OPERATOR REVISION OF A KY FAN TYPE INEQUALITY

J. ROOIN1, A. ALIKHANI1

Abstract. Let $\mathcal{H}$ be a complex Hilbert space and $A, B \in \mathfrak{B}(\mathcal{H})$ such that $0 < A, B \leq \frac{1}{2}I$. Setting $A' := I - A$ and $B' := I - B$, we prove

$$A' \nabla_{\lambda} B' - A' \lambda B' \leq A \nabla_{\lambda} B - A \lambda B,$$

where $\nabla_{\lambda}$ and $\lambda$ denote the weighted arithmetic and harmonic operator means, respectively. This inequality is the natural extension of a Ky Fan type inequality due to H. Alzer. Some parallel and related results are also obtained.

1. Introduction

Let $n \geq 2$ and $\lambda_1, \cdots, \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$. For $n$ arbitrary real numbers $x_1, \cdots, x_n > 0$, we denote by $A_n, G_n$ and $H_n$ the arithmetic, geometric and harmonic means of $x_1, \cdots, x_n$ respectively, i.e.

$$A_n = \sum_{i=1}^{n} \lambda_i x_i \quad G_n = \prod_{i=1}^{n} x_i^{\lambda_i} \quad H_n = \frac{1}{\sum_{i=1}^{n} \lambda_i x_i}.$$  \hfill (1.1)

Also when $x_i \in (0, \frac{1}{2}]$ we denote by $A'_n, G'_n$ and $H'_n$ the arithmetic, geometric and harmonic means of $x'_1 := 1 - x_1, \cdots, x'_n := 1 - x_n$ respectively, i.e.

$$A'_n = \sum_{i=1}^{n} \lambda_i x'_i \quad G'_n = \prod_{i=1}^{n} x'_i^{\lambda_i} \quad H'_n = \frac{1}{\sum_{i=1}^{n} \lambda_i x_i}.$$  \hfill (1.2)

The Ky Fan’s inequality

$$\frac{A_n'}{G_n'} \leq \frac{A_n}{G_n}.$$  \hfill (1.3)

was introduced for the first time in [4]. From then several mathematicians attained new proofs, extensions, refinements and various related results. For more information about Ky Fan and Ky Fan type inequalities see [2, 7]. In 1988, an additive analogue of (1.3) presented by H. Alzer [3] as

$$A'_n - G'_n \leq A_n - G_n.$$  \hfill (1.4)
In both of (1.3) and (1.4), equality holds if and only if \( x_1 = \cdots = x_n \).

Another interesting additive analogue of Ky Fan’s inequality was discovered by H. Alzer \[1\] in 1993, as follows

\[
A'_n - H'_n \leq A_n - H_n
\]  \hspace{1cm} (1.5)

with equality holding if and only if \( x_1 = \cdots = x_n \). For a new refinement and converse of (1.5) refer to \[8\]. If we divide two sides of (1.5) by \( A'_n H'_n \), and using this fact that \( A'_n H'_n \geq A_n H_n \), we conclude the following inequality due to J. Sandor \[9\]

\[
\frac{1}{H'_n} - \frac{1}{A'_n} \leq \frac{1}{H_n} - \frac{1}{A_n}.
\]  \hspace{1cm} (1.6)

Similarly, dividing two sides of (1.5) by \( H'_n \) and note that \( H'_n \geq H_n \), we obtain

\[
\frac{A'_n}{H'_n} \leq \frac{A_n}{H_n} \]  \hspace{1cm} (1.7)

Throughout this paper, let \( \mathcal{B}(\mathcal{H}) \) denote the \( C^* \)-algebra of all bounded linear operators acting on a complex Hilbert space \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) and \( I \) be the identity operator. An operator \( A \in \mathcal{B}(\mathcal{H}) \) is called positive if \( \langle Ax, x \rangle \geq 0 \) holds for every \( x \in \mathcal{H} \) and then we write \( A \geq 0 \). If \( A \) is positive and invertible, we write \( A > 0 \). For self-adjoint operators \( A, B \), we say \( A \leq B \) if \( B - A \geq 0 \). If \( A \geq 0 \) and \( X \in \mathcal{B}(\mathcal{H}) \), then \( X^*AX \geq 0 \). We define the weighted arithmetic mean \( \nabla_{\lambda} \), the weighted geometric mean (the \( \lambda \)-power mean) \( \sharp_{\lambda} \) and the weighted harmonic mean \( !_{\lambda} \) for \( A, B > 0 \) and \( 0 \leq \lambda \leq 1 \) by

\[
A \nabla_{\lambda} B := (1 - \lambda)A + \lambda B \\
A^*_{\lambda}B := A^\frac{1}{2}(A^{-\frac{1}{2}}B^\frac{1}{2}A^{-\frac{1}{2}})^{\lambda}A^\frac{1}{2} \\
A!_{\lambda}B := (1 - \lambda)A^{-1} + \lambda B^{-1} \]

In particular, in the case of \( \lambda = \frac{1}{2} \), the usual arithmetic, geometric and harmonic means of \( A, B \) simply denoted by \( A \nabla B, A^*_B \) and \( A!B \), respectively. For more information on operator inequalities and operator means see \[5\].

In \[6\] an operator revision of (1.4) for \( n = 2 \) in the case of commutative operators are obtained. The aim of this paper is to generalize the inequalities (1.5), (1.6) and (1.7) for operators on Hilbert spaces. In this way, we obtain natural direct operator version for the inequality (1.5). Also we give operator extensions of (1.6) and (1.7).
2. Main results

We start with the following useful lemma.

**Lemma 2.1.** Let $T \in \mathcal{B}(\mathcal{H})$ be a strictly positive operator and $\lambda \in [0, 1]$. Then

\begin{align*}
(i) & \quad I \nabla_\lambda T - I!_\lambda T = \lambda(1 - \lambda)(I - T)(T \nabla \lambda I)^{-1}(I - T). \\
(ii) & \quad T^{\frac{1}{2}}(I \nabla_\lambda T^{-1} - I!_\lambda T^{-1})T^{\frac{1}{2}} = \lambda(1 - \lambda)(T - I)(I \nabla_\lambda T)^{-1}(T - I). \\
(iii) & \quad (I!_\lambda T)^{-\frac{1}{2}}(I \nabla_\lambda T)(I!_\lambda T)^{-\frac{1}{2}} - I = \lambda(1 - \lambda)(I - T)T^{-1}(I - T).
\end{align*}

**Proof.** (i) Since the operators $I$ and $T$ commute, we have

\[
I \nabla_\lambda T - I!_\lambda T = (1 - \lambda)I + \lambda T - ((1 - \lambda)I + \lambda T^{-1})^{-1} = (1 - \lambda)I + \lambda T - ((1 - \lambda)T + \lambda I)^{-1}T = (1 - \lambda)T + \lambda I)^{-1}T = \lambda(1 - \lambda)((1 - \lambda)T + \lambda I)^{-1}(I - 2T + T^2) = \lambda(1 - \lambda)(I - T)((1 - \lambda)T + \lambda I)^{-1}(I - T) = \lambda(1 - \lambda)(I - T)(T \nabla_\lambda I)^{-1}(I - T).
\]

(ii) Changing $T$ by $T^{-1}$ in (2.1), we have

\[
I \nabla_\lambda T^{-1} - I!_\lambda T^{-1} = \lambda(1 - \lambda)(I - T^{-1})(T^{-1} \nabla_\lambda I)^{-1}(I - T^{-1}) = \lambda(1 - \lambda)(T - I)T^{-1}(I \nabla_\lambda T)^{-1}(T - I).
\]

Now multiplying both sides of (2.4) from left and right by $T^{\frac{1}{2}}$, we deduce (2.2).

(iii) First, we multiply both sides of (2.1) from left and right by $(I!_\lambda T)^{-\frac{1}{2}}$. Since

\[
(I!_\lambda T)^{-1}(T \nabla_\lambda I)^{-1} = ((1 - \lambda)I + \lambda T^{-1})(1 - \lambda)T + \lambda I)^{-1}T^{-1} = T^{-1},
\]

so we obtain (iii). \qed

**Theorem 2.2.** Let $A, B$ be two strictly positive operators and $\lambda \in [0, 1]$. Then

\begin{align*}
(i) & \quad A \nabla_\lambda B - A!_\lambda B = \lambda(1 - \lambda)(A - B)(B \nabla_\lambda A)^{-1}(A - B). \\
(ii) & \quad (A!_\lambda B)[(A!_\lambda B)^{-1} - (A \nabla_\lambda B)^{-1}](A!_\lambda B) = \lambda(1 - \lambda)(B - A)(A \nabla_\lambda B)^{-1}(B - A). \\
(iii) & \quad A\left(A^{-1}_\lambda B(A!_\lambda B)^{-1}\right)(A \nabla_\lambda B)(A^{-1}_\lambda B(A!_\lambda B)^{-1})A - A \]

\[
= \lambda(1 - \lambda)(A - B)B^{-1}(A - B).
\]
Proof. (i) Considering strictly positive operator $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and putting it in (2.1) we obtain
\[
\left((1 - \lambda)I + \lambda A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) - \left((1 - \lambda)I + \lambda A\overline{A}^{-1}A\overline{\lambda}\right)^{-1}
= \lambda(1 - \lambda)(I - A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\left((1 - \lambda)A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + \lambda I\right)^{-1}(I - A^{-\frac{1}{2}}BA^{-\frac{1}{2}}). \tag{2.8}
\]

Multiplying both sides of (2.8) from left and right by $A\overline{\lambda}$, yields
\[
((1 - \lambda)A + \lambda B) - ((1 - \lambda)A^{-1} + \lambda B^{-1})^{-1}
= \lambda(1 - \lambda)(A\overline{A}^{-\frac{1}{2}} - BA\overline{A}^{-\frac{1}{2}})\left((1 - \lambda)A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + \lambda I\right)^{-1}(A\overline{A}^{-\frac{1}{2}} - A\overline{\lambda} B)
= \lambda(1 - \lambda)(A - B)\overline{A}^{-\frac{1}{2}}\left((1 - \lambda)A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + \lambda I\right)^{-1}A\overline{A}^{-\frac{1}{2}}(A\overline{A}^{-\frac{1}{2}} - A\overline{\lambda} B)
= \lambda(1 - \lambda)(A - B)((1 - \lambda)B + \lambda A)^{-1}(A - B).
\]

So the proof of (i) is complete.

(ii) Similar to (i), putting $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in (2.2), we have
\[
A^{-\frac{1}{2}}BA\overline{\lambda}^{-\frac{1}{2}}\left((1 - \lambda)I + \lambda A\overline{A}^{-1}A\overline{\lambda} - ((1 - \lambda)I + \lambda A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-1}\right)(A\overline{A}^{-\frac{1}{2}} - I)
= \lambda(1 - \lambda)(A\overline{A}^{-\frac{1}{2}} - BA\overline{A}^{-\frac{1}{2}} - I)((1 - \lambda)I + \lambda A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-1}(A\overline{A}^{-\frac{1}{2}} - I)
\]
or
\[
A^{-\frac{1}{2}}BA\overline{\lambda}^{-\frac{1}{2}}\left((1 - \lambda)A^{-1} + \lambda B^{-1} - ((1 - \lambda)A + \lambda B)^{-1}\right)A\overline{A}^{-\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-1}
= \lambda(1 - \lambda)(A - B)((1 - \lambda)A + \lambda B)^{-1}(B - A)A\overline{A}^{-\frac{1}{2}}. \tag{2.9}
\]

Multiplying both sides of (2.9) from left and right by $A\overline{\lambda}$ we deduce (ii).

(iii) Putting $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in (2.3) we get
\[
((1 - \lambda)I + \lambda A\overline{A}^{-\frac{1}{2}}B^{-1}A\overline{\lambda})^{-\frac{1}{2}}((1 - \lambda)I + \lambda A^{-\frac{1}{2}}BA^{-\frac{1}{2}})((1 - \lambda)I + \lambda A\overline{A}^{-1}A\overline{\lambda})^{-\frac{1}{2}} - I
= \lambda(1 - \lambda)(I - A^{-\frac{1}{2}}BA^{-\frac{1}{2}}A\overline{A}^{-\frac{1}{2}}B^{-1}A\overline{\lambda})(I - A^{-\frac{1}{2}}BA^{-\frac{1}{2}}).
\]

Therefore
\[
\left(A\overline{A}^{-\frac{1}{2}}((1 - \lambda)A^{-1} + \lambda B^{-1})A\overline{\lambda}\right)\overline{\lambda}^{-\frac{1}{2}}A^{-\frac{1}{2}}((1 - \lambda)A + \lambda B)A\overline{\lambda}^{-\frac{1}{2}}\left(A\overline{A}^{-\frac{1}{2}}((1 - \lambda)A^{-1} + \lambda B^{-1})A\overline{\lambda}\right)^{-\frac{1}{2}} - I
= \lambda(1 - \lambda)A\overline{\lambda}^{-\frac{1}{2}}(A - B)B^{-1}(A - B)A\overline{A}^{-\frac{1}{2}}.
\]

This is equivalent to
\[
\left(A\overline{A}^{-\frac{1}{2}}((\lambda A^{-1}B)^{-1}A\overline{\lambda})\overline{\lambda}^{-\frac{1}{2}}A^{-\frac{1}{2}}(A\overline{\lambda}B)\right)^{-\frac{1}{2}} - I
= \lambda(1 - \lambda)A\overline{\lambda}^{-\frac{1}{2}}(A - B)B^{-1}(A - B)A\overline{A}^{-\frac{1}{2}}. \tag{2.10}
\]
Multiplying both sides of (2.10) from left and right by $A^{\frac{1}{2}}$ we deduce (iii).

The next theorem gives a natural operator version of the Ky Fan type inequality (1.5) in the case of $n = 2$. Operator extensions of (1.6) and (1.7), are also presented. For the sake of brevity, we set $A' := I - A$ and $B' := I - B$.

**Theorem 2.3.** Let $A, B \in \mathfrak{B}(\mathcal{H})$ such that $0 < A, B \leq \frac{1}{2}I$ and $\lambda \in [0, 1]$. Then we have

(i) $A' \nabla \lambda B' - A'^\dagger \lambda B' \leq A \nabla \lambda B - A \lambda B.$

(ii) $(A' \nabla B')((A'^\dagger \lambda B')^{-1} - (A' \nabla B')^{-1})(A' \nabla B') \leq (A \nabla B)(A \nabla B)^{-1} - (A \nabla B) - 1) (A \nabla B).

(iii) $A' \left(A^{-1} \nabla B B' \lambda A^{-1} \nabla B B' \right)(A' \nabla B') \left(A^{-1} \nabla B B' \lambda A^{-1} \nabla B B' \right) A' - A' \leq A \left(A^{-1} \nabla B B' \lambda A^{-1} \nabla B B' \right)(A \nabla B)(A \nabla B)^{-1} A - A.$

**Proof.** (i) Since $A', B' > 0$, substituting $A$ and $B$ with $A'$ and $B'$ in (2.5) respectively, we get

$$A' \nabla \lambda B' - A'^\dagger \lambda B' = \lambda(1 - \lambda)(A - B)(B' \nabla \lambda A')^{-1}(A - B).$$

Now since $0 < A \leq A'$ and $0 < B \leq B'$, we have

$$0 < B \nabla A = (1 - \lambda)B + \lambda A \leq (1 - \lambda)B' + \lambda A' = B' \nabla A',$$

and so

$$(B' \nabla A')^{-1} \leq (B \nabla A)^{-1}.$$

Hence

$$(A - B)(B' \nabla A')^{-1}(A - B) \leq (A - B)(B \nabla A)^{-1}(A - B).$$

Now considering (2.5) and (2.14) we obtain (2.11). The proofs of (ii) and (iii) are similar to (i) and we omit the details.

**Remark 2.4.** (i) In the process of proving (2.6) and (2.7), we used (2.2) and (2.3). If in (2.4), instead of multiplying $T^{\frac{1}{2}}$ from left and right, multiplying $T$ only from right or only from left respectively, and then substituting $T = A^{\frac{1}{2}}BA^{-\frac{1}{2}}$ and multiplying $A^{\frac{1}{2}}$ from left and right, we obtain the following chain of identities

$$A \nabla B \left[(A ! \lambda B)^{-1} - (A \nabla B)^{-1}\right] A = \lambda(1 - \lambda)(B - A)(A \nabla B)^{-1}(B - A).$$
Similarly related to (2.7), instead of multiplying (2.1) from left and right by $(I_\lambda T)^{-\frac{1}{2}}$, multiplying (2.1) only form right or only from left by $(I_\lambda T)^{-1}$ and continuing in the same manner, yield

\[ A\left( A^{-1}\sharp(\lambda!\lambda B)^{-1}\right)(A\nabla\lambda B)\left( A^{-1}\sharp(\lambda!\lambda B)^{-1}\right)A = A(\lambda!\lambda B)^{-1}(A\nabla\lambda B) \]

\[ = (A\nabla\lambda B)(\lambda!\lambda B)^{-1}A = \lambda(1-\lambda)(A-B)B^{-1}(A-B) + A. \quad (2.16) \]

(ii) In the case that $A, B$ are two commutative operators such that $0 < A, B \leq \frac{1}{2}I$ and $\lambda \in [0, 1]$, the identities (2.6) and (2.7) transform to

\[ (\lambda!\lambda B)^{-1} - (A\nabla\lambda B)^{-1} = \lambda(1-\lambda)(B-A)(A\sharp B)^{-2}(A\nabla\lambda B)^{-1}(B-A) \quad (2.17) \]

\[ (\lambda!\lambda B)^{-1}(A\nabla\lambda B) - I = \lambda(1-\lambda)(A-B)A^{-1}B^{-1}(A-B). \quad (2.18) \]

Now changing $A, B$ by $A', B'$ we obtain

\[ (\lambda!\lambda B')^{-1} - (A'\nabla\lambda B')^{-1} = \lambda(1-\lambda)(B-A)(A'\sharp B')^{-2}(A'\nabla\lambda B')^{-1}(B-A) \quad (2.19) \]

\[ (\lambda!\lambda B')^{-1}(A'\nabla\lambda B') - I = \lambda(1-\lambda)(A-B)A'^{-1}B'^{-1}(A-B). \quad (2.20) \]

Comparing (2.17) and (2.19), yields

\[ (\lambda!\lambda B')^{-1} - (A'\nabla\lambda B')^{-1} \leq (\lambda!\lambda B)^{-1} - (A\nabla\lambda B)^{-1}. \quad (2.21) \]

Similarly, comparing (2.18) and (2.20) consequences

\[ (\lambda!\lambda B')^{-1}(A'\nabla\lambda B') \leq (\lambda!\lambda B)^{-1}(A\nabla\lambda B). \quad (2.22) \]

If $x_1, x_2 \in (0, \frac{3}{2}]$ and $\lambda \in [0, 1]$, putting $A = x_1I$ and $B = x_2I$ in (2.11), (2.21) and (2.22), we get (1.5), (1.6) and (1.7) when $n = 2$. These show that (2.11), (2.12) and (2.13) are operator extensions of (1.5), (1.6) and (1.7).

References

[1] H. Alzer, "An inequality for arithmetic and harmonic means", Aequations Math, vol. 46, pp. 257–163, 1993.
[2] H. Alzer, The inequality of Ky Fan and related results, Acta Appl. Math. 38 (1995), 305–354.
[3] H. Alzer, Ungleichungen für geometrische und arithmetische Mittelwerte, Proc. Kon. Nederl. Akad. Wetensch. 91 (1988), 365–374.
[4] E. F. Beckenbach and R. Bellman, Inequalities, Springer–Verlag, Berlin, 1961.
[5] T. Furuta, J. Mićić Hot, J.E. Pečarić and Y. Seo, Mond–Pečarić Method in Operator Inequalities, Element, Zagreb, 2005.
[6] A. Morassaei and F. Mirzapour, Alzer inequality for Hilbert spaces operators, J. Appl. Funct. Anal. 8 (2013), no. 2, 229-234.
[7] M.S. Moslehian, Ky Fan inequalities, Linear Multilinear Algebra, 60 (2012), no. 11–12, 1313-1325.
[8] J. Rooin, *On a Ky Fan type inequality due to H. Alzer*, J. Math. Inequal. 7 (2013), no. 3, 487–493.

[9] J. Sandor, *On an inequality of Ky Fan II*, Inter. J. Math. Educ. Tech. 22 (1991), 326-328.

1Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan 45137-66731, Iran

E-mail address: rooin@iasbs.ac.ir, akram.alikhani88@gmail.com