A NOTE ON THE ABELIANIZATIONS
OF FINITE-INDEX SUBGROUPS
OF THE MAPPING CLASS GROUP

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Abstract. For some $g \geq 3$, let $\Gamma$ be a finite index subgroup of the mapping class group of a genus $g$ surface (possibly with boundary components and punctures). An old conjecture of Ivanov says that the abelianization of $\Gamma$ should be finite. In the paper we prove two theorems supporting this conjecture. For the first, let $T_x$ denote the Dehn twist about a simple closed curve $x$. For some $n \geq 1$, we have $T_x^n \in \Gamma$. We prove that $T_x^n$ is torsion in the abelianization of $\Gamma$. Our second result shows that the abelianization of $\Gamma$ is finite if $\Gamma$ contains a “large chunk” (in a certain technical sense) of the Johnson kernel, that is, the subgroup of the mapping class group generated by twists about separating curves.

1. Introduction

Let $\Sigma_{g,b}^p$ be an oriented genus $g$ surface with $b$ boundary components and $p$ punctures and let $\text{Mod}(\Sigma_{g,b}^p)$ be its mapping class group, that is, the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,b}^p$ that fix the boundary components and punctures pointwise (we will omit $b$ or $p$ when they are zero). A long-standing conjecture of Ivanov (see [6] for a recent discussion) says that for $g \geq 3$, the group $\text{Mod}(\Sigma_{g,b}^p)$ does not virtually surject onto $\mathbb{Z}$. In other words, if $\Gamma$ is a finite-index subgroup of $\text{Mod}(\Sigma_{g,b}^p)$, then $H_1(\Gamma; \mathbb{R}) = 0$.

The goal of this paper is to offer some evidence for this conjecture. If $G$ is a group and $g \in G$, then we will denote by $[g]_G$ the corresponding element of $H_1(G; \mathbb{R})$. Also, for a simple closed curve $\gamma$ on $\Sigma_{g,b}^p$, we will denote by $T_\gamma$ the corresponding right Dehn twist. Observe that if $\Gamma$ is any finite-index subgroup of $\text{Mod}_{g,b}^p$, then $T_\gamma^n \in \text{Mod}_{g,b}^p$ for some $n \geq 1$. Our first result is the following.

**Theorem A** (Powers of twists vanish). For some $g \geq 3$, let $\Gamma < \text{Mod}(\Sigma_{g,b}^p)$ satisfy $[\text{Mod}(\Sigma_{g,b}^p) : \Gamma] < \infty$ and let $\gamma$ be a simple closed curve on $\Sigma_{g,b}^p$. Pick $n \geq 1$ such that $T_\gamma^n \in \Gamma$. Then $[T_\gamma^n]_\Gamma = 0$.

**Remark.** After this paper was written, Bridson informed us that in unpublished work he had proven a result about mapping class group actions on CAT(0) spaces that implies Theorem A. Bridson’s work will appear in [3].
We use this to verify Ivanov’s conjecture for a class of examples. For a long time, the only positive evidence for Ivanov’s conjecture was a result of Hain [5] that says that it holds for all finite-index subgroups containing the Torelli group $T_{g,b}^p$, that is, the kernel of the action of $\text{Mod}(\Sigma_{g,b}^p)$ on $H_1(\Sigma_g;\mathbb{Z})$ induced by filling in all the punctures and boundary components. The group $T_{g,b}^p$ contains the Johnson kernel $K_{g,b}^p$, which is the subgroup generated by Dehn twists about separating curves. A result of Johnson [7] says that $K_{g,b}^p$ is an infinite-index subgroup of $T_{g,b}^p$.

For a subgroup $\Gamma$ of $\text{Mod}(\Sigma_{g,b}^p)$, denote by $K(\Gamma)$ the subgroup of $\Gamma \cap K_{g,b}^p$ generated by the set

$$\{T^n_\gamma \mid \gamma \text{ a separating curve, } n \in \mathbb{Z}, \text{ and } T^n_\gamma \in \Gamma\}.$$ 

If $K_{g,b}^p < \Gamma$, then $K(\Gamma) = \Gamma \cap K_{g,b}^p$, but the converse does not hold. Our second result is the following.

**Theorem B** (Subgroups containing large pieces of Johnson kernel). *For some $g \geq 3$, let $\Gamma < \text{Mod}(\Sigma_{g,b}^p)$ satisfy $[\text{Mod}(\Sigma_{g,b}^p) : \Gamma] < \infty$. Assume that $[\Gamma \cap K_{g,b}^p : K(\Gamma)] < \infty$. Then $H_1(\Gamma;\mathbb{R}) = 0$.*

As a corollary, we obtain the following result, which was recently proven by Boggi [2] via a difficult algebro-geometric argument under the assumption $b = p = 0$.

**Corollary C** (Subgroups containing Johnson kernel). *For some $g \geq 3$, let $\Gamma < \text{Mod}(\Sigma_{g,b}^p)$ satisfy $[\text{Mod}(\Sigma_{g,b}^p) : \Gamma] < \infty$. Assume that $K_{g,b}^n < \Gamma$. Then $H_1(\Gamma;\mathbb{R}) = 0$.*

**Remark.** McCarthy [11] proved that Ivanov’s conjecture fails in the case $g = 2$.

## 2. Notation and basic facts about group homology

If $M$ is a $G$-module, then $M_G$ will denote the coinvariants of the action, that is, the quotient of $M$ by the submodule generated by the set $\{x - g(x) \mid x \in M, g \in G\}$. This appears in the 5-term exact sequence [4] Corollary VII.6.4], which asserts the following. If

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1$$

is a short exact sequence of groups, then for any ring $R$, there is an exact sequence

$$H_2(G;R) \longrightarrow H_2(Q;R) \longrightarrow (H_1(K;R))_Q \longrightarrow H_1(G;R) \longrightarrow H_1(Q;R) \longrightarrow 0.$$ 

If $G_2 < G_1$ are groups satisfying $[G_1 : G_2] < \infty$ and $R$ is a ring, then for all $k$ there exists a transfer map of the form $t : H_k(G_1;R) \rightarrow H_k(G_2;R)$ (see, e.g., [4] Chapter III.9]). The key property of $t$ (see [4] Proposition III.9.5]) is that if $i : H_k(G_2;R) \rightarrow H_k(G_1;R)$ is the map induced by the inclusion, then $i \circ t : H_k(G_1;R) \rightarrow H_k(G_1;R)$ is multiplication by $[G_1 : G_2]$. In particular, if $R = \mathbb{R}$, then we obtain a right inverse $\frac{1}{[G_1 : G_2]} t$ to $i$. This yields the following standard lemma.

**Lemma 2.1.** *Let $G_2 < G_1$ be groups satisfying $[G_1 : G_2] < \infty$. For all $k$, the map $H_k(G_2;\mathbb{R}) \rightarrow H_k(G_1;\mathbb{R})$ is surjective.*
3. Proof of Theorem A

Let $n \geq 1$ be the smallest integer such that $T^n_\beta \in \Gamma$.

We first claim that there exists a subsurface $S \hookrightarrow \Sigma^p_{g,b}$ whose genus is at least 2 with the following property. Let $i: \text{Mod}(S) \to \text{Mod}(\Sigma^p_{g,b})$ be the induced map ("extend by the identity"). Then there exists some boundary component $\beta$ of $S$ such that $i(T_\beta) = T_\gamma$. There are two cases. If $\gamma$ is nonseparating, then let $S$ be the complement of a regular neighborhood of $\gamma$. Observe that $S \cong \Sigma^p_{g-1,b+2}$, so the genus of $S$ is at least 2. If instead $\gamma$ is separating, then let $S$ be the component of $\Sigma^p_{g,b}$ cut along $\gamma$ whose genus is maximal. Since $g \geq 3$, this subsurface must have genus at least 2. The claim follows.

Define $\Gamma' = i^{-1}(\Gamma)$. We have $T^n_\beta \in \Gamma'$, and it is enough to show that $[T^n_\beta]_{\Gamma'} = 0$. Let $\Sigma$ be the result of gluing a punctured disc to $\beta$ and let $\text{Mod}(\Sigma)$ be the image of $\Gamma'$ in $\text{Mod}(\Sigma)$. There is a diagram of central extensions

$$
\begin{array}{cccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Gamma' & \longrightarrow & \text{Mod}(\Sigma) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \text{Mod}(S) & \longrightarrow & \text{Mod}(\Sigma) & \longrightarrow & 1
\end{array}
$$

with $\mathbb{Z} < \text{Mod}(S)$ and $\mathbb{Z} < \Gamma'$ generated by $T_\beta$ and $T^n_\beta$, respectively. The last 4 terms of the corresponding diagram of 5-term exact sequences are

$$
\begin{array}{cccccc}
H_2(\Gamma'; \mathbb{R}) & \longrightarrow & \mathbb{R} & \longrightarrow & H_1(\Gamma'; \mathbb{R}) & \longrightarrow & H_1(\text{Mod}(\Sigma); \mathbb{R}) & \longrightarrow & 0 \\
\downarrow f_2 & & \downarrow \cong & & \downarrow & & \downarrow & & \\
H_2(\text{Mod}(\Sigma); \mathbb{R}) & \longrightarrow & \mathbb{R} & \longrightarrow & H_1(\text{Mod}(S); \mathbb{R}) & \longrightarrow & H_1(\text{Mod}(\Sigma); \mathbb{R}) & \longrightarrow & 0
\end{array}
$$

We remark that there are no nontrivial coinvariants in these sequences since our extensions are central. We must show that $f_1$ is a surjection. Since $S$ has genus at least 2, we have $H_1(\text{Mod}(S); \mathbb{R}) = 0$ (see, e.g., [10]), so $f_3$ is a surjection. Since $[\text{Mod}(\Sigma); \Gamma'] < \infty$, Lemma 3.1 implies that $f_2$ is a surjection, so $f_1$ is a surjection, as desired.

4. Proof of Theorem B

4.1. Two facts about $\text{Sp}_{2g}(\mathbb{Z})$. We will need two standard facts about finite-index subgroups $\Gamma$ of $\text{Sp}_{2g}(\mathbb{Z})$, both of which follow from the fact that $\Gamma$ is a lattice in $\text{Sp}_{2g}(\mathbb{R})$.

For the first, since $\text{Sp}_{2g}(\mathbb{R})$ is a connected simple Lie group with finite center and real rank $g$, the group $\Gamma$ has Kazhdan’s property (T) when $g \geq 2$ (see, e.g., [13 Theorem 7.1.4]). One standard property of groups with property (T) is that they have no nontrivial homomorphisms to $\mathbb{R}$ (see, e.g., [13 Theorem 7.1.7]). Combining these facts, we obtain the following theorem.

**Theorem 4.1.** For some $g \geq 2$, let $\Gamma < \text{Sp}_{2g}(\mathbb{Z})$ satisfy $[\text{Sp}_{2g}(\mathbb{Z}); \Gamma] < \infty$. Then $H_1(\Gamma; \mathbb{R}) = 0$.

For the second, since $\text{Sp}_{2g}(\mathbb{R})$ is a connected noncompact simple real algebraic group, we can apply the Borel density theorem (see, e.g., [13 Theorem 3.2.5]) to deduce that $\Gamma$ is Zariski dense in $\text{Sp}_{2g}(\mathbb{R})$. This implies that any finite-dimensional
nontrivial irreducible $\text{Sp}_{2g}^p(\mathbb{R})$-representation $V$ must also be an irreducible $\Gamma$-representation; indeed, if $V'$ was a nontrivial proper $\Gamma$-submodule of $V$, then the subgroup of $\text{Sp}_{2g}^p(\mathbb{R})$ preserving $V'$ would be a proper subvariety of $\text{Sp}_{2g}^p(\mathbb{R})$ containing $\Gamma$. Recall that the ring of coinvariants $V_\Gamma$ of $V$ under $\Gamma$ is the quotient $V/K$, where $K = \{x - g(x) \mid x \in V, g \in \Gamma\}$. Since $K \neq 0$, we can apply Schur’s lemma to deduce that $K = V$, i.e. that $V_\Gamma = 0$. We record this fact as the following theorem.

**Theorem 4.2.** For some $g \geq 1$, let $\Gamma < \text{Sp}_{2g}^p(\mathbb{Z})$ satisfy $[\text{Sp}_{2g}^p(\mathbb{Z}) : \Gamma] < \infty$ and let $V$ be a nontrivial irreducible $\text{Sp}_{2g}^p(\mathbb{R})$-representation. Then $V_\Gamma = 0$.

4.2. Two preliminary lemmas. We will need two lemmas. The first is the following, which slightly generalizes a theorem of Johnson [8].

**Lemma 4.3.** For $g \geq 3$, we have $\mathcal{T}^p_{g,b}/\mathcal{K}^p_{g,b} \cong (\wedge^3 H)/H \oplus H^{p+1}$, where $H = H_1(\Sigma_g; \mathbb{Z})$.

**Proof.** Since $\mathcal{K}^p_{g,b}$ contains all twists about boundary curves, we can assume that $b = 0$.

Building on work of Johnson [8], Hain [5] proved that

$$H_1(\mathcal{T}^p_{g,b}; \mathbb{R}) \cong (\wedge^3 H)/H \oplus H^p,$$

where $H = H_1(\Sigma_g; \mathbb{R})$. Also, Johnson [9, Lemma 2] proved that for $x \in \mathcal{K}^p_g$, we have $[x]_x^p = 0$. (Johnson only considered the case where $p = 0$, but his argument works in general.) It follows that

$$H_1(\mathcal{T}^p_{g,b}/\mathcal{K}^p_{g,b}; \mathbb{R}) \cong (\wedge^3 H)/H \oplus H^p.$$

We will prove the lemma by induction on $p$. The base case $p = 0$ is a theorem of Johnson [8]. Assume now that $p > 0$ and that the lemma is true for all smaller $p$. Fixing a puncture $*$ of $\Sigma_g$, work of Birman [1] and Johnson [9] gives an exact sequence

$$1 \longrightarrow \pi_1(\Sigma_g^{p-1}; *) \longrightarrow \mathcal{T}^p_g \longrightarrow \mathcal{T}^{p-1}_g \longrightarrow 1,$$

where the map $\mathcal{T}^p_g \rightarrow \mathcal{T}^{p-1}_g$ comes from “forgetting the puncture *”. Quotienting out by $\mathcal{K}^p_g$, we obtain an exact sequence

$$1 \longrightarrow \pi_1(\Sigma_g^{p-1}; *)/(\pi_1(\Sigma_g^{p-1}; *) \cap \mathcal{K}^p_g) \longrightarrow \mathcal{T}^p_g/\mathcal{K}^p_g \longrightarrow \mathcal{T}^{p-1}_g/\mathcal{K}^{p-1}_g \longrightarrow 1.$$

By induction, we have

$$\mathcal{T}^{p-1}_g/\mathcal{K}^{p-1}_g \cong (\wedge^3 H)/H \oplus H^{p-1}.$$

Set $A = \pi_1(\Sigma_g^{p-1}; *)/(\pi_1(\Sigma_g^{p-1}; *) \cap \mathcal{K}^p_g)$. We will prove that $A$ is a quotient of $H$. We will then be able to conclude that $\mathcal{T}^{p-1}_g/\mathcal{K}^{p-1}_g$ acts trivially on $A$, so $\mathcal{T}^p_g/\mathcal{K}^p_g$ is the abelian group

$$(\wedge^3 H)/H \oplus H^{p-1} \oplus A.$$

Using [1], a simple dimension count will then imply that $A$ cannot be a proper quotient of $H$, and the lemma will follow.

The element of $\mathcal{T}^p_g$ corresponding to $\delta \in \pi_1(\Sigma_g^{p-1}, *)$ “drags” $*$ around $\delta$. As shown in Figure [1]a–b, a simple closed curve $\gamma \in \pi_1(\Sigma_g^{p-1}, *)$ corresponds to $T_\gamma T^{-1}_\gamma \in \mathcal{T}^p_g$, where $\gamma_1$ and $\gamma_2$ are the boundary components of a regular neighborhood of $\gamma$. In particular, if $\gamma$ is a simple closed separating curve, then as shown in Figure [1]c–d, the corresponding element of $\mathcal{T}^p_g$ is a product of separating twists. Since $[\pi_1(\Sigma_g^{p-1}, *), \pi_1(\Sigma_g^{p-1}, *)]$ is generated by simple closed separating curves (see,
e.g., [12 Lemma A.1]), we deduce that \([\pi_1(\Sigma_g^{-1}, *), \pi_1(\Sigma_g^{-1}, *)] \subset \pi_1(\Sigma_g^{-1}, *) \cap K_g^p\).

Thus \(A = \pi_1(\Sigma_g^{-1}, *)/(\pi_1(\Sigma_g^{-1}, *) \cap K_g^p)\) is a quotient of \(H_1(\Sigma_g^{-1}; \mathbb{Z})\). Finally, as shown in Figure 1e–f, all simple closed curves that are homotopic into punctures are also contained in \(\pi_1(\Sigma_g^{-1}, *) \cap K_g^p\), so we conclude that \(A\) is a quotient of \(H = H_1(\Sigma_g; \mathbb{Z})\), as desired. \(\square\)

For the second lemma, define \(Q_{g,b}^p = \text{Mod}_{g,b}^p / K_{g,b}^p\).

**Lemma 4.4.** For some \(g \geq 3\), let \(Q' < Q_{g,b}^p\) satisfy \([Q_{b,b}^p : Q'] < \infty\). Then \(H_1(Q'; \mathbb{R}) = 0\).

**Proof.** Restricting the short exact sequence

\[
1 \longrightarrow T_{g,b}^p / K_{g,b}^p \longrightarrow Q_{b,b}^p \longrightarrow \text{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1
\]

to \(Q'\), we obtain a short exact sequence

\[
1 \longrightarrow B \longrightarrow Q' \longrightarrow \overline{Q} \longrightarrow 1,
\]

where \(B\) and \(\overline{Q}\) are finite-index subgroups of \(T_{g,b}^p / K_{g,b}^p\) and \(\text{Sp}_{2g}(\mathbb{Z})\), respectively. The last 3 terms of the associated 5-term exact sequence are

\[
(H_1(B; \mathbb{R}))_{\overline{Q}} \longrightarrow H_1(Q'; \mathbb{R}) \longrightarrow H_1(\overline{Q}; \mathbb{R}) \longrightarrow 0.
\]

By Theorem 4.1 we have \(H_1(\overline{Q}; \mathbb{R}) = 0\). Letting \(H = H_1(\Sigma_g; \mathbb{Z})\), Lemma 4.3 says that

\[
T_{g,b}^p / K_{g,b}^p \cong (\wedge^3 H) / H \oplus H^{b+p}.
\]

Since \(B\) is a finite-index subgroup of \(T_{g,b}^p / K_{g,b}^p\), we get that \(B\) is itself abelian and

\[
H_1(B; \mathbb{R}) \cong B \otimes \mathbb{R} \cong (T_{g,b}^p / K_{g,b}^p) \otimes \mathbb{R} \cong (\wedge^3 H_R) / H_R \oplus H_R^{b+p},
\]

where \(H_R = H_1(\Sigma_g; \mathbb{R})\). Both \((\wedge^3 H_R) / H_R\) and \(H_R^{b+p}\) are nontrivial finite-dimensional irreducible representations of \(\text{Sp}_{2g}(\mathbb{R})\), so Theorem 4.2 implies that \((H_1(B; \mathbb{R}))_{\overline{Q}} = 0\), and we are done. \(\square\)

### 4.3. The proof of Theorem 3

The last 3 terms of the 5-term exact sequence associated to the short exact sequence

\[
1 \longrightarrow \Gamma \cap K_{g,b}^p \longrightarrow \Gamma \longrightarrow \Gamma / (\Gamma \cap K_{g,b}^p) \longrightarrow 1
\]

are

\[
(H_1(\Gamma \cap K_{g,b}^p; \mathbb{R}))_{\Gamma / (\Gamma \cap K_{g,b}^p)} \longrightarrow H_1(\Gamma; \mathbb{R}) \longrightarrow H_1(\Gamma / (\Gamma \cap K_{g,b}^p); \mathbb{R}) \longrightarrow 0.
\]

By assumption, \([\Gamma \cap K_{g,b}^p : K(\Gamma)] < \infty\), so Lemma 2.1 implies that the map \(H_1(K(\Gamma); \mathbb{R}) \rightarrow H_1(\Gamma \cap K_{g,b}^p; \mathbb{R})\) is surjective. Since \(K(\Gamma)\) is generated by powers of twists, Theorem A allows us to deduce that \(i = 0\). Also, \(\Gamma / (\Gamma \cap K_{g,b}^p)\) is a finite-index subgroup of \(Q_{g,b}^p\), so Lemma 4.4 implies that \(H_1(\Gamma / (\Gamma \cap K_{g,b}^p); \mathbb{R}) = 0\), and we are done.
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