ON TRIPLE COVERINGS OF IRRATIONAL CURVES

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ABSTRACT. Given a triple covering $X$ of genus $g$ of a general (in the sense of Brill-Noether) curve $C$ of genus $h$, we show the existence of base-point-free pencils of degree $d$ which are not composed with the triple covering for any $d \geq g - \lceil \frac{3h+1}{2} \rceil - 1$ by utilizing some enumerative methods and computations. We also discuss about the sharpness of our main result and the so-called Castelnuovo-Severi bound by exhibiting some examples.

0. Introduction

In this paper, we investigate the problem of the existence of base-point-free pencils of relatively low degree on a triple covering $X$ of genus $g$ of a general curve $C$ of genus $h > 0$. Such a problem is classical and the picture is rather well known for the degree range close to the genus $g$. On the other hand, by a simple application of the Castelnuovo-Severi inequality one can easily see that there does not exist a base-point-free pencil of degree less than or equal to $\frac{g-3h}{2}$ other than the pull-backs from the base curve $C$; while for the degree beyond this range not many things have been known about the existence of such a pencil which is not composed with the given triple covering. The main result of this paper is the following.

Theorem A. Let $X$ be a smooth algebraic curve of genus $g$, over an algebraic closed field $k$ with $\text{char}(k) = 0$, which admits a three sheeted covering onto a general curve $C$ of genus $h \geq 1$, $g \geq (2\lceil \frac{3h+1}{2} \rceil + 1)(\lceil \frac{3h+1}{2} \rceil + 1)$. Then there exists a base-point-free pencil of any degree $d \geq g - \lceil \frac{3h+1}{2} \rceil - 1$ which is not composed with the given triple covering.

This paper is organized as follows. In §1, we prove Theorem A by using the enumerative methods and computations in $H^*(X_\alpha, \mathbb{Q})$ of various sub-loci of the symmetric product $X_\alpha$ of the given triple covering $X$, while we defer proving the key lemma which are necessary to prove Theorem A. Specifically we compare the fundamental class of $X^1_\alpha := \{ D \in X_\alpha : \dim |D| \geq 1 \}$ with the class of all irreducible components of $X^1_\alpha$ whose general elements correspond to pencils on $X$ with base points. This argument works because the latter components are all induced from
the base curve \( C \) and \( X^1_\alpha \) has the expected dimension. In §2, we proceed to prove Lemma 2.1. In proving Lemma 2.1, we need to utilize and carry out some computations based on the fundamental result of Rick Miranda about the triple covering of algebraic varieties in general [M]. In the last section we discuss about the sharpness of our main result and the so-called Castelnuovo-Severi bound by considering some examples.

1. Existence of base-point-free pencils on triple coverings

The purpose of this section is to prove Theorem A. We begin with the following dimension theoretic statement about the variety of special linear systems on an algebraic curve; cf. [CKM] Corollary 3.1.2.

**Lemma 1.1.** Let \( X \) be an algebraic curve of genus \( g \). Let \( n \in \mathbb{N}, g \geq (2n-3)(n-1) \) (and if \( n \leq 2 \) let \( g \geq 2n-1 \)). Assume that \( \dim W^1_{n+1}(X) < 1 \). Then \( W^1_d(X) \) is equi-dimensional of the minimal possible dimension \( \rho(d, g, 1) := 2d - 2 - g \) for all \( d \) such that \( g - n < d \leq g \).

The following lemma is a special case of the Castelnuovo-Severi bound which one can prove easily by using the adjunction formula and the Hodge index theorem.

**Lemma 1.2** (Castelnuovo-Severi bound). Let \( X \) be a smooth algebraic curve of genus \( g \) which admits a triple covering onto a curve \( C \) of genus \( h \). Let \( \pi : X \to C \) be the triple covering. For any integer \( n \leq \frac{g-3h}{2} \), a base-point-free pencil \( g^1_n \) (possibly incomplete) is composed with \( \pi \), or equivalently the morphism \( X \to \mathbb{P}^1 \) induced by the \( g^1_n \) always factors through \( \pi \).

**Proof of Theorem A.** Let \( n = \lfloor \frac{3h+1}{2} \rfloor + 2 \). Then \( n+1 = \lfloor \frac{3h+1}{2} \rfloor + 3 \leq \frac{g-3h}{2} \) by the numerical hypothesis on the genus \( g \) of \( X \). Assume that there exists a \( g^1_{n+1} \) on \( X \). By Lemma 1.2, it follows that every \( g^1_{n+1} \) is composed with \( \pi \) and

\[
(A.1) \quad W^1_{n+1}(X) = \pi^*W^1_m(C) + W^1_{n+1-3m}(X),
\]

for some \( m \) with \( n+1 - 3m \geq 0 \). On the other hand, since \( C \) is general

\[
(A.2) \quad \dim W^1_m(C) = \dim \pi^*W^1_m(C) = \rho(m, h, 1) := 2m - 2 - h \geq 0.
\]

By (A.1) and (A.2), we have

\[
\dim W^1_{n+1}(X) < 1.
\]

Accordingly by Lemma 1.1, \( W^1_d(X) \) is equi-dimensional of the minimal possible dimension \( \rho(d, g, 1) := 2d - 2 - g \) for all \( d \) such that \( g - n < d \leq g \). Furthermore, since

\[
\dim W^1_d(X) = \rho(d, g, 1) > \dim (\pi^*W^1_m(C) + W^1_{d-3m}(X))
\]
(a) $h$ is even: Set $h = 2e$, then $d = g - \left\lceil \frac{3h+1}{2} \right\rceil - 1 = g - 3e - 1$. Let $\Sigma$ be a component of $W_{g-3e-1}^1(X)$ whose general element has a base point. Then $\Sigma = \Sigma_{\beta}^1 + W_{g-3e-1-\beta}(X)$ for some $\beta$ with $4 \leq \beta \leq g - 3e - 2$, where $\Sigma_{\beta}^1$ is a subvariety of $W_{\beta}^1(X)$ whose general element is base-point-free. Note that $\dim \Sigma = \rho(g - 3e - 1, g, 1) = g - 6e - 4$, hence $\beta - 3e - 3 = \dim \Sigma_{\beta}^1 \geq 0$.

Let $L \in \Sigma_{\beta}^1$ be a general element. By the standard description of the Zariski tangent space to the variety $W_{d}^r(X)$ of any curve $X$ in general, we have

$$\dim(\Im \mu_0) = \dim T_L(W_{\beta}^1(X)) \geq \dim \Sigma_{\beta}^1 = \beta - 3e - 3$$

where $\mu_0 : H^0(X, L) \otimes H^0(X, K_X \otimes L^{-1}) \to H^0(X, K_X)$ is the usual cup product map. By the base-point-free pencil trick ([ACGH], p126), we have

$$\dim(\Im \mu_0) = g - \dim(\Im \mu_0) = g - h^0(X, L)h^1(X, L) + \dim(\Ker \mu_0)$$

$$= g - 2(g - \beta + 1) + h^0(X, K_X \otimes L^{-2})$$

$$= h^0(X, L^2) - 3 \geq \beta - 3e - 3.$$ 

Hence $h^0(X, L^2) \geq \beta - 3e \geq 3$ which implies $\dim W_{2\beta-3e-1}^1(X) \geq \beta - 3e - 3$. By reducing to pencils if necessary we have

$$\dim W_{\beta+3e+2}^1(X) = \dim W_{2\beta-(\beta-3e-2)}^1(X) \geq \beta - 3e - 3 + (\beta - 3e - 2) = 2(\beta - 3e) - 5.$$ 

Note that $\beta \leq g - 3e - 2$, hence $\beta + 3e + 2 \leq g$. We now consider the following two cases:

(1) If $\beta + 3e + 2 = g$, then by passing to residual series

$$\dim W_{\beta+3e+2}^1(X) = \dim W_{g-2}^1(X) = g - 2 \geq 2(\beta - 3e) - 5,$$

hence, $12e \geq g - 7$ which contradicts the genus assumption.

(2) If $\beta + 3e + 2 \leq g - 1$, we have

$$2(\beta - 3e) - 5 \leq \dim W_{\beta+3e+2}^1(X) \leq (\beta + 3e + 2) - 2 - 1,$$

hence, $\beta \leq 9e + 4$ by H.Martens-Mumford’s theorem; cf. [ACGH] Ch. 4, §5.

On the other hand by the assumption on $h$ and $g$ we have $\beta \leq 9e + 4 \leq \frac{g-3h}{2}$. Hence by the Castelnuovo-Severi inequality, every element of $\Sigma_{\beta}^1$ is a pull-back of a $g_{\beta/3}^1$ on $C$, i.e. $\Sigma_{\beta}^1 = \pi^*(\Sigma_{\beta/3}^1(C))$, where $\Sigma_{\beta/3}^1(C)$ is a component of $W_{\beta/3}^1(C)$, i.e.

$$\Sigma_{\beta}^1 \subset W_{\beta}^1(X) = \pi^* W_{\beta/3}^1(C)$$

and $W_{\beta}^1(X) = \pi^* W_{\beta/3}^1(C)$ has dimension $\rho(\frac{\beta}{3}, h, 1) = 2 \cdot \frac{\beta}{3} - 2 - 2e$ since $C$ is general. We also have

$$\rho(\frac{\beta}{3}, h, 1) = \dim \Sigma = \dim(\Sigma_{\beta}^1 + W_{\beta-3e-1}(X)).$$
From this we get \( \frac{\beta}{3} = e + 1 \) and every component of \( W_{g-3e-1}^1(X) \) whose general element has a base point is of the form \( \pi^*\Sigma' + W_{g-6e-4}(X) \), where \( \Sigma' \) is a component of \( W_{e+1}^1(C) \).

Since \( C \) is general and the Brill-Noether number \( \rho(e + 1, h, 1) \) is zero, \( \Sigma' \) is a single point in \( g_{e+1}^1(C) \) of \( W_{e+1}^1(C) \) and there are exactly \( s := h! \left( \frac{1}{(h-e)!} \cdot \frac{1}{(h-e+1)!} \right) \) reduced points in \( W_{e+1}^1(C) \), which is Castelnuovo’s count for the number of \( g_d^1 \)’s when \( \rho = 0 \); cf. [ACGH] Theorem 1.3, p212.

Let \( \tilde{\Sigma}' \) be the locus in \( C_{e+1}^1 \) corresponding to \( g_{e+1}^1(C) = \Sigma' \). We will show in the second section (Lemma 2.1) that \( W_{g-3e-1}^1(X) \) is reduced at a general point of \( \pi^*(W_{e+1}^1(C)) + W_{g-6e-4}(X) \), which in turn says that \( X_{g-3e-1}^1 \) is reduced at a general point of \( \pi^*(\tilde{\Sigma}') + X_{g-6e-4} \).

We now recall some of the notations and conventions used in [ACGH], especially in Chapter VIII. Let \( u : X_d \rightarrow J(X) \) be the abelian sum map and let \( \theta \) be the class of the theta divisor in \( J(X) \). Let \( u^* : H^*(J(X), \mathbb{Q}) \rightarrow H^*(X_d, \mathbb{Q}) \) be the homomorphism induced by \( u \). By abusing notation, we use the same letter \( \theta \) for the class \( u^*\theta \). By fixing a point \( P \) on \( X \), one has the map \( \iota : X_{d-1} \rightarrow X_d \) defined by \( \iota(D) = D + P \). We denote the class of \( \iota(X_{d-1}) \) in \( X_d \) by \( x \).

By our discussion so far, the only components of \( W_{g-3e-1}^1(X) \) whose general element has a base point is of the form \( \pi^*\Sigma' + W_{g-6e-4}(X) \), where \( \Sigma' \) is a component of \( W_{e+1}^1(C) \).

We denote \( \sigma \) and \( \tilde{\sigma} \) by the class of \( \pi^*C_{e+1}^1 + X_{g-6e-4} \) in \( X_{g-3e-1} \) and the class of \( \pi^*C_{e+1}^1 \) in \( X_{3e+3} \) respectively. Because \( X_{g-3e-1} \) is of pure and expected dimension \( \rho(g - 3e - 1, g, 1) + 1 = g - 6e - 3 \), the class \( x_{g-3e-1}^1 \) of \( X_{g-3e-1}^1 \) in \( X_{g-3e-1} \) is well known (cf. [ACGH] p326), namely

\[
(A.3) \quad x_{g-3e-1}^1 = \frac{1}{(3e + 1)!(3e + 2)!} \left( (3e + 1)!\theta^{3e+2} - (3e + 2)!x\theta^{3e+1} \right).
\]

Let’s also recall that for a cycle \( Z \) in \( X_d \), the assignments

\[
Z \mapsto A_k(Z) := \{ E \in X_{d+k} : E - D \geq 0 \text{ for some } D \in Z \}
\]

\[
Z \mapsto B_k(Z) := \{ E \in X_{d-k} : D - E \geq 0 \text{ for some } D \in Z \}
\]

induce maps

\[
A_k : H^{2m}(X_d, \mathbb{Q}) \rightarrow H^{2m}(X_{d+k}, \mathbb{Q})
\]

\[
B_k : H^{2m}(X_d, \mathbb{Q}) \rightarrow H^{2m-2k}(X_{d-k}, \mathbb{Q})
\]

and the so called push-pull formulas for symmetric products hold (cf. [ACGH], p367-369).

First note that \( X_{g-3e-1}^1 \) and \( \Sigma' \) have dimension \( g - 6e - 3 \) in \( X_{g-3e-1} \). On the other hand, by the push-pull formulas

\[
B_{g-6e-4}(x^{g-6e-3}) = (g - 6e - 3)x
\]
by noting the fact that \((\bar{\sigma} \cdot x)X_{3e+3} = s := h! \left( \frac{1}{(h-e)!} \cdot \frac{1}{(h-e+1)!} \right) \).

By (A.3)

\[
\left( x_{g-3e-1}^1 \cdot x^{g-6e-3} \right)_{X_{g-3e-1}} = \left( \frac{1}{(3e+1)(3e+2)} \right) (3e+1)! \cdot \frac{1}{(3e+1)!} \cdot \frac{1}{(3e+2)!} \cdot \frac{1}{(g-3e-1)!} \cdot \frac{1}{g!} \cdot x^{g-6e-3}
\]

where the last equality comes from the fact that \((x^{d-a} \cdot \theta^a)X_d = g!/(g-\alpha)!\), which is a consequence of Poincaré’s formula; cf. [ACGH], p328.

Finally an easy calculation yields

\[
\left( x_{g-3e-1}^1 \cdot x^{g-6e-3} \right)_{X_{g-3e-1}} > (\sigma \cdot x^{g-6e-3})_{X_{g-3e-1}}
\]

and this shows that there exists a component in \(X_{g-3e-1}^1\) other than those of the form \(\pi^*(C_{e+1}^1) + X_{g-6e-4}\) which in turn proves that there exists a divisor of degree \(g-3e-1\) on \(X\) which moves in a base-point-free pencil and whose complete linear series is not composed with the given triple covering.

We now proceed to handle the case when \(h = 2e+1\). Even though the method is almost the same, we will present somewhat detailed argument and the computations for the convenience of the reader.

(b) \(h\) is odd: Set \(h = 2e+1\), then \(d = g - \left[\frac{3h+1}{2}\right] - 1 = g - 3e - 3\). Let \(\Sigma\) be a component of \(W_{g-3e-3}^1(X)\) whose general element has a base point. Then \(\Sigma = \Sigma_\beta^1 + W_{g-3e-3-\beta}(X)\) for some \(\beta\) with \(4 \leq \beta \leq g - 3e - 4\), where \(\Sigma_\beta^1\) is a subvariety of \(W_{\beta}^1(X)\) whose general element is base-point-free. Note that \(\dim \Sigma = \rho(g-3e-3, g, 1) = g - 6e - 8\), hence \(\beta - 3e - 5 = \dim \Sigma_\beta^1 \geq 0\).

Let \(L \in \Sigma_\beta^1\) be a general element. Again by the description of the Zariski tangent space to the variety of special linear systems, we have

\[
\dim(\text{Im } \mu_0)^\perp = \dim T_L(W_{\beta}^1(X)) \geq \dim \Sigma_\beta^1 \geq \beta - 3e - 5.
\]

By the base-point-free pencil trick, we have

\[
\dim(\text{Im } \mu_0)^\perp = g - \dim(\text{Im } \mu_0) = g - h^0(X, L)h^1(X, L) + \dim(\text{Ker } \mu_0)
\]
\[
= g - 2(g - \beta + 1) + h^0(X, K_X \otimes L^{-2})
\]
\[
= h^0(X, L^2) - 3 \geq \beta - 3e - 5.
\]

Hence \(h^0(X, L^2) > \beta - 3e - 2 \geq 3\) which implies \(W_{\beta}^{g-3e-3}(X) > \beta - 3e - 5\). By
(1) If $\beta + 3e + 4 = g$, then by passing to residual series
\[
\dim W_{\beta+3e+4}^1(X) = \dim W_{g-2}(X) = g - 2 \geq 2(\beta - 3e) - 9,
\]
hence $12e \geq g - 15$, contradictory to the genus assumption.

(2) If $\beta + 3e + 4 \leq g - 1$, we have
\[
2(\beta - 3e) - 9 \leq \dim W_{\beta+3e+4}^1(X) \leq (\beta + 3e + 4) - 2 - 1,
\]
hence, $\beta \leq 9e + 10$ by H.Martens-Mumford’s theorem.

On the other hand by the assumption on $h$ and $g$ we have $\beta \leq 9e + 10 \leq \frac{g-3h}{2}$.
Hence by the Castelnuovo-Severi inequality, every element of $\Sigma_{\beta}$ is a pull-back of a $g_{\beta/3}^1$ on $C$, i.e. $\Sigma_{\beta} = \pi^*(\Sigma_{\beta/3}^1(C))$, where $\Sigma_{\beta/3}^1(C)$ is a component of $W_{\beta/3}^1(C)$,
\[
\Sigma_{\beta}^1 \subset W_{\beta}^1(X) = \pi^*W_{\frac{\beta}{3}}^1(C)
\]
and $W_{\beta}^1(X) = \pi^*W_{\frac{\beta}{3}}^1(C)$ has dimension $\rho(\frac{\beta}{3}, h, 1) = 2 \cdot \frac{\beta}{3} - 2 - (2e + 1)$ since $C$ is general. We also have
\[
g - 6e - 8 = \dim \Sigma = \dim (\Sigma_{\beta}^1 + W_{g-3e-3-\beta}(X)) \\
\leq \dim (W_{\beta}^1(X) + W_{g-3e-3-\beta}(X)) \\
= (2 \cdot \frac{\beta}{3} - 2 - 2e - 1) + (g - 3e - 3 - \beta) \\
\leq \dim W_{g-3e-3}(X) = g - 6e - 8
\]
From this we get $\frac{\beta}{3} = e + 2$ and every component of $W_{g-3e-3}^1(X)$ whose general element has a base point is of the form $\pi^*\Sigma' + W_{g-6e-9}(X)$, where $\Sigma'$ is a component of $W_{e+2}^1(C)$.

Since $C$ is general and the Brill-Noether number $\rho(e + 2, h, 1) = 1 > 0$, $W_{e+2}^1(C)$ is irreducible of dimension 1 which follows from Gieseker’s theorem by Fulton and Lazarsfeld’s connectedness theorem together with Brill-Noether theorem by Griffiths-Harris; cf. [ACGH] p214. Hence we have
\[
\Sigma' = W_{e+2}^1(C) \quad \text{and} \quad \Sigma = \pi^*W_{e+2}^1(C) + W_{g-6e-9}(X).
\]
By Lemma 2.1, $W_{g-3e-3}^1(X)$ is reduced at a general point of $\Sigma$, which in turn says that $X_{g-3e-3}^1$ is reduced at a general point of $\pi^*C_{e+2}^1 + X_{g-6e-9}$.

We denote $\sigma$ and $\tilde{\sigma}$ by the class of $\pi^*C_{e+2}^1 + X_{g-6e-9}$ in $X_{g-3e-3}$ and the class of $\pi^*C_{e+2}^1$ in $X_{3e+6}$ respectively. Because $X_{g-3e-3}^1$ is of pure and expected dimension $\rho(g - 3e - 3, g, 1) + 1 = g - 6e - 7$, the fundamental class $x_{g-3e-3}^1$ of $X_{g-3e-3}^1$ in $X_{g-3e-3}$ is
Lemma 2.1. Assume that $C$ is a general curve of genus $h$.

(i) Assume $h = 2e$ and $g > 6h + 4$. Then a general element $\mathcal{L}$ of $\pi^*(W_{e+1}^1 + W_{g-6e-4}(X))$ is a reduced point of $W_{g-3e-3}(X)$.

(ii) Assume $h = 2e + 1$ and $g > 6h + 7$. Then a general element $\mathcal{L}$ of $\pi^*(W_{e+2}^1 + W_{g-6e-9}(X))$ is a reduced point of $W_{g-3e-3}(X)$.

In this section, our aim is to prove the following key lemma.

Let $X = A_g$ be a general curve of genus $g$ and $C = C_{g-3e-3}^1$ be a component of $C_{g-3e-3}^1$. Then

$$B_{g-6e-9}(x_{g-6e-7}) = \frac{(g - 6e - 7)}{(g - 6e - 9)}x^2$$

and

$$(\sigma \cdot x_{g-6e-7})X_{g-3e-3} = (A_{g-6e-9}(\bar{\sigma}) \cdot x_{g-6e-7})X_{g-3e-3}$$

$$= (\bar{\sigma} \cdot B_{g-6e-9}(x_{g-6e-7}))X_{g-3e-3} = \left(\frac{g - 6e - 7}{g - 6e - 9}\right)(\bar{\sigma} \cdot x^2)X_{g-3e-3}$$

$$= \left(\frac{g - 6e - 7}{g - 6e - 9}\right) \left(\frac{(2e + 1)!}{e!(e + 1)!} - \frac{(2e + 1)!}{(e - 1)!(e + 2)!}\right)$$

where the last equality comes from Poincaré’s formula, the fact that $C$ is general and the class $c_{e+2}^1$ of $C_{e+2}^1$ in $C_{e+2}$ is

$$c_{e+2}^1 = \frac{\theta^e}{e} - \frac{x\theta^{e-1}}{(e - 1)!}$$

On the other hand, by (A.4) and Poincaré’s formula

$$(x_{g-3e-3}^1 \cdot x_{g-6e-7}^1)X_{g-3e-3}$$

$$= \frac{1}{(3e + 3)!(3e + 4)!} \left((3e + 3)!\theta^{3e+4} - (3e + 4)!x\theta^{3e+3}\right) \cdot x_{g-6e-7}$$

$$= \frac{g!(g - 6e - 7)}{(3e + 4)!(g - 3e - 3)!}$$

Finally an easy calculation yields

$$(x_{g-3e-3}^1 \cdot x_{g-6e-7}^1)X_{g-3e-3} > (\sigma \cdot x_{g-6e-7}^1)X_{g-3e-3}$$

which shows that there exists a component in $X_{g-3e-3}^1$ other than those of the form $\pi^*(C_{e+2}^1) + X_{g-6e-9}$ which in turn proves that there exists a divisor of degree $g - 3e - 3$ on $X$ which moves in a base-point-free pencil and whose complete linear series is not composed with the given triple covering. □

2. The proof of Lemma 2.1

(ii) Assume $h = 2e + 1$ and $g > 6h + 7$. Then a general element $\mathcal{L}$ of $\pi^*(W_{e+2}^1 + W_{g-6e-9}(X))$ is a reduced point of $W_{g-3e-3}(X)$.
Remark 2.2. For the rest of this section, we assume, again, that $n = 3$, i.e. a triple covering. Therefore $\pi$ is a finite separable morphism of degree 3 and $\mathcal{E}$ is a locally free $\mathcal{O}_C$-module of rank 2. As $\pi_* (\mathcal{O}_X)$ is a commutative $\mathcal{O}_C$-algebra, we have a map defined by the multiplication,

$$\pi_* (\mathcal{O}_X) \otimes_{\mathcal{O}_C} \pi_* (\mathcal{O}_X) \rightarrow \pi_* (\mathcal{O}_X).$$

Hence the multiplication induces $\mathcal{O}_C$-module homomorphisms

$$\phi_1 : S^2 \mathcal{E} \rightarrow \mathcal{O}_C \quad \text{and} \quad \phi_2 : S^2 \mathcal{E} \rightarrow \mathcal{E}.$$

Furthermore, the algebra structure on $\pi_* (\mathcal{O}_X) = \mathcal{O}_C \oplus \mathcal{E}$ is written as follows:

$$(2.A) \quad (a, b) \cdot (a', b') = (aa' + \phi_1 (bb'), ab' + a'b + \phi_2 (bb')),$$

where $a, a' \in \mathcal{O}_C$ and $b, b' \in \mathcal{E}$. Locally, $\phi_2$ is written as the following form [M]:

$$\phi_2 (z^2) = az + bw, \phi_2 (zw) = -dz - aw, \phi_2 (w^2) = cz + dw$$

where $z, w \in \mathcal{E}$ is a local frame of $\mathcal{E}$ and $a, b, c, d \in \mathcal{O}_C$. Such a map $\phi_2$ is called a triple covering homomorphism due to R. Miranda. We denote $\text{TCHom}_{\mathcal{O}_C} (S^2 \mathcal{E}, \mathcal{E})$ by the set of all triple covering homomorphisms on $\mathcal{E}$ and there is a functorial homomorphism

$$F : \text{TCHom}_{\mathcal{O}_C} (S^2 \mathcal{E}, \mathcal{E}) \rightarrow \text{Hom}_{\mathcal{O}_C} (S^2 \mathcal{E}, \mathcal{O}_C)$$

such that $\phi_2 \in \text{TCHom}_{\mathcal{O}_C} (S^2 \mathcal{E}, \mathcal{E})$ and $\phi_1 = F (\phi_2)$ gives an $\mathcal{O}_C$-algebra structure on $\mathcal{O}_C \oplus \mathcal{E}$ by the relation (2.A); [M], p.1131 Proposition 3.5.

Also for the triple covering $\pi : X \rightarrow C$ there is a functorial isomorphism

$$\text{Hom}_{\mathcal{O}_X} (\pi^* \pi_* (\mathcal{O}_X), \mathcal{O}_X) \cong \text{Hom}_{\mathcal{O}_C} (\pi_* (\mathcal{O}_X), \pi_* (\mathcal{O}_X)).$$

We take a $\sigma \in \text{Hom}_{\mathcal{O}_X} (\pi^* \pi_* (\mathcal{O}_X), \mathcal{O}_X)$ which corresponds to the identity map: $\pi_* (\mathcal{O}_X) \rightarrow \pi_* (\mathcal{O}_X)$. Since $\pi_* (\mathcal{O}_X) = \mathcal{O}_C \oplus \mathcal{E}$, we can write $\sigma = (\sigma_1, \sigma_2)$ where $\sigma_1 : \mathcal{O}_X \rightarrow \mathcal{O}_X$ and $\sigma_2 : \pi^* \mathcal{E} \rightarrow \mathcal{O}_X$. Because $\sigma$ is an $\mathcal{O}_X$-algebra homomorphism, $\sigma_1$ is the identity map. As for $\sigma_2$, we have the following result.

**Lemma 2.3.** The image of $\sigma_2$ is an invertible sheaf.

**Proof.** Note that $\text{Im} (\sigma_2)$ is an ideal sheaf of $\mathcal{O}_C$. Thus, if $\sigma_2$ is not a zero map then $\text{Im} (\sigma_2)$ is an invertible sheaf, because $\text{Im} (\sigma_2)$ is an ideal sheaf of the divisor $D = \text{Spec} (\mathcal{O}_C / \text{Im} (\sigma_2))$ and $X$ is a non-singular curve. $D$ is a Cartier divisor on $X$. Therefore $\text{Im} (\sigma_2)$ is an invertible sheaf.
where \( \tau = \mu \oplus (\phi_1, \phi_2) \), \( \mu : M \otimes_A A \to B \) is a multiplication map, \( \phi_2 \) is the triple covering homomorphism corresponding to the \( A \)-algebra structure of \( B \) and \( \phi_1 \) is the homomorphism defined in Remark 2.2. Clearly \( \mu \) is a non-zero map, therefore \( \sigma_2 \) is not a zero map either. \( \square \)

In general, one knows that to give a morphism of \( X \) to the projective bundle \( \mathbb{P}(\mathcal{E}) \) over \( C \) is equivalent to give an invertible sheaf \( L \) on \( X \) and a surjective homomorphism \( \pi^* \mathcal{E} \to L \); cf. [H], p162 Proposition 7.12. Hence by Lemma 2.3, we have a morphism \( f : X \to \mathbb{P}(\mathcal{E}) \) such that the following diagram commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & \mathbb{P}(\mathcal{E}) \\
\pi \downarrow & & \circlearrowleft \varphi \\
C & & \\
\end{array}
\]

With these preparations we now prove the following lemma.

**Lemma 2.4.** \( f \) is embedding.

**Proof.** Let \( U = \text{Spec}(A) \) be an affine open subset of \( C \). Let \( Z = \pi^{-1}(U) = \text{Spec}(B) \), \( \Gamma(U, \pi_* (\mathcal{O}_X)) = A \oplus A s_1 \oplus A s_2 \) and \( Z_i = \{ P \in Z | s_i(P) \neq 0 \} \), \( i = 1, 2 \). Because \( \pi \) is a finite morphism, \( Z \) is an affine open set and hence \( Z_i \) is also an affine open set for \( i = 1, 2 \). Moreover \( Z_i = \text{Spec}(B_{s_i}) \). Therefore \( B_{s_i} \) is generated by \( 1, \frac{s_j}{s_i} \) where \( j \neq i \). By a criterion for a morphism to a projective space to be an embedding ([H] p.151 Proposition 7.2), \( f|_{\pi^{-1}(U)} \) is embedding for every affine open set \( U \subset C \) and hence \( f \) is embedding. \( \square \)

We now consider the ruled surface \( \varphi : \mathbb{P}(\mathcal{E}) \to C \). Let \( C_0 \) be a divisor on \( \mathbb{P}(\mathcal{E}) \) such that \( \varphi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(C_0)) = \mathcal{E} \). Let \( \bar{C} \) be a minimal section of \( \varphi : \mathbb{P}(\mathcal{E}) \to C \), \( \bar{\mathcal{E}} = \varphi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}((\bar{C})) \) and \( \delta = -(\bar{C}^2) \).

**Lemma 2.5.** There exist an exact sequence

\[
0 \to \mathcal{O}_C \to \bar{\mathcal{E}} \to \wedge^2 \bar{\mathcal{E}} \to 0
\]

and an isomorphism

\[
\mathcal{E} \cong \bar{\mathcal{E}} \otimes_{\mathcal{O}_C} \mathcal{M}
\]

for some line bundle \( \mathcal{M} \) on \( C \).

**Proof.** Let \( s \in \Gamma(C, \bar{\mathcal{E}}) \) be a non-zero section. Then \( s \) determines an injection \( 0 \to \mathcal{O}_C \to \bar{\mathcal{E}} \). Let \( \mathcal{P} \) be the cokernel of \( 0 \to \mathcal{O}_C \to \bar{\mathcal{E}} \). \( \mathcal{P} \) is of rank 1 because \( \mathcal{E} \) is of rank 2. Therefore it is enough to show that \( \mathcal{P} \) is a torsion free sheaf because \( C \) is non-singular. Now we assume that \( \mathcal{P} \) has a torsion part. Let \( \mathcal{F} \) be the inverse image of \( \mathcal{P} \) by the map \( \bar{\mathcal{E}} \to \mathcal{P} \). Since \( \mathcal{P} \) is torsion, \( \mathcal{F} \) is also a torsion sheaf. This contradicts the assumption that \( \mathcal{P} \) is a torsion-free sheaf. Therefore \( \mathcal{P} \) is torsion-free, and hence \( \mathcal{E} \cong \bar{\mathcal{E}} \otimes_{\mathcal{O}_C} \mathcal{M} \). \( \square \)
We take an invertible sheaf \( M \) such that \( \mathcal{E} \cong \mathcal{E} \otimes M \) and let \( n = \deg(\mathcal{M}) \). Then we have \( C_0 \sim \bar{C} + \pi^*(\mathcal{M}) \) (linearly equivalent) by Lemma 2.5, therefore \( C_0 \equiv \bar{C} + nF \) (where \( \equiv \) means numerical equivalence and \( F \) is a fibre of \( \varphi \)) and \( (C_0^2) = 2n - \delta = \deg(\wedge^2 \mathcal{E}) \). One also knows the following numerical equivalence;

\[(2.B)\quad K_{\mathbb{P}(\mathcal{E})} \equiv -2C_0 + (2h - 2) + \deg(\wedge^2 \mathcal{E})F,\]

see [H], p.372 Proposition 2.8 and p.374 Corollary 2.11.

By (2.B), we have the following.

**Lemma 2.6.** We have \( n = \frac{(C_0^2) + \delta}{2} = \frac{\delta - g + 3h - 2}{2} \), and we also have

\[
f(X) \equiv 3C_0 - 2 \deg(\wedge^2 \mathcal{E})F \equiv 3\bar{C} + \frac{3\delta + g - 3h + 2}{2} F
\]

for numerical equivalence.

**Proof.** By the Riemann-Hurwitz relation ([H], p.127 Exercise 5.16), we have

\[
2g - 2 = 3(2h - 2) + \deg(\wedge^2 \mathcal{E}^\otimes -2)
\]

and hence

\[
\deg(\wedge^2 \mathcal{E}) = \frac{3(2h - 2) - (2g - 2)}{2}.
\]

As \( C_0 \equiv \bar{C} + nF \), we have \( n = \frac{\delta - g + 3h - 2}{2} \) because \( (C_0^2) = 2n - \delta = \deg(\wedge^2 \mathcal{E}) \). By Lemma 2.4, \( f : X \to \mathbb{P}(\mathcal{E}) \) is embedding and \( \pi : X \to C \) is a finite morphism of degree 3. Therefore we have \( f(X) \sim 3C_0 + \varphi^*(T) \) for some divisor \( T \) on \( C \) and \( 2g - 2 = (K_{\mathbb{P}(\mathcal{E})} + f(X)).f(X) \) by the adjunction formula. Hence \( \deg(T) = -2\deg(\wedge^2 \mathcal{E}) \) by (2.B) and we have

\[
f(X) \equiv 3C_0 - 2 \deg(\wedge^2 \mathcal{E})F
\]

\[
\equiv 3(\bar{C} + \frac{\delta - g + 3h - 2}{2} F) - 2(\frac{3(2h - 2) - (2g - 2)}{2})F
\]

\[
\equiv 3\bar{C} + \frac{3\delta + g - 3h + 2}{2} F. \quad \square
\]

**Proposition 2.7.** There is an exact sequence

\[
0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{L} \to 0,
\]

where \( \deg(\mathcal{M}) = -(\frac{g - 3h}{2} + \frac{2}{\delta} - \frac{2}{2}) \) and \( \deg(\mathcal{L}) = -(\frac{g - 3h}{2} + \frac{2}{\delta} + \frac{2}{2}) \).

**Proof.** By Lemma 2.5, we have \( \mathcal{E} \cong \bar{\mathcal{E}} \otimes \mathcal{M} \) hence \( \wedge^2 \mathcal{E} \cong \wedge^2 \bar{\mathcal{E}} \otimes \mathcal{M} \otimes^2 \) and \( \deg(\wedge^2 \bar{\mathcal{E}}) = -\delta \). Again by the Riemann-Hurwitz relation,

\[
\deg(\wedge^2 \mathcal{E}) = \frac{3(2h - 2) - (2g - 2)}{2}.
\]

Therefore we have \( \deg(\mathcal{M}) = -(\frac{g - 3h}{2} + \frac{2}{\delta} - \frac{2}{2}) \). Also by Lemma 2.5, we have the exact sequence

\[
0 \to \mathcal{O}_C \to \bar{\mathcal{E}} \to \wedge^2 \bar{\mathcal{E}} \to 0.
\]
Lemma 2.8. \(-h \leq \delta \leq \frac{g-3h+2}{3}\).

Proof. If \(\delta \leq 0\), then \(-h \leq \delta\) by a theorem of Nagata;\(^N\) p.191 Theorem 1. Therefore we may assume that \(\delta \geq 0\). As \(f(X) \equiv 3C + \frac{3\delta + g - 3h + 2}{2}F\) by Lemma 2.6 and \(f(X)\) is irreducible, we have \(3\delta \leq \frac{3\delta + g - 3h + 2}{2}\) by \([H]\) p.382 Proposition 2.20. Therefore we have \(\delta \leq \frac{g-3h+2}{3}\). □

Proposition 2.9. \(\deg(M) \leq \frac{-g+3h-2}{3}\) and \(\deg(L) \leq \frac{-g+4h-2}{2}\).

Proof. By Proposition 2.7, we have the exact sequence

\[
0 \to M \to E \to L \to 0
\]

such that \(\deg(M) = -\left(\frac{g-3h+2}{2} + \frac{2-\delta}{2}\right)\) and \(\deg(L) = -\left(\frac{g-3h+2}{2} + \frac{2+\delta}{2}\right)\). Therefore we have

\[
\deg(M) = -\left(\frac{g-3h+2}{2} + \frac{2-\delta}{2}\right) = -\frac{g-3h+2}{2} + \frac{\delta}{2}
\]

\[
\leq -\frac{g-3h+2}{2} + \frac{g-3h+2}{6} \quad \text{by Lemma 2.8}
\]

\[
= -\frac{2g+6h-4}{6} = -\frac{-g+3h-2}{3}
\]

and

\[
\deg(L) = -\left(\frac{g-3h+2}{2} + \frac{2+\delta}{2}\right) = -\frac{g-3h+2}{2} - \frac{\delta}{2}
\]

\[
\leq -\frac{g-3h+2}{2} + \frac{h}{2} \quad \text{by Lemma 2.8}
\]

\[
= -\frac{-g+4h-2}{2}. \quad \square
\]

With all these preparations we now start to prove Lemma 2.1.

Proof of Lemma 2.1. We will only provide the proof of the case \(h = 2e\) because the odd genus case can be done in the same way.

Let \(\mathcal{N} = \mathcal{O}_X(\pi^*(D) + \Delta)\) be a general element of a component of \(\pi^*(W^1_{e+1}(C)) + W_{g-6e-4}(X)\), where \(\mathcal{O}_X(\Delta) \in W_{g-6e-4}(X)\) is general and \(\mathcal{O}_C(D) \in W^1_{e+1}(C)\). Since \(C\) is general, \(\mathcal{O}_C(D)\) is a base-point-free pencil by the Brill-Noether theorem for general curves. Hence we deduce that \(\mathcal{O}_X(\pi^*(D))\) is also base-point-free and the base locus of \(\mathcal{N}\) is just \(\Delta\) by the general choice of \(\Delta\). Since every component of \(\pi^*(W^1_{e+1}(C)) + W_{g-6e-4}(X)\) has the expected dimension, to show the reducedness at \(\mathcal{N}\) it is sufficient to show the injectivity of the Brill-Noether map \(\Gamma(X,\mathcal{N}) \otimes \Gamma(X,\omega_X \otimes \mathcal{N}^{-1}) \to \Gamma(X,\omega_X)\) by the description of the Zariski tangent space to
We will show that
\[
\dim_k(\Gamma(X, \omega_X \otimes \pi^*(\mathcal{O}_C(-D)^{\otimes 2}))) \leq g - 6e - 4,
\]
which will imply \(\Gamma(X, \omega_X \otimes N^{\otimes -2} \otimes \mathcal{O}_X(\Delta)) = 0\) by the general choice of \(\Delta\).

On the other hand, since \(C\) is a general curve \(\mathcal{O}_C(D)\) is a reduced point of \(W_{e+1}(C)\) (cf. [ACGH] p.214) and hence by the base-point-free pencil trick again, we have
\[
\dim_k(\Gamma(C, \mathcal{O}_C(D)^{\otimes 2})) = 3.
\]
Moreover we have the exact sequence
\[
0 \to \mathcal{M} \otimes \mathcal{O}_C(2D) \to \mathcal{E} \otimes \mathcal{O}_C(2D) \to \mathcal{L} \otimes \mathcal{O}_C(2D) \to 0
\]
and
\[
\deg(\mathcal{M}) \leq \frac{-g + 3h - 2}{3} \quad \text{and} \quad \deg(\mathcal{L}) \leq \frac{-g + 4h - 2}{2}
\]
by Proposition 2.7 and Proposition 2.9. Therefore we have
\[
\deg(\mathcal{M} \otimes \mathcal{O}_C(2D)) \leq \frac{-g + 6h + 4}{3} < 0
\]
and
\[
\deg(\mathcal{L} \otimes \mathcal{O}_C(2D)) \leq \frac{-g + 6h + 2}{2} < 0
\]
by the assumption on \(g\) and \(h\). Therefore we have \(\Gamma(C, \mathcal{E} \otimes \mathcal{O}_C(2D)) = 0\). This implies \(\dim_k(\Gamma(X, \pi^*(\mathcal{O}_C(2D)))) = 3\) and this finishes the proof of Lemma 2.1 by the Riemann-Roch theorem. □

Remark 2.10. (i) One may show the result of Lemma 2.1 (with a slightly weaker bound on \(g\)) in the following way. Here we sketch an alternative argument for \(h = 2e + 1\) only; the \(h = 2e\) case is similar. \(W_{e+2}(C)\) is reduced at a general point \(\mathcal{L}\), hence \(h^0(C, 2\mathcal{L}) = 4\). Suppose \(\pi^*\mathcal{L}\) is a non-reduced point on \(\pi^*(W_{e+2}(C)) = W_{3e+6}(X)\). Then \(h^0(X, 2\pi^*\mathcal{L}) \geq 5\) and since \(\pi^*|2\mathcal{L}| \subsetneq |2\pi^*\mathcal{L}|\), it follows that \(|2\pi^*\mathcal{L}|\) is not composed with \(\pi\). Thus \(X\) has a base-point-free \(g^1_x\) with \(x \leq 6e + 9\) not composed with \(\pi\) by subtracting 3 generically chosen points on \(X\). And this is contradictory to Lemma 1.2 if \(g \geq 18e + 21\). Hence \(W^1_{3e+6}(X)\) is reduced at a general point \(\pi^*\mathcal{L}\). Finally, one can assert that \(\pi^*\mathcal{L} \otimes \mathcal{O}_X(\Delta)\) is a reduced point of \(W^1_{g-3e-3}(X)\) where \(\mathcal{O}_X(\Delta) \in W_{g-6e-9}\) is general, by invoking a result of Coppens [C], Corollary 8.

(ii) On the other hand, one may recognize the importance of the method (i.e. utilizing the Rick Miranda’s triple covering result) adopted in the proof of Lemma 2.1 as follows:
forthcoming paper where the methods we used in Lemma 2.1 play some role in the paper.

(2) The reducedness result we proved in Lemma 2.1 has somewhat better bound (of the genus \(g\) of the curve \(X\) with respect to the genus \(h\) of the base curve \(C\)) than the one we outlined in (i) above; \(g \geq 12e + 14\) vs. \(g \geq 18e + 21\) for \(h = 2e + 1\) and \(g \geq 12e + 5\) vs. \(g \geq 18e + 6\) for \(h = 2e\).

3. Examples

In this section, we shall give examples of cyclic triple coverings of curves. One example suggests that there is a possibility of an improvement of our bounds given in Section 1. Another assures that the Castelnuovo-Severi bound is not sharp in some sense, i.e. the bound of degree of base-point-free pencils, which is not composed with the triple covering projection, is greater than the Castelnuovo-Severi bound. Both examples are given by cyclic triple coverings. We begin with the following remark about cyclic triple coverings.

**Remark 3.1.** Let \(C\) be a curve of genus \(h\) and \(X\) be a cyclic triple covering of \(C\). Let \(X\) be of genus \(g\) and let \(\sigma\) be the automorphism of \(X\) of order 3 so that \(X/\langle \sigma \rangle\) is equivalent to \(C\). Let \(\pi\) be the projection map of \(X\) to \(C\). Since the order of \(\sigma\) is prime, every fixed point of \(\sigma\) is a total ramification point of \(\pi\) and they are all. By the Riemann-Hurwitz formula, there are \(g - 3h + 2\) fixed points, \(P_1, \ldots, P_{g - 3h + 2} \in X\), of \(\sigma\).

Since the order of \(\sigma\) is 3, the space \(H = H^1(X, O)\) can be decomposed into the direct sum \(H = H_0 \oplus H_1 \oplus H_2\), where

\[
H_j = \{ \omega \in H : \omega \circ \sigma = \zeta^j \omega \}, \quad (j = 0, 1, 2), \quad \zeta = e^{2\pi i/3}.
\]

It is known that every \(\omega \in H_0\) is the pullback \(\pi^* \omega'\) of some \(\omega' \in H^1(C, O_C)\). So \(\dim H_0 = h\) and for \(\omega \in H_0\), the divisor \((\omega) = 2P_1 + \cdots + 2P_{g - 3h + 2} + \pi^*(Q_1 + \cdots + Q_{2h - 2})\) where \(Q_1 + \cdots + Q_{2h - 2}\) is a canonical divisor on \(C\) (cf. [L]).

For a fixed point \(P_i \in X\) of \(\sigma\), one can take a local parameter \(z_i\) at \(P_i\) such that \(\sigma : z_i \mapsto \zeta z_i\) or \(\sigma : z_i \mapsto \zeta^2 z_i\). The exponent of \(\zeta\) only depends on \(P_i\).

For \(\eta \in H_1\), let the Taylor expansion of \(\eta\) near \(P_i\) be given by

\[
\eta = (a_0 + a_1 z_i + a_2 z_i^2 + \cdots)dz_i.
\]

In case \(\sigma : z_i \mapsto \zeta z_i\),

\[
\eta \circ \sigma = (a_0 + a_1 \zeta z_i + a_2 \zeta^2 z_i^2 + \cdots)\zeta dz_i.
\]

By the definition of \(H_1\), \(\eta \circ \sigma = \zeta \eta\), hence we have \(a_j = 0\) if \(j \equiv 1, 2 \pmod{3}\). Hence, \(\eta\) has a zero of order 0 modulo 3 at \(P_i\).

In case \(\sigma : z_i \mapsto \zeta^2 z_i\), using a similar argument as above, one obtains that \(\eta\) has
where $D_1$ is an effective divisor on $C$ which depends only on $\eta$. Similarly, for every $\tau \in H_2$,

\begin{equation}
(\tau) = P_{t+1} + \cdots + P_{g-3h+2} + \pi^*(D_2),
\end{equation}

where $D_2$ is an effective divisor on $C$ which depends only on $\tau$.

Let $k_j = \deg D_j$. Since $\eta, \tau \in H$, $\deg \eta = \deg \tau = 2g - 2$. Hence, $t + 3k_1 = g - 3h + 2 - t + 3k_2 = 2g - 2$. Since $0 \leq t \leq g - 3h + 2$,

$$3k_j \geq 2g - 2 - (g - 3h + 2) = g + 3h - 4.$$ 

If $g + 3h - 4 \geq 6h - 5$, i.e. $k_j > 2h - 2$, then $h^0(C, D_j) = \dim H_j = k_j - h + 1$. We may also assume $t \geq g - 3h + 2 - t$, i.e. $2t \geq g - 3h + 2$. If it is not the case, we take $\zeta = e^{4\pi i/3}$ instead of $e^{2\pi i/3}$.

**Example 3.2.** Let $X, C, \pi, \sigma, t$ be as above. If $g \geq 3h - 1$, then there exists a complete base-point-free pencil $g^1_n$ which is not composed with $\pi$, for some integer $n \leq \max\{t, g + 2 - t\}$.

**Proof.** It is enough to show the existence of a linear series $g^1_{\max\{t, g+2-t\}}$ which is not composed with $\pi$.

Take an $\eta \in H_1$. Let the divisor of $\eta$ be

$$(\eta) = P_1 + \cdots + P_t + \pi^*(Q_1 + \cdots + Q_{k_1}),$$

$Q_1, \ldots, Q_{k_1} \in C$. By (3.1.2), for every $\tau \in H_2$, the divisor of $\tau$ is of the form

$P_{t+1} + \cdots + P_{g-3h+2} + \pi^*(D_2)$,

$\deg D_2 = k_2$ and $h^0(D_2) = k_2 - h + 1$.

In case $t \leq \frac{g+2}{2}$, we have $2t = g+2-3h+3k_2-3k_1 \leq g+2$, hence $k_2-h \leq k_1$. So we may assume that

$$D_2 = Q_1 + \cdots + Q_{k_2-h} + Q'_{k_2-h+1} + \cdots + Q'_{k_2},$$

for $Q'_{k_2-h+1}, \ldots, Q'_{k_2} \in C$. Then, $\frac{\eta}{\pi}$ is a meromorphic function on $X$ whose divisor is

$$P_1 + \cdots + P_t + \pi^*(Q_{k_2-h+1} + \cdots + Q_{k_1})$$

$$- (P_{t+1} + \cdots + P_{g-3h+2}) - \pi^*(Q'_{k_2-h+1} + \cdots + Q'_{k_2}).$$

Hence, the linear series $|P_1 + \cdots + P_t + \pi^*(Q_{k_2-h+1} + \cdots + Q_{k_1})|$ is of degree $t + 3(k_1 - k_2 + h) = g + 2 - t$ and is not composed with $\pi$.

In case $t > \frac{g+2}{2}$, we have $k_2 - h > k_1$. In this case, we take

$$D_2 = Q_1 + \cdots + Q_{k_1} + Q'_{k_1+1} + \cdots + Q'_{k_2},$$

Then, the divisor of $\frac{\eta}{\pi}$ is
Example 3.3. Let \( X, C, \pi, \sigma, t \) be as above. Assume that \( g \geq 3h - 1 \). If
\[ n < \frac{g - 3h + 2 + t}{3} \]
then every \( g^1 \) is composed with \( \pi \). Furthermore, if \( g \geq 7h - 4 \),
then for every \( C \) and every \( t \) such that \( \frac{g - 3h + 2}{2} \leq t \leq g - 3h + 2 \) and \( t \equiv 2g - 2 \mod 3 \), there exists a cyclic triple covering \( \tilde{X} \) of \( C \) on which there exists a complete
base point free linear series \( g^1 \) not composed with the triple covering.

Proof. Let \( f \) be a meromorphic function on \( X \) such that \( f \circ \sigma = \zeta f \), \( \deg(f) = N_1 \)
and the support of \( (f) \) does not contain any \( P_j \). Then \( f(P_j) = 0 \). For \( \omega \in H_0 \),
\( \omega' = f\omega \) is a 1-form which satisfies \( \omega' \circ \sigma = \zeta \omega' \). Hence, the order of zero of \( \omega' \) at
\( P_j \), denoted by \( \text{ord}_{P_j} \omega' \), is

\[
\text{ord}_{P_j} \omega' = \begin{cases} 1 & (j = 1, \ldots, t) \\ 0 & (j = t + 1, \ldots, g - 3h + 2) \end{cases} \mod 3.
\]

Hence,

\[
\text{ord}_{P_j} f = \begin{cases} 2 & (j = 1, \ldots, t) \\ 1 & (j = t + 1, \ldots, g - 3h + 2) \end{cases} \mod 3.
\]

So, \( N_1 = \deg(f) = \deg(f) \geq g - 3h + 2 + t \).

Applying a similar argument for a meromorphic function \( f \) on \( X \) such that
\( f \circ \sigma = \zeta^2 f \), \( \deg(f) = N_2 \) and the support of \( (f) \) does not contain any \( P_j \), we have \( N_2 \geq 2(g - 3h + 2) - t \).

Let \( f \) be a meromorphic function on \( X \) such that \( \deg(f) = m \) and \( f \circ \sigma \neq f \).
Considering a linear fraction, if necessary, we may assume neither the support of
\( (f) \) nor \( (f) \) contains any \( P_j \).

Let \( f_i = f + \zeta^2 f \circ \sigma + \zeta^i f \circ \sigma^2 \), \( (i = 0, 1, 2) \). Then \( f_i \circ \sigma = \zeta^i f_i \), \( (i = 0, 1, 2) \) and
\( 3f = f_0 + f_1 + f_2 \). Since \( f(P_j) \neq 0 \) and \( f_i(P_j) = 0 \) \( (i = 1, 2, j = 1, \ldots, g - 3h + 2) \), we have \( f_0(P_j) \neq 0 \). Further we may assume neither \( f_1 \) nor \( f_2 \) vanishes identically. If
it is not the case, say \( f_2 \) vanishes identically (denoted by \( f_2 \equiv 0 \)), one can consider
\( F = \frac{1}{f} \) instead of \( f \). Again, \( F \) can be decomposed as \( F = F_0 + F_1 + F_2 \) such that
\( F_i \circ \sigma = \zeta^i F_i \) \( (i = 0, 1, 2) \). Assume \( F_1 \equiv 0 \), then

\[
3 = 3fF = (f_0 + f_1)(F_0 + F_2) = (f_0 F_0 + f_1 F_2) + f_1 F_0 + f_0 F_2.
\]

Here, \( (f_0 F_0 + f_1 F_2) \circ \sigma = f_0 F_0 + f_1 F_2, f_1 F_0 \circ \sigma = \zeta f_1 F_0 \) and \( f_0 F_2 \circ \sigma = \zeta^2 f_0 F_2 \).
Hence, \( f_1 F_0 \equiv f_0 F_2 \equiv 0 \) and \( f_1 \equiv F_2 \equiv 0 \). A contradiction. In case \( F_2 \equiv 0 \), again
we have a contradiction.

By the definition of \( f_i \), we have \( (f_i) \leq (f) + (f \circ \sigma) + (f \circ \sigma^2) \) \( (i = 0, 1, 2) \).
Hence, \( \deg(f_i) \leq 3m \). This implies that

\[
3m \geq \max\{g - 3h + 2 + t, 2(g - 3h + 2) - t\}.
\]

\[ g - 3h + 2 + t \]
$P_{t+1}, \ldots, P_{g-3h+2}, Q_1, \ldots, Q_{\ell}$ on $C$. Since $g - 3h + 2 - t + 3\ell = t \geq \frac{g - 3h + 2}{2} \geq 2h - 1$, there is a $y \in K(C)$ such that

$$(y)_{\infty} = P_{t+1} + \cdots + P_{g-3h+2} + 3(Q_1 + \cdots + Q_{\ell}).$$

We may assume the support of the zero divisor of $y$ consists of distinct $t$ points $P_1, \ldots, P_t \in C$. Let $x$ be a triple valued function on $C$ satisfying $y = x^3$ and let $X$ be a curve so that $K(X) = K(C)(x)$. For $P \in C$, taking a local parameter $z_P$ at $P$, we have an automorphism $\sigma$ of $X$ such that $\sigma : (z_P, x(z_P)) \mapsto (z_P, \zeta x(z_P))$. Then $X/\langle \sigma \rangle$ is equivalent to $C$. Thus, $X$ is a cyclic triple covering of $C$ which is branched over $P_1, \ldots, P_{g-3h+2}$.

Since $x \in K(X)$ satisfies $x \circ \sigma = \zeta x$, we may divide $P_1, \ldots, P_{g-3h+2}$ into $P_1, \ldots, P_t$ and $P_{t+1}, \ldots, P_{g-3h+2}$ as desired. It is noted that in this case we have $g_1^1$ which is not composed with $\pi$. It is a better estimate in case $t < \frac{g+2}{2}$. □

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