UNIVERSAL BOUNDS FOR EIGENVALUES OF A BUCKLING PROBLEM II

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ABSTRACT. In this paper, we investigate universal estimates for eigenvalues of a buckling problem. For a bounded domain in a Euclidean space, we solve partially a conjecture proposed in [7]. For a domain in the unit sphere, we give an important improvement on the results of Wang and Xia [16].

1. Introduction

Let $M$ be an $n$-dimensional complete Riemannian manifold and $\Omega \subset M$ a bounded domain in $M$ with piecewise smooth boundary $\partial \Omega$. A Dirichlet eigenvalue problem of Laplacian is given by

\begin{equation}
\begin{cases}
\Delta u = -\lambda u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\end{equation}

which is also called a fixed membrane problem, where $\Delta$ denotes the Laplacian on $M$. The spectrum of this eigenvalue problem is real and discrete.

The following eigenvalue problem of a biharmonic operator is called a buckling problem:

\begin{equation}
\begin{cases}
\Delta^2 u = -\Lambda \Delta u & \text{in } \Omega, \\
|\partial \Omega = \frac{\partial u}{\partial \nu} |_{\partial \Omega} = 0,
\end{cases}
\end{equation}

which describes the critical buckling load of a clamped plate subjected to a uniform compressive force around its boundary, where $\nu$ is the outward unit normal vector field of the boundary $\partial \Omega$. It is known that the spectrum of the buckling problem is also real and discrete.

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When $\Omega \subset \mathbb{R}^n$ be a bounded domain in an $n$-dimensional Euclidean space $\mathbb{R}^n$, Payne, Pólya and Weinberger [14] and [15] proved the following inequality for eigenvalues of the eigenvalue problem (1.1): for $k = 1, 2, \cdots$,

$$
\lambda_{k+1} - \lambda_k \leq \frac{4}{kn} \sum_{i=1}^{k} \lambda_i.
$$

One calls it a universal inequality since it does not depend on the domain $\Omega$.

On the other hand, Payne, Pólya and Weinberger [14] and [15] also studied eigenvalues of the buckling problem for a bounded domain $\Omega$ in $\mathbb{R}^n$ and intended to derive a universal inequality for eigenvalues of the buckling problem. But it is very hard to deal with this problem. They only proved, for $n = 2$,

$$
\Lambda_2 \leq 3\Lambda_1.
$$

As an open problem, Payne, Pólya and Weinberger [14] and [15] proposed the following:

**Problem.** Whether can one obtain a universal inequality for eigenvalues of the buckling problem (1.2) on a bounded domain in a Euclidean space, which is similar to the universal inequality (1.3) for the eigenvalues of the fixed membrane problem (1.1)?

Although many mathematicians have intended to attack this problem, there are no any progresses except for lower order eigenvalues. For lower order eigenvalues, Hile and Yeh [13] and so on improved the result of Payne, Pólya and Weinberger to

$$
\Lambda_2 \leq \frac{n^2 + 8n + 20}{(n + 2)^2} \Lambda_1.
$$

Furthermore, Ashbaugh [2] (cf. [1]) has obtained

$$
\sum_{i=1}^{n} \Lambda_{i+1} \leq (n + 4)\Lambda_1
$$

and he has commented that to obtain a universal inequality for eigenvalues of the buckling problem remains a challenge for mathematicians since 1955.

Recently, by introducing a new method to construct trial functions for the buckling problem, Cheng and Yang [7] have obtained the following universal inequality for eigenvalues of the buckling problem (1.2):

$$
\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4(n + 2)}{n^2} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)\Lambda_i.
$$

Thus, the problem proposed by Payne, Pólya and Weinberger has been solved affirmatively. Furthermore, it is very important to prove a sharp universal inequality for eigenvalues of the buckling problem. The following has been conjectured in [7]:
Conjecture. Eigenvalues of the buckling problem on a bounded domain in a Euclidean space $\mathbb{R}^n$ satisfy the following universal inequality:

$$\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)\Lambda_i.$$ 

The first purpose in this paper is to attack the above conjecture, we prove the following:

**Theorem 1.1.** Let $\Lambda_i$ be the $i$-th eigenvalue of the buckling problem (1.2) for a bounded domain $\Omega \subset \mathbb{R}^n$. Then, we have

$$\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4(n + \frac{4}{3})}{n^2} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)\Lambda_i.$$ 

**Remark 1.1.** Since our universal inequality is a quadratic inequality of the eigenvalue $\Lambda_{k+1}$, we can conclude an upper bound of $\Lambda_{k+1}$ and an upper bound of the gap between two consecutive eigenvalues as in [7] from (1.5). We will not give it in details.

When $M$ is an $n$-dimensional unit sphere $S^n(1)$, Wang and Xia [16] have studied the buckling problem for a domain $\Omega$ in $S^n(1)$. They have obtained a universal inequality for eigenvalues of the buckling problem, namely, they have proved that eigenvalues of the buckling problem (1.2) for a domain $\Omega$ in the unit sphere $S^n(1)$ satisfy

$$2 \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2$$

$$\leq \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left\{ \delta \Lambda_i + \frac{\delta^2 (\Lambda_i - (n - 2))}{4(\delta \Lambda_i + n - 2)} \right\}$$

$$+ \frac{1}{\delta} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)(\Lambda_i + \frac{(n - 2)^2}{4}),$$

where $\delta$ is an arbitrary positive constant.

The second purpose in this paper is to give an important improvement for the result of Wang and Xia.
Theorem 1.2. Eigenvalues $\Lambda_i$'s of the buckling problem (1.2) for a domain $\Omega$ in the unit sphere $S^n(1)$ satisfy

$$
2 \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 + (n - 2) \sum_{i=1}^{k} \frac{(\Lambda_{k+1} - \Lambda_i)^2}{\Lambda_i - (n - 2)} \leq \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \{ \Lambda_i - \frac{n - 2}{\Lambda_i - (n - 2)} \} \delta_i 
$$

$$
+ \sum_{i=1}^{k} \frac{(\Lambda_{k+1} - \Lambda_i)}{\delta_i} (\Lambda_i + \frac{(n - 2)^2}{4}) \}
$$

for an arbitrary positive non-increasing monotone sequence $\{\delta_i\}_{i=1}^{k}$.

Remark 1.2. It is obvious that our result is sharper than one of Wang and Xia [16] even if we take $\delta_i = \delta$ for any $i$. Since our universal inequality is a quadratic inequality of $\Lambda_{k+1}$, we can obtain an explicit upper bound for the eigenvalue $\Lambda_{k+1}$ from (1.7).

In particular, when $n = 2$, we have

Corollary 1.1. Eigenvalues $\Lambda_i$'s of the buckling problem (1.2) for a domain $\Omega$ in the unit sphere $S^2(1)$ satisfy

$$
\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)\Lambda_i^2.
$$

Proof. Since $n = 2$, from the theorem 1.2 and taking $\delta_i = \frac{1}{\Lambda_i}$, for $i = 1, 2, \cdots, k$, for which $\{\delta_i\}_{i=1}^{k}$ is a positive non-increasing monotone sequence, we finish the proof of the corollary 1.1. \(\Box\)

Remark 1.3. About the recent developments in universal inequalities for eigenvalues of the Dirichlet eigenvalue problem of the Laplacian and the clamped plate problem, readers can see [3], [4], [5], [6], [8], [11], [12] and [17].

2. Proof of the theorem 1.1

For the convenience of readers, we review the method for constructing trial functions introduced by Cheng and Yang [7]. In this section, $\Omega$ is assumed to be a bounded domain in $\mathbb{R}^n$. For functions $f$ and $h$, we define Dirichlet inner product $(f, h)_D$ of $f$ and $h$ by

$$
(f, h)_D = \int_{\Omega} \langle \nabla f, \nabla h \rangle.
$$

Dirichlet norm of a function $f$ is defined by

$$
||f||_D = \{(f, f)_D\}^{1/2} = \left( \int_{\Omega} \sum_{\alpha=1}^{n} |\nabla_{\alpha} f|^2 \right)^{1/2}.
$$
Let \( u_i \) be the \( i \)-th orthonormal eigenfunction of the buckling problem (1.2) corresponding to the eigenvalue \( \Lambda_i \), namely, \( u_i \) satisfies

\[
\begin{align*}
\Delta^2 u_i &= -\Lambda_i \Delta u_i \quad \text{in} \quad \Omega, \\
\frac{\partial u_i}{\partial \nu} &= 0 \\
(u_i, u_j)_D &= \int_\Omega (\nabla u_i, \nabla u_j) = \delta_{ij}.
\end{align*}
\]

(2.1)

\( H^2_2(\Omega) \) defined by

\[
H^2_2(\Omega) = \{ f : f, \nabla \alpha f, \nabla \alpha \nabla \beta f \in L^2(\Omega), \quad \alpha, \beta = 1, \ldots, n \}
\]

is a Hilbert space with norm \( \| \cdot \|_2 \):

\[
\| f \|_2 = \left( \int_\Omega |f|^2 + \int_\Omega |\nabla f|^2 + \sum_{\beta, \alpha = 1}^n (\nabla \alpha \nabla \beta f)^2 \right)^{1/2}.
\]

Let \( H^2_{2,D}(\Omega) \) be a subspace of \( H^2_2(\Omega) \) defined as

\[
H^2_{2,D}(\Omega) = \left\{ f \in H^2_2(\Omega) : f|_{\partial M} = \frac{\partial}{\partial \nu} f \bigg|_{\partial \Omega} = 0 \right\}.
\]

The biharmonic operator \( \Delta^2 \) defines a self-adjoint operator acting on \( H^2_{2,D}(\Omega) \) with discrete eigenvalues \( 0 < \Lambda_1 \leq \Lambda_2 \leq \cdots \leq \Lambda_k \leq \cdots \) for the buckling problem (1.2) and the eigenfunctions defined in (2.1)

\[
\{ u_i \}_{i=1}^\infty = \{ u_1, u_2, \ldots, u_k, \ldots \}
\]

form a complete orthogonal basis for Hilbert space \( H^2_{2,D}(\Omega) \). We define an inner product \( \langle f, h \rangle \) for vector-valued functions \( f = (f^1, f^2, \ldots, f^n) \in \mathbb{R}^n \) and \( h = (h^1, h^2, \ldots, h^n) \in \mathbb{R}^n \) by

\[
(f, h) \equiv \int_\Omega \langle f, h \rangle = \int_\Omega \sum_{\alpha=1}^n f^\alpha h^\alpha.
\]

The norm of \( f \) is defined by

\[
\| f \| = (f, f)^{1/2} = \left( \int_\Omega \sum_{\alpha=1}^n (f^\alpha)^2 \right)^{1/2}.
\]

Denote a Hilbert space \( H^2_1(\Omega) \) of the vector-valued functions as

\[
H^2_1(\Omega) = \{ f : f^\alpha, \nabla \beta f^\alpha \in L^2(\Omega), \quad \alpha, \beta = 1, \ldots, n \}
\]

with norm \( \| \cdot \|_1 \):

\[
\| f \|_1 = \left( \| f \|^2 + \int_\Omega \sum_{\alpha, \beta = 1}^n |\nabla \alpha f^\beta|^2 \right)^{1/2}.
\]

Let \( H^2_{1,D}(\Omega) \subset H^2_1(\Omega) \) be a subspace of \( H^2_1(\Omega) \) spanned by the vector-valued functions \( \{ \nabla u_i \}_{i=1}^\infty \), which form a complete orthonormal basis of \( H^2_{1,D}(\Omega) \).

It is easy to see that for any \( f \in H^2_{2,D}(\Omega) \), \( \nabla f \in H^2_{1,D}(\Omega) \) and for any \( h \in H^2_{1,D}(\Omega) \), there exists a function \( f \in H^2_{2,D}(\Omega) \) such that \( h = \nabla f \).
Let \( x^p \) for \( p = 1, 2, \cdots, n \) be the \( p \)-th coordinate function of \( \mathbb{R}^n \). For the vector-valued function \( x^p \nabla u_i, i = 1, \ldots, k \), we decompose it into

\[
(2.2) \quad x^p \nabla u_i = \nabla h_{pi} + w_{pi},
\]

where \( h_{pi} \in H^2_{2,D}(\Omega) \) and \( \nabla h_{pi} \) is the projection of \( x^p \nabla u_i \) onto \( H^2_{1,D}(\Omega) \) and \( w_{pi} \perp H^2_{1,D}(\Omega) \). Thus,

\[
(2.3) \quad (w_{pi}, \nabla u) = \int_{\Omega} \sum_{j=1}^n w_{pj}^j \nabla_j u = 0, \quad \text{for any } u \in H^2_{2,D}(\Omega).
\]

Therefore, since \( H^2_{2,D}(\Omega) \) is dense in \( L^2(\Omega) \) and \( C^1(\Omega) \) is dense in \( L^2(\Omega) \), we have, for any function \( h \in C^1(\Omega) \cap L^2(\Omega) \),

\[
(2.4) \quad (w_{pi}, \nabla h) = 0.
\]

Hence, from the definition of \( w_{pi} \) and (2.4), we have

\[
(2.5) \quad \begin{cases}
    w_{pi}|_{\partial \Omega} = 0, \\
    \| \text{div} w_{pi} \|^2 = 0, \quad (\text{div} w_{pi} \equiv \sum_{j=1}^n \nabla_j w_{pj}^j).
\end{cases}
\]

We define function \( \varphi_{pi} \) by

\[
(2.6) \quad \varphi_{pi} = h_{pi} - \sum_{j=1}^k b_{pj} u_j,
\]

where

\[
b_{pj} = \int_{\Omega} x^p \langle \nabla u_i, \nabla u_j \rangle = b_{pji}.
\]

It is easy to check, from the definition (2.2) of \( h_{pi} \), that \( \varphi_{pi} \) satisfies

\[
(2.7) \quad \varphi_{pi}|_{\partial \Omega} = \frac{\partial \varphi_{pi}}{\partial \nu}|_{\partial \Omega} = 0 \quad \text{and} \quad (\varphi_{pi}, u_j)_D = (\nabla \varphi_{pi}, \nabla u_j) = 0,
\]

for any \( j = 1, 2, \cdots, k \). Hence, we know that \( \varphi_{pi} \) is a trial function.

In order to prove our theorem 1.1, we prepare three lemmas.

**Lemma 2.1.** For any \( p \) and \( i \), we have

\[
(2.8) \quad 1 + 2\| \langle \nabla x^p, \nabla u_i \rangle \|^2 = 2 \int x^p u_i \langle \nabla x^p, \nabla (\Delta u_i) \rangle.
\]

**Proof.** From the Stokes’ formula, we have

\[
\begin{align*}
\int \langle x^p u_i, \nabla x^p, \nabla (\Delta u_i) \rangle \\
&= -\int \text{div}(x^p u_i \nabla x^p) \Delta u_i \\
&= -\int u_i \Delta u_i - \int x^p \Delta u_i \langle \nabla x^p, \nabla u_i \rangle,
\end{align*}
\]
\[\int x^p \Delta u_i \langle \nabla x^p, \nabla u_i \rangle\]
\[= - \int \langle \nabla x^p, \nabla u_i \rangle^2 - \int x^p \langle \nabla u_i, \nabla \langle x^p, \nabla u_i \rangle \rangle \]
\[= -\| \langle \nabla x^p, \nabla u_i \rangle \|^2 + \int \text{div}(x^p \nabla \langle x^p, \nabla u_i \rangle) u_i \]
\[= -\| \langle \nabla x^p, \nabla u_i \rangle \|^2 + \int \langle \nabla x^p, \nabla \langle \nabla x^p, \nabla u_i \rangle \rangle u_i + \int x^p u_i \Delta \langle \nabla x^p, \nabla u_i \rangle \]
\[= -\| \langle \nabla x^p, \nabla u_i \rangle \|^2 - \int \langle \nabla x^p, \nabla u_i \rangle^2 + \int x^p u_i \langle \nabla x^p, \nabla (\Delta u_i) \rangle.\]

Since \(\| \nabla u_i \|^2 = 1\), we have
\[1 + 2 \| \langle \nabla x^p, \nabla u_i \rangle \|^2 = 2 \int x^p u_i \langle \nabla x^p, \nabla (\Delta u_i) \rangle.\] \(\Box\)

According to \(x^p \nabla u_i = \nabla h_{pi} + w_{pi}\) and \(\nabla (x^p u_i) \in H^2_{1,D}(\Omega)\), we have
\[\text{(2.9)}\]
\[u_i \nabla x^p = \nabla (x^p u_i) - \nabla h_{pi} - w_{pi} = \nabla q_{pi} - w_{pi}\]
with \(\nabla q_{pi} = \nabla (x^p u_i) - \nabla h_{pi}\) and \(q_{pi} \in H^2_{2,D}(\Omega)\). Hence, we derive
\[\text{(2.10)}\]
\[\| u_i \|^2 = \| \nabla q_{pi} \|^2 + \| w_{pi} \|^2.\]

**Lemma 2.2.** For any \(p\) and \(i\),
\[\text{(2.11)}\]
\[3 \| \langle \nabla x^p, \nabla u_i \rangle \|^2 - 2 \Lambda_i \| \nabla q_{pi} \|^2 = \frac{1}{2} - \frac{1}{2} \Lambda_i \| u_i \|^2.\]

**Proof.** Since, from the Stokes’ formula,
\[\int x^p u_i \langle \nabla x^p, \nabla (\Delta u_i) \rangle\]
\[= \int \Delta (x^p u_i) \langle \nabla x^p, \nabla u_i \rangle\]
\[= - \int \langle u_i \nabla x^p, \nabla (\Delta (x^p u_i)) \rangle\]
\[= - \int \langle \nabla q_{pi}, \nabla (\Delta (x^p u_i)) \rangle \quad \text{(from (2.4) and (2.9))}\]
\[= \int q_{pi} \Delta^2 (x^p u_i)\]
\[= \int q_{pi} \left(4 \langle \nabla x^p, \nabla (\Delta u_i) \rangle - \Lambda_i x^p \Delta u_i \right)\]
\[= -4 \int \Delta u_i \langle \nabla q_{pi}, \nabla x^p \rangle - \Lambda_i \int q_{pi} x^p \Delta u_i\]
and

\[- \Lambda_i \int q_{pi} x^p \Delta u_i \]

\[= \Lambda_i \int \langle \nabla q_{pi}, x^p \nabla u_i \rangle + \Lambda_i \int q_{pi} \langle \nabla x^p, \nabla u_i \rangle \]

\[= \Lambda_i \int \langle \nabla q_{pi}, x^p \nabla u_i \rangle - \Lambda_i \int \langle \nabla q_{pi}, u_i \nabla x^p \rangle \]

\[= \Lambda_i \int \langle \nabla q_{pi}, x^p \nabla u_i \rangle - \Lambda_i \| \nabla q_{pi} \|^2, \]

we obtain

(2.12)

\[
\int x^p u_i \langle \nabla x^p, \nabla (\Delta u_i) \rangle = 4 \| \nabla x^p, \nabla u_i \| ^2 + \Lambda_i \int \langle \nabla q_{pi}, x^p \nabla u_i \rangle - \Lambda_i \| \nabla q_{pi} \|^2. \]

From the lemma 2.1 and the above equality, we have

(2.13) \[6 \| \nabla x^p, \nabla u_i \| ^2 - 2 \Lambda_i \| \nabla q_{pi} \| ^2 - 1 = -2 \Lambda_i \int \langle \nabla q_{pi}, x^p \nabla u_i \rangle. \]

Furthermore, from (2.4), \( x^p \nabla u_i = \nabla h_{pi} + w_{pi} \) and \( \nabla q_{pi} = \nabla (x^p u_i) - \nabla h_{pi} \), we have

(2.14)

\[
\int \langle \nabla q_{pi}, x^p \nabla u_i \rangle = \int \langle \nabla q_{pi}, \nabla h_{pi} \rangle \]

\[= \int \langle \nabla q_{pi}, \nabla (x^p u_i) - \nabla q_{pi} \rangle \]

\[= \int \langle \nabla q_{pi}, \nabla (x^p u_i) \rangle - \| \nabla q_{pi} \|^2 \]

\[= \int \langle u_i \nabla x^p, \nabla (x^p u_i) \rangle - \| \nabla q_{pi} \|^2 \]

\[= \| u_i \|^2 + \int \langle u_i \nabla x^p, x^p \nabla u_i \rangle - \| \nabla q_{pi} \|^2. \]
Since
\[
\int \langle u_i \nabla x^p, x^p \nabla u_i \rangle = -\|u_i\|^2 - \int \langle u_i \nabla x^p, x^p \nabla u_i \rangle,
\]
we obtain
\[
\int \langle u_i \nabla x^p, x^p \nabla u_i \rangle = -\frac{1}{2}\|u_i\|^2.
\]
According to (2.13) and (2.14), we have
\[
3\|\langle \nabla x^p, \nabla u_i \rangle\|^2 - 2\Lambda_i \|\nabla q_{pi}\|^2 = \frac{1}{2} - \frac{1}{2} \Lambda_i \|u_i\|^2.
\]
It finishes the proof of the lemma 2.2. \qed

Lemma 2.3. For any \(i\),

\[\tag{2.15} \Lambda_i \sum_{p=1}^{n} \|w_{pi}\|^2 \geq (n - 1) \]

holds.

**Proof.** Since
\[\tag{2.16} \nabla_\beta (x^p \nabla_\alpha u_i) - \nabla_\alpha (x^p \nabla_\beta u_i) = \nabla_\beta w^\alpha_{pi} - \nabla_\alpha w^\beta_{pi}, \]
where \(w^\alpha_{pi} = x^p \nabla_\alpha u_i - \nabla_\alpha h_{pi}\) denotes the \(\alpha\)-th component of \(w_{pi}\), we infer, from \(\text{div}(w_{pi}) = 0\),

\[\begin{align*}
\|\nabla w_{pi}\|^2 &= \sum_{\alpha,\beta=1}^{n} \|\nabla_\alpha w^\beta_{pi}\|^2 \\
&= \frac{1}{2} \sum_{\alpha,\beta=1}^{n} \|\nabla_\beta w^\alpha_{pi} - \nabla_\alpha w^\beta_{pi}\|^2 + \|\text{div}(w_{pi})\|^2 \\
&= \frac{1}{2} \sum_{\alpha,\beta=1}^{n} \|\nabla_\beta (x^p \nabla_\alpha u_i) - \nabla_\alpha (x^p \nabla_\beta u_i)\|^2 \\
&= 1 - \|\nabla p u_i\|^2.
\end{align*}\]

Furthermore, we have
\[
\begin{align*}
\Delta w^\alpha_{pi} &= \Delta (x^p \nabla_\alpha u_i - \nabla_\alpha h_{pi}) \\
&= \Delta (x^p \nabla_\alpha u_i) - \nabla_\alpha \left(\text{div}(\nabla h_{pi})\right) \\
&= \Delta (x^p \nabla_\alpha u_i) - \nabla_\alpha \left(\text{div}(x^p \nabla u_i)\right) \\
&= \nabla_p \nabla_\alpha u_i - \nabla_\alpha x^p \Delta u_i.
\end{align*}
\]

Thus, we obtain
\[\tag{2.18} \Delta w_{pi} = \nabla \langle \nabla x^p, \nabla u_i \rangle - \Delta u_i \nabla x^p. \]
For any positive constant $\epsilon_i$, we have
\[
\|\nabla w_{pi}\|^2 = -\int \langle w_{pi}, \Delta w_{pi} \rangle
\]
\[
= -\int \langle w_{pi}, \nabla(\nabla x^p, \nabla u_i) - \Delta u_i \nabla x^p \rangle
\]
\[
\leq \frac{\epsilon_i}{2} \|w_{pi}\|^2 + \frac{1}{2\epsilon_i} \|\nabla(\nabla x^p, \nabla u_i) - \Delta u_i \nabla x^p\|^2.
\]

Since, from (2.17),
\[
\sum_{p=1}^{n} \|\nabla w_{pi}\|^2 = n - 1, \quad \sum_{p=1}^{n} \|\nabla(\nabla x^p, \nabla u_i)\| = \Lambda_i,
\]
by taking sum on $p$ from 1 to $n$ for (2.19), we have
\[
(n - 1) \leq \frac{\epsilon_i}{2} \sum_{p=1}^{n} \|w_{pi}\|^2 + \frac{n - 1}{2\epsilon_i} \Lambda_i.
\]

Putting
\[
\epsilon_i = \sqrt{\frac{(n - 1)\Lambda_i}{\sum_{p=1}^{n} \|w_{pi}\|^2}},
\]
we obtain
\[
\Lambda_i \sum_{p=1}^{n} \|w_{pi}\|^2 \geq (n - 1).
\]

It completes the proof of the lemma 2.3. \qed

**Proof of Theorem 1.1.** Since $\varphi_{pi}$ is a trial function, from the Rayleigh-Ritz inequality, we have
\[
\Lambda_{k+1} \|\nabla \varphi_{pi}\|^2 \leq \int \varphi_{pi} \Delta^2 \varphi_{pi} = -\int \nabla \varphi_{pi} \cdot \nabla (\Delta \varphi)_{pi}.
\]

By making use of the same arguments as in Cheng and Yang [7], we have, for any $p$ and $i$,
\[
(\Lambda_{k+1} - \Lambda_i) \|\nabla \varphi_{pi}\|^2 \leq 1 + 3 \|\nabla u_i\|^2 - \Lambda_i (\|u_i\|^2 - \|w_{pi}\|^2) + \sum_{j=1}^{k} (\Lambda_i - \Lambda_j) b_{p_{ij}}.
\]

(2.22) \[1 + 2 \sum_{j=1}^{k} b_{p_{ij}} c_{p_{ij}} = -2 \int_{\Omega} \langle \nabla \varphi_{pi}, \nabla(\nabla x^p, \nabla u_i) \rangle,
\]
where
\[
c_{p_{ij}} = \int \langle \nabla(\nabla x^p, \nabla u_i), \nabla u_j \rangle = -c_{p_{ji}}.
\]
Hence, we have, for any positive constant $\delta_i$,

\[
(\Lambda_{k+1} - \Lambda_i)^2 (1 + 2 \sum_{j=1}^{k} b_{pij} c_{pij})
\]

\[
= (\Lambda_{k+1} - \Lambda_i)^2 \int_{\Omega} -2(\nabla \varphi_{pi}, \nabla (x^p, \nabla u_i) - \sum_{j=1}^{k} c_{pij} \nabla u_j)
\]

\[
\leq \delta_i (\Lambda_{k+1} - \Lambda_i)^3 \|\nabla \varphi_{pi}\|^2 + \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \left( \|\nabla (x^p, \nabla u_i)\|^2 - \sum_{j=1}^{k} c_{pij}^2 \right).
\]

From (2.21) and $\|u_i\|^2 = \|\nabla q_{pi}\|^2 + \|w_{pi}\|^2$, we obtain

\[
(\Lambda_{k+1} - \Lambda_i)^2 (1 + 2 \sum_{j=1}^{k} b_{pij} c_{pij})
\]

\[
\leq \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \left( 1 + 3 \| \nabla p u_i \|^2 - \Lambda_i \| \nabla q_{pi} \|^2 + \sum_{j=1}^{k} (\Lambda_i - \Lambda_j) b_{pij}^2 \right)
\]

\[
+ \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \left( \|\nabla (x^p, \nabla u_i)\|^2 - \sum_{j=1}^{k} c_{pij}^2 \right).
\]

By taking sum on $p$ from 1 to $n$, we derive

\[
(\Lambda_{k+1} - \Lambda_i)^2 (n + 2 \sum_{p=1}^{n} \sum_{j=1}^{k} b_{pij} c_{pij})
\]

\[
\leq \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \left( n + 3 - \Lambda_i \sum_{p=1}^{n} \| \nabla q_{pi} \|^2 + \sum_{p=1}^{n} \sum_{j=1}^{k} (\Lambda_i - \Lambda_j) b_{pij}^2 \right)
\]

\[
+ \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \left( \Lambda_i - \sum_{p=1}^{n} \sum_{j=1}^{k} c_{pij}^2 \right).
\]

From the lemma 2.2, the lemma 2.3 and

\[
\|u_i\|^2 = \|\nabla q_{pi}\|^2 + \|w_{pi}\|^2,
\]

we infer

\[
\Lambda_i \sum_{p=1}^{n} \| \nabla q_{pi} \|^2 \geq \frac{5}{3}.
\]
Thus, we obtain, for any $i$,

$$
(\Lambda_{k+1} - \Lambda_i)^2(n + 2 \sum_{p=1}^{n} \sum_{j=1}^{k} b_{p|j} c_{p|j})
\leq \delta_i(\Lambda_{k+1} - \Lambda_i)^2 \left(n + \frac{4}{3} + \sum_{p=1}^{n} \sum_{j=1}^{k}(\Lambda_i - \Lambda_j)b_{p|j}^2\right)
+ \frac{1}{\delta_i}(\Lambda_{k+1} - \Lambda_i) \left(\Lambda_i - \sum_{p=1}^{n} \sum_{j=1}^{k} c_{p|j}^2\right).
$$

By taking sum for $i$ from 1 to $k$ and noticing that $b_{p|j}$ is symmetric and $c_{p|j}$ is antisymmetric on $i, j$, we have

$$
n \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 - 2 \sum_{p=1}^{n} \sum_{i,j=1}^{k} (\Lambda_{k+1} - \Lambda_i)(\Lambda_i - \Lambda_j)b_{p|j}c_{p|j}
\leq \left(n + \frac{4}{3}\right) \sum_{i=1}^{k} \delta_i(\Lambda_{k+1} - \Lambda_i)^2 + \sum_{i=1}^{k} \frac{1}{\delta_i}(\Lambda_{k+1} - \Lambda_i)\Lambda_i
- \sum_{p=1}^{n} \sum_{i,j=1}^{k} \delta_i(\Lambda_{k+1} - \Lambda_i)(\Lambda_i - \Lambda_j)^2b_{p|j}^2
+ \sum_{p=1}^{n} \sum_{i,j=1}^{k} \delta_i(\Lambda_{k+1} - \Lambda_i)^2(\Lambda_i - \Lambda_j)b_{p|j}^2
+ \sum_{p=1}^{n} \sum_{i,j=1}^{k} \delta_i(\Lambda_{k+1} - \Lambda_i)^2(\Lambda_i - \Lambda_j)c_{p|j}^2.
$$

Since, for a non-increasing monotone sequence $\{\delta_i\}_{i=1}^{k}$,

$$
\sum_{p=1}^{n} \sum_{i,j=1}^{k} \delta_i(\Lambda_{k+1} - \Lambda_i)(\Lambda_i - \Lambda_j)^2b_{p|j}^2 + \sum_{p=1}^{n} \sum_{i,j=1}^{k} \delta_i(\Lambda_{k+1} - \Lambda_i)^2(\Lambda_i - \Lambda_j)b_{p|j}^2
= \frac{1}{2} \sum_{p=1}^{n} \sum_{i,j=1}^{k} (\Lambda_{k+1} - \Lambda_i)(\Lambda_{k+1} - \Lambda_j)(\Lambda_i - \Lambda_j)(\delta_i - \delta_j)b_{p|j}^2 \leq 0.
$$

We conclude from (2.26) and the above formula, for a non-increasing monotone sequence $\{\delta_i\}_{i=1}^{k}$,

$$
n \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \left(n + \frac{4}{3}\right) \sum_{i=1}^{k} \delta_i(\Lambda_{k+1} - \Lambda_i)^2 + \sum_{i=1}^{k} \frac{1}{\delta_i}(\Lambda_{k+1} - \Lambda_i)\Lambda_i.
$$

In particular, putting

$$
\delta_i = \sqrt{\frac{\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)\Lambda_i}{\left(n + \frac{4}{3}\right) \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2}}.
$$
for any $i$, we obtain
$$
\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4(n + \frac{4}{3})}{n^2} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \Lambda_i.
$$
This finishes the proof of the theorem 1.1.

\[\square\]

**Remark 2.1.** If one can prove, for any $i$,
$$
\Lambda_i \sum_{p=1}^{n} \|\nabla q_{pi}\|^2 \geq 3,
$$
one will infer
$$
\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \Lambda_i,
$$
which solves the conjecture.

### 3. Proof of the theorem 1.2

For the unit sphere
$$
S^n(1) = \left\{ (x^1, x^2, \cdots, x^{n+1}) \in \mathbb{R}^{n+1}; \sum_{i=1}^{n+1} (x^i)^2 = 1 \right\},
$$
we denote the induced metric on $S^n(1)$ by the canonical metric $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{n+1}$ also. For any $p$, we have
$$
\nabla_i \nabla_j x^p = -g_{ij} x^p, \quad \Delta x^p = -nx^p,
$$
where $g_{ij}$ denotes components of the metric tensor of $S^n(1)$. Let $u_i$ be the $i$-th orthonormal eigenfunction of the buckling problem (1.2) corresponding to the eigenvalue $\Lambda_i$, namely, $u_i$ satisfies
$$
\begin{cases}
\Delta^2 u_i = -\Lambda_i \Delta u_i \quad \text{in} \quad \Omega, \\
u_i \big|_{\partial \Omega} = \frac{\partial u_i}{\partial n} \big|_{\partial \Omega} = 0 \\
(u_i, u_j)_D = \int_{\Omega} \langle \nabla u_i, \nabla u_j \rangle = \delta_{ij}.
\end{cases}
$$

For constructing trial functions, we use the same notations as in the section 2. We would like to remark that vector-valued functions in this section have $n + 1$ components. Although the orders of differentiations of functions in the Euclidean space can be exchanged freely, we must do it very carefully for the covariant differentiations of functions in the case of the unit sphere.

Since $x^p$ for $p = 1, 2, \cdots, n + 1$ is a coordinate function of $\mathbb{R}^{n+1}$, for the vector-valued function $x^p \nabla u_i$, $i = 1, \ldots, k$, we decompose it into
$$
x^p \nabla u_i = \nabla h_{pi} + w_{pi},
$$
where $h_{pi} \in H^2_{2, D}(\Omega)$ and $\nabla h_{pi}$ is the projection of $x^p \nabla u_i$ onto $H^2_{1, D}(\Omega)$ and $w_{pi} \perp H^2_{1, D}(\Omega)$. Thus, we have, for any function $h \in C^1(\Omega) \cap L^2(\Omega)$,
$$
(w_{pi}, \nabla h) = 0.
$$
Hence, \( w_{pi} \) satisfies
\[
\begin{cases}
  w_{pi}\big|_{\partial \Omega} = 0, \\
  \| \text{div} w_{pi} \|^2 = 0.
\end{cases}
\]

We define function \( \varphi_{pi} \) by
\[
\varphi_{pi} = h_{pi} - \sum_{j=1}^{k} b_{ pij} u_j,
\]
where
\[
b_{pij} = \int x^p \langle \nabla u_i, \nabla u_j \rangle = b_{ pji}.
\]
It is easy to check that \( \varphi_{pi} \) satisfies
\[
\varphi_{pi}\big|_{\partial \Omega} = \frac{\partial \varphi_{pi}}{\partial \nu}\big|_{\partial \Omega} = 0 \quad \text{and} \quad (\varphi_{pi}, u_j)_D = (\nabla \varphi_{pi}, \nabla u_j) = 0,
\]
for any \( j = 1, 2, \cdots, k \), that is, \( \varphi_{pi} \) is a trial function. Since \( \sum_{p=1}^{n+1} (x^p)^2 = 1 \), from (3.3), we have, for any \( i \),
\[
1 = \sum_{p=1}^{n+1} \| \nabla h_{pi} \|^2 + \sum_{p=1}^{n+1} \| w_{pi} \|^2.
\]

**Lemma 3.1.** For any \( i \), we have
\[
\sum_{p=1}^{n+1} \| w_{pi} \|^2 \leq \frac{\Lambda_i - (n - 1)}{\Lambda_i - (n - 2)}.
\]

**Proof.** From \( \sum_{p=1}^{n+1} (x^p)^2 = 1 \), we have
\[
1 = \sum_{p=1}^{n+1} \| \langle \nabla x^p, \nabla u_i \rangle \|^2
\]
\[
= - \sum_{p=1}^{n+1} \int x^p \text{div}\{ \langle \nabla x^p, \nabla u_i \rangle \nabla u_i \}
\]
\[
= - \sum_{p=1}^{n+1} \int x^p \langle \nabla x^p, \nabla u_i \rangle \Delta u_i - \sum_{p=1}^{n+1} \int \langle x^p \nabla u_i, \nabla \langle \nabla x^p, \nabla u_i \rangle \rangle
\]
\[
= - \sum_{p=1}^{n+1} \int \langle \nabla h_{pi}, \nabla \langle \nabla x^p, \nabla u_i \rangle \rangle.
\]
For any positive constant \( \epsilon_i \), we have
\[
1 \leq \epsilon_i \sum_{p=1}^{n+1} \| \nabla h_{pi} \|^2 + \frac{1}{4 \epsilon_i} \sum_{p=1}^{n+1} \| \langle \nabla x^p, \nabla u_i \rangle \|^2
\]
According to the following Bochner formula for a smooth function $f$:
\[
\frac{1}{2} \Delta |\nabla f|^2 = |\nabla^2 f|^2 + \langle \nabla f, \nabla (\Delta f) \rangle + \text{Ric}(\nabla f, \nabla f)
\]
\[
= |\nabla^2 f|^2 + \langle \nabla f, \nabla (\Delta f) \rangle + (n-1)|\nabla f|^2,
\]
where Ric and $\nabla^2 f$ denote the Ricci tensor of $S^n(1)$ and the Hessian of $f$, respectively, we can derive, from (3.1) and by making use of a direct computation,
\[
\Delta \langle \nabla x^p, \nabla u_i \rangle = -2x^p \Delta u_i + \langle \nabla x^p, \nabla (\Delta u_i) \rangle + (n-2)\langle \nabla x^p, \nabla u_i \rangle.
\]
Hence, we have
\[
\sum_{p=1}^{n+1} \|\nabla \langle \nabla x^p, \nabla u_i \rangle \|^2
\]
\[
= -\sum_{p=1}^{n+1} \int \langle \nabla x^p, \nabla u_i \rangle \Delta \langle x^p, \nabla u_i \rangle
\]
\[
= -\sum_{p=1}^{n+1} \int \langle \nabla x^p, \nabla u_i \rangle \left\{ -2x^p \Delta u_i + \langle \nabla x^p, \nabla (\Delta u_i) \rangle + (n-2)\langle \nabla x^p, \nabla u_i \rangle \right\}
\]
\[
= -\sum_{p=1}^{n+1} \left\{ \int \langle \nabla x^p, \nabla u_i \rangle \langle \nabla x^p, \nabla (\Delta u_i) \rangle + (n-2)\langle \nabla x^p, \nabla u_i \rangle^2 \right\}
\]
\[
= -\int \langle \nabla u_i, \nabla (\Delta u_i) \rangle - (n-2)\|\nabla u_i\|^2
\]
\[
= \Lambda_i - (n-2),
\]
that is,
\[
\sum_{p=1}^{n+1} \|\nabla \langle \nabla x^p, \nabla u_i \rangle \|^2 = \Lambda_i - (n-2).
\]
Here we have used
\[
\sum_{p=1}^{n+1} \int \langle \nabla x^p, \nabla u_i \rangle \langle \nabla x^p, \nabla (\Delta u_i) \rangle = \int \langle \nabla u_i, \nabla (\Delta u_i) \rangle.
\]
Therefore, from (3.9), we obtain
\[
1 \leq \epsilon_i \sum_{p=1}^{n+1} \|\nabla h_{pi} \|^2 + \frac{1}{4\epsilon_i} \left( \Lambda_i - (n-2) \right)
\]
From (3.7), we have
\[
1 + \epsilon_i \sum_{p=1}^{n+1} \|w_{pi} \|^2 \leq \epsilon_i + \frac{1}{4\epsilon_i} \left( \Lambda_i - (n-2) \right).
\]
Taking
\[
\epsilon_i = \frac{\Lambda_i - (n-2)}{2},
\]
we complete the proof of the lemma 3.1. \qed
Proof of Theorem 1.2. By making use of the trial function $\varphi_{pi}$ and the same arguments as in Wang and Xia [16], we have, for any $p$ and $i$,

\begin{equation}
(\Lambda_{k+1} - \Lambda_i)\|\nabla \varphi_{pi}\|^2 \leq P_{pi} + \|\langle \nabla x^p, \nabla u_i \rangle\|^2 + \Lambda_i\|w_{pi}\|^2 + \sum_{j=1}^{k} (\Lambda_i - \Lambda_j)b_{pij}^2,
\end{equation}

where

\[ P_{pi} = \int \langle \nabla (x^p)^2, u_i \nabla (\Delta u_i) + \Lambda_i u_i \nabla u_i \rangle. \]

Defining

\[ Z_{pi} = \nabla \langle \nabla x^p, \nabla u_i \rangle - \frac{n-2}{2} x^p \nabla u_i, \]

\[ c_{pij} = \int \langle \nabla u_j, Z_{pi} \rangle = -c_{pji}, \]

has been proved in Wang and Xia [16]. Since

\[ \gamma_{pi} = -2 \int \langle x^p \nabla u_i, Z_{pi} \rangle \]

\[ = -2 \int \langle \nabla h_{pi} + w_{pi}, Z_{pi} \rangle \]

\[ = -2 \int \langle \nabla \varphi_{pi} + \sum_{j=1}^{k} b_{pij} \nabla u_j + w_{pi}, Z_{pi} \rangle \]

\[ = -2 \int \langle \nabla \varphi_{pi}, Z_{pi} - \sum_{j=1}^{k} c_{pij} \nabla u_j \rangle - 2 \sum_{j=1}^{k} b_{pij} c_{pij} + (n-2)\|w_{pi}\|^2, \]

we have

\[ \gamma_{pi} + 2 \sum_{j=1}^{k} b_{pij} c_{pij} = -2 \int \langle \nabla \varphi_{pi}, Z_{pi} - \sum_{j=1}^{k} c_{pij} \nabla u_j \rangle + (n-2)\|w_{pi}\|^2. \]

Hence, for any positive constant $\delta_i$, we have, according to (3.12),

\begin{equation}
(\Lambda_{k+1} - \Lambda_i)^2 \left( \gamma_{pi} + 2 \sum_{j=1}^{k} b_{pij} c_{pij} \right) - (n-2)(\Lambda_{k+1} - \Lambda_i)^2\|w_{pi}\|^2
\leq \delta_i(\Lambda_{k+1} - \Lambda_i)^3\|\nabla \varphi_{pi}\|^2 + \frac{1}{\delta_i}(\Lambda_{k+1} - \Lambda_i) \left( \|Z_{pi}\|^2 - \sum_{j=1}^{k} c_{pij}^2 \right)
\end{equation}

\begin{equation}
\leq \delta_i(\Lambda_{k+1} - \Lambda_i)^2 \left\{ P_{pi} + \|\langle \nabla x^p, \nabla u_i \rangle\|^2 + \Lambda_i\|w_{pi}\|^2 + \sum_{j=1}^{k} (\Lambda_i - \Lambda_j)b_{pij}^2 \right\}
+ \frac{1}{\delta_i}(\Lambda_{k+1} - \Lambda_i) \left( \|Z_{pi}\|^2 - \sum_{j=1}^{k} c_{pij}^2 \right).
\end{equation}
By taking sum on \( p \) from 1 to \( n \), we derive

\[
(\Lambda_{k+1} - \Lambda_i)^2 \sum_{p=1}^{n+1} \left( \gamma_{pi} + 2 \sum_{j=1}^{k} b_{pij} c_{pij} \right) - (n - 2)(\Lambda_{k+1} - \Lambda_i)^2 \sum_{p=1}^{n+1} \|w_{pi}\|^2 \\
\leq \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \sum_{p=1}^{n+1} \left\{ P_{pi} + \|\nabla x^p, \nabla u_i\|^2 \right\} \\
+ \Lambda_i \|w_{pi}\|^2 + \sum_{j=1}^{k} (\Lambda_i - \Lambda_j) b_{pj}^2 \right\} \\
+ \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \sum_{p=1}^{n+1} \left( \|Z_{pi}\|^2 - \sum_{j=1}^{k} c_{pj}^2 \right).
\]

(3.14)

Since

\[
\gamma_{pi} = -2 \int \langle x^p \nabla u_i, Z_{pi} \rangle \\
= -2 \int \langle x^p \nabla u_i, \nabla \langle \nabla x^p, \nabla u_i \rangle - \frac{n - 2}{2} x^p \nabla u_i \rangle \\
= 2 \int \langle \nabla x^p, \nabla u_i \rangle^2 + 2 \int \Delta u_i \langle x^p \nabla x^p, \nabla u_i \rangle + (n - 2) \int (x^p)^2 \langle \nabla u_i, \nabla u_i \rangle,
\]

we have

\[
\sum_{p=1}^{n+1} \gamma_{pi} = n
\]

From the definition of \( Z_{pi} \), we have

\[
\sum_{p=1}^{n+1} \|Z_{pi}\|^2 \\
= \sum_{p=1}^{n+1} \int |\nabla \langle \nabla x^p, \nabla u_i \rangle - \frac{n - 2}{2} x^p \nabla u_i|^2 \\
= \sum_{p=1}^{n+1} \left\{ \|\nabla \langle \nabla x^p, \nabla u_i \rangle\|^2 - (n - 2) \int \langle \nabla \langle \nabla x^p, \nabla u_i \rangle, x^p \nabla u_i \rangle + \frac{(n - 2)^2}{4} \|x^p \nabla u_i\|^2 \right\} \\
= \Lambda_i + \frac{(n - 2)^2}{4} \quad \text{(from (3.11)).}
\]

Since \( P_{pi} = \int \langle \nabla (x^p)^2, u_i \nabla (\Delta u_i) + \Lambda_i u_i \nabla u_i \rangle \), we have

\[
\sum_{p=1}^{n+1} P_{pi} = 0.
\]
From the lemma 3.1 and (3.14), we obtain

\[
(A_{k+1} - A_i)^2 \left( n + 2 \sum_{p=1}^{n+1} \sum_{j=1}^{k} b_{prij} c_{prij} \right) - (n - 2)(A_{k+1} - A_i)^2 \frac{A_i - (n - 1)}{A_i - (n - 2)} \leq \delta_i (A_{k+1} - A_i)^2 \left\{ 1 + A_i \frac{A_i - (n - 1)}{A_i - (n - 2)} + \sum_{p=1}^{n+1} \sum_{j=1}^{k} (A_i - A_j) b_{prij}^2 \right\} \\
+ \frac{1}{\delta_i} (A_{k+1} - A_i) \left( A_i + \frac{(n - 2)^2}{4} \right) - \frac{1}{\delta_i} (A_{k+1} - A_i) \sum_{p=1}^{n+1} \sum_{j=1}^{k} c_{prij}^2,
\]

that is,

\[
2(A_{k+1} - A_i)^2 + (n - 2) \frac{(A_{k+1} - A_i)^2}{A_i - (n - 2)} \leq \delta_i (A_{k+1} - A_i)^2 \left\{ A_i - \frac{(n - 2)}{A_i - (n - 2)} \right\} + \frac{1}{\delta_i} (A_{k+1} - A_i) \left( A_i + \frac{(n - 2)^2}{4} \right) \\
- 2(A_{k+1} - A_i)^2 \sum_{p=1}^{n+1} \sum_{j=1}^{k} b_{prij} c_{prij} + \delta_i (A_{k+1} - A_i)^2 \sum_{p=1}^{n+1} \sum_{j=1}^{k} (A_i - A_j) b_{prij}^2 \\
- \frac{1}{\delta_i} (A_{k+1} - A_i) \sum_{p=1}^{n+1} \sum_{j=1}^{k} c_{prij}^2,
\]

(3.15)

Since, for a non-increasing monotone sequence \( \{\delta_i\}^k_{i=1} \),

\[
\sum_{p=1}^{n} \sum_{i,j=1}^{k} \delta_i (A_{k+1} - A_i) (A_i - A_j)^2 b_{prij}^2 + \sum_{p=1}^{n} \sum_{i,j=1}^{k} \delta_i (A_{k+1} - A_i) (A_i - A_j)^2 (A_i - A_j) b_{prij}^2 \\
= \frac{1}{2} \sum_{p=1}^{n} \sum_{i,j=1}^{k} (A_{k+1} - A_i) (A_{k+1} - A_j) (A_i - A_j) (\delta_i - \delta_j) b_{prij}^2 \leq 0
\]

and

\[
- 2 \sum_{i=1}^{k} (A_{k+1} - A_i)^2 \sum_{p=1}^{n+1} \sum_{j=1}^{k} b_{prij} c_{prij} - \sum_{i=1}^{k} \sum_{p=1}^{n+1} \sum_{j=1}^{k} (A_i - A_j)^2 b_{prij}^2 \\
- \sum_{i=1}^{k} \frac{1}{\delta_i} (A_{k+1} - A_i) \sum_{p=1}^{n+1} \sum_{j=1}^{k} c_{prij}^2 \\
= - \sum_{p=1}^{n} \sum_{i,j=1}^{k} \left( \sqrt{\delta_i (A_{k+1} - A_i) (A_i - A_j) b_{prij}^2} - \frac{1}{\sqrt{\delta_i}} \sqrt{(A_{k+1} - A_i) c_{prij}} \right)^2 \leq 0,
\]
by taking sum on $i$ from 1 to $k$ for (3.15), we obtain

\[
2 \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 + (n - 2) \sum_{i=1}^{k} \frac{(\Lambda_{k+1} - \Lambda_i)^2}{\Lambda_i - (n - 2)} \leq \sum_{i=1}^{k} \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \left\{ \Lambda_i - \frac{(n - 2)}{\Lambda_i - (n - 2)} \right\} \\
+ \sum_{i=1}^{k} \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \left( \Lambda_i + \frac{(n - 2)^2}{4} \right).
\]

(3.16)

It completes the proof of the theorem 1.2.

\[\square\]

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