EXISTENCE AND UNIQUENESS OF MILD SOLUTIONS
FOR QUASI-LINEAR FRACTIONAL
INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. We discuss the existence and uniqueness of mild solutions for a class of quasi-linear fractional integro-differential equations with impulsive conditions via Hausdorff measures of noncompactness and fixed point theory in Banach space. Mild solution controllability is discussed for two particular cases.

1. Introduction. The aim of this paper is to discuss the existence and uniqueness of a class of mild solutions of quasi-linear fractional integro-differential equations with impulsive and nonlocal conditions of the form

\[
\begin{aligned}
C^D_0^+ \theta(t) + A(t, \theta(t))\theta(t) &= \Phi(t, \theta(t)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(t, s, \theta(s)) ds, \\
\theta(0) + \Xi(\theta) &= \theta_0 \\
\Delta \theta(t_i) &= I_i(\theta(t_i)), \quad i = 1, \ldots, n, 0 < t_1 < \ldots < t_n < b
\end{aligned}
\]

where \( C^D_0^+ (\cdot) \) is the Caputo fractional derivative of order \( 0 < \alpha < 1 \), \( t \in J = [0, b] \), \( t \neq t_i \), \( A : J \times \Lambda \to \Lambda \) is a closed linear operator, the generator of an \((\alpha, \theta)\)-resolvent family; \( \theta_0 \in \Lambda \) (\( \Lambda \) is a Banach space); \( \Phi : J \times \Lambda \to \Lambda \); \( g : \Omega \times \Lambda \to \Lambda \); \( \Xi : PC(J, \Lambda) \times \Lambda \to \Lambda \) and \( \Delta \theta(t_i) = \theta(t_i^+) - \theta(t_i^-) \) constitutes an impulsive condition. Here \( \Omega = (t, s); 0 \leq s \leq t \leq b \).

For a long time, the quest for results about the existence, uniqueness, controllability and other properties of solutions of differential and integro-differential equations has been the target of investigations of paramount importance in mathematics, among other areas of knowledge [11, 32, 36]. Tools as measures of noncompactness and fixed point theory are very useful in discussing the theory of differential equations [4, 6, 7, 18].

On the other hand, the theory of impulsive differential equations appears as a natural description of several real processes subject to certain disturbances whose duration is insignificant as compared with the duration of the process. For many
years, Lakshmikantham et al. [11] and Ntouyas and Tsamatos [32] discussed numerous properties of their solutions. Other important researchers who have developed and are developing other theories can be found in [19, 21, 23] and references therein.

In 2012, Arjunan et al. [28] investigated the existence of solutions of impulsive differential equations with nonlocal conditions in a real Banach space $X$ and are developing other theories can be found in [19, 21, 23] and references therein.

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\[
\begin{aligned}
\begin{cases}
  u'(t) &= Au(t) + f(t, u(t)), t \in J, t \neq t_i \\
  u(0) &= g(u) \\
  \Delta u(t_i) &= I_i(u(t_i)), i = 1, \ldots, n, 0 < t_1 < \ldots < t_n < b
\end{cases}
\end{aligned}
\]

where $A : D(A) \subset X$ is a non-densely defined operator and $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$, where $u(t_i^-)$, $u(t_i^+)$ denote the right and left limits of $u$ at $t_i$, respectively.

Over the years, fractional calculus acquired unquestionable importance and relevance for several fields of knowledge, particularly physics, engineering, medicine and biology [1, 2, 3, 40, 29, 31, 33, 39, 55]. Today, the theory of fractional calculus is well established and there is a growing number of researchers who use tools discussed in fractional calculus, applying it to other areas and providing this important link [12, 13, 27, 24, 35, 38]. Here, we will highlight the fractional differential equations, which has been the subject of study by several researchers [16, 25, 26, 56].

The theory of fractional differential equations is of paramount importance in both theoretical and practical aspects. In the theoretical sense, we can cite the names of Trujillo, Nieto, Donal O’Regan, Benchahra, N’Guérékata and Debbouche, among many researchers who discovered important results that enabled the growth and strengthening of the area [8, 9, 41, 42]. Recently, Sousa and Oliveira [44, 45, 46, 48, 49, 50] presented some results on the existence, uniqueness, stability of solutions of fractional differential equations by means of the $\psi$-Hilfer fractional derivative. Other works involving the $\psi$-Hilfer fractional derivative are [51, 52, 53].

We can also highlight the important work by Hu et al. [22] on the existence and uniqueness of mild solutions for semilinear integro-differential equations of fractional order with nonlocal initial conditions and delays, in 2009. This work is of paramount importance for the theory of fractional differential equations, especially equations with delay. In addition, Hu, Ren and Sakthivel provided important results on the existence, uniqueness and controllability of solutions of fractional differential equations [5, 17, 37, 54]. See too [30, 34].

In 2010, Debbouche [14] discussed the existence and uniqueness of local mild and classic solutions of a class of nonlinear fractional integro-differential evolution systems with nonlocal conditions of the form

\[
\begin{aligned}
\begin{cases}
  \frac{d^\alpha}{dt^\alpha} u(t) + A(t)u(t) &= f(t, u(t)) + \int_0^t B(t-s)g(s, u(s))ds, \\
  u(t_0) + h(u) &= u_0
\end{cases}
\end{aligned}
\]

in a Banach space $X$, where $0 < \alpha \leq 1$, $0 \leq t_0 < t$, $-A(t)$ is a closed linear operator defined on a dense domain $D(A) \subset X$ into $X$ such that $D(A)$ is independent of $t$. It is also assumed that $-A(t)$ generates an evolution operator in Banach space $X$, the function $B$ is real valued and locally integrable on $[t_0, \infty]$, the nonlinear maps $f$ and $g$ are defined on $[t_0, \infty] \times X$ into $X$ and $h : C(J, X) \to \overline{D(A)}$ is a given function. In addition to these important results, Debbouche and Baleanu [15] obtained results on controllability of solutions of fractional integro-differential equations.

In 2017, Gou and Li [20] investigated the local and global existence of mild solutions to an impulsive fractional semilinear integro-differential equation with
noncompact semigroup
\[\begin{align*}
u^\alpha(t) + Au(t) &= f(t, u(t)) + \int_0^t q(t - s)g(s, u(s))ds, \quad t > 0, \alpha \in (0, 1) \\
u(0) &= u_0 \in X \\
\Delta u|_{t = t_k} &= I_k(u(t_k)), \quad k = 1, ..., m
\end{align*}\]
where \(u^\alpha(\cdot)\) is the Caputo fractional derivative of order \(\alpha\) \((\alpha \in (0, 1))\), \(A : D(A) \subset X \rightarrow X\) is a closed linear operator and \(-A\) generates a uniformly bounded \(c_0\)-semigroup \(T(t) \ (t \geq 0)\) in \(X\), the nonlinear maps \(f, g : [0, \infty] \times X \rightarrow X\), and \(q : I \rightarrow X\) are continuous, \(I = [0, T), \ 0 < T \leq \infty, u_0 \in X\). \(I_k : X \rightarrow X, 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T, \ \Delta u|_{t = t_k} = u(t^+_k) - u(t^-_k), u(t^+_k) = \lim_{n\rightarrow\infty}u(t_k + n)\) and \(u(t^-_k) = \lim_{n\rightarrow\infty}u(t_k - n)\) represent the right and left limits of \(u(t)\) at \(t = t_k\) respectively.

Therefore, many papers of paramount importance and high quality have been published over the years. However, works involving an \((\alpha, \theta)\)-resolvent family of problems of fractional differential and integro-differential equations with impulses are still restricted and few good and interesting works can be found in the literature. Motivated by the papers above, by the restrict number of works in the area and in the search to provide new results, in this paper we discuss results on the existence and uniqueness of a class of mild solutions for the fractional problem Eq.(1). To make the development of the paper simple and clear, we highlight the main results discussed in the sequel. It is also important to note that the results discussed in this paper are based on conditions that will be presented throughout the paper, which makes the discussions presented interesting and important to completely understand the results obtained. This paper is basically divided in three stages: existence, uniqueness and applications.

In the first stage we discuss the existence of mild solutions for Eq.(1). This discussion comprehends three different cases, corresponding to three distinct sets of conditions called \(\text{EC}_i\). So, we first assume the validity of conditions \(\text{EC}_i\) for \(i = 1, ..., 6\) in order to prove our first result about existence:

**Theorem 1.1.** Assume that conditions \(\text{EC}_i\), \(i = 1, ..., 6\), hold. Then the impulsive nonlocal problem Eq.(1) has at least one mild solution.

Next, we remove conditions \(\text{EC}_i\) for \(i = 3, ..., 6\), and impose conditions \(\text{EC}_i\) for \(i = 7, 8\), which yields the second result about existence:

**Theorem 1.2.** Suppose that the \(\text{EC}_i\) for \(i = 1, 2\) and \(\text{EC}_i\) for \(i = 7, 8\) are satisfied. Then Eq.(1) has at least one mild solution if

\[\lim_{n \rightarrow \infty} \sup \frac{M_0}{r} \left( \varphi(r) + \phi(r) \int_0^r p(s)ds + \frac{G_1 \chi(r)}{\Gamma(\alpha)} \int_0^r (t - s)^{\alpha - 1}q(s)ds + \sum_{i=1}^n d_i \right) < 1,\]

where \(\varphi(r) = \sup \{||\Xi(\theta)||, ||\theta|| < r\}\).

Finally, to complete the first stage, we assume the validity of all conditions \(\text{EC}_i\) for \(i = 1, ..., 9\) and discuss the following two results.

**Theorem 1.3.** Let \(0 < \alpha < 1\) and suppose that assumptions \(\text{EC}_i\) for \(i = 1, ..., 9\) are satisfied. Then Eq.(1) has at least one mild solution, provided that

\[M_0^2 \left[ L_0 + 4 \int_0^t \left( k_1(s) + \frac{2G_0k_3(s)(t - s)^{\alpha - 1}}{\Gamma(\alpha)} \right) ds + \sum_{i=1}^n l_i \right] \leq 1.\]
Theorem 1.4. Suppose that assumptions EC$_i$ for $i = 1, ..., 9$ are satisfied. Then Eq.(1) has at least one mild solution if Eq.(26) and the condition
\[
M_0^6 L_0 + \lim_{n \to \infty} \frac{M_0^6}{r} \left( \phi(r) \int_0^t p(s)ds + \frac{\lambda(r) G_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{q}(s)ds + \sum_{i=1}^{n} d_i \right) < 1.
\]
(2)
are satisfied.

The second contribution of this paper (second stage), is the discussion about the uniqueness of solutions for Eq.(1), that is, the following result:

Theorem 1.5. Let $\theta_0 \in \Lambda$ and let $B_r = \{ \theta \in PC(J, \Lambda); \|\theta\| \leq r \}$, $r > 0$. If assumptions UC$_i$ for $i = 1, ..., 5$ are satisfied, then Eq.(1) has a unique mild solution.

The conditions EC$_i$ for $i = 1, ..., 9$ and UC$_i$ for $i = 1, ..., 5$ are presented throughout the paper.

The paper is divided as follows. In section 2 we present the fundamental concepts of Riemann-Liouville fractional integral and Caputo fractional derivative. We also present some fundamental results on the noncompactness measure and the fundamental concept of mild solution. In section 3 we discuss the existence of solutions to the main problem (1). In section 4, the second main result of this work, the uniqueness of solutions to the main problem (1). Finally, in section 5, we present two examples involving Theorem 3.2 and Theorem 4.1. Concluding remarks close the paper.

2. Preliminaries. In this section, we present fundamental concepts of integrals and fractional derivatives of Riemann-Liouville and Caputo, respectively. In this sense, the concept of noncompactness measure and some fundamental results that will be used throughout the paper are presented.

Definition 2.1. [43, 47] Let $(a,b)$ ($-\infty \leq a < b \leq \infty$) be a finite or infinite interval of the real line $\mathbb{R}$ and let $\alpha > 0$. In addition, let $\psi(t)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(t)$ on $(a,b)$. The left-sided fractional integral of function $\theta$ with respect to another function $\psi$ on $[a,b]$ is defined by
\[
I^{\alpha;\psi}_{a+} \theta(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \theta(s)ds.
\]
(3)

The right-sided fractional integral is defined in an analogous form [43, 47].

Choosing $\psi(t) = t$ in Eq.(3), we have the Riemann-Liouville fractional integral given by [43, 47]
\[
I^{\alpha}_{a+} \theta(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \theta(s)ds,
\]
(4)
where $\Gamma(\cdot)$ is the gamma function and $f \in L^1([a,b], \mathbb{R})$.

If $a = 0$, we can write $I^{\alpha}_{0+} \theta(t) = (g_{\alpha} * \theta)(t)$, where
\[
g_{\alpha}(t) := \begin{cases} 
\frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0 \\
0, & t \leq 0
\end{cases}
\]
and as usual $*$ denotes convolution of functions, also we have $\lim_{\alpha \to 0} g_{\alpha}(t) = \delta(t)$. From choosing $\psi(\cdot)$, we have another fractional integrals.
Here, we restrict to the Riemann-Liouville fractional integral to discuss the results of this paper. However, other formulations of fractional integrals can be obtained by choosing \( \psi(t) \) \([43, 47]\).

Also, we begin with the definition of the \( \psi \)-Hilfer fractional derivative.

**Definition 2.2.** \([43, 47]\) Let \( n - 1 < \alpha < n \), with \( n \in \mathbb{N} \), let \( I = [a, b] \) be an interval such that \( -\infty \leq a < b \leq \infty \) and let \( \theta, \psi \in C^n([a, b], \mathbb{R}) \) be two functions, such that \( \psi \) is increasing and \( \psi'(t) \neq 0 \), for all \( t \in I \). The left-sided \( \psi \)-Hilfer fractional derivative \( H_{D_{a+}^\alpha}^{\beta; \psi} (\cdot) \) of a function \( \theta \), of order \( \alpha \) and type \( 0 \leq \beta \leq 1 \) is defined by

\[
H_{D_{a+}^\alpha}^{\beta; \psi} \theta(t) := I_{a+}^{\beta(n-\alpha);\psi} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_{a+}^{(1-\beta)(n-\alpha);\psi} \theta(t). \tag{6}
\]

The right-sided \( \psi \)-Hilfer fractional derivative is defined in an analogous form \([43, 47]\).

Choosing \( \psi(t) = t \) and taking the limit \( \beta \to 1 \), on both sides of the Eq.(6), we have the Caputo fractional derivative given by \([43, 47]\)

\[
C_{D_{a+}^\alpha} \theta(t) = I_{a+}^{(n-\alpha);\psi} \left( \frac{d}{dt} \right)^n \theta(t) = I_{a+}^{(n-\alpha);\psi} \theta^{(n)}(t). \tag{7}
\]

To investigate our main result, we use Caputo fractional derivative as in Eq.(7).

Let \( \Lambda \) a Banach space with norm \( \| \cdot \| \). Let \( \mathcal{PC}(J; \Lambda) \) consist of functions \( \theta \) from \( J \) into \( \Lambda \), such that \( \theta(t) \) is continuous at \( t \neq t_i \) and let continuous at \( t = t_i \) and the right \( \lim \theta(t_i^+) \) exists for \( i = 1, ..., n \). Evidently \( \mathcal{PC}(J; \Lambda) \) is a Banach space, where

\[
\| \theta \|_{\mathcal{PC}} = \sup_{t \in J} \| \theta(t) \|
\]

and denoted \( \mathcal{L}(J, \Lambda) \) by the space of \( \Lambda \)-valued Bochner integrable functions on \( J \) with the norm \([56]\)

\[
\| \theta \|_{\mathcal{L}} = \int_0^b \| \theta(t) \| dt.
\]

We will present below the definition of Hausdorff measure of noncompactness and some results of paramount importance in the discussion of the main results of this paper.

The Hausdorff measure of noncompactness \( \mu_\Omega(\cdot) \) is defined by

\[
\mu_\Omega(B) = \inf\{ r > 0, B \text{ can be covered by a finite number of balls with radii } r \}
\]

for bounded set \( B \) in a Banach space \( \Omega \) \([56]\).

**Lemma 2.3.** \([56]\) Let \( \Omega \) be a real Banach space and \( B, E \subseteq \Omega \) be bounded, with the following properties:

1. \( B \) is precompact if and only if \( \mu_\Omega(B) = 0 \).
2. \( \mu_\Omega(B) = \mu_\Omega(\overline{B}) = \mu_\Omega(\text{con } B) \), where \( B \) and \( \text{con } B \) mean the closure and convex hull of \( B \), respectively.
3. \( \mu_\Omega(B) \leq \mu_\Omega(E) \), where \( B \subseteq E \).
4. \( \mu_\Omega(B + E) \leq \mu_\Omega(B) + \mu_\Omega(E) \), where \( B + E = \{ x + y; x \in B, y \in E \} \).
5. \( \mu_\Omega(B \cup E) \leq \max\{\mu_\Omega(B), \mu_\Omega(E)\} \).
6. \( \mu_\Omega(\lambda B) \leq |\lambda| \mu_\Omega(B) \), for any \( \lambda \in \mathbb{R} \).
7. If the map \( \Theta : D(\Theta) \subseteq \Omega \to \mathbb{Z} \) is a Lipschitz continuous with constant \( r \), then \( \mu_\Omega(\Theta(B)) \leq r \mu_\Omega(B) \) for any bounded subset \( B \subseteq D(\Theta) \), where \( \mathbb{Z} \) being a Banach space.
Lemma 2.4. [56] If $\mathbb{W} \subseteq \Omega$ is bounded, closed and convex, the continuous map $\Theta : \mathbb{W} \rightarrow \mathbb{W}$ is a $\mu_\Omega$-contraction, the map $\Theta$ has at least one fixed point in $\mathbb{W}$.

We denote by $\mu$ the Hausdorff measure of noncompactness of $\Lambda$ and denote $\mu_c$ by the Hausdorff measure of noncompactness of $\mathcal{P}(J;\Lambda)$.

Lemma 2.5. [56] If $\mathbb{W}$ is bounded, then for each $\varepsilon > 0$, there exists a sequence $\{\theta_n\}_{n=1}^\infty \subset \mathbb{W}$, such that

$$\mu(\mathbb{W}) \leq 2\mu\left(\left\{\theta_n\right\}_{n=1}^\infty\right) + \varepsilon.$$ 

Lemma 2.6. [56] If $\mathbb{W} \subseteq \mathcal{P}(J;\Lambda)$ is bounded, then $\mu(\mathbb{W}(t)) \leq \mu_c(\mathbb{W})$, for all $t \in J$, where $\mathbb{W}(t) = \{\theta(t); \theta \in \mathbb{W}\} \subseteq \Lambda$. Furthermore, if $\mathbb{W}$ is equicontinuous on $J$, then $\mu(\mathbb{W}(t))$ is continuous on $J$ and $\mu_c(\mathbb{W}) = \sup\{\mu(\mathbb{W}(t)), t \in J\}$.

Lemma 2.7. [56] If $\{\theta_n\}_{n=1}^\infty \subset \mathcal{L}^1(J;\Lambda)$ is uniformly integrable, then the function $\mu(\{\theta_n\}_{n=1}^\infty)$ is measurable and

$$\mu\left(\left\{\int_0^t \theta_n(s)ds\right\}_{n=1}^\infty\right) \leq 2\int_0^t \mu(\{\theta_n(s)\}_{n=1}^\infty) ds.$$ 

Lemma 2.8. [56] If $\mathbb{W} \subseteq \mathcal{P}(J;\Lambda)$ is bounded and equicontinuous, then $\mu(\mathbb{W}(t))$ is continuous and

$$\mu\left(\int_0^t \mathbb{W}(s)ds\right) \leq \int_0^t \mu(\mathbb{W}(s)) ds,$$

for all $t \in J$, where

$$\int_0^t \mu(\mathbb{W}(s)) ds = \left\{\int_0^t \theta(s) ds; \theta \in \mathbb{W}\right\}.$$ 

The $C_0$ semigroup $U_\theta(t,s)$ is said to be equicontinuous if $(t,s) \rightarrow \left\{U_\theta(t,s)\theta(s); \theta \in \mathbb{B}\right\}$ is equicontinuous for $t > 0$ for all bounded set $\mathbb{B}$ in $\Lambda$. So, follows the lemma.

Lemma 2.9. [56] If the evolution family $\left\{U_\theta(t,s)\right\}_{0 \leq s \leq t \leq b}$ is continuous and $\eta \in \mathcal{L}(J;\mathbb{R}^+)$, then the set $\left\{\int_0^t U_\theta(t,s)\theta(s) ds\right\}$, $\|\theta(s)\| \leq \eta(s)$ for a.e. $s \in J$ is equicontinuous for $t \in J$.

From Eq.(6) we know that for any fixed $u \in \mathcal{P}(J;\Lambda)$ there exists a unique continuous function $U_\theta : J \times J \rightarrow \mathbb{B}(\Lambda)$ defined on $J \times J$ such that [36]

$$U_\theta(t,s) = I + \int_s^t A_\theta(w) U_\theta(w,s) dw,$$

(8)
where $\mathcal{B}(\Lambda)$ denote the Banach space of bounded linear operator from $\Lambda$ to $\Lambda$ with the norm $\|\Theta\| = \sup\{\|\Theta(t)\|; \|\Theta\| = 1\}$ and $I$ stands for the identity operator on $\Lambda$, $\mathcal{A}_0 = \mathcal{A}(t, \theta(t))$, we have [36]

\[
U_\theta(t,t) = I, \quad U_\theta(t,s)U_\theta(s,r) = U_\theta(t,r), \quad (t,s,r) \in J \times J \times J
\]

and

\[
\frac{\partial U_\theta(t,s)}{\partial t} = \mathcal{A}_0(t)U_\theta(t,s), \quad \text{for almost all } t, s \in J.
\]

Let $\mathcal{E}$ be the Banach space formed from $D(\mathcal{A})$ with the graph norm. Since, $\mathcal{A}(t)$ is a closed operator, it follows that $\mathcal{A}(t)$ is in the set bounded from $\mathcal{E}$ to $\Lambda$.

**Definition 2.10.** [14, 15] Let $\mathcal{A}(t, \theta)$ be a closed and linear operator with domain $D(\mathcal{A})$ defined on a Banach space $\Lambda$ and $\alpha > 0$. Let $\rho(\mathcal{A}(t, \theta))$ be the resolvent set of $\mathcal{A}(t, \theta)$. We call $\mathcal{A}(t, \theta)$ the generator of an $(\alpha, \theta)$-resolvent family if there exists $w \geq 0$ and a strongly continuous function $\mathcal{R}_{(\alpha, \theta)} : \mathbb{R}_+^2 \to \mathcal{L}(\Lambda)$ such that \{\lambda^\alpha : \text{Re}(\lambda) > w\} $\subset \rho(\mathcal{A})$ and for $0 \leq s \leq t \leq \infty$,

\[
\left(\lambda^\alpha I - \mathcal{A}(s, \theta)\right)^{-1} \nu = \int_0^\infty e^{-\lambda(t-s)}\mathcal{R}_{(\alpha, \theta)}(t,s) \nu \, dt, \quad \text{Re}(\lambda) > w, \quad (\theta, \nu) \in \Lambda^2.
\]

In this case, $\mathcal{R}_{(\alpha, \theta)}(t,s)$ is called the $(\alpha, \theta)$-resolvent family generated by $\mathcal{A}(t, \theta)$.

**Remark 1.**

- We can deduce that Eq.(1) is well posed if and only if $\mathcal{A}(t, \theta)$ is the generator of $(\alpha, \theta)$-resolvent family.
- Here $\mathcal{R}_{(\alpha, \theta)}(t,s)$ can be extracted from the evolution operator of the generator $\mathcal{A}(t, \theta)$.
- The $(\alpha, \theta)$-resolvent family is similar to the evolution for non-autonomous differential equation in a Banach space.

3. **The existence of mild solution.** In this section, we discuss the existence of a mild solution for the quasi-linear fractional integro-differential equation with impulsive and nonlocal conditions in the sense of Caputo fractional derivative, using Hausdorff of noncompactness measure.

**Definition 3.1.** [14, 15] By a mild solution of Eq.(1) we mean a function $\theta \in \mathcal{PC}(J, \Lambda)$ with values in $\Omega$ satisfying the integral equation

\[
\theta(t) = \mathcal{R}_{(\alpha, \theta)}(t,0)[\theta_0 - \Xi(\theta)] + \sum_{0 \leq t_i < t} \mathcal{R}_{(\alpha, \theta)}(t,t_i) I_i(\theta(t_i)) + \int_0^t \mathcal{R}_{(\alpha, \theta)}(t,s) \left(\Phi(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-x)^{\alpha-1}g(s, x, \theta(x))dx\right) ds
\]

$t \in J$ for all $\theta_0 \in \Lambda$ and $0 < \alpha < 1$.

In this work, we denote

\[
\mathcal{M}_0^\alpha = \sup\{\mathcal{R}_{(\alpha, \theta)}(t,s); (t,s) \in J \times J, 0 < \alpha < 1\}
\]

for all $\theta \in \Lambda$. Without loss of generality, we let $\theta_0 = 0$.

To investigate the main result of this paper, we need some conditions, namely:

**EC1.** The evolution family $\left\{\mathcal{R}_{(\alpha, \theta)}(t,s)\right\}_{0 \leq s \leq t \leq \infty}$ is called the $(\alpha, \theta)$-resolvent generated by $\mathcal{A}(t, \theta(t))$ if it is equicontinuous and $\|\mathcal{R}_{(\alpha, \theta)}(t,s)\| \leq \mathcal{M}_0^\alpha$ for almost every $t, s \in J$ and $0 < \alpha < 1$.

**EC2.** The function $\Xi : \mathcal{PC}(J, \Lambda) \to \Lambda$ is continuous and compact.

**EC2.2** There exists $\mathcal{N}_0 > 0$ such that $\|\Xi(\theta)\| \leq \mathcal{N}_0$. 
EC3.1. The nonlinear function $\Phi : J \times \Lambda \to \Lambda$ satisfies the Caratheodory-type conditions, i.e., $\Phi(\cdot, \theta)$ is measurable for all $\theta \in \Lambda$ and $\Phi(t, \cdot)$ is continuous for almost every $t \in J$.

EC3.2. There exists a function $\xi \in \mathcal{L}(J; \mathbb{R}^+)$ such that for every $\theta \in \Lambda$, we have $\|\Phi(t, \theta)\| \leq \xi(t)(1 + \|\theta\|)$, a.e. $t \in J$.

EC3.3. There exists a function $k_1 \in \mathcal{L}(J, \mathbb{R}^+)$ such that, for every bounded $D \subset \Lambda$, we have

$$\mu(\Phi(t, D)) \leq k_1(t) \mu(D), \text{ a.e. } t \in J.$$  \hspace{1cm} (10)

EC4.1. The nonlinear function $g : J \times J \times \Lambda \to \Lambda$ satisfies the Caratheodory-type conditions, i.e., $g(\cdot, \cdot, \theta)$ is continuous for a.e. $t \in J$.

EC4.2. There exist two functions $\beta_1 \in \mathcal{L}(J, \mathbb{R}^+)$ and $\beta_2 \in \mathcal{L}(J, \mathbb{R}^+)$ such that for every $\theta \in \Lambda$, we have

$$\|g(t, s, \theta(s))\| \leq \beta_1(t)\beta_2(t)(1 + \|\theta(s)\|), \text{ a.e. } t \in J.$$  \hspace{1cm} (11)

EC4.3. There exist functions $k_2, k_3 \in \mathcal{L}(J, \mathbb{R}^+)$ such that, for every bounded $D \subset \Lambda$, we have

$$\mu(g(t, s, D)) \leq k_2(t)k_3(t) \mu(D), \text{ a.e. } t \in J.$$  \hspace{1cm} (12)

EC5.1. Assume that the finite bound of $\int_0^t k_2(t)ds$ is $G_0$.

EC5.2. For every $t \in J$ there exists positive constants $N_1$ and $N_2$, the scalar equation

$$m^\alpha(t) = M_0^\alpha N_1$$

$$+M_0^\alpha N_2 \left[ \frac{1}{\Gamma(\alpha)} \int_0^t (\xi(t) + C_0(t - s)^{\alpha - 1}\beta_2(s))(1 + m^\alpha(s)) \, ds + \sum_{i=1}^n d_i \right],$$

for $0 < \alpha < 1$.

EC6.1. $I_i : \Lambda \to \Lambda$ is continuous. There exists a constant $d_i > 0$, $i = 1, \ldots, n$ such that

$$\|I_i(\theta(t_i))\| \leq \sum_{i=1}^n d_i, \quad i = 1, \ldots, n.$$  \hspace{1cm} (13)

For any bounded subset $D \subset \Lambda$, there is a constant $l_i > 0$ such that

$$\mu(I_i(D)) \leq \sum_{i=1}^n l_i \mu(D), \quad i = 1, \ldots, n.$$  \hspace{1cm} (14)

The first result that we will discuss, is Theorem 3.2 with some conditions as previously presented. In this sense, the other results that we will discuss in this section, we use the conditions EC$_i$ for $i = 1, \ldots, 6$ and impose additional ones.

**Theorem 3.2.** Assumptions EC$_i$ for $i = 1, \ldots, 6$, hold then the impulsive nonlocal problem Eq.(1) has at least one mild solution.

**Proof.** Let $m^\alpha(t)$ be a solution of the scalar equation

$$m^\alpha(t) = M_0^\alpha N_0$$

$$+M_0^\alpha \left[ \frac{1}{\Gamma(\alpha)} \int_0^t (\xi(t) + C_0(t - s)^{\alpha - 1}\beta_2(s))(1 + m^\alpha(s)) \, ds + \sum_{i=1}^n d_i \right]$$

and assume that the finite bound of $\int_0^t \beta_1(s) \, ds$ is $C_0$ for $t \in J$. 


Consider the map $\Theta : \mathcal{PC}(J, \Lambda) \to \mathcal{PC}(J, \Lambda)$ defined by

$$
\Theta(\theta)(t) = R_{(a,\theta)}(t,0)[\theta_0 - \Xi(\theta)] + \sum_{0 < t_i < t} R_{(a,\theta)}(t,t_i)I_i(\theta(t_i))
+ \int_0^t R_{(a,\theta)}(t,s) \left(\Phi(s,\theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-x)^{\alpha-1}g(s,x,\theta(x))dx\right) ds,
$$

for all $\theta \in \mathcal{PC}(J, \Lambda)$ and $W_0 = \{ \theta \in \mathcal{PC}(J, \Lambda), \|\theta(t)\| \leq m^\alpha(t), \text{ for all } t \in J \}$. Then, $W_0 \subseteq \mathcal{PC}(J, \Lambda)$ is bounded and convex.

On the other hand, we define $W_1 = \overline{conv} \Theta(W_0)$, where $\overline{conv}$ means the closure of the convex hull in $\mathcal{PC}(J, \Lambda)$ is bounded due to Lemma 2.6, using the assumptions, $W_1 \subseteq \mathcal{PC}(J, \Lambda)$ is bounded closed convex nonempty and equicontinuous on $J$.

Now, for $\theta \in \Theta(W_0)$, yields

$$
\|\theta(t)\| \leq \|R_{(a,\theta)}(t,0)\Xi(\theta)\| + \sum_{0 < t_i < t} \|R_{(a,\theta)}(t,t_i)I_i(\theta(t_i))\|
+ \int_0^t \left\|R_{(a,\theta)}(t,s) \left(\Phi(s,\theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-x)^{\alpha-1}g(s,x,\theta(x))dx\right)\right\| ds
\leq \|R_{(a,\theta)}(t,0)\| \|\Xi(\theta)\| + \sum_{0 < t_i < t} \|R_{(a,\theta)}(t,t_i)\| \|I_i(\theta(t_i))\|
+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^s \left\|R_{(a,\theta)}(t,s)\right\| (s-x)^{\alpha-1} \|g(s,x,\theta(x))\| dx ds
+ \int_0^t \left\|R_{(a,\theta)}(t,s)\right\| \|\Phi(s,\theta(s))\| ds.
$$

Using the conditions $\text{EC}_{2.1}$, $\text{EC}_{2.2}$, $\text{EC}_{3.2}$, $\text{EC}_{4.2}$ and $\text{EC}_{6.1}$, we obtain

$$
\|\theta(t)\|
\leq M_0^\alpha N_0 + M_0^\alpha \int_0^t \xi(s)(1 + m^\alpha(s))ds
+ M_0^\alpha C_0 \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \beta_2(s)(1 + \|\theta(s)\|)ds + M_0^\alpha \sum_{i=1}^n d_i
\leq M_0^\alpha N_0 + M_0^\alpha \left(\frac{1}{\Gamma(\alpha)} \int_0^t (\xi(s) + C_0(t-s)^{\alpha-1}\beta_2(s))(1 + m^\alpha(s))ds + \sum_{i=1}^n d_i\right)
= m^\alpha(t).
$$

It follows that $W_1 \subseteq W_0$. We define $W_{n+1} = \overline{conv} \Theta(W_n)$, for $n = 1, 2, 3, \ldots$.

From above we know that $\{W_n\}_{n=1}^\infty$ is a decreasing sequence of bounded, closed, convex, equicontinuous on $J$ and nonempty subsets in $\mathcal{PC}(J, \Lambda)$.

Now for $n \geq 1$ and $t \in J$, $W_n(t)$ and $\Theta(W_n(t))$ are bounded subsets of $\Lambda$, hence, for any $\varepsilon > 0$, there is a sequence $\{\theta_k\}_{k=1}^\infty \subseteq W_n$, using the Lemma 2.5, Lemma
2.6, Lemma 2.7 and Lemma 2.8, such that
\[
\mu(\mathbb{W}_{n+1}(t)) = \mu(\Theta(\mathbb{W}_n(t))) \\
\leq 2\mu \left( \int_0^t \mathcal{R}(\alpha, \theta_k)(t, s)\Phi(s, \{\theta_k\}_k=1)ds \right) + 2\mu \left( \int_0^t \mathcal{R}(\alpha, \theta_k)(t, s) \frac{1}{\Gamma(\alpha)}(s-x)^{\alpha-1}g(s, x, \{\theta_k(x)\}_k=1)dxds \right) + \varepsilon
\]
\[
\leq 4M_0^3 \int_0^t k_1(s)\mu(\{\theta_k\}_k=1)ds + 4M_0^3 \sum_{i=1}^\infty l_i\mu(\{\theta_k\}_k=1) + \varepsilon
\]
\[
\leq 4M_0^3 \left( \int_0^t k_1(s)\mu(\mathbb{W}_n(s))ds + \frac{2G_0}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}k_3(s)\mu(\mathbb{W}_n(s))ds \right) + \sum_{i=1}^\infty l_i\mu(\mathbb{W}_n(t_i)) + \varepsilon.
\]
(17)

Since \( \varepsilon > 0 \) is arbitrary, from inequality (17), we get
\[
\mu(\mathbb{W}_{n+1}(t)) \leq 4M_0^3 \left( \int_0^t k_1(s)\mu(\mathbb{W}_n(s))ds + \frac{2G_0}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}k_3(s)\mu(\mathbb{W}_n(s))ds + \sum_{i=1}^\infty l_i\mu(\mathbb{W}_n(t_i)) \right) \quad (18)
\]
for all \( t \in J \) and \( 0 < \alpha \leq 1 \). Note that \( w_n \) is decreasing for \( n \), yields
\[
\mathbf{H}(t) = \lim_{n \to \infty} \mu(\mathbb{W}_n(t)) \quad (19)
\]
for all \( t \in J \). So, by means of the inequality (18) and Eq.(19), we have
\[
\mathbf{H}(t) = \lim_{n \to \infty} \mu(\mathbb{W}_n(t)) \leq 4M_0^3 \left( \int_0^t \left( k_1(s) + \frac{2G_0k_3(s)}{\Gamma(\alpha)} \right)\mathbf{H}(s)ds + \sum_{i=1}^n l_i\mathbf{H}(t_i) \right) \quad (20)
\]
for \( t \in J \), which implies that \( \mathbf{H}(t) = 0 \) for all \( t_i \in J \).

Using Lemma 2.6, we have \( \lim_{n \to \infty} \mu(\mathbb{W}_n(t)) = 0 \). On the other hand, by means of Lemma 2.3, we have \( \mathbb{W} = \bigcap_{n=1}^\infty \mathbb{W}_n \) is convex compact and nonempty in \( \mathcal{PC}(J, \Lambda) \) and \( \Theta(\mathbb{W}) \subset \mathbb{W} \). Finally, using the Schauder fixed point theorem, there exist at least one mild solution \( u \) of the initial value problem Eq.(1), where \( \theta \in \mathbb{W} \) is a fixed point of the continuous map \( \Theta \).

We finished the proof of the Theorem 3.2 (existence) assuming the conditions \( \mathbf{EC}_1 \) - \( \mathbf{EC}_6 \). So, we have the following question: If we remove the conditions \( \mathbf{EC}_3 \) - \( \mathbf{EC}_6 \) and impose other conditions, it is possible to obtain solutions for the Eq.(1). A priory, the answer is yes. Note that if the functions \( \Phi, g \) and \( I_i \) are compact or
Lipschitz continuous, then $\text{EC}_3 - \text{EC}_5$ are automatically satisfied (see for example [10]). On the other hand, in the references highlighted in the introduction and the above results, note that we assume that $\Xi$ is an application uniformly bounded. Indeed, if $\Xi$ is compact, then it should be bounded in a bounded set. Therefore, we do not need $\Xi$ to be uniformly bounded, to discuss the existence of solutions for Eq.(1), only to impose growth conditions for $\Phi$, $g$ and $f$. In this sense, we will impose two new conditions and consequently discuss through Theorem 3.3, another result of existence.

1. **EC$_{7.1}$**. There exists a function $p \in \mathcal{L}(J, \mathbb{R}^+)$ and an increasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|\Phi(t, \theta)\| \leq p(t)\phi(\|\theta\|)$$

for a.e. $t \in J$ and for all $u \in \mathcal{PC}(J, \Lambda)$.

2. **EC$_{8.1}$**. There exists two functions $q \in \mathcal{L}(J, \mathbb{R}^+)$ and $\hat{q} \in \mathcal{L}(J, \mathbb{R}^+)$ and an increasing function $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|g(t, s, \theta)\| \leq q(t)\hat{q}(s)\chi(\|\theta\|),$$

for a.e. $t \in J$ and for all $\theta \in \mathcal{PC}(J, \Lambda)$. Assume that the finite bound of

$$\int_0^t g(s)ds$$

is $G_1$.

**Theorem 3.3.** Suppose that the EC$_1$-EC$_2$ and EC$_{7}$-EC$_8$ are satisfied, then the Eq.(1) has at least one mild solution if

$$\lim_{n \to \infty} \sup \frac{M_0^\alpha}{r} \left( \varphi(r) + \phi(r) \int_0^t p(s)ds + \frac{G_1\chi(r)}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}\hat{q}(s)ds + \sum_{i=1}^n d_i \right) < 1$$

where $\varphi(r) = \sup\{\|\Xi(\theta)\|, \|\theta\| < r\}$.

**Proof.** By means of the inequality (21) implies that there exists a constant $r > 0$ such that

$$M_0^\alpha \left[ \varphi(r) + \phi(r) \int_0^t p(s)ds + \frac{G_1\chi(r)}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}\hat{q}(s)ds + \sum_{i=1}^n d_i \right] < r.$$  

(22)

Now, we consider the following set $\mathcal{W}_0 = \{\theta \in \mathcal{PC}(J, \Lambda), \|\theta\| \leq r\}$ and $\mathcal{W}_1 = \sup\text{Θ}(\mathcal{W}_0)$. Then for any $\theta \in \mathcal{W}_1$, and using the conditions EC$_1$-EC$_2$ and EC$_{7}$-EC$_8$, yields

$$\|\theta(t)\| \leq \|\mathcal{R}_{(\alpha, \theta)}(t, 0)\|\|\Xi(\theta)\| + \int_0^t \|\mathcal{R}_{(\alpha, \theta)}(t, s)\|\|\Phi(s, \theta(s))\| ds$$

$$\quad + \sum_{0 < t_i < t} \|\mathcal{R}_{(\alpha, \theta)}(t, t_i)\|\|I_i(\theta(t_i))\|$$

$$\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^s \|\mathcal{R}_{(\alpha, \theta)}(t, s)\|(s - x)^{\alpha-1}\|g(x, s, \theta(x))\| dx ds$$
\[ M_0^2 \varphi(r) + M_0^3 \int_0^t p(s)\phi(\|\theta(s)\|)ds + M_0^5 \sum_{i=1}^n d_i \]
\[ + \frac{M_0^2 G_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\hat{q}(s)\chi(\|\theta(s)\|)ds \]
\[ \leq M_0^2 \varphi(r) + M_0^2 \phi(r) \int_0^t p(s)ds + M_0^5 \sum_{i=1}^n d_i \]
\[ + \frac{M_0^2 G_1 \chi(r)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\hat{q}(s)ds. \]

In this sense, we have
\[ \|\theta(t)\| \leq M_0^2 \left[ \varphi(r) + \phi(r) \int_0^t p(s)ds + \frac{G_1 \chi(r)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\hat{q}(s)ds + \sum_{i=1}^n d_i \right] < r \]
for \( t \in J \). It means that \( \mathbb{W}_1 \subset \mathbb{W}_2 \). So we can complete the proof similarly to Theorem 3.2.

Now let’s investigate the existence of the mild solution for the function \( \Xi \) being Lipschitz, however \( \Phi, g \) and \( I_i \) are not Lipschitz. First, let’s admit the following condition:

1. \( \text{EC}_{9,1} \). The function \( \Xi \) is Lipschitz continuous in \( \Lambda \), there exists a constant \( L_0 > 0 \) such that
\[ ||\Xi(\theta) - \Xi(\nu)|| \leq L_0 ||\theta - \nu||, \theta, \nu \in \mathcal{P}C(J, \Lambda). \]

**Theorem 3.4.** Let \( 0 < \alpha < 1 \). Suppose that the assumptions \( \text{EC}_i \) for \( i = 1, ..., 9 \) are satisfied, then Eq.(1) has at least one mild solution provided that
\[ M_0^2 \left[ \left( L_0 + 4 \int_0^t \left( k_1(s) + \frac{2G_0k_3(s)(t-s)^{\alpha-1}}{\Gamma(\alpha)} \right)ds + \sum_{i=1}^n l_i \right) \right] < 1. \]

**Proof.** For the proof, we consider the map \( \Theta : \mathcal{P}C(J, \Lambda) \to \mathcal{P}C(J, \Lambda) \) defined by \( \Theta = \Theta_1 + \Theta_2 \), where
\[ (\Theta_1 \theta)(t) = \mathcal{R}_{(\alpha, \theta)}(t,0) \Xi(\theta) \]
and
\[ (\Theta_2 \theta)(t) = \int_0^t \mathcal{R}_{(\alpha, \theta)}(t,s)\Phi \left( (s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-x)^{\alpha-1}g(s,x,\theta(x))dx \right)ds \]
\[ + \sum_{0 < t_i < t} \mathcal{R}_{(\alpha, \theta)}(t, t_i)I_i(\theta(t_i)) \]
for \( \theta \in \mathcal{P}C(J, \Lambda) \).

As introduced in the proof of Theorem 3.2, we consider \( \mathbb{W}_0 = \{ \theta \in \mathcal{P}C(J, \Lambda); ||\theta(t)|| \leq m^*(t), \text{for all } t \in J \} \) and let \( \mathbb{W} = \overline{\text{conv}}(\mathbb{W}_0) \). Then from the proof of the Theorem 3.2 we know that \( \mathbb{W} \) is a bounded, closed, convex and equicontinuous subset of \( \mathcal{P}C(J, \Lambda) \) and \( \Theta(\mathbb{W}) \subset \mathbb{W} \). We shall prove that \( \Theta \) is \( \mu_c \)-contraction on \( \mathbb{W} \). Then Darbo-Sadovskii's fixed point theorem can be used to get a fixed point of \( \Theta \) in \( \mathbb{W} \), which is a mild solution of the Eq.(1). First, for every bounded subset \( B \subset \mathbb{W} \),
from Eq.(1) and by Lemma 2.3, we have
\[
\mu_c(\Theta_2(B)) = \mu_c\left(\mathcal{R}_{(\alpha,B)}(t,0) \Xi(B)\right)
\leq M_0^c \mu_c(\Xi(B))
\leq M_0^c L_0 \mu_c(B). \tag{23}
\]

For every bounded subset \( B \subset \mathcal{W} \), for \( t \in J \) and every \( \varepsilon > 0 \), there exists the sequence \( \{\theta_k\}_{k=1}^\infty \subset B \), such that
\[
\mu(\Theta_2(B(t))) \leq 2\mu(\{\Theta_2(\theta_k(t))\}_{k=1}^\infty) + \varepsilon. \tag{24}
\]

Note that \( B \) and \( \Theta_2(B) \) are equicontinuous. So, by means of Lemmas 2.3, 2.5, 2.7, 2.8 and using the conditions EC, for \( i = 1, \ldots, 9 \), yields
\[
\mu(\Theta_2(\theta_k(t))) \leq 2\mu\left(\int_0^t \mathcal{R}_{(\alpha,\theta)}(t,s)\Phi(s,\{\theta_k(s)\}_{k=1}^\infty) ds\right)
+ 2\mu\left(\sum_{0 < \xi < t} \mathcal{R}_{(\alpha,\theta)}(t,\xi)I_i(\{\theta_k(\xi)\}_{k=1}^\infty)\right)
+ \sum_{0 < \xi < t} \mu(\{\theta_k(\xi)\}_{k=1}^\infty)
+ 2\mu\left(\int_0^t \mathcal{R}_{(\alpha,\theta)}(t,s)(x-s)^{\alpha-1}g(s, x, \{\theta_k(x)\}_{k=1}^\infty) ds\right)
\leq 4M_0^c \int_0^t k_1(s)\mu(\{\theta_k(s)\}_{k=1}^\infty) ds + 4M_0^c \sum_{0 < \xi < t} l_i \mu(\{\theta_k(\xi)\}_{k=1}^\infty)
+ \sum_{0 < \xi < t} \mu(\{\theta_k(\xi)\}_{k=1}^\infty)
\leq 4M_0^c \left(\int_0^t k_1(s)\mu(B) ds + \frac{2G_0}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}k_3(s) \mu(B) ds + \sum_{i=1}^n l_i \mu(B)\right) + \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, using above inequality \( (\mu_c(\mathcal{W}) = \sup\{\mu(\mathcal{W}(t)), t \in J\}) \), we have
\[
\mu_c(\Theta_2(\theta_k(t))) \leq 4M_0^c \left[\int_0^t \left( k_1(s) + \frac{2G_0}{\Gamma(\alpha)} (t-s)^{\alpha-1}k_3(s) \right) ds + \sum_{i=1}^n l_i\right] \mu_c(B) \tag{25}
\]
for any bounded \( B \subset \mathcal{W} \).

Now, for any subset \( B \subset \mathcal{W} \), using Lemma 2.4 and inequalities (23) and (25), we obtain
\[
\mu_c(\Theta(B)) = \mu_c(\Theta_1(B) + \Theta_2(B)) \leq \mu_c(\Theta_1(B)) + \mu_c(\Theta_2(B))
\leq M_0^c L_0 \mu_c(B) + 4M_0^c \left(\int_0^t \left( k_1(s) + \frac{2G_0}{\Gamma(\alpha)} (t-s)^{\alpha-1}k_3(s) \right) ds + \sum_{i=1}^n l_i\right) \mu_c(B)
= M_0^c \left( L_0 + 4 \int_0^t \left( \frac{2k_1(s)G_0}{\Gamma(\alpha)} (t-s)^{\alpha-1}k_3(s) \right) ds + \sum_{i=1}^n l_i\right) \mu_c(B). \tag{26}
\]

So, from Eq.(26), we obtain \( \Theta \) a \( \mu_c \)-contraction on \( \mathcal{W} \). Therefore, by Lemma 2.4, there is a fixed point \( \theta \) of \( \Theta \) in \( \mathcal{W} \), which is a solution of Eq.(1).

Finally, the next Theorem 3.5, also aims to discuss the existence of mild solution using the conditions EC, for \( i = 1, \ldots, 9 \).
Theorem 3.5. Suppose that the assumption EC\textsubscript{i} for \( i = 1, \ldots, 9 \) are satisfied, then Eq.(1) has at least one mild solution if Eq.(26) and the following condition are satisfied,

\[
M_0 r \limsup_{n \to \infty} \sup_{s \leq t} \frac{\alpha(t)}{r} \left( \phi(r) \int_0^t p(s) ds + \frac{\chi(r)G_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{q}(s) ds + \sum_{i=1}^n d_i \right) < 1.
\]  

Proof. Using the condition (27) and the fact that \( L_0 < 1 \), there exists a constant \( r > 0 \) such that

\[
M_0 r + \| \Xi(0) \| + \phi(r) \int_0^t p(s) ds + \frac{\chi(r)G_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{q}(s) ds + \sum_{i=1}^n d_i < r.
\]

Now, we consider \( \mathcal{W}_0 = \{ \theta \in \mathcal{PC}(J, \Lambda) \mid \| \theta(t) \| \leq r, \text{ for all } t \in J \} \). Then for every \( \theta \in \mathcal{W}_0 \) and using the condition EC\textsubscript{9}, yields

\[
\| \Theta \theta(t) \| \\ \leq \left\| [\mathcal{R}_{(\alpha, \theta)}(t, 0) \Xi(\theta)] + \sum_{0 < \eta_i < t} \mathcal{R}_{(\alpha, \theta)}(t, \eta_i) I_i(\theta(t_i)) \right\| \\ + \left\| \int_0^t \mathcal{R}_{(\alpha, \theta)}(t, s) \left( \Phi(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-x)^{\alpha-1} g(s, x, \theta(x)) dx \right) ds \right\| \\ \leq M_0^2 \| \Xi(\theta) \| + M_0 \int_0^t p(s) \phi(\| \theta(s) \|) ds \\ + \frac{M_0^2}{\Gamma(\alpha)} \int_0^t \int_0^s (s-x)^{\alpha-1} q(s) \hat{q}(x) \chi(\| \theta(x) \|) dxds + M_0 \sum_{i=1}^n d_i \\ \leq M_0 \left( L_0 r + \| \Xi(0) \| + \int_0^t p(s) \phi(\| \theta(s) \|) ds \\ + \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^s (s-x)^{\alpha-1} q(s) \hat{q}(x) \chi(\| \theta(x) \|) dxds + \sum_{i=1}^n d_i \right) \\ \leq M_0^2 \left( L_0 r + \| \Xi(0) \| + \phi(r) \int_0^t p(s) ds + \frac{\chi(r)G_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{q}(s) ds \\ + \sum_{i=1}^n d_i \right). \tag{28}
\]

Thus, we have

\[
\| \Theta \theta(t) \| \leq M_0^2 \left( L_0 r + \| \Xi(0) \| + \phi(r) \int_0^t p(s) ds + \frac{\chi(r)G_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{q}(s) ds + \sum_{i=1}^n d_i \right) < r,
\]

for all \( t \in J \). This means that \( \Theta(\mathcal{W}_0) \subset \mathcal{W}_0 \).

Define \( \mathcal{W} = \text{conv} \Theta(\mathcal{W}_0) \). The above proof also implies that \( \Theta(\mathcal{W}) \subset \mathcal{W} \). So we can prove the theorem similar with Theorem 3.4 and hence we omit it. \( \square \)

Noting this section is dedicated to discussing existence of a mild solution for Eq.(1), discussed via four theorems. We note the importance of each condition in its respective result. In addition, we also highlight that, from changes in initial
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conditions $EC_1$-$EC_6$, we can still get the desired result. On the other hand, one of the direct consequences of fractional calculus, in particular, the discussion of fractional differential equations, is to discuss the special case $\alpha = 1$. So, in this sense, we will present the results for the entire case, for the problem Eq.(1). Then, taking the limit $\alpha \to 1$ in the Eq.(1), we get

$$
\begin{cases}
\theta'(t) + A(t, \theta(t))\theta(t) = \Phi(t, \theta(t)) + \int_0^t g(t, s, \theta(s))ds, \ t \in J, t \neq t_i \\
\theta(0) + \Xi(\theta) = \theta_0 \\
\Delta \theta(t_i) = I_i(\theta(t_i)), \ i = 1, ..., n, 0 < t_1 < ... < t_n < b
\end{cases}.
$$

Consequently, taking the limit $\alpha \to 1$ in Eq.(9), we have the mild solution for Eq.(29), given by

$$
\theta(t) = U_\theta(t, 0)[\theta_0 - \Xi(\theta)] + \int_0^t U_\theta(t, s) \left(\Phi(s, \theta(s)) + \int_0^s g(s, x, \theta(x))dx\right)ds
+ \sum_{0 < t_i < t} U_\theta(t, t_i)I_i(\theta(t_i))
$$

$0 \leq t \leq b$.

The following results are direct consequences of the existence results previously investigated. In this sense, their respective statements are omitted.

**Theorem 3.6.** Assumptions $EC_i$ for $i = 1, ..., 6$ hold, then the impulsive nonlocal problem Eq.(29) has at least one mild solution.

**Proof.** Direct consequence of Theorem 3.2. □

**Theorem 3.7.** Suppose that assumptions $EC_1$-$EC_2$ and $EC_7$-$EC_8$, are satisfied, then Eq.(29) has at least one mild solution if

$$
\lim_{r \to \infty} \sup \frac{M_0}{r} \left(\varphi(r) + \phi(r) \int_0^t p(s)ds + G_1 \chi(r) \int_0^t \hat{q}(s)ds + \sum_{i=1}^n d_i\right) < 1,
$$

where $\varphi(r) = \sup\{\|\Xi(\theta)\|, \|\theta\| < r\}$.

**Proof.** Direct consequence of Theorem 3.3. □

**Theorem 3.8.** Suppose that $EC_i$ for $i = 1, ..., 9$, are satisfied, then Eq.(29) has at least one mild solution provided that

$$
M_0 \left(L_0 + 4 \int_0^t (k_1(s) + 2G_0k_3(s)) ds + \sum_{i=1}^n l_i\right) < 1.
$$

**Proof.** Direct consequence of Theorem 3.4. □

**Theorem 3.9.** Suppose that $EC_i$ for $i = 1, ..., 9$, are satisfied, then Eq.(29) has at least one mild solution if Eq.(26) (with $\alpha \to 1$) and the following condition is satisfied

$$
M_0 L_0 + \lim_{r \to \infty} \frac{M_0}{r} \left(\varphi(r) \int_0^t p(s)ds + \chi(r)G_1 \int_0^t \hat{q}(s)ds + \sum_{i=1}^n d_i\right) < 1.
$$

**Proof.** Direct consequence of Theorem 3.5. □
We conclude the section on existence of mild solutions for quasi-linear fractional integro-differential equations with impulsive and nonlocal conditions given by Eq. (1), discussing some necessary and sufficient conditions. In addition, as a particular case, we discuss the entire case. In this sense, the next section is intended to investigate the uniqueness of mild solutions for Eq. (1).

4. The uniqueness of mild solution. In this section, we discuss the uniqueness of mild solutions for Eq. (1), using the fixed point technique. However, first, we need some conditions before we attack the main purpose of this section. So, consider:

1. **UC$_1$.** $\Phi : J \times \Lambda \to \Lambda$ is continuous and there exists constant $\lambda_A > 0$, and $\lambda_0 > 0$ such that
   \[
   \|\Phi(t, \theta) - \Phi(t, \nu)\| \leq \lambda_A \left(\|\theta - \nu\|\right), \quad \theta, \nu \in \Lambda \text{ and } \lambda_0 = \max_{t \in J} \|\Phi(t, 0)\|.
   \]

2. **UC$_2$.** $g : J \times \Omega \to \Lambda$ is continuous and there exists constant $H_A > 0$, and $H_0 > 0$ such that
   \[
   \int_0^t \|g(t, s, \theta) - g(t, s, \nu)\| ds \leq H_A \|\theta - \nu\|, \quad H_0 = \max \left\{ \int_0^t \|g(t, s, 0)\| ds; \; t \in J \right\}.
   \]

3. **UC$_3$.** $\Xi : \mathcal{PC}(J, \Lambda) \to \Omega$ is Lipschitz continuous in $\Lambda$ and there exists a constant $G_A > 0$ such that
   \[
   \|\Xi(\theta) - \Xi(\nu)\| \leq G_A \|\theta - \nu\|, \quad \theta, \nu \in \mathcal{PC}(J, \Lambda).
   \]

4. **UC$_4$.** $I_i : \Omega \to \Omega$ is continuous and there exists a constant $l_i > 0$, $i = 1, \ldots, n$ such that
   \[
   \|I_i(\theta) - I_i(\nu)\| \leq l_i \|\theta - \nu\|, \quad \theta, \nu \in \Lambda.
   \]

5. **UC$_5$.** Let
   \[
   \rho = \left\{ k_0 b r + 2 k_0 b (G_A r + \|g(0)\|) + M_0^a \lambda_A b + \frac{M_0^a}{\Gamma(\alpha)} H_A b + k_0 b r \sum_{i=1}^n l_i + M_0^a \sum_{i=1}^n l_i \right\}
   \]
   be such that $0 < \rho < 1$ and $b > 0$ a constant.

   Further there exists a constant $k_0$ such that for every $\theta, \nu \in \mathcal{PC}(J, \Lambda)$ and $y \in \Lambda$ we have
   \[
   \|R_{(\alpha, \theta)}(t, s)y - R_{(\alpha, \nu)}(t, s)y\| \leq k_0 b \|y\|, \|\theta - \nu\| \leq \rho \|y\| \Lambda
   \]
   for $0 < \alpha < 1$.

**Theorem 4.1.** Let $\theta_0 \in \Lambda$ and let $B_r = \left\{ \theta \in \mathcal{PC}(J, \Lambda); \; \|\theta\| \leq r \right\}$, $r > 0$. If the assumptions **UC$_1$-UC$_5$** are satisfied, then Eq. (1) has a unique mild solution.

**Proof.** Consider $\theta_0 \in \Lambda$ (fixed) and the operator $\Theta$ on $\mathcal{PC}(J, \Lambda)$, given by
   \[
   (\Theta \theta)(t) = R_{(\alpha, \theta)}(t, 0)[\theta_0 - \Xi(\theta)] + \sum_{0 < t_i < t} R_{(\alpha, \theta)}(t, t_i) I_i(\theta(t_i))
   \]

   \[
   + \int_0^t R_{(\alpha, \theta)}(t, s) \left( \Phi(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s - x)^{\alpha - 1} g(s, \theta(x)) dx \right) ds.
   \]
Note that $\Theta : \mathcal{PC}(J, \Lambda) \longrightarrow \mathcal{PC}(J, \Lambda)$. Also, we have

$$
\| (\Theta \theta)(t) - (\Theta \nu)(t) \|
\leq \| (R_{\alpha, \theta}(t, 0) - R_{\alpha, \nu}(t, 0)) \theta_t \| + \| R_{\alpha, \theta}(t, 0) \Xi(\theta) - R_{\alpha, \nu}(t, 0) \Xi(\theta) \|
+ \int_0^t \| R_{\alpha, \theta}(t, s) \left( \Phi(s, \theta(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-x)^{\alpha-1} g(s, x, \theta(x)) \, dx \right) \| \, ds
- \int_0^t \| R_{\alpha, \nu}(t, s) \left( \Phi(s, \nu(s)) + \frac{1}{\Gamma(\alpha)} \int_0^s (s-x)^{\alpha-1} g(s, x, \nu(x)) \, dx \right) \| \, ds
+ \sum_{0 < t_i < t} \| R_{\alpha, \theta}(t, t_i) I_i(\theta(t_i)) \| - \sum_{0 < t_i < t} \| R_{\alpha, \nu}(t, t_i) I_i(\nu(t_i)) \|.
$$

Using

$$
\| R_{\alpha, \theta}(t, s) \| \leq M_0^\alpha, \| g(\theta) \| \leq G_A\| \theta \| \mathcal{P}_C
$$

and

$$
\| R_{\alpha, \theta}(t, s)y - R_{\alpha, \nu}(t, s)y \| \leq k_0 \| y \|_\Lambda \| \theta - \nu \| \mathcal{P}_C
$$

with $k_0$ a constant, yields

$$
\| R_{\alpha, \nu}(t, 0)g(\nu) - R_{\alpha, \theta}(t, 0)g(\theta) \| \leq \| R_{\alpha, \nu}(t, 0) \| \| g(\nu) \| + \| R_{\alpha, \theta}(t, 0) \| \| g(\theta) \|
\leq M_0^\alpha \| g(\nu) \| + M_0^\alpha \| g(\theta) \|
\leq M_0^\alpha G_A \| \theta - \nu \|.
$$

On the other hand, we obtain

$$
\| R_{\alpha, \theta}(t, 0)g(\theta) - R_{\alpha, \nu}(t, 0)g(\theta) \| \leq k_0 \| g(\theta) \| \| \theta - \nu \|
\leq k_0 \| g(\theta) \| G_A \| \theta - \nu \|.
$$

Using Eq.(31)-Eq.(32), yields

$$
M_0^\alpha G_A \left( \| \theta \| + \| \nu \| \right) \leq k_0 G_A \| \theta \| \left( \| \theta \| + \| \nu \| \right).
$$

Then we conclude that

$$
\| R_{\alpha, \nu}(t, 0)g(\nu) - R_{\alpha, \theta}(t, 0)g(\theta) \| \leq \| R_{\alpha, \theta}(t, 0)g(\theta) - R_{\alpha, \nu}(t, 0)g(\nu) \|.
$$
Now, using the inequality (33), we obtain
\[
\begin{align*}
&\| R_{(\alpha, \theta)}(t, 0) g(\theta) - R_{(\alpha, \nu)}(t, 0) g(\nu) \| \\
&\leq \| R_{(\alpha, \theta)}(t, 0) g(\theta) - R_{(\alpha, \nu)}(t, 0) g(\nu) \| + \| R_{(\alpha, \nu)}(t, 0) g(\nu) - R_{(\alpha, \theta)}(t, 0) g(\theta) \| \\
&\quad + \| R_{(\alpha, \theta)}(t, 0) g(\theta) - R_{(\alpha, \nu)}(t, 0) g(\nu) \| + \| R_{(\alpha, \theta)}(t, 0) g(\theta) - R_{(\alpha, \nu)}(t, 0) g(\nu) \| \\
&\leq 2 \| R_{(\alpha, \theta)}(t, 0) g(\theta) - R_{(\alpha, \nu)}(t, 0) g(\nu) \| + 2 \| R_{(\alpha, \theta)}(t, 0) g(\theta) - R_{(\alpha, \nu)}(t, 0) g(\nu) \| \\
&\leq 2k_0b \| g(\theta) \|_A \| \theta - \nu \|_{PC} + 2k_0b \| g(0) \|_A \| \theta - \nu \|_{PC} \\
&= 2k_0b (G_A \| \theta \| + \| g(0) \|) \| \theta - \nu \|_{PC}.
\end{align*}
\]
Now, returning to the inequality (30), we have
\[
\begin{align*}
&\| (\Theta \theta)(t) - (\Theta \nu)(t) \| \\
&\leq k_0 b \| \theta \| \| \theta - \nu \|_{PC} + 2k_0b (G_A \| \theta \| + \| g(0) \|) \| \theta - \nu \|_{PC} \\
&\quad + M_5^* \int_0^t (\| \Phi(s, \theta(s)) - \Phi(s, 0) \| + \| \Phi(s, 0) \|) \, ds \\
&\quad - M_5^* \int_0^t \int_0^s (s-x)^{\alpha-1} \left( \| g(s, x, \theta(x)) - g(s, 0, 0) \| + g(s, 0, 0) \right) \, dx \, ds \\
&\quad - M_5^* \int_0^t (\| \Phi(s, \theta(s)) - \Phi(s, 0) \| + \| \Phi(s, 0) \|) \, ds \\
&\quad - M_5^* \int_0^t \int_0^s (s-x)^{\alpha-1} \left( \| g(s, x, \nu(x)) - g(s, 0, 0) \| + g(s, 0, 0) \right) \, dx \, ds \\
&\quad + \sum_{i=1}^{n} k_0 b \| \theta - \nu \|_{PC} l_i \| \theta(t_i) \| + M_5^* \sum_{i=1}^{n} l_i \| \theta - \nu \|_{PC} \\
&\leq k_0 b \| \theta \| \| \theta - \nu \|_{PC} + 2k_0b (G_A \| \theta \| + \| g(0) \|) \| \theta - \nu \|_{PC} \\
&\quad + M_5^* \lambda_A \int_0^t (\| \theta(s) \| - \| \nu(s) \|) \, ds + M_5^* \int_0^t (\| \theta \| - \| \nu \|) \, ds \\
&\quad + \sum_{i=1}^{n} k_0 b \| \theta - \nu \|_{PC} l_i \| \theta(t_i) \| + M_5^* \sum_{i=1}^{n} l_i \| \theta - \nu \|_{PC} \\
&\leq k_0 b r \| \theta - \nu \|_{PC} + 2k_0b (G_A r + \| g(0) \|) \| \theta - \nu \|_{PC} \\
&\quad + M_5^* \lambda_A \int_0^t \| \theta - \nu \|_{PC} \, ds + M_5^* \int_0^t \| \theta \| - \| \nu \| \, ds \\
&\quad + \sum_{i=1}^{n} k_0 b \| \theta - \nu \|_{PC} l_i \, r + M_5^* \sum_{i=1}^{n} l_i \| \theta - \nu \|_{PC} \\
&= \left[ k_0 b r + 2k_0b (G_A r + \| g(0) \|) + M_5^* \lambda_A b \right] \| \theta - \nu \|_{PC} \\
&\quad + M_5^* \int_0^t \| \theta - \nu \|_{PC} \, ds \\
&= \rho \| \theta - \nu \|_{PC}.
\end{align*}
\]
From this inequality it follows that for any $t \in J$, we have
\[
\| (\Theta \theta)(t) - (\Theta \nu)(t) \| \leq \rho \| \theta - \nu \|_{PC}.
\]
As presented in the previous section, here we also have the special case, when we choose $\alpha = 1$, given by the following Theorem.

**Theorem 4.2.** Let $\theta_0 \in \Lambda$ and let $B_r = \{ \theta \in \mathcal{PC}(J, \Lambda); \|\theta\| \leq r \}$, $r > 0$. If the assumptions $UC_1-UC_5$ are satisfied, then Eq.(1) has a unique mild solution.

**Proof.** Follows from Theorem 4.1. \qed

5. **Application.** In this section, we discuss the controllability of a fractional system Eq.(34), via Theorem 3.2. In addition, an example on the uniqueness of mild solutions via the Theorem 4.1, is presented.

So, let’s consider the system (1) with a control parameter such as

$$
\begin{cases}
\mathcal{D}^\alpha_{0+} \theta(t) + A(t, \theta(t)) \theta(t) = \Phi(t, \theta(t)) + Q \mu(t) + \int_0^t \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)} g(t, s, \theta(s)) ds \\
\theta(0) + \Xi(\theta) = \theta_0 \\
\Delta \theta(t_i) = I_i(\theta(t_i)), \ i = 1, ..., n, 0 < t_1 < ... < t_n < b
\end{cases}
$$

(34)

where $A, \Phi, \Xi, \theta$ are as before and $Q$ is a bounded linear operator from a Banach space $V$ into $\Lambda$ and $\mu \in L^2(J, V)$. The mild solution of the system Eq.(34), is given by a mild solution of Eq.(1) means a function $\theta \in \mathcal{PC}(J, \Lambda)$ with values in $\Omega$ satisfying the integral equation

$$
\theta(t) = R_{(\alpha, \theta)}(t, 0)[\theta_0 - \Xi(\theta)] + \sum_{0 < t_i < t} R_{(\alpha, \theta)}(t, t_i) I_i(\theta(t_i)) \\
+ \int_0^t R_{(\alpha, \theta)}(t, s) \left( \Phi(s, \theta(s)) + Q \mu(s) + \int_0^s \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)} g(s, x, \theta(x)) dx \right) ds
$$

t \in J \quad \text{for all } \theta_0 \in \Lambda \text{ and } 0 < \alpha < 1.

Let

$$
Z_{\delta} = \{ \theta : \theta \in \mathcal{PC}(J, \Lambda); \theta(0) + \Xi(\theta) = \theta_0, \Delta \theta(t_i) = I_i(\theta(t_i)), \ \|\theta\| \leq \delta \}
$$

for $t \in J$, $\delta > 0$, $\theta_0 \in \Lambda$ and $i = 1, ..., m$.

**Definition 5.1.** The system Eq.(34) is said to be controllable on the interval $J$ if every $\theta_0, \theta_1 \in \Lambda$, exists a control $\mu \in L^2(J, V)$ such that the mild solution $\theta(\cdot)$ of the system Eq.(34), satisfies

$$
\theta(0) + \Xi(\theta) = \theta_0 \text{ and } \theta(b) = \theta_1.
$$

To study the controllability, we need the following additional condition $EC_{10}$. The linear operator $W : L^2(J, V) \to \Lambda$, defined by

$$
W_{\mu} = \int_0^b R_{(\alpha, \theta)}(b, s) Q \mu(s) ds,
$$

induces an inverse operator $\widetilde{W}$ defined on $L^2(J, V) / \ker W$ and there exists a positive constant $M_1 > 0$ such that

$$
\left\| Q \widetilde{W}^{-1} \right\| \leq M_1.
$$

**Theorem 5.2.** If the conditions $EC_i$ with $i = 1, 2, ..., 10$ are satisfied, then the system Eq.(34) is controllable on $J$. 

Then we have the problem, given by

\[ \mu(t) = W^{-1} \left[ \theta_1 - R_{(\alpha, \theta)}(t, 0)[\theta_0 - \Xi(\theta)] - \sum_{0 < t_i < t} R_{(\alpha, \theta)}(t_i) I_i(\theta(t_i)) \right. \\
\left. - \int_0^t R_{(\alpha, \theta)}(t, s) \left( \Phi(s, \theta(s)) + \int_0^s (s-x)^{\alpha-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} g(s, x, \theta(x)) dx \right) ds \right]. \]

Proof. Using the condition for any arbitrary function \( \theta(\cdot) \) define the control

\[ (P \theta_\mu)(t) = R_{(\alpha, \theta)}(t, 0)[\theta_0 - \Xi(\theta)] + \sum_{0 < t_i < t} R_{(\alpha, \theta)}(t_i) I_i(\theta(t_i)) \]

\[ + \int_0^t R_{(\alpha, \theta)}(t, s) \left( \Phi(s, \theta(s)) + Q \mu(s) + \int_0^s (s-x)^{\alpha-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} g(s, x, \theta(x)) dx \right) ds \]

has a fixed point. This fixed point is, then a solution of the system Eq.(34). Clearly \( (P \theta_\mu)(b) = \theta_1 \), which means that the control \( \mu \) steers system Eq.(34). From the initial state \( \theta_0 \) to \( \theta_1 \), in time \( b \), provided we can obtain a fixed point of the nonlinear operator \( P \). The remaining part of the proof is similar to Theorem 3.2 and hence, it is omitted. Any fixed point of \( P \) is a mild solution of the system Eq.(34) on \( J \) which satisfies \( \theta(b) = \theta_1 \). Thus, system (5) is controllable on \( J \).

The next example discuss the uniqueness of solutions for the following fractional problem, given by

\[
\begin{aligned}
C^D_{0+}^\alpha \theta(t) &= \frac{1}{100} \sin \theta(t) \theta(t) + \frac{e^{-t} \theta(t)}{(49 + e^t)(1 + \theta(t))} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\frac{s}{4}} \theta(s) ds \\
\theta(0) + \frac{e^{-t}}{25} \theta(t) &= \theta_0 \\
\end{aligned}
\]

where \( 0 < \alpha \leq 1 \). Taking \( \Lambda = \mathbb{R}^+ \), \( t \in [0, 1] \) and so \( b = 1 \). Set \( A(t, \theta(t)) = \frac{1}{100} \sin \theta(t) \theta(t) \), \( \Xi(\theta) = e^{-t} \frac{\theta(t)}{25} \), \( \Phi(t, \theta(t)) = \frac{e^{-t} \theta(t)}{(49 + e^t)(1 + \theta(t))} \), \( g(t, s, \theta) = e^{-\frac{s}{4}} \theta(s) \), \( I_i(\theta) = \frac{\theta_i}{\alpha+\theta} \) and \( \theta(0) = 1 \).

Let \( \theta, \nu \in C(J, \Lambda) \) and \( J \), we will check the conditions to apply Theorem 4.1. Then we have

\[
\| \Phi(t, \theta) - \Phi(t, \nu) \| = \left\| \frac{e^{-t} \theta}{(49 + e^t)(1 + \theta)} - \frac{e^{-t} \nu}{(49 + e^t)(1 + \nu)} \right\| \\
= \frac{e^{-t}}{(49 + e^t)} \left\| \frac{\theta}{1 + \theta} - \frac{\nu}{1 + \nu} \right\| \\
\leq \frac{1}{50} \| \theta - \nu \| \\
= \lambda A \| \theta - \nu \| \ (\lambda A > 0). 
\]
Note that,
\[
\int_0^t ||g(t, s, \theta) - g(t, s, \nu)|| ds = \int_0^t \left| e^{-\frac{\theta(s)}{4}} - e^{-\frac{\nu(s)}{4}} \right| ds \\
\leq \frac{1}{4} ||\theta - \nu|| \\
= H_A ||\theta - \nu|| \ (H_A > 0).
\]

On the other hand,
\[
||\Xi(\theta) - \Xi(\nu)|| = \left| e^{-\frac{\theta}{25}} - e^{-\frac{\nu}{25}} \right| \\
\leq \frac{1}{25} ||\theta - \nu|| \\
= G_A ||\theta - \nu|| \ (G_A > 0)
\]
\[
e
\]
\[
||I_i(\theta) - I_i(\nu)|| = \left| \frac{\theta}{9 + \theta} - \frac{\nu}{9 + \nu} \right| \\
\leq \frac{1}{9} ||\theta - \nu|| \\
= l_i ||\theta - \nu|| \ (l_i > 0)
\]

So, we check the conditions UC_1 - UC_4. Finally, choosing \( r = 1, n = 1 \) and \( k_0 = \frac{1}{4} \), shall check that condition
\[
\rho = \left( \frac{1}{4} + \frac{1}{2} \left( \frac{1}{25} + 1 \right) + \frac{M_0^\alpha}{50} + \frac{M_0^\alpha}{\Gamma(\alpha)} \frac{1}{4} + \frac{1}{36} + \frac{M_0^\alpha}{9} \right) < 1
\]
implies \( \frac{359}{450} + M_0^\alpha \left( \frac{1}{50} + \frac{1}{9} + \frac{1}{4\Gamma(\alpha)} \right) < 1 \). Note that, we have conditions over \( \Gamma(\alpha) \) e \( M_0^\alpha \). So, let’s impose that \( \Gamma(\alpha) > \frac{2}{5} \), so, we have to \( M_0^\alpha < 0.267 \). Hence, by means of Theorem 4.1, the problem has a unique solution on \([0, 1]\) for \( \theta_0 \).

6. Concluding remarks. We conclude this paper, with the proposed objectives achieved, that is, through conditions EC_1 - EC_9, we discussed four results about the existence of a mild solution for a class of the quasi-linear fractional integro-differential equations with impulsive in the sense of Caputo fractional derivative. We attack the second main result of the paper, the guarantee that the mild solution of the main problem Eq.(1), is unique. We emphasize that at the limit \( \alpha \to 1 \), the study done here, is valid for the classic integer case. Finally, two examples on the existence and uniqueness of the fractional problem Eq.(1), were discussed in particular, involving controllability, in order to elucidate one of the results discussed in the paper.

These results are part of a research project (Imecc-Unicamp) PNPD-CAPES, which aims to discuss issues involving fractional differential equations. A natural sequence of this work, is to discuss conditions to obtain regularity and continuous dependence on mild solutions to the main problem Eq.(1), and its consequences. The work also involves the Caputo fractional derivative.

On the other hand, some open questions that need to be answered, namely:
1. Is it possible to discuss the existence and uniqueness of mild solutions to the problem Eq.(1), involving the $\psi$-Hilfer fractional derivative?

2. What would a formulation of the mild solution for the problem in item 1 look like? What conditions are necessary and sufficient to obtain such results?

One way is to use a Laplace transform that involves the function $\psi(\cdot)$. These questions don’t have an answer and studies in this regard are also ongoing and are part of the continuity and future work of this paper.

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