The Hadwiger number, chordal graphs and $ab$-perfection

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Abstract

A graph is chordal if every induced cycle has three vertices. The Hadwiger number is the order of the largest complete minor of a graph. We characterize the chordal graphs in terms of the Hadwiger number and we also characterize the families of graphs such that for each induced subgraph $H$, (1) the Hadwiger number of $H$ is equal to the maximum clique order of $H$, (2) the Hadwiger number of $H$ is equal to the achromatic number of $H$, (3) the $b$-chromatic number is equal to the pseudoachromatic number, (4) the pseudo-$b$-chromatic number is equal to the pseudoachromatic number, (5) the Hadwiger number of $H$ is equal to the Grundy number of $H$, and (6) the $b$-chromatic number is equal to the pseudo-Grundy number.

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1 Introduction

Let $G$ be a finite graph. A $k$-coloring of $G$ is a surjective function $\varsigma$ that assigns a number from the set $[k] := \{1, \ldots, k\}$ to each vertex of $G$. A $k$-coloring $\varsigma$ of $G$ is called proper if any two adjacent vertices have different colors, and $\varsigma$ is called complete if for each pair of different colors $i, j \in [k]$ there exists an edge $xy \in E(G)$ such that $x \in \varsigma^{-1}(i)$ and $y \in \varsigma^{-1}(j)$. A $k$-coloring $\varsigma$ of a connected graph $G$ is called connected if for all $i \in [k]$, each color class $\varsigma^{-1}(i)$ induces a connected subgraph of $G$.

The chromatic number $\chi(G)$ of $G$ is the smallest number $k$ for which there exists a proper $k$-coloring of $G$. The Hadwiger number $h(G)$ is the maximum $k$ for which a connected and complete coloring of a connected graph $G$ exists, and it is defined as the maximum $h(H)$ among the connected components $H$ of a disconnected graph $G$ (it is also known as the connected-pseudoachromatic number, see [1]).

A graph $H$ is called a minor of the graph $G$ if and only if $H$ can be formed from $G$ by deleting edges and vertices and by contracting edges. Suppose that $K_k$ is a minor of a connected graph $G$. If $V(K_k) = [k]$ then there exists a natural corresponding complete $k$-coloring $\varsigma : G \to [k]$ for which $\varsigma^{-1}(i)$ is exactly the set of vertices of $G$ which contract to vertex $i$ in $K_k$. The Hadwiger number $h(G)$ of a graph $G$ is the largest $k$ for which $K_k$ is a minor of $G$. Clearly,

$$\omega(G) \leq h(G) \tag{1}$$

where $\omega(G)$ denotes the clique number of $G$: the maximum clique order of $G$.

The Hadwiger number was introduced by Hadwiger in 1943 [14] together the Hadwiger conjecture which states that $\chi(G) \leq h(G)$ for any graph $G$.

The following definition is an extension of the notion of perfect graph, introduced by Berge [4]: Let $a, b$ be two distinct parameters of $G$. A graph $G$ is called $ab$-perfect if for every induced subgraph $H$ of $G$, $a(H) = b(H)$. Note that, with this definition a perfect graph is denoted by $\omega\chi$-perfect. The concept of the $ab$-perfect graphs was introduced by Christen and Selkow in [8] and extended in [3, 2, 6, 18, 19, 20].

A graph $G$ without an induced subgraph $H$ is called $H$-free. A graph $H_1$-free, $H_2$-free,... is called $(H_1, H_2, \ldots)$-free. A chordal graph is a $(C_4, C_5, \ldots)$-free one.

Some known results are the following: Lóvasz proved in [17] that a graph $G$ is $\omega\chi$-perfect if and only if its complement is $\omega\chi$-perfect. Chudnovsky, Robertson, Seymour and Thomas proved in [9] that a graph $G$ is $\omega\chi$-perfect if and only if $G$ and its complement are $(C_5, C_7, \ldots)$-free.
This paper is organized as follows: In Section 2 we prove that the families of chordal graphs and the family of $\omega h$-perfect graphs are the same. In Section 3 we give some consequences of the Section 2 as characterizations of other graph families related to complete colorings.

2 Chordal graphs and $\omega h$-perfect graphs

We will use the following chordal graph characterization to prove Theorem 2.2:

**Theorem 2.1** (Hajnal, Surányi [15] and Dirac [10]). A graph $G$ is chordal if and only if $G$ can be obtained by identifying two complete subgraphs of the same order in two chordal graphs.

Now, we characterize the chordal graphs and the $\omega h$-perfect ones. The following proof is based on the standard proof of the chordal graph perfection (see [7]).

**Theorem 2.2.** A graph $G$ is $\omega h$-perfect if and only if $G$ is chordal.

**Proof.** Assume that $G$ is $\omega h$-perfect. Note that if a cycle $H$ is one of four or more vertices then $\omega(H) = 2$ and $h(H) = 3$. Hence, every induced cycle of $G$ has at the most 3 vertices and the implication is true.

Now, we verify the converse. Since every induced subgraph of a chordal graph is also a chordal graph, it suffices to show that if $G$ is a connected chordal graph, then $\omega(G) = h(G)$. We proceed by induction on the order $n$ of $G$. If $n = 1$, then $G = K_1$ and $\omega(G) = h(G) = 1$. Assume, therefore, that $\omega(H) = h(H)$ for every induced chordal graph $H$ of order less than $n$ for $n \geq 2$ and let $G$ be a chordal graph of order $n$. If $G$ is a complete graph, then $\omega(G) = h(G) = n$. Hence, we may assume that $G$ is not complete. By Theorem 2.1 $G$ can be obtained from two chordal graphs $H_1$ and $H_2$ by identifying two complete subgraphs of the same order in $H_1$ and $H_2$. Let $S$ denote the set of vertices in $G$ that belong to $H_1$ and $H_2$. Thus the induced subgraph $\langle S \rangle_G$ in $G$ by $S$ is complete and no vertex in $V(H_1) \setminus S$ is adjacent to a vertex in $V(H_2) \setminus S$. Hence,

$$\omega(G) = \max\{\omega(H_1), \omega(H_2)\} = k.$$ 

Moreover, according to the induction hypothesis, $\omega(H_1) = h(H_1)$ and $\omega(H_2) = h(H_2)$, then

$$\max\{\omega(H_1), \omega(H_2)\} = \max\{h(H_1), h(H_2)\} = k.$$ 

On the other hand, since $S$ is a clique cut then each walk between $V(H_1) \setminus S$ and $V(H_2) \setminus S$ contains at least one vertex in $S$. Let $\varsigma$ be a pseudo-connected $h(G)$-coloring of $G$, and suppose there exist two color classes such that one is completely contained in $V(H_1) \setminus S$, and the other one is completely contained in $V(H_2) \setminus S$. Clearly these two color classes do not intersect, which contradicts our choice of $\varsigma$. Moreover, each color class with vertices both in
\( V(H_1) \setminus S \) and in \( V(H_2) \setminus S \), contains vertices in \( S \). Consequently, every pair of color classes having vertices both in \( V(H_1) \setminus S \) and in \( V(H_2) \setminus S \) must have an incidence in \( \langle S \rangle_G \). Thus,\
\[
    h(G) \leq \max\{h(H_1), h(H_2)\} = k.
\]

By Equation 1, \( \omega(G) = k = h(G) \) and the result follows.

It is known that every chordal graph is a \( \omega \chi \)-perfect one (see [7]). The following corollary is a consequence of the chordal graph perfection.

**Corollary 2.3.** Every \( \omega h \)-perfect graph is \( \omega \chi \)-perfect.

### 3 Other classes of \( ab \)-perfect graphs

In this section, we give a new characterization of several family of \( ab \)-perfect graphs related to complete colorings.

#### 3.1 Achromatic and pseudoachromatic numbers

Firstly, the \( \text{pseudoachromatic number} \ \psi(G) \) of \( G \) is the largest number \( k \) for which there exists a complete \( k \)-coloring of \( G \) [13], and it is easy to see that

\[
    \omega(G) \leq h(G) \leq \psi(G).
\]  

Secondly, the \( \text{achromatic number} \ \alpha(G) \) of \( G \) is the largest number \( k \) for which there exists a proper and complete \( k \)-coloring of \( G \) [16], and it is not hard to see that

\[
    \omega(G) \leq \alpha(G) \leq \psi(G).
\]  

Complete bipartite graphs have achromatic number two (see [7]) but their Hadwiger number can be arbitrarily large, while the graph formed by the union of \( K_2 \) has Hadwiger number two but its achromatic number can be arbitrarily large. Therefore, \( \alpha \) and \( h \) are two non comparable parameters. We will use the following characterization in the proof of Corollary 3.2.

**Theorem 3.1** (Araujo-Pardo, R-M [3, 2]). A graph \( G \) is \( \omega \psi \)-perfect if and only if \( G \) is \( (C_4, P_4, P_3 \cup K_2, 3K_2) \)-free.
Corollary 3.2 is an interesting result because it gives a characterization of two non comparable parameters.

**Corollary 3.2.** A graph $G$ is $\alpha_h$-perfect if and only if $G$ is $\omega_\psi$-perfect.

*Proof.* Since $h(C_4) = \alpha(P_4) = \alpha(P_3 \cup K_2) = \alpha(3K_2) = 3$ and $\alpha(C_4) = h(P_4) = h(P_3 \cup K_2) = h(3K_2) = 2$ (see Figure 1) then a $\alpha_h$-perfect graph is $(C_4, P_4, P_3 \cup K_2, 3K_2)$-free. By Theorem 3.1, $G$ is $\omega_\psi$-perfect.

For the converse, if $G$ is $\omega_\psi$-perfect, then by Equation 2, $G$ is a $\omega_h$-perfect graph, thus, the implication follows. $\square$

**Corollary 3.3.** Every $\omega_\psi$-perfect graph is $\omega_\chi$-perfect.

*Proof.* If a graph $G$ is $\omega_\psi$-perfect then Equation 2 implies that $G$ is $\omega_h$-perfect, and by Theorem 2.2 $G$ is chordal, therefore $G$ is $\omega_\chi$-perfect. $\square$

The following corollary is a consequence of the perfection of $\omega_\psi$-perfect graphs.

**Corollary 3.4.** Every $\alpha_h$-perfect graph is $\omega_\chi$-perfect.

### 3.2 $b$-chromatic and pseudo-$b$-chromatic numbers

On one hand, a coloring such that every color class contains a vertex that has a neighbor in every other color class is called dominating. The pseudo-$b$-chromatic number $B(G)$ of a graph $G$ is the largest integer $k$ such that $G$ admits a dominating $k$-coloring.

On the other hand, the $b$-chromatic number $b(G)$ of $G$ is the largest number $k$ for which there exists a proper and dominating $k$-coloring of $G$ [6], therefore

$$\omega(G) \leq b(G) \leq B(G) \leq \psi(G).$$

We get the following characterizations:

**Corollary 3.5.** For any graph $G$ the following are equivalent: (1) $G$ is $\omega_\psi$-perfect, (2) $G$ is $b_\psi$-perfect, (3) $G$ is $B_\psi$-perfect and (4) $G$ is $(C_4, P_4, P_3 \cup K_2, 3K_2)$-free.

*Proof.* The proofs of (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) immediately follow from [4]. To prove (3) $\Rightarrow$ (4) note that, if $H \in \{C_4, P_4, P_3 \cup K_2, 3K_2\}$ then $B(H) \neq \psi(H)$, hence the implication is true, see Figure 1. The proof of (4) $\Rightarrow$ (1) is a consequence of Theorem 3.1. $\square$
The following corollary is a consequence of Corollaries 3.3 and 3.5.

**Corollary 3.6.** The $b\psi$-perfect graphs and the $B\psi$-perfect ones are $\omega\chi$-perfect.

Corollary 3.5 is related to the following theorem:

**Theorem 3.7** (Christen, Selkow [8] and Blidia, Ikhlef, Maffray [6]). For any graph $G$ the following are equivalent: (1) $G$ is $\omega\alpha$-perfect, (2) $G$ is $b\alpha$-perfect and (3) $G$ is $(P_4, P_3 \cup K_2, 3K_2)$-free.

### 3.3 Grundy and pseudo-Grundy numbers

First, a coloring of $G$ is called *pseudo-Grundy* if each vertex is adjacent to some vertex of each smaller color. The *pseudo-Grundy number* $\gamma(G)$ is the maximum $k$ for which a pseudo-Grundy $k$-coloring of $G$ exists (see [5, 7]).

Second, a proper pseudo-Grundy coloring of $G$ is called *Grundy*. The *Grundy number* $\Gamma(G)$ (also known as the *first-fit chromatic number*) is the maximum $k$ for which a Grundy $k$-coloring of $G$ exists (see [7, 12]). From the definitions, we have that

$$\omega(G) \leq \Gamma(G) \leq \gamma(G). \quad (5)$$

The following characterization of the graphs call *trivially perfect graphs*, it will be used in the proof of Corollary 3.9.

**Theorem 3.8** (R-M [19]). A graph $G$ is $\omega\gamma$-perfect if and only if $G$ is $(C_4, P_4)$-free.

It is known that a trivially perfect graph is chordal (see [11]). The following corollary also gives a characterization of two non comparable parameters.

**Corollary 3.9.** A graph $G$ is $\Gamma h$-perfect if and only if $G$ is $\omega\gamma$-perfect.

**Proof.** A $\Gamma h$-perfect graph is $(C_4, P_4)$-free because $\Gamma(C_4) = h(P_4) = 2$ and $\Gamma(P_4) = h(C_4) = 3$ (see Figure 1) then by Theorem 3.8 $G$ is $\omega\gamma$-perfect.

For the converse, let $G$ be a $\omega\gamma$-perfect graph. If $H$ is an induced graph of $G$, by Equation 5 $\omega(H) = \Gamma(H)$. Since $G$ is a chordal graph, $\omega(H) = h(H)$, so the implication follows.

The following corollary is a consequence of the perfection of $\omega\gamma$-perfect graphs.

**Corollary 3.10.** Every $\Gamma h$-perfect graph is $\omega\chi$-perfect.
3.4 The $b\gamma$-perfect graphs

Finally, we will use the following characterization of the proof of Theorem 3.12.

**Theorem 3.11** (Blidia, Ikhlef, Maffray [6]). A graph $G$ is $b\Gamma$-perfect if and only if $G$ is $(P_4,3P_3,2D)$-free.

We get the following characterization.

**Theorem 3.12.** A graph $G$ is $b\gamma$-perfect if and only if $G$ is $(C_4,P_4,3P_3,2D)$-free.

**Proof.** Note that, if $H \in \{C_4,P_4,3P_3,2D\}$ then $b(H) \neq \gamma(H)$, hence, the implication is true (see Figure 1).

For the converse, a $(C_4,P_4,3P_3,2D)$-free graph $G$ is $\omega\gamma$-perfect (by Theorem 3.8) and $b\Gamma$-perfect (by Theorem 3.11). Then, for every induced subgraph $H$ of $G$, $\omega(H) = \gamma(H) = \Gamma(H)$ by Equation 5 and $b(H) = \Gamma(H)$. Therefore, $b(H) = \gamma(H)$ and the result follows.

![Figure 1: Graphs with a complete coloring with numbers and a connected coloring with symbols.](image)

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