Holomorphic extensions of representations: (II) geometry and harmonic analysis

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Introduction

Akhiezer and Gindikin [AG90] proposed the existence of a natural complexification of a Riemannian symmetric space of noncompact type, $G/K$. They sought a complex $G$-space with proper $G$ action and which contained $G/K$ as a totally real embedded submanifold. The obvious candidate for a complexification, $G_{\mathbb{C}}/K_{\mathbb{C}}$, fails to have the $G$ action proper, nevertheless in it they identified a candidate,

$$\Xi \subseteq G_{\mathbb{C}}/K_{\mathbb{C}}.$$ 

The open subset $\Xi$ inherits a complex structure from $G_{\mathbb{C}}/K_{\mathbb{C}}$, and it contains $G/K$ as a totally real submanifold via the natural inclusion $G/K \hookrightarrow G_{\mathbb{C}}/K_{\mathbb{C}}$.

Almost simultaneously a related development was taking place in complex differential geometry. In the 60’s much work had been done determining which geometric structures on a manifold $M$ prolong to similar geometric structures on $TM$. In the 90’s a significant refinement of Grauert’s embedding theorem was developed. A real analytic, compact manifold with a complete Riemannian metric, $(M, g)$, was shown to prolong to further structure. Namely, one obtains on an open neighborhood of the zero section in $TM$, a so-called Grauert domain, a unique complex structure having the property that the canonical lift of any geodesic to a map from $T\mathbb{R} \cong \mathbb{C} \to TM$ is holomorphic. Using the metric one can identify the cotangent bundle with the tangent bundle. If one combines this complex structure with the transported symplectic structure, one obtains a Kähler metric on the Grauert domain.

The Akhiezer-Gindikin domain can easily be shown to be a Grauert domain for the Riemannian symmetric space $G/K$. One of the first results in this paper, a consequence of Theorem 2.4, is that $\Xi$ is a $G$-invariant domain of holomorphy. The proof involves a major improvement of one of the main results from [KS04]. Briefly, there we identified a natural complexification for semisimple groups $G \subset G_{\mathbb{C}}$ as an open $G - K_{\mathbb{C}}$ double coset of the form $G \exp(i\Omega)K_{\mathbb{C}}$. The importance of this domain is that we showed that any $K$-finite matrix coefficient of an irreducible unitary representation of $G$ has holomorphic extension to it, and blows-up at specific points of the boundary of $\Omega$. Fortuitously, this construction and that of Akhiezer-Gindikin are compatible in that $G \exp(i\Omega)K_{\mathbb{C}}/K_{\mathbb{C}}$ and $\Xi$ are biholomorphic. Then to obtain the result, we show in Theorem 2.4 that for spherical unitary representations the holomorphic extension blows-up at all points of the boundary of $\Xi$.

Hence any symmetric space of noncompact type $X = G/K$ prolongs to a canonical complex manifold $\Xi$ contained in the tangent bundle $TX$, and $\Xi$ is a natural domain of holomorphy for

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harmonic analysis on $G/K$. One of the goals in the paper is to formulate some fundamental harmonic analysis on $G/K$ in terms of complex analysis on $\Xi \subset TX$. The other main goal of the paper is to give a detailed description of the complex differential geometry arising from several natural Kähler structures defined on $\Xi$ or natural subdomains of it. Besides the Kähler structure associated to the canonical symplectic structure on the cotangent bundle, there are two more surprising Kähler structures that we identified and whose associated metric holomorphically extends that of $G/K$.

For applications to harmonic analysis one needs the detailed structure of the Riemannian metric on $\Xi$, the Riemannian measure on $\Xi$, etc. This we do in the first part of the paper for the canonical Kähler structure on $\Xi$. In §2 we show that $\Xi$ has an ample supply of plurisubharmonic exhaustion functions, and from this obtain an easy proof that $\Xi$ is Stein. While one always has existence of the canonical Kähler structure on the Grauert domain, one seldom has an explicit formula for it. In §4 we describe in Lie theoretic terms the canonical Kähler structure on $\Xi$. Similarly for the associated Riemannian structure, we present an explicit expression for the Riemannian measure in terms of roots. For this Riemannian structure we identify the element in the universal enveloping algebra that induces the Laplace-Beltrami operator on $\Xi$. The results are consistently presented from the point of view of possible applications to harmonic analysis, i.e. using Lie structure theory.

The reformulation of harmonic analysis in complex analytic terms originated in [KS04] and was presented there in completeness for classical groups. In §1 this is extended to exceptional groups. The holomorphic extension was critical for the recent successes on the Gelfand-Gindikin program in [GKÔ03] and [GKÔ04]. Here, in a direction very different from that topic, we present new connections of representation theory with complex differential geometry. For example, we show in §3 that $\hat{G}_s$, the $K$-spherical irreducible unitary dual of $G$, embeds in a specific moduli space of Kähler structures on $\Xi$. While tempted to conjecture that the spherical unitary dual is parametrized by such Kähler metrics, the natural approach to this problem would require information about solutions with prescribed singularities to a Monge-Ampère equation on a noncompact manifold. Further applications to harmonic analysis are presented in §5 and §6. The use of Hilbert spaces of holomorphic functions is a familiar technique in the study of highest weight representations. In §5 we show that $K$-spherical unitary representations are similarly realizable by Hilbert spaces of holomorphic functions - a result new even for $SL(2, \mathbb{R})$. The heat kernel associated to the Riemannian Laplace operator is fundamental to harmonic analysis. In §6 we give a holomorphic extension of the heat kernel on $G/K$ to $\Xi$. One immediate application is to define a $G$-equivariant holomorphic heat kernel transform (i.e. Bargmann-Segal transform) for $G/K$.

Whereas there is only one complex structure on $\Xi$ that we should consider, there are several Kähler metrics whose associated Riemannian metric will extend the metric on $G/K$. These other Kähler metrics need not be defined on all of $\Xi$. In §7 we define a subdomain, denoted $\Xi_0$, that we show is bi-holomorphic to a Hermitian symmetric space but with the inherited metric not isometric to it. We also identify which domains have $\Xi = \Xi_0$. For these $\Xi$ one has then an unexpected hidden symmetry group not predictable from their structure theoretic definition, and one can impose on $\Xi$ a Hermitian symmetric metric which also extends the metric on $G/K$. That eigenfunctions of the invariant differential operators on $G/K$ and, more generally, quotients of $G/K$ by lattices in $G$ have holomorphic extension to a Hermitian symmetric space seems very intriguing. Finally, in §8 we describe and characterize a subdomain $\Xi_1$ of $\Xi_0$ and produce a canonical Kähler metric on it using Jordan algebra formulas. This metric too gives an extension of the Riemannian metric on $G/K$ but to a different subdomain of $\Xi$. What role this domain and metric has in harmonic analysis or automorphic functions is not at all clear to us.
Since our first lecture on our results in April 2000 at Oberwolfach, and since the original submission of this manuscript in February 2002, various individuals obtained, frequently with different methods, some results presented here. We cite the work of which we are aware at the appropriate point in this paper.

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§1 Holomorphic extensions

In [KS04] we gave a proof of the maximal holomorphic extension of the orbit map of a $K$-finite vector of an irreducible Hilbert representation of a connected classical semisimple Lie group. Conjecture A in [KS04] was the technical result needed to extend the proof to all connected semisimple Lie groups. Now from results in [Hu02] or [M03], see also a related result in [B03], this is known to be proved. For the readers convenience and because of its importance to the paper we sketch how the holomorphic extension follows from it.

Let $G$ be a connected reductive Lie group contained in a complexification $G_C$. We denote by $g$ (resp. $g_C$) the Lie algebra of $G$ (resp. $G_C$). Let $K$ be a maximal compact subgroup of $G$ and let $\mathfrak{k}$ be its Lie algebra. Attached to $\mathfrak{k}$ and the Killing form is the Cartan decomposition $g = \mathfrak{k} \oplus \mathfrak{p}$. Take $a \subseteq p$ a maximal abelian subspace and let $\Sigma = \Sigma(\mathfrak{g}, a) \subseteq a^*$ be the corresponding system of roots. The simultaneous eigenspaces of $\text{ad}(H), H \in a$, give the root space decomposition $g = a \oplus m \oplus \bigoplus_{\alpha \in \Sigma} g^\alpha$, here $m = \mathfrak{z}_K(a)$ and $g^\alpha = \{X \in g : (\forall H \in a)[H, X] = \alpha(H)X\}$. For each choice of positive roots, $\Sigma^+ \subseteq \Sigma$, one obtains a nilpotent Lie algebra $n = \bigoplus_{\alpha \in \Sigma^+} g^\alpha$ and an Iwasawa decomposition on the Lie algebra level $g = \mathfrak{t} \oplus a \oplus n$.

Denote by $A$ and $N$ the analytic subgroups of $G$ corresponding to $a$ and $n$. These choices give an Iwasawa decomposition for $G$; namely, the multiplication map

$$N \times A \times K \to G, \quad (n, a, k) \mapsto nak$$

is an analytic diffeomorphism. Thus, every element $g \in G$ can be written uniquely as $g = n(g)a(g)\kappa(g)$, with each of the maps $n(g) \in N$, $a(g) \in A$, $\kappa(g) \in K$ depending analytically on $g \in G$. The last piece of structure theory we shall recall is the Weyl group of $\Sigma(\mathfrak{g}, a)$, $\mathcal{W} = N_K(a)/Z_K(a)$.

With Akhiezer-Gindikin we define a domain in $G_C$ using the restricted roots. Set

$$\Omega = \{X \in a : (\forall \alpha \in \Sigma) |\alpha(X)| < \frac{\pi}{2}\}.$$

Then $\Omega$ is convex and $\mathcal{W}$-invariant. Set

$$T_\Omega = A \exp(i\Omega).$$

$T_\Omega$ is open in $A_C$ and diffeomorphic to $A \times \Omega$, which, in turn, is diffeomorphic to $a \oplus i\Omega$. In other words $T_\Omega$ is diffeomorphic to a tube domain in a complex vector space.

We showed in [KS04] that the domain $G \exp(i\Omega)K_C$ is an open $G$-$K_C$ double coset in $G_C$. For classical groups we also showed that $G \exp(i\Omega)K_C \subseteq N_C A_C K_C$, and that the Iwasawa projection, $a: G \to A$, extends holomorphically to a map $a: G \exp(i\Omega)K_C \to A_C$ such that
$x \in N_C a(x) K_C$. Now, as a consequence of the complex convexity theorem of Gindikin-Krötz (cf. [GK02b]), we have more precisely that

$$G \exp(i\Omega) K_C \subseteq N_C T_{\Omega} K_C,$$

giving a holomorphic extension of the Iwasawa projection $\alpha: G \exp(i\Omega) K_C \to A_C$ with values in $T_{\Omega}$. Exactly as in [KS04, Th. 3.1] one obtains

**Theorem 1.1.** Let $(\pi, \mathcal{H})$ be an admissible Hilbert representation of $G$. Then for any $K$-finite vector $v \in \mathcal{H}$, the orbit map

$$G \to \mathcal{H}, \ g \mapsto \pi(g)v$$

deextends to a $G$-equivariant holomorphic map on $G \exp(i\Omega) K_C$.

From Theorem 1.1 follows immediately

**Corollary 1.2.** Let $(\pi, \mathcal{H})$ be an admissible Hilbert representation of $G$. Then for all $K$-finite vectors $v \in \mathcal{H}$ and for all hyperfunction vectors $\eta \in \mathcal{H}^{\omega}$, the generalized matrix coefficient

$$\pi_v,\eta: G \to \mathbb{C}, \ g \mapsto \langle \pi(g)v, \eta \rangle = \overline{\eta(\pi(g)v)}$$

deextends to a holomorphic function on $G \exp(i\Omega) K_C$.

If one takes right $K_C$ cosets of $G \exp(i\Omega) K_C$, one obtains the domain

$$\Xi := G \exp(i\Omega) K_C / K_C,$$

introduced in [AG90] and dubbed by Gindikin the complex crown of the Riemannian symmetric space $G/K$.

$\Xi$, as an open subset of $G_C / K_C$, inherits the structure of a complex manifold. In this paper we will obtain several facts about the complex structure as well as geometric properties of this domain and domains related to it. Also, the relationship of harmonic analysis on $G/K$ and complex analysis on $\Xi$ will be investigated. The following is crucial to these developments.

Let $\mathbb{D}(G/K)$ denote the algebra of $G$-invariant differential operators on $G/K$. Specializing Corollary 1.2 to $K$-fixed vectors, and using the solution by [K-T78] to the Helgason conjecture, one obtains

**Proposition 1.3.** Every eigenfunction $\varphi$ on $G/K$ of the algebra of $G$-invariant differential operators $\mathbb{D}(G/K)$ extends holomorphically to $\Xi$.

§2 Plurisubharmonic functions on $\Xi$

From Proposition 1.3, for $(\pi_{\lambda}, \mathcal{H}_{\lambda})$ a unitary spherical principal series representation with parameter $\lambda \in a_C^*$ and with non-zero $K$-fixed vector $v_{0}$, it follows that the spherical function $\varphi_{\lambda}$ extends to a holomorphic function on $\Xi$. In [KS04] we showed that this holomorphic extension blows-up in specific directions at the boundary. In this section we shall show how the holomorphic extension of the spherical function $\varphi_{\lambda}$ provides $\Xi$ blows-up at all points of the boundary and provides $\Xi$ with a supply of strictly pluriharmonic exhaustion functions.

For $Z \in g_C$ we define a vector field $\tilde{Z}$ on $\Xi$ by

$$(\tilde{Z} f)(x) = \frac{d}{dt} \bigg|_{t=0} f(\exp(tZ)x),$$

here $f$ is a function on $\Xi$ differentiable at $x$. Let $J$ denote the linear operator on $g_C$ corresponding to multiplication by $i$ on $g_C$. To $Z$ one can associate the familiar Cauchy-Riemann operators

$$\partial_Z := \frac{i}{2}(\tilde{Z} - iJZ) \quad \text{and} \quad \overline{\partial_Z} := \frac{i}{2}(\tilde{Z} + iJZ).$$

Our first observation relating the unitary dual to complex analysis on $\Xi$ is given in the following result.
Proposition 2.1. Let $(\pi, \mathcal{H})$ be a non-trivial $K$-spherical unitary irreducible representation of $G$. Let $v_0$ be a normalized $K$-spherical vector. Then the functions
\[ s_\pi: \Xi \to \mathbb{R}, \quad xK_C \mapsto \|\pi(x)v_0\|^2, \]
and
\[ \log(s_\pi): \Xi \to \mathbb{R}, \quad xK_C \mapsto 2 \log \|\pi(x)v_0\| \]
are $G$-invariant and strictly plurisubharmonic.

Proof. Before we start the proof, let us recall some standard facts on plurisubharmonic functions. Let $M$ be a complex manifold, $\mathcal{H}$ a Hilbert space and $f: M \to \mathcal{H}$ a holomorphic map. Then both
\[ z \mapsto \|f(z)\|^2, \quad z \mapsto \log \|f(z)\| \]
are plurisubharmonic functions on $M$. The first one is strictly plurisubharmonic if and only if $f$ is an immersion. The second map is strictly plurisubharmonic if and only if for all $m \in M$ the vector $df(m)(Z)$ is transversal to $f(m)$ for all $Z \neq 0$ in a fixed totally real subspace of $T_mM$.

From Theorem 1.1 the orbit map from $G$ to $\mathcal{H}$, $g \mapsto \pi(g)v_0$, admits a holomorphic extension to a $G$-equivariant map from $G \exp(i\Omega)K_C$ to $\mathcal{H}$, $x \mapsto \pi(x)v_0$. Clearly this map is right $K_C$-invariant, so factors to a holomorphic $G$-equivariant map
\[ f: \Xi \to \mathcal{H}, \quad xK_C \mapsto \pi(x)v_0. \]
The $G$-invariance of $s_\pi$ and $\log(s_\pi)$ follows from the unitarity of $\pi$.

In view of our comments at the beginning, the proof of the proposition will be complete once we have established the transversality of $f$. For that, fix $m = xK_C \in \Xi$ and identify the tangent space $T_m\Xi$ with $\text{Ad}(x)p_C$. Then $\text{Ad}(x)p$ becomes a totally real subspace of $T_m\Xi$. We compute $df(m)(Z) = d\pi(Z)\pi(x)v_0$. For $Z = \text{Ad}(x)Y$ with $Y \in p$, we thus get $df(m)(Z) = \pi(x)d\pi(Y)v_0$. Assume that $f$ is not transversal. Then for some $Y \in p$, $Y \neq 0$, we would have $\pi(x)d\pi(Y)v_0 \in \mathbb{C}\pi(x)v_0$ or, equivalently, $d\pi(Y)v_0 \in \mathbb{C}v_0$. Now $\mathfrak{l} := \{X \in \mathfrak{g}: d\pi(X)v_0 \in \mathbb{C}v_0\}$ is a subalgebra of $\mathfrak{g}$ and it contains $\mathfrak{t}$ and $Y \in p$. But then $\mathfrak{l} = \mathfrak{g}$, and so $\pi$ is the trivial representation - a contradiction. So $f$ is transversal.

We continue with some properties of the function $\log(s_\pi)$.

Proposition 2.2. Let $(\pi, \mathcal{H})$ be a non-trivial $K$-spherical unitary irreducible representation of $G$. Then the function
\begin{itemize}
\item[(i)] $\log(s_\pi): \Xi \to \mathbb{R}$ is $G$-invariant and strictly plurisubharmonic;
\item[(ii)] $\log(s_\pi) \geq 0$;
\item[(iii)] $G/K = \{z \in \Xi: \log(s_\pi)(z) = 0\}$.
\end{itemize}

Proof. (i) was already established in Proposition 2.1.

Moving on to (ii) and (iii). As $\log(s_\pi)$ is $G$-invariant it is specified by its values on $T_0\Omega$. But by $G$-invariance and $\mathcal{W}$-invariance of $\log(s_\pi)$ we may consider $\log(s_\pi)$ as an $\{A, \mathcal{W}\}$-invariant function on the abelian tube domain $T_0\Omega = A\exp(i\Omega)$. Then $\psi_\pi = \log(s_\pi) \circ \exp|\Omega$ defines a strictly convex $\mathcal{W}$-invariant function on $\Omega$. Clearly, (ii) and (iii) will be proved if we can show that $\psi_\pi$ has a unique absolute minimum at 0. Let us first show that 0 is a minimum of $\psi_\pi$. Since $\psi_\pi = \psi_\pi \circ w$ for all $w \in \mathcal{W}$, and $w(0) = 0$, we have
\[ d\psi_\pi(0) = d\psi_\pi(0) \circ w \]
for all $w \in \mathcal{W}$. Hence $d\psi_\pi(0) = 0$ and 0 is a minimum, as $\psi_\pi$ is strictly convex. To conclude the proof it suffices to show that 0 is the only minimum of $\psi_\pi$. If not, then there exists an $X_0 \in \Omega$, $X_0 \neq 0$ such that $\psi_\pi$ has a local minimum at $X_0$, say $\psi_\pi(X_0) = m_0$. As $X_0 \neq 0$ we find a $w \in \mathcal{W}$ such that $X_1 = w(X_0) \neq X_0$. Since $\psi_\pi(X_1) = \psi_\pi(X_0) = m_0$, restricting $\psi_\pi$ to the line segment connecting $X_0$ and $X_1$ we obtain a strictly convex function with local minima at the endpoints - impossible. Hence there is no minimum other than 0.
We denote the topological boundary of $\Xi$ in $G_C/K_C$ by $\partial \Xi$. We wish to investigate the boundary behaviour of the plurisubharmonic functions $s_\pi$ and $\log(s_\pi)$. We start with a geometrical fact.

**Lemma 2.3.**

(i) $G \exp(i\partial \Omega)K_C/K_C \subseteq \partial \Xi$.

(ii) Let $\{x_n = g_n \exp(iX_n)K_C \mid g_n \in G, X_n \in \Omega\}$ be a sequence in $\Xi$ with $x_n \to x \in \partial \Xi$. Then $X_n \to X \in \partial \Omega$.

**Proof.**

(i) This is proved in [AG90, p. 8-9].

(ii) Let $\theta$ also denote the complex linear extension to $G_C$ of the Cartan involution on $G$. Recall from (1.1) that $G \exp(i\Omega) \subseteq N_C A_C K_C$. Then $g_n \exp(iX_n) \in N_C A_C K_C$ and

$$y_n = g_n \exp(i2X_n) \theta(g_n)^{-1} \in N_C A_C N_C,$$

where, as usual, $N_C = \theta(N_C)$. Write $y \in G_C$ for the limit of $(y_n)_{n \in \mathbb{N}}$.

Suppose that there is a subsequence of $(X_n)_{n \in \mathbb{N}}$ which converges to an element in $\Omega$. Replacing the original sequence by the corresponding subsequence, we may assume that $X_n \to X \in \Omega$. Write $g_n = b_n k_n$ for $b_n \in AN$ and $k_n \in K$. As $K$ is compact, we may assume that $k_n \to k \in K$. Set $z_n = k_n \exp(2X_n)k_n^{-1}$. Then $\{z_n\}$ converges, say to $z = k \exp(2X)k^{-1}$, and it follows from (1.1) that $z \in N_C A_C N_C$. If $(b_n)_{n \in \mathbb{N}}$ were unbounded, then from [KS04, Lemma 1.6] we would have $(y_n)_{n \in \mathbb{N}}$ an unbounded sequence in $G_C$ - a contradiction, as $y_n \to y$. Hence $(b_n)_{n \in \mathbb{N}}$ is bounded. Thus we may assume that $(g_n)_{n \in \mathbb{N}}$ converges to some element $g \in G$. But then $x$ must be in $\Xi$, contradicting $x \in \partial \Xi$. \[\square\]

We come now to the main result in this section.

**Theorem 2.4.** Let $(\pi, \mathcal{H})$ be a non-trivial $K$-spherical unitary irreducible representation of $G$. Then there exists a constant $C_\pi > 0$ such that

$$(2.1) \quad (\forall X \in \partial \Omega) \quad s_\pi(\exp(i(1 - \varepsilon)X)K_C) \geq C_\pi |\log \varepsilon|$$

for all $0 < \varepsilon \leq 1$. In particular, one has

$$\lim_{\varepsilon \to 0^+ \atop \varepsilon \in \mathbb{R}} s_\pi(x) = \infty.$$

**Proof.** The last assertion of the theorem follows from (2.1) and Lemma 2.3(ii). So we are left with proving (2.1).

Let $X \in \partial \Omega$. It follows from the definition of $\Omega$ that there exists an $\alpha \in \Sigma$ such that $\alpha(X) = \frac{\pi}{2}$. Write $H_\alpha \in a$ for the co-root of $\alpha$. Take $X_\alpha \in g^\alpha$ such that $[X_\alpha, \theta X_\alpha] = -H_\alpha$. Then

$$g_0 = \text{span}_\mathbb{R}\{H_\alpha, X_\alpha, \theta X_\alpha\}$$

is a $\theta$-stable subalgebra of $g$ isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Let $a_\alpha = \alpha^\perp$ be the kernel of $\alpha$. Notice that

$$g_1 : = g_0 + a_\alpha$$

is in fact a Lie algebra direct sum. Denote by $G_1$, $G_0$ and $A_\alpha$ the analytic subgroups of $G$ with Lie algebras $g_1$, $g_0$ and $a_\alpha$. One has

$$G_1 = G_0 \times A_\alpha.$$
Now if $\frac{2}{\pi}H_\alpha \in \partial \Omega$, then a mild generalization of Proposition 4.5 in [KS04] (simply replace $G_0$ by $G_1$ in the proof) gives

$$s_\pi(\exp(i(1 - \varepsilon)X)) \geq C_\alpha |\log \varepsilon|,$$

for all $0 < \varepsilon \leq 1$ and a constant $C_\alpha > 0$ depending only on $\alpha$. Thus it is enough to show:

(2.2) Among all $\alpha \in \Sigma$ with $\alpha(X) = \frac{2}{\pi}$ there is one with $\frac{2}{\pi}H_\alpha \in \partial \Omega$.

We will verify (2.2) by considering the different types of root systems. Of course we may assume that $\mathfrak{g}$ is simple. (2.2) would be a consequence of the following statement (2.3), however, (2.3) is not valid for all root systems.

(2.3) For all $\alpha \in \Sigma$, $\frac{2}{\pi}H_\alpha \in \partial \Omega$.

We proceed to consider the various root systems.

Case 1: $\Sigma$ is of type $A_n$, $C_n$ or $D_n$.
In all these cases (2.3) holds (cf. [KS04, Sect. 4]), and thus so does (2.2).

Case 2: $\Sigma$ is of type $BC_n$.
This can be reduced to the $C_n$ case since any root $\alpha \in \Sigma(BC_n)$ with $\alpha(X) = \frac{2}{\pi}$ must lie in the natural $\Sigma(C_n) \subseteq \Sigma(BC_n)$.

Case 3: $\Sigma$ is of type $B_n$.
Recall that the roots of $B_n$, in the standard notation, are

$$\Sigma(B_n) = \{\pm \varepsilon_i \pm \varepsilon_j : 1 \leq i \neq j \leq n\} \cup \{\pm \varepsilon_i : 1 \leq i \leq n\},$$

and for a basis of $\Sigma(B_n)$ we may take

$$\Pi = \{\alpha_1, \ldots, \alpha_n\} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}.$$

Define $e_j \in a$ by $\varepsilon_j(e_i) = \delta_{ij}$. Take $X \in \partial \Omega$. By the Weyl group invariance of (2.1) we may assume that $\alpha_i(X) = \frac{2}{\pi}$ for some $1 \leq i \leq n$. Now $\frac{2}{\pi}H_{\alpha_i} \in \partial \Omega$ for $1 \leq i \leq n - 1$, while $\frac{2}{\pi}H_\alpha \notin \partial \Omega$. Thus we have to consider only the case where $\alpha_n(X) = \frac{2}{\pi}$. But this determines $X$ uniquely to be $X = \frac{2}{\pi}e_n$ (see the proof of [KS04, Lemma 2.8]). Hence $-\alpha_{n-1}(X) = \frac{2}{\pi}$, and (2.2) is established in this case.

Case 4: $\Sigma$ is of type $E_6$, $E_7$ or $E_8$.
We will show that (2.3) holds for all these cases. We have to consider only the case of $\Sigma(E_8)$ since $\Sigma(E_6) \subseteq \Sigma(E_7) \subseteq \Sigma(E_8)$. Now the roots of $E_8$ are

$$\Sigma(E_8) = \{\pm \varepsilon_i \pm \varepsilon_j : 1 \leq i \neq j \leq n\} \cup \{\frac{1}{2}\sum_{j=1}^{8}(-1)^{n(j)}\varepsilon_j : \sum_{j=1}^{n}n(j) \text{ even}\},$$

and for a basis of $\Sigma(E_8)$ we may take

$$\Pi = \{\alpha_1, \ldots, \alpha_8\} = \{\frac{1}{2}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 - \varepsilon_5 - \varepsilon_4 - \varepsilon_3 - \varepsilon_2 + \varepsilon_1),$$

$$\varepsilon_2 + \varepsilon_1, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \varepsilon_4 - \varepsilon_3, \varepsilon_5 - \varepsilon_4, \varepsilon_6 - \varepsilon_5, \varepsilon_7 - \varepsilon_6\}.$$ 

In order to establish (2.3) it is enough to show that $\frac{2}{\pi}H_{\alpha_i}$ is in $\partial \Omega$ for all $1 \leq i \leq 8$. Now $\alpha_2, \ldots, \alpha_7$ span a root subsystem $\Sigma_0$ of $\Sigma(E_8)$ of type $D_7$, and (2.3) holds for such a system. From this it is easy to see that $\frac{2}{\pi}H_{\alpha_i} \in \partial \Omega$ for all $\alpha \in \Sigma_0$. It remains to consider

$$H_{\alpha_1} = \frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1).$$
Here one readily checks that $\frac{2}{4} H_{\alpha_1} \in \partial \Omega$.

Case 5: $\Sigma$ is of type $G_2$.

With

$$a = \{\sum_{j=1}^{3} x_j e_j : x_1 + x_2 + x_3 = 0\},$$

we have

$$\Sigma(G_2) = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_1 - \varepsilon_3) \pm (\varepsilon_2 - \varepsilon_3)\} \cup \{\pm(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3), \pm(2\varepsilon_2 - \varepsilon_1 - \varepsilon_3) \pm (2\varepsilon_3 - \varepsilon_1 - \varepsilon_2)\}.$$  

For a basis of $\Sigma(G_2)$ we take

$$\Pi = \{\alpha_1, \alpha_2\} = \{\varepsilon_1 - \varepsilon_2, -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}.$$  

Let $X \in \partial \Omega$. Arguing as in Case 3 we may assume that $\alpha_1(X) = \frac{\pi}{2}$ or $\alpha_2(X) = \frac{\pi}{2}$. First we show that $\alpha_1(X) = \frac{\pi}{2}$ is impossible. Write $X = \sum_{j=1}^{3} x_j e_j$ with $\sum_{j=1}^{3} x_j = 0$. Now if $\alpha_1(X) = \frac{\pi}{2}$, we have $X = \frac{\pi}{4}(e_1 - e_2)$ - but then $X \in \partial \Omega$. This is not possible since

$$(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3)(X) > \frac{\pi}{2}.$$  

Suppose $\alpha_2(X) = \frac{\pi}{2}$. From the explicit description of $\Sigma(G_2)$ we see that

$$\frac{\pi}{4} H_{\alpha_2} = \frac{\pi}{4} \frac{1}{2}(-2e_1 + e_2 + e_3) \in \partial \Omega,$$

thus establishing (2.2) for $\Sigma(G_2)$.

Case 6: $\Sigma$ is of type $F_4$.

Here we have

$$\Sigma(F_4) = \{\pm \varepsilon_i : 1 \leq i \leq 4\} \cup \{\pm \varepsilon_i : 1 \leq i \leq 4\} \cup \{\frac{1}{2}(\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\},$$

and for a basis we take

$$\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{\frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4), \varepsilon_4, \varepsilon_3 - \varepsilon_4, \varepsilon_2 - \varepsilon_3\}.$$  

As explained in Case 3 we may assume that $\alpha_i(X) = \frac{\pi}{2}$ for some $1 \leq i \leq 4$. Now it is easy to check that $\frac{2}{4} H_{\alpha_j} \in \partial \Omega$ for $j = 1, 3, 4$ but $\frac{2}{4} H_{\alpha_2} \notin \partial \Omega$. Thus the crucial case is when $\alpha_2(X) = \frac{\pi}{2}$. But this determines $X$ uniquely, namely $X = \frac{\pi}{2} e_4$. Then $-\alpha_3(X) = \frac{\pi}{2}$ and (2.2) is proved for the case of $\Sigma(F_4)$. $\blacksquare$

Recall that $\Xi$ is open in $G_C/K_C$, so it inherits a complex structure. For $\pi_\lambda$ a $K$-spherical representation let $\varphi_\lambda(gK) = \langle \pi_\lambda(g) v_0, v_0 \rangle$ ($g \in G$) be the associated spherical function. In [KS04, §4] we showed that this function extends holomorphically to $\Xi$ and its restriction to $A$ extends holomorphically to $T(2\Omega)$. Denote these holomorphic extensions also by $\varphi_\lambda$. The relation with $s_{\pi_\lambda}$ is $(\forall a \in \exp(\Omega))$ $\varphi_\pi(a^2) = s_\pi(a)$. Then from Theorem 2.4 it follows that $\Xi$ is a $G$-invariant domain of holomorphy. See also [F03] and [B03].

Later we explain why this complex structure is the adapted complex structure arising from the Riemannian structure on $G/K$, but for now we are concerned with properties related to the homogeneous nature of $\Xi$. One knows that $G$ acts properly on $\Xi$ ([AG90, Prop. 3]). So if $\Gamma$ is a torsion-free discrete subgroup of $G$, the quotient $\Gamma \backslash \Xi$ is a complex manifold.

As an immediate consequence of Theorem 2.4 we have
Proposition 2.5. Let $\Gamma < G$ be a torsion free co-compact subgroup. Then the complex manifold $\Gamma \backslash \Xi$ is Stein.

Proof. Let $s_\pi: \Xi \rightarrow \mathbb{R}$ be the function from Proposition 2.1. Since $s_\pi$ is left $G$-invariant, this function factors to a strictly plurisubharmonic function on $\Gamma \backslash \Xi$ denoted $s^\Gamma_\pi$. In order to show that $\Gamma \backslash \Xi$ is Stein, it suffices to show that $s^\Gamma_\pi$ is proper. For that let $(x_n)_{n \in \mathbb{N}}$ be a sequence which tends to infinity in $\Gamma \backslash \Xi$. Then Theorem 2.4 implies

$$\lim_{n \to \infty} s^\Gamma_\pi(x_n) = \lim_{n \to \infty} s_\pi(\exp(iY_n)) = \infty.$$ 

Proposition 2.6. $\Xi$, with its inherited complex structure from $G_\mathbb{C}/K_\mathbb{C}$, is Stein.

Proof. We only have to construct a strictly plurisubharmonic exhaustion function on $\Xi$. Since $G_\mathbb{C}/K_\mathbb{C}$ is Stein, there exists a plurisubharmonic exhaustion function on $G_\mathbb{C}/K_\mathbb{C}$, $\psi: G_\mathbb{C}/K_\mathbb{C} \rightarrow \mathbb{R}^+$. Now define a function $\varphi: \Xi \rightarrow \mathbb{R}^+$ by

$$\varphi = \psi|_\Xi + s_\pi,$$

where $s_\pi$ is a function as in Proposition 2.1. As $\varphi$ is a sum of two strictly plurisubharmonic functions, $\varphi$ is strictly plurisubharmonic. It remains to show that $\varphi$ is proper. Suppose that $x_n \rightarrow \infty$. That $x_n \rightarrow \infty$ means either $x_n \rightarrow x \in \partial \Xi$ or $x_n \rightarrow \infty$ in $G_\mathbb{C}/K_\mathbb{C}$. As both $\psi$ and $s_\pi$ are positive, in the first case it follows from (2.1) that $\lim_{n \to \infty} \varphi(x_n) = \infty$, while in the latter case it follows from the fact that $\psi$ is an exhaustion function.

Remark 2.7. The result presented in Proposition 2.6 has been approached from several directions. We refer the reader to [AG90], [B03], [BHH03], [GK02b], [GM01] and [Hu02].

§3 A moduli space based on $\Xi$

We shall denote by $\hat{G}_s$ the $K$-spherical unitary dual of $G$. In this section we show that $\hat{G}_s$ provides the complex manifold $\Xi$ with a space of Kähler potentials.

For $\pi \in \hat{G}_s - \{1\}$ we write $g^\pi$ for the Riemannian metric associated to the $G$-invariant strictly plurisubharmonic function $\log(s_\pi)$. Explicitly it is given by

$$g^\pi_z(Z,\overline{Y}) = \text{Re} \left( \partial_Z \overline{\partial_Y} \log(s_\pi) \right)(z) \quad (z \in \Xi, Z, Y \in g_\mathbb{C}).$$

Below we will show that $g^\pi$ is complete. For that it is useful to recall some facts on the infinite dimensional projective space.

Let $\mathcal{H}$ denote a complex Hilbert space and let $\mathbb{P}(\mathcal{H})$ be its associated projective space. For $v \in \mathcal{H} \setminus \{0\}$ we write $[v] = \mathbb{C}v$ for the corresponding line through $v$. Then there is the projection mapping

$$p: \mathcal{H} \setminus \{0\} \rightarrow \mathbb{P}(\mathcal{H}), \quad v \mapsto [v]$$

and we equip $\mathbb{P}(\mathcal{H})$ with the quotient topology of $p$. The projective space $\mathbb{P}(\mathcal{H})$ becomes a Hilbert manifold as follows. For $v \in \mathcal{H}$ a unit vector we define an open neighborhood of $[v]$ in $\mathbb{P}(\mathcal{H})$ by

$$U_v = \{[w] \in \mathbb{P}(\mathcal{H}); \langle v, w \rangle \neq 0\}.$$

Then the mapping
ψ|v⟩: U|v⟩ → v⊥, [w] → \frac{u}{(w,v)} − v

is a homeomorphism with inverse

ψ⁻¹|v⟩: v⊥ → U|v⟩, u → [v + u].

A Riemannian metric on $\mathcal{H}\setminus\{0\}$ is defined by

$$g_{v}(u, w) = \text{Re}(\frac{u, w}{\|v\|^2}) \quad (v \in \mathcal{H}\setminus\{0\})(u, w \in T_u(\mathcal{H}\setminus\{0\}) = \mathcal{H}).$$

This metric is invariant under multiplication by $\mathbb{C}^*$ hence pushes down to a metric on $\mathbb{P}(\mathcal{H})$. One obtains the familiar Fubiny-Study metric on $\mathbb{P}(\mathcal{H})$ which we shall denote by $g^{FS}$. Explicitly it is given by

$$g_{[v]}^{FS} \left( \frac{d}{dt} \big|_{t=0} [v + tv], \frac{d}{dt} \big|_{t=0} [v + tv] \right) = \frac{\|v\|^2(u, u) − |(u, v)|^2}{\|v\|^4}.$$  

We note that $(\mathbb{P}(\mathcal{H}), g^{FS})$ is a metrically and geodesically complete space.

Associated to $\pi \in \hat{G}_s - \{1\}$ we consider the mapping

$$F_{\pi}: \Xi \to \mathbb{P}(\mathcal{H}), \ xK_C \mapsto [\pi(x)v_0].$$

This mapping is holomorphic as the orbit mapping $\Xi \to \mathcal{H}, \ xK_C \mapsto \pi(x)v_0$ is holomorphic. Then we have the relation

$$g^\pi = 2F_{\pi}^*g^{FS}$$  

which is a consequence of formula (3.1) and the standard computation:

$$(\partial_Z \overline{\partial}_Z \log(s_{\pi}))(xK_C) = \partial_Z \left( \frac{2 \langle \pi(x)v_0, d\pi(Z)\pi(x)v_0 \rangle}{\|\pi(x)v_0\|^2} \right) = 2 \langle \pi(x)v_0, \pi(x)v_0 \rangle \frac{d\pi(Z)(\pi(x)v_0, d\pi(Z)\pi(x)v_0) − |(d\pi(Z)\pi(x)v_0, \pi(x)v_0)|^2}{\|\pi(x)v_0\|^4}.$$  

**Proposition 3.1.** Let $\pi \in \hat{G}_s - \{1\}$. Then the map $F_{\pi}: \Xi \to \mathbb{P}(\mathcal{H}), \ xK_C \mapsto [\pi(x)v_0]$ is proper. To be more precise, for every chart $\psi|v⟩: U|v⟩ → v⊥$ and every bounded closed ball $B \subseteq v⊥$ the subset $F_{\pi}^{-1}(\psi|v⟩^{-1}(B))$ is compact in $\Xi$.

**Proof.** Fix a unit vector $v \in \mathcal{H}$ and a closed ball of finite radius $B \subseteq v⊥$. Then for all $xK_C \in F_{\pi}^{-1}(\psi|v⟩^{-1}(B))$ we have

$$\psi|v⟩(F_{\pi}(xK_C)) = \frac{\pi(x)v_0}{\langle \pi(x)v_0, v \rangle} − v.$$  

To obtain a contradiction let us assume that there exists a sequence $x_nK_C$ in $F_{\pi}^{-1}(\psi|v⟩^{-1}(B))$ such that $x_nK_C → \infty$. We can write $x_n = g_n\exp(iX_n)$ for $g_n \in G$ and $X_n \in \Omega$. As $\Omega$ is relatively compact, we may assume that $X_n → X \in \Omega$. From $x_nK_C → \infty$ we either deduce $X \in \partial\Omega$ or $g_n → \infty$. If $X \in \partial\Omega$, then Theorem 2.4 together with (3.3) shows that $\psi|v⟩(F_{\pi}(x_nK_C))$ becomes unbounded. Hence $X_n → X \in \Omega$ and so $g_n → \infty$. Thus $\pi(\exp(iX_n)v_0) \in B$ lies in a compact subset $Q \subseteq \mathcal{H}$. As $g_n → \infty$, the Howe-Moore theorem on vanishing of matrix coefficients implies

$$\lim_{n → \infty} |\langle \pi(x_n)v_0, v \rangle| ≤ \lim_{n → \infty} \sup_{w \in Q} |\langle \pi(g_n^{-1})v, w \rangle| = 0.$$  

In view of (3.3), this shows that $\psi|v⟩(F_{\pi}(x_nK_C))$ becomes unbounded. Again a contradiction, completing the proof.  

\[\Box\]
Corollary 3.2. Let \( \pi \in \hat{G}_s - \{1\} \). Then \( \text{Im} F_\pi \) is a closed submanifold of \( \mathbb{P}(\mathcal{H}) \). In particular, the \( G \)-invariant Riemannian metric \( g^\pi \) associated to \( \log(s_\pi) \) is complete.

**Proof.** It follows from Proposition 3.1 that \( \text{im} F_\pi \) is a closed submanifold of the complete Riemannian manifold \( (\mathbb{P}(\mathcal{H}), g^{FS}) \). Now the assertion follows from (3.2).

Our discussion in this section leads to a formulation in complex-analytic terms of the non-trivial \( K \)-spherical unitary dual of \( G \). We hope to expand on this result in a subsequent publication.

Theorem 3.3. The map \( \pi \mapsto \omega_\pi = \frac{i}{2} \partial \bar{\partial} \log(s_\pi) \) identifies \( \hat{G}_s - \{1\} \) with positive Kähler forms on \( \Xi \) whose associated Riemannian metric is complete.

**Proof.** The geometric properties of \( \omega_\pi \) have been verified in Proposition 2.1 and Corollary 3.2. It remains to show that the assignment \( \pi \mapsto \omega_\pi \) is injective. Let \( \pi, \pi' \in \hat{G}_s - \{1\} \) such that \( \omega_\pi = \omega_{\pi'} \). We recall that \( \Xi \) is contractible (cf. Remark 4.5(b) below). Using Theorem 2.4 we showed in Proposition 2.6 that \( \Xi \) is Stein. Thus the potentials \( \log s_\pi \) and \( \log s_{\pi'} \) for \( \omega_\pi \) and \( \omega_{\pi'} \) differ by the real part of a holomorphic function:

\[
(3.4) \quad \log s_\pi - \log s_{\pi'} = f
\]

where \( f = \text{Re} F \) for some \( F \in \mathcal{O}(\Xi) \). Notice that \( f \) is \( G \)-invariant as the left hand side of (3.4) is. In the sequel we identify \( T(\Omega) \) with its image in \( \Xi \). Set \( H = F \mid _{T(\Omega)} \) and \( h = \text{Re} H \). As \( h \) is \( A \)-invariant pluriharmonic function we obtain:

\[
H(\exp(Z)) = c + i \lambda(Z) \quad (Z \in \mathfrak{a} + i \Omega),
\]

where \( \lambda \in \mathfrak{a}^* \) and \( c \in \mathbb{C} \) is a constant. As \( f \) is \( G \)-invariant, it follows in addition that \( h \) is \( \mathcal{W} \)-invariant. Thus \( \lambda = 0 \) and so \( H = c \) is constant. It follows that \( F = c \) is constant by the \( G \)-invariance of \( f \).

We conclude from (3.4) that \( s_\pi = C s_{\pi'} \) for \( C = e^{\text{Re} c} > 0 \). Actually we have \( C = 1 \) as \( s_\pi(K_C) = s_{\pi'}(K_C) = 1 \). Thus

\[
(3.5) \quad s_\pi = s_{\pi'}.
\]

From (3.5) one can show that \( \pi = \pi' \) as follows. For \( \pi \in \hat{G}_s \) denote by

\[
\varphi_\pi(gK) = \langle \pi(g)v_0, v_0 \rangle \quad (g \in G)
\]

the associated spherical function. This function extends holomorphically to \( \Xi \) and its restriction to \( A \) extends holomorphically to \( T(2\Omega) \) (cf. [KS04, §4]). Denote these holomorphic extensions also by \( \varphi_\pi \). The relation with \( s_\pi \) is

\[
(\forall a \in \exp(i\Omega)) \quad \varphi_\pi(a^2) = s_\pi(a).
\]

Thus (3.5) implies \( \varphi_\pi = \varphi_{\pi'} \) and so \( \pi = \pi' \).

§4 Geometric analysis on \( \Xi \)

In this section we will introduce a canonical \( G \)-invariant Kähler structure on \( \Xi \). Lie theoretically, the Kähler structure is determined by the choice of Cartan-Killing form on \( \mathfrak{g} \), the Levi-Civita connection on \( G/K \) and their natural extensions to \( \mathfrak{g}_C \). It will be easy to see that they agree with the geometric description given in the introduction. We begin the section with a useful parametrization of \( \Xi \).
Proposition 4.1. The map
\[ \Phi: G \times \Omega \to \Xi, \quad (g, X) \mapsto g \exp(iX)K \]
is a continuous surjection. Moreover, \( \Phi(g, X) = \Phi(g', X') \) for \( g, g' \in G \) and \( X, X' \in \Omega \) if and only if there exists \( k \in Z_K(X) \) and \( w \in N_K(a) \) such that
\[ g = g'wk \quad \text{and} \quad X = \text{Ad}(w^{-1})X'. \]

Proof. Essentially this is Proposition 4 in [AG90].

Write \( \Omega' \) for the regular elements in \( \Omega \), and \( \Omega^+ \) for a connected component of \( \Omega' \). As usual we set \( M = Z_K(a) \) and \( N = \theta(N) \). Then we have the following refinement of Proposition 4.1.

Corollary 4.2.
(i) The map
\[ \Phi': G/M \times \Omega^+ \to \Xi, \quad (g, X) \mapsto g \exp(iX)K \]
is an analytic diffeomorphism onto its open and dense image \( \Xi' \).
(ii) The map
\[ \Phi'': N\Lambda N \times \Omega^+, \quad (g, X) \mapsto g \exp(iX)K \]
is an analytic diffeomorphism onto its open and dense image \( \Xi'' \).

Proof. (i) Later, in the proof of Proposition 4.6, we will show that \( \Phi' \) has everywhere regular differential. (i) will then be a direct consequence of Proposition 4.1.
(ii) In view of the Bruhat decomposition
\[ G = \bigsqcup_{w \in W} \nabla wMAN, \]
with \( \nabla MAN \) the open dense cell, (ii) follows immediately from (i).

The canonical Kähler structure
As before, for \( X \in \mathfrak{g}_C \) we have defined a vector field \( \tilde{X} \) on \( G_C/K_C \) by
\[ (\tilde{X}f)(x) = \left. \frac{d}{dt} \right|_{t=0} f(\exp(tX)x). \]

Let \( p: G_C \to G_C/K_C \) be the natural projection and denote by \( p_\#: \mathfrak{g}_C \to \mathfrak{p}_C \) the linear projection along \( \mathfrak{t}_C \). Let \( B(\cdot, \cdot) \) be the Cartan-Killing form on \( \mathfrak{g} \) as well as its complex linear extension to \( \mathfrak{g}_C \). With respect to the real form \( \mathfrak{g} \) of \( \mathfrak{g}_C \) we have the complex conjugation, \( X \mapsto \overline{X} \).

Using Proposition 4.1 we write \( z \in \Xi \) as \( z = gaK \), for some \( g \in G \) and \( a = \exp(iX) \), \( X \in \Omega \). Let \( T_z\Xi \) denote the real tangent space to \( \Xi \) at \( z \) and \( \overline{Y}_z, \overline{Z}_z \in T_z\Xi \). We claim that
\[ (\overline{Y}_z, \overline{Z}_z) \mapsto \Re B(p_\#(\text{Ad}(ga)^{-1}Y), p_\#(\text{Ad}(ga)^{-1}Z)) \]
gives a well-defined Hermitian bilinear form on \( T_z\Xi \), which we shall denote
\[ h_z: T_z\Xi \times T_z\Xi \to \mathbb{R}. \]
In fact, $\bar{\bar{Y}}_z = 0$ means that $Y \in \text{Ad}(ga)^{-1}K_C$, so $p_\#(\text{Ad}(ga)^{-1}Y) = 0$. To see that the definition does not depend on the choice of $g$ and $a$ which represent $z$, suppose that $z = g'a'K_C$. Then Proposition 4.1 gives that $g'a' = gak$ for some $k \in K$. Then

$$B(p_\#(\text{Ad}(g'a')^{-1}Y), p_\#(\text{Ad}(gak)^{-1}Z)) = B(p_\#(\text{Ad}(gak)^{-1}Y), p_\#(\text{Ad}(gak)^{-1}Z))$$

$$= B(p_\#(\text{Ad}(k)^{-1} \text{Ad}(ga)^{-1}Y), p_\#(\text{Ad}(gak)^{-1}Z))$$

$$= B(p_\#(\text{Ad}(ga)^{-1}Y), p_\#(\text{Ad}(ga)^{-1}Z)),$$

where we used the fact that $\text{Ad}(k)^{-1}$ commutes with $p_\#$ and the complex conjugation, and that $B$ is invariant under $\text{Ad}(k)^{-1}$.

Taking real and imaginary parts we get

$$(4.1)\quad h_z = g_z + i\omega_z,$$

with $g_z = \text{Re } h_z$ symmetric, and $\omega_z = \text{Im } h_z$ skew-symmetric.

Recall the open dense subdomain $\Xi''$ from Corollary 4.2(ii). Write the elements $z \in \Xi''$ as $z = gaK_C$ with $g \in \mathcal{N}AN$ and $a \in \exp(i\Omega^+)$.

For every $X \in g_C$ we define an auxiliary vector field $\tilde{X}$ on $\Xi'$ by

$$(\tilde{X}f)(z) = \frac{d}{dt} \bigg|_{t=0} f(ga \exp(tX)K_C).$$

That $\tilde{X}$ is well-defined follows from the uniqueness of the parametrization. Also observe that $T_z\Xi = \{\tilde{X}z : X \in p_C \}$ ($z \in \Xi''$).

**Lemma 4.3.** For all $z \in \Xi''$ and $X, Y \in g_C$ we have

$$h_z(\tilde{X}z, \tilde{Y}z) = B(p_\#(X), p_\#(Y)).$$

**Proof.** This is immediate from the definition of $(h_z(\cdot, \cdot))_{z \in \Xi}$ and the auxiliary vector fields $\tilde{X}$.

For every $h \in G$ we let $\lambda_h$ denote left translation by $h$ on $G_C/K_C$,

$$\lambda_h : G_C/K_C \to G_C/K_C, \quad x \mapsto hx.$$

**Theorem 4.4.** The Hermitian metric $h_z(\cdot, \cdot)$ on $\Xi$ is a $G$-invariant Kähler metric.

**Proof.** Clearly this structure is smooth so from the above $h_z(\cdot, \cdot)$ is an Hermitian metric. To show $(h_z(\cdot, \cdot))_{z \in \Xi}$ is $G$-invariant we have to show that

$$h_{hz}(d\lambda_h(\tilde{Y}z), d\lambda_h(\tilde{Z}z)) = h_z(\tilde{Y}z, \tilde{Z}z)$$

holds for all $z \in \Xi$ and $h \in G$. Since $d\lambda_h(\tilde{Y}z) = \overline{\text{Ad}(h)Y_{hz}}$ and $d\lambda_h(\tilde{Z}z) = \overline{\text{Ad}(h)Z_{hz}}$, we have

$$h_{hz}(d\lambda_h(\tilde{Y}z), d\lambda_h(\tilde{Z}z)) = h_{hz}(\overline{\text{Ad}(h)Y_{hz}}, \overline{\text{Ad}(h)Z_{hz}})$$

$$= B(p_\#(\text{Ad}(hga)^{-1} \text{Ad}(h)Y), p_\#(\text{Ad}(hga)^{-1} \text{Ad}(h)Z))$$

$$= B(p_\#(\text{Ad}(ga)^{-1}Y), p_\#(\text{Ad}(ga)^{-1}Z))$$

$$= h_z(\tilde{Y}z, \tilde{Z}z).$$
Next we show the structure \( (g_z(\cdot, \cdot))_{z \in \Xi} \) is Riemannian. By \( G \)-invariance it is enough to show for every \( a \in \exp(i\Omega) \) that

\[
g_a: T_a \Xi \times T_a \Xi \to \mathbb{R}
\]

is positive definite, where, by abuse of notation, we write \( a \) for \( aK_C \in \Xi \). For that, first we observe that

\[
T_a \Xi = \{ \tilde{Y}_a : Y \in p_C \}.
\]

Indeed, every \( Y \in p_C \) can be written as

\[
Y = Y_0 + \sum_{\alpha \in \Sigma^+} (Y_\alpha - \theta Y_\alpha)
\]

for \( Y_0 \in aC \) and \( Y_\alpha \in g_C^\alpha \). So

\[
\text{Ad}(a)^{-1}Y = Y_0 + \sum_{\alpha \in \Sigma^+} (a^{-\alpha}Y_\alpha - a^\alpha \theta Y_\alpha)
\]

\[
= Y_0 + \sum_{\alpha \in \Sigma^+} \frac{a^\alpha + a^{-\alpha}}{2}(Y_\alpha - \theta Y_\alpha) + \sum_{\alpha \in \Sigma^+} \frac{a^\alpha - a^{-\alpha}}{2}(Y_\alpha + \theta Y_\alpha)
\]

But then

\[
p_\#(\text{Ad}(a)^{-1}Y) = Y_0 + \sum_{\alpha \in \Sigma^+} \frac{a^\alpha + a^{-\alpha}}{2}(Y_\alpha - \theta Y_\alpha).
\]

Using the definition of \( \Omega \) we get for any \( a = \exp(iX), X \in \Omega \),

\[
\frac{a^\alpha + a^{-\alpha}}{2} = \cos \alpha(X) > 0.
\]

From this it follows that \( T_a \Xi = \{ \tilde{Y}_a : Y \in p_C \} \). In addition, for \( Y \neq 0 \) we get an expression for \( g_a(\tilde{Y}_a, \tilde{Y}_a) \) showing that \( g_a(\cdot, \cdot) \) is positive definite, namely

\[
g_a(\tilde{Y}_a, \tilde{Y}_a) = \text{Re} B(Y_0, Y_0) + \sum_{\alpha \in \Sigma^+} \left( \frac{a^\alpha + a^{-\alpha}}{2} \right)^2 \text{Re} B(Y_\alpha - \theta Y_\alpha, Y_\alpha - \theta Y_\alpha)
\]

\[
= \text{Re} B(Y_0, Y_0) + 2 \sum_{\alpha \in \Sigma^+} \left( \frac{a^\alpha + a^{-\alpha}}{2} \right)^2 \text{Re} B(Y_\alpha, -\theta Y_\alpha) > 0.
\]

Finally, to see that \( (h_z(\cdot, \cdot))_{z \in \Xi} \) is Kähler we have to show that \( (\omega_z(\cdot, \cdot))_{z \in \Xi} \) is closed. Recall the open dense subdomain \( \Xi'' \) and the auxiliary vectorfields \( \hat{X} \) on \( \Xi'' \) for \( X \in g_C \). From Lemma 4.3 we have

\[
\omega_z(\hat{X}_z, \hat{Y}_z) = \text{Im} B(p_\#(X), p_\#(Y)).
\]

By the density of \( \Xi'' \) in \( \Xi \), it suffices to show that \( d\omega_z(\hat{X}_z, \hat{Y}_z, \hat{Z}_z) = 0 \) for all \( z \in \Xi'' \) and \( X, Y, Z \in p_C \). Now

\[
d\omega(\hat{X}, \hat{Y}, \hat{Z}) = \hat{X}\omega(\hat{Y}, \hat{Z}) \quad - \omega([\hat{X}, \hat{Y}], \hat{Z}) + \omega([\hat{X}, \hat{Z}], \hat{Y}) - \omega([\hat{Y}, \hat{Z}], \hat{X})
\]

On \( \Xi'' \) the first three terms on the right hand side are zero by (4.2), while the last three terms are zero by (4.2) and \( [p_C, p_C] \subseteq \xi_C \).
Remark 4.5. (a) One can see from the previous proof that \((g_z(\cdot, \cdot))_{z \in \Xi}\) extends the \(G\)-invariant metric on \(G/K\).

(b) The Riemannian structure \((g_z(\cdot, \cdot))_{z \in \Xi}\) is the Kähler metric coming from the adapted complex structure. The adapted complex structure associated to a complete Riemannian manifold \(M\) can be characterized as the unique complex structure on a neighborhood of the zero section of the tangent bundle of \(M\) satisfying: for any geodesic \(\gamma: \mathbb{R} \to M, \ t \mapsto \gamma(t)\) the map

\[\varphi_{\gamma}: \mathbb{C} \to TM, \ t + is \mapsto (\gamma(t), s\gamma'(t))\]

is holomorphic.

For \(M = G/K\) we have the isomorphism

\[T(G/K) = G \times_K p\]

and the analytic \(G\)-equivariant map

\[\Phi: T(G/K) \to G_{\mathbb{C}}/K_{\mathbb{C}}, \ [g, X] \mapsto g \exp(iX)K_{\mathbb{C}}.\]

If \(\gamma: \mathbb{R} \to G/K\) is a geodesic then \(\gamma(t) = g \exp(tX)K\) for some \(g \in G\) and \(X \in p\). So

\[\Phi(\varphi_{\gamma}(s, t)) = \Phi([\gamma(t), sX]) = g \exp((t + is)X)K_{\mathbb{C}}.\]

In particular, for all geodesics \(\gamma\) the maps \(\Phi \circ \varphi_{\gamma}\) are holomorphic.

Define \(W = \text{Ad}(K)(\Omega)\) and note that \(W \subseteq p\) is a convex open set (easy consequence from the Kostant convexity theorem). Recall from [AG90, Prop. 4] that the restriction of \(\Phi\) to \(G \times_K W\)

\[G \times_K W \to \Xi, \ [g, X] \mapsto g \exp(iX)K_{\mathbb{C}}\]

is an analytic diffeomorphism. From our calculation above it now follows that \(\Xi\) carries the adapted complex structure, see also [BHH03].

(c) The metric is not complete, since for every \(X \in \overline{\Omega}\), the curve \(\gamma(t) = \exp(itX), \ t \in [0, 1]\) has finite length.

The Riemannian measure

For the remainder of this section we consider \(\Xi\) as a Riemannian manifold equipped with the Riemannian structure \((g_z(\cdot, \cdot))_{z \in \Xi}\). We write \(\mu_{\Xi}\) for the Riemannian measure on \(\Xi\). The next proposition gives a simple description of the Riemannian measure in terms of coordinates. For this purpose, for \(\alpha \in \Sigma\) set \(m_\alpha = \dim g^\alpha\). We remind the reader of the notation \(\Omega^+ \subseteq \Omega\) for the regular chamber.

Proposition 4.6. For \(f \in L^1(\Xi)\)

\[\int_{\Xi} f(z) \, d\mu_{\Xi}(z) = c \int_{G/M} \int_{\Omega^+} f(g \exp(iX)K_{\mathbb{C}}) \prod_{\alpha \in \Sigma^+} |\sin 2\alpha(X)|^{m_\alpha} \, dX \, d\mu_{G/M}(gM),\]

where \(c > 0\) depends only on the normalization of the Haar measure \(\mu_{G/M}\) on \(G/M\).

Proof. Recall that the parametrization
\( \Phi': G/M \times \Omega^+ \to \Xi, \ (g, X) \mapsto g \exp(iX)K_C \)

is a smooth injective map. We show that \( \Phi' \) is everywhere regular and we compute the Jacobian. For \( gM \in G/M \) and \( X \in \Omega^+ \) we write \( z = g \exp(iX)K_C \). We use the following identifications of tangent spaces:

\[
T_{(gM,X)}(G/M \times \Omega^+) \simeq \mathfrak{g}/\mathfrak{m} \simeq \mathfrak{m}^\perp \times \mathfrak{a}
\]

and

\[
T_z\Xi \simeq \mathfrak{p}_C, \ \hat{Y}_z \leftrightarrow Y \quad (Y \in \mathfrak{p}_C).
\]

With these and Lemma 4.3 we describe an inner product on \( \mathfrak{m}^\perp \times \mathfrak{a} \) (independent of \( g \) and \( X \)) such that the map

\[
T_{(gM,X)}(G/M \times \Omega^+) \to T_z\Xi = \mathfrak{p}_C
\]

(4.3)

\[
(Y, Z) \mapsto \begin{cases} 
Y + iZ & \text{for } Y \in \mathfrak{a} \\
Y - \theta Y & \text{for } Z = 0 \text{ and } Y \in \mathfrak{g}^a, \alpha \in \Sigma^+ \\
iY - i\theta Y & \text{for } Z = 0 \text{ and } Y \in \mathfrak{g}^a, \alpha \in \Sigma^-
\end{cases}
\]

becomes an isometric isomorphism of tangent spaces. With respect to this basis and identifications we compute the Jacobian \( |\det d\Phi(gM, X)| \) of the differential \( d\Phi(gM, X) \). By \( G \)-equivariance we have

\[
|\det d\Phi(gM, X)| = |\det d\Phi(1, X)|,
\]

so we may assume that \( g = 1 \). Fix \( X \in \Omega^+ \), set \( a = \exp(iX) \) and \( z = aK_C \). Then for \( Y \in \mathfrak{m}^\perp \) and \( Z \in \mathfrak{a} \) we have

\[
d\Phi(1, X)(Y, Z) = \left. \frac{d}{dt} \right|_{t=0} \exp(itY) \exp(X)K_C + \left. \frac{d}{dt} \right|_{t=0} \exp(iX) \exp(itZ)K_C
\]

\[
= \Ad(a)^{-1}Y_z + i\hat{Z}_z \leftrightarrow \Ad(a)^{-1}Y + iZ.
\]

As \( Y \) is in \( \mathfrak{m}^\perp \) it can be written \( Y = Y_0 + \sum_{\alpha \in \Sigma} Y_\alpha \) with \( Y_0 \in \mathfrak{a} \) and \( Y_\alpha \in \mathfrak{g}^a \). For any \( W \in \mathfrak{p}_C \)

\[
g_z(d\Phi(1, X)(Y, Z), \hat{W}_z) = \Re B(p_#(\Ad(a)^{-1}Y + iZ), W)
\]

(4.5)

\[
= \Re B(p_#(\Ad(a)^{-1}Y + iZ), W)
\]

\[
= \Re B(iZ, W) + \Re B(Y_0, W) + \sum_{\alpha \in \Sigma} \Re B(p_#(a^\alpha Y_\alpha), W).
\]

Putting together the identifications of tangent spaces, (4.3), (4.4) and (4.5) one obtains

\[
|\det d\Phi(1, X)| = c \prod_{\alpha \in \Sigma^+} |\det d\Phi(1, X)|_{(\mathfrak{g}^a \oplus \mathfrak{g}^{-a}) \times \{0\}}.
\]

With \( Y_\alpha \in \mathfrak{g}^a \) a unit vector (4.3)-(4.5) give

\[
|\det d\Phi(1, X)|_{(\mathfrak{g}^a \oplus \mathfrak{g}^{-a}) \times \{0\}} =
\]

\[
= \det \begin{pmatrix} 
\Re B(p_#(a^{-\alpha} Y_\alpha), Y_\alpha - \theta Y_\alpha) & \Re B(p_#(a^{-\alpha} Y_\alpha), iY_\alpha - i\theta Y_\alpha) \\
\Re B(p_#(a^{\alpha} Y_\alpha), Y_\alpha - \theta Y_\alpha) & \Re B(p_#(a^{\alpha} Y_\alpha), iY_\alpha - i\theta Y_\alpha)
\end{pmatrix} |^{m_\alpha}
\]

\[
= \det \left( \begin{vmatrix} \Re a^{-\alpha} \cos \alpha(X) & \sin \alpha(X) \\ \Re a^{\alpha} \sin \alpha(X) & -\sin \alpha(X) \end{vmatrix} \right) |^{m_\alpha}
\]

\[
= |\sin 2\alpha(X)|^{m_\alpha}.
\]
So from (4.6) we get
\[
| \det d\Phi(1, X) | = c \prod_{\alpha \in \Sigma^+} | \sin 2\alpha(X) |^{m_\alpha} \neq 0,
\]
concluding the proof of the proposition.

**Remark 4.7.** From the formula in Proposition 4.6 one sees that the Riemannian measure $\mu_\Xi$ equals the restriction of a Haar measure $\mu_{G_C/K_C}$ to $\Xi$. Similarly, one sees that the Riemannian measure $\mu_\Xi$ vanishes at infinity, i.e. the density vanishes at the boundary $\partial \Xi$. This, and the precise order of the decay, will be used later.

Our next goal is to identify the Laplace-Beltrami operator on $\Xi$ corresponding to this Riemannian structure. For that we have to recall some facts about invariant differential operators on $G_C/K_C$.

**Invariant differential operators on $G_C/K_C$**

We begin with some standard facts about the complexification of complex Lie algebras. Let $\mathfrak{g}_R$ denote $\mathfrak{g}_C$ considered as a real Lie algebra. $\mathfrak{g}_R$ has two natural (almost) complex structures corresponding to multiplication by $i$ and by $-i$. With respect to the first $\mathfrak{g}_R$ is canonically isomorphic to $\mathfrak{g}_C$. Denote by $\mathfrak{g}_C$ the complex Lie algebra $\mathfrak{g}_R$ equipped with the conjugate complex structure. Then $\mathfrak{g}_C$ with the second complex structure is canonically isomorphic to $\mathfrak{g}_C$. Let $\mathfrak{g}_C$ denote the complexification of the real Lie algebra $\mathfrak{g}_R$. As a complex vector space $\mathfrak{g}_C$ has a complex structure, say $J_C$. Relative to these structures the map
\[
(\mathfrak{g}_C, J_C) \rightarrow \mathfrak{g}_C \oplus \overline{\mathfrak{g}_C}, \quad X + J_C Y \mapsto (X + i Y, X - i Y)
\]
is an isomorphism of complex Lie algebras.

Let $\text{Diff}_L(G_C)$ denote the left $G_C$-invariant differential operators on $G_C$. Any $X \in \mathfrak{g}_C$ gives a left invariant differential operator $L_X$ by
\[
(L_X f)(x) = \frac{d}{dt} \bigg|_{t=0} f(g \exp(tX)),
\]
for $x \in G_C$ and $f$ a function on $G_C$ differentiable at $x$.

Then the map
\[
\mathfrak{g}_C \rightarrow \text{Diff}_L(G_C), \quad X + J_C Y \mapsto L_X + i L_Y
\]
extends to an algebra isomorphism
\[
\mathcal{U}(\mathfrak{g}_C) \simeq \text{Diff}_L(G_C).
\]

As $K_C$ is reductive in $G_C$, a left invariant differential operator on $G_C$, viewed as an element in $\mathcal{U}(\mathfrak{g}_C)$, projects to an operator on $G_C/K_C$ if and only if it is $\text{Ad}(K_C)$-invariant. In other words,
\[
\frac{\mathcal{U}(\mathfrak{g}_C)^{K_C}}{\mathcal{U}(\mathfrak{g}_C)^{K_C} \cap \mathcal{U}(\mathfrak{g}_C)^{K_C}} \simeq \text{Diff}_L(G_C/K_C).
\]

Let $X_1, \ldots, X_n$ be an orthonormal basis of $\mathfrak{p}$ and set $Y_j = i X_j \in i \mathfrak{p}$. With
\[
Z_j = \frac{1}{2} (X_j - J_C Y_j) \quad \text{and} \quad \overline{Z}_j = \frac{1}{2} (X_j + J_C Y_j)
\]
and using the identification (4.7) we set, as before, 

\[ L_{Z_j} = \frac{1}{2}(L_{X_j} - iL_{Y_j}) \quad \text{and} \quad L_{\overline{Z}_j} = \frac{1}{2}(L_{X_j} + iL_{Y_j}). \]

Consider the differential operators on \( G_C \) defined by

\[ \Delta = \sum_{j=1}^{n} L_{X_j}^2, \]
\[ \Delta_C = \sum_{j=1}^{n} L_{X_j}^2 + \sum_{j=1}^{n} L_{Y_j}^2, \]
\[ \Box = \sum_{j=1}^{n} L_{Z_j}^2, \]
and
\[ \overline{\Box} = \sum_{j=1}^{n} L_{\overline{Z}_j}^2. \]

**Proposition 4.9.** Each of the differential operators \( \Delta, \Delta_C, \Box \) and \( \overline{\Box} \) induces a differential operator on \( G_C/K_C \).

**Proof.** We consider first the operator \( \Delta \). Since \( X_1, \ldots, X_n \) is an orthonormal basis of \( \mathfrak{p} \), \( \sum_{j=1}^{n} X_j^2 \in \mathcal{U}(\mathfrak{g}) \) is \( \text{Ad}(K) \)-invariant. In particular, \( \sum_{j=1}^{n} X_j^2 \in \mathcal{U}(g)_K \) and, since \( K_C \) is reductive,

\[ \sum_{j=1}^{n} X_j^2 \in \mathcal{U}(g)_K. \]

In view of (4.8), it follows that \( \Delta \) projects to a differential operator on \( G_C/K_C \). One then shows mutatis mutandis that \( \Delta_C \) factors to \( G_C/K_C \).

For the operators \( \Box \) and \( \overline{\Box} \) we use the identification \( \mathcal{U}(g_C^K) \simeq \mathcal{U}(g_C) \times \mathcal{U}(\overline{g}_C) \) induced by (4.7). Then \( \Box \) and \( \overline{\Box} \) become identified with the operators

\[ \Box \leftrightarrow (0, \sum_{j=1}^{n} X_j^2) \quad \text{and} \quad \overline{\Box} \leftrightarrow (\sum_{j=1}^{n} X_j^2, 0). \]

As observed above it follows that \( \Box \) and \( \overline{\Box} \) are represented by \( \text{Ad}(K_C) \)-invariant elements in \( \mathcal{U}(g_C^K) \), hence induce differential operators on \( G_C/K_C \).

By abuse of notation we use \( \Delta, \Delta_C, \Box \) and \( \overline{\Box} \) to denote either differential operators on \( G_C/K_C \) or on \( G_C \). Their use should be clear from the context.

**Proposition 4.10.** For \( \Delta, \Delta_C, \Box \) and \( \overline{\Box} \) considered as elements in \( \text{Diff}_L(G_C/K_C) \) the following assertions hold:

(i) \( [\Box, \Box] = 0; \)
(ii) \( [\Delta_C, \Box + \overline{\Box}] = 0; \)
(iii) \( \Delta = \frac{1}{4} \Delta_C + (\Box + \overline{\Box}). \)

**Proof.** (i) This is immediate from (4.9).

(ii) In view of (4.8), the assertion in (ii) can be phrased as...
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\[ \left( \sum_{j=1}^{n} X_j^2 + \sum_{j=1}^{n} Y_j^2, \sum_{j=1}^{n} Z_j^2 + \sum_{j=1}^{n} Z_j^2 \right) \in U(g_{\mathbb{C}}) t_{\mathbb{C}}. \]

An easy computation shows that

\[ \square + \square = \frac{1}{2} \left( \sum_{j=1}^{n} L_{X_j}^2 - \sum_{j=1}^{n} L_{Y_j}^2 \right) \leftrightarrow \frac{1}{2} \left( \sum_{j=1}^{n} X_j^2 - \sum_{j=1}^{n} Y_j^2 \right). \]

Substituting (4.11) into the left side of (4.10) we obtain

\[ \left( \sum_{j=1}^{n} X_j^2 + \sum_{j=1}^{n} Y_j^2, \sum_{j=1}^{n} Z_j^2 + \sum_{j=1}^{n} Z_j^2 \right) = \frac{1}{2} \left( \sum_{j=1}^{n} X_j^2 \right) = \sum_{j,k=1}^{n} [X_j, Y_k]. \]

Since \( X_j, Y_k \in p_{\mathbb{C}}, [X_j, Y_k] \in t_{\mathbb{C}} \). An easy computation shows that \( [X_j, Y_k] \in U(g_{\mathbb{C}}) t_{\mathbb{C}} \) and so (4.12) implies (4.10), completing the proof of (ii).

(iii) This is immediate from (4.11) and the definition of \( \Delta \) and \( \Delta_C \).

**Remark 4.11.** The operators \( \square, \square \) and \( \Delta \) can be related through the previously described holomorphic extension result. Indeed, for \( x \in G/K \) let \( f \) be a function defined in a connected neighborhood \( U \) of \( x \) which extends holomorphically to a connected complex neighborhood \( \mathcal{U}_{\mathbb{C}} \subseteq G_{\mathbb{C}}/K_{\mathbb{C}} \) with \( \mathcal{U} \cap G/K = U \). Write \( f^\sim \) for the holomorphic extension of \( f \) to \( \mathcal{U}_{\mathbb{C}} \). Then \( \square, \square \) and \( \Delta \) are related through

\[ (\Delta f)^\sim (z) = (\square f^\sim) (z) \quad (\forall z \in \mathcal{U}_{\mathbb{C}}) \]

and

\[ (\Delta f)^\sim (z) = (\square f^\sim) (z) \quad (\forall z \in \mathcal{U}_{\mathbb{C}}). \]

**The Laplace-Beltrami operator on \( \Xi \)**

We write \( \Delta_{\Xi} \) for the restriction of \( \Delta_C \) to \( \Xi \). The next result establishes the naturality of the previously defined Riemannian metric on \( \Xi \).

**Theorem 4.12.** The operator \( \Delta_{\Xi} \) is the Laplace-Beltrami operator for the Riemannian metric \( \left( g_z (\cdot, \cdot) \right)_{z \in \Xi} \) on \( \Xi \).

**Proof.** We use the open dense subdomain \( \Xi'' \). In particular, it follows from Lemma 4.3 that for \( z = gaK_{\mathbb{C}} \in \Xi'' \) and \( X, Y \in p_{\mathbb{C}} \) we have

\[ g_z(\hat{X}_z, \hat{Y}_z) = \text{Re} \, B(X, Y). \]

Thus with \( X_1, \ldots, X_n \) and \( Y_1, \ldots, Y_n \) as before, the Laplace-Beltrami operator on \( \Xi'' \) is given by

\[ \sum_{j=1}^{n} \hat{X}_j^2 + \sum_{j=1}^{n} \hat{Y}_j^2. \]

But this operator evidently coincides with \( \Delta_{\Xi} |_{\Xi''} = \Delta_C |_{\Xi''} \). Since \( \Xi'' \) is open dense, and \( \Delta_C \) has analytic coefficients, the theorem follows.
§5 Invariant Hilbert spaces of holomorphic functions on $\Xi$

The holomorphic extension result in Theorem 1.1 has as a consequence a realization of each representation in $\hat{G}_s$ as a Hilbert space of holomorphic functions on $\Xi$. In this section we draw some natural conclusions from this point of view. We shall use some facts about Hilbert spaces of holomorphic functions for which we refer the reader to [FT99] for a recent account.

For $M$ a second countable complex manifold, we write $\mathcal{O}(M)$ for the Fréchet space of holomorphic functions on $M$. A Hilbert space of holomorphic functions on $M$ is a Hilbert space $\mathcal{H} \subseteq \mathcal{O}(M)$ such that the inclusion mapping $\mathcal{H} \hookrightarrow \mathcal{O}(M)$ is continuous. For an equivariant version suppose that $G$ is a locally compact group that acts by holomorphic automorphisms on $M$. One says that $\mathcal{H}$ is a $G$-invariant Hilbert space of holomorphic functions on $M$ if the left regular representation of $G$ on $\mathcal{H}$ is a unitary representation of $G$.

If $\mathcal{H} \subseteq \mathcal{O}(M)$ is a Hilbert space of holomorphic functions, then the point evaluation map, $f \mapsto f(z)$, is continuous for all $z \in M$. In the usual way one obtains a reproducing kernel for $\mathcal{H}$

$$K: M \times M \to \mathbb{C},$$

holomorphic in the first variable and anti-holomorphic in the second variable. Unitarity of $G$ on $\mathcal{H}$ becomes $G$-invariance for $K$, $K(gz,gw) = K(z,w)$ for all $g \in G$ and $z, w \in M$.

As is well known, this use of the Riesz representation theorem allows one to establish an equivalence between $G$-invariant Hilbert spaces of holomorphic functions on $M$ and such kernels. Indeed, if we let $\overline{M}$ denote $M$ with the conjugate complex structure, then the kernel $K$ is in $\mathcal{O}(M \times \overline{M})$, and, if $\mathcal{P}_G(M) \subseteq \mathcal{O}(M \times \overline{M})$ is the cone of $G$-invariant holomorphic positive definite kernels on $M$, then $K \in \mathcal{P}_G(M)$. Now $\mathcal{P}_G(M)$ can be shown to be a closed convex conuclear cone in the Fréchet space $\mathcal{O}(M \times \overline{M})$. Let $\text{Ext}(\mathcal{P}_G(M))$ be the cone of extremal rays in $\mathcal{P}_G(M)$. The elements in $\text{Ext}(\mathcal{P}_G(M))$ correspond to the irreducible $G$-invariant Hilbert spaces of holomorphic functions on $M$. Then reducibility questions concerning the action of $G$ on $\mathcal{H}$ are re-formulated into so-called Choquet theory.

Now we specialize this by taking for $M$ the domain $\Xi \subseteq G_\mathbb{C}/K_\mathbb{C}$ on which $G$ acts by holomorphic automorphisms.

As before, let $\hat{G}_s$ denote the $K$-spherical unitary dual of $G$. Let $[\pi] \in \hat{G}_s$ and $(\pi, \mathcal{H}_\pi)$ a representative of it with $v_0 \in \mathcal{H}_\pi$ a normalized $K$-spherical vector. Then Corollary 1.2 implies the existence of a continuous $G$-equivariant embedding

$$(5.1) \quad \iota_\pi: \mathcal{H}_\pi \hookrightarrow \mathcal{O}(\Xi), \ v \mapsto f_v; \ f_v(xK_\mathbb{C}) = \langle \pi(x^{-1})v, v_0 \rangle.$$

In this way $\iota_\pi(\mathcal{H}_\pi)$ equipped with the topology of $\mathcal{H}_\pi$ becomes a $G$-invariant Hilbert space of holomorphic functions. From (5.1) it follows that the reproducing kernel $K_\pi$ of $\iota_\pi(\mathcal{H}_\pi)$ is given by

$$(5.2) \quad K_\pi: \Xi \times \Xi \to \mathbb{C}, \quad (xK_\mathbb{C}, yK_\mathbb{C}) \mapsto \langle \pi(\overline{y})v_0, \pi(\overline{x})v_0 \rangle,$$

where $x \mapsto \overline{x}$ denotes the complex conjugation on $G_\mathbb{C}$ with respect to the real form $G$. The re-formulation of $\hat{G}_s$ into kernels as provided by the general theory gives

$$\hat{G}_s \leftrightarrow \text{Ext}(\mathcal{P}_G(\Xi)) = \coprod_{[\pi] \in \hat{G}_s} \mathbb{R}^+ K_\pi.$$

The decomposition of invariant Hilbert spaces of holomorphic functions on $\Xi$ into irreducibles is a direct consequence of results in [FT99] or [K99]. For this reason we simply sketch
the argument. Independently Faraut [F03] made the same observation with the same argument in a lecture at MSRI, '01.

Identifying $\hat{G}_s$ with a subset of $a_\ast^+$, we write $(\pi_\lambda, H_\lambda)$ for a representative of $\lambda \in \hat{G}_s \subseteq a_\ast^+$. Similarly we write $K_\lambda$ for $K_{\pi_\lambda}$. It is known that the Borel structure on $\hat{G}_s$ induced from the hull-kernel topology on $\hat{G}$ is the same as the Borel structure on $a_\ast^+$ induced from the Euclidean topology. Also, the map

$$\hat{G}_s \to \text{Ext}(P_G(\Xi)), \quad \lambda \mapsto K_\lambda$$

is continuous and hence constitutes an admissible parametrization of $\text{Ext}(P_G(\Xi))$ in the sense of [K99]. The following result is an immediate consequence of the abstract results in [FT99] or [K99].

**Theorem 5.1.** Let $\mathcal{H}$ be a $G$-invariant Hilbert space of holomorphic functions on $\Xi$. Let $K(z, w)$ be the reproducing kernel of $\mathcal{H}$. Then there exists a unique Borel measure $\mu$ on $\hat{G}_s$ such that

$$K(z, w) = \int_{\hat{G}_s} K_\lambda(z, w) \ d\mu(\lambda) \quad (z, w \in \Xi)$$

with the right hand side converging absolutely on compact subsets of $\Xi \times \Xi$.

Theorem 5.1 implies a unitary $G$-equivalence of the left regular representation of $G$ on $\mathcal{H}$ with a direct integral of representations from $\hat{G}_s$ relative to a spectral measure $\mu$.

$$\left( L, \mathcal{H} \right) \simeq \left( \int_{\hat{G}_s}^\oplus \pi_\lambda d\mu(\lambda), \int_{\hat{G}_s}^\oplus H_\lambda d\mu(\lambda) \right).$$

Shortly we shall give a criterion for a Borel measure $\mu$ on $\hat{G}_s$ to be a spectral measure of a $G$-invariant Hilbert space of holomorphic functions on $\Xi$. But first we present some examples of invariant Hilbert spaces on $\Xi$.

**Example 5.2.** (a) The $G$-invariance of the Riemannian measure $\mu_{\Xi}$ allows one to define the Bergmann space of $\Xi$,

$$B^2(\Xi) = \{ f \in \mathcal{O}(\Xi): \| f \|^2 = \int_{\Xi} |f(z)|^2 \ d\mu_{\Xi}(z) < \infty \}.$$

Certainly $B^2(\Xi)$ is a $G$-invariant Hilbert space of holomorphic functions on $\Xi$. We conjecture that that $B^2(\Xi) \neq \{0\}$.

As usual, more generally one can define for every $G$-invariant weight function $p: \Xi \to \mathbb{R}^+$ a weighted Bergman space $B^2(\Xi, p)$. Again, these weighted Bergman spaces are $G$-invariant Hilbert spaces of holomorphic functions on $\Xi$. A characterization of the spectral measure follows from Faraut’s Gutzmer Formula ([F02, 03]).

(b) Using notation to be introduced in §7, we suppose that $\Xi = \Xi_0$ and that the distinguished boundary of $\Xi$ (cf. [GK02a]) is a symmetric space $G/H$. For such spaces one can define a Hardy space $H^2(\Xi)$ on $\Xi$. Then $H^2(\Xi)$ is a $G$-invariant Hilbert-space of holomorphic functions on $\Xi$. This Hardy space has a boundary value map $H^2(\Xi) \to L^2(G/H)$ whose image is a full multiplicity one subspace of the most continuous spectrum of $L^2(G/H)$. The spectral measure here is nothing other than the contribution to the most continuous spectrum of the Plancherel measure. For all this see [GKO03, 04].

We shall present a necessary condition for a Borel measure to be a spectral measure for a $G$-invariant Hilbert space of holomorphic functions on $\Xi$. We start with a simple observation.
Lemma 5.3. Let $\lambda \in \widehat{G}_s$. Then for all $g \in G$ and $a = \exp(iX)$, $X \in \Omega$ we have

(i) if $v \in \mathcal{H}_\lambda$, then
$$|f_v(gaK_C)| \leq \|\pi(\lambda^{-1})v_0\| \cdot \|v\|;$$

(ii) $K(\lambda)(gaK_C, gaK_C) = \|\pi(\lambda^{-1})v_0\|^2$.

Proof. (i) Using the unitarity of $\pi_\lambda$ and the Cauchy-Schwarz inequality one gets
$$|f_v(gaK_C)| = |\langle \pi_\lambda(a^{-1}g^{-1})v, v_0\rangle| = |\langle \pi_\lambda(g^{-1})v, \pi_\lambda(a^{-1})v_0\rangle|$$
$$\leq \|\pi_\lambda(g^{-1})v\| \cdot \|\pi_\lambda(a^{-1})v_0\| = \|\pi_\lambda(a^{-1})v_0\| \cdot \|v\|.$$

(ii) From the definition of $K_\lambda$ (cf. (5.2)) we have
$$K(\lambda)(gaK_C, gaK_C) = \langle \pi_\lambda(a^{-1})v_0, \pi_\lambda(a^{-1})v_0\rangle = \|\pi_\lambda(a^{-1})v_0\|^2,$$
as was to be shown.

Let us define a norm $|\cdot|_\Omega$ on $a^*$ by
$$|\lambda|_\Omega := \sup_{X \in \Omega} |\lambda(X)| \quad (\lambda \in a^*).$$

Recall from [KS04, Sect. 4] that if $\varphi_\lambda$ is the analytically continued spherical function on $G/K$ with parameter $\lambda$, and $a \in \exp(i\Omega)$,

(5.4) $\|\pi_\lambda(a^{-1})v_0\|^2 = \varphi_\lambda(a^2)$.

Let $Q \subseteq \Omega$ be a compact subset. Then it follows from (5.4), (1.1) and [KS04, Th. 4.2] that there exist constants $C_Q > 0$ and $\varepsilon_Q$, $0 \leq \varepsilon_Q < 1$ such that

(5.5) $\|\pi_\lambda(a^{-1})v_0\|^2 \leq C_Q e^{(1-\varepsilon_Q)2|\text{Im} \lambda|_\Omega}$

holds for all $a \in \exp(iQ)$.

Proposition 5.4. Let $\mu$ be a Borel measure on $\widehat{G}_s$ such that

$$(\forall 0 \leq c < 2) \quad \int_{\widehat{G}_s} e^{c|\text{Im} \lambda|_\Omega} \, d\mu(\lambda) < \infty.$$

Then $\mu$ is the spectral measure of a $G$-invariant Hilbert space $\mathcal{H}_{[\mu]}$ of holomorphic functions on $\Xi$.

Proof. Since reproducing kernels are dominated by their values on the diagonal, we have to show only that
$$K_{[\mu]}(z, z) = \int_{\widehat{G}_s} K_\lambda(z, z) \, d\mu(\lambda)$$
converges uniformly on compact subsets of $\Xi$. But this is immediate from Lemma 5.3(ii) and (5.5).
§6 Applications to harmonic analysis on $G/K$

We shall give a holomorphic extension of the heat kernel on $G/K$ associated to the Riemannian structure from §4. This can be done quite naturally from the view point of our holomorphic extensions of spherical functions and it illustrates some complex analytic methods in the harmonic analysis on $G/K$. For use in a future paper we use this to formulate an analog of the Bargmann-Segal transform (cf. [St99] for a discussion on compact symmetric spaces).

Holomorphic extension of the heat kernel

Let $G/K$ be the Riemannian symmetric space associated to $G$ and $K$. In §4 we took $X_1, \ldots, X_n$ an orthonormal basis of $\mathfrak{p}$ and considered the differential operator

$$\Delta = \sum_{j=1}^{n} L_{X_j}^2$$

on $G_\mathbb{C}/K_\mathbb{C}$. It is well known that the Laplace-Beltrami operator on $G/K$ is given by the restriction of $\Delta$ to functions on $G/K$. We remark that with this convention $\Delta$ is a negative operator. Denote also by $\Delta$ the self-adjoint extension of $\Delta|_{C^\infty(G/K)}$. Fix a left $G$-invariant positive measure on $G/K$, say $\mu_{G/K}$, and let $L^2(G/K)$ be the corresponding $L^2$-space. We will denote the identity coset in $G/K$ by $x_0$.

Consider the heat equation for $\Delta$:

\[ \frac{\partial}{\partial t} u = \Delta u \quad (u \in C^\infty(\mathbb{R}^+ \times G/K)). \]

(6.1)

We write $k_t(x), t \in \mathbb{R}^+, x \in G/K$, for the heat kernel normalized by

\[ \lim_{t \to 0} \int_{G/K} f(x) k_t(x) \, d\mu_{G/K}(x) = f(x_0) \]

for $f \in C_c(G/K)$. Recall from [G68] the formula

\[ k_t(x) = \frac{c}{|W|} \int_{a^*} e^{-t(|\lambda|^2 + |\rho|^2)} \varphi_\lambda(x) \frac{d\lambda}{|c(\lambda)|^2} \quad (x \in G/K). \]

(6.3)

Here $|\cdot|$ denotes the inner product on $\mathfrak{a}_c^*$ obtained from the Cartan-Killing form, $c(\lambda)$ is the Harish-Chandra $c$-function on $G/K$ while $c$ is a positive constant. Substituting the integral formula for spherical functions into (6.3) we get

\[ k_t(x) = \frac{c}{|W|} \int_{a^*} \int_{K} e^{-t(|\lambda|^2 + |\rho|^2)} a(kx)^{\rho - \lambda} \, dk \, \frac{d\lambda}{|c(\lambda)|^2} \quad (x \in G/K). \]

(6.4)

For $f \in L^p(G/K), 1 \leq p < \infty$,

\[ u_f(x,t) = (k_t * f)(x) = \int_{G/K} k_t(g^{-1}x) f(gK) \, d\mu_{G/K}(gK) \]

is a solution of (6.1). From the normalization condition (6.2), $u_f$ satisfies the initial condition $u_f(0,x) = f(x)$. Rephrased in terms of the generator for the heat semigroup, it becomes $e^{t\Delta} f = k_t * f$ for all $f \in L^p(G/K), 1 \leq p < \infty$. 

\[ u_f(x,t) = (k_t * f)(x) = \int_{G/K} k_t(g^{-1}x) f(gK) \, d\mu_{G/K}(gK) \]
Theorem 6.1. Let $G/K$ be a Riemannian symmetric space and $k_t(x)$ its heat kernel. Then $k_t$ has an extension to a function on $\mathbb{R}^+ \times \Xi$ which is holomorphic on $\Xi$.

Proof. First recall from (1.1) that the Iwasawa projection $a: G/K \to A$ extends holomorphically to $a: \Xi \to T_\Omega$. Hence the theorem will follow from (6.4) provided we can show that for any compact subset $Q \subseteq \Xi$ there exists a constant $C_Q > 0$ such that

\[(\forall \lambda \in i\alpha^*) \sup_{z \in Q} |a(z)^{\rho-\lambda}| \leq C_Q e^{\lambda |\alpha|}.\]

Since $Q$ is compact, we have that

\[\sup_{z \in Q} |\text{Re} \log a(z)| < \infty.\]

But then (6.5) is immediate from (1.1) and the theorem follows.

If $f$ is a function on $G/K$ which extends holomorphically to $\Xi$, we write $f^\sim$ for the holomorphic extension.

In view of Theorem 6.1, the map

\[(6.6) \quad H_t: C_c(G/K) \to O(\Xi), \quad f \mapsto H_t(f); \quad H_t(f)(z) = \int_{G/K} k_t^\sim(g^{-1}z) f(gK) \, d\mu_{G/K}(gK)\]

is a well-defined $G$-equivariant map. The map $H_t$ is a $G$-equivariant version of the familiar Bargmann-Segal transform which we refer to as the heat kernel transform. Notice that we have

\[H_t(f) = (k_t * f)^\sim\]

for all $f \in C_c(G/K)$.

The Fourier transform on $G/K$ and the Paley-Wiener space

Helgason [He94, Ch. III, §1] defines the Fourier-transform on $G/K$ by

\[\mathcal{F}: L^2(G/K) \to L^2(K/M \times i\alpha^*, \, d\mu_{K/M} \otimes \frac{d\lambda}{|c(\lambda)|^2})\]

\[(\mathcal{F}f)(kM, \lambda) = \int_{G/K} f(gK) \, a(k^{-1}g)^{\rho-\lambda} \, d\mu_{G/K}(gK).\]

This is an isomorphism of Hilbert spaces with inverse

\[\mathcal{F}^{-1}(f)(gK) = \int_{K/M \times i\alpha^*} F(kM, \lambda) \, a(k^{-1}g)^{\rho+\lambda} \, d\mu_{K/M}(kM) \, \frac{d\lambda}{|c(\lambda)|^2}.\]

Define the Paley-Wiener space by

\[\text{PW}(G/K) = \mathcal{F}^{-1}(C^\omega(K/M) \otimes C^\infty_c(i\alpha^*)).\]

Then this is a dense subspace of $L^2(G/K)$. 

Lemma 6.2. All functions in $\text{PW}(G/K)$ extend holomorphically to $\Xi$, i.e., we have a $G$-equivariant injective mapping

$$\text{PW}(G/K) \to \mathcal{O}(\Xi), \ f \mapsto f^\sim.$$  

Proof. Given the definition of the Paley-Wiener space and the Fourier transform as explained above, the proof reduces to the estimate (6.5) which was established in the proof of Theorem 6.1.

By the definition of the heat kernel we have

$$(6.7) \quad \mathcal{F}(k_t)(\lambda, kM) = e^{-t(|\lambda|^2 + |\rho|^2)}.$$  

Lemma 6.3. Let $t > 0$. Then the following assertions hold:

(i) $k_t * \text{PW}(G/K) \subseteq \text{PW}(G/K);$  
(ii) If $f \in \text{PW}(G/K)$, then $k_t * f$ extends holomorphically to $\Xi$. In particular, the heat kernel transform is well defined on $\text{PW}(G/K)$ and we obtain a $G$-equivariant map

$$H_t: \text{PW}(G/K) \to \mathcal{O}(\Xi), \ f \mapsto (k_t * f)^\sim.$$  

Proof. In view of Lemma 6.2, (ii) follows from (i). Thus it suffices to prove (i). For that let $f \in \text{PW}(G/K)$. By definition of $\text{PW}(G/K)$ we have $f = \mathcal{F}^{-1}(F)$ for some $F \in C^\omega(K/M) \otimes C^\infty_c(i^\ast)$. From (6.7) we obtain for all $g \in G$ that

$$(k_t * f)(gK) = \mathcal{F}^{-1}(\mathcal{F}(k_t * f)) = \mathcal{F}^{-1}(\mathcal{F}(k_t) \mathcal{F}(f)) = \mathcal{F}^{-1}(\mathcal{F}(k_t)F)$$

$$= \int_{K/M} \int_{i^\ast} e^{-t(|\lambda|^2 + |\rho|^2)} F(kM, \lambda) a(k^{-1}g)^{\sigma + \lambda} \frac{d\lambda}{|e(\lambda)|^2} \ d\mu_{K/M}(kM).$$

Then the assertion in (i) is immediate from this.

Consider $\mathcal{O}(\Xi)$ as a Fréchet space with the topology of compact convergence. From (6.5) and (6.8) one can show that the heat kernel transform $H_t$ is well-defined on $L^2(G/K)$ for $t > 0$. Moreover,

$$H_t: L^2(G/K) \to \mathcal{O}(\Xi), \ f \mapsto (k_t * f)^\sim$$

is an injective $G$-equivariant continuous mapping. Consequently, $\mathcal{G}_t(\Xi) := H_t(L^2(G/K)$ equipped with the topology of $L^2(G/K)$ is a $G$-invariant Hilbert space of holomorphic functions on $\Xi$ (cf. Section 5), hence has a reproducing kernel

$$\mathcal{K}^t: \Xi \times \Xi \to \mathbb{C}, \ (z, w) \mapsto \mathcal{K}^t(z, w)$$

holomorphic in the first and antiholomorphic in the second variable. Using the technique to prove Theorem 5.1 one readily obtains

Theorem 6.4. The reproducing kernel for $\mathcal{G}_t(\Xi), \mathcal{K}^t(z, w)$ is given by

$$\mathcal{K}^t(z, w) = \int_{G/K} k_t(g^{-1}z) \overline{k_t(g^{-1}w)} \ d\mu_{G/K}(gK) \quad (z, w \in \Xi).$$

Moreover,

$$\mathcal{K}^t(z, w) = \int_{i^\ast} K_\lambda(z, w) e^{-2t(|\lambda|^2 + |\rho|^2)} \frac{d\lambda}{|c(\lambda)|^2}$$

uniformly in $z, w \in \Xi$.  


§7 Hermitian symmetric subdomain of $\Xi$

In addition to the intrinsic importance of the domain $\Xi$, it also contains some natural subdomains whose properties we think should be investigated. An introduction to these subdomains will be the topic of these last two sections. In this section we shall show that $\Xi$ coming from a classical group $G$ contains a large $G$-invariant subdomain, $\Xi_0 \subseteq \Xi$, that is $G$-biholomorphic to a Hermitian symmetric space of tube type. In some cases $\Xi_0 = \Xi$, and these we identify. For such groups $G$ one has the intriguing situation that the maximal Grauert domain of a Riemannian symmetric space of non-compact type, $G/K$, is bi-holomorphic though not isometric to a Hermitian symmetric space. Hence, from Proposition 1.3, the harmonic analysis on $G/K$ extends holomorphically to this Hermitian symmetric space. These results were explicitly mentioned in [KS04] and the point of this section is to present the detailed argument.

The setup for this structure originates, we believe, with the paper by Nagano ([N65]) and subsequently has had many incarnations as, e.g., symmetric R-spaces, real forms of Hermitian symmetric spaces, real forms of simple Jordan triple systems, causal symmetric spaces, etc. We shall move freely between these different equivalent versions as necessary, but shall begin from the viewpoint of Jordan theory.

We consider first Riemannian symmetric spaces $G/K$ associated to Euclidean Jordan algebras in order to illustrate how $\Xi_0$ arises, but more importantly because this will lead to another domain in §8 seemingly more related to algebraic geometry.

$G/K$ associated to Euclidean Jordan algebras

We recall some basic facts about Jordan algebras, referring the reader to [FK94] for a particularly nice presentation of the details. Let $V$ be a Euclidean Jordan algebra. For $x \in V$ define $L(x) \in \text{End}(V)$ by $L(x)y = xy$, $y \in V$. We denote by $e$ the identity element of $V$. The symmetric cone $W \subseteq V$ can be defined by

$$W = \text{int}\{x^2; x \in V\},$$

where $\text{int}\{\cdot\}$ denotes the topological interior of $\{\cdot\}$. Let $G = \text{Aut}_0(W)$ be the identity component of the automorphism group of $W$. Then $G$ is a reductive subgroup of $\text{Gl}(V)$. The isotropy group at $e$

$$K = G_e = \{g \in G; g(e) = e\}$$

is a maximal compact subgroup of $G$, and the map

$$G/K \rightarrow W, \quad gK \mapsto g(e)$$

is a homeomorphism.

Complexify $V$ to $V_C = V \oplus iV$ and consider the tube domain

$$T_W = V + iW \subseteq V_C.$$

The identity component of the complex automorphism group of $T_W$ we denote by $G^h$. Set $K^h = G^h_{ie}$, the stabilizer of $ie \in T_W$. Then $K^h$ is a maximal compact subgroup of $G^h$ and contains $K$. The map

$$G^h/K^h \rightarrow T_W, \quad gK^h \mapsto g(ie)$$

is a biholomorphism of the Hermitian symmetric space $G^h/K^h$ onto $T_W$ giving an explicit realization of $G^h/K^h$ as a tube domain. Clearly one has a natural inclusion of the real symmetric space $G/K$ into the Hermitian space $G^h/K^h$. 
For the convenience of the reader we list the irreducible Euclidean Jordan algebras and their associated groups $G$ and $G^h$ (cf. [FK94, p. 213]).

**Table I**

| $V$                      | $G$                           | $G^h$                    |
|-------------------------|-------------------------------|--------------------------|
| $\text{Symm}(n, \mathbb{R})$ | $\text{Sl}(n, \mathbb{R}) \times \mathbb{R}^+$ | $\text{Sp}(n, \mathbb{R})$ |
| $\text{Herm}(n, \mathbb{C})$ | $\text{Sl}(n, \mathbb{C}) \times \mathbb{R}^+$ | $\text{SU}(n, n)$        |
| $\mathbb{R} \times \mathbb{R}^n$ | $\text{SO}(1, n) \times \mathbb{R}^+$ | $\text{SO}(2, n + 1)$   |
| $\text{Herm}(3, \mathbb{O})$ | $\text{E}_6(-26) \times \mathbb{R}^+$ | $\text{E}_7(-25)$        |
| $\mathbb{C} \times \mathbb{R}^n$ | $\text{SU}(n, n)$ | $\text{SO}^*(4n)$        |

Next let $c_1, \ldots, c_l$ be a Jordan frame (cf. [FK94, p. 44]) of $V$ and set $V^0 = \bigoplus_{j=1}^l \mathbb{R} c_j$. Then $K(V^0) = V$ (cf. [FK94, Cor. IV.2.7]).

**Lemma 7.1.** $T_W = K \cdot ((V^0) \cap T_W)$.

**Proof.** The inclusion $\supseteq$ is clear. Conversely, notice that 

$$((V^0) \cap T_W) = V^0 + i(W \cap V^0).$$

So let $z = x + iy \in T_W$, $x \in V$, $y \in W$. Then we find a $g \in G$ such that $g(y) = e$ and thus $g(z) = g(x) + ie$. Since $V = K(V^0)$, we find a $k \in K$ such that $k(g(x)) \in V^0$. As $k(e) = e$ we get $(kg)(z) = (kg)(x) + ie \in ((V^0) \cap T_W)$, providing the other containment.

The choice $a = \bigoplus_{j=1}^l \mathbb{R} L(c_j)$ defines a maximal abelian subspace orthogonal to $k$. As can be seen from Table I, the root system $\Sigma = \Sigma(g, a)$ is classical and of type $A_{l-1}$. If we define $\varepsilon_j \in a^*$ by $\varepsilon_j(L(c_i)) = \delta_{ij}$, then we have

$$\Sigma = \left\{ \frac{1}{2}(\varepsilon_i - \varepsilon_j) : i \neq j \right\}$$

(cf. [FK94, Prop. VI.3.3]). Thus

$$\Omega = \left\{ x = \sum_{j=1}^l x_j L(c_j) : x_j \in \mathbb{R}, |x_i - x_j| < \pi \right\}.$$ 

In particular,

$$\Omega_0 = \bigoplus_{j=1}^l \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] L(c_j) \subseteq \Omega.$$ 

From Lemma 1.4 in [KS04] it follows that the domain

$$\Xi_0 = G \exp(i\Omega_0) K_C / K_C \subseteq G_C / K_C$$

is a $G$-invariant open subdomain of $\Xi$. The interest in $\Xi_0$ is provided by the next result.

**Theorem 7.2.** The map

$$\Xi_0 \rightarrow T_W, \ g K_C \mapsto g(ie)$$

is a biholomorphism, i.e. $\Xi_0$ is biholomorphic to a Hermitian symmetric space.

**Proof.** In view of Lemma 7.1 it suffices to show that

$$T_{\Omega}(ie) = ((V^0) \cap T_W).$$

But since $((V^0) \cap T_W) = \bigoplus_{j=1}^l (\mathbb{R} + i\mathbb{R}^+) c_j$, this is immediate from the definition of $T_{\Omega}$. 


Compactly causal symmetric spaces

In fact, the characterization of \( \Xi_0 \) in the previous subsection will be seen to be a special case of results in this section. For the more general results one could use Jordan triple systems. Instead, we find it easier to use compactly causal symmetric spaces \( S/G \).

Let \( \mathfrak{s} \) be a semisimple real Lie algebra and \( \theta \) a Cartan involution on \( \mathfrak{s} \) with Cartan decomposition \( \mathfrak{s} = \mathfrak{u} \oplus \mathfrak{p}^* \). Let \( \tau : \mathfrak{s} \to \mathfrak{s} \) be an involution on \( \mathfrak{s} \) which, as we may, commutes with \( \theta \). Write \( \mathfrak{s} = \mathfrak{g} \oplus \mathfrak{q} \) for the \( \tau \)-eigenspace decomposition corresponding to the \( \tau \)-eigenvalues \(+1\) and \(-1\). Set \( \mathfrak{k} = \mathfrak{g} \cap \mathfrak{u} \) and \( \mathfrak{p} = \mathfrak{g} \cap \mathfrak{p}^* \). Then \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) is a Cartan decomposition of \( \mathfrak{g} \).

The symmetric Lie algebra \((\mathfrak{s}, \tau)\) is called irreducible if the only \( \tau \)-invariant ideals of \( \mathfrak{s} \) are \( \{0\} \) and \( \mathfrak{s} \). An irreducible semisimple symmetric Lie algebra \((\mathfrak{s}, \tau)\) is called compactly causal if, in the notation above, \( \mathfrak{z}(\mathfrak{u}) \cap \mathfrak{q} \neq \{0\} \).

For the rest of the section \( \mathfrak{s} \) will denote a compactly causal Lie algebra. For the convenience of the reader we recall the various types of compactly causal Lie algebras.

**Remark 7.3.** (a) Suppose that \( \mathfrak{s} \) is simple. Then \( \mathfrak{z}(\mathfrak{u}) \neq \{0\} \) implies \( \mathfrak{s} \) is Hermitian, i.e. the symmetric space associated to \( \mathfrak{s} \) is Hermitian. In this case we rename \( \mathfrak{s} \) to be \( \mathfrak{g}^h \). The compactly causal symmetric pairs \((\mathfrak{g}^h, \mathfrak{g})\) are in Table II (cf. [HÓ96, Th. 3.2.8]).

| \( \mathfrak{g} \) | \( \mathfrak{g}^h \) |
|-----------------|-----------------|
| \( \mathfrak{sl}(n, \mathbb{R}) \oplus \mathbb{R} \) | \( \mathfrak{sp}(n, \mathbb{R}) \) |
| \( \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R} \) | \( \mathfrak{su}(n, n) \) |
| \( \mathfrak{so}(n, \mathbb{H}) \oplus \mathbb{R} \) | \( \mathfrak{so}^*(4n) \) |
| \( \mathfrak{so}(1, n) \oplus \mathbb{R} \) | \( \mathfrak{so}(2, n + 1) \) |
| \( \mathfrak{e}_6(-26) \oplus \mathbb{R} \) | \( \mathfrak{e}_7(-25) \) |
| \( \mathfrak{so}(p, q) \) | \( \mathfrak{su}(p, q) \) |
| \( \mathfrak{sp}(p, q) \) | \( \mathfrak{su}(2p, 2q) \) |
| \( \mathfrak{so}(n, \mathbb{C}) \) | \( \mathfrak{so}^*(2n) \) |
| \( \mathfrak{sp}(n, \mathbb{C}) \) | \( \mathfrak{sp}(2n, \mathbb{R}) \) |
| \( \mathfrak{so}(p, 1) \oplus \mathfrak{so}(q, 1) \) | \( \mathfrak{so}(2, p + q) \) |
| \( \mathfrak{sp}(2, 2) \) | \( \mathfrak{e}_6(-14) \) |
| \( \mathfrak{e}_8(-20) \) | \( \mathfrak{e}_6(-14) \) |
| \( \mathfrak{su}^*(8) \) | \( \mathfrak{e}_7(-25) \) |

Notice that the first 5 listings in Table II are those \( \mathfrak{g} \) which arise as automorphism groups of the cone in a Euclidean Jordan algebra. One calls such \((\mathfrak{g}^h, \mathfrak{g})\) compactly causal symmetric pairs of Cayley type.

(b) Suppose that \( \mathfrak{s} \) is not simple. Then it turns out that \( \mathfrak{s} = \mathfrak{g} \oplus \mathfrak{g} \) with \( \tau \) the flip involution \( \tau(X, Y) = (Y, X) \) and \( \mathfrak{g} \cong \mathfrak{g}^h \) a simple Hermitian Lie algebra (cf. [HÓ96, Lemma 1.3.7(2)]). We shall refer to this as the "group case". Of course the familiar simple Hermitian Lie algebras are, up to isomorphism,

\[
\begin{align*}
\mathfrak{su}(p, q) & \quad \mathfrak{so}^*(2n) \\
\mathfrak{sp}(n, \mathbb{R}) & \quad \mathfrak{sp}(2, n) \quad \mathfrak{so}(2n) \quad \mathfrak{e}_6(-14) \quad \mathfrak{e}_7(-25).
\end{align*}
\]

(c) From the tables in (a) and (b) one can observe the folklore fact that up to possibly a direct summand of \( \mathbb{R} \), every classical simple real Lie algebra is the fixed point algebra \( \mathfrak{g} \) in a compactly causal
causal symmetric Lie algebra \((\mathfrak{s}, \tau)\), or said differently, every Riemannian symmetric space of noncompact type, \(G/K\), with \(G\) a classical group is the fixed point set of an anti-holomorphic involution on some Hermitian symmetric space.

To generalize the previous subsection we need to define the analog of \(\Omega_0\) and then \(\Xi_0\). Unavoidably this will entail a little structure theory. Let \(t \subseteq u\) be a \(\tau\)-stable compact Cartan subalgebra of \(\mathfrak{s}\) and \(\Delta = \Delta(\mathfrak{g}_C, \mathfrak{t}_C)\) the associated root system. Let \(X_0 \in \mathfrak{g}(u) \cap \mathfrak{q}\) be normalized by Spec(\(\text{ad} X_0\)) = \{-i, 0, i\}. We follow the classic treatment of Hermitian symmetric spaces by Harish-Chandra. Choose a positive system \(\Delta^+\) of \(\Delta\) such that \(\Delta^+ \subseteq \{\alpha \in \Delta: i\alpha(X_0) \geq 0\}\). The set of non-compact roots in \(\Delta\), i.e., the set of roots \(\alpha\) for which \(\mathfrak{g}_s^\alpha \subseteq \mathfrak{p}_s\) is denoted \(\Delta_n\), and, with \(\mathfrak{p}^\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_s^\alpha\), we have \(\mathfrak{p}_s = \mathfrak{p}^+ \oplus \mathfrak{p}^-\). Then

\[\mathfrak{g}_s = \mathfrak{p}^- \oplus \mathfrak{uc} \oplus \mathfrak{p}^+\]

Take \(\Gamma = \{\gamma_1, \ldots, \gamma_r\} \subseteq \Delta^+_n\) a maximal set of long strongly orthogonal roots. By [HÓ96, Lemma 4.1.7] we may assume that \(\Gamma\) is \(-\tau\)-invariant. Let \(\mathfrak{a}_s \subseteq \mathfrak{p}_s\) be the maximal abelian subspace of \(\mathfrak{p}_s\) which is constructed from \(\Gamma\). Since \(\Gamma\) is \(-\tau\)-invariant, \(\mathfrak{a}_s\) is \(\tau\)-invariant. Hence \(\mathfrak{a}_s = \mathfrak{a} \oplus \mathfrak{f}\) with \(\mathfrak{a} = \mathfrak{a}_s \cap \mathfrak{g}\) and \(\mathfrak{f} = \mathfrak{a}_s \cap \mathfrak{q}\). Now the crucial observation is that \(\mathfrak{a}\) is maximal abelian in \(\mathfrak{p}\) by [HÓ96, Lemma 4.1.9] so we define restricted root systems \(\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})\) and \(\tilde{\Sigma} = \tilde{\Sigma}(\mathfrak{s}, \mathfrak{a})\). Let \(\sigma_1, \ldots, \sigma_s\) be the set of restrictions of \(\Gamma \circ C\) to \(\mathfrak{a}\) where \(C: \mathfrak{s}_C \to \mathfrak{s}_C\) is the Cayley transform with \(C^{-1}(\mathfrak{a}) \subseteq i(\mathfrak{t} \cap \mathfrak{g})\). Now it turns out that either \(r = s\) (as for Cayley type spaces \((\mathfrak{g}^h, \mathfrak{g})\)) or \(r = 2s\) (as in the case \((\mathfrak{s}, \mathfrak{g}) = (\mathfrak{g}^h \oplus \mathfrak{g}^h, \mathfrak{g}^h))\).

From various results of Olafsson one concludes that the restricted root system \(\tilde{\Sigma}\) is either of type \(C_s\) or \(BC_s\), (for a proof cf. [K01, Th. 4.4]). Then

\[\tilde{\Sigma} = \left\{\frac{1}{2}(\pm \sigma_i, \pm \sigma_j): 1 \leq i, j \leq s\right\} \cup \{0\}\]

or

\[\tilde{\Sigma} = \left\{\frac{1}{2}(\pm \sigma_i, \pm \sigma_j): 1 \leq i, j \leq s\right\} \cup \left\{\pm \frac{1}{2} \sigma_i: 1 \leq i \leq s\right\}\]

We are ready to define a domain \(\Omega_0\) in \(\mathfrak{a}\). Set

\[\Omega_0 = \Omega(\tilde{\Sigma}) = \{X \in \mathfrak{a}: (\forall \alpha \in \tilde{\Sigma}) |\alpha(X)| < \frac{\pi}{2}\}\]

Since \(\Sigma \subseteq \tilde{\Sigma}\) it is clear that \(\Omega_0 \subseteq \Omega = \Omega(\Sigma)\). Also, if we set \(T_{\Omega_0} = A \exp(i\Omega_0)\) then \(T_{\Omega_0} \subseteq T_{\Omega}\).

Lemma 7.4.

\[\Omega_0 = \{X \in \mathfrak{a}: (\forall \sigma_i) |\sigma_i(X)| < \frac{\pi}{2}\}\]

Proof. Since \(\tilde{\Sigma}\) is of type \(C_s\) or \(BC_s\), this is immediate from the definition of \(\Omega_0\).}

To define \(\Xi_0\) we remain with a compactly causal Lie algebra \(\mathfrak{s}\). As seen from examples (a) and (b) in Remark 7.3 \(s\) will be either a simple Hermitian algebra or the product of two such. Let \(S\) be a connected Lie group with Lie algebra \(\mathfrak{s}\) and let \(G, K, U\) the analytic subgroups of \(S\) with Lie algebras \(\mathfrak{g}, \mathfrak{t}\) and \(\mathfrak{u}\). We will assume that \(S\) is contained in its complexification \(S_C\). Let \(P^\pm\) be the analytic subgroup of \(S_C\) corresponding to \(\mathfrak{p}^\pm\).

Write \(D \subseteq \mathfrak{p}^+\) for the Harish-Chandra realization of \(S/U\) as a bounded symmetric domain. It is convenient for us to consider \(D\) as an open subset in the flag manifold \(S_C/U_C P^-\). Hence \(0 \in D\) becomes identified with the identity coset in \(S_C/U_C P^-\).

We define the domain

\[\Xi_0 = G \exp(i\Omega_0) K_C / K_C\]

Since \(\Omega_0 \subseteq \Omega\), \(\Xi_0\) is contained in \(\Xi\) and by the result in [KS04] is a \(G\)-invariant open domain. Clearly, \(\Xi_0\) contains \(T_{\Omega_0}\).
Theorem 7.5. Let $S/G$ be a compactly causal symmetric space and let $D$ be the Harish-Chandra realization of the Hermitian symmetric space $S/U$. Then the map
\[ \Phi: \Xi_0 \to D, \quad xK_c \mapsto x(0) \]
is a $G$-equivariant biholomorphism. In particular, $\Xi_0$ is Stein.

Proof. First we show that $\Phi(T_{\Omega_0}) \subseteq D$. So define elements $H_j \in a, \ 1 \leq j \leq s$, by $\sigma_k(H_j) = 2\delta_{jk}$. Recall from Lemma 7.4 that
\[ \Omega_0 = \sum_{j=1}^{s} -\frac{\pi}{4}, \frac{\pi}{4} [H_j]. \]
In [HÓ96, Lemma 5.1.5] it is shown that there exists pairwise orthogonal elements $E_1, \ldots, E_s$ of $p^+ \cap (g \oplus iq)$ (if $r = s$, then $E_j \in g_{C}^{\gamma_4}$ while if $r = 2s$, then, after a possible renumbering of the elements in $\Gamma$, one has $E_j \in g_{C}^{\gamma_{2r-1}} \oplus g_{C}^{\gamma_{2r}}$) such that for $x_j \in \mathbb{R}$
\[ \exp(\sum_{j=1}^{s} x_jH_j)(0) = \sum_{j=1}^{s} (tanh x_j)E_j. \]
Since $a + i\Omega_0 = \sum_{j=1}^{s} \mathbb{R}H_j + i\left\{-\frac{\pi}{4}, \frac{\pi}{4}\right\}[H_j]$ and the map
\[ \tanh: \mathbb{R} + i\left\{-\frac{\pi}{4}, \frac{\pi}{4}\right\} \to \{z \in \mathbb{C} : |z| < 1\}, \quad z \mapsto \tanh z \]
is a biholomorphism,
\[ T_{\Omega_0}(0) = \left\{ \sum_{j=1}^{s} z_jE_j : z_j \in \mathbb{C}, \ |z_j| < 1\right\}. \]
But then $T_{\Omega}(0) \subseteq D$.

We still have to show that $\Phi$ is onto. From [S84, Prop. 7.1.2] the map
\[ p \times (p_\ast \cap q) \to S/U, \quad (X,Y) \mapsto \exp(X)\exp(Y)U \]
is a diffeomorphism (here $p = p_\ast \cap q$). Let $\mathfrak{e}$ be a maximal abelian subspaces of $p_\ast \cap q$. All such $\mathfrak{e}$ are conjugate under $Ad(K)$. Hence
\[ D = G\exp(\mathfrak{e})(0). \]
So it would be enough to show there exists an $\mathfrak{e}$ with $\exp(\mathfrak{e})(0) \subseteq T_{\Omega}(0)$. Recall the element $X_0 \in \mathfrak{g}(u) \cap q$ and define an automorphism $J$ of $\mathfrak{s}$ by $J = e^{ad \frac{1}{2}X_0}$. Of course $J$ also induces multiplication by $i$ on $D$. A simple computation shows that $J(p) = p_\ast \cap q$ (cf. [Ó91, Lemma 1.4 (2)]). In particular, $\mathfrak{e} = J(\mathfrak{a})$ defines a maximal abelian subspace $\mathfrak{e}$ in $p_\ast \cap q$. So we get
\[ \exp(\mathfrak{e})(0) = i\exp(\mathfrak{a})(0). \]
As $i\exp(\mathfrak{a})(0) \subseteq T_{\Omega}(0)$, the proof of the theorem is complete. $\blacksquare$

An immediate consequence of Theorem 7.5 in conjunction with Proposition 1.3 is the following result. T. Kobayashi informs us that independently he and Faraut had also obtained this result in uncirculated personal notes.
Theorem 7.6. Assume that a Riemannian symmetric space \( G/K \) is a totally real form of a Hermitian symmetric space \( S/U \) via a \( G \)-equivariant embedding \( G/K \hookrightarrow S/U \). Then all eigenfunctions on \( G/K \) for the algebra of \( G \)-invariant differential operators \( \mathcal{D}(G/K) \) extend holomorphically to \( S/U \). ■

Example - the group case. Here we have \( S = G \times G \) and \( G = G^h \). In particular \( u = t \times k \) and \( U = K \times K \). Further \( t = t_G \times t_G \) with \( t_G \) a compact Cartan algebra of \( t \) in \( g \). The element \( X_0 \in \mathfrak{g}(u) \cap \mathfrak{q} \) is such that \( X_0 = (X'_0, -X'_0) \) with \( X'_0 \in \mathfrak{g}(t) \) such that \( \text{Spec ad}_g(X'_0) = \{-i, 0, i\} \).

Let \( g_C = p_G \oplus t_C \oplus p_G^\perp \) be the triangular decomposition of \( g_C \) with respect to \( X'_0 \). Then we have

\[
p^+ = p_G^+ \times p_G^+ \quad \text{and} \quad p^- = p_G^\perp \times p_G^+.
\]

Write \( \mathcal{D}_G \) for the Harish-Chandra realization of \( G/K \) in \( p_G^+ \) and \( \overline{\mathcal{D}}_G \) for the Harish-Chandra realization of \( G/K \) in \( p_G^\perp \). Then

\[
\mathcal{D} = \mathcal{D}_G \times \overline{\mathcal{D}}_G.
\]

Write \( \mathcal{D}^{opp}_G \) for \( \mathcal{D}_G \) equipped with the opposite complex structure. Write \( X \mapsto \overline{X} \) for the complex conjugation in \( g_C \) with respect to the real form \( g \). Note that

\[
\overline{\mathcal{D}}_G \to \mathcal{D}^{opp}_G, \quad X \mapsto \overline{X}
\]
is a \( G \)-equivariant biholomorphism. Denote by 0 the origin in \( \mathcal{D} \).

Obviously we have \( \hat{\Sigma} = \Sigma \) in these cases. Thus \( \Xi = \Xi_0 \) and Theorem 7.5 reads as:

Theorem 7.7. Assume that \( G/K \) is a Hermitian symmetric space. Then the map

\[
\Phi: \Xi \to \mathcal{D}_G \times \mathcal{D}^{opp}_G, \quad xK_C \mapsto (x(0), x(0))
\]
is a \( G \)-equivariant biholomorphism. In particular, \( \Xi \) is Stein. ■

The cases where \( \Xi = \Xi_0 \)

We have already seen that \( \Xi = \Xi_0 \) in the group case. Here we give some simple criteria to determine when \( \Xi = \Xi_0 \) in general. An independent and more differential geometric approach to this result is in [BHH03].

Theorem 7.8. The following are equivalent:

1. \( \Xi = \Xi_0 \);
2. \( \Omega_0 = \Omega \);
3. \( \Sigma \) is of type \( C_s \) or \( BC_s \) for \( s \geq 2 \) or \( \Sigma = \hat{\Sigma} \) if \( \Sigma \) has rank one;
4. \( \text{rank}_{\mathbb{R}} g = \frac{1}{2} \text{rank}_{\mathbb{R}} \mathfrak{s} \).

Proof. (1) \( \iff \) (2) is just the definition.

(2) \( \iff \) (3): In the following we write \( \Omega = \Omega(\Sigma) \) in order to make the dependence clear on the root system. For a classical root system \( \Sigma \) one easily verifies the following facts:

- \( \Omega(C_n) = \Omega(BC_n) \) for all \( n \geq 1 \).
- \( \Omega(B_n) = \Omega(D_n) \) for all \( n \geq 3 \).
• \(\Omega(C_n) \subseteq \Omega(A_n)\) for \(n \geq 2\).
• \(\Omega(C_n) \subseteq \Omega(B_n)\) for \(n \geq 3\).

Recall that the root system \(\hat{\Sigma}\) is of type \(C_n\) or \(BC_n\). A quick look at Table II tells us that the root system \(\Sigma\) is always classical for all \(g\) in question. From that the equivalence (2) \(\iff\) (3) is easily verified if rank \(\Sigma \geq 2\). The various rank \(\Sigma = 1\) cases one checks separately with Table II.

(3) \(\iff\) (4) This is easily checked with Table II.

From Theorem 7.8 we arrive at the following table:

| \(g\)   | \(g^h = s\) |
|--------|--------------|
| \(sp(p, q)\) | \(su(2p, 2q)\) |
| \(sp(n, \mathbb{C})\) | \(sp(2n, \mathbb{R})\) |
| \(so(1, p)\) | \(so(2, p)\) |

Table III
\(\Xi = \Xi_0\) and \(s\) simple

Remark 7.9. (a) Suppose that \(G/K\) is such that there is a Hermitian symmetric space \(S/U\) containing \(G/K\) as a totally real submanifold. Using the proof of Theorem 7.5 and the structure of \(\Sigma\), one can show that not only the subdomain \(\Xi_0\), biholomorphic to \(S/U\), embeds in the projective variety \(S_C/U_C P^-\), but so does \(\Xi\) embed in it.

(b) In [BHH03] the spaces with \(\Xi \neq \Xi_0\) are called rigid because, as they show, \(\text{Aut}_0(\Xi_0) = G\) if \(\Xi \neq \Xi_0\). With hindsight and the observation in (a), this can be deduced easily using that the generic \(S\)-orbits in \(S_C/U_C P^-\) are open. Motivated by (a) we pose what seems to us also an interesting question. Consider the contraction semigroup of \(\Xi\), namely

\[ \Gamma = \{ s \in S_C : s\Xi \subseteq \Xi \}. \]

Then we ask if \(\Gamma\) reduces to \(G\) when \(\Xi \neq \Xi_0\)?

(c) In an Appendix at the end of the paper we illustrate the main results in this section with two explicit compactly causal spaces.

§8 \(\Xi^\downarrow\) - the “square root” of \(\Xi\)

In this section we will describe another interesting domain associated to \(G/K\)’s coming from Euclidean Jordan algebras \(V\) which we call the square root of \(\Xi\), and denote by \(\Xi^\downarrow\). The square root domain \(\Xi^\downarrow\) is strictly smaller than \(\Xi\) and \(\Xi_0\) but arises naturally as the maximal domain of definition for the natural polarization of the metric on the symmetric cone \(W \simeq G/K\).

Characterization of \(\Xi^\downarrow\)

Throughout this section \(G/K\) is associated to a Euclidean Jordan algebra \(V\) as described earlier. We will identify \(G_C/K_C\) as a subset of \(V_C\) by means of the orbit map

\[ G_C/K_C \to V_C, \quad gK_C \mapsto g(e). \]

We denote by (\(\cdot|\cdot\)) the Hermitian extension of the Jordan inner product on \(V\) to \(V_C\). Let \(z \mapsto \overline{z}\) be the complex conjugation with respect to the real form \(V\). The technical aspects of the proofs will use the Pierce decomposition of a Euclidean Jordan algebra (cf. [FK94, Th. IV.2.1]). We
recall it briefly. For every \( \lambda \in \mathbb{R} \) and \( x \in V \) we write \( V(x, \lambda) \) for the \( \lambda \)-eigenspace of the symmetric operator \( L(x) \in \text{End}(V) \). Set
\[
V_{ij} = V(c_i, \frac{1}{2}) \cap V(c_j, \frac{1}{2}) \quad (1 \leq i < j \leq n)
\]
and \( V_{ii} = \mathbb{R}c_i \) for \( 1 \leq i \leq n \). Then
\[
(8.1) \quad V = \bigoplus_{i \leq j} V_{ij}
\]
and
\[
(8.2) \quad V_{ij} \cdot V_{ij} \subseteq V_{ii} + V_{jj} \quad V_{ij} \cdot V_{jk} \subseteq V_{ik} \quad \text{if } i \neq k \quad V_{ij} \cdot V_{kl} = \{0\} \quad \text{if } \{i,j\} \cap \{k,l\} = \emptyset.
\]
Here \( V_{ij} \perp V_{kl} \) if \( (i,j) \neq (k,l) \).

The quadratic representation (cf. [FK94, Ch. II, §3]) of the Jordan algebra \( V_\mathbb{C} \) is defined by
\[
P(z) = 2L(z)^2 - L(z^2) \in \text{End}(V_\mathbb{C}) \quad (z \in V_\mathbb{C}),
\]
and its polarized version is defined by
\[
P(z, w) = L(z)L(w) + L(w)L(z) - L(zw) \quad (z, w \in V_\mathbb{C}).
\]
For this we have the transformation property
\[
(8.3) \quad P(gz, gw) = gP(z, w)g^t \quad (\forall g \in G, \ z, w \in V_\mathbb{C})
\]
(cf. [FK94, Prop. VIII.2.4]).

**Lemma 8.1.** For every \( z \in V_\mathbb{C} \) the operator \( P(z, \overline{z}) \) is Hermitian.

**Proof.** For \( A \in \text{End}(V_\mathbb{C}) \) let \( A \mapsto A^* \) be the conjugate transpose map. Then \( L(z)^* = L(\overline{z}) \) for all \( z \in V_\mathbb{C} \). From this and the definition of \( P(z, w) \) the assertion follows. \( \square \)

Set \( \Omega^\frac{1}{2} = \frac{1}{2}\Omega \) and define the square root of \( \Xi \) by
\[
\Xi^\frac{1}{2} = G \exp(i\Omega^\frac{1}{2})K_\mathbb{C}/K_\mathbb{C} = G \exp(i\Omega^\frac{1}{2})(e).
\]
Again from [KS04] one has that \( \Xi^\frac{1}{2} \) is a \( G \)-invariant open subdomain in \( G_\mathbb{C}/K_\mathbb{C} \). From the definition of \( \Xi_0 \) one can see that \( \Xi^\frac{1}{2} \subseteq \Xi_0 \). The next result gives an algebro-geometric characterization of \( \Xi^\frac{1}{2} \).

**Proposition 8.2.**
\[
\Xi^\frac{1}{2} = \{ z \in GA_\mathbb{C}K_\mathbb{C}/K_\mathbb{C} \subseteq V_\mathbb{C}: P(z, \overline{z}) \text{ is positive definite} \}_0
\]
where \( \{\cdot\}_0 \) denotes the connected component of \( \{\cdot\} \) which contains \( e \).

**Proof.** In view of the transformation property (8.3), we have to show only that
\[
A \exp(i\Omega^\frac{1}{2})K_\mathbb{C}/K_\mathbb{C} = \{ z \in A_\mathbb{C}(e) \subseteq (V^0)_\mathbb{C}: P(z, \overline{z}) \text{ is positive definite} \}_0.
\]
Fix $z = \sum_{j=1}^{l} z_j c_j \in (V^0)_{C}$, $z_j \in C$. Let $V = \bigoplus_{i,j} V_{ij}$ be the Pierce decomposition of $V$ (cf. (8.1), (8.2)) and $V_C = \bigoplus_{i,j} (V_{ij})_{C}$ its complexification. Let $v \in (V_{ij})_{C}$. If $i \neq j$, then we have $L(c_k)v = \frac{1}{2}(\delta_{ik} + \delta_{jk})v$, while for $i = j$ we have $L(c_k)v = \delta_{ik}v$. So the definition of $P(z, w)$ gives

$$P(z, \overline{w})v = \begin{cases} \frac{1}{2}(z_i\overline{z_j} + \overline{z}_iz_j) & \text{for } i \neq j \\ |z_i|^2v & \text{for } i = j. \end{cases}$$

Then $P(z, \overline{w})$ is invertible if and only if

$$z_i\overline{z}_j + \overline{z}_iz_j \neq 0 \quad (\forall 1 \leq i, j \leq l).$$

Since $P(e) = \text{id}$ is positive definite, it follows from (i) and the continuity of the spectrum that

$$\{z \in A_C(e): P(z, \overline{w}) \text{ is positive definite} \} \subseteq \{z \in A_C(e): P(z, \overline{w}) \text{ is invertible} \}.$$  

Suppose now that $z \in A_C K_C / K_C$. Then $z = \exp(\sum_{j=1}^{l} w_j L(c_j))e$ for $w_j \in C$ and we have $z_j = e^{w_j}$. Hence (8.4) implies that $P(z, \overline{w})$ is invertible if and only if

$$|\text{Im}(w_i - w_j)| \not\in \frac{\pi}{2} + \mathbb{Z}\pi \quad (i \neq j).$$

In view of the definition of $\Omega^+$, this completes the proof.

---

The natural exhaustion function

Proposition 8.2 implies in particular that the function

$$\varphi: \Xi^+ \to \mathbb{R}, \quad z \mapsto \log \det P(z, \overline{w})^{-1}$$

is well defined and analytic. Our goal is to show that $\varphi$ is plurisubharmonic.

Let $Z \in V_{C}$ and $f$ a differentiable function defined on some open subset in $V_{C}$. Then we set

$$(\delta_Z f)(z) = \frac{d}{dt} \big|_{t=0} f(z + tZ).$$

For $Z \in V_{C}$ we define the usual Cauchy-Riemann operators

$$\partial_Z = \frac{1}{2}(\delta_Z - i\delta_{iZ}) \quad \text{and} \quad \overline{\partial}_Z = \frac{1}{2}(\delta_Z + i\delta_{iZ}).$$

**Lemma 8.3.** For all $Z_1, Z_2 \in V_{C}$ and $z \in \Xi^+$ we have

$$\langle \partial_Z, \overline{\partial}_{Z_2}, \varphi \rangle = \text{tr} \left[ P(z, \overline{w})^{-1} P(Z_1, \overline{w}) P(z, \overline{w})^{-1} P(Z_2, \overline{w}) \right] - \text{tr} \left[ P(z, \overline{w})^{-1} P(Z_1, \overline{w}) P(Z_2, \overline{w}) \right].$$

**Proof.** First notice that the definition of $\varphi$ implies that

$$(\partial_Z, \overline{\partial}_{Z_2}, \varphi)(z) = -\frac{d}{dt} \big|_{t=0} \frac{d}{ds} \big|_{s=0} \log \det P(z + tZ_1, \overline{w} + sZ_2).$$

For all $t \in \mathbb{R}$ we have

$$-\frac{d}{ds} \big|_{s=0} \log \det P(z + tZ_1, \overline{w} + sZ_2) = -\frac{1}{\text{det} P(z + tZ_1, \overline{w})} \cdot \text{det} P(z + tZ_1, \overline{w}) \cdot \text{tr} \left[ P(z + tZ_1, \overline{w})^{-1} \frac{d}{ds} \big|_{s=0} P(z + tZ_1, \overline{w}) \right].$$
Since
\[ \frac{d}{ds} \Big|_{s=0} P(z + tZ_1, \overline{\sigma} + s\overline{Z}_2) = L(z + tZ_1) L(\overline{Z}_2) + L(\overline{Z}_2) L(z + tZ_1) - L((z + tZ_1) \overline{Z}_2), \]
we obtain
\[ -\frac{d}{ds} \big|_{s=0} \log \det P(z + tZ_1, \overline{\sigma} + s\overline{Z}_2) = -\text{tr} \left[ P(z + tZ_1, \overline{\sigma})^{-1} L(z + tZ_1) L(\overline{Z}_2) + L(\overline{Z}_2) L(z + tZ_1) - L((z + tZ_1) \overline{Z}_2) \right]. \]
With that, finally we get
\[ (\partial_{Z_1} \overline{Z}_2 \varphi)(z) = -\frac{d}{dt} \big|_{t=0} \text{tr} \left[ P(z + tZ_1, \overline{\sigma})^{-1} L(z + tZ_1) L(\overline{Z}_2) + L(\overline{Z}_2) L(z + tZ_1) - L((z + tZ_1) \overline{Z}_2) \right] \]
concluding the proof of the lemma.

**Lemma 8.4.** For all \( g \in G, \; z \in \mathbb{H}^+ \) and \( Z \in V_C \) we have
\[ (\partial_{gZ} \overline{gZ} \varphi)(g z) = (\partial_{Z} \overline{Z} \varphi)(z). \]

**Proof.** For all \( g \in G \) we have by the transformation property (8.3) that
\[ \varphi(gz) = -\log \det P(gz, \overline{g\sigma}) = -\log \det g P(z, \overline{\sigma}) g^t = -\log \det P(z, \overline{\sigma}) - \log \det g g^t = \varphi(z) - \log \det g g^t. \]
Therefore
\[ (\partial_{gZ} \overline{gZ} \varphi)(gz) = (\partial_{Z} \overline{Z} \varphi \circ g)(z) = \varphi(z). \]

**Lemma 8.5.** Let \( V = \bigoplus_{1 \leq i \leq j \leq l} V_{ij} \) be the Pierce decomposition of a Euclidean Jordan algebra \( V \). Let \( 1 \leq i,j,k,l,r,s \leq l \) be integers such that \( (i,j) \neq (k,l) \) and \( i \neq j \) or \( k \neq l \). Then the following assertions hold:
(i) \( V_{ij}(V_{kl} \cdot V_{rs}) \perp V_{rs} \).
(ii) \( (V_{ij} \cdot V_{kl})V_{rs} \perp V_{rs} \).

**Proof.** (i) By the invariance of the inner product on \( V \) we have
\[ V_{ij}(V_{kl} \cdot V_{rs}) \perp V_{rs} \iff V_{kl} \cdot V_{rs} \perp V_{ij} \cdot V_{rs} \iff V_{kl} \perp V_{rs}(V_{rs} \cdot V_{ij}). \]
If \( i \neq j \), then (i) follows from \( V_{rs}(V_{rs} \cdot V_{ij}) \subseteq V_{ij} \), which is seen as follows. If \( r = s \) or \( (r,s) = (i,j) \), then this is clear from (8.2). So assume \( r \neq s \) and \( (r,s) \neq (i,j) \). Also we may assume that \( V_{rs} \cdot V_{ij} \neq \{ 0 \} \). Then \( V_{rs}(V_{rs} \cdot V_{ij}) \perp V_{ij} \) again by the invariance of \( \langle \cdot, \cdot \rangle \). Since \( (r,s) \neq (i,j) \) we obtain from (8.2) that \( V_{rs} \cdot V_{ij} = V_{uv} \) with \( (u,v) \neq (r,s) \). Hence \( V_{rs} \cdot V_{uv} \subseteq V_{xy} \) and so \( (x,y) = (i,j) \).

Let now \( i = j \). Then \( k \neq r \) by assumption. It is enough to do the case \( r = i \). Then
\[ V_{rs}(V_{rs} \cdot V_{ij}) = V_{is}(V_{is} \cdot V_{ii}) \subseteq V_{is} \cdot V_{is} \subseteq V_{ii} + V_{ss}. \]
Now \( V_{ii} + V_{ss} \perp V_{kl} \) since \( k \neq l \) concluding the proof of (i).
(ii) Again by the invariance of the inner product on \( V \) we get
\[ (V_{ij} \cdot V_{kl})V_{rs} \perp V_{rs} \iff V_{ij} \perp V_{kl} \perp V_{rs} \cdot V_{rs}. \]
But \( V_{rs} \cdot V_{rs} \subseteq V^0 \) and \( V_{ij} \cdot V_{kl} \subseteq (V^0)^\perp \) by (8.2) and the assumptions in the Lemma.

The main result of this section is
Theorem 8.6. The function
\[ \varphi: \mathbb{C}^+ \to \mathbb{R}, \quad z \mapsto \log \det P(z, \overline{z})^{-1} \]
is plurisubharmonic.

Proof. We have to show that \((\partial_{\overline{z}} \partial_{z} \varphi) (z) \geq 0\) for all \(Z \in \mathbb{V}_C\) and \(z \in \mathbb{C}^+\). In view of Lemma 8.4, we may assume that \(z \in \exp(\i \Omega) K_{\mathbb{C}} / K_{\mathbb{C}} \subseteq (\mathbb{V}^0)_{\mathbb{C}}\).

For \(Z_1, Z_2 \in \mathbb{V}_C\) we define linear operators on \(\mathbb{V}_C\) by
\[ A(Z_1, \overline{Z}_2) = P(z, \overline{z})^{-1} P(Z_1, \overline{z}) P(z, \overline{z})^{-1} P(z, \overline{Z}_2) \]
and
\[ B(Z_1, \overline{Z}_2) = P(z, \overline{z})^{-1} P(Z_1, \overline{Z}_2). \]

Then Lemma 8.3 implies that
\[ (\partial_{\overline{z}} \partial_{z} \varphi) (z) = \text{tr} A(Z_1, \overline{Z}_2) - \text{tr} B(Z_1, \overline{Z}_2). \]

Note that \(P(z, \overline{z})^{-1}\) preserves all \((V_{ij})_{\mathbb{C}}\). In the proof of Proposition 8.2 we have already shown that
\[ P(z, \overline{z})^{-1} v = \begin{cases} \frac{2}{|z_i|^2 v} & \text{for } v \in V_{ij}, \ i \neq j, \\ \frac{1}{|z_i|^2 v} & \text{for } v \in V_{ii}. \end{cases} \]

Then \(V = V^0 \oplus \bigoplus_{i \neq j} V_{ij}\) with \(V^0 = \bigoplus_{j=1}^n \mathbb{C} c_j\). Let now \(Z = U + \sum_{i \neq j} Z_{ij}\) with \(U \in (\mathbb{V}^0)_{\mathbb{C}}\) and \(Z_{ij} \in (V_{ij})_{\mathbb{C}}\). So
\[ (\partial_{\overline{z}} \partial_{z} \varphi) (z) = \text{tr} A(U, \overline{U}) - 2 \text{tr} \text{Re} B(U, \overline{U}) - \sum_{i \neq j} \text{tr} B(Z_{ij}, \overline{Z}_{ij}). \]

First we claim that \(\text{tr} A(U, \overline{Z}_{ij}) = \text{tr} B(Z_{ij}, \overline{U}) = \text{tr} B(Z_{ij}, \overline{Z}_{kl}) = 0\). In fact, this is an immediate consequence of the complexified version of Lemma 8.5. Hence we get that
\[ (\partial_{\overline{z}} \partial_{z} \varphi) (z) = \text{tr} A(U, \overline{U}) - 2 \text{tr} \text{Re} B(U, \overline{U}) - \sum_{i \neq j} \text{tr} (A(Z_{ij}, \overline{Z}_{ij}) - B(Z_{ij}, \overline{Z}_{ij})). \]

We show separately that \(\text{tr} A(U, \overline{U}) - B(U, \overline{U}) \geq 0\) and \(\text{tr} A(Z_{ij}, \overline{Z}_{ij}) - B(Z_{ij}, \overline{Z}_{ij}) \geq 0\).

We begin by showing that \(\text{tr} A(U, \overline{U}) - B(U, \overline{U}) \geq 0\). Write \(U = \sum_{j=1}^n u_j c_j\) for \(u_j \in \mathbb{C}\). Now \(B(U, \overline{U}) = P(z, \overline{z})^{-1} P(U, \overline{U})\) and
\[ P(U, \overline{U}) v = \begin{cases} |u_i|^2 v & \text{if } v \in V_{ii}, \\ \frac{1}{2} (u_i \overline{m}_j + u_j \overline{m}_i) v & \text{if } v \in V_{ij}, \ i \neq j. \end{cases} \]

From (8.5) we obtain
\[ \text{tr} B(U, \overline{U}) = \sum_{j=1}^n |u_j|^2 |z_j|^2 + \sum_{i < j} \frac{1}{2} \frac{u_i \overline{m}_j + u_j \overline{m}_i}{z_i z_j + z_j z_i} \dim V_{ij}. \]
Next since

\[ P(U, \bar{v}) = L(U)L(\bar{v}) + L(\bar{v})L(U) - L(U\bar{v}) \]

we have

\[ P(U, \bar{v})v = \begin{cases} (u_i\bar{v}_i)v & \text{if } v \in V_{ii}, \\ \frac{1}{2}(u_i\bar{v}_j + u_j\bar{v}_i)v & \text{if } v \in V_{ij}, \ i \neq j. \end{cases} \]

Similarly one has

\[ P(z, \bar{U})v = \begin{cases} (\overline{u_i}z_i)v & \text{if } v \in (V_{ii})_c, \\ \frac{1}{2}(\overline{u_i}z_j + \overline{u_j}z_i)v & \text{if } v \in (V_{ij})_c, \ i \neq j. \end{cases} \]

Combined with (8.5) we get

\[ \text{tr } A(U, \bar{U}) = \sum_{j=1}^n |u_j|^2 + \sum_{i<j} |u_i\overline{v}_j + u_j\overline{v}_i|^2 \text{dim } V_{ij}. \]

So

\[ \text{tr } A(U, \bar{U}) - \text{tr } B(U, \bar{U}) = \sum_{i<j} \frac{\text{dim } V_{ij}}{z_i\overline{v}_j + z_j\overline{v}_i} [ (u_i\overline{v}_j + u_j\overline{v}_i)^2 - (u_i\overline{v}_j + u_j\overline{v}_i)(z_i\overline{v}_j + z_j\overline{v}_i) ] \]

\[ = \sum_{i<j} \frac{\text{dim } V_{ij}}{z_i\overline{v}_j + z_j\overline{v}_i} |u_i\overline{v}_j - u_j\overline{v}_i|^2 \geq 0 \]

as we claimed.

Next we show that \( \text{tr } A(Z_{ij}, \overline{Z}_{ij}) - B(Z_{ij}, \overline{Z}_{ij}) \geq 0 \) for \( Z_{ij} \in (V_{ij})_C \). It is enough to do the case \( (i, j) = (1, 2) \). Set \( e_{12} = c_1 + c_2 \). If \( Z_1, Z_2 \in (V_{12})_C \), then one has

\[ Z_1 \cdot Z_2 = \frac{1}{2}(Z_1\overline{Z}_2)e_{12}. \]

Fix \( Z \in (V_{12})_C \). We can normalize \( Z \) such that \( ZZ = e_{12} \). Fix \( 0 \neq Z \in (V_{12})_C \) and set

\[ Z^{\perp 12} = \{ v \in (V_{12})_C | (v|Z) = 0 \}. \]

Similarly define \( Z^{\perp 12} \). We claim that \( ZQ = \overline{Z}Q = 0 \) for all \( Q \in Z^{\perp 12} \cap \overline{Z}^{\perp 12} \). In fact \( ZQ, \overline{Z}Q \in (V_0')_C \) by (8.2). Let now \( x \in (V_0')_C \) be arbitrary. Then \( x\overline{Z} = \lambda \overline{Z} \) for some \( \lambda \in \mathbb{C} \) and so

\[ (Q|x) = \overline{Q|x Z} = \overline{Q|Z} = 0 \]

for all \( x \in V_0' \). Thus \( QZ = 0 \). Similarly one shows that \( Q\overline{Z} = 0 \), completing the proof of our claim.

Polarizing [FK94, Lemma IV.2.2] gives

\[ (L(Z)L(\overline{Z}) + L(\overline{Z})L(Z))v = \frac{1}{4} |Z|^2 \quad \text{for all } v \in V_{ij}, \ i \in \{1, 2\}, \ j \geq 3. \]

Now \( |Z|^2 = 2 \), and since \( ZZ = e_{12} \) we have\( P(Z, \overline{Z}) = L(Z)L(\overline{Z}) + L(\overline{Z})L(Z) - L(e_{12}) \). So

\[ P(Z, \overline{Z})v = \begin{cases} c_2 & \text{if } v = c_1, \\ c_1 & \text{if } v = c_2, \\ 0 & \text{if } v \in \bigoplus_{j=3}^t \mathbb{R}c_j \\ \frac{1}{2}(Z|Z)Z & \text{if } v = \overline{Z}, \\ \frac{1}{2}(\overline{Z}|Z)Z & \text{if } v = Z, \\ 0 & \text{if } v \in V_{ij}, \ i \neq j, \ (i, j) \neq (1, 2). \end{cases} \]
Using (8.5) we get
\[ \text{tr } B(Z, \overline{Z}) = \frac{2}{z_1 \overline{z}_2 + z_2 \overline{z}_1} (\frac{1}{2} (Z|\overline{Z})(\overline{Z}|Z) - \dim Z^{12} \cap \overline{Z}^{12}). \]

Again, \((Z|Z) = 2\) together with the Cauchy-Schwarz inequality yields
\[ -\text{tr } B(Z, \overline{Z}) \geq \frac{2}{z_1 \overline{z}_2 + z_2 \overline{z}_1} (-2 + \dim Z^{12} \cap \overline{Z}^{12}). \]

To compute \(\text{tr } A(Z, \overline{Z})\) first notice that
\[ P(z, \overline{Z}) = L(z) L(\overline{Z}) + L(\overline{Z}) L(z) - \frac{z_1 + z_2}{2} L(\overline{Z}). \]

A simple calculation now shows that
\[ P(z, \overline{Z})v = \begin{cases} \frac{z_1}{z_2} Z & \text{if } v = c_1, \\ \frac{z_2}{z_1} Z & \text{if } v = c_2, \\ 0 & \text{if } v \in V_{ij}, \{i, j\} \cap \{1, 2\} = \emptyset, \\ 0 & \text{if } v \in Z^{12}, \\ z_1 c_1 + z_2 c_2 & \text{if } v = Z, \\ z_j Zv & \text{if } v \in V_{ij}, i \in \{1, 2\}, j \geq 3. \end{cases} \]

Similarly we obtain that
\[ P(Z, \overline{Z})v = \begin{cases} \frac{z_1}{z_2} Z & \text{if } v = c_1, \\ \frac{z_2}{z_1} Z & \text{if } v = c_2, \\ 0 & \text{if } v \in V_{ij}, \{i, j\} \cap \{1, 2\} = \emptyset, \\ 0 & \text{if } v \in Z^{12}, \\ z_1 c_1 + z_2 c_2 & \text{if } v = Z, \\ z_j Zv & \text{if } v \in V_{ij}, i \in \{1, 2\}, j \geq 3. \end{cases} \]

With (8.5) this implies that
\[ A(Z, \overline{Z})v = \begin{cases} \frac{1}{z_1 z_2 + \overline{z}_1 \overline{z}_2} c_1 + \frac{1}{z_1 z_2 + \overline{z}_1 \overline{z}_2} \overline{c}_2 & \text{if } v = c_1, \\ \frac{1}{z_1 z_2 + \overline{z}_1 \overline{z}_2} \overline{c}_2 + \frac{1}{z_1 z_2 + \overline{z}_1 \overline{z}_2} c_1 & \text{if } v = c_2, \\ 0 & \text{if } v \in V_{ij}, \{i, j\} \cap \{1, 2\} = \emptyset, \\ 0 & \text{if } v \in Z^{12}, \\ \overline{z}_j |z_j|^{2} v & \text{if } v = Z, \\ \overline{z}_j |z_j|^{2} v & \text{if } v \in V_{ij}, i \in \{1, 2\}, j \geq 3. \end{cases} \]

Hence we get that
\[ \text{tr } A(Z, \overline{Z}) = \frac{4}{z_1 \overline{z}_2 + z_2 \overline{z}_1} + \sum_{i \in \{1, 2\}} \sum_{j \geq 3} \frac{|z_j|^2 \dim V_{ij}}{(z_1 \overline{z}_j + z_j \overline{z}_1)(z_2 \overline{z}_j + z_j \overline{z}_2)}. \]

and so
\[ (8.7) \quad \text{tr } A(Z, \overline{Z}) - B(Z, \overline{Z}) = \frac{2 \dim (Z^{12} \cap \overline{Z}^{12})}{z_1 \overline{z}_2 + z_2 \overline{z}_1} + \sum_{i \in \{1, 2\}} \sum_{j \geq 3} \frac{|z_j|^2 \dim V_{ij}}{(z_1 \overline{z}_j + z_j \overline{z}_1)(z_2 \overline{z}_j + z_j \overline{z}_2)} > 0. \]

This concludes the proof of the theorem.
\( \Xi^{1,1} - \text{the commutator square root domain} \)

The function \( \varphi: \Xi^{1,1} \to \mathbb{R} \) is almost strictly plurisubharmonic, in the sense that it is strictly plurisubharmonic on a codimension one subdomain.

Write \( \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] \) for the commutator subalgebra of \( \mathfrak{g} \). Then \( \mathfrak{g} = \mathfrak{g}^1 \times \mathbb{R} \). With \( \mathfrak{p}^1 = \mathfrak{g}^1 \cap \mathfrak{p} \) we obtain a Cartan decomposition of \( \mathfrak{g}^1 \), \( \mathfrak{g}^1 = \mathfrak{t} \oplus \mathfrak{p}^1 \). Denote by \( \mathcal{G}^1 \) the analytic subgroup of \( \mathcal{G} \) corresponding to \( \mathfrak{g}^1 \) and by \( G^1_\mathcal{C} \) its complexification. Then one has the canonical isomorphism \( \mathcal{G}^1_\mathcal{C}/\mathcal{K}_\mathcal{C} \simeq G^1_\mathcal{C}/K \times \mathbb{C}^* \) and the canonical embedding 
\[ \iota: \mathcal{G}^1_\mathcal{C}/\mathcal{K}_\mathcal{C} \hookrightarrow \mathcal{G}^1_\mathcal{C}/\mathcal{K}_\mathcal{C}, \quad x\mathcal{K}_\mathcal{C} \mapsto (x\mathcal{K}_\mathcal{C}, 1). \]

Henceforth we identify \( G^1_\mathcal{C}/\mathcal{K}_\mathcal{C} \) as a complex submanifold of \( \mathcal{G}^1_\mathcal{C}/\mathcal{K}_\mathcal{C} \) via the embedding \( \iota \).

Set \( \Omega^1 = \Omega \cap \mathfrak{g}^1 \) and notice that \( \Omega = \Omega^1 \times \mathbb{R} \). Then we define the \textit{commutator square root domain} by 
\[ \Xi^{1,1} = G^1 \exp(i \frac{1}{2} \Omega^1)K\mathcal{C}/\mathcal{K}_\mathcal{C}. \]
Clearly 
\[ \Xi^{1,1} \simeq \Xi^{1,1} \times \mathbb{C}^*. \]

We shall consider \( \Xi^{1,1} \) as a submanifold of \( \Xi^{1,1} \) via the embedding \( \iota \).

Set \( V^1_\mathcal{C} = \{ z \in \mathcal{V}_\mathcal{C}: \det z = 1 \} \). Note that \( V^1_\mathcal{C} \) is a complex hypersurface in \( \mathcal{V}_\mathcal{C} \) and \( \Xi^{1,1} \) is open in \( V^1_\mathcal{C} \).

Consider the function 
\[ \varphi^1: \Xi^{1,1} \to \mathbb{R}, \quad z \mapsto \log \det P(z, \overline{z})^{-1}, \]
the restriction of \( \varphi \) to \( \Xi^{1,1} \).

\( i \)From the proof of Theorem 8.6 we can obtain

**Theorem 8.7.** The function 
\[ \varphi^1: \Xi^{1,1} \to \mathbb{R}, \quad z \mapsto \log \det P(z, \overline{z})^{-1} \]
is strictly plurisubharmonic.

**Proof.** We have to show that 
\[ (8.8) \quad (\partial Z \overline{Z}) \varphi(z) > 0 \]
for all \( z \in \Xi^{1,1} \) and \( Z \in T_z \Xi^{1,1}, \ Z \neq 0 \). By Lemma 8.4 we may assume that \( z \in \exp(i \frac{1}{2} \Omega^1)(e) \), hence \( z = \sum_{j=1}^{l} z_j c_j \) with \( z_j \in \mathbb{C} \). Since \( \Xi^{1,1} \) is open in \( V^1_\mathcal{C} \), we have 
\[ T_z \Xi^{1,1} = T_z V^1_\mathcal{C}. \]

Further \( T_z V^1_\mathcal{C} = \{ u \in \mathcal{V}_\mathcal{C}: \text{tr} \ u = 0 \} \) and so 
\[ T_z \Xi^{1,1} = dL_z(e) (T_z V^1_\mathcal{C} = \{ zu \in \mathcal{V}_\mathcal{C}, \text{tr} \ u = 0 \}). \]
In particular, we obtain that 
\[ T_z \Xi^{1,1} = \bigoplus_{i \neq j} (V^1_\mathcal{C})_i \oplus \left\{ \sum_{j=1}^{l} z_j u_j c_j: u_j \in \mathbb{C}, \sum_{j=1}^{l} u_j = 0 \right\}. \]
Now (8.8) follows from (8.6) (which is positive now if \( U = \sum_{j=1}^{l} z_j u_j c_j \neq 0 \) and \( \sum_{j=1}^{l} u_j = 0 \)) and (8.7).
We give some applications of Theorem 8.6 to the complex analysis of the square root domain $\Xi^\frac{1}{2}$. In particular, we will show that $\Xi^\frac{1}{2}$ is Stein and that there is a natural $G$-invariant Kähler structure on $\Xi^\frac{1}{2}$.

**Theorem 8.8.** The square root domain $\Xi^\frac{1}{2} \subseteq G_C/K_C$ is Stein.

**Proof.** In view of Grauert’s solution of the Levi problem (cf. [Hö73, Th. 5.2.10]), it suffices to find a strictly plurisubharmonic exhaustion function of $\Xi^\frac{1}{2}$. Evidently

$$\varphi_0 : \Xi^\frac{1}{2} \rightarrow \mathbb{R}^+, \ z \mapsto \|z\|^2$$

is strictly plurisubharmonic. Using the $\varphi$ of Theorem 8.6 we set

$$\varphi_1(z) = \exp \varphi(z).$$

Since $\varphi$ is plurisubharmonic, $\varphi_1$ is plurisubharmonic. Since $\varphi_1$ is positive

$$\psi = \varphi_0 + \varphi_1$$

is a positive strictly plurisubharmonic function on $\Xi^\frac{1}{2}$. It remains to see that $\psi$ is proper. For that let $(z_n)_{n \in \mathbb{N}}$ be a sequence which tends to $\infty$ in $\Xi^\frac{1}{2}$. Then we have either $z_n \rightarrow \infty$ in $V_C$ or $z_n \rightarrow z_0 \in \partial \Xi^\frac{1}{2}$. In the first case $\varphi_0$ blows up, while in the second case $\varphi_1$ blows up. This concludes the proof of the theorem.

The Kähler structure

**Proposition 8.9.** For $z \in \Xi^\frac{1}{2}$ the family of Hermitian forms $H_z(\cdot, \cdot)$ on $T_z \Xi^\frac{1}{2}$ defined by

$$H_z(v, w) = (P(z, \overline{z})^{-1} v|w) \quad (v, w \in V_C \simeq T_z \Xi^\frac{1}{2})$$

defines a $G$-invariant Hermitian metric on $\Xi^\frac{1}{2}$. Moreover the associated Riemannian structure $G_z(\cdot, \cdot) = \text{Re} H_z(\cdot, \cdot)$ is complete.

**Proof.** It follows from Proposition 8.2 and the transformation property (8.3) that $H_z(\cdot, \cdot)$ defines a $G$-invariant Hermitian structure on $\Xi^\frac{1}{2}$. That the associate Riemannian metric is complete follows from the fact that the metric blows up at the boundary.

**Remark 8.10.** (a) If $z \in V_C$, write $z = x + iy$ with $x, y \in V$. Then a simple computation shows that

$$P(z, \overline{z}) = P(x) + P(y). \quad (8.9)$$

In particular, as a refinement of Proposition 8.2 we have the square root domain $\Xi^\frac{1}{2}$ equivalently defined as

$$\Xi^\frac{1}{2} = \{z = x + iy \in V_C; P(x) + P(y) \in \text{End}(V) \text{ is positive definite}\}.$$

(b) With $H_z(\cdot, \cdot)$ as in Proposition 8.9 set $H_z = G_z + i\Omega_z$, $\Omega_z$ the imaginary part of $H_z$. We ask whether the Hermitian metric $H_z(\cdot, \cdot)$ is Kähler. From (8.9) we see that $P(z, \overline{z})$ preserves the real form $V$. Identify $V_C$ with $V \times V$ via $V \times V \rightarrow V_C$, $(x, y) \mapsto x + iy$. Then in matrix notation we have

$$G_z = \begin{pmatrix} P(z, \overline{z})^{-1} & 0 \\ 0 & P(z, \overline{z})^{-1} \end{pmatrix} \quad \text{and} \quad \Omega_z = \begin{pmatrix} 0 & P(z, \overline{z})^{-1} \\ -P(z, \overline{z})^{-1} & 0 \end{pmatrix}.$$
If $V$ is irreducible, then one can show that the Hermitian metric $H_z$ is Kähler if and only if $V = \mathbb{R}$.

c) The $G$-invariant Riemannian metric on the cone $W$ is known [FK94, Th. III.5.3] to be given by

$$g_x(v, w) = (P(x)^{-1}v, w) \quad (x \in W ; v, w \in V \simeq T_xW).$$

Since $P(z, z) = P(z)$, we see that the Hermitian metric $H_z$ is a polarization of the Riemannian metric on $W$ to the square root domain $\Xi^{\frac{1}{2}}$.

d) We recall from [FK94, p. 15-16 and Prop. III.4.3] that the metric $(g_x)_{x \in W}$ is obtained from a potential function

$$\psi: W \to \mathbb{R}, \quad x \mapsto \frac{1}{2} \log \det P(x)^{-1}$$

in such a way that

$$g_x(u, v) = \left( \partial_u \partial_v \psi \right)(x) \quad (x \in W; u, v \in V \simeq T_xW).$$

One might now expect that the Hermitian metric $H_z$ is obtained in a similar manner through the potential function

$$\varphi: \Xi^{\frac{1}{2}} \to \mathbb{R}, \quad z \mapsto \frac{1}{2} \log \det P(z, \overline{z})^{-1}$$

and the associated Hermitian metric

$$h_z(u, v) = \left( \partial_u \overline{\partial}_v \varphi \right)(z) \quad (z \in \Xi^{\frac{1}{2}}; u, v \in V_{\mathbb{C}} \simeq T_z \Xi^{\frac{1}{2}}).$$

However, it turns out that $h_z \neq H_z$. The easiest example to observe this phenomenon is for $V = \mathbb{R}$. Then $W = \mathbb{R}^+$ and $G = \mathbb{R}^+$ is the group of scaling transformations. Here we have

$$P(x) = x^2, \quad \psi(x) = -\log x \quad \text{and} \quad g_x(u, v) = \frac{1}{x^2} uv \quad (x \in \mathbb{R}^+ = W; u, v \in \mathbb{R}).$$

On the other hand we have $\Xi^{\frac{1}{2}} = \mathbb{C}^*$ and $P(z, \overline{z}) = |z|^2$. Hence $\varphi(z) = -\log |z|$. Since $\varphi$ is harmonic, we thus obtain $h_z = 0$ and so evidently $h_z \neq H_z$.

Finally we construct the $G$-invariant Kähler structure on $\Xi^{1,\frac{1}{2}}$. 

**Theorem 8.11.** Define $h_z: T_z \Xi^{1,\frac{1}{2}} \times T_z \Xi^{1,\frac{1}{2}} \to \mathbb{C}$ by

$$(Z_1, Z_2) \mapsto \left( \partial Z_1, \overline{\partial} Z_2 \phi_1 \right)(z) \quad (z \in \Xi^{1,\frac{1}{2}}).$$

Then $h_z(\cdot, \cdot)$ is a $G^1$-invariant positive Kähler structure on $\Xi^{1,\frac{1}{2}}$ whose associated Riemannian structure $(g_z)_{z \in \Xi^{1,\frac{1}{2}}} = (\text{Re} h_z)_{z \in \Xi^{1,\frac{1}{2}}}$ is complete.

**Proof.** This is immediate from Theorem 8.7 and Lemma 8.4. The completeness of the associated Riemannian structure follows from the fact that the metric blows up at the boundary (cf. the formulas (8.6) and (8.7)).

**Remark 8.12.** (a) To us it seems an interesting question to determine if the Kähler metric $h_z$ is Einstein-Kähler.

(b) Certainly there should be a generalization of the results in this section to the other groups $G$ whose Lie algebra appears in Table II.
Appendix - Examples of §7

The example of $G = \text{SO}(p, q)$. We assume that $p \leq q$. Choose $K = S(O(p) \times O(q))$ as a maximal compact subgroup of $G$. For $G^h$ and $K^h$ we have $G^h = \text{SU}(p, q)$ and $K^h = S(U(p) \times U(q))$. The involution $\tau$ on $G^h$ which has $G$ as fixed point group is

$$\tau: G^h \to G^h, \ g \mapsto \overline{g}$$

the complex conjugation, i.e., if $g = x + iy, \ x, y \in M(n, \mathbb{R})$, then $\overline{g} = x - iy$.

An appropriate maximal abelian subspace of $\mathfrak{p}$ in $\mathfrak{g} = \mathfrak{so}(p, q)$ is

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0_{pp} & I_{t_1, \ldots, t_p} \\ I_{t_1, \ldots, t_p} & 0_{qq} \end{pmatrix} : t_1, \ldots, t_p \in \mathbb{R} \right\},$$

where

$$I_{t_1, \ldots, t_p} = \begin{pmatrix} 0_{p, p-q} & \cdots & t_1 \\ \vdots & \ddots & \vdots \\ 0_{p, q} & \cdots & 0_{p, p} \end{pmatrix} \in M(p \times q; \mathbb{R}).$$

Let $n = p + q$. For all $1 \leq i, j \leq n$ define $E_{ij} \in M_n(\mathbb{R})$ by $E_{ij} = (\delta_{ki}\delta_{jl})_{k,l}$. Then $\mathfrak{a} = \bigoplus_{j=1}^{p} \mathbb{R} e_j$ with

$$e_j = E_{j,p+q+1-j} + E_{p+q+1-j,j}.$$

Define $\varepsilon_j \in \mathfrak{a}$ by $\varepsilon_j(e_k) = \delta_{jk}$. Then the root system $\Sigma = \Sigma(\mathfrak{a}, \mathfrak{g})$ is given by

$$\Sigma = \begin{cases} \{ \pm \varepsilon_i \pm \varepsilon_j : 1 \leq i \neq j \leq p \} \cup \{ \pm \varepsilon_i : 1 \leq i \leq p \} & \text{for } 1 < p < q, \\
\{ \pm \varepsilon_i \pm \varepsilon_j : 1 \leq i \neq j \leq p \} & \text{for } 1 < p = q, \\
\{ \pm \varepsilon_1 \} & \text{for } p = 1.\end{cases}$$

In all cases one easily sees that

$$\Omega_0 = \bigoplus_{j=1}^{p} \frac{\pi}{4} \varepsilon_j$$

lies in $\Omega$.

The Harish-Chandra realization of $G^h/K^h$ is given by

$$\mathcal{D} = \{ Z \in M(p \times q; \mathbb{C}) : I_q - Z^*Z >> 0 \}$$

where $G^h$ acts on $\mathcal{D}$ via

$$g(Z) = (AZ + B)(CZ + D)^{-1} (g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G^h, Z \in \mathcal{D}).$$

Define

$$\mathcal{D}_\mathbb{R} = \mathcal{D} \cap M(p \times q; \mathbb{R}).$$

Note that $\tau$ induces an involution on $\mathcal{D} \simeq G^h/K^h$ which is given by complex conjugation. Hence $G/K$ defines a real subspace of $G^h/K^h$ and since $G^h/G$ is compactly causal we even have

$$\mathcal{D}_\mathbb{R} \simeq G/K$$

by [KNÓ97, Th. II.9].
Let \( a = \exp(\sum_{j=1}^{p} z_j e_j) \) with \( z_j \in \mathbb{R} + i \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \). Then an easy calculation shows that

\[
a(0) = \begin{pmatrix}
0_{p,p-q} & \cdots & \tanh z_1 \\
\tanh z_p & \ddots & \\
& & \tanh z_p
\end{pmatrix}
\]

and so

\[
a(0)^* a(0) = \begin{pmatrix}
|\tanh z_p|^2 & \cdots & z_1 \\
\cdots & & \cdots \\
& & |\tanh z_1|^2
\end{pmatrix}.
\]

Hence

\[
\Phi(A \exp(i\Omega) K_C / K_C) = \left\{ \begin{pmatrix}
0_{p,p-q} & \cdots & z_1 \\
\cdots & & \\
0 & \cdots & 0
\end{pmatrix} : z_i \in \mathbb{C}, |z_i| < 1 \right\} \subseteq D.
\]

The maximal abelian subspace \( \mathfrak{e} = J(a) \) of \( \mathfrak{p} \cap \mathfrak{q} \) is given by

\[
\mathfrak{e} = \left\{ \begin{pmatrix}
0 & i I_{t_1, \ldots, t_p} \\
-i I_{t_1, \ldots, t_p} & 0
\end{pmatrix} : t_1, \ldots, t_n \in \mathbb{R} \right\}.
\]

Then

\[
\exp(\mathfrak{e})(0) = \left\{ \begin{pmatrix}
0_{p,p-q} & \cdots & ix_1 \\
\cdots & & \cdots \\
x_p & \cdots & 0
\end{pmatrix} : x_j \in \mathbb{R}, |x_j| < 1 \right\},
\]

and so \( \exp(\mathfrak{e})(0) \subseteq \Phi(A \exp(i\Omega) K_C / K_C) \) which was the crucial step in the proof of Theorem 7.5.

**The example of** \( G = \text{SO}(n, \mathbb{C}) \). We let \( G^h = \text{SO}^*(2n) \) realized as

\[
G^h = \{ g \in \text{SU}(n, n) g' \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \}.
\]

A maximal compact subgroup of \( G^h \) is given by

\[
K^h = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in U(n) \right\} \simeq U(n).
\]

The Lie algebra \( \mathfrak{g}^h \) of \( G^h \) is given by

\[
\mathfrak{g}^h = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A, B \in M(n, \mathbb{C}) \right\} \simeq \mathfrak{so}(n, \mathbb{C}).
\]

We equip \( G^h \) with the involution \( \tau : G^h \to G^h, \ g \mapsto \overline{g} \) of complex conjugation and set \( G = G^h_0 \) for the connected component of \( \tau \)-fixed points. Then the Lie algebra \( \mathfrak{g} \) of \( G \) is given by

\[
\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A, B \in M(n, \mathbb{R}) \right\} \simeq \mathfrak{so}(n, \mathbb{R}).
\]

Hence \( \mathfrak{g} \simeq \mathfrak{so}(n, \mathbb{C}) \) and it is easy to check that \( G \simeq \text{SO}(n, \mathbb{C}) \). Then the Harish-Chandra realization of \( G^h / K^h \) is given by

\[
D = \{ Z \in M(n, \mathbb{C}) : Z^t = -Z, I - Z^* Z > 0 \}\]
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with $G^h$ acting on $D$ via

$$g(Z) = (AZ + B)(CZ + D)^{-1} \quad (g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G^h, Z \in D).$$

Set $D_\Re = D \cap M(n, \Re)$. Now $\tau$ induces an involution on $D \simeq G^h/K^h$ by complex conjugation. Hence $G/K$ defines a real subspace of $G^h/K^h$ and, since $G^h/G$ is compactly causal, we have $D_\Re \simeq G/K$ by [KNÖ97, Th. II.9].

We have to distinguish the cases of $n$ even and odd. Suppose first that $n = 2m$ is even. Then as a maximal abelian subspace $a$ of $p \subseteq g$ we choose

$$a = \{ \begin{pmatrix} 0 & J_{t_1, \ldots, t_m} \\ -J_{t_1, \ldots, t_m} & 0 \end{pmatrix} : t_1, \ldots, t_m \in \Re \},$$

where

$$J_{t_1, \ldots, t_p} = \begin{pmatrix} 0 & t_1 \\ -t_1 & 0 \\ & \ddots \\ & & 0 & t_m \\ & & & -t_m & 0 \end{pmatrix}.$$  

Then $a = \bigoplus_{j=1}^m \Re e_j$ with

$$e_j = E_{2j-1,m+2j} - E_{2j,m+2j-1} + E_{m+2j,2j-1} - E_{m+2j-1,2j}$$

If $n = 2m + 1$, $m \geq 1$, is odd, then we choose

$$a = \{ \begin{pmatrix} 0 & J'_{t_1, \ldots, t_m} \\ -J'_{t_1, \ldots, t_m} & 0 \end{pmatrix} : t_1, \ldots, t_m \in \Re \},$$

where

$$J'_{t_1, \ldots, t_p} = \begin{pmatrix} 0 & t_1 \\ -t_1 & 0 \\ & \ddots \\ & & 0 & t_m \\ & & & -t_m & 0 \end{pmatrix}.$$  

In this case $a = \bigoplus_{j=1}^m \Re e_j$.

Define $\varepsilon_j \in a^*$ by $\varepsilon_j(e_k) = \delta_{jk}$. Then the root system $\Sigma = \Sigma(a, g)$ is given by

$$\Sigma = \begin{cases} \{ \pm \varepsilon_i, \pm \varepsilon_j : 1 \leq i \neq j \leq m \} & \text{for } n = 2m, \\ \{ \pm \varepsilon_i, \pm \varepsilon_j : 1 \leq i \neq j \leq m \} \cup \{ \pm \varepsilon_i : 1 \leq i \leq m \} & \text{for } n = 2m + 1. \end{cases}$$

In both cases we see that the set

$$\Omega_0 = \bigoplus_{j=1}^m \left[ -\frac{\pi}{4}, -\frac{\pi}{4} \right] [e_j].$$

In the following we assume for simplicity that $n = 2m$ is even. Let now $a = \exp(\sum_{j=1}^m z_j e_j)$ with $z_j \in \Re + i \left[ -\frac{\pi}{4}, \frac{\pi}{4} \right]$. Then by a straightforward calculation one gets that

$$a(0) = J_{\tanh z_1, \ldots, \tanh z_m}.$$
and so
\[ T_{\Omega}(0) = \{ J_{z_1, \ldots, z_m} : z_j \in \mathbb{C}, \, |z_j| < 1 \}. \]

As a maximal abelian subspace \( \varepsilon \subseteq q \cap p \) we choose
\[ \varepsilon = \left\{ \begin{pmatrix} 0 & iJ_{t_1, \ldots, t_m} \\ iJ_{t_1, \ldots, t_m} & 0 \end{pmatrix} : t_1, \ldots, t_m \in \mathbb{R} \right\}. \]

Then one easily shows that
\[ \exp(\varepsilon)(0) = \{ iJ_{x_1, \ldots, x_m} : x_j \in \mathbb{R}, \, |x_j| < 1 \} \]
and in particular \( \exp(\varepsilon)(0) \subseteq A \exp(i\Omega_0)(0) \).

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