ON COMPACTNESS AND $L^p$-REGULARITY IN THE $\overline{\partial}$-NEUMANN PROBLEM

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ABSTRACT. Let $\Omega$ be a $C^4$-smooth bounded pseudoconvex domain in $\mathbb{C}^2$. We show that if the $\overline{\partial}$-Neumann operator $N_1$ is compact on $L^2_{(0,1)}(\Omega)$ then the embedding operator $J : \text{Dom}(\overline{\partial}) \cap \text{Dom}(\overline{\partial}^*) \to L^2_{(0,1)}(\Omega)$ is $L^p$-regular for all $2 \leq p < \infty$.

1. INTRODUCTION

Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$, $1 \leq q \leq n$, and let $\text{Dom}^2(\overline{\partial})$ and $\text{Dom}^2(\overline{\partial}^*)$ denote the domains of the densely defined operators $\overline{\partial}$ and $\overline{\partial}^*$ in $L^2_{(0,q)}(\Omega)$, respectively. On bounded pseudoconvex domains, Hörmander in [Hör65] proved the following basic estimate,

$$\|f\|_{L^2} \lesssim \|\overline{\partial}f\|_{L^2} + \|\overline{\partial}^*f\|_{L^2}$$

for all $(0,q)$-forms $f \in \text{Dom}^2(\overline{\partial}) \cap \text{Dom}^2(\overline{\partial}^*) \subset L^2_{(0,q)}(\Omega)$. The sum on the right hand side is called the $L^2$-graph norm of the $(0,q)$-form $f$. In other words, the embedding operator

$$J : \text{Dom}^2(\overline{\partial}) \cap \text{Dom}^2(\overline{\partial}^*) \to L^2_{(0,q)}(\Omega)$$

is bounded, where the space on the left hand side is endowed with the graph norm.

Let $1 < p, \tilde{p} < \infty$ such that $\frac{1}{p} + \frac{1}{\tilde{p}} = 1$. We define $\text{Dom}^p(\overline{\partial}) = \{f \in L^p_{(0,q)}(\Omega) : \overline{\partial}f \in L^p_{(0,q+1)}(\Omega)\}$. We define $\text{Dom}^p(\overline{\partial}^*)$ as follows: we say $f \in \text{Dom}^p(\overline{\partial}^*)$ if $f \in L^p_{(0,q)}(\Omega)$ and there exists $C > 0$ such that

$$|\langle f, \overline{\partial}g \rangle| \leq C\|g\|_{L^{\tilde{p}}}$$

for all $g \in L^p_{(0,q-1)}(\Omega)$ with $\overline{\partial}g \in L^p_{(0,q)}(\Omega)$. Finally, we define the space

$$D^p_{(0,q)}(\Omega) = \text{Dom}^p(\overline{\partial}) \cap \text{Dom}^p(\overline{\partial}^*) \subset L^p_{(0,q)}(\Omega)$$

and endow it with the $L^p$-graph norm $\|\cdot\|_{G^p}$ defined as

$$\|f\|_{G^p} = \|\overline{\partial}f\|_{L^p} + \|\overline{\partial}^*f\|_{L^p}$$

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for \( f \in D^p_{(0,q)}(\Omega) \). We note that on bounded pseudoconvex domains, \( \| \cdot \|_{G^p} \) is a norm because \( \overline{\partial} f = 0 \) and \( \overline{\partial}^* f = 0 \) imply that \( f = 0 \) for \( 1 \leq q \leq n \) (see, for example, [CS01, (4.4.2) in section 4.4]).

**Definition 1.** We say that the operator \( J \) is \( L^p \)-regular on \( D^p_{(0,q)}(\Omega) \) if there exists \( C > 0 \) such that

\[
\| Jf \|_{L^p} = \| f \|_{L^p} \leq C \| f \|_{G^p} = C \left( \| \overline{\partial} f \|_{L^p} + \| \overline{\partial}^* f \|_{L^p} \right)
\]

for all \( f \in D^p_{(0,q)}(\Omega) \).

That is, whenever \( J : D^p_{(0,q)}(\Omega) \to L^p_{(0,q)}(\Omega) \) is a bounded embedding we say that it is \( L^p \)-regular. In particular, by Hörmander’s basic estimate above, \( J \) is \( L^2 \)-regular on bounded pseudoconvex domains. We note that \( D^p_{(0,q)}(\Omega) \) is a Banach space (for \( 1 \leq q \leq n \) with the graph norm \( \| \cdot \|_{G^p} \) when \( J \) is \( L^p \)-regular.

The operator \( J \) is related to the \( \overline{\partial} \)-Neumann operator \( N \), the bounded inverse of the complex Laplacian \( \overline{\partial} \overline{\partial}^* + \overline{\partial}^* \overline{\partial} \) on \( L^2_{(0,q)}(\Omega) \), as \( N = J J^* \) (see, for example, [Str10, Proof of Theorem 2.9]). Hence, \( N \) is compact if and only if \( J \) is compact. In this note, we show that compactness of \( N \) implies \( L^p \)-regularity of \( J \) for \( 2 \leq p < \infty \). We also note that it is not yet clear if \( J \) is \( L^p \)-regular for \( 1 < p < 2 \) under the compactness assumption. We further note that the question of whether the \( \overline{\partial} \)-Neumann operator or the Bergman projection are bounded in \( L^p \)-norm whenever \( J \) is compact is open as well.

Although the mapping properties of the canonical operators relate well in the \( L^2 \)-Sobolev setting, similar equivalences in the \( L^p \) setting are less clear. In [BS91], Bonami and Sibony obtained \( L^p \) estimates for the solutions of \( \overline{\partial} \)-problem under some Sobolev estimates. In [HZ19] Harrington and Zeytuncu obtained some \( L^p \) estimates on the canonical operators under the assumption of the existence of good weight functions. Both assumptions are more stringent than the compactness of \( N \) and hence the \( L^p \) estimates are more general. Also, recently, Haslinger in [Has16, Theorem 2.2] showed that if \( J \) gains regularity in the \( L^p \) scale then \( N \) is compact. In this paper, we observe a property that is less general than the ones in [BS91, HZ19] under a weaker assumption, and that is in the converse direction of the result in [Has16]. Namely, in Theorem 1 below, we show that compactness of \( N_1 \) (at the \( L^2 \) level) induces \( L^p \)-regularity of \( J \) for \( 2 \leq p < \infty \).

**Theorem 1.** Let \( \Omega \) be a \( C^4 \)-smooth bounded pseudoconvex domain in \( \mathbb{C}^2 \). Assume that \( N_1 \) is compact on \( L^2_{(0,1)}(\Omega) \) (or, equivalently, \( J \) is compact on \( D^2_{(0,1)}(\Omega) \)). Then \( J \) is \( L^p \)-regular on \( D^p_{(0,1)}(\Omega) \) for all \( 2 \leq p < \infty \).

We note that the \( L^p \) boundedness is not an automatic consequence of compactness on \( L^2 \); as we demonstrate with Example 1, in which we present an operator that is compact on the \( L^2 \) space but unbounded on any \( L^p \) spaces for \( p \neq 2 \).
In the rest of the paper, we use the symbol \( x \lesssim y \) to mean that there exists \( C > 0 \) such that \( x \leq Cy \). Furthermore, when we write a family of inequalities depending on a parameter \( \varepsilon \)

\[ x \lesssim \varepsilon y, \]

we mean that there exists \( C > 0 \) that is independent of \( \varepsilon \) such that \( x \leq C\varepsilon y \).

### 2. Proof of Theorem 1

One can prove the following density lemma similarly as in [CS01, Lemma 4.3.2] (see also [Str10, Proposition 2.3]) using an \( L^p \) version of Friedrichs Lemma (see, for instance, [BLD01, Lemma 3.1]).

**Lemma 1.** Let \( \Omega \) be a \( C^{k+1} \)-smooth bounded domain in \( \mathbb{C}^n \), \( 1 \leq q \leq n \), and \( 1 < p < \infty \). Then \( C^k(\overline{\Omega}) \cap \text{Dom}(\partial^*) \) is dense in \( \text{Dom}(\partial^*) \) in the graph norm \( f \rightarrow \|f\|_{L^p} + \|\bar{\partial} f\|_{L^p} + \|\partial^* f\|_{L^p} \). The statement also holds with \( k \) and \( k+1 \) replaced with \( \infty \).

We will need the following lemma which is a corollary of [JK95, Theorem 1.1].

**Lemma 2** (Jerison-Kenig). Let \( \Omega \) be a \( C^1 \)-smooth bounded domain in \( \mathbb{R}^n \) and \( 1 < p < \infty \). Then there exists \( C > 0 \) such that

\[ \|u\|_{W^{1,p}} \leq C \|\Delta u\|_{W^{-1,p}} \]

for all \( u \in W^{1,p}_0(\Omega) \).

Using the lemmas above together with the proof of [Str10, Lemma 2.12] one can prove the following estimate on the normal component of forms. We note that, in the lemma below, \( f_{\text{norm}} \) denotes the normal component of \( f \) (see (2.86) in [Str10]).

**Lemma 3.** Let \( \Omega \) be a \( C^4 \)-smooth bounded pseudoconvex domain in \( \mathbb{C}^n \), \( 1 \leq q \leq n \), and \( 1 < p < \infty \). There exists \( C > 0 \) such that if \( f \in D^p(\Omega) \) then \( f_{\text{norm}} \in W^{1,p}_0(\Omega) \) and

\[ \|f_{\text{norm}}\|_{W^{1,p}} \leq C \left( \|\bar{\partial} f\|_{L^p} + \|\bar{\partial}^* f\|_{L^p} + \|f\|_{L^p} \right). \]

We will use Lemma 2 to also prove the following \( L^p \) version of [CS01, Proposition 5.1.1].

**Proposition 1.** Let \( \Omega \) be a \( C^2 \)-smooth bounded domain in \( \mathbb{C}^n \), \( 1 < p < \infty \), \( 1 \leq q \leq n \), and \( \phi \in C^1(\overline{\Omega}) \) such that \( \phi = 0 \) on \( b\Omega \). Then there exists \( C > 0 \) such that

\[ \|\phi f\|_{W^{1,p}} \leq C(\|\bar{\partial} f\|_{L^p} + \|\bar{\partial}^* f\|_{L^p} + \|f\|_{L^p}) \]

for \( f \in D^p(\Omega) \).

**Proof.** First we assume that \( g \in D^p(\Omega) \) with coefficient functions in \( W^{1,p}_0(\Omega) \). Then we have

\[ \|g\|_{W^{1,p}} \lesssim \|\Delta g\|_{W^{-1,p}} \lesssim \|\bar{\partial} g\|_{L^p} + \|\bar{\partial}^* g\|_{L^p}. \]
Then we substitute \( g = \phi f \) for \( f \in C^1_{(0,q)}(\Omega) \cap \text{Dom}(\overline{\partial}^*) \) in the inequality above
\[
\|\phi f\|_{W^{1,p}} \lesssim \|\overline{\partial}(\phi f)\|_{L^p} + \|\overline{\partial}^*(\phi f)\|_{L^p} \lesssim \|\overline{\partial} f\|_{L^p} + \|\overline{\partial}^* f\|_{L^p} + \|f\|_{L^p}.
\]
Then we use Lemma 1 to conclude the proof. \( \square \)

The interpolation inequality for Sobolev spaces together with Proposition 1 imply the following corollary.

**Corollary 1.** Let \( \Omega \) be a \( C^2 \)-smooth bounded domain in \( \mathbb{C}^n \), \( 1 < p < \infty \), \( 1 \leq q \leq n \), and \( \phi \in C(\overline{\Omega}) \) such that \( \phi = 0 \) on \( b\Omega \). Then the multiplication operator \( M_\phi : D^p_{(0,q)}(\Omega) \to L^p_{(0,q)}(\Omega) \) is compact.

In other words, in the terminology of [Sz09], continuous functions on \( \overline{\Omega} \) that vanish on the boundary are compactness multipliers.

We note that even though [Str10, Lemma 4.3] is stated for Hilbert spaces the proof works for Banach spaces as well. In the proof of i) of Lemma 4 below one uses the facts that on reflective Banach spaces bounded sequences have weakly convergent subsequences (see [Yos95, Theorem 1 on pg 126]) as well as compact operators map weakly convergent sequences to convergent sequences. Therefore, proof of [Str10, Lemma 4.3] (see also exercise 6.13 in [Bre11]) implies the following lemma.

**Lemma 4.** Let \( X, Y \) be Banach spaces and \( T : X \to Y \) be a bounded linear map.

i. Assume that \( X \) is reflexive, \( T \) is compact and \( K : X \to Z \) is an injective bounded linear map. Then for any \( \epsilon > 0 \) there exists \( C_\epsilon > 0 \) such that
\[
\|Tx\|_Y \leq \epsilon \|x\|_X + C_\epsilon \|Kx\|_Z
\]
for all \( x \in X \).

ii. Assume that for all \( \epsilon > 0 \) there exist a Banach space \( Z_\epsilon \) and a compact linear map \( K_\epsilon : X \to Z_\epsilon \) such that
\[
\|Tx\|_Y \leq \epsilon \|x\|_X + \|K_\epsilon x\|_{Z_\epsilon}
\]
for all \( x \in X \). Then \( T \) is compact.

**Proof of Theorem 1.** We define \( K : D^p_{(0,1)}(\Omega) \to L^p_{(0,1)}(\Omega) \) as \( Kf = \rho f \) where \( \rho(z) = \text{dist}(z, b\Omega) \) is the distance of \( z \) to the boundary of \( \Omega \). Then Corollary 1 implies that \( K \) is a compact for all \( 1 < p < \infty \). We note that \( K \) is an injection as well.

Using i) in Lemma 4 we get the following: for any \( \epsilon > 0 \) there exists \( C_\epsilon > 0 \) such that
\[
\|f\|_{L^2} \leq \epsilon (\|\overline{\partial} f\|_{L^2} + \|\overline{\partial}^* f\|_{L^2}) + C_\epsilon \|Kf\|_{L^2}
\]
for all \( f \in D^2_{(0,1)}(\Omega) \).

First we show how to get \( L^4 \)-regularity. Let \( F = f_1 \overline{\omega}_1 + f_2 \overline{\omega}_2 \) be in \( D^4_{(0,1)}(\Omega) \subset L^4_{(0,1)}(\Omega) \) such that \( f_2 \) is the normal component. Because of Lemma 1, without loss of generality, we
may assume that $f_1$ and $f_2$ are $C^3$-smooth on $\Omega$. We denote $F_2 = f_1^2 \overline{w}_1 + f_2^2 \overline{w}_2$. Since $f_2$ vanishes on the boundary we have $F_2 \in D^2_{(0,1)}(\Omega)$. Then $\|F\|_{L^4}^4 \approx \|F_2\|_{L^2}^2 < \infty$ and

$$\|F_2\|_{L^2} \leq \varepsilon (\|\overline{\partial} F_2\|_{L^2} + \|\overline{\partial}^* F_2\|_{L^2}) + C_{\varepsilon} \|K F_2\|_{L^2}$$

$$\leq \varepsilon (\|f_1 \overline{\partial} f_1 - f_2 \overline{\partial} f_2\|_{L^2} + \|f_1 \overline{\partial} f_2 - f_2 \overline{\partial} f_2\|_{L^2})$$

$$+ C_{\varepsilon} \|K F_2\|_{L^2}$$

$$\leq \varepsilon (\|f_1 \overline{\partial} f_1 - f_1 \overline{\partial} f_2\|_{L^2} + \|f_1 \overline{\partial} f_2 - f_2 \overline{\partial} f_2\|_{L^2})$$

$$+ \varepsilon (\|f_1 \overline{\partial} f_1 - f_1 \overline{\partial} f_2\|_{L^2} + \|f_1 \overline{\partial} f_2 - f_2 \overline{\partial} f_2\|_{L^2})$$

$$+ C_{\varepsilon} \|K F_2\|_{L^2}$$

By absorbing the terms that are multiple of $\|F_2\|_{L^2}$ into the left hand side we get

$$(2) \quad \|F_2\|_{L^2} \leq \varepsilon (\|f_1 \overline{\partial} F\|_{L^2} + \|(f_1 - f_2) \overline{\partial} f_2\|_{L^2} + \|f_1 \overline{\partial}^* F\|_{L^2} + \|(f_1 - f_2) L \overline{\partial} f_2\|_{L^2})$$

$$+ C_{\varepsilon} \|K F_2\|_{L^2}.$$
for $F \in D^8_{(0,1)}(\Omega)$. That is, $J : D^8_{(0,1)}(\Omega) \to L^8_{(0,1)}(\Omega)$ is a compact linear map. Inductively, we show that $J : D^p_{(0,1)}(\Omega) \to L^p_{(0,1)}(\Omega)$ is a compact linear map for $p \in \mathbb{Z}^+$.

Note that for any $p \in \mathbb{Z}^+$, we have $D^p_{(0,1)}(\Omega) \cap D^{p+1}_{(0,1)}(\Omega) = D^{p+1}_{(0,1)}(\Omega)$ and $D^p_{(0,1)}(\Omega) + D^{p+1}_{(0,1)}(\Omega) \subset D^p_{(0,1)}(\Omega)$. In other words, for $2^p < q < 2^{p+1}$ we get

$$D^p_{(0,1)}(\Omega) \subset D^q_{(0,1)}(\Omega) \subset D^{p+1}_{(0,1)}(\Omega)$$

and since the graph norm is the sum of $L^p$ norms we conclude that $D^q_{(0,1)}(\Omega)$ is an intermediate space ([BL76, Definition 2.4.1]) for two Banach spaces $D^p_{(0,1)}(\Omega)$ and $D^{p+1}_{(0,1)}(\Omega)$. Now, by the complex interpolation theorem ([BL76, Chapter 4]) we conclude that $J : D^p_{(0,1)}(\Omega) \to L^p_{(0,1)}(\Omega)$ is $L^p$-regular for all $2 \leq p < \infty$. □

Remark 1. The proof of Theorem 1 shows that we have the same result for $(p, n - 1)$-forms on $C^4$-smooth bounded pseudoconvex domains in $\mathbb{C}^n$.

We note that $\text{Ker}(\bar{\partial})$ and $A^2(\Omega)^\perp$ denote the set of $\bar{\partial}$-closed forms and the orthogonal complement of the Bergman space $A^2(\Omega) \subset L^2(\Omega)$, respectively.

Proposition 2. Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, and $1 < p \leq 2$. Assume that $J$ is $L^p$-regular on $D^p_{(0,1)}(\Omega)$. Then the following operators are bounded

i. $\bar{\partial}^* N_2 : L^p_{(0,2)}(\Omega) \cap L^2_{(0,2)}(\Omega) \cap \text{Ker}(\bar{\partial}) \to L^p_{(0,1)}(\Omega)$,

ii. $\partial N_0 : L^p(\Omega) \cap L^2(\Omega) \cap A^2(\Omega)^\perp \to L^p_{(0,1)}(\Omega)$.

Proof. Since $J$ is $L^p$-regular and there exists $C > 0$ such that

(5) $\|f\|_{L^p} \leq C\left(\|\bar{\partial}f\|_{L^p} + \|\bar{\partial}^* f\|_{L^p}\right)$

for $f \in D^p_{(0,1)}(\Omega)$. Note that $\partial N_0 g \in \text{Dom}^2(\bar{\partial}^*) \subset \text{Dom}^p(\bar{\partial}^*)$ for $g \in L^p(\Omega) \cap L^2(\Omega) \cap A^2(\Omega)^\perp$ and $p \leq 2$. Then applying the estimate (5) to $\partial N_0 g$ we get

$$\|\partial N_0 g\|_{L^p} \leq C\|\bar{\partial}^* \partial N_0 g\|_{L^p} = C\|g\|_{L^p}$$

for $g \in L^2(\Omega) \cap A^2(\Omega)^\perp$.

Similarly, if we apply (5) to $\bar{\partial}^* N_2 h$ with $h \in L^p_{(0,2)}(\Omega) \cap L^2_{(0,2)}(\Omega) \cap \text{Ker}(\bar{\partial})$ we get

$$\|\bar{\partial}^* N_2 h\|_{L^p} \leq C\|\bar{\partial}^* \bar{\partial}^* N_2 h\|_{L^p} = C\|h\|_{L^p}.$$

Hence the proof of the proposition is complete. □

The following example shows that the $L^p$ boundedness of an operator $T$ is not an automatic consequence of the compactness of $T$ on $L^2$. 
Example 1. Set

\[ \phi(z) = \exp \left( \frac{-1}{1 - |z|} \right) \]

and consider the weighted Bergman space \( A^2(\mathbb{D}, \phi) \) on the unit disc. The weighted Bergman projection \( B_\phi \) is studied in [Dos04, Dos07, Zey13], and it was noted that \( B_\phi \) is unbounded on \( L^p(\mathbb{D}, \phi) \) for any \( p \neq 2 \).

We define an operator \( T \) on \( L^2(\mathbb{D}, \phi) \) by

\[ T : L^2(\mathbb{D}, \phi) \to L^2(\mathbb{D}, \phi) \]

\[ Tf(z) = B_\phi(f)(z)(1 - |z|^2)^2. \]

The operator \( T \) is bounded, linear and self-adjoint. Furthermore, we show that \( T \) is compact on \( L^2(\mathbb{D}, \phi) \) yet it is unbounded on \( L^p(\mathbb{D}, \phi) \) for any \( p \neq 2 \).

First we show that \( T \) is compact. For \( \varepsilon > 0 \) there exists a compact set \( K_\varepsilon \subset \mathbb{D} \) such that \( (1 - |z|^2)^2 < \varepsilon \) on \( \mathbb{D} \setminus K \). We define \( S_\varepsilon f = \chi_{K_\varepsilon}Tf \) where \( \chi_{K_\varepsilon} \) is the characteristic function of \( K_\varepsilon \). Montel’s theorem implies that \( S_\varepsilon \) is compact.

\[ \|Tf\|^2 = \|Tf\|_{L^2(\mathbb{D} \setminus K, \phi)}^2 + \|Tf\|_{L^2(K, \phi)}^2 \leq \varepsilon \|B_\phi f\|^2 + \|S_\varepsilon f\|^2. \]

That is, \( T \) satisfies compactness estimates and by Lemma 4 it is a compact operator on \( L^2(\mathbb{D}, \phi) \) (see also [D'02, Proposition V.2.3] or [Str10, Lemma 4.3]).

Next we show that \( T \) is unbounded on \( L^p(\mathbb{D}, \phi) \) for any \( p \neq 2 \). Let \( 0 < p < 2 \) and

\[ f_n(z) = z^n z^{n-p} \]

where \( k \) is a positive integer to be determined later. Then one can compute that

\[ Tf_n(z) = a_n z^{kn-n}(1 - |z|^2)^2 \]

where

\[ a_n = \frac{\int_{\mathbb{D}} |z|^{2kn-2n} \phi(z) dA(z)}{\int_{\mathbb{D}} |z|^{2(k-1)n} \phi(z) dA(z)}. \]

Furthermore,

\[ \frac{\|Tf_n\|_p}{\|f_n\|_p} = \left( \frac{\int_{\mathbb{D}} |z|^{2kn-2n} \phi(z) dA(z)}{\int_{\mathbb{D}} |z|^{2(k-1)n} \phi(z) dA(z)} \right)^p \frac{\int_{\mathbb{D}} |z|^{2kn-2n} \phi(z) dA(z)}{\int_{\mathbb{D}} |z|^{2(k-1)n} \phi(z) dA(z)}. \]

We need the following asymptotic [Dos07, Lemma 1]

\[ \int_{\mathbb{D}} |z|^t (1 - |z|^2)^{2s} \phi(z) dA(z) \sim (t + 1)^{s-\frac{3}{2}} \exp(-2\sqrt{t+1}) \]

as \( t \to \infty \).
We have the following asymptotic computations

\[
\frac{\|Tf_n\|_p^p}{\|f_n\|_p^p} = \left( \frac{\int_D |z|^{2kn} \phi(z) dA(z)}{\int_D |z|^{2(k-1)n} \phi(z) dA(z)} \right)^p \frac{\int_D |z|^{pkn-pn}(1-|z|^2)^{2p} \phi(z) dA(z)}{\int_D |z|^{pkn+pn} \phi(z) dA(z)} \sim \frac{(2kn + 1)^{-3p/4} \exp(-2p\sqrt{2kn + 1})}{(2kn - 2n + 1)^{-3p/4} \exp(-2p\sqrt{2kn - 2n + 1})} \times \frac{(pkn - pn + 1)^{-4p-3/4} \exp(-2\sqrt{pkn - pn + 1})}{(pkn + pn + 1)^{-3/4} \exp(-2\sqrt{pkn + pn + 1})} \sim C_{k,p} n^{-p} \exp(2D_{k,p,n})
\]

as \(n \to \infty\) where

\[C_{k,p} = \frac{(k + 1)^{3/4}}{p^p k^{3p/4} (k - 1)^{(3+p)/4}}\]

and

\[D_{k,p,n} = -p\sqrt{2kn + 1} + p\sqrt{2kn - 2n + 1} - \sqrt{pkn - pn + 1} + \sqrt{pkn + pn + 1}
\]

\[\geq \frac{-2pn}{\sqrt{2kn + 1} + \sqrt{2kn - 2n + 1}} + \frac{2pn}{\sqrt{pkn - pn + 1} + \sqrt{pkn + pn + 1}}
\]

\[\geq \frac{pn}{\sqrt{pkn + pn + 1}} - \frac{pn}{\sqrt{2kn - 2n + 1}}
\]

\[\geq p\sqrt{n} \left( \frac{1}{\sqrt{pk + p + 1}} - \frac{1}{\sqrt{2k - 2}} \right).
\]

Then one can show that for large \(k\) we have

\[\frac{1}{\sqrt{pk + p + 1}} - \frac{1}{\sqrt{2k - 2}} \geq \frac{1}{2} \left( \frac{1}{\sqrt{p}} - \frac{1}{\sqrt{2}} \right) > 0.
\]

Therefore, for large \(k\) we have

\[D_{k,p,n} \geq \frac{\sqrt{np}}{2} \left( \frac{1}{\sqrt{p}} - \frac{1}{\sqrt{2}} \right).
\]

Therefore, for \(k\) large enough we have \(C_{k,p} n^{-p} \exp(2D_{k,p,n}) \to \infty\) as \(n \to \infty\). Then we conclude that \(\|Tf_n\|_p \to \infty\) as \(n \to \infty\). Hence \(T\) is not bounded on \(L^p(\mathbb{D}, \phi)\) for any \(p < 2\). Furthermore, the fact that \(T\) is self-adjoint implies that \(T\) is unbounded on \(L^p(\mathbb{D}, \phi)\) for any \(p \neq 2\).

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