CONGRUENCES FOR THE $k$ DOTS BRACELET PARTITION FUNCTIONS

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Abstract. By finding the congruent relations between the generating function of the 5 dots bracelet partitions and that of the 5-regular partitions, we get some new congruences modulo 2 for the 5 dots bracelet partition function. Moreover, for a given prime $p$, we study the arithmetic properties modulo $p$ of the $k$ dots bracelet partitions.

1. Introduction

In [1], Andrews and Paule studied the broken $k$-diamond partitions by using MacMahon's partition analysis, and gave the generating function of $\Delta_k(n)$ which denotes the number of the broken $k$-diamond partitions of $n$:

$$
\sum_{n=0}^{\infty} \Delta_k(n)q^n = \frac{(-q; q)_\infty}{(q; q^k)_{\infty}^{\frac{1}{2}}(-q^{2k+1}; q^{2k+1})_\infty}.
$$

In [1], They proved the following arithmetic theorem for $\Delta_1(n)$.

Theorem 1.1. [1, Theorem 5] For $n \geq 0$,
$$
\Delta_1(2n + 1) \equiv 0 \pmod{3}.
$$

Meanwhile, they posed some conjectures related to $\Delta_2(n)$. For other study of the arithmetic of the broken $k$-diamond partitions, see [3,7,9,12,14,17]. In [4], Fu found a combinatorial proof of Theorem [1,1] for $\Delta_1(n)$, and introduced a generalization of the broken $k$-diamond partitions which he called the $k$ dots bracelet partitions. The number of this kind of partitions of $n$ is denoted by $B_k(n)$, and the generating function of $B_k(n)$ is stated as follows.

$$
\sum_{n=0}^{\infty} B_k(n)q^n = \frac{(-q; q)_\infty}{(q; q^{k-1})_{\infty}(-q^k; q^k)_\infty}, \quad k \geq 3.
$$

In [4], Fu proved the following congruences for $B_k(n)$.

Theorem 1.2. [4, Theorem 3.3] For $n > 0$, $k \geq 3$, if $k = p^r$ is a prime power, we have
$$
B_k(2n + 1) \equiv 0 \pmod{p}.
$$

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Theorem 1.3. [4, Theorem 3.5] For any $k \geq 3$, $s$ an integer between 1 and $p - 1$ such that $12s + 1$ is a quadratic nonresidue modulo $p$, and any $n \geq 0$, if $p \mid k$ for some prime $p \geq 5$ say $k = pm$, then we have
\[
\mathcal{B}_k(pn + s) \equiv 0 \pmod{p}.
\]

Theorem 1.4. [4, Theorem 3.6] For $n \geq 0$, $k \geq 3$ even, say $k = 2^m\ell$, where $\ell$ is odd, we have
\[
\mathcal{B}_k(2n + 1) \equiv 0 \pmod{2^m}.
\]

Later, in [13], Radu and Sellers found some new congruences for $\mathcal{B}_k(n)$.

Theorem 1.5. [13, Theorem 1.4] For all $n \geq 0$,
\[
\begin{align*}
\mathcal{B}_5(10n + 7) &\equiv 0 \pmod{5^2}, \\
\mathcal{B}_7(14n + 11) &\equiv 0 \pmod{7^2}, \text{ and} \\
\mathcal{B}_{11}(22n + 21) &\equiv 0 \pmod{11^2}.
\end{align*}
\]

In this paper, we continue to study the arithmetic of the $k$ dots bracelet partitions. First, we recall two kinds of partitions which are used in this paper.

A partition of a positive integer $n$ is a nonincreasing sequence of positive integers whose sum is $n$. Let $p(n)$ denote the number of partitions of $n$. We know that
\[
\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_{\infty}}. \tag{1.2}
\]

If $\ell$ is a positive integer, then a partition is called $\ell$-regular partition if there is no part divisible by $\ell$. Let $b_{\ell}(n)$ denote the number of $\ell$-regular partitions of $n$. The generating function of $b_{\ell}(n)$ is stated as follows.
\[
\sum_{n=0}^{\infty} b_{\ell}(n)q^n = \frac{(q^{\ell}; q^{\ell})_{\infty}}{(q; q)_{\infty}}. \tag{1.3}
\]

In section 2, based on an identity given by Ramanujan in [16] and a congruence for the generating function of $b_5(2n)$ given by Hirschhorn and Sellers in [8], we obtain two congruences modulo 2 for $B_5(n)$. Meanwhile, by finding a congruent relation between the generating function of $\mathcal{B}_5(n)$ and that of $b_5(n)$, we get many infinite family of congruences modulo 2 for $\mathcal{B}_5(n)$. In section 3, for a given prime $p$, by means of a $p$-dissection identity of $f(-q)$ given by the authors in [6] and three classical congruences for $p(n)$ given by Ramanujan in [15,16], we get more congruences modulo $p$ for $\mathcal{B}_k(n)$.

In the following, we list some definitions and identities which are frequently used in this paper.

As usual, we follow the notation and terminology in [5]. For $|q| < 1$, the $q$-shifted factorial is defined by
\[
(a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k) \quad \text{and} \quad (a; q)_{n} = \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}}, \text{ for } n \in \mathbb{C}.
\]
The Legendre symbol is a function of $a$ and $p$ defined as follows:

$$\left( \frac{a}{p} \right) = \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \pmod{p}, \\ -1, & \text{if } a \text{ is a quadratic non-residue modulo } p, \\ 0, & \text{if } a \equiv 0 \pmod{p}. \end{cases}$$

Jacobi’s triple product identity [2, Theorem 1.3.3]: for $z \neq 0$ and $|q| < 1$,

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq, -q/z, q^2; q^2)_\infty. \quad (1.4)$$

Ramanujan’s general theta function $f(a, b)$ is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab| < 1.$$

A special case of $f(a, b)$ is stated as follows.

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q; q)_\infty. \quad (2.3)$$

$$\sum_{n=0}^{\infty} b_5(2n)q^n \equiv (q^2, q^2)_\infty \pmod{2}. \quad (2.3)$$

By means of (2.2) and (2.3), we have the following results for $\mathcal{B}_5(n)$.

**Theorem 2.1.** For $n \geq 0$, we have

$$\mathcal{B}_5(10n + 6) \equiv 0 \pmod{2},$$

$$\mathcal{B}_5(10n + 8) \equiv 0 \pmod{2}.$$
Proof. First, we have
\[
\sum_{n=0}^{\infty} \mathcal{B}_5(n)q^n = \frac{(-q; q)_\infty}{(q; q)^4(-q^5; q^5)_\infty} \\
= \frac{(q^2; q^2)_\infty(q^5; q^5)_\infty}{(q; q)^4(q_{10}; q_{10})_\infty} \\
\equiv \frac{1}{(q^4; q^4)(q^4; q^4)} (\text{mod 2}) \\
= \frac{1}{(q^2; q^2)(q^2; q^2)} \cdot \sum_{n=0}^{\infty} b_5(n)q^n.
\]

Then,
\[
\sum_{n=0}^{\infty} \mathcal{B}_5(2n)q^n \equiv \frac{1}{(q; q)(q^5; q^5)} \cdot \sum_{n=0}^{\infty} b_5(2n)q^n (\text{mod 2}) \\
\equiv \frac{(q^5; q^5)}{(q; q)(q^5; q^5)} (\text{mod 2}) \quad \text{by (2.3)} \\
\equiv \frac{(q; q)}{(q^5; q^5)} (\text{mod 2}).
\]

According to (2.2), we have
\[
\sum_{n=0}^{\infty} \mathcal{B}_5(2n)q^n \equiv \frac{(q^{25}; q^{25})}{(q^5; q^5)(a(q) - q - q^2b(q))} (\text{mod 2}). \quad (2.4)
\]

Therefore, we get
\[
\mathcal{B}_5(2(5n + 3)) = \mathcal{B}_5(10n + 6) \equiv 0 \quad (\text{mod 2}), \\
\mathcal{B}_5(2(5n + 4)) = \mathcal{B}_5(10n + 8) \equiv 0 \quad (\text{mod 2}).
\]

Lemma 2.2. We have
\[
\sum_{n=0}^{\infty} \mathcal{B}_5(10n + 2)q^n \equiv \sum_{n=0}^{\infty} b_5(n)q^n \quad (\text{mod 2}).
\]

Proof. Due to (2.3), we get
\[
\sum_{n=0}^{\infty} \mathcal{B}_5(2(5n + 1))q^n = \sum_{n=0}^{\infty} \mathcal{B}_5(10n + 2)q^n \equiv \frac{(q^5; q^5)}{(q; q)} = \sum_{n=0}^{\infty} b_5(n)q^n \quad (\text{mod 2}).
\]

In [6], the authors found many infinite family of congruences modulo 2 for \(b_5(n)\).
Theorem 2.3. [6, Theorem 3.17] For any prime \( p \geq 5 \), \( \left( \frac{-10}{p} \right) = -1 \), \( \alpha \geq 1 \), and \( n \geq 0 \), we have
\[
 b_5(4 \cdot p^{2\alpha} n + \frac{(24i + 7p)p^{2\alpha-1} - 1}{6}) \equiv 0 \pmod{2},
\]
where \( i = 1, 2, \ldots, p-1 \).

Theorem 2.4. [6, Theorem 3.20] For \( \alpha \geq 0 \) and \( n \geq 0 \), we have
\[
 b_5(4 \cdot 5^{2\alpha+1} n + \frac{31 \cdot 5^{2\alpha} - 1}{6}) \equiv 0 \pmod{2},
\]
\[
 b_5(4 \cdot 5^{2\alpha+1} n + \frac{79 \cdot 5^{2\alpha} - 1}{6}) \equiv 0 \pmod{2},
\]
\[
 b_5(4 \cdot 5^{2\alpha+2} n + \frac{83 \cdot 5^{2\alpha+1} - 1}{6}) \equiv 0 \pmod{2},
\]
\[
 b_5(4 \cdot 5^{2\alpha+2} n + \frac{107 \cdot 5^{2\alpha+1} - 1}{6}) \equiv 0 \pmod{2}.
\]

Therefore, combining Lemma 2.2 with Theorem 2.3 and Theorem 2.4, we obtain some more congruences for \( \mathbb{B}_5(n) \).

Theorem 2.5. For any prime \( p \geq 5 \), \( \left( \frac{-10}{p} \right) = -1 \), \( \alpha \geq 1 \), and \( n \geq 0 \), we have
\[
 \mathbb{B}_5(40 \cdot p^{2\alpha} n + \frac{5 \cdot (24i + 7p)p^{2\alpha-1} + 1}{3}) \equiv 0 \pmod{2},
\]
where \( i = 1, 2, \ldots, p-1 \).

For example, by setting \( p = 17, i = 6, \) and \( \alpha = 1 \) in Theorem 2.5, we have the following congruence.
\[
 \mathbb{B}_5(11560n + 7452) \equiv 0 \pmod{2}.
\]

Theorem 2.6. For \( \alpha \geq 1 \) and \( n \geq 0 \), we have
\[
 \mathbb{B}_5(8 \cdot 5^{2\alpha} n + \frac{31 \cdot 5^{2\alpha-1} + 1}{3}) \equiv 0 \pmod{2},
\]
\[
 \mathbb{B}_5(8 \cdot 5^{2\alpha} n + \frac{79 \cdot 5^{2\alpha-1} + 1}{3}) \equiv 0 \pmod{2},
\]
\[
 \mathbb{B}_5(8 \cdot 5^{2\alpha+1} n + \frac{83 \cdot 5^{2\alpha} + 1}{3}) \equiv 0 \pmod{2},
\]
\[
 \mathbb{B}_5(8 \cdot 5^{2\alpha+1} n + \frac{107 \cdot 5^{2\alpha} + 1}{3}) \equiv 0 \pmod{2}.
\]

3. Congruences modulo \( p \) for \( \mathbb{B}_k(n) \)

In [6], the authors studied a \( p \)-dissection identity of \( f(-q) \) for a given prime \( p \geq 5 \).
Theorem 3.1. [1] Theorem 2.2] For any prime $p \geq 5$, we have

$$f(-q) = \sum_{k = -\frac{p-1}{2}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2 + k}{2}} f(-q^{\frac{3p^2 + (6k+1)p}{2}}, -q^{\frac{3p^2 - (6k+1)p}{2}}) + (-1)^{\frac{p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^p),$$

where $\pm$ depends on the condition that $(\pm p - 1)/6$ should be an integer. Meanwhile, we claim that $(3k^2 + k)/2$ and $(p^2 - 1)/24$ are not in the same residue class modulo $p$ for $-(p-1)/2 \leq k \leq (p-1)/2$ and $k \neq (\pm p - 1)/6$.

According to the above theorem, we have the following result.

Lemma 3.2. For any prime $p \geq 5$, $n \geq 0$, and $r \geq 1$, if $k = p^r$ is a prime power, then for $1 \leq \alpha \leq (r + 1)/2$, we have

$$\sum_{n=0}^{\infty} \mathcal{B}_k(p^{2\alpha-1}n + \frac{p^{2\alpha} - 1}{12}) q^n \equiv \left((-1)^{\frac{p-1}{6}}\right)^\alpha (q^{2p}; q^{2p})_\infty (q^{2p^{-(2\alpha - 1)}}; q^{2p^{-(2\alpha - 1)}})_\infty \pmod{p},$$

where $\pm$ depends on the condition that $(\pm p - 1)/6$ should be an integer.

Proof. We prove the lemma by induction on $\alpha$. For $k = p^r$, in [1], Fu stated the following fact

$$\sum_{n=0}^{\infty} \mathcal{B}_k(n) q^n = \frac{(-q; q)_\infty}{(q; q)_\infty^k (-q^k; q^k)_\infty} = \frac{(q^2; q^2)_\infty}{(q; q)_\infty^k (-q^k; q^k)_\infty} \equiv (q^k; q^k)_\infty (-q^k; q^k)_\infty \pmod{p} = \frac{(q^2; q^2)_\infty}{(q^{2k}; q^{2k})_\infty} \pmod{p}.$$

Due to Theorem 3.1, for any prime $p \geq 5$, we have

$$\sum_{n=0}^{\infty} \mathcal{B}_k(pn + \frac{p^2 - 1}{12}) q^n \equiv (-1)^{\frac{p-1}{6}} (q^{2p}; q^{2p})_\infty (q^{2p^{-(2\alpha - 1)}}; q^{2p^{-(2\alpha - 1)}})_\infty \pmod{p}.$$ 

That means the lemma holds for $\alpha = 1$. Suppose that lemma holds for $\alpha$. Now we prove the case for $\alpha + 1$. For

$$\sum_{n=0}^{\infty} \mathcal{B}_k(p^{2\alpha-1}n + \frac{p^{2\alpha} - 1}{12}) q^n \equiv \left((-1)^{\frac{p-1}{6}}\right)^\alpha (q^{2p}; q^{2p})_\infty (q^{2p^{-(2\alpha - 1)}}; q^{2p^{-(2\alpha - 1)}})_\infty \pmod{p}.$$ 

Then

$$\sum_{n=0}^{\infty} \mathcal{B}_k(p^{2\alpha-1}(pn) + \frac{p^{2\alpha} - 1}{12}) q^n = \sum_{n=0}^{\infty} \mathcal{B}_k(p^{2\alpha}n + \frac{p^{2\alpha} - 1}{12}) q^n \equiv \left((-1)^{\frac{p-1}{6}}\right)^\alpha (q^2; q^2)_\infty (q^{2p^{-(2\alpha)}}; q^{2p^{-(2\alpha)}})_\infty \pmod{p}. \quad (3.1)$$
Using Theorem 3.1 again, we have
\[
\sum_{n=0}^{\infty} B_k(p^{2\alpha}(pn + \frac{p^2 - 1}{12}) + \frac{p^{2\alpha} - 1}{12})q^n = \sum_{n=0}^{\infty} B_k(p^{2\alpha+1}n + \frac{p^{2\alpha+2} - 1}{12})q^n \\
\equiv \left(\left(-1\right)^{\frac{p-1}{6}}\right)^{\alpha+1} \frac{(q^{2p}; q^{2p})_\infty}{(q^{2p^{\alpha-(2\alpha+1)}}; q^{2p^{\alpha-(2\alpha+1)}})_\infty} \pmod{p}.
\]
Therefore the lemma holds for \( \alpha + 1 \).
\[\square\]

According to Lemma 3.2, we have the following results.

**Theorem 3.3.** For any prime \( p \geq 5 \), \( n \geq 0 \), and \( r \geq 1 \), if \( k = p^r \) is a prime power, then we have the following two cases:

1. For \( i = 1, 2, \ldots, p - 1 \) and \( 1 \leq \alpha \leq r/2 \), we have
   \[
   B_k(p^{2\alpha}(pn + (12i + p)p^{2\alpha-1} - 1) + \frac{p^{2\alpha} - 1}{12}) \equiv 0 \pmod{p}.
   \]
2. Let \( j \) be an integer between \( 1 \) and \( p - 1 \) and \( 12j + 1 \) is a quadratic nonresidue modulo \( p \). For \( n \geq 0 \) and \( 1 \leq \alpha \leq (r - 1)/2 \), we have
   \[
   B_k(p^{2\alpha+1}n + (12j + 1)p^{2\alpha} - 1 + \frac{p^{2\alpha}}{12}) \equiv 0 \pmod{p}.
   \]

**Proof.** According to Lemma 3.2, when \( 1 \leq \alpha \leq r/2 \), for \( i = 1, 2, \ldots, p - 1 \), we have
\[
B_k(p^{2\alpha-1}(pn + i) + \frac{p^{2\alpha} - 1}{12}) = B_k(p^{2\alpha}n + \frac{(12i + p)p^{2\alpha-1} - 1}{12}) \equiv 0 \pmod{p}.
\]

For \( 1 \leq \alpha \leq (r - 1)/2 \), according to (3.1) and Theorem 3.1, we know that the powers of \( q \) modulo \( p \) congruent to \( 2 \cdot (3k^2 + k)/2 \) for \( -(p - 1)/2 \leq k \leq (p - 1)/2 \) in the expansion of \( (q^2; q^2)_\infty \). So we have
\[
j \equiv 2 \cdot \frac{3k^2 + k}{2} \pmod{p},
12j + 1 \equiv (6k + 1)^2 \pmod{p}.
\]
Therefore, if \( 12j + 1 \) is a quadratic nonresidue modulo \( p \), then we have
\[
\sum_{n=0}^{\infty} B_k(p^{2\alpha}(pn + j) + \frac{p^{2\alpha} - 1}{12})q^n \equiv 0 \pmod{p}.
\]
\[\square\]

Based on Lemma 3.2 and the generating functions of \( p(n) \) and \( b_k(n) \), we get the following congruent relations.
Theorem 3.4. For any prime \( p \geq 5 \), \( \alpha \geq 1 \), and \( n \geq 0 \), if \( k = p^{2\alpha-1} \) is a prime power, then we have

\[
\sum_{n=0}^{\infty} \mathcal{B}_k(2p^{2\alpha-1}n + \frac{2\alpha-1}{12})q^n \equiv \left(-1\right)^{\frac{p-1}{6}} \sum_{n=0}^{\infty} b_p(n)q^n \pmod{p}, \tag{3.2}
\]

\[
\sum_{n=0}^{\infty} \mathcal{B}_k(2p^{2\alpha-1}n + \frac{2\alpha-1}{12})q^n \equiv \left(-1\right)^{\frac{p-1}{6}} \left(q^p; q^p\right)_{\infty} \sum_{n=0}^{\infty} p(n)q^n \pmod{p}. \tag{3.3}
\]

Proof. Set \( r = 2\alpha - 1 \) in Lemma 3.2. Then \( k = p^{2\alpha-1} \). So we get

\[
\sum_{n=0}^{\infty} \mathcal{B}_k(p^{2\alpha-1}n + \frac{2\alpha-1}{12})q^n \equiv \left(-1\right)^{\frac{p-1}{6}} \left(q^p; q^p\right)_{\infty} \pmod{p}.
\]

Therefore,

\[
\sum_{n=0}^{\infty} \mathcal{B}_k(p^{2\alpha-1}(2n) + \frac{2\alpha-1}{12})q^n = \sum_{n=0}^{\infty} \mathcal{B}_k(2p^{2\alpha-1}n + \frac{2\alpha-1}{12})q^n
\]

\[
\equiv \left(-1\right)^{\frac{p-1}{6}} \left(q^p; q^p\right)_{\infty} \pmod{p}.
\]

\( \square \)

Combining (3.3) with the three famous congruences for \( p(n) \) given by Ramanujan in \([15,16]\)

\[
p(5n + 4) \equiv 0 \pmod{5}, \tag{3.4}
\]

\[
p(7n + 5) \equiv 0 \pmod{7}, \tag{3.5}
\]

\[
p(11n + 6) \equiv 0 \pmod{11}, \tag{3.6}
\]

we get the following results.

Corollary 3.5. For \( \alpha \geq 1 \) and \( n \geq 0 \), we have

\[
\mathcal{B}_{52\alpha-1}(2 \cdot 5^{2\alpha}n + \frac{101 \cdot 5^{2\alpha-1} - 1}{12}) \equiv 0 \pmod{5},
\]

\[
\mathcal{B}_{72\alpha-1}(2 \cdot 7^{2\alpha}n + \frac{127 \cdot 7^{2\alpha-1} - 1}{12}) \equiv 0 \pmod{7},
\]

\[
\mathcal{B}_{112\alpha-1}(2 \cdot 11^{2\alpha}n + \frac{155 \cdot 11^{2\alpha-1} - 1}{12}) \equiv 0 \pmod{11}.
\]

Proof. According to (3.3), we have

\[
\sum_{n=0}^{\infty} \mathcal{B}_{52\alpha-1}(2 \cdot 5^{2\alpha-1}n + \frac{5^{2\alpha} - 1}{12})q^n \equiv (-1)^{\alpha}(q^5; q^5)_{\infty} \sum_{n=0}^{\infty} p(n)q^n \pmod{5},
\]

\[
\sum_{n=0}^{\infty} \mathcal{B}_{72\alpha-1}(2 \cdot 7^{2\alpha-1}n + \frac{7^{2\alpha} - 1}{12})q^n \equiv (-1)^{\alpha}(q^7; q^7)_{\infty} \sum_{n=0}^{\infty} p(n)q^n \pmod{7},
\]

\[
\sum_{n=0}^{\infty} \mathcal{B}_{112\alpha-1}(2 \cdot 11^{2\alpha-1}n + \frac{11^{2\alpha} - 1}{12})q^n \equiv (q^{11}; q^{11})_{\infty} \sum_{n=0}^{\infty} p(n)q^n \pmod{11}.
\]
Based on (3.4), (3.5), and (3.6), we get

\[
\begin{align*}
\mathfrak{B}_{5^{2\alpha-1}}(2 \cdot 5^{2\alpha-1}(5n + 4) + \frac{5^{2\alpha} - 1}{12}) & \equiv 0 \pmod{5}, \\
\mathfrak{B}_{7^{2\alpha-1}}(2 \cdot 7^{2\alpha-1}(7n + 5) + \frac{7^{2\alpha} - 1}{12}) & \equiv 0 \pmod{7}, \\
\mathfrak{B}_{11^{2\alpha-1}}(2 \cdot 11^{2\alpha-1}(11n + 6) + \frac{11^{2\alpha} - 1}{12}) & \equiv 0 \pmod{11}.
\end{align*}
\]

□

Another congruence modulo \( p \) for \( \mathfrak{B}_k(n) \) can be directly obtained from Lemma 3.2.

**Theorem 3.6.** For any prime \( p \geq 5 \), \( \alpha \geq 1 \), and \( n \geq 1 \), if \( k = p^{2\alpha} \) is a prime power, then we have

\[
\mathfrak{B}_k(p^{2\alpha-1}n + \frac{p^{2\alpha} - 1}{12}) \equiv 0 \pmod{p}.
\]

**Proof.** Set \( r = 2\alpha \) in Lemma 3.2. Then \( k = p^{2\alpha} \). So we have

\[
\sum_{n=0}^{\infty} \mathfrak{B}_k(p^{2\alpha-1}n + \frac{p^{2\alpha} - 1}{12})q^n \equiv \left(\frac{-1}{p}\right)^{\frac{p-1}{6}}(\frac{-1}{p})^\alpha (\pmod{p}).
\]

□

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