GLOBAL EXISTENCE THEOREM FOR A MODEL GOVERNING
THE MOTION OF TWO CELL POPULATIONS

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Abstract. This article is concerned with the existence of a weak solution to
the initial boundary problem for a cross-diffusion system which arises in the
study of two cell population growth. The mathematical challenge is due to the
fact that the coefficient matrix is non-symmetric and degenerate in the sense
that its determinant is 0. The existence assertion is established by exploring
the fact that the total population density satisfies a porous media equation.

1. Introduction. Let Ω be a bounded domain in \( \mathbb{R}^d \) with Lipschitz boundary \( \partial \Omega \)
and \( T \) any positive number. We consider the initial boundary value problem
\[
\frac{\partial u_1}{\partial t} - \mu \text{div} (u_1 \nabla w) = R_1 \quad \text{in } \Omega_T \equiv \Omega \times (0, T),
\]
\[
\frac{\partial u_2}{\partial t} - \nu \text{div} (u_2 \nabla w) = R_2 \quad \text{in } \Omega_T,
\]
\[
u_1 \nabla w \cdot n = 0 \quad \text{on } \Sigma_T \equiv \partial \Omega \times (0, T),
\]
\[
u_2 \nabla w \cdot n = 0 \quad \text{on } \Sigma_T,
\]
\[
(u_1(x, 0), u_2(x, 0)) = (u_1^{(0)}(x), u_2^{(0)}(x)) \quad \text{on } \Omega,
\]
where \( n \) is the unit outward normal to \( \partial \Omega \),
\[
w = u_1 + u_2,
\]
\[
R_1 = u_1 F_1(w) + u_2 G_1(w), \quad \text{and}
\]
\[
R_2 = u_1 F_2(w) + u_2 G_2(w).
\]

Assume:
(H1) \( F_i, G_i, i = 1, 2, \) are all continuous functions with the properties
\[
F(w) \equiv F_1(w) + F_2(w) \leq 0 \quad \text{on } [w_p, \infty),
\]
\[
G(w) \equiv G_1(w) + G_2(w) \leq 0 \quad \text{on } [w_p, \infty), \quad \text{and}
\]
\[
E(w) \equiv \min\{F_2(w), G_1(w)\} \geq 0 \quad \text{on } [0, w_p) \text{ for some } w_p > 0;
\]
(H2) \( \mu, \nu \in (0, \infty), \gamma > 1; \)
(H3) \( u_1^{(0)}(x) \geq 0, \quad u_2^{(0)}(x) \geq 0, \) and
\[
w(x, 0) \leq w_p \quad \text{on } \Omega.
\]

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This model was first introduced in [12] to describe interacting populations that disperse to avoid crowding in the biological context. It has also been used to model the interaction between two cell populations which respond primarily to the pressure in terms of their motions but undergo different growth/death rates in the study of tissue growth (see [13] and the references therein). In this case the function $w^\gamma$ represents the pressure. The second terms on the left hand sides of the two equations (1) and (2) model the tendency of cells to move down pressure gradients and rely on the definition of the cell velocity fields through Darcy’s law [4]. The parameters $\mu, \nu$ stand for the mobilities (i.e. the quotient of permeability and viscosity) of the two cells, respectively. If $\mu \neq \nu$, then the two cell populations are characterized by different mobilities. Assumptions (9) and (10) mean that competition for space decreases the cell division rate according to the local pressure. The parameter $(w_p)^\gamma$ models the threshold pressure above which dividing cells are entering a quiescent state (i.e. the so-called homeostatic pressure) [5]. As we shall see, (9) and (10) imply that the total density $w$ cannot exceed $w_p$, while the non-negativity of $u_1, u_2$ follows from (11).

The objective of this paper is to investigate an approximation to the initial boundary value problem when $\mu$ and $\nu$ may be of different values and existence when $\mu = \nu$.

**Definition 1.1.** We say that $(u_1, u_2)$ is a weak solution to (1)-(5) if:

(D1) $u_1, u_2$ are non-negative and bounded with

$$\partial_t u_1, \quad \partial_t u_2 \in L^2(0, T; (W^{1,2}(\Omega))^*)$$

$$w^\gamma \in L^2(0, T; W^{1,2}(\Omega)),$$

where $w$ is given as in (6) and $(W^{1,2}(\Omega))^*$ denotes the dual space of $W^{1,2}(\Omega)$;

(D2) there hold

$$- \int_{\Omega_T} u_1 \partial_t \varphi dx dt + \int_{\Omega_T} u_1 \nabla w^\gamma \cdot \nabla \varphi dx dt$$

$$= \int_{\Omega_T} R_1 \varphi dx dt - \langle u_1(\cdot, T), \varphi(\cdot, T) \rangle + \int_{\Omega} u_1(0)(x) \varphi(x, 0) dx$$

for each $\varphi \in H^1(0, T; W^{1,2}(\Omega))$ and

$$- \int_{\Omega_T} u_2 \partial_t \psi dx dt + \int_{\Omega_T} u_2 \nabla w^\gamma \cdot \nabla \psi dx dt$$

$$= \int_{\Omega_T} R_2 \psi dx dt - \langle u_2(\cdot, T), \psi(\cdot, T) \rangle + \int_{\Omega} u_2(0)(x) \psi(x, 0) dx$$

for each $\psi \in H^1(0, T; W^{1,2}(\Omega))$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{1,2}(\Omega)$ and $(W^{1,2}(\Omega))^*$ and

$$H^1(0, T; W^{1,2}(\Omega)) = \{ v \in L^2(0, T; W^{1,2}(\Omega)) : \partial_t v \in L^2(0, T; W^{1,2}(\Omega)) \}.$$

To see that the two equations in (D2) make sense, we can conclude from (D1) that $u_1, u_2 \in C([0, T]; (W^{1,2}(\Omega))^*)$. Since $w$ is bounded and $\gamma \geq \frac{\gamma+1}{2}$, we also have $w^\gamma \in L^2(0, T; W^{1,2}(\Omega)).$

**Theorem 1.2.** Let (H1)-(H3) be satisfied. Assume:

(H4) $u_1^{(0)}, u_2^{(0)} \in W^{1,2}(\Omega)$;

(H5) $\mu = \nu$.

Then there is a weak solution to (1)-(5).
In general, the rigorous mathematical analysis of nonlinear differential equations depends primarily upon deriving estimates, but typically also upon using these estimates to justify limiting procedures of various sorts. The two issues are closely related. Our system here is a cross-diffusion one, and the mathematical analysis of systems of this type has attracted a lot of attention recently. One approach (see, e.g., [7, 8]) is to seek a possibly convex function \( \psi \) on \( \mathbb{R}^2 \) so that \( t \to \int_{\Omega} \psi(u_1(x,t),u_2(x,t))dx \) is a Lyapunov functional [6] along the solutions to (1)-(4). Unfortunately, this so-called entropy method by itself is not enough for an existence assertion here. To see this, we formally calculate

\[
\frac{d}{dt} \int_{\Omega} \psi(u_1,u_2)dx = \int_{\Omega} (\psi_{u_1} \partial_t u_1 + \psi_{u_2} \partial_t u_2) dx
\]

\[
= -\mu \gamma \int_{\Omega} u_1 w^{\gamma-1} \nabla w \cdot (\psi_{u_1} \nabla u_1 + \psi_{u_2} \nabla u_2) dx
\]

\[
-\nu \gamma \int_{\Omega} u_2 w^{\gamma-1} \nabla w \cdot (\psi_{u_1} \nabla u_1 + \psi_{u_2} \nabla u_2) dx
\]

\[
+ \int_{\Omega} (R_1 \psi_{u_1} + R_2 \psi_{u_2}) dx
\]

\[
= -\gamma \int_{\Omega} w^{\gamma-1} (a|\nabla u_1|^2 + (a + b) \nabla u_1 \cdot \nabla u_2 + b|\nabla u_2|^2) dx
\]

\[
+ \int_{\Omega} (R_1 \psi_{u_1} + R_2 \psi_{u_2}) dx, \tag{13}
\]

where

\[
a = \mu u_1 \psi_{u_1} + \nu u_2 \psi_{u_2},
\]

\[
b = \mu u_1 \psi_{u_1} + \nu u_2 \psi_{u_2}.
\]

**Proposition 1.3.** *The quadratic term* \( A |\xi|^2 + B \xi \cdot \eta + C |\eta|^2 \geq 0 \) *for all* \( \xi, \eta \in \mathbb{R}^d \) *if and only if*

\[
A \geq 0, \quad C \geq 0, \quad \text{and} \quad B^2 \leq 4AC. \tag{14}
\]

**Proof.** If \( A |\xi|^2 + B \xi \cdot \eta + C |\eta|^2 \geq 0 \) *for all* \( \xi, \eta \in \mathbb{R}^d \) *then we must have*

\[
A \geq 0 \quad \text{and} \quad C \geq 0.
\]

If \( A = 0 \), then we must have \( B = 0 \). Suppose this is not the case. Then take \( \xi = -\frac{2C}{B} \eta \). This immediately yields \( C = 0 \). If both \( A \) and \( C \) are 0, then \( B \) must be 0, a contradiction.

If \( A > 0 \), then

\[
A |\xi|^2 + B \xi \cdot \eta + C |\eta|^2 = \left( \sqrt{A} \xi + \frac{B}{2\sqrt{A}} \eta \right)^2 + \left( C - \frac{B^2}{4A} \right) |\eta|^2 \geq 0. \tag{15}
\]

Taking any \( \eta \) with \( |\eta| = 1 \) and \( \xi = -\frac{B}{2\sqrt{A}} \eta \), we obtain (14). The converse is an easy consequence of (15). \( \square \)

Thus to ensure

\[
a|\nabla u_1|^2 + (a + b) \nabla u_1 \cdot \nabla u_2 + b|\nabla u_2|^2 \geq 0,
\]

we must choose \( \psi \) so that

\[
a \geq 0 \quad \text{and} \quad a = b.
\]
Then (13) reduces to
\[
\frac{d}{dt} \int_\Omega \psi(u_1, u_2) dx + \gamma \int_\Omega w^{\gamma-1} a |\nabla w|^2 dx = \int_\Omega (R_1 \psi u_1 + R_2 \psi u_2) dx,
\]
which can only give us an estimate on the gradient of the sum of $u_1$ and $u_2$. This is not very surprising because our system is degenerate in the sense that the coefficient matrix
\[
\gamma w^{\gamma-1} \begin{pmatrix} \mu u_1 & \mu u_1 \\ \nu u_2 & \nu u_2 \end{pmatrix}
\]
has determinant 0. As we shall see, (16) is an important equation to us. But this alone is not enough for an existence assertion. To gain more information, we are forced to assume $\mu = \nu$. Under this assumption, the total density $w$ satisfies a porous media equation, and we wish to take advantage of this fact. To be more specific, we employ a so-called weak convergence method[10]. That is, construct an approximation and then pass to the limit. The central issue is how to take the limit in the product
\[
u_i^\varepsilon \nabla \left( w^\varepsilon \right)^\gamma, \quad i = 1, 2,
\]
in our approximate problems. We are basically faced with two choices: One can prove either the precompactness of the two population densities or the same for the pressure gradients. The former was done in [6] in the one-dimensional case, while the latter was considered in [13]. To prove the precompactness of $\{w^\varepsilon\}^{\gamma}$ in $L^2(0,T;W^{1,2}(\mathbb{R}^d)), \ d > 1$, the authors of [13] developed an extension of the Aronson-Benilan regularizing effect for porous media equations which provided estimates for the Laplacian of the pressure term $w^{\gamma}$. In our case we obtain the precompactness of $\{w^\varepsilon\}^{\gamma+1}$ in $L^2(0,T;W^{1,2}(\Omega))$ and show that this is enough to justify passing to the limit. Our proof seems to be more direct and also simpler and requires weaker assumptions. For example, we do not impose any assumptions on the second order partial derives of the initial data as done in [13]. Moreover, condition (7) in [13], which imposes restrictions on space dimensions and the growth of $|F(w) - G(w)|$ near 0, has also been removed.

If the initial data $u^{(1)}_0(x), u^{(2)}_0(x)$ have disjoint supports, a result in [6] indicates that they can remain disjoint for all $t > 0$ at least in the case $d = 1$. That is to say, the two cell populations are segregated. Our assumption (H3) does not exclude this possibility here. We also refer the reader to [16] and the references therein for numerical results that deal with how the mobilities change the morphology of the interfaces between the two cell populations and analytical study of traveling wave solutions with composite shapes and discontinuities for cell-density models of avascular tumor growth. The limit of $(u_1, u_2)$ as $\gamma \to \infty$ is very interesting from the modeling perspective ans has been investigated in [3].

This paper is organized as follows. In Section 2 we fabricate an approximation scheme for (1)-(5) and prove an existence assertion for the approximate problems. Here we allow the possibility that $\mu \neq \nu$. In Section 3 we prove Theorem 1.2. Then we give a brief indication on how to extend Theorem 1.2 to the case $\Omega = \mathbb{R}^d$. Since all these are done under the assumption $\mu = \nu$, the case where $\mu \neq \nu$ remains open.

2. The approximate problems. In this section we design an approximation scheme for (1)-(5) from a totally different perspective than the one in [1, 6, 13], and then prove the existence of a solution to the approximate problems. Before we begin, we will need the following two classical lemmas.
Then the function \( t \rightarrow \int_{\Omega} h(f(x,t))dx \) is absolutely continuous on \([0,T]\) and
\[
\frac{d}{dt} \int_{\Omega} h(f)dx = \langle \partial_t f, h'(f) \rangle.
\] (17)

If \( h(s) = s^2 \), this lemma is a special case of the well known Lions-Magenes lemma ([18], p.176–177). Formula (17) is trivial if \( f \) is smooth. The general case can be established by suitable approximation. We shall omit the details.

**Lemma 2.2** (Lions-Aubin). Let \( X_0, X \) and \( X_1 \) be three Banach spaces with \( X_0 \subseteq X \subseteq X_1 \). Suppose that \( X_0 \) is compactly embedded in \( X \) and that \( X \) is continuously embedded in \( X_1 \). For \( 1 \leq p, q \leq \infty \), let
\[
W = \{ u \in L^p([0,T]; X_0) : \partial_t u \in L^q([0,T]; X_1) \}.
\]
Then:
(i) If \( p < \infty \), then the embedding of \( W \) into \( L^p([0,T]; X) \) is compact.
(iii) If \( p = \infty \) and \( q > 1 \), then the embedding of \( W \) into \( C([0,T]; X) \) is compact.

The proof of this lemma can be found in [17]. We mention in passing that Lemmas 2.1 and 2.2 imply that \( W(0,T) \) is contained in \( C([0,T]; L^2(\Omega)) \).

Our approximate problem is:
\[
\begin{align*}
\partial_t w - \gamma \text{div} \left( (\mu u_1 + \nu u_2) w^{\gamma - 1} \nabla w \right) - \varepsilon \Delta w &= \frac{R_1 + R_2}{R} \text{ in } \Omega_T, \quad (18) \\
\partial_t u_1 - \gamma \text{div} (u_1 w^{\gamma - 1} \nabla w) - \varepsilon \Delta u_1 &= R_1 \text{ in } \Omega_T, \quad (19) \\
\partial_t u_2 - \gamma \text{div} (u_2 w^{\gamma - 1} \nabla w) - \varepsilon \Delta u_2 &= R_2 \text{ in } \Omega_T, \quad (20) \\
\nabla w \cdot n - \nabla u_1 \cdot n &= \nabla u_2 \cdot n = 0 \text{ on } \Sigma_T, \quad (21) \\
(w, u_1, u_2)|_{t=0} &= (w^{(0)}(x), u_1^{(0)}(x), u_2^{(0)}(x)) \text{ on } \Omega, \quad (22)
\end{align*}
\]
where \( \varepsilon > 0 \) and
\[
w^{(0)}(x) = u_1^{(0)}(x) + u_2^{(0)}(x).
\]

**Theorem 2.3.** Let \((H1)-(H3)\) be satisfied. Then there exists a triplet \((w, u_1, u_2)\) in the function space \(W(0,T))^3\) such that
(1) \( u_1 \geq 0, u_2 \geq 0, \) and \( w = u_1 + u_2 \) with \( w \leq w_p; \)
(2) Equations (18)-(22) are all satisfied in the usual weak sense.

This theorem will be established via the Leray-Schauder fixed point theorem ([11], p.280). For this purpose, we introduce the function
\[
\theta_p(s) = \begin{cases} 
0 & \text{if } s \leq 0, \\
s & \text{if } 0 < s < w_p, \\
w_p & \text{if } s \geq w_p,
\end{cases} \quad (23)
\]
where \( w_p \) is given as in \((H1)\). We define an operator \( M \) from \((L^2(\Omega_T))^3\) into itself as follows: Let \((w_1, v_1, v_2) \in (L^2(\Omega_T))^3\). We first consider the initial boundary value problem
\[
\partial_t w = \gamma \text{div} \left[ (\mu \theta_p(v_1) + \nu \theta_p(v_2)) (\theta_p(v_1) + \theta_p(v_2))^{\gamma - 1} \nabla w \right] + \varepsilon \Delta w
\]
Form the following two initial boundary problems.

We can conclude from the classical result ([15], Chap. III) that there is a unique weak solution \( w \) to (24)-(26) in the space \( W(0, T) \). Use the function \( w \) so obtained to form the following two initial boundary problems.

\[
\partial_t u_1 - \varepsilon \Delta u_1 = \gamma \mu \operatorname{div} \left[ \theta_p(v_1)(\theta_p(v_1) + \theta_p(v_2))^{\gamma - 1} \nabla w \right] \\
\varepsilon \nabla u_1 \cdot \mathbf{n} = 0 \quad \text{on } \Sigma_T, \\
u_1(x, 0) = u^{(0)}_1(x) \quad \text{on } \Omega.
\]

Each of the two problems here has a unique solution in \( W(0, T) \). We define \((w, u_1, u_2) = M(w_1, v_1, v_2)\). Evidently, \( M \) is well-defined.

**Lemma 2.4.** For each fixed \( \varepsilon > 0 \), the operator \( M \) is compact, i.e., \( M \) is continuous and maps bounded sets into precompact ones.

**Proof.** Use \( w \) as a test function in (24) to obtain

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega w^2 dx + \varepsilon \int_\Omega |\nabla w|^2 dx
+ \gamma \int_\Omega (\mu \theta_p(v_1) + \nu \theta_p(v_2))(\theta_p(v_1) + \theta_p(v_2))^{\gamma - 1} |\nabla w|^2 dx
\leq 2pM_0 \int_\Omega w dx \leq \int_\Omega w^2 dx + 4p^2 M_0^2,
\]

where

\[
M_0 = \max \left\{ \max_{w \in [0, w_p]} F(w), \max_{w \in [0, w_p]} F_2(w), \max_{w \in [0, w_p]} G(w), \max_{w \in [0, w_p]} G_1(w) \right\}.
\]

Use Gronwall’s inequality in (33) to obtain

\[
\sup_{0 \leq t \leq T} \int_\Omega w^2 dx + \int_\Omega |\nabla w|^2 dx dt \leq c.
\]

This together with (24) implies that \( w \) is bounded in \( W(0, T) \) for all \((w_1, v_1, v_2) \in \left( L^2(\Omega_T) \right)^3 \). The same is true of \( u_1, u_2 \), from whence follows that the range of \( M \) is a bounded set in \( W(0, T) \). Thus we can conclude from Lemma 2.2 that \( M \) maps bounded sets into precompact ones. The continuity of \( M \) is based upon the observation that if any subsequence of a sequence has a further convergent subsequence and all its convergent subsequences have the same limit then the whole sequence converges. Suppose

\[
(w_1^{(k)}, v_1^{(k)}, v_2^{(k)}) \to (w_1, v_1, v_2, ) \quad \text{strongly in } \left( L^2(\Omega_T) \right)^3.
\]
Then for any bounded continuous function $H$ on $\mathbb{R}$ we have
\[
(H(w_1^{(k)}), H(v_1^{(k)}), H(v_2^{(k)})) \to (H(w_1), H(v_1), H(v_2))
\]
strongly in $(L^p(\Omega_T))^3$ for each $p \geq 1$. (37)

To see this, we can conclude from (36) that there is a subsequence $\{v_1^{(k)}_i\}$ of $\{v_1^{(k)}\}$ such that
\[
v_1^{(k)}_i \to v_1 \text{ a.e. on } \Omega_T.
\]

Subsequently,
\[
H(v_1^{(k)}_i) \to H(v_1) \text{ a.e. on } \Omega_T.
\]

This combined with Egoroff’s theorem implies that
\[
H(v_1^{(k)}) \to H(v_1) \text{ strongly in } (L^p(\Omega_T))^3 \text{ for each } p \geq 1.
\]

That is, any subsequence of $\{H(v_1^{(k)})\}$ has a further subsequence which converges to $H(v_1)$. Thus the whole sequence converges to $H(v_1)$. This gives (37).

We have
\[
\theta_p(v_1^{(k)}) \to \theta_p(v_i),
\]
\[
F_i(\theta_p(w_1^{(k)})) \to F_i(\theta_p(w_1)),
\]
\[
G_i(\theta_p(w_1^{(k)})) \to G_i(\theta_p(w_1))
\]
strongly in $(L^p(\Omega_T))^3$ for each $p \geq 1$ and $i = 1, 2$.

Set
\[
(w^{(k)}, u_1^{(k)}, u_2^{(k)}) = \mathcal{M}(w_1^{(k)}, v_1^{(k)}, v_2^{(k)}).
\]

That is,

\[
\partial_t w^{(k)} - \varepsilon \Delta w^{(k)} = \gamma \text{div} \left[ \mu \theta_p(v_1^{(k)}) + \nu \theta_p(v_2^{(k)}) \right] \left( \theta_p(v_1^{(k)}) + \theta_p(v_2^{(k)}) \right)^{\gamma-1} \nabla w^{(k)}
\]
\[
+ \theta_p(v_1^{(k)}) F_i(\theta_p(w_1^{(k)})) + \theta_p(v_2^{(k)}) G_i(\theta_p(w_1^{(k)})) \text{ in } \Omega_T, \tag{38}
\]
\[
\nabla w^{(k)} \cdot n = 0 \text{ on } \Sigma_T, \tag{39}
\]
\[
w^{(k)}(x, 0) = w^{(0)}(x) \text{ on } \Omega, \tag{40}
\]
\[
\partial_t u_1^{(k)} - \varepsilon \Delta u_1^{(k)} = \gamma \mu \text{div} \left[ \theta_p(v_1^{(k)}) \left( \theta_p(v_1^{(k)}) + \theta_p(v_2^{(k)}) \right)^{\gamma-1} \nabla w^{(k)} \right]
\]
\[
+ \theta_p(v_1^{(k)}) F_i(\theta_p(w_1^{(k)})) + \theta_p(v_2^{(k)}) G_i(\theta_p(w_1^{(k)})) \text{ in } \Omega_T, \tag{41}
\]
\[
\nabla u_1^{(k)} \cdot n = 0 \text{ on } \Sigma_T, \tag{42}
\]
\[
u_1^{(k)}(x, 0) = u_1^{(0)}(x) \text{ on } \Omega, \tag{43}
\]
\[
\partial_t u_2^{(k)} - \varepsilon \Delta u_2^{(k)} = \gamma \text{div} \left[ \theta_p(v_2^{(k)}) \left( \theta_p(v_1^{(k)}) + \theta_p(v_2^{(k)}) \right)^{\gamma-1} \nabla w^{(k)} \right]
\]
\[
+ \theta_p(v_1^{(k)}) F_i(\theta_p(w_1^{(k)})) + \theta_p(v_2^{(k)}) G_i(\theta_p(w_1^{(k)})) \text{ in } \Omega_T, \tag{44}
\]
\[
\nabla u_2^{(k)} \cdot n = 0 \text{ on } \Sigma_T, \tag{45}
\]
\[
u_2^{(k)}(x, 0) = u_2^{(0)}(x) \text{ on } \Omega. \tag{46}
\]

We can easily conclude from the proof of (35) that $\{w^{(k)}\}$ is bounded in $W(0, T)$. This together with (41) and (44) implies that $\{u_1^{(k)}\}$ and $\{u_2^{(k)}\}$ are also bounded in
Thus we can pass to the limit in (38)-(46) to derive Lemma 2.5.

Furthermore,\[ \nabla u_1^{(k)} \to \nabla u_1 \text{ weakly in } L^2(0, T, (L^2(\Omega))^d), \]
\[ \nabla u_2^{(k)} \to \nabla u_2 \text{ weakly in } L^2(0, T, (L^2(\Omega))^d), \text{ and} \]
\[ \nabla w^{(k)} \to \nabla w \text{ weakly in } L^2(0, T, (L^2(\Omega))^d). \]

Thus we can pass to the limit in (38)-(46) to derive
\[
\begin{align*}
\partial_t w - \varepsilon \Delta w &= \gamma \text{div} \left[ (\mu \theta_p(v_1) + \nu \theta_p(v_2))(\theta_p(v_1) + \theta_p(v_2))^{\gamma-1} \nabla w \right] \\
&\quad + \theta_p(v_1) F(\theta_p(w_1)) + \theta_p(v_2) G(\theta_p(w_1)) \text{ in } \Omega, \\
\nabla w \cdot n &= 0 \text{ on } \Sigma_T, \\
\n w(x, 0) &= w_0(x) \text{ on } \Omega, \\
\partial_t u_1 - \varepsilon \Delta u_1 &= \gamma \mu \text{div} \left[ \theta_p(v_1)(\theta_p(v_1) + \theta_p(v_2))^{\gamma-1} \nabla w \right] \\
&\quad + \theta_p(v_1) F_1(\theta_p(w_1)) + \theta_p(v_2) G_1(\theta_p(w_1)) \text{ in } \Omega, \\
\nabla u_1 \cdot n &= 0 \text{ on } \Sigma_T, \\
\n u_1(x, 0) &= u_1^0(x) \text{ on } \Omega, \\
\partial_t u_2 - \varepsilon \Delta u_2 &= \gamma \nu \text{div} \left[ \theta_p(v_2)(\theta_p(v_1) + \theta_p(v_2))^{\gamma-1} \nabla w \right] \\
&\quad + \theta_p(v_1) F_2(\theta_p(w_1)) + \theta_p(v_2) G_2(\theta_p(w_1)) \text{ in } \Omega, \\
\nabla u_2 \cdot n &= 0 \text{ on } \Sigma_T, \\
\n u_2(x, 0) &= u_2^0(x) \text{ on } \Omega.
\end{align*}
\]

The solution $w$ to (47)-(49) is unique, and $u_1$ and $u_2$ are uniquely determined by $w$. That is, there is only one solution to (47)-(55). This means that any subsequence of $(w^{(k)}, u^{(k)}_1, u^{(k)}_2)$ has the same limit $M(w_1, v_1, v_2)$. Thus the whole sequence also converges to it. This completes the proof. \hfill \Box

**Lemma 2.5.** There is a positive number $c$ such that
\[
\| (w, u_1, u_2) \|_{(L^2(\Omega))^3} \leq c
\]
for all $(w, u_1, u_2) \in (L^2(\Omega))^3$ and $\sigma \in (0, 1)$ satisfying $(w, u_1, u_2) = \sigma M(w_1, v_1, v_2)$.

**Proof.** It is easy to see that the above equation is equivalent to the following problem
\[
\begin{align*}
\partial_t w &= \gamma \text{div} \left[ (\mu \theta_p(u_1) + \nu \theta_p(u_2))(\theta_p(u_1) + \theta_p(u_2))^{\gamma-1} \nabla w \right] + \varepsilon \Delta w \\
&\quad + \sigma \theta_p(u_1) F(\theta_p(w)) + \sigma \theta_p(u_2) G(\theta_p(w)) \text{ in } \Omega, \\
\nabla w \cdot n &= 0 \text{ on } \Sigma_T, \\
w(x, 0) &= \sigma w_0(x) \text{ on } \Omega, \\
\partial_t u_1 &= \gamma \mu \text{div} \left[ \theta_p(u_1)(\theta_p(u_1) + \theta_p(u_2))^{\gamma-1} \nabla w \right] + \varepsilon \Delta u_1 \\
&\quad + \sigma \theta_p(u_1) F_1(\theta_p(w)) + \sigma \theta_p(u_2) G_1(\theta_p(w)) \text{ in } \Omega, \\
\nabla u_1 \cdot n &= 0 \text{ on } \Sigma_T, \\
\end{align*}
\]
\[ u_1(x,0) = \sigma u_1^0(x) \text{ on } \Omega, \]
\[ \partial_t u_2 = \gamma \nu \text{div} \left[ \theta_p(u_2)(\theta_p(u_1) + \theta_p(u_2))^{-1} \nabla w \right] + \varepsilon \Delta u_2 + \sigma \theta_p(u_1) F_2(\theta_p(w)) + \theta_p(u_2) G_2(\theta_p(w)) \text{ in } \Omega_T, \]
\[ \nabla u_2 \cdot n = 0 \text{ on } \Sigma_T, \]
\[ u_2(x,0) = \sigma u_2^0(x) \text{ on } \Omega. \]

Add (59) to (58) and subtract the resulting equation from (57) to derive
\[ \partial_t (w - (u_1 + u_2)) - \varepsilon \Delta (w - (u_1 + u_2)) = 0 \text{ in } \Omega_T. \]

Recall the initial boundary conditions for \((w - (u_1 + u_2))\) to deduce
\[ w = u_1 + u_2. \]

Use \((w - w_p)^+\) as a test function in (58) to obtain
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega [(w - w_p)^+]^2 dx \]
\[ + \gamma \int_\Omega [(\mu \theta_p(u_1) + \nu \theta_p(u_2))(\theta_p(u_1) + \theta_p(u_2))^{-1} \nabla (w - w_p)^+]^2 dx \]
\[ = \int_\Omega (\sigma \theta_p(u_1) F(\theta_p(w)) + \sigma \theta_p(u_2) G(\theta_p(w)))(w - w_p)^+ dx \leq 0. \]

The last step is due to the definition of \(\theta_p\) (23) and (H1). Integrate with respect to \(t\) to yield
\[ w \leq w_p. \]

Note that
\[ \theta_p(u_1)u_1^- = 0 \text{ and } \theta_p(u_2)G_1(w) \geq 0 \text{ because } w \leq w_p \text{ and } (11) \text{ holds.} \]

With this in mind, we use \(u_1^-\) as a test function in (58) to derive
\[ -\frac{1}{2} \frac{d}{dt} \int_\Omega (u_1^-)^2 dx - \varepsilon \int_\Omega |\nabla u_1^-|^2 dx = \sigma \int_\Omega \theta_p(u_2) G_1(w) u_1^- dx \geq 0. \]

Consequently,
\[ u_1 \geq 0. \]

By the same token,
\[ u_2 \geq 0. \]

In view of (23), (62), and (63), we have
\[ \theta_p(w) = w, \quad \theta_p(u_1) = u_1, \quad \theta_p(u_2) = u_2. \]

By the proof of (35), we have
\[ \sup_{0 \leq t \leq T} \int_\Omega w^2 dx + \int_{\Omega_T} |\nabla w|^2 dx dt \leq c. \]

This together with (57) implies that \(w\) is bounded in \(W(0,T)\). Similarly, we can show that \(u_1, u_2\) are bounded in \(W(0,T) \subset L^2(\Omega_T)\). This completes the proof. \(\square\)

Theorem 2.3 is a consequence of the preceding two lemmas and the Leray-Schauder fixed point theorem.
3. **Proof of Theorem 1.2.** The proof of Theorem 1.2 is divided into several lemmas.

For each $1 \geq \varepsilon > 0$ we denote by $(w(\varepsilon), u^1(\varepsilon), u^2(\varepsilon))$ the solution to (18)-(22) constructed earlier. Subsequently, we have

$$u^1(\varepsilon) \geq 0, \quad u^2(\varepsilon) \geq 0, \quad w(\varepsilon) = u^1(\varepsilon) + u^2(\varepsilon).$$

Moreover,

$$w(\varepsilon) \leq w_p.$$

Set

$$R^{(\varepsilon)} = u^1(\varepsilon) F(w(\varepsilon)) + u^2(\varepsilon) G(w(\varepsilon)),$$

$$R_1^{(\varepsilon)} = u^1(\varepsilon) F_1(w(\varepsilon)) + u^2(\varepsilon) G_1(w(\varepsilon)),$$

$$R_2^{(\varepsilon)} = u^1(\varepsilon) F_2(w(\varepsilon)) + u^2(\varepsilon) G_2(w(\varepsilon)).$$

It follows that

$$0 \leq R^{(\varepsilon)}, \quad R_1^{(\varepsilon)}, \quad R_2^{(\varepsilon)} \leq w_p M_0,$$

where $M_0$ is given as in (34). We can write (18)-(22) in the form

$$\partial_t w(\varepsilon) - \text{div} \left( \left( \mu u^1(\varepsilon) + \nu u^2(\varepsilon) \right) \nabla \left( w(\varepsilon) \right) \right) - \varepsilon \Delta w(\varepsilon) = R^{(\varepsilon)} \text{ in } \Omega_T,$$

$$\partial_t u^1(\varepsilon) - \mu \text{div} \left[ u^1(\varepsilon) \nabla \left( w(\varepsilon) \right) \right] - \varepsilon \Delta u^1(\varepsilon) = R_1^{(\varepsilon)} \text{ in } \Omega_T,$$

$$\partial_t u^2(\varepsilon) - \nu \text{div} \left[ u^2(\varepsilon) \nabla \left( w(\varepsilon) \right) \right] - \varepsilon \Delta u^2(\varepsilon) = R_2^{(\varepsilon)} \text{ in } \Omega_T,$$

$$\nabla w(\varepsilon) \cdot n = \nabla u^1(\varepsilon) \cdot n = 0 \text{ on } \Sigma_T,$$

$$(w^0(x, 0), u^1_1(x, 0), u^2_2(x, 0)) = (w^0(0), u^1_1(0), u^2_2(0)) \text{ on } \Omega.$$

**Lemma 3.1.** We have

$$\int_{\Omega_T} \left| \nabla \left( w(\varepsilon) \right) \right|^{\frac{p+1}{2}} dx dt + \varepsilon \int_{\Omega_T} \left( \left| \nabla u^1(\varepsilon) \right|^2 + \left| \nabla u^2(\varepsilon) \right|^2 \right) dx dt \leq c.$$

**Proof.** Pick $\tau > 0$. Use $\frac{1}{\mu} \ln(u^1(\varepsilon) + \tau)$ as a test function in (67) to derive

$$\frac{1}{\mu} \int \frac{d}{dt} \left( (u^1(\varepsilon) + \tau) \ln(u^1(\varepsilon) + \tau) - u^1(\varepsilon) \right) dx + \int \frac{u^1(\varepsilon)}{u^1(\varepsilon) + \tau} \nabla \left( w(\varepsilon) \right) \nabla u^1(\varepsilon) dx$$

$$+ \frac{\varepsilon}{\mu} \int \frac{1}{u^1(\varepsilon) + \tau} \left| \nabla u^1(\varepsilon) \right|^2$$

$$= \frac{1}{\mu} \int \left( u^1(\varepsilon) F_1(w(\varepsilon)) + u^2(\varepsilon) G_1(w(\varepsilon)) \right) \ln(u^1(\varepsilon) + \tau) dx$$

$$\leq \frac{1}{\mu} \int \left( u^1(\varepsilon) F_1(w(\varepsilon)) + u^2(\varepsilon) G_1(w(\varepsilon)) \right) \ln(u^1(\varepsilon) + \tau) dx$$

$$\leq M_0 \int \left( u^1(\varepsilon) F_1(w(\varepsilon)) + u^2(\varepsilon) G_1(w(\varepsilon)) \right) \ln(u^1(\varepsilon) + \tau) dx.$$

Integrate, note that $\sup_{\Omega_T} |u^1(\varepsilon)| \cdot \ln(u^1(\varepsilon)) \leq c$, and take $\tau \to 0$ to get

$$\int_{\Omega_T} \nabla \left( w(\varepsilon) \right) \cdot \nabla u^1(\varepsilon) dx dt + \frac{4\varepsilon}{\mu} \int \left| \nabla u^1(\varepsilon) \right|^2 dx dt \leq c.$$
Similarly,

\[ \int_{\Omega_T} \nabla \left( w^{(c)} \right)^{\gamma} \cdot \nabla u^{(c)}_2 \, dx dt + \frac{4\varepsilon}{\nu} \int_{\Omega_T} \left| \nabla \sqrt{u^{(c)}_2} \right|^2 \, dx dt \leq c. \]

Add up the two preceding inequalities to obtain the desired result. \( \square \)

**Lemma 3.2.** The sequence \{w^{(c)}\} is precompact in \( L^p(\Omega_T) \) for each \( p \geq 1 \).

**Proof.** Without loss of generality, we may assume

\[ w^{(c)} \geq \varepsilon. \]  

(71)

This can be achieved easily by replacing \( u^{(0)}_1 \) with \( u^{(0)}_1 + \varepsilon \) in our approximate problems. From here on, we assume that we have already done so. Use \( (\varepsilon - w^{(c)})^+ \) as a test function in (66) to get

\[ -\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left[ (\varepsilon - w^{(c)})^+ \right]^2 \, dx \]

\[ -\gamma \int_{\Omega} (\mu w^{(c)}_1 + \nu w^{(c)}_2) \left( w^{(c)} \right)^{\gamma-1} |\nabla (\varepsilon - w^{(c)})^+|^2 \, dx - \varepsilon \int_{\Omega} |\nabla (\varepsilon - w^{(c)})^+|^2 \, dx = \int_{\Omega} R^{(c)}(\varepsilon - w^{(c)})^+ \, dx \geq 0. \]

This gives (71).

We derive from (66) that

\[ \partial_t \left( w^{(c)} \right)^{\frac{\gamma+1}{2}} = \frac{\gamma+1}{2} \left( w^{(c)} \right)^{\frac{\gamma+1}{2}-1} \partial_t w^{(c)} \]

\[ = \frac{\gamma+1}{2} \text{div} \left[ (\mu w^{(c)}_1 + \nu w^{(c)}_2) \left( w^{(c)} \right)^{\frac{\gamma+1}{2}-1} \nabla \left( w^{(c)} \right)^{\gamma} \right] \]

\[ -\frac{\gamma+1}{2} (\mu w^{(c)}_1 + \nu w^{(c)}_2) \nabla \left( w^{(c)} \right)^{\frac{\gamma+1}{2}-1} \cdot \nabla \left( w^{(c)} \right)^{\gamma} \]

\[ + \frac{(\gamma+1)\varepsilon}{2} \text{div} \left[ \left( w^{(c)} \right)^{\frac{\gamma+1}{2}-1} \nabla w^{(c)} \right] \]

\[ -\frac{(\gamma+1)\varepsilon}{2} \nabla \left( w^{(c)} \right)^{\frac{\gamma+1}{2}-1} \cdot \nabla w^{(c)} + \frac{\gamma+1}{2} \left( w^{(c)} \right)^{\frac{\gamma+1}{2}-1} R^{(c)} \]

\[ = \gamma \text{div} \left[ (\mu w^{(c)}_1 + \nu w^{(c)}_2) \left( w^{(c)} \right)^{\gamma-1} \nabla \left( w^{(c)} \right)^{\frac{\gamma+1}{2}} \right] \]

\[ -\frac{\gamma(\gamma-1)}{\gamma+1} (\mu w^{(c)}_1 + \nu w^{(c)}_2) \left( w^{(c)} \right)^{\frac{\gamma+1}{2}-2} \left| \nabla \left( w^{(c)} \right)^{\frac{\gamma+1}{2}} \right|^2 \]

\[ + \varepsilon \Delta \left( w^{(c)} \right)^{\frac{\gamma+1}{2}} - (\gamma^2 - 1)\varepsilon \left( w^{(c)} \right)^{\frac{\gamma+1}{2}-1} \left| \nabla \sqrt{w^{(c)}} \right|^2 \]

\[ + \frac{\gamma+1}{2} \left( w^{(c)} \right)^{\frac{\gamma+1}{2}-1} R^{(c)}. \]  

(72)

Remember that \( \frac{\gamma+1}{2} - 1 > 0 \). It follows that

\[ (\mu w^{(c)}_1 + \nu w^{(c)}_2) \left( w^{(c)} \right)^{\frac{\gamma+1}{2}-2} \leq \max\{\mu, \nu\} \left( w^{(c)} \right)^{\frac{\gamma+1}{2}-1} \leq c. \]
We can conclude from Lemma 3.1 that the sequence \( \{ \partial_t (w^{(e)})^{\frac{\gamma+1}{\gamma}} \} \) is bounded in \( L^2(0, T; (W^{1,2} (\Omega))^*) \) + \( L^1(\Omega_T) \equiv \{ \psi_1 + \psi_2 : \psi_1 \in L^2(0, T; (W^{1,2} (\Omega))^*), \psi_2 \in L^1(\Omega_T) \} \). Now we are in a position to use (i) in Lemma 2.2, thereby obtaining the precompactness of \( \{ (w^{(e)})^{\frac{\gamma+1}{\gamma}} \} \) in \( L^2(\Omega_T) \). Then the lemma follows from (65). 

We may extract a subsequence of \( \{ (w^{(e)}, u_1^{(e)}, u_2^{(e)}) \} \), still denoted by the same notation, such that
\[
\begin{align*}
  w^{(e)} &\to w \text{ a.e. in } \Omega_T \text{ and strongly in } L^p(\Omega_T) \text{ for each } p \geq 1, \\
u_1^{(e)} &\to u_1 \text{ weak* in } L^\infty(\Omega_T), \\
u_2^{(e)} &\to u_2 \text{ weak* in } L^\infty(\Omega_T), \\
\left( w^{(e)} \right)^{\frac{\gamma+1}{\gamma}} &\to w^{\frac{\gamma+1}{\gamma}} \text{ weakly in } L^2(0, T; W^{1,2}(\Omega)).
\end{align*}
\]

Since \( \{ w^{(e)} \} \) is bounded, we also have
\[
\left( w^{(e)} \right)^p \to w^p \text{ weakly in } L^2(0, T; W^{1,2}(\Omega)) \text{ for each } p \geq \frac{\gamma+1}{2}.
\]

This combined with (66) implies
\[
\partial_t w^{(e)} \to \partial_t w \text{ weakly in } L^2(0, T; (W^{1,2}(\Omega))^*).
\]

In view of (37), we infer
\[
\begin{align*}
  F_i(w^{(e)}) &\to F_i(w) \text{ strongly in } L^p(\Omega_T) \text{ for each } p \geq 1 \text{ and } (77) \\
  G_i(w^{(e)}) &\to G_i(w) \text{ strongly in } L^p(\Omega_T) \text{ for each } p \geq 1, i = 1, 2. (78)
\end{align*}
\]

Subsequently,
\[
R^{(e)} \to R, \quad R_1^{(e)} \to R_1, \quad R_2^{(e)} \to R_2 \text{ weak* in } L^\infty(\Omega_T). (79)
\]

Our key result is the following.

**Lemma 3.3.** If \( \mu = \nu \), then
\[
\left( w^{(e)} \right)^{\gamma+1} \to w^{\gamma+1} \text{ strongly in } L^2(0, T; W^{1,2}(\Omega)).
\]

**Proof.** In this case, we have
\[
\mu u_1^{(e)} + \nu u_2^{(e)} \nabla \left( w^{(e)} \right)^\gamma = \mu w^{(e)} \nabla (w^{(e)})^\gamma = \frac{\mu \gamma}{\gamma+1} \nabla \left( w^{(e)} \right)^{\gamma+1}. (80)
\]

Thus we can write (66) in the form
\[
\partial_t w^{(e)} - \frac{\mu \gamma}{\gamma+1} \Delta w_1^{(e)} = R^{(e)}, (81)
\]

where
\[
w_1^{(e)} = \left( w^{(e)} \right)^{\gamma+1} + \frac{\epsilon (\gamma+1)}{\mu \gamma} w^{(e)}.
\]

We may assume that \( w^{(e)} \) is a classical solution to (81) because it can be viewed as the limit of a sequence of classical approximate solutions. Use \( \partial_t w_1^{(e)} \) as a test function in (81) to derive
\[
\int_{\Omega} \partial_t w^{(e)} \partial_t w_1^{(e)} dx + \frac{\mu \gamma}{\gamma+1} \int_{\Omega} \nabla w_1^{(e)} \cdot \nabla \partial_t w_1^{(e)} dx = \int_{\Omega} R^{(e)} \partial_t w_1^{(e)} dx (82)
\]
We proceed to evaluate each integral in the above equation as follows:

\[
\int_{\Omega} \partial_t w^{(e)} \partial_t w_1^{(e)} \, dx = (\gamma + 1) \int_{\Omega} \left( w^{(e)} \right)^{\gamma} \left( \partial_t w^{(e)} \right)^2 \, dx \\
+ \frac{\varepsilon(\gamma + 1)}{\mu \gamma} \int_{\Omega} \left( \partial_t w^{(e)} \right)^2 \, dx,
\]

\[
\int_{\Omega} \nabla w_1^2 \cdot \nabla \partial_t w_1^{(e)} \, dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left| \nabla w_1^{(e)} \right|^2 \, dx,
\]

\[
\int_{\Omega} R^{(e)} \partial_t w_1^{(e)} \, dx = (\gamma + 1) \int_{\Omega} R^{(e)} \left( w^{(e)} \right)^{\gamma} \partial_t w^{(e)} \, dx \\
+ \frac{\varepsilon(\gamma + 1)}{\mu \gamma} \int_{\Omega} R^{(e)} \partial_t w^{(e)} \, dx \\
\leq \frac{\gamma + 1}{2} \int_{\Omega} \left( w^{(e)} \right)^{\gamma} \left( \partial_t w^{(e)} \right)^2 \, dx \\
+ \frac{\gamma + 1}{2} \int_{\Omega} \left( w^{(e)} \right)^{\gamma} \left( R^{(e)} \right)^2 \, dx \\
+ \frac{\varepsilon(\gamma + 1)}{2 \mu \gamma} \int_{\Omega} \left( \partial_t w^{(e)} \right)^2 \, dx + \frac{\varepsilon(\gamma + 1)}{2 \mu \gamma} \int_{\Omega} \left( R^{(e)} \right)^2 \, dx
\]

Plug the preceding three results into (82) and integrate to derive

\[
\int_{\Omega_T} \left( \partial_t \left( w^{(e)} \right)^{\frac{\gamma + 1}{2}} \right)^2 \, dx + \varepsilon \int_{\Omega_T} \left( \partial_t w^{(e)} \right)^2 \, dx + \sup_{0 \leq t \leq T} \int_{\Omega} \left| \nabla w_1^{(e)} \right|^2 \, dx \leq c.
\]

Note

\[
\partial_t \left( w^{(e)} \right)^{\gamma + 1} = 2 \left( w^{(e)} \right)^{\frac{\gamma + 1}{2}} \partial_t \left( w^{(e)} \right)^{\frac{\gamma + 1}{2}},
\]

\[
\nabla \left( w^{(e)} \right)^{\gamma + 1} = (\gamma + 1) \left( w^{(e)} \right)^{\gamma} \nabla w^{(e)}.
\]

On account of (65), \{\partial_t \left( w^{(e)} \right)^{\gamma + 1}\} is bounded in \(L^2(\Omega_T)\), while \{(w^{(e)})^{\gamma + 1}\} is bounded in \(L^\infty(0,T; W^{1,2}(\Omega))\). By (ii) in Lemma 2.2, the sequence \{(w^{(e)})^{\gamma + 1}\} is precompact in \(C([0,T], L^2(\Omega))\). Consequently, \{(w^{(e)})^{\gamma + 1}\} is precompact in \(C([0,T], L^p(\Omega))\) for each \(p \geq 1\). This asserts

\[
\int_{\Omega} \left( w^{(e)}(x,t) \right)^q \, dx \rightarrow \int_{\Omega} w^q(x,t) \, dx \text{ for each } t \in [0,T] \text{ and each } q \geq \gamma + 1. \quad (85)
\]

Take \(\varepsilon \rightarrow 0\) in (81) to obtain

\[
\partial_t w - \frac{\mu \gamma}{\gamma + 1} \Delta w^{\gamma + 1} = R.
\]

Subtract this equation from (81) and keep (80) in mind to get

\[
\partial_t (w^{(e)} - w) - \frac{\mu \gamma}{\gamma + 1} \Delta \left[ \left( w^{(e)} \right)^{\gamma + 1} - w^{\gamma + 1} \right] - \varepsilon \Delta w^{(e)} = R^{(e)} - R. \quad (86)
\]

Use \(w^{(e)})^{\gamma + 1} - w^{\gamma + 1}\) as a test function in (86) to derive

\[
\frac{\mu \gamma}{\gamma + 1} \int_{\Omega_T} \left| \nabla \left[ \left( w^{(e)} \right)^{\gamma + 1} - w^{\gamma + 1} \right] \right|^2 \, dx + \varepsilon \int_{\Omega_T} \nabla w^{(e)} \cdot \nabla \left[ \left( w^{(e)} \right)^{\gamma + 1} - w^{\gamma + 1} \right] \, dx
\]
\[ \int_{\Omega} (R(\varepsilon) - R) \left[ (w(\varepsilon))^{\gamma+1} - w^{\gamma+1} \right] dxdt \\
- \int_{0}^{T} \left< \partial_t (w(\varepsilon) - w), (w(\varepsilon))^{\gamma+1} - w^{\gamma+1} \right> dt. \quad (87) \]

We will show that the last three terms in the above equation all go to 0 as \( \varepsilon \to 0 \).

It is easy to see from Lemma 3.1 that
\[
\left| \varepsilon \int_{\Omega_T} \nabla w(\varepsilon) \cdot \nabla \left[ (w(\varepsilon))^{\gamma+1} - w^{\gamma+1} \right] dxdt \right| \\
= 4\varepsilon \left| \int_{\Omega_T} \sqrt{w(\varepsilon)} \nabla \sqrt{w(\varepsilon)} \cdot \left[ (w(\varepsilon))^{\gamma+1} - w^{\gamma+1} \right] dxdt \right| \\
\leq c \sqrt{\varepsilon} \to 0 \text{ as } \varepsilon \to 0. \quad (88) \]

By (73) and (79), we have
\[
\int_{\Omega_T} (R(\varepsilon) - R) \left[ (w(\varepsilon))^{\gamma+1} - w^{\gamma+1} \right] dxdt \to 0 \text{ as } \varepsilon \to 0. \]

Finally, we compute from Lemma 2.1 and (85) that
\[
\int_{0}^{T} \left< \partial_t (w(\varepsilon) - w), (w(\varepsilon))^{\gamma+1} - w^{\gamma+1} \right> dt \\
= \frac{1}{\gamma + 2} \int_{0}^{T} \left[ \frac{d}{dt} \int_{\Omega} (w(\varepsilon))^{\gamma+2} dx + \frac{d}{dt} \int_{\Omega} w^{\gamma+2} dx \right] dt \\
- \int_{0}^{T} \left< \partial_t w(\varepsilon), (w(\varepsilon))^{\gamma+1} \right> dt - \int_{0}^{T} \left< \partial_t w, (w(\varepsilon))^{\gamma+1} \right> dt \\
= \frac{1}{\gamma + 2} \left[ \int_{\Omega} (w(\varepsilon)_{x,T})^{\gamma+2} dx + \int_{\Omega} w^{\gamma+2} dx \right] \\
- \frac{2}{\gamma + 2} \left[ \int_{\Omega} (w(0)(x))^{\gamma+2} dx \right] \\
- \int_{0}^{T} \left< \partial_t w, (w(\varepsilon))^{\gamma+1} \right> dt \\
\to \frac{2}{\gamma + 2} \int_{\Omega} w^{\gamma+2} dx - \frac{2}{\gamma + 2} \int_{\Omega} (w(0)(x))^{\gamma+2} dx - 2 \int_{0}^{T} \left< \partial_t w, w^{\gamma+1} \right> dt \\
= 0. \]

This completes the proof. \( \square \)

**Proof of Theorem 1.2.** Equipped with this lemma, we can complete the proof of Theorem 1.2. Keeping (71) in mind, we can set
\[
\eta_{1}(\varepsilon) = \frac{u_{1}(\varepsilon)}{w(\varepsilon)}, \quad \eta_{2}(\varepsilon) = \frac{u_{2}(\varepsilon)}{w(\varepsilon)}. \]

Suppose \( \eta_{1}(\varepsilon) \to \eta_{1}, \quad \eta_{2}(\varepsilon) \to \eta_{2} \text{ weak* in } L^\infty(\Omega_T). \)

We calculate
\[
u_{1}(\varepsilon) \nabla (w(\varepsilon)) = \eta_{1}(\varepsilon) w(\varepsilon) \nabla (w(\varepsilon))^{\gamma} \]
We obtain $\mu^{13}$. For this purpose, we set $\eta_1^{(c)}$. This implies that Theorem 1.2.

We claim that $(90)$. Similarly, we can show $\eta_1 = 0$, then $\eta_1 = 0$, and we still have $u_1 = \eta_1$. This completes the proof of (90). Similarly, we can show $u_2 = \eta_1$ on the set $\{w > 0\}$. If $w = 0$, then $u_1 = 0$, and we still have $u_1 = \eta_1$. This completes the proof of Theorem 1.2.

Finally, we remark that we can extend Theorem 1.2 to the case considered in [13]. For this purpose, we set $\mu = 1$ and let $w, R, R_1, R_2$ be given as before.

**Corollary 3.4.** Let (H1), (H2), (H5) be satisfied, and assume that (H3) holds for $\Omega = \mathbb{R}^d$ and $u_1^{(0)}, u_2^{(0)} \in W^{1,2}_{\text{loc}}(\mathbb{R}^d)$. Then there is a weak solution to the initial value problem

\begin{align*}
\partial_t u_1 - \text{div} (u_1 \nabla u^\gamma) &= R_1 \quad \text{in } \mathbb{R}^d \times (0, T), \\
\partial_t u_2 - \text{div} (u_2 \nabla u^\gamma) &= R_2 \quad \text{in } \mathbb{R}^d \times (0, T), \\
(u_1, u_2)|_{t=0} &= (u_1^{(0)}(x), u_2^{(0)}(x)) \quad \text{on } \mathbb{R}^d,
\end{align*}

in the following sense:

(R1) $u_1, u_2$ are non-negative and bounded with $w^{\frac{\gamma + 1}{\gamma}} \in L^2(0, T; W^{1,2}_{\text{loc}}(\mathbb{R}^d))$;

(R2) there hold

\begin{align*}
- \int_{\mathbb{R}^d \times (0, T)} u_1 \partial_t \varphi dx + \int_{\mathbb{R}^d \times (0, T)} u_1 \nabla u^\gamma \cdot \nabla \varphi dx dt &= \int_{\mathbb{R}^d \times (0, T)} R_1 \varphi dx dt + \int_{\mathbb{R}^d} u_1^{(0)}(x) \varphi(x, 0) dx \quad \text{and} \\
- \int_{\mathbb{R}^d \times (0, T)} u_2 \partial_t \psi dx + \int_{\mathbb{R}^d \times (0, T)} u_2 \nabla u^\gamma \cdot \nabla \psi dx dt &= \int_{\mathbb{R}^d \times (0, T)} R_2 \psi dx dt + \int_{\mathbb{R}^d} u_2^{(0)}(x) \psi(x, 0) dx,
\end{align*}
for each pair of smooth functions $\varphi, \psi$ with compact support and $\varphi(x, T) = \psi(x, T) = 0$.

We will give a brief outline of the proof. To this end, we set

\[ B_k(0) = \{ x \in \mathbb{R}^d : |x| < k \}, \quad k = 1, 2, \ldots. \]

We replace $\Omega$ in (1)-(5) by $B_k(0)$ and denote the resulting solution by $(u_1^{(k)}, u_2^{(k)})$. That is, we have

\begin{align*}
\partial_t u_1^{(k)} - \text{div} \left[ u_1^{(k)} \nabla \left( w^{(k)} \right) \right] &= R_1^{(k)} \quad \text{in } B_k(0) \times (0, T), \quad (91) \\
\partial_t u_2^{(k)} - \text{div} \left[ u_2^{(k)} \nabla \left( w^{(k)} \right) \right] &= R_2^{(k)} \quad \text{in } B_k(0) \times (0, T), \quad (92) \\
\left. u_1^{(k)} \nabla \left( w^{(k)} \right) \cdot n \right|_{t=0} &= 0 \quad \text{on } \partial B_k(0) \times (0, T), \quad (93) \\
\left. u_2^{(k)} \nabla \left( w^{(k)} \right) \cdot n \right|_{t=0} &= 0 \quad \text{on } \partial B_k(0) \times (0, T), \quad (94) \\
\left. (u_1^{(k)}, u_2^{(k)}) \right|_{t=0} &= (u_1^{(0)}(x), u_2^{(0)}(x)) \quad \text{on } B_k(0). \quad (95)
\end{align*}

Of course, here

\[ w^{(k)} = u_1^{(k)} + u_2^{(k)}, \]

\[ R_1^{(k)} = u_1^{(k)} F_1(w^{(k)}) + u_2^{(k)} G_1(w^{(k)}), \]

\[ R_2^{(k)} = u_1^{(k)} F_2(w^{(k)}) + u_2^{(k)} G_2(w^{(k)}). \]

Moreover,

\[ u_1^{(k)} \geq 0, \quad u_2^{(k)} \geq 0, \quad w^{(k)} \leq w_p. \quad (96) \]

Adding (92) to (91) yields

\[ \partial_t u^{(k)} - \frac{\gamma}{\gamma + 1} \Delta \left( w^{(k)} \right)^{\gamma+1} = R_1^{(k)} + R_2^{(k)} \equiv R^{(k)} \quad \text{in } B_k(0) \times (0, T). \quad (97) \]

Pick a smooth cut-off function $\zeta$ on $\mathbb{R}^d$. From here on, we assume that $k$ is so large that

\[ \text{supp } \zeta \subset B_k(0). \quad (98) \]

By using $\zeta^2 \ln(w^{(k)} + \tau), \tau > 0$, as a test function in (97), we can infer from the proof of Lemma 3.1 that

\[ \int_{\mathbb{R}^d \times (0, T)} \zeta^2 \left| \nabla \left( w^{(k)} \right)^{\frac{\gamma+1}{\gamma}} \right|^2 \, dx \, dt \leq c. \quad (99) \]

Here $c$ depends on both $T$ and $\zeta$. Similarly, use $\zeta^2 \partial_t \left( w^{(k)} \right)^{\gamma+1}$ as test function in (97) to derive

\[ \int_{\mathbb{R}^d \times (0, T)} \zeta^2 \left( \frac{\partial_t \left( w^{(k)} \right)^{\frac{\gamma+2}{\gamma}}}{\gamma+2} \right)^2 \, dx \, dt + \sup_{0 \leq t \leq T} \int_{\mathbb{R}^d} \zeta^2 \left| \nabla \left( w^{(k)} \right)^{\gamma+1} \right|^2 \, dx \leq c. \quad (100) \]

It is not difficult to see that Lemmas 3.2 and 3.3 still hold with $L^p(\Omega_T)$ being replaced by $L^p(0, T; L^p_{\text{loc}}(\mathbb{R}^d))$ and $L^2(0, T; W^{1,2}(\Omega))$ by $L^2(0, T; W^{1,2}_{\text{loc}}(\mathbb{R}^d))$, respectively. Take $k \to \infty$ in (91)-(95) suitably to conclude the corollary.
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