PERFECT FLUID SPACE-TIMES ADMITTING A
3-DIMENSIONAL CONFORMAL GROUP ACTING
ON NULL ORBITS

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Abstract

Space-times admitting a 3-dimensional Lie group of conformal motions $C_3$ acting on null orbits are studied. Coordinate expressions for the metric and the conformal Killing vectors (CKV) are then provided (irrespectively of the matter content) and all possible perfect fluid solutions are found, although none of them verifies the weak and dominant energy conditions over the whole space-time manifold.

1 Introduction

So far in the literature, the study of null orbits has been restricted to isometries only. The groups $G_r$, $r \geq 4$, on $N_3$ have at least one subgroup $G_3$ which may act on $N_3$, $N_2$ or $S_2$. For $G_3$ on $S_2$, one obtains special cases of the LRS models, $G_r$ admitting either a group $G_3$ on $N_3$ or a null Killing vector. The case $G_3$ on $N_2$ was studied by Barnes, the group $G_3$ is then of Bianchi type II and perfect fluid solutions are excluded since the metric leads to a Ricci tensor whose Segre type is not that of a perfect fluid. Another case that has been considered in the literature is that of $G_3$ on $N_3$, such case is subject to the condition $R_{ab}k^a k^b = 0$ and this condition excludes perfect fluids with $\mu + p \neq 0$. Perfect fluid solutions cannot admit a non-twisting ($w = 0$) null Killing vector except if $\mu + p = 0$. The algebraically special perfect fluid solutions with twisting null Killing vectors are treated by Wainwright and they admit an Abelian group $G_2$.

This paper will deal with space-times admitting a 3-dimensional Lie group of conformal motions $C_3$ acting on null orbits. In the beginning one could get the feeling that this kind of space-times would be empty for perfect fluid solutions, since the line element of these space-times is, by the theorem of Defrise-Carter, conformally related to one admitting a $G_3$ acting on null orbits and these ones, as we have pointed out above, do not admit perfect fluid solutions. But, as we will show, this is not the case, since indeed a conformal scaling does change the Ricci tensor, but there are just a few solutions.

2 Space-times admitting CKVs acting on null orbits

We shall concern ourselves with space-times $(M, g)$ that admit a three-parameter conformal group $C_3$ containing an Abelian two-parameter subgroup of isometries $G_2$, whose orbits $S_2$
are spacelike, diffeomorphic to $\mathbb{R}^2$ and admit orthogonal two-surfaces; furthermore, we shall assume that the $C_3$ acts transitively on null orbits $N_3$.

The classification of all possible Lie algebra structures for $C_3$ under the previous hypothesis was given in [3] where coordinates were adapted so that the line element associated with the assumed energy-momentum tensor.

In all of these cases

$$F, Q, H \text{ and } W \text{ are all functions of } t \text{ and } r \text{ alone.}$$

If the conformal algebra $C_3$ belongs to the family (A), it was shown in [3] that, for null conformal orbits, one can always bring $X$ to the form

$$X = \partial_t + \partial_r + X^y(y, z)\partial_y + X^z(y, z)\partial_z,$$

where $X^y(y, z)$ and $X^z(y, z)$ are linear functions of their arguments to be determined from the commutation relations of $X$ with the Killing vectors. Specializing now the conformal equations to the CKV (2) and the metric (1), for each possible case, one has the following forms for $X$ and the metric functions $F$, $Q$, $H$, and $W$ appearing in [3]

\begin{align*}
(I) & \quad Q = q(t - r), \quad H = h(t - r), \quad W = w(t - r), \\
& \quad X = \partial_t + \partial_r.
\end{align*}

\begin{align*}
(II) & \quad Q = q(t - r), \quad H = h(t - r), \quad W = w(t - r) - \frac{t + r}{2}, \\
& \quad X = \partial_t + \partial_r + z\partial_y.
\end{align*}

\begin{align*}
(III) & \quad Q = e^{\frac{t}{1 + p}}q(t - r), \quad H = e^{\frac{t}{1 + p}}h(t - r), \quad W = e^{\frac{t}{1 + p}}w(t - r), \\
& \quad X = \partial_t + \partial_r + y\partial_y.
\end{align*}

\begin{align*}
(IV) & \quad Q = e^{-(t + r)}q(t - r), \quad H = h(t - r), \quad W = w(t - r) - \frac{t + r}{2}, \\
& \quad X = \partial_t + \partial_r + (y + z)\partial_y + z\partial_z.
\end{align*}

\begin{align*}
(V) & \quad Q = e^{-(t + r)}q(t - r), \quad H = h(t - r), \quad W = w(t - r), \\
& \quad X = \partial_t + \partial_r + y\partial_y + z\partial_z.
\end{align*}

\begin{align*}
(VI) & \quad Q = e^{-(1 + p)\frac{t + r}{2}}q(t - r), \quad H = e^{(1 - p)\frac{t + r}{2}}h(t - r), \quad W = e^{(1 - p)\frac{t + r}{2}}w(t - r), \\
& \quad X = \partial_t + \partial_r + y\partial_y + pz\partial_z \quad (p \neq 0, 1).
\end{align*}

\begin{align*}
(VII) & \quad Q = e^{-\frac{t + r}{2}}q(t - r), \quad c = c(t - r), \quad g = g(t - r), \\
& \quad H = \frac{\sqrt{1 - p^2}}{\sqrt{1 + c^2 + g^2 + c\cos(\sqrt{4 - p^2\frac{t + r}{2}}) + g\sin(\sqrt{4 - p^2\frac{t + r}{2}})}}, \\
& \quad W = \frac{p + \sqrt{4 - p^2}[c\sin(\sqrt{4 - p^2\frac{t + r}{2}}) - g\cos(\sqrt{4 - p^2\frac{t + r}{2}})]}{\sqrt{1 + c^2 + g^2 + c\cos(\sqrt{4 - p^2\frac{t + r}{2}}) + g\sin(\sqrt{4 - p^2\frac{t + r}{2}})}}, \\
& \quad X = \partial_t + \partial_r - z\partial_y + (y + pz)\partial_z \quad (p^2 < 4).
\end{align*}

In all of these cases $F = F(t, r)$ and the conformal factor $\Psi$ is given by

$$\Psi = F_t + F_r .$$

Furthermore one can prove that family (B) cannot admit conformal Killing vectors acting on null orbits (the proof can be found in [3]).

Note that these results are completely independent of the Einstein field equations and therefore of the assumed energy-momentum tensor.
3 Perfect fluid solutions

For perfect fluid solutions the study is exhausted. For a maximal $C_3$, with a proper CKV, all possible solutions have been found (see [8] for details). They correspond only to the types $III$ and $VI$, although none of them satisfies the weak and dominant energy conditions over the whole space-time manifold.

**Type VI** (this includes the type $III$ for $p = 0$)

We make the coordinate transformation $u = t + r$ and $v = t - r$, so that we have $h = h(v)$ and $q = q(v)$. The field equations yield

\[ W = 0 , \]  
\[ F = f(x) + \frac{1}{2} \frac{1 + p}{1 - p} \ln h - \frac{1}{2} \ln q , \quad x \equiv u - \frac{2}{1 - p} \ln h , \]  
\[ 0 = \left\{ \frac{q_x h_v}{qh} + \frac{h_{vv}}{h} \right\} \Sigma_0 + \left( \frac{h_v}{h} \right)^2 \Sigma_1 , \]  
where

\[ \Sigma_0 \equiv -1 + p^4 + 4 f_x - 4 p f_x + 4 p^2 f_x - 4 p^3 f_x + 8 f^2_x - 8 p^2 f^2_x \]  
\[- 32 f^3_x + 32 p f^3_x - 8 f_{xx} + 8 p^2 f_{xx} + 32 f_{xx} f_x - 32 p f_{xx} f_x . \]  
\[ \Sigma_1 \equiv 2 + 2 p + 2 p^2 + 2 p^3 - 16 f_x - 8 p f_x - 16 f^2_x - 8 p^3 f_x + 32 f^2_x + 16 p f^2_x \]  
\[ + 48 p^2 f^2_x - 64 p f^3_x - 16 f_{xx} + 16 p f_{xx} - 32 p f_{xx} x + 64 f_{xx} f_x . \]  

$h_{,v} = 0$ is excluded since the solution does not correspond to a perfect fluid. Therefore, two possibilities arise:

i) $\Sigma_0 = 0, \quad \Sigma_1 = 0$

ii) $\frac{q_x h_v}{qh} + \frac{h_{vv}}{h} = a \left( \frac{h_v}{h} \right)^2 \quad (a = \text{const}) .$

In the first case $f_x$ must be a constant, and therefore the CKV is not proper. For the second case we have

\[ \frac{q_v}{q} = a \frac{h_v}{h} - \frac{h_{,vv}}{h_{,v}} , \]  
which can be integrated to give

\[ q = \frac{h^a}{h_{,v}} , \]  
and equation (13) reduces to:

\[ 1 = \frac{f_{,xx}[f_x 32(ap - a - 2p) + 8(2 - p^2a - 2p + 4p^2 + a)]}{[4f_x - p - 1][f_x 8(ap - a - 2p) + f_{,x} 32p(2 + 1) + a - ap + ap^2 - ap^3 - 2 - 2p^2]} . \]  

It is convenient to divide the analysis into three sub-cases.

**Sub-case (a):** $a = 2p/(p - 1)$.

Equation (18) can be readily integrated to give

\[ f = \frac{p + 1}{4} x - \frac{(1 - p)^2}{p^2 + 1} \ln |x| + c , \quad c = \text{const} . \]  

We notice that for \( p = -1 \) there exists a third Killing vector of the form

\[
\zeta = \left( \frac{1}{2} + \frac{1}{2} \frac{h}{h_v} \right) \partial_t + \left( \frac{1}{2} - \frac{1}{2} \frac{1}{h_v} \right) \partial_r + y \partial_y - z \partial_z.
\]  

(20)

Sub-case (b): \( a = 2/(1 - p) \).

When \( p = -1 \) the solution is a particular case of sub-case (a). The remaining cases may now be integrated giving:

\[
f = - \ln |1 - e^{-(1+p)x/4}| + c, \quad c = \text{const}.
\]  

(21)

We note that this sub-case admits the further Killing vector

\[
\zeta = \left( \frac{1}{2} + \frac{1 - p}{4} \frac{h}{h_v} \right) \partial_t + \left( \frac{1}{2} - \frac{1 - p}{4} \frac{h}{h_v} \right) \partial_r + \frac{1 - p}{2} y \partial_y - \frac{1 - p}{2} z \partial_z,
\]  

(22)

which violates our requirement of a maximal three-dimensional conformal group \( C_3 \).

Sub-case (c): we finally consider the possibility \( a \neq 2p/(p - 1) \) and \( a \neq 2/(1 - p) \). The solution of (18) is then given implicitly by

\[
x = \gamma_1 \ln |f_{,x} - \beta_0| + \gamma_2 \ln |f_{,x} - \beta_+| + \gamma_3 \ln |f_{,x} - \beta_-|
\]  

(23)

where

\[
\beta_0 = \frac{p + 1}{4}, \quad \beta_\pm = \frac{-2(p^2 + 1) \pm \sqrt{2(p^2 + 1)(1 - p)^2(a^2 - 2a + 2)}}{4(ap - a - 2p)}.
\]  

(24)

\( \gamma_i, i = 1, 2, 3 \) being constants.

A careful analysis of the energy conditions shows that for all cases (i.e., for all values of the parameters \( a \) and \( p \)) the solutions can only satisfy the energy conditions over certain open domains of the manifold (see [8]).

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