SCHEMES OF FINITE EXPANSION AND UNIVERSALLY CLOSED CURVES

MATTHIAS JOHANN STEINER

Abstract. In algebraic geometry there is a well-known categorical equivalence between the category of normal proper integral curves over a field $k$ and the category of finitely generated field extensions of $k$ of transcendence degree 1. In this paper we generalize this equivalence to the category of normal quasi-compact universally closed separated integral $k$-schemes of dimension 1 and the category of field extensions of $k$ of transcendence degree 1. Our key technique are morphisms of finite expansion which can be considered as relaxation of morphisms of finite type. Since the schemes in the generalized category have many properties similar to normal proper integral curves, we call them normal integral universally closed curves over $k$.

Keywords. algebra, algebraic geometry, curves, universally closed curves

Introduction

The following theorem is a well-known result in the theory of curves.

Theorem ([GW20, Thm. 15.21]). Let $k$ be a field. There is a contravariant equivalence between the categories of

(i) normal proper integral curves over $k$ (with non-constant morphisms),
(ii) extension fields $K$ of $k$, finitely generated and of transcendence degree 1 (with $k$-homomorphisms),

given by mapping a curve $C$ as in (i) to its function field $K(C)$.

In this paper we will extend this equivalence of categories to arbitrary field extensions of transcendence degree 1. To do so we will have to relax the property of finite type of schemes in (i). Although this property is utilized in several critical steps of the proof of the theorem it limits us to finitely generated field extensions of $k$. In [Ham19] Paul Hamacher introduced a suitable relaxation for of finite type, he called it of finite expansion. In a nutshell a $R$-algebra $A$ is of finite expansion, if the structure homomorphism decomposes into a homomorphism of finite presentation followed by an integral homomorphism. Indeed, every finite type $k$-algebra is of finite expansion by Noether normalization, but we will see that the new notion of finite expansion also contains many non-finite type algebras.

With morphisms of finite expansion we will be able to prove the following theorem.
Theorem (see Theorem 3.7). Let $k$ be a field. There is a contravariant equivalence between the categories of

(i) normal quasi-compact universally closed separated integral $k$-schemes of dimension 1 (with non-constant morphisms),
(ii) extension fields $K$ of $k$ of transcendence degree 1 (with $k$-homomorphisms),
given by mapping a scheme $X$ as in (i) to its function field $K(X)$.

During the development of the proof we will see that the schemes in (i) have many properties very similar to normal proper integral curves. Therefore, we will call the schemes in (i) normal integral universally closed curves.

This paper is a condensed version of the author’s master’s thesis [Ste21].

Acknowledgments. I would like to thank my master’s thesis advisor, Dr. Paul Hamacher, for his support, explanations and patience.

1. Morphisms of finite expansion

In his preprint [Ham19] Paul Hamacher developed an étale cohomology theory for universally closed morphism of schemes. Previously it was only possible to define the étale cohomology group with compact support of a scheme if the scheme is of finite type. With his new cohomology theory Hamacher also introduced a new notion of morphisms of schemes, namely morphisms of finite expansion. Morphisms of finite expansion generalize morphisms of finite type, they can be viewed as relaxation of the finiteness condition. Nevertheless, they behave very similar to morphisms of finite type. One of the most important of these similarities is that they are compactifiable. Compactification in our sense means that we can decompose a separated morphism of finite expansion into an open immersion followed by a universally closed separated morphism.

Since [Ham19] is only available as preprint, we will restate all necessary definitions and results for a comprehensive treatment.

1.1. Algebras of finite expansion. We start with the definition of finite expansion for algebras.

Definition 1.1 ([Ham19] Def. 1.1). Let $R$ be a ring and $A$ be an $R$-algebra. A family $(a_i)_{i \in I}$ of elements in $A$ is called a quasi-generating system of $A$, if $A$ is integral over $R[a_i \mid i \in I]$; the $a_i$ are called quasi-generators. If there exists a finite quasi-generating system of $A$, we say that $A$ is of finite expansion over $R$.

Remark 1.2 ([Ham19] Rem. 1.2]). Alternatively we could say an $R$-algebra $A$ is of finite expansion if and only if there exists an integral morphism $R[x_1, \ldots, x_n] \to A$. In particular, the structure morphism decomposes into a morphism of finite presentation and an integral morphism.

Lemma 1.3 ([Ham19] Lemma 1.3]). We fix a ring $R$ and an $R$-algebra $A$. 

(1) Let $B$ be an $A$-algebra and assume that $A$ is of finite expansion over $R$. Then $B$ is of finite expansion over $R$ if and only if it is of finite expansion over $A$.

(2) Let $f_1, \ldots, f_m \in A$ such that $(f_1, \ldots, f_m)_A = A$. Then $A$ is of finite expansion over $R$ if and only if $A_{f_i}$ is of finite expansion over $R$ for every $i$.

(3) Let $R'$ be another $R$-algebra and assume that $A$ is of finite expansion over $R$. Then $R' \otimes_R A$ is finite expansion over $R'$.

(4) Let $R'$ be a faithfully flat $R$-algebra and assume that $R' \otimes A$ is of finite expansion over $R'$. Then $A$ is of finite expansion over $R$.

We provide two examples that should convince the reader that our new notion of finite expansion includes non-finite type algebras.

**Example 1.4.** Let $k$ be a field.

(1) The $k$-domain $k[x^n \mid n \in \mathbb{N}]$ is of finite expansion over $k$ and $x$ is the quasi-generator.

(2) Let $x$ be a transcendental element over $k$, and let $k(x) \subset K$ be an algebraic field extension. Then the integral closure of $k[x]$ in $K$ is of finite expansion over $k$.

1.2. Schemes of finite expansion. We now generalize Definition 1.1 to schemes and introduce the first fundamental results for morphisms of finite expansion.

**Corollary/Definition 1.5** ([Ham19, Cor./Def. 1.4]). We call a morphism $f : X \to Y$ of schemes locally of finite expansion if the following equivalent conditions are satisfied.

(a) For every affine open subscheme $V \subset Y$ and every affine open subscheme $U \subset f^{-1}(V)$, the $O_Y(V)$-algebra $O_X(U)$ is of finite expansion.

(b) There exists a covering $Y = \bigcup V_i$ by open affine subschemes $V_i \cong \text{Spec}(R_i)$ and a covering $f^{-1}(V_i) = \bigcup U_{i,j}$ by open affine subschemes $U_{i,j} \cong \text{Spec}(A_{i,j})$ such that for all $i, j$ the $R_i$-algebra $A_{i,j}$ is of finite expansion.

We say that a morphism $f : X \to Y$ is of finite expansion if it is locally of finite expansion and quasi-compact.

**Corollary 1.6** ([Ham19 Cor. 1.5]).

(1) The properties “locally of finite expansion” and “of finite expansion” of morphisms of schemes are stable under composition, base change and faithfully flat descent, and are local on the target. The property “of finite expansion” is also local on the source.

\footnote{A spelling mistake in the original statement was corrected.}
Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If $g \circ f$ is locally of finite expansion (resp. of finite expansion and $g$ is quasi-separated), then $f$ is locally of finite expansion (resp. of finite expansion).

If we impose mild conditions, it is also possible to decompose a morphism of finite expansion into an integral morphism and a morphism of finite presentation.

**Proposition 1.7** ([Ham19, Prop. 1.8]). Let $f : X \to Y$ be a separated morphism of qcqs schemes.

1. If $f$ is universally closed then it can be decomposed as $f = h \circ g$ with $g$ integral and $h$ proper.
2. If $f$ is of finite expansion then it can be decomposed as $f = h \circ g$ with $g$ integral and $h$ of finite presentation.

A key result for our paper is the following corollary.

**Corollary 1.8** ([Ham19, Cor. 1.9]). Every separated universally closed morphism between qcqs schemes is of finite expansion.

As already mentioned in the introduction morphisms of finite expansion are compactifiable in the following sense.

**Theorem 1.9** ([Ham19, Thm. 1.17]). Let $f : X \to Y$ be a separated morphism of finite expansion between qcqs schemes. Then $f$ can be written as composition $f = \overline{f} \circ j$ where $j : X \to \overline{X}$ is an open embedding and $\overline{f} : \overline{X} \to Y$ is separated and universally closed.

We conclude this section with the definition of universally closed curves over a field. We introduce this definition to simplify notation and to emphasize the relation between normal proper integral curves and normal quasi-compact universally closed separated integral schemes of dimension 1 in the coming sections.

**Definition 1.10** (Universally closed curves). Let $k$ be a field. A non-empty connected $k$-scheme $X$ is called a universally closed curve if it satisfies the following conditions.

1. $X$ is quasi-compact, universally closed, separated and of dimension 1.
2. $\text{trdeg}_k \left( \kappa(\eta) \right) = 1$ for every generic point $\eta$ of an irreducible component of $X$.

By Corollary 1.8 every universally closed curve is of finite expansion over $\text{Spec}(k)$ and we will often use this fact without explicitly referring to the corollary.
1.2.1. Extending morphisms of finite expansion. For schemes locally of finite presentation it is well-known that one can extend a local ring homomorphism to a morphism of schemes (cf. [GW20, Prop. 10.52] and [Stacks, Tag 0BX6]). We will extend this property to schemes locally of finite expansion.

Lemma 1.11. Let $R$ be a ring, and let $A$ and $B$ be $R$-domains. Assume that $A$ is of finite expansion, that $B$ is a normal, and let $p \subset B$ be a prime ideal. Suppose a homomorphism $\phi : A \to B_p$ is given. Then $\phi$ splits as $A \to B_f \to B_p$ for some $f \in B \setminus p$.

Proof. For simplicity we assume that $\phi$ is injective. By assumption we can find elements $a_1, \ldots, a_n \in A$ such that $R[a_1, \ldots, a_n] \to A$ is an integral ring homomorphism. Via $\phi$ we have that $a_i \in B_p$ for all $i$, thus we can write $a_i$ as $a_i = \frac{x_i}{y_i}$, where $x_i \in B$ and $y_i \in B \setminus p$. We define the element $f \in B$ as $f = \prod_{i=1}^n y_i$. It is easy to see that $\phi |_{R[a_1, \ldots, a_n]}$ factors through $B_f$, and by [Kem11, Prop. 8.10] $B_f$ is also normal. Denote with $C$ the integral closure of $R[a_1, \ldots, a_n]$ in $\text{Frac}(B)$. It is obvious that $A \subset C$ and by normality of $B_f$ we must also have that $C \subset B_f$. Thus we have found the following split for $\phi$:

$$\begin{array}{ccc}
A & \xrightarrow{\phi} & B_f \\
\downarrow & & \downarrow \\
& \longrightarrow & B_p
\end{array}$$

If $\phi$ is not injective, we replace $A$ by $\phi(A)$ in the above arguments. This concludes the proof. □

In the next proposition we provide the geometric formulation of this lemma. We omit the proof.

Proposition 1.12. Let $X$ and $Y$ be integral $S$-schemes, and let $x \in X$, $y \in Y$ be points lying over the same point $s \in S$. Suppose that $X$ is normal and that $Y$ is locally of finite expansion over $S$. Let $\phi_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ be a local $\mathcal{O}_{S,s}$-homomorphism. Then there exists an open neighborhood $U$ of $x$ and a $S$-morphism $f : U \to Y$ with $f(x) = y$ and such that the homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ induced by $f$ is $\phi_x$.

Proof. Replacing $S$, $X$ and $Y$ by suitable affine open subschemes, we may assume that $S = \text{Spec}(R)$, $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$ are affine. The points $x$ and $y$ correspond to prime ideals $q \subset B$ and $p \subset A$. Via $\phi_x$ we immediately obtain the following $R$-homomorphism

$$\begin{array}{ccc}
A & \xrightarrow{\phi_x} & A_p \\
\downarrow & & \downarrow \\
& \longrightarrow & B_q
\end{array}$$

By Lemma 1.11 this homomorphism factors through $\phi : A \to B_t$ for some $t \in B \setminus q$. It is clear that the corresponding morphism $D_{\text{Spec}(B)}(t) \to \text{Spec}(A)$ maps $x$ to $y$ and induces the morphism $\phi_x$ on the stalks. □
1.3. Quasi-generating systems and polynomial rings. Let \( k \) be a field, and let \( A \) be a non-empty \( k \)-algebra of finite expansion. By assumption we can find elements \( a_1, \ldots, a_n \in A \) such that \( k[a_1, \ldots, a_n] \subset A \) is an integral extension of \( k \)-algebras. Via Noether normalization [Kem11, Thm. 8.19] we can find algebraically independent elements \( a'_1, \ldots, a'_m \in k[a_1, \ldots, a_n] \) such that \( k[a'_1, \ldots, a'_m] \subset k[a_1, \ldots, a_n] \) is also an integral extension. Thus, by the tower property of integral extensions [Kem11, Cor. 8.6] \( (a'_1, \ldots, a'_m) \) is an algebraically independent quasi-generating system of \( A \). Further \( k[a'_1, \ldots, a'_m] \sim k(x_1, \ldots, x_n) \), where the latter ring denotes the polynomial ring in \( n \) variables over \( k \). Let \( (b_1, \ldots, b_l) \) be another algebraically independent quasi-generating system of \( A \). Then by [Kem11, Cor. 8.13] we have the following equality:

\[
m = \dim(k[a'_1, \ldots, a'_m]) = \dim(A) = \dim(k[b_1, \ldots, b_l]) = l.
\]

Thus all algebraically independent quasi-generating systems are of the same size.

This observation justifies the following assumption for our notation in the coming chapters: If \( \mathfrak{a} \) is a quasi-generating system of \( A \), then we can always assume that \( \mathfrak{a} \) is algebraically independent over \( k \). Further, \( k[\mathfrak{a}] \) is isomorphic to a polynomial ring over \( k \). To exemplify this isomorphism we will denote the elements of \( \mathfrak{a} \) by indeterminate variables of a polynomial ring. I.e., we will write \( \mathfrak{a} = (x_1, \ldots, x_n) \).

Equipped with our new notation we conclude this section by demonstrating that a \( k \)-algebra of finite expansion is a Jaffard ring.

**Lemma 1.13.** Let \( k \) be a field, and let \( A \) be a \( k \)-algebra of finite expansion. Then \( A \) is a Jaffard ring, i.e., \( \dim(A[x_1, \ldots, x_n]) = n + \dim(A) \).

**Proof.** First note that if \( R \subset S \) is an integral extension of rings, then \( R[x] \subset S[x] \) is also an integral extension of rings.

Let \( m = \dim(A) \). By assumption we can find algebraically independent elements \( x_1, \ldots, x_m \in A \) such that \( k[x_1, \ldots, x_m] \subset A \) is an integral extension. Now let \( n \geq 1 \) and consider the polynomial ring \( A[y_1, \ldots, y_n] \). By our remark at the beginning

\[
k[x_1, \ldots, x_m][y_1, \ldots, y_n] \subset A[y_1, \ldots, y_n]
\]

is also an integral extension. Further we have that

\[
k[x_1, \ldots, x_m][y_1, \ldots, y_n] \cong k[y_1, \ldots, y_n, y_{n+1}, \ldots, y_{m+n}],
\]

and combined with [Kem11] Cor. 5.7, Cor. 8.13 we conclude that

\[
\dim(A[y_1, \ldots, y_m]) = \dim(k[y_1, \ldots, y_n, y_{n+1}, \ldots, y_{m+n}]) = m + n. \quad \square
\]

2. Schemes of finite expansion over fields

In the first part of this section we introduce dimension formulae for schemes of finite expansion over a field \( k \). In the second part we demonstrate that a
normal $k$-domain of finite expansion is a Prüfer domain. We will then use this property to show that for a normal integral universally closed curve over $k$ we have a bijection between the closed points of the curve and valuation rings inside the function field.

2.1. **Dimension of schemes of finite expansion.** For an affine scheme of finite expansion it is straightforward to compute its dimension.

**Proposition 2.1.** Let $k$ be a field, and let $X = \text{Spec}(A)$ be an affine $k$-scheme of finite expansion. Then we have that $\dim(X) = d$, where $d$ is the number of algebraically independent quasi-generators of $A$.

**Proof.** By assumption we can find algebraically independent elements $x_1, \ldots, x_d \in A$ such that $k[x_1, \ldots, x_d] \subset A$ is an integral extension of rings. The claim follows then from [Kem11, Cor. 5.7, Cor. 8.13]. \hfill $\square$

Before investigating the general case, we observe that the property (locally) of finite expansion can be passed to the reduced subscheme.

**Lemma 2.2.** Let $k$ be a field.

1. Let $A$ be a $k$-algebra of finite expansion. Then $A/\mathcal{N}$ is also of finite expansion over $k$, where $\mathcal{N}$ denotes the nilradical of $A$.

2. Let $X$ be a $k$-scheme (locally) of finite expansion. Then $X_{\text{red}}$ is also (locally) of finite expansion over $k$.

**Proof.** For (1), by assumption we have an integral ring extension $f : k[x_1, \ldots, x_n] \hookrightarrow A$ for some $n \in \mathbb{N}$. Via the projection $A \rightarrow A/\mathcal{N}$ we can extend $f$ to an integral homomorphism $\bar{f} : k[\bar{x}_1, \ldots, \bar{x}_n] \hookrightarrow A/\mathcal{N}$, where $\bar{x}_i$ denotes the projection of $x_i$ into $A/\mathcal{N}$.

For (2), the reduced subscheme is defined as $X_{\text{red}} = (X, \mathcal{O}_X/\mathcal{N})$, where $\mathcal{N}$ is the nilradical of $\mathcal{O}_X$. The claim now follows from Corollary/Definition [1.5] and part (1). \hfill $\square$

The following theorem connects the dimension of an integral $k$-scheme locally of finite expansion with the transcendence degree of its function field. It is an adaptation of the locally of finite type case presented in [GW20, Thm. 5.22].

**Theorem 2.3.** Let $k$ be a field. Let $X$ be an irreducible $k$-scheme locally of finite expansion with generic point $\eta$.

1. $\dim(X) = \text{trdeg}_k(\kappa(\eta))$.

2. Let $f : Y \rightarrow X$ be a morphism of $k$-schemes of finite expansion such that $f(Y)$ contains the generic point $\eta$ of $X$. Then $\dim(Y) \geq \dim(X)$. In particular we have $\dim(U) = \dim(X)$ for any non-empty open subscheme $U$ of $X$.

**Proof.** (1) We may assume that $X$ is reduced, and covering $X$ by non-empty open affine subschemes $U$ we may assume that $X = \text{Spec}(A)$, where $A$ is
a $k$-domain of finite expansion. Thus we have that $\kappa(\eta) = \text{Frac}(A)$. Let $(x_1, \ldots, x_d)$ be an algebraically independent quasi-generating system of $A$, then $k[x_1, \ldots, x_d] \subset A$ is an integral extension of integral domains, hence $\kappa(\eta)$ is an algebraic field extension of $K = k(x_1, \ldots, x_d)$. With the tower property of the transcendence degree (cf. [KM17, Satz 23.4]) we conclude that:

$$\text{trdeg}_k(\kappa(\eta)) = \text{trdeg}_k(K) + \text{trdeg}_K(\kappa(\eta)) = \text{trdeg}_k(K) = d.$$ 

By Proposition 2.1 we also have that $\dim(A) = d$, so the claim $\dim(A) = \text{trdeg}_k(\kappa(\eta))$ follows.

(2) By hypothesis there exists $\theta \in Y$ such that $f(\theta) = \eta$. Therefore $f$ induces a $k$-embedding $\kappa(\eta) \hookrightarrow \kappa(\theta)$. Denote with $Z$ the closure of $\theta$. Then

$$\dim(X) = \text{trdeg}_k(\kappa(\eta)) \leq \text{trdeg}_k(\kappa(\theta)) = \dim(Z) \leq \dim(Y). \quad \square$$

With this theorem we can immediately conclude that for an integral universally closed curve $X$ over $k$ we have that $\text{trdeg}_k(K(X)) = 1$.

Similar one can extend dimension formulae for products and extensions of the base field. Since the proof is analog to [GW20, Prop. 5.37, 5.38] we skip it.

**Proposition 2.4.** Let $k$ be field. Let $X, Y$ be non-empty $k$-schemes locally of finite expansion, and let $K$ be a field extension of $k$. Then

(1) $\dim(X \times_k Y) = \dim(X) + \dim(Y),$

(2) $\dim(X) = \dim(X \otimes_k K)$.

2.2. **Normal one-dimensional domains of finite expansion are Prüfer domains.** Let $C$ be a normal proper integral curve over a field $k$. It is well-known that for a closed point $x \in C$ the local ring $\mathcal{O}_{C,x}$ is a discrete valuation ring (cf. [GW20, Rem. 15.23]). In this section we will establish a similar result for schemes of finite expansion over $k$.

**Theorem 2.5.** Let $k$ be a field, and let $A$ be a normal $k$-domain of finite expansion of dimension $1$. Then $A$ is a Prüfer domain. I.e., if $m \subset A$ is a maximal ideal, then $A_m$ is a valuation ring.

**Proof.** This is an application of [FS01, Ch. III, §1, Thm. 1.2]. \quad \square

We provide counterexamples that normality is a necessary assumption and that in general we do not obtain a discrete valuation ring.

**Example 2.6.** Let $k$ be a field.

(1) The $k$-algebra $A = k[x^2, x^3] \subset k[x]$ is of finite expansion and of dimension $1$. It is not normal, because $x \in \text{Frac}(A)$ is integral over $A$ however $x \notin A$. If we localize $A$ at $m = (x^2, x^3)$ we again have that $x, x^{-1} \notin A_m$, so $A_m$ is not a valuation ring.
Let \( B = k[x^{1/n} \mid n \in \mathbb{N}] \). Then \( B \) is of finite expansion, normal and of dimension 1. Consider the maximal ideal \( \mathfrak{m} = (x^{1/n} \mid n \in \mathbb{N}) \). By Theorem 2.5 \( B_{\mathfrak{m}} \) is a valuation ring and it has value group \( \mathbb{Q} \). Thus \( B_{\mathfrak{m}} \) is not a discrete valuation ring.

In the next Corollary we provide the geometric formulation of Theorem 2.5.

**Corollary 2.7.** Let \( k \) be a field, and let \( X \) be a \( k \)-scheme which is normal, integral, of finite expansion and of dimension 1. Let \( x \in X \) be a closed point. Then \( \mathcal{O}_{X,x} \) is a valuation ring.

**Proof.** This is an application of Theorem 2.3 (2) and Theorem 2.5. \( \square \)

### 2.3. Closed points correspond to valuation rings

Let \( C \) be a normal proper integral curve over a field \( k \), then one has a bijection between the closed points of \( C \) and discrete valuation rings inside the function field \( K(C) \) (cf. [GW20, Rem. 15.23]). We can extend this result in a similar fashion to universally closed curves.

**Theorem 2.8.** Let \( k \) be a field, and let \( X \) be a normal integral universally closed curve over \( k \). Then one has a bijection between the sets

\[
\{\text{closed points of } X\} \leftrightarrow \left\{ \begin{array}{c}
\text{valuation rings } \\
\mathcal{O} \subset K(X) \\
\text{with } k^\times \subset \mathcal{O}^\times
\end{array} \right\}.
\]

**Proof.** First we prove that every valuation ring in \( K(X) \) comes from a closed point.

Let \( \mathcal{O} \subset K(X) \) be a valuation ring with \( k^\times \subset \mathcal{O}^\times \). Then we can construct a commutative diagram

\[
\begin{array}{ccc}
\text{Spec}(K(X)) & \rightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec}(\mathcal{O}) & \rightarrow & \text{Spec}(k),
\end{array}
\]

where \( \iota \) maps the unique point of \( \text{Spec}(K(X)) \) to the generic point of \( X \). By the generalized valuative criterion (cf. [GW20, Thm. 15.8]) there exists a unique morphism \( v : \text{Spec}(\mathcal{O}) \rightarrow X \), which preserves commutativity in the diagram. Suppose the unique closed point \( z \in \text{Spec}(\mathcal{O}) \) maps to the closed point \( x \in X \), i.e., \( v(z) = x \). Then we obtain the following local homomorphism of local rings

\[
v_z^\#: (v^{-1}\mathcal{O}_X)_z \rightarrow \mathcal{O}_{\text{Spec}(\mathcal{O}),z}
\]

\[\Rightarrow v_z^\#: \mathcal{O}_{X,x} \rightarrow \mathcal{O}.
\]

For commutativity of the above diagram we must have that the generic point of \( \mathcal{O} \) is mapped to the generic point of \( X \). I.e., \( v \) is a dominant morphism, but this makes \( v_z^\# \) into an injective local homomorphism (see [Stacks, Tag 0CC1]). Hence
\( \mathcal{O} \) dominates \( \mathcal{O}_{X,x} \). From Corollary 1.8 and Corollary 2.7 it follows that \( \mathcal{O}_{X,x} \) is also a valuation ring. Valuation rings are maximal among the domination order, thus \( \mathcal{O} = \mathcal{O}_{X,x} \).

Injectivity follows similar. Let \( x, x' \in X \) be such that \( \mathcal{O}_{X,x} \cong \mathcal{O}_{X,x'} \). Again by the valuative criterion we obtain a morphism \( v : \text{Spec}(\mathcal{O}_{X,x}) \cong \text{Spec}(\mathcal{O}_{X,x'}) \to X \). If \( z \) and \( z' \) are the unique closed points of \( \text{Spec}(\mathcal{O}_{X,x}) \) and \( \text{Spec}(\mathcal{O}_{X,x'}) \), then we have that \( x = v(z) = v(z') = x' \).

\begin{flushright}
\( \square \)
\end{flushright}

\textbf{Remark 2.9.} From the theorem we can also conclude that for a normal proper integral curve \( C \) over a field \( k \) all valuation rings \( \mathcal{O} \) contained in the function field \( K(C) \) with \( k^\times \subset \mathcal{O}^\times \) are discrete.

3. Universally closed curves and extensions of transcendence degree 1

We have now developed all necessary tools to prove the main theorem. The proof is developed in a similar way as the original proof for curves presented in [EGAII][Rem. 7.4.19].

\textbf{3.1. A homomorphism of function fields induces a morphism of universally closed curves.} On objects the association \( X \mapsto K(X) \) is clear, but we must also check that there is a contravariant association of morphisms. In the first lemma we will establish this for dominant morphisms between integral schemes of finite expansion. Later in Theorem 3.5 we will prove that a morphism between integral universally closed curves is either constant or surjective.

\textbf{Lemma 3.1.} Let \( k \) be a field, let \( X \) and \( Y \) be integral \( k \)-schemes of finite expansion, and let \( f : X \to Y \) be a dominant morphism of \( k \)-schemes.

\begin{enumerate}
\item \( f \) induces a homomorphism of fields \( f^* : K(Y) \to K(X) \) in a functorial way.
\item If \( \dim(X) = \dim(Y) \), then \( K(X) \) becomes via \( f^* \) an algebraic field extension of \( K(Y) \).
\end{enumerate}

\textbf{Proof.} For a dense morphism the generic point of \( X \) is mapped to the generic point of \( Y \), and the morphism \( f^* \) is the induced morphism on stalks of the structure sheaves. It is a homomorphism of fields, hence injective. Under the assumptions of (2) the transcendence degrees over \( k \) agree by Theorem 2.3 (1), thus the field extension is algebraic. \( \square \)

The first major step is to establish that a homomorphism of function fields of universally closed curves is induced by a unique morphism of universally closed curves. This can be seen as adaption of [EGAII Cor. 7.4.13)] to universally closed curves.

\textbf{Theorem 3.2.} Let \( k \) be a field. Let \( X \) be a normal separated integral \( k \)-scheme of finite expansion of dimension 1, and let \( Y \) be an integral universally closed
curve over $k$. Then every $k$-homomorphism $\alpha : K(Y) \to K(X)$ is of the form $f^*$ for a uniquely determined morphism $f : X \to Y$.

**Proof.** We immediately obtain a morphism

$$\text{Spec}(\alpha) : \text{Spec}(K(X)) \to \text{Spec}(K(Y)) \to Y,$$

where the second morphism corresponds to the inclusion of the generic point of $Y$. Let $x \in X$ be a closed point and let $U = \text{Spec}(B) \subset X$ be an affine open neighborhood of $x$. By assumption $U$ is normal, integral, of finite expansion and one-dimensional, further $x$ corresponds to a maximal ideal $m_x \subset B$. Hence by Corollary 2.7 $O_{X,x} \cong B_{m_x}$ is a valuation ring. Now we consider the following commutative diagram of schemes:

$$
\begin{array}{ccc}
\text{Spec}(K(X)) & \xrightarrow{\text{Spec}(\alpha)} & Y \\
\downarrow & & \downarrow \\
\text{Spec}(O_{X,x}) & \longrightarrow & \text{Spec}(k).
\end{array}
$$

By the generalized valuative criterion (see [GW20, Thm. 15.8]) there exists a unique morphism $g_x : \text{Spec}(O_{X,x}) \to Y$.

Let $V = \text{Spec}(A)$ be an affine open neighborhood of finite expansion of $g_x(x)$ in $Y$. Then $g_x^{-1}(V)$ is open in $\text{Spec}(O_{X,x})$ and contains $x$, thus it is equal to $\text{Spec}(O_{X,x})$. So we obtain an induced homomorphism of rings $A \to B_{m_x}$.

By Lemma 1.11 the homomorphism $A \to B_{m_x}$ splits as $A \to B_{\omega} \to B_{m_x}$ for some $\omega \in B \setminus m_x$. Thus on the level of spectra we obtain an extension of $g_x$ to $\tilde{g}_x : D_U(\omega) \to V$, where $D_U(\omega)$ is an affine open neighborhood of $x$ in $X$.

For varying $x$ we would like to glue the extensions $\tilde{g}_x$ to a unique morphism $f : X \to Y$. According to [GW20, Prop. 3.5] it is enough to show that $\tilde{g}_x : D_U(\omega) \to Y$ and $\tilde{g}_{x'} : D_U'(\omega') \to Y$ coincide on $D_U(\omega) \cap D_U'(\omega')$. Let us consider the equalizer $\text{Eq}(\tilde{g}_x, \tilde{g}_{x'})$, since $Y$ is separated the equalizer is a closed subscheme of $D_U(\omega) \cap D_U'(\omega')$ (cf. [GW20, Def. and Prop. 9.7]). By construction $\tilde{g}_x$ and $\tilde{g}_{x'}$ preserve commutativity in Diagram 3.1 so we must have that $\text{Spec}(K(X)) \subset \text{Eq}(\tilde{g}_x, \tilde{g}_{x'})$ and hence $\text{Eq}(\tilde{g}_x, \tilde{g}_{x'}) = D_U(\omega) \cap D_U'(\omega')$. Now we glue these extensions to a unique morphism $f : X \to Y$. □

Analog to curves we can define the degree of a morphism between integral universally closed curves (cf. [GW20, p. 498]).

**Definition 3.3** (Degree of a morphism). Let $X$ and $Y$ be integral universally closed curves over a field $k$, and let $f : X \to Y$ be a morphism. If $f$ has dense image we define the degree of $f$ as

$$\deg(f) := \left[ K(X) : f^*(K(Y)) \right].$$

If the image of $f$ is not dense we define the degree to be 0.
The next corollary can be seen as generalization of [EGAII, Cor. 7.4.16].

**Corollary 3.4.** Let $k$ be a field. Let $f : X \to Y$ be a morphism of normal integral universally closed curves over $k$ which is of degree 1. Then $f$ is an isomorphism.

**Proof.** $f^*$ induces a field extension of degree 1, hence it is an isomorphism. By Theorem 3.2 its inverse comes from a unique morphism $g : Y \to X$. Using the equalizer we conclude that $g$ is indeed the inverse of $f$. \hfill $\square$

3.2. Properties of morphisms of universally closed curves. It is well-known that a non-constant morphism between normal proper integral curves is either constant or surjective, finite and flat (cf. [GW20, Prop. 15.16] and [Stacks, Tag 0CCK]). We will now prove the analog statement for universally closed curves.

**Theorem 3.5.** Let $k$ be a field, and let $f : X \to Y$ be a morphism between integral universally closed curves over $k$. Then $f$ is constant or quasi-compact, separated, universally closed and surjective. With additional assumptions the following assertions hold if $f$ is not constant.

1. If $X$ is normal then $f$ is integral.
2. If $Y$ is normal then $f$ is flat.

**Proof.** Since $X$ and $Y$ are universally closed curves we can conclude by cancellation that $f$ is separated, quasi-compact and universally closed. In particular it follows that $f(X)$ is closed in $Y$. Images of irreducible sets under continuous maps are again irreducible, hence $f(X)$ is also irreducible. As $Y$ is integral and quasi-compact an irreducible closed subscheme is either a closed point or the whole scheme $Y$. Thus $f$ is either constant or surjective.

Now let us prove the additional assertion (1). By [Stacks, Tag 01WM] it is equivalent to show that $f$ is affine and universally closed. Recall that by Lemma 3.1 $f^* : K(Y) \to K(X)$ induces an algebraic field extension. Now let $\text{Spec}(A) \subset Y$ be a non-empty affine open subscheme. Then $A$ is of finite expansion over $k$ and $\text{Frac}(A) \cong K(Y) \subset K(X)$. Denote with $Z$ the normalization of $\text{Spec}(A)$ in $K(X)$. Then by [GW20, Prop. 12.43] $Z \cong \text{Spec}(B)$, $K(Z) \cong K(X)$ and $B$ is also of finite expansion over $k$. So by Theorem 3.2 there exists a unique morphism $g : Z \to X$. By construction of $Z$ the image of $f \circ g$ lies in $\text{Spec}(A)$, hence we have an induced morphism $g : Z \to f^{-1}(\text{Spec}(A))$. To conclude the proof we must show that this is an isomorphism, so let us construct an inverse. We choose an affine open covering $f^{-1}(\text{Spec}(A)) = \bigcup_i \text{Spec}(B_i)$. Then $\text{Frac}(B_i) \cong K(X) \cong K(Z)$ for all $i$. Let $b \in B$, then $b$ is integral over $A$ and thus also over $B_i$. But $\text{Spec}(B_i)$ is an affine open of $X$, so it is normal. Thus $b \in B_i$ and $B \subset B_i$ for all $i$. We obtain morphisms $h_i : \text{Spec}(B_i) \to \text{Spec}(B) = Z$ which are induced by inclusions of the coordinate rings into $K(X)$, thus we can glue them to a morphism $h : f^{-1}(\text{Spec}(A)) \to Z$. Now we consider the
equalizer \(\text{Eq}(h \circ g, \text{id}_Z)\) which is a closed subscheme of \(Z\). Since on function fields \(h^*\) is inverse to \(g^*\) we must have that \(h \circ g(\eta_Z) = \text{id}_Z(\eta_Z)\), but this implies that \(\eta_Z \in \text{Eq}(h \circ g, \text{id}_Z)\) and thus \(\text{Eq}(h \circ g, \text{id}_Z) = Z\). Therefore \(h \circ g = \text{id}_Z\). As an open subscheme of a separated scheme \(f^{-1}(\text{Spec}(A))\) is also separated over \(\text{Spec}(k)\), therefore we can conclude by a symmetric argument that \(g \circ h = \text{id}_{f^{-1}(\text{Spec}(A))}\).

For assertion (2), pick points \(x \in X\) and \(y \in Y\) such that \(f(x) = y\). The local ring \(O_{Y,y}\) is either a field or a valuation ring by Corollary 2.7. Further, \(f\) is a dominant morphism between integral schemes, hence the induced homomorphism on local rings \(O_{Y,y} \to O_{X,x}\) is injective by [Stacks, Tag 0CC1]. Therefore \(O_{X,x}\) is torsion free as an \(O_{Y,y}\)-module and by [Stacks, Tag 0539] this proves that \(O_{X,x}\) is a flat \(O_{Y,y}\)-module. \(\square\)

3.3. Construction of universally closed curves starting from field extensions.

Starting with a field extension \(K\) of \(k\) of transcendence degree \(1\) we will now construct a universally closed curve \(X\) with \(K(X) \cong \text{Spec}(k)\). This construction can be seen as generalization of the first part of [EGAII, Prop. 7.4.18].

**Theorem 3.6.** Let \(k\) be a field, and let \(K\) be a field extension of \(k\) of transcendence degree \(1\). Then there is a normal integral universally closed curve \(X\) over \(k\) with \(K(X) \cong \text{Spec}(k)\). It is unique up to isomorphism.

**Proof.** Let \(x \in K\) be such that \(x\) is transcendent over \(k\). We denote with \(\nu : U \to \text{Spec}(k[x])\) the normalization of \(\text{Spec}(k[x])\) in \(K\). By [GW20, Prop. 12.43] the scheme \(U\) has the following properties:

1. The scheme \(U\) is integral and normal, and \(K(U) = K\).
2. The morphism \(\nu\) is integral and surjective and \(\dim(U) = \dim(\text{Spec}(k[x])) = 1\).
3. \(\text{Spec}(k[x])\) is affine, thus \(U = \text{Spec}(A)\), where \(A\) is the integral closure of \(k[x]\) in \(K\).

Naturally we can regard \(U\) also as \(k\)-scheme, we denote the \(k\)-structure morphism by \(\pi : U \to \text{Spec}(k)\). From the above properties we can immediately conclude that \(U\) is a one-dimensional, normal, quasi-compact, separated, integral \(k\)-scheme of finite expansion.

Now we apply Theorem 1.9 to write \(\pi\) as composition \(\pi = \tilde{\pi} \circ j\), where \(j : U \to X\) is an open immersion and \(\tilde{\pi} : X \to \text{Spec}(k)\) is separated and universally closed. We consider \(\tilde{\pi}\) and \(X\) as the compactification of \(\pi\) and \(U\) respectively. A priori it may not be clear that this is a meaningful notion of compactification for our purpose, therefore we will now establish that \(j\) is dominant and that we can consider \(X\) to be integral, normal and quasi-compact.

**j is dominant:** We restate the arguments presented in [Con07, Rem, 4.2]: \(\pi\) is quasi-compact and \(\tilde{\pi}\) is separated, so by cancellation \(j\) is also quasi-compact. Thus, the scheme-theoretic closure of \(U\) in \(X\) exists and
we rename it as \( X \). Obviously a closed subscheme of a separated and universally closed scheme is also separated and universally closed. So \( j \) is dominant.

**\( X \) is integral:** Closures and images under continuous maps of irreducible sets are again irreducible. Thus \( X = \overline{j(U)} \) is irreducible.

As \( U \) is reduced \( j \) factors as \( (GW20\text{, Prop. 3.51]) \)

\[
\begin{array}{ccc}
    U & \xrightarrow{j_{\text{red}}} & X_{\text{red}} \\
    \downarrow j & & \downarrow \iota \\
    X
\end{array}
\]

where \( \iota : X_{\text{red}} \rightarrow X \) is the canonical inclusion of the reduced subscheme.

As the topological spaces of \( X \) and \( X_{\text{red}} \) agree \( j_{\text{red}} \) must also be dense. We have an isomorphism between \( U \) and \( j(U) \), so \( j(U) \) is reduced too. Therefore \( \iota \) defines an isomorphism between \( j(U) \) and \( j_{\text{red}}(U) \).

Hence \( j_{\text{red}} \) is also an open immersion. The reduced structure morphism \( \tilde{\pi}_{\text{red}} : X_{\text{red}} \rightarrow \text{Spec}(k) \) is again separated and universally closed by cancellation. Now we rename \( X_{\text{red}} \) as \( X \).

**\( X \) is normal:** Denote with \( \pi' : X_{\text{norm}} \rightarrow X \) the normalization of \( X \) in \( K(X) \).

By \( GW20\text{, Prop. 12.44] the morphism } \pi' \text{ is integral and dominant and we obtain a unique morphism } j' : U \rightarrow X_{\text{norm}} \text{ such that } j = \pi' \circ j' \text{. Also note that by } GW20\text{, Rem. 12.46] the restriction } \pi'^{-1}(j(U)) \rightarrow j(U) \text{ is an isomorphism. The open immersion } j \text{ defines an isomorphism between } U \text{ and } j(U) \text{ and } j' \text{ defines an isomorphism between } U \text{ and } j'(U) \text{. The composition of isomorphisms is an isomorphism, so we have that }

\[
U \cong j(U) \cong j'(U) \cong \pi'^{-1}(j(U)) .
\]

I.e., \( j'(U) \) is isomorphic to an open subscheme of \( X_{\text{norm}} \), therefore \( j' \) is an open immersion. Further, it is also clear that \( j' \) is dominant, else the equality

\[
\overline{j(U)} = X = \pi'(j'(U))
\]

would not hold. Integral morphisms are separated and universally closed and both properties are stable under composition, thus \( \tilde{\pi}_{\text{norm}} : X_{\text{norm}} \rightarrow \text{Spec}(k) \) is also separated and universally closed. Now we can rename \( X_{\text{norm}} \) as \( X \) and \( j' \) as \( j \).

**\( X \) is quasi-compact:** A universally closed morphism is also quasi-compact by \( \text{Stacks Tag 04XU} \). So \( \tilde{\pi} \) is quasi-compact.

Finally, we have the following equalities for dimensions and function fields

\[
\dim(X_{\text{norm}}) = \dim(X) = \dim(U) = 1 , \quad K(X_{\text{norm}}) \cong K(X) \cong K(U) \cong K .
\]
To sum it up, we have constructed a $k$-scheme $X$ which is quasi-compact, universally closed, separated, integral, normal, of dimension 1 and with $K(X) \cong K$.

Let $X'$ be another such scheme, then we have an isomorphism $K(X) \cong K \cong K(X')$. By Theorem 3.2 and Corollary 3.4 it induces an isomorphism $X' \to X$. □

3.4. **Proof of the main theorem.** We are now able to prove the central result of this thesis.

**Theorem 3.7.** Let $k$ be a field. There is a contravariant equivalence between the categories of

(i) normal integral universally closed curves over $k$ (with non-constant morphisms),

(ii) extension fields $K$ of $k$ of transcendence degree 1 (with $k$-homomorphisms),

given by mapping a scheme $X$ as in (i) to its function field $K(X)$.

**Proof.** The association $X \mapsto K(X)$ indeed defines a contravariant functor. For objects this is obvious, for morphisms we use Lemma 3.1 and Theorem 3.5. Conversely, given a $k$-homomorphism $K(Y) \to K(X)$ of function fields we can construct with Theorem 3.2 a unique morphism $X \to Y$ of normal integral universally closed curves. By Corollary 3.4 this is an isomorphism if the fields are isomorphic, so the functor is fully faithful. Essentially surjective follows from Theorem 3.6. So by [Awo10] Prop. 7.26 the functor gives rise to a contravariant categorical equivalence. □

**References**

[Awo10] Steve Awodey. *Category Theory*. 2nd ed. Oxf. Log. Guides. Oxford University Press, 2010. ISBN: 978-0-19-923718-0.

[Con07] Brian Conrad. “Deligne’s notes on Nagata compactifications.” In: *J. Ramanujan Math. Soc.* 22.3 (2007), pp. 205–257.

[EGAII] Alexander Grothendieck and Jean Dieudonné. “Eléments de géométrie algébrique, II: Étude globale élémentaire de quelques classes de morphismes.” In: *Publ. Math. de l'IHÉS* 8 (1961), pp. 5–222. URL: [http://www.numdam.org/item/PMIHES_1961__8__5_0/](http://www.numdam.org/item/PMIHES_1961__8__5_0/).

[FS01] László Fuchs and Luigi Salce. *Modules over Non-Noetherian Domains*. Math. surv. and monogr. American Mathematical Society, 2001. ISBN: 978-0-8218-1963-0. DOI: [https://doi.org/10.1090/surv/084](https://doi.org/10.1090/surv/084).

[GW20] Ulrich Görtz and Torsten Wedhorn. *Algebraic Geometry I: Schemes*. Sec. ed. Springer Studium Mathematik - Master. Springer Fachmedien Wiesbaden, 2020. ISBN: 978-3-658-30732-5. DOI: [https://doi.org/10.1007/978-3-658-30733-2](https://doi.org/10.1007/978-3-658-30733-2).
[Ham19] Paul Hamacher. *On the generalisation of cohomology with compact support to non-finite type schemes*. 2019. arXiv: [1902.04831](https://arxiv.org/abs/1902.04831) [math.AG]

[Kem11] Gregor Kemper. *A Course in Commutative Algebra*. Grad. Texts in Math. Springer Berlin Heidelberg, 2011. ISBN: 978-3-642-03544-9. DOI: [https://doi.org/10.1007/978-3-642-03545-6](https://doi.org/10.1007/978-3-642-03545-6)

[KM17] Christian Karpfinger and Kurt Meyberg. *Algebra*. 4. Auflage. Springer Berlin Heidelberg, 2017. ISBN: 978-3-662-54721-2. DOI: [https://doi.org/10.1007/978-3-662-54722-9](https://doi.org/10.1007/978-3-662-54722-9)

[Stacks] The Stacks Project Authors. *Stacks Project*. [https://stacks.math.columbia.edu](https://stacks.math.columbia.edu), 2018.

[Ste21] Matthias Johann Steiner. “Schemes of Finite Expansion and Universally Closed Curves.” MA thesis. Technische Universität München, Apr. 2021.

Matthias Steiner - 9020 Klagenfurt am Wörthersee, Österreich

*Email address:* steiner.matthias@gmx.at