On Some Exponential Sums Related to the Coulter’s Polynomial

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Abstract

In this paper, the formulas of some exponential sums over finite field, related to the Coulter’s polynomial, are settled based on the Coulter’s theorems on Weil sums, which may have potential application in the construction of linear codes with few weights.

Index Terms

Gauss sum, exponential sum over finite field, the Coulter’s polynomial, linear code, set cardinality.

I. INTRODUCTION

THROUGHOUT this paper, we fix the following notations:

• \( p \) is an odd prime, and \( q = p^e \) with \( e \) a positive integer.
• \( \alpha \) is a positive integer, and \( d = \gcd(\alpha, e) \).
• \( F_q \) denotes the Galois field with \( q \) elements, whose cyclic multiplicative group is \( F_q^* \).
• \( f(x) \) is the complex conjugate of the function \( f(x) \) over \( F_q \), while \( \hat{f}(x) \) is the same function but with \( x \in F_p \).
• \( \zeta = e^{2\pi \sqrt{-1}} \).
• \( a \in F_q^* \).

Let \( a \in F_q^* \), \( C(x) = q^{\alpha}x^p + ax \in F_q[x] \) is called the Coulter’s polynomial over \( F_q \), based on which is or not a permutation polynomial of \( F_q \), and on how whose solution set is for a given \( b \in F_q^* \) such that \( C(x) = -b^\alpha \), Coulter gave the evaluation of two exponential sums of type Weil \([8], [9]\), which have important applications in the coding theory, information security, and combinatorics, for instance see \([5], [7], [10]–[12]\).

Let \( a, c \in F_p \), and \( b \in F_q^* \). The two exponential sums related to the Coulter’s polynomial, and to be studied in this paper, are defined below.

\[
A_\alpha(a) = \sum_{y \in F_p} \sum_{x \in F_q} \zeta_p^{ay + yTr(x^{p\alpha} + 1)},
\]

\[
B_\alpha(a, c) = \sum_{y \in F_p} \sum_{z \in F_p} \zeta_p^{-ay - cz} \sum_{x \in F_q} \zeta_p^{Tr(yx^{p\alpha} + 1 + zb)},
\]

The algebraic evaluation of (1) and (2) is crucial in the determination of the cardinalities of some subsets of \( F_q \), which are involved in the construction of linear codes with few weights, especially by using the generic method called defining set method, firstly introduced by Ding et al \([1], [2]\), and much studied later, see, for instance \([4]–[7]\). In this paper, we are concerned with the following two subsets of \( F_q \):

\[
D_\alpha(a) = \{ x \in F_q : Tr(x^{p\alpha} + 1) = a \},
\]

\[
M_\alpha(a, c) = \{ x \in F_q : Tr(x^{p\alpha} + 1) = a \text{ and } Tr(bx) = c \}.
\]
The subset (3) was used in (7) as the defining set for case \( \frac{d}{a} \) being even. If both \( a = 0 \) and \( c = 0 \), the defining set (3) may be made smaller by a factor \( p - 1 \), to derive the punctured version of the original linear codes [2], [5].

The cardinalities of (3) and (4) are defined below:

\[
n_\alpha(a) = \begin{cases} |D_\alpha(a) \cup \{0\}|, & \text{if } a = 0, \\ |D_\alpha(a)|, & \text{if } a \neq 0, \end{cases}
\]

and

\[
N_\alpha(a, c) = |M_\alpha(a, c)|.
\]

In this paper, it is aimed at establishing the algebraic formulas to the exponential sums (1) and (2), and then applying them to evaluate the cardinalities of the subsets (3) and (4), i.e., \( n_\alpha(a) \) and \( N_\alpha(a, c) \), for case \( \frac{d}{a} \) being odd. The rest of the paper is organized as follows: in Section II, mathematical background such as group characters, Gauss sum, and the Coulter’s formulae, is presented. The formulae for the exponential sums (1) and (2), and for the set cardinalities (3) and (4), are given in Section III, along with their proofs. Finally, a brief concluding remarks are given in Section IV.

II. MATHEMATICAL BACKGROUND

An additive character of \( \mathbb{F}_q \), \( \chi \), is a nonzero function from \( \mathbb{F}_q \) to a set of nonzero complex numbers such that for any pair \( (x, y) \in \mathbb{F}_q \times \mathbb{F}_q \), \( \chi(x + y) = \chi(x)\chi(y) \). In this paper, the complex conjugate of \( \chi \) is denoted by \( \bar{\chi} \). For each \( b \in \mathbb{F}_q \), an additive character of \( \mathbb{F}_q \) can be defined below

\[
\chi_b(c) = e^{2\pi i bc}, \text{ for all } c \in \mathbb{F}_q,
\]

where \( \zeta_p = e^{\frac{2\pi i}{p-1}} \). In (7), when setting \( b = 0 \), the resultant character \( \chi_0 \) is said to be trivial since for all \( c \in \mathbb{F}_q \), \( \chi_0(c) = 1 \). The character \( \chi_1 \) is called the canonical additive character. It was shown that any additive character of \( \mathbb{F}_q \) can be written as \( \chi_b(x) = \chi_1(bx) \) [13, Chapter 5]. In this paper, the canonical additive character is used and its subscript omitted.

The orthogonal property of the additive character over \( \mathbb{F}_q \) is resumed in the following [13, Chapter 5]:

\[
\sum_{x \in \mathbb{F}_q} \chi(bx) = \begin{cases} q, & \text{if } b = 0, \\ 0, & \text{otherwise}. \end{cases}
\]

A multiplicative character \( \psi \) of \( \mathbb{F}_q \) over \( \mathbb{F}_q^* \) is a nonzero function from \( \mathbb{F}_q^* \) to a set of nonzero complex number such that for any \( (x, y) \in \mathbb{F}_q^* \times \mathbb{F}_q^* \), \( \psi(xy) = \psi(x)\psi(y) \). Let \( \theta \) be a primitive element of \( \mathbb{F}_q^* \). Then, any multiplicative character over \( \mathbb{F}_q^* \) can be written as

\[
\psi_j(\theta^k) = e^{2\pi i jk/(q-1)},
\]

where \( 0 \leq j, k \leq q - 2 \). The multiplicative character \( \psi_{(q-1)/2} \) is called the quadratic character of \( \mathbb{F}_q \), denoted by \( \eta \). In this paper, suppose that \( \eta(0) = 0 \). With the canonical additive character and the quadratic character, the quadratic Gauss sum over \( \mathbb{F}_q \) can be defined by

\[
G(\eta, \chi) = \sum_{x \in \mathbb{F}_q} \eta(x)\chi(x).
\]

In order not to confuse with the complex conjugation, let \( \hat{\chi} \) denote the canonical additive character over \( \mathbb{F}_p^* \), and \( \hat{\eta} \) the quadratic character over \( \mathbb{F}_p^* \), respectively.

**Lemma 2.1 ([13], Theorem 5.15):** Let the symbols be that presented before. Then,

\[
\begin{align*}
G(\eta, \chi) &= (-1)^{(p-1)^2/4} \sqrt{q}, \\
G(\hat{\eta}, \hat{\chi}) &= \sqrt{-1}^{(p-1)^2/4} \sqrt{p}.
\end{align*}
\]
In some situation, it is necessary to know what is the value of \( \eta(x) \) when \( x \in \mathbb{F}_p^* \), which is answered by the following lemma [5, Lemma 7]:

**Lemma 2.2:** Let the symbols be that presented before. Then,

\[
\eta(x) = \begin{cases} 
1, & \text{if } e \text{ is even}, \\
\hat{\eta}(x), & \text{if } e \text{ is odd}, 
\end{cases}
\]

where \( x \in \mathbb{F}_p^* \).

Next lemma [13, Theorem 5.33] gives the explicit evaluation of exponential sum of the quadratic polynomial over finite field, \( f(x) = a_2x^2 + a_1x + a_0 \in \mathbb{F}_q[x] \), with \( a_2 \neq 0 \):

**Lemma 2.3:**

\[
\sum_{x \in \mathbb{F}_q} \chi(f(x)) = \chi(a_0 - a_1^2(4a_2)^{-1})\eta(a_2)G(\eta, \chi).
\]

Recall that \( q = p^e \) where \( p \) is an odd prime and \( e \) a positive integer, \( d = \gcd(e, \alpha) \). Let \( \mathcal{E}(x) = ap^\alpha x^{p^\alpha} + ax \in \mathbb{F}_q[x] \). We call \( \mathcal{E}(x) \) the Coulter’s polynomial, based on which is or not a permutation polynomial over \( \mathbb{F}_q \), Coulter [9] completely determined the following Weil sum:

\[
S_\alpha(a, b) = \sum_{x \in \mathbb{F}_q} \chi(ax^{p^\alpha+1} + bx),
\]

where \( a \in \mathbb{F}_q^*, b \in \mathbb{F}_q \).

The explicit formulae to evaluate (9) are resumed in the next two lemmas:

**Lemma 2.4 (9, Theorem 1):** Let \( q \) be odd and suppose \( \mathcal{E}(x) \) is a permutation polynomial over \( \mathbb{F}_q \). Let \( x_0 \) be the unique solution of the equation \( \mathcal{E}(x) = -b^{p^\alpha}, b \neq 0 \). The evaluation of (9) partitions into the following two cases.

1. If \( e/d \) is odd then

\[
S_\alpha(a, b) = \begin{cases} 
(-1)^{e-1} \sqrt{q}\eta(-a)\chi(ax_0^{p^\alpha+1}) & \text{if } p \equiv 1 \pmod{4} \\
(-1)^e \sqrt{-1}^{3e} \sqrt{q}\eta(-a)\chi(ax_0^{p^\alpha+1}) & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]  

(10)

2. If \( e/d \) is even then \( e = 2m \), \( a(q-1)/(p^d+1) \neq (-1)^{m/d} \) and

\[
S_\alpha(a, b) = \omega(-1)^{m/d} p^m \chi(ax_0^{p^\alpha+1}).
\]

**Lemma 2.5 (9, Theorem 2):** Let \( q = p^e \) be odd and suppose \( \mathcal{E}(x) \) is not a permutation polynomial over \( \mathbb{F}_q \). Then for \( b \neq 0 \) we have \( S_\alpha(a, b) = 0 \) unless the equation \( \mathcal{E}(x) = -b^{p^\alpha} \) is solvable. If this equation is solvable, with some solution \( x_0 \) say, then

\[
S_\alpha(a, b) = \omega(-1)^{m/d} p^m \chi(ax_0^{p^\alpha+1}).
\]

To ease symbol manipulation, in this paper we define the following constant, relative to the formulae to evaluate (9) when \( \frac{e}{d} \) is odd:

\[
\kappa = \begin{cases} 
(-1)^{e-1} \sqrt{q}, & \text{if } p \equiv 1 \pmod{4}, \\
(-1)^e \sqrt{-1}^{3e} \sqrt{q}, & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]  

(11)

The following lemma that computes the constant in (11) is straightforward:

**Lemma 2.6:** Let the symbols be that of (11) and before, then

- If \( e = 2m \), then

\[
\kappa = \begin{cases} 
-p^m, & \text{if } p \equiv 1 \pmod{4}, \\
-(1)^m p^m, & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]  

(12)
If \( e \) is odd, then
\[
\kappa = \begin{cases} 
  p^{\frac{e}{2}}, & \text{if } p \equiv 1 \pmod{4}, \\
  -\sqrt{-1} p^{\frac{e}{2}}, & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\] (13)

A special case of (9), \( S_\alpha(a, 0) \) where \( a \in \mathbb{F}_p^* \), was studied in [8], resumed in the following two lemmas:

**Lemma 2.7 ([8], Theorem 1):** Let \( \frac{e}{d} \) be odd. Then,
\[
S_\alpha(a, 0) = \begin{cases} 
  (\lambda)^{e-1} \sqrt{q\eta}(a), & \text{if } p \equiv 1 \pmod{4} \\
  (\lambda)^{e-1} \sqrt{q\eta}(a), & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]

**Lemma 2.8 ([8], Theorem 2):** Let \( \frac{e}{d} \) be even. Then,
\[
S_\alpha(a, 0) = \begin{cases} 
  p^m, & \text{if } a^{(q-1)/(p^d+1)} \neq (-1)^{m/d} \text{ and } m/d \text{ even} \\
  -p^m, & \text{if } a^{(q-1)/(p^d+1)} \neq (-1)^{m/d} \text{ and } m/d \text{ odd} \\
  p^{m+d}, & \text{if } a^{(q-1)/(p^d+1)} = (-1)^{m/d} \text{ and } m/d \text{ odd} \\
  -p^{m+d}, & \text{if } a^{(q-1)/(p^d+1)} = (-1)^{m/d} \text{ and } m/d \text{ even}.
\end{cases}
\]

### III. Main results and their proofs

This section is devoted to the presentation of the main results of this paper, and their proofs by using the series of lemmas of the previous section. Lemma 3.1 and 3.2 compute the exponential sums [11] and [12], respectively; while Theorem 3.1 and 3.2 explicitly give the set cardinalities of (3) and (4).

**Lemma 3.1:** Let \( q = p^e \) where \( p \) is an odd prime and \( e \) a positive integer, \( a \in \mathbb{F}_p^* \), and \( d = \gcd(\alpha, e) \). Suppose \( \frac{e}{d} \) to be odd. Then,
- if \( e \) is even with \( e = 2m \), we have
  \[
  A_\alpha(0) = \begin{cases} 
  -(p-1)p^m, & \text{if } p \equiv 1 \pmod{4} \\
  -(-1)^m(p-1)p^m, & \text{if } p \equiv 3 \pmod{4}.
  \end{cases}
  \]
  and
  \[
  A_\alpha(a) = \begin{cases} 
  p^m, & \text{if } p \equiv 1 \pmod{4} \\
  (-1)^mp^m, & \text{if } p \equiv 3 \pmod{4}.
  \end{cases}
  \]
- If \( e \) is odd, then
  \[
  A_\alpha(0) = \begin{cases} 
  0, & \text{if } p \equiv 1 \pmod{4} \\
  0, & \text{if } p \equiv 3 \pmod{4},
  \end{cases}
  \]
  and
  \[
  A_\alpha(a) = \begin{cases} 
  \tilde{\eta}(a)p^{\frac{e+1}{2}}, & \text{if } p \equiv 1 \pmod{4} \\
  -\sqrt{-1} \tilde{\eta}(a)p^{\frac{e+1}{2}}, & \text{if } p \equiv 3 \pmod{4}.
  \end{cases}
  \]

**Proof:** From Lemma 2.7 and (11), it is easy to obtain that
\[
A_\alpha(a) = \sum_{y \in \mathbb{F}^*_{p}} \zeta_{p}^{-ay} \sum_{x \in \mathbb{F}_q} \zeta_{p}^{y Tr(x^{p^m} \alpha)} = \sum_{y \in \mathbb{F}^*_{p}} \zeta_{p}^{-ay} S_\alpha(y, 0)
\]
\[
= \begin{cases} 
  \sum_{y \in \mathbb{F}^*_{p}} \zeta_{p}^{-ay}(-1)^{e-1} \sqrt{q\eta}(y), & \text{if } p \equiv 1 \pmod{4} \\
  \sum_{y \in \mathbb{F}^*_{p}} \zeta_{p}^{-ay}(-1)^{e-1} \sqrt{q\eta}(y), & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\] (14)

- Let \( e \) be even with \( e = 2m \). By Lemma 2.2, \( \eta(y) = 1 \) for all \( y \in \mathbb{F}^*_{p} \). From (14), we have
  \[
  A_\alpha(a) = \begin{cases} 
  -p^m \sum_{y \in \mathbb{F}^*_{p}} \zeta_{p}^{-ay}, & \text{if } p \equiv 1 \pmod{4} \\
  -(-1)^m p^m \sum_{y \in \mathbb{F}^*_{p}} \zeta_{p}^{-ay}, & \text{if } p \equiv 3 \pmod{4}.
  \end{cases}
  \]
If \( a \neq 0 \), by the orthogonal property of the additive character of \( \mathbb{F}_q \), \( \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-ay} = -1 \). Hence,

\[
A_\alpha(a) = \begin{cases} 
    p^m & \text{if } p \equiv 1 \pmod{4}, \\
    (-1)^m p^m & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]

If \( a = 0 \), \( \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-ay} = p - 1 \). Hence,

\[
A_\alpha(0) = \begin{cases} 
    -p^m(p - 1) & \text{if } p \equiv 1 \pmod{4}, \\
    -(1)^m p^m(p - 1) & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]

- Let \( e \) be odd. By Lemma 2.2, \( \eta(y) = \hat{\eta}(y) \) for all \( y \in \mathbb{F}_p^* \). From (14) for \( a \neq 0 \), we have

\[
A_\alpha(a) = \begin{cases} 
    \hat{\eta}(a)p^{e+1} \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-ay} \hat{\eta}(-ay) & \text{if } p \equiv 1 \pmod{4}, \\
    -\sqrt{-1} \hat{\eta}(a)p^{e+1} \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-ay} \hat{\eta}(-ay) & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]

In (15) is used the fact that \( \hat{\eta}(-1) = 1 \) if \( p \equiv 1 \pmod{4} \), and \( \hat{\eta}(-1) = -1 \) if \( p \equiv 3 \pmod{4} \). It is clear that \( \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-ay} \hat{\eta}(-ay) = G(\hat{\eta}, \hat{\chi}) \). By Lemma 2.1, \( G(\hat{\eta}, \hat{\chi}) = \sqrt{p} \) if \( p \equiv 1 \pmod{4} \), and \( G(\hat{\eta}, \hat{\chi}) = -\sqrt{p} \) if \( p \equiv 3 \pmod{4} \). Combining so far the mentioned fact and identities, and (15), we can obtain

\[
A_\alpha(a) = \begin{cases} 
    \hat{\eta}(a)p^{e+1} & \text{if } p \equiv 1 \pmod{4}, \\
    -\sqrt{-1} \hat{\eta}(a)p^{e+1} & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]

In (14), set \( a = 0 \) and \( \eta(y) = \hat{\eta}(y) \). It is easy to see that \( A_\alpha(0) = 0 \) for both \( p \equiv 1 \pmod{4} \) and \( p \equiv 3 \pmod{4} \), since \( \sum_{y \in \mathbb{F}_p^*} \hat{\eta}(y) = 0 \).

**Lemma 3.2:** Let \( q = p^e \) where \( p \) is an odd prime and \( e \) a positive integer, and \( d = \gcd(\alpha, e) \). In addition, suppose \( \frac{e}{d} \) to be odd. For the given \( b \in \mathbb{F}_q^* \), let \( \gamma \) be the unique solution of \( f(x) = x^{p^{2e}} + x = -b^{p^e} \).

**Case 1** \( a = c = 0 \).

- Let \( e \) be even with \( e = 2m \).
  - If \( p \equiv 1 \pmod{4} \), then
    \[
    B_\alpha(0,0) = \begin{cases} 
        -p^m(p - 1)^2, & \text{if } Tr(\gamma^{p^{2m}+1}) = 0, \\
        p^m(p - 1), & \text{if } Tr(\gamma^{p^{2m}+1}) \neq 0.
    \end{cases}
    \]
  - If \( p \equiv 3 \pmod{4} \), then
    \[
    B_\alpha(0,0) = \begin{cases} 
        -(-1)^m p^m(p - 1)^2, & \text{if } Tr(\gamma^{p^{2m}+1}) = 0, \\
        (-1)^m p^m(p - 1), & \text{if } Tr(\gamma^{p^{2m}+1}) \neq 0.
    \end{cases}
    \]

- Let \( e \) be odd.
  - If \( p \equiv 1 \pmod{4} \), then
    \[
    B_\alpha(0,0) = \begin{cases} 
        0, & \text{if } Tr(\gamma^{p^{2m}+1}) = 0, \\
        \hat{\eta}(Tr(\gamma^{p^{2m}+1}))p^{e+1}(p - 1), & \text{if } Tr(\gamma^{p^{2m}+1}) \neq 0.
    \end{cases}
    \]
  - If \( p \equiv 3 \pmod{4} \), then
    \[
    B_\alpha(0,0) = \begin{cases} 
        0, & \text{if } Tr(\gamma^{p^{2m}+1}) = 0, \\
        -\sqrt{-1}^{e+1} \hat{\eta}(Tr(\gamma^{p^{2m}+1}))p^{e+1}(p - 1), & \text{if } Tr(\gamma^{p^{2m}+1}) \neq 0.
    \end{cases}
    \]

**Case 2** \( a = 0, \ c \neq 0 \).

- Let \( e \) be even with \( e = 2m \).
- If $p \equiv 1 \pmod{4}$, then
  \[ B_\alpha(0, c) = \begin{cases} p^m(p - 1), & \text{if } Tr(\gamma^{p^\alpha+1}) = 0, \\ -p^m, & \text{if } Tr(\gamma^{p^\alpha+1}) \neq 0. \end{cases} \]

- If $p \equiv 3 \pmod{4}$, then
  \[ B_\alpha(0, c) = \begin{cases} (-1)^mp^m(p - 1), & \text{if } Tr(\gamma^{p^\alpha+1}) = 0, \\ -(-1)^mp^m, & \text{if } Tr(\gamma^{p^\alpha+1}) \neq 0. \end{cases} \]

  - Let $e$ be odd.
    - If $p \equiv 1 \pmod{4}$, then
      \[ B_\alpha(0, c) = \begin{cases} 0, & \text{if } Tr(\gamma^{p^\alpha+1}) = 0, \\ -\hat{\eta}(Tr(\gamma^{p^\alpha+1}))p^{e+1}, & \text{if } Tr(\gamma^{p^\alpha+1}) \neq 0. \end{cases} \]
    - If $p \equiv 3 \pmod{4}$, then
      \[ B_\alpha(0, c) = \begin{cases} 0, & \text{if } Tr(\gamma^{p^\alpha+1}) = 0, \\ \sqrt{-1}^{e+1}\hat{\eta}(Tr(\gamma^{p^\alpha+1}))p^{e+1}, & \text{if } Tr(\gamma^{p^\alpha+1}) \neq 0. \end{cases} \]

**Case 3** $a \neq 0, c = 0$.

  - Let $e$ be even with $e = 2m$.
    - If $p \equiv 1 \pmod{4}$, then
      \[ B_\alpha(a, 0) = \begin{cases} p^m(p - 1), & \text{if } Tr(\gamma^{p^\alpha+1}) = 0, \\ -(1 + p\hat{\eta}(aTr(\gamma^{p^\alpha+1})))p^m, & \text{if } Tr(\gamma^{p^\alpha+1}) \neq 0. \end{cases} \]
    - If $p \equiv 3 \pmod{4}$, then
      \[ B_\alpha(a, 0) = \begin{cases} (-1)^mp^m(p - 1), & \text{if } Tr(\gamma^{p^\alpha+1}) = 0, \\ -(-1)^m(1 - p\hat{\eta}(aTr(\gamma^{p^\alpha+1})))p^m, & \text{if } Tr(\gamma^{p^\alpha+1}) \neq 0. \end{cases} \]

  - Let $e$ be odd.
    - If $p \equiv 1 \pmod{4}$, then
      \[ B_\alpha(a, 0) = \begin{cases} \hat{\eta}(a)(p - 1)p^{\frac{e+1}{2}}, & \text{if } Tr(\gamma^{p^\alpha+1}) = 0, \\ -(\hat{\eta}(a) + \hat{\eta}(Tr(\gamma^{p^\alpha+1})))p^{\frac{e+1}{2}}, & \text{if } Tr(\gamma^{p^\alpha+1}) \neq 0. \end{cases} \]
    - If $p \equiv 3 \pmod{4}$, then
      \[ B_\alpha(a, 0) = \begin{cases} -\sqrt{-1}^{e+1}\hat{\eta}(a)(p - 1)p^{\frac{e+1}{2}}, & \text{if } Tr(\gamma^{p^\alpha+1}) = 0, \\ \sqrt{-1}^{e+1}(\hat{\eta}(a) + \hat{\eta}(Tr(\gamma^{p^\alpha+1})))p^{\frac{e+1}{2}}, & \text{if } Tr(\gamma^{p^\alpha+1}) \neq 0. \end{cases} \]

**Case 4** $ac \neq 0$.

  - Let $e$ be even with $e = 2m$.
    - If $p \equiv 1 \pmod{4}$, then
      \[ B_\alpha(a, c) = \begin{cases} -p^m, & \text{if } Tr(\gamma^{p^\alpha+1}) = 0, \\ -p^m, & \text{if } Tr(\gamma^{p^\alpha+1}) = c^2/(4a), \\ -(1 + p\hat{\eta}(e^2 - 4aTr(\gamma^{p^\alpha+1})))p^m, & \text{otherwise}. \end{cases} \]
– If \( p \equiv 3 \pmod{4} \), then
\[
B_\alpha(a, c) = \begin{cases} 
-(-1)^m p^m, & \text{if } Tr(\gamma^{\alpha+1}) = 0, \\
-(-1)^m p^m, & \text{if } Tr(\gamma^{\alpha+1}) = c^2/(4a), \\
-(-1)^m (1 + p\tilde{\eta}(c^2 - 4aTr(\gamma^{\alpha+1})))p^m, & \text{otherwise.}
\end{cases}
\]

• Let \( \alpha \) be odd.
  – If \( p \equiv 1 \pmod{4} \), then
\[
B_\alpha(a, c) = \begin{cases} 
-\tilde{\eta}(a)p^{\frac{\alpha+1}{2}}, & \text{if } Tr(\gamma^{\alpha+1}) = 0, \\
(\tilde{\eta}(Tr(\gamma^{\alpha+1}))(p-1) - \tilde{\eta}(a))p^{\frac{\alpha+1}{2}}, & \text{if } Tr(\gamma^{\alpha+1}) = c^2/(4a), \\
-(\tilde{\eta}(Tr(\gamma^{\alpha+1}))) + \tilde{\eta}(a))p^{\frac{\alpha+1}{2}}, & \text{otherwise.}
\end{cases}
\]

– If \( p \equiv 3 \pmod{4} \), then
\[
B_\alpha(a, c) = \begin{cases} 
\sqrt{-1}(-1)^{\frac{\alpha+1}{2}}\tilde{\eta}(a)p^{\frac{\alpha+1}{2}}, & \text{if } Tr(\gamma^{\alpha+1}) = 0, \\
-\sqrt{-1}(-1)^{\frac{\alpha+1}{2}}(\tilde{\eta}(Tr(\gamma^{\alpha+1}))(p-1) - \tilde{\eta}(a))p^{\frac{\alpha+1}{2}}, & \text{if } Tr(\gamma^{\alpha+1}) = c^2/(4a), \\
\sqrt{-1}(-1)^{\frac{\alpha+1}{2}}(\tilde{\eta}(Tr(\gamma^{\alpha+1}))) + \tilde{\eta}(a))p^{\frac{\alpha+1}{2}}, & \text{otherwise.}
\end{cases}
\]

**Proof:** We only prove Case \( ac \neq 0 \) since the proofs of other cases are similar and easier. Since \( \frac{\xi}{\eta} \) is odd, \( f(x) = x^{p^{2\alpha}} + x = 0 \) has no solution in \( \mathbb{F}_q^* \), and is a permutation polynomial over \( \mathbb{F}_q \) [3], [9], [14]. Hence, for the given \( b \in \mathbb{F}_q^* \), the equation \( f(x) = -b^{p^{\alpha}} \) has an unique solution in \( \mathbb{F}_q^* \). Denote the unique solution as \( \gamma \), and let \( y, z \in \mathbb{F}_q^* \). It is easy to see that \( y^{-1}\gamma z \) is the unique solution of the equation \( y^{p^\alpha}x^{p^{2\alpha}} + yx = -(bz)^{p^{\alpha}} \). From (10) and (11), we have
\[
S_\alpha(y, bz) = \kappa\eta(-y)\chi(y(y^{-1}\gamma z)^{p^{\alpha+1}}) = \kappa\eta(-y)\chi(y^{-1}\gamma^{p^{\alpha+1}}z^2) = \kappa\eta(-y)\zeta_p^{-y^{-1}z^2Tr(\gamma^{\alpha+1})}.
\] (16)

From the definition of \( B_\alpha(a, c) \) in (2), and (16), we have
\[
B_\alpha(a, c) = \sum_{y \in \mathbb{F}_q^*} \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-ay-cz} \sum_{x \in \mathbb{F}_p^*} \chi(yx^{p^{\alpha+1}} +bzx)
= \sum_{y \in \mathbb{F}_q^*} \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-ay-cz} S_\alpha(y, bz)
= \sum_{y \in \mathbb{F}_q^*} \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-ay-cz} \kappa\eta(-y)\zeta_p^{-y^{-1}z^2Tr(\gamma^{\alpha+1})}
= \kappa \sum_{y \in \mathbb{F}_q^*} \eta(-y)\zeta_p^{-ay} \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-y^{-1}Tr(\gamma^{\alpha+1})z^2-cz}.
\] (17)

If \( Tr(\gamma^{\alpha+1}) = 0 \), then, from the last equality of (17), we have
\[
B_\alpha(a, c) = \kappa \sum_{y \in \mathbb{F}_q^*} \eta(-y)\zeta_p^{-ay} \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-cz}
= -\kappa \sum_{y \in \mathbb{F}_q^*} \eta(-y)\zeta_p^{-ay}.
\] (18)

Recall that the Gauss sum over \( \mathbb{F}_p^* \), \( G(\tilde{\eta}, \tilde{\chi}) \), is equal to \( \sqrt{p} \) if \( p \equiv 1 \pmod{4} \), and \( \sqrt{-p} \) if \( p \equiv 3 \pmod{4} \) (see Lemma 2.1). The values of the constant occurring in the Coulter’s formulae (10) are explicitly listed in (12) and (13) to ease computation.
For $Tr(\gamma^{p^{\alpha}+1}) = 0$, two cases are distinguished:

- Let $e$ be even with $e = 2m$. From Lemma 2.2, $\eta(y) = 1$. Thus, (18) becomes
  
  \begin{align*}
  B_\alpha(a, c) &= -\kappa \sum_{y \in F_p^*} \zeta_p^{-ay} \\
  &= \kappa \\
  &= \begin{cases} 
  -p^m & \text{if } p \equiv 1 \pmod{4}, \\
  -(-1)^mp^m & \text{if } p \equiv 3 \pmod{4}.
  \end{cases}
  \end{align*}
  \tag{19}

- Let $e$ be odd. From Lemma 2.2, $\eta(-y) = \hat{\eta}(-y)$. Thus, (18) becomes
  
  \begin{align*}
  B_\alpha(a, c) &= \kappa \sum_{y \in F_p^*} \eta(-y)\zeta_p^{-ay} \sum_{z \in F_p^*} \zeta_p^{-cz} \\
  &= -\kappa \sum_{y \in F_p^*} \hat{\eta}(-y)\zeta_p^{-ay} \\
  &= -\kappa \hat{\eta}(a) \sum_{y \in F_p^*} \hat{\eta}(-ay)\zeta_p^{-ay} \\
  &= -\kappa \hat{\eta}(a)G(\hat{\eta}, \hat{\chi}) \\
  &= \begin{cases} 
  -\hat{\eta}(a)p^{e+1} & \text{if } p \equiv 1 \pmod{4}, \\
  \sqrt{-1}^{e+1}\hat{\eta}(a)p^{e+1} & \text{if } p \equiv 3 \pmod{4}.
  \end{cases}
  \end{align*}
  \tag{20}

Now, suppose that $Tr(\gamma^{p^{\alpha}+1}) \neq 0$. Following the last equality of (17), we have

\begin{align*}
B_\alpha(a, c) &= \kappa \sum_{y \in F_p^*} \eta(-y)\zeta_p^{-ay} \sum_{z \in F_p^*} \zeta_p^{-y^{-1}Tr(\gamma^{p^{\alpha}+1})z^2-cz} \\
&= \kappa \sum_{y \in F_p^*} \eta(-y)\zeta_p^{-ay}(\sum_{z \in F_p^*} \zeta_p^{-y^{-1}Tr(\gamma^{p^{\alpha}+1})z^2-cz} - 1).
\end{align*}
\tag{21}

By applying Lemma 2.3 to the sum $\sum_{z \in F_p^*} \zeta_p^{-y^{-1}Tr(\gamma^{p^{\alpha}+1})z^2-cz}$ in (21) with $a_2 = -y^{-1}Tr(\gamma^{p^{\alpha}+1}), a_1 = -c$, and $a_0 = 0$, we obtain

\begin{align*}
&\sum_{z \in F_p^*} \zeta_p^{-y^{-1}Tr(\gamma^{p^{\alpha}+1})z^2-cz} \\
&= \sum_{z \in F_p^*} \hat{\chi}(-y^{-1}Tr(\gamma^{p^{\alpha}+1})z^2 - cz) \\
&= \hat{\chi}(-\frac{c^2y}{4Tr(\gamma^{p^{\alpha}+1})})\hat{\eta}(-y)\frac{Tr(\gamma^{p^{\alpha}+1})}{y}G(\hat{\eta}, \hat{\chi}) \\
&= \hat{\eta}(Tr(\gamma^{p^{\alpha}+1}))G(\hat{\eta}, \hat{\chi})\hat{\eta}(-y)\zeta_p^{\frac{c^2y}{4Tr(\gamma^{p^{\alpha}+1})}}.
\end{align*}
\tag{22}

Substitute the last equality of (22) into (21), we have

\begin{align*}
B_\alpha(a, c) &= \kappa \sum_{y \in F_p^*} \eta(-y)\zeta_p^{-ay}(\hat{\eta}(Tr(\gamma^{p^{\alpha}+1}))G(\hat{\eta}, \hat{\chi})\hat{\eta}(-y)\zeta_p^{\frac{c^2y}{4Tr(\gamma^{p^{\alpha}+1})}} - 1) \\
&= \kappa \hat{\eta}(Tr(\gamma^{p^{\alpha}+1}))G(\hat{\eta}, \hat{\chi}) \sum_{y \in F_p^*} \eta(-y)\hat{\eta}(-y)\zeta_p^{\frac{c^2y}{4Tr(\gamma^{p^{\alpha}+1})}} - \kappa \sum_{y \in F_p^*} \eta(-y)\zeta_p^{-ay}.
\end{align*}
\tag{23}
Consider case $Tr(\gamma^{p^\alpha+1}) = c^2/(4a)$. Then, from (23), we get

$$B_\alpha(a, c) = \kappa\tilde{n}(Tr(\gamma^{p^\alpha+1}))G(\tilde{\eta}, \tilde{\chi}) \sum_{y \in \mathbb{F}_p^*} \tilde{n}(-y)\tilde{n}(-y) - \kappa \sum_{y \in \mathbb{F}_p^*} \eta(-y)\zeta_p^{-ay}$$

(24)

For $Tr(\gamma^{p^\alpha+1}) = c^2/(4a)$, two cases are distinguished:

- Let $e$ be even with $e = 2m$. From Lemma \ref{lem:2.2}, $\eta(-y) = 1$. Thus, (24) becomes

$$B_\alpha(a, c) = \kappa\tilde{n}(Tr(\gamma^{p^\alpha+1}))G(\tilde{\eta}, \tilde{\chi}) \sum_{y \in \mathbb{F}_p^*} \tilde{n}(-y)\tilde{n}(-y) - \kappa \sum_{y \in \mathbb{F}_p^*} \eta(-y)\zeta_p^{-ay}$$

$$= \kappa\tilde{n}(Tr(\gamma^{p^\alpha+1}))G(\tilde{\eta}, \tilde{\chi}) \cdot 0 - \kappa \cdot (-1)$$

$$= \kappa$$

(25)

- Let $e$ be odd. From Lemma \ref{lem:2.2}, $\eta(-y) = \tilde{n}(-y)$. Thus, (24) becomes

$$B_\alpha(a, c) = \kappa\tilde{n}(Tr(\gamma^{p^\alpha+1}))G(\tilde{\eta}, \tilde{\chi}) \sum_{y \in \mathbb{F}_p^*} \tilde{n}(-y)\tilde{n}(-y) - \kappa \sum_{y \in \mathbb{F}_p^*} \eta(-y)\zeta_p^{-ay}$$

$$= \kappa\tilde{n}(Tr(\gamma^{p^\alpha+1}))G(\tilde{\eta}, \tilde{\chi})(p - 1) - \kappa\tilde{n}(a)\tilde{n}(ay)\zeta_p^{-ay}$$

(26)

$$= \kappa\tilde{n}(Tr(\gamma^{p^\alpha+1}))G(\tilde{\eta}, \tilde{\chi})(p - 1) - \tilde{n}(a))$$

$$\begin{align*}
&= \left\{ \begin{array}{ll}
-\sqrt{-1}^{a+1}\tilde{n}(Tr(\gamma^{p^\alpha+1}))(p - 1) - \tilde{n}(a) & \text{if } p \equiv 1 \pmod{4}, \\
-\sqrt{-1}^{a+1}\tilde{n}(Tr(\gamma^{p^\alpha+1}))(p - 1) - \tilde{n}(a) & \text{if } p \equiv 3 \pmod{4}.
\end{array} \right.
\end{align*}$$

Now consider the most general case in (23): $Tr(\gamma^{p^\alpha+1}) \neq 0$ and $Tr(\gamma^{p^\alpha+1}) \neq c^2/(4a)$. Two cases must be dealt with:

- Let $e$ be even with $e = 2m$. From Lemma \ref{lem:2.2}, $\eta(-y) = 1$. Thus, (23) becomes

$$B_\alpha(a, c) = \kappa\tilde{n}(Tr(\gamma^{p^\alpha+1}))G(\tilde{\eta}, \tilde{\chi}) \sum_{y \in \mathbb{F}_p^*} \tilde{n}(-y)\tilde{n}(-y) - \kappa \sum_{y \in \mathbb{F}_p^*} \eta(-y)\zeta_p^{-ay}$$

$$= \kappa\tilde{n}(-c^2 - 4aTr(\gamma^{p^\alpha+1}))(c^2 - 4aTr(\gamma^{p^\alpha+1}))y \frac{(c^2 - 4aTr(\gamma^{p^\alpha+1}))y}{4Tr(\gamma^{p^\alpha+1})} - \kappa \cdot (-1)$$

$$= \kappa\tilde{n}(-c^2 - 4aTr(\gamma^{p^\alpha+1}))(c^2 - 4aTr(\gamma^{p^\alpha+1}))y \frac{(c^2 - 4aTr(\gamma^{p^\alpha+1}))y}{4Tr(\gamma^{p^\alpha+1})}$$

$$= \kappa(1 + \tilde{n}(-c^2 - 4aTr(\gamma^{p^\alpha+1})))G(\tilde{\eta}, \tilde{\chi})^2$$

$$\begin{align*}
&= \left\{ \begin{array}{ll}
-p^m(1 + \tilde{n}(c^2 - 4aTr(\gamma^{p^\alpha+1}))p) & \text{if } p \equiv 1 \pmod{4}, \\
-(-1)^mp^m(1 + \tilde{n}(c^2 - 4aTr(\gamma^{p^\alpha+1}))p) & \text{if } p \equiv 3 \pmod{4}.
\end{array} \right.
\end{align*}$$

(27)
• Let \( e \) be odd. From Lemma 2.2, \( \eta(-y) = \tilde{\eta}(-y) \). Thus, (23) becomes

\[
B_\alpha(a, c) = \kappa \tilde{\eta}(Tr(\gamma^{p^\alpha+1}))G(\tilde{\eta}, \tilde{x}) \sum_{y \in F_p^*} \tilde{\eta}(-y)^2 \zeta_p^{\frac{(e^2 - 4atTr(\gamma^{p^\alpha+1}))y}{4Tr(\gamma^{p^\alpha+1})}} - \kappa \sum_{y \in F_p^*} \tilde{\eta}(-y) \zeta_p^{-ay}
\]

\[
= \kappa \tilde{\eta}(Tr(\gamma^{p^\alpha+1}))G(\tilde{\eta}, \tilde{x}) \sum_{y \in F_p^*} \zeta_p^{\frac{(e^2 - 4atTr(\gamma^{p^\alpha+1}))y}{4Tr(\gamma^{p^\alpha+1})}} - \kappa \tilde{\eta}(a) \sum_{y \in F_p^*} \tilde{\eta}(a)y \zeta_p^{-ay}
\]

\[\text{(28)}\]

The proof is completed by gathering the results of (19)-(20) and (25)-(28).

The following theorem determines the cardinality of the subset (3), i.e., \( n_\alpha(a) \) of (5):

**Theorem 3.1:** Let \( q = p^e \) where \( p \) is an odd prime and \( e \) a positive integer, and \( d = \gcd(\alpha, e) \). In addition, suppose \( \frac{e}{d} \) to be odd. Then, \( n_\alpha(a) \) with \( a \in F_p \) of (5), can be determined as follows:

• Let \( e \) be even with \( e = 2m \).
  - If \( p \equiv 1 \pmod{4} \), then
    \[
    \begin{align*}
    n_\alpha(0) &= p^{e-1} - (p - 1)p^{m-1}, \\
    n_\alpha(a) &= p^{e-1} + p^{m-1}.
    \end{align*}
    \]
  - If \( p \equiv 3 \pmod{4} \), then
    \[
    \begin{align*}
    n_\alpha(0) &= p^{e-1} - (-1)^m(p - 1)p^{m-1}, \\
    n_\alpha(a) &= p^{e-1} + (-1)^m p^{m-1}.
    \end{align*}
    \]

• Let \( e \) be odd.
  - If \( p \equiv 1 \pmod{4} \), then
    \[
    \begin{align*}
    n_\alpha(0) &= p^{e-1}, \\
    n_\alpha(a) &= p^{e-1} + \tilde{\eta}(a)p^{\frac{e-1}{2}}.
    \end{align*}
    \]
  - If \( p \equiv 3 \pmod{4} \), then
    \[
    \begin{align*}
    n_\alpha(0) &= p^{e-1}, \\
    n_\alpha(a) &= p^{e-1} - (\sqrt{-1})^{e+1}\tilde{\eta}(a)p^{\frac{e-1}{2}}.
    \end{align*}
    \]

Where \( a \in F_p^* \).

**Proof:** From (5), we have

\[
n_\alpha(a) = \frac{1}{p} \sum_{x \in F_q^*} \left( \sum_{y \in F_p^*} \zeta_p(yTr(x^{p^\alpha+1}) - a) \right)
\]

\[
= p^{e-1} + \frac{1}{p} \sum_{y \in F_p^*} \zeta_p^{-ay} \sum_{x \in F_q^*} \zeta_p Tr(x^{p^\alpha+1})
\]

\[
= p^{e-1} + p^{-1} A_\alpha(a).
\]

(29)

The actual theorem can be easily proven by combining (29) and the formulae of \( A_\alpha(a) \) in Lemma 3.1.

Next theorem lists the formulae that compute the cardinality of the set (4), i.e., \( N_\alpha(a, c) \) of (6):

**Theorem 3.2:** Let \( q = p^e \) where \( p \) is an odd prime and \( e \) a positive integer, and \( d = \gcd(\alpha, e) \). In addition, suppose \( \frac{e}{d} \) to be odd. For the given \( b \in F_q^* \), let \( \gamma \) be the unique solution of \( f(x) = x^{p^2\alpha} + x = -b^{p^\alpha} \).
Case 1 \(a = 0, c = 0\).

- Let \(e\) be even with \(e = 2m\).
  - If \(p \equiv 1 \pmod{4}\), then
    \[
    N_\alpha(0, 0) = \begin{cases} 
    p^{e-2} - (p - 1)p^{m-1}, & \text{if } Tr(\gamma^{p^{a+1}}) = 0, \\
    p^{e-2}, & \text{if } Tr(\gamma^{p^{a+1}}) \neq 0. 
    \end{cases}
    \]
  - If \(p \equiv 3 \pmod{4}\), then
    \[
    N_\alpha(0, 0) = \begin{cases} 
    p^{e-2} - (-1)^{m}(p - 1)p^{m-1}, & \text{if } Tr(\gamma^{p^{a+1}}) = 0, \\
    p^{e-2}, & \text{if } Tr(\gamma^{p^{a+1}}) \neq 0. 
    \end{cases}
    \]
- Let \(e\) be odd.
  - If \(p \equiv 1 \pmod{4}\), then
    \[
    N_\alpha(0, 0) = \begin{cases} 
    p^{e-2}, & \text{if } Tr(\gamma^{p^{a+1}}) = 0, \\
    p^{e-2} + \eta(Tr(\gamma^{p^{a+1}}))(p - 1)p^{\frac{e-3}{2}}, & \text{if } Tr(\gamma^{p^{a+1}}) \neq 0. 
    \end{cases}
    \]
  - If \(p \equiv 3 \pmod{4}\), then
    \[
    N_\alpha(0, 0) = \begin{cases} 
    p^{e-2}, & \text{if } Tr(\gamma^{p^{a+1}}) = 0, \\
    p^{e-2} - (\sqrt{-1})^{e+1}\eta(Tr(\gamma^{p^{a+1}}))(p - 1)p^{\frac{e-3}{2}}, & \text{if } Tr(\gamma^{p^{a+1}}) \neq 0. 
    \end{cases}
    \]

Case 2 \(a = 0, c \neq 0\).

- Let \(e\) be even with \(e = 2m\).
  - If \(p \equiv 1 \pmod{4}\), then
    \[
    N_\alpha(0, c) = \begin{cases} 
    p^{e-2}, & \text{if } Tr(\gamma^{p^{a+1}}) = 0, \\
    p^{e-2} - p^{m-1}, & \text{if } Tr(\gamma^{p^{a+1}}) \neq 0. 
    \end{cases}
    \]
  - If \(p \equiv 3 \pmod{4}\), then
    \[
    N_\alpha(0, c) = \begin{cases} 
    p^{e-2}, & \text{if } Tr(\gamma^{p^{a+1}}) = 0, \\
    p^{e-2} - (-1)^{m}p^{m-1}, & \text{if } Tr(\gamma^{p^{a+1}}) \neq 0. 
    \end{cases}
    \]
- Let \(e\) be odd.
  - If \(p \equiv 1 \pmod{4}\), then
    \[
    N_\alpha(0, c) = \begin{cases} 
    p^{e-2}, & \text{if } Tr(\gamma^{p^{a+1}}) = 0, \\
    p^{e-2} - \eta(Tr(\gamma^{p^{a+1}}))p^{\frac{e-3}{2}}, & \text{if } Tr(\gamma^{p^{a+1}}) \neq 0. 
    \end{cases}
    \]
  - If \(p \equiv 3 \pmod{4}\), then
    \[
    N_\alpha(0, c) = \begin{cases} 
    p^{e-2}, & \text{if } Tr(\gamma^{p^{a+1}}) = 0, \\
    p^{e-2} + (\sqrt{-1})^{e+1}\eta(Tr(\gamma^{p^{a+1}}))p^{\frac{e-3}{2}}, & \text{if } Tr(\gamma^{p^{a+1}}) \neq 0. 
    \end{cases}
    \]

Case 3 \(a \neq 0, c = 0\).

- Let \(e\) be even with \(e = 2m\).
  - If \(p \equiv 1 \pmod{4}\), then
    \[
    N_\alpha(a, 0) = \begin{cases} 
    p^{e-2} + p^{m-1}, & \text{if } Tr(\gamma^{p^{a+1}}) = 0, \\
    p^{e-2} - \eta(aTr(\gamma^{p^{a+1}}))p^{m-1}, & \text{if } Tr(\gamma^{p^{a+1}}) \neq 0. 
    \end{cases}
    \]
Case 4 \(ac \neq 0\).

- Let \(e\) be odd.
  - If \(p \equiv 1 \pmod{4}\), then
    \[
    N_\alpha(a, c) = \begin{cases}
    p^{e-2}, & \text{if } Tr(\gamma^{p+1}) = 0, \\
    p^{e-2} + (\sqrt{-1})^{e+1} \tilde{\eta}(Tr(\gamma^{p+1}))p^{\frac{e+3}{2}}, & \text{if } Tr(\gamma^{p+1}) = c^2/(4a), \\
    p^{e-2} - \tilde{\eta}(c^2 - 4aTr(\gamma^{p+1}))p^{m-1}, & \text{otherwise}.
    \end{cases}
    \]
  - If \(p \equiv 3 \pmod{4}\), then
    \[
    N_\alpha(a, c) = \begin{cases}
    p^{e-2}, & \text{if } Tr(\gamma^{p+1}) = 0, \\
    p^{e-2} + (\sqrt{-1})^{e+1} \tilde{\eta}(Tr(\gamma^{p+1}))p^{\frac{e+3}{2}}, & \text{if } Tr(\gamma^{p+1}) = c^2/(4a), \\
    p^{e-2} - (\sqrt{-1})^{e+1} \tilde{\eta}(Tr(\gamma^{p+1}))p^{\frac{e+3}{2}}, & \text{otherwise}.
    \end{cases}
    \]

- Let \(e\) be even with \(e = 2m\).
  - If \(p \equiv 1 \pmod{4}\), then
    \[
    N_\alpha(a, c) = \begin{cases}
    p^{e-2}, & \text{if } Tr(\gamma^{p+1}) = 0, \\
    p^{e-2} + (\sqrt{-1})^{e+1} \tilde{\eta}(Tr(\gamma^{p+1}))p^{\frac{e+3}{2}}, & \text{if } Tr(\gamma^{p+1}) = c^2/(4a), \\
    p^{e-2} - \tilde{\eta}(c^2 - 4aTr(\gamma^{p+1}))p^{m-1}, & \text{otherwise}.
    \end{cases}
    \]
  - If \(p \equiv 3 \pmod{4}\), then
    \[
    N_\alpha(a, c) = \begin{cases}
    p^{e-2}, & \text{if } Tr(\gamma^{p+1}) = 0, \\
    p^{e-2} + (\sqrt{-1})^{e+1} \tilde{\eta}(Tr(\gamma^{p+1}))p^{\frac{e+3}{2}}, & \text{if } Tr(\gamma^{p+1}) = c^2/(4a), \\
    p^{e-2} - \tilde{\eta}(c^2 - 4aTr(\gamma^{p+1}))p^{m-1}, & \text{otherwise}.
    \end{cases}
    \]

**Proof:** From the definition of \(N_\alpha(a, c)\) in (6), it is easy to see that

\[
N_\alpha(a, c) = p^{-2} \sum_{x \in F_\mathbb{P}} \left( \sum_{y \in F_\mathbb{P}} \zeta_p(Tr(x\gamma^{p+1}) - a) \right) \left( \sum_{z \in F_\mathbb{P}} \zeta_p(Tr(bx) - c) \right)
= p^{-2} \sum_{x \in F_\mathbb{P}} \left( 1 + \sum_{y \in F_\mathbb{P}} \zeta_p(Tr(x\gamma^{p+1}) - a) \right) \left( 1 + \sum_{z \in F_\mathbb{P}} \zeta_p(Tr(bx) - c) \right)
= p^{e-2} + p^{-2} \sum_{y \in F_\mathbb{P}} \sum_{x \in F_\mathbb{P}} \zeta_p(Tr(x\gamma^{p+1}) - a) + p^{-2} \sum_{z \in F_\mathbb{P}} \sum_{x \in F_\mathbb{P}} \zeta_p(Tr(bx) - c)
+ p^{-2} \sum_{y \in F_\mathbb{P}} \sum_{z \in F_\mathbb{P}} \sum_{x \in F_\mathbb{P}} \zeta_p Tr(yx\gamma^{p+1} + zbx) - ay - cz
= p^{e-2} + p^{-2} A_\alpha(a) + p^{-2} B_\alpha(a, c).
\]
The actual theorem can be proven by substituting the formulae of both $A_\alpha(a)$ and $B_\alpha(a, c)$ from Lemma 3.1 and 3.2 into the last equality of (30).

IV. CONCLUDING REMARKS

In this paper, two exponential sums over finite field related to the Coulter’s polynomial, and the cardinalities of two subsets of $\mathbb{F}_q$, are settled for case $\frac{q}{d}$ odd, that may have potential application in the construction of linear codes with few weights.

REFERENCES

[1] C. Ding, X. Wang, A coding theory construction of new systematic authentication codes, Theoretical Computer Science 330 (2005) 81-99.
[2] C. Ding, J. Luo, H. Niederreiter, Two weight codes punctured from irreducible cyclic codes, in: Proc. of the First International Workshop on Coding Theory and Cryptography (2008) 119-124, Singapore, World Scientific.
[3] C. Ding, H. Niederreiter, Cyclotomic linear codes of order 3, IEEE Trans. Inform. Theory 53 (6) (2007) 2274-2277.
[4] K. Ding, C. Ding, Binary linear codes with three weights, IEEE Communication Letters 18(11) (2014) 1879-1882.
[5] K. Ding, C. Ding, A class of two-weight and three-weight codes and their applications in secret sharing, IEEE Trans. Inform. Theory 61 (11) (2015) 5835-5842.
[6] Q. Wang, K. Ding, R. Xue, Binary linear codes with two weights, IEEE Communication Letters 19 (7) (2015) 1097-1100.
[7] Q. Wang, F. Li, K. Ding, D. Lin, Complete weight enumerators of two classes of linear codes, Discrete Mathematics 340 (3) (2017) 467-480.
[8] R.S. Coulter, Explicit evaluation of some Weil sums, Acta Arith 83 (1998) 241-251.
[9] R.S. Coulter, Further evaluation of Weil sums, Acta Arith 86 (1998) 217-226.
[10] H. Hu, Q. Zhang, S. Shao, On the dual of the Coulter-Matthews bent functions, IEEE Trans. Inform. Theory 63 (4) (2017) 2454-2463.
[11] C. Tang, N. Li, Y. Qi, Z. Zhou, T. Helleseth, Linear codes with two or three weights from weakly regular bent functions, IEEE Trans. Inform. Theory 62 (3) (2016) 1166-1176.
[12] Z. Zhou, C. Ding, Seven Classes of Three-Weight Cyclic Codes, IEEE Trans. Inform. Theory 61 (10) (2013) 0090-6778.
[13] R. Lidl, H. Niederreiter, Finite fields, Cambridge University Press, New York, 1997.
[14] B. Courteau, J. Wolfmann, On triple-sum-sets and two or three weights codes, Discrete Mathematics 50 (1984) 179-191.
[15] H. Cohn, Advanced Number Theory, Dover, 1980.