Differential structure associated to axiomatic Sobolev spaces

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Abstract

The aim of this note is to explain in which sense an axiomatic Sobolev space over a
general metric measure space (à la Gol’dshtein-Troyanov) induces – under suitable locality
assumptions – a first-order differential structure.

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Introduction

An axiomatic approach to the theory of Sobolev spaces over abstract metric measure spaces
has been proposed by V. Gol’dshtein and M. Troyanov in [6]. Their construction covers
many important notions: the weighted Sobolev space on a Riemannian manifold, the Hajlasz
Sobolev space [7] and the Sobolev space based on the concept of upper gradient [2,3,8,9].

A key concept in [6] is the so-called D-structure: given a metric measure space (X, d, m)
and an exponent p ∈ (1, ∞), we associate to any function u ∈ Lp loc(X) a family D[u] of non-
negative Borel functions called pseudo-gradients, which exert some control from above on the
variation of u. The pseudo-gradients are not explicitly specified, but they are rather supposed

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to fulfil a list of axioms. Then the space \(W^{1,p}(X, d, m, D)\) is defined as the set of all functions in \(L^p(m)\) admitting a pseudo-gradient in \(L^p(m)\). By means of standard functional analytic techniques, it is possible to associate to any Sobolev function \(u \in W^{1,p}(X, d, m, D)\) a uniquely determined minimal object \(Du \in D[u] \cap L^p(m)\), called minimal pseudo-gradient of \(u\).

More recently, the first author of the present paper introduced a differential structure on general metric measure spaces (cf. [4,5]). The purpose was to develop a second-order module theory for \(L^p\) to fulfil a list of axioms. Then the space \(Sf\) to fulfil a list of axioms. It is possible to associate to any Sobolev function \(u \in W^{1,p}(X, d, m, D)\) a uniquely determined minimal object \(Du \in D[u] \cap L^p(m)\), called minimal pseudo-gradient of \(u\).

The main result of this paper – namely Theorem 3.2 – says that any \(L^p\)-normed \(L^\infty\)-modules, among which a special role is played by the cotangent module, denoted by \(L^2(T^*X)\). Its elements can be thought of as ‘measurable 1-forms on \(X\).

Roughly speaking, the cotangent module allows us to represent minimal pseudo-gradients as pointwise norms of suitable linear objects. More precisely, this theory provides the existence of an abstract differential \(d : W^{1,p}(X, d, m, D) \rightarrow L^p(T^*X, D)\), which is a linear operator such that the pointwise norm \(|du| \in L^p(m)\) of \(du\) coincides with \(Du\) in the \(m\)-a.e. sense for any function \(u \in W^{1,p}(X, d, m, D)\).

1 General notation

For the purpose of the present paper, a metric measure space is a triple \((X, d, m)\), where

\[
\begin{align*}
(X, d) & \quad \text{is a complete and separable metric space,} \\
m & \neq 0 \quad \text{is a non-negative Borel measure on } X, \text{ finite on balls.}
\end{align*}
\]

Fix \(p \in [1, \infty)\). Several functional spaces over \(X\) will be used in the forthcoming discussion:

- \(L^0(m)\) : the Borel functions \(u : X \rightarrow \mathbb{R}\), considered up to \(m\)-a.e. equality.
- \(L^p(m)\) : the functions \(u \in L^0(m)\) for which \(|u|^p\) is integrable.
- \(L^p_{\text{loc}}(m)\) : the functions \(u \in L^0(m)\) with \(u|_B \in L^p(m|_B)\) for any \(B \subseteq X\) bounded Borel.
- \(L^\infty(m)\) : the functions \(u \in L^0(m)\) that are essentially bounded.
- \(L^0(m)^+\) : the Borel functions \(u : X \rightarrow [0, +\infty]\), considered up to \(m\)-a.e. equality.
- \(L^p(m)^+\) : the functions \(u \in L^0(m)^+\) for which \(|u|^p\) is integrable.
- \(L^p_{\text{loc}}(m)^+\) : the functions \(u \in L^0(m)^+\) with \(u|_B \in L^p(m|_B)^+\) for any \(B \subseteq X\) bounded Borel.
- \(\text{LIP}(X)\) : the Lipschitz functions \(u : X \rightarrow \mathbb{R}\), with Lipschitz constant denoted by \(\text{Lip}(u)\).
- \(Sf(X)\) : the functions \(u \in L^0(m)\) that are simple, i.e. with a finite essential image.

Observe that for any \(u \in L^p_{\text{loc}}(m)^+\) it holds that \(u(x) < +\infty\) for \(m\)-a.e. \(x \in X\). We also recall that the space \(Sf(X)\) is strongly dense in \(L^p(m)\) for every \(p \in [1, \infty)\).
Remark 1.1 In [6, Section 1.1] a more general notion of $L^p_{loc}(m)$ is considered, based upon the concept of $K$-set. We chose the present approach for simplicity, but the following discussion would remain unaltered if we replaced our definition of $L^p_{loc}(m)$ with the one of [6].

2 Axiomatic theory of Sobolev spaces

We begin by briefly recalling the axiomatic notion of Sobolev space that has been introduced by V. Gol’dshtein and M. Troyanov in [6, Section 1.2]:

Definition 2.1 (D-structure) Let $(X,d,m)$ be a metric measure space. Let $p \in [1,\infty)$ be fixed. Then a D-structure on $(X,d,m)$ is any map $D$ associating to each function $u \in L^p_{loc}(m)$ a family $D[u] \subseteq \mathbb{L}^0(m)^+$ of pseudo-gradmrents of $u$, which satisfies the following axioms:

A1 (Non triviality) It holds that $\text{Lip}(u) \chi_{\{u>0\}} \in D[u]$ for every $u \in L^p_{loc}(m)^+ \cap \text{LIP}(X)$.

A2 (Upper linearity) Let $u_1, u_2 \in L^p_{loc}(m)$ be fixed. Consider $g_1 \in D[u_1]$ and $g_2 \in D[u_2]$. Suppose that the inequality $g \geq |\alpha_1|g_1 + |\alpha_2|g_2$ holds $m$-a.e. in $X$ for some $g \in \mathbb{L}^0(m)^+$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Then $g \in D[\alpha_1 u_1 + \alpha_2 u_2]$.

A3 (Leibniz rule) Fix a function $u \in L^p_{loc}(m)$ and a pseudo-gradient $g \in D[u]$ of $u$. Then for every $\varphi \in \text{LIP}(X)$ bounded it holds that $g \sup_X |\varphi| + \text{Lip}(\varphi) |u| \in D[\varphi u]$.

A4 (Lattice property) Fix $u_1, u_2 \in L^p_{loc}(m)$. Given any $g_1 \in D[u_1]$ and $g_2 \in D[u_2]$, one has that $\max\{g_1, g_2\} \in D\left[\max\{u_1, u_2\}\right] \cap D\left[\min\{u_1, u_2\}\right]$.

A5 (Completeness) Consider two sequences $(u_n)_n \subseteq L^p_{loc}(m)$ and $(g_n)_n \subseteq L^p(m)$ that satisfy $g_n \in D[u_n]$ for every $n \in \mathbb{N}$. Suppose that there exist $u \in L^p_{loc}(m)$ and $g \in L^p(m)$ such that $u_n \rightarrow u$ in $L^p_{loc}(m)$ and $g_n \rightarrow g$ in $L^p(m)$. Then $g \in D[u]$.

Remark 2.2 It follows from axioms A1 and A2 that $0 \in D[c]$ for every constant map $c \in \mathbb{R}$. Moreover, axiom A2 grants that the set $D[u] \cap L^p(m)$ is convex and that $D[\alpha u] = |\alpha| D[u]$ for every $u \in L^p_{loc}(m)$ and $\alpha \in \mathbb{R} \setminus \{0\}$, while axiom A5 implies that each set $D[u] \cap L^p(m)$ is closed in the space $L^p(m)$.

Given any Borel set $B \subseteq X$, we define the $p$-Dirichlet energy of a map $u \in L^p(m)$ on $B$ as

$$\mathcal{E}_p(u|B) := \inf \left\{ \int_B g^p \, dm \mid g \in D[u] \right\} \in [0, +\infty].$$

(2.1)

For the sake of brevity, we shall use the notation $\mathcal{E}_p(u)$ to indicate $\mathcal{E}_p(u|X)$.

Definition 2.3 (Sobolev space) Let $(X,d,m)$ be a metric measure space. Let $p \in [1,\infty)$ be fixed. Given a D-structure on $(X,d,m)$, we define the Sobolev class associated to $D$ as

$$S^p(X) = S^p(X,d,m,D) := \left\{ u \in L^p_{loc}(m) : \mathcal{E}_p(u) < +\infty \right\}.$$  

(2.2)

Moreover, the Sobolev space associated to $D$ is defined as

$$W^{1,p}(X) = W^{1,p}(X,d,m,D) := L^p(m) \cap S^p(X,d,m,D).$$

(2.3)
Theorem 2.4  The space $W^{1,p}(X,d,m,D)$ is a Banach space if endowed with the norm
\[ \|u\|_{W^{1,p}(X)} := \left( \|u\|_{L^p(m)}^p + E_p(u) \right)^{1/p} \]
for every $u \in W^{1,p}(X)$.  \hfill (2.4)

For a proof of the previous result, we refer to [6, Theorem 1.5].

Proposition 2.5 (Minimal pseudo-gradient) Let $(X,d,m)$ be a metric measure space and let $p \in (1,\infty)$. Consider any $D$-structure on $(X,d,m)$. Let $u \in S^p(X)$ be given. Then there exists a unique element $Du \in D[u]$, which is called the minimal pseudo-gradient of $u$, such that $E_p(u) = \|Du\|_{L^p(m)}^p$.

Both existence and uniqueness of the minimal pseudo-gradient follow from the fact that the set $D[u] \cap L^p(m)$ is convex and closed by Remark 2.2 and that the space $L^p(m)$ is uniformly convex; see [6, Proposition 1.22] for the details.

In order to associate a differential structure to an axiomatic Sobolev space, we need to be sure that the pseudo-gradients of a function depend only on the local behaviour of the function itself, in a suitable sense. For this reason, we propose various notions of locality:

Definition 2.6 (Locality) Let $(X,d,m)$ be a metric measure space. Fix $p \in (1,\infty)$. Then we define five notions of locality for $D$-structures on $(X,d,m)$:

L1 If $B \subseteq X$ is Borel and $u \in S^p(X)$ is $m$-a.e. constant in $B$, then $E_p(u|B) = 0$.

L2 If $B \subseteq X$ is Borel and $u \in S^p(X)$ is $m$-a.e. constant in $B$, then $Du = 0$ $m$-a.e. in $B$.

L3 If $u \in S^p(X)$ and $g \in D[u]$, then $\chi_{\{u>0\}} g \in D[u^+]$.

L4 If $u \in S^p(X)$ and $g_1, g_2 \in D[u]$, then $\min\{g_1, g_2\} \in D[u]$.

L5 If $u \in S^p(X)$ then $Du \leq g$ holds $m$-a.e. in $X$ for every $g \in D[u]$.

Remark 2.7 In the language of [6, Definition 1.11], the properties L1 and L3 correspond to locality and strict locality, respectively.

We now discuss the relations among the several notions of locality:

Proposition 2.8 Let $(X,d,m)$ be a metric measure space. Let $p \in (1,\infty)$. Fix a $D$-structure on $(X,d,m)$. Then the following implications hold:

\[
\begin{align*}
L3 & \implies L2 \implies L1, \\
L4 & \iff L5 \\
L1 + L5 & \implies L2 + L3.
\end{align*}
\]

Proof.

L2 $\implies$ L1. Simply notice that $E_p(u|B) \leq \int_B (Du)^p \, dm = 0$. 

\textbf{L3} \implies \textbf{L2}. Take a constant \( c \in \mathbb{R} \) such that the equality \( u = c \) holds \( m \)-a.e. in \( B \). Given that \( D u \in D[u - c] \cap D[c - u] \) by axiom \textbf{A2} and Remark 2.2, we deduce from \textbf{L3} that

\[
\chi_{\{u > c\}} D u \in D[(u - c)^+], \\
\chi_{\{u < c\}} D u \in D[(c - u)^+].
\]

Given that \( u - c = (u - c)^+ - (c - u)^+ \), by applying again axiom \textbf{A2} we see that

\[
\chi_{\{u \neq c\}} D u = \chi_{\{u > c\}} D u + \chi_{\{u < c\}} D u \in D[u - c] = D[u].
\]

Hence the minimality of \( D u \) grants that

\[
\int_X (D u)^p \, dm \leq \int_{\{u \neq c\}} (D u)^p \, dm,
\]

which implies that \( D u = 0 \) holds \( m \)-a.e. in \( \{u = c\} \), thus also \( m \)-a.e. in \( B \). This means that the \( D \)-structure satisfies the property \textbf{L2}, as required.

\textbf{L4} \implies \textbf{L5}. We argue by contradiction: suppose the existence of \( u \in S^p(X) \) and \( g \in D[u] \) such that \( m(\{D u > g\}) > 0 \), whence \( h := \min\{D u, g\} \in L^p(m) \) satisfies \( \int h^p \, dm < \int (D u)^p \, dm \). Since \( h \in D[u] \) by \textbf{L4}, we deduce that \( \epsilon_p(u) < \int (D u)^p \, dm \), getting a contradiction.

\textbf{L5} \implies \textbf{L4}. Since \( D u \leq g_1 \) and \( D u \leq g_2 \) hold \( m \)-a.e., we see that \( D u \leq \min\{g_1, g_2\} \) holds \( m \)-a.e. as well. Therefore \( \min\{g_1, g_2\} \in D[u] \) by \textbf{A2}.

\textbf{L1} + \textbf{L5} \implies \textbf{L2} + \textbf{L3}. Property \textbf{L1} grants the existence of \( (g_n)_n \subseteq D[u] \) with \( \int_B (g_n)^p \, dm \to 0 \). Hence \textbf{L5} tells us that \( \int_B (D u)^p \, dm \leq \lim_n \int_B (g_n)^p \, dm = 0 \), which implies that \( D u = 0 \) holds \( m \)-a.e. in \( B \), yielding \textbf{L2}. We now prove the validity of \textbf{L3}: it holds that \( D[u] \subseteq D[u^+] \), because we know that \( h = \max\{h, 0\} \in D[\max\{u, 0\}] = D[u^+] \) for every \( h \in D[u] \) by \textbf{A4} and \( 0 \in D[0] \), in particular \( u^+ \in S^p(X) \). Given that \( u^+ = 0 \) \( m \)-a.e. in the set \( \{u \leq 0\} \), one has that \( D u^+ = 0 \) holds \( m \)-a.e. in \( \{u \leq 0\} \) by \textbf{L2}. Hence for any \( g \in D[u] \) we have \( D u^+ \leq \chi_{\{u > 0\}} g \) by \textbf{L5}, which implies that \( \chi_{\{u > 0\}} g \in D[u^+] \) by \textbf{A2}. Therefore \textbf{L3} is proved.

\textbf{Definition 2.9 (Pointwise local)} Let \((X, d, m)\) be a metric measure space and \( p \in (1, \infty) \). Then a \( D \)-structure on \((X, d, m)\) is said to be pointwise local provided it satisfies \textbf{L1} and \textbf{L5} (thus also \textbf{L2}, \textbf{L3} and \textbf{L4} by Proposition 2.8).

We now recall other two notions of locality for \( D \)-structures that appeared in the literature:

\textbf{Definition 2.10 (Strong locality)} Let \((X, d, m)\) be a metric measure space and \( p \in (1, \infty) \). Consider a \( D \)-structure on \((X, d, m)\). Then we give the following definitions:

\begin{enumerate}
  \item[i)] We say that \( D \) is strongly local in the sense of Timoshin provided
  \[
  \chi_{\{u_1 < u_2\}} g_1 + \chi_{\{u_2 < u_1\}} g_2 + \chi_{\{u_1 = u_2\}} (g_1 \wedge g_2) \in D[u_1 \wedge u_2]
  \] \hspace{1cm} (2.6)
  whenever \( u_1, u_2 \in S^p(X), g_1 \in D[u_1] \) and \( g_2 \in D[u_2] \).
\end{enumerate}
ii) We say that $D$ is strongly local in the sense of Shanmugalingam provided
\[
\chi_B g_1 + \chi_{X\setminus B} g_2 \in D[u_2] \quad \text{for every } g_1 \in D[u_1] \text{ and } g_2 \in D[u_2]
\] (2.7)
whenever $u_1, u_2 \in S^p(X)$ satisfy $u_1 = u_2$ m.a.e. on some Borel set $B \subseteq X$.

The above two notions of strong locality have been proposed in [11] and [10], respectively. We now prove that they are actually both equivalent to our pointwise locality property:

**Lemma 2.11** Let $(X, d, m)$ be a metric measure space and $p \in (1, \infty)$. Fix any $D$-structure on $(X, d, m)$. Then the following are equivalent:

i) $D$ is pointwise local.

ii) $D$ is strongly local in the sense of Shanmugalingam.

iii) $D$ is strongly local in the sense of Timoshin.

**Proof.**

i) $\implies$ ii) Fix $u_1, u_2 \in S^p(X)$ such that $u_1 = u_2$ m.a.e. on some $E \subseteq X$ Borel. Pick $g_1 \in D[u_1]$ and $g_2 \in D[u_2]$. Observe that $D(u_2 - u_1) + g_1 \in D[(u_2 - u_1) + u_1] = D[u_2]$ by A2, so that we have $(D(u_2 - u_1) + g_1) \land g_2 \in D[u_2]$ by L4. Since $D(u_2 - u_1) = 0$ m.a.e. on $B$ by L2, we see that $\chi_B g_1 + \chi_{X\setminus B} g_2 \geq (D(u_2 - u_1) + g_1) \land g_2$ holds m.a.e. in $X$, whence accordingly we conclude that $\chi_B g_1 + \chi_{X\setminus B} g_2 \in D[u_2]$ by A2. This shows the validity of ii).

ii) $\implies$ i) First of all, let us prove L1. Let $u \in S^p(X)$ and $c \in \mathbb{R}$ satisfy $u = c$ m.a.e. on some Borel set $B \subseteq X$. Given any $g \in D[u]$, we deduce from ii) that $\chi_{X\setminus B} g \in D[u]$, thus accordingly $\mathcal{E}_p(u|B) \leq \int_B (\chi_{X\setminus B} g)^p \, dm = 0$. This proves the property L1.

To show property L4, fix $u \in S^p(X)$ and $g_1, g_2 \in D[u]$. Let us denote $B := \{g_1 \leq g_2\}$. Therefore ii) grants that $g_1 \land g_2 = \chi_B g_1 + \chi_{X\setminus B} g_2 \in D[u]$, thus obtaining L4. By recalling Proposition 2.8, we conclude that $D$ is pointwise local.

i) + ii) $\implies$ iii) Fix $u_1, u_2 \in S^p(X)$, $g_1 \in D[u_1]$ and $g_2 \in D[u_2]$. Recall that $g_1 \lor g_2 \in D[u_1 \land u_2]$ by axiom A4. Hence by using property ii) twice we obtain that
\[
\chi_{\{u_1 \leq u_2\}} g_1 + \chi_{\{u_1 > u_2\}} (g_1 \lor g_2) \in D[u_1 \land u_2],
\]
\[
\chi_{\{u_2 \leq u_1\}} g_2 + \chi_{\{u_2 > u_1\}} (g_1 \lor g_2) \in D[u_1 \land u_2].
\] (2.8)

The pointwise minimum between the two functions that are written in (2.8) – namely given by $\chi_{\{u_1 < u_2\}} g_1 + \chi_{\{u_2 < u_1\}} g_2 + \chi_{\{u_1 = u_2\}} (g_1 \land g_2)$ – belongs to the class $D[u_1 \land u_2]$ as well by property L4, thus showing iii).

iii) $\implies$ i) First of all, let us prove L1. Fix a function $u \in S^p(X)$ that is m.a.e. equal to some constant $c \in \mathbb{R}$ on a Borel set $B \subseteq X$. By using iii) and the fact that $0 \in D[0]$, we have that
\[
\chi_{\{u < c\}} g \in D[(u - c) \land 0] = D[0] = D[(u - c)^+],
\]
\[
\chi_{\{u > c\}} g \in D[(c - u) \land 0] = D[0] = D[(c - u)^+].
\] (2.9)
Since \( u - c = (u - c)^+ - (c - u)^+ \), we know from A2 and (2.9) that
\[
\chi_{\{u\neq c\}} g = \chi_{\{u<c\}} g + \chi_{\{u>c\}} g \in D[u-c] = D[u],
\]
whence \( \mathcal{E}_p(u|B) \leq \int_B (\chi_{\{u\neq c\}} g)^p \, dm = 0 \). This proves the property L1.

To show property \( \text{L4} \), fix \( u \in S^p(X) \) and \( g_1, g_2 \in D[u] \). Hence (2.6) with \( u_1 = u_2 = u \) simply reads as \( g_1 \land g_2 \in D[u] \), which gives L4. This proves that \( D \) is pointwise local. \( \square \)

**Remark 2.12 (L1 does not imply L2)** In general, as we are going to show in the following example, it can happen that a \( D \)-structure satisfies L1 but not L2.

Let \( G = (V, E) \) be a locally finite connected graph. The distance \( d(x, y) \) between two vertices \( x, y \in V \) is defined as the minimum length of a path joining \( x \) to \( y \), while as a reference measure \( m \) on \( V \) we choose the counting measure. Notice that any function \( u : V \to \mathbb{R} \) is locally Lipschitz and that any bounded subset of \( V \) is finite. We define a \( D \)-structure on the metric measure space \( (V, d, m) \) in the following way:
\[
D[u] := \left\{ g : V \to [0, +\infty) \mid |u(x) - u(y)| \leq g(x) + g(y) \text{ for any } x, y \in V \text{ with } x \sim y \right\} \tag{2.10}
\]
for every \( u : V \to \mathbb{R} \), where the notation \( x \sim y \) indicates that \( x \) and \( y \) are adjacent vertices, i.e. that there exists an edge in \( E \) joining \( x \) to \( y \).

We claim that \( D \) fulfills L1. To prove it, suppose that some function \( u : X \to \mathbb{R} \) is constant on some set \( B \subseteq V \), say \( u(x) = c \) for every \( x \in B \). Define the function \( g : V \to [0, +\infty) \) as
\[
g(x) := \begin{cases} 0 & \text{if } x \in B, \\ |c| + |u(x)| & \text{if } x \in V \setminus B. \end{cases}
\]
Hence \( g \in D[u] \) and \( \int_B g^p \, dm = 0 \), so that \( \mathcal{E}_p(u|B) = 0 \). This proves the validity of L1.

On the other hand, if \( V \) contains more than one vertex, then L2 is not satisfied. Indeed, consider any non-constant function \( u : V \to \mathbb{R} \). Clearly any pseudo-gradient \( g \in D[u] \) of \( u \) is not identically zero, thus there exists \( x \in V \) such that \( Du(x) > 0 \). Since \( u \) is trivially constant on the set \( \{x\} \), we then conclude that property L2 does not hold. \( \blacksquare \)

Hereafter, we shall focus our attention on the pointwise local \( D \)-structures. Under these locality assumptions, one can show the following calculus rules for minimal pseudo-gradients, whose proof is suitably adapted from analogous results that have been proved in [2].

**Proposition 2.13 (Calculus rules for \( Du \))** Let \( (X, d, m) \) be a metric measure space and let \( p \in (1, \infty) \). Consider a pointwise local \( D \)-structure on \( (X, d, m) \). Then the following hold:

i) Let \( u \in S^p(X) \) and let \( N \subseteq \mathbb{R} \) be a Borel set with \( L^1(N) = 0 \). Then the equality \( Du = 0 \) holds m-a.e. in \( u^{-1}(N) \).

ii) Chain rule. Let \( u \in S^p(X) \) and \( \varphi \in \text{LIP}(\mathbb{R}) \). Then \( |\varphi'| \circ u Du \in D[\varphi \circ u] \). More precisely, \( \varphi \circ u \in S^p(X) \) and \( D(\varphi \circ u) = |\varphi'| \circ u Du \) holds m-a.e. in \( X \).
iii) **Leibniz rule.** Let \( u, v \in \mathcal{S}^p(X) \cap L^\infty(m) \). Then \( |u| \frac{Dv}{|v|} + |v| \frac{Du}{|u|} \in D[uv] \). In other words, \( uv \in \mathcal{S}^p(X) \cap L^\infty(m) \) and \( D(uv) \leq |u| \frac{Dv}{|v|} + |v| \frac{Du}{|u|} \) holds \( m \)-a.e. in \( X \).

**Proof.**

**Step 1.** First, consider \( \varphi \) affine, say \( \varphi(t) = \alpha t + \beta \). Then \( |\varphi'| \circ u Du = |\alpha| Du \in D[\varphi \circ u] \) by Remark 2.2 and A2. Now suppose that the function \( \varphi \) is piecewise affine, i.e. there exists a sequence \( (a_k)_{k \in \mathbb{Z}} \subseteq \mathbb{R} \), with \( a_k < a_{k+1} \) for all \( k \in \mathbb{Z} \) and \( a_0 = 0 \), such that each \( \varphi|_{[a_k, a_{k+1}]} \) is an affine function. Let us denote \( A_k := u^{-1}([a_k, a_{k+1}]) \) and \( u_k := (u \vee a_k) \wedge a_{k+1} \) for every index \( k \in \mathbb{Z} \). By combining L3 with the axioms A2 and A5, we can see that \( \chi_{A_k} Du \in D[u_k] \) for every \( k \in \mathbb{Z} \). Called \( \varphi_k : \mathbb{R} \to \mathbb{R} \) that affine function coinciding with \( \varphi \) on \([a_k, a_{k+1}]\), we deduce from the previous case that \( |\varphi'_k| \circ u_k Du_k \in D[\varphi_k \circ u_k] = D[\varphi \circ u_k] \), whence we have that \( |\varphi'| \circ u_k \chi_{A_k} Du \in D[\varphi \circ u_k] \) by L5, A2 and L2. Let us define \((v_n)_n \subseteq \mathcal{S}^p(X)\) as

\[
v_n := \varphi(0) + \sum_{k=0}^{n} (\varphi \circ u_k - \varphi(k)) + \sum_{k=-n}^{-1} (\varphi \circ u_k - \varphi(k)) \quad \text{for every } n \in \mathbb{N}.
\]

Hence \( g_n := \sum_{k=-n}^{n} |\varphi'| \circ u_k \chi_{A_k} Du \in D[v_n] \) for all \( n \in \mathbb{N} \) by A2 and Remark 2.2. Given that there exists \( \psi \in \mathcal{S}^p(X) \cap L^p(m) \) and \( g_n \to |\varphi'| \circ u Du \) in \( L^p(m) \) as \( n \to \infty \), we finally conclude that \( |\varphi'| \circ u Du \in D[\varphi \circ u] \), as required.

**Step 2.** We aim to prove the chain rule for \( \varphi \in C^1(\mathbb{R}) \cap \text{LIP(} \mathbb{R}) \). For any \( n \in \mathbb{N} \), let us denote by \( \varphi_n \) the piecewise affine function interpolating the points \((k/2^n, \varphi(k/2^n))\) with \( k \in \mathbb{Z} \). We call \( D \subseteq \mathbb{R} \) the countable set \( \{k/2^n : k \in \mathbb{Z}, n \in \mathbb{N}\} \). Therefore \( \varphi_n \) uniformly converges to \( \varphi \) and \( \varphi_n(t) \to \varphi(t) \) for all \( t \in \mathbb{R} \setminus D \). In particular, the functions \( g_n := |\varphi'_n| \circ u Du \) converge \( m \)-a.e. to \( |\varphi'| \circ u Du \) by L2. Moreover, \( \text{Lip}(\varphi_n) \leq \text{Lip}(\varphi) \) for every \( n \in \mathbb{N} \) by construction, so that \((g_n)_n \) is a bounded sequence in \( L^p(m) \). This implies that (up to a not relabeled subsequence) \( g_n \to |\varphi'| \circ u Du \) weakly in \( L^p(m) \). Now apply Mazur lemma: for any \( n \in \mathbb{N} \), there exists \( (\alpha_i^j)_{i,j} \subseteq [0,1] \) such that \( \sum_{i=1}^{N_n} \alpha_i^j = 1 \) and \( h_n := \sum_{i=1}^{N_n} \alpha_i^j g_n \to |\varphi'| \circ u Du \) strongly in \( L^p(m) \). Given that \( g_n \in D[\varphi_n \circ u] \) for every \( n \in \mathbb{N} \) by Step 1, we deduce from axiom A2 that \( h_n \in D[\psi_n \circ u] \) for every \( n \in \mathbb{N} \), where \( \psi_n := \sum_{i=1}^{N_n} \alpha_i^j \varphi_i \). Finally, it clearly holds that \( \psi_n \circ u \to \varphi \circ u \) in \( L^p(\mu) \), whence \( |\varphi'| \circ u Du \in D[\varphi \circ u] \) by A5.

**Step 3.** We claim that

\[
Du = 0 \quad m \text{-a.e. in } u^{-1}(K), \quad \text{for every } K \subseteq \mathbb{R} \text{ compact with } L^1(K) = 0. \tag{2.11}
\]

For any \( n \in \mathbb{N} \setminus \{0\} \), define \( \psi_n := n d(\cdot, K) \wedge 1 \) and denote by \( \varphi_n \) the primitive of \( \psi_n \) such that \( \varphi_n(0) = 0 \). Since each \( \psi_n \) is continuous and bounded, any function \( \varphi_n \) is of class \( C^1 \) and Lipschitz. By applying the dominated convergence theorem we see that the \( L^1 \)-measure of the \( \varepsilon \)-neighbourhood of \( K \) converges to 0 as \( \varepsilon \downarrow 0 \), thus accordingly \( \varphi_n \) uniformly converges to \( \text{id}_K \) as \( n \to \infty \). This implies that \( \varphi_n \circ u \to u \) in \( L^p(m) \). Moreover, we know from Step 2 that \( |\psi_n| \circ u Du \in D[\varphi_n \circ u] \), thus also \( \chi_{X \setminus u^{-1}(K)} Du \in D[\varphi_n \circ u] \). Hence \( \chi_{X \setminus u^{-1}(K)} Du \in D[u] \) by A5, which forces the equality \( Du = 0 \) to hold \( m \)-a.e. in \( u^{-1}(K) \), proving (2.11).

**Step 4.** We are in a position to prove i). Choose any \( m' \in \mathcal{S}(X) \) such that \( m \ll m' \ll m \) and call \( \mu := u_{\ast} m' \). Then \( \mu \) is a Radon measure on \( \mathbb{R} \), in particular it is inner regular. We can thus
find an increasing sequence of compact sets $K_n \subseteq N$ such that $\mu(N \setminus \bigcup_n K_n) = 0$. We already know from Step 3 that $Du = 0$ holds m-a.e. in $\bigcup_n u^{-1}(K_n)$. Since $u^{-1}(N) \setminus \bigcup_n u^{-1}(K_n)$ is m-negligible by definition of $\mu$, we conclude that $Du = 0$ holds m-a.e. in $u^{-1}(N)$. This shows the validity of property i).

**STEP 5.** We now prove ii). Let us fix $\varphi \in \text{LIP}(\mathbb{R})$. Choose some convolution kernels $(\rho_n)_n$ and define $\varphi_n := \varphi * \rho_n$ for all $n \in \mathbb{N}$. Then $\varphi_n \to \varphi$ uniformly and $\varphi'_n \to \varphi'$ pointwise $\mathcal{L}^1$-a.e., whence accordingly $\varphi_n \circ u \to \varphi \circ u$ in $L^1_{\text{loc}}(m)$ and $|\varphi'_n| \circ u Du \to |\varphi'| \circ u Du$ pointwise m-a.e. in $X$. Since $|\varphi'_n| \circ u Du \leq \text{Lip}(\varphi) Du$ for all $n \in \mathbb{N}$, there exists a (not relabeled) subsequence such that $|\varphi'_n| \circ u Du \rightharpoonup |\varphi'| \circ u Du$ weakly in $L^p(m)$. We know that $|\varphi'_n| \circ u Du \in D[\varphi_n \circ u]$ for all $n \in \mathbb{N}$ because the chain rule holds for all $\varphi_n \in C^1(\mathbb{R}) \cap \text{LIP}(\mathbb{R})$, hence by combining Mazur lemma and A5 as in Step 2 we obtain that $|\varphi'| \circ u Du \in D[\varphi \circ u]$, so that $\varphi \circ u \in \mathcal{S}^p(X)$ and the inequality $Du(\varphi \circ u) \leq |\varphi'| \circ u Du$ holds m-a.e. in $X$.

**STEP 6.** We conclude the proof of ii) by showing that one actually has $Du(\varphi \circ u) = |\varphi'| \circ u Du$. We can suppose without loss of generality that $\text{Lip}(\varphi) = 1$. Let us define the functions $\psi_\pm(t) := \pm t - \varphi(t)$ for all $t \in \mathbb{R}$. Then it holds m-a.e. in $u^{-1}(\{\pm \varphi' \geq 0\})$ that

$$Du = D(\pm u) \leq D(\varphi \circ u) + D(\psi \circ u) \leq (|\varphi'| \circ u + |\psi| \circ u) Du = Du,$$

which forces the equality $Du(\varphi \circ u) = \pm \varphi' \circ u Du$ to hold m-a.e. in the set $u^{-1}(\{\pm \varphi' \geq 0\})$. This grants the validity of $Du(\varphi \circ u) = |\varphi'| \circ u Du$, thus completing the proof of item ii).

**STEP 7.** We consider the case in which $u, v \geq c$ is satisfied m-a.e. in $X$, for some $c > 0$. Call $\varepsilon := \min\{c, c^2\}$ and note that the function log is Lipschitz on the interval $[\varepsilon, +\infty)$, then choose any Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$ that coincides with log on $[\varepsilon, +\infty)$. Now call $C$ the constant $\|uv\|_{L^\infty(m)}$ and choose a Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi = \exp$ on the interval $[\log \varepsilon, C]$. By applying twice the chain rule ii), we thus deduce that $uv \in \mathcal{S}^p(X)$ and the m-a.e. inequalities

$$Du(uv) \leq |\psi| \circ u Du(uv) \leq |uv| (Du du + Du log v) = |uv| (\frac{Dv}{|v|} + \frac{Du}{|u|}) = |u| Du + |v| Du.$$ 

Therefore the Leibniz rule iii) is verified under the additional assumption that $u, v \geq c > 0$.

**STEP 8.** We conclude by proving item iii) for general $u, v \in \mathcal{S}^p(X) \cap L^\infty(m)$. Given any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, let us denote $I_{n,k} := [k/n, (k + 1)/n)$. Call $\varphi_{n,k} : \mathbb{R} \to \mathbb{R}$ the continuous function that is the identity on $I_{n,k}$ and constant elsewhere. For any $n \in \mathbb{N}$, let us define

$$u_{n,k} := u - \frac{k - 1}{n}, \quad \tilde{u}_{n,k} := \varphi_{n,k} \circ u - \frac{k - 1}{n} \quad \text{for all } k \in \mathbb{Z},$$

$$v_{n,\ell} := v - \frac{\ell - 1}{n}, \quad \tilde{v}_{n,\ell} := \varphi_{n,\ell} \circ v - \frac{\ell - 1}{n} \quad \text{for all } \ell \in \mathbb{Z}.$$ 

Notice that the equalities $u_{n,k} = \tilde{u}_{n,k}$ and $v_{n,\ell} = \tilde{v}_{n,\ell}$ hold m-a.e. in $u^{-1}(I_{n,k})$ and $v^{-1}(I_{n,\ell})$, respectively. Hence $Du_{n,k} = D\tilde{u}_{n,k} = Du$ and $Dv_{n,\ell} = D\tilde{v}_{n,\ell} = Dv$ hold m-a.e. in $u^{-1}(I_{n,k})$ and $v^{-1}(I_{n,\ell})$, respectively, but we also have that

$$Du_{n,k}v_{n,\ell} = D(\tilde{u}_{n,k} \tilde{v}_{n,\ell}) \quad \text{is verified m-a.e. in } u^{-1}(I_{n,k}) \cap v^{-1}(I_{n,\ell}).$$
Moreover, we have the m-a.e. inequalities $1/n \leq \tilde{u}_{n,k}, \tilde{v}_{n,\ell} \leq 2/n$ by construction. Therefore for any $k, \ell \in \mathbb{Z}$ it holds m-a.e. in $u^{-1}(I_{n,k}) \cap v^{-1}(I_{n,\ell})$ that

$$D(uv) \leq D(\tilde{u}_{n,k} \tilde{v}_{n,\ell}) + \frac{|k-1|}{n} D\tilde{v}_{n,\ell} + \frac{|\ell-1|}{n} D\tilde{u}_{n,k}$$

$$\leq |\tilde{v}_{n,\ell}| D\tilde{u}_{n,k} + |\tilde{u}_{n,k}| D\tilde{v}_{n,\ell} + \frac{|k-1|}{n} Dv_{n,\ell} + \frac{|\ell-1|}{n} Du_{n,k}$$

$$\leq \left(|v| + \frac{4}{n}\right) Du + \left(|u| + \frac{4}{n}\right) Dv,$$

where the second inequality follows from the case $u, v \geq c > 0$, treated in STEP 7. This implies that the inequality $D(uv) \leq |u| Dv + |v| Du + 4(Du + Dv)/n$ holds m-a.e. in $X$. Given that $n \in \mathbb{N}$ is arbitrary, the Leibniz rule iii) follows.

\[\square\]

### 3 Cotangent module associated to a $D$-structure

It is shown in [4] that any metric measure space possesses a first-order differential structure, whose construction relies upon the notion of $L^p(m)$-normed $L^\infty(m)$-module. For completeness, we briefly recall its definition and we refer to [4,5] for a comprehensive exposition of this topic.

**Definition 3.1 (Normed module)** Let $(X, d, m)$ be a metric measure space and $p \in [1, \infty)$. Then an $L^p(m)$-normed $L^\infty(m)$-module is any quadruplet $(\mathcal{M}, \| \cdot \|_\mathcal{M}, \cdot, | \cdot |)$ such that

i) $(\mathcal{M}, \| \cdot \|_\mathcal{M})$ is a Banach space,

ii) $(\mathcal{M}, \cdot)$ is an algebraic module over the commutative ring $L^\infty(m)$,

iii) the pointwise norm operator $| \cdot | : \mathcal{M} \to L^p(m)^+$ satisfies

$$|f \cdot v| = |f| |v| \text{ m-a.e. for every } f \in L^\infty(m) \text{ and } v \in \mathcal{M},$$

$$\|v\|_\mathcal{M} = \|v\|_{L^p(m)} \text{ for every } v \in \mathcal{M}. \quad (3.1)$$

A key role in [4] is played by the cotangent module $L^2(T^*X)$, which has a structure of $L^2(m)$-normed $L^\infty(m)$-module; see [5] Theorem/Definition 1.8] for its characterisation. The following result shows that a generalised version of such object can be actually associated to any $D$-structure, provided the latter is assumed to be pointwise local.

**Theorem 3.2 (Cotangent module associated to a $D$-structure)** Let $(X, d, m)$ be any metric measure space and let $p \in (1, \infty)$. Consider a pointwise local $D$-structure on $(X, d, m)$. Then there exists a unique couple $(L^p(T^*X; D), d)$, where $L^p(T^*X; D)$ is an $L^p(m)$-normed $L^\infty(m)$-module and $d : S^p(X) \to L^p(T^*X; D)$ is a linear map, such that the following hold:

i) the equality $|du| = Du$ is satisfied m-a.e. in $X$ for every $u \in S^p(X)$,

ii) the vector space $\mathcal{V}$ of all elements of the form $\sum_{i=1}^n X_{B_i} du_i$, where $(B_i)_i$ is a Borel partition of $X$ and $(u_i)_i \subseteq S^p(X)$, is dense in the space $L^p(T^*X; D)$. 


Consider any element \( \varphi \), first of all, let us define the grants that \( \Phi \) is well-defined, in the sense that it does not depend on the particular way of representing \( \varphi \), and that \( \Phi : V \to M \) preserves the pointwise norm. In particular, one has that the map \( \Phi : V \to M \) is (linear and) continuous. Since \( V \) is dense in \( L^p(T^*X; D) \), we can uniquely extend \( \Phi \) to a linear and continuous map \( \Phi : L^p(T^*X; D) \to M \), which also preserves the pointwise norm. Moreover, we deduce from the very definition of \( \Phi \) that the identity \( \Phi(h \varphi) = h \Phi(\varphi) \) holds for every \( \varphi \in V \) and \( h \in S^f(X) \), whence the \( L^p(M) \)-linearity of \( \Phi \) follows by an approximation argument. Finally, the image \( \Phi(V) \) is dense in \( M \), which implies that \( \Phi \) is surjective. Therefore \( \Phi \) is the unique isomorphism satisfying \( \Phi \circ \mathcal{D} = \mathcal{D}' \).

**Uniqueness.** Consider any element \( \varphi \in V \) written as \( \varphi = \sum_{i=1}^n \chi_{B_i} \, du_i \), with \( (B_i)_i \) Borel partition of \( X \) and \( u_1, \ldots, u_n \in S^p(X) \). Notice that the requirements that \( \Phi \) is \( L^\infty(M) \)-linear and \( \Phi \circ \mathcal{D} = \mathcal{D}' \) force the definition \( \Phi(\varphi) := \sum_{i=1}^n \chi_{B_i} \, d' u_i \). The \( M \)-a.e. equality

\[
|\Phi(\varphi)| = \sum_{i=1}^n \chi_{B_i} |d' u_i| = \sum_{i=1}^n \chi_{B_i} \mathcal{D} u_i = \sum_{i=1}^n \chi_{B_i} |du_i| = |\varphi|
\]

grants that \( \Phi(\varphi) \) is well-defined, in the sense that it does not depend on the particular way of representing \( \varphi \), and that \( \Phi : V \to M \) preserves the pointwise norm. In particular, one has that the map \( \Phi : V \to M \) is (linear and) continuous. Since \( V \) is dense in \( L^p(T^*X; D) \), we can uniquely extend \( \Phi \) to a linear and continuous map \( \Phi : L^p(T^*X; D) \to M \), which also preserves the pointwise norm. Moreover, we deduce from the very definition of \( \Phi \) that the identity \( \Phi(h \varphi) = h \Phi(\varphi) \) holds for every \( \varphi \in V \) and \( h \in S^f(X) \), whence the \( L^p(M) \)-linearity of \( \Phi \) follows by an approximation argument. Finally, the image \( \Phi(V) \) is dense in \( M \), which implies that \( \Phi \) is surjective. Therefore \( \Phi \) is the unique isomorphism satisfying \( \Phi \circ \mathcal{D} = \mathcal{D}' \).

**Existence.** First of all, let us define the pre-cotangent module as

\[
P_{cm} := \left\{ \left\{ (B_i, u_i) \right\}_{i=1}^n \left| n \in \mathbb{N}, u_1, \ldots, u_n \in S^p(X), (B_i)_{i=1}^n \text{ Borel partition of } X \right. \right\}.
\]

We define an equivalence relation on \( P_{cm} \) as follows: we declare that \( \left\{ (B_i, u_i) \right\}_i \sim \left\{ (C_j, v_j) \right\}_j \) provided \( \mathcal{D}(u_i - v_j) = 0 \) holds \( M \)-a.e. on \( B_i \cap C_j \) for every \( i, j \). The equivalence class of an element \( \left\{ (B_i, u_i) \right\}_i \) of \( P_{cm} \) will be denoted by \( [B_i, u_i]_i \). We can endow the quotient \( P_{cm}/\sim \) with a vector space structure:

\[
[B_i, u_i] + [C_j, v_j] := [B_i \cap C_j, u_i + v_j], \quad \lambda [B_i, u_i] := [B_i, \lambda u_i],
\]

for every \( [B_i, u_i], [C_j, v_j] \in P_{cm}/\sim \) and \( \lambda \in \mathbb{R} \). We only check that the sum operator is well-defined; the proof of the well-posedness of the multiplication by scalars follows along the same lines. Suppose that \( \left\{ (B_i, u_i) \right\}_i \sim \left\{ (B'_k, u'_k) \right\}_k \) and \( \left\{ (C_j, v_j) \right\}_j \sim \left\{ (C'_\ell, v'_\ell) \right\}_\ell \), in other words \( \mathcal{D}(u_i - u'_k) = 0 \) \( M \)-a.e. on \( B_i \cap B'_k \) and \( \mathcal{D}(v_j - v'_\ell) = 0 \) \( M \)-a.e. on \( C_j \cap C'_\ell \) for every \( i, j, k, \ell \), whence accordingly

\[
\mathcal{D}((u_i + v_j) - (u'_k + v'_\ell)) \leq \mathcal{D}(u_i - u'_k) + \mathcal{D}(v_j - v'_\ell) = 0 \quad \text{holds } M \text{-a.e. on } (B_i \cap C_j) \cap (B'_k \cap C'_\ell).
\]

This shows that \( \left\{ (B_i \cap C_j, u_i + v_j) \right\}_i \sim \left\{ (B'_k \cap C'_\ell, u'_k + v'_\ell) \right\}_k \), thus proving that the sum operator defined in (3.2) is well-posed. Now let us define

\[
\| [B_i, u_i] \|_{L^p(T^*X; D)} := \left( \sum_{i=1}^n \left( \int_{B_i} |\mathcal{D} u_i|^p \, dm \right) \right)^{1/p} \quad \text{for every } [B_i, u_i] \in P_{cm}/\sim.
\]
Such definition is well-posed: if \( \{(B_i, u_i)\}_i \sim \{(C_j, v_j)\}_j \) then for all \( i, j \) it holds that

\[
|D_{u_i} - D_{v_j}|_1^5 \leq D(u_i - v_j) = 0 \quad \text{m.a.e. on } B_i \cap C_j,
\]

i.e. that the equality \( D_{u_i} = D_{v_j} \) is satisfied m.a.e. on \( B_i \cap C_j \). Therefore one has that

\[
\sum_i \left( \int_{B_i} (D_{u_i})^p \, dm \right)^{1/p} = \sum_{i, j} \left( \int_{B_i \cap C_j} (D_{u_i})^p \, dm \right)^{1/p} = \sum_{i, j} \left( \int_{B_i \cap C_j} (D_{v_j})^p \, dm \right)^{1/p}
\]

which grants that \( \| \cdot \|_{L^p(T^*X; D)} \) in (3.3) is well-defined. The fact that it is a norm on \( Pcm/\sim \) easily follows from standard verifications. Hence let us define

\[
L^p(T^*X; D) := \text{completion of } (Pcm/\sim, \| \cdot \|_{L^p(T^*X; D)}),
\]

\[
d : S^p(X) \to L^p(T^*X; D), \quad du := [X, u] \text{ for every } u \in S^p(X).
\]

Observe that \( L^p(T^*X; D) \) is a Banach space and that \( d \) is a linear operator. Furthermore, given any \( [B_i, u_i]_i \in Pcm/\sim \) and \( h = \sum_j \lambda_j x_{C_j} \in Sf(X) \), where \( (\lambda_j)_j \subseteq \mathbb{R} \) and \( (C_j)_j \) is a Borel partition of \( X \), we set

\[
[[B_i, u_i]]_i := \sum_i x_{B_i} D_{u_i},
\]

\[
h[B_i, u_i]_i := [B_i \cap C_j, \lambda_j u_i]_{i, j}.
\]

One can readily prove that such operations, which are well-posed again by the pointwise locality of \( D \), can be uniquely extended to a pointwise norm \( \| \cdot \| : L^p(T^*X; D) \to L^p(m)^+ \) and to a multiplication by \( L^\infty \)-functions \( L^\infty(m) \times L^p(T^*X; D) \to L^p(T^*X; D) \), respectively. Therefore the space \( L^p(T^*X; D) \) turns out to be an \( L^p(m) \)-normed \( L^\infty(m) \)-module when equipped with the operations described so far. In order to conclude, it suffices to notice that

\[
|du| = [[X, u]] = D_u \quad \text{holds m.a.e. for every } u \in S^p(X)
\]

and that \( [B_i, u_i]_i = \sum_i x_{B_i} du_i \) for all \( [B_i, u_i]_i \in Pcm/\sim \), giving i) and ii), respectively. \( \square \)

In full analogy with the properties of the cotangent module that is studied in [4], we can show that the differential \( d \) introduced in Theorem 3.2 is a closed operator, which satisfies both the chain rule and the Leibniz rule.

**Theorem 3.3 (Closure of the differential)** Let \( (X, d, m) \) be a metric measure space and let \( p \in (1, \infty) \). Consider a pointwise local \( D \)-structure on \( (X, d, m) \). Then the differential operator \( d \) is closed, i.e. if a sequence \( (u_n)_n \subseteq S^p(X) \) converges in \( L^p_{loc}(m) \) to some \( u \in L^p_{loc}(m) \) and \( du_n \rightharpoonup \omega \) weakly in \( L^p(T^*X; D) \) for some \( \omega \in L^p(T^*X; D) \), then \( u \in S^p(X) \) and \( du = \omega \).
Proof. Since $d$ is linear, we can assume with no loss of generality that $du_n \to \omega$ in $L^p(T^*X; D)$ by Mazur lemma, so that $d(u_n - u_m) \to \omega - du_m$ in $L^p(T^*X; D)$ for any $m \in \mathbb{N}$. In particular, one has $u_n - u_m \to u - u_m$ in $L^p_T(m)$ and $D(u_n - u_m) = \|d(u_n - u_m)\| \to |\omega - du_m|$ in $L^p(m)$ as $n \to \infty$ for all $m \in \mathbb{N}$, whence $u - u_m \in S^p(X)$ and $D(u - u_m) \leq |\omega - du_m|$ holds $m$-a.e. for all $m \in \mathbb{N}$ by A5 and L5. Therefore $u = (u - u_0) + u_0 \in S^p(X)$ and

$$\lim_{m \to \infty} \|du - du_m\|_{L^p(T^*X; D)} = \lim_{m \to \infty} \|D(u - u_m)\|_{L^p(m)} \leq \lim_{m \to \infty} \|\omega - du_m\|_{L^p(T^*X; D)}$$

which grants that $du_m \to du$ in $L^p(T^*X; D)$ as $m \to \infty$ and accordingly that $du = \omega$. \hfill \Box

Proposition 3.4 (Calculus rules for $du$) Let $(X, d, m)$ be any metric measure space and let $p \in (1, \infty)$. Consider a pointwise local $D$-structure on $(X, d, m)$. Then the following hold:

i) Let $u \in S^p(X)$ and let $N \subseteq \mathbb{R}$ be a Borel set with $\mathcal{L}^1(N) = 0$. Then $\chi_{u^{-1}(N)} du = 0$.

ii) Chain rule. Let $u \in S^p(X)$ and $\varphi \in \text{LIP}(\mathbb{R})$ be given. Recall that $\varphi \circ u \in S^p(X)$ by Proposition 2.13. Then $d(\varphi \circ u) = \varphi' \circ u du$.

iii) Leibniz rule. Let $u, v \in S^p(X) \cap L^\infty(m)$ be given. Recall that $uv \in S^p(X) \cap L^\infty(m)$ by Proposition 2.13. Then $d(uv) = u dv + v du$.

Proof.

i) We have that $|du| = Du = 0$ holds $m$-a.e. on $u^{-1}(N)$ by item i) of Proposition 2.13, thus accordingly $\chi_{u^{-1}(N)} du = 0$, as required.

ii) If $\varphi$ is an affine function, say $\varphi(t) = \alpha t + \beta$, then $d(\varphi \circ u) = d(\alpha u + \beta) = \alpha du = \varphi' \circ u du$. Now suppose that $\varphi$ is a piecewise affine function. Say that $(I_n)_n$ is a sequence of intervals whose union covers the whole real line $\mathbb{R}$ and that $(\psi_n)_n$ is a sequence of affine functions such that $\varphi|_{I_n} = \psi_n$ holds for every $n \in \mathbb{N}$. Since $\varphi'$ and $\psi'_n$ coincide $\mathcal{L}^1$-a.e. in the interior of $I_n$, we have that $d(\varphi \circ f) = d(\psi_n \circ f) = \psi'_n \circ f df = \varphi' \circ f df$ holds $m$-a.e. on $f^{-1}(I_n)$ for all $n$, so that $d(\varphi \circ u) = \varphi' \circ u du$ is verified $m$-a.e. on $\bigcup_n u^{-1}(I_n) = X$.

To prove the case of a general Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$, we want to approximate $\varphi$ with a sequence of piecewise affine functions: for any $n \in \mathbb{N}$, let us denote by $\varphi_n$ the function that coincides with $\varphi$ at $\{k/2^n : k \in \mathbb{Z}\}$ and that is affine on the interval $\left[k/2^n, (k+1)/2^n\right]$ for every $k \in \mathbb{Z}$. It is clear that $\text{Lip}(\varphi_n) \leq \text{Lip}(\varphi)$ for all $n \in \mathbb{N}$. Moreover, one can readily check that, up to a not relabeled subsequence, $\varphi_n \to \varphi$ uniformly on $\mathbb{R}$ and $\varphi'_n \to \varphi'$ pointwise $\mathcal{L}^1$-almost everywhere. The former grants that $\varphi_n \circ u \to \varphi \circ u$ in $L^p_{loc}(m)$. Given that $|\varphi'_n - \varphi'|^p \circ u (Du)^p \leq 2^p \text{Lip}(\varphi)^p (Du)^p \in L^1(m)$ for all $n \in \mathbb{N}$ and $|\varphi'_n - \varphi'|^p \circ u (Du)^p \to 0$ pointwise $m$-a.e. by the latter above together with i), we obtain $\int |\varphi'_n - \varphi'|^p \circ u (Du)^p dm \to 0$ as $n \to \infty$ by the dominated convergence theorem. In other words, $\varphi'_n \circ u du \to \varphi' \circ u du$ in the strong topology of $L^p(T^*X; D)$. Hence Theorem 3.3 ensures that $d(\varphi \circ u) = \varphi' \circ u du$, thus proving the chain rule ii) for any $\varphi \in \text{LIP}(\mathbb{R})$.  

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iii) In the case $u, v \geq 1$, we argue as in the proof of Proposition 2.13 to deduce from ii) that
\[
\frac{d(uv)}{uv} = d \log(uv) = d \left( \log(u) + \log(v) \right) = d \log(u) + d \log(v) = \frac{du}{u} + \frac{dv}{v},
\]
whence we get $d(uv) = u \, dv + v \, du$ by multiplying both sides by $uv$.

In the general case $u, v \in L^\infty(m)$, choose a constant $C > 0$ so big that $u + C, v + C \geq 1$.

By the case treated above, we know that
\[
d((u+C)(v+C)) = (u+C) \, d(v+C) + (v+C) \, d(u+C)
= (u+C) \, dv + (v+C) \, du
= u \, dv + v \, du + C \, d(u+v),
\] (3.4)
while a direct computation yields
\[
d((u+C)(v+C)) = d(\left(uv + C(u+v) + C^2\right) = d(uv) + C \, d(u+v).
\] (3.5)
By subtracting (3.5) from (3.4), we finally obtain that $d(uv) = u \, dv + v \, du$, as required. This completes the proof of the Lebniz rule iii). \(\square\)

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