THE DENOMINATORS OF NORMALIZED R-MATRICES OF TYPES $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, $B_n^{(1)}$ AND $D_{n+1}^{(2)}$

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ABSTRACT. Denominators of normalized $R$-matrices provide important information on finite dimensional integrable representations over quantum affine algebras, and over quiver Hecke algebras by the generalized quantum affine Schur-Weyl duality functors. We compute the denominators of all normalized $R$-matrices between fundamental representations of types $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, $B_n^{(1)}$ and $D_{n+1}^{(2)}$. Thus we can conclude that the normalized $R$-matrices of types $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, $B_n^{(1)}$ have only simple poles, and of type $D_{n+1}^{(2)}$ have double poles under certain conditions.

INTRODUCTION

Let $\mathfrak{g}$ be an affine Kac-Moody algebra and $U'_q(\mathfrak{g})$ be the quantum affine algebra corresponding to $\mathfrak{g}$. The finite dimensional integrable representations over $U'_q(\mathfrak{g})$ have been investigated by many authors during the past twenty years from different perspectives (see [1, 3, 4, 10, 11, 23, 26]). Among these aspects, we focus on the theory of $R$-matrices which has deep relationship with $q$-analysis, operator algebras, conformal field theories, statistical mechanical models, etc.

The purpose of this paper is to compute the denominators of normalized $R$-matrices between the fundamental representations $V(\varpi_{ik})$'s over $U'_q(\mathfrak{g})$. Knowing the denominators is quite crucial to study the finite dimensional integrable representations by the following theorem:

Theorem [1, 23] Let $M$ be a finite dimensional irreducible integrable $U'_q(\mathfrak{g})$-module $M$. Then, there exists a finite sequence $\{(i_1, a_1), \ldots, (i_l, a_l)\}$ in $\{1, 2, \ldots, n\} \times \mathbb{C}^\times$ such that

- $d_{i_k, i_k'}(a_{k'}) 
eq 0$ for $1 \leq k < k' \leq l$ and
- $M$ is isomorphic to the head of $\otimes_{k=1}^{l} V(\varpi_{ik})_{a_k}$.

Moreover, such a sequence $\{(i_1, a_1), \ldots, (i_l, a_l)\}$ is unique up to permutation. Here $\mathbb{k} = \overline{\mathbb{Q}(q)} \subset \cup_{m>0} \mathbb{C}((q^{1/m}))$ and $d_{i_k, i_k'}(z) \in \mathbb{k}[z]$ denotes the denominator of the normalized $R$-matrix

$$R_{i_k, i_k'}^{\text{norm}}(z) := R_{V(\varpi_{ik}), V(\varpi_{ik'})}^{\text{norm}}(z) : V(\varpi_{ik}) \otimes V(\varpi_{ik'}) \rightarrow \mathbb{k}(z) \otimes \mathbb{k}[z^{\pm 1}] \left(V(\varpi_{ik'})_{z} \otimes V(\varpi_{ik})\right)$$

satisfying

$$d_{i_k, i_k'}(z) R_{i_k, i_k'}^{\text{norm}}(z)(V(\varpi_{ik}) \otimes V(\varpi_{ik'})_{z}) \subset V(\varpi_{ik'})_{z} \otimes V(\varpi_{ik})$$

Thus the study of denominators is one of the first step to study the category $\mathcal{C}_\mathfrak{g}$ consisting of finite dimensional integrable representations over $U'_q(\mathfrak{g})$.

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On the other hand, Kang, Kashiwara and Kim [18, 19] recently constructed the quantum affine Schur-Weyl duality functor $\mathcal{F}$ by observing zeros of denominators of normalized $R$-matrices. The way of constructing $\mathcal{F}$ can be described as follows: Let $\{V_s\}_{s \in \mathcal{S}}$ be a family of fundamental representations over $U_q'(\mathfrak{g})$. For an index set $J$ and two maps $X : J \rightarrow k^\times$, $s : J \rightarrow \mathcal{S}$, we can define a quiver $Q^J = (Q^J_0, Q^J_1)$ associated with $(J, X, s)$ as (i:vertices) $Q^J_0 = J$, (ii:arrows) for $i, j \in J$, we put $d_{ij}$ many arrows from $i$ to $j$, where $d_{ij}$ is the order of the zero of $d_{s(i), s(j)}$ at $X(j)/X(i)$.

Then we obtain a symmetric Cartan matrix $A^J = (a^J_{ij})_{i,j \in J}$ associated with $(J, X, s)$ by

$$a^J_{ij} = 2 \quad \text{if} \quad i = j \quad \text{and} \quad a^J_{ij} = -d_{ij} - d_{ji} \quad \text{if} \quad i \neq j.$$  

Let $R^J$ be the quiver Hecke algebras associated with the symmetric Cartan matrix $A^J$ and the parameters ([24, 25, 29])

$$Q_{i,j}(u, v) = (u - v)^{d_{ij}}(v - u)^{d_{ji}} \quad \text{if} \quad i \neq j \quad \text{and} \quad Q_{i,i}(u, v) = 0 \quad \text{for all} \quad i \in J.$$  

**Theorem** [18] There exists a functor

$$\mathcal{F} : \text{Rep}(R^J) \rightarrow \mathcal{C}_g$$  

where $\text{Rep}(R^J)$ denotes the category of finite dimensional representations over $R^J$. Moreover, the functor enjoys the following properties:

(a) $\mathcal{F}$ is a tensor functor; that is, there exist $U_q'(\mathfrak{g})$-module isomorphisms

$$\mathcal{F}(R^J(0)) \simeq k \quad \text{and} \quad \mathcal{F}(M_1 \circ M_2) \simeq \mathcal{F}(M_1) \otimes \mathcal{F}(M_2)$$  

for any $M_1, M_2 \in \text{Rep}(R^J)$.

(b) If the Cartan matrix $A^J$ is of type $A_n(n \geq 1)$, $D_n(n \geq 4)$, $E_6$, $E_7$ or $E_8$, then the functor $\mathcal{F}$ is exact.

Thus the generalized quantum affine Schur-Weyl duality functor provides the way of investigating the category $\mathcal{C}_g$ via the category $\text{Rep}(R^J)$ and the other way around (see [20]).

Note that $A^J$ depends on the choice of $(J, X, s)$ and the denominators. Hence one may expect various exact functors defined on $\text{Rep}(R^J)$ for a fixed algebra $R^J$. In the forthcoming papers by the author and his collaborators ([21, 22]), they will consider such situations, and the denominator formulas given in this paper will play an important role.

The denominators of all normalized $R$-matrices $R_{k,l}^{\text{norm}}(z)$ for $A_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$ were studied in [11, 6, 19] and the denominators of normalized $R$-matrix $R_{1,1}^{\text{norm}}(z)$ (resp. $R_{n,n}^{\text{norm}}(z)$) between vector representations (resp. spin representations) for all classical affine types are given in [17, 27]. On the other hand, the explicit forms of normalized $R$-matrix $R_{k,l}^{\text{norm}}(z)$ for all classical affine types were studied in [7, 13, 14, 15]. With these results, we will compute the denominators $d_{k,l}(z)$ of all normalized $R$-matrices $R_{k,l}^{\text{norm}}(z)$ by employing the framework given in [19, Appendix A].

Our main results are

$$d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z^t - (-q^t)^{k-l+2s}(z^t - (p^*)^t(-q^t)^{2s-k-l})$$

if $V(\varpi_k)$ and $V(\varpi_l)$ are not spin representations, and

$$d_{k,n}(z) = \prod_{s=1}^{k} (z - (-1)^{n+k}q^{2n-2k-1+4s}) \quad \text{if} \quad g = D_n^{(1)} \quad \text{and} \quad k < n,$$

$$d_{k,n}(z) = \prod_{s=1}^{k} (z^2 + (-q^2)^{n-k+2s}) \quad \text{if} \quad g = D_{n+1}^{(2)} \quad \text{and} \quad k < n.$$
Here,
\[
t = \begin{cases} 
2 & \text{if } g = D^{(2)}_{n+1}, \\
1 & \text{otherwise,}
\end{cases}
\]
for the null root $\delta$ (see (1.2)). Hence, we can conclude that
(a) $R^{\text{dim}}_{k,l}(z)$ of $A^{(2)}_{2n-1}$, $A^{(2)}_{2n}$ or $B^{(1)}_n$ has only simple poles,
(b) $R^{\text{dim}}(z)$ of $D^{(2)}_{n+1}$ has a double pole at $z = (-q^2)^{s/2}$ if
\[2 \leq k, l \leq n - 1, \quad k + l > n, \quad 2n + 2 - k - l \leq s \leq k + l \text{ and } s \equiv k + l \mod 2,
\]
(c) $R^{\text{dim}}_{k,l}(z)$ has a pole at $\pm (-q^k)^{l/t}$ only if $k \in \mathbb{Z}$ such that $2 \leq t \leq (\rho, \delta)$ (see [8]).

This paper is organized as follows. In the first section, we recall the notion of quantum affine algebras and $R$-matrices, briefly. In the next section, we give the $U'_q(g)$-module structure of the vector representations and spin representations over $U'_q(g)$. In the third section, we study morphisms in $\text{Hom}_{U'_q(g)}(V(\varpi_i)_a \otimes V(\varpi_j)_b, V(\varpi_k)_c)$, called the Dorey’s type morphisms. After that, we prove the existence of certain surjective homomorphisms which can be understood as $D^{(2)}_{n+1}$-analogue of [13] Lemma A.3.2. In the last section, we propose the general framework for computing the denominators, which is originated from [19] Appendix A. Then we compute $d_{1,n}(z)$ for $g = D^{(2)}_{n+1}$ and the unknown denominators $d_{k,l}(z)$ of normalized $R$-matrices for $g = A^{(2)}_{2n-1}$, $A^{(2)}_{2n}$, $B^{(1)}_n$ and $D^{(2)}_{n+1}$, by using the results in the previous sections. In the appendix, we provide a table of $d_{k,l}(z)$ for all classical affine types for reader’s convenience.

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1. Quantum affine algebras and $R$-matrices

In this section, we briefly recall the backgrounds and theories on quantum affine algebras, their finite dimensional integral representations and $R$-matrices. We refer to [1] [13] [23] for precise statements and definitions.

1.1. Quantum affine algebras and their representations. Let $I = \{0, 1, \ldots, n\}$ be a set of indices and set $I_0 := I \setminus \{0\}$. An affine Cartan datum is a quadruple $(A, P, \Pi, \Pi')$ consisting of
(a) a matrix $A$ of corank 1, called the affine Cartan matrix satisfying
\[
(i) \quad a_{ii} = 2 \quad (i \in I), \quad (ii) \quad a_{ij} \in \mathbb{Z}_{\leq 0}, \quad (iii) \quad a_{ij} = 0 \text{ if } a_{ji} = 0
\]
with $D = \text{diag}(d_i \in \mathbb{Z}_{>0} \mid i \in I)$ making $DA$ symmetric,
(b) a free abelian group $P$ of rank $n + 2$, called the weight lattice,
(c) $\Pi = \{\alpha_i \mid i \in I\} \subset P$, called the set of simple roots,
(d) $\Pi' = \{h_i \mid i \in I\} \subset P' := \text{Hom}(P, \mathbb{Z})$, called the set of simple coroots,
which satisfy
\[
(1) \quad \langle h_i, \alpha_j \rangle = a_{ij} \text{ for all } i, j \in I,
(2) \quad \Pi \text{ and } \Pi' \text{ are linearly independent sets},
(3) \quad \text{for each } i \in I, \text{ there exists } \Lambda_i \in P \text{ such that } \langle h_i, \Lambda_j \rangle = \delta_{ij} \text{ for all } j \in I.
\]
We set $Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i$, $Q_+ = \bigoplus_{i \in I} \mathbb{Z}_{>0} \alpha_i$, $Q^\vee = \bigoplus_{i \in I} \mathbb{Z} h_i$ and $Q^\vee_+ = \bigoplus_{i \in I} \mathbb{Z}_{>0} h_i$. We choose the imaginary root $\delta = \sum_{i \in I} a_i \alpha_i \in Q_+$ and the center $c = \sum_{i \in I} c_i h_i \in Q^\vee_+$ such that \([16 \text{ Chapter 4}])

$$\{\lambda \in Q \mid \langle h_i, \lambda \rangle = 0 \text{ for every } i \in I\} = \mathbb{Z} \delta$$

and

$$\{h \in Q^\vee \mid \langle h, \alpha_i \rangle = 0 \text{ for every } i \in I\} = \mathbb{Z} c.$$

Set $h = Q \otimes \mathbb{Z} P^\vee$. Then there exists a symmetric bilinear form $(\cdot, \cdot)$ on $h^*$ satisfying

$$\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$$

for any $i \in I$ and $\lambda \in h^*$.

We normalize the bilinear form by

$$\langle c, \lambda \rangle = (\delta, \lambda)$$

for any $\lambda \in h^*$.

Let us denote by $g$ the affine Kac-Moody Lie algebra associated with $(A, P, \Pi, \Pi^\vee)$ and by $W$ the Weyl group of $g$, generated by $(s_i)_{i \in I}$. We define $g_0$ the subalgebra of $g$ generated by the chevalley generators $e_i, f_i,$ and $h_i$ for $i \in I_0$. Then $g_0$ is a finite dimensional simple Lie algebra.

Let $\gamma$ be the smallest positive integer such that

$$\gamma(\alpha_i, \alpha_i)/2 \in \mathbb{Z}$$

for any $i \in I$.

Let $q$ be an indeterminate. For $m, n \in \mathbb{Z}_{>0}$ and $i \in I,$ we define $q_i = q^{(\alpha_i, \alpha_i)/2}$ and

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \frac{m}{n} = \frac{[m]_i!}{[m-n]_i! [n]_i!}.$$

**Definition 1.1.** The quantum affine algebra $U_q(g)$ associated with $(A, P, \Pi, \Pi^\vee)$ is the associative algebra over $\mathbb{Q}(q^{1/\gamma})$ with 1 generated by $e_i, f_i$ ($i \in I$) and $q^h$ ($h \in \gamma^{-1} \Pi^\vee$) satisfying following relations:

1. $q^0 = 1, q^h q^{h'} = q^{h+h'}$ for $h, h' \in \gamma^{-1} \Pi^\vee$,
2. $q^h e_i q^{-h} = q^{(h, \alpha_i)} e_i, \quad q^h f_i q^{-h} = q^{-(h, \alpha_i)} f_i$ for $h \in \gamma^{-1} \Pi^\vee$, $i \in I$,
3. $e_i f_j - f_j e_i = \delta_{ij} K_i - K_i^{-1},$ where $K_i = q_i^{h_i}$,
4. $\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{1-a_{ij}-k} f_j e_i^{(k)} = 0$ for $i \neq j$,

where $e_i^{(k)} = e_i^k/[k]_i!$ and $f_i^{(k)} = f_i^k/[k]_i!$.

We denote by $U'_q(g)$ the subalgebra of $U_q(g)$ generated by $e_i, f_i, K_i^{\pm 1}$ ($i \in I$) and we call it also the quantum affine algebra. Throughout this paper, we mainly deal with $U'_q(g)$.

For $U'_q(g)$-modules $M$ and $N$, $M \otimes N$ becomes a $U'_q(g)$-module by the coproduct $\Delta$ of $U'_q(g)$:

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i.$$

Set $P_{cl} := P/\mathbb{Z} \delta$ and $cl: P \to P_{cl}$ as the canonical projection.

We say that a $U'_q(g)$-module $M$ is integrable provided that

1. $M$ decomposes into $P_{cl}$-weight spaces; that is,

$$M = \bigoplus_{\mu \in P_{cl}} M_{\mu},$$

where $M_{\mu} := \{v \in M \mid K_i v = q^{(h_i, \mu)} v\}$,
2. $e_i$ and $f_i$ ($i \in I$) act on $M$ nilpotently.
For $i \in I_0$, the level 0 fundamental weight $\varpi_i$ is defined by

$$\varpi_i := \gcd(c_0, c_i)^{-1}(c_0 \Lambda_i - c_i \Lambda_0) \in \mathcal{P}.$$  

Then $\{\cl(\varpi_i) \mid i \in I_0\}$ forms a basis for the space of classical integral weight level 0, denoted by $\mathcal{P}_\cl^0$, which is defined as follows:

$$\mathcal{P}_\cl^0 = \{ \lambda \in \mathcal{P}_\cl \mid (c, \lambda) = 0 \}.$$  

The Weyl group $W_0$ of $\mathfrak{g}_0$, generated by $(s_i)_{i \in I_0}$, acts on $\mathcal{P}_\cl^0$ (see [1, §1.2]). We denote by $w_0$ the longest element of $W_0$.

**Definition 1.2.** [1, §1.3] For $i \in I_0$, the $i$th fundamental module is a unique finite dimensional integrable $U_q'(\mathfrak{g})$-module $V(\varpi_i)$ satisfying the following properties:

1. The weights of $V(\varpi_i)$ are contained in the convex hull of $W_0 \cl(\varpi_i)$.
2. $V(\varpi_i)_{\cl(\varpi_i)} = \mathbb{C}(q)v_{\varpi_i}$. (We call the vector $v_{\varpi_i}$ a dominant integral weight vector.)
3. For any $\mu \in W_0 \cl(\varpi_i)$, we can associate a non-zero vector $u_\mu$, called an extremal vector of weight $\mu$, such that

$$S_i \cdot u_\mu := u_{s_i \mu} = \begin{cases} f_{q_i}^{\langle h_i, \mu \rangle} u_\mu & \text{if } \langle h_i, \mu \rangle \geq 0, \\ e_{q_i}^{\langle -h_i, \mu \rangle} u_\mu & \text{if } \langle h_i, \mu \rangle \leq 0, \end{cases}$$

for any $i \in I$.

4. $v_{\varpi_i}$ generates $V(\varpi_i)$ as a $U_q'(\mathfrak{g})$-module.

Let $k$ be an algebraic closure of $\mathbb{C}(q)$ in $\cup_{m>0} \mathbb{C}((q^{1/m}))$. When we deal with $U_q'(\mathfrak{g})$-modules, we regard the base field as $k$.

For a $U_q'(\mathfrak{g})$-module $M$, we denote by $^* M$ the right dual and $M^*$ the left dual of $M$, if there exist $U_q'(\mathfrak{g})$-homomorphisms

$$M^* \otimes M \overset{\text{tr}}{\longrightarrow} k, \quad k \longrightarrow M \otimes M^* \quad \text{and} \quad M \otimes ^* M \overset{\text{tr}}{\longrightarrow} k, \quad k \longrightarrow M^* \otimes M.$$  

Recall that $V(\varpi_i)$ is finite dimensional and has the right dual and left dual as follows:

$$V(\varpi_i)^* := V(\varpi_i^*)_{\rho^*}, \quad *V(\varpi_i) := V(\varpi_i^*)_{p^*} \quad \text{and} \quad p^* := (-1)^{\rho^*} q^{\langle \rho, \delta \rangle}.$$  

where

- $^*$ is the involution of $I_0$ defined by the image of $\varpi_i$ under the action $w_0$; i.e., $w_0(\varpi_i) = -\varpi_i^*$,
- $\rho$ is defined by $\langle h_i, \rho \rangle = 1$ and $\rho^*$ is defined by $\langle \rho^*, \alpha_i \rangle = 1$ for all $i \in I$.

We call an integrable $U_q'(\mathfrak{g})$-module $M$ good if $M$ satisfies certain properties. In this paper, the whole definition of the good module is not needed. Thus we refer [23] for the precise definition of good module. However, we would like to emphasize one condition of the good module:

A good module $M$ contains the unique (up to constant) weight vector $v_M$ of weight $\lambda$, such that

$$\text{wt}(M) \subset \lambda + \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \cl(\alpha_i).$$  

We call $v_M$ the dominant extremal weight vector and $\lambda$ dominant extremal weight. For instance, the $i$th fundamental representation is a good module.

For an indeterminate $z$ and a $U_q'(\mathfrak{g})$-module $M$, let us denote by $M_z = \{ u_z \mid u \in M \}$ the $U_q'(\mathfrak{g})$-module $k[z^{\pm 1}] \otimes M$ with the action of $U_q'(\mathfrak{g})$ given by

$$e_i(u_z) = z^{\delta_i, 0} e_i(u) z, \quad f_i(u_z) = z^{-\delta_i, 0} f_i(u) z, \quad K_i(u_z) = (K_i u) z.$$  

We sometimes use the notation $u$ for $u_z$ to simplify equations. (For example, see the proof of Proposition 4.7)
1.2. Normalized and universal \( R \)-matrices. We call a \( k[z^\pm 1] \otimes U'_q(\mathfrak{g}) \)-module homomorphism between \( M \otimes N_z \) and \( N_z \otimes M \) as an **intertwiner**. It is known that, for finite dimensional integral \( U'_q(\mathfrak{g}) \)-modules \( M \) and \( N \), there exists an intertwiner

\[
R_{M,N}^{\text{univ}}(z) : M \otimes N_z \rightarrow N_z \otimes M
\]

which satisfies

\[
R_{M,N}^{\text{univ}} \otimes R_{N',N}^{\text{univ}}(z) = (N_z \otimes R_{M,N}^{\text{univ}}(z)) \circ (R_{M,N}^{\text{univ}}(z) \otimes N'_z).
\]

We call \( R_{M,N}^{\text{univ}} \) the **universal \( R \)-matrix** \([9]\).

**Definition 1.3.** For good modules \( M \) and \( N \), the **normalized \( R \)-matrix** \( R_{M,N}^{\text{norm}} \) is the \( U'_q(\mathfrak{g}) \)-module homomorphism

\[
R_{M,N}^{\text{norm}} : M_\mathbb{Z}M \otimes N_{\mathbb{Z}N} \rightarrow k(z_M, z_N) \otimes _{k[z_M^\pm 1, z_N^\pm 1]} N_{\mathbb{Z}N} \otimes M_\mathbb{Z}M
\]

which satisfies

\[
R_{M,N}^{\text{norm}} \circ z_M = z_M \circ R_{M,N}^{\text{norm}}, \quad R_{M,N}^{\text{norm}} \circ z_N = z_N \circ R_{M,N}^{\text{norm}} \quad \text{and} \quad R_{M,N}^{\text{norm}}(v_M \otimes v_N) = v_N \otimes v_M.
\]

\([1] \text{ Corollary 2.5} \) tells that, for good modules \( M \) and \( N \)

\[
\text{Hom}_{k(z) \otimes U'_q(\mathfrak{g})}(M \otimes N_z, N_z \otimes M) \simeq k(z),
\]

and hence there exists \( a_{M,N}(z) \in k(z) \) such that

\[
a_{M,N}(z)R_{M,N}^{\text{norm}}(z).
\]

Note that

\[
R_{M,N}^{\text{norm}}(z)(M \otimes N_z) \subset k(z) \otimes_{k[z^\pm 1]} (N_z \otimes M)
\]

and there exists a unique monic polynomial \( d_{M,N}(z) \in k[z] \) such that

\[
d_{M,N}(z)R_{M,N}^{\text{norm}}(z)(M \otimes N_z) \subset (N_z \otimes M).
\]

We call \( d_{M,N}(u) \) the **denominator** of \( R_{M,N}^{\text{norm}}(z) \).

**Lemma 1.4** (**[1]** Lemma C.15). Let \( V', V'' \), \( V \) and \( W \) be irreducible \( U'_q(\mathfrak{g}) \)-modules. Assume that we have a surjective \( U'_q(\mathfrak{g}) \)-homomorphism

\[
V' \otimes V'' \rightarrow V.
\]

Then we have

\[
\frac{d_{W,V'}(z)d_{W,V''}(z)a_{W,V}(z)}{d_{W,V}(z)a_{W,V'}(z)a_{W,V''}(z)} \quad \text{and} \quad \frac{d_{V',W}(z)d_{V'',W}(z)a_{V',W}(z)}{d_{V,W}(z)a_{V',W}(z)a_{V'',W}(z)} \in k[z^\pm 1].
\]

2. **Vector and spin representation.**

In this section, we record the \( U'_q(\mathfrak{g}) \)-module structure of

- \( V(\varnothing_1) \), called the **vector representation**,
- \( V(\varnothing_n) \), called the **spin representation**, for \( \mathfrak{g} = B_n^{(1)} \) or \( D_n^{(2)} \).

As a vector space, the vector representation can be expressed as follows (**[12]** Chapter 11):

\[
V(\varnothing_1) = (\bigoplus_{j=1}^n k\varnothing_j) \oplus (\bigoplus_{j=1}^n k\varnothing_j) \oplus W
\]

where

| \( \mathfrak{g} \) | \( A_{2n-1}^{(2)} \) | \( B_n^{(1)} \) | \( A_{2n}^{(2)} \) | \( D_{n+1}^{(2)} \) |
|---|---|---|---|---|
| \( W \) | \( \emptyset \) | \( k\varnothing_0 \) | \( k\varnothing_0 \) | \( k\varnothing_0 \oplus k\varnothing_0 \) |
The actions of $e_i$, $f_i$ and $q^h$ are defined by follows:

$$q^h \cdot v_j = q^{(h, \text{wt}(v_j))}v_j \quad \text{for} \quad h \in P_{cl}^\vee,$$

| $\mathfrak{g}$ | $e_i v_j$ | $f_i v_j$ |
|----------------|------------|-----------|
| $A_{2n-1}^{(2)}$ | $v_i$ if $j = i + 1$ and $i \neq n$, $v_{i+1}$ if $j = 1$ and $i \neq n$, $v_n$ if $j = \overline{i}$ and $i = n$, $v_{n-1}$ if $j = 1$ and $i = 0$, $v_{n+1}$ if $j = 2$ and $i = 0$, $0$ otherwise, | $v_{i+1}$ if $j = i$ and $i \neq n$, $v_7$ if $j = i + 1$ and $i \neq n$, $v_{n+1}$ if $j = n$ and $i = n$, $v_{n-1}$ if $j = \overline{i}$ and $i = 0$, $v_1$ if $j = 2$ and $i = 0$, $0$ otherwise, |
| $A_{2n}^{(2)}$ | $v_i$ if $j = i + 1$ and $i \neq n$, $v_{i+1}$ if $j = 7$ and $i \neq n$, $v_0$ if $j = 1$ and $i = 0$, $0$ otherwise, | $v_{i+1}$ if $j = i$ and $i \neq n$, $v_7$ if $j = i + 1$ and $i \neq n$, $v_0$ if $j = n$ and $i = n$, $v_{n+1}$ if $j = \overline{i}$ and $i = 0$, $v_1$ if $j = 2$ and $i = 0$, $0$ otherwise, |
| $B_n^{(1)}$ | $v_i$ if $j = i + 1$ and $i \neq n$, $v_{i+1}$ if $j = i + 1$ and $i \neq n$, $v_n$ if $j = 0$ and $i = n$, $v_{n-1}$ if $j = 1$ and $i = 0$, $v_{n+1}$ if $j = 2$ and $i = 0$, $0$ otherwise, | $v_{i+1}$ if $j = i$ and $i \neq n$, $v_7$ if $j = i + 1$ and $i \neq n$, $v_0$ if $j = n$ and $i = n$, $v_{n+1}$ if $j = \overline{i}$ and $i = 0$, $v_1$ if $j = \overline{i}$ and $i = 0$, $0$ otherwise, |
| $D_{n+1}^{(2)}$ | $v_i$ if $j = i + 1$ and $i \neq n$, $v_{i+1}$ if $j = i + 1$ and $i \neq n$, $v_n$ if $j = 0$ and $i = n$, $v_{n-1}$ if $j = 1$ and $i = 0$, $v_{n+1}$ if $j = 2$ and $i = 0$, $0$ otherwise, | $v_{i+1}$ if $j = i$ and $i \neq n$, $v_7$ if $j = i + 1$ and $i \neq n$, $v_0$ if $j = n$ and $i = n$, $v_{n+1}$ if $j = \overline{i}$ and $i = 0$, $v_1$ if $j = \overline{i}$ and $i = 0$, $0$ otherwise, |

where

$$\text{wt}(v_j) = e_j, \quad \text{wt}(v_{\overline{j}}) = -e_j \quad \text{for} \quad j = 1, \ldots, n \quad \text{and} \quad \text{wt}(v_0) = \text{wt}(v_{\overline{0}}) = 0.$$

For $\mathfrak{g} = B_n^{(1)}$ or $\mathfrak{g} = D_{n+1}^{(2)}$, the spin representation $V(\omega_n)$ is the $k$-vector space with a basis

$$\mathcal{B}_{sp} = \{(m_1, \ldots, m_n) ; m_i = + \text{ or } - \}.$$

Its $U_q'(\mathfrak{g})$-module structure is given by defining the action of $e_i$, $f_i$ and $q^h$ as follows:

$$q^h v = q^{(h, \text{wt}(v))}v \quad \text{for} \quad h \in P_{cl}^\vee, \text{ where } \text{wt}(v) = \frac{1}{2} \sum_{k=1}^n m_k e_k,$$
3. Surjective Homomorphisms Between Integrable $U_q(g)$-Modules

In this section, we first study the morphisms in

$$\text{Hom}_{U_q(g)}(V(\varpi_i)_a \otimes V(\varpi_j)_b, V(\varpi_k)_c) \quad \text{for } i, j, k \in I_0 \text{ and } a, b, c \in k^\times.$$ 

These kinds of morphisms are known as Dorey’s type morphisms and have been investigated in [5] for the classical untwisted affine types $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$. In the last part of this section, we study the surjective homomorphisms which can be understood as $D_{n+1}^{(2)}$-analogue of the surjective homomorphisms given in [19] (A.17)

Hereafter, we will use the following convention frequently:

For a statement $P$, $\delta(P)$ is 1 if $P$ is true and 0 if $P$ is false.

By the result on $B_n^{(1)}$ in [5], it suffices to consider when $g = A_{2n-1}^{(2)}$ ($n \geq 3$), $A_{2n}^{(2)}$ ($n \geq 2$) and $D_{n+1}^{(2)}$ ($n \geq 2$).

The finite Dynkin diagrams of $g_0$ associated with $g$ are given as follows:

$$C_n : \frac{\epsilon_1 - \epsilon_2}{n-1} \cdots \frac{\epsilon_{n-1} - \epsilon_n}{n-1} \frac{\epsilon_n}{n} (A_{2n-1}^{(2)}, A_{2n}^{(2)}) \quad B_n : \frac{\epsilon_1 - \epsilon_2}{n-1} \cdots \frac{\epsilon_{n-1} - \epsilon_n}{n} \frac{2\epsilon_n}{n} (D_{n+1}^{(2)}).$$

We denote by $V_0(\varpi_i)$ for $i \in I_0$, the highest weight $U_q(g_0)$-module with the highest weight $\varpi_i$.

Throughout this paper, we set

$$(3.1) \quad t = \begin{cases} 2 & \text{if } g = D_{n+1}^{(2)}, \\ 1 & \text{otherwise,} \end{cases} \quad \vartheta = \begin{cases} 1 & \text{if } g = B_n^{(1)} \text{ or } D_{n+1}^{(2)}, \\ 0 & \text{otherwise.} \end{cases}$$
3.1. \(i + j = k \leq n - \vartheta\). Recall that there exists an injective \(U_q(\mathfrak{g}_0)\)-module homomorphism (see, [12], Chapter 8]) \[
\Phi_{i,j} : V_0(\varpi_{i+j}) \rightarrow V_0(\varpi_i) \otimes V_0(\varpi_j) \quad \text{for} \quad i + j \leq n - \vartheta
\]
given by
\[
(3.2) \quad u_\lambda \mapsto v_\lambda = \sum_{\lambda = \mu + \xi} C_{\mu,\xi}^\lambda u_\mu \otimes u_\xi \quad (C_{\mu,\xi}^\lambda \in k)
\]
where \(\lambda \in W_0 \cdot \varpi_{i+j}\) and \(\mu\) (resp. \(\xi\)) runs over the elements in \(W_0 \cdot \varpi_i\) (resp. \(W_0 \cdot \varpi_j\)).

For a positive integer \(l \leq n - \vartheta\), we sometimes write \(\lambda \in \text{wt}(V_0(\varpi_l))\) as a sequence \((\lambda_1, \ldots, \lambda_n) \in \{1, 0, -1\}^n\) such that \(\lambda = \sum_{k=1}^n \lambda_k \epsilon_k\).

In [32], since \(\Phi_{i,j}\) is a \(U_q(\mathfrak{g}_0)\)-homomorphism and \(V(\varpi_{i+j})\) is generated by \(u_{\varpi_{i+j}}\), we can observe that
\[
(3.3) \quad \lambda_k \geq 0 \implies \mu_k, \xi_k \geq 0 \quad \text{and} \quad \lambda_k \leq 0 \implies \mu_k, \xi_k \leq 0.
\]
Since \(\lambda_k \in \{1, 0, -1\}\), we can conclude that
\[
(3.4) \quad \mu_k \xi_k = 0 \quad \text{for all} \quad 1 \leq k \leq n.
\]
From the observation (3.3), \(C_{\mu,\xi}^\lambda\) must be the same as \(C_{s_k \mu, s_k \xi}^{s_k \lambda} \) whenever \(\langle h_k, \lambda \rangle \neq 0\).

**Proposition 3.1.** Set \[
(3.5) \quad c_{\mu,\xi}^\lambda = \# \{(a, b) \mid a < b, (\mu_a, \xi_a) = (0, 1), \mu_b \neq 0\}
\]
\[
\quad \quad \quad \quad + \# \{(a, b) \mid a < b, (\mu_a, \xi_a) = (-1, 0), \xi_b \neq 0\}.
\]
Then the \(C_{\mu,\xi}^\lambda\) in (3.2) is given as follows:
\[
C_{\mu,\xi}^\lambda = (-q_1)^{c_{\mu,\xi}^\lambda}.
\]

**Proof.** First, we check that \(C_{s_k \mu, s_k \xi}^{s_k \lambda} = c_{\mu,\xi}^\lambda\) whenever \(\langle h_k, \lambda \rangle \neq 0\). To do this, it suffices to consider \((a, b) = (k, k + 1)\). Then one can easily check that
\[
\# \{(a, b) \mid a < b, (\mu_a, \xi_a) = (0, 1), \mu_b \neq 0\}
\]
\[
\quad \quad \quad \quad + \# \{(a, b) \mid a < b, (\mu_a, \xi_a) = (-1, 0), \xi_b \neq 0\}
\]
\[
\quad \quad \quad = \# \{(a, b) \mid a < b, ((s_k \mu)_a, (s_k \xi)_a) = (0, 1), (s_k \mu)_b \neq 0\}
\]
\[
\quad \quad \quad \quad + \# \{(a, b) \mid a < b, ((s_k \mu)_a, (s_k \xi)_a) = (-1, 0), (s_k \xi)_b \neq 0\}.
\]
Thus we can assume that \(\lambda = \varpi_{i+j}\). If \(k \geq i + j\), then \(e_k v_\lambda = 0\), trivially. When \(1 \leq k < i + j\),
\[
0 = e_k v_\lambda = \sum_{(\mu_k, \mu_{k+1}) = (0, 1)} C_{\mu_k,\xi_k}^{\lambda, \mu_k} v_{s_k \mu} \otimes v_{s_k \xi} + \sum_{(\mu_k, \mu_{k+1}) = (1, 0)} C_{\mu_k,\xi_k}^{\lambda, \mu_k} v_{s_k \mu} \otimes v_{s_k \xi}.
\]
Thus, for \((\mu_k, \mu_{k+1}) = (0, 1)\) and \((\xi_k, \xi_{k+1}) = (1, 0)\), we have
\[
c_{\mu,\xi}^\lambda = c_{s_k \mu, s_k \xi}^{s_k \lambda} + 1
\]
which implies our assertion. \(\Box\)

Now we shall determine \(x, y \in k^\times\) such that there exists an injective \(U_q'(\mathfrak{g})\)-module homomorphism \[
(3.6) \quad V(\varpi_{i+j}) \rightarrow V(\varpi_i)_x \otimes V(\varpi_j)_y.
\]
The strategy in this subsection can be explained as follows: As a \(U_q(\mathfrak{g}_0)\)-module, we have an injection \(V_0(\varpi_{i+j}) \rightarrow V(\varpi_i)_x \otimes V(\varpi_j)_y\).
Proposition 3.3. Let \( \lambda, \mu \) and \( \xi \) be extremal weights and \( \langle h_0, \lambda \rangle \neq 0 \). Let \( f(x) \) and \( g(y) \) be functions arising from the action of \( e_0 \) or \( f_0 \) on \( V(\varpi_i)_x \) or \( V(\varpi_j)_y \).

Recall the notion \( S_k \cdot u_\mu = u_{s_k \mu} \) for an extremal weight \( \mu \) in [111].

**Proposition 3.2.** Let \( g = A^{(2)}_{2n-1} \) \((n \geq 3)\). Then the \( x, y \) in (3.6) are given as follows:

\[
\begin{align*}
\lambda &= (-q)^j \\
\mu &= (-q)^{-i}.
\end{align*}
\]

**Proof.** The Dynkin diagram of \( A^{(2)}_{2n-1} \) is given as follows:

\[
\begin{array}{c}
\varepsilon_1 - \varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_{n-1} - \varepsilon_n \\
\varepsilon_n
\end{array}
\]

It suffices to consider \( \lambda \in W_0 \cdot \varpi_{i+j} \) such that \( \lambda_1, \lambda_2 \geq 0 \). Thus it is enough to consider when \( \mu_1, \mu_2, \xi_1, \xi_2 \geq 0 \). Then we have

\[
S_0 \cdot v_\lambda = v_{s_0 \lambda} = \sum_{\mu, \xi} C^\lambda_{\mu, \xi} \cdot x^{\delta(\mu_1=1)+\delta(\mu_2=1)} y^{\delta(\xi_1=1)+\delta(\xi_2=1)} v_{s_0 \mu} \otimes v_{s_0 \xi}.
\]

Here \( s_0(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n) = s_0(-\varepsilon_2, -\varepsilon_1, \varepsilon_3, \ldots, \varepsilon_n) \) \((\varepsilon_i \in \{-1, 0, 1\})\). Thus

\[
C^\lambda_{s_0 \mu, s_0 \xi} = x^{\delta(\mu_1=1)+\delta(\mu_2=1)} y^{\delta(\xi_1=1)+\delta(\xi_2=1)} C^\lambda_{\mu, \xi}.
\]

On the other hand, by (3.3),

\[
\begin{align*}
C^\lambda_{s_0 \mu, s_0 \xi} - C^\lambda_{\mu, \xi} &= +\delta(\mu_1 = 1) \times \# \{ b > 1 \mid \xi_b \neq 0 \} + \delta(\mu_2 = 1) \times \# \{ b > 1 \mid \xi_b \neq 0 \} \\
&- \delta(\xi_1 = 1) \times \# \{ b > 1 \mid \mu_b \neq 0 \} - \delta(\xi_2 = 1) \times \# \{ b > 1 \mid \mu_b \neq 0 \} \\
&= \delta(\mu_1 = 1) \times j + \delta(\mu_2 = 1) \times (j - \delta(\xi_2 = 1)) \\
&- \delta(\xi_1 = 1) \times i - \delta(\xi_2 = 1) \times (i - \delta(\mu_2 = 1)).
\end{align*}
\]

By (3.4), \( \mu_i \xi_i = 0 \) \((i = 1, 2)\) and hence we can conclude that

\[
C^\lambda_{s_0 \mu, s_0 \xi} - C^\lambda_{\mu, \xi} = -\delta(\xi_1 = 1) + \delta(\xi_2 = 1) \times i + \delta(\mu_1 = 1) + \delta(\mu_2 = 1) \times j.
\]

Thus we have

\[
\begin{align*}
\lambda &= (-q)^j \\
\mu &= (-q)^{-i}.
\end{align*}
\]

\[
\square
\]

**Proposition 3.3.** Let \( g = A^{(2)}_{2n} \) \((n \geq 2)\). Then the \( x, y \) in (3.6) are given as follows:

\[
\begin{align*}
\lambda &= (-q)^j \\
\mu &= (-q)^{-i}.
\end{align*}
\]

**Proof.** The Dynkin diagram of \( A^{(2)}_{2n} \) is given as follows:

\[
\begin{array}{c}
\varepsilon_1 - \varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_{n-1} - \varepsilon_n \\
\varepsilon_n
\end{array}
\]

It suffices to consider \( \lambda \in W_0 \cdot \varpi_{i+j} \) such that \( \langle h_0, \lambda \rangle < 0 \) and hence \( \lambda_1 = 1 \). Then we have

\[
S_0 \cdot v_\lambda = v_{s_0 \lambda} = e^{(2)}_0 v_\lambda = C^\lambda_{\mu, \xi} \cdot x^{\delta(\mu_1=1)} y^{\delta(\xi_1=1)} v_{s_0 \mu} \otimes v_{s_0 \xi}.
\]
Here $s_0(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = s_0(-\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$. Thus
\[
C_{s_0, s_0 \mu, s_0 \xi}^{\delta(\mu_1 = 1) y^{\delta(\xi_1 = 1)}} = x^\delta(\mu_1 = 1) y^{\delta(\xi_1 = 1)} C_{\mu, \xi}^\lambda.
\]
On the other hand, by (3.5),
\[
C_{s_0 \mu, s_0 \xi}^{\delta(\mu_1 = 1) \# \{ b > 1 \mid \xi_b \neq 0 \}} = (-q)^{\delta(\mu_1 = 1) \# \{ b > 1 \mid \mu_b \neq 0 \}} C_{\mu, \xi}^\lambda.
\]
Thus we can conclude that
\[
x = (-q)^j \quad \text{and} \quad y = (-q)^{-i}.
\]

**Proposition 3.4.** Let $g = D_{n+1}^{(2)}$. Then the $x$, $y$ in (3.6) are given as follows:
\[
x = (-q^2)^{j/2} \quad \text{and} \quad y = (-q^2)^{-i/2}.
\]

**Proof.** The Dynkin diagram of $D_{n+1}^{(2)}$ is given as follows:
\[
\begin{array}{c}
\varepsilon_1 \quad \varepsilon_2 \quad \cdots \quad \varepsilon_{n-1} \quad \varepsilon_n \\
0 \quad 1 \quad \cdots \quad n-1 \quad n
\end{array}
\]
It suffices to consider $\lambda \in W_0 \cdot \varpi_{i+j}$ such that $\langle h_0, \lambda \rangle < 0$. Thus we assume that $\lambda_1 = 1$. Note that $q_1 = q^2$. Then we have
\[
S_0 \cdot v_{\lambda} = v_{\lambda} = e_0^{(2)} v_{\lambda} = \sum C_{\mu, \xi}^{\lambda} x^{2\delta(\mu_1 = 1)} y^{2\delta(\xi_1 = 1)} v_{s_0 \mu} \otimes v_{s_0 \xi}.
\]
Here $s_0(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = s_0(-\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$. Thus
\[
C_{s_0, s_0 \mu, s_0 \xi}^{\delta(\mu_1 = 1) y^{\delta(\xi_1 = 1)}} = x^{2\delta(\mu_1 = 1)} y^{2\delta(\xi_1 = 1)} C_{\mu, \xi}^\lambda.
\]
On the other hand, by (3.5),
\[
C_{s_0 \mu, s_0 \xi}^{\delta(\mu_1 = 1) \# \{ b > 1 \mid \xi_b \neq 0 \}} = (-q)^{\delta(\mu_1 = 1) \# \{ b > 1 \mid \mu_b \neq 0 \}} C_{\mu, \xi}^\lambda.
\]
Thus we can conclude that
\[
x^2 = (-q^2)^{j} \quad \text{and} \quad y^2 = (-q^2)^{-i},
\]
which yield our assertion. 

**Theorem 3.5.** For $i + j = k \leq n - \vartheta$, there exists a surjective $U_q'(g)$-module homomorphism
\[
p_{i,j}: V(\varpi_i)(-q^2)^{j-\vartheta} \otimes V(\varpi_j)(-q^2)^{i-\vartheta} \rightarrow V(\varpi_k).
\]
By taking dual, there exists an injective $U_q'(g)$-module homomorphism
\[
\iota_{i,j}: V(\varpi_k) \rightarrow V(\varpi_i)(-q^2)^{j-\vartheta} \otimes V(\varpi_j)(-q^2)^{-i-\vartheta}.
\]

**Proof.** The proof immediately comes from the previous propositions.
3.2. \( i = j = n, \ k < n \) for \( g = D_{n+1}^{(2)} \). In this subsection, we fix \( g = D_{n+1}^{(2)} \). Recall that there exists an injective \( U_q(B_n) \)-module homomorphism (see. [12, Chapter 8])

\[
V_0(\varpi_i) \rightarrow V_0(\varpi_n) \otimes V_0(\varpi_n)
\]
given by

\[
u_\lambda \longmapsto v_\lambda = \sum_{\lambda = \mu + \xi} C_{\mu, \xi}^\lambda u_\mu \otimes u_\xi
\]

where \( \lambda \in W_0 \cdot \varpi_i \) and \( \mu, \xi \in W_0 \cdot \varpi_n \).

We sometimes write \( \mu \in \text{wt}(V_0(\varpi_n)) \) as a sequence \( (\mu_1, \ldots, \mu_n) \in \{+,-\}^n \) such that

\[
\mu = \sum_{k=1}^n \frac{\mu_k}{2} \epsilon_k.
\]

**Proposition 3.6.** Set

\[
1c_{\mu, \xi}^\lambda = \# \{(a, b) \mid a < b, \ (\mu_a, \xi_a) = (-, +), \ (\mu_b, \xi_b) = (+, -)\},
\]

\[
2c_{\mu, \xi}^\lambda = \# \{a \mid (\mu_a, \xi_a) = (-, +)\},
\]

\[
\varphi(c) = (-q)^c(-q^2)^{\frac{k-1}{2}}.
\]

Then \( C_{\mu, \xi}^\lambda \) in (3.9) is given as follows:

\[
C_{\mu, \xi}^\lambda = (-q^2)^{1c_{\mu, \xi}^\lambda} \varphi(2c_{\mu, \xi}^\lambda).
\]

**Proof.** As in Proposition 3.1, one can check that \( C_{s_k \mu, s_k \xi}^{s_k \lambda} = C_{\mu, \xi}^\lambda \) whenever \( (h_k, \lambda) \neq 0 \). Thus we can assume that \( \lambda = \varpi_i \). If \( k \leq i \), then \( e_k u_\lambda = 0 \), trivially. Thus, for \( k > i \), we have

\[
0 = e_k v_\lambda = \begin{cases}
\sum_{(\mu_k, \mu_{k+1}) = (+, -)} C_{\mu, \xi}^\lambda (q^2)^{-1} u_{s_k \mu} \otimes u_\xi \\
+ \sum_{(\mu_k, \mu_{k+1}) = (+, -)} C_{\mu, \xi}^\lambda u_\mu \otimes u_{s_k \xi} & \text{if } i < k < n,
\sum_{(\mu_n, \xi_n) = (-, +)} C_{\mu, \xi}^\lambda q^{-1} u_{s_n \mu} \otimes u_\xi \\
+ \sum_{(\mu_n, \xi_n) = (+, -)} u_\mu \otimes u_{s_n \xi} & \text{if } k = n.
\end{cases}
\]

Thus we have

\[
C_{\mu, \xi}^\lambda = \begin{cases}
-q^2 C_{s_k \mu, s_k \xi}^{s_k \lambda} & \text{if } i < k < n \text{ and } (\mu_k, \xi_k) = (-, +),
(-q)^{-1} C_{s_n \mu, s_n \xi}^{s_n \lambda} & \text{if } k = n \text{ and } \mu_n = +.
\end{cases}
\]

On the other hand, for \( i < k < n \) and \( (\mu_k, \xi_k) = (-, +) \), we have

\[
1c_{\mu, \xi}^\lambda = \begin{cases}
1c_{s_k \mu, s_k \xi}^{s_k \lambda} - 1 & \text{if } i < k < n \text{ and } (\mu_k, \xi_k) = (-, +),
1c_{\mu, \xi}^\lambda = 1c_{s_k \mu, s_k \xi}^{s_k \lambda} + 2c_{\mu, \xi}^\lambda & \text{if } k = n \text{ and } \mu_n = +,
\end{cases}
\]

\[
2c_{\mu, \xi}^\lambda = \begin{cases}
2c_{s_k \mu, s_k \xi}^{s_k \lambda} & \text{if } i < k < n \text{ and } (\mu_k, \xi_k) = (-, +),
2c_{\mu, \xi}^\lambda = 2c_{s_k \mu, s_k \xi}^{s_k \lambda} - 1 & \text{if } k = n \text{ and } \mu_n = +.
\end{cases}
\]

which yield our assertion. \( \square \)

**Theorem 3.7.** For \( k \leq n - 1 \), there exists a surjective \( U_q'(D_{n+1}^{(2)}) \)-module homomorphism

\[
p_{n,k} : V(\varpi_n) \oplus \sqrt{-1}(-q^2)^{\frac{n-k}{2}} V(\varpi_n) \rightarrow V(\varpi_k).
\]
By taking dual, there exists an injective $U_q'(D_{n+1}^{(2)})$-module homomorphism

$$\iota_{n,k}: V(\varpi_k) \rightarrow V(\varpi_n) \pm \sqrt{-1} (-q^2)^{\frac{a_{n-k}}{2}} \otimes V(\varpi_n) \pm \sqrt{-1} (-q^2)^{-\frac{a_{n-k}}{2}}. \tag{3.12}$$

**Proof.** We apply the same strategy of (3.11) i.e., we determine the $x$ and $y$ in (3.6). As in Proposition 3.4, we first consider $\lambda \in W_0 \cdot \varpi_k$ with $\lambda_1 = 1$ and hence $\mu_1 = \xi_1 = +$. In this case, we have

$$S_0 \cdot v_\lambda = u_{s_0 \lambda} = e_0^{(2)} v_\lambda = \sum C_{x,y}^\lambda u_{s_0 \mu} \otimes u_{s_0 \xi}.$$

On the other hand

$$1 c_{x,y}^\lambda = 1 c_{s_0 \mu, s_0 \xi}^\lambda, \quad 2 c_{x,y}^\lambda = 2 c_{s_0 \mu, s_0 \xi}^\lambda,$$

Thus we conclude that

$$xy = 1.$$

Consider $\lambda \in W_0 \cdot \varpi_i$ with $\langle h_0, \lambda \rangle = 0$. Equivalently $\lambda_1 = 0$ and hence $-\mu_1 = \xi_1$. In this case,

$$0 = e_0 v_\lambda = \sum_{(\mu_1, \xi_1) = (+,-)} C_{x,y}^\lambda q^{-1} x u_{s_0 \mu} \otimes u_{s_0 \xi} + \sum_{(\mu_1, \xi_1) = (-,+)} C_{x,y}^\lambda u_{s_0 \mu} \otimes u_{s_0 \xi}.$$

Thus, for $\mu_1 = +$, we have

$$C_{s_0 \mu, s_0 \xi}^\lambda = C_{x,y}^\lambda (-q)^{-1} x = C_{x,y}^\lambda (-q)^{-1} x^2.$$

On the other hand,

$$1 c_{s_0 \mu, s_0 \xi}^\lambda = 1 c_{x,y}^\lambda + \# \{b \mid (\mu_b, \xi_b) = (+,-) \} \quad \text{and} \quad 2 c_{s_0 \mu, s_0 \xi}^\lambda = 2 c_{x,y}^\lambda + 1.$$

Thus we have

$$x^2 = (-q^2)^{n-k},$$

which yields our assertion. \hfill \Box

### 3.3. $j = 1$ and $i = k = n$ for $\mathfrak{g} = A_{2n}^{(2)}$

In this subsection, we show that there exists a surjective $U_q'(A_{2n}^{(2)})$-homomorphism

$$V(\varpi_n)_{(-q)^{-1}} \otimes V(\varpi_1)_{(-q)^n} \rightarrow V(\varpi_n). \tag{3.13}$$

Indeed, we do not use (3.13) in this paper. However, for the forthcoming works, we present the existence of such a homomorphism.

Similar to the previous subsections, we determine the relations among $a$, $b$ and $c$ such that

$$V(\varpi_n)_a \rightarrow V(\varpi_1)_b \otimes V(\varpi_n)_c. \tag{3.14}$$

Recall that for $k \in I_0$ (see [28 Table 1]),

$$V(\varpi_k) \simeq \bigoplus_{j=0}^k V_0(\varpi_j)$$

as a $U_q(C_n)$-module.

Here $V_0(\varpi_0)$ is the trivial $U_q(\mathfrak{g})$-module $\mathbf{k}$. Thus

$$V_0(\varpi_n)^{\otimes 2} \rightarrow V(\varpi_1) \otimes V(\varpi_n)$$

as a $U_q(C_n)$-module.

The crystal graph of $V(\varpi_1)$ is given by (see [12 Example 11.1.4])

$$\begin{array}{cccccccccc}
0 & 1 & 2 & 2 & \ldots & n-2 & n-1 & n \\
\begin{array}{cccccccccc}
0 & 1 & 2 & 2 & \ldots & n-2 & n-1 & n \\
\end{array}
\end{array}$$
We denote by
\[
\mathbf{u}
\]
the dominant integral weight vector of \(V(\varpi_n)\) with its weight \(\varpi_n = \sum_{i \in I_0} \epsilon_i\).

For \(i_1, \ldots, i_k, j_1, \ldots, j_l \in I_0\), we set \(u[i_1, \ldots, i_k, j_1, \ldots, j_l]\) the vector in \(V_0(\varpi_{n-l})\) with its weight given by
\[
\text{wt}(u[i_1, \ldots, i_k, j_1, \ldots, j_l]) = \text{wt}(u) - \sum_{s=1}^k 2\epsilon_{i_s} - \sum_{t=1}^l \epsilon_{j_t},
\]
if such a weight vector exists in \(V_0(\varpi_{n-l})\).

The map \((3.14)\), if it exists, sends \(u\) to the following vector, say \(v\):
\[
u \mapsto v = v_0 \otimes u + (-q^{-1}c) \left( \sum_{k=1}^n (-q)^{-1} v_k \otimes u[\hat{k}] \right),
\]
which is unique (up to constant) in the sense that it satisfies \(e_i v = 0\) for \(i \in I_0\), and \(f_0 u = 0\).

In \(V(\varpi_n)\), we have
\[
(3.15) \quad S_0 \cdot u = a S_w \cdot u \quad \text{where} \quad S_w = S_1 S_2 \cdots S_n \quad \text{for} \quad w = s_1 s_2 \cdots s_n \in W_0.
\]

On the other hand,
\[
S_0 \cdot v = e_0^{(2)} v = c v_1 \otimes u[1] - q c v_0 \otimes u[\hat{1}] - q b^{-1} c \sum_{k \neq 1} (-q)^{k-1} v_k \otimes u[\hat{1}, \hat{k}],
\]
\[
S_w \cdot v = f_1^{(2)} f_2^{(2)} \cdots f_{n-1}^{(2)} f_n v = v_0 \otimes u[\hat{1}] + (-q^{-1}c) \left( \sum_{k \neq n} (-q)^{k-1} v_{k+1} \otimes u[\hat{1}, \hat{k}+1] \right)
\]
\[
+ (-q^{-1}c)(-q)^{n-1} v_0 \otimes u[\hat{1}],
\]
where \(d\) is an element in \(k^\times\) such that
\[
(3.16) \quad e_0^{(2)} u[\hat{k}] = d \times u[\hat{1}, \hat{k}] \quad \text{for} \quad k \neq 1 \quad \text{in} \quad V(\varpi_n).c
\]

By \((3.15)\), we can conclude that
\[
(3.17) \quad a = -q c, \quad b = a(-q)^n, \quad d = c.
\]

Now, it suffices to show that \(d = c = 1\).

**Proposition 3.8.** For \(1 \neq k \in I_0\), the coefficient \(d\) in \((3.16)\) must be equal to 1; i.e.,
\[
e_0^{(2)} u[\hat{k}] = u[\hat{1}, \hat{k}] \quad \text{in} \quad V(\varpi_n).c.
\]

**Proof.** By Definition \([3.1]\) (3), we have
\[
f_1 e_0 u[2] = e_0 f_1 u[2] = e_0 u[1] = [2]_0 u[\hat{1}].
\]

Thus
\[
e_1 e_0 u[2] = [2]_0 e_1^{(2)} u[\hat{1}] = [2]_0 u[\hat{2}].
\]

From the actions \(e_i\) \((i \in I)\) on \(V(\varpi_n).c\), we have
\[
e_0 e_1 e_0^{(2)} u[\hat{2}] = c e_0 e_1 u[\hat{1}, \hat{2}] = c e_0 u[\hat{2}, \hat{1}] = c [2]_0 u[\hat{1}, \hat{2}].
\]
Since all vectors in $V(\varpi_n)$ are annihilated by the action $e_0^{(3)}$, the relation in Definition 1.1 (4) implies that

\[(3.19) \quad e_0 e_1^{(2)} u[2] = (e_1^{(2)} + e_0^{(2)} e_1 - e_0^{(3)} e_1) u[2] = e_0^{(2)} e_1 e_0 u[2] = [2]_0 e_0^{(2)} u[2] = [2]_0 u[1, 2].\]

From (3.17), (3.18) and (3.19), we can conclude that $d = c = 1$.

Now, we have the following theorem.

**Theorem 3.9.** There exists a surjective $U'_q(A^{(2)}_{2n})$-module homomorphism

\[(3.20) \quad p_{1,n} : V(\varpi_n)_{(-q)^{-1}} \otimes V(\varpi_1)_{(-q)^n} \twoheadrightarrow V(\varpi_n).\]

By taking dual, there exists an injective $U'_q(A^{(2)}_{2n})$-module homomorphism

\[(3.21) \quad \iota_{1,n} : V(\varpi_n) \rightarrow V(\varpi_1)_{(-q)^n} \otimes V(\varpi_n)_{(-q)^{-1}}.\]

3.4. $D^{(2)}_{n+1}$-analogue of the surjective homomorphisms given in [19] (A.17)]. This subsection is devoted to proving the following lemma.

**Lemma 3.10.** Let $\eta, \eta' \in \{\sqrt{-1}, -\sqrt{-1}\}$ and $1 \leq k, l \leq n - 1$ such that $k + l = n$. Then there exists a surjective $U'_q(D^{(2)}_{n+1})$-module homomorphism

\[V(\varpi_k)^{\eta(-q^2)^{-\frac{1}{2}}} \otimes V(\varpi_l)^{\eta'(-q^2)^{\frac{1}{2}}} \twoheadrightarrow V(\varpi_n)_{(-q)^{n-k-l}} \otimes V(\varpi_n).\]

**Proof.** Note that $\eta/\eta' = \pm 1$. By Theorem 3.7 there are two injective $U'_q(D^{(2)}_{n+1})$-homomorphisms

\[\psi_1 : V(\varpi_k)^{\eta(-q^2)^{-\frac{1}{2}}} \rightarrow V(\varpi_n)_{(-q^2)^{-n-k-l}},\]

\[\psi_2 : V(\varpi_l)^{\eta'(-q^2)^{\frac{1}{2}}} \rightarrow V(\varpi_n)_{(-q^2)^{n-k-l}} \otimes V(\varpi_n),\]

by taking dual. Then we can obtain $\varphi = (\text{id}_{V(\varpi_n)_{(-q)^{-1}}} \otimes \text{tr} \otimes \text{id}_{V(\varpi_n)_{-1}}) \circ (\psi_1 \otimes \psi_2)$,

\[\varphi : V(\varpi_k)^{\eta(-q^2)^{-\frac{1}{2}}} \otimes V(\varpi_l)^{\eta'(-q^2)^{\frac{1}{2}}} \rightarrow V(\varpi_n)_{(-q^2)^{-n-k-l}} \otimes V(\varpi_n),\]

since $V(\varpi_n)_{(-q^2)^{-n-k-l}}$ and $V(\varpi_n)_{(-q^2)^{n-k-l}}$ are dual to each other.

Applying the argument of [19] Lemma A.3.2], we have

\[\varphi(v \otimes w) \equiv \text{tr}(u_{-\varpi_n} \otimes u_{\varpi_n}) v_1 \otimes w_1 \mod \bigoplus_{\lambda \neq -\varpi_k + \varpi_n} \left(V(\varpi_n)_{(-q^2)^{-\frac{1}{2}}} \otimes V(\varpi_n)_{-\varpi_k + \varpi_l - \lambda}\right),\]

where

- $v$ is the $U_q(B_n)$-lowest weight vector of $V(\varpi_k)^{\eta(-q^2)^{-\frac{1}{2}}}$ of weight $-\varpi_k$,
- $w$ is the $U_q(B_n)$-highest weight vector of $V(\varpi_l)^{\eta'(-q^2)^{\frac{1}{2}}}$ of weight $\varpi_l$,
- $v_1$ is a non-zero vector of $V(\varpi_n)_{(-q^2)^{-1}}$ of weight $-\varpi_k + \varpi_n$,
- $w_1$ is a non-zero vector of $V(\varpi_n)$ of weight $\varpi_l - \varpi_n$.

Thus $\varphi$ is non-zero. Then our assertion follows from the fact that $V(\varpi_n)_{(-q^2)^{-1}} \otimes V(\varpi_n)$ is irreducible. $\square$
4. The computation of denominators between fundamental representations

For simplicity, we write $R^\text{norm}_{k,l}$ for $R^\text{norm}_{V(\varpi_k), V(\varpi_l)}$ in (1.4), $d_{k,l}$ for $d_{V(\varpi_k), V(\varpi_l)}$ in (1.6) and $a_{k,l}$ for $a_{V(\varpi_k), V(\varpi_l)}$ in (1.5).

By the result of [1, Appendix A] and [2], the denominator $d_{k,l}(z)$ and the element $a_{k,l}(z) \in \mathfrak{k}(z)$ are symmetric with respect to the indices $k$ and $l$; that is,
\begin{equation}
    d_{k,l}(z) = d_{l,k}(z) \quad \text{and} \quad a_{k,l}(z) = a_{l,k}(z).
\end{equation}

4.1. General framework. In this subsection, we propose the strategy for computing $d_{k,l}(z)$, which is originated from [19, Appendix A].

Note that we have a surjective homomorphism
\begin{equation}
p_{l-1,1}: V(\varpi_{l-1})(-q^t)^{-1/t} \otimes V(\varpi_1)(-q^t)^{-1/t} \to V(\varpi_l) \quad \text{if } l \leq n - d,
\end{equation}
by the previous section.

Assumption 4.1. Assume the followings:

(A) We know $a_{k,l'}(z)$ for $k \in I_0$ and $l' \leq l - 1$.

(B) We know $d_{1,1}(z)$ for all $g$, and $d_{1,n}(z)$ for $g = B^{(1)}_n$ or $g = D^{(2)}_{n+1}$.

With these assumptions and (1.3), consider the following commutative diagram:
\begin{equation}
\begin{array}{c}
V(\varpi_k) \otimes V(\varpi_{l-1})(-q^t)^{-1/t} \otimes V(\varpi_1)(-q^t)^{-1/t} \otimes V(\varpi_k) \otimes V(\varpi_l)
\\
\downarrow R_{k,l-1}^{\text{inv}}((q^t)^{-1/t}) \otimes V(\varpi_1)(-q^t)^{-1/t}
\\
V(\varpi_{l-1})(-q^t)^{-1/t} \otimes V(\varpi_k) \otimes V(\varpi_1)(-q^t)^{-1/t}
\\
\downarrow V(\varpi_{l-1})(-q^t)^{-1/t} \otimes V(\varpi_k)(-q^t)^{-1/t} \otimes V(\varpi_k)
\\
\downarrow V(\varpi_{l-1})(-q^t)^{-1/t} \otimes V(\varpi_1)(-q^t)^{-1/t} \otimes V(\varpi_k) \otimes V(\varpi_k)
\\
\end{array}
\end{equation}

Then we have
\begin{equation}
v[1,\ldots,k] \otimes v[1,\ldots,l-1] \otimes v_l \quad \text{and} \quad a_{k,l-1}((q^t)^{-1/t})v[1,\ldots,l-1] \otimes v[1,\ldots,k] \otimes v_l
\end{equation}
\begin{equation}
a_{k,l-1}((q^t)^{-1/t})v[1,\ldots,l-1] \otimes v[1,\ldots,k] \otimes v_l
\end{equation}
\begin{equation}
a_{k,l-1}((q^t)^{-1/t})v[1,\ldots,l-1] \otimes v[1,\ldots,k] \otimes v_l
\end{equation}
\begin{equation}
a_{k,l-1}((q^t)^{-1/t})v[1,\ldots,l-1] \otimes w \quad \text{and} \quad a_{k,l}(z)v[1,\ldots,l-1] \otimes v[1,\ldots,k],
\end{equation}
where
- $v[1,\ldots,a]$ is the dominant extremal weight vector of $V(\varpi_a)$ for $a \in I_0$,
- $w = R^\text{norm}_{k,1}((q^t)^{-1/t})(v[1,\ldots,k] \otimes v_l)$.

By observing the vector $w$, we can get an equation explaining the relationship between
\begin{equation}
a_{k,l-1}(-q^{-1}z)a_{k,1}((-q)^{-1/z}) \quad \text{and} \quad a_{k,l}(z).
\end{equation}

By Assumption 4.1 (A), we can compute $a_{k,l}(z)$ by using an induction.
After getting \( a_{k,l}(z) \), we use the formulas in Lemma 4.3 by applying two surjective homomorphisms in Section 3.

\[
\begin{align*}
 p_{k-1,1} : V(\varpi_{k-1})(-q^i)^{-1/t} \otimes V(\varpi_1)(-q^j)^{k-1/t} & \to V(\varpi_k), \\
p_{k-1,1}^* : V(\varpi_k)(-q^i)^{-1/t} \otimes V(\varpi_1)(-q^j)^{k-1/t} & \to V(\varpi_{k-1}),
\end{align*}
\]

and setting \( W = V(\varpi_l) \) or \( V(\varpi_n) \), to get two elements in \( k[z^\pm 1] \) which are described in terms of \( d_{k,l}(z) \)'s and \( a_{k,l}(z) \)'s. Here (4.6) is the composition of \( U'_q(\mathfrak{g}) \)-homomorphisms given as follows:

\[
V(\varpi_k)(-q^i)^{-1/t} \otimes V(\varpi_1)(-q^j)^{k-1/t} \to V(\varpi_k) \otimes V(\varpi_1)(-q^j)^{k-1/t} \\
\to V(\varpi_{k-1}) \otimes k \simeq V(\varpi_{k-1}).
\]

Since we know the forms of \( a_{k,l}(z) \)'s, two elements in \( k[z^\pm 1] \) can be described in terms of \( d_{k,l}(z) \)'s and polynomials in \( k[z] \) (up to constant multiple of \( k[z^\pm 1]^\times \)).

By the assumptions, we know \( d_{1,1}(z) \), \( d_{1,n}(z) \) and hence we can compute \( d_{k,l}(z) \) and \( d_{k,n}(z) \), by manipulating the two elements in \( k[z^\pm 1] \) and using inductions.

The denominator \( d_{1,1}(z) \) of \( R_{1,1}^{\text{norm}}(z) : V(\varpi_1) \otimes V(\varpi_1)_z \to V(\varpi_1)_z \otimes V(\varpi_1) \) are computed in [17] (see also [13] for \( g = A_2(2) \)) as follows:

\[
d_{1,1}(z) = (z^t - (q^2)^t)(z^t - (p^*)^t).
\]

The denominator \( d_{1,n}(z) \) of \( R_{1,n}^{\text{norm}}(z) : V(\varpi_1) \otimes V(\varpi_n)_z \to V(\varpi_n)_z \otimes V(\varpi_1) \) for \( g = B_n(1) \) is computed in [17] as follows:

\[
d_{1,n}(z) = d_{n,1}(z) = z - (1)^{n+1}q^2n+1.
\]

Considering Assumption 4.11, the only missing part is the denominator \( d_{1,n}(z) \) for \( g = D_{n+1}^{(2)} \).

4.2. The denominator \( d_{1,n}(z) \) for \( g = D_{n+1}^{(2)} \). To compute the denominator \( d_{1,n}(z) \) for \( g = D_{n+1}^{(2)} \), we follow the notations and arguments given in [17] Section 4.

By the \( U'_q(D_{n+1}^{(2)}) \)-module structure of \( V(\varpi_1) \) and \( V(\varpi_n) \) in Section 2 we have

\[
V(\varpi_1) \simeq V_0(\varpi_1) \oplus V_0(0) \quad \text{and} \quad V(\varpi_n) \simeq V_0(\varpi_n) \quad \text{as} \quad U_q(B_n)\text{-modules}.
\]

Here \( V_0(\varpi_n) \) (resp. \( V_0(0) \)) is the highest \( U_q(B_n) \)-module with the highest weight \( \varpi_n \) (resp. 0). Thus we have

\[
V(\varpi_n) \otimes V(\varpi_1) \simeq V_0(\lambda) \oplus V_0(\varpi_n)^{\otimes 2} \quad \text{as} \quad U_q(B_n)\text{-module},
\]

where \( \lambda = (\begin{smallmatrix} 1 & 2 & \ldots & n \\ \frac{1}{2} & \frac{1}{2} & \ldots & \frac{1}{2} \end{smallmatrix}) \). Let

\[
m^+_n = (+, \ldots, +) \quad \text{and} \quad m^i = (+, \ldots, +, -^i, +, \ldots, +) \quad (1 \leq i \leq n)
\]

be the elements in \( V(\varpi_n) \). Then we have the following lemmas by the direct calculation:

Lemma 4.2. Let \( u_\lambda, u_{\varpi_n}^1 \) and \( u_{\varpi_n}^2 \) be the \( U_q(B_n) \)-highest weight vectors with the weight \( \lambda, \varpi_n \) and \( \varpi_n \) in \( V(\varpi_n)_x \otimes V(\varpi_1)_y \) respectively. Then we have

(a) \( u_\lambda = (m^+_n) \otimes v_1 \),
(b) \( u_{\varpi_n}^1 = [2]_0^{-1}(m^+_n) \otimes v_0 \),
(c) \( u_{\varpi_n}^2 = \sum_{k=1}^{n-1}(-1)^kq^{2k}(m^{n+1-k}) \otimes v_{n+1-k} + [2]_n^{-1}(m^+_n) \otimes v_0 \).

Lemma 4.3. Let \( \bar{u}_\lambda, \bar{u}_{\varpi_n}^1 \) and \( \bar{u}_{\varpi_n}^2 \) be the \( U_q(B_n) \)-highest weight vectors with the weight \( \lambda, \varpi_n \) and \( \varpi_n \) in \( V(\varpi_1)_y \otimes V(\varpi_n)_x \) respectively. Then we have
Lemma 4.6. \( \tilde{u}_\lambda = (1) \otimes (m^+_n) \),
(b) \( \tilde{u}^i_{\omega_n} = \sum_{k=1}^{n} (-1)^{n+1-k} q^{-2(n+1-k)} v_k \otimes (m^k) + q^{-1}[2]_{1} v_0 \otimes (m^+_n) \).
(c) \( u^2_{\omega_n} = \sum_{k=1}^{n} (-1)^{n-1-k} q^{-2(n-1-k)} v_k ^{-1} \).

Hence \( R^{\text{norm}}_{1,n} : V(\omega_1)_y \otimes V(\omega_n)_x \rightarrow V(\omega_n)_x \otimes V(\omega_1)_y \) can be expressed by
\[ R^{\text{norm}}_{1,n}(\tilde{u}_\lambda) = u_\lambda \quad \text{and} \quad R^{\text{norm}}_{1,n}(\tilde{u}^i_{\omega_n}) = \sum_{j=1}^{2} a^{\omega_n}_{ij} u^i_{\omega_n}. \]

The following lemmas can be obtained by direct calculations.

Lemma 4.4. For the highest weight vectors defined in Lemma 4.2, we have
(a) \( f_0(u^1_{\omega_n}) = x^{-1} y^{-1} (q^{-1} x) u_\lambda \),
(b) \( f_0(u^2_{\omega_n}) = x^{-1} y^{-1} ((-1)^{n} q^{2 n-2} y) u_\lambda \),
(c) \( e_1 \cdot e_{n-1} e_n^{(2)} e_{n-1} \cdot e_2 e_1 e_0(u^1_{\omega_n}) = (y) u_\lambda \),
(d) \( e_1 \cdot e_{n-1} e_n^{(2)} e_{n-1} \cdot e_2 e_1 e_0(u^2_{\omega_n}) = (q^{-1} x) u_\lambda \),
in \( V(\omega_n)_x \otimes V(\omega_1)_y \).

Lemma 4.5. For the highest weight vectors defined in Lemma 4.3, we have
(a) \( f_0(u^1_{\omega_n}) = x^{-1} y^{-1} (x) \tilde{u}_\lambda \),
(b) \( f_0(u^2_{\omega_n}) = x^{-1} y^{-1} ((-1)^{n} q^{2 n-2} y) \tilde{u}_\lambda \),
(c) \( e_1 \cdot e_{n-1} e_n^{(2)} e_{n-1} \cdot e_2 e_1 e_0(u^1_{\omega_n}) = (q^{-1} y) \tilde{u}_\lambda \),
(d) \( e_1 \cdot e_{n-1} e_n^{(2)} e_{n-1} \cdot e_2 e_1 e_0(u^2_{\omega_n}) = (q^{-1} x) \tilde{u}_\lambda \),
in \( V(\omega_1)_y \otimes V(\omega_n)_x \).

From these lemmas, we obtain
\[
\begin{pmatrix}
q^{-1} y^{-1} & (-1)^{n} q^{2 n-1} x^{-1} \\
(y^{-1} q^{-1} y) & q^{-1} x
\end{pmatrix} (a^{\omega_n}_{ij}) = \begin{pmatrix}
y^{-1} & (-1)^{n} q^{-2 n-2} x^{-1} \\
q^{-1} x & q^{-1} x
\end{pmatrix},
\]
and hence
\[
(a^{\omega_n}_{ij}) = \frac{1}{z^2 + (-q^2)^{n+1}} \begin{pmatrix}
z q^2 - (-1)^{n} q^{2 n+1} & (-1)^{n} (q^{-2 n-1} - q^{2 n+1}) \\
1 - q^2 & z^2 - (-1)^{n} q^{-2 n}
\end{pmatrix},
\]
where \( z = x y^{-1}. \)

Hence we can conclude that
\[
d_{1,n}(z) = d_{n,1}(z) = z^2 + (-q^2)^{n+1} \quad \text{for} \ g = D^{(2)}_{n+1}.
\]

4.3. Denominators between fundamental representations. Write
\[
d_{k,l}(z) = \prod_{\nu}(z - x_\nu).
\]

For rational functions \( f, g \in k(z) \), we write \( f \equiv g \) if there exists an element \( a \in k[z^{\pm 1}] \) such that \( f = ag. \)

Lemma 4.6. For \( k, l \in I_0 \), we have
\[
ak_{k,l}(z) a_{k,l}(p^*)^{-1} z \equiv \frac{d_{k,l}(z)}{d_{k,l}(p^* z^{-1})},
\]
\[
ak_{k,l}(z) = q^{(\omega_k, \omega_l)} \prod_{\nu}(p^* x_\nu z; p^* z^{-1} z; p^2 z; p^2)_{\infty},
\]
\[
+ (x_\nu z; p^2)_{\infty}(p^2 x_\nu z^{-1} z; p^* z; p^2)_{\infty},
\]
where \((z;q)_\infty = \prod_{n=0}^{\infty}(1 - q^n z)\).

Now we list a table for triple \((\delta,c,p^*)\) for each \(g\):

| \(g\)         | \(\delta\)                                    | \(c\)                                      | \(p^*\)                |
|---------------|-----------------------------------------------|--------------------------------------------|------------------------|
| \(A^{(2)}_{2n-1}\) | \(\alpha_0 + \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{n-1}) + \alpha_n\) | \(h_0 + h_1 + 2(2h_2 + \cdots + h_n)\) | \(-(-q)^{2n}\)         |
| \(A^{(2)}_{2n}\)   | \(2(\alpha_0 + \cdots + \alpha_{n-1}) + \alpha_n\) | \(h_0 + 2(h_1 + \cdots + h_n)\)         | \(-(-q)^{2n+1}\)      |
| \(B_{n+1}^{(1)}\)  | \(\alpha_0 + \alpha_1 + 2(\alpha_2 + \cdots + \alpha_n)\) | \(h_0 + h_1 + 2(2h_2 + \cdots + h_{n-1}) + h_n\) | \(-(-q)^{2n-1}\)      |
| \(D^{(2)}_{n+1}\)  | \(\alpha_0 + \alpha_1 + \cdots + \alpha_n\) | \(h_0 + 2(h_1 + \cdots + h_{n-1}) + h_n\) | \(-(-q)^n\)           |

Table 1. \((\delta,c,p^*)\) for each affine type

By Lemma 4.6 and (4.7), we can compute \(a_{1,1}(z)\) for all \(g\) as follows:

\[
a_{1,1}(z) = \begin{cases} 
q^{(2n+2)/2n-2} [4n][0] & \text{if } g = A^{(2)}_{2n-1}, \\
q^{(2n+3)/2n-1} [4n+2][0] & \text{if } g = A^{(2)}_{2n}, \\
q^{(2n+3)/2n-1} [4n+2][0] & \text{if } g = B_{n+1}^{(1)}, \\
q^{(2n+3)/2n-1} [4n+2][0] & \text{if } g = D^{(2)}_{n+1}, \\
q^{(n+1)n} / \{1\{2n-1\}} & \text{if } g = D^{(2)}_{n+1}, \\
\end{cases}
\]

(4.11)

where, for \(a \in \mathbb{Z}\) and \(b \in \frac{1}{2} \mathbb{Z}\),

\[a = ((-q)^a z; p^{*2})_\infty, \quad \langle a \rangle = (-(q)^a z; p^{*2})_\infty\] and \(\{b\} = ((-q)^b z; p^{*2})_\infty \times (-(q)^b z; p^{*2})_\infty\).

Note that, for \(a \in \mathbb{Z}\) and \(b \in \frac{1}{2} \mathbb{Z}\), we have

\[
[a]/[a + 4n] \equiv z - (-q)^{-a} \quad \text{and} \quad \langle a \rangle / (a + 4n) \equiv z + (-q)^{-a} \quad \text{if } g = A^{(2)}_{2n-1},
\]

\[
[a]/[a + 4n + 2] \equiv z - (-q)^{-a} \quad \text{if } g = A^{(2)}_{2n},
\]

\[
[a]/[a + 4n - 2] \equiv z - (-q)^{-a} \quad \text{if } g = B_{n+1}^{(1)},
\]

\[
\{b\} / (b + 2n) \equiv z^2 - (-q)^{-2b} \quad \text{if } g = D^{(2)}_{n+1}.
\]

Following [15, 7, (3.12)] and [14, (3.7)], we recall the image of \(v_k \otimes v_l\) \((k \neq l \in I_0)\) under the normalized \(R\)-matrix

\[
R \text{ }^\text{norm}_{1,1}(z) : V(\varpi_1) \otimes V(\varpi_1)_z \rightarrow V(\varpi_1)_z \otimes V(\varpi_1),
\]

which is given by

\[
R \text{ }^\text{norm}_{1,1}(z)(v_k \otimes v_l) = \frac{(1 - (q^2)^t)z^t \times \delta(k \prec l)}{z^t - (q^2)^t}(v_k \otimes v_l) + \frac{q^t(z^t - 1)}{z^t - (q^2)^t}(v_l \otimes v_k).
\]

Here \(\prec\) is the linear order on the labeling set of the basis of \(V(\varpi_1)\) (see [12, Section 8]).
Proposition 4.7. For $1 \leq k, l \leq n - \vartheta$, we have

\[
(4.14) \quad a_{k, l}(z) = \left\{
\begin{array}{ll}
\frac{[k-l][4n-k-l]}{[k+l][4n-k-l]}(2n+k+l)(2n-k-l) & \text{if } g = A_{2n-1}^{(2)}, \\
\frac{[k-l][4n+2-k-l]}{[k+l][4n-k-l]}(2n+k+l)(2n-k-l) & \text{if } g = A_{2n}^{(2)}, \\
\frac{[k-l][4n+1+k+l]}{[k+l][4n-k-l]}(2n+1+k+l)(2n-k-l) & \text{if } g = B_{n+1}^{(2)}, \\
\frac{[k-l][2n+k-l-1]}{[k+l][2n-k-l-1]}(2n+k+l-1)(2n-k-l-1) & \text{if } g = D_{n+1}^{(2)}.
\end{array}
\right.
\]

Proof. We prove only for the case when $g$ is of type $A_{2n-1}^{(2)}$. For the other $g$, one can apply the same argument to prove our assertion. We first consider when $k = 1$.

By (4.11), our assertion for $k = l = 1$ holds. Applying the commutative diagram (4.3) for $k = 1$, we have

\[
(4.15) \quad a_{1, 1-1}((-q)^{-1}z)a_{1, 1}((-q)^{-1}z)v_{[1, ..., l-1]} \otimes w \mapsto a_{1, l}(z)v_{[1, ..., l-1]} \otimes v_1,
\]
where

\[
w = R_{1, 1}^{\text{norm}}((-q)^{-1}z)v_{1} \otimes v_1 = \frac{q((-q)^{-1}z-1)}{(-q)^{-1}z-q^2}v_1 \otimes v_1 + \frac{(1-q^2)}{(-q)^{-1}z-q^2}v_1 \otimes v_1.
\]

Since $v_{[1, ..., l-1]} \otimes v_1$ vanishes under the map $p_{l-1, 1}$, (4.15) indicates that

\[
a_{1, l}(z) = a_{1, l-1}((-q)^{-1}z)a_{1, 1}((-q)^{-1}z)q((-q)^{-1}z-1)
\]

\[
\times \frac{(l-1)}{(-q)^{-1}z-q^2} \left\{ \frac{4n+l-3}{4n+l+3} \right\}.
\]

Hence our assertion for $k = 1$ follows from an induction on $l$:

\[
(4.16) \quad a_{1, l}(z) = a_{1, 1}(z) = \left\{ \frac{[l-1][4n-l+1]}{[l+1][4n-l-1]}(2n-l-1)(2n+l+1) \right\}.
\]

By (4.1), we now assume $2 \leq l \leq k \leq n$. By the direct calculation, one can show that

\[
(f_{l-1}f_{l-2} \cdots f_1(v_{1, ..., k}) \otimes v_1) = v_{1, ..., k} \otimes v_1 \quad \text{and} \quad (f_{l-1}f_{l-2} \cdots f_1(v_1 \otimes v_{1, ..., k})) = v_1 \otimes v_{1, ..., k}.
\]

Since $R_{k, 1}^{\text{norm}}$ is a $U_q'(g)$-homomorphism and sends $v_{1, ..., k} \otimes v_1$ to $v_1 \otimes v_{1, ..., k}$, we have

\[
R_{k, 1}^{\text{norm}}(z)(v_{1, ..., k} \otimes v_1) = v_1 \otimes v_{1, ..., k}.
\]

Thus, the image in (4.4),

\[
a_{k, l-1}((-q)^{-1}z)a_{k, 1}((-q)^{-1}z)v_{[1, ..., l-1]} \otimes w \mapsto a_{k, l}(z)v_{[1, ..., l-1, l]} \otimes v_{[1, ..., k]}
\]

for $w = R_{k, 1}^{\text{norm}}((-q)^{-1}z)v_{[1, ..., k]} \otimes v_1 = v_1 \otimes v_{1, ..., k}$, implies that

\[
(4.17) \quad a_{k, l}(z) = a_{k, l-1}((-q)^{-1}z)a_{k, 1}((-q)^{-1}z) \quad (2 \leq l \leq k \leq n).
\]

Hence one can obtain our assertion by applying an induction on $l$. \hfill \square

Theorem 4.8. For $1 \leq k, l \leq n - \vartheta$, we have

\[
(4.18) \quad d_{k, l}(z) = \prod_{s=1}^{\min(k,l)} (z^s - (-q^s)^{k-l+2s})(z^s - (p^*)^s(-q^t)^{2s-k})�.
\]
Proof. For $1 \leq k, l \leq n - \vartheta$, set

$$D_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z^t - (-q^t)^{|k-l|+2s})(z^t - (p^*)^t(-q^2)^{2s-k-l}).$$

Then we can observe that $D_{k,l}(z)$ behaves similar to $d_{k,l}(z)$. Namely, (cf. (4.11), (4.7) and (4.10))

$$D_{1,1}(z) = d_{1,1}(z), \quad D_{k,l}(z) = D_{l,k}(z),$$

$$\frac{D_{k,l}(z)}{D_{k,l}(p^* z^{-1})} \equiv a_{k,l}(z)a_{k,l}((p^*)^{-1}z) \equiv \frac{d_{k,l}(z)}{d_{k,l}(p^* z^{-1})}.$$ 

By calculations, one can check that

$$D_{k,l}(z) = D_{k,l-1}((-q^t)^{-1/t} z)D_{k,1}((-q^t)^{1-t/2} z) \quad \text{for } 2 \leq k \leq n - \vartheta,$$

which is similar to (4.17), also.

Now we give a proof for $g = D_{n+1}^{(2)}$, since this case is most complicated. For the other $g$, one can apply the similar argument to prove.

We shall show that $D_{k,l}(z) = d_{k,l}(z)$ indeed. Our assertion for $k = l = 1$ is presented in (4.9). Assume that $1 \leq k \leq n - 1$ and $2 \leq l \leq n - 1$.

From the a surjective homomorphism in Theorem 3.5

$$\pi_{l-1,1}: V(\varpi_{l-1}) \otimes V(\varpi_1) \rightarrow V(\varpi_l),$$

the first formula in Lemma 1.4 with setting $W = V(\varpi_k)$ yields an element in $k[z^\pm 1]$ as follows:

$$\frac{d_{k,l-1}((-q^2)^{-1/2} z)d_{k,1}((-q^2)^{1/2} z)}{d_{k,l}(z)} \equiv \frac{a_{k,l}(z)}{a_{k,l-1}((-q^2)^{-1/2} z)a_{k,1}((-q^2)^{1/2} z)} \in k[z^\pm 1],$$

for $1 \leq k \leq n - 1$ and $2 \leq l \leq n - 1$. In particular, if $2 \leq l \leq k \leq n - 1$,

$$\frac{d_{k,l-1}((-q^2)^{-1/2} z)d_{k,1}((-q^2)^{1/2} z)}{d_{k,l}(z)} \in k[z^\pm 1],$$

since (cf. (4.17))

$$\frac{a_{k,l}(z)}{a_{k,l-1}((-q^2)^{-1/2} z)a_{k,1}((-q^2)^{1/2} z)} \in k[z^\pm 1]^x$$

by the computation using (4.14).

Using (4.14) once again, for $k = 1 < l$, one can compute that

$$\frac{a_{1,l}(z)}{a_{1,l-1}((-q^2)^{-1/2} z)a_{1,1}((-q^2)^{1/2} z)} = \frac{(z^2 - (-q^2)^{1-l})}{(z^2 - (-q^2)^{3-l})} \quad \text{for } 2 \leq l \leq n - 1.$$

Set $k = 1$ and then replace $l$ with $k$ in (4.23). Then (4.23) becomes

$$\frac{d_{1,k-1}((-q^2)^{-1/2} z)D_{1,1}((-q^2)^{1/2} z)(z^2 - (-q^2)^{-1-k})}{d_{1,k}(z)} \equiv \frac{d_{1,k-1}((-q^2)^{-1/2} z)(z^2 - (-q^2)^{2n-k+1})(z^2 - (-q^2)^{1-k})}{d_{1,k}(z)} \in k[z^\pm 1] \quad \text{for } 2 \leq k \leq n - 1,$$

since $D_{1,1}(z) = d_{1,1}(z)$.
On the other hand, from the surjective homomorphism
\[
V(\varpi_k)(-q^2)^{-\frac{j}{2}} \otimes V(\varpi_l)(-q^2)^{2n-k} \to V(\varpi_{k-1}),
\]
the second formula in Lemma 1.3 with setting \( W = V(\varpi_l) \) yields an element in \( k[z^\pm 1] \) as follows:
\[
(4.26) \quad \frac{a_{k-l}(z)}{a_{k-l}((-q^2)\frac{1}{2} z)a_{k-1,l}((-q^2)\frac{1}{2} z)} \in k[z^\pm 1].
\]

By computation using (4.14), we have
\[
\frac{a_{k-l}(z)}{a_{k-l}((-q^2)\frac{1}{2} z)a_{k-1,l}((-q^2)\frac{1}{2} z)} = \begin{cases} (2 - (q^2)^n)_{n-k-l-1} & \text{if } 1 \leq l < k \leq n - 1, \\ (2 - (q^2)^n)_{n-k-l+1} & \text{if } 2 \leq l = k \leq n - 1. \end{cases}
\]

Thus the element (4.26) in \( k[z^\pm 1] \) can be written as follows:
\[
(4.27) \quad \frac{d_{k-l}((-q^2)\frac{k}{2} z)d_{k-l}((-q^2)\frac{1}{2} z)}{d_{k-1,l}(z)} \in k[z^\pm 1] \quad \text{if } 1 \leq l < k \leq n - 1,
\]
and
\[
(4.28) \quad \frac{d_{k-l}((-q^2)\frac{k}{2} z)d_{k-1,l}((-q^2)\frac{1}{2} z) (2 - (q^2)^n)_{n-k-l-1} (2 - (q^2)^n)_{n-k-l+1}}{(2 - (q^2)^n)_{n-k+1}} \in k[z^\pm 1]
\]
if \( 2 \leq l = k \leq n - 1. \)

Setting \( l = 1 \) in (4.27), we obtain
\[
(4.29) \quad \frac{D_{1,l}((-q^2)\frac{k}{2} z)d_{k-1,l}(z)}{d_{k-1,l}((-q^2)^\frac{-1}{2} z)} \in k[z^\pm 1] \quad \text{for } 2 \leq k \leq n - 1.
\]

Now we claim that
\[
d_{1,k}(z) = D_{1,k}(z) = (2 - (q^2)^{k+1})(2 - (q^2)^{2n-k+1}) \quad \text{for } 2 \leq k \leq n - 1.
\]

With (4.20), we can start an induction on \( k \). Thus (4.25) can be written in the following form:
\[
(4.30) \quad \frac{D_{1,k-1}(z) (2 - (q^2)^{2n-k+1})(2 - (q^2)^{k-1})}{d_{1,k}(z)} = \frac{(2 - (q^2)^{k+1})(2 - (q^2)^{2n-k+3})(2 - (q^2)^{2n-k+1})(2 - (q^2)^{k-1})}{d_{1,k}(z)} \in k[z^\pm 1].
\]

Now we claim that
\[
(4.31) \quad k = \pm (q^2)^{\frac{k}{2}}, \pm (q^2)^{2n-k+1} \quad \text{are not zero of } d_{1,k}(z).
\]

If (4.31) is true, we have
\[
(4.32) \quad \frac{D_{1,k}(z)}{d_{1,k}(z)} = \frac{(2 - (q^2)^{k+1})(2 - (q^2)^{2n-k+1})}{d_{1,k}(z)} \in k[z^\pm 1] \quad (2 \leq k \leq n - 1).
\]

Since \( \frac{1-k}{2} \leq 0, \pm (q^2)^{\frac{k}{2}} \notin \mathbb{C}[[q]]. \) Then [19, Theorem 2.2.1 (i)] tells that \( \pm (q^2)^{\frac{k}{2}} \) cannot be a zero of \( d_{1,k}(z). \)
If \( z = \pm (-q^2)^{\frac{k-3}{2}} \) is a zero of \( d_{1,k}(z) \), we have a contradiction to the fact that the element \((4.32)\) is in \( k[z^{\pm 1}] \). Thus we know that \( z = \pm (-q^2)^{\frac{k-3}{2}} \) is not a zero of \( d_{1,k}(z) \). Since \( \frac{D_{k,l}(z)}{d_{k,l}(z)} = \frac{D_{k,l}(p^sz^{-1})}{d_{k,l}(p^sz^{-1})} \) by \((4.21)\), one can check that

- \( z = \pm (-q^2)^{\frac{k-3}{2}} \) is not a pole of \( D_{1,k}(z)/d_{1,k}(z) \),
- \( z = \pm (-q^2)^{\frac{2n-k+3}{2}} \) is not a pole of \( D_{1,k}(-(-q^2)^n z^{-1})/d_{1,k}(-(-q^2)^n z^{-1}) \).

Thus \( \pm (-q^2)^{\frac{2n-k+3}{2}} \) cannot be zero of \( d_{1,k}(z) \) and hence the claim in \((4.31)\) holds.

By an induction on \( k \) in \((4.29)\), we also obtain

\[
\frac{d_{1,k}(z)}{(z^2 - (-q^2)^{k+1})(z^2 - (-q^2)^{2n+1-k})} \in k[z^{\pm 1}] \quad \text{if } k \neq n - 1,
\]
\[
\frac{d_{1,k}(z)}{(z^2 - (-q^2)^{2n+1-k})} \in k[z^{\pm 1}] \quad \text{if } k = n - 1.
\]

By Theorem \(3.5\) and Lemma \(3.10\) \( d_{1,k}(z) \) has zeros at \( \pm (-q^2)^{\frac{k-3}{2}} \) for \( 1 \leq k \leq n - 1 \). Thus we have

\[
(4.33) \quad \frac{d_{1,k}(z)}{D_{1,k}(z)} = \frac{d_{1,k}(z)}{(z^2 - (-q^2)^{k+1})(z^2 - (-q^2)^{2n+1-k})} \in k[z^{\pm 1}] \quad (2 \leq k \leq n - 1).
\]

By considering \((4.32)\) and \((4.33)\) together, our assertion for \( k = 1 \) holds:

\[
(4.34) \quad \frac{d_{k,l-1}((-q^2)^{\frac{k-3}{2}} z) d_{k,l}((-q^2)^{\frac{k-3}{2}} z)}{d_{k,l}(z)} = \frac{D_{k,l-1}((-q^2)^{\frac{k-3}{2}} z) D_{k,l}((-q^2)^{\frac{k-3}{2}} z)}{D_{k,l}(z)} = \frac{D_{k,l}(z)}{d_{k,l}(z)} \in k[z^{\pm 1}]
\]

for \( 2 \leq l \leq k \leq n - 1 \).

Let \( \phi_{k,l}(z) \) be the elements in \( k[z^{\pm 1}] \) satisfying \( D_{k,l}(z) = d_{k,l}(z) \phi_{k,l}(z) \). We claim that

\[
\phi_{k,l}(z) = 1 \quad \text{for} \quad 2 \leq l \leq k \leq n - 1.
\]

Note that

\[
\frac{D_{1,l}((-q^2)^{\frac{k-3}{2}} z)D_{k,l}((-q^2)^{\frac{k-3}{2}} z)}{D_{k-1,l}(z)} \left( \frac{z^2 - (-q^2)^{2n-k-l+1}}{(z^2 - (-q^2)^{2n-k-l+1})} \right) = \begin{cases} 
\left( z^2 - (-q^2)^{4n-k-l+1} \right) (z^2 - (-q^2)^{2n-k-l+1}) & \text{if } l < k, \\
\left( z^2 - (-q^2)^{4n-k-l+1} \right) (z^2 - (-q^2)^{2n-k-l+1}) (z^2 - (-q^2)^{2n+1}) (z^2 - (-q^2)^{1}) & \text{if } l = k.
\end{cases}
\]

By \((4.27), (4.28)\) and an induction on \( k + l \), the above elements are written in the following form:

\[
\frac{(z^2 - (-q^2)^{4n-k-l+1})(z^2 - (-q^2)^{2n-k-l+1})}{\phi_{k,l}((-q^2)^{\frac{k-3}{2}} z)} \in k[z^{\pm 1}] \quad \text{if } l < k,
\]
\[
\frac{(z^2 - (-q^2)^{4n-k-l+1})(z^2 - (-q^2)^{2n-k-l+1}) (z^2 - (-q^2)^{2n+1})(z^2 - (-q^2)^{1})}{\phi_{k,l}((-q^2)^{\frac{k-3}{2}} z)} \in k[z^{\pm 1}] \quad \text{if } l = k.
\]

Since \( \phi_{k,l}((-q^2)^{\frac{k-3}{2}} z) \) divides \( D_{k,l}((-q^2)^{\frac{k-3}{2}} z) \), we conclude that

\[
\phi_{k,l}(z) = 1 \quad \text{if } k + l < n,
\]
\[
(4.34) \quad \frac{(z^2 - (-q^2)^{2n-k-l})}{\phi_{k,l}(z)} \in k[z^{\pm 1}] \quad \text{if } k + l \geq n.
\]
Now our assertion holds if \( z = \pm (q^2)^{\frac{2n-k-1}{2}} \) is not a zero of \( \phi_{k,l}(z) \) for \( k + l \geq n \). From (4.21), one can see that \( \phi_{k,l}(-q^2n^{-1}) = \phi_{k,l}(z) \). Thus we suffice to prove that \( z = \pm (q^2)^{\frac{k+l}{2}} \) is not a zero of \( \phi_{k,l}(z) \) for \( k + l \geq n \). If \( k + l > n \), then we have \( n > 2n - k - l \) and hence \( \phi_{k,l}(z) = 1 \).

Now we consider when \( k + l = n \). Then Lemma 3.10 tells that \( d_{k,l}(z) \) has zeros at \( z = \pm (q^2)^{\frac{k+l}{2}} \). By the definition of \( D_{k,l}(z) \), \( \pm (q^2)^{\frac{k+l}{2}} \) is a zero of multiplicity 1. Thus \( \pm (q^2)^{\frac{k+l}{2}} \) cannot be a zero of \( \phi_{k,l}(z) \) when \( k + l = n \).

Now we shall compute \( d_{k,n}(z) \) for \( g = B_n^{(1)} \) and \( g = D_n^{(2)} \). By Lemma 4.6, (4.37) and (4.38), we have

\[
a_{1,n}(z) = \begin{cases} \frac{2n - 3}{2[n+1]} & \text{if } g = B_n^{(1)}, \\ \frac{2n + 1}{2[n+1]} & \text{if } g = D_n^{(2)}, \\ \frac{3n + 1}{2[n+1]} & \text{if } g = D_n^{(2)}. \end{cases}
\]

where, for \( a, k \in \mathbb{Z} \) and \( b \in \frac{1}{2}\mathbb{Z} \),

\[
[a]_k = ((-1)^k q^a z; p^{-2})_\infty \quad \text{and} \quad [b] = (-\sqrt{-1}(-q^2)^{b}; p^{2})_\infty (\sqrt{-1}(-q^2)^{b}; p^{2})_\infty.
\]

Now, we give a proof only for \( g = B_n^{(1)} \). For \( g = D_n^{(2)} \), one can apply the same arguments.

**Proposition 4.9.** For \( 1 \leq l \leq n - 1 \), we have

\[
a_{l,n}(z) = \begin{cases} \frac{2n - 2l - 1}{2[n+1]} & \text{if } g = B_n^{(1)}, \\ \frac{2n + 2l - 1}{2[n+1]} & \text{if } g = D_n^{(2)}, \\ \frac{3n + 1}{2[n+1]} & \text{if } g = D_n^{(2)}. \end{cases}
\]

**Proof.** By (4.35), it suffices to consider when \( 2 \leq l \leq n - 1 \). Applying the commutative diagram (4.3) with setting \( k = n \), (4.4) tells that we have

\[
a_{n,l-1}(-q^{-1}z) a_{n,1}((-q)^{-1}z) v_{[1,\ldots,l-1]} \otimes w \longrightarrow a_{n,l}(z) v_{[1,\ldots,l-1]} \otimes m_n^+, \]

where \( w = R_{n,1}^{\text{norm}}((-q)^{-1}z)(m_n^+ \otimes v_l) \) for the highest weight vector \( m_n^+ \) of \( V(\varpi_n) \).

Since \( m_n^+ \) vanishes by the action \( f_k \) (\( 1 \leq i \leq l - 1 \)), as in the proof of Proposition 4.7

\[
w = R_{n,1}^{\text{norm}}((-q)^{-1}z)(m_n^+ \otimes v_l) = v_l \otimes m_n^+,
\]

and hence

\[
a_{n,l}(z) = a_{n,l-1}(-q^{-1}z) a_{n,1}((-q)^{-1}z) \quad \text{for } 2 \leq l \leq n - 1.
\]

By (4.35) and an induction on \( l \), our assertion follows.

**Theorem 4.10.** For \( 1 \leq k \leq n - 1 \), we have

\[
d_{k,n}(z) = \begin{cases} \prod_{s=1}^{k} (z - (-1)^{n+k} q^2) & \text{if } g = B_n^{(1)}, \\ \prod_{s=1}^{k} (z^2 + (-q^2)^{n-k+2s}) & \text{if } g = D_n^{(2)}. \end{cases}
\]
Proof. By (4.8), it suffices to consider when $2 \leq k \leq n - 1$. From the surjective homomorphism in Theorem 3.5, we have

$$V(\varpi_{k-1})(-q)^{-1} \otimes V(\varpi_1)(-q)^{k-1} \rightarrow V(\varpi_k),$$

the first formula in Lemma 1.4 with $W = V(\varpi_n)$ yields an element in $k[z^{\pm 1}]$ as follows:

$$\frac{d_{k-1,n}(-q^{-1}z)d_{1,n}(-q^{k-1}z)}{d_{k,n}(z)} \frac{a_{k,n}(z)}{a_{k-1,n}(-q^{-1}z)a_{1,n}((-q)^{k-1}z)} \in k[z^{\pm 1}].$$

By (4.37), the element is written in more simplified form as follows:

$$\frac{d_{k-1,n}(-q^{-1}z)d_{n,1}((-q)^{k-1}z)}{d_{k,n}(z)} \equiv \frac{d_{k-1,n}(-q^{-1}z)(z - (-1)^{n+k}q_s^{2n-2k+3})}{d_{k,n}(z)} \in k[z^{\pm 1}].$$

On the other hand, for each $2 \leq k \leq n - 1$, we have a surjective homomorphism

$$V(\varpi_k)_q^{-1} \otimes V(\varpi_1)_{(-q)^{2n-1-k}} \rightarrow V(\varpi_{k-1}).$$

Then the second formula in Lemma 1.4 with $W = V(\varpi_n)$ yields an element in $k[z^{\pm 1}]$ as follows:

$$\frac{d_{1,n}((-q)^{k+1-2n}z)d_{k,n}(-q^z)}{d_{k-1,n}(z)} \frac{a_{k-1,n}(z)}{a_{1,n}((-q)^{k+1-2n}z)a_{k,n}(-q^z)} \in k[z^{\pm 1}].$$

Using (4.36), the second factor of (4.39) can be written as

$$\frac{a_{k-1,n}(z)}{a_{1,n}((-q)^{k+1-2n}z)a_{k,n}(-q^z)} \equiv \frac{z - (-1)^{n+k+1}q_s^{2n-2k-3}}{z - (-1)^{n+k+1}q_s^{2n-2k+1}}.$$

and hence (4.40) becomes

$$\frac{d_{k,n}(z)(z - (-1)^{n+k+1}q_s^{2n-2k+1})(z - (-1)^{n+k+1}q_s^{2n-2k-3})}{d_{k-1,n}(z)(z - (-1)^{n+k+1}q_s^{2n-2k+1})} \in k[z^{\pm 1}].$$

By the induction hypothesis, $z = (-1)^{n+k+1}q_s^{2n-2k+1}$ and $(-1)^{n+k+1}q_s^{2n-2k-3}$ are not zeros of $d_{k-1,n}(z)$. Hence we can conclude that

$$\frac{d_{k,n}(-q^z)}{d_{k-1,n}(z)(z - (-1)^{n+k+1}q_s^{2n-2k+1})} \in k[z^{\pm 1}],$$

which is equivalent to

$$\frac{d_{k,n}(z)}{d_{k-1,n}(-q^{-1}z)(z - (-1)^{n+k}q_s^{2n-2k+3})} \in k[z^{\pm 1}].$$

Considering (4.39) and (4.41) together, our assertion follows:

$$d_{k,n}(z) \equiv d_{k-1,n}(-q^{-1}z)(z - (-1)^{n+k}q_s^{2n-2k+3}) = \prod_{s=1}^{k}(z - (-1)^{n+k}q_s^{2n-2k-1+4s}).$$

\[\square\]

Remark 4.11. In conclusion, we can observe that

for all $1 \leq k \leq n$, $R_{k,l}^{\text{norm}}(z)$ has only simple poles unless $g = D_{n+1}^{(2)}$.

For $g = D_{n+1}^{(2)}$, $R_{k,l}^{\text{norm}}(z)$ has a double pole at $z = \pm(-q^2)^{s/2}$ if

$$2 \leq k, l \leq n - 1, \ k + l > n, \ 2n + 2 - k - l \leq s \leq k + l \text{ and } s \equiv k + l \mod 2.$$
### A. The table of denominators.

| Type   | $n$  | $k, l$               | Denominators                                                                 |
|--------|------|----------------------|-------------------------------------------------------------------------------|
| $A_n^{(1)}$ | $n \geq 1$ | $1 \leq k, l \leq n$ | $d_{k,l}(z) = \prod_{s=1}^{\min(k,l,n+1-k,n+1-l)} (z - (-q)^{2s+k-l})$         |
| $B_n^{(1)}$ | $n \geq 3$ | $1 \leq k, l \leq n-1$ | $d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z - (-q)^{k-l+2s})(z + (-q)^{2n-k-l+2s})$ |
|         | $q_s^2 = q$ | $1 \leq k \leq n-1$ | $d_{k,n}(z) = \prod_{s=1}^{k} (z - (-1)^{n+k}q_{s}^{2n-2k-1+4s})$             |
|         |         | $k = l = n$         | $d_{n,n}(z) = \prod_{s=1}^{n} (z - (q_{s})^{4s-2})$                           |
| $C_n^{(1)}$ | $n \geq 2$ | $1 \leq k, l \leq n-2$ | $d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z - (-q)^{k-l+2s})(z - (-q)^{2n-2k-l+2s})$ |
|         |         | $1 \leq k \leq n-2$ | $d_{k,n-1}(z) = d_{k,n}(z) = \prod_{s=1}^{k} (z - (-q)^{n-k-1+2s})$           |
|         |         | $\{k, l\} = \{n, n-1\}$ | $d_{n,n-1}(z) = d_{n-1,n}(z) = \prod_{s=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (z - (-q)^{4s})$ |
|         |         | $k = l \in \{n, n-1\}$ | $d_{n,n}(z) = d_{n-1,n-1}(z) = \prod_{s=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (z - (-q)^{4s-2})$ |
| $A_{2n-1}^{(2)}$ | $n \geq 3$ | $1 \leq k, l \leq n$ | $d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z - (-q)^{k-l+2s})(z + (-q)^{2n-2k-l+2s})$ |
| $A_{2n}^{(2)}$ | $n \geq 1$ | $1 \leq k, l \leq n$ | $d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z - (-q)^{k-l+2s})(z + (-q)^{2n+1-k-l+2s})$ |
| $D_{n+1}^{(2)}$ | $n \geq 2$ | $1 \leq k, l \leq n-1$ | $d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z^2 - (-q^2)^{k-l+2s})(z^2 - (-q^2)^{2n-k-1+l+2s})$ |
|         |         | $1 \leq k \leq n-1$ | $d_{k,n}(z) = \prod_{s=1}^{k} (z^2 + (-q^2)^{n-k+2s})$                        |
|         |         | $k = l = n$         | $d_{n,n}(z) = \prod_{s=1}^{n} (z + (-q^2)^{s})$                              |

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