Seesaw in the Bulk

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A five-dimensional seesaw framework is analyzed with the lepton-number-violating propagator of bulk right-handed neutrinos. That can bypass summing up the effects of heavy Majorana particles whose masses and wavefunctions are not exactly known. The propagator method makes it easier to evaluate the seesaw-induced neutrino mass for various boundary conditions of bulk neutrinos and in a general background geometry, including the warped extra dimension. It is also found that the higher-dimensional seesaw gives a natural framework for the inverse seesaw suppression of low-energy neutrino masses.

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§1. Introduction

The discovery of nonzero neutrino masses is one of the most impressive developments in particle physics recently made. In addition to cosmological observations, the flux measurements of solar and atmospheric neutrinos indicate that neutrino masses are tiny compared to the other fermion masses.\(^1\) The smallness of neutrino masses may be regarded as an indication of higher energy scale than the electroweak one, possibly connected with deeper concepts such as grand unification and other anticipated scenarios beyond the Standard Model (SM).

As a feasible paradigm to address problems in the SM, theories with extra spatial dimensions have been widely studied over the past decade. For example, the gauge hierarchy problem is solved by large volume of the extra space which makes the Planck scale suppressed down to TeV.\(^2\) The localized gravity with the warped metric\(^3\) also provides a possible interpretation of the gauge hierarchy by small overlap between matter and gravitational fields. Interestingly, these mechanisms for the hierarchy problem also control the neutrino mass. As in the same way that the gravitational flux is diluted, the neutrino mass is suppressed if gauge-singlet neutrinos propagate in the bulk.\(^4\) For the warped extra dimension, the localization of bulk neutrinos produces tiny Dirac neutrino masses.\(^5\) Thus the connection between neutrino physics and extra dimensions has been a subject of great interests to particle physics.\(^6\)

Motivated by this phenomenological connection, we discuss the seesaw mechanism in higher-dimensional theory where the right-handed neutrinos spread over the extra space. Our main emphasis in this paper is on the lepton-number violating propagator which serves as a useful tool to calculate the low-energy neutrino mass.

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Fig. 1. A sketch of the model. The SM fields are localized at \( y = L \) while the right-handed neutrinos \( \Psi(x, y) \) spread over the extra space with bulk Dirac and Majorana masses.

induced by the seesaw operation. This paper covers several issues which have been left untouched in the previous study;\(^7\) the derivation of propagators, the equivalence to the mode expansion, various applications of propagators including the multi-generation case, etc. We also classify possible seesaw mass formulas and survey the whole picture of the five-dimensional seesaw with bulk Majorana and Dirac masses. That gives a useful insight into the seesaw in the bulk; for example, it generally realizes the inverse seesaw at low-energy effective theory, which could make the scenario testable at future experiments.

This paper is organized as follows. In §2, we discuss the setup of the higher-dimensional seesaw and formulate the bulk field propagator. In §3, we present various applications of the propagator method to higher-dimensional seesaw models. Section 4 is devoted to summarizing the results. Appendices A and B show our convention for Lorentz spinors and the derivation of propagators in various situations.

§2. Seesaw in five dimensions

In this section, we introduce the framework of higher-dimensional seesaw mechanism and illustrate the Kaluza-Klein (KK) expansion and the propagator method to obtain low-energy neutrino masses.

2.1. Setup

The fifth dimension \( y \) is compactified on the \( S^1/Z_2 \) orbifold such that there are two fixed points at \( y = 0 \) and \( L \). The four-component bulk fermions \( \Psi(x, y) \) are introduced as right-handed neutrinos and obey the boundary conditions associated with two operations on the \( S^1/Z_2 \) space; the translation \( \hat{T} : y \to y + 2L \) and the parity \( \hat{Z} : y \to -y \). These conditions are written as

\[
\Psi(x, -y) = (Z \otimes \gamma_5)\Psi(x, y), \quad \Psi(x, y + 2L) = T\Psi(x, y),
\]  

(2.1)
where $Z$ and $T$ are the matrices acting on the field space. The translation and parity imply that $Z^2 = 1$ and $ZT = T^{-1}Z$ should be satisfied. Instead of the translation $T$ in (2.1), another parity $Z' = TZ$ can be used to define the boundary conditions:

$$
\Psi(x, -y) = (Z \otimes \gamma_5)\Psi(x, y), \quad \Psi(x, L - y) = (Z' \otimes \gamma_5)\Psi(x, L + y).
$$

The parity $Z'$ is the reflection with respect to $y = L$ and Eq. (2.2) choose the Dirichlet or Neumann condition at each boundary. In this section, we consider the standard condition $Z = 1$ and $Z' = 1$ with which the upper (right-handed) component has the zero mode. The other possibilities and their physical implication will be discussed in §3 and Appendix B.

The SM fields including the left-handed neutrinos $N = P_L N = (\nu^0_L)$ and the Higgs field $H$ are assumed to be localized at $y = L$ (Fig. 1). This SM-field profile gives an example and the analysis below is applied to other cases in a similar manner. The seesaw mechanism in five dimensions is described by the following bulk and boundary Lagrangians:

$$
L_{\text{bulk}} = i\bar{\Psi} T^M \partial_M \Psi - m_d \theta(y) \bar{\Psi} \Psi - \frac{1}{2} \left( M \bar{\Psi}^c \Psi + \text{h.c.} \right),
$$

$$
L_{\text{bound}} = -\left( \frac{m}{\sqrt{\Lambda}} \bar{\Psi} N + \text{h.c.} \right) \delta(y - L),
$$

where $\Lambda$ stands for the fundamental scale of the theory. We have introduced the bulk Dirac mass $m_d$ with the step function $\theta(y)$ which is needed to implement the $Z_2$ invariance. The mass parameters $m_d$ and $M$ are assumed to be flavor diagonal in this section, while that will be relaxed later. The boundary Dirac mass $m$ is made out of the neutrino Yukawa coupling and the vacuum expectation value of the Higgs field $\langle H \rangle = (0_v)$. The charge-conjugate spinor $\Psi^c$ is defined by $\Psi^c \equiv \Gamma^3 \Gamma^1 \Psi^T$. Our convention for the gamma matrices and Lorentz spinors are given in Appendix A.

2.2. The KK expansion

There are two physically equivalent, but different prescriptions to derive four-dimensional effective theory from the original five-dimensional Lagrangian: the KK expansion and the propagator method for bulk fields.

With the KK expansion, the neutrino spectrum is obtained by the diagonalization of the infinite-dimensional mass matrix spanned by KK modes and the SM neutrinos. A bulk field $\Psi(x, y)$ is expanded as

$$
\Psi(x, y) = \left( \begin{array}{c}
\sum_n \chi^R_n(y) \psi^R_n(x) \\
\sum_n \chi^L_n(y) \psi^L_n(x)
\end{array} \right),
$$

with the orthogonal systems $\chi^R_{nR,L}(y)$. It is convenient to choose them to satisfy

$$
\left[ \partial_y + m_d \theta(y) \right] \chi^R_n = +M_{Kn} \chi^L_n, \quad (2.6)
$$

$$
\left[ \partial_y - m_d \theta(y) \right] \chi^L_n = -M_{Kn} \chi^R_n, \quad (2.7)
$$

and the normalization conditions $\int_0^L dy \chi^R_{nR,L} \chi^R_{nR,L} = \delta_{mn}$. Under the boundary conditions $Z = 1$ and $Z' = 1$, the expansion functions $\chi^R_{nR,L}$ and the KK mass $M_{Kn}$ are
The mass spectrum of Majorana neutrinos is obtained by diagonalizing $\chi_R^n$, where $N_0 = \sqrt{m_dL/(1-e^{-2m_dL})}$ and $N_n = (\frac{n\pi}{L})/M_{K_n}$. The zero-mode function $\chi_R^0$ is localized at $y = 0$ ($y = L$) due to the bulk Dirac mass if $m_d > 0$ ($m_d < 0$).

By substituting the KK expansion into the five-dimensional Lagrangian and integrating it over the extra space, we have the four-dimensional effective Lagrangian

$$\mathcal{L}_4 = i\mathcal{N}^\dagger \sigma^\mu \partial_\mu \mathcal{N} - \frac{1}{2} (\mathcal{N}^T \epsilon \mathcal{M} \mathcal{N} + \text{h.c.}),$$

where $\mathcal{M} = \begin{pmatrix} 0 & \mathcal{M}_D^T \\ \mathcal{M}_D & \mathcal{M}_H \end{pmatrix}$, $\mathcal{M}_D = \begin{pmatrix} m_0^T \\ m_1 \\ \vdots \end{pmatrix}$, $\mathcal{M}_H = \begin{pmatrix} m_0 \\ -M_{R0}^T \\ \vdots \end{pmatrix}$, $\mathcal{N} = \begin{pmatrix} \nu_L \\ \epsilon \psi_R^0 \\ \epsilon \psi_R^1 \end{pmatrix}$.

$$M_{Rnm} = \int_0^L dy \chi_R^{nT} M \chi_R^m = M_{\delta nm}, \quad M_{Lnm} = \int_0^L dy \chi_L^{nT} M \chi_L^m = M_{\delta nm},$$

$$m_n = \frac{m}{\sqrt{A}} \chi_R^{nT} (L) = \begin{cases} m \sqrt{\frac{2}{A}} N_0 e^{-m_dL}, & (n = 0) \\ m \sqrt{\frac{2}{A}} N_n (-1)^{n+1}, & (n \geq 1) \end{cases}$$

The mass spectrum of Majorana neutrinos is obtained by diagonalizing $\mathcal{M}$. For $\mathcal{M}_D \ll \mathcal{M}_H$, the seesaw mechanism works and the Majorana mass of light neutrinos $M_\nu$ is approximately given by

$$M_\nu = -M_D^T \mathcal{M}_H^{-1} \mathcal{M}_D.$$
where \( \tilde{M} = \sqrt{m^2_d + |M|^2} \). We will discuss physical implications of this result in §3.

2.3. Propagators for bulk Majorana fermions

In the propagator method, the low-energy neutrino mass is calculated by treating the bulk-boundary mixing (2.4) as a small perturbation. Let us start with the functional integral

\[
Z = \int \mathcal{D}\Psi^* \mathcal{D}\Psi \exp \left[ i \int d^5x \left( \mathcal{L}_{\text{bulk}} + \mathcal{L}_{\text{bound}} \right) \right]
\]

\[
= \int \mathcal{D}\Psi^* \mathcal{D}\Psi \left[ 1 + \frac{1}{2} \left( i \int d^5x \mathcal{L}_{\text{bound}} \right)^2 + \ldots \right] \exp \left( i \int d^5x \mathcal{L}_{\text{bulk}} \right). \tag{2.17}
\]

After integrating out the bulk fermions \( \Psi(x, y) = \left( \xi(x,y) \right. \right. \) \( \left. \left. \eta(x,y) \right) \), the lepton-number-violating part in the quadratic term is given by

\[
\frac{(i)^2}{2} \int d^4xd^4x' \left[ -N_c(x) \frac{m^T}{\sqrt{\Lambda}} \int \frac{d^4p}{(2\pi)^4} \langle \psi^c(p, L) \psi(p, L) \rangle e^{ip(x-x')} \frac{m}{\sqrt{\Lambda}} N(x') \right] + \text{h.c.},
\]

where \( \Psi(p, y) = \int d^4x \Psi(x, y)e^{ipx} \). From this expression, the seesaw-induced tiny mass of light Majorana neutrinos is found in the low-energy regime \( p \to 0 \);

\[
M_\nu = -\frac{m^T}{\sqrt{\Lambda}} \left. \langle i\epsilon \xi^*(p, L) \xi^\dagger(p, L) \rangle \frac{m}{\sqrt{\Lambda}} \right|_{p=0}. \tag{2.18}
\]

The left-handed component \( \eta(x, y) \) does not join in the seesaw operation as it obeys the Dirichlet conditions at the boundaries. Higher-order contributions in \( p^2 \) are negligible if the energy scale of interest is much smaller than the masses of intermediate states.

The lepton-number-violating part of the correlator is obtained by inverting the five-dimensional Dirac operator in the presence of bulk Majorana masses. The differential equations for the propagators become in the mixed position-momentum space

\[
\left[ p^2 - m^2_d - |M|^2 + \partial_y^2 - 2m_d[\delta(y) - \delta(y - L)] \right] \langle i\epsilon \xi^*(p, y) \xi^\dagger(p, y') \rangle = M \delta(y - y'), \tag{2.19}
\]

\[
\left[ p^2 - m^2_d - |M|^2 + \partial_y^2 + 2m_d[\delta(y) - \delta(y - L)] \right] \langle i\epsilon \xi^*(p, y) \xi^\dagger(p, y') \rangle = M \delta(y - y'). \tag{2.20}
\]
Solving these equations under the boundary conditions \( Z = 1 \) and \( Z' = 1 \), we find

\[
\langle i \epsilon \eta^*(p, y) \eta^\dagger(p, y') \rangle = \frac{\sinh(q y_<) \sinh(q y_> - qL)}{q \sinh(qL)} m, \tag{2.21}
\]

\[
\langle i \epsilon \xi^*(p, y) \xi^\dagger(p, y') \rangle = \frac{1}{(m_d^2 - q^2)q \sinh(qL)} \left[ q \cosh(q y_<) - m_d \sinh(q y_<) \right] \times \left[ q \cosh(q y_> - qL) - m_d \sinh(q y_> - qL) \right] m, \tag{2.22}
\]

where \( y_<(y_<) \) stands for the lesser (greater) of \( y \) and \( y' \), and \( q \equiv \sqrt{m_d^2 + |M|^2 - p^2} \).

Appendix B shows specific details of the derivation of propagators.

With the propagators for bulk Majorana fermions at hand, the seesaw neutrino mass is obtained by taking the low-energy limit \( p \to 0 \) and setting \( y = y' = L \) in the propagators. Thus the neutrino mass reads

\[
M_\nu = \frac{1}{\lambda L} \frac{\tilde{M} L \cosh(\tilde{M} L) - m_d L \sinh(\tilde{M} L)}{\sinh(\tilde{M} L)} \frac{m^T m}{M^*}, \tag{2.23}
\]

that exactly agrees with the previous result (2.16). Once the propagator is found, one needs neither to work on infinite-size KK mass matrices nor to sum up the KK contributions in the seesaw formula.

2.4. Equivalence of two methods

The correspondence between the KK expansion and the propagator method is clarified with the mode expansion \( \xi(p, y) = \sum_n \chi_R^n(y) \psi_R^n(p) \):

\[
\frac{m^T}{\sqrt{\lambda}} \langle -i \epsilon \xi^*(p, L) \xi^\dagger(p, L) \rangle \frac{m}{\sqrt{\lambda}} = \sum_{k,n} \frac{m^T}{\sqrt{\lambda}} \chi_R^k(L)^* \langle -i \epsilon \psi_R^k(p)^* \psi_R^n(p)^\dagger \rangle \chi_R^n(L)^\dagger \frac{m}{\sqrt{\lambda}}
\]

\[
= \sum_{k,n} m_k^T \left( \frac{M_H^*}{p^2 - M_H^* M_H} \right) R_{kn} m_n, \tag{2.24}
\]

where \( R_{kn} \) means the matrix element for the KK right-handed neutrinos \( \psi_R^{k,n} \). Taking \( p \to 0 \) limit exactly reproduces the seesaw formula (2.15) in the KK expansion method. A schematic view of the correspondence is shown in Fig. 2. The propagator method avoids the jobs of mode expansion and re-summation, and simplifies the model analysis.

The mass spectrum in the four-dimensional effective theory is found from the poles of (2.21) and (2.22). For example, \( p^2 = |M|^2 \) in the denominator of (2.22) corresponds to the chiral zero mode \( \psi_R^0 \) with the mass \( M \). The other mass eigenvalues appear at \( qL = \pm n \pi \) in (2.21) and (2.22). These poles, \( p^2 = m_d^2 + |M|^2 + \left( \frac{n \pi}{L} \right)^2 \), are consistent with the KK expansion discussed in §2.2; the \( n \)-th \( 2 \times 2 \) block in the heavy-sector mass matrix (2.13) is diagonalized as

\[
\left( \begin{array}{cc}
-M^* & \sqrt{m_d^2 + \left( \frac{n \pi}{L} \right)^2} \\
\sqrt{m_d^2 + \left( \frac{n \pi}{L} \right)^2} & M
\end{array} \right) \to \left( \begin{array}{cc}
\sqrt{m_d^2 + |M|^2 + \left( \frac{n \pi}{L} \right)^2} & 0 \\
0 & -\sqrt{m_d^2 + |M|^2 + \left( \frac{n \pi}{L} \right)^2}
\end{array} \right).
\]
Thus the propagator method also reduces the efforts of expansion and (generally complicated) diagonalization to have physical mass eigenvalues.

§3. Applications of the propagator

In this section, we apply the propagator to the higher-dimensional seesaw model described in §2.1. We discuss neutrino-mass phenomenology under various types of boundary conditions, applications to flavor models, and the seesaw mechanism in the warped geometry.

3.1. The standard seesaw-induced mass

For the standard boundary condition \((Z = 1\) and \(Z' = 1\)), the neutrino mass after the seesaw operation is given by (2.23). Let us first study a simple case of vanishing bulk Dirac mass \(m_d = 0\). The seesaw-induced mass becomes

\[
M_\nu = \frac{1}{\Lambda L} \frac{|M|L}{\tanh (|M|L)} m^T m \frac{M^*}{M^*}.
\]

(3.1)

The effect of extra dimension is evident in the appearance of hyperbolic factor. If the contribution from the KK-excited modes is negligible, the seesaw with only the zero mode gives a usual four-dimension-like formula

\[
M_\nu^{(0)} = \frac{m_0 m_0}{M^*} = \frac{1}{\Lambda L} \frac{m^T m}{M^*}.
\]

(3.2)

To see how the zero-mode approximation is related to the complete formula (3.1), consider two extreme cases in (3.1);

\[
M_\nu \simeq \frac{1}{\Lambda L} \frac{m^T m}{M^*} \quad \text{for} \quad ML \ll 1,
\]

(3.3)

\[
M_\nu \simeq \frac{1}{\Lambda L} \frac{m^T m}{1/L} \quad \text{for} \quad ML \gg 1.
\]

Fig. 2. The seesaw mechanism in views of the five-dimensional propagator and the KK summation in four-dimensional effective theory.
In the former limit, the KK-excited modes are decoupled, more exactly, each nonzero KK level has much smaller Majorana mass (lepton number violation) than its mass eigenvalue and gives negligible contribution to the seesaw Majorana mass. The result hence coincides with the zero-mode seesaw (3·2). In the latter case, the effects of higher KK modes \((n' \gtrsim ML)\) are dropped due to the same reason as the former limit, and the lower-mode contributions are piled up to giving \(\frac{1}{L} \times n' \simeq \frac{1}{L} M\). In any case, the heavy mass scale in the seesaw mechanism is determined by a smaller one between \(M\) and \(1/L\). If the neutrino Yukawa couplings are of order unity, \(\Lambda > 10^{14}\) GeV \((\Lambda \sim 10^{14}\) GeV\) for \(O(\text{eV})\) neutrino masses in the former (latter) case. The other two scales, \(M\) and \(1/L\), vary widely depending on the cutoff \(\Lambda\).

For the full expression (2·23) including the bulk Dirac mass, an interesting case is that \(m_d\) is much larger than the other mass scales in the bulk, \(M\) and \(1/L\). In the regime \(m_d \gg M\) and \(\tilde{M} \gg 1/L\), the seesaw-induced mass turns out to be

\[
M_\nu \simeq \frac{1}{\Lambda L} M \frac{m_T m}{m_d} 1/L . \tag{3·4}
\]

The obtained neutrino mass decreases as the lepton number violation (the right-handed Majorana neutrino mass) decreases. That is the opposite behavior to the usual seesaw mechanism. In view of the KK expansion, the inverse seesaw\(^8)\) takes place in each KK level.

An essential point for realizing the inverse seesaw is pairing two KK neutrinos in each level into one pseudo Dirac fermion. That is, Majorana masses of two spinors should be much smaller than their lepton-number-conserving Dirac mixing. It seems unlikely that the inverse seesaw occurs since a chiral zero mode exists and does not belong to the vector-like KK tower. However the zero mode has a localized wavefunction which can make its seesaw contribution irrelevant. In the regime \(m_d \gg M\) and \(\tilde{M} \gg 1/L\), the bulk Dirac mass \(m_d\) is large so that the zero mode is localized towards the \(y = 0\) boundary and its bulk–boundary interaction at \(y = L\) is exponentially suppressed. On the other hand, the KK-excited modes have sizable bulk-boundary interactions compared to the zero mode [see the wavefunctions (2·14)]. Subtracting the zero mode, the KK mass matrix \(M\) is effectively written as

\[
M \simeq \begin{pmatrix}
m_1^T & \cdots & m_1^{T} \\
m_1 & M_{K_1} & \cdots \\
& & & \ddots
\end{pmatrix} . \tag{3·5}
\]

Mode by mode, the inverse seesaw takes place if \(M_{K_n} = \sqrt{m_d^2 + (n\pi/L)^2} \gg |M|\). The inverse seesaw effect is not available unless the zero mode is localized away from the SM boundary. This is consistent with the fact that a positive \(m_d\) is needed for (3·4) in the propagator method.

For the inverse seesaw case (3·4), a large scale such as the Planck or grand unification scale is not necessary for producing \(\text{eV}-\text{scale neutrino masses. For example, if }\Lambda\) and \(m_d\) are around TeV, the bulk Majorana mass is \(M \sim 10^2\) eV for the neutrino...
Yukawa coupling of order unity. Thus the model could be reconciled with low-cutoff scenarios such as the large extra dimensions.*

3.2. Twisted boundary conditions

Next we study other types of boundary conditions for bulk right-handed neutrinos. The seesaw setup is the same as before (Fig. 1). Let us consider the case $Z = -1$ and $Z' = +1$, namely, different reflection profiles are assigned at two boundaries. In this case, a non-trivial twist à la Scherk-Schwarz\(^{9}\) is generated by the translation along the extra dimension; $T = Z'Z = -1$. The seesaw mass formula (2.18) is unchanged and only difference is the form of propagator. Using the propagator presented in Appendix B, we find the seesaw-induced neutrino mass for $Z = -1$ and $Z' = +1$,

$$M_\nu = \frac{1}{AL} \frac{(|M|L)^2 \sinh(\tilde{M}L)}{ML \cosh(ML) + m_d L \sinh(ML)} \frac{m^T m}{M^*}. \quad (3.6)$$

A vanishing Dirac mass, $m_d = 0$, reveals an essential difference from the standard boundary condition. For the present twisted boundary condition, we have

$$M_\nu = \frac{1}{AL} |M|L \tanh (|M|L) \frac{m^T m}{M^*}. \quad (3.7)$$

Contrary to (3.1), the neutrino mass has a tanh factor in the numerator and behaves as

$$M_\nu \simeq \frac{1}{AL} (ML)^2 \frac{m^T m}{M} \quad \text{for} \quad ML \ll 1,$$

$$M_\nu \simeq \frac{1}{AL} \frac{m^T m}{1/L} \quad \text{for} \quad ML \gg 1. \quad (3.8)$$

The latter case [$\tanh(|M|L) \simeq 1$] leads to the same result as the standard one. This is because, in the large-size limit of extra dimension, the difference of boundary conditions at $y = 0$ is irrelevant to the physics at $y = L$ where the SM fields reside. The former case $ML \ll 1$ shows up an interesting feature of the twisted boundary condition; the seesaw-induced mass is proportional to the Majorana mass parameter of bulk heavy neutrinos.

In terms of the KK expansion, such unusual seesaw behavior is understood as the inverse seesaw suppression, which is similar to the previous standard case (3.4). In the present case, the inverse seesaw is achieved by the twisted boundary condition which forbids no-winding wavefunctions. That is, the zero mode is absent in the effective theory. (In the standard case, the zero-mode effect is suppressed by its localized wavefunction.) With the twisted boundary condition $Z = -1$ and $Z' = +1$, the KK wavefunctions and masses are given by

$$\chi^n_R = \sqrt{\frac{2}{L}} \cos (MK_n y), \quad \chi^n_L = \sqrt{\frac{2}{L}} \sin (MK_n y), \quad MK_n = \left(n - \frac{1}{2}\right) \frac{\pi}{L}, \quad (3.9)$$

\(^{*}\) In this case, the present setup is interpreted as a subspace of higher dimensions not to conflict with experimental bounds.
Table I. The seesaw-induced neutrino masses $M_\nu$ in various situations. The entries with symbols “*” have further suppression factors beyond the standard seesaw up to the volume factor $1/AL$.

| Type   | Condition | Boundary Conditions | $\Lambda L$ | $1/AL$ |
|--------|-----------|---------------------|-------------|--------|
| Type A | $m_d \gg M, 1/L$ | $(Z, Z') = (+1, +1)$ | $* \frac{M_d}{L} \frac{m_1^2}{\Lambda L}$ | $* \frac{M_d}{L} \frac{m_1^2}{\Lambda L}$ |
| Type B | $1/L \gg M, m_d$ | $\frac{m_d^2}{L}$ | $\frac{m_d^2}{L}$ | $\frac{1}{\Lambda L} (ML)^2 \frac{m_d^2}{L}$ |
| Type C | $M \gg m_d, 1/L$ | $\frac{m_d^2}{L}$ | $\frac{m_d^2}{L}$ | $\frac{1}{\Lambda L} (ML)^2 \frac{m_d^2}{L}$ |

for $n \geq 1$. The neutrino mass matrix is then given by the same form as (3.5) and the inverse seesaw takes place if $ML \ll 1$.

Table I shows the seesaw-induced neutrino mass $M_\nu$ for various limits and boundary conditions. The columns represent two patterns of boundary conditions $(Z, Z')$ and the rows possible hierarchies among the mass parameters $M$, $1/L$, and $m_d$. The other two conditions $(Z, Z') = (+1, -1)$ and $(-1, -1)$ correspond to the exchanges of right- and left-handed components of bulk fermions. If $1/L$ is the largest, it turns out that the neutrino mass does not depend on the ordering of $M$ and $m_d$. Further, if $1/L$ is not the largest, only the hierarchy between $M$ and $m_d$ affects the results. Therefore three ordering patterns (labeled by Type A, B, C in Table I) are practically relevant. The entries with symbols “*” are viable patterns for the inverse seesaw suppression.

Out of six general possibilities, three patterns have suppression factors by the inverse seesaw. Two of them are already discussed: Type A with the standard condition [Eq. (3-4)] and Type B with the opposite parity at the distant brane [Eq. (3-8)]. The third case belongs to the Type A hierarchy with the twisted boundary condition, and results in the same form as the non-twisted case. In any case, a small bulk Majorana mass $M$ is a key to obtain suppressed neutrino masses.

3.3. Flavor symmetry breaking

We have so far focused on the eigenvalues of seesaw-induced masses. The higher-dimensional seesaw also gives an interesting possibility for the generation structure of light neutrinos, e.g., the boundary condition breaking of flavor symmetry. We will show in this section that the propagator method simplifies the previous KK-mode analysis and also makes it clear how flavor symmetry is broken down.

The setup is the same as before and the Lagrangian is written down with the generation indices $(i, j = 1, 2, 3)$

$$
\mathcal{L}_{\text{bulk}} = i \bar{\Psi}_j \Gamma^M \partial_M \Psi_j - m_{d_{ij}} \theta(y) \bar{\Psi}_i \Psi_j - \frac{1}{2} (M_{ij} \bar{\Psi}_i \Psi_j + \text{h.c.}),
$$

(3-10)

$$
\mathcal{L}_{\text{boundary}} = - \left( \frac{m_{ij}}{\sqrt{A}} \bar{\Psi}_i N_j + \text{h.c.} \right) \delta(y - L).
$$

(3-11)

The general boundary conditions are

$$
\Psi_i(x, -y) = (Z_{ij} \otimes \gamma_5) \Psi_j(x, y), \quad \Psi_i(x, L - y) = (Z_{ij}' \otimes \gamma_5) \Psi_j(x, L + y).
$$

(3-12)

When the Lagrangian is invariant under some flavor symmetry, $Z$ and $Z'$ are allowed.
to be identified as some elements of the symmetry group. The matrices $Z$ and $Z'$ represent the parity operations in the field space and should satisfy $Z^2 = I$ and $Z'^2 = I$.

As an example of flavor symmetry, we adopt the $S_3$ permutation, which has been widely studied in the literature. The $S_3$ group is the simplest non-abelian discrete group with six elements: the identity $I$, two cyclic permutations $R_{1,2}$, and three permutations $P_{1,2,3}$. The irreducible representations are the doublet $2$, pseudo singlet $1'$ and singlet $1$. The representation matrices for $3 = 2 + 1$ are given by

$$I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$P_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

(3.13)

The three generations are treated democratically under the flavor symmetry, and its breaking is a key to account for the observed mass differences and mixing angles in the lepton sector.

Suppose that the bulk fields $\Psi_i(x,y)$ and the SM neutrinos $N_i(x)$ belong to $3$ representations of $S_3$. The symmetry-invariant mass parameters are given by the combinations of the identity $I$ and the democratic matrix $D$ whose all elements are one thirds:

$$M = M_1 I + M_2 D, \quad m_d = \delta_1 I + \delta_2 D, \quad m = \mu_1 I + \mu_2 D.$$  

(3.14)

Each mass matrix is described by two parameters, which reflects the fact that the tensor product of two $3$'s contains two singlet components.

Now let us fix the boundary condition by identifying $Z$ and $Z'$ as the $S_3$ group elements. Note that the cyclic permutations $R_{1,2}$ do not satisfy the parity conditions and are excluded. We then consider

$$Z = P_1, \quad Z' = I.$$  

(3.15)

The translation is twisted as $T = Z'Z = P_1$ and becomes a typical (discrete) example of the original Scherk-Schwarz theory. The boundary conditions (3.15) imply that the flavor symmetry breaking occurs in a separate place from the (symmetry-preserving) SM boundary and is mediated by bulk right-handed neutrinos through the seesaw mechanism.

It is convenient to move onto the basis where $P_1$ and the mass matrices (3.14) are diagonal; they become $M' = \text{diag}(M_1, M_1, M_1 + M_2)$, $m'_d = \text{diag}(\delta_1, \delta_1, \delta_1 + \delta_2)$, $m' = \text{diag}(\mu_1, \mu_1, \mu_1 + \mu_2)$, and the boundary conditions are given by $Z = \text{diag}(1, -1, 1)$ and $Z' = \text{diag}(1, 1, 1)$. It is straightforward in this basis to perform the five-dimensional seesaw using the results in §2.3 and Appendix B. On the other hand, the propagator in the original basis is useful for intuitive understanding of the boundary condition breaking of flavor symmetry. It turns out that

$$\langle i\xi(p,y)\xi^\dagger(p,y') \rangle = Z^{++}_p(y, y', \delta_1 + \delta_2, M_1 + M_2) D$$

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\[-Z_{p}^{++}(y, y', \delta_1, M_1) E + Z_{p}^{-+}(y, y', \delta_1, M_1) F, \quad (3.16)\]

where

\[E \equiv \frac{1}{6} \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix}, \quad F \equiv \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \quad (3.17)\]

Here the notation for \(Z_p\)’s follows Appendix B. Flavor symmetry breaking is clearly seen in (3.16); the matrices \(E\) and \(F\) are not invariant under the general permutations, while they are invariant under the exchange of the second and third generations. With the twisted boundary condition imposed, the original \(S_3\) is broken down to \(S_2\).

The seesaw-induced mass \(M_\nu\) is computed by taking the low-energy limit \(p \to 0\) and setting \(y = y' = L\) in the propagator (3.16), and by multiplying the boundary mass matrix \(m\) given in (3.14). For a simple case with vanishing bulk Dirac mass, \(M_\nu\) reads

\[M_\nu = \frac{M_1 + M_2}{|M_1 + M_2|} (\mu_1 + \mu_2)^2 D + \frac{M_1}{|M_1|} (\mu_1)^2 E + \frac{M_2}{|M_2|} (\mu_2)^2 F. \quad (3.18)\]

The mass matrix has the same structure of generations mixing as the propagator. An important property of (3.18) is that it is diagonalized by the tri-bimaximal mixing matrix\(^{12}\)

\[V_{\text{tri}} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad (3.19)\]

for which various theoretical approaches have been discussed.\(^{13}\) Note that the diagonalization is free from the parameters involved in the Lagrangian, and the tri-bimaximal mixing is a rigid prediction of the flavor twisting. The prediction is not disturbed by nonzero bulk Dirac masses since the flavor structure is independent of the parameters \(\delta_1, \delta_2\) [see the propagator (3.16)].

The boundary condition (3.15) is an example of all possible choices. However the tri-bimaximal mixing is also induced by many other types of boundary conditions.\(^{10}\) Thus the prediction is not a special feature of (3.15), but it is rather common outcome of the present setup and twisted flavors.

3.4. Seeaw in the warped geometry

The higher-dimensional seesaw is calculable with the propagator method not only in the flat space but also for a generic class of curved geometry. Even if the background is so complicated that KK wavefunctions and mass eigenvalues cannot be found, the seesaw-induced neutrino mass is analytically obtained.

Let the five-dimensional seesaw setup be placed on the gravitational background

\[ds^2 = e^{-2k|y|} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2, \quad (3.20)\]
where \( k \) stands for the anti de-Sitter curvature and \( \eta_{\mu\nu} \) is the four-dimensional Minkowski metric. The previous Lagrangian is modified such that it incorporates the gravity;

\[
\mathcal{L} = \sqrt{g} \left[ \mathcal{D} \mathcal{D} \Psi - m_d \theta(y) \bar{\Psi} \Psi - \left( \frac{1}{2} M \bar{\Psi} c \Psi + \frac{m}{\sqrt{A}} \bar{\Psi} N \delta(y - L) + \text{h.c.} \right) \right].
\] (3.21)

The covariant derivative \( \mathcal{D} \) includes the spin connection, and the generation indices are suppressed. The SM fields are assumed to reside in the \( y = L \) boundary to solve the gauge hierarchy problem with the warp factor.\(^3\) In the above Lagrangian, the SM neutrinos \( N \) and the Higgs field \( H \) have been rescaled for their kinetic terms being canonical, and therefore \( m \) is a parameter of the electroweak scale.

The non-trivial metric factor modifies the equations of propagators. After the (non-canonical) rescaling \( \Psi \rightarrow e^{2k|y|} \Psi \), the lepton-number-violating parts of bulk neutrino propagators are determined by

\[
\left[ e^{2k|y|} p^2 - m_d^2 - M M^* + \partial_y^2 - 2m_d \delta(y) - \delta(y - L) \right] \langle i\epsilon \eta^*(p, y) \eta^\dagger(p, y') \rangle - k\theta(y) e^{k|y|} p^\mu \sigma_\mu \langle i\epsilon \xi^*(p, y) \xi^\dagger(p, y') \rangle = M \delta(y - y'),
\] (3.22)

\[
\left[ e^{2k|y|} p^2 - m_d^2 - M M^* + \partial_y^2 + 2m_d \delta(y) - \delta(y - L) \right] \langle i\epsilon \xi^*(p, y) \xi^\dagger(p, y') \rangle + k\theta(y) e^{k|y|} p^\mu \sigma_\mu \langle i\epsilon \eta^*(p, y) \eta^\dagger(p, y') \rangle = M \delta(y - y').
\] (3.23)

Unlike (2.19) and (2.20) in the flat background, these are the coupled equations due to the non-vanishing curvature. Further, in the presence of the exponential factor, it seems difficult to solve the above equations. However the low-energy behavior \((p \rightarrow 0)\) of the solutions is sufficient for the seesaw mechanism.\(^*\) It is found from (3.22) and (3.23) that, in the low-energy limit, the warp factor vanishes away from the problem and the propagators (with the non-canonical rescaling) are found to have the same forms as in the flat extra dimension. Note that the low-energy limit \( p \rightarrow 0 \) is allowed before solving the propagator equations only if the solutions are non-singular in that limit. The regularity is ensured in the seesaw theory where the bulk Majorana mass lifts the chiral zero modes (right-handed neutrinos) which are otherwise massless even in the presence of bulk Dirac masses. In the end, the seesaw-induced mass in the warped geometry is evaluated with the propagator in the flat space and the couplings in the rescaled basis.

The procedure for acquiring \( M_\nu \) goes parallel to the flat case. The only difference is the appearance of warped metric factors, which count the mass dimensions of couplings. Let us incorporate the generation structure as before by supposing that the Lagrangian respects the \( S_3 \) flavor symmetry and the bulk fermions obey the boundary conditions \( Z = P_1 \) and \( Z' = I \). The flavor symmetry requires the Lagrangian mass parameters \( M, m_d, \) and \( m \) to have the form (3.14). After all, the

\(^*\) To be precise, the following procedure is valid if, at any point in the bulk, \( e^{k|y|} p \) is smaller than the fundamental scale of the theory.
seesaw-induced mass matrix is found (for vanishing bulk Dirac masses)

\[ M_\nu = \frac{M_1 + M_2}{|M_1 + M_2|} (\mu_1 + \mu_2)^2 D + \frac{M_1}{|M_1|} (\mu_1)^2 E + \frac{M_1}{|M_1|} (\mu_1)^2 F, \]

where \( \Lambda' = \Lambda e^{-kL} \). Comparing this to the previous result (3.18), one finds that the warped geometry modifies the neutrino mass only by an overall factor of each matrix, and the details of propagators (or KK wavefunctions) do not affect the flavor structure of low-energy neutrinos. That is the geometry-free nature of seesaw-induced masses in higher dimensions.\(^7\) The conclusion is unchanged by the non-vanishing bulk Dirac masses.

If the warp factor is used to solve the gauge hierarchy problem, the effective seesaw scale \( \Lambda' \) is around TeV and the neutrino mass of \( \mathcal{O}(eV) \) requires tiny values of neutrino Yukawa couplings. A way to ameliorate this problem is to consider bulk Majorana masses of intermediate scale which generate additional suppression via the inverse seesaw. For example, when \( M_1, 1/L \ll m_d \) (Type A in Table I), the neutrino masses are given by \( m_{1,3} \sim \frac{M_1}{\delta_1} \frac{\mu_1^2}{\Lambda'} \) and \( m_2 \sim \frac{M_1 + M_2}{\delta_1 + \delta_2} \frac{(\mu_1 + \mu_2)^2}{\Lambda'} \). Thus a small ratio \( M/\delta \) is used to make a tuning of Yukawa couplings reduced. Another way to have mild Yukawa hierarchy is to extend the SM neutrinos (the lepton doublets) into the extra dimension and to utilize the localization effect. It is however noted that the wavefunction suppression by the left-handed neutrinos cannot be arbitrarily strong as it also brings down the charged-lepton mass scale. For example, in the case that the right-handed tau resides on the SM boundary, its wavefunction lowers the neutrino masses by the factor of \( (m_\tau/\Lambda')^2 \).

\[ \text{§4. Summary} \]

We have studied the higher-dimensional seesaw mechanism with two methods: the KK-mode expansion and the five-dimensional propagators. The propagator is derived for various types of boundary conditions and mass parameters of bulk right-handed neutrinos. The propagator method simplifies the calculation of seesaw-induced masses and clarifies the physical implications. That can skip identifying KK eigenfunctions, evaluating (infinite-dimensional) mass matrices, and summing up the KK contributions to the seesaw-induced mass. Noticing that the neutrino mass is estimated in the low-energy limit, its explicit form is obtained even when the background geometry is non-trivial and a suitable KK expansion is not viable. The propagator method is also useful to capture symmetry-breaking effects by boundary conditions. As an application of these facts, we have discussed the Scherk-Schwarz breaking of flavor symmetry in the flat and warped extra dimensions. The neutrino mass matrix in the warped case is calculated in the same fashion as in the flat case, with the same propagator and rescaled couplings. The two results differ only by the overall metric factor.

The higher-dimensional seesaw realizes various structures in low-energy effective theory, in particular, suitable for the inverse seesaw suppression of neutrino masses. For instance, by taking the bulk Dirac mass such that the zero-mode wavefunction...
is localized away from the SM fields, its seesaw contribution is suppressed and the
seesaw mediator is played by vector-like pairs of KK-excited modes with almost Dirac
nature. In this case, the seesaw-induced mass is proportional to the (lepton-number-
violating) bulk Majorana mass. Alternatively, the Dirichlet boundary condition for
right-handed neutrinos forbids the existence of zero mode and the inverse seesaw is
realized naturally. The possible forms of seesaw-induced mass in various limits are
summarized in Table I.

Besides several examples discussed in this paper, there may be other broad us-
age of the (lepton-number-violating) propagator in higher-dimensional theory, e.g.,
for seesaw collider phenomenology,\textsuperscript{14) leptogenesis,\textsuperscript{15) and so on. Such phenomeno-
nological applications remain to be studied in future work.

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Appendix A

 '\textit{Lorentz Spinors and Gamma Matrices} \textit{'}

\[
\{\Gamma^M, \Gamma^N\} = 2\eta^{MN} = 2 \text{diag}(+1, -1, -1, -1, -1), \quad (A.1)
\]

\[
\Gamma_\mu = \gamma_\mu = \begin{pmatrix} \sigma_\mu \\ \bar{\sigma}_\mu \end{pmatrix}, \quad i\Gamma_4 = \gamma_5 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad (A.2)
\]

where $\sigma_\mu = (1, \sigma_i)$ and $\bar{\sigma}_\mu = (1, -\sigma_i)$. A 4-component spinor is written in terms of
2-component spinors as

\[
\Psi = \begin{pmatrix} \xi_\alpha \\ \eta^\dot{\alpha} \end{pmatrix}. \quad (A.3)
\]

The Dirac and charge conjugates for $\Psi$ are given by

\[
\overline{\Psi} = (\eta^\alpha \xi_\alpha), \quad \Psi^c = C_5 \overline{\Psi}^T = \begin{pmatrix} -\epsilon_{\alpha\beta} \eta_{\dot{\beta}} \\ -\epsilon_{\dot{\alpha}\dot{\beta}} \xi_{\dot{\beta}} \end{pmatrix}, \quad (A.4)
\]

where $C_5$ is the charge conjugation matrix in five dimensions: $C_5 = i\gamma^2\gamma^0 \gamma_5$. The
antisymmetric tensors are

\[
\epsilon^{\alpha\beta} = \epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\beta}\dot{\alpha}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (A.5)
\]
Appendix B

--- Propagators for Bulk Majorana Fermions ---

To find the lepton-number-violating part of the propagator, it is convenient to rewrite the bulk Lagrangian (2.3) as

$$\mathcal{L}_{\text{bulk}} = \frac{1}{2} \left( \overline{\Psi} \mathcal{D} \overline{\Psi} + \overline{\Psi} \mathcal{D}^\dagger \Psi \right),$$  \hspace{1cm} (B.1)

where

$$\mathcal{D} = \begin{pmatrix} i\partial - \gamma_5 \partial_y - m_d \theta(y) & -M^* \\ -M & i\partial - \gamma_5 \partial_y + m_d \theta(y) \end{pmatrix}. \hspace{1cm} (B.2)$$

The propagator is given by the inverse of $\mathcal{D}$;

$$\mathcal{D}G(x, x', y, y') = i\delta^4(x-x')\delta(y-y'), \hspace{1cm} (B.3)$$

where

$$G(x, x', y, y') = \begin{pmatrix} \langle \Psi(x, y)\overline{\Psi}(x', y') \rangle & \langle \Psi(x, y)\overline{\Psi}(x', y') \rangle \\ \langle \Psi^c(x, y)\overline{\Psi}^c(x', y') \rangle & \langle \Psi^c(x, y)\overline{\Psi}^c(x', y') \rangle \end{pmatrix} \hspace{1cm} (B.4)$$

$$= \begin{pmatrix} \langle \xi \eta^\dagger \rangle & \langle \xi \xi^\dagger \rangle & \langle \xi \xi^\dagger \rangle & \langle \xi \xi^\dagger \rangle \\ \langle -\eta \eta^\dagger \rangle & \langle -\eta \xi^\dagger \rangle & \langle -\eta \xi^\dagger \rangle & \langle -\eta \xi^\dagger \rangle \\ \langle -\xi \xi^\dagger \rangle & \langle -\xi \eta^\dagger \rangle & \langle -\xi \eta^\dagger \rangle & \langle -\xi \eta^\dagger \rangle \\ \langle -\xi \eta^\dagger \rangle & \langle -\xi \xi^\dagger \rangle & \langle -\xi \xi^\dagger \rangle & \langle -\xi \xi^\dagger \rangle \end{pmatrix}. \hspace{1cm} (B.5)$$

The upper-right and the lower-left blocks violate the lepton number. These two blocks are related as $\langle \Psi(x, y)\overline{\Psi}^c(x', y') \rangle = \Gamma^0 \langle \Psi^c(x, y)\overline{\Psi}^c(x', y') \rangle \Gamma^0 |_{x,y \rightarrow x',y'}$.

Equation (B.3) is written in the mixed position-momentum space as

$$\langle p - \gamma_5 \partial_y - m_d \theta(y) \rangle \langle \Psi(p, y)\overline{\Psi}(p, y') \rangle - M^* \langle \Psi^c(p, y)\overline{\Psi}^c(p, y') \rangle = i\delta(y-y'), \hspace{1cm} (B.6)$$

$$\langle p - \gamma_5 \partial_y + m_d \theta(y) \rangle \langle \Psi^c(p, y)\overline{\Psi}^c(p, y') \rangle - M \langle \Psi(p, y)\overline{\Psi}(p, y') \rangle = 0. \hspace{1cm} (B.7)$$

By eliminating the lepton-number-conserving part $\langle \Psi\overline{\Psi} \rangle$, one obtains

$$[p^2 - m_d^2 - MM^* + \partial_y^2 + 2m_d[\delta(y) - \delta(y-L)]]Z_p(y, y') = M\delta(y-y'), \hspace{1cm} (B.8)$$

$$[p^2 - m_d^2 - MM^* + \partial_y^2 - 2m_d[\delta(y) - \delta(y-L)]]H_p(y, y') = M\delta(y-y'). \hspace{1cm} (B.9)$$

Here we have introduced the notation

$$Z_p(y, y') \equiv \langle i\xi^\dagger(p, y)\xi^\dagger(p, y') \rangle, \hspace{1cm} H_p(y, y') \equiv \langle i\eta^\dagger(p, y)\eta^\dagger(p, y') \rangle. \hspace{1cm} (B.10)$$

The general solutions in the bulk are

$$Z_p(y, y') = A_Z(y')\sinh(qy) + B_Z(y')\cosh(qy), \hspace{1cm} (B.11)$$

$$H_p(y, y') = A_H(y')\sinh(qy) + B_H(y')\cosh(qy), \hspace{1cm} (B.12)$$
with \( q = \sqrt{m_d^2 + MM^* - p^2} \). The coefficients \( A_{Z,H} \) and \( B_{Z,H} \) are determined by the boundary conditions and matching in the following.

Let us first consider \( Z = +1 \) and \( Z' = +1 \) that the right-(left-)handed component obeys the Neumann (Dirichlet) conditions at both boundaries. The lepton-number-violating propagators then satisfy

\[
\begin{align*}
\partial_y Z_p^\leq(y, y') & \bigg|_{y=0} + m_d Z_p^\leq(0, y') = 0, \\ \partial_y Z_p^\geq(y, y') & \bigg|_{y=L} + m_d Z_p^\geq(L, y') = 0, \\ H_p^\leq(0, y') &= 0, \\ H_p^\geq(L, y') &= 0,
\end{align*}
\]

where the superscripts \( \leq \) and \( \geq \) represent the solutions for \( y < y' \) and \( y > y' \), respectively. The Neumann conditions follow from the integration of (B.8) over the infinitesimal regions around \( y = 0 \) and \( y = L \), and the continuity of wavefunction. The derivative of \( Z_p \) is jumped at both boundaries due to the existence of source terms. The solutions with respect to \( y \) are found up to normalizations;

\[
\begin{align*}
Z_p^\leq(y, y') &= C^\leq Z_p(y'), \\ Z_p^\geq(y, y') &= C^\geq Z_p(y'), \\ H_p^\leq(y, y') &= C^\leq H_p(y'), \\ H_p^\geq(y, y') &= C^\geq H_p(y').
\end{align*}
\]

The functions \( C_{Z,H}^\leq, \geq \) are determined by the conditions which connect the solutions in two regions, i.e.,

\[
\begin{align*}
Z_p^\leq &= Z_p^\geq, & \partial_y Z_p^\leq &= \partial_y Z_p^\geq - M, \\ H_p^\leq &= H_p^\geq, & \partial_y H_p^\leq &= \partial_y H_p^\geq - M,
\end{align*}
\]

at \( y = y' \). The discontinuities of the slopes follow from the integration of (B.8) and (B.9) around \( y = y' \). The final result is as follows;

\[
Z = +1, \quad Z' = +1
\]

\[
\begin{align*}
Z_p^{++}(y, y', m_d, M) &= \frac{1}{(m_d^2 - q^2)q \sinh(qL)} [q \cosh(qy_\leq) - m_d \sinh(qy_\leq)] \\
&\quad \times [q \cosh(qy_\geq - qL) - m_d \sinh(qy_\geq - qL)] M, (B.23) \\
H_p^{++}(y, y', m_d, M) &= \frac{\sinh(qy_\leq) \sinh(qy_\geq - qL)}{q \sinh(qL)} M, (B.24)
\end{align*}
\]

where \( y_\leq \) \( y_\geq \) stands for the lesser (greater) of \( y \) and \( y' \). The superscript “++” is attached to indicate that the propagators satisfy the boundary conditions \( Z = +1 \) and \( Z' = +1 \).

The mass spectrum in four-dimensional effective theory is extracted from the poles of these propagators. First, \( q^2 = m_d^2 \) in (B.23) corresponds to the chiral zero
mode with the mass $M$. The other poles, $qL = in\pi$, in both (B.23) and (B.24) give the masses of KK-excited states; $m_d^2 + |M|^2 + \left(\frac{n\pi}{L}\right)^2$ ($n \geq 1$).

The lepton-number-violating propagators for the other boundary conditions can be derived in parallel ways to the above:

$Z = +1, Z' = -1$

$$Z^+(y, y', m_d, M) = \frac{q \cosh(qy') - m_d \sinh(qy')}{q \cosh(qy) - m_d \sinh(qL)} \sinh(qy - qL) M, \quad (B.25)$$

$$H^+(y, y', m_d, M) = -\frac{\sinh(qy') \left[q \cosh(qy) - qL\right] + m_d \sinh(qy) - qL}{q \cosh(qL) - m_d \sinh(qL)} M. \quad (B.26)$$

$Z = -1, Z' = +1$

$$Z^-(y, y', m_d, M) = -\frac{\sinh(qy') \left[q \cosh(qy) - qL\right] - m_d \sinh(qy) - qL}{q \cosh(qL) + m_d \sinh(qL)} M, \quad (B.27)$$

$$H^-(y, y', m_d, M) = \frac{q \cosh(qy') + m_d \sinh(qy')}{q \cosh(qL) + m_d \sinh(qL)} \sinh(qy - qL) M. \quad (B.28)$$

$Z = -1, Z' = -1$

$$Z^-(y, y', m_d, M) = \frac{\sinh(qy') \sinh(qy) - qL}{q \sinh(qL)} M, \quad (B.29)$$

$$H^-(y, y', m_d, M) = \frac{1}{(m_d^2 - q^2)q \sinh(qL)} \left[q \cosh(qy') + m_d \sinh(qy')\right]$$

$$\times \left[q \cosh(qy) - qL + m_d \sinh(qy) - qL\right] M. \quad (B.30)$$

The last case with $Z = -1$ and $Z' = -1$ gives the same mass spectrum as that for $Z = +1$ and $Z' = +1$. For the other two cases with $Z = \pm 1$ and $Z' = \mp 1$, the positions of poles are at $p^2 = m_d^2 + |M|^2 + (\pm x^\pm_n/L)^2$ where $x^\pm_n$ are determined by the equations $\tan x^\pm = \pm x^\pm/m_dL$. For small Dirac mass $m_dL \ll 1$, the KK indices $x^\pm_n$ approach to $(n - \frac{1}{2})\pi$, which just correspond to (3.9). In the opposite limit $m_dL \gg 1$, the indices become $x^\pm_n \approx n\pi$ for low-lying modes. A special case is $m_dL = 1$ that leads to the eigenvalues $x^\pm_n$:

| $n$ | $x^+ / \pi$ | $x^- / \pi$ |
|-----|--------------|--------------|
| 1   | 0            | 0.65         |
| 2   | 1.43         | 1.56         |
| 3   | 2.46         | 2.54         |
| :  | :            | :            |

A remark is the appearance of “zero mode” $x^+_n = 0$. It is seen from the propagators (B.25) and (B.26) that $q = 0$ becomes a pole only if this special relation $m_dL = 1$ is satisfied. A similar pole $x^-_n = 0$ appears for $m_dL = -1$. 

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