CRYSTAL GRAPHS FOR BASIC REPRESENTATIONS OF THE QUANTUM AFFINE ALGEBRA $U_q(C^{(1)}_2)$

JIN HONG AND SEOK-JIN KANG*

Abstract. We give a realization of crystal graphs for basic representations of the quantum affine algebra $U_q(C^{(1)}_2)$ in terms of new combinatorial objects called the Young walls.

1. Introduction

In [5, 6], Kashiwara developed the theory of crystal bases for integrable modules over the quantum groups associated with symmetrizable Kac-Moody algebras. The crystal bases can be viewed as bases at $q = 0$ and they are given a structure of colored oriented graph, called the crystal graphs, with arrows defined by Kashiwara operators. The crystal graphs have many nice combinatorial features reflecting the internal structure of integrable representations of quantum groups. For instance, the characters of integrable representations can be computed by counting the elements in the crystal graphs with a given weight. Moreover, the tensor product decomposition of integrable modules into a direct sum of irreducible submodules is equivalent to decomposing the tensor product of crystal graphs into a disjoint union of connected components. Therefore, to understand the combinatorial nature of integrable representations, it is essential to find realizations of crystal graphs in terms of nice combinatorial objects.

In [8], Misra and Miwa constructed the crystal graphs for basic representations of quantum affine algebras $U_q(A^{(1)}_n)$ using Young diagrams with colored boxes. Their idea was extended to construct crystal graphs for irreducible highest weight $U_q(A^{(1)}_n)$-modules of arbitrary higher level [1]. The crystal graphs constructed in [8] can be parametrized by certain paths which arise naturally in the theory of solvable lattice models. Motivated by this observation, Kang, Kashiwara, Misra, Miwa, Nakashima and Nakayashiki developed the theory of perfect crystals for general quantum affine algebras and gave a realization of crystal graphs for irreducible highest weight modules of arbitrary higher level in terms of paths. In this way, the theory of vertex models can be explained in the language of representation theory of quantum affine algebras and the 1-point function of the vertex model was expressed as the quotient of the string function and the character of the corresponding irreducible highest weight representation.

The purpose of this paper is to give a realization of crystal graphs for basic representations of the quantum affine algebra $U_q(C^{(1)}_2)$ using some new combinatorial objects which we call the Young walls. The Young walls consist of colored blocks that are built on the given ground-state and can be viewed as generalizations of

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Young diagrams. The rules for building Young walls are quite similar to playing with LEGO blocks and the Tetris game. The crystal graphs for basic representations are characterized as the set of all reduced proper Young walls. The weight of a Young wall can be computed easily by counting the number of colored blocks that have been added to the ground-state. Hence the weight multiplicity is just the number of all reduced proper Young walls of given weight.

One can define the $U_q(C^{(1)}_2)$-action on the space spanned by all proper Young walls and can give an algorithm for finding global basis (or canonical basis) associated with each reduced proper Young wall [2]. It still remains to extend the results of this paper to quantum affine algebras $U_q(C^{(1)}_n)$ for $n \geq 3$ and to the higher level integrable representations of $C^{(1)}_n$ ($n \geq 2$).

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2. THE QUANTUM AFFINE ALGEBRA OF TYPE $C^{(1)}_2$

Let $I = \{0, 1, 2\}$ be the index set. Consider the generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ of affine type $C^{(1)}_2$ and its Dynkin diagram:

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

Let $P^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \mathbb{Z}h_2 \oplus \mathbb{Z}d$ be a free abelian group, called the dual weight lattice and set $\mathfrak{h} = \mathbb{C} \otimes_\mathbb{Z} P^\vee$. We define the linear functionals $\alpha_i$ and $\Lambda_i$ ($i \in I$) on $\mathfrak{h}$ by

$$\alpha_i(h_j) = a_{ji}, \quad \alpha_i(d) = \delta_{0,i},$$

$$\Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d) = 0 \quad (i,j \in I).$$

The $\alpha_i$ (resp. $h_i$) are called the simple roots (resp. simple coroots) and the $\Lambda_i$ are called the fundamental weights. We denote by $\Pi = \{\alpha_i | i \in I\}$ (resp. $\Pi^\vee = \{h_i | i \in I\}$) the set of simple roots (resp. simple coroots).

Let $c = h_0 + h_1 + h_2$ and $\delta = \alpha_0 + 2\alpha_1 + \alpha_2$. Then we have $\alpha_i(c) = 0, \delta(h_i) = 0$ for all $i \in I$ and $\delta(d) = 1$. We call $c$ (resp. $\delta$) the canonical central element (resp. null root). The free abelian group $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\Lambda_2 \oplus \mathbb{Z}\delta$ is called the weight lattice and the elements of $P$ are called the affine weights.

We denote by $q^h$ ($h \in P^\vee$) the basis elements of the group algebra $\mathbb{C}(q)[P^\vee]$ with the multiplication $q^h q^{h'} = q^{h+h'}$ ($h, h' \in P^\vee$). Set $q_0 = q_2 = q^2$, $q_1 = q$ and $K_0 = q^{2\alpha_0}$, $K_1 = q^{h_1}$, $K_2 = q^{2h_2}$. We will also use the following notations.

$$[k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad e_i^{(n)} = e_i^n/[n]_i!, f_i^{(n)} = f_i^n/[n]_i!.$$

Definition 2.1. The quantum affine algebra $U_q(C^{(1)}_2)$ of type $C^{(1)}_2$ is the associative algebra with 1 over $\mathbb{C}(q)$ generated by the symbols $e_i$, $f_i$ ($i \in I$) and $q^h$ ($h \in P^\vee$) subject to the following defining relations:
$q^0 = 1, \quad q^h q^{h'} = q^{h + h'} \quad (h, h' \in P^\vee)$,
$q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \quad (h \in P^\vee, i \in I),
\begin{align*}
e_i f_j - f_j e_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \quad (i, j \in I), \\
e_0^2 e_1 - (q^2 + q^{-2}) e_0 e_1 e_0 + e_1 e_0^2 &= 0, \\
f_0^2 f_1 - (q^2 + q^{-2}) f_0 f_1 f_0 + f_1 f_0^2 &= 0, \\
e_0^2 e_0 - (q^2 + 1 + q^{-2}) e_0^2 e_0 e_1 + (q^2 + 1 + q^{-2}) e_1 e_0 e_1^2 - e_0 e_1^2 &= 0, \\
f_0^2 f_0 - (q^2 + 1 + q^{-2}) f_0^2 f_0 f_1 + (q^2 + 1 + q^{-2}) f_1 f_0 f_0^2 - f_0 f_1^3 &= 0, \\
e_0^2 e_1 - (q^2 + 1 + q^{-2}) e_1 e_2 e_1 + (q^2 + 1 + q^{-2}) e_1 e_2 e_1^2 - e_1 e_2 e_1^2 &= 0, \\
f_0^2 f_2 - (q^2 + 1 + q^{-2}) f_0^2 f_2 f_1 + (q^2 + 1 + q^{-2}) f_1 f_2 f_0^2 - f_2 f_1^3 &= 0, \\
e_0^2 e_1 - (q^2 + q^{-2}) e_2 e_1 e_2 + e_1 e_2^2 &= 0, \\
f_0^2 f_1 - (q^2 + q^{-2}) f_0 f_1 f_2 + f_1 f_2 f_0^2 &= 0, \\
e_0 e_2 &= e_2 e_0, \quad f_0 f_2 = f_2 f_0.
\end{align*}

We call $(A, II, IV, P, P^\vee)$ the Cartan datum associated with the quantum affine algebra $U_q(C_2^{(1)})$.

The subalgebra of $U_q(C_2^{(1)})$ generated by $e_i$, $f_i$, $K_i^{\pm 1}$ $(i \in I)$ is denoted by $U_q'(C_2^{(1)})$. It is also called the quantum affine algebra of type $C_2^{(1)}$. Let $\bar{P}^\vee = \mathbb{Z} h_0 \oplus \mathbb{Z} h_1 \oplus \mathbb{Z} h_2$ and $\bar{P} = C \otimes \mathbb{Z} \bar{P}^\vee$. Consider $\alpha_i$ and $\Lambda_i$ $(i \in I)$ as linear functionals on $\bar{P}$ and set $P = \mathbb{Z} \alpha_0 \oplus \mathbb{Z} \Lambda_1 \oplus \mathbb{Z} \Lambda_2$. The elements of $P$ are called the classical weights. We call $(A, II, IV, P, P^\vee)$ the Cartan datum for the quantum affine algebra $U_q'(C_2^{(1)})$.

3. Crystal bases

In this section, we review the crystal basis theory for quantum affine algebras $U_q(C_2^{(1)})$ and $U_q'(C_2^{(1)})$. A $U_q(C_2^{(1)})$-module (resp. $U_q'(C_2^{(1)})$-module) $M$ is called integrable if

(i) $M = \bigoplus_{\lambda \in \bar{P}} M_{\lambda}$ (resp. $M = \bigoplus_{\lambda \in \bar{P}} M_{\lambda}$), where

\[ M_{\lambda} = \{ v \in M \mid q^h v = q^{\lambda(h)} v \text{ for all } h \in P^\vee \text{ (resp. } h \in \bar{P}^\vee) \}, \]

(ii) for each $i \in I$, $M$ is a direct sum of finite dimensional irreducible $U_i$-modules, where $U_i$ denotes the subalgebra generated by $e_i$, $f_i$, $K_i^{\pm 1}$ which is isomorphic to $U_q(\mathfrak{sl}_2)$.

Fix $i \in I$. By the representation theory of $U_q(\mathfrak{sl}_2)$, any element $v \in M_{\lambda}$ may be written uniquely as

\[ v = \sum_{k \geq 0} f_i^{(k)} v_k, \]

where $v_k \in \ker e_i \cap M_{\lambda + k\alpha_i}$. We define the endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$ on $M$, called the Kashiwara operators, by

\[ \tilde{e}_i v = \sum_{k \geq 1} f_i^{(k-1)} v_k, \quad \tilde{f}_i v = \sum_{k \geq 0} f_i^{(k+1)} v_k. \]
Let $A$ be the subring of $C(q)$ consisting of the rational functions in $q$ that are

regular at $q = 0$.

**Definition 3.1.**

(a) A free $A$-submodule $L$ of an integrable $U_q$-module $M$, stable under $\tilde{e}_i$ and $\tilde{f}_i$, is called a crystal lattice if $M \cong C(q) \otimes_A L$ and $L = \bigoplus_{\lambda \in P} L_\lambda$, where $L_\lambda = L \cap M_\lambda$.

(b) A crystal basis of an integrable module $M$ is a pair $(L, B)$ such that

(i) $L$ is a crystal lattice of $M$,

(ii) $B$ is a $C$-basis of $L/qL$,

(iii) $B = \cup_{\lambda \in P} B_\lambda$, where $B_\lambda = B \cap (L_\lambda/qL_\lambda)$,

(iv) $\tilde{e}_i B \subset B \cup \{0\}$, $\tilde{f}_i B \subset B \cup \{0\}$,

(v) for $b, b' \in B$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$.

The set $B$ is given a colored oriented graph structure by defining $\xrightarrow{b \mapsto b'}$ if and only if $b' = \tilde{f}_i b$. The graph $B$ is called the crystal graph of $M$ and it reflects the combinatorial structure of $M$. For instance, we have $\dim_{C(q)} M_\lambda = \#B_\lambda$ for all $\lambda \in P$ (or $\lambda \in \check{P}$). By extracting properties of the crystal graphs, we define the notion of abstract crystals as follows.

**Definition 3.2** ([7]). An affine crystal (resp. classical crystal) is a set $B$ together with the maps $\wt : B \to P$ (resp. $\wt : B \to \check{P}$), $\varepsilon_i : B \to \mathbb{Z} \cup \{-\infty\}$, $\varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$, $\tilde{e}_i : B \to B \cup \{0\}$, and $\tilde{f}_i : B \to B \cup \{0\}$, satisfying the following conditions:

(i) $\langle \wt(b), h_i \rangle = \varphi_i(b) - \varepsilon_i(b)$ for all $b \in B$,

(ii) $\wt(\tilde{e}_i b) = \wt(b) + \alpha_i$ for $b \in B$ with $\tilde{e}_i b \in B$,

(iii) $\wt(\tilde{f}_i b) = \wt(b) - \alpha_i$ for $b \in B$ with $\tilde{f}_i b \in B$,

(iv) $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$ for $b, b' \in B$,

(v) $\tilde{e}_i b = \tilde{f}_i b = 0$ if $\varepsilon_i(b) = -\infty$.

The crystal graph of an integrable $U_q(C^{(1)}_2)$-module (resp. $U'_q(C^{(1)}_2)$-module) is an affine crystal (resp. a classical crystal).

Let $B_1$ and $B_2$ be (affine or classical) crystals. A morphism $\psi : B_1 \to B_2$ of crystals is a map $\psi : B_1 \cup \{0\} \to B_2 \cup \{0\}$ such that

(i) $\psi(0) = 0$,

(ii) if $b \in B_1$ and $\psi(b) \in B_2$, then $\wt(\psi(b)) = \wt(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, $\varphi_i(\psi(b)) = \varphi_i(b)$,

(iii) if $b, b' \in B_1$, $\psi(b), \psi(b') \in B_2$ and $\tilde{f}_i b = b'$, then $\tilde{f}_i \psi(b) = \psi(b')$.

The tensor product $B_1 \otimes B_2$ of $B_1$ and $B_2$ is the set $B_1 \times B_2$ whose crystal structure is defined by

$\wt(b_1 \otimes b_2) = \wt(b_1) + \wt(b_2)$,

$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - (\wt(b_1), h_i))$,

$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varepsilon_i(b_1) + (\wt(b_2), h_i))$,

$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$

$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$
4. Perfect crystals

Let \( B \) be a classical crystal. For \( b \in B \), we write \( \varepsilon(b) = \sum_i \varepsilon_i(b) \Lambda_i \) and \( \varphi(b) = \sum_i \varphi_i(b) \Lambda_i \). Note that \( \text{wt}(b) = \varphi(b) - \varepsilon(b) \). Set \( \bar{P}^+ = \{ \lambda \in \bar{P} \mid \langle \lambda, h_i \rangle \geq 0 \text{ for all } i \in I \} \) and \( \bar{P}^+_l = \{ \lambda \in \bar{P}^+ \mid \langle \lambda, c \rangle = l \} \).

**Definition 4.1** ([4]). For \( l \in \mathbb{Z}_{\geq 0} \), we say that a classical crystal \( B \) is a *perfect crystal of level* \( l \) if

(i) \( B \otimes \bar{B} \) is connected,

(ii) there exists some \( \lambda_0 \in \bar{P} \) such that \( \text{wt}(B) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \alpha_i \) and \( \#(B_{\lambda_0}) = 1 \),

(iii) for any \( b \in B \), we have \( (\varepsilon(b), c) \geq l \),

(iv) the maps \( \varepsilon, \varphi : B^{\text{min}} = \{ b \in B \mid \langle \varepsilon(b), c \rangle = l \} \to \bar{P}^+_l \) are bijective.

A finite dimensional \( U'_q(\mathfrak{g}) \)-module \( V \) is called a *perfect representation of level* \( l \) if it has a crystal basis \( (L, B) \) such that \( B \) is a perfect crystal of level \( l \).

Consider the space

\[
V = C(q)v_{1,2} \oplus C(q)v_{1,\bar{1}} \oplus C(q)v_{2,\bar{1}} \oplus C(q)v_{2,1} \oplus C(q)v_{1,\bar{1}} \oplus C(q)v_{2,\bar{1}},
\]

with the action of \( U'_q(\mathfrak{g}) \) defined by

\[
e_0v_{i,j} = \begin{cases} 
  v_{2,\bar{1}} & \text{if } (i, j) = (1, 2), \\
  v_{2,\bar{1}} & \text{if } (i, j) = (1, \bar{2}), \\
  0 & \text{otherwise},
\end{cases}
\]

\[
f_0v_{i,j} = \begin{cases} 
  v_{1,2} & \text{if } (i, j) = (2, 1), \\
  v_{1,2} & \text{if } (i, j) = (2, \bar{1}), \\
  0 & \text{otherwise},
\end{cases}
\]

\[
K_0v_{i,j} = \begin{cases} 
  q^2v_{i,j} & \text{if } (i, j) = (2, \bar{1}) \text{ or } (2, 1), \\
  q^{-2}v_{i,j} & \text{if } (i, j) = (1, 2) \text{ or } (1, \bar{2}), \\
  v_{i,j} & \text{otherwise},
\end{cases}
\]

\[
e_1v_{i,j} = \begin{cases} 
  (q + q^{-1})v_{1,2} & \text{if } (i, j) = (2, \bar{2}), \\
  v_{2,\bar{1}} & \text{if } (i, j) = (2, 1), \\
  0 & \text{otherwise},
\end{cases}
\]

\[
f_1v_{i,j} = \begin{cases} 
  v_{2,\bar{1}} & \text{if } (i, j) = (1, 2), \\
  (q + q^{-1})v_{2,\bar{1}} & \text{if } (i, j) = (2, 2), \\
  0 & \text{otherwise},
\end{cases}
\]

\[
K_1v_{i,j} = \begin{cases} 
  q^2v_{1,\bar{1}} & \text{if } (i, j) = (1, \bar{2}), \\
  q^{-2}v_{2,\bar{1}} & \text{if } (i, j) = (2, \bar{1}), \\
  v_{i,j} & \text{otherwise},
\end{cases}
\]

\[
e_2v_{i,j} = \begin{cases} 
  v_{1,2} & \text{if } (i, j) = (1, 2), \\
  v_{2,\bar{1}} & \text{if } (i, j) = (2, 1), \\
  0 & \text{otherwise},
\end{cases}
\]
\[
\begin{align*}
    f_{2v_{i,j}} &= \begin{cases} 
        v_{1,2} & \text{if } (i,j) = (1,2), \\
        v_{2,1} & \text{if } (i,j) = (2,1), \\
        0 & \text{otherwise,}
    \end{cases} \\
    K_{2v_{i,j}} &= \begin{cases} 
        q^2v_{i,j} & \text{if } (i,j) = (1,2) \text{ or } (2,1), \\
        q^{-2}v_{i,j} & \text{if } (i,j) = (1,\bar{2}) \text{ or } (\bar{2},1), \\
        v_{i,j} & \text{otherwise.}
    \end{cases}
\end{align*}
\]

**Theorem 4.2 ([4]).** The space \( V \) is an irreducible \( \mathcal{U}^\prime_q(C_2^{(1)}) \)-module whose crystal graph \( B \) is perfect of level 1 as is shown below:

![Crystal Graph](image)

For a dominant integral weight \( \Lambda \in \tilde{P}^+ \), we denote by \( B(\Lambda) \) the crystal graph for the irreducible highest weight \( \mathcal{U}^\prime_q(C_2^{(1)}) \)-module \( V(\Lambda) \). As the canonical central element for \( C_2^{(1)} \) is given by \( c = h_0 + h_1 + h_2 \), the dominant integral weights of level one (i.e., those \( \Lambda \in \tilde{P}^+ \) such that \( \Lambda(c) = 1 \)) are of the form \( \Lambda = \Lambda_i \) (i.e., \( i = 0,1,2 \)). The level one irreducible highest weight representations are called the basic representations.

**Theorem 4.3 ([3]).** There exist isomorphisms of \( \mathcal{U}^\prime_q(C_2^{(1)}) \)-crystals:

\[
\begin{align*}
    B(\Lambda_0) \otimes B & \cong B(\Lambda_2), \quad u_{\Lambda_0} \otimes \begin{array}{c} 2 \\ 1 \end{array} \mapsto u_{\Lambda_2}, \\
    B(\Lambda_1) \otimes B & \cong B(\Lambda_1), \quad u_{\Lambda_1} \otimes \begin{array}{c} 2 \\ 1 \end{array} \mapsto u_{\Lambda_1}, \\
    B(\Lambda_2) \otimes B & \cong B(\Lambda_0), \quad u_{\Lambda_2} \otimes \begin{array}{c} 2 \\ 1 \end{array} \mapsto u_{\Lambda_0}.
\end{align*}
\]

5. **Path realization of crystal graphs**

Using the crystal isomorphisms given in Theorem 4.3, we will give a realization of the crystal graphs \( B(\Lambda_i) \) in terms of paths. A path is an infinite sequence

\[ p = (p(k))_{k=0}^\infty = (\cdots, p(k+1), p(k), \cdots, p(1), p(0)) \]

with \( p(k) \in B \) for all \( k \geq 0 \). Among these, we single out the following three distinguished ones, called the ground-state paths:

\[
\begin{align*}
    p_{\Lambda_0} &= (\cdots, 1 2 1 2 1 2 \frac{1}{2},) \\
    p_{\Lambda_1} &= (\cdots, 2 2 2 2 2 2 2, ) \\
    p_{\Lambda_2} &= (\cdots, 2 1 2 1 2 1 2,).
\end{align*}
\]

The ground-state paths are determined as the image of highest weight vectors in \( B(\Lambda_i) \) under taking the compositions of the inverses of the crystal isomorphisms.
given in Theorem 4.3. For example, under the isomorphism

\[
B(\Lambda_0) \cong B(\Lambda_2) \otimes B \\
\cong B(\Lambda_0) \otimes B \otimes B \\
\cong B(\Lambda_2) \otimes B \otimes B \otimes B \\
\cong \ldots,
\]

the highest weight vector \( u_{\Lambda_0} \) is mapped onto the ground-state path \( p_{\Lambda_0} \).

Let

\[
P(\Lambda_i) = \{ p = (p(k))_{k=0}^{\infty} \mid p(k) = p_{\Lambda_i}(k) \text{ for all } k \gg 0 \}.
\]

The elements of \( P(\Lambda_i) \) are called the \( \Lambda_i \)-paths. These are infinite sequence of elements from \( B \) whose tail is identical to the ground-state path \( p_{\Lambda_i} \). We will define a crystal structure on \( P(\Lambda_i) \) as follows. We need to define the action of \( \tilde{f}_i \) and \( \tilde{e}_i \) to each sequence. First, replace each element of the sequence with some 0’s and 1’s as shown below:

|   | 2 | 2 | 2 | 2 | 2 |
|---|---|---|---|---|---|
| \( i = 0 \) | 1 | 1 | 0 | 0 |   |
| \( i = 1 \) | 0 | 0 | 1 | 1 |   |
| \( i = 2 \) | 0 | 1 | 0 | 1 |   |

The blank spaces above mean to replace them with nothing. Having done so, we remove each 0 1 pair occurring in the resulting sequence of 0’s and 1’s. Reading from the left, we would obtain a finite sequence of 1’s followed by some finite sequence of 0’s. If we want to apply \( \tilde{e}_i \), we apply it to the element corresponding to the right-most 1 to obtain another sequence of elements from \( B \). For \( \tilde{f}_i \), we apply it to the element corresponding to the left-most 0. Then the set \( P(\Lambda_i) \) becomes a \( U'_q(C_2^{(1)}) \)-crystal. Moreover, we have:

**Theorem 5.1 ([4]).** For each \( i = 0, 1, 2 \), there exists an isomorphism of crystals

\[
P(\Lambda_i) \cong B(\Lambda_i).
\]

The ground-state path \( p_{\Lambda_i} \) is mapped onto the highest weight vector \( u_{\Lambda_i} \) under this crystal isomorphism.

Hence we obtain a realization of \( B(\Lambda_i) \) in terms of paths. Let us see some examples.
Example 5.2. The highest parts of the crystal graph $B(\Lambda_0)$ may be drawn as below.

Example 5.3. The highest parts of the crystal graph $B(\Lambda_1)$ may be drawn as below.

6. The Young walls

The main purpose of this paper is to give a realization of crystal graph $B(\Lambda_i)$ using new combinatorial objects called the Young walls. In this section, we will explain the notion of Young walls.

The Young walls will be built of three kinds of blocks; the 0-block \( [0] \), the 1-block \( [1] \), and the 2-block \( [2] \). They are supposed to be colored by elements from the index set $I$ and we do not allow rotations of the blocks. The 0-block and 2-block are of unit height, unit width, and half thickness. The 1-block is of half height, unit width, and unit thickness. With these blocks, we will build a wall of unit thickness, extending infinitely to the left, like playing with LEGO blocks. There will be many rules we must adhere to in building the wall. The base of the wall may not be arbitrary, but must be chosen from one of the following three.
The drawings are meant to extend infinitely to the left. The dotted lines in the front parts of $Y_{\Lambda_0}$ and $Y_{\Lambda_2}$ signify where other blocks that will build up the wall of unit thickness may be placed. These will be called the ground-state walls. As it is quite awkward drawing these, and since we can’t see the blocks lying to the back, we will simplify the drawing as follows. The drawing on the right shows an example.

We will now list the rules for building the wall with colored blocks.

**Rules 6.1.**

(i) The wall must be built on top of one of the ground-state walls.

(ii) The blocks should be stacked in columns. No block may be placed on top of a column of half thickness.

(iii) The top and the bottom of the 0-block and 2-block may only touch a 1-block. The sides of the 0-block may only touch a 2-block and vice versa.

(iv) Placement of 0-block and 2-block to either the front or the back of the wall must be consistent for each column. The 0-block and 2-block should always be placed at heights which are multiples of the unit length.

(v) Stacking more than two 1-blocks consecutively on top of another is not allowed.

(vi) Except for the right-most column, there should be no free space to the right of any block.

(vii) In the right-most column of a wall built on $Y_{\Lambda_1}$, the 0-block should be placed to the front and the 2-block should be placed to the back.

We will give some examples illustrating the rules for building the walls. From now on, the ground-state wall extending infinitely to the left will be omitted and what remains will be shaded in the drawings.

**Example 6.2.** Good walls.
Example 6.3. Bad walls.

A column in the wall is a full column if the height of the column is a multiple of the unit length and the top of the column is of unit thickness. The first wall in Example 6.2 has two full columns. The right-most column in the second wall is not full but the other two are. Neither the third nor the forth wall contain full columns.

Definition 6.4.

(a) A wall satisfying Rules 6.1 is called a Young wall of ground state $\Lambda_i$, if it is built on the ground state wall $Y_{\Lambda_i}$ and the height of the columns are weakly decreasing as we go to the left.

(b) A Young wall is said to be proper if no two full columns of the wall are of the same height.

7. The Crystal Structure

In this section, we will define a crystal structure on the set of all proper Young walls. The action of Kashiwara operators will be described in a way similar to playing the Tetris game.

Definition 7.1.

(a) A block in a proper Young wall is removable if the wall remains a proper Young wall after removing the block.

(b) A place in a proper Young wall, where one may add a block to obtain another proper Young wall is called an admissible slot.

(c) A column in a proper Young wall is said to contain a removable $i$-block, if we may remove a 0-block, a 2-block, and two 1-blocks from the column in some order and still obtain a proper Young wall.

(d) A proper Young wall is reduced if none of its columns contain a removable $i$-block.

The second and the last wall of Example 6.2 are reduced. The first wall is not proper and the third wall contains one removable $\delta$. The set of all reduced proper Young walls of ground-state $\Lambda_i$ is denoted by $\mathcal{Y}(\Lambda_i)$.

We now defined the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$. A column in a proper Young wall is called $i$-removable if the top block of that column is a removable $i$-block. A column is $i$-admissible if the top of that column is an $i$-admissible slot. The action of Kashiwara operators is defined as follows.

1. Go through each column and write a 0 under each $i$-admissible column and a 1 under each $i$-removable column.
2. If we are dealing with the $i = 1$ case, there could be columns from which two 1-blocks may be removed. Place 1 1 under them. Under the columns that are 1-removable and at the same time 1-admissible, place 1 0. Write 0 0 under the columns that are twice 1-admissible.
3. From the (half-)infinite list of 0’s and 1’s, cancel out each 01 pair to obtain a finite sequence of 1’s followed by some 0’s (reading from left to right).
4. For $\hat{e}_i$, remove the $i$-block corresponding to the right-most 1 remaining. Set it to zero if no 1 remains.
5. For $\hat{f}_i$, add an $i$-block to the column corresponding to the left-most 0 remaining. Set it to zero if no 0 remains.

**Lemma 7.2.** The set of all reduced proper Young walls is stable under the Kashiwara operators defined above. That is, if $Y$ is a reduced proper Young wall, then $\hat{e}_iY$ and $\hat{f}_iY$ are either reduced proper Young walls or 0 for all $i \in I$.

**Proof.** By definition of removable blocks and admissible slots, it is obvious that the Kashiwara operators preserve proper Young walls. We need to verify whether they preserve reduced proper Young walls. We can think of two cases:

(i) adding a block may create a removable $\delta$,
(ii) removing a block may make an existing $\delta$ removable.

Suppose we had added a 0-block to create a removable $\delta$. The rules for the Kashiwara operators show that the column standing to the left of the created $\delta$ cannot be 0-admissible. So the following are essentially all possible ways the top parts of the two columns may stand after the addition of the 0-block.

In all cases, the column containing $\delta$ already contained a removable $\delta$ before the addition of the 0-block, contrary to our assumption. We may proceed similarly with other types of blocks.

Similarly, consider the case of removing an $i$-block to make an existing $\delta$ removable. The column standing to the right of the removed block must contain a removable $\delta$. The rules for the Kashiwara operators show that the $\delta$ column cannot be $i$-removable. Again, we consider all possible cases subject to these two conditions only to find that all were $\delta$-removable before the block was taken away.

Therefore we have defined a crystal structure on the set $\mathcal{Y}(\Lambda_i)$ of all reduced proper Young walls. We now state the main result of this paper: the realization of the crystal graph $B(\Lambda_i)$ in terms of reduced proper Young walls.

**Theorem 7.3.** For each $i = 0, 1, 2$, we have the isomorphism of crystals

$$\mathcal{Y}(\Lambda_i) \cong B(\Lambda_i).$$

**Proof.** We will prove this by giving a crystal isomorphism between $\mathcal{Y}(\Lambda_i)$ and $\mathcal{P}(\Lambda_i)$. Theorem 5.1 tells us that this is enough. To each element of $\mathcal{Y}(\Lambda_i)$, we map an element of $\mathcal{P}(\Lambda_i)$ by reading off just the top unit cube of each column and sending
It is clear that this map sends the ground-state walls to the appropriate ground-state paths and that the image does indeed lie in the set of $\Lambda_i$-paths. It is easy to see that this map is surjective. Injectivity follows from the fact that the set $\mathcal{Y}(\Lambda_i)$ consists of reduced proper Young walls. It remains to show that this map commutes with the action of Kashiwara operators.

To do this, we first go through each possible case and check that the rules for finding out which column to act on is the same as the corresponding rules for the paths. Let us try just one part of the $i = 0$ case. Consider the cube $\square_2$. If it is a 0-admissible slot, we would place a 0 under the column, which is exactly what we would do with the corresponding element $\begin{array}{c} \overset{0}{\square}_2 \\ \overset{0}{\square} \end{array}$. Suppose that it is not a 0-admissible slot. Then, the column to the right of the cube in consideration has to be a full column of the same height. The top cube will be $\begin{array}{c} \overset{2}{\square} \\ \overset{0}{\square} \end{array}$. This column is neither 0-removable nor 0-admissible. So under the two columns, we would place nothing. The corresponding two elements of $\mathcal{B}$ are $\begin{array}{c} \overset{1}{\square}_2 \\ \overset{2}{\square} \end{array}$, under which we would place a 0 and a 1, respectively. These cancel out when removing the 0 1 pair to give nothing. So the two rules agree. We could do the same work with other unit cubes. The $i = 1$ case is somewhat more tedious, but still possible.

Knowing that the rules for finding the column or element to act on are the same, it suffices to check that the addition or removal of blocks match up well with the action on the perfect crystal $\mathcal{B}$. For example, adding a 2-block to $\overset{0}{\square}$ makes it into $\begin{array}{c} \overset{2}{\square} \\ \overset{0}{\square} \end{array}$. This is in good correspondence with $\begin{array}{c} \overset{2}{\square} \\ \overset{0}{\square} \end{array}$. This completes the proof.

In the next examples, we redraw the crystal graphs $B(\Lambda_0)$ and $B(\Lambda_1)$ given in Example 5.2 and Example 5.3, this time, in terms of reduced proper Young walls.
Example 7.4. Crystal graph of $B(\Lambda_0)$.

Example 7.5. Crystal graph of $B(\Lambda_1)$. 
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Department of Mathematics, Seoul National University, Seoul 151-742, Korea
E-mail address: jhong@math.snu.ac.kr, sjkang@math.snu.ac.kr