DELIGNE’S CONJECTURE AND MIRROR SYMMETRY

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Abstract. In this paper, we will study the connections between mirror symmetry and Deligne’s conjecture on the values of the $L$-functions of critical pure motives. Using mirror symmetry, we will give an explicit method to compute Deligne’s period for a smooth fiber in the deformation of a one-parameter Calabi-Yau threefold. We will give examples to show how this method works and express Deligne’s period in terms of the classical periods of the threeform of Calabi-Yau threefolds and its derivative. We will also compute the Deligne’s period of the Calabi-Yau threefolds studied in a recent paper by Candelas, de la Ossa, Elmi and van Straten, and verify that Deligne’s conjecture is satisfied based on the numerical results in their paper.

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1. Introduction

In the paper [5], Deligne formulated a profound conjecture about the relation between the value of the $L$-function of an algebraic variety at a critical point and the classical periods of the same variety. It is a generalization of the BSD conjecture, while further prompt Beilinson to formulate a much more general conjecture about the special values of $L$-functions and algebraic geometry [16]. Deligne’s conjecture is extremely hard to prove. In fact, given a variety $X$, even the computation of its Deligne’s period is far from trivial. The main result of this paper is that the mirror symmetry of Calabi-Yau threefolds will facilitate the computations of Deligne’s periods. We will explicitly show how this method works for an example. We will also compute the Deligne’s period of the Calabi-Yau threefolds that have been studied in the recent paper [3]. Based on their numerical results, we will explicitly verify Deligne’s conjecture.

Given a smooth variety $X$ defined over $\mathbb{Q}$, the pure motive $h^i(X)$ has three realizations:
(1) The Betti realization $H^i_B(X)$, which is the cohomology group $H^i(X(\mathbb{C}), \mathbb{Q})$ of the complex manifold defined by $X$ [13]. It has a pure Hodge structure on it.

(2) The de Rham realization $H^{i\text{dR}}(X)$, which is defined only by the algebraic data of $X$, e.g. the algebraic forms over $\mathbb{Q}$ [20].

(3) The étale realization $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ [12].

The étale realization of $h^i(X)$ is a continuous representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, which allows us to define its $L$-function $L(h^i(X), s)$. The $L$-function of its Tate twist $h^i(X)(n), n \in \mathbb{Z}$, satisfies

$$L(h^i(X)(n), s) = L(h^i(X), n + s).$$

Therefore, in order to study the value of $L(h^i(X), s)$ at $s = n$, it is sufficient to study the value of $L(h^i(X)(n), s)$ at $s = 0$. The pure motive $h^i(X)(n)$ is called critical if its Hodge numbers satisfy a condition that can be explicitly checked [5]. For example, if $X$ is Calabi-Yau threefold, then $h^3(X)(2)$ is critical. Deligne’s conjecture claims that for a critical motive $h^i(X)(n)$, the value $L(h^i(X), 0)$ is a rational multiple of Deligne’s period $c^+(h^i(X)(n))$, which is determined by the Betti realization and de Rham realization of $h^i(X)(n)$ [5].

In order to compute Deligne’s period $c^+(h^i(X)(n))$, we will need a rational basis of the Betti realization $H^i_B(X)(n)$ and the matrix of $F_\infty$, an involution defined by the complex conjugation of the points of $X(\mathbb{C})$, with respect to this basis. We will also need a rational basis of the de Rham realization $H^{i\text{dR}}(X)(n)$, and express its Hodge filtration explicitly using this basis. In practice, it is very difficult to have all these data at hand, but we will see that all of them have already appeared in the mirror symmetry of Calabi-Yau threefolds. More concretely, given a deformation of Calabi-Yau threefolds

$$\pi : \mathcal{X} \rightarrow \mathbb{P}^1_{\mathbb{Q}},$$

where the Hodge number $h^{2,1}$ of a smooth fiber is 1. If this deformation is the mirror family of a mirror pair, then mirror symmetry will provide all the data needed to explicitly compute the Deligne’s period of a smooth rational fiber. We will show how to do the computations for an example in this paper.

In the paper [3], the authors have studied two one-parameter mirror pairs of Calabi-Yau threefold that have the same Picard-Fuchs equation. In the mirror family, there is a special rational fiber $\mathcal{X}_{-1/7}$ that has very interesting properties. The zeta functions of the pure motive $h^3(\mathcal{X}_{-1/7})$ have been numerically computed by them for small prime numbers, through which they are able to identify the $L$-function of $h^3(\mathcal{X}_{-1/7})$ as

$$L(h^3(\mathcal{X}_{k,-1/7}), s) = L(f_2, s - 1)L(f_4, s).$$

Here $f_2$ is a weight-2 modular form for the modular group $\Gamma_0(14)$ that is designated as 14.2.a.a in LMFDB. While $f_4$ is a weight-4 modular form also for the modular group $\Gamma_0(14)$, which is designated as 14.4.a.a in LMFDB [3]. The authors have numerically computed the values of $L(f_2, 1), L(f_4, 1)$ and $L(f_4, 2)$. They also have numerically computed the values of the canonical periods of the threeform and its derivatives at the point $-1/7$. Interestingly, they are able to express these values in terms of that of $L(f_2, 1), L(f_4, 1), L(f_4, 2)$ and $v^\perp$, where $v^\perp$ is a number appears in the paper [3] that has close connections with the modular curve $X_0(14)$. The authors have speculated the connection of their results with Deligne’s
conjecture. Nevertheless they have not computed the Deligne’s period, hence they have not
directly checked whether Deligne’s conjecture is satisfied or not. Using the method we have
developed in this paper, we are able to compute the Deligne’s period $c^+(h^3(\mathcal{X}_{-1/7})(2))$ for
the critical pure motive $h^3(\mathcal{X}_{-1/7})(2)$, and from the numerical results in [3], we find
\[
c^+(h^3(\mathcal{X}_{-1/7})(2)) = -\frac{2401}{32} L(f_2, 1) L(f_4, 2),
\]
which of course depends on a special basis chosen in our computation. In this way, we
explicitly verify that Deligne’s conjecture is satisfied by the critical pure motive
$h^3(\mathcal{X}_{-1/7})(2)$.

We have also found that the period $c^-(h^3(\mathcal{X}_{k,-1/7}))$, defined by Deligne in [5], is of the form
\[
c^-(h^3(\mathcal{X}_{-1/7})) = \frac{1029}{32} \pi^3 L(f_4, 1) L(f_2, 1),
\]
while a better explanation to this equation is left to the interested readers.

The outline of this paper is as follows. In Section 2, we will briefly review the theory
of pure motives through the Betti, de Rham and étale realizations. In Section 3, we will
discuss the $L$-functions associated to pure motives. In Section 4, we will introduce the
construction of Deligne’s period and Deligne’s profound conjecture. In Section 5, we will
briefly review the theory of the mirror symmetry of Calabi-Yau threefolds. In Section 6, we
will develop a method to compute the Deligne’s periods for Calabi-Yau threefolds based on
mirror symmetry, which is the main result of this paper. In Section 7, we will compute the
Deligne’s period of the special Calabi-Yau threefolds studied in [3], and we will explicitly
verify Deligne’s conjecture based on their numerical results. In Section 8, we will summarize
the results of this paper.

2. THE CLASSICAL REALIZATIONS OF PURE MOTIVE

In this section, we will briefly introduce pure motives through their realizations. This
section completely consists of standard materials, and it is included here only to let the
readers become familiar with the notations. Suppose $X$ is a smooth projective variety
defined over $\mathbb{Q}$, and let $M$ be the following pure motive associated to $X$
\[
M := h^i(X)(n), \ i, n \in \mathbb{Z},
\]
whose meaning will be explained through its realizations. $M$ has three important realizations:

(1) The Betti realization. The $\mathbb{C}$-valued points (classical points) of $X$, denoted by $X(\mathbb{C})$, form a smooth projective complex manifold. Intuitively, $X(\mathbb{C})$ is just the complex
manifold defined by the rational polynomials that define $X$. The Betti realization of
$M$ is the following cohomology group
\[
M_B := H^i(X(\mathbb{C}), \mathbb{Q}(n)) = H^i(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}(n),
\]
where $\mathbb{Q}(n)$ is the rational vector space $(2\pi i)^n \mathbb{Q}$. Moreover, $\mathbb{Q}(n)$ has a pure Hodge
structure of the Hodge type $(-n, -n)$ [15]. From Hodge theory, there exists a pure
Hodge structure on $M_B$ with weight $w := i - 2n$, i.e. it has a Hodge decomposition
of the form
\[
M_B \otimes \mathbb{Q} \mathbb{C} = \bigoplus_{p+q=w} H^{p,q}.
\]
The Hodge number $h^{p,q}$ is by definition

$$h^{p,q} := \dim_{\mathbb{C}} H^{p,q}. \quad (2.4)$$

The complex conjugation $c \in \text{Gal}(\mathbb{C}/\mathbb{R})$ defines an action on the points of $X(\mathbb{C})$, which further induces an involution $c^*$ on $M_B$. Define $F_{\infty}$ to be the involution on $M_B$ induced by the action of complex conjugation on both the points $(X \times_{\sigma} \mathbb{C})(\mathbb{C})$ and the coefficient ring $\mathbb{Q}(n)$. The conjugate-linear involution $F_{\infty} \otimes c$ preserves the Hodge decomposition of $M_B \otimes \mathbb{C}$, i.e. it sends $H^{p,q}$ to $H^{p,q}$.

(2) The de Rham realization. Over the variety $X$, there exists a complex of sheaves of algebraic differential forms [8]

$$\Omega^*_X : 0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim(X)}_X \to 0. \quad (2.5)$$

However, in order to define a ‘reasonable’ cohomology theory, we have to choose an injective resolution $\Omega^*_X \to I^*$ in the abelian category of the complex of sheaves on $X$, then the algebraic de Rham cohomology of $X$ is just the hypercohomology of the shifted complex of sheaves $\Omega^*_X[n]$

$$M_{\text{dR}} := \mathbb{H}^i(X_{\text{Zar}}, \Omega^*_X[n]), \quad \text{where } (\Omega^*_X[n])^i = \Omega^{i+n}. \quad (2.6)$$

which is also called the hypercohomology of $\Omega^*_X$. Here $X_{\text{Zar}}$ means the Zariski topology on $X$. The de Rham realization of $M$ is just the hypercohomology of the shifted complex of sheaves $\Omega^*_X[n]$

$$F^pM_{\text{dR}} := \mathbb{H}^i(X_{\text{Zar}}, F^p\Omega^*_X[n]), \quad (2.7)$$

where the complex $F^p\Omega^*_X[n]$ is given by

$$F^p \Omega^*_X[n] : 0 \to \cdots \to 0 \to \Omega^{p+n} \xrightarrow{d} \Omega^{p+1+n} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim X} \to 0. \quad (2.8)$$

(3) The $\ell$-adic realization. Suppose $\ell$ is a prime number, then the $\ell$-adic cohomology of $X$ is defined by the inverse limit

$$H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) := \lim_{\leftarrow n} H^i((X \times_{\mathbb{Q}} \overline{\mathbb{Q}})_{\text{ét}}, \mathbb{Z}/\ell^n \mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \quad (2.9)$$

where $(X \times_{\mathbb{Q}} \overline{\mathbb{Q}})_{\text{ét}}$ is the étale topology on the $\overline{\mathbb{Q}}$-variety $X_{\overline{\mathbb{Q}}} := X \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ and $\mathbb{Z}/\ell^n \mathbb{Z}$ is the constant étale torsion sheaf on $(X \times_{\mathbb{Q}} \overline{\mathbb{Q}})_{\text{ét}}$. The $\ell$-adic cyclotomic character $\mathbb{Q}_\ell(1)$ is defined by the inverse limit

$$\mathbb{Q}_\ell(1) := \lim_{\leftarrow n} \mu_{\ell^n}(\overline{\mathbb{Q}}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \quad (2.10)$$

where $\mu_{\ell^n}(\overline{\mathbb{Q}})$ consists of the $\ell^n$-th roots of unity which admits a natural action by $\mathbb{Z}/\ell\mathbb{Z}$. Let $\mathbb{Q}_\ell(n)$ be the $n$-fold tensor product $\mathbb{Q}_\ell(1)^{\otimes n}$, which is a continuous representation of the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ [19]. The $\ell$-adic realization of $M$ is

$$M_\ell := H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(n), \quad (2.11)$$
which is also a continuous representation of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) \cite{12}.

There exist standard comparison isomorphisms between the three realizations \cite{13}:

1. There is an isomorphism $I_\infty$ between the Betti realization and de Rham realization

$$I_\infty : M_B \otimes_{\mathbb{Q}} \mathbb{C} \to M_{\text{dR}} \otimes_{\sigma} \mathbb{C},$$

(2.13)

which sends $\bigoplus_{k \geq p} H^{k,w-k} \to F^p M_{\text{dR}} \otimes_{\sigma} \mathbb{C}$. It is very important that the comparison isomorphism $I_\infty$ sends the involution $\phi_\sigma \otimes c$ on the left hand side to the involution $1 \otimes c$ on the right hand. This property will be crucial when we compute Deligne’s periods of Calabi-Yau threefolds.

2. Suppose $\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ is an embedding, then there is an isomorphism $I_{\ell,\infty}$ between the Betti realization and the $\ell$-adic realization

$$I_{\ell,\infty} : M_B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to M_\ell,$$

(2.14)

which however depends on the choice of $\infty$ up to an isomorphism. Moreover, the complex conjugation $c$ also defines an element of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$). Then $I_{\ell,\infty}$ sends the involution $F_\infty \otimes 1$ on the left hand side to the involution $c$ on the right hand side.

The two comparison isomorphisms immediately imply

$$\dim_{\mathbb{Q}}(M_B) = \dim_{\mathbb{Q}}(M_{\text{dR}}) = \dim_{\mathbb{Q}_\ell}(M_\ell),$$

(2.15)

and the common dimension is denoted by $\dim(M)$, which is called the rank of $M$.

**Example 2.1.** The Tate motive $\mathbb{Q}(1)$ is by definition the dual of the Lefschetz motive $h^2(\mathbb{P}^1_\mathbb{Q})$, whose classical realizations are:

1. $\mathbb{Q}(1)_B = 2\pi i \mathbb{Q}$, which has a pure Hodge structure of type $(-1, -1)$.
2. $\mathbb{Q}(1)_{\text{dR}} = \mathbb{Q}$, with filtrations $F^0 = 0$ and $F^{-1} = \mathbb{Q}$.
3. $\mathbb{Q}(1)_\ell = \mathbb{Q}_\ell(1)$.

The Tate motive $\mathbb{Q}(n)$ is the $n$-fold tensor product $\mathbb{Q}(1)^\otimes n$.

The twist of the pure motive $M$ by the Tate motive $\mathbb{Q}(m)$ is the tensor product

$$M(m) := M \otimes \mathbb{Q}(m),$$

(2.16)

so $M$ can also be written as

$$M = h^i(X) \otimes \mathbb{Q}(n).$$

(2.17)

There exist a Poincaré duality and a hard Lefschetz theorem in each of the three classical realizations, which are compatible under the standard comparison isomorphisms, therefore the dual of $M$ is given by

$$M^\vee = h^i(X)^\vee(-n) = h^{2\dim X-i}(X)(\dim X - n) = h^i(X)(i - n) = M(w),$$

(2.18)

where $w = i - 2n$ is the weight of $M$ \cite{13}.
3. The L-functions of pure motives

In this section, we will discuss the L-functions associated to pure motives, which is also a section that completely consists of standard materials. The readers who are familiar with these materials can skip this section completely.

The ℓ-adic realization $M_\ell$ of $M$ is a continuous representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Suppose $I_p$ is the inertia group in $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of the prime number $p$, then we say $M_\ell$ is unramified at $p$ if the action of $I_p$ on $M_\ell$ is trivial. In this case, the geometric Frobenius element has a well-defined action on $M_\ell$ that will be denoted by $\text{Fr}_p$. Since $X$ is a smooth projective variety defined over $\mathbb{Q}$, its ℓ-adic realization $M_\ell$ is pure of weight $w$. Here 'pure' means that there exists a finite set $S$ consists of prime numbers such that for a prime number $p \not\in S$ that does not divide $\ell$, $M_\ell$ is unramified at $p$. Moreover, all the eigenvalues of $\text{Fr}_p$ are algebraic numbers with absolute values equal to $p^{w/2}$ [6]. For a prime number $p \neq \ell$, let $M_\ell^{Ip}$ be the subspace of $M_\ell$ that is invariant under the action of $I_p$. Then the geometric Frobenius element also has a well-defined action on $M_\ell^{Ip}$. The characteristic polynomial of $\text{Fr}_p$ at $p$ is

$$P_p(M, T) = \det \left(1 - T \text{Fr}_p|M_\ell^{Ip}\right), \quad \ell \neq p.$$  (3.1)

From Deligne’s proof of Weil conjectures [6], if $X$ has good reduction at $p$, we have:

1. $P_p(M, T)$ is an integral polynomial of $\mathbb{Z}[T]$ and it is independent of the choice of $\ell$.
2. $P_p(M, T)$ has a factorization of the form

$$P_p(M, T) = \prod_{j=1}^{\dim(M)} \left(1 - \alpha_j T\right),$$  (3.2)

where $\alpha_j$ is an algebraic integer with $|\alpha_j| = p^{w/2}$ for every $j$.

$X$ has bad reduction at only finitely many primes, and Serre has a conjecture about the property of $P_p(M, T)$ at bad primes [18].

**Conjecture 3.1.** Given an arbitrary prime number $p$, $P_p(M, T)$ is integral that does not depend on the choice of $\ell$. It has a factorization

$$P_p(M, T) = \prod_{j=1}^{\dim(M)^{Ip}} \left(1 - \alpha_j T\right),$$  (3.3)

where $\alpha_j$ is an algebraic integer with absolute value

$$|\alpha_j| = p^{w_j/2}, \ 0 \leq w_j \leq w.$$  (3.4)

The local L-factor of $M$ at $p$ is defined by

$$L_p(M, s) := \frac{1}{P_p(M, p^{-s})},$$  (3.5)

while the L-function of $M$ is defined by

$$L(M, s) := \prod_p L_p(M, s),$$  (3.6)
where the product is over all the prime numbers. The local $L$-factor $L_p(M, s)$ satisfies the following properties [13]

$$L_p(M(m), s) = L_p(M, m + s), \quad L_p(M_1 \oplus M_2, s) = L_p(M_1, s)L_p(M_2, s), \quad (3.7)$$

and the $L$-function of $M$ satisfies similar properties, i.e.

$$L(M(m), s) = L(M, m + s), \quad L(M_1 \oplus M_2, s) = L(M_1, s)L(M_2, s). \quad (3.8)$$

Deligne’s theorem and Conjecture 3.1 imply that the $L(M, s)$ converges absolutely when $\text{Re}(s) > w/2 + 1$, thus $L(M, s)$ is a nowhere vanishing holomorphic function in this region. However the existen of a meromorphic extension of $L(M, s)$ to the whole complex plane is still a conjecture [5, 16].

**Conjecture 3.2.** $L(M, s)$ has a meromorphic extension to the whole complex plane, and the only possible pole occurs at $s = w/2 + 1$ when $w$ is an even integer. Moreover, the value $L(M, w/2 + 1)$ is non-zero if $s = w/2 + 1$ is not a pole.

The archimedean prime of $\mathbb{Z}$ is given by the natural embedding of $\mathbb{Q}$ into $\mathbb{C}$, and we will denote this archimedean prime by $\infty$. There is also the local $L$-factor associated to this archimedean prime [14, 5]. For simplicity, let us define

$$\Gamma_\mathbb{R}(s) := \pi^{-s/2} \cdot \Gamma(s/2), \quad \Gamma_\mathbb{C}(s) := \Gamma_\mathbb{R}(s) \cdot \Gamma_\mathbb{R}(s + 1) = 2 \cdot (2\pi)^{-s} \cdot \Gamma(s). \quad (3.9)$$

The local $L$-factor $L_\infty(M, s)$ only depends on the pure Hodge structure on $M_\mathbb{R} \otimes_\mathbb{Q} \mathbb{R}$, which is carefully discussed in [18] and section 5.2 of [5]:

1. If the weight $w$ of $M$ is odd, $L_\infty(M, s)$ is defined by

$$L_\infty(M, s) = \prod_{p < q} \Gamma_\mathbb{C}(s - p)^{h^p,q} \cdot \Gamma_\mathbb{R}(s - w/2)^{\text{dim}H^{w/2,+}} \cdot \Gamma_\mathbb{R}(s - w/2 + 1)^{\text{dim}H^{w/2,-}}. \quad (3.10)$$

2. If the weight $w$ of $M$ is even, and the subspace $H^{w/2,w/2}$ decomposes into the direct sum

$$H^{w/2,w/2} = H^{w/2,+} \oplus H^{w/2,-}; \quad F_\infty|H^{w/2,+} = (-1)^{w/2}, \quad F_\infty|H^{w/2,-} = (-1)^{w/2+1}, \quad (3.11)$$

then $L_\infty(M, s)$ is defined by

$$L_\infty(M, s) = \prod_{p < q} \Gamma_\mathbb{C}(s - p)^{h^p,q} \cdot \Gamma_\mathbb{R}(s - w/2)^{\text{dim}H^{w/2,+}} \cdot \Gamma_\mathbb{R}(s - w/2 + 1)^{\text{dim}H^{w/2,-}}. \quad (3.12)$$

The local $L$-factor $L_\infty(M, s)$ also satisfies [5]

$$L_\infty(M(m), s) = L_\infty(M, m + s), \quad L_\infty(M_1 \oplus M_2, s) = L_\infty(M_1, s) \cdot L_\infty(M_2, s). \quad (3.13)$$

The full $L$-function of $M$ is defined by

$$\Lambda(M, s) = L(M, s) \cdot L_\infty(M, s). \quad (3.14)$$

**Conjecture 3.3.** $\Lambda(M, s)$ satisfies the following functional equation [5, 18],

$$\Lambda(M, s) = \varepsilon(M, s) \Lambda(M', 1 - s), \quad (3.15)$$

where $\varepsilon(M, s)$ is of the form $a \cdot b^s$ with $a$ and $b$ as non-zero complex numbers. From equation 2.18, this functional equation can also be rewritten as

$$\Lambda(M, s) = \varepsilon(M, s) \Lambda(M, w + 1 - s). \quad (3.16)$$
4. Deligne’s conjecture

In this section, we will introduce Deligne’s conjecture on the values of the $L$-functions of critical motives. We will follow the paper [5] closely. Let us first introduce the definition of a critical pure motive $M$.

**Definition 4.1.** Given an integer $n$, it is called critical for $M$ if neither $L_\infty(M, s)$ nor $L_\infty(M^\vee, 1-s)$ has a pole at $s = n$.

Deligne’s conjecture is a conjecture about the value of $L(M, s)$ at a critical integer $s = n$. From the formulas 3.8 and 3.13, we only need to consider the case where $s = 0$. $M$ is said to be critical if 0 is critical for $M$. In fact, $M$ is critical if its Hodge numbers $h^{p,q}$ satisfy the following conditions:

1. If the pair $(p, q)$ satisfies $p \neq q$ and $h^{p,q} \neq 0$, then we must have $p \leq -1, q \geq 0$ or $p \geq 0, q \leq -1$.
2. If the weight of $M$ is even, then the action of $F_\infty$ on $H^{p,p}$ is 1 if $p < 0$ and $-1$ if $p \geq 0$.

From the paper [5], if $M$ is critical, then $L(M, 0)$ is not $\infty$, i.e. $s = 0$ is not a pole of $L(M, s)$. For simplicity, in the rest of this paper, we will only consider the case where the weight $w$ of the pure motive $M$ is odd.

Let $M_B^+$ (resp. $M_B^-$) be the subspace of $M_B$ that is fixed by $F_\infty$ (resp. $F_\infty = -1$), then we define $d^+(M) = \dim Q M_B^+$ (resp. $d^-(M) = \dim Q M_B^-$). Since $F_\infty$ exchanges $H^{p,q}$ and $H^{q,p}$, we deduce that if the weight $w$ of $M$ is odd, then we have

$$d^+(M) = d^-(M) = \frac{1}{2} \dim Q (M_B).$$

(4.1)

On the other hand, let $F^+$ (resp. $F^-$) be the subspace of $M_{\text{dR}}$ occurring in its Hodge filtration that has the same dimension as $M_B^+$ (resp. $M_B^-$). More concretely, via the comparison isomorphism between Betti and de Rham realizations, $F^+ \otimes \mathbb{C}$ corresponds to

$$\bigoplus_{p \geq q} H^{p,q}(M) \text{ with } p + q = w.$$  

(4.2)

Since we have assumed the weight $w$ of $M$ is odd, we will have $F^- = F^+$. Next we define

$$M_{\text{dR}}^+ = M_{\text{dR}}^+/F^-, \ M_{\text{dR}}^- = M_{\text{dR}}/F^+.$$  

(4.3)

In fact, we have $M_{\text{dR}}^+ = M_{\text{dR}}^-$, and

$$\dim Q M_{\text{dR}}^+ = \dim Q M_{\text{dR}}^- = d^+(M) = \frac{1}{2} \dim Q (M_B).$$

(4.4)

The comparison isomorphism $I_\infty$ between Betti and de Rham realizations together with the natural projection map $M_{\text{dR}} \to M_{\text{dR}}^+$ induces the following composition of maps

$$I_\infty^+: M_B^+ \otimes \mathbb{C} \hookrightarrow M_B \otimes \mathbb{C} \xrightarrow{I_\infty} M_{\text{dR}} \otimes \mathbb{C} \to M_{\text{dR}}^+ \otimes \mathbb{C}.$$  

(4.5)

Since $F_\infty$ exchanges $H^{p,q}$ and $H^{q,p}$, the homomorphism $I_\infty^+$ is in fact an isomorphism. Now we choose a rational basis of $M_B^+$ and a rational basis of $M_{\text{dR}}^+$. With respect to them, we can compute the determinant of $I_\infty^+$, which is by definition Deligne’s period

$$c^+(M) = \det(I_\infty^+).$$

(4.6)
Notice that Deligne’s period $c^+(M)$ is only well-defined up to a nonzero rational multiple. Similarly, there exists an isomorphism

$$I_\infty : M_B^- \otimes \mathbb{C} \to M_{dR}^- \otimes \mathbb{C}. \tag{4.7}$$

Now a rational basis of $M_B^-$ and a rational basis of $M_{dR}^-$ allow us to define another period by

$$c^-(M) = \det(I_\infty). \tag{4.8}$$

It is also well-defined up to a nonzero rational multiple. In fact Deligne’s period $c^+(M)$ (resp. $c^-(M)$) can be expressed in terms of classical periods. More precisely, the dual of $M_{dR}^+$ is the subspace $F^+$ of $M_{dR}^\vee$ that is denoted by $F^+ M_{dR}^\vee$. Choose a basis $\{\omega_i\}$ of $F^+ M_{dR}^\vee$ and a basis $\{\rho_i\}$ of $M_B^+$, then the matrix of $I_\infty^+$ with respect to these two bases is $\langle \omega_i, \rho_j \rangle$, where $\langle \cdot, \cdot \rangle$ means the pairing defined by Poincaré duality. Deligne’s period $c^+(M)$ is given by

$$c^+(M) = \det(\langle \omega_i, \rho_j \rangle). \tag{4.9}$$

Similarly we can also express $c^-(M)$ in terms of classical periods.

**Deligne’s Conjecture:** If the pure motive $M$ is critical, then $L(M,0)$ is a rational multiple of $c^+(M)$.

### 5. Mirror symmetry of Calabi-Yau threefold

In this section, we will briefly review the theory of mirror symmetry of Calabi-Yau threefolds [2, 4, 7, 10]. For the purpose of this paper, we will only focus on one-parameter mirror pairs. Given a mirror pair $(X^\vee, X)$, mirror symmetry studies the complexified Kähler moduli space of $X^\vee$ and the complex moduli space of $X$. One-parameter means that their Hodge numbers satisfy

$$h^{1,1}(X^\vee) = h^{2,1}(X) = 1. \tag{5.1}$$

#### 5.1. Picard-Fuchs equation

For the purpose of this paper, we will assume the mirror threefold $X$ has an algebraic deformation defined over $\mathbb{Q}$ of the form

$$\pi : \mathcal{X} \to \mathbb{P}^1_{\mathbb{Q}}. \tag{5.2}$$

From now on, $X$ will mean the underlying differential manifold of a smooth fiber of this family. The coordinate of the base $\mathbb{P}^1_{\mathbb{Q}}$ has been chosen to be $\varphi$. We will also assume that for each smooth fiber $\mathcal{X}_\varphi$, there exists a nowhere-vanishing algebraic threeform $\Omega_\varphi$ that varies algebraically with respect to $\varphi$. Moreover, as a form on a smooth open subvariety of $\mathcal{X}$, $\Omega$ is defined over $\mathbb{Q}$. In particular, for a rational point $\varphi$, $\Omega_\varphi$ is defined over $\mathbb{Q}$ [4, 7, 10].

From Griffiths transversality, the threeform $\Omega_\varphi$ satisfies a fourth-order Picard-Fuchs equation of the form

$$\mathcal{L} \Omega_\varphi = 0, \tag{5.3}$$

where $\mathcal{L}$ is a differential operator with polynomial coefficients $R_i(\varphi) \in \mathbb{Q}[\varphi]$

$$\mathcal{L} = R_4(\varphi) \partial^4 + R_3(\varphi) \partial^3 + R_2(\varphi) \partial^2 + R_1(\varphi) \partial + R_0(\varphi), \text{ with } \partial = \varphi \frac{d}{d\varphi}. \tag{5.4}$$

We will also assume $\varphi = 0$ is the large complex structure limit, the meaning of which will be explained now. The point $\varphi = 0$ is called the large complex structure limit if the
monodromy about it is maximally unipotent. More precisely, given a small disc \( \Delta \) of \( \varphi = 0 \), the Picard-Fuchs operator \( \mathcal{L} \) has four canonical solutions of the form

\[
\varpi_0 = f_0, \\
\varpi_1 = \frac{1}{2\pi i} (f_0 \log \varphi + f_1), \\
\varpi_2 = \frac{1}{(2\pi i)^2} (f_0 \log^2 \varphi + 2 f_1 \log \varphi + f_2), \\
\varpi_3 = \frac{1}{(2\pi i)^3} (f_0 \log^3 \varphi + 3 f_1 \log^2 \varphi + 3 f_2 \log \varphi + f_3),
\]

where \( \{f_j\}_{j=0}^3 \) are power series in \( \mathbb{Q}[[\varphi]] \) that converge on \( \Delta \). We will further impose the condition

\[
f_0(0) = 1, \ f_1(0) = f_2(0) = f_3(0) = 0,
\]

under which the four canonical solutions 5.5 are unique. The canonical period vector \( \varpi \) is the column vector defined by

\[
\varpi := (\varpi_0, \varpi_1, \varpi_2, \varpi_3)^	op.
\]

**Remark 5.1.** In this paper, the multi-valued homomorphic function \( \log \varphi \) satisfy

\[
\log(1) = 0, \ \log(-1) = \pi i.
\]

Poincaré duality implies the existence of a unimodular skew symmetric pairing on \( H_3(X, \mathbb{Z}) \) (modulo torsion), which allows us to choose a symplectic basis \( \{A_0, A_1, B_0, B_1\} \) that satisfy the following intersection pairing [2, 4, 7]

\[
A_a \cdot A_b = 0, \ B_a \cdot B_b = 0, \ A_a \cdot B_b = \delta_{ab}.
\]

Suppose the dual of this basis is \( \{\alpha^0, \alpha^1, \beta^0, \beta^1\} \), i.e. the only non-trivial pairings are

\[
\alpha^a(A_b) = \delta_{ab}, \ \beta^a(B_b) = \delta_{ab},
\]

and they form a basis of \( H^3(X, \mathbb{Z}) \) (modulo torsion). From Poincaré duality, we have [9]

\[
\int_X \alpha^a \smile \beta^b = \delta_{ab}, \ \int_X \alpha^a \smile \alpha^b = 0, \ \int_X \beta^a \smile \beta^b = 0.
\]

**Remark 5.2.** The torsions of homology or cohomology groups will be ignored in this paper.

The integration of the threeform \( \Omega_\varphi \) over the symplectic basis \( \{A_a, B_a\}_{a=0}^1 \) give us the integral periods

\[
z_a(\varphi) = \int_{A_a} \Omega_\varphi, \ G_b(\varphi) = \int_{B_b} \Omega_\varphi,
\]

which are multi-valued holomorphic functions [2, 4, 7]. Now we define the integral period vector \( \Pi(\varphi) \) by

\[
\Pi(\varphi) := (G_0(\varphi), G_1(\varphi), z_0(\varphi), z_1(\varphi))^	op.
\]

For later convenience, let us also define the row vector \( \beta \) by

\[
\beta := (\beta^0, \beta^1, \alpha^0, \alpha^1).
\]
Under the comparison isomorphism, $\Omega(\varphi)$ has an expansion given by

$$\Omega(\varphi) = \beta \cdot \Pi(\varphi) = G_0(\varphi) \beta^0 + G_1(\varphi) \beta^1 + z_0(\varphi) \alpha^0 + z_1(\varphi) \alpha^1. \quad (5.15)$$

Since the integral period vector $\Pi$ form another basis of the solution space of 5.3, there exists a matrix $S \in \text{GL}(4, \mathbb{C})$ such that

$$\Pi = S \cdot \varpi. \quad (5.16)$$

The transformation matrix $S$ is crucial in this paper, and it will be determined by mirror symmetry.

5.2. **Prepotential on the Kähler side.** In the one-parameter case, the complexified Kähler moduli space $\mathcal{M}_K(X^\vee)$ has a very simple description \[7\]

$$\mathcal{M}_K(X^\vee) = (\mathbb{R} + i \mathbb{R}_{>0})/\mathbb{Z} = \mathbb{H}/\mathbb{Z}, \quad (5.17)$$

where $\mathbb{H}$ is the upper half plane of $\mathbb{C}$. Now let $e$ be a basis of $H^2(X^\vee, \mathbb{Z})$ (modulo torsion) that lies in the Kähler cone of $X^\vee$ \[7\], then every point of $\mathcal{M}_K(X^\vee)$ is represented by $te$, $t \in \mathbb{H}$, while $e t$ is equivalent to $e (t + 1)$ under the quotient by $\mathbb{Z}$. Conventionally $t$ is called the flat coordinate of $\mathcal{M}_K(X^\vee)$ by physicists \[2, 4, 7\]. In mirror symmetry, the prepotential $F$ on the Kähler side admits an expansion near $i \infty$ that is of the form \[2, 4\]

$$F = \frac{1}{6} Y_{111} t^3 - \frac{1}{2} Y_{011} t^2 - \frac{1}{2} Y_{001} t - \frac{1}{6} Y_{000} + F_{np}, \quad (5.18)$$

where $F_{np}$ is the non-perturbative instanton correction. $F_{np}$ is invariant under the translation $t \rightarrow t + 1$ and it is exponentially small when $t \rightarrow i \infty$, i.e. it admits a series expansion in $\exp 2\pi i t$ of the form

$$F_{np} = \sum_{n=1}^{\infty} a_n \exp 2\pi i n t. \quad (5.19)$$

The coefficient $Y_{111}$ in 5.18 is the topological intersection number given by \[2, 4, 7\]

$$Y_{111} = \int_M e \wedge e \wedge e, \quad (5.20)$$

which is a positive integer. In most cases, $e$ can be represented by a divisor of $X^\vee$, then $Y_{111}$ is the triple intersection number of this divisor with itself. The coefficients $Y_{011}$ and $Y_{001}$ are rational numbers \[10\]. In all examples of mirror pairs, $Y_{000}$ is always of the form \[2\]

$$Y_{000} = -3 \chi(X^\vee) \frac{\zeta(3)}{(2\pi i)^3}, \quad (5.21)$$

where $\chi(X^\vee)$ is the Euler characteristic of $X^\vee$. A detailed study of the appearance of $\zeta(3)$ from the motivic point of view is presented in the paper \[10\].
5.3. **Mirror symmetry.** In all examples of one-parameter mirror pairs, there exists an integral symplectic basis \{A_0, A_1, B_0, B_1\} of \(H^3(X, \mathbb{Z})\) such that

\[ z_i(\varphi) = \lambda(2\pi i)^3 w_i(\varphi), \quad i = 0, 1; \lambda \in \mathbb{Q}^\times. \tag{5.22} \]

Let us denote the quotient \(\varpi_1/\varpi_0\) by \(t_c\)

\[ t_c = \frac{z_1}{z_0} = \frac{\varpi_1}{\varpi_0} = \frac{1}{2\pi i} \log \varphi + \frac{f_1(\varphi)}{f_0(\varphi)}, \tag{5.23} \]

and under the action of monodromy induced by \(\log \varphi \to \log \varphi + 2\pi i\), it transforms in the way

\[ t_c \to t_c + 1. \tag{5.24} \]

**Definition 5.3.** The mirror map is defined by the identification

\[ t \equiv t_c. \tag{5.25} \]

The normalization of \(\Pi\) is defined to be

\[ \Pi_A = (\mathcal{G}_0/z_0, \mathcal{G}_1/z_0, 1, z_1/z_0)^\top. \tag{5.26} \]

On the Kähler side, the mirror period vector \(\Pi\) is defined by [2, 10]

\[ \Pi = (\mathcal{F}_0, \mathcal{F}_1, 1, t)^t, \quad \text{with} \quad \mathcal{F}_0 = 2 \mathcal{F} - t \frac{\partial \mathcal{F}}{\partial t}, \quad \mathcal{F}_1 = \frac{\partial \mathcal{F}}{\partial t}. \tag{5.27} \]

The mirror symmetry conjecture claims that under the mirror map in the formula 5.25, we have

\[ \Pi = \Pi_A. \tag{5.28} \]

Now we are ready to compute the matrix \(S\) in the formula 5.16. Near the large complex structure limit, formula 5.6 implies

\[ t_c = \frac{1}{2\pi i} \log \varphi + \mathcal{O}(\varphi), \tag{5.29} \]

therefore the large complex structure limit on the complex side corresponds to \(t = i \infty\) on the Kähler side [2, 4, 10]. In the limit \(t \to i \infty\), the leading parts of \(\Pi_A\) and \(\varpi\) are given by

\[ \Pi_A \equiv \Pi \sim \begin{pmatrix} \frac{1}{6} Y_{111} t^3 - \frac{1}{2} Y_{001} t - \frac{1}{3} Y_{000} \\ -\frac{1}{2} Y_{111} t^2 - Y_{011} t - \frac{1}{2} Y_{101} \\ 1 \\ t \end{pmatrix}, \quad \varpi \sim \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}, \tag{5.30} \]

from which the matrix \(S\) can be easily evaluated [10]

\[ S = \lambda(2\pi i)^3 \begin{pmatrix} -\frac{1}{2} Y_{000} & -\frac{1}{2} Y_{001} & 0 & \frac{1}{6} Y_{111} \\ -\frac{1}{2} Y_{001} & -Y_{011} & -\frac{1}{2} Y_{111} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \lambda \in \mathbb{Q}^\times. \tag{5.31} \]
6. The computation of Deligne’s period

In this section, we will give a method to compute Deligne’s period for a smooth fiber \( \mathcal{X}_\varphi \) in the family 5.2 where \( \varphi \in \mathbb{Q} \) using mirror symmetry. We will give an example to explain how this method works.

The algebraic de Rham cohomology \( H^3_{\text{dr}}(\mathcal{X}_\varphi) \) of \( \mathcal{X}_\varphi \) is a four dimensional vector space over \( \mathbb{Q} \) and it has a very explicit description. By definition, we have

\[
\Omega_\varphi \in F^3 H^3_{\text{dr}}(\mathcal{X}_\varphi).
\]  

(6.1)

Since \( \Omega_\varphi \) is nowhere-vanishing, it forms a basis for the one dimensional rational vector space \( F^3 H^3_{\text{dr}}(\mathcal{X}_\varphi) \). As \( \mathcal{X}_\varphi \) is a smooth fiber, \( \{\Omega_\varphi, \Omega'_\varphi\} \) form a basis of \( F^2 H^3_{\text{dr}}(\mathcal{X}_\varphi) \), where we have used Griffiths transversality. Here the derivative of \( \Omega_\varphi \) is with respect to \( \varphi \). Similarly, \( \{\Omega_\varphi, \Omega'_\varphi, \Omega''_\varphi, \Omega'''_\varphi\} \) form a basis of \( F^0 H^3_{\text{dr}}(\mathcal{X}_\varphi) \). Under the comparison isomorphism between Betti cohomology and algebraic de Rham cohomology, we have

\[
\Omega_\varphi^{(n)} = \beta \cdot S \cdot \omega^{(n)}, \quad n = 0, 1, 2, 3;
\]

(6.2)

where we have used the formulas 5.15 and 5.16. Now we define the Wronskian \( W \) by

\[
W = \begin{pmatrix}
\omega_0 & \omega'_0 & \omega''_0 & \omega'''_0 \\
\omega_1 & \omega'_1 & \omega''_1 & \omega'''_1 \\
\omega_2 & \omega'_2 & \omega''_2 & \omega'''_2 \\
\omega_3 & \omega'_3 & \omega''_3 & \omega'''_3 \\
\end{pmatrix},
\]

(6.3)

which does not vanish at a smooth point \( \varphi \) [21]. The rational basis \( (\Omega_\varphi, \Omega'_\varphi, \Omega''_\varphi, \Omega'''_\varphi) \) of \( H^3_{\text{dr}}(\mathcal{X}_\varphi) \otimes \mathbb{C} \) is mapped to the basis \( \beta \cdot S \cdot W \) of \( H^3(\mathcal{X}_\varphi(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C} \) under the comparison isomorphism between Betti cohomology and algebraic de Rham cohomology.

The action of the map \( F_\infty \) on the Betti cohomology \( H^3(\mathcal{X}_\varphi(\mathbb{C}), \mathbb{Q}) \), which is just \( H^3(X, \mathbb{Q}) \), can be computed explicitly. The key is that under the comparison isomorphism between Betti cohomology and algebraic de Rham cohomology, the map \( F_\infty \otimes c \) on \( H^3(X, \mathbb{Q}) \otimes \mathbb{C} \) corresponds to \( 1 \otimes c \) on \( H^3_{\text{dr}}(\mathcal{X}_\varphi) \otimes \mathbb{C} \). From it, we immediately deduce that

\[
\beta \cdot S \cdot W = \beta \cdot F_\infty \cdot \overline{S} \cdot \overline{W},
\]

(6.4)

where \( F_\infty \) also means the matrix of \( F_\infty \) with respect to the basis \( \beta \) of \( H^3(X, \mathbb{Q}) \). Here \( \overline{S} \) (resp. \( \overline{W} \)) means the complex conjugation of \( S \) (resp. \( W \)). Thus we have found that the matrix \( F_\infty \) is given by

\[
F_\infty = S \cdot W \cdot \overline{W}^{-1} \cdot \overline{S}^{-1}.
\]

(6.5)

The dual of the algebraic de Rham cohomology \( H^3_{\text{dr}}(\mathcal{X}_\varphi) \) is given by

\[
H^3_{\text{dr}}(\mathcal{X}_\varphi)^\vee = H^3_{\text{dr}}(\mathcal{X}_\varphi) \otimes \mathbb{Q}(3).
\]

(6.6)

The subspace \( F^+(H^3_{\text{dr}}(\mathcal{X}_\varphi) \otimes \mathbb{Q}(3)) \) is given by

\[
F^+(H^3_{\text{dr}}(\mathcal{X}_\varphi) \otimes \mathbb{Q}(3)) = F^{-1}(H^3_{\text{dr}}(\mathcal{X}_\varphi) \otimes \mathbb{Q}(3)),
\]

(6.7)
which is spanned by $\Omega_\varphi$ and $\Omega'_\varphi$. Suppose the subspace of $H^3(X, \mathbb{Q})$ where $F_\infty$ act as $1$ has a basis $(\gamma_0^+, \gamma_1^+)$, then Deligne’s period $c^+(h^3(\mathcal{X}_\varphi))$ is given by

$$c^+(h^3(\mathcal{X}_\varphi)) = \det \left( \frac{1}{(2\pi i)^3} \int_X \Omega_\varphi \sim \gamma_0^+ \quad \frac{1}{(2\pi i)^3} \int_X \Omega'_\varphi \sim \gamma_0^+ \right).$$

(6.8)

Notice that the additional factor $(2\pi i)^{-3}$ comes from the fact that the dual of $H^3(X, \mathbb{Q})$ is $H^3(X, \mathbb{Q}) \otimes \mathbb{Q}(3)$, and the pairing

$$H^3(X, \mathbb{Q}) \times (H^3(X, \mathbb{Q}) \otimes \mathbb{Q}(3)) \to \mathbb{Q}$$

(6.9)

is defined by

$$\langle \phi_1, \phi_2 \rangle = \frac{1}{(2\pi i)^3} \int_X \phi_1 \sim \phi_2.$$  

(6.10)

The period $c^-(h^3(\mathcal{X}_\varphi))$ can be computed similarly. Suppose the subspace of $H^3(X, \mathbb{Q})$ where $F_\infty$ act as $-1$ has a basis $(\gamma_0^-, \gamma_1^-)$, then we have

$$c^-(h^3(\mathcal{X}_\varphi)) = \det \left( \frac{1}{(2\pi i)^3} \int_X \Omega_\varphi \sim \gamma_0^- \quad \frac{1}{(2\pi i)^3} \int_X \Omega'_\varphi \sim \gamma_0^- \right).$$

(6.11)

Since the Hodge numbers of $\mathcal{X}_\varphi$ satisfy

$$h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1,$$

(6.12)

the pure motive $h^3(\mathcal{X}_\varphi)(n)$ is critical if and only if when $n = 2$. The basis of the Betti cohomology $H^3(X, \mathbb{Q}) \otimes \mathbb{Q}(2)$ is given by $(2\pi i)^2 \beta$. But $(2\pi i)^2$ is real, so the subspace of $H^3(X, \mathbb{Q}) \otimes \mathbb{Q}(2)$ where $F_\infty$ act as $1$ has a basis $((2\pi i)^2 \gamma_0^+, (2\pi i)^2 \gamma_1^+)$. Therefore we deduce that [5]

$$c^+(h^3(\mathcal{X}_\varphi)(2)) = (2\pi i)^4 c^+(h^3(\mathcal{X}_\varphi)).$$

(6.13)

Similarly, we also have

$$c^-(h^3(\mathcal{X}_\varphi)(2)) = (2\pi i)^4 c^-(h^3(\mathcal{X}_\varphi)).$$

(6.14)

**Example 6.1.** If $\varphi \in \mathbb{Q}$ is a small positive number such that the power series $f$, in 5.5 converges at it, then we have $W = V \cdot W$, where $V$ is the matrix

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$  

(6.15)

The matrix $F_\infty$ is given by

$$F_\infty = S \cdot V^{-1} \cdot S^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2Y_{011} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

(6.16)

The subspace of $H^3(X, \mathbb{Q})$ where $F_\infty$ act as $1$ has a basis

$$\beta^0, \alpha^1 - Y_{011} \beta^1,$$  

(6.17)
while subspace of $H^3(X, \mathbb{Q})$ where $F_{\infty}$ act as -1 has a basis
\[ \alpha^0, \beta^1. \]

Under the comparison isomorphism, we have
\[ \Omega_{\varphi} = \beta \cdot S \cdot \varpi, \quad \Omega'_{\varphi} = \beta \cdot S \cdot \varpi', \]
and we have
\[ \int_X \Omega_{\varphi} \sim \beta^0 = \lambda (2\pi i)^3 \varpi_0, \]
\[ \int_X \Omega_{\varphi} \sim (\alpha^1 - Y_{011} \beta^1) = \lambda (2\pi i)^3 \left( \frac{1}{2} Y_{001} \varpi_0 + \frac{1}{2} Y_{111} \varpi_2 \right). \]

Hence we deduce Deligne’s period $c^+(h^3(\mathcal{X}_{\varphi}))$ is given by
\[ c^+(h^3(\mathcal{X}_{\varphi})) = \frac{1}{2} \lambda^2 Y_{111} (\varpi_0 \varpi'_2 - \varpi_2 \varpi'_0), \]
where $\lambda$ is a non-zero rational number and $Y_{111}$ is a positive integer. Similarly we have
\[ c^-(h^3(\mathcal{X}_{\varphi})) = \frac{1}{6} \lambda^2 Y_{111} \left( \left( \frac{2Y_{000}}{Y_{111}} \varpi_0 - \varpi_3 \varpi'_1 \right) - \left( \frac{2Y_{000}}{Y_{111}} \varpi'_0 - \varpi'_3 \varpi_1 \right) \right). \]

Remark 6.1. The numbers $Y_{111}$ and $Y_{000}$ are determined by the topological data of $X$, which is independent of the choice of the symplectic basis $\beta$. While the numbers $Y_{011}$ and $Y_{001}$ does depend on the choice of the symplectic basis $\beta$. It is very interesting to notice that $Y_{011}$ and $Y_{001}$ do not appear in the periods $c^+(h^3(\mathcal{X}_{\varphi}))$ and $c^-(h^3(\mathcal{X}_{\varphi}))$.

7. Examples for Deligne’s Conjecture

In this section, we will use the method in the previous section to compute the Deligne’s periods for the two special Calabi-Yau threefolds that have been studied in the paper [3]. Based on their numerical results, we will directly verify Deligne’s conjecture.

In the paper [3], the authors have constructed two one-parameter mirror pairs $(X_k^\vee, X_k)$, $k = 1, 2$, of Calabi-Yau threefolds. There exists an algebraic deformation of $X_k$ defined over $\mathbb{Q}$, which will be denoted by
\[ \pi_k : \mathcal{X}_k \to \mathbb{P}^1_{\mathbb{Q}}, \quad k = 1, 2. \]

The zeta functions of the fiber of $\mathcal{X}_k$ over the point $\varphi = -1/7$ have been numerically computed for small primes, from which the authors are able to determine the $L$-functions of the pure motive $h^3(\mathcal{X}_{k,-1/7})$. The numerical values of the canonical periods (and its derivatives) at $\varphi = -1/7$ have been computed by them to a very high precision, and they are able to express these values in terms of the special values of $L$-functions. They have speculated the connections of their numerical results with Deligne’s conjecture. But Deligne’s period $c^+(h^3(\mathcal{X}_{k,-1/7})(2))$ for the critical motive $h^3(\mathcal{X}_{k,-1/7})(2)$ has not been computed, and Deligne’s conjecture has not been numerically verified. In this section, we will compute Deligne’s period $c^+(h^3(\mathcal{X}_{k,-1/7})(2))$ using the method developed in Section 6, and verify that Deligne’s conjecture has been satisfied by $h^3(\mathcal{X}_{k,-1/7})(2)$. We will also compute the period $c^-(h^3(\mathcal{X}_{k,-1/7}))$, and study its properties.
7.1. **Review the results of AESZ34.** In this section, we will review the results in the paper [3], and in fact every result in this section is from their paper. The readers are referred to it for more details. The two families in the formula 7.1 have the same Picard-Fuchs equation

\[ \mathcal{D} = \theta^4 - \varphi(35\theta^4 + 70\theta^3 + 63\theta^2 + 28\theta + 5) + \varphi^2(\theta + 1)^2(259\theta^2 + 518\theta + 285) \]

\[ - 225\varphi^3(\theta + 1)^2(\theta + 2)^2, \quad \theta = \frac{d}{d\varphi}, \]

which is listed as AESZ34 in the paper [1]. The Picard-Fuchs operator \( \mathcal{D} \) has five regular singularities at the points

\[ \varphi = 0, 1/25, 1/9, 1, \infty, \]

while \( \varphi = 0 \) is the large complex structure limit. The canonical period \( \varpi_0 \) is given by

\[ \varpi_0 = 1 + \sum_{n=1}^{\infty} a_n \varphi^n; \quad a_n = \sum_{i+j+k+l+m=n} \left( \frac{n!}{i!j!k!l!m!} \right)^2. \]

The numbers that appear in the prepotential of \( X_k^\vee \) have also been computed in [3], and they are given by

\[ Y_{111} = 12k, \quad Y_{011} = 0, \quad Y_{001} = -k, \quad Y_{000} = 24k \frac{\zeta(3)}{(2\pi i)^3}, \quad k = 1, 2. \]  

**Remark 7.1.** The Hodge diamond of \( X_k \) is of the form [3]

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 4k + 1 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & 4k + 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{array}
\]

The zeta function of the pure motive \( h^3(\mathcal{X}_{k,-1/7}) \) at a good prime \( p \) is of the form

\[ (1 - a_p(pT) + p(pT)^2)(1 - b_p T + p^3 T^2), \]

which is the same for both \( k = 1 \) and \( k = 2 \). Moreover, \( a_p \) is the \( p \)-th coefficient of the \( q \)-expansion of a weight-2 modular form \( f_2 \) for the modular group \( \Gamma_0(14) \), which is designated as 14.2.a.a in LMFDB. While \( b_p \) is the \( p \)-th coefficient of the \( q \)-expansion of a weight-4 modular form \( f_4 \) also for the modular group \( \Gamma_0(14) \), which is designated as 14.4.a.a in LMFDB. This has been numerically checked by them for small prime numbers [3]. Hence the \( L \)-function of the pure motive \( h^3(\mathcal{X}_{k,-1/7}) \) is given by

\[ L(h^3(\mathcal{X}_{k,-1/7}), s) = L(f_2, s - 1)L(f_4, s). \]

In particular, the special value \( L(h^3(\mathcal{X}_{k,-1/7}), 2) \) is equal to \( L(f_2, 1)L(f_4, 2) \). In the paper [3], both \( L(f_2, 1) \) and \( L(f_4, 2) \) have been numerically computed to a high decision

\[ L(f_2, 1) = 0.3302236593448053902826194612283487754045234078189 \ldots, \]

\[ L(f_4, 2) = 0.1930674266912115653914356907939249680895763199044 \ldots. \]
The series expansions of the canonical periods $\varpi_i$ do not converge at $\varphi = -1/7$, nevertheless their values can be numerically computed to a very high precision by numerically solving the Picard-Fuchs equation. In [3], the Wronskian $W$ of the canonical periods $\varpi_i$ at $\varphi = -1/7$ has been numerically computed, and its entries can be numerically expressed in terms of the special values $L(f_2, 1)$, $L(f_4, 1)$, $L(f_4, 2)$ and $v^\perp$. The numerical value of $L(f_4, 1)$ has also been computed by the authors

$$L(f_4, 1) = 0.67496319716994177129269568273091339919322842904407 \cdots. \quad (7.9)$$

The number $v^\perp$ is very interesting, and its numerical value is

$$v^\perp = 0.37369955695472976697672927524994632117655651682 \cdots. \quad (7.10)$$

The authors have found that the $j$-value of $\tau^\perp := \frac{1}{2} + i v^\perp$ is

$$j(\tau^\perp) = \left( \frac{215}{28} \right)^3. \quad (7.11)$$

They also find that LMFDB includes only one rationally defined elliptic curve with the above $j$-invariant which also has the weight-2 modular form $14.2.a.a$ as its eigenform. In fact, this curve is defined by

$$y^2 + xy + y = x^3 + 4x - 6, \quad (7.12)$$

which is the modular curve $X_0(14)$. The readers are referred to [3] for more details.

### 7.2. Computation of Deligne’s period.

In this section, we will compute Deligne’s period $c^+(h^3(X^*_{k, -1/7})(2))$ for the critical motive $h^3(X^*_{k, -1/7})(2)$ and verify that Deligne’s conjecture is satisfied by it based on the numerical results of [3], which have been reviewed in the previous section. We will also compute the period $c^-(h^3(X^*_{k, -1/7}))$, and study its interesting properties.

First, let us compute the matrix of the involution $F_\infty$, which is given by the formula 6.5. From the formula 7.5 and the numerical values of Wronskian $W$ at $\varphi = -1/7$, formula 6.5 gives us

$$F_\infty = \begin{pmatrix} 1 & 1 & -3k & 6k \\ 0 & -1 & 6k & -12k \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad k = 1, 2. \quad (7.13)$$

The two linearly independent eigenvectors associated to the eigenvalue 1 are

$$v^+_1 = (1, 0, 0, 0), \quad v^+_2 = (0, -6k, 0, 1), \quad (7.14)$$

hence the subspace of $H^3(X_k, \mathbb{Q})$ that $F_\infty$ acts as 1 is spanned by

$$\beta^0 \text{ and } -6k \beta^1 + \alpha^1. \quad (7.15)$$

From Section 6, Deligne’s period $c^+(h^3(X^*_{k, -1/7})(2))$ is given by

$$c^+(h^3(X^*_{k, -1/7})(2)) = (2\pi i)^4 \frac{1}{(2\pi i)^4} \det \left( \begin{array}{cccc} \int_{X_k} \Omega_{-1/7} \beta^0 & \int_{X_k} \Omega_{-1/7} \beta^0 \\ \int_{X_k} \Omega_{-1/7} \beta^0 & \int_{X_k} \Omega_{-1/7} \beta^0 \end{array} \right) \left( \begin{array}{cccc} \int_{X_k} \Omega_{-1/7} (-6k \beta^1 + \alpha^1) \\ \int_{X_k} \Omega_{-1/7} (-6k \beta^1 + \alpha^1) \end{array} \right), \quad (7.16)$$
which is equal to
\[
c^+(h^3(\mathcal{X}_{k,-1/7})(2)) = 6k \lambda^2 (2\pi i)^4 \det \begin{pmatrix} \varpi_0(-1/7) & -\varpi_1(-1/7) + \varpi_2(-1/7) \\ \varpi'_0(-1/7) & -\varpi'_1(-1/7) + \varpi'_2(-1/7) \end{pmatrix},
\]
(7.17)
where \( \lambda \) is a nonzero rational number. Since Deligne’s period is only well-defined up to a nonzero rational multiple, we can just let \( c^+(h^3(\mathcal{X}_{k,-1/7})(2)) \) be
\[
c^+(h^3(\mathcal{X}_{k,-1/7})(2)) = \pi^4 \det \begin{pmatrix} \varpi_0(-1/7) & -\varpi_1(-1/7) + \varpi_2(-1/7) \\ \varpi'_0(-1/7) & -\varpi'_1(-1/7) + \varpi'_2(-1/7) \end{pmatrix}.
\]
(7.18)
Plug in the numerical values of \( \varpi_i^{(n)}(-1/7) \), we find that
\[
c^+(h^3(\mathcal{X}_{k,-1/7})(2)) = -\frac{2401}{32} L(f_2, 1)L(f_4, 2) = L(h^3(\mathcal{X}_{k,-1/7})(2), 0),
\]
(7.19)
which indeed satisfies Deligne’s conjecture.

The subspace of \( H^3(X_k, \mathbb{Q}) \) that \( F_\infty \) acts as \(-1\) is spanned by
\[
2\alpha^0 + \alpha^1 \quad \text{and} \quad -\beta^0 + 2\beta^1.
\]
(7.20)
The period \( c^-(h^3(\mathcal{X}_{k,-1/7})) \) is given by
\[
c^-(h^3(\mathcal{X}_{k,-1/7})) = \det \begin{pmatrix} 32 - \frac{1}{(2\pi i)^4} & 1 \end{pmatrix} \begin{pmatrix} \varpi_0 - 2\varpi_1 + 12\varpi_2 - 8\varpi_3, & -\varpi_0 + 2\varpi_1 \\ \varpi'_0 - 2\varpi'_1 + 12\varpi'_2 - 8\varpi'_3, & -\varpi'_0 + 2\varpi'_1 \end{pmatrix},
\]
(7.21)
where the value of this determinant is taken at \( \varphi = -1/7 \). Plug in the numerical values of \( \varpi_i^{(n)}(-1/7) \), we find that
\[
c^-(h^3(\mathcal{X}_{k,-1/7})) = \frac{1029}{32} \pi^3 \frac{L(f_4, 1)L(f_2, 1)}{v^\perp}.
\]
(7.22)
It would be very interesting to have a better understanding of this relation in terms of the existing conjectures about the special values of \( L \)-functions, which will however not be pursued in this paper. We will leave this open question to interested readers.

8. Conclusion

In this paper, we have developed an explicit method to compute Deligne’s periods for one-parameter Calabi-Yau threefolds in mirror symmetry, which is our main result. We also illustrate how this method works by explicitly computing an example. It should be noticed that mirror symmetry plays a crucial role in our computations. We have also computed Deligne’s periods for the Calabi-Yau threefolds studied in the paper [3], and we explicitly verify that Deligne’s conjecture is satisfied based on the numerical results of [3].

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