Orbifold regularity of weak Kähler-Einstein metrics

Chi Li
Stony Brook University
Gang Tian†
Beijing University and Princeton University

1 Introduction

In the resolution of the YTD conjecture on the existence of Kähler-Einstein metrics on Fano manifolds (see [23] and also [5]), a crucial tool is a compactness result. In its simplest form, this result says that the Gromov-Hausdorff limit of a sequence of smooth Kähler-Einstein manifolds \((X_i, \omega_i, \text{KE})\) is a normal Fano variety \(X := X_\infty\) with klt singularities and that there is a weak Kähler-Einstein metric \(\omega_{\infty, \text{KE}}\) on \(X_\infty\). The existence of a Gromov-Hausdorff limit follows from Gromov’s compactness theorem. So the important information in this statement is about the regularity of \(X_\infty\). It was the second author ([20], [22], see also [15]) who first pointed out the route to prove that \(X_\infty\) is an algebraic variety is to establish a so-called partial \(C^0\)-estimate. He demonstrated in [20] how to achieve this when the complex dimension \(n\) is equal to 2 by showing that a sequence of Kähler-Einstein surfaces converges to a Fano orbifold with a smooth orbifold Kähler-Einstein metric. Note that when \(n = 2\), klt singularities are nothing but quotient singularities or orbifold singularities. Two key ingredients to prove the partial \(C^0\)-estimate in dimension 2 are orbifold compactness result of Einstein 4-manifolds and Hörmander’s \(L^2\)-estimates.

Recently, Donaldson-Sun [7] and the second author [21] generalized the partial \(C^0\)-estimate to higher dimensional Kähler-Einstein manifolds. Here they need to rely on compactness results of higher dimensional Einstein manifolds developed by Cheeger-Colding and Cheeger-Colding-Tian (see [4] and the reference therein). Compared to the complex dimension 2 case, the second author also conjectured that \(\omega_{\infty, \text{KE}}\) is a smooth orbifold metric away from analytic subvarieties of complex codimension 3. Note that in [4], it was proved that the (metric) singular set of \(X_\infty\) has complex codimension at least 2.

It can be shown that, by partial \(C^0\)-estimate, there is a uniform \(C^2\)-estimate of the potential of \(\omega_{\infty, \text{KE}}^w\) on \(X_\infty^\text{reg}\). Then the Evans-Krylov theory or Calabi’s 3rd derivative estimate allows one to show that \(\omega_{\infty, \text{KE}}^w\) is smooth on \(X_\infty^\text{reg}\) (see [20], [7], [23]). Alternatively using Păun’s Laplacian estimate in [16] and Evans-Krylov theory, Berman-Boucksom-Eyssidieux-Guedj-Zeriahi [1] showed directly that any weak Kähler-Einstein metric \(\omega_{\text{KE}}^w\) on a klt Fano variety \(X_\infty\) is smooth on \(X_\infty^\text{reg}\). The purpose of this note is to answer the question by the second author about the regularity of \(\omega_{\text{KE}}^w\) on the orbifold locus \(X_\infty^\text{orb}\) of \(X_\infty\). First, if \((X, -K_X)\) is a klt Fano variety, then by [8, Proposition 9.3] there exists a closed subset \(Z \subset X\) with \(\text{codim}_X Z \geq 3\) such that \(X \setminus Z\) has quotient singularities. So we just need to show the following regularity result. For the definition of weak Kähler-Einstein metric, see Definition 1.

**Theorem 1.** Assume that \(\omega_{\text{KE}}^w\) is a weak Kähler-Einstein metric on \(X_\infty\). Then \(\omega_{\text{KE}}^w\) is a smooth orbifold metric on \(X_\infty^\text{orb}\).
Our proof now uses the existence of an orbifold resolution, i.e., Theorem 3 which is proved by algebraic method. However, we believe that it is not necessary. There should be a purely differential geometric proof of Theorem 1 which does not rely on Theorem 3. In a subsequent paper, we will analyze further structures of singularities of higher codimension. We believe that our analysis can be used to yield a complete understanding of the singularity for any 3-dimensional weak Kähler-Einstein metrics.

2 Regularity on the orbifold locus

From now on we will denote by $X$ any $\mathbb{Q}$-Fano variety with klt singularities. Assume $\iota : X \to \mathbb{P}^N$ is an embedding given by the linear system $|−mK_X|$ for $m > 0 \in \mathbb{Z}$ sufficiently large and divisible. Let $h_0 = (\iota^*h_{FS})^{1/m}$ be the pull back of the Fubini-Study Hermitian metric $h_{FS}$ on $\mathcal{O}_{\mathbb{P}^N}(1)$ normalized to be a Hermitian metric on $-K_X$. The Chern curvature form of $h_0$ is

$$\omega_0 = -\sqrt{-1} \partial \bar{\partial} \log h_0$$

which is a positive $(1,1)$-current on $X$. $\omega_0$ is a smooth positive definite $(1,1)$-form on $X^{\text{reg}}$. However, on the singular locus $X^{\text{sing}}$, $\omega_0$ in general is not canonically related to the local structure of $X$. Assume $p \in X^{\text{orb}}$ is a quotient singularity. By this, we mean that there exists a small neighborhood $U_p$ which is isomorphic to a quotient of a smooth manifold by a finite group. In other words, there exists a branched covering map $U_p \to \tilde{U}_p/G \cong U_p$. The lifting of metric $\omega_0$ to the cover $\tilde{U}_p$ in general is degenerate.

Now we define an adapted volume for on $X$ by

$$\Omega = |v|^2h_0 (v \wedge \bar{v})^{1/m}.$$  

Here $v$ is any local generator of $\mathcal{O}(mK_X)$ and $v^*$ is the dual generator of $\mathcal{O}(-mK_X)$. The Kähler-Einstein equation

$$\text{Ric}(\omega_0) = \omega_0.$$  

(1)



can be transformed into a complex Monge-Ampère equation:

$$(\omega_0 + \sqrt{-1} \partial \bar{\partial} \phi)^n = e^{-\phi} \Omega.$$  

(2)

**Definition 1.** A weak solution to the (2) is a bounded function $\phi \in L^\infty(X) \cap \text{PSH}(X, \omega)$ satisfying (2) in the sense of pluripotential theory.

Let’s first recall the method to prove the regularity of $\phi$ on $X^{\text{reg}}$ following [1]. One first chooses a resolution $\pi : \tilde{X} \to X$ with simple normal crossing exceptional divisor $E = \pi^{-1}(X^{\text{sing}})$ such that $\pi$ is an isomorphism over $X^{\text{reg}}$. Then we can pull back the equation (2) to $\tilde{X}$ and get:

$$(\pi^* \omega_0 + \sqrt{-1} \partial \bar{\partial} \psi)^n = e^{-\psi} \pi^* \Omega.$$  

(3)

On the other hand we can write:

$$K_{\tilde{X}} = \pi^*K_X + \sum_{i=1}^r a_i E_i - \sum_{j=1}^s b_j F_j,$$

such that $E = \cup_{i=1}^r E_i \cup \cup_{j=1}^s F_j$ and $a_i > 0$, $b_j > 0$. The klt property implies: $a_i > 0$, and $0 < b_j < 1$. Analytically, choosing a smooth Kähler metric $\eta$ on $\tilde{X}$, there exists $f \in C^\infty(\tilde{X})$ such that:

$$\pi^* \Omega = e^f \prod_{i=1}^r |\sigma_i|^{2a_i} \prod_{j=1}^s |\sigma_j|^{2b_j} \eta^n.$$  


where \( s_i \) and \( \sigma_j \) are defining sections of \( E_i \) for \( F_j \) respectively and \( |s_i|^2 \) and \( |\sigma_j|^2 \) are some fixed hermitian norms of them. So we have:

\[
(\pi^*\omega_0 + \sqrt{-1} \partial \bar{\partial} \psi)^n = e^{-\psi_+ + \sum_i a_i \log |s_i|^2 - \sum_j b_j \log |\sigma_j|^2} \eta^n = e^{\psi_+ - \psi_-} \eta^n, \tag{4}
\]

Here we have denoted

\[
\psi_+ = f + \sum_i a_i \log |s_i|^2, \quad \psi_- = \psi = \sum_j b_j \log |\sigma_j|^2.
\]

It’s easy to see that they satisfy the quasi-plurisubharmonic condition:

\[
\sqrt{-1} \partial \bar{\partial} \psi_+ \geq -C \eta, \quad \sqrt{-1} \partial \bar{\partial} \psi_- \geq -C \eta, \tag{5}
\]

for some uniform constant \( C > 0 \). To get Laplacian estimate of \( \psi \) away from \( Z \), we can first regularize (4) to

\[
(\omega_\epsilon + \sqrt{-1} \partial \bar{\partial} \psi_\epsilon)^n = e^{\psi_+ - \psi_-} \eta^n, \tag{6}
\]

where \( \omega_\epsilon = \pi^*\omega_0 - \epsilon \theta_E \) is a Kähler metric on \( \tilde{X} \), and \( \psi_\pm, \epsilon \in C^\infty(\tilde{X}) \) converges to \( \psi_\pm \) in \( L^p(\tilde{X}) \cap L^\infty(\tilde{X}\setminus Z) \) for some \( p > 1 \). Using (5) and cleverly modifying the \( C^2 \)-estimate of Aubin-Yau-Siu, Păun [16] proved the Laplacian estimate for the solution \( \psi_\epsilon \) away from \( Z \). More precisely, for any compact set \( K \subset \tilde{X}\setminus Z \), there exists a constant \( A = A(\|\psi\|_\infty, K) \), such that

\[
\Delta_\eta \psi_\epsilon \leq A(\|\psi\|_\infty, K) e^{-\psi_-}.
\]

From this estimate, we know that the right-hand side of (6) is uniformly \( C^{1,\alpha} \) on \( \tilde{X}\setminus Z \). By Evan-Krylov’s theory ([3]), we know that \( \psi_\epsilon \) is uniformly \( C^{2,\alpha} \) and hence by bootstrapping, \( C^{k,\alpha} \) on \( \tilde{X}\setminus Z \). Now because \( \psi_\epsilon \) converges to \( \psi \) in \( C^k \) norm uniformly away from \( Z \), we get that \( \psi \) is smooth on \( \tilde{X}\setminus Z \).

One can also prove the regularity on \( X^\text{reg} \) with the help of Kähler-Ricci flow. Starting from the work in [6], this idea has been used several times in the literature to prove the regularity of weak solutions to complex Monge-Ampère equations. Recall that the Kähler-Ricci flow is a solution to the following equation:

\[
\frac{\partial \omega}{\partial t} = -Ric(\omega) + \omega; \quad \omega(0) = \omega_0, \tag{7}
\]

As in the elliptic case, this equation can be transformed into the following Monge-Ampère flow

\[
\frac{\partial \phi}{\partial t} = \log \frac{\omega + \sqrt{-1} \partial \bar{\partial} \phi}{\omega_0}, \quad \phi(0, \cdot) = \phi_0. \tag{8}
\]

To define a solution to this Monge-Ampère flow on the singular variety \( X \), Song-Tian [19] pulled up the flow equation in (8) to \( \tilde{X} \) to get:

\[
\frac{\partial \hat{\phi}}{\partial \tau} = \log \frac{\pi^*\omega + \sqrt{-1} \partial \bar{\partial} \hat{\phi}}{\pi^*\omega_0} + \hat{\phi}; \quad \hat{\phi}(0, \cdot) = \pi^* \phi_0. \tag{9}
\]

**Theorem 2** ([19]). Let \( \phi_0 \in PSH_p(X, \omega_0) \) for some \( p > 1 \). Then the Monge-Ampère flow (9) on \( \tilde{X}\setminus E \) has a unique solution \( \phi \in C^\infty((0, T_0) \times \tilde{X}\setminus E) \cap C^0([0, T_0) \times \tilde{X}\setminus E) \) such that for all \( t \in [0, T_0) \), \( \phi(t, \cdot) \in L^\infty(\tilde{X}) \cap PSH(\tilde{X}, \pi^*\omega_0) \).

Since \( \tilde{\phi} \) is constant along (connected) fibre of \( \pi \), \( \tilde{\phi} \) descends to a solution \( \phi \in C^\infty((0, T_0) \times X^\text{reg}) \cap C^0([0, T_0] \times X^\text{reg}) \) of the Monge-Ampère flow.

Now suppose \( \omega_{KE}^w = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi_{KE}^w \) is a weak solution to the equation (2). If one can prove that the solution \( \phi(t) \) to (8) with the initial condition \( \phi(0) = \phi_{KE}^w \) is stationary, then it follows
from Theorem (2) that $\omega_{KE}^{w}$ is smooth on $X_{\text{reg}}^\infty$. The idea to prove stationarity in [6] is to show that the energy functional is decreasing along the flow solution $\phi(t)$ and to use the uniqueness of weak Kähler-Einstein metrics. These are indeed true in the current case by the work of [1].

To prove Theorem 1, the main observation is that the above arguments can be used to prove the regularity of $\omega_{KE}^{w}$ on $X^{\text{orb}}$ as long as one can find a partial resolution by orbifolds: $\pi^{\text{par}} : X^{\text{par}} \to X$. Indeed, by the next section, there exist orbifold (partial) resolutions. If $\pi^{\text{par}} : X^{\text{par}} \to X$ is an orbifold resolution, then we can write:

$$K_{X^{\text{par}}} = (\pi^{\text{par}})^{*}K_{X} + \sum_{i} a_{i}E_{i} - \sum_{j=1}^{s} b_{j}F_{j},$$

where $E = \bigcup_{i=1}^{r}E_{i} \bigcup \bigcup_{j=1}^{s}F_{j}$ is now a simple normal crossing divisor within orbifold category (in the sense of Satake [17, 18]). The klt property of $X$ again implies $a_{i} > 0$ and $0 < b_{i} < 1$. Then the similar argument as in the proof of regularity of $\omega_{KE}^{w}$ on $X^{\text{reg}}$ carries over to the orbifold setting to prove the orbifold regularity of $\omega_{KE}^{w}$ on $X^{\text{orb}}$.

Note that it was already observed in [19, Section 4.3] that if $X$ has only orbifold singularities, then the Kähler-Ricci flow smooths out initial metric to become genuine smooth orbifold metric immediately when $t > 0$.

### 3 Orbifold partial resolution

The results in this section were communicated to us by Chenyang Xu.

**Lemma 1** (Resolution of Deligne-Mumford stacks). Let $X$ be an integral Deligne-Mumford stack which is of finite type over $\mathbb{C}$. Then there exists a birational proper representable morphism $g^{\text{sm}} : X^{\text{sm}} \to X$ from a smooth Deligne-Mumford stack $X^{\text{sm}}$. Furthermore, we can assume that $g^{\text{sm}}$ is isomorphic over the smooth locus of $X$, and the exceptional locus of $g^{\text{sm}}$ is a normal crossing divisorial closed substacks of $X^{\text{sm}}$.

**Proof.** This follows from the functoriality property of resolution of singularities (see [26], [12], [2], [24]).

**Lemma 2** (Blow up the indeterminacy locus). Let $X$ be a projective scheme. Let $X$ be a normal Deligne-Mumford stack with a dense open set $U \subset X$, such that $U$ admits a morphism $U \to X$. Then we can blow up an ideal $I \subset \mathcal{O}_{X}$ to obtain a Deligne-Mumford stack $\tilde{X}$ such that $\tilde{X} \to X$ is isomorphic over $U$ and $f_{\tilde{X}}$ extends to a morphism $f : \tilde{X} \to X$.

**Proof.** We can replace $X$ by $\mathbb{P}^{N}$. Let $D \subset U$ be the pull back of a hyperplane section $H$ which does not vanish along $U$, and we let $I \subset \mathcal{O}_{X}$ be the ideal of the closure of $D$ in $\mathcal{O}_{X}$. Then the rest of the proof follows from the cases for schemes as in [9, II.7.17.3].

**Theorem 3.** Let $X$ be a quasi-projective normal variety. Let $X^{\text{orb}}$ be the locus where $X$ only has orbifold singularity. Then there exists $f^{\text{par}} : X^{\text{par}} \to X$ a proper birational morphism, such that $X^{\text{par}}$ only has quotient singularity and $f^{\text{par}}$ is an isomorphic over $X^{\text{orb}}$.

**Proof.** After taking the closure of $X \subset \mathbb{P}^{N}$, we can assume $X$ is projective.

By [25, 2.8], we know there is a smooth Deligne-Mumford stack $X^{0}$ whose coarse moduli space is $X^{\text{orb}}$. It follows from [14, Theorem 4.4] that $X^{0} = [Z/G]$ for some quasi-projective scheme $Z$ and linear algebraic group $G$. Actually, $Z$ can be taken as the frame bundle of $X^{\text{orb}}$ and $G = GL_{n}(\mathbb{C})$. Then by [14, Theorem 5.3], there is a proper Deligne-Mumford stack $X$, such that $X^{0} \subset X$ is a dense open set.
Consider the rational map $f : \mathcal{X} \to X$, by Lemma 2 we know that there is a blow up $\mathcal{Y} \to \mathcal{X}$ along the indeterminacy locus of $f$, such that there is a morphism $g : \mathcal{Y} \to X$. Moreover, by the construction, we know over $X^{\text{orb}}$,

$$\mathcal{Y}^0 := g^{-1}(X^{\text{orb}}) \cong \mathcal{X}^0.$$  

By Lemma 1, we know that there is a smooth Deligne-Mumford stack $h : \mathcal{Y}^{\text{sm}} \to \mathcal{Y}$, where $h$ is a representable proper birational morphism which is isomorphic over the smooth locus of $\mathcal{Y}$. In particular, $h$ is isomorphic over $\mathcal{Y}^0$.

As $\mathcal{X}$ has finite stabilizer and $\mathcal{Y}^{\text{sm}} \to \mathcal{Y} \to \mathcal{X}$ is proper, we know that $\mathcal{Y}^{\text{sm}}$ has also finite stabilizer. Thus it follows from [11] that $\mathcal{Y}^{\text{sm}}$ admits a coarse moduli space, which we denote by $\mathcal{X}^{\text{par}}$. It has a morphism $f^{\text{par}} : \mathcal{X}^{\text{par}} \to X$ by the universal property. And we easily check that they satisfy all the properties.

\[\Box\]

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