High Energy Scattering in 2+1 QCD: A Dipole Picture

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A dipole picture of high energy scattering is developed in the 2+1 dimensional QCD, following Mueller. A generalized integral equation for the dipole density with a given separation and center of mass position is derived, and meson-meson non-forward scattering amplitude is therefore calculated. We also calculate the amplitude due to two pomeron exchange, and the triple pomeron coupling. We compare the result obtained by this method to our previous result based on an effective action approach, and find the two results agree at the one pomeron exchange level.
1. Introduction

One of the most striking aspects of high-energy hadron-hadron scattering is the continued increase of the total cross section $\sigma_T$ with the energy. Traditional approaches to high energy near-forward hadronic collisions invariably involve the notion of a Pomeron exchange. There are currently two seemingly conflicting interpretations of Pomeron in QCD: One based on perturbative leading log approximation (LLA) \cite{1} \cite{2} and another based on nonperturbative (large-$N_c$ and/or phenomenological) considerations \cite{3} \cite{4}. Thus one of the main puzzles for our understanding of high energy hadronic collisions in QCD is the relation, if any, between the perturbative (Lipatov) Pomeron and the nonperturbative (soft) Pomeron. In a perturbative treatment, a Pomeron loosely corresponds to the color-singlet bound state of two (reggeized) gluons. In a nonperturbative treatment, the Pomeron is thought to correspond to nonperturbative gluon exchanges having the topology of a closed string. It has been emphasized in \cite{3} that a “two-gluon ladder” also has the topological structure of a “cylinder” in the color space. This fact offers the possibility of a unified approach to both perturbative and nonperturbative Pomerons in a systematic large-$N_c$ treatment.

Two recent studies in QCD at high energies, together with the small-$x$ results coming from HERA, have heightened interests in this area of research. Formally, a high energy hadronic collision in the near-forward limit corresponds to the mixing of a “short-distance” phenomenon in the longitudinal coordinates with a “long-distance” phenomenon in the transverse coordinates. By treating the longitudinal and transverse degrees of freedom separately, one could hope to achieve a “dimensional reduction”, simplifying QCD at high energies to an effective two-dimensional field theory.\cite{6} An interesting attempt in this direction has been made recently by Verlinde and Verlinde \cite{7}. The effective theory involves several fields and two coupling constants, the original gauge coupling $g$ and an effective coupling $e^2 \sim g^2 \log s$. Like previous attempts by Lipatov and collaborators, the emphasis is on first considering individual quark-quark scattering through which one reconstructs physical hadronic scattering amplitudes.

The second important development is due to Mueller\cite{8} \cite{9}, who provided an intuitively attractive dipole picture for the high energy hard processes. In this approach, which is based on a large-$N_c$ analysis, the basic scattering is between a pair of color dipoles through a two-gluon exchange mechanism and the whole complexity of high energy hadronic collisions lies in calculating the wave function and the “dipole density” in the transverse space created
by a fast moving hadron. Interestingly, the BFKL pomeron is rederived at the stage of calculating the dipole density. For related work, see [10]. One possible advantage of this new approach is the relative ease in deriving the BFKL pomeron and generalizing it to multi-pomerons case.

In this paper, we carry out a study for QCD in 2+1 dimensions using this dipole picture. One of the purposes of this study is to clarify as well as to amplify the large-$N_c$ cylinder structure, i.e., closed-string like structure, of the Lipatov Pomeron. Since much simpler analytic calculations are involved, the resulting physical picture becomes clearer. Secondly, since we have previously carried out a similar analysis for 2+1 QCD using Verlindes’ approach[11], this study also helps to check on the validity of this alternative approach. One intriguing result of Ref.[11] is the fact that the total cross section due to one-Pomeron exchange does not grow with energy. We shall reproduce this result using this dipole approach and identify its physical origin as due to color-screening in 2+1 dimensions. Even more importantly, this study potentially could help to clarify and contrast the relation of a perturbative approach to that of a nonperturbative soft Pomeron, which also relies on a large-$N_c$ expansion.

In Sec. 2 we first derive an integral equation governing the light-cone meson wave function with emission of soft gluons. A physical interpretation for the virtual correction factor involved in the integral equation is also provided. We emphasize the surface interpretation of the dipole picture in Sec. 3, thus providing a possible link between this perturbative scheme and other large $N_c$ nonperturbative schemes. The integral equation for the dipole density is solved in Sec. 3, and the meson-meson scattering amplitude due to one Pomeron exchange is calculated and found, in agreement with results of Ref. [11]. We proceed in Sec. 4 to derive and then solve the integral equation for dipole pair density. A surprisingly simple result is obtained which allows one to calculate the contribution to an elastic amplitude due to two Pomeron exchange. The triple Pomeron coupling is discussed in Sec. 5. A brief discussion of our results is given in Sec. 6.

2. The meson wave function

In this section we shall write down the integral equation governing the generating functional of probability amplitudes of soft gluons in a given fast moving meson in 2+1 dimensions. This is a preparation for deriving an integral equation governing the dipole
density in a meson state. We adopt the approach of [8] and try to stay as close as possible with notations introduced in [9].

Let $\psi(k, z)$ be the light-cone wave function of the meson containing no soft gluons, where $k$ is the relative transverse momentum between quark and anti-quark and there is only one component in our situation, $z$ is the fraction of total longitudinal momentum carried by quark. We shall ignore spinor indices, since they are irrelevant in the following derivation. The light-cone wave function with one soft gluon is

$$\psi^a(k_1, k_2, z_1, z_2) = 2gT_a(\psi(k_1, z_1) - \psi(k_1 + k_2, z_1)) \frac{k_2}{k_2^2 + \mu^2},$$

(2.1)

where $k_1$ is the transverse momentum carried by quark, $k_2$ is the transverse momentum carried by the soft gluon, $z_2$ is the fraction of total longitudinal momentum carried by this soft gluon and is assumed to be much smaller than $z_1$. The formula is nearly identical to that obtained in [8] and is schematically represented by diagrams in figure 1. Note that in 2+1 dimensions, there is only one polarization and we assumed the polarization constant be unity in the above formula. A infrared cut-off $\mu$ is also introduced.

![Diagram](image)

**Fig.1**

It proves convenient to express the expression (2.1) in coordinate space,

$$\psi^a(x_0, x_1, x_2, z_1, z_2) = igT_a\psi(x_0, x_1, z_1)[\epsilon(x_2 - x_0) - \epsilon(x_2 - x_1)],$$

(2.2)

where $\epsilon$ is the step function, *i.e.*, $\epsilon(x) = 1$ if $x > 0$ and $\epsilon(x) = -1$ if $x < 0$. Just as in the 3+1 dimensional case, the wave function with one soft gluon factorizes: It is the product of the wave function of valence quarks and an elementary function. It follows that the probability for one soft gluon emission is then

$$\phi(1)(x_0, x_1, x_2, z_1, z_2) = g^2\pi C_F K(x_{20}, x_{21})\phi(0)(x_0, x_1, z_1),$$

(2.3)
where \( C_F \equiv 1/2(N_c - 1/N_c) \to N_c/2 \) in the large \( N_c \) limit, the factor \( 1/\pi \) comes from the longitudinal integration measure, and

\[
K(x_{20}, x_{21}) = (1/4)[\epsilon(x_{20}) - \epsilon(x_{21})]^2. \tag{2.4}
\]

However, unlike the situation of 3+1 dimensions, the kernel \( K(x_{20}, x_{21}) \) vanishes for \( x_2 \) outside of the interval \((x_0, x_1)\), (we assume that \( x_1 > x_0 \)). This special property of one dimensional propagator implies that, due to color screening, no gluon can be emitted outside of the “parent” dipole in the transverse impact space. This fact is ultimately responsible for our final result that the total cross section does not grow with energy.

Similarly, the probability of finding two soft gluons in the meson is

\[
\phi^{(2)}(x_i, z_i) = \lambda(K(x_{32}, x_{30}) + K(x_{31}, x_{32}))\lambda K(x_{20}, x_{21})\phi^{(0)}(x_0, x_1, z_1), \tag{2.5}
\]

where \( \lambda = g^2 N_c/(2\pi) \). Since in the large-\( N_c \) limit the color structure of a gluon can be identified with that of a quark-antiquark pair, \((2.3)\) has a simple interpretation in a dipole picture: The second gluon at \( x_3 \) is either emitted from the dipole consisting of one valence quark and the anti-quark part in the first gluon, or emitted from the dipole consisting of the valence anti-quark and the quark part of the first gluon. We emphasize that one could speak of splitting a gluon into a pair of quark and anti-quark only in a large-\( N_c \) language. In this next section, a clearer discussion on the relation of this picture to the usual large-\( N_c \) planar graph analysis will be provided.

As in \[8\], we next introduce a generating functional \( \Phi(x_0, x_1, z_1, u(x, z)) \) whose \( n \)th-order coefficient in an expansion in \( u(x, z) \) is the probability amplitude of finding \( n \) soft gluons in the meson. Clearly, \( \Phi \) factorizes into \( \phi^{(0)}(x_0, x_1, z_1)Z(x_0, x_1, u(x, z)) \). One can of course obtain these coefficients iteratively. More directly, it follows formally from \((2.3)\) and \((2.3)\) that \( Z \) satisfies the following integral equation:

\[
Z(x_0, x_1, z_1, u) = 1 + \lambda \int_{x_0}^{x_1} dx_2 \int_{z_0}^{z_1} dz_2 \, u(x_2, z_2) Z(x_2, x_1, z_2, u) Z(x_0, x_2, z_2, u). \tag{2.6}
\]

The structure of this integral equation can also be seen more clearly by examining figure 2. Note that a cut-off \( z_0 \) to the longitudinal integration over \( z_2 \) has been introduced.
However, (2.6) as it stands is inconsistent, as pointed out in [8] for the analogous equation in the case of 3+1 dimensions. If we define $\phi^{(0)}(x_0, x_1, z_1)$ as the inclusive probability of finding the valence quark and anti-quark in the meson, then production of soft gluons should not change the probability. Consequently one has $\phi^{(0)}(x_0, x_1, z_1) = \Phi(x_0, x_1, z_1, u = 1)$, or, $Z(x_0, x_1, z_1, u = 1) = 1$. However, this condition is not satisfied by (2.6). To rectify the situation, we notice that the “bare” wave function $\phi^{(0)}$ used in (2.3) must be replaced by $\phi^{(0)}A(x_0, x_1, z_1)$ where the factor $A$ takes into account virtual corrections to the emission of one real gluon. A similar factor must be used in each dipole constructed from emitted gluons. Since $Z(x_0, x_1, z_1, 0) = A(x_0, x_1, z_1)$, it follows that it should appear as the inhomogeneous term in (2.3). Introducing rapidities $y = \ln(z_1/z_0)$, $y' = \ln(z_2/z_0)$, it is convenient to treat the soft gluon emission as an evolution process. Normalizing $A(x_0, x_1, z_0) = 1$, it can be interpreted as the survival probability of no soft gluon emission at $y > 0$. It follows that (2.3) should be further modified by inserting a factor $A(x_0, x_1, z_1 z_0/z_2)$, i.e., the probability of no gluon emission as one evolves from $y'$ to $y$.

This survival probability function satisfies a first order differential equation $dA/dy = -\lambda \int dx_2 K(x_{20}, x_{21}) A$. It follows that $A = \exp(-\lambda x_{10} y)$. The integral equation (2.6) is thus modified to become

$$Z(x_0, x_1, y, u(x, y)) = e^{-\lambda x_{10} y} + \lambda \int_{x_0}^{x_1} dx_2 \int_0^y e^{-\lambda x_{10}(y-y')} dy' u(x, y')$$

$$Z(x_2, x_1, y', u)Z(x_0, x_2, y', u).$$

(2.7)

Note that the desired normalization condition at $u = 1$ is satisfied by (2.7). There is no ultraviolet divergence in the virtual corrections in 2+1 dimensions since QCD is super-renormalizable, unlike the situation for 3+1 dimensions.
3. One Pomeron exchange

3.1. Topological structure of a meson state

Knowing the integral equation for the generating functional $Z$, it is straightforward to write down the integral equation for the dipole density. Before doing so, let us first provide an intuitive picture of such a density function from the point of view of large-$N_c$ expansion where the dominant topological structure is that of closed-string exchange, figure 3. The cylindrical surface in figure 3 is understood to have been populated with arbitrary number of gluons, provided that they are connected in a “planar” fashion in their color structure. This can best be represented if each gluon is labeled in a “double-line” representation, i.e., as far as color is concerned, a gluon can be thought of as a pair of quark and antiquark. In this fashion, the cylindrical surface is “tiled” by polygons whose edges are gluon lines. The color structure is such that the boundary of each polygon corresponds to a closed “quark loop”, formed by identifying a quark line from each gluon of the polygon edges.

Figure 3 can also be obtained by joining of a closed-string propagator, i.e., the long neck, to wave functions at both ends of the graph. By construction, this closed-string propagator corresponds to a color-neutral exchange. (In what follows, it will be represented by the color-neutral component of two-gluon exchange. Here we shall concentrate on the topological structure of the wave-functions at each end.) Each wave function, before the closed-string exchange, topologically has the structure of a disk. In order to join to the closed-string, a “window”, or more precisely, a “quark-line” boundary must be cut out. In
a light-cone wave function approach, this quark-loop represents the propagation of a color-dipole. The “distribution” of this quark-loop can then be understood as the desired dipole density. The quark-gluon diagram and the corresponding surface diagram are illustrated in figure 4.

![Diagram of quark-gluon and surface diagrams](image)

Fig.4

3.2. The dipole density in a meson state

Let us now return to our treatment of the light-cone wave function of a fast meson. Let $n(x_0, x_1, x, \bar{x}, y)$ be the dipole density in a meson with valence quark at $x_0$ and anti-quark at $x_1$. The dipole has a separation $x$ and center of mass position $\bar{x}$. The density is defined for dipoles with the lowest longitudinal momentum greater than $\exp(-y)p_+$, where $p_+$ is the total longitudinal momentum of the meson. By keeping track of the center of mass, our treatment here represents an extension of that treated in [8] and [9].
From graphs in figure 5, it is easy to see that the integral equation governing the dipole density is

\[ n(x_0, x_1, x, \bar{x}, y) = e^{-\lambda x_{10} y} \delta(|x| - x_{10}) \delta(\bar{x} - \bar{x}_{10}) \]

\[ + \lambda \int_{x_0}^{x_1} dx_2 \int_{y_0}^{y} dy' e^{-\lambda x_{10} (y - y')} [n(x_0, x_2, x, \bar{x}, y') + n(x_2, x_1, x, \bar{x}, y')], \]

(3.1)

where \( \bar{x}_{10} \) is the center of mass position \( 1/2(x_0 + x_1) \) of valence quarks. The delta function in the first term represents the contribution from the valence quarks. Again we assume \( x_1 > x_0 \).

To solve Eq. (3.1), use

\[ n(x_0, x_1, x, \bar{x}, y) = e^{\frac{y \omega}{2\pi i}} n_\omega(x_0, x_1, x, \bar{x}), \]

(3.2)

where the contour of the integral runs parallel to the imaginary axis and to the right of all singularities in \( n_\omega \). The integral equation for \( n_\omega \) becomes

\[ (\omega + \lambda x_{10}) n_\omega(x_0, x_1, x, \bar{x}) = \delta(|x| - x_{10}) \delta(\bar{x} - \bar{x}_{10}) \]

\[ + \lambda \int_{x_0}^{x_1} dx_2 [n_\omega(x_0, x_2, x, \bar{x}) + n_\omega(x_2, x_1, x, \bar{x})]. \]

(3.3)

The above equation can be further simplified by taking a Fourier transform with respect to the center of mass \( \bar{x} \),

\[ n_\omega(x_0, x_1, x, q) = \int d\bar{x} e^{iq\bar{x}} n_\omega(x_0, x_1, x, \bar{x}). \]

One of our key results is an exact solution to (3.3),

\[ e^{-iqx_{10}} n_\omega(x_0, x_1, x, q) = \delta(|x| - x_{10}) + \frac{4\lambda^2}{(\omega + \lambda|x|)^2 q} \sin \left( \frac{x_{10} - |x|}{2q} \right) \]

\[ + \frac{2\lambda}{(\omega + \lambda|x|)^2} \cos \left( \frac{x_{10} - |x|}{2q} \right). \]

(3.4)
This solution can be understood intuitively. The first term is due to the dipole formed by valence quarks, the parent dipole. The second term comes from dipoles formed by emitted gluons. In particular, the step function in the second term tells us that the size of induced dipoles cannot be greater than $x_{10}$. This is a direct consequence of the fact that no gluon is emitted outside of the parent dipole. (We stress that $n_\omega(x_0, x_1, x, q)$ is not to be confused with a dipole density at a given transverse momentum $q$. Indeed, it is not even positive definite.)

Transforming $n_\omega$ in (3.4) back to the rapidity variable, one finds

$$e^{-iq\bar{x}_{10}}n(x_0, x_1, x, q, y) = \delta(|x| - x_{10})e^{-\lambda x_{10}y} + \theta(x_{10} - |x|)e^{-\lambda|x|y}\left(\frac{4\lambda^2 y^2}{q} \sin\left(\frac{x_{10} - |x|}{2q}\right) + 2\lambda y \cos\left(\frac{x_{10} - |x|}{2q}\right)\right). \quad (3.5)$$

Surprisingly, the transform of (3.5) back into the center of mass coordinate takes on a very simple form:

$$n(x_0, x_1, x, \bar{x}, y) = \delta(|x| - x_{10})\delta(\bar{x} - \bar{x}_{10})e^{-\lambda x_{10}y} + \theta(x_{10} - |x|)e^{-\lambda|x|y}[\lambda^2 y^2$$

$$\left(\epsilon(x_1 - \bar{x} - |x|/2) - \epsilon(x_0 - \bar{x} + |x|/2)\right) + \lambda y \left(\delta(x_1 - \bar{x} - |x|/2) + \delta(x_0 - \bar{x} + |x|/2)\right)]. \quad (3.6)$$

It is easy to see that this expression for the density is positive definite, as it should.

### 3.3. Meson-meson non-forward scattering

Consider meson $A$ collides with meson $B$, in their center of mass system. Denote the wave function of meson $A$ by $\Psi_A(x_{10}, z)\exp(ip\bar{x}_{10})$ and the wave function of meson $B$ by $\Psi_B(x'_{10}, z')\exp(-ip\bar{x}'_{10})$. After integrating out their center of mass positions, the scattering amplitude between these two mesons can be expressed as

$$A(s, t) = is \int dx_10 dx'_10 \Phi_A(x_{10})\Phi_B(x'_{10})A(x_{10}, x'_{10}, Y, q), \quad (3.7)$$

where $\Phi_A(x_{10}) = \int dz |\Psi_A(x_{10}, z)|^2$, $\Phi_B(x'_{10}) = \int dz' |\Psi_B(x'_{10}, z')|^2$, and $A(x_{10}, x'_{10}, Y, q)$ is a scattering amplitude between two mesons with fixed dipole sizes. Under a two gluon exchange approximation, it becomes

$$A(x_{10}, x'_{10}, Y, q) = 8g^4 \int \frac{dk}{2\pi k^2(q-k)^2} \int dx dx' n(x_0, x_1, x, q, Y/2)n(x_0', x_1', x', q, Y/2)$$

$$\sin(kx/2)\sin((q-k)x/2)\sin(kx'/2)\sin((q-k)x'/2)), \quad (3.8)$$
where $Y = \ln s$, and $t = -q^2$. Thus, the amplitude given in Eq. (3.7) can be associated with a (hard) Pomeron exchange. Indeed as we shall see shortly, the leading contribution to $A(s, t)$ in Eq. (3.7) comes from a logarithmic branch point in the complex angular momentum plane.

Let us begin by first evaluating the following integral:

$$
\int dx_n(x_0, x_1, x, q, Y/2) \sin(kx/2) \sin((q-k)x/2).
$$

We first point out that this quantity depends on the center of mass position $\bar{x}_{10}$ only through the factor $\exp(iq\bar{x}_{10})$ as one can easily seen from formula (3.5). However, this dependence, together the similar factor arising from the second dipole density in (3.8), will be removed when the integration over center of mass positions is carried out in arriving at (3.7). (The net result is a momentum conservation factor which has been dropped from our formula (3.7).) We henceforth ignore the dependence on $\bar{x}_{10}$ in the dipole density. Next observe that, when $Y$ large, the leading term in (3.5) is proportional to $Y^2$. It follows that

$$
\int dx_n(x_0, x_1, x, q, Y/2) \sin(kx/2) \sin((q-k)x/2) = 2\lambda^2 Y^2 \frac{k(q-k)}{q} \Im \left( \frac{\lambda Y + iq}{(\lambda^2 Y^2 + 2i\lambda qY)(\lambda^2 Y^2 + 2i\lambda qY - 4k(q-k))} \right),
$$

where $\Im(F)$ denotes the imaginary part.

In arriving at (3.9), we have initially assumed $x_1 > x_0$. However, it is easy to see (3.9) depends only on $|x_{10}|$. It follows that the corresponding integral in (3.8) involving the second dipole density can be obtained directly from (3.9) by letting $q \rightarrow -q$, $k \rightarrow -k$ and $x \rightarrow x'$. Substituting these results into (3.8) and performing the integral over $k$, we find

$$
A(x_{10}, x'_{10}, Y) = -g^4 \frac{1}{q^2} \left( \frac{2\lambda Y \cos(|x_{10}| - |x'_{10}|)q/2}{\lambda^2 Y^2 + 4q^2} \right.
$$

$$
- \lambda^2 Y^2 \left( \frac{\exp(iq(|x_{10}| + |x'_{10}|)/2)}{(\lambda Y + iq)^2(\lambda Y + iq)} + \text{c.c.} \right). \tag{3.10}
$$

This amplitude is non-singular in the limit $q \rightarrow 0$. When $|q| << \lambda Y$, it becomes

$$
A(x_{10}, x'_{10}, Y) = \frac{4g^4}{\lambda q^2 Y} \sin(qx_{10}/2) \sin(qx'_{10}/2), \tag{3.11}
$$

and, in the limit $q = 0$, it behaves as

$$
A(x_{10}, x'_{10}, Y) = \frac{g^4 |x_{10}|x'_{10}|}{\lambda Y}. \tag{3.12}
$$
We point out that the amplitude decreases as $1/Y$ at high energies. This result is in agreement with a calculation done in Ref. [11], using an effective action approach. Note in the dipole picture, the important ingredient of virtual correction is derived rather indirectly. In [11], however, the decrease of scattering amplitude is calculated with an effective action without additional input. (3.11) can also interpreted as composed of a Pomeron “propagator” $1/(\lambda Y)$ and its coupling to dipoles $x_{10}$ and $x'_{10}$. In this language, the coupling of a BFKL Pomeron to a dipole of size $x_{10}$ is given by

$$\beta(q) = \frac{2g^2}{q} \sin(q x_{10}/2).$$

Finally substituting Eq. (3.10) back to (3.7), one arrives at

$$A(s, t) = -isg^4 \frac{4}{q^2} \frac{\lambda Y \Phi_A(q/2) \Phi_B(-q/2)}{\lambda^2 Y^2 + 4q^2} - \frac{\lambda^2 Y^2 \Phi_A(q/2) \Phi_B(q/2)}{(\lambda Y + 2iq)^2(\lambda Y + iq)} + c.c,$$

where

$$\Phi_A(q/2) = \int_0^\infty dx \Phi_A(x)e^{iqx/2}, \quad \Phi_B(q/2) = \int_0^\infty dx \Phi_B(x)e^{iqx/2}$$

and $\Phi_A(-q/2), \Phi_B(-q/2)$ are their complex conjugates. (We assume that both $\Phi_A(x)$ and $\Phi_B(x)$ are even functions.) For the forward limit $q = 0$, one can use (3.12) with $|x_{10}|$ and $|x'_{10}|$ replaced by the effective sizes of meson $A$ and meson $B$. Note Also that the high energy behavior (3.12) can be understood as due to a logarithmic branch point in the complex angular momentum plane, i.e.,

$$A(x_{10}, x'_{10}, Y) = -g^4 |x_{10}| |x'_{10}| \int \frac{d\omega}{2\pi i} s^\omega \ln(\omega).$$

The branch point is located at $\omega = 0$.

Physically, the asymptotic decrease of the scattering amplitude is due to a suppression of the dipole density at large rapidity. This can be understood as follows. For 2+1 dimensions, since there is only one transverse dimension, the emitted gluons in a meson is confined between the valence quarks, and the branching process of emission of soft gluons must compete with a suppression factor $\exp(-\lambda |x|y)$ due to virtual corrections. In 3+1 dimensions, this competition is won by the branching process, hence a BFKL Pomeron emerges to the right of $J = 1$ in the complex angular momentum plane. In 2+1 dimensions, however, the branching process loses the competition and the suppressing factor due to virtual corrections dominates, thus leading to a logarithmic branch point at $J = 1$. 

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4. Two Pomeron exchange

4.1. Dipole pair density

The next level contribution to the scattering amplitude in the leading logarithmic approximation is from exchanges of four gluons between two mesons. In order to calculate this contribution, we need to know the density of two dipoles in a given meson state. Let \( n_2(x_0, x_1, x_a, x_b, \bar{x}_a, \bar{x}_b, y) \) be the density of two dipoles with respective center of mass positions \( \bar{x}_a \) and \( \bar{x}_b \) and separations \( x_a \) and \( x_b \). Again this density is defined for dipoles with lowest longitudinal momentum greater than \( e^{-y} p_+ \), where \( p_+ \) is the meson longitudinal momentum. (In deriving an integral equation for \( n_2 \), we will first focus on its dependences on \( x_0, x_1 \) and \( y \). In the following, for compactness, we shall sometimes drop its dependences on \( x_a, \bar{x}_a, x_b, \) and \( \bar{x}_b \). They will be re-inserted when necessary.)

After a careful calculation, we have found that the leading term in the double density is simply the product of two single dipole densities. This will be demonstrated below. This surprising result implies that, to the leading order, there is no correlation between two single dipole densities. This is not the case for 3+1 dimensions. We take this as one of the remarkable features of the Pomeron physics in three dimensions.

The integral equation can be written down readily by examining the graphical equation in figure 6:

\[
\begin{align*}
n_2(x_0, x_1, y) = & \lambda \int_{x_0}^{x_1} dx_2 \int_{0}^{y} e^{-\lambda x_{10} (y-y')} dy' [n(x_2, x_1, x_a, x_a, y') n(x_0, x_2, x_b, \bar{x}_b, y') \\
& + n(x_2, x_1, x_b, \bar{x}_b, y') n(x_0, x_2, x_a, \bar{x}_a, y') + n_2(x_2, x_1, y') + n_2(x_0, x_2, y')],
\end{align*}
\]

where again we assumed \( x_1 > x_0 \). The first two terms come from finding two dipoles in separated dipoles, as indicated in figure 6, which serve as the inhomogeneous term in the integral equation. Note that the single dipole density depends on the separation of the dipole only through its absolute value, the same is true for the double dipole density, as evident from the above equation.

![Fig.6]
The single dipole density, as given in (3.6), is already lengthy. Fortunately, in order to calculate the leading term in the double dipole density, it is enough to only keep the leading term in the single dipole density as the input. This term, from (3.6), is

\[
n(x_0, x_1, x, \bar{x}, y) = \lambda y^2 e^{-\lambda xy} \theta(x_{10} - x)(\epsilon(x_1 - \bar{x} - x/2) - \epsilon(x_0 - \bar{x} + x/2)) = 2\lambda y^2 e^{-\lambda xy} \theta(x_{10} - x) \theta(x_1 - \bar{x} - x/2) \theta(\bar{x} - x_0 - x/2). \tag{4.2}
\]

Plugging this into (4.1), the integral equation reduces to

\[
n_2(x_0, x_1, y) = n_2^{(0)}(x_0, x_1, y) + \lambda \int_{x_0}^{x_1} dx_2 \int_0^y dy' e^{-\lambda x_{10}(y-y')} (n_2(x_2, x_1, y') + n_2(x_0, x_2, y')), \tag{4.3}
\]

where

\[
n_2^{(0)}(x_0, x_1, y) = 4\lambda^5 e^{-\lambda x_{10}y} \int_0^y y'^4 e^{\lambda(x_{10} - x_a - x_b)y'} dy' \theta(x_a - x_b - (x_a + x_b)/2) \theta(x_1 - \bar{x}_a - x_a/2) \theta(\bar{x}_b - x_0 - x_b/2) + (a \leftrightarrow b). \tag{4.4}
\]

The presence of various step functions assures that both dipoles are found only within the parent dipole, and they do not overlap. Eq. (4.3) can be solved by taking a Laplace transform with respect to \(y\), and this is carried in the Appendix. The leading order term in \(y\) takes on a surprisingly simple form

\[
n_2(x_0, x_1, x_a, \bar{x}_a, x_b, \bar{x}_b, y) = 4\lambda^4 y^4 e^{-\lambda(x_a + x_b)y} \theta(x_a - \bar{x}_a - (x_a + x_b)/2) \theta(x_1 - \bar{x}_a - x_a/2) \theta(\bar{x}_b - x_0 - x_b/2) + (a \leftrightarrow b). \tag{4.5}
\]

Comparing this result to the single dipole density in Eq. (4.2), we find that the leading term of the double dipole density, as given in (4.5), is just the product of two single dipole densities, the only additional requirement is that the two dipoles do not overlap. The first term corresponds to the situation that the center of mass position of dipole \(a\) is above the center of mass position of dipole \(b\). The second term represents the other possibility. This result is not at all obvious before we solve the integral equation (4.3). Note that the inhomogeneous term in (4.3), given in the first graph on the r.h.s. of figure 6, is different from (4.5), since its leading term in \(y\) is proportional to \(\bar{x}_a - \bar{x}_b - (x_a + x_b)/2\), which is absent in (4.5). With this factor, the density is not homogeneous regarding to the center of mass positions. The result in (4.5) is rather homogeneous in the center of mass positions.
4.2. Meson-meson amplitude due to two Pomeron exchange

With formula (4.5) of the dipole pair density, we can calculate the scattering amplitude due to four gluon exchanges between a pair of dipoles in one meson and another pair of dipoles in the other meson. The scattering amplitude with zero momentum transfer, for given parent dipole sizes \(x_{10}\) and \(x_{10}'\), is given by

\[
A_2(x_{10}x_{10}', q = 0, Y) = \frac{2!}{2! \times 2!} \int \frac{dk}{2\pi} dx_a dx_b dx'_a dx'_b n_2(x_0, x_1, x_a, k, x_b, -k, Y/2) \]

\[
n_2(x'_0, x'_1, x'_a, k, x'_b, -k, Y/2)[8g^4 \frac{dk_a}{2\pi k_a^2} d(k_a, k, x_a) d(k_a, k, x'_a)]
\]

\[
[8g^4 \frac{dk_b}{2\pi k_b^2} d(k_b, -k, x_b) d(k_b, -k, x'_b)],
\]

where the term in the first bracket is from the exchange of two gluons between dipoles \(x_a\) and \(x'_a\), and the term in the second bracket is from the exchange of two gluons between dipoles \(x_b\) and \(x'_b\). In (4.6), \(d(k_a, k, x_a) = \sin(k_a x_a/2) \sin((k - k_a)x_a/2)\), and the dipole density \(n_2(x_0, x_1, x_a, k, x_b, -k, Y/2)\) is the Fourier transform of \(n_2(x_0, x_1, x_a, x'_a, x_b, x'_b, Y/2)\) with respect to the center of mass positions, (relative to the center of mass of the parent dipole). Finally, the numerical factor \(2!\) is from possible pairing among two pairs of dipoles, and two factors \(2!\) in the denominator are due to indistinguishability of two dipoles in a meson. A version of Eq.(4.6) in two transverse dimensions has been derived in the Appendix of [9].

Were the dipole density \(n_2\) more complicated than given in Eq. (4.5), it would be a hard task to perform the integral in (4.7). Fortunately, Eq. (4.3) tells us that in the leading order, the dipole pair density is the product of two single dipole densities, up to the requirement that the two dipoles do not overlap. Recall in our calculation of the amplitude due to one Pomeron exchange, the major contribution is from dipole size \(x \sim 1/(\lambda Y)\), which is very small at high energies. The smallness of dipole sizes then allows us to neglect the requirement that two dipoles do not overlap in a given meson, then the dipole pair density is simply the product of two single dipole densities. The integral in (4.6) factorizes. Thus, the two Pomeron exchange amplitude is given by

\[
A_2(x_{10}, x_{10}', q = 0, Y) = \frac{1}{2} \int \frac{dk}{2\pi} A(x_{10}, x'_1, k, Y) A(x_{10}, x_{10}', -k, Y) = \frac{1}{2} \int \frac{dk}{2\pi} A^2(x_{10}, x_{10}', k, Y),
\]
where the last equality derives from the fact that the single Pomeron exchange amplitude, as given in (3.10), is an even function of $k$. The above formula allows us to calculate the forward scattering amplitude $A_2$ from the knowledge of $A$. Performing the integral over $k$ in (4.7) and keeping only the leading term, we have

$$A_2(x_{10}, x'_{10}, q = 0, Y) = \frac{g^8}{6\lambda^2 Y^2} \left( \frac{1}{2}|x_{10} - x'_{10}|^3 + \frac{1}{2}(x_{10} + x'_{10})^3 - x_{10}^3 - (x'_{10})^3 \right). \quad (4.8)$$

So the amplitude is suppressed by the factor $1/(\ln s)^2$, comparable to the sub-leading term in the single Pomeron exchange amplitude. The dependence of $A_2$ on the dipole sizes is cubic, as the correct dimensionality requires. In principle one can also compute $A_2$ for a finite momentum transfer. The formula is given by

$$A_2(x_{10}, x'_{10}, q, Y) = \frac{1}{2} \int \frac{dk}{2\pi} A(x_{10}, x'_{10}, k; Y)A(x_{10}, x'_{10}, q - k, Y). \quad (4.9)$$

5. The triple Pomeron coupling

We have found a simple formula for the scattering amplitude of two Pomeron exchange, indicating the simplicity of Pomeron physics in the three dimensional QCD. The next important object to calculate is the triple Pomeron coupling. Armed with knowledge of all these, one may attempt to construct the Regge field theory which helps one to study high energy scattering systematically. It is interesting, as one may expect, that once again the triple Pomeron coupling can be calculated exactly to the leading order, unlike the situation in the four dimensional QCD.

The physical process involving triple Pomeron coupling is $A + B \rightarrow C + X$, where $A$, $B$ and $C$ are mesons and $X$ is anything. The generalized optical theorem states that the partial cross section of this so-called one-particle inclusive process is given by the imaginary part of amplitude of the elastic process $A + B + \bar{C} \rightarrow A + B + \bar{C}$ [12]. To be more precise, let $p$ be the out-going momentum of $C$, then $E_C d\sigma/d^2p = \frac{1}{2s(2\pi)^3} \Im(A_6)$, where $A_6$ is the connected part of the forward scattering amplitude of $A + B + \bar{C} \rightarrow A + B + \bar{C}$. To extract information concerning the triple Pomeron coupling, we consider the case $C = A$ and the one particle inclusive process is dominated by a Pomeron exchange. Instead of considering some definite meson states, let us consider mesons with a fixed dipole size. Let $x_{10}$ the dipole size of the $B$ meson and $x'_{10}$ the dipole size of the $A$ meson. Furthermore, we define $M^2$ be the mass squared of $X$, rapidity $\bar{y} = \ln(s/M^2)$, where $s$ is the total energy squared.
Also define \( Y = \ln(s/m_T^2) \), \( m_T \) is the transverse mass of the outgoing \( A \) meson. With these notations, it is easy to see that

\[
E_C \frac{d\sigma}{d^2p} = s \frac{d\sigma}{dq dM^2},
\]

where \( q \) is the momentum transfer. Consider large \( \bar{y} \), the process is then dominated by a single Pomeron exchange, so

\[
s \frac{d\sigma}{dq^2 dM^2} \sim \frac{1}{s} \frac{g^4}{q^2} \left( \sin(q x_{10}/2) \right)^2 \frac{4}{\lambda^2 y^2} \left| \sum_X \beta_{BX} \right|^2
\]

where we have used the Pomeron “propagator” and its coupling to meson \( A \), \( \beta_{BX} \) denotes the Pomeron coupling to \( B \) and \( X \). Again applying the optical theorem, \( \left| \sum_X \beta_{BX} \right|^2 \) can be written as a product of the triple coupling coupling, the Pomeron propagator at \( Y - \bar{y} = \ln(M^2/m_T^2) \) and the Pomeron coupling to meson \( B \). Note that this coupling to \( B \) is the one at zero momentum transfer. Putting everything together, one has

\[
E_C \frac{d\sigma}{d^2p} \sim \frac{M^2}{s} \frac{8g^6 x_{10}}{\lambda^2 \bar{y}(Y - \bar{y}) |q|} \left( \sin(q x_{10}/2) \right)^2 V_{3P}(q), \tag{5.1}
\]

where \( V_{3P} \) shall be referred to as the “triple Pomeron coupling”.

Next, we calculate the 6-point amplitude \( A_6 \) and use the optical theorem to relate this amplitude to the above quantity, and finally read off \( V_{3P} \). One can imagine that such amplitude is due to one Pomeron exchange between \( A \) and \( B \) and one Pomeron exchange between another \( A \) and \( B \). So for meson \( B \) we need a dipole pair density. The rapidity available for one Pomeron exchange is \( \bar{y} \). However for the dipole pair in meson \( B \), the rapidity available is different from \( \bar{y} \), and \( \bar{y} \) serves only as a rapidity cut. We then follow [9] to define the dipole pair density depending on both \( Y \) and \( \bar{y} \), here \( Y \) is the maximum rapidity.

The desired equation again can be readily read off from a graphical equation, which we spare here:

\[
n_2(x_0, x_1, y, \bar{y}) = \lambda \int_{x_0}^{x_1} dx_2 e^{-\lambda x_{10}(y - \bar{y})} [n(x_2, x_1, x_2, \bar{x}_a, \bar{y})n(x_0, x_2, x_b, \bar{x}_b, \bar{y})
\]

\[
+ (a \leftrightarrow b)] + \lambda \int_{x_0}^{x_1} dx_2 \int_{\bar{y}}^{y} dy' e^{-\lambda x_{10}(y - y')} [n_2(x_2, x_1, y', \bar{y}) + n_2(x_0, x_2, y', \bar{y})], \tag{5.2}
\]
where all quantities with subscript $a$ are those associated to meson $A$, and those with subscript $b$ are associated to meson $B$. The solution to this equation can be obtained in a similar way as we solved Eq. (4.1). We skip the details and directly write down the answer

$$n_2(x_0, x_1, y, \bar{y}) = 4\lambda^5 \bar{y}^4 e^{-\lambda(x_a - x_b)(y - \bar{y}) - \lambda(x_a + x_b)(y + \bar{y})/2} (\bar{x}_a - \bar{x}_b - (x_a + x_b)/2)
\theta(\bar{x}_a - \bar{x}_b - (x_a + x_b)/2)\theta(x_1 - \bar{x}_a - x_a/2)\theta(\bar{x}_b - x_0 - x_b/2) + (a \leftrightarrow b).
$$ (5.3)

We remark that this solution is exact with the input of the single dipole density given in (4.2). Next we need to transform this density into the momentum space, which is the quantity needed in calculate the 6-point amplitude. Assuming $q = q_a = -q_b$ and ignoring the dipole sizes in various theta functions, we find

$$
n_2(x_0, x_1, q, -q, y, \bar{y}) = \int d\bar{x}_a d\bar{x}_b e^{iq(\bar{x}_a - \bar{x}_b)} n_2(x_0, x_1, \bar{x}_a, \bar{x}_b, y, \bar{y}) = 4\lambda^5 \bar{y}^4 e^{-\lambda(x_a + x_b)(y + \bar{y})/2} x_{10}^{-1} [e^{-\lambda(y - \bar{y})} + 1] + \frac{2}{\lambda(y - \bar{y}) + iq} (e^{-\lambda(y - \bar{y}) + iq} x_{10}^{-1} - 1) + c.c.].
$$ (5.4)

This formula is still a bit too long. Indeed the physical situation corresponds to $\lambda(y - \bar{y}) x_{10} >> 1$, and this simplifies the above formula a lot:

$$n_2(x_0, x_1, q, -q, y, \bar{y}) = 8\lambda^5 \bar{y}^4 x_{10}^{-1} \frac{\lambda^2(y - \bar{y})^2 - q^2}{(\lambda^2(y - \bar{y})^2 + q^2)} e^{-\lambda(x_a + x_b)(y + \bar{y})/2}.
$$ (5.5)

Now we are ready to calculate the 6-point amplitude. By definition, this amplitude is given by

$$A_6(q) = iM^2 \int dx_a dx_b dx'_a dx'_b n_2(x_0, x_1, x_a, q, x_b, -q, Y - \frac{\bar{y}}{2}, \frac{y}{2}) n(x'_{10}, x'_a, q, \frac{\bar{y}}{2}) \frac{dk_a}{2\pi k_a^2 (q - k_a)^2} d(k_a, k, x_a) d(k_a, k, x'_a)
+ \frac{dk_b}{2\pi k_b^2 (q - k_b)^2} d(k_b, -k, x_b) d(k_b, -k, x'_b),
$$ (5.6)

where $Y - \bar{y} = \ln M^2$ and other notations are the same as in the previous section. Recall that the single dipole density in the momentum space is

$$n(x'_{10}, x'_a, q, \bar{y}) = 4\lambda^2 \bar{y}^2 \frac{1}{q} \sin(q x'_a/2) e^{-\lambda x'_a \bar{y}},$$

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again we ignored the dipole separation as compared to $x'_{10}$. Substituting this formula and the formula in Eq.(5.3) into (5.6), we find

$$A_6(q) = \frac{i}{2} \lambda^6 M^2 g^8 \frac{\lambda^2 (Y - \bar{y})^2 - q^2}{\lambda^2 (Y - \bar{y})^2 + q^2} \frac{x_{10}}{q^2} (\sin(qx'_{10}/2))^2$$

$$\left(8g^4 \int dx_a dx'_a e^{-\lambda x_a Y/2 - \lambda x'_a \bar{y}/2} \frac{dk_a}{2\pi k_a^2 (q - k_a)^2} d(k_a, q, x_a) d(k_a, q, x'_a) \right)^2.$$  

Finally performing the integral in the above equation and one is led to a quite lengthy formula

$$A_6(q) = 2ig^8 M^2 \lambda^7 g^8 \frac{x_{10}}{(Y + \bar{y})^2 q^2} (\sin(qx'_{10}/2))^2$$

$$\frac{(\lambda^2 (Y - \bar{y})^2 - q^2)}{(\lambda^2 (Y - \bar{y})^2 + q^2)^2 (\lambda^2 Y^2 + q^2)^2 (\lambda^2 \bar{y}^2 + q^2)^2}.$$  

This formula is actually not valid when $q \sim \lambda Y$, since we have ignored the dipole size in the sine factors in the single dipole density as well as in the dipole pair density. This is to be contrasted with the case for one Pomeron exchange amplitude, (3.10), where the dependence on dipole sizes is not ignored. (If one so desires, one can of course take the dependence on dipole sizes into account, leading to a much more complicated function of $q$.) When $|q| << \lambda Y$ where Eq. (5.7) is valid, it simplifies to

$$A_6(q) = 2ig^8 M^2 \frac{\bar{y}^4 x_{10}}{\lambda^3 Y^4 (Y + \bar{y})^2 (Y - \bar{y})^2} \frac{1}{q^2} (\sin(qx'_{10}/2))^2.$$  

(5.8)

Upon using the optical theorem, $E_c d\sigma/d^2 p = \frac{1}{2s(2\pi)^2} \Im(A_6)$, and comparing (5.8) to (5.1), we find that the triple Pomeron coupling depends on rapidities:

$$V_{3P} \sim \frac{g^2}{|q|} \frac{\bar{y}^6}{Y^4 (Y + \bar{y})^2 (Y - \bar{y})}.$$  

(5.9)

It tends to zero when both $Y$ and $\bar{y}$ are large. It is also singular when $\sqrt{-t} = |q| \to 0$, just as in the 3+1 dimensional case [9]. However the origin of this singularity in the present case is purely kinematic, as Eq. (5.1) shows. If the one particle inclusive cross section factorizes, one expects that $V_{3P}$ is independent of rapidities. Our result indicates that the inclusive cross section does not factorize and the triple Pomeron coupling vanishes as $Y^{-1}$, with $Y - \bar{y}$ fixed.
6. Discussion

As we have seen, one remarkable property of high energy scattering processes in our toy world, 2+1 dimensional QCD, is the fact that no soft on-shell gluon can escape from hadron emitting it. This is due to the quantum interference of quark and anti-quark. This property is also the reason underlying the fact that the total cross section decreases at high energies, as was first pointed out in [11]. It is also responsible for the interesting fact that, in the leading order, there is no correlation between a pair of dipoles in a meson state, so the amplitude due to two Pomeron exchange is simply given by Eq. (4.9). Physically, one expects that this holds true even for multi-dipole density. Thus, in the leading order, the amplitude due to n-Pomeron exchange is given by

\[
A_n(x_{10}, x'_{10}, q) = \frac{1}{n!} \int \prod_{i=1}^{n-1} \left( \frac{dk_i}{2\pi} A(x_{10}, x'_{10}, k_i) \right) A(x_{10}, x'_{10}, q - \sum_i k_i),
\]

(6.1)

where \(A(x_{10}, x'_{10}, k)\) is the amplitude due one Pomeron exchange between dipole \(x_{10}\) and dipole \(x'_{10}\). (6.1) is a straightforward generalization of (4.9). Going to the impact parameter space, Eq. (6.1) implies that the eikonal formula for summing over all possible number of Pomeron exchanges is valid in the leading order. However, as can be seen in Sec. 3, there are sub-leading terms on each level of Pomeron exchanges, and these terms are comparable to the leading term on the next level of Pomeron exchanges.

Of course, to incorporate the whole complexity of high energy scattering, one needs to include various couplings among Pomerons. The triple Pomeron coupling is already calculated in this work. It is interesting to see whether one can develop a Reggeon field theory based on our results.

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Appendix

Eq. (4.3) can be simplified by a Laplace transform. Denoting the transform for $n_2$ by $n_{2,\omega}(x_0, x_1)$, where the variable $\omega$ plays the role of a “complex angular momentum”, the integral over $y'$ can be undone. Let

$$\tilde{n}_{2,\omega}(x_0, x_1) = \frac{1}{4 \times 4!} (\omega + \lambda (x_a + x_b))^5 n_{2,\omega}(x_0, x_1),$$

we obtain the integral equation for $\tilde{n}_{2,\omega}$:

$$(\omega + \lambda x_0)\tilde{n}_{2,\omega}(x_0, x_1) = \lambda^5 [(\bar{x}_a - \bar{x}_b - (x_a + x_b)/2)\theta(\bar{x}_a - \bar{x}_b - (x_a + x_b)/2)$$

$$\theta(x_1 - \bar{x}_a - x_a/2)\theta(\bar{x}_b - x_0 - x_b/2) + (a \leftrightarrow b)] + \lambda \int_{x_0}^{x_1} dx_2 (\tilde{n}_{2,\omega}(x_2, x_1) + \tilde{n}_{2,\omega}(x_0, x_1)).$$

(A.1)

To solve (A.1), we use the following ansatz

$$\tilde{n}_{2,\omega}(x_0, x_1) = \lambda^5 [(\bar{x}_a - \bar{x}_b - (x_a + x_b)/2)\theta(\bar{x}_a - \bar{x}_b - (x_a + x_b)/2)\theta(x_1 - \bar{x}_a - x_a/2)$$

$$\theta(\bar{x}_b - x_0 - x_b/2)f_\omega(x_0, x_1, x_a, \bar{x}_a, x_b, \bar{x}_b) + (a \leftrightarrow b).$$

(A.2)

The unknown function $f_\omega$ then satisfies

$$(\omega + \lambda x_0)f_\omega(x_0, x_1) = 1 + \lambda \int_{x_0}^{\bar{x}_b - x_b/2} dx_2 f_\omega(x_2, x_1) + \int_{x_a + x_a/2}^{x_1} dx_2 f_\omega(x_0, x_2).$$

The solution is

$$f_\omega = [\omega + \lambda (\bar{x}_a - \bar{x}_b + (x_a + x_b)/2)]^{-1},$$

which is independent of $x_0$ and $x_1$. Substituting this result in (A.2) and recalling the definition for $\tilde{n}_{2,\omega}$, we find

$$n_{2,\omega}(x_0, x_1) = 4 \times 4! \lambda^5 [\omega + \lambda (x_a + x_b)]^{-5} [\omega + \lambda (\bar{x}_a - \bar{x}_b + (x_a + x_b)/2)]^{-1} (\bar{x}_a - \bar{x}_b$$

$$- (x_a + x_b)/2)\theta(\bar{x}_a - \bar{x}_b - (x_a + x_b)/2)\theta(x_1 - \bar{x}_a - x_a/2)\theta(\bar{x}_b - x_0 - x_b/2) + (a \leftrightarrow b).$$

(A.3)

It is easy to transform $n_{2,\omega}$ back to arrive at $n_2(x_0, x_1, y)$. There are many terms each proportional to a certain power of $y$. The leading term takes on the following simple form:

$$n_2(x_0, x_1, x_a, \bar{x}_a, x_b, \bar{x}_b, y) = 4\lambda^4 y^4 e^{-\lambda (x_a + x_b)y} \theta(\bar{x}_a - \bar{x}_b - (x_a + x_b)/2)$$

$$\theta(x_1 - \bar{x}_a - x_a/2)\theta(\bar{x}_b - x_0 - x_b/2) + (a \leftrightarrow b).$$

(A.4)
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**Figure captions**

Fig.1: A soft gluon emitted either from the quark or the anti-quark of a dipole. The dashed line denotes the gluon.

Fig.2: A graphical illustration of Eq. (2.6) for the generating functional $Z$.

Fig.3: A string picture of meson-meson scattering by a closed string, a Pomeron, exchange.

Fig.4: Quark-gluon diagram of the process of two gluon exchange and its corresponding surface picture, obtained by replacing a gluon line by two quark lines.

Fig.5: A graphical equation for the dipole density, again dashed lines are soft gluon lines.

Fig.6: A graphical equation for the dipole pair density.
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