A Generalization of the Clifford Index and
Determinantal Equations for Curves and Their
Secant Varieties

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Abstract

This paper gives a geometric characterization of the Clifford index of a curve $C$, in terms of the existence of determinantal equations for $C$ and its secant varieties in the bicanonical embedding. The key idea is the generalization of Shiffer deformations to Shiffer variations supported on an arbitrary effective divisor $D$. The definition of Shiffer variations and Clifford index is then generalized to an arbitrary very ample line bundle, $L$, on $C$. This allows one to give geometric characterizations of the generalized Clifford index in many cases. In particular it allows one to show the existence of determinantal equations for $C$ and $\text{Sec}^k(C)$ for $k < \text{Cliff}(C,L)$ in the embedding of $C$ in the projective $\mathbb{P}(H^0(L^2)^\vee)$. We then explain how the results can be generalized to embeddings of $C$ in $\mathbb{P}(H^0(L_1 \otimes L_2)^\vee)$ where $L_1$ and $L_2$ are very ample. For line bundles of large degree this proves a conjecture of Eisenbud, Koh, and Stillman relating the existence of a determinantal presentation for $\text{Sec}^k(C)$ to $\text{deg}(L)$.

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1  Introduction  

When Brendan Hassett was a post doc at the University of Chicago I spoke to him a lot about a paper of Griffiths [11], which involved the concept of Shiffer variations. These are infinitesimal deformations of a curve, which have rank one when viewed (via cup product) as homomorphisms from $H^0(K_C) \to H^1(\mathcal{O}_C)$. They are in fact parametrized by points of the curve. Initially I was trying to understand on which curves $C$, every rank $j$ infinitesimal deformations is just the sum of $j$ rank one Shiffer variations. Theorem 2.1 shows this holds only for $j < \text{Cliff}(C)$.  

In hindsight, it became clear to me that the construction was of a geometric nature and really had nothing to do with infinitesimal deformations per se. The construction could be made on any very ample line bundle. In its most general form the idea is that whenever one can factor a line bundle $L$ as $L_1 \otimes L_2$ then one can consider $C$ as embedded in a space of matrices $\text{Hom}(H^0(C, L_1)^\vee, H^0(C, L_2) = H^0(C, L_1) \otimes H^0(C, L_2) \to H^0(C, L)$ and that this gives one equations defining the curve. Roughly, because if we let $t_{ij}$ be the image of $u_i \otimes v_j$ in $H^0(C, L)$, then we get the equality $t_{ij}t_{kl} - t_{ik}t_{jl} = 0$. One may recognize these as equations defining a ‘Segre embedding’. This certainly is not a new idea. It was explained for example in [4]. What is new, is the notion of a Clifford index for any line bundle on a curve and the explanation of how it controls whether the curve and its secant varieties are defined by equations of ‘Segre type’. One can show that for $j < c$, where $c$ is a ’Clifford index’ defined for $L_1 \otimes L_2$, it is the case that the equations defining $\text{Sec}^j(C)$ are the equations of $\text{Sec}^j(\mathbb{P}(H^0(C, L_1)^\vee) \times \mathbb{P}(H^0(C, L_2)^\vee))$ (restricted to the linear subspace $\mathbb{P}(H^0(C, L)^\vee)$. In general terms, the goal of this thesis is to analyze when the equations defining $\text{Sec}^j(C)$ in $\mathbb{P}(H^0(C, L_1 \otimes L_2))$ are the equations defining of $\text{Sec}^j(\mathbb{P}(H^0(C, L_1)^\vee \otimes \mathbb{P}(H^0(C, L_2)))) \in \mathbb{P}(H^0(C, L_1) \otimes H^0(C, L_2))$. It is standard to say that $\text{Sec}^j(C)$ is determinantal defined in this circumstance.  

The arrangement of the thesis is as follows. In the first section we review the definition of the Clifford index and its relationship to the geometry of
a curve $C$. Shiffer variations are defined and then generalized to Shiffer variations supported on a divisor $D$. We then prove a theorem (2.5) which explains how Shiffer variations are related to the Clifford index. The main result of this section is a geometric characterization of the Clifford index. Namely we show that in the bicanonical embedding, Sec$^j(C)$ is in fact set theoretically the locus of all infinitesimal deformations or rank $j + 1$ if and only if $j < c - 1$. Recall that points in Sec$^j(C)$ are the linear combinations of $j + 1$ points of $C$ and hence are the sum of $j + 1$ Shiffer variations. Since the sum of $j + 1$ rank one matrices is of rank at most $j + 1$, Sec$^j(C)$ consists of deformations of rank at most $j + 1$. Scheme theoretic equality which implies that Sec$^j(C)$ is defined by equations of degree $j + 2$ for $j < (\text{Cliff}(C) - 1)$ is true and is proven in section 6. This implies that Sec$^j(C)$ is determinantally defined. The methods used are different and deferring the proof until later allows us to prove a more general result.

Sections 3 and 4 are devoted to generalizing the machinery of the Clifford index and Shiffer variations to arbitrary line bundles. The definitions make sense for arbitrary line bundles, but are only useful as far as we can tell, when the line bundle is very ample. One sign that this machinery is of general use is the fact that it gives a very nice way of describing line bundles in terms of their embedding properties. As another application we give a very short proof in a special case of the existence of $d$ pointed $d - 2$ planes. This result is in ([2]) but the proof contained in there is exceedingly technical. This proof, which perhaps can be generalized to other cases is very elementary. Sections 5 and 6 are the heart of the matter. If $L$ is a very ample quadratically normal line bundle with $h^1(L) = 0$ then one has an injection $\mathbb{P}(H^0(C, L^{\otimes 2})) \rightarrow \mathbb{P}(\text{Sym}^2(H^0(C, L)))$ realizing elements of $\mathbb{P}(H^0(C, L^{\otimes 2}))$ as symmetric matrices. This is true once $\deg(L) \geq 2g + 1$. From our work on Shiffer variations we show that points on the curve represent rank one transformations and hence (since the sum of $j$ rank one matrices is a matrix of rank at most $j$) we get an inclusion of Sec$^{j-1}(C) \rightarrow R^j$ where $R^j$ represents the locus of matrices in $\mathbb{P}(H^0(C, L^{\otimes 2}))$ of rank $\leq j$. In section 5 applying the ideas of sections 3 and 4 we show that these two schemes are the same set theoretically as long as $j < \text{Cliff}(C, L)$ where $\text{Cliff}(C, L)$ is the clifford index of $L$ as defined in section 3. Section 6 is devoted to proving the scheme theoretic equality of Sec$^j(C)$ and $R^j$. Essentially by definition the variety Sec$^j(C)$ is reduced, so the trick is to show that $R^j$ is also reduced. The hard work is to show that in fact $R^j$ is normal (and hence reduced!) which seems to be of independent interest. The technique used is to find a resolution $\tilde{R}^j$ which in fact is a vector bundle over Sym$^j(C)$. This idea is one I learned from ([2]), but in fact seems to be related to the ‘Basic Theorem’
of [27]. Since the varieties $R^j$ are determinantal by definition, it follows that $\text{Sec}^{j-1}(C)$ is determinantal.

Up until this point the results have been on embeddings in $L \otimes L$ and its obvious factorization as $L \otimes L$. In section 7 we take up the question of what happens when $L$ factors as $L_1 \otimes L_2$. All the machinery developed in sections 3 and 4 generalize and one can prove a theorem stating that for $j < \text{Cliff}(C, L_1, L_2)$ the secant varieties are determinantly defined. Rarely is it the case that a divisor is ‘special’ for both $L_1$ and $L_2$ and hence one gets improved bounds on the circumstances under which one can say that the secant varieties are determinantal. In particular for $j = 0$ that is to say for the case of the curve $C$ itself, one recovers the main result of [5] in the case of smooth curves, which is if $\text{deg}(L_i) \geq 2g + 1$ and $L_1 \neq L_2$ if $\text{deg}(L_1) = \text{deg}(L_2) = 2g + 1$ then for $L = L_1 \otimes L_2$, $C$ is determinantly defined in $L$.

In section 8 we discuss the results that occur when $h^1(L) = 1$. First as an application of the ideas of Clifford index we give another proof of the theorem of Green and Lazarsfeld giving a bound in terms of $\text{deg}(L)$ and $\text{Cliff}(C)$ as to when an embedding by a very ample line bundle is quadratically normal. In essence, a very ample line bundle satisfying $\text{Cliff}(C, L) > 0$ is quadratically normal, and one can only realize a line bundle of Clifford index zero as the projection from a linear space of dimension $p - 1$ of a line bundle of Clifford index $p$. A computation finishes the proof. We then take up the question of bounding when $\text{Sec}^j(C)$ is determinantly defined. The same bounds hold as in the case $h^1(L) = 0$ but it is much harder to prove, that for $j > \text{Cliff}(C, L)$ that the two varieties, the rank locus and the secant variety differ. Ironically this means that for special line bundles one is far more likely to have a stronger result than for line bundles with $h^1(L) = 0$.

Finally in the last section, we consider the question of the relationship between these results and Green’s conjecture. The story here is incomplete, as one would like to be able to use these results to prove the conjecture. What we can say is that, the classes we produce in Theorem 2.5 for $D$ a base point free divisor, do give rise to non-trivial Koszul cohomology classes. This applies in particular to divisors which calculate the Clifford index of $C$. Further these classes are ‘decomposable’ (see section 9 for details) and that there are no such ‘decomposable’ Koszul cohomology classes in $K_{p,2}(C, K_C)$ for $c < \text{Cliff}(C)$. This hardly settles the matter though.

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2 The Clifford Index

Let $C$ be a smooth curve of genus $g \geq 3$. The canonical map $\Phi_1 : C \rightarrow \mathbb{P}^{g-1}$ has been studied since the 19th century. Two of the most important theorems with regard to this map are

A:) (Max Noether) $\Phi_1$ is a projectively normal embedding unless $C$ is
hyperelliptic.

B:) (Petri) Assume $\Phi_1$ embeds $C \hookrightarrow \mathbb{P}^{g-1}$, then $C$ is cut out by the
quadrics through $C$ unless $C$ is trigonal or a plane quintic.

We wish to rephrase these theorems in terms of the Clifford index of C. Recall that the Clifford index of $C$, Cliff($C$) is defined as follows:

**Definition 2.1.** Let $L$ be any line bundle.
Cliff($L$) = $\deg(L) - 2(h^0(C, L) - 1)$

**Definition 2.2.** The Clifford index of $C$ written Cliff($C$) is
$\min\{\text{Cliff}(L) | h^0(L) \geq 2 \}$
Then Cliffords theorem may be restated as Cliff(C) ≥ 0 with equality if and only if C is hyperelliptic. Further one has that Cliff(C) = 1 if and only if C is trigonal or a plane quintic (see [2]), that is to say C has a g_3^1 or g_2^3. This leads to the restatement of Thm A as Φ_1 is a projectively normal embedding if and only if Cliff(C) > 0; and Thm B as Φ_1(C) ⊂ P^{g-1} is cut out by quadrics if and only if Cliff(C) > 1.

Mark Green has proposed algebraic generalizations of these statements to arbitrary Clifford index which state that the Clifford index should control the syzygies of the ideal sheaf of C ⊂ P^{g-1}.

In this section we will give a geometric generalization of Thms A and B that characterizes the Clifford index.

To state our theorem we must recall some more notation. Recall that if ξ ∈ H^1(C, T_C) then via cup product we get a map ξ : H^0(C, K_C) → H^1(C, O_C) called the Kodaira-Spencer map. This induces a map

κ : H^1(C, T_C) → Hom(H^0(C, K_C), H^1(C, O_C)).

Recalling that Φ_2 : C → P^{3g-4} = P(H^0(K_C^⊗2)) we get a commutative diagram:

\[\begin{array}{ccc}
P^{g-1} = P(H^0(K_C)) & \xrightarrow{\nu} & P^{g-1} = P(H^0(K_C^⊗2)) \\
C & \xleftarrow{\Phi_1} & \xrightarrow{i} P^{3g-4} = P(H^0(K_C^⊗2)^\vee)
\end{array}\]

Here ν is the Veronese map and i is induced from the map H^0(K_C) ⊗ H^0(K_C) → H^0(K_C^⊗2) which is the Kodaira-Spencer map suitably dualized. In otherwords i(ξ) is the symmetric matrix representing ξ : H^0(K_C) → H^1(O_C).

Identifying Hom^{sym}(H^0(C, K_C), H^1(C, O_C)) with Sym^2(H^0(K_C)^\vee), we see that the maps κ and i are the same map. As long as C is not hyperelliptic, the maps i and κ are injective and we abuse notation and write ξ when we mean κ(ξ) or i(ξ). We have a filtration 0 ⊂ R^1 ⊂ R^2 ⊂ R^g of P^{3g-4} via \[R^k(C) = R^k = \{ξ ∈ H^1(T_C) | \text{rank}(ξ) ≤ k\}.\]

Via Shiffer variations we can
see that $\Phi_2(C) \subset R^1(C)$ (Roughly $p \in C$ kills $H^0(K_C(-p))$ and sends the
1 dimensional quotient to the point $\Phi_1(c) \in \mathbb{P}^{g-1} = \mathbb{P}(H^1(O_C))$ and hence
that the $k$-secant variety $\text{Sec}^k(C) \subset R^k(C)$. We may roughly restate Thm A
as $R^0(C) = \emptyset \iff \text{Cliff}(C) > 0$ and Thm B as $\nu(\mathbb{P}^{g-1}) \cap i(\mathbb{P}^{3g-4}) = i\Phi_2(C) \iff
R^1(C) = \text{Sec}^0(C) = C \iff \text{Cliff}(C) > 1$. As such the generalization we
propose is

**Theorem 2.1** (Geometric Characterization of the Clifford Index).

Let $C$ be a smooth curve then, $\text{Sec}^{j-1}(C) = R^j(C)$ for $j < \text{Cliff}(C)$ and
$\text{Sec}^j(C) \subset R^j(C)$ for $j \geq \text{Cliff}(C)$. This result is than weaker, but connected to, Green’s conjecture. We
will discuss the exact relationship in the last section of this thesis. Notice
in the case $\text{Cliff}(C) = 0$ our theorem merely states that $\text{Cliff}(C) > 0 \iff H^0(K_C) \otimes H^0(K_C) \implies H^0(K_C^{\otimes 2})$ is surjective and in the case $\text{Cliff}(C) = 1$ we get only set theoretic and not scheme theoretic results. The theorem
only asserts a set theoretic equality. Scheme theoretic equality is true, but
will be discussed later as it involves some different ideas.

**Notation**

$C$ will always be a smooth curve of genus at least three defined over an
algebraically closed field. $D = \sum_i^d p_i$ will be a divisor defining a $g^r_d$ (i.e.
h$f^0(C, O_C(D)) = r + 1$). We will almost always use the convention that $\mathbb{P}(V)$
is the space of lines in $V$. That is because we will be dealing with classes
that naturally live in $H^1$ of various line bundles.

The first order of business will be to define Shiffer variations and Shiffer
cohomology classes. Shiffer variations were classically defined analytically
as deformations that only disturbed the complex structure at one point
$p \in C$. If $z$ is a local analytic coordinate near $p$ then $\frac{1}{z} \frac{\partial}{\partial z}$ defines a Cech 1
cocycle of $T_C$ and hence a class in $H^1(T_C)$. Call it $\tau_p$. Notice $\tau_p$ depends
upon the choice of local parameter, but $\tau_p \in \mathbb{P}(H^1(T_C)) = \mathbb{P}(H^0(K_C^{\otimes 2}))$ is
independent of the choice of local parameter. This is the Shiffer variation
associated to $p \in C$. For convenience we recast this definition in an algebraic
mode.

**Definition 2.3.** Recall that $H^0(T_C(p)) = 0 \forall p \in C$ when $g \geq 2$. Hence in
the long exact sequence of cohomology associated to

$$0 \rightarrow T_C \rightarrow T_C(p) \rightarrow T_C(p)|_p \rightarrow 0$$

$\partial(H^0(T_C(p)|_p)) \subset H^1(T_C)$ is a line and hence defines a unique class in
$\mathbb{P}(H^1(T_C(p)))$. This is the Shiffer variation (associated to $p$).
Remark 1: The choice of a local parameter \( z \) at \( p \) amounts to the choice of a generator \( \frac{1}{z} \) in \( H^0(T_C(p)_{|p}) \). When the choice of the local parameter is irrelevant we may speak of the Shiffer variation \( \tau_p \in H^1(T_C) \).

Remark 2: The map \( p \mapsto \tau_p \in \mathbb{P}(H^1(T_C)) = \mathbb{P}(H^0(K_C^\otimes 2)^\vee) \) is the bicanonical embedding.

We will need this same notion regarding classes in \( H^1(O_C) \) so we note the following:

Definition 2.4. Consider the exact sequence

\[
0 \to O_C \to O_C(p) \to O_C(p)_{|p} \to 0.
\]

The class \( \partial(H^0(O_C(p))) \subset \mathbb{P}(H^1(O_C)) \) is the Shiffer cohomology class (associated to \( p \)) and is denoted by \( \sigma_p \) (or \( \sigma \) if the choice of \( p \) is irrelevant).

Remark 3: As before, the assignment \( p \mapsto \sigma_p \in \mathbb{P}(H^0(K_C)^*) \) is the canonical embedding.

We wish to generalize the preceding definitions to reduced divisors. Let \( D = \sum_{i=1}^d p_i \) with the \( p_i \) distinct.

Definition 2.5. \( \langle \sigma_i \rangle \subset H^1(O_C) \) is the vector space spanned by the \( \sigma_{p_i} = \sigma_i \) i.e. pick local parameters \( z_i \) around \( p_i \) and let \( \sigma_i = \partial \left( \frac{1}{z_i} \right) \in H^0(O_C(p)_{|p}) \), then \( \langle \sigma_i \rangle = \left\{ \sum_{i=1}^d a_i \sigma_i \mid a_i \in k \right\} \).

Similarly let \( \tau_i = \partial \left( \frac{1}{z_i} \frac{\partial}{\partial z_i} \right) \in H^0(T_C(p)_{|p}) \) then

Definition 2.6. \( T(D) = \left\{ \sum_{i=1}^d a_i \tau_i \mid a_i \in k \right\} \). \( T(D) \) is the set of Shiffer variations supported on \( D \).

Notice with this notation

\( \langle \sigma_i \rangle = \partial(H^0(O_D(D))) \subset H^1(O_C) \) and \( T(D) = \partial(H^0(T_C(D)_{|D})) \subset H^1(T_C) \)

Unless the degree of \( D \) is large, and in particular if \( \deg(D) \leq 2g - 3 \), then \( H^0(T_C(D)) = 0 \). Hence in this case, \( \dim T(D) = \deg D \).

Recall that any \( \xi \in H^1(T_C) \) induces \( \xi : H^0(K_C) \to H^1(O_C) \). In terms of Shiffer variations we can describe the action of \( T(D) \) as follows:
Lemma 2.2. Pick local coordinates $z_i$ around $p_i$ and let $\tau_i = \frac{1}{z_i} \partial_{z_i} \in H^1(T_C)$, $\sigma_i = \frac{1}{z_i} \in H^1(O_C)$. Suppose $\omega \in H^0(K_C)$ has a local representation $f(z_i)dz_i$ with $f(0) = a_i$. Then, for $\tau = \sum_{i=1}^d \tau_i$

$$\tau(\omega) = \sum_{i=1}^d a_i \sigma_i \in \langle \sigma_i \rangle \subset H^1(O_C)$$

Proof. Both $\sigma_i$ and $\tau_i$ are defined in terms of boundaries (Say $\sigma_i = \partial(\tau_i)$ and $\tau_i = \partial(\tau_i)$) so denoting by $\tau_i = \frac{1}{z_i} \partial_{z_i} \in H^0(T_C(p_i))$ to $H^0(T_C(p_i))$ for any $\omega \in H^0(T_C)$.

$$H^0(T_C(D)) \xrightarrow{\cup \omega} H^0(O_D(D))$$

$$\downarrow \partial \quad \quad \quad \downarrow \partial$$

$$H^1(T_C) \xrightarrow{\cup \omega} H^1(O_C)$$

But $\tau_i \cup \omega = \frac{1}{z_i} \partial_{z_i} \cup f(z_i)dz_i = f(z_i)/z_i = a_i + \text{holomorphic fnc.} f(z_i)/z_i$ in $H^0(O_D(D))$ so denoting by $\tau_i(\omega) = \partial(\tau_i \cup \omega) = \partial(a_i) = a_i \sigma_i$. Since $T_C(D)\mid_D$ and $O_D(D)$ are skyscraper sheaves it follows that $\tau(\omega) = \sum_{i=1}^d a_i \sigma_i(\omega) = \sum_{i=1}^d a_i \sigma_i$ as claimed.

Corollary 2.3. $\text{ker} \ (\tau_i) = H^0(K_C(-p_i))$ and for $\tau \in T(D)$ $\text{ker}(\tau) \subset H^0(K_C(-D))$.

Proof. $\tau_i(\omega) = a_i \sigma_i$ where $\omega = f(z_i)dz_i$ and $f(0) = a_i$. Hence $\tau_i(\omega) = 0 \iff a_i = 0 \iff \omega \in H^0(K_C(-p_i))$. If $\tau = \sum b_i \tau_i$ then

$$\text{ker}(\tau) \supset \bigcap_{i=1}^d \text{ker}(b_i \tau_i) = \bigcap_{i=1}^d \text{ker}(\tau_i) = \bigcap_{i=1}^d H^0(K_C(-p_i)) = H^0(K_C(-D))f.$$
Lemma 2.4. If $\xi = \sum_{i=1}^{k} b_i \tau_i$ where $\tau_i$ are Shiffer variations then $\text{rank}(\xi) \leq k$ since each $\tau_i$ has rank 1; i.e. $\text{Sec}^{k-1}(C) \subset R^k(C)$ where $\text{Sec}^{k-1}(C)$ is the $k$ secant variety to $\Phi_2(C) \subset \mathbb{P}^{3g-4}$. $\text{Sec}^k(C)$ is the closure in $\mathbb{P}^{3g-4}$ of all $k$ planes $\langle \tau_i \rangle_{i=0}^k$ where $\tau_i$ are distinct Shiffer variations. To give a complete description we need to describe what happens to $T(D)$ when $D$ has points of multiplicity greater than one.

Definition 2.8. Suppose $E = kp$ is a divisor. Then we define generalized Shiffer variations as follows: let $\tau^j_p = \partial(\tau^j_p) 1 \leq j \leq k$ where $\tau^j_p = \frac{1}{z^j} \frac{\partial}{\partial z} \in H^0(T_C(kp)_{kp})$. If $D = \sum_{i=1}^{d'} k_ip_i$ then

$$T(D) = \left\{ \sum_{i=1}^{d'} \sum_{j=1}^{k_i} b_{ij}\tau^j_i \mid \tau^j_i \text{ are generalized variations and } b_{ij} \in \mathbb{C} \right\}$$

From the definition it follows that $T(D)$ is the linear space spanned by the divisor $D$. Thus the following is clear.

Claim: $\text{Sec}^j(C) = \bigcup_{D \in \text{Sym}^j(C)} T(D)$

Proof: We just mean the set theoretic statement that if $\varphi_j : \text{Sym}^j(C) \rightarrow G(j - 1, 3g - 4)$ is the map which takes $j$ points $p_1, \ldots, p_j \in \Phi_2(C)$ to the $j - 1$ plane they span in $\mathbb{P}^{3g-4}$ and if $U_j \subset \text{Sym}^j(C) \times \mathbb{P}^{3g-4} = \{(D, x) \mid D = \sum_{i=1}^{j} p_i \text{ and } x \in \langle p_1 \ldots p_j \rangle \}$ then $\pi_2(U_j) = \text{Sec}^j(C)$. Since $T(D)$ is the linear span of the divisor $D$ we are done. In fact we are only interested in the case where $d \leq 2g - 2$ and hence $d$ points always define a $d - 1$ plane.

The purpose of this thesis is to show that the relationship between $\text{Sec}^{j-1}(C)$ and $R^j(C)$ is controlled by the Clifford index. This will follow from an analysis of Shiffer variations and classes. To that end we need to understand for any $\tau \in T(D)$ what the possible kernels and images can be. Since $\text{ker}(\tau) \supset H^0(K_C(-D))$ in any event set $W_D = W = H^0(K_C)/H^0(K_C(-D))$ and $S = \langle \sigma_i \rangle_{i \in D}$ so $\tau : W \rightarrow S$. Note $\dim(W) = g - (g - d + r) = d - r$ since $D$ defines a $g^r_d$. We calculate $\dim(S)$:

Lemma 2.4. i) $\dim(S) = d - r$

ii) $\sum a_i \sigma_i = 0 \iff \exists f \in H^0(O_C(D))$ s.t. in the same local coordinates $z_i$ used to define $\sigma_i$ one has $f(z_i) = a_i/z_i + \text{holomorphic function.}$
Proof. Consider the exact sequence

\[ 0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_C(D) \rightarrow \mathcal{O}_D(D) \rightarrow 0. \]

It gives rise to:

\[ 0 \rightarrow H^0(\mathcal{O}_C) \rightarrow H^0(\mathcal{O}_C(D)) \xrightarrow{\rho} H^0(\mathcal{O}_D(D)) \xrightarrow{\partial} H^1(\mathcal{O}_C) \]

since \( \partial(H^0(\mathcal{O}_D(D))) = \langle \sigma_i \rangle \) so i) follows from \( h^0(\mathcal{O}_D(D)) = d \) and \( \dim(\text{Im}(\rho)) = r \) (\( D \) is a \( g^r_d \)). ii) This is just an explication of i). Choosing local coordinates \( z_i \) around \( p_i \) and identifying \( H^0(\mathcal{O}_{p_i}(p_i)) \) with \( \frac{1}{z_i} \). Then if \( f = \frac{a_i}{z_i} \) holomorphic function near \( p_i \) then \( \rho(f) = \bigoplus_{i \in D} \frac{a_i}{z_i} \) so \( \partial(\rho(f)) = 0 = \sum a_i \sigma_i \).

Remark: If for \( D = \sum_{i=1}^{d'} k_i p_i \) with \( \sum k_i = d \) and we define \( \tilde{\sigma}_i^j \) \( 1 \leq j \leq k_i \) to be the class of \( \frac{1}{z_i} \in \mathcal{O}_{k_i p}(k_i p) \) \( \sigma_i^j = \partial(\tilde{\sigma}_i^j) \) then the lemma extends in a straigt forward manner to non-reduced divisors.

Main Result.

The main theorem of this section is Theorem 2.1 which characterizes the rank filtration in terms of the Clifford index. I recall the statement.

**Main Theorem** (Theorem 2.1)

- \( \text{Sec}^j(C) = R^j(C) \) for \( j < \text{Cliff}(C) \)
- \( \text{Sec}^j(C) \subseteq R^j(C) \) for \( j \geq \text{Cliff}(C) \).

This theorem is a geometric characterization of the Clifford index. Griffiths [11] has commented upon the importance of understanding the rank filtration in terms of doing Hodge theory, but perhaps there is more to be said algebraically.

The main tool used is a theorem that characterizes the possible ranks of a generalized Shiffer variation supported on \( D \) in terms of data about \( D \), in particular the Clifford index \( d - 2r \) of \( D \). A calculation of the dimension of \( \text{Sec}^j(C) \) which allows one to bound the number of Shiffer variations needed to express any element of \( H^1(T_C) \) finishes the proof. The relationship between the rank filtration and the Clifford index is governed by:

**Theorem 2.5.** Let \( \tau = \sum_{p_i \in D} a_i \tau_i \in T(D) \) \( a_i \neq 0 \) be a generalized Shiffer variation. Then:

i) \( d - 2r \leq \text{rank}(\tau) \leq d - r \)
ii) The upper bound is always achieved and the lower bound is achieved if $D$ and $K_C(-D)$ are base point free.

**Remarks:** Notice that if $D$ computes the Clifford index of $C$ then $D$ satisfies ii). Also note the choice of local coordinates $z_i$ is irrelevant as different choices of $z_i$ just scale $\sigma_i$ and $\tau_i$ differently.

**Proof.** Since $\tau : W \to S, \text{rk}(\tau) \leq \dim(S) = d - r$. Let $p_1, \ldots, p_{d-r} \in D$ be such that $h^0(\mathcal{O}_C(p_1, \ldots, p_{d-r})) = 1$. That is to say that the points $p_1, \ldots, p_{d-r}$ are base point free. One can check easily by induction that such points exist. If we take $\tau = \sum_{i=1}^{d-r} \tau_i$ then rank $\tau = d - r$ because if $\omega \in H^0(K_C)$ then $\tau(\omega) = \sum_{i=1}^{d-r} b_i \sigma_i$ where $b_i \in k$ is the value of $\omega$ at $z_i$, that is to say that locally $\omega = f_i(z_i)dz_i$ where $f_i(0) = b_i$. Thus $\tau(\omega) = 0$ if and only if $b_i = 0$ for all $i$, since the $\sigma_i$ were constructed to be linearly independent. Hence $\tau = \sum_{i=1}^{d-r} \tau_i$ achieves the upper bound.

Next we show the lower bound $d - 2r \leq \text{rank}(\tau)$. Recall that the action of a Shiffer variation $\tau$ on one forms can be calculated by considering the action of $\tilde{\tau} \in H^0(T_C(D)|_D)$ representing $\tau$. In fact we have the following commutative diagram.

\[
\begin{array}{ccc}
H^0(\mathcal{O}_C(D)) & \xrightarrow{} & H^0(\mathcal{O}_D(D)) \\
\downarrow & & \downarrow \\
H^0(K_C)/H^0(K_C(-D)) & \xrightarrow{\tau} & H^0(\omega_C|D) & \xrightarrow{\tilde{\tau}} & H^0(\mathcal{O}_D(D)) \\
\downarrow & & \downarrow \partial & & \downarrow \\
H^1(\mathcal{O}_C) & & & & \\
\end{array}
\]

Here $r$ is the restriction map and $\tilde{\tau}$ is such that $\partial(\tilde{\tau}) = \tau$. The key point is if $D = \sum_{j=1}^{d-1} k_j p_j$ and $\tilde{\tau} = \sum_{i=1}^{d-r} \sum_{j=1}^{d-1} b_{ij} \tau_i$ then $\tilde{\tau}$ is an isomorphism if and only if $b_{ik_i} \neq 0 \forall i$; i.e. the highest pole order terms are nonzero. This means that all coefficients are non-zero if $D$ is reduced. To see this it’s clearly enough to check at any point $p_i$ that the map $H^0(K_C|_{k_i p}) \xrightarrow{\tilde{\tau}_i} H^0(\mathcal{O}_{k_i p}(k_i p))$ is an isomorphism $\iff b_{ik_i} \neq 0$. Working in local coordinates since $z_i^j \tau_i^k = \tau_i^{k-j}$ one sees that rank $\tau_i^j = j$ and Im $\tau_i^j = z_i \text{Im}(\tau_i^{j+1})$ so $\sum_{j=1}^{d-1} b_{ij} \tau_i^j$ has rank $m \iff b_{im} \neq 0$ and $b_{ij} = 0$ if $j > m$ and hence rank $\tilde{\tau}_i = k_i \iff b_{ik_i} \neq 0$.  

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Now returning to our diagram, the map $\tau$ is an isomorphism, and $r$ is injective, so $\ker(\tau) = \ker(\partial \circ \tau \circ r) = \ker(\partial_{\text{Im}(\tau r)})$. Since $\dim(\ker \partial) = r \geq \dim(\ker(\partial_{\text{Im}(\tau r)}))$ so $\dim \ker(\tau) \leq r$ and hence $\text{rank}(\tau) = \dim(W) - \dim(\ker(\tau)) \geq (d - r) - r = d - 2r$.

To finish we must show that if $K_C(-D)$ is base point free there exists a $\tau$ with $\text{rank}(\tau) = d - 2r$. Because $D$ moves and is base point free, $(H^0(\mathcal{O}(D)) = 1$ is the case $r = 0$) we may find $D'$ linearly equivalent to $D$ and reduced, $D' = \sum_{i=1}^{d} p_i$ where $p_i \neq p_j$ for $i \neq j$. We may set $D = D'$ since we are only interested in existence. Now since $K_C(-D)$ is base point free there exists $\omega \in H^0(K_C(-D))$ such that $\omega$ has simple zeroes at $p_i$ for all $i$. Pick local coordinates s.t. $z_i$ on $U_i \ni p_i$ with $\omega(z_i) = z_i dz_i$ and set $\sigma_i = \partial \left( \frac{1}{z_i} \right) \in H^1(\mathcal{O}_C)$ and $\tau_i = \partial \left( \frac{1}{z_i \partial z_i} \right) \in H^1(T_C)$. Let $f_1 \ldots f_r$ be a basis for $H^0(\mathcal{O}_C(D))/H^0(\mathcal{O}_C)$. From our choices we have $f_i \omega = \eta_i \in H^0(K_C)$ and $\tau_i(\eta_j) = a^{(j)}_i$ where $f_i|U_i = a^{(j)}_i/z_i + \text{hol.}$ function. Note that $\eta_i$ are linearly independent elements of $H^0(K_C)$ because if $\sum_{j=1}^{r} R_j \eta_j = 0$ for $R_j \in k$ then $\left( \sum_{j=1}^{r} R_j f_j \right) \omega = 0$. But $\omega$ is a holomorphic one form so this can only happen if $\sum_{j=1}^{r} R_j f_j = 0$ which contradicts the linear independence of $f_j$. Now let $\tau = \sum_{i=1}^{d} \tau_i$. $\tau(\eta_j) = \sum_{i=1}^{d} a^{(j)}_i \tau_i = 0$ (because $\tau_i(\eta_j) = a^{(j)}_i$ by Lemma 2.2). Hence we have exhibited $r$ linearly independent elements of $\ker \tau$ i.e. $\text{rank}(\tau) \leq d - 2r$ hence $\text{rank}(\tau) = d - 2r$.

**Remark:** The construction of the second part is related to classical ideas about constructing quadrics containing the canonical curve. In fact general $\omega \in H^0(K_C(-D))$ have simple zeroes on $D$ and if $\omega'$ and $\eta_i'$ are another such choice then $\omega' - \eta_i' \omega \in H^0(K_C)^{\otimes 2}$ is a quadric containing $C$ i.e. $\eta_i \omega' = f_i \omega \cdot \omega' = f_i \omega' \cdot \omega = \eta_i' \omega$ so $\eta_i \omega' - \eta_i' \omega \in \ker(H^1(K_C)^{\otimes 2} \rightarrow H^0(K_C^{\otimes 2}))$.

Clearly this is close to Theorem 2.1. We need to see only that any $\tau \in H^1(T_C)$ can be written as an element of $T(D)$ with $D$ special (or degree $D$ small). This is a consequence of a calculation of the dimension of the secant variety. The fact that the secant variety of a curve is non-degenerate is well-known. We include a proof for completeness. The theorem and the proof have nothing to do with the dimension of the projective space involved. Nonetheless for concreteness we consider only the case of $\mathbb{P}(V) = \mathbb{P}(H^1(T_C))$

**Theorem 2.6.** $\dim(\text{Sec}^j(C)) = \min(2j + 1, 3g - 4)$
Proof. Recall that for \( j \leq 2g - 3 \) and any divisor \( D \) of \( \deg(D) = j \) we have \( H^0(T_C(D)) = 0 \). Hence, for any divisor \( D \in \text{Sym}^j(C) \), \( D \) spans a \( j - 1 \) plane in \( \mathbb{P}^{3g-4} \) which we will denote by \( \langle D \rangle \subset \mathbb{P}^{3g-4} \). Thus the map \( D \mapsto \langle D \rangle \) is a map \( \varphi_j : \text{Sym}^j(C) \to G(j-1, 3g-4) \). Let \( U_j \subset \text{Sym}^j(C) \times \mathbb{P}^{3g-4} = \{(D, p) \mid p \in \langle D \rangle \subset \mathbb{P}^{3g-4}\} \) be the incidence correspondence relation. Notice \( \pi_1 : U_j \to \text{Sym}^j(C) \) is smooth with fibers \( \mathbb{P}^{j-1} \) and \( \pi_2(U_j) = \text{Sec}^j(C) \) so its enough to show \( \pi_2 \) is quasifinite.

To show \( \pi_2 \) is quasifinite it is enough to show:

\( (*) \ \exists U \subset \text{Sym}^j(C) \) open and non-trivial s.t. for \( D \in U \ \exists \) an open subset \( V_D \subset \text{Sym}^j(C) \) such that \( D \in V_D \) and \( \forall E \in V_D \ H^0(T_C(D + E)) = 0 \).

Statement \( (*) \) may be translated as saying for general \( D \in \text{Sym}^j(C) \) and any nearby \( E \subset \text{Sym}^j(C) \langle D \rangle \cap \langle E \rangle = \emptyset \). But by \( (*) \) if \( D_1 \) and \( D_2 \in U \) are distinct divisors then for \( p_i \in \pi_i^{-1}(D_i) \) for \( i = 1, 2 \) then \( \pi_2(p_1) \neq \pi_2(p_2) \). Hence \( \pi_2 \) is quasifinite on the open set \( \pi_1^{-1}(U) \). However \( (*) \) is clear. Since \( j \leq \frac{3g-5}{2} \), \( \deg(T_C(E + D)) \leq g - 3 \) for \( D, E \in \text{Sym}^j(C) \) and the general divisor of degree \( g - 3 \) is not effective. By semi-continuity for general \( D, E \) \( h^0(\mathcal{O}_C(D+E)) = 0 \) once it is true for one special \( D, E \). \( \square \)

Finally we will show how Theorem 2.6 implies Theorem 2.1.

Proof. By Theorem 2.6, if \( (2j + 1) \geq 3g - 4 \) or equivalently that \( j \geq \frac{3g-5}{2} \) then \( \text{Sec}^j(C) = \mathbb{P}(H^1(T_C)) \) and hence every Shiffer variation can be written as the sum of at most \( \frac{3g-5}{2} \) rank one variations. In other words every \( \tau \in H^1(T_C) \) is in a \( T(D) \) with \( \deg(D) \leq \frac{3g-5}{2} \). If \( D \) is eligible to compute \( \text{Cliff}(C) \), then by Theorem 2.5, any \( \tau \in T(D) \) has rank \( \geq \text{Cliff}(C) \). If \( D \) doesn’t move, that is \( h^0(\mathcal{O}_C(D)) = 1 \), then \( \text{rk}(\tau) = \text{deg}(\tau) \) for any \( \tau \in T(D) \) so we may assume that \( D \) moves and hence, since \( D \) is not eligible to compute \( \text{Cliff}(C) \) that \( h^1(\mathcal{O}_C(C(D))) = 1 \). However \( \text{Cliff}(D) = \text{Cliff}(K_C(-D)) = \text{deg}(K_C(-D)) \), since \( h^1(\mathcal{O}_C(D)) = 1 \) implies \( h^0(K_C(-D)) = 1 \). However we have \( \deg(D) \leq \frac{3g-5}{2} \) and hence \( \text{deg}(K_C(-D)) \geq \frac{2g+1}{2} \). But \( K_C(-D) \) doesn’t move, so we have that \( \text{rk}(\tau) = \text{deg}(K_C(-D)) \) for all \( \tau \in T(D) \) that have all non-zero coefficients. But we always have \( \text{Cliff}(C) \geq \frac{2g+1}{2} \). Thus in this case too, if \( \text{Sec}^{j-1}(C) \subsetneq R^j(C) \) then \( j \geq \text{Cliff}(C) \). \( \square \)
3 Geometric Riemann-Roch and the Definition of the Clifford Index of a General Line Bundle

In the previous section we have proved that the Clifford index of a curve can be characterized in terms of the geometry of the curve, specifically the geometry of the bicanonical embedding. The purpose of this section is to generalize these ideas to a much wider class of line bundles. The Clifford index can be given a more geometric interpretation which allows one to make sense of the notion of a Clifford index for any very ample line bundle. In fact the definition makes sense for any line bundle, but it doesn’t seem to be useful unless the line bundle is very ample.

The main result we are aiming to prove is one that compares secant varieties of curves to certain rank loci for non special line bundles of large degree. While secant varieties have nice geometric properties, rank loci have the important property that their equations are (by definition!) determinants of a given degree. For example to say that a curve in some given embedding is a rank one locus is to say that the curve is defined by the vanishing of two by two minors of some matrix. This brings to the forefront the issue of whether the two scheme structures defined by the secant structure and the rank loci structure coincide. We show that this is true in a large range of circumstances. Finally we relate this to earlier work of Eisenbud, Koh, and Stillman on determinantal presentations of curves and their secant varieties.

The key idea is as follows. First use Geometric Riemann-Roch to define a general Clifford index. For a very ample line bundle $L$ interpret $L^\otimes 2$ as giving an embedding of C in a space of matrices. Finally use the Clifford Index to bound from below the rank of a matrix. This allows one to say that up to a given integer $d$, $\text{Sec}^d(C)$ is set-theoretically a determinantal locus. Scheme theoretic equality then comes from a different argument.

Let $C$ be a smooth curve, $L$ a very ample line bundle. Set $P \equiv \mathbb{P}(H^0(C,L)^\vee)$ and let $D$ be a divisor of degree $d$.

**Definition 3.1.** Denote by $\overline{D}$ the span of $D$ in $P$. If $D$ has no multiple points this is clear. In general if $V \subset H^0(C,L)$ is of codimension $m$ the $H^0(C,L) \rightarrow H^0(C,L)/V$ determines a $m-1$ dimensional subspace. $\overline{D}$ is the space corresponding to $V = H^0(L(-D))$.

**Theorem 3.1** (Geometric Riemann-Roch).

(i) $\overline{D} \cong \mathbb{P}^{d-1-r}$ if and only $h^0(L(-D)) = h^0(L) - d + r$; i.e. $D$ imposes $d - r$ conditions on $L$. 

(ii) If \( s \in H^0(\mathcal{O}_C(D)) \) is a section which vanishes on \( D \) (i.e. "1" \( \in H^0(\mathcal{O}_C) \mapsto H^0(\mathcal{O}_C(D))) \) then under the natural multiplication map\( \mathcal{O}_C(D) \otimes L(-D) \to L \) we can identify \( I(D) = \{ \xi \in H^0(L) \mid \xi|_D = 0 \} \) with \( s \otimes H^0(L(-D)) \).

Proof.

(i) This is the definition since \( h^0(L(-D)) = h^0(L) - d + r \) means that codimension of \( H^0(L(-D)) \) in \( H^0(L) \) is \( d - r \).

(ii) We have an exact sequence

\[
0 \to \mathcal{O}_C \xrightarrow{s} \mathcal{O}_C(D) \to \mathcal{O}_C(D)|_D \to 0.
\]

Tensoring this with \( L(-D) \) and taking global sections give us (ii).

\( \square \)

If \( L = K_C \) then the criterion that a divisor \( D \) be eligible to compute \( \text{Cliff}(C) \) is that i) \( h^0(\mathcal{O}_C(D)) \geq 2 \) and ii) \( h^1(\mathcal{O}_C(D)) \geq 2 \). By Serre Duality ii) is equivalent to \( h^0(K_C(-D)) \geq 2 \). The first condition is that the points of \( D \) do not impose independent conditions on \( \mathbb{P} = \mathbb{P}(H^0(C,L)^\vee) \), and the second condition is that the span of \( D \) has codimension at least 2. This second condition is necessary because if we take any hyperplane \( H, H \cap C \) is a divisor of \( \text{deg}(L) \) and \( \text{deg}(L) > h^0(L) \). Therefore we can always find divisors supported on hyperplanes that fail to impose independent conditions. In other words, such divisors aren’t really special.

Thus we may rephrase the definition of the Clifford index as

\[
\text{Cliff}(C) = \min \{ d - 2r \mid D \text{ is a divisor of degree } d \}
\]

where \( D \) is such that \( \overline{D} = \mathbb{P}^{d-1-r} \) with \( r > 0 \) and \( \overline{D} \) is not a hyperplane in \( \mathbb{P}^{g-1} \). This suggests the following definitions for \( L \) an arbitrary very ample line bundle.

**Definition 3.2.** \( r_L(D) = r \geq 0 \) is the integer such that \( \overline{D} = \mathbb{P}^{d-1-r} \).

**Definition 3.3.** \( \text{Cliff}(L, D) = d - 2r_L(D) \).

And finally,

**Definition 3.4.** \( \text{Cliff}(L, C) = \min \{ \text{Cliff}(L, D) \mid r_L(D) > 0 \text{ and the span of } D \text{ is of codimension two or greater} \} \)
**Remark** It is a standard notation that Cliff$(D) = \text{Cliff}(K_C, D)$. This definition is specific for curves. As far as I can tell, the important point is that $D$ consists of points not divisors. We now record the basic properties of $r_L(D)$ and Cliff$(L, D)$.

**Lemma 3.2.**

(i) $r_L(D) = \text{codim} (\text{im} (H^0(L) \to H^0(L, D)))$.

(ii) $r_L(D) = h^0(K_C \otimes L^{-1}(D)) - h^0(K_C \otimes L^{-1})$.

(iii) Cliff$(L, D) = \text{Cliff}(L(-D)) - \text{Cliff}(L)$.

(iv) Cliff$(L, D) = \text{Cliff}(L, L \otimes K^{-1}_C(-D))$.

**Proof.** $r_L(D)$ is defined by $h^0(L(-D)) = h^0(L) - d + r_L(D)$. Using

$$0 \to \mathcal{O}(-D) \to L \to L|_D \to 0,$$

we get a long exact sequence

$$0 \to H^0(L(-D)) \to H^0(L) \to H^0(L|_D) \to H^1(L(-D)) \to H^1(L) \to 0$$

(since $H^1(L|_D) = 0$) and $h^0(L|_D) = d$, we see that dim $\text{im} (H^0(L) \to H^0(L|_D))) = d - r_L(D)$ which is (i).

(ii) From (i) we get $r_L(D) = h^1(L(-D)) - h^1(L)$. Applying Serre duality the result follows.

(iii) $\text{Cliff}(L, D) = d - 2r_L(D) = (d - r_L(D)) - r_L(D)$, which by (i) and (ii) becomes

$$(h^0(L) - h^0(L(-D))) - h^1(L(-D)) = (h^0(L) + h^1(L)) - (h^0(L(-D)) + h^1(L(-D))).$$

(iii) now follows from the well known

**Lemma 3.3.** $h^0(L) + h^1(L) = g + 1 - \text{Cliff}(L)$.

**Proof.** If $h^0(L) = r + 1$, $h^1(L) = h^0(K_C \otimes L^{-1}) = g - \ell + r$, where $\ell = \text{deg}(L)$, so $h^0(L) + h^1(L) = r + 1 + g - \ell + r = g + 1 - (\ell - 2r) = g + 1 - \text{Cliff}(L)$.

(iv) By (iii), Cliff$(L, D) = \text{Cliff}(L(-D)) - \text{Cliff}(L) = \text{Cliff}(K_C \otimes L^{-1}(D)) - \text{Cliff}(L)$, since Cliff$(L) = \text{Cliff}(K_C \otimes L^{-1})$. Now $K_C \otimes L^{-1}(D) = L \otimes (L \otimes K_C(-D))^{-1}$ and the result follows.
We now compute some examples of the Clifford index \( \text{Cliff}(L, C) \) to give a feeling for the ideas involved and to show how it is related to the geometry of the embedding of \( C \) in \( \mathbb{P}(H^0(L)^*) \). A key idea is that if \( h^1(L) = 0 \), then a divisor \( D \) fails to impose independent conditions on the line bundle \( L \) \( \iff \) \( h^1(L(-D)) > 0 \), which by Serre duality means \( h^0(K_C \otimes L^{-1}(D)) > 0 \).

This means that there exists an injection \( O_C \to K_C \otimes L^{-1}(D) \). If \( i \) vanishes on the divisor \( E \) then \( L(-D) \cong K_C(-E) \), that is to say, \( L \cong K_C(D - E) \). We exploit this representation as well as information about how special \( E \) can be for \( K_C \) – which is given to us by Clifford’s Theorem.

We first consider the case when \( \deg(L) = 2g - 2 \). Depending on how “far away” \( L \) is from \( K_C \), its behavior becomes more “very ample”. “Far away” means more general in a sense to be explained below. I am sure that most, if not all, of this material is well known to the experts. I could not find references to these specific statements, so I am including the proofs.

**Theorem 3.4.** Suppose \( \deg(L) = 2g - 2 \) and \( L \neq K_C \)

i. \( L \) is base-point free unless \( L = K_C(p - q), p, q \in C \). \( K_C(p - q) \) has a unique base point unless \( C \) is hyperelliptic.

ii. Suppose \( L \) is base-point free, then \( L \) is very ample unless \( L = K_C(D - E), D, E \in \text{Sym}^2(C) \).

iii. Suppose \( L \) is very ample, then \( C \) is not defined by quadrics if \( L = K_C(D_1 - D_2), \) where \( \deg(D_i) = 3 \) and \( D_i \) is general. That is, \( L \) has a 3-secant line.

**Proof.**  
i. Suppose \( p \in C \) is a base point. Then \( H^0(C, L) = H^0(C, L(-p)) \) and hence \( h^1(L(-p)) = 1 \). As mentioned above, this means \( L(-p) = K_C(-E) \) (with \( E \) effective). It follows that \( \deg(E) = 1 \), so \( L(-p) = K_C(-q) \) and \( L = K_C(p - q) \).

For any \( p, q \in C \), \( K_C(p - q) \) has a base point at \( p \). Furthermore, if \( p_1 \neq p \) is another base point, then \( K_C(p - q - p_1) = K_C(-q_1) \), so \( \mathcal{O}_C \cong \mathcal{O}_C(p - q + q_1 - p_1) \) for some choice of \( p_1 \in C \) which means that there exists a function on \( C \) with two poles, i.e. \( C \) is hyperelliptic.

ii. If \( L \) is not very ample, there is a divisor \( D \) of degree 2, such that \( h^0(L(-D)) > h^0(L) - 2 \). This is just the usual criterion that a divisor is very ample if and only if it separates any two points, including infinitely near points, on \( C \) (see [13] p.152). Since \( L \) is base-point free
if $D = p_1 + p_2$,
\[ h^0(L(-D)) = h^0(L(-p_1)) = h^0(L(-p_2)) = g - 2, \]
hence $h^1(L(-D)) = 1$ and hence $L(-D) = K_C(-E)$ with $\deg(E) = 2$ and $E$ effective, so $L = K_C(D - E)$.

iii. The condition that $L = K_C(D_1 - D_2)$ with $D_i$ general is equivalent to a line which intersects $C$ in 3 points. This is because $h^0(L) = g - 1$ and
\[ h^0(L(-D_1)) = h^0(K_C(-D_2)) = g - 3 \]
(since $D_2$ is general). So denoting $\overline{D}_1$ the span of $D_1$ in $\mathbb{P}(H^0(C, L)^\vee)$ we see that
\[ h^0(O_{\overline{D}_1}(1)) = h^0(L) - h^0(L(-D_1)) = 2, \]
so $D_1$ is a line. Finally, any quadric containing $C$ contains $\overline{D}_1$, since if $Q$ is any such quadric $\#(\overline{D}_1 \cap Q) \geq 3$.

Remarks:

i. The map $C^d \times C^d \to Pic^{2d}(C)$ given by $(E, D) \mapsto E - D$ has 2d dimensional image in $P_{2d}(C)$ ([2]). So for $g \geq 3$, a general line bundle is base-point free, and for $g \geq 5$, a general bundle of degree $2g - 2$ is very ample. This is because the above theorem shows that the space of line bundles with a base point is two dimensional and the space of non ample line bundles is of dimension 4.

ii One can check that for an embedding $C \hookrightarrow \mathbb{P}(H^0(C, L))$, where $\deg(L) = 2g - 2$ and $C$ is general, $C$ is defined by quadrics if and only if there does not exist a 3-pointed line. Once one has defined Shiffer variations for an arbitrary line bundle $L$, then one sees that, just as in the case of the canonical embedding, that the rank one locus, consisting of divisors $D$, such that $\text{Cliff}(L, D) = 1$ corresponds to the intersection of all the quadrics through $C$. Because $\text{Cliff}(L) = 2$ in this case one sees by Lemma 3.2 part iii that a divisor with $\text{Cliff}(L, D)$ corresponds to a divisor $E$ with $\text{Cliff}(E) = 3$. On a general curve the only such divisors are non-special and then $L \cong K_C(D - E)$ where $L$ and $E$ are divisors of degree 3. Then on $L$, $D$ corresponds to a 3 pointed line.
Notice how this corresponds to the case of \( L = K_C \). For the canonical embedding a non-hyperelliptic curve is the intersection of the quadrics containing \( C \) if and only if \( C \) is neither a trigonal or a plane quintic. Trigonal means that there exists a divisor \( D \) of degree 3 with \( h^0(O_C(D)) = 2 \). That is, \( D \) spans a line in \( \mathbb{P}(H^0(C, K_C)^\vee) \).

Before moving on to the the Clifford index of line bundles of degree \( \geq 2g - 2 \), I would like to give one more application of this idea. One says a divisor \( D \) of degree \( d \) defines a \( d \)-pointed \( j \) secant if \( D \) is of degree \( d \) and spans a projective space of dimension \( j \). By Geometric Riemann-Roch, \( D \) spans a \( d - 1 - r \) plane with \( r \geq 0 \). A general set of points has \( r = 0 \). One can ask what is the smallest \( d \) such that there exists a \( d \)-pointed \( d - 1 - r \) plane. [2] gives the formula that there exists a \( d \)-pointed \( d - 1 - r \) plane if: \( d \geq r(h^0(L) - d + r) \) This holds for any very ample \( L \) such that \( h^1(L) = 0 \). The techniques used in [2] are sophisticated.

For \( r = 1 \) and \( \text{deg}(L) = 2g - 2 \), we can prove this very simply. In this case, the result is

**Proposition 3.5.** If \( d \geq h^0(L) - d + 1 \), then there exists a \( d \)-pointed \( d - 2 \) plane.

**Proof.** Rearranging terms, we need to show that if \( 2d \geq h^0(L) + 1 \), then there exists a divisor \( D \) of degree \( d \), such that \( h^1(L(-D)) \geq 0 \). Using the same argument of [2] as before, if \( g \) is even, the map \( \text{Sym}^{g+1}(C) \times \text{Sym}^{g+1}(C) \rightarrow \text{Pic}^g(C) \), or if \( g \) is odd, the map \( \text{Sym}^{g+1}(C) \times \text{Sym}^{g+1}(C) \rightarrow \text{Pic}^g(C) \) \( (D, E) \mapsto D - E - p_0 \) for a fixed \( p_0 \) are surjective. In either case we get that for any divisor \( L_0 \) of degree 0, we may write \( L_0 = O_C(D - E) \), with \( \text{deg}(D) \leq \frac{g+1}{2} \). Since \( L = K_C \otimes L_0 \) for some \( L_0 \), \( L(-D) = K_C(-E) \), and hence for the embedding given by \( L \), \( D \) is a \( d \)-secant \( d - 2 \) plane.

We now consider the case where \( \text{deg}(L) \geq 2g + 1 \). We distinguish between the case when \( L \) contains \( K_C \) as a subsheaf and when \( L \) doesn’t. Of course if \( \text{deg}(L) \geq 3g - 2 \), then the canonical bundle will always be a subsheaf. These results are used in calculating the bounds for which one can say that curves of high degree, and certain of their secant varieties, are determinantal defined.

**Theorem 3.6.** If \( L = K_C(D) \) with \( D \) effective of degree \( d \geq 2 \), then \( \text{Cliff}(L, C) = d - 2 \). Unless \( C \) is hyperelliptic, \( D \) uniquely achieves this bound.
Proof. $h^0(L) = g - 1 + d$ and $h^0(L(-D)) = h^0(K_C) = g$, so $D$ spans a
$d$ secant $d - 2$ plane, that is, $r_L(D) = 1$ and $\text{Cliff}(L, D) = d - 2$. If $E$
satisfies $r_L(E) > 0$ then $h^1(L(-E)) > 0$, i.e. $L(-E) = K_C(-E_1)$. Let $e = \deg(E)$, $e_1 = \deg(E_1)$, and set $h^0(\mathcal{O}_C(E_1)) = r_1 + 1$. Then $h^1(L(-E)) = h^1(K_C(-E_1)) = g - e + r_1 = g + d - e + r_1 = g + d - 1 - e + (r_1 + 1)$, and hence $r_L(E) = r_1 + 1$. Hence we get $\text{Cliff}(L, E) = \deg(E) - 2r_L(E)$. $d + e_1 - 2(r_1 + 1) = d - 2 + e_1 - 2r_1 = d - 2 + \text{Cliff}(E_1)$. By Clifford’s theorem: $\text{Cliff}(E_1) \geq 0$, with equality $\iff E_1 = 0$, $K_C$, or $C$ is hyperelliptic and $E_1 = ng_2^1$ a multiple of the $g_2^1$. $E_1 = 0$ means $E = D$; $E_1 = K_C$ means $E = L$. Clearly $E = D + ng_2^1$ will give a divisor with $\text{Cliff}(L, E) = d - 2$.

**Theorem 3.7.** Suppose $\deg(L) = 2g - 2 + d$, $L = K_C(D)$, $\deg(D) = d > 1$
and $h^0(\mathcal{O}_C(D)) = 0$, then $\text{Cliff}(C, D) \geq d - 1$ unless $C$ is hyperelliptic, in
which case $\text{Cliff}(C, L) = d - 2$ is achieved.

**Proof.** If $r_L(E) > 0$, then $H^1(L(-E)) > 0$, so $L(-E) = K_C(-E_1)$, with $E_1$
effective. Again let $\deg(E) = e$, $\deg(E_1) = e_1$, and suppose $h^0(\mathcal{O}_C(E_1)) = r_1 + 1$, then $r_L(E) = r_1 + 1$ and $\text{Cliff}(L, E) = e - 2r_L(E) = d + e_1 - 2(r_1 + 1) = d - 2 + (e_1 - 2r_1)$). Again by Clifford’s theorem, $e - 2r_1 > 0$ unless $C$ is
hyperelliptic. If $C$ is hyperelliptic, then taking $L = K_C(D - E)$ with $D$
general and $E$ a multiple of the $g_2^1$ will produce $L$ with $\text{Cliff}(C, L) = d - 2$.

**Corollary 3.8.** If $L = K_C(D)$, $\deg(D) = d > 0$, then $\text{Cliff}(L, C) \geq d - 2$.

**4 Generalized Shiffer Variations**

The goal of this section is to generalize Shiffer variations to an arbitrary very
ample line bundle. We wish to prove the same sort of theorem for arbitrary
line bundles as we have proven for $K_C$. That is we wish to define Shiffer
variations and relate their ranks to $\text{Cliff}(C, L)$. In this section we will show
how Shiffer variations may be defined in general and discuss the geometric
information that is necessary to relate a curve and its secant varieties to
rank loci.

As with the case $L = K_C$ it is convenient to use the geometric version
of projective spaces of lines in the dual vector space. That is to say, we
consider the space of lines in $H^0(C, L)^\vee = H^1(C, K_C \otimes L^{-1})$, rather than
hyperplanes in $H^0(C, L)$.
Consider the exact sequence

\[ 0 \to K_C \otimes L^{-1} \to K_C \otimes L^{-1}(D) \to K_C \otimes L^{-1}(D)|_D \to 0. \]

This gives rise to a boundary map:

\[ \partial_D : H^0 \left( K_C \otimes L^{-1}(D)|_D \right) \to H^1 \left( K_C \otimes L^{-1} \right) \]

**Theorem 4.1.** \( \partial_D (H^0 \left( K_C \otimes L^{-1}(D)|_D \right)) \subset H^1 \left( K_C \otimes L^{-1} \right) \) corresponds to the linear subspace \( \overline{D} \subset \mathbb{P} \left( H^0 (L) \right) \).

**Proof.** \( \overline{D} \) is defined by

\[
\overline{D} = \{ x \vee \in \mathbb{P} \left( H^0 (L) \right) \mid x \vee |_{H^0 (L(-D))} = 0 \}
\]

\[
= \{ x \vee \in H^1 \left( K_C \otimes L^{-1} \right) \mid x \vee |_{H^0 (L(-D))} = 0 \}
\]

\[
= \{ x \vee \in H^1 \left( K_C \otimes L^{-1} \right) \mid x \to 0 \text{ in } H^0 (L(-D)) = H^1 \left( K_C \otimes L^{-1} \right) \}
\]

\[
= \{ x \in H^1 \left( K_C \otimes L^{-1} \right) \mid x = \partial_D(y) \text{ in some } y \in H^0 \left( K_C \otimes L^{-1}(D)|_D \right) \}
\]

where the last statement follows by the exactness of the long exact sequence.

We now turn our attention to defining Shiffer variations in general. Just as elements of \( H^1 (T_C) \) can be considered as elements of \( \text{Hom} \left( H^0 (C, K_C), H^1 (C, \mathcal{O}_C) \right) \)

(via the cup product), we can consider elements of \( H^1 \left( C, K_C \otimes L^{-2} \right) = H^0 \left( C, L^{-2} \right) \) as elements of \( \text{Hom} \left( H^0 (C, L), H^1 (K_C \otimes L^{-1}) \right) \). To make sense of the geometry we must assume that the map

\[ A : H^1 \left( K_C \otimes L^{-2} \right) \to \text{Hom} \left( H^0 (L), H^1 (K_C \otimes L^{-1}) \right) \]

is injective. Notice:

**Fact:** \( A \) is injective \( \iff \) \( H^0 (C, L) \) is quadratically normal, that is \( H^0 (L) \otimes H^0 (L) \to H^0 (L^\otimes 2) \) is surjective.

**Proof.** \( \text{Hom} \left( H^0 (L), H^1 (K_C \otimes L^{-1}) \right) = H^0 (L)^\vee \otimes H^0 (L)^\vee \) by Serre duality. The natural map \( H^1 \left( K_C \otimes L^{-2} \right) \to H^0 (L)^\vee \otimes H^0 (L)^\vee \) is injective \( \iff \) the dual map \( H^0 (L) \otimes H^0 (L) \to H^1 \left( K_C \otimes L^{-2} \right)^\vee = H^0 (L^\otimes 2) \) is surjective.

\( \square \)
Consider the exact sequence
\[0 \rightarrow K_C \otimes \mathcal{L}^{-2} \rightarrow K_C \otimes \mathcal{L}^{-2}(D) \rightarrow K_C \otimes \mathcal{L}^{-2}(D)|_D \rightarrow 0.\]
Let \( T_L(D) = \partial(H^0(K_C \otimes \mathcal{L}^{-2}(D)|_D)) \subset H^1(K_C \otimes \mathcal{L}^{-2}). \)

**Definition 4.1.** \( T_L(D) \) are the Shiffer variations (for \( L \)) supported on \( D \).

**Remark:** By Theorem 4.1 the vector space \( T_L(D) \) corresponds to the linear space spanned by \( D \) in \( \mathbb{P}(\mathcal{H}^1(K_C \otimes \mathcal{L}^{-2})) \). By an abuse of notation we will frequently make no distinction between the affine elements of \( T_L(D) \subset H^0(K_C \otimes \mathcal{L}^{-2}) \) and the projective elements \( T_L(D) = \overline{D} \subset \mathbb{P}(\mathcal{H}^1(K_C \otimes \mathcal{L}^{-2})) \). If \( p \in D \), and \( z \) is a local parameter, and \( \ell_z \) a local generator of \( L \) at \( p \), then \( \frac{\partial}{\partial z} \otimes \ell_z^2 \) is an element of \( T_L(D) \) supported at \( p \). Any other representative differs from this one by a scalar.

Our calculations relate to rank and are independent of the choice of local parameter. We will generally proceed by choosing a local parameter and making our calculation “locally”. We need one more piece of notation.

**Definition 4.2.** \( S_L(D) \subset H^1(K_C \otimes L) \) is the affine cone over the span of \( D \) in \( \mathbb{P}(\mathcal{H}^1(K_C \otimes L)) \).

The next theorem shows how to compute Shiffer deformations. Just as in the case of \( L = K_C \) they can be computed locally, and hence the exact same proof applies.

**Theorem 4.2.** Let \( \xi \in T_L(D) \subset H^1(K_C \otimes \mathcal{L}^{-2}) \). Denote by \( \rho \) the natural restriction \( H^0(C, L) \xrightarrow{\rho} H^0(C, L|_D) \). Denote by
\[ \partial_1 : H^0(K_C \otimes \mathcal{L}^{-1}(D)|_D) \rightarrow H^1(K_C \otimes \mathcal{L}^{-1}) \]
the boundary map in the long exact sequence of cohomology coming from the exact sequence
\[0 \rightarrow K_C \otimes \mathcal{L}^{-1} \rightarrow K_C \otimes \mathcal{L}^{-1}(D) \rightarrow K_C \otimes \mathcal{L}^{-1}(D)|_D \rightarrow 0,\]
and denote by \( \partial_2 : H^0(K_C \otimes \mathcal{L}^{-2}(D)|_D) \rightarrow H^1(K_C \otimes \mathcal{L}^{-2}) \) the boundary map in the long exact sequence of cohomology coming from the short exact sequence
\[0 \rightarrow K_C \otimes \mathcal{L}^{-2} \rightarrow K_C \otimes \mathcal{L}^{-2}(D) \rightarrow K_C \otimes \mathcal{L}^{-2}(D)|_D \rightarrow 0.\]

Let \( \tilde{\xi} \in H^0(K_C \otimes \mathcal{L}^{-2}(D)|_D) \) be a lifting of \( \xi \), i.e. \( \xi = \partial_2(\tilde{\xi}) \). Then the following diagram factors the map \( \cup \xi : H^0(C, L) \rightarrow H^1(C, K_C \otimes \mathcal{L}^{-1}) \) as
\[ H^0(C, L) \xrightarrow{\rho} H^0(L|_D) \xrightarrow{\cup \tilde{\xi}} H^0(K_C \otimes \mathcal{L}^{-1}(D)|_D) \xrightarrow{\partial_1} H^1(K_C \otimes \mathcal{L}^{-1}) \]
Proof. We can use Cech cohomology to compute all the maps. Let $V_1$ be an open set such that $D \subset V_1$, and $V_2 = C - D$. We use this open cover. Write $D = \sum_{i=1}^n n_i p_i$.

A class $\xi \in T_L(D)$ is exactly $\partial_2 (\tilde{\xi})$, $\tilde{\xi} \in H^0 (K_C \otimes L^{-2}(D)|_D)$. Letting $s_i$ be a local section of $K_C \otimes L^{-2}$ around $p_i$.

\begin{equation*}
(1) \quad \tilde{\xi} = \sum_{i=1}^n s_i \left( \sum_{j=1}^{n_i} a_{ij}^j z_i^j \right) \text{ where } z_i \text{ is a local parameter at } p_i.
\end{equation*}

Lifting $\tilde{\xi}$ to a section of $\Gamma(V_1 \cap V_2, K_C \otimes L^{-2})$ gives a representation of $\xi \in C^1(C, K_C \otimes L^{-2})$. With this description it is clear that $H^0 (L) \cup \xi \hookrightarrow H^1 (K_C \otimes L^{-1})$ factors as

\begin{equation*}
H^0 (L) \cup \xi \rightarrow H^0 (K_C \otimes L^{-1}(D)|_D) \rightarrow H^1 (K_C \otimes L^{-1}).
\end{equation*}

From this and 4.1 we get:

**Corollary 4.3.** Let $\xi \in T_L(D)$. Then $\ker(\xi) \supset H^0 (L(-D))$ and $\text{im}(\xi) \subset S_L(D)$, the affine cone over $\overline{D}$.

When $L = K_C$, Theorem 2.1 characterized the Clifford index as the smallest integer $j$ for which $\text{Sec}^j(C)$ is not the full rank $j$ locus. Here is the set-up for general $L$. We assume for now only that $L$ is very ample and quadratically normal.

\begin{equation*}
\begin{array}{c}
\mathbb{P} (H^1 (K_C \otimes L^{-1})) \\
\mathbb{P} (H^1 (K_C \otimes L^{-2}))
\end{array}
\end{equation*}

\begin{equation*}
\begin{array}{c}
\phi_L \\
\phi_{2L}
\end{array}
\end{equation*}

\begin{equation*}
\begin{array}{c}
\mathbb{P} (\text{Sym}^2 (H^1 (K_C \otimes L^{-1}))) \\
\\hat{v}
\end{array}
\end{equation*}

\begin{equation*}
\begin{array}{c}
v \\
i
\end{array}
\end{equation*}

$\phi_L$ and $\phi_{2L}$ are the maps determined by the linear system $|L|$ and $|2L|$ and $i$ is the inclusion $\mathbb{P} (H^1 (K_C \otimes L^{-2}))$ as a linear subspace, which is true since $L$ is quadratically normal. Finally, $v$ is the Veronese map. $v$ is usually defined as a map to $V \otimes V$, but on the algebra level it is $x \in V \mapsto x \otimes x \in$
$V \otimes V$ which is clearly in the $\text{Sym}^2(V)$. In other words, $\nu$ embeds $\mathbb{P}(V)$ as the rank one symmetric matrices in $\mathbb{P}(V \otimes V)$ Our general set-up is summarized by

**Theorem 4.4.** Let $V$ be any vector space, $M$ a linear subspace of $\text{Sym}^2(V)$ and $X$ a subvariety of $\mathbb{P}(V)$ such that $X_2 = \phi_{2L}(X) \subset \mathbb{P}(M)$.

(i) Viewing $\text{Sym}^2(V)$ as a linear subspace of $\text{Hom}(V^*, V)$, then $v(\mathbb{P}(V))$ is exactly the rank 1 matrices in $\mathbb{P}(\text{Sym}^2(V))$. $\text{Sec}^k(v(\mathbb{P}(V)))$, which is the closure of the variety swept out by linear spans of $(k+1)$ points of $\text{Sec}^0(\mathbb{P}(V)) = v(\mathbb{P}(V))$, is exactly the rank $(k+1)$ locus.

(ii) $\text{Sec}^k(X_2) \subset \mathbb{P}(M) \cap \text{Sec}^k(v(\mathbb{P}(V)))$; in short, the variety of rank $(k+1)$ matrices in $\mathbb{P}(M)$ contains the $k$ secants to $X_2$.

**Proof.** (i) This is classical- elements of $\text{Sym}^2(V)$ may be viewed as quadrics. The only invariant of a quadric is its rank (see for example [12] p.792). This means given a quadric $Q$ of rank $k+1$ there is a basis $\langle q_1, \ldots, q_n \rangle$ of $V$ such that in these coordinates $Q = \sum_{i=1}^{k+1} a_i q_i^2$. This shows that $Q$ is in $\text{Sec}^k(v(\mathbb{P}(V)))$ as each $q_i^2 = v(q_i)$ is in $v(\mathbb{P}(V))$.

(ii) is clear because any element of $v(X)$ is of rank 1. As many people have observed the sum of $k$ rank 1 matrices is of rank $\leq k$.

By definition $\text{Sec}^0(X) = X$. So that if points of $X$ correspond to rank one matrices, then elements of $\text{Sec}^k(X)$ correspond to sums of $k+1$ elements and hence to matrices of rank $k+1$ or less. Because $\text{Sec}^k(v(\mathbb{P}(V)))$ is exactly rank $k+1$ locus in $\mathbb{P}(\text{Sym}^2(V))$ means $\text{Sec}^k(v(\mathbb{P}(V))) \cap \mathbb{P}(M)$ is the rank $k+1$ locus in $M$, and hence the equality $\text{Sec}^k(v(\mathbb{P}(V)) \cap M = \text{Sec}^k(X_2)$ means that every element of $M$ of rank $k+1$ (or less) can be written as the sum of $k+1$ (or less) elements of $v(X)$. In short, we have for $v(X)$ that $\text{Sec}^k(v(X))$, considered as a subvariety of $\mathbb{P}(\text{Hom}(V^*, V))$, is contained in the rank $k+1$ locus, and we wish to know when it is exactly the rank $k+1$ locus. This rank $k+1$ locus is harder to understand geometrically than the secant variety but is determinantal-ly defined.

We are interested in when one has equality. We want to know when is $\text{Sec}^j(X_2) = \text{Sec}^j(v(\mathbb{P}(V))) \cap \mathbb{P}(M)$. We are only discussing the issue of set theoretic equality. We use this space to record a criteria of Hassett and a generalization.

**Theorem 4.5.** Let $X \subset \mathbb{P}(V)$. Assume $X$ is projective. Let $v_d : \mathbb{P}(V) \rightarrow \mathbb{P}(\text{Sym}^d(V))$. Assume $L = \mathcal{O}_X(1)$ is linearly and d normal, i.e. $H^0(X, \mathcal{O}_X(1)) = \text{dim} V = d$.
\[ V \text{ and } H^0(X, L^{\otimes d}) \rightarrow H^0(X, L^{\otimes d}) = M \text{ is surjective. Then we get a commutative diagram} \]

\[
\begin{array}{ccc}
X \subset & \mathbb{P}(V) \\
\varphi_L \downarrow & & \downarrow \psi_d \\
\mathbb{P}(M) & \rightarrow & \mathbb{P}(\text{Sym}^d(V))
\end{array}
\]

Let \( X_d \) be the image of \( X \) under \( \varphi_{L^{\otimes d}} \) and let \( R^j = \text{Sec}^{j-1}(v_d(\mathbb{P}(V))) \cap \mathbb{P}(M) \) \((j \geq 1)\). Suppose that \( m < d \) and that \( R^j = \text{Sec}^{j-1}(v_d(\mathbb{P}(V))) \cap \mathbb{P}(M) \) \( \forall j < m \). Then \( \text{Sec}^{m-1}(X_d) \subseteq R^m \iff \exists \) an \( m - 1 \) dimensional subspace \( \Lambda \subset \mathbb{P}(V) \) such that

(i) \( X \cap \Lambda = \emptyset \), and

(ii) the map \( H^0(\mathbb{P}(V), I_X(d)) \rightarrow H^0(\Lambda, \mathcal{O}_\Lambda(d)) \) is not surjective.

**Proof.** This is a set theoretic statement. First notice \( H^0(V, I_X(d)) \subset H^0(V, \mathcal{O}_V(d)) \) corresponds to the hyperplanes \( H \) on \( \mathbb{P}(\text{Sym}^d(V)) \) such that \( H \supset \mathbb{P}(M) \), i.e. \( \mathbb{P}(M) = \bigcap V(H), H \in H^0(I_X(d)) \). If \( m = 0 \) the theorem states that \( X \) is defined by degree \( d \) homogeneous polynomials, that is to say \( X_d = \text{Sec}^{0}(X_d) = \mathbb{P}(M) \cap v_d(\mathbb{P}(V)) \) if and only if \( \forall p \notin X \exists \varphi \in H^0(V, I_X(d)) \) such that \( \varphi(p) \neq 0 \). This is clear.

The proof in the general case is exactly the same as for when \( d = 2 \) and \( X \) is a curve, the case Hassett did. We include it for completeness.

Firstly suppose \( \exists p \in \text{Sec}^m(v_d(V)) \cap \mathbb{P}(M) \setminus \text{Sec}^m(X_d) \). Since \( p \in \mathbb{P}(M) \), \( p \in V(H), \forall H \in I_X(d) \). Further since \( p \in \text{Sec}^m \), \( \exists p_1, \ldots, p_m \in v(\mathbb{P}(V)) \) such that \( p = \sum_{i=1}^m \lambda_i p_i \), i.e. each \( p_i = x_i^d \) so \( p = \sum_{i=1}^m \lambda_i x_i^d \).

If we let \( \Lambda = \langle p_1, \ldots, p_m \rangle \), then \( H^0(V, I_X(d)) \) vanished at \( p \), i.e.

\[ H^0(V, I_X(d)) \twoheadrightarrow H^0(\Lambda, \mathcal{O}_\Lambda(d)). \]

If \( \Lambda \cap X \neq \emptyset \), say some \( y \in \Lambda \cap X \), then writing \( \sum_{i=1}^m \lambda_i x_i^d = Q + \mu y^d \) with rank \( Q \leq m - 1 \), say \( Q = \sum_{i=1}^{m-1} \tilde{\lambda}_i \tilde{y}_i \), we would have \( \langle y_1, \ldots, y_{m-1} \rangle \in \text{Sec}^{m-1}(\mathbb{P}(V)) \cap \mathbb{P}(M) / \text{Sec}^{m-1}(X_d), \) contradicting our hypothesis.

Conversely, given \( \Lambda \) such that \( Q \in H^0(\Lambda, \mathcal{O}_\Lambda(d)) \) not in the image of \( H^0(\mathbb{P}(V), I_X(d)) \), such a \( Q \) can be written as \( \sum_{i=1}^m \lambda_i x_i^d \), then \( \sum_{i=1}^d \lambda_i v_d(x_i) \) is a point in \( \text{Sec}^m(v_d(V)) \cap \mathbb{P}(M) \). Clearly it is not in \( \text{Sec}^m(X_d) \) since \( \Lambda \cap X = \emptyset \) by hypothesis.

\[ \square \]
We now return to Shiffer variations. For \( L = K_C \), we used the Clifford index to put lower bounds on the rank of elements of \( T_L(D) \). Namely we had \( \text{rk}(\tau) \geq d - 2r \) if all the coefficients were non-zero. That translated to, after eliminating the possibilities of using elements of \( T_L(D) \) where \( D \) is not eligible to compute \( \text{Cliff}(C) \), the fact that the \( j \) secant varieties of \( C \) in \( \mathbb{P}\left(H^0\left(C, K_C^{\otimes n}\right)\right) \) are exactly the rank \( j \) varieties. The first part goes through unchanged.

**Theorem 4.6.** Let \( \tau \in T_L(D) \), \( D = \sum_{i=1}^n n_i p_i \), and let \( z_i \) be a local parameter at \( p_i \) and let \( l_i \) be a local generator for \( L \) at \( z_i \). Suppose that \( \tau \) corresponds to \( \tilde{\xi} \in \left(\sum_{i=1}^n \sum_{j=1}^{n_i} \beta_{ij} z_i^{-j}\right) \otimes l_i \), with all \( \beta_{ij} \neq 0 \), \( \forall i, j \), or at least \( \forall i, \beta_{in_i} \neq 0 \). Then \( d - 2r \leq \text{rk}(\tau) \leq d - r \), where \( r = \text{Cliff}(L,D) \).

**Proof.** The condition on the coefficients of the Shiffer variations being non-zero is exactly the condition that the Shiffer variation is supported on \( D \), but not on some subdivisor of \( D \). By Corollary 4.3 \( \text{im}(\tau) \subset |D| \cong \mathbb{P}^{d-r-1} \) and hence \( \text{rk}(\tau) \leq d - r \). Recall that by Theorem 4.2 we have the factorization

\[
H^0(L) \longrightarrow H^0(L|_D) \xrightarrow{\cup \tilde{\xi}} H^0(K_C \otimes L^{-1}|_D) \longrightarrow H^1(K_C \otimes L^{-1}).
\]

Assume temporarily that \( \cup \tilde{\xi} \) is an isomorphism of \( V = H^0(L|_D) \) with \( H^0(K_C \otimes L^{-1}|_D) = V \vee \), then \( \tilde{\xi} \) restricted to \( \text{im}(H^0(L) \longrightarrow H^0(L|_D)) \) maps to \( H^0(L)^\vee = H^1(K_C \otimes L^{-1}) \) and the result follows from the linear algebra fact:

**Lemma 4.7.** Let \( \varphi : V \longrightarrow V^\vee \) be a linear map which is an isomorphism. Let \( W \subset V \) be of codimension \( r \), then \( \varphi|_W : W \longrightarrow W^\vee \) is of rank \( g \geq d - 2r \).

**Proof.** Because \( \varphi \) is an isomorphism, \( \varphi_1 : W \longrightarrow V^\vee \) has rank \( d - r \) and since \( \pi : V^\vee \longrightarrow W^\vee \) has an \( r \) dimensional kernel, \( \varphi|_W = \pi \circ \varphi_1 \) has rank at least \( (d - r) - r = d - 2r \).

\[\Box\]

Hence we are reduced to showing \( \tilde{\xi} \) is an isomorphism. Since all the sheaves are skyscraper sheaves the question is local and we may assume that \( D = np \). Our diagram now looks like

\[
H^0(C, L|_{np}) \xrightarrow{\cup \tilde{\xi}} H^0(K_C \otimes L^{-1}(np)|_{np}).
\]

Since we are local over \( p \) we can identify \( H^0(C, L|_{np}) \) with \( k[z]/z^n \), where \( k \) is a field of definition for \( C \) and we can identify \( H^0(K_C \otimes L^{-1}(np)|_{np}) \)
with $\bigoplus_{j=0}^{n-1} z^{-j}k$. Under this identification $\xi = \sum_{j=0}^{n-1} \beta_j z^j$ corresponds to the matrix
\[
\begin{pmatrix}
\beta_{n-1} & \beta_{n-2} & \cdots & \beta_0 \\
\beta_{n-1} & \cdots & \beta_1 \\
\vdots & & \ddots & \\
\beta_{n-1} & & & \beta_1
\end{pmatrix}
\]
which has determinant $(\beta_{n-1})^n$.

As mentioned in Theorem 4.6 the condition $\beta_{i,n} \neq 0$ is exactly the condition that $\tau \in T_L(D)$ is not an element of $T_L(D')$ for some $D' \subset D$. That is to say that $\tau \in \text{Sec}^{d-1}(C)/\text{Sec}^{d-2}(C)$. Obviously elements of $T_L(D)$ can have low rank by lying in a low dimensional secant variety. For example $\text{rk}(\tau_p) = 1$ for any $p \in C$. Since we are only interested in the ranks of generic elements of $T_L(D)$ we make the following definition.

**Definition 4.3.** $T^*_L(D) = \{ \tau \in T_L(D) | \beta_{i,n} \neq 0 \forall i \}$ That is $T^*_L(D) = T_L(D) \cap (\text{Sec}^{d-1}(C)/\text{Sec}^{d-2}(C))$.

When $L = K_C$ one checked, using Shiffer variations, that one had equality $\text{Sec}^{j-1}(C) = R^j$ for $j < \text{Cliff}(C)$. The same sort of result is true in general, but the results have different flavors depending on whether $h^1(L) > 0$ or $h^1(L) = 0$. Roughly speaking for $h^1(L) > 2$, I cannot say anything general. For $h^1(L) = 1$ a version of the theorem is true, but weaker, as the lower bound does not generally occur. If $h^1(L) = 0$ and $\text{deg}(L) \geq 2g - 2$, one can get strong results with the strongest results being for $\text{deg}(L) \geq 2g + 1$. We first consider the case $h^1(L) = 0$ and $\text{deg}(L) > 2g + 1$.

## 5 Geometric Characterization of the Clifford Index for Line Bundles of Large Degree

Throughout this section $L$ will be a very ample line bundle of degree $\geq 2g + 1$. We restrict to this case because in this range $L$ is always a very ample, quadratically normal line bundle. Recall that in section 3 we have calculated the Clifford index of any line bundle of degree $\geq 2g + 1$. We recall this and explain its relationship with secant varieties now.

**Theorem 5.1.** Suppose $\text{deg}(L) = 2g - 2 + d$ with $d \geq 3$. Then $\text{Cliff}(C, L) \geq d - 2$. Let $C_2 = \phi_{2L}(C) \subset \mathbb{P}(H^0(C, L^{\otimes 2}))$, then for $j < \text{Cliff}(C, L)$, $\text{Sec}^{j-1}(C) = R^j$.

For $c = \text{Cliff}(C, L)$ and generic $L$, $R^c \not\subseteq \text{Sec}^{c-1}$. 

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By Theorem 3.6 we know that if $L = K_C(D)$ with $D$ effective, then $\text{Cliff}(C,L) = d - 2$ and by Theorem 3.7 generically $\text{Cliff}(C,L) = d - 1$. In both cases by Lemma 5.4 we can find special Shiffer variations of rank equal to $\text{Cliff}(C,L)$. We are only claiming set theoretic equality (and inequality) at the moment. We will break the proof down into two pieces, the equality and the inequality.

Recall our basic setup: $L$ is very ample.

That this diagram commutes follows if $L$ is very ample and quadratically normal.

Recall that via the inclusion

we can endow $\mathbb{P}(H^1(K_C \otimes L - 2))$ with two filtrations $\text{Sec}^{j-1}(C) = \cup$ (span of $j$ points of $C$) and $R^j$, which is the set of all elements of $\mathbb{P}(H^1(K_C \otimes L - 2))$ which are of rank $\leq j$ when viewed as elements of $\mathbb{P}(\text{Sym}^2(H^1(K_C \otimes L - 1))) \hookrightarrow \mathbb{P}(\text{Hom}(H^0(L), H^0(L)^\vee))$. Since every $p \in C$ corresponds to rank one elements, namely the rank one Shiffer deformation $\sigma_p^2$, and since the sum of $j$ rank one elements is of rank $\leq j$, we have

$$\text{Sec}^j(C) \subset R^j(C) \subset \mathbb{P}(H^1(K_C \otimes L - 2)).$$

**Theorem 5.2.** Suppose that $j < \text{Cliff}(C,L) = c$. Then as sets $\text{Sec}^{j-1}(C) = R^j(C)$. Further if $L$ is generic then $\text{Sec}^{c-1}(C) \nleq R^c(C)$.

**Proof.** Let $E$ be any divisor on $C$. Using the factorization for the map $\xi \in T_L(E)$ given by theorem 4.2 we can factor $\xi \in T_L(E)$ as,

$$H^0(L)/H^0(L(-E)) \hookrightarrow H^0(L|_E) \xrightarrow{\cup} H^0(K_C \otimes L^{-1}(E)|_E) \xrightarrow{\partial_E} H^1(K_C \otimes L^{-1}).$$

If $h^1(L(-E)) = h^0(K_C \otimes L^{-1}(E)) = 0$, we see $\partial_E$ is injective, and hence if $\xi \in T_L(E)$ is represented by $\xi = \partial(\tilde{\xi})$, with $\tilde{\xi} \in H^0(K_C \otimes L^{-2}(E)|_E)$, then $\xi$ is injective $\iff \tilde{\xi}$ is injective.

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By theorem 4.6 $\xi$ is injective if $\xi \in T^*_L(E)$. This is exactly the statement that $\xi \in \text{Sec}^e(C)/\text{Sec}^{e-1}(C)$, with $e = \text{deg}(E)$. If $h^1(L(-E)) > 0$ then $\text{rk}(\xi) \geq \text{Cliff}(L,E) \geq \text{Cliff}(C,L) = c$, since even divisors ineligible to compute $\text{Cliff}(C,L)$ have Clifford index greater than $\text{Cliff}(C,L)$. Thus we see that any $\xi \in H^1(K_C \otimes L^{-2})$ of rank $< c$ must be in $T_L(E)$ with $\text{deg}(E) = \text{rk}(\xi)$, and all the highest order terms of $\xi$ are non-zero. This last statement is that $R^j(C) = \text{Sec}^c(C)$, for $j < c$. $\text{Sec}^c \neq R^c$ follows from two lemmas:

**Lemma 5.3.** If $r_L(D) = r > 0$ and $r_{L \otimes 2}(D) = 0$, we can find a $\tau \in T_L(D)$, $\text{rk}(\tau) = d - 2$.

Our assumption is that $D$ fails to impose independent conditions on the linear system given by $L$, but does impose independent conditions of $L \otimes 2$. Hence $T_L(D)$ gives rise to a $d$-dimensional subspace of $H^1(K_C \otimes L^{-2})$, and hence we have $d$ rank one elements of $T_L(D)$ whose image lies in at most a $(d-1)$ dimensional space. In fact, $\tau$ are distinct as elements of

$$\text{Hom}(H^0(L)/H^0(L(-D)), \ker(H^1(K_C \otimes L^{-1})) \rightarrow H^1(K_C \otimes L^{-1}(D)))$$

and so the result follows from:

**Lemma 5.4.** Let $x_1, \ldots, x_d \in V$ with $\dim(V) = d - 1$, $x_1, \ldots, x_{d-1}$ a base, $x_d = \sum_{i=1}^{d-1} a_i x_i$, with $a_i \neq 0$. Then identifying $\text{Sym}^2(V)$ with $\text{Hom}_{\text{sym}}(V^\vee V)$, there exists $\lambda \neq 0$ such that $\text{rk}(\sum_{i=1}^{d-1} x_i^2 + \lambda x_d^2) \leq d - 2$.

**Proof.** Let $f(\lambda) = \det(x_1^2 + \cdots + x_{d-1}^2 + \lambda x_d^2)$, so $f(0) = 1$. Unless $f$ is constant there exists $\lambda$ such that $f(\lambda) = 0$, i.e. $\text{rk}(x_1^2 + \cdots + x_{d-1}^2 + \lambda x_d^2) \leq d - 2$ as desired. But as a polynomial in $\lambda$ it has leading term $\lambda^{d-1} \prod_{i=1}^{d-1} a_i^2$, and hence the polynomial is non-constant.

**Lemma 5.5.** Let $L = K_C(D)$. $\exists \tau \in T_L(D)$ such that i) $\text{rk}(\tau) = d - 2$, ii) $\text{im}(\tau) \cap C = \emptyset$.

**Proof.** By the lemma above $\exists \tau$ of rank $< d - 1$ and $\text{rk}(\tau) \geq d - 2 = \text{Cliff}(C,D)$ in any event, so $\exists \tau$ such that $\text{rk}(\tau) = d - 2$. By Hassett’s criterion(4.5), since $\text{Sec}^{d-3} = R^{d-3}$, if $\tau \notin \text{Sec}^{d-2}(C)$, then $\text{im}(\tau) \cap C \neq \emptyset$. But if $\tau \notin \text{Sec}^{d-2}(C)$, then $\exists \tau' \in T_L(D')$ with $\text{deg}(D') < d - 2$ such that $\tau = \tau'$, i.e. $D' = D - R$, with $R$ effective. This would mean $\sigma_1^2 + \cdots + \sigma_{d-1}^2 + \lambda \sigma_d^2 = \sum_{i=1}^{d-1} \lambda_i \sigma_i^2$, i.e. $H^0(K_C \otimes L^{-2}(D)) \neq 0$, i.e. $H^0(L^{-1}) \neq 0$ which is absurd. (Any relation $\sum_{\sigma_i \in D} \lambda_i \sigma_i^2 = 0 \in H^0(K_C \otimes L^{-2})$ implies $H^0(K_C \otimes L^{-2}(D)) \neq 0$.)
The scheme structure of $R^p$ for $p < \text{Cliff}(L, C)$. 

Throughout this section $L$ will either be $K_C$ or $\text{deg}(L) \geq 2g + 1$. If $L = K_C$ then we will further assume that $g(C) \geq 3$ and $\text{Cliff}(C) \geq 1$. In particular $L$ will always be very ample and quadratically normal. Let $n = h^0(L)$, $V = H^0(C,L)\vee$ and let $M_2 = H^0(L^{\otimes 2})\vee$. Denote by 

$$R^p = R^p(L) \subset \mathbb{P}(H^1(K_C \otimes L^{-2})) \subset \mathbb{P}(\text{Sym}^2(H^1(K_C \otimes L^{-1})))$$

the rank $p$ locus. We consider $\mathbb{P}(\text{Sym}^2(H^1(K_C \otimes L^{-1})))$ as a subspace of $\mathbb{P}(\text{Hom}(V, V))$ and follow [2] in defining the scheme structure of $R^p$.

Let $H = \text{Hom}(V, W)$ where $V$ and $W$ are finite dimensional vector spaces. The space of rank $p$ matrices in $H$ is defined by the vanishing of all the $(p+1) \times (p+1)$ minors with respect to some choice of a basis for $V$ and $W$. It is denoted by $H^p$. It is known that $H^p$ is Cohen–Macaulay and smooth away from $H^p - 1$. The Cohen-MacCauley statement can be found in [27] p.175 for example. The proof of the smoothness statement follows from the existence of a canonical desingularization, $\tilde{H}^p$, of $H^p$, and the calculation of the tangent space to $\tilde{H}^p$ at a general point. For completeness we sketch the construction. The details may be found in [2].

If $\varphi \in H^p$, then $\dim(\ker(\varphi)) \geq n - p$ and so we can consider the set of all pairs $(\varphi, A) \subset \tilde{H}^p \times G(n-p,n)$ such that $A \subset \ker(\varphi)$. One checks that this is smooth by showing that the projection onto $G(p,n)$ has fiber over $A$ equal to $\text{Hom}(V/A, W)$ and hence is a vector bundle. This is the definition of $\tilde{H}^p$. Further if $\varphi \in H^p \setminus H^p - 1$ then $T_{\varphi,H} = \{ \psi \in H | \psi : \ker(\varphi) \rightarrow \text{im}(\varphi) \}$.

**Definition 6.1.** Let $M \hookrightarrow \text{Hom}(V, W) = H$ be an inclusion of vector spaces. The scheme $R^p(M) = R^p$ is defined as $i^*(H^p)$. With this scheme structure, the ideal of $R^p$ in $M$ is the pull back of the ideal defining $H^p$, which is all the $(p+1) \times (p+1)$ minors. The same definition holds in the projective case, $\mathbb{P}(M) \hookrightarrow \mathbb{P}(\text{Hom}(V, W) = \mathbb{P}(H))$

**Remark 1:** In our case $V = W^\vee = H^0(C,L)$ and $M = H^0(C,L^{\otimes 2})$ and in fact $M \hookrightarrow \text{Sym}^2(W) \hookrightarrow \text{Hom}(V, W)$. We will need this greater generality to deal with the scheme structures that occur in section 7.
Remark 2: We will see later that in this case, \( M = H^0(C, L^\otimes 2) \), that \( R^p \setminus R^{p-1} \) is smooth if \( p < \text{Cliff}(C, L) \).

For later reference we include:

**Lemma 6.1.** With the notation of Definition 6.1, let \( \varphi \in R^p \setminus R^{p-1} \), then \( T_{\varphi, M} = \{ \psi \in M | \psi : \ker(\varphi) \rightarrow \text{im}(\varphi) \} \)

**Proof.** Let \( T \) denote the tangent space to \( \varphi \) in \( \text{Hom}(V, W) \). We have previously identified \( T \) with \( \{ \psi \in H | \psi : \ker(\varphi) \rightarrow \text{im}(\varphi) \} \). Since \( M \) is a subspace of \( H \), \( T_{\varphi, M} = \{ \psi \in T | \psi(I) = 0 \} \) where \( I = I(M) \), is the ideal of \( M \). Since \( M \) is a linear space, the condition on \( \psi \) is that \( \psi \in M \)

Recall that Theorem 5.1 identified \( \text{Sec}^{p-1} \) with \( R^p(L) \) as sets. We have given a scheme structure to \( R^p \) and to show scheme theoretic equality we recall the definition of the scheme \( \text{Sec}^{p-1}(C) \). We are interested in the secant varieties to \( C \) in \( \mathbb{P}(H^0(L^\otimes 2)) \). However, the definitions apply to any embedding of \( C \) by a very ample line bundle \( M \) in \( \mathbb{P}(H^0(C, M))^\vee \). The definition of the scheme structure of the Secant variety goes back to [16], but we follow [3]. In essence one considers the open set of points in \( \text{Sym}^k(C) \) which define \( k - 1 \) planes, take their image in \( \mathbb{P}(V) \) and take the reduced scheme structure on the closure of this set. The actual definition requires some notation.

We recall the notations and results of [3] and construct a rank \( p \) bundle \( B^{p-1}(M) \) over \( \text{Sym}^p(C) \). Informally it is the rank \( p \) bundle \( D \rightarrow H^0(C, M_{|D}) \). The actual construction is given below. \( M \) will denote any very ample line bundle with \( h^0(M) = n \) and satisfying \( h^0(M(-D)) = h^0(M) - d \) for all divisors \( D \) with \( \text{deg}(D) = d \leq p \). The last condition is that \( M \) separates \( p \) points.

Let \( D_p \hookrightarrow C \times \text{Sym}^p(C) \) be the universal divisor, \( \pi_2 : C \times \text{Sym}^p(C) \rightarrow \text{Sym}^p(C) \), then \( B^{p-1}(M) = \pi_2^* (\pi_1^*(M)|_{D_p}) \).

Since \( \pi_2^* \pi_1^*(M) = H^0(M) \otimes \mathcal{O}_{\text{Sym}^p(C)} \) and \( M \) separate \( p \) points, the map \( H^0(M) \otimes \mathcal{O}_{\text{Sym}^p(L)} \rightarrow B^{p-1}(M) \) is surjective. Hence we get an inclusion of projective bundles,

\[
\beta_{p-1} : \mathbb{P}(B^{p-1}(M)) \hookrightarrow \mathbb{P}(\pi_2^* \pi_1^*(M)) = \mathbb{P}(H^0(C, M)) \times \text{Sym}^p(C)
\]

We now define the Secant variety.

**Definition 6.2.** The secant variety \( \text{Sec}^{p-1}(C) \) is the scheme theoretic image of \( B^{p-1}(M) \) in \( \mathbb{P}(H^0(C, M)) \) under the projection onto the first factor in equation 2. We refer to \( B^{p-1}(M) \) as the full secant bundle.
Remark 1) Since $B^{p−1}(M)$ is smooth, and so reduced, the scheme theoretic image of the projection is also reduced. See [13] p.92 for details.

Remark 2) Since $H^0(M) \otimes O_{\text{Sym}^p(C)} \rightarrow B^{p−1}(M)$ is surjective, we see that $B^{p−1}(M)$ is a rank $p$ bundle over $\text{Sym}^p(C)$ generated by $n$ global sections. Hence $B^{p−1}(L)$ gives rise to map $g : \text{Sym}^p(C) \rightarrow G(p,n)$ This map takes a divisor $D \in \text{Sym}^p(C)$ to the $p−1$ dimensional subspace spanned by the divisor $D$. As such, $B^{p−1}(M)$ is the pullback of the universal subbundle on $G(p,n)$ and hence is the incidence correspondence $\{(x, \mathcal{D})| x \in \mathcal{D}\} \subset \mathbb{P}(H^0(C,M)) \times \text{Sym}^p(C)$.

Remark 3) We have used the fact the the line bundle $L$ separates $p$ points in the definition. It is possible to define a the Secant variety without this extra condition, see for example [3].

For the rest of this section $L$ will also satisfy Cliff($C,L$) > $p$. Notice that if Cliff($C,L$) > $p$ then $L$ separates $p$ points. Recall that $R^p$ is defined in equation 1.

Lemma 6.2. We have a scheme theoretic inclusion $\text{Sec}^{p−1}(L) \hookrightarrow R^p$.

Proof. To show scheme theoretic inclusion we need to show an inclusion of the ideal sheaves: $I_{R^p} \hookrightarrow I_{\text{Sec}}$. Since $R^p$ is defined by the vanishing of all the $(p+1) \times (p+1)$ minors of the generic matrix, we need to show that $\text{Sec}^{p−1}(C)$ vanishes on all $(p+1) \times (p+1)$ minors. Roughly speaking, any point $x \in \text{Sec}^{p−1}(C)$ is a linear sum of $p$ elements of $C$. Each point of $C$ represents a rank one linear transformation. Hence each $x \in \text{Sec}^{p−1}(C)$ has a representation as the sum of $p$ rank one transformations and hence is of rank at most $p$. Thus every point $x \in \text{Sec}^{p−1}(C)$ will vanish at all $(p+1) \times (p+1)$ minors and hence scheme theoretically lies in $R^p$. Because this point comes up frequently, I will give it a separate formal proof. The result is well-known (see ([5]) for example).

First we fix some notation. Let $W \subset \text{Hom}(V_1,V_2)$ be a linear space and let $R \subset W$ be a subset consisting of rank one transformations. This means that for every $r \in R$, the kernel of $r$ is of codimension one and that the image of $r$ is of dimension one. By $\text{Sec}^{p−1}(R)$ we mean all linear combinations of $p$ elements of $R$.

Lemma 6.3. Every element $r_p \in \text{Sec}^{p−1}(R)$ is of rank at most $p$. That is if we fix a basis for $V_1$ and $V_2$ every $(p+1) \times (p+1)$ minor of $r_p$ vanishes. Informally: every sum of at most $p$ rank one matrices is of rank at most $p$. 

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Proof. If \( p + 1 > \min(\dim(V_1), \dim(V_2)) \) the result is trivial. So we assume that \( p + 1 \leq \min(\dim(V_1), \dim(V_2)) \). Once we have picked a basis we can represent any \( r \in R \) as a matrix \((a_{ij})\) with a unique \( a_{ij} \neq 0 \). Hence any \( r_p \in \text{Sec}^{p-1} \) can have at most \( p \) columns (or rows) with a non-zero entry. If we take a \((p + 1) \times (p + 1)\) submatrix and expand along a column (or row) with all zeros we see that the determinant vanishes.

Since \( \text{Sec}^{p-1}(C) \) and \( R^p \) agree as sets, if \( R^p \) is reduced then since \( R^p \) is a thickening of \( \text{Sec}^{p-1}(C) \) both varieties are isomorphic having the reduced scheme structure.

There is a fair amount of literature on the scheme structure of \( \text{Sec}^p(C) \). See [3], [16], [25] for example. It is known to be normal in many circumstances. The first theorem on the subject is due to Bertram.

**Theorem 6.4.** If \( L \) separates \( 2p \) points then \( \text{Sec}^{p-1}(C) \) is normal and smooth away from \( \text{Sec}^{p-2}(C) \).

*Proof.* see [3]

**Remark:** In case \( L = K_C \) or \( \deg(L) \geq 2g + 1 \) and \( p < \text{Cliff}(C, L) \), one easily checks that \( L^{\otimes 2} \) separates \( 2p \) points and hence that \( \text{Sec}^{p-1}(C) \) is normal.

The fact that \( \text{Sec}^{p-1}(C) = R^p \), a rank locus, may suggest that in fact these varieties are Cohen-Macaulay. Very little is known about this. Recently Sidman and Vermeire [18] have proven that if \( \deg(L) \geq 2g + 3 \) then \( \text{Sec}^1(C) \) is Cohen-Macaulay. Further Vermeire in [21] has shown that if \( \deg(L) \) is sufficiently large, then \( \text{Sec}(C) \) is generated by cubics. His bound is better than ours in complete analogy to the fact that, once \( \deg(L) \geq 2g + 2 \) then \( C \) is generated by quadrics, but one needs \( \deg(L) \) to be about \( 4g + 4 \) before the equations defining \( C \) are 'determinantly presented'.

Our strategy is to construct a desingularization \( \tilde{R}^p \) of \( R^p \), which is analogous to the canonical desingularization presented in [2]. We show that the variety we construct, \( \tilde{R}^p \), coincides with the full secant variety \( B^{p-1}(L^{\otimes 2}) \). It follows that \( \tilde{R}^p \) is smooth. We then show directly by analyzing the fibers of \( p : \tilde{R}^p \to R^p \) that \( R^p \) is normal. This involves a reasonably complicated tangent space calculation to show that the scheme theoretic fibers of \( p \) are reduced. Once \( R^p \) is normal, it is reduced and must agree scheme theoretically with \( \text{Sec}^{p-1}(C) \).

The methods of this section are similar to the 'geometric technique' of Kempf. This provides for a way to desingularize rank loci. I first learned
of these ideas in [2]. Weyman in his book [27] proves a theorem giving one a resolution of the structure sheaf of \( \tilde{R}^p \). Under some circumstances this resolution may be used to construct a resolution of \( R^p \). This may allow one to prove that \( R^p \) is Cohen-Macauley.

**Remark** If it could be directly shown that \( R^p \) is reduced, then one could conclude directly the scheme theoretic equality. Using Bertram’s Theorem, one would have normality too. I have not been able to create a simple argument to prove this simpler fact. The current proof does have the advantage of explicating the geometry of \( R^p \). Namely, it shows that the resolution is isomorphic to the full secant variety and that it has reduced, and in fact smooth, fibers.

The first step in the construction of \( \tilde{R}^p \) is the fact that \( R^p \) is smooth away from \( R^{p-1} \). We first note:

**Lemma 6.5.** Suppose \( L = K_C \) or \( \text{deg}(L) \geq 2g + 1 \), then \( \text{Cliff}(C, L) > p \) implies \( \text{Cliff}(C, L^{\otimes 2}) > 2p \).

**Proof.** If \( L = K_C \), then \( \text{Cliff}(K_C) \leq \frac{g-1}{2} \) and \( K_C^{\otimes 2} \) separates any \( g-1 \) points.

If \( L \neq K_C \) then if \( \text{deg}(L) = 2g - 2 + d \) with \( d < g \), then \( \text{Cliff}(C, L) \leq d - 1 \) and if \( d \geq g \) then \( \text{Cliff}(C, L) = d - 2 \). Since \( \text{deg}(L^{\otimes 2}) = 4g - 4 + 2d \), \( \text{Cliff}(C, L^{\otimes 2}) = 2g - 4 + 2d - 2 > 2(d - 1) \).

We can now prove:

**Theorem 6.6.** If \( p < \text{Cliff}(C, L) \) and \( \varphi \in R^p \setminus R^{p-1} \), then \( R^p \) is smooth at \( \varphi \). In fact \( T_{\varphi, R^p} = 2D_{L^{\otimes 2}} \), that is, the tangent space to \( R^p \) at \( \varphi \) is the \( 2p - 1 \) dimensional space spanned by the divisor \( 2D \).

**Proof.** If \( \varphi \in R^p \setminus R^{p-1} \) with \( p < \text{Cliff}(C, L) \) then \( \varphi \in T^*_D \) with \( \text{deg}(D) = p \) for some divisor \( D \). Hence \( \ker(\varphi) = H^0(C, L(-D)) \) and \( \text{im}(\varphi) = S_L(D) \).

We will work with \( \tilde{\varphi} \), a lift of \( \varphi \) to \( H^1(C, K_C \otimes L^{-2}) \) and show that the tangent space \( T_{\tilde{\varphi}, R^p} \subset H^1(C, K_C \otimes L^{-2}) \) is the correct \( 2p \) dimensional space. By Theorem 4.1 we identify \( \text{im}(\varphi) \) with \( \{ x \in H^1(K_C \otimes L^{-1}) \mid x \rightarrow 0 \in H^1(K_C \otimes L^{-1}(D)) \} \) and hence, if we denote by \( r_D \) the map \( r_D : H^1(K_C \otimes L^{-2}) \rightarrow \text{Hom}(H^0(L(-D)), H^1(K_C \otimes L^{-1}(D))) \), we may identify \( T_{\tilde{\varphi}, R^p} \) with \( \ker(r_D) \). We can factor \( r_D \) as

\[
\begin{align*}
H^1(K_C \otimes L^{-2}) & \xrightarrow{\alpha} H^1(K_C \otimes L^{-2}(2D)) \\
& \xrightarrow{\beta} \text{Hom}(H^0(L(-D)), H^1(K_C \otimes L^{-1}(D))).
\end{align*}
\]
Notice that \( \ker(\alpha) = \partial(\mathcal{H}^0 \left( K_C \otimes L^{-2}(2D)|_{2D} \right)) \) that is to say \( \overline{\mathcal{D}}_{L^2} \). Since \( \ker(\alpha) \subset \ker(r_D) \)' to finish we need to check that \( \beta \) is injective. We have \( p < \text{Cliff}(C,L) \) and hence \( \text{Cliff}(C,L(-D)) \geq 1 \). In particular \( L(-D) \) is arithmetically normal, which is equivalent to \( \beta \) being injective by Serre Duality.

The next step is to construct a resolution \( \tilde{R}^p \) of \( R^p \)

**Theorem 6.7.** Let \( \tilde{R}^p = \{(\varphi, \lambda) | \im(\varphi) \subset \lambda \} \subset R^p \times G(p,n) \). Then \( \tilde{R}^p \) is smooth.

**Proof.** \( \tilde{R}^p = \{(\varphi, \lambda) | \im(\varphi) \subset \lambda \} \). Of course we mean by this that if we can represent \( \lambda \) as \( \sigma_1 \wedge \cdots \wedge \sigma_p \in \Lambda^p(V) \) then \( \im(\varphi) \subset \langle \sigma_1, \ldots, \sigma_p \rangle \). Notice that \( \tilde{R}^p \subset R^p \times G(p,n) \subset R^p \times \mathbb{P}(\Lambda^p(V)) \subset \mathbb{P}(M_2) \times \mathbb{P}(\Lambda^p(V)) \). There is a Koszul map

\[
\mathbb{P}(\text{Sym}^2(V)) \times \mathbb{P}(\Lambda(V)) \rightarrow \mathbb{P}(V \otimes \Lambda(V))
\]

given by:

\[
\text{Sym}^2(V) \otimes \Lambda(V) \rightarrow V \otimes \Lambda(V)
\]

\[
(v_1 \cdot v_2) \otimes \lambda \rightarrow v_1 \otimes (v_2 \wedge \lambda) + v_2 \otimes (v_1 \wedge \lambda) \quad (2).
\]

The inclusions \( R^p \hookrightarrow \mathbb{P}(M_2) \hookrightarrow \mathbb{P}(\text{Sym}^2(V)) \) induce a map

\[
R^p \times G(p,n) \rightarrow \mathbb{P}(V \otimes \Lambda(V))
\]

(3)

We first observe that \( \tilde{R}^p \) can be considered as an incidence variety.

**Lemma 6.8.** With this notation \( \tilde{R}^p = \{(\varphi, \lambda) | \varphi \cup \lambda = 0 \} \). This is true as long as we are not working in characteristic 2.

**Proof.** Fix some \( \lambda \) say \( \lambda = v_1 \wedge \cdots \wedge v_p \) with \( v_i \in V \) linearly independent. Let \( \langle v_1 \ldots v_p \rangle = W \subset V \). From equation (3) if \( \varphi \in \text{Sym}^2(W) \) then \( \varphi \cup \lambda = 0 \) since \( \varphi = \sum_{i=0}^p a_i v_i^2 \) with \( a_i \) constants (this is true after changing our basis of \( W \)). So we may assume that \( \varphi \) contains no terms entirely in \( W \). Extend \( \langle v_1 \ldots v_p \rangle \) to a basis \( \langle v_1 \ldots v_n \rangle \) of \( V \). Write \( \varphi = \sum_{i=p+1}^n a_i l_i \otimes v_i + \sum_{i \geq j \geq (p+1)} b_{ij} v_i \otimes v_j \) with \( l_i \in W \). Then

\[
\varphi \cup \lambda = \sum_{i=p+1}^n a_i l_i \otimes v_i \wedge \lambda + \sum_{i \geq j \geq (p+1)} b_{ij} (v_i \otimes v_j \wedge \lambda + v_j \otimes v_i \wedge \lambda) \quad (4)
\]
Each of the terms are linearly independent and (as long as char(k) ≠ 2) \( \varphi \cup \lambda \) vanishes if and only if all the \( a_i \) and all the \( b_{ij} \) are zero. If char(k)= 2 then the coefficients of \( b_{ii} \) are zero because they are divisible by two.

To proceed we formalize the idea that we can consider \( \text{Sym}^p(C) \to \mathbb{P}(\mathbb{P}^p(V)) \), via the association of a degree \( p \) divisor to the \( p-1 \) dimensional space \( \overline{D} \) it spans.

**Lemma 6.9.** Let \( g : \text{Sym}^p(C) \to G(p,n) \) be the natural map associating \( D \to \overline{D} \); then for \( p \leq \text{Cliff}(L,C) \) this map is an embedding.

**Proof.** Following [3] we construct a rank \( p \) bundle \( B^{p-1}(L) \) over \( \text{Sym}^p(C) \). Let \( D_p \to C \times \text{Sym}^{p}(C) \) be the universal divisor, \( \pi_2 : C \times \text{Sym}^p(C) \to \text{Sym}^p(C) \), then \( E = \pi_{2*} (\pi_1^*(L)|_{D_p}) \). Informally this is the rank \( p \) bundle \( D \to H^0(C,L|_D) \). Since \( \pi_{2*} \pi_1^*(L) = H^0(L) \otimes \mathcal{O}_{\text{Sym}^p(C)} \) and \( p \leq \text{Cliff}(C) \) implies \( L \) is at least \( (p+1) \) spanned the map \( H^0(L) \otimes \mathcal{O}_{\text{Sym}^p(L)} \to \mathcal{E}_{|L} \) is surjective. Since a map \( g : \text{Sym}^p(C) \to G(p,n) \) is given by a rank \( p \) bundle and \( n \) sections generating the bundle, this gives a map \( g : \text{Sym}^p(C) \to G(p,n) \). Again because \( L \) separates at least \( p+1 \) points, \( \overline{D_1} \neq \overline{D_2}, \forall D_1, D_2 \in \text{Sym}^p(C) \) and so the map is set theoretically one to one. Using the identification of [2], we identify \( T_{D,\text{Sym}^p(C)} \) with \( H^0(C,\mathcal{O}_D(D)) \) and \( T_{D,G(p,n)} \) with \( \text{Hom}(H^0(L(-D)), H^0(L|_D)) \). Then (cf. [2]) the map on tangent spaces, is the map “cup product”, i.e.

\[
  t \in H^0(C,\mathcal{O}_D(D)) \mapsto \cup t : H^0(C,L(-D)) \to H^0(C,L|_D).
\]

Since \( L \) separates \( (p+1) \) points, this map is an injection as required. If \( D \) is smooth this is well known. We prove the case \( D = kq \) for completeness. Let \( z \) be a local parameter at \( q \). \( H^0(\mathcal{O}_D(D)) = \langle z^{-1}, \ldots, z^{-k} \rangle \), \( H^0(L|_D) = \langle l, z^l, \ldots, z^{k-1} l \rangle \) where \( l \) is a local section of \( L \) at \( q \). Since \( L \) is at least \( (k+1) \) ample, there exists a section of \( L \) which locally at \( q \) looks like \( z^k l \) and the cup product is now multiplication. Since \( z^{-i} z^k l = z^{k-i} l \) for \( 1 \leq i \leq h \), multiplication gives rise to linearly independent elements of \( H^0(L|_D) \) the map is injective.

\[ \square \]

**Corollary 6.10.** Let \( \lambda_D \in \Lambda^p(V) \) represent the image of \( D \in \text{Sym}^p(C) \) in \( G(p,n) \). Then \( B^{p-1}(L) \subset \mathbb{P}(V) \times \mathbb{P}(\Lambda^p(V)) = \{(v, \lambda_D) | v \wedge \lambda_D = 0 \in \Lambda^{p+1}(V) \} \).

**Proof.** \( B^{p-1}(L) \) is the pullback to \( \text{Sym}^p(C) \) of the universal subbundle on \( G(p,n) \). That is \( B^{p-1}(L) = g^*(\mathcal{S}) \) where \( \mathcal{S} = \{(x, \lambda) \in V \times \Lambda^p(V) | x \wedge \lambda = \)
0 ∈ ∧^{p+1}(V)$. Since $B^{p-1}(L)$ is just the restriction of $S$ to the embedding of $\text{Sym}^p(C)$ in $G(p,n)$ this is clear.

\[ \square \]

Remark: this map is not the usual Gauss map. We can include $C$ in $\text{Sym}^p(C)$ via the diagonal, i.e. $q \mapsto pq$. As Voisin proved cf. [24] $H^0(C, \Lambda^p(B^{p-1}(L))) = \Lambda^p(H^0(C,L))$ whereas the rank $p$ bundle associated with the Gauss map is $P^{p-1}(L)$ (the jet bundle) and for example for $p = 2 \Lambda^2(P^1(L)) \simeq L^2 \otimes K_C$ and so the two bundles have different global sections.

To finish the proof that $\tilde{R}^p$ is smooth, we will identify $\tilde{R}^p$ with $B^{p-1}(L^2)$ as schemes. We have characterized $B^{p-1}(L^2)$ in Corollary 6.10 so by Lemma 6.8 it is enough to prove the following.

Lemma 6.11. Let $\tilde{\lambda}_D$ denote the image of $D \in \text{Sym}^p(C)$ in $G(p,V)$ and let $\lambda_D$ denote the image of $D \in \text{Sym}^p(C)$ in $G(p,M_2)$. Then for $\varphi \in M_2$, we have $\varphi \cup \tilde{\lambda} = 0$ if and only if $\varphi \wedge \lambda = 0$.

Proof. Suppose $D = \sum n_i q_i$ where $n_i$ are integers such that $\sum n_i = p$ and $q_i \in C$. Then $\varphi \wedge \lambda_D = 0$ if and only if $\varphi \in T_L(D)$. Since $\varphi$ is symmetric, this is true if and only if $\text{im}(\varphi) \subset \tilde{\lambda}_D$.

We now have a scheme theoretic identification of $B^{p-1}(L^2)$ with $\tilde{R}^p$. Since $B^{p-1}(L^2)$ is smooth, so is $\tilde{R}^p$.

To finish the proof we show that $R^p$ is normal. We use a result that we learned in [23] namely:

Lemma 6.12. Let $f : X \to Y$ be a proper surjective morphism of irreducible varieties over an algebraically closed field, with reduced and connected fibers. If $X$ is normal, then $Y$ is normal.

Proof. See [23] for details.

We will actually prove that the fibers of $p : \tilde{R}^p \to R^p$ are smooth.

Theorem 6.13. Let $\varphi \in R^k \setminus R^{k-1}$, then $p^{-1}(\varphi)$ is scheme theoretically isomorphic to $\text{Sym}^{b-k}(C)$.

Proof. Because $k \leq p < \text{Cliff}(C,L)$ we can write $\varphi = \sum_{i=1}^{k} \sigma^2_{p_i}$ where $\sigma^2_{p_i}$ is a Shiffer variation supported at $p_i \in C$. Denote by $D$ the divisor spanned by $\langle p_1, \ldots, p_k \rangle$. We analyze $p^{-1}(\varphi)$ as follows:
$\cup \varphi : \bigwedge^P(V) \to \bigwedge^{p+1} \otimes V$

is the map on points $\lambda \mapsto 2(\sum_{i=1}^k \sigma_i \otimes (\sigma_i \wedge \lambda))$ First extend $\sigma_1, \ldots, \sigma_k$ to a basis $\sigma_1, \ldots, \sigma_k, \ldots, \sigma_n$ of $V$, and set $V' = \langle \sigma_{k+1}, \ldots, \sigma_n \rangle$, so $\ker(\varphi) \subseteq \sigma_1 \wedge \cdots \wedge \sigma_k \otimes \bigwedge^{p-k}(V') = \hat{V}$. We also have a map of schemes:

$$\text{Sym}^{p-k}(C) \xrightarrow{\varphi_D} \text{Sym}^p(C)E \hookrightarrow E + D$$

$\varphi_D$ identifies $\ker(\cup \varphi) \cap \text{Sym}^p(C)$ with $\text{Sym}^{p-k}(C)$ because if $\lambda \in \text{Sym}^p(C)$, considered as a subset of $G(p, n)$ then $\sum_{i=1}^k \sigma_i^2 \cup \lambda = \sum_{i=1}^k \sigma_i \otimes \sigma_i \wedge \lambda = 0$ if and only if $\sigma_i \wedge \lambda = 0$ for all $i$. We need to see that this is an equality of schemes. By induction it will be enough to consider the case $k = 1$ because we can write a general $\varphi_D$ as a composition of $\varphi_{\sigma_{p_i}}$ for the different $p_i \in D$.

So we assume that $\varphi = \sigma^2$ for a specific $\sigma = \sigma_{p_i}$ with $p_i \in C$.

Recall the relevant identifications of tangent spaces. $T_{\text{Sym}^p(C),D} = H^0(\mathcal{O}(D)|_D)$ and $T_{G,D} = \text{Hom}(H^0(L(-D)), H^0(L|_D))$. Suppose $p$ occurs with multiplicity $k$ in $D$. Then in fact our fiber is the scheme theoretic intersection of $\text{Sym}^p(C)$ and $\mathbb{P}_\sigma$ where $\sigma$ is any element in the span of the divisor $D$ that is not in the span of $D - p$ and $\mathbb{P}_\sigma$ denotes $\{x \in \bigwedge^p(V)| \sigma \wedge x = 0\}$.

To show that the intersection is smooth, we must find an element of the tangent space of $\text{Sym}^p(C)$ which is not in the tangent space of $\mathbb{P}_\sigma$. This forces the tangent space of the intersection to be at most of dimension $p - 1$ and hence smooth, since the intersection is set theoretically $\text{Sym}^{p-1}(C)$. If $\varphi$ is in $T_{\mathbb{P}_\sigma} \cap T_{G(p,n)}$ then $\varphi \in \text{Hom}(H^0(L(-D))), H^0(L|_D))$, is also in $T_{\mathbb{P}_\sigma}$ if and only if $\sigma \notin \varphi(H^0(L(-D)) \subset H^0(L|_D)$.

The element $\frac{1}{x} \in H^0(\mathcal{O}_D(D))$ contain such $\sigma$ in its image and hence is not in the tangent space of the fiber $p^{-1}(\varphi)$.

**Corollary 6.14.** As schemes, $\text{Sec}^{p-1}(C) = R^p$.

**Proof.** $B^{p-1}(L^\otimes 2)$ and $\tilde{R}^p$ are defined as subsets of $\mathbb{P}(M_2) \times \mathbb{P}(\bigwedge^p(V))$. $\text{Sec}^{p-1}(C)$ and $R^p$ are defined as the projection onto $\mathbb{P}(M_2)$ of these two schemes. Since the two schemes are identified, so are their (reduced) projections. □

**Remark:** The conclusion of the theorem is probably as strong as possible. We know for generic $L$ with $\deg(L) < 3g - 2$ and for all $L$ with $\deg(L) \geq 3g - 2$ the theorem is sharp as the two schemes do not even underly the same set in this case. It is probably the case that this is always so. □
The Clifford Index of a Pair of Line Bundles and a Conjecture of Eisenbud, Koh, and Stillman

In their paper ([5]), Eisenbud, Koh and Stillman considered the following situation. Let \( L_i = (L_i, V_i) \) for \( i = 1, 2 \) be two linear series on a curve – that is \( L_i \) is a line bundle on \( C \) and \( V_i \subset H^0(C, L_i) \) is a linear subspace. Let \( L_1 \cdot L_2 \) be the linear series \( (L_1 \otimes L_2, V = \text{im}(V_1 \otimes V_2 \xrightarrow{\mu} H^0(C, L_1 \otimes L_2)) \). That is \( L_1 \cdot L_2 \) represents the line bundle \( L_1 \otimes L_2 \) with the sub-linear series generated by \( V_1 \otimes V_2 \). By the linear series generated by \( V_1 \otimes V_2 \) we mean the natural map \( H^0(C, L_1) \otimes H^0(C, L_2) \xrightarrow{\mu} H^0(C, L_1 \otimes L_2) \) restricts to a map \( \mu : V_1 \otimes V_2 \rightarrow H^0(C, L_1 \otimes L_2) \). \( \mu(V_1 \otimes V_2) \subset H^0(C, L_1 \otimes L_2) \) is the linear series generated by \( V_1 \otimes V_2 \).

If \( \{e_i\} \) and \( \{f_j\} \) are bases for \( V_1 \) and \( V_2 \), then \( M = \{\mu(e_i \otimes f_j)\} \) can be considered as a matrix of linear form with entries in \( H^0(C, L_1 \otimes L_2) \) which represents a basis for \( \mu(V_1 \otimes V_2) \). Writing \( I_2(M) \) for the ideal of \( 2 \times 2 \) minors of \( M \) in \( S = \text{Sym}(V) \), one has that \( I_2(M) \subset I_2(C) \), where \( I_2(X) \) is equations of degree 2 in the ideal of \( X \). (Proof: Let \( m_{ij} = \mu(e_i \otimes f_j) \) then the equations of \( I_2(M) \) are \( m_{ij} \cdot m_{kl} - m_{ik} \cdot m_{jl} = 0 \) which clearly vanish on \( C \).)

We now assume that \( V_i = H^0(C, L_i) \).

**Definition 7.1 ([5];).** If \( C \) is defined by quadratic equations and \( I_2(M) = I_2(C) \), we say \( L_1 \cdot L_2 \) is a determinantal presentation of \( C \) and that \( C \) is determinantly presented.

Eisenbud-Koh-Sullivan then prove if deg\((L_i)\) are “large” then \( L_1 \otimes L_2 \) is determinantly presented. For ease of reading we will denote \( L_1 \otimes L_2 \) by \( L_{12} \). Notice that if \( C \) is embedded in \( \mathbb{P}(H^0(C, L)) \), the question of whether \( C \) is determinantly presented depends on the factorization of \( L = L_1 \otimes L_2 \), in particular there can be infinitely many such presentations. See [5] for a fuller discussion.

We explain how this result is related to our theorem and how their result can be extended to \( \text{Sec}^k(C) \) in an appropriate range of \( k \).

We can create a diagram for \( L_{12} \) which generalizes our standard diagram.
Here \( \varphi_i : C \to \mathbb{P}(V) \) and \( \psi : C \to \mathbb{P}(V) \) are the maps given by the linear series; \( \iota \) is the inclusion induced by the surjection of \( V_1 \otimes V_2 \to V \), \( \mu \) is surjective since \( V \) is generated by \( \mu(e_i \otimes f_j) \); and finally, \( \sigma \) is the Segre map.

If we identify \((V_1 \otimes V_2)^\vee \) with \( \text{Hom}(V_2, V_1^\vee) \), then \( \sigma(\mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee)) \) can be identified with the set of “rank one” matrices in \( \mathbb{P}(V_1^\vee \otimes V^\vee) \). That is to say the matrices such that all their \( 2 \times 2 \) minors vanish, or alternatively the subvariety of all \( \varphi \) such that \( \dim(\text{im}(\varphi)) = \text{codim}(\ker(\varphi)) = 1 \).

In general, we will define the rank \( j \) locus for \( j \leq \min\{\dim(V_1), \dim(V_2)\} \) as the variety defined by all the \((j+1) \times (j+1)\) minors. If \( L_1 = L_2 = L \), \( V_i = H^0(C, L_i) \) we recover our standard diagram.

The basic observation of [5] is that because of this diagram, \( \psi(C) \subset \mathbb{P}(V^\vee) \cap \sigma(\mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee)) = R^1(C) \). Since \( \psi(C) \) is contained in the rank one locus, the ideal of \( \psi(C) \) contains the ideal of \( \sigma(\mathbb{P}(V_1^\vee) \times \mathbb{P}(V_2^\vee)) \), which is generated by determinantal quadrics! The main theorem of [5] is:

**Theorem 7.1 ([5]):** Let \( \deg(L_1), \deg(L_2) \geq 2g + 1 \), and if \( \deg(L_1) = \deg(L_2) = 2g + 1 \), assume \( L_1 \) and \( L_2 \) are not isomorphic. Then \( \psi(C) = \iota(\mathbb{P}(V^\vee)) \cap \sigma(\mathbb{P}(V_1) \times \mathbb{P}(V_2)) \) as schemes, and so \( \psi(C) \) is determinantal presented.

Further in ([5]) it was conjectured:

**Conjecture 7.2.** Let \( C \) be a curve of genus \( g \), and \( L \) a line bundle which can be factored as \( L = L_1 \otimes L_2 \) for some choice of line bundles \( L_1 \) and \( L_2 \). Then there is a constant \( k_0 \) depending on the genus of \( C \) and the degrees of the \( L_i \) such that the variety \( \text{Sec}^k(C) \) is determinantly presented for \( k \leq k_0 \).

Theorem 7.1 can be interpreted as a version of our theorem for \( L_1 \neq L_2 \) and for \( C = \text{Sec}^0(C) = R^1 \). M.S. Ravi in [17] gave a partial answer to this conjecture.

**Theorem 7.3** (Ravi). Suppose \( \deg(L_1), \deg(L_2) \geq 2g + 1 + k \) and \( \deg(L_1 \otimes L_2) \geq 4g + 3 + 2k \). Then set-theoretically, \( \text{Sec}^k(C) \) is defined by \( I_{k+2}(M) \), that is, \( \text{Sec}^k(\psi(C)) = R^{k+1}(C) \).

By generalizing the techniques of Shiffer variations and the Clifford index of a line bundle from the case of \( L^{\otimes 2} = L \otimes L \) to the case of \( M = L_{12} \) we can improve this result.

**Theorem 7.4.** Suppose \( \deg(L_1), \deg(L_2) \geq 2g + 1 + k \) and \( \deg(L_1 \otimes L_2) \geq 4g + 2 + 2k \). Then as a scheme \( \text{Sec}^{k-1}(C) \) is defined by \( I_{k+1}(M) \). That is, as schemes, \( \text{Sec}^{k-1}(C) = R^k(C) \), the rank \( k \) locus.
As in the case $L_1 = L_2$, the proof proceeds in a number of steps. We first define and prove the basic properties of Shiffer variations for $L_1 \neq L_2$. Then one proves a set theoretic equality of the schemes $\text{Sec}^j(C)$ and $R^{j+1}$. As before the scheme $\text{Sec}^j(C)$ is reduced and includes in $R^{j+1}$, so the only issue is to show that $R^{j+1}$ is reduced. The proof proceeds exactly as in the case of $L_1 = L_2$, by showing that one has two resolutions of $R^g$ that agree scheme theoretically. Finally since $\deg(L)$ is very large, one can show the existence of an appropriate factorization of $L$ so that Theorem 7.4 is true for some choice $L_1$ and $L_2$ and one obtains,

**Corollary 7.5.** Suppose $C \hookrightarrow \mathbb{P}(H^0(C,L))$ is embedded by the complete linear system $|L|$ and that $\deg(L) \geq 4g + 2k$. Then for any $j \leq k$, $\text{Sec}^j(C)$ is determinantly presented and so is defined by equations of degree $j + 2$.

**Remark** More formally, we let $S^1 \subset \mathbb{P}(V_1 \vee V_2 \vee)$ be $\sigma(\mathbb{P}(V_1 \vee) \times \mathbb{P}(V_2 \vee))$, and set $S^j = \text{Sec}^{j-1}(\sigma(\mathbb{P}(V_1 \vee) \times \mathbb{P}(V_2 \vee))) \subset \mathbb{P}(V_1 \vee \otimes V_2 \vee)$. $S^j$ is defined by the vanishing of all the $(j + 1) \times (j + 1)$ minors. We then set $R^j(C, L_1, L_2) = S^j \cap \mathbb{P}(H^0(C, L_1 \otimes L_2)^\vee)$. When $C, L_1, L_2$, etc. are clear we will usually just write $R^j$ or $R^j(C)$. The corollary follows by taking $L_1$ to be any line bundle of degree $\frac{1}{2} \deg(L)$ and $L_2 = L \otimes L^{-1}$. Notice this gives rise to infinitely many determinantal presentations.

The basic idea is the same as for $L_1 = L_2$. For any $p \in C$, the image of $\psi(p)$ in $\mathbb{P}(H^1(C, K_C \otimes L_1^{-1} \otimes L_2^{-1}))$ will correspond to a rank one matrix that has kernel $H^0(C, L_1(-P))$ and image equal to $\partial(H^0(C, K_C \otimes L_1^{-1}(P)|p)) \subset H^1(K_C \otimes L_2^{-1})$. If $D$ is any divisor we will define Shiffer variations supported on $D$ and the rank of any such matrix will depend on $d$, $r_{L_1}(D)$, and $r_{L_2}(D)$.

We work out an example before doing things in general.

Let $L_1$ and $L_2$ be line bundles of degree $2g + 1$, and let $D = p_1 + p_2 + p_3$, $p_i \in C$. Let $s_1, s_2, s_3 \in H^1(K_C \otimes L_1^{-1})$, $v_1, v_2, v_3 \in H^1(C, K_C \otimes L_2^{-1})$, and $t_1, t_2, t_3 \in H^1(C, K_C \otimes L_1^{-1} \otimes L_2^{-1})$ be elements representing $p_1, p_2, p_3$. Recall that for any line bundle $M$ on $C$ an element of $m \in H^1(K_C \otimes M^{-1}) = H^0(M)^\vee$ represents the point $p \in C$ means that that $m$ kills $H^0(M(-p))$.

**Lemma 7.6.** The natural map $H^0(C, L_1) \otimes H^0(C, L_2) \longrightarrow H^0(C, L_1 \otimes L_2)$ dualizes to give a map:

$$H^1(C, K_C \otimes L_{12}) \xrightarrow{\mu} H^1(C, K_C \otimes L_1) \otimes H^1(C, K_C \otimes L_2)$$

Then as long as $\text{char}(k) \neq 2$, $t_i = \mu(s_i \otimes v_i)$. 42
Proof. The map $\mu$ can also be considered as a map

$$
\mu : H^1(C, K_C \otimes L_{12}) \rightarrow \text{Hom}(H^0(L_1), H^1(K_C \otimes L_{12}^{-1}))
$$

With this description it is clear that $\mu(s_i \otimes v_i)$ kills $H^0(L_1(-p_i))$ and has image generated by $v_i$ which is the action of $t_i$.

We consider 3 cases:

(i) $h^0(L_i(-D)) = g + 2 - 3 = g - 1$, for $i = 1, 2$.

(ii) $h^0(L_1(-D)) = g - 1$, $h^0(L_2(-D)) = g$.

(iii) $h^0(L_1(-D)) = h^0(L_2(-D)) = g$.

Case (i) is the generic case. Case (ii) can occur if $L_1 = K_C(p_1 + p_2 + p_3)$, with $L_2$ general of degree $2g - 2$. Case (iii) can occur if $L_1 = L_2 = K_C(p_1 + p_2 + p_3)$.

In case (i), $t = t_1 + t_2 + t_3$ is of rank 3 as $p_1, p_2, p_3, \in D$ are linearly independent in both embeddings. That is to say that the $t_i$ represent linearly independent rank one transformations. In case (ii), $D$ spans only a line in $\mathbb{P}(H^1(K_C \otimes L_{12}^{-1}))$ and $t$ is of rank 2, because the image of the $t_i$ lie in a 2-dimensional space. In case (iii), we cannot be sure that rank of $t$ is greater than one.

This picture generalizes without difficulty. The notation is a bit cumbersome. Let $D = \sum_{i=1}^{d} n_i p_i$ be a divisor of degree $d$. Let $L_1$ and $L_2$ be line bundles such that

$$2g + 1 \leq \deg(L_1) \leq \deg(L_2).$$

We have an exact sequence

$$0 \rightarrow K_C \otimes L_{12}^{-1} \rightarrow K_C \otimes L_{12}^{-1}(D) \rightarrow K_C \otimes L_{12}^{-1}(D)|_D \rightarrow 0.$$ 

Denote by $T_{L_{12}}(D) = \partial(H^0(C, K_C \otimes L_{12}^{-1}(D)|_D)) \subset H^1(C, K_C \otimes L_{12}^{-1})$.

Definition 7.2. $T_{L_{12}}(D)$ are the Shiffer variation for $(L_1, L_2)$ supported on $D$.

Definition 7.3. $\text{Cliff}(L_1; L_2, D) = d - r_{L_1}(D) - r_{L_2}(D)$.

Definition 7.4. $\text{Cliff}(C, L_1; L_2) = \min\{\text{Cliff}(L_1; L_2, D) \mid r_{L_1}(D) > 0 \text{ or } r_{L_2}(D) > 0\}$. 

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Proposition 7.7. (i) \(\min\{\text{Cliff}(C, L_1), \text{Cliff}(C, L_2)\} \leq \text{Cliff}(C, L_1; L_2)\)

(ii) Suppose \(\deg(L_1) = \deg(L_2)\), then \(\text{Cliff}(C, L_1, L_2) \geq d - 1\), unless \(L_1 = L_2 = K_C(D)\), with \(D\) effective or \(C\) is hyperelliptic and \(L_1 = K_C(D - E_1), L_2 = K_C(D - E_2)\), with \(E_1, E_2\) both multiples of the \(g_2^1\) on \(C\).

(iii) Suppose \(\deg(L_1) = \deg(L_2) \leq 3g - 3\) and \(L_1 \neq L_2\) are generic line bundles, then \(\text{Cliff}(C, L_1, L_2) \geq d - 1\).

Proof. (i) Suppose \(\text{Cliff}(C, L_1; L_2)\) is computed by \(D\). Then if \(r_{L_1}(D) \leq r_{L_2}(D)\), then

\[d - 2r_{L_2}(D) \leq d - r_{L_2}(D) - r_{L_1}(D)\]

(ii) Follows from (i) and Corollary 3.8 since those are the only cases, for which \(\text{Cliff}(C, L_1) = d - 2\). Notice that if \(L_1 = K_C(D_1)\) and \(L_2 = K_C(D_2)\) with \(D_1 \neq D_2\) then for example, \(r_{L_1}(D_2) = 0\) so that \(\text{Cliff}(C, L_1, L_2) = d - 1\).

(iii) We must eliminate the cases in (ii) which are possible exceptions. The only case to consider is \(C\) is hyperelliptic and \(L_1 = K_C(D - E_i)\) where \(D\) computes \(\text{Cliff}(C, L_1, L_2)\). But \(\deg(L_1) = \deg(L_2)\) forces \(\deg(E_1) = \deg(E_2)\) which means that \(E_1 = E_2\) since the \(g_2^1\) is unique on \(C\). Hence \(L_1 = L_2\).

The results about the rank of a matrix in \(T_{L_12}\), and the relationship between \(R^j(C)\) and \(\text{Sec}^{j-1}(C)\) are the same as in the case \(L_1 = L_2\). We will state the results and sketch the proofs.

Theorem 7.8. Let \(\xi \in T_{L_12}(D) \subset H^1(C, K_C \otimes L_{12}^{-1})\). Denote by \(\rho\) the natural restriction \(H^0(C, L_1) \rightarrow H^0(C, L_1|_D)\). Denote by \(\partial_1 : H^0(K_C \otimes L_{21}^{-1}(D)|_D) \rightarrow H^1(K_C \otimes L_{21}^{-1})\) the boundary map in the long exact sequence

\[0 \rightarrow K_C \otimes L_{21}^{-1} \rightarrow K_C \otimes L_{21}^{-1}(D) \rightarrow K_C \otimes L_{21}^{-1}(D)|_D \rightarrow 0\]

and denote by \(\partial_2 : H^0(K_C \otimes L_{12}^{-1}(D)|_D) \rightarrow H^1(K_C \otimes L_{12}^{-1})\) the boundary map in the definition of \(T_{L_12}(D)\). Let \(\tilde{\xi} \in H^0(C, K_C \otimes L_{12}^{-1}(D)|_D)\) be an element lifting \(\xi\), i.e. \(\partial_2(\tilde{\xi}) = \xi\). Then \(\cup\xi : H^0(C, L_1) \rightarrow H^1(C, K_C \otimes L_{21}^{-1})\) factors as:

\[H^0(C, L_1) \xrightarrow{\rho} H^0(C, L_1|_D) \xrightarrow{\cup\tilde{\xi}} H^0(K_C \otimes L_{21}^{-1}(D)|_D) \xrightarrow{\partial_1} H^1(C, K_C \otimes L_{21}^{-1}).\]
Proof. The proof is exactly the same. Pick an affine open cover of $C$ by $V_i$ such that $V_1 \supset D$ and $V_2 = C - D$. If one uses this Čech cover to compute the cup product then $\xi \in \Gamma (V_1 \cap V_2, K_C \otimes L_1^{-1} \otimes L_2^{-1})$ is given by $\tilde{\xi}$ where $\tilde{\xi}$ is a lifting of $\xi$ to $\Gamma (V_1 \cap V_2, K_C \otimes L_1^{-1} \otimes L_2^{-1})$. So given $s_1 \in H^0 (C, L_1)$, $s_1 \cup \xi$ is represented by $s_1 \cdot \xi \in \Gamma (V_1 \cap V_2, K_C \otimes L_2^{-1})$ which is $\partial_1 (\tilde{\xi} \cdot s_1)$. \hfill $\blacksquare$

Corollary 7.9. Let $\xi \in T_{L_{11}}(D)$. Then $\ker(\xi) \supset H^0 (L_1(-D))$ and $\text{im}(\xi) \subset S_L(D)$, the affine cone over $D$ in $\mathbb{P} (H^1 (K_C \otimes L_2^{-1}))$.

Proof. This follows from Theorem 4.1 and the above description. \hfill $\blacksquare$

The next theorem calculates the rank of an element $\tau \in T_{L_{11}}(D)$. Suppose that $r_{L_{11}}(D) = r_2 \geq r_{L_1}(D) = r_1$.

Theorem 7.10. Let $\xi \in T_{L_{11}}^*(D)$, then:

$$d - r_1 - r_2 \leq \text{rk}(\xi) \leq d - r_2$$

Proof. The condition that $\xi \in T_{L_{11}}^*(D)$ is: write $D = \sum_{i=1}^n n_i p_i$ and choose $z_i$ a local parameter at $p_i$, then we can write a lifting of $\xi$ to $H^0 (K_C \otimes L_1^{-1} \otimes L_2^{-1}(D))$ as $\tilde{\xi} = \sum_{i=1}^n \left( \sum_{j=1}^{n_i} \beta_{i,j} z_i^j \right)$ with $\beta_{i,n_i} \neq 0$ for $1 \leq i \leq n$. By cor 7.9 $\text{im}(\xi) \subset S_L(D)$, which is a linear space of dimension $d - r_2$. This gives the upper bound. $\cup \xi$ is an isomorphism by the same argument as in Theorem 4.6 so we are done by the following algebraic fact. \hfill $\blacksquare$

Lemma 7.11. Let $\varphi : V_1 \rightarrow V_2$ be a linear isomorphism between two vector spaces of dimension $d$. Let $W_1 \subset V_1$ be a subspace of codimension $r_1$ and let $V_2 \rightarrow W_2$ be a surjection onto a space of dimension $d - r_2$. Then $\varphi : W_1 \rightarrow W_2$ has rank $\geq d - r_1 - r_2$.

Proof. Because $\varphi$ is an isomorphism, $\overline{\varphi}_1 : W_1 \rightarrow V_2$ has rank $d - r_1$ and $p : V_2 \rightarrow W_2$ has kernel of rank $r_2$. $\ker (p \circ \overline{\varphi}_1) \subset \ker (p)$ since $\overline{\varphi}_1$ is injective and hence $\dim (\ker \varphi) \leq r_2$ and so $\text{rk} (\varphi) = \text{rk} (\overline{\varphi}_1) - \dim \ker (p \circ \overline{\varphi}_1) \geq d - r_1 - r_2$. \hfill $\blacksquare$

We next show set-theoretic equality of the appropriate secant varieties and rank-loci. Having established the generalized notation, the proofs go
exactly as in the case of $L_1 = L_2$. Recall our setup

$$P \left( \text{Hom} (H^1 (K_C \otimes L_1^{-1}), H^1 (K_C \otimes L_2^{-1})) \right) \times P \left( \text{Hom} (H^1 (K_C \otimes L_2^{-1}), \sigma) \right)$$

$$P (H^1 (K_C \otimes L_1^{-1}) \otimes H^1 (K_C \otimes L_2^{-1}))$$

$$\varphi_1 : C \rightarrow P (H^1 (C, K_C \otimes L_i^{-1}))$$ and $\psi : C \rightarrow P (H^1 (C, K_C \otimes L_1^{-1} \otimes L_2^{-1}))$ are the maps given by the complete linear series, $\iota$ is the inclusion induced by the surjection

$$H^0 (C, L_1) \otimes H^0 (C, L_2) \rightarrow H^0 (C, L_{12})$$

and $\sigma$ is the Segre embedding.

We view $P (H^1 (K_C \otimes L_1^{-1}) \otimes H^1 (K_C \otimes L_2^{-1}))$ as the projectivization of the space of matrices

$$\text{Hom} (H^0 (C, L_1), H^1 (C, K_C \otimes L_2^{-1}))$$

and we define $S^j \subset P (H^1 (K_C \otimes L_1^{-1}) \otimes H^1 (K_C \otimes L_2^{-1}))$ as the rank $j$ locus given by the vanishing of all the $(j+1) \times (j+1)$ minors. Then $R^j = R^j (C) = \iota^* (S^j)$ consists of the rank $j$ locus in $P (H^1 (K_C \otimes L_1^{-1} \otimes L_2^{-1}))$. Since $p \in C$ represents a rank one matrix with kernel $H^0 (C, L_1(p))$ and image $\partial (H^0 (C, L_2(p))|_p) \subset H^1 (C, K_C \otimes L_2^{-1})$, we have $C = \text{Sec}^0 (C) \subset R^1$ and hence $\text{Sec}^{j-1} (C) \subset R^j$ because, as stated in Lemma 6.3, a sum of $j$ rank one matrices is of rank $\leq j$.

**Theorem 7.12.** Suppose $\text{Cliff} (C, L_1, L_2) = c \geq 2$. Then, as sets, $\text{Sec}^{j-1} (C) = R^j$ for $j < c$.

**Proof.** The proof is essentially identical to the case $L_1 = L_2$. For completeness we give details.

Since $\dim \left( \text{Sec}^j (C) \right) = 2j + 1$,

$$P (H^1 (K_C \otimes L_{12}^{-1})) = \bigcup_{D \in \text{Sym}^k (C)} T_{L_{12}} (D)$$

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for \( k \geq \frac{h^0(C,L_{12})+1}{2} \). In other words, every element can be written as a Shiffer variation in some \( T_{L_{12}}(D) \) for some \( D \) of degree \( k \). Let

\[
T_{L_{12}}^*(D) = T_{L_{12}}(D) \setminus \left( \bigcup_{D' \subset D} T_{L_{12}}(D') \right)
\]

be the elements of maximal rank. Set \( r_1 = r_{L_1}(D) \) and \( r_2 = r_{L_2}(D) \). By Theorem 7.10, if \( t \in T_{L_{12}}^*(D) \),

\[
d - r_1 - r_2 \leq \text{rk}(C) \leq d - \max\{r_1, r_2\}.
\]

If \( r_1 = r_2 = 0 \), then \( \text{rk}(t) = d \). That is \( t \in \text{Sec}^{d-1}(C) \) as desired. We have \( r_1 = r_2 = 0 \) for \( \text{deg}(D) < c \). Further, if \( r_1 > 0 \) or \( r_2 > 0 \), then \( d - r_1 - r_2 \geq c \) so any \( t \in H^1(K_C \otimes L_1^{-1} \otimes L_2^{-1}) \) with \( \text{rk}(t) < c \) can be written as the sum of \( \text{rk}(t) \) matrices of rank 1. This is the statement \( \text{Sec}^{j-1}(C) = R^j \) for \( j < c \).

Finally we need to check that the set theoretic equality is a scheme theoretic equality.

**Theorem 7.13.** \( \text{Sec}^{j-1}(C) = R^j \) as schemes for \( j < c \).

**Proof.** We check the same information as before.

1. \( \text{Sec}^{j-1}(C) \) is normal and smooth away from \( \text{Sec}^{j-2}(C) \) with tangent space generated by the span of the divisor \( 2D \) for \( q \) a general point in the span of \( D \). The exact same proof holds.

2. By exactly the same argument as in Lemma 6.2 we get an inclusion of schemes: \( \text{Sec}^{p-1} \hookrightarrow R^p \).

3. Let \( \bar{R}^p \subset R^p \times G(p,n) = \{(\varphi, \lambda)| \text{im}(\varphi) \subset \lambda\} \), then \( \bar{R}^p \) is smooth. We make the minor modification that now \( \bar{R}^p \subset \mathbb{P}(V_1 \otimes V_2) \), \( V_i = H^0(C, L_i)^\vee \) and the linear map is

\[
v : V_1 \otimes V_2 \otimes \bigwedge^p (V_2) \rightarrow V_1 \otimes \bigwedge^{p+1} (V_2)
\]

and the fibers are linear spaces. Since \( p < \text{Cliff}(C,L_1,L_2) \), we can identify \( \bar{R}^p \) with the incidence correspondence defining the full secant variety, \( B^{p-1}(L_{12}) \). Again as in the case \( L_1 = L_2 \) it follows that the projections are the same and that \( \text{Sec}^{p-1}(C) = R^p \).

\( \square \)
Since Sec\(^{p-1}\) = R\(^p\) as schemes they have the same tangent spaces at all points. We present an independent calculation of the tangent space to R\(^p\) at a smooth point, which is to say, at a point of rank exactly p

**Theorem 7.14.** Let \(p < \text{Cliff}(C, L)\) \(\varphi \in R^p \setminus R^{p-1}\), so \(\varphi \in T_{L^p_{12}}(D)\) for some \(D\) of degree \(p\). The tangent space \(T_{\varphi,R_p}\) is the projectivization of

\[
\partial \left( H^0 \left( K_C \otimes L_{12}^{-1}(2D) \right) \right) \subset H^1 \left( K_C \otimes L_{12}^{-1} \right)
\]

**Proof.** Let \(\tilde{T} \subset H^1 \left( K_C \otimes L_1^{-1} \otimes L_2^{-1} \right)\) be the cone over \(T_{\varphi,R_p}\). Then \(\tilde{T}\) is the tangent space to the affine rank \(p\) locus. For \(\varphi \in R^p \setminus R^{p-1}\) this tangent space is described in ([2] see page 68) as the matrices which map the kernel of \(\varphi\) into the image of \(\varphi\). That is to say: \(\tilde{T} = \{ \psi \in H^1 \left( K_C \otimes L_1^{-1} \otimes L_2^{-1} \right) | \psi : \ker \varphi \rightarrow \text{im} \varphi \}\). But, \(\ker \varphi = H^0 \left( C, L_1(-D) \right)\) and \(\text{im} \varphi = (\{ x \in H^1 \left( K_C \otimes L_1^{-1} \right) | x_{H^0(L(-D))} = 0 \})\) so

\(\tilde{T} = \ker \left( H^1 \left( K_C \otimes L_{12}^{-1} \right) \rightarrow \text{Hom} \left( H^0 \left( L_1(-D) \right), H^0 \left( L_2(-D) \right) \right) \right)\)

Clearly

\[T_L(2D) \subset \ker \left( H^1 \left( K_C \otimes L_{12}^{-1} \right) \rightarrow H^1 \left( K_C \otimes L_{12}^{-1}(D) \right) \right) \subset \tilde{T}\]

so we must show

\[H^1 \left( C, K_C \otimes L_{12}^{-1}(2D) \right) \rightarrow \text{Hom} \left( H^0 \left( L_1(-D) \right), H^0 \left( L_2(-D) \right) \right) \]

is injective or by Serre duality (and using \(\text{Hom}(A \vee, B) = A \otimes B\)) that

\[H^0 \left( C, L_1(-D) \right) \otimes H^0 \left( nC, L_2(-D) \right) \rightarrow H^0 \left( C, L_{12}(-2D) \right)\]

is surjective, which is true since \(\text{deg}(L_i(-D)) \geq 2g - 1\).

**Corollary 7.15.** \(R^p\) is smooth at \(\varphi \in R^p \setminus R^{p-1}\)

**Proof.** Since \(p < \text{Cliff}(C, L)\) by the above argument \(T_{\varphi,R_p} = T_{L_{12}}(D)\) which is of dimension \(2p + 1\).

This concludes the proof of Theorem 7.4, since for any factorization \(L = L_1 \otimes L_2\) with \(\text{deg}(L_i) \geq 2g + 1 + k\) one has \(\text{Cliff}(C, L_i) \geq k + 1\). For the case \(L_1 = L_2\) the result is sharp. By Lemma 5.5 we can for \(L_1 = L_2 = L = K_C(D)\) find an element \(\tau \in T_{L_{12}}(D)\) with \(d - 2 = \text{rk}(\tau) < \text{deg}(\tau) = d - 1\). In the language of Theorem 7.4 \(d = k + 3\) and \(\text{Cliff}(C, L) = k + 1\). In particular, for \(k = 0\) this gives a weaker result than 7.1, the result proved in ([5]). However, the theorem can be ‘tweaked’ to get,
Lemma 7.16. If \( L_1 \neq L_2 \), but \( \deg(L_i) \geq 2g + k + 1 \), then \( \text{Sec}^j(C) = R^j \) for \( j \leq (k + 1) \) then

Proof. By Proposition 7.7 (iii) we have \( \text{Cliff}(C, L_1, L_2) \geq k + 2 \). The lemma follows from Theorem 7.13.

8 The Clifford Index for line bundles with \( h^1(L) = 1 \)

Throughout this section we will write \( L = K_C(-P) \), where \( P \) is an effective divisor of degree \( p < c = \text{Cliff}(C) \). Since \( \deg(P) < c \), \( h^0(O_C(P)) = 1 \) (else \( \text{Cliff}(C) \leq p - 2 \)), so \( h^1(L) = h^0(K_C \otimes L^{-1}) = h^0(O_C(P)) = 1 \). This is the only case we will discuss as it is the only situation in which I can say something meaningful about \( \text{Cliff}(C, L) \).

Lemma 8.1. Suppose that \( L = K_C(-P) \) with \( \deg(P) = p < c \) then

1. \( \text{Cliff}(L, D) = \text{Cliff}(D + P) - p \).
2. \( c - p \leq \text{Cliff}(C, L) \leq c \).
3. \( \text{Cliff}(C, L) = c - p \), except possibly in the case where \( \text{Cliff}(C) = \left\lceil \frac{g - 1}{2} \right\rceil \) and \( p = \text{Cliff}(C) - 1 \).

Proof. 1. First notice that \( r_L(D) = r_{K_C}(D + P) \) since

\[
\begin{align*}
r_L(D) &= h^0(K_C \otimes L^{-1}(D)) - h^0(K_C \otimes L^{-1}) \\
&= h^0(O_C(D + P)) - h^0(O_C(P)) \\
&= h^0(O_C(D + P)) - 1 \\
&= r_{K_C}(D + P).
\end{align*}
\]

Setting \( d = \deg(D) \) we have:

\[
\begin{align*}
\text{Cliff}(L, D) &= d - 2r_L(D) \\
&= (d + p - 2r_L(D)) - p \\
&= \deg(D + P) - 2r_{K_C}(D + P) - p \\
&= r_{K_C}(D + P) - p.
\end{align*}
\]
2. If $D$ computes $\text{Cliff}(C, L)$, then it spans at most a codimension 2 plane in $\mathbb{P}(H^0(C, L^\vee))$, so $h^0(I_{T}) = h^0(L(-D)) \geq 2$. But $L(-D) = K_C(-P - D)$, so $D + P$ is eligible to compute $\text{Cliff}(C)$, and hence by 1. $\text{Cliff}(L, D) = \text{Cliff}(D + P) - p \geq c - p$. Considering the divisor $D + P$ where $D$ computes $\text{Cliff}(C)$ gives the other inequality.

3. Suppose $D$ computes $\text{Cliff}(C)$ and $\deg(D) < \left[\frac{g-1}{2}\right]$. Consider the divisor $L(-D)$. It is effective except possibly in the case when $\text{Cliff}(C) = \left[\frac{g-1}{2}\right]$ and $p = \text{Cliff}(C) - 1$. By 1, $\text{Cliff}(C, L(-D)) = c - p$.

Remark: In general I would not expect it to be the case these divisors give rise to Shiffer variations of low rank. This is in complete analogy to the fact that a curve being non-arithmetic normalization gives rise to a divisor of degree $2n + 2$, spanning $n$ planes, but the existence of an $2n + 2$ pointed $n$ plane do not necessarily imply that the embedding is not arithmetically normal.

Remark: We postpone the discussion of the possible pathology that can occur until after our one positive result.

Notice that when $L = K_C(-P)$, $L$ is always very ample. If $D$ was a divisor of degree 2 such that $h^0(L(-D)) \geq h^0(L) - 1$, then $h^1(L(-D)) \geq 2$, and hence $\text{Cliff}(P + D) \leq p + 2 - 2 = p < \text{Cliff}(C)$. Since

$$\deg(P + D) = \deg(P) + 2 < c + 2 < \frac{g-1}{2} + 2 < g,$$

$P + D$ is eligible to compute $\text{Cliff}(C)$, and we would have a contradiction since $p < c$.

To apply our standard setup we need to know that $L$ is quadratically normal. This is a special case of a theorem of Green and Lazarsfeld proven in [10]

**Theorem 8.2** (Green-Lazarsfeld). Suppose $L$ is very ample with $\deg(L) \geq 2g + 1 - 2h^1(L) - \text{Cliff}(C)$, then $L$ is projectively normal.

In our case $h^1(L) = 1$ and $\deg(L) = 2g - 2 - p > 2g - 2 - \text{Cliff}(C)$. We will actually give a proof of the theorem, as the use of Shiffer deformation and Clifford index gives (to us!) a conceptual proof of the theorem. As [G-L] points out, cubic and higher normality follow from the base-point-free pencil trick and the only issue is to prove quadratic normality.

A simple argument will show that the failure of quadratic normality implies the existence of a divisor of Clifford index zero. From our point of
view it is natural to restate the inequality of the theorem as
\[ \deg(L) \geq (2g - 2h^1(L)) - (\text{Cliff}(C) - 1). \]

This makes clear the basic idea, the only way to get a divisor of Clifford index zero is as the “projection” from a plane of dimension \((c - 1)\) of a divisor of Clifford index \(c\). This is the geometry behind the proof. Incidentally one can check that the inequality of the theorem implies \(h^1(L) \leq 1\).

**Proof.** Firstly, if \(L\) is not quadratically normal, then the map
\[ H^0(L) \otimes H^0(L) \rightarrow H^0(L^\otimes 2) \]
is not surjective; or dually the map
\[ H^1(K_C \otimes L^{-2}) \rightarrow \text{Hom}(H^0(L), H^1(K_C \otimes L^{-1})) \]
is not injective. If \(\xi \in H^1(K_C \otimes L^{-2})\) is in the kernel of this map, then, viewing \(\xi\) as a matrix in \(\text{Hom}(H^0(L), H^1(K_C \otimes L^{-1}))\) we have \(\text{rk}(\xi) = 0\). Viewing \(\xi\) as a Shiffer variation, there is a divisor \(D\) on \(C\) such that \(\xi \in T_L(D)\) and setting \(d = \deg(D)\), \(r = r_L(D)\), we have \(d - 2r \leq 0\). Adding points to \(D\) if necessary we have a divisor \(D\) such that \(\text{Cliff}(L, D) = 0\).

We can bound the degree of \(D\) such that \(\xi \in T_L(D)\). Namely, if \(L = K_C(-P)\) we have \(\deg(D) \leq \frac{3g - 5}{2} - p\). To prove this, we use the same idea as for \(L = K_C\). Namely the embedding \(C \hookrightarrow \mathbb{P}(H^0(C, L^\otimes 2)^\vee)\) in a projective space of dimension \(3g - 4 - 2p\). This is because \(p < \text{Cliff}(C) \leq \frac{g - 1}{2}\) implies \(\deg(L^\otimes 2) = 4g - 4 - 2p \geq 3g - 3\), so \(L^\otimes 2\) isn’t special. Now \(\dim(\text{Sec}(C)) = 2j + 1\) implies that as long as \(j \geq \frac{3g - 5}{2} - p\), \(\text{Sec}(C) = \mathbb{P}(H^0(C, L^\otimes 2)^\vee)\), so any \(\xi \in H^1(K_C \otimes L^{-2})\) is the sum of at most \(\frac{3g - 5}{2} - p\) Shiffer variations.

We consider separately the cases of \(h^1(L) = 0\) and \(h^1(L) = 1\). If \(h^1(L) = 1\), \(L = K_C(-P)\), where \(P\) is a divisor of degree \(p\) and our inequality is that \(p < c = \text{Cliff}(C)\). \(\text{Cliff}(L, D) = 0\) means \(d - 2r_L(D) = 0\) and \(r_L(D) = h^0(\mathcal{O}_C(D + P)) - 1 = \frac{d}{2}\), and hence \(\text{Cliff}(K_C, \mathcal{O}_C(D + P)) = d + p - 2\left(\frac{d}{2}\right) = p < c\). Thus \(D + P\) cannot be used to compute \(\text{Cliff}(C)\) and hence must span a hyperplane. That is, \(h^0(K_C(-D - P)) = 1\) and so \(\text{Cliff}(K_C(-D - P)) = \deg(K_C(-D - P))\). But then
\[
\text{Cliff}(\mathcal{O}_C(D + P)) = \text{Cliff}(K_C(-D - P)) = \deg(K_C(-D - P)) = 2g - 2 - (d + p).
\]
But recall, \(d \leq \frac{3g - 3}{2} - p\), so
\[
\deg(K_C(-D - P)) \geq 2g - 2 - \frac{3g - 5}{2} \geq \frac{g + 1}{2} > \text{Cliff}(C),
\]

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and hence $\text{Cliff}(L, D) = \text{Cliff}(\mathcal{O}_C(D + P) - p > \text{Cliff}(C) - p > 0$. This is a contradiction.

Now assume $h^1(L) = 0$ and $\text{Cliff}(L, D) = 0$, $h^1(L(-D)) = r_L(D) > 0$ so $L(-D) = K_C(-E)$. Let $\ell = \deg(L)$, $d = \deg(D)$, $e = \deg(E)$, and $r = r_{K_C}(D)$, so $r + 1 = r_L(D)$. From $D = L \otimes K_C^{-1}(E)$ we get $d = \ell + e - (2g - 2)$. Now $d - 2(r + 1) = 0$ can be rewritten as $l + e - (2g - 2) - 2(r + 1) = 0$, or $\ell + (e - 2r) = 2g$, or $\ell = 2g - \text{Cliff}(E)$. If $E$ satisfies $r > 0$ and $h^1(\mathcal{O}_C(E)) \geq 2$, then $\text{Cliff}(E) \geq \text{Cliff}(C)$, and $\ell \leq 2g - \text{Cliff}(C)$ as desired. If $r = 0$, then $d = 2g$ and $L$ isn’t very ample! The only case left to rule out is $h^1(\mathcal{O}_C(E)) \leq 1$. That is $E = K_C(-P)$ with $P$ general. Since $P$ is general, $\text{Cliff}(E) = \text{Cliff}(P) = \deg(P)$. We can bound $\deg(D)$ from above and hence $\deg(P)$ from below. If $\deg(L) = \ell$, $h^0(L \otimes 2) = 2\ell - g + 1$, and by the usual argument we can assume $\deg(D) \leq \frac{2\ell - g - 1}{2}$. Since $K_C(-E) = L(-D)$,

$$\deg(P) = \deg(K_C(-E)) = \deg(L(-D)) \geq \ell - \frac{2\ell - g - 1}{2} \geq \frac{g + 1}{2}.$$ 

So $\text{Cliff}(E) \geq \frac{2g + 1}{2} \geq \text{Cliff}(C)$ and so

$$l = 2g - \text{Cliff}(E) \geq 2g - \text{Cliff}(C)$$

as desired.

**Corollary 8.3.** Suppose $L$ is very ample, $\deg(L) = 2g$ and $L$ is not arithmetically normal. Then $C$ is hyperelliptic.

**Proof.** From the proof of the theorem we see that $\deg(L) = 2g$ implies $e - 2r = 0$. By Clifford’s theorem, after easily ruling out the cases of $E = \mathcal{O}_C$ or $K_C$, we see that $C$ is hyperelliptic and $E$ is a multiple of the $g^1_2$ on $C$.

$$\square$$

When $L = K_C$ we have seen that $\text{Cliff}(C)$ characterizes on the nose the degree to which secant varieties are rank loci. For $L = K_C(-P)$ this is no longer true. We can always guarantee the same bound, but unlike for $L$ with $h^1(L) = 0$ or $L = K_C$, the existence of a divisor with $\text{Cliff}(L, D) = c$ does not seem to imply there exists a $\xi \in T_L(D)$ with $\text{rk}(\xi) = \text{Cliff}(L, D)$. Nonetheless, the more important lower bound always holds. Consider the
standard diagram:

\[ P(H^1(C,K \otimes L^{-1})) \]

\[ C \xrightarrow{\Phi_L} \mathbb{P}(\text{Sym}^2(H^1(C,K \otimes L^{-1}))) \]

\[ \mathbb{P}(H^1(K \otimes L^{-2})) \]

\[ \mathbb{P}(H^1(K \otimes L^{-2})) \]

As always, \( v \) is the Veronese map, \( i \) is an inclusion since \( L \) is quadratically normal, and \( \Phi_L, \Phi_{2L} \) are the maps associated to the linear systems \( L \) and \( L^{\otimes 2} \). We set \( \text{Sec}^j(C) = \text{Sec}^j(\Phi_{2L}(C)) \subset \mathbb{P}(H^1(K \otimes L^{-2})), R^j(C) = R^j = \text{Sec}^j(v(C)) \cap \mathbb{P}(H^1(K \otimes L^{-2})), \) so \( R^j \) is the rank \( j \) locus.

**Theorem 8.4.** Suppose \( j < \text{Cliff}(C,L) \); then \( R^j(C) = \text{Sec}^{j-1}(C) \) as sets.

**Proof.** We use the same argument as before in the case \( L = K_C \). As in the proof of the quadratic normality, any \( \xi \in H^1(K_C \otimes L^{-2}) \) can be written as \( \xi \in T_L(D) \) where \( \deg(D) \leq (3g - 5)/2 - p \). If \( D \) is eligible to compute \( \text{Cliff}(C,L) \) then \( \text{Cliff}(L,D) \geq \text{Cliff}(C,L) \). If \( D \) is not eligible to compute \( \text{Cliff}(C,L) \) then either \( r_L(D) = 0 \) or \( h^1(L(-D)) \leq 1 \). In the first case, \( \xi \in \text{Sec}^{d-1}(C) \setminus \text{Sec}^{d-2}(C) \). In the later case we may assume \( r_L(D) > 0 \). Again exactly as in the proof of quadratic normality theorem and recalling \( K_C \otimes L^{-1} = O_C(P) \) we see that

\[
\text{Cliff}(L,D) = d - 2 \left( h^0(O_C(D + P) - 1) \right) = \text{Cliff}(K_C, D + P) - p.
\]

But

\[
\text{Cliff}(K_C, D + P) = \text{Cliff}(K_C, K_C(-D - P)) \geq 2g - 2 - \deg(D) - \deg(P) \geq 2g - 2 - \frac{3g - 5}{2} \geq \frac{g + 1}{2}
\]

Thus

\[
\text{Cliff}(L,D) \geq \frac{g + 1}{2} - p > \text{Cliff}(L,C)
\]

\[ \square \]

**Remarks:**

1. Scheme theoretic equality should follow exactly as in the cases of \( L = K_C \) or \( h^1(L) = 0 \). I have not checked the details.
2. In case $L = K_C$ the existence of $\xi$ with $\text{rk}(\xi) = \text{Cliff}(C)$ was delicate. If $D$ computed $\text{Cliff}(C)$ one could find an $\eta \in H^0(K_C(-D))$ such that there was a $\tau \in T_{K_C}(D)$ with the property that $\tau$ vanished on $1 \otimes H^0(K_C(-D))$ as well as vanishing on $f_i \otimes \eta_1$ for $1 < i \leq n$ and so $\text{rk}(\tau) = \text{Cliff}(D) = \text{Cliff}(C)$.

If $D$ computes $\text{Cliff}(C)$ then $K_C(-D)$ is base point free and that is the key point.

In the case $L = K_C(-P)$ it is the twisted linear system, $K_C \otimes L^{-1}(D)$, not $O_C(D)$ which comes into play. One needs the base point freeness of $L^\otimes 2 \otimes K_C^{-1}(-D) = L(-D-P)$.

In the case of the divisor used to compute $\text{Cliff}(C, L)$ being $D = L(-E)$ where $E$ computes $\text{Cliff}(C)$ and $\text{deg}(E)$ small, $L^\otimes 2 \otimes K_C^{-1}(-D) = E - P$ which satisfies $h^0(O_C(E - P)) < 1$. I cannot provide a proof, but I think that these divisors do not give rise to Shiffer variations $\tau$, satisfying $\text{rk}(\tau) = \text{Cliff}(L, D)$. I suspect that on a general line bundle $L = K_C(-P)$ that $\text{Sec}^j(C) = \mathcal{R}^j$ for $j \leq \text{Cliff}(C) - 2$ as opposed to holding for $j \leq \text{Cliff}(C) - p - 1$. The moral is that projecting from a general point should not affect the Clifford index, but projection from special points should. Here is a small positive result.

**Theorem 8.5.** Suppose $\text{deg}(E) < (g - 1)$, $h^0(O_C(E)) \geq 2$. Suppose, $L(-E)$ is base point free, $h^0(O_C(E - P)) = 1$ and $\text{deg}(P) < \text{Cliff}(C)$. Let $D = E - P$, $L = K_C(-P)$. Then

1. $\text{Cliff}(L, D) = \text{Cliff}(E) - p$.

2. There exists a Shiffer variation $\xi$, of rank $c = \text{Cliff}(L, D)$, but such that $\xi \notin \text{Sec}^{c-1}(C)$.

**Proof.**

1. Since $K_C \otimes L^{-1}(D) = E$, $\text{Cliff}(L, D) = \text{Cliff}(E) - p$. By degree considerations $D$ cannot span a hyperplane and $r_L(D) = r_{K_C}(E)$.

2. $L(-E)$ is base point free so the same argument as for $L = K_C$ works. We can find $\eta \in H^0(L(-E))$ which vanishes on exactly $E$. If $\{1, f_1, \ldots, f_n\}$ is a basis for $H^0(O_C(E))$ then since $E - D = P$ and any Shiffer variations in $T_L(D)$ kills $H^0(L(E - P)) \supset H^0(L(-E))$ and $\eta \otimes f_i$ are not in $H^0(L(E - P))$ and killed by some $\tau \in T_L(D)$, specifically by $
\tau = \sum_{p_i \in D} \tau_{p_i}$. $

\square$

**Remark:** I believe that one should have $L(-E)$ base point free always when $\text{deg}(E) < g - 1$ and $E$ computes $\text{Cliff}(C)$. I do not have a proof.
In general the technique will fail since $L(-E)$ need not be base point free. To produce this example, we also give an example of a divisor not in the Petri locus. That is a curve $C$ and a divisor $D$ such that $(r+1)(g-d+r) > g$ but the Petri map is not surjective. After conversations with L. Ein and I. Coskun it is clearly not hard to find such examples. I do not know of examples in the literature though. Firstly we prove:

Lemma 8.6. Suppose $D$ is base point free. The following two assertions are equivalent:

1. The Petri map $H^0(O_C(D)) \otimes H^0(K_C(-D)) \rightarrow H^0(K_C)$ is not surjective.

2. The natural map $H^0(O_C(D)) \otimes H^0(O_C(D)) \rightarrow H^0(O_C(2D))$ is not surjective.

Proof. Now by the base point free pencil trick c.f. [ACGH, p126] we have an exact sequence:

$$0 \rightarrow O_C(-D) \rightarrow O_C(D) \rightarrow O_C(D) \rightarrow 0$$

Twisting by $K_C(-D)$ we get:

$$0 \rightarrow K_C(-2D) \rightarrow K_C(-D)^{\oplus 2} \rightarrow O_C(D) \rightarrow 0$$

and $m$ is the Petri map. So $h^0(m)$ is not surjective if and only if $h^1(\iota)$ is not injective if and only if (by Serre duality) $H^0(O_C(D))^{\oplus 2} \rightarrow H^0(O_C(2D))$ is not surjective. \hfill \Box

I speculate that the curves carrying a $g^1_n$ such that the Petri map is not surjective will be represented by a cohomology class in $M_{g,n}$ that is not an intersection of divisors or in the cohomology ring of ordinary Brill-Noether loci.

To construct such curves recall that if $f : C \rightarrow \mathbb{P}^1$ is a $n$-gonal map then $f^*(O_{\mathbb{P}^1}(1)) = O_C(D)$ where $D$ is the $g^1_n$. If $O_C(2D)$ is to have extra sections we need $h^0(O_C(2D)) \neq h^0(O_{\mathbb{P}^1}(2))$. But by the projection formula $H^0(C, O_C(2D)) = H^0(C, g^*(O_{\mathbb{P}^1}(2))) = H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(2) \otimes g^*(O_C))$ and we can write $g_*(O_C) = \bigoplus_{i=0}^{n-1} O_{\mathbb{P}^1}(-a_i)$, with $a_0 = 0$ and $a_i > 0$ for $i \geq 1$. For $h^0(O_C(2D)) > 3$ we need some $a_i \leq 2$. In fact $a_i > 1$ because otherwise using the projection formula we see $h^0(O_C(D)) \geq 3$. Hence $h^0(O_C(2D)) = 3 + \# \{i|a_i = 2\}$. The simplest example of this phenomena is to take a cyclic cover of $\mathbb{P}^1$ of
degree 4 branched at 8 points. By construction $g_*(\mathcal{O}_C) \cong \bigoplus_{i=0}^3 \mathcal{O}_{\mathbb{P}^1}(-2i)$ (as an $\mathcal{O}_{\mathbb{P}^1}$ module). A simple computation with Hurwitz’s formula shows $g(C) = 9$ and $\mathcal{O}_C(F) = g^*(\mathcal{O}_{\mathbb{P}^1}(1))$ is by construction a $\mathcal{O}_{\mathbb{P}^1}$ on $C$. That is, $h_{\mathcal{O}_C}(2F) = 4$. This already provides an example of a curve $C$ and line bundle $F$ such that the Petri map $H^0(\mathcal{O}_C(F)) \otimes H^0(\mathcal{O}_C(K_C - F)) \longrightarrow H^0(K_C)$ is not surjective. Further Cliff$(F) = 2$ and we can check that $C$ is not hyperelliptic so that Cliff$(C) = 2$. Let $F = P_1 + P_2 + P_3 + P_4$. Then since $F$ (by construction as a cyclic cover) moves in a base point free pencil, $h^0(\mathcal{O}_C(F - P_1)) = 1$ and hence any 3 points of a fiber span a plane, hence any two points span a line and so $h^0(K_C(-P_1 - P_2)) = g - 2 = 7$. If $P_1$ and $P_2$ are two points in separate fibers they clearly span a line in $\mathbb{P}(H^0(C,K_C)^*)$ as they map to separate points of $\mathbb{P}^1$. This means that $C$ is not hyperelliptic. For this example it is not clear how to construct a $D$ such that $L^2 \otimes K_C^{-1}(-D)$ has base points. Here is a procedure to do this in general.

Suppose $C$ is any curve with a $g^1_n$ ($n \geq 4$), call it $F$, and such that $h^0(\mathcal{O}_C(2F)) = 5$. Let $\sum_{i=1}^n P_i \in |F|$, and set $P = \sum_{i=1}^{n-3} P_i$, $D = \sum_{i=n-2}^n P_i$, and $L = K_C(-P)$. Then in general $L$ will be very ample and if $L$ is very ample, $h^0(L) = g - n + 3$, $h^0(L(-D)) = g - n + 1$, so $D$ spans a 3-secant line.

$L^\otimes 2 \otimes K_C^{-1}(-D) = L(-P - D) = K_C(-F - P)$.

By construction $h^0(K_C(-F - P)) = g - 2n + 4 = h^0(K_C(-2E))$, so $L(-F)$ has base points on $D$, and hence the map

$H^0(L(-E)) \otimes H^0(\mathcal{O}_C(E)) \longrightarrow H^0(L)$

lands in $H^0(L(-D))$. For such a curve and linear system, the method of construction of Shiffer variations used in the case of $L = K_C$ won’t work!

We now construct such $C$ and $F$. We merely iterate the previous construction. Namely on $C$ we have the linear system $g^*(\mathcal{O}(1))$, which is of degree 32. We take the 4-fold cyclic cover of $C$ branched on $g^*(\mathcal{O}(1))$, call this $C_2$. By Hurwitz formula we calculate $g(C_2) = \frac{4(16) + 3(32)}{2} + 1 = 81$. Let $f$ be the composed map,
then \( h_*(\mathcal{O}_{C_2}) = \bigoplus_{i=0}^{3} g^*(\mathcal{O}_{\mathbb{P}^1}(-2i)) \), and so

\[
 f_*(\mathcal{O}_{C_2}) = g_* h_*(\mathcal{O}_{C_2}) = g_* \left( \bigoplus_{i=0}^{3} g^*(\mathcal{O}_{\mathbb{P}^1}(-2i)) \right) 
 = g_*(\mathcal{O}_{C_1}) \otimes \bigoplus_{i=0}^{3} \mathcal{O}_{\mathbb{P}^1}(-2i) = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-4) \oplus \ldots
\]

so setting \( F = f^*(\mathcal{O}(1)) \) we see that \( h^0(\mathcal{O}_{C}(F)) = 2, h^0(\mathcal{O}_{C}(2F)) = 5. \)

Since \( F \) is a \( \mathcal{O}_{\mathbb{P}^1} \) we get \( H^0(K_{C}(-F)) = 81 - 16 + 1 = 66, \) and \( h^0(K_{C}(-2F)) = 81 - 32 + 4 = 53. \) In particular, \( F \) imposes 13 conditions on the linear system \( K_{C}(-F). \) Take \( F = \sum_{i=1}^{16} P_i \) and \( P = \sum_{i=1}^{13} P_i, \) such that \( P \) imposes independent conditions on \( K_{C}(-F). \) Let \( L = K_{C}(-P). \) \( L \) is very ample because if not there would be a divisor \( A \) of degree 2, such that \( h^0(\mathcal{O}_{C}(F - P + A)) = 2. \) This would mean these 15 points are linearly dependent. Exactly as in the case of showing \( K_{C} \) is very ample, the 15 points cannot lie on one fiber, and if they lie on different fibers, they clearly are linearly independent. This completes the proof.

While pathological behavior occurs, the generic case is fine. For example, we have seen that if \( r_L(D) = 1, \) for simple linear algebra reasons we can construct a \( \tau \in T_L(D) \) with \( \text{rk}(\tau) = \text{Cliff}(L, D) = d - 2. \) So if \( D \) computes \( \text{Cliff}(C, L) \) we can construct a Shiffer variation of rank equal to \( \text{Cliff}(C, L). \)

### 9 Connections with Koszul Cohomology and Green’s Conjecture

The standard reference for Koszul cohomology is [8]. However I have also profited from reading [9], [24], and [7]. We will not define these groups in their greatest generality. We begin by reviewing with brief proofs some of the basic facts about Koszul cohomology. We will assume throughout that \( k \) is an algebraically closed field of characteristic \( \neq 2, 3. \)

Let \( W \) be a vector space over \( k \) and let \( S = \bigoplus_{k=0}^{\infty} \text{Sym}^k(W) \) be the symmetric algebra. Thus \( k \) is the quotient of \( S \) by the irrelevant ideal. Let \( M = \bigoplus_{q=0}^{\infty} M^q \) be a graded \( S \) module. Define

\[
 K_{p,q} = \bigwedge^p(W) \otimes M^q \tag{5}
\]

and let

\[
 \delta : K_{p,q} \longrightarrow K_{p-1,q+1} \tag{6}
\]
be defined by
\[ \delta(w_1 \wedge \cdots \wedge w_p \otimes m) = \sum_{i=1}^{p} (-1)^i (w_1 \wedge \cdots \wedge w_{i-1} \wedge w_{i+1} \cdots w_p) \otimes (w_i m) \] (7)

One checks that \( \delta^2 = 0 \) by a direct computation. Therefore we get a complex and we can compute cohomology.

**Definition 9.1.** The Koszul cohomology group \( K_{p,q}(S,M) \) is the cohomology of the complex:

\[ K_{p+1,q-1} \xrightarrow{\delta} K_{p,q} \xrightarrow{\delta} K_{p-1,q+1} \]

The most common case is when \( W = H^0(C,L) \) and \( M^q = H^0(C,L \otimes q) \). In that case we will denote the Koszul cohomology group as \( K_{p,q}(C,L) \). Koszul cohomology groups are useful because they can be used to compute free resolutions of graded modules over S. For completeness we include a brief description of this phenomena. The information can be found in any of the sources mentioned above, as well as Eisenbud's book 'Commutative Algebra' [6].

**Definition 9.2.** A free resolution \( F^\bullet \) of \( M \) is an exact sequence of the form:

\[ 0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0 \]

with the \( F_i \) free S modules.

The maps \( F_{i+1} \rightarrow F_i \) are given by a matrix of homogenous polynomials, call it \( f_i \). The most important case is when the maps are given by non constant polynomials.

**Definition 9.3.** A free resolution is said to be minimal if all the matrices \( f_i \) contain no non-zero constant terms.

The two basic facts are that free resolutions always exist and are (essentially unique). Many important properties of a module \( M \), and geometry of the curve embedded in \( \mathbb{P}(W^*) \) can be read off from the minimal free resolution. For example, \( M_0 = S \) is equivalent to the curve being linearly normal. If we write \( M_1 = \sum_{i=1}^{n} S(-a_i) \) then the \( a_i \)'s represent the degrees of the minimal generators of the ideal of \( C \) in the projective embedding. The connection between the Koszul complexes and the minimal free resolution comes as follows.

Let \( M = \sum_{i=0}^{\infty} M_i \) be a graded S module with a free resolution \( F^\bullet \) with \( F_p = \sum q (V_{p,q} \otimes S(-q)) \): ie \( V_{p,q} \) is a vector space that keeps track of how many \( S(-q) \)'s appear in \( F_p \).
Theorem 9.1. \( \dim(K_{p,q}(S, M)) = \dim(V_{p,p+q}) \)

Proof. The proof of this is well known and in all the above sources. I will sketch it for the sake of completeness. Firstly one has the Koszul resolution of \( k \):

\[
0 \rightarrow \bigwedge^n(V) \otimes S \rightarrow \ldots \rightarrow V \otimes S \rightarrow S \rightarrow k \rightarrow 0
\]

One can tensor this resolution with \( M \) and one sees by inspection that the maps are the boundary maps \( \delta \) from equation 7. Hence one has \( K_{p,q}(S, M) = \text{Tor}_{p+q}^p(M, k) \). On the other hand, we can take the free resolution \( F^\bullet \) of \( M \) and tensor it with \( k \) (viewed as the quotient of \( S \) by its maximal ideal). By the commutivity of \( \text{Tor} \), this gives the same answer as above. On the other hand, since \( F^\bullet \) is minimal, when we tensor with \( k \) all the boundary maps are zero and hence we get \( V_{p,p+q} \otimes S \) is the degree \( p+q \) piece of \( F_p \).

To relate our work to Koszul cohomology we introduce some new notation. This is needed as we are going to work with the dual of the Koszul complex. This is the complex which, term by term, is the vector space dual of the Koszul complex. There is another notion of a dual Koszul complex which involves divided powers (see the appendix to [6] for details). As long as \( \text{char}(k) \) is sufficiently large, these are the same. The reason we do this is that our Shiffer variations live naturally in the dual of the Koszul complex we describe.

Let \( V = W^*, \ W = H^0(C, L) \) where \( L \) is a very ample line bundle. We denote by \( M_t \) the dual of \( H^0(C, L^t) \) so that \( M_1 = V \).

We also assume that the line bundle \( L \) is quadratically normal which means that \( M_3 \hookrightarrow \text{Sym}^2(V) \) is injective. We frequently consider \( \varphi \in M_2 \) as an element of \( \text{Sym}^2(V) \) or equivalently as an element of \( \text{Hom}^{\text{Sym}}(W, V) \).

The Koszul cohomology group can be calculated as the cohomology of:

\[
\bigwedge^{p+1}(W) \otimes H^0(C, L) \longrightarrow \bigwedge^p(W) \otimes H^0(C, L^2) \longrightarrow \bigwedge^{p-1}(W) \otimes H^0(C, L^3)
\]

and so the dual cohomology groups are calculated by the complex:

\[
M_3 \otimes \bigwedge^{p-1}(V) \xrightarrow{\delta_{p-1}} M_2 \otimes \bigwedge^p(V) \xrightarrow{\delta_p = \delta} M_1 \otimes \bigwedge^{p+1}(V)
\]

We are generally only interested in determining if these groups are zero or non-zero, so calculating the dual group is good enough. Any \( v = \varphi \otimes \lambda \in M_2 \otimes \bigwedge^p(V) \) can be viewed as an element of \( \text{Hom}(\bigwedge^p(W), M_2) \) and \( \delta(v) \) as an element of \( \text{Hom}(\bigwedge^{p+1}(W), V) \). If \( w \in \bigwedge^{p+1}(W) \) then \( \delta(v)(w) \) is calculated,
by first contracting $\lambda \wedge w$ via the standard map $\Lambda^p(V) \otimes \Lambda^{p+1}(W) \to W$ (recall that $W$ and $V$ are dual) and then letting $\varphi$ act on $\lambda \wedge \nu$.

We start out with the basic observation:

\textbf{Lemma 9.2.} Let $\varphi \in M_2$ and let $\lambda \in \Lambda^p(V)$ be decomposable. If $\text{Im}(\varphi) \subset \lambda$, then $\delta(\varphi \otimes \lambda) = 0$.

\textbf{Proof.} By $\lambda$ decomposable we mean that $\lambda = v_1 \wedge \cdots \wedge v_p$ and hence defines a subspace $V_p \subset V$. We mean that viewing $\varphi$ as an element of $\text{Hom}(W,V)$, that $\varphi(W) \subset V_p$. First extend $v_1, \ldots, v_p$ to a basis $\langle v_1 \ldots v_n \rangle$, of $V$. Let $w_1, \ldots, w_p, \ldots, w_n$ be a dual basis so that $v_i(w_j) = \delta_{ij}$. First note that because $\varphi$ is symmetric, $\ker(\varphi) \supset \langle w_{p+1} \ldots w_n \rangle$. Further, one has $v_1 \wedge \ldots \wedge v_p \wedge w_j = 0$ (under the standard contraction) if $j > p$, and so $\lambda \wedge w_1 \cdots \wedge w_n = 0$ unless $\langle w_1, \ldots, w_n \rangle = \langle w_1 \ldots w_p, w_j \rangle$ with $j > p$. Then $\lambda \wedge \Lambda^{p+1}(W) \subset W_{>p}$, the subspace of $W$ generated by $\langle w_{p+1} \ldots w_n \rangle$ and hence is killed by $\varphi$. \hfill $\Box$

Since $C \subset \mathbb{P}(M_3)$ is non degenerate, $C$ generates $M_3$. That means we can construct a basis of $M_3$ of the form $\sigma_{p_1}^3 \cdots \sigma_{p_m}^3$, where the $p_i$ are points of $C$ and $m = h^0(M^{\otimes 3})$. From this it follows that $\partial(M_3 \otimes \Lambda^{p-1}(V))$ is generated by elements of the form $\sigma_{p_i}^2 \otimes \lambda \wedge \sigma_{p_i}$ with $\lambda \in \Lambda^{p-1}(V)$. Any element of $\partial(M_3 \otimes \Lambda^{p-1}(V))$ is then a sum of elements of the form $(\sum_{i=1}^{k \leq p} a_i \sigma_{p_i}^2) \otimes \sigma_{p_i} \wedge \cdots \wedge \sigma_{p_k} \wedge \lambda$ where $k \leq p$ and $\lambda \in \Lambda^{p-k}(V)$. Thus any element of $\partial(M_3 \otimes \Lambda^{p-1}(V))$ can be written as a sum, $\sum \varphi_i \otimes \lambda_i$ where not only $\text{rk}(\varphi) \leq p$ but in fact $\varphi \in \text{Sec}^{p-1}(C)$. We do not claim that this representation is unique, that is to say the $\lambda_i$ are not necessarily a basis for $\Lambda^p(V)$. Nonetheless, this means that Shiffer variations that live in $R^d$ but not $\text{Sec}^{d-1}(C)$ are candidates to give non-trivial Koszul cohomology classes. To fix ideas we consider the case of $L = K_C$.

\textbf{Theorem 9.3.} Let $D$ compute $\text{Cliff}(C)$ and let $\varphi \in T_{K_C}(D)$ be a Shiffer variation in $\text{Sec}^{d-1}(C)/\text{Sec}^{d-2}(C)$ with $\text{rk}(\tau) = \text{Cliff}(C)$. Let $\text{Im}(\varphi) = \langle \sigma_1, \ldots, \sigma_c \rangle \subset V$ and set $\sigma = \sigma_1 \wedge \cdots \wedge \sigma_c$. Then $\varphi \otimes \sigma$ represents a non-trivial Koszul cohomology class in $K_{p,2}(C, K_C)$.

\textbf{Proof.} By construction $\delta(\varphi \otimes \sigma) = 0$ since $\sigma = \text{Im}(\varphi)$. The incoming boundary map is from $M_3 \otimes \Lambda^{p-1}(V)$. Since $C$ is embedded in $\mathbb{P}(M_3)$ by the complete linear system $|3K_C|$, the points of $C$ span $\mathbb{P}(M_3)$ and hence we can find a basis of $M_3$ consisting of $\sigma_{p_i}^3$, with $p_i \in C$. That is to say $\sigma_{p_i}^3$ represents the point $p_i \in C$. Up to a scalar (we are not in characteristic 3!) $\varphi \otimes \sigma = \delta(\sum_{i=1}^{c} (-1)^i \sigma_{p_i}^3 \otimes \sigma_1 \wedge \cdots \sigma_{i-1} \wedge \sigma_{i+1} \cdots \wedge \sigma_c)$. This element cannot
lie in \( M_3 \otimes \Lambda^{p-1}(V) \) as \( \sigma_i^q \notin M_3 \) because by Hassett’s criteria (see Theorem 4.5), the only rank one elements of \( \mathbb{P}(M_3) \) are in \( C \) and \( C \cap (\sigma_1 \ldots \sigma_c) = \emptyset \). Adding a coboundary in \( M_4 \otimes \Lambda^{p-2}(V) \) cannot produce an element that lives in \( M_3 \).

We comment on how this compares to the construction of Green and Lazarsfeld in the appendix to [8]. Their idea is that if one can factor a line bundle \( L \) as \( L_1 \otimes L_2 \), where \( h^0(L_1) = r_1 + 1 \) and \( h^0(L_2) = r_2 + 1 \) with \( r_i \geq 1 \) then one can produce a non-trivial class in \( K_{r_1+r_2-1,1}(C,L) \).

There is a duality of Koszul cohomology groups and for \( L = K_C \) the group \( K_{g-p-2,1} \) is dual to \( K_{p,2} \). If \( L_1 = O_C(D) \), then \( L_2 = K_C(-D) \) and by Riemann-Roch \( r_2 = g - d + r_1 - 1 \) so that \( r_1 + r_2 - 1 = g - d + 2r_1 - 2 = g - \text{Cliff}(D) - 2 \). If \( D \) computes \( \text{Cliff}(C) = c \) the cohomology class lives in \( K_{g-c-2,1}(C,L) \) which is dual to \( K_{c,2}(C,L) \). Their construction amounts to the following: if \( s_i \in H^0(C,L_i) \) corresponds to a divisor \( D_i \) then the linear space corresponding to \( D_1 \cap D_2 \) is used to construct the cohomology class in \( K_{r_1+r_2-1,1}(C,L) \). Eisenbud ([9] ch.8) has given a different version of their construction.

The class I have constructed also lies in a group dual to \( K_{c,2} \). This is the situation of Theorem 9.3. Let \( D \) be a divisor used to compute \( \text{Cliff}(C) \). If \( H^0(O_C(D)) = \{f_0, \ldots, f_r\} \) and \( H^0(K_C(D)) = \{\eta_1, \ldots, \eta_{g-d+r}\} \) then we constructed a Shiffer variation which vanished on \( \{f_0 \otimes \eta_i\} \) and \( \{f_i \otimes \eta_i\} \) which is exactly the intersection of the linear spaces spanned by \( D \) and \( K_C(-D) \). By this I mean that the \( \{f_i\} \) generate the ideal of the linear space corresponding to \( K_C(-D) \) and the \( \{\eta_i\} \) generate the ideal of the linear space corresponding to \( D \).

**Remark 1** Suppose \( C \) is a generic curve of genus 5 so that \( \text{Cliff}(C) = 2 \) and \( C \) has a \( g_1^1 \) which computes which computes \( \text{Cliff}(C) \), call it \( D \). By Riemann-Roch, \( h^0(O(D)) = h^0(K_C(-D)) = 2 \). Let \( \{1, f\} \) be a basis for \( H^0(O(D)) \) and let \( \{\eta, \omega\} \) be a basis for \( H^0(K_C(-D)) \). By a slight abuse of notation (suppressing the tensor signs) we get four elements of \( H^0(K_C) \) which we denote by \( \{1, f, \eta, \omega\} \) and further we have that \( \{1, f, \eta\} \) are linearly independent in \( H^0(K_C) \). Let \( p_1, \ldots, p_4 \in C \) be the elements of \( D \) and denote by \( \tau_i \) the corresponding Shiffer variations. Then by Theorem 2.5 some sum, \( \tau = \sum a_i \tau_i \) with \( a_i \neq 0 \) contains \( \{1, f, \eta\} \) in its kernel, which is to say it is of rank one. Hassett’s criteria (see 4.5) says such a \( \tau \) corresponds to a line \( L \subset \mathbb{P}^4 \) such that \( C \cap L = \emptyset \) and \( H^0(I_C(2)) \rightarrow H^0(O_L(2)) \) is not surjective. Notice that \( L = I(1,f,\eta) \) and the extra quadric in \( H^0(I_C(2)) \) that vanishes on \( L \) is \( 1 \times \omega - f \times \eta \). Eisenbud’s version of the Green-Lazarsfeld construction produces a class in \( K_{1,1}(C,K_C) = H^0(I_C(2)) \) and
this class is by construction the quadric, $1 \times \omega - f \times \eta$. It appears that the two constructions are connected.

**Remark 2** Notice how the formula: $h^0(O_C(D)) + h^0(K_C(-D)) = g + 1 - \text{Cliff}(D)$ comes into play in these constructions. In essence, whenever we factor $K_C$ as $O_C(D) \otimes K_C(-D)$, we should expect to find a linear subspace of dimension = Cliff(D) and a Shiffer variation supported on this subspace. Our construction will produce such a Shiffer variation and hence such a cohomology class as long as one of the line bundles is base point free. We have discussed this more extensively in the section line bundles with $h^1(L) = 1$. As long as that is the case, we can non-trivial Koszul cohomology classes. They should be essentially the same as the classes constructed by Green and Lazarsfeld.

**Remark 3** The construction of [8] is not the only way to construct non-trivial classes. For example [1] a more general construction is presented. In all these cases the natural location of these classes is in a group of the form $K_{p,1}$. Our classes always live in $K_{p,2}$. Thus all previous methods speak to the length of the linear strand of the minimal free resolution of $S_C$, whereas our methods possibly give information on where the quadratic strand may start. Recall that the linear strand of a variety (defined by quadrics) is the piece in degree $p$ composed of $O(-p - 1)$’s and the quadratic strand is the piece composed of $O(-p - 2)$’s. In the special case $L = K_C$ there is duality between $K_{p,2}$ and $K_{p-p-2,1}$ which doesn’t exist for other line bundles allows one to translate results about the linear strand to results about the quadratic strand.

We also do not always require that the bundle factor as the tensor product of two bundles, both of which have at least 2 sections. This is necessary for the construction of Green and Lazarsfeld. For example, let $L = K_C(D)$ where $D$ is effective. Then by Theorem 3.6 $\text{Cliff}(C,L) = d - 2$ and by 5.4 there exists a $\tau \in T_L(D)$ of rank $d - 2$ such that $\tau \in \text{Sec}^{d-1}$ but $\tau \notin \text{Sec}^{d-2}$. If $D$ is general of with deg$(D) < g$, then $H^0(O_C(D)) = 0$ so the method of Green and Lazarsfeld doesn’t produce anything interesting. However:

**Theorem 9.4.** The class $\tau$ produced above gives a nontrivial cohomology class in $K_{d-2,2}$.

**Proof.** The proof goes exactly as in Theorem 9.3. Namely by Hassett’s criteria, the $d-2$ plane, $Im(\tau)$ cannot meet $C$. This means for any expression $\tau = \sum_{i=1}^{i=d-2} t_i^2$, the $t_i^3$ cannot lie in $M_3$ and hence $\tau$ cannot be a boundary.

The final issue to be discussed in this thesis is the relationship between
our theorems and Green’s conjecture. Because $K_{0,2}(C, K_C) = 0$ if and only if $\text{Cliff}(C) > 0$ and if $K_{0,2}(C, K_C) = 0$, then $K_{1,2}(C, K_C) = 0$ iff $\text{Cliff}(C) > 1$, Green conjectured that $K_{p,2}(C, K_C) = 0$ for $p < \text{Cliff}(C)$. There has been a lot of progress on this issue. After partial results mainly by Schreyer (see [14] and [15]), Voisin proved Green’s conjecture for a generic curve in [24] and [26]. Needless to say her techniques are very different from the ones of this thesis. In particular she uses the duality between $K_{p,2}$ and $K_{g-p-2,1}$ which is particular to the case of $L = K_C$. She then proves the vanishing on a specific curve. Generic vanishing follows by semi-continuity. The techniques of this thesis work for a larger class of line bundles and prove results that hold for all curves carrying such a line bundle. However the results given here do not seem to prove Green’s conjecture on the nose.

We do have a positive result. Suppose $p < \text{Cliff}(C, L)$. We consider the dual of $K_{p,2}(C, L) = 0$. Recall that the dual cohomology group is the cohomology of the complex:

$$
M_3 \otimes \bigwedge^{p-1} (V) \rightarrow M_2 \otimes \bigwedge^p (V) \rightarrow M_1 \otimes \bigwedge^{p+1} (V)
$$

**Theorem 9.5.** Suppose that $\varphi \otimes \lambda \in M_2 \otimes \bigwedge^p (V)$ with $\lambda$ decomposable and that $\delta(\varphi \otimes \lambda) = 0$, then $\varphi \otimes \lambda = \delta(\mu)$ for some $\mu \in M_3 \otimes \bigwedge^{p-1} (V)$. That is to say, in the group dual to the Koszul cohomology group, any decomposable element is trivial.

**Proof.** Write $\lambda = \sigma_1 \wedge \cdots \wedge \sigma_p$. We first claim that $\text{im}(\varphi) \subset \langle \sigma_1 \ldots \sigma_p \rangle$. If not let $\tau \in \text{im}(\varphi)$ be such that $\tau \wedge \lambda \neq 0$ and let $\tau \vee \wedge \lambda \vee$ be dual to this element. Then $\delta(\varphi \otimes \lambda)(\tau \vee \wedge \lambda \vee) = \varphi(\tau \vee) \neq 0$.

Now since $\text{rk}(\varphi) \leq p$ we can write $\varphi = \sum_{i=1}^k \sigma_{p_i}^2$ where $k \leq p$ and $\sigma_{p_i}^2$ is a Shiffer variation associated to the point $p_i \in C$. Set $\mu = \frac{1}{2} \sum_{i=1}^k (-1)^{k-1} \sigma_i \otimes \sigma_1 \wedge \cdots \wedge \sigma_{i-1} \wedge \sigma_{i+1} \cdots \wedge \sigma_k$. Then $\mu \in M_3 \otimes \bigwedge^{p-1} (V)$ and $\delta(\mu) = \varphi \otimes \lambda$. \[\square\]

At this point we are left with the words of Ludwig Bemelmans, “And thats all there is– there isn’t anymore”.

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