Cubic Pencils and Painlevé Hamiltonians

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Abstract We present a simple heuristic method to derive the Painlevé differential equations from the corresponding geometry of rational surfaces. We also give a direct relationship between the cubic pencils and Seiberg-Witten curves.

1. Introduction

For each Painlevé equation, there exists an associated rational surface called the “space of initial conditions”. This surface was introduced by Okamoto\textsuperscript{1}, and further studied by Takano and his collaborators. By the work of Sioda and Takano\textsuperscript{2}, the corresponding Painlevé equation was characterized as the unique Hamiltonian system satisfying certain holomorphy properties on the surface. Hence, in principle, one can recover the Painlevé equations from geometry.

This geometric approach to the Painlevé equations has been extended to the difference (or discrete) cases, from which the difference Painlevé equations (and their Bäcklund transformations) arise naturally as Cremona automorphisms of the surfaces\textsuperscript{3}. Compared with the difference cases, however, the way how the differential Painlevé equations appear is rather indirect. The known method used so far to recover the differential Painlevé equations from geometry is either to take suitable continuous limit of discrete ones or to employ a deformation theory\textsuperscript{4}. The aim of this note is to present yet another way, which is heuristic but much simpler.

The main idea of our method is to use cubic pencils. In our previous work\textsuperscript{5}, it is clarified that the cubic pencils play the essential role in the discrete Painlevé equation. It is natural to expect that they are also important in the differential Painlevé equations. Indeed, we find that the cubic pencils are directly related to the symplectic forms and Hamiltonians.

In Section 2, we explain our method in the case of the sixth Painlevé equation $P_{VI}$. All the other degenerate cases are treated in Section 3. Finally, a relation of our cubic pencils and the Seiberg-Witten curves are discussed in Appendix A.

2. Procedure to obtain Hamiltonian

In this section, using the sixth Painlevé equation $P_{VI}$ as an example, we explain a procedure to obtain the symplectic 2-form $\omega$ and the Hamiltonian $H$ from the datum of the surface: the configuration of nine points on $\mathbb{P}^2$. The parameterization of the points is borrowed from\textsuperscript{3}.

Case $P_{VI}$: (Fig\textsuperscript{4}, Add 4)

The configuration of the nine points for $P_{VI}$ is given as follows,

$P_1 = (0 : 1 : 0)$, $P_2 = (1 : -a_2 : 1)$, $P_3 = (1 : -a_1 - a_2 : 1)$,
$P_4 = (0 : 0 : 1)$, $P_5 = (0 : a_3 : 1)$, $P_6 = (1 : 0 : 0)$,
$P_7 = (1 : a_4 : 0)$, $P_8 = ((s - 1)\epsilon : 1 : s\epsilon)$, $P_9 = ((s - 1)\epsilon : 1 : s\epsilon - sa_0\epsilon^2)$. 

This cubic determines the symplectic form \( \omega \)

\[ \omega \] 

\[ \omega_{G}(4) \] 

\[ C_{P}(6) \] 

Similarly, \( \lambda F \) through the nine points forms a pencil (one parameter family) Fig. 2.

In terms of the canonical variables \( f, g \), the pencil equation \( \lambda F + \mu G = 0 \) can be written as \( \lambda H + \mu = 0 \) where

\[ H = f(f-1)(f-s)g^2 + [(a_1 + 2a_2)(f-1)f + a_3(s-1)f + a_4s(f-1)]g + a_2(a_1 + a_2)(f-s). \]

Note that the choice of \( F \) involves the ambiguity such as \( F \to c_1 F + c_2 G \) where \( c_1, c_2 \) are constants. This ambiguity, however, results only in changing \( H \) as \( H \to c_1 H + c_2 \).
At this stage, we drop the condition \( \delta = 0 \) by hand. We recognize then that \( H \) is a Hamiltonian for \( P_{VI} \), namely

**Theorem 2.1.** With the above Hamiltonian \( H \), the system of differential equation

\[
D_t f = \frac{\partial H}{\partial g}, \quad D_t g = -\frac{\partial H}{\partial f}, \quad D_t = s(s-1) \frac{d}{dt}, \quad \frac{ds}{dt} = \delta,
\]

gives a Hamiltonian form of the sixth Painlevé equation \( P_{VI} \):

\[
\frac{d^2 f}{dt^2} = \frac{1}{2} \left( \frac{1}{f} + \frac{1}{f-1} + \frac{1}{f-s} \right) \left( \frac{df}{dt} \right)^2 - \delta \left( \frac{1}{s} + \frac{1}{s-1} + \frac{1}{f-s} \right) \frac{df}{dt} \\
+ \frac{f(f-1)(f-s)}{s^2(s-1)^2} \left( \frac{a_1^2}{2} - \frac{a_2^2}{2} + \frac{a_3^2}{2} + \frac{s-1}{(f-1)^2} + \frac{\delta^2-a_0^2 s(s-1)}{2} \right).
\]

In the next section, we will show similar results for all the cases in Table 1.

3. Degenerate cases

In this section, we consider the degenerate cases Add 5-11 in [3]. The constructions are essentially the same as the previous section (Add 4) and we give only the relevant data.

**Case** \( P_V \): (Fig 3 Add 5)

Condition for the cubic: \( F(P_i) = 0 \) (\( i = 3, 4, 5, 6, 7 \)) and \( F(P_{1289}) = 0 \),

\[
F(1, -a_2, 1) = F(0, 0, 1) = F(0, a_1, 1) = F(1, 0, 0) = F(1, a_3, 0) = 0,
\]

\[
F(\epsilon, 1, \epsilon + se^2 + s(s-a_0)e^3) = O(\epsilon^4).
\]

Pencil: \( \lambda F + \mu G = 0 \), (\( \delta = a_0 + a_1 + a_2 + a_3 = 0 \))

\[
F = a_3x^2y - xy^2 - a_2sx^2z + (a_1 - a_3 - s)xyz + y^2z - a_1yz^2,
\]

\[
G = xz(z - x).
\]

\(^1\)For the autonomous case (\( \delta = 0 \)) the pencil is invariant.
| Painlevé eq. | Sakai’s list | configuration | symmetry |
|-------------|--------------|---------------|----------|
| $P_{VI}$    | Add 4        | $D_4^{(1)}$(Fig 1) | $D_4^{(1)}$ |
| $P_{V}$     | Add 5        | $D_5^{(1)}$(Fig 3) | $A_3^{(1)}$ |
| $P_{III}^{D_{3}}$ | Add 6 | $D_6^{(1)}$(Fig 4) | $(2A_1)^{(1)}$ |
| $P_{II}^{D_{3}}$ | Add 7 | $D_7^{(1)}$(Fig 5) | $A_1^{(1)}$ |
| $P_{III}^{D_{5}}$ | Add 8 | $D_8^{(1)}$(Fig 6) | $E_2$ |
| $P_{IV}$    | Add 9        | $E_6^{(1)}$(Fig 7) | $A_2^{(1)}$ |
| $P_{II}$    | Add 10       | $E_7^{(3)}$(Fig 8) | $A_1^{(1)}$ |
| $P_{I}$     | Add 11       | $E_8^{(3)}$(Fig 9) | $-$ |

**Table 1.** The Painlevé equations

![Diagram](image)

**Figure 3.** Configuration for $P_{V}$

Hamiltonian $H$ and canonical variables $f, g$:

$$H = f(f-1)(g+s) - (a_1 + a_3)fg + a_1g + a_2sf,$$

$$f = \frac{x}{x-z}, \quad g = \frac{y(x-z)}{xz}. \tag{14}$$

**Theorem 3.1.** With the above Hamiltonian $H$, the system of differential equation

$$D_t f = \frac{\partial H}{\partial g}, \quad D_t g = -\frac{\partial H}{\partial f}, \quad D_t = s \frac{d}{dt}, \quad \frac{ds}{dt} = \delta. \tag{15}$$

gives a Hamiltonian form of the fifth Painlevé equation $P_{V}$: \((y = 1-1/f)\)

$$\frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right) \left(\frac{dy}{dt}\right)^2 - \frac{\delta}{s} \frac{df}{dt}$$

$$+ \frac{(y-1)^2}{s^2} \left(\frac{a_1^2}{2y} - \frac{a_2}{2y} \right) + (a_0 - a_2)\frac{y}{s} - \frac{1}{2} \frac{y(y+1)}{y-1}. \tag{16}$$

**Case $P_{III}^{D_{3}}$** (Fig 11 Add 6)

Condition for the cubic: $F(P_i) = 0$ \((i=4, 5, 6, 7)\) and $F(P_{12389}) = 0$,

$$F(0,0,1) = F(0,a_1,1) = F(1,0,0) = F(1,b_1,0) = 0, \tag{17}$$

$$F(\epsilon, 1, \epsilon + se^3 + s(b_1 - a_0)e^4) = O(\epsilon^5).$$

Pencil: $\lambda F + \mu G = 0, (\delta = a_0 + a_1 = 0)$

$$F = -b_1 x^2 y + xy^2 + sx^2 z + (b_1 - a_1)xyz - y^2 z + a_1 y z^2, \quad G = xz(x-z). \tag{18}$$
Hamiltonian $H$ and canonical variables $f, g$:

\[ H = f^2 g^2 + \left[ f^2 - (a_1 + b_1) f - s \right] g - a_1 f, \]

\[ f = y \frac{z-x}{xz}, \quad g = \frac{x}{z-x}. \]

**Theorem 3.2.** With the above Hamiltonian $H$, the system of differential equation

\[ D_t f = \frac{\partial H}{\partial g}, \quad D_t g = -\frac{\partial H}{\partial f}, \quad D_t = s \frac{d}{dt}, \quad \frac{ds}{dt} = \delta, \]

gives a Hamiltonian form of the third Painlevé equation $P_{III}^{(1)}$:

\[ \frac{d^2 f}{dt^2} = \frac{1}{f} \left( \frac{df}{dt} \right)^2 - \delta \frac{df}{dt} + \frac{f^2}{s^2} (f + a_1 - b_1) - \frac{1}{f} - \frac{a_0 + 2a_1 + b_1}{s}. \]

**Case $P_{III}^{(1)}$:** (Fig. 5 Add 7)

Condition for the cubic: $F(P_{46}) = F(P_{57}) = F(P_{12389}) = 0,$

\[ F(\epsilon, 0, 1) = O(\epsilon^2), \quad F(\epsilon, 1 + a_1 \epsilon, 1) = O(\epsilon^2), \quad F(\epsilon, 1, s \epsilon^3 - a_0 s \epsilon^4) = O(\epsilon^5). \]

Pencil: $\lambda F + \mu G = 0,$ $(\delta = a_0 + a_1 = 0)$

\[ F = -s x^3 - a_1 x y z + y^2 z - y z^2, \quad G = x^2 z. \]

Hamiltonian $H$ and canonical variables $f, g$:

\[ H = f^2 g^2 + (a_1 f + s) g - f, \]

\[ f = \frac{yz}{x^2}, \quad g = -\frac{x}{z}. \]

**Theorem 3.3.** With the above Hamiltonian $H$, the system of differential equation

\[ D_t f = \frac{\partial H}{\partial g}, \quad D_t g = -\frac{\partial H}{\partial f}, \quad D_t = s \frac{d}{dt}, \quad \frac{ds}{dt} = \delta, \]
gives a Hamiltonian form of the third Painlevé equation \( P_{III} \):

\[
\frac{d^2 f}{dt^2} = \frac{1}{f} \left( \frac{df}{dt} \right)^2 - \frac{\delta}{s} \frac{df}{dt} + 2 \frac{f^2}{s^2} - \frac{1}{f} + \frac{a_0}{s}.
\]

Case \( P_{III} \): (Fig. 6 Add 8)

| 12389 | 7 | 12 | 9 |
|-------|---|----|---|
| 4567  |   |    |   |

**Figure 6.** Configuration for \( P_{III} \)

Condition for the cubic: \( F(P_{4567}) = F(P_{12389}) = 0 \),

\[
F(\epsilon^2, \epsilon, 1) = O(\epsilon^4), \quad F(\epsilon, 1, se^3 - ase^4) = O(\epsilon^5).
\]

Pencil: \( \lambda F + \mu G = 0 \), \( \delta = a = 0 \)

\[
F = -sx^3 + y^2z - xz^2, \quad G = x^2z.
\]

Hamiltonian \( H \) and canonical variables \( f, g \):

\[
H = f^2g^2 - f - \frac{s}{f}, \quad f = \frac{z}{x}, \quad g = \frac{y}{z}.
\]

**Theorem 3.4.** With the above Hamiltonian \( H \), the system of differential equation

\[
D_t f = \frac{\partial H}{\partial g}, \quad D_t g = -\frac{\partial H}{\partial f}, \quad D_t s = \frac{ds}{dt}, \quad \frac{ds}{dt} = \delta,
\]

gives a Hamiltonian form of the third Painlevé equation \( P_{III} \):

\[
\frac{d^2 f}{dt^2} = \frac{1}{f} \left( \frac{df}{dt} \right)^2 - \frac{\delta}{s} \frac{df}{dt} + 2 \frac{f^2}{s^2} - \frac{2}{s}.
\]

Case \( P_{IV} \): (Fig. 7 Add 9)

| 12789 | 12 | 123 | 78 |
|-------|----|-----|----|
| 4 | 36 |     |    |

**Figure 7.** Configuration for \( P_{IV} \)

Condition for the cubic: \( F(P_4) = F(P_5) = F(P_{36}) = F(P_{12789}) = 0 \),

\[
F(0, 0, 1) = F(0, a_1, 1) = 0, \quad F(1, -a_2 \epsilon, \epsilon) = O(\epsilon^2), \quad F(\epsilon, 1, \epsilon^2 + se^3 + (s^2 - a_0)\epsilon^4) = O(\epsilon^5).
\]
Pencil: \( \lambda F + \mu G = 0 \), \((\delta = a_0 + a_1 + a_2 = 0)\)

\[ F = -x^2y - a_2x^2z - sx y z + y^2z - a_1yz^2, \quad G = xz^2. \]  

Hamiltonian \( H \) and canonical variables \( f, g \):

\[ H = fg(g - f - s) - a_2f - a_1g, \]

\[ f = \frac{x}{z}, \quad g = \frac{y}{x}. \]

**Theorem 3.5.** With the above Hamiltonian \( H \), the system of differential equation

\[ D_t f = \frac{\partial H}{\partial g}, \quad D_t g = -\frac{\partial H}{\partial f}, \quad D_t = \frac{d}{dt}, \quad \frac{ds}{dt} = \delta, \]

gives a Hamiltonian form of the fourth Painlevé equation \( P_{IV} \):

\[ \frac{d^2 f}{dt^2} = 1 + 2f^3 + 3f^2 + \frac{1}{2} [s^2 + 2(a_2 - a_0)] f - \frac{a_2^2}{2f}. \]

**Case** \( PI \): (Fig. 8, Add 10)

**Figure 8.** Configuration for \( P_I \)

Condition for the cubic:

\[ F(P_4) = F(P_5) = F(P_{1236789}) = 0, \]

\[ F(0, 0, 1) = F(0, a_1, 1) = 0, \quad F(\epsilon, 1, \epsilon^3 - se^5 - a_0e^6) = O(e^7). \]

Pencil: \( \lambda F + \mu G = 0 \), \((\delta = a_0 + a_1 = 0)\)

\[ F = x^3 - sx^2z - y^2z + a_1yz^2, \quad G = xz^2. \]

Hamiltonian \( H \) and canonical variables \( f, g \):

\[ H = g^2 + (f^2 + s)g + a_1f, \quad f = \frac{y}{x}, \quad g = -\frac{x}{z}. \]

**Theorem 3.6.** With the above Hamiltonian \( H \), the system of differential equation

\[ D_t f = \frac{\partial H}{\partial g}, \quad D_t g = -\frac{\partial H}{\partial f}, \quad D_t = \frac{d}{dt}, \quad \frac{ds}{dt} = \delta, \]

gives a Hamiltonian form of the second Painlevé equation \( P_{II} \):

\[ \frac{d^2 f}{dt^2} = 2f^3 + 2sf + (a_0 - a_1). \]

**Case** \( PI \): (Fig. 9, Add 11)

Condition for the cubic:

\[ F(P_{123456789}) = 0, \]

\[ F(\epsilon, 1, \epsilon^3 + se^5 + ae^8) = O(e^9). \]

Pencil: \( \lambda F + \mu G = 0 \): \((\delta = a = 0)\)

\[ F = -x^3 + y^2z - sxz^2, \quad G = z^3. \]

Hamiltonian \( H \) and canonical variables \( f, g \):

\[ H = g^2 - f^3 - sf, \quad f = \frac{x}{z}, \quad g = \frac{y}{z}. \]
Theorem 3.7. With the above Hamiltonian $H$, the system of differential equation
\begin{equation}
D_t f = \frac{\partial H}{\partial g}, \quad D_t g = -\frac{\partial H}{\partial f}, \quad D_t = \frac{d}{dt}, \quad ds = \delta,
\end{equation}
gives a Hamiltonian form of the first Painlevé equation $P_I$:
\begin{equation}
\frac{d^2 f}{dt^2} = 6f^2 + 2s.
\end{equation}

Appendix A. Relation to Seiberg-Witten curves

It may be interesting to note that the cubic pencils we considered in this paper are directly related with the Seiberg-Witten curves appearing in the $N = 2$ supersymmetric gauge theory with SU(2) gauge group. The following is the Seiberg-Witten curves given in [6] and [7] (with some parameters rescaled).
\begin{align}
D_8 & : y^2 = x^3 - ux^2 + 2\Lambda^2_1x,
D_7 & : y^2 = x^3 - 2ux - 2Mu + M^3 - 4m_1^3x,
D_6 & : y^2 = x^3 - 2(Mu + c_2)x - u^2 - \frac{M^3}{3}u + \frac{M^6}{108} - \frac{2M^2}{3}c_2 + \frac{8}{3}c_3,
\end{align}
c_k = m_1^2 + m_2^2 + m_3^2 \quad (c_1 = 0).

The correspondence between our cubic pencils and the above Seiberg-Witten curves is a direct consequence of their definition/construction [8] [9] [10]. In fact, by comparing the Weierstrass canonical form of both curves, the relations of the parameters are explicitly determined as in Table 2.
| Painlevé | SW curve | Relation of parameters |
|---------|----------|------------------------|
| $P_I$   | $E_8$    | $s = -2M$              |
| $P_{II}$| $E_7$    | $a_1 = 4m_1, \quad s = -3M$ |
| $P_{IV}$| $E_6$    | $a_1 = 2(m_1 - m_2), \quad a_2 = 2(m_2 - m_3), \quad s = 2M$ |
| $P_{III}^{D_8}$ | $D_8$ | $s = 2\Lambda_0^4$ |
| $P_{III}^{D_7}$ | $D_7$ | $a_1 = 2m_1, \quad s = 2\Lambda_1^3$ |
| $P_{III}^{D_6}$ | $D_6$ | $a_1 = m_1 - m_2, \quad a_2 = m_1 + m_2, \quad s = -2\Lambda_2^2$ |
| $P_V$   | $D_5$    | $a_1 = -(m_1 + m_3), \quad a_2 = m_1 + m_2, \quad a_3 = m_3 - m_1, \quad s = 2\Lambda_3$ |
| $P_{VI}$| $D_4$    | $a_1 = m_3 + m_4, \quad a_2 = m_2 - m_3, \quad a_3 = m_1 - m_2, \quad a_4 = m_3 - m_4, \quad s = \frac{\beta}{\alpha}$ |

Table 2. Painlevé equation and Seiberg-Witten curve

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