MINIMAL FREE RESOLUTION OF MONOMIAL IDEALS
BY ITERATED MAPPING CONE

LEILA SHARIFAN

Abstract. In this paper we study minimal free resolutions of some classes of monomial ideals. We first give a sufficient condition to check the minimality of the resolution obtained by the mapping cone. Using it, we obtain the Betti numbers of max-path ideals of rooted trees and ideals containing powers of variables. In particular, we discuss about resolutions of ideals of the form $J_H + (x_1^{i_1}, \ldots, x_m^{i_m})$ where $J_H$ is the edge ideal of a hypergraph $H$.

Keywords: Mapping cone, regularity, max-path ideal, edge ideal of hypergraph, independent number.

MSC(2010): Primary: 13D02; Secondary: 05E40, 05C65.

1. Introduction

Let $k$ be a field, $R = k[x_1, \ldots, x_n]$ the polynomial ring in $n$ variables, and $I$ a graded ideal. Finding algebraic properties of $I$ like regularity, projective dimension and depth is a central problem in commutative algebra and algebraic geometry. Computing the (graded) minimal free resolution of $I$ is the key to find these invariants. However, describing the precise minimal free resolution of an ideal, even in the case that $I$ is a square-free monomial ideal is not an easy problem and when $I$ is not a square monomial ideal the problem is more difficult. An standard tool to compute a free resolution of an ideal is iterated mapping cone. In the monomial case, several well known resolution arise as iterated mapping cone. For example, the Taylor resolution [13], the Eliahou-Kervaire resolution of stable monomial ideals [4] and resolution of monomial ideals with linear quotients [9].

In this paper, by iterated mapping cone, we study minimal free resolution of some class of monomial ideals. Note that, in general the result of the mapping cone is not a minimal free resolution. The importance of our work is that we find a sufficient condition for minimality of the resolution obtained by this tool. Then we focus to the monomial case and study the particular classes max-path ideals of rooted trees and monomial ideals containing some powers of variables.

The paper proceeds as follows. After reviewing some algebraic tools in Section 2, in Theorem 2.4 we show that for a graded ideal $I$ and a homogeneous polynomial $f$ which does not belong to $I$, the minimal free resolution
of $R/I + (f)$ is obtained by the mapping cone provided that we can decompose $f$ as $f = h_1 h_2$ where $h_1, h_2$ are homogeneous polynomials, $\deg(h_2) > 0$ and $(I : f) = (I : h_1)$. Theorem 2.4 leads us to introduce the class of monomial ideals of decreasing type. We say $I = (u_1, \ldots, u_m)$ is of decreasing type with respect to the order $u_1, \ldots, u_m$ of its generators, if for each $u_j$ there exists $x_j \in \text{supp}(u_j)$ such that $\deg_{x_j}(u_j) > \deg_{x_j}(u_r)$ for all $r < j$. In this situation the minimal free resolution of $R/I$ is obtained by iterated mapping cone (see Corollary 2.8).

In the next sections we apply Theorem 2.4 and Corollary 2.8 to study homological properties of max-path ideals of rooted trees and monomial ideals containing powers of some variables. Beside this goal we present some other interesting properties of the mentioned classes of ideals.

When $I$ is a square-free monomial ideal, it is possible to associate to $I$ a combinatorial object such as graph or hypergraph and encode algebraic properties of $I$ in terms of combinatorial properties of corresponding object. It is also natural to start by a combinatorial object and associate to it an ideal. The classes of path ideals of graphs in [2] and max-path ideals of trees in [12] are defined in this way.

Let $T$ be a rooted tree, the max-path ideal of $T$, denoted $PI(T)$, is defined as

$$PI(T) = (x_{i_1} \cdots x_{i_t} ; \ i_1, \ldots, i_t \text{ is a maximal path in } T) \subseteq R,$$

where by a maximal path we mean a path between the root of tree and one of its leaves. In Theorem 3.4 we give an interesting application of Corollary 2.8. We show that $PI(T)$ is of decreasing type and compute Betti numbers, regularity, and projective dimension of $R/PI(T)$ in terms of the number of vertices of $T$ and the number of its leaves.

Next, we consider $PI(T)$ as the facet ideal of a simplicial complex, denoting by $\Delta_{PI(T)}$ the simplicial complex corresponding to $PI(T)$, in Theorem 3.6 we show that $\Delta_{PI(T)}$ is a simplicial tree. This shows that $R/PI(T)$ is sequentially Cohen-Macaulay and so, $PI(T)^\vee$ is a componentwise linear ideal (Theorem 3.7).

Section 4 is devoted to the study of monomial ideals that contain some powers of some variables. Assume that $I = J + (x_{i_1}^{a_1}, \ldots, x_{i_m}^{a_m})$ where $J$ is a monomial ideal and $G(I) = G(J) \cup \{x_{i_1}^{a_1}, \ldots, x_{i_m}^{a_m}\}$. In Theorem 4.1 we give a formula for the graded Betti numbers of $R/I$. We remark that this result is a straightforward consequence of [11, Theorem 6.1] (see also [11, Theorem 2.1]). Here we give an easier proof for it as an application of Theorem 2.4.

Next we apply Theorem 4.1 to study monomial ideals of the form $I = J + (x_{i_1}^2, \ldots, x_{i_m}^2)$ where $J$ is a square-free monomial ideal. We consider $J$ as edge ideal of a hypergraph $H$. In Theorem 4.3 we compute the graded Betti numbers of $R/I$ in terms of the graded Betti numbers of $R/J_H$ and the graded Betti numbers of $R/J_{H'}$ for some hypergraphs $H'$ associated to $H$. We believe that this approach can be more efficient than the technique
of polarization in many cases. For example, when $\mathcal{H}$ is a graph, we just need to consider the edge ideals of the graph and some induced subgraphs of it instead of working in a larger polynomial ring. To see an application of our approach, in Theorem 4.7 and Theorem 4.8 we focus to the particular case $I = J_G + (x_1^2, \ldots, x_n^2)$ when $G = K_{n_1, \ldots, n_t}$. We compute the graded Betti numbers of $R/I$ and show that the property of being a complete $t-$partite graph for $G$ depends only to the last Betti numbers of $R/I$.

Another interesting consequence of Theorem 4.3 is given in Corollary 4.4. There, we study the last (graded) Betti numbers of $R/I$ and relate these invariants to the maximal independent sets of $\mathcal{H}$. In Corollary 4.5, for the case $I = J_H + (x_1^2, \ldots, x_n^2)$, we show that $\beta_{n,j}(R/I)$ is equal to the number of facets of size $j - n$ in the independent complex of $H$. As an important consequence of it we have $\text{reg}(R/I) = \alpha(H)$ where $\alpha(H)$ is the independence number of $H$. Note that the formula of regularity, just in the case that $\mathcal{H}$ is a graph, also obtained by [15, Theorem 20 and Lemma 21].

2. Preliminaries

Throughout this paper, $m = (x_1, \ldots, x_n)$ is the unique maximal graded ideal of $R$ and the set $\{1, \ldots, n\}$ is denoted by $[n]$.

For a graded $R-$module $M$, let $\{\beta_{i,j}(M)\}$ be the sequence of the graded Betti numbers of $M$, the Castelnuovo-Mumford regularity of $M$ is defined as

$$\text{reg}(M) = \max\{j - i ; \beta_{i,j}(M) \neq 0\},$$

and the projective dimension of $M$ is defined as

$$\text{pd}(M) = \max\{i ; \beta_{i,j}(M) \neq 0 \text{ for some } j\}.$$  

By Auslander-Buchsbaum formula (see [3, Theorem 19.9], one has $\text{pd}(M) + \text{depth}(M) = n$.

Remark 2.1. For a squarefree monomial ideal $I \subsetneq m$, since $m$ does not belong to the set of associated primes of $I$, we always have $\text{depth}(R/I) > 0$ and consequently, $\text{pd}(R/I) < n$.

Let $I = (x_{11} \cdots x_{1m_1}, \ldots, x_{lt} \cdots x_{lm_t})$ be a squarefree monomial ideal, the Alexander dual ideal of $I$, denote $I^\vee$, is defined as

$$I^\vee = (x_{11}, \ldots, x_{1m_1}) \cap \cdots \cap (x_{lt}, \ldots, x_{lm_t}).$$

For a graded $R$-module $M$ and $d \in \mathbb{Z}$ we write $M_{<d>}$ for the submodule of $M$ which is generated by all homogeneous elements of $M$ with degree $d$. We say that $M$ has a $d$-linear resolution if $\beta_{i,j}(M) = 0$ for $j \neq d + i$ and we say $M$ is componentwise linear if for all integers $d$ the module $M_{<d>}$ has a $d$-linear resolution.

Definition 2.2. A graded $R-$module $M$ is called sequentially Cohen-Macaulay if there exists a finite filtration of graded $R-$modules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$
such that each $M_i/M_{i-1}$ is Cohen-Macaulay and
\[
\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).
\]

**Theorem 2.3.** Let $I$ be a squarefree monomial ideal. Then

1. $\text{pd}(R/I) = \text{reg}(I')$ ([14, Theorem 2.1]).
2. $R/I$ is sequentially Cohen-Macaulay if and only if $I'$ is componentwise linear ([6], see also [7, Theorem 8.2.20]).

**Iterated mapping cone.** In the following we recall the mapping cone technique from [9]. Let $\{f_1, \ldots, f_m\}$ be a homogeneous system of generators for $I$, and $I_j = (f_1, \ldots, f_j)$. Then for $j = 2, \ldots, m$ there are exact sequences
\[
0 \to R/(I_{j-1} : f_j) \to R/I_{j-1} \to R/I_j \to 0.
\]
Assuming that a free $R$–resolution $(F_i, \delta_i)$ of $R/I_{j-1}$ and a free $R$–resolution $(G_i, d_i)$ of $R/(I_{j-1} : f_j)$ are known, we can obtain a resolution $(M(\psi), \gamma_i)$ of $R/I_j$ as a mapping cone of a complex homomorphism $\psi : G_i \to F_i$ which is a lifting of the map $R/(I_{j-1} : f_j) \to R/I_{j-1}$. The mapping cone $M(\psi)$ is the complex such that
\[
(M(\psi))_i = F_i \oplus G_{i-1},
\]
with the differential maps
\[
\gamma_i(x, y) = (\psi_{i-1}(y) + \delta_i(x), -d_{i-1}(y))
\]
where $x \in F_i$ and $y \in G_{i-1}$. This complex is exact (see [3, Page 650 and Proposition A3.19]), so, it is a free resolution for $R/I_j$.

Of course, in general, such a resolution may be non-minimal. But in any case this method yields an inductive procedure to compute a resolution of $R/I$ provided for each $j$, a resolution of $R/(I_{j-1} : f_j)$ is known as well as the comparison map.

Next, we give a sufficient condition to check the minimality of the resolution obtained by the mapping cone technique for $R/I + (f)$ where $I$ is a graded ideal and $f$ is a homogeneous polynomial.

We remark that this result is a generalization of [1, Theorem 2.7] where the authors study the minimal free resolution of the path ideal of a rooted tree.

**Theorem 2.4.** Let $I$ be a graded ideal of $R$ and $f$ is a homogeneous polynomial of degree $d$ which does not belong to $I$ then we have the following graded short exact sequence
\[
0 \to R/(I : f)(-d) \to R/I \to R/I + (f) \to 0.
\]
Assuming that the minimal free resolution of the modules $R/(I : f)$ and $R/I$ are already known. Then the minimal free resolution of $R/I + (f)$ is obtained by the mapping cone provided that $f = h_1h_2$ where $h_1$ and $h_2$ are homogeneous polynomials, $\deg(h_2) > 0$ and $(I : f) = (I : h_1)$.
and in this case

(a):

\[ \beta_{ij}(R/I + (f)) = \beta_{ij}(R/I) + \beta_{i-1j-d}(R/(I : f)), \]

(b):

\[ \text{reg}(R/(I + (f)) = \max\{\text{reg}(R/I), \text{reg}(R/(I : f)) + d - 1\} \]

(c):

\[ \text{pd}(R/(I + (f)) = \max\{\text{pd}(R/I), \text{pd}(R/(I : f)) + 1\}. \]

Proof. Let \((F, \delta)\) be the minimal free resolution of \(R/I\), \((G_, d.)\) be the minimal free resolution of \(R/I : h_1\) shifted by \(\text{deg}(h_1)\) and \(\psi : G \to F\). be the complex graded homomorphism which is a lifting of the map \(R/I : h_1 \to R/I\). Since \(I : f = I : h_1\), if we denote by \((G', d')\) the shifted by \(\text{deg}(h_2)\) of the graded complex \((G_, d.)\), clearly we get the minimal free resolution of \(R/(I : f)\) shifted by \(d\). Moreover \(\psi' = h_2\psi : G' \to F\). is the complex graded homomorphism which is a lifting of the map \(R/I : u(-d) \to R/I\).

Let for each \(r\), \(M_r\) (resp. \(N_r\)) be the matrix of \(\delta_r\) (resp. \(d'_r\)) with respect to the canonical basis of \(F_r\) and \(F_{r-1}\) (resp. \(G'_r\) and \(G'_{r-1}\)). Also assume that for each \(r\), \(O_r\) be the matrix of \(\psi'_r : G'_r \to F_r\). Then, by mapping cone construction, the matrix of \(\gamma_r\), with respect to the canonical basis of \(F_r \oplus G'_{r-1}\) and \(F_{r-1} \oplus G'_{r-2}\), is denoted by \(M'_r\) has the following shape;

\[
M'_r = \begin{pmatrix}
M_r & O_{r-1} \\
0 & -N_{r-1}
\end{pmatrix}
\]

So, the result of the mapping cone is the minimal free resolution if and only if \(\text{Im}(\psi') \subset mF\). This clearly holds since \(\psi' = h_2\psi\), and \(h_2 \in m\). \(\square\)

Example 2.5. Let \(I = (x^3y^5, xy^3z^6) \subset R = k[x, y, z]\) and \(f = xy^2z^7 - z^9\). Then \(f = z^2(xy^3 - z^7), I : f = ((I : z^7) : (xy - z^2)) = ((xy^3) : xy - z^2) = (xy^5) = I : z^2\). So Theorem 2.4 shows that we can compute the minimal free resolution of \(R/I + (f)\) by the mapping cone technique. Note that \(I\) is a monomial ideal generated by \(x^3y^5\) and \(xy^3z^6\). It is easy to see that the set \(\{(g_1, g_2) \in R^2 : g_1x^3y^5 + g_2xy^3z^6 = 0\}\) is the submodule of \(R^2\) generated by \((z^6, -x^2)\). So the minimal free resolution of \(R/I\) is

\[ 0 \to R(-14) \to R(-8) \oplus R(-12) \to R \to R/I \to 0. \]

It is also clear that the minimal free resolution of \(R/I : f(-\text{deg}(f)) = R/(xy^5)(-9)\) is

\[ 0 \to R(-15) \to R(-9) \to R/I : f(-\text{deg}(f)) \to 0. \]

So, by the mapping cone, the minimal free resolution of \(R/I + (f)\) is

\[ 0 \to R(-14) \oplus R(-15) \to R(-8) \oplus R(-12) \oplus R(-9) \to R \to R/I + (f) \to 0. \]
We remark that if $u = x_1^{a_1} \cdots x_n^{a_n} \in R$ is a monomial, then $\text{supp}(u) = \{ i : \alpha_i > 0 \}$ and $\deg_x(u) = \alpha_i$. For a monomial ideal $I$, the unique minimal system of generators for $I$ denoted by $G(I)$. In the following, when we write $I = (u_1, \ldots, u_m)$ it means that $I$ is a monomial ideal and $G(I) = \{ u_1, \ldots, u_m \}$.

**Corollary 2.6.** If $I$ is a monomial ideal of $R$ and $u$ is a monomial which does not belong to $I$, then the minimal free resolution of $R/I + (u)$ is given by the mapping cone technique provided that

$$\exists x_i \in \text{supp}(u) \text{ such that } \forall v \in G(I) \ deg_x(u) > deg_x(v).$$

**Proof.** By assumption, for some $x_i \in \text{supp}(u)$ we can decompose $u$ as $u = u_1 x_i$ such that $I : u = I : u_1$. So the result follows by Theorem 2.4. $\square$

**Definition 2.7.** Let $I = (u_1, \ldots, u_m)$ be a monomial ideal. We say $I$ is of decreasing type with respect to the order $u_1, \ldots, u_m$ of its generators, if for each $u_j$ there exists $x_i \in \text{supp}(u_j)$ such that $\deg_x(u_j) > \deg_x(u_r)$ for all $r < j$.

For example if $I = (x_1 x_2, x_2 x_3, x_3^2, x_3 x_4) \subset k[x_1, x_2, x_3, x_4]$, then $I$ is of decreasing type with respect to $x_1 x_2, x_2 x_3, x_3^2, x_3 x_4$.

Note that being of decreasing type depends to the ordering of the generators and when we say $I = (u_1, \ldots, u_m)$ is of decreasing type, it means that it is of decreasing type with respect to the order $u_1, \ldots, u_m$.

The following theorem is an immediate consequence of Corollary 2.6.

**Corollary 2.8.** Let $I = (u_1, \ldots, u_m)$ be a monomial ideal of decreasing type. Then the minimal free resolution of $I$ is given by iterated mapping cone.

In the next sections we apply Corollary 2.6 and Theorem 2.8 in different situations to study the minimal free resolution of some classes of monomial ideals.

### 3. Max-path ideals of rooted trees

A tree is a graph in which there exists a unique path between every pair of distinct vertices; a rooted tree is a tree together with a fixed vertex called the root with the property that there exists a unique path from the root to any given vertex. So a rooted tree is a directed graph by assigning to each edge the direction that goes away from the root. Also an isolated vertex is considered as a trivial rooted tree. If $\{i, j\}$ is an edge in a rooted tree $T$, then we write $(i, j)$ for the directed edge whose direction is from $i$ to $j$. A directed path is a sequence of distinct vertices $i_1, \ldots, i_t$, in which $(i_j, i_{j+1})$ is the directed edge from $i_j$ to $i_{j+1}$ for any $j = 1, \ldots, t - 1$.

We need the following definitions for a rooted tree $T$.

**Definition 3.1.** Let $T$ be a rooted tree. A vertex $y$ is called a child of $x$ if $(x, y)$ is a directed edge in $T$. A vertex $y \neq x$ is a descendant of $x$ if there is a directed path from $x$ to $y$. The vertex $x$ is called a leaf of $T$ if $x$ has no child.
Definition 3.2. Let $T$ be a rooted tree. An induced subtree (or forest) of $T$ is a directed subtree (or forest) that is also an induced subgraph of $T$. Let $x$ be a vertex in $T$. The induced subtree rooted at $x$ of $T$ is the induced subtree of $T$ on the vertex set $\{x\} \cup \{y : y \text{ is a descendant of } x\}$.

Next we define and study the class of max-path ideals of rooted trees. This class of ideals first defined and studied in [12] for an arbitrary tree. This class of ideals has interesting properties as we see later.

Definition 3.3. Let $T$ be a rooted tree on the vertex set $[n]$. The max-path ideal of $T$ is defined as

$$PI(T) = (x_{i_1} \cdots x_{i_t} : i_1, \ldots, i_t \text{ is a maximal path in } T) \subseteq R,$$

where by a maximal path we mean a directed path between the root of tree and one of its leaves.

Here, we show that $PI(T)$ is a monomial ideal of decreasing type and we study the numerical invariants of its minimal free resolution by using Corollary 2.8.

Theorem 3.4. Let $T$ be a rooted tree on the vertex set $[n]$. Then

(i) The max-path ideal of $T$ is of decreasing type. So the minimal free resolution of $R/PI(T)$ is obtained by the iterated mapping cone.

(ii) $\dim(R/PI(T)) = n - 1$.

(iii) Let $m = \text{The number of leaves of } T$. Then

(a) $\beta_i(R/PI(T)) = \binom{m}{i}$.

(b) $\text{pd}(R/PI(T)) = m$ and $\text{depth}(R/PI(T)) = n - m$.

(c) $\text{reg}(R/PI(T)) = n - m$.

(iv) $R/PI(T)$ is Cohen-Macaulay if and only if $T$ is a directed path.

Proof. (i): Let 1 be the root of $T$ and $L(T) = \{i_1, \ldots, i_m\}$ be the set of leaves of $T$. For each $1 \leq j \leq m$, let $u_j$ be the monomial corresponding to the maximal path from 1 to $i_j$. It is clear that $\deg_{x_i}(u_r) > \deg_{x_i}(u_j)$ for each $j \neq r$. So, $PI(T)$ is a monomial ideal of decreasing type and by Corollary 2.8, the minimal free resolution of $R/PI(T)$ is obtained by the iterated mapping cone.

(ii): By definition of $PI(T)$, it is clear that $(x_1)$ is an associated prime of $PI(T)$. So $\dim(R/PI(T)) = n - 1$.

(iii): By induction on $m$ and using the mapping cone technique we compute the desired formulas.

Let $m = 1$ and $i_1$ be the only leaf of $T$. So $T$ is just a directed path and $PI(T)$ is a principal monomial ideal. So it is clear that $\beta_i(R/PI(T)) = \binom{1}{i}$, $\text{pd}(R/PI(T)) = 1$, $\text{depth}(R/PI(T)) = n - 1$ and $\text{reg}(R/PI(T)) = n - 1$.

Now assume that the result is true for each rooted tree whose number of leaves are less than $m$ and assume that $T$ is a rooted tree with $m$ leaves. Let $u = x_{r_1} \cdots x_{r_k} \in G(I)$ where $r_1 = 1$ is the root of $T$, each $r_j$ is a child of $r_{j-1}$ and $r_k = i_m$ is a leaf. Then $PI(T) = PI(T') + \langle u \rangle$ where $T'$ is the rooted tree that $G(PI(T')) = G(PI(T)) \setminus \{x_{r_1} \cdots x_{r_k}\}$. Note that
\[ L(T') = \{i_1, \ldots, i_{m-1}\} \] and \[ V(T') \subseteq V(T) \setminus \{i_m\}. \] For each \( 1 \leq j \leq k - 1 \), let \( C_j = \{x \in V(T) \mid x \text{ is a child of } r_j\} \setminus \{r_{j+1}\} \) and \( C = \bigcup_{j=1}^{k-1} C_j \).

It is easy to see that \( PI(T') : u = \sum_{l=1}^{\ell} PI(T_l) \) where \( \ell = |C| \) and each \( T_l \) is an induced subtree rooted at a vertex of \( C \). Moreover \( \bigcup_{l=1}^{\ell} L(T_l) = L(T') \) and \( \bigcup_{l=1}^{\ell} V(T_l) = V(T) \setminus \{r_1, \ldots, r_k\} \).

Now let \( R_0 = k[x_i, i \in [n] \setminus \bigcup_{l=1}^{\ell} V(T_l)] \) and for each \( 1 \leq l \leq \ell, R_l = k[x_i, i \in V(T_l)] \). Then

\[
R/(PI(T') : u) = R_0 \bigotimes_{l=1}^{\ell} R_l/PI(T_l).
\]

By induction hypothesis we have:

\[
\beta_i(R/(PI(T') : u)) = \sum_{l_1 + \ldots + l_\ell = i} \beta_{l_1}(R_1/PI(T_1)) \times \cdots \times \beta_{l_\ell}(R_\ell/PI(T_\ell))
\]

\[
= \sum_{l_1 + \ldots + l_\ell = i} \binom{|L(T_1)|}{l_1} \times \cdots \times \binom{|L(T_\ell)|}{l_\ell}
\]

\[
= \binom{\sum_{l=1}^{\ell} |L(T_l)|}{i} = \binom{|L(T)| - 1}{i}
\]

\[
= \binom{m - 1}{i},
\]

\[
pd(R/(PI(T') : u)) = \sum_{l=1}^{\ell} pd(R_l/PI(T_l))
\]

\[
= \sum_{l=1}^{\ell} |L(T_l)|
\]

\[
= |L(T)| - 1
\]

\[
= m - 1,
\]

and

\[
reg(R/(PI(T') : u)) = \sum_{l=1}^{\ell} reg(R_l/PI(T_l))
\]

\[
= \sum_{l=1}^{\ell} (|V(T_l)| - |L(T_l)|)
\]

\[
= n - k - (m - 1).
\]

Also, for \( R/PI(T') \) we have
\[
\beta_i(R/(PI(T'))) = \binom{|L(T')|}{i} = \binom{m-1}{i},
\]
and
\[
\text{pd}(R/(PI(T'))) = |L(T')| = m - 1,
\]
and
\[
\text{reg}(R/(PI(T'))) = |V(T')| - (m - 1) \leq n - 1 - (m - 1) \leq n - m.
\]

Now we apply mapping cone to the short exact sequence
\[0 \to R/(PI(T') : u)(-k) \to R/(PI(T')) \to R/(PI(T)) \to 0.\]

By Theorem 2.4 we get
\[
\beta_i(R/(PI(T)) = \beta_i(R/(PI(T'))) + \beta_{i-1}(R/(PI(T') : u))
\]
\[
= \binom{m-1}{i} + \binom{m-1}{i-1}
\]
\[
= \binom{m}{i},
\]
\[
\text{reg}(R/(PI(T))) = \max\{\text{reg}(R/(PI(T'))), \text{reg}(R/(PI(T') : u)) + k - 1\}
\]
\[
= n - m,
\]
and
\[
\text{pd}(R/(PI(T))) = \max\{\text{pd}(R/(PI(T'))), \text{pd}(R/(PI(T') : u)) + 1\}
\]
\[
= m.
\]
So the result follows.

(iv): By parts (ii) and (iii), \(R/PI(T)\) is Cohen-Macaulay if and only if \(m = 1\). So the result is clear. \(\square\)

In the following we are going to find some nice properties of \(PI(T)\). We first need to recall the definition of a simplicial tree. Simplicial trees have the nice property that whose facet ideals are sequentially Cohen-Macaulay (see [5, Corollary 5.6]).

**Definition 3.5.** A simplicial complex \(\Delta\) on the vertex set \(V(\Delta) = \{x_1, \ldots, x_n\}\) is a collection of subsets of \(V(\Delta)\) such that if \(F \in \Delta\) and \(G \subset F\), then \(G \in \Delta\.

An element in \(\Delta\) is called a face of \(\Delta\), and \(F \in \Delta\) is said to be a facet if \(F\) is maximal with respect to the inclusion. Let \(F_1, \ldots, F_q\) be all the facets of a simplicial complex \(\Delta\), we write \(\Delta = \langle F_1, \ldots, F_q \rangle\).

The facet ideal of \(\Delta\) is
\[
I(\Delta) = \left( \prod_{x \in F} x : F \text{ is a facet of } \Delta \right).
\]

Let \(T\) be a rooted tree. Then \(PI(T)\) can be considered as the facet ideal of the following simplicial complex
\[ \Delta_{PI(T)} = \{ x_{r_1}, \ldots, x_{r_k} \}; \ r_1, \ldots, r_k \ \text{is a maximal path of} \ T \]

A leaf of a simplicial complex \( \Delta \) is a facet \( F \) of \( \Delta \) such that either \( F \) is the only facet of \( \Delta \), or there exists a facet \( G \) in \( \Delta \), \( G \neq F \), such that \( F \cap F' \subseteq F \cap G \) for every facet \( F' \in \Delta \), \( F' \neq F \). A simplicial complex \( \Delta \) is called a simplicial tree if \( \Delta \) is connected and every non-empty subcomplex \( \Delta' \) contains a leaf. By a subcomplex, we mean any simplicial complex of the form \( \Delta' = \{ F_{i_1}, \ldots, F_{i_q} \} \), where \( \{ F_{i_1}, \ldots, F_{i_q} \} \) is a subset of the set of all facets of \( \Delta \).

We next see that \( R/PI(T) \) is sequentially Cohen-Macaulay. This is an immediate consequence of the following theorem which shows that \( \Delta_{PI(T)} \) is a simplicial tree.

**Theorem 3.6.** Let \( T \) be a rooted tree. Then \( \Delta_{PI(T)} \) is a simplicial tree.

**Proof.** We show that each facet of \( \Delta_{PI(T)} \) is a leaf. Let \( P: r_1, \ldots, r_k \) be a maximal path of \( T \) where \( r_1 = 1 \) is the root of \( T \) and each \( r_j \) is a child of \( r_{j-1} \). So \( r_k \) is a leaf of \( T \). Let \( F \) be the facet of \( \Delta_{PI(T)} \) corresponding to \( P \). For each \( 1 \leq i \leq k-1 \), let \( C_i = \{ x ; x \text{ is a child of } r_i \} \setminus \{ r_{i+1} \} \) and \( \ell = \max\{ i ; C_i \neq \emptyset \} \). Let \( G \) be the facet corresponding to a maximal path \( P': r_1, \ldots, r_{\ell'}, r_{\ell'+1}', \ldots, r_k' \) where \( r_{\ell'+1}' \) is a child of \( r_{\ell'} \), \( r_{\ell'+1}' \neq r_{\ell+1} \) and each \( r_j' \) is a child of \( r_j \). It is easy to see that \( F' \cap F' \subseteq F \cap G \) for every facet \( F' \in \Delta_{PI(T)}, \ F' \neq F \).

Now let \( \Delta' = \{ F_{i_1}, \ldots, F_{i_q} \} \) be a subcomplex of \( \Delta_{PI(T)} \) and \( V' = V(\Delta') \). If \( T' \) is the induced subtree of \( T \) on the vertex set \( V' \), then \( \Delta' = \Delta_{PI(T')} \).

So by the previous paragraph, each facet of \( \Delta' \) is a leaf. So \( \Delta_{PI(T)} \) is a simplicial tree. \( \square \)

**Corollary 3.7.** Let \( T \) be a rooted tree. Then
- \( R/PI(T) \) is sequentially Cohen-Macaulay.
- \( PI(T)^{\vee} \) is componentwise linear.
- \( \text{reg}(PI(T)^{\vee}) = m \) where \( m \) is the number of leaves in \( T \).
- \( \text{pd}(PI(T)^{\vee}) = n - m \).

**Proof.** By Theorem 3.6, \( \Delta_{PI(T)} \) is a simplicial tree and \( PI(T) = I(\Delta_{PI(T)}) \).

By ([5, Corollary 5.6]), \( R/PI(T) \) is sequentially Cohen-Macaulay. Other parts follows by Theorem 2.3 and the fact that \( PI(T)^{\vee} = PI(T) \). \( \square \)

4. Monomial ideals containing some powers of variables

Let \( J \) be a monomial ideal and \( I = J + (x_{i_1}^{a_{i_1}}, \ldots, x_{i_m}^{a_{i_m}}) \) where \( a_{i_j} \) are positive integers and \( G(I) = G(J) \cup \{ x_{i_1}^{a_{i_1}}, \ldots, x_{i_m}^{a_{i_m}} \} \). In this section we are going to study the minimal free resolution of \( R/I \) using Theorem 2.4.

First, we compute the graded Betti numbers of \( R/I \) in terms of the graded Betti numbers of \( R/J \) and the graded Betti numbers of some other modules associated to \( R/J \). This result has been proved in [11] by applying mapping...
cone to a long exact sequence. Here, using Theorem 2.4, we give an easier proof with more details for it. Next we focus to the case that \( J \) is a square-free monomial ideal.

**Theorem 4.1.** Let \( J \) be a monomial ideal, \( I = J + \langle x_{11}^{a_{i_1}}, \ldots, x_{im}^{a_{i_m}} \rangle \) and \( G(I) = G(J) \cup \{ x_{11}^{a_{i_1}}, \ldots, x_{im}^{a_{i_m}} \} \). Then

1. The minimal free resolution of \( R/I \) is obtained by iterated mapping cone starting from the minimal free resolution of \( R/J \).
2. \( \dim(R/I) \leq n - m \),
3. \( \beta_{i,j}(R/I) = \sum_{r=0}^{m} \sum_{|\sigma|=r} \beta_{r-j-\ell,\sigma}(R/(J : \prod_{j \in \sigma} x_j^{a_j})) \)

where \( \sigma \subseteq \{i_1, \ldots, i_m\} \), \( \ell = \sum_{t \in \sigma} a_t \).

**Proof.**

1) For each \( 1 \leq j \leq m \) and \( u \in G(J) \), \( \deg_{x_{i_j}}(u) < a_{i_j} \), so part 1 is an immediate consequence of Theorem 2.4.

2) Since \( \langle x_{11}^{a_{i_1}}, \ldots, x_{im}^{a_{i_m}} \rangle \subseteq I \), it is clear that \( \dim(R/I) \leq \dim(R/\langle x_{11}^{a_{i_1}}, \ldots, x_{im}^{a_{i_m}} \rangle) \leq n - m \).

3) We compute the Betti numbers of \( R/I \) by induction on \( m \). Let \( m = 1 \). So, \( I = J + \langle x_{11}^{a_{i_1}} \rangle \) and \( G(I) = G(J) \cup \{ x_{11}^{a_{i_1}} \} \). Therefore \( \deg_{x_{i_1}}(x_{11}^{a_{i_1}}) > \deg_{x_{i_1}}(u) \) for each \( u \in G(J) \), and by Theorem 2.4, the minimal free resolution of \( R/I \) is obtained by the mapping cone corresponding to the following short exact sequence

\[ 0 \rightarrow R/(J : \langle x_{11}^{a_{i_1}} \rangle)(-a_{i_1}) \rightarrow R/J \rightarrow R/I \rightarrow 0. \]

So

\[ \beta_{i,j}(R/I) = \beta_{i,j}(R/J) + \beta_{i-1,j-a_{i_1}}(R/J : \langle x_{11}^{a_{i_1}} \rangle). \]

which coincides to the Equation (4.1) for the case \( m = 1 \). Now assume that \( m > 1 \) and the result is true for all \( k \) smaller than \( m \). We prove it for \( m \). So assume that \( I = J + \langle x_{11}^{a_{i_1}}, \ldots, x_{im}^{a_{i_m}} \rangle \) and \( G(I) = G(J) \cup \{ x_{11}^{a_{i_1}}, \ldots, x_{im}^{a_{i_m}} \} \).

Let \( J' = J + \langle x_{11}^{a_{i_1}}, \ldots, x_{im-1}^{a_{i_m-1}} \rangle \). It is clear that

\[ G(J') = G(J) \cup \{ x_{11}^{a_{i_1}}, \ldots, x_{im-1}^{a_{i_m-1}} \}, \]

\[ I = J' + \langle x_{im}^{a_{i_m}} \rangle \text{ and } G(I) = G(J') \cup \{ x_{im}^{a_{i_m}} \}. \] Therefore,

\[ \beta_{i,j}(R/I) = \beta_{i,j}(R/J') + \beta_{i-1,j-a_{im}}(R/J' : \langle x_{im}^{a_{i_m}} \rangle). \]

Moreover, \( J' : \langle x_{im}^{a_{i_m}} \rangle = (J : \langle x_{im}^{a_{i_m}} \rangle) + \langle x_{11}^{a_{i_1}}, \ldots, x_{im-1}^{a_{i_m-1}} \rangle \). It is easy to see that \( G(J' : \langle x_{im}^{a_{i_m}} \rangle) = G(J : \langle x_{im}^{a_{i_m}} \rangle) \cup \{ x_{11}^{a_{i_1}}, \ldots, x_{im-1}^{a_{i_m-1}} \} \). So by induction hypothesis for the case \( m - 1 \),
\[
\beta_{i,j}(R/J') = \sum_{r=0}^{m-1} \sum_{|\sigma|=r} \beta_{i-r,j-\ell_{\sigma}}(R/(J : \prod_{j \in \sigma} x_{j}^{a_{j}}))
\]

where \(\sigma \subseteq \{i_1, \ldots, i_{m-1}\}\), \(\ell_{\sigma} = \sum_{t \in \sigma} a_{t}\),

and

\[
\beta_{i-1,j-a_{im}}(R/(J' : (x_{im}^{a_{im}}))) = \sum_{r=0}^{m-1} \sum_{|\sigma|=r} \beta_{i-1-r,j-a_{im}-\ell_{\sigma}}(R/(J : (x_{im}^{a_{im}})) : \prod_{j \in \sigma} x_{j}^{a_{j}}))
\]

where \(\sigma \subseteq \{i_1, \ldots, i_{m-1}\}\), \(\ell_{\sigma} = \sum_{t \in \sigma} a_{t}\).

For each \(\sigma \subseteq \{i_1, \ldots, i_{m-1}\}\), we let \(\sigma' = \sigma \cup \{i_{m}\}\). It is clear that \(a_{im} + \ell_{\sigma} = \ell_{\sigma'}\), and \((J : (x_{im}^{a_{im}})) : \prod_{j \in \sigma} x_{j}^{a_{j}} = J : \prod_{j \in \sigma'} x_{j}^{a_{j}}\). So the Equation (4.4) can be written as:

\[
\beta_{i-1,j-a_{im}}(R/(J' : (x_{im}^{a_{im}}))) = \sum_{r=1}^{m} \sum_{|\sigma|=r} \beta_{i-1-r,j-\ell_{\sigma}}(R/(J : \prod_{j \in \sigma} x_{j}^{a_{j}}))
\]

where \(\{i_{m}\} \subseteq \sigma \subseteq \{i_1, \ldots, i_{m}\}\), \(\ell_{\sigma} = \sum_{t \in \sigma} a_{t}\).

Now it is enough to replace (4.3) and (4.5) in (4.2) to get Equation (4.1). \(\square\)

In the following we are going to apply Theorem 4.1 to the case that \(J\) is a square-free monomial ideal. We remark that an arbitrary square-free monomial ideal can be considered as edge ideal of a hypergraph.

Let \(X\) be a finite set and \(E = \{E_1, \ldots, E_s\}\) a finite collection of nonempty subsets of \(X\). The pair \(H = (X, E)\) is called a hypergraph on \(X\). The elements of \(X\) and \(E\), respectively, are called the vertices and the edges of the hypergraph. A hypergraph is called simple if \(|E_i| \geq 2\) for all \(i = 1, \ldots, s\) and \(E_j \subset E_i\) only if \(i = j\). In the following we assume that \(H\) is a simple hypergraph.

Let \(H\) be a hypergraph on the vertex set \(X\). We recall that \(W \subseteq X\) is an independent set if \(W\) does not contain any edge of \(H\). The size of an independent set is the number of vertices it contains.

A maximal independent set is either an independent set such that adding any other vertex to the set forces the set to contain an edge or the set of all
vertices of the empty hypergraph. In the following we denote by \( \text{max}(\mathcal{H}) \) the set of all maximal independent subsets of \( \mathcal{H} \).

A maximum independent set is an independent set of largest possible size for a given hypergraph \( \mathcal{H} \). This size is called the independence number of \( \mathcal{H} \), and denoted \( \alpha(\mathcal{H}) \).

For a hypergraph \( \mathcal{H} \) on the vertex set \( X \), the independence complex of \( \mathcal{H} \) is defined as:

\[
\Delta(\mathcal{H}) = \{ W \subseteq X \mid W \text{ is an independent set} \}.
\]

For a hypergraph \( \mathcal{H} \) with vertex set \([n]\) the edge ideal of \( \mathcal{H} \) in the polynomial ring \( R \) is defined as:

\[
J_\mathcal{H} = ( \prod_{x \in E} x; \ E \text{ is an edge of } \mathcal{H}).
\]

Note that the edge ideal of a hypergraph is defined in the same way as the edge ideal of a graph. We also remark that we can consider \( J_\mathcal{H} \) as Stanley-Reisner ideal of \( \Delta(\mathcal{H}) \).

**Remark 4.2.** Let \( \mathcal{H} \) be a hypergraph on the vertex set \([n]\). Assume that \( J = J_\mathcal{H} \). For each \( \sigma \subseteq [n] \) let \( N(\sigma) = \{ i \in [n] \mid \sigma \cup \{i\} \text{ is not independent} \} \), and \( \mathcal{H}_\sigma \) be the simple hypergraph on the vertex set \([n] \setminus (\sigma \cup N(\sigma)) \) with \( \mathcal{E}(\mathcal{H}_\sigma) = \{ E \setminus \sigma : E \subseteq \mathcal{E}(\mathcal{H}) \setminus \sigma \subseteq V(\mathcal{H}_\sigma) \} \).

Assume that for each \( j \in \sigma, a_j > 0 \). If \( \sigma \) is not an independent set, then it is clear that \( J : \prod_{j \in \sigma} x_j^{a_j} = R \). If \( \sigma \) is an independent set, then \( J : \prod_{j \in \sigma} x_j^{a_j} = (x_i : i \in N(\sigma)) + J_{\mathcal{H}_\sigma} \). In particular, if \( \sigma \) is a maximal independent set, then \( J : \prod_{j \in \sigma} x_j^{a_j} = (x_i : i \in N(\sigma)) = (x_i : i \in [n] \setminus \sigma) \).

If \( I = J_\mathcal{H} + (x_{i_1}^{a_{i_1}}, \ldots, x_{i_m}^{a_{i_m}}) \), then by Theorem 4.1 and Remark 4.2, we can write the graded Betti numbers of \( R/I \) in terms of the graded Betti numbers of \( R/J_\mathcal{H} \) and \( R/J_{\mathcal{H}} \) for some hypergraphs associated to \( \mathcal{H} \). In the following we discuss the case that \( \forall j, a_{i_j} = 2 \).

**Theorem 4.3.** Let \( \mathcal{H} \) be a hypergraph on the vertex set \([n]\). Assume that \( I = J_\mathcal{H} + (x_{i_1}^2, \ldots, x_{i_m}^2) \). Then

\[
\beta_{i,j}(R/I) = \sum_{r=0}^{m} \sum_{|\sigma|=r} \beta_{i-r,j-2r}(R/(J_\mathcal{H} : \prod_{j \in \sigma} x_j^{a_j}))
\]

where \( \sigma \subseteq \{i_1, \ldots, i_m\} \), \( \sigma \in \Delta(\mathcal{H}) \) and

\[
R/J_\mathcal{H} : \prod_{j \in \sigma} x_j^2 = k[x_i : i \in \sigma \cup N(\sigma)]/(x_i : i \in N(\sigma)) \otimes k[V(\mathcal{H}_\sigma)]/J_{\mathcal{H}_\sigma}.
\]

**Proof.** First note that \( G(I) = G(J) \cup \{x_{i_1}^2, \ldots, x_{i_m}^2\} \). So we can apply Theorem 4.1.

By Remark 4.2, in order to compute the Betti numbers of \( R/I \), it is enough to consider all \( \sigma \subseteq \{i_1, \ldots, i_m\} \) where \( \sigma \in \Delta(\mathcal{H}) \). Also, if \( \sigma \) is an
independent set,

\[ R/J_H : \prod_{j \in \sigma} x_j^2 = k[x_i \mid i \in \sigma \cup N(\sigma)]/(x_i \mid i \in N(\sigma)) \otimes k[V(H_\sigma)]/J_{H_\sigma}. \]

Corollary 4.4. Let \( \mathcal{H} \) be a hypergraph on the vertex set \([n]\). Assume that \( I = J_H + (x_{i_1}^2, \ldots, x_{i_m}^2) \). Then

\[ \beta_{n,j}(R/I) = |\{\sigma \mid \sigma \in \max(\mathcal{H}), |\sigma| = j - n, \text{ and } \sigma \subseteq \{i_1, \ldots, i_m\}\}|. \]

Therefore,

- \( \beta_n(R/I) = |\{\sigma \mid \sigma \in \max(\mathcal{H}) \text{ and } \sigma \subseteq \{i_1, \ldots, i_m\}\}|. \)
- \( \text{depth}(R/I) = 0 \) if and only if \( \{i_1, \ldots, i_m\} \) is containing a maximal independent set.

Proof. \( \beta_{n,j}(R/I) \) can be computed by Equation (4.6). If \( \sigma \subseteq \{i_1, \ldots, i_m\} \) is an independent, we have

\[ \text{pd}(R/J_H : \prod_{j \in \sigma} x_j^2) = \text{pd}(k[x_i \mid i \in \sigma \cup N(\sigma)]/(x_i \mid i \in N(\sigma)) + \text{pd}(k[V(H_\sigma)]/J_{H_\sigma}), \]

(where in the above formula \( k[V(H_\sigma)]/J_{H_\sigma} \) appears when \( V(H_\sigma) \neq \emptyset \) and in this case, by Remark 2.1, \( \text{pd}(k[V(H_\sigma)]/J_{H_\sigma} \leq |V(H_\sigma)| - 1) \). Therefore, if \( \sigma \) is a maximal independent set then

\[ \text{pd}(R/J_H : \prod_{j \in \sigma} x_j^2) = |N(\sigma)| = n - |\sigma|, \]

and if \( \sigma \) is not a maximal independent set

\[ \text{pd}(R/J_H : \prod_{j \in \sigma} x_j^2) = |N(\sigma)| + \text{pd}(k[V(H_\sigma)]/J_{H_\sigma}) \leq |N(\sigma)| + |V(H_\sigma)| - 1 \]

\[ \leq |N(\sigma)| + n - |\sigma| - |N(\sigma)| - 1 = n - |\sigma| - 1. \]

So

\[ \beta_{n,j}(R/I) = \sum_{r=0}^{m} \sum_{|\sigma| = r} \beta_{n-r,j-2r}(R/(J_H : \prod_{j \in \sigma} x_j^2)) \]

(where \( \sigma \subseteq \{i_1, \ldots, i_m\}, \sigma \in \max(\mathcal{H}) \))

\[ = \sum_{\sigma \subseteq \{i_1, \ldots, i_m\}, \sigma \in \max(\mathcal{H})} \beta_{n-|\sigma|,j-2|\sigma|}(R/(x_i \mid x_i \in N(\sigma))) \]

\[ = |\{\sigma \mid \sigma \subseteq \{i_1, \ldots, i_m\}, \sigma \in \max(\mathcal{H}) \text{ and } 2|\sigma| + |N(\sigma)| = j\}| \]

\[ = |\{\sigma \mid \sigma \subseteq \{i_1, \ldots, i_m\}, \sigma \in \max(\mathcal{H}), |\sigma| = j - n\}|. \]
Therefore
\[ \beta_n(R/I) = \sum_j \beta_{n,j}(R/I) = |\{\sigma \in \max(H) : \sigma \subseteq \{i_1, \ldots, i_m\}\}|. \]

\( \square \)

**Corollary 4.5.** Let \( H \) be a hypergraph on the vertex set \([n]\). Assume that \( I = J_H + (x^2_i, \ldots, x^2_n) \). Then

1. \( \beta_{n,j}(R/I) = |\{\sigma \in \max(H) : |\sigma| = j - n\}| \)
2. \( \beta_n(R/I) = \) the number of maximal independent sets for \( H \)
   \hspace{1cm} = \) the number of facets of \( \Delta(H) \).
3. \( R/I \) is a level ring if and only if \( \Delta(H) \) is a pure simplicial complex
   if and only if \( J_H \) is an unmixed ideal.
4. \( \operatorname{reg}(R/I) = \alpha(H) \).

**Proof.** (1) and (2) are immediate consequences of Corollary 4.4.

To see (3) note that \( R/I \) is a level ring if and only if the last nonzero graded free module of its graded minimal free resolution, is of the form \( R^{a}\langle-s\rangle \), for some positive integers \( a \) and \( s \). So by part (1), \( R/I \) is a level ring if and only if all maximal independent sets of \( H \) are of the same size. Also note that \( J_H \) is unmixed if all minimal vertex covers of \( H \) have the same cardinality. So the conclusion follows from the fact that \( C \subset [n] \) is a minimal vertex cover if and only if \( [n] \setminus C \) is a maximal independent set.

To prove (4) it is enough to notice that \( \dim(R/I) = 0 \) and therefore
\[ \operatorname{reg}(R/I) = \max\{j : \beta_{n,j}(R/I) \neq 0\} = \alpha(H). \]

\( \square \)

**Remark 4.6.** Let \( I \) be a monomial ideal generated in degree 2 and \( I_s \) be the square-free part of \( I \). It is clear that there exists a graph \( G \) on the vertex set \([n]\) in such a way that \( I_s = J_G \). So \( I = J_G + (x^2_i, \ldots, x^2_m) \) for some \( \{i_1, \ldots, i_m\} \subseteq [n] \). Theorem 4.3 shows that we can compute the graded Betti numbers of \( R/I \) in terms of the graded Betti numbers of \( R/J_G \) and the graded Betti numbers of \( R/J_H \) (For some induced subgraphs \( H \) of \( G \)).

It is also possible to study the Betti numbers of \( R/I \) by the idea of polarization (see [7, Corollary 1.6.3]). Note that if \( I \) and \( G \) be as above, and
\[ J = I_s + (x_{i_1}y_1, \ldots, x_{i_m}y_m) \subset k[x_1, \ldots, x_n, y_1, \ldots, y_m] \]
be its polarization, then we can view \( J \) as the edge ideal of the graph \( H \) that is defined as \( V(H) = [n] \cup \{-1, \ldots, -m\} \) and \( E(H) = E(G) \cup \{\{i_1, -1\}, \ldots, \{i_m, -m\}\} \). It means that \( J = J_H \). Here, \( G \) is an induced subgraph of \( H \). The idea of attaching the graph \( H \) to the ideal \( I \) in order
to study the Betti numbers has been used in [8] where the authors studied the class of monomial ideals with 2-linear resolution (see [8, Section 2]).

Note that by [7, Corollary 1.6.3],
\[
\forall i, j, \beta_{i,j}(R/I) = \beta_{i,j}(k[x_1, \ldots, x_n, y_i, \ldots, y_m]/J).
\]

By [15, Theorem 20 and Lemma 21] \( \text{reg}(k[x_1, \ldots, x_n, y_i, \ldots, y_m]/J) = \alpha(G) \). Since \( \text{reg}(R/I) = \text{reg}(k[x_1, \ldots, x_n, y_i, \ldots, y_m]/J) \), part (4) of Corollary 4.5 is a generalization of the mentioned result of [15] to the case that \( I = J_H + (x_1^2, \ldots, x_n^2) \) and \( H \) is a hypergraph on the vertex set \([n]\).

Finally, we are going to apply Theorem 4.3 and Corollary 4.5 to the case that \( G \) is a complete \( r \)-partite graph.

**Theorem 4.7.** Let \( G = K_{n_1, \ldots, n_t} \) be a complete \( t \)-partite graph on the vertex set \([n]\) and let \( I = J_G + (x_1^2, \ldots, x_n^2) \). Then
\[
\beta_{i,j}(R/I) = \beta_{i,j}(R/J_G) + \sum_{\ell=1}^{t} \binom{n_{\ell}}{j-i} \binom{n-n_{\ell}}{2i-j}
\]
where
\[
\beta_{i,j}(R/J_G) = \begin{cases} 
\sum_{\ell=2}^{t} \left( \sum_{i=0}^{\ell-1} \alpha_1 + \ldots + \alpha_\ell = i + 1, j_1 < \ldots < j_\ell, \alpha_1, \ldots, \alpha_\ell \geq 1 \right) \binom{n_{j_\ell}}{\alpha_1} \cdots \binom{n_{j_\ell}}{\alpha_\ell}, & \text{if } j = i + 1 \\
0, & \text{if } j \neq i + 1.
\end{cases}
\]

**Proof.** Assume that \( G \) is a \( t \)-partite graph with partitions \( V_1, \ldots, V_t \) where \( |V_\ell| = n_\ell \). Then \( \sigma \subseteq [n] \) is an independent set if and only if \( \sigma \subseteq V_\ell \) for some \( 1 \leq \ell \leq t \). So if \( \sigma \neq \emptyset \) is a dependent set, then for some \( 1 \leq \ell \leq t \), \( N(\sigma) = [n] \setminus V_\ell \) and \( G_\sigma \) is the empty graph on the vertex set \( V_\ell \setminus \sigma \).

Now if \( \sigma \subseteq V_\ell \) and \( |\sigma| = r \), by Theorem 4.3 we have
\[
\beta_{i-r, j-2r}(R/J_G : \prod_{j \in \sigma} x_j^2) = \beta_{i-r, j-2r}(k[x_i ; i \in [n] \setminus (V_\ell \setminus \sigma)]/(x_i ; i \in [n] \setminus V_\ell)).
\]

Thus \( \beta_{i-r, j-2r}(R/J_G : \prod_{j \in \sigma} x_j^2) \neq 0 \) if and only if \( j - 2r = i - r = 2i - j \) and if this is the case, then \( r = j - i \) and \( \beta_{i-r, j-2r}(R/J_G : \prod_{j \in \sigma} x_j^2) = \binom{n-n_i}{2i-j} \).

Now the result follows from Theorem 4.3 and [10, Theorem 5.3.8]. \( \Box \)

**Theorem 4.8.** Let \( G \) be a graph on the vertex set \([n]\), \( I = J_G + (x_1^2, \ldots, x_n^2) \) and \( \beta_n(R/I) = t \). Then
\[
\sum_{j \in \mathbb{N}} \beta_{n,j}(R/I)(j - n) = n \iff G \text{ is complete } t \text{-partite graph}.
\]

**Proof.** If \( G \) is a complete \( t \)-partite graph with partitions \( V_1, \ldots, V_t \), then \( V_1, \ldots, V_t \) are the only maximal independent sets of \( G \). So by Corollary 4.5, we have \( \sum_{j \in \mathbb{N}} \beta_{n,j}(R/I)(j - n) = n \).

Conversely, Let \( G \) be a graph on the vertex set \([n]\) with maximal independent sets \( V_1, \ldots, V_t \). Assume that \( \sum_{j \in \mathbb{N}} \beta_{n,j}(R/I)(j - n) = n \). Since each vertex of the graph belongs to at least one independent set, this equality
beside Corollary 4.5 show that each vertex belongs to exactly one of the independent sets. So $G$ is a complete $t$–partite graph whose partitions are $V_1,\ldots,V_t$. □

Acknowledgments

This research was in part supported by a grant from IPM (No. 94130058). The author would like to thank Rashid Zaare-Nahandi and Somayeh Moradi for reading an earlier version of the paper and for helpful comments and remarks.

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(Leila Sharifan) Faculty of Mathematics and Computer Sciences, Hakim Sabzevari University, P.O. Box 397, Sabzevar, Iran, and School of Mathematics, Institute for research in Fundamental Sciences (IPM), P. O. Box: 19395-5746, Tehran, Iran.

E-mail address: leila-sharifan@aut.ac.ir