ON DERIVATION OF EULER-LAGRANGE EQUATIONS FOR INCOMPRESSIBLE ENERGY-MINIMIZERS

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Abstract. We prove that any distribution $q$ satisfying the equation
$$\nabla q = \text{div} f$$
for some tensor $f = (f_i^j)$, $f_i^j \in h^r(U)$ ($1 \leq r < \infty$) -the local Hardy space, $q$ is in $h^r$, and is locally represented by the sum of singular integrals of $f_i^j$ with Calderón-Zygmund kernel. As a consequence, we prove the existence and the local representation of the hydrostatic pressure $p$ (modulo constant) associated with incompressible elastic energy-minimizing deformation $u$ satisfying $|\nabla u|^2, |\text{cof} \nabla u|^2 \in h^1$. We also derive the system of Euler-Lagrange equations for incompressible local minimizers $u$ that are in the space $K_{1,3}^{1,3}$ (defined in (1.2)); partially resolving a long standing problem. For Hölder continuous pressure $p$, we obtain partial regularity of area-preserving minimizers.

1. Introduction

Let $\Omega \subset \mathbb{R}^n, n \geq 2$ be a bounded Lipschitz material body. For Mooney-Rivlin or Neo-Hookean materials [Ba 77], [TO 81], [Og 84], such as vulcanized rubber, in the equilibrium state, one is interested in minimizing the elastic energy

$$E[w] := \int_{\Omega} L(\nabla w(x)) dx,$$

for incompressible $W^{1,2}$-deformations $w : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, subject to its own boundary condition, and corresponding to a given smooth bulk energy $L : M_n^{n \times n} \to \mathbb{R}$. Let us define the subspace $K^{1,r}$ for $1 \leq r < \infty$, by

$$K^{1,r}(\Omega, \mathbb{R}^n) := \{w \in W^{1,r}(\Omega, \mathbb{R}^n) : \text{cof} \nabla w \in L^r(\Omega, M_n^{n \times n})\},$$

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where $W^{1,r}$ denotes the usual Sobolev spaces (see for example, [GT 97, Chapter 7]) and $\text{cof} P$ is the cofactor matrix, whose $ij$-th entry is $(-1)^{i+j}$ times the determinant of $(n-1) \times (n-1)$ submatrix obtained by deleting the $i$-th row and the $j$-th column from the $n \times n$ matrix $P$. Using the identity $P^t \text{cof} P = \text{Id}_n \det P$, it follows that $\det \nabla w \in L^1$ for any $w \in K^{1,2}$. Since $|P| = |\text{cof} P|$ for any $P \in \mathbb{M}^{2 \times 2}$, the function spaces $K^{1,r}$ and $W^{1,r}$ are equal in $\mathbb{R}^2$. Let us denote the admissible set of deformations

\begin{equation}
\mathcal{A} := \{ w \in K^{1,2}(\Omega, \mathbb{R}^n) : \det \nabla w = 1 \text{ a.e. in } \Omega \},
\end{equation}

We call $u \in \mathcal{A}$ to be a local minimizer of $E[\cdot]$ if and only if

\begin{equation}
E[u] \leq E[w] \text{ for all } w \in \mathcal{A} \text{ and } \text{supp} (w - u) \subset \Omega.
\end{equation}

Under the hypothesis that the energy density $L$ is smooth, polyconvex (convex function of minors) [Ba 77] and satisfies the growth condition

\begin{equation}
C_1 (|X|^2 + |\text{cof} X|^2) - C_2 \leq L(X) \leq C_3 (1 + |X|^2 + |\text{cof} X|^2),
\end{equation}

for all $X \in \mathbb{M}^{n \times n}$, for some $C_1 > 0$, $C_2 \geq 0$, $C_3 > 0$, where $|X|^2 := \text{trace}(X^t X)$, using direct methods in the calculus of variations together with weak continuity of the determinant, J. Ball [Ba 77] proved the existence of local minimizers $u \in \mathcal{A}$ of the energy $E[\cdot]$. An example of polyconvex $L$ satisfying the growth condition (1.5) is the stored-energy for incompressible isotropic Mooney-Rivlin materials in $\mathbb{R}^3$, given by

\begin{equation}
L(X) = \frac{\mu_1}{2} (I_1(X) - 3) + \frac{\mu_2}{2} (I_2(X) - 3),
\end{equation}

where $I_1(X) := \text{trace}(C) = |X|^2$, $I_2(X) := \frac{1}{2} \left[ (\text{trace}(C))^2 - \text{trace}(C^2) \right] = |\text{cof} X|^2$, are the first two principle invariants of the right Cauchy-Green strain tensor $C := X^t X$ and $\mu_1, \mu_2$ are positive material constants.

Though the existence of the local minimizers of $E[\cdot]$ in $\mathcal{A}$ is known for over 30 years, the existence of integrable hydrostatic pressure associated with such minimizers, the derivation of system of Euler-Lagrange equations, and the partial regularity for such minimizers remains a challenging open problem. In this article we prove the following results:

(Ι) The $h^r$ ($1 \leq r < \infty$) -integrability and local representation of any distribution $q$ satisfying the equation $\nabla q = f$, where $f := (f^i_j)$, $f^i_j \in h^r$, the local $r$-Hardy spaces. (Theorem 2.2)

(ΙΙ) The existence of a pressure $p \in L^r_{\text{loc}}$ if the minimizer is $u \in K^{1,2r}_{\text{loc}}$ for some $r > 1$. (Theorem 3.1)
(III) The existence of a pressure $p \in h^1$ if the minimizer $u$ satisfies the conditions $|\nabla u|^2, |\text{cof} \nabla u|^2 \in h^1$. (Theorem 3.1)

(IV) The validity of the Euler-Lagrange equations if the minimizer is $u \in K_{\text{loc}}^{1,3}$. (Theorem 4.1). The pair $(u, p)$ satisfies the system

$$\text{div} [DL(\nabla u(x)) - p(x) \text{cof} (\nabla u(x))] = 0 \quad \text{in } \Omega,$$

where the divergence is taken in each rows.

(V) The partial regularity of $W^{1,3}$ area-preserving minimizers $u$ for which the hydrostatic pressure $p$ is Hölder continuous with exponent $0 < \alpha < 1$. (Theorem 5.1)

The $L^2$-version of the result in (I) is classical (see, [Te 01, Remark 1.4, p 11]), and plays an important role in incompressible fluids [Te 01]. The result in (I) is a crucial ingredient in proving (II) & (III). The $h^1$-version of (I) is quite delicate and to the best of our knowledge, it is new and may be of independent interest. For the case $r > 1$, it follows that $\nabla q \in W^{-1,r}$, and adapting the classical functional-analytic approach demonstrated for $r = 2$ (see [Te 01], [TO 81]), or arguing directly by duality, and solving the equation of the type

$$\text{div} w = f \quad \text{in } V \subset \subset U, \quad w = 0 \quad \text{in } \partial V,$$

[Ev 98, p. 472-474], one can prove that $q \in L^r_{\text{loc}}$. However, both of these approaches fail to give informations for the critical case $r = 1$ and does not give a representation of $q$. Whereas, our unified singular integral approach is self-contained, simple and provides the local $h^r$-estimate, as well as the local representation of $q$. The main ideas in our proof is to represent the localized-mollified distribution $q$ in terms of the Newtonian potential in $\mathbb{R}^n$ and finding its uniform bound in $h^r$, by using Calderón-Zygmund estimate [FS 72], [CZ 52]. Finally we show that the local representation of $q$ consists the sum of Calderón-Zygmund type singular integrals of the tensor $f$ (see equation (2.27) in Section 4).

For the case $n = 2$, under the stronger hypothesis that the local minimizers of $E[\cdot]$ are classical ($C^{1,\alpha}$-diffeomorphism), namely in the Sobolev space $W^{2,r}$ for some $r > 2$, LeTallec and Oden [TO 81] established the system of equations in (1.7). For $n = 2$, Bauman, Owen and Phillips [BOP 92] proved that if a minimizer is in $W^{2,r}$ for some $r > 2$, then it is smooth. For such $W^{2,r}$, $r > 2$ minimizers, the authors in [BOP 92] argued directly on the level of the Euler-Lagrange equations exploring the existence of integrable hydrostatic pressure. Evans and Giariepy [EG 99]
proved that any non-degenerate, Lipschitz area-preserving local minimizers of $E[\cdot]$ are $C^{1,\alpha}(\Omega_0)$, for some $0 < \alpha < 1$ for a dense open subset $\Omega_0 \subset \Omega$. We believe that the Euler-Lagrange equations (1.7) that we derived for $K^{1,3}$-minimizers may be useful in understanding the partial regularity of such minimizers, as evidenced by the result in (V).

In order to prove the existence of an integrable pressure $p$ associated with the local minimizer $u$, we only require the additional mild assumption $|\nabla u|^2 \log(2 + |\nabla u|^2), |\text{cof}\nabla u|^2 \log(2 + |\text{cof}\nabla u|^2) \in L^1_{\text{loc}}$. For $n = 2$, to derive the system of equilibrium equations (1.7) for $(u, p)$ in $\Omega$, we need $u$ to be in $W^{1,3}$, whereas the best-known previous result in this direction were for $W^{2,r}$-minimizers for some $r > 2$.

We organize the paper as follows. In Section 2 we prove (I); in Section 3 we prove (II) & (III); in Section 4 we prove (IV), and finally in Section 5 we prove (V). Throughout this article $C$ is a generic absolute constant depending on $n$, $U$, $\Omega$, $u(\Omega)$, $V \subset \subset u(\Omega)$, $r$, and $L$. Its value can vary from line to line, but each line is valid with $C$ being a pure positive number.

2. Local integrability of solutions $\nabla q = \text{div}f$

We recall some of the basic definitions and terminologies of Hardy spaces. Let $1 \leq r < \infty$. A distribution $f$ belongs to $H^r(\mathbb{R}^n)$ if and only if $f \in L^r(\mathbb{R}^n)$ and $R_j(f) \in L^r(\mathbb{R}^n)$ (see for example, [St 93, Proposition 3, p. 123]) for $j = 1, \cdots, n$, where $R_j$ is the Riesz transform of $f$ given by

$$R_j(f)(x) := \lim_{\varepsilon \to 0} c_n \int_{|y| \geq \varepsilon} \frac{y_j}{|y|^{n+1}} f(x - y) \, dy, \quad c_n := \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}},$$

so that $\widehat{R_j(f)}(\xi) = i\xi_j \hat{f}$. In short, we will write $H^r(\mathbb{R}^n)$ as simply $H^r$. For $f \in H^r$, the norm is defined as

$$\|f\|_{H^r} := \|f\|_{L^r} + \sum_{j=1}^n \|R_j(f)\|_{L^r}.$$

A standard result [St 70, p. 237] states that a positive function $f$, the Riesz transform $R_j f \in L^1_{\text{loc}}$ if and only if $f \log(2 + f) \in L^1_{\text{loc}}$. For $1 < r < \infty$, a classical result asserts that $f \in H^r$ if and only if $f \in L^r$, see [St 70, p. 220]. The celebrated Fefferman duality theorem [Fe 71], [FS 72, Theorem 2], [St 93, Theorem 1, p. 142] asserts that the dual of $H^1$ is the BMO, the functions of bounded mean oscillations. The following
Theorem 2.1 (Calderón-Zygmund, Fefferman-Stein). Let $1 \leq r < \infty$ and $f \in H^r$. Let $G$ be a $C^1$ function on $\mathbb{R}^n \setminus \{0\}$ homogeneous of degree 0 with mean value 0 over the unit sphere $S^{n-1}$, that is
\[
\int_{S^{n-1}} G(x) \, d\sigma(x) = 0.
\]
Then the function defined as
\[
T_0f(x) := \lim_{\delta \to 0} \int_{|y| \geq \delta} \frac{G(y)}{|y|^n} f(x - y) \, dy
\]
exists a.e. and furthermore,
\[
\|T_0f\|_{H^r} \leq C_{n,r} \|f\|_{H^r}.
\]

In particular, $R_j$'s are bounded linear operator on $H^r$, for any $1 \leq r < \infty$. Let us recall the definition of local Hardy spaces introduced by Goldberg [Go 79]. A distribution $f$ on $\mathbb{R}^n$ is said to be in the local $r$-Hardy space, written as $f \in h^r$, if and only if the maximal function
\[
M_{\text{loc}} f(x) := \sup_{0<\varepsilon<1} |(\rho_{\varepsilon} \ast f)(x)|
\]
is in $L^r$, where $\rho_{\varepsilon} := \varepsilon^{-n}\rho(x/\varepsilon)$, is a standard approximation of the identity. The $h^r$ norm of $f$ is defined to be the $L^r$ norm of the maximal function $M_{\text{loc}} f$. It follows that if $f \in h^r$ then $\eta f \in h^r$ for any smooth cut-off function and $H^r \subset h^r$. For bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, we adopt the definition of Hardy spaces $h^r(\Omega)$ introduced by Miyachi [Mi 90]. A distribution $f$ on $\Omega$ is said to be in $h^r(\Omega)$ if $f$ is the restriction to $\Omega$ of a distribution $F$ in $h^r(\mathbb{R}^n)$, i.e.,
\[
h^r(\Omega) := \{f \in \mathcal{D}'(\Omega) : \exists F \in h^r(\mathbb{R}^n), \text{ such that } F|_{\Omega} = f\}
\]  
\[= h^r(\mathbb{R}^n)/\{F \in h^r(\mathbb{R}^n) : F = 0 \text{ on } \Omega\}.
\]
The norm on this space is the quotient norm: the infimum of $h^r$ norms of all possible extensions of $f$ in $\mathbb{R}^n$. For $1 < r < \infty$ the spaces $h^r(\Omega)$ is equivalent to $L^r(\Omega)$. For smooth bounded domains $\Omega$, the Theorem 2.1 is valid for $f \in h^1(\Omega)$, see [Mi 90], [CKS 93].
Theorem 2.2. Let $U \subset \mathbb{R}^n$, $n \geq 2$ be a bounded Lipschitz domain and $1 \leq r < \infty$. Let $f = (f^i_j)$ such that $f^i_j \in h^r(U)$, for $1 \leq i, j \leq n$. Then the distribution $q : C^\infty_0(U) \to \mathbb{R}$ defined by

$$\nabla q = \text{div } f \iff \langle \nabla q, \nu \rangle = - \int_U f(x) : \nabla \nu(x) \, dx$$

for $\nu \in C^\infty_0(U, \mathbb{R}^n)$, is in $h^r(V)$, for any $V \subset\subset U$ where $A : B := \text{trace}(A^tB) = \sum_{ij} a^i_j b^j_i$, for $A, B \in \mathbb{M}^{n \times n}$. Furthermore, $q$ is locally represented by sum of singular integrals of $f^i_j$ (see equation (2.27)), and for any $V \subset\subset U$, there exists $C > 0$, depending only on $U$, $V$ and $r$ such that

$$\|q\|_{h^r(V)} \leq C \|f\|_{h^r(V)}.$$

Proof of Theorem 2.2. Let $U \subset \mathbb{R}^n$, $n \geq 2$ be a Lipschitz domain. Let $f := (f^i_j) \in \mathbb{M}^{n \times n}$ and $f^i_j \in h^r(U)$, for $1 \leq r < \infty$ and $1 \leq i, j \leq n$. Let $q \in D'(U)$, such that

$$\nabla q = \text{div } f \quad \text{in } D'(U).$$

Our idea is to mollify the equations in (2.5) and obtain uniform bound for the mollified $q$, by using Calderón-Zygmund estimate. Let $V \subset\subset U$ be a sub-domain and $0 < \varepsilon < \text{dist}(V, \partial U)$. Let $\rho_\varepsilon$ be the usual mollification kernel, and define convolution $q_\varepsilon : V \to \mathbb{R}$ by

$$q_\varepsilon(x) = (q * \rho_\varepsilon)(x) := \langle q, (\rho_\varepsilon)_x \rangle \quad \text{for } x \in V, \quad \text{where } (\rho_\varepsilon)_x(y) := \rho_\varepsilon(y-x), \ y \in U$$

Then by the standard properties of the mollification [DL 88, Proposition 1, p492], $q_\varepsilon$ is smooth and for any $1 \leq i \leq n$

$$\frac{\partial}{\partial x_i}(q * \rho_\varepsilon) = \frac{\partial q}{\partial x_i} * \rho_\varepsilon = q * \frac{\partial \rho_\varepsilon}{\partial x_i}.$$ 

Hence mollifying the system of equations in (2.5), we obtain

$$\nabla q_\varepsilon = \text{div } f_\varepsilon \quad \text{in } V,$$

where the divergence is taken in each rows of $f_\varepsilon := \left( (f^i_j)_\varepsilon \right)$, and $(f^i_j)_\varepsilon := f^i_j * \rho_\varepsilon$ is the mollification of $f$. Since $f^i_j \in h^r(U)$, we conclude that

$$(f^i_j)_\varepsilon \to f^i_j \quad \text{strongly in } h^r(V) \quad \text{as } \varepsilon \to 0,$$

for all $1 \leq i, j \leq n$. Applying the divergence operator to the both sides of the above equation, we obtain

$$\Delta q_\varepsilon = \text{div}(\text{div } f_\varepsilon) \quad \text{in } V.$$
Since there is no control on the boundary values, we need to localize the equation (2.8). Let $W \subset V \subset U$. Let $\eta \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \eta \leq 1$ be a cut-off function such that $\eta \equiv 1$ in $W$ and $\eta \equiv 0$ outside $V$. Let $q_\varepsilon := \eta q_\varepsilon$ be the localization of $q_\varepsilon$. Then $q_\varepsilon$ is the solution of Poisson equation (2.9)
\[ \Delta q_\varepsilon = \bar{f}_\varepsilon \quad \text{in} \quad \mathbb{R}^n, \]
where
\[ \bar{f}_\varepsilon := \eta \Delta q_\varepsilon + 2\langle \nabla q_\varepsilon, \nabla \eta \rangle + q_\varepsilon \Delta \eta. \]

Therefore $q_\varepsilon$ is represented by the Newtonian potential of in $\mathbb{R}^n$. In other words,
\[ q_\varepsilon(x) = -\int_{\mathbb{R}^n} \Phi(x - y) f_\varepsilon(y) \, dy, \]
where $\Phi$ is fundamental solution of the Laplace equation in $\mathbb{R}^n$ and is given by
\[ \Phi(x) := \begin{cases} -\frac{1}{2\pi} \log |x| & \text{if } n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & \text{if } n \geq 3, \end{cases} \]
for $x \in \mathbb{R}^n \setminus \{0\}$, and $\alpha(n) := \frac{\pi^n/2}{\Gamma(\frac{n}{2}+1)}$ is the volume of the unit ball in $\mathbb{R}^n$. Using (2.10) in (2.11), we obtain
\[ q_\varepsilon(x) = -\int_{\mathbb{R}^n} \eta(y) \Phi(x - y) \, dy \\
+ 2\int_{\mathbb{R}^n} (\langle \nabla f_\varepsilon, \nabla \eta \rangle + q_\varepsilon \Delta \eta) \Phi(x - y) \, dy \\
:= \ -I_1^1(x) - 2I_2^2(x) - I_3^3(x), \]
where
\[ I_1^1(x) := \int_{\mathbb{R}^n} \eta(y) \Phi(x - y) \, dy \\
I_2^2(x) := \int_{\mathbb{R}^n} (\langle \nabla f_\varepsilon, \nabla \eta \rangle \Phi(x - y) \, dy \\
I_3^3(x) := \int_{\mathbb{R}^n} q_\varepsilon(y) \Delta \eta(y) \Phi(x - y) \, dy \]
By direct computations, observe that, for \(1 \leq i, j \leq n\)

\[
(\eta \Phi)_{y_i} = \eta y_i \Phi(y) - \frac{1}{\omega_n} \eta \frac{y_i}{|y|^n},
\]

\[
(\eta \Phi)_{y_i y_j} = \eta y_i y_j \Phi(y) - \frac{1}{\omega_n} \left( \frac{y_i y_j}{|y|^n} \right) \frac{\eta}{|y|^n},
\]

where \(\delta_{ij}\) is the Kronecker delta and \(\omega_n := n\alpha_n\) is the surface area of the unit sphere \(S^{n-1}\). We now establish an uniform local \(h^r\)-estimates \((1 \leq r < \infty)\) for \(q_\varepsilon\) through the following steps.

**Step 1: Limit of \(I^3_\varepsilon\).** Let us fix \(x \in W \subset \subset V \subset \subset U\). Since \(\Delta \eta = 0\) on \(W\), the integrand in \(I^3_\varepsilon(x)\) is smooth. Since \(q_\varepsilon\) is determined up to a constant, we can add a constant to \(y \mapsto \Delta \eta(y) \Phi(x - y)\), if necessary, to ensure that it has vanishing integral. For each fixed \(x \in W\), let \(v_x : V \to \mathbb{R}^n\) be the solution of the Dirichlet problem

\[
\begin{align*}
\text{div} v_x(y) &= \Delta \eta(y) \Phi(x - y) \quad \text{for } y \in V \\
v_x &= 0 \quad \text{on } \partial V.
\end{align*}
\]

Then using (2.19), integrating by parts, and the convergence of \(f_\varepsilon\) in (2.16), we obtain

\[
I^3_\varepsilon(x) = \int_{\mathbb{R}^n} q_\varepsilon(y) \Delta \eta(y) \Phi(x - y) \, dy
\]

\[
= \int_{\mathbb{R}^n} q_\varepsilon(y) \text{div} v_x(y) \, dy
\]

\[
= - \int_{\mathbb{R}^n} \langle \nabla q_\varepsilon(y), v_x(y) \rangle \, dx
\]

\[
= - \int_{\mathbb{R}^n} \langle \text{div} f_\varepsilon(y), v_x(y) \rangle \, dy
\]

\[
= \int_{\mathbb{R}^n} f_\varepsilon(y) : \nabla y v_x(y) \, dy
\]

\[
\to \int_{\mathbb{R}^n} f(y) : \nabla y v_x(y) \, dy \quad \text{as } \varepsilon \to 0
\]

\[
:= I^3_0(x) \quad \text{for } x \in W \subset \subset V.
\]

Since \(f_\varepsilon \to f\) strongly in \(h^r(V, M^{n \times n})\), it follows that \(I^3_\varepsilon \to I^3_0\) strongly in \(h^r(W)\).
**Step 2: Limit of** $I_\varepsilon^2$. Let us fix $x \in W \subset V \subset U$. Integrating by parts, invoking (2.17) and letting $\varepsilon \to 0$ we have

\begin{align*}
I_\varepsilon^2(x) &= \int_{\mathbb{R}^n} \left\langle \text{div} f_\varepsilon(y), \Phi(x-y) \nabla \eta(y) \right\rangle dy \\
&= - \int_{\mathbb{R}^n} f_\varepsilon \cdot \nabla_y \left( \Phi(x-y) \nabla \eta \right) dy \\
&= - \int_{\mathbb{R}^n} f_\varepsilon \cdot \left( \Phi(x-y) \nabla^2 \eta - \frac{(y-x) \otimes \nabla \eta}{\omega_n |y-x|^n} \right) dy \\
&\to - \int_{\mathbb{R}^n} f \cdot \left( \Phi(x-y) \nabla^2 \eta - \frac{(y-x) \otimes \nabla \eta}{\omega_n |y-x|^n} \right) dy \\
&:= I_0^2(x) \quad x \in W.
\end{align*}

Using the strong convergence of $f_\varepsilon$ in $h^r(V)$, again it follows that $I_\varepsilon^2 \to I_0^2$ in $h^r(W)$.

**Step 3: Limit of** $I_\varepsilon^1$. Integrating by parts twice the integral in (2.14) and using (2.18)

\begin{align*}
I_\varepsilon^1(x) &= \int_{\mathbb{R}^n} \text{div} \text{div} f_\varepsilon(y) \eta(y) \Phi(x-y) dy \\
&= \int_{\mathbb{R}^n} f_\varepsilon(y) \cdot \nabla_y^2 \left( \eta(y) \Phi(x-y) \right) dy \\
&= \int_{\mathbb{R}^n} f_\varepsilon(y) \cdot \left( \Phi(x-y) \nabla^2 \eta(y) - \frac{1}{\omega_n} \nabla \eta \otimes (y-x) + \frac{(y-x) \otimes \nabla \eta}{|x-y|^n} \right) dy \\
&\quad - \frac{1}{\omega_n} \int_{\mathbb{R}^n} f_\varepsilon(y) \cdot \left( \text{Id}_n - n \frac{(y-x) \otimes (y-x)}{|x-y|^2} \right) \frac{\eta}{|x-y|^n} dy \\
&:= I_{11}^\varepsilon(x) + I_{12}^\varepsilon(x), \quad x \in W,
\end{align*}

where $\text{Id}_n$ is the $n \times n$ identity matrix. Using the convergence of $f_\varepsilon$, observe that as $\varepsilon \to 0$,

\begin{align*}
I_{11}^\varepsilon(x) := & \int_{\mathbb{R}^n} f_\varepsilon \cdot \left( \Phi(x-y) \nabla^2 \eta - \frac{\nabla \eta \otimes (y-x) + (y-x) \otimes \nabla \eta}{\omega_n |x-y|^n} \right) dy \\
&\to \int_{\mathbb{R}^n} f \cdot \left( \Phi(x-y) \nabla^2 \eta - \frac{\nabla \eta \otimes (y-x) + (y-x) \otimes \nabla \eta}{\omega_n |x-y|^n} \right) dy \\
&:= I_{11}^0(x), \quad x \in W.
\end{align*}

In order to estimate $I_{12}^\varepsilon$, define the kernels $\Omega_{ij} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by

\begin{align*}
\Omega_{ij}(y) := & \delta_{ij} - n \frac{y_i y_j}{|y|^2}, \quad y \in \mathbb{R}^n \setminus \{0\}, \quad i,j = 1, \cdots, n.
\end{align*}
Since \( n\alpha_n = \omega_n \), integrating by parts, observe that for any \( i, j = 1, \cdots, n \),
\[
\int_{\mathbb{S}^{n-1}} \Omega_{ij}(y) \, d\sigma(y) = \int_{\mathbb{S}^{n-1}} (\delta_{ij} - ny_i y_j) \, d\sigma(y)
\]
\[
= \omega_n \delta_{ij} - n \int_{\mathbb{S}^{n-1}} y_i y_j \, d\sigma(y)
\]
\[
= \omega_n \delta_{ij} - n \int_{B_1} \frac{\partial}{\partial y_j} y_i \, dy
\]
\[
= \omega_n \delta_{ij} - n \delta_{ij} \alpha_n
\]
\[
= 0.
\]
Hence each \( \Omega_{ij} \) satisfies all the conditions of Calderón-Zygmund Kernel [St 70]. Therefore,

\[
I_{12}^\varepsilon(x) := -\frac{1}{\omega_n} \int_{\mathbb{R}^n} \eta f \varepsilon : \left( Id_n - n \frac{(y-x) \otimes (y-x)}{|x-y|^2} \right) \frac{dy}{|x-y|^n}
\]
is the sum of Calderón-Zygmund type singular integrals with the homogeneous kernel \( \Omega_{ij} \). Since \( f \in h^r(U, M^{n \times n}) \), \( 1 \leq r < \infty \), by Theorem 2.1 \( I_{12}^\varepsilon \in h^r(W) \). Furthermore, the following sum of singular integrals

\[
I_0^{12}(x) := -\frac{1}{\omega_n} \int_{\mathbb{R}^n} \eta f : \left( Id_n - n \frac{(y-x) \otimes (y-x)}{|x-y|^2} \right) \frac{dy}{|x-y|^n}
\]
exists for almost every \( x \in W \subset W \) and is in \( h^r(W) \). From the singular integrals (2.24) and (2.25), by Theorem 2.1, we have

\[
I_{12}^\varepsilon(x) - I_0^{12}(x) = -\frac{1}{\omega_n} \sum_{i,j=1}^n \int_{\mathbb{R}^n} \left( \eta(f_j^i)\varepsilon(y) - \eta f_j^i(y) \right) \frac{\Omega_{ij}(x-y)}{|x-y|^n} \, dy.
\]
Hence there exists \( C := C(V, W, r) > 0 \) such that

\[
(2.26) \quad \| I_{12}^\varepsilon - I_0^{12} \|_{h^r(W)} \leq C \sum_{j=1}^n \| (f_j^i)\varepsilon - f_j^i \|_{h^r(V)} \to 0 \quad \text{as } \varepsilon \to 0.
\]

**Step 4: Explicit representation of \( q \).** To complete the proof, let us define the potential \( q : W \to \mathbb{R} \) by

\[
q(x) := -(I_{11}^1(x) + I_0^{12}(x) + 2I_0^2(x) + I_0^3(x)).
\]
Then from (2.20), (2.21), (2.22), and (2.26), we conclude that \( q_\varepsilon \to q \) strongly in \( h^r_{\text{loc}} \) for any \( 1 \leq r < \infty \), and hence and \( q \) is represented as

\[
q(x) = \int_U f : \left( \Phi(x-y)\nabla^2 \eta - \nabla_y v_x \right) dy + \frac{1}{\omega_n} \int_U f : \left( \nabla \eta \otimes (y-x) - (y-x) \otimes \nabla \eta \right) \frac{dy}{|x-y|^n} + \frac{1}{\omega_n} \int U \eta f : \left( Id_n - n \frac{(y-x) \otimes (y-x)}{|x-y|^2} \right) \frac{dy}{|x-y|^n}
\]

for any \( x \in W \). Since \( q \) is the strong limit of the family \( q_\varepsilon \) in \( W \), it is independent of the choice of the cut-off function \( \eta \). This completes the proof of Theorem 1.1. \( \square \)

3. **First Variation of Energy and the existence of hydrostatic pressure**

Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \) be a smooth, simply connected and bounded domain and let \( L : \mathbb{M}^{n \times n} \to \mathbb{R} \) be smooth function. We are now in a position to establish the existence of integrable hydrostatic pressure associated with the local minimizers of the energy

\[
E[w] := \int_\Omega L(\nabla w(x)) dx,
\]

for incompressible \( W^{1,2} \)-deformations \( w : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n \). By direct computation, observe that Mooney-Rivlin bulk-energy given by

\[
L(X) = \frac{\mu_1}{2} (|\nabla u|^2 - 3) + \frac{\mu_2}{2} (|\text{cof}\nabla u|^2 - 3),
\]

satisfies the following,

\[
DL = \mu_1 P + \mu_2 \begin{pmatrix}
\text{cof}(SQ)^1 : (SP)^1 & -\text{cof}(SQ)^2 : (SQ)^1 & \text{cof}(SQ)^3 : (SP)^1 \\
-\text{cof}(SQ)^2 : (SP)^2 & \text{cof}(SQ)^3 : (SP)^2 & -\text{cof}(SQ)^3 : (SP)^2 \\
\text{cof}(SQ)^3 : (SP)^3 & -\text{cof}(SQ)^3 : (SP)^3 & \text{cof}(SQ)^3 : (SP)^3
\end{pmatrix},
\]

where \( Q := \text{cof} P \), and \( (SX)^i_j \) is the \( 2 \times 2 \) submatrix obtained by deleting the \( i \)-th row and the \( j \)-th column of the matrix \( X \in \mathbb{M}^{3 \times 3} \). Furthermore,
satisfies the integral identity for all \( E \) for some equation \( \nabla \). Let Theorem 3.1. Let \( L : M^{n \times n} \to \mathbb{R} \) be smooth and satisfies the growth condition (3.3). Assume that \( u \in A \) be a continuous and injective local minimizer of \( E[\cdot] \), such that \( |\nabla u|^2, |\text{cof} \nabla u|^2 \in h_{\text{loc}}^r(\Omega) \) for some \( 1 \leq r < \infty \). Then there exists a scalar function \( q \in h_{\text{loc}}^r(u(\Omega)) \), such that

\[
\|q\|_{h^r(V)} \leq C \left( \|\nabla u\|^2_{h^{r-1}(V)} + \||\text{cof} \nabla u\|^2_{h^{r-1}(u^{-1}(V))} \right), \quad V \subset \subset u(\Omega),
\]

for some \( C > 0 \) (depending on \( r, V, n \) and \( u(\Omega) \)) and the pair \((u, q)\) satisfies the integral identity

\[
\int_{\Omega} DL(\nabla u(x)) : \nabla(v_\circ u) \, dx = \int_{u(\Omega)} q(y) \, \text{div} v(y) \, dy
\]

for all \( v \in C_0^\infty(u(\Omega), \mathbb{R}^n) \), where \( A : B := \text{tr}(A^tB) = \sum_{i,j=1}^n a_{ij} b_{ij} \) is the scalar product on \( M^{n \times n} \).
Remark 3.2. Let \( W \subset \subset V \subset \subset \mathbf{u}(\Omega) \), and \( \eta \in C^\infty_0(V) \) be a cut-off function such that \( \eta \equiv 1 \) on \( W \). Then \( q \) is locally represented as

\[
q(x) = \int_V \tilde{\sigma} : (\Phi(x - y) \nabla^2 \eta - \nabla_y \mathbf{v}_x) \, dy
+ \frac{1}{\omega_n} \int_V \frac{\eta \tilde{\sigma} : \left( \nabla \eta \otimes (y - x) - (y - x) \otimes \nabla \eta \right)}{|y - x|^n} \, dy
+ \frac{1}{\omega_n} \int_V \frac{\eta \tilde{\sigma} : \left( \text{Id}_n - n \frac{(y - x) \otimes (y - x)}{|y - x|^2} \right)}{|y - x|^n} \, dy,
\]

for any \( x \in W \), where \( \Phi \) is Newtonian potential in \( \mathbb{R}^n \) defined in (2.12) and \( \mathbf{v}_x \) as defined in (2.19).

Remark 3.3. In the study of regularity of finite energy deformations, Šverák [Sv 88] proved that for any \( W^{1,n} \)-deformation \( \mathbf{w} \) with \( \det \nabla \mathbf{w}(x) > 0 \), a.e., there exists a continuous function \( \omega \) on \( \mathbb{R} \) with \( \omega(0) = 0 \) such that

\[
|\mathbf{w}(x) - \mathbf{w}(y)| \leq \omega(|x - y|), \quad \text{for any } x, y \in \Omega \subset \subset \mathbb{R}^n.
\]

It is also well-known any \( W^{1,n} \)-deformation \( \mathbf{w} \) for which the distortion function \( K(\cdot, \mathbf{w}) := |\nabla \mathbf{w}(\cdot)|^n / \det \nabla \mathbf{w}(\cdot) \in L^r \) for some \( r > n - 1 \), is a homeomorphism. Thus in particular, area-preserving \( W^{1,r} \) (\( r > 2 \))-deformations in the plane are continuous and open maps. However, in general for \( n \geq 3 \), any deformation \( \mathbf{w} \in K^{1,2} \) may be totally discontinuous, see [Sv 88, p. 119].

In order to prove Theorem 3.1, we establish the following first variation of the energy integral \( E[\cdot] \).

Lemma 3.4. First Variation. Let \( \mathbf{u} \in \mathcal{A} \) be a local minimizer of \( E[\cdot] \). We further assume that \( \mathbf{u} \) is a continuous and an injective map. Then \( \mathbf{u} \) satisfies the following integral identity

\[
\int_\Omega DL(\nabla \mathbf{u}(x)) : \nabla (\mathbf{v} \circ \mathbf{u})(x) \, dx = 0,
\]

for all smooth, compactly supported and divergence free vector fields \( \mathbf{v} \) on \( \mathbf{u}(\Omega) \).

Proof: By the invariance of domain \( \mathbf{u}(\Omega) \) is open and \( \mathbf{u} : \Omega \to \mathbf{u}(\Omega) \) is a homeomorphism. Let \( \mathbf{v} \in C_0^\infty(\mathbf{u}(\Omega), \mathbb{R}^n) \) be a vector field with \( \text{div} \mathbf{v} = 0 \). For each \( y \in \mathbf{u}(\Omega) \), consider the unique smooth flow \( \phi(y, \cdot) : \mathbb{R} \to \mathbf{u}(\Omega) \) given by

\[
\frac{d\phi}{dt}(y, t) = \mathbf{v}(\phi(y, t)) \quad \text{in } \mathbb{R}, \quad \phi(y, 0) = y.
\]
Using the relations $\frac{\partial}{\partial P^i_j}\det P = (\text{cof } P)^i_j$ and $P (\text{cof } P)^t = Id_n \det P$, by a direct calculations we observe that

\begin{equation}
\frac{d}{dt}(\det \nabla_y \phi(y, t)) = \det \nabla_y \phi(y, t) \text{ div } v = 0. \tag{3.8}
\end{equation}

Since $\det \nabla_y \phi(y, 0) = 1$, from (3.8) it follows that $\det \nabla_y \phi(y, t) = 1$ for all $t \in \mathbb{R}$ and $y \in u(\Omega)$. Consider the map $w : \Omega \times \mathbb{R} \to u(\Omega)$ defined by

\[ w(x, t) := \phi(\cdot, t) \circ u(x) = \phi(u(x), t) \quad \text{for any } t \in \mathbb{R}, \ x \in \Omega. \]

Let $V := \text{supp } v \subset u(\Omega)$, then $v(u(x)) = 0$ for $u(x) \not\in V$. This in conjunction with the uniqueness of $\phi$ implies that $\phi(u(x), t) = u(x)$ for all points $x$ such that $u(x) \not\in V$. Since $\Omega$ is bounded, $u$ is continuous and $V$ is compact, $\Omega' = u^{-1}(V)$ is a compact subset of $\Omega$. Hence $\text{supp}(w(x, t) - u(x)) \subset \Omega'$. Furthermore, $\det \nabla_x w(x, t) = \det \nabla_y \phi(y, t) \det \nabla u(x) = 1$. Therefore, $w(\cdot, t) \in A$ and $\text{supp}(u - w(\cdot, t)) \subset \Omega$ for all $t \in \mathbb{R}$. Since $u$ is a local minimizer of $E[\cdot],$

\[ E[u] \leq E[w(\cdot, t)] \quad \text{for all } \ t \in \mathbb{R}. \]

Thus in particular,

\[ 0 = \frac{d}{dt} \int_{\Omega} L(\nabla w(x, t)) \, dx \bigg|_{t=0} = \sum_{i,j=1}^{2} \int_{\Omega} L^i_j(\nabla w(x, t)) \frac{d}{dt} \left( \frac{\partial w^i}{\partial x_j}(x, t) \right) \, dx \bigg|_{t=0} = \sum_{i,j=1}^{2} \int_{\Omega} L^i_j(\nabla w(x, t)) \frac{\partial}{\partial x_j} \left( \frac{d\phi^i}{dt}(u(x), t) \right) \, dx \bigg|_{t=0} = \sum_{i,j=1}^{2} \int_{\Omega} L^i_j(\nabla u(x)) \frac{\partial}{\partial x_j} (v^i(u(x))) \, dx = \sum_{i,j=1}^{2} \int_{\Omega} L^i_j(\nabla u(x)) \frac{\partial}{\partial x_j} (v^i(u(x))) \, dx = \int_{\Omega} DL(\nabla u(x)) : \nabla (v \circ u)(x) \, dx, \]

for all smooth, compactly supported and divergence free vector fields on $u(\Omega)$, where $L^i_j(P) := \frac{\partial L}{\partial p^i_j}(P)$. This proves the Theorem. \hfill \Box
**Proof of Theorem 3.1:** Let $1 \leq r < \infty$ and $U' \subset \subset U$. Let $u \in \mathcal{A}$ be a local minimizer of $E[\cdot]$ such that $|\nabla u|^2 \in h^r$ and $|\text{cof} \nabla u|^2 \in h^r(U')$ for some $1 \leq r < \infty$. Assume further that $u : \Omega \rightarrow u(\Omega)$ is continuous and bijective map.

Now define $g = (g^1, \cdots, g^n) : C^1_0(\Omega, \mathbb{R}^n) \rightarrow \mathbb{R}$ by

$$
(3.9) \quad \langle g, v \rangle := \int_\Omega DL(\nabla u(x)) : \nabla (v \circ u)(x) \, dx,
$$

for all $v = (v^1, \cdots, v^n) \in C^1_0(\Omega, \mathbb{R}^n)$. In view of the volume constraint and growth condition (3.3), it follows that

$$
(3.10) \quad |\langle g, v \rangle| \leq C(1 + \|\nabla u\|_{L^2(\Omega)} + \|\text{cof} \nabla u\|_{L^2(\Omega)}) \|\nabla v\|_{L^\infty(\Omega)},
$$

for any $v \in C^1_0(\Omega, \mathbb{R}^n)$. Hence $g$ is a continuous linear functional on $C^1_0(\Omega, \mathbb{R}^n)$. Using the first variation (3.6), we conclude that

$$
(3.11) \quad \langle g, v \rangle = 0 \quad \forall \ v \in C^1_0(\Omega, \mathbb{R}^n), \ \text{div} \ v = 0.
$$

Hence there exists $q \in D'(\Omega)$ (see [Te 01, Proposition 1.1, p10]), such that

$$
(3.12) \quad g = -\nabla q \quad \text{in} \ D'(\Omega, \mathbb{R}^n)
$$

modulo translation of a constant. In order to obtain $h^r$ estimates of $q$, for $1 \leq i, j \leq n$, let us define $\sigma^i_j : \Omega \rightarrow \mathbb{R}$ by

$$
(3.13) \quad \sigma^i_j(x) := \sum_{k=1}^n L_k^i(\nabla u(x)) \frac{\partial u^j}{\partial x_k}(x) \quad \text{for} \ x \in \Omega,
$$

so that, the Cauchy-Green strain tensor on $\Omega$ is given by

$$
(3.14) \quad \sigma := (\sigma^i_j) = (DL(\nabla u))' \nabla u
$$

Define the $ij$-th component of the Cauchy-Green Strain tensor $\tilde{\sigma}^i_j$ on the deformed domain $u(\Omega)$ by

$$
(3.15) \quad \tilde{\sigma}^i_j := \sigma^i_j \circ u^{-1} \quad \text{on} \ u(\Omega), \ i, j = 1, \cdots, n.
$$

The growth condition $|\sigma^i_j| \leq C(|\nabla u|^2 + |\text{cof} \nabla u|^2)$ and $|\nabla u|^2, |\text{cof} \nabla u|^2 \in L \log L$ yields $\tilde{\sigma}^i_j \in h^1(V)$. If $u \in K^1_0(\Omega, \mathbb{R}^n), 1 < r < \infty$, from the definition of $\sigma^i_j, \tilde{\sigma}^i_j$, and the condition (3.3) on $L$, it follows that

$$
(3.16) \quad \int_V |(\tilde{\sigma}^i_j)^r = \int_{u^{-1}(V)} |\sigma^i_j|^r
\leq C \left( \|\nabla u\|_{L^{2r}(u^{-1}(V))}^{2r} + \|\text{cof} \nabla u\|_{L^{2r}(u^{-1}(V))}^{2r} \right),
$$
for any $V \subset\subset u(\Omega)$. Therefore, if $|\nabla u|^2 \in h^r$ and $|\text{cof } \nabla u|^2 \in h^r_{\text{loc}}$ for some $1 \leq r < \infty$, from (3.16), we have

$$\sigma := (\sigma_j^i) \in h^r_{\text{loc}}(\Omega, M^{n \times n}) \quad \text{and} \quad \tilde{\sigma} := (\tilde{\sigma}_j^i) \in h^r_{\text{loc}}(u(\Omega), M^{n \times n}).$$

Observe that, from the definition of $g$ in (3.9), $\sigma_j^i$ in (3.13), $\tilde{\sigma}_j^i$ in (3.15), and change of variables,

$$\langle g, v \rangle = \sum_{i,k=1}^n \int_{\Omega} L^i_k(\nabla u(x)) \frac{\partial}{\partial x_k} (v^i \circ u)(x) \, dx$$

$$= \sum_{i,j,k=1}^n \int_{\Omega} L^i_j(\nabla u(x)) \frac{\partial v^i}{\partial y_j}(u(x)) \frac{\partial u^i}{\partial x_k}(x) \, dx$$

$$= \sum_{i,j=1}^n \int_{\Omega} \sigma^i_j(x) \frac{\partial v^i}{\partial y_j}(u(x)) \, dx$$

$$= \int_{\Omega} \sigma(x) : \nabla v(u(x)) \, dx$$

$$= \int_{\Omega} \tilde{\sigma}(y) : \nabla v(y) \, dy$$

$$= -\langle \text{div } \tilde{\sigma}, v \rangle$$

for any $v \in C^1_0(u(\Omega), \mathbb{R}^n)$. Hence

$$g = -\text{div } \tilde{\sigma} \quad \text{in } \mathcal{D}'(u(\Omega), M^{n \times n})$$

where the divergence is taken in each rows. Therefore, combining (3.12) and (3.18), we get

$$\nabla q = \text{div } \tilde{\sigma} \quad \text{in } \mathcal{D}'(u(\Omega), M^{n \times n}).$$

By taking $f = \tilde{\sigma}$, and $U = V \subset\subset u(\Omega)$ in (3.19), from Theorem 2.2, we conclude that $q \in h^r_{\text{loc}}(u(\Omega))$, it satisfies the local representation (3.5), and

$$\|q\|_{h^r(U)} \leq C\|\tilde{\sigma}\|_{h^r(U)}$$

$$\leq C \left( \|\nabla u\|^2_{h^r(u^{-1}(V))} + \|\text{cof } \nabla u\|^2_{h^r(u^{-1}(V))} \right),$$

for any $V \subset\subset u(\Omega)$, for some $C > 0$, depending on $r$, $V$, $n$ and $u(\Omega)$. Since $q \in L^1_{\text{loc}}$, from (3.12), it follows that

$$\langle g, v \rangle = -\langle \nabla q, v \rangle = \langle q, \text{div } v \rangle = \int_{u(\Omega)} q(y) \text{div } v(y) \, dy.$$
for any \( v \in C_0^1(\mathbf{u}(\Omega), \mathbb{R}^n) \). Hence
\[
(3.21) \quad \int_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla(v \circ \mathbf{u})(x) \, dx = \int_{\mathbf{u}(\Omega)} q(y) \, \text{div} \, v(y) \, dy,
\]
for any \( v \in C_0^1(\mathbf{u}(\Omega), \mathbb{R}^n) \). This proves the Theorem.

\[ \Box \]

4. Derivation of Euler-Lagrange Equations

**Theorem 4.1.** Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \), be a smooth, simply connected and bounded domain. Let \( \mathbf{u} \in \mathcal{A} \cap K^{1,s}_{\text{loc}}(\Omega, \mathbb{R}^n) \) for some \( s \geq 3 \) be a continuous and injective local minimizer of \( \text{E}[:\cdot:] \). Then the hydrostatic pressure \( p := q \circ \mathbf{u} \in L^{s/2}_{\text{loc}}(\Omega) \), and the pair \( (\mathbf{u}, p) \) satisfies
\[
(4.1) \quad \int_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla \phi(x) \, dx = \int_{\Omega} p(x) \, \text{cof} \,(\nabla \mathbf{u}(x)) : \nabla \phi(x) \, dx,
\]
for all \( \phi \in C^1_0(\Omega, \mathbb{R}^n) \), where \( q \in L^{s/2}_{\text{loc}}(\mathbf{u}(\Omega)) \) as in Theorem 3.1. In other words, the pair \( (\mathbf{u}, p) \) satisfies the system of Euler-Lagrange equations
\[
\text{div} \left[ DL(\nabla \mathbf{u}(x)) - p(x) \, \text{cof} \,(\nabla \mathbf{u}(x)) \right] = 0 \quad \text{in} \ \Omega,
\]
in the sense of distribution, where the divergence is taken in each rows.

**Proof.** Let \( \Omega \subset \mathbb{R}^n \) be a smooth, simply connected domain. Recall that \( K^{1,s} := \{ w \in W^{1,s} : \text{cof} \, \nabla w \in L^s \} \) and \( \mathcal{A} := \{ w \in K^{1,2}(\Omega, \mathbb{R}^n) : \det \nabla w = 1 \ \text{a.e.} \} \). Let \( \mathbf{u} \in \mathcal{A} \cap K^{1,s}_{\text{loc}}(\Omega, \mathbb{R}^n) \), \( s \geq 3 \) be a continuous injective local minimizer of the functional \( \text{E}[\cdot] \). By Theorem 3.1, there exists \( q \in L^{s/2}_{\text{loc}} \) such that the pair \( (\mathbf{u}, q) \) satisfies the identity (3.21). Let \( \mathbf{u}^{-1} : \mathbf{u}(\Omega) \rightarrow \Omega \) be the inverse of \( \mathbf{u} \). Then using the volume-constraint we obtain
\[
\nabla_y \mathbf{u}^{-1}(y) = (\nabla_x \mathbf{u}(x))^{-1} = (\text{cof} \, \nabla \mathbf{u}(x))^t, \quad y = \mathbf{u}(x),
\]
and hence by the change of variables
\[
\int_{\mathbf{u}(\Omega)} |\nabla \mathbf{u}^{-1}(y)|^2 \, dy = \int_{\Omega} |\text{cof} \, \nabla \mathbf{u}(x)|^2 \, dx < \infty.
\]
Using the relation \( \text{cof} \,(XY) = \text{cof} \, X \, \text{cof} \, Y \), for \( X, Y \in \mathbb{M}^{n \times n} \), observe that
\[
\text{Id}_n = \text{cof} \,(\nabla_y \mathbf{u}^{-1} \nabla \mathbf{u}) = \text{cof} \,(\nabla_y \mathbf{u}^{-1}) \, \text{cof} \,(\nabla \mathbf{u}) = \text{cof} \,(\nabla_y \mathbf{u}^{-1})(\nabla \mathbf{u})^{-t},
\]
and hence
\[
\text{cof} \,(\nabla \mathbf{u}^{-1}) = (\nabla \mathbf{u})^t.
\]
Since \( \mathbf{u} \in K^{1,s}_{\text{loc}}(\Omega, \mathbb{R}^n) \), it follows that \( \mathbf{u}^{-1} \in K^{1,s}_{\text{loc}}(\mathbf{u}(\Omega), \Omega) \) for \( s \geq 3 \). Let \( V \subset \subset \mathbf{u}(\Omega) \) and \( \phi \in C^1_0(\mathbf{u}^{-1}(V), \mathbb{R}^n) \). Then the composition \( \phi \circ \mathbf{u}^{-1} \in \)}
$W^{1,s}_0(V, \mathbb{R}^n)$. Hence there exists $v_\varepsilon \in C^1_0(V, \mathbb{R}^n)$ such that $v_\varepsilon \to \psi := \phi \circ u^{-1}$ strongly in $W^{1,s}(V, \mathbb{R}^n)$ as $\varepsilon \to 0$. Let $U := u^{-1}(V)$. Then Hölder inequality yields

$$
\int_U DL(\nabla u) : \left( \nabla (v_\varepsilon \circ u) - \nabla (\psi \circ u) \right) dx \\
\quad \quad \quad \quad = \int_U (\nabla u)^t DL(\nabla u) : \left( \nabla_z v_\varepsilon(u) - \nabla_z \psi(u) \right) dx \\
\quad \quad \quad \quad \quad \quad \quad \leq C \|\nabla u\|_{L^{2s'}(U)} \|\nabla (v_\varepsilon - \psi)\|_{L^s(V)},
$$

where $s' := s/(s-1)$. Notice that $s \geq 3$ yields $2s' \leq s$ and hence $\nabla u \in L^s_{loc}(\Omega) \subseteq L^{2s'}_{loc}(\Omega)$. Therefore, from (3.9) we obtain

(4.2) $\langle g, v_\varepsilon \rangle = \int_{u^{-1}(V)} DL(\nabla u(x)) : \nabla (v_\varepsilon \circ u)(x) \, dx$

$$
\to \int_{u^{-1}(V)} DL(\nabla u(x)) : \nabla (\phi \circ u^{-1} \circ u)(x) \, dx \quad \text{as} \quad \varepsilon \to 0
$$

$$
= \int_{u^{-1}(V)} DL(\nabla u(x)) : \nabla \phi(x) \, dx.
$$

Since $\nabla u$, $\text{cof} \nabla u \in L^q_{loc}$, $q \in L^{s/2}_{loc}$ and $L^{s/2}_{loc} \subseteq L^{s/(s-1)}_{loc}$ for $s \geq 3$, applying change of variables in (3.21), and letting $\varepsilon \to 0$ we obtain

(4.3) $\langle g, v_\varepsilon \rangle = \int_V q(y) \text{trace} (\nabla v_\varepsilon(y)) \, dy$

$$
= \int_{u^{-1}(V)} q(u(x)) \text{trace} \left( \nabla_u v_\varepsilon(u(x)) \right) \, dy
$$

$$
= \int_{u^{-1}(V)} q(u(x)) \text{trace} \left( \nabla (v_\varepsilon \circ u)(x) \left( \text{cof} \nabla u(x) \right)^t \right) \, dx
$$

$$
= \int_{u^{-1}(V)} q(u(x)) \text{cof} (\nabla u(x)) : \nabla (v_\varepsilon \circ u)(x) \, dx,
$$

$$
\to \int_{u^{-1}(V)} q(u(x)) \text{cof} (\nabla u(x)) : \nabla (\phi \circ u^{-1} \circ u)(x) \, dx
$$

$$
= \int_{u^{-1}(V)} q(u(x)) \text{cof} (\nabla u(x)) : \nabla \phi(x) \, dx.
$$

Hence from (4.2) and (4.3) we obtain

$$
\int_{u^{-1}(V)} DL(\nabla u(x)) : \nabla \phi(x) \, dx = \int_{u^{-1}(V)} q(u(x)) \text{cof} (\nabla u(x)) : \nabla \phi(x) \, dx,
$$
for any $\phi \in C^1_0(u^{-1}(V), \mathbb{R}^n)$. Finally choose a sequence of smooth, simply connected sets $V_k \subset \subset V_{k+1} \subset \subset u(\Omega)$ sub-domains such that $u(\Omega) = \bigcup_{k=1}^{\infty} V_k$. Utilizing the foregoing arguments, there exists $q_k \in L^{s/2}(V_k)$, $k \geq 1$ such that

$$\int_{u^{-1}(V_k)} DL(\nabla u) : \nabla \phi = \int_{u^{-1}(V_k)} q_k(u) \cof(\nabla u) : \nabla \phi,$$

for $\phi \in C^1_0(u^{-1}(V_k), \mathbb{R}^n)$. Since $u$ is locally volume-preserving homeomorphism, $\Omega = \bigcup_{k=1}^{\infty} u^{-1}(V_k)$ is an open covering of $\Omega$ and $u^{-1}(V_k) \subset \subset u^{-1}(V_{k+1})$. Using the identity $\text{div} \cof \nabla u(x) = 0$ and invertibility of $\nabla u(x)$, from (4.4) it follows that $q_k$ is unique up to a translation of a constant. Thus adding constant terms as necessary to each $q_k$, we deduce from (4.4) that for each fixed $k \geq 1$

$$q_i(z) = q_k(z) \quad \text{for } z \in V_i, \quad 1 \leq i \leq k.$$

We finally define $q : u(\Omega) \to \mathbb{R}$ as $q(z) := q_k(z)$, for $z \in V_k$, so that $q \in L^{s/2}_{\text{loc}}(u(\Omega))$. This proves that for any $\phi \in C^1_0(\Omega, \mathbb{R}^n)$, the pair $(u, q)$ satisfies

$$\int_{\Omega} DL(\nabla u(x)) : \nabla \phi(x) \, dx = \int_{\Omega} q(u(x)) \cof(\nabla u(x)) : \nabla \phi(x) \, dx.$$

Now let us define the pressure $p$ on $\Omega$ by

$$p(x) := q(u(x)) \quad \text{for } x \in \Omega.$$

Then for any $k \geq 1$,

$$\int_{u^{-1}(V_k)} |p(x)|^{s/2} = \int_{u^{-1}(V_k)} |q(u(x))|^{s/2} \, dx = \int_{V_k} |q(z)|^{s/2} \, dz < \infty,$$

and hence $p \in L^{s/2}_{\text{loc}}(\Omega)$ and the pair $(u, p)$ satisfies

$$\int_{\Omega} DL(\nabla u(x)) : \nabla \phi(x) \, dx = \int_{\Omega} p(x) \cof(\nabla u(x)) : \nabla \phi(x) \, dx,$$

for any $\phi \in C^1_0(\Omega, \mathbb{R}^n)$. In other words, $(u, p)$ satisfies the system of Euler-Lagrange equations

$$\text{div}[DL(\nabla u(x)) - p(x) \cof(\nabla u(x))] = 0, \quad \text{in } \Omega.$$

in the sense of (4.5). This completes the proof.
5. Partial Regularity of area-preserving minimizers

For \( n = 2 \), as a consequence of the Euler-Lagrange equations (1.7), together with the standard elliptic estimates [GM 79], we establish the following theorem.

**Theorem 5.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a smooth, bounded simply connected domain and let \( L : \mathbb{M}^{2 \times 2} \to \mathbb{R} \) be smooth, uniformly convex, such that \( DL \) has linear growth and \( D^2L \) is bounded. Let \( u \in W^{1,3}(\Omega, \mathbb{R}^2) \) be an area-preserving minimizer of the energy \( E[\cdot] \). Furthermore, assume that the associated hydrostatic pressure \( q \) on the deformed domain \( u(\Omega) \) is \( C^{0,\alpha} \) for some \( 0 < \alpha < 1 \). Then \( \nabla u \) are H"older continuous on a dense open set \( \Omega_0 \subset \Omega \).

**Proof.** Since \( u \in W^{1,3}(\Omega, \mathbb{R}^2) \) and \( u \) is area-preserving, \( u(\Omega) \) is open and \( u \) is a homeomorphism from \( \Omega \) to \( u(\Omega) \). By Theorem 4.1, there exists \( q \in L^{3/2}(u(\Omega)) \) and the pair \( (u, q \circ u) \) satisfies the system

\[
\sum_{j=1}^{2} \frac{\partial}{\partial x_j} \left( \frac{\partial L}{\partial p_i^j}(\nabla u) - p(x)(\text{cof} \nabla u)^i_j \right) = 0, \quad \text{in } \Omega, \quad i = 1, 2,
\]

where \( p := q \circ u \). Assume that \( q \in C^{0,\alpha}(u(\Omega)) \). Since \( u \in W^{1,3} \), Sobolev imbedding theorem yields \( u \in C^{1/3} \), and hence \( p(x) = q(u(x)) \) is H"older continuous with the exponent \( \alpha/3 \). Let \( F : \Omega \times \mathbb{M}^{2 \times 2} \to \mathbb{R} \) be the free-energy defined as

\[
F(x, P) := L(P) - p(x) \det P \quad x \in \Omega, \quad P \in \mathbb{M}^{2 \times 2},
\]

so that we can rewrite the nonlinear system (5.1) as

\[
\sum_{j=1}^{2} \frac{\partial}{\partial x_j} (A^i_j(x, \nabla u)) = 0, \quad \text{in } \Omega, \quad i = 1, 2,
\]

where

\[
A^i_j(x, P) := \frac{\partial F}{\partial p^i_j}(x, P) = \frac{\partial L}{\partial p^i_j}(P) - p(x)(\text{cof} P)^i_j.
\]

Let \( U \subset \subset \Omega \). Since \( |\text{cof} P| = |P| \) for any \( P \in \mathbb{M}^{2 \times 2} \), \( |DL(P)| \leq C(1 + |P|) \) and \( D^2L(P) \) is bounded,

\[
|A^i_j(x, P)| \leq C(1 + |P|), \quad \left| \frac{\partial A^i_j}{\partial p^k_l}(x, P) \right| \leq C,
\]
for any \( x \in U, \ P \in \mathbb{M}^{2 \times 2} \). By Hölder continuity of \( p \), it follows that
\[
\frac{|A_j^i(x, P) - A_j^i(y, P)|}{1 + |P|} = \frac{|p(x) - p(y)|}{1 + |P|} \leq C|x - y|^{\alpha/3},
\]
for any \( x \in U, \ P \in \mathbb{M}^{2 \times 2} \). By direct calculations and the ellipticity of \( L \) it follows that
\[
\frac{\partial A_i^j}{\partial p_k^l}(x, P)\xi_{ij}\xi_{kl} = \frac{\partial^2 F}{\partial p_j^i p_k^l}(x, P)\xi_{ij}\xi_{kl} = \frac{\partial^2 L}{\partial p_j^i p_k^l}(P)\xi_{ij}\xi_{kl} - 2p(x) \det \xi \geq \lambda_0 |\xi|^2 - 2p(x) \det \xi := I(x, \xi), \text{ for } P = (p_j^i), \ \xi = (\xi_{ij}) \in \mathbb{M}^{2 \times 2},
\]
where \( \lambda_0 > 0 \) is the ellipticity constant of \( L \). Completing squares, observe that
\[
I(x, \xi) = |\xi|^2 - 2p(x) \frac{\lambda_0}{\lambda_0} \det \xi \geq \lambda_0 |\xi|^2 - 2p(x) \det \xi \geq \lambda_0 |\xi|^2 - 2p(x) \det \xi := I(x, \xi), \text{ for } P = (p_j^i), \ \xi = (\xi_{ij}) \in \mathbb{M}^{2 \times 2},
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where \( \lambda_0 > 0 \) is the ellipticity constant of \( L \). Completing squares, observe that
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\]
where \( \lambda_0 > 0 \) is the ellipticity constant of \( L \). Completing squares, observe that
\[
I(x, \xi) = |\xi|^2 - 2p(x) \frac{\lambda_0}{\lambda_0} \det \xi \geq \lambda_0 |\xi|^2 - 2p(x) \det \xi \geq \lambda_0 |\xi|^2 - 2p(x) \det \xi := I(x, \xi), \text{ for } P = (p_j^i), \ \xi = (\xi_{ij}) \in \mathbb{M}^{2 \times 2},
\]
Thus from (5.6) and (5.8), it follows that the map \( P \mapsto A(\cdot, P) \) is strongly elliptic if there exists \( \mu_0 > 0 \) such that
\[
\frac{\partial L_i^j}{\partial p_i^k}(x, P)\xi_j\xi_k \geq \frac{\lambda_0}{2} \left( 1 - \frac{p^2}{\lambda_0^2} \right) |\xi|^2 \geq \mu_0 |\xi|^2, \quad \text{for } x \in \Omega, \ P, \xi \in M^{2 \times 2},
\]
which is equivalent to assume that
\[
(5.9) \quad p^2 \leq \lambda_0^2 - 2\lambda_0\mu_0 \iff (p - \mu_0)^2 \leq (\lambda_0 - \mu_0)^2.
\]
Since \( p \) is defined up to addition of arbitrary constant, thus the inequality (5.9) is satisfied in subdomain \( U \subset \subset \Omega \) if and only if
\[
(5.10) \quad \text{osc}_U p < \lambda_0.
\]
Since \( p \) is Hölder continuous, the estimate (5.10) holds for any subdomain \( U \subset \subset \Omega \) with sufficiently small diameter. Hence \( A(x, P) \) is strongly elliptic in \( P \) for each \( x \in U \subset \subset \Omega \), for sufficiently small diameter. This proves that \( A^i_j(x, P) \) satisfies all the conditions of Giaquinta-Modica in [GM 79] on \( U \subset \subset \Omega \), with diameter of \( U \) being small. Hence by [GM 79, Theorem 1], we conclude that \( \nabla u \) is Hölder continuous on a dense open subset \( U_0 \) of \( U \). By standard covering arguments we conclude the proof. \( \square \)

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