Inverse source problem in a forced network

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Abstract

We address the nonlinear inverse source problem of identifying a time-dependent source occurring in one node of a network governed by a wave equation. We prove that time records of the associated state taken at a strategic set of two nodes yield uniqueness of the two unknown elements: the source position and the emitted signal. We develop a non-iterative identification method that localizes the source node by solving a set of well posed linear systems. Once the source node is localized, we identify the emitted signal using a deconvolution problem or a Fourier expansion. Numerical experiments on a 5 node graph confirm the effectiveness of the approach.

1 Introduction

Networks play a crucial role in the transmission of numerous quantities. These can be miscible like fluids [11] and electric power [2], or non miscible like information packets [3] in telecommunications. The mathematical structure describing a network is a graph composed of nodes connected by edges. The graph can be directed as for traffic flow or nerve propagation [4] or undirected as for miscible flows; in these two situations the equations differ.

Miscible flows satisfy discrete conservation laws; for electrical circuits, these are the well-known Kirchoff and Ohm’s laws. When dissipation is absent from the edges, miscible flows obey a “graph wave equation” where the graph Laplacian matrix replaces the ordinary Laplacian [11, 21]. This is a very general model, essentially Newton’s equation, linking

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acceleration to the forces applied on the system. It is valid for acoustic waves in a fluid or normal modes in a molecule \[5\]. It is a linear limit of the so-called Kuramoto model for an electrical network \[27\]. In that context, the variable is the phase of a generator or load, located at each node of the network. Note that one could introduce damping at each node of the system leading to a so-called reaction-diffusion type dynamics \[4\].

For such a miscible flow network, the forward problem consists in finding the state of the transmitted entity in each node of the network, given the physical parameters and the initial state. For many applications, the inverse problem is important. There, we aim to identify some unknown parameters affecting the state of the system using measurements at some observation nodes of the network. For example, we would like to detect the presence of a failing node on the network -typically an unsynchronized generator in the electrical grid context- and possibly localize its position to take appropriate action. Many authors analyzed source problems in networks; two examples are the articles by Rodriguez and Sahah \[6, 7\] who -in an effort to contain an epidemic or contamination- analyzed the influence of the propagation network and of the unknown source, assuming known the state in all the nodes.

In practice, it is usually infeasible to record the state of the system at all the nodes of the network and recent studies address this constraint. That on a used network only a few number of its nodes can be observed. For example, Pinto et al \[22\] propose a probabilistic approach to estimate the location of a node source using measurements collected at sparsely-placed observation nodes in the network. The authors conclude that such a sparse deployment of sensors provides an effective alternative to recording the state at all nodes of the network. However, the position of the observation nodes strongly affects the performance of the approach and thus optimal strategies for placing the sensors need to be found. In \[23\], the authors developed the so-called Short-Fat Tree algorithm to identify a node diffusion source in networks; they proved limit results on the asymptotic probability of their approach in identifying the source node with respect to the infection duration. In the power systems context, Nudell and Chakrabortty \[20\] propose a graph theoretic algorithm to find forced oscillation inputs. Minimal information is assumed to be known in order to estimate the transfer function.

In the present study, we consider a dissipation less graph wave equation modeling a general miscible flow, forced by an unknown signal generated at an unknown node. We develop a constructive deterministic approach based on time records of the transmitted entity in a strategic set of two nodes of the network. We show that, with this method, we localize the node source and identify its unknown emitted signal in a unique manner. We introduce an adjoint problem formed by convoluting the equation with an appropriate boundary value problem, yielding an over-determined system of \(N\) equations and \(N - 2\) unknowns for a \(N\) node network. The right hand side of this system contains the unknown final state \(X(T)\) of the network. A first step is to estimate \(X(T)\); for this it is crucial that the observation nodes form a strategic set. Once \(X(T)\) is found, we show how to solve the over-determined linear system and find the node source. Graph theory arguments provide guidelines on how to choose the observation nodes for a given network. Then we give two methods to estimate the unknown signal.

The paper is organized as follows. Section two introduces the model and the mathematical framework. Section 3 presents the adjoint problem and the over-determined linear system.
We establish the identifiability of the source location and signal in section 4 and develop
an identification method in section 5. Numerical calculations on a 5 node graph are shown
in section 6 and we conclude in section 7.

2 Mathematical Modelling and technical results

Consider a network defined over N nodes labelled \( k = 1, \ldots, N \) on which propagates a
miscible entity. The state of this entity at time \( t \in (0, T) \) in the different nodes of the
network is given by the following vector
\[
X(t) = (x_1(t), \ldots, x_N(t))^\top,
\]
where \( T > 0 \) is a final monitoring time. Neglecting friction in the nodes and the edges,
the evolution of the state \( X \) is governed by the so-called graph wave equation, see [11][12]:
\[
\begin{align*}
\ddot{X}(t) - \Delta X(t) &= \lambda(t) S & \text{in} \ (0, T) \\
X(0) &= a \in \mathbb{R}^N & \text{and} \quad \dot{X}(0) = b \in \mathbb{R}^N,
\end{align*}
\]
where \( a \) and \( b \) represent, respectively, the initial state and velocity in all nodes of the
network, \( \Delta \) is the \( N \times N \) graph Laplacian matrix [9] and \( \lambda S \) designates a node source
of the network located at \( s \) and forcing the state of the transmitted entity by emitting a
time-dependent signal \( \lambda \). We assume \( \lambda \) and \( S \) to be elements of the admissible set:
\[
A := \{ \lambda \in L^2(0, T) \mid \exists T^0 \in (0, T) : \lambda(t) = 0, \ \forall t \in (T^0, T) \ \text{and} \}
\]
\[
S = (s_1, \ldots, s_N)^\top \text{ where for } k = 1, \ldots, N, \ s_k \in \{0, 1\} \ \text{and} \quad \sum_{k=1}^N s_k = 1\}
\]
Since the graph wave equation (2) is a system of linear second order differential equations
it follows that for all given two admissible elements \( \lambda \) and \( S \) defining a node source \( \lambda S \),
the system (2) admits a unique solution \( X = (x_1, \ldots, x_N)^\top \) such that \( x_k \in H^2(0, T) \) for
\( k = 1, \ldots, N \). That allows to define the following observation operator:
\[
M[\lambda, S] := \{x_i(t), x_j(t); \quad \text{for } 0 < t < T\},
\]
where \( i \neq j \in \{1, \ldots, N\} \) are two distinct nodes of the network. This is the so-called
forward problem.

The inverse problem with which we are concerned here is: given time records \( d_i \) and \( d_j \)
in \( (0, T) \) of the two local state components \( x_i \) and \( x_j \) taken at the two distinct nodes \( i \) and
\( j \) of the network, determine the two unknown elements \( \lambda \) and \( S \) of \( A \) that yield
\[
M[\lambda, S] = \{d_i(t), d_j(t); \quad \text{for } 0 < t < T\}
\]
Besides, the graph Laplacian matrix \( \Delta \) is a \( N \times N \) real symmetric and negative semi-
definite matrix, for more details see [9]. In the remainder, we assume the eigenvalues of
\( \Delta \) to be \( N \) distinct nonpositive real numbers ordered and denoted as follows:
\[
0 = -\omega_1^2 > -\omega_2^2 > \cdots > -\omega_N^2.
\]
The set of eigenvectors \( \{ v^1, \ldots, v^N \} \) associated to these eigenvalues is an orthogonal base of \( \mathbb{R}^N \) and can be chosen orthonormal. Thus, the state \( X \) solution of the system (2) can be written in \( (0, T) \) as follows:

\[
X(t) = \sum_{n=1}^{N} y_n(t) v^n \quad \text{where} \quad y_n(t) = \langle X(t), v^n \rangle \quad \text{for} \quad n = 1, \ldots, N,
\]

(6)

where \( \langle , \rangle \) designates the inner product in \( \mathbb{R}^N \).

We now introduce the definition of a strategic set of nodes.

**Definition 2.1** A set of \( \{ k_1, \ldots, k_\ell \} \) nodes is called strategic if for all vector \( v^n \), where \( n = 1, \ldots, N \) there exists at least one element \( k \in \{ k_1, \ldots, k_\ell \} \) such that \( v^n_k \neq 0 \).

Therefore, according to Definition 2.1, a single node \( i \in \{ 1, \ldots, N \} \) is called strategic if \( v^n_i \neq 0 \) for all \( n = 1, \ldots, N \). This is the opposite of a "soft node" that was introduced in our previous work [12]. Also, for spatially continuous systems Jai and Pritchard stressed the importance of these strategic sets [25].

A set of two distinct nodes \( \{ i, j \} \) is strategic if for all \( n = 1, \ldots, N \), it holds \( |v^n_i| + |v^n_j| \neq 0 \). Hence, if \( i \) is a strategic node then, all set of nodes containing \( i \) is also strategic. This leads to the following technical result:

**Lemma 2.2** Let \( T^* \in (T^0, T) \), \( X \) be fulfilling the following system:

\[
\begin{cases}
\dot{X}(t) - \Delta X(t) = 0 \quad \text{in} \ (T^*, T) \\
X(T) \in \mathbb{R}^N \quad \text{and} \quad \dot{X}(T) \in \mathbb{R}^N,
\end{cases}
\]

(7)

and \( \{ k_1, \ldots, k_\ell \} \) be a strategic set of nodes. We have

\[
x_{k_1}(t) = \cdots = x_{k_\ell}(t) = 0, \ \forall \ t \in (T^*, T) \implies X(T) = \dot{X}(T) = \vec{0}
\]

(8)

**Proof.** See the appendix.

### 3 Adjoint problem

To solve the inverse problem for (2), we introduce an adjoint problem. For that, we consider the Sturm-Liouville eigenvalue problem:

\[
\begin{cases}
-\ddot{\varphi}_m(t) = \mu_m \varphi_m(t) \quad \text{in} \ (0, T) \\
\varphi_m(0) = \varphi_m(T) = 0.
\end{cases}
\]

(9)
Then, the normalized eigenfunctions $\varphi_m$ and their associated eigenvalues $\mu_m$ are defined for all $m \in \mathbb{N}^*$ by

$$\varphi_m(t) = \sqrt{\frac{2}{T}} \sin \left( \frac{m\pi}{T} t \right) \quad \text{and} \quad \mu_m = \left( \frac{m\pi}{T} \right)^2 > 0$$

The set $\{\varphi_m\}$ forms a complete orthonormal family of $L^2(0,T)$ for the standard inner product in $L^2(0,T)$ i.e., $\langle f, g \rangle_{L^2(0,T)} = \int_0^T f(t)g(t)dt$.

To obtain the adjoint problem, we multiply each equation of the system (12) by $\varphi_m$ and integrate by parts over $(0,T)$, to get the following linear system:

$$-(\Delta + \mu_m I) \bar{X}_m = \lambda_m S + P_m,$$

where $I$ is the $N \times N$ identity matrix and

$$\bar{X}_m = (\bar{x}_1, \ldots, \bar{x}_N)^\top \quad \text{with} \quad \bar{x}_k = \langle x_k, \varphi_m \rangle_{L^2(0,T)}$$

$$P_m = \dot{\varphi}_m(T)X(T) - \varphi_m(0)X(0) \quad \text{and} \quad \lambda_m = \langle \lambda, \varphi_m \rangle_{L^2(0,T)}$$

Notice that since the final state $X(T)$ of the system (2) is unknown, the first term defining the vector $P_m$ introduced in (12) is also unknown. Estimating $X(T)$ will then be the first step in the identification process.

Furthermore, the choice of the boundary conditions on $\varphi_m$ solution of the auxiliary problem (11) allowed to eliminate from the linear system (11) the unknown term $\dot{X}(T)$. If we had chosen other boundary conditions for $\varphi_m$ we would always have one unknown term among $X(T)$ and $\dot{X}(T)$.

To simplify our presentation we introduce for all $m \in \mathbb{N}^*$ the notation

$$A_m := \Delta + \mu_m I$$

Furthermore, according to the observation operator $M[\lambda, S]$ in (3) it follows that the two components $\bar{x}_i = \langle x_i, \varphi_m \rangle_{L^2(0,T)}$ and $\bar{x}_j = \langle x_j, \varphi_m \rangle_{L^2(0,T)}$ of the vector $\bar{X}_m$ are known. Thus, (11) is reduced to the following $N \times (N-2)$ linear system:

$$-A_m^{ij} \bar{X}_m^{ij} = P_m^{ij} + \lambda_m S,$$

where $A_m^{ij}$ is the $N \times (N-2)$ matrix obtained by removing the two columns $i$ and $j$ from the $N \times N$ matrix $A_m$ introduced in (13). $\bar{X}_m^{ij} \in \mathbb{R}^{N-2}$ is the unknown vector defined by removing the two known components $\bar{x}_i$ and $\bar{x}_j$ from the vector $\bar{X}_m$ in (12) and

$$P_m^{ij} = \dot{\varphi}_m(T)X(T) - \varphi_m(0)X(0) + A_m(:, i)\bar{x}_i + A_m(:, j)\bar{x}_j,$$

where $A_m(:, i)$ is the $i^{th}$ column vector of the matrix $A_m$ in (13). At this stage, $\lambda_m$, $S$ and $P_m^{ij}$ are unknown, they need to be determined to solve the linear system (14).

### 4 Identifiability

In this section, we prove that -under some reasonable assumptions- the observation operator $M[\lambda, S]$ introduced in (3) determines uniquely the two unknown elements $\lambda$ and $S$ of the admissible set $\mathcal{A}$ defining the source $\lambda S$ occurring in the system (2).
4.1 Estimation of $X(T)$

As indicated above, the first step is to determine the unknown final state $X(T)$ defining the vector $P_m$ in [11]-[12] from the measurements $d_k(t)$ at the nodes $k$ of a strategic set. We prove the following result:

**Theorem 4.1** Provided $\lambda$ and $S$ are two elements of the admissible set $A$, time records in $(0, T)$ of $X$ the solution of (2) taken in a strategic set of nodes identify uniquely $X(T)$.

**Proof.** For $n = 1, 2$, let $\lambda^{(n)}$ and $S^{(n)}$ be two elements of the admissible set $A$ and $X^{(n)}$ be the solution of the system (2) with the node source $\lambda^{(n)} S^{(n)}$ instead of $\lambda S$. Then, the variable $X = X^{(2)} - X^{(1)}$ solves the system:

$$\begin{cases} \ddot{X}(t) - \Delta X(t) = \lambda^{(2)}(t)S^{(2)} - \lambda^{(1)}(t)S^{(1)} & \text{in } (0, T) \\ X(0) = \dot{X}(T) = 0 \in \mathbb{R}^N \end{cases}$$

(16)

Since $\lambda^{(2)}$ and $\lambda^{(1)}$ are two admissible time-dependent source intensities, it follows that $\lambda^{(2)} = \lambda^{(1)} = 0$ in $(T^0, T)$. Thus, the variable $X$ satisfies the system (16) for all $T^* \in (T^0, T)$. Moreover, if the local time records of $X^{(2)}$ and of $X^{(1)}$ taken in a strategic set of nodes coincide in $(0, T)$ then the values of the state $X$ in all nodes of a strategic set is null in $(0, T)$. Therefore, from applying Lemma 2.2 we conclude that $X(T) = 0$ which means $X^{(2)}(T) = X^{(1)}(T)$.

Then, using the solution of (16) given in (51) it follows that $X(t) = (x_1(t), \ldots, x_N(t))^\top$ is redefined for all $t \in (T^*, T)$ by

$$X(t) = \left(y_1(T) + (t - T)\dot{y}_1(T)\right)v^1 + \sum_{n=2}^{N} \left(y_n(T) \cos(\omega_n(t - T)) + \frac{\dot{y}_n(T)}{\omega_n} \sin(\omega_n(t - T))\right)v^n$$

(17)

According to Theorem 4.1 we determine the coefficients $y_n(T)$ and $\dot{y}_n(T)$ for $n = 1, \ldots, N$ defining in (17) the state $X$ in $(T^*, T)$ by solving the minimization problem:

$$\min_{Y \in \mathbb{R}^{2N}} \frac{1}{2} \sum_{k \in \mathcal{E}} \left\|x_k - d_k\right\|^2_{L^2(T^*, T)}$$

(18)

where $x_k(t)$ is the $k$th component of $X(t)$, $Y = (y_1(T), \ldots, y_N(T), \dot{y}_1(T), \ldots, \dot{y}_N(T))^\top$ and $\mathcal{E}$ is a strategic set of nodes.

**Remark 4.2** From the proof of Theorem 4.1, it follows that a single strategic node is enough to identify uniquely $X(T)$ via the minimization problem (18).
4.2 Identifiability of the source

This result is given by the following identifiability theorem:

**Theorem 4.3** Let $m \in \mathbb{N}^*$ and $A_m$ be the $N \times N$ matrix introduced in \([13]\). Let $i, j$ be two nodes. If

1. the set \( \{i, j\} \) is strategic,
2. \( \langle \lambda, \varphi_m \rangle_{L^2(0, T)} \neq 0 \),
3. all \((N - 2) \times (N - 2)\) matrices defined by removing from $A_m$ its $i^{th}$, $j^{th}$ columns and any 2 rows are invertible,

then, the observation operator $M[\lambda, S]$ in \([2]\) identifies uniquely the two unknown elements $S$ and $\lambda$ of the admissible set $\mathcal{A}$ defining the source $\lambda S$ occurring in the system \([2]\).

**Proof.** For $n = 1, 2$, let $\lambda^{(n)}$ and $S^{(n)}$ be two elements of $\mathcal{A}$ and $X^{(n)}$ be the solution of \([2]\) with the source $\lambda^{(n)} S^{(n)}$ instead of $\lambda S$. We denote $X = (x_1, \ldots, x_N)^T$ the variable defined by $X = X^{(2)} - X^{(1)}$. Then, $X$ solves the system \([16]\) and in view of \([3]\), we have

$$M[\lambda^{(2)}, S^{(2)}] = M[\lambda^{(1)}, S^{(1)}] \implies x_i(t) = x_j(t) = 0 \forall t \in (0, T) \quad (19)$$

The right hand side of the implication in \((19)\) leads to the following results:

1. Applying Lemma \([2, 2]\), we get $X(T) = X(T) = 0$.
2. From \([12]\), we find $\bar{x}_i = \bar{x}_j = 0$.

Therefore, the adjoint system associated to $X$ is

$$-A_m^{i,j} \bar{X}_m^{i,j} = \lambda_m^{(2)} S^{(2)} - \lambda_m^{(1)} S^{(1)}, \quad (20)$$

where $\lambda_m^{(n)} = \langle \lambda^{(n)}, \varphi_m \rangle_{L^2(0, T)} \neq 0$ for $n = 1, 2$. The linear system \((20)\) contains $N$ equations with $(N - 2)$ unknowns. Furthermore, since $S^{(n=1,2)}$ belong to $\mathcal{A}$, the right hand side in \((20)\) contains at most two non-null components. Therefore, by taking out two equations containing the two eventual non-null right hand side terms, we end up with an homogeneous $(N - 2) \times (N - 2)$ linear system. Then, the $(N - 2) \times (N - 2)$ matrix is invertible from the third hypothesis of Theorem \([13]\) so that $\bar{X}_m^{i,j} = 0$ and thus,

$$\lambda_m^{(2)} S^{(2)} = \lambda_m^{(1)} S^{(1)} \implies S^{(2)} = S^{(1)} \quad \text{and} \quad \lambda_m^{(2)} = \lambda_m^{(1)} \quad (21)$$

We set $S^{(2)} = S^{(1)} = S$ in the system \([16]\). To prove that we have also $\lambda^{(2)} = \lambda^{(1)}$ in $(0, T)$, we expand the solution $X$ of \([16]\) in the orthonormal family $\{v^1, \ldots, v^N\}$ as in \([6]\) i.e.,

$$X(t) = \sum_{n=1}^{N} y_n(t)v^n.$$

Then, $y_{n=1,\ldots,N}$ solve

$$\begin{cases}
\dot{y}_n(t) + \omega_n^2 y_n(t) = (\lambda^{(2)}(t) - \lambda^{(1)}(t)) \langle S, v^n \rangle & \text{in} \ (0, T) \\
y_n(0) = \dot{y}_n(0) = 0
\end{cases} \quad (22)$$
Besides, we determine the two following fundamental solutions:

\[ y_1^0(t) = H(t) t \quad \text{solves} \quad \ddot{y}_1^0(t) = \delta(t) \]
\[ y_n^0(t) = \frac{1}{\omega_n} H(t) \sin(\omega_n t) \quad \text{solves} \quad \ddot{y}_n^0(t) + \omega_n^2 y_n^0(t) = \delta(t), \quad \forall n = 2, \ldots, N \]

(23)

where \( H \) is the Heaviside function and \( \delta \) is the Dirac mass. Thus, (23) gives the solution of (22) for all \( n = 1, \ldots, N \) and then, \( X \) the solution of (16) for \( S(2) = S(1) = S \) as follows:

\[
X(t) = \langle S, v^1 \rangle \int_0^t (\lambda^{(2)}(s) - \lambda^{(1)}(s))(t-s) ds \quad v^1 \\
+ \sum_{n=2}^{N} \left( \frac{\langle S, v^n \rangle}{\omega_n} \int_0^t (\lambda^{(2)}(s) - \lambda^{(1)}(s)) \sin(\omega_n(t-s)) ds \right) v^n \quad \text{in } (0, T)
\]

(24)

As from (19) we have \( x_{k=i,j}(t) = 0 \) for all \( t \in (0, T) \), it follows in view of (24) that

\[
\int_0^t (\lambda^{(2)}(s) - \lambda^{(1)}(s)) \Phi_{k=i,j}(t-s) ds = 0, \quad \forall t \in (0, T)
\]

where:

\[
\Phi_k(t) = \langle S, v^1 \rangle t v^1_k + \sum_{n=2}^{N} \frac{\langle S, v^n \rangle}{\omega_n} \sin(\omega_n t) v^n_k
\]

(25)

Suppose that \( \Phi_{k=i,j} = 0 \) in \( (t_k, \bar{t}_k) \) where \( 0 < t_k < \bar{t}_k < T \). Then, using similar techniques as employed in (51)-(57) it comes from (25) that \( \langle S, v^n \rangle v^n_{k=i,j} = 0 \) for all \( n = 1, \ldots, N \). That’s absurd since the set \( \{i, j\} \) is strategic. Hence, it follows that for \( k = i \) and/or \( k = j \) we have \( \Phi_k \neq 0 \) a.e. in \( (0,T) \). Therefore, from (25) and according to Titchmarsh Theorem on convolution of \( L^1 \) functions it comes that \( \lambda^{(2)} = \lambda^{(1)} \) in \( (0, T) \).

\[\blacksquare\]

### 4.3 Appropriate observation nodes

The third condition of Theorem 4.3 i.e. that all \( (N-2) \times (N-2) \) submatrices defined by removing from the \( N \times N \) matrix \( (\Delta + \mu_n I) \) the two columns \( i, j \) and any two rows are invertible may not always be satisfied. This depends on the topology of the graph and the choice of the observation nodes \( i, j \). In this subsection, we establish a result for a particular kind of graph showing that an arbitrary choice of observation nodes may not satisfy the third condition of Theorem 4.3. This result will guide the selection of the observation nodes. We observed numerically that selecting those nodes following the established guidelines leads to fulfill that required condition.

We introduce a joint for a graph.

**Definition 4.4** A joint is a node whose removal increases the number of disconnected subgraphs.
Figure 1: A connected graph (left) and a graph with two disconnected subgraphs (right).

In the example of Fig. 1, 4 is a joint of the graph on the left, since removing it gives a graph with two disconnected subgraphs. We have the following result.

**Theorem 4.5** Consider a graph $G$ with a joint at node $k$. Without loss of generality, we label the nodes $1, 2, \ldots N$ such that the sets $\{1, 2, \ldots k-1\}$ and $\{k+1, \ldots N\}$ correspond to two disconnected subgraphs of $G$. Choose two indices $i, j \leq k$ and two other indices $p, q \geq k$. Then the $(N-2) \times (N-2)$ matrix $A_{ij}^pq$ obtained by taking out the two columns $i, j$ and the two lines $p, q$ from $\Delta + \mu_m I$ is not invertible.

**Proof**
We assume that $k$ is the joint node. The matrix $A_m = \Delta + \mu_m I$ has the form

$$A_m = \begin{bmatrix}
1 & \cdots & k & k+1 & \cdots & N \\
& B & 0 & & & \\
& & 0 & & k-1 & k \\
& & & C & & \\
& & & & 0 & \\
& & & & &
\end{bmatrix}$$

There are no links between nodes in $\{1, \ldots, k-1\}$ and nodes in $\{k+1, \ldots, N\}$, hence there are two blocks of zeros in $A_m$. After removing columns $i$ and $j$ ($i, j \leq k$) and lines $p$ and $q$ ($p, q \geq k$) from $A_m$, we get the matrix $A_{ij}^{pq}$.
where \(d\) is a \(1 \times (k - 2)\) vector, \(e\) is a \(1 \times (N - k - 1)\) vector. The matrix \(A_{ij}^{pq}\) is block triangular. The coefficient 0 on the diagonal of \(A_{ij}^{pq}\) at position \((k - 1, k - 1)\) comes from the coefficient 0 at position \((k - 1, k + 1)\) of \(A_m\). The block triangular matrix \(A_{ij}^{pq}\) with a 0 block has a zero determinant and is therefore singular. ■

As a consequence of this theorem, the third condition of the identifiability theorem 4.3 will not be satisfied unless we have an observation node in each part of the graph that will appear as a disconnected subgraph once a joint is deleted.

Such part of the graph is called a maximal bi-connected component in the graph theory literature [17]. All the joints of a graph can be efficiently determined using Tarjan’s algorithm [18] in time \(O(M)\) where \(M\) is the number of edges.

An example is the graph shown in Fig. 2 whose node 5 is a joint. As a result, putting sensors on any two nodes on the left, say \(i = 1, j = 2\), will yield a singular matrix in theorem 4.3 if we take out two lines in the set \{6, \ldots, 9\}. However, placing the two observation nodes such that one node is on each side of the joint, for example \(i = 1\) and \(j = 7\), fulfills the third condition of Theorem 4.3.

In Fig. 3 we show another example where nodes 3 and 4 are joints. Taking out the joints yields four disconnected subgraphs 1, 2, 5, 6. Then we will need to put sensors on nodes 1, 2, 5, 6. In other words, there should be a sensor for each maximal bi-connected component of the graph.
Figure 3: A connected graph with two joints.

5 Identification

Let $M[\lambda, S]$ be the observation operator introduced in (3) defined from recording the state $X$ solution of the system (2) in the set of two distinct nodes $\{i, j\}$. We consider that the final state $X(T)$ has been determined by the least-square procedure of section 4.1 so that the vector $P_m$ in (12) is known. Assuming the three conditions of Theorem 4.3 to be satisfied, we focus in this section on developing an identification method that determines the two unknown elements $\lambda$ and $S$ of the admissible set $A$ defining the source $\lambda S$ occurring in the system (2). To this end, we proceed as follows: Firstly, we localize the node source position $S$ in the network. Secondly, we propose two different algorithms to identify its emitted signal $\lambda \in L^2(0, T)$.

5.1 Localization of the node source position $S$

Since the second assumption of Theorem 4.3 holds, it follows that all reduced $(N - 2) \times (N - 2)$ matrix obtained by removing any two rows from the rectangular $N \times (N - 2)$ matrix $A_{i,j}^{m}$ in (14) is invertible. Therefore, each $(N - 2)$ equations of the linear system (14)-(15) admit a unique solution. Moreover, as the vector $\bar{X}_{i,j}^{m}$ defined from the solution $X$ of the system (2) solves the $N$ equations in (14)-(15) then, all reduced systems of $(N - 2)$ equations from (14)-(15) admit the same unique solution $\bar{X}_{i,j}^{m}$.

Hence, to localize the node source position $S$ we proceed as follows: given $m \in \mathbb{N}^*$ such that $\langle \lambda, \varphi_m \rangle_{L^2(0,T)} \neq 0$, repeat solving a reduced $(N - 2) \times (N - 2)$ linear system obtained by removing two equations labelled $l_1$ and $l_2$ from the system (14)-(15) and omitting the source term $\lambda_m S$ i.e., solving only $(N - 2)$ equations from the following system:

$$-A_{m}^{i,j} \bar{X}_{m}^{i,j} = P_{m}^{i,j}$$  \hspace{1cm} (26)

As long as different choices $l_1, l_2$ and $l'_1, l'_2$ give the same solution $\bar{X}_{m}^{i,j}$, this means the source equation i.e., the equation containing in its right hand side the unique non-null component of the vector $\lambda_m S$, doesn’t belong to the selected $(N - 2)$ equations. However, as soon as a choice $l'_1, l'_2$ gives a solution different from the one obtained with $l_1, l_2$, then the source equation is among the $(N - 2)$ solved equations. This process leads to determine the label $l$ of the source equation in (14). Moreover, $l$ corresponds to the position of the unique non-null component in $S$ which defines the node source.
5.2 Identification of the time-dependent signal $\lambda$

Assuming to be known the position $S$ of the node source $\lambda S$ occurring in the problem (2), we develop two different methods for identifying the unknown time-dependent signal $\lambda \in L^2(0,T)$ emitted by the involved node source: The first method transforms the identification of $\lambda$ into solving a deconvolution problem whereas the second method consists of determining the expansion of $\lambda$ in the orthonormal family of $L^2(0,T)$ made by the normalized eigenfunctions $\{\varphi_m\}$ of the Sturm-Liouville problem introduced in (9)-(10).

5.2.1 First method: Deconvolution

We split the solution $X$ of the system (2) as follows:

$$X(t) = X^S(t) + X^0(t) \quad \text{for all } t \in (0,T)$$

where the variables $X^S$ and $X^0$ solve the two following systems:

$$\begin{cases}
\begin{align*}
\ddot{X}^S(t) - \Delta X^S(t) &= \lambda(t)S \quad \text{in } (0,T) \\
X^S(0) &= \dot{X}^S(0) = \vec{0}
\end{align*}
\end{cases} \quad \text{and} \quad \begin{cases}
\ddot{X}^0(t) - \Delta X^0(t) &= 0 \quad \text{in } (0,T) \\
X^0(0) &= a \quad \text{and} \quad \dot{X}^0(0) = b
\end{cases}$$

where $a, b$ are exactly the initial conditions of the original problem (2). Using the fundamental solutions obtained in (23) and expanding the variable $X^S$ in the orthonormal family $\{v^1, \ldots, v^n\}$ it follows, as in (24), that

$$X^S(t) = \langle S, v^1 \rangle \int_0^t \lambda(s)(t-s)ds v^1 + \sum_{n=2}^N \frac{\langle S, v^n \rangle}{\omega_n} \int_0^t \lambda(s) \sin(\omega_n(t-s))ds v^n, \quad \text{in } (0,T) \quad (27)$$

Afterwards, in view of (25) and according to (28), it comes that for all $k \in \{1, \ldots, N\}$ the $k^{th}$ component $x^S_k$ of the variable $X^S = (x^S_1, \ldots, x^S_N)^\top$ is given by

$$x^S_k(t) = \int_0^t \lambda(s)\Phi_k(t-s)ds \quad \forall \, t \in (0,T)$$

where:

$$\Phi_k(t) = \langle S, v^1 \rangle tv^1_k + \sum_{n=2}^N \frac{\langle S, v^n \rangle}{\omega_n} \sin(\omega_n t)v^n_k$$

Therefore, from selecting an observation node $k \in \{i, j\}$ such that $\Phi_k(t) \neq 0$ for almost all $t \in (0,T)$ and since we have: $x^S_k(t) = x_k(t) - x^0_k(t)$ for all $t \in (0,T)$ then, the unknown signal $\lambda \in L^2(0,T)$ occurring in the problem (2) is subject to:

$$\int_0^t \lambda(s)\Phi_k(t-s)ds = x_k(t) - x^0_k(t) \quad \forall t \in (0,T)$$

(30)
where \(x_k^0\) is the \(k^{th}\) component of \(X^0\). Hence, the identification of \(\lambda\) can be transformed into solving the deconvolution problem associated to (30). Indeed, given a desired number of time steps \(M\), we employ the regularly distributed discrete times: \(t_m = m\Delta t\) for \(m = 0, \ldots, M\) where \(\Delta t = T/M\). Then, using the trapezoidal rule we get

\[
\int_0^{t_{m+1}} \lambda(s)\Phi_k(t_{m+1} - s)ds = \sum_{\ell=0}^{m} \int_{t_{\ell}}^{t_{\ell+1}} \lambda(s)\Phi_k(t_{m+1} - s)ds \\
\approx \frac{\Delta t}{2} \sum_{\ell=0}^{m} \left( \lambda(t_{\ell})\Phi_k(t_{m+1} - t_{\ell} + \lambda(t_{\ell+1})\Phi_k(t_{m+1} - t_{\ell}) \right) \tag{31}
\]

where according to (29), we used \(\Phi_k(0) = 0\) and assumed that \(\lambda(0) = 0\). Afterwards, using the notation \(\lambda_m \approx \lambda(t_m)\) and in view of (31) we obtain a discrete version of the deconvolution problem associated to (30) that leads to the following recursive formula:

\[
\lambda_m = \frac{1}{\Phi_k(t_1)} \left( \frac{x_k(t_{m+1}) - x_k^0(t_{m+1})}{\Delta t} - \sum_{\ell=1}^{m-1} \lambda_\ell \Phi_k(t_{m+1} - t_\ell) \right), \quad \forall \ m \geq 1 \tag{32}
\]

Moreover, by denoting \(\Lambda = (\lambda_1, \ldots, \lambda_M)^\top\) and \(Q = (x_k(t_2) - x_k^0(t_2), \ldots, x_k(t_M) - x_k^0(t_M))^\top\) it follows from (31) that the identification of the unknown emitted signal \(\lambda\) is transformed into solving the linear system:

\[
B\Lambda = \frac{1}{\Delta t}Q \tag{33}
\]

where \(B\) is the \(M \times M\) lower triangular matrix defined for \(m = 1, \ldots, M\) by

\[
\begin{cases}
B_{m\ell} = \Phi_k(t_{m+1} - t_\ell), & \text{for } \ell = 1, \ldots, m \\
B_{m\ell} = 0, & \text{for } \ell = m + 1, \ldots, M 
\end{cases} \tag{34}
\]

In practice, the identification of \(\Lambda\) using the recursive formula in (32) which corresponds to the straightforward solution of the triangular linear system (33) could not yield a stable approximation of the unknown source signal \(\lambda\). To regularize the system, we employed the Tikhonov method that replaces the linear system (33)-(34) by a penalized least-squares problem under the form:

\[
\min_{\Lambda \in \mathbb{R}^M} \frac{1}{2} ||B\Lambda - \frac{1}{\Delta t}Q||^2_2 + \frac{r}{2} ||\Lambda||^2_2 \tag{35}
\]

where \(r > 0\) is the regularization parameter, see for example [19]. A simpler regularization adding \(r\) to the diagonal of the triangular matrix \(B\) gave the best results.
5.2.2 Second method: Expansion in a Fourier basis

This method consists in determining a number of coefficients \( \lambda_m = \langle \lambda, \varphi_m \rangle_{L^2(0,T)} \) defining the expansion of the unknown emitted signal \( \lambda \) in the orthonormal family \( \{ \varphi_m \} \) \(^{(9)-(10)}\). Given \( m \in \mathbb{N}^* \) a sufficiently large number of \( m \in \mathbb{N}^* \) such that the third condition of Theorem 4.3 is satisfied we can estimate

\[
\lambda(t) \approx \sum_m \lambda_m \varphi_m(t). \tag{36}
\]

The procedure to obtain \( \lambda_m \) is the following

1. Since \( S \) is known, select from \(^{(14)}\): \( (N - 2) \) equations that do NOT contain the source equation i.e., the equation involving \( \lambda_m \) in its right-hand side term.
2. Solve the selected \( (N - 2) \times (N - 2) \) linear system and determine \( \bar{X}_{i,j}^m \).
3. Inject the computed \( \bar{X}_{i,j}^m \) in the source equation of \(^{(14)}\) and deduce \( \lambda_m \).

5.3 Algorithm

For the clarity of our presentation, we summarize in the following algorithm the different steps describing the identification method developed in the present paper to determine from the observation operator \( M[\lambda, S] \) introduced in \(^{(3)}\) the two unknown elements \( \lambda \) and \( S \) defining the source \( \lambda S \) occurring in the problem \(^{(2)}\).

Algorithm

1. Determine a strategic set of two nodes \( \{i, j\} \) satisfying the conditions of Theorem 4.3.
2. Compute \( X(T) \) by solving the minimization problem \(^{(18)}\) and form the rhs of the over-determined linear system \(^{(26)}\).
3. Repeat solving \( (N - 2) \) equations from the linear system \(^{(26)}\) until determining the source vector \( S \) as explained in subsection 5.1.
4. Identify the time-dependent emitted signal \( \lambda \) using one of the two methods suggested in subsection 5.2.

6 Numerical experiments

To check the effectiveness of the graph source identification method developed in the present study, we carry out numerical experiments. We first select the following \( N = 5 \) nodes graph and choose 1, 2 as our observation nodes. This will enable us to study the influence of the different parameters.
The associated $5 \times 5$ graph Laplacian matrix is given by

$$
\Delta = \begin{pmatrix}
-2 & 0 & 0 & 1 & 1 \\
0 & -3 & 1 & 1 & 1 \\
0 & 1 & -2 & 1 & 0 \\
1 & 1 & 1 & -4 & 1 \\
1 & 1 & 0 & 1 & -3 \\
\end{pmatrix}.
$$

(37)

Furthermore, the eigenvalues of (37) are:

$$0, \ -3 + \sqrt{2}, \ -3, \ -3 - \sqrt{2}, \ -5.$$

so that the $\omega_n, \ n = 1, \ldots, N$ are

$$\omega_1 = 0, \ \omega_2 = 1.259, \ \omega_3 = 1.732, \ \omega_4 = 2.10 \ \text{and} \ \omega_5 = 2.236 \quad (38)$$

The corresponding normalized eigenvectors $v^n, \ n = 1, \ldots, 5$ of (37) are

$$v^1 = \frac{1}{\sqrt{5}}(1,1,1,1,1)^T, \quad v^2 = \frac{1}{2\sqrt{2} - \sqrt{2}}(1,1 - \sqrt{2},-1,0,-1 + \sqrt{2})^T,$$

$$v^3 = \frac{1}{2}(1,-1,0,1)^T, \quad v^4 = \frac{1}{2\sqrt{2} + \sqrt{2}}(1,1 + \sqrt{2},-1,0,-1 - \sqrt{2})^T,$$

$$v^5 = \frac{1}{2\sqrt{5}}(1,1,1,-4,1)^T. \quad (39)$$

Besides, according to Definition 2.1 the set of two nodes $\{i = 1, j = 2\}$ is strategic. In fact, we observe from (39) that $v^n_{i=1,\ldots,N} \neq 0$ and $v^n_{j=1,\ldots,N} \neq 0$ which implies that the node $i = 1$ as well as the node $j = 2$ are both strategic. To carry out numerical experiments, we generate synthetic state records by solving the forward problem (2) using the time-dependent source intensity function defined by

$$\lambda(t) = \frac{\beta}{2} \left( 1 + \tanh\left(\frac{t - t_L}{w}\right) - (1 + \tanh\left(\frac{t - t_R}{w}\right)) \right), \quad (40)$$

Figure 4: 5 nodes graph
for all $t \in (0, T_0)$ whereas $\lambda(t) = 0$ for all $t \geq T_0$. We set the observation operator $M[\lambda, S]$ at the strategic set of two nodes $\{i = 1, j = 2\}$ as in (3). To this end, we used in (40) the coefficients:

$$\beta = 3, \quad t_L = 0.3T, \quad w = 0.01T, \quad t_R = 0.6T, \quad T = 100, \quad s = 3.$$  \hfill (41)

Then the time active limit $T_0$ in (3) is $T_0 = 70$. Moreover, given a desired number of time steps $\mathcal{M}$, we employ the regularly distributed discrete times $t_m = m\Delta t$ for $m = 0, \ldots, \mathcal{M}$ where $\Delta t = T/\mathcal{M}$ and solve the forward problem (2), (40) using a variable step Runge-Kutta solver of order 4-5.

First, we consider the first step of the Algorithm., the estimation of the final state $X(T)$ using least squares.

### 6.1 Least squares fit

The two nodes $i = 1$ and $j = 2$ defining the observation operator $M[\lambda, S]$ are both strategic. Then it follows from Remark 4.2 that using only the state records $x_k(t)$ for $t \in (T^*, T)$ and $k = 1$ or $k = 2$ yields uniqueness of the sought final state vector $X(T)$. Then, using the regularly distributed discrete times: $t_m = m\Delta t$ for $m = 0, \ldots, \mathcal{M}$ where $\Delta t = T/\mathcal{M}$ and assuming there exists $\mathcal{M}^*$ such that $T^* = \mathcal{M}^*\Delta t$, we sample the observed data $d_k(t) \equiv x_k(t)$ as $d_k(t_m)$ for $m = 1, \ldots, \mathcal{M}$. Therefore, we aim to identify the final state vector $X(T) = (x_1(T), x_2(T), \ldots, x_N(T))^\top$ by solving the least squares problem (18).

We generate synthetic state records $D_k$ by solving the problem (2) for a source located at node $s = 3$ with the time-dependent intensity function $\lambda$ introduced in (40). For simplicity, we set in the least squares problem (18): $\mathcal{M} = 100$ and $T^* = T_0$ which implies $\mathcal{M}^* = 70$. The relative error $r_k$ between the generated (exact) final state vector $X_{ex}(T)$ and the identified final state vector $X_{id}(T)$

$$r_k = \frac{\|X_{id}(T) - X_{ex}(T)\|_2}{\|X_{ex}(T)\|_2},$$  \hfill (42)

is presented in Table 1 for different observation nodes $k$ in single and double precision.

| Observation node | $k = 1$ | $k = 2$ | $k = 3$ | $k = 5$ |
|------------------|--------|--------|--------|--------|
| $r_k$ double     | $9.87 \times 10^{-6}$ | $1.08 \times 10^{-5}$ | $1.46 \times 10^{-5}$ | $9.74 \times 10^{-6}$ |
| $r_k$ single     | $4 \times 10^{-5}$ | $3.59 \times 10^{-5}$ | $4.19 \times 10^{-5}$ | $4 \times 10^{-5}$ |

Table 1: Relative error $r_k$ in the least-square estimation of $X(T)$ for different observed nodes $k = 1, 2, 3, 5$ in single and double precision.

The results presented in Table 1 confirm our claim in Remark 4.2; they show that the state records at a single strategic node $k$ lead to identify with accuracy the sought final state vector $X(T)$. 

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16
However, if the node $k$ is not strategic, we show with the following example that the solution of the non regularized least-square problem from [13] may not exist or not be unique. Consider the node $k = 4$ which is not strategic because from (32) $v_k^2 = v_k^3 = v_k^4 = 0$. The least square consists in minimizing with respect to the parameters $y_n(T), \dot{y}_n(T), n = 1, \ldots, N$ the following function:

$$E_k(y_{n=1,\ldots,N}(T), \dot{y}_{n=1,\ldots,N}(T)) = \sum_{m=M^*}^{M} (x_k(t_m) - d_k(t_m))^2,$$

where $x_k$ is defined as in [17] by

$$x_k(t) = \left(y_1(T) + (t - T)\dot{y}_1(T)\right)v_k^1 + \sum_{n=2}^{N} \left(y_n(T) \cos(\omega_n(t - T)) + \frac{\dot{y}_n(T)}{\omega_n} \sin(\omega_n(t - T))\right)v_k^n,$$

The function $E_k$ in (50) can be written as

$$E_k(Y) = \|A_kY - D_k\|_2^2,$$

where $Y = (y_1(T), \dot{y}_1(T), \ldots, y_N(T), \dot{y}_N(T))^\top$, $D_k = (d_k(t_{M^*}), \ldots, d_k(t_M))^\top$ and

$$A_k = \begin{pmatrix}
v_k^1 & v_k^1 & \cos(\omega_1(t_{M^*} - T))v_k^2 & \sin(\omega_1(t_{M^*} - T))v_k^3 & \cdots & \cos(\omega_N(t_{M^*} - T))v_k^N & \sin(\omega_N(t_{M^*} - T))v_k^N \\
v_k^1 & v_k^1 & \cos(\omega_2(t_{M^* + 1} - T))v_k^2 & \sin(\omega_2(t_{M^* + 1} - T))v_k^3 & \cdots & \cos(\omega_N(t_{M^* + 1} - T))v_k^N & \sin(\omega_N(t_{M^* + 1} - T))v_k^N \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
v_k^1 & v_k^1 & \cos(\omega_N(t_{M} - T))v_k^2 & \sin(\omega_N(t_{M} - T))v_k^3 & \cdots & \cos(\omega_N(t_{M} - T))v_k^N & \sin(\omega_N(t_{M} - T))v_k^N \\
\end{pmatrix}.$$ (46)

For $k = 4$, the matrix $A_{k=4}$ is exactly singular because it has three columns of zeros, corresponding to $v_4^2, v_4^3$ and $v_4^4$. Then the linear system has no solution or an infinite number of solutions and no information can be obtained on $y_2, \dot{y}_2, y_3, \dot{y}_3, y_4, \dot{y}_4$. This means that the least square problem for the $k = 4$ non strategic node cannot be solved.

### 6.2 Localization of the unknown node source

We now assume the final state vector $X(T)$ to be known and focus on the localization of the unknown node source using the time records in $(0, T)$ of the state $x_1(t)$ and $x_2(t)$ at the two strategic nodes $\{i = 1, j = 2\}$.

The adjoint problem [14] is the following linear system of $N = 5$ equations and $N - 2 = 3$ unknowns:

$$-A^{1,2}X^{1,2}_m = P^{1,2}_m + \lambda_m S,$$ (47)

where $X^{1,2}_m \in \mathbb{R}^{N-2}$ and the coefficient $P^{1,2}_m$ is given by [15].

We carried out numerical experiments on the localization of the node source from state records $x_1(t)$ and $x_2(t)$ for $t \in (0, T)$ generated by a source occurring at node $s = 3$ i.e., $S = (0, 0, 1, 0, 0)^\top$ forcing the graph with the time-dependent intensity function $\lambda$ introduced in [10]. We applied the procedure described above i.e., repeat taking out two equations $l_1$ and $l_2$ from the $N$ equations in [17] and solve the remaining $(N - 2)$
equations. The results are presented in the following table where for each couple \((l_1, l_2)\) we give the solution \(\bar{X}_{m}^{1,2}\) together with the error between the computed solution \(\bar{X}_{m}^{1,2}\) and the exact solution \(\bar{X}_{ex}^{1,2}\) generated by solving the direct problem (2) with \(\lambda\) and \(S\).

|   |   | \(\bar{X}_{m}^{1,2}\) | \(\|\bar{X}_{m}^{1,2} - \bar{X}_{ex}^{1,2}\|_2\) |
|---|---|----------------------|---------------------------------|
| 1 | 2 | (2020.704778 , 2020.72667 , 2020.732166 )\(^T\) | 3.58 \(10^{-3}\) |
| 1 | 4 | (2023.289145 , 2025.89286 , 2022.454794 )\(^T\) | 2.75 \(10^{-3}\) |
| 1 | 5 | (2021.652750 , 2022.621687 , 2027.362363 )\(^T\) | 4.19 \(10^{-3}\) |
| 2 | 4 | (2021.652808 , 2022.621803 , 2021.364083 )\(^T\) | 2.97 \(10^{-3}\) |
| 2 | 5 | (2020.985776 , 2021.288396 , 2022.697489 )\(^T\) | 3.43 \(10^{-3}\) |
| 4 | 5 | (2027.650916 , 2034.612098 , 2009.373787 )\(^T\) | 8.39 \(10^{-3}\) |

Table 2: Identification of a source at node \(s = 3\): solution \((\bar{X}_3, \bar{X}_4, \bar{X}_5)\) of the overdetermined system (26) when lines \(l_1, l_2\) are taken out. The calculations were done with 64 bit arithmetics and the parameters is \(m = 1\).

The numerical results presented in Table 2 show that the source localization procedure developed in the present study leads to identify with accuracy the node source generating the state records \(x_1(t), x_2(t)\) for \(t \in (0, T)\). Indeed, the first part in Table 2 shows that as long as the source equation \(s = 3\) is not among the two equations \(l_1\) and \(l_2\) taken out from the \(N\) equations in (17), the solutions of the remaining \((N - 2)\) equations are different and the error, about \(10^{-3}\) is relatively important. However, when the source equation \(s = 3\) is among the two lines \(l_1\) or \(l_2\) as in the second part of Table 2, we observe that all remaining \((N - 2)\) equations admit about the same solution \(\bar{X}_{m}^{1,2}\) that fits closely the exact solution \(\bar{X}_{ex}^{1,2}\) because the error is about \(10^{-5}\). Similar calculations done with 32 bits arithmetic yield an error of \(10^{-5}\) when \(l_1 = s\) or \(l_2 = s\), otherwise the error is about \(10^{-3}\), see Table 3.

The results were obtained for a source at node \(s = 3\). Placing the source at the other nodes of the 5-nodes graph yielded similar results to those presented in Table 2.

We now examine the influence of the order \(m\) of the eigenfunction \(\varphi_m\) needed to form the adjoint problem. We repeated the calculations shown in Table 2 for \(m = 5\) and \(m = 10\).
and the results are shown in tables 3 and 4 respectively.

Table 3: Localization of the node source $s = 3$ with $m = 5$; the other parameters are the same as in Table 2.
Table 4: Localization of the node source \( s = 3 \) with \( m = 10 \); the other parameters are the same as in Table 2.

Compared to Table 2, the numerical results of Table 3 show that the error between the exact solution and the identified \( \bar{X}_{12} \) has increased from \( 10^{-7} \) to \( 10^{-5} \). For \( m = 10 \) shown in Table 4, there is no significant difference between the error when the source line \( s = 3 \) is included and when it is not. This suggests to use small values of \( m \) for the source localization procedure. In fact, a large \( m \) corresponds to a higher frequency eigenfunction \( \varphi_m \) from (10) so that the integrands defining the terms involved in the right hand side \( P_{l_{1},j}^{m} \) in (26)-(15) are highly oscillatory. The integrals become therefore less accurate.

To conclude this section, note that to identify the unknown source node, we developed a simple algorithm where for all lines \( l_1 \), we pick two other lines \( l_2 \) and \( l_3 \). For the two couples of lines, \((l_1, l_2)\) and \((l_1, l_3)\), we extract the two lines of (14) and solve the reduced system. If the norm of the difference of the solutions is smaller than a threshold, we retain \( l_1 \) as the node source. This yields a complexity \( \mathcal{O}(N^4) \), \( N \) for the sweep in \( l_1 \) and \( N^3 \) for solving the reduced linear system. We used this naive algorithm successfully in preliminary blind tests -where the source was not known to the user- on a 14-node network.

### 6.3 Identification of the time-dependent source signal \( \lambda \)

In this last part of our numerical experiments, we assume the source position \( S \) to be now known and aim to identify its time-dependent intensity function \( \lambda \) using the two
developed methods: Deconvolution and expansion in the complete orthonormal family \( \{\varphi_m\}_m \). For both methods, we use the same state records \( x_1(t) \) and \( x_2(t) \) for \( t \in (0,T) \) generated by a source occurring in node \( s = 3 \) i.e., \( S = (0,0,1,0,0)^\top \) forcing the graph with the time-dependent intensity function \( \lambda \) introduced in (10). In addition, we use the already introduced regularly distributed discrete times: \( t_m = m\Delta t \) for \( m = 0, \ldots, M \) where \( \Delta t = T/M \) for \( M = 100 \).

### 6.3.1 Identification of \( \lambda \) using deconvolution

This first method identifies the time-dependent source intensity function \( \lambda \) by solving the discrete version of the deconvolution problem obtained in (30). This leads to determine \( \lambda \) in (49) i.e., applying the deconvolution method using simulated measures generated by \( \lambda(t_\tau) \) where \( \tau \) is given by (38) are much

\[
\begin{align*}
\begin{pmatrix}
\Phi_k(t_1) & 0 & \ldots & 0 \\
\Phi_k(t_2) & \Phi_k(t_1) & 0 & \ldots & 0 \\
\Phi_k(t_3) & \Phi_k(t_2) & \Phi_k(t_1) & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\Phi_k(t_M) & \Phi_k(t_{M-1}) & \Phi_k(t_{M-2}) & \Phi_k(t_{M-3}) & \ldots & \Phi_k(t_1)
\end{pmatrix}
\begin{pmatrix}
\lambda^1 \\
\lambda^2 \\
\lambda^3 \\
\vdots \\
\lambda^M
\end{pmatrix}
= 
\begin{pmatrix}
b_k^2 \\
b_k^3 \\
b_k^4 \\
\vdots \\
b_k^{M+1}
\end{pmatrix}
\end{align*}
\] (48)

where the right hand side is given by the state record \( x_k(t) \) for \( t \in (0,T) \) where \( k = 1 \) or \( k = 2 \), \( b_k^m = (x_k(t_m) - x_k^0(t_m))/\Delta t \) (from (30)) and where \( \Phi_k(t_m) \) is given by (29).

We carry out numerical experiments on the identification of the time-dependent signal \( \lambda \) emitted by the already localized node source \( S \) using the deconvolution method developed in (32)-(35). For each experiment, we generate simulated measures by solving the system (2) with a different kind of signal \( \gamma = \lambda(t) \), \( t \in \{1,2,3\} \) where \( \lambda_1 \) is given by (10) and (11) and

- \( \lambda_2(t) = \sin(\pi t/T^0) \), \( \forall t \in (0,T^0) \) and \( \lambda_2(t) = 0 \), \( \forall t \geq T^0 \)
- \( \lambda_3(t) = \sum_{n=1}^{3} c_n e^{-\alpha_n(t-\tau_n)^2}, \forall t \in (0,T^0) \) and \( \lambda_3(t) = 0 \), \( \forall t \geq T^0 \) (49)

where \( \beta = 2 \) and \( c_1 = 1.2, c_2 = 0.4, c_3 = 0.6, \alpha_1 = 10^{-6}, \alpha_2 = 5 \times 10^{-5}, \alpha_3 = 10^{-6} \) and \( \tau_1 = 4500, \tau_2 = 6500, \tau_3 = 8500 \). We use here a large time interval and the same sampling rate as in the previous sections,

\[
T = 14400, \quad T^0 = 10800 \quad \text{and} \quad M = 14400.
\]

Note that here the typical periods of oscillation of the network as given by (35) are much smaller than \( T \).

We start by presenting the numerical results on the identification of the first kind of signal in (49) i.e., applying the deconvolution method using simulated measures generated by \( \lambda = \lambda_1 \) and emitted by a source located in the node 3 i.e., \( S = (0,0,1,0,0)^\top \). First, we
present in Figures 5 and 6 the curve, labeled "Simulation signal", of \( \lambda_1 \) introduced in (40) as well as the curve of the identified signal using measures taken at the observation node \( k = 1 \) and subject to different intensities of a Gaussian noise.

![Figure 5](image5.png)  
**Figure 5:** (a) Gaussian Noise 0%, Error \( \lambda_1 = 0.04 \)  
(b) Gaussian Noise 1%, Error \( \lambda_1 = 0.08 \)

![Figure 6](image6.png)  
**Figure 6:** (c) Gaussian Noise 3%, Error \( \lambda_1 = 0.09 \)  
(d) Gaussian Noise 5%, Error \( \lambda_1 = 0.11 \)

Second, we identify \( \lambda_1 \) using measures taken for each experiment in a different observation node \( k \in \{2, 3, 4, 5\} \) and subject to a Gaussian noise of intensity 3%. The results are presented in Table 5 where we indicate for each observation node \( k \) the corresponding \( L^2 \) relative error:

\[
\text{Error}_l = \frac{\| \Lambda_l - \Lambda_l^{\text{ident}} \|_2}{\| \Lambda_l \|_2} \tag{50}
\]

where \( \| \cdot \|_2 \) designates the euclidean norm, \( \Lambda_l = (\lambda_l(t_1), \ldots, \lambda_l(t_M)) \) and \( \Lambda_l^{\text{ident}} = (\lambda_1^l, \ldots, \lambda_M^l) \) with \( \lambda_i^l \) is the identified value of \( \lambda_i(t_i) \).
The numerical results presented in Figures 5 and 6 show that the identification method based on deconvolution developed in the present paper leads to identify the emitted signal $\lambda_1$ and is relatively stable with respect to the introduction of a noise on the measures. Then, we give in the following table the $L^2$ relative error computed from (50) between the ”Simulation signal” and the identified signal while using measures taken at a different observation node $k$:

| Observor node | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ |
|---------------|---------|---------|---------|---------|---------|
| Error$_1$     | 0.093   | 0.086   | 0.072   | -       | 0.096   |

Table 5: Identification of $\lambda_1$: Source $S = (0, 0, 1, 0, 0)^\top$ and Gaussian Noise 3%

Figures 7 and 8 present the curve, labeled ”Simulation signal”, of the second kind of signal $\lambda_2$ defined in (49) and the curve of the identified signal obtained by applying the deconvolution method on measures, taken at the observation node $k = 5$, generated by $\lambda = \lambda_2$ and a source located at the node 2 i.e., $S = (0, 1, 0, 0, 0)^\top$.

![Figure 7](image)

Figure 7: (a) Gaussian Noise 0%, Error$_2 = 0.04$  
(b) Gaussian Noise 1%, Error$_2 = 0.07$
The $L^2$ relative error on the identification of $\lambda_2$ while changing the observation node $k$ is given by

| Observer node | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ |
|---------------|---------|---------|---------|---------|---------|
| Error$_2$     | 0,095   | 0,128   | 0,111   | -       | 0,109   |

Table 6: Identification of $\lambda_2$: Source $S = (0, 1, 0, 0, 0)^\top$ and Gaussian Noise 3%
The numerical results in Figures 9 and 10 give the curve, labeled "Simulation signal", of the third kind of signal $\lambda_3$ introduced in (49) and the curve of the identified signal obtained by applying the deconvolution method on measures, taken in the observation node $k = 3$, generated by $\lambda = \lambda_3$ and a source located at the node 1 i.e., $S = (1, 0, 0, 0, 0)^\top$. The $L^2$ relative error on the identification of $\lambda_3$ while changing the observation node $k$ is given by

| Observer node | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ |
|---------------|---------|---------|---------|---------|---------|
| Error$_3$     | 0.098   | 0.088   | 0.071   | -       | 0.113   |

Table 7: Identification of $\lambda_3$: Source $S = (1, 0, 0, 0, 0)^\top$ and Gaussian Noise 3%

The numerical results obtained from the identification of the different signals $\lambda_{l=1,2,3}$ introduced in (49) show that the identification method based on deconvolution developed in the present study enables to identify the unknown emitted signal $\lambda$ using measures taken in any observation node $k$ except the node $k = 4$. In fact, the numerical difficulties encountered while identifying the emitted signal using the observation node $k = 4$ were expected since the latter is not a strategic node and in view of the application of Titchmarsh’s Theorem in the last part of the proof of Theorem 4.2.

The noisy tails observed for added noise are due to the amplification of the noise by the numerical errors. Despite them, the signal $\lambda(t)$ is well approximated.

The numerical results presented in this subsection on the identification of the emitted signal $\lambda$ were obtained solving the triangular linear system (48). The diagonal of this system is $\Phi_k(t_1)$ which is close to 0. To obtain convergence of the iteration, we replaced the term $\Phi_k(t_1)$ by $\Phi_k(t_1) + r$ with $r > 0$ which can be seen as a regularisation parameter. In the absence of noise, $r \approx 1$ while with the noise present we took $r \approx 10$. 
At the beginning of the subsection, we indicate that $\lambda(t)$ was chosen to vary slowly compared to the typical periods of the network. This is the most favorable situation to reconstruct $\lambda(t)$. When this forcing varies on time scales comparable to the typical periods, the identification of $\lambda(t)$ can also be done but it is less accurate.

### 6.3.2 Identification of $\lambda$ using Fourier basis

As introduced in section 4, the second way of identifying the source signal $\lambda$ consists in using the over-determined linear system (26)-(15) once the source position $S$ is localized to compute the coefficients $\lambda_m = \langle \lambda, \varphi_m \rangle_{L^2(0,T)}$ for $m = 1, \ldots, M$. To this end, since the source position was determined in the previous subsection i.e., $S = (0, 0, 1, 0, 0)^T$ it follows that in the linear system (17)-(15) the third equation contains the source term $\lambda_m$. Therefore, we proceed as follows:

For $m = 1$ to $M$ do

- Compute the right-hand side vector $P_{m}^{1,2}$ in (15).
- Solve 3 equations other than the third equation (which contains the source term) from the over-determined linear system (17) and compute $\tilde{X}_{m}^{1,2}$.
- Use $\tilde{X}_{m}^{1,2}$ in the third equation of the system (17) to deduce the coefficient $\lambda_m$.
- End For

Hence, the identified source signal is defined by

$$\lambda(t) = \sum_{m=1}^{M} \lambda_m \varphi_m(t),$$

for all $t \in (0, T)$. We applied this procedure to the data generated in section 6.2 and obtained an approximation of the signal, using the first seven harmonics. For this rapid $\lambda(t)$, both the Fourier and the deconvolution methods provided an estimate of the signal.

In the numerical experiments that we carried out, the Fourier method of reconstructing the unknown source intensity function $\lambda$ appeared to work well. Nevertheless, this second method admits the two following disadvantages with respect to the deconvolution method. First, to obtain an accurate approximation of $\lambda$, we need a relatively large number $M$ of eigenfunctions $\varphi_m$. This involves functions of high frequency which in turn affect the accuracy of the method. Another disadvantage is that to determine each component $\lambda_m = \langle \lambda, \varphi_m \rangle_{L^2(0,T)}$, we need to solve a linear system of $(N - 2)$ equations. This increases the total reconstruction cost of $\lambda$ using this second method.

On the other hand, we noted some convergence problems for the deconvolution method on both the fast signal of section 6.2 and the slow signal of section 6.3.1. The linear system (33) needs to be regularized to obtain convergence. The Fourier method appears to be more fail-proof. A systematic study is needed to determine -depending on the typical periods of oscillation of the network and the period of $\lambda(t)$- what reconstruction method will be more appropriate. Both methods should be used in practical cases.
7 Conclusion

We studied the nonlinear inverse source problem of localizing a node source emitting an unknown time-dependent signal in a network defined over a $N$-node graph, on which propagates a miscible flow.

We proved under reasonable assumptions that time records of the associated state taken in a strategic set of two nodes yield uniqueness of the two unknown elements defining the occurring node source. Graph theory arguments guide us in the choice of observation nodes for a given network.

The article also presents a direct identification method that localizes the position of the node source in the network by solving a set of well posed linear systems and proposes two different algorithms to identify its unknown emitted signal.

Numerical experiments on a 5-node graph showed the effectiveness of the identifications of the node source and the signal.

Due the wide spectrum of applications covered by the inverse source problem studied in this article, perspectives are numerous. A challenge will be to find an efficient numerical procedure to identify sources on networks of large size.

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References

[1] J. Snow, The cholera near Golden-square and at Deptford, Media Times and Gazette (1854).

[2] J. Chow et al., eds. Systems and Control Theory for Power Systems, New York: Springer-Verglas (1995).

[3] J. Tsitsiklis J and D. Bertsekas, Distributed asynchronous optimal routing in data networks, Automatic Control IEEE transactions 31:4 pp. 325-332 (1986).

[4] A. C. Scott, Nonlinear Science: Emergence and Dynamics of Coherent Structures. Oxford Texts in Applied and Engineering Mathematics 2nd edn, Oxford-New York: Oxford University Press (2003).

[5] L. D. Landau and E.M. Lifshitz, ”Mechanics”, Third Edition: Volume 1 Course of Theoretical Physics, Butterworth-Heinemann; 3rd edition, (1976).
[6] M. Rodriguez, J. Leskovec and A. Krause, Inferring Networks of Diffusion and Influence, Proceeding ACM SIGKDD Conference 1019 (2010).

[7] D. Saha and T. Zaman, Detecting sources of computer viruses in networks: Theory and experiment. SIGMETRICS’10, New York (2010).

[8] R.S. Varga, Matrix Iterative Analysis (Prentice Hall, Englewood Cliffs, NJ. 1962).

[9] T. Biyikoglu, J. Leydold and P. F. Stadler "Laplacian Eigenvectors of Graphs", Springer (2007).

[10] D. Cvetkovic, P. Rowlinson and S. Simic, "An Introduction to the Theory of Graph Spectra", London Mathematical Society Student Texts (No. 75), (2001).

[11] C. Maas, "Transportation in graphs and the admittance spectrum", Discrete Applied Mathematics, 16, 32-49, (1987).

[12] J.G. Caputo, A. Knippel and E. Simo, "Oscillations of simple networks: the role of soft nodes", J. Phys. A: Math. Theor. 46, 035100 (2013).

[13] J. Friedman and J.-P. Tillich, "Wave equations for graphs and the edge-based Laplacian", Pacific J. of Mathematics, vol. 216, nb. 2, (2004).

[14] Yue Wu, Kin Keung Lai and Yongjin Liu, "Deterministic global optimization approach to steady-state distribution gas pipeline networks", Optim. Eng. 8, 259-275, (2007).

[15] R. Burioni, D. Cassi, M. Rasetti, P. Sodano and A. Vezzani, "Bose-Einstein condensation on inhomogeneous complex networks", J. Phys. B 34, 4697-4710, (2001).

[16] R. Burioni, D. Cassi, P. Sodano, A. Trombettoni and A. Vezzani, "Soliton propagation on chains with simple nonlocal defects", Physica D 216, 71-76, (2006).

[17] M. Gondran and M. Minoux, "Graphs and Algorithms ", Wiley Series in Discrete Mathematics and Optimization, (1984).

[18] R. E. Tarjan, "Depth-first search and linear graph algorithms", SIAM Journal on Computing, vol. 1, no 2, (1972), p. 14616

[19] M. Donatelli, A. Neuman and L. Reichel, "Square regularization matrices for large linear discrete ill-posed problems", Numer. Linear Algebra Appl., 19:896-913, (2012).

[20] T. R. Nudell and A. Chakrabortty (2014) A Graph-Theoretic Algorithm for Localization of Forced Harmonic Oscillation Inputs in Power System Networks, American Control Conference (ACC).

[21] B. Mohar, "The Laplacian spectrum of graphs", in "Graph Theory, Combinatorics and Applications", vol. 2 Ed. Y. Alavi, G. Chartrand, O. R. Oellermannand A. J. Schwenk, Wiley, (1991).

[22] P. C. Pinto, P. Thiran and M. Vetteli (2012) Locating the Source of Diffusion in Large-Scale Networks, Phys. Rev. Letters 109, 068702.
In the sequel, we establish the proof of Lemma 2.2.

**Proof.** From expanding the solution $X$ of the system (7) in the orthonormal family \( \{v^1, \ldots, v^N\} \) of \( \mathbb{R}^N \) and using the notations: \( y_n(T) = \langle X(T), v^n \rangle \) and \( \dot{y}_n(T) = \langle \dot{X}(T), v^n \rangle \) for \( n = 1, \ldots, N \), we obtain for all \( t \in (T^*, T) \)

\[
X(t) = \left( y_1(T) + (t - T)\dot{y}_1(T) \right)v^1 + \sum_{n=2}^{N} \left( y_n(T) \cos \left( \omega_n(t - T) \right) + \frac{\dot{y}_n(T)}{\omega_n} \sin \left( \omega_n(t - T) \right) \right)v^n
\]

(51)

Since, we have \( x_k(t) = 0 \) for all \( t \in (T^*, T) \) and \( k \in \{k_1, \ldots, k_\ell\} \) it follows in view of (51) and by analytic continuation that for all \( t \in [T^*, +\infty[ \) and all \( k \in \{k_1, \ldots, k_\ell\} \),

\[
\left( y_1(T) + (t - T)\dot{y}_1(T) \right)v^1_k + \sum_{n=2}^{N} \left( y_n(T) \cos \left( \omega_n(t - T) \right) + \frac{\dot{y}_n(T)}{\omega_n} \sin \left( \omega_n(t - T) \right) \right)v^n_k = 0
\]

(52)

Let \( \tau > 0 \). By integrating the equation (52) over \( (T, T + \tau) \), we find

\[
\left( \tau y_1(T) + \frac{\tau^2}{2} \dot{y}_1(T) \right)v^1_k + \sum_{n=2}^{N} \left( y_n(T) \frac{\sin \left( \omega_n \tau \right)}{\omega_n} + \frac{\dot{y}_n(T)}{\omega_n} \frac{1 - \cos \left( \omega_n \tau \right)}{\omega_n} \right)v^n_k = 0, \ \forall \tau > 0
\]

(53)

Then, from dividing by \( \tau^2 \) the left hand side in (53) and setting the limit when \( \tau \) tends to \( +\infty \), we obtain: \( \dot{y}_1(T)v^1_k = 0 \) for all \( k \in \{k_1, \ldots, k_\ell\} \). Furthermore, using this last result in (53) and dividing now by \( \tau \) with setting the limit when \( \tau \) tends to \( +\infty \) gives:
Thus, in (52) the term involving $v_k^1$ vanishes. Let $n_0 \in \{2, \ldots, N\}$. From multiplying (52) by $\cos(\omega_{n_0}(t - T))$ and integrating over $(T, T + \tau)$ then, dividing the obtained result by $\tau$ and setting the limit when $\tau$ tends to $+\infty$ we get:

$$y_{n_0}(T) v_k^{n_0} = 0 \text{ for all } k \in \{k_1, \ldots, k_\ell\}. \quad (53)$$

Therefore, we find for all $n \in \{1, \ldots, N\}$

$$y_n(T) v_k^n = 0, \quad \forall \; k \in \{k_1, \ldots, k_\ell\} \quad \implies \quad y_1(T) = \cdots = y_N(T) = 0 \quad (54)$$

The two implications in (54) are obtained since the set $\{k_1, \ldots, k_\ell\}$ is strategic. In addition, using (54) in (51) and deriving the obtained form with respect to $t$ gives

$$\dot{X}(t) = \sum_{n=2}^N \dot{y}_n(T) \cos(\omega_n(t - T)) v^n, \quad \forall \; t \in (T^*, T) \quad (55)$$

Moreover, as we have: $\dot{x}_k(t) = 0$ for all $t \in (T^*, T)$ and $k \in \{k_1, \ldots, k_\ell\}$ then, by analytic continuation it follows from (55) that for all $k \in \{k_1, \ldots, k_\ell\}$

$$\sum_{n=2}^N \dot{y}_n(T) \cos(\omega_n(t - T)) v^n = 0, \quad \forall \; t \in ]T^*, +\infty[ \quad (56)$$

Let $n_0 \in \{2, \ldots, N\}$ and $\tau > 0$. From multiplying (56) by $\cos(\omega_{n_0}(t - T))$ and integrating over $(T, T + \tau)$ then, dividing the obtained result by $\tau$ and setting the limit when $\tau$ tends to $+\infty$ we obtain $\dot{y}_{n_0}(T) v_k^{n_0} = 0$ for all $k \in \{k_1, \ldots, k_\ell\}$. Hence, we have for all $n \in \{2, \ldots, N\}$:

$$\dot{y}_n(T) v_k^n = 0, \quad \forall \; k \in \{k_1, \ldots, k_\ell\} \quad \implies \quad \dot{y}_2(T) = \cdots = \dot{y}_N(T) = 0 \quad (57)$$

The implication in (57) is obtained since $\{k_1, \ldots, k_\ell\}$ is strategic. Therefore, (54) and (57) imply that $X(T) = \dot{X}(T) = \vec{0}$ which is the result announced in (8). ■