbb{C}^n) = S^0(\mathbb{R}^4;\mathbb{C}^n)$$

is the nuclear subspace of all those functions in $S(\mathbb{R}^4;\mathbb{C}^n)$ which together with all their derivatives vanish at zero, so that $S^{00}(\mathbb{R}^4;\mathbb{C}^n)$ is the nuclear space of Fourier transforms of all such functions, compare Subsections 5.2-5.5 of [59].

For QED it is sufficient to confine attention to just one case of all $n_i = 4$ in [44] and the case of integral kernel operators in the tensor product of two Fock liftings of the standard Gelfand triples $S_{A_i}(\mathbb{R}^3;\mathbb{C}^4) \subset L^2(\mathbb{R}^3;\mathbb{C}^4) \subset S_{A_i}(\mathbb{R}^3;\mathbb{C}^4)^*$, $i = 1, 2$, both over $L^2(\mathbb{R}^3;\mathbb{C}^4)$. Namely: one fermi Fock lifting of the standard triple in (10), corresponding to the Dirac field, with the standard operators $A_1 = \oplus H_{(4)}, B_1 = \oplus H_{(4)}$ defined above, and one boson Fock lifting of the standard triple in (272) of Subsect. 5.8 of [59], corresponding to the electromagnetic potential field with the standard operators $A_2 = \oplus A^{(3)}, B_2 = \mathcal{F}^{-1} \oplus A^{(4)} \mathcal{F}$ constructed in Subsection 5.8 of [59]. Then we consider the standard Hida space $(\mathbf{E}) = (E_1) \otimes (E_2)$ as arising from the standard (with nuclear inverse) operator $\Gamma_{\text{Fermi}}(A_1) \otimes \Gamma_{\text{Bose}}(A_2)$ in the tensor product Fock space $\Gamma_{\text{Fermi}}(L^2(\mathbb{R}^3;\mathbb{C}^4)) \otimes \Gamma_{\text{Bose}}(L^2(\mathbb{R}^3;\mathbb{C}^4))$ and equal to the tensor product of the Hida spaces

$$(E_i) = (S_{A_i}(\mathbb{R}^3;\mathbb{C}^4)).$$

The corresponding boson Hida differential operators acting on $(E_2) \subset \Gamma_{\text{Bose}}(L^2(\mathbb{R}^3;\mathbb{C}^4))$ (constructed in Section 5 of [59]) we denote here by $\partial_{s,p}, \mu \in \{0, 1, 2, 3\}, p \in \mathbb{R}^3$. We use the greek indices notation for the discrete parameter $\mu$ in order to distinguish them from the fermi Hida differential operators $\partial_{s,p}$ acting on $(E_1) \subset \Gamma_{\text{Fermi}}(L^2(\mathbb{R}^3;\mathbb{C}^4))$. In fact the Hida differential operators as acting on $(\mathbf{E}) = (E_1) \otimes (E_2) \subset \Gamma_{\text{Fermi}}(L^2(\mathbb{R}^3;\mathbb{C}^4)) \otimes \Gamma_{\text{Bose}}(L^2(\mathbb{R}^3;\mathbb{C}^4))$ should be understood respectively as equal $\partial_{s,p} \otimes 1$ and $1 \otimes \partial_{s,p}$. However in order to simplify notation we will likewise write for them simply $\partial_{s,p}$ and $\partial_{s,p}$. Of course in this notation $E_1$, $\mathcal{H}'_1$ is the standard nuclear space $E_1 = S_{\mathbb{E}H_{(3)}}(\mathbb{R}^3;\mathbb{C}^4)$ and the single particle Hilbert space $\mathcal{H}'$ in (10); and $E_2$, $\mathcal{H}'_2$ is the nuclear space $E_2 = S_{\mathbb{E}A^{(3)}}(\mathbb{R}^3;\mathbb{C}^4)$ and the single particle Hilbert space $\mathcal{H}'$ in (272) of Subsection 5.8 of [59].

Of course one can consider the generalization of [38] for vector-valued kernels for integral kernel operators on tensor product of any finite number of standard fermi and/or boson Fock spaces with the respective tensor product of the corresponding standard Gelfand triples. Having in view only the QED case we confine attention to the tensor product of just two mentioned above Fock spaces and the tensor product of the corresponding standard Gelfand triples (10) (of this Subsection) and (272) (of Subsect. 5.8 of [59]). We consider integral kernel
operators $\Xi_{l,m}(\kappa_{l,m})$ for general $\mathcal{L}(\mathcal{E}, \mathcal{C}) \cong \mathcal{E}^*$-valued kernel

$$\kappa_{l,m} \in \mathcal{L}\left(\mathcal{S}_{A_{i_1}}(\mathbb{R}^3, \mathbb{C}^4) \otimes \cdots \otimes \mathcal{S}_{A_{i_{l+m}}}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}(\mathcal{E}, \mathcal{C})\right),$$

with

$$A_{i_k} = A_1 = \oplus_1^3 H(3) \text{ or } A_{i_k} = A_2 = \oplus_3^3 A(3) \text{ on } L^2(\mathbb{R}^3; \mathbb{C}^4) = \oplus L^2(\mathbb{R}^3; \mathbb{C}).$$

In this case $\Xi_{l,m}(\kappa_{l,m})$, if expressed as integral kernel operator

$$\Xi_{l,m}(\kappa_{l,m})$$

$$= \sum_{s_{i_k}, \mu_{i_k}(\mathbb{R}^3)^{l+m}} \int_{s_{i_k}, \mu_{i_k}(\mathbb{R}^3)^{l+m}} \kappa_{l,m}\left(\sum_{s_{i_k}, \mu_{i_k} \in \mathbb{R}^3, \mu_{i_k}} s_{i_k}, \mu_{i_k}, \ldots, \mu_{i_k}, p_{i_k}, \ldots, p_{i_k}, \ldots, p_{i_k}\right) \times$$

$$\times \partial_{s_{i_k}, \mu_{i_k}, \ldots, \mu_{i_k}}^{s_{i_k}, \mu_{i_k}, \ldots, \mu_{i_k}} \partial_{s_{i_k}, \mu_{i_k}, \ldots, \mu_{i_k}}^{s_{i_k}, \mu_{i_k}, \ldots, \mu_{i_k}} d^3 p_{i_1} \ldots d^3 p_{i_l} d^3 p_{i_{l+1}} \ldots d^3 p_{i_{l+m}},$$

transforming $(\mathcal{E}) \otimes \mathcal{E}$ into $(\mathcal{E})$, is understood as follows (compare [33]): the operators $\partial_{s_{i_k}, \mu_{i_k}}$ and $\partial_{s_{i_k}, \mu_{i_k}}$ as operators on $(\mathcal{E}) \otimes \mathcal{E} = (E_1) \otimes (E_2) \otimes \mathcal{E}$ are, respectively, shortened notation for $\left((\partial_{s_{i_k}, \mu_{i_k}}) \otimes \mathbb{1}_c\right)^*, \left((\partial_{s_{i_k}, \mu_{i_k}}) \otimes \mathbb{1}_c\right)^*$ and $(\partial_{s_{i_k}, \mu_{i_k}}) \otimes \mathbb{1}_c$, $(1 \otimes \partial_{s_{i_k}, \mu_{i_k}}) \otimes \mathbb{1}_c$, and $\kappa_{l,m}$ is an $\mathcal{L}(\mathcal{E}, \mathcal{C}) \cong \mathcal{E}^*$-valued distribution on $(\mathbb{R}^3)^{(l+m)}$, i.e. on the test space $E_{l_1} \otimes \cdots \otimes E_{l_{l+m}} ((l + m)\text{-fold tensor product})$ and this distribution $\kappa_{l,m}$ in the above formula for the integral kernel operator should be identified with $\mathbb{1}_{(\mathcal{E})} \otimes \kappa_{l,m}$.

Now any element $\Phi \in (\mathcal{E}) = (E_1) \otimes (E_2)$ has the unique absolutely convergent decomposition (compare [33], Prop. A.7)

$$\Phi = \sum_{n=0}^{\infty} \Phi_n, \quad \Phi_n \in \bigoplus_{n_1 + n_2 = n} E_1^{\otimes n_1} \otimes E_2^{\otimes n_2},$$

(66)

(here the tensor product $E_1^{\otimes n_1}$ is antisymmetrized $\otimes$ and symmetrized $\otimes$ in $E_2^{\otimes n_2}$). For any element

$$\Phi \otimes \phi \in (\mathcal{E}) \otimes \mathcal{E} = (E_1) \otimes (E_2) \otimes \mathcal{E}$$
and any $\mathcal{L}(\mathcal{E}, \mathbb{C})$-valued distribution

\[ \kappa_{l,m} \in \mathcal{L}(E_{l_1} \otimes \cdots \otimes E_{l_{i+m}}, \mathcal{L}(\mathcal{E}, \mathbb{C})) \cong \mathcal{L}(E_{l_1} \otimes \cdots \otimes E_{l_{i+m}}, \mathcal{E}^*) \]

we put after [38]

\[ \Xi_{l,m}(\kappa_{l,m})(\Phi \otimes \phi) = \sum_{n=0}^{\infty} \kappa_{l,m} \otimes_m (\Phi_{n+m} \otimes \phi). \]

Note that here $\otimes_m$ denotes the $m$-contraction of $\Phi_{n+m} \otimes \phi$ with the $\mathcal{L}(\mathcal{E}, \mathbb{C})$-valued distribution uniquely determined (after [38]) by the formula

\[ \langle \kappa_{l,m} \otimes_m (f_0 \otimes \phi), g_0 \rangle = \langle \kappa_{l,m}(g_0 \otimes_n f_0), \phi \rangle, \]

\[ f_0 \in E_{l_1} \otimes \cdots \otimes E_{l_m} \otimes E_{l_1} \otimes E_{l_{n+1}}, \]

\[ g_0 \in E_{l_1} \otimes \cdots \otimes E_{l_m} \otimes E_{l_1} \otimes \cdots \otimes E_{l_n}, \quad \phi \in \mathcal{E}. \]

It follows that for any

\[ \kappa_{l,m} \in \mathcal{L}(E_{l_1} \otimes \cdots \otimes E_{l_{i+m}}, \mathcal{L}(\mathcal{E}, \mathbb{C})) \cong \mathcal{L}(E_{l_1} \otimes \cdots \otimes E_{l_{i+m}}, \mathcal{E}^*) \cong \mathcal{L}(\mathcal{E}, (E_{l_1} \otimes \cdots \otimes E_{l_{i+m}})^*), \]

the operator $\Xi_{l,m}(\kappa_{l,m})$, defined by contraction $\otimes_m$ with $\kappa_{l,m}$, belongs to

\[ \mathcal{L}(\mathcal{E}) \otimes \mathcal{E}^*, (\mathcal{E}^*) \cong \mathcal{L}(\mathcal{E}, (\mathcal{E}^*)) \]

with a precise norm estimation (compare Thms. 3.6 and 3.9 of [38]). Moreover $\Xi_{l,m}(\kappa_{l,m})$ is uniquely determined by the formula

\[ \langle \langle \Xi_{l,m}(\kappa_{l,m})(\Phi \otimes \phi), \Psi \rangle \rangle = \langle \kappa_{l,m} (\eta_{\Phi, \Psi}), \phi \rangle, \quad \Phi, \Psi \in (E), \phi \in \mathcal{E}, \quad (67) \]

or equivalently

\[ \langle \langle \Xi_{l,m}(\kappa_{l,m})(\Phi \otimes \phi), \Psi \rangle \rangle = \langle \kappa_{l,m}(\phi), \eta_{\Phi, \Psi} \rangle = \langle \kappa_{l,m}(\eta_{\Phi, \Psi}), \phi \rangle, \quad \Phi, \Psi \in (E), \phi \in \mathcal{E}, \quad (68) \]

for $\kappa_{l,m}$ understood as an element of

\[ \mathcal{L}(E_{l_1} \otimes \cdots \otimes E_{l_{i+m}}, \mathcal{E}^*) \] or \[ \mathcal{L}(\mathcal{E}, (E_{l_1} \otimes \cdots \otimes E_{l_{i+m}})^*) \cong \mathcal{L}(E_{l_1} \otimes \cdots \otimes E_{l_{i+m}}, \mathcal{E}^*) \]

respectively in the first case (67) and in the second case (68). Here

\[ \eta_{\Phi, \Psi}(w_{i_1}, \ldots, w_{i_l}, w_{i_{l+1}}, \ldots, w_{i_{l+m}}) = \langle \langle \partial_{w_{i_1}}^* \cdots \partial_{w_{i_l}}^* \partial_{w_{i_{l+1}}} \cdots \partial_{w_{i_{l+m}}} \Phi, \Psi \rangle \rangle, \]

and $w_{i_k} = (s_{i_k}, k_{i_k})$ if $E_{i_k} = E_1$ or $w_{i_k} = (\mu_{i_k}, k_{i_k})$ if $E_{i_k} = E_2$. 

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Note that 

\[ \eta \Phi, \Psi \in E_{i_1} \otimes \cdots \otimes E_{i_{l+m}}. \]

The formula (67), or equivalently (68), justifies the identification of \( \Xi_{l,m}(\kappa_{l,m}) \), defined through the \( m \)-contraction \( \otimes_m \) with vector valued distribution \( \kappa_{l,m} \), with the integral kernel operator

\[
\Xi_{l,m}(\kappa_{l,m}) = \int_{(\mathbb{R}^3)^{l+m}} \kappa_{l,m}(w_{i_1}, \ldots, w_{i_l}, w_{i_{l+1}}, \ldots, w_{i_{l+m}}) \\
\times \frac{\partial}{\partial w_{i_1}} \cdots \frac{\partial}{\partial w_{i_l}} \frac{\partial}{\partial w_{i_{l+1}}} \cdots \frac{\partial}{\partial w_{i_{l+m}}} \, dw_{i_1} \cdots dw_{i_{l+m}}
\]

defined by \( \mathcal{L}(\mathcal{E}, \mathcal{C}) \)-valued distribution kernel \( \kappa_{l,m} \). Here of course

\[
\int_{\mathbb{R}^3} f(w) \, dw \overset{df}{=} \sum_{s=1}^4 \int_{\mathbb{R}^3} f(s, p) \, dl^3 p \quad \text{for} \quad w = (s, p),
\]

\[
\int_{\mathbb{R}^3} f(w) \, dw \overset{df}{=} \sum_{\mu=0}^3 \int_{\mathbb{R}^3} f(\mu, p) \, dl^3 p \quad \text{for} \quad w = (\mu, p),
\]

and we have put \( u_{jk} = w_{i_{s+k}}, k = 1,2,\ldots,m. \)

In our work we are especially interested in (the generalization of) Thm. 3.13 of [38], which gives necessary and sufficient condition for the \( \mathcal{L}(\mathcal{E}, \mathcal{C}) \cong \mathcal{E}^* \)-valued distribution \( \kappa_{l,m} \) in order that the corresponding \( \Xi_{l,m}(\kappa_{l,m}) \) be a continuous operator from \( (E) \otimes \mathcal{E} \) into \( (E) \), thus belonging to

\[
\mathcal{L}
\]

and thus determining a well defined operator-valued distribution on the test space \( \mathcal{E} \).

We formulate the generalization of Thm. 3.13 over to our tensor product of Fock spaces and the corresponding tensor product of Gelfand triples ([40] (of this Subsect.) and (272) (of Subsection 5.8 of [59]). We will use the (generalization of) Theorem 3.13 and Proposition 3.12 of [38] for the construction of free fields and in Subsection 2.6 and Section 3 when analysing the perturbative corrections (within the causal method of St"uckelberg-Bogoliubov) to interacting fields, as integral kernel operators with \( \mathcal{E}^* \)-valued kernels, in QED.

Exactly as for the analysis of integral kernel operators with scalar valued kernels, also the results and proofs of [38] for integral kernel operators with vector-valued kernels can be easily adopted to the fermi case, as well as for the more general case of several bose and fermi fields on the tensor product of the corresponding Fock spaces.

We have the following generalization of Thm. 3.13 of [38]:

\[
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\]
THEOREM 4. Let

\( \kappa_{l,m} \in \mathcal{L} \left( \bigotimes_{i=1}^{l+m} E_i, \mathcal{L}(\mathcal{E}, C) \right) \cong \mathcal{L} \left( E_i \otimes \cdots \otimes E_{i+m}, \mathcal{E}^* \right) \).

Then

\[ \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L} \left( (E) \otimes \mathcal{E}, (E) \right) \cong \mathcal{L} \left( \mathcal{E}, \mathcal{L}(\mathcal{E}, (E)) \right) \]

if and only if the bilinear map

\[ \xi \times \eta \mapsto \kappa_{l,m}(\xi \otimes \eta) \]

first \( l \) terms \( E_{i,j} \), \( i, j \in \{1, 2\} \)

\( \xi \in \bigotimes_{i=1}^{l} E_i \)

last \( m \) terms \( E_{i,j} \), \( i, j \in \{1, 2\} \)

\( \eta \in \bigotimes_{i=l+1}^{l+m} E_i \)

can be extended to a separately continuous bilinear map from

\[ \left( \bigotimes_{i=1}^{l} E_i \right)^* \times \left( \bigotimes_{i=l+1}^{l+m} E_i \right) \rightarrow \mathcal{L}(\mathcal{E}, C) = \mathcal{E}^* \]

This is the case if and only if for any \( k \geq 0 \) there exist \( r \in \mathbb{R} \) such that

\[ |\kappa_{l,m}|_{l, m, k, r, k} < \infty ; \text{ and moreover in this case for any } k \in \mathbb{R} \text{ and } q_0 < q_1 < q \text{ we have} \]

\[ \| \Xi_{l,m}(\kappa_{l,m})(\Phi \otimes \phi) \|_k \leq \rho^{q/2} \delta^{-1} \sigma^2 \sqrt{l^m \Delta^{(l+m)/2}} \times |\kappa_{l,m}|_{l, m, k+1, -(k+q+1), k+1} \| \Phi \|_{k+q+2}, \Phi \in (E), \phi \in \mathcal{E}. \]

Here for any linear map

\( \kappa_{l,m} : \bigotimes_{i=1}^{l+m} E_i \rightarrow \mathcal{L}(\mathcal{E}, C) = \mathcal{E}^* \)

and \( k, q, r \in \mathbb{R} \) we put (after [33]):

\[ |\kappa_{l,m}|_{l, m, k, q, r} = \sup \left\{ \sum_{i,j} |(\kappa_{l,m}(e(i) \otimes e(j)), \phi)|^2 |e(i)|^2 |e(j)|^2, \phi \in \mathcal{E}, |\phi|_r \leq 1 \right\}^{1/2} \cdot \]

Note that we are using the multiindex notation

\[ e(i) = e_{i_1} \otimes \cdots \otimes e_{i_l} \in E_{i_1} \otimes \cdots \otimes E_{i_l}, i = (i_1, \ldots, i_l) \]
\( e(j) = e_{j_1} \otimes \cdots \otimes e_{j_m} = e_{i_1} \otimes \cdots \otimes e_{i_m} \in E_{i_{i_1}} \otimes \cdots \otimes E_{i_{i_m}}, \)
\( j = (j_1, \ldots, j_m) = (i_{i_1+1}, \ldots, i_{i_m+1}), \)

but now \( e_{i_k} \) is the element of the complete orthonormal system of eigenvectors of the standard operator \( A_1 \) whenever \( e_{i_k} \in E_{i_k} = E_1 \) or of the standard operator \( A_2 \) whenever \( e_{i_k} \in E_{i_k} = E_2 \). Note also that with the system of eigenvalues (counted with multiplicity)

\[ \lambda_{i_0}, \lambda_{i_1}, \lambda_{i_2}, \ldots \text{ of } A_i, \]

we have put here

\[ \delta = \max_{i=1,2} \left( \sum_{j=0}^{\infty} \lambda_{ij} \right)^{1/2} = \|A_i^{-1}\|_{\text{HS}} < \infty \]

for the maximum of the Hilbert-Schmidt norms of the nuclear operators \( A_i^{-1}, i = 1, 2 \). Similarly here

\[ \rho = \max_{i=1,2} \|A_i^{-1}\|_{\text{op}} \]

for the operator norm \( \| \cdot \|_{\text{op}} \). Here

\[ \Delta_q = \max_{i=1,2} \Delta_{q,i}, \quad q > \max_{i=1,2} q_{0i} = q_0 \]

where for \( i = 1, 2 \)

\[ \Delta_{q,i} = \frac{\delta_i}{-c q^{q/2} \ln(\delta_i^2 \rho_i^q)}, \quad q > q_{0i} = \inf \{ q > 0, \delta_i^2 \rho_i^q \leq 1 \} \]

is a finite constant uniquely determined by the standard operator \( A_i, i = 1, 2 \), if \( q > q_{0i} \) for the positive constant \( q_{0i} \) again depending on \( A_i \), compare [38], p. 210. Recall that

\[ \delta_i = \left( \sum_{j=0}^{\infty} \lambda_{ij} \right)^{1/2} = \|A_i^{-1}\|_{\text{HS}} \quad \rho_i = \|A_i^{-1}\|_{\text{op}}. \]

Finally

\[ \sigma = (\inf \text{ Spec } B)^{-1} = \|B^{-1}\|_{\text{op}} \]

for the standard operator \( B = B_{p_1} \otimes \cdots \otimes B_{p_M}, p_k \in \{1, 2\} \) on \( \otimes_{k=1}^{M} L^2(\mathbb{R}^4; \mathbb{C}^q_k) \), defining the nuclear test space

\[ \mathcal{E}' = S_B(\cup \mathbb{R}^{4M}; \mathbb{C}) \]
\[ = S_{B_{p_1}}(\mathbb{R}^4; \mathbb{C}^{q_1}) \otimes \cdots \otimes S_{B_{p_M}}(\mathbb{R}^4; \mathbb{C}^{q_M}) \subset L^2(\cup \mathbb{R}^{4M}; \mathbb{C}) = \otimes_{k=1}^{M} L^2(\mathbb{R}^4; \mathbb{C}^{q_k}) \]

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Recall once more that here
\[ B_{p_k} = \mathcal{F}^{-1} \oplus A^{(4)} \mathcal{F} \quad \text{on} \quad \oplus_{k=1}^{Q_0} L^2(\mathbb{R}^4; \mathbb{C}) = L^2(\mathbb{R}^4; \mathbb{C}^{Q_0}), \quad \text{for} \quad p_k = 2 \]
with the Hamiltonian operator \( H_{(4)} \) on \( L^2(\mathbb{R}^4; \mathbb{C}) \) of the 4-dimensional oscillator, compare Appendix 9 of [59].

The standard operator \( A^{(4)} \) on \( L^2(\mathbb{R}^4; \mathbb{C}) \) is defined in Subsection 5.3 of [59].

\[ \mathcal{E}_{p_k} = \mathcal{S}_{B_{p_k}}(\mathbb{R}^4; \mathbb{C}^{Q_0}) = \mathcal{S}_{\mathcal{H}(\mathcal{A}^{(k)})}(\mathbb{R}^4; \mathbb{C}^{Q_0}), \quad p_k = 1 \]
\[ \mathcal{E}_{p_k} = \mathcal{S}_{B_{p_k}}(\mathbb{R}^4; \mathbb{C}^{Q_0}) = \mathcal{S}_{\mathcal{F} \oplus \mathcal{A}^{(k)}}(\mathbb{R}^4; \mathbb{C}^{Q_0}), \quad p_k = 2. \]  

Recall that
\[ |\phi|_r \triangleq |B^{-r}\phi|_0 = |(B_{p_1} \otimes \cdots \otimes B_{p_M})^{-r}\phi|_0 = |(B_{p_1} \otimes \cdots \otimes B_{p_M})^{-r}\phi|_{\oplus_{k=1}^{Q_0} L^2(\mathbb{R}^4; \mathbb{C}^{Q_0})}, \quad \phi \in \mathcal{E}, r \in \mathbb{R}. \]

Recall that in computation of the operator or Hilbert-Schmidt norm the unitary Fourier transform \( \mathcal{F} \) in definition of \( B_2 \) can be ignored and the respective norms can be simply computed for \( \oplus A^{(4)} \).

From Thm. 4 we obtain the following

**COROLLARY 2.** The Dirac free field

\[ \psi = \chi_{0,1}(\kappa_{0,1}) + \chi_{1,0}(\kappa_{1,0}) \in \mathcal{L}((E) \otimes \mathcal{E}, (E)^*) \cong \mathcal{L}(\mathcal{E}, \mathcal{L}((E), (E)^*)) \]

understood as integral kernel operator with vector-valued distributions

\[ \kappa_{0,1}, \kappa_{1,0} \in \mathcal{L}(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4), \mathcal{E}^*) \cong \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^* \otimes \mathcal{E}^* \]

belongs to \( \mathcal{L}((E) \otimes \mathcal{E}, (E)) \equiv \mathcal{L}(\mathcal{E}, \mathcal{L}((E), (E))), \) i.e.

\[ \psi = \chi_{0,1}(\kappa_{0,1}) + \chi_{1,0}(\kappa_{1,0}) \in \mathcal{L}((E) \otimes \mathcal{E}, (E)) \cong \mathcal{L}(\mathcal{E}, \mathcal{L}((E), (E))), \]

if and only if the map \( \phi \mapsto \kappa_{1,0}(\phi) \) belongs to

\[ \mathcal{L}(\mathcal{E}, \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)), \]

i.e. if and only if \( \kappa_{1,0} \) can be extended to a map belonging to

\[ \mathcal{L}(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^*, \mathcal{E}^*) \cong \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^* \otimes \mathcal{E}^* \]

\[ \cong \mathcal{E}^* \otimes \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) \cong \mathcal{L}(\mathcal{E}, \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)). \]

Here of course we have the special case of Thm. 4 with the tensor product of the two Fock spaces (corresponding to the Dirac field and the electromagnetic...
potential field) degenerated to just one Fock space – that corresponding to the Dirac field, and with the Hida space \( (E) = (E_1) \otimes (E_2) \) degenerated to just the Hida space \((E_1) \overset{df}{=} (S_A(\mathbb{R}^3; \mathbb{C}^4)) = (S_{\otimes H_2}(\mathbb{R}^3; \mathbb{C}^4)) \) corresponding to the Dirac field, with the standard operator \( A_1 = A_1 = \oplus H_3 \) given by (30); and finally with \( M = 1 \) and \( B \) degenerated to \( B_1 \) with the nuclear test space \( \mathcal{E} \) degenerated to

\[
\mathcal{E} = \mathcal{S}_B(l \mathbb{R}^4; \mathbb{C}) = \mathcal{S}_D(\mathbb{R}^4; \mathbb{C}^4) = \mathcal{S}_{\otimes H_4}(\mathbb{R}^4; \mathbb{C}^4) = \mathcal{S}(\mathbb{R}^4; \mathbb{C}^4) = \mathcal{E}_1
\]

of (70).

Equivalently we may consider here the integral kernel operator \( \psi = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0}) \) as acting in the said tensor product of two Fock spaces, having the form of sum of tensor product operators on \((E) = (E_1) \otimes (E_2)\) with the second factor operators acting on the second factor \((E_2)\) trivially as the unit operator, in accordance with the identification of the operator

\[
\partial_w = \begin{cases} 
\partial_{s,p} \otimes 1, & \text{if } w = (s,p) \text{ refers to fermi variables}, \\
1 \otimes \partial_{\mu,p}, & \text{if } w = (\mu,p) \text{ refers to bose variables},
\end{cases}
\]

in the general formula (69). But now we have to replace the general formula (69) defining the operators \( \Xi_{0,1}(\kappa_{0,1}) \), \( \Xi_{1,0}(\kappa_{1,0}) \) giving the Dirac field, with another one in which the integration variables are restricted only to the fermi variables. This is not the special case of (69) for \( l = 0, m = 1 \) (or \( l = 1, m = 0 \)) of an integral operator in the tensor product of Fock spaces, because this is not true that the kernels \( \kappa_{0,1}, \kappa_{1,0} \) inserted into the general formula (69) cancel out the unwanted boson variables. Thus \( \psi = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0}) \) considered as acting in the said tensor product of two Fock spaces is a special integral kernel operator with integration variables restricted to fermion variables. Similarly we have for the electromagnetic potential field, if considered as integral kernel operator in the said tensor product of Fock spaces: it is an exceptional integral kernel operator with the integration variables in the general formula (69) restricted only to boson variables.

From the Corollary 2 and Lemma 4 it follows

**COROLLARY 3.** Let

\[
\psi = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0}) \in \mathcal{L}(\mathbb{E} \otimes \mathcal{E}, (E)^* \otimes \mathbb{E}^*) \cong \mathcal{L}(\mathcal{E}, \mathcal{L}(\mathbb(E), (E)^*))
\]

be the Dirac field understood as an integral kernel operator with vector-valued kernels

\[
\kappa_{0,1}, \kappa_{1,0} \in \mathcal{L}(S_A(\mathbb{R}^3; \mathbb{C}^4), \mathcal{E}^*) \cong S_A(\mathbb{R}^3, \mathbb{C}^4)^* \otimes \mathcal{E}^*,
\]

defined by (71) and (72). Then the Dirac field operator

\[
\psi = \psi^- + \psi^+ = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0}),
\]

belongs to \( \mathcal{L}(\mathbb{E} \otimes \mathcal{E}, (E)) \cong \mathcal{L}(\mathcal{E}, \mathcal{L}(\mathbb(E), (E))) \), i.e.

\[
\psi = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0}) \in \mathcal{L}(\mathbb{E} \otimes \mathcal{E}, (E)) \cong \mathcal{L}(\mathcal{E}, \mathcal{L}(\mathbb(E), (E)))
\]
which means in particular that the Dirac field $\psi$, understood as a sum $\psi = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0})$ of two integral kernel operators with vector-valued kernels, defines an operator valued distribution through the continuous map

$$\delta' \ni \varphi \mapsto \Xi_{0,1}(\kappa_{0,1}(\varphi)) + \Xi_{1,0}(\kappa_{1,0}(\varphi)) \in \mathcal{L}((E), (E)).$$

Note here that the last Corollary follows immediately from the proved equality \[60\], i.e. Lemma \[3\] and continuity of the restriction to the orbit $\mathcal{O}_{m, 0, 0, 0}$ regarded as a map $\mathcal{S}(\mathbb{R}^4; \mathbb{C}) \to \mathcal{S}(\mathbb{R}^4; \mathbb{C})$.

We have introduced the decomposition of the Dirac field operator $\psi$ into the positive and negative frequency parts after the classic physical tradition

$$\psi(-) \overset{df}{=} \Xi_{0,1}(\kappa_{0,1}), \quad \psi(+) \overset{df}{=} \Xi_{1,0}(\kappa_{1,0}).$$

Thus as a Corollary to Thm. \[4\] we have obtained the Dirac field $\psi$ as a sum of two integral kernel operators with vector valued kernels $\kappa_{0,1}, \kappa_{1,0}$ \[61\] and \[62\]. But as we have seen the (free) Dirac field $\psi$ (and in general a quantum free field understood as sum of integral kernel operators with vector-valued kernels) is naturally an integral kernel operator with well defined kernel equal to (scalar) integral kernel operator

$$\psi^a(x) = \sum_{s=1}^{4} \int_{\mathbb{R}^3} \kappa_{0,1}(s, p; a, x) \partial_{s,p} \, d^3p + \sum_{s=1}^{4} \int_{\mathbb{R}^3} \kappa_{1,0}(s, p; a, x) \partial_{s,p}^* \, d^3p$$

$$= \sum_{s=1}^{2} \int_{\mathbb{R}^3} \frac{1}{2\|p_0(p)\|} v_x^a(p)e^{-ip\cdot x} \, d^3p + \sum_{s=1}^{2} \int_{\mathbb{R}^3} \frac{1}{2\|p_0(p)\|} v_x^a(p)e^{ip\cdot x} \, d^3p$$

$$= \sum_{s=1}^{2} \int_{\mathbb{R}^3} \frac{1}{2\|p_0(p)\|} v_x^a(p)e^{-ip\cdot x} a_x(p) \, d^3p + \sum_{s=1}^{2} \int_{\mathbb{R}^3} \frac{1}{2\|p_0(p)\|} v_x^a(p)e^{ip\cdot x} a_{x+2}(p)^+ \, d^3p$$

$$= \sum_{s=1}^{2} \int_{\mathbb{R}^3} \frac{1}{2\|p_0(p)\|} v_x^a(p)e^{-ip\cdot x} b_x(p) \, d^3p + \sum_{s=1}^{2} \int_{\mathbb{R}^3} \frac{1}{2\|p_0(p)\|} v_x^a(p)e^{ip\cdot x} d_x(p)^+ \, d^3p.$$  \hspace{1cm} (71)

with $p = (\|p_0(p)\|, p) \in \mathcal{O}_{m, 0, 0, 0}$,

and where we have put $b_{s=1}(p), b_{s=2}(p), d_{s=1}(p), d_{s=2}(p)$, respectively, for the operators $b_{s=1}(p), b_{s=1}(p), d_{s=1}(p), d_{s=1}(p)$ used in \[46\], p. 82, just changing the names of the summation index from $\{1, -1\}$ into $\{1, 2\}$. Here the expressions in \[71\], for each fixed space-time point $x$, are not merely symbolic, but they are meaningful integral kernel operators transforming continuously the Hida space $(E)$ into its strong dual $(E)^*$, and moreover even the integral signs in these expressions are not merely symbolic, but are meaningful (point-wise) Pettis integrals (compare \[27\], or Subsection 5.8 of \[59\]).
We see that there is an additional weight \( |p_0(p)|^{-1} \) factor under the integration sign in our formula for the local free Dirac field \( \psi(x) \) in our formula (71) in comparison to the standard formula for the free quantum Dirac field used in other books, compare [46] formula (2.2.33) or the formula (7.32) of [6] (with the respective amplitudes \( a_\psi \) replaced with the creation-annihilation operators). Our field \( \psi \) (71) and the standard Dirac field, given by the formula (2.2.33) of Subsection 2.8, although not equal, are mutually unitary isomorphic in a sense explained in Subsection 2.8. Nonetheless there are important differences between these two realizations of the field \( \psi \). We explain them in more details in Subsection 2.8.

2.7 Fundamental rules for computations involving free fields understood as integral kernel operators with vector-valued kernels

In this Subsection we give several useful computational rules, performed upon integral kernel operators \( \Xi_{l,m}(\kappa_{l,m}) \) determined by \( \mathcal{L}(\mathcal{E}, \mathbb{C}) \)-valued distributions, \( \kappa_{l,m} \), respecting the extendibility condition of Thm. 4 of the preceding Subsection 2.6 (or resp. of Thm. 3.13 of [38]). This property allows to treat such \( \Xi_{l,m}(\kappa_{l,m}) \) as well defined operator-valued distributions on the standard nuclear test space \( \mathcal{E} \), which in our case will always be equal to the tensor product

\[
\mathcal{E} = \mathcal{E}_{n_1} \otimes \cdots \otimes \mathcal{E}_{n_M}, \quad n_k \in \{1, 2\},
\]

of \( M \) space-time test spaces \( \mathcal{E}_1, \mathcal{E}_2 \) given by (70), Subsection 2.6 with \( M = 1 \) and \( p_k \) put equal \( n_k \). We encounter the cases with \( M = 1 \) and (operator-valued distributions with one space-time variable) or with \( M > 1 \) space-time variables. In fact the integral kernel operators which are of importance for us are of still more special character, being obtainable from the integral kernel operators defined by the free fields underlying the considered Quantum Field Theory, as a result of special operations: composition of Wick product, differentiation, integration and convolution with pairing functions.

Having in view the causal perturbative QED we confine attention to integral kernel operators \( \Xi_{l,m}(\kappa_{l,m}) \) in the tensor product of just two Fock spaces – the first one fermionic and corresponding to the Dirac field and the second one bosonic and corresponding to the electromagnetic potential field, compare Subsection 2.6. Thus considered here integral kernel operators \( \Xi_{l,m}(\kappa_{l,m}) \) act on the Hida space \( (\mathcal{E}) = (E_1) \otimes (E_2) \subset \Gamma_{\text{Fermi}}(L^2(\mathbb{R}^3; \mathbb{C}^4)) \otimes \Gamma_{\text{Bose}}(L^2(\mathbb{R}^3; \mathbb{C}^4)) \), constructed as in the previous Subsection 2.6. We have also formulated the Thm. 4 Subsection 2.6 for the said tensor product of the two mentioned above Fock spaces. Of course analogous Theorem and corresponding rules of calculation with integral kernel operators \( \Xi_{l,m}(\kappa_{l,m}) \) are valid on tensor product of more than just two indicated Fock spaces.

\[\text{In the formula (2.2.33) of [46] the summation sign over } s \text{ has been lost (of course by a trivial misprint), and the additional irrelevant constant factors equal to the respective powers of } 2\pi \text{ appear in the literature which are lost in our formula because we have not normalized the measures when using Fourier transformations.}\]
The space \( E_1 = S_{A_1}(\mathbb{R}^3; \mathbb{C}^4) = S(\mathbb{R}^3; \mathbb{C}^4) \) with index 1 and the standard operator \( A_1 = A \) refers to the standard nuclear space in (10), corresponding to the Dirac field, with the space-time test space \( \delta_1 = S_{\mathcal{B}H_{\mu}}(\mathbb{R}^4; \mathbb{C}^4) = S(\mathbb{R}^4; \mathbb{C}^4) \). The space \( E_2 = S_{A_2}(\mathbb{R}^3; \mathbb{C}^4) = S^0(\mathbb{R}^3; \mathbb{C}^4) \) with index 2 is the nuclear space \( E \) determined by the standard operator \( A_2 = \oplus^3 A^{(3)} = A \), which enters the triple in (272) of Subsect. 5.8 of [59], and which serves to define the free quantum electromagnetic potential field, Subsection 5.8 of [59], with the space-time test space \( \delta_2 = S_{\mathcal{B}H_{\mu,A^{(0)}}}(\mathbb{R}^4; \mathbb{C}^4) = S^{00}(\mathbb{R}^4; \mathbb{C}^4) \).

The vector-valued distributions \( \kappa_{0,1}, \kappa_{1,0} \in \mathcal{L}(E_1, \delta^+_1) \) determined by the plane wave kernels (61) and (62), defining the free Dirac field as the integral kernel operator
\[
\psi = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0}) = \psi^-(\kappa) + \psi^+(\kappa),
\]
and in general the vector-valued plane-wave distributions \( \kappa_{0,1}, \kappa_{1,0}, \ldots \) defining all free quantum fields of the theory play a fundamental role in the theory. In QED we encounter besides the plane waves (61) and (62) the plane waves \( \kappa_{0,1}, \kappa_{1,0} \in \mathcal{L}(E_2, \delta^+_2) \) (120). Subsection 2.10 defining the free quantum electromagnetic potential field:
\[
A = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0}) = A^-(\kappa) + A^+(\kappa),
\]
if we change slightly the convention (used by mathematicians) of Subsection 2.10 and use for \( \partial^*_w \) in the general integral kernel operator (109), on the tensor product of Fock spaces of the Dirac field \( \psi \) and the electromagnetic potential field \( A \), the operators \( \eta \partial^*_{\mu,p} \eta \) whenever \( w = (\mu, p) \) corresponds to the photon variables \( \mu, p \) in (92), instead of the ordinary transposed operators \( \partial^*_{\mu,p} \). Here \( \eta \) is the Gupta-Bleuler operator. This convention fits well with notation used by physicists, as they are using the Krein-adjoined annihilation operators of the photon variables in Fock normal expansions.

Indeed in terms of these kernels \( \kappa_{0,1}, \kappa_{0,1}, \ldots \) all important quantities of the theory are expressed:

1) The Wick polynomials of free fields are expressed through (symmetrized in boson variables or respectively antisymmetrized in fermi variables) tensor product operation performed upon the plane wave kernels \( \kappa_{0,1}, \kappa_{1,0}, \ldots \) defining the free fields of the theory,

2) Wick polynomial of free fields at the same space-time point are expressed through the symmetrized or antisymmetrized in \( \xi_1, \ldots, \xi_M \) operation of pointwise product \( \kappa_{l_1,m_1}(\xi_1) \cdot \kappa_{l_1,m_1}(\xi_1) \cdot \ldots \cdot \kappa_{l_M,m_M}(\xi_M) \) utilizing the fact that \( \kappa_{0,1}(\xi), \kappa_{1,0}(\xi), \kappa_{0,1}(\xi), \kappa_{1,0}(\xi), \ldots \) belong to the algebra of multipliers of the respective nuclear algebra \( \delta_i = S_{\mathcal{B}i}(\mathbb{R}^4; \mathbb{C}^4) \) (equal \( S(\mathbb{R}^4; \mathbb{C}^4) \) or respectively \( S^{00}(\mathbb{R}^4; \mathbb{C}^4) \)) of spaces of space-time test functions, and the fact that the maps
\[
E_i \times E_j \ni \xi \times \zeta \mapsto \kappa_{l,0}(\xi) \cdot \kappa_{l,0}(\zeta) \in \delta^*_k,
\]
\[i, j, k \in \{1, 2\},\]
are jointly continuous in the ordinary nuclear topology on $E_i$ and strong dual topology on $E_k^*$ which secures the Wick product to be a well defined integral kernel operator belonging to

$$\mathcal{L}((E) \otimes \mathcal{S}, (E)^*)$$

for $\mathcal{S}$ equal to the test function space $\mathcal{S}_1 = \mathcal{S}(\mathbb{R}^4)$ as well as for $\mathcal{S}_2 = \mathcal{S}^{00}(\mathbb{R}^4)$. Moreover if among the integral kernel operators defined by the plane waves defining free fields there are no factors corresponding to zero mass free fields, then

$$E_i^* \times E_j^* \subset E_i \times E_j \ni \xi \times \zeta \mapsto \kappa_{1,0}(\xi) \cdot \kappa'_{1,0}(\zeta) \in E_k^*,$$

$$i,j,k \in \{1,2\},$$

defined through ordinary point-wise product $\cdot$, are hypocontinuous in the topology inherited from the strong dual topology on $E_i^*$, and strong dual topology on $E_j^*$, which secures in this case the Wick product to be an integral kernel operator which belongs even to

$$\mathcal{L}((E) \otimes \mathcal{S}, (E)^*) \cong \mathcal{L}(\mathcal{S}, \mathcal{L}((E),(E)))$$

for $\mathcal{S}$ equal to the test function space $\mathcal{S}_1 = \mathcal{S}(\mathbb{R}^4)$ as well as for $\mathcal{S}_2 = \mathcal{S}^{00}(\mathbb{R}^4)$.

3) The perturbative contributions to interacting fields are expressed through convolutions of the kernels corresponding to Wick polynomials of free fields with the respective pairing “generalized functions”, and utilizing the fact that $\kappa_{0,1}(\xi_{n_1}), \kappa_{0,1}(\xi_{n_2}), \kappa'_{0,1}(\xi_{n_2}), \ldots$, and their pointwise products with $\xi_{n_k} \in \mathcal{S}_{A_{n_k}}(\mathbb{R}^3, \mathbb{C}^4)$ belong to the algebra of convolutors of the respective nuclear algebra $\mathcal{S}_{n_k}$ ($n_k \in \{1,2\}$).

In all these constructions we apply the Theorem and check validity of the condition stated in this Theorem, asserting that the constructed integral kernel operator belongs to

$$\mathcal{L}((E) \otimes \mathcal{S}, (E)) \cong \mathcal{L}(\mathcal{S}, \mathcal{L}((E),(E)))$$

and defines an operator-valued distribution on the corresponding test space $\mathcal{S}$. Alternatively we check that the constructed operator $\Xi(\kappa)$ has the kernel which respect weaker condition $[64]$.

$$\kappa \in \mathcal{L}(E_{n_1}, \ldots, E_{n_{i+m}}, \mathcal{S}^*)$$

which means by the generalization to tensor product of Fock spaces of Thm. 3.9 (compare Subsection 2.6) that the integral kernel operator belongs to

$$\mathcal{L}((E) \otimes \mathcal{S}, (E)^*) \cong \mathcal{L}(\mathcal{S}, \mathcal{L}((E),(E))^*).$$

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In general it cannot be asserted\footnote{At least the author has been not able to prove that the Wick product of integral kernel operators corresponding to zero mass fields or their derivatives belongs to $\mathcal{L}((E) \otimes \mathcal{E}, (E))$.} that the integral kernel operator $\Xi$ represented by the Wick product $\Xi$ of integral kernel operators defined by free fields belonging to $\mathcal{L}((E) \otimes \mathcal{E}, (E))$, belongs to $\mathcal{L}((E) \otimes \mathcal{E}, (E))$. This would be true only for the Wick product (at the fixed space-time point) $\Xi$ of integral kernel operators corresponding to massive free fields (such as Dirac field) or their derivatives. But if among the factors in the Wick product there are present integral kernel operators corresponding to zero mass fields (or their derivatives), then their Wick product (at the fixed space-time point) $\Xi$ represents a general integral kernel operator (with vector valued kernel) $\Xi(\kappa)$ which belongs to $\mathcal{L}((E) \otimes \mathcal{E}, (E)^*)$.

Therefore for any test function $\phi \in \mathcal{E}$ this Wick product operator $\Xi(\kappa)$ can be evaluated $\langle \langle \Xi(\kappa)(\Phi \otimes \phi), \Psi \rangle \rangle = \langle \langle \Xi(\kappa(\phi))\Phi, \Psi \rangle \rangle$ at $\Phi \otimes \phi$ and $\Phi, \Psi \in (E)$, and for fixed $\Phi, \Psi \in (E)$ represents a scalar distribution (as a function of $\phi \in \mathcal{E}$ compare (67) or (67)). Otherwise: for any test function $\phi \in \mathcal{E}$ the Wick product operator $\Xi(\kappa(\phi))$ can be evaluated at $\Phi, \Psi \in (E)$, and gives the value $\langle \langle \Xi(\kappa(\phi))\Phi, \Psi \rangle \rangle$, which is equal to a distribution (as a functional of the space-time test function $\phi$). This is what might have been expected since the very work of Wick himself or from the analysis of Bogoliuov and Shirkov [6], which already suggested that the general Wick product of free fields determines, at each fixed space-time point, is a well defined sesquilinear form for states ranging over a suitable dense domain.

But what is most important each order contribution to interactig Dirac and electromagnetic potential field, has the form of a finite sum of integral kernel operators

$$\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}((E) \otimes \mathcal{E}, (E)^*)$$

respectively with $\mathcal{E}_i^*$ -valued kernels $\kappa_{l,m}$, $i = 1, 2$, exactly as for the Wick polynomials of free fields (at fixed space-time point), ant thus represent objects of the same class as the Wick polynomials of free fields, i.e. finite sums of well defined integral kernel operators with vector-valued kernels. Moreover the full interactig Dirac field and the interacting electromagnetic field (in all orders) have the form of Fock expansions (in the sense of [38])

$$\sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}),$$

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which can be subject to computationally effective convergence criteria of \cite{38}, utilizing symbol calculus of Obata.

Thus all operators considered by the theory: free fields, Wick products of their derivatives, and contributions to interacting fields are all finite sums of integral kernel operators in the sense of Obata \cite{38} introduced in Subsection 2.6. Among them the free field operators, their derivatives and Wick polynomials of derivatives of massive fields behave most “smoothly” and belong to

\[ \mathcal{L}'((E) \otimes \mathcal{E}', (E)) \cong \mathcal{L}(\mathcal{E}, \mathcal{L}((E),(E))). \]

General Wick polynomials of derivatives of free fields (including zero mass fields) and contributions to interacting fields, of which we can say that belong to the general class of integral kernel operators, belong to

\[ \mathcal{L}'((E) \otimes \mathcal{E}', (E)^\star) \cong \mathcal{L}(\mathcal{E}, \mathcal{L}((E),(E)^\star)), \]

and are in this sense slightly more singular integral kernel operators than the free fields themselves. In particular we cannot say that they are operator-valued distributions in the white noise sense but nonetheless, when evaluated at fixed elements of Hida subspace of the Fock space, they represent scalar-valued distributions on the space-time test function space \( \mathcal{E}_2 \) and \( \mathcal{E}_2 \).

Thus we start with the fundamental integral kernel operators \( \Xi_{0,1}(\kappa_{0,1}), \Xi_{1,0}(\kappa_{1,0}) \) defined by the free fields of the theory. But we should distinguish the free field integral kernel operators

\[ \psi = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0}), \quad A = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0}), \]

acting in their own (resp. fermionic or bosonic) Fock spaces from the corresponding free field integral kernel operators

\[ \psi = \Xi_{0,1}(^1\kappa_{0,1}) + \Xi_{1,0}(^1\kappa_{1,0}), \quad A = \Xi_{0,1}(^2\kappa_{0,1}) + \Xi_{1,0}(^2\kappa_{1,0}), \]

both acting in the tensor product Fock space. In the last case the integral kernel operators \( \Xi_{0,1}(^1\kappa_{0,1}), \Xi_{1,0}(^1\kappa_{1,0}) \) are defined by the integral formula \( (69) \) in which the integration is restricted to fermi variables \( w \) only, and the operators \( \Xi_{0,1}(^1\kappa_{0,1}), \Xi_{1,0}(^1\kappa_{1,0}) \) act trivially as unit operators on the second factor. Here \( ^1\kappa_{0,1}, ^1\kappa_{0,1} \) are exactly the kernels \( (61) \) and \( (62) \) corresponding to the Dirac field, and denoted with the additional left-handed-superscript \( 1 \), in order to distinguish them from the kernels \( ^2\kappa_{0,1}, ^2\kappa_{0,1} \) \( (126) \). Subsection 2.10 in \( A = \Xi_{0,1}(^2\kappa_{0,1}) + \Xi_{1,0}(^2\kappa_{1,0}) \) acting trivially on the first factor in the tensor product of Fock spaces, and defined by the formula \( (69) \) in which the integration is restricted to bose variables \( w \) only.

And generally kernels \( \kappa_{0,1}, \kappa_{1,0} \) respecting the condition of Lemma 4 Subsection 2.6 corresponding to integral kernel operators which act trivially as unit operators on the second bosonic Fock space factor with integration in their definition restricted to fermi variables, will be denoted by \( ^1\kappa_{0,1}, ^1\kappa_{0,1} \) with the additional superscript \( 1 \); and vice versa for kernels corresponding to integral...
kernel operators acting trivially on the first fermionic Fock space factor with integration in their definition restricted to boson variables, denoted by $2\kappa_{0,1}, 2\kappa_{0,1}$ with the additional left-handed superscript $2$.

Thus we start with the following fundamental integral kernel operators

$$
\Xi_{0,1}(1\kappa_{0,1}), \Xi_{1,0}(1\kappa_{1,0}), \Xi_{0,1}(2\kappa_{0,1}), \Xi_{1,0}(2\kappa_{1,0}),
$$
determined by the free fields of the theory and their derivatives, corresponding to vector-valued distributions

$$
1\kappa_{0,1}, 1\kappa_{1,0} \in \mathcal{L}(E_1, \mathcal{E}_1^*) \cong E_1 \otimes \mathcal{E}_1^*,
$$

$$
2\kappa_{0,1}, 2\kappa_{1,0} \in \mathcal{L}(E_2, \mathcal{E}_2^*) \cong E_2 \otimes \mathcal{E}_2^*,
$$

which have the property that they can be (uniquely) extended to elements (denoted by the same symbols)

$$
1\kappa_{0,1}, 1\kappa_{1,0} \in \mathcal{L}(E_1^*, \mathcal{E}_1^*) \cong E_1 \otimes \mathcal{E}_1^*,
$$

$$
2\kappa_{0,1}, 2\kappa_{1,0} \in \mathcal{L}(E_2^*, \mathcal{E}_2^*) \cong E_2 \otimes \mathcal{E}_2^*,
$$

$$
1\kappa_{0,1}(\xi), 1\kappa_{1,0}(\xi) \in \mathcal{O}_C = \mathcal{O}_{CB_1} \subset \mathcal{O}_{CB_2}^* \text{ if } \xi \in E_1,
$$

$$
2\kappa_{0,1}(\xi), 2\kappa_{1,0}(\xi) \in \mathcal{O}_C \subset \mathcal{O}_{CB_2}^* \text{ if } \xi \in E_2,
$$

compare Lemma 4, Subsection 2.6 (for the kernels defining Dirac field), and respectively Lemma 9, Subsection 2.9 (for the kernels defining the electromagnetic potential field. Here $\mathcal{O}_C(\mathbb{R}^4), \mathcal{O}_{CB_1}^*(\mathbb{R}^4)$ denote the algebras of convolutors, respectively, of $S_{B_1}(\mathbb{R}^4) = S(\mathbb{R}^4), S_{B_2}(\mathbb{R}^4) = S^{\text{even}}(\mathbb{R}^4)$, and $\mathcal{O}_C(\mathbb{R}^4), \mathcal{O}_{CB_2}^*(\mathbb{R}^4)$ are their preduals, compare Appendix B. Because all the spaces $E_i, E_i^*, \mathcal{E}_i, \mathcal{E}_i^*$, $i = 1, 2$, are nuclear then we have natural topological inclusions

$$
\mathcal{L}(E_i^*, \mathcal{E}_i^*) \cong E_i \otimes \mathcal{E}_i^* \subset E_i^* \otimes \mathcal{E}_i^* \cong \mathcal{L}(E_i, \mathcal{E}_i^*), \quad i = 1, 2
$$

induced by the natural topological inclusions $E_i \subset E_i^*$ in both cases: if we endow $E_i$ with the topologies on $E_i$ inherited from $E_i^*$ and with their ordinary nuclear topologies, compare Prop. 43.7 and its Corollary in [58]. In the first case we obtain isomorphic inclusions by the cited Proposition, as in case of nuclear spaces the projective tensor product coincides with the equicontinuous and thus with the essentially unique tensor product in this category of linear topological spaces, compare [58]. Therefore we simply have

$$
1\kappa_{0,1}, 1\kappa_{1,0} \in \mathcal{L}(E_1^*, \mathcal{E}_1^*) \cong E_1 \otimes \mathcal{E}_1^*,
$$

$$
2\kappa_{0,1}, 2\kappa_{1,0} \in \mathcal{L}(E_2^*, \mathcal{E}_2^*) \cong E_2 \otimes \mathcal{E}_2^*,
$$

$$
1\kappa_{0,1}(\xi), 1\kappa_{1,0}(\xi) \in \mathcal{O}_C = \mathcal{O}_{CB_1} \text{ if } \xi \in E_1,
$$

$$
2\kappa_{0,1}(\xi), 2\kappa_{1,0}(\xi) \in \mathcal{O}_C \subset \mathcal{O}_{CB_2}^* \text{ if } \xi \in E_2.
$$

Recall that in case of kernels $1\kappa_{0,1}, 1\kappa_{1,0}$, respectively, $2\kappa_{0,1}, 2\kappa_{1,0}$, defining the free fields $\psi, A$ we have the spacetime test spaces $\mathcal{E}_1$, respectively, $\mathcal{E}_2$, given by
the formula (70) with \( p_k = n_k = 1 \), and respectively, \( p_k = n_k = 2 \) and with \( q_k = 4 \) and \( M = 1 \) in (70).

In fact we have two possible realizations of the free Dirac field \( \psi \), having different commutation functions and pairings, which nonetheless are a priori equally good form the point of view of causal perturbative approach. This will be explained in Subsection 2.8. Thus besides the plane wave distributions \( 1_{\kappa_0,1}, 1_{\kappa_0,1} \) defined by (61) and (62), Subsect. 2.6, we can use (104) and (105) of Subsection 2.8. Similarly we have two possibilities for the realization of the free electromagnetic potential field \( A \), both having the same commutation and pairing functions, but with slightly different behaviour in the infrared regime. This will be explained in Subsection 2.9. Namely besides the formulas (126) for \( 2_{\kappa_0,1}, 1_{\kappa_0,1} \) we can use (119), Subsection 2.9. Correspondingly we have a priori four versions of perturbative QED, and although it seems that they all should be essentially equivalent, they all should be subject to a systematic investigation.

Here we give definition and general rules in forming Wick product of integral kernel operators

\[
\Xi_{l_1,m_1}^{n_1 l_1,m_1} \cdots \Xi_{l_M,m_M}^{n_M l_M,m_M}
\]

with general (not necessary equal to plane wave distributions defining the free fields, as we have in view e.g. also their spatio-temporal-derivative fields)

\[
n_k^{l_k,m_k} \in \mathcal{L}(E_{n_k}, E^*_{n_k}) \cong E^*_{n_k} \otimes E^*_{n_k}, \quad k = 1, 2, \ldots M
\]

extendible to

\[
n_k^{l_k,m_k} \in \mathcal{L}(E^*_{n_k}, E^*_{n_k}) \cong E_{p_k} \otimes E^*_{n_k}
\]

and with the property that

\[
n_k^{l_k,m_k}(\xi) \in \mathcal{O}_C, \quad \xi \in E_{n_k}^*.
\]

Here

\[
n_k = \begin{cases} 1 & \text{or } \quad (l_k, m_k) = \begin{cases} (0, 1) & \text{or } \\ (1, 0) \end{cases} \\ 2 \end{cases}
\]

and the integral kernel operator

\[
\Xi_{l_k,m_k}^{n_k l_k,m_k},
\]

regarded as the operator on the said tensor product of Fock spaces, has the exceptional form (similarly as for the operators defined by the free fields \( A \) and \( \psi \)) that the integration in the general formula (69) for this operator is restricted to fermion variables, if \( n_k = 1 \), or to bose variables, if \( n_k = 2 \).
We then define the Wick product

\[ : \Xi_{l_1, m_1}^{\kappa_{l_1, m_1}} \cdots \Xi_{l_M, m_M}^{\kappa_{l_M, m_M}} : \]

of \( M \) such operators as the ordinary product of these operators, but rearranged in such a manner that all operators

\[ \Xi_{l_k, m_k}^{\kappa_{l_k, m_k}} \]

with \((l_k, m_k) = (1, 0)\) stand to the left of all operators

\[ \Xi_{l_k, m_k}^{\kappa_{l_k, m_k}} \]

with \((l_k, m_k) = (0, 1)\), multiplied in addition by the factor \((-1)^p\) with \( p \) equal to the parity of the permutation performed upon fermi operators, having \( n_k = 1 \) and corresponding to the fermi variables, required to bring the operators into the required “normal” order.

**RULE I**

We have the following computational rule

\[ : \Xi_{l_1, m_1}^{\kappa_{l_1, m_1}} \cdots \Xi_{l_M, m_M}^{\kappa_{l_M, m_M}} : \]

\[ = \Xi_{l, m}(\kappa_{l, m}), \]

\[ l = l_1 + \cdots + l_M, \quad m = m_1 + \cdots + m_M \]

where

\[ \kappa_{l, m} = \bigotimes_{i=1}^{n_1} \bigotimes_{j=l_1+\cdots+l_M}^{n_M} \bigotimes_{k=1}^{m_1} \bigotimes_{i=l_1+\cdots+l_M}^{m_M} \]

stands for the ordinary tensor product

\[ \bigotimes_{i=1}^{n_1} \bigotimes_{j=l_1+\cdots+l_M}^{n_M} \bigotimes_{k=1}^{m_1} \bigotimes_{i=l_1+\cdots+l_M}^{m_M} \in E_{n_1} \otimes E^*_{n_1} \otimes \cdots \otimes E_{n_M} \otimes E^*_{n_M} \]

\[ \cong E_{n_1} \otimes \cdots \otimes E_{n_M} \otimes E^*_{n_1} \otimes \cdots \otimes E^*_{n_M} \cong \mathcal{L}(E^*_{n_1} \otimes \cdots \otimes E^*_{n_M}, E_{n_1} \otimes \cdots \otimes E_{n_M}) \]

1) separately symmetrized with respect to all bose variables, lying among the first \( l \) variables, 2) separately symmetrized with respect to all bose variables, lying among the last \( m \) variables, 3) separately antisymmetrized with respect to all fermi variables which lie among the first \( l \) variables, 4) separately antisymmetrized with respect to all fermi variables lying among the last \( m \) variables, finally 5) the result multiplied by the factor \((-1)^p\), where \( p \) is the parity of the permutation performed upon the fermi operators necessary to rearrange them into the order in which they stand in the general formula \((69)\) for \( \Xi_{l, m}(\kappa_{l, m}) \).

Here by definition \( n_k = 1 \) counted among the first \( l \) variables iff the corresponding
\((l_k, m_k) = (1, 0)\), and \(n_k\) is counted among last \(m\) variables iff the corresponding \((l_k, m_k) = (0, 1)\).

This is effective computational rule because in practical situations, e.g. for the Wick product of integral kernel operators defined by free fields of the theory, the tensor product of the corresponding kernels may be represented by ordinary products of the functions representing kernels:

\[
\left( ^{n_1}_{l_1, m_1} \kappa \right) \otimes \cdots \otimes \left( ^{n_M}_{l_M, m_M} \kappa \right)(w_1, \ldots, w_M; X_1, \ldots, X_M)
= \left( ^{n_1}_{l_1, m_1} \kappa \right)(w_1, X_1) \cdots \left( ^{n_M}_{l_M, m_M} \kappa \right)(w_M, X_M),
\]

\(X_k = \begin{cases} (a_k, x_k), & \text{for } X_k \text{ corresponding to fermi variables } w_k = (s_k, p_k) \\ (\mu_k, x_k), & \text{for } X_k \text{ corresponding to bose variables } w_k = (\nu_k, p_k) \end{cases}\),

\(w_k = \begin{cases} (s_k, p_k), & \text{for fermi variables } w_k \\ (\nu_k, p_k), & \text{for bose variables } w_k \end{cases}\),

\(x_k\) denotes for each \(k\) spacetime coordinates variable,

\(s_k \in \{1, 2, 3, 4\}, \mu_k, \nu_k \in \{0, 1, 2, 3\}, a_k \in \{1, 2, 3, 4\}\).

In case of Wick product integral kernel operators corresponding to fixed components of the fields, the respective values of \(\mu_k\) and \(a_k\) will be correspondingly fixed, and the test spaces \(E_{n_k}\) will be equal \((70)\) with \(q_k = 1\), i.e. scalar test spaces. Thus the symmetrized/antisymmetrized tensor product \(\otimes\) of the kernels corresponding to free fields can be easily and explicitly computed, by the indicated symmetrizations and antisymmetrizations applied to the kernel functions:

\[
\left( ^{n_1}_{l_1, m_1} \kappa \right) \otimes \cdots \otimes \left( ^{n_M}_{l_M, m_M} \kappa \right)(w_1, \ldots, w_M; X_1, \ldots, X_M),
\]

remembering that the variable \((w_k, X_k)\) is counted among the first \(l\) variables iff \((l_k, m_k) = (1, 0)\), and the variable \((w_k, X_k)\) is counted among the last \(m\) variables iff \((l_k, m_k) = (0, 1)\).

The Rule I can be justified by utilizing the fact that

\[
\Xi_{l_k, m_k} \left( ^{n_k}_{l_k} \kappa \right)(X_k),
\]

exist point-wisely as Pettis integral for each fixed point \(X_k\), with the scalar distribution

\[
^{n_k}_{l_k} \kappa(X_k)
\]

(with fixed \(X_k\)) represented by the scalar function

\[
w_k \mapsto ^{n_k}_{l_k} \kappa(w_k, X_k)
\]

kernel, as in the proof of Bogoliubov-Shirkov Hypothesis in Subsection 5.9 of [59].
From the Rule I it easily follows that the Wick product of the class of integral kernel operators (72), subsuming free field operators, is a well defined (sum of) integral kernel operator(s) Ξ(κ_{l,m}) with the kernel(s)

\[ \kappa_{l,m} \in \mathcal{L}(E^*_{n_1} \otimes \cdots \otimes E^*_{n_M}, \mathcal{E}^*_{n_1} \otimes \cdots \otimes \mathcal{E}^*_{n_M}), \quad M = l + m \quad (73) \]

and thus with

\[ \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}((E) \otimes \mathcal{E}, (E)) \cong \mathcal{L}(\mathcal{E}, \mathcal{L}((E), (E))) \]

by Thm. [4] Subsection [2.6] for

\[ E = E^*_{n_1} \otimes \cdots \otimes E^*_{n_M} \quad (74) \]

In particular it defines an operator-valued distribution on the tensor product (74) of space-time test function spaces \( \mathcal{E} \) with \( \mathcal{L}(E^*_{n_1} \otimes \cdots \otimes E^*_{n_M}) \sim \mathcal{L}(E^*_{n_{-1}} \otimes \cdots \otimes E^*_{n_{-M}}) \)

It is easily seen that we get in this way a Wick graded algebra which subsumes in particular all finite sums of integral kernel operators Ξ(κ_{l,m}) with kernels κ_{l,m} having the property (75). Let

\[ \Xi_{l,m}^\prime(\kappa_{l,m}^\prime) \quad \text{and} \quad \Xi_{l,m}^\prime\prime(\kappa_{l,m}^\prime\prime) \]

be two such operators with

\[ \kappa_{l,m}^\prime \in \mathcal{L}(E^*_{n_1} \otimes \cdots \otimes E^*_{n_M}, \mathcal{E}^*_{n_1} \otimes \cdots \otimes \mathcal{E}^*_{n_M}), \]
\[ \kappa_{l,m}^\prime\prime \in \mathcal{L}(E^*_{n_1} \otimes \cdots \otimes E^*_{n_M}, \mathcal{E}^*_{n_1} \otimes \cdots \otimes \mathcal{E}^*_{n_M}) \]

It is easily seen that we have the following rule for Wick product of such operators

\[ : \Xi_{l,m}^\prime(\kappa_{l,m}^\prime) \Xi_{l,m}^\prime\prime(\kappa_{l,m}^\prime\prime) := \Xi_{l,m}(\kappa_{l,m}), \quad l = l' + l'' \quad m = m' + m'', \]

where

\[ \kappa_{l,m} = \kappa_{l,m}^\prime \otimes \kappa_{l,m}^\prime\prime \]

is equal to the ordinary tensor product

\[ \kappa_{l,m}^\prime \otimes \kappa_{l,m}^\prime\prime \]
\[ \in E_{n_1}^* \otimes \cdots \otimes E_{n_M}^* \otimes E_{n_1} \otimes \cdots \otimes E_{n_M}, \mathcal{E}^*_{n_1} \otimes \cdots \otimes \mathcal{E}^*_{n_M} \otimes \mathcal{E}_{n_1} \otimes \cdots \otimes \mathcal{E}_{n_M} \]
\[ \cong \mathcal{L}(E^*_{n_1} \otimes \cdots \otimes E^*_{n_M}, \mathcal{E}^*_{n_1} \otimes \cdots \otimes \mathcal{E}^*_{n_M} \otimes \mathcal{E}_{n_1} \otimes \cdots \otimes \mathcal{E}_{n_M}) \]

1) multiplied by \((-1)^p\) where \( p \) is the parity of the permutation which has to be applied to the fermi operators lying among the Hida operators put in the order

\[ \partial^*_{w_1} \cdots \partial^*_{w_{M'}} \partial_{w_1} \cdots \partial_{w_{M''}} \]

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in which the Hida operators are put formally together in the order in which they stand in the general formula (69) for \( \Xi_{l,m}^r(\kappa_{l,m}^r) \) (first) and in the general formula (69) for \( \Xi_{l,m}^{r''}(\kappa_{l,m}^{r''}) \) (second), in order to rearrange them into the order in which they stand in the general formula (69) for \( \Xi_{l,m}(\kappa_{l,m}) \)

2) separately symmetrized with respect to all bose variables which lie within the first \( l \) variables,

3) separately symmetrized with respect to all bose variables which lie within the last \( m \) variables,

4) separately antisymmetrized with respect to all fermi variables which lie among the first \( l \) variables,

5) separately antisymmetrized with respect to all fermi variables which lie among the last \( m \) variables,

6) the \( n_1^r \)-th or respectively \( n_m^{r''} \)-th variable is counted as lying among the first \( l \) variables if it lies among the first \( l' \) variables in \( \kappa_{l,m}^r \) or among the first \( l'' \) variables of the kernel \( \kappa_{l,m}^{r''} \). The remaining variables are counted as the last \( m \) variables.

In fact Wick product is well defined on a much larger class of integral kernel operators \( \Xi_{l,m}(\kappa_{l,m}) \), because for its validity it is sufficient that the kernels \( \kappa_{l,m} \) respect the condition of Theorem 4, considerably weaker than the condition (75). In this wider class of operators the last rule for computation of the Wick product remains true.

A much more interesting case we encounter when among the integral kernel operators (72) there are present such, which are equal to Wick polynomials of free fields at one and the same space-time point. Now we give general definition of such a Wick product of (fixed components of) free fields at one and the same space-time point, and show that the corresponding integral kernel operator lies among the class which can be placed into the above Wick product. The resulting integral kernel operator \( \Xi \) will be a finite sum of well defined integral kernel operators \( \Xi(\kappa_{l,m}) \) with the kernel(s)

\[
\kappa_{l,m} \in \mathcal{L}(E_{n_1} \otimes \cdots \otimes E_{n_M}, \mathcal{E}_{n_1}^* \otimes \cdots \otimes \mathcal{E}_{n_M}^*), \quad M = l + m
\]

and thus with

\[
\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(\mathcal{E}, \mathcal{E}^* \otimes \mathcal{L}((E) \otimes (E)^*)) \approx \mathcal{L}(\mathcal{E}, \mathcal{L}((E) \otimes (E)^*))
\]

by the generalization of Thm. 3.9 of [38] to the tensor product of Fock spaces, compare Subsection 2.4. Therefore the Wick product of free fields (or their derivatives) \( \Xi \) at the fixed space-time point belongs to the general class of finite sums of integral kernel operators with vector-valued kernels, which in general does not belong to

\[
\mathcal{L}(\mathcal{E}, \mathcal{E}^* \otimes \mathcal{L}((E) \otimes (E)^*)) \approx \mathcal{L}(\mathcal{E}, \mathcal{L}((E) \otimes (E)))
\]

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if among the factors in the Wick product (at fixed point) there are zero mass fields or their derivatives. But if among the factors there are no factors corresponding to zero mass fields (or their derivatives) then the resulting integral kernel operator $\Xi$ – Wick product at fixed point – will be a finite sum of well-defined integral kernel operators $\Xi(\kappa_{l,m})$ with the kernels respecting the condition of Thm. 4, i.e. with

$$\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}\left((E) \otimes \mathcal{S}, (E)\right) \cong \mathcal{L}\left(\mathcal{S}, \mathcal{L}\left((E), (E)\right)\right)$$

by the generalization of Thm. 3.13 of [38] to the tensor product of Fock spaces, compare Thm. 4 of Subsection 2.6 and with $\mathcal{E}_1$-valued or respectively $\mathcal{E}_2$-valued distribution kernels, for both nuclear space-time test function spaces: $\mathcal{E}_1$ and for $\mathcal{E}_2$ given by the special case of (70) with $M = 1$ and $q_k = 1$ in it, i.e.

$$\mathcal{E}_1 = \mathcal{S}(\mathbb{R}^4; \mathbb{C}) = \mathcal{S}(\mathbb{R}^4; \mathbb{C}) \text{ or}$$

$$\mathcal{E}_2 = \mathcal{S}_{\mathcal{F}(\mathbb{R}^4)}(\mathbb{R}^4; \mathbb{C}) = \mathcal{S}^{(0)}(\mathbb{R}^4; \mathbb{C}).$$

For the need of causal perturbative construction of interacting fields it is sufficient to confine attention to integral kernel operators representing the respective components of free fields, of their spatio-temporal derivatives, their Wick products, their integrals with pairing functions (e.g. convolutions of Wick products, their integrals with pairing functions”). Therefore we confine ourselves to fixed components of the free fields and of their spatio-temporal derivatives and thus to scalar-valued space-time test function spaces $\mathcal{E}_1 = \mathcal{S}(\mathbb{R}^4; \mathbb{C})$ or respectively $\mathcal{E}_2 = \mathcal{S}^{(0)}(\mathbb{R}^4; \mathbb{C})$. Correspondingly to this we consider integral kernel operators with the vector-valued kernels corresponding to fixed components of free fields which can be represented by the functions

$$1\kappa_{0,1}(w; X) = 1\kappa_{0,1}(s, p; a, x), \quad 1\kappa_{1,0}(w; X) = 1\kappa_{1,0}(s, p; a, x) \text{ or}$$

$$2\kappa_{0,1}(w; X) = 2\kappa_{0,1}(\nu, p; \mu, x), \quad 2\kappa_{1,0}(w; X) = 2\kappa_{1,0}(\nu, p; \mu, x),$$

with fixed values of the discrete indices $a, \mu$. To this class (76) of kernels we add their spatio-temporal derivatives

$$\partial^{\alpha} 1\kappa_{0,1}(w; X) = \partial^{\alpha} 1\kappa_{0,1}(s, p; a, x), \quad \partial^{\alpha} 1\kappa_{1,0}(w; X) = \partial^{\alpha} 1\kappa_{1,0}(s, p; a, x) \text{ or}$$

$$\partial^{\alpha} 2\kappa_{0,1}(w; X) = \partial^{\alpha} 2\kappa_{0,1}(\nu, p; \mu, x), \quad \partial^{\alpha} 2\kappa_{1,0}(w; X) = \partial^{\alpha} 2\kappa_{1,0}(\nu, p; \mu, x),$$

where

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^4 \text{ and } \partial^{\alpha} = \frac{\partial^{\alpha_0}}{(\partial x_0)^{\alpha_0}} \frac{\partial^{\alpha_1}}{(\partial x_1)^{\alpha_0}} \frac{\partial^{\alpha_2}}{(\partial x_2)^{\alpha_2}} \frac{\partial^{\alpha_3}}{(\partial x_3)^{\alpha_2}}$$

**DEFINITION 1.** The class $\mathcal{K}_0$ of kernels we are considering in the sequel consists of the plane wave kernels (76) defining the free fields of the theory and of their spatio-temporal derivatives (77), with fixed values of the indices $a, \mu, \alpha$. 

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Upon the integral kernel operators determined by the vector valued kernels \( \mathcal{K}_0 \) we perform the operations of Wick product (Rule I), Wick products at the same space-time point (Rule II), spatio-temporal derivations (Rule III), integrations (IV and V) and finally convolutions with pairing functions (Rule VI). Correspondingly to each of the said operations there exists the corresponding Rule performed upon the kernels, corresponding to the operators. Of course the operations performed upon the kernels in \( \mathcal{K}_0 \) and determined by the Rules will extend the initial class \( \mathcal{K}_0 \). We use a general notation

\[
^n k_i, m_i (s, p; x), \quad n = 1
\]

for a kernel

\[
\partial^a \, ^1 k_i, m_i (s, p; a, x), \quad (l, m) = (0, 1) \text{ or } (1, 0)
\]

with fixed indices \( a, \alpha \) and with \( ^1 k_{0,1} (s, p; a, x) \) equal to the plane wave kernel defining the free Dirac field. Similarly we will denote simply by

\[
^n k_i, m_i (\nu, p; x), \quad n = 2
\]

the kernel

\[
\partial^a \, ^2 k_i, m_i (\nu, p; \mu, x), \quad (l, m) = (0, 1) \text{ or } (1, 0)
\]

with fixed indices \( \mu, \alpha \) and with \( ^2 k_{1,0} (\nu, p; \mu, x) \) equal to the plane wave kernel defining the free electromagnetic potential field.

Assuming

\[
^n k_i, m_k, \in \mathcal{K}_0, \quad k = 1, \ldots, M,
\]

we consider the following Wick monomials, i.e. Wick products at the same space-time point, of the following operators

\[
\Xi_{l_1, m_1} \, ^{n_1} k_{l_1, m_1} \cdots \Xi_{l_M, m_M} \, ^{n_M} k_{l_M, m_M}
\]

with general (not necessary equal to plane wave distributions defining the free fields, as we have in view also their spatio-temporal-derivative fields) kernels

\[
^n k_i, m_k \in \mathcal{L}(E_{n_k}, E_{n_k}^* \otimes \delta_{n_k}^*), \quad k = 1, 2, \ldots, M
\]

representable by ordinary functions, respecting the conditions expressed in Lemma 4, Subsection 2.6 or respectively Lemma 9, Subsection 2.9, i.e. extendible to elements

\[
^n k_i, m_k \in \mathcal{L}(E_{n_k}^*, E_{n_k}^* \otimes \delta_{n_k}^*), \quad k = 1, 2, \ldots, M
\]

with the property that

\[
^n k_i, m_k (\xi) \in \mathcal{O}_C (\mathbb{R}^4; \mathbb{C}), \quad \xi \in E_{n_k}.
\]

Here

\[
n_k = \begin{cases} 
1 & \text{or } (l_k, m_k) = (0, 1) \\
2 & \text{or } (1, 0)
\end{cases}
\]
and the integral kernel operator

\[ \Xi_{l_k,m_k}(n_k^{l_k,m_k}(x)), \]

regarded as the operator on the said tensor product of Fock spaces, has the exceptional form (similarly as for the operators defined by the free fields \( A \) and \( \psi \)) that the integration in the general formula (69) for this operator is restricted to fermion variables, if \( n_k = 1 \), or to bose variables, if \( n_k = 2 \).

Validity of (79) and (80) for spatio-temporal derivatives of the plane wave kernels (76) can be proved exactly as for kernels (76) themselves by repeating the argument of the proof of Lemma 4, Subsection 2.6 or respectively Lemma 9, Subsection 2.9.

In fact in construction of interacting fields in the standard spinor QED it would be sufficient to consider only the kernels (76) and the kernels which arise by performing upon them the respective operations determined by the Rules I - VI, except the III-rd, given below. This is because no spatio-temporal derivatives of free fields enter the interaction lagrangian in spinor QED, but only free fields themselves. But in case of scalar QED the interaction lagrangian contains derivatives of free fields, so in that case spatio-temporal derivatives of the kernels determining the scalar free field has to be taken into consideration.

So let

\[ n_k^{l_k,m_k} \in \mathcal{H}_0, \ k = 1, \ldots, M. \]

Then for each fixed space-time point \( x \) the scalar integral kernel operators

\[ \Xi_{l_1,m_1}(n_1^{l_1,m_1}(x)) \cdots \Xi_{l_M,m_M}(n_M^{l_M,m_M}(x)) \]

(81)
determined by scalar kernel functions

\[ n_k^{l_k,m_k}(x) : w_n \mapsto n_k^{l_k,m_k}(w_n;x), \]

are well defined generalized operators transforming continuously the Hida space \((E)\) into its strong dual \((E)^*\), and exist point-wisely as Pettis integrals (69) with integration in (69) restricted to fermi variables, if \( n_k = 1 \), or to bose variables, if \( n_k = 2 \), compare Subsection 5.9 of [59]. Moreover for each fixed \( x \) there exist a well defined Wick product of the operators (81)

\[ \Xi_{l_1,m_1}(n_1^{l_1,m_1}(x)) \cdots \Xi_{l_M,m_M}(n_M^{l_M,m_M}(x)) : \]

(82)
defined as the ordinary product of these operators, but rearranged in the so called “normal” order, in which all operators

\[ \Xi_{l_k,m_k}(n_k^{l_k,m_k}(x)) \]

(83)
with \((l_k,m_k) = (1,0)\) stand to the left of all operators

\[ \Xi_{l_k,m_k}(n_k^{l_k,m_k}(x)) \]

(84)

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with \((l_k, m_k) = (0, 1)\), multiplied in addition by the factor \((-1)^p\) with \(p\) equal to the parity of the permutation performed upon fermi operators, having \(n_k = 1\) and corresponding to the fermi variables, required to bring the operators into the required “normal” order.

RULE II

We have the following computational rule

\[
: \Xi_{l_1, m_1} \left( \kappa_{l_1, m_1}(x) \right) \cdots : \Xi_{l_M, m_M} \left( \kappa_{l_M, m_M}(x) \right) : = \Xi_{l, m}(\kappa_{lm}(x)), \quad l = l_1 + \cdots l_M, \quad m = m_1 + \cdots m_M
\]

where the ordinary function representing the kernel \(\kappa_{l, m}\)

\[
\kappa_{l, m}(w_1, \ldots, w_M; x) = \left( \kappa_{l_1, m_1}(w_1; x) \right) \odot \cdots \odot \left( \kappa_{l_M, m_M}(w_M; x) \right)
\]

is equal to the ordinary product

\[
\left( \kappa_{l_1, m_1}(w_1; x) \right) \odot \cdots \odot \left( \kappa_{l_M, m_M}(w_M; x) \right) = \left( \kappa_{l_1, m_1}(w_1; x) \right) \cdots \left( \kappa_{l_M, m_M}(w_M; x) \right)
\]

1) separately symmetrized with respect to all bose variables, lying among the first \(l\) variables, 2) separately symmetrized with respect to all bose variables, lying among the last \(m\) variables, 3) separately antisymmetrized with respect to all fermi variables which lie among the first \(l\) variables, 4) separately antisymmetrized with respect to all fermi variables lying among the last \(m\) variables, finally 5) the result multiplied by the factor \((-1)^p\), where \(p\) is the parity of the permutation performed upon the fermi operators necessary to rearrange them into the order in which they stand in the general formula for \(\Xi_{l, m}(\kappa_{l, m})\).

Here by definition \(n_k\) is counted among the first \(l\) variables iff the corresponding \((l_k, m_k) = (1, 0)\), and \(n_k\) is counted among last \(m\) variables iff the corresponding \((l_k, m_k) = (0, 1)\).

Again the Rule II can be justified by using the fact that the operators exist point-wisely as Pettis integrals, and represent operators mapping continuously the strong dual \((E)^*\) of the Hida space into its strong dual \((E)^*\) (continuous as well as operators \((E) \to (E)^*\)), and similarly we have for the operators, representing continuous operators \((E) \to (E)\) (as well continuous as operators \((E) \to (E)^*\)). The proof, using essentially the same arguments as that used in the proof of Bogoliubov-Shirkov Hypothesis in Subsection 5.9 of [59], can be omitted, compare Subsection 5.9 of [59].

From the Rule II it easily follows that the Wick product determines integral kernel operator

\[
\Xi_{l, m}(\kappa_{l, m}) = \Xi_{l, m} \left( \kappa_{l_1, m_1}(w_1; x) \right) \cdots \left( \kappa_{l_M, m_M}(w_M; x) \right)
\]
with vector valued kernel

\[ \kappa_{l,m} = \left( n_1^{i_1} \kappa_{i_1,m_1} \right) \otimes \cdots \otimes \left( n_M^{i_M} \kappa_{i_M,m_M} \right) \]

\[ \in E_{n_1}^* \otimes \cdots \otimes E_{n_M}^* \otimes E_{i}^* \cong \mathcal{L}(E_{n_1} \otimes \cdots \otimes E_{n_M}, E_{i}^*), \quad i = 1, 2 \quad (85) \]

and, when all \( n_k = 1 \) (i.e. all \( \kappa_{k,i_k,m_k} \) are the plane wave kernels corresponding to derivatives of the Dirac field), defines the bilinear map

\[ \xi \times \eta \mapsto \kappa_{l,m}(\xi \otimes \eta), \]

first \( l \) terms \( E_{i_j}, \ i_j \in \{1, 2\} \)

\[ \xi \in \underbrace{E_{i_1} \otimes \cdots \otimes E_{i_l}}_{\text{first } l \text{ terms } E_{i_j}, \ i_j \in \{1, 2\}} \]

last \( m \) terms \( E_{i_j}, \ i_j \in \{1, 2\} \)

\[ \eta \in \underbrace{E_{i_{l+1}} \otimes \cdots \otimes E_{i_{l+m}}}_{\text{last } m \text{ terms } E_{i_j}, \ i_j \in \{1, 2\}} \quad (86) \]

which can be extended to a separately continuous bilinear map from

\[ \left( \underbrace{E_{i_1} \otimes \cdots \otimes E_{i_l}}_{\text{first } l \text{ terms } E_{i_j}} \right)^* \times \left( \underbrace{E_{i_{l+1}} \otimes \cdots \otimes E_{i_{l+m}}}_{\text{last } m \text{ terms } E_{i_j}} \right)^* \] into \( \mathcal{L}(\mathcal{E}, \mathbb{C}) = \mathcal{E}^* \). \quad (87)

Thus in each case

\[ \Xi_{l,m}(\kappa_{l,m}) = \Xi_{l,m}\left( \left( n_1^{i_1} \kappa_{i_1,m_1} \right) \otimes \cdots \otimes \left( n_M^{i_M} \kappa_{i_M,m_M} \right) \right) \]

\[ \in \mathcal{L}\left( (E) \otimes \mathcal{E}_i, (E)^* \right) \cong \mathcal{L}\left( \mathcal{E}_i, \mathcal{L}\left( (E), (E)^* \right) \right), \quad i = 1, 2 \]

by Theorem 3.9 of [38] (or its generalization to the case of tensor product of Fock spaces, compare Subsection 2.6).

In case in which there are no factors

\[ \Xi_{k,m_k}\left( \kappa_{i_k,m_k} \right) \] with \( n_k = 2 \)

i.e. no factors corresponding to the (derivatives) of the zero mass free fields of the theory, e.g. of the electromagnetic potential field in case of QED, we have

\[ \Xi_{l,m}(\kappa_{l,m}) = \Xi_{l,m}\left( \left( n_1^{i_1} \kappa_{i_1,m_1} \right) \otimes \cdots \otimes \left( n_M^{i_M} \kappa_{i_M,m_M} \right) \right) \]

\[ \in \mathcal{L}\left( (E) \otimes \mathcal{E}_i, (E) \right) \cong \mathcal{L}\left( \mathcal{E}_i, \mathcal{L}\left( (E), (E) \right) \right), \quad i = 1, 2 \]

by Theorem 4, Subsection 2.6 (generalization of Thm. 3.13 in [38]).

Indeed we use several technical Lemmas which allow us to show \( (85) \) as well as the extedibility \( (87) \) property of the bilinear map \( (86) \) in case in which the zero mass terms are absent. We need the following technical definition
DEFINITION 2. Let \( \mathcal{S}_i \), \( i = 1, 2 \), denote the family of subsets of \( E_i \subset E_i^* \) which are bounded in the topology on \( E_i \) induced by the strong dual topology on \( E_i^* \). Otherwise: \( \mathcal{S}_i \) is the family of intersections of all sets bounded in the strong dual space \( E_i^* \) with the subset \( E_i \) of \( E_i^* \).

LEMMA 5. Let \( \kappa_{1,0}, \kappa_{1,0} \in \mathcal{R}_0 \), i.e. let the above two kernels be equal to fixed components of plane wave kernels defining the massive free fields of the theory (i.e. the Dirac field in case of QED), or to their spatio-temporal derivatives \( \partial^\alpha \) with fixed value of the multi-index \( \alpha \in \mathbb{N}_0^4 \). Then the map

\[
E_i^* \times E_i^* \ni \kappa_{1,0}(\xi_1) \cdot \kappa_{1,0}(\xi_2) \quad \mapsto \quad (\kappa_{1,0}(\xi_1) \cdot \kappa_{1,0}(\xi_2)) \in \mathcal{E}_k^*,
\]

is \((\mathcal{S}_1, \mathcal{S}_1)\)-hypocontinuous as a map

\[
E_i \times E_i \longrightarrow \mathcal{E}_k^*, \quad k = 1, 2
\]

with the topology on \( E_i \subset E_i^* \), induced by the strong dual topology on \( E_i^* \), and with the strong dual topology on \( \mathcal{E}_k^* \), \( k = 1, 2 \).

\[\blacksquare\] (An outline of the proof) \( \mathcal{E}_2 = \mathcal{S}^{00}(\mathbb{R}^4; \mathbb{C}) \) is continously inserted into \( \mathcal{S}(\mathbb{R}^4; \mathbb{C}) \), and thus the strong dual \( \mathcal{E}_1^* = \mathcal{S}(\mathbb{R}^4; \mathbb{C})^* \) is continously inserted into the strong dual \( \mathcal{E}_2^* = \mathcal{S}^{00}(\mathbb{R}^4; \mathbb{C})^* \), for the proof compare Subsection 5.5 of [59]. It is therefore sufficient to prove the Lemma for the case \( \mathcal{E}_1^* = \mathcal{S}(\mathbb{R}^4; \mathbb{C})^* \) with \( k = 1 \).

Consider for example the case of the plane wave kernel \( \kappa_{1,0} \) given by the formula (102), Subsect. 2.6 or 103 of Subsection 2.8 which defines (one of the two a priori possible) Dirac free fields (the analysis of their fixed spatio-temporal derivation components is identical).

Recall that for \( \phi \in \mathcal{D}_1 = \mathcal{S}(\mathbb{R}^4; \mathbb{C}) \), \( \xi_1, \xi_2 \in E_1 = \mathcal{S}(\mathbb{R}^3; \mathbb{C}^4) \) (here we fix once for all the spinor indices \( \alpha_1, \alpha_2 \) and in case of spatio-temporal derivatives \( \partial^{\alpha_1} \kappa_{1,0} \) and \( \partial^{\alpha_2} \kappa_{1,0} \) the additional multiindices \( \alpha_1, \alpha_2 \in \mathbb{N}_0^4 \) would also be fixed) we have

\[
\langle \kappa_{1,0}(\xi_1) \cdot \kappa_{1,0}(\xi_2), \phi \rangle = \sum_{s_1, s_2} \int_{\mathbb{R}^4} \kappa_{1,0}(s_1, p_1; a_1, x) \cdot \kappa_{1,0}(s_2, p_2; a_2, x) \cdot \xi_1(s_1, p_1) \cdot \xi_2(s_2, p_2) \cdot \phi(x) \, d^3p_1 \, d^3p_2 \, d^4x.
\]

\[
k_{1,0}(\xi_1)(a_1, x) = \sum_{s_1} \int_{\mathbb{R}^3} \kappa_{1,0}(s_1, p_1; a_1, x) \cdot \xi_1(s_1, p_1) \, d^3p_1,
\]

\[
k_{1,0}(\xi_2)(a_2, x) = \sum_{s_2} \int_{\mathbb{R}^3} \kappa_{1,0}(s_2, p_2; a_2, x) \cdot \xi_2(s_2, p_2) \, d^3p_2.
\]

Next we show that if \( \xi_1 \in E_1 = \mathcal{S}(\mathbb{R}^3; \mathbb{C}) \) ranges over a set \( S \in \mathcal{S}_1 \), i.e. over \( S \subset E_1 \subset E_1^* \) bounded in the strong dual topology on \( E_1^* \), and if \( \phi \in 101 \)
\( \delta_1 = \mathcal{S}(\mathbb{R}^4; \mathbb{C}) \) ranges over a set \( B \subset \delta_1 = \mathcal{S}(\mathbb{R}^4; \mathbb{C}) \) bounded in \( \delta_1 = \mathcal{S}(\mathbb{R}^4; \mathbb{C}) \) (with respect to the ordinary nuclear Schwartz topology on \( \mathcal{S}(\mathbb{R}^4; \mathbb{C}) \), then the set \( B^+(S, B) \) of functions (spinor indices \( a_1, a_2 \) are fixed)

\[
(s_2, p_2) \mapsto \sum_{s_1} \int_{\mathbb{R}^3 \times \mathbb{R}^4} \kappa_{1,0}(s_1, p_1; a_1, x) \kappa_{1,0}(s_2, p_2; a_2, x) \xi_1(s_1, p_1) \phi(x) \, d^3p_1 \, d^4x
\]

and the set \( B^+(B, S) \) of functions

\[
(s_1, p_1) \mapsto \sum_{s_1} \int_{\mathbb{R}^3 \times \mathbb{R}^4} \kappa_{1,0}(s_1, p_1; a_1, x) \kappa_{1,0}(s_2, p_2; a_2, x) \xi_2(s_2, p_2) \phi(x) \, d^3p_2 \, d^4x
\]

with \( \xi_2 \) ranging over \( S \in \mathcal{S}_1 \) and \( \phi \in B \) are bounded in \( E_1 = \mathcal{S}(\mathbb{R}^3; \mathbb{C}^4) \). The proof, being a simple verification of definition of boundedness, can be omitted, but we encourage the reader to perform the computations explicitly.

Next we observe that for any \( S \in \mathcal{S}_1 \) and any strong zero-neighborhood \( W(B, \epsilon) \) in \( \delta_1 = \mathcal{S}(\mathbb{R}^4; \mathbb{C})^* \), determined by a bounded set \( B \in \delta_1 = \mathcal{S}(\mathbb{R}^4; \mathbb{C}) \) and \( \epsilon > 0 \), for the strong zero-neighborhoods \( V(B^+(S, B), \epsilon) \) and \( V(B^+(B, S), \epsilon) \) we have

\[
|\langle \kappa_{1,0}(\xi_1) \cdot \kappa_{1,0}(\xi_2), \phi \rangle| < \epsilon
\]

whenever

\[
\xi_1 \in S, \quad \xi_2 \in V(B^+(S, B), \epsilon)
\]

or whenever

\[
\xi_1 \in V(B^+(B, S), \epsilon), \quad \xi_2 \in S.
\]

Put otherwise

\[
\kappa_{1,0}(S) \cdot \kappa_{1,0}


\[
\kappa_{1,0}

\]

\[
\kappa_{1,0}(S) \subset W(B, \epsilon), \quad \kappa_{1,0}(V(B^+(B, S), \epsilon)) \subset W(B, \epsilon).
\]

**Lemma 6.** 1) Let \( \phi \in \delta_1 = \mathcal{S}(\mathbb{R}^4; \mathbb{C}) \) and let \( \tilde{\phi} \) be equal to its Fourier transform

\[
\tilde{\phi}(p) = \int_{\mathbb{R}^4} \phi(x) e^{ip \cdot x} \, d^4x.
\]

Then if \( \phi \in \mathcal{S}(\mathbb{R}^4; \mathbb{C}) \) ranges over a bounded set \( B \) in the Schwartz space \( S \), equivalently, if \( \phi \) ranges over a bounded set \( B \) in \( \mathcal{S}(\mathbb{R}^4; \mathbb{C}) \), then there exists a constant \( C_B \) depending on \( B \) such that

\[
|\tilde{\phi}(p \pm p', p_0(p) \pm p'_0(p'))| \leq C_B, \quad p, p' \in \mathbb{R}^3, \phi \in B
\]

in each case

\[
p_0(p) = \sqrt{|p|^2 + m}, \quad \text{or} \quad p_0(p) = \sqrt{|p|^2} = |p|
\]

\[
p'_0(p') = \sqrt{|p'|^2 + m}, \quad \text{or} \quad p'_0(p') = \sqrt{|p'|^2} = |p'|.
\]

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2) Let
\[ n_1^{\kappa_1, m_1}, n_2^{\kappa_2, m_2} \in \mathfrak{C}_0, \quad (l_k, m_k) \in \{(0, 1), (1, 0)\}, \quad n_k \in \{1, 2\}, \quad k = 1, 2, \]
i.e. let the above two kernels be equal to fixed components of plane wave kernels defining free fields of the theory, or to their spatio-temporal derivatives \( \partial^\alpha \) with fixed value of the multiindex \( \alpha \in \mathbb{N}_0^4 \). Then the map
\[ E_{n_1} \times E_{n_2} \ni \xi_1 \times \xi_2 \mapsto n_1^{\kappa_1, m_1}(\xi_1) \cdot n_2^{\kappa_2, m_2}(\xi_2) \in \mathcal{E}_k^*, \]
is continuous as a map
\[ E_{n_1} \times E_{n_2} \longrightarrow \mathcal{E}_k^*, \quad k = 1, 2 \]
with the ordinary nuclear topology on \( E_{n_k}, k = 1, 2 \), and with the strong dual topology on \( \mathcal{E}_k^*, k = 1, 2 \).

The first part 1) is obvious.
Concerning 2) we will use the the following two facts.

I) The functions
\[ p \mapsto \frac{P(p)}{p_0(p)} = \frac{P(p)}{\sqrt{|p|^2 + m}}, \quad m \neq 0 \]
with \( P(p) \) being equal to polynomials in four real variables \((p, p_0(p)) = (p_1, p_2, p_3, \sqrt{|p|^2 + m})\) are multipliers of the Schwartz algebra \( S(\mathbb{R}^3; \mathbb{C}) \), compare [17] or Appendix 5.

II) The functions
\[ p \mapsto \frac{P(p)}{p_0(p)} = \frac{P(p)}{|p|}, \]
with \( P(p) \) being equal to polynomials in four real variables \((p, p_0(p)) = (p_1, p_2, p_3, |p|)\) are multipliers of the nuclear algebra \( S^0(\mathbb{R}^3; \mathbb{C}) \), for a proof compare Subsections 5.2-5.5 of [59].

Recall that in case of QED we have
\[ E_1 = S_{A_1}(\mathbb{R}^3; \mathcal{C}^4) = S(\mathbb{R}^3; \mathcal{C}^4) = \oplus S(\mathbb{R}^3; \mathcal{C}) \] and
\[ E_2 = S_{A_2}(\mathbb{R}^3; \mathcal{C}^4) = S^0(\mathbb{R}^3; \mathcal{C}^4) = \oplus S^0(\mathbb{R}^3; \mathcal{C}). \]
with \( A_2 = \oplus_0^3 A^{(3)} \) and \( A^{(3)} \) on \( L^2(\mathbb{R}^3; \mathbb{C}) \) constructed in Subsection 5.3 of [59], and with \( A_1 = \oplus_1^3 H_{(3)} \) equal to the direct sum of four copies of the three dimensional oscillator hamiltonian, i.e. \( A_1 \) is equal to the operator \( A \) given by [59].

In particular let us consider the distribution defined by the kernel
\[ \kappa_{1, 0} \cdot \kappa_{1, 0}(\nu_1, p_1, \nu_2, p_2; x) = \kappa_{1, 0}(\nu_1, p_1; \mu, x) \cdot \kappa_{1, 0}(\nu_2, p_2; \lambda, x), \quad \text{with fixed } \mu, \lambda \]
(88)
and with \( \kappa_{1,0} \) equal to the plane wave kernel defining the free electromagnetic potential field, and given by the formula (126), Subsection 2.10. For each \( \xi_1, \xi_2 \in E_2 = S^0(\mathbb{R}^4; \mathbb{C}) \) the value of the distribution

\[
\kappa_{1,0} \cdot \kappa_{1,0}(\xi_1 \otimes \xi_2)(x) = \kappa_{1,0}(\xi_1)(\mu, x) \cdot \kappa_{1,0}(\xi_2)(\lambda, x) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{d^3p_1 d^3p_2}{|p_1| |p_2|} \xi^\mu_1(p_1) \xi^\lambda_2(p_2) e^{i(p_1 + p_2) \cdot x},
\]

where

\[
\xi_1 \otimes \xi_2(p_1 \times p_2) = \xi_1(p_1) \xi_2(p_2)
\]
on \( \phi \in S(\mathbb{R}^4; \mathbb{C}) \) is equal

\[
\langle \kappa_{1,0}(\xi_1) \cdot \kappa_{1,0}(\xi_2), \phi \rangle = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{d^3p_1 d^3p_2}{|p_1| |p_2|} \xi^\mu_1(p_1) \xi^\lambda_2(p_2) \bar{\phi}(p_1 + p_2, |p_1| + |p_2|).
\]

Now let \( \xi_1, \xi_2 \) range respectively over the bounded sets \( B_1 \) and \( B_2 \) in \( E_2 = S^0(\mathbb{R}^4; \mathbb{C}^4) \). Let \( \phi \) range over a bounded set \( B \) in \( S(\mathbb{R}^4; \mathbb{C}) \), equivalently, \( \phi \) range over a bounded set \( \widetilde{B} \) in \( S(\mathbb{R}^4; \mathbb{C}) \). Because the function

\[
p \mapsto \frac{1}{|p|}
\]
is a multiplier of the nuclear algebra \( S^0(\mathbb{R}^3; \mathbb{C}) \) (Subsections 5.4 and 5.5 of [59]) then the sets of functions

\[
B'_1 = \{ \xi'_1, \xi_1 \in B_1 \} \quad \text{where} \quad \xi'_1(p_1) = \frac{\xi_1(p_1)}{|p_1|},
\]

\[
B'_2 = \{ \xi'_2, \xi_2 \in B_2 \} \quad \text{where} \quad \xi'_2(p_2) = \frac{\xi_2(p_2)}{|p_2|},
\]
are bounded in \( E_2 = S(\mathbb{R}^4; \mathbb{C}^4) \), and the set \( B'_1 \otimes B'_2 \) is bounded in \( E_2 \otimes E_2 \). This in particular means that each of the norms (values of the indeces \( \mu, \nu \in \{0, 1, 2, 3\} \) are fixed and \( \zeta^{(q)} \) denotes derivative of \( q \)-th order \( q \in \mathbb{N}_0 \) of a function \( \zeta \) on \( \mathbb{R}^3 \))

\[
|\xi^\mu_1 \otimes \xi^\lambda_2| \mid_m \equiv \sup_{|q| \leq m} (1 + |p_1 \times 2|)^m \left| \left( \xi^{(q)}_1 \otimes \xi^{(q)}_2 \right) \right| (q)
\]
is separately bounded on \( B'_1 \otimes B'_2 \), i.e. for each \( m = 0, 1, 2, \ldots \) there exists a finite constant \( C'_m \) such that

\[
|\xi^\mu_1 \otimes \xi^\lambda_2| \mid_m \leq C'_m, \quad \xi_1 \in B'_1, \xi_2 \in B'_2,
\]
and moreover for each \( m = 0, 1, 2, \ldots \) there exists \( m'(m) \in \mathbb{N}_0 \) and \( C(m) < \infty \) such that

\[
\left| \left( \frac{1 + |p_1 \times p_2|^2}{|p_1| |p_2|} \right)^m \xi_1 \otimes \xi_2 \right| \mid_m \leq C(m) \mid \xi_1 \mid_{m'} \mid \xi_2 \mid_{m'}
\]

(89)
where \{1 ∗ m\}_{m∈N_0} is one of the equivalent systems of norms defining \(S^0(\mathbb{R}^3; \mathbb{C})\) and given in Subsection 5.5 of [59].

Now using the part 1) of the Lemma and the inequality \(\mathcal{S}(9)\) we obtain the following inequalities (with fixed values of the indeces \(µ\) and \(λ\) in each factor \(κ_{1,0}(ξ_1)\) and \(κ_{1,0}(ξ_1)\))

\[
|⟨κ_{1,0}(ξ_1) ∗ κ_{1,0}(ξ_2), φ⟩| = \left| \int_{\mathbb{R}^3 × \mathbb{R}^3} \frac{d^3p_1 d^3p_2}{|p_1||p_2|} ξ_1^µ(p_1)ξ_2^λ(p_2) \tilde{φ}(p_1 + p_2, |p_1| + |p_2|) \right|
\]

\[
\leq \int_{\mathbb{R}^3 × \mathbb{R}^3} \frac{d^3p_1 d^3p_2}{|p_1||p_2|} |ξ_1^µ(p_1)ξ_2^λ(p_2)| |φ(p_1 + p_2, |p_1| + |p_2|)|
\]

\[
\leq CB \int_{\mathbb{R}^3 × \mathbb{R}^3} \frac{d^3p_1 d^3p_2}{|p_1||p_2|} |ξ_1^µ(p_1)ξ_2^λ(p_2)|
\]

\[
\leq CB \left| \frac{1}{(1 + |p_1 × p_2|^2)^4} \right|_{L^2(κ^6)} \left| (1 + |p_1 × p_2|^2)^4 \right|_{L^2(κ^6)} ξ_1^µ ⊗ ξ_2^λ
\]

\[
\leq C' \left[ (1 + |p_1 × p_2|^2)^4 \right]_{L^2(κ^6)} ξ_1^µ ⊗ ξ_2^λ
\]

\[
\leq C'C(4) |ξ_1^µ|_{m'} |ξ_2^λ|_{m'}, \quad (90)
\]

for some finite \(m' ∈ N_0\).

Therefore for any strong zero-neighborhood \(V(B, ε)\) in \(S(\mathbb{R}^4; \mathbb{C})^*\) determined by a bounded subset \(B\) in \(S(\mathbb{R}^4; \mathbb{C})\) and \(ε > 0\) there exist zero-neighborhoods \(V_1\) and \(V_2\) in \(E_2 = S^0(\mathbb{R}^3; \mathbb{C}^4)\) such that

\[
|⟨κ_{1,0}(ξ_1) ∗ κ_{1,0}(ξ_2), φ⟩| ≤ ε, \quad ξ_1 ∈ V_1, ξ_2 ∈ V_2, φ ∈ B,
\]

or equivalently

\[
κ_{1,0}(ξ_1) ∗ κ_{1,0}(ξ_2) ∈ V(B, ε), \quad ξ_1 ∈ V_1, ξ_2 ∈ V_2,
\]

if we define

\[
V_1 = \left\{ ξ, |ξ^µ|_{m'} < \sqrt{\frac{ε}{C'C(4)}} \right\}, \quad V_2 = \left\{ ξ, |ξ^λ|_{m'} < \sqrt{\frac{ε}{C'C(4)}} \right\},
\]

which follows from the inequalities \(\mathcal{S}(9)\).

The same proof holds if we replace one or both the kernels \(κ_{1,0}\) by the kernel \(κ_{0,1}\) defined by \(\mathcal{S}(25)\), Subsection 2.10 or by their derivatives because for any polynomial \(P(p_1, p_2)\) in eight real variables

\[
(P(p_1, p_10(p_1), p_2, p_20(p_2)) = (p_1, |p_1|, p_2, |p_2|)
\]

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and for each $m = 0, 1, 2, \ldots$ there exists $m'(m) \in \mathbb{N}_0$ and $C(m) < \infty$ such that

$$\left|\left(1 + \frac{|p_1 \times p_2|^2}{|p_1||p_2|}\right)^n P(p_1, p_2)\right|_{m} \leq C(m) \left|\left[m_1\right], \left[m_2\right]\right|.$$  

(91)

Analogous proof can be repeated for all $\kappa_{1,0}, \kappa_{0,1}$ defined by (119), Subsection 2.3 (for plane wave kernels defining the free electromagnetic potential field) and their derivatives; or for plane wave kernels (101) and (102), Subsect. 2.6 or (104) and (105) of Subsection 2.8 (for kernels defining the Dirac field) and their derivatives. We have to remember that if the kernel corresponds to the electromagnetic potential field then the nuclear space on which it is defined is equal $E_2 = \mathcal{S}(\mathbb{R}^3; \mathbb{C}^4)$ and if the kernel corresponds to the Dirac field then it is defined on the nuclear space $E_1 = \mathcal{S}(\mathbb{R}^3; \mathbb{C})$. In the last case we can use the standard system of norms defining the Schwartz topology on $\mathcal{S}(\mathbb{R}^3; \mathbb{C})$. In particular if both factors $\kappa_{1,0}, \kappa_{0,1}$ in the pointwise product $\kappa_{1,0}(\xi_1) \cdot \kappa_{0,1}(\xi_2)$ correspond to kernels defining a fixed component of the Dirac field (or its fixed component derivative) then we are using the inequality (91) with the same system of norms defining the nuclear topology on $\mathcal{S}(\mathbb{R}^3; \mathbb{C})$, replaced by the standard system of norms defining the Schwartz topology on $\mathcal{S}(\mathbb{R}^3; \mathbb{C})$ and with

$$\left(1 + \frac{|p_1 \times p_2|^2}{|p_1||p_2|}\right)^n P(p_1, p_2)$$

in (91) replaced by

$$\left(1 + \frac{|p_1 \times p_2|^2}{|p_1|^2 + m\sqrt{|p_2|^2 + m}}\right)^n P(p_1, p_2) \quad \text{or} \quad \left(1 + |p_1 \times p_2|^2\right)^n P(p_1, p_2)$$

with $P(p_1, p_2)$ equal to any polynomial in eight real variables

$$(p_1, p_{10}(p_1), p_2, p_{20}(p_2)) = (p_1, \sqrt{|p_1|^2 + m}, p_2, \sqrt{|p_2|^2 + m}).$$

If the first factor $\kappa_{1,0}(\xi_1)$ corresponds to a fixed component of the Dirac field (or its fixed component derivative) and the second factor $\kappa_{0,1}(\xi_2)$ to a fixed component of the electromagnetic potential field (or its fixed component derivative) then we are using the inequality (91), with the the same system of norms $\left\{\left[m\right]\right\}_{m \in \mathbb{N}_0}$ on the left hand side, the same system of norms $\left\{\left[m\right]\right\}_{m \in \mathbb{N}_0}$ defining the nuclear topology $\mathcal{S}(\mathbb{R}^3; \mathbb{C})$ (inherited from $\mathcal{S}(\mathbb{R}^3; \mathbb{C})$, compare Subsections 5.2-5.5 of [59]), but with the system of norms $\left\{\left[m\right]\right\}_{m \in \mathbb{N}_0}$ replaced by any standard which defines the Schwartz topology on $\mathcal{S}(\mathbb{R}^3; \mathbb{C})$, and with

$$\left(1 + \frac{|p_1 \times p_2|^2}{|p_1||p_2|}\right)^n P(p_1, p_2)$$

\[13\] $(l_k, m_k) = (1, 0)$ or $(l_k, m_k) = (0, 1)$ for $k = 1, 2$. 

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in (91) replaced by
\[
\frac{(1 + |p_1 \times p_2|^2)^4 P(p_1, p_2)}{\sqrt{|p_1|^2 + m^2 |p_2|}} \quad \text{or} \quad \frac{(1 + |p_1 \times p_2|^2)^4 P(p_1, p_2)}{|p_2|}
\]
with \(P(p_1, p_2)\) equal to any polynomial in eight real variables
\[
(p_1, p_{10}(p_1), p_2, p_{20}(p_2)) = (p_1, \sqrt{|p_1|^2 + m}, p_2, |p_2|).
\]

\[\Box\]

**Lemma 7.** Let
\[
k_k \kappa_{k,m_k} \in \mathfrak{K}_0, \quad k = 1, \ldots, M.
\]
i.e. we have the kernels belonging to the class \(\mathfrak{K}_0\).

1) Then it follows in particular that
\[
k_k \kappa_{k,m_k} \in \mathcal{L}(E_{n_k}^*, \mathcal{E}_{n_k}^*) \cong E_{n_k}^* \otimes \mathcal{E}_{n_k}^*, \quad k = 1, \ldots, M,
\]
are regular vector-valued distributions defined by ordinary functions, which fulfill the condition (77), i.e. are extendible to elements
\[
k_k \kappa_{k,m_k} \in \mathcal{L}(E_{n_k}^*, \mathcal{E}_{n_k}^*) \cong E_{n_k}^* \otimes \mathcal{E}_{n_k}^*, \quad k = 1, \ldots, M,
\]
and have the property (78) that
\[
k_k \kappa_{k,m_k}(\xi) \in \mathcal{O}_C(\mathbb{R}^4; \mathbb{C}), \quad \xi \in E_{n_k}^*.
\]

2) The “point-wise” multiplicative tensor product \(\hat{\otimes}\) of these distributions, defined as in Rule II, gives a vector valued kernel
\[
k_{i,m} = \left( {}_1^n \kappa_{1,m_1} \right) \otimes \cdots \otimes \left( {}_m^n \kappa_{M,m_M} \right) \in \mathcal{O}_C(\mathbb{R}^4; \mathbb{C}).
\]

3) The “point-wise” multiplicative tensor product \(\hat{\otimes}\) of these distributions, defined as in Rule II, gives a vector valued kernel
\[
k_{i,m} = \left( {}_1^n \kappa_{1,m_1} \right) \otimes \cdots \otimes \left( {}_m^n \kappa_{M,m_M} \right)
\]
\[
\in E_{n_1}^* \otimes \cdots \otimes E_{n_M}^* \otimes \mathcal{E}_{i}^* \cong \mathcal{L}(E_{n_1}^* \otimes \cdots \otimes E_{n_M}^*, \mathcal{E}_{i}^*), \quad i = 1, 2.
\]

4) If all \(n_1, \ldots, n_M\) are equal 1, i.e. if all factors
\[
\Xi_{k,m_k} \left( {}_k^n \kappa_{1,m_1} \right) \quad \text{with} \quad n_k = 1
\]
\[\text{Recall that each element of } \mathfrak{K}_0 \text{ is equal to a component of a plane wave kernel defining free field of the theory or to its spatio-temporal derivative } \partial^\alpha \text{ with fixed } \alpha, \text{ compare Definition } 1.\]
correspond to (derivatives) of the free massive fields of the theory (i.e. derivatives of the Dirac free field in case of spinor QED), then the bilinear map

\[ \xi \times \eta \mapsto \kappa_{l,m}(\xi \otimes \eta), \]

where

- first \( l \) terms \( E_{i,j}, i_j \in \{1, 2\} \)
- last \( m \) terms \( E_{i,j}, i_j \in \{1, 2\} \)

\[ \xi \in \underbrace{E_{i_1} \otimes \cdots \otimes E_{i_l}}_{\text{first } l \text{ terms } E_{i,j}}, \]
\[ \eta \in \underbrace{E_{i_{l+1}} \otimes \cdots \otimes E_{i_{l+m}}}_{\text{last } m \text{ terms } E_{i,j}}, \]

can be extended to a separately continuous bilinear map from

\[ \left( \underbrace{E_{i_1} \otimes \cdots \otimes E_{i_l}}_{\text{first } l \text{ terms } E_{i,j}} \right)^* \times \left( \underbrace{E_{i_{l+1}} \otimes \cdots \otimes E_{i_{l+m}}}_{\text{last } m \text{ terms } E_{i,j}} \right) \]

into \( \mathcal{L}(\mathcal{D}, \mathbb{C}) = \mathcal{D}^* \).

\[ \square \]

The first two parts 1) and 2) can be proved exactly as Lemma 4, Subsection 2.6 or respectively Lemma 9, Subsection 2.9.

Concerning 3) it is sufficient to consider the case \( M = 2 \). But the case \( M = 2 \) follows immediately from the part 2) of Lemma 6.

Concerning 4) it is sufficient to consider the case \( M = 2 \). Let us consider first the case in which the first factor has \((l_1, m_1) = (1, 0)\) and the second \((l_2, m_2) = (0, 1)\). That the map

\[ \xi_1 \times \xi_2 \mapsto \kappa_{1,0} \otimes \kappa_{0,1}(\xi_1 \otimes \xi_2) = \kappa_{1,0}(\xi_1) \otimes \kappa_{0,1}(\xi_2) \]

can be extended to a map which is separately continuous as a map

\[ E_1^* \times E_2 \mapsto \mathcal{D}_k^*, \ k = 1, 2 \]

follows immediately from the extendibility property (79) asserted in the first part of our Lemma and from the property (80) which assures that

\[ \kappa_{l_k,m_k}(\xi) \in \mathcal{O}_C(\mathbb{R}^4; \mathbb{C}), \ \xi \in E_{n_k}. \]

and in particular assures that

\[ \kappa_{l_k,m_k}(\xi), \ \xi \in E_{n_k} \]

is contained within the algebra of multipliers of \( \mathcal{D}_k, k = 1, 2 \) and of \( \mathcal{D}_k^* \). This is because \( \mathcal{O}_C(\mathbb{R}^4; \mathbb{C}) \) is contained in both the algebras of multipliers \( \mathcal{O}_{MB_1} = \mathcal{O}_M, \mathcal{O}_{MB_2} \), respectively, of \( \mathcal{D}_1, \mathcal{D}_2 \), compare Subsections 5.4, 5.5 of [59] and Appendix 5. In particular the operator of pointwise multiplication by a fixed

\[ \kappa_{l_k,m_k}(\xi), \ \xi \in E_{n_k} \]

transforms continuously \( \mathcal{D}_k, k = 1, 2 \) and \( \mathcal{D}_k^*, k = 1, 2 \) into themselves.
Let us consider now the case $M = 2$ in which both factors have $(l_1, m_1) = (l_2, m_2) = (1, 0)$:

$$\xi_1 \times \xi_2 \mapsto 1_{1,0} \otimes 2_{1,0} (\xi_1 \otimes \xi_2) = 1_{1,0} (\xi_1) \cdot 2_{1,0} (\xi_2)$$

(92)

and the plane wave kernels

$$1_{1,0} \otimes 2_{1,0},$$

correspond to some fixed components of the Dirac field or its fixed component derivative. In this case the above map (92) coincides with a particular case of the map of Lemma 5. From Lemma 5 and the Proposition of Chap III §5.4, it follows that the $(\mathcal{S}_{n_1}, \mathcal{S}_{n_2})$-hypocontinuous map

$$E_{n_1} \times E_{n_2} \rightarrow \mathcal{E}_k^*, \quad k = 1, 2$$

of Lemma 5 can be uniquely extended to $(\mathcal{S}_{n_1}^*, \mathcal{S}_{n_2}^*)$-hypocontinuous map

$$E_{n_1}^* \times E_{n_2}^* \rightarrow \mathcal{E}_k^*, \quad k = 1, 2$$

with respect to the strong dual topology on each indicated space, where $\mathcal{S}_{n_k}^*$, $k = 1, 2$, is the family of all bounded sets on strong dual space $E_{n_k}$, which simply means that the map of Lemma 5 can be uniquely extended to a hypocontinuous map

$$E_{n_1}^* \times E_{n_2}^* \rightarrow \mathcal{E}_k^*, \quad k = 1, 2$$

or in particular to separately continuous map

$$E_{n_1}^* \times E_{n_2}^* \rightarrow \mathcal{E}_k^*, \quad k = 1, 2$$

with respect to the strong dual topology. Because $E_{n_k}^*, \mathcal{E}_k^*, k = 1, 2$ are all equal to strong dual spaces of reflexive Fréchet spaces $E_{n_k}, \mathcal{E}_k$, then by Thm. 41.1 the map of Lemma 5 can be uniquely extended to (jointly) continuous map

$$E_{n_1}^* \times E_{n_2}^* \rightarrow \mathcal{E}_k^*, \quad k = 1, 2$$

with respect to the strong dual topology.

Before continuing we give a commentary concerning the proof of 4), case $M = 2$ of the last Lemma. Namey in this proof we can proceed as in the proof of the second part of Lemma 4, Subsection 2.6 or respectively of Lemma 9, Subsection 2.9. Namely

$$1_{1,0} \otimes 2_{1,0}$$

we can treat as an element of

$$\mathcal{L}(\mathcal{E}_i^*, E_{n_1}^* \otimes E_{n_2}^*) \cong \mathcal{L}(E_{n_1} \otimes E_{n_2}, \mathcal{E}_i)$$

Assertion 4), case $M = 2$, will be proved if we show that

$$1_{1,0} \otimes 2_{0,1} \in \mathcal{L}(\mathcal{E}_i, E_{n_1}^* \otimes E_{n_2}^*)$$

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actually belongs to
\[ \mathcal{L}(E, E_{n_1} \otimes E_{n_2}). \]

Similarly
\[ ^2\kappa_{1,0} \otimes ^2\kappa_{0,1} \in \mathcal{L}(E, E^* \otimes E^*) \cong \mathcal{L}(E_{n_1} \otimes E_{n_2}, E^*). \]

would be extensible to an element of
\[ \mathcal{L}(E^* \otimes E^*, E^*) \cong \mathcal{L}(E_i, E_{n_1} \otimes E_{n_2}) \]

if
\[ ^2\kappa_{1,0} \otimes ^2\kappa_{0,1} \in \mathcal{L}(E_i, E_{n_1} \otimes E_{n_2}) \]

actually belongs to
\[ \mathcal{L}(E_i, E_{n_1} \otimes E_{n_2}). \]

This however is impossible because if both kernels \(^2\kappa_{1,0} \otimes ^2\kappa_{0,1}\) are associated to a fixed component of the free zero mass electromagnetic potential field (or its derivative), then easy computation shows that \(^2\kappa_{1,0} \otimes ^2\kappa_{0,1}(\phi), \phi \in \mathcal{E}_2\), has the following general form
\[
^2\kappa_{1,0} \otimes ^2\kappa_{0,1}(\phi)(p_1, p_2) = M_{1\nu}(p_1)M_{2\mu}(p_2)\tilde{\phi}(p_1 + p_2, p_{10}(p_1) + p_{20}(p_2)),
\]

where \(M_{1\nu}\) is a multiplier of \(E_{n_i}, i = 1, 2\), and
\[
p_{10}(p_1) = |p_1|, \quad p_{20}(p_2) = |p_2|.
\]

We can now easily see that
\[ ^2\kappa_{1,0} \otimes ^2\kappa_{0,1}(\phi) \]

cannot even belong to \(\mathcal{C}^\infty(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{C}^8)\), so all the more it cannot belong to
\(S(\mathbb{R}^3; \mathbb{C}^4) \otimes S(\mathbb{R}^3; \mathbb{C}^4) = E_1 \otimes E_1\) or to \(S^0(\mathbb{R}^3; \mathbb{C}^4) \otimes S^0(\mathbb{R}^3; \mathbb{C}^4) = E_2 \otimes E_2\) or to \(E_1 \otimes E_2\) or finally to \(E_2 \otimes E_1\). In particular
\[
\phi \mapsto ^2\kappa_{1,0} \otimes ^2\kappa_{0,1}(\phi) \quad (93)
\]
cannot be continuous as a map
\[ \mathcal{E}_i \mapsto E_{n_1} \otimes E_{n_2}. \]

From this it follows that
\[ ^2\kappa_{1,0} \otimes ^2\kappa_{0,1} \]
cannot be extended to an element of
\[ \mathcal{L}(E^* \otimes E^*, \mathcal{E}^*). \]
Of course from the last Lemma, part 3), it follows that the Wick product at
the same point of any number of zero mass or massive fields is a well defined
integral kernel operator belonging to
\[ \mathcal{L}(\mathcal{E}, \mathcal{L}(\mathcal{E}, \mathcal{E}^*)) \cong \mathcal{L}(\mathcal{E} \otimes \mathcal{E}, \mathcal{E}^*) \]
in the sense of Obata [38] with vector-valued kernel. We therefore have the
following

**PROPOSITION.**

1) For the Wick product at the same space-time point \( x \)
\[ : \Xi_{l_1, m_1}^{(n_1} \kappa_{l_1, m_1}(x) \cdots \Xi_{l_M, m_M}^{(n_M} \kappa_{l_M, m_M}(x)) : \]
\[ = \Xi_{l, m}(\kappa_{l}(x)), \ k \kappa_{l_k, m_k} \in \mathbb{K} \]
of the integral kernel operators corresponding to the free fields of the theory
or their derivatives we have
\[ \kappa_{l, m} = \left( \kappa_{l_k, m_k} ^{n_1} \otimes \cdots \otimes \kappa_{l_M, m_M} ^{n_M} \right) \]
\[ \in \mathcal{E} ^* \otimes \cdots \otimes \mathcal{E} _{n_M} ^* \otimes \mathcal{E} _{i} ^* \cong \mathcal{L}(\mathcal{E} ^1 \otimes \cdots \otimes \mathcal{E} _{n_M}) = \mathcal{L}(\mathcal{E} ^1, \mathcal{E} ^{*}) \]

Thus by (the generalization to tensor product of Fock spaces of) Thm. 3.9
of [38]
\[ : \Xi_{l_1, m_1}^{(n_1} \kappa_{l_1, m_1}(x) \cdots \Xi_{l_M, m_M}^{(n_M} \kappa_{l_M, m_M}(x)) : \]
\[ = \Xi_{l, m}(\kappa_{l}(x)) \in \mathcal{L}(\mathcal{E}, \mathcal{E} ^{*}) \cong \mathcal{L}(\mathcal{E}, \mathcal{E} ^{*}) \]

2) If all \( n_k = 1 \), i.e. among the factors
\[ \Xi_{l_1, m_1}^{(n_1} \kappa_{l_1, m_1}(x) \cdots \Xi_{l_M, m_M}^{(n_M} \kappa_{l_M, m_M}(x)) : \]
there are no integral kernel operators corresponding to mass less free fields
(electromagnetic potential field in case of QED) or their derivatives, then
(by 4) of the preceding Lemma) the bilinear map
\[ \xi \times \eta \mapsto \kappa_{l}(\xi \otimes \eta), \]
\[ \xi \in \mathcal{E} _{1} ^0 \otimes \cdots \otimes \mathcal{E} _{i} ^0, \ \text{first } l \text{ terms } E, i_j \in \{1, 2\}, \]
\[ \eta \in \mathcal{E} _{i+1} ^0 \otimes \cdots \otimes \mathcal{E} _{i+m} ^0, \ \text{last } m \text{ terms } E, i_j \in \{1, 2\}, \]
can be extended to a separately continuous bilinear map from
\[ \left( \mathcal{E} _1 ^0 \otimes \cdots \otimes \mathcal{E} _i ^0 \right) ^* \times \left( \mathcal{E} _{i+1} ^0 \otimes \cdots \otimes \mathcal{E} _{i+m} ^0 \right) \rightarrow \mathcal{L}(\mathcal{E}, \mathbb{C}) = \mathcal{E} ^{*}. \]
Thus by Thm. 4, Subsection 2.6:

\[ \Xi_{l,m_1}(\kappa_{l,m_1}) \cdots \Xi_{l,M}(\kappa_{l,M,m_M}) : \]

\[ = \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}((\mathcal{E}) \otimes \mathcal{E}^\ast, \mathcal{E}) \cong \mathcal{L}(\mathcal{E}, \mathcal{L}((\mathcal{E}), (\mathcal{E}))) \]

Now we pass to the operation of differentiation with respect to space-time coordinates. Suppose we have an integral kernel operator \( \Xi_{l,m}(\kappa_{l,m}) \) with vector-valued kernel

\[ \kappa_{l,m} \in \mathcal{L}(\mathcal{E^r}, (E_{i_1} \otimes \cdots \otimes E_{i_{i+m}})^\ast) \cong \mathcal{L}(E_{i_1} \otimes \cdots \otimes E_{i_{i+m}}, \mathcal{E}^\ast) \]

with the operator

\[ \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(\mathcal{E}, \mathcal{L}((\mathcal{E}), (\mathcal{E})^\ast)) \cong \mathcal{L}((\mathcal{E}) \times \mathcal{E}, (\mathcal{E})^\ast) \]

uniquely determined by

\[ \langle \langle \Xi_{l,m}(\kappa_{l,m})(\Phi \otimes \phi), \Psi \rangle \rangle = \langle \langle \Xi_{l,m}(\kappa_{l,m}(\phi)) \Phi, \Psi \rangle \rangle = \langle \kappa_{l,m}(\phi) \eta_{\phi, \psi} = \kappa_{l,m}(\eta_{\phi, \psi}), \phi \rangle, \Phi, \Psi \in (\mathcal{E}), \phi \in \mathcal{E}, \]

compare (68) Subsection 2.6. Suppose moreover that

\[ \mathcal{E} = \mathcal{E}_1 = S_{H(1)}(\mathbb{R}_4; \mathbb{C}) = S(\mathbb{R}_4; \mathbb{C}) \text{ or} \]

\[ \mathcal{E} = \mathcal{E}_2 = S_{\mathcal{A}(1)}(\mathbb{R}_4; \mathbb{C}) = S^{\mathbb{C}_0}(\mathbb{R}_4; \mathbb{C}). \]

Let for \( \kappa_{l,m} \) understood as an element of

\[ \mathcal{L}(E_{i_1} \otimes \cdots \otimes E_{i_{i+m}}, \mathcal{E}^\ast) \cong \mathcal{L}(\mathcal{E}, (E_{i_1} \otimes \cdots \otimes E_{i_{i+m}})^\ast) \]

we have

\[ \kappa_{l,m}(\xi_1 \otimes \cdots \otimes \xi_{i+m}) \in \mathcal{O}_{\mathbb{C}}(\mathbb{R}_4; \mathbb{C}), \quad \xi_k \in E_{i_k}, i_k \in \{1,2\}. \]

We moreover include into consideration the special cases of integral kernel operators

\[ \Xi_{0,1}(1^{1}\kappa_{0,1}), \Xi_{0,1}(1^{1}\kappa_{1,0}), \Xi_{0,1}(2^{1}\kappa_{1,0}), \Xi_{1,0}(1^{2}\kappa_{1,0}), \Xi_{1,0}(1^{2}\kappa_{1,0}), \Xi_{1,0}(2^{2}\kappa_{1,0}), \]

(94)
determined by the free fields of the theory with the integration in the general formula (69) is restricted, respectively, only to fermi or only to bose variables, and the Wick products of (94) at the same space-time point (representing ordinary integral kernel operators (69) with vector-valued kernels and integration with integration in general ranging over both, bose and fermi, variables if the Wick product involves both, bose and fermi, field components).

Then we can define the space-time derivative

\[ \left( \frac{\partial}{\partial x^\mu} \Xi_{l,m}(\kappa_{l,m}) \right) \]

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as the integral kernel operator uniquely determined by the condition

\[
\langle \left( \frac{\partial}{\partial x^\mu} \Xi_{l,m} \right) (\kappa_{l,m}), (\Phi \otimes \phi) \rangle = \langle \Xi_{l,m} \left( \left( \frac{\partial}{\partial x^\mu} \kappa_{l,m} \right) (\phi) \right), \Psi \rangle
\]

\[
\begin{align*}
&= -\langle \Xi_{l,m} \left( \left( \frac{\partial}{\partial x^\mu} \kappa_{l,m} \right) (\phi) \right), \eta_{\Phi, \Psi} \rangle \\
&= -\langle \kappa_{l,m} \left( \left( \frac{\partial}{\partial x^\mu} \phi \right) \right), \eta_{\Phi, \Psi} \rangle,
\end{align*}
\]

and thus by

\text{RULE III}'

We have the following computational rule

\[
\left( \frac{\partial}{\partial x^\mu} \Xi_{l,m} \right) (\kappa_{l,m}) = \Xi_{l,m} \left( \frac{\partial}{\partial x^\mu} \kappa_{l,m} \right)
\]

for \( \kappa_{l,m} \) understood as an element of

\[
\mathcal{L}(\mathcal{S}, (E_{i_1} \otimes \cdots \otimes E_{i_l+m})^*) \cong \mathcal{L}(E_{i_1} \otimes \cdots \otimes E_{i_l+m}, \mathcal{S}^*).
\]

Thus the operation of space-time differentiation performed on \( \Xi(\kappa_{l,m}) \) corresponds, via the Rule III, to the operation of differentiation performed upon the vector-valued distributional kernel \( \kappa_{l,m} \), understood as an \( (E_{i_1} \otimes \cdots \otimes E_{i_l+m})^* \)-valued distribution on the test function space \( \mathcal{S} \). Again the Rule III can be justified by utilizing the fact that

\[
\Xi_{l,m}(\kappa_{l,m}(x)) = \int_{(\mathbb{R}^3)^{(l+m)}} \kappa_{l,m}(w_{i_1}, \ldots, w_{i_l}, w_{i_{l+1}}, \ldots, w_{i_{l+m}}; x) \\
\times \partial_{w_{i_1}}^* \cdots \partial_{w_{i_l}}^* \partial_{w_{i_{l+1}}} \cdots \partial_{w_{i_{l+m}}} dw_{i_1} \cdots dw_{i_l} dw_{i_{l+1}} \cdots dw_{i_{l+m}} = \int_{(\mathbb{R}^3)^{(l+m)}} \kappa_{l,m}(w_{i_1}, \ldots, w_{i_l}, u_{j_1}, \ldots, u_{j_m}; x) \\
\times \partial_{w_{i_1}}^* \cdots \partial_{w_{i_l}}^* \partial_{u_{j_1}} \cdots \partial_{u_{j_m}} dw_{i_1} \cdots dw_{i_l} du_{j_1} \cdots du_{j_m}
\]

exists pointwisely as a Pettis integral, just repeating the arguments in construction of space-time derivatives of the free electromagnetic potential field during the proof of Bogoliubov-Shirkov Quantization Postulate, compare Subsection 5.9 of [59]. Moreover during this proof we have given justification of the following Rules IV, V and VI.

For the integral kernel operator \( \Xi_{l,m} \) we have

\text{RULE IV'}

\[
\int_{\mathbb{R}^4} \Xi_{l,m} \left( \kappa_{l,m}(x) \right) d^4x = \Xi_{l,m} \left( \int_{\mathbb{R}^4} \kappa_{l,m}(x) d^4x \right).
\]

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RULE V*  

$$\int_{\mathbb{R}^4} \Xi_{l,m}(\kappa_{l,m}(x, x_0)) \, d^3x = \Xi_{l,m} \left( \int_{\mathbb{R}^4} \kappa_{l,m}(x, x_0) \, d^3x \right).$$  

Let $S \in \mathcal{S}(\mathbb{R}^4; \mathbb{C})^*$ then

RULE VI  

$$S \ast \Xi_{l,m}(\kappa_{l,m})(x) = \int_{\mathbb{R}^4} S(x - y) \Xi_{l,m}(\kappa_{l,m}(y)) \, d^4y = \Xi_{l,m} \left( \int_{\mathbb{R}^4} S(x - y) \kappa_{l,m}(y) \, d^4y \right) = \Xi_{l,m}(S \ast \kappa_{l,m}(x)).$$  

Here

$$S \ast \kappa_{l,m}(\xi_1, \ldots, \xi_{l+m})(x) = \int_{\mathbb{R}^4} S(x - y) \kappa_{l,m}(w_{i_1}, \ldots, w_{i_l+m}; y) \xi_{i_1}(w_{i_1}), \ldots, \xi_{i_{l+m}}(w_{i_{l+m}}) \, d^4y, \quad \xi_{i_k} \in E_{i_k}$$

is well defined because

$$\kappa_{l,m}(\xi_1 \otimes \cdots \otimes \xi_{l+m}) \in \mathcal{O}_C(\mathbb{R}^4; \mathbb{C}) \subset \mathcal{O}'_C(\mathbb{R}^4; \mathbb{C}),$$

and by definition is equal to the (kernel of the) distribution $S* (\kappa_{l,m}(\xi_1, \ldots, \xi_{l+m}))$, compare Appendix 5.

The Rules III*, IV*, V*, VI are also valid in case of more than just one space-time variable $x$. In order to see it we can repeat the proof replacing $E$ (previously equal to $E_1 = \mathcal{S}(\mathbb{R}^4; \mathbb{C})$ or $E_2 = \mathcal{S}^{00}(\mathbb{R}^4; \mathbb{C})$) by $E$ equal to tensor product of several $E_1$ or $E_2$. In this case we would obtain more generally with

$$\kappa_{l,m}(\xi_1 \otimes \cdots \otimes \xi_{l+m}; x_1, \ldots, x_n) \in \mathcal{O}_C(\mathbb{R}^{4n}; \mathbb{C})$$

the integral kernel operator

$$\Xi_{l,m}(\kappa_{l,m}(x_1, \ldots, x_n)) = \int_{(\mathbb{R}^4)^{l+m}} \kappa_{l,m}(w_{i_1}, \ldots, w_{i_l}, w_{i_{l+1}}, \ldots, w_{i_{l+m}}; x_1, \ldots, x_n) \times$$

$$\times \partial_{w_{i_1}}^* \cdots \partial_{w_{i_l}}^* \partial_{w_{i_{l+1}}} \cdots \partial_{w_{i_{l+m}}} \, dw_{i_1} \cdots dw_{i_l} \, dw_{i_{l+1}} \cdots dw_{i_{l+m}} =$$

$$= \int_{(\mathbb{R}^4)^{l+m}} \kappa_{l,m}(w_{i_1}, \ldots, w_{i_l}, u_{j_1}, \ldots, u_{j_m}; x_1, \ldots, x_n) \times$$

$$\times \partial_{w_{i_1}}^* \cdots \partial_{w_{i_l}}^* \partial_{u_{j_1}} \cdots \partial_{u_{j_m}} \, dw_{i_1} \cdots dw_{i_l} \, du_{j_1} \cdots du_{j_m} \quad (96)$$

existing pointwisely as a Pettis integral and with the following Rules:
RULE III

\[ \left( \frac{\partial^n}{\partial x_1^\mu_1 \cdots \partial x_n^\mu_n} \Xi_{\ell,m} \right)(\kappa_{\ell,m}) = \Xi_{\ell,m} \left( \frac{\partial^n}{\partial x_1^\mu_1 \cdots \partial x_n^\mu_n} \kappa_{\ell,m} \right) \]

for \( \kappa_{\ell,m} \) understood as an element of \( \mathcal{L}(E, \left( E_{i_1} \otimes \cdots \otimes E_{i_{l+m}} \right)^*) \) with \( E = E_{n_1} \otimes \cdots \otimes E_{n_n}, \quad n_k \in \{1, 2\} \).

RULE IV

\[ \int_{\mathbb{R}^n} \Xi_{\ell,m}(x_1, \ldots, x_n) \, d^4x_1 \ldots d^4x_n = \Xi_{\ell,m} \left( \int_{\mathbb{R}^n} \kappa_{\ell,m}(x_1, \ldots, x_n) \, d^4x_1 \ldots d^4x_n \right) . \]

RULE V

\[ \int_{\mathbb{R}^n} \Xi_{\ell,m}(x_1, \ldots, x_n) \, d^3x_1 \ldots d^3x_n = \Xi_{\ell,m} \left( \int_{\mathbb{R}^4} \kappa_{\ell,m}(x_1, \ldots, x_n, x_{n+1}) \, d^3x_1 \ldots d^3x_n \right) . \]

Now concerning the Rule VI for more space-time variables we can repeatedly combine the convolutions of several distributions \( S \in \mathcal{S}(\mathbb{R}^4; \mathbb{C})^* \) each in one space-time variable, with the Wick product operation provided the corresponding kernels \( \kappa_{\ell,m} \) obtained in the intermediate steps are well defined elements of \( \mathcal{L}(E_{i_1} \otimes \cdots \otimes E_{i_{l+m}}, \mathcal{E}^*) \) with \( \kappa_{\ell,m}(\xi_{i_1} \otimes \cdots \otimes \xi_{i_{l+m}})(x_{n_1}, \ldots) \in \mathcal{O}_C \).

Namely we have the following useful Lemma which allows us to operate with convolutions of integral kernel operators with tempered distributions \( S \in \mathcal{S}(\mathbb{R}^4; \mathbb{C})^* \):

**Lemma 8.** Let \( S \in \mathcal{S}(\mathbb{R}^4; \mathbb{C})^* \), and let

\[ \kappa_{\ell,m} \in \mathcal{L}(E_{i_1} \otimes \cdots \otimes E_{i_{l+m}}) \]

with \( \kappa_{\ell,m}(\xi_1 \otimes \cdots \otimes \xi_{l+m}) \in \mathcal{O}_C(\mathbb{R}^4; \mathbb{C}), \quad \xi_k \in E_{i_k}, \quad i_k \in \{1, 2\} \).

In particular this is the case (compare 1), 2), and 3) of Lemma 7) for the kernel

\[ \kappa_{\ell,m} = \left( ^{n_1}_{1} \kappa_{1,m_1} \right) \otimes \cdots \otimes \left( ^{n_M}_{M} \kappa_{M,m_M} \right) \]
corresponding to the Wick product (at the same space-time point $x$)

\[
\Xi_{l,m}(\kappa_{l,m}(x)) = \Xi_{l_1,m_1}(\kappa_{l_1,m_1}(x)) \cdots \Xi_{l_M,m_M}(\kappa_{l_M,m_M}(x)),
\]

of the integral kernel operators

\[
\Xi_{l_k,m_k}(\kappa_{l_k,m_k}(x)), \quad \kappa_{l_k,m_k} \in \mathfrak{K}.
\]

Let the integral kernel $S * \kappa_{l,m}$ be equal

\[
\langle S \ast \kappa_{l,m}(\xi_1 \otimes \cdots \otimes \xi_{l+m}), \phi \rangle = \int_{\mathbb{R}^4} S * \kappa_{l,m}(\xi_1, \ldots, \xi_{l+m})(x) \phi(x) \, d^4x
\]

\[
\int_{\mathbb{R}^4 \times \mathbb{R}^4} S(x-y)\kappa_{l,m}(w_1, \ldots, w_{l+m}; y) \xi_1(w_1), \ldots, \xi_{l+m}(w_{l+m}) \, dw_1 \cdots dw_{l+m} \, d^4y \, d^4x,
\]

$\xi_{l_k} \in E_{l_k}, \phi \in \mathcal{E} = \mathcal{S}(\mathbb{R}^4; \mathbb{C})$ or $\mathcal{E} = \mathcal{S}^{00}(\mathbb{R}^4; \mathbb{C})$.

Then

1) the kernel $S * \kappa_{l,m} \in \mathcal{L}(E_{l_1} \otimes \cdots \otimes E_{l+m}, \mathcal{E}^*)$;

2) and if

\[
\kappa_{l,m} = \left( \kappa_{l_1,m_1} \right)_{1} \otimes \cdots \otimes \left( \kappa_{l_M,m_M} \right)_{M}, \quad \kappa_{l_k,m_k} \in \mathfrak{K}
\]

then

\[
S \ast \kappa_{l,m}(\xi_1, \ldots, \xi_{l+m}) \in \mathcal{O}_C(\mathbb{R}^4; \mathbb{C}) \subset \mathcal{O}'_C(\mathbb{R}^4; \mathbb{C}).
\]

It is sufficient to consider the case $\mathcal{E} = \mathcal{E}_1 = \mathcal{S}(\mathbb{R}^4; \mathbb{C})$, because $\mathcal{E}_1^*$ is continuously embedded into $\mathcal{E}_2^* = \mathcal{S}^{00}(\mathbb{R}^4; \mathbb{C})^*$, compare Subsection 5.5 of [59].

Because the Schwartz’ algebra $\mathcal{O}_C(\mathbb{R}^4; \mathbb{C})$ of convolutors of $\mathcal{S}(\mathbb{R}^4; \mathbb{C})^*$ (for definition of $\mathcal{O}_C$ compare e.g. [47] or Appendix 5) is dense in $\mathcal{S}(\mathbb{R}^4; \mathbb{C})^*$ in the strong dual topology, then for $\epsilon > 0$ we can find $S_\epsilon \in \mathcal{O}_C$ such that

\[
\lim_{\epsilon \to 0} S_\epsilon = S
\]

in the strong topology of the dual space $\mathcal{S}(\mathbb{R}^4; \mathbb{C})^*$ of tempered distributions.

Let $\xi$ be any element of

$E_{i_1} \otimes \cdots \otimes E_{i_{l+m}}$.

For $\epsilon > 0$ we define the following linear operator $\Lambda_\epsilon$

\[
\Lambda_\epsilon(\xi) \overset{\text{df}}{=} S_\epsilon * \kappa_{l,m}(\xi), \quad \xi \in E_{i_1} \otimes \cdots \otimes E_{i_{l+m}}.
\]

on

$E_{i_1} \otimes \cdots \otimes E_{i_{l+m}}$.
Because $S_{\epsilon} \in \mathcal{O}_{C}'$, $\epsilon > 0$, and because

$$\kappa_{l,m} \in \mathcal{L}(E_{i_1} \otimes \cdots \otimes E_{i_{l+m}}, \mathcal{E}^*)$$

then for each $\epsilon > 0$ the operator

$$\Lambda_{\epsilon} : E_{i_1} \otimes \cdots \otimes E_{i_{l+m}} \rightarrow \mathcal{E}^*$$

is continuous, i.e.

$$\Lambda_{\epsilon} \in \mathcal{L}(E_{i_1} \otimes \cdots \otimes E_{i_{l+m}}, \mathcal{E}^*)$$

For each $\xi \in E_{i_1} \otimes \cdots \otimes E_{i_{l+m}}$

$$\kappa_{l,m}(\xi) \in \mathcal{O}_{C} \subset \mathcal{O}_{C}'$$

and

$$\lim_{\epsilon \to 0} S_{\epsilon} = S$$

in strong dual topology of $S(\mathbb{R}^4)^* = \mathcal{E}^*$

so for each $\xi \in E_{i_1} \otimes \cdots \otimes E_{i_{l+m}}$

$$\lim_{\epsilon \to 0} \Lambda_{\epsilon}(\xi) = \lim_{\epsilon \to 0} S_{\epsilon} \ast \kappa_{l,m}(\xi)$$

in strong dual topology of $\mathcal{E}^*$ exists and is equal

$$\lim_{\epsilon \to 0} \Lambda_{\epsilon}(\xi) = S \ast \kappa_{l,m}(\xi)$$

(compare Appendix 5 and references cited there).

Because $E_{i_1} \otimes \cdots \otimes E_{i_{l+m}}$ is a complete Fréchet space then by the Banach-Steinhaus theorem (e.g. Thm. 2.8 of [44]) it follows that $S \ast \kappa_{l,m}$ is a continuous linear operator $E_{i_1} \otimes \cdots \otimes E_{i_{l+m}} \rightarrow \mathcal{E}^*$, i.e.

$$S \ast \kappa_{l,m} \in \mathcal{L}(E_{i_1} \otimes \cdots \otimes E_{i_{l+m}}, \mathcal{E}^*)$$

If $\mathcal{E} = S^{00}(\mathbb{R}^4; \mathbb{C})$ then $S$ can be extended over to an element of $S^{00}(\mathbb{R}^4; \mathbb{C})^*$ (Hahn-Banach theorem), and the above proof can be repeated, because the algebra of convolutors of $S^{00}(\mathbb{R}^4; \mathbb{C})^*$ is dense in $S^{00}(\mathbb{R}^4; \mathbb{C})^*$ and contains $\mathcal{O}_{C}(\mathbb{R}^4; \mathbb{C})$ (compare Subsection 5.4, 5.5 of [59] and Appendix 5). This completes the proof of part 1).

The assertion 2) follows by an explicit verification and essentially repetition of the proof of the analogue assertion of Lemma 4, Subsection 2.6 or respectively Lemma 9, Subsection 2.9. ■

REMARK. We should emphasize here that the mere assumption

$$\kappa_{l,m}(\xi_1 \otimes \cdots \otimes \xi_{l+m}) \in \mathcal{O}_{C}(\mathbb{R}^4; \mathbb{C}), \ \xi_k \in E_{i_k}, \ i_k \in \{1, 2\}$$

would be insufficient for

$$S \ast \kappa_{l,m}(\xi_1, \ldots, \xi_{l+m}) \overset{\text{df}}{=} S \ast (\kappa_{l,m}(\xi_1, \ldots, \xi_{l+m}))$$
to be an element of $O_C \subset O'_C$. Indeed it is the special property of the plane wave distribution kernels defining the free fields which assures the validity of the assertion 2). Moreover the fact that the space $E_2$ is equal

$$S^0(\mathbb{R}^3; \mathbb{C}^4) \neq S(\mathbb{R}^3; \mathbb{C}^4)$$

intervenes here nontrivially. For the wrong space $S(\mathbb{R}^3; \mathbb{C}^4)$ used for $E_2$ the assertion 2) would be false. But both parts, 1) and 2), are important for the construction of higher order contributions to interacting fields understood as well defined integral kernel operators with vector-valued kernels. Analogue situation we encounter for any other zero mass field for which the corresponding space $E_2$ must be equal $S^0(\mathbb{R}^3; \mathbb{C}^4)$.

From the Rule VI and Lemma 8 it follows the following

**PROPOSITION.** If

$$\kappa_{l,m} \in \mathcal{L}(\mathcal{E}, (E_{i_1} \otimes \cdots \otimes E_{i_l + m})^*) \cong \mathcal{L}(E_{i_1} \otimes \cdots \otimes E_{i_l + m}, \mathcal{E}^*)$$

with

$$\kappa_{l,m}(\xi_1 \otimes \cdots \otimes \xi_{l+m}; x) \in O_C(\mathbb{R}^4; \mathbb{C}), \quad \xi_k \in E_{i_k}, \quad i_k \in \{1, 2\},$$

and $S \in S(\mathbb{R}^4; \mathbb{C}^*)$, then the operator

$$S \ast \Xi_{l,m}(\kappa_{l,m})(x) = \int_{\mathbb{R}^4} S(x - y)\Xi_{l,m}(\kappa_{l,m}(y)) \, dy$$

defines integral kernel operator

$$\Xi_{l,m}(S \ast \kappa_{lm}) \in \mathcal{L}(\mathcal{E}, (E_{i_1} \otimes \cdots \otimes E_{i_l + m})^*) \cong \mathcal{L}(\mathcal{E}, \mathcal{L}(E_{i_1} \otimes \cdots \otimes E_{i_l + m}, \mathcal{E}^*))$$

with the vector-valued kernel

$$S \ast \kappa_{lm} \in \mathcal{L}(\mathcal{E}, (E_{i_1} \otimes \cdots \otimes E_{i_l + m})^*) \cong \mathcal{L}(E_{i_1} \otimes \cdots \otimes E_{i_l + m}, \mathcal{E}^*).$$

If moreover

$$\kappa_{l,m} = \left(\begin{array}{cc} n_1 \kappa_{l_1,m_1} \\ \vdots \\ n_M \kappa_{l_M,m_M} \end{array} \right) \otimes \left(\begin{array}{c} n_k \kappa_{l_k,m_k} \end{array} \right), \quad n_k \kappa_{l_k,m_k} \in \mathbb{R}_0$$

then

$$S \ast \kappa_{lm}(\xi_1, \ldots, \xi_{l+m}) \in O_C(\mathbb{R}^4; \mathbb{C}) \subset O'_C(\mathbb{R}^4; \mathbb{C}).$$

**THEOREM 5.** Let

$$\psi(x) = \Xi_{0,1}(\kappa_{0,1}(x)) + \Xi_{1,0}(\kappa_{1,0}(x)), \quad A = \Xi_{0,1}(\kappa_{0,1}(x)) + \Xi_{1,0}(\kappa_{1,0}(x)),$$
be the integral kernel operators defining the free fields of the spinor QED. Let

\[
\psi^a_{\text{out}}(g = 1, x) = \psi^a(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^n} d^4x_1 \cdots d^4x_n \psi^a(n)(x_1, \ldots, x_n; x),
\]

with

\[
\psi^{a(1)}(x_1; x) = e S_{\text{ret}}^a \gamma^{a_1 a_2} \psi^{a_2}(x_1) A_{\nu_1}(x_1),
\]

\[
\psi^{a(2)}(x_1, x_2; x) = e^2 \left\{ S_{\text{ret}}^a (x - x_1) \gamma^{a_1 a_2} S_{\text{ret}}^a (x_1 - x_2) \gamma^{a_3 a_4} : \psi^{a_4}(x_2) A_{\nu_1}(x_1) A_{\nu_2}(x_2) : - S_{\text{ret}}^a (x - x_1) \gamma^{a_1 a_2} : \psi^{a_2}(x_1) \psi^{a_3}(x_2) \gamma^{a_3 a_4} \psi^{a_4}(x_2) : D^a_{\mu}(x_1 - x_2) + S_{\text{ret}}^a (x - x_1) \Sigma^{a_1 a_2} (x_1 - x_2) \psi^{a_2}(x_2) \right\} + \left\{ x_1 \leftrightarrow x_2 \right\}.
\]

and let

\[
A_{\text{out} \mu}(g = 1, x) = A_{\mu}(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^n} d^4x_1 \cdots d^4x_n A^{(n)}_{\mu}(x_1, \ldots, x_n; x),
\]

with

\[
A^{(1)}_{\mu}(x_1; x) = - e D^a_{\mu}(x_1 - x) : \overline{\psi}^a(x_1) \gamma^{a_1 a_2} \psi^{a_2}(x_1) :,
\]

\[
A^{(2)}_{\mu}(x_1, x_2; x) = e^2 \left\{ : \overline{\psi}^a(x_1) \left( \gamma^{a_1 a_2} S_{\text{ret}}^a (x_1 - x_2) \gamma^{a_3 a_4} D^a_{\mu}(x_1 - x) A_{\nu_1}(x_2) + \gamma^{a_1 a_2} S_{\text{ret}}^a (x_1 - x_2) \gamma^{a_3 a_4} D^a_{\mu}(x_2 - x) A_{\nu_1}(x_1) \right) \psi^{a_4}(x_2) : + D^a_{\mu}(x_1 - x) \Pi^{a_{\nu_1}}_{\mu}(x_2 - x) A_{\nu_1}(x_2) \right\} + \left\{ x_1 \leftrightarrow x_2 \right\}
\]

be equal to the formulas for (fixed components \(a\) and \(\mu\)) of interacting Dirac and electromagnetic fields \(\psi_{\text{int}}\) and \(A_{\text{int}}\) in the causal Stückelberg-Bologoluivob spinor QED, [13], [9] or [40], in which the intensity-of-interaction function \(g\) is put equal to the constant 1.

If the free fields \(\psi(x), A(x)\) in these formulas for \(\psi_{\text{out}}\) and \(A_{\text{out}}\) are understood as integral kernel operators

\[
\psi(x) = \Xi_{0,1}(^1 \kappa_{0,1}(x)) + \Xi_{1,0}(^1 \kappa_{1,0}(x)), \quad A = \Xi_{0,1}(^2 \kappa_{0,1}(x)) + \Xi_{1,0}(^2 \kappa_{1,0}(x)),
\]

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and correspondingly the operations of Wick product $: \cdots :$ and integrations $\int_4 x_1 \cdots \int_4 x_n$ involved in the formulas for $\psi_{\alpha g}$ and $A_{\alpha g}$ are understood as Wick products and integrations of integral kernel operators with vector valued distributional kernels (which as we know have the properties expressed by the Rules I-VI), then each $n$-th order term contribution

$$\psi_{\alpha g}^{(n)}(g = 1, x) = \frac{1}{n!} \int_{\mathbb{R}^{4n}} \mathrm{d}^4 x_1 \cdots \mathrm{d}^4 x_n \psi_{\alpha g}^{(n)}(x_1, \ldots, x_n; x),$$

$$A_{\alpha g \mu}^{(n)}(g = 1, x) = \frac{1}{n!} \int_{\mathbb{R}^{4n}} \mathrm{d}^4 x_1 \cdots \mathrm{d}^4 x_n A_{\alpha g \mu}^{(n)}(x_1, \ldots, x_n; x),$$

respectively, to the interacting field $\psi_{\alpha g}^{a}(g = 1, x)$ and $A_{\alpha g \mu}(g = 1, x)$ is equal to a finite sum

$$\sum_{l,m} \Xi(\kappa_{l,m}(x)) \text{ respectively } \sum_{l,m} \Xi(\kappa'_{l,m}(x))$$

of integral kernel operators

$$\Xi_{l,m}(\kappa_{l,m}(x)), \text{ respectively } \Xi(\kappa'_{l,m}(x))$$

which define integral kernel operators

$$\Xi_{l,m}(\kappa_{l,m}(x)) \in \mathcal{L}((E) \otimes E_1, E_1^*) \cong \mathcal{L}(E_1, \mathcal{L}((E), (E)^*)),$$

respectively

$$\Xi_{l,m}(\kappa'_{l,m}(x)) \in \mathcal{L}((E) \otimes E_2, E_2^*) \cong \mathcal{L}(E_1, \mathcal{L}((E), (E)^*))$$

with vector-valued distributional kernels

$$\kappa_{l,m} \in \mathcal{L}(E_1 \otimes \cdots \otimes E_{i_l+m}, E_1^*)$$

$$\kappa'_{l,m} \in \mathcal{L}(E_1 \otimes \cdots \otimes E_{i_l+m}, E_2^*).$$

Thus each $n$-th order term contribution $\psi_{\alpha g}^{a,(n)}(g = 1)$ and $A_{\alpha g \mu}^{(n)}(g = 1)$, respectively, to interacting fields $\psi_{\alpha g}^{a}(g = 1)$ and $A_{\alpha g \mu}(g = 1)$ is equal

$$\psi_{\alpha g}^{a,(n)}(g = 1) = \sum_{l,m} \Xi(\kappa_{l,m}),$$

$$A_{\alpha g \mu}^{(n)}(g = 1) = \sum_{l,m} \Xi(\kappa'_{l,m}),$$

to a finite sum of well defined integral kernel operators $\Xi(\kappa_{l,m}), \Xi(\kappa'_{l,m})$ with vector-valued distributional kernels $\kappa_{l,m}, \kappa'_{l,m}$ in the sense of Obata [38] (compare Subsection 2.6).

- The proof follows by induction and the repeated application of the Rules I-VI and the fundamental Lemma [8].
REMARK. Note that each $n$-th order contribution $\psi^{(n)}_{\mu}(g = 1)$ and $A^{(n)}_{\mu}(g = 1)$ to interacting fields $\psi^{a}(g = 1)$ and $A_{\mu}(g = 1)$ belongs to the same general class of (finite sums of) integral kernel operators (with vector-valued kernels) as the Wick products (at fixed space-time point) of mass less fields. In fact some of the contributions to interacting fields are finite sums of integral kernel operators which even belong to a much better behaved class of integral kernel operators, which belong to

$$\mathcal{L}((E) \otimes \mathcal{E}_1, (E)) \cong \mathcal{L}(\mathcal{E}_1, \mathcal{L}((E), (E)))$$

respectively

$$\mathcal{L}((E) \otimes \mathcal{E}_2, (E)) \cong \mathcal{L}(\mathcal{E}_2, \mathcal{L}((E), (E))).$$

In particular one can show that the first order contribution $A^{(1)}_{\mu}(g = 1)$ to the interacting electromagnetic potential field $A_{\mu}(g = 1)$ belongs to

$$\mathcal{L}((E) \otimes \mathcal{E}_2, (E)) \cong \mathcal{L}(\mathcal{E}_2, \mathcal{L}((E), (E))).$$

Let us emphasize here that the Wick product (at the same space-time point) of mass less free fields (or containing such among the factors) does not belong to

$$\mathcal{L}((E) \otimes \mathcal{E}_1, (E)) \cong \mathcal{L}(\mathcal{E}_1, \mathcal{L}((E), (E)))$$

respectively

$$\mathcal{L}((E) \otimes \mathcal{E}_2, (E)) \cong \mathcal{L}(\mathcal{E}_2, \mathcal{L}((E), (E))).$$

But we know that such product, as an integral kernel operator with vector-valued kernel, belongs to

$$\mathcal{L}((E) \otimes \mathcal{E}_1, (E^*)) \cong \mathcal{L}(\mathcal{E}_1, \mathcal{L}((E), (E^*)))$$

respectively

$$\mathcal{L}((E) \otimes \mathcal{E}_2, (E^*)) \cong \mathcal{L}(\mathcal{E}_2, \mathcal{L}((E), (E^*))).$$

Similarly we know that each order term contribution to interacting fields is a finite sum of integral kernel operators which belong to

$$\mathcal{L}((E) \otimes \mathcal{E}_1, (E^*)) \cong \mathcal{L}(\mathcal{E}_1, \mathcal{L}((E), (E^*)))$$

respectively

$$\mathcal{L}((E) \otimes \mathcal{E}_2, (E^*)) \cong \mathcal{L}(\mathcal{E}_2, \mathcal{L}((E), (E^*))).$$

But at least some of them, e.g. the first order contribution $\psi^{a}_{\nu}(g = 1)$ to the interacting Dirac field $\psi^{a}(g = 1)$, do not belong to

$$\mathcal{L}((E) \otimes \mathcal{E}_1, (E)) \cong \mathcal{L}(\mathcal{E}_1, \mathcal{L}((E), (E))).$$

Nonetheless the contributions to interacting fields are finite sums of integral kernel operators which belong to the same general class as the integral kernel
operators which are equal to Wick products (at the same space-time point) of mass less free fields.

One can even show that if the Wick products (at the same space-time point) of free fields (including mass less fields) were equal to finite sums of integral kernel operators belonging to

\[ \mathcal{L}((E \otimes \delta_1, (E)) \cong \mathcal{L}(\delta_1, \mathcal{L}((E), (E))), \]

respectively

\[ \mathcal{L}((E \otimes \delta_2, (E)) \cong \mathcal{L}(\delta_2, \mathcal{L}((E), (E))), \]

then the same would be true of the contributions to interacting fields. But the assumption about the Wick product necessary to infer this conclusion is however false (compare the corresponding Proposition of this Subsection).

2.8 Comparison with the standard realization of the free Dirac field \( \psi \). Bogoliubov-Shirkov quantization postulate

In our formula (71) for the free Dirac field \( \psi(x) \):

\[
\psi(x) = \sum_{s=1}^{2} \frac{1}{2|p_0(p)|} \int_{\mathbb{R}^3} u_s(p) e^{-ip \cdot x} b_s(p) \, d^3p
\]

\[
+ \sum_{s=1}^{2} \int_{\mathbb{R}^3} \frac{1}{2|p_0(p)|} v_s(p) e^{ip \cdot x} d_s(p)^+ \, d^3p. \quad (97)
\]

we have an additional weight \(|2p_0(p)|^{-1}\) in comparison to the standard formula which can be found e.g. in [40] or [6], as well as in the classic works of Dirac. Of course this weight may be absorbed to the corresponding solutions \( u_s(p), v_s(-p), s = 1, 2 \), constructed as in Appendix 4. But this redefinition of \( u_s(p), v_s(-p) \) would have changed the orthonormality conditions (134) into the following conditions

\[
u_s(p)^+u_{s'}(p) = \frac{1}{(2|p_0(p)|)^2} \delta_{ss'}, \quad v_s(p)^+v_{s'}(p) = \frac{1}{(2|p_0(p)|)^2} \delta_{ss'}, \quad u_s(p)^+v_{s'}(-p) = 0. \quad (98)
\]

But because the same standard orthonormalization conditions (134) are also assumed in [40], pp. 38-41 (even exatly the same \( u_s(p), v_s(-p) \) are used there as we do for the standard representation of Dirac gamma matrices, compare Appendix 4), and the same we have in [6], formula (7.16) p. 67, (and the same is assumed in the classic works of the very founders of QED) we see that the difference between our formula (97) and the standard formula:

\[
\psi(x) = \sum_{s=1}^{2} \int_{\mathbb{R}^3} u_s(p) e^{-ip \cdot x} b_s(p) \, d^3p + \sum_{s=1}^{2} \int_{\mathbb{R}^3} v_s(p) e^{ip \cdot x} d_s(p)^+ \, d^3p. \quad (99)
\]
of $\mathbb{H}$ or $\mathbb{H}^\prime$, cannot be explained by any redefinition of $u_s(p), v_s(-p)$.

Nonetheless the standard quantum Dirac field $\Psi$ given by (69), is unitarily isomorphic to the Dirac field $\psi$ given by (97). Indeed the unitary equivalence between our $\Psi$ and (69) is realized by the lifting to the Fock space of the unitary operator $U$, and its inverse $U^{-1}$, of point-wise multiplication by the function $p \mapsto |2p_0(p)|^{-1}$ and respectively $p \mapsto |2p_0(p)|$ regarded as unitary operators on the respective single particle Hilbert spaces of the realizations of the field $\psi$: first is the space $\mathcal{H}' = \mathcal{H}_{m,0}^\oplus \oplus \mathcal{H}_{m,0}^{\c}$ used by us and the second $\mathbb{U}\mathcal{H}'$ is almost identical with ours, the only change is that we are using the additional factor $1/2p_0(p)$ instead of $\frac{d^3p}{|2p_0(p)|^2}$, in constructing Hilbert spaces of bispinors whose Fourier transforms are continuous respectively on $\mathcal{O}_{m,0,0,0}, \mathcal{O}_{m,0,0,0}$ and are component-wise square summable with respect to $d^3p$. Therefore the corresponding function $|2p_0(p)|^{-1}$ is just equal to the square root of the Radon-Nikodym derivation of the measure $\frac{d^3p}{|2p_0(p)|}$ on the orbits $\mathcal{O}_{m,0,0,0}, \mathcal{O}_{m,0,0,0}$ used by us (compare Subsection 2.1 of [59]) with respect to the known one $d^3p$. Under this redefinition of measure on the orbits the formulas for $u_s(p), v_s(-p)$ remain unchanged, similarly as the formulas for the projectors $P_\oplus, P_\oplus^\prime(p), P_\oplus^\prime, P_\oplus(p), E_0^\prime, E_0^\prime(p)$ (compare Appendix 4) remain unchanged. The nuclear space $E$ in the corresponding Gelfand triples (40) will remain unchanged with the single particle Hilbert space $\mathcal{H}'$ replaced of course by $\mathbb{U}\mathcal{H}'$. The formula (97) for the unitary isomorphism $U$ joining the Gelfand triple $E \subset \mathbb{U}\mathcal{H}' \subset E^*$ with the standard Gelfand triple $\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4) \subset L^2(\mathbb{R}^3; \mathbb{C}^4) \subset \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)^*$ will remain almost the same with the only difference that the additional factor $1/|2p_0(p)|$ will be absent in it, and accordingly the factor $2|p_0(p)|$ will be absent in the formula for $U^{-1}$. It is readily seen now that the construction of Subsection 2.7 with the mentioned modification of the measure, will indeed produce the standard formula (99) for the Dirac field.

Note that the unitary operators $U$, and $\Gamma(U)$, are well defined as unitary isomorphisms for fields understood as integral kernel operators with vector-valued kernels, because the operator $U$ of multiplication by the function $p \mapsto 2p_0(p)$ transforms $\mathcal{S}(\mathbb{R}^3; \mathbb{C})$ continuously, and even isomorphically, into itself and induces the isomorphism of the Gelfand triples

$$\mathcal{H}_{m,0}^\oplus \oplus \mathcal{H}_{m,0}^{\c}$$

$$E \subset \mathcal{H}' \subset E^*$$

$$\downarrow \uparrow \uparrow \mathbb{U} \downarrow \uparrow \mathbb{U}^{-1} \downarrow \uparrow$$

$$E \subset \mathcal{H}'' = \mathbb{U}\mathcal{H}' \subset E^*$$

Let us denote the standard annihilation and creation operators over the Fock space $\Gamma(\mathbb{U}\mathcal{H}')$ by $a^\alpha(u \mp v), a^\alpha(u \pm v)^\dagger$. They are constructed exactly as the operators $a^\alpha(u \mp v), a^\alpha(u \mp v)^\dagger$ in Subsections 2.2, 2.4 with the only change that the weight $1/|2p_0(p)|^2$ in the inner products will be absent, and analogously
we extend them over to \( u \oplus v \in E^* \) using the corresponding isomorphism

\[
L^2(\mathbb{R}^3; \mathbb{C}) \ni \quad S_A(\mathbb{R}^3; \mathbb{C}^4) \subset L^2(\mathbb{R}^3; \mathbb{C}^4) \subset S_A(\mathbb{R}^3; \mathbb{C}^4)^* ,
\]

\[
\downarrow \uparrow \quad U \downarrow \uparrow U^{-1} \quad \downarrow \uparrow \quad E \subset \mathcal{H}' \subset E^*
\]

of the triple \( E \subset \mathcal{U}\mathcal{H}' \subset E^* \) with the standard Gelfand triple, and with \( U, U^{-1} \)
given by the formula \( \psi = \omega U^* \) with the factors \( 1/|p_0(p)| \) (resp. \( 2|p_0(p)| \)) removed. Then if \( \psi \) is the standard Dirac field \( \psi \) we have

\[
\psi(f) = a''(P^\oplus f|_{\mathcal{E}_{m,0},0,0} \oplus 0) + a''\left(0 \oplus \left( P^\oplus \tilde{f}\right)|_{\mathcal{E}_{m,0},0,0}\right)^+, \quad f \in \mathcal{S}(\mathbb{R}^4; \mathbb{C}^4) \quad (100)
\]

correspondingly to the formula

\[
\psi(f) = a'(P^\oplus \tilde{f}|_{\mathcal{E}_{m,0,0,0}} \oplus 0) + a'(0 \oplus \left( P^\oplus \tilde{f}\right)|_{\mathcal{E}_{m,0,0,0}})^+, \quad f \in \mathcal{S}(\mathbb{R}^4; \mathbb{C}^4) \quad (101)
\]

for the free Dirac field \( \psi \) constructed in Subsection 2.3 and with the following isomorphism

\[
a'(\mathbb{U}^+(u \oplus v)) = a''(u \oplus v), \quad a'(\mathbb{U}^+(u \oplus v))^+ = a''(u \oplus v)^+, \quad u \oplus v \in E^*, \quad (102)
\]

\[
a'(\mathbb{U}^{-1}(u \oplus v)) = a''(u \oplus v), \quad a'(\mathbb{U}^{-1}(u \oplus v))^+ = a''(u \oplus v)^+, \quad u \oplus v \in E \subset E^*. \quad (103)
\]

joining the Hida operators \( a'(u \oplus v) \) and \( a''(u \oplus v) \).

Of course the plane waves defining the vector-valued distributional kernels \( \kappa_{0,1}, \kappa_{1,0} \) defining the standard Dirac field \( \psi \) as integral kernel operator

\[
\psi = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0})
\]

are equal

\[
\kappa_{0,1}(s, p; a, x) = \begin{cases} 
\varrho^n(p) e^{-ipx} & \text{with } p = (|p_0(p)|, p) \in \mathcal{E}_{m,0,0,0} \quad \text{if } s = 1, 2 \\
0 & \text{if } s = 3, 4
\end{cases},
\]

\[
\kappa_{1,0}(s, p; a, x) = \begin{cases} 
\varrho^n(p) e^{ipx} & \text{with } p = (|p_0(p)|, p) \in \mathcal{E}_{m,0,0,0} \quad \text{if } s = 1, 2 \\
0 & \text{if } s = 3, 4
\end{cases}
\]
We claim that if the orthonormality conditions (134) for \( u_s(p), v_s(-p), s = 1, 2 \) (compare Appendix 11) are to be preserved, then it is the formula (97) for the free Dirac field \( \psi(x) \) which defines the Dirac field with the local and unitary transformation formula, as an immediate consequence of the locality of the transformation law (26) and (27) of Subsect. 2.1 of [59]. The locality of (26) and (27) of [59] is in turn an immediate consequence of the fact that there are no momentum dependent multipliers in the transformation law (24) and (25) of Subsect. 2.1 of [59], acting on the Fourier transforms of bispinors concentrated respectively on \( O_{m,0,0,0} \) or \( O_{-m,0,0,0} \) (elements of \( H^\oplus_{m,0} \) or \( H^\cap_{m,0} \)).

Namely recall that that the representation \( U(a,\alpha) \) of \( (a,\alpha) \in T_4 \tilde{\otimes} SL(2,\mathbb{C}) \) acts on the Fourier transform \( \tilde{\phi} \in H^\oplus_{m,0} \) (concentrated on \( O_{m,0,0,0} \)) of bispinor \( \phi \) through the formulas (24) and (25) of [59], and on \( \phi \) through (26) and (27) of [59]. Similarly \( U'(a,\alpha)^c \) act on \( (\tilde{\phi}')^c \in H^\cap_{m,0} \) by the conjugation of the representation \( U'(a,\alpha) \) acting on the bispinor \( \tilde{\phi}' \in H^\cap_{m,0} \) by the same formula (24) and (25) of [59] and on \( \phi' \) through the formula (26) and (27) of [59].

Subsect. 2.1. On writing \( U(a,\alpha) = U(a,\alpha) \oplus U'(a,\alpha)^c \) for the representation of \( (a,\alpha) \in T_4 \tilde{\otimes} SL(2,\mathbb{C}) \) acting in the single particle Hilbert space \( H^\oplus_{m,0} \oplus H^\cap_{-m,0} \) of the field (97), we have

\[
\Gamma(U(a,\alpha))\psi(f)\Gamma(U(a,\alpha))^{-1} = \psi(U(a,\alpha)f)
\]

(106)

where \( U(a,\alpha) \) acts on \( f \in \mathcal{S}(\mathbb{R}^4;\mathbb{C}^4) \) and gives \( U(a,\alpha)f \) in the same fashion as in (26) and (27) of [59]. In particular

\[
U(a) f(x) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} f(x\Lambda(\alpha^{-1})) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{*^{-1}} \end{pmatrix} f(\Lambda(\alpha)x),
\]

(107)

\[
T(a)f(x) = f(x-a).
\]

(108)

In particular the field (97) transforms locally, and in particular translations act on (97) in the standard fashion

\[
\Gamma(U(a,0))\psi(f)\Gamma(U(a,0))^{-1} = \psi(U(a,0)f) = \psi(T(a)f)
\]

(109)

It is easily seen that the operator of multiplication by the function \( p \mapsto |p_0(p)|^{-1} \) in action on \( H^\oplus_{m,0} \) and on \( H^\cap_{m,0} \) (compare Subsect. 2.1 of [59]) commutes with the translation operator (25) of Subsect. 2.1 of [59] and with the operators (24) of Subsect. 2.1 of [59] representing spatial rotations (because \( |p_0(p)| = \sqrt{|p|^2 + m^2} \) is invariant under rotations). Therefore both the free Dirac fields: ours (71) and the standard one (99), transform locally and identically under translations and spatial rotations. Namely for \( (a,\alpha) = (a,0) \in T_4 \tilde{\otimes} SL(2,\mathbb{C}) \) or for \( (a,\alpha) = (0,\alpha) \in T_4 \tilde{\otimes} SU(2,\mathbb{C}) \subset T_4 \tilde{\otimes} SL(2,\mathbb{C}) \) i.e. for translations or spatial rotations, we have

\[
\Gamma(UU(a,\alpha)UU^{-1})\psi(f)\Gamma(UU(a,0)UU^{-1})^{-1} = \psi(U(a,\alpha)f)
\]

15Recall that here \( \Lambda : \alpha \to \Lambda(\alpha) \) is an antihomomorphism.
with the standard local formula for the transformation formula (107), (108) for space-time transformed bispinor \( U(a, \alpha)f \), and for the standard Dirac quantum field [99] with the representation

\[ \Gamma(U(a, \alpha)U^{-1}) \]

acting in its Fock space

\[ \Gamma(U) \left( \mathcal{H}^{\oplus}_{m,0} \oplus \mathcal{H}^{\ominus}_{-m,0} \right) = \Gamma(U) \left( \mathcal{H}^{\oplus}_{m,0} \oplus \mathcal{H}^{\ominus}_{-m,0} \right), \]

and with the representation

\[ \mathbb{U} U(a, \alpha)U^{-1} \]

acting in its single particle Hilbert space

\[ \mathcal{H}'' = U(\mathcal{H}^{\oplus}_{m,0} \oplus \mathcal{H}^{\ominus}_{-m,0}) = \mathbb{U} \mathcal{H}'. \]

Note that for the bispinor \( \tilde{\phi} = U \tilde{\varphi}, \tilde{\varphi} \in \mathcal{H}^{\oplus}_{m,0} \), such that \( \tilde{\varphi} \oplus 0 \in \mathcal{H}'' \), concentrated on \( \mathcal{O}_{m,0,0,0} \), or \( 0 \oplus \tilde{\varphi} \in \mathcal{H}'' \), \( \tilde{\varphi} = U \tilde{\varphi}, \tilde{\varphi} \in \mathcal{H}^{\ominus}_{-m,0} \), concentrated on \( \mathcal{O}_{-m,0,0,0} \), we have

\[ \mathbb{U} U(\alpha)U^{-1} \tilde{\varphi}(p) = \left| \frac{p_0(\Lambda(\alpha)p)}{p_0(p)} \right| \left( \begin{array}{cc} \alpha & 0 \\ 0 & \alpha^{-1} \end{array} \right) \tilde{\varphi}(\Lambda(\alpha)p), \]

\[ \mathbb{U} T(\alpha)U^{-1} \tilde{\varphi}(p) = e^{ia \cdot \vec{p}} \tilde{\varphi}(p). \]

Therefore for the Lorentz transformations (24) of Subsect. 2.1 of [59] situation is different for the two mentioned realizations of the Dirac free field. Namely our field [97] by construction transforms locally as a bispinor field also under Lorentz transformations. But the operator \( \mathbb{U} \) of point-wise multiplication by the function \( p \mapsto \frac{p_0(\Lambda(\alpha)p)}{p_0(p)} \) does not commute with the operator \( U(\alpha) \) for \( \alpha \notin SU(2, \mathbb{C}) \) given by (24) of [59], and moreover it is immediately seen that transformation formula \( \mathbb{U} U(\alpha)U^{-1} \) gains non-trivial momentum dependent multiplier

\[ | \frac{p_0(\Lambda(\alpha)p)}{p_0(p)} | \neq 1 \]

for \( \alpha \notin SU(2, \mathbb{C}) \). This additional multiplier means that \( \mathbb{U} U(a, \alpha)U^{-1} \) in action on the elements of \( \mathcal{H}'' \), viewed as distributional Fourier transforms of positive (respectively conjugations of negative) energy solutions \( \mathcal{F}^{-1} \tilde{\varphi} \) of Dirac equation, concentrated respectively on \( \mathcal{O}_{m,0,0,0} \) or \( \mathcal{O}_{-m,0,0,0} \), induce nonlocal transformation law on \( \mathcal{F}^{-1} \tilde{\varphi} \). Alternatively this additional multiplier, however, can be viewed as coming from the non-invariance of the ordinary euclidean measure \( d^3p \) under Lorentz transformation on the respective orbits \( \mathcal{O}_{m,0,0,0} \) and \( \mathcal{O}_{-m,0,0,0} \), which assures locality of Lorentz transformations not for the ordinary inverse
Fourier transformed elements of $H''$ but for the inverse Fourier transform of the elements $U^{-1} \tilde{\phi}, \tilde{\phi} \in H''$. Namely consider the following formula

$$
\phi(x) = \int_{\mathcal{E}_{m,0,0,0}} \tilde{\phi}(p)e^{-ip \cdot x} d\mu_{\mathcal{E}_{m,0,0,0}}(p) = \int_{\mathbb{R}^3} \frac{\tilde{\phi}(p, p_0(p))}{p_0(p)} e^{-ip \cdot x} d^3p
$$

$$
= \int_{\mathbb{R}^3} U\tilde{\phi}(p)e^{-ip \cdot x} d^3p = \int_{\mathbb{R}^3} \tilde{\phi}(p)e^{-ip \cdot x} d^3p,
$$

for the positive energy solutions. We have analogue formula for negative energy solutions. Consider now the local transformation formula for $U(\alpha)\phi$ with $\phi$ expressed by the above formula. We will get

$$
U(\alpha)\phi(x) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^s \end{pmatrix} \phi(\Lambda(\alpha)x)
$$

$$
= \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^s \end{pmatrix} \int_{\mathbb{R}^3} \tilde{\phi}(p)e^{-ip \cdot \Lambda x} d^3p
$$

$$
= \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^s \end{pmatrix} \int_{\mathbb{R}^3} \tilde{\phi}(p)e^{-ip \cdot \Lambda x} d^3p
$$

$$
= \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^s \end{pmatrix} \int_{\mathbb{R}^3} \tilde{\phi}(p)e^{-ip \cdot x} \left| \frac{d^3\Lambda p}{|p_0(\Lambda p)|} \right| d^3p.
$$

Taking into account the invariance property

$$
\frac{d^3\Lambda p}{|p_0(\Lambda p)|} = \frac{d^3p}{|p_0(p)|} \iff \left| \frac{d^3\Lambda p}{d^3p} \right| = \frac{|p_0(\Lambda p)|}{|p_0(p)|},
$$

we obtain

$$
U(\alpha)\phi(x) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^s \end{pmatrix} \int_{\mathbb{R}^3} \tilde{\phi}(p)e^{-ip \cdot x} \left| \frac{p_0(\Lambda p)}{|p_0(p)|} \right| d^3p, \quad p \in \mathcal{O}_{m,0,0,0},
$$

i.e. again the assertion that the transformation $UU(\alpha)U^{-1}\tilde{\phi}$ of $\tilde{\phi} = U\tilde{\phi}$ is accompanied by the ordinary local bispinor transformation $U(\alpha)\phi$ of $\phi$, but not of $\mathcal{F}^{-1}\tilde{\phi}$. Similar relation we obtain for the conjugations of the negative energy solutions whose Fourier transforms are concentrated on $\mathcal{E}_{-m,0,0,0}$. Therefore if $f \in \mathcal{S}(\mathbb{R}^4; \mathbb{C}^4)$ is a space-time test bispinor, then the transformation $UU(\alpha)U^{-1}$ (or its conjugation) in action on

$$
P^\Phi Uf|_{\mathcal{E}_{m,0,0,0}} \text{ or resp. } (P^\odot Uf|_{\mathcal{E}_{-m,0,0,0}})^c
$$

127
induces local bispinor transformation on $f$. This would be false for the action of $UU(\alpha)U^{-1}$ (or its conjugation) on

$$P^\oplus \mathcal{F}_{m,0,0,0}^c \text{ or resp. } (P^\oplus \mathcal{F}_{-m,0,0,0}^c)^c.$$

Thus we see again that it is the field (97), or equivalently the field (101), which transforms locally as ordinary bispinor under the Fock lifting of $U(\alpha)$ (summed up with its conjugation). The field (99), or equivalently the field (100), transforms non-locally under the Fock lifting of the unitary representation $UU(\alpha)U^{-1}$ (summed up with its conjugation). Correspondingly the standard Dirac quantum field (99) transforms non-locally under Lorentz transformations if the unitarity of the transformation is to be preserved. Locality under proper Lorentz transformations of the standard field (99) can be restored, but then the unitarity of the Lorentz transformations will have to be abandoned. Below in this Subsection we explain this fact together with its connection to the so called Noether theorem for free fields.

Although the Dirac free fields (97) and (99) are unitarily isomorphic, in the sense of the isomorphism (102) or (103), joining the corresponding Hida operators $a', a''$, there are some important differences between them.

The first concerns locality under the proper Lorentz transformations, already explained. The field (97) is constructed from the direct sum of two (equivalent) irreducible representations, giving the local transformation law for the elements of the single particle Hilbert space regarded as the space of (regular distributional) solutions of the Dirac equation, whose Fourier transforms compose $\mathcal{H}'$ and are concentrated on the orbit $\mathcal{O}_{m,0,0,0}$ or eventually are equal to conjugations of bispinors concentrated on the orbit $\mathcal{O}_{-m,0,0,0}$. The standard field (99) is constructed from the slightly different representation, but unitary equivalent with it, which assures the local transformation law of the elements of the single particle space, understood as solutions of the Dirac equation, but only under the translation subgroup or spatial rotations. It is a general paradigm that the locality of the transformation under the full $T_4 \otimes SL(2, \mathbb{C})$ is the fundamental assumption, and whenever we are able to construct a free field out of a representation of $T_4 \otimes SL(2, \mathbb{C})$ it is customary to put the additional requirement of locality of the transformation law induced by the representation. But it turns out that, at least in the realm of causal perturbative approach to QFT, that it is the covariance under translations (with the standard local transformation formula) which plays the important role in the construction of the causal perturbative series, e.g. for interacting fields. The local Lorentz covariance and its unitarity turns out to be optional (which is of course a nontrivial fact). Moreover it is known that also for determination of the commutation rules for free fields according to the classic procedure due to Pauli-Bogoliubov-Shirkov, it is the the so-called Noether theorem for translations which is sufficient in derivation of these rules (compare [6], where it is understood as an example of the Bohr’s correspondence principle). Therefore at least from the causal perturbative approach, both (97) and (99) are equally well.

Although (71) and (99) are unitarily isomorphic, they have different “com-
mutation generalized functions” as well as different “pairing functions”, which enter the causal perturbative series accordingly to different anti-commutation rules

\[ \{ a'(u \oplus v), a'(u' \oplus v')^\dagger \} = (u \oplus v, u' \oplus v')_{\mu'\nu'}, \quad u \oplus v \in E, \]

\[ \{ a''(u \oplus v), a''(u' \oplus v')^\dagger \} = (u \oplus v, u' \oplus v')_{\mu'\nu'}, \quad u \oplus v \in E \]

with different inner products: with the additional weight \(|2p_0(p)|^{-2}\) in the formula for \((\cdot, \cdot)_{\mu'\nu'}\) in comparison to \((\cdot, \cdot)_{\mu'\nu'}\), where the weight \(|2p_0(p)|^{-2}\) is absent. Because of the isomorphism between the Hida operators \(a', a''\) defining respectively the fields \(\text{(97)}\) and \(\text{(99)}\) we expect that both these fields should be physically equivalent, in giving the same physical quantities, although it is still non trivial (nontriviality follows e.g. by the difference in commutation and pairing functions contributing to the perturbative series). At the present stage of the theory we should be careful and keep in mind both possibilities \(\text{(97)}\) and \(\text{(99)}\) for the free Dirac field.

That locality and unitarity under Lorentz transformations cannot be reconciled for the standard Dirac field \(\text{(99)}\) has so far been unnoticed, because of the rather heuristic approach in its construction, which either does not enter the theory of representations of \(T(\mathbb{R}) \otimes SL(2, \mathbb{C})\) at all or recalls to it, but in a rather disrespectful manner. The lack of the adequate group theoretical construction of the Dirac field has been noted e.g. by Haag \(\text{(22)},\) p. 48.

But there is also another difference between \(\text{(97)}\) and \(\text{(99)}\), which can be invariantly expressed by recalling to the first Noether theorem applied to the free quantum fields. We devote the rest part of this Subsection to the Noether theorem restricted to translations and Lorentz transformations and its relation to the fields \(\text{(97)}\) and \(\text{(99)}\).

Let us recall the Noether theorem for free fields after \(\text{(6)},\) Chap. 2, §9.4 (in 1980 Ed.), where it is called the Quantization Postulate:

*The operators for the energy-momentum four-vector \(P\), and the angular momentum tensor \(M\), the charge \(Q\), and so on, which are the generators of the corresponding symmetry transformations of state vectors, can be expressed in terms of the operator functions of the fields by the same relations as in classical field theory with the operators arranged in the normal order.*

Let us start our analysis with translations.

Here we confine our attention to the Dirac field \(\psi\) given by \(\text{(99)}\) (and respectively \(\text{(97)}\)). Let \(T^{0\mu}\) be the \(0 - \mu\)-components of the energy-momentum tensor for the free “classic” Dirac field \(\psi\) corresponding to translations via Emmy Noether theorem (compare \(\text{(3)}\)) expressed in terms of \(\psi(x)\) and of its derivatives \(\partial_\mu \psi(x)\). According to this theorem the spatial integral

\[ \int T^{0\mu} \, d^3x = \frac{i}{2} \int \left( \overline{\psi(x)} \gamma^0 \frac{\partial \psi}{\partial x_\mu}(x) - \frac{\partial \overline{\psi}}{\partial x_\mu}(x) \gamma^0 \psi(x) \right) \, d^3x, \]

is equal to the conserved integral corresponding to the translational symmetry, i.e. energy-momentum components of the field \(\psi\). Here \(\overline{\psi}(x)\) stands for the
Dirac adjoint \( \psi(x)^+ \gamma^0 \), and not for the complex conjugation, as usual. We replace the classical field \( \psi \) in the above integral formally by the quantum field \( \psi \) with the counterpart of Dirac adjoint appropriately defined (see below) and with the product under the integral sign defined as the Wick product of the fields at the same space-time point (compare preceding Subsection 2.7).

Recall that in both cases, (97) and (99), we realize the field operators as the integral kernel operators with the corresponding vector-valued distributions \( \kappa_{0,1}, \kappa_{1,0} \), over the standard Gelfand triple \( E_1 = \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4) \subset L^2(\mathbb{R}^3; \mathbb{C}^4) \subset E_1^* \) in both cases (97) and (99).

Thus we are going to check if

\[
\frac{i}{2} \int : T^{0\mu} : d^3x = P^\mu = d\Gamma(P^\mu),
\]

where \( P^\mu, \mu = 0, 1, 2, 3 \), are the translation generators of the representation \( \mathcal{U}(\mathbf{a}, \alpha) \mathcal{U}^{-1} \), acting in \( \mathcal{U} \mathcal{H}' = L^2(\mathbb{R}^3; \mathbb{C}^4) \) (in the first case (97)) or \( \mathcal{U} \mathcal{U}(\mathbf{a}, \alpha) \mathcal{U}^{-1} \mathcal{U}^{-1} \) in the same \( \mathcal{U}\mathcal{U} \mathcal{H}' = L^2(\mathbb{R}^3; \mathbb{C}^4) \) standard Hilbert space (in the second case (99)), and with \( P^\mu = d\Gamma(P^\mu), \mu = 0, 1, 2, 3 \), equal to the generators of translations of the representation \( \Gamma \left( \mathcal{U}(\mathbf{a}, \alpha) \mathcal{U}^{-1} \right) \) or \( \Gamma \left( \mathcal{U} \mathcal{U}(\mathbf{a}, \alpha) \mathcal{U}^{-1} \mathcal{U}^{-1} \right) \) of \( \mathcal{T}_4 \otimes \mathcal{S}(2, \mathbb{C}) \), both acting in the Fock space \( \Gamma(\mathcal{U} \mathcal{H}') = \Gamma(L^2(\mathbb{R}^3; \mathbb{C}^4)) \) (in the second case corresponding to (99) we also have \( \Gamma(\mathcal{U} \mathcal{U} \mathcal{H}') = \Gamma(L^2(\mathbb{R}^3; \mathbb{C}^4)) \) with the isomorphism \( U \) given by the modification of (37) in which we remove the factor \( 1/p_0(p) \), with the removal being compensated by the presence of \( U \).

Note that in the first case (97) the unitary operator is given by the formula (37), and in the second case \( U \) is given by the similar formula with the weight factor \( 1/p_0(p) \) omitted.

Equivalently Bogoliubov-Shirkov Quantization Postulate for \( \psi \) demands the equality

\[
\frac{i}{2} \int : T^{0\mu} : d^3x = P^\mu = d\Gamma(P^\mu), \text{ in this order!}
\]  

(110)

to hold.

The whole point about the Quantization Postulate (or Emmy Noether theorem for free fields) is that the operators \( P^\mu = d\Gamma(P^\mu) \) may be computed in terms of Wick polynomials in free fields – integral kernel operators – to which we know how to apply the perturbative series in the sense of Bogoliubov-Epstein-Glaser. In checking its validity for the Dirac field we proceed in two steps. In the first step we show that for each \( \mu = 0, 1, 2, 3 \), there exist a distribution \( \kappa^\mu \in E_1 \otimes E_1^* \) such that the corresponding integral kernel operator \( \Xi_{1,1}(\kappa^\mu) \) is equal to \( P^\mu = d\Gamma(P^\mu) \). Then according to the rule giving the Wick product of free fields at the same point as integral kernel operator with vector
valued kernel as well as the rule giving its spatial integral as an integral kernel operator with scalar kernel, given in the preceding Subsection, we show that the left hand side integral kernel operator is equal to the right hand side integral kernel operator \( \Xi_{1,1}(\kappa^\mu) \) in \((110)\) for the standard field \((99)\). It turns out that \((110)\) does not hold for the local field \((97)\).

It is easily seen that the representors \(UU(a,\alpha)U^{-1}\) and respectively

\[ UUU(a,\alpha)U^{-1}U^{-1} \]

are continuous as operators \( E_1 \to E_1 \), in case of both the representations of \( T_3(S)SL(2,\mathbb{S}) \):

1) for the representation \( UU(a,\alpha)U^{-1} \) acting in \( UH' = L^2(\mathbb{R}^3; \mathbb{C}^4) \), with \( U \) given by \((37)\), corresponding to the field \((97)\).

2) for the representation \( UUU(a,\alpha)U^{-1}U^{-1} \), acting in \( UUH' = L^2(\mathbb{R}^3; \mathbb{C}^4) \), with \( U \) given by \((37)\) without the factor \( 1/p_0(p) \), which is compensated here by the operator \( U \), and corresponding to the field \((99)\).

In particular this holds for the translation subgroup representors. And the translation representors in both of the representations are unitary and act identically on the common nuclear space \( E_1 = S_A(\mathbb{R}^3; \mathbb{C}^4) \). Therefore the translation subgroup in both cases of representations compose the subgroup of the Yoshizawa group \( U(E_1; L^2(\mathbb{R}^3; \mathbb{C}^4)) \). The Yoshizawa group \( U(E_1; L^2(\mathbb{R}^3; \mathbb{C}^4)) \) is the group of unitary operators on \( L^2(\mathbb{R}^3; \mathbb{C}^4) \) which induce homeomorphisms of the test function space \( E_1 = S_A(\mathbb{R}^3; \mathbb{C}^4) \) with respect to the nuclear topology of \( E_1 \). In other words the translation representors in both representations compose automorphisms of the Gelfand triple \( E_1 \subset L^2(\mathbb{R}^3; \mathbb{C}^4) \subset E_1^* \).

Moreover any one parameter subgroup \( \{T_\theta\}_{\theta \in \mathbb{R}} \) of translations in both considered representations is differentiable, i.e. \( \lim_{\theta \to 0}(T_\theta \xi - \xi)/\theta = X\xi \) converges in \( E_1 \). Let us consider the one parameter subgroup of translations along the \( \mu \)-th axis and write in this case \( X^\mu \) for \( X \), where in our case \( X^\mu \) is the operator \( M_{ip^\mu} \) of multiplication by the function \( p \to ip^\mu(p) \), and where \( (p^0(p),...p^3(p)) = (\sqrt{p \cdot p + m^2}, p) \in \mathcal{O}_{(1,0,0,1)} \). Existence of the limit is equivalent to

\[
\lim_{\theta \to 0} \frac{|T_\theta \xi - \xi - X^\mu \xi|_k^2}{\theta} = \lim_{\theta \to 0} \int \left( \frac{A^k(e^{i\theta p^\mu} - 1 - i\theta p^\mu)}{\theta} \xi(p) , \frac{A^k(e^{i\theta p^\mu} - 1 - i\theta p^\mu)}{\theta} \xi(p) \right)_{\mathbb{C}^4} d^3 p = 0,
\]

\( k = 0,1,2,..., \xi \in E_1, \quad (111) \)

where \( p^\mu, \mu = 0,1,2,3, \) in the exponent are the functions \( p \to (p^\mu(p)) = (\sqrt{p \cdot p + m^2}, p) \) and where \( A \) is the standard operator \((33)\) used in the construction of the standard Gelfand triple \( E_1 = S_A(\mathbb{R}^3; \mathbb{C}^4) \subset L^2(\mathbb{R}^3; \mathbb{C}^4) \subset E_1^* \).
Explicit calculation shows that (111) is fulfilled. Therefore \( \{T_\theta \}_{\theta \in \mathbb{R}} \) is differentiable subgroup and by the Banach-Steinhaus theorem the linear operators \( X^\mu, \mu = 0, 1, 2, 3 \), are continuous as operators \( E_1 \to E_1 \) and finally by Proposition 3.1 of [26] every such subgroup is regular in the sense of [26], §3.

For every operator \( X \) which is continuous as the operator \( E_1 \to E_1 \) we define \( \Gamma(X) \) and \( d\Gamma(X) \) on \( (E_1) \). Let \( \Phi \in (E_1) \) be any element of the Hida space with decomposition (41) corresponding to the Gelfand triple in particular for any of the translations subgroup along the direction of the one parameter translation subgroups of the mentioned representations, of Proposition 4.2 and Theorem 4.3 of [26] is applicable in the fermi case to any into itself.

In this situation it is not difficult to see that for each \( \mu = 0, 1, 2, 3 \), the proof of Proposition 4.2 and Theorem 4.3 of [26] is applicable in the fermi case to any of the one parameter translation subgroups of the mentioned representations, in particular for any of the translation subgroup along the direction of the \( \mu \)-th axis, \( \mu = 0, 1, 2, 3 \), there exists a symmetric distribution \( \kappa^\mu \in E_1 \otimes E_1^* \) such that

\[
d\Gamma(X^\mu) = \Xi_{1,1}(\kappa^\mu) = \sum_{s,s'} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa^\mu(p',s',p,s) \partial_{p',s'} \partial_{p,s} d^3p d^3p',
\]

(112)

and \( \kappa^\mu \in E_1 \otimes E_1^* \) fulfills

\[
\langle \kappa^\mu, \zeta \otimes \xi \rangle = \langle \zeta, X^\mu \xi \rangle, \quad \zeta, \xi \in E_1.
\]

(113)

Because the pairings \( \langle \cdot, \cdot \rangle \) in the formula are induced by the inner product \( \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^3; \mathbb{C}^4)} \) in \( L^2(\mathbb{R}^3; \mathbb{C}^4) \), and because \( X^\mu \) is the operator of multiplication by \( ip^\mu(p) \), we have

\[
\langle \zeta, X^\mu \xi \rangle_{L^2(\mathbb{R}^3)} = \langle \zeta, \xi \rangle_{L^2(\mathbb{R}^3)} = \langle \zeta, X^\mu \xi \rangle = \langle \xi, X^\mu \zeta \rangle, \quad \zeta, \xi \in E,
\]

so that

\[
\langle \kappa^\mu, \zeta \otimes \xi \rangle = \langle \kappa^\mu, \xi \otimes \zeta \rangle, \quad \zeta, \xi \in E,
\]

and \( \kappa^\mu \) is indeed symmetric.

On the other hand the pairing \( \langle \cdot, \cdot \rangle \) on left hand side of (113) expressed in terms of the kernel \( \kappa^\mu(p',p) \) is likewise induced by the inner product \( \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^3)} \) in \( L^2(\mathbb{R}^3; \mathbb{C}^4) \). Therefore we have

\[
\langle \kappa^\mu, \zeta \otimes \xi \rangle = \sum_{s,s'} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa^\mu(p',s',p,s) \zeta(p',s') \xi(p,s) d^3p' d^3p.
\]

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Joining this with (113) we obtain
\[ \kappa^\mu(p', s', p, s) = ip^\mu(p)\delta_{s,s'}(p' - p). \]

Therefore we get
\[ P^\mu = d\Gamma(P^\mu) = \sum_{s,s'} \int_{\mathbb{R}^3 \times \mathbb{R}^3} p^\mu(p) \delta_{s,s'}(p' - p) \delta_{p',s'} \delta_{p,s} d^3p'd^3p, \quad (114) \]

which is customary to be written as
\[ P^0 = d\Gamma(P^0) = \sum_s \int_{\mathbb{R}^3} |p^0(p)| \delta_{p,s} \delta_{p,s} d^3p \]
\[ = \sum_{s=1,2} \int_{\mathbb{R}^3} |p^0(p)| b_s(p)^+ b_s(p) d^3p + \sum_{s=1,2} \int_{\mathbb{R}^3} |p^0(p)| d_s(p)^+ d_s(p) d^3p, \quad (115) \]

\[ P^i = d\Gamma(P^i) = \sum_s \int_{\mathbb{R}^3} p^i(p) \delta_{p,s} \delta_{p,s} d^3p \]
\[ = \sum_{s=1,2} \int_{\mathbb{R}^3} p^i(p) b_s(p)^+ b_s(p) d^3p + \sum_{s=1,2} \int_{\mathbb{R}^3} p^i(p) d_s(p)^+ d_s(p) d^3p. \quad (116) \]

Both operators \( d\Gamma(P^\mu) \) and \( \Xi_{1,1}(-i\kappa^\mu) \) transform (continuously) the nuclear, and thus perfect, space \((E_1)\) into itself and both being equal and symmetric on \((E_1)\) have self-adjoint extension to self-adjoint operator in the Fock space \(\Gamma(L^2(\mathbb{R}^3; \mathbb{C}^4))\), again by the classical criterion of [43] (p. 120 in Russian Ed. 1954). In general the criterion of Riesz-Szökefalvy-Nagy does not exclude existence of more than just one self-adjoint extension, but in our case it is unique. Indeed because for each \( \mu = 0, 1, 2, 3 \), the one-parameter unitary group generated by \( d\Gamma(P^\mu) \) leaves invariant the dense nuclear space \((E_1)\), then by general theory, e.g. Chap. 10.3., it follows that \( d\Gamma(P^\mu) \) with domain \((E_1)\) is essentially self-adjoint (admits unique self adjoint extension).

Now applying the Rules II and V' of Subsection 2.7 to the left hand side of (110) with \( \psi \) equal to the standard Dirac free field (99), understood as an integral kernel operator
\[ \psi = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0}), \]

with the kernels \( \kappa_{0,1}, \kappa_{1,0} \), (104) and (105), we immediately get the result equal to (114) or equivalently (115), (116). Thus we arrive at the following

**PROPOSITION.** The standard free Dirac field \( \psi \), equal (99), satisfies the Bogoliubov-Shirkov Quantization Postulate (110) for translations:
\[ \frac{i}{2} \int \left( \begin{array}{c} \overline{\psi}(x)\gamma^0 \frac{\partial \psi}{\partial x_\mu}(x) - \frac{\partial \overline{\psi}}{\partial x_\mu}(x)\gamma^0 \psi(x) \end{array} \right) : d^3x = d\Gamma(P^\mu). \]

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On the other hand if we apply the Rules II and V’ of Subsection 2.7 to the left hand side of (110) with \( \psi \) equal to the local Dirac free field (97), understood as an integral kernel operator

\[
\psi = \Xi(\kappa_{0,1}) + \Xi(\kappa_{1,0})
\]

with the kernels \( \kappa_{0,1}, \kappa_{1,0} \), we obtain an integral kernel operator not equal to (114) or, equivalently, not equal to (115), (116). Thus we arrive at the following

**Proposition.** The Bogoliubov-Shirkov Quantization Postulate (110) for translations is not satisfied by the local Dirac field (97).

Now let us consider Lorentz transformations. The Noether integral generator corresponding to Lorentz transformations is equal

\[
\frac{i}{2} \int : \left( \psi(x)^+ x^\mu \frac{\partial \psi}{\partial x_\nu}(x) - \psi(x)^+ x^\nu \frac{\partial \psi}{\partial x_\mu}(x) + \frac{1}{2} \psi(x)^+ \gamma^\mu \gamma^\nu \psi(x) \right) : dx^3 = M^{\mu\nu}
\]

Again applying the Rules II and V’ of Subsection 2.7 we arrive at the following (infinitesimal form of) local transformation formula

\[
i[M^{\mu\nu}, \psi^a] = \sum_b \delta^{a b} \psi^b + (x^\mu \partial^\nu - x^\nu \partial^\mu) \psi^a
\]

for the standard Dirac free field \( \psi \). It generates the ordinary local bispinor transformation formula \( U(a, \alpha) \) in the single particle Hilbert space \( \mathcal{H}' \) of the standard Dirac field \( \psi \), which does not coincide with the unitary representation \( \mathbb{U}U(a, \alpha)\mathbb{U}^{-1} \), and which is not unitary if regarded as representation in the single particle Hilbert space \( \mathcal{H}' = \mathbb{U}\mathcal{H}' \). In particular \( M^{\mu\nu} = d\Gamma(M^{\mu\nu}) \), regarded as an operator in the Fock space \( \Gamma(\mathcal{H}') \) of the standard Dirac free field \( \psi \), generates a nonunitary transformation. Therefore the generator \( M^{\mu\nu} = d\Gamma(M^{\mu\nu}) \) given by the Noether integral (117) corresponding to the Lorentz transformations, and computed for the standard Dirac field \( \psi \), is not self-adjoint.

We therefore have the following alternative: we can save locality of the transformation of the standard Dirac field \( \psi \), with the generators of the local representation given by the Noether integrals (with Wick ordered products), but unitarity of the Lorentz transformations have to be abandoned. Alternatively we have the unitary representation \( \Gamma(\mathbb{U}U(a, \alpha)\mathbb{U}^{-1}) \) in the Fock space \( \Gamma(\mathcal{H}') \) of the standard Dirac field \( \psi \), but locality of the Lorentz transformations is lost.

This alternative has not been discovered before. One reason lies in the fact that there are the white noise technics which allow us to construct equal time integrals of Wick products of free fields, and to investigate their self-adjointness. As far as we know nobody has applied them before to the realistic fields, and in particular to the analysis of Wick product fields and their Cauchy integrals. On the other hand the aproach more popular among mathematical physicists, i.e. due to Wightman-Gårding, is not effective here, which was recognized by Segal...
In particular non-self-adjointness of the Lorentz transformations generator $M^{\mu\nu}$ for the standard Dirac field (99) given by the Noether integral formula (117), could have not been discovered by such founders of Quantum Field Theory like Pauli or Schwinger. This alternative explains, among other things, also the fact why we do not encounter the standard Dirac field (99) among the free fields whose construction is based on the unitary and local representations. In particular it escaped the classification of free fields based on local unitary representations of the double covering of the Poincaré group given in [33] or [34]. This fact was also recognized by Haag [22], p. 48. The local bispinor field (97) has the standard local and unitary bispinor transformation formula, but it does not coincide with the standard Dirac field (99). Note that the standard Dirac field (99) is a field which is obtained through the canonical quantization, i.e. it is uniquely determined by the condition that it satisfies the Bogoliubov-Shirkov Quantization Postulate (110) for translations. It seems that also the local bispinor field (97) has not been constructed before and appears here for the first time.

Note that the Wick product of the Dirac field components is skew-commutative, therefore the order is important in (110).

We end this Subsection with a remark on the Pauli theorem on spin-statistics relation. It is based on the properties of the “classical”, i.e. before “quantization”, fields. Essentially it says that the energy component of the Noether energy-momentum tensor is not positive definite for half-odd-integer free “classical” fields, compare e.g. [16] and Pauli’s book cited there. Technically speaking, generic half-odd-integer spin field (solution of equations of motion), when Fourier decomposed and inserted into the Noether energy integral, gives formally the expression (115), but with operators $b_{s}(p), d_{s}(p)$ replaced with the Fourier coefficients and with the opposite sign at the second term in (115). Pauli then joined this result with the canonical quantization procedure, equivalent to the Pauli-Bogoliubov-Shirkov Quantization Postulate (110) for translations. Because the Wick product of fermi fields in (114) repeats the sign of the second term in the ‘classical” counterpart of (115), Pauli arrived at the spin-statistics relation: half-odd-integer spin “classical” (free) fields should be quantized with the canonical anticommutation relations.

The so called “spin-statistics theorem” due to Wightman is different and in fact gives the relation between the commutation relation of smeared out fields, within his axiomatic definition of a quantum field, and the representation defining a local transformation rule of the field. In Wightman’s proof no relation with “classical” fields and with positivity of the energy-momentum of “classical” fields intervenes. In this sense Pauli’s spin-statistics theorem is different pointing out that such relation exists, and in this sense reveals what is untouched in the Wightman’s version of spin-statistics theorem.
2.9 The quantum electromagnetic potential field $A$ as an integral kernel operator with vector-valued distributional functions

Recall that the formula (294) of Subsection 5.9 of [59]:

$$A^\mu(x) = \int_\mathbb{R}^3 d^3p \left\{ \frac{1}{\sqrt{2p^0(p)}} \sqrt{B(p, p^0(p))}_{\lambda} a^\lambda(p) e^{-ip \cdot x} + \frac{1}{\sqrt{2p^0(p)}} \sqrt{B(p, p^0(p))}_{\lambda} \eta a^\lambda(p) \right\}$$  

(118)

gives a well defined generalized operator transforming continuously the Hida space $(E)$ into its strong dual $(E)^*$, where $(E)$ is the Hida space of the Gelfand triple $(E) \subset \Gamma(H^{'}) \subset (E)^*$ defining the electromagnetic potential field $A$ within the white noise setup. Recall that $E = S_A(\mathbb{R}^3; \mathbb{C}^4) = S_{B(A)}(\mathbb{R}^3; \mathbb{C}^4)$ is defined by the standard operator $A = \oplus_{\lambda} A^{(3)}_{\lambda}$ on the standard Hilbert space $L^2(\mathbb{R}^3; \mathbb{C}^4)$, with the operator $A^{(3)}_{\lambda}$ defined as in Subsection 5.3 of [59]. Recall that the integral (118) exists pointwisely as the Pettis integral, compare (294), Subsection 5.9 of [59]. Nonetheless the potential field $A$ is naturally a sum of two integral kernel operators

$$A = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0}) \in \mathcal{L}(\mathcal{E}, \mathcal{E}^*) \cong \mathcal{L}(\mathcal{E}, \mathcal{L}((E), (E)^*))$$

with vector valued kernels $\kappa_{0,1}, \kappa_{1,0} \in \mathcal{L}(E, E^*)$ for

$$\mathcal{E} = S_{\mathcal{F}_{\mathcal{B}A}}(\mathbb{R}^4; \mathbb{C}^4) = \mathcal{F} \left[ S_{\mathcal{B}A}(\mathbb{R}^4; \mathbb{C}^4) \right] = S_{\mathcal{B}A}(\mathbb{R}^4; \mathbb{C}^4),$$

in the sense of Obata [38] explained in Subsection 2.6. The vector valued distributions $\kappa_{0,1}, \kappa_{1,0}$ are defined by the following plane waves

$$\kappa_{0,1}(\nu, p; \mu, x) = \frac{\sqrt{B(p, p^0(p))}_{\lambda}}{\sqrt{2p^0(p)}} \eta a^\lambda(p) e^{-ip \cdot x}, \quad p = (|p^0(p)|, p) \in \mathcal{O}_{1,0,0,1},$$

$$\kappa_{1,0}(\nu, p; \mu, x) = (-1)^{\nu} \frac{\sqrt{B(p, p^0(p))}_{\lambda}}{\sqrt{2p^0(p)}} \eta a^\lambda(p) e^{ip \cdot x}, \quad p = (|p^0(p)|, p) \in \mathcal{O}_{1,0,0,1},$$

with

$$(-1)^{\nu} \begin{cases} -1 & \text{if } \mu = 0, \\ 1 & \text{if } \mu = 1, 2, 3. \end{cases}, \quad p^0(p) = |p|.$$ 

The above stated formulas for $\kappa_{0,1}, \kappa_{1,0}$ can be immediately read off from the formula (118) and the commutation rules (219) of [59] for the Gupta-Bleuler operator $\eta$ and the Hida operators $\partial_{\mu,p} = a_\mu(p)$:

$$a_0(p)\eta = -\eta a_0(p), \quad a_i(p)\eta = \eta a_i(p), \quad i = 1, 2, 3, \quad \eta^2 = 1.$$ 

Here we are using the standard convention of Subsection 2.6 that in the general integral kernel operator [69] in the tensor product of the Fock space of the Dirac
field \( \mathbf{p} \) and of the electromagnetic potential field \( A \) we have the ordinary Hida operators in the normal order with the ordinary adjoint (linear transpose) \( \partial^*_\mu, \mathbf{p} = a_\mu(\mathbf{p})^+ \) corresponding to photon variables \( \mu, \mathbf{p} \). This is the convention assumed in mathematical literature concerning integral kernel operators. But physicists never use the ordinary adjoint \( \partial^*_\mu, \mathbf{p} = a_\mu(\mathbf{p})^+ \) whenever using expansions into normally ordered creation-annihilation operators for the variables corresponding to the electromagnetic field, but instead they are using the “Krein-adjointed” operators \( \eta \partial^*_\mu, \mathbf{p} \eta = \eta a_\mu(\mathbf{p})^+ \eta \) instead, as in the formula (115). Therefore it is more convenient, when adopting the integral kernel operators to QED (in Gupta-Bleuler gauge), to change slightly the convention of Subsection 2.6 and use for \( \partial^*_\mu \) in the general integral kernel operator (69), on the tensor product of Fock spaces of the Dirac field \( \psi \) and the electromagnetic potential field \( A \), the operators \( \eta \partial^*_\mu, \mathbf{p} \eta \) whenever \( \mathbf{w} = (\mu, \mathbf{p}) \) corresponds to the photon variables \( \mu, \mathbf{p} \) in (69), instead of the ordinary transposed operators \( \partial^*_\mu, \mathbf{p} \). With this convention of physicists we will have the following formulas

\[
\begin{align*}
\kappa_{0,1}(\nu, \mathbf{p}; \mu, x) &= \frac{\sqrt{B(\mathbf{p}, \mathbf{p}^2(\mathbf{p}))}}{2\rho^0(\mathbf{p})} e^{-ip^x} \quad p \in \mathcal{O}_{1,0,0,1}, \\
\kappa_{1,0}(\nu, \mathbf{p}; \mu, x) &= \frac{\sqrt{B(\mathbf{p}, \mathbf{p}^2(\mathbf{p}))}}{2\rho^0(\mathbf{p})} e^{ip^x} \quad p \in \mathcal{O}_{1,0,0,1},
\end{align*}
\]

(119)

without the additional factor \((-1)^{(\nu)}\). In fact presence of the factors

\((-1)^{(\nu_1)} \cdots (-1)^{(\nu_3)}\)

for the kernels of the corresponding integral kernel operators is the only difference between the two conventions, and which are absorbed coincisely by the Gupta-Bleuler operator \( \eta \).

In other words: we will show that for the plane wave kernels (119) we have

\[
A(\varphi) = a^*(\tilde{\varphi}|_{\mathcal{E}_{1,0,0,1}}) + \eta a^*(\tilde{\varphi}|_{\mathcal{E}_{1,0,0,1}})^+ \eta
\]

\[
= a\left(U(\tilde{\varphi}|_{\mathcal{E}_{1,0,0,1}})\right) + \eta a\left(U(\tilde{\varphi}|_{\mathcal{E}_{1,0,0,1}})\right)^+ \eta
\]

\[
= a(\sqrt{B} \tilde{\varphi}|_{\mathcal{E}_{1,0,0,1}}) + \eta a(\sqrt{B} \tilde{\varphi}|_{\mathcal{E}_{1,0,0,1}})^+ \eta
\]

\[
= \sum_{\nu=0}^3 \int \kappa_{0,1}(\nu, \mathbf{p}) \partial^*_\nu, \mathbf{p} d^3 \mathbf{p} + \sum_{\nu=0}^3 \int \kappa_{1,0}(\nu, \mathbf{p}) \eta \partial^*_\nu, \mathbf{p} \eta d^3 \mathbf{p}
\]

\[
= \Xi_{0,1}(\kappa_{0,1}(\varphi)) + \Xi_{1,0}(\kappa_{1,0}(\varphi)), \quad \varphi \in \mathcal{E} = \mathcal{S}(\mathbb{R}^4; \mathbb{C}^4). \quad (120)
\]

Moreover we will show that the kernels \( \kappa_{0,1}, \kappa_{1,0} \) defined by (119) can be (uniquely) extended to the elements (and denoted by the same \( \kappa_{0,1}, \kappa_{1,0} \))

\[
\kappa_{0,1}, \kappa_{1,0} \in \mathcal{L}(E^*, \mathcal{E}^*),
\]

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so that by Thm 3.13 of [38] (or Thm. 4 of Subsection 2.6),

\[ A = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0}) \in \mathcal{L}(\mathcal{E} \otimes \mathcal{E}', (E)) \cong \mathcal{L}(\mathcal{E}', \mathcal{L}(E, (E))) \]

and \( A \), understood as an integral kernel operator with vector-valued distributional kernels \([119]\), determines a well defined operator-valued distribution on the space-time nuclear test space

\[ \mathcal{E}' = \mathcal{F}\left[ \mathcal{S}_{\Xi}(\mathbb{R}^4, \mathbb{C}^4) \right] = \mathcal{S}^{\text{tr}}(\mathbb{R}^4, \mathbb{C}^4). \]

In the formula \([120]\) \( \kappa_{0,1}(\phi), \kappa_{1,0}(\phi) \) denote the kernels representing distributions in \( E^* = \mathcal{S}_{\Xi}(\mathbb{R}^3, \mathbb{C}^4)^* \) which are defined in the standard manner

\[ \kappa_{0,1}(\phi)(\nu, p) = \sum_{\mu=0}^{3} \int_{\mathbb{R}^4} \kappa_{0,1}(\nu, p; \mu, x) \phi_{\mu}(x) \, d^4x \]

and analogously for \( \kappa_{1,0}(\phi) \), where \( \kappa_{0,1}, \kappa_{1,0} \) are understood as elements of

\[ \mathcal{L}(\mathcal{E}', E^*) \cong \mathcal{L}(E, \mathcal{L}(\mathcal{E}', \mathbb{C})) \cong \mathcal{L}(E, \mathcal{E}^*). \]

Similarly we have

\[ \kappa_{0,1}(\xi)(\mu, x) = \sum_{\nu=0}^{3} \int_{\mathbb{R}^3} \kappa_{0,1}(\nu, p; a, x) \xi(\nu, p) \, d^3p, \quad \xi \in E, \]

and analogously for \( \kappa_{1,0}(\xi)(\mu, x) \), with \( \kappa_{0,1}, \kappa_{1,0} \) understood as elements of

\[ \mathcal{L}(E, \mathcal{L}(\mathcal{E}', \mathbb{C})) \cong \mathcal{L}(E, \mathcal{E}^*) \cong \mathcal{L}(\mathcal{E}', E^*); \]

with pairings

\[ \langle \kappa_{0,1}(\phi), \xi \rangle = \sum_{\mu=0}^{3} \int_{\mathbb{R}^4 \times \mathbb{R}^3} \kappa_{0,1}(\phi)(\mu, p) \xi(s, p) \, d^3p \]

\[ = \sum_{s=1}^{4} \sum_{a=1}^{4} \int_{\mathbb{R}^3} \kappa_{0,1}(s, p; a, x) \phi_{a}(x) \xi(s, p) \, d^4x \, d^3p = \langle \kappa_{0,1}(\xi), \phi \rangle, \quad \xi \in E, \phi \in \mathcal{E}, \]

defined through the ordinary Lebesgue integrals.

\( U \) is the unitary isomorphism (and its inverse \( U^{-1} \))

\[ U : \mathcal{H} \ni \xi \mapsto \sqrt{B} \xi \in L^2(\mathbb{R}^3, \mathbb{C}^4), \]

\[ U^{-1} : L^2(\mathbb{R}^3, \mathbb{C}^4) \ni \zeta \mapsto \sqrt{B}^{-1} \zeta \in \mathcal{H} ', \]

joining the Gelfand triples \((272)\) of Subsection 2.1 of \([59]\) defining the field \( A \) through its Fock lifting, and is defined as point-wise multiplication

\[ \sqrt{B} \xi(p) \overset{df}{=} \frac{1}{\sqrt{2p^0(p)}} \sqrt{B(p, p^0(p))} \xi(p), \]

\[ \sqrt{B}^{-1} \zeta(p) \overset{df}{=} \sqrt{2p^0(p)} \sqrt{B(p, p^0(p))^{-1}} \zeta(p) \]

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by the matrix (and respectively its inverse)
\[
\frac{1}{\sqrt{2p^0(p)}}\sqrt{B(p, p^0(p))},
\]
\hspace{1cm} (121)
the same which is present in the formula (118), with the matrix \(\sqrt{B(p)}\), \(p \in \mathcal{O}_{1,0,0,1}\) defined by (200) of Subsection 4.1 of [59].

Note here that the Gelfand triples (272) of [59] with the joining unitary isomorphism \(U\) in (272) of [59] plays the same role in the construction of the field \(A\) in Subsection 5.8 of [59] as does the triples (10) joined by the unitary isomorphism (37) in the construction of the Dirac field \(\psi\), Subsection 2.6.

Concerning the equality (120) note that the first equality in (120) follows by definition, second by the fact that \(U\) is the unitary isomorphism joining the standard Gelfand triple
\[
E = \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4) \subset L^2(\mathbb{R}^3; \mathbb{C}^4) \subset \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4)^*\]
with the triple
\[
E \subset \mathcal{H}' \subset E^*\]
over the single particle Hilbert space of the field \(A\) (the analogue of the unitary isomorphism (37) of Subsection 2.6). The Fock lifting of the standard triple serves to construct the standard Hida operators \(a(\zeta)\), and the Fock lifting of the second triple serves to construct the Hida operators \(a'(\xi)\). Therefore we obtain the second equality (the analogue of the isomorphism (36)), compare also Subsection 5.8 of [59]. Third equality in (120) follows by definition of the isomorphism \(U\). Finally note that it follows almost immediately from definition (119) of \(\kappa_{0,1}, \kappa_{1,0}\) that
\[
\kappa_{0,1}(\varphi) = \sqrt{B\bar{\varphi}}|_{\mathcal{O}_{1,0,0,1}}, \quad \kappa_{1,0}(\varphi) = \sqrt{B\bar{\varphi}}|_{\mathcal{O}_{1,0,0,1}}.
\]
\hspace{1cm} (122)
Thus the fourth equality in (120) follows by Prop. 4.3.10 of [59] (compare also the fermi analogue of Prop. 4.3.10 of [59] – the Corollary 1 of Subsection 2.6).

Let \(\mathcal{O}'_C, \mathcal{O}_M\) be the algebras of convolutors and multipliers of the ordinary Schwartz algebra \(\mathcal{S}(\mathbb{R}^4; \mathbb{C}^4)\), defined by Schwartz [47], compare also Appendix 5. If the elements of \(\mathcal{O}'_C\) (resp. \(\mathcal{O}_M\)) are understood as continuous linear operators \(\mathcal{S} \rightarrow \mathcal{S}\) of convolution with distributions in \(\mathcal{O}'_C\) (or respectively as continuous operators of multiplication by an element of \(\mathcal{O}_M\)) then we can endow \(\mathcal{O}'_C, \mathcal{O}_M\) with the operator topology of uniform convergence on bounded sets (after Schwartz). The Fourier exchange theorem of Schwartz then says that the Fourier transform becomes a topological isomorphism of \(\mathcal{O}_M\) onto \(\mathcal{O}'_C\), which exchanges pointwise multiplication product defined by pointwise multiplication of functions in \(\mathcal{O}_M\) (representing the corresponding tempered distributions) with the convolution product, defined through the composition of the corresponding convolution operators in \(\mathcal{L}'(\mathcal{S}, \mathcal{S})\), compare [47], or Appendix 5.

Let \(\mathcal{O}_C\) be the predual (a smooth function space determined explicitly by Horváth) of the Schwartz convolution algebra \(\mathcal{O}'_C\) endowed with the above
Schwartz operator topology of uniform convergence on bounded sets on $O'_C$ (strictly stronger than the topology inherited from the strong dual space $S^*$ of tempered distributions), compare Appendix 5.

Let $O'_{CB_2}$ be the algebra of convolutors of the algebra

$$
E = S^{00}(\mathbb{R}^4; \mathbb{C}^4) = \mathcal{F} \left[ S^0(\mathbb{R}^4; \mathbb{C}^4) \right] = \mathcal{F} \left[ S_{\oplus A^{(4)}}(\mathbb{R}^4; \mathbb{C}^4) \right] = S_{B_2}(\mathbb{R}^4; \mathbb{C}^4),
$$

where we have used the standard operator

$$
B_2 = \mathcal{F} \oplus_0^3 A^{(4)} \mathcal{F}^{-1} \text{ on } \oplus_0^3 L^2(\mathbb{R}^4; \mathbb{C}) = L^2(\mathbb{R}^4; \mathbb{C}^4),
$$

introduced in Subsection 2.6 and further used in Subsection 2.7. Recall that the standard operators $A^{(n)}$ on $L^2(\mathbb{R}^n; \mathbb{C})$ have been constructed in Subsection 5.3 of [59].

Let $O'_{MB_2}$ be the algebra of multipliers of the nuclear algebra

$$
S^0(\mathbb{R}^4; \mathbb{C}^4) = S_{\oplus A^{(4)}}(\mathbb{R}^4; \mathbb{C}^4) = S_{B_2}(\mathbb{R}^4; \mathbb{C}^4).
$$

All the spaces $O_C, O_M, O_{MB_2}$ equipped with the Horváth inductive limit or respectively Schwartz operator topology of uniform convergence on bounded sets, and their strong duals $O'_C, O'_M, O'_{MB_2}$, equipped with the Schwartz operator topology of uniform convergence on bounded sets, are nuclear.

We have:

$$
O_M \subset O_{MB_2},
$$

$$
O'_C \subset O'_{CB_2},
$$

$$
O_C \subset O'_C \subset O'_{CB_2},
$$

by the results of Subsections 5.2-5.5 of [59].

Recall that here $O_M(\mathbb{R}^m; \mathbb{C}^n)$ is understood as the pointwise multiplication algebra of $\mathbb{C}^n$-valued functions on $\mathbb{R}^3$ in $O_M(\mathbb{R}^m; \mathbb{C}^n)$, with the elements of $O_M(\mathbb{R}^m; \mathbb{C}^n)$, $S(\mathbb{R}^m; \mathbb{C}^n)$ understood as $\mathbb{C}$-valued functions on the disjoint sum $\sqcup \mathbb{R}^m$ of $n$ copies of $\mathbb{R}^m$, compare Subsection 2.7. The translation $T_b, b \in \mathbb{R}^m$ is understood as acting on $(a, x) \in \sqcup \mathbb{R}^m$, $a \in \{1, 2, \ldots, n\}$, in the following manner $T_b(a, x) = (a, x+b)$. Equivalently $f \in O_M(\mathbb{R}^m; \mathbb{C}^n)$ (or $f \in O_C(\mathbb{R}^m; \mathbb{C}^n)$) means that each component of $f$ belongs to $O_M(\mathbb{R}^m; \mathbb{C})$ (or resp. to $O_C(\mathbb{R}^m; \mathbb{C})$).

We need the following Lemma (analogously as in Subsection 2.6 for the Dirac field).

**Lemma 9.** For the $\mathcal{L}(\mathcal{E}, \mathbb{C})$-valued (or $\mathcal{E}^*$-valued) distributions $\kappa_{0,1}, \kappa_{1,0}$, given by (119), in the equality (120) defining the electromagnetic potential field
A we have

\[
\left( \mu, x \right) \mapsto \sum_{\nu} \int_{\mathbb{R}^3} \kappa_{0,1}(\mu, p; \mu, x) \xi(\nu, p) \, d^3 p
\]
\[\in \mathcal{O}_C \subset \mathcal{O}_M \subset \mathcal{E}_* \cup \mathcal{E}_*, \xi \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4),\]

\[
\left( \mu, x \right) \mapsto \sum_{\nu} \int_{\mathbb{R}^3} \kappa_{1,0}(\mu, p; \mu, x) \xi(s, p) \, d^3 p
\]
\[\in \mathcal{O}_C \subset \mathcal{O}_M \subset \mathcal{E}_* \cup \mathcal{E}_*, \xi \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4),\]

\[
\left( \nu, p \right) \mapsto \sum_{\mu} \int_{\mathbb{R}^4} \kappa_{0,1}(\nu, p; \nu, x) \varphi^{\mu}(x) \, d^4 x
\]
\[\in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4), \varphi \in \mathcal{E},\]

\[
\left( \nu, p \right) \mapsto \sum_{\mu} \int_{\mathbb{R}^4} \kappa_{1,0}(\nu, p; \nu, x) \varphi^{\mu}(x) \, d^4 x
\]
\[\in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4), \varphi \in \mathcal{E}.\]

Moreover the maps

\[
k_{0,1} : \mathcal{E} \ni \varphi \rightarrow k_{0,1}(\varphi) \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4),
\]

\[
k_{1,0} : \mathcal{E} \ni \varphi \rightarrow k_{1,0}(\varphi) \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)
\]

are continuous, with \( k_{0,1}, k_{1,0} \) understood as maps in

\[
\mathcal{L}(\mathcal{E}, (\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4))^\ast) \cong \mathcal{L}(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}(\mathcal{E}, \mathbb{C}))
\]

and, equivalently, the maps \( \xi \mapsto k_{0,1}(\xi), \xi \mapsto k_{1,0}(\xi) \) can be extended to continuous maps

\[
k_{0,1} : \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^\ast \ni \xi \rightarrow k_{0,1}(\xi) \in \mathcal{E}^*,
\]

\[
k_{1,0} : \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^\ast \ni \xi \rightarrow k_{1,0}(\xi) \in \mathcal{E}^*,
\]

(for \( k_{0,1}, k_{1,0} \) understood as maps \( \mathcal{L}(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}(\mathcal{E}, \mathbb{C})) \cong \mathcal{L}(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4), \mathcal{E}^*) \). Therefore not only \( k_{0,1}, k_{1,0} \in \mathcal{L}(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}(\mathcal{E}, \mathbb{C})) \), but both \( k_{0,1}, k_{1,0} \) can be (uniquely) extended to elements of

\[
\mathcal{L}(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^\ast, \mathcal{L}(\mathcal{E}, \mathbb{C})) \cong \mathcal{L}(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^\ast, \mathcal{E}^*) \cong \mathcal{L}(\mathcal{E}, \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)).
\]

\[\blacksquare\]

That for each \( \xi \in \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) \) the functions \( k_{0,1}(\xi), k_{1,0}(\xi) \) given by (here \( x = (x_0, \mathbf{x}) \))

\[
\left( \mu, x \right) \mapsto \sum_{\nu=0}^{3} \int_{\mathbb{R}^3} \kappa_{0,1}(\nu, p; \mu, x) \xi(\nu, p) \, d^3 p
\]
\[= \sum_{\nu=0}^{3} \int_{\mathbb{R}^3} \frac{\sqrt{B(p, p^\mu(p))} \nu!}{\sqrt{2p_0(p)}} \xi(\nu, p) e^{-ip_0(p)x_0 + ip \cdot \mathbf{x}} \, d^3 \mathbf{x},\]

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Thus the said integrals defining \( \kappa \) belong to \( \mathcal{O}_C \subset \mathcal{O}_M \subset \mathcal{E}^* \) is immediate. Indeed, that they are smooth is obvious, similarly as it is obvious the existence of such a natural \( N \) (it is sufficient to take here \( N = 0 \)) that for each multiindex \( \alpha \in \mathbb{N}^4 \) the functions

\[
(a, x) \mapsto (1 + |x|^2)^{-N} |D^\alpha_0 \kappa_{0,1}(a, x)|, \quad (a, x) \mapsto (1 + |x|^2)^{-N} |D^\alpha_0 \kappa_{1,0}(a, x)|
\]

are bounded (of course for fixed \( \xi \)). Here \( D^\alpha_0 \kappa_{l,m}(\xi) \) denotes the ordinary derivative of the function \( \kappa_{l,m}(\xi) \) of \( |\alpha| = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 \) order with respect to space-time coordinates \( x = (x_0, x_1, x_2, x_3) \); and here \( |x|^2 = (x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2 \). Recall that by the results of Subsections 5.4 and 5.5 of [59], the operation of point-wise multiplication by the matrix \([121]\) is a multiplier of the nuclear algebra \( \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4) \), similarly multiplication by the function \( |p_0(p)|^k = |p|^k \), \( k \in \mathbb{Z} \), is a multiplier of this algebra, by the same Subsections. Thus the said integrals defining \( \kappa_{0,1}(\xi), \kappa_{1,0}(\xi) \) are convergent, similarly as the integrals defining their space-time derivatives with the obviously preserved mentioned above boundedness.

Consider now the functions

\[
\varphi \mapsto \kappa_{0,1}(\varphi) = \sqrt{p_0(p)}_{\kappa_{1,0,0,1}}, \quad \varphi \mapsto \kappa_{1,0}(\varphi) = \sqrt{p_0(p)}_{\kappa_{1,0,0,1}},
\]

with \( \varphi \in \mathcal{S}^{00}(\mathbb{R}^4; \mathbb{C}^4) \). It is obvious that both functions \( \kappa_{0,1}(\varphi), \kappa_{1,0}(\varphi) \) belong to \( \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4) = \mathcal{S}^{00}(\mathbb{R}^3; \mathbb{C}^4) \) whenever \( \varphi \in \mathcal{S}^{00}(\mathbb{R}^4; \mathbb{C}^4) \), by the results of Subsections 5.4 and 5.5 of [59]. That both functions \( \kappa_{0,1}(\varphi), \kappa_{1,0}(\varphi) \) depend continuously on \( \varphi \) as maps

\[
\mathcal{E} = \mathcal{S}^{00}(\mathbb{R}^4; \mathbb{C}^4) \longrightarrow \mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4) = \mathcal{S}^{00}(\mathbb{R}^3; \mathbb{C}^4)
\]

follows from: 1) the results of Subsection 5.5 of [59] and continuity of the Fourier transform as a map on the Schwartz space, 2) from the continuity of the restriction to the orbits \( \mathcal{E}_{1,0,0,1} \) and \( \mathcal{E}_{-1,0,0,1} \) regarded as a map from

\[
\mathcal{S}^{0}(\mathbb{R}^4; \mathbb{C}) = \mathcal{S}_{\odot A}(4)(\mathbb{R}^4; \mathbb{C}^4)
\]

into

\[
\mathcal{S}^{0}(\mathbb{R}^3; \mathbb{C}) = \mathcal{S}_{\odot A}(3)(\mathbb{R}^3; \mathbb{C}^4),
\]

compare the second Proposition of Subsection 5.6 of [59], and finally 3) from the fact that the operators of point-wise multiplication by the matrix \([121]\) are multipliers of the nuclear algebra

\[
\mathcal{S}_A(\mathbb{R}^3; \mathbb{C}^4) = \mathcal{S}_{\odot A}(3)(\mathbb{R}^3; \mathbb{C}^4) = \mathcal{S}^{0}(\mathbb{R}^3; \mathbb{C})
\]

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COROLLARY 4. Let $E = S_A(\mathbb{R}^3; \mathbb{C}^4) = S_{R^A(\mathbb{R}^3; \mathbb{C}^4)}$. Let

$$A = \Xi_{1,0}(\kappa_1,0) + \Xi_{1,0}(\kappa_1,0) \in \mathcal{L}((E) \otimes \mathcal{E}, (E)^\ast) \cong \mathcal{L}(\mathcal{E}, \mathcal{L}((E), (E)^\ast))$$

be the free quantum electromagnetic potential field understood as an integral kernel operator with vector-valued kernels

$$\kappa_{0,1}, \kappa_{1,0} \in \mathcal{L}(S_A(\mathbb{R}^3, \mathbb{C}^4), \mathcal{E}^\ast) \cong S_A(\mathbb{R}^3, \mathbb{C}^4)^* \otimes \mathcal{E}^* = E^* \otimes \mathcal{E}^*,$$

defined by \[119\]. Then the electromagnetic potential field operator

$$A = A^{(-)} + A^{(+)} = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0}),$$

belongs to $\mathcal{L}((E) \otimes \mathcal{E}, (E)) \cong \mathcal{L}(\mathcal{E}, \mathcal{L}((E), (E)))$, i.e.

$$A = \Xi_{0,1}(\kappa_0,1) + \Xi_{1,0}(\kappa_1,0) \in \mathcal{L}((E) \otimes \mathcal{E}, (E)) \cong \mathcal{L}(\mathcal{E}, \mathcal{L}((E), (E)))$$

which means in particular that the electromagnetic potential field $A$, understood as a sum $A = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0})$ of two integral kernel operators with vector-valued kernels, defines an operator valued distribution through the continuous map

$$\mathcal{E} \ni \varphi \mapsto \Xi_{0,1}(\kappa_0,1(\varphi)) + \Xi_{1,0}(\kappa_1,0(\varphi)) \in \mathcal{L}((E), (E)).$$

Note that the last Corollary likewise follows from:

1) the equality \[120\],
2) from Thm. 2.2 and 2.6 of \[29\],
3) continuity of the Fourier transform as a map on the Schwartz space,
4) continuity of the restriction to the orbit $\mathcal{E}_{1,0,0,1}$ regarded as a map $S^0(\mathbb{R}^4) \rightarrow S^0(\mathbb{R}^3)$ and finally
5) from continuity of the multiplication by the matrix \[121\], regarded as a map $S^0(\mathbb{R}^3; \mathbb{C}^4) \rightarrow S^0(\mathbb{R}^3; \mathbb{C}^4)$.

It is important to emphasize here that by the Thm. 3.13 of \[38\] (or Thm. 4 of Subsection 2.6) the continuity of the map $\varphi \mapsto \kappa_{1,0}(\varphi)$, regarded as a map $\mathcal{E} \rightarrow E = S_A(\mathbb{R}^3; \mathbb{C}^4)$, equivalent to the continuous unique extendibility of $\kappa_{1,0}$ to an element of $\mathcal{L}(E^*, \mathcal{E}^*)$, is a necessary and sufficient condition for the operator $A = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0})$ to be an element of

$$\mathcal{L}((E) \otimes \mathcal{E}, (E)) \cong \mathcal{L}(\mathcal{E}, \mathcal{L}((E), (E))).$$
i.e. for $A$ being a sum of integral kernel operators with vector-valued kernels which defines an operator-valued distribution on $\mathcal{E}$. On the other hand the continuity of the map

$$\mathcal{E} \ni \varphi \mapsto \kappa_{1,0}(\varphi) \in E = S_A(\mathbb{R}^3; \mathbb{C}^4)$$

is equivalent, as we have seen, to the continuity of the restriction to the cone $\mathcal{O}_{1,0,0,1}$, regarded as a map

$$\mathcal{E} \rightarrow E = S_A(\mathbb{R}^3; \mathbb{C}^4),$$

followed by the multiplication by the matrix $121$, and regarded as a map $E \rightarrow E$. From this it follows that

$$\mathcal{E} \neq S(\mathbb{R}^4), \ E \neq S(\mathbb{R}^3)$$

for the space-time test space of the zero mass field $A$ determined by a representation pertinent to the cone orbit $\mathcal{O}_{1,0,0,1}$, because restriction to the cone $\mathcal{O}_{1,0,0,1}$ is not continuous as a map $S(\mathbb{R}^4) \rightarrow S(\mathbb{R}^3)$, nor the multiplication by the matrix $121$ regarded as a map $S(\mathbb{R}^3) \rightarrow S(\mathbb{R}^3)$. This is in general the case for any zero mass (free) field. Namely we have the following

**THEOREM 6.** For any zero mass field, pertinent to the cone orbit $\mathcal{O}_{1,0,0,1}$, such as the electromagnetic potential field, which can be regarded as an integral kernel operator

$$\Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0})$$

with vector-valued kernels

$$\kappa_{0,1}, \kappa_{1,0} \in \mathcal{L}(S_A(\mathbb{R}^3, \mathbb{C}^4), \mathcal{E}^*) \cong S_A(\mathbb{R}^3, \mathbb{C}^4)^* \otimes \mathcal{E}^* = E^* \otimes \mathcal{E}^*,$$

extendible to

$$\kappa_{0,1}, \kappa_{1,0} \in \mathcal{L}(S_A(\mathbb{R}^3, \mathbb{C}^4)^*, \mathcal{E}^*) \cong S_A(\mathbb{R}^3, \mathbb{C}^4) \otimes \mathcal{E}^* = E \otimes \mathcal{E}^*,$$

and defined by plane waves

$$\kappa_{0,1}(s, p; a, x) = u^0(s, p) e^{-ip \cdot x}, \ p = (p_0(p), p) \in \mathcal{O}_{1,0,0,1},$$

$$\kappa_{1,0}(s, p; a, x) = v^0(s, p) e^{ip \cdot x}, \ p = (p_0(p), p) \in \mathcal{O}_{1,0,0,1},$$

$s, a = 1, 2, \ldots N$

the space-time test space $\mathcal{E}$ cannot be equal to the ordinary Schwartz space $S(\mathbb{R}^4; \mathbb{C}^N)$ but instead it has to be equal

$$\mathcal{E} = S^{00}(\mathbb{R}^4; \mathbb{C}^N) = \mathcal{F} \left[ S^{00}(\mathbb{R}^4; \mathbb{C}^N) \right] = \mathcal{F} \left[ S_{\oplus A(4)}(\mathbb{R}^4; \mathbb{C}^N) \right],$$

where $A^{(4)}$ is the standard operator on $L^2(\mathbb{R}^4; \mathbb{C})$ constructed in Subsection 5.3 of [27], and $\oplus A^{(4)}$ denotes direct sum of $N$ copies of the operator $A^{(4)}$ acting on

$$L^2(\mathbb{R}^4; \mathbb{C}^N) = \oplus^N_1 L^2(\mathbb{R}^4; \mathbb{C}).$$
In particular this Theorem holds for all zero mass free gauge fields $A$ of the Standard Model.

Let us stress once more that the conclusion of the last Theorem is inapplicable to zero-mass fields in the sense of Wightman, which allows the ordinary Schwartz space as the space-time test space. This follows immediately from the fact that the integration of the restriction of the test function to the cone orbit $O_{1,0,0,1}$ along $O_{1,0,0,1}$ with respect to the measure induced by the ordinary measure of the ambient space $\mathbb{R}^4$, is a well defined continuous functional on the ordinary Schwartz space $S(\mathbb{R}^4; \mathbb{C})$. We have also used this fact in extending the zero mass Pauli-Jordan function from $S^{00}(\mathbb{R}^4)$ over to a functional on $S(\mathbb{R}^4)$, with preservation of the homogeneity and its degree, compare Subsection 5.6 of [59].

2.10 Equivalent realizations of the free local electromagnetic potential quantum field. Comparison with the realization used by other authors

Let $U^{*-1} = WU^{(1,0,0,1)}LW^{-1}$ and $U = [WU^{(1,0,0,1)}LW^{-1}]^{*-1}$ be the Lopuszański representation and its conjugation $U$ acting in the single particle space of the quantum field $A$ realization of Sections 4 and 5 of [59]. Both $U^{*-1}$, and $U$ transform continuously the nuclear space $E_C$ into itself (let us write simply $E$ instead $E_C$ for simplicity). Similarly the lifting $\Gamma(U)$ of $U$ acting in the Krein-Fock space $\Gamma(\mathcal{H}', \Gamma(\mathcal{J}''))$ transforms continuously the nuclear Hida's test space $(E)$ onto itself, and is Krein isometric in the Krein-Fock space of the field $A$.

We can consider different such realizations of $A$, with the representations $U$ and $\Gamma(U)$ restricted to the translation subgroup commuting with the Krein fundamental symmetry $\mathcal{J}'$, and resp. $\Gamma(U)$ commuting with the Gupta-Bleuler operator $\Gamma(\mathcal{J}')$, and thus with translations being represented by unitary and Krein-unitary operators. The natural equivalence for such realizations is the existence of Krein isometric mapping transforming bi-uniquelly and bi-continuously $E$, resp. $(E)$, onto itself, and which intertwines the representations. It is easily seen that in case of ordinary non gauge fields with unitary representations, this equivalence reduces to the ordinary unitary equivalence of the realizations of the fields. In case of gauge mass-less fields, such as electromagnetic potential field $A$, where $U$ and $\Gamma(U)$ are unbounded (and Krein-isometric) the equivalence is weaker, although preserves the pairing functions of the field, the linear equation it fulfills and its local transformation formula. Nonetheless the analytic properties of the representation may be substantially different for equivalent realizations of the field $A$, especially the behaviour of the restriction of the representation $U$ or $\Gamma(U)$ of $T_4 \otimes SL(2, \mathbb{C})$ to the subgroup $SL(2, \mathbb{C})$, as is no very surprising as the representors of the Lorentz hyperbolic rotations are unbounded, contrary to the representors of translations, which are bounded (even unitary and Krein-unitary).

We illustrate this phenomena on a concrete example of different equivalent realizations of the free field $A$. Although the example is concrete it can be
shown that the construction encountered is generic, and that the general class of equivalent realizations may be constructed without any substantial modification. The general construction of a realization of the free field \( A \) is equivalent to the construction of the most general intertwining operator bi-uniquely and bi-continuously mapping the nuclear spaces, where the initial spaces and representations are these given in Sections 4 and 5 of [59] for the realization of \( A \) given there. We give a concrete example of such an intertwining operator, in case where the nuclear spaces corresponding to different realizations are identical. Because this assumption is not relevant, and because the construction of the general intertwining operator is general for the case where the nuclear spaces are identical, we prefer to give the concrete example instead of going immediately into a general situation, which would be less transparent.

On the single particle space \((\mathcal{H}', \mathcal{J}')\) of the realization of \( A \) of Sect. 4 and 5 of [59] there exists, besides \( U, U^{-1} \), the Krein-isometric representation

\[
\text{ass}
U(0, \alpha) \tilde{\varphi}(p) = \sqrt{B(p)^{-1}} V(\alpha) \sqrt{B(p)} \tilde{\varphi}(\Lambda(\alpha)p) = \sqrt{B(p)^{-1}} \Lambda(\alpha^{-1}) \sqrt{B(p)} \tilde{\varphi}(\Lambda(\alpha)p),
\]

\[
\text{ass}
U(a, 1) \tilde{\varphi}(p) = \sqrt{B(p)^{-1}} T(a) \sqrt{B(p)} \tilde{\varphi}(p) = e^{ia\cdot p} \tilde{\varphi}(p),
\]

(124)

associated to the Lopuszański representation \( U^{*-1} = WU(1,0,0,1) \mathcal{L}_W^{-1} \), where \( \sqrt{B(p)} \) is the (positive) square root of the (positive) matrix \( B(p) \), \( p \in \mathcal{O}_{1,0,0,1} \) (198) of [59], equal (200) of Subsection 4.1 of [59]. Recall that for each fixed point \( p \in \mathcal{O}_{1,0,0,1} \), the matrices \( \sqrt{B(p)}, B(p), \mathcal{J}_p' = V(\beta(p))^{-1} \mathcal{J}_p V(\beta(p)) = \mathcal{J}_p B(p) \), are all Krein-unitary in the Krein space \((\mathbb{C}^4, \mathcal{J}_p)\), where \( \mathcal{J}_p \) is the constant matrix (185) of Subsection 4.1 of [59]. In other words all the matrices \( \sqrt{B(p)}, B(p), \mathcal{J}_p' \) are Lorentz matrices preserving the the Lorentz metric \( g^{\mu\nu} = \text{diag}(-1,1,1,1) \).

This representation is Krein-isometrically equivalent to the Lopuszański representation \( U^{*-1} = WU(1,0,0,1) \mathcal{L}_W^{-1} \) given by (187) of Subsection 4.1 of [59]. (Analogously its conjugation is equivalent to the conjugation \( U \) of the Lopuszański representation \( U^{*-1} \)). Indeed the intertwining operator \( C \), understood as an operator \((\mathcal{H}', \mathcal{J}') \rightarrow (\mathcal{H}', \mathcal{J}')\), acting in the single particle space is equal

\[
C \tilde{\varphi}(p) = \sqrt{B(p)^{-1}} \tilde{\varphi}(p), \quad C^{-1} \tilde{\varphi}(p) = \sqrt{B(p)} \tilde{\varphi}(p),
\]

and \( C \) transforms bi-uniquely and bi-continuously the nuclear space \( E \) onto itself (compare the first Proposition of Subsect. 5.6 of [59]) and the intertwining operator \( \Gamma(C) \) transforms bi-uniquely and bi-continuously \( (E) \) onto itself \( (E) \), [26], [39]. One easily checks that that \( C \) indeed intertwines \( U^{*-1} \) and \( \text{ass} U ^* U \):

\[
CU^{*-1}C^{-1} = \text{ass} U
\]

and thus that \( \Gamma(C) \) intertwines \( \Gamma(U^{*-1}) \) and \( \Gamma(\text{ass} U) \).

Let us introduce another operator \( K \):

\[
K \tilde{\varphi}(p) = \sqrt{B(p)} \tilde{\varphi}(p), \quad K^{-1} \tilde{\varphi}(p) = \sqrt{B(p)^{-1}} \tilde{\varphi}(p),
\]

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understood as a Krein-unitary operator mapping the Krein space \((\mathcal{H}', J')\) onto the Krein space \((KH', J_\mathcal{H}'K^{-1}) = (KH', J_p)\), where the Krein fundamental symmetry in the Krein space \((KH', J_p)\) is equal to the operator of multiplication by the constant matrix \(J_p\) equal (185) of Subsection 4.1 of [59]. Recall that the Krein fundamental symmetry operator \(J'\) in the single paricle Krein space \((H', J')\) is equal to the operator of multiplication by the matrix (193) of Subsection 4.1 of [59]:

\[ J'_p = V(\beta(p))^{-1} J_p V(\beta(p)) = J_p B(p), \]

where \(B(p)\) is equal to the matrix (198) of [59]. The operator \(K\) gives a Krein-unitary equivalence between the representation \(\hat{\omega} U\) acting on the Krein space \((\mathcal{H}', J')\) and defined by the formula (124) with the dense nuclear domain \((E)\), and the Krein-isometric representation given by formula (187) of [59] identical as for the Lopuszański representation \(U^{*^{-1}}\) on \((E)\), but on the Krein space \((KH', J_x)\) and with the nuclear domain \((E)\), which differs from the Krein space of Sections 4 and 5 of [59] by the replacement of the Lorentz matrices \(\sqrt{B(p)}\) and \(B(p)\) everywhere with the constant unit matrix 1. Because on the other hand the Lopuszański representation \(U^{*^{-1}}\), defined by (187) of [59], and the representation \(\hat{\omega} U\), both acting on the Krein space \((\mathcal{H}', J')\) are Krein isometric equivalent (with \(C\) defining the equivalence), then it follows that the Lopuszański representation, defined by (187) of [59], with the nuclear domain \(E\), on the Krein space \((\mathcal{H}', J')\) (with the matrix \(B(p) \neq 1\) and equal (198) of [59] is equivalent to the Krein isometric representation defined by the same formula (187) of [59] and the same nuclear domain \(E\), but on the Krein space in which the operators \(B(p)\) and \(\sqrt{B(p)}\) are everywhere replaced by the constant unital matrices 1.

In this way we have obtained two equivalent realizations of the free quantum field \(A\). The first one is obtained as in Sections 4 and 5 of [59]. The other is obtained exactly as in Sections 4 and 5 of [59] by the replacement everywhere in the formulas of the positive Lorentz matrices \(B(p)\) and \(\sqrt{B(p)}\) by the unit 4\(\times\)4-matrix. A simple inspection shows that all proofs remain valid if we replace \(B(p), \sqrt{B(p)}\) by 1 in Sections 4 and 5 of [59]. In particular we obtain in this way a local mass-less quantum four-vector field \(A\), fulfilling d’Alembert equation with the pairing equal to the zero mass Pauli-Jordan distribution function multiplied by the Minkowski metric components. In particular this realization should be identified with the one used e.g. in [46], [9]–[12]. In particular replacement of the matrix

\[ \sqrt{B(p, p^0(p))^{\mu}_{\lambda}} \]

by the unit 4\(\times\)4 matrix in the formula (294) of [59]:

\[ A^\mu(x) = \int d^3p \left\{ \frac{1}{\sqrt{2p^0(p)}} \sqrt{B(p, p^0(p))^{\mu}_{\lambda}} a^\lambda(p) e^{-ip\cdot x} + \frac{1}{\sqrt{2p^0(p)}} \sqrt{B(p, p^0(p))^{\mu}_{\lambda}} \eta a^\lambda(p) + \eta e^{ip\cdot x} \right\} \]

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gives exactly the formula (2.11.45):

\[ A^\mu(x) = \int_{\mathbb{R}^3} d^3p \left\{ \frac{1}{\sqrt{2p^0(p)}} a^\mu(p) e^{-ip\cdot x} + \frac{1}{\sqrt{2p^0(p)}} \eta a^\mu(p)^+ \eta e^{ip\cdot x} \right\} \]  

(125)
of [46] (the lack of the additional constant factor \((2\pi)^{-3/2}\) in our formula comes from the fact that we have discarded the normalization factor for the measures in the Fourier transforms, in order to simplify notation). Similarly for other operator-valued distributions, or ordinary operators, which we obtain by inserting the unit matrix for \(\sqrt{B(p)}\).

However the explicit formula for the Krein-isometric representation of \(T_4 \otimes SL(2, \mathbb{C})\) is lacking in the cited works as well as in other works (as to the knowledge of the author) using the Gupta-Bleuler or BRST method. Moreover any analysis of the electromagnetic potential field in the Gupta-Bleuler approach, giving the linkage to the (generalized) induced representation theory of Mackey necessary uses the operator \(\sqrt{B(p)} \neq 1\). In particular no explicit construction of the representation of \(T_4 \otimes SL(2, \mathbb{C})\) would be possible and its immediate linkage to the induced Lopuszański representation, without the analysis using explicitly the realization of the field \(A\) with the matrix \(B(p)\) equal (198) of Subsection 4.1 of [59]. We can pass to the (apparently) simpler formulas only after using the intertwining operators, \(C, K\), defined again with the help of \(\sqrt{B(p)}\), and starting with the realization of \(A\) presented in 4 and 5 of [59].

Perhaps we should emphasize that the two realizations of the free electromagnetic potential quantum field \(A\): 1) the one with with \(\sqrt{B(p)} \neq 1\) equal (200) of [59] and presented in Sect. 4, 4 of [59] and 2) the one with \(\sqrt{B(p)} = 1\), differ substantially. In particular we have the following

**PROPOSITION.** Consider the restriction of the Krein-isometric representations of \(T_4 \otimes SL(2, \mathbb{C})\) to the subgroup \(SL(2, \mathbb{C})\), acting in the single particle Krein-Hilbert spaces in the two realizations, 1) and 2). Then for the second realization 2) (with \(\sqrt{B(p)} = 1\)) the restriction can be decomposed into ordinary Hilbert space direct integral of subrepresentations \(U^\chi\) each acting in the Hilbert space of generalized homogeneous of degree \(\chi\) eigenstates \(\in E^*\) (distributions) of the scaling operator \(S_\lambda\):

\[ S_\lambda \tilde{\varphi}(p) = \tilde{\varphi}(\lambda p), \tilde{\varphi} \in E, \]

where \(\lambda\) is a fixed positive real number.

No such decomposition is possible for the 1) realization of \(A\) (with \(\sqrt{B(p)} \neq 1\) and equal (200) in [59]).

**REMARK.** The statement of the last Proposition can be easily lifted to the Fock-Krein spaces of the realizations 1) and 2) of the field \(A\), therefore we consider the statement and the proof only for the single particle Krein-Hilbert spaces.

■ (Proof of the Proposition. An outline.) We consider the two versions of the Lopuszański representation \(U^{*1}\) with \(\sqrt{B(p)}\) equal respectively (200)
of [59] or 1 in case 1) or 2). The results for its conjugation \( U \) actually acting in the single particle space will follow as a consequence from the result for the Lopuszański representation \( U^{*-1} \) itself.

Note that in both realizations the operator \( S_\lambda \) (checking of which we leave as an easy exercise) has (unique) bounded extension to a normal operator, i.e. commuting with its adjoint \( S_\lambda^* \) (with respect to the ordinary Hilbert space inner product \( \langle \cdot, \cdot \rangle \), and not with respect to the Krein-inner product \( \langle \cdot, \tilde{\lambda}' \rangle \)).

The point is that the operators \( S_\lambda, S_\lambda^* \), both commute with the Lopuszański representation \( U^{*-1} \) in the second realization 2) (with \( \sqrt{B(p)} = 1 \)) and with the operator \( \tilde{\lambda}' \) (which in the realization 2) with \( \sqrt{B(p)} = 1 \) reduces to the constant matrix operator \( \tilde{\lambda}_p \) equal to (185) of [59]. But in the first realization 1) (with \( \sqrt{B(p)} \) equal (200) of [59], although \( S_\lambda \) commutes with the Lopuszański representation \( U^{*-1} \), the adjoint operator \( S_\lambda^* \) does not commute with the Lopuszański representation \( U^{*-1} \), nor with the operator \( \tilde{\lambda}' \). Checking the commutation rules we again leave as an easy exercise to the reader.

The proof of the statement of the Proposition can now be essentially reduced to the application of Theorems 1 and 2, [51], with the commutative decomposition \(*\)-algebra \( C \) of Thm. 2 in [51] equal to the one generated by the commuting operators \( S_\lambda, S_\lambda^* \).

In both realizations, 1) and 2), the operators \( S_\lambda, S_\lambda^* \) transform continuously the nuclear space \( E \) into itself, which follows easily by the results of Section 5 of [59] (compare the proof of the first Proposition of Subsection 5.6 of [59]). On the other hand \( E \), the single particle Krein-Hilbert space \( \mathcal{H}' \) and \( E^* \), compose the Gefand triple \( E \subset \mathcal{H}' \subset E^* \) (or a rigged Hilbert space). Thus the decomposition of \( U \) (restricted to \( SL(2, \mathbb{C}) \)) in the realization 2), is precisely the decomposition corresponding to the decomposition corresponding of the normal operator \( S_\lambda \), into the direct integral of subspaces of generalized eigen-subspaces of generalized eigenvectors in \( E^* \) of \( S_\lambda \), constructed as in Chap. I.4. of [18].

Using the formula (125) for the electromagnetic potential field operator, regarded as the sum of integral kernel operators

\[
A = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0})
\]

with vector-valued distributional plane wave kernels

\[
\kappa_{0,1}, \kappa_{1,0} \in \mathcal{L}(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4), \mathcal{E}^*) \cong \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^* \otimes \mathcal{E}^* = E^* \otimes \mathcal{E}^*,
\]

we will have the following formula for the plane wave kernels:

\[
\begin{align*}
\kappa_{0,1}(\nu, p; \mu, x) &= \frac{\delta_{\nu \mu}}{\sqrt{2p^0(p)}} e^{-ip \cdot x}, \quad p \in \mathcal{E}_{1,0,0,1}, \\
\kappa_{1,0}(\nu, p; \mu, x) &= \frac{\delta_{\nu \mu}}{\sqrt{2p^0(p)}} e^{ip \cdot x}, \quad p \in \mathcal{E}_{1,0,0,1}.
\end{align*}
\]  

(126)

defining the distributions \( \kappa_{0,1}, \kappa_{1,0} \) instead of \( [119] \). Proof that they can be (uniquely) extended to elements

\[
\kappa_{0,1}, \kappa_{1,0} \in \mathcal{L}(\mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4)^*, \mathcal{E}^*) \cong \mathcal{S}_A(\mathbb{R}^3, \mathbb{C}^4) \otimes \mathcal{E}^*,
\]

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remains the same as for the kernels (119) in Lemma 9, Subsection 2.9. Thus by Thm. 3.13 of [38] (or Thm. 4 of Subsection 2.6) we obtain the corollary that

\[ A = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0}) \in \mathcal{L}((E) \otimes \mathcal{S}, (E)) \cong \mathcal{L}(\mathcal{S}, \mathcal{L}((E), (E))) \]

with \( \kappa_{0,1}, \kappa_{1,0} \) defined by (120). Thus the field \( A = \Xi_{0,1}(\kappa_{0,1}) + \Xi_{1,0}(\kappa_{1,0}) \), understood as integral kernel operator defines an operator-valued distribution through the continuous map

\[ \mathcal{S} \ni \varphi \mapsto \Xi_{0,1}(\kappa_{0,1}(\varphi)) + \Xi_{1,0}(\kappa_{1,0}(\varphi)) \in \mathcal{L}((E), (E)). \]

3 Higher order contributions \( A_{\text{int}}^{\mu(n)}(g = 1, x) \) and \( \psi_{\text{int}}^{(n)}(g = 1, x) \) to the interacting fields \( A_{\text{int}}^{\mu}(g = 1, x) \) and \( \psi_{\text{int}}(g = 1, x) \)

The only modification which we introduce into the causal perturbative approach to spinor QED, which goes back to Stückelberg and Bogoliubov is that we are using the white noise construction of free fields of the theory.

This allows us to treat each free field at specified space-time point as a well defined generalized Hida operator, but moreover each free field gains the mathematical interpretation of an integral kernel operator with vector-valued kernel in the sense of Obata [38]. We have constructed the free Dirac and electromagnetic potential fields as integral kernel operators with vector-valued kernels in the sense of Obata [38]. The operations of Wick product, differentiation, integration, convolution with tempered distributions, which can be performed upon field operators understood as integral kernel operators in the sense of Obata, have been described in Subsection 2.7. Construction of the free fields as integral kernel operators opens us to the general and effective theory of integral kernel operators due to Hida-Obata-Saitô. In particular we can treat the Wick product (compare the so called “Wick theorem” in the book [6]) in the rigorous mathematically controllable fashion, necessary for the needs of the causal method (note here that in particular Wightman’s definition is not effective here). The whole causal method is left completely untouched. We just put the free fields, understood as integral kernel operators, into the formulas for the causal perturbative series using the computational Rules for the Wick product, integration and convolution with tempered distributions, which are given in Subsection 2.7. The only nontrivial point is the splitting of the causal distributions. Namely (if the free fields are understood as integral kernel operators) each contribution to the causal scattering matrix is a finite sum

\[ \sum_{l, m} \Xi_{l,m}(\kappa_{l,m}) \]

of well defined integral kernel operators (which almost immediately follows from
the our results summarized in Subsection 2.7

$$\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}\left( (E) \otimes \mathcal{E}, (E^*) \right) \cong \mathcal{L}\left( \mathcal{E}, \mathcal{L}( (E), (E) ) \right)$$

with vector-valued kernels

$$\kappa_{l,m} \in \mathcal{L}\left( E_{i_1} \otimes \cdots \otimes E_{i_{l+m}}, \mathcal{E}^* \right) \cong E_{i_1}^* \otimes \cdots \otimes E_{i_{l+m}}^* \otimes \mathcal{E}^*$$

in the sense of Obata, compare Subsections 2.6 and 2.7 where the the Hida subspace $E$ in the tensor product of the Fock spaces of the Dirac field and the electromagnetic potential field is constructed.

Here

$$\mathcal{E} = \mathcal{E}_{n_1} \otimes \cdots \otimes \mathcal{E}_{n_d}, \ n_k \in \{1, 2\}$$

is equal to the tensor product of several space-time test function s

$$\mathcal{E}_1 = \mathcal{S}(\mathbb{R}^4; \mathbb{C}) \text{ or } \mathcal{E}_2 = \mathcal{S}^{00}(\mathbb{R}^4; \mathbb{C})$$

correspondingly to the massive or mass less component field (compare Subsections 2.6 and 2.7). The nontrivial task in construction is the splitting of vector valued causal distribution kernels $\kappa_{l,m}$ into retarded and advanced parts, which in practical computation reduces to the splitting of causal distributions in

$$\mathcal{E} = \mathcal{E}_{n_1}^* \otimes \cdots \otimes \mathcal{E}_{n_d}^*, \ n_k \in \{1, 2\}$$

causally supported into retarded and advanced parts. This problem has been solved by Epstein and Glaser [14] but for the case where all factors $E_{n_k}$ are equal to the ordinary Schwartz space $\mathcal{S}(\mathbb{R}^4; \mathbb{C})$. But, as we have already explained in Subsection 5.8 of [59] and in Subsections 2.9, 2.7 of this work, the modification of the space-time test space into the space $\mathcal{E}_2 = \mathcal{S}^{00}(\mathbb{R}^4; \mathbb{C})$ is necessary for the white noise construction of free mass less field to be possible. Moreover the white noise construction allows us to construct and control the Wick product and allows rigorous formulation and proof of the “Wick theorem” of Bogoliubov-Shirkov [6], necessary for the causal method, compare Subsection 2.7. Therefore we need the splitting to be extended over to causal elements of

$$\mathcal{E} = \mathcal{E}_{n_1}^* \otimes \cdots \otimes \mathcal{E}_{n_d}^*, \ n_k \in \{1, 2\}$$

in which some of the factors $E_{n_k}^*$ are equal $\mathcal{S}^{00}(\mathbb{R}^4; \mathbb{C})^*$. The test space $\mathcal{S}^{00}(\mathbb{R}^4; \mathbb{C})$ in turn is much less flexible concerning localization, in particular it contains no non trivial elements with compact support. Fortunately the Pauli-Jordan functions of mass less fields (e.g. of the free electromagnetic potential field) are by definition homogeneous. This means that the causal distributions in

$$\mathcal{E} = \mathcal{E}_{n_1}^* \otimes \cdots \otimes \mathcal{E}_{n_d}^*, \ n_k \in \{1, 2\}$$

which are to be split into retarded and advanced parts have the factors in $E_{n_k}^* = \mathcal{S}^{00}(\mathbb{R}^4; \mathbb{C})^*$ which are homogeneous and for homogeneous distributions we have enough elements in $\mathcal{S}^{00}(\mathbb{R}^4; \mathbb{C})$ to realize the splitting of homogeneous and causal
distributions, compare Subsection 5.7 of [59]. Moreover all of the homogeneous factors in $E^*_{n_k} = S^{00}(\mathbb{R}^4; \mathbb{C})^*$ which we encounter in practice can be extended over $S(\mathbb{R}^4; \mathbb{C})^*$ with the preservation of homogeneity. Thus the splitting problem for causal distributions (homogeneous over the factors $E^*_{n_k} = S^{00}(\mathbb{R}^4; \mathbb{C})^*$) in

$$\mathcal{E} = \mathcal{E}^*_{n_1} \otimes \cdots \otimes \mathcal{E}^*_{n_{\lambda_d}}, \quad n_k \in \{1, 2\}$$

can in fact be reduced to the splitting of Epstein-Glaser, compare Subsection 5.7 of [59].

Summing up we can insert the free fields, understood as integral kernel operators in the sense of Obata, into the formulas for the causal perturbative series for interacting fields. The necessary operations of Wick product, splitting, integrations, have a rigorous meaning as operations performed upon integral kernel operators explained in Subsection 2.7. The computation being essentially simple can therefore be omitted. We give only the final formulas for the interacting fields (compare [9], [13], [16]).

$$\psi^{a^1}_\text{tot}(g, x) = \psi^a(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^{4n}} d^4x_1 \cdots d^4x_n \psi^{a(n)}(x_1, \ldots, x_n; x) g(x_1) \cdots g(x_n),$$

with

$$\psi^{a^1}(x_1; x) = eS^a_{\text{tot}} \gamma^\nu_1 a_1 a_2 \psi^{a^2}(x_1) A_{\nu_1}(x_1),$$

$$\psi^{a^2}(x_1, x_2; x) =$$

$$e^2 \left\{ S^a_{\text{tot}}(x_1 - x) \psi^{a_1}(x_1) A_{\nu_1}(x_1) A_{\nu_2}(x_2) : \psi^{a^2}(x_1) A_{\nu_2}(x_2) - S^a_{\text{tot}}(x_1 - x) \psi^{a_2}(x_1) A_{\nu_1}(x_1) A_{\nu_2}(x_2) : D_0^{a^2}(x_1 - x_2) \right\}$$

$$+ \left\{ x_1 \leftrightarrow x_2 \right\},$$

e. t. c.

and

$$A_{\text{tot} \mu}(g, x) = A_{\mu}(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^{4n}} d^4x_1 \cdots d^4x_n A^{(n)}_{\mu}(x_1, \ldots, x_n; x) g(x_1) \cdots g(x_n),$$

with

$$A^{(1)}_{\mu}(x_1; x) = -eD^{a^2}_{0}(x_1 - x) : \psi^{a^1}(x_1) \gamma^\mu_1 a_1 a_2 \psi^{a^2}(x_1) :,$$

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\[ A^{(2)}_{\mu}(x_1, x_2; x) = e^2 \left\{ \bar{\psi} a_1(x_1) \left( \gamma_{\mu} a_1 a_2 S_{\text{int}}(x_1 - x_2) \gamma_{\nu_1} a_3 a_4 D^0_{\nu}(x_1 - x) A_{\nu_1}(x_2) \right) \right. \\
+ \left. \gamma_{\nu_1} a_1 a_2 S_{\text{av}}(x_1 - x_2) \gamma_{\nu_2} a_3 a_4 D^0_{\nu}(x_2 - x) A_{\nu_1}(x_1) \right) \psi a_4(x_2) : \\
+ D^0_{\nu}(x_1 - x) \Pi_{\text{av}}^{\nu_{\nu_1}}(x_2 - x_1) A_{\nu_1}(x_2) \right\} + \left\{ x_1 \leftrightarrow x_2 \right\} \]

e. t. c.

where \( g \) is the intensity-of-interaction function over space-time which is assumed to be an element of the ordinary Schwartz space \( \mathcal{S}(\mathbb{R}^4; \mathbb{C}) \), and which plays a technical role in realizing the causality condition in the form we have learned from Bogoliubov and Shirkov [6], compare [9], [46], [13]. This intensity function \( g \) modifies the interaction into unphysical in the regions which lie outside the domain on which \( g \) is constant and equal to 1. It is therefore important problem to pass to a "limit" case of physical interaction with \( g = 1 \) everywhere over the space-time.

\[ \psi a_{\text{int}}^{(n)}(g, x) = \frac{1}{n!} \int_{\mathbb{R}^{4n}} \text{d}^4 x_1 \cdots \text{d}^4 x_n \psi a^{(n)}(x_1, \ldots, x_n; x), \]
\[ A_{\text{int}}^{(n)}(g, x) = \frac{1}{n!} \int_{\mathbb{R}^{4n}} \text{d}^4 x_1 \cdots \text{d}^4 x_n A a^{(n)}(x_1, \ldots, x_n; x), \]

are the repetitive \( n \)-th order contributions to the interacting Dirac and electromagnetic potential fields.

Here in the above formulas for the \( n \)-th order contributions to interacting fields the free Dirac and electromagnetic fields \( \psi \) and \( A \) we understood as integral kernel operators with vector-valued kernels as explained in 2.6 and 2.9. Correspondingly the Wick product and the integrations in these formulas are understood in a rigorous sense as operations performed upon integral kernel operators, and summarized in the Rules of Subsection 2.7. It turns out that each order contribution is equal

\[ \psi a_{\text{int}}^{(n)}(g) = \sum_{l,m} \Xi(\kappa_{l,m}), \]
\[ A_{\text{int}}^{(n)}(g) = \sum_{l,m} \Xi(\kappa'_{l,m}), \]

to a finite sum of well defined integral kernel operators \( \Xi(\kappa_{l,m}), \Xi(\kappa'_{l,m}) \) with vector-valued distributional kernels \( \kappa_{l,m}, \kappa'_{l,m} \) in the sense of Obata [38] (compare Subsection 2.7).

But the main and the whole point is that if the free fields are understood as integral kernel operators in the sense of Obata, then the above formulas for each \( n \)-th order contribution to interacting fields, preserve their rigorous mathematical meaning even if we put \( g = 1 \) everywhere: namely for \( g \) put everywhere equal
to 1 the formulas for each order contributions to interacting fields represent well defined integral kernel operators in the sense of Obata. This we have proved as Theorem 5 Subsection 2.7. Free fields are of course understood as integral kernel operators in the formulas for contributions to interacting fields, and the respective operations of Wick product and integrations with pairing functions are understood as performed upon integral kernel operators according to the Rules of Subsection 2.7.

Thus each order contribution to interacting fields in the adiabatic limit $g = 1$ of physical interaction is well defined integral kernel operator and belongs to the same general class of integral kernel operators as the Wick product at the same space-time point of free mass less fields (such as the free electromagnetic potential field). Thus the construction of the free fields within the white noise setup as integral kernel operators allows us to solve the adiabatic limit problem in the causal perturbative and spinor QED.

Presented method of solution of this problem is general enough to be applicable to other more general and realistic QFT, provided they can be formulated within the causal perturbative approach, which is for example the case for the Standard Model with the Higgs field \([10], [11]\).

Moreover the interacting fields are given through Fock expansions

$$\sum_{l,m} \Xi(\kappa_{l,m})$$

into integral kernel operators in the sense of \([38]\) which can be subject to a precise and computable convergence criteria, which utilize the symbol calculus of Obata, compare \([38], [37], [39]\). This allows us to verify the convergence of the perturbative series with the tools which were beyond our reach before.

3.1 Example 1: kernels $\kappa_{l,m}$ corresponding to $A_{\text{int}}^{\mu(1)}(g = 1, x)$

Here we give explicit formula for the (finite set of) kernels $\kappa'_{l,m}$ for which

$$A_{\text{int},\mu}^{(1)}(g = 1) = \sum_{l,m} \Xi(\kappa'_{l,m}),$$

i. e. which define (finite set of) integral kernel operators, (finite) sum of which gives the first order contribution to the interacting electromagnetic potential field in the adiabatic limit $g = 1$. More explicitly (using the notation of Sub-
sections \(2.6\) and \(2.9\)

\[
A_{m\mu}^{(1)}(g = 1, x) =
\]

\[
= \sum_{s,s' = 1}^{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa'_{2,0}(p', s', p, s; \mu, x) \partial^*_{s', p} \partial^*_{s, p} d^3 p' d^3 p
\]

\[
+ \sum_{s,s' = 1}^{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa'_{1,1}(p', s', p, s; \mu, x) \partial^*_{s', p} \partial^*_{s, p} d^3 p' d^3 p
\]

\[
+ \sum_{s,s' = 1}^{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa'_{0,2}(p', s', p, s; \mu, x) \partial^*_{s', p} \partial^*_{s, p} d^3 p' d^3 p
\]

or otherwise (according to the notation for the Hida operators \(\partial_{s,p}, \partial_{\nu,p}\) i.e. the annihilation operators \(a_s(p), a_\mu(p)\) introduced in Subsection \(2.6\))

\[
A_{m\mu}^{(1)}(g = 1, x) =
\]

\[
= \sum_{s,s' = 1}^{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa'_{2,0}(p', s', p, s; \mu, x) a_{s'}(p')^+ a_s(p)^+ d^3 p' d^3 p
\]

\[
+ \sum_{s,s' = 1}^{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa'_{1,1}(p', s', p, s; \mu, x) a_{s'}(p')^+ a_s(p) d^3 p' d^3 p
\]

\[
+ \sum_{s,s' = 1}^{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa'_{0,2}(p', s', p, s; \mu, x) a_{s'}(p') a_s(p) d^3 p' d^3 p
\]

or using still another notation for the annihilation and creation operators (used e.g. in \([10]\), compare notation for \(a_s(p), a_\mu(p)\) introduced in Subsection \(2.6\))

\[
A_{m\mu}^{(1)}(g = 1, x) =
\]

\[
= \sum_{s,s' = 1}^{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa'^{++}_{2,0}(p', s', p, s; \mu, x) b_{s'}(p')^+ b_s(p)^+ d^3 p' d^3 p
\]

\[
+ \sum_{s,s' = 1}^{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa'^{+-}_{1,1}(p', s', p, s; \mu, x) b_{s'}(p')^+ b_s(p) d^3 p' d^3 p
\]

\[
+ \sum_{s,s' = 1}^{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa'^{--}_{1,1}(p', s', p, s; \mu, x) d_{s'}(p')^+ d_s(p) d^3 p' d^3 p
\]

\[
+ \sum_{s,s' = 1}^{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa'^{--}_{0,2}(p', s', p, s; \mu, x) d_{s'}(p') b_s(p) d^3 p' d^3 p
\]
where we have put

\[
\kappa'_{2,0}(p', s', p, s; \mu, x) = \begin{cases} 
\kappa'_{2,0}^{++}(p', s', p, s - 2; \mu, x) & s' = 1, 2, s = 3, 4 \\
0 & \text{otherwise}
\end{cases},
\]

\[
\kappa'_{1,1}(p', s', p, s; \mu, x) = \begin{cases} 
\kappa'_{1,1}^{+-}(p', s', p, s; \mu, x) & s' = 1, 2, s = 1, 2 \\
\kappa'_{1,1}^{-+}(p', s' - 2, p, s - 2; \mu, x) & s' = 3, 4, s = 3, 4 \\
0 & \text{otherwise}
\end{cases},
\]

\[
\kappa'_{0,2}(p', s', p, s; \mu, x) = \begin{cases} 
\kappa'_{0,2}^{-+}(p', s' - 2, p, s; \mu, x) & s' = 3, 4, s = 1, 2 \\
0 & \text{otherwise}
\end{cases}.
\]

Let us assume the standard plane wave distribution kernels, \(\kappa_{0,1}\) and \(\kappa_{1,0}\), namely \([99]\), \([104]\), Subsect. 2.8 and \([105]\), Subsection 2.10, which define, respectively, the free standard Dirac \([99]\) and standard electromagnetic potential \([125]\) fields as sums of two integral kernel operators with vector valued kernels \(\kappa_{0,1}\) and \(\kappa_{1,0}\).

Application of the Rules II, IV and VI immediately gives the following result

\[
\langle \kappa'_{2,0}(\zeta, \chi), \varphi \rangle \overset{df}{=} \begin{align*}
&= -e \sum_{s, s' \in \mathbb{R}^3 \times \mathbb{R}^3} \int d^3 p' d^3 p u_{s'}(p') v_{s}(p) \frac{\bar{\varphi}(p + p', E(p) + E'(p')) \zeta(s', p') \chi(s, p)}{|p + p'|^2 - (E(p) + E'(p'))^2} \\
&= -e \sum_{s, s' \in \mathbb{R}^3 \times \mathbb{R}^3} \int d^3 p' d^3 p u_{s'}(p') v_{s}(p) \frac{\bar{\varphi}(p - p', E(p') - E(p)) \zeta(s', p') \chi(s, p)}{|p - p'|^2 - (E'(p') - E(p))^2} \\
&= -e \sum_{s, s' \in \mathbb{R}^3 \times \mathbb{R}^3} \int d^3 p' d^3 p v_{s'}(p') u_{s}(p) \frac{\bar{\varphi}(- (p + p'), -(E(p) + E'(p'))) \zeta(s', p') \chi(s', p')}{|p + p'|^2 - (E(p) - E'(p'))^2}
\end{align*}
\]
with
\[ \zeta, \chi \in \mathcal{S}(\mathbb{R}^3; \mathbb{C}^2), \ \varphi \in \mathcal{S}(\mathbb{R}^4; \mathbb{C}), \ \tilde{\varphi} \in \mathcal{F}\mathcal{S}(\mathbb{R}^4; \mathbb{C}), \]
and with the convention that \( \mathcal{S}(\mathbb{R}^3; \mathbb{C}^2) \subset \mathcal{S}(\mathbb{R}^3; \mathbb{C}^4) = E_1 \) with the convention that only two components of \( \zeta \) or \( \chi \) are non zero when \( \xi, \chi \) are regarded as elements of \( E_1 \). Here
\[ E(p) = |p|, \ E(p') = |p'|. \]

It follows from the general Theorem 5 of Subsection 2.7 that
\[ \kappa'_{2,0}, \kappa'_{1,1}, \kappa'_{0,2} \in \mathcal{L}(E_1 \otimes E_2, \mathcal{E}_2^*), \]
so that (compare generalization of Thm 3.9 of [38], and Subsection 2.6)
\[ \Xi_{l,m}(\kappa'_{l,m}) \in \mathcal{L}((E) \otimes \mathcal{E}, (E)^*) \cong \mathcal{L}(\mathcal{E}, \mathcal{L}((E), (E)^*)) \]

But (127) can also be shown with the help of the explicit formulas for the kernels \( \kappa'_{l,m} \) by repeating the proof of Lemma 6, Subsection 2.7. Moreover we have the following

**Proposition.** 1) The bilinear map
\[ \xi \times \eta \mapsto \kappa'_{1,1}(\xi \otimes \eta), \ \xi, \eta \in E_1, \]
can be extended to a separately continuous bilinear map from
\[ E_1^* \times E_1 \text{ into } \mathcal{L}(\mathcal{E}, \mathcal{C}) = \mathcal{E}^*. \]

2) The bilinear map
\[ \xi \times \eta \mapsto \kappa'_{2,0}(\xi \otimes \eta), \ \xi, \eta \in E_1, \]
can be extended to a continuous bilinear map from
\[ E_1^* \times E_1^* \text{ into } \mathcal{L}(\mathcal{E}, \mathcal{C}) = \mathcal{E}^*. \]

Therefore
\[ \Xi_{l,m}(\kappa'_{l,m}) \in \mathcal{L}((E) \otimes \mathcal{E}, (E))^* \cong \mathcal{L}(\mathcal{E}, \mathcal{L}((E), (E)^*)) \]
and
\[ A^{(1)}_{\gamma \mu}(g = 1) = \sum_{l,m} \Xi(\kappa'_{l,m}) \in \mathcal{L}((E) \otimes \mathcal{E}, (E)) \cong \mathcal{L}(\mathcal{E}, \mathcal{L}((E), (E)^*)), \]
by Thm. 4, Subsection 2.6

The same holds for all other possible choices, (61), (62), Subsect. 2.6 and (119), Subsection 2.9, of the plane wave distribution kernels \( \kappa_{0,1}, \kappa_{1,0} \) defining the free fields \( \psi, A \) of the theory.
3.2 Example 2: kernels \( \kappa_{l,m} \) corresponding to \( \psi^{(1)}_{\text{int}}(g = 1, x) \)

Here we give explicit formula for the (finite set of) kernels \( \kappa'_{l,m} \) for which

\[
\psi^{\text{a}}_{\text{int}}(g = 1) = \sum_{l,m} \Xi(\kappa_{l,m}).
\]

i.e. which define (finite set of) integral kernel operators, (finite) sum of which gives the first order contribution to the interacting Dirac field in the adiabatic limit \( g = 1 \). More explicitly (using the notation of Subsections 2.6 and 2.9)

\[
\psi^{\text{a}}_{\text{int}}(g = 1) = \sum_{\nu' = 0}^{3} \sum_{s = 0}^{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa_{2,0}(p', \nu', s; a, x) \eta \tilde{\eta}_{s,p}^* \eta \partial_{s}^* d^3p' d^3p
\]

\[
\quad + \sum_{\nu' = 0}^{3} \sum_{s = 0}^{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa_{1,1}(p', \nu', s; a, x) \eta \tilde{\eta}_{s,p}^* \eta \partial_{s}^* d^3p' d^3p
\]

\[
\quad + \sum_{\nu' = 0}^{3} \sum_{s = 0}^{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa_{1,1}(p', s', s; a, x) \eta \tilde{\eta}_{s'}^* \eta \partial_{s'}^* d^3p' d^3p
\]

\[
\quad + \sum_{\nu' = 0}^{3} \sum_{s = 0}^{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa_{0,2}(p', \nu', s; a, x) \eta \tilde{\eta}_{s,p}^* \eta \partial_{s}^* d^3p' d^3p
\]

or otherwise (according to the notation for the Hida operators \( \partial_{s,p}, \partial_{\nu,p} \) i.e. the annihilation operators \( a_s(p), a_\mu(p) \) introduced in Subsection 2.6)

\[
\psi^{\text{a}}_{\text{int}}(g = 1) = \sum_{\nu' = 0}^{3} \sum_{s = 0}^{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa_{2,0}(p', \nu', s; a, x) \eta a_{\nu'}(p') + \eta a_s(p)^+ d^3p' d^3p
\]

\[
\quad + \sum_{\nu' = 0}^{3} \sum_{s = 0}^{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa_{1,1}(p', \nu', s; a, x) \eta a_{\nu'}(p') + \eta a_s(p)^+ d^3p' d^3p
\]

\[
\quad + \sum_{\nu' = 0}^{3} \sum_{s = 0}^{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa_{1,1}(p', s'; s; a, x) a_{s'}(p') + a_{\nu'}(p)^+ d^3p' d^3p
\]

\[
\quad + \sum_{\nu' = 0}^{3} \sum_{s = 0}^{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \kappa_{0,2}(p', \nu', s; a, x) a_{s'}(p') + a_{\nu'}(p)^+ a_s(p) d^3p' d^3p
\]
or using still another notation for the annihilation and creation operators (used e.g. in [46], compare Subsection 2.6)

\[ \psi_{\text{int}}^\alpha (g=1) = \]
\[ = \sum_{\nu'=0}^{3} \sum_{s=1}^{2} \int \kappa_{2,0}^{+\pm}(p', \nu', p; s; a, x) \eta a_{\nu'}(p')^+ \eta d_{s}(p')^+ \, d^3p' \, d^3p \]
\[ + \sum_{\nu'=0}^{3} \sum_{s'=1}^{2} \int \kappa_{1,1}^{+\mp}(p', \nu', p; s; a, x) \eta a_{\nu'}(p')^+ \eta b_{s}(p) \, d^3p' \, d^3p \]
\[ + \sum_{\nu'=0}^{3} \sum_{s'=1}^{2} \int \kappa_{1,1}^{-\pm}(p', s', p; \nu; a, x) \eta d_{s'}(p')^+ a_{\nu}(p) \, d^3p' \, d^3p \]
\[ + \sum_{\nu'=0}^{3} \sum_{s'=1}^{2} \int \kappa_{0,2}^{-\pm}(p', \nu', p; s; a, x) a_{\nu}(p')^+ b_{s}(p) \, d^3p' \, d^3p \]

where we have put

\[ \kappa_{2,0}(p', \nu', p; s; a, x) = \begin{cases} \kappa_{2,0}^{+\pm}(p', \nu', p; s-2; a, x) & s = 3, 4 \\ 0 & \text{otherwise} \end{cases} \]
\[ \kappa_{1,1}(p', \nu', p; s; a, x) = \begin{cases} \kappa_{1,1}^{+\mp}(p', \nu', p; s; a, x) & s = 1, 2 \\ 0 & \text{otherwise} \end{cases} \]
\[ \kappa_{1,1}(p', s', p; \nu; a, x) = \begin{cases} \kappa_{1,1}^{-\pm}(p', s'-2; p; \nu; a, x) & s' = 3, 4 \\ 0 & \text{otherwise} \end{cases} \]
\[ \kappa_{0,2}(p', \nu', p; s; a, x) = \begin{cases} \kappa_{0,2}^{-\mp}(p', \nu', p; s; a, x) & s = 1, 2 \\ 0 & \text{otherwise} \end{cases} \]

Let us assume the standard plane wave distribution kernels, \( \kappa_{0,1} \) and \( \kappa_{1,0} \), namely \([104], [105] \), Subsect. 2.3 and \([129] \), Subsection 2.10 which define, respectively, the free standard Dirac \([99] \) and standard electromagnetic potential \([125] \) fields as sums of two integral kernel operators with vector valued kernels \( \kappa_{0,1} \) and \( \kappa_{1,0} \).

Application of the Rules II, IV and VI immediately gives the following result

\[ \langle \kappa_{2,0}^{\mp}(\zeta, \chi), \phi \rangle \]
\[ \overset{\text{df}}{=} \sum_{\nu'=0}^{3} \sum_{s=1}^{2} \int \kappa_{2,0}^{+\mp}(p', \nu', p; s; a, x) \zeta(s', p') \chi(s, p) \phi(x) d^3p' d^3p d^4x \]
\[ = e \sum_{\nu'=0}^{3} \sum_{s=1}^{2} \int d^3p' d^3p d^4p s(p) \left( -(p' + p) \cdot \tilde{\gamma}_{ab} + (E'(p') + E(p))^0 + 1_{ab} m \right) \gamma_{\nu'} \times \]
\[ \frac{\tilde{\phi}(p + p', E(p) + E'(p')) \zeta(\nu', p') \chi(s, p)}{2|p'|^2(p'|E| - \langle p'|p \rangle)} \]

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\[
\langle \kappa_1^+ (\zeta, \chi), \phi \rangle = \\
e \sum_{\nu'=0}^3 \sum_{s=1}^2 \int d^3 p' d^3 p u^c_s(p)(-\langle p' - p \rangle \cdot \vec{\gamma}_{ab} + (E'(p') - E(p)) \gamma^0 + 1_{ab}m) \gamma_{bc} \times \\
\times \frac{\tilde{\phi}(\langle p' - p \rangle \cdot (E'(p') - E(p)) \gamma^0 + 1_{ab}m) \gamma_{bc}}{2|p'|(|p'|^2 - |p'|^4E(p))}
\]

\[
\langle \kappa_1^- (\zeta, \chi), \phi \rangle = \\
e \sum_{\nu'=0}^3 \sum_{s=1}^2 \int d^3 p' d^3 p u^c_s(p)(\langle p' - p \rangle \cdot \vec{\gamma}_{ab} + (E'(p') - E(p)) \gamma^0 + 1_{ab}m) \gamma_{bc} \times \\
\times \frac{\tilde{\phi}(\langle p' - p \rangle \cdot (E'(p') - E(p)) \gamma^0 + 1_{ab}m) \gamma_{bc}}{2|p'|(|p'|^2 - |p'|^4E(p))}
\]

\[
\langle \kappa_2^- (\zeta, \chi), \phi \rangle = \\
e \sum_{\nu'=0}^3 \sum_{s=1}^2 \int d^3 p' d^3 p u^c_s(p)(\langle p' - p \rangle \cdot \vec{\gamma}_{ab} - (E'(p') + E(p)) \gamma^0 + 1_{ab}m) \gamma_{bc} \times \\
\times \frac{\tilde{\phi}(\langle p' - p \rangle \cdot \gamma^0 + 1_{ab}m) \gamma_{bc}}{2|p'|(|p'|^2 - |p'|^4E(p) - \langle p'|^2\rangle)}
\]

with summation over repeated spinor indices \( b, c \{1, 2, 3, 4 \} \) and with

\( \zeta \in S^0(\mathbb{R}^3; \mathbb{C}^4) = E_2, \ \chi \in S(\mathbb{R}^3; \mathbb{C}^2), \ \phi \in \mathcal{D}_1 = S(\mathbb{R}^4; \mathbb{C}), \)

and with the convention that \( S(\mathbb{R}^3; \mathbb{C}^2) \subset S(\mathbb{R}^3; \mathbb{C}^4) = E_1 \) with the convention that only two components of \( \chi \) are non-zero when \( \chi \) is regarded as an element of \( E_1. \)

It follows from the general Theorem 5 of Subsection 2.7 that

\[
\kappa_{2,0}, \kappa_{1,1}, \kappa_{0,2} \in \mathcal{L}(E_1 \otimes E_2, \mathcal{D}_1^*),
\]

so that (compare generalization of Thm 3.9 of \( \text{[38]}, \) and Subsection 2.6)

\[
\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(E \otimes \mathcal{D}, (E)^*) \cong \mathcal{L}(\mathcal{D}, \mathcal{L}(E), (E)^*)
\]

But \( \text{[128]} \) can also be shown with the help of the explicit formulas for the kernels \( \kappa_{l,m} \) by repeating the proof of Lemma \( \text{[38]} \) Subsection 2.7.
Thus the first order contribution to the interacting Dirac field is equal to a finite sum

$$\psi^{\text{int}}_{\text{free}}(g = 1) = \sum_{l,m} \Xi(\kappa_{l,m}) \in \mathcal{L}((E) \otimes \delta, (E)^*) \cong \mathcal{L}\left(\delta, \mathcal{L}(\delta, (E), (E)^*)\right)$$

of well defined integral kernel operators $\Xi(\kappa_{l,m})$ with vector-valued distributional kernels in the sense of Obata, compare [38] or Subsections 2.6 and 2.7.

However

$$\psi^{\text{int}}_{\text{free}}(g = 1) = \sum_{l,m} \Xi(\kappa_{l,m}) \notin \mathcal{L}((E) \otimes \delta, (E)) \cong \mathcal{L}\left(\delta, \mathcal{L}(\delta, (E), (E))\right)$$

similarly as for Wick products of free mass less fields (such as $A_{\mu}(x)$) at the same space-time point $x$ which do belong to

$$\mathcal{L}((E) \otimes \delta, (E)^*) \cong \mathcal{L}\left(\delta, \mathcal{L}(\delta, (E), (E)^*)\right),$$

but do not belong to

$$\mathcal{L}((E) \otimes \delta, (E)) \cong \mathcal{L}\left(\delta, \mathcal{L}(\delta, (E), (E))\right).$$

The same holds for all other possible choices, (61), (62), Subsect. 2.6 and (119), Subsection 2.9, of the plane wave distribution kernels $\kappa_{0,1}, \kappa_{1,0}$ defining the free fields $\psi, A$ of the theory.

### 4 APPENDIX: Fourier transforms $u_s(p)$ and $v_s(-p)$ of a complete system of distributional solutions of the homogeneous Dirac equation

As we have seen in Subsection 2.1 of [59] the Hilbert spaces $\mathcal{H}^{\oplus}_{m,0}$ and $\mathcal{H}^{\ominus}_{-m,0}$ of Fourier transforms of bispinor solutions of the Dirac equation, concentrated respectively on the orbit $\mathcal{O}_{m,0,0,0}$ and $\mathcal{O}_{-m,0,0,0}$, are equal to the images of the corresponding projection operators $P^{\oplus}$ and $P^{\ominus}$ – the multiplication operators by the corresponding orthogonal projections $P^{\oplus}(p), p \in \mathcal{O}_{m,0,0,0}$ and $P^{\ominus}(p), p \in \mathcal{O}_{-m,0,0,0}$ – compare Subsection 2.1 of [59]. Recall that

$$\text{rank} P^{\oplus}(p) = 2, p \in \mathcal{O}_{m,0,0,0}, \quad \text{rank} P^{\ominus}(p) = 2, p \in \mathcal{O}_{-m,0,0,0}.$$ 

It is therefore possible to choose at each point $p = (p, p_0(p)) = (p, E(p) = \sqrt{|p|^2 + m^2})$ of the orbit $\mathcal{O}_{m,0,0,0}$ (specified uniquely by $p \in \mathbb{R}^3$) a pair of vectors $u_s(p)$, $s = 1, 2$, which span the image $\text{Im} P^{\oplus}(p, p_0(p)) = \text{Im} P^{\oplus}(p, E(p))$ of $P^{\oplus}(p) = P^{\oplus}(p, p_0(p))$. Similarly for each point $p = (p, p_0(p)) = (p, -E(p) = -\sqrt{|p|^2 + m^2})$ of the orbit $\mathcal{O}_{-m,0,0,0}$ (specified by $p \in \mathbb{R}^3$) we can find a pair of two vectors $v_s(p)$, $s = 1, 2$, which span the image $\text{Im} P^{\ominus}(p, p_0(p)) = \text{Im} P^{\ominus}(p, -E(p)), E(p) = \sqrt{|p|^2 + m^2}$ for $p = (p, -E(p)) = (p, -\sqrt{|p|^2 + m^2}) \in \mathbb{R}^3$. 

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We choose these vectors in such a manner that their components depend smoothly on \( p \) and are multipliers and even convolutors of the Schwartz nuclear algebra \( S(\mathbb{R}^3; \mathbb{C}) \). Moreover we choose them in such a manner that \( p \mapsto u_s(p) \) and \( p \mapsto v_s(-p) \) represent Fourier transforms of certain solutions of the free Dirac equation concentrated respectively on the orbit \( O_{m,0,0,0} \) and \( O_{-m,0,0,0} \). That \( p \mapsto v_s(-p) \), \( s = 1, 2 \), represent the Fourier transforms of solutions of the Dirac equation and not simply \( p \mapsto v_s(p) \), \( s = 1, 2 \), is a matter of tradition and does not have any deeper justification. Of course there is a whole infinity of different choices for \( u_s(p) \) and \( v_s(p) \), giving unitary equivalent constructions of the Dirac field.

In this Appendix we construct one useful example of \( u_s(p) \) and \( v_s(p) \), \( s = 1, 2 \) for the chiral representation of the Clifford algebra generators (Dirac matrices)

\[
\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix},
\]

(129)

which we have used in Subsection 2.1 of [59] as well as for the so called standard representation

\[
\gamma^0 = C \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} C^{-1} = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix},
\gamma^k = C \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix} C^{-1} = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix},
\]

(130)

of the Dirac matrices, where

\[
C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_2 & 1_2 \\ 1_2 & -1_2 \end{pmatrix} = C^+ = C^{-1}
\]

is unitary involutive \( 4 \times 4 \) matrix.

THE SOLUTIONS \( u_s(p) \) AND \( v_s(p) \) IN THE CHIRAL REPRESENTATION

Let us start with the chiral representation (used in Subsection 2.1 of [59]). Recall that

\[
P^\pm(p) = \frac{1}{2} \begin{pmatrix} 1 & \beta(p)^{-2} \\ \beta(p)^2 & 1 \end{pmatrix}, \quad p \in \mathcal{O}_{m,0,0,0}
\]

with \( \beta(p) \) (chosen correspondingly to the chiral representation, as there is infinitum of other possible choices of \( \beta(p) \), compare Subsect. 2.1 of [59]) corresponding to the orbit \( \mathcal{O}_{m,0,0,0} \), i.e.

\[
\beta(p)^{-2} = \frac{1}{m} (p^0 \mathbf{1} + \vec{p} \cdot \vec{\sigma}), \quad p^0(p) = \sqrt{\vec{p} \cdot \vec{p} + m^2} = E(p),
\beta(p)^2 = \frac{1}{m} (p^0 \mathbf{1} - \vec{p} \cdot \vec{\sigma}), \quad p^0(p) = \sqrt{\vec{p} \cdot \vec{p} + m^2} = E(p).
\]

(131)

Similarly recall that here

\[
P^\pm(p) = \frac{1}{2} \begin{pmatrix} 1 & -\beta(p)^{-2} \\ -\beta(p)^2 & 1 \end{pmatrix}, \quad p \in \mathcal{O}_{-m,0,0,0}
\]

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with \( \beta(p) \) corresponding to the orbit \( \mathcal{O}_{-m,0,0,0} \), i.e.

\[
\beta(p)^{-2} = \frac{1}{m} (-p^0 \mathbf{1} - \mathbf{p} \cdot \mathbf{\sigma}), \quad p^0(\mathbf{p}) = -\sqrt{\mathbf{p} \cdot \mathbf{p} + m^2} = -E(\mathbf{p}),
\]

\[
\beta(p)^2 = \frac{1}{m} (-p^0 \mathbf{1} + \mathbf{p} \cdot \mathbf{\sigma}), \quad p^0(\mathbf{p}) = -\sqrt{\mathbf{p} \cdot \mathbf{p} + m^2} = -E(\mathbf{p}),
\]

(132)

compare Subsection 2.1 of [59]. In this case (of chiral representation [129]) one can put

\[
u_s(p) = \frac{1}{\sqrt{2}} \sqrt{\frac{E(p) + m}{2E(p)}} \begin{pmatrix} \chi_s + \frac{p \cdot \sigma}{E(p) + m} \chi_s \\ \chi_s - \frac{p \cdot \sigma}{E(p) + m} \chi_s \end{pmatrix},
\]

\[
\nu_s(p) = \frac{1}{\sqrt{2}} \sqrt{\frac{E(p) + m}{2E(p)}} \begin{pmatrix} \chi_s + \frac{p \cdot \sigma}{E(p) + m} \chi_s \\ -\left(\chi_s - \frac{p \cdot \sigma}{E(p) + m} \chi_s \right) \end{pmatrix},
\]

(133)

where

\[
\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Here \( \beta(p) \) in the formula for \( u_s(p) \) is that (131) corresponding to the orbit \( \mathcal{O}_{m,0,0,0} \) and in the formula for \( \nu_s(p) \) the matrix function \( \beta(p) \) equals (132) correspondingly to the orbit \( \mathcal{O}_{-m,0,0,0} \), so that by construction the solutions \( u_s(p), \nu_s(-p) \) have the general form (with the respective \( \beta(p) \) corresponding to the respective orbit \( \mathcal{O}_{\pm m,0,0,0} \))

\[
u_s(p) \overset{\text{df}}{=} u_s(p_0(p), \mathbf{p}) = \begin{pmatrix} \bar{\varphi}_{s+}(p) \\ \beta(p)^2 \bar{\varphi}_{s+}(p) \end{pmatrix}, \quad p = (p_0(p), \mathbf{p}) \in \mathcal{O}_{m,0,0,0},
\]

\[
u_s(-p) \overset{\text{df}}{=} u_s(p_0(p), -\mathbf{p}) = \begin{pmatrix} \bar{\varphi}_{s-}(-p) \\ -\beta(p)^2 \bar{\varphi}_{s-}(-p) \end{pmatrix}, \quad p = (p_0(p), \mathbf{p}) \in \mathcal{O}_{-m,0,0,0},
\]

with

\[
\bar{\varphi}_{s+}(p = (p_0(p), \mathbf{p})) = \chi_s + \frac{p \cdot \sigma}{E(p) + m} \chi_s, \quad p = (p_0(p), \mathbf{p}) \in \mathcal{O}_{m,0,0,0},
\]

\[
\bar{\varphi}_{s-}(p = (p_0(p), \mathbf{p})) = \chi_s - \frac{p \cdot \sigma}{E(p) + m} \chi_s, \quad p = (p_0(p), \mathbf{p}) \in \mathcal{O}_{-m,0,0,0}.
\]

as expected by construction of \( H_{m,0}^\oplus \) and \( H_{-m,0}^\ominus \) in Subsection 2.1 of [59].

The vectors \( u_s(p) \) and \( \nu_s(p) \), \( s = 1,2 \), respect the following orthogonality relations:

\[
u_s(p)^* u_s'(p) = \delta_{s s'}, \quad \nu_s(p)^* v_s'(p) = \delta_{s s'}, \quad u_s(p)^* u_s'(-p) = 0.
\]

(134)
By construction we have

\[
E_+(p) = \sum_{s=1,2} u_s(p) u_s(p)^+ = \frac{1}{2E(p)} (E(p)1 + p \cdot \alpha + \beta m), \quad E(p) = \sqrt{|p|^2 + m^2}
\]

\[
E_-(p) = \sum_{s=1,2} v_s(p) v_s(p)^+ = \frac{1}{2E(p)} (E(p)1 + p \cdot \alpha - \beta m), \quad E(p) = \sqrt{|p|^2 + m^2}.
\]

(135)

Here

\[
\begin{align*}
\sigma &= (\sigma_1, \sigma_2, \sigma_3), \quad \alpha = (\alpha^1, \alpha^2, \alpha^3), \\
p \cdot \sigma &= \sum_{i=1}^3 p_i \sigma_i, \quad p \cdot \alpha = \sum_{i=1}^3 p_i \alpha^i, \\
\alpha^i &= \gamma^0 \gamma^i, \quad \beta = \gamma^0.
\end{align*}
\]

Note that \(E_+(p)\) and \(E_-(-p)\) are mutually orthogonal projectors on \(\mathbb{C}^4\) such that \(E_+(p) + E_-(-p) = 1\) and such that the operators \(E_+\) and \(E_-\) of Subsection 2.1 are equal to the operators of point-wise multiplications by the matrices \(E_\pm(\pm p)\) on the Hilbert spaces \(\mathcal{H}_{m,0}^\oplus\) and \(\mathcal{H}_{-m,0}^\ominus\) of bispinors concentrated respectively on \(\mathcal{O}_{m,0,0,0}\) and \(\mathcal{O}_{-m,0,0,0}\) (with the point \(p = (p_0(p), \mathbf{p})\) of the respective orbit identified with its cartesian coordinates \(\mathbf{p}\)).

Moreover, recall that for any element \(\tilde{\phi} \in \mathcal{H}_{m,0}^\oplus\) the following algebraic relation holds (summation with respect to \(i = 1, 2, 3\))

\[
p_0 \gamma^0 \tilde{\phi}(p) = [p_i \gamma^i + m1] \tilde{\phi}(p), \quad p \in \mathcal{O}_{m,0,0,0},
\]

compare Subsection 2.1 of [59], so that

\[
E(p) \tilde{\phi}(p) = [p \cdot \alpha + m \beta] \tilde{\phi}(p), \quad p = (p_0(p), \mathbf{p}) \in \mathcal{O}_{m,0,0,0},
\]

for all \(\tilde{\phi} \in \mathcal{H}_{m,0}^\oplus\) and thus

\[
E_+(p) \tilde{\phi}(p) = \left( \sum_{s=1,2} u_s(p) u_s(p)^+ \right) \tilde{\phi}(p)
\]

\[
= \frac{1}{2E(p)} (E(p)1 + p \cdot \alpha + \beta m) \tilde{\phi}(p) = \tilde{\phi}(p),
\]

\[p = (p_0(p), \mathbf{p}) \in \mathcal{O}_{m,0,0,0}, \quad \quad (136)\]

for each \(\tilde{\phi} \in \mathcal{H}_{m,0}^\oplus\).

Similarly for any element \(\tilde{\phi} \in \mathcal{H}_{-m,0}^\ominus\) the following algebraic relation holds (summation with respect to \(i = 1, 2, 3\))

\[
p_0 \gamma^0 \tilde{\phi}(p) = [p_i \gamma^i + m1] \tilde{\phi}(p), \quad p = (p_0(p), \mathbf{p}) = (-E(p), \mathbf{p}) \in \mathcal{O}_{-m,0,0,0},
\]

p = (p_0(p), \mathbf{p}) = (-E(p), \mathbf{p}) \in \mathcal{O}_{-m,0,0,0},

\]

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From the formulas (138) or (139) it follows in particular that
\[ -E(p)\tilde{\phi}(-E(p), p) = [p \cdot \alpha + m \beta] \tilde{\phi}(-E(p), p), \]
\[ p = (p_0(p), p) = (-E(p), p) \in \mathcal{O}_{-m,0,0,0}, \]
for all \( \tilde{\phi} \in \mathcal{H}^{\mathbb{C}}_{-m,0} \) and thus
\[ E(p)\tilde{\phi}(-E(p), -p) = (p \cdot \alpha - \beta m) \tilde{\phi}(-E(p), -p), \quad \tilde{\phi} \in \mathcal{H}^{\mathbb{C}}_{-m,0}. \]
Therefore we have
\[ E_- (p) \tilde{\phi}(-E(p), -p) = \left( \sum_{s=1,2} v_s(p)v_s(p)^+ \right) \tilde{\phi}(-E(p), -p) \]
\[ = \frac{1}{2E(p)} (E(p)1 + p \cdot \alpha - \beta m) \tilde{\phi}(-E(p), -p) = \tilde{\phi}(-E(p), -p), \]
\[ p = (p_0(p), p) \in \mathcal{O}_{-m,0,0,0}, \quad (137) \]
for each \( \tilde{\phi} \in \mathcal{H}^{\mathbb{C}}_{-m,0} \).

By construction we have
\[ P^\oplus (E(p), p) u_s(p) = u_s(p), \quad P^\ominus (-E(p), p) v_s(-p) = v_s(-p) \quad (138) \]
or
\[ P^\ominus (-E(p), -p) v_s(p) = v_s(p), \quad (139) \]
and
\[ P^\oplus (E(p), p) \tilde{\phi}((E(p), p) = \tilde{\phi}((E(p), p), \quad \tilde{\phi} \in \mathcal{H}^{\mathbb{C}}_{m,0}, \]
\[ P^\ominus (-E(p), p) \tilde{\phi}(-E(p), p) = \tilde{\phi}(-E(p), p), \quad \tilde{\phi} \in \mathcal{H}^{\mathbb{C}}_{-m,0}. \quad (140) \]

From the formulas (138) or (139) it follows in particular that
\[ u_s(p)^+ \tilde{\phi}(E(p), p) = \sum_{a=1}^4 u_s^2(p) \tilde{\phi}^a(E(p), p) = \left( u_s(p), \tilde{\phi}(E(p), p) \right)_{C^4} \]
\[ = \left( P^\oplus (E(p), p) u_s(p), \tilde{\phi}(E(p), p) \right)_{C^4} = \left( u_s(p), P^\oplus (E(p), p) \tilde{\phi}(E(p), p) \right)_{C^4} \]
\[ = u_s(p)^+ \left( P^\oplus (E(p), p) \tilde{\phi}(E(p), p) \right) = u_s(p)^+ \left( P^\oplus \tilde{\phi}(E(p), p) \right), \]
for any smooth \( \tilde{\phi} \) \quad (141)
\[ v_s(p)^+ \tilde{\phi}(-E(p), -p) = \sum_{a=1}^{4} v_s^a(p) \phi_a(-E(p), -p) \]
\[ \left( v_s(p), \tilde{\phi}(-E(p), -p) \right)_{C^4} = \left( P^\square(-E(p), -p)v_s(p), \tilde{\phi}(-E(p), -p) \right)_{C^4} \]
\[ = v_s(p)^+ (P^\square(-E(p), -p)\tilde{\phi}(-E(p), -p)) = v_s(p)^+ (P^\square\tilde{\phi})(-E(p), -p), \]
for any smooth \( \tilde{\phi} \). (142)

It should be stressed that the formulas (141) and (142) are valid for any \( \tilde{\phi} \) not necessary belonging to \( \mathcal{H}^\oplus_{m,0} \) or \( \mathcal{H}^\ominus_{-m,0} \).

It is obvious that the projectors \( P^\square(p) \), \( p \in \mathcal{O}_{m,0,0,0} \) and \( P^\square(p) \), \( p \in \mathcal{O}_{-m,0,0,0} \) can be expressed in the following manifestly covariant form
\[ P^\square(p) = \frac{1}{2m} [g_{\nu\rho} \gamma^\nu \gamma^\rho + m1_s] = \frac{1}{2m} [\phi + m], \ p \in \mathcal{O}_{-m,0,0,0}. \]
\[ P^\square(p) = \frac{1}{2m} [g_{\nu\rho} \gamma^\nu \gamma^\rho + m1_s] = \frac{1}{2m} [\phi + m], \ p \in \mathcal{O}_{-m,0,0,0}. \] (143)

Finally let us give the formulas useful in computation of the commutation functions and pairing functions for the Dirac field and its Dirac adjoined field. To this end let us recall that for a bispinor \( u(p) \) the Dirac adjoint \( \overline{u}(p) \) is defined to be equal \( u(p)^+ \gamma^0 \). This (common) notation is somewhat unfortunate, because the Dirac adjoint may be mislead with the ordinary complex conjugation, which we have already agreed to be denoted by overset bar (which also is a traditional notation for complex conjugation). It must be explicitly stated what is meant in each case in working with bispinors. When working with quantum Dirac field \( \psi(x) \) the overset bar \( \overline{\psi}(x) \) will always mean the Dirac adjoint. Denoting here \( \overline{u}_s(p), \overline{v}_s(-p) \) the Dirac adjoints of the complete system of solutions \( u_s(p), v_s(-p) \), we get (summation with respect to \( i = 1, 2, 3 \))
\[ \sum_{s=1,2} u_s(p)\overline{u}_s(p) = \frac{1}{2E(p)} (E(p)1 - p_i \gamma^i + 1m), \ E(p) = \sqrt{|p|^2 + m^2} \]
\[ \sum_{s=1,2} v_s(p)\overline{v}_s(p) = \frac{1}{2E(p)} (E(p)\gamma^0 - p_i \gamma^i - 1m), \ E(p) = \sqrt{|p|^2 + m^2}, \]
on multiplying the formulas (136) for \( E_\pm(p) \) by \( \gamma^0 \) on the right, and which is frequently written as
\[ \sum_{s=1,2} u_s(p)\overline{u}_s(p) = \frac{\phi + m}{2E(p)} = \frac{p_\mu \gamma^\mu + m}{2E(p)}, \ E(p) = \sqrt{|p|^2 + m^2} \]
\[ \sum_{s=1,2} v_s(p)\overline{v}_s(p) = \frac{\phi - m}{2E(p)} = \frac{p_\mu \gamma^\mu - m}{2E(p)}, \ E(p) = \sqrt{|p|^2 + m^2}. \] (144)
THE SOLUTIONS \(u_s(p)\) AND \(v_s(p)\) IN THE STANDARD REPRESENTATION

Now let us give the formulas for the fundamental solutions \(u_s(p), v_s(-p)\), \(s = 1, 2\), and projections \(P^\oplus, P^\ominus E_+, E_-\), in the so called standard representation \([130]\) of the Dirac gamma matrices. It is not necessary to start the whole analysis with unitary Mackey's induced representations using the other choice of the functions \(\beta(p)\) corresponding to the orbits \(\mathcal{O}_{m,0,0,0}\) and \(\mathcal{O}_{-m,0,0,0}\), which determines the Hilbert spaces of solutions of the Dirac equation with the standard Dirac matrices \([130]\). Indeed in order to determine the corresponding solutions \(u_s(p), v_s(-p)\) it is sufficient to apply the unitary operator of multiplication by \(C\)

\[
\begin{align*}
    u_s(p) &= C \frac{1}{\sqrt{2}} \sqrt{\frac{E(p) + m}{2E(p)}} \left( \chi_s + \frac{p \cdot \sigma}{E(p) + m} \chi_s \right) \\
    &= \sqrt{\frac{E(p) + m}{2E(p)}} \left( \frac{\chi_s}{\bar{E}(p) + m} \right), \\
    v_s(p) &= C \frac{1}{\sqrt{2}} \sqrt{\frac{E(p) + m}{2E(p)}} \left( \chi_s - \frac{p \cdot \sigma}{E(p) + m} \chi_s \right) \\
    &= \sqrt{\frac{E(p) + m}{2E(p)}} \left( \frac{\chi_s}{\bar{E}(p) + m} \right)
\end{align*}
\]

(145)

to the complete system of solutions in the chiral representation. For the corresponding projectors in the standard representation \([130]\) we thus have

\[
\begin{align*}
    P^\oplus(p) &= C^{\frac{1}{2}} \left( \begin{array}{c} \frac{1}{\beta(p)^2} \\ \frac{\beta(p)^{-2}}{12} \end{array} \right) C \\
    &= \frac{1}{2} \left( \begin{array}{cc} \frac{m + E(p)}{m} & \frac{p \cdot \sigma}{m - E(p)} \\ \frac{p \cdot \sigma}{m} & \frac{m - E(p)}{m} \end{array} \right), \quad p = (E(p), p) \in \mathcal{O}_{m,0,0,0},
\end{align*}
\]

(here with \(\beta(p)\) equal \([131]\)) and similarly for \(P^\ominus(-E(p), p)\) (with \(\beta(p)\) equal \([132]\) in the formula below)

\[
\begin{align*}
    P^\ominus(p) &= C^{-\frac{1}{2}} \left( \begin{array}{c} \frac{1}{\beta(p)^2} \\ \frac{-\beta(p)^{-2}}{12} \end{array} \right) C \\
    &= \frac{1}{2} \left( \begin{array}{cc} \frac{m - E(p)}{m} & \frac{-p \cdot \sigma}{m + E(p)} \\ \frac{-p \cdot \sigma}{m} & \frac{m + E(p)}{m} \end{array} \right), \quad p = (-E(p), p) \in \mathcal{O}_{-m,0,0,0}.
\end{align*}
\]

Of course we have the analogous formulas for \(E_{\pm}(p)\) but we have to remember that with the corresponding matrices \(\alpha^i = \gamma^0 \gamma^i\) in the standard representation \([130]\). By construction the (Fourier transforms) \(u_s(p), v_s(-p)\) of solutions in the standard representation \([130]\) respect the analogous relations \([134]-[144]\).
ON THE UNITARY ISOMORPHISM $U$ OF SUBSECTION 2.6 FOR THE DIRAC FIELD

Note that the unitary isomorphism operator $U$, defined by (37) in Subsection 2.6, can be regarded as the operator of pointwise multiplication by the matrix

$$U(p) = \frac{1}{2|p_0(p)|} \begin{pmatrix} u_1(p) & u_2'(p) & 0 & 0 \\
v_1(p) & u_2(p) & u_2'(p) & 0 \\
v_2(p) & u_2(p) & u_2'(p) & 0 \\
u_2(p) & 0 & 0 & 0 \end{pmatrix}$$

acting on the element $\tilde{\phi} \oplus (\tilde{\phi}')^{c} \in \mathcal{H}_{m,0}^{\oplus} \oplus \mathcal{H}_{m,0}^{\oplus} C$, where the value $(\tilde{\phi} \oplus (\tilde{\phi}')^{c})(|p_0(p)|, p)$ at $p = ([|p_0(p)|, p]) \in \mathcal{O}_{m,0,0,0}$ of $\tilde{\phi} \oplus (\tilde{\phi}')^{c}$ is written as a column vector

$$\begin{pmatrix} \tilde{\phi}(|p_0(p)|, p) \\
(\tilde{\phi}')^{c}(|p_0(p)|, p) \end{pmatrix}^{T}.$$ 

Similarly the inverse $U^{-1}$ of the isomorphism (37), Subsection 2.6, can be regarded as the operator of pointwise multiplication by the matrix

$$U^{-1}(p) = 2|p_0(p)| \begin{pmatrix} u_1(p) & u_2'(p) & 0 & 0 \\
v_1(p) & u_2(p) & u_2'(p) & 0 \\
v_2(p) & u_2(p) & u_2'(p) & 0 \\
u_2(p) & 0 & 0 & 0 \end{pmatrix}$$

with the value $(\tilde{\phi}_1 \oplus (\tilde{\phi})_2 \oplus (\tilde{\phi})_3 \oplus (\tilde{\phi})_4)(p)$ of the element

$$(\tilde{\phi}_1 \oplus (\tilde{\phi})_2 \oplus (\tilde{\phi})_3 \oplus (\tilde{\phi})_4) \in \mathcal{O}_{m,0} L^2(\mathbb{R}^3; \mathbb{C}) = L^2(\mathbb{R}^3; \mathbb{C}^4)$$

regarded as a column

$$\begin{pmatrix} (\tilde{\phi}_1)(p) \\
(\tilde{\phi}_2)(p) \\
(\tilde{\phi}_3)(p) \\
(\tilde{\phi}_4)(p) \end{pmatrix}.$$ 

Note that

$$U(p)U^{-1}(p) = 1_4, \quad U^{-1}(p)U(p) = \begin{pmatrix} E_+(p) & 0_4 \\
0 & E_-(p)^T \end{pmatrix}.$$ 

Note also that

$$\begin{pmatrix} E_+(p) & 0 \\
0 & E_-(p)^T \end{pmatrix} \begin{pmatrix} \tilde{\phi}(|p_0(p)|, p) \\
(\tilde{\phi}')^{c}(|p_0(p)|, p) \end{pmatrix} = \begin{pmatrix} \tilde{\phi}(|p_0(p)|, p) \\
(\tilde{\phi}')^{c}(|p_0(p)|, p) \end{pmatrix}^{T}$$

for $\tilde{\phi} \oplus (\tilde{\phi}')^{c} \in \mathcal{H} = \mathcal{H}_{m,0}^{\oplus} \oplus \mathcal{H}_{m,0}^{\oplus} C'$, which follows from (13) and (37).
5 APPENDIX: Schwartz’ spaces of convolutors $O'_C$ and multipliers $O_M$ of $S$

Schwartz [47] introduced the following linear function spaces (in this Appendix we use notation of Schwartz including his notation $E$ for $C^\infty(R^n;\mathbb{C})$ and its strong dual space $E'$ of distributions with compact support, which should not be mislead with our notation $E$ for a class of countably-Hilbert nuclear space-time test spaces $S(R^1;\mathbb{C}^m)$ or $S^{00}(R^1;\mathbb{C}^m))$

$D = \{ \varphi \in C^\infty(R^n;\mathbb{C}), \text{supp}\varphi \text{ compact}\}$,

$S = S_{H(0)}(R^n;\mathbb{C}) = S(R^n;\mathbb{C}) = \{ \varphi \in C^\infty(R^n;\mathbb{C}), \forall \alpha, \beta \in N_0^n : x^\alpha \partial^\beta \varphi \in C_0 \}$,

$D_{L_p} = \{ \varphi \in C^\infty(R^n;\mathbb{C}), \forall \alpha \in N_0^n : \partial^\alpha \varphi \in L^p \}$ (Sobolev space $W^{\infty,p}$)

$B = D_{L^\infty} = \{ \varphi \in C^\infty(R^n;\mathbb{C}), \forall \alpha \in N_0^n : \partial^\alpha \varphi \in L^\infty \}$,

$\mathcal{B} = \{ \varphi \in C^\infty(R^n;\mathbb{C}), \forall \alpha \in N_0^n : \partial^\alpha \varphi \in C_0 \}$,

$O_C = \{ \varphi \in C^\infty(R^n;\mathbb{C}), \exists k \in N_0 \forall \alpha \in N_0^n : (1 + |x|^2)^{-k} \partial^\alpha \varphi \in C_0 \}$ (very slowly increasing functions),

$O_M = \{ \varphi \in C^\infty(R^n;\mathbb{C}), \exists k \in N_0 \forall \alpha \in N_0^n : (1 + |x|^2)^{-k} \partial^\alpha \varphi \in C_0 \}$ (slowly increasing functions),

$E = C^\infty(R^n;\mathbb{C})$;

and their strong duals, which we will denote in this Appendix (after Schwartz [47]) with the prime sign ‘($\cdot$)’

$D'$ (distributions),

$S'$ (tempered distributions, denoted by us $S(R^n;\mathbb{C})^*$),

$D'_{L_p} = \{ T \in D', \exists m \in N_0 : T = \sum_{|\alpha| \leq m} \partial^\alpha f_\alpha \text{ with } f_\alpha \in L^p \}$,

$O'_C = \{ T \in D', \forall k \in N_0 \exists m \in N_0^n : (1 + |x|^2)^k T = \sum_{|\alpha| \leq m} \partial^\alpha f_\alpha \text{ with } f_\alpha \in L^\infty \}$ (rapidly decreasing distributions),

$O'_M = \{ T \in D', \exists m \in N_0^n \forall k \in N_0 : (1 + |x|^2)^k T = \sum_{|\alpha| \leq m} \partial^\alpha f_\alpha \text{ with } f_\alpha \in L^\infty \}$ (very rapidly decreasing distributions),

$E'$ (distributions with compact support).

Here $C_0$ is the space of continous $\mathbb{C}$-valued functions on $R^n$, tending to zero at infinity.

All these linear topological spaces together with the topology were constructed in [47], except the space $O_C$ – the predual of the Schwartz convolutor algebra $O'_C$ of rapidly decreasing distributions. The function space $O_C$
together with its inductive limit topology such that $O_C'$, with the Schwartz operator topology of uniform convergence on bounded sets, becomes the strong dual of $O_C$, has been determined by Horváth. Namely $O_C' = \{ T \in S' : T \text{ extends uniquely to a continuous linear functional } \tilde{T} \text{ on } O_C \}$, with the operator Schwartz topology of uniform convergence on bounded sets, becoming the strong dual of $O_C$.

We have the following topological inclusions (with $E \subset F$ meaning that the topology of $E$ is finer than that of $F$):

$$1 \leq p \leq q$$

$\mathcal{D} \subset S \subset \mathcal{D}_{L^p} \subset \mathcal{D}_{L^q} \subset \mathcal{B} \subset B \subset O_M \subset \mathcal{E}$,

$\mathcal{E}' \subset O_C' \subset \mathcal{D}_{L^p}' \subset \mathcal{D}_{L^q}' \subset \mathcal{B}' \subset B' \subset S' \subset \mathcal{D}'$,

$\mathcal{O}_C \subset O_C'$,

$\mathcal{D}, \mathcal{S}, \mathcal{D}_{L^p}, \mathcal{E}', \mathcal{D}_{L^p}', \mathcal{O}_M', \mathcal{O}_C'$,

Therefore elements of all indicated spaces (except the whole of $\mathcal{E} = \mathcal{E}^\infty$ and $\mathcal{D}'$)

$\mathcal{D}, \mathcal{S}, \mathcal{D}_{L^p}, \mathcal{E}', \mathcal{D}_{L^p}', \mathcal{O}_M', \mathcal{O}_C'$,

can be naturally regarded as tempered distributions, i.e. as elements of $S'$. But we should emphasize that the topology of each individual space is strictly stronger than the topology induced from the topology of the strong dual space $S'$ of tempered distributions.

Let us recall that the Fourier transform $\mathcal{F}$ maps isomorphically $\mathcal{S}$ onto $\mathcal{S}$. The Fourier transform is defined on the space of tempered distributions $S'$ through the linear transpose (dual) of the Fourier transform on $\mathcal{S}$, which by the general properties of the linear transpose [58] defines a continuous linear isomorphism $S' \rightarrow S'$ for the strong dual topology on $S'$, and denoted by the same symbol $\mathcal{F}$.

Because the elements of the linear spaces

$\mathcal{D}, \mathcal{S}, \mathcal{D}_{L^p}, \mathcal{E}', \mathcal{D}_{L^p}', \mathcal{O}_M', \mathcal{O}_C'$,

are naturally identified with elements of $S'$ then in particular the Fourier transform is a well defined linear map on these spaces (although in general it leads us out of the particular space in question).

Recall further that the operator $M_S$ of multiplication by any element $S$ of $O_M$ maps isomorphically $S \rightarrow S$. Thus elements $S$ of $O_M$ are naturally identified
with continuous multiplication operators \( M_S \) mapping continuously \( S \) into \( S \), i.e. with elements of \( \mathcal{L}(S,S) \). Therefore we can introduce \( \mathcal{O}_M \) after Schwartz [47] the topology of uniform convergence on bounded sets induced from \( \mathcal{L}(S,S) \).

Further recall that translation
\[
T_b : \varphi \rightarrow T_b \varphi, \quad T_b \varphi(x) \overset{df}{=} \varphi(x-b)
\]
maps isomorphically \( S \rightarrow S \). Again by duality we define
\[
S * \varphi(x) \overset{df}{=} \langle S, T_x \varphi \rangle = S(T_x \varphi),
\]
where \( \langle \cdot, \cdot \rangle \) stands for the canonical bilinear form on \( S' \times S = S^* \times S \), i.e. the pairing defined by taking the value of the functional. It turns out that if \( S \in S' \) then the operator
\[
C_S : \varphi \mapsto S * \varphi = C_S(\varphi)
\]
of convolution with \( S \in S' \) corresponding to \( S \) maps continuously \( S \rightarrow \mathcal{O}_C \), i.e. \( C_S \in \mathcal{L}(S, \mathcal{O}_C) \). Moreover \( S \in \mathcal{O}_C \) if and only if the corresponding convolution operator \( C_S \in \mathcal{L}(S, S) \), i.e. if and only if \( C_S \) maps (continuously) the Schwartz space \( S \) into itself. Moreover if \( S \in \mathcal{O}_C \), then \( C_S \in \mathcal{L}(\mathcal{O}_C, \mathcal{O}_C) \), where \( \mathcal{S} \) is the unique extension of the functional \( S \) on \( S \) over \( \mathcal{O}_C \).

Therefore we can, again after Schwartz [47], introduce the topology on \( \mathcal{O}_C' \) induced from the topology of uniform convergence on bounded sets on \( \mathcal{L}(S,S) \).

These are the Schwartz operator topologies on \( \mathcal{O}_M \) and \( \mathcal{O}_C' \). These spaces become nuclear with these topologies, (quasi-) complete and barreled. For their definitions as induced by systems of semi-norms we refer the reader to the classic work [47] or [28], [31], [30]. In fact all indicated spaces are barreled, although all of them are endowed with topology strictly stronger than the topology induced by the strong dual topology of \( S' \) (for all of them except the whole of the space \( \mathcal{E} \) and \( \mathcal{D}' \) which cannot be naturally included into \( S' \)).

**THEOREM.** Let \( S' \) be endowed with the strong dual topology, and \( \mathcal{O}_M, \mathcal{O}_C' \) with the Schwartz’ operator topologies defined as above. On the space \( S' \) we can define the operation of multiplication by \( S \in \mathcal{O}_M \) through the linear transpose of the map \( M_S \), which maps continuously \( S' \rightarrow S' \) and defines a bilinear hypocontinuous multiplication map \( S' \times \mathcal{O}_M \rightarrow S' \). Similarly on the space \( S' \) we can define the operation of convolution by \( S \in \mathcal{O}_C' \) through the linear transpose of the map \( C_S \), which maps continuously \( S' \rightarrow S' \) and defines a bilinear hypocontinuous convolution map \( S' \times \mathcal{O}_C' \rightarrow S' \).

Compare [47], Thm. X and Thm. XI, Chap. VII, §5, pp. 245-248.

On the space \( \mathcal{O}_M \) we can define the commutative multiplication operation \( S_1 \cdot S_2 \):
\[
\mathcal{O}_M \times \mathcal{O}_M \ni S_1 \times S_2 \mapsto S_1 \cdot S_2 \in \mathcal{O}_M
\]
through the composition of the corresponding multiplication operators \( M_{S_1} \circ M_{S_2} = M_{S_2} \circ M_{S_1} = M_{S_1 \cdot S_2} \), which corresponds to the ordinary pointwise multiplication of functions \( f_1, f_2 \in \mathcal{O}_M \) representing the corresponding tempered
distributions $S_1, S_2 \in \mathcal{O}_M \subset \mathcal{S}'$. Similarly we can define commutative convolution operation $S_1 * S_2$:

\[ \mathcal{O}'_C \times \mathcal{O}'_C \ni S_1 \times S_2 \mapsto S_1 * S_2 \in \mathcal{O}'_C \]

through the composition of the corresponding convolution operators $C_{S_1} \circ C_{S_2} = C_{S_2} \circ C_{S_1} = C_{S_1 * S_2}$, which coincides with the ordinary convolution $f_1 \ast f_2$ of functions $f_1, f_2$ if the tempered distributions $S_1, S_2, S_1 * S_2 \in \mathcal{O}_M \subset \mathcal{S}'$ can be represented by ordinary functions $f_1, f_2, f_1 \ast f_2$.

**THEOREM.**

1) The multiplication $S_1 \cdot S_2$ operation is not only hypocontinuous as a map $\mathcal{O}_M \times \mathcal{O}_M \to \mathcal{O}_M$, but likewise (jointly) continuous.

2) The convolution $S_1 * S_2$ operation is not only hypocontinuous as a map $\mathcal{O}'_C \times \mathcal{O}'_C \to \mathcal{O}'_C$, but likewise (jointly) continuous.

Compare [47], Remark on page 248, or [31], Proposition 5.

Similarly we define a function to be a multiplier (convolutor) of the indicated function space if the corresponding multiplication (convolution) operator maps the space continuously into itself. Similarly we define by duality the multipliers (convolutors) of the strong dual of the indicated function space.

Recall the Schwartz’ Fourier exchange Theorem ([47], Chap. VII.8, Thm. XV)

**THEOREM.** If linear topological spaces $\mathcal{O}_M$ and $\mathcal{O}'_C$ are endowed with the Schwartz’ operator topologies, defined as above, then the Fourier transform $\mathcal{F}$, regarded as a map on $\mathcal{S}'$ restricted to $\mathcal{O}'_C$, transforms isomorphically $\mathcal{O}'_C$ onto $\mathcal{O}_M$, and the following formula

\[ \mathcal{F}(S * T) = \mathcal{F}S \cdot \mathcal{F}T, \]

is valid for any $S \in \mathcal{O}'_C$ and $T \in \mathcal{S}'$.

All cited results in this Appendix are essentially contained in the classic work [47] of L. Schwartz. Some of the results are only remarked there or sometimes formulated without (detailed) proofs, but the reader will find all details in the subsequent literature on distribution theory. In particular a topological supplement to the proof of the Fourier exchange Theorem XV (Chap. VII.8 [47]) can be found e.g. in [29], but a full and systematic treatement of this theorem can be found in [30], where a detailed construction of the predual $\mathcal{O}_C$ of $\mathcal{O}'_C$ is also given. For further details on the indicated spaces and their multipliers and convolutors compare [47], [63], [31], [32], [28].

**REMARK.** Note that the multiplication $\cdot$ map $\mathcal{O}_M \times \mathcal{O}_M \to \mathcal{O}_M$ (as well as the convolution $\ast$ map: $\mathcal{O}'_C \times \mathcal{O}'_C \to \mathcal{O}'_C$) is not hypocontinuous with respect to the topology on $\mathcal{O}_M$ (resp. on $\mathcal{O}'_C$) induced from the strong dual topology on $\mathcal{S}'$. Indeed if it was hypocontinuous then by the well known extension theorem, compare the Proposition of Chap. III, §5.4, p.90 in [47], a hypocontinuous extension of the multiplication to a product $\mathcal{S}' \times \mathcal{S}' \to \mathcal{S}'$ (resp. extension of
the convolution) could have been constructed, which coincides with the ordinary function point-wise multiplication (resp. convolution) product if the distributions can be represented by functions. Because $S'$ is the strong dual of a reflexive Fréchet space $S$, then by Thm. 41.1 of [58], we could have obtained in this way a continuous extension of the product of distributions respecting the natural algebraic laws under multiplication and differentiation and coinciding with the ordinary point-wise multiplication (resp. convolution) product of functions whenever the distributions coincide with ordinary functions. But this would be in contradiction to the classic result of Schwartz, which says that such extension is impossible, compare [48] or [47], Chap. V.1. Similarly we can show that the extension of the convolution product on the convolution algebra of $S^0(\mathbb{R}^n; \mathbb{C})$ is not hypocontinuous with respect to the topology inherited from the strong dual $S^0(\mathbb{R}^n; \mathbb{C})^*$, because of the topological inclusions $S^0(\mathbb{R}^n; \mathbb{C}) \subset S(\mathbb{R}^n; \mathbb{C})$ and $S(\mathbb{R}^n; \mathbb{C})^* \subset S^0(\mathbb{R}^n; \mathbb{C})^*$, with the topology on $S^0(\mathbb{R}^n; \mathbb{C})$ coinciding with that inherited from $S(\mathbb{R}^n; \mathbb{C})$, compare Subsection 5.5 of [59]. Equivalently: the point-wise multiplication product defined on the multiplier algebra of $S^{00}(\mathbb{R}^n; \mathbb{C})$ is not hypocontinuous with respect to the topology inherited from the strong dual $S^{00}(\mathbb{R}^n; \mathbb{C})^*$.

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