Fixed Point Semantics for Stream Reasoning

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Abstract

Reasoning over streams of input data is an essential part of human intelligence. During the last decade stream reasoning has emerged as a research area within the AI-community with many potential applications. In fact, the increased availability of streaming data via services like Google and Facebook has raised the need for reasoning engines coping with data that changes at high rate. Recently, the rule-based formalism LARS for non-monotonic stream reasoning under the answer set semantics has been introduced. Syntactically, LARS programs are logic programs with negation incorporating operators for temporal reasoning, most notably window operators for selecting relevant time points. Unfortunately, by preselecting fixed intervals for the semantic evaluation of programs, the rigid semantics of LARS programs is not flexible enough to constructively cope with rapidly changing data dependencies. Moreover, we show that defining the answer set semantics of LARS in terms of FLP reducts leads to undesirable circular justifications similar to other ASP extensions. This paper fixes all of the aforementioned shortcomings of LARS. More precisely, we contribute to the foundations of stream reasoning by providing an operational fixed point semantics for a fully flexible variant of LARS and we show that our semantics is sound and constructive in the sense that answer sets are derivable bottom-up and free of circular justifications.

Keywords: Dynamic Data; Answer Set Programming; Stream Reasoning

1. Introduction

Reasoning over streams of input data is an essential part of human intelligence. During the last decade stream reasoning has emerged as a research area within the AI-community with many potential applications, e.g., web of things, smart cities, and social media analysis (cf. Valle et al. (2009); Mileo et al. (2017); Aglio et al. (2017)). In fact, the increased availability of streaming data via services like Google and Facebook has raised the need for reasoning engines coping with data that changes at high rate.
Logic programs are rule-based systems with the rules and facts being written in a sublanguage of predicate logic extended by a unary operator ‘∼’ denoting negation-as-failure (or default negation) (Clark, 1978). While each monotone (i.e., negation-free) logic program has a unique least Herbrand model (with the least model semantics (van Emden and Kowalski, 1976) being the accepted semantics for this class of programs), for general logic programs a large number of different purely declarative semantics exist. Many of it have been introduced some 20 years ago, among them the answer set semantics (Gelfond and Lifschitz, 1991) and the well-founded semantics (van Gelder et al., 1991). The well-founded semantics, because of its nice computational properties (computing the unique well-founded model is tractable), plays an important role in database theory. However, with the emergence of efficient solvers such as DLV (Leone et al., 2006), Smodels (Simons et al., 2002), Cmodels (Giunchiglia et al., 2006), and Clasp (Gebser et al., 2012), programming under answer set semantics led to a predominant declarative problem solving paradigm, called answer set programming (or ASP) (Marek and Truszczynski, 1999; Lifschitz, 2002). Answer set programming has a wide range of applications and has been successfully applied to various AI-related subfields such as planning and diagnosis (for a survey see Brewka et al. (2011); Eiter et al., 2009; Baral, 2003). Driven by this practical needs, a large number of extensions of classical answer set programs have been proposed, e.g. aggregates (cf. Faber et al., 2004, 2011; Pelov, 2004), choice rules (Niemelä et al., 1999), dl-atoms (Eiter et al., 2008), and general external atoms (Eiter et al., 2005). For excellent introductions to the field of answer set programming we refer the reader to Brewka et al. (2011); Baral, 2003; Eiter et al., 2009.

Beck et al. (2018) introduced LARS, a Logic-based framework for Analytic Reasoning over Streams, where the semantics of LARS has been defined in terms of FLP-style answer sets (Faber et al., 2011). Syntactically, LARS programs are logic programs with negation as failure incorporating operators for temporal reasoning, most notably window operators for selecting relevant time points. Unfortunately, by preselecting fixed intervals for the semantic evaluation of programs, the rigid semantics of LARS programs is not flexible enough to constructively cope with rapidly changing data dependencies. For example, sentences of the form “a holds at time point t if b holds at every relevant time point” are not expressible within LARS (cf. Example 2.3), as the interval of ’relevant’ time points changes dynamically, whereas LARS preselects a static interval. Our first step therefore is to refine and simplify Beck et al.’s (2018) semantics in Section 2.4 by employing dynamic time intervals.

Extensions of the answer set semantics adhere to minimal models or, even more restricting, to models free of unfoundedness. However, FLP-answer sets of stream logic programs may permit undesirable circular justifications similar to other ASP extensions (cf. Shen et al. (2014); Antić et al. (2013)). Fixed point semantics of logic programs (cf. Fitting (2002)), on the other hand, are constructive by nature, which suggests to define a fixed point semantics for stream logic programs targeted for foundedness, by recasting suitable operators in such a way that the FLP semantics can be reconstructed or refined, in the sense that a subset of the respective answer sets are selected (sound “approximation”). The benefit is twofold: by coinciding semantics, we get operable fixed point constructions, and by refined semantics, we obtain a sound approximation that is constructive. For this we recast two well-known fixed point oper-
ators from ordinary to stream logic programs, namely the van Emden-Kowalski operator (van Emden and Kowalski, 1976) and the Fitting operator (Fitting, 2002). This task turns out to be non-trivial due to the intricate properties of windows (Arasu et al., 2006; Beck et al., 2018) and other modal operators occurring in rule heads. We show that the so obtained operators inherit the following characteristic properties: models of a program are characterized by the prefixed points of its associated van Emden-Kowalski operator, and the Fitting operator is monotone with respect to a suitable ordering which guarantees the existence of certain least fixed points, namely the so obtained constructive answer sets. We then show the constructiveness of our fixed point semantics in terms of level mappings (Shen et al., 2014). Specifically, we prove that our semantics captures those answer sets which possess a level mapping or, equivalently, which are free of circular justifications, which is regarded as a positive feature.

The rest of the paper is structured as follows. In Section 2 we define the syntax and semantics of stream logic programs first in the vein of Beck et al. (2018) (Section 2.3) followed by our refined semantics in Section 2.4. Sections 3 and 4 constitute the main part of the paper. More precisely, in Section 3.1 we define a novel (partial) model operator for the evaluation of rule heads, and in Section 3.2 and 3.3 we recast the well-known van Emden-Kowalski operator $T_P$ and the Fitting operator $\Phi_P$ from ordinary to stream logic programs and prove some non-trivial properties. In Section 4 we then define a fixed point semantics for stream logic programs in terms of the (extended) Fitting operator and prove in our Main Theorem 4.7 the soundness of our approach. Afterwards, in Section 5 we characterize our semantics in terms of level mappings and conclude that our semantics is sound, constructive, and free of circular justifications.

2. Stream Logic Programs

We denote the set $\mathbb{N} \cup \{\infty\}$ by $\mathbb{N}^\infty$. A partially ordered set (or poset) is a pair $\langle L, \leq \rangle$ where $L$ is a set and $\leq$ is a reflexive, antisymmetric, and transitive binary relation on $L$. A lattice is a poset $\langle L, \leq \rangle$ where every pair of elements $x, y \in L$ has a unique greatest lower bound and least upper bound in $L$. We call $\langle L, \leq \rangle$ complete if every subset has a greatest lower bound and a least upper bound. For any two elements $x, y \in L$, we define the interval $[x, y] = \{z \in L | x \leq z \leq y\}$. Given a mapping $f : L \to L$, we call $x \in L$ a prefixed point of $f$ if $f(x) \leq x$, and we call $x$ a fixed point of $f$ if $f(x) = x$. Moreover, we call $f$ monotone if $x \leq y$ implies $f(x) \leq f(y)$, for all $x, y \in L$. In case $f$ has a least fixed point, we denote it by $\text{lp}f$. Moreover, for a mapping $g : L \times L \to L$ we denote by $g(\cdot,y)$ the function mapping every $x \in L$ to $g(x,y) \in L$.

2.1. Streams and Windows

In the rest of the paper, $\Sigma$ will denote a finite nonempty set of propositional atoms containing the special symbol $\top$.

A formula (over $\Sigma$) is defined by the grammar

$$\alpha ::= a \mid \neg \alpha \mid \alpha \land \alpha \mid \alpha \lor \alpha \mid \alpha \to \alpha \mid \Diamond \alpha \mid \Box \alpha \mid \Diamond_1 \alpha \mid \Box_{[e,f]} \alpha$$
where \( a \in \Sigma, t \geq 1, \) and \( \ell, r \in \mathbb{N}^\infty, \ell \leq r. \) We call \( \alpha \) (i) \( \square \)-free if it does not contain \( \square; \) (ii) monotone if it does not contain \( \neg, \rightarrow, \square; \) and (iii) normal if it does not contain \( \neg, \lor, \rightarrow, \odot. \)

A stream (over \( \Sigma \)) is an infinite sequence \( I = I_1 I_2 \ldots \) of subsets of \( \Sigma, \) i.e., \( I_t \subseteq \Sigma \) for all time points \( t \geq 1. \) We call a stream \( J = J_1 J_2 \ldots \) a substream of \( I, \) in symbols \( J \subseteq I, \) if \( J_t \subseteq I_t \) for all \( t \geq 1. \) In the sequel, we omit empty sets in a sequence and write, e.g., \( I_1 I_2 \) instead of \( I_1 \emptyset I_2 \emptyset \ldots, \) and we denote the empty sequence \( \emptyset \emptyset \ldots \) simply by \( \emptyset. \)

We define the support of \( I, \) in symbols \( \text{supp}_{I}, \) to be the tightest interval \([t_1, t_2]\) containing \( \{t \geq 1 \mid I_t \neq \emptyset\}; \) formally, \( t_1 = \min\{t \geq 1 \mid I_t \neq \emptyset\} \) and \( t_2 = \max\{t \geq 1 \mid I_t \neq \emptyset\} \) in case \( I \neq \emptyset, \) and \( \text{supp}_{\emptyset} = \emptyset. \)

A window \( \rho \) is a function \([.\) mapping every stream \( I = I_1 I_2 \ldots \) to the substream \( I[\ell, r; t] = I_{\max(0, \ell - t)} \ldots I_{r+t}, \) of \( I, \) where \( \ell, r \in \mathbb{N}^\infty, \ell \leq r. \) Note that \([.\) and \( \text{supp}_{\cdot} \) are monotone functions, that is, \( I \subseteq J \) implies \( I[\ell, r; t] \subseteq J[\ell, r; t] \) and \( \text{supp}_{I} \subseteq \text{supp}_{J}, \) for all \( \ell, r \in \mathbb{N}^\infty \) and \( t \geq 1. \)

### 2.2 Syntax

A (stream logic) program \( \mathcal{P} \) is a finite nonempty set of rules of the form

\[
\alpha \leftarrow \beta_1, \ldots, \beta_j, \sim \beta_{j+1}, \ldots, \sim \beta_k, \quad k \geq j \geq 1,
\]

where \( \alpha \) is a normal \( t \)-formula, \( \beta_1, \ldots, \beta_k \) are formulas, and \( \sim \) denotes negation-as-failure (Clark, 1978). We will often write \( \lnot \) in expressions of the form \([.\) to make the name of the rule explicit. For convenience, we define for a rule \( \rho \) of the form \([.\)

\( H(\rho) = \alpha, \) and \( B(\rho) = \beta_1 \land \ldots \beta_j \land \sim \beta_{j+1} \land \ldots \land \sim \beta_k. \) As is customary in logic programming, we will interpret every finite set \( A \) of formulas as the conjunction \( \land A \) over all formulas in \( A. \) We call a rule \( \rho \) a fact if \( B(\rho) = \top, \) and we call \( \rho \) ordinary if \( \alpha, \beta_1, \ldots, \beta_k \in \Sigma. \) Moreover, we define \( H(\mathcal{P}) \) to be the conjunction of all rule heads occurring in \( \mathcal{P}, \) that is, \( H(\mathcal{P}) = \land_{\rho \in \mathcal{P}} H(\rho). \)

### 2.3 Semantics of Beck et al. (2018)

We now recall the FLP-style answer set semantics (Faber et al., 2011) as defined in Beck et al. (2018) and we show that their semantics yields counter-intuitive answer sets (cf. Example 2.2).

Let \( T \) be a closed interval in \( \mathbb{N} \) and let \( B \subseteq \Sigma \) be a finite set, called the background data. We define the entailment relation \( \models_B, \) with respect to \( B, \) for all streams \( I, a \in \Sigma - \{\top\}, \) formulas \( \alpha, \beta, \) and all time points \( t \in T: \)

1. \( I, T, t \models_B \top; \)
2. \( I, T, t \models_B a \text{ if } a \in I_t \cup B; \)
3. \( I, T, t \models_B \lnot \alpha \text{ if } I, T, t \not\models_B \alpha; \)

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Faber et al. (2011) employed more sophisticated windows and called them window functions; for simplicity, we consider here only the windows defined above and note that our results are independent of the particular choice of windows.
4. $1, T, t \models_B \alpha \land \beta$ if $I, T, t \models_B \alpha$ and $I, T, t \models_B \beta$;
5. $I, T, t \models_B \alpha \lor \beta$ if $I, T, t \models_B \alpha$ or $I, T, t \models_B \beta$;
6. $I, T, t \models_B \alpha \rightarrow \beta$ if $I, T, t \not\models_B \alpha$ or $I, T, t \models_B \beta$;
7. $I, T, t \models_B \Diamond \alpha$ if $I, T, t' \models_B \alpha$, for some $t' \in T$;
8. $I, T, t \models_B \Box \alpha$ if $I, T, t' \models_B \alpha$, for all $t' \in T$;
9. $I, T, t \models_B @ \rho \alpha$ if $I, T, t' \models_B \alpha$, and $t' \in T$;
10. $I, T, t \models_B \Box_{[\ell, r]} \alpha$ if $I[\ell, r; t], T, t \models_B \alpha$.

In case $I, T, t \models_B \alpha$, we call $I$ a $(t, T)$-model of $\alpha$.

We wish to evaluate $P$ with respect to some fixed stream $D$, called the data stream. We call a stream $I$ an interpretation stream for $D$ if $D \subseteq I$, and we say that such an interpretation stream $I$ is a $(t, T)$-model of $P$ if $I, T, t \models_B B(\rho) \rightarrow H(\rho)$, for all rules $\rho \in P$. The reduct of $P$ with respect to $I$ and $T$ at time point $t$ is given by

$$P^{1, T, t}_{I} = \{ \rho \in P \mid I, T, t \models_B B(\rho) \}.$$  

**Definition 2.1** [Beck et al. (2018)] Let $T$ be a closed interval in $\mathbb{N}$ and let $t \in T$. An interpretation stream $I$ for $D$ is a $(t, T)$-answer stream of $P$ (for $D$) if $I$ is a $(t, T)$-model of $P^{1, T, t}_{I}$ and there is no $(t, T)$-model $J$ of $P^{1, T, t}_{I}$ (for $D$) with $J \subseteq I$.

Note that the minimality condition in **Definition 2.1** is given with respect to the same interval $T$, which is crucial. In fact, the following example shows that as a consequence of **Definition 2.1** trivial programs may have infinitely many answer streams which is counter-intuitive from an answer set programming perspective.

**Example 2.2** The ordinary program $P$ consisting of a single fact $a$ has the single answer set $\{a\}$. Given some arbitrary time point $t \geq 1$ for the evaluation of $P$ within the LARS context defined above, we therefore expect $P$ to have the single answer stream $\{a\}$. Unfortunately, under Beck et al.’s (2018) semantics, $P$ has infinitely many answer streams: the $(t, [t, t])$-answer stream $\{a\}$, the $(t, [t, t+1])$-answer stream $\{a\} \cup \{b\}$, the $(t, [t, t+2])$-answer stream $\{a\} \cup \{b\} \cup \{c\}$ and so on.

The reason for the existence of the infinitely many answer streams for the trivial program in Example 2.2 is the preselection of the fixed interval $T$ in **Definition 2.1** for the semantic evaluation of programs. As a negative consequence of this choice, which appears to be an artificial simplification of the semantics of programs, is that some specifications which occur in practice cannot be expressed within the LARS language as is demonstrated by the following example.

**Example 2.3** Let $T$ be some interval and let $t \in T$ be some time point. According to **Definition 2.1** the statement “$a$ holds at $t$ if $b$ holds at every time point in $T$” is formalized by the single rule $a \leftarrow \Box b$ evaluated at time point $t$. Now consider the slightly different statement “$a$ holds at $t$ if $b$ holds at every relevant time point in the
of the input data.” As natural as this statement seems, it is not expressible within the LARS language. The intuitive reason is that the support function is flexible and depends on the data, whereas the preselected interval $T$ is fixed by the programmer and therefore does not depend on the data. In a sense, preselecting fixed intervals for the semantic evaluation of programs contradicts the very idea of stream reasoning which aims at coping with data that changes at a high rate by incorporating window operators on a syntactic level for selecting relevant time points. Arguably, it is therefore more natural to formalize the first statement by the rule $a ← b\top T\Box b$ thus syntactically encoding the restricted interval $T$, and interpreting $a ← b\Box b$ as a formalization of the second statement (cf. Example 2.7).

2.4. Refined Semantics

We refine the FLP-style semantics of Beck et al. (2018) (cf. Definition 2.1) by employing dynamic intervals. For this we first refine the entailment relation by using the support function in the definition of $\Box$ and $\Diamond$ for dynamically computing intervals instead of the fixed interval $T$ used by Beck et al. (2018):

1. $I, t \models_B \Diamond \alpha$ if $I, t' \models_B \alpha$, for some $t' \in \supp I$;
2. $I, t \models_B \Box \alpha$ if $I, t' \models_B \alpha$, for all $t' \in \supp I$;
3. $I, t \models_B @_T \alpha$ if $I, t' \models_B \alpha$, for $t' \geq 1$.

In case $I, t \models_B \alpha$, we call $I$ a $t$-model of $\alpha$, and we call $\alpha$ $t$-consistent (resp., $t$-inconsistent) if $\alpha$ has at least one (resp., no) $t$-model. For convenience, we call $\alpha$ a $t$-formula if $\alpha$ is $t$-consistent.

Example 2.4 The formula $\boxplus_02a$ is inconsistent since $@_2$ is a reference to time point 2 which is outside the scope of the window $\boxplus_{[0,0]}$ evaluated at time point 1. More precisely, let $I = I_1I_2\ldots$ be an arbitrary stream and compute $I[0,0;1] = I_1$ which implies $I_1,2 /\models_B a$—so $I$ is not a 1-model of $\alpha$.

Remark 1 Note that $t$-inconsistency of normal formulas can be easily verified by a syntactic check as in the example above and in the rest of the paper we assume that (normal) formulas occurring in rule heads are $t$-consistent, for all relevant $t$.

We can now refine and simplify the definition of answer streams by omitting the reference to interval $T$ which gives a more natural minimality condition.

Definition 2.5 An interpretation stream $I$ for $D$ is a $t$-answer stream of $P$ (for $D$) if $I$ is a substream minimal $t$-model of $\mathcal{P}$. 

Example 2.6 The ordinary program $P$ of Example 2.2 consisting of the single fact $a$ has the single $t$-answer stream $\{a\}$, as expected.
2.4 Refined Semantics

Example 2.7 The two statements in Example 2.3 are formalized according to Definition 2.5 by the two rules $a \leftarrow \lozenge_T \square b$ and $a \leftarrow \square b$, respectively, as desired.

We now illustrate the above concepts in more detail with the following running example.

Example 2.8 Consider the program $P$ consisting of the following rules:

\begin{align*}
@_2 a & \leftarrow \rho_1 \sim @_2 c \\
\lozenge_{[0,\infty)} a & \leftarrow \rho_3 \sim @_2 a \\
\lozenge_{[2,3]} a \& b & \leftarrow \rho_4 \lozenge_{[0,1]} c, \square d.
\end{align*}

Let the background data $B$ be given by

$$D = \{a\}_1 \{a, b\}_5 \{c\}_{10}.$$ 

That is, the propositions $a$ and $b$ hold at time point 5 and so on. Then, the 5-answer streams of $P$ (for $D$) are given by:

$$I = \{a\}_1 \{a, b\}_3 \{a, b, c\}_4 \{a, b, c\}_5 \{a, b, c\}_6 \{a, b, c\}_7 \{a, b, c\}_8 \{c\}_9 \{c\}_{10};$$

$$J = \{a\}_1 \{a\}_2 \{a\}_3 \{a, b\}_4 \{a, b\}_5 \{c\}_6.$$

For instance, we verify that $I$ is indeed a 5-answer stream of $P$. First of all, note that $I$ is a 5-model of $P$ (for $D$): (i) as $c$ holds in $I$ at time points 5 and 7, we have $I, 5 \not\models_B \sim @_2 c$ and $I, 5 \models_B \sim c$ which implies $I, 5 \models_B \rho_1$ and $I, 5 \models_B \rho_2$; (ii) as $c$ holds at every time point in the interval $[4, 10]$, we have $I, 5 \models_B \lozenge_{[1, \infty]} \square c$ which implies $I, 5 \models_B \rho_3$; and (iii) as $a$ and $b$ hold at every time point in $[3, 8]$, we have $I, 5 \models_B \lozenge_{[2, 3]} \square (a \& b)$ which implies $I, 5 \models_B \rho_4$.

Now we argue that $I$ is a minimal 5-model of $P^I$ for $D$ with $D \subseteq I \subseteq I$. Then, since $I', 5 \models_B B(\rho_3)$ and $I', 5 \models_B H(\rho_4)$, for $I'$ to be a 5-model of $P$ we must have $I', 5 \models_B H(\rho_3)$ and $I', 5 \models_B H(\rho_4)$; but this is equivalent to $I'[1, \infty; 5], t \models_B c$, for all $t \in [4, 10]$, and $I'[2, 3; 5], t' \models_B a \& b$, for all $t' \in [3, 8]$ where $I'[1, \infty; 5] = I'_4 \ldots I'_9$ and $I'[2, 3; 5] = I'_3 \ldots I'_5$, respectively.

That is, we have $c \in I'_t$, for all $t \in [4, 10]$, and $a, b \in I'_t$, for all $t' \in [3, 8]$.—but this, together with $D \subseteq I' \subseteq I$, immediately implies $I' = I$ which shows that $I$ is indeed a minimal 5-model of $P^I$ and therefore a 5-answer stream of $P$.

It is important to emphasize that we can capture Beck et al. (2018)'s semantics as follows.

Proposition 2.9 Let $T = [t_1, t_2]$ be an interval and let $t \in T$ be some time point. An interpretation stream $I$ for $D$ is a $(t, T)$-answer stream of $P$ if, and only if, $I \cup \{#\}_1 \ldots \{#\}_{t_2}$ is a $t$-answer stream of

$$\lozenge_T P \cup \{@, \# \mid t \in T\},$$

where $\#$ is a special symbol not occurring in $\Sigma$ and $\lozenge_T P$ consists of all rules of the form

$$\lozenge_T \rho = \lozenge_T a \leftarrow \lozenge_T b_1, \ldots, \lozenge_T b_j, \sim \lozenge_T b_{j+1}, \ldots, \lozenge_T b_k, \rho \in P.$$
At this point, we have successfully extended the FLP-style answer set semantics from ordinary to stream logic programs by refining Beck et al. (2018)'s semantics. Unfortunately, as for other program extensions (cf. Shen et al. (2014); Antić et al. (2013)), our FLP-style semantics may permit circular justifications as is demonstrated by the following example.

**Example 2.10** Consider the program \( \mathcal{R} \) consisting of the following two rules:

\[
\begin{align*}
    a & \leftarrow \Box b \\
    b & \leftarrow \Box a.
\end{align*}
\]

We argue that the \( t \)-model \( \{a, b\}_t \) of \( \mathcal{R} \) is a \( t \)-answer stream of \( \mathcal{R} \), for every \( t \geq 1 \) (and \( D = B = \emptyset \)): (i) The empty stream \( \emptyset \) is not a \( t \)-model of \( \mathcal{R}^{\{a, b\}, t} = \mathcal{R} \) since both rules fire in \( \emptyset \); (ii) the stream \( \{a\}_t \) is not a \( t \)-model of \( \mathcal{R} \) since \( \rho_2 \) fires; (iii) the stream \( \{b\}_t \) is not a \( t \)-model of \( \mathcal{R} \) since \( \rho_1 \) fires. This shows that \( \{a, b\}_t \) is indeed a subset minimal \( t \)-model of \( \mathcal{R} \), and, hence, a \( t \)-answer stream of \( \mathcal{R} \).

In the next two sections, we will develop the tools for formalizing the reasoning in Example 2.8 in an operational setting (cf. Example 4.4) while avoiding circular justifications.

### 3. Fixed Point Operators

In this section, we recast the following well-known fixed point operators from ordinary to stream logic programs: (i) the van Emde-Kowalski operator \( T \) (van Emde and Kowalski, 1976), and (ii) the Fitting operator \( \Phi \) (Fitting, 2002). This task turns out to be non-trivial due to the intricate properties of windows and other modal operators occurring in rule heads.

In the rest of the paper, let \( I \) be a stream, let \( D \) be some data stream, let \( B \) be some background data, and let \( t \geq 1 \) be some fixed time point.

#### 3.1. The Model Operator

In this subsection, we define an operator for the evaluation of rule heads. Specifically, given a normal \( t \)-formula \( \alpha \), we wish to construct a \( t \)-model of \( \alpha \) which is in some sense minimal with respect to a given stream \( I \) (cf. Theorem 3.9).

**Definition 3.1** For normal \( t \)-formulas \( \alpha \) and \( \beta \), and for \( a \in \Sigma \), we define the partial model operator \( M_{I,t} \) at time point \( t \) and with respect to \( I \), inductively as follows:

\[
\begin{align*}
    M_{I,t}(a) &= \begin{cases} 
        \{a\}_t & \text{if } a \not\in B, \\
        \emptyset & \text{if } a \in B
    \end{cases} \\
    M_{I,t}(\alpha \land \beta) &= M_{I,t}(\alpha) \cup M_{I,t}(\beta); \\
    M_{I,t}(\Box \alpha) &= \bigcup_{t' \in \text{supp} I} M_{I,t'}(\alpha); \\
    M_{I,t}(\Diamond \alpha) &= \bigcup_{t' \in \text{supp} I} M_{I,t'}(\alpha); \\
    M_{I,t}(\Box \Diamond \alpha) &= M_{I,t'}(\alpha); \\
    M_{I,t}(\Diamond \Box \alpha) &= M_{I,t}(\alpha).
\end{align*}
\]
Finally, define the model operator $\text{MM}_{I_f}$ to be the twofold application of $\text{M}_{I_f}$, that is,

$$\text{MM}_{I_f}(\alpha) = \text{M}_{\text{M}_{I_f}(\alpha)}(\alpha).$$

One can easily derive the following computation rules for the model operator:

$$\text{MM}_{I_f}(a) = \text{M}_{I_f}(a);$$
$$\text{MM}_{I_f}(\neg \alpha) = \text{M}_{I_f}(\neg \alpha);$$
$$\text{MM}_{I_f}(\square \alpha) = \text{M}_{\text{M}_{I_f}(\alpha)}(\alpha).$$

In case $\alpha$ is $\square$-free, we will often write $\text{M}_{I_f}(\alpha)$ instead of $\text{MM}_{I_f}(\alpha)$ to indicate that the evaluation of $\text{M}_{I_f}$ does not depend on $I$.

**Example 3.2** Let $\alpha$ be the $\square$-free normal 1-formula $\exists_{0,0} \mathbin{@}_1 a \land \mathbin{@}_2 b$, and compute

$$\text{M}_{\emptyset,1}(\exists_{0,0} \mathbin{@}_1 a \land \mathbin{@}_2 b) = \text{M}_{\emptyset,0,1}(a) \cup \text{M}_{\emptyset,0,2}(b) = \{a\}_1 \{b\}_2$$

which is a 1-model of $\alpha$. On the other hand, for the normal 1-formula $\beta = \square a \land b$ containing $\square$, we obtain

$$\text{M}_{\emptyset,1}(\square a \land b) = \text{M}_{\emptyset,1}(\square a) \cup \text{M}_{\emptyset,1}(b) = \text{M}_{\emptyset,1}(b) = \{b\}_1$$

which is *not* a 1-model of $\beta$; however, by applying $\text{M}_{\emptyset,1}$ twice, we do obtain a 1-model of $\beta$:

$$\text{MM}_{\emptyset,1}(\square a \land b) = \text{M}_{\{b\}_1,1}(\square a \land b) = \text{M}_{\{b\}_1,1}(a) \cup \text{M}_{\{b\}_1,1}(b) = \{a, b\}_1.$$

Intuitively, to obtain a 1-model of $\beta$, we have to apply $\text{M}_{\emptyset,1}$ twice as the subformula $b$ induces an expansion of the support of the generated stream which has to be taken into account for the generation of a 1-model for $\square a$ (note that conjunctions are treated separately by the partial model operator).

Example 3.2 indicates that $\square$ requires a special treatment. In fact, if $\alpha$ is $\square$-free then $\text{M}_{I_f}$ does not depend on $I$ and, consequently, in this case $\text{M}_{I_f}(\alpha)$ and $\text{MM}_{I_f}(\alpha)$ coincide which simplifies the matters significantly.

**Proposition 3.3** For every $\square$-free normal 1-formula $\alpha$, $\text{M}_{I_f}(\alpha) = \text{M}_{J_f}(\alpha)$ holds for all streams $I$ and $J$; consequently, $\text{M}_{I_f}(\alpha) = \text{MM}_{I_f}(\alpha)$.

**Proof.** By definition of the partial model operator, $\text{M}_{I_f}(\alpha)$ depends on $I$ only if $\alpha$ contains $\square$. The second assertion follows from the first with $J = \text{M}_{I_f}(\alpha)$ and

$$\text{MM}_{I_f}(\alpha) = \text{M}_{\text{M}_{I_f}(\alpha)}(\alpha).$$

In the next two propositions, we show some monotonicity properties of the (partial) model operator.

**Proposition 3.4** For every normal 1-formula $\alpha$ and all streams $K$ and $I, I \subseteq J$ implies $\text{M}_{K_f}(\alpha) \subseteq \text{M}_{J_f}(\alpha)$ and $\text{MM}_{K_f}(\alpha) \subseteq \text{MM}_{J_f}(\alpha)$. Moreover, $A \subseteq B$ implies $\text{M}_{I_f}(A) \subseteq \text{M}_{I_f}(B)$ and $\text{MM}_{I_f}(A) \subseteq \text{MM}_{I_f}(B)$, for all finite sets $A$ and $B$ of normal 1-formulas.
3.1 The Model Operator

The first inclusion can be shown by a straightforward structural induction on \( \alpha \), so we prove here only the case \( \alpha = \square \beta \), for some normal \( t \)-formula \( \beta \):

\[
M_{I,J}(\square \beta) = \bigcup_{t' \in \text{supp } I} M_{I,J'}(\beta) \subseteq \bigcup_{t' \in \text{supp } J} M_{I,J'}(\beta) \subseteq \bigcup_{t' \in \text{supp } J} M_{J,J'}(\beta) = M_{J,J}(\square \beta).
\]

The second inclusion, \( MM_{I,J}(\alpha) \subseteq MM_{I,J}(\alpha) \), is a direct consequence of the first. Finally, the second part of the proposition is an immediate consequence of the first part and the definition of the (partial) model operator.

**Proposition 3.5** For every normal \( t \)-formula \( \alpha \), \( \text{supp } M_{I,J}(\alpha) = \text{supp } MM_{I,J}(\alpha) \) and \( M_{I,J}(\alpha) \subseteq MM_{I,J}(\alpha) \).

**Proof.** The first identity can be proved by a straightforward structural induction on \( \alpha \). We prove the inclusion by structural induction on \( \alpha \). The only non-trivial case is \( \alpha = \square \beta \), for some normal \( t \)-formula \( \beta \):

\[
M_{I,J}(\square \beta) = \bigcup_{t' \in \text{supp } I} M_{I,J'}(\beta) \subseteq \bigcup_{t' \in \text{supp } I} MM_{I,J'}(\beta) = M_{I,J}(\square \beta).
\]

where the second and third inclusion follows from Proposition 3.4 together with

\[
M_{I,J}(\square \beta) \subseteq MM_{I,J}(\square \beta) \quad \text{for all } t' \in \text{supp } I,
\]

and the last equality holds since:

\[
MM_{I,J}(\square \beta) = MM_{I,J'}(\square \beta) \quad \text{for all } t' \in \text{supp } I. \tag{2}
\]

To prove 2, first note that, by definition, \( M_{I,J}(\square \beta) = M_{I,J'}(\square \beta) \) holds for all \( t' \in \text{supp } I \);
3.1 The Model Operator

consequently:

\[ \text{MM}_{I,t}(□β) = \text{M}_{\text{M}_{I,t}(□β),I}(□β) \]

\[ = \bigcup_{t'' \in \text{supp} \text{M}_{I,t}(□β)} \text{M}_{\text{M}_{I,t}(□β),t''}(β) \]

\[ = \bigcup_{t'' \in \text{supp} \text{M}_{I,t}(□β)} \text{M}_{\text{M}_{I,t}(□β),t''}(β) \]

\[ = \text{M}_{\text{M}_{I,t}(□β),t''}(□β) \]

\[ = \text{MM}_{I,t'}(□β). \]

It will often be convenient to separate a proof into a □-free and a general case. Therefore we define the translation \( α_{I,t} \) of \( α \) with respect to \( I \) at time point \( t \) to be the homomorphic extension to all normal \( t \)-formulas of:

\[(□α)_{I,t} = \bigwedge_{t' \in \text{supp} I} @_{t'} α_{I,t'}; \]

\[(□[(t',r)α])_{I,t} = □[(t',r)α] \]

where we interpret the empty conjunction in \((□α)_{θ,I} \) as \( T \). Intuitively, \( α_{I,t} \) eliminates every \( □ \) occurring in \( α \) while preserving the meaning of \( α \) in the following sense.

**Proposition 3.6** For every normal \( t \)-formula \( α \), and for all streams \( I \) and \( J \) with \( \text{supp} I = \text{supp} J \), we have \( I, t \models B α \text{ if, and only if, } I, t \models B α_{I,t} \).

**Proof.** A straightforward structural induction on \( α \).

Interestingly, the next proposition shows that we can simulate the model operator by the partial model operator applied to an appropriate translation of the input formula.

**Proposition 3.7** For every normal \( t \)-formula \( α \), \( M_{I,t}(α) = M_{I,t}(α_{I,t}) \) and, consequently, \( \text{MM}_{I,t}(α) = M_{I,t}(α_{I,t}) \) with \( M = M_{I,t}(α) \).

**Proof.** The first identity can be proved by a straightforward structural induction on \( α \), and the second identity follows from the first, i.e., \( \text{MM}_{I,t}(α) = M_{M,I,t}(α_{I,t}) = M_{I,t}(α_{I,t}) \).

Monotone formulas inherit their name from the following property.

**Proposition 3.8** For every monotone formula \( α \), \( I, t \models B α \text{ implies } J, t \models B α \), for all streams \( I \subseteq J \).

We are now ready to prove our first theorem which shows that \( \text{MM}_{I,t}(α) \) is a \( t \)-model of \( α \) which is in some sense “minimal” (with respect to \( I \)).

**Theorem 3.9** For every normal \( t \)-formula \( α \), \( \text{MM}_{I,t}(α) \) is a \( t \)-model of \( α \), that is,

\( \text{MM}_{I,t}(α), t \models B α. \)

In case \( α \) is □-free, a single application of \( M_{I,t} \) suffices, that is, \( M_{I,t}(α), t \models B α \).

Moreover, if \( I \) is a \( t \)-model of \( α \), then \( \text{MM}_{I,t}(α) \subseteq I \).
3.2 The van Emden-Kowalski Operator

**Theorem 3.11** A stream \(I\) is a t-model of \(\mathcal{P}\) if, and only if, \(I\) is a prefixed point of \(T_{\mathcal{P},D_I}\).

**Proof.** We start with the second assertion and prove by structural induction on \(\alpha\) that in case \(\alpha\) is \(\square\)-free, \(M_{I,\alpha}(t) \models_B \alpha\). The induction hypothesis \(\alpha = \sigma \in \Sigma\), and the case \(\alpha = \oplus_{\ell,r} \beta\) are straightforward. In what follows, let \(\beta \) and \(\gamma\) denote normal \(t\)-formulas. For \(\alpha = \beta \land \gamma\), we have \(M_{I,\alpha}(\beta \land \gamma) = M_{I,\beta}(\beta) \cup M_{I,\gamma}(\gamma)\) and, by induction hypothesis, \(M_{I,\beta}(\beta), t \models_B \beta\) and \(M_{I,\gamma}(\gamma), t \models_B \gamma\). Since \(\beta\) and \(\gamma\) are \(\square\)-free, we have \(M_{I,\beta}(\beta) \cup M_{I,\gamma}(\gamma), t \models_B \beta \land \gamma\) as a consequence of Proposition 3.8. Finally, a straightforward structural induction on \(\alpha\) shows \(M_{I,\alpha}(t) \models_B \alpha\) if \(\alpha\) is \(\square\)-free, we know from the first part of the proof that

\[M_{I,\alpha}(M_{I,\alpha}(t)), t \models_B \alpha_{M_I}.\]  

(3)

By Proposition 3.7

\[MM_{I,\alpha}(t) = M_{I,\alpha_{M_I}}.\]  

(4)

From (3) and (4) we infer

\[MM_{I,\alpha}(t), t \models_B \alpha_{M_I}.\]

Now since \(\text{supp} M = \text{supp} MM_{I,\alpha}\) (cf. Proposition 3.5), Proposition 3.6 proves our claim.

Finally, a straightforward structural induction on \(\alpha\) shows \(MM_{I,\alpha}(\alpha) \subseteq I\).

**Remark 2** We want to emphasize that the requirement in Theorem 3.9 of \(\alpha\) being \(t\)-consistent is essential. For instance, reconsider the \(1\)-inconsistent normal formula \(\alpha = \oplus_{[0,1]} \oplus_{2} a\) of Remark 7 and compute \(MM_{I,1}(\alpha) = \{a\}_2\) which is not a \(1\)-model of \(\alpha\).

3.2 The van Emden-Kowalski Operator

We are now ready to extend the well-known van Emden-Kowalski operator to the class of stream logic programs.

**Definition 3.10** We define the van Emden-Kowalski operator \(T_{\mathcal{P},D_I}\) of \(\mathcal{P}\) (for \(D\) at time point \(t\)), for every stream \(I\), by

\[T_{\mathcal{P},D_I}(I) = D \cup MM_{I,\mathcal{P}}(\langle H(\rho) \mid \rho \in \mathcal{P} : I, t \models B(\rho) \rangle).\]

As for ordinary logic programs (van Emden and Kowalski 1976), prefixed points of the van Emden-Kowalski operator \(T_{\mathcal{P},D_I}\) characterize the models of \(\mathcal{P}\) (for \(D\) at time point \(t\)).

**Theorem 3.11** A stream \(I\) is a \(t\)-model of \(\mathcal{P}\) if, and only if, \(I\) is a prefixed point of \(T_{\mathcal{P},D_I}\).
3.2 The van Emden-Kowalski Operator

Proof. Suppose $I$ is a $t$-model of $P$ and note that this is equivalent to

$$I, t \models B (P^{L_J}).$$

Moreover, note that we can rewrite the van Emden-Kowalski operator more compactly as

$$T_{P, D_J}(I) = D \cup MM_{L_J}(H(P^{L_J})).$$

So we have to show $MM_{L_J}(H(P^{L_J})) \subseteq I$ (recall that $D \subseteq I$ holds by assumption), but this follows directly from $[5]$ together with the last part of Theorem 3.9.

For the other direction, we show that $MM_{L_J}(H(P^{L_J})) \subseteq I$ implies $I, t \models B (P^{L_J})$. Let $M = MM_{L_J}(H(P^{L_J}))$. By Proposition 3.5

$$M = MM_{L_J}(H(P^{L_J})) \subseteq MM_{L_J}(H(P^{L_J})) = M$$

On the other hand, $M \subseteq I$ and Proposition 3.4 imply

$$MM_{L_J}(H(P^{L_J})) \subseteq M.$$  

Consequently, from $[6]$ and $[7]$ we infer

$$M_{L_J}(H(P^{L_J})) = MM_{L_J}(H(P^{L_J})).$$

Intuitively, $[8]$ means that in case $MM_{L_J}(H(P^{L_J})) \subseteq I$, one application of $M_{L_J}$ suffices (cf. Theorem 3.9 and Example 3.2). Moreover, Proposition 3.7 implies

$$M_{L_J}(H(P^{L_J})) = M_{L_J}(H(P^{L_J})) \subseteq I.$$  

Since $M_{L_J}(H(P^{L_J}))$ is a $t$-model of $H(P^{L_J})$ (cf. Theorem 3.9), $M_{L_J}(H(P^{L_J}))$ holds by $[9]$ and $H(P^{L_J})$ is monotone, Proposition 3.8 and $[9]$ imply

$$I, t \models B (P^{L_J}).$$

Finally, Proposition 3.6 and $[10]$ imply $I, t \models B (P^{L_J})$.

Example 3.12 Reconsider the program $P$ of Example 2.8 consisting of the following rules:

- $\forall a \in P$ \quad $\neg a \in @_2 a$
- $\forall a \in P$ \quad $\neg a \in P\lor a$
- $\forall a \in P$ \quad $\neg a \in \square c$
- $\forall a \in P\lor a$ \quad $\neg a \in [0,1] \diamond c$
- $\forall a \in P\lor a$ \quad $\neg a \in [0,1] \triangleright c$
- $\forall a \in P\lor a$ \quad $\neg a \in [0,1] \triangleright d$

We have argued in Example 2.8 that the interpretation stream

$$I = \{a\}_1 \{a, b\}_3 \{a, b, c\}_4 \{a, b, c\}_5 \{a, b, c\}_6 \{a, b, c\}_7 \{a, b, c\}_8 \{c\}_9 \{c\}_10$$

of $P$ for $D = \{a\}_1 \{a, b\}_3 \{c\}_10$ and $B = \{d\}$ is a 5-model of $P$. Now we want to rigorously prove that $I$ is a 5-model of $P$ by showing that $I$ is a prefixed point of $T_{P, D, S}$. 

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3.3 The Fitting Operator

We compute:

\[ M = M_{L5}(\{H(\rho_3), H(\rho_4)\}) \]
\[ = M_{L5}(\square [1, \infty] c \land \square [2, 3] (a \land b)) \]
\[ = M_{L5}(\square [1, \infty] c) \cup M_{L5}(\square [2, 3] (a \land b)) \]
\[ = \bigcup_{t \in \text{supp} I[1, \infty]} M_{I[1, \infty], t}(c) \cup \bigcup_{t \in \text{supp} I[2, 3]} M_{I[2, 3], t}(a \land b) \]
\[ = \{a, b\}_3 \{a, b, c\}_4 \{a, b, c\}_5 \{a, b, c\}_6 \{a, b, c\}_7 \{a, b, c\}_8 \{c\}_9 \{c\}_{10} \]

and

\[ T_{P, D, 5}(I) = D \cup M_{L5}(\{H(\rho_3), H(\rho_4)\}) \]
\[ = D \cup M_{M, 5}(\square [1, \infty] c) \cup M_{M, 5}(\square [2, 3] (a \land b)) \]
\[ = D \cup \bigcup_{t \in \text{supp} M[1, \infty]} M_{M[1, \infty], t}(c) \cup \bigcup_{t \in \text{supp} M[2, 3]} M_{M[2, 3], t}(a \land b) \]
\[ = D \cup M \]
\[ = \{a\}_1 \{a, b\}_3 \{a, b, c\}_4 \{a, b, c\}_5 \{a, b, c\}_6 \{a, b, c\}_7 \{a, b, c\}_8 \{c\}_9 \{c\}_{10} \]
\[ = I. \]

3.3. The Fitting Operator

In the presence of negation, the van Emden-Kowalski operator is non-monotonic and cannot be iterated bottom-up. We therefore extend the (3-valued) Fitting operator \cite{Fitting2002} from ordinary to stream logic programs as follows. Firstly, we define a 3-valued stream to be a pair of streams \((I, J)\) with \(I \subseteq J\) or, equivalently, a sequence of pairs \((I_1, J_1)(I_2, J_2)\ldots\) with \(I_t \subseteq J_t\) for all \(t \geq 1\), with the intuitive meaning that every \(a \in I_t\) (resp., \(a \notin J_t\)) is true (resp., false) at time point \(t\), whereas every \(a \in J_t - I_t\) is undefined at \(t\).

We then define the precision ordering \(\subseteq_p\) on the set of all 3-valued streams by

\[(I, J) \subseteq_p (I', J') \iff I \subseteq I' \text{ and } J' \subseteq J.\]

Intuitively, \((I, J) \subseteq_p (I', J')\) means that \((I', J')\) is a “tighter” interval inside \((I, J)\). The maximal elements with respect to \(\subseteq_p\) are exactly the \((2\text{-valued})\) streams where we identify each stream \(I\) with \((I, I)\). Note that since distinct streams have no upper bound with respect to the precision ordering, the set of all 3-valued streams is not a lattice.

We extend the entailment relation to 3-valued streams as follows.

**Definition 3.13** For every 3-valued stream \((I, J)\) and formula \(\alpha\),

\[(I, J).t \models_B \alpha \iff K.t \models_B \alpha \text{ for every } K \in [I, J].\]

---

\(^3\)The precision ordering corresponds to the knowledge ordering \(\leq_k\) in \cite{Fitting2002}, cf. \cite{Deneckeretal2004}. 

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The intuition behind Definition 3.13 is as follows. Recall that a formula \( \alpha \) containing \( \neg, \rightarrow, \) or \( \Box \) may be non-monotone in the sense of Proposition 3.8 and in this case we have to take all possible extensions \( K \in [I, J] \) of \( I \) into account.

Now define the Fitting operator \( \Phi_{P, D, t} \) of \( P \) for \( D \) at time point \( t \), for every 3-valued stream \( (I, J) \), by

\[
\Phi_{P, D, t}(I, J) = D \cup \text{MM}_{Lt}(\{H(\rho) \mid \rho \in P : (I, J), t \models B(\rho)\}).
\]

The only difference between \( \Phi_{P, D, t} \) and \( T_{P, D, t} \) is that \( \Phi_{P, D, t} \) evaluates the body of a rule in a 3-valued stream, which guarantees the monotonicity of \( \Phi_{P, D, t} \) with respect to the precision ordering (cf. Proposition 4.1). As for ordinary logic programs, the Fitting operator encapsulates the van Emden-Kowalski operator.

**Proposition 3.14** For every stream \( I \), \( \Phi_{P, D, t}(I, I) = T_{P, D, t}(I) \).

### 4. Fixed Point Semantics

In this section, we define a fixed point semantics for the class of stream logic programs in terms of the Fitting operator defined above. More precisely, we first show that the Fitting operator is monotone with respect to the precision ordering, and conclude that certain least fixed points, the so-called \( \Phi_{P, D, t} \)-answer streams, exist (cf. Definition 4.2). Then we compare our constructive semantics to the FLP-style semantics of Beck et al. [2018] (cf. Theorem 4.7 and Theorem 5.1).

**Proposition 4.1** The Fitting operator \( \Phi_{P, D, t} \) is monotone.

**Proof.** Let \((I, J)\) and \((I', J')\) be 3-valued streams with \((I, J) \subseteq P (I', J')\). For an arbitrary rule \( \rho \in P, (I, J), t \models B(\rho) \) implies \((I', J'), t \models B(\rho)\) as a direct consequence of Definition 3.13. Finally, Proposition 3.4 implies \( \Phi_{P, D, t}(I, J) \subseteq \Phi_{P, D, t}(I', J')\).

A consequence of Proposition 4.1 is that in case \( I \) is a \( t \)-model of \( P \), \( \Phi_{P, D, t}(I, I) \) is a monotone operator on the complete lattice \([0, I]\), since for every \( K \in [0, I] \),

\[
\Phi_{P, D, t}(K, I) \subseteq \Phi_{P, D, t}(I, I) = T_{P, D, t}(I) \subseteq I
\]

holds by Proposition 3.14 and Theorem 3.11.

Define the operator \( \Phi_{P, D, t}^+ \) on the set of all \( t \)-models of \( P \) by

\[
\Phi_{P, D, t}^+(I) = \text{lfp } \Phi_{P, D, t}(I, I).
\]

The soundness of \( \Phi_{P, D, t}^+ \) is justified by the well-known Knaster-Tarski theorem which guarantees the existence of least fixed points of monotone operators on complete lattices.

We are now ready to formulate our fixed point semantics.

**Definition 4.2** We call every \( t \)-model \( I \) of \( P \) (for \( D \)) a \( \Phi_{P, D, t} \)-answer stream if \( I \) is a fixed point of \( \Phi_{P, D, t}^+ \).
For readers not familiar with the fixed point theory of logic programming, we briefly recall the basic intuitions behind Definition 4.2 in the setting of ordinary logic programs. Let \( P \) be an ordinary program, and let \( I \) be an interpretation of \( P \). The Gelfond-Lifschitz reduct of \( P \) with respect to \( I \) is defined by

\[
P^I = \{ H(\rho) \leftarrow B^+(\rho) \mid \rho \in P : I \cap \{ \} = \emptyset \}.
\]

We call \( I \) an answer set (Gelfond and Lifschitz, 1991) of \( P \) if \( I \) is the least model of \( P^I \), which coincides with \( \text{lfp} T_{P^I} \). So the computation of answer sets according to Gelfond and Lifschitz (1991) is a two-step process, and Fitting (2002) showed how these two steps can be emulated by a single (monotone) operator, namely the Fitting operator \( \Phi_P \). Specifically, the identity

\[
\Phi_P(I, I) = T_{P^I}
\]

implies that \( I \) is an answer set of \( P \) if and only if \( I \) is the least fixed point of \( \Phi_P(I, I) \) or, equivalently, if \( I \) is a fixed point of \( \Phi_P^+ \). It is now clear that Definition 4.2 is an extension of the ordinary answer set semantics to stream logic programs.

Proposition 4.3 For an ordinary program \( P \), \( I \) is an answer set of \( P \) if, and only if,

\[
I = I_t \text{ with } I_t = I \text{ is a } t\text{-answer stream of } P, \text{ for every } t \geq 1.
\]

We illustrate our fixed point semantics with the following example.

Example 4.4 Reconsider the program \( P \) of Example 2.8 consisting of the following rules:

\[
\begin{align*}
\mathcal{G} & 2a \leftarrow \neg c \\
\mathcal{B} & 3a \leftarrow \neg c \\
\mathcal{G} & 4a \leftarrow \neg c \\
\mathcal{B} & 5a \leftarrow \neg c \\
\mathcal{G} & 6a \leftarrow \neg c \\
\mathcal{B} & 7a \leftarrow \neg c \\
\mathcal{G} & 8a \leftarrow \neg c \\
\mathcal{B} & 9a \leftarrow \neg c \\
\mathcal{G} & 10a \leftarrow \neg c.
\end{align*}
\]

We have argued in Example 2.8 that the interpretation stream

\[
I = \{a\}_1 \{a, b\}_3 \{a, b, c\}_4 \{a, b, c\}_5 \{a, b, c\}_6 \{a, b, c\}_7 \{a, b, c\}_8 \{c\}_9 \{c\}_10
\]

of \( P \) for \( D = \{a\}_1 \{a, b\}_5 \{c\}_10 \) and \( B = \{d\} \) is a 5-answer stream of \( P \). We now want to apply our tools from above to rigorously prove that \( I \) is a \( \Phi_P^+ \)-answer stream by computing \( \Phi_{P, D, 5}(I) \) bottom-up as follows. We start the computation with \( I_0 = \emptyset \):

\[
\Phi_{P, D, 5}(\emptyset, I) = D \cup \text{MM}_{0,5}(H(\rho_3)) = D \cup \text{MM}_{0,5}(\square \{a\}_1 \{c\}_3) = D \cup \text{MM}_{0,5}[1, \infty; 5 \{c\}] = D
\]

where the last equality follows from \( \emptyset[1, \infty; 5 \{c\}] = \emptyset \) and \( \text{MM}_{0,5}(\square \{c\}) = \emptyset \). Then we

\[\]
Finally, we verify that \( \Phi \) is a fixed point of \( \Phi^\dagger \) or, equivalently, a \( \Phi_{\mathcal{P}, \mathcal{D}, 5} \)-answer stream.

We now wish to relate our fixed point semantics from above to the FLP-style semantics of Beck et al. (2018) presented in Section 2. Firstly, we prove some auxiliary lemmata.

Lemma 4.5 Let \( (I, J) \) be a 3-valued stream, and let \( K \in [I, J] \). Then, \( \Phi_{\mathcal{P}, \mathcal{K}, \mathcal{D}, 5}(I, J) = \Phi_{\mathcal{P}, \mathcal{D}, 5}(I, J) \).
4 FIXED POINT SEMANTICS

PROOF. Define \( \mathcal{P}^{(I,J)}_t = \{ \rho \in \mathcal{P} | (I,J), t \models_B \mathcal{P}(\rho) \} \). As a direct consequence of 3-valued entailment (cf. Definition 3.13), we have the following inclusions:

\[
\mathcal{P}^{(I,J)}_t \subseteq \mathcal{P}^K_j \subseteq \mathcal{P}.
\]  

(11)

By the monotonicity of \(MM_I\) (cf. Proposition 3.4), \( \Phi_{R_D,I,J}(I,J) \subseteq \Phi_{P,D,I,J}(I,J) \) holds whenever \( R \subseteq \mathcal{P} \), for all programs \( \mathcal{P} \) and \( R \). Therefore, we can conclude from \( \Phi_{P,D,I,J}(I,J) \subseteq \Phi_{R,D,I,J}(I,J) \).

Note that by definition, we have \( \Phi_{P,D,I,J}(I,J) = \Phi_{P,D,I,J}(I,J) \) which together with \( \Phi_{P,D,I,J}(I,J) = \Phi_{P,D,I,J}(I,J) \) entails \( \Phi_{P,D,I,J}(I,J) = \Phi_{P,D,I,J}(I,J) \).

Lemma 4.6 For every prefixed point \( K \subseteq I \) of \( T_{P,D,I,J} \), \( \Phi_{P,D,I,J}(I,J) \subseteq K \).

PROOF. We compute \( \Phi_{P,D,I,J}(I,J) \) bottom-up. Clearly, \( K_0 = 0 \subseteq I \). Since \( \Phi_{P,D,I,J}(I,J) \) is monotone, we have

\[
K_1 = \Phi_{P,D,I,J}(\emptyset, I) \subseteq \Phi_{P,D,I,J}(K, K) = T_{P,D,I,J}(K) \subseteq K.
\]

Similarly, we can compute \( K_2 = \Phi_{P,D,I,J}(K_1, I) \subseteq K \) and so on, which shows that the limit \( \Phi_{P,D,I,J}(I,J) \) is contained in \( K \), i.e., \( \Phi_{P,D,I,J}(I,J) \subseteq K \).

We are now ready to prove the main result of this paper.

Theorem 4.7 Every \( \Phi_{P,D,I,J} \)-answer stream is a \( t \)-answer stream of \( \mathcal{P} \).

PROOF. By assumption, we have \( \Phi_{P,D,I,J}(I,J) = I \) which implies

\[
\Phi_{P,D,I,J}(I,J) = I.
\]

(13)

by Lemma 4.5, that is, \( I \) is a \( \Phi_{P,D,I,J} \)-answer stream. Since every \( \Phi_{P,D,I,J} \)-answer stream is a \( t \)-model of \( \mathcal{P}^{(I,J)}_t \), it remains to show that \( I \) is a minimal \( t \)-model of \( \mathcal{P}^{(I,J)}_t \). For this suppose there exists some stream \( K \) with \( K \subseteq I \) such that \( K \) is a \( t \)-model of \( \mathcal{P}^{(I,J)}_t \).

Then, by Theorem 3.11 we have \( T_{P,D,I,J}(K) \subseteq K \subseteq I \) which implies

\[
\Phi_{P,D,I,J}(I,J) \subseteq K \subseteq I
\]

by Lemma 4.6—a contradiction to \( \Phi_{P,D,I,J}(I,J) = I \).

Theorem 4.7 shows that our fixed point semantics is sound with respect to our FLP-style semantics. However, the next example shows that the converse of Theorem 4.7 fails in general.

Example 4.8 Reconsider the program \( \mathcal{R} \) of Example 2.10 consisting of the rules

\[
a \leftarrow \square b
\]

\[
b \leftarrow \square a.
\]

In Example 2.10 we have seen that \( \{a,b\} \) is a \( t \)-answer stream of \( \mathcal{R} \), for every \( t \geq 1 \) (and \( D = B = \emptyset \)). On the other hand, we have \( \Phi_{\mathcal{R},i}(\emptyset, \{a,b\}_t) = \emptyset \) which shows that \( \{a,b\} \) is not a \( \Phi_{\mathcal{R},i} \)-answer stream.
5. Level Mappings

In this section, we define level mappings for stream logic programs in the vein of Shen et al. (2014), and prove in Theorem 5.4 that $\Phi_{P,D,I}$-answer streams characterize those $t$-models which possess a level mapping or, equivalently, which are free of circular justifications.

Firstly, we recast the notion of a partitioning to stream logic programs.

**Definition 5.1** A partitioning of a stream $I$ is a sequence of streams $S = (S_0, S_1, ..., S_m)$, $m \geq 1$, where $S_0 = \emptyset$, $S_1 \cup ... \cup S_m = I$, $S_i \neq \emptyset$ for every $i \geq 1$, and $S_i \cap S_j = \emptyset$ for every $i \neq j \neq 0$.

We now define level mappings over such partitionings.

**Definition 5.2** A $t$-level mapping of a stream $I$ with respect to $P$ (for $D$) is a partitioning $S = (S_0, S_1, ..., S_m)$ of $I$ such that for all $1 \leq i \leq m$,

$$S_i \subseteq D \cup \text{MM}_{S_i \cup \ldots \cup S_{i-1}} \left( \{ \rho \in P : (S_1 \cup \ldots \cup S_{i-1}, I), t \models B(\rho) \} \right).$$

We call $S$ a total $t$-level mapping of $I$ if in addition $I = S_0 \cup \ldots \cup S_m$ is a $t$-model of $P$.

The intuition behind Definition 5.2 is as follows. A partitioning $S = (S_0, S_1, ..., S_m)$ with $S_i = S_i \cup S_{i+1}$, $1 \leq i \leq m$, is a $t$-level mapping of $I$ if each proposition $a \in S_i$ (i.e., $a$ holds in level $i$ at time point $t_i$) is non-circularly justified by the rules in $P$, i.e., either $a \in D_i$ or there exists a rule $\rho \in P$ justifying $a$ at time point $t_i$, that is, $a$ occurs in the head of $\rho$ and the body of $\rho$ “fires” in a level smaller than $i$. For $S$ to be called total, we additionally require $S_1 \cup \ldots \cup S_m = I$ to contain every proposition occurring in a rule head which is derivable from $I$, i.e., $D \cup \text{MM}_{L_I}(H(P^I)) \subseteq I$ which, by Theorem 3.11, is equivalent to $I$ being a $t$-model of $P$. Clearly, a stream $I$ possessing a $t$-level mapping is free of circular justifications.

Note that we can rewrite (14) more compactly as

$$S_i \subseteq \Phi_{P,D,I}(S_1 \cup \ldots \cup S_{i-1}, I)$$

which shows the direct relationship between $t$-level mappings and the Fitting operator.

**Example 5.3** Once again, reconsider the program $P$ of Example 2.8. In Example 4.4, we have seen that

$$I = \{a\}_1 \{a,b\}_3 \{a,b,c\}_4 \{a,b,c\}_5 \{a,b,c\}_6 \{a,b,c\}_7 \{a,b,c\}_8 \{c\}_9 \{c\}_10$$

is a $\Phi_{P,D,I}$-answer stream for $D = \{a\}_1 \{a,b\}_5 \{c\}_10$. We construct the total 5-level mapping $S = (S_0, S_1, S_2, S_3)$ of $I$ for $P$ as follows:\footnote{By “−” we mean here the point-wise relative complement, e.g., $\{a\}_1 \{b\}_2 - \{b\}_2 = \{a\}_1$.}

- $S_0 = I_0 = \emptyset$
- $S_1 = I_1 - I_0 = D = \{a\}_1 \{a,b\}_5 \{c\}_10$
- $S_2 = I_2 - I_1 = \{c\}_4 \{c\}_5 \{c\}_6 \{c\}_7 \{c\}_8 \{c\}_9$
- $S_3 = I_3 - I_2 = \{a,b\}_3 \{a,b\}_4 \{a,b\}_6 \{a,b\}_7 \{a,b\}_8$

5 LEVEL MAPPINGS
where \( I_0 = \emptyset, I_1 = D, I_2, \) and \( I_3 = I \) are the intermediate results in the bottom-up computation of \( \Phi^t_{P,D,I}(I) \) (cf. Example 4.4).

We can characterize the \( \Phi_{P,D,I} \)-answer streams in terms of \( t \)-level mappings as follows.

**Theorem 5.4** A stream \( I \) is a \( \Phi_{P,D,I} \)-answer stream if, and only if, there is a total \( t \)-level mapping \( S \) of \( I \) with respect to \( P \).

**Proof.** For the direction from left to right, we construct the total \( t \)-level mapping \( S \) of the \( \Phi_{P,D,I} \)-answer stream \( I \) as in Example 5.3. Let \( I_0 = \emptyset, I_1, \ldots, I_m = I \) be the intermediate results of the bottom-up computation of \( \Phi^t_{P,D,I}(I) = I \), i.e.,

\[
\Phi^t_{P,D,I}(I_{i-1}, I) = I_i, \quad 1 \leq i \leq m,
\]

and define \( S_0 = \emptyset \) and \( S_i = I_i - I_{i-1} \), for all \( 1 \leq i \leq m \). By construction, we have \( I_i = S_1 \cup \ldots \cup S_i \), for all \( 1 \leq i \leq m \), which directly yields the inclusion in \( 15 \) moreover, since \( I \) is a \( t \)-model of \( P \), \( S \) is a total \( t \)-level mapping of \( I \) with respect to \( P \).

For the opposite direction, let \( S = (S_0, S_1, \ldots, S_m), m \geq 1, \) be a total \( t \)-level mapping of \( I \) with respect to \( P \). We need to show that \( I = \bigcup S \), with \( \bigcup S = S_1 \cup \ldots \cup S_m \), is a fixed point of \( \Phi^t_{P,D,I} \). Since \( S \) is total, \( I \) is a \( t \)-model of \( P \), so we have by \( 15 \) the monotonicity of \( \Phi^t_{P,D,I} \) (cf. Proposition 3.4, Proposition 3.14 and Theorem 3.11)

\[
I = \bigcup S \subseteq \Phi^t_{P,D,I}(S_1 \cup \ldots \cup S_{m-1}, I) \subseteq \Phi^t_{P,D,I} \left( \bigcup S, I \right) = \Phi^t_{P,D,I}(I, I) \subseteq \bigcup I.
\]

So \( I \) is a fixed point of \( \Phi^t_{P,D,I}(\ldots, I) \) and it remains to show that there is no fixed point \( K \subseteq I \) of \( \Phi^t_{P,D,I}(\ldots, I) \). Suppose, towards a contradiction, that for some \( K \subseteq I \),

\[
\Phi^t_{P,D,I}(K, I) = K.
\]

Since \( K \subseteq I \) there is some \( i, 1 \leq i \leq m, \) such that \( S_1 \cup \ldots \cup S_i \subseteq K \subseteq S_1 \cup \ldots \cup S_i \). So by \( 15 \) and Proposition 4.1, we have

\[
S_i \subseteq \Phi^t_{P,D,I}(S_1 \cup \ldots \cup S_{i-1}, I) \subseteq \Phi^t_{P,D,I}(K, I) = K.
\]

Consequently,

\[
K = \Phi^t_{P,D,I}(K, I) \subseteq \Phi^t_{P,D,I}(S_1 \cup \ldots \cup S_i, I) \subseteq \Phi^t_{P,D,I}(S_1 \cup \ldots \cup S_{i-1} \cup K, I) = \Phi^t_{P,D,I}(K, I) = K,
\]

which implies

\[
\Phi^t_{P,D,I}(S_1 \cup \ldots \cup S_i, I) = K.
\]
From \( \Phi_{P, D, t} \) (cf. Proposition 4.1) we infer

\[
S_{i-1} \subseteq \Phi_{P, D, t}(S_1 \cup \ldots \cup S_{i-2}, I) \subseteq \Phi_{P, D, t}(S_1 \cup \ldots \cup S_i, I) = K;
\]

\[
S_{i+1} \subseteq \Phi_{P, D, t}(S_1 \cup \ldots \cup S_i, I) = K.
\]

Hence, \( S_j \subseteq K \) for all \( 1 \leq j \leq m \), and so \( \bigcup S \subseteq K \subseteq I \)—a contradiction to \( \bigcup S = I \).

**Example 5.5** Reconsider the program \( \mathcal{R} \) of Example 2.10 consisting of the following two rules:

\[
a \leftarrow \Box b
\]

\[
b \leftarrow \Box a.
\]

In Example 2.10 we have seen that for every \( t \geq 1 \), \( I = \{a, b\} \) is a \( t \)-answer stream of \( \mathcal{R} \). Note that \( a \) and \( b \) are circularly justified in \( \mathcal{R} \). As \( I \) is not a \( \Phi_{P, D, t} \)-answer stream (cf. Example 4.8), there is no total \( t \)-level mapping of \( I \) by Theorem 5.4.

Note that Theorem 5.4 together with Theorem 4.7 (and Example 4.8) characterize our semantics as the strict constructive subclass of our FLP-style semantics.

### 6. Conclusion

This paper contributed to the foundations of stream reasoning (Valle et al., 2009; Mileo et al., 2017; Aglio et al., 2017) by providing a sound and constructive extension of the answer set semantics from ordinary to stream logic programs. For this we refined the FLP-style semantics of Beck et al. (2018). Moreover, we extended the van Emde-Kowalski and Fitting operators from ordinary to stream logic programs. As a result of our investigations, we obtained constructive semantics of stream logic programs with nice properties. More precisely, it turned out that our fixed point semantics can be characterized in terms of level mappings or, equivalently, is free of circular justifications, which is regarded as a positive feature. Moreover, the algebraic nature of our fixed point semantics yields computational proofs which are satisfactory from a mathematical point of view.

As our fixed point semantics hinges on the (extended) Fitting operator, it can be reformulated within the algebraic framework of Approximation Fixed Point Theory (AFT) (Denecker et al., 2004, 2012), which is grounded in the work of Fitting on biforms in logic programming (cf. Fitting (2002), and which captures a number of related (non-monotonic) formalisms (e.g., Denecker et al. (2003); Antić et al. (2013)).

In the future, we wish to apply the full framework of AFT to LARS, which provides a well-founded semantics (van Gelder et al. 1991), a notion of strong and uniform equivalence (Truszczynski 2000), a bottom-up semantics for disjunctive programs (Antić et al. 2013), and a recently introduced algebraic notion of groundedness (Bogaerts et al., 2015).

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