Harmonic Tutte polynomials of matroids

Himadri Shekhar Chakraborty · Tsuyoshi Miezaki · Manabu Oura

Abstract
In the present paper, we introduce the concept of harmonic Tutte polynomials of matroids and discuss some of their properties. In particular, we generalize Greene’s theorem, thereby expressing harmonic weight enumerators of codes as evaluations of harmonic Tutte polynomials.

Keywords Tutte polynomials · Weight enumerators · Matroids · Codes · Harmonic functions

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1 Introduction
In 1954, Tutte [28] introduced a celebrated graph polynomial which is originally called the dichromatic polynomial (also see [29]), now known as the Tutte polynomial. It plays a key role in the study of the graph properties related to counting problems. In 1969, Crapo [12] generalized the Tutte polynomial for graphs with respect to matroids. Later, Greene [14] proved a remarkable connection between the weight enumerator $W_C(x, y)$ of an $[n, k]$ code $C$ and the Tutte polynomial of its matroid $M_C$; this identity, which is known as the Greene
identity, is as follows:

\[ WC(x, y) = (x - y)^k y^{n-k} T \left( MC; \frac{x + (q - 1)y}{x - y}, \frac{x}{y} \right). \]

As an application of the above relation, Greene [14] gave an alternative proof of the MacWilliams identity for the weight enumerator of an \([n, k]\) code.

Delsarte [13] introduced discrete harmonic functions on a finite set. Bachoc [2, 3] associated the discrete harmonic functions to linear codes by introducing the harmonic weight enumerator \( W_{C,f} \) of a linear code \( C \). Bachoc [3] and Tanabe [27] independently both proved a MacWilliams identity for the harmonic weight enumerators of linear codes over \( \mathbb{F}_q \).

In this paper, we introduce the notion of the harmonic Tutte polynomials associated with a harmonic function of a certain degree, and give a Greene-type identity which we call the generalized Greene identity, which relates the harmonic weight enumerator of a code and the harmonic Tutte polynomial of the matroid corresponding to the code. Moreover, as an application of the generalized Greene identity, we give a combinatorial proof of Bachoc and Tanabe’s MacWilliams-type identity which is stated in Theorem 2.5.

This paper is organized as follows: In Sect. 2, we present the basic definitions and properties in coding theory and matroid theory used in this paper. In Sect. 3, we define the harmonic Tutte polynomial, and obtain a relation between the harmonic Tutte polynomials of a matroid and its dual (Theorem 3.2). Moreover, we reinterpret the definition of harmonic weight enumerators of codes (Theorem 3.10). In Sect. 4, we give a generalization of Greene identity (Theorem 4.2) with an application in the proof of the MacWilliams identity for harmonic weight enumerator. Finally, in Sect. 5, we conclude the paper with some remarks.

2 Basic definitions and notions

In this section, we give some basic definitions and properties of codes and matroids that are necessary for this paper. We follow [15, 18, 24, 25, 27] for the discussions. Moreover, we recall some definitions and properties of the (discrete) harmonic functions; see [2, 13, 27] for more detail.

2.1 Discrete harmonic functions

Let \( E := \{1, 2, \ldots, n\} \) where \( n \) is a positive integer. Let \( 2^E \) denote the set of all subsets of \( E \). We define \( E_d := \{X \in 2^E \mid |X| = d\} \) for \( d = 0, 1, \ldots, n \). We denote by \( \mathbb{R}2^E \) and \( \mathbb{R}E_d \) the real vector spaces spanned by the elements of \( 2^E \) and \( E_d \), respectively. An element of \( \mathbb{R}E_d \) is denoted by

\[ f := \sum_{Z \in E_d} f(Z)Z \]

with coefficients \( f(Z) \in \mathbb{R} \). Thus \( \mathbb{R}E_d \) is identified with the real-valued function on \( E_d \) given by \( Z \mapsto f(Z) \). Such an element \( f \in \mathbb{R}E_d \) can be extended to an element \( \tilde{f} \in \mathbb{R}2^E \) by setting, for all \( X \in 2^E \),

\[ \tilde{f}(X) := \sum_{Z \in E_d, Z \subset X} f(Z). \]
Note that \( \tilde{f}(X) = 0 \) for any \( X \in \mathbb{R}^E \) such that \( |X| < d \). If an element \( g \in \mathbb{R}^E \) is equal to \( \tilde{f} \) for some \( f \in \mathbb{R}E_d \), then we say that \( g \) has degree \( d \). We call the vector space \( \mathbb{R}E_d \) the homogeneous space of degree \( d \), and denote it by \( \text{Hom}_d(n) \). The differentiation \( \gamma \) is the operator on \( \mathbb{R}^E \) defined by linearity from the identity
\[
\gamma(Z) := \sum_{Y \in E_{d-1}, Y \subset Z} Y
\]
for all \( Z \in E_d \) and for all \( d = 0, 1, \ldots, n \). Also, \( \text{Harm}_d(n) \) is the kernel of \( \gamma \):
\[
\text{Harm}_d(n) := \ker \left( \gamma' |_{\mathbb{R}E_d} \right).
\]

**Remark 2.1** ([2, 13]) Let \( f \in \text{Harm}_d(n) \). Then \( \gamma^{d-i}(f) = 0 \) for all \( 0 \leq i \leq d - 1 \). This means from definition (3) that
\[
\sum_{X \in E_i} \left( \sum_{Z \in E_d, X \subset Z} f(Z) \right) X = 0.
\]
This implies that \( \sum_{Z \in E_d, X \subset Z} f(Z) = 0 \) for any \( X \in E_i \).

**Remark 2.2** Let \( f \in \text{Harm}_d(n) \). Since \( \sum_{Z \in E_d} f(Z) = 0 \), then it is easy to check from (3) that \( \sum_{X \in E_i} \tilde{f}(X) = 0 \), where \( 1 \leq d \leq t \leq n \).

To make the above definitions and remarks related to harmonic functions easier to understand, we give the following example.

**Example 2.3** Let \( E = \{1, 2, 3\} \) and \( d = 1 \). Let \( f \in \mathbb{R}E_1 \) be the element
\[
f = a[1] + b[2] + c[3],
\]
where \( a = f([1]) \), \( b = f([2]) \) and \( c = f([3]) \). Then \( \gamma(f) = (a + b + c) \theta \). Suppose that \( f \in \text{Harm}_1(3) \). This implies \( \gamma(f) = 0 \). So, \( a + b + c = 0 \). That is, \( c = -(a + b) \). Hence
\[
\text{Harm}_1(3) \ni f = a[1] + b[2] - (a + b)[3].
\]
For \( f \in \text{Harm}_1(3) \), we have the following \( \tilde{f} \)'s by (2):
\[
\tilde{f}(\emptyset) = 0, \quad \tilde{f}([1]) = a, \quad \tilde{f}([2]) = b, \quad \tilde{f}([3]) = -(a + b)
\]
\[
\tilde{f}([1, 2]) = a + b, \quad \tilde{f}([1, 3]) = -b, \quad \tilde{f}([2, 3]) = -a, \quad \tilde{f}([1, 2, 3]) = 0.
\]

### 2.2 Linear codes

Let \( \mathbb{F}_q \) be a finite field of order \( q \), where \( q \) is a prime power. Then \( V := \mathbb{F}_q^n \) denotes the vector space of dimension \( n \) with the ordinary inner product:
\[
\mathbf{u} \cdot \mathbf{v} := u_1 v_1 + \cdots + u_n v_n
\]
for \( \mathbf{u}, \mathbf{v} \in V \), where \( \mathbf{u} = (u_1, \ldots, u_n) \) and \( \mathbf{v} = (v_1, \ldots, v_n) \). Let \( \text{supp}(\mathbf{u}) := \{ i \in E \mid u_i \neq 0 \} \) and \( \text{wt}(\mathbf{u}) := |\text{supp}(\mathbf{u})| \) for \( \mathbf{u} \in V \). Let \( V_d := \{ \mathbf{u} \in V \mid \text{wt}(\mathbf{u}) = d \} \). An element \( f \in \mathbb{R}E_d \) can be extended to an element \( \hat{f} \in \mathbb{R}V \) by setting, for all \( \mathbf{u} \in V \),
\[
\hat{f}(\mathbf{u}) := \sum_{\substack{\mathbf{v} \in V_d, \supp(\mathbf{v}) \subseteq \supp(\mathbf{u})}} f(\text{supp}(\mathbf{v})).
\]
An $\mathbb{F}_q$-linear code of length $n$ is a linear subspace of $V$. An $\mathbb{F}_q$-linear code of length $n$ with dimension $k$ is called an $[n, k]$ linear code. Let $C$ be an $\mathbb{F}_q$-linear code. We denote by $C^\perp$ the dual code of $C$, defined as:

$$C^\perp := \{ u \in V \mid u \cdot v = 0 \text{ for all } v \in C \}.$$

The weight distribution of $C$ is the sequence $\{ A_i \mid i = 0, 1, \ldots, n \}$, where $A_i$ is the number of codewords of weight $i$. The polynomial

$$WC(x, y) := \sum_{u \in C} x^{n-\text{wt}(u)} y^{\text{wt}(u)} = \sum_{i=0}^{n} A_i x^{n-i} y^i$$

is called the weight enumerator of $C$ and satisfies the MacWilliams identity:

$$WC(x, y) = \frac{1}{|C|} WC(x + (q - 1)y, x - y).$$

Bachoc [2] introduced the concept of a harmonic weight enumerator for a binary code which was later defined for an $\mathbb{F}_q$-linear code by Tanabe [27] as follows.

**Definition 2.4** Let $C$ be an $\mathbb{F}_q$-linear code of length $n$. Let $f \in \text{Harm}_d(n)$. The harmonic weight enumerator associated with $C$ and $f$ is

$$WC_f(x, y) := \sum_{u \in C} f(u) x^{n-\text{wt}(u)} y^{\text{wt}(u)}.$$

**Theorem 2.5** ([27], MacWilliams type identity) Let $WC_f(x, y)$ be the harmonic weight enumerator of an $\mathbb{F}_q$-linear code $C$ associated to $f \in \text{Harm}_d(n)$. Then

$$WC_f(x, y) = (xy)^d ZC_f(x, y),$$

where $ZC_f$ is a homogeneous polynomial of degree $n - 2d$, and satisfies

$$ZC_f(x, y) = (-1)^d \frac{q^{n/2}}{|C|} ZC_f \left( \frac{x + (q - 1)y}{\sqrt{q}}, \frac{x - y}{\sqrt{q}} \right).$$

Let $C$ be an $\mathbb{F}_q$-linear code of length $n$ and let $f \in \text{Harm}_d(n)$. Then the weight distribution of $C$ associated to $f$ is defined as

$$A_{i, f} := \sum_{u \in C, \text{wt}(u)=i} f(u).$$

Therefore the harmonic weight enumerator of $C$ associated with $f$ can be rewritten as

$$WC_f(x, y) = \sum_{i=0}^{n} A_{i, f} x^{n-i} y^i.$$

Now from the above definition, we have by Theorem 2.5,

$$ZC_f(x, y) = \sum_{i=0}^{n} A_{i, f} x^{n-i-d} y^{i-d}.$$

**Remark 2.6** If $\deg f = 0$, then we have $A_{i, f} = A_i$, that is, $WC_f(x, y)$ becomes the usual weight enumerator $WC(x, y)$. 
Example 2.7 Let $C$ be a binary linear code of length 3. The elements of $C$ are listed as follows:

$$(0, 0, 0), (0, 0, 1), (1, 1, 0), (1, 1, 1).$$

We consider $f \in \text{Harm}_1(3)$ as computed in Example 2.3. Then by direct computation, we have the harmonic weight enumerator of $C$ associated to $f$ is as follows

$$W_{C,f} = -(a + b)x^2y + (a + b)xy^2 = (xy)^1Z_{C,f},$$

where $Z_{C,f} = (a + b)(y - x)$.

2.3 Matroids

The matroids can be defined in several equivalent ways. We prefer the definition which is in terms of independent sets. A (finite) matroid $M$ is an ordered pair $(E, \mathcal{I})$ consisting of set $E$ and a collection $\mathcal{I}$ of subsets of $E$ satisfying the following conditions:

(I1) $\emptyset \in \mathcal{I}$,

(I2) if $I \in \mathcal{I}$ and $J \subset I$, then $J \in \mathcal{I}$, and

(I3) if $I, J \in \mathcal{I}$ with $|I| < |J|$, then there exists $j \in J \setminus I$ such that $I \cup \{j\} \in \mathcal{I}$.

The elements of $\mathcal{I}$ are called the independent sets of $M$, and $E$ is called the ground set of $M$. A subset of the ground set $E$ that does not belong to $\mathcal{I}$ is called dependent. An independent set is called a basis if it is not contained in any other independent set.

It follows from axiom (I3) that the cardinalities of all bases in a matroid $M$ are equal; this cardinality is called the rank of $M$. The rank $\rho(J)$ of an arbitrary subset $J$ of $E$ is the size of the largest independent subset of $J$. That is, $\rho(J) := \max\{|I| : I \in \mathcal{I} \text{ and } I \subset J\}$. In particular, $\rho(\emptyset) = 0$. We call $\rho(E)$ the rank of $M$. We refer the readers to [25] for more information on matroids.

Definition 2.8 Let $M$ be a matroid on the set $E$ having a rank function $\rho$. The Tutte polynomial of $M$ is defined as follows:

$$T(M; x, y) := \sum_{J \subset E} (x - 1)^{\rho(E) - \rho(J)}(y - 1)^{|J| - \rho(J)}.$$

Definition 2.9 Let $A$ be a $k \times n$ matrix over a finite field $\mathbb{F}_q$. This gives a matroid $M[A]$ on the set $E$ in which a set $I$ is independent if and only if the family of columns of $A$ whose indices belong to $I$ is linearly independent. Such a matroid is called a vector matroid.

For an $\mathbb{F}_q$-linear code $C$, $M_C$ denotes the vector matroid that corresponds to $C$. In the rest of this note, we prefer to call $M_C$ as a matroid instead of a vector matroid. Next we recall this construction, which is treated in [16]. Let $G$ be a $k \times n$ matrix with rank $k$ over the finite field $\mathbb{F}_q$. The set $E$ is indexing the columns of $G$. Let $\mathcal{I}_G$ be the family of all subsets $I$ of $E$ for which the columns of $G$ indexed by $I$ are independent. Then $M_G := (E, \mathcal{I}_G)$ is a matroid. If $G_1$ and $G_2$ are generator matrices of an $\mathbb{F}_q$-linear code $C$, then $(E, \mathcal{I}_{G_1}) = (E, \mathcal{I}_{G_2})$. Therefore, the matroid $M_C := (E, \mathcal{I}_C)$ of an $\mathbb{F}_q$-linear code $C$ is well defined by $(E, \mathcal{I}_G)$ for any generator matrix $G$ of $C$.

Example 2.10 Let $E = \{1, 2, 3\}$. Let $C$ be a $[3, 2]$ code over $\mathbb{F}_2$ (given in Example 2.7) with generator matrix as follows:

$$G = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
The columns of the matrix $G$ are indexed by $E$. Now we compute a family $\mathcal{I}$ of subsets $I \subset E$ such that the columns of $G$ indexed by $I$ are independent:

$$\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}, \{2, 3\}\}.$$ 

Then the matroid corresponding to $C$ is $M_C = (E, \mathcal{I})$.

### 3 Harmonic generalizations of polynomials

#### 3.1 Harmonic Tutte polynomials

In this section, we define the Tutte polynomials of a (finite) matroid $M$ associated with a harmonic function. We also present a very useful relation between the Tutte polynomial of a matroid and its dual associated to a harmonic function.

**Definition 3.1** Let $M = (E, \mathcal{I})$ be a matroid with rank function $\rho$, and $f \in \text{Hom}_d(n)$ be a real-valued function of degree $d$. Then the weighted Tutte polynomial of $M$ associated to $f$ is defined as follows:

$$T(M, f; x, y) := \sum_{J \subset E} \tilde{f}(J)(x - 1)^{\rho(E) - \rho(J)}(y - 1)^{|J| - \rho(J)}.$$ 

In particular, if $f \in \text{Harm}_d(n)$, then we call the weighted Tutte polynomial $T(M, f; x, y)$ the harmonic Tutte polynomial associated with $f$.

We define

$$\mathcal{I}^* := \{I \in 2^E | I \subset E \setminus A \text{ for some } A \in B(M)\},$$

where $B(M)$ be the collection of all bases of $M$. It is clear by [25, Theorem 2.1.1] that $\mathcal{I}^*$ is the set of independent sets of a matroid on $E$. This matroid $M^* := (E, \mathcal{I}^*)$ is called the dual matroid of $M$. It is well known that if $\rho$ is the rank function of a matroid $M = (E, \mathcal{I})$, then the rank function of $M^* = (E, \mathcal{I}^*)$ is given as follows: for any $J \subset E$,

$$\rho^*(J) := |J| + \rho(E \setminus J) - \rho(E)$$

(see [25, Proposition 2.1.9]). In particular, $\rho^*(E) + \rho(E) = |E|$. The correspondence between the harmonic Tutte polynomial of a matroid $M$ and its dual $M^*$ associated to a harmonic function is given as follows:

**Theorem 3.2** Let $M = (E, \mathcal{I})$ be a matroid with a rank function $\rho$, and let $f \in \text{Harm}_d(n)$. Then $T(M^*, f; x, y) = (-1)^d T(M, f; y, x)$.

Before giving a proof of the above theorem, we need to know about the following technical lemma on harmonic functions from [2].

**Lemma 3.3** ([2]) Let $f \in \text{Harm}_d(n)$ and $J \subset E$. Let

$$f^{(i)}(J) := \sum_{\substack{Z \in E_d, \, |J \cap Z| = i}} f(Z).$$

Then for all $0 \leq i \leq d$, $f^{(i)}(J) = (-1)^{d-i} \binom{d}{i} \tilde{f}(J)$. 

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Lemma 3.4 ([27]) Let $f \in \text{Harm}_d(n)$, and let $J \in 2^E$. Then $\tilde{f}(J) = (-1)^d \tilde{f}(E \setminus J)$. Furthermore, $\tilde{f}(J) = 0$ if $|J| > n - d$.

**Proof** Let $I, J \in 2^E$ such that $I = E \setminus J$. Then

$$\tilde{f}(J) = \sum_{Z \in E_d, \ Z \subseteq J} f(Z) = \sum_{Z \in E_d, \ |Z \setminus I| = 0} f(Z) = f^{(0)}(I) = (-1)^d \tilde{f}(E \setminus J), \tag{5}$$

which is immediate by Lemma 3.3. Again, if $|J| > n - d$, then $|I| = |E \setminus J| < d$. Therefore, by (2) and (5) we have $\tilde{f}(J) = 0$.

**Proof of Theorem 3.2** Let $M$ be a matroid on $E$ with rank function $\rho$. Then $M^*$ is the dual matroid of $M$ with rank function $\rho^*(J) = |J| + \rho(E \setminus J) - \rho(E)$ for any $J \subseteq E$. Therefore,

$$T(M^*, f; x, y) = \sum_{J \subseteq E} \tilde{f}(J)(x - 1)^{\rho^*(E) - \rho^*(J)}(y - 1)^{|J| - \rho^*(J)}$$

$$= \sum_{J \subseteq E} \tilde{f}(J)(x - 1)^{\rho^*(E) - |J| - \rho(E \setminus J) + \rho(E)}(y - 1)^{|J| - |J| - \rho(E \setminus J) + \rho(E)}$$

$$= \sum_{J \subseteq E} \tilde{f}(J)(x - 1)^{|E \setminus J| - \rho(E \setminus J)}(y - 1)^{\rho(E) - \rho(E \setminus J)}$$

$$= (-1)^d \sum_{J \subseteq E} \tilde{f}(E \setminus J)(y - 1)^{\rho(E) - \rho(E \setminus J)}(x - 1)^{|E \setminus J| - \rho(E \setminus J)}$$

$$= (-1)^d T(M, f; y, x).$$

This completes the proof.

**Example 3.5** We assume that $E = \{1, 2, 3\}$. From Example 2.3 we have

$$\text{Harm}_1(3) \ni f = a\{1\} + b\{2\} - (a + b)\{3\}$$

is a harmonic function of degree 1, where $f(\{1\}) = a$, $f(\{2\}) = b$ and $f(\{3\}) = -(a + b)$. Let $M_C = (E, \mathcal{I})$ be a matroid corresponding to a $[3, 2]$ code $C$ that is discussed in Example 2.10. Now we can easily find the ranks of all the subsets of $E$ as follows:

$$\rho(\emptyset) = 0, \quad \rho(\{1\}) = 1, \quad \rho(\{2\}) = 1, \quad \rho(\{3\}) = 1$$

$$\rho(\{1, 2\}) = 1, \quad \rho(\{1, 3\}) = 2, \quad \rho(\{2, 3\}) = 2, \quad \rho(\{1, 2, 3\}) = 2.$$

By direct calculation, we have

$$T(M_C, f; x, y) = \sum_{J \subseteq E} \tilde{f}(J)(x - 1)^{\rho(E) - \rho(J)}(y - 1)^{|J| - \rho(J)}$$

$$= (a + b)(x - 1)(y - 1) - (a + b).$$

### 3.2 Harmonic weight enumerator

In this section, we introduce a new approach to define the harmonic weight enumerators of an $\mathbb{F}_q$-linear code. This formulation is inspired by Jurrius and Pellikaan [16].

**Definition 3.6** Let $E = \{1, 2, \ldots, n\}$ be a finite set. Again let $C$ be an $\mathbb{F}_q$-linear code of length $n$. Then for an arbitrary subset $J \subseteq E$, we define
Lemma 3.7 ([16]) Let $C$ be an $[n, k]$ linear code with generator matrix $G$. Assume that the columns of $G$ are indexed by the set $E$. Let $G_J$ be the $k \times t$ submatrix of $G$ consisting of the columns of $G$ indexed by $J \in E_t$, and let $\rho(J)$ be the rank of $G_J$. Then $\ell(J) = k - \rho(J)$.

Now we have the following proposition.

Proposition 3.8 Let $f \in \text{Harm}_d(n)$ and $J \subset E$. Define

$$B_t, f := \sum_{J \in E_t} \tilde{f}(J) B_J.$$ 

Then we have the following relation between $B_t, f$ and $A_i, f$ as follows:

$$B_t, f = (-1)^d \sum_{i=d}^{n-t} \binom{n-d-i}{t-d} A_i, f,$$

if $d \leq t \leq n - d$; otherwise $B_t, f = 0$.

Proof It is immediate from (2) and Lemma 3.4 that $B_t, f = 0$ for $0 \leq t < d$ and $n - d < t \leq n$.

We now focus on $t$ with $d \leq t \leq n - d$. By Lemma 3.4,

$$B_t, f = \sum_{J \in E_t} \tilde{f}(J) B_J$$

$$= (-1)^d \sum_{J \in E_t} \tilde{f}(E \setminus J) B_J$$

$$= (-1)^d \binom{t}{d}^{-1} \sum_{J \in E_t, X \in E_d, X \subset J} \tilde{f}(E \setminus J) B_J.$$

Therefore, it is sufficient to show that for $d \leq t \leq n - d$,

$$\binom{t}{d}^{-1} \sum_{J \in E_t, X \in E_d, X \subset J} \tilde{f}(E \setminus J) B_J = \sum_{i=d}^{n-t} \binom{n-d-i}{t-d} A_i, f.$$

Now following the definition of $B_J$, we can easily observe that

$$\sum_{J \in E_t, X \in E_d, X \subset J} \tilde{f}(E \setminus J) B_J = \sum_{J \in E_t, X \in E_d, \text{supp}(c) \cap J = \emptyset} \sum_{c \in C, \text{wt}(c) = w} \tilde{f}(E \setminus J)$$

$$= \sum_{c \in C, J \in E_t, X \in E_d, \text{supp}(c) \cap J = \emptyset} \tilde{f}(E \setminus J)$$

$$= \sum_{w=1}^{n-t} \sum_{c \in C, J \in E_t, X \in E_d, \text{wt}(c) = w} \tilde{f}(E \setminus J).$$
For $X \in E_d$ and $c \in C$, let

$$A_X(c) := \{ J \in E_t \mid X \subset J \text{ and } \text{supp}(c) \cap J = \emptyset \}.$$ 

Then from (2) we have

$$\sum_{J \in E_t, X \in E_d, \text{supp}(c) \cap J = \emptyset, X \subset J} \tilde{f}(E \setminus J) = \sum_{X \in E_d, J \in A_X(c), \text{supp}(c) \cap X = \emptyset} \tilde{f}(E \setminus J)$$

$$= \sum_{X \in E_d, J \in A_X(c)} \sum_{Z \in E_d, \text{supp}(c) \cap Z = \emptyset} f(Z)$$

$$= \sum_{T \in E_t, X \in E_d, J \in A_X(c), Z \in E_d, \text{supp}(c) \cap T = \emptyset, X \subset T} f(Z)$$

$$= \left( \frac{t}{d} \right) \frac{n - d - w}{t - d} \sum_{T \in E_t, Z \in E_d, \text{supp}(c) \cap T = \emptyset, Z \subset E \setminus T} f(Z).$$

For $c \in C$ with $\text{wt}(c) = w$ such that $0 < w < d$, by Remark 2.1, we have

$$\sum_{T \in E_t, Z \in E_d, \text{supp}(c) \cap T = \emptyset, Z \subset E \setminus T} f(Z)$$

$$= \sum_{i=0}^{w} \sum_{Y \in E_t, Y \subset \text{supp}(c)} \binom{w}{i} \binom{n - t - w}{d - i} \sum_{Z \in E_d, Y \subset Z} f(Z)$$

$$= 0.$$

Now for $c \in C$ with $\text{wt}(c) = w$ such that $d \leq w \leq n - t$, by Remark 2.1, we have

$$\sum_{T \in E_t, Z \in E_d, \text{supp}(c) \cap T = \emptyset, Z \subset E \setminus T} f(Z)$$

$$= \sum_{Z \in E_d, Z \subset \text{supp}(c)} f(Z) + \sum_{i=0}^{d-1} \sum_{Y \in E_t, Y \subset \text{supp}(c)} \binom{w}{i} \binom{n - t - w}{d - i} \sum_{Z \in E_d, Y \subset Z} f(Z)$$

$$= \sum_{Z \in E_d, Z \subset \text{supp}(c)} f(Z) + 0$$

$$= \tilde{f}(c).$$

Therefore,

$$\sum_{J \in E_t, X \in E_d, \text{supp}(c) \cap J = \emptyset} \tilde{f}(E \setminus J) B_J \tilde{f}(E) = \sum_{w=d}^{n-t} \sum_{c \in C, \text{wt}(c) = w} \left( \frac{t}{d} \right) \frac{n - d - w}{t - d} \tilde{f}(c)$$

$$= \sum_{w=d}^{n-t} \left( \frac{t}{d} \right) \frac{n - d - w}{t - d} A_i.$$

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This completes the proof.

**Example 3.9** Let $C$ be a binary linear code of length 3 given in Example 2.7. We consider $f \in \text{Harm}_1(3)$ and its $\tilde{f}$’s as in Example 2.3. Now we compute $B_t, f$ for $1 \leq t \leq 2$ in two ways: by direct computation and by using Proposition 3.8.

\[
B_{1, f} = \sum_{J \in E_1} \tilde{f}(J) J \\
= a.1 + b.1 - (a + b).1 \\
= 0
\]

\[
B_{1, f} = (-1)^1 \sum_{i=1}^{2} \begin{pmatrix} 2 - i \end{pmatrix} A_i, f \\
= -A_1, f - A_2, f \\
= (a + b) - (a + b) \\
= 0
\]

\[
B_{2, f} = \sum_{J \in E_2} \tilde{f}(J) J \\
= (a + b).1 - b.0 - a.0 \\
= a + b
\]

\[
B_{2, f} = (-1)^1 \sum_{i=1}^{1} \begin{pmatrix} 2 - i \end{pmatrix} A_i, f \\
= -A_1, f \\
= a + b
\]

Now we have the following result.

**Theorem 3.10** Let $C$ be an $\mathbb{F}_q$-linear code of length $n$, and let $f \in \text{Harm}_d(n)$. Then

\[
Z_{C, f}(x, y) = (-1)^d \sum_{t=d}^{n-d} B_t, f (x - y)^{t-d} y^{n-t-d}.
\]

**Proof** By using Proposition 3.8 and using the binomial expansion of $x^{n-i} = ((x - y) + y)^{n-i}$ we have

\[
(-1)^d \sum_{t=d}^{n-d} B_t, f (x - y)^{t-d} y^{n-t-d} \\
= \sum_{t=d}^{n-d} \sum_{i=d}^{n-t} \binom{n-d-i}{t-d} A_i, f (x - y)^{t-d} y^{n-t-d} \\
= \sum_{i=d}^{n-d} A_i, f \left( \sum_{t=d}^{n-i} \binom{n-d-i}{t-d} (x - y)^{t-d} y^{(n-d-i)-(t-d)} \right) y^{i-d} \\
= \sum_{i=d}^{n-d} A_i, f x^{n-d-i} y^{i-d} \\
= \sum_{i=0}^{n-d} A_i, f x^{n-i-d} y^{i-d} \\
= Z_{C, f}(x, y),
\]

since $A_i, f = 0$ for $i < d$ and $i > n - d$.  

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4 Generalized Greene’s identity

Let $M_C$ be a matroid associated to an $\mathbb{F}_q$-linear code $C$ of length $n$. It is immediate from [11, 14, 16] that $(M_C)^* = M_{C^\perp}$. Now we have the following proposition.

**Proposition 4.1** Let $C$ be an $[n, k]$ code and $M_C$ be its matroid. Let $f$ be a harmonic function of degree $d$. Then

$$T(M_C, f; x, y) = \sum_{t=d}^{n-d} \sum_{J \in E_t} \tilde{f}(J) (x - 1)^{\ell(J)} (y - 1)^{\ell(J)-(k-t)}.$$

**Proof** The proposition follows from $\ell(J) = k - \rho(J)$ for any $J \in E_t$ by Lemma 3.7, and $\rho(E) = k$.

Now we have the following harmonic generalization of Greene’s identity.

**Theorem 4.2** Let $C$ be an $[n, k]$ code and $f$ be a harmonic function with degree $d$. Then

$$Z_{C, f}(x, y) = (-1)^d (x - y)^{k-d} y^{n-k-d} T \left( M_C, f; \frac{x + (q - 1)y}{x - y}, \frac{x}{y} \right).$$

**Proof** By using Proposition 4.1 and Remark 2.2, we can write

$$T \left( M_C, f; \frac{x + (q - 1)y}{x - y}, \frac{x}{y} \right) = \sum_{t=d}^{n-d} \sum_{J \in E_t} \tilde{f}(J) \left( \frac{qy}{x - y} \right)^{\ell(J)} \left( \frac{x - y}{y} \right)^{\ell(J)-(k-t)}$$

$$= \sum_{t=d}^{n-d} \sum_{J \in E_t} \tilde{f}(J) q^{\ell(J)} \left( \frac{y}{x - y} \right)^{\ell(J)} \left( \frac{x - y}{y} \right)^{\ell(J)-(k-t)}$$

$$= \sum_{t=d}^{n-d} \sum_{J \in E_t} \tilde{f}(J) ((q^{\ell(J)} - 1) + 1)(x - y)^{-(k-t)} y^{k-t}$$

$$= \sum_{t=d}^{n-d} \sum_{J \in E_t} \tilde{f}(J) (B_J + 1)(x - y)^{-(k-t)} y^{k-t}$$

$$= \left( \sum_{J \in E_t} \tilde{f}(J) B_J \right) (x - y)^{-(k-t)} y^{k-t}$$

$$= \sum_{t=d}^{n-d} \left( B_t, f + 0 \right) (x - y)^{-(k-t)} y^{k-t}$$

$$= \sum_{t=d}^{n-d} B_t, f (x - y)^{-(k-t)} y^{k-t}.$$
\((-1)^d (x - y)^{k-d} y^{n-k-d} T \left( M_C, f; \frac{x + (q - 1)y}{x - y}, \frac{x}{y} \right) \)
\[= (-1)^d \sum_{t=d}^{n-d} B_t, f (x - y)^{t-d} y^{n-t-d} \]
\[= Z_{C,f} (x, y). \]

This completes the proof.

Now we give an alternative proof of the \(\mathbb{F}_q\)-analogue of Bachoc’s MacWilliams type identity (see [2]) stated in Theorem 2.5 as an application of Theorem 4.2.

**Proof of Theorem 2.5** Let \(C\) be an \([n, k]\) code, and \(M_C\) be its matroid. Then

\[-1\rightd \frac{q^{n/2}}{|C|} Z_{C,f} \left( \frac{x + (q - 1)y}{\sqrt{q}}, \frac{x - y}{\sqrt{q}} \right) \]
\[= (-1)^d \frac{q^{n/2}}{q^k} \left( \frac{qy}{\sqrt{q}} \right)^{k-d} \left( \frac{x - y}{\sqrt{q}} \right)^{n-k-d} T \left( M_C, f; \frac{x + (q - 1)y}{x - y} \right) \]
\[= (x - y)^{n-k-d} y^{k-d} T \left( M_C, f; \frac{x + (q - 1)y}{x - y} \right) \]
\[= (-1)^d (x - y)^{(n-k)} y^{-(n-k)-d} T \left( MC^{-1}, f; \frac{x + (q - 1)y}{x - y} \right) \]
\[= (-1)^d (x - y)^{\dim C - d} y^{n - \dim C - d} T \left( MC^{-1}, f; \frac{x + (q - 1)y}{x - y} \right) \]
\[= Z_{C^{-1}, f} (x, y). \]

Hence Theorem is proved.

## 5 Concluding remarks

Let \(n, k, t\) and \(\lambda\) be non-negative integers such that \(n \geq k \geq t\) and \(\lambda \geq 1\). A \(t-(n, k, \lambda)\) design (in short, \(t\)-design) is a pair \(D := (E, B)\), where \(E\) is a finite set of points of cardinality \(n\), and \(B\) is a collection of \(k\)-element subsets of \(E\) called blocks, with the property that any \(t\) points are contained in precisely \(\lambda\) blocks. Some properties of combinatorial \(t\)-designs obtained from codes were discussed in [1, 2, 4, 9, 10, 17, 21–23, 26] and their analogies in the theory of lattices and vertex operator algebras were discussed in [4–7, 19–21].

The harmonic functions have many applications; particularly, the relations between design theory and coding theory were stated in Bachoc [2]: the set of codewords for every given weight in a binary code \(C\) forms a \(t\)-design if and only if \(W_{C,f} (x, y) = 0\) for all \(f \in \text{Harm}_d(n)\), \(1 \leq d \leq t\). Now we have the following design theoretical remark that gives an application of the harmonic function connecting the \(t\)-designs with matroids.

**Remark 5.1** If \(T (MC, f; x, y) = 0\) for all \(f \in \text{Harm}_d(n)\), \(1 \leq d \leq t\), then the set of codewords for every given weight in an \(\mathbb{F}_q\)-linear code \(C\) forms a \(t\)-design.

We will more precisely discuss a relation between matroids and combinatorial designs with respect to “harmonic Tutte polynomials” in a forthcoming paper. Moreover, we give the generalizations of the results in this paper in [8].
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