Hyers-Ulam stability of parabolic Möbius difference equation

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Abstract

The linear fractional map \( g(z) = \frac{az+b}{cz+d} \) with complex number coefficients on the Riemann sphere where \( ad - bc = 1 \) and \( a + d = \pm 2 \) is called parabolic Möbius map. Let \( \{b_n\}_{n \in \mathbb{N}_0} \) be the solution of the parabolic Möbius difference equation \( b_{n+1} = g(b_n) \) for every \( n \in \mathbb{N}_0 \). Then the sequence \( \{b_n\}_{n \in \mathbb{N}_0} \) has no Hyers-Ulam stability.

1 Introduction

In 1940, Ulam suggested a problem about the stability of approximate homomorphism between metric groups [10]. In detail, if \( f \) is a map from the metric group to itself and it satisfies that

\[ d(f(xy), f(x)f(y)) < \varepsilon \]

for all \( x, y \) in the given group \( G \), then does the homomorphism \( h \) exist such that \( d(h(x), f(x)) < \delta \) for all \( x \in G \)? Hyers gave an affirmative answer [6] for Cauchy’s additive equation in Banach space. Hyers-Ulam stability has been searched in the field of functional equations and differential equations for decades. More recently, this stability has been considered for difference equations. For instance, see [4, 7, 8, 9, 11].

Denote the set of natural number by \( \mathbb{N} \) and denote the set \( \mathbb{N} \cup \{0\} \) by \( \mathbb{N}_0 \).

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Let \( \{b_n\}_{n \in \mathbb{N}_0} \) be the sequence determined by the difference equation

\[
b_{n+1} = F(n, b_n)
\]  
(1.1)

where \( F \) is the map from \( \mathbb{N}_0 \times \mathbb{C} \) into \( \mathbb{C} \) with an initial point \( b_0 \in \mathbb{C} \) for \( n \in \mathbb{N}_0 \). If the map \( F(n, \cdot) \) is independent of \( n \), then we use the notion \( F(b_n) \) instead of \( F(n, b_n) \).

**Definition 1.1.** Let \( \{a_n\}_{n \in \mathbb{N}_0} \) be the complex valued sequence which satisfies the inequality

\[
|a_{n+1} - F(n, a_n)| \leq \varepsilon
\]  
(1.2)

for a given \( \varepsilon > 0 \) and for all \( n \in \mathbb{N}_0 \) where \( |\cdot| \) is the absolute value of complex number. For every sequence \( \{a_n\}_{n \in \mathbb{N}_0} \) satisfying (1.2) if there exists a sequence \( \{b_n\}_{n \in \mathbb{N}_0} \) satisfying (1.1) for each \( n \in \mathbb{N}_0 \) and \( |a_n - b_n| \leq K(\varepsilon) \) for all \( n \in \mathbb{N} \) where \( K(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), then the difference equation (1.1) is called that it has **Hyers-Ulam stability**.

### Parabolic Möbius map

The linear fractional map \( z \mapsto \frac{az+b}{cz+d} \) on the Riemann sphere, \( \hat{C} = \mathbb{C} \cup \{\infty\} \) is called Möbius map where \( a, b, c \) and \( d \in \mathbb{C} \) and \( ad - bc \neq 0 \). Every Möbius map has the matrix representation \( (\begin{array}{cc} a & b \\ c & d \end{array}) \). Since the map \( z \mapsto \frac{az+b}{cz+d} \) is the same as \( z \mapsto \frac{aqz+qb}{qc+qd} \), the matrix \( (\begin{array}{cc} a & b \\ c & d \end{array}) \) and \( (\begin{array}{cc} qa & qb \\ qc & qd \end{array}) \) represent the same map. Thus we may assume that the determinant of matrix representation, namely, \( ad - bc \) is one. Denote the trace of the matrix representation is \( a + d \) under the condition \( ad - bc = 1 \). The map \( g(z) = \frac{az+b}{cz+d} \) where \( ad - bc = 1 \) and \( a + d = \pm 2 \) is called **parabolic** Möbius map. If \( c \neq 0 \), then we define \( g(\infty) = \frac{a}{c} \) and \( g\left(-\frac{d}{c}\right) = \infty \).

### Non stability

Let \( g(z) = \frac{az+b}{cz+d} \) be the linear fractional map for \( c \neq 0 \). Thus \( g\left(-\frac{d}{c}\right) = \infty \). However, the meaning of Hyers-Ulam stability is not clear to the inequality \( |a_{n+1} - g(a_n)| \leq \varepsilon \) where \( a_n = -\frac{d}{c} \) for some \( n \in \mathbb{N} \). Then we consider that some region in \( \mathbb{C} \) which contains the whole sequence \( \{a_n\}_{n \in \mathbb{N}_0} \) and is disjoint
from \( \{ g^{-n}(\infty) \mid n \in \mathbb{N} \} \) for Hyers-Ulam stability. This region is dependent on \(-\frac{d}{c}\) which is determined by the map. Then the common region of all \(g\) is not considered unless some additional specific condition guarantees the existence of common region for all sequences from different maps. Similarly, the number \(K(\varepsilon)\) depends on each sequence. The non stability in the sense of Hyers-Ulam is discussed in [2, 3]. For difference equations non stability means that

"At any given \(\varepsilon > 0\), there is no region which contains the sequence \(\{a_n\}_{n \in \mathbb{N}_0}\) satisfying the definition of Hyers-Ulam stability".

The initial point \(a_0\) may be chosen arbitrarily on some dense subset of \(\mathbb{C}\) for non stability in the Hyers-Ulam.

**Main content**

In Section 2 and Section 3, invariant circles and the extended line (defined later) are constructed and the convergence of \(g^n(z)\) to the unique fixed point is proved as \(n \to \pm \infty\) along invariant circle. In Section 4, we show the non-stability of the sequence from parabolic Möbius difference equation in the sense of Hyers-Ulam. In particular, a periodic sequence \(\{a_n\}_{n \in \mathbb{N}_0}\) which satisfies (1.2) is constructed and it is compared with any sequence \(\{b_n\}_{n \in \mathbb{N}_0}\) defined by the equation (1.1). In Section 5 and Section 6, we show the non-stability of real parabolic Möbius difference equation defined on the extended real line. Without any finite invariant circle, the proof requires different calculation for the real parabolic Möbius map.

### 2 Horocycles

Let \(g(z) = \frac{az+b}{cz+d}\) be the parabolic Möbius map where \(ad - bc = 1, c \neq 0\). The extended line is defined as the union of the straight line and \(\{\infty\}\) in the Riemann sphere. We define horocycle at the fixed point of \(g\) in Definition 2.3 and show that each of these is invariant under \(g\).

**Lemma 2.1.** Let \(g(z) = \frac{az+b}{cz+d}\) be Möbius map where \(ad - bc = 1, c \neq 0\). If \(g\) is the parabolic Möbius map, that is, \(a + d = \pm 2\), then \(g\) has the unique fixed point, say \(\alpha\), and \(\alpha = \frac{a-d}{2c}\).
Proof. The fixed point of $g$ satisfies the equation $cz^2 - (a - d)z - b = 0$. The unique solution of the above quadratic equation is $\frac{a - d}{2c}$. \qed

Lemma 2.2. Let $g(z) = \frac{az + b}{cz + d}$ be the parabolic Möbius map for $c \neq 0$. Then the extended line which contains $\frac{a}{c}$, $-\frac{d}{c}$ and $\infty$, say $L_\infty$, is invariant under $g$.

Proof. The image of circle or line under Möbius map is circle or line. The extended line $L_\infty$ contains the fixed point, $\alpha = \frac{a - d}{2c}$ because $\alpha = \frac{1}{2} (\frac{a}{c} + (-\frac{d}{c}))$. Observe that $g(\infty) = \frac{a}{c}$ and $g(-\frac{d}{c}) = \infty$. Thus since $g(L_\infty)$ contains $\infty, \alpha$ and $\frac{a}{c}$, $g(L_\infty)$ is the extended line and it is the same as $L_\infty$. \qed

Definition 2.3. Horocycle at the fixed point of parabolic Möbius map is defined as follows

1. the extended line $L_\infty$ which contains $\frac{a}{c}$ and $-\frac{d}{c}$ or
2. Every circle which intersects $L_\infty$ at the fixed point of $g$ tangentially.

By the above definition, the center of each horocycle is contained in the straight line which contains $\alpha$ and is perpendicular to $L_\infty$. Define the straight line as follows

$$\ell = \left\{ z : \left| z + \frac{d}{c} \right| = \left| z - \frac{a}{c} \right| \right\}. \quad (2.1)$$

Then every horocycle is the circle $S_p = \{z : |z - p| = |\alpha - p|\}$ where $p$ is a point in $\ell$.

Remark 2.4. The horocycle in the above definition is slightly different from the usual definition in hyperbolic geometry \cite{1} as the set in the complex plane. Our definition of horocycle contains the extended line for convenience.

Proposition 2.5. Let $g(z) = \frac{az + b}{cz + d}$ be the Möbius map where $ad - bc = 1$ and $c \neq 0$. Suppose that $g$ is parabolic, that is, $a + d = \pm 2$. Then every horocycle at $\alpha$ of $g$ is invariant under $g$, that is, $z$ satisfies that $|z - p| = |p - \alpha|$ if and only if $|g(z) - p| = |\alpha - p|$.

The proof of the above proposition requires the following lemmas.
Lemma 2.6. Let \( g(z) = \frac{az + b}{cz + d} \) be the parabolic Möbius map for \( c \neq 0 \). Then
\[
 cp - a = -(cp + d)
\]
where \( p \in \ell \) defined in (2.1).

Proof. The fact that \( p \in \ell \) implies that \( |p + \frac{d}{c}| = |p - \frac{a}{c}| \), that is, \( |cp - a| = |cp + d| \). However, the difference of these complex numbers is
\[
 cp - a - (cp + d) = -a - d = \pm 2
\]
the non zero real number. Then \( \text{Im} (cp - a) = \text{Im} (cp + d) \). However, since \( |cp - a| = |cp + d| \), we obtain that \( \text{Re} (cp - a) = -\text{Re} (cp + d) \). Hence, \( cp - a = -(cp + d) \).

Corollary 2.7. Let \( g(z) = \frac{az + b}{cz + d} \) be the parabolic Möbius map for \( c \neq 0 \). Then \( c(p - \alpha) \) is a purely imaginary number.

Proof. The \( \alpha = \frac{a - d}{2c} \) is the fixed point of \( g \). Thus
\[
 c(p - \alpha) = cp - c\alpha = cp - \frac{a - d}{2} = \frac{1}{2} [(cp - a) + (cp + d)].
\]
However, Lemma 2.6 implies that \( \text{Re} (cp - a) = -\text{Re} (cp + d) \). Then the sum \( cp - a + cp + d \) is a purely imaginary number. Hence, so is \( c(p - \alpha) \).
Corollary 2.8. Let $g(z) = \frac{az+b}{cz+d}$ be the parabolic Möbius map for $c \neq 0$. Then the equation

$$\frac{d}{c} + \frac{cp-a}{c} = -p$$

holds.

Proof. Lemma 2.6 implies that

$$\frac{d}{c} + \frac{cp-a}{c} = \frac{d}{c} - \frac{cp+d}{c} = -p.$$

Lemma 2.9. Let $g(z) = \frac{az+b}{cz+d}$ be the parabolic Möbius map for $c \neq 0$. Let $\alpha$ be the fixed point of $g$ and $p$ is contained in $\ell$. Then the equation

$$|cp-a|^2 = |c(p - \alpha)|^2 + 1$$

holds.

Proof. The fixed point $\alpha$ is $\frac{a-d}{2c}$ and $a + d = \pm 2$. Thus

$$c\alpha - a = \frac{c(a-d)}{2c} - a = \frac{-a + d}{2} = \pm 1$$

(2.2)

which is the real number. Corollary 2.7 implies that $c(p - \alpha)$ is the purely imaginary number. Since $cp - a = cp - c\alpha + c\alpha - a$, the Pythagorean theorem holds

$$|cp-a|^2 = |cp-c\alpha|^2 + |c\alpha-a|^2.$$  

Hence, the equation (2.2) implies that $|cp-a|^2 = |cp-c\alpha|^2 + 1$.  

Lemma 2.10. Let $A$ and $B$ be the complex numbers satisfying that $|A| \neq |B|$. Then $\left| \frac{1}{z} + A \right| = |B|$ if and only if

$$\left| z + \frac{\overline{A}}{|A|^2 - |B|^2} \right| = \frac{|B|}{|A|^2 - |B|^2}.$$  

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Proof. The following equivalent equation completes the proof

\[
\begin{align*}
\left| \frac{1}{z} + A \right| &= |B| \\
\iff |1 + Az|^2 &= |Bz|^2 \\
\iff 1 + Az + \overline{A}z + |A|^2 &= |Bz|^2 \\
\iff (|A|^2 - |B|^2)|z|^2 + Az + \overline{A}z &= -1 \\
\iff |z|^2 + \frac{Az}{|A|^2 - |B|^2} + \frac{\overline{A}z}{|A|^2 - |B|^2} &= -\frac{1}{|A|^2 - |B|^2} \\
\iff \left| z + \frac{\overline{A}}{|A|^2 - |B|^2} \right|^2 &= \frac{|A|^2}{(|A|^2 - |B|^2)^2} - \frac{1}{|A|^2 - |B|^2} \\
\iff \left| z + \frac{\overline{A}}{|A|^2 - |B|^2} \right| &= \frac{|B|}{|A|^2 - |B|^2}.
\end{align*}
\]

\[\square\]

Proof of Proposition 2.5. Suppose that \(|g(z) - p| = |p - \alpha|\). The following equations are equivalent

\[
\begin{align*}
\left| \frac{az + b}{cz + d} - p \right| &= |p - \alpha| \\
\iff \left| \frac{a}{c(z + d)} - \frac{1}{c} \right| &= |p - \alpha| \\
\iff \left| \frac{1}{cz + d} + cp - a \right| &= |c||p - \alpha|.
\end{align*}
\]

Lemma 2.10, Lemma 2.9 and Corollary 2.8 implies that the equation as follows

\[
\begin{align*}
\iff \left| \frac{cz + d}{|cp - a|} \right| &= \left| \frac{c(p - \alpha)}{|p - \alpha|} \right| \quad \text{by Lemma 2.9} \\
\iff \left| \frac{cz + d}{c} + \frac{cp - a}{c} \right| &= |p - \alpha| \quad \text{by Corollary 2.8} \\
\iff |z - p| &= |p - \alpha|
\end{align*}
\]
Then we obtain that the equation $|g(z) - p| = |p - \alpha|$ is satisfied if and only if the equation $|z - p| = |p - \alpha|$ is so. Moreover, the extended line $L_\infty$ is invariant under $g$ by Lemma 2.2. Hence, any point $q$ is contained in a horocycle if and only if $g(q)$ is contained in the same horocycle.

3 Conjugation between parabolic Möbius map and translation

The map $h(z) = \frac{a + b}{cz + d}$ is the conjugation between parabolic Möbius map and the translation $z \mapsto z + 1$, that is, $h \circ g(z) = h(z) + 1$. Let the set $\{g(z), g^2(z), \ldots, g^n(z), \ldots\}$ be the (forward) orbit of $z$ under $g$. We show that the forward orbit of any point in $\mathbb{C}$ under $g$ converges to the fixed point. Moreover, in this section we show that each point of the orbit, $g^n(z)$ is arranged with either clockwise or anticlockwise direction along a horocycle as $n$ increases.

**Lemma 3.1.** Let $g(z) = \frac{az + b}{cz + d}$ be the parabolic Möbius map for $ad - bc = 1$ and $c \neq 0$. Let $h$ be the map, $h(z) = \frac{1}{c(z - \alpha)}$ where $\alpha$ is the unique fixed point of $g$. Then the map $h \circ g \circ h^{-1}$ is the translation $z \mapsto z + 1$ where $a + d = 2$ or $h \circ g \circ h^{-1}$ is the translation $z \mapsto z - 1$ where $a + d = -2$.

**Proof.** Recall that any composition of two Möbius map is also Möbius map. Thus if a Möbius map has $\infty$ as a fixed point, then it is a linear map. Denote $h \circ g \circ h^{-1}$ by $f$. The straightforward calculation implies that $h^{-1}(w) = \alpha + \frac{1}{cw}$. Recall that $\alpha = \frac{a - d}{2c}$. Suppose that $a + d = 2$ firstly. The following is an intermediate calculations

$$\alpha + \frac{1}{c} = \frac{a - d}{2c} + \frac{1}{c} = \frac{a}{c}, \quad \alpha - \frac{1}{c} = \frac{a - d}{2c} - \frac{1}{c} = -\frac{d}{c}. \quad (3.1)$$

Then we have the equations as follows using (3.1)

$$f(\infty) = h \circ g \circ h^{-1}(\infty) = h \circ g(\alpha) = h(\alpha) = \infty$$

$$f(0) = h \circ g \circ h^{-1}(0) = h \circ g(\infty) = h \left( \frac{a}{c} \right) = h \left( \alpha + \frac{1}{c} \right) = 1$$

$$f(-1) = h \circ g \circ h^{-1}(-1) = h \circ g \left( \alpha - \frac{1}{c} \right) = h \left( -\frac{d}{c} \right) = h(\infty) = 0.$$
Since $f$ is a Möbius map, the fact that $f(\infty) = \infty$, $f(0) = 1$ and $f(-1) = 0$ implies that $f(z) = z + 1$. In the case that $a + d = -2$, the proof which is similar to the above one is applicable. Thus the remaining proof is left to the readers.

Lemma 3.2. Let $g(z) = \frac{az + b}{cz + d}$ be the parabolic Möbius map for $ad - bc = 1$ and $c \neq 0$. Let $h$ be the map, $h(z) = \frac{1}{c(z - \alpha)}$. Then $h$ maps each horocycle at $\alpha$ of $g$ to the extended line parallel to the $x$-axis in $\mathbb{C}$.

Proof. Any point in a horocycle at $\alpha$ of $g$ satisfies the equation $|z - p| = |\alpha - p|$ for some $p \in \ell$. Denote $h(z)$ by $w$. Recall that $h^{-1}(w) = z = \alpha + \frac{1}{cw}$. Thus the following equations are equivalent

$$
|z - p| = |\alpha - p| \iff \left| \frac{1}{cw} + \alpha - p \right| = |\alpha - p|
$$

$$
\iff \left| \frac{1}{cw} + \alpha - p \right|^2 = |\alpha - p|^2
$$

$$
\iff \frac{1}{|cw|^2} + \frac{\alpha - p}{cw} + \frac{\alpha - p}{cw} + |\alpha - p|^2 = |\alpha - p|^2
$$

$$
\iff \frac{1}{|cw|^2} \left( 1 + \frac{(\alpha - p)cw + (\alpha - p)cw}{cw} \right) = 0
$$

$$
\iff (p - \alpha)cw + (p - \alpha)cw = 1.
$$

Corollary 2.8 implies that $c(p - \alpha)$ is a purely imaginary number. Thus the equation

$$
c(p - \alpha) = -c(p - \alpha)
$$

holds. The equation (3.2) implies that

$$
1 = \overline{c(p - \alpha)w} + c(p - \alpha)w = c(p - \alpha)[ - \overline{w} + w] = c(p - \alpha)(2 \text{Im } w)i
$$

Hence, the imaginary part of $w$ is constant as follows

$$
\text{Im } w = \frac{1}{2c(p - \alpha)i}.
$$

It completes the proof. \qed

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Corollary 3.3. The point $z$ satisfies that $|z + \frac{d}{c}| = |z - \frac{a}{c}|$ if and only if the real part of $h(z)$ is zero, that is, $\text{Re} \, h(z) = 0$.

Proof. Denote $h(z)$ by $w$. Observe that for any $z \in \mathbb{C}$ there exists a horocycle at $\alpha$ which contains $z$. Thus we may assume that $\text{Im} \, w = \frac{1}{2c(p-\alpha)}$ by the equation (3.3) in Lemma 3.2. Suppose that $w = i \text{Im} \, w$. Then since $h(z) = \frac{1}{c(z-\alpha)}$, the following equation holds for $z$

$$\frac{1}{c(z-\alpha)} = \frac{1}{2c(p-\alpha)}.$$ 

Thus we obtain that $z = 2(p - \alpha) + \alpha$. Recall that $\alpha = \frac{a-d}{2c}$ and $a + d = \pm 2$. Thus

$$c\alpha + d = c \cdot \frac{a-d}{2c} + d = \frac{a+d}{2} = 1 \quad \text{or} \quad -1$$

$$c\alpha - a = c \cdot \frac{a-d}{2c} - a = -\frac{a+d}{2} = -1 \quad \text{or} \quad 1.$$ 

Then both $c\alpha + d$ and $c\alpha - a$ are real numbers and the sum of these two numbers is zero. Recall also that Corollary 2.7 implies that $c(p - \alpha)$ is a purely imaginary number. Thus for any real number $r$, distance between $c(p - \alpha)$ and $r$ is the same as the distance between $c(p - \alpha)$ and $-r$. Then we obtain that

$$|z + \frac{d}{c}| = \left|2(p - \alpha) + \alpha + \frac{d}{c}\right|$$

$$= \frac{1}{|c|} |2c(p-\alpha) + c\alpha + d|$$

$$= \frac{1}{|c|} |2c(p-\alpha) + c\alpha - a|$$

$$= \left|2(p - \alpha) + \alpha - \frac{a}{c}\right|$$

$$= \left|z - \frac{a}{c}\right|.$$ 

The map $h$ is the bijection between each horocycle and the corresponding extended line in Lemma 3.2. The line $\ell$ defined in (2.1)

$$\ell = \left\{ z : \left|z + \frac{d}{c}\right| = \left|z - \frac{a}{c}\right| \right\}.$$ 

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is the straight line which goes through $\alpha$ and the center of every horocycles. Thus the intersection of a single horocycle and $\ell$ is the set of two points, one of which is $\alpha$. The other point is the unique point $z_0$ satisfying $h(z_0) = \frac{1}{2c(p-\alpha)}$, which is determined by the equation $|p - \alpha| = |z_0 - p|$ where $p \in \ell$. Since $c(p-\alpha)$ is a purely imaginary number, so is $h(z_0)$. This completes the proof.

Let $q\bar{s}$ be the circular arc of which end points are $q$ and $s$. The points $q$ and $s$ are called the end points of $q\bar{s}$.

(3.a) We consider only circular arc in $\mathbb{C}$ which is homeomorphic to the closed interval.

(3.b) The definition of circular arc is extended to the closed interval, which is a subset of the extended line.

(3.c) The embedded homeomorphic image of the circular arc in $\mathbb{C}$ is called arc and the notation of arc is the same as that of circular arc.

(3.d) The arc $z_0\bar{z}_1\bar{z}_2 \ldots \bar{z}_{n-1}\bar{z}_n$ is defined as the arc $\bar{z}_0\bar{z}_n$ which contains the points $z_1, z_2, \ldots, z_{n-1}$ where $\bar{z}_i\bar{z}_{i+1}$ is disjoint from $\bar{z}_j\bar{z}_{j+1}$ only if $0 < i + 1 < j < n$ for $n \geq 4$.

Lemma 3.4. Let $h$ be a homeomorphism on the Riemann sphere. Let the points $z_0, z_1, \ldots, z_n$ in $\mathbb{C}$ be in the single circular arc $\bar{z}_0\bar{z}_n$ and $w_j$ be $h(z_j)$ in $\mathbb{C}$ for $0 \leq j \leq n$. Suppose that the arc $\bar{z}_0\bar{z}_1\bar{z}_2 \ldots \bar{z}_{n-1}\bar{z}_n$ satisfies the condition (3.d). Then $h(\bar{z}_0\bar{z}_1\bar{z}_2 \ldots \bar{z}_{n-1}\bar{z}_n)$ is either $\bar{w}_0\bar{w}_1\bar{w}_2 \ldots \bar{w}_{n-1}\bar{w}_n$ or $\bar{w}_0\bar{w}_1\bar{w}_2 \ldots \bar{w}_{n-1}\bar{w}_n$.

Proof. Suppose that $h(\bar{z}_0\bar{z}_1\bar{z}_2 \ldots \bar{z}_n) = \bar{w}_0\bar{w}_1\bar{w}_2 \ldots \bar{w}_{n} = \bar{w}_0\bar{w}_1\bar{w}_2 \ldots \bar{w}_{n}$ where $0 < i < j < n$ for $n \geq 4$. Then by the condition of the arc, $\bar{z}_0\bar{z}_i$ is disjoint from the arc $\bar{z}_j\bar{z}_n$. However, the intersection, $\bar{w}_0\bar{w}_i \cap \bar{w}_j\bar{w}_n = \bar{w}_j\bar{w}_n$ is not empty. It contradicts that $h$ is the homeomorphism. For the case $n = 3$, suppose that $\bar{z}_0\bar{z}_1\bar{z}_2$ is mapped to $\bar{w}_0\bar{w}_1\bar{w}_2$ or $\bar{w}_2\bar{w}_1\bar{w}_0$, that is, $w_2$ is an end point of $h(\bar{z}_0\bar{z}_1\bar{z}_2)$. However, $\bar{z}_0\bar{z}_1\bar{z}_2 \setminus \{z_2\}$ is disconnected but $\bar{w}_0\bar{w}_1\bar{w}_2 \setminus \{w_2\}$ or $\bar{w}_2\bar{w}_1\bar{w}_0 \setminus \{w_2\}$ is connected. It contradicts that $h$ is the homeomorphism.

Let $S_p$ be a horocycle which is contained in $\mathbb{C}$ as follows

$$S_p = \{z : |z - p| = |\alpha - p|\} \quad (3.4)$$
Figure 2: Image of horocycle under $h$

where $p$ satisfies that $|p + \frac{d}{c}| = |p - \frac{a}{c}|$. Let the principal argument of $\alpha - p$ be the argument between $-\pi$ to $\pi$, that is, $-\pi < \text{Arg}(\alpha - p) \leq \pi$.

(3.e) Let $z_0, z_1, \ldots, z_n$ be the points contained in $S_p \setminus \{\alpha\}$. Let $\theta_j$ be the argument of $z_j - p$ where $\text{Arg}(\alpha - p) < \theta_j < \text{Arg}(\alpha - p) + 2\pi$ for every $1 \leq j \leq n$.

Thus the arc $z_1 z_2 \ldots z_{n-1} z_n$ satisfies that the condition [3.d] if and only if the arguments of $z_j$ for $1 \leq j \leq n$ satisfies that $\theta_1 < \theta_2 < \cdots < \theta_n$ or $\theta_n < \theta_{n-1} < \cdots < \theta_1$. Similarly, if $w_1, w_2, \ldots, w_n$ are contained in the line parallel to real axis, that is, $w_j \in \{w: \text{Im} w = \text{const.}\}$ for all $1 \leq j \leq n$, then the arc $w_1 w_2 \ldots w_{n-1} w_n$ satisfies the condition [3.d] if and only if the real part of $w_j$ for $1 \leq j \leq n$ satisfies that $\text{Re} w_1 < \text{Re} w_2 < \cdots < \text{Re} w_n$ or $\text{Re} w_n < \text{Re} w_{n-1} < \cdots < \text{Re} w_1$. Then the above statements and Lemma 3.4 implies the following lemma.

**Lemma 3.5.** Let $z_0, z_1, \ldots, z_n$ be the points contained in $S_p \setminus \{\alpha\}$ where $S_p$ is the horocycle at $\alpha$ in (3.4). Let $\theta_j$ be the argument of $z_j - p$ where $\text{Arg}(\alpha - p) < \theta_j < \text{Arg}(\alpha - p) + 2\pi$ for every $1 \leq j \leq n$. Let $w_1, w_2, \ldots, w_n$ be points in the set $h(S_p \setminus \{\alpha\})$ where $h(z) = \frac{1}{c(z - \alpha)}$. Then $\theta_1 < \theta_2 < \cdots < \theta_n$ or $\theta_n < \theta_{n-1} < \cdots < \theta_1$ if and only if $\text{Re} w_1 < \text{Re} w_2 < \cdots < \text{Re} w_n$.

**Proof.** The $h$ maps horocycle at $\alpha$ of $g$ to the extended line parallel to the $x$-axis in $\hat{C}$ by Lemma 3.2. Moreover, Lemma 3.4 and the relation between arc and the argument of $z_j$ for $1 \leq j \leq n$ completes the proof. □
Proposition 3.6. Let $S_p$ be the horocycle at $\alpha$ of $g$ where $g(z) = \frac{az+b}{cz+d}$ with $ad - bc = 1$ and $c \neq 0$. For a point $z_0 \in S_p \setminus \{\alpha\}$, denote $g^n(z_0)$ by $z_n$ for $n \in \mathbb{Z}$. Then

$$\lim_{n \to \pm\infty} z_n = \alpha.$$ 

Moreover, either $\Arg(\alpha - p) < \theta_i < \theta_j < \Arg(\alpha - p) + 2\pi$ where $i < j$ or $\Arg(\alpha - p) < \theta_i < \theta_j < \Arg(\alpha - p) + 2\pi$ where $i > j$ for $i, j \in \mathbb{Z}$.

Proof. Lemma 3.1 implies that $h \circ g \circ h^{-1}(w) = w + 1$ where $h(z) = \frac{1}{c(z-\alpha)}$. Denote the translation $w \mapsto w + 1$ by $f$. Let $w_j = h(z_j)$ for every $j \in \mathbb{Z}$. Since every point $z_j$ is in a single horocycle for all $j \in \mathbb{Z}$ by Proposition 2.5, every $w_j$ is also in a single straight line $\{w: \text{Im} = \text{const.}\}$ for all $j \in \mathbb{Z}$. Moreover, by the conjugation $h$ we have that

$$w_{j+1} = h(z_{j+1}) = h \circ g(z_j) = h \circ g \circ h^{-1}(w_j) = f(w_j) = w_j + 1.$$ 

Thus $\text{Re} w_i < \text{Re} w_j$ where $i < j$ for each $i, j \in \mathbb{Z}$. Lemma 3.5 implies that either $\theta_i < \theta_j$ for every integer $i < j$ or $\theta_i > \theta_j$ for every integer $i < j$ where the argument of each point $z_j$ in the horocycle is defined in (3.e). By induction, the equation $w_n = w_0 + n$ holds for $n \in \mathbb{Z}$. The map $h$ is the continuous bijection on $\hat{\mathbb{C}}$ under spherical metric. Then

$$h(\alpha) = \infty = \lim_{n \to \pm\infty} w_n = \lim_{n \to \pm\infty} h(z_n) = h\left(\lim_{n \to \pm\infty} z_n\right).$$

Hence, $\alpha = \lim_{n \to \pm\infty} z_n$. \hfill $\Box$

Remark 3.7. The orbit of the any point $z_0$ in $\mathbb{C} \setminus \{\alpha\}$, namely, the set $\{g^n(z_0): n \in \mathbb{Z}\}$ is contained in a single horocycle at $\alpha$ of $g$. Proposition 3.6 implies that every point, $g^n(z_0)$ for $n \in \mathbb{Z}$ are positioned with clockwise or counterclockwise direction.

4 No Hyers-Ulam stability of parabolic Möbius difference equation

Let $\{b_n\}_{n \in \mathbb{N}_0}$ be the sequence satisfying that $b_{n+1} = g(b_n)$ for $n \in \mathbb{N}_0$ where $g$ is the parabolic Möbius map $g(z) = \frac{az+b}{cz+d}$ for $ad - bc = 1$. We show that
the above sequence \( \{b_n\}_{n \in \mathbb{N}_0} \) for any initial point \( b_0 \in \mathbb{C} \) has no Hyers-Ulam stability. The result holds for the both cases that \( c \neq 0 \) or \( c = 0 \).

**Proposition 4.1.** Let \( g \) be the parabolic Möbius map which does not fix \( \infty \), that is, \( g(z) = \frac{az + b}{cz + d} \) for \( ad - bc = 1 \), \( a + d = \pm 2 \) and \( c \neq 0 \). Let \( \{b_n\}_{n \in \mathbb{N}_0} \) be the sequence satisfying that \( b_{n+1} = g(b_n) \) for every \( n \in \mathbb{N}_0 \). Then for any initial point \( b_0 \in \mathbb{C} \), the sequence \( \{b_n\}_{n \in \mathbb{N}_0} \) has no Hyers-Ulam stability.

**Proof.** For any given \( \varepsilon > 0 \) we show that there exists a sequence \( \{a_n\}_{n \in \mathbb{N}_0} \) which satisfies the following properties.

1. \( |a_{n+1} - g(a_n)| \leq \varepsilon \) for all \( n \in \mathbb{N}_0 \),
2. the sequence \( \{a_n\}_{n \in \mathbb{N}_0} \) is pre-periodic one, and
3. \( |a_{N_0} - b_{N_0}| \geq 1 + \varepsilon \) for some big enough \( N_0 \in \mathbb{N} \).

Then \( |a_n - b_n| \geq 1 \) for infinitely many \( n \in \mathbb{N} \) due to the periodicity of the sequence \( \{a_n\}_{n \in \mathbb{N}_0} \). Hence, \( \{b_n\}_{n \in \mathbb{N}_0} \) has no Hyers-Ulam stability. In the rest of proof, we construct the sequence \( \{a_n\}_{n \in \mathbb{N}_0} \) satisfying the above properties.

For any given \( z_0 \in \mathbb{C} \), \( \lim_{n \to \pm \infty} g^n(z_0) = \alpha \) by Proposition 3.6. Thus for big enough \( N_1 \) the points \( g^n(z_0) \) in the disk, \( B(\alpha, \frac{\varepsilon}{2}) \) for all \( n \geq N_1 \). Thus there exists a point \( q \) in the disk \( B(\alpha, \frac{\varepsilon}{2}) \) which satisfies that

\[
(4.1) \quad |q - g^{N_1}(z_0)| \leq \varepsilon,
\]

\[
(4.2) \quad g^{-k}(q) \in B(\alpha, \frac{\varepsilon}{2}) \quad \text{for all} \quad k \in \mathbb{N}_0,
\]

\[
(4.3) \quad \text{the point} \quad q \quad \text{is in the horocycle} \quad S_p = \{z : |z - p| = |\alpha - p|\} \quad \text{in} \quad \mathbb{C} \quad \text{where} \quad p \quad \text{is contained in the line} \quad \ell \quad \text{defined in (2.1)} \quad \text{for} \quad |\alpha - p| \geq 1 + 2\varepsilon \quad \text{and}
\]

\[
(4.4) \quad \text{for some} \quad N_2 > 0, \quad g^{N_2}(q) \quad \text{is the point in the intersection of} \quad \ell \cap S_p \setminus \{\alpha\}.
\]

Then the following sequence

\[
\{z_0, g(z_0), g^2(z_0), \ldots, g^{N_1}(z_0), q, g(q), \ldots, g^{N_2}(q), \ldots, g^{2N_2}(q), q, g(q), \ldots\}
\]

(4.1)

is pre-periodic sequence with period \( 2N_2 \). For the given sequence \( \{b_n\}_{n \in \mathbb{N}_0} \) satisfying \( b_{n+1} = g(b_n) \) for all \( n \in \mathbb{N}_0 \), define \( \{a_n\}_{n \in \mathbb{N}_0} \) as the sequence (4.1), that is, \( a_0 = z_0 \), \( a_n = g^n(z_0) \) for \( 1 \leq n \leq N_1 \), \( a_{N_1+1} = q \), \( a_{m+N_1+1} = g^m(q) \) for \( 1 \leq m \leq 2N_2 \) and \( a_{k+2N_2} = a_k \) for every \( k \geq N_1 + 1 \) where \( z_0 = b_0 \).
Then \( \{a_n\}_{n \in \mathbb{N}_0} \) is the pre-periodic sequence satisfying \( |a_{n+1} - g(a_n)| \leq \varepsilon \) for all \( n \in \mathbb{N}_0 \). Moreover, the distance between \( a_{N_2+N_1+1} \) and \( \alpha \) is the diameter of the horocycle \( S_p \). Then

\[
|a_{N_2+N_1+1} - b_{N_2+N_1+1}| \geq |a_{N_2+N_1+1} - \alpha| - |\alpha - b_{N_2+N_1+1}| \geq 1 + 2\varepsilon - \varepsilon > 1
\]

By the periodicity of the sequence \( \{a_n\}_{n \in \mathbb{N}_0} \) for \( n \geq N_1 + 1 \), we obtain that

\[
|a_{N_2+N_1+1+2kN_2} - b_{N_2+N_1+1}| > 1
\]
for all \( k \in \mathbb{N}_0 \). Hence, the sequence \( \{b_n\}_{n \in \mathbb{N}_0} \) does not have Hyers-Ulam stability.

Assume that the parabolic Möbius map \( g(z) = \frac{az+b}{cz+d} \) for \( ad-bc = 1 \) fixes the infinity, that is, \( c = 0 \). Since \( a+d = \pm 2 \) and \( ad = 1 \), we obtain that either \( a = d = 1 \) or \( a = d = -1 \). Thus the Möbius map is the translation, \( g(z) = z \pm b \). If \( b = 0 \), then the sequence \( \{b_n\}_{n \in \mathbb{N}_0} \) is the constant sequence and it has no Hyers-Ulam stability. Thus without loss of generality we may assume that \( b \neq 0 \).

**Proposition 4.2.** Let \( g(z) = z + q \) where \( q \) is a non-zero complex number. Then the sequence \( \{b_n\}_{n \in \mathbb{N}_0} \) satisfying \( b_{n+1} = g(b_n) \) for any given initial point \( b_0 \in \mathbb{C} \) has no Hyers-Ulam stability.

**Proof.** For a given \( \varepsilon > 0 \), define the sequence \( \{a_n\}_{n \in \mathbb{N}_0} \) with its elements as follows

\[
a_n = a_0 + n(\varepsilon + q)
\]
for every \( n \in \mathbb{N}_0 \). Thus

\[
|a_{n+1} - g(a_n)| = |a_0 + (n+1)(\varepsilon + q) - [a_0 + n(\varepsilon + q) + q]| = \varepsilon.
\]

By induction each element of the sequence \( \{b_n\}_{n \in \mathbb{N}_0} \) is of the following form, \( b_n = b_0 + nq \). Then we obtain that

\[
|b_n - a_n| = |b_0 + nq - [a_0 + n(\varepsilon + q) + q]| = |b_0 - a_0 - n\varepsilon| \\
\geq |b_0 - a_0| - n\varepsilon.
\]

(4.2)

However, \( |a_n - b_n| \to \infty \) as \( n \to \infty \) by the inequality (4.2). Hence, \( \{b_n\}_{n \in \mathbb{N}_0} \) has no Hyers-Ulam stability. \( \square \)
Proposition 4.1 and Proposition 4.2 implies the following theorem.

**Theorem 4.3.** Let $g$ be the parabolic Möbius map, that is, $g(z) = \frac{az+b}{cz+d}$ for $ad - bc = 1$ and $a + d = \pm 2$. Let $\{b_n\}_{n \in \mathbb{N}_0}$ be the sequence satisfying that $b_{n+1} = g(b_n)$ for every $n \in \mathbb{N}_0$. Then for any initial point $b_0 \in \mathbb{C}$, the sequence $\{b_n\}_{n \in \mathbb{N}_0}$ has no Hyers-Ulam stability.

**Remark 4.4.** The proof of Proposition 4.1 uses the horocycle and pre periodic sequence. In [5] the periodicity of sine function is used to proving the lack of Hyers-Ulam stability of the linear isometry in general metric space.

## 5 Real parabolic Möbius difference equation

Let $g$ be the real parabolic Möbius map, that is, $g(x) = \frac{ax+b}{cx+d}$ on the extended real line, $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ where $a, b, c, d$ are real numbers for $ad - bc = 1$ and $a + d = \pm 2$. The extended real line corresponds the extended line $L_\infty$ of the parabolic Möbius map on the Riemann sphere. However, $\hat{\mathbb{R}}$ does not contain any horocycle with finite diameter. In the case that $c = 0$, the real parabolic Möbius map $g$ is the translation on the real line. Then Proposition 4.2 is applicable to the translation on $\mathbb{R}$, which does not have Hyers-Ulam stability. So we may assume that real parabolic Möbius map does not fix $\infty$. In this section, we separate the real line to subintervals of which endpoints are $-\frac{a}{c}$, $\alpha$ and $\frac{a}{c}$. Moreover, we calculate the image of each intervals under parabolic Möbius map.

**Remark 5.1.** Real parabolic Möbius map can be realized as the restriction of the parabolic Möbius map on the extended real line. Thus real parabolic map is continuous under spherical metric on $\hat{\mathbb{R}}$.

We use the notation $+\infty$ as the unbounded limit which is greater than any positive number and $-\infty$ as the unbounded limit which is less than any negative number. The following auxiliary lemma is for later use.

**Lemma 5.2.** Let $g$ be the parabolic Möbius map which does not fix $\infty$, that is, $g(x) = \frac{ax+b}{cx+d}$ for $ad - bc = 1$, $a + d = \pm 2$ and $c \neq 0$. Then the following equations hold

\[
\frac{a + d}{2} = \frac{2}{a + d}, \quad \alpha + d = \frac{a + d}{2} \quad \text{and} \quad \alpha = \frac{a}{c} - \frac{2}{c(a + d)}.
\]
Proof. Observe that $\frac{a + d}{2} = \pm 1$. Thus
\begin{equation}
1 = \left(\frac{a + d}{2}\right)^2 = \frac{a + d}{2} \cdot \frac{2}{a + d}.
\end{equation}
(5.1)

Thus $\frac{a + d}{2} = \frac{2}{a + d}$. Recall that $\alpha$ is the unique fixed point of the parabolic Möbius map $g$ and $\alpha = \frac{a - d}{2c}$. Then the following equation holds. Hence, $\alpha = \frac{a}{c} - \frac{2}{c(a + d)}$.

Real parabolic Möbius map has two cases, one of which is $(a + d)c > 0$ and the other is $(a + d)c < 0$. Lemma 5.2 implies that $\alpha < \frac{a}{c}$ if and only if $(a + d)c > 0$. We deal with the case $(a + d)c > 0$ in the following lemmas.

**Lemma 5.3.** Let $g$ be the real parabolic Möbius map, that is, $g(x) = \frac{ax + b}{cx + d}$ where $a, b, c$ and $d$ are real numbers for $ad - bc = 1$, $a + d = \pm 2$ and $c \neq 0$. Suppose that $(a + d)c > 0$. Then $\alpha < x < +\infty$ if and only if $\alpha < g(x) < \frac{a}{c}$. The inequality $-\infty < x < -\frac{d}{c}$ holds if and only if $\frac{a}{c} < g(x) < +\infty$.

**Proof.** The following equivalent conditions prove the first part of the lemma. Observe that $\frac{ax + b}{cx + d} = \frac{a}{c} - \frac{1}{c(cx + d)}$. Lemma 5.2 is applied to the followings.

\begin{align*}
+\infty > x > \alpha &\iff +\infty > c^2x + cd > c^2\alpha + cd \\
&\iff +\infty > c(cx + d) > c(c\alpha + d) = \frac{c(a + d)}{2} > 0 \\
&\iff 0 < \frac{1}{c(cx + d)} < \frac{2}{c(a + d)} < +\infty \\
&\iff 0 > -\frac{1}{c(cx + d)} > -\frac{2}{c(a + d)} \\
&\iff \frac{a}{c} > \frac{a}{c} - \frac{1}{c(cx + d)} > \frac{a}{c} - \frac{2}{c(a + d)} = \alpha \\
&\iff \frac{a}{c} > \frac{ax + b}{cx + d} > \alpha.
\end{align*}
Hence, \( \alpha < x < +\infty \) if and only if \( \alpha < g(x) < \frac{a}{c} \). Moreover, the following equivalent condition completes the proof

\[
-\infty < x < -\frac{d}{c} \iff -\infty < c^2 x + cd < 0 \\
\iff -\infty < \frac{1}{c(cx + d)} < 0 \\
\iff 0 < -\frac{1}{c(cx + d)} < +\infty \\
\iff \frac{a}{c} < \frac{a - c}{c(cx + d)} < +\infty \\
\iff \frac{a}{c} < \frac{ax + b}{cx + d} < +\infty.
\]

Hence, \( -\infty < x < -\frac{d}{c} \) if and only if \( \frac{a}{c} < g(x) < +\infty \). \( \square \)

**Corollary 5.4.** Let \( g \) be the real parabolic Möbius map, which is \( g(x) = \frac{ax + b}{cx + d} \) where \( a, b, c \) and \( d \) are real numbers for \( ad - bc = 1 \), \( a + d = \pm 2 \) and \( c \neq 0 \). Suppose that \((a + d)c > 0\). If \( -\infty < x < -\frac{d}{c} \), then \( \alpha < g^2(x) < \frac{a}{c} \).

**Lemma 5.5.** Let \( g \) be the real parabolic Möbius map, which is \( g(x) = \frac{ax + b}{cx + d} \) where \( a, b, c \) and \( d \) are real numbers for \( ad - bc = 1 \), \( a + d = \pm 2 \) and \( c \neq 0 \). Suppose that \((a + d)c > 0\). If \( x \) satisfies the inequality \( \alpha < x < \frac{a}{c} \), then the inequality

\[
\alpha < \cdots < g^n(x) < g^{n-1}(x) < \cdots < g(x) < x
\]

holds and \( \lim_{n \to +\infty} g^n(x) = \alpha \).

**Proof.** The following equivalent conditions hold

\[
\alpha < x < \frac{a}{c} \iff c^2 \alpha + cd < c^2 x + cd < c^2 \frac{a}{c} + cd \\
\iff (c\alpha + d) < c(cx + d) < c(a + d).
\]

Since \( c(a + d) > 0 \) and \( c(c\alpha + d) = \frac{c(a + d)}{2} \), the inequality \( 0 < c(cx + d) \) is satisfied. Thus \( g(x) - x \) is as follows

\[
g(x) - x = \frac{ax + b}{cx + d} - x = -\frac{cx^2 - (a - d)x - b}{cx + d} = -\frac{c(x - \alpha)^2}{cx + d}.
\]
Figure 3: Iterated images under real parabolic Möbius map, \( g(z) = \frac{az+b}{cz+d} \)

Thus the equation \( g(x) - x = -\frac{c^2(x-a)^2}{c(x+d)} \) holds. Since both \( c^2 \) and \( c(x+d) \) are positive number, \( g(x) - x < 0 \) for every \( x \) in the interval \((\alpha, \frac{a}{c})\). Lemma 5.3 implies that \( g((\alpha, +\infty)) = (\alpha, \frac{a}{c}) \). Thus we obtain that \( g \left( (\alpha, \frac{a}{c}) \right) \subset (\alpha, \frac{a}{c}) \).

By induction, for any \( x_0 \in (\alpha, \frac{a}{c}) \) the set \( \{g^k(x_0) \mid k \in \mathbb{N}\} \) is contained in the same interval. Denote \( g^n(x_0) \) by \( x_n \) for each \( n \in \mathbb{N} \). Thus the following inequality

\[
g(x_{n-1}) - x_{n-1} = g \circ g^{n-1}(x_0) - g^{n-1}(x_0) = g^n(x_0) - g^{n-1}(x_0) < 0
\]

holds. Thus inductively we obtain that

\[ \alpha < \cdots < g^n(x) < g^{n-1}(x) < \cdots < g(x) < x. \]

The sequence \( \{g^n(x)\}_{n \in \mathbb{N}_0} \) is a decreasing sequence bounded below by \( \alpha \). Then there exists \( \lim_{n \to +\infty} g^n(x) \), say \( \beta \). However, by the continuity of \( g \), \( \beta \) is a fixed point of \( g \). The uniqueness of the fixed point of \( g \) implies that \( \beta = \alpha \). Hence, \( \lim_{n \to +\infty} g^n(x) \) is \( \alpha \).

We obtain the similar result if the inequality \( c(a + d) < 0 \) holds in the following lemma.

**Lemma 5.6.** Let \( g \) be the real parabolic Möbius map, which is \( g(x) = \frac{ax+b}{cx+d} \) where \( a, b, c \) and \( d \) are real numbers for \( ad - bc = 1 \), \( a + d = \pm 2 \) and \( c \neq 0 \). Suppose that \( (a + d)c < 0 \). Then

- \( -\infty < x < \alpha \) if and only if \( \frac{a}{c} < g(x) < \alpha \),
- \( -\frac{d}{c} < x < +\infty \) if and only if \( -\infty < g(x) < \frac{a}{c} \) and
- if \( \frac{a}{c} < x < \alpha \), then \( x < g(x) < g^2(x) < \cdots < g^n(x) < \cdots < \alpha \) and moreover, \( \lim_{n \to +\infty} g^n(x) = \alpha \).
Proof. The following equivalent conditions prove the first part of the lemma.

\[-\infty < x < \alpha \iff -\infty < c^2 x + cd < c^2 \alpha + cd\]
\[\iff -\infty < c(cx + d) < c(c\alpha + d) = \frac{c(a + d)}{2} < 0\]
\[\iff 0 > \frac{1}{c(cx + d)} > \frac{2}{c(a + d)} > -\infty\]
\[\iff 0 < -\frac{1}{c(cx + d)} < -\frac{2}{c(a + d)}\]
\[\iff \frac{a}{c} < a - \frac{1}{c} < \frac{a}{c} - \frac{2}{c} < -\infty\]

Since \(\frac{a}{c} - \frac{1}{c} = \frac{ax + b}{cx + d}\) and \(\frac{a}{c} - \frac{2}{c} = \alpha\) by Lemma 5.2, the condition \(-\infty < x < \alpha\) is equivalent to \(\frac{a}{c} < g^{-1}(x) < \alpha\). The proof of the second and third parts is similar to that of Lemma 5.3 and Lemma 5.5 as well as the first part of the lemma. Then the remaining proofs are left to the reader. \(\Box\)

Lemma 5.7. Let \(g\) be the real parabolic Möbius map, which is \(g(x) = \frac{ax + b}{cx + d}\) where \(a, b, c\) and \(d\) are real numbers for \(ad - bc = 1\), \(a + d = \pm 2\) and \(c \neq 0\). Suppose that \((a + d)c > 0\). Then \(-\infty < x < \alpha\) if and only if \(-\frac{d}{c} < g^{-1}(x) < \alpha\). The point \(x\) satisfies the inequality \(-\frac{d}{c} < x < \alpha\), then the inequality

\[x < g^{-1}(x) < g^{-2}(x) < \cdots < g^{-n}(x) < \cdots < \alpha\]

holds and \(\lim_{n \to +\infty} g^{-n}(x) = \alpha\).

Proof. The straightforward calculation implies that \(g^{-1}(x) = \frac{dx - b}{cx + a}\), which is also real parabolic Möbius map. \(\alpha\) is the fixed point of \(g^{-1}\) and \(-c(a + d) < 0\). Replace \(a, b, c\) and \(d\) of the map \(g\) by \(d, -b, -c\) and \(a\) respectively. Then apply the proof of Lemma 5.6 to the proof for the map \(x \mapsto \frac{dx - b}{cx + a}\). It completes the proof. \(\Box\)

Corollary 5.8. Let \(g\) be the real parabolic Möbius map, which is \(g(x) = \frac{ax + b}{cx + d}\) where \(a, b, c\) and \(d\) are real numbers for \(ad - bc = 1\), \(a + d = \pm 2\) and \(c \neq 0\). For every \(x \in (-\frac{d}{c}, \alpha)\), the number \(g^{N_1}(x)\) is contained in the interval \((-\infty, -\frac{d}{c})\) for some \(N_1 \in \mathbb{N}\).

In the case that \((a + d)c < 0\), the lemma holds as follows.
Lemma 5.9. Let \( g \) be the real parabolic Möbius map, which is \( g(x) = \frac{ax + b}{cx + d} \) where \( a, b, c \) and \( d \) are real numbers for \( ad - bc = 1 \), \( a + d = \pm 2 \) and \( c \neq 0 \). Suppose that \( (a + d)c < 0 \). Then \( -\infty < x < \alpha \) if and only if \( \alpha < g^{-1}(x) < -\frac{d}{c} \). If the inequality \( \alpha < x < \infty \) holds, then the inequality

\[
\alpha < \cdots < g^{-n}(x) < g^{-(n-1)}(x) < \cdots < g^{-1}(x) < x
\]

holds and \( \lim_{n \to +\infty} g^{-n}(x) = \alpha \).

6 Non stability of real parabolic Möbius map

In this section, we prove the non stability of the sequence \( \{b_n\}_{n \in \mathbb{N}_0} \) with for any initial point \( b_0 \in \mathbb{R} \) satisfying \( b_{n+1} = g(b_n) \) for every \( n \in \mathbb{N}_0 \).

Theorem 6.1. Let \( g \) be the real parabolic Möbius map, which is \( g(x) = \frac{ax + b}{cx + d} \) where \( a, b, c \) and \( d \) are real numbers for \( ad - bc = 1 \), \( a + d = \pm 2 \) and \( c \neq 0 \). Let \( \{b_n\}_{n \in \mathbb{N}_0} \) be the sequence satisfying \( b_{n+1} = g(b_n) \) for every \( n \in \mathbb{N}_0 \). Then for any \( b_0 \in \mathbb{R} \) and for any given \( \varepsilon > 0 \), the sequence \( \{b_n\}_{n \in \mathbb{N}_0} \) has no Hyers-Ulam stability.

Proof. Assume first that \( (a+d)c > 0 \). The proof for the case that \( (a+d)c < 0 \) is similar. Corollary 5.4 and Corollary 5.8 imply that for each \( x \in \mathbb{R} \) there exists \( N \in \mathbb{N} \) such that \( g^N(x) \in (\alpha, \alpha + \frac{\varepsilon}{2}) \). Lemma 5.5 and Lemma 5.7 imply that \( \lim_{n \to +\infty} g^n(x) = \alpha \). Using same method in the proof of Proposition 4.1, we show that there exists a sequence \( \{a_n\}_{n \in \mathbb{N}_0} \) which satisfies the following properties

1. \( |a_{n+1} - g(a_n)| \leq \varepsilon \) for all \( n \in \mathbb{N}_0 \),
2. the sequence \( \{a_n\}_{n \in \mathbb{N}_0} \) is pre-periodic one, and
3. \( |a_{N_0} - b_{N_0}| \geq 1 + \varepsilon \) for some big enough \( N_0 \in \mathbb{N} \).

Then \( |a_n - b_n| \geq 1 \) for infinitely many \( n \in \mathbb{N} \) due to the periodicity of the sequence \( \{a_n\}_{n \in \mathbb{N}_0} \). Hence, \( \{b_n\}_{n \in \mathbb{N}_0} \) has no Hyers-Ulam stability. We construct the sequence \( \{a_n\}_{n \in \mathbb{N}_0} \) satisfying the above properties.

For every \( x \in \mathbb{R} \), for some \( N_0 \in \mathbb{N} \), \( g^{N_0}(x) \) is contained in the interval \( (\alpha, \alpha + \frac{\varepsilon}{2}) \) by Lemma 5.5. Observe that the distance between any point \( x' \in (\alpha - \frac{\varepsilon}{2}, \alpha) \) and \( g^{N_0}(x) \) is less than \( \varepsilon \). Moreover, the point \( x' \) satisfies
that $g^{N_1}(x') \in (-\infty, -\frac{d}{c})$ by Corollary 5.8. Since $g^{N_1}(x')$ can be an arbitrary point in the interval $(-\infty, -\frac{d}{c})$ and the distance between $\alpha$ and $-\frac{d}{c}$ is $\frac{1}{|c|}$, we may choose the point $q$ such that $q \in (\alpha - \epsilon, \alpha)$ and $|g^{N_1}(q) - \alpha| > \max\left\{\frac{1}{|c|}, 1 + \epsilon\right\}$. Thus $g^{N_1}(q) \in (\alpha, \alpha + \frac{\epsilon}{2})$ for some $N_2 \in \mathbb{N}$.

Define the pre-periodic sequence $\{a_n\}_{n \in \mathbb{N}_0}$ as follows

$\{a_0, g(a_0), g^2(a_0), \ldots, g^{N_0}(a_0), q, g(q), \ldots, g^{N_1}(q), \ldots, g^{N_1+N_2}(q), q, g(q), \ldots\}$.

Thus $|a_{n+1} - g(a_n)| \leq \epsilon$ for all $n \in \mathbb{N}_0$ and $a_{N_1+N_2+k} = a_{k+1}$ for every $k \geq N_0$. Furthermore, since $g^{N_1}(q) = a_{N_0+N_1+2}$ and $b_n \in (\alpha, \alpha + \frac{\epsilon}{2})$ for all big enough $n \geq N$, the inequality

$$|a_{N_0+N_1+2+k(N_1+N_2)} - b_{N_0+N_1+2}| = |a_{N_0+N_1+2} - b_{N_0+N_1+2}|$$

$$\geq |a_{N_0+N_1+2} - \alpha| - |\alpha - b_{N_0+N_1+2}|$$

$$\geq 1 + \epsilon - \frac{\epsilon}{2}$$

$$> 1$$

for all $k \geq N_0 + N$. Then $|b_n - a_n| > 1$ for infinitely many $n \in \mathbb{N}$. Hence, the sequence $\{b_n\}_{n \in \mathbb{N}_0}$ does not have Hyers-Ulam stability.

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