A new class of positive recurrent functions

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Abstract. In [Sa] Sarig has introduced and explored the concept of positively recurrent functions. In this paper we construct a natural wide class of such functions and we show that they have stronger ergodic properties than the general functions considered in [Sa].

1. Preliminaries

In [Sa] Sarig has introduced and explored the concept of positively recurrent functions. In this paper, using the concept of an iterated function system, we construct a natural wide class of positively recurrent functions and we show that they have stronger properties than the general functions considered in [Sa]. In some parts our exposition is similar and follows the approach developed in [MU1] and [Wa], where also the idea of embedding the infinite dimensional shift space into a compact metric space and the Shauder-Tichonov fixed-point theorem have been used. To begin with, let $\mathcal{N}$ be the set of positive integers and let $\Sigma = \mathcal{N}^\infty$ be the infinitely dimensional shift space equipped with the product topology. Let $\sigma : \Sigma \to \Sigma$ be the shift transformation (cutting out the first coordinate), $\sigma(\{x_n\}_{n=1}^\infty) = (\{x_n\}_{n=2}^\infty)$. Fix $\beta > 0$. If $\phi : \Sigma \to \mathbb{R}$ and $n \geq 1$, we set

$$V_n(\phi) = \sup\{\|\phi(x) - \phi(y)\| : x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n\}.$$ 

The function $\phi$ is said to be Hölder continuous of order $\beta$ if and only if

$$V(\phi) = \sup_{n \geq 1}\{e^{\beta n} V_n(\phi)\} < \infty.$$

We also assume that

$$\sup_{\omega \in \Sigma} \sum_{\tau \in \sigma^{-1}(\omega)} e^{\phi(\tau)} < \infty.$$ 

This assumption allows us to introduce the Perron-Frobenius-Ruelle operator $L_\phi : C_b(\Sigma) \to C_b(\Sigma)$,

$$L_\phi(g)(\omega) = \sum_{\tau \in \sigma^{-1}(\omega)} e^{\phi(\tau)} g(\tau).$$

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acting on \( C_b(\Sigma) \), the space of all bounded continuous real-valued functions on \( \Sigma \) equipped with the norm \( || \cdot ||_0 \), where \( ||k||_0 = \sup_{x \in \Sigma} |k(x)| \). Moreover,

\[
||L_0||_0 \leq L_\phi(\mathbb{1}) = \sup_{\omega \in \Sigma} \sum_{\tau \in \sigma^{-1}(\omega)} e^{\phi(\tau)} < \infty.
\]

We extend the standard definition of topological pressure by setting

\[
(1.2) \quad P(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \left( \sum_{|\omega|=n} \sup_{\tau \in [\omega]} \exp \left( \sum_{j=0}^{n-1} \phi \circ \sigma^j(\tau) \right) \right),
\]

where \([\omega] = \{ \rho \in \Sigma : \rho_1 = \omega_1, \rho_2 = \omega_2, \ldots, \rho_{|\omega|} = \omega_{|\omega|} \}\). Notice that the limit exists since the partition functions

\[
Z_n(\phi) = \sum_{|\omega|=n} \sup_{\tau \in [\omega]} \exp \left( \sum_{j=0}^{n-1} \phi \circ \sigma^j(\tau) \right)
\]

form a subadditive sequence. Notice also that our definition of pressure formally differs from that provided by Sarig in [Sa] which reads that given \( i \in \mathbb{N} \)

\[
(1.3) \quad P(\phi) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(\phi, i),
\]

where

\[
Z_n(\phi, i) = \sum \exp \left( \sum_{j=0}^{n-1} \phi \circ \sigma^j(\omega) \right)
\]

and the summation is taken over all elements \( \omega \) satisfying \( \sigma^n(\omega) = \omega \) and \( \omega_1 = i \). However in [Sa] Sarig proves Theorem 2 which says that \( P(\phi) = \sup\{P(\phi|_Y)\} \), where the supremum is taken over all topologically mixing subshifts of finite type \( Y \subset \Sigma \) and the same proof goes through with (1.3) replaced by (1.2) (comp. Theorem 3.1 of [MU2]). Thus we have the following.

**Lemma 1.1.** The definitions of topological pressures given by (1.2) and (1.3) coincide.

Here is a direct proof of this lemma communicated to us by Sarig Omri: Fix \( i \in \mathbb{N} \). Using Hölder continuity of the function \( \phi \) we can write

\[
Z_n(\phi) \asymp \sum_{|\omega|=n} \exp(\sum_{j=0}^{n-1} \phi(\omega^\infty)) \asymp \sum_{|\omega|=n} \exp(\sum_{j=0}^{n} \phi((i\omega)^\infty)) = Z_{n+1}(\phi, i).
\]

Thus the lemma is proved.

Following the definition 2 of [Sa] we call the function \( \phi : \Sigma \to \mathbb{R} \) positive recurrent if for every \( i \in \mathbb{N} \) there exists a constant \( M_i \) and an integer \( N_i \) such that for all \( n \geq N_i \)

\[
M_i^{-1} \leq Z_n(\phi, i) \lambda^{-n} \leq M_i
\]

for some \( \lambda > 0 \). As we already have said the main purpose of this paper is to provide a wide natural class of examples of positive recurrent potential which additionally satisfy much stronger properties than those claimed in Theorem 4 of [Sa]. In order to describe our setting let \((X, d)\) be a compact metric space and let \( \phi_i : X \to X, i \in \mathbb{N} \), be a family of uniform contractions, i.e. \( d(\phi_i(x), \phi_i(y)) \leq sd(x, y) \) for all \( i \in \mathbb{N} \),
Given $\omega \in \Sigma$ consider the intersection $\bigcap_{n \geq 1} \phi_{\omega_n}(X)$, where $\phi_{\omega_n} = \phi_1 \circ \ldots \circ \phi_{\omega_n}$. Since $\phi_{\omega_n}(X)$, $n \geq 1$, form a descending family of compact sets, this intersection is non-empty and since the maps $\phi_i$, $i \in \mathbb{N}$, are uniform contractions, it is a singleton. So, we have defined a projection map $\pi : \Sigma \to X$ given by the formula

$$\{\pi(\omega)\} = \bigcap_{n \geq 1} \phi_{\omega_n}(X).$$

$J$, the range of $\pi$, is said to be the limit set of the iterated function system $\phi_i : X \to X$, $i \in \mathbb{N}$. Let now $\phi^{(i)} : X \to \mathbb{R}$, $i \in \mathbb{N}$, be a family of continuous functions such that

$$\sup_{x} \sum_{i \in \mathbb{N}} e^{\phi^{(i)}(x)} < \infty. \quad (1.4)$$

We define a function $\phi : \Sigma \to \mathbb{R}$ by setting

$$\phi(\omega) = \phi^{(\omega_1)}(\pi(\sigma^i(\omega))). \quad (1.5)$$

It easily follows from (1.4) that $P(\phi) < \infty$. In the next section we shall prove the following.

**Theorem 1.2.** Suppose that the function $\phi : \Sigma \to \mathbb{R}$ defined by (1.5) and satisfying (1.4) is Hölder continuous. Let $L^*_\phi$ be the operator conjugate to $L_\phi$. Then $\phi$ is positive recurrent with $\lambda = e^{P(\phi)}$. Moreover there exists $M > 0$ such that $M^{-1} \leq \lambda^{-n} L^*_\phi(\mathbb{1}) \leq M$ for all $n \geq 1$. Suppose additionally that $\phi_i(X) \cap \phi_j(X) = \emptyset$ for all $i, j \in \mathbb{N}$, $i \neq j$. Then there are a probability measure $\nu$ on $\Sigma$ and a bounded away from zero and infinity Hölder continuous function $h : \Sigma \to (0, \infty)$ such that $L^*_\phi(\nu) = \lambda \nu$, $L_\phi(h) = \lambda h$, $\nu(h) = 1$ and $\lambda^{-n} L^*_\phi(g) \to (\int g d\nu)h$ uniformly for every uniformly continuous bounded function $g$. Additionally $\lambda^{-n} L^*_\phi(g) \to (\int g d\nu)h$ exponentially fast for each Hölder continuous bounded function $g$.

**2. Proof of Theorem 1.2.**

Define first an auxiliary Perron-Frobenius operator $\tilde{L}_\phi : C(X) \to C(X)$ given by the formula

$$\tilde{L}_\phi(g)(x) = \sum_{i \in \mathbb{N}} \phi^{(i)}(x) g(\phi_i(x)).$$

$\tilde{L}_\phi$ is continuous, positive and $||\tilde{L}_\phi||_0 \leq \sup_X \sum_{i \in \mathbb{N}} e^{\phi^{(i)}(x)} < \infty$. Let $\hat{L}^*_\phi : C(X)^* \to C(X)^*$ be the conjugate operator and following Bowen’s approach from [Bo] consider the map

$$\mu \mapsto \frac{\hat{L}^*_\phi(\mu)}{L^*_\phi(\mu)(\mathbb{1})},$$

of the space of Borel probability measures on $X$ into itself. This map is continuous in the weak-* topology of measures and therefore, in view of the Schauder-Tichonov theorem, it has a fixed point, say $m_\phi$. Thus

$$\hat{L}^*_\phi(m_\phi) = \lambda m_\phi \quad (2.1)$$

with $\lambda = \hat{L}^*_\phi(m_\phi)(\mathbb{1})$.

Given $n \geq 1$ and $\omega \in \mathbb{N}^n$, denote $\sum_{j=1}^n \phi^{(\omega_j)} \circ \phi_{\sigma^j \omega}$ by $S_\omega(\phi)$. Let us then prove the following.
Lemma 2.1. If \( x, y \in \phi_\tau(X) \) for some \( \tau \in I^* \), then for all \( \omega \in I^* \)
\[
|S_\omega(\phi)(x) - S_\omega(\phi)(y)| \leq \frac{V(\phi)}{1 - e^{-\beta}} e^{-\beta|\tau|}
\]

Proof. Let \( n = |\omega| \). Write \( x = \phi_\tau(u) \), \( y = \phi_\tau(w) \), where \( u, w \in X \). By (2.1) we get
\[
\left| \sum_{j=1}^{n} \phi^{(\omega_j)}(\phi_{\tau_j\omega_j}(x)) - \sum_{j=1}^{n} \phi^{(\omega_j)}(\phi_{\tau_j\omega_j}(y)) \right| = \left| \sum_{j=1}^{n} \phi^{(\omega_j)}(\phi_{\tau_j}(u)) - \sum_{j=1}^{n} \phi^{(\omega_j)}(\phi_{\tau_j}(w)) \right| \\
\leq \sum_{j=1}^{n} V(\phi) e^{-\beta(n+|\tau|-j)} \\
\leq \frac{V(\phi)}{1 - e^{-\beta}} e^{-\beta|\tau|}
\]
The proof is finished.

Remark 2.2. We allow in Lemma 2.1 \( \tau \) to be the empty word \( \emptyset \). Then \( \phi_\emptyset = \text{Id}_X \) and \( |\emptyset| = 0 \).

Set
\[
Q = \exp \left( V(\phi) \frac{e^{-\beta}}{1 - e^{-\beta}} \right).
\]
We shall prove the following.

Lemma 2.3. The eigenvalue \( \lambda \) (see 2.1) of the dual Perron-Frobenius operator is equal to \( e^{P(\phi)} \).

Proof. Iterating (2.1) we get
\[
\lambda^n = \lambda^n m_\phi(\mathbb{I}) = \tilde{L}_\phi^n(\mathbb{I}) = \int_X \tilde{L}_\phi^n(\mathbb{I}) dm_\phi \\
= \int_X \sum_{|\omega|=n} \exp(S_\omega(\phi)(x)) \leq \sum_{|\omega|=n} \| \exp(S_\omega(\phi)) \|_0.
\]
So,
\[
\log \lambda \leq \lim_{n \to \infty} \frac{1}{n} \log \left( \sum_{|\omega|=n} \| \exp(S_\omega(\phi)) \|_0 \right) = P(\phi).
\]
Fix now \( \omega \in I^n \) and take a point \( x_\omega \) where the function \( S_\omega(\phi) \) takes on its maximum. In view of Lemma 2.1, for every \( x \in X \) we have
\[
\sum_{|\omega|=n} \exp(S_\omega(\phi)(x)) \geq Q^{-1} \sum_{|\omega|=n} \exp(S_\omega(\phi)(x_\omega)) = Q^{-1} \sum_{|\omega|=n} \| \exp(S_\omega(\phi)) \|_0.
\]
Hence, iterating (2.1) as before,

$$
\lambda^n = \int_X \sum_{|\omega|=n} \exp(S_\omega(\phi))dm_\phi \geq Q^{-1} \sum_{|\omega|=n} \|\exp(S_\omega(\phi))\|_0.
$$

So, log $\lambda \geq \lim_{n \to \infty} \frac{1}{n} \log(\sum_{|\omega|=n} \|\exp(S_\omega(\phi))\|_0) = P(\phi)$. The proof is finished. \qed

Let $\hat{L}_0$ and $L_0$ denote the corresponding normalized Perron-Frobenius operators, i.e. $\hat{L}_0 = e^{-P(\phi)}\hat{L}_\phi$ and $L_0 = e^{-P(\phi)}L_\phi$. We shall prove the following.

**Theorem 2.4.** $m_\phi(J) = 1$.

**Proof.** Since by (2.1)

$$
\hat{L}_0^*(m_\phi) = m_\phi
$$

and consequently $\hat{L}_0^*(m_\phi) = m_\phi$ for all $n \geq 0$, we have

$$
\int_X \sum_{|\omega|=n} \exp(S_\omega(\phi) - P(\phi)n) \cdot (f \circ \phi_\omega)dm_\phi = \int_X f dm_\phi
$$

for all $n \geq 0$ and all continuous functions $f : X \to \mathbb{R}$. Since this equality extends to all bounded measurable functions $f$, we get

$$
m_\phi(A) = \sum_{\tau \in \mathbb{T}^n} \int \exp(S_\tau(\phi) - P(\phi)n) \cdot \mathbb{1}_{\phi_\omega(A)} \circ \phi_\tau dm_\phi
$$

(2.4) \geq \int_A \exp(S_\omega(\phi) - P(\phi)n) dm_\phi

for all $n \geq 0$, all $\omega \in \mathbb{T}^n$, and all Borel sets $A \subset X$. Now, for each $n \geq 1$ set $X_n = \bigcup_{|\omega|=n} \phi_\omega(X)$. Then $\mathbb{1}_{X_n} \circ \phi_\omega = \mathbb{1}$ for all $\omega \in \mathbb{N}_n$. Thus applying (2.3) to the function $f = \mathbb{1}_{X_n}$ and later to the function $f = \mathbb{1}$, we obtain

$$
m_\phi(X_n) = \int_X \sum_{|\omega|=n} \exp(S_\omega(\phi) - P(\phi)n) \cdot (\mathbb{1}_{X_n} \circ \phi_\omega) dm_\phi
$$

$$
= \int_X \sum_{|\omega|=n} \exp(S_\omega(\phi) - P(\phi)n) dm_\phi = \int \mathbb{1} dm_\phi = 1.
$$

Hence $m_\phi(J) = m_\phi(\bigcap_{n \geq 1} X_n) = 1$. The proof is complete. \qed

**Theorem 2.5.** For all $n \geq 1$

$$
Q^{-1} \leq \hat{L}_0^n(\mathbb{1}) \leq Q.
$$

**Proof.** Given $n \geq 1$ by (2.3) there exits $x_n \in X$ such that $\hat{L}_0^n(\mathbb{1})(x_n) \leq 1$. It then follows from Lemma 2.1 that for every $x \in X$, $\hat{L}_0^n(\mathbb{1}) \leq Q$. Similarly by (2.3) there exists $y_n \in X$ such that $\hat{L}_0^n(\mathbb{1}) \geq 1$. It then follows from Lemma 2.1 that for every $x \in X$, $\hat{L}_0^n(\mathbb{1}) \geq Q^{-1}$. The proof is finished. \qed

So far we have worked downstairs in the compact space $X$. It is now time to lift our considerations up to the shift space $\Sigma$.

**Lemma 2.6.** There exists a unique Borel probability measure $\tilde{m}_\phi$ on $\mathbb{N}^\infty$ such that $\tilde{m}_\phi([\omega]) = \int \exp(S_\omega(\phi) - P(\phi)n) dm_\phi$ for all $\omega \in \mathbb{N}^\ast$. 
PROOF. In view of (2.4) \( \int \exp(S_\omega(\phi) - P(\phi)n)dm_\phi = 1 \) for all \( n \geq 1 \) and therefore one can define a Borel probability measure \( m_n \) on \( C_n \), the algebra generated by the cylinder sets of the form \( [\omega], \omega \in \mathbb{N}^n \), putting \( m_n(\omega) = \int \exp(S_\omega(\phi) - P(\phi)n)dm_\phi \). Hence, applying (2.4) again we get for all \( \omega \in \mathbb{N}^n \).

\[
m_{n+1}(\omega) = \sum_{i \in \mathbb{N}} m_{n+1}([\omega]) = \sum_{i \in \mathbb{N}} \int \exp(S_{\omega i}(\phi) - P(\phi)n)dm_\phi
\]

\[
= \int \sum_{i \in \mathbb{N}} \exp \left( \sum_{j=1}^n \phi(\omega_j) \circ \phi_{\sigma^j(\omega)} - L_\phi \right) dm_\phi
\]

\[
= \int \sum_{i \in \mathbb{N}} \exp(S_\omega \circ \phi_i - P(\phi)n) \exp(\phi(\omega)) - P(\phi))dm_\phi
\]

\[
= \int \tilde{L}_\phi \exp(S_\omega(\phi) - P(\phi))dm_\phi = \int \exp(S_\omega(\phi) - P(\phi))dm_\phi = m_n(\omega)
\]

and therefore in view of Kolmogorov’s extension theorem there exists a unique probability measure \( \tilde{m}_\phi \) on \( \mathbb{N}^\infty \) such that \( \tilde{m}_\phi([\omega]) = \tilde{m}_\phi([\omega]) \) for all \( \omega \in \mathbb{N}^n \). The proof is complete. \( \square \)

Now we are ready to prove that the function \( \phi \) is positive recurrent. Let us first notice that

\[
L_{\phi}(\mathbb{I}) (\omega) = \sum_{\tau \in \sigma^{-1}(\omega)} e^{\phi(\tau)} = \sum_{\tau \in \sigma^{-1}(\omega)} \exp(\phi(\tau))(\pi(\tau)) = \sum_{i \in \mathbb{N}} e^{\phi(i)}(\pi(\sigma)) = \tilde{L}_{\phi}(\mathbb{I})(\pi(\omega)).
\]

Since \( \tilde{L}_\phi = e^{-P(\phi)}\tilde{L}_{\phi} \), it then follows from Theorem 2.4 that as \( M \) we can take \( Q \).

In order to demonstrate that the function \( \phi \) is positive recurrent we first show that

\[
\frac{Z_n(\phi, i)}{L_{\phi}^n(\mathbb{I})(\omega)} \leq M_i
\]

for all \( n \geq 1 \), \( \omega \in \Sigma \), and some constant \( M_i > 0 \). So fix \( \omega \in \Sigma \). We shall define an injection \( j \) from \( \{ \rho \in \Sigma : \sigma^n(\rho) = \rho \} \) into \( \sigma^{-n}(\omega) \) as follows: \( j(\rho) = \rho_1 \rho_2 \cdots \rho_n \omega \). Now, by Lemma 2.1

\[
\left| \sum_{j=0}^{n-1} \phi(\sigma^j(\rho)) - \sum_{j=0}^{n-1} \phi(\sigma^j(j(\rho))) \right| \leq \log Q
\]

and therefore \( Z_n(\phi, i) \leq Q L_\phi^n(\mathbb{I})(\omega) \). Thus by Theorem 2.4 and the definition of the operators \( \tilde{L}_\phi \) and \( L_0 \), \( Z_n(\phi, i) \leq M_i \lambda^n \), where \( M_i = Q^2 \). Now we shall prove that \( Z_n(\phi, i) \geq M_i' \lambda^n \) for some constant \( M_i' \) and all \( n \geq 1 \). We demonstrate first that for all \( n \geq 1 \) and all \( i \in \Sigma \)

\[
L_0(\mathbb{I}[i]) \geq \tilde{m}_\phi([i]).
\]
Indeed, since \( \int L_0(\mathbb{I}_i) \, d\tilde{m}_\phi = \int \mathbb{I}_i \, d\tilde{m}_\phi = \tilde{m}_\phi([i]) > 0 \), there exists \( \tau \in \Sigma \) such that \( L_0(\mathbb{I}_i)(\tau) \geq \tilde{m}_\phi([i]) \). It the follows from Lemma 2.1 that for every \( \omega \in \Sigma \)

\[
L^n_0(\mathbb{I}_i)(\omega) = \sum_{\rho \in \sigma^n(\omega)} \exp \left( \sum_{j=0}^{n-1} \phi \circ \sigma^j(\rho) \mathbb{I}_i(\rho) \right)
\]

\[
\geq Q^{-1} \sum_{\rho \in \sigma^n(\tau)} \exp \left( \sum_{j=0}^{n-1} \phi \circ \sigma^j(\rho) \mathbb{I}_i(\rho) \right) = Q^{-1} L_0(\mathbb{I}_i)(\tau)
\]

Hence \( L^n_0(\mathbb{I}_i)(\omega) \geq \lambda^n \tilde{m}_\phi([i]) \). So, in order to conclude the proof that \( \phi \) is positively recurrent it suffices now to show that

\[
\frac{Z_n(\phi,i)}{L^n_0(\mathbb{I}_i)(\omega)} \geq M_i''
\]

for all \( n \geq 1 \), all \( \omega \in \Sigma \) and some constant \( M_i'' > 0 \). Indeed, we shall define an injection \( k \) from \( \sigma^{-n}(\omega) \cap [i] \) to \( \{ \rho : \Sigma : \sigma^n(\rho) = \rho \text{ and } \rho_1 = i \} \) by taking as \( k(\tau) \) the infinite concatenation of the first \( n \) words of \( \tau \). Then by Lemma 2.1,

\[
\left| \sum_{j=0}^{n-1} \phi(\sigma^j(\tau)) - \sum_{j=0}^{n-1} \phi(\sigma^j(k(\tau))) \right| \leq \log Q
\]

and therefore

\[
L^n_0(\mathbb{I}_i)(\omega) = \sum_{\rho \in \sigma^{-n}(\omega)} \exp \left( \sum_{j=0}^{n-1} \phi \circ \sigma^j(\rho) \mathbb{I}_i(\rho) \right)
\]

\[
= \sum_{\rho \in \sigma^{-n}(\omega) \cap [i]} \exp \left( \sum_{j=0}^{n-1} \phi \circ \sigma^j(\rho) \right)
\]

\[
\leq \sum_{\rho \in \sigma^{-n}(\omega) \cap [i]} \exp \left( \sum_{j=0}^{n-1} \phi \circ \sigma^j(k(\rho)) + \log Q \right)
\]

\[
\leq Q \sum_{\rho \in \sigma^{-n}(\omega) \cap [i]} \exp \left( \sum_{j=0}^{n-1} \phi \circ \sigma^j(\rho) \right) \leq QZ_n(\phi,i),
\]

where the last summation is taken over all elements \( \omega \) satisfying \( \sigma^n(\omega) = \omega \) and \( \omega_1 = i \). So, the proof of the positive recurrence of \( \phi \) is complete taking \( Q^{-1} \) as \( M_i'' \).

Now we pass to proving the existence of the measure \( \nu \) and the function \( h \). We begin with the following two facts.

**Lemma 2.7.** The measures \( m_\phi \) and \( \tilde{m}_\phi \circ \pi^{-1} \) are equal.

**Proof.** Let \( A \subset J \) be an arbitrary closed subset of \( J \) and for every \( n \geq 1 \) let \( A_n = \{ \omega \in \mathcal{I}^n : \phi_\omega(X) \cap A \neq \emptyset \} \). In view of (2.3) applied to the characteristic
function \( I_A \) we have for all \( n \geq 1 \)

\[
m_\phi(A) = \sum_{\omega \in \mathbb{N}^n} \int \exp(S_\omega(\phi) - P(\phi)|\omega|) (I_A \circ \phi_\omega)\, dm_\phi
\]

\[
= \sum_{\omega \in A_n} \int \exp(S_\omega(\phi) - P(\phi)|\omega|) (I_A \circ \phi_\omega)\, dm_\phi
\]

\[
\leq \sum_{\omega \in A_n} \int \exp(S_\omega(\phi) - P(\phi)|\omega|)\, dm_\phi = \sum_{\omega \in A_n} \tilde{m}_\phi(\omega) = \tilde{m}_\phi(\bigcup \omega)
\]

Since the family of sets \( \{ \bigcup_{\omega \in A_n} \omega : n \geq 1 \} \) is descending and \( \bigcap_{n \geq 1} \bigcup_{\omega \in A_n} \omega = \pi^{-1}(A) \) we therefore get \( m_\phi(A) \leq \lim_{n \to \infty} \tilde{m}_\phi(\bigcup_{\omega \in A_n} \omega) = \tilde{m}_\phi(\pi^{-1}(A)). \) Since the limit set \( J \) is a metric space, using the Baire classification of Borel sets we easily see that this inequality extends to the family of all Borel subsets of \( J \). Since both measures \( m_\phi \) and \( \tilde{m}_\phi \circ \pi^{-1} \) are probabilistic we get \( m_\phi = \tilde{m}_\phi \circ \pi^{-1}. \) The proof is finished. \( \square \)

We recall that an invariant measure of a metric dynamical system is said to be totally ergodic if it is ergodic with respect to all the iterates of the system under consideration.

**Theorem 2.8.** There exists a unique totally ergodic \( \sigma \)-invariant probability measure \( \tilde{m}_\phi \) absolutely continuous with respect to \( m_\phi \). Moreover \( \tilde{m}_\phi \) is equivalent with \( m_\phi \) and \( Q^{-1} \leq \frac{d\tilde{m}_\phi}{dm_\phi} \leq Q. \)

**Proof.** First notice that, using Lemma 2.5, for each \( \omega \in \mathbb{N}^n \) and each \( n \geq 0 \) we have

\[
\tilde{m}_\phi(\sigma^{-n}([\omega])) = \sum_{\tau \in \mathbb{N}^n} \tilde{m}_\phi([\tau \omega]) = \sum_{\tau \in \mathbb{N}^n} \int \exp(S_{\tau\omega}(\phi) - P(\phi)|\tau\omega|)\, dm
\]

\[
\geq \sum_{\tau \in \mathbb{N}^n} Q^{-1} \| \exp(S_{\tau}(\phi) - P(\phi)|\tau) \|_0 \exp(S_{\omega}(\phi - P(\phi)|\omega))\, dm_\phi
\]

\[
= Q^{-1} \int \exp(S_\omega(\phi - P(\phi)|\omega))\, dm_\phi \sum_{\tau \in \mathbb{N}^n} \| \exp(S_{\tau}(\phi - P(\phi)|\tau)) \|_0
\]

\[
\geq Q^{-1} \tilde{m}_\phi([\omega]) m_\phi(\mathbb{N}^\infty) = Q^{-1} \tilde{m}_\phi([\omega])
\]

and

\[
\tilde{m}_\phi(\sigma^{-n}([\omega])) = \sum_{\tau \in \mathbb{N}^n} \tilde{m}_\phi([\tau \omega]) = \sum_{\tau \in \mathbb{N}^n} \int \exp(S_{\tau\omega}(\phi - P(\phi)|\tau\omega))\, dm_\phi
\]

\[
\leq \sum_{\tau \in \mathbb{N}^n} \| \exp(S_{\tau}(\phi - P(\phi)|\tau)) \|_0 \int \exp(S_{\omega}(\phi - P(\phi)|\omega))\, dm_\phi
\]

\[
= \exp(S_{\omega}(\phi - P(\phi)|\omega))\, dm_\phi \sum_{\tau \in \mathbb{N}^n} \| \exp(S_{\tau}(\phi - P(\phi)|\tau)) \|_0
\]

\[
\leq Q \tilde{m}_\phi([\omega]).
\]

Let now \( L \) be a Banach limit defined on the Banach space of all bounded sequences of real numbers. We define \( \mu([\omega]) = L(\tilde{m}_\phi(\sigma^{-n}([\omega])))_{n \geq 0}. \) Hence \( Q^{-1} \tilde{m}_\phi([\omega]) \leq \mu([\omega]) \leq Q \tilde{m}_\phi([\omega]) \) and therefore it is not difficult to check that the formula \( \mu(A) = L(\tilde{m}_\phi(\sigma^{-n}(A)))_{n \geq 0} \) defines a finite non-zero finitely additive measure on Borel
sets of \( \mathcal{N}^\infty \) satisfying \( Q^{-1}\tilde{m}_\phi(A) \leq \mu(A) \leq Q\tilde{m}_\phi(A) \). Using now a theorem of Calderon (Theorem 3.13 of [Fr]) and its proof one constructs a Borel probability (\( \sigma \)-additive) measure \( \tilde{\mu}_\phi \) on \( \mathcal{N}^\infty \) satisfying the formula

\[
Q^{-1}\tilde{m}_\phi(A) \leq \tilde{\mu}_\phi(A) \leq Q\tilde{m}_\phi(A)
\]

for every Borel set \( A \subset \mathcal{N}^\infty \) with, perhaps, a larger constant \( Q \). Thus, to complete the proof of our theorem we only need to show total ergodicity of \( \tilde{\mu}_\phi \) or equivalently of \( \tilde{m}_\phi \). Toward this end take a Borel set \( A \subset \mathcal{N}^\infty \) with \( \tilde{m}_\phi(A) > 0 \). Since the nested family of sets \( \{ \} : \tau \in \mathcal{N}^* \) generates the Borel \( \sigma \)-algebra on \( \mathcal{N}^\infty \), for every \( n \geq 0 \) and every \( \omega \in \mathcal{N}^n \) we can find a subfamily \( Z \) of \( \mathcal{N}^* \) consisting of mutually incomparable words and such that \( A \subset \bigcup \{ \} : \tau \in Z \} \) and \( \sum_{\tau \in Z} \tilde{m}_\phi(\{ \omega \tau \}) \leq 2\tilde{m}_\phi(\omega A) \), where \( \omega A = \{ \rho \omega : \rho \in A \} \). Then

\[
\tilde{m}_\phi(\sigma^{-n}(\mathcal{N}^\infty \setminus A) \wedge [\omega]) = \tilde{m}_\phi([\omega] \wedge \sigma^{-n}(A) \wedge [\omega]) = \tilde{m}_\phi([\omega]) - \tilde{m}_\phi(\sigma^{-n}(A) \wedge [\omega]) \leq (1 - (2Q)^{-1}\tilde{m}_\phi(A))\tilde{m}_\phi([\omega]).
\]

Hence for every Borel set \( A \subset \mathcal{N}^\infty \) with \( \tilde{m}_\phi(A) < 1 \), for every \( n \geq 0 \), and for every \( \omega \in \mathcal{N}^n \) we get

\[
\tilde{m}_\phi(\sigma^{-n}(\mathcal{N}^\infty \setminus A) \wedge [\omega]) \leq (1 - (2Q)^{-1}(1 - \tilde{m}_\phi(A)))\tilde{m}_\phi([\omega]).
\]

In order to conclude the proof of total ergodicity of \( \sigma \) suppose that \( \sigma^{-r}(A) = A \) for some integer \( r \geq 1 \) and some Borel set \( A \) with \( 0 < \tilde{m}_\phi(A) < 1 \). Put \( \gamma = 1 - (2Q)^{-1}(1 - \tilde{m}_\phi(A)) \). Note that \( 0 < \gamma < 1 \). In view of (2.5), for every \( \omega \in (\mathcal{N}^r)^* \) we get \( \tilde{m}_\phi(A \cap [\omega]) = \tilde{m}_\phi(\sigma^{-w}(A) \cap [\omega]) \leq \gamma \tilde{m}_\phi([\omega]) \). Take now \( \eta > 1 \) so small that \( \gamma \eta < 1 \) and choose a subfamily \( R \) of \( (\mathcal{N}^r)^* \) consisting of mutually incomparable words and such that \( A \subset \bigcup \{ [\omega] : \omega \in R \} \) and \( \tilde{m}_\phi(\bigcup [\omega] : \omega \in R) \leq \eta \tilde{m}_\phi(A) \). Then \( \tilde{m}_\phi(A) \leq \sum_{\omega \in R} \tilde{m}_\phi(A \cap [\omega]) \leq \sum_{\omega \in R} \gamma \tilde{m}_\phi([\omega]) = \gamma \tilde{m}_\phi(\bigcup [\omega] : \omega \in R) \leq \gamma \eta \tilde{m}_\phi(A) < \tilde{m}_\phi(A) \). This contradiction finishes the proof. \( \square \)

Set \( \nu = \tilde{m}_\phi \). Clearly our assumption \( \phi_i(X) \cap \phi_j(X) = \emptyset \) for \( i, j \in \mathcal{N}, i \neq j \) implies that \( \pi : \Sigma \to J \) is a homeomorphism; in particular, in view of Lemma 2.6, it establishes a measure preserving isomorphism between measure spaces \( (\Sigma, \nu) \) and
(J,mφ). To check that $L^*_φ(ν) = λν$ take $g ∈ C_b(Σ)$ and compute

$$
\int gdL^*_0(ν) = \int L_0(g)dν = \int L_0(g)(π^{-1}(x))dν \circ π^{-1}(x) = \int L_0(g)(π^{-1}(x))dmφ
$$

$$
= \int τ∈σ^{-1}(π^{-1}(x)) \exp(φ(τ) - P(φ))dmφ
$$

$$
= \int \sum_{i∈N} \exp(φ(i)(x) - P(φ))g \circ π^{-1}(φ_i(x))dmφ(x)
$$

$$
= \int L_0(g)dmφ = \int g \circ π^{-1}dmφ = \int gdν.
$$

Thus $L_0(ν) = ν$ and by the definition of $L_0$ and $L^*_0$, $L^*_φ(ν) = λν$. The fact that $L_0(h)(ν) = λh$ follows immediately from the definition of the operator $L_0$ and Theorem 2.7, where $h = d̃μφ/d̃mφ$. Theorem 2.7 also implies that $h$ is bounded away from zero and infinity. In order to obtain Hölder continuity of the function $h$ and two convergence statements claimed in Theorem 1.2 one may argue as follows: A well-known computation (see [DU], comp. [MU1]) shows that $L_0$ acts on the Banach space of bounded uniformly continuous functions on $\mathbb{N}^\infty$ as an almost periodic operator (see [Ly], comp. [DU] and [MU1]). Using Theorem 2.7 and the theory of positive operators on lattices (see [Sc]) one then proves as in [DU] that 1 is the only spectral point of modulus 1 and additionally that 1 is a simple eigenvalue of $L_0$. These facts and almost periodicity imply the first convergence statement of Theorem 1.2 and uniform continuity of $h$. A similar computation as above produces constants $0 < γ < 1$, $n ≥ 1$ and $C ≥ 0$ such that

$$
||L^n_0(γ)||_β ≤ C||γ||_0 + γ||g||_β,
$$

where $||γ||_β = V_β(γ) + ||g||_0$. This is so called the Ionescu-Tulcea and Marinescu inequality. Using this inequality and Theorem 2.4 one checks that the assumptions of the theorem of Ionescu-Tulcea and Marinescu (see [IM], comp. [PU]) are satisfied. This theorem gives a nice spectral decomposition of the operator $L_0$ acting on the space $H_β$ of bounded Hölder continuous functions of order $β$. Having this, a relatively straightforward reasoning (comp. [PU]) shows Hölder continuity of $h$ and the second convergence statement of Theorem 1.2.

3. Equilibrium states

In this section we further investigate the $σ$-invariant measure $μφ$ introduced in Theorem 2.7. We begin with the following technical result.

**Lemma 3.1.** The following 3 conditions are equivalent (a) $∫ −φdμφ < ∞$.
(b) $∑_{i∈N} \inf(-φ(i)) \exp(\inf φ(i)) < ∞$.
(c) $H_α(μφ) < ∞$, where $α = \{[i] : i ∈ \mathbb{N}\}$ is the partition of $Σ$ into initial cylinders of length 1.
Thus suppose that \( \sum_{i \in \mathbb{N}} -\phi d\tilde{\mu}_\phi < \infty \) and consequently

\[
\infty > \sum_{i \in \mathbb{N}} \inf(-\phi|_i) \int_{[i]} d\tilde{\mu}_\phi = \sum_{i \in \mathbb{N}} \inf(-\phi|_i) \int_{[i]} h d\tilde{m}_\phi
\]

\[
\geq Q^{-1} \sum_{i \in \mathbb{N}} \inf(-\phi|_i) \tilde{m}_\phi([i]) = Q^{-1} \sum_{i \in \mathbb{N}} \inf(-\phi|_i) \int_X \exp(\phi^{(i)}(x) - P(\phi)) dm_\phi(x)
\]

\[
= Q^{-1} e^{-P(\phi)} \sum_{i \in \mathbb{N}} \inf(-\phi|_i) \int_X \exp(\phi^{(i)}(x)) dm_\phi(x)
\]

Thus

\[
\infty > \sum_{i \in \mathbb{N}} \inf(-\phi|_i) \int_X \exp(\phi^{(i)}(x)) dm_\phi(x) \geq \sum_{i \in \mathbb{N}} \inf(-\phi|_i) \exp(\inf(\phi|_i))
\]

\[
= \sum_{i \in \mathbb{N}} \inf(-\phi|_i) \exp(\inf(\phi|_i))
\]

Now suppose that \( \sum_{i \in \mathbb{N}} \inf(-\phi|_i) \exp(\inf(\phi|_i)) < \infty \). We shall show that \( H_{\tilde{\mu}_\phi}(\alpha) < \infty \). So,

\[
H_{\tilde{\mu}_\phi}(\alpha) = \sum_{i \in \mathbb{N}} -\tilde{\mu}_\phi([i]) \log \tilde{\mu}_\phi([i]) \leq \sum_{i \in \mathbb{N}} -Q\tilde{m}_\phi([i]) (\log \tilde{m}_\phi([i]) - \log Q).
\]

But \( \sum_{i \in \mathbb{N}} -Q\tilde{m}_\phi([i]) (-\log Q) = Q \log Q \), so it suffices to show that

\[
\sum_{i \in \mathbb{N}} -\tilde{m}_\phi([i]) \log \tilde{m}_\phi([i]) < \infty.
\]

But

\[
\sum_{i \in \mathbb{N}} -\tilde{m}_\phi([i]) \log \tilde{m}_\phi([i]) = \sum_{i \in \mathbb{N}} -\tilde{m}_\phi([i]) \log \left( \int_X \exp(\phi^{(i)} - P(\phi)) \right) dm_\phi
\]

\[
\leq \sum_{i \in \mathbb{N}} -\tilde{m}_\phi([i]) (\inf_X \phi^{(i)} - P(\phi)).
\]

But \( \sum_{i \in \mathbb{N}} \tilde{m}_\phi([i]) P(\phi) = P(\phi) \), so it suffices to show that \( \sum_{i \in \mathbb{N}} -\tilde{m}_\phi([i]) \inf_X \phi^{(i)} < \infty \). And indeed, using Lemma 2.1 we get

\[
\sum_{i \in \mathbb{N}} -\tilde{m}_\phi([i]) \inf_X \phi^{(i)} = \sum_{i \in \mathbb{N}} \tilde{m}_\phi([i]) \sup_X (-\phi^{(i)}) \leq \sum_{i \in \mathbb{N}} \tilde{m}_\phi([i]) (\inf_X (-\phi^{(i)}) + \log Q).
\]

Since \( \sum_{i \in \mathbb{N}} \tilde{m}_\phi([i]) \log Q = \log Q \), it is enough to show that

\[
\sum_{i \in \mathbb{N}} \tilde{m}_\phi([i]) \inf_X (-\phi^{(i)}) < \infty.
\]

And indeed,

\[
\sum_{i \in \mathbb{N}} \tilde{m}_\phi([i]) \inf_X (-\phi^{(i)}) = \sum_{i \in \mathbb{N}} \int_X \exp(\phi^{(i)} - P(\phi)) dm_\phi \inf_X (-\phi^{(i)})
\]

But in view of (1.4) \( \phi^{(i)} \) are negative everywhere for all \( i \) large enough, say \( i \geq k \). Then using Lemma 2.1 again we get

\[
\sum_{i \geq k} \tilde{m}_\phi([i]) \inf_X (-\phi^{(i)}) \leq e^{-P(\phi)} Q \sum_{i \geq k} \exp(\inf_X (\phi^{(i)})) \inf_X (-\phi^{(i)})
\]
which is finite due to our assumption. Hence, \( \sum_{i \in \mathbb{N}} \tilde{m}_\phi ([i]) \inf_X (-\phi ([i])) < \infty \). Finally suppose that \( H_{\tilde{\mu}_\phi} (\alpha) < \infty \). We need to show that \( \int -\phi d\tilde{\mu}_\phi < \infty \). We have

\[
\infty > H_{\tilde{\mu}_\phi} (\alpha) = \sum_{i \in \mathbb{N}} -\tilde{m}_\phi ([i]) \log (\tilde{m}_\phi ([i])) \leq \sum_{i \in \mathbb{N}} -\tilde{m}_\phi ([i]) (\inf (\phi ([i]) - P(\phi) - \log Q)).
\]

Hence \( \sum_{i \in \mathbb{N}} -\tilde{m}_\phi ([i]) \inf (\phi ([i])) < \infty \) and therefore

\[
\int -\phi d\tilde{\mu}_\phi = \sum_{i \in \mathbb{N}} \int_{[i]} -\phi d\tilde{\mu}_\phi \leq \sum_{i \in \mathbb{N}} \sup_{\phi} (\phi ([i]) \tilde{m}_\phi ([i])) = \sum_{i \in \mathbb{N}} -\inf (\phi ([i]) \tilde{m}_\phi ([i])) < \infty.
\]

The proof is complete.

By Theorem 3 of [Sa] we know that \( \sup \{ h_\mu (\sigma) + \int \phi d\mu \} = P(\phi) \), where the supremum is taken over all \( \sigma \)-invariant probability measures such that \( \int -\phi d\mu < \infty \). We call a \( \sigma \)-invariant probability measure \( \mu \) an equilibrium state of the potential \( \phi \) if \( h_\mu (\sigma) + \int \phi d\mu = P(\phi) \). We shall prove the following.

**Theorem 3.2.** If \( \sum_{i \in \mathbb{N}} \inf (-\phi ([i])) \exp (\inf (\phi ([i]))) < \infty \), then \( \tilde{\mu}_\phi \) is an equilibrium state of the potential \( \phi \) satisfying \( \int -\phi d\tilde{\mu}_\phi < \infty \).

**Proof.** It follows from Lemma 3.1 that \( \int -\phi d\tilde{\mu}_\phi < \infty \). To show that \( \tilde{\mu}_\phi \) is an equilibrium state of the potential \( \phi \) consider \( \alpha = \{ [i] : i \in \mathbb{N} \} \), the partition of \( \Sigma \) into initial cylinders of length one. By Lemma 3.1, \( H_{\tilde{\mu}_\phi} (\alpha) < \infty \). Applying the Breiman-Shannon-McMillan theorem and the Birkhoff ergodic theorem we therefore get for \( \tilde{\mu}_\phi \)-a.e. \( \omega \in \Sigma \)

\[
h_\mu_\phi (\sigma) \geq h_\mu_\phi (\sigma, \alpha) = \lim_{n \to \infty} \frac{-1}{n} \log (\exp (S_n (\phi)))
= \lim_{n \to \infty} \frac{-1}{n} \log \left( \int \exp (\sum_{j=0}^{n-1} \phi (\sigma^j (\omega))) d\mu_\phi (\omega) - P(\phi) n \right)
\geq \lim_{n \to \infty} \frac{-1}{n} \log \left( \int \exp (\sum_{j=0}^{n-1} \phi (\sigma^j (\omega))) + \log Q - P(\phi) n \right)
= \lim_{n \to \infty} \frac{-1}{n} \sum_{j=0}^{n-1} \phi (\sigma^j (\omega)) + P(\phi) = - \int \phi d\tilde{\mu}_\phi + P(\phi).
\]

Hence \( h_\mu_\phi (\sigma) + \int \phi d\tilde{\mu}_\phi \geq P(\phi) \), which in view of the variational principle (see Theorem 3 in [Sa]), implies that \( \tilde{\mu}_\phi \) is an equilibrium state for the potential \( \phi \). The proof is finished.

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