CONCENTRATION OF DISCREPANCY–BASED APPROXIMATE BAYESIAN COMPUTATION VIA RADEMACHER COMPLEXITY

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There has been an increasing interest on summary–free versions of approximate Bayesian computation (ABC), which replace distances among summaries with discrepancies between the empirical distributions of the observed data and the synthetic samples generated under the proposed parameter values. The success of these solutions has motivated theoretical studies on the limiting properties of the induced posteriors. However, current results (i) are often tailored to a specific discrepancy, (ii) require, either explicitly or implicitly, regularity conditions on both the data generating process and the assumed statistical model, and (iii) yield concentration bounds depending on sequences of control functions that are not made explicit. As such, there is still the lack of a theoretical framework for summary–free ABC that (i) is unified, instead of discrepancy–specific, (ii) does not necessarily require to constrain the analysis to data generating processes and statistical models meeting specific regularity conditions, but rather facilitates the derivation of limiting properties that hold uniformly, and (iii) relies on verifiable assumptions that provide more explicit concentration bounds clarifying which factors govern the limiting behavior of the ABC posterior. We address this gap via a novel theoretical framework that introduces the concept of Rademacher complexity in the analysis of the limiting properties for discrepancy–based ABC posteriors, including in non–i.i.d. and misspecified settings. This yields a unified theory that relies on constructive arguments and provides more informative asymptotic results and uniform concentration bounds, even in settings not covered by current studies. These advancements are obtained by relating the limiting properties of summary–free ABC posteriors to the behavior of the Rademacher complexity associated with the chosen discrepancy within the family of integral probability semimetrics (IPS). The IPS class extends summary–based ABC, and also includes the widely–implemented Wasserstein distance and maximum mean discrepancy (MMD), among others. As clarified through a focus of these results on the MMD case and via illustrative simulations, this novel theoretical perspective yields an improved understanding of summary–free ABC.

1. Introduction. The increasing complexity of statistical models in modern applications not only yields intractable likelihoods, but also raises substantial challenges in the identification of effective summary statistics (e.g., Fearnhead & Prangle, 2012; Marin et al., 2014; Frazier et al., 2018). Such drawbacks have motivated an increasing adoption of ABC solutions, along with a shift away from summary–based implementations (e.g., Marin et al., 2012) and towards summary–free strategies relying on discrepancies among the empirical distributions of the observed and synthetic data (e.g., Drovandi & Frazier, 2022). These solutions only require that simulating from the model is feasible, and provide samples from an approximate posterior distribution for the parameter of interest \( \theta \in \Theta \subseteq \mathbb{R}^p \). This is achieved

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by retaining all those values of \( \theta \), drawn from the prior, which produced synthetic samples \( z_{1:m} = (z_1, \ldots, z_m) \) from the model \( \mu_\theta \) whose empirical distribution is sufficiently close to the one of the observed data \( y_{1:n} = (y_1, \ldots, y_n) \), under the chosen discrepancy.

Notable examples of the above implementations are ABC versions that employ maximum mean discrepancy (MMD) (Park et al., 2016), Kullback–Leibler (KL) divergence (Jiang et al., 2018), Wasserstein distance (Bernton et al., 2019), energy statistic (Nguyen et al., 2020), the Hellinger and Cramer–von Mises distances (Frazier, 2020), and the \( \gamma \)-divergence (Fujisawa et al., 2021); see also Gutmann et al. (2018), Forbes et al. (2021) and Wang et al. (2022) for additional examples of summary–free ABC strategies. By overcoming the need to preselect summaries, all these solutions reduce the potential information loss of summary–based ABC, thus yielding improved performance in simulation studies and illustrative applications. These promising empirical results have motivated active research on the theoretical properties of the induced ABC posterior, with a main focus on the limiting behavior under different asymptotic regimes for the tolerance threshold and the sample size (Jiang et al., 2018; Bernton et al., 2019; Nguyen et al., 2020; Frazier, 2020; Fujisawa et al., 2021). Among these regimes, of particular interest are the two situations in which the ABC threshold is either fixed or progressively shrinks as both \( n \) and \( m \) diverge. In the former case, the theory focuses on assessing whether the control on the discrepancy among the empirical distributions established by the selected ABC threshold yields a pseudo–posterior whose asymptotic form guarantees the same threshold–control on the discrepancy among the underlying truths (e.g., Jiang et al., 2018). The latter addresses instead the more challenging theoretical question on whether a suitably–decaying ABC threshold can provide meaningful rates of concentration for the sequence of ABC posteriors around those \( \theta \) values yielding a \( \mu_\theta \) close to the data generating process \( \mu^\ast \), as \( n \) and \( m \) diverge (e.g., Bernton et al., 2019).

Available contributions along these lines of research have the merit of providing theoretical support to several important versions of summary–free ABC. However, current theory is often tailored to the specific discrepancy analyzed, and generally relies on difficult–to–verify existence assumptions and concentration inequalities that constrain the analysis, either implicitly or explicitly, to data generating processes and statistical models which meet suitable regularity conditions, thereby lacking results that hold uniformly. Recalling Bernton et al. (2019) and Nguyen et al. (2020), this yields also concentration bounds involving sequences of control functions which are not made explicit. As such, although convergence and concentration can still be proved, the core factors that govern these limiting properties remain yet unexplored, thus limiting the methodological impact of current theory, while hindering the derivation of novel informative results in more challenging settings. For example, the available theoretical studies pre–assume that the discrepancy among the empirical distributions \( \hat{\mu}_{z_{1:m}} \) and \( \hat{\mu}_{y_{1:n}} \) suitably converges to the one among the corresponding truths \( \mu_\theta \) and \( \mu^\ast \), or, alternatively, that both \( \hat{\mu}_{z_{1:m}} \) and \( \hat{\mu}_{y_{1:n}} \) converge, under the selected discrepancy, to \( \mu_\theta \) and \( \mu^\ast \), respectively. While these assumptions can be verified under suitable conditions and for specific discrepancies, as clarified within Sections 2 and 3, an in–depth theoretical understanding of summary–free ABC necessarily requires relating convergence to the learning properties of the selected discrepancy, rather than pre–assuming it. Such a more precise
but yet–unexplored theoretical treatment has the potentials to shed light on the factors that
govern the limiting behavior of discrepancy–based ABC posteriors. In addition, it could pos-
sibly provide more general, verifiable and explicit sufficient conditions under which popular
discrepancies are guaranteed to ensure that the previously pre–assumed convergence and
concentration hold uniformly over models and data generating processes, while facilitating
the study of the limiting behavior of discrepancy–based ABC posteriors in more general sit-
tuations where these convergence guarantees may lack.

In this article, we address the aforementioned endeavor by introducing an innovative the-
oretical framework which analyses the limiting properties of discrepancy–based ABC poste-
rors, under a unified perspective and for different asymptotic regimes, through the concept
of Rademacher complexity (e.g., Wainwright, 2019, Chapter 4), within the general class of
integral probability semimetrics (IPS) (see e.g., Müller, 1997; Sriperumbudur et al., 2012).
This class naturally generalizes summary–based ABC and includes the widely–implemented
MMD and Wasserstein distance, among others. As clarified in Sections 2, 3 and 6, such a per-
spective, which to the best of our knowledge is novel in ABC, allows to derive unified, infor-
mative and uniform concentration bounds for ABC posteriors under several discrepancies, in
possibly misspecified and non–i.i.d. contexts. Moreover, it relies on more constructive argu-
ments that clarify under which sufficient conditions a discrepancy within the IPS class guar-
antees uniform convergence and concentration of the induced ABC posterior; i.e., without
necessarily requiring suitable regularity conditions for the underlying data generating pro-
cess $\mu^*$ and the assumed statistical model. This yields an important theoretical and method-
ological advancement, provided that $\mu^*$ is often unknown in practice and, hence, verifying
regularity conditions on the data generating process is generally unfeasible.

Crucially, the theoretical framework we introduce allows to prove informative theoretical
results even in yet–unexplored settings that possibly lack those convergence guarantees as-
sumed in available literature. More specifically, in these settings we derive novel upper and
lower bounds for the limiting acceptance probabilities that clarify in which contexts the ABC
posterior is still well–defined for $n$ large enough. When this is the case, it is further possible
to obtain informative supersets for the support of such a posterior. These show that, when
relaxing standard convergence assumptions employed in state–of–the–art theoretical stud-
ies, the control established by a fixed ABC threshold on the discrepancy among the empirical
distributions does not necessarily translate, asymptotically, into the same control on the dis-
crepancy among the corresponding truths, but rather yields an upper bound equal to the sum
between the ABC threshold and a multiple of the Rademacher complexity, namely a measure
of richness of the class of functions that uniquely identify the chosen IPS; see Section 3.1.

The above results clarify the fundamental relation among the limiting behavior of ABC pos-
teriors and the learning properties of the chosen discrepancy, when measured via Rademacher
complexity. Moreover, the bounds derived clarify that a sufficient condition to recover a lim-
itig pseudo–posterior with the same threshold–control on the discrepancy among the truths
as the one enforced on the corresponding empirical distributions, is that the selected discrep-
ancy has a Rademacher complexity vanishing to zero in the large–data limit. As proved in
Section 3.2, this setting also allows constructive derivations of novel, informative and uni-
form concentration bounds for discrepancy–based ABC posteriors in the challenging regime where the threshold shrinks towards zero as both $m$ and $n$ diverge. This is facilitated by the existence of meaningful upper bounds for the Rademacher complexity of popular ABC discrepancies, along with the availability of constructive conditions for the derivation of these bounds (e.g., Sriperumbudur et al., 2012) which leverage fundamental connections among such a complexity measure and other key quantities in statistical learning theory, such as the Vapnik–Chervonenkis (VC) dimension and the notion of uniform Glivenko–Cantelli classes (see e.g., Wainwright, 2019, Chapter 4). This yields an improved understanding of the factors that govern the concentration of discrepancy–based ABC posteriors under a unified perspective that further allows to (i) quantify rates of concentration and (ii) directly translate any advancement on Rademacher complexity into novel ABC theory. The former advantage is illustrated within Section 4 through a specific focus on MMD with routinely–implemented bounded and unbounded kernels, whereas the latter is clarified in Section 6, where we extend the theory from Section 3 to non–i.i.d. settings, leveraging results in Mohri & Rostamizadeh (2008) on Rademacher complexity under $\beta$–mixing processes (e.g., Doukhan, 1994).

The illustrative simulation studies within Sections 5 and 7 show that the theoretical results derived in Sections 3, 4 and 6 find empirical evidence in practice, including also in scenarios based on model misspecification, data contamination and non–i.i.d. data generating processes. These findings suggest that discrepancies with guarantees of uniformly–vanishing Rademacher complexity provide a robust and sensible choice when the assumed statistical model and/or the underlying data generating process do not necessarily meet specific regularity conditions, or it is not possible to verify these conditions. This is a common situation in applications, provided that the data generating process is often unknown in practice.

As discussed in Section 8, the unexplored bridge between discrepancy–based ABC and the Rademacher complexity introduced in this article can also be leveraged to derive even more general theory by exploiting the active literature on the Rademacher complexity. For example, combining our novel perspective with the recently–derived unified treatment of IPS and $f$–divergences (Agrawal and Horel, 2021; Birrell et al., 2022) might set the premises to derive similarly–tractable, interpretable and general results for other discrepancies employed within ABC, such as the Kullback–Leibler divergence (Jiang et al., 2018) and Hellinger distance (Frazier, 2020). More generally, our contribution can have direct implications even beyond ABC, in particular on generalized Bayesian inference via pseudo–posteriors based on discrepancies (e.g., Bissiri et al., 2016; Chérief-Abdellatif & Alquier, 2020; Matsubara et al., 2022). The proofs of the theoretical results can be found in the Supplementary Material.

2. Integral probability semimetrics (IPS) and Rademacher complexity. Let $y_{1:n} = (y_1, \ldots, y_n) \in \mathcal{Y}^n$ denote an i.i.d. sample from $\mu^* \in \mathcal{P}(\mathcal{Y})$, where $\mathcal{P}(\mathcal{Y})$ is the space of probability measures on $\mathcal{Y}$, and assume that $\mathcal{Y}$ is a metric space endowed with the distance $\rho$; see Section 6 for extensions to non–i.i.d. settings. Given a statistical model $\{\mu_\theta : \theta \in \Theta \subseteq \mathbb{R}^p\}$ within $\mathcal{P}(\mathcal{Y})$ and a prior distribution $\pi$ on $\theta$, rejection ABC iteratively samples $\theta$ from $\pi$, draws a synthetic i.i.d. sample $z_{1:m} = (z_1, \ldots, z_m)$ from $\mu_\theta$, and retains $\theta$ as a sample from the ABC posterior $\pi_{\theta}(z_{1:m})$ if the discrepancy $\Delta(z_{1:m}, y_{1:n})$ between the synthetic and the ob-
served data is below a preselected threshold $\varepsilon_n \geq 0$. Although $m$ may differ from $n$, here we follow common practice in ABC theory (Bernton et al., 2019; Frazier, 2020) and set $m = n$. Rejection ABC does not sample from the exact posterior distribution $\pi_n(\theta) \propto \pi(\theta) \mu_\theta^n(y_{1:n})$, but rather from the ABC posterior

$$\pi_n^{(\varepsilon_n)}(\theta) \propto \pi(\theta) \int_{\mathcal{Y}^n} 1\{\Delta(z_{1:n}, y_{1:n}) \leq \varepsilon_n\} \mu_\theta^n(dz_{1:n}),$$

whose properties clearly depend on the chosen discrepancy $\Delta(\cdot, \cdot)$. Within summary–based ABC, $\Delta(z_{1:n}, y_{1:n})$ is a suitable distance, typically Euclidean, among summaries computed from the synthetic and observed data. However, recalling, e.g., Marin et al. (2014) and Frazier et al. (2018), the identification of summaries that do not lead to information loss is challenging for those complex models requiring ABC implementations.

To overcome these challenges, ABC literature has progressively moved towards the adoption of discrepancies $D : \mathcal{P}(\mathcal{Y}) \times \mathcal{P}(\mathcal{Y}) \to [0, \infty]$ between the empirical distributions of the synthetic and observed data, that is

$$\Delta(z_{1:n}, y_{1:n}) = D(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}}) = D(n^{-1}\sum_{i=1}^n \delta_{z_i}, n^{-1}\sum_{i=1}^n \delta_{y_i}),$$

where $\delta_x$ denotes the Dirac delta at a generic $x \in \mathcal{Y}$. Popular examples are ABC versions based on MMD, KL divergence, Wasserstein distance, energy statistic, Hellinger and Cramer–von Mises distances, and $\gamma$–divergence, whose limiting properties have been studied in Park et al. (2016), Jiang et al. (2018), Bernton et al. (2019), Nguyen et al. (2020), Frazier (2020) and Fujisawa et al. (2021) under different asymptotic regimes and with a common reliance on existence assumptions to ease the proofs. As a first step toward our unified and constructive theoretical framework, we shall emphasize that, although most of the above contributions treat discrepancies separately, some of these choices share, in fact, a common origin. For example, MMD, Wasserstein distance and the energy statistic all belong to the IPS class (Müller, 1997) in Definition 2.1. Such a class, also includes summary–based distances.

**Definition 2.1** (Integral probability semimetric – IPS). Let $(\mathcal{Y}, \mathcal{A})$ denote a measure space, $g : \mathcal{Y} \to [1, \infty)$ a measurable function, and $\mathcal{B}_g$ the set of all measurable functions $f : \mathcal{Y} \to \mathbb{R}$ such that $\|f\|_g := \sup_{y \in \mathcal{Y}} |f(y)|/g(y) < \infty$. Then, for any $\mathcal{F} \subseteq \mathcal{B}_g$, a generic integral probability semimetric $D_{\mathcal{F}}$ between $\mu_1, \mu_2 \in \mathcal{P}(\mathcal{Y})$ is defined as

$$D_{\mathcal{F}}(\mu_1, \mu_2) := \sup_{\mathcal{F}} \int f d\mu_1 - \int f d\mu_2,$$

where $\sup_{\mathcal{F}}$ denotes the sup over the prespecified class of functions $\mathcal{F}$.

Examples 2.1–2.3 show that routinely–employed discrepancies both in summary–free and summary–based ABC (Park et al., 2016; Bernton et al., 2019; Nguyen et al., 2020; Drovandi & Frazier, 2022) are, in fact, IPS with a known characterizing family $\mathcal{F}$ that uniquely identifies each discrepancy.
Example 2.1. (Wasserstein distance). Let us consider the Lipschitz seminorm defined as $\|f\|_L := \sup \{|f(x) - f(x')|/\rho(x, x') : x \neq x' \text{ in } \mathcal{Y}\}$. If $\mathcal{F} = \{f : \|f\|_L \leq 1\}$, then $\mathcal{D}_\mathcal{F}$ coincides with the Kantorovich metric. Recalling the Kantorovich–Rubinstein theorem (e.g., Dudley, 2018), when $\mathcal{Y}$ is separable this metric coincides with the dual representation of the Wasserstein distance, which is therefore recovered as an example of IPS. Recall also that such a distance is defined on the space of probability measures having finite first–order moments (e.g., Sriperumbudur et al., 2012; Bernton et al., 2019).

Example 2.2. (MMD and energy distance). Given a positive–definite kernel $k(\cdot, \cdot)$ on $\mathcal{Y} \times \mathcal{Y}$, let $\mathcal{F} = \{f : \|f\|_H \leq 1\}$, where $H$ is the reproducing kernel Hilbert space corresponding to $k(\cdot, \cdot)$. Then $\mathcal{D}_\mathcal{F}$ is the MMD (e.g., Muandet et al., 2017). When $k(\cdot, \cdot)$ is characteristic, then $\mathcal{D}_\mathcal{F}$ is not only a semimetric, but also a metric — i.e., $\mathcal{D}_\mathcal{F}(\mu_1, \mu_2) = 0$ implies $\mu_1 = \mu_2$. Relevant examples of routinely–implemented characteristic kernels in $\mathcal{Y} = \mathbb{R}^d$ are the Gaussian $\exp(-\|x - x'\|^2/\sigma^2)$ and Laplace $\exp(-\|x - x'\|/\sigma)$ ones (Muandet et al., 2017). Note that MMD is also directly related to energy distance, due to the correspondence among positive–definite kernels and negative–definite functions (Sejdinovic et al., 2013).

Example 2.3. (Summary–based distances). Classical summary–based ABC employs distances among a finite set of summaries $f_1, \ldots, f_K$ (e.g., Drovandi & Frazier, 2022). Interestingly, these distances can be recovered as a special case of MMD with suitably–defined kernel and, hence, are guaranteed to belong to the IPS class. In particular, for a pre–selected vector of summaries $f(x) = [f_1(x), \ldots, f_K(x)] \in H = \mathbb{R}^K$ equipped with the standard Euclidean norm $\|f(x)\|_H^2 = \langle f(x), f(x) \rangle = f_1^2(x) + \cdots + f_K^2(x)$, one can define the kernel $k(x, x') = \langle f(x), f(x') \rangle$ to obtain classical summary–based ABC recasted within the MMD framework. As a result, popular kernels, such as the Gaussian one, relying on infinite feature maps can be interpreted as infinite dimensional versions of summary–based ABC.

Although Examples 2.1–2.3 characterize the most popular instances of IPS discrepancies employed in ABC, it shall be emphasized that other interesting semimetrics belong to the IPS class (e.g., Sriperumbudur et al., 2012; Birrell et al., 2022). Two relevant ones are the total variation (TV) and Kolmogorov–Smirnov distances discussed in the Supplementary Materials (see Examples A.1.1–A.1.2).

While ABC based on discrepancies, such as those presented above, overcomes the need to preselect summaries, it still requires the choice of a discrepancy. This motivates theory on the asymptotic properties of the induced ABC posterior — under general discrepancies — for $n \to \infty$ and relevant thresholding schemes. More specifically, considering here a unified perspective which applies to the whole IPS class, current theory focuses both on fixed $\varepsilon_n = \varepsilon$ settings, and also on $\varepsilon_n \to \varepsilon^*$ regimes, where $\varepsilon^* = \inf_{\theta \in \Theta} \mathcal{D}_\mathcal{F}(\mu_\theta, \mu^*)$ is the lowest attainable discrepancy between the assumed model and the data generating process (Jiang et al., 2018; Bernton et al., 2019; Nguyen et al., 2020; Frazier, 2020; Fujisawa et al., 2021). Under these settings, available theory investigates whether the $\varepsilon_n$–control on $\mathcal{D}_\mathcal{F}(\mu_{y_{1:n}}, \hat{\mu}_{y_{1:n}})$ established by rejection–ABC translates into meaningful convergence results, and upper bounds on the rates of concentration for the sequence of discrepancy–based ABC posteriors around the true
data generating process $\mu^*$, both under correctly specified models where $\mu^* = \mu_\theta^*$ for some $\theta^* \in \Theta$, and in misspecified contexts where $\mu^*$ does not belong to $\{\mu_\theta : \theta \in \Theta \subseteq \mathbb{R}^p\}$.

A common strategy to derive the aforementioned results is to study convergence of the empirical distributions $\hat{\mu}_{z_1:n}$ and $\hat{\mu}_{y_{1:n}}$ to the corresponding truths $\mu_\theta$ and $\mu^*$, in the chosen discrepancy $\mathcal{D}_\mathcal{F}$, after noticing that, since $\mathcal{D}_\mathcal{F}$ is a semimetric, we have

\begin{equation}
\mathcal{D}_\mathcal{F}(\mu_\theta, \mu^*) \leq \mathcal{D}_\mathcal{F}(\hat{\mu}_{z_1:n}, \mu_\theta) + \mathcal{D}_\mathcal{F}(\hat{\mu}_{z_1:n}, \hat{\mu}_{y_{1:n}}) + \mathcal{D}_\mathcal{F}(\hat{\mu}_{y_{1:n}}, \mu^*).\end{equation}

To this end, available theoretical studies pre–assume suitable convergence results for the empirical measures along with specific concentration inequalities regulated by non–explicit sequences of control functions (see e.g., Assumptions 1–2 in Proposition 3 of Bernton et al. (2019), and Assumptions A1–A2 in Nguyen et al. (2020)). Alternatively, it is possible to directly require that $\mathcal{D}_\mathcal{F}(\hat{\mu}_{z_1:n}, \hat{\mu}_{y_{1:n}}) \rightarrow \mathcal{D}_\mathcal{F}(\mu_\theta, \mu^*)$ almost surely as $n \rightarrow \infty$ (see Jiang et al., 2018; Nguyen et al., 2020, Theorems 1 and 2, respectively). However, as highlighted by the same authors, these conditions (i) can be difficult to verify for several discrepancies, (ii) do not allow to assess whether some of these discrepancies can achieve convergence and concentration uniformly over $\mathcal{P}(\mathcal{Y})$, and (iii) often yield bounds which hinder an in–depth understanding of the factors regulating these limiting properties. In addition, when the above assumptions are not met, the asymptotic behavior of the induced ABC posteriors remains yet unexplored.

As a first step towards addressing the above issues, notice that the aforementioned results, when contextualized within the IPS class, are inherently related to the richness of the known class of functions $\mathcal{F}$ that identifies each $\mathcal{D}_\mathcal{F}$, see Definition 2.1. Intuitively, if $\mathcal{F}$ is too rich, it is always possible to find a function $f \in \mathcal{F}$ yielding large discrepancies even when $\hat{\mu}_{z_1:n}$ and $\hat{\mu}_{y_{1:n}}$ are arbitrarily close to $\mu_\theta$ and $\mu^*$, respectively. Hence, $\mathcal{D}_\mathcal{F}(\hat{\mu}_{z_1:n}, \mu_\theta)$ and $\mathcal{D}_\mathcal{F}(\hat{\mu}_{y_{1:n}}, \mu^*)$ will remain large with positive probability, making the previous triangle inequality of limited interest, since a low $\mathcal{D}_\mathcal{F}(\hat{\mu}_{z_1:n}, \hat{\mu}_{y_{1:n}})$ will not necessarily imply a small $\mathcal{D}_\mathcal{F}(\mu_\theta, \mu^*)$. In fact, in this context, it is clearly even not guaranteed that $\mathcal{D}_\mathcal{F}(\hat{\mu}_{z_1:n}, \hat{\mu}_{y_{1:n}})$ will converge to $\mathcal{D}_\mathcal{F}(\mu_\theta, \mu^*)$. This suggests that the limiting properties of the ABC posterior are inherently related to the richness of $\mathcal{F}$. In Sections 3 and 6 we prove that this is the case when such a richness is measured via the notion of Rademacher complexity clarified in Definition 2.2; see also Chapter 4 in Wainwright (2019) for an introduction.

**Definition 2.2 (Rademacher complexity).** Given a sample $x_{1:n} = (x_1, \ldots, x_n)$ of i.i.d. realizations from $\mu \in \mathcal{P}(\mathcal{Y})$, the Rademacher complexity $\mathcal{R}_{\mu,n}(\mathcal{F})$ of a class $\mathcal{F}$ of real–valued measurable functions is

\[
\mathcal{R}_{\mu,n}(\mathcal{F}) = \mathbb{E}_{x_{1:n},\epsilon_{1:n}} \left[ \sup_{f \in \mathcal{F}} \left| (1/n) \sum_{i=1}^n \epsilon_i f(x_i) \right| \right],
\]

where $\epsilon_1, \ldots, \epsilon_n$ denote i.i.d. Rademacher variables, that is $\mathbb{P}(\epsilon_i = 1) = \mathbb{P}(\epsilon_i = -1) = 1/2$.

As is clear from the above definition, high values of the Rademacher complexity $\mathcal{R}_{\mu,n}(\mathcal{F})$ mean that $\mathcal{F}$ is rich enough to contain functions that can closely interpolate, on average, even full–noise vectors. Hence, if $\mathcal{R}_{\mu,n}(\mathcal{F})$ is bounded away from zero for every $n$, then $\mathcal{F}$
might be excessively rich for statistical purposes. Conversely, if \( \mathcal{R}_{\mu,n}(\mathcal{F}) \) goes to zero as \( n \) diverges, then \( \mathcal{F} \) is a more parsimonious class. Lemma 2.1 formalizes this intuition.

**Lemma 2.1** (Theorem 4.10 and Proposition 4.12 in Wainwright (2019)). Let \( x_{1:n} \) be i.i.d. from some distribution \( \mu \in \mathcal{P}(\mathcal{Y}) \). Then, for any \( b \)-uniformly bounded class \( \mathcal{F} \), i.e., any class \( \mathcal{F} \) of functions \( f \) such that \( ||f||_{\infty} \leq b \), any integer \( n \geq 1 \) and scalar \( \delta \geq 0 \), it holds that

\[
P_{x_{1:n}}[D_{\mathcal{F}}(\hat{\mu}_{x_{1:n}}, \mu) \leq 2\mathcal{R}_{\mu,n}(\mathcal{F}) + \delta] \geq 1 - \exp(-n\delta^2/2b^2),
\]

and

\[
P_{x_{1:n}}[D_{\mathcal{F}}(\hat{\mu}_{x_{1:n}}, \mu) \geq \mathcal{R}_{\mu,n}(\mathcal{F})/2 - \sup_{f \in \mathcal{F}} ||E(f)||/2n^{1/2} - \delta] \geq 1 - \exp(-n\delta^2/2b^2).
\]

Lemma 2.1 provides general bounds for the probability that \( D_{\mathcal{F}}(\hat{\mu}_{x_{1:n}}, \mu) \) takes values below a multiple and above a fraction of the Rademacher complexity of \( \mathcal{F} \). Recalling the previous discussion on the concentration of ABC posteriors, this result is fundamental to study the convergence of both \( \hat{\mu}_{x_{1:n}} \) and \( \mu_{1:n} \) to \( \mu_\theta \) and \( \mu^* \), respectively. Our theory in Section 3 proves that this is possible when \( \mathcal{R}_n(\mathcal{F}) := \sup_{\mu \in \mathcal{P}(\mathcal{Y})} \mathcal{R}_{\mu,n}(\mathcal{F}) \) goes to 0 as \( n \to \infty \). According to Lemma 2.1, this is, in fact, a necessary and sufficient condition for \( D_{\mathcal{F}}(\hat{\mu}_{x_{1:n}}, \mu) \to 0 \) as \( n \to \infty \), in \( P_{x_{1:n}} \)–probability, uniformly over \( \mathcal{P}(\mathcal{Y}) \). See Section 6 for non–i.i.d. extensions.

### 3. Asymptotic properties of discrepancy–based ABC posterior distributions.

As anticipated in Section 2, the asymptotic results we derive in this article leverage Lemma 2.1 to connect the properties of the ABC posterior under an IPS discrepancy \( D_{\mathcal{F}} \) with the behavior of the Rademacher complexity for the associated family \( \mathcal{F} \).

Section 3.1 clarifies the importance of such a bridge by studying the limiting properties of the ABC posterior when \( \varepsilon_n = \varepsilon \) is fixed and \( n \to \infty \), including in yet-unexplored situations where convergence guarantees for the empirical measures and for the discrepancy among such measures are not necessarily available. The theoretical results we derive in this setting suggest that the use of discrepancies whose Rademacher complexity decays to zero in the limit is a sufficient condition to obtain strong convergence guarantees. On the basis of these findings, Section 3.2 focuses on the regime \( \varepsilon_n \to \varepsilon^* \) as \( n \to \infty \), and derives unified and informative concentration bounds that crucially hold uniformly over \( \mathcal{P}(\mathcal{Y}) \), and are based on more constructive conditions than those employed in the available discrepancy–specific theory. More specifically, to prove our theoretical results in Sections 3.1 and 3.2, we will rely on some or all of the following assumptions:

1. (I) the observed data \( y_{1:n} \) are i.i.d. from a data generating process \( \mu^* \);
2. (II) there exist positive \( L \) and \( c_\pi \) such that, for \( \bar{\varepsilon} \) small, \( \pi(\{\theta: D_{\mathcal{F}}(\mu_\theta, \mu^*) \leq \varepsilon^* + \bar{\varepsilon}\}) \geq c_\pi \bar{\varepsilon}^L \);
3. (III) \( \mathcal{F} \) is \( b \)-uniformly bounded; i.e., there exists \( b \in \mathbb{R}^+ \) such that \( ||f||_{\infty} \leq b \) for any \( f \in \mathcal{F} \);
4. (IV) \( \mathcal{R}_n(\mathcal{F}) := \sup_{\mu \in \mathcal{P}(\mathcal{Y})} \mathcal{R}_{\mu,n}(\mathcal{F}) \) goes to zero as \( n \to \infty \).

Condition (I) is the only assumption made on the data generating process and is present in, e.g., Nguyen et al. (2020) and in the supplementary materials of Bernton et al. (2019). Although the theory we derive in Section 6 relaxes (I) to study convergence and concentration
also beyond the i.i.d. context, it shall be emphasized that some of the assumptions considered in the literature may not hold even in i.i.d. settings. Hence, an improved understanding of the ABC properties under (I) is crucial to clarify the range of applicability and potential limitations of available existence theory under more complex, possibly non–i.i.d., regimes. In fact, as shown in Section 3.1, certain discrepancies may yield posteriors that are either not well–defined in the limit or lack strong convergence guarantees even in i.i.d. settings. Assumption (II) is standard in Bayesian asymptotics and defined in the limit or lack strong convergence guarantees even in i.i.d. settings. In fact, as shown in Section 3.1, certain discrepancies may yield posteriors that are either not well–defined in the limit or lack strong convergence guarantees even in i.i.d. settings. Assumption (II) is standard in Bayesian asymptotics and defined in the limit or lack strong convergence guarantees even in i.i.d. settings. Assumption (II) is standard in Bayesian asymptotics and defined in the limit or lack strong convergence guarantees even in i.i.d. settings. As clarified in Section 3.3, conditions (III)–(IV) can be broadly verified leveraging known bounds on the Rademacher complexity, and hold for the relevant IPS discrepancies. Moreover, these conditions are generally not stronger than those implicit in current theory on discrepancy–based ABC, and are inherently related with the notion of uniform Glivenko–Cantelli classes (see e.g., Wainwright, 2019, Chapter 4), arguably a minimal sufficient requirement to establish convergence and concentration properties that, unlike those currently derived, hold uniformly over \( \mathcal{P}(\mathcal{Y}) \). It shall be also emphasized that the convergence theory derived in Section 3.1 does not necessarily require a vanishing Rademacher complexity to obtain informative results. However, it also clarifies that, when this condition is verified, the limiting pseudo–posterior inherits the same threshold–control on \( D_\delta(\theta^*, \mu^*) \) as the one established by the ABC procedure on \( D_\delta(\hat{\theta}_{1:n}, \hat{\mu}_{1:n}) \).

As anticipated above, Sections 3.1–3.2 consider the two scenarios where the sample size \( n \) diverges to infinity, while the ABC threshold \( \varepsilon_n \) is either fixed to \( \varepsilon \) or progressively shrinks to \( \varepsilon^* = \inf_{\theta \in \Theta} D_\delta(\theta, \mu^*) \). If the model is well–specified, i.e., \( \mu^* = \mu_\theta \) for some, possibly not unique, \( \theta^* \in \Theta \), then \( \varepsilon^* = 0 \). Although the regime characterized by \( \varepsilon_n \rightarrow 0 \) with fixed \( n \) is also of interest, this setting can be readily addressed under the unified IPS class via a direct adaptation of available results in, e.g., Jiang et al. (2018) and Bernton et al. (2019); see also Miller and Dunson (2019) for results in the context of coarsened posteriors under the regime with \( n \rightarrow \infty \) and fixed \( \varepsilon_n = \varepsilon \), related to those in Jiang et al. (2018).

3.1. Limiting behavior with fixed \( \varepsilon_n = \varepsilon \) and \( n \rightarrow \infty \). Current ABC theory for the setting \( \varepsilon_n = \varepsilon \) and \( n \rightarrow \infty \), studies convergence of the discrepancy–based ABC posterior

\[
\pi_n(\varepsilon)(\theta) \propto \pi(\theta) \int_{\mathcal{Y}^n} \mathbb{1}\{D_\delta(\hat{\theta}_{1:n}, \hat{\mu}_{1:n}) \leq \varepsilon\} \mu_\theta^n(dz_{1:n}),
\]

under the key assumption that \( D_\delta(\hat{\theta}_{1:n}, \hat{\mu}_{1:n}) \rightarrow D_\delta(\theta^*, \mu^*) \) almost surely as \( n \rightarrow \infty \) (see, e.g., Jiang et al., 2018, Theorem 1). When such a condition is met, it can be shown that
\[
\lim_{n \to \infty} \pi_n^{(\varepsilon)}(\theta) \propto \pi(\theta) \mathbb{I}\{\mathcal{D}_\varepsilon(\mu_\theta, \mu^*) \leq \varepsilon\}. \]

This means that the limiting ABC–posterior coincides with the prior constrained within the support \(\{\theta : \mathcal{D}_\varepsilon(\mu_\theta, \mu^*) \leq \varepsilon\}\), where \(\varepsilon\) is the same threshold employed for \(\mathcal{D}_\varepsilon(\hat{\mu}_{1:n}, \hat{\mu}_{y_{1:n}})\).

Although this is a key result, verifying the convergence assumption behind the above finding can be difficult for several discrepancies (Jiang et al., 2018). More crucially, such a theory fails to provide informative results in those situations where \(\mathcal{D}_\varepsilon(\hat{\mu}_{1:n}, \hat{\mu}_{y_{1:n}})\) is not guaranteed to converge to \(\mathcal{D}_\varepsilon(\mu_\theta, \mu^*)\). In fact, under such a setting, the asymptotic properties of discrepancy–based ABC posteriors have been overlooked to date, and it is not even clear whether a well–defined \(\pi_n^{(\varepsilon)}\) exists in the large–data limit. Indeed, for specific choices of the discrepancy and threshold, the ABC posterior may not be well–defined even for fixed \(n\). For example, when \(\mathcal{D}_\varepsilon\) is the TV distance (see Example A.1.1), then \(\mathcal{D}_\varepsilon(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}}) = 1\), almost surely, whenever \(z_{1:n}\) and \(y_{1:n}\) are from two continuous distributions \(\mu_\theta\) and \(\mu^*\), respectively. Hence, for any \(\varepsilon < 1\), \(\pi_n^{(\varepsilon)}\) is not defined, even if \(\mu_\theta\) and \(\mu^*\) are not mutually singular. Note that, as clarified in Example A.2.1, the Rademacher complexity of the TV distance is always bounded away from zero in such a setting. In this case and, more generally, for IPS discrepancies whose \(\mathcal{R}_n(\mathcal{G})\) does not vanish to zero, Lemma 2.1 together with the triangle inequality in (2.1) still allow to derive new informative upper and lower bounds for limiting acceptance probabilities which further guarantee that \(\mathcal{D}_\varepsilon(\mu_\theta, \mu^*)\) can be bounded from above, for a sufficiently large \(n\), by \(\varepsilon + 4\mathcal{R}_n(\mathcal{G})\), even without assuming that \(\mathcal{D}_\varepsilon(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}}) \to \mathcal{D}_\varepsilon(\mu_\theta, \mu^*)\) almost surely for \(n \to \infty\) as in the current convergence theory. Theorem 3.1 formalizes this intuition for the whole class of ABC posteriors arising from discrepancies in the IPS family.

**Theorem 3.1.** Let \(\mathcal{D}_\varepsilon\) be an IPS as in Definition 2.1. Moreover, assume (I), (III), and let \(c_\varepsilon = 4 \lim \sup \mathcal{R}_n(\mathcal{G})\). Then, the acceptance probability \(p_n = \int_y \int_y \mathbb{I}\{\mathcal{D}(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}}) \leq \varepsilon\} \mu^*_n(dz_{1:n}) \pi(d\theta)\) of the rejection–based ABC routine with discrepancy \(\mathcal{D}_\varepsilon\) satisfies

\[
\pi\{\theta : \mathcal{D}_\varepsilon(\mu_\theta, \mu^*) \leq \varepsilon - c_\varepsilon\} + o(1) \leq p_n \leq \pi\{\theta : \mathcal{D}_\varepsilon(\mu_\theta, \mu^*) \leq \varepsilon + c_\varepsilon\} + o(1),
\]

almost surely with respect to \(y_{1:n}\) \(\text{i.i.d.} \sim \mu^*\), as \(n \to \infty\), for any fixed \(\varepsilon > 0\).

In particular, whenever \(\varepsilon > \tilde{\varepsilon} + c_\varepsilon\), with \(\tilde{\varepsilon} = \inf\{\varepsilon > 0 : \pi\{\theta : \mathcal{D}_\varepsilon(\mu_\theta, \mu^*) \leq \varepsilon\} > 0\}\), the ABC acceptance probability \(p_n\) is bounded away from 0, for \(n\) large enough. This implies that the ABC posterior \(\pi_n^{(\varepsilon)}\) is always well–defined for a large–enough \(n\) and its support is almost surely included asymptotically within the set \(\{\theta : \mathcal{D}_\varepsilon(\mu_\theta, \mu^*) \leq \varepsilon + 4\mathcal{R}_n(\mathcal{G})\}\). Namely

\[
\pi_n^{(\varepsilon)}\{\theta : \mathcal{D}_\varepsilon(\mu_\theta, \mu^*) \leq \varepsilon + 4\mathcal{R}_n(\mathcal{G})\} \to 1,
\]

almost surely with respect to \(y_{1:n}\) \(\text{i.i.d.} \sim \mu^*\), as \(n \to \infty\).

Theorem 3.1 clarifies that, even when relaxing the convergence assumptions behind current theory, the newly–introduced Rademacher complexity framework still allows to obtain guarantees on the acceptance probabilities, existence and limiting support of the ABC posterior. Such a relaxation crucially highlights the key role of the richness of \(\mathcal{G}\) in driving the asymptotic behavior of \(\pi_n^{(\varepsilon)}\). As previously discussed, the higher is \(\mathcal{R}_n(\mathcal{G})\) the lower are the
guarantees that $\mathcal{D}_\theta(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}})$ is close enough to $\mathcal{D}_\theta(\mu_\theta, \mu^*)$ in the large data limit. In fact, recalling Section 2, when $\mathcal{R}_n(\hat{\theta})$ does not vanish with $n$ there are no guarantees that the convergence of $\mathcal{D}_\theta(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}})$ to $\mathcal{D}_\theta(\mu_\theta, \mu^*)$ assumed in the current literature is verified. Nonetheless, the inequalities in Lemma 2.1 still yield informative results which correctly translate these arguments into an inflation of the superset that contains the support of the limiting ABC posterior, and in a more conservative choice for the threshold $\varepsilon$ to ensure that $\pi_n^{(\varepsilon)}$ is well–defined for large $n$. Notice that in Theorem 3.1 the dependence on the dimensions $d$ and $p$ is embedded within $\mathcal{D}_\theta(\mu_\theta, \mu^*)$; see Equation (6) in the Appendix of Chérif–Abdellatif and Alquier (2022) for an example making such a dependence more explicit.

Recall that, for $\pi_n^{(\varepsilon)}$ to be well–defined in the limit (with a fixed threshold $\varepsilon > 0$), the acceptance probability $p_n$ needs to be bounded away from zero. According to Theorem 3.1, as $n \to \infty$ this is guaranteed whenever $\varepsilon > \bar{\varepsilon} + c_{\bar{\theta}}$. For instance, recalling Section 2, in misspecified models $\mathcal{D}_\theta(\mu_\theta, \mu^*) \geq \varepsilon^*$. As a consequence, $\varepsilon > c_{\bar{\theta}}$ is not sufficient to guarantee that $\pi\{\theta : \mathcal{D}_\theta(\mu_\theta, \mu^*) \leq \varepsilon - c_{\bar{\theta}}\}$ in (3.1) is bounded away from zero. In fact, when $c_{\bar{\theta}} < \varepsilon < c_{\bar{\theta}} + \varepsilon^*$ the lower bound in (3.1) is exactly 0. For this event to happen, $\varepsilon - c_{\bar{\theta}}$ needs to define a ball with radius not lower than the one to which the prior is already guaranteed to assign strictly positive mass. This implies $\varepsilon > \bar{\varepsilon} + c_{\bar{\theta}}$, with $c_{\bar{\theta}} = 4 \limsup \mathcal{R}_n(\hat{\theta})$.

The results in Theorem 3.1 also clarify that, when $\mathcal{R}_n(\hat{\theta}) \to 0$ as $n \to \infty$ (i.e., Assumption (IV) is satisfied), then, for any $\varepsilon > \bar{\varepsilon}$, it holds that

$$\pi_n^{(\varepsilon)}(\theta) \to \pi(\theta \mid \mathcal{D}_\theta(\mu_\theta, \mu^*) \leq \varepsilon) \propto \pi(\theta) \mathbb{1}\{\mathcal{D}_\theta(\mu_\theta, \mu^*) \leq \varepsilon\},$$

almost surely with respect to $y_{1:n}$ i.i.d. $\sim \mu^*$, as $n \to \infty$.

Corollary 3.1 clarifies that to obtain within the whole IPS class a convergence result in line with those stated in the current theory it is not necessary to assume that $\mathcal{D}_\theta(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}})$ converges almost surely to $\mathcal{D}_\theta(\mu_\theta, \mu^*)$. In fact, it is sufficient to verify that $\mathcal{R}_n(\hat{\theta}) \to 0$ as $n \to \infty$. Crucially, this condition is imposed directly on the selected discrepancy within the IPS class, rather than on the model or the underlying data generating process and, therefore, can be constructively verified for all the discrepancies in Examples 2.1–2.3. For instance, while Jiang et al. (2018) is unable to provide conclusive guidelines on whether (3.2) holds under Wasserstein distance and MMD, our Corollary 3.1 can be directly applied to prove convergence for both these divergences, leveraging the results and discussions in Section 3.3.

Unlike current theory, Corollary 3.1 accounts also for misspecified models. In fact, the limiting pseudo–posterior in (3.2) is well–defined only when $\varepsilon > \bar{\varepsilon}$. For example, when $\mu^*$...
is not within \( \{ \mu_\theta : \theta \in \Theta \subseteq \mathbb{R}^p \} \), then \( D_\delta (\mu_\theta, \mu^*) \geq \varepsilon^* \). Hence, for any \( \varepsilon \leq \varepsilon^* \), we have that \( \mathbb{I} \{ D_\delta (\mu_\theta, \mu^*) \leq \varepsilon \} = 0 \) for any \( \theta \in \Theta \) and the limiting pseudo–posterior is not well–defined.

Motivated by the convergence result in Corollary 3.1, the theory in Section 3.2 provides an in–depth study of the concentration properties for the ABC posterior when also the tolerance threshold progressively shrinks. As is clear from (3.2), by employing a vanishing threshold might guarantee that, in the limit, the ABC posterior \( \pi_n(\varepsilon) \) concentrates all its mass around \( \mu^* \), in correctly specified settings, or around the distribution \( \mu_\theta^* \) closest to \( \mu^* \), among those in \( \{ \mu_\theta : \theta \in \Theta \subseteq \mathbb{R}^p \} \) for misspecified regimes. These results and the associated rates of concentration are provided in Section 3.2 by leveraging, again, Rademacher complexity theory.

3.2. Concentration as \( \varepsilon_n \to \varepsilon^* \) and \( n \to \infty \). Theorem 3.2 states our main concentration result. As for related ABC theory (see e.g., Proposition 3 in Bernton et al. (2019) and Nguyen et al. (2020)), we also leverage the triangle inequality (2.1). However, we crucially avoid pre–assuming convergence of \( D_\delta (\hat{\mu}_{y_1:n}, \mu^*) \), and we do not rely on concentration inequalities for \( D_\delta (\hat{\mu}_{z_1:n}, \mu_\theta) \) regulated by non–explicit sequences of control functions \( c_n(\theta) f_n(\varepsilon_n) \) (Bernton et al., 2019) and \( c_n(\theta) s_n(\varepsilon_n) \) (Nguyen et al., 2020). Rather, we leverage Lemma 2.1 under the proposed Rademacher complexity framework to obtain more direct and informative results, that are also guaranteed to hold uniformly over \( \mathcal{P}(\mathcal{Y}) \). Indeed, Lemma 2.1 ensures that both \( D_\delta (\hat{\mu}_{y_1:n}, \mu^*) \) and \( D_\delta (\hat{\mu}_{z_1:n}, \mu_\theta) \) exceed \( 2\mathcal{R}_n(\varepsilon) \) with a vanishing probability. Therefore, when \( \mathcal{R}_n(\varepsilon) \to 0 \), we have that \( D_\delta (\hat{\mu}_{z_1:n}, \hat{\mu}_{y_1:n}) \approx D_\delta (\mu_\theta, \mu^*) \), by combining the triangle inequality in (2.1) with

\[
D_\delta (\mu_\theta, \mu^*) \geq -D_\delta (\hat{\mu}_{z_1:n}, \mu_\theta) + D_\delta (\hat{\mu}_{z_1:n}, \hat{\mu}_{y_1:n}) - D_\delta (\hat{\mu}_{y_1:n}, \mu^*).
\]

This means that, if \( D_\delta (\hat{\mu}_{z_1:n}, \hat{\mu}_{y_1:n}) \) is small, then \( D_\delta (\mu_\theta, \mu^*) \) is also small with a high probability. This clarifies the importance of Assumption (IV), which is further supported by the fact that, if \( \mathcal{R}_n(\varepsilon) \) does not shrink towards zero, by the second inequality in Lemma 2.1, the two empirical measures \( \hat{\mu}_{y_1:n} \) and \( \hat{\mu}_{z_1:n} \) do not converge to \( \mu^* \) and \( \mu_\theta \), respectively, and hence there is no guarantee that a small discrepancy \( D_\delta (\hat{\mu}_{z_1:n}, \hat{\mu}_{y_1:n}) \) would necessarily imply a vanishing \( D_\delta (\mu_\theta, \mu^*) \).

**THEOREM 3.2.** Let \( \bar{\varepsilon}_n \to 0 \) when \( n \to \infty \), with \( n\bar{\varepsilon}_n^2 \to \infty \) and \( \bar{\varepsilon}_n / \mathcal{R}_n(\varepsilon) \to \infty \). Then, if \( D_\delta \) is from the IPS class in Definition 2.1 and Assumptions (I)–(IV) hold, the ABC posterior with threshold \( \varepsilon_n = \varepsilon^* + \bar{\varepsilon}_n \) satisfies

\[
\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)} \left( \left\{ \theta : D_\delta (\mu_\theta, \mu^*) > \varepsilon^* + \frac{4\bar{\varepsilon}_n}{3} + 2\mathcal{R}_n(\varepsilon) + \left[ \frac{2b^2}{n} \log n \frac{\varepsilon^*}{\bar{\varepsilon}_n^2} \right]^{1/2} \right\} \right) \leq \frac{2 \cdot 3^L}{c_\pi n},
\]

with \( \mathbb{P}_{y_1:n} \)–probability going to 1 as \( n \to \infty \).

The proof of Theorem 3.2 is provided in the Supplementary Material and follows similar arguments considered to establish the ABC concentration results in Bernton et al. (2019) and Nguyen et al. (2020), which, in turn, extend those in Frazier et al. (2018). However, as mentioned above, those proofs are specific to a single discrepancy, preassume the convergence of
\(D_{\hat{\theta}}(\hat{\mu}_{y_{1:n}}, \mu^*)\), and rely on concentration inequalities for \(D_{\hat{\theta}}(\hat{\mu}_{z_{1:n}}, \mu_\theta)\) that depend on non–explicit sequences of control functions. Theorem 3.2 overcomes these issues and proves a unified theory based on the single concentration inequality within Lemma 2.1. This yields a number of technical differences in the proof and, more importantly, it introduces a novel mathematical perspective for the analysis of concentration properties of discrepancy–based ABC posteriors. Such a perspective facilitates the derivation of novel theory and proofs (see e.g., Theorem 3.1), along with the extension of current theoretical results to more challenging settings (see Section 6).

Notice that, within Theorem 3.2, the constant \(b\) can be typically set equal to 1 either by definition or upon normalization of the class of \(b\)–uniformly bounded functions. Moreover, as clarified within the Supplementary Material, Theorem 3.2 also holds when replacing \(n/\bar{\varepsilon}_n^L\) and \(c_\pi M_n\) with \(M_n/\bar{\varepsilon}_n^L\) and \(c_\pi M_n\), respectively, for any sequence \(M_n > 1\). Nonetheless, to ensure concentration, it suffices to let \(M_n = n\). In such a case, the quantities \(4\bar{\varepsilon}_n/3, 2R_n(\tilde{\varepsilon}), [(2b^2/n) \log(n/\bar{\varepsilon}_n^L)]^{1/2}\) and \(2 \cdot 3^L/(c_\pi n)\) converge to 0 as \(n \to \infty\), under the settings of Theorem 3.2. This implies that the ABC posterior asymptotically concentrates around those \(\theta\) values yielding a \(\mu_\theta\) within discrepancy \(\varepsilon^*\) from \(\mu^*\).

In contrast to state–of–the–art theory, this concentration result is guaranteed to hold uniformly over \(\mathcal{P}(\mathcal{Y})\), and replaces the currently–employed non–explicit control functions with a known and well–studied quantity, i.e., the Rademacher complexity. Notice that, although \(\mathcal{R}_n(\tilde{\varepsilon})\), which is specific to each discrepancy \(D_{\tilde{\theta}}\) and plays a fundamental role in controlling the rate of concentration of the ABC posterior. In particular, to make the bound as tight as possible, we must choose an \(\bar{\varepsilon}_n\) such that \(4\bar{\varepsilon}_n/3\) and \([(2b^2/n) \log(n/\bar{\varepsilon}_n^L)]^{1/2}\) are of the same order. By neglecting all the terms in \(\log \log n\), such a choice leads to setting \(\bar{\varepsilon}_n\) of the order \([\log(n)/n]^{1/2}\). In this case, the constraint \(\bar{\varepsilon}_n/\mathcal{R}_n(\tilde{\varepsilon}) \to \infty\) may not be satisfied, thereby requiring a larger \(\bar{\varepsilon}_n\), such as \(\mathcal{R}_n(\tilde{\varepsilon}) \log \log(n)\). Summarizing, when \(\bar{\varepsilon}_n = \max\{[\log(n)/n]^{1/2}, \mathcal{R}_n(\tilde{\varepsilon}) \log \log(n)\}\), it follows, under the conditions of Theorem 3.2, that

\[
\pi_n^{(\varepsilon^* + \bar{\varepsilon})}(\{\theta : D_{\tilde{\theta}}(\mu_\theta, \mu^*) > \varepsilon^* + \mathcal{O}(\bar{\varepsilon}_n)\}) \leq 2 \cdot 3^L/(c_\pi n).
\]

Notice that the conditions of interest \(n\bar{\varepsilon}_n^2 \to \infty\) and \(\bar{\varepsilon}_n/\mathcal{R}_n(\tilde{\varepsilon}) \to \infty\) do not allow to set \(\bar{\varepsilon}_n < 1/\sqrt{n}\). Although this regime is of interest, we are not aware of explicit results in the discrepancy–based ABC literature for such a setting. In fact, available studies rely on similar restrictions for some function \(f_n(\bar{\varepsilon}_n)\) which is not made explicit except for specific examples that still point toward setting \(\bar{\varepsilon}_n = [\log(n)/n]^{1/2} > 1/\sqrt{n}\) (e.g., Bernton et al., 2019, Supplementary Materials). Faster rates for the ABC threshold have been considered by Li and Fearnhead (2018) within the context of summary–based ABC, but with a substantially different theoretical focus relative to the one considered here. Recalling the previous discussion, it shall be also emphasized that \(\bar{\varepsilon}_n < 1/\sqrt{n}\) would not yield a faster concentration rate in Theorem 3.2 because of the terms \(2\mathcal{R}_n(\tilde{\varepsilon})\) and \([(2b^2/n) \log(n/\bar{\varepsilon}_n^L)]^{1/2}\) in the bound.

Theorem 3.2 holds under both well–specified and misspecified models. In the former case, \(\mu^* = \mu_{\theta^*}\) for some \(\theta^* \in \Theta\). Therefore, \(\varepsilon^* = 0\) and the ABC posterior concentrates around those \(\theta\) yielding \(\mu_\theta = \mu^*\). Conversely, when the model is misspecified, the ABC posterior
concentrates on those $\theta$ yielding a $\mu_\theta$ within discrepancy $\varepsilon^*$ from $\mu^*$. Since $\varepsilon^*$ is the lowest attainable discrepancy between $\mu_\theta$ and $\mu^*$, this implies that the ABC posterior will concentrate on those $\theta$ such that $D_\delta(\mu_\theta, \mu^*) = \varepsilon^*$. Example A provides an explicit bound for $\varepsilon^*$ in a simple yet noteworthy class of misspecified models.

**Example A.** (Huber contamination model). In this model, with probability $1 - \alpha_n$, the data are from a distribution $\mu_\theta^*$ belonging to the assumed model $\{\mu_\theta : \theta \in \Theta\}$, while with probability $\alpha_n$ arise from the contaminating distribution $\mu_C$. Therefore, the data generating process is $\mu^* = (1 - \alpha_n)\mu_\theta^* + \alpha_n\mu_C$, with $\alpha_n \in [0, 1]$ controlling the amount of contamination. In such a context, Definition 2.1 and Assumption (III) imply

$$\varepsilon^* \leq D_\delta(\mu_\theta^*, \mu^*) = D_\delta(\mu_\theta^*, (1 - \alpha_n)\mu_\theta^* + \alpha_n\mu_C) = \alpha_n D_\delta(\mu_\theta^*, \mu_C) \leq 2b\alpha_n.$$  

By plugging this bound into Theorem 3.2, we can obtain the same statement with $\varepsilon^*$ replaced by $2b\alpha_n$, where $\alpha_n \in [0, 1)$ is the amount of contamination. This means that the ABC posterior asymptotically contracts in a neighborhood of the contaminated model $\mu^*$ of radius at most $2b\alpha_n$. The previous bound on $\varepsilon^*$ combined with a triangle inequality also implies

$$D_\delta(\mu_\theta, \mu^*) \geq D_\delta(\mu_\theta, \mu_\theta^*) - D_\delta(\mu_\theta^*, \mu^*) \geq D_\delta(\mu_\theta, \mu_\theta^*) - 2b\alpha_n.$$  

Therefore, by replacing $D_\delta(\mu_\theta, \mu^*)$ with $D_\delta(\mu_\theta, \mu_\theta^*)$ yields a similar statement as in Theorem 3.2 which guarantees concentration also around the uncontaminated model $\mu_\theta^*$, with $\varepsilon^*$ replaced by $2b\alpha_n + 2b\alpha_n = 4b\alpha_n$, thus ensuring robustness to Huber contamination.

Before concluding our analysis of the concentration properties of discrepancy–based ABC posteriors under the general IPS class, it is important to notice that Theorem 3.2 is stated for neighborhoods in the space of distributions. Although such a perspective is in line with the overarching focus of current theory for discrepancy–based ABC (e.g., Jiang et al., 2018; Bernton et al., 2019; Nguyen et al., 2020; Frazier, 2020; Fujisawa et al., 2021), it shall be emphasized that similar results can be also derived in the space of parameters. To this end, it suffices to adapt Corollary 1 in Bernton et al. (2019) to our general framework, under the same additional assumptions, which are adapted below to the whole IPS class.

(V) The minimizer $\theta^*$ of $D_\delta(\mu_\theta, \mu^*)$ exists and is well separated, meaning that for any $\delta > 0$ there is a $\delta' > 0$ such that $\inf_{\theta \in \Theta : \d(\theta, \theta^*) > \delta} D_\delta(\mu_\theta, \mu^*) > D_\delta(\mu_\theta^*, \mu^*) + \delta'$;

(VI) The parameters $\theta$ are identifiable, and there exist positive constants $K > 0$, $\nu > 0$ and an open neighborhood $U \subset \Theta$ of $\theta^*$ such that, for any $\theta \in U$, it holds that $\d(\theta, \theta^*) \leq K [D_\delta(\mu_\theta, \mu^*) - \varepsilon^*]^{-\nu}$.

Assumptions (V)–(VI) essentially require that the parameters in $\theta$ are identifiable, sufficiently well–separated, and that the distance between parameter values has some reasonable correspondence with the discrepancy among the associated distributions. Although these two assumptions introduce a condition on the model, it shall be emphasized that (V)–(VI)...
are not specific to our framework (e.g., Frazier et al., 2018; Bernton et al., 2019; Frazier, 2020). On the contrary, these identifiability conditions are arguably customary and minimal requirements in parameter inference. Moreover, these two assumptions have been checked in Chérief-Abdellatif and Alquier (2022) for MMD and in Bernton et al. (2019) for Wasserstein distance; arguably the two most remarkable examples of IPS employed in the ABC context. Under (V) and (VI), it is possible to state Corollary 3.2.

**Corollary 3.2.** Assume (I)–(IV) along with (V)–(VI), and that \( \mathcal{D}_\varnothing \) denotes a discrepancy within the IPS class in Definition 2.1. Moreover, take \( \bar{\varepsilon}_n \to 0 \) as \( n \to \infty \), with \( n\bar{\varepsilon}_n^2 \to \infty \) and \( \bar{\varepsilon}_n / \mathfrak{R}_n(\varnothing) \to \infty \). Then, the ABC posterior with threshold \( \varepsilon_n = \varepsilon^* + \bar{\varepsilon}_n \) satisfies

\[
\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)} \left( \left\{ \theta : d(\theta, \theta^*) > K \left[ \frac{4\bar{\varepsilon}_n}{3} + 2\mathfrak{R}_n(\varnothing) + \left[ \frac{2b^2}{n} \log \frac{n}{\bar{\varepsilon}_n^2} \right]^{1/2} \right] \nu \right\} \right) \leq \frac{2 \cdot 3^L}{c_n},
\]

with \( \mathbb{P}_{y_{1:n}} \)–probability going to 1 as \( n \to \infty \).

As for Theorem 3.2, also Corollary 3.2 holds more generally when replacing both \( n/\bar{\varepsilon}_n^L \) and \( c_n \) with \( M_n/\bar{\varepsilon}_n^L \) and \( c_n M_n \), respectively, for any \( M_n > 1 \). The proof of Corollary 3.2 follows directly from Theorem 3.2 and Assumptions (V)–(VI), thereby allowing to inherit the previous discussion after Theorem 3.2, also when the concentration is measured directly within the parameter space via \( d(\theta, \theta^*) \). For instance, when \( d(\theta, \theta') \) is the Euclidean distance and \( \nu = 1 \), this implies that whenever \( \mathfrak{R}_n(\varnothing) = O(n^{-1/2}) \) the contraction rate will be in the order of \( O(\left[ \log(n)/n \right]^{1/2}) \), which is the expected rate in parametric models.

### 3.3. Validity of the assumptions.

The main theorems in Section 3.1 and 3.2 — i.e., Theorems 3.1–3.2 — leverage Assumptions (I)–(IV). As anticipated within Section 3, condition (I) is useful to formally clarify the range of applicability along with the possible limitations of current existence theory even in simple i.i.d. settings, and it will be relaxed in Section 6 to study uniform convergence and concentration of discrepancy–based ABC posteriors also beyond the i.i.d. context. Assumption (II) is not specific to our framework. Rather, it defines a standard minimal requirement routinely employed in Bayesian asymptotics and ABC theory (e.g., Bernton et al., 2019; Nguyen et al., 2020; Frazier, 2020). Conversely, Assumptions (III)–(IV) provide sufficient conditions on the discrepancy \( \mathcal{D}_\varnothing \) in the IPS class to obtain guarantees of uniform convergence and concentrations under the proposed Rademacher complexity perspective. These two conditions essentially replace the pre–assumed convergence for \( \mathcal{D}_\varnothing(\hat{\mu}_{y_{1:n}}, \mu^*) \) and the non–explicit bounds within the concentration inequalities for \( \mathcal{D}_\varnothing(\hat{\mu}_{z_{1:n}}, \mu_0) \) leveraged by current discrepancy–specific theory. While these latter assumptions may implicitly require regularity conditions on \( \{ \mu_\theta : \theta \in \Theta \subset \mathbb{R}^p \} \) and on the unknown data generating process \( \mu^* \), Assumptions (III)–(IV) are made directly on the user–selected discrepancy \( \mathcal{D}_\varnothing \) and allow to state results that hold uniformly over \( \mathcal{P}(Y) \).

Notice that, to derive these uniform convergence and concentration properties, an arguably minimal sufficient requirement is that \( \varnothing \) is a uniform Glivenko–Cantelli class (see e.g., Dud-
ley et al., 1991, Proposition 10), i.e.,

$$\sup_{\mu \in \mathcal{P}(\mathcal{Y})} \left| \frac{1}{n} \sum_{i=1}^{n} f(x_i) - \int f \, d\mu \right| = \sup_{\mu \in \mathcal{P}(\mathcal{Y})} \mathcal{D}(\hat{\mu}_{x_1:n}, \mu) \to 0$$

in $\mathbb{P}_{x_1:n}$ probability as $n \to \infty$. In fact, as already discussed in Section 2, the lack of uniform convergence guarantees for $\mathcal{D}(\hat{\mu}_{y_1:n}, \mu^*)$ and $\mathcal{D}(\hat{\mu}_{z_1:n}, \mu_\theta)$ would fail to ensure that the control established by the ABC threshold on $\mathcal{D}(\hat{\mu}_{z_1:n}, \hat{\mu}_{y_1:n})$ would directly apply, asymptotically, to $\mathcal{D}(\mu_\theta, \mu^*)$, uniformly over $\mathcal{P}(\mathcal{Y})$. Interestingly, although the uniform Glivenko–Cantelli property seems more general and weaker than (III)–(IV), by the upper and lower bounds in Lemma 2.1 along with the subsequent discussion, it follows immediately that (3.3) is exactly equivalent to (IV), under (III); see also Chapter 4 in Wainwright (2019). Regarding (III), notice that, as discussed in Dudley et al. (1991), if $\mathcal{F}$ is a uniform Glivenko–Cantelli class, $\mathcal{F} := \{ \bar{f} := f - \inf_x f(x), f \in \mathcal{F} \}$ is uniformly bounded and $\mathcal{D}(\mathcal{F}) = \mathcal{D}(\mathcal{F})$. Indeed, for any $f \in \mathcal{F}$, $\int f \, d\mu_1 - \int f \, d\mu_2 = \int [f - \inf_x f(x)] \, d\mu_1 - \int [f(x) - \inf_x f(x)] \, d\mu_1 = \int f \, d\mu_1 - \int f \, d\mu_2$. Thus, when the uniform Glivenko–Cantelli property in (3.3) holds, it is always possible change the definition of $\mathcal{F}$, without affecting $\mathcal{D}(\mathcal{F})$, in order to ensure that (III) is verified, and hence also (IV) as a consequence of the above discussion.

The above connection clarifies that Assumptions (III) and (IV) are arguably at the core of the uniform convergence and concentration properties of discrepancy–based ABC posteriors. Moreover, although (3.3) is inherently related to (III) and (IV), such a uniform Glivenko–Cantelli property only states a convergence in probability result which can be crucially refined through the notion of Rademacher complexity under the more precise concentration inequalities in Lemma 2.1. Recalling the theoretical results in Sections 3.1–3.2, this allows not only to state convergence and concentration of specific ABC posteriors, but also to clarify the factors governing these limiting properties and possibly derive the associated rates.

As clarified in Examples 3.1–3.3, Assumptions (III)–(IV) can be generally verified for the key IPS discrepancies presented in Examples 2.1–2.3 leveraging known upper bounds on the Rademacher complexity, along with connections between such a measure and other well–studied quantities in statistical learning theory; see, in particular, Chapter 4.3 of Wainwright (2019) for an overview of several useful techniques for upper–bounding the Rademacher complexity via, e.g., the notion of polynomial discrimination and the VC dimension. The validity of (III)–(IV) for other instances of IPS (i.e., total variation distance and Kolmogorov–Smirnov distance) is discussed in Examples A.2.1–A.2.2 within the Supplementary Materials. Notice that, albeit interesting, these two discrepancies are less common in ABC implementations relative to Wasserstein distance, MMD and summary–based distances.

Example 3.1. (Wasserstein distance). When $\mathcal{Y}$ is a bounded subset of $\mathbb{R}^d$, Assumptions (III)–(IV) hold without further constraints under the Wasserstein distance. In particular, (III) follows immediately from the definition $\mathcal{F} = \{ f : \|f\|_L \leq 1 \}$, together with the fact that the diameter of $\mathcal{Y}$ is finite in this case (see e.g., Villani, 2021, Remark 1.15). Assumption (IV) is instead a direct consequence of the bounds in Sriperumbudur et al. (2012). Although
it would be desirable to remove such a constraint on $\mathcal{Y}$, it shall be emphasized that this condition is ubiquitous in state–of–the–art concentration results of empirical measures, under the Wasserstein distance, that are guaranteed to hold uniformly over $\mathcal{P}(\mathcal{Y})$ (see e.g., Talagrand, 1994; Sriperumbudur et al., 2012; Ramdas et al., 2017; Weed & Bach, 2019). One possibility to preserve (III) and (IV) under the Wasserstein distance beyond bounded $\mathcal{Y}$ is to consider a variable transformation via a monotone function $g(\cdot)$ (e.g., logistic transform) mapping from $\mathcal{Y} = \mathbb{R}^d$ to a bounded subset of $\mathbb{R}^d$. In the original unbounded $\mathcal{Y}$, this transformation induces a Wasserstein distance based on a bounded $\bar{\rho}(x, x') = \rho(g(x), g(x'))$. As such, when $\mathcal{Y}$ is not a bounded subset of $\mathbb{R}^d$, defining $\mathfrak{F}$ as the class of Lipschitz functions with respect to $\bar{\rho}(x, x') = \rho(g(x), g(x'))$ satisfies (III) and (IV). This direction requires, however, care and further research on how $g(\cdot)$ affects the properties of the induced discrepancy.

**Example 3.2. (MMD).** The properties of MMD inherently depend on the selected kernel $k(\cdot, \cdot)$. This is evident from the inequalities $\mathfrak{R}_{\mu,n}(\mathfrak{F}) \leq \frac{\mathbb{E}_x k(x, x)/n}{1/2}$, with $x \sim \mu$ (e.g., Lemma 22 in Bartlett & Mendelson, 2002), and $|f(x)| \leq [k(x, x)]^{1/2} ||f||_H$ for every $x \in \mathcal{Y}$ (e.g., equation 16 in Hofmann et al., 2008). By these two inequalities, all bounded kernels, including, e.g., the commonly–implemented Gaussian $\exp(-\|x - x'\|^2/\sigma^2)$ and the Laplace $\exp(-\|x - x'\|/\sigma)$ ones, ensure that Assumptions (III) and (IV) are always met without requiring additional regularity conditions on $\mu \in \mathcal{P}(\mathcal{Y})$, nor constraints on $\mathcal{Y}$. Instead, when $k(\cdot, \cdot)$ is unbounded, the inequality $\mathfrak{R}_{\mu,n}(\mathfrak{F}) \leq \frac{\mathbb{E}_x k(x, x)/n}{1/2}$ is only informative for those $\mu$ such that $\mathbb{E}_x k(x, x) < \infty$, with $x \sim \mu$, whereas the bound on $|f(x)|$ does not hold in general, unless additional conditions are made. Due to the relevance of these MMD instances and the direct connections with the classical summary–based ABC implementations leveraging unbounded summaries, Proposition 4.1 in Section 4 derives specific theory proving that concentration results similar to those in Theorem 3.2 can be derived, under different assumptions, also for unbounded kernels.

**Example 3.3. (Summary–based distance).** As discussed in Example 2.3, classical ABC implementations relying on a finite set of summaries $f_1, \ldots, f_K$ with $K < \infty$, can be seen as a special case of MMD by letting $f(x) = [f_1(x), \ldots, f_K(x)]$ and $k(x, x') = \langle f(x), f(x') \rangle$. Leveraging this bridge and the results for MMD discussed in Example 3.2, it is clear that, if $\sup_{x \in \mathcal{Y}} \langle f(x), f(x) \rangle$ is finite — i.e., the induced kernel is bounded — then (III)–(IV) are satisfied without requiring regularity conditions for $\mu$ or constraints on $\mathcal{Y}$. While this result clarifies that ABC with bounded summaries achieves uniform convergence and concentration, classical ABC implementations often employ unbounded summaries, such as moments, i.e., $f(x) = [x, x^2, \ldots, x^K]$. In this case (III) is not satisfied. In fact, recalling again Example 2.3, such a setting is a special case of MMD with unbounded kernel and, hence, lacks guarantees that (III)–(IV) hold, unless further conditions are imposed, e.g., on $\mathcal{Y}$. Nonetheless, this connection also clarifies that the concentration theory we derive in Proposition 4.1 for MMD with unbounded kernels directly applies to classical ABC with unbounded summaries.

Examples 3.1–3.3 show that (III)–(IV) can be realistically verified for the key instances of IPS discrepancies presented in Section 2, and generally hold under either no additional
conditions or for suitable constraints on \( \mathcal{Y} \) which can be directly checked simply on the basis of the support of the data analyzed. From a practical perspective, this is an important gain relative to the need of verifying more sophisticated regularity conditions on the assumed model and on the unknown data generating process. Notice that the boundedness condition on \( \mathcal{Y} \) in Example 3.1 interestingly relates to Assumptions 1 and 2 of Bernton et al. (2019) which have been verified when \( \mathcal{Y} \) is a bounded subset of \( \mathbb{R}^d \) by, e.g., Weed & Bach (2019). In this context, our Rademacher complexity perspective further refines the important results in Bernton et al. (2019) by clarifying that it is possible to derive convergence and concentration results for Wasserstein–ABC that are regulated by a known complexity measure and hold uniformly over the space of probability measures defined on a bounded \( \mathcal{Y} \).

It shall be emphasized that an alternative possibility to verify Assumptions 1–2 in Bernton et al. (2019) is to leverage the results in Fournier and Guillin (2015) who replace the boundedness condition in Weed & Bach (2019) with assumptions on the existence of exponential moments; see also the supplementary materials in Bernton et al. (2019). A similar direction within our Rademacher complexity framework would be to assume that the class of functions \( \mathcal{F} \) defining the Wasserstein distance admits a uniform Glivenko–Cantelli property over a subset \( \bar{\mathcal{P}}(\mathcal{Y}) \) of \( \mathcal{P}(\mathcal{Y}) \) that comprises probability measures on \( \mathcal{Y} \) meeting suitable regularity conditions. Recalling the previous discussion on the connection among (3.3) and our assumptions, this would imply a relaxation of (III)–(IV) allowing the theory in Section 3.1 and 3.2 to still hold for statistical models and data generating processes belonging to \( \bar{\mathcal{P}}(\mathcal{Y}) \).

Such a relaxation is general and applies to every IPS discrepancy. Nonetheless, it requires checking that \( \{\mu_\theta : \theta \in \Theta \subseteq \mathbb{R}^p \} \) and \( \mu^* \) belong to \( \bar{\mathcal{P}}(\mathcal{Y}) \), which can be difficult since, again, \( \mu^* \) is generally not known in practice. Conversely, when (III)–(IV) hold without constraints on \( \mathcal{P}(\mathcal{Y}) \) (e.g., in MMD with bounded kernels and Wasserstein distance in bounded subsets of \( \mathbb{R}^d \)) the convergence and concentration results in Section 3.1–3.2 are guaranteed without the need to worry about the peculiar properties of the assumed model and of the generally–unknown data generating process.

4. Asymptotic properties of ABC with maximum mean discrepancy. As discussed in Sections 1–3, MMD stands out as a prominent example of discrepancy within summary–free ABC. Nonetheless, an in–depth and comprehensive study on the limiting properties of MMD–ABC is still lacking. In fact, no theory is available in the original article proposing MMD–ABC methods (Park et al., 2016), while convergence in the fixed \( \varepsilon_n = \varepsilon \) and \( n \to \infty \) regime is explored in Jiang et al. (2018) but without conclusive results. Nguyen et al. (2020) study convergence and concentration of ABC with the energy distance in both fixed and vanishing \( \varepsilon_n \) settings. Recalling Example 2.2, the direct correspondence between MMD and the energy distance would allow to translate these results within the MMD framework. However, as highlighted by the same authors, the theory derived relies on difficult–to–verify existence assumptions which yield bounds depending on control functions that are not made explicit.

Leveraging Theorems 3.1–3.2 and Corollary 3.1 along with the available upper bounds on the Rademacher complexity of MMD — see Example 3.2 — Corollaries 4.1–4.2 substantially refine and expand available knowledge on the limiting properties of MMD–ABC with
routinely–implemented bounded kernels. Crucially, these discrepancies automatically satisfy (III)–(IV) without additional constraints on the model or on the data generating process. Notice that Corollaries 4.1–4.2 also hold for summary–based distances with bounded summaries as a direct consequence of the discussion in Examples 2.3 and 3.3.

**Corollary 4.1.** Consider the MMD with a bounded kernel $k(\cdot, \cdot)$ defined on $\mathbb{R}^d$, where $|k(x, x')| \leq 1$ for any $x \in \mathbb{R}^d$. Then, under (I), the acceptance probability $p_n$ of the rejection–based ABC routine employing discrepancy $D_{\text{MMD}}$ satisfies

$$p_n \to \pi\{\theta : D_{\text{MMD}}(\mu_\theta, \mu^*) \leq \varepsilon\},$$

almost surely with respect to $y_{1:n} \stackrel{\text{i.i.d.}}{\sim} \mu^*$, as $n \to \infty$, for any $\mu_\theta, \mu^* \in \mathcal{P}(\mathcal{Y})$. Moreover, let $\bar{\varepsilon}$ be defined as in Theorem 3.1. Then, for any $\varepsilon > \bar{\varepsilon}$, it holds that

$$\pi_n^{(\varepsilon)}(\theta) \to \pi(\theta \mid D_{\text{MMD}}(\mu_\theta, \mu^*) \leq \varepsilon) \propto \pi(\theta) \mathbb{I}\{D_{\text{MMD}}(\mu_\theta, \mu^*) \leq \varepsilon\},$$

almost surely with respect to $y_{1:n} \stackrel{\text{i.i.d.}}{\sim} \mu^*$, as $n \to \infty$.

The above result applies, e.g., to routinely–implemented Gaussian $k(x, x') = \exp(-\|x - x'\|^2/\sigma^2)$ and Laplace $k(x, x') = \exp(-\|x - x'\|/\sigma)$ kernels on $\mathbb{R}^d$, which are both bounded by 1, thereby implying $\|f\|_\infty \leq 1$ and $\mathfrak{R}_n(\mathfrak{F}) \leq n^{-1/2}$. These results are also crucial to prove the concentration statement in Corollary 4.2 below.

**Corollary 4.2.** Consider the MMD with a bounded kernel $k(\cdot, \cdot)$ defined on $\mathbb{R}^d$, where $|k(x, x)| \leq 1$ for any $x \in \mathbb{R}^d$. Then, for any $\mu_\theta, \mu^* \in \mathcal{P}(\mathcal{Y})$, under (I)–(II) and the settings of Theorem 3.2, with $\bar{\varepsilon}_n = [\log(n)/n]^{1/2}$, we have that

$$\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)}(\{\theta : D_{\text{MMD}}(\mu_\theta, \mu^*) > \varepsilon^* + \left[\frac{10}{3} + (L + 2)^{1/2}\right] \cdot \left[\frac{\log n}{n}\right]^{1/2}\}) \leq \frac{2 \cdot 3^L}{c_\pi n},$$

with $\mathbb{P}_{y_{1:n}}$–probability going to 1 as $n \to \infty$.

Notice that $\mathfrak{R}_n(\mathfrak{F}) \leq n^{-1/2}$ implies $\mathfrak{R}_n(\mathfrak{F}) \log n \leq \log n / n^{1/2} \leq [\log n / n]^{1/2}$ for any $n \geq 1$, and hence, as a consequence of the previous discussion, the concentration rate is essentially minimized by setting $\bar{\varepsilon}_n = [\log(n)/n]^{1/2}$.

Corollaries 4.1–4.2 are effective examples of the potentials of Theorems 3.1–3.2 and Corollary 3.1, which can be readily specialized to any discrepancy in the IPS class. For instance, in the context of MMD with Gaussian and Laplace kernels, Corollary 4.2 ensures informative posterior concentration without requiring additional assumptions on $\mu_\theta$ or $\mu^*$. Similar results can be obtained for all IPS discrepancies as long as (III)–(IV) are satisfied and $\mathfrak{R}_n(\mathfrak{F})$ admits explicit upper bounds. For example, when $\mathcal{Y}$ is bounded, this is possible for the Wasserstein distance in Example 3.1, leveraging the upper bounds for $\mathfrak{R}_n(\mathfrak{F})$ derived by Sriperumbudur et al. (2012).

While (III) and (IV) hold for MMD with a bounded kernel without additional assumptions, the currently–available bounds on the Rademacher complexity ensure that MMD with an
unbounded kernel meets the above conditions only under specific models and data generating processes, even within the i.i.d. setting. In this context, it is however possible to revisit the results for the Wasserstein case in Proposition 3 of Bernton et al. (2019) under the new Rademacher complexity framework introduced in the present article. In particular, as shown in Proposition 4.1, under MMD with an unbounded kernel, the existence Assumptions 1 and 2 in Bernton et al. (2019) can be directly related to constructive conditions on the kernel, inherently related to our Assumption (IV). This in turn yields informative concentration inequalities that are reminiscent of those in Theorem 3.2 and Corollary 4.2. Notice that these inequalities also hold for summary–based ABC with routinely–used unbounded summaries (e.g., moments) as a direct consequence of the discussion in Example 3.3.

**Proposition 4.1.** Consider the MMD with an unbounded kernel $k(\cdot, \cdot)$ on $\mathbb{R}^d$. Assume (II) along with (A1) $E_y[k(y, y)] < \infty$, (A2) $\int_\Theta E_z[k(z, z)] \pi(d\theta) < \infty$, and (A3) there exist constants $\delta_0 > 0$ and $c_0 > 0$ such that $E_z[k(z, z)] < c_0$ for any $\theta$ satisfying $(E_{z,z'}[k(z, z')] − 2E_{z,y}[k(y, z)] + E_{y,y'}[k(y', y')])^{1/2} \leq \varepsilon^* + \delta_0$, where $z, z' \sim \mu_\theta$ and $y, y' \sim \mu^*$. Then, when $n \to \infty$ and $\bar{\varepsilon}_n \to 0$, for some $C \in (0, \infty)$ and any $M_n \in (0, \infty)$, it holds that

$$\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)}(\{\theta : D_{\text{MMD}}(\mu_\theta, \mu^*) > \varepsilon^* + \frac{4\bar{\varepsilon}_n}{3} + \left[\frac{M_n}{n\bar{\varepsilon}_n^2}\right]^{1/2}\}) \leq \frac{C}{M_n},$$

with $\mathbb{P}_{y_{1:n}}$–probability going to 1 as $n \to \infty$.

A popular example of unbounded kernel is provided by the polynomial one, which is defined as $k(x, x') = (1 + a \langle x, x' \rangle)^q$ for some integer $q \in \{2, 3, \ldots\}$, and constant $a > 0$. Under such a kernel, it can be easily shown that if $E_y[\|y\|^q] \leq \infty$, and $\theta \mapsto E_z(\|z\|^p)$ is $\pi$–integrable, then Assumptions (A1)–(A3) are satisfied. The latter conditions essentially require that the kernel has finite expectation under both $\mu^*$ and $\mu_\theta$ for suitable $\theta \in \Theta \subseteq \Theta$, and is uniformly bounded for those $\mu_\theta$ close to $\mu^*$. Recalling Example 2.2 and the bound $\mathfrak{R}_{\mu, n}(\Theta) \leq [E_{x \sim \mu} k(x, x)/n]^{1/2}$, (A1)–(A3) are inherently related to (IV), which however additionally requires that these expectations are finite for any $\mu \in \mathcal{P}(\mathcal{Y})$.

Recalling Proposition 4.1, a sensible setting for $\bar{\varepsilon}_n$ in the unbounded–kernel case would be $\bar{\varepsilon}_n = (M_n/n)^{1/(2 + L)}$. This yields

$$\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)}(\{\theta : D_{\text{MMD}}(\mu_\theta, \mu^*) > \varepsilon^* + (7/3)(M_n/n)^{1/(2 + L)}\}) \leq C/M_n,$$

which is essentially the tighter possible order of magnitude for the bound. In the unbounded–kernel setting, $M_n = n$ would not be suitable, but any $M_n \to \infty$ slower than $n$ can work; e.g., $M_n = n^{1/2}$ yields $\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)}(\{\theta : D_\Theta(\mu_\theta, \mu^*) > \varepsilon^* + (7/3)(1/n)^{1/(2L + 4)}\}) \leq C/n^{1/2}$.

5. **Illustrative simulation in i.i.d. settings.** Before extending the results in Section 3.1–3.2 to non–i.i.d. settings, let us illustrate the theory we derived in the i.i.d. context through a simple, yet insightful, simulation. Notice that several empirical studies have already compared the performance of summary–free ABC under different discrepancies and complex non–i.i.d. models. Recalling the thorough overview in Drovandi & Frazier (2022), all these
analyses clarify the practical feasibility of ABC based on a wide variety of discrepancies, including those within the IPS class, and in several i.i.d. and non–i.i.d. scenarios which require ABC procedures. This feasibility is also supported by recently–released softwares (see e.g., Dutta et al., 2021).

Rather than replicating already–available empirical comparisons on commonly–assessed benchmark examples, we complement here the current quantitative evidence on summary–free ABC performance by focusing on a misspecified and contaminated scenario, that clarifies the possible challenges in convergence and concentration encountered by ABC even in basic univariate i.i.d. models. Crucially, as clarified within Table 1 and in Figure 1, such a scenario also illustrates in practice an important consequence of the novel theoretical results in Sections 2–4. Namely that effective IPS discrepancies with guarantees of uniform convergence and concentration provide a safe and sensible choice in the absence of knowledge on the specific properties of the underlying data generating process.

Recalling Example A, we consider, in particular, an uncontaminated bivariate Student’s $t$ distribution $\mu_{\theta_0}$ with 3 degrees of freedom, mean vector $(1, 1)$, and dispersion matrix having entries $\sigma_{11} = \sigma_{22} = 1$ and $\sigma_{21} = \sigma_{21} = 0.5$. Such an uncontaminated data generating process is then perturbed with three different levels $\alpha_n = \alpha \in \{0.05, 0.10, 0.15\}$ of contamination from a Student’s $t$ distribution $\mu_C$ having the same parameters as $\mu_{\theta_0}$, except for the mean vector which is set to $(20, 20)$. As such, the data $y_{1:n}$ from $\mu^* = (1 - \alpha)\mu_{\theta_0} + \alpha\mu_C$ are obtained by sampling $n = 100$ pairs from the bivariate Student’s $t$ distribution $\mu_{\theta_0}$ and then replacing the $(100 \cdot \alpha)\%$ of these draws with a sample of size $(100 \cdot \alpha)\%$ from the contaminating Student’s $t$ distribution $\mu_C$. For Bayesian inference, we focus on the parameter $\theta \in \mathbb{R}$ defining the unknown location vector $[1, 1]^{\top}\theta$, and consider a misspecified bivariate Gaussian model $\mu_\theta$ with mean vector $[1, 1]^{\top}\theta$ and known covariance matrix coinciding with that of the uncontaminated Student’s $t$ data generating process. Such a choice is interesting in providing a model that is slightly misspecified even when the data are not contaminated. Notice that, although this model does not necessarily require an ABC approach to allow Bayesian inference, as discussed above the issues outlined in Table 1 and Figure 1 for certain discrepancies, even in this basic example, provide a useful empirical insight that complements those in the extensive quantitative studies already available in the literature for more complex settings.

In performing approximate Bayesian computation under the above model and for the different discrepancies of interest, we employ rejection–based ABC with $m = n$ and a $N(0, 1)$

|                | $\alpha = 0.05$ | $\alpha = 0.10$ | $\alpha = 0.15$ | time   |
|----------------|-----------------|-----------------|-----------------|--------|
| (IPS) MMD      | 0.024           | 0.027           | 0.031           | < 0.01”|
| (IPS) Wasserstein | 0.027       | 0.067           | 0.122           | < 0.01”|
| (IPS) summary (mean) | 0.841     | 2.648           | 2.835           | < 0.01”|
| (non–IPS) KL   | 0.073           | 0.076           | 0.077           | < 0.01”|
prior for $\theta$. Following standard practice in comparing discrepancies (Drovandi & Frazier, 2022), we specify a common budget of $T = 25,000$ simulations and define the ABC threshold in order to retain, for every discrepancy analyzed, the 1% (i.e., 250) values of $\theta$ that generated the synthetic data closest to the observed ones, under such a discrepancy. Although the theory in Section 3 can potentially guide the choice of the ABC threshold, similar guidelines are not yet available beyond the IPS class. Hence, to ensure a fair assessment across all discrepancies analyzed, we rely on the aforementioned recommended practice in comparing the different ABC implementations. As clarified in Table 1 and Figure 1, the discrepancies assessed are those most commonly used in ABC implementations, namely Wasserstein distance as in Example 2.1 (Bernton et al., 2019) — i.e., Wasserstein–1 — MMD with Gaussian kernel defined in Example 2.2 (Park et al., 2016; Nguyen et al., 2020), and a summary–based distance leveraging the sample mean as the summary statistics, see also Example 2.3. For comparison, we also consider a similarly–popular discrepancy that does not belong to the IPS class, i.e., the Kullback–Leibler divergence (Jiang et al., 2018). All these discrepancies can be effectively implemented in R, leveraging, in particular, the libraries transport and eummd. For MMD, the choice of the length–scale parameter $\sigma^2$ is based on the median heuristic (Gretton et al., 2022) automatically implemented in the function mmd($\cdot$).

Leveraging the samples from the ABC posterior for $\theta$ under each discrepancy analyzed, we first estimate the mean squared error $\hat{\mathbb{E}}_{ABC}(\theta - \theta_0)^2$ with respect to the location $\theta_0 = 1$ of the uncontaminated Student’s $t$, under the different discrepancies, thereby assessing performance with a focus on a common metric on the space of parameters. Table 1 displays these error estimates, under each discrepancy and level of contamination, further averaged over 50 simulated datasets to obtain a comprehensive assessment based on replicated studies. The results in Table 1 together with the graphical representation in Figure 1 of the ABC posteriors
for one simulated dataset from the contaminated model with varying $\alpha \in \{0.05, 0.10, 0.15\}$ effectively illustrate an important practical consequence of the theory derived in Sections 2–4. Namely, when one is not sure, or cannot check, whether the assumed statistical model and/or the underlying data generating process meet specific regularity conditions, effective IPS discrepancies with guarantees of uniform convergence and concentration (e.g., MMD with bounded kernel) provide a robust and safe default choice. When the level of contamination is mild ($\alpha = 0.05$), Wasserstein–ABC achieves a comparable performance, which suggests that some conditions under which bounds on the Rademacher complexity for the Wasserstein distance are currently derived might be relaxed under specific settings, although arguably not holding in general for unbounded $\mathcal{Y}$ — as discussed in Example 3.1. In fact, when the amount of contamination grows to $\alpha = 0.10$ and $\alpha = 0.15$ the performance of Wasserstein–ABC tends to slightly deteriorate, mainly due to a location shift in the induced posterior. Such a shift is even more evident for summary–based ABC relying on the sample mean, which is not robust to location contaminations. Conversely, the Kullback–Leibler divergence preserves robustness but at the expense of a lower concentration of the induced posterior. Notice that all the discrepancies analyzed in Table 1 and Figure 1 are well–defined under the Student’s $t$ and Gaussian distributions considered in this simulation study. In particular, since the Student’s $t$ has 3 degrees of freedom, its mean and variance exist and are finite. As such, the Wasserstein–1 is well–defined for both the assumed model and the underlying data generating process.

Considering the running times, as displayed in Table 1 all the discrepancies under analysis can be evaluated in the order of milliseconds for a sample of size $n = m = 100$ on a standard laptop. This enables scalable and effective R implementations.

6. Extension to non–i.i.d. settings. Although the theoretical results in Section 2–4 provide an improved understanding of the limiting properties of discrepancy–based ABC posteriors, the i.i.d. assumption in (I) rules out important settings which typically require ABC implementations. A remarkable case is that of time–dependent observations (e.g., Fearnhead & Prangle, 2012; Bernton et al., 2019; Nguyen et al., 2020; Drovandi & Frazier, 2022). Section 6.1 clarifies that the theory derived under i.i.d. assumptions in Section 2–4 can be naturally extended to these non–i.i.d. settings leveraging results for the Rademacher complexity of $\beta$–mixing stochastic processes (Mohri & Rostamizadeh, 2008).

6.1. Convergence and concentration beyond i.i.d. settings. Let us assume again that $\mathcal{Y}$ is a metric space endowed with distance $\rho$. However, unlike the i.i.d. setting considered in Section 2, we now focus on the situation in which the observed data $y_{1:n} = (y_1, \ldots, y_n) \in \mathcal{Y}^n$ are dependent and drawn from the joint distribution $\mu^{s(n)} \in \mathcal{P}(\mathcal{Y}^n)$, where $\mathcal{P}(\mathcal{Y}^n)$ is the space of probability measures on $\mathcal{Y}^n$. Under this more general framework, the i.i.d. case is recovered by assuming that $\mu^{s(n)}$ can be expressed as a product, i.e., $\mu^{s(n)} = \prod_{i=1}^n \mu^s$.

In the following, the above product structure is not imposed. Instead, we only assume that the marginal of $\mu^{s(n)}$ is constant, and is denoted with $\mu^s$. Such an assumption is met whenever $y_{1:n}$ is extracted from a stationary process $(y_t)_{t \in \mathbb{Z}}$, thus embracing a broader variety of
applications of direct interest. Under these settings, a statistical model is defined as a collection of distributions in \( \mathcal{P}(\mathcal{Y}^n) \), i.e., \( \{\mu_\theta^{(n)} : \theta \in \Theta \subseteq \mathbb{R}^p\} \), with a constant marginal distribution denoted by \( \mu_\theta \). Notice that these assumptions of constant marginals \( \mu^* \) and \( \mu_\theta \) are made also in available concentration theory under non–i.i.d. settings (Bernton et al., 2019; Nguyen et al., 2020) when requiring convergence of \( \mathcal{D}_{\hat{\Theta}}(\hat{\mu}_{y1:n}, \mu^*) \) and suitable concentration inequalities for \( \mathcal{D}_{\hat{\Theta}}(\hat{\mu}_{z1:n}, \mu_\theta) \). As such, the settings we consider are not more restrictive than those addressed in discrepancy–specific theory. In fact, both Bernton et al. (2019) and Nguyen et al. (2020) explicitly refer to stationary processes when discussing the validity of the assumptions on \( \mathcal{D}_{\hat{\Theta}}(\hat{\mu}_{y1:n}, \mu^*) \) and \( \mathcal{D}_{\hat{\Theta}}(\hat{\mu}_{z1:n}, \mu_\theta) \) in non–i.i.d. contexts.

Given the above statistical model, a prior \( \pi \) on \( \theta \) and a generic IPS discrepancy \( \mathcal{D}_{\hat{\Theta}} \), the ABC posterior with threshold \( \varepsilon_n \geq 0 \) is defined as

\[
\pi_n^{(\varepsilon_n)}(\theta) \propto \pi(\theta) \int_{\mathcal{Y}^n} \mathbb{I}\{\mathcal{D}_{\hat{\Theta}}(z_{1:n}, y_{1:n}) \leq \varepsilon_n \} \mu_\theta^{(n)}(dz_{1:n}).
\]

Notice that the above definition is the same as the one provided in Section 2, with the only difference that \( \mu_\theta^{(n)} = \prod_{i=1}^n \mu_\theta \) is replaced by the joint distribution \( \mu_\theta^{(n)} \), since in this case the data are no more assumed to be independent.

In order to extend the convergence result in Corollary 3.1 together with the concentration statement in Theorem 3.2 to the above framework, we require an analog of equation (2.2) in Lemma 2.1 for time–dependent data. This generalization can be derived leveraging results in Mohri & Rostamizadeh (2008) under the notion of \( \beta \)–mixing coefficients.

**Definition 6.1 (\( \beta \)–mixing; e.g., Mohri & Rostamizadeh (2008)).** Consider a stationary sequence \( (x_t)_{t \in \mathbb{Z}} \) of random variables, and let \( \sigma^t_j \) be the \( \sigma \)–algebra generated by the random variables \( x_k, j \leq k \leq j' \), for any \( j, j' \in \mathbb{Z} \cup \{-\infty, +\infty\} \). Then, for any positive integer \( k \), the \( \beta \)–mixing coefficient of \( (x_t)_{t \in \mathbb{Z}} \) is defined as

\[
\beta(k) = \sup_{t \in \mathbb{Z}} \mathbb{E}[\sup_{A \in \sigma^t_{-\infty}} |\mathbb{P}(A | \sigma^t_{-\infty}) - \mathbb{P}(A)|].
\]

If \( \beta(k) \to 0 \) as \( k \to \infty \), then the stochastic process \( (x_t)_{t \in \mathbb{Z}} \) is said to be \( \beta \)–mixing.

Intuitively, \( \beta(k) \) measures the dependence between the past (before \( t \)) and the future (after \( t + k \)) of the process. When such a dependence is weak, we expect that \( \beta(k) \) will decay to 0 fast when \( k \to \infty \). In the most extreme case, when the \( x_t \)'s are i.i.d., we have \( \beta(k) = 0 \) for all \( k > 0 \). More generally, as clarified in Definition 6.1, a process having \( \beta(k) \to 0 \) when \( k \to \infty \) is named \( \beta \)–mixing. We refer the reader to Doukhan (1994) for an in–depth study of the main properties of \( \beta \)–mixing processes along with a more comprehensive discussion of relevant examples. The most remarkable ones will be also presented in the following.

Leveraging the above notion of \( \beta \)–mixing coefficient, Lemma 6.1 extends Lemma 2.1 to the dependent setting. The proof of Lemma 6.1 can be found in the Supplementary Material and combines Proposition 2 and Lemma 2 in Mohri & Rostamizadeh (2008). For readability, let us also introduce the notation \( s_n = \lfloor n/(2\lfloor \sqrt{n} \rfloor) \rfloor \). Note that \( s_n \sim \sqrt{n}/2 \) as \( n \to \infty \), and thus \( s_n \to \infty \).
Lemma 6.1. Define \( s_n = \lceil n/(2\sqrt{n}) \rceil \). Moreover, consider the stationary stochastic process \((x_t)_{t \in \mathbb{Z}}\) and denote with \( \beta(k), k \in \mathbb{N} \), its \( \beta \)-mixing coefficients. Let \( \mu^{(n)} \) be the joint distribution of a sample \( x_{1:n} \) extracted from \((x_t)_{t \in \mathbb{Z}}\) and denote with \( \mu = \mu^{(1)} \) its constant marginal. Then, for any \( b \)-uniformly bounded class \( \mathcal{F} \), any integer \( n \geq 1 \) and scalar \( \delta \geq 0 \),

\[
\mathbb{P}_{x_{1:n}} \left[ \mathcal{D}_\beta(\hat{\mu}_{x_{1:n}}, \mu) \leq 2\mathcal{R}_{\mu,s_n}(\mathcal{F}) + \frac{4b}{\sqrt{n}} + \delta \right] \geq 1 - 2\exp \left[ -\frac{s_n\delta^2}{2b^2} \right] - 2s_n\beta(\lceil \sqrt{n} \rceil),
\]

where \( \mathcal{R}_{\mu,s_n}(\mathcal{F}) \) corresponds to the Rademacher complexity in Definition 2.2 for an i.i.d. sample of size \( s_n \) from \( \mu \).

Equation (6.1) extends (2.2) beyond the i.i.d. setting. This extension crucially provides a bound that still depends on the Rademacher complexity introduced in Definition 2.2 for an i.i.d. sample — in this case from the common marginal \( \mu \) of the stochastic process \((x_t)_{t \in \mathbb{Z}}\). As such, Assumption (IV) requires no modifications and, hence, no additional validity checks relative to those discussed in Section 3.3. This result suggests that the Rademacher complexity framework might also be leveraged to derive improved convergence and concentration results for discrepancy-based ABC posteriors in more general situations which do not necessarily meet Assumption (I). To prove these results we leverage again Assumptions (II), (III) and (IV), and replace (I) with condition (VII) below.

(VII) The data \( y_{1:n} \) are from a \( \beta \)-mixing stochastic process \((y_t)_{t \in \mathbb{Z}}\) with mixing coefficients \( \beta(k) \leq C_\beta e^{-\gamma k^\xi} \) for some \( C_\beta, \gamma, \xi > 0 \), common marginal \( \mu \), and generic joint \( \mu^{(n')} \) for a sample \( y_{1:n'} \) from \((y_t)_{t \in \mathbb{Z}}\) for any \( n' \in \mathbb{N} \). The same \( \beta \)-mixing conditions hold also for the stochastic process \((z_t)_{t \in \mathbb{Z}}\) associated with the synthetic data \( z_{1:n} \) generated from the assumed model. In this case, the joint distribution for a generic sample \( z_{1:n'} \) is \( \mu^{(n')}_{\theta}, \theta \in \Theta \), whereas the common marginal is denoted by \( \mu_{\theta} \).

Assumption (VII) is clearly more general than (I). As discussed previously, it embraces several stochastic processes of substantial interest in practical applications, including those in Examples 6.1–6.2 below; see Doukhan (1994) for additional examples and discussion.

Example 6.1 (Doeblin–recurrent Markov chains). Let \((x_t)_{t \in \mathbb{Z}}\) be a Markov chain on \( \mathcal{Y} \subset \mathbb{R}^d \) with transition kernel \( P(\cdot, \cdot) \). Such a Markov chain is said to be Doeblin–recurrent if there exists a probability measure \( q \), a constant \( 0 < c < 1 \) and an integer \( r > 0 \) such that, for any measurable set \( A \) and any \( x \in \mathbb{R}^d \), \( P^r(x, A) > cq(A) \). When this is the case, \((x_t)_{t \in \mathbb{Z}}\) is \( \beta \)-mixing with \( \beta(k) \leq 2(1 - c)^{k/r} \); see e.g., Theorem 1 in page 88 of Doukhan (1994).

Example 6.2 (Hidden Markov chains). Assume \((x_t)_{t \in \mathbb{Z}}\) is a \( \beta \)-mixing stochastic process with coefficients \( \beta_x(k), k \in \mathbb{N} \). If \( \tilde{x}_t = F(x_t, \varepsilon_t) \) with \( \varepsilon_t \) i.i.d., then the \( \beta \)-mixing coefficients of \((\tilde{x}_t)_{t \in \mathbb{Z}}\) satisfy \( \beta_{\tilde{x}}(k) = \beta_x(k) \). Therefore, \((\tilde{x}_t)_{t \in \mathbb{Z}}\) is also \( \beta \)-mixing and inherits the bounds on \( \beta_x(k) \). These processes are often used in practice with \((x_t)_{t \in \mathbb{Z}}\) being a Markov chain. In this case \((\tilde{x}_t)_{t \in \mathbb{Z}}\) is called a Hidden Markov chain.
Section 2.4.2 of Doukhan (1994) also provides conditions on $F$ and on the i.i.d. sequence $(\varepsilon_t)_{t \in \mathbb{Z}}$ ensuring that a stationary process $(x_t)_{t \in \mathbb{Z}}$ satisfying $x_t = F(x_{t-1}, \ldots, x_{t-k}, \varepsilon_t)$ exists and is $\beta$–mixing. Lemma 6.2 specializes such a result in the context of Gaussian AR(1) processes, which will be considered in the empirical study in Section 7.

**Lemma 6.2 (Gaussian AR(1) process).** Consider a generic sequence $(\varepsilon_t)_{t \in \mathbb{Z}}$ of i.i.d random variables from a $N(0, \sigma^2)$. Moreover, let $-1 < \theta < 1$ and $\psi \in \mathbb{R}$. Then the stationary solution to $x_t = \psi + \theta x_{t-1} + \varepsilon_t$ is $\beta$–mixing and has coefficients $\beta(k) \leq |\theta|^k / (2\sqrt{1 - \theta^2}) = (2\sqrt{1 - \theta^2})^{-1} \exp(-k \log(1/|\theta|))$, $k \in \mathbb{N}$, thus meeting (VII).

Note that in the empirical study in Section 7 the focus will be on Bayesian inference for the parameter $\theta$. Clearly, in this case it is not sufficient to focus on the marginal distribution of each $x_t$. Rather, one should leverage the bivariate distribution for the pairs $\tilde{x}_t := (x_t, x_{t+1})$; see also Bernton et al. (2019) where such a strategy is named delay reconstruction. This procedure simply changes the focus to the bivariate stochastic process $(\tilde{x}_t)_{t \in \mathbb{Z}}$, but does not alter the mixing properties. In particular, if $\beta_x(k)$ and $\beta_{\tilde{x}}(k)$ are the mixing coefficients of $(x_t)_{t \in \mathbb{Z}}$ and $(\tilde{x}_t)_{t \in \mathbb{Z}}$, respectively, then from Definition 6.1 we have $\beta_{\tilde{x}}(k) = \beta_x(k-1)$, for $k \geq 1$.

Leveraging Lemma 6.1 along with the newly–introduced assumption, Proposition 6.1 states convergence of the ABC posterior when $\varepsilon_n = \varepsilon$ is fixed and $n \to \infty$.

**Proposition 6.1.** Under Assumptions (III), (IV) and (VII), for any $\varepsilon > \bar{\varepsilon}$, it holds that

$$
\pi_n^{(\varepsilon)}(\theta) \to \pi(\theta \mid \mathcal{D}_3(\mu_\theta, \mu^*) \leq \varepsilon) \propto \pi(\theta) \mathbb{I} \{\mathcal{D}_3(\mu_\theta, \mu^*) \leq \varepsilon\},
$$

almost surely with respect to $y_{1:n} \sim \mu^{*(n)}$, as $n \to \infty$.

According to Proposition 6.1, replacing Assumption (I) with (VII), does not alter the uniform convergence properties of the ABC posterior originally stated in Corollary 3.1 under the i.i.d. assumption. This allows to inherit the discussion after Corollary 3.1 also beyond i.i.d. settings, while suggesting that similar extensions would be possible in the regime $\varepsilon_n \to \varepsilon^*$ and $n \to \infty$. These extensions are stated in Theorem 6.1, which provides an important generalization of Theorem 3.2 beyond the i.i.d. case.

**Theorem 6.1.** Let $\varepsilon_n = \varepsilon^* + \bar{\varepsilon}_n$, and assume (II), (III), (IV) and (VII). Then, if $\bar{\varepsilon}_n \to 0$ is such that $\sqrt{n}\bar{\varepsilon}_n^2 \to \infty$ and $\bar{\varepsilon}_n / R_{s_n}(\mathcal{F}) \to \infty$, with $s_n = \lfloor n / (2[\sqrt{n}]) \rfloor$, we have

$$
\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)}\left(\left\{\theta : \mathcal{D}_3(\mu_\theta, \mu^*) > \varepsilon^* + \frac{4\bar{\varepsilon}_n}{3} + 2R_{\varepsilon_n}(\mathcal{F}) + \frac{4b^2}{\sqrt{n} \varepsilon_n} + \left[\frac{2b^2}{s_n} \log \frac{n}{\bar{\varepsilon}_n} \right]^{1/2}\right\}\right) \leq \frac{4 \cdot 3L}{c_n},
$$

with probability going to 1 as $n \to \infty$, where $R_{s_n} = \sup_{\mu \in \mathcal{P}(\mathcal{Y})} R_{\mu, s_n}$.

As for Proposition 6.1, also Theorem 6.1 shows that informative concentration inequalities similar to those derived in Section 3.2, can be obtained beyond the i.i.d. setting. These results provide comparable insights to those in Theorem 3.2 with the only difference that in
this case we require $\sqrt{n}\bar{\varepsilon}_n^2 \to \infty$ rather than $n\bar{\varepsilon}_n^2 \to \infty$ and the term $2b^2/n$ within the bound in Theorem 3.2 is now replaced by $2b^2/s_n$ with $s_n \sim \sqrt{n/2}$ as $n \to \infty$. This means that $\bar{\varepsilon}_n$ must shrink to zero with a rate at least $n^{1/4}$ slower than the one allowed in the i.i.d. setting. This is an interesting result which clarifies that by moving beyond i.i.d. regimes concentration can still be achieved, although with a slower rate. Such a rate might be pessimistic in some models and we believe it may be improved under future refinements of Lemma 6.1.

Notice that Assumption (VII) could be relaxed to include $\beta$–mixing processes whose coefficients $\beta(k)$ vanish to zero, but at a non–exponential rate, for example, $\beta(k) \sim 1/(k+1)^{\xi}$ for some $\xi > 0$. In this case, we could still use Lemma 6.1 to prove concentration, but with a smaller $s_n$, which would lead to an even slower concentration. However, we did not provide the most general result for the sake of readability. As for processes that are not $\beta$–mixing, we are not aware of results similar to Lemma 6.1 in this context. This is an important direction for future research.

7. Illustrative simulation in non–i.i.d. settings. We conclude by illustrating the results in Section 6 on a simple simulation study based on a contaminated Gaussian $\text{AR}(1)$ process. More specifically, the uncontaminated data are generated from the model $y_t^* = 0.5y_{t-1}^* + \varepsilon_t$ for $t = 1, \ldots, 100$ with $\varepsilon_t \sim \text{N}(0, 1)$ independently, and initial state $y_0^* \sim \text{N}(0, 1)$. Then, similarly to the simulation study in Section 5, these data are contaminated with a growing fraction $\alpha \in \{0.05, 0.10, 0.15\}$ of independent realizations from a $\text{N}(20, 1)$. As such, each observed data point $y_t$ is either equal to $y_t^*$ or to a sample from $\text{N}(20, 1)$, for $t = 1, \ldots, 100$. For Bayesian inference, we assume a Gaussian $\text{AR}(1)$ model $z_t = \theta z_{t-1} + \varepsilon_t$ for $t = 1, \ldots, 100$ with $\varepsilon_t \sim \text{N}(0, 1)$, independently, and focus on learning $\theta$ via discrepancy–based ABC under a uniform prior in $[-1, 1]$ for such a $\theta$.

Rejection ABC is implemented under the same settings and discrepancies considered in Section 5. However, as discussed in Section 6, in this case we focus on distances among the empirical distributions of the $n = m = 100$ observed $(y_0, y_1, y_2, \ldots, (y_{99}, y_{100})$ and synthetic $(z_0, z_1, \ldots, (z_{99}, z_{100})$ pairs. This is consistent with the delayed reconstruction strategy in Bernton et al. (2019) and is motivated by the fact that information on $\theta$ is in the bivariate distributions, rather than the marginals. For the same reason, in implementing summary–based ABC we consider the sample covariance rather than the sample mean.

Table 2 summarizes the concentration achieved by the different discrepancies analyzed under the aforementioned non–i.i.d. data generating process and model, at varying contam-

| Discrepancy | MSE ($\alpha = 0.05$) | MSE ($\alpha = 0.10$) | MSE ($\alpha = 0.15$) | Time |
|-------------|------------------------|------------------------|------------------------|------|
| (IPS) MMD   | 0.029                  | 0.036                  | 0.049                  | < 0.01” |
| (IPS) Wasserstein | 0.043                | 0.091                  | 0.180                  | < 0.01” |
| (IPS) summary (covariance) | 0.575              | 0.998                  | 1.001                  | < 0.01” |
| (non–IPS) KL | 0.058                  | 0.060                  | 0.061                  | < 0.01” |
ination \(\alpha \in \{0.05, 0.10, 0.15\}\). The results are coherent with those displayed in Table 1 for the i.i.d. scenario and further clarify that discrepancies with guarantees of uniform convergence and concentration generally provide a robust choice, including in non–i.i.d. contexts.

8. Discussion. This article yields important theoretical advancements with respect to the recent literature on the limiting properties of discrepancy–based ABC posteriors by connecting these properties with the asymptotic behavior of the Rademacher complexity associated with the chosen discrepancy. As clarified in Sections 2, 3, 4 and 6, although the concept of Rademacher complexity had never been considered within ABC before, this notion yields a powerful and promising framework to derive general, informative and uniform limiting properties of discrepancy–based ABC posteriors.

While the above contribution already provides key advancements, the proposed perspective based on Rademacher complexity has broader scope and sets the premises for additional future research. For example, as clarified in Sections 2–4 and 6, any novel result and bound on the Rademacher complexity of specific discrepancies can be directly applied to ABC theory through our framework. This may yield tighter or more explicit bounds, possibly holding under milder assumptions and more general discrepancies. For example, to our knowledge, informative bounds for the Rademacher complexity of the Wasserstein distance are currently available only for bounded \(\mathcal{Y}\) and, hence, it would be of interest to leverage future findings on the unbounded \(\mathcal{Y}\) case to broaden the range of models for which our theory, when specialized to Wasserstein distance, applies. To this end, it might also be promising to explore available results on local Rademacher complexities (e.g., Bartlett et al., 2005), along with the proposed variable–transformation strategy in Example 3.1. Such a latter solution requires the choice of a mapping \(g(\cdot)\) which clearly influences the learning properties of the induced distance and, as such, requires further investigation. Finally, although Section 6 provides an important extension beyond the i.i.d. setting, other relaxations of this assumption could be of interest, such as, for instance, the case of independent but not identically distributed data. This setting could be arguably addressed via the residual reconstruction strategy (e.g., Bernstein et al., 2019, Section 4.2.3.) that would imply studying discrepancies among empirical distributions of residuals for which the i.i.d. assumption can be again reasonably made.

We shall also emphasize that, although our focus is on the actively–studied convergence and concentration theory for discrepancy–based ABC, other properties such as accuracy in uncertainty quantification of credible intervals and the limiting shapes of ABC posteriors, in correctly–specified models, have attracted recent interest in the context of summary–based ABC (e.g., Frazier et al., 2018), and in a few summary–free implementations (e.g., Frazier, 2020; Wang et al., 2022). While this direction goes beyond the scope of our article, extending and unifying such results, as done for concentration properties, is worth future studies.

Although the IPS class is broad, it does not cover all the discrepancies employed in ABC. For example, the KL divergence (Jiang et al., 2018) and the Hellinger distance (Frazier, 2020) are not IPS but rather belong to the class of \(f\)–divergences. While this latter family has important differences relative to the IPS class, it is worth investigating \(f\)–divergences in the light of the results in Sections 2–4 and 6. To accomplish this goal, a possible direction is to
explore the recent unified treatments of these two classes in, e.g., Agrawal and Horel (2021) and Birrell et al. (2022). More generally, our results could also stimulate methodological and theoretical advancements even beyond discrepancy–based ABC, especially within generalized likelihood–free Bayesian inference via discrepancy–based pseudo–posteriors (e.g., Bissiri et al., 2016; Jewson et al., 2018; Miller and Dunson, 2019; Chérief-Abdellatif & Alquier, 2020; Matsubara et al., 2022).

Finally, notice that — for the sake of simplicity and ease of comparison with related studies — we have focused on rejection ABC, and we have constrained the number $m$ of synthetic samples to be equal to the sample size $n$ of the observed data. While these settings are standard in state–of–the–art theory (Bernton et al., 2019; Frazier, 2020), other ABC routines and alternative scenarios where $m$ grows, e.g., sub–linearly, with $n$ deserve further investigation. This latter regime would be especially of interest in settings where the simulation of synthetic data is computationally expensive.

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Supplementary Materials

APPENDIX A: ADDITIONAL INTEGRAL PROBABILITY SEMIMETRICS

A.1. Two additional remarkable examples of IPS discrepancies. While MMD, Wasserstein and summary–based distances provide the most noticeable examples of IPS discrepancies employed in ABC, two other relevant IPS instances are the total variation distance and the Kolmogorov–Smirnov distance, discussed below.

Example A.1.1. (Total variation distance). Although the total variation (TV) distance is not a common choice within discrepancy–based ABC, it still provides a notable example of IPS, obtained when $\mathcal{F}$ is the class of measurable functions whose sup–norm is bounded by 1; i.e. $\mathcal{F} = \{f : \|f\|_\infty \leq 1\}$.

Example A.1.2. (Kolmogorov–Smirnov distance). When $\mathcal{Y} = \mathbb{R}$ and $\mathcal{F} = \{1_{(-\infty, a]}\}_{a \in \mathbb{R}}$, then $\mathcal{D}_\mathcal{F}$ is the Kolmogorov–Smirnov distance, which can also be written as $\mathcal{D}_\mathcal{F}(\mu_1, \mu_2) = \sup_{x \in \mathcal{Y}} |F_1(x) − F_2(x)|$, where $F_1$ and $F_2$ are the cumulative distribution functions associated with $\mu_1$ and $\mu_2$, respectively.

A.2. Validity of (III)–(IV) for TV distance and Kolmogorov–Smirnov distance. Examples A.2.1–A.2.2 verify in detail the validity of assumptions (III) and (IV) under the TV distance and the Kolmogorov–Smirnov distance, respectively.

Example A.2.1. (Total variation distance). As mentioned in Example A.1.1 the TV distance is not a common choice in discrepancy–based ABC implementations. Such a distance satisfies (III) by definition, but generally not Assumption (IV), unless the cardinality $|\mathcal{Y}|$ of $\mathcal{Y}$ is finite. In fact, when $\mathcal{Y} = \mathbb{R}$ and $\mu \in \mathcal{P}(\mathcal{Y})$ is continuous, the probability that there exists an index $i \neq i'$ such that $x_i = x_{i'}$, is zero. Hence, with probability 1, for any vector $\epsilon_{1:n}$ of Rademacher variables there always exists a function $f_\epsilon$ from $\mathcal{Y}$ to $\{0; 1\}$ such that $f_\epsilon(x) = 1_{\{\epsilon_i = 1\}}$. Therefore, $\sup_{f \in \mathcal{F}} |(1/n) \sum_{i=1}^n \epsilon_i f(x_i)| \geq (1/n) \sum_{i=1}^n 1_{\{\epsilon_i = 1\}}$, which implies that the Rademacher complexity $R_{\mu,n}(\mathcal{F})$ is bounded below by $(1/n) \sum_{i=1}^n \mathbb{P}(\epsilon_i = 1) = 1/2$. Nonetheless, as mentioned above, the TV distance can still satisfy (IV) in specific contexts.
For instance, leveraging the bound in Lemma 5.2 of Massart (2000), when the cardinality \(|\mathcal{Y}|\) of \(\mathcal{Y}\) is finite, there will be replicates in \([f(x_1), \ldots, f(x_n)]\) whenever \(n > |\mathcal{Y}|\). Hence, as \(n \to \infty\), it will be impossible to find a function in \(\mathcal{F}\) which can interpolate any noise vector of Rademacher variables with \([f(x_1), \ldots, f(x_n)]\), thus ensuring \(\mathfrak{A}_n \to 0\).

**Example A.2.2.** (Kolmogorov–Smirnov distance). The KS distance meets (III) by definition and, similarly for MMD with bounded kernels, also condition (IV) is satisfied without the need to impose additional constraints on the model \(\mu_0\) or on the data–generating process. More specifically, Assumption (IV) follows from the inequality \(\mathfrak{R}_{\mu, n}(\mathfrak{F}) \leq 2[\log(n + 1)/n]^{1/2}\) in Chapter 4.3.1 of Wainwright (2019). This is a consequence of the bounds on \(\mathfrak{R}_{\mu, n}(\mathfrak{F})\) when \(\mathfrak{F}\) is a class of \(b\)-uniformly bounded functions such that, for some \(\nu \geq 1\), it holds \(\text{card}\{f(x_{1:n}) : f \in \mathfrak{F}\} \leq (n + 1)\nu\) for any \(n\) and \(x_{1:n}\) in \(\mathcal{Y}^n\). When \(\mathfrak{F} = \{1_{(-\infty, a]}\}_{a \in \mathbb{R}}\) each \(x_{1:n}\) would divide the real line in at most \(n + 1\) intervals and every indicator function within \(\mathfrak{F}\) will take value 1 for all \(x \leq a\) and zero otherwise, meaning that \(\text{card}\{f(x_{1:n}) : f \in \mathfrak{F}\} \leq (n + 1)\). Therefore, by applying equation (4.24) in Wainwright (2019), with \(b = 1\) and \(\nu = 1\), yields \(\mathfrak{R}_{\mu, n}(\mathfrak{F}) \leq 2[\log(n + 1)/n]^{1/2}\) for any \(\mu \in \mathcal{P}(\mathcal{Y})\), which implies that Assumption (IV) is met. These derivations clarify the usefulness of the available techniques for upper bounding the Rademacher complexity (e.g., Wainwright, 2019, Chapter 4.3), leveraging, in this case, the notion of polynomial discrimination and the closely–related VC dimension.

**APPENDIX B:** PROOFS OF THEOREMS, COROLLARIES AND PROPOSITIONS

**Proof of Theorem 3.1.** Note that, by leveraging the first inequality in Lemma 2.1, we have \(\mathbb{P}_{y_{1:n}}[\mathcal{D}_\mathfrak{F}(\hat{\mu}_{y_{1:n}}, \mu^*) > 2\mathfrak{R}_{\mu^*, n}(\mathfrak{F}) + \delta] \leq \exp(-n\delta^2/2b^2)\). Hence, setting \(\delta = 1/n^{1/4}\), and recalling that \(\mathfrak{R}_{\mu^*, n}(\mathfrak{F}) \leq \mathfrak{R}_n(\mathfrak{F})\), it follows \(\mathbb{P}_{y_{1:n}}[\mathcal{D}_\mathfrak{F}(\hat{\mu}_{y_{1:n}}, \mu^*) > 2\mathfrak{R}_n(\mathfrak{F}) + 1/n^{1/4}] \leq \exp(-\sqrt{n}/2b^2)\); note that \(\sum_{n \geq 0} \exp(-\sqrt{n}/2b^2) < \infty\). As a direct consequence of this result, if we define the event

\[
(E_n^c) = \{\theta : \mathcal{D}_\mathfrak{F}(\hat{\mu}_{y_{1:n}}, \mu^*) \leq 2\mathfrak{R}_n(\mathfrak{F}) + 1/n^{1/4}\},
\]

then \(1\{E_n^c\} \to 0\) almost surely with respect to \(y_{1:n} \overset{i.i.d.}{\sim} \mu^*\) as \(n \to \infty\). Now, notice that

\[
\pi_n^{(e)}\{\theta : \mathcal{D}_\mathfrak{F}(\mu_\theta, \mu^*) \leq \varepsilon\} = \pi_n^{(e)}\{\theta : \mathcal{D}_\mathfrak{F}(\hat{\mu}_{y_{1:n}}, \mu^*) \leq \varepsilon\} 1\{E_n\} + \pi_n^{(e)}\{\theta : \mathcal{D}_\mathfrak{F}(\hat{\mu}_{y_{1:n}}, \mu^*) \leq \varepsilon\} 1\{E_n^c\}.
\]

Hence, in the following we will focus on \(\pi_n^{(e)}\{\theta : \mathcal{D}_\mathfrak{F}(\hat{\mu}_{y_{1:n}}, \mu^*) \leq \varepsilon\} 1\{E_n\}\). To study such a quantity, recall that, by definition,

\[
\pi_n^{(e)}(\theta) \propto \pi(\theta) \int 1\{\mathcal{D}_\mathfrak{F}(\hat{\mu}_{y_{1:n}}, \hat{\mu}_{z_{1:n}}) \leq \varepsilon\} \mu_\theta^n(dz_{1:n}) = \pi(\theta) \int 1\{\mathcal{D}_\mathfrak{F}(\hat{\mu}_{y_{1:n}}, \hat{\mu}_{z_{1:n}}) \leq \varepsilon + W_\mathfrak{F}(z_{1:n})\} \mu_\theta^n(dz_{1:n}) =: \pi(\theta)p_n(\theta),
\]

where \(W_\mathfrak{F}(z_{1:n}) = \mathcal{D}_\mathfrak{F}(\mu_\theta, \mu^*) - \mathcal{D}_\mathfrak{F}(\hat{\mu}_{y_{1:n}}, \hat{\mu}_{z_{1:n}})\), whereas \(p_n(\theta)\) denotes the probability of generating a sample \(z_{1:n}\) from \(\mu_\theta^n\) which leads to accept the parameter value \(\theta\). Note that, by
applying the triangle inequality twice, we have
\[-\mathcal{D}_\mathcal{G}(\hat{\mu}_{z_{1:n}}, \mu_\theta) - \mathcal{D}_\mathcal{G}(\hat{\mu}_{y_{1:n}}, \mu^*) \leq \mathcal{W}_\mathcal{G}(z_{1:n}) \leq \mathcal{D}_\mathcal{G}(\hat{\mu}_{z_{1:n}}, \mu_\theta) + \mathcal{D}_\mathcal{G}(\hat{\mu}_{y_{1:n}}, \mu^*),\]
and, hence, \(|\mathcal{W}_\mathcal{G}(z_{1:n})| \leq \mathcal{D}_\mathcal{G}(\hat{\mu}_{z_{1:n}}, \mu_\theta) + \mathcal{D}_\mathcal{G}(\hat{\mu}_{y_{1:n}}, \mu^*)\). This implies that the quantity \(p_n(\theta)\) can be bounded below and above as follows
\[
\int 1 \{\mathcal{D}_\mathcal{G}(\mu_\theta, \mu^*) \leq \varepsilon - \mathcal{D}_\mathcal{G}(\hat{\mu}_{z_{1:n}}, \mu_\theta) - \mathcal{D}_\mathcal{G}(\hat{\mu}_{y_{1:n}}, \mu^*)\} \mu_\theta^n(dz_{1:n}) \\
\leq p_n(\theta) \leq \int 1 \{\mathcal{D}_\mathcal{G}(\mu_\theta, \mu^*) \leq \varepsilon + \mathcal{D}_\mathcal{G}(\hat{\mu}_{z_{1:n}}, \mu_\theta) + \mathcal{D}_\mathcal{G}(\hat{\mu}_{y_{1:n}}, \mu^*)\} \mu_\theta^n(dz_{1:n}).
\]

Applying again Lemma 2.1 yields \(\mathbb{P}_{z_{1:n}}[\mathcal{D}_\mathcal{G}(\hat{\mu}_{z_{1:n}}, \mu_\theta) > 2\mathcal{R}_{\mu_\theta, n}(R) / n] \leq \exp(-n\delta^2 / 2b^2)\). Therefore, setting \(\delta = 1/n^{1/4}\), and recalling that \(\mathcal{R}_{\mu_\theta, n}(R) \leq \mathcal{R}_n(R)\) and that we are on the event given in (B.1), it follows
\[
\exp(-\sqrt{n}/2b^2) + 1 \{\mathcal{D}_\mathcal{G}(\mu_\theta, \mu^*) \leq \varepsilon - 4\mathcal{R}_n(R) - 2/n^{1/4}\} \\
\leq p_n(\theta) \leq \exp(-\sqrt{n}/2b^2) + 1 \{\mathcal{D}_\mathcal{G}(\mu_\theta, \mu^*) \leq \varepsilon + 4\mathcal{R}_n(R) + 2/n^{1/4}\}.
\]

Now, notice that the acceptance probability is defined as \(p_n = \int p_n(\theta) \pi(d\theta)\). Hence, integrating with respect to \(\pi(\theta)\) in the above inequalities yields, for \(n \to \infty\),
\[
\pi\{\theta : \mathcal{D}_\mathcal{G}(\mu_\theta, \mu^*) \leq \varepsilon - c_R\} + o(1) \leq p_n \leq \pi\{\theta : \mathcal{D}_\mathcal{G}(\mu_\theta, \mu^*) \leq \varepsilon + c_R\} + o(1),
\]
where \(c_R = 4 \limsup \mathcal{R}_n(R)\), as in equation (3.1), thus concluding the first part of the proof.

To proceed with the second part of the proof, notice that, by the definition of \(\bar{\varepsilon} = \inf\{\varepsilon > 0 : \pi\{\theta : \mathcal{D}_\mathcal{G}(\mu_\theta, \mu^*) \leq \varepsilon\} > 0\}\), the left part of the above inequality is bounded away from zero for \(n\) large enough, whenever \(\varepsilon - c_R > \bar{\varepsilon}\). This implies that also the acceptance probability \(p_n\) is strictly positive. As a consequence, for such \(n\), it follows that
\[
\pi_n^{(\varepsilon)}(A) = \frac{\int p_n(\theta) 1_A(\theta) \pi(d\theta)}{\int p_n(\theta) \pi(d\theta)} = \frac{\int p_n(\theta) 1_A(\theta) \pi(d\theta)}{p_n},
\]
is well-defined for any event \(A\). Then, leveraging the upper bound in (B.2) yields
\[
\pi_n^{(\varepsilon)}\{\theta : \mathcal{D}_\mathcal{G}(\mu_\theta, \mu^*) > \varepsilon + 4\mathcal{R}_n(R)\} = \frac{\int p_n(\theta) 1 \{\mathcal{D}_\mathcal{G}(\mu_\theta, \mu^*) > \varepsilon + 4\mathcal{R}_n(R)\} \pi(d\theta)}{\int p_n(\theta) \pi(d\theta)} \\
\leq \frac{\int 1 \{\mathcal{D}_\mathcal{G}(\mu_\theta, \mu^*) \leq \varepsilon + 4\mathcal{R}_n(R) + 2/n^{1/4}\} 1 \{\mathcal{D}_\mathcal{G}(\mu_\theta, \mu^*) > \varepsilon + 4\mathcal{R}_n(R)\} \pi(d\theta)}{p_n} \\
+ \frac{\int \exp(-\sqrt{n}/2b^2) 1 \{\mathcal{D}_\mathcal{G}(\mu_\theta, \mu^*) > \varepsilon + 4\mathcal{R}_n(R)\} \pi(d\theta)}{p_n}.
\]

To conclude the proof it is now necessary to control both terms. Note that we already proved that the denominator \(p_n\) is bounded away from zero for \(n\) large enough. Both numerators are
bounded by 1, and going to 0 when \( n \to \infty \). Thus, by the dominated convergence theorem, both summands in the above upper bound for \( \pi_n(\mu \mid \theta) = \{ \theta : D_\theta(\mu, \mu^*) > \varepsilon + 4R_n(\theta) \} \to 0 \). This implies, as a direct consequence, that \( \pi_n(\mu \mid \theta) \to 0 \) almost surely with respect to \( y_{1:n} \) as \( n \to \infty \), thereby concluding the proof.

**Proof of Corollary 3.1.** Note that by combining Equation (2.2) in Lemma 2.1 with the result \( \sum_{n>0} \exp[-n \delta^2/(2b^2)] < \infty \), the Borel–Cantelli Lemma implies that both \( D_\theta(\hat{\mu}_{z_{1:n}}, \mu_\theta) \) and \( D_\theta(\hat{\mu}_{y_{1:n}}, \mu_\theta) \) converge to 0 almost surely when \( R_n(\theta) \to 0 \) as \( n \to \infty \). Hence, since

\[
-D_\theta(\hat{\mu}_{z_{1:n}}, \mu_\theta) - D_\theta(\hat{\mu}_{y_{1:n}}, \mu_\theta) \leq D_\theta(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}}) - D_\theta(\mu_\theta, \mu_\star) \leq D_\theta(\hat{\mu}_{z_{1:n}}, \mu_\theta) + D_\theta(\hat{\mu}_{y_{1:n}}, \mu_\theta),
\]

it follows that \( D_\theta(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}}) \to D_\theta(\mu_\theta, \mu_\star) \) almost surely as \( n \to \infty \).

Combining the above result with the proof of Theorem 1 in Jiang et al. (2018) yields the statement of Corollary 3.1. Notice that, as discussed in Section 3.1, the limiting pseudo-posterior in Corollary 3.1 is well–defined only for those \( \varepsilon \geq \bar{\varepsilon} \), with \( \bar{\varepsilon} \) as in Theorem 3.1.

**Proof of Theorem 3.2.** Since Lemma 2.1 and \( R_n(\theta) = \sup_{\theta \in \mathcal{P}(\mathcal{Y})} R_{\theta, n}(\theta) \geq R_{\mu, n}(\theta) \) hold for every \( \mu \in \mathcal{P}(\mathcal{Y}) \), then, for every integer \( n \geq 1 \) and any scalar \( \delta \geq 0 \), Equation (2.2) implies \( \mathbb{P}_{x_{1:n}}[D_\theta(\hat{\mu}_{x_{1:n}}, \mu) \leq 2R_n(\theta) + \delta] \geq 1 - \exp(-n \delta^2/2b^2) \). Moreover, since this result holds for any \( \delta \geq 0 \), it follows that \( \mathbb{P}_{x_{1:n}}[D_\theta(\hat{\mu}_{x_{1:n}}, \mu) \leq 2R_n(\theta) \to \infty] \geq 1 - \exp[-n(c_1 - 2R_n(\theta))^2/2b^2] \), for any \( c_1 \geq 2R_n(\theta) \). Hence,

\[
\mathbb{P}_{x_{1:n}}[D_\theta(\hat{\mu}_{x_{1:n}}, \mu) \leq c_1] \geq 1 - \exp[-n(c_1 - 2R_n(\theta))^2/2b^2].
\]

Recalling the settings of Theorem 3.2, consider the sequence \( \bar{\varepsilon}_n \to 0 \) as \( n \to \infty \), with \( n \bar{\varepsilon}_n^2 \to \infty \) and \( \bar{\varepsilon}_n/R_n(\theta) \to \infty \), which is possible by the Assumption (IV). These regimes imply that \( \bar{\varepsilon}_n \) goes to zero slower than \( R_n(\theta) \) and, therefore, for \( n \) large enough, \( \bar{\varepsilon}_n/3 > 2R_n(\theta) \). Therefore, under Assumptions (I)–(III) it is now possible to apply (B.3) to \( y_{1:n} \), by setting \( c_1 = \bar{\varepsilon}_n/3 \), which yields

\[
\mathbb{P}_{y_{1:n}}[D_\theta(\hat{\mu}_{y_{1:n}}, \mu^*) \leq \bar{\varepsilon}_n/3] \geq 1 - \exp[-n(\bar{\varepsilon}_n/3 - 2R_n(\theta))^2/2b^2].
\]

Since \( -n(\bar{\varepsilon}_n/3 - 2R_n(\theta))^2 = -n\bar{\varepsilon}_n^2[1/9 + 4(R_n(\theta)/\bar{\varepsilon}_n)^2 - (4/3)R_n(\theta)/\bar{\varepsilon}_n] \), it follows that \( -n(\bar{\varepsilon}_n/3 - 2R_n(\theta))^2 \to -\infty \) when \( n \to \infty \). From the above settings we also have that \( n\bar{\varepsilon}_n^2 \to \infty \) and \( R_n(\theta)/\bar{\varepsilon}_n \to 0 \), when \( n \to \infty \). Therefore, as a consequence, we obtain

\[
1 - \exp[-n(\bar{\varepsilon}_n/3 - 2R_n(\theta))^2/2b^2] \to 1 \text{ as } n \to \infty.
\]

Hence, in the rest of this proof, we will restrict to the event \( \{D_\theta(\hat{\mu}_{y_{1:n}}, \mu^*) \leq \bar{\varepsilon}_n/3\} \). Denote with \( \mathbb{P}_{\theta, z_{1:n}} \) the joint distribution of \( \theta \sim \pi \) and \( z_{1:n} \) i.i.d. from \( \mu_\theta \). By definition of conditional probability, for any \( c_2 \), including \( c_2 > 2R_n(\theta) \), it follows that

\[
\pi_n^{(\varepsilon + \bar{\varepsilon}_n)} \big\{ \theta : D_\theta(\mu_\theta, \mu^*) > \varepsilon + 4\bar{\varepsilon}_n/3 + c_2 \big\}
\]

\[
= \frac{\mathbb{P}_{\theta, z_{1:n}}[D_\theta(\mu_\theta, \mu^*) > \varepsilon + 4\bar{\varepsilon}_n/3 + c_2, D_\theta(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}}) \leq \varepsilon^* + \bar{\varepsilon}_n]}{\mathbb{P}_{\theta, z_{1:n}}[D_\theta(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}}) \leq \varepsilon^* + \bar{\varepsilon}_n]}.
\]

(B.4)
To derive an upper bound for the above ratio, we first identify an upper bound for its numerator. In addressing this goal, we leverage the triangle inequality $D_\Sigma(\mu_\theta, \mu^*) \leq D_\Sigma(\hat{\mu}_{1:n}, \mu_\theta) + D_\Sigma(\hat{\mu}_{1:n}, \hat{\mu}_{y_1:n}) + D_\Sigma(\hat{\mu}_{y_1:n}, \mu^*)$, since $D_\Sigma$ is a semimetric, and the previously-proved result that the event $\{D_\Sigma(\hat{\mu}_{y_1:n}, \mu^*) \leq \xi_n/3\}$ has $\mathbb{P}_{y_1:n}$-probability going to 1, thereby obtaining

$$\mathbb{P}_{\theta,z_1:n}[D_\Sigma(\hat{\mu}_{y_1:n}, \mu^*) > \xi^* + 4\xi_n/3 + c_2, D_\Sigma(\hat{\mu}_{1:n}, \hat{\mu}_{y_1:n}) \leq \xi^* + \xi_n] \leq \mathbb{P}_{\theta,z_1:n}[D_\Sigma(\hat{\mu}_{1:n}, \mu_\theta) + D_\Sigma(\hat{\mu}_{y_1:n}, \mu_\theta) > \xi^* + 4\xi_n/3 + c_2,$$

$$D_\Sigma(\hat{\mu}_{1:n}, \hat{\mu}_{y_1:n}) \leq \xi^* + \xi_n] \leq \mathbb{P}_{\theta,z_1:n}[D_\Sigma(\hat{\mu}_{1:n}, \mu_\theta) + D_\Sigma(\hat{\mu}_{y_1:n}, \mu^*) > \xi_n/3 + c_2] \leq \mathbb{P}_{\theta,z_1:n}[D_\Sigma(\hat{\mu}_{1:n}, \mu_\theta) > c_2].$$

Rewriting $\mathbb{P}_{\theta,z_1:n}[D_\Sigma(\hat{\mu}_{1:n}, \mu_\theta) > c_2]$ as $\int_{\theta \in \Theta} \mathbb{P}_{z_1:n}[D_\Sigma(\hat{\mu}_{1:n}, \mu_\theta) > c_2 | \theta] \pi(\theta) d\theta$ and applying (B.3) to $z_{1:n}$ yields

$$\int_{\theta \in \Theta} \mathbb{P}_{z_1:n}[D_\Sigma(\hat{\mu}_{1:n}, \mu_\theta) > c_2 | \theta] \pi(\theta) d\theta = \int_{\theta \in \Theta} \left(1 - \mathbb{P}_{z_1:n}[D_\Sigma(\hat{\mu}_{1:n}, \mu_\theta) \leq c_2 | \theta]\right) \pi(\theta) d\theta \leq \int_{\theta \in \Theta} \exp[-n(c_2 - 2R_n(\xi))]^2/2b^2] \pi(\theta) d\theta = \exp[-n(c_2 - 2R_n(\xi))]^2/2b^2].$$

Therefore, the numerator of the ratio in equation (B.4) can be upper bounded by $\exp[-n(c_2 - 2R_n(\xi))]^2/2b^2$ for any $c_2 > 2R_n(\xi)$. As for the denominator, defining the event $E_n := \{\theta \in \Theta : D_\Sigma(\mu_\theta, \mu^*) \leq \xi^* + \xi_n/3\}$ and applying again the triangle inequality, we have that

$$\mathbb{P}_{\theta,z_1:n}[D_\Sigma(\hat{\mu}_{1:n}, \mu_\theta) \leq \xi^* + \xi_n] \geq \int_{E_n} \mathbb{P}_{z_1:n}[D_\Sigma(\hat{\mu}_{1:n}, \mu_\theta) \leq \xi^* + \xi_n | \theta] \pi(\theta) d\theta$$

$$\geq \int_{E_n} \mathbb{P}_{z_1:n}[D_\Sigma(\hat{\mu}_{1:n}, \mu^*) + D_\Sigma(\mu_\theta, \mu^*) + D_\Sigma(\hat{\mu}_{1:n}, \mu_\theta) \leq \xi^* + \xi_n | \theta] \pi(\theta) d\theta$$

$$\geq \int_{E_n} \mathbb{P}_{z_1:n}[D_\Sigma(\hat{\mu}_{1:n}, \mu_\theta) \leq \xi_n/3 | \theta] \pi(\theta) d\theta,$$

where the last inequality follows directly from the fact that it is possible to restrict to the event $\{D_\Sigma(\hat{\mu}_{1:n}, \mu^*) \leq \xi_n/3\}$, and that we are integrating over $E_n := \{\theta \in \Theta : D_\Sigma(\mu_\theta, \mu^*) \leq \xi^* + \xi_n/3\}$. Applying again Equation (B.3) to $z_{1:n}$, with $c_1 = \xi_n/3 > 2R_n(\xi)$, the last term of the above inequality can be further lower bounded by

$$\int_{E_n} (1 - \exp[-n(\xi_n/3 - 2R_n(\xi))]^2/2b^2]) \pi(\theta) d\theta = \pi(E_n)(1 - \exp[-n(\xi_n/3 - 2R_n(\xi))]^2/2b^2])$$

with $\pi(E_n) \geq c_\pi(\xi_n/3)^L$ by (II), and, as shown before, $1 - \exp[-n(\xi_n/3 - 2R_n(\xi))]^2/2b^2] \to 1$, when $n \to \infty$, which implies, for $n$ large enough, $1 - \exp[-n(\xi_n/3 - 2R_n(\xi))]^2/2b^2] > 1/2$. Leveraging both results, the denominator in (B.4) is lower bounded by $(c_\pi/2)(\xi_n/3)^L$. 


Let us now combine the upper and lower bounds derived, respectively, for the numerator and the denominator of the ratio in (B.4), to obtain

\[(B.5) \quad \pi_n(\varepsilon + \bar{\varepsilon}_n)(\{\theta \in \Theta : D_\delta(\mu_\theta, \mu^*) > \varepsilon^* + 4\bar{\varepsilon}_n/3 + c_2\}) \leq \frac{\exp[-n(c_2 - 2\mathcal{R}_n(\bar{\delta}))^2/2b^2]}{(c\pi/2)(\bar{\varepsilon}_n/3)^L},\]

with \(P_{y_1,n}\)–probability going to 1 as \(n \to \infty\). To conclude the proof it suffices to replace \(c_2\) in (B.5) with \(2\mathcal{R}_n(\bar{\delta}) + \sqrt{2b^2/n} \log(M_n/\bar{\varepsilon}_n^L)\), which is never lower than \(2\mathcal{R}_n(\bar{\delta})\). Finally, setting \(M_n = n\) yields the statement of Theorem 3.2.

**Proof of Corollary 3.2.** Corollary 3.2 follows by replacing the bounds in the proof of Corollary 1 by Bernton et al. (2019) with the newly–derived ones in Theorem 3.2. □

**Proof of Corollary 4.1.** Recall that in the case of MMD with bounded kernels we have \(\mathcal{R}_n(\bar{\delta}) \leq n^{-1/2}\). Hence, for the upper and lower bounds on \(p_n\) in (3.1) it holds

\[
\pi\{\theta : D_{\text{MMD}}(\mu_\theta, \mu^*) \leq \varepsilon - c_{\delta}\} \geq \pi\{\theta : D_{\text{MMD}}(\mu_\theta, \mu^*) \leq \varepsilon - 4/\sqrt{n}\}, \\
\pi\{\theta : D_{\text{MMD}}(\mu_\theta, \mu^*) \leq \varepsilon + c_{\delta}\} \leq \pi\{\theta : D_{\text{MMD}}(\mu_\theta, \mu^*) \leq \varepsilon + 4/\sqrt{n}\}.
\]

Combining the above inequalities with the result in (3.1), and taking the limit for \(n \to \infty\), proves the first part of the statement. The second part is a direct application of Corollary 3.1 to the case of MMD with bounded kernels, after noticing that the aforementioned inequality \(\mathcal{R}_n(\bar{\delta}) \leq n^{-1/2}\) implies \(\mathcal{R}_n(\bar{\delta}) \to 0\) as \(n \to \infty\). □

**Proof of Corollary 4.2.** To prove Corollary 4.2, it suffices to plug \(\bar{\varepsilon}_n = \lfloor(\log n)/n\rfloor^{1/2}\) and \(b = 1\) into the statement of Theorem 3.2, and then upper–bound the resulting radius via the inequalities \(\mathcal{R}_n(\bar{\delta}) \leq n^{-1/2}\) and \(\log n \geq 1\). The latter holds for any \(n \geq 3\) and hence for \(n \to \infty\). □

**Proof of Proposition 4.1.** We first show that, under (A1)–(A3), Assumptions 1–2 in Bernton et al. (2019) are satisfied when \(f_n(\bar{\varepsilon}_n) = 1/(n\bar{\varepsilon}_n^2)\) and \(c(\theta) = \mathbb{E}_Z[k(z, z)]\), with \(z \sim \mu_\theta\). To this end, first recall that, by standard properties of MMD,

\[(B.6) \quad D_{\text{MMD}}^2(\mu_1, \mu_2) = \mathbb{E}_{x_1, x'_1}[k(x_1, x'_1)] - 2\mathbb{E}_{x_1, x_2}[k(x_1, x_2)] + \mathbb{E}_{x_2, x'_2}[k(x_2, x'_2)],\]

with \(x_1, x'_1 \sim \mu_1\) and \(x_2, x'_2 \sim \mu_2\), all independently; see e.g., Chérief-Abdellatif and Alquier (2022). Since \(k(x, x') = \langle \phi(x), \phi(x') \rangle_\mathcal{H}\) (e.g., Muandet et al., 2017), the above result implies

\[(B.7) \quad D_{\text{MMD}}^2(\mu_1, \mu_2) = \mathbb{E}_{x_1, x'_1}[(\phi(x_1), \phi(x'_1))]_\mathcal{H}^2 - 2\mathbb{E}_{x_1, x_2}[(\phi(x_1), \phi(x_2))]_\mathcal{H} + \mathbb{E}_{x_2, x'_2}[(\phi(x_2), \phi(x'_2))]_\mathcal{H}.
\]
Leveraging Equations (B.6)–(B.7) and basic Markov inequalities, for any \(\bar{\varepsilon}_n \geq 0\), it holds
\[
\begin{align*}
\mathbb{P}_{y_1:n}[\mathcal{D}_{\text{MMD}}(\hat{\mu}_{y_1:n}, \mu^*) > \bar{\varepsilon}_n] & \leq (1/\bar{\varepsilon}^2_n)\mathbb{E}_{y_1:n}[\mathcal{D}_{\text{MMD}}^2(\hat{\mu}_{y_1:n}, \mu^*)] \\
& = (1/\bar{\varepsilon}^2_n)\mathbb{E}_{y_1:n}[(1/(n\varepsilon^2_n)\sum_{i=1}^{n}[\phi(y_i) - \mathbb{E}_y[\phi(y)]]^2)] \\
& \leq 1/(n\varepsilon^2_n)\mathbb{E}_{y_1:n}[\sum_{i=1}^{n}[\phi(y_i)]]^2 + \mathbb{E}_{y_1:n}[\sum_{i=1}^{n}[\phi(y_i)]]^2 \\
& \leq [1/(n\varepsilon^2_n)]\mathbb{E}_{y_1:n}[\phi(y_1)]^2 + [1/(n\varepsilon^2_n)]\mathbb{E}_{y_1:n}[k(y_1,y_1)] = [1/(n\varepsilon^2_n)]\mathbb{E}_{y}[k(y,y)],
\end{align*}
\]
with \(y \sim \mu^*\). Since \([1/(n\varepsilon^2_n)]\mathbb{E}_{y}[k(y,y)] \to 0\) as \(n \to \infty\) by condition (A1), we have that \(\mathcal{D}_{\text{MMD}}(\hat{\mu}_{y_1:n}, \mu^*) \to 0\) in \(\mathbb{P}_{y_1:n}\)–probability as \(n \to \infty\), thus meeting Assumption 1 in Bernton et al. (2019). Moreover, as a direct consequence of the above derivations,
\[
\mathbb{P}_{z_1:n}[\mathcal{D}_{\text{MMD}}(\hat{\mu}_{z_1:n}, \mu_\theta) > \bar{\varepsilon}_n] \leq 1/(n\varepsilon^2_n)\mathbb{E}_{z}[k(z,z)].
\]
Thus, setting \(1/(n\varepsilon^2_n) = f_n(\bar{\varepsilon}_n)\) and \(\mathbb{E}_{z}[k(z,z)] = c(\theta)\), with \(z \sim \mu_\theta\), ensures that
\[
\mathbb{P}_{z_1:n}[\mathcal{D}_{\text{MMD}}(\hat{\mu}_{z_1:n}, \mu_\theta) > \bar{\varepsilon}_n] \leq c(\theta)f_n(\bar{\varepsilon}_n),
\]
with \(f_n(u) = 1/(nu^2)\) strictly decreasing in \(u\) for any fixed \(n\), and \(f_n(u) \to 0\) as \(n \to \infty\), for fixed \(u\). Moreover, by Assumptions (A2)–(A3), \(c(\theta) = \mathbb{E}_{z}[k(z,z)]\) is \(\pi\)–integrable and there exist a \(\delta_0 > 0\) and a \(c_0 > 0\) such that \(c(\theta) < c_0\) for any \(\theta\) satisfying \((\mathbb{E}_{z,z'}[k(z,z')] - 2\mathbb{E}_{z,y}[k(y,z)] + \mathbb{E}_{y,y'}[k(y,y')])^{1/2} = \mathcal{D}_{\text{MMD}}(\mu_\theta, \mu^*) \leq \varepsilon^* + \delta_0\). This ensures that Assumption 2 in Bernton et al. (2019) holds.

Under the assumptions in Proposition 4.1 it is, therefore, possible to apply Proposition 3 in Bernton et al. (2019) with \(f_n(\bar{\varepsilon}_n) = 1/(n\varepsilon^2_n)\), \(c(\theta) = \mathbb{E}_{z}[k(z,z)]\), and \(R = M_n\), which yields the concentration result in Proposition 4.1.

\[\square\]

**Proof of Lemma 6.1.** Let \((x_t)_{t \in \mathbb{Z}}\) be a stochastic process with \(\beta\)–mixing coefficients \(\beta(k)\), for \(k \in \mathbb{N}\). Moreover, denote with \(\mu^{(n)}\) and \(\mu\) the joint distribution of a sample \(x_{1:n}\) from the process \((x_t)_{t \in \mathbb{Z}}\) and its constant marginal \(\mu^{(1)}\), respectively. Then, by combining Proposition 2 and Lemma 2 in Mohri & Rostamizadeh (2008) under our notation, we have that for any \(b\)–uniformly bounded class \(\mathfrak{S}\), any integer \(n \geq 1\) and any scalar \(\delta \geq 0\), the inequality
\[
\mathbb{P}_{x_{1:n}}[\mathcal{D}_{\mathfrak{S}}(\hat{\mu}_{x_{1:n}}, \mu) > 2\mathfrak{R}_{\mu,n/(2K)}(\mathfrak{S}) + \delta] \leq 2\exp(-n\delta^2/Kb^2) + 2(n/2K - 1)\beta(K'),
\]
holds for every integer \(K > 0\) such that \(n/(2K) \in \mathbb{N}\), where \(\mathfrak{R}_{\mu,n/(2K)}\) is the Rademacher complexity based on the i.i.d. sample of size \(n/(2K)\) from \(\mu\); see Definition 2.2. The above concentration inequality also implies
\[
\begin{align*}
\mathbb{P}_{x_{1:n}}[\mathcal{D}_{\mathfrak{S}}(\hat{\mu}_{x_{1:n}}, \mu) \leq 2\mathfrak{R}_{\mu,n/(2K)}(\mathfrak{S}) + \delta] & \geq 1 - 2\exp(-n\delta^2/Kb^2) - 2(n/2K - 1)\beta(K) \\
& > 1 - 2\exp(-n\delta^2/(4Kb^2)) - 2(n/2K)\beta(K).
\end{align*}
\]
Notice that, in order to ensure that both \(2\exp(-n\delta^2/4Kb^2)\) and \(2(n/2K)\beta(K)\) vanish to zero (under Assumption (VII) for \(\beta(K')\)), it is tempting to apply (B.8) with \(K = n^\alpha\) for some
\(\alpha < 1\). Unfortunately, there is no reason for such a \(K\) to be an integer. A solution, would be to let \(K = \lfloor n^\alpha \rfloor\), but in this case \(n/(2K)\) might not be an integer. To address such issues, it is necessary to consider a careful modification of (B.8). To this end write the Euclidean division \(n = n' + r\) where \(n' = 2Kh \leq n\), \(h = \lfloor n/(2K)\rfloor\) and \(0 \leq r < 2K\). Then, under the common marginal assumption and recalling also Proposition 1 in Mohri & Rostamizadeh (2008) together with the triangle inequality and the fact that the functions \(f\) within \(\mathcal{F}\) are \(\beta\)-uniformly bounded, we have that

\[
D_{\hat{\mathcal{F}}} (\hat{\mu}_{x_{1:n}}, \mu) = \sup_{f \in \hat{\mathcal{F}}} \left| \frac{1}{n} \sum_{i=1}^{n} \left[ f(x_i) - \mathbb{E}_{\mu} f(x) \right] \right| \\
\leq \sup_{f \in \hat{\mathcal{F}}} \left| \frac{1}{n} \sum_{i=1}^{n'} \left[ f(x_i) - \mathbb{E}_{\mu} f(x) \right] \right| + \frac{1}{n} \sup_{f \in \hat{\mathcal{F}}} \left| \sum_{i=n'+1}^{n'+r} \left[ f(x_i) - \mathbb{E}_{\mu} f(x) \right] \right| \\
\leq \sup_{f \in \hat{\mathcal{F}}} \left| \frac{1}{n'} \sum_{i=1}^{n'} \left[ f(x_i) - \mathbb{E}_{\mu} f(x) \right] \right| + \frac{2br}{n} \leq D_{\hat{\mathcal{F}}} (\hat{\mu}_{x_{1:n'}}, \mu) + \frac{4bK}{n},
\]

where the last inequality follows directly from the definition of \(D_{\hat{\mathcal{F}}} (\hat{\mu}_{x_{1:n'}}, \mu)\) together with the fact that \(0 \leq r < 2K\). Applying now (B.8) to \(D_{\hat{\mathcal{F}}} (\hat{\mu}_{x_{1:n'}}, \mu) + 4bK/n\) yields

\[
\mathbb{P}_{x_{1:n}} \left[ D_{\hat{\mathcal{F}}} (\hat{\mu}_{x_{1:n'}}, \mu) + 4bK/n \leq 2\mathcal{M}_{\mu,h}(\hat{\mathcal{F}}) + \delta + 4bK/n \right] \geq 1 - 2 \exp(-h\delta^2/2b^2) - 2h\beta(K).
\]

Therefore, since \(D_{\hat{\mathcal{F}}} (\hat{\mu}_{x_{1:n'}}, \mu) \leq D_{\hat{\mathcal{F}}} (\hat{\mu}_{x_{1:n'}}, \mu) + 4bK/n\) we also have that

\[
\mathbb{P}_{x_{1:n}} \left[ D_{\hat{\mathcal{F}}} (\hat{\mu}_{x_{1:n'}}, \mu) \leq 2\mathcal{M}_{\mu,h}(\hat{\mathcal{F}}) + \delta + 4bK/n \right] \geq 1 - 2 \exp(-h\delta^2/2b^2) - 2h\beta(K).
\]

To conclude the proof, notice that to prove convergence and concentration of the ABC posterior it will be sufficient to let \(K = \lfloor \sqrt{n} \rfloor\). Therefore, by replacing \(K = \lfloor \sqrt{n} \rfloor\) in the above inequality and within the expression for \(h = \lfloor n/(2K) \rfloor\) we have

\[
\mathbb{P}_{x_{1:n}} \left[ D_{\hat{\mathcal{F}}} (\hat{\mu}_{x_{1:n}}, \mu) \leq 2\mathcal{M}_{\mu,s_n}(\hat{\mathcal{F}}) + \frac{4b}{\sqrt{n}} + \delta \right] \geq 1 - 2 \exp(-s_n\delta^2/2b^2) - 2s_n\beta(\lfloor \sqrt{n} \rfloor)
\]

where \(s_n = \lfloor n/(2\lfloor \sqrt{n} \rfloor) \rfloor\) and the term \(4b/\sqrt{n}\) follows directly from the fact that \(\lfloor \sqrt{n} \rfloor/n \leq \sqrt{n}/n = 1/\sqrt{n}\).

**Proof of Lemma 6.2.** The proof follows the arguments used in Chapter 2 of Doukhan (1994) to study general Markov chains. In particular, when \((x_t)_{t \in \mathbb{Z}}\) is a stationary Markov chain with invariant distribution \(\pi(\cdot)\) and transition kernel \(P(\cdot, \cdot)\), a result proven in Davydov (1974) and recalled in page 87–88 of Doukhan (1994) gives

\[
\beta(k) = \mathbb{E}_{x \sim \pi} \| P^k(x, \cdot) - \pi(\cdot) \|_{TV}.
\]

When \(-1 < \theta < 1\), standard results for the AR(1) model in Lemma 6.2 lead to the invariant distribution \(\pi = N(\psi/(1-\theta), \sigma^2/(1-\theta^2))\).
As for $P^k(x, \cdot)$, notice that, under such an AR(1) model, we can write
\[
x_k = \theta x_{k-1} + \psi + \varepsilon_k = \theta(\theta x_{k-2} + \psi + \varepsilon_{k-1}) + \psi + \varepsilon_k = \ldots
\]
\[
= \theta^k x + \psi \sum_{l=0}^{k-1} \theta^l + \sum_{l=0}^{k-1} \theta^l \varepsilon_{k-l}.
\]
Therefore, $P^k(x, \cdot) = N(\theta^k x + \psi \sum_{l=0}^{k-1} \theta^l, \sigma^2 \sum_{l=0}^{k-1} \theta^{2l})$. Moreover, notice that, by direct application of standard properties of finite power series, we have $\sum_{l=0}^{k-1} \theta^l = (1 - \theta^k)/(1 - \theta)$ and $\sum_{l=0}^{k-1} \theta^{2l} = (1 - \theta^{2k})/(1 - \theta^2)$. Since our goal is to derive an upper bound for $\beta(k)$ and provided that the KL divergence among Gaussian densities is available in closed form, let us first consider the Pinsker’s inequality
\[
\|P^k(x, \cdot) - \pi(\cdot)\|_{TV} \leq [D_{KL}(P^k(x, \cdot), \pi(\cdot))/2]^{1/2}
\]
where $D_{KL}$ stands for the KL divergence. Since both $P^k(x, \cdot)$ and $\pi(\cdot)$ are Gaussian, then
\[
D_{KL}(P^k(x, \cdot), \pi(\cdot)) = \frac{1}{2} \left[ \frac{\sigma^2(1 - \theta^{2k})}{\sigma^2} - 1 + \frac{[\theta^k x + \psi(1 - \theta^k)/(1 - \theta) - \psi/(1 - \theta)]^2}{\sigma^2/(1 - \theta^2)} + \log \frac{\sigma^2}{\sigma^2(1 - \theta^2)} \right]
\]
\[
= \frac{1}{2} \left[ (1 - \theta^{2k}) - 1 + \frac{\theta^k [x - \psi/(1 - \theta)]^2}{\sigma^2/(1 - \theta^2)} + \log \left( 1 + \frac{\theta^k}{1 - \theta^{2k}} \right) \right]
\]
\[
= \frac{1}{2} \left[ -\theta^{2k} + \frac{\theta^k [x - \psi/(1 - \theta)]^2}{\sigma^2/(1 - \theta^2)} + \frac{\theta^k}{1 - \theta^{2k}} \right] \leq \frac{\theta^k}{2} \left[ -1 + \frac{[x - \psi/(1 - \theta)]^2}{\sigma^2/(1 - \theta^2)} + \frac{1}{1 - \theta^2} \right].
\]
Therefore, by leveraging the above result, together with standard properties of the expectation, we have
\[
\beta(k) \leq \mathbb{E}_{x \sim \pi}[D_{KL}(P^k(x, \cdot), \pi(\cdot))/2]^{1/2} \leq [\mathbb{E}_{x \sim \pi}D_{KL}(P^k(x, \cdot), \pi(\cdot))/2]^{1/2}
\]
\[
\leq \left[ \frac{\theta^k}{4} \left[ -1 + \mathbb{E}_{x \sim \pi}[x - \psi/(1 - \theta)]^2 \frac{1}{\sigma^2/(1 - \theta^2)} + \frac{1}{1 - \theta^2} \right] \right]^{1/2} = \sqrt{\frac{\theta^k}{4(1 - \theta^2)}} = \frac{|\theta|^k}{2\sqrt{1 - \theta^2}},
\]
which concludes the proof. \hfill \square

**Proof of Proposition 6.1.** Under Assumption (VII), for any fixed $\delta > 0$, we have that $\sum_{n>0}[2 \exp(-s_n\delta^2/2b^2) + 2s_nC_\beta \exp(-\gamma\sqrt{n}/\delta)] < \infty$. Therefore, combining Lemma 6.1 with Assumption (IV), both $D_\delta(\hat{\mu}_{21:n}, \mu_\theta)$ and $D_\delta(\hat{\mu}_{21:n}, \mu^*)$ converge to 0 almost surely as $n \to \infty$, by the Borel–Cantelli Lemma. As a result, since $-D_\delta(\hat{\mu}_{21:n}, \mu_\theta) - D_\delta(\hat{\mu}_{21:n}, \mu^*) \leq D_\delta(\hat{\mu}_{21:n}, \mu_\theta) - D_\delta(\hat{\mu}_{21:n}, \mu^*) \leq D_\delta(\hat{\mu}_{21:n}, \mu_\theta) + D_\delta(\hat{\mu}_{21:n}, \mu^*)$, it holds that $D_\delta(\hat{\mu}_{21:n}, \hat{\mu}_{21:n}) \to D_\delta(\mu_\theta, \mu^*)$ almost surely as $n \to \infty$. To conclude it suffices to apply again the proof of Theorem 1 in Jiang et al. (2018); see also proof of Corollary 3.1. \hfill \square
PROOF OF THEOREM 6.1. To prove Theorem 6.1 we will follow the same line of reasoning as in the proof of Theorem 3.2. However, in this case we leverage Lemma 6.1 instead of Lemma 2.1. To this end, letting $\delta = c_1 - 2 \mathcal{R}_{s_n}(\bar{\theta}) - 4 b / \sqrt{n}$ with $c_1 \geq 2 \mathcal{R}_{s_n}(\bar{\theta}) + 4 b / \sqrt{n}$ and $\mathcal{R}_{s_n} = \sup_{\mu \in \mathcal{P}(\mathcal{Y})} \mathcal{R}_{s_n,\mu}$, we obtain, under Assumption (VII), Equation (B.9) below, instead of (B.3).

\[(B.9) \quad \mathbb{P}_{x_{1:n}}[\mathcal{D}_{\bar{\theta}}(\hat{\mu}_{x_{1:n}}, \mu) \leq c_1] \geq 1 - 2 \exp[-s_n(c_1 - 2 \mathcal{R}_{s_n}(\bar{\theta}) - 4 b / \sqrt{n})^2/2b^2] - 2s_n C_\beta \exp(-\gamma \sqrt{n})^\xi].\]

As in Theorem 3.2, let $c_1 = \bar{\varepsilon}_n / 3$ and notice that, by the settings of Theorem 6.1, for $n$ large enough $\bar{\varepsilon}_n / 3 > 2 \mathcal{R}_{s_n}(\bar{\theta}) + 4 b / \sqrt{n}$. Therefore, applying (B.9) to $y_{1:n}$, with $c_1 = \bar{\varepsilon}_n / 3$, leads to the following upper bound

\[
\mathbb{P}_{y_{1:n}}[\mathcal{D}_{\bar{\theta}}(\hat{\mu}_{y_{1:n}}, \mu^*) \leq \bar{\varepsilon}_n / 3] \geq 1 - 2 \exp[-s_n(\bar{\varepsilon}_n / 3 - 2 \mathcal{R}_{s_n}(\bar{\theta}) - 4 b / \sqrt{n})^2/2b^2] - 2s_n C_\beta \exp(-\gamma \sqrt{n})^\xi).
\]

Recall that, under the settings of Theorem 6.1, we have that $\sqrt{n} \bar{\varepsilon}_n^2 \to \infty$ and $\bar{\varepsilon}_n / \mathcal{R}_{s_n}(\bar{\theta}) \to \infty$ and, therefore, $s_n(\bar{\varepsilon}_n / 3 - 2 \mathcal{R}_{s_n}(\bar{\theta}) - 4 b / \sqrt{n})^2 \sim s_n \bar{\varepsilon}_n^2/9 \sim \sqrt{n} \bar{\varepsilon}_n^2 \to \infty$. Combining this result with the fact that $2s_n C_\beta \exp(-\gamma \sqrt{n})^\xi) \to 0$ as $n \to \infty$ (recall (VII)), it follows that the right end side of the above equation goes to 1 as $n \to \infty$. Hence, in the remaining part of the proof, we will restrict to the event $\{\mathcal{D}_{\bar{\theta}}(\hat{\mu}_{y_{1:n}}, \mu^*) \leq \bar{\varepsilon}_n / 3\}$.

Let $\mathbb{P}_{\theta, z_{1:n}}$ corresponds to the joint distribution of $\theta \sim \pi$ and $z_{1:n}$ from $\mu_{\theta}^{(n)}$. Then, as a direct consequence of the definition of conditional probability, for every positive $c_2$, including $c_2 > 2 \mathcal{R}_{s_n}(\bar{\theta}) + 4 b / \sqrt{n}$, it follows that

\[
\pi_n^{(\varepsilon^* + \bar{\varepsilon}_n)}(\{\theta : \mathcal{D}_{\bar{\theta}}(\mu_{\theta}, \mu^*) > \varepsilon^* + 4 \bar{\varepsilon}_n / 3 + c_2\}) \geq \mathbb{P}_{\theta, z_{1:n}}[\mathcal{D}_{\bar{\theta}}(\hat{\mu}_{y_{1:n}}, \mu^*) \leq \varepsilon^* + \bar{\varepsilon}_n] - \mathbb{P}_{\theta, z_{1:n}}[\mathcal{D}_{\bar{\theta}}(\hat{\mu}_{z_{1:n}}; \hat{\mu}_{y_{1:n}}) \leq \varepsilon^* + \bar{\varepsilon}_n] \tag{B.10}
\]

To upper bound the ratio in (B.10), let us first derive an upper bound for the numerator. To this end, consider the triangle inequality $\mathcal{D}_{\bar{\theta}}(\mu_{\theta}, \mu^*) \leq \mathcal{D}_{\bar{\theta}}(\hat{\mu}_{z_{1:n}}, \mu_{\theta}) + \mathcal{D}_{\bar{\theta}}(\hat{\mu}_{y_{1:n}}, \hat{\mu}_{z_{1:n}}) + \mathcal{D}_{\bar{\theta}}(\hat{\mu}_{y_{1:n}}, \mu^*)$ (recall that $\mathcal{D}_{\bar{\theta}}$ is a semimetric), along with the previously-proved result that the event $\{\mathcal{D}_{\bar{\theta}}(\hat{\mu}_{y_{1:n}}, \mu^*) \leq \bar{\varepsilon}_n / 3\}$ has $\mathbb{P}_{y_{1:n}}$-probability going to 1. Therefore, for large $n$, we have

\[
\mathbb{P}_{\theta, z_{1:n}}[\mathcal{D}_{\bar{\theta}}(\mu_{\theta}, \mu^*) > \varepsilon^* + 4 \bar{\varepsilon}_n / 3 + c_2, \mathcal{D}_{\bar{\theta}}(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}}) \leq \varepsilon^* + \bar{\varepsilon}_n] \leq \mathbb{P}_{\theta, z_{1:n}}[\mathcal{D}_{\bar{\theta}}(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}}) > \varepsilon^* + 4 \bar{\varepsilon}_n / 3 + c_2, \mathcal{D}_{\bar{\theta}}(\hat{\mu}_{y_{1:n}}, \hat{\mu}_{z_{1:n}}) \leq \varepsilon^* + \bar{\varepsilon}_n] \leq \mathbb{P}_{\theta, z_{1:n}}[\mathcal{D}_{\bar{\theta}}(\hat{\mu}_{y_{1:n}}, \mu_{\theta}) + \mathcal{D}_{\bar{\theta}}(\hat{\mu}_{y_{1:n}}, \mu^*) > \bar{\varepsilon}_n / 3 + c_2] \leq \mathbb{P}_{\theta, z_{1:n}}[\mathcal{D}_{\bar{\theta}}(\hat{\mu}_{z_{1:n}}, \mu_{\theta}) > c_2].
\]
Now, note that $\mathbb{P}_{\theta, z_{1:n}}[\mathcal{D}_h(\hat{\mu}_{z_{1:n}}, \mu_\theta) > c_2] = \int_{\theta \in \Theta} \mathbb{P}_{z_{1:n}}[\mathcal{D}_h(\hat{\mu}_{z_{1:n}}, \mu_\theta) > c_2 \mid \theta] \pi(d\theta)$. Therefore, by applying (B.9) to $z_{1:n}$ yields,

$$
\int_{\theta \in \Theta} \mathbb{P}_{z_{1:n}}[\mathcal{D}_h(\hat{\mu}_{z_{1:n}}, \mu_\theta) > c_2] \pi(d\theta) = \int_{\theta \in \Theta} (1 - \mathbb{P}_{z_{1:n}}[\mathcal{D}_h(\hat{\mu}_{z_{1:n}}, \mu_\theta) \leq c_2 \mid \theta]) \pi(d\theta) \\
\leq 2 \exp[-s_n(c_2 - 2\Re_{s_n}(\mathcal{F}) - 4b/\sqrt{n})^2/2b^2] + 2s_nC_\beta \exp(-\gamma[\sqrt{n}]\xi).
$$

This controls the numerator in (B.10). As for the denominator, defining the event $E_n := \{\theta \in \Theta : \mathcal{D}_h(\mu_\theta, \mu^*) \leq \varepsilon + \bar{\varepsilon}_n/3\}$ and applying again the triangle inequality, we have that

$$
\mathbb{P}_{\theta, z_{1:n}}[\mathcal{D}_h(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}}) \leq \varepsilon + \bar{\varepsilon}_n] \geq \int_{E_n} \mathbb{P}_{z_{1:n}}[\mathcal{D}_h(\hat{\mu}_{z_{1:n}}, \hat{\mu}_{y_{1:n}}) \leq \varepsilon + \bar{\varepsilon}_n \mid \theta] \pi(d\theta) \\
\geq \int_{E_n} \mathbb{P}_{z_{1:n}}[\mathcal{D}_h(\hat{\mu}_{y_{1:n}}, \mu^*) + \mathcal{D}_h(\mu_\theta, \mu^*) + \mathcal{D}_h(\hat{\mu}_{z_{1:n}}, \mu_\theta) \leq \varepsilon + \bar{\varepsilon}_n \mid \theta] \pi(d\theta) \\
\geq \int_{E_n} \mathbb{P}_{z_{1:n}}[\mathcal{D}_h(\hat{\mu}_{z_{1:n}}, \mu_\theta) \leq \bar{\varepsilon}_n/3 \mid \theta] \pi(d\theta).
$$

The last inequality follows from that fact that we can restrict to the event $\{\mathcal{D}_h(\hat{\mu}_{y_{1:n}}, \mu^*) \leq \bar{\varepsilon}_n/3\}$, and that we are integrating over $E_n := \{\theta \in \Theta : \mathcal{D}_h(\mu_\theta, \mu^*) \leq \varepsilon + \bar{\varepsilon}_n/3\}$. Let us now apply again equation (B.9) to $z_{1:n}$, with $c_1 = \bar{\varepsilon}_n/3$, to further lower bound the last term of the above inequality by

$$
\int_{E_n} [1 - 2 \exp[-s_n(\bar{\varepsilon}_n/3 - 2\Re_{s_n}(\mathcal{F}) - 4b/\sqrt{n})^2/2b^2] - 2s_nC_\beta \exp(-\gamma[\sqrt{n}]\xi)] \pi(d\theta) \\
= \pi(E_n)[1 - 2 \exp[-s_n(\bar{\varepsilon}_n/3 - 2\Re_{s_n}(\mathcal{F}) - 4b/\sqrt{n})^2/2b^2] - 2s_nC_\beta \exp(-\gamma[\sqrt{n}]\xi)].
$$

Note that, by Assumption (II), $\pi(E_n) \geq c_\pi(\bar{\varepsilon}_n/3)^L$. Moreover, as shown before, the quantity $1 - 2 \exp[-s_n(\bar{\varepsilon}_n/3 - 2\Re_{s_n}(\mathcal{F}) - 4b/\sqrt{n})^2/2b^2] - 2s_nC_\beta \exp(-\gamma[\sqrt{n}]\xi)$ goes to 1 when $n \to \infty$, which also implies that, for a large enough $n$, $1 - 2 \exp[-s_n(\bar{\varepsilon}_n/3 - 2\Re_{s_n}(\mathcal{F}) - 4b/\sqrt{n})^2/2b^2] - 2s_nC_\beta \exp(-\gamma[\sqrt{n}]\xi) > 1/2$. Therefore, leveraging both results, the denominator in (B.10) can be lower bounded by $(c_\pi/2)(\bar{\varepsilon}_n/3)^L$. To proceed with the proof, let us combine the upper and lower bounds derived, respectively, for the numerator and the denominator of the ratio in (B.10). This yields, for any integer $K$,

$$
\pi_n^{(\varepsilon + \bar{\varepsilon}_n)}(\{\theta : \mathcal{D}_h(\mu_\theta, \mu^*) > \varepsilon + 4\bar{\varepsilon}_n/3 + c_2\}) \\
\leq \frac{2 \exp[-s_n(c_2 - 2\Re_{s_n}(\mathcal{F}) - 4b/\sqrt{n})^2/2b^2] + 2s_nC_\beta \exp(-\gamma[\sqrt{n}]\xi)}{(c_\pi/2)(\bar{\varepsilon}_n/3)^L},
$$

with $\mathbb{P}_{y_{1:n}}$–probability going to 1 as $n \to \infty$. 

(B.11)
Replacing $c_2$ in (B.11) with $2\bar{R}_s(\bar{\gamma}) + \sqrt{(2b^2/s_n)}\log(n/\bar{\varepsilon}_n) + 4b/\sqrt{n}$, gives

$$
\pi_n^{(\varepsilon^*+\bar{\varepsilon}_n)}\left( \{ \theta: \mathcal{D}_{\bar{\gamma}}(\mu_\theta, \mu^*) > \varepsilon^* + \frac{4\bar{\varepsilon}_n}{3} + 2\bar{R}_s(\bar{\gamma}) + \frac{4b}{\sqrt{n}} + \left[ \frac{2b^2}{s_n} \log \frac{n}{\bar{\varepsilon}_n} \right]^{1/2} \} \right)
\leq \frac{4 \cdot 3L}{nc_\pi} \left[ 1 + \frac{ns_nC_\beta \exp(-\gamma\sqrt{n})}{\bar{\varepsilon}_n} \right].
$$

To conclude, note that $ns_nC_\beta \exp(-\gamma\sqrt{n})/\bar{\varepsilon}_n^L = n^{1+L/2}s_nC_\beta \exp(-\gamma\sqrt{n})/((n\bar{\varepsilon}_n^2)^{L/2})$ where the numerator goes to 0 and the denominator goes to $\infty$ when $n \to \infty$, under the setting of Theorem 6.1. Therefore, for $n$ large enough,

$$
\pi_n^{(\varepsilon^*+\bar{\varepsilon}_n)}\left( \{ \theta: \mathcal{D}_{\bar{\gamma}}(\mu_\theta, \mu^*) > \varepsilon^* + \frac{4\bar{\varepsilon}_n}{3} + 2\bar{R}_s(\bar{\gamma}) + \frac{4b}{\sqrt{n}} + \left[ \frac{2b^2}{s_n} \log \frac{n}{\bar{\varepsilon}_n} \right]^{1/2} \} \right) \leq \frac{4 \cdot 3L}{nc_\pi},
$$

concluding the proof. \qed