LOOP-ERASED PARTITIONING OF A GRAPH

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Abstract. We consider a random partition of the vertex set of an arbitrary graph that can be efficiently sampled using loop-erased random walks stopped at a random independent exponential time of parameter $q > 0$. The related random blocks tend to cluster nodes visited by the random walk—with generator minus the graph Laplacian—on time scale $1/q$, with $q$ being the tuning parameter. We explore the emerging macroscopic structure by analyzing 2-point correlations. To this aim, it is defined an interaction potential between pair of vertices, as the probability that they do not belong to the same block of the random partition. This interaction potential can be seen as an affinity measure for “densely connected nodes” and capture well-separated regions in network models presenting non-homogeneous landscapes. In this spirit, we compute the potential and its scaling limits on a complete graph and on a non-homogeneous weighted version with community structures. For the latter we show a phase-transition for detectability as a function of the tuning parameter and the edge weights.

1. Intro: Loop-erasure and random partitioning

Consider an arbitrary simple undirected weighted graph $G = (V, E, w)$ on $N = |V|$ vertices where $E = \{e = (x, y) : x, y \in V\}$ stands for the edge set and $w : E \to [0, \infty)$ is a given weight function. We call the Simple Random Walk (SRW) associated to $G$ the continuous-time Markov chain $(X_t)_{t \geq 0}$ with state space $V$ and minus the graph Laplacian as infinitesimal generator, i.e., the $N \times N$ matrix:

$$-L = A - D,$$

(1.1)

where for any $x, y \in [N] := \{1, 2, \ldots, N\}$, $A(x, y) = w(x, y)1_{\{x \neq y\}}$ is the weighted adjacency matrix and $D(x, y) = 1_{\{x = y\}} \sum_{z \in [N] \setminus \{x\}} w(x, z)$ is the diagonal matrix guaranteeing that the entries of each row in $L$ sum up to 0.

The goal of this paper is to explore the following probability measure on the set of partitions $P(V)$ of the vertex set $V$.

Definition 1 (Loop-erased partitioning). For a given graph $G = (V, E, w)$, fix a positive parameter $q > 0$. We call loop-erased a partition of $V$ in $m \leq N$ blocks sampled according to the following probability measure:

$$\mu_q(\Pi_m) = \frac{q^m \times \sum_{F : \Pi(F) = \Pi_m} w(F)}{Z(q)}, \quad \Pi_m \in P(V),$$

(1.2)

where the sum is over spanning rooted forests $F$’s of $G$, $\Pi(F)$ stands for the partition of $V$ induced by a forest $F$, $w(F) := \prod_{e \in F} w(e)$ for the forest weight, and $Z(q)$ is a normalizing constant. We denote by $\Pi_q$ a random variable in $P(V)$ with law $\mu_q$.

In the above definition a spanning rooted forest of a graph is a collection of rooted trees spanning its vertex set. Denoting by $F$ the set of spanning rooted forests of $G$, we notice that—due to the Markov tree
Theorem—the normalizing constant in Eq. (1.2) can be expressed as the characteristic polynomial of the matrix $-L$ evaluated at $q$, i.e.

$$Z(q) := \sum_{F \in \mathcal{F}} w(F) = \det[q + L].$$

Furthermore, the number of blocks in $\Pi_q$, denoted by $|\Pi_q|$, is distributed as the sum of $N$ independent Bernoulli random variables with success probabilities $\frac{q}{q + \lambda_i}$, for $i \leq N$, with $\lambda_i$’s being the eigenvalues of $L$. We refer the reader to [5, Prop. 2.1] for the latter statements.

1.1. Small Vs “fat” clusters. The first factor $q^m$ in Eq. (1.2) favors partitions having many small blocks as $q$ grows, while as $q$ vanishes, the measure degenerates into a one-block partition. The second combinatorial factor concentrates instead on partitions with a few “fat” blocks. Indeed in the unweighted case this second factor is counting how many rooted trees can be arranged in each block. For example, in the simple setup of a complete graph with $w \equiv 1$, the measure in Definition 1 reduces to

$$\mu_q(\Pi_m) = \frac{q^m \times \prod_{i=1}^m n_i^{n_i-1}}{Z(q)},$$

(1.3)

for a partition $\Pi_m = \{B_1, \ldots, B_m\} \in P(V)$ constituted of $m$ blocks with sizes $|B_i| = n_i$, $i \leq m$. Eq. (1.3) holds true because, by Cayley’s formula, $n_i^{n_i-2}$ unrooted trees can cover block $B_i$, and since we are dealing with rooted trees, an extra volume factor $n_i$ for the possible roots is needed. This competition between “many small and few fat” blocks depends on the delicate interplay among the tuning parameter $q$, the underlying geometry and the weight function $w$.

1.2. Sampling algorithm and Loop-Erased RW (LERW). An attractive feature of this measure is that there exists a simple efficient\(^1\) sampling algorithm. Originally due to Wilson [19] and based on the associated LERW killed at random times. The LERW with killing is the process obtained by running the SRW, erasing cycles as soon as they appear, and stopping the resulting self-avoiding trajectory at an independent time $\tau_q$ with law an exponential of parameter $q$. The algorithm can be described as follows:

(1) pick any arbitrary vertex in $V$ and run a LERW up to time $\tau_q \sim \exp(q)$. Call $\gamma_1$ the obtained self-avoiding trajectory.

(2) pick any arbitrary vertex in $V$ that does not belong to $\gamma_1$. Run a LERW until $\min\{\tau_q, \tau_{\gamma_1}\}$, $\tau_{\gamma_1}$ being the first time the SRW hits a vertex in $\gamma_1$. Call $\gamma_2$ the union of $\gamma_1$ and the new self-avoiding trajectory obtained in this step. Notice that if the killing occurs before $\tau_{\gamma_1}$, then $\gamma_2$ is a rooted forest in $G$, else $\gamma_2$ is a rooted tree.

(3) Iterate step (2) with $\gamma_{\ell+1}$ in place of $\gamma_\ell$ until exhaustion of the vertex set $V$.

When this algorithm stops, it produces a spanning rooted forest $F \in \mathcal{F}$, where the roots are the points where the involved LERWs were killed along the algorithm steps. The resulting forest $F$ on $G$ induces the partition $\Pi(F)$ of the vertex set $V$, where each block is identified by vertices belonging to the same tree. It can be shown that the probability to obtain a given rooted spanning forest $F$ is proportional to $q$ to the power of the number of trees, times the forest weight $w(F)$. It follows that the induced partition is distributed as $\Pi_q$ in Definition 1. We refer the reader to [5] for the proof of the latter and for more detailed aspects of this algorithm, including dynamical variants. In the sequel we will denote by $\mathbb{P}$ a probability measure on an abstract probability space sufficiently rich for the randomness required in the steps of this algorithm.

\(^1\)The averaged running time is given by the sum of the inverse of the eigenvalues of the graph Laplacian, see [14].
1.3. **Partition detecting metastable landscapes.** The sampling algorithm described above shows that the resulting partition has the tendency to cluster in the same block (tree) points that can be visited by the SRW with high probability on time scale $\tau_q$. In this sense the loop-erased partition has the tendency to capture metastable-like regions (blocks), namely, regions of points from which it is difficult for the SRW to escape on time scale $1/q$. This makes the probability $\mu_q$ an interesting measure for randomized clustering procedures, see in this direction [2] and [3, Sec. 5]. Still, it is not a-priori clear how strong and stable is this feature of capturing metastable landscapes, since it heavily depends on the underlying geometry (weighted adjacency matrix) and the choice of the killing parameter, $q$. The goal of this paper is to start making precise this heuristic by analyzing 2-points correlations associated to $\mu_q$ on the simplest informative geometries.

1.4. **Two-point correlations.** Consider the probability that two points in $V$ belong to different blocks in $\Pi_q$. As we will see, such a 2-point correlation function turns out to be analyzable by means of LERW explorations, and it encodes relevant information on how the loop-erased partition looks like on the underlying graph as a function of the parameters. Here is the formal definition together with an operative characterization.

**Definition 2 (Pairwise interaction potential).** For a given $q > 0$ and $G$, fix $x, y \in V$, we call pairwise interaction potential the following probability:

$$U_q(x, y) := \mathbb{P}(x \text{ and } y \text{ are in different blocks of } \Pi_q) = \sum_{\gamma} \mathbb{P}^{LE}_x(\Gamma = \gamma) \mathbb{P}^{y}(\tau_\gamma > \tau_q)$$

(1.4)

where $\mathbb{P}^{LE}_x$ and $\mathbb{P}_y$ stand for the laws of the LERW killed at rate $q$ and of the SRW, respectively, starting from $x \in V$, and the above sum runs over all possible loop-erased paths $\gamma$'s starting at $x$.

The representation in Eq. (1.4) is a consequence of the sampling algorithm in Section 1.2 and it holds true since, remarkably, in steps (1) and (2) of the algorithm the starting points can be chosen arbitrarily.

Furthermore, we notice that, as for any general random partitioning of a vertex set, such an interaction potential defines a distance on the vertex set. This specific metric $U_q(x, y)$ can be interpreted as an affinity measure capturing how densely connected vertices $x$ and $y$ are in the graph $G$. Thus providing a further motivation to analyze it.

1.5. **Related literature.** Several properties of the forest measure associated to the loop-erased partitioning have been derived in the recent [5, 6]. Based on these results, in [3, Prop. 6] and [4, Sect. 5.2], the authors proposed an approach making use of the loop-erased partition and so-called intertwining dualities to describe the evolution of local equilibria of a finite state space Markov chain presenting traps.

From a broad perspective, the presence of the first factor in Eq. (1.2) shows that this measure has the flavor of the celebrated Random Cluster Model, or so-called FK-percolation, see e.g. [10]. Nonetheless, these objects are quite different. Indeed, our clusters are identified by oriented spanning trees rather than arbitrary undirected subgraphs, and from a practical view-point, unlike the FK-percolation, we do have at disposal an efficient exact sampling method.

As mentioned before, this sampling method based on LERW is originally due to Wilson [19] and shows that the measure considered herein is intimately related to the well-known Uniform Spanning Tree (UST) measure. Actually the measure on spanning rooted forests mentioned in Section 1.2 can be seen as a non-homogenous variant of the UST measure. Therefore the results presented in the next section are along the line of the flourishing literature on scaling limits of the UST and LERW, see e.g. [1, 7, 8, 9, 11, 13, 12, 15, 18].

A detailed exact and asymptotic analysis of observables related to Wilson’s algorithm on a complete graph have been pursued in [16]. The derivation of our results is in this spirit, though, we deal with the additional
randomness given by the presence of the killing parameter, which in turns makes the combinatorics more involved.

1.6. Paper overview. Our main theorems are presented in Section 2 and identify the pairwise-potential and its scaling limits on a complete graph, Theorem 1, and on a non-homogeneous complete graph with two communities, Theorems 2 and 3. Some basic consequences on the macroscopic emergent partition on these mean-field models are derived in Corollary 1.

The concluding Sections 3 and 4 are devoted to the proofs for the complete graph and the community model, respectively.

1.7. Basic standard notation. In what follows we will use the following standard asymptotic notation. For given sequences \( f(N) \) and \( g(N) \), we write:

- \( f(N) = o(g(N)) \) if \( \lim_{N \to \infty} \frac{f(N)}{g(N)} = 0 \).
- \( f(N) = O(g(N)) \) if \( \lim \sup_{N \to \infty} \frac{f(N)}{g(N)} < \infty \).
- \( f(N) = \omega(g(N)) \) if \( \lim \inf_{N \to \infty} \frac{f(N)}{g(N)} = \infty \).
- \( f(N) = \Omega(g(N)) \) if \( \lim \inf_{N \to \infty} \frac{f(N)}{g(N)} > 0 \).
- \( f(N) = \Theta(g(N)) \) if \( 0 < \lim \inf_{N \to \infty} \frac{f(N)}{g(N)} \leq \lim \sup_{N \to \infty} \frac{f(N)}{g(N)} < \infty \).
- \( f(N) \sim g(N) \) if \( \lim_{N \to \infty} \frac{f(N)}{g(N)} = 1 \).

For \( k \leq n \in \mathbb{N} \) we will denote by \( (n)_k := n(n-1)(n-2)\cdots(n-k) \) the descendent factorial. Furthermore, we denote by \( I \) the identity matrix, \( 1 \) and \( 1' \), respectively, for the row and column vectors of all 1’s, where the dimensions will be clear from the context. We will write \( A^T \) for the transpose of a matrix \( A \).

2. Results: Potential and its scaling on mean-field models

Our first result gives the characterization of the pairwise interaction potential in absence of geometry for finite \( N \), and shows that this probability is asymptotically non-degenerate at scale \( \sqrt{N} \):

**Theorem 1. (Mean-field potential and limiting law)** Fix \( q > 0 \) and let \( K_N \) be a complete graph on \( N \geq 1 \) vertices with constant edge weight \( w > 0 \). Then, for all \( x \neq y \in [N] \),

\[
U_q^{(N)}(x,y) = U_q^{(N)} = \sum_{h=1}^{N-1} \frac{q}{q + Nw} \left( \frac{Nw}{q + Nw} \right)^{h-1} \prod_{k=2}^{h} \left( 1 - \frac{k}{N} \right),
\]

(2.1)

Furthermore, if \( q = \Theta(wz\sqrt{N}) \), for fixed \( z > 0 \) and \( w = \Theta(1) \), then

\[
U_q := \lim_{N \to \infty} U_q^{(N)} = \sqrt{2\pi} z e^{\frac{-z^2}{2}} P(Z > z),
\]

(2.2)

with \( Z \) being a standard Gaussian random variable.

Our second result is the analogous of Eq. (2.1) when still every vertex is accessible from any other, but the edge weights are non-homogeneous and give rise to a community structure. In this sense we will informally refer to this graph as of a mean-field-community model. Formally, for given positive reals \( w_1 \) and \( w_2 \), we denote by \( K_2N(w_1, w_2) \) the graph \( G \) with \( V = [2N] \), and \( w(e) = w_1 \) if \( e = (x,y) \) is such that either \( x, y \in [N] \) or \( x,y \in [2N] \setminus [N] \), and \( w(e) = w_2 \) otherwise. Thus, the weight \( w_1 \) measures the pairwise connection intensity within the same community, while \( w_2 \) between pairs of nodes belonging to different communities.

Given the symmetry of the model, we will use the notation \( U_q^{(N)}(out) \) to refer to the potential \( U_q^{(N)}(x,y) \), for \( x \) and \( y \) in different communities. Conversely, we set \( U_q^{(N)}(in) \) for the potential associated to two nodes belonging to the same community.
Theorem 2. (Potential for mean-field-community model) Fix $q, w_1, w_2 > 0$ and consider a two-community-graph $K_{2N}(w_1, w_2)$. Let $T_q \geq 1$ be a geometric random variable with success parameter

$$
\alpha := \frac{q}{q + N(w_1 + w_2)}
$$

and let $\left(\tilde{X}_n\right)_{n \in \mathbb{N}_0}$ be a discrete-time Markov chain with state space $\{1, 2\}$ and transition matrix

$$
\tilde{P} = \begin{pmatrix}
p & 1 - p \\
1 - p & p
\end{pmatrix}, \quad p = \frac{w_1}{w_1 + w_2}.
$$

Denote by $\ell(t) = \sum_{s \leq t} 1\{\tilde{X}_s = 1\}$ the corresponding local time in state $1$ up to time $t$ and by $\tilde{P}_\ell$ the corresponding path measure starting from $1$.

For $x \in [N]$, set $* = \text{in}$ if $y \in [N]$, and $* = \text{out}$ if $y \in [2N] \setminus [N]$, then

$$
U_q^{(N)}(x, y) = U_q^{(N)}(\bullet) := \sum_{n \geq 1} \tilde{P}(T_q = n) \sum_{k=1}^{n} \tilde{P}_\ell(\ell(n) = k) N^{-n+1} \hat{f}(n, k) \theta(n, k) P_\bullet^!(n, k) \tag{2.3}
$$

where

$$
\hat{f}(n, k) = (N - 2)k - 1(N - 1)n - k, \quad \theta(n, k) = \frac{(q - \lambda_1(n, k)) (q - \lambda_2(n, k))}{q(q + 2Nw_2)} \tag{2.4}
$$

and

$$
P_\bullet^!(n, k) = \frac{q(q + k_\bullet(w_1 - w_2) + w_2 N)}{[q + kw_1][q + (n - k)w_1] + Nw_2(2q + nw_1) + w_2[Nn - k(n - k)]} \times \eta_\bullet \tag{2.6}
$$

with

$$
k_\bullet := \begin{cases} k, & \text{if } \bullet = \text{out}, \\
-n-k, & \text{if } \bullet = \text{in}
\end{cases} \quad \eta_\bullet := \begin{cases} (N - 1)(N - n + k - 1), & \text{if } \bullet = \text{out}, \\
N(N - k - 1), & \text{if } \bullet = \text{in}.
\end{cases} \tag{2.7}
$$

The above theorem is saying that the pairwise potential can be seen as the double-expectation of the function $g_\bullet(n, k) = N^{-n+1} \left(\hat{f}P_\bullet^!(n, k)\right)$ in Eq. (2.3) with respect to the geometric time $T_q$ and to the local time of the coarse-grained RW $\{\tilde{X}_n\}_{n \in \mathbb{N}_0}$. As can be seen in the proof, the analysis of this model can be in fact reduced to the study of such a coarse-grained RW jumping between the two “clumped communities” up to the independent random time $T_q$. The function $g_\bullet$ is the crucial combinatorial term encoding in the different parameter regimes the most likely trajectories for such a stopped two-state macroscopic walk.

Remark 1. (Extensions to many communities of arbitrary sizes and weights) The formula in Eq. (2.3) can be derived also for the general model with arbitrary number of communities of variable compatible sizes and arbitrary weights within and among communities. The corresponding statement and proof are more involved but they follow exactly the same scheme of this equal-size-two-community case captured in the above theorem. We refer the reader interested in such an extension to [17].

The next theorem gives the scaling limit of the potential computed in Theorem 2, the resulting scenario is summarized in the phase-diagram in Fig. 1.
Figure 1. The above diagram describes at glance the limiting behavior of the interaction potential as captured in Theorem 3. The \textit{detectability} region corresponds to the regimes where the difference of the \textit{in}- and \textit{out}-potential is maximal. In this case, indeed, the SRW does not manage to exit its starting community within time scale $1/q$ and hence it is “confined with high probability to its local universe”. In the \textit{dust region} both \textit{in}- and \textit{out}-potential degenerates to 1, it is in fact a regime where the killing rate is sufficiently large (recall from Eq. (2.2) that $\sqrt{N}$ is the critical scale for the complete graph) to produce “dust” as emerging partition. Finally, the \textit{global mixing region} is the other degenerate regime where the RW “mixes globally” in the sense that it changes community many times within time scale $1/q$, hence loosing memory of its starting community. The separating lines correspond to the delicate critical phases where the competition of the above behaviors occurs. This will become transparent in the proof in Section 4.2 where such boundaries will deserve a more detailed asymptotic analysis.

\textbf{Theorem 3. (Detectability and phase diagram for two communities)} Under the assumptions of Theorem 2, set $w_1 = 1$, $w_2 = N^{-\beta}$ and $q = N^\alpha$ for some $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^+$. Then:

\begin{enumerate}[(a)]
  \item if $1 - \beta < \alpha = \frac{1}{2}$, $\lim_{N \to \infty} U_q^{(N)}(\text{out}) = 1$ and $\lim_{N \to \infty} U_q^{(N)}(\text{in}) = \varepsilon_0(\beta) \in (0, 1)$.
  \item if $1 - \beta < \alpha < \frac{1}{2}$, $\lim_{N \to \infty} U_q^{(N)}(\text{out}) = 1$ and $\lim_{N \to \infty} U_q^{(N)}(\text{in}) = 0$.
  \item if $\alpha = 1 - \beta < \frac{1}{2}$, $\lim_{N \to \infty} U_q^{(N)}(\text{out}) = \varepsilon_2(\alpha, \beta) \in (0, 1)$ and $\lim_{N \to \infty} U_q^{(N)}(\text{in}) = 0$.
  \item if $\alpha < \min\{\frac{1}{2}, 1 - \beta\}$, $\lim_{N \to \infty} U_q^{(N)}(\ast) = 0$, $\ast \in \{\text{in}, \text{out}\}$.
  \item if $\alpha = \frac{1}{2} < 1 - \beta$, $\lim_{N \to \infty} U_q^{(N)}(\ast) = \varepsilon_1(\alpha, \beta) \in (0, 1)$, $\ast \in \{\text{in}, \text{out}\}$.
\end{enumerate}
(f) if $\alpha > \frac{1}{2}$, $\lim_{N \to \infty} U_q^{(N)}(\ast) = 1, \ast \in \{\text{in, out}\}$.

**Remark 2. (Anticommunities for negative $\beta$)** The above theorem is stated for arbitrary $\alpha \in \mathbb{R}$ and $\beta > 0$. We notice that while for $\beta = 0$ we are back to the complete graph with constant weight 1, for $\beta < 0$, it would be more appropriate to speak about “anticommunities” rather than communities. In fact in this case, at every step, the SRW prefers to change community rather than staying in its original one. Thus, it is somewhat artificial to see what the loop-erased partitioning captures. This is the reason why the plot in Fig. 1 is restricted to $\beta \geq 0$. However, the theorem still remains valid and not surprisingly the difference between the in and out potentials turns out to be zero.

**Remark 3. (Community detection)** We notice that this two-point correlation function is sufficient to detect the underlying communities in a sub-region where the ratio of the out and in weights is bigger than $\sqrt{N}$. This suggests that estimating the probabilities in Definition 2 could be a valuable cheap method to design a community detection algorithm for well-separated regions. Nonetheless, there might be other observables associated to $\Pi_q$ which perform better, in the sense that they can be used for detection beyond regions (a)-(c) in Fig. 1. It is not the scope of this paper to explore the practical implication of this loop-erased partitioning in the context of community detection. For this reason we will omit this and similar type of algorithmic considerations. As mentioned in the introduction, the main goal is rather to start understanding analytically the measure $\mu_q$ as a functions of the tuning parameter.

The last statement below collects some simple consequences, deduced from these two-point correlations, on the macroscopic structure of $\Pi_q$ on such mean-field models. We recall that $|\Pi_q|$ stands for the number of blocks in the random partition $\Pi_q$.

**Corollary 1. (Macroscopic emergent structure)** Under the assumption of Theorem 3, the following scenarios hold true. If $\beta > 0$, there exists $c > 0$ depending only on $\alpha$ and $\beta$ s.t.

$$\mathbb{P}(|\Pi_q| = cN^{\alpha\Lambda}(1 + o(1))) = 1 - o(1).$$

Moreover:

(a) if $1 - \beta < \alpha = \frac{1}{2}$ then whp there are two blocks of linear size s.t. each block has a fraction $(1 - o(1))$ of vertices from the same community.

(b) if $1 - \beta < \alpha < \frac{1}{2}$ then whp there are two blocks of size $N(1 - o(1))$ s.t. each block has a fraction $(1 - o(1))$ of vertices from the same community.

(c) if $\alpha = 1 - \beta < \frac{1}{2}$ then whp there is at least a block of linear size.

(d) if $\alpha < \min\{\frac{1}{2}, 1 - \beta\}$ then whp there is one block of size $2N(1 - o(1))$.

(e) if $\alpha = \frac{1}{2} < 1 - \beta$ then whp there is at least a block of linear size.

(f) if $\alpha > \frac{1}{2}$ then whp blocks of linear size do not exist.

3. **Proofs of Theorem 1: homogeneous complete graph**

**Proof of Eq. (2.1).** For convenience, we consider a discretization of the continuous time Markov process with generator

$$-L = A - D, \quad \text{with} \quad A = w(11' - I) \quad \text{and} \quad D = -(n - 1)wI. \quad (3.1)$$

Set $L = \frac{1}{\alpha}L$ with $\alpha = Nw$, so that $L = I - \frac{1}{n}11'$ and the associated transition matrix is given by

$$P = L - I = \frac{1}{n}11' \quad (3.2)$$
If we consider the killing as an absorbing state within the state space of the Markov chain extended from \( V \) to \( V \cup \{ \Delta \} \), \( \Delta \) denoting this absorbing state, we get the adjacency matrix
\[
\hat{A} = \begin{pmatrix} w \mathbf{1}' & q \mathbf{1} \\ 0' & 0 \end{pmatrix},
\]
and generator
\[
-\hat{\mathcal{L}} = \hat{A} - \hat{D}, \quad \hat{D} = \begin{pmatrix} -[(N - 1)w + q]I & 0 \\ 0 & 0 \end{pmatrix}.
\]
We can then normalize it by setting
\[
-\hat{\mathcal{L}} = -\frac{1}{\alpha + q} \hat{\mathcal{L}} = \begin{pmatrix} \frac{w}{Nw + q} \mathbf{1}' - I & \frac{q}{Nw + q} \mathbf{1} \\ 0' & 0 \end{pmatrix}
\]
and get a discrete RW with associate transition matrix given by
\[
\hat{P} = I - \hat{\mathcal{L}} = \begin{pmatrix} \frac{w}{Nw + q} \mathbf{1}' & \frac{q}{Nw + q} \mathbf{1} \\ 0' & 1 \end{pmatrix} = \begin{pmatrix} (1 - p) \frac{1}{N} \mathbf{1}' & p \mathbf{1} \\ 0' & 1 \end{pmatrix},
\]
where
\[
p := \frac{q}{\alpha + q}.
\]
It should be clear that a sample of a LE-path starting at a given vertex can be obtained as the output of the following procedure:

- With probability \( p \) the discrete process reaches the absorbing state. In particular we set \( T_q \) for a geometric random variable of parameter \( q/(\alpha + q) \).
- With probability \( 1 - p \) the LERW moves accordingly to the law \( P(v, \cdot) \) where \( v \) is the last reached node.
- We call \( H_n \) the vertices covered by the LE-path up to time \( n \). Then, if at time \( n + 1 \) the transition \( X_n \to X_{n+1} \) takes place and the vertex \( X_{n+1} \not\in H_n \), then \( H_{n+1} = H_n \cup \{ X_{n+1} \} \). Conditioning on \( |H_n| \), the latter event occurs with probability \( \frac{N - |H_n|}{N} \). Conversely, if \( X_{n+1} \in H_n \), then we remove from \( H_n \) all the vertices that has been visited by the LERW since its last visit to \( X_{n+1} \). As consequence the quantity \( |H| \) reduces. One can then compute that the reductions occur with law
\[
P(|H_{n+1}| = h \mid |H_n| \geq h, T_q > n + 1) = \frac{1}{N}.
\]
It would be easier to look at the quantity \( |H_n| \) by using the following metaphor. We interpret \( |H_n| \) as the height from which a bear fall down while moving on a stair of height \( n \). In particular, we will assume that

- The bear starts with probability \( 1 \) from the first step.
- At each time the bear select a step of the stair uniformly at random, including also the step he currently stands on.
- If the choice made by the bear is a lower step (or the current one), he moves to that step.
- If he chooses an upper step, then he walks in the upper direction by a single step.
- Before doing each step, there is a probability \( p \) as in Eq. (3.7) that the bear “falls down.”
Let us next fix $q = 0$, that is, $p = 0$, so that we can study the bear’s dynamic independently of his falling. By setting $Z(n)$ for the position of the bear at time $n \in \mathbb{N}$, we get

\[
\mathbb{P}(Z(0) = \cdot) = (1, 0, 0, \ldots, 0) \tag{3.9}
\]

\[
\mathbb{P}(Z(1) = \cdot) = \left(\frac{1}{N}, 1 - \frac{1}{N}, 0, 0, \ldots, 0\right) \tag{3.10}
\]

\[
\mathbb{P}(Z(2) = \cdot) = \left(\frac{1}{N}, \left(1 - \frac{1}{N}\right) \frac{2}{N}, \left(1 - \frac{1}{N}\right) \left(1 - \frac{2}{N}\right), 0, \ldots, 0\right) \tag{3.11}
\]

\[
\mathbb{P}(Z(3) = \cdot) = \left(\frac{1}{N}, \left(1 - \frac{1}{N}\right) \frac{2}{N}, \left(1 - \frac{1}{N}\right) \left(1 - \frac{2}{N}\right) \frac{3}{N}, \left(1 - \frac{1}{N}\right) \left(1 - \frac{2}{N}\right) \left(1 - \frac{3}{N}\right), \ldots, 0\right) \tag{3.12}
\]

\[
\mathbb{P}(Z(n) = \cdot) = \begin{cases} 
(1 - \frac{1}{N}) (1 - \frac{2}{N}) \cdots (1 - \frac{n-1}{N}) \frac{h}{N} & \text{if } n \geq h \\
(1 - \frac{1}{N}) (1 - \frac{2}{N}) \cdots (1 - \frac{n-1}{N}) & \text{if } n = h - 1 \\
0 & \text{if } n < h - 1.
\end{cases} \tag{3.13}
\]

The latter implies that at time $n = h$ we reached the ergodic measure over the first $h$ steps of the stair, while at time $n = N$ the probability measure is exactly the ergodic one.

It is interesting to notice that an easier expression can be written for the cumulative distribution of the variable $Z(n)$, i.e.

\[
\mathbb{P}(Z(n) \geq h) = \begin{cases} 
(1 - \frac{1}{N}) (1 - \frac{2}{N}) \cdots (1 - \frac{n-1}{N}) & \text{if } n \geq h - 1 \\
0 & \text{if } n < h - 1.
\end{cases} \tag{3.14}
\]

Next, calling $T^-$ the time immediately before the bear falls, we get

\[
\mathbb{P} (Z(T^-) \geq h) = \mathbb{P} (T^- < h - 1) \mathbb{P} (Z(T^-) \geq h | T^- < n - 1) + \mathbb{P} (T^- \geq h - 1) \mathbb{P} (Z(T^-) \geq h | T^- \geq n - 1) \\
= 0 + (1 - p)^{h-1} \left(1 - \frac{1}{N}\right) \left(1 - \frac{2}{N}\right) \cdots \left(1 - \frac{h-1}{N}\right) \tag{3.15}
\]

which gives us the distribution of the last step of the bear before his failing. Recall that this is equivalent to the length of the original LERW starting on $x \in \mathcal{K}_N$, when the walk is stopped at an exponential time of rate $q$. Hence, we are now left to compute the probability that another walker, starting on $y \neq x$, is killed before it hits the previously sampled LERW.

Thanks to the bear metaphor, for the size of the LE-trajectory we get:

\[
\mathbb{P}_{x}^{LE_q} (|\Gamma| \geq h) = (1 - p)^{h-1} \prod_{i=1}^{h-1} \left(1 - \frac{i}{N}\right) \tag{3.16}
\]
and by explicit computation, setting \( T_\Gamma \) for the first hitting time of the LE-path \( \Gamma \),

\[
U_q^{(N)}(x, y) = \sum_{h \geq 1} \mathbb{P}_x^{LE}(|\Gamma| = h) \mathbb{P}_y(T_q < T_\Gamma| |\Gamma| = h)
\]

\[
= \sum_{h \geq 1} \mathbb{P}_x^{LE}(|\Gamma| = h)[\mathbb{P}_y(T_q < T_\Gamma| |\Gamma| = h, y \in \Gamma) \mathbb{P}(y \notin \Gamma| |\Gamma| = h)]
\]

\[
= \sum_{h \geq 1} \mathbb{P}_x^{LE}(|\Gamma| = h) \left( \frac{q}{q + hw} \right) \frac{N - h}{N - 1}
\]

\[
= \sum_{h = 1}^{N - 1} \left[ \left( \frac{Nw}{q + Nw} \right)^{h-1} \prod_{i=1}^{h-1} \left( 1 - \frac{i}{N} \right) \right] \left( \frac{q}{q + hw} \right) \frac{N - h}{N - 1} + \sum_{h = 1}^{N - 1} \left[ \left( \frac{Nw}{q + Nw} \right)^{h} \prod_{i=1}^{h-1} \left( 1 - \frac{i}{N} \right) \right] \left( \frac{q}{q + hw} \right) \frac{N - h}{N - 1}
\]

\[
= \sum_{h = 1}^{N - 1} \left[ \left( \frac{Nw}{q + Nw} \right)^{h} \prod_{i=1}^{h-1} \left( 1 - \frac{i}{N} \right) \right] \left( \frac{q}{q + hw} \right) \frac{N - h}{N - 1} + \sum_{h = 1}^{N - 1} \left[ \left( \frac{Nw}{q + Nw} \right)^{h+1} \prod_{i=1}^{h} \left( 1 - \frac{i}{N} \right) \right] \left( \frac{q}{q + hw} \right) \frac{N - h}{N - 1}
\]

\[
= \sum_{h = 0}^{N - 2} \left[ \left( \frac{Nw}{q + Nw} \right)^{k+1} \prod_{i=2}^{k+1} \left( 1 - \frac{i}{N} \right) \right] \left( \frac{q}{q + \alpha} \right)
\]

\[
\square
\]

**Proof of Eq. (2.2).** Let

\[
\xi_q := \frac{q}{Nw + q}
\]

and notice that if \( q = x\sqrt{N} \), with \( x, w = \Theta(1) \), then

\[
q = \frac{Nw\xi_q}{N - \xi_q} \Rightarrow q \sim w\xi_q.
\]
Call
\[
f(k, N) := \prod_{i=2}^{k} \left(1 - \frac{i}{N}\right),
\]
(3.19)
in order to rewrite
\[
U_q^{(N)} = \sum_{k=0}^{N-2} \left(\frac{\xi_q}{N}\right) \left(1 - \frac{\xi_q}{N}\right)^k \prod_{i=2}^{k+1} \left(1 - \frac{i}{N}\right)
\]
(3.20)
and notice that the first term in this last sum is the probability that the geometric random variable \(T_q \sim Geom \left(\frac{\xi_q}{N}\right)\) assumes value \(k\). Moreover it trivially holds that
\[
f(k + 1, N) \leq 1, \quad \forall k \in \mathbb{N}, \quad f(k + 1, N) = 0, \quad \forall k \geq N - 1.
\]
(3.21)
Hence,
\[
U_q^{(N)} = \mathbb{E}[f(T_q + 1, N)].
\]
(3.22)
Let us approximate \(\ln f(k + 1, N)\) at the first order as follows
\[
\ln f(k + 1, N) = \sum_{i=2}^{k+1} \ln \left(1 - \frac{i}{N}\right) = - \sum_{i=2}^{k+1} \frac{i}{N} + O \left(\frac{i^2}{N^2}\right)
\]
\[
= - \frac{(k+1)(k+2)}{2N} + kO \left(\frac{k^2}{N^2}\right) = - \frac{k^2}{2N} + O \left(\frac{k^3}{N^2}\right)
\]
(3.23)
Next, set \(Y \sim exp(x)\) and \(Z \sim \mathcal{N}(0,1)\), notice that \(\mathbb{E}[e^{\frac{Y^2}{2}}] = \sqrt{2\pi}xe^{\frac{x^2}{2}}\mathbb{P}(Z > x)\) and that
\[
\lim_{N \to \infty} \left|\mathbb{E}[e^{-\frac{T_q^2}{2N}}] - \mathbb{E}[e^{\frac{Y^2}{2}}]\right| = 0,
\]
(3.24)
since \(T_q/\sqrt{N}\) converges in distribution to \(Y\) as \(N\) diverges. In view of the latter together with Eq. (3.22), we can estimate
\[ U_q^{(N)}(x) - \sqrt{2\pi x} e^{\frac{x^2}{2}} \mathbb{P}(Z > x) \leq \left| \mathbb{E}[f(T_q + 1, N)] - \mathbb{E}[e^{-\frac{T_q^2}{2\pi}}] \right| + o(1) \]

\[ \leq \left| \mathbb{E}[f(T_q + 1, N)] - \sum_{k=0}^{\lfloor N^\delta \rfloor} \mathbb{P}(T_q = k)e^{-\frac{k^2}{2\pi}} e^{c_N(k)} \right| + o(1) \]

\[ + \sum_{k=0}^{\lfloor N^\delta \rfloor} \mathbb{P}(T_q = k)e^{-\frac{k^2}{2\pi}} e^{c_N(k)} - \mathbb{E}[e^{-\frac{T_q^2}{2\pi}}] \] + o(1)

\[ \leq \sum_{k=\lfloor N^\delta \rfloor + 1}^\infty \mathbb{P}(T_q = k) + \left| \sum_{k=0}^{\lfloor N^\delta \rfloor} \mathbb{P}(T_q = k)e^{-\frac{k^2}{2\pi}} e^{c_N(k)} - \sum_{k=0}^{\lfloor N^\delta \rfloor} \mathbb{P}(T_q = k)e^{-\frac{k^2}{2\pi}} \right| + o(1) \]

where the last inequality holds true by choosing any \( \delta \in \left( \frac{1}{2}, \frac{2}{3} \right) \) which in particular guarantees that \( c_N(k) = o(1) \).

\[ \square \]

4. Proofs for mean-field-communities

4.1. Proof of Theorem 2. We use here the same line of argument used in the proof of Theorem 1. We will consider the process having state space \( V = V_1 \sqcup V_2 \), where

\[ V_1 = \{1, \ldots, N_1\}, \quad V_2 = \{N_1 + 1, \ldots, N_1 + N_2\}, \]

and generator

\[ -\mathcal{L}(x, y) = \begin{cases} w_1 & \text{if } x \neq y \text{ and } x, y \in \text{the same community} \\ w_2 & \text{if } x \neq y \text{ and } x, y \not\in \text{the same community} \\ -(N_1 - 1)w_1 - N_2w_2 & \text{if } x = y \text{ and } x \in V_1 \\ -(N_2 - 1)w_1 - N_1w_2 & \text{if } x = y \text{ and } x \in V_2. \end{cases} \] (4.1)

We will specialize later on the case \( N_1 = N_2 = N \).

We now consider a killed LERW \( \Gamma \), and we denote by \( \Gamma_i \) the set of points of the \( i \)-th community belonging to \( \Gamma \), i.e.,

\[ \Gamma_i = \Gamma \cap V_i, \quad i = 1, 2. \] (4.2)

We can write

\[ \mathbb{P}_x^{LE}\left(|\Gamma_1| = k_1, |\Gamma_2| = k_2\right) = \sum_{\gamma:|\gamma_1|=k_1,|\gamma_2|=k_2} \mathbb{P}_x^{LE}(\gamma), \] (4.3)

and we assume, without loss of generality, that \( x \in V_1 \); then, by conditioning, we get for \( y \neq x \) with \( y \in V_j \), \( j = 1, 2 \)

\[ U_q^{(N)}(x, y) = \sum_{k_1=1}^{N_1-1} \sum_{k_2=0}^{N_2-1} \mathbb{P}_x^{LE}\left(|\Gamma_1| = k_1, |\Gamma_2| = k_2\right) \cdot \mathbb{P}_y \left(T_q < T_\Gamma | \Gamma\right), \] (4.4)

\( T_\Gamma \) being the hitting time of \( \Gamma \).
The LERW starting from \( x \). A result due to Marchal [14] provides the following explicit expression for the probability of a loop erased trajectory:

\[
\mathbb{P}^{LE}_x(\Gamma = \gamma) = \prod_{i=1}^{|\gamma|} w(x_{i-1}, x_i) \frac{\det_{V\setminus\gamma} (qI + L)}{\det (qI + L)}.
\]  

By looking closely at the latter formula we distinguish two parts: a product over the weights of the edges of the path and an algebraic part containing the ratio of two determinants which encodes the “loop-erased” feature of the process. In particular we notice that the former contains all the details about the trajectory, while the latter only depends on the number of points visited in each community. Let \( j_1 \) (respectively, \( j_2 \)) be the number of jumps from the first community to the second (from the second to the first, respectively) along the LE-path. We have

\[
\mathbb{P}^{LE}_x(|\Gamma_1| = k_1, |\Gamma_2| = k_2 | x \in V_1, y \in V_2) =
\sum_{\gamma: |\gamma_1| = k_1, |\gamma_2| = k_2} \mathbb{P}^{LE}_x(\Gamma = \gamma)
= \binom{N_1 - 1}{k_1 - 1} \binom{N_2 - 1}{k_2} \cdot (k_1 - 1)! (k_2)! \cdot \sum_{j_1=0}^{\min(k_1,k_2)} \sum_{j_2=j_1+1}^{j_1} (j_1 - 1, j_1) \cdot (j_2 - 1, j_2) \cdot
w_1^{k_1+j_2-(j_1+j_2)-1} w_2^{j_1+j_2} q^{\det_{V\setminus\{1,2,...,k_1,N_1+1,N_1+2,...,N_1+k_2\}} (qI + L)} \frac{\det (qI + L)}{\det_{V\setminus\gamma} (qI + L)}.
\]

where

- The first binomial coefficients stay for the \( k_1 - 1 \) possible choices for the points in \( G_1 \) (one of those must be \( x \)) over the possible \( N_1 - 1 \) points of the first community (except \( x \)). In the second community we can choose any \( k_2 \) vertices over the possible \( N_2 - 1 \) vertices of the second community (except \( y \)).
- The factorials stay for the possible ordering of the nodes covered in each community. Notice that the path on the first community must start by \( x \).
- We sum over all the possible jumps from the first community to the second, \( j_1 \), and from the second to the first, \( j_2 \) (notice that if \( j_2 \) must be equal or one smaller than \( j_1 \)).
- For any choice over the product of the previous three terms we have a path that has probability as given by the Marchal formula.

In the case in which we condition on having both \( x \) and \( y \) in the same (first, say) community we have

\[
\mathbb{P}^{LE}_x(|\Gamma_1| = k_1, |\Gamma_2| = k_2 | x \in V_1, y \in V_1) =
\sum_{\gamma: |\gamma_1| = k_1, |\gamma_2| = k_2} \mathbb{P}^{LE}_x(\Gamma = \gamma)
= \binom{N_1 - 2}{k_1 - 1} \binom{N_2}{k_2} \cdot (k_1 - 1)! (k_2)! \cdot \sum_{j_1=0}^{\min(k_1,k_2)} \sum_{j_2=j_1+1}^{j_1} (j_1 - 1, j_1) \cdot (j_2 - 1, j_2) \cdot
w_1^{k_1+k_2-(j_1+j_2)-1} w_2^{j_1+j_2} q^{\det_{V\setminus\{1,2,...,k_1,N_1+1,N_1+2,...,N_1+k_2\}} (qI + L)} \frac{\det (qI + L)}{\det_{V\setminus\gamma} (qI + L)}.
\]

Namely, only the first combinatorial term changes.
The ratio of determinants. In our mean-field setup, the terms in Eq. (4.6) and Eq. (4.7) coming from Eq. (4.5) can be explicitly computed. We consider here the two communities case, i.e. $V = V_1 \cup V_2$, where the communities possibly have different sizes, $|V_1| = N_1$ and $|V_2| = N_2$. Now, consider the matrix obtained by erasing $k_1$ ($k_2$) rows and corresponding columns in the first community (the second one, respectively) in $-\mathcal{L}$. We are left with a square matrix made of two square blocks on the diagonal of size $N_1 - k_1 =: K_1$ (respectively $N_2 - k_2 =: K_2$). We will denote this matrix by

$$- M = \begin{pmatrix}
    d_1 & \cdots & w_1 & w_2 & \cdots & w_2 \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    w_1 & \cdots & d_1 & w_2 & \cdots & w_2 \\
    w_2 & \cdots & w_2 & d_2 & \cdots & w_1 \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    w_2 & \cdots & w_1 & w_1 & w_1 & d_2
\end{pmatrix} = \begin{pmatrix}
    A_1 & B \\
    B^T & A_2
\end{pmatrix},$$

(4.8)

where the elements on the diagonal are given by

$$d_1 = -((N_1 - 1)w_1 + N_2w_2), \quad d_2 = -((N_2 - 1)w_1 + N_1w_2).$$

(4.9)

We want to find $K_1 + K_2$ solutions of the problem

$$- M v = \lambda v$$

(4.10)

First we consider eigenvectors of the form $v = (x_1, x_1, \ldots, x_1, x_2, \ldots, x_2)^T$, where the upper component has length $K_1$ and the lower one has length $K_2$. If we write explicitly Eq. (4.10) we get the following linear system:

$$- \begin{pmatrix}
    d_1 + (K_1 - 1)w_1 \\
    K_1w_2 \\
    d_2 + (K_2 - 1)w_1
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_1 \\
    x_2
\end{pmatrix} = \lambda \begin{pmatrix}
    x_1 \\
    x_1 \\
    x_2
\end{pmatrix},$$

(4.11)

from which we get two eigenvalues, which we will refer to as $\lambda_1$ and $\lambda_2$. Then we consider $v = (x_1, x_2, \ldots, x_{K_1}, 0, \ldots, 0)^T$, with this choice we are left with the system

$$- \begin{pmatrix}
    d_1 & \cdots & w_1 \\
    \vdots & \ddots & \vdots \\
    w_1 & \cdots & d_1
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    \vdots \\
    x_{K_1}
\end{pmatrix} = \lambda \begin{pmatrix}
    x_1 \\
    \vdots \\
    x_{K_1}
\end{pmatrix}, \quad w_2(x_1 + \cdots + x_{K_1}) = 0$$

(4.12)

and we have to find $K_1 - 1$ eigenvalues that are associated with eigenvector orthogonal to constants. By direct computation, $A_1$ has eigenvalue $\lambda_1' := (N_1w_1 + N_2w_2)$ with multiplicity $K_1 - 1$. With the opposite choice, namely $v = (0, \ldots, 0, x_1, \ldots, x_{K_2})^T$, we get

$$- \begin{pmatrix}
    d_2 & \cdots & w_1 \\
    \vdots & \ddots & \vdots \\
    w_1 & \cdots & d_2
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    \vdots \\
    x_{K_2}
\end{pmatrix} = \lambda \begin{pmatrix}
    x_1 \\
    \vdots \\
    x_{K_2}
\end{pmatrix}, \quad w_2(x_1 + \cdots + x_{K_2}) = 0.$$

(4.13)

Namely, there is an eigenvalue $\lambda_2' := (N_2w_1 + N_1w_2)$ with multiplicity $K_2 - 1$. So the spectrum of $M$ is

$$\text{spec}(M) = (\lambda_1, \lambda_2, \lambda_1', \lambda_2')$$

(4.14)

with multiplicity denoted by $\mu_M(\cdot)$:

$$\mu_M(\lambda_1) = 1, \quad \mu_M(\lambda_2) = 1, \quad \mu_M(\lambda_1') = K_1 - 1, \quad \mu_M(\lambda_2') = K_2 - 1.$$  

(4.15)

Therefore, we can see that the ratio of determinants in Eq. (4.6) and Eq. (4.7) can be written explicitly. Indeed, at the denominator we have

$$\det(qI + \mathcal{L}) = q(q + Nw_2)(q + N_1w_1 + N_2w_2)^{N_1 - 1}(q + N_2w_1 + N_1w_2)^{N_2 - 1},$$

(4.16)
while at the numerator we are left with
\[ \det_{V \setminus \{1,2,\ldots,k_1,N_1+1,N_1+2,\ldots,N_1+k_2\}} (qI + L) = (q + \lambda_1)(q + \lambda_2)(q + \lambda'_1)^{N_1-k_1-1}(q + \lambda'_2)^{N_2-k_2-1} \] (4.17)
where
\[ \lambda'_1 := N_1w_1 + N_2w_2, \quad \lambda'_2 := N_1w_2 + N_2w_1, \] (4.18)
while \( \lambda_1 \) and \( \lambda_2 \) are the two solutions of the system in Eq. (4.11). In particular, if we specialize in the case \( N_1 = N_2 = N \) we can conclude that the ratio of determinants is given by
\[ \theta(k_1, k_2) := \frac{(q - \lambda_1(k_1, k_2))(q - \lambda_2(k_1, k_2))}{q(q + 2Nw_2)(q + \alpha)^{k_1+k_2}} \] (4.19)
where we defined
\[ \alpha := N(w_1 + w_2), \] (4.20)
and
\[ \lambda_i(k_1, k_2) := -\frac{1}{2} \left[ w_1(k_1 + k_2) + 2Nw_2 + (-1)^i \sqrt{w_1^2(k_1 - k_2)^2 + 4(N - k_1)(N - k_2)w_2^2} \right], \quad i = 1, 2. \]

The path starting from \( y \). Now we have to consider the second path starting from \( y \) which decides the root at which \( y \) will be connected in the forest generated by the algorithm. The latter corresponds to the second factor in Eq. (4.4). Notice that it is sufficient to consider such path in the simpler fashion, i.e. without erasing the loops, since we are only concerned with the absorption of the walker: either in \( \gamma \) or killed at rate \( q \). Moreover, we can exploit again the symmetry of the model to reduce it to a Markov chain with state space \( \{1,2,3,4\} \) corresponding to the sets \( \{V_1 \setminus \gamma_1, V_2 \setminus \gamma_2, \gamma_1 \cup \gamma_2, \Delta\} \), where \( \Delta \) is again the absorbing state, i.e., the “state-independent” exponential killing. We will assume that
\[ |\gamma_i| = k_i, \quad |V_i| = N_i, \quad i = 1, 2. \]

Hence, the transition matrix we are interested in is given by
\[ \tilde{P} := \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}, \] (4.21)
where
\[ Q := D^{-1} \begin{pmatrix} (N_1 - k_1 - 1)w_1 & (N_2 - k_2 - 1)w_2 \\ (N_1 - k_1)w_2 & (N_2 - k_2)w_1 \end{pmatrix}, \] (4.22)
\[ D^{-1} := \begin{pmatrix} (q + \alpha_1 - w_1)^{-1} & 0 \\ 0 & (q + \alpha_2 - w_1)^{-1} \end{pmatrix}, \quad R := D^{-1} \begin{pmatrix} k_1w_1 + k_2w_2 & q \\ k_1w_2 + k_2w_1 & q \end{pmatrix}. \] (4.23)
with
\[ \alpha_1 := N_1w_1 + N_2w_2, \quad \alpha_2 := N_1w_2 + N_2w_1. \] (4.24)

The states represent:
(1) nodes of the 1st community that have not been covered by the LE-path started at \( x \).
(2) nodes of the 2nd community that have not been covered by the LE-path started at \( x \).
(3) nodes of both communities that have been covered by the LE-path started at \( x \).
(4) the absorbing state \( \Delta \).
Called $T_{abs}$ the hitting time of the absorbing set $\{3,4\}$, we want to compute the probability that the process $\bar{X}$ is absorbed in state 4 and not in 3. In terms of our original process, this means that the process is killed before the hitting of the LE-path starting at $x$. By direct computation

$$
\mathbb{P}_2(\bar{X}(T_{abs}) = \bar{4}) = \sum_{k=0}^{\infty} \bar{P}^k(2,1) \frac{q}{q + \alpha_1 - w_1} + \sum_{k=0}^{\infty} \bar{P}^k(2,2) \frac{q}{q + \alpha_2 - w_1}
$$

$$=

\left(\sum_{k=0}^{\infty} Q^k\right) D^{-1} \left(\begin{array}{c} q \\ q \end{array}\right)(2)

=(I - Q)^{-1} D^{-1} \left(\begin{array}{c} q \\ q \end{array}\right)(2)

=: P^i(2)

(4.25)

notice that the first component of the vector $P^i \in \mathbb{R}^2$ corresponds to the intra-community case $\{x, y\} \in V_i$ for some $i$, i.e., $U_q^{(N)}(in)$, while the second one to the inter-community case, namely $U_q^{(N)}(out)$.

If we now use the assumption that $N_1 = N_2 = N$, the steps above allow us to write the following formulas

$$
U_q^{(N)}(out) = \sum_{k_1=1}^{N} \sum_{k_2=0}^{N-1} \binom{N-2}{k_1-1} \binom{N-1}{k_2} (k_1 - 1)!(k_2)! \theta(k_1, k_2) P^i(2)

\cdot \sum_{j_1=0}^{\min(k_1,k_2)} \sum_{j_2=j_1-1}^{\min(k_1,k_2)} \left( f_1(j_1, j_2) \right) \left( f_2(j_1, j_2) \right) w_1^{k_1 + k_2 - 1 - j_1 - j_2} w_2^{j_1 + j_2} q

(4.26)

$$

$$
U_q^{(N)}(in) = \sum_{k_1=1}^{N-1} \sum_{k_2=0}^{N} \binom{N-2}{k_1-1} \binom{N}{k_2} (k_1 - 1)!(k_2)! \theta(k_1, k_2) P^i(1)

\cdot \sum_{j_1=0}^{\min(k_1,k_2)} \sum_{j_2=j_1-1}^{\min(k_1,k_2)} \left( f_1(j_1, j_2) \right) \left( f_2(j_1, j_2) \right) w_1^{k_1 + k_2 - 1 - j_1 - j_2} w_2^{j_1 + j_2} q

(4.27)

$$

where

$$
f_1(j_1, j_2) := j_1 - 1_{\{j_1 \neq j_2\}}, \quad f_2(j_1, j_2) := j_2 - 1_{\{j_1 = j_2\}},

\theta(k_1, k_2) \text{ as in Eq. (4.19) and}

P^i = \frac{1}{q + \alpha - w_1} (I - Q)^{-1} \left(\begin{array}{c} q \\ q \end{array}\right).

(4.29)

By direct computation we see that

$$
P^i = \frac{q}{c} \left( q + k_2(w_1 - w_2) + 2w_2N \right)

\cdot \left( q + k_1(w_1 - w_2) + 2w_2N \right)

\cdot \frac{1}{c} \left( q + k_1(w_1 - w_2) + 2w_2N \right)

(4.30)

where

$$
c := (q + k_1 w_1)(q + k_2 w_1) + Nw_2(2q + (k_1 + k_2)w_1) + w_2^2[N(k_1 + k_2) - k_1 k_2].

(4.31)
Local time interpretation. Now consider the part of the formula concerning the jumps among the two communities of the killed-LE-path starting at $x$, i.e.

$$
\sum_{j_1=0}^{\min(k_1,k_2)} \sum_{j_2=j_1-1}^{j_1} \left( k_1 - 1 \right) \left( f_1(j_1,j_2) \right) \left( k_2 - 1 \right) \left( f_2(j_1,j_2) \right) w_1^{k_1 + k_2 - 1 - j_1 - j_2} w_2^{j_1 + j_2}.
$$

(4.32)

The latter can be thought of as a function of a Markov Chain $(\tilde{X}_n)_{n \in \mathbb{N}}$ on the state space $\{1, 2\}$, with transition matrix

$$
\tilde{P} = \left( \begin{array}{cc} p & 1 - p \\ 1 - p & p \end{array} \right), \quad p = \frac{w_1}{w_1 + w_2}
$$

(4.33)

where the $j$-th state stays for the $i$-th community. Indeed, we can rewrite Eq. (4.32) as

$$(w_1 + w_2)^{k_1 + k_2 - 1} \sum_{j_1=0}^{\min(k_1,k_2)} \sum_{j_2=j_1-1}^{j_1} \left( k_1 - 1 \right) \left( f_1(j_1,j_2) \right) \left( k_2 - 1 \right) \left( f_2(j_1,j_2) \right) \left( \frac{w_1}{w_1 + w_2} \right)^{k_1 + k_2 - 1 - j_1 - j_2} \left( \frac{w_2}{w_1 + w_2} \right)^{j_1 + j_2} =
$$

$$
= (w_1 + w_2)^{k_1 + k_2 - 1} \tilde{P}_1(\ell(k_1 + k_2) = k_1)
$$

(4.34)

with $\ell$ being the local time as in the statement of Theorem 2.

Geometric smoothing. From the previous steps we get the following expression

$$
U_q^{(N)}(out) = \sum_{k_1=1}^{N} \sum_{k_2=0}^{N-1} (N-1)^{k_1} (N-1)^{k_2} \frac{(q - \lambda_1(k_1,k_2))(q - \lambda_2(k_1,k_2))}{q(q + \alpha)k_1 + k_2}.
$$

(4.35)

Next, we would like to make appear a geometric term as in the complete and uniform case of Theorem 1. Notice that multiplying and dividing by $N^{k_1 + k_2 - 1}$ one obtains

$$
U_q^{(N)}(out) = \sum_{k_1=1}^{N} \sum_{k_2=0}^{N-1} N^{-k_1 - k_2} (N-1)^{k_1} (N-1)^{k_2} \frac{(q - \lambda_1(k_1,k_2))(q - \lambda_2(k_1,k_2))}{q(q + \alpha)(k_1 + k_2)} \cdot q(w_1 + w_2)^{k_1 + k_2 - 1} \tilde{P}_1(\ell(k_1 + k_2) = k_1)P^1(2).
$$

(4.36)

We can then define

$$
\xi_q := q = \frac{q}{q + \alpha} = \frac{q}{q + N(w_1 + w_2)}
$$

(4.37)

in order to obtain

$$
U_q^{(N)}(out) = \sum_{k_1=1}^{N} \sum_{k_2=0}^{N-1} N^{-k_1 - k_2} (N-1)^{k_1} (N-1)^{k_2} \frac{(q - \lambda_1(k_1,k_2))(q - \lambda_2(k_1,k_2))}{q(q + 2Nw_2)} \cdot P(T_q = k_1 + k_2)\tilde{P}_1(\ell(k_1 + k_2) = k_1)P^1(2)
$$

(4.38)

and

$$
U_q^{(N)}(in) = \sum_{k_1=1}^{N} \sum_{k_2=0}^{N-1} N^{-k_1 - k_2} (N-2)^{k_1} (N)^{k_2} \frac{(q - \lambda_1(k_1,k_2))(q - \lambda_2(k_1,k_2))}{q(q + 2Nw_2)} \cdot P(T_q = k_1 + k_2)\tilde{P}_1(\ell(k_1 + k_2) = k_1)P^1(1)
$$

(4.39)

where $T_q$ is an independent random variable with law $Geom(\frac{\xi_q}{N})$. 
Conclusions. One can ideally divide the formulas in Eqs. (4.38) and (4.39) in five terms, namely

1. The entropic term

\[ N^{-(k_1+k_2-1)} (N-2)_{k_1-1} (N)_{k_2} \quad \text{or} \quad N^{-(k_1+k_2-1)} (N-1)_{k_1-1} (N-1)_{k_2} \]  

was already present in the complete and uniform case Eq. (2.1). Indeed

\[ \prod_{h=2}^{k} \left( 1 - \frac{h}{N} \right) = N^{-(k-1)} (N-2)_{k-2}. \]  

2. The term related to the spectrum of the size 2 matrix presented in Eq. (4.11), i.e.

\[ \frac{(q - \lambda_1(k_1, k_2))(q - \lambda_2(k_1, k_2))}{q(q + 2Nw_2)} \]  

which is the same in both in e out community cases. It can be rewritten as the ratio between two parabolas in q, i.e.,

\[ \frac{q^2 + [(k_1 + k_2)w_1 + 2Nw_2]q + (w_1 + w_2)[(k_1 + k_2)Nw_2 + k_1k_2(w_1 - w_2)]}{q^2 + 2Nw_2q} \]  

3. The term related to the geometric random variable of parameter \( \frac{\xi}{N} \), which was present also in the case of the uniform graph, Eq. (2.1).

4. The term related to the local times of the 2-states Markov chain \( \tilde{P} \), in Eq. (4.33).

5. The term related to the absorption probability, i.e., to the quantity \( P^\dagger \), see Eq. (4.25), as a function of the process \( \tilde{P} \) presented in Eq. (4.21).

It is worth noticing that the \( P^\dagger \) above is slightly different from the \( P^\dagger \) in the statement of Theorem 2 which contains the extra factor \( \eta_* \). At this point by setting

\[ g^\prime_{out}(k_1, k_2) := N^{-(k_1+k_2-1)} (N-1)_{k_1-1} (N-1)_{k_2} \frac{(q - \lambda_1(k_1, k_2))(q - \lambda_2(k_1, k_2))}{q(q + 2Nw_2)}, \]  

\[ g^\prime_{in}(k_1, k_2) := N^{-(k_1+k_2-1)} (N-2)_{k_1-1} (N)_{k_2} \frac{(q - \lambda_1(k_1, k_2))(q - \lambda_2(k_1, k_2))}{q(q + 2Nw_2)} \]  

we can write

\[ U_q^{(N)}(out) = \sum_{k_1=1}^{N} \sum_{k_2=0}^{N-1} g^\prime_{out}(k_1, k_2) P(T_q = k_1 + k_2) \tilde{P}_{\perp} (\ell(k_1 + k_2) = k_1) \]  

\[ = \sum_{n=1}^{2N} \sum_{k_1+k_2=n} g^\prime_{out}(k_1, k_2) P(T_q = n) \tilde{P}_{\perp} (\ell(n) = k_1), \]  

and

\[ U_q^{(N)}(in) = \sum_{k_1=1}^{N} \sum_{k_2=0}^{N-1} g^\prime_{in}(k_1, k_2) P(T_q = k_1 + k_2) \tilde{P}_{\perp} (\ell(k_1 + k_2) = k_1) \]  

\[ = \sum_{n=1}^{2N} \sum_{k_1+k_2=n} g^\prime_{in}(k_1, k_2) P(T_q = n) \tilde{P}_{\perp} (\ell(n) = k_1), \]  

which is equivalent to the statement in Theorem 2.
4.2. Proof of Theorem 3. Proofs of (a) and (b): $1 - \beta < \alpha < (\leq) \frac{1}{2}$ (detectability)

As expressed in the following lemma in this regime the SRW is confined to its starting community for the entire life-time.

**Lemma 1** (RW is confined to its community up to dying). Let $1 > \alpha > 1 - \beta$ and for $x \in [2N]$, consider the event

$$E_x := \{ T_q > T_x^{\text{out}} \}$$

where $T_x^{\text{out}}$ is the first time in which the SRW moves out of the community in which $x$ lies.

Then, as $N \to \infty$,

$$\mathbb{P}_x(E_x) = o(1).$$

**Proof.** Let $Z$ be a r.v. that can assume values in the set $\{\text{Out}, \text{In}, \Delta\}$ with probabilities:

$$\mathbb{P}(Z = \text{Out}) = \frac{N^{1-\beta}}{N^{\alpha} + N + N^{1-\beta}} =: a_N,$$

$$\mathbb{P}(Z = \text{In}) = \frac{N}{N^{\alpha} + N + N^{1-\beta}} =: b_N$$

and $\mathbb{P}(Z = \Delta) = 1 - (a_N + b_N)$.

Let $(Z_n)_{n \in \mathbb{N}}$ be a sequence of i.i.d. r.v.s with the same law of $Z$ and notice that

$$\mathbb{P}(T_q < T_x^{\text{out}}) = \mathbb{P}(\min\{n \geq 0 \mid Z_n = \Delta\} < \min\{n \geq 0 \mid Z_n = \text{Out}\}).$$

Therefore

$$\mathbb{P}_x(E_x) = \mathbb{P}_x(T_q > T_x^{\text{out}}) = \sum_{n=1}^{\infty} \mathbb{P}_x(T_x^{\text{out}} = n, T_q > n)$$

$$= \sum_{n=1}^{\infty} b_n^{n-1} a_N$$

$$= \frac{a_N b_N}{1 - b_N} \sim N^{1-\beta-\alpha},$$

from which the claim. \qed

In view of the decomposition in Eq. (1.4) and the above lemma, we can write for any $x \neq y$

$$U_q^{(N)}(x, y) = \sum_{\gamma} \mathbb{P}_x^{LE}(\gamma) \mathbb{P}_y(T_{\gamma} > T_q | E_x^c) \mathbb{P}_y(E_x^c) + \mathbb{P}_y(T_{\gamma} > T_q | E_x^c) \mathbb{P}_y(E_x^c)$$

$$= o(1) + (1 - o(1)) \sum_{\gamma} \mathbb{P}_x^{LE}(\gamma) \mathbb{P}_y(T_{\gamma} > T_q | E_x^c)$$

$$\sim \sum_{\gamma} \mathbb{P}_x^{LE}(\gamma) \mathbb{P}_y(T_{\gamma} > T_q | E_x^c). \quad (4.48)$$

Let us first consider $U_q^{(N)}(\text{out})$. In this case, by Lemma 1, for any $\alpha \leq 1/2$ and uniformly in $\gamma$, we have that

$$\mathbb{P}_y(T_{\gamma} < T_q | E_x^c) \leq \mathbb{P}_y(T_q^{\text{out}} < T_q | E_x^c)$$

$$= \mathbb{P}_y(E_y)$$

$$= o(1).$$
As a consequence \( P_y(T > T_q | E_y^c) \geq 1 - o(1) \), and by plugging this estimate in Eq. (4.48), we get \( U_q^{(N)}(\text{out}) \to 1 \).

Concerning \( U_q^{(N)}(\text{in}) \), one has to notice that, for every LERW \( \gamma \) starting from \( x \) and ending at the absorbing state, we can consider the event

\[
E_{\gamma,y} = \{ T^{\text{out}}_y < \min(T_{\gamma}, T_q) \}.
\]

Once more, uniformly in \( \gamma \), we get by Lemma 1 that

\[
P_y(E_{\gamma,y}) \leq P_y(E_y) = o(1)
\]

Thus, for \( x, y \in [N] \), by Eq. (4.48), we can estimate

\[
U^{(N)}(x,y) = o(1) + (1 - o(1)) \sum_\gamma P^{LE}_x(\gamma | E_x^c) P_y(T_{\gamma} > T_q | E^c_x, E^c_{\gamma,y})
\]

Notice that, under such conditioning, the sum can be read as the probability that two vertices in a complete graph with \( N \) vertices end up in two different trees. Therefore, this reduces to Eq. (2.2), which in turns gives \( U^{(N)}(\text{in}) \to 0 \) for \( \alpha < 1/2 \) and \( U^{(N)}(\text{in}) \to \varepsilon_0(\alpha) \) else.

**Proof of (f) : \( \alpha > \frac{1}{2} \) (high killing region)**

We will only show that \( U^{(N)}(\text{in}) \to 1 \), this will suffice since e.g. by direct computation one can check that \( U^{(N)}(\text{in}) \geq U^{(N)}(\text{out}) \).

Observe first that being \( \alpha > \frac{1}{2} \), the length of the Loop-Erased path \( \Gamma \) must be “small” with high probability. In particular we can bound

\[
P^{LE}_x(|\Gamma| > \sqrt{N}) \leq P(T_q > \sqrt{N}) = \left(1 - \frac{N^\alpha}{N + N^{1-\beta} + N^\alpha}\right)^{\sqrt{N}} = o(1),
\]

hence

\[
U^{(N)}(\text{in}) = o(1) + \sum_{\gamma: |\gamma| \leq \sqrt{N}} P^{LE}_x(\Gamma = \gamma) P_y(T_{\gamma} > T_q)
\]

\[
\geq \sum_{\gamma: |\gamma| \leq \sqrt{N}} P^{LE}_x(\Gamma = \gamma) \frac{N^\alpha}{\sqrt{N} + N^\alpha} = 1 - o(1).
\]

We next prove the remaining items in Theorem 3 for which we will implement a similar strategy which we start explaining. In all remaining regimes we need to show that \( U^{(N)}(\bullet) \), \( \bullet \in \{\text{in, out}\} \) either vanishes or stays bounded away from zero. To this aim, we will use the representation in Eq. (2.3).

Depending on the parameter regimes, we will split the sum over \( t \) in different pieces to be treated according to the asymptotic behavior of the involved factors. To simplify the exposition we will restrict in what follows to the positive quadrant \( \alpha, \beta > 0 \). We stress however that, as the reader can check, the following estimates hold true and actually converge faster even outside of the positive quadrant.
Let us start with a few observations. We notice that \( \hat{f}(n, k) \leq 1 \) for every choice of \( k, N, n \), moreover \( \hat{f}(t, n) = 0 \) if \( n \geq N \). Furthermore, for each \( N \),

\[
\sum_{n=1}^{\infty} \mathbb{P}(T_q = n) \sum_{k=1}^{n} \mathbb{P}_1(\ell(n) = k) = \sum_{n=1}^{\infty} \mathbb{P}(T_q = n) = 1,
\]

and while estimating the involved factors it will be crucial the behavior of the product \( \left( \hat{f} \theta P^1 \right)(n, k) \) for which we can in general observe the following facts.

(A) For any \( \varepsilon > 0 \), if \( n > N_{1/2+\varepsilon} \), then it follows from Eq. (3.23) that \( N \mapsto \hat{f}_N \) decays to zero, uniformly in \( k \), faster than any polynomial as \( N \to \infty \). For such \( n \)'s, since \( N \mapsto \theta N P^1 \) is polynomially bounded (uniformly in \( n, k \)), the contribution in Eq. (2.3) of such terms can be neglected.

(B) Whenever we consider \( n \)'s for which \( \theta P^1 = o(1) \), because of Eq. (4.49) and the uniform control on \( \hat{f} \), the contribution of such terms in Eq. (2.3) can also be neglected.

(C) For \( n \)'s for which neither Item A nor Item B hold, we will estimate the asymptotics of such part of the sum by controlling the mass of the geometric time \( T_q \) against \( \theta P^1 \), and in the most delicate cases (on the separation lines in Fig. 1), taking into account the behavior of the local time too.

We are now ready to treat the remaining parameter regimes using such facts.

**Proof of (d):** \( \alpha < \min\left\{ \frac{1}{2}, 1 - \beta \right\} \) (changing-communities before dying)

In this regime, the overall picture resembles the phenomenology of the complete graph. In particular, the SRW will manage to change community before being killed and up to the killing time scale, it will forget its starting community. Moreover, with high probability a single tree of size \( 2N(1 - o(1)) \) will be formed, so that, given any two points \( x, y \), they will end up in the same tree with high probability independently on their communities.

To prove the claim notice that, uniformly in \( n, k \),

\[
P^1(n, k) \sim \frac{N^{1-\beta+\alpha} + N^n k^n}{2N^{1-\beta+\alpha} + nN^{1-\beta} + k(n-k)} = \frac{N^{1-\beta+\alpha}}{2N^{1-\beta+\alpha} + nN^{1-\beta} + k(n-k)} + O \left( \frac{1}{N^{1-\beta-\alpha}} \right).
\]

As a consequence the asymptotics of \( U_q^{(N)}(*) \) will be independent of \( * \). To show that such a limit is zero we argue as follows. Within this parameter region:

\[
\theta(n, k) \sim 1 + \frac{nN^{\alpha} + 2k(n-k)}{2N^{1-\beta+\alpha}},
\]

which together with Eq. (4.50) leads to

\[
\theta P^1(n, k) = \frac{N^{1-\beta+\alpha}}{2N^{1-\beta+\alpha} + nN^{1-\beta} + k(n-k)} + \frac{k(n-k)}{2N^{1-\beta+\alpha} + nN^{1-\beta} + k(n-k)} + O \left( \frac{k(n-k)}{N^{2(1-\beta)}} \right) + O \left( \frac{nN^{\alpha}}{N^{2(1-\beta)}} \right)
\]

\[
= \theta P^1(n, k) + \theta P^1_{II}(n, k) + \theta P^1_{II}(n, k) + \theta P^1_{IV}(n, k).
\]

We can now plug in this asymptotic representation of \( \theta P^1 \) in Eq. (2.3), and separately treat the four resulting terms.

For the first term, namely the sum in Eq. (2.3) with \( \theta P^1_1 \) in place of \( \theta P^1 \), we split the sum in \( n \) into two parts at \( N^{\alpha+\varepsilon} \), for small \( \varepsilon > 0 \), and show that they both goes to zero, by using Item C and Item B,
respectively. In fact, with this “cut” we see that:

\[
(1) := \sum_{n=1}^{\infty} P(T_q = n) \sum_{k=1}^{n} \hat{P}_1(\ell(n) = k) \hat{f}(n,k) \theta P_{I}^l(n,k) \quad (4.53)
\]

\[
= \sum_{n < N^{\alpha + \varepsilon}} P(T_q = n) \sum_{k=1}^{n} \hat{P}_1(\ell(n) = k) \cdot 1 \cdot \Theta(1) + \sum_{n \geq N^{\alpha + \varepsilon}} P(T_q = n) \sum_{k=0}^{n} \hat{P}_1(\ell(n) = k) \cdot 1 \cdot o(1) \quad (4.54)
\]

\[
= \Theta \left( \sum_{n < N^{\alpha + \varepsilon}} P(T_q = n) \right) + o(1) = o(1). \quad (4.55)
\]

Analogously, for the second term we split the sum over \( n \) into two parts at \( N^{1/2 + \varepsilon} \), with small \( \varepsilon > 0 \).

Using Item C for the first part and Item A for the second one, we see that:

\[
(II) := \sum_{n=1}^{\infty} P(T_q = n) \sum_{k=1}^{n} \hat{P}_1(\ell(n) = k) \hat{f}(n,k) \theta P_{II}^l(n,k) \quad (4.56)
\]

\[
= \sum_{n < N^{1/2 + \varepsilon}} P(T_q = n) \sum_{k=1}^{n} \hat{P}_1(\ell(n) = k) \cdot 1 \cdot O(1) + o(1) \quad (4.57)
\]

\[
= O \left( \sum_{n < N^{1/2 + \varepsilon}} P(T_q = n) \right) + o(1) \quad (4.58)
\]

\[
= o(1). \quad (4.59)
\]

For the third term we need to split the corresponding sum into three parts at \( T_1 := N^{1-\beta-\varepsilon} \) and \( T_2 := N^{1/2 + \varepsilon} \), which will be controlled by Item B, Item C and Item A, respectively. That is:

\[
(III) := \sum_{n=1}^{\infty} P(T_q = n) \sum_{k=1}^{n} \hat{P}_1(\ell(n) = k) \hat{f}(n,k) \theta P_{III}^l(n,k) \quad (4.60)
\]

\[
\leq \sum_{n < T_1} P(T_q = n) \sum_{k=1}^{n} \hat{P}_1(\ell(n) = k) \cdot 1 \cdot o(1) + \sum_{n = T_1}^{T_2} P(T_q = n) \sum_{k=1}^{n} \hat{P}_1(\ell(n) = k) \cdot 1 \cdot O(N^{-1+2\beta+2\varepsilon}) + o(1) \quad (4.61)
\]

\[
= o(1) + O \left( N^{\alpha - \beta - \varepsilon} \cdot 1 \cdot N^{-1+2\beta+2\varepsilon} + o(1) \right) \quad (4.62)
\]

Finally, for the last term, we split the sum at \( N^{1/2 + \varepsilon} \). Indeed we see that: on the one hand, for \( n \leq N^{1/2 + \varepsilon} \), we can use Item C since

\[
\theta P_{IV}^l(n,k) = O \left( N^{\frac{1}{2} + \varepsilon + \alpha - 2(1-\beta)} \right) \quad \text{and} \quad P \left( T_q \leq N^{\frac{1}{2} + \varepsilon} \right) = O \left( N^{\frac{1}{2} + \alpha + \varepsilon} \right).
\]
On the other hand, for $n \geq N^{1/2+\varepsilon}$, we can argue as in Item A. Hence,

\[
(IV) := \sum_{n=1}^{\infty} \mathbb{P}(T_q = n) \sum_{k=1}^{n} \mathbb{P}(\ell(n) = k) \hat{f}(n, k) \theta P^1_{IV}(n, k) \tag{4.63}
\]

\[
\leq \sum_{n=1}^{N^{1/2+\varepsilon}} \mathbb{P}(T_q = n) \sum_{k=1}^{n} \mathbb{P}(\ell(n) = k) \cdot 1 \cdot O\left(N^{\frac{1}{2}+\varepsilon+\alpha-2(1-\beta)}\right) + o(1) \tag{4.64}
\]

\[
= O\left(N^{-\frac{1}{2}+\alpha+\varepsilon} \cdot 1 \cdot 1 \cdot N^{\frac{1}{2}+\varepsilon+\alpha-2(1-\beta)}\right) + o(1) = o(1) \tag{4.65}
\]

\[
\square
\]

**Proofs of (c) and (e) (high-entropy separating lines)**

We start by proving (e), i.e.

\[
\text{if } \alpha = \frac{1}{2} < 1 - \beta \implies \exists \varepsilon > 0 \text{ s.t. } \lim_{N \rightarrow \infty} U_q^N(in) = U_q^N(out) = \varepsilon. \tag{4.66}
\]

Start noting that under our assumptions on $\alpha$ and $\beta$ we have that

\[
\theta(n, k) \sim \frac{n \sqrt{N} + 2N^{\frac{1}{2}-\beta} + 2k(n - k)}{2N^{\frac{1}{2}-\beta}}, \tag{4.67}
\]

and

\[
\theta P^1_{IV}(n, k) \sim \frac{k \sqrt{N} + N^{\frac{1}{2}-\beta}}{2N^{\frac{1}{2}-\beta} + nN^{1-\beta} + k(n - k)}. \tag{4.68}
\]

We are going to split the sum over $n$ in Eq. (2.3) in three parts:

- $n \leq N^{\frac{1}{2}-\varepsilon}$. For such $n$’s we have that the product $\theta P^1_{IV}(n, k)$ is of order 1. Hence we can neglect this part by using Item C together with the estimate

\[
\mathbb{P}(T_q \leq N^{\frac{1}{2}-\varepsilon}) = O\left(N^{-\frac{1}{2}+\alpha-\varepsilon}\right).
\]

- $n > N^{\frac{1}{2}+\varepsilon}$. Also this part can be neglected thanks to the argument of Item A.

- $N^{\frac{1}{2}-\varepsilon} < n \leq N^{\frac{1}{2}+\varepsilon}$. This is the delicate non-vanishing part. We start by noticing that, due to Eq. (4.67) and Eq. (4.68), the leading term in $\theta P^1_{IV}$ does not involve $k$, so that —at first order— $U_q^N(in)$ must equal $U_q^N(out)$. In order to show that the latter are asymptotically bounded away from zero, we fix $c \in (0, 1)$ and consider

\[
U_q^N(*) \geq \sum_{n \in c\sqrt{N}}^{\sqrt{N}/c} \mathbb{P}(T_q = n) \sum_{k=1}^{n} \mathbb{P}(\ell(n) = k) \theta(n, k) \hat{f}(n, k) \tag{4.69}
\]

\[
\hat{f} = \Theta(1) \Rightarrow = \Omega\left(\sum_{n \in c\sqrt{N}}^{\sqrt{N}/c} \mathbb{P}(T_q = n) \sum_{k=1}^{n} \mathbb{P}(\ell(n) = k) \theta(t, k) P^1_{IV}(n, k)\right) \tag{4.70}
\]

\[
\theta P^1_{IV}(n, k) \in \left[\frac{1}{2 + c^{-1}}, \frac{1}{2 + c}\right] \Rightarrow = \Omega\left(\sum_{n \in c\sqrt{N}}^{\sqrt{N}/c} \mathbb{P}(T_q = n)\right) = \Omega(1). \tag{4.71}
\]

Moreover, thanks to Eq. (4.71) we can easily deduce that the limit is strictly smaller than $\frac{1}{2}$. 
We next conclude by giving the proof of (e), i.e., we are going to show that

$$\text{if } \alpha = 1 - \beta < \frac{1}{2} \implies \exists \varepsilon > 0 \text{ s.t. } \lim_{N \to \infty} U_q^{(N)}(in) = 0 \text{ while } \lim_{N \to \infty} U_q^{(N)}(out) = \varepsilon. \quad (4.72)$$

Observe that, under our assumptions on $\alpha$ and $\beta$, we have that

$$\theta(n, k) \sim \frac{3N^{2\alpha} + nN^{\alpha} + 2k(n - k)}{3N^{2\alpha}}, \quad (4.73)$$

and

$$P^+_1(n, k) \sim \frac{N^{2\alpha} + kN^{\alpha}}{3N^{2\alpha} + 2nN^{\alpha} + k(n - k)}, \quad (4.74)$$

hence, their product behaves asymptotically as

$$\theta P^+_1(n, k) = \Theta \left(1 + \frac{k}{N^\alpha}\right). \quad (4.75)$$

To evaluate the asymptotic behavior of $U_q^{(N)}(*)$, we split the sum over $n$ in Eq. (2.3) in three pieces:

- $n \leq N^{\alpha + \varepsilon}$: where, thanks to Eq. (4.75), we know that $\theta P^+_1(n, k) = O(N^\varepsilon)$. We argue as in Item C, obtaining

$$\sum_{n \leq N^{\alpha + \varepsilon}} \mathbb{P}(T_q = n) \sum_{k=1}^n \mathbb{P}_1(\ell(n) = k)\theta(n, k)P^+_1(n, k)\hat{f}(n, k) \leq O\left(N^\varepsilon \sum_{n \leq N^{\alpha + \varepsilon}} \mathbb{P}(T_q = n)\right) \leq O\left(N^{-1+2\alpha}\right) \quad (4.76)$$

- $n > N^{1/2 + \varepsilon}$: in this case we can argue as in Item A.

- $N^{\alpha + \varepsilon} < n \leq N^{1/2 + \varepsilon}$: in this case we have to distinguish between $U_q^{(N)}(in)$ and $U_q^{(N)}(out)$. Consider first $U_q^{(N)}(in)$. We call $E_n$ the following event concerning the Markov chain $(\hat{X}_n)_{n \in \mathbb{N}}$

$$E_n := \{\text{At least one jump occurs before time } n\}. \quad (4.78)$$

Notice that if $N^{\alpha + \varepsilon} < n \leq N^{1/2 + \varepsilon}$ then the event $E_n^c$ occurs with high probability. Hence, for any choice of $n \in [1, N]$ and $k \in [1, N]$ we can write

$$\hat{P}_1(\ell(n) = k) = \hat{P}_1(\ell(n) = k|E_n^c)\hat{P}_1(E_n^c) + \hat{P}_1(\ell(n) = k|E_n)\hat{P}_1(E_n) = \delta_{k,n} + o(1), \quad (4.79)$$

$\delta_{k,n}$ being the Kronecker delta. Hence

$$\sum_{n = N^{1/2 + \varepsilon}}^{N^{1/2 + \varepsilon}} \mathbb{P}(T_q = n) \sum_{k=1}^n \hat{P}_1(\ell(n) = k)\theta P^+_1(n, k)\hat{f}(n, k) = \Theta \left(\sum_{n = N^{1/2 + \varepsilon}}^{N^{1/2 + \varepsilon}} \mathbb{P}(T_q = n) \sum_{k=1}^n \delta_{k,n} \left(n - k \sqrt{\frac{n}{N^\alpha}} + 1\right)\right) \quad (4.80)$$

$$= \Theta \left(\sum_{n = N^{1/2 + \varepsilon}}^{N^{1/2 + \varepsilon}} \mathbb{P}(T_q = n)\right) = o(1). \quad (4.81)$$

Concerning $U_q^{(N)}(out)$, it is easy to get a lower bound via a soft argument by considering the events

$$B_x = \{\text{The LERW starting at } x \text{ never changes community}\} \quad (4.82)$$

$$B'_y = \{\text{The RW starting at } y \text{ does not change community before dying}\}. \quad (4.83)$$
Indeed,
\[
U_q^{(N)}(\text{out}) \geq \mathbb{P}(B_x) \mathbb{P}(B'_y) = \left( \frac{N^\alpha}{N^\alpha + N^{1-\beta}} \right)^2 = \frac{1}{4}.
\]

Finally, we are left to show that \(U_q^{(N)}(\text{out})\) is asymptotically bounded away from 1. We consider the further split
\[
U_q^{(N)}(\text{out}) \leq o(1) + \sum_{n=N^{\alpha+\epsilon}}^{N^{1/2}} \mathbb{P}(T_q = n) \sum_{k=1}^{n} \tilde{P}_q(\ell(n) = k)(\hat{f}_\theta \mathbb{P}^\dagger_{\text{out}})(n, k) + \sum_{n=N^{1/2}}^{N^{1+\epsilon}} \mathbb{P}(T_q = n) \sum_{k=1}^{n} \tilde{P}_q(\ell(n) = k)(\hat{f}_\theta \mathbb{P}^\dagger_{\text{out}})(n, k).
\]

Focusing on the first sum in the latter display, thanks to Eq. (4.75), we have that
\[
\sum_{n=N^{\alpha+\epsilon}}^{N^{1/2}} \mathbb{P}(T_q = n) \sum_{k=1}^{n} \tilde{P}_q(\ell(n) = k)(\hat{f}_\theta \mathbb{P}^\dagger_{\text{out}})(n, k) \leq \sum_{n=N^{\alpha+\epsilon}}^{N^{1/2}} \mathbb{P}(T_q = n) \frac{n}{N^\alpha} + \sum_{n=N^{1/2}}^{N^{1+\epsilon}} \mathbb{P}(T_q = n)
\]
\[
= \frac{1}{N} \sum_{n=N^{\alpha+\epsilon}}^{N^{1/2}} \left(1 - \frac{1}{N^{1-\alpha}}\right)^n + o(1)
\]
\[
\leq \frac{1}{N} \left( \frac{\sqrt{N}(\sqrt{N} + 1)}{2} \right) \sim \frac{1}{2}.
\]

Concerning the second sum, we have
\[
\sum_{n=\sqrt{N}}^{N^{1/2+\epsilon}} \mathbb{P}(T_q = n) \sum_{k=1}^{n} \tilde{P}_q(\ell(n) = k)(\hat{f}_\theta \mathbb{P}^\dagger_{\text{out}})(n, k) = O \left( \sum_{n=\sqrt{N}}^{N^{1/2+\epsilon}} \mathbb{P}(T_q = n) \hat{f}(n, n) \frac{n}{N^\alpha} \right)
\]
\[
= O \left( \frac{1}{N} \sum_{n=\sqrt{N}}^{N^{1/2+\epsilon}} ne^{-\frac{n^2}{2N}} \right)
\]
\[
= O \left( \frac{1}{\sqrt{N}} \sum_{m=1}^{\sqrt{N}} me^{-\frac{m^2}{2N}} \right)
\]
\[
= O \left( \frac{\sqrt{N}}{\sqrt{N}} \sum_{m=1}^{\infty} e^{-\frac{m^2}{2N}} \right) = o(1).
\]

4.3. **Proof of Corollary 1.** Let \(0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{2N-1}\) be the eigenvalues of \(\mathcal{L}\). As shown in [5, Prop. 2.1], the number of blocks of the induced partition, \(|\Pi_q|\), is distributed as the sum of \(2N\) independent Bernoulli random variables with success probabilities \(\frac{q}{q + \lambda_i}\). That is
\[
|\Pi_q| \overset{\text{d}}{=} \sum_{i=0}^{2N-1} X^{(q)}_i, \quad \text{with} \quad X^{(q)}_i \overset{\text{d}}{=} \text{Ber} \left( \frac{q}{q + \lambda_i} \right), \quad i \in \{0, \ldots, 2N - 1\}
\]

In case of the mean-field-two-communities model we have
\[
\lambda_0 = 0, \quad \lambda_1 = 2N^{1-\beta}, \quad \lambda_i = N(1 + N^{-\beta}), \quad i \in \{2, \ldots, 2N - 1\}.
\]
Therefore\[|\Pi_q| \overset{d}{\sim} 1 + X + \sum_{i=1}^{2(N-1)} Y_i,\]
where\[X \overset{d}{\sim} \text{Ber}\left(\frac{N^\alpha}{2N^{1-\beta} + N^\alpha}\right) \quad \text{and} \quad Y_i \overset{d}{\sim} \text{Ber}\left(\frac{N^\alpha}{N(1 + N^{-\beta}) + N^\alpha}\right), \quad i \in \{1, \ldots, 2(N-1)\}.
\]
Hence\[E[|\Pi_q|] \sim 1 + \frac{N^\alpha}{N^{1-\beta} + N^\alpha} + \frac{2N^{\alpha+1}}{N^\alpha + N} = \Theta(N^{\alpha+1}).\]
Moreover, we can prove the concentration result claimed in the first part of the statement by using the multiplicative version of the Chernoff bound on the sum of $Y_i$’s. Indeed, denoting by\[S := \sum_{i=1}^{2(N-1)} Y_i\]
we have that\[P\left(|S - E[S]| \geq \epsilon E[S]\right) \leq 2 \exp\left(-\frac{\epsilon^2 E[S]}{3}\right),\]
and since\[E[S] \sim \frac{2N^{\alpha+1}}{N^\alpha + N} = \omega(1)\]
we can deduce the concentration of $|\Pi_q|$. Notice also that the second part of the statement is a trivial consequence of the detectability result of Theorem 3.
\[\square\]

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References

[1] D. Aldous, The Continuum Random Tree. I. Ann. Probab. 19, 1–28 (1991).
[2] L. Avena, F. Castell, A. Gaudillère and C. Mélot, Intertwining wavelets or multiresolution analysis on graphs through random forests, ACHA DOI:10.1016/j.acha.2018.09.006 (2018).
[3] L. Avena, F. Castell, A. Gaudillère and C. Mélot, Random Forests and Networks Analysis, J. Stat. Phys. 173, 985–1027 (2018).
[4] L. Avena, F. Castell, A. Gaudillère and C. Mélot, Approximate and exact solutions of intertwining equations through random spanning forests, ArXiv:1702.05992 (2017).
[5] L. Avena and A. Gaudillière, Two applications of random spanning forests, J. Theor. Probab. 31, 1975–2004 (2018).
[6] L. Avena and A. Gaudillère, A proof of the transfer-current theorem in absence of reversibility, Stat. Probab. Lett. 142, 17–22 (2018).
[7] I. Benjamini and G. Kozma, Loop-erased random walk on a torus in dimensions 4 and above, Comm. Math. Phys. 259, 257–286 (2005).
[8] R. Burton and R. Pemantle, Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedances, Ann. Probab. 21, 1329–1371 (1993).
[9] G. R. Grimmett, Random labelled trees and their branching networks, *J. Aust. Math. Soc. Ser. A* 30, 229–237 (1980).
[10] G. R. Grimmett, *The Random-Cluster Model*. Berlin: Springer-Verlag (2009).
[11] G. Kozma, The scaling limit of loop-erased random walk in three dimensions, *Acta Math.* 199, 29–152 (2007).
[12] G. F. Lawler, O. Schramm and W. Werner, Conformal invariance of planar loop-erased random walks and uniform spanning trees, *Ann. Probab.* 32, 939–995 (2004).
[13] X. Li and D. Shiraishi, Convergence of three-dimensional loop-erased random walk in the natural parametrization, ArXiv:1811.11685 (2018).
[14] P. Marchal, Loop-erased random walks, spanning trees and Hamiltonian cycles, *Elect. Comm. Probab.* 5, 39–50 (2000).
[15] J. Pitman, *Combinatorial stochastic processes*. Lecture notes in Mathematics, Ecole d’Eté de Probabilités de Saint-Flour XXXII, Springer-Verlag Berlin/Heidelberg (2002).
[16] B. Pittel, Note on exact and asymptotic distributions of the parameters of the loop-erased random walk on the complete graph, In: Chauvin B., Flajolet P., Gardy D., Mokkadem A. (eds) *Mathematics and Computer Science II. Trends in Mathematics*, Birkhäuser, Basel (2002).
[17] M. Quattropani, *Spectral techniques for community detection: a probabilistic perspective*. Master thesis, Leiden University (2016).
[18] O. Schramm, Scaling limits of loop-erased random walks and uniform spanning trees, *Israel J. Math.* 118, 221–288 (2000).
[19] D. Wilson, Generating random spanning trees more quickly than the cover time, *Proceedings of the twenty-eight annual ACM symposium on the theory of computing*, 296–303 (1996).

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