A new super-Hopf algebra, denoted by $H$, is obtained by using the standard method (the RTT-relation) with an $R$-matrix which is a solution of the quantum Yang-Baxter equation.

Keywords: Yang-Baxter equation, super-Hopf algebra, quantum supergroup.

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GL(1|2) Süper Grubunun Bir İki-parametreli Deformasyonu

ÖZET: Kuantum Yang-Baxter denkleminin çözümü olan bir $R$-matrisi yardımcıla, standard RTT-bağntısı kullanılarak ile gösterilen yeni bir süper-Hopf cebiri elde edilmiştir.

Keywords: Yang-baxter equation, super-hopf cebiri, kuantum super grup.

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INTRODUCTION

Quantum groups (Drinfeld, 1986) have a rich mathematical structure (Klimyk and Schmüdgen, 1997), (Majid, 1995). The standard method to construct a new algebra from a solution of the quantum Yang-Baxter equation (Yang, 1967) was initiated by Faddeev et al. in 1990. With this method, we will introduce a new superalgebra related to a $\mathbb{Z}_2$-graded $R$-matrix with two-parameter. The RTT-relation for the quantum supergroups has the same form as in the (Faddeev et al. in 1990), but matrix tensor product contains a factor (-1), as additional to the (Kulish and Sklyanin, 1982) related to $\mathbb{Z}_2$-grading (Berezin, 1987).

The tensor product of two even matrices $U$ and $V$ has the signs

$$(U \otimes V)_{ij,kl} = (-1)^{\tau(i) + \tau(k)} U_{ik} V_{jl}$$

where $\tau(U) = \tau(i) + \tau(j)$. Because of this description, a matrix in the form $I \otimes U$ has the same block-diagonal form as in the standard (no-grading) case while a matrix in the form $U \otimes I$ contains the factor (-1) for odd elements standing at odd rows of blocks. To give a little explanation, we consider the matrix $U = \begin{pmatrix} a & \alpha \\ \gamma & b \end{pmatrix}$ appearing as $T_{33}$ on page 7, line 5. Then the tensor product of the matrices $U$ and $I = (\delta_{ij})$ has the signs

$$(U \otimes I)_{ij,kl} = (-1)^{\tau(i) + \tau(k)} U_{ik} \delta_{jl}$$ and $$(I \otimes U)_{ij,kl} = (-1)^{\tau(i) + \tau(k)} \delta_{ik} U_{jl}$$

where $\delta_{ij}$ denotes the kronecker delta. So, we have, for example

$$(U \otimes I)_{11,21} = U_{12} \delta_{11} = \alpha, \quad (U \otimes I)_{12,22} = -U_{12} \delta_{22} = -\alpha, \quad (U \otimes I)_{22,12} = -U_{21} \delta_{12} = -\gamma, \text{ etc.}$$

In this paper, we construct a two-parameter deformation of the supergroup $GL(1|2)$ denoted by $GL_{\mu,\nu}(1|2)$.
MATERIAL AND METHODS

Let \( a,b,c,d,e,\alpha,\beta,\gamma,\delta \) be generators of an algebra \( A \), where the generators \( a,b,c,d,e \) are of grade 0 and the generators \( \alpha,\beta,\gamma,\delta \) are of grade 1. Let \( O(M(1|2)) \) be defined as the polynomial algebra \( k[a,b,c,d,e,\alpha,\beta,\gamma,\delta] \). It will be sometimes more convenient and more illustrative to write a point \((a,b,c,d,e,\alpha,\beta,\gamma,\delta)\) of \( O(M(1|2)) \) in the matrix form, as a supermatrix,

\[
T = \begin{pmatrix}
a & \alpha & \beta \\
\gamma & b & c \\
\delta & d & e \\
\end{pmatrix} = (t_{ij}). \tag{1}
\]

We consider the \( R \)-matrix

\[
R = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^{-1} & p^{-1} & 0 & 0 & 0 & 1 - p^{-1} \\
0 & 1 - p^{-1} & 0 & q p^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q p^{-1} & 0 & p^{-1} - 1 \\
0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p^{-1} \\
\end{pmatrix}
\]

where \( p, q \in \mathbb{C} - \{0\} \). This matrix satisfies the graded Yang-Baxter equation

\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \quad \text{where} \quad R_{12} = R \otimes I_3, \quad \text{etc} \quad \text{with the 3x3 identity matrix} \quad I_3.
\]

The matrix \( \hat{R} \) satisfies the \( \mathbb{Z}_2 \)-graded braid relation

\[
\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}
\]
and the $\mathbb{Z}_2$-graded Hecke condition

$$(\hat{R} - I_q)(\hat{R} + p^{-1}I_q) = 0.$$ 

The eigenvalues of $\hat{R}$ are 1 and $-p^{-1}$ and it can be written, as a sum of projectors, in the form

$$\hat{R} = -p^{-1}P_{-} + P_{+},$$

where

$$P_{-} = \frac{-\hat{R} + I_q}{1 + p^{-1}}, \quad P_{+} = \frac{\hat{R} + p^{-1}I_q}{1 + p^{-1}} \quad (2)$$

provided that $1 + p^{-1} \neq 0$. The projectors obey $P_{i}P_{j} = \delta_{ij}P_{j}$ (no summation) and $P_{-} + P_{+} = I_q$.

RESULTS AND DISCUSSION

In this section, we get the $(p,q)$–commutation relations of the elements of the supermatrix $T$ given in (1) and show that the algebra $O(GL_{p,q}(1|2))$ is a super-Hopf algebra.

**Theorem 3.1.** A 3x3-supermatrix $T$ is a $\mathbb{Z}_2$-graded quantum matrix if and only if

$$\hat{R}T_2T_2 = T_1T_2\hat{R} \quad (3)$$

where $T_2 = I_3 \otimes T$, $T_1 = PT_2P$ and $\hat{R} = PR$ with the super permutation matrix $P$. As a result of (3), the elements of the supermatrix $T$ satisfy the relations
\[ ab = ba + q(1-p^{-1}) \gamma a, \quad ac = p^{-1}ca, \quad ad = p da, \]
\[ ae = ea + q(1-p) \delta b, \quad bc = q cb, \quad bd = pq^{-1} db, \]
\[ be = eb + q^{-1}(p-1) dc, \quad cd = pq^{-2} dc, \quad ce = pq^{-1} ec, \quad de = q ed, \]
\[ aa = pq^{-1}aa, \quad a\beta = q^{-1}p^{-1} \beta a, \quad a\gamma = q \gamma a, \quad a\delta = q \delta a, \]
\[ ba = pq^{-1}ab, \quad b\beta = \beta b + q^{-1}(p-1) ac, \quad b\gamma = q \gamma b, \quad (4) \]
\[ b\delta = \delta b + q(1-p) \gamma d, \quad ca = pq ac, \quad c\beta = pq \beta c, \quad c\gamma = q \gamma c, \]
\[ c\delta = p \delta c, \quad da = q^{-1}p^{-1} ad, \quad db = p^{-1} bd, \quad d\gamma = q^{2}p^{-1} \gamma d, \quad d\delta = q \delta d, \]
\[ ea = q^{-2} ae + q^{-1}(p^{-1}-1) \beta d, \quad e\beta = q^{-1}p^{-1} \beta e, \quad e\gamma = q^{2} \gamma e + q(1-p) \delta c, \]
\[ e\delta = q \delta e, \quad a\beta = -q \beta \alpha, \quad a\gamma = -q^{2} p^{-1} \gamma a, \quad a\delta = -q^{2} \delta a + q(p-1) \alpha, \]
\[ \beta \gamma = -q^{2} \gamma \beta + q(1-p) ac, \quad \beta \delta = -p q^{2} \delta \beta, \quad \gamma \delta = -q^{-1} \delta \gamma, \]
\[ \alpha^{2} = \beta^{2} = \gamma^{2} = \delta^{2} = 0. \]

**Proof.** Results can be obtained by making direct calculations. \( \square \)

One can see that when \( p = q \), these relations coincide with those of \( GL_{p,q}(1|2) \) given in (Celik, 2016).

**Definition 3.1.** The superalgebra \( O(M_{p,q}(1|2)) \) is the quotient of the free algebra \( k <a,b,c,d,e,\alpha,\beta,\gamma,\delta> \) by the two-sided ideal \( J_{p,q} \) constituted by the relations in (4) of Theorem 3.1.

Let \( A \) and \( B \) be two superalgebras. Then their tensor product \( A \otimes B \) is a superalgebra with respect to tensor product of \( A \) and \( B \). The product rule for tensor product of superalgebras is given in the following definition. We denote by \( \tau(a) \) the grade of an element \( a \in A \).

**Definition 3.2.** If \( A \) is a superalgebra, then the product rule in the superalgebra \( A \otimes A \) is described by
\[(a_i \otimes a_j)(a_i \otimes a_j) = (-1)^{\tau(a_i)\tau(a_j)} a_ia_j \otimes a_ia_j\]

where \(a_i\)'s are homogeneous elements in the superalgebra \(A\).

The quantum superdeterminant for the supermatrix \(T\) in the block form is given by (cf. Kobayashi and Uematsu, 1992)

\[s \det(T) = \det(A - BD^{-1}C)(\det(D))^{-1}\]

and it is not a central element. If the inverse of the quantum superdeterminant \(s \det(T)\) exists, then the algebra \(O(GL_{p,q}(112))\) has a super-Hopf algebra structure. The super-Hopf algebra structure of \(O(GL_{p,q}(112))\) is given in below.

**Theorem 3.2.** The algebra \(O(GL_{p,q}(112))\) has a unique super-Hopf algebra structure with co-maps \(\Delta, \varepsilon\) and \(S\) such that

\[\Delta(t_{ij}) = \sum_{j=1}^{3} t_{ik} \otimes t_{kj}, \quad \varepsilon(t_{ij}) = \delta_{ij}\] and \(S(T) = T^{-1}\).

**Proof.** The following properties of the co-structures can easily verified:

The comultiplication \(\Delta\) is coassociative in the sense that

\[(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta\]

where \(\text{id} : A \rightarrow A\) denotes the identity map and \(\Delta(uv) = \Delta(u)\Delta(v)\), \(\Delta(1) = 1 \otimes 1\).

The counit \(\varepsilon\) has the property

\[m \circ (\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = m \circ (\text{id} \otimes \varepsilon) \circ \Delta\]

where \(m : A \otimes A \rightarrow A\) and \(\varepsilon(uv) = \varepsilon(u)\varepsilon(v)\), \(\varepsilon(1) = 1\).
The coinverse $S$ satisfies

$$m \circ (S \otimes \text{id}) \circ \Delta = \varepsilon = m \circ (\text{id} \otimes S) \circ \Delta$$

and $S(uv) = (-1)^{r(u) r(v)} S(v)S(u)$, $S(I) = 1$. □

**Definition 3.3.** The super-Hopf algebra $O(GL_{p,q}(1|2))$ is called the coordinate algebra of the quantum supergroup $GL_{p,q}(1|2)$.

**A discussion of some submatrices**

Here are a few comments about some submatrices of $T$.

1. Let us first consider the even 2x2-submatrix $T_{33} = \begin{pmatrix} a & \alpha \\ \gamma & b \end{pmatrix}$ which forms subgroup $GL_{p,q}(111)$ with the commutation rules

   $$aa = pq^{-1}aa, \quad a\gamma = qa, \quad b\alpha = pq^{-1}ab, \quad b\gamma = q\gamma b,$$

   $$ab = ba + q(1 - p^{-1})\gamma a, \quad a\gamma = -q^2p^{-1}\gamma a, \quad \alpha^2 = \gamma^2 = 0.$$

These relations coincide with relations in (Dabrowski and Wang, 1991) when $p$ is replaced by $pq$. If we assume that the formal inverse $b^{-1}$ of $b$ exists, then the quantum superdeterminant is given by the expression

$$s \det(T_{33}) = ab^{-1} - ab^{-1}\gamma b^{-1}$$

and it is a central element of the quantum superalgebra $O(GL_{p,q}(111))$.

It can be seen in a similar way that the even 2x2-submatrix $T_{22} = \begin{pmatrix} a & \beta \\ \delta & e \end{pmatrix}$ forms subgroup $GL_{p,q}(111)$ with the defining commutation relations.
2. We now consider an algebra $A$ generated by the elements $a, \alpha, \delta, d$ and defining commutation rules

$$
a\beta = q^{-1}p\beta a, \quad a\delta = q\delta a, \quad e\beta = q^{-1}p\beta e, \quad e\delta = q\delta e,
$$

$$
ae = ea + q(1 - p)\delta\beta, \quad \beta\delta = -p q^2\delta\beta, \quad \beta^2 = \delta^2 = 0.
$$

Obviously these relations represent a two-parameter deformation of the algebra $A$. Here the generators $a$ and $d$ are almost even (bosonic) and the generators $\alpha$ and $\delta$ are almost odd (fermionic). Indeed, as $p, q \to 1$ the algebra $A$ with these relations becomes a superalgebra. However, submatrices of the form $T_{23} = \begin{pmatrix} a & \alpha \\ \delta & d \end{pmatrix}$ with the defined relations (except for $p=q=1$) do not form a subgroup $GL_{p,q}(11)$. It seems that such matrices are related to the super braided matrices (Majid, 1991). If so, this will be addressed in another study.

3. The $2\times2$-submatrix $T_{23} = \begin{pmatrix} b & c \\ d & e \end{pmatrix}$ forms subgroup $GL_{p,q}(2)$ subject to the relations

$$
bc = qcb, \quad bd = pq^{-1}db, \quad ce = pq^{-1}ec, \quad de = qed,
$$

$$
be = eb + q^{-1}(p - 1)dc, \quad cd = pq^{-2}dc.
$$

These relations coincide with relations given in (Schirrmacher et al., 1991) when $q$ is replaced by $p$ and $pq^{-1}$ is replaced by $q$. The quantum determinant is given by

$$
\det(T_{11}) = be - q cd = eb - q^{-1}dc
$$

and it is not in the centre of the algebra $O(GL_{p,q}(2))$, but it becomes central if $p = q^2$. 

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CONCLUSION

An $R$-matrix satisfying quantum Yang-Baxter equation was found, and using this matrix, deformation of the supergroup with a two-parameter was obtained and it shown that has a super-Hopf algebra structure, as usual.

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