Refined asymptotics for Landau-de Gennes minimizers on planar domains

Dmitry Golovaty¹ · Jose Alberto Montero²

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Abstract
In our previous work Golovaty and Montero (Arch Ration Mech Anal 213(2):447–490, 2014), we studied asymptotic behavior of minimizers of the Landau-de Gennes energy functional on planar domains as the nematic correlation length converges to zero. Here we improve upon those results, in particular by sharpening the description of the limiting map of the minimizers. We also provide an expression for the energy valid for a small, but fixed value of the nematic correlation length.

Mathematics Subject Classification
35Q92 · 35J50

In this paper we revisit some of the conclusions we obtained in [13]. In that paper we considered the Landau-de Gennes energy functional, which can be expressed as

$$E_\varepsilon(u) = \int_\Omega \left( \frac{|\nabla u|^2}{2} + \frac{W_\beta(u)}{\varepsilon^2} \right).$$  \hspace{1cm} (0.1)

Here $\Omega \subset \mathbb{R}^2$ is a bounded, smooth, simply-connected open subset of the plane and $\varepsilon > 0$ is a small parameter known as the nematic correlation length. In [13] we considered the functional $E_\varepsilon$ among maps $u \in W^{1,2}(\Omega, M^{3,1}_s(\mathbb{R}))$, where $M^{3,1}_s(\mathbb{R})$ denotes the set of symmetric, $3 \times 3$ matrices with real entries and trace equal to 1. The potential $W_\beta$ can be expressed as

$$W_\beta(u) = \frac{(1 - |u|^2)^2}{4} - \beta \det(u),$$

where $1 \leq \beta < 3$; here and throughout the paper, for two matrices $A, B$ of the same size, we consider the inner product $\langle A, B \rangle = \text{tr}(B^T A)$, along with its induced norm $|A|^2 = \langle A, A \rangle$.
For $\beta \in [1, 3]$ the potential $W_\beta$ is minimized [13] by the elements of the set

$$\mathcal{P} = \{ P \in M^{3,1}_{s,1} : P^2 = P \}$$

of $3 \times 3$, rank-1, orthogonal projection matrices.

Note that, although our expression (0.1) for the Landau-de Gennes energy functional differs from the standard form considered in the physics literature, the corresponding variational formulations are in fact equivalent. Indeed, an easy calculation shows that one recovers the usual Landau-de Gennes energy density for a traceless $Q$-tensor by setting $Q = s (u - \frac{1}{3} I)$ for some constant $s \in \mathbb{R}$.

Our aim in [13] was to study the global minimizers of $E_\varepsilon$, in the limit $\varepsilon \to 0$, among maps $u \in W^{1,2}(\Omega, M^{3,1}_{s,1}(\mathbb{R}))$ that satisfy the boundary condition $u = u_b$ on $\partial \Omega$. A crucial hypothesis in [13] was that $u_b : \partial \Omega \to \mathcal{P}$ represents a non-contractible curve in $\mathcal{P}$.

In the Appendix 5 we give a more detailed description of the results in [13], but roughly speaking, if $u_\varepsilon$ denotes a global minimizer of $E_\varepsilon$ under the conditions we just described, the results of [13] show that, along subsequences denoted by $\varepsilon_n \to 0$, there is a single interior point $a \in \Omega$ such that $u_{\varepsilon_n} \to u$ in $W^{1,2}(\Omega \setminus \{ a \})$ for any fixed $r > 0$, where $u : \Omega \to \mathcal{P}$ is a projection-valued map. Furthermore, the limit map $u$ locally minimizes the Dirichlet integral in $\Omega \setminus \{ a \}$, so it is a $\mathcal{P}$-valued harmonic map. Finally, if we write $[A, B] = AB - BA$ for the commutator of the matrices $A$ and $B$, we showed that the current vector of $u$, defined by

$$j(u) = [u, \nabla u] := \left( u, \frac{\partial u}{\partial x}, u, \frac{\partial u}{\partial y} \right)$$

splits as a sum of a meromorphic function with an explicit singular part, plus a map in $W^{1,2}$.

We have a few reasons to consider this current vector. First, in the study of harmonic maps $p : \Omega \to S^{n-1}$ with values in the unit sphere $S^{n-1} \subset \mathbb{R}^n$, the vector field $j = (j_1, j_2)$ with coordinates

$$j_k = p \times \frac{\partial p}{\partial x_k}$$

is very convenient because it allows to rewrite a corresponding PDE in a linear, divergence-free form (see for instance [14], Section 1.3.1). Similarly, for complex-valued maps in the context of the Ginzburg-Landau energy from superconductivity, say $v : \Omega \to \mathbb{C}$, the current vector defined as

$$j(v) = \text{Im}(\bar{u} \nabla u),$$

and in particular its distributional curl, has been widely employed in the analysis of singularities for the so-called extreme type II superconductors, as can be seen for example in [23] and [1].

To illustrate the utility of (0.2) in the present case, let

$$u_0 = \frac{1}{2} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) + \left( \begin{array}{ccc} \cos \theta & \sin \theta & 0 \\ -\sin \theta & -\cos \theta & 0 \\ 0 & 0 & 0 \end{array} \right),$$

where $\theta$ is the standard polar angle centered at some $a \in \Omega$. It is easy to check that this is a projection-valued map, and we will refer to this map as a canonical flat map. Denoting by
\( \hat{\theta} = (-\sin \theta, \cos \theta) \), a straightforward computation shows that

\[
j(u_0) = \frac{\hat{\theta}}{2r} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
suggesting that the current vector, as defined in (0.2), is a natural generalization of objects that have proven to be quite useful in previous studies.

The second reason to consider the current vector is that, for projection valued maps \( u : \Omega \to P \), the current vector \( j(u) \) can be thought of as a set of differential equations satisfied by \( u \). Indeed, for projection valued maps, we have the identity

\[
\nabla u = [u, j(u)],
\]
which can be verified by differentiating the equation \( u^2 = u \)—which by definition holds for projection valued maps—and inserting the result into the commutator \([u, j(u)]\). We interpret this as saying that the current vector \( j(u) \) contains all the information one needs to reconstruct the map \( u \) as can be shown by elementary means for the canonical flat map defined in (0.3).

The present paper will show that this intuition is largely true not only for canonical flat maps, but also for limits of minimizers of the Landau-de Gennes energy where much information about the map \( u \) can be obtained through the analysis of the current vector \( j(u) \).

As in [13], in this paper we consider the minimizers of (0.1) among all maps \( u \in W^{1,2}(\Omega, M^3_{S, 1}) \) which satisfy the condition \( u = u_b \) on \( \partial\Omega \), where

\[
u_b : \partial\Omega \to P
\]
is a fixed, non-contractible curve in \( P \). We improve upon the results from [13] in two ways. First, we find a more detailed description for the limits \( u \) of global minimizers \( u_\varepsilon \) of \( E_\varepsilon \).

Second, we use this refined description to provide an expansion of the energy \( E_\varepsilon (u_\varepsilon) \) of global minimizers, valid for small, but fixed, \( \varepsilon > 0 \).

The first result giving a better description of the limits of global minimizers of \( E_\varepsilon \) as \( \varepsilon \to 0 \) is summarized in the following proposition (here and elsewhere in the paper \( S^k \) denotes the unit sphere in \( \mathbb{R}^{k+1} \)):

**Proposition 0.1** Let \( a \in \Omega \) be the distinguished point given by Theorem 5.1 in the Appendix, \( \varepsilon_n \to 0 \), and \( u_{\varepsilon_n} \in W^{1,2}(\Omega, M^3_{S, 1}) \) be a sequence of global minimizers of \( E_{\varepsilon_n} \) such that \( u_{\varepsilon_n} \to u \) in \( W^{1,2}_{\text{loc}}(\Omega \setminus \{a\}, M^3_{S, 1}) \). There is a unit vector-valued map \( k \in W^{1,2}(\Omega, S^2) \) such that \( u(x) k(x) = 0 \) for all \( x \in \Omega \setminus \{a\} \).

Furthermore, if we define \( \gamma_r : S^1 \to P \) by

\[
\gamma_r(\omega) = u(a + r \omega),
\]
then there is a closed geodesic \( \gamma_0 : S^1 \to P \) such that \( \gamma_r \to \gamma_0 \) as \( r \to 0 \) in \( W^{1,2}(S^1, P) \).

**Remark 0.2** We emphasize that in the above proposition the convergence \( \gamma_r \to \gamma_0 \) in \( W^{1,2}(S^1, P) \) is for \( r \to 0 \), not along a particular sequence \( r_n \to 0 \). Results in this spirit appear in [13] and [26], but only for sequences \( r_n \to 0 \).

It is also worth mentioning that this proposition confirms the intuition that, while the first two eigenvectors of the limit map \( u \) are singular at \( a \in \Omega \), the third eigenvector should be smooth—although at this point we can only prove that it is in \( W^{1,2} \).

Our next result makes use of the Hopf differential of the map \( u \). Let us first recall its definition. For this we will need to switch to complex derivatives in the plane. We will use
the usual
\[ \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right). \]
We will also denote
\[ j_C(u) = \left[ u, \frac{\partial u}{\partial \bar{z}} \right] \quad (0.4) \]
for the complex-valued current vector. We have the relation
\[ j(u) = 2 \text{Re} \left( j_C(u) \right) e_1 - 2 \text{Im} \left( j_C(u) \right) e_2, \]
where \( \text{Re}(z), \text{Im}(z) \) denote the real and imaginary parts of \( z \), respectively.

**Definition 0.3** For our limit of minimizers \( u \), its Hopf differential is the function
\[ \omega_u(z) = \text{tr} \left( \left( j_C(u) \right)^2 \right). \]
By the properties of projection-valued maps, the Hopf differential can also be defined by
\[ \omega_u(z) = -\text{tr} \left( \left( \frac{\partial u}{\partial \bar{z}} \right)^2 \right) \]
Here \( \text{tr}(A) \) denotes the trace of the matrix \( A \), and \( A^2 \) is the standard product of the matrix \( A \) with itself.

**Theorem 0.4** Let \( \omega_u \) denote the Hopf differential of the map \( u \), which we assume to be a limit of global minimizers of \( E_\varepsilon \). We have
\[ \omega_u(z) = -\frac{1}{8(z - a)^2} + h(z), \]
where \( h \) is a holomorphic map in all of \( \Omega \).
Let now \( Z_{\omega_u} \) denote the set of zeros of \( \omega_u \) in \( \Omega \). Under the hypothesis that \( Z_{\omega_u} = \emptyset \), there are
(1) a fixed orthogonal basis \( \Lambda_1, \Lambda_2, \Lambda_3 \) of the set \( M_3^3(\mathbb{R}) \) of \( 3 \times 3 \) anti-symmetric matrices that satisfies
\[ \{\Lambda_1, \Lambda_2\} = \Lambda_3, \quad \Lambda_3^j = -\Lambda_j, \quad j = 1, 2, 3, \]
(2) a real-valued function \( g : \Omega \setminus \{a\} \to \mathbb{R} \), defined up to a sign,
(3) a fixed projection \( P \in \mathcal{P} \) and
(4) a multi-valued map \( S : \Omega \setminus \{a\} \to O(3) \)
such that
\[ u = SPS^T \quad \text{and} \quad j_C(u) = \mu_u S(\cosh(g) \Lambda_1 + i \sinh(g) \Lambda_2)S^T, \]
where \( -2\mu_u^2 = \omega_u. \) Furthermore, letting \( \Gamma_j = S\Lambda_j S^T, \quad j = 1, 2, 3, \) the function \( g \) satisfies
\[ -\frac{i}{4}(\Delta g) \Gamma_3 = \frac{i |\omega_u|}{4} \sinh(2g) \Gamma_3 = \frac{1}{2} \left[ j_C(u), j_C(u) \right] \text{ locally in } \Omega. \quad (0.5) \]
When \( Z_{\omega_u} \neq \emptyset \), this set is discrete in \( \Omega \) and all conclusions of the theorem remain valid locally away from \( Z_{\omega_u} \). This is due to the fact that the equation \( -2\mu_u^2 = \omega_u \) is no longer valid globally in \( \Omega \setminus \{a\} \).
Finally, regardless of the nature of \( Z_{\omega_u} \), we also have
\[ \int_\Omega |\omega_u| \sinh^2(g) < +\infty. \]
Before we state our next results, several comments are in order.

Intuitively, in this work a map is considered to be multi-valued if, while we follow its values by traveling once along a curve that encircles a distinguished point, we come back to the starting point with a different value than the one that we started with. This is what happens, for example, with the map $S : \Omega \rightarrow O(3)$ mentioned in the previous theorem. One can think of this map $S$ as a lift of $u$ through $O(3)$. Using the properties of $P$-valued harmonic maps, we will prove that, in any small $V \subset \Omega \setminus \{a\}$ where we can take $S : V \rightarrow O(3)$ to be single-valued, we have

$$\frac{\partial S^T}{\partial z} = -i \frac{\partial g}{\partial z} \Lambda_3 + \mu_u (\cosh(g) \Lambda_1 + i \sinh(g) \Lambda_2),$$

where $g$ and $\Lambda_j$, $j = 1, 2, 3$ are as in Theorem 0.4. This last equation can be thought of as a set of over-determined differential equations satisfied by $S$. Then, the equation

$$-\Delta g = |\omega_u| \sinh(2g)$$

is the compatibility condition for the over-determined equations satisfied by $S$.

Regarding the real-valued function $g$ which appears in the last theorem, we stated that this function is only defined up to a sign. By this we mean that replacing $g$ by $-g$ does not affect the validity of the results. However, Theorem 0.4 establishes that $g$ satisfies the equations (0.5). Since the right-most term in (0.5) depends only on the limit map $u$, we conclude that, while the function $g$ is only defined up to a sign, the product $\sinh(2g) \Gamma_3$, or equivalently, the product $g \Gamma_3$, is well-defined in all of $\Omega \setminus \{a\}$.

It is useful to compare Theorem 0.4 with the corresponding statement for the non-magnetic Ginzburg-Landau energy studied in [3]. In the Ginzburg-Landau case, the role played in this work by the manifold $P$, is taken by the unit circle $S^1$. Among many others, an obvious difference between $S^1$ and $P$ is that $S^1$ is one-dimensional, whereas $P$ has dimension 2. In particular, the images of maps from $\Omega \subset \mathbb{C}$ into $S^1$ can never have positive 2-dimensional area, whereas the images of maps $u : \Omega \setminus \{a\} \rightarrow P$ considered in Theorem 0.4 can. For such maps, the 2-dimensional area content of their images is completely determined by the real valued function $g$. Indeed, looking at the equations (0.5), we notice in particular that $g = 0$ identically if and only if $[\frac{\partial u}{\partial z}, \frac{\partial u}{\partial \bar{z}}] = -[\gamma_{\mathbb{C}}(u), j_\mathbb{C}(u)] = 0$ in all of $\Omega$. Thinking of $u : \Omega \setminus \{a\} \rightarrow P$ locally as a parameterization of a portion of $P$, then $[\gamma_{\mathbb{C}}(u), j_\mathbb{C}(u)]$ gives the area factor of this parameterization. Therefore $g$ vanishes identically if and only if the image of $u$ has zero 2-dimensional area. Our results show that in this case the image of $u$ is contained in a closed geodesic of $P$, and the map $u$ has the same structure that the canonical harmonic maps of [3]. In other words, Theorem 0.4 shows that many of the features of what is called canonical harmonic maps in [3] are also present in the limit maps studied here. It also shows that important differences between these two situations boil down to the function $g$, which is driven by the area content of our limit maps.

Regarding the set $Z_{\omega}$ of zeros of the Hopf differential $\omega_u$, in principle we view it as a possible obstruction to obtain a square root of $\omega_u$ defined in the whole of $\Omega \setminus \{a\}$. However, it is not hard to see that, for a limit map in Theorem 0.4, there is a small enough radius $r > 0$ such that the Hopf differential does have well-defined square root in $B_r(a) \setminus \{a\}$. Furthermore, if we consider a projection-valued map of the form (0.3), a direct computation shows that the Hopf differential has the form

$$\omega_v = -\frac{1}{4} \left( \frac{\partial \theta}{\partial z} \right)^2.$$
These facts lead us to believe that, in our situation, the Hopf differential of any limit map considered in Theorem 0.4 has a square root defined globally in \( \Omega \setminus \{a\} \). In other words, the set \( Z_{\omega u} \) is not an obstruction to the existence of a square root of the Hopf differential. However, as of now we do not have a proof for this fact.

We should also point out that the properties of the Hopf differential, particularly Proposition 1.9, were pointed out in [27]. For the sake of completeness, we provide a proof of this, which follows closely that of [3].

Having formulated the results concerning the limit map \( u \), we state now an expansion of the energy valid for a family \( u_{\varepsilon n} \) of converging global minimizers of \( E_{\varepsilon n} \).

**Theorem 0.5** Let \( \varepsilon_n \to 0 \), let \( u_n \in W^{1,2}(\Omega, M_{3,1}^3(\mathbb{R})) \) be a global minimizer of \( E_{\varepsilon_n} \), and assume \( u_n \to u \) in \( W^{1,2}(\Omega \setminus B_r(a), M_{3,1}^3(\mathbb{R})) \) for every fixed \( r > 0 \). Let also \( \omega_n \) be as in Eq. (0.3), and let \( g : \Omega \setminus \{a\} \to \mathbb{R} \) be the function described in Theorem 0.4. We have the expansion

\[
\int_{\Omega} e_{\varepsilon_n}(u_n) = I(r, \varepsilon_n) + 2 \int_{\Omega \setminus B_r(a)} |\omega_n| + 2 \int_{\Omega \setminus B_r(a)} |\omega_n| \sinh^2(g) + o(1) + q(r). \tag{0.6}
\]

Here,

\[
I(r, \varepsilon) = \inf \{ E_{\varepsilon}(w) : w \in W^{1,2}(B_r(a), M_{3,1}^3(\mathbb{R})), w \text{ is canonical flat on } \partial B_r(a) \},
\]

where \( E_{\varepsilon}(u) \) is defined in (0.1), \( M_{3,1}^3(\mathbb{R}) \) is the set of \( 3 \times 3 \) symmetric matrices with trace 1 and real entries, and a canonical flat map is as defined in Eq. (0.3). Further, \( o(1) \) represents a quantity that goes to 0 as \( n \to \infty \), and \( q(r) \) represents a quantity that is independent of \( \varepsilon > 0 \) and such that \( q(r) \to 0 \) as \( r \to 0 \).

**Remark 0.6** Regarding the terms on the right hand side of Eq. (0.6), \( I(\varepsilon, r) \) is clearly independent of the limiting map \( u \), and can be thought of as a fixed cost associated with the singularity at the point \( a \in \Omega \). The second term in (0.6) originates from the fact that our boundary data in general is not canonical flat. Both of these terms do not depend on the surface area of the image of \( u \), and are exactly what we would expect for complex-valued maps if the analysis were carried out in the framework of [3]. The third term, however, is completely driven by the 2-dimensional area content of the image of \( u \), through its dependence on the function \( g \).

**Proof of Theorem 0.5.** We split

\[
\int_{\Omega} e_{\varepsilon_n}(u_n) = \int_{B_r(a)} e_{\varepsilon_n}(u_n) + \int_{\Omega \setminus B_r(a)} e_{\varepsilon_n}(u_n).
\]

The estimate of the difference

\[
\int_{B_r(a)} e_{\varepsilon_n}(u_n) - I(r, \varepsilon_n)
\]

is contained in Theorem 2.1.

Next, by the results in the appendix we have

\[
\int_{\Omega \setminus B_r(a)} e_{\varepsilon_n}(u_n) = \int_{\Omega \setminus B_r(a)} \frac{\|\nabla u\|^2}{2} + o(1).
\]

The results in Theorem 0.4 show that

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\[
\int_{\Omega \setminus B_r(a)} \frac{|\nabla u|^2}{2} = 2 \int_{\Omega \setminus B_r(a)} |\omega_u| + 2 \int_{\Omega \setminus B_r(a)} |\omega_u| \sinh^2(g).
\]

In the reference [13] we showed that the current vector of a limit map \( u \) has the expression
\[
j(u) = \nabla \perp \left( \frac{1}{2} \ln \left( \frac{1}{|z - a|} \right) \Lambda + \phi(z) \right),
\]
where \( \Lambda \) is a constant, \( 3 \times 3 \) anti-symmetric matrix normalised so that \( \Lambda^3 = -\Lambda \), and \( \phi \in (W^{1,2} \cap L^\infty)(\Omega, M^3_a(\mathbb{R})) \). So far we have been unable to show that the map \( \phi \) is smooth. However, if we assume this, we can give another expression for the energy of global minimizers that converge in \( W^{1,2}_{loc}(\Omega \setminus \{a\}) \) to a projection-valued map \( u \). This is the content of our next theorem. To state it, we assume the current vector of \( u \) can be written in the form (0.8), where \( \phi \in W^{1,\infty}(\Omega, M^3_a(\mathbb{R})) \). Note that by adding and subtracting the regular part of the Green’s function for \( \Omega \) we can write
\[
j(u) = \nabla \perp (\pi G(x, a) \Lambda + \phi_1),
\]
where \( G(x, y) \) is the Green’s function for \( \Omega \), and we then re-state our hypothesis as \( \phi_1 \in W^{1,\infty}(\Omega) \). We now present our last theorem.

**Theorem 0.7** Let \( \varepsilon_n \to 0 \), let \( u_n \in W^{1,2}(\Omega, M^3_{s,1}(\mathbb{R})) \) be a minimizer of \( E_{\varepsilon_n} \), and assume \( u_n \to u \) in \( W^{1,2}(\Omega \setminus B_r(a), M^3_{s,1}(\mathbb{R})) \) for every fixed \( r > 0 \). With the notation above we have
\[
\int_{\Omega} e_{\varepsilon_n}(u_n) = I(r, \varepsilon_n) + \frac{\pi}{2} \ln \left( \frac{1}{r} \right) + \frac{R(a, a)}{2} + \int_{\Omega} G(x, a) \left( \Lambda, \left[ \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \right] \right) dx + \frac{1}{2} \int_{\Omega} |\nabla \phi_1|^2 + q(r) + o(1).
\]
Here, as in Theorem 0.5, \( I(r, \varepsilon) \) is as defined in Eq. (0.7), \( o(1) \) represents a quantity that goes to 0 as \( n \to \infty \), and \( q(r) \) is independent of \( \varepsilon > 0 \) and such that \( q(r) \to 0 \) as \( r \to 0 \). Finally,
\[
R(x, y) = G(x, y) - \frac{1}{2\pi} \ln \left( \frac{1}{|x - y|} \right)
\]
is the regular part of the Green’s function for \( \Omega \).

Many authors have studied the Landau-de Gennes energy (0.1) in the last decade, particularly in the limit limit as \( \varepsilon \to 0 \). The authors of [15, 16] and [25] all provide descriptions of the global minimizers of \( E_{\varepsilon} \) in the limit \( \varepsilon \to 0 \), in a 3-dimensional domain. Several other problems related to (0.1) in 3-d domains have been studied, from homogenization via \( \Gamma \)-convergence, to stability of particular solutions, to the appearance of line defects in the minimizers in the limit \( \varepsilon \to 0 \), in [5–7, 19] and [28]. In 2 dimensions, [9, 10, 20–22, 24], among other results, prove existence and multiplicity of symmetric solutions under appropriate boundary conditions, and study stability of point defects.

The study [4] is perhaps the closest to the issues considered in the present work. In [4] the author considers a family of energy functionals that contain (0.1) as a particular case, and establishes convergence of minimizers in the \( \varepsilon \to 0 \) limit, among other results.

Also related to our work is that of [26] and [27]. There, the authors consider an energy that is significantly more general than \( E_{\varepsilon} \) from (0.1). They analyze singular limit \( \varepsilon \to 0 \), find
a $\Gamma$-limit for this energy, and obtain an energy for the location of the singularities that appear in minimizers of $E_\varepsilon$ as $\varepsilon \to 0$. Their results apply to a wide range of manifolds, that include $\mathcal{P}$ as a particular case. Because of this generality, however, their results for the $\Gamma$-limit of $E_\varepsilon$ are rather implicit, and can be made explicit only for very special boundary conditions.

Some of the tools used in [9] are similar to ours, but this work deals with a completely different regime. In particular, in this paper the authors consider families of functions $u_\varepsilon \in W^{1,2}(\Omega, M_3^2(\mathbb{R}))$ such that $E_\varepsilon(u_\varepsilon) \leq C$ as $\varepsilon \to 0$, for some constant $C > 0$ independent of $\varepsilon$. Another important difference with our work is that, throughout [9] the authors assume that their boundary data $u : \partial\Omega \to \mathcal{P}$ satisfies $ue = 0$, where $e \in \mathbb{S}^2$ is a fixed unit vector. They also assume that their boundary condition $u : \partial\Omega \to \mathcal{P}$ can be lifted through a smooth $n : \partial\Omega \to \mathbb{S}^2$, in the sense that

$$u = nn^T \text{ on } \partial\Omega.$$  

We do not assume either of these hypotheses in our work.

Perhaps the main contributions of our paper are the proofs of the results we present here. A first crucial fact we appeal to is that, for projection-valued maps $u : \Omega \to \mathcal{P}$, the current vector

$$j_C(u) = \left[ u, \frac{\partial u}{\partial z} \right]$$

can be thought of as a set of differential equations satisfied by $u$. Concretely, for the projection-valued map $u$ we have

$$\frac{\partial u}{\partial z} = [u, j_C(u)],$$

and this equation holds pointwise in $\Omega$. This fact, along with the decomposition (0.8) of $j_C(u)$ we found in [13], and arguments from [3], allow us to derive several of our conclusions.

A second important fact we use is that there is an integrable system that appears naturally in the study of projection valued maps that arise as limits of minimizers of $E_\varepsilon$. Indeed, for such a projection-valued map we have

$$\frac{\partial}{\partial z} j_C(u) = -[j_C(u), j_C(u)].$$ (0.9)

Lifting $u$ locally through a map $S : \Omega \to O(3)$, in the sense that $u = SPS^T$ for a fixed $P \in \mathcal{P}$, we can re-write the current vector of the limiting map $u$ as

$$j_C(u) = S \eta S^T.$$  

Here, $\eta$ is an anti-symmetric matrix valued map, and we will prove in Sect. 3 that, in appropriate coordinates, $\eta$ satisfies the equation

$$\frac{\partial \eta}{\partial z} = -i \frac{\partial g}{\partial z} [\Lambda, \eta],$$ (0.10)

where $\Lambda$ is a constant, $3 \times 3$ anti-symmetric matrix, and $g$ is the real-valued function from Theorem 0.4. This system can be solved explicitly for $\eta$ in terms of $g$, which is why we call (0.10) an integrable system (although there does not seem to be a universally accepted definition of an integrable system; see for instance the first few lines of the Introduction in [17]). Now, we can use Eq. (0.10) to derive a differential system for $S$ that gives us Theorem 0.4. This in turn allows us to derive Theorem 0.5.
Equation (0.10) is well-known to geometers (see for instance chapters 3 and 4 of [29]), and has been used in the study of harmonic maps into more general manifolds than \( P \), for instance in [11]. To the best of our knowledge, however, Eq. (0.10) has not been used to this point in the Landau-de Gennes literature.

We believe that our methods, while currently restricted to the manifold \( P \), should lead to more explicit expressions, particularly for the energy for the location of singularities, than those currently available. It is also worth mentioning that our methods seem to provide a natural generalization to the complex-valued methods used in [3].

In the remainder of the paper we provide the proofs of the results we have just described. Section 2 contains the analysis of the limiting map \( u \). In Sect. 3 we analyze the energy of a sequence global minimizers \( u_{\varepsilon_n}, \varepsilon_n \to 0 \), near the singular point \( a \in \Omega \). In Sect. 4 we prove Theorem 0.4. In Sect. 5, we use numerical simulations to illustrate our results. Finally, the Appendix contains details of results from [13] and elsewhere that we need in this work.

1 The limiting map

Throughout this section \( u : \Omega \setminus \{a\} \to \mathcal{P} \) will denote a limit of minimizers of the Landau-de Gennes energy, and \( a \in \Omega \) will be the unique singularity of \( u \) in \( \Omega \). We will assume throughout that \( a = 0 \in \mathbb{R}^2 \cong \mathbb{C} \). Let us recall here that the harmonic map equation for the \( \mathcal{P} \)-valued map \( u \) is

\[
-\Delta u = 2 \left( |\nabla u|^2 u - \left( \frac{\partial u}{\partial x} \right)^2 - \left( \frac{\partial u}{\partial y} \right)^2 \right),
\]

where \( A^2 = AA \) for any square matrix \( A \). One way to obtain this equation is to observe that the map \( u \) can be locally lifted through a unit-vector valued \( p \), in the sense that we can write

\[
u = pp^T,
\]

and then observe that in this lifting \( u \) is harmonic if and only if \( p \) is. Then, using the well-known harmonic map equation for \( p \), one obtains the equation for \( u \) above. From this we also obtain that the right hand side of harmonic map equation for \( u \) commutes with \( u \). Hence, we have

\[ [u, \Delta u] = 0. \]

We know from [13] (see Theorem 5.2 in the Appendix) that

\[
j(u) = \frac{\hat{\theta}}{2r} \Lambda + \nabla^\perp \phi.
\]

Here \( r = |x - a| \) is the distance to the singularity, \( \hat{\theta} \) is the standard unit vector from polar coordinates centered at \( a = 0 \in \mathbb{R}^2 \), \( \phi \in (L^\infty \cap W^{1,2})(\Omega, M_3^3(\mathbb{R})) \), and \( \Lambda \in M_3^3(\mathbb{R}) \) is a constant anti-symmetric matrix, the representative of a closed geodesic in \( \mathcal{P} \) in the language of [13]. By selecting appropriate coordinates in \( \mathcal{P} \) we can choose

\[
\Lambda = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
Let us consider now the domain

\[ U = \{ \xi \in \mathbb{C} : e^\xi \in \Omega \}. \]

The set \( U \) is the preimage of \( \Omega \setminus \{ a \} \) by the exponential map (recall we assume \( a = 0 \)). Defining \( v : U \to \mathcal{P} \) and \( \psi : U \to M_3^a(\mathbb{R}) \) by

\[ v(\xi) = u(e^\xi) \quad \text{and} \quad \psi(\xi) = \phi(e^\xi), \]

we see that both \( v(\xi) = v(\xi + 2\pi i) \) and \( \psi(\xi) = \psi(\xi + 2\pi i) \) whenever \( e^\xi \in \Omega \), that is, both \( v \) and \( \psi \) are \( 2\pi i \)-periodic. Denote now

\[ H = \{ \xi \in U : -\pi < \text{Im}(\xi) < \pi \}, \]

and

\[ H_\lambda = \{ \xi \in H : \text{Re}(\xi) \leq \lambda \}, \]

for \( \lambda \in \mathbb{R} \). We observe that, since \( \phi \in (W^{1,2} \cap L^\infty)(\Omega, M^3_a(\mathbb{R})) \), then \( \psi \in L^\infty(H, M^3_a(\mathbb{R})) \) and \( \nabla \psi \in L^2(H, M^3_a(\mathbb{R})) \). We will now prove the following

**Proposition 1.1** The map \( \psi : U \to M^3_a(\mathbb{R}) \) satisfies \( \nabla \psi \in (W^{1,2} \cap L^\infty)(H, M^3_a(\mathbb{R})) \), and

\[ \nabla \psi(\xi_1, \xi_2) \to 0 \quad \text{as} \quad \xi_1 \to -\infty, \]

uniformly in \( \xi_2 \).

**Remark 1.2** Notice that this proposition, along with the relation \( \psi(\xi) = \phi(e^\xi) \), allow us to conclude that

\[ \lim_{r \to 0} \left( r \sup_{x \in \partial B_r(a)} |\nabla \phi(x)| \right) = 0. \]

**Proof of Proposition 1.1** The proof follows an argument that appears in [2].

**Step 1.** The map \( \psi \) satisfies the inequality

\[ -\Delta (|\nabla \psi|^2) + |D^2 \psi|^2 \leq C(|\nabla \psi|^2 + |\nabla \psi|^4) \]

for some constant \( C > 0 \).

**Proof of Step 1.** Direct computations show that

\[ j(v) = [v, \nabla v] = \frac{e_2}{2} \Lambda + \nabla^\perp \psi, \quad \text{for} \quad e_2 = (0, 1). \]

where \( e_2 = (0, 1) \) is the second vector of the canonical basis in \( \mathbb{R}^2 \). Now \( v \) is a projection-valued map. This has the consequence that

\[ \begin{bmatrix} \frac{\partial v}{\partial \xi_1} & \frac{\partial v}{\partial \xi_2} \end{bmatrix} = -2 \begin{bmatrix} \frac{\partial v}{\partial \xi_1}, \frac{\partial v}{\partial \xi_2} \end{bmatrix}, \]

and from here we deduce

\[ -\Delta \psi = \nabla^\perp \cdot j(v) = 2 \begin{bmatrix} \frac{\partial v}{\partial \xi_1}, \frac{\partial v}{\partial \xi_2} \end{bmatrix} = -2 \begin{bmatrix} \frac{\partial v}{\partial \xi_1}, \frac{\partial v}{\partial \xi_2} \end{bmatrix}. \]

From the Eq. (1.1) we obtain

\[ \Delta \psi = 2 \begin{bmatrix} \frac{\partial \psi}{\partial \xi_2}, \\frac{\Lambda}{2} - \frac{\partial \psi}{\partial \xi_1} \end{bmatrix} = 2 \begin{bmatrix} \frac{\partial \psi}{\partial \xi_1}, \frac{\partial \psi}{\partial \xi_2} \end{bmatrix} + \begin{bmatrix} \frac{\partial \psi}{\partial \xi_2}, \Lambda \end{bmatrix}. \]
This leads to the identity

\[-\Delta \left( \frac{|\nabla \psi|^2}{2} \right) + |D^2 \psi|^2 = 2 \left( \left[ \frac{\partial \psi}{\partial \xi_1}, \frac{\partial \psi}{\partial \xi_2} \right], \Delta \psi \right) + \left( \Lambda, \left[ \frac{\partial \psi}{\partial \xi_1}, \frac{\partial^2 \psi}{\partial \xi_1 \partial \xi_2} \right] + \left[ \frac{\partial \psi}{\partial \xi_2}, \frac{\partial^2 \psi}{\partial \xi_2^2} \right] \right) \right]. \tag{1.2}\]

We now observe that

\[\left( \left[ \frac{\partial \psi}{\partial \xi_1}, \frac{\partial \psi}{\partial \xi_2} \right], \Delta \psi \right) \leq \left\| \left[ \frac{\partial \psi}{\partial \xi_1}, \frac{\partial \psi}{\partial \xi_2} \right] \right\|_2 |\Delta \psi| \leq C(\delta) |\nabla \psi|^4 + \delta |D^2 \psi|^2.\]

Similarly,

\[\left( \Lambda, \left[ \frac{\partial \psi}{\partial \xi_1}, \frac{\partial^2 \psi}{\partial \xi_1 \partial \xi_2} \right] + \left[ \frac{\partial \psi}{\partial \xi_2}, \frac{\partial^2 \psi}{\partial \xi_2^2} \right] \right) \leq C |\nabla \psi| |D^2 \psi| \leq C(\delta) |\nabla \psi|^2 + \delta |D^2 \psi|^2.\]

For \(\delta > 0\) small enough, we can absorb the terms \(\delta |D^2 \psi|^2\) in the term \(|D^2 \psi|^2\) on the left hand side of (1.2). The conclusion of Step 1 follows from here. \(\square\)

**Step 2.** \(\nabla \psi \in W^{1,2}(H, M_a^3(\mathbb{R})).\)

**Proof of Step 2.** Let \(M > L + 1 > 0\), and consider a cut-off function \(\chi_{L,M} \in C_0^\infty(\mathbb{C})\) such that \(0 \leq \chi_{L,M} \leq 1\) and

\[\chi_{L,M}(\xi) = \begin{cases} 1 & \text{for } \xi_1 \in [-M, -L] \text{ and } \xi_2 \in [-\pi, \pi] \\ 0 & \text{if } \xi_1 \not\in [-M - 1, -L + 1] \text{ or } \xi_2 \not\in [-2\pi, 2\pi]. \end{cases}\]

We choose \(M, L\) large enough so that the support \(\text{supp}(\chi_{L,M}) \subset U\). Note that the definition of \(\chi_{L,M}\) allows us to require that the derivatives of \(\chi_{L,M}\), up to order 2, be bounded uniformly in \(L, M > 0\). From Step 1 we obtain

\[\int_U \chi_{L,M}^2 |D^2 \psi|^2 \leq C \int_{\text{supp}(\chi_{L,M})} |\nabla \psi|^2 + C \int_U \chi_{L,M}^2 |\nabla \psi|^4. \tag{1.3}\]

We will estimate the last integral in this inequality using Gagliardo-Nirenberg. To this end, we first notice that

\[\int_{B_r(\alpha)} |\nabla \phi|^2 = \int_{H_{\ln(r)}} |\nabla \psi|^2,\]

where we recall that

\[H_{\ln(r)} = \{ \xi \in \mathbb{C} : e^\xi \in \Omega, \xi_1 \leq \ln(r), \xi_2 \in [-\pi, \pi] \}.\]

Since \(\phi \in W^{1,2}(\Omega)\), it holds

\[\lim_{r \to 0} \int_{B_r(\alpha)} |\nabla \phi|^2 = 0,\]

and hence

\[\lim_{L \to -\infty} \left( \lim_{M \to -\infty} \int_{\text{supp}(\chi_{L,M})} |\nabla \psi|^2 \right) = 0.\]
Next, we recall that Gagliardo-Nirenberg inequality establishes that
\[ \int_C f^2 \leq \left( \int_C |\nabla f| \right)^2 \]
for any function \( f \in C_0^1(C) \). We apply this estimate to \( f = \chi_{L,M} |\nabla \psi|^2 \). Clearly
\[ |\nabla \psi| \leq C(\|\text{supp}(\chi_{L,M})\| |\nabla \psi|^2 + \chi_{L,M} |\nabla \psi| |D^2 \psi|) \]
where \( 1_A \) denotes the characteristic function of the set \( A \). From here we obtain
\[ \int_U \chi_{L,M}^2 |\nabla \psi|^4 \leq C \left( \int_{\text{supp}(\chi_{L,M})} |\nabla \psi|^2 \right)^2 + C \int_{\text{supp}(\chi_{L,M})} |\nabla \psi|^2 \int_U \chi_{L,M}^2 |D^2 \psi|^2. \]

Since \( \lim_{L \to -\infty} \left( \lim_{M \to -\infty} \int_{\text{supp}(\chi_{L,M})} |\nabla \psi|^2 \right) = 0 \), choosing \( L > 0 \) large enough the last estimate and (1.3) yield
\[ \int_U \chi_{L,M}^2 |D^2 \psi|^2 \leq C \left( \int_{\text{supp}(\chi_{L,M})} |\nabla \psi|^2 \right)^2 + C \int_{\text{supp}(\chi_{L,M})} |\nabla \psi|^2. \quad (1.4) \]

Finally, we let \( M \to -\infty \) in (1.4), and recall that \( \psi \) is \( 2\pi i \)-periodic. We obtain
\[ \int_{H_{-L}} |D^2 \psi|^2 \leq C \left( \int_{H_{-L+1}} |\nabla \psi|^2 \right)^2 + C \int_{H_{-L+1}} |\nabla \psi|^2 \]
for some constant \( C > 0 \) that remains bounded as \( L \to -\infty \). Now, we know the map \( \phi \in W^{1,2}(\Omega, M^3_a(\mathbb{R})) \) is smooth away from \( a \in \Omega \). We conclude that \( D^2 \psi \in L^2(H, M^3_a(\mathbb{R})) \), which completes the proof of Step 2.

\( \square \)

**Step 3.** \( \nabla \psi(\xi_1, \xi_2) \to 0 \) as \( \xi_1 \to -\infty \), uniformly in \( \xi_2 \in \mathbb{R} \).

**Proof of Step 3.** To show this we first recall that in our last step we proved the estimate
\[ \int_{H_{-L}} |D^2 \psi|^2 \leq C \left( \int_{H_{-L+1}} |\nabla \psi|^2 \right)^2 + C \int_{H_{-L+1}} |\nabla \psi|^2. \]

We can use this inequality to conclude that
\[ \lim_{L \to -\infty} \int_{H_{-L}} |D^2 \psi|^2 = 0. \]

Now the map \( \psi \) is \( 2\pi i \)-periodic, which is to say that it is \( 2\pi \)-periodic in the variable \( \xi_2 \). Let \( \xi_0 = (\xi_{0,1}, \xi_{0,2}) \in H_{-\lambda} \) and \( R > 0 \) be such that \( B_{2R}(\xi_0) = B_{2R}((\xi_{0,1}, \xi_{0,2})) \subset H_{-\lambda} \). This implies that for a fixed \( R > 0 \), we have
\[ \lim_{\xi_{0,1} \to -\infty} \int_{B_{2R}((\xi_{0,1}, \xi_{0,2}))} \left( |D^2 \psi|^2 + |\nabla \psi|^2 \right) = 0. \]

By standard Sobolev embeddings, we conclude that
\[ \lim_{\xi_{0,1} \to -\infty} \int_{B_{2R}((\xi_{0,1}, \xi_{0,2}))} |\nabla \psi|^p = 0 \]
for any \( 1 < p < \infty \). Next we recall from Step 1 that
\[ -\Delta (|\nabla \psi|^2) + |D^2 \psi|^2 \leq C(|\nabla \psi|^2 + |\nabla \psi|^4) \]
for some constant \( C > 0 \). Fixing \( p > 2 \), Theorem 8.17 of [12] yields

\[
\sup_{B_R((\xi_0, 1, \xi_0, 2))} |\nabla \psi|^2 \leq C \int_{B_{2R}((\xi_0, 1, \xi_0, 2))} |\nabla \psi|^{2p} + C \int_{B_{2R}((\xi_0, 1, \xi_0, 2))} |\nabla \psi|^{4p},
\]

for some \( C > 0 \) that depends on \( R > 0 \) and \( p > 2 \). Since

\[
\lim_{\xi_0, 1 \to -\infty} \int_{B_{2R}((\xi_0, 1, \xi_0, 2))} |\nabla \psi|^p = 0
\]

for any \( 1 < p < \infty \), this proves Step 3, and concludes the proof of the proposition. \( \square \)

Now recall that

\[
\Lambda = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

and let \( u_0 \) be a canonical flat map as defined in (0.3). A direct computation shows that

\[
j(u_0) = [u_0, \nabla u_0] = \frac{\hat{\theta}}{2r} \Lambda,
\]

We will refer to \( u_0 \) as a canonical flat map \( u_0 : \Omega \setminus \{a\} \to \mathcal{P} \) represented by \( \Lambda \). This is the same as saying that the image of \( u_0 \) in \( \mathcal{P} \) is a closed geodesic. Observe that the image \( u_0(\partial B_r(a)) \) of every circle centered at \( a \) by \( u_0 \) is a closed geodesic in \( \mathcal{P} \).

Let us observe that the images of the matrices \( u_0(x), x \in \Omega \setminus \{a\} \), are all contained in a fixed plane, that we will denote by \( S_0 \). Now we state the following proposition.

**Proposition 1.3** With the notation above, we have

\[
\frac{1}{|x-a|} [\Lambda, [u, u_0]] \in L^2(\Omega, M^3_a(\mathbb{R})) \quad \text{and} \quad [u, u_0] \in W^{1,2}(\Omega, M^3_a(\mathbb{R})).
\]

**Proof** We start by recalling that

\[
\nabla u = [u, j(u)], \quad \nabla u_0 = [u_0, j(u_0)].
\]

Because of this we obtain

\[
\nabla[u, u_0] = [\nabla u, u_0] + [u, \nabla u_0]
\]

\[
= \left[ u, \frac{\hat{\theta}}{2r} \Lambda + \nabla^\perp \phi \right], u_0 \right] + \left[ u, [u_0, \frac{\hat{\theta}}{2r} \Lambda \right], u_0 \right],
\]

\[
= \frac{\hat{\theta}}{2r} ([u, \Lambda], u_0] + [[\Lambda, u_0], u]) + [[u, \nabla^\perp \phi], u_0]
\]

\[
= \frac{\hat{\theta}}{2r} [[u, u_0], \Lambda] + [[u, \nabla^\perp \phi], u_0],
\]

where the last identity follows by the Jacobi identity for commutators. Summarizing, we have

\[
\nabla[u, u_0] = \frac{\hat{\theta}}{2r} [[u, u_0], \Lambda] + [[u, \nabla^\perp \phi], u_0]. \quad (1.5)
\]

Since \( \phi \in W^{1,2}(\Omega, M^3_a(\mathbb{R})) \), this last identity shows that

\[
\frac{1}{|x-a|} [\Lambda, [u, u_0]] \in L^2(\Omega, M^3_a(\mathbb{R})) \quad \Leftrightarrow \quad [u, u_0] \in W^{1,2}(\Omega, M^3_a(\mathbb{R})).
\]
Hence, we concentrate on proving $\frac{1}{|x-a|} [\Lambda, [u, u_0]] \in L^2(\Omega, M^3_u(\mathbb{R}))$.

Take the commutator of (1.5) with $\Lambda$, and then take the inner product of the resulting equation with $[\Lambda, [u, u_0]]$. We obtain

$$\nabla \left( \frac{[\Lambda, [u, u_0]]^2}{2} \right) = \left( [\Lambda, [u, u_0]], [\Lambda, [u, \nabla \perp \phi], u_0] \right). \quad (1.6)$$

Next consider $r > 0$ such that $B_r(a) \subset \Omega$, and observe that at some $x_r \in \partial B_r(a)$ the image of $u(x_r)$ will be contained in the plane $S_0$ in which the images of $u_0(x)$ are contained. Because of this, we deduce that

$$[\Lambda, [u(x_r), u_0(x_r)]] = 0.$$

For any $x \in \partial B_r(a)$, let $\gamma_{x_r,x}$ be an arc of $\partial B_r(a)$ from $x_r$ to $x$. Integrating (1.6) over $\gamma_{x_r,x}$ we obtain

$$\frac{[\Lambda, [u, u_0]]^2}{2}(x) = \int_{\gamma_{x_r,x}} \left( [\Lambda, [u, u_0]], [\Lambda, [u, \nabla \perp \phi], u_0] \right) \cdot \tau \, dl,$$

which gives

$$[\Lambda, [u, u_0]]^2(x) \leq C \int_{\partial B_r(a)} [\Lambda, [u, u_0]] \left| [u, \nabla \perp \phi] \right| \, dl$$

for every $x \in \partial B_r(a)$. Integrating the last inequality over $\partial B_r(a)$, and dividing by $r^2$, we find

$$\frac{1}{r^2} \int_{\partial B_r(a)} [\Lambda, [u, u_0]]^2 \leq \frac{C}{r} \int_{\partial B_r(a)} [\Lambda, [u, u_0]] \left| [u, \nabla \perp \phi] \right|^2.$$

From here we obtain

$$\frac{1}{r^2} \int_{\partial B_r(a)} [\Lambda, [u, u_0]]^2 \leq C \int_{\partial B_r(a)} [u, \nabla \perp \phi]^2$$

for every $r > 0$ such that $B_r(a) \subset \Omega$ and some $C > 0$ independent of $r > 0$. Since $\phi \in W^{1,2}(\Omega, M^3_u(\mathbb{R}))$, and $|u| \leq 1$, this last inequality implies

$$\frac{1}{|x-a|} [\Lambda, [u, u_0]] \in L^2(\Omega, M^3_u(\mathbb{R})).$$

This concludes the proof of the proposition. \qed

Now recall that we denote by $S_0$ the plane that contains all the images of $u_0(x), x \in \Omega \setminus \{a\}$, and denote by $P_3$ the orthogonal projection onto the 1-dimensional subspace in $\mathbb{R}^3$ orthogonal to $S_0$. The matrix $P_3$ can be characterized as the orthogonal projection onto the kernel of $\Lambda$, or alternatively, as the only element $P_3 \in \mathcal{P}$ such that $P_3 \Lambda = \Lambda P_3 = 0$. In particular, $[P_3, \Lambda] = 0$. A similar argument to the one we used in the last proposition yields the following

**Proposition 1.4** With the notation above, we have

$$\frac{1}{r} [u, P_3] \in L^2(\Omega, M^3_u(\mathbb{R})), \quad [u, P_3] \in W^{1,2}(\Omega, M^3_u(\mathbb{R})),$$

and

$$\lim_{r \to 0} \sup_{x \in \partial B_r(a)} [u(x), P_3] = 0.$$
**Proof** We begin by recalling that
\[
\nabla u = \frac{\partial}{\partial r} [u, \Lambda] + [u, \nabla^\perp \phi].
\]
Taking commutator of this last identity with \(P_3\) we obtain
\[
\nabla [u, P_3] = \frac{\partial}{\partial r} [[u, P_3], \Lambda] + [[u, \nabla^\perp \phi], P_3],
\]
where in the first term of the right hand side we used Jacobi’s identity for commutators, plus the fact that \([\Lambda, P_3] = 0\). As in the proof of the previous proposition, this last identity shows that
\[
\frac{1}{|x - a|} \left[ [u, P_3], \Lambda \right] \in L^2(\Omega, M^3_a(\mathbb{R})) \iff [u, P_3] \in W^{1,2}(\Omega, M^3_a(\mathbb{R})).
\]
We will, in fact, prove the stronger statement
\[
\frac{1}{|x - a|} [u, P_3] \in L^2(\Omega, M^3_a(\mathbb{R})).
\]
(1.8)
Clearly this will establish the first two statements of the proposition. To demonstrate (1.8), take the inner product of (1.7) with \([u, P_3]\), to obtain
\[
\nabla \left( \frac{[u, P_3]^2}{2} \right) = \left( [[u, \nabla^\perp \phi], P_3], [u, P_3] \right).
\]
(1.9)
Now let \(r > 0\) be such that \(B_r(a) \subset \Omega\), and recall that at some \(x_r \in \partial B_r(a)\) the image of \(u(x_r)\) will belong to the plane \(S_0\) in which the images of \(u_0(x)\) are contained. Because of this, we deduce that
\[
[u(x_r), P_3] = 0.
\]
For any \(x \in \partial B_r(a)\), let \(\gamma_{x_r,x}\) be an arc of \(\partial B_r(a)\) from \(x_r\) to \(x\). Integrating Eq. (1.9) over \(\gamma_{x_r,x}\) we get
\[
\frac{[u, P_3]^2}{2}(x) = \int_{\gamma_{x_r,x}} \left( [[u, P_3], [[u, \nabla^\perp \phi], P_3]] \cdot \tau \, dl \leq C \int_{\partial B_r(a)} [u, P_3] \left| [u, \nabla^\perp \phi] \right|,
\]
for every \(x \in \partial B_r(a)\). A similar argument to the one we used in the previous proposition yields
\[
\frac{1}{|x - a|} [u, P_3] \in L^2(\Omega, M^3_a(\mathbb{R})).
\]
Also from
\[
\frac{[u, P_3]^2}{2}(x) \leq C \int_{\partial B_r(a)} [u, P_3] \left| [u, \nabla^\perp \phi] \right|
\]
we obtain
\[
[u, P_3]^2(x) \leq C \int_{\partial B_r(a)} \left| \nabla^\perp \phi \right| \leq \frac{C}{r} \int_{\partial B_r(a)} |y - a| \left| \nabla^\perp \phi \right| \, dl(y)
\]
for all \(x \in \partial B_r(a)\). By Remark 1.2, this concludes the proof of the proposition. \(\square\)

Our next result estimates the inner product \(\langle u, P_3 \rangle\).
Proposition 1.5  With the notation above we have
\[
\left\langle u, P_3 \right\rangle(x) \quad \left| x - a \right|^2 \in L^1(\Omega).
\]

Furthermore
\[
\lim_{r \to 0} \sup_{x \in \partial B_r(a)} \left| \left\langle u, P_3 \right\rangle(x) \right| = 0.
\]

Proof  From our previous computations we know that
\[
\nabla u = \hat{\theta} \left[ u, \Lambda \right] + [u, \nabla \phi].
\]

Now recall the triple product formula for square matrices
\[
\langle [A, B], C \rangle = \langle [C^T, A], B^T \rangle = \langle [B, C^T], A^T \rangle.
\]

Taking the inner product of \( \nabla u = \hat{\theta} [u, \Lambda] + [u, \nabla \phi] \) with \( P_3 \), and using the triple product formula for the first term of the right hand side, plus the fact that \( [\Lambda, P_3] = 0 \), we obtain
\[
\nabla \left\langle u, P_3 \right\rangle = \left\langle [u, P_3], \nabla \phi \cdot \tau \right\rangle.
\]

Recall now that for \( B_r(a) \subset \Omega \) there is \( x_r \in \partial B_r(a) \) such that \( u(x_r) \in S_0 \). If \( \gamma_{x_r} \) is an arc of \( \partial B_x(a) \) from \( x_r \) to \( x \in \partial B_r(a) \), we obtain
\[
\left\langle u, P_3 \right\rangle(x) = \int_{\gamma_{x_r}} \left\langle [u, P_3], \nabla \phi \cdot \tau \right\rangle.
\]

We deduce that
\[
\left| \left\langle u, P_3 \right\rangle(x) \right| \leq \int_{\partial B_r(a)} \left| [u, P_3] \right| \left| \nabla \phi \right|
\]
for all \( x \in \partial B_r(a) \). By Remark 1.2, this implies \( \lim_{r \to 0} \sup_{x \in \partial B_r(a)} \left| \left\langle u, P_3 \right\rangle(x) \right| = 0 \). Now, integrating the last inequality over \( \partial B_r(a) \) and dividing by \( r^2 \) we obtain
\[
\frac{1}{r^2} \int_{\partial B_r(a)} \left| \left\langle u, P_3 \right\rangle \right| \leq 2\pi \int_{\partial B_r(a)} \frac{\left| [u, P_3] \right|}{r} \left| \nabla \phi \right|.
\]

Integrating over \( r \in [0, R] \) for \( R > 0 \) such that \( B_R(a) \subset \Omega \), we find that
\[
\int_{B_R(a)} \left| \left\langle u, P_3 \right\rangle \right| \left| x - a \right|^2 \leq \int_{B_R(a)} \frac{\left| [u, P_3] \right|}{\left| x - a \right|} \left| \nabla \phi \right|.
\]

Since both \( \nabla \phi, \frac{[u, P_3]}{\left| x - a \right|} \in L^2(\Omega, M^3_3(\mathbb{R})) \), this concludes the proof.

Now we show that the length of the curves \( \gamma_r(e^{i\theta}) = u(r e^{i\theta}) \) approaches the length of closed geodesics in \( \mathcal{P} \) as \( r \to 0 \). This is the content of the next proposition.

Proposition 1.6  Let \( L \) denote the length of a closed geodesic in \( \mathcal{P} \). We have
\[
\lim_{r \to 0} \int_0^{2\pi} \left| \gamma_r \right| = L.
\]
Proof Let us first recall that

\[ L = \int_0^{2\pi} \left| \frac{du_0}{d\theta} \right| (\theta) \, d\theta = \sqrt{2\pi}. \]

From \([13]\) we have that

\[ \int_{\partial B_r(a)} |\nabla u \cdot \tau|^2 = \int_{\partial B_r(a)} |\nabla u_0 \cdot \tau|^2 + \int_{\partial B_r(a)} |\nabla u \cdot v|^2, \]

where \(\tau\) and \(v\) denote the tangent and outer normal to \(\partial B_r(a)\), respectively, and \(u_0\) denotes a canonical flat map. Now, we also know that

\[ j(u) = \hat{\gamma} r / \Lambda_1 + \nabla \perp \phi, \]

from which we deduce that

\[ \nabla u = \hat{\gamma} [u, \nabla \perp \phi] + [u, \nabla \perp \phi]. \]

We obtain on \(\partial B_r(a)\) that \(\nabla u \cdot v = [u, \nabla \perp \phi \cdot v]\). With all this we now have

\[ \frac{L^2}{2\pi r} = \frac{1}{2\pi r} \left( \int_{\partial B_r(a)} |\nabla u \cdot \tau| \right)^2 \leq \int_{\partial B_r(a)} |\nabla u \cdot \tau|^2 = \int_{\partial B_r(a)} |\nabla u_0 \cdot \tau|^2 + \int_{\partial B_r(a)} |\nabla u \cdot v|^2 = \frac{\pi}{r} + \int_{\partial B_r(a)} |[u, \nabla \perp \phi \cdot v]|^2 = \frac{L^2}{2\pi r} + \int_{\partial B_r(a)} |[u, \nabla \perp \phi \cdot v]|^2. \]

Therefore

\[ L^2 \leq \left( \int_0^{2\pi} \left| \frac{dy_r}{d\theta} \right| d\theta \right)^2 \leq L^2 + \frac{2\pi}{r} \int_{\partial B_r(a)} |y - a|^2 |[u, \nabla \perp \phi \cdot v]|^2 \, dl(y). \]

We now appeal to Remark 1.2 to finish the proof.

Next we use the last proposition to write the limit of minimizers \(u\) in terms of two angles near the singularity. Neither of these angles will be single-valued in \(\Omega \setminus \{a\}\), hence we do this for the map \(v(\xi) = u(e^{\xi})\) in the domain

\[ G_{-\lambda} = \{ \xi \in \mathbb{C} : \text{Re}(\xi) < -\lambda \}. \]

Here we choose \(\lambda > 0\) at least large enough for \(G_{-\lambda} \subset U = \{ \xi \in \mathbb{C} : e^{\xi} \in \Omega \}\). An additional condition on \(\lambda\) will appear in the next proposition.

Proposition 1.7 There is \(\lambda > 0\) large enough, and real valued functions \(\alpha, \beta : G_{-\lambda} \to \mathbb{R}\) such that

\[ v(\xi) = n(\xi)n^T(\xi), \]

where

\[ n(\xi) = \begin{pmatrix} \cos(\beta) \cos(\alpha) \\ \cos(\beta) \sin(\alpha) \\ \sin(\beta) \end{pmatrix}. \]
Furthermore, $\beta(\xi + 2\pi i) = -\beta(\xi)$ and $\alpha(\xi + 2\pi i) = \alpha(\xi) + \pi$ for all $\xi \in G_{-\lambda}$. Finally, the function $\alpha$ can be written as $\alpha(\xi) = \frac{\xi_1 + \alpha_1(\xi)}{2}$ for some real valued function $\alpha_1 : G_{-\lambda} \to \mathbb{R}$ such that $\alpha_1(\xi + 2\pi i) = \alpha_1(\xi)$.

**Remark 1.8** Notice that the last proposition shows that the vector field

$$k(\xi) = \begin{pmatrix} -\sin(\beta) \cos(\alpha) \\ -\sin(\beta) \sin(\alpha) \\ \cos(\beta) \end{pmatrix}$$

satisfies $v(\xi)k(\xi) = 0$ for $\xi \in G_{-\lambda}$. Also, since $\beta(\xi + 2\pi i) = -\beta(\xi)$, and $\alpha(\xi + 2\pi i) = \alpha(\xi) + \pi$, it follows that $k(\xi + 2\pi i) = k(\xi)$. By our change of variables, this gives a map, that we still denote by $k \in W^{1,2}(\Omega_{1}, \mathbb{S}^2)$, such that $u(x)k(x) = 0$ for all $x \in \Omega_{1} \setminus \{a\}$.

By our change of variables, and the properties of $\alpha$ and $\beta$ stated in Step 1 of Theorem 2.1, this defines a single-valued map, that we still denote by $k \in W^{1,2}(\Omega_{1}, \mathbb{S}^2)$, such that $u(x)k(x) = 0$ for all $x \in \Omega_{1} \setminus \{a\}$.

This proves the first claim of Proposition 0.1.

**Proof of Proposition 1.7.** First observe that Proposition 1.5 ensures that for $s = \frac{1}{2}$, there is $r > 0$ small such that

$$|\langle u, P_3 \rangle|(x) < s,$$

for all $x \in B_r(a) \setminus \{a\}$. Let us now recall the definition $v(\xi) = u(e^{i\xi})$, and note that this last condition implies the existence of $\lambda > 0$ large enough such that

$$|\langle v, P_3 \rangle|(\xi) < s$$

for every $\xi \in G_{-\lambda}$ where

$$G_{-\lambda} = \{ \xi \in \mathbb{C} : \text{Re}(\xi) < -\lambda \}.$$ 

Now $G_{-\lambda}$ is simply-connected, so we can lift $v$ through $n : G_{-\lambda} \to \mathbb{S}^2$, that is, we can find a map $n : G_{-\lambda} \to \mathbb{S}^2$ such that

$$v(\xi) = n(\xi)n(\xi)^T$$

for every $\xi \in G_{-\lambda}$. Now let

$$n(\xi) = \begin{pmatrix} n_1(\xi) \\ n_2(\xi) \\ n_3(\xi) \end{pmatrix},$$

and observe that $n_3^2(\xi) = \langle v(\xi), P_3 \rangle < s = \frac{1}{2}$ for every $\xi \in G_{-\lambda}$. In particular, we can define $\beta : G_{-\lambda} \to \mathbb{R}$ by the condition $\sin(\beta(\xi)) = n_3(\xi)$. Since $n_3^2(\xi) = \langle v(\xi), P_3 \rangle < s = \frac{1}{2}$, we conclude that $\cos(\beta(\xi)) > \frac{1}{\sqrt{2}}$ for every $\xi \in G_{-\lambda}$. In particular, the map $\zeta : G_{-\lambda} \to \mathbb{S}^1$ defined by

$$\zeta(\xi) = \begin{pmatrix} \sec(\beta) n_1 \\ \sec(\beta) n_2 \end{pmatrix}$$
is well defined for $\xi \in G_{-\lambda}$. Again, since $G_{-\lambda}$ is simply-connected, we can lift $\zeta$ through a real valued function $\alpha$ in such a way that

$$\zeta(\xi) = \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix}.$$ 

Summarizing so far, we have found real valued functions $\alpha, \beta : G_{-\lambda} \to \mathbb{R}$ such that

$$v(\xi) = n(\xi)n^T(\xi),$$

where

$$n(\xi) = \begin{pmatrix} \cos(\beta(\xi)) \cos(\alpha(\xi)) \\ \cos(\beta(\xi)) \sin(\alpha(\xi)) \end{pmatrix}.$$ 

To prove the last conclusions of the proposition let us recall that $u : \Omega \setminus \{a\} \to P$ restricted to any circle $\partial B_r(a) \subset \Omega$ represents a non-contractible curve in $P$. Since $v(\xi) = u(e^{\xi}) = n(\xi)n^T(\xi)$, we conclude that $n(\xi + 2\pi i) = -n(\xi)$. This shows that $\beta(\xi + 2\pi i) = -\beta(\xi)$.

For the last conclusion let us recall the canonical flat map

$$u_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ \sin(\theta) & -\cos(\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

By hypothesis, we know that $u_0$ is homotopic to $u$ in $\Omega \setminus \{a\}$. This implies that $v(\xi) = u(e^{\xi})$ is homotopic to

$$v_0(\xi) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \cos(\xi_2/2) & \sin(\xi_2/2) & 0 \\ \sin(\xi_2/2) & -\cos(\xi_2/2) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in $G_{-\lambda}$. Furthermore, a direct computation shows that $v_0$ lifts through $S^2$ by the map

$$n_0(\xi) = \begin{pmatrix} \cos(\xi_2/2) \\ \sin(\xi_2/2) \\ 0 \end{pmatrix}.$$ 

Now in $G_{-\lambda}$ we have $\sin^2(\beta) < s = \frac{1}{2}$. Hence

$$\zeta(\xi) = \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix}$$

is homotopic in $G_{-\lambda}$ to

$$\zeta_0(\xi) = \begin{pmatrix} \cos(\xi_2/2) \\ \sin(\xi_2/2) \end{pmatrix}.$$ 

We conclude that the function $\frac{\alpha(\xi)}{2} = \alpha(\xi) - \frac{\xi_2}{2}$ satisfies $\alpha(\xi + 2\pi i) = \alpha(\xi)$. This concludes the proof of this proposition. \hfill $\square$

In our next step we recall a property of the Hopf differential (see Definition 0.3).

**Proposition 1.9** For the Hopf differential of $u$ we have

$$\omega_u(z) = -\frac{1}{8z^2} + h,$$

where $h$ is a holomorphic map in all of $\Omega$. 

\[ \square \] Springer
Proof To prove this proposition let us recall that our map \( u \) is in fact
\[
 u = \lim_{\epsilon \to 0} u_\epsilon,
\]
where the maps \( u_\epsilon \) are global minimizers of the LdG energy in \( \Omega \), and the convergence is strong in \( W^{1,2}_{loc}(\Omega \setminus \{a\}, \mathcal{M}_3^2(\mathbb{R})) \). Because of their minimizing character, the maps \( u_\epsilon \) satisfy
\[
 -\frac{\partial^2 u_\epsilon}{\partial z \partial \overline{z}} + \frac{1}{4\epsilon^2} (\nabla u \cdot W) (u_\epsilon) = \lambda_\epsilon I_3,
\]
where \( W(u) \) is the potential term in the energy, and \( \lambda_\epsilon \) is the Lagrange multiplier associated to the restriction \( \text{tr}(u) = 1 \). Multiplying this equation by \( \frac{\partial u_\epsilon}{\partial z} \), and taking trace of the resulting equation, we obtain
\[
 -\frac{\partial \omega_{u_\epsilon}}{\partial \overline{z}} + \frac{\partial}{\partial z} \frac{W(u_\epsilon)}{2\epsilon^2} = 0, \tag{1.10}
\]
where we are using the notation
\[
 \omega_{u_\epsilon}(z) = -\text{tr} \left( \left( \frac{\partial u_\epsilon}{\partial z} \right)^2 \right).
\]
Let now \( G_\epsilon(z) \) be the convolution of \( \frac{W(u_\epsilon)}{\epsilon^2} \) with the Newtonian potential of the plane. Note that this is well defined, since \( W(u_\epsilon) = 0 \) on \( \partial \Omega \) by our boundary conditions there. Note also that
\[
 -\frac{\partial^2 G_\epsilon(z)}{\partial z \partial \overline{z}} = \frac{W(u_\epsilon)}{2\epsilon^2}.
\]
This and (1.10) yield
\[
 \frac{\partial}{\partial \overline{z}} \left( \omega_{u_\epsilon} + \frac{\partial^2 G_\epsilon}{\partial z^2} \right) = 0.
\]
We deduce that the function
\[
 h_\epsilon = \omega_{u_\epsilon} + \frac{\partial^2 G_\epsilon}{\partial z^2}
\]
is holomorphic in \( \Omega \). To conclude the proof we show that \( h_\epsilon \) is uniformly bounded on compact sets \( K \subset \Omega \setminus \{a\} \). Since \( h_\epsilon \) is holomorphic, this shows that \( h_\epsilon \) is in fact locally bounded in \( \Omega \), which then allows us to conclude that \( h_\epsilon \to h \) along a subsequence for some \( h \) holomorphic in \( \Omega \). To show that \( h_\epsilon \) is uniformly bounded in compact sets \( K \subset \Omega \setminus \{a\} \) we observe that the properties of \( u_\epsilon \) we list in the appendix allow us to apply Step 1 of the proof of Theorem VII.1 of [3]. This concludes the proof of the proposition. \( \square \)

2 Estimates near the singularity

We now will choose \( r > 0 \) small but independent of \( \epsilon > 0 \), and compare the energy of a minimizer of LdG in \( B_r(a) \) with canonical flat data, to the energy of one of our minimizers in the same ball. We recall that by canonical flat we mean data of the form given in Eq. (0.3). We will use the following notation:
\[
 I(r, \epsilon) = \inf \left\{ \int_{B_r(a)} e_\epsilon(u) : u \text{ is canonical flat on } \partial B_r(a) \right\}. \tag{2.1}
\]
Theorem 2.1 For $u$ our limit of minimizers, along a subsequence $\varepsilon_n \to 0$ we can choose $r > 0$ small but independent of $\varepsilon_n$ so that

$$\left| I(r, \varepsilon_n) - \int_{B_r(a)} e_{\varepsilon_n}(u_{\varepsilon_n}) \right| \leq o(1) + q(r),$$

where $o(1) \to 0$ as $\varepsilon_n \to 0$, and

$$q(r) \leq C \int_{B_{2r}(a)} \left( |\nabla \alpha_1|^2 + |\nabla \beta|^2 + \frac{|\nabla \alpha_1|}{|x - a|} \right) + \frac{C}{r^2} \int_{B_{2r}(a)} (\alpha_1^2 + \beta^2),$$

for some constant $C > 0$ independent of $\varepsilon_n$ and $r > 0$.

Remark 2.2 In the terminology of the previous section, Step 2 in the following proof, along with Proposition 1.5 show that $\beta(\xi_1, \xi_2) \to 0$ as $\xi_1 \to -\infty$, uniformly in $\xi_2$. This, and Step 4 of the following proof, give the second claim of Proposition 0.1. This also shows that the quantity $q(r)$ that appears in the statement of this theorem has $q(r) \to 0$ as $r \to 0$.

Proof We will prove the statement of the theorem in several steps. For most of the proof we will let $H_{-\lambda} = \{ \xi \in \mathbb{C} : \text{Re}(\xi) < -\lambda, -\pi \leq \text{Im}(\xi) < \pi \}$ be the lift of $B_r(a)$ through the exponential map. In particular, $\lambda = \ln(\frac{1}{r})$ will be chosen in the course of the proof. We also write $u = \lim_{\varepsilon_n \to 0} u_{\varepsilon_n}$. We know this convergence is strong in $W^{1,2}(\Omega \setminus B_r(a), M^3_{\mathbb{S}^1}(\mathbb{R}))$.

Step 1. In this first step we assume $\lambda > 0$ chosen as in Proposition 1.7. For such a $\lambda > 0$, the functions $\alpha_1, \beta$ from Proposition 1.7 satisfy $\nabla \beta, \nabla \alpha_1 \in L^2(H_{-\lambda})$. ☐

Proof of Step 1. To prove this, let us recall from Proposition 1.7 that $v(\xi) = u(e^\xi) = n(\xi)n_T(\xi)$, where

$$n(\xi) = \begin{pmatrix} \cos(\beta(\xi)) \cos(\alpha(\xi)) \\ \cos(\beta(\xi)) \sin(\alpha(\xi)) \\ \sin(\beta(\xi)) \end{pmatrix} = \cos(\beta)n_0(\alpha(\xi)) + \sin(\beta(\xi))e_3.$$

Here

$$n_0(\alpha) = \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \\ 0 \end{pmatrix} \quad \text{and} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We will also write

$$m_0(\alpha) = \begin{pmatrix} -\sin(\alpha) \\ \cos(\alpha) \\ 0 \end{pmatrix}.$$

A direct computation shows that

$$j(v) = \left( n \frac{\partial n_T}{\partial \alpha} - n \frac{\partial n}{\partial \alpha} n^T \right) \nabla \alpha + \left( n \frac{\partial n_T}{\partial \beta} - n \frac{\partial n}{\partial \beta} n^T \right) \nabla \beta$$

$$= \left( \cos^2(\beta) \Lambda + \cos(\beta) \sin(\beta)(e_3m_0(\alpha)^T - m_0(\alpha)e_3^T) \right) \nabla \alpha$$

$$+ \left( n_0(\alpha)e_3^T - e_3n_0(\alpha)^T \right) \nabla \beta.$$
We know that $\alpha(\xi) = \frac{\xi_2 + \alpha_1(\xi)}{2}$, so
\[
\nabla \alpha = \frac{e_2 + \nabla \alpha_1}{2}.
\]
However, we also know from [13] that
\[
j(v) = \frac{e_2}{2} \Lambda + \nabla \psi.
\]
We deduce that
\[
\left( \cos^2(\beta) \Lambda + \cos(\beta) \sin(\beta)(e_3m_0(\alpha)^T - m_0(\alpha)e_3^T) \right) \nabla \alpha_1 + \left( n_0(\alpha)e_3^T - e_3n_0(\alpha)^T \right) \nabla \beta
\]
so that
\[
\cos^2(\beta) |\nabla \alpha_1|^2 + |\nabla \beta|^2 \leq C (\sin^2(\beta) + |\nabla \psi|^2).
\]
Now recall that in the proof of Proposition 1.7 we chose $\lambda > 0$ so that $\sin^2(\beta) < \frac{1}{2}$ in $H_{-\lambda}$, so for $\xi \in H_{-\lambda}$ we have $\cos^2(\beta) > \frac{1}{2}$. Finally, recall from Proposition 1.5 that $\langle v, P_3 \rangle \in L^1(\Omega)$. By our change of variables, this implies that $\langle v, P_3 \rangle \in L^1(H_{-\lambda})$. Since $\langle v, P_3 \rangle = \sin^2(\beta)$ in $H_{-\lambda}$, this concludes the proof of this step. 

**Step 2.** We have $\nabla \alpha_1, \nabla \beta \in W^{1,2}(H_{-\lambda})$ and
\[
\nabla \alpha_1(\xi_1, \xi_2) \to 0 \quad \text{and} \quad \nabla \beta(\xi_1, \xi_2) \to 0
\]
as $\xi_1 \to -\infty$, both uniformly in $\xi_2 \in [-\pi, \pi]$.

**Proof of Step 2.** A direct computation shows that in $H_{-\lambda}$ we have
\[
|\nabla v|^2 = \frac{\cos^2(\beta)}{2} |e_2 + \nabla \alpha_1|^2 + 2 |\nabla \beta|^2.
\]
Since $u$ locally minimizes the Dirichlet integral in $\Omega \setminus \{a\}$, and our change of variables is holomorphic, we conclude that $v(\xi) = u(e^\xi)$ also locally minimizes the Dirichlet integral in $H_{-\lambda}$. Because of this, $\alpha_1$ and $\beta$ satisfy
\[
\Delta \alpha_1 = 2 \tan(\beta)(e_2 + \nabla \alpha_1) \cdot \nabla \beta
\]
and
\[
\Delta \beta = -\frac{\sin(2\beta)}{8} |e_2 + \nabla \alpha_1|^2,
\]
respectively. From here we find
\[
\Delta \left( \frac{|\nabla \alpha_1|^2}{2} \right) = |D^2 \alpha_1|^2 + 2 \sec^2(\beta) \nabla \alpha_1 \cdot \nabla (e_2 + \nabla \alpha_1) \cdot \nabla \beta
\]
\[
+ 2 \tan(\beta) \sum_{k=1}^2 \frac{\partial \alpha_1}{\partial \xi_k} \nabla \left( \frac{\partial \alpha_1}{\partial \xi_k} \right) \cdot \nabla \beta
\]
\[
+ 2 \tan(\beta) \sum_{k=1}^2 \frac{\partial \alpha_1}{\partial \xi_k} (e_2 + \nabla \alpha_1) \cdot \nabla \left( \frac{\partial \beta}{\partial \xi_k} \right)
\]
and
\[
\Delta \left( \frac{|\nabla \beta|^2}{2} \right) = |D^2 \beta|^2 - \frac{\cos(2\beta)}{4} \left( |\nabla \alpha_1|^2 + 1 + 2 \frac{\partial \alpha_1}{\partial \xi_2} \right) |\nabla \beta|^2 \\
- \frac{\sin(2\beta)}{4} \sum_{k=1}^2 \frac{\partial \beta}{\partial \xi_k} \nabla \left( \frac{\partial \alpha_1}{\partial \xi_k} \right) \cdot (e_2 + \nabla \alpha_1).
\]

We conclude that
\[
- \Delta \left( \frac{|\nabla \alpha_1|^2}{2} + \frac{|\nabla \beta|^2}{2} \right) + (1 - \delta)(|D^2 \alpha_1|^2 + |D^2 \beta|^2) \\
\leq C(\delta)(|\nabla \alpha_1|^2 + |\nabla \alpha_1|^4 + |\nabla \beta|^2 + |\nabla \beta|^4).
\]

We now follow Steps 2 and 3 of the proof of Proposition 1.1 to complete the proof of this step. \[\square\]

**Step 3.** \(\nabla \alpha_1 \in L^1(H_{-\lambda}).\) By our change of variables, this implies that \(\nabla \alpha_1 \bigg|_{x-a} \in L^1(B_r(a)).\)

**Proof of Step 3.** Recall from Proposition 1.9 that the Hopf differential of \(u\) satisfies
\[
\omega_u(z) = -\frac{1}{8z^2} + h(z),
\]
where \(h\) is a holomorphic map in all of \(\Omega\). Now, by our change of variables from \(\Omega\) to \(U\), for the Hopf differential of \(v\) we have
\[
\omega_v(\xi) = e^{2\xi} \omega_u(e^\xi) = -\frac{1}{8} + e^{2\xi} h(e^\xi).
\]

On the other hand, a direct computation in terms of \(\alpha = \frac{\xi + \alpha_1}{2}\) and \(\beta\) shows that
\[
j_C(v) = \left( n \frac{\partial n^T}{\partial \alpha} - \frac{\partial n}{\partial \alpha} n^T \right) \frac{\partial \alpha}{\partial \xi} + \left( n \frac{\partial n^T}{\partial \beta} - \frac{\partial n}{\partial \beta} n^T \right) \frac{\partial \beta}{\partial \xi} \\
= \cos(\beta) \left( \cos(\beta) \Lambda + \sin(\beta) (e_3 m_0(\alpha)^T - m_0(\alpha) e_3^T) \right) \frac{\partial \alpha}{\partial \xi} \\
+ \left( n_0(\alpha) e_3^T - e_3 n_0(\alpha)^T \right) \frac{\partial \beta}{\partial \xi},
\]
where \(j_C\) was defined in (0.4). Since
\[
\frac{\partial \alpha}{\partial \xi} = -\frac{i}{4} + \frac{\partial \alpha_1}{\partial \xi},
\]
we also have that
\[
\omega_v(\xi) = \text{tr} \left( (j_C(v))^2 \right) \\
= 2 \cos^2(\beta) \left( -\frac{i}{4} + \frac{\partial \alpha_1}{\partial \xi} \right)^2 + 2 \left( \frac{\partial \beta}{\partial \xi} \right)^2 \\
= -\frac{\cos^2(\beta)}{8} - i \cos^2(\beta) \frac{\partial \alpha_1}{\partial \xi} + 2 \cos^2(\beta) \left( \frac{\partial \alpha_1}{\partial \xi} \right)^2 + 2 \left( \frac{\partial \beta}{\partial \xi} \right)^2.
\]
We conclude that

\[-i \cos^2(\beta) \frac{\partial \alpha_1}{\partial \xi} = -\frac{\sin^2(\beta)}{8} - 2 \cos^2(\beta) \left( \frac{\partial \alpha_1}{\partial \xi} \right)^2 - 2 \left( \frac{\partial \beta}{\partial \xi} \right)^2 + e^{2\xi} h(e^\xi).\]

Now observe that Propositions 1.5 and (1.7), along with our change of variables, imply that \( \sin^2(\beta) \in L^1(H_{-\lambda}) \), whereas \( \nabla \alpha_1, \nabla \beta \in L^2(H_{-\lambda}) \) by Step 1. Since \( \cos^2(\beta) \geq \frac{1}{2} \) in \( H_{-\lambda} \), this concludes the proof of Step 3.

**Step 4.** There is a constant \( \alpha^* \in \mathbb{R} \) such that

\[ \alpha_1(\xi_1, \xi_2) \to \alpha^* \quad \text{as} \quad \xi_1 \to -\infty, \]

uniformly in \( \xi_2 \in \mathbb{R} \).

**Proof of Step 4.** Let

\[ \overline{\alpha}_1(\xi_1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \alpha_1(\xi_1, \xi_2) d\xi_2, \]

and observe that, for \( \xi_{1,1} < \xi_{1,2} < -\lambda \), we have

\[ \overline{\alpha}_1(\xi_{1,2}) - \overline{\alpha}_1(\xi_{1,1}) = \frac{1}{2\pi} \int_{\xi_{1,1}}^{\xi_{1,2}} \left( \int_{-\pi}^{\pi} \frac{\partial \alpha_1}{\partial \xi_1}(s, t) dt \right) ds. \]

Hence

\[ \left| \overline{\alpha}_1(\xi_{1,2}) - \overline{\alpha}_1(\xi_{1,1}) \right| \leq \frac{1}{2\pi} \int_{\xi_{1,1}}^{\xi_{1,2}} \int_{-\pi}^{\pi} |\nabla \alpha_1| \cdot (s, t) dt ds. \]

By Step 3, \( \overline{\alpha}_1(\xi_1) \) is Cauchy as \( \xi_1 \to -\infty \). Since \( (\nabla \alpha_1)(\xi_1, \xi_2) \to 0 \) as \( \xi_1 \to -\infty \), uniformly in \( \xi_2 \in \mathbb{R} \), this proves Step 4.

**Step 5.** We have the lower bound

\[ \int_{B_r(a)} e_\varepsilon(u_\varepsilon) \geq I(r, \varepsilon) - o(1) \]

\[ -C \int_{B_r(a)} \left( |\nabla \alpha_1|^2 + |\nabla \beta|^2 + \frac{|\nabla \alpha_1|}{|x-a|} \right) - \frac{C}{r^2} \int_{B_r(a)} (\alpha_1^2 + \beta^2), \]

where \( o(1) \to 0 \) as \( \varepsilon \to 0 \).

**Proof of Step 5.** In this step we will work in \( B_r(a) \). Then, by Step 4, we can apply a fixed rotation to \( u \) so as to obtain \( \alpha_1(x) \to 0 \) as \( x \to a \).

To prove the claim in this step we build a suitable comparison map. Recall that \( v_\varepsilon \) denotes the nearest point projection of \( u_\varepsilon \) on \( \mathcal{P} \). We will denote \( \Pi(\omega) \) the nearest point projection of \( \omega \in M^3_{s,1}(\mathbb{R}) \) onto \( \mathcal{P} \), whenever this projection is well defined and unique. Our comparison map is

\[ w_\varepsilon(x) = \begin{cases} 
  u_\varepsilon(x) & \text{for } |x-a| < \frac{r}{2}, \\
  \left( 4 - \frac{6|x-a|}{r} \right) u_\varepsilon + \frac{6|x-a|}{r} - 3 \right) v_\varepsilon & \text{for } \frac{r}{2} < |x-a| < \frac{2r}{3}, \\
  \left( 5 - \frac{6|x-a|}{r} \right) u_\varepsilon + \frac{6|x-a|}{r} - 4 \right) u & \text{for } \frac{2r}{3} < |x-a| < \frac{5r}{6}, \\
  u \left( \theta + \frac{6(r-|x-a|)}{r} \alpha_1; \frac{6(r-|x-a|)}{r} \beta \right) & \text{for } \frac{5r}{6} < |x-a| < r.
\]
Note that \( w_\varepsilon \) has canonical flat data on \( \partial B_r(a) \). Hence
\[
\int_{B_r(a)} e_\varepsilon(w_\varepsilon) \geq I(r, \varepsilon).
\]

We now estimate \( \int_{B_r(a)} e_\varepsilon(w_\varepsilon) \) in each of the intervals for \( |x - a| \) that appear in the definition of \( w_\varepsilon \).

In the range \( |x - a| < \frac{r}{2} \), clearly we have
\[
\int_{B_{\frac{r}{2}}(a)} e_\varepsilon(w_\varepsilon) = \int_{B_{\frac{r}{2}}(a)} e_\varepsilon(u_\varepsilon).
\]

Let now \( \frac{r}{2} < |x - a| < 2\frac{r}{3} \). In this case we first observe that
\[
\text{dist}(w_\varepsilon, \mathcal{P}) \leq \text{dist}(u_\varepsilon, \mathcal{P}).
\]

Since
\[
4W_\beta(w_\varepsilon) \leq (1 - |w_\varepsilon|^2)^2 \leq (\text{dist}(w_\varepsilon, \mathcal{P}))^2 \leq (\text{dist}(u_\varepsilon, \mathcal{P}))^2 \leq CW_\beta(u_\varepsilon),
\]
we obtain
\[
\int_{B_{\frac{r}{3}}(a) \setminus B_{\frac{r}{2}}(a)} \frac{W_\beta(w_\varepsilon)}{\varepsilon^2} = o(1).
\]

This last claim follows from the end of the proof of Lemma 8 of [13], which shows that minimizers \( u_\varepsilon \) satisfy
\[
\limsup_{\varepsilon \to 0} \int_{\Omega \setminus B_r(a)} \frac{W(u_\varepsilon)}{\varepsilon^2} = 0.
\]

Next, from [13] we also know that
\[
\int_{B_{\frac{r}{3}}(a) \setminus B_{\frac{r}{2}}(a)} |\nabla v_\varepsilon|^2 \leq \int_{B_{\frac{r}{3}}(a) \setminus B_{\frac{r}{2}}(a)} |\nabla u_\varepsilon|^2 + o(1).
\]

All this gives us
\[
\int_{B_{\frac{r}{3}}(a) \setminus B_{\frac{r}{2}}(a)} e_\varepsilon(w_\varepsilon) \leq \int_{B_{\frac{r}{3}}(a) \setminus B_{\frac{r}{2}}(a)} e_\varepsilon(u_\varepsilon) + o(1).
\]

For the range \( \frac{2r}{3} < |x - a| < \frac{5r}{6} \), we first observe that
\[
\Pi \left( 5 - \frac{6|x - a|}{r} \right) v_\varepsilon + \left( \frac{6|x - a|}{r} - 4 \right) u = \Pi \left( u + \left( 5 - \frac{6|x - a|}{r} \right) (v_\varepsilon - u) \right).
\]

Set
\[
z_{\varepsilon, r} = u + \left( 5 - \frac{6|x - a|}{r} \right) (v_\varepsilon - u),
\]
so that \( w_\varepsilon = \Pi(z_{\varepsilon, r}) \). We have
\[
\frac{\partial w_\varepsilon}{\partial x_k} = (D\Pi)(z_{\varepsilon, r}) \left( \frac{\partial z_{\varepsilon, r}}{\partial x_k} \right)
\]
\[
= \frac{\partial u}{\partial x_k} + ((D\Pi)(z_{\varepsilon, r}) - (D\Pi)(u)) \left( \frac{\partial u}{\partial x_k} \right) + (D\Pi)(z_{\varepsilon, r}) \left( \frac{\partial z_{\varepsilon, r}}{\partial x_k} - \frac{\partial u}{\partial x_k} \right).
\]
Because \( u_\varepsilon \to u \) and \( v_\varepsilon \to u \) strongly in \( B_r(a) \setminus B_{\frac{3r}{\varepsilon}}(a) \), using the facts described in the appendix we obtain
\[
\int_{B_{\frac{3r}{\varepsilon}}(a) \setminus B_{\frac{2r}{\varepsilon}}(a)} |\nabla w_\varepsilon|^2 \leq \int_{B_{\frac{3r}{\varepsilon}}(a) \setminus B_{\frac{2r}{\varepsilon}}(a)} |\nabla u|^2 + o(1) \leq \int_{B_{\frac{3r}{\varepsilon}}(a) \setminus B_{\frac{2r}{\varepsilon}}(a)} |\nabla u_e|^2 + o(1).
\]
Since \( W_\beta(w_\varepsilon) = 0 \), all this gives us
\[
\int_{B_{\frac{3r}{\varepsilon}}(a) \setminus B_{\frac{2r}{\varepsilon}}(a)} e_\varepsilon(w_\varepsilon) \leq \int_{B_{\frac{3r}{\varepsilon}}(a) \setminus B_{\frac{2r}{\varepsilon}}(a)} e_\varepsilon(u_\varepsilon) + o(1).
\]
Finally, let \( \frac{5r}{6} < |x - a| < r \). We observe that
\[
\nabla w_\varepsilon = \left( \nabla \theta + \frac{6(r - |x - a|)}{r} \nabla \alpha_1 - \frac{6}{r} \alpha_1 \nabla \beta \right) \frac{\partial u}{\partial \alpha} + \frac{6}{r} \beta \nabla \beta \frac{\partial u}{\partial \beta},
\]
where we use the notation \( \nabla \beta = \frac{x - a}{|x - a|} \). From this identity we obtain
\[
\nabla w_\varepsilon = \nabla u + \left( \frac{5r - 6|x - a|}{r} \nabla \alpha_1 - \frac{6}{r} \alpha_1 \nabla \beta \right) \frac{\partial u}{\partial \alpha} + \frac{6}{r} \beta \nabla \beta \frac{\partial u}{\partial \beta}.
\]
Since \( \left( \frac{\partial u}{\partial \alpha}, \frac{\partial u}{\partial \beta} \right) = 0 \), it follows that
\[
|\nabla w_\varepsilon|^2 \leq |\nabla u|^2 + C \int_{B_r(a)} \left( |\nabla \alpha_1|^2 + |\nabla \beta|^2 + \frac{|\nabla \alpha_1|}{|x - a|} \right) + \frac{C}{r^2} \int_{B_r(a)} (\alpha_1^2 + \beta^2).
\]
Since \( W_\beta(w_\varepsilon) = 0 \), we finally get
\[
\int_{B_r(a) \setminus B_{\frac{3r}{\varepsilon}}(a)} e_\varepsilon(w_\varepsilon) \leq \int_{B_r(a) \setminus B_{\frac{3r}{\varepsilon}}(a)} e_\varepsilon(u_\varepsilon) + C \int_{B_r(a)} \left( |\nabla \alpha_1|^2 + |\nabla \beta|^2 + \frac{|\nabla \alpha_1|}{|x - a|} \right) + \frac{C}{r^2} \int_{B_r(a)} (\alpha_1^2 + \beta^2).
\]
Putting together the estimates in the various ranges for \( |x - a| \), we conclude the proof of this step. \( \square \)

**Step 6.** We have the upper bound
\[
\int_{B_r(a)} e_\varepsilon(u_\varepsilon) \leq I(r, \varepsilon) + o(1)
\]
\[
+ C \int_{B_r(a)} \left( |\nabla \alpha_1|^2 + |\nabla \beta|^2 + \frac{|\nabla \alpha_1|}{|x - a|} \right) + \frac{C}{r^2} \int_{B_r(a)} (\alpha_1^2 + \beta^2).
\]

**Proof of Step 6.** Let \( \zeta_\varepsilon \) be a minimizer of LdG in \( B_r(a) \) with canonical flat data, and define
\[
w_\varepsilon(x) = \begin{cases} 
\zeta_\varepsilon & \text{for } r < |x - a| < \frac{4r}{3} \\
\Pi \left( \left( \frac{5 - 3|x - a|}{r} \right) u + \left( \frac{3|x - a| - 4}{r} \right) v_\varepsilon \right) & \text{for } \frac{4r}{3} < |x - a| < \frac{5r}{3} \\
\left( \frac{6 - 3|x - a|}{r} \right) u + \left( \frac{3|x - a| - 5}{r} \right) u_\varepsilon & \text{for } \frac{5r}{3} < |x - a| < 2r.
\end{cases}
\]
Note that $\zeta_\varepsilon = u_\varepsilon$ on $\partial B_{2r}(a)$. Since $u_\varepsilon$ minimizes the LdG energy with respect to its own boundary data, we obtain
\[
\int_{B_{2r}(a)} e_\varepsilon(\zeta_\varepsilon) \geq \int_{B_{2r}(a)} e_\varepsilon(u_\varepsilon).
\]
Furthermore, by definition we have
\[
\int_{B_r(a)} e_\varepsilon(\zeta_\varepsilon) = I(r, \varepsilon).
\]
To conclude we follow essentially the same strategy we used in Step 5 to show that
\[
\int_{B_{2r}(a) \setminus B_r(a)} e_\varepsilon(\zeta_\varepsilon) \leq \int_{B_{2r}(a) \setminus B_r(a)} e_\varepsilon(u_\varepsilon) + o(1)
+ C \int_{B_{2r}(a)} \left( |\nabla \alpha_1|^2 + |\nabla \beta|^2 + \frac{|\nabla \alpha_2|}{|x - a|} \right) + \frac{C}{r^2} \int_{B_{2r}(a)} (\alpha_1^2 + \beta^2).
\]
This concludes the proof of the theorem. $\square$

3 Estimates away from the singularity

In this section we prove Theorem 0.4. Recall that we are assuming $a = 0$, and denote the zero set of the Hopf differential $\omega_\mu$ of $u$ by
\[
Z_{\omega_\mu} = \{ z \in \Omega \setminus \{a\} : \omega_0(z) = 0 \}.
\]
From Proposition 1.9 we have
\[
\omega_\mu = \frac{1}{8z^2} + h,
\]
where $h$ is holomorphic in $\Omega$. Our main result in this section gives an expression for $j(u)$ in the case of $Z_{\omega_\mu} = \emptyset$. In this situation the function $1 + 8z^2h$ does not vanish in $\Omega$. Because $\Omega$ is simply-connected, we can extract a square root of $1 + 8z^2h$, and hence of $\omega_0$. For convenience, let $\mu_\mu$ be a (necessarily meromorphic) function that satisfies $-2\mu_\mu^2 = \omega_\mu$. We observe, however, that when $Z_{\omega_\mu} \neq \emptyset$, the conclusions are still valid, but only locally away from $Z_{\omega_\mu}$.

**Proof of Theorem 0.4.** Consider the exponential map $e : \mathbb{C} \rightarrow \mathbb{C} \setminus \{0\}$, and let
\[
H_{\Omega^*} = e^{-1}(\Omega \setminus \{0\}).
\]
In other words, $H_{\Omega^*}$ is the lift of the punctured domain $\Omega \setminus \{0\}$. It is well known that $H_{\Omega^*}$ along with the (complex) exponential map is the universal covering space of $\Omega \setminus \{0\}$. Any map $u : \Omega \setminus \{0\} \rightarrow X$, into any topological space, defines a map $v : H_{\Omega^*} \rightarrow X$ via $v(\xi) = u(e^\xi)$. Observe that $v$ is $2\pi i$-periodic in $H_{\Omega^*}$. On the other hand, any map $v : H_{\Omega^*} \rightarrow X$, into the topological space $X$, that is $2\pi i$-periodic, induces a unique map $u : \Omega \setminus \{0\} \rightarrow X$ such that $v(\xi) = u(e^\xi)$. We will abuse the notation and say that a map $v : H_{\Omega} \rightarrow X$, that is not $2\pi i$-periodic, is a multivalued map in $\Omega \setminus \{0\}$.

Next, for a fixed $P \in \mathcal{P}$, we define
\[
Q_P(A) = AP + PA
\]
for any $A \in M^3_3(\mathbb{R})$. It turns out that
\[ Q_P : M_3^3(\mathbb{R}) \to M_3^3(\mathbb{R}) \]
is an orthogonal projection with respect to the inner product $\langle A, B \rangle = \text{tr}(B^T A)$ for $A, B \in M_3^3(\mathbb{R})$. Denoting the image of this projection by $A_P$, it is easy to check that $A_P$ is also isomorphic to $T_P P$.

Let now $u : \Omega \setminus \{a\} \to P$ be a limit of minimizers and lift it to $H_{\Omega^*}$ through $v(\xi) = u(e^\xi)$. Recall that $v$ is $2\pi i$-periodic in $H_{\Omega^*}$. As before, we define
\[ j_\mathcal{C}(v) = \left[ v ; \frac{\partial v}{\partial \xi} \right]. \]

We now recall that $u$ satisfies a minimality condition. Since the change of variables to go from $\Omega \setminus \{a\}$ to $H_{\Omega^*}$ is conformal, $v$ also satisfies a minimality condition so that
\[ [v; \Delta v] = 0. \quad (3.2) \]
This, plus the fact that $v$ is $\mathcal{P}$-valued, implies that
\[ \frac{\partial j_\mathcal{C}(v)}{\partial \xi} = -\left[ j_\mathcal{C}(v); j_\mathcal{C}(v) \right]. \quad (3.3) \]

We next recall that the map $L_P : SO(3) \to \mathcal{P}$, where $P \in \mathcal{P}$ is fixed, defined through
\[ L_P(R) = RPR^T, \]
is onto. It is not, however, even locally injective. In fact, its stabiliser
\[ O_P(3) = \{ R \in SO(3) : RPR^T = P \}, \]
is non-trivial. Although $L_P$ is not a covering map, we can still lift $v$ with some $R : H_{\Omega^*} \to SO(3)$, so that
\[ v = RPR^T. \]
This can be seen by building the map $R$ locally around any $v(\xi), \xi \in H_{\Omega^*}$, and extending it.

The lifting $R$ need not be unique. However, since $v(\xi + 2\pi i) = v(\xi)$, we must have
\[ R^T(\xi)R(\xi + 2\pi i) \in O_P(3) \]
for all $\xi \in H_{\Omega^*}$. Next observe that, since $P$ is constant,
\[ \frac{\partial v}{\partial \xi} = \frac{\partial R}{\partial \xi} PR^T + RP \frac{\partial R^T}{\partial \xi}. \]

Now $R$ takes values in $SO(3)$. Hence
\[ \text{Re} \left( R^T \frac{\partial R}{\partial \xi} \right), \text{Im} \left( R^T \frac{\partial R}{\partial \xi} \right) \in M_3^3(\mathbb{R}). \]
Since $P \in \mathcal{P}$, we obtain
\[ j_\mathcal{C}(v) = R \left( P \frac{\partial R^T}{\partial \xi} R - R^T \frac{\partial R}{\partial \xi} P \right) R^T. \]
Denote
\[ B(\xi) = \frac{\partial R^T}{\partial \xi} R. \]
If we set
\[ \eta(\xi) = Q_P(B), \]
then so far we only have that
\[ j_\mathbb{C}(v) = R\eta R^T. \]
We will prove next that, in fact,
\[ \frac{\partial \eta}{\partial \bar{\xi}} = [\bar{B} - \eta; \eta]. \]
To this end, we observe that \( R : H^*_\Omega \rightarrow O(3) \) has real entries. Hence
\[ \frac{\partial R}{\partial \xi} = \frac{\partial R}{\partial \bar{\xi}}. \]
From here we obtain
\[ \frac{\partial j_\mathbb{C}(v)}{\partial \bar{\xi}} = R \left( \frac{\partial \eta}{\partial \bar{\xi}} + R^T \frac{\partial R}{\partial \bar{\xi}} \eta + \eta R^T \frac{\partial \bar{\eta}}{\partial \bar{\xi}} R \right) R^T = R \left( \frac{\partial \eta}{\partial \bar{\xi}} - [\bar{B}; \eta] \right) R^T, \]
where again we used the fact that \( R \) has real entries. Next, we already know that
\[ \frac{\partial j_\mathbb{C}(v)}{\partial \bar{\xi}} = -R[\bar{\eta}; \eta]R^T. \]
This proves
\[ \frac{\partial \eta}{\partial \bar{\xi}} = [\bar{B} - \eta; \eta]. \]

Let now \( \Lambda_1 \in M^3_A(\mathbb{R}) \) be (constant and) such that \( Q_P(\Lambda_1) = 0 \) and
\[ [\Lambda_1; [\Lambda_1; A]] = -Q_P(A) \quad \text{for all} \quad A \in M^3_A(\mathbb{R}). \]
Let also \( \Lambda_2, \Lambda_3 \in A_P \) be such that \( \{\Lambda_1, \Lambda_2, \Lambda_3\} \) is an orthogonal basis in \( M^3_A(\mathbb{R}) \) with the additional property
\[ [\Lambda_1, \Lambda_2] = \Lambda_3, [\Lambda_2, \Lambda_3] = \Lambda_1 \quad \text{and} \quad [\Lambda_3, \Lambda_1] = \Lambda_2. \]
By the definition of \( \eta \), there is a function \( \alpha_1 : H^*_\Omega \rightarrow \mathbb{C} \) such that
\[ B(\xi) = \alpha_1 \Lambda_1 + \eta. \]
What we know so far can be expressed as
\[ \frac{\partial \eta}{\partial \bar{\xi}} = \overline{\alpha_1} T_1(\eta), \]
where \( T_1(B) = T_{\Lambda_1}(B) = [\Lambda_1; B] \).

Let now \( a_1 : H^*_\Omega \rightarrow \mathbb{C} \) be such that
\[ \frac{\partial a_1}{\partial \bar{\xi}} = \bar{\alpha_1}. \]
To see that such \( a_1 \) should exist, we write \( \alpha_1 = s + it \), and seek real-valued functions \( f, g \) such that
\[ \frac{\partial^2}{\partial \xi \partial \bar{\xi}} f = s, \quad \frac{\partial^2}{\partial \xi \partial \bar{\xi}} g = t. \]
Such $f$ and $g$ always exist in a half-plane by Theorem 3.6.4 in [18]. Since $H_{\Omega^*}$ is conformal to a half space, $f$ and $g$ also exist in $H_{\Omega^*}$. With these functions, we set

$$a_1 = \frac{\partial f}{\partial \xi} - i \frac{\partial g}{\partial \xi}.$$ 

By construction,

$$\frac{\partial a_1}{\partial \xi} = \overline{a}_1.$$ 

But then

$$\frac{\partial \eta}{\partial \xi} = \frac{\partial a_1}{\partial \xi} [\Lambda_1, \eta],$$

and we deduce that

$$\frac{\partial}{\partial \xi} \left( e^{-a_1 \Lambda_1} \eta e^{a_1 \Lambda_1} \right) = 0,$$

where $e^{a_1 \Lambda_1}$ is the standard exponential of a matrix. Because of the definition of $\Lambda_2$, $\Lambda_3$, there are holomorphic functions $z_2, z_3 : H_{\Omega^*} \to \mathbb{C}$ such that

$$\eta = e^{a_1 \Lambda_1} (z_2 \Lambda_2 + z_3 \Lambda_3) e^{-a_1 \Lambda_1}. \quad (3.4)$$

Define now

$$\omega_v(\xi) = \langle j_{\mathbb{C}}(v), j_{\mathbb{C}}(v) \rangle. \quad (3.5)$$

One checks directly that

$$\omega_v = -2(z_1^2 + z_2^2).$$

Let now $\mu_v$ be a holomorphic map such that $2\mu_v^2 = -\omega_v$. Lemma 3.1 allows us then to find a holomorphic $\zeta$ such that

$$z_2 = \mu_v \cos(\zeta), \quad z_3 = \mu_v \sin(\zeta).$$

This, in particular, shows that

$$z_2 \Lambda_2 + z_3 \Lambda_3 = \mu_v e^{\zeta T_1}(\Lambda_2), \quad (3.6)$$

where $T_1(A) = [\Lambda_1, A]$. Along with (3.4) the Eq. (3.6) gives

$$\eta = \mu_v e^{(a_1 + \zeta) T_1}(\Lambda_2). \quad (3.7)$$

We now observe the following: if $f : H_{\Omega^*} \to \mathbb{R}$ is any function, then

$$S = \text{Re} f \Lambda_1 \in O(3)$$

also satisfies

$$v = S P S^T.$$ 

Furthermore, setting

$$B_S(\xi) = \frac{\partial S^T}{\partial \xi} S,$$

it is not hard to see that

$$\eta_S = Q_\rho(B_S) = S^T R \eta R^T S = e^{-f \Lambda_1} \eta e^{f \Lambda_1} = e^{-f T_1}(\eta),$$
and $e^{fA_1}$ is the exponential of a matrix, whereas $e^{fT_1}$ is the exponential of an operator in $M^3_a(\mathbb{R})$ (which incidentally can also be written as a $3 \times 3$ matrix with respect to the appropriate basis in $M^3_a(\mathbb{R})$). This and (3.7) yield

$$\eta_S = \mu_v e^{(a_1 + \zeta - f)T_1}(A_2).$$

Since the function $f$ is arbitrary, except for the fact that it must be real-valued, we set

$$f = \text{Re}(a_1 + \zeta), \quad g = \text{Im}(a_1 + \zeta).$$

We conclude that

$$\eta_S = \mu_v e^{igT_1}(A_2) = \mu_v (\cosh(g)A_2 + i \sinh(g)A_3).$$

For the final conclusion we notice that a direct computation shows that

$$B_S = -i \frac{\partial g}{\partial \xi} A_1 + \eta_S.$$

Recall now that

$$B_S = \frac{\partial S^T}{\partial \xi} S,$$

and that $S$ has real entries. Because of this

$$\text{Im} \left( \frac{\partial^2 S^T}{\partial \xi \partial \bar{\xi}} \right) = 0.$$

Since

$$\frac{\partial S^T}{\partial \xi} = B_S S^T,$$

we obtain from here that

$$\text{Im} \left( \left( \frac{\partial B_S}{\partial \bar{\xi}} + B_S \overline{B_S} \right) S^T \right) = 0.$$

Observing that

$$B_S \overline{B_S} = \frac{1}{2} [B_S; \overline{B_S}] + \frac{1}{2} (B_S \overline{B_S} + \overline{B_S} B_S),$$

and

$$\text{Im} \left( \frac{1}{2} (B_S \overline{B_S} + \overline{B_S} B_S) \right) = 0,$$

we obtain

$$\text{Im} \left( \frac{\partial B_S}{\partial \bar{\xi}} + \frac{1}{2} [B_S; \overline{B_S}] \right) = 0.$$

Inserting everything we have obtained so far into this last equation we obtain

$$-\frac{\partial^2 g}{\partial \xi \partial \bar{\xi}} = \frac{|\mu_v|^2}{2} \sinh(2g) = \frac{|\omega_v|}{4} \sinh(2g).$$

Lastly, we observe that

$$[j_C(v), j_C(v)] = S[\overline{\eta_S}, \eta_S] S^T = \frac{|\omega_v|}{4} \sinh(2g) \Gamma_1,$$
because \( \Gamma_j = S \Lambda_j S^T \). The conclusions of the theorem now follow by changing variables back from \( H_{\Omega^*} \) to \( \Omega \).

We now present the proof of a simple lemma that we used during this proof.

**Lemma 3.1** Let \( D \subset \mathbb{C} \) be a simply-connected open set. For any two holomorphic functions \( a, b \) in \( D \) that satisfy

\[
a^2 + b^2 = 1 \text{ in } D,
\]

there is a holomorphic function \( \beta \) in \( D \) such that

\[
a = \cos(\beta) \text{ and } b = \sin(\beta).
\]

**Proof** Differentiating \( a^2 + b^2 = 1 \) we obtain

\[
a \frac{\partial a}{\partial z} + b \frac{\partial b}{\partial z} = 0.
\]

However, \( a \) and \( b \) cannot be zero simultaneously. Hence, at least one of the sides of the identity

\[
\frac{-1}{a} \frac{\partial b}{\partial z} = \frac{1}{b} \frac{\partial a}{\partial z}
\]

always makes sense, and is holomorphic. Set

\[
\alpha = -\frac{1}{a} \frac{\partial b}{\partial z} = \frac{1}{b} \frac{\partial a}{\partial z}.
\]

Observe next that

\[
\frac{\partial}{\partial z} \begin{pmatrix} a \\ b \end{pmatrix} = \alpha \begin{pmatrix} -b \\ a \end{pmatrix} = \alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.
\]

Call

\[
T_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]

and let \( \beta_1 \) be any holomorphic antiderivative of \( \alpha \). What we know so far can be written as

\[
\frac{\partial}{\partial z} \left( e^{-\beta_1 T_0} \begin{pmatrix} a \\ b \end{pmatrix} \right) = 0.
\]

Since the expression inside the derivative above is holomorphic, then the expression is constant. In other words, there are complex constants \( c_1, c_2 \in \mathbb{C} \) such that

\[
\begin{pmatrix} a \\ b \end{pmatrix} = e^{\beta_1 T_0} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 \cos(\beta_1) - c_2 \sin(\beta_1) \\ c_1 \sin(\beta_1) + c_2 \cos(\beta_1) \end{pmatrix}.
\]

Observe next that

\[
1 = a^2 + b^2 = c_1^2 + c_2^2.
\]

We finish the proof by picking a constant \( \beta_2 \in \mathbb{C} \) with

\[
c_1 = \cos(\beta_2) \text{ and } c_2 = \sin(\beta_2).
\]
This will imply that
\[
\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos(\beta_1 + \beta_2) \\ \sin(\beta_1 + \beta_2) \end{pmatrix}.
\]

Setting \( \beta = \beta_1 + \beta_2 \) we obtain the conclusion of the Lemma.

To show that we can pick \( \beta_2 \), we first observe that this is easy to do if either \( c_1 = 0 \) or \( c_2 = 0 \). If neither is 0, let us choose first \( \beta_3 \) such that \( \cos(\beta_3) = c_1 \). To do this recall that
\[
\cos(\beta_3) = \frac{e^{i\beta_3} + e^{-i\beta_3}}{2},
\]
so the equation \( c_1 = \cos(\beta_3) \) is equivalent to the equation
\[
e^{2i\beta_3} - 2c_1e^{i\beta_3} + 1 = 0.
\]
This is a quadratic equation for \( \lambda = e^{i\beta_3} \). Its solutions are
\[
\lambda = \frac{2c_1 \pm \sqrt{4c_1^2 - 4}}{2},
\]
and it is easy to check that neither of these can be 0. Since the image of the exponential map is \( \mathbb{C} \setminus \{0\} \), there always is a \( \beta_3 \) such that \( e^{i\beta_3} = \lambda \).

With this choice of \( \beta_3 \) we have \( c_1 = \cos(\beta_3) \). Now this implies that
\[
1 = c_1^2 + c_2^2 = \cos^2(\beta_3) + c_2^2 = 1 - \sin^2(\beta_3) + c_2^2.
\]
We conclude that either \( c_2 = \sin(\beta_3) \) or \( c_2 = -\sin(\beta_3) \). In the first case we set \( \beta_2 = \beta_3 \) and we are done. In the second case we set \( \beta_2 = -\beta_3 \). Since \( \cos(\beta_2) = \cos(-\beta_3) = \cos(\beta_3) \), we are done in this case as well.

\[\square\]

4 Numerical simulations

To visualize the results established in the preceding sections, we simulated the gradient flow for the energy functional \( (0.1) \) using the off-the-shelf finite element analysis solver COMSOL [8] in order to arrive at local minimizers of \( (0.1) \). The computations were performed in a

Fig. 1 Eigenvalue \( \lambda_1 \) (left) and eigenvector \( e_1 \) (right) of the minimizer \( u_\varepsilon \) of \( (0.1) \). The vector field plot zooms in on the region near the singularity of \( u \), represented by a red circle.
square domain \( \Omega \) with the side of length 1, assuming that \( \varepsilon = 0.01 \) and using the boundary data of degree 1/2 with values deviating from a geodesic in \( P \).

Figures 1, 2 and 3 show the eigenvalues and eigenvectors fields for the computed (local) minimizer \( u_\varepsilon \) of (0.1). Because the degree of \( u_\varepsilon \) on the boundary is equal to 1/2, there is a single point in \( \Omega \) where the eigenvalues of \( u_\varepsilon \) should cross and this point should be located near the singularity of the limiting map \( u \). To make the subsequent discussion simpler, we will identify the eigenvalues crossover point of \( u_\varepsilon \) with the singular point of \( u \) in what follows.

In agreement with Proposition 0.1, the third eigenvalue of \( u_\varepsilon \) is asymptotically close to 0, while the corresponding eigenvector field is smooth everywhere in \( \Omega \), including the singular point of \( u \). The first and the second eigenvalues of \( u_\varepsilon \) are equal to 1 and 0, respectively, away from the singularity of \( u \), while at the core of the singularity both of these eigenvalues are close to 1/2.

Further, both the first and the second eigenvectors of \( u_\varepsilon \) have degree 1/2 singularity at the singular point of \( u \)—this is an expected behavior because \( u_\varepsilon \) near the singularity is essentially an \( \mathbb{R}P^1 \)-valued map.

In Fig. 4 we plot the distribution of the entire eigenframe of \( u_\varepsilon \) in a vicinity of the singular point of \( u \).

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Fig. 2  Eigenvalue \( \lambda_2 \) (left) and eigenvector \( e_2 \) (right) of the minimizer \( u_\varepsilon \) of (0.1). The vector field plot zooms in on the region near the singularity of \( u \), represented by a red circle.

Fig. 3  The eigenvalue \( \lambda_3 \) (left) and the eigenvector \( e_3 \) (right) of the minimizer \( u_\varepsilon \) of (0.1). The vector field plot zooms in on the region near the singularity of \( u \), represented by a red circle.
Fig. 4 Eigenframe distribution of the minimizer $u_\varepsilon$ of (0.1). The location of the singularity is marked by a red circle.

Fig. 5 Plot of $|(z - a)\mu_u|$, where $a \in \Omega$ is the location of the singularity of $u$ and $\mu_u$ is as defined in Theorem 0.4.

Figures 5, 6 approximate the behavior of $\mu_u$ that appears in Theorem 0.4 as it is computed using $u_\varepsilon$, rather than $u$. From the statement of Theorem 0.4, in particular from the asymptotic expression for $\omega_u$ and the definition of $\mu_u$, it follows that $|(z - a)\mu_u| \sim 0.25$ near the singularity $a$ of $u$. Indeed, this is what is observed in Fig. 5, except that the approximation of $|(z - a)\mu_u|$ plunges to 0 at $a$, because $\mu_u$ computed using $u_\varepsilon$ instead of $u$ is bounded at $a$. From Theorem 0.4, it also follows that $1/|\mu_u|$ should be linear in the radial coordinate centered at $a$ and this is confirmed by the plot in Fig. 6.
Fig. 6 Plot of $1/|\mu_u|$, where $\mu_u$ is as defined in Theorem 0.4

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5 Appendix

In this appendix we provide a short summary of previous results that are relevant to the present manuscript. We start with the following theorem that was proved in [13].

**Theorem 5.1** Let $g : \partial \Omega \to \mathcal{P}$ be a non-contractible curve in $\mathcal{P}$ and suppose that $u_\varepsilon \in W^{1,2}(\Omega; F_1)$ is a minimizer of $E_\varepsilon$ among functions $u \in W^{1,2}(\Omega; F_1)$ that satisfy the Dirichlet boundary condition $u = g$ on $\partial \Omega$. First, the minimizers $u_\varepsilon$ take values in the convex envelope of $\mathcal{P}$; in particular they are uniformly bounded in $\varepsilon$. Second, there is a single point $a$ in the interior of $\Omega$ such that the $u_\varepsilon$ converge strongly (along a subsequence) to $u_0 \in W^{1,2}(\Omega \setminus B_R(a); \mathcal{P})$ in $W^{1,2}(\Omega \setminus B_R(a); F_1)$ as $\varepsilon \to 0$ for any fixed $R > 0$. Finally, for any open set $U \subset \subset \Omega \setminus \{a\}$, $u_0$ minimizes $\int_U |\nabla v|^2$ among functions $v \in W^{1,2}_{loc}(\Omega \setminus \{a\}; \mathcal{P})$ satisfying $v = u_0$ on $\partial U$.

To describe the structure of $u_0$, let $M^3_a(\mathbb{R})$ be the set of antisymmetric $3 \times 3$ matrices and let $[A; B] = AB - BA$ denote the commutator of matrices $A$ and $B$. It turns out that one can consider a vector field $j(u_0)$ with matrix entries

$$j(u_0) = \left(\begin{bmatrix} u_0, \frac{\partial u_0}{\partial x} \\ u_0, \frac{\partial u_0}{\partial y} \end{bmatrix}, \begin{bmatrix} u_0, \frac{\partial u_0}{\partial x} \\ u_0, \frac{\partial u_0}{\partial y} \end{bmatrix}\right),$$
instead of \( u_0 \) because \( u_0 \) can always be recovered from \( j(u_0) \) (the reason for this reduces to the following standard fact: if \( A : [0, T] \to M^3_d(\mathbb{R}) \), then the solution of the initial value problem

\[
\gamma' = [\gamma, A], \quad \gamma(0) \in \mathcal{P},
\]

takes values in \( \mathcal{P} \). In light of this observation, the following theorem \([13]\) gives a rough description of the limiting map \( u_0 \) described in Theorem 5.1.

**Theorem 5.2** Let \( u_0 \) be as in Theorem 5.1. There is a function

\[
\psi_0 \in (W^{1,2} \cap L^\infty)(\Omega; M^3_d(\mathbb{R}))
\]

and a constant anti-symmetric matrix \( \Lambda_0 \) such that

\[
j(u_0) = \frac{1}{2r}(\hat{\theta} \Lambda_0) + \nabla \perp \psi_0
\]

in \( \Omega \). Here \( a \in \Omega \) is as defined in Theorem 5.1, \( r \) and \( \hat{\theta} \) are the radial variable and the unit vector in an angular direction for polar coordinates centered at \( a \) respectively, and we interpret \((\hat{\theta} \Lambda_0)\) and \( \nabla \perp \psi_0 \) as matrix-valued vector fields according to (5.2) and (5.1) respectively. Further, \( \psi_0 \) satisfies

\[
\Delta \psi_0 = 2 \left[ \frac{\partial \psi_0}{\partial x}, \frac{\partial \psi_0}{\partial y} \right] + \frac{1}{\pi r} \left[ \nabla \psi_0 \cdot \hat{\theta}, \Lambda_0 \right],
\]

in \( \Omega \), where we interpret \( \nabla \psi_0 \cdot \hat{\theta} \) according to (5.3), subject to boundary conditions

\[
-\nabla \psi_0 \cdot \nu = \left[ g, \frac{dg}{d\tau} \right] - \frac{\hat{\theta} \cdot \tau}{2\pi r} \Lambda_0,
\]

on \( \partial \Omega \), where \( \nu \) and \( \tau \) are the outward unit normal and unit tangent vector to \( \partial \Omega \), respectively. Finally, the function \( Z_{u_0}(x) := \frac{1}{2\pi r} \left( \Lambda_0 - u_0 \Lambda_0 - \Lambda_0 u_0 \right) \in L^2(\Omega; M^3_d(\mathbb{R})) \).

Notice that in the preceding theorem we deal with matrix-valued functions \( u : \Omega \to M^3(\mathbb{R}) \) and matrix-valued vector fields \( F : \Omega \to (M^3(\mathbb{R}))^2, F = (F_1, F_2) \). For a matrix-valued function \( u \), the gradient and its perpendicular are given by the matrix-valued vector fields

\[
\nabla u = \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right), \quad \nabla \perp u = \left( \frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x} \right),
\]

respectively. For matrix-valued vector fields, the divergence and curl

\[
\nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}, \quad \nabla \perp \cdot F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}
\]

are matrix-valued functions. When \( z : \Omega \to \mathbb{R}^2 \) and \( A : \Omega \to M^3(\mathbb{R}) \), the matrix-valued vector field \( zA \) has the entries

\[
zA = (z_1A, z_2A).
\]

(5.2)

On the other hand, if \( F \) is a matrix-valued vector field and \( e = (e_1, e_2) \in \mathbb{R}^2 \), we set

\[
F \cdot e = e_1F_1 + e_2F_2,
\]

(5.3)

which is a matrix-valued function. We emphasize the difference between \( F \cdot e \) and \( zA \) defined in (5.2).

In the next proposition we summarize the properties of the potential we use.
Proposition 5.3 For \( u \in M^3_{s,1}(\mathbb{R}) \), define
\[
W_\beta(u) = \frac{1}{4}(1 - |u|^2)^2 - \beta \det(u).
\] (5.4)

We have
\[
W_\beta(u) = \frac{1}{4}(1 - |u|^2)^2 - \frac{\beta}{6}(1 - 3|u|^2 + 2\text{tr}(u^3)).
\] (5.5)

Next, for \( u \in M^3_{s,1}(\mathbb{R}) \) such that \( \langle u, P \rangle \geq 0 \) for all \( P \in \mathcal{P} \), we have
\[
\frac{(1 - |u|^2)^2}{3} \leq |u - u^2|^2 \leq \frac{(1 - |u|^2)^2}{2}.
\] (5.6)

Lastly, we have
\[
W_\beta(u) \geq \frac{(3 - \beta)}{6} \left( \text{dist}(u, \mathcal{P}) \right)^2.
\] (5.7)

Proof We start by recalling that Cayley-Hamilton theorem for matrices \( u \in M^3_{s,1}(\mathbb{R}) \) tells us
\[
u^3 - u^2 + \frac{(1 - |u|^2)}{2}u - \det(u)I = 0.
\]
Applying this in (5.4) gives us (5.5).

Next, again from Cayley-Hamilton theorem, for \( u \in M^3_{s,1}(\mathbb{R}) \), we obtain
\[
|u - u^2|^2 + 2\det(u) = \frac{(1 - |u|^2)^2}{2}.
\] (5.8)

In particular, if \( u \in M^3_{s,1}(\mathbb{R}) \) has \( \langle u, P \rangle \geq 0 \) for all \( P \in \mathcal{P} \), then
\[
\frac{(1 - |u|^2)^2}{3} \leq |u - u^2|^2 \leq \frac{(1 - |u|^2)^2}{2}.
\] This is (5.6)

Under these conditions, if \( \text{dist}(u, \mathcal{P}) \leq \delta \leq \frac{1}{3} \), it is not hard to check that
\[
\frac{2}{3} \text{dist}(u, \mathcal{P}) \leq 2 \frac{|u - u^2|}{1 - \delta} \leq \frac{|1 - |u|^2|}{1 - \delta}.
\]
Furthermore, again for \( u \in M^3_{s,1}(\mathbb{R}) \) such that \( \langle u, P \rangle \geq 0 \) for all \( P \in \mathcal{P} \), a lengthy, but ultimately straight forward minimization shows that
\[
(1 - |u|^2)^2 \geq 12 \det(u).
\]
Hence, for \( 1 \leq \beta < 3 \), and \( u \in M^3_{s,1}(\mathbb{R}) \) such that \( \langle u, P \rangle \geq 0 \) for all \( P \in \mathcal{P} \), we have
\[
W_\beta(u) = \frac{1}{4}(1 - |u|^2)^2 - \beta \det(u) = \frac{3 - \beta}{12}(1 - |u|^2)^2 + \beta(\frac{1}{12}(1 - |u|^2)^2 - \det(u))
\]
\[
\geq \frac{3 - \beta}{12}(1 - |u|^2)^2 \geq \frac{(3 - \beta)}{6} \left( \text{dist}(u, \mathcal{P}) \right)^2.
\]
This is (5.7). \( \square \)
Remark 5.4 The potential $W_{\beta}$ in [13] is written as

$$W_{\beta}(u) = \frac{1}{2} (1 - |u|^2)^2 - \beta \det(u).$$

As can be seen from (5.8), in order for this potential to be equal $|u - u^2|^2$ we need to choose $\beta = 2$. In [13] we erroneously stated that our results were valid for $2 < \beta < 6$ while the correct range should be $2 \leq \beta < 6$. In this paper, however, we use the expression given in (5.4).

Proposition 5.5 Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded, simply-connected open set, and $u_\varepsilon : \Omega \rightarrow M_{3,1}^3(\mathbb{R})$ be a minimizer of the LdG energy with non-contractible boundary data in $\mathcal{P}$. Let $a \in \Omega$ be the distinguished point in $\Omega$ that Theorem 5.1 shows exist. For $r > 0$ such that $B_{2r}(a) \subset \Omega$, there is $\varepsilon_0 > 0$ and a constant $C > 0$ such that

$$|\nabla u_\varepsilon(x)| + \frac{(1 - |u_\varepsilon|^2)}{\varepsilon^2} \leq C$$

for all $x \in \Omega \setminus B_r(a)$, and all $0 < \varepsilon \leq \varepsilon_0$.

Proof The proof follows [3]. Let us observe that the end of the proof of Lemma 8 of [13] shows that minimizers $u_\varepsilon$ satisfy

$$\limsup_{\varepsilon \to 0} \int_{\Omega \setminus B_r(a)} \frac{W(u_\varepsilon)}{\varepsilon^2} = 0.$$ 

We now appeal to Steps A.2 and B.2 of the proof of Theorem 1 of [2], to conclude that $W(u_\varepsilon) \to 0$ uniformly in $\Omega \setminus B_r(a)$. In particular, for $\delta > 0$ we can choose $\varepsilon_0 > 0$ such that

$$0 \leq 1 - |u_\varepsilon|^2 \leq \delta$$

for all $x \in \Omega \setminus B_r(a)$ and all $0 < \varepsilon \leq \varepsilon_0$.

We next recall from the appendix of [13] that

$$\frac{4 - \beta}{\varepsilon^2} (1 - |u|^2) - 4 W_{3\beta} \frac{\Delta}{\varepsilon} = -\frac{|u|^2}{2} + |\nabla u|^2 = -\langle u, \Delta u \rangle.$$ 

We know from [13] that $\langle u_\varepsilon, P \rangle \geq 0$ for all $P \in \mathcal{P}$, so we deduce

$$4 W_{3\beta}(u_\varepsilon) \leq (1 - |u|^2)^2.$$ 

Hence, we can choose $\varepsilon_0 > 0$ small enough for

$$\frac{4 - \beta}{\varepsilon^2} (1 - |u_\varepsilon|^2) - 4 W_{3\beta}(u_\varepsilon) \geq \delta (1 - |u_\varepsilon|^2)$$

in $\Omega \setminus B_r(a)$, for all $0 < \varepsilon \leq \varepsilon_0$. Since $|u_\varepsilon| \leq 1$, we conclude that

$$|\Delta u_\varepsilon| \geq \delta (1 - |u_\varepsilon|^2)$$

in $\Omega \setminus B_r(a)$, for all $0 < \varepsilon \leq \varepsilon_0$.

From the Euler-Lagrange equation for $u_\varepsilon$, we obtain

$$-\Delta \frac{\partial u_\varepsilon}{\partial x_k} + \frac{1}{\varepsilon^2} (D^2 u W)(u) \left( \frac{\partial u_\varepsilon}{\partial x_k} \right) = \frac{\partial \lambda_\varepsilon}{\partial x_k},$$

\[ Springer \]
and then
\[ \Delta |\nabla u_\varepsilon|^2 = 2 |D_\varepsilon^2 u|^2 + \frac{2}{\varepsilon^2} \sum_{k=1}^2 \langle (D^2 W)(u) \left( \frac{\partial u_\varepsilon}{\partial x_k} \right), \frac{\partial u_\varepsilon}{\partial x_k} \rangle. \]

Now, writing \( v_\varepsilon \) for the nearest element of \( P \) to \( u_\varepsilon \), we have
\[
\langle (D^2 W_\beta)(u_\varepsilon) \left( \frac{\partial u_\varepsilon}{\partial x_k} \right), \frac{\partial u_\varepsilon}{\partial x_k} \rangle = \langle (D^2 W_\beta)(v_\varepsilon) \left( \frac{\partial u_\varepsilon}{\partial x_k} \right), \frac{\partial u_\varepsilon}{\partial x_k} \rangle + \langle (D^2 W_\beta)(u_\varepsilon) \left( \frac{\partial u_\varepsilon}{\partial x_k} \right) - (D^2 W_\beta)(v_\varepsilon) \left( \frac{\partial u_\varepsilon}{\partial x_k} \right), \frac{\partial u_\varepsilon}{\partial x_k} \rangle.
\]

Now \( v_\varepsilon \in P \), which is the set of minimizers of \( W_\beta \), and it is easy to check that \( \text{tr} \left( \frac{\partial u_\varepsilon}{\partial x_k} \right) = 0 \).

Hence
\[
\langle (D^2 W_\beta)(v_\varepsilon) \left( \frac{\partial u_\varepsilon}{\partial x_k} \right), \frac{\partial u_\varepsilon}{\partial x_k} \rangle \geq 0.
\]

We deduce that
\[
\langle (D^2 W_\beta)(u_\varepsilon) \left( \frac{\partial u_\varepsilon}{\partial x_k} \right), \frac{\partial u_\varepsilon}{\partial x_k} \rangle \geq -C |u_\varepsilon - v_\varepsilon| \left\| \frac{\partial u_\varepsilon}{\partial x_k} \right\|^2 \geq -C \left(1 - |u_\varepsilon|^2 \right) \left\| \frac{\partial u_\varepsilon}{\partial x_k} \right\|^2,
\]
where the last inequality holds because from the comments before the proposition we have
\[ |u_\varepsilon - v_\varepsilon| = \text{dist}(u_\varepsilon, P) \leq C (1 - |u|^2). \]

We conclude that
\[
\Delta |\nabla u_\varepsilon|^2 \geq 2 |D_\varepsilon^2 u|^2 - C \frac{1 - |u_\varepsilon|^2}{\varepsilon^2} |\nabla u|^2 \geq 2 |D_\varepsilon^2 u|^2 - C |\Delta u_\varepsilon| |\nabla u|^2.
\]

Since this implies
\[
\Delta |\nabla u_\varepsilon|^2 \geq |D_\varepsilon^2 u_\varepsilon|^2 - C |\nabla u|^4
\]
in \( \Omega \setminus B_r(a) \), we can apply Steps A.4 and B.3 of the proof of Theorem 1 of [2] to conclude that
\[ |\nabla u_\varepsilon| \leq C \]
in \( \Omega \setminus B_r(a) \), for some constant independent of \( \varepsilon \in ]0, \varepsilon_0[ \).

Finally, we recall from [13] that
\[
\Delta \frac{|u_\varepsilon|^2}{2} = \frac{4}{\varepsilon^2} W_{3\beta}(u_\varepsilon) - \frac{4 - \beta}{\varepsilon^2} (1 - |u_\varepsilon|^2) + |\nabla u_\varepsilon|^2.
\]

From here, \( \zeta = 1 - |u_\varepsilon|^2 \) satisfies
\[
-\Delta \zeta + \frac{4 - \beta}{\varepsilon^2} \zeta = |\nabla u_\varepsilon|^2.
\]

Steps A.5 and B.4 of Theorem 1 of [2] give us the last conclusion of the proposition. \( \Box \)
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