UNIFORM ARTIN-REES BOUNDS FOR SYZYGIES

IAN M. ABERBACH, ALINE HOSRY, AND JANET STRIUlli

ABSTRACT. Let $(R, m)$ be a local Noetherian ring, let $M$ be a finitely generated $R$-module and let $(F_{\bullet}, \partial_{\bullet})$ be a free resolution of $M$. We find a uniform bound $h$ such that the Artin-Rees containment $I^n F_i \cap \text{Im} \partial_{i+1} \subseteq I^{n-h} \text{Im} \partial_{i+1}$ holds for all integers $i \geq d$, for all integers $n \geq h$, and for all ideals $I$ of $R$. In fact, we show that a considerably stronger statement holds. The uniform bound $h$ holds for all ideals and all resolutions of $d$th syzygy modules. In order to prove our statements, we introduce the concept of Koszul annihilating sequences.

1. Introduction

Let $R$ be a Noetherian commutative ring. To paraphrase Huneke in [9], the unsubtle finiteness condition that each ideal of $R$ is finitely generated often, in fact, implies subtle forms of finiteness in the ring, many of which are “uniform.” By “uniform” we mean, for instance, that one element may annihilate the homology of an entire class of complexes, or that an integer known to exist for a certain finite set of data (e.g., a finitely generated module and submodule, along with an ideal), may actually apply uniformly to an infinite data set. The Main Theorem, Theorem (1.1) below, is of the latter form, but in order to prove it, we have been led to the notion of what we call a Koszul annihilating sequence, which shows a form of finiteness of the former type.

Let $R$ be a Noetherian ring and let $M \subseteq N$ be finitely generated $R$-modules. For an ideal $I \subseteq R$, the classical Artin-Rees Lemma states that there exists an integer $h$, depending on the ideal $I$ and the inclusion $M \subseteq N$, such that for all integers $n \geq h$, the equality

$$I^n N \cap M = I^{n-h} (I^h N \cap M)$$

holds. A weaker form of this statement, which is often used in applications, is that for all integers $n \geq h$, the following inclusion holds:

$$I^n N \cap M \subseteq I^{n-h} M.$$  

The proof of the equality in equation (1.0.1) uses the fact that the associated graded ring of $R$ with respect to $I$ and the associated graded modules of $M$ and $N$ with respect to $I$ are Noetherian. With this approach, the integer $h$ above, which is referred to as the Artin-Rees number, clearly depends on all of the data, that is, on $I$ and on $M \subseteq N$.

Eisenbud and Hochster, in [6], first raised the question of whether or not there might be an integer $h$ as in equality (1.0.1) which, given $M \subseteq N$, works for all the maximal ideals in the ring $R$. In this paper, the term “uniform Artin-Rees” was used for the first time, and an example in a non-excellent ring was given to show that some condition on the ring is required in order to find a uniform Artin-Rees number that works for all maximal ideals. In fact, for excellent rings, Duncan and O’Carroll in [4] obtained a uniform Artin-Rees number for which equation (1.0.1) holds for every maximal ideal, answering Eisenbud and Hochster’s question. Their work was proceeded by O’Carroll, who proved in [11] that an integer $h$ can be chosen to uniformly satisfy...
We define a module $M$ exists a uniform integer $h$ in inclusion (1.0.2) for all maximal ideals of an excellent ring. In [9], Huneke showed that, given the inclusion $M \subseteq N$, the integer $h$ in inclusion (1.0.2) can be chosen independently of the ideal $I$ under mild assumptions on the ring, for example when the ring is essentially of finite type over a local ring (by “local” we include the condition of being Noetherian). The integer $h$ is then called a Uniform Artin-Rees (UAR) number for $M \subseteq N$. Huneke’s result gives a lot of information even in a local ring, whereas the O’Carroll and Duncan/O’Carroll results have content only in rings with infinitely many maximal ideals. The focus of this paper remains on local rings, with an emphasis on uniform results over the set of all ideals.

The impetus for our results is a “uniform Artin-Rees” problem raised by Eisenbud and Huneke in [7], concerning uniform bounds of Artin-Rees type on free resolutions. Let $M$ be a finitely generated $R$-module and let $(F_\bullet, \partial_\bullet)$ be a free resolution of $M$ by finitely generated free modules:

$$
\cdots \longrightarrow F_{i+1} \xrightarrow{\partial_{i+1}} F_i \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \longrightarrow M \longrightarrow 0.
$$

We define a module $M$ to be syzygetically Artin-Rees with respect to a family of ideals $\mathcal{I}$, if there exists a uniform integer $h$ such that for every $n \geq h$, for every $i \geq 0$, and for every $I \in \mathcal{I}$,

$$
I^n F_i \cap \text{Im} \partial_{i+1} \subseteq I^{n-h} \text{Im} \partial_{i+1}.
$$

This definition does not depend on the free resolution (see for example [13] Lemma 2.1)]. Eisenbud and Huneke raise the question ([7] Question B]) of whether or not, given a local ring $(R, m)$ and an ideal $I \subseteq R$, every module is syzygetically Artin-Rees with respect to the family $\mathcal{I} = \{I\}$.

In [7], the authors show that given a local ring $(R, m)$, a module $M$ is syzygetically Artin-Rees with respect to an ideal $I \subseteq R$ under the hypotheses that $M$ has finite projective dimension and constant rank on the punctured spectrum. In [13], the third author of this paper observed that much stronger statements could be made if one looks at high enough syzygies. She proved that every second syzygy module in a local ring of dimension at most two has a certain uniform annihilation property with respect to the family of $m$-primary ideals, see [13] Theorem 5.4]. Although it is never formally stated in her paper, one can deduce from her results that in a local ring of dimension two, every finitely generated second syzygy is syzygetically Artin-Rees with respect to the set of $m$-primary ideals, and that the integer $h$ in inclusion (1.0.3) can be chosen independently of the module $M$, i.e., we have a double uniform Artin-Rees theorem. This leads us to an extended definition: We say that a family of modules $\mathcal{M}$ is syzygetically Artin-Rees with respect to a family of ideals $\mathcal{I}$ if there exists an $h$ such that inclusion (1.0.3) holds for every module of $\mathcal{M}$ and every ideal of $\mathcal{I}$. It is in this form that we prove our main theorem.

(1.1) **Main Theorem.** Let $(R, m)$ be a local Noetherian ring of dimension $d$. Then, the family of finitely generated $d$th syzygy modules is syzygetically Artin-Rees with respect to the family of all ideals.

It quickly becomes apparent, when considering this problem over infinite sets of ideals and modules, that no uniform statement is possible when we consider the beginning of a resolution, even in very nice rings. Suppose that $(R, m)$ is Cohen-Macaulay (or even regular) and $x_1, \ldots, x_d$ is a system of parameters. The Koszul complex $K_*(x_1^t, \ldots, x_d^t; R)$ is a free resolution of the module $R/(x_1^t, \ldots, x_d^t)$. This complex will have Artin-Rees number at least $t$ for the ideal $(x_1, \ldots, x_d)$. But there is a uniform behavior for all such complexes past $d$, since all higher syzygies are zero.

The syzygetically Artin-Rees property is intertwined with finding uniform annihilators for homology modules. In fact, given a free resolution $(F_\bullet, \partial_\bullet)$ of an $R$-module $M$, there is an isomorphism

$$
\frac{I^n F_{i-1} \cap \text{Im} \partial_i}{I^n \text{Im} \partial_i} \cong \text{Tor}_i^R(R/I^n, M), \quad \text{for every } i \geq 1.
$$
Our work to find uniform annihilator elements for the homology module displayed above is inspired by the ideas contained in the monograph [8], which also relate to important homological conjectures. In this monograph, the authors show that elements that annihilate higher homology modules of Koszul complexes on sequences that are part of a system of parameters can often, after taking a power, annihilate higher homology modules of a much wider class of complexes. Huneke’s proof of his uniform Artin-Rees and uniform Briançon-Skoda theorems also relies heavily on such elements.

In order to prove our Main Theorem, given a local ring \((R, m)\), we need a sequence of elements which annihilate higher Koszul homology modules of parameters. As we need this sequence to itself form a system of parameters, we are forced to choose elements that annihilate higher Koszul homology of part of system of parameters of appropriate length. This leads us to introduce the notion of \(K_{\text{AS}}\) sequence (KAS), whose precise definition, Definition (2.3), is given in Section 2, together with the proof of its existence under mild hypotheses on the ring, Theorem (2.6). We believe that KAS sequences may offer a significant tool in problems where controlling higher homology modules for classes of free complexes is important.

Section 3 is devoted to showing how Koszul annihilating sequences uniformly annihilate the homology modules of a much larger class of complexes.

Finally, in Section 4, after proving several lemmas, we give the proof of Theorem (4.8) of which the Main Theorem is a corollary.

2. Koszul annihilating sequences

Throughout this section, \(R\) denotes a Noetherian ring of dimension \(d\). Let \(x = x_1, \ldots, x_n\) be a sequence of elements in \(R\), and let \(M\) be an \(R\)-module. We denote the Koszul complex on the sequence \(x\) and the module \(M\) by \(K_\bullet(x; M) = K_\bullet(x_1, \ldots, x_n; M)\), and we denote its \(i\)th homology by \(H_i(x : m) = H_i(x_1, \ldots, x_n; M)\).

This section is devoted to showing, given a local ring \((R, m)\), the existence of a system of parameters, which we call a Koszul annihilating sequence (or KAS), whose elements uniformly annihilate higher homology modules of the Koszul complex on sequences that are part of a system of parameters for \(R\) against all modules that are high syzygies. The precise definition of KAS is given in Definition (2.3), and the existence of such a sequence for the class of \(d\)th syzygies is the content of Theorem (2.6).

As mentioned in the introduction, our work is inspired by the monograph of Hochster and Huneke. The first result we need to modify is [8, Theorem 2.16], which we state below as (2.1). For a finitely generated module \(M\), the height of \(I\) on \(M\), \(\text{ht}_M I\), is the height of the ideal \(I\) in the ring \(R/\text{Ann}_R M\).

(2.1) Let \(R\) be a Noetherian ring, not necessarily local. Let \(x_1, \ldots, x_n\) be elements in the ring \(R\) and suppose that \(M\) is a finitely generated \(R\)-module of dimension \(d\) such that the inequality \(\text{ht}_M(x_1, \ldots, x_i) \geq i\) is satisfied for all \(i = 1, \ldots, n\). Let \(c\) be an element of \(R\) such that, for \(i = 0, \ldots, n - 1\), the equality \(cH^i_{pM}(M_p) = 0\) holds for any prime ideal \(p\) containing the elements \(x_1, \ldots, x_{i+1}\). Then the equality \(cE(d, n, \delta)H_{n-1}(x_1, \ldots, x_n; M) = 0\) holds for \(0 \leq t \leq n - 1\), where, for integers \(\delta \geq \nu > \tau \geq 0\), the function \(E(\delta, \nu, \tau)\) is defined recursively as follows:

\[
\begin{align*}
E(\delta, \nu, 0) &= \delta - \nu + 1 \\
E(\delta, \nu, \tau) &= \delta + (\delta + 2)E(\delta, \nu - 1, \tau - 1), \quad \tau \geq 1.
\end{align*}
\]

We provide here an alternative version of result (2.1), with essentially the same proof as given in [8, Theorem 2.16]. With mild hypotheses on \(R\), we may look only at the height of the sequence on \(R\) (not on the module \(M\), and we may drop the dimension assumption on \(M\). This is valuable
for our applications to syzygies as we do not always know that syzygies have the full dimension of
the ring. On the other hand, the hypotheses on the ring are satisfied when we apply our theorem
to Gorenstein rings.

In our case, we define a new function \( E_1(\delta, \nu, \tau) \) recursively by

\[
\begin{align*}
E_1(\delta, \nu, 0) &= \delta - \nu + 1 \\
E_1(\delta, \nu, \tau) &= \delta + (\delta + 2)E_1(\delta - 1, \nu - 1, \tau - 1), \quad \tau \geq 1.
\end{align*}
\]

(2.2) Theorem. Let \( R \) be a Noetherian ring of dimension \( d \) which is catenary and locally equidi-
men-tional at each maximal ideal. Let \( x = x_1, \ldots, x_n \) be a sequence of elements of \( R \) for
which the inequality \( \text{ht}_R(x_1, \ldots, x_i)R \geq i \) holds for all integers \( i \) with \( 1 \leq i \leq n \). Let \( M \) be a finitely
generated \( R \)-module, and let \( c \in R \) be an element such that the equality \( cH^i_p(R_M) = 0 \) holds
for any prime ideal \( p \) containing the sequence \( x_1, \ldots, x_{i+1} \) and for \( 0 \leq i \leq n - 1 \).

Then, the equality \( c^{E_1(d, n, t)}H_{n-t}(x; M) = 0 \) holds for \( 0 \leq t \leq n - 1 \).

Proof. We use induction on \( d, n \), and \( t \). The property of being equidimensional at maximal ideals
localizes and the statement is a local one on \( R \) at maximal ideals. Moreover, if \( m \) is a maximal
ideal of \( R \) then \( \text{ht}_R m \leq d \) and for \( d' \leq d \) we have \( E_1(d', n, t) \leq E_1(d, n, t) \). Finally, since a
module is zero if and only if it is zero after you localize at every maximal ideal, without loss of
generality we may assume that \( (R, m) \) is a local Noetherian catenary and equidimensional ring
of dimension \( d \).

If \( t = 0 \), we shall show that \( H_n(x; M) \) is annihilated by \( c^{d-n+1} \). We can assume that \( n \geq 1 \).
If \( n > d \), then \( (x)R = R \) and all Koszul homologies vanish. If \( d = n \), then \( (x) \) generates an
\( m \)-primary ideal and \( H_n(x; M) \subseteq H^n_m(M) \), which is annihilated by \( c \) by hypothesis. We may
thus assume that \( n < d \). Consider \( c^{d-n}H_n(x; M) \). By the induction hypothesis on \( d \), this
module vanishes on the punctured spectrum. The module is also a submodule of \( M \), and hence \( c^{d-n}H_n(x; M) \subseteq H^n_m(M) \). By hypothesis, \( c \) is in the annihilator of the module \( c^{d-n}H_n(x; M) \).

We now assume that \( t > 0 \). Let \( N = \text{ann}_M(x_1) = H_1(x_1; M) \). We may apply the case \( t = 0 \)
and \( n = 1 \) to see that \( c^d \) is in the annihilator of the module \( N \). Consider the two exact sequences
\[
0 \to N \to M \to M' \to 0, \quad \text{and}
\]
\[
0 \to M' \to M \to M'/x_1M \to 0, \quad \text{where } M' = x_1M \cong M/N.
\]

Let \( r \) be an integer with \( 0 \leq r \leq n - 1 \) and \( p \) a prime ideal containing \( x_1, \ldots, x_{r+1} \).

The short exact sequence (2.2.1) yields the following long exact sequence:

\[
\cdots \to H^i_p(R_p)(M_p) \to H^i_p(R_p)(M'_p) \to H^{i+1}_{pR_p}(N_p) \to \cdots.
\]

Since the equality \( c^dN = 0 \) holds, then \( c^d \) is in the annihilator of the third module displayed in
(2.2.3) for all \( i \geq 0 \); by hypothesis \( c \) annihilates the first module displayed in (2.2.3) for all \( i \leq r \).

Thus, for \( i \leq r \), \( c^{d+1}H^i_{pR_p}(M'_p) = 0 \).

In particular, \( c^{d+1} \) kills the third term displayed below for all \( i < r \)
\[
\cdots \to H^i_{pR_p}(M_p) \to H^i_{pR_p}(M/x_1M_p) \to H^{i+1}_{pR_p}(M'_p) \to \cdots,
\]

where the long exact sequence comes from the short exact sequence (2.2.2). Hence, for all \( i < r \),
\( c^{d+2} \) kills the middle term in (2.2.4). Since by hypothesis \( cH^1_{pR_p}(M_p) = 0 \) for \( i \leq r \).

Because \( R \) is equidimensional and catenary of dimension \( d \), the ring \( R/x_1R \) is equidimensional
and catenary of dimension \( d - 1 \), and \( \text{ht}_R((x_2, \ldots, x_{i+1})(R/x_1R)) \geq i \). We apply the induction
hypothesis with \( M/x_1M \) replacing \( M \), \( R/x_1R \) replacing \( R \), \( x_2, \ldots, x_n \) replacing the original
sequence, \( c^{d+2} \) replacing \( c \), and \( d - 1, n - 1, t - 1 \) replacing \( d, n, \) and \( t \) respectively to conclude
that \( c^{(d+2)E_1(d-1,n-1,t-1)} \) kills \( H_{n-t}(x_2, \ldots, x_n; M/x_1M) \).
Finally, via the degeneration of a spectral sequence for change of rings for $\text{Tor}$ (see [12, p. V-17]) there is a long exact sequence
\[ \cdots \to H_{i-1}(x_2, \ldots, x_n; N) \to H_i(\mathbf{x}; M) \to H_i(x_2, \ldots, x_n; M/x_1 M) \to \cdots. \]
Let $i = n - t$. Since $c^d$ annihilates the first displayed term and $c^{(d+2) \mathcal{E}_i(n-1,t-1)}$ annihilates the third term, we obtain that $c^{\mathcal{E}_i(d,n,t)} H_i(\mathbf{x}; M) = 0$, as required. \hfill $\square$

We are now ready to give the definition of a Koszul annihilating sequence.

(2.3) **Definition.** Let $(R, \mathfrak{m})$ be a local ring of dimension $d$ and let $M$ be a finitely generated $R$-module. Given an integer $i$ such that $0 \leq i < d$, a sequence of elements, $c_i, c_{i+1}, \ldots, c_d \in \mathfrak{m}$ is a *Koszul annihilating sequence* (KAS) for $M$ (of length $d - i + 1$), if

1. the elements $c_i, \ldots, c_d$ are part of a system of parameters for $R$, and
2. for all integers $k$ such that $d \geq k \geq i - 1$, if $x_1, \ldots, x_k, c_{k+1}, \ldots, c_d$ are a system of parameters for $R$, then the equality
   \[ c_v H_k(x_1, \ldots, x_k, c_{k+1}, \ldots, c_j; M) = 0 \]
   holds for all $v \geq j \geq k$ and for all $n, t \geq 1$.

If $M$ is a family of finitely generated $R$-modules, and $c_i, \ldots, c_d$ is a KAS for all $M \in \mathcal{M}$, then we say that the sequence is a KAS for $\mathcal{M}$.

(2.4) **Remark.** It is clear from the definition that if we modify a KAS by replacing each element by a (possibly different) power of itself, we still have a KAS.

Moreover a KAS of length $\dim R$ is a system of parameters for $R$.

Theorem (2.6) below shows that for any local image of a Gorenstein ring, it is always possible to find a KAS of length $d$ for the class of modules which are $d$th syzygies. Before proving the theorem, we introduce some ideals that will be used in the proof.

(2.5) Let $(R, \mathfrak{m})$ be a local ring. For an integer $i$ such that $0 \leq i < d$, set $a_i = a_i(R) = \text{Ann}_R H^i_d(R)$ and $b_i = b_i(R) = a_0(R) \cdots a_i(R)$. It is a simple induction to see that if $M$ is a $j$th syzygy then for $0 \leq i < j$, $b_i \subseteq \text{Ann}_R H^i(M)$. In particular, for every $d$th syzygy, and for every $0 \leq i < d$, $b_i \subseteq \text{Ann}_R H^i_d(M)$. We also note that $b_0 \supseteq b_1 \supseteq \cdots \supseteq b_d$ and, when $R$ is an image of a Gorenstein ring, then $\dim R/b_i \leq i$.

(2.6) **Theorem.** Let $(R, \mathfrak{m})$ be a local ring of dimension $d$ which is the image of a Gorenstein ring. Then, there exists a KAS sequence, $c_1, \ldots, c_d$, for the family of finitely generated modules which are $d$th syzygies over $R$.

Moreover, the KAS sequence $c_1, \ldots, c_d$ can be chosen such that the following holds
\[ \dim R/(0 :_R (0 :_R (c_{d-1}))) \leq d - i - 1, \]
for all $1 \leq i \leq d - 1$.

**Proof.** We begin by choosing $c'_d \in b_{d-1} - \bigcup_{p \in \text{Assh}_R p} p$, where $\text{Assh} R$ denotes the set of associated prime ideals $p$ of maximal dimension, i.e., $\dim R/p = \dim R$. This choice is possible, since $\dim R/b_{d-1} \leq d - 1$ by (2.5), and therefore, by prime avoidance, $b_{d-1}$ is not contained in any minimal prime of $R$ of maximal dimension.

Inductively, having chosen $c'_d, \ldots, c'_{i+1}, c'_i$ we pick
\[ c'_i \in b_{i-1} - \bigcup \left( \{ p \mid p \in \text{Assh}_R R/(c'_{i+1}, \ldots, c'_d) \} \cup \{ p \mid p \in \text{Ass} R, \dim R/p \geq i \} \right). \]

We can avoid all the above primes in question since $\dim R/b_{i-1} \leq i - 1$ by (2.5).
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\[ I^n F_i \cap \text{Im} \partial_{i+1} \subseteq I^{n-h} \text{Im} \partial_{i+1} \]

holds for all integers \(i \geq d\), for all integers \(n \geq h\), and for all ideals \(I\) of \(R\). In fact, we show that a considerably stronger statement holds. The uniform bound \(h\) holds for all ideals and all resolutions of \(d\)th syzygy modules. In order to prove our statements, we introduce the concept of Koszul annihilating sequences.

1. Introduction

Let \(R\) be a Noetherian commutative ring. To paraphrase Huneke in [9], the unsubtle finiteness condition that each ideal of \(R\) is finitely generated often, in fact, implies subtle forms of finiteness in the ring, many of which are “uniform.” By “uniform” we mean, for instance, that one element may annihilate the homology of an entire class of complexes, or that an integer known to exist for a certain finite set of data (e.g., a finitely generated module and submodule, along with an ideal), may actually apply uniformly to an infinite data set. The Main Theorem, Theorem (1.1) below, is of the latter form, and in order to prove it, we have been led to the notion of what we call a Koszul annihilating sequence, which shows a form of finiteness of the former type.

Let \(R\) be a Noetherian ring and let \(M \subseteq N\) be finitely generated \(R\)-modules. For an ideal \(I \subseteq R\), the classical Artin-Rees Lemma states that there exists an integer \(h\), depending on the ideal \(I\) and the inclusion \(M \subseteq N\), such that for all integers \(n \geq h\), the equality

\[ I^n N \cap M = I^{n-h}(I^h N \cap M) \]

holds. A weaker form of this statement, which is often used in applications, is that for all integers \(n \geq h\), the following inclusion holds:

\[ I^n N \cap M \subseteq I^{n-h} M. \]

The proof of the equality in equation (1.0.1) uses the fact that the associated graded ring of \(R\) with respect to \(I\) and the associated graded modules of \(M\) and \(N\) with respect to \(I\) are Noetherian. With this approach, the integer \(h\) above, which is referred to as the Artin-Rees number, clearly depends on all of the data, that is, on \(I\) and on \(M \subseteq N\).

Eisenbud and Hochster, in [6], first raised the question of whether or not there might be an integer \(h\) as in equality (1.0.1) which, given \(M \subseteq N\), works for all the maximal ideals in the ring \(R\). In this paper, the term “uniform Artin-Rees” was used for the first time, and an example in a non-excellent ring was given to show that some condition on the ring is required in order to find a uniform Artin-Rees number that works for all maximal ideals. In fact, for excellent rings, Duncan and O’Carroll in [4] obtained a uniform Artin-Rees number for which equation (1.0.1) holds for every maximal ideal, answering Eisenbud and Hochster’s question. Their work was preceeded by O’Carroll, who proved in [11] that an integer \(h\) can be chosen to uniformly satisfy

\[ I^n N \cap M \subseteq I^{n-h} M. \]

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complexes described, for instance, in \[8\] Definition 3.16 and we study their relations with KAS sequences.

Let \((G_\bullet, \partial_\bullet)\) be a complex of finitely generated free modules:

\[
0 \longrightarrow G_n \xrightarrow{\partial_n} G_{n-1} \longrightarrow \cdots \longrightarrow G_1 \xrightarrow{\partial_1} G_0 \longrightarrow 0
\]

where \(n\) is the length of the complex. For a positive integer \(r\), denote by \(I_r(\partial_i)\) the ideal generated by the \(r \times r\) minors of a matrix that represents the map \(\partial_i\) after a choice of basis (the ideal is independent of the choice). The rank of the homomorphism \(\partial_i\) is given by

\[
\text{rank} \, \partial_i = \max \{ r \mid I_r(\partial_i) \neq 0 \}.
\]

Denote by \(I(\partial_i)\) the ideal \(I_{\text{rank} \, \partial_i}(\partial_i)\).

The complex \((G_\bullet, \partial_\bullet)\) satisfies the standard condition on rank if the following equality holds for \(i = 1, \ldots, n:\n
\text{rank}_R G_i = \text{rank} \, \partial_i + \text{rank} \, \partial_{i+1}.
\]

(3.1) The standard condition on rank plays a very important role on the exactness of a complex. Buchsbaum and Eisenbud show in \[3\] that a complex of free modules

\[
0 \longrightarrow G_n \xrightarrow{\partial_n} G_{n-1} \longrightarrow \cdots \longrightarrow G_1 \xrightarrow{\partial_1} G_0 \longrightarrow 0
\]

is exact if and only if it satisfies the standard condition on rank, and \(\text{depth}_R I(\partial_i) \geq i\) for all integers \(i \geq 1\).

The proof of (3.1) uses several lemmas. We include two of them for easy reference. One may consult \[2\] Proposition 1.4.10 or refer to the original paper \[8\] for a proof.

(3.2) Let \(\partial : F_1 \to F_0\) be a homomorphism between two free \(R\)-modules. Denote by \(M\) the cokernel of \(\partial\). Then \(I(\partial) = R\) if and only if \(M\) is a free module of rank equal to \(\text{rank}(F_0 - \text{rank}(\partial))\).

(3.3) Let \(0 \to G_n \to \cdots \to G_0\) be a complex of free modules that satisfies the standard condition on rank. Assume that \(I(\partial_i) = R\) for all \(i = 1, \ldots, n\). Then the complex is split exact.

Our first proposition shows how a KAS sequence annihilates the homology of complexes which are the tensor product of high syzygies and certain complexes satisfying the standard conditions on rank and height. We use the following result, which is \[8\] Proposition 3.1. The statement in the original publication has a typo (for the upper bound of \(i\) in item (1) below), so we restate it here.

(3.4) Let \(R\) be an arbitrary commutative ring and let \(M_\bullet\) be a complex

\[
0 \to M_n \to \cdots \to M_1 \to M_0 \to 0
\]

of arbitrary \(R\)-modules. Let \(x = x_1, \ldots, x_n\) be a sequence of elements in \(R\), and let \(\ell, \ell_0, \ell_1, \ldots, \ell_{n-2}\) be elements of \(R\) such that

1. \(\ell_i\) kills \(H_{n-i}(M_\bullet)\), for \(0 \leq i \leq n - 2\) and
2. \(\ell\) kills \(H_{n-j}(x_{1}, \ldots, x_n; M_{j+1})\), for \(1 \leq j \leq n - 1\).

Then \(D = (\ell_0 \ell_1 \cdots \ell_{n-2}) \ell^n\) kills \(\text{Hom}_R(R/(x_1, \ldots, x_n), H_1(M_\bullet))\).

(3.5) **Proposition.** Let \((R, \mathfrak{m})\) be a local ring of dimension \(d\) and let \(c_1, \ldots, c_d\) be a KAS for the family of modules which are \(d\)th syzygies. Then there exists an integer \(t \geq 0\) such that

\[
c_{i+j}^t H_{d-i+j}(G_\bullet \otimes M) = 0,
\]

for all integers \(i\) and \(j\) such that \(i \geq 1, 0 \leq j \leq d - n\), for every \(d\)th syzygy \(M\), and for every complex \((G_\bullet, \partial_\bullet)\) of length \(n \leq d\) of finitely generated free \(R\)-modules with following properties:
(1) $G_\bullet$ satisfies the standard condition on rank,
(2) For $1 \leq i \leq n$, the ideal $I(\partial_i) + (c_{i+1}, \ldots, c_d)$ is $m$-primary.

Proof. Denote by $b_i$ the rank of the free module $G_i$. We use reverse induction on the homological degree of the complex $G_\bullet$.

At homological degree $n$, the complex looks like $0 \to G_n \to G_{n-1}$, with $\partial_n : G_n \to G_{n-1}$. By hypothesis, $I(\partial_n) + (c_{n+1}, \ldots, c_d)$ is $m$-primary. Choose elements $y_1, \ldots, y_n \in I(\partial_n)$ such that the sequence $y_1, \ldots, y_n, c_{n+1}, \ldots, c_d$ forms a system of parameters. Then for each $1 \leq i \leq n$, we obtain $I(\partial_n)_{y_i} = R_{y_i}$. The content of (3.2) implies that the cokernel of $(\partial_n)_{y_i}$ is a free module of rank equal to $b_{n-1} = b_n - b_{n-1}$, where the last equality holds since the complex $G_\bullet$ satisfies the standard condition on rank. In particular, the sequence

$$0 \to (G_n)_{y_i} \to (G_{n-1})_{y_i} \to \text{Coker}(\partial_n)_{y_i} \to 0$$

is a split exact sequence.

Let $M$ be a $d$th syzygy and let $z \in H_n(G_\bullet \otimes M) \subseteq G_n \otimes M$. Then $(G_n \otimes M)_{y_i} \to (G_{n-1} \otimes M)_{y_i}$ is injective for each $i$ such that $0 \leq i \leq n$. so there is an integer $s$ such that $y_i^s z = 0$ for each $i$ such that $1 \leq i \leq n$, and therefore $z \in H_n(y_1^s, \ldots, y_n^s; G_n \otimes M)$. By Corollary (2.8), each of $c_n, \ldots, c_d$ annihilates $H_n(G_\bullet \otimes M)$. Thus $c_n, \ldots, c_d$ annihilate all the results that are used in the next section.

Remark. (3.6) Suppose that the conclusion holds for all complexes of length $n$, at each homological degree greater than $m$ (with $1 \leq m < n$). In particular, there exists an integer $k$ such that $c_n^k, \ldots, c_d^k$ annihilate $H_i(G_\bullet \otimes M)$ for all $i > m$. We need to show that there is a power of the elements $c_n, \ldots, c_d$ that annihilates $H_m(G_\bullet \otimes M)$.

Pick $y_1, \ldots, y_m \in I(\partial_m)$ such that $y_1, \ldots, y_m, c_{m+1}, \ldots, c_d$ are a system of parameters. For each $1 \leq i \leq m$, the localized subcomplex $0 \to (G_n)_{y_i} \to \cdots \to (G_m)_{y_i} \to (G_{m-1})_{y_i}$ is exact, since the complex satisfies the hypothesis of (3.3). Hence, given a $d$th syzygy $M$, there is a power $z$ such that $(y_1^z, \ldots, y_m^z)$ kills the homology $H_m(G_\bullet \otimes M)$. This implies that there is an isomorphism that sends the identity in $R/(y_1^z, \ldots, y_m^z)$ to any element $z \in H_m(G_\bullet \otimes M)$. As the elements $c_i$’s are part of a KAS, we may now apply the content of (3.4) to the complex $(0 \to G_n \to \cdots \to G_{m-1} \to 0) \otimes M$ with each $\ell_0 = \cdots = \ell_{n-m-1} = c_j^k$ and $\ell = c_j$ for $j \geq n$. Thus $c_j^{(n-m)k + m + 1}$ kills the desired homology.

We now look at the particular complex we use.

(3.6) Let $x_1, \ldots, x_h$ be a sequence of elements of a ring $R$. Consider the $n \times (n + h - 1)$ matrix

$$B = \begin{pmatrix}
x_1 & x_2 & \cdots & x_h & 0 & \cdots & 0 \\
0 & x_1 & x_2 & \cdots & x_h & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & x_h
\end{pmatrix}.$$

The $n \times n$ minors of $B$ are the generators of the ideal $J^n$, where $J = (x_1, \ldots, x_h)$. In [5], the authors construct the Eagon-Northcott complex $(B_{\bullet}^{I,n}, \partial_\bullet)$ for which $H_0(B_{\bullet}^{I,n}) = R/J^n$. By [5, Theorem 2], if $x$ is a regular sequence, then the complex $(B_{\bullet}^{I,n}, \partial_\bullet)$ is a free resolution of $R/J^n$ of length $h$. Moreover, $J \subseteq \sqrt{I(\partial_i)}$ for all $i = 1, \ldots, h$.

Using the Eagon-Northcott complex above, we are finally able to list in the following proposition all the results that are used in the next section.

(3.7) Proposition. Let $(R, m, k)$ be a local ring of dimension $d$, let $\mathcal{M}$ be the family of all finitely generated $R$-modules that are $d$th syzygies, and let $c = c_1, \ldots, c_d$ be a KAS sequence for $\mathcal{M}$. Let $x_1, \ldots, x_d$ be a system of parameters well-suited to $c$ and denote by $I_j$ the ideal $(x_1, \ldots, x_j)$. Let
t be the integer given by Proposition (3.3) for c. For every i and j such that 1 ≤ j ≤ i ≤ n and for any positive integers t+1, . . . , ti, the following hold:

1. The complex (B+1, n, i, n, θ) • K(c+i−1, 1, . . . , c+i−1; R) (where if j = i there is no Koszul complex) satisfies conditions (1) and (2) of Proposition (3.5);
2. c+i−11, . . . , c+i−1) (I+j−1), n, (c+j−1, 1, . . . , c+j−1)), M) = 0, for all M ∈ M and for all k = i, . . . , d;
3. c+i−11, . . . , c+i−1) (I+j−1, n, (c+j−1, 1, . . . , c+j−1)) N, for all modules N and free modules G such that N ⊆ G and G/N ∈ M, and for all k = i, . . . , d;
4. c+i−11, . . . , c+i−1)M : c+i−1) ⊆ (I+j−1, n, (c+j−1, 1, . . . , c+j−1)) M, for all k = i, . . . , d.

Proof. For (1), let S = Z[X1, . . . , Xd, C1, . . . , Cd] be the polynomial ring in 2d variables and define the ring homomorphism ψ : S → R such that ψ(Xj) = xj and ψ(Cj) = cj. Denote by X for the sequence X1, . . . , Xd. By Remark (3.3), the complex B+1, n, i, n, K(c+i−1, 1, . . . , c+i−1) is a free resolution of S over the ideal (X1, . . . , Xd)n. Since the elements C1, . . . , Cd are a regular sequence over the S-module H0(B+1, n, i, n) = S/(X1, . . . , Xd)n, the complex S ⊗ K(c+i−1, 1, . . . , c+i−1; S) is a free resolution of S and therefore it satisfies the standard condition on rank by (3.1). By induction on the length of the Koszul complex and using (3.3), one can see that each ideal of rank-size minors is, up to radical, the ideal (X1, . . . , Xd, C1, . . . , Cd). Base change to R clearly preserves the rank in this case, as well as condition (2) of Proposition (3.5).

For (2), denote by F, a minimal free resolution of R/Ij and c+i−11, . . . , c+i−1) to F, that lifts the identity between the free modules of homological degree 1. This chain map induces the following surjection

Hj (B+1, n, i, n, K(c+i−1, 1, . . . , c+i−1; R) ⊗ R M) → Torj1(R/Ij + (c+j−1, 1, . . . , c+j−1), M).

So it is enough to show that

(3.7.1) c+i−1 Hj (B+1, n, i, n, K(c+i−1, 1, . . . , c+i−1; R) ⊗ R M) = 0.

This follows from part (1) and Proposition (3.3).

Statement (3) follows from (2) by isomorphism (1.1.1), for i = 1.

For (4), denote the first and the second maps of the complex B+1, n, i, n, ⊗ K(c+i−1, 1, . . . , c+i−1; S) by Φ1 and Φ2. Such complex is a free resolution of S/(X1, . . . , Xd)n, C+i, 1, . . . , C+i and therefore the entries of the last row of a matrix representing Φ2 generate the ideal ((X1, . . . , Xd)n, C+i, 1, . . . , C+i−1 : C+i) which is equal to ((X1, . . . , Xd)n, C+i, 1, . . . , C+i−1) for each value of t, since the elements X1, . . . , Xd form a regular sequence.

Let z ∈ (I+j−1, n, (c+j−1, 1, . . . , c+j−1))M : c+i−1, for which z is the last entry. Notice that equation (3.7.1) reads

c+i−1Ker(Φ1 ⊗ R IdM) ⊆ Ker(Φ2 ⊗ R IdM).

Therefore, c+i−1z ⊆ Ker(Φ2 ⊗ R IdM) by reading this inclusion componentwise, we obtain that

c+i−1z ∈ ((X1, . . . , Xd)n, C+i, 1, . . . , C+i−1) ⊗ S M = (I+j−1, n, (c+j−1, 1, . . . , c+j−1))M,

which finishes the proof of the proposition. □
4. Proof of the Main Theorem

In this section, \((R, \mathfrak{m})\) is a local ring of dimension \(d\). If \((S, \mathfrak{n})\) is a faithfully flat local extension of \(R\) then the conclusion of the Main Theorem descends from \(S\) to \(R\). Hence we may always assume that \(R\) has an infinite residue field. The idea behind the proof of the Main Theorem is to first pick a KAS, then reduce the statement to families of \(\mathfrak{m}\)-primary ideals generated by minimal reductions (which are then systems of parameters) and then, by picking a generic generating set for the reduction, change the calculation from the reduction to the KAS.

Recall that given an ideal \(I \subseteq R\), a reduction of \(I\) is an ideal \(J \subseteq I\) such that there exists an integer \(k\) satisfying the equality \(I^{k+1} = J J^k\), which in turn implies that \(I^n \subseteq J^{n-k}\), for all \(n \geq k\). If \(J \subseteq I\) is a reduction of \(I\) then \(T = \overline{T}\). When \((R, \mathfrak{m})\) has an infinite residue field, minimal reductions of an ideal always exist. For a comprehensive treatment, one may consult \[10\].

We rely heavily on Huneke’s Uniform Briançol-Skoda Theorem, see \[9\] Theorem 4.13. We present it in a less general form which is adequate for our needs.

(4.1) Let \((R, \mathfrak{m}, k)\) be a complete reduced Noetherian ring. There exists a positive integer \(k\) such that for all ideals \(I \subseteq R\), the inclusion \(I^n \subseteq I^{n-k}\) holds for all \(n \geq k\).

Our next proposition shows that for any complete local ring, the integer \(k\) in the definition of a reduction can be uniformly chosen to work for all ideals and for all reductions, even in the case that the ring is not reduced.

(4.2) Proposition. Let \((R, \mathfrak{m}, k)\) be a complete local ring. There exists a positive integer \(k\) such that for all ideals \(I\) and all reductions \(J \subseteq I\), the inclusion \(I^n \subseteq I^{n-k}\) holds for all \(n \geq k\).

Proof. Denote by \(N\) the nilradical of the ring \(R\). If \(N = 0\), we are done by Theorem (4.1), since \(I^n \subseteq J^n\). By Noetherian induction we may assume that, for any non-zero element \(x \in N\), the result holds, since the property of being a reduction is preserved modulo nilpotent elements. We can assume that \(x^2 = 0\). Let \(k_1\) be an integer which works mod \(xR\), and let \(k_2\) be the uniform Artin-Rees number for \(xR \subseteq R\). Then \(I^n \subseteq J^{n-k_1} + (xR \cap J^{n-k_1}) \subseteq J^{n-k_1} + xI^{n-k_1-k_2} \subseteq J^{n-k_1} + x(J^{n-2k_1-k_2} + xR) \subseteq J^{n-2k_1-k_2} \). \(\square\)

We need a notion of reduction that relates to the notion of KAS sequences.

(4.3) Definition. Let \(I\) be an \(\mathfrak{m}\)-primary ideal and let \(c = c_1, \ldots, c_d\) be a KAS sequence. A special reduction of \(I\) with respect to \(c\) is a sequence \(x_1, \ldots, x_d\) which is well-suited to \(c\) and verifies the following, for all integers \(i\) such that \(0 \leq i \leq d - 1\):

1. \(I_{d-i-1}\) is a reduction of \(I_{d-i}\) modulo \((c_{d-i}, \ldots, c_d)\)
2. \(I_{d-i-1}\) is a reduction of \(I_{d-i}\) modulo \((0 : (0 : c_{d-i}))\),

where \(I_k\) denotes the ideal generated by \(x_1, \ldots, x_k\). We also set \(I_0 = 0\).

(4.4) Remark. Special reductions exist if the residue field is infinite. We need to pick the KAS sequence such that \(\dim R/(0 : (0 : c_{d-i})) \leq d - i - 1\), and this is possible by Theorem (2.6). Suppose that we have picked a KAS, \(c = c_1, \ldots, c_d\). There are only finitely many ideals generated by the subsets of \(\{c_1, \ldots, c_d\}\). Thus, with an infinite field, one can choose \(d\) general generators of the ideal \(I\) that are both well-suited to \(c\) and satisfy the reduction conditions (1) and (2) of Definition (4.3).

We now collect the steps that are needed to reduce the proof of the Main Theorem from the family of all ideals to the family of ideals generated by sequences that are special reductions.

(4.5) The following statements hold:
(1) If the Main Theorem holds for the family of all \( m \)-primary ideals then it holds for the family of all ideals. Indeed, given modules \( N \subseteq M \), if there exists an integer \( k \) such that \( K^k M \cap N \subseteq K^{n-k} N \) for all \( m \)-primary ideals \( K \), then the following inclusions hold for all ideals \( I \):

\[
I^n M \cap N \subseteq (I + m^\ell)M \cap N \subseteq (I + m^\ell)^{n-k} N \subseteq I^{n-k} N + m^\ell N.
\]

Taking intersections over \( \ell \), we need to show that

\[
\bigcap_\ell \left( \frac{m^\ell N}{I^{n-k} N} \right) = 0,
\]

which is true by the Krull Intersection Theorem, see for example [11, Corollary 10.19].

(2) In view of Proposition (4.2), we can reduce the proof of our Main Theorem to a family of ideals that are special reductions of \( m \)-primary ideals.

We now present the technique that allows us to use an induction on the number of elements in the sequences. First we need a technical lemma.

(4.6) **Lemma.** Let \((R, m)\) be a local ring and let \( N \) be a \( d \)-th syzygy. Let \( c = c_1, \ldots, c_d \) be a KAS sequence with respect to the family of modules that are \( d \)-th syzygies, and let \( t \) be the integer given by Proposition (4.2) for \( \mathbf{c} \). Assume that the sequence of elements \( x_1, \ldots, x_d \) is well-suited to \( \mathbf{c} \). Let \( w \) be an element in \( N \). If there exists an integer \( h \) such that

\[
C^t_j w \in (x_1, \ldots, x_{j-1})^{n-h} N + (c_1^{(i-j+1)}t, \ldots, c_t^{(i-j+1)}, c_{j+1}) (x_1, \ldots, x_j)^{n-h} N,
\]

where \( i > j \) and \( 1 \leq i \leq d \), then there exists

\[
w_1 \in (x_1, \ldots, x_j)^{n-h} N
\]

such that

\[
c_j^t (w - w_1) \in (x_1, \ldots, x_{j-1})^{n-h} N + (c_1^{(i-j)}t, \ldots, c_t^{(i-j)}, c_{j+1}) (x_1, \ldots, x_j)^{n-h} N.
\]

**Proof:** For notational simplicity, replace each \( C_k^t \) by \( C_k \), and let \( I_k \) be the ideal \((x_1, \ldots, x_k)\) for every integer \( k \) such \( 1 \leq k \leq d \).

Let \( v_i \in (x_1, \ldots, x_j)^{n-h} N \) be the coefficient of \( c_i^{(i-j+1)} \). Then

\[
c_i^{(i-j+1)} v_i \in c_j N + I_j^{n-h} N + (c_{i-1}^{(i-j)}, \ldots, c_{j+1}^{(i-j)}) I_j^{n-h} N.
\]

In particular, \( v_i \in \left( I_j^{n-h} N + (c_{i-1}^{(i-j)}, \ldots, c_{j+1}^{(i-j)}, c_j) N \right) : c_i^{(i-j+1)} \). By Proposition (3.7) (4),

\[
c_i v_i \in \left( I_j^{n-h} N + (c_{i-1}^{(i-j)}, \ldots, c_{j+1}^{(i-j)}, c_j) N \right) : c_i^{(i-j+1)}
\]

We claim that for every integer \( r \) such that \( 1 \leq r \leq i - j + 1 \), we have

\[
c_i v_i \in I_j^{n-h} N + (c_{i-1}^{(i-j)}, \ldots, c_{j+1}^{(i-j)}, c_j) I_j^{n-h} N + (c_{i-r}^{(i-j-r+2)}, \ldots, c_{j+1}^{(i-j-r+2)}, c_j) N,
\]

and we prove the claim by induction on \( r \).

The claim is true for \( r = 1 \) by (4.6.1). Assume the claim is true for \( r \geq 1 \).

Let \( n_{i-r} \in N \) be the coefficient of \( c_{i-r}^{(i-j-r+1)} \) in (4.6.2). Then

\[
n_{i-r} \in \left( I_j^{n-h} + (c_{i-r-1}^{(i-j-r-1)}, \ldots, c_{j+1}^{(i-j-r-1)}, c_j) N \right) : c_{i-r}^{(i-j-r+1)}.
\]

By Proposition (3.7) (4),

\[
c_i n_{i-r} \in \left( I_j^{n-h} + (c_{i-r-1}^{(i-j-r-1)}, \ldots, c_{j+1}^{(i-j-r-1)}, c_j) N \right).
\]
Multiplying (4.6.2) by $c_i$ and substituting for $c_i \mathbf{n}_{i-r}$ yields
\[c_i^{i+1} \mathbf{v}_i \in I_{j-1}^{n-h} N + (c_i^{i-j}, \ldots, c_i^{i-j+r+2}, c_i^{i-j+r+1})I_j^{n-h} N + (c_i^{i-j-r+1}, \ldots, c_{i+1}, c_j)N,\]
as desired. Therefore, the claim is true for all $r = 1, \ldots, i-j+1$.

In particular, for $r = i-j+1$, we obtain
\[c_i^{i-j+1} \mathbf{v}_i \in I_{j-1}^{n-h} N + (c_i^{i-j}, \ldots, c_j)I_j^{n-h} N.\]
By hypothesis, we have the following containment
\[c_j \mathbf{w} \in I_{j-1}^{n-h} N + c_i^{i-j+1} \mathbf{v}_i + (c_i^{i-j}, \ldots, c_{i+1})I_j^{n-h} N,
\]
which, together with the previous containment, gives
\[c_j \mathbf{w} \in I_{j-1}^{n-h} N + (c_i^{i-j}, \ldots, c_{i+1}, c_j)I_j^{n-h} N.\]
We conclude that there exists $w_1 \in I_j^{n-h} N$ such that
\[c_j(w - w_1) \in I_{j-1}^{n-h} N + (c_i^{i-j}, \ldots, c_{i+1})I_j^{n-h} N,
\]
which concludes the proof.

We now focus our attention on choosing the sequence $x_1, \ldots, x_d$ to which we can apply the previous lemma.

With the same notation as in (4.3), we can state the main reduction that is used to prove the Main Theorem.

(4.7) Proposition. Let $(R, \mathfrak{m}, k)$ be a local ring of dimension $d$ with an infinite residue field. Let $\mathbf{c} = c_1, \ldots, c_d$ be a KAS-sequence of $R$, and let $0 \to N \to G \to M \to 0$ be a short exact sequence of finitely generated $R$-modules where $M$ is a $d$th syzygy and $G$ is a free $R$-module.

There exists an integer $h$, depending only on $\mathbf{c}$, such that if $I$ is an $\mathfrak{m}$-primary ideal and $x_1, \ldots, x_d$ is a special reduction of $I$, then for all integers $i$ such that $0 \leq i \leq d-1$ and for all $n \geq h$, the following inclusion holds
\[I_d^n - G \cap N \subseteq I_{d+1}^{n-h} N + I_{d+1}^{n-h} G \cap N.
\]

Proof. Let $t$ be the exponent for $\mathbf{c}$ given by Proposition (3.5). For simplicity of notation, we replace each $c_i^t$ by $c_i$, for each integer $i$ such that $1 \leq i \leq d$.

By Proposition (3.7) (3) with $i = j$, $c_{d-i}(I_{d-i}^n G \cap N) \subseteq I_{d-i}^n N$. In particular, given an element $\mathbf{w} \in I_d^n G \cap N$, we obtain $c_{d-i} \mathbf{w} \in I_{d-i}^n N$. Since $I_{d-i}^n$ is a reduction of $I_{d-i}$ modulo the ideal $(c_i^{i+1}, \ldots, c_{d-i+1}, c_{d-i})$, there exists an integer $h_1$, which, by Proposition (4.3), depends only on the KAS sequence $\mathbf{c}$, such that
\[(4.7.1) \quad I_{d-i}^n \subseteq I_{d-i}^{n-h_1} + (c_i^{i+1}, \ldots, c_{d-i+1}, c_{d-i})I_{d-i}^{n-h_1},\]
for all $n \geq h_1$.

By the uniform Artin-Rees property, there exists an integer $h_2$, depending only on the KAS sequence $\mathbf{c}$, such that
\[(c_i^{i+1}, \ldots, c_{d-i+1}, c_{d-i})I_{d-i}^{n-h_1} \subseteq (c_i^{i+1}, \ldots, c_{d-i+1}, c_{d-i})I_{d-i}^{n-h_1-h_2}.
\]
Combining this last inclusion with inclusion (4.7.1), and setting $h_3$ equal to $h_1 + h_2$, we obtain
\[I_{d-i}^n \subseteq I_{d-i}^{n-h_3} + (c_i^{i+1}, \ldots, c_{d-i+1}, c_{d-i})I_{d-i}^{n-h_3},\]
for all $n \geq h_3$. Thus the containment $c_{d-i} \mathbf{w} \in I_{d-i}^{n-h_3} N + (c_i^{i+1}, \ldots, c_{d-i+1}, c_{d-i})I_{d-i}^{n-h_3} N$ holds.

It follows that there exists an element $w_1 \in I_{d-i}^{n-h_3} N$ such that
\[c_{d-i}(w - w_1) \in I_{d-i}^{n-h_3} N + (c_i^{i+1}, \ldots, c_{d-i+1})I_{d-i}^{n-h_3} N.
\]
By a repeated application of Lemma (4.6), there are elements \( w_2, \ldots, w_{i+1} \in I_{d-i}^{n-h} N \) satisfying
\[
c_{d-i}(w - w_1 - w_2 - \cdots - w_{i+1}) \in I_{d-i-1}^{n-h} N \subseteq I_{d-i-1}^{n-h} G \cap c_{d-i} G.
\]
By the uniform Artin-Rees property, there exists an integer \( h_4 \geq h_3 \), depending only on the element \( c_{d-i} \), such that \( I_{d-i-1}^{n-h} G \cap c_{d-i} G \subseteq c_{d-i} I_{d-i-1}^{n-h} G \). So we obtain
\[
c_{d-i}(w - w_1 - w_2 - \cdots - w_{i+1}) \in c_{d-i} I_{d-i-1}^{n-h} G,
\]
for all \( n \geq h_4 \).

Therefore there exists an element \( f \in I_{d-i}^{n-h} G \) such that
\[
c_{d-i}(w - w_1 - w_2 - \cdots - w_{i+1} - f) = 0,
\]
and then \( w - w_1 - w_2 - \cdots - w_{i+1} = f + z \), where \( z \in (0 : G c_{d-i}) G \). It follows that
\[
z \in I_{d-i}^{n-h} G \cap (0 : G c_{d-i}).
\]
By the uniform Artin-Rees property, there exists an integer \( h_5 \geq h_4 \), which depends only on the element \( c_{d-i} \), such that \( I_{d-i}^{n-h} G \cap (0 : G c_{d-i}) \subseteq I_{d-i}^{n-h} (0 : G c_{d-i}) \). Thus \( z \in I_{d-i}^{n-h} (0 : G c_{d-i}) \), for all \( n \geq h_5 \).

As \( I_{d-i-1} \) is a reduction of \( I_{d-i} \) modulo \((0 : R (0 : R c_{d-i}))\), there exists an integer \( h_6 \), depending only on \( c_{d-i} \), such that \( I_{d-i}^{n-h} \subseteq I_{d-i-1}^{n-h} + (0 : R (0 : R c_{d-i})) \), for all \( n \geq h_6 \). Thus, by setting \( h_7 \) equal to \( h_5 + h_6 \), we obtain
\[
I_{d-i}^{n-h} (0 : G c_{d-i}) \subseteq I_{d-i-1}^{n-h} (0 : G c_{d-i}) \subseteq I_{d-i-1}^{n-h} G,
\]
which implies that \( f + z \in I_{d-i-1}^{n-h} G \).

As \( w - w_1 - w_2 - \cdots - w_{i+1} \in N \) and \( w - w_1 - w_2 - \cdots - w_{i+1} = f + z \), we have \( w - w_1 - \cdots - w_{i+1} \in I_{d-i-1}^{n-h} G \cap N \). Thus, \( w \in I_{d-i}^{n-h} N + I_{d-i-1}^{n-h} G \cap N \) as desired. \( \square \)

We are finally ready to prove the theorem, from which the Main Theorem in the introduction follows as a corollary.

(4.8) Theorem. Let \((R, m, k)\) be a local ring of dimension \(d\). There exists an integer \(h\) such that, for all ideals \(I \subseteq R\) and for all short exact sequences of finitely generated modules \(0 \to A \to B \to M \to 0\) with \(M\) a \(d\)th syzygy,
\[
I^n B \cap A \subseteq I^{n-h} A.
\]

Proof. If there exists such a bound for any faithfully flat extension of \(R\) then the same bound holds for \(R\). Thus, without loss of generality, we may assume that \(R\) is complete and has an infinite residue field.

We can reduce to the case that the middle module is free as follows. Let \(G\) be a free module mapping onto \(B\) and let \(N\) be the kernel of the composite map from \(G\) to \(M\). Then \(G/N \cong M\) and \(N\) maps to \(A\) (via the snake lemma). If \(w \in I^n B \cap A\), then we may lift \(w\) to \(w_1 \in I^n G\). Since \(w\) maps to \(0\) in \(M\), so does \(w_1\), i.e., \(w_1 \in I^n G \cap N\). So, if \(w_1 \in I^{n-h} N\) then \(w \in I^{n-h} A\). For the rest of the proof, we keep the notation that \(G\) is free and \(G/N \cong M\).

Fix a KAS sequence \(c\) for the class of finitely generated \(d\)th syzygies of \(R\), as given by Theorem (2.6) (\(R\) is complete, so by the Cohen Structure Theorem, is the image of a Gorenstein ring). By (4.5), it is enough to find an integer \(h\) such that \(J^n G \cap N \subseteq J^{n-h} N\) for all ideals \(J\) that are special reductions of \(m\)-primary ideals with respect to the KAS sequence \(c\).

Let \(I = I_d = \langle x_1, \ldots, x_d \rangle\) be a special reduction. Let \(k\) be the integer given by Proposition (4.7). By \(d\) applications of Proposition (4.7),
\[ I^kG \cap N \subseteq I^{n-k}N + (I_d^{n-k}G \cap N) \subseteq \cdots \]
\[ \subseteq I^{n-k}N + I_d^{n-2k}N + \cdots + I_d^{2-\ell(k)}N \subseteq I_d^{n-\ell(k)}G \cap N \]
Hence \( h = dk \) suffices.

(4.9) **Corollary.** Let \((R, m, k)\) be a local ring. Then every finitely generated module is syzygetically Artin-Rees with respect to every ideal in \(R\).

**Proof.** By Theorem (4.8), there exists an integer \( h \) such that if \( I \) is any ideal and if \((F_\bullet, \partial_\bullet)\) is a free resolution of \(M\), then \( I^hF_i \cap \partial_{i+1}(F_{i+1}) \subseteq I^{n-h}\partial_{i+1}(F_{i+1}) \) for \( i \geq d = \dim_R R \). Once \( I \) is fixed, then there are also Artin-Rees numbers for the earlier syzygies, so the maximum of \( h \) and these numbers works for \( I \) and \( M \).

A second corollary has to do with perturbations of resolutions. Let \((R, \mathfrak{m})\) be a local ring. A complex \((G_\bullet, d_\bullet)\) is a perturbation of \((G_\bullet, d_n)\) to orders \( q_1, q_2, \ldots \) if the free modules in each homological degree are the same and the difference \( d'_{n-i} - d_n \) of the \( n \)th differentials maps \( G_n \) to \( \mathfrak{m}^qG_{n-1} \). Eisenbud and Huneke raise the following question, [7] Question C]: If \( G_\bullet \) is a minimal free resolution of the f nitely generated module \( M \), is there a number \( q \) such that any complex \((G'_\bullet, d'_\bullet)\) which is a perturbation of \( G_\bullet \) to order \( q, q, \ldots \) is exact?

(4.10) **Corollary.** Let \((R, \mathfrak{m})\) be a local ring of dimension \( d \). Let \((F_\bullet, \partial_\bullet)\) be a free resolution of an \( R \)-module. There exists an integer \( q \) such that any perturbation of \((F_\bullet, \partial_\bullet)\) of order \( q, q, q, \ldots \) is exact. Moreover, for the class of \( d \)th syzygy modules, the integer \( q \) may be chosen depending only on the ring.

**Proof.** We refer the reader to [7] Proposition 1.1 for details of how the syzygetic Artin-Rees property is connected to the perturbation question.

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