FPTAS for barrier covering problem with equal circles in 2D

Adil Erzin\textsuperscript{1,2} and Natalya Lagutkina\textsuperscript{2}

\textsuperscript{1} Sobolev Institute of Mathematics, Novosibirsk, Russia,  
adilerzin@math.nsc.ru,  
\textsuperscript{2} Novosibirsk State University, Novosibirsk, Russia,  
lagutnat@yandex.ru

Abstract. In this paper, we consider a problem of covering a straight line segment using circles that are arbitrarily placed on a plane by moving their centers into a segment so that the segment is completely covered and the total length of the paths of movement of the circles would be minimal. In the case when the circles are equal, the complexity status of the problem is not known. In this paper, for the case when all circles are equal and touch each other in the cover, we propose an FPTAS with the complexity $O\left(n^{2.5} \log^2 n \varepsilon^2 \right)$, where $n$ is the number of circles.

Keywords: barrier coverage, mobile sensors, FPTAS

1 Introduction

Wireless sensor networks (WSN) consist of autonomous sensors, the operation time of which depends on the capacity of the batteries. The task of the WSN is to collect and transfer data within a certain area. The time during which the network performs these functions is called a WSN’s lifetime. Energy efficient functioning of the WSN allows extending the lifetime. One of the important applications of the WSN is the monitoring of extended objects (borders, roads, pipelines, perimeters of buildings, etc.), which are often called the barriers.

Monitoring can be carried out using both stationary \textsuperscript{[7,8,12,16,17,18]} and mobile sensors \textsuperscript{[1,2,3,4,6,9,10,11,12,13,14]}. The problem of energy efficient covering a segment with stationary sensors with adjustable circular sensing areas initially located on a segment is considered in \textsuperscript{[12,17]}. In \textsuperscript{[12]} it is proved that such a problem is NP-hard.

The authors of \textsuperscript{[17]} consider the formulation when the sensors are located at a short distance from the segment. It is proved that the problem is also NP-hard, and FPTAS with time complexity equals $O\left(\frac{n^3}{\varepsilon^2} \right)$ is proposed for it, where $n$ is the number of sensors and $0 < \varepsilon < 1$ is the accuracy.

If the sensor is mobile, then the moving energy consumption is directly proportional to the length of the path traveled. In this case, the Problem of Barrier Monitoring by Mobile Sensors (PBMMMS) with circular coverage areas is to move the sensors so that each barrier point is in the coverage area of at least one
sensor, and the total length of the movement paths of the sensors involved in the covering is minimal [11]. In general, the initial arrangement of the sensors can be arbitrary.

In the literature, two main classes of barrier monitoring problems with mobile devices are considered. The first class is a one-dimensional (1D) problem, when the sensors are initially on the line containing the segment to be covered \[1,2,3,6,10,12,14\]. In the two-dimensional (2D) case, the problem is considered when the sensors are arbitrarily located on the plane \[4,9,11\]. In [10] it is proved that even for the one-dimensional case when the radii of the circles are different, the PBMMS is NP-hard. In the case when the radii of the circles coincide, a polynomial algorithm for constructing a solution with the complexity of \(O(n^2)\) is proposed.

The complexity status of the problem of covering the barrier with circles of the same radius in the two-dimensional case is open. At the same time, a \(O(n^2)\)-time algorithm for constructing an optimal solution in the \(L_1\) metric, which is a \(\sqrt{2}\)-approximate solution in the Euclidean metric, is proposed in [9]. This solution has the property of preserving the order when the initial order of the circles after moving to the barrier does not change. This result was improved in [11], where an algorithm for constructing an order preserving cover (OPC) was proposed, having the complexity \(O(n^2)\).

The [3] presents the results for 1D problems when the circles are different, and the centers of circles are initially outside the barrier. In the case when the sensors lie on one side of the segment, FPTAS is proposed with complexity \(O(n^2)\), and for the case when the sensors are located on both sides of the segment, FPTAS is proposed with the complexity of \(O(n^2)\). If the radii of the circles coincide, the problem is solved with the complexity \(O(n \log_2 n)\) [1], which is an improvement in the result of the [10].

The authors of the work [4] consider the problem of covering a circle with mobile sensors with the same circular coverage areas. A PTAS with a complexity of \(O(n^3)\) is proposed when the circle is covered with equal circles, which are evenly spaced. This result is improved in [6], where PTAS is proposed, the complexity of which is \(\left(\frac{1}{\epsilon}O(1)\right)n^{2+\epsilon'}\), \(\epsilon' > 0\).

In [2], the barrier is covered with circles, initially located on the line containing the barrier. This takes into account that the sensors consume energy on both movement and monitoring, and two problems are considered: (a) when the radii of the sensors are fixed, (b) when the radii of the sensors can be adjusted. In case (b), it is required to find the optimal radius for each circle, which is the sensor coverage area. For case (b), FPTAS is proposed with the complexity \(O\left(\frac{n^3}{\epsilon^{4(1+\alpha)}}\right)\), where \(\alpha \geq 1\). For the case of \(\alpha = 1\) in [12], FPTAS was proposed with the complexity \(O(n^3)\). These results are improved in [14], which considers the 1D problem of covering a segment with circles of different radii, each of which has its own weight. A weighting factor was added in order to get closer to reality, when different sensors may need different amounts of energy for their
work (for example, due to the wear and obsolescence of some of them). FPTAS is proposed for the case when the sensors are initially located on one side of the covered segment, the complexity of which is $O\left(\frac{n^3}{\varepsilon^2}\right)$. For the general case, when the sensors are initially located on both sides of the segment, the FPTAS proposed has complexity $O\left(\frac{n^5}{\varepsilon^3}\right)$. In this paper, it is proved that the weighted 1D problem of covering a segment with circles of different radii is NP-hard even when the sensors are initially located on one side of the barrier.

In this paper, the problem of covering the barrier, which is represented by a straight line segment, and mobile sensors – by the centers of circles on a plane, is considered. We assume that the initial location of the sensors is known and they all have the same circular coverage areas. It is required to move the centers of the circles to a segment so as to cover the segment with touching circles and the total length of the paths of movement of the sensors would be minimal. For this problem, we propose an FPTAS, the complexity of which is $O\left(\frac{n^2}{\varepsilon^2} \log^3 n\right)$.

As far as we know, this is the first FPTAS for the problem under consideration.

The paper is organized as follows. Section 2 presents the mathematical formulation of the problem under consideration. Section 3 is devoted to the description of the FPTAS scheme and the proof of the complexity of the scheme. Section 4 summarizes this work.

## 2 Problem Formulation

Let a barrier in the form of a line segment of length $L > 0$ and a set $S (|S| = n)$ of arbitrarily arranged circles be given on the plane. It is required to cover the barrier by moving the centers of the circles on the segment. We introduce a coordinate system so that the segment is located on the abscissa axis between the points $(0, 0)$ and $(L, 0)$. Let the points $p_i = (x_i, y_i)$ be the initial coordinates of the centers of the circles, and $r_i \geq 0$ be the radius of the circle $i \in S$. Enumerate the circles from left to right according to the values $x_i$, $i = 1, \ldots, n$.

In the PBMMS, it is necessary to move the circles so that each point of the segment is covered and the total length of the paths of movement is minimal.

**Definition 1.** The function $\hat{\mathbf{p}} : S \rightarrow \mathbb{R}^2$ that determines the final position of the sensors $\hat{\mathbf{p}}_i = (\hat{x}_i, \hat{y}_i)$, $i \in S$, at which the barrier is completely covered, determines the covering.

Let $d(p_i, \hat{p}_i)$ is the distance between the points $p_i$ and $\hat{p}_i$. Then the PBMMS is to search for the function $\hat{\mathbf{p}}^*$, which is the solution to the problem

$$
\text{cost}(\hat{\mathbf{p}}^*) = \min_{\hat{\mathbf{p}}} \text{cost}(\hat{\mathbf{p}}) = \min_{\hat{\mathbf{p}}} \sum_{i=1}^{n} d(p_i, \hat{p}_i). \quad (1)
$$

In the general case, when the radii of the circles are different, the problem (1) is NP-hard [10]. In the case of identical radii of circles, the complexity status of the problem is unknown [9].
In this paper, we consider a PBMMS with identical circles, whose centers need to be moved to a segment in such a way that the circles touch each other in the cover.

Remark 1. The requirement that the circles touch each other is a special case of a uniform placement of sensors. If we require that the distance between adjacent sensors be the same, but less than the diameter of the circle, then we obtain a covering not of the line, but of the strip.

In the next section, we propose FPTAS for this problem.

3 FPTAS

Let all circles have a radius of 1.

Theorem 1. For problem (1), in which it is necessary to find a covering with circles touching each other, there is an FPTAS, the complexity of which is \( O\left(\frac{n^2 \log n}{\varepsilon^2}\right) \).

Proof. Since a cover is being sought in which the circles do not intersect, it will depend on the length \( \Delta \in (0, 2] \) of the segment, which is covered by the first circle on the left. Knowing \( \Delta \), it is possible to unambiguously divide the covered segment into subsegments (let’s call them the “cells”) of length 2 (perhaps, except for the first and last cells). As an auxiliary problem, we solve a generalized assignment problem (2). To do this, we number the circles \( i = 1, \ldots, n \) and cells from left to right \( j = 1, \ldots, m \) and denote by \( c_{ij} = d(p_i, \hat{p}_i) \) the distance from the initial position of the center of the circle \( i \) to its final position. For all cells except the first and last, the final positions of the centers of the circles will coincide with the midpoints of the cells. For the first and last cells, the points \( \hat{p}_i \) are closest to \( p_i \) lying on the segment and the corresponding cell is covered. Then the generalized assignment problem has the following form:

\[
W(\Delta) = \min_{x_{ij}} \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij}
\]

\[
\begin{align*}
\sum_{i=1}^{n} x_{ij} & = 1, j = 1, \ldots, m \\
\sum_{j=1}^{m} x_{ij} & \leq 1, i = 1, \ldots, n
\end{align*}
\]

However, the value of \( \Delta \) (the length of the first cell) can take any value from the interval \((0, 2]\). We introduce a grid on the interval \([0, 2]\) with a step \( \delta^2 \) and for each \( \Delta = \delta^2 k, k = 1, \ldots, 2/\delta^2 \) solve the generalized assignment problem (2).

Let \( W(\Delta^*) = \text{cost}(\hat{p}^*) \) be the minimal value of the problem functional. If \( \Delta^* \) is a multiple of \( \delta^2 \), then the optimal solution to (2) is the optimal solution to (1). Suppose that \( \Delta^* \) is not a multiple of \( \delta^2 \), that is, does not coincide with \( \delta^2 k \) for any \( k \in \mathbb{N} \). Denote by \( \overline{\Delta} = \delta^2 k^* \) the closest value to \( \Delta^* \). Then
\[ |\delta^2 k - \Delta^*| = \min_k |\delta^2 k - \Delta^*| \leq \frac{\delta^2}{2}. \]

Shift all the circles of the optimal coverage by an equal distance (left or right) so that the first circle covers the segment \([0, \Delta]\) and touches the second circle at the point \(\Delta\) (see Fig. 1). Note that in the approximate solution constructed as a result of solving the problem (2) with \(\Delta = \Delta\), the segment \([\Delta, L]\) is optimally covered. Then

\[ W(\Delta) - W(\Delta^*) \leq \frac{\delta^2}{2} \]

and

\[ \frac{W(\Delta) - W(\Delta^*)}{W(\Delta^*)} \leq \frac{\delta^2}{2W(\Delta^*)}. \]

\[ \text{Fig. 1.} \] Dotted black circles indicate an approximate solution, and red solid circles represent the optimal coverage.

Two cases need to be considered:
1. In the optimal coverage, at least one circle moves by at least \(\delta\) distance.
2. In the optimal coverage, none of the circles moves a distance of \(\delta\) or more.

In the first case, we have \(W(\Delta^*) \geq \delta\) and, therefore

\[ \frac{W(\Delta) - W(\Delta^*)}{W(\Delta^*)} \leq \frac{\delta^2}{2W(\Delta^*)} \leq \frac{\delta}{2}. \]

In the second case, circles horizontally move no more than a distance \(\delta\). This means that the centers of neighboring circles participating in the optimal coverage are initially located at a distance of at least \(2 - \delta\) and at most \(2 + \delta\) (circles that do not participate in the covering can be excluded). Thus, only the order preserving covering (OPC) can be optimal, and OPC is constructed with the complexity \(O(n^2)\) [11].
In FPTAS, it is necessary that the inequality \( \frac{W(\Delta)}{W(\Delta^*)} \leq \varepsilon \) be satisfied. Therefore, it suffices to require that \( \delta^2 \leq 4\varepsilon^2 \). The generalized assignment problem for each \( \Delta \) is solved with the complexity \( O\left(n^{2.5} \log_2 n\right) \) \( [15] \). The number of such problems is limited to \( 2/\delta^2 \). As a result, the best of the solutions built by two algorithms (dynamic programming builds an OPC, and by solving assignment problems \( [2] \) for different \( \Delta \) another solution is found) has a relative error of no more than \( \varepsilon \). The complexity of constructing the OPC is \( O(n^2) \) \( [11] \), and the complexity of solving the problem \( [2] \) is \( O(n^{2.5} \log_2 n) \) \( [15] \).

The theorem is proved.

4 Conclusion

In the paper, we consider the problem of covering a straight line segment by arbitrary located equal circles on a plane by moving their centers to a segment in such a way that the circles involved in the covering touch each other and the total length of the circles movements would be minimal. The complexity status of this problem is not known. We propose an FPTAS, the complexity of which is \( O\left(n^{2.5} \log_2 n / \varepsilon^2\right) \), where \( n \) is the number of circles. As far as we know, this is the first FPTAS for the problem under consideration.

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References

1. Andrews A.M., Wang H.: Minimizing the aggregate movements for interval coverage. Algorithmica, 78(1): pp. 47-85 (2017)
2. Bar-Noy A., Rawitz D. and Terlecky P.: “Green” barrier coverage with mobile sensors. In Algorithms and Complexity: 9th International Conference, CIAC 2015, Paris, France. Proceedings, pp. 3346. Springer International Publishing (2015)
3. Benkoczi R., Friggstad Z., Gaur D. and Mark Thom M.: Minimizing total sensor movement for barrier coverage by non-uniform sensors on a line. In Proc. of the 11th International Symposium on Algorithms and Experiments for Wireless Sensor Networks. Lecture Notes in Computer Science, vol 9536, pp.98-111 (2015)
4. Bhattacharya B.K., Burmeister M., Hu Y., Kranakis E., Shi Q., and A. Wiese A.: Optimal movement of mobile sensors for barrier coverage of a planar region. Theor. Comput. Sci., 410(52): pp. 5515-5528 (2008)
5. Carmi P., Katz M.J., Saban R., Stein Y.: Improved PTASs for Convex Barrier Coverage. In: Solis-Oba R., Fleischer R. (eds) Approximation and Online Algorithms. WAOA 2017. Lecture Notes in Computer Science, vol 10787. Springer, Cham (2017)
6. Carmi P., Katz M.J., Saban R., Stein Y.: Improved PTASs for Convex Barrier Coverage. In: Solis-Oba R., Fleischer R. (eds) Approximation and Online Algorithms. WAOA 2017. Lecture Notes in Computer Science, vol 10787. Springer, Cham (2017)
7. Chen A., Kumar S., Lai T.H.: Designing localized algorithms for barrier coverage. In: Proceedings of the 13th annual ACM international conference on Mobile computing and networking, pp. 63-74 (2007)
8. Chen A., Lai T.H., Xuan D.: Measuring and guaranteeing quality of barrier-coverage in wireless sensor networks. In: Proceedings of the ACM International Symposium on Mobile Ad Hoc Networking and Computing (MobiHoc), pp. 421-430 (2008)
9. Cherry A., Gudmundsson J., Mestre J.: Barrier coverage with uniform radii in 2D. Fernandez Anta A. et al. (eds.) ALGOSENSORS 2017, LNCS 10718, pp. 57-69 (2017)
10. Czyzowicz J. et al.: On Minimizing the Sum of Sensor Movements for Barrier Coverage of a Line Segment. In: Nikolaidis I., Wu K. (eds) Ad-Hoc, Mobile and Wireless Networks. Ad Hoc, Mobile and Wireless Networks. LNSC, vol. 6288, pp. 29-42. Springer, Berlin, Heidelberg (2010)
11. Erzin A., Lagutkina N.: Barrier Coverage Problem in 2D // LNCS (2018) (in press)
12. Fan H., Li M., Sun X., Wan P., Zhao Y.: Barrier coverage by sensors with adjustable ranges. ACM Transactions on Sensor Networks 11(1) (2014)
13. Saipulla A., Westphal C., Liu B., Wang J.: Barrier coverage with line-based deployed mobile sensors. Ad Hoc Networks, pp. 1381-1391 (2010)
14. Thom M.: Investigation on two classes of covering problems // PhD thesis, University of Lethbridge, Canada (2017)
15. Vaidya P., Geometry helps in matching, SIAM J. Comput. 18, pp. 12011225 (1989)
16. Wu F., Gui Y., Wang Z., Gao X., Chen G.: A survey on barrier coverage with sensors. Front. Comput. Sci., 10(6), pp. 968-984 (2016)
17. Zhang X.: Algorithms for Barrier Coverage with Wireless Sensors // PhD thesis, City University of Hong Kong, Hong Kong (2016)
18. Zhao L., Bai G., Shen H., Tang Z.: Strong Barrier Coverage of Directional Sensor Networks with Mobile Sensors. Int. J. of Distributed Sensor Networks, 14(2) (2018) DOI: 10.1177/1550147718761582.