Quenched local central limit theorem for random walks in a
time-dependent balanced random environment

Jean-Dominique Deuschel\textsuperscript{1} \& Xiaoqin Guo\textsuperscript{2,3}

Received: 6 December 2019 / Revised: 25 April 2021 / Accepted: 7 November 2021 / Published online: 4 December 2021 © The Author(s) 2021

Abstract
We prove a quenched local central limit theorem for continuous-time random walks in $\mathbb{Z}^d$, $d \ge 2$, in a uniformly-elliptic time-dependent balanced random environment which is ergodic under space-time shifts. We also obtain Gaussian upper and lower bounds for quenched and (positive and negative) moment estimates of the transition probabilities and asymptotics of the discrete Green’s function.

Keywords Parabolic Harnack inequality \cdot Random walks in random environments \cdot Non-divergence form operators \cdot Balanced environments \cdot Local limit theorem

Mathematics Subject Classification 60K37

Contents

1 Introduction .............................................1 1 2
2 Auxiliary probability estimates ................................. 1 1 8
3 A local volume-doubling property and its consequences .......... 1 2 4
3.1 Volume-doubling properties ................................ 1 2 5
3.2 $A_p$ property and proof of the PHI for $\mathcal{L}_\omega$ ................. 1 3 1
4 Estimates of caloric functions near the boundary ................ 1 3 4
4.1 An elliptic-type Harnack inequality .......................... 1 3 4
4.2 A boundary Harnack inequality .............................. 1 3 6
5 Proof of the PHI for the adjoint operator (Theorem 6) .......... 1 3 7
6 Proof of Theorems 4, 11, and Corollary 12 ....................... 1 4 1

\textcopyright Jean-Dominique Deuschel
deuscher@math.tu-berlin.de
Xiaoqin Guo
guoxq@ucmail.uc.edu

1 Institut für Mathematik, Technische Universität Berlin, Berlin, Germany
2 Department of Mathematics, University of Wisconsin-Madison, Madison, USA
3 Department of Mathematical Sciences, University of Cincinnati, Cincinnati, USA
1 Introduction

In this article we consider a random walk in a balanced uniformly-elliptic time-dependent random environment on $\mathbb{Z}^d$, $d \geq 2$.

For $x, y \in \mathbb{Z}^d$, we write $x \sim y$ if $|x - y|_2 = 1$. Denote by $P$ the set (of nearest-neighbor transition rates on $\mathbb{Z}^d$)

$$P := \left\{ v : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty) \left| v(x, y) = 0 \text{ if } x \sim y \right\} \right. .$$

Equip $P$ with the product topology and the corresponding Borel $\sigma$-field. We denote by $\Omega_1 \subset P^\mathbb{R}$ the set of all measurable functions $\omega : t \mapsto \omega_t$ from $\mathbb{R}$ to $P$ and call every $\omega \in \Omega$ a time-dependent environment. For $\omega \in \Omega$, we define the parabolic difference operator

$$\mathcal{L}_\omega u(x, t) = \sum_{y : y \sim x} \omega_t(x, y)(u(y, t) - u(x, t)) + \partial_t u(x, t)$$

for every bounded function $u : \mathbb{Z}^d \times \mathbb{R} \to \mathbb{R}$ which is differentiable in $t$. Let $(\hat{X}_t)_{t \geq 0} = (X_t, T_t)_{t \geq 0}$ denote the continuous-time Markov chain on $\mathbb{Z}^d \times \mathbb{R}$ with generator $\mathcal{L}_\omega$. Note that almost surely, $T_t = T_0 + t$. We say that $(X_t)_{t \geq 0}$ is a continuous-time random walk in the environment $\omega$ and denote by $P^{x, t}_\omega$ its law (called the quenched law) with initial state $(x, t) \in \mathbb{Z}^d \times \mathbb{R}$.

We equip $\Omega \subset P^\mathbb{R}$ with the induced product topology and let $\mathbb{P}$ be a probability measure on the Borel $\sigma$-field $\mathcal{B}(\Omega)$ of $\Omega$. An environment $\omega \in \Omega$ is said to be balanced if

$$\omega_t(x, x + e) = \omega_t(x, x - e) \quad \text{for all } e \in \mathbb{Z}^d \text{ with } |e| = 1.$$

and uniformly elliptic if there is a constant $\kappa \in (0, 1)$ such that

$$\kappa < \omega_t(x, y) \frac{1}{\kappa} \quad \text{for all } t \in \mathbb{R}, x, y \in \mathbb{Z}^d \text{ with } x \sim y.$$

Let $\Omega_\kappa \subset \Omega$ denote the set of balanced and uniformly elliptic environments with ellipticity constant $\kappa \in (0, 1)$. The measure $\mathbb{P}$ is said to be balanced and uniformly elliptic if $\mathbb{P}(\omega \in \Omega_\kappa) = 1$ for some $\kappa \in (0, 1)$.
For each \((x, t) \in \mathbb{Z}^d \times \mathbb{R}\) we define the space-time shift \(\theta_{x,t} : \Omega \to \Omega\) by

\[
(\theta_{x,t} \omega)(y, z) := \omega_{x+t}(y + x, z + x).
\]

We assume that the law \(\mathbb{P}\) of the environment is translation-invariant and ergodic under the space-time shifts \(\{\theta_{x,t} : x \in \mathbb{Z}^d, t \geq 0\}\). I.e., \(P(A) \in \{0, 1\}\) for any \(A \in \mathcal{B}(\Omega)\) such that \(\mathbb{P}(A \Delta \theta_{\hat{x}}^{-1}A) = 0\) for all \(\hat{x} \in \mathbb{Z}^d \times [0, \infty)\).

Given \(\omega\), the environmental process

\[
\tilde{\omega}_t := \theta_{\hat{x}_t} \omega, \quad t \geq 0,
\]

with initial state \(\tilde{\omega}_0 = \omega\) is a Markov process on \(\Omega\). With abuse of notation, we use \(P^0_\omega\) to denote the quenched law of \(\tilde{\omega}_t\).

**Assumptions:** throughout this paper, we assume that \(\mathbb{P}\) is balanced, ergodic, and uniformly elliptic with ellipticity constant \(\kappa > 0\).

We recall the quenched central limit theorem (QCLT) in [14].

**Theorem 1** [14, Theorem 1.2] *Under the above assumptions of \(\mathbb{P}\),

(a) there exists a unique invariant measure \(Q\) for the process \((\tilde{\omega}_t)_{t \geq 0}\) such that \(Q \ll \mathbb{P}\) and \((\tilde{\omega}_t)_{t \geq 0}\) is an ergodic flow under \(Q \times P^0_\omega\). Let

\[
\rho(\omega) := dQ/d\mathbb{P}.
\]

Then we have \(\rho > 0\), \(\mathbb{P}\)-almost surely, and

\[
E_{\mathbb{P}}[\rho^{(d+1)/d}] < \infty.
\]

(b) (QCLT) For \(\mathbb{P}\)-almost all \(\omega\), \(P^0_\omega\)-almost surely, \((X_{n+1}/n)_{t \geq 0}\) converges weakly, as \(n \to \infty\), to a Brownian motion with deterministic non-degenerate covariance matrix \(\Sigma = \text{diag}(2E_Q[\omega_0(0, e_i)], i = 1, \ldots, d)\).

In the special case where the environment is time-independent, i.e., \(\mathbb{P}(\omega_t = \omega_s\text{ for all } t, s \in \mathbb{R}) = 1\), we say that the environment is static.

**Remark 2** For balanced random walks in a static, uniformly-elliptic, ergodic random environment on \(\mathbb{Z}^d\), the QCLT has been first shown by Lawler [24], which is a discrete version of the result of Papanicolaou and Varadhan [27]. It is then generalized to static random environments with weaker ellipticity assumptions in [9,19].

**Remark 3** Write \(\|f\|_{L^p(\mathbb{P})} := (E_{\mathbb{P}}[|f|^p])^{1/p}\) for \(p \in \mathbb{R}\). At the end of the proof of [14, Theorem 1.2], it is shown that \(E_Q[g] \leq C\|g\|_{L^{d+1}(\mathbb{P})}\) for any bounded continuous function \(g\), which implies (2).

For \((x, t) \in \mathbb{Z}^d \times \mathbb{R}\), set

\[
\rho_{\omega}(x, t) := \rho(\theta_{x,t}\omega).
\]

Since \(\Omega\) is equipped with a product \(\sigma\)-field, for any fixed \(\omega \in \Omega\), the map \(\mathbb{R} \to \Omega\) defined by \(t \mapsto \theta_{0,t}\omega\) is measurable. Hence for almost-all \(\omega\), the function \(\rho_{\omega}(x, t)\) is measurable in \(t\). Moreover, \(\rho_{\omega}\) possesses the following properties. For \(\mathbb{P}\)-almost all \(\omega\),
(i) \( \rho_\omega(x, t)\delta_x \, dt \) is an invariant measure for the process \( \hat{X}_t \) under \( P_\omega \);
(ii) \( \rho_\omega(x, t) > 0 \) is the unique density (with respect to \( \delta_x \, dt \)) for an invariant measure of \( \hat{X} \) that satisfies \( E_P[\rho_\omega(0, 0)] = 1; \)
(iii) \( \rho_\omega \) has a version which is absolutely continuous with respect to \( t \) with
\[
\partial_t \rho_\omega(x, t) = \sum_y \rho_\omega(y, t) \omega_t(y, x) \tag{3}
\]
for almost every \( t \), where \( \omega_t(x, x) := -\sum_y: y \sim x \omega_t(y, x) \).

The proof of these properties, which is rather standard, is given in Sect. A.1 for the purpose of completeness.

As a main result of our paper, we will present the following local limit theorem (LLT), which is a finer characterization of the local behavior of the random walk than the QCLT. Let
\[
\hat{0} := (0, 0) \in \mathbb{Z}^d \times \mathbb{R}.
\]

For \( \hat{x} = (x, t), \hat{y} = (y, s) \in \mathbb{Z}^d \times \mathbb{R}, t \leq s \), define
\[
p^{o}(\hat{x}, \hat{y}) := P^{x,t}_\omega(X_{s-t} = y), \quad q^{o}(\hat{x}, \hat{y}) = \frac{p^{o}(\hat{x}, \hat{y})}{\rho_\omega(\hat{y})}. \tag{4}
\]

**Theorem 4 (LLT)** For \( \mathbb{P} \)-almost all \( \omega \) and any \( T > 0 \),
\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}^d, t > T} \abs{n^d q^{\omega}(\hat{0}; [nx], n^2 t) - p^\Sigma_T(0, x)} = 0.
\]

Here \( p^\Sigma_T(0, x) = [(2\pi t)^d \det \Sigma]^{-1/2} \exp(-x^T \Sigma^{-1} x / 2t) \) is the transition kernel of the Brownian motion with covariance matrix \( \Sigma \) and starting point 0, and \( [x] := ([x_1], \ldots, [x_d]) \in \mathbb{Z}^d \) for \( x \in \mathbb{R}^d \).

The proof of the LLT follows from Theorem 1 and a localization of the heat kernel \( q^{\omega}(\hat{0}, \cdot) \), an argument already implemented in [7] and [4] in the context of random conductance models. For this purpose, the regularity of \( \hat{x} \mapsto q^{\omega}(\hat{0}, \hat{x}) \) is essential. We use an analytical tool from classical PDE theory: the parabolic Harnack inequality (PHI) which yields not only Hölder continuity of \( q^{\omega}(\hat{0}, \cdot) \) but also very sharp heat kernel estimates.

To state the PHI, we need some notations. For \( \hat{x} = (x, t) \in \mathbb{Z}^d \times \mathbb{R}, \) let
\[
\omega_t^*(x, y) := \frac{\rho_\omega(y, t) \omega_t(y, x)}{\rho_\omega(x, t)} \quad \text{for } x \sim y \in \mathbb{Z}^d,
\]
and define the the adjoint operator \( \mathcal{L}_\omega^* \) by
\[
\mathcal{L}_\omega^* v(\hat{x}) := \sum_{y:y \sim x} \omega_t^*(x, y) [v(y, t) - v(\hat{x})] - \partial_t v(\hat{x}).
\]
We say that a function \( u \) is \( \omega\)-caloric (resp. \( \omega^*\)-caloric) on \( D \subset \mathbb{Z}^d \times \mathbb{R} \) if \( \mathcal{L}_\omega u = 0 \) (resp. \( \mathcal{L}^*_\omega u = 0 \)) on \( D \).

Throughout this paper, unless otherwise specified, \( C, c \) denote generic positive constants that depend only on \( (d, \kappa) \), and which may differ from line to line. If two functions \( f \) and \( g \) satisfy \( cg \leq f \leq Cg \), we write

\[
\exists f \asymp g.
\]

First, let us state the PHI for the operator \( \mathcal{L}_\omega \). For \( r > 0, x \in \mathbb{R}^d \), let

\[
B_r(x) = \{ y \in \mathbb{Z}^d : |x - y|_2 < r \}, \quad B_r = B_r(0).
\]

**Proposition 5** (*PHI for \( \mathcal{L}_\omega \)) Assume \( \omega \in \Omega_\kappa \) and \( \theta > 0 \). Let \( u \) be a non-negative \( \omega\)-caloric function on \( B_R \times (0, \theta R^2) \). Then, for \( 0 < \theta_1 < \theta_2 < \theta_3 < \theta \), there exists a constant \( C = C(\kappa, d, \theta_1, \theta_2, \theta_3, \theta) \) such that

\[
\sup_{B_{R/2} \times (\theta_2 R^2, \theta_3 R^2)} u \leq C \inf_{B_{R/2} \times [0, \theta_1 R^2]} u.
\]

We remark that for dynamical environments in the discrete time setting, the PHI is obtained by Kuo and Trudinger for the so-called implicit form operators, see [23, (1.16)]. For discrete-time random walks in a static environment, the PHI is shown by Lawler [25] for uniformly elliptic operators, and by Berger and Criens [8] (see also [11]) for a genuinely \( d\)-dimensional i.i.d. environment which is not necessarily elliptic.

Observe that for fixed \( \hat{x} = (x, t) \), the function \( \hat{y} \mapsto \omega^*(\hat{y}, \hat{x}) \) is \( \omega\)-caloric on \( \mathbb{Z}^d \times (-\infty, t) \). Whereas, the heat kernel \( \hat{x} \mapsto \omega^*(0, \hat{x}) \) is \( \omega^*\)-caloric on \( \mathbb{Z}^d \times (0, \infty) \).

Hence, to obtain the regularity of the heat kernel, we need to prove, instead of the PHI for \( \mathcal{L}_\omega \), the following PHI for \( \mathcal{L}^*_\omega \), which is our second main result.

**Theorem 6** (*PHI for \( \mathcal{L}^*_\omega \)) For \( \mathbb{P}\)-almost all \( \omega \), any non-negative \( \omega^*\)-caloric function \( v \) on \( B_{2R} \times (0, 4R^2] \) satisfies

\[
\sup_{B_R \times (R^2, 2R^2)} v \leq C \inf_{B_R \times (3R^2, 4R^2]} v.
\]

As a standard consequence of the PHI for \( \mathcal{L}^*_\omega \), we get the following Hölder estimate for \( \omega^*\)-caloric functions. (See a proof in Sect. A.2.)

**Corollary 7** Let \( (x_0, t_0) \in \mathbb{Z}^d \times \mathbb{R} \) and \( R > 0 \). There exists \( \gamma = \gamma(d, \kappa) \in (0, 1] \) such that, \( \mathbb{P}\)-almost surely, any non-negative \( \omega^*\)-caloric function \( u \) on \( B_R(x_0) \times (t_0 - R^2, t_0] \) satisfies

\[
|u(\hat{x}) - u(\hat{y})| \leq C \left( \frac{r}{R} \right)^\gamma \sup_{B_R(x_0) \times (t_0 - R^2, t_0]} u
\]

for all \( \hat{x}, \hat{y} \in B_r(x_0) \times (t_0 - r^2, t_0] \) and \( r \in (0, R) \).
The main challenge in proving Theorem 6 is that $\omega^*$ is neither balanced nor uniformly elliptic, and so the PHI for $L_\omega$ (Proposition 5) is not immediately applicable. This is the main difference with the random conductance model with symmetric jump rates where $\omega_t(x, y) = \omega_t(y, x) = \omega_t^*(x, y)$, in which case the PHI for $L_\omega$ is the same as PHI for $L_\omega^*$. See [1,2,12,13,20].

In PDE, the Harnack inequality for the adjoint of non-divergence form elliptic differential operators was first proved by Bauman [6], and was generalized to the parabolic setting by Escauriaza [15]. Our proof of Theorem 6 follows the main idea of [15].

Let us explain the main idea for the proof of Theorem 6. An important observation is that $\omega^*$-caloric functions can be expressed in terms of hitting probabilities of the time-reversed process. Thus to compare values of an

$\omega^*$-caloric function at different points, one only needs to estimate hitting probabilities of the original process that starts from the boundary. To this end, we will use a “boundary Harnack inequality” (Proposition 30) which compares

$\omega$-caloric functions near the boundary. We will also need the following parabolic volume-doubling property (VDP) for the invariant measure to control the change of probabilities due to time-reversal.

**Theorem 8** (parabolic VDP) $\mathbb{P}$-almost surely, for every $r \geq 1/2$,

$$\sup_{t:|t| \leq r^2} \rho_\omega(B_{2r}, t) \leq C \rho_\omega(B_r, 0).$$

**Remark 9** For time discrete random walks in a static environment, Theorem 6 was shown by Mustapha [26]. His argument follows basically [15], and uses the PHI [23, Theorem 4.4] of Kuo and Trudinger in the time discrete situation. Note that for static environments, the PHI for $L_\omega^*$ follows from the PHI (for $L_\omega$) and a representation formula (See Remark 33), and a VDP is not needed. However, in our dynamical setting, the parabolic VDP is crucially employed. To this end, we adapt ideas of Safonov-Yuan [28] and results in the references therein [6,16,18] into our discrete space setting.

**Remark 10** For adjoint solutions of non-divergence form elliptic PDE, a VDP was first established by Fabes and Stroock [17]. It was then generalized by Safonov and Yuan [28] to the parabolic case.

Recall the heat kernel $q_\omega$ in (4). For any $A \subset \mathbb{R}^d$ and $s \in \mathbb{R}$, let

$$\rho_\omega(A, s) = \sum_{x \in A \cap \mathbb{Z}^d} \rho_\omega(x, s).$$

We write the $\ell^2$-norm of $x \in \mathbb{R}^d$ as $|x| = |x|_2$. For $r \geq 0, t > 0$, define

$$\mathfrak{h}(r, t) = \frac{r^2}{t + r} + r \log(\frac{r}{t} \lor 1).$$

(5)

Note that $\mathfrak{h}(c_1 r, c_2 t) \asymp \mathfrak{h}(r, t)$ for constants $c_1, c_2 > 0$. 

\[ \mathfrak{h} \] Springer
Our third main results are the following heat kernel estimates (HKE). Note that for a general balanced environment, the density $\rho_\omega$ does not have deterministic (positive) upper and lower bounds, thus one cannot expect deterministic Gaussian bounds for $p^\omega(\hat{0}, \hat{x})$. However, our HKE below shows that $p^\omega(\hat{0}, \hat{x})$ has both $L^{(d+1)/d}(\mathbb{P})$ and $L^{-p}(\mathbb{P})$ moment bounds.

**Theorem 11 (HKE)** For $\mathbb{P}$-almost every $\omega$ and all $\hat{x} = (x, t) \in \mathbb{Z}^d \times (0, \infty)$,

\[
\frac{c}{\rho_\omega(B_{\sqrt{t}}(y), s)} e^{-C |x|^2 \frac{t}{2}} \leq q^\omega(\hat{0}, \hat{x}) \leq \frac{C}{\rho_\omega(B_{\sqrt{t}}(y), s)} e^{-C h(|x|, t)}
\]  

(6)

for all $s \in [0, t]$ and $y$ with $|y| \leq |x| + c\sqrt{t}$. Moreover, recalling the definition of $L^p(\mathbb{P})$ in Remark 3, there exists $p = p(d, \kappa) > 0$ such that

\[
\|P^0_\omega(0, x)\|_{L^{(d+1)/d}(\mathbb{P})} \leq \frac{C}{(t + 1)^{d/2}} e^{-C h(|x|, t)}
\]  

(7)

and

\[
\|P^0_\omega(0, x)\|_{L^{-p}(\mathbb{P})} \geq \frac{c}{(t + 1)^{d/2}} e^{-C |x|^2 \frac{t}{2}}
\]  

(8)

for all $(x, t) \in \mathbb{Z}^d \times (0, \infty)$. As a consequence, setting $G^\omega(0, x) = \int_0^\infty P^0_\omega(0, X_t = x)dt$, we have for $d \geq 3$ and $x \in \mathbb{Z}^d$,

\[
\|G^\omega(0, x)\|_{L^{(d+1)/d}(\mathbb{P})} \asymp \|G^\omega(0, x)\|_{L^{-p}(\mathbb{P})} \asymp (|x| + 1)^{2-d}.
\]  

(9)

Furthermore, we can characterize asymptotics of the Green’s function of the RWRE. Recall the notations $\Sigma$ (in Theorem 1 (b)), $p^\Sigma_t$, and $[x]$ (in Theorem 4).

**Corollary 12** The following statements are true for $\mathbb{P}$-almost every $\omega$.

(i) There exists $t_0(\omega) > 0$ such that for any $\hat{x} = (x, t) \in \mathbb{Z}^d \times (t_0, \infty)$,

\[
\frac{c}{t^{d/2}} e^{-C |x|^2 \frac{t}{2}} \leq q^\omega(\hat{0}, \hat{x}) \leq \frac{C}{t^{d/2}} e^{-C h(|x|, t)},
\]

where $h$ is as in (5). As a consequence, the RWRE is recurrent when $d = 2$ and transient when $d \geq 3$.

(ii) When $d = 2$, for all $x \in \mathbb{R}^d \setminus \{0\}$,

\[
\lim_{n \to \infty} \frac{1}{\log n} \int_0^\infty \left[ q^\omega(\hat{0}; 0, t) - q^\omega(\hat{0}; \lfloor nx \rfloor, t) \right] dt = \frac{1}{\pi \sqrt{\det \Sigma}}.
\]

(iii) When $d \geq 3$, for all $x \in \mathbb{R}^d \setminus \{0\}$,

\[
\lim_{n \to \infty} n^{d-2} \int_0^\infty q^\omega(\hat{0}; \lfloor nx \rfloor, t) dt = \int_0^\infty p^\Sigma_t(0, x) dt.
\]
Similar results as Corollary 12(ii)(iii) are also obtained recently for the conductance model [5].

Our major technical novelties and main results can be summarized as follows. (a) Using probability estimates, we solve the difficult analytic problem of obtaining a parabolic VDP (for the density of the invariant measure) in a time-dependent balanced environment. (b) Using the parabolic VDP, we established the $A_p$ bounds, and as a consequence proved the PHI for $\omega$-caloric functions. The latter proof, which is of interest on its own, can be viewed as the parabolic version of Fabes and Stroock’s [17] proof in the elliptic static setting. (c) Interpreting $\omega^*$-caloric functions in terms of a time reversed RWRE, and using the parabolic VDP and boundary PHI estimates, we prove the PHI for the adjoint operator. (d) As applications, we obtain LLT, quenched HKE, positive and negative $L^p$ bounds for the heat kernel, and Green’s function asymptotics for the RWRE.

The organization of this paper is as follows. Section 2 contains probability estimates that are used in the later sections. In Sect. 3, we obtain the parabolic VDP and an $A_p$ inequality for $\rho_\omega$, and prove the PHI for $L_\omega$. In Sect. 4, we establish estimates of $\omega$-caloric functions near the boundary, showing both the interior elliptic-type and boundary PHI’s. We prove the PHI for the adjoint operator (Theorem 6) in Sect. 5. Finally, with the adjoint PHI, we prove Theorems 4, 11, and Corollary 12 in Sect. 6. Some classical estimates and standard arguments can be found in the Appendix.

2 Auxiliary probability estimates

This section contains probability estimates that are crucial in the rest of the paper. For a finite subgraph $D \subset \mathbb{Z}^d$, let

$$\partial D = \{ y \in \mathbb{Z}^d \setminus D : y \sim x \text{ for some } x \in D \}, \quad \bar{D} := D \cup \partial D.$$ 

For $\mathcal{D} \subset \mathbb{Z}^d \times \mathbb{R}$, define the parabolic boundary of $\mathcal{D}$ as

$$\partial^P \mathcal{D} := \{ (x, t) \notin \mathcal{D} : (B_{1+\epsilon}(x) \times (t - \epsilon, t]) \cap \mathcal{D} \neq \emptyset \text{ for all } \epsilon > 0 \}. \quad (10)$$

In the special case $\mathcal{D} = D \times [0, T)$ for some finite $D \subset \mathbb{Z}^d$, it is easily seen that $\partial^P \mathcal{D} = (\partial D \times [0, T)) \cup (\bar{D} \times \{T\})$. See Fig. 1.

Recall the definition of the function $h(r, t)$ in (5).

Lemma 13 Assume $\omega \in \Omega_\kappa$. Then for $t > 0$, $r > 0$,

$$P_{\omega}^{0,0} \left( \sup_{0 \leq s \leq t} |X_s| \geq r \right) \leq C \exp (-c h(r, t)) .$$

Proof Let $x(i), i = 1, \ldots, d$, denotes the $i$-th coordinate of $x \in \mathbb{R}^d$. It suffices to show that for $i = 1, \ldots, d$,

$$P_{\omega}^{0,0} \left( \sup_{0 \leq s \leq t} |X_s(i)| > r \right) \leq C \exp (-c h(r, t)) .$$
We will prove the statement for $i = 1$. Let $\tilde{N}_t := \#\{0 \leq s \leq t : X_s(1) \neq X_s(-1)\}$ be the number of jumps in the $e_1$ direction before time $t$. Let $(S_n)$ be the discrete time simple random walk on $\mathbb{Z}$, then $X_t(1) = S_{\tilde{N}_t}$. Note that $\tilde{N}_t$ is stochastically dominated by a Poisson process $N_t$ with rate $c_0 := 2d/\kappa$, and so $P_0^\omega(\sup_{0 \leq s \leq t} |X_s(1)| > r) \leq P_0^\omega(\max_{0 \leq m \leq N_t} |S_m| > r)$. Hence,

$$P_0^\omega(\sup_{0 \leq s \leq t} |X_s(1)| > r) \leq P(N(t) \geq 2c_0(t \vee r)) + P(\max_{0 \leq m \leq 2c_0(t \vee r)} |S_m| > r) \leq e^{-c(t \vee r)} + Ce^{-cr^2/(t \vee r)} \leq Ce^{-cr^2/(t \vee r)},$$

where we used Hoeffding’s inequality in the second inequality. On the other hand, since the random walk is in a discrete set $\mathbb{Z}$, we have, for any $\theta > 0$,

$$P_0^\omega(\sup_{0 \leq s \leq t} |X_s(1)| > r) \leq P(N(t) > r) \leq E[\exp(\theta N(t) - \theta r)] = \exp[c_0 t(e^\theta - 1) - \theta r].$$

When $r \geq 9c_0^2 t$, taking $\theta = \log(\frac{r}{c_0})$, we get an upper bound $\exp[-\frac{r}{2} \log(\frac{r}{t})].$ Hence, letting $f(r, t) = \frac{r^2}{t \vee r} \mathbb{1}_{r < 9c_0^2 t} + r \log(\frac{r}{t}) \mathbb{1}_{r \geq 9c_0^2 t}$, we obtain

$$P_0^\omega(\sup_{0 \leq s \leq t} |X_s(1)| > r) \leq C \exp(-cf(r, t)).$$

Since $f(r, t) \asymp \frac{r^2}{t \vee r} + r \log(\frac{r}{t}) \mathbb{1}_{r \geq 9c_0^2 t} \asymp \mathcal{h}(r, t)$, our proof is complete. \hfill $\Box$

**Corollary 14** Assume $\omega \in \Omega_\kappa$ and $\theta_2 > \theta_1 > 0$. There exist $C, c$ depending on $(d, \kappa, \theta_1, \theta_2)$ such that for $\theta \in (\theta_1, \theta_2)$, $(x, t) \in \mathbb{Z}^d \times (0, \infty)$,

$$P_0^{0,0}(X_t \in B_{\sqrt{\theta t}}(x)) \leq C \exp[-c\mathcal{h}(|x|, t)].$$

**Proof** Since $\mathcal{h}(0, t) = 0$, we only need to consider the case $x \neq 0$.\hfill $\square$
If $0 < t \leq 1$, then $P^0_\omega(X_t \in B_{\sqrt{\omega} t}(x)) = P^0_\omega(X_t = x) \leq P^0_\omega(\sup_{0 \leq s \leq t} |X_s| \geq |x|) \leq C \exp[-c\theta(|x|, t)]$ by Lemma 13.

If $0 < t > 1$ and $1 \leq |x| \leq 2\sqrt{\omega}t$, then $|x| \leq |x|^2 \leq 4\theta t$ and so $h(|x|, t) \sim h(|x|, 4\theta t) \sim \frac{|x|^2}{t^2}$. In particular, $h(|x|, t) \leq C\frac{|x|^2}{t} \leq C$. Hence, trivially, $P^0_\omega(X_t \in B_{\sqrt{\omega} t}(x)) \leq 1 \leq C \exp(-c\theta(|x|, t)).$

It reminds to consider $|x| > 2\sqrt{\omega}t > 2$. In this case, by Lemma 13, $P^0_\omega(X_t \in B_{\sqrt{\omega} t}(x)) \leq P^0_\omega(\sup_{0 \leq s \leq t} |X_s| \geq |x|/2) \leq C \exp[-c\theta(|x|, t)].$ \hfill \Box

For any $A \subset \mathbb{Z}^d$, $s \in \mathbb{R}$, define the stopping time

$$\Delta(A, s) = \inf\{t \geq 0 : \hat{X}_t \notin A \times (-\infty, s)\}. \quad (11)$$

**Lemma 15** Let $0 < \theta_1 < \theta_2$, $R > 0$ and $\omega \in \Omega_k$. Recall the stopping time $\Delta$ in (11). There exists a constant $\alpha = \alpha(\kappa, d, \theta_1, \theta_2) > 1$ such that for any $s \in (\theta_1 R^2, \theta_2 R^2)$, and $\sigma > 0$

$$\min_{x \in B_R} P^x_\omega(\Delta(B_{2R}, s) \in B_{\sigma R}) \geq \left(\frac{\sigma}{2}\right)^\alpha.$$

**Proof** It suffices to consider the case $\sigma \in (0, 1)$ and $R > K_1$, where $K_1 = K_1(\theta_1, \theta_2, \kappa, d)$ is a large constant to be determined. Indeed, if $R < K_1$, then by uniform ellipticity, for any $x \in B_R$,

$$P^x_\omega(\Delta(B_{2R}, s) \in B_{\sigma R}) \geq P^0_\omega(X_s = 0, \Delta(B_{2R}, s) = s) \geq C(\kappa, d, \theta_1, \theta_2).$$

Further, for $R \geq K_1$, if suffices to consider the case $\sigma R \geq \sqrt{K_1}$. Indeed, assume the lemma is proved for $R \geq K_1$ and $\sigma R \geq \sqrt{K_1}$, then, when $\sigma R < \sqrt{K_1}$ and $x \in B_R$, by uniform ellipticity,

$$P^x_\omega(\Delta(B_{2R}, s) \in B_{\sigma R}) \geq P^x_\omega(\Delta(B_{2R}, s-K_1) \in B_{\sqrt{K_1}}) \min_{y \in B_{\sqrt{K_1}}} P^y_\omega(\Delta(B_{2R}, s) = K_1)$$

$$\geq \frac{\sqrt{K_1}}{2R}^\alpha C(K_1, \kappa, d) \geq C(\frac{\sigma}{2})^\alpha.$$

Hence in what follows we only consider the case $R \geq K_1$ and $\sigma R \geq \sqrt{K_1}$.

For $(x, t) \in \mathbb{R}^d \times \mathbb{R}$, set

$$\psi_0(t) = 1 - \frac{(\sigma/2)^2}{2R^2} t, \quad \tilde{\psi}_1(x, t) = \psi_0 - \frac{|x|^2}{4R^2}, \quad \psi_1 = \tilde{\psi}_1 \vee 0,$$

and, for some large constant $q \geq 2$ to be chosen,

$$\psi(x, t) := \psi_1^q \psi_0^{1-q}, \quad w(x, t) = (\sigma/2)^{2q-4} \psi(x, t).$$
Let \( U := \{ \hat{x} \in B_{2R} \times [0, s) : \psi_1(\hat{x}) > 0 \} \). We will show that for \( \hat{x} \in U \),

\[
w(\hat{x}) \leq v(\hat{x}) := P_0^t(X_\tau \in B_{\sigma R}).
\]

Recall the parabolic boundary \( \partial^p \) in (10). We first show that \( w \) satisfies

\[
\begin{aligned}
   & w|_{\partial^p U} \leq 1_{x \in B_{\sigma R}, s=t} \\
   & \min_{x \in B_R} w(x, 0) \geq \frac{1}{2}(\sigma/2)^{2q-4}, \\
   & \mathcal{L}_\omega w \geq 0 \quad \text{in } U, \text{ for } q \text{ large.}
\end{aligned}
\]

The first two properties in (12) are obvious. For the third property, note that

\[
\partial_t \psi = R^{-2}\psi_0^{-q}\left[\frac{1-(\sigma/2)^2}{s/R^2}(\psi_1\psi_0/q - 2)\psi_1\right] \quad \text{in } U.
\]

For any unit vector \( e \in \mathbb{Z}^d \), let

\[
\nabla^2_e u(x, t) := u(x + e, t) + u(x - e, t) - 2u(x, t).
\]

When \( \hat{x} \in U_1 := \{(z, s) \in U : (y, s) \in U \text{ for all } y \sim z\} \), then \( \nabla^2_e[\psi_1^2(\hat{x})] = \nabla^2_e[\psi_1^2(\hat{x})] \). When \( \hat{x} = (x, t) \in U \setminus U_1 \), then for some \( |e| = 1 \), either \( (x + e, t) \) or \( (x - e, t) \) is not in \( U \). Say, \( (x + e, t) \not\in U \), then \( x \cdot e \geq 1 \) and \( \exists \delta \in (0, 1) \) such that \( \psi_1(x + \delta e, t) = 0 \). In both cases, there exists \( \delta \in (0, 1) \) such that

\[
\nabla^2_e[\psi_1(x, t) \hat{e}^2]
\]

\[
\begin{aligned}
   &= \psi_1^2(x + \delta e, t) + \psi_1^2(x - e, t) - 2\psi_1^2(\hat{x}) \\
   &= -\frac{[1 + \delta^2 + 2x \cdot e(\delta - 1)]\psi_1}{2R^2} + \frac{1 + \delta^2 + 4(\delta^2 + 1)(x \cdot e)^2 + 4(\delta^3 - 1)x \cdot e}{16R^4} \\
   &\geq -\frac{\psi_1}{R^2} + \frac{(x \cdot e)^2}{4R^4} - \frac{\psi_0^{1/2}}{2R^3},
\end{aligned}
\]

where in the last inequality we used the fact \( 1 \leq x \cdot e \leq |x| \leq 2R\psi_0^{1/2} \). Thus, letting \( \xi := \psi_1/\psi_0 \in [0, 1] \), we have for \( \hat{x} = (x, t) \in \hat{U} \),

\[
R^2\psi_0^{1/2-1}\mathcal{L}_\omega \psi(\hat{x}) = R^2\left(\sum_{i=1}^d \omega_i(x, x + e_i)\nabla^2_e[\psi_1^2]/\psi_0 + \psi_0^{-1}\partial_t \psi\right)
\]

\[
\begin{aligned}
   &\geq \frac{c|x|^2}{R^2\psi_0} - C\xi - \frac{C}{R\psi_0^{1/2}} + \frac{1-(\sigma/2)^2}{s/R^2}(q\xi - 2)\xi \\
   &\geq Cq\xi^2 - c_1\xi + c_2 - c_3/K_1^{1/2},
\end{aligned}
\]
where in the last inequality we used $|x|^2/(4R^2\psi_0) = 1 - \xi$ and $\psi_0^{1/2} \geq \sigma/2 \geq K_1^{1/2}/(2R)$. Taking $q$ and $K_1$ large enough, we have $\mathcal{L}_\omega \psi \geq 0$ in $U$. The third property in (12) is proved.

Finally, we set $v(\hat{x}) = P^\delta_\omega (X_{\Delta(B_{2r},s)} \in B_\sigma R)$. By (12), $v(\hat{X}_t) - w(\hat{X}_t)$ is a supermartingale for $t \leq T_U := \inf \{s \geq 0 : \hat{X}_s \in U \}$ and $(v - w)_{\partial P U} \geq 0$. Hence, the optional-stopping theorem yields

$$v(\hat{x}) - w(\hat{x}) \geq E_{\hat{x}}^\delta [v(X_{T_U}) - w(X_{T_U})] \geq 0 \quad \text{for } \hat{x} \in U.$$ 

In particular, $\min_{x \in B_R} v(x, 0) \geq \min_{x \in B_R} w(x, 0) \geq (\sigma/2)^2 q - 4/2$. \hfill \(\Box\)

**Corollary 16** Assume that $\omega \in \Omega_\kappa$, $R/2 > r > 1/2$, $\theta > 0$. There exists $c = c(d, \kappa, \theta) \in (0, 1)$ such that for any $y \in \partial B_R$ with $B_{2r}(y) \cap B_R \neq \emptyset$,

$$\min_{x \in B_r(y) \cap B_R} P^x_\omega (X_{t \in B_{2r}(y) \cap B_R} \text{ from } \partial B_R \text{ before time } \theta r^2) > c.$$ 

**Proof** By uniform ellipticity, it suffices to consider $r \geq 10$.

Let $z = \frac{y}{|y|} + y \in \mathbb{R}^d$. Note that $B_r(y) \subset B_{5r/4}(z) \subset B_{3r/2}(z) \subset B_{2r}(y)$ and $B_{r/5}(z) \subset \mathbb{Z}^d \setminus B_r$. Recall $\Delta$ in (11). Then, by Lemma 15,

$$\min_{x \in B_r(y) \cap B_R} P^x_\omega (X_{\Delta(B_{2r}(y)) \cap B_R, \theta r^2) \in \partial B_R) \geq \min_{x \in B_{5r/4}(z)} P^x_\omega (X_{\Delta(B_{3r/2}(z), \theta r^2) \in B_{r/5}(z)) \geq c(\theta, d, \kappa).$$ 

The corollary is proved. \hfill \(\Box\)

**Lemma 17** Assume $\omega \in \Omega_\kappa$, $\beta \in (0, 1)$. Let $\tau_{\beta,1} = \tau_{\beta,1}(R) = \inf \{t \geq 0 : X_t \notin B_R \cup \bar{B}_{\beta R} \}$. Then if $y \in B_R \setminus \bar{B}_{\beta R} \neq \emptyset$ and $\theta > 0$, we have

$$P^y_\omega (X_{\tau_{\beta,1} \leq \theta R^2} \in \partial B_{\beta R}, \tau_{\beta,1} \leq \theta R^2) \geq C \frac{\text{dist}(y, \partial B_R)}{R},$$
where \( C = C(\kappa, d, \beta, \theta) \).

**Proof** It suffices to prove the lemma for \( R > \alpha^2 \), where \( \alpha = \alpha(\kappa, d, \beta, \theta) \) is a large constant to be determined. We only need to consider \( y \) with \( \mathrm{dist}(y, \partial(B_R \setminus \bar{B}_R)) \geq 2 \) in which case \( R - |y| \asymp \mathrm{dist}(y, \partial B_R) \).

For \( \hat{x} = (x, t) \), let \( g(\hat{x}) = \exp(-\beta |x|^2 - \alpha t/R^2) \). Using the inequalities \( e^a + e^{-a} \geq 2 + a^2 \) and \( e^a \geq 1 + a \), we get for \( x \in B_R \setminus \bar{B}_R, t \in \mathbb{R} \),

\[
\mathcal{L}_\omega g(\hat{x}) = g(\hat{x}) \left( \sum_{i=1}^d \omega_i (x, x + e_i) [e^{-\alpha R^2 (1 + 2x_i)} + e^{-\alpha R^2 (1 - 2x_i)} - 2] - \frac{\alpha}{\theta R^2} \right) \\
\geq g(\hat{x}) \left( \sum_{i=1}^d \omega_i (x, x + e_i) [e^{-\alpha/R^2 (2 + 4\alpha^2 x_i^2 / R^4)} - 2] - \frac{\alpha}{\theta R^2} \right) \\
\geq g(\hat{x})(-C \frac{\alpha}{R^2} + c \frac{\alpha^2 |x|^2}{R^4} - \frac{\alpha}{\theta R^2}) \\
\geq \frac{\alpha}{R^2} g(\hat{x})(c\alpha \beta^2 - C) > 0
\]

if \( \alpha \) is chosen to be large enough. Hence \( g(\hat{X}_t) \) is a submartingale for \( t \leq \tau_{\beta, 1} \).

Recall the definition of the stopping time \( \Delta \) in (11). Let

\[
v(\hat{x}) := \frac{g(\hat{x}) - e^{-\alpha}}{e^{-\alpha \beta^2} - e^{-\alpha}} \quad \text{and} \quad u(\hat{x}) := P^{\hat{x}}_\omega X_{\Delta(B_R \setminus \bar{B}_R, \theta R^2)} \in \partial B_R R).
\]

Set \( \mathcal{D} = (B_R \setminus \bar{B}_R) \times [0, \theta R^2) \). Since \( (u - v)|_{\partial \mathcal{D}} \geq 0 \) and \( u(\hat{X}_t) \) is a martingale in \( \mathcal{D} \), by the optional-stopping theorem we conclude that \( u \geq v \) in \( \mathcal{D} \). In particular, \( u(x, 0) \geq v(x, 0) \geq C(R^2 - |x|^2) / R^2 \) for \( x \in B_R \setminus \bar{B}_R \).

**Lemma 18** Let \( \beta \in (0, 1) \), and let \( \tau_{\beta, 1} \) be as in Lemma 17. For \( \theta > 0 \), there exists a constant \( C = C(\beta, \kappa, d, \theta) \) such that, if \( x \in B_R \setminus \bar{B}_R \neq \emptyset \),

\[
P^{x, 0}_\omega (X_{\theta R^2} \wedge \tau_{\beta, 1} \notin \partial B_R) \leq C \mathrm{dist}(x, \partial B_R)/R.
\]

**Proof** Set \( \mathcal{D} := (B_R \setminus \bar{B}_R) \times [0, \theta R^2) \). It suffices to consider the case \( R > k^2 \), where \( k = k(\beta, \kappa, d, \theta) > \log 2 / \log(2 - \beta^2) \) is a large constant to be determined. Let \( h(x, t) = 2 - |x|^2 / (R + 1)^2 + t / (\theta R^2) \).

Recall the notation \( \nabla^2 \) in (13). For \( \hat{x} = (x, t) \in \mathcal{D} \), note that \( 1 \leq h(\hat{x}) \leq 3 \) and \( |\nabla^2_{\hat{x}} (h^{-k})(\hat{x}) - \partial_{ii} (h^{-k})(\hat{x})| \leq C k^3 R^{-3} h^{-k}(x, t) \). Hence for any \( \hat{x} = (x, t) \in \mathcal{D} \), when \( k \) is sufficiently large,

\[
\mathcal{L}_\omega (h^{-k})(\hat{x}) \\
\geq \sum_{i=1}^d \omega_i (x, x + e_i) \partial_{ii} (h^{-k}) - C k^3 R^{-3} h^{-k} + \partial_t (h^{-k})
\]

\( \square \) Springer
\[ \geq c \sum_{i=1}^{d} \left[ \frac{x_i^2}{(R+1)^2} h^{-k-2} + \frac{k}{(R+1)^2} h^{-k-1} \right] - C k^3 R^{-3} h^{-k} - \frac{k}{\partial R} h^{-k-1} \]

\[ \geq c k h^{-k} R^{-2} [k - C - C k^2 R^{-1}] > 0, \]

which implies that \( h(\hat{X}_t)^{-k} \) is a submartingale inside the region \( \mathcal{D} \).

Next, set (Recall the stopping time \( \Delta_1 \) in (11).)

\[ u(\hat{x}) = P^{\hat{x}}_{\omega}(X_{\Delta(B_R \setminus \bar{B}_{\beta R}, \hat{\theta} R^2)} \notin \partial B_R). \]

Then \( u(\hat{X}_t) + (2 - \beta^2)^k h(\hat{X}_t)^{-k} \) is a submartingale in \( \mathcal{D} \). Since

\[ \begin{cases} h^{-k}_{|x \in \partial B_R} \leq (2 - 1 + 0)^{-k} = 1 \\ h^{-k}_{|x \in \partial B_{\beta R}} \leq (2 - \beta^2)^{-k} \\ h^{-k}_{|t = \theta R^2} \leq (2 - 1 + 1)^{-k} \leq (2 - \beta^2)^{-k} \end{cases} \]

by the optional stopping theorem, we have for \( x \in B_R \setminus \bar{B}_{\beta R} \),

\[ u(x, 0) + (2 - \beta^2)^k h(x, 0)^{-k} \leq \sup_{\partial^{p}\mathcal{D}} [u + (2 - \beta^2)^k h^{-k}] \leq (2 - \beta^2)^k. \]

Therefore, for any \( x \in B_R \setminus \bar{B}_{\beta R} \),

\[ u(x, 0) \leq (2 - \beta^2)^k (1 - h(x, 0)^{-k}) \leq C (h(x, 0) - 1) = C [1 - |x|^2/(R + 1)^2] \leq C \text{dist}(x, \partial B_R)/R. \]

Our proof of Lemma 18 is complete. \( \square \)

3 A local volume-doubling property and its consequences

The purpose of this section is to obtain the parabolic VDP (Theorem 8) and a negative moment estimate (Theorem 26) for the density \( \rho_{\omega} \). The former is an essential part for the proof of the PHI for \( L^{*}_{\omega} \), while the latter will imply the negative moment bound (8) for the heat kernel. Their proofs rely crucially on a VDP for hitting probabilities restricted in a finite ball (Lemma 19), which is an improved version of [28, Theorem 1.1] by Safonov and Yuan in the PDE setting.

As a by-product, we obtain a new proof of the classical PHI of Krylov and Safonov [22] in the lattice (Proposition 5). Our proof can be viewed as the parabolic version of Fabes and Stroock’s [17] proof of the elliptic HI in the static PDE setting.

In the course of our proof, we also use the maximum principle (Theorem A.3.1) and mean-value inequality (Theorem A.4.1) both of which are standard results for \( \omega \)-caloric functions. Their statements and proofs are included in Sects. A.3 and A.4 for completeness.
3.1 Volume-doubling properties

By the optional stopping theorem, for any \((x, t) \in \mathcal{D} \subset \mathbb{Z}^d \times \mathbb{R}\) and any bounded integrable function \(u\) on \(\mathcal{D} \cup \partial \mathcal{D}\),

\[
u(x, t) = -E_{\omega}^{x,t} \left[ \int_0^\tau L_\omega u(\hat{X}_r) dr \right] + E_{\omega}^{x,t}[u(\hat{X}_\tau)],
\]

where \(\tau = \inf \{ r \geq 0 : (X_r, T_r) \notin \mathcal{D} \} \).

To prove Theorem 8, a crucial estimate is a VDP (Lemma 19) for the hitting measure of the random walk, which we will obtain by adapting some ideas of Safonov and Yuan [28] in the PDE setting. In contrast to [28, Theorem 1.1], our proof relies on a probabilistic estimate (Lemma 15) rather than the PHI (Proposition 5).

Lemma 19 Assume \(\omega \in \Omega_\kappa\). Recall \(\Delta(A, s)\) in (11). There exists \(k_0 = k_0(d, \kappa)\) such that for any \(k \geq k_0, m \geq 2, r, s > 0\) and \(y \in B_{k\sqrt{s}}\), we have

\[
P_{\omega}^{y,0}(X_{\Delta(B_{mk\sqrt{s}}, s)} \in B_{2r}) \leq C_k P_{\omega}^{y,0}(X_{\Delta(B_{mk\sqrt{s}}, s)} \in B_r).
\]

Here \(C_k\) depends only on \((k, d, \kappa)\). In particular, for any \(k \geq 1, |y| \leq k\sqrt{s}\),

\[
P_{\omega}^{y,0}(X_s \in B_{2r}) \leq C_k P_{\omega}^{y,0}(X_s \in B_r).
\]

Proof of Lemma 19: Since \(B_1 = \{0\}\), we only consider \(r \geq 1/2\). Fix \(s, r, m\) and let \(k_0 \geq 1\) be a large constant to be determined. For \(\rho \geq 0, k \geq k_0, \) define \(L_{k, \rho} = B_{k\rho} \times \{s - \rho^2\}\) and (See Fig. 3.)

\[
D_{k, \rho} = \bigcup_{R \leq \rho} L_{k, R} = \left\{ (x, t) \in \mathbb{Z}^d \times (-\infty, s) : |x|/k \leq \sqrt{s} - t \leq \rho \right\}.
\]

For any \(R \leq \rho\), by Lemma 15, there exists \(\alpha_k > 0\) depending on \((k, \kappa, d)\) such that

\[
\min_{\hat{x} \in L_{k, R}} P_{\omega}^\hat{x}(X_{\Delta(B_{mk\rho}, s)} \in B_r) \geq \min_{x \in B_{kR}} P_{\omega}^x(X_{\Delta(B_{2kR}, s)} \in B_r)
\]

\[
\geq \left( \frac{r}{2kR} \wedge \frac{1}{2} \right)^{\alpha_k}.
\]

Let \(\beta_k > 1\) be a large constant to be determined later. Then, letting

\[
v_\rho(\hat{x}) = (8k)^{\alpha_k} (\beta_k + 1) P_{\omega}^\hat{x}(X_{\Delta(B_{mk\rho}, s)} \in B_r) - P_{\omega}^\hat{x}(X_{\Delta(B_{mk\rho}, s)} \in B_2r),
\]

we get \(\inf_{D_{k, R}} v_\rho \geq (\beta_k + 1) \left( \frac{4r}{R} \wedge 2k \right)^{\alpha_k} - 1\) for \(0 < R \leq \rho\). In particular, \(\inf_{D_{k, (4r)} \wedge \rho} v_\rho \geq \beta_k\) for \(\rho \geq 0\). Set

\[
R_\rho = \sup\{ R \in [0, m\rho] : \inf_{D_{k, R}} v_\rho \geq 0 \}.
\]
Fig. 3 The shaded region is $D_{k,\rho}$. Clearly, $R_\rho \geq (4r) \wedge \rho$. We will prove that

$$R_\rho \geq \rho \text{ for all } \rho > 0. \tag{15}$$

Assuming (15) fails, then $R_\rho < \rho$ for some $\rho > 4r$. We will show that this is impossible via contradiction. First, for such $\rho > 4r$, we claim that there exists a constant $\gamma = \gamma(d, \kappa) > 0$ such that

$$\min_{L_{1,R}} v_\rho \geq \beta_k \left( \frac{r}{R} \right)^\gamma$$

for all $R \in [2r, R_\rho)$. \tag{16}

By Lemma 15, $g(R) := \min_{s \in B_R} P^x,s,R^2(X_{\Delta(B_{2R},s,(R/2)^2)} \in B_R) \geq C$. Further, by the Markov property and that $R_\rho < \rho$, for $R \in [2r, R_\rho)$, $n \geq 1$,

$$\min_{s \in B_R} P^x,s,R^2(X_{\Delta(D_{k,\rho},s,(R/2)^2)} \in B_{R/2^n}) \geq g(R) \cdots g(R/2^n-1) \geq C^n,$$

Since $v_\rho(\hat{X}_t)$ is a martingale in $B_{mk\rho} \times (-\infty, s)$ and that $v_\rho \geq 0$ in $D_{k,\rho}$, choosing $n$ such that $R/2^n \leq r < R/2^n-1$, the above inequality yields

$$v_\rho(x, s - R^2) \geq P^x,s,R^2(X_{\Delta(D_{k,\rho},s,(R/2)^2)} \in B_{R/2^n}) \inf_{D_{k,\rho}} v_\rho$$

$$\geq C^n \beta_k \geq \beta_k \left( \frac{r}{R} \right)^c$$

for $R \in [2r, R_\rho)$ and $x \in B_R$. Display (16) is proved.

Next, we will show that for $R \in [2r, R_\rho)$,

$$f_\rho(R) := \sup_{\partial B_{k\rho} \times [s-R^2,s]} v^-_\rho \leq \left( \frac{r}{R} \right)^{-c/\log q_k}, \tag{17}$$

where $v^-_\rho = \max\{0, -v_\rho\}$, $q_k = 1 - \frac{c_1}{k}$ and $c_1 > 0$ is a constant to be determined. Noting that $v^-_\rho = 0$ in $D_{k,\rho} \cup (B_{2r}^c \times \{s\})$ and that $v^-_\rho(\hat{X}_t)$ is a sub-martingale, we know that $f_\rho(R)$ is a decreasing function for $R \in \left( \frac{2r}{k}, R_\rho \right)$. Further, for any
(x, t) ∈ ∂B_{kR} × [s − R^2, s) with q_k R ∈ (\frac{2r}{R}, R_\rho), by the optional stopping lemma, in view of Lemma 13,

\[ v_\rho^-(x, t) \leq P_{\omega}^{x, t}(X_a ∈ \partial B_{kq_k R} \text{ for some } a ∈ [0, R^2]) \leq \frac{f(q_k R)}{\sqrt{R}} \leq \frac{f(q_k R)}{R} \leq 2^{−n} f(q_k R) \leq \left( \frac{R}{R} \right)^{−c/\log q_k} f_\rho(r). \]

Inequality (17) then follows from the fact \( v_\rho^- \leq 1 \).

Finally, if \( R_\rho < \rho \) for some \( \rho > 4r \), let \( \tau = \inf\{t \geq 0 : \hat{X}_t \notin (B_{mk_\rho} × (−\infty, s)) \setminus (B_{kR/2} × [s − (R/2)^2, s])\}. \) Since \( v_\rho = 0 \) on \( \partial\rho(B_{mk_\rho} × (−\infty, s)) \setminus (B_{2r} × [s]) \), by the optional stopping lemma, for \( R ∈ [R_\rho, 2R_\rho] \) and \( x ∈ B_{kR} \),

\[ v_\rho(x, s − R^2) = E_{\omega}^{x, s − R^2}[v_\rho(\hat{X}_\tau) 1_{\hat{X}_\tau ∈ B_{kR/2} × [s − (R/2)^2] \text{ or } \hat{X}_\tau ∈ \partial B_{kR/2} × [s − (R/2)^2, s]}] \geq P_{\omega}^{x, s − R^2}(X_{\Delta(B_{kR/2} × [s − (R/2)^2])} ∈ B_{R/2}) \min_{L_{1, R/2}} v_\rho − f_\rho(R/2) \]

\[ \geq A_k \beta_k \left( \frac{2r}{R} \right)^{2r} − \left( \frac{2r}{R} \right)^{−c/\log q_k}, \]

where \( A_k \) depends on \( (k, \kappa, d) \). Taking \( k_0 > c_1 \) to be big enough such that \( −c/\log q_k > γ \) for \( k ≥ k_0 \) and choosing \( \beta_k > A_k^{−1} \), the above inequality then implies \( \inf_{D_{k,2R_\rho}} v ≥ 0 \), which contradicts our definition of \( R_\rho \). Display (15) is proved, and therefore, \( \min_{x ∈ B_{kR/\sqrt{s}}} v_\sqrt{s}(x, 0) ≥ 0 \). The lemma follows.

**Corollary 20** Let \( \omega ∈ \Omega_\kappa \) and \( k_0 \) as in Lemma 19. For any \( r > 0, k ≥ k_0, m ≥ 2, s > 0 \) and \( y ∈ B_{kR/\sqrt{s}} \), we have

\[ \sup_{t ≥ 0: |t − s| ≤ r^2} P_{\omega}^{y, 0}(X_{\Delta(B_{mkR/\sqrt{s}}, t)} ∈ B_{2r}) ≤ C_k P_{\omega}^{y, 0}(X_{\Delta(B_{mkR/\sqrt{s}}, s)} ∈ B_r), \]

where \( C_k \) depends on \( (k, \kappa, d) \). In particular, for any \( k ≥ 1, |y| ≤ k\sqrt{s} \),

\[ \sup_{t ≥ 0: |t − s| ≤ r^2} P_{\omega}^{y, 0}(X_t ∈ B_{2r}) ≤ C_k P_{\omega}^{y, 0}(X_{s} ∈ B_r). \]
Proof It suffices to consider \( r < \sqrt{s} \), because otherwise, by Lemma 15, the right side is bigger than a constant. When \( t \in [0 \vee (s - r^2), s] \),

\[
\min_{y \in B_{k\sqrt{s}}} P^y_\omega (X_{\Delta(B_{mk\sqrt{s}}, s)} \in B_{4r}) \\
\geq \min_{y \in B_{k\sqrt{s}}} P^y_\omega (X_{\Delta(B_{mk\sqrt{s}}, t)} \in B_{2r}) \min_{x \in B_{2r}} P^x_\omega (X_{s-t} \in B_{2r}(x))
\]

Lemma \[ \geq C \min_{y \in B_{k\sqrt{s}}} P^y_\omega (X_{\Delta(B_{mk\sqrt{s}}, t)} \in B_{2r}). \]

By Lemma 19, we can replace \( 4r \) in the above inequality by \( r \).

When \( t \in [s, s + r^2] \), for any \( y \in B_{k\sqrt{s}} \),

\[
P^y_\omega (X_{\Delta(B_{mk\sqrt{s}}, t)} \in B_{2r}) \\
\leq \sum_{n=0}^{\infty} \sum_{x:|x| \in [2^n r, 2^{n+1} r)} P^y_\omega (X_{\Delta(B_{mk\sqrt{s}}, s)} = x) P^x_\omega (X_{t-s} \in B_{2r})
\]

Corollary \[ \leq C \sum_{n=0}^{\infty} P^y_\omega (X_{\Delta(B_{mk\sqrt{s}}, s)} \in B_{2^n r})(e^{-c2^n r} + e^{-c4^n}). \]

Observing that (cf. Lemma 19)

\[
P^y_\omega (X_{\Delta(B_{mk\sqrt{s}}, s)} \in B_{2^n r}) \leq C^n P^y_\omega (X_{\Delta(B_{mk\sqrt{s}}, s)} \in B_r),
\]

our proof is complete. \( \square \)

We define, for \( \hat{x} = (x, t) \in \mathbb{R}^d \times \mathbb{R} \), the parabolic balls

\[
Q_r(\hat{x}) = B_r(x) \times [t, t + r^2), \quad Q_r = Q_r(\hat{0}). \quad (18)
\]

Proof of Theorem 8: Let \( k_0 \geq 2 \) be as in Lemma 19. Recall \( \bar{\omega}_t, \Delta, Q_r \) in (1), (11), (18). For fixed \( \xi \in \Omega_K \), define a probability measure \( \mathbb{Q}_R = \mathbb{Q}_R^\xi \) on \( \{\theta \xi \xi : \hat{x} \in Q_R\} \) such that for any bounded measurable \( f \in \mathbb{R}^\Omega \),

\[
E^{\mathbb{Q}_R}[f] = \frac{1}{C_R} E^{\xi}_{\hat{x}} \int_0^{\Delta(B_{2k_0 R}, R^2)} f(\tilde{\xi}_s) 1_{\hat{x}_s \in Q_R} ds,
\]

where \( C_R \) is a normalization constant such that \( \mathbb{Q}_R \) is a probability.
First, we claim that \( C_R \approx R^2 \). Clearly, \( C_R \leq 2R^2 \). On the other hand,

\[
C_R = E_\xi^{0, -R^2} \left[ \int_0^{\Delta(B_{2k_0R}, R^2)} \mathbb{1}_{\hat{X}_t \in \mathcal{Q}_R} \, ds \right] \\
\geq P_\xi^{0, -R^2} (X_{\Delta(B_{2k_0R}, 0)} \in B_{R/2}) \min_{x \in B_{R/2}} E_\xi^{x, 0} [\Delta(B_R, R^2)]
\] \\

**Lemma 15** \[
\geq C \min_{x \in B_{R/2}} E_\xi^{x, 0} [\Delta(B_R, R^2)].
\]

Since \( |X_t - X_0|^2 - \frac{d}{k} t \) is a supermartingale, denoting \( \tau = \Delta(B_R, R^2) \), we have

\[
0 \geq E_\xi^{x, 0} [|X_\tau - x|^2 - \frac{d}{k} \tau].
\]

Hence for any \( x \in B_{R/2} \),

\[
E_\xi^{x, 0} [\tau] \geq c E_\xi^{x, 0} [|X_\tau - x|^2] \geq CR^2 P_\xi^{x, 0} (\tau < R^2),
\]

which implies \( E_\xi^{x, 0} [\tau] \geq c R^2 \). Thus \( C_R \geq CR^2 \) and so \( C_R \approx R^2 \).

Next, since \( \Omega \) is pre-compact, by Prohorov’s theorem, there is a subsequence of \( \mathcal{Q}_R \) that converges weakly, as \( R \to \infty \), to a probability measure \( \hat{\Omega} \) on \( \Omega \). We will show that \( \hat{\Omega} \) is an invariant measure of the process \( (\hat{\omega}_t) \). Indeed, let \( p_R = p_{R, \xi} \) denote the kernel \( p_R(\hat{x}; y, s) := p_\xi^s (\hat{X}_{\Delta(B_{2k_0R}, s)} = (y, s)) \). Then, letting \( \mathcal{L} f(\omega) = \sum_{e} \omega_0(0, e) [f(\theta_e, \epsilon_0) - f(\omega)] + \partial_t f(\theta_0, \omega)|_{t=0} \) denote the generator of the process \( (\omega_t) \), and \( \hat{y} := (y, s) \), we have

\[
E_{\mathcal{Q}_R} [\mathcal{L} f(\omega)] = C_R^{-1} \sum_{y \in B_R} \int_0^{R^2} p_R(0, -R^2; \hat{y}) \mathcal{L} f(\theta_0 \xi) \, ds \tag{19}
\]

for \( f \in \operatorname{dom}(\mathcal{L}) \), where \( \operatorname{dom}(\mathcal{L}) \) denotes the domain of the generator \( \mathcal{L} \). Note that similar to \( \rho_\omega \), the function \( v(\hat{x}) = p_\rho(0, -R^2; \hat{x}) \) satisfies the equality (3): \( \mathcal{L}^T v(\hat{x}) = 0 \) for \( \hat{x} \in B_{2R} \times (-R^2, R^2) \), where \( \mathcal{L}^T v(\hat{x}) = \sum_y v(y, t) \omega_t(y, x) - \partial_t v(x, t) \). Hence, using integration by parts,

\[
\left| \sum_{y \in B_R} \int_0^{R^2} p_R(0, -R^2; \hat{y}) \mathcal{L} f(\theta_0 \xi) \, ds \right|
\leq C \|f\|_\infty \int_0^{R^2} \sum_{y \in \overline{B}_R \setminus \hat{B}_R} p_R(0, -R^2; \hat{y}) \, ds + 2 \|f\|_\infty \tag{20}
\]

for all \( f \in \operatorname{dom}(\mathcal{L}) \), where \( \overline{B}_R = \{ x \in B_R : x \sim \partial B_R \} \). Observe that

\[
u(\hat{x}) = \int_0^{R^2} \sum_{y \in \overline{B}_R \setminus B_R} p_R(\hat{x}; \hat{y}) \, ds = E_\xi^{\hat{x}} \int_0^{\Delta(B_{2k_0R}, R^2)} \mathbb{1}_{\hat{X}_t \in \overline{B}_R \setminus B_R \times (0, R^2)} \, dt
\]
satisfies \( L_\xi u(\hat{x}) = -\mathbb{1}_{\hat{x} \in B_R \setminus B_{2\eta R} \times (0,R^2)} \) for \( \hat{x} \in \mathcal{D} := B_{2\eta R} \times [-R^2, R^2] \) and \( u|_{\partial \mathcal{D}} = 0 \). By the parabolic maximum principle (Theorem A.3.1), we get \( u(0, -R^2) \leq CR^{(2d+1)/(d+1)} \). Hence, by (19), (20), and \( C \sim R^2 \),

\[
\lim_{R \to \infty} E_{Q_R}[\mathcal{L} f] = 0 \quad \forall \text{ bounded function } f \in \text{dom}(\mathcal{L}),
\]

and so \( E_{\tilde{Q}}[\mathcal{L} f] = 0 \), which implies that \( \tilde{Q} \) is an invariant measure of \( (\tilde{\omega}_t) \).

Furthermore, we will show that \( \tilde{Q} \ll \mathbb{P} \). Notice that the function

\[
w(\hat{x}) := E_{\tilde{x}}^\xi [\int_0^{\Delta(B_{2\eta R}, R^2)} f(\tilde{\xi}_s) \mathbb{1}_{\tilde{X}_s \in Q_R} \, ds]
\]

satisfies \( L_\xi w(\hat{x}) = -f(\hat{x} \xi) \mathbb{1}_{\hat{x} \in Q_R} \in \mathcal{D} \) and \( w|_{\partial \mathcal{D}} = 0 \). By Theorem A.3.1, for any bounded measurable \( f \in \mathbb{R}^\Omega \),

\[
E_{Q_R}[f] \leq CR^{-2} w(0, -R^2) \leq C \left[ \int_0^R \sum_{x \in B_R} |f(\theta_{x,t} \xi)|^{d+1} \, dt \right]^{1/(d+1)},
\]

which, by the multi-dimensional ergodic theorem, yields \( E_{\tilde{Q}}[f] \leq C \| f \|_{L^{d+1}(\mathbb{P})} \) as we take \( R \to \infty \). So \( \tilde{Q} \ll \mathbb{P} \). By Theorem 1, \( \tilde{Q} = Q \).

Finally, since \( Q_R \Rightarrow Q \), for any bounded measurable \( f \in \mathbb{R}^\Omega \),

\[
E_{\mathbb{P}}[\rho_\omega(B_r, t) f] = \sum_{x \in B_r} E_{Q}\{ f(\theta_{x,-t}\omega) \}
\]

\[
= \lim_{R \to \infty} \sum_{x \in B_r, y \in B_r} \int_0^R \int_0^{R^2} P_{\xi}^{0,-R^2}(X_{\Delta(B_{2\eta R}, s)} = y) f(\theta_{x+y,s-t\xi}) \, ds / CR. \quad (21)
\]

Hence, for any measurable function \( f \geq 0, |t| \leq r^2 \), and \( \mathbb{P} \)-a.a. \( \xi \),

\[
E_{\mathbb{P}}[\rho_\omega(B_r, 0) f] \geq \lim_{R \to \infty} \sum_{z \in B_{2R-r}} \int_0^R \int_0^{R^2} P_{\xi}^{0,-R^2}(X_{\Delta(B_2, s)} \in B_r(z)) f(\theta_{z,s}\xi) \, ds / CR
\]

\[
\geq C \lim_{R \to \infty} \sum_{z \in B_{2R-r}} \int_0^R \int_0^{R^2} P_{\xi}^{0,-R^2}(X_{\Delta(B_{2\eta R}, s+t)} \in B_{2r}(z)) f(\theta_{z,s}\xi) \, ds / CR
\]

\[
\geq C \lim_{R \to \infty} \sum_{x \in B_2, y \in B_{2R-3r}} \int_0^R \int_0^{R^2} P_{\xi}^{0,-R^2}(X_{\Delta(B_{2\eta R}, s)} = y) f(\theta_{x+y,s-t\xi}) \, ds / CR
\]

\[
(21) \quad = C E_{\mathbb{P}}[\rho_\omega(B_{2r}, 0) f].
\]

Since \( f \) is arbitrary, the theorem follows. \( \square \)
**Remark 21** By Theorem 8, for any \( r \geq 1, \)
\[
\frac{c}{r^2} \int_0^{r^2} \rho_\omega(B_r, s) ds \leq \rho_\omega(B_r, 0) \leq \frac{C}{r^2} \int_0^{r^2} \rho_\omega(B_r, s) ds.
\] (22)

Hence, by the multi-dimensional ergodic theorem, for \( \mathbb{P} \)-almost every \( \omega \),
\[
c \leq \lim_{r \to \infty} \frac{1}{|B_r|} \rho_\omega(B_r, 0) \leq \lim_{r \to \infty} \frac{1}{|B_r|} \rho_\omega(B_r, 0) \leq C.
\] (23)

Display (23) will be used in the Proof of Theorem 4 in Sect. 6.

### 3.2 \( A_p \) property and proof of the PHI for \( \mathcal{L}_\omega \)

The goal of this subsection is to obtain a negative moment bound for the density \( \rho_\omega \) and to prove the PHI for \( \mathcal{L}_\omega \). We will first obtain a reverse Hölder inequality for adjoint solutions, and then use it to imply the negative moment bounds and the PHI.

We endow \( \mathbb{Z}^d \) with the discrete topology and counting measure, and equip \( \mathbb{Z}^d \times \mathbb{R} \) with the corresponding product topology and measure (where \( \mathbb{R} \) has the usual topology and measure). For \( \mathcal{D} \subset \mathbb{Z}^d \times \mathbb{R} \), let \( |\mathcal{D}| \) be its measure, and denote the integration over \( \mathcal{D} \) by \( \int_{\mathcal{D}} f \). For instance,
\[
\int_{B_R \times [0, T]} f = \sum_{x \in B_R} \int_0^T f(x, t) dt,
\] (24)

and \( |\mathcal{D}| = \int_{\mathcal{D}} 1 \). For \( p > 0 \), we define a norm
\[
\| f \|_{p, p} := \left( \int_{\mathcal{D}} |f|^p / |\mathcal{D}| \right)^{1/p}.
\] (25)

We write \( f(\mathcal{D}) := \int_{\mathcal{D}} f \).

**Definition 22** A function \( v \in \mathbb{R}^{\mathbb{Z}^d \times \mathbb{R}} \) is called an adjoint solution of \( \mathcal{L}_\omega \) in \( \mathcal{D} = B_R \times [T_1, T_2] \) if \( \int_{\mathcal{D}} v \mathcal{L}_\omega \phi = 0 \) for any test function \( \phi(x, t) \in \mathbb{R}^{\mathbb{Z}^d \times \mathbb{R}} \) that is supported on \( B_R \times (T_1, T_2) \) and smooth in \( t \).

For \( \hat{x} = (x_1, \ldots, x_d, t) \), define parabolic cubes with side-length \( r > 0 \) as
\[
K_r(\hat{x}) = (\prod_{i=1}^d [x_i - r, x_i + r] \cap \mathbb{Z}^d) \times [t, t + r^2), \quad K_r = K_r(\hat{0}).
\] (26)

We say that a function \( w \in \mathbb{R}^{\mathbb{Z}^d \times \mathbb{R}} \) satisfies the reverse Hölder inequality \( RH_q(\mathcal{D}) \), \( 1 < q < \infty \), if for any parabolic subcube \( K \) of \( \mathcal{D} \),
\[
\|w\|_{K, q} \leq C \|w\|_{K, 1}.
\] (RH_q)
We say that \( w \) belongs to the \( A_p(\mathcal{D}) \) class (with \( A_p \) bound \( A \)), \( 1 < p < \infty \), if there exists \( A < \infty \) such that, for any parabolic subcube \( K \) of \( \mathcal{D} \),

\[
\|w\|_{K,1} \|w/K,1/(p-1) \leq A \quad (A_p)
\]

The following lemma is useful in the derivation of reverse Hölder inequalities for adjoint solutions.

**Lemma 23** Recall \( \|\cdot\|_{\mathcal{D},p} \) in (25) and the parabolic balls \( Q_r \) in (18). Let \( \omega \in \Omega_\kappa \). For any non-negative adjoint solution \( v \) of \( L_\omega \) in \( Q_{2r} \), \( r > 10 \),

\[
\|v\|_{Q_r,(d+1)/d} \leq C \|v\|_{Q_{5r/2},1}.
\]

**Proof** Denote the balls of radius \( r \) by

\[
O_r = \{ x \in \mathbb{R}^d : |x| < r \} \quad \text{and} \quad O_r(y) = y + O_r, \quad y \in \mathbb{R}^d.
\]

Let \( \phi_0 \geq 0 \) be a smooth (with respect to \( t \)) function supported on \( O_{3/2} \times [0, 9/4) \) with \( \phi_0|_{O_1 \times (0,1)} = 1 \) and set \( \phi(x,t) = \phi_0(x/r, t/r^2) \). Let \( f \) be any non-negative smooth function supported on \( Q_r \) with \( \|f\|_{Q_r,d+1} = 1 \) and let \( u \in [0, \infty)^{Z_d \times \mathbb{R}} \) be supported on \( Q_{9r/5} \) with \( L_\omega u = -f \) in \( Q_{9r/5} \). Since

\[
0 = \int v L_\omega(\phi u) = \int v \phi L_\omega u + \int v u L_\omega \phi + \sum_{x,y} \int_{\mathbb{R}} v(x,t) \omega_t(x,y) \nabla_{x,y} u \nabla_{x,y} \phi dt,
\]

where \( \nabla_{x,y} u(\cdot, t) := u(x,t) - u(y,t) \) and (cf. (24)) \( \int = \int_{Z_d \times \mathbb{R}} \), we get

\[
\int v \phi f = \int v u L_\omega \phi + \sum \int_{\mathbb{R}} v(x,t) \omega_t(x,y) \nabla_{x,y} u \nabla_{x,y} \phi dt =: I + II.
\]

By the maximum principle Theorem A.3.1, \( u \leq Cr^2 \|f\|_{Q_r,d+1} \leq Cr^2 \). Thus, using \( |L_\omega \phi| \leq C/r^2 \), we get \( |I| \leq Cv(Q_{5r/2}) \). Further, noting that

\[
| \sum_{x,y} \int_{\mathbb{R}} v(x,t) \omega_t(x,y) (\nabla_{x,y} u)^2 dt | = | \int v L_\omega(u^2) - 2 \int v u L_\omega u | = 2 | \int v u L_\omega u | \leq Cr^2 \int vf,
\]
we have
\[ |\Pi| \leq \left( \sum_{x,y} \int_{\mathbb{R}} v(x, t) \omega_t(x, y)(\nabla_x, \nabla_y \phi)^2 \, dt \right)^{1/2} \left( \sum_{x,y} \int_{\mathbb{R}} v(x, t) \omega_t(x, y)(\nabla_x, \nabla_y u)^2 \, dt \right)^{1/2} \]
\[ \leq C v(Q_{3r/2})^{1/2} \left( \int v f \right)^{1/2}. \]

Hence we obtain
\[ \int v f \leq \int v \phi f \leq C v(Q_{3r/2}) + C v(Q_{3r/2})^{1/2} \left( \int v f \right)^{1/2} \]
and
\[ v(Q_{3r/2}) \geq c \int v f. \]
The lemma follows by taking supremum over all \( f \) with \( \|f\|_{Q_r, d+1} = 1 \).

Recall the stopping time \( \Delta \) in (11). For \( R > 0, \hat{y} \in B_{2R} \times \mathbb{R} \), let
\[ g_R(\hat{y}; x, t) = P_{\omega}(X_{\Delta(B_{2R}, t)} = x). \] (28)

Using Lemma 23, we obtain the following reverse Hölder inequalities for the functions \( \rho \omega(\cdot) \) and \( g_R(\hat{y}; \cdot) \).

**Corollary 24** Let \( \omega \in \Omega_\kappa, R > 0. \) Recall \( k_0 \) in Lemma 19.

(i) \( \rho \omega \) satisfies RH\( d+1/d \)\( (\mathbb{Z}^d \times \mathbb{R}) \).

(ii) For any \( y \in B_R, v_y(\hat{x}) = g_R(y, 0; \hat{x}) \) satisfies
\[ RH_{d+1/d}(B_{2R}/2 \times [R^2/(2k_0^2), R^2/k_0^2]). \]

**Proof** Note that \( \rho \omega, v_y \) are adjoint solutions with volume-doubling properties Theorem 8 and Corollary 20. The corollary follows from Lemma 23. \( \square \)

As a classical result in harmonic analysis, reverse Hölder inequalities imply \( (A_p) \). See e.g., [10, pg.246-249], [29, pg. 213-214]. We state this fact for our discrete setting as below, and include its proof in Sect. A.6.

**Lemma 25** Let \( K^0 \subset \mathbb{Z}^d \times \mathbb{R} \) be a parabolic cube with side-length \( r > 0. \) If a function \( w > 0 \) on \( K^0 \) satisfies RH\( q \)\( (K^0) \), \( q > 1 \), then

(i) \( w \in A_p(K^0) \) for some \( 1 < p < \infty \);

(ii) \( \frac{w(E)}{w(K)} \geq C \frac{|E|}{|K|} \) for all \( E \subset K \) where \( K \neq \emptyset \) is a subcube of \( K^0 \).

With Lemma 25 and the reverse Hölder inequalities for \( \rho \omega \) and \( g_R(y, 0; \cdot) \), the following \( A_p \) bounds and measure estimate follow immediately.

**Theorem 26** Let \( \omega, R, k_0, v_y \) be the same as in Corollary 24. There exist \( p = p(d, \kappa) > 1, A = A(d, \kappa) \) such that, for \( \mathbb{P} \)-a.e. \( \omega \),

(a) \( \rho \omega \in A_p(\mathbb{Z}^d \times \mathbb{R}) \) with \( A_p \) bound \( A \). As a consequence,
\[ E_\mathbb{P}[\rho \omega_1/(p-1)] < \infty; \] (29)
(b) For any $y \in B_R$, $v_y$ belongs to $A_p(B_{R/2} \times [R^2/(2k_0^2), R^2/k_0^2])$ with $A_p$ bound $A$. Moreover, for any $E \subset K$ where $K$ is a parabolic subcube of $B_{R/2} \times [R^2/(2k_0^2), R^2/k_0^2]$, 
\[ g_R(y, 0; E) \geq C \left( \frac{|E|}{|K|} \right)^c. \] (30)

**Remark 27** In the elliptic non-divergence form PDE setting, the $A_p$ inequality for adjoint solutions was proved by Bauman [6], and estimate of the form (30) was used by Fabes and Stroock [17] to obtain a short proof of the elliptic Harnack inequality. Using (30), we will prove the PHI (Proposition 5). Our proof follows the ideas of [17].

**Proof of Proposition 5** Let $\ell_0 = 1/k_0^2$ and $\mathcal{D} = \{ x : |x|_\infty < R/\sqrt{d} \} \times [\ell_0 R^2/2, \ell_0 R^2]$. We only prove a weaker version $\sup_{\mathcal{D}} u \leq C \min_{x \in B_{R/\sqrt{d}}} u(x, 0)$. The PHI then follows by iteration. Indeed, assume $\min_{x \in B_{R/\sqrt{d}}} u(x, 0) = u(y, 0) = 1$ for $y \in B_{R/\sqrt{d}}$. Let $E_\lambda = \{ \hat{x} \in \mathcal{D} : u(\hat{x}) \geq \lambda \}$. By Lemma 15, $g_R(y, 0; \mathcal{D}) > CR^2$. Moreover, for $s \in [\ell_0 R^2/2, \ell_0 R^2]$, $1 = u(y, 0) \geq \lambda g_R(y, 0; E_\lambda \cap \{(x, t) : t = s\})$, and so
\[ 1 \geq C \lambda g_R(y, 0; E_\lambda)/R^2 \geq C \lambda (|E_\lambda|/|\mathcal{D}|)^c. \]

Hence $|E_\lambda|/|\mathcal{D}| \leq C \lambda^{-\gamma}$ for some $\gamma > 0$. Therefore, for $0 < p < \gamma/2$,
\[ \|u\|_{\mathcal{D}, p} \leq \left[ 1 + p \int_{|E_\lambda||\mathcal{D}|}^{\infty} \lambda^{p-1} |E_\lambda||\mathcal{D}|d\lambda \right]^{1/p} < C' = C' \min_{x \in B_{R/\sqrt{d}}} u(x, 0) < \infty. \]

This inequality, together with the mean value inequality (Theorem A.4.1), completes our proof. \(\square\)

## 4 Estimates of caloric functions near the boundary

The purpose of this section is to establish estimates (Propositions 28 and 30) of $\omega$-caloric functions near the parabolic boundary. These estimates are important tools for our proof of the PHI for $L^*_\omega$ in Sect. 5.

For $x \in \mathbb{Z}^d$, $A \subset \mathbb{Z}^d$, let
\[ \text{dist}(x, A) := \min_{y \in A} |x - y|_1. \]

### 4.1 An elliptic-type Harnack inequality

**Proposition 28** (Interior elliptic-type Harnack inequality) Assume $\omega \in \Omega_k$, $R \geq 2$. Suppose $u \geq 0$ is an $\omega$-caloric function on $Q_R$ with $u = 0$ on $\partial B_R \times [0, R^2)$. Then
for $0 < \delta \leq \frac{1}{4}$, letting $Q^\delta_R := B(1-\delta)R \times [0, (1 - \delta^2)R^2)$, there exists a constant $C = C(d, \kappa, \delta)$ such that

$$\sup_{Q^\delta_R} u \leq C \inf_{Q^\delta_R} u.$$ (31)

To prove Proposition 28, we need a so-called Carlson-type estimate. For parabolic differential operators in non-divergence form, this kind of estimate was first proved by Garofalo [18] (see also [16, Theorem 3.3]).

Lemma 29 (Carlson-type estimate) Assume $\omega \in \Omega_\kappa$, $R > 2r > 0$. Suppose $u \geq 0$ is an $\omega$-caloric function on $(B_R \setminus \bar{B}_{R-2r}) \times [0, 3r^2)$ with $u = 0$ on $\partial B_R \times [r^2, 3r^2)$. Then, with the convention $\sup \emptyset = -\infty$, we have

$$\sup_{(B_R \setminus B_{R-r}) \times [r^2, 2r^2)} u \leq C \min_{y \in \partial B_{R-r}} u(y, 0).$$ (31)

Proof Set $\mathcal{D} = (B_R \setminus \bar{B}_{R-2r}) \times [r^2, 3r^2)$. For $\hat{x} = (x, t) \in \mathcal{D}$, let $d_1(\hat{x}) = \sup\{\rho \geq 0 : B_\rho(x) \subset B_R \setminus \bar{B}_{R-2r}\} \geq 1$.

First, we show that there exists $\gamma = \gamma(d, \kappa)$ such that

$$\sup_{\hat{x} \in \mathcal{D}} \left(\frac{d_1(\hat{x})}{r}\right)^\gamma u(\hat{x}) \leq C \min_{y \in \partial B_{R-r}} u(y, 0).$$ (32)

Indeed, for any $\hat{x} = (x, t) \in \mathcal{D}$, we can find a sequence of $n \leq C \log(r/d_1(\hat{x}))$ balls with increasing radii $r_k := c2^kd_1(\hat{x})$:

$$B_{r_1}(x_1) \subset B_{r_2}(x_2) \subset \cdots \subset B_{r_n}(x_n) \subset B_R \setminus \bar{B}_{R-2r}$$

such that $x_1 = x$, $\text{dist}(x_n, \partial B_{R-r}) \leq r/2$, and $t - r_n^2 \geq r^2/2$. By Proposition 5,

$$u(x, t) \leq Cu(x_1, t - r_1^2) \leq \cdots \leq C^n u(x_n, t - r_n^2) \leq C \left(\frac{r}{d_1(\hat{x})}\right)^C \min_{y \in \partial B_{R-r}} u(y, 0),$$
where in the last inequality we applied Proposition 5 to a chain of parabolic balls with spatial centers at \( \partial B_{R-r} \) and radius \( cr \). Display (32) is proved.

Next, with \( \gamma \) as in (32), letting \( d_0(\hat{x}) = \sup\{\rho \geq 0 : Q_{\rho}(\hat{x}) \subset (\mathbb{Z}^d \setminus \bar{B}_{R-2r}) \times [r^2, 3r^2] \} \), we claim that

\[
\sup_{\hat{x} \in \mathcal{D}} d_0(\hat{x})^{-\gamma} u(\hat{x}) \leq \epsilon^{-\gamma} \sup_{\hat{y} \in \mathcal{D}} d_1(\hat{y})^{-\gamma} u(\hat{y}),
\]

where \( \epsilon = \epsilon(d, \kappa) \in (0, 1/5) \) is to be determined. It suffices to show that \( \sup_{\mathcal{D}} d_0^{-\gamma} u \) is achieved at \( \hat{x} \in \mathcal{D} \) with \( \epsilon d_0(\hat{x}) \leq d_1(\hat{x}) \). Indeed, if \( \epsilon d_0(\hat{x}) > d_1(\hat{x}) \), then \( B_{2d_1(\hat{x})} \setminus B_R \neq \emptyset \), and for any \( \hat{y} = (y, s) \in Q_{2d_1(\hat{x})}(\hat{x}) \cap \mathcal{D} \),

\[
d_0(\hat{x}) \leq d_0(\hat{y}) + |x - y| + |t - s|^{1/2} \leq d_0(\hat{y}) + 4d_1(\hat{x}) \leq d_0(\hat{y}) + 4\epsilon d_0(\hat{x})
\]

and so \( d_0(\hat{x}) \leq (1 - 4\epsilon)^{-1} d_0(\hat{y}) \). Moreover, by Corollary 16,

\[
d_0(\hat{x})^{-\gamma} u(\hat{x}) \leq [1 - P^o_0(X, \text{ exits } B_{2d_1(\hat{x})}(x) \cap B_R \text{ from } \partial B_R \text{ before time } d_1^2(\hat{x}))]
\times d_0(\hat{x})^{-\gamma} \sup_{(B_{2d_1(\hat{x})}(x) \cap B_R) \times [r^2, 3r^2]} u
\]

\[
\leq (1 - c_0)(1 - 4\epsilon)^{-\gamma} \sup_{\mathcal{D}} d_0^{-\gamma} u
\]

for a constant \( c_0 \in (0, 1) \). Thus, when \( \epsilon d_0(\hat{x}) > d_1(\hat{x}) \), choosing \( \epsilon > 0 \) so that \( (1 - c_0)(1 - 4\epsilon)^{-\gamma} < 1 - \frac{c_0}{2} \), we get \( d_0(\hat{x})^{-\gamma} u(\hat{x}) < (1 - \frac{c_0}{2}) \sup_{\mathcal{D}} d_0^{-\gamma} u \). Display (33) is proved. Inequality (31) follows from (32) and (33).

**Proof of Proposition 28:** Since \( u = 0 \) on \( \partial B_R \times [0, R^2) \),

\[
\sup_{Q_R^\delta} u \leq \sup_{Q_R^\delta} u
\]

\[
\leq C \sup_{B_{(1-\delta)R} \times [R^2 - \frac{1}{2}(\delta R)^2]} u
\]

\[
\leq C(d, \kappa, \delta) \inf_{Q_R^\delta} u,
\]

where we used Lemma 29 and Proposition 5 in the second inequality.

**4.2 A boundary Harnack inequality**

For positive caloric functions with zero values on the spatial boundary, the following boundary PHI compares values near the spatial boundary and values inside, with time coordinates appropriately shifted.
Proposition 30 (Boundary PHI) Let $R > 0$. Suppose $u$ is a nonnegative $\omega$-caloric function on $(B_4 \setminus B_2) \times (-2R^2, 3R^2)$, and $u|_{\partial B_4 \times \mathbb{R}} = 0$. Then for any $\hat{x} = (x, t) \in (B_4 \setminus B_3) \times (-R^2, R^2)$, we have

$$C \frac{\text{dist}(x, \partial B_4)}{R} \max_{y \in \partial B_3} u(y, t + R^2) \leq u(\hat{x}) \leq C \frac{\text{dist}(x, \partial B_4)}{R} \min_{y \in \partial B_3} u(y, t - R^2).$$

Proposition 30 is a lattice version of [18, (3.9)]. In what follows we offer a probabilistic proof.

Proof of Proposition 30: Our proof uses the fact that $u(\hat{X}_t)$ is a martingale before exiting the region $\mathcal{D} := (B_4 \setminus B_2) \times (-2R^2, 3R^2)$.

For the lower bound, let $\tau_{3,4} := \inf\{s > 0 : X_s \notin B_4 \setminus \bar{B}_3\}$. By the optional stopping lemma, $u(\hat{x}) = E^x_\omega[u(\hat{X}_{\tau_{3,4} \wedge 0.5R^2})]$, and so

$$u(\hat{x}) \geq P^x_\omega(\tau_{3,4} < R^2/2, X_{\tau_{3,4}} \in \partial B_3) \inf_{\partial B_3 \times [t, t + 0.5R^2]} u \geq C \frac{\text{dist}(x, \partial B_4)}{R} \max_{y \in \partial B_3} u(y, t + R^2)$$

where in the last inequality we used Lemma 17 and applied Proposition 5 (to a chain of parabolic balls). The lower bound is obtained.

To obtain the upper bound, note that for $\hat{x} \in (B_4 \setminus \bar{B}_3) \times (-R^2, R^2)$,

$$u(\hat{x}) \leq \left[ \max_{z \in B_{4R} \setminus \bar{B}_3} u(z, t + \frac{R^2}{2}) \right] P^x_\omega(\tau_{3,4} \wedge 0.5R^2 \notin \partial B_4) \leq C \left[ \max_{z \in B_{3R} \setminus \bar{B}_3} u(z, t - \frac{R^2}{2}) \right] P^x_\omega(\tau_{3,4} \wedge 0.5R^2 \notin \partial B_4) \leq C \min_{z \in \partial B_3} u(z, t - R^2) \text{dist}(x, \partial B_4)/R,$$

where in the last inequality we applied Lemma 18 and used an iteration of the PHI for $\omega$-caloric functions (Proposition 5). \(\square\)

5 Proof of the PHI for the adjoint operator (Theorem 6)

In this section we will prove the PHI for $\mathcal{L}_\omega^*$. Our proof relies on a representation formula for $\omega^*$-caloric functions (Proposition 31), the parabolic volume-doubling property of $\rho_\omega$ (Theorem 8), the PHI (Proposition 5) and boundary PHI (Proposition 30) for $\omega$-caloric functions.

We define $\hat{Y}_t = (Y_t, S_t)$ to be the continuous-time Markov chain on $\mathbb{Z}^d \times \mathbb{R}$ with generator $\mathcal{L}_\omega^*$. The process $\hat{Y}_t$ can be interpreted as the time-reversal of $\hat{X}_t$. Denote by $P^y_{\omega^*}$ the quenched law of $\hat{Y}$ starting from $\hat{Y}_0 = (y, s)$ and by $E^y_{\omega^*}$ the corresponding expectation. Note that $S_t = S_0 - t$. 

\(\circled{3}\) Springer
For \( R > 0 \), \( \hat{x} = (x, t), \hat{y} = (y, s) \in B_R \times \mathbb{R} \) with \( s > t \), set
\[
\begin{align*}
p^x_0(\hat{x}; \hat{y}) &= P^x_{\omega}(X_{s-t} = y, s-t < \tau_R(\hat{X})), \\
p^y_0(\hat{y}; \hat{x}) &= P^{y,s}_{\omega}(Y_{s-t} = x, s-t < \tau_R(\hat{Y})),
\end{align*}
\]
where
\[
\tau_R(\hat{X}) := \inf\{t \geq 0 : X_t \notin B_R\}
\]
and \( \tau_R(\hat{Y}) \) is defined similarly. Note that
\[
p^{y,0}_R(\hat{y}; \hat{x}) = \frac{\rho_\omega(\hat{x})}{\rho_\omega(\hat{y})} p^x_0(\hat{x}; \hat{y}).
\]

**Proposition 31** For any \( \hat{y} = (y, s) \in B_R \times (0, \infty) \) and any non-negative \( \omega^s \)-caloric function \( v \) on \( B_R \times (0, s) \),
\[
v(\hat{y}) = \sum_{x \in \partial B_R, z \in B_R, x \sim z} \int_0^s \frac{\rho_\omega(x, t)}{\rho_\omega(\hat{y})} \omega_t(z, x)p^x_0(z, t; \hat{y})v(x, t)dt \\
+ \sum_{x \in B_R} \frac{\rho_\omega(x, 0)}{\rho_\omega(\hat{y})} p^x_0(x, 0; \hat{y})v(x, 0).
\]

**Proof** Write the two summations in the proposition as I and II. Clearly, \( II = E^\hat{y}_\omega[v(\hat{Y}_s)1_{\tau_R > s}] \). Since \( (v(\hat{Y}_t))_{t \geq 0} \) is a martingale, we have
\[
v(y, s) = E^\hat{y}_\omega[v(\hat{Y}_{\tau_R})1_{\tau_R \leq s}] + E^\hat{y}_\omega[v(\hat{Y}_s)1_{\tau_R > s}].
\]
So it remains to show \( I = E^\hat{y}_\omega[v(\hat{Y}_{\tau_R})1_{\tau_R \leq s}] \). We claim that for \( x \in \partial B_R \),
\[
P^y_\omega(Y_{\tau_R} = x, \tau_R \in dt) = \sum_{z \in B_R, z \sim x} \frac{\rho_\omega(x, s-t)}{\rho_\omega(\hat{y})} \omega_{s-t}(z, x)p^{x,0}_\omega(z, s-t; \hat{y})dt.
\]
Indeed, for \( h > 0 \) small enough, \( x \in \partial B_R \) and almost every \( t \in (0, s) \),
\[
P^{y,s}_\omega(Y_{\tau_R} = x, \tau_R \in (t-h, t+h)) \\
= \sum_{z \in B_R, z \sim x} P^\hat{y}_\omega(Y_{t-h} = z, \tau_R > t-h)p^{z,s-t+h}_\omega(Y_{2h} = x) + o(h) \\
= \sum_{z \in B_R, z \sim x} p^\hat{y}_\omega(\hat{y}; z, s-t) \int_{-h}^h \omega^{z,s-t+r}_\omega(z, x)dr + o(h).
\]
Dividing both sides by $2h$ and taking $h \to 0$, display (35) follows by Lebesgue’s differentiation theorem. Applying (35) to
\[
E_{\omega^s}^{y,s}[v(\hat{Y}_{\tau_R})1_{\tau_R \leq s}] = \sum_{x \in \partial B_R} \int_0^s v(x, s - t) P_{\omega^s}(Y_{\tau_R} = x, \tau_R \in dt),
\]
we obtain $I = E_{\omega^s}^{y,s}[v(\hat{Y}_{\tau_R})1_{\tau_R \leq s}]$ with a change of variable. \hfill $\square$

For fixed $\hat{y} := (y, s) \in B_R \times \mathbb{R}$, set $u(\hat{x}) := p_{2R}^{\omega^0}(\hat{x}, \hat{y})$. Then $\mathcal{L}_\omega u = 0$ in $B_R \times (-\infty, s) \cup (B_2 \setminus B_R) \times \mathbb{R}$ and $u(x, t) = 0$ when $x \in \partial B_2$ or $t > s$. By Proposition 30 and Proposition 28, for any $(x, t) \in B_2 \times (s - 4R^2, s)$,

\[
u(x, t) \leq Cu(o, s - R^2) \text{dist}(x, \partial B_2)/R, \tag{36}
\]
and, for any $(x, t) \in (B_2 \setminus B_{3R/2}) \times (s - 4R^2, s),

\[
u(x, t) \leq Cu(o, s - R^2) \text{dist}(x, \partial B_2)/R. \tag{37}
\]

**Lemma 32** Let $v \geq 0$ satisfies $\mathcal{L}_\omega^s v = 0$ in $B_2 \times (0, 4R^2]$, then for any $\bar{Y} = (\bar{y}, \bar{s}) \in B_R \times (3R^2, 4R^2]$ and $Y = (y, s) \in B_R \times (R^2, 2R^2)$, we have

\[
\frac{v(\bar{Y})}{v(Y)} \geq C \int_0^{R^2} \frac{\rho_\omega(\partial B_2, t)dt}{\int_0^{R^2} \rho_\omega(\partial B_2, t)dt} + \sum_{x \in B_2} \frac{\rho_\omega(x, 0) \text{dist}(x, \partial B_2)}{\rho_\omega(\partial B_2)}.
\]

**Proof** Write $\hat{x} := (x, t)$ and set $\tilde{u}(\hat{x}) := p_{2R}^{\omega^0}(\hat{x}; \bar{Y})$, $u(\hat{x}) := p_{2R}^{\omega^0}(\hat{x}; \bar{Y})$. By Proposition 31 and (36),

\[
v(\bar{Y}) \geq C \sum_{x \in \partial B_2} \int_0^{\bar{s}} \frac{\rho_\omega(\hat{x})}{\rho_\omega(\bar{Y})} \tilde{u}(z, t)v(\hat{x})dt + C \sum_{x \in B_2} \frac{\rho_\omega(x, 0)}{\rho_\omega(\bar{Y})} \tilde{u}(0, \bar{s} - R^2) \text{dist}(x, \partial B_2)/R, v(x, 0)
\]
\[
\geq C \frac{\tilde{u}(0, \bar{s} - R^2)}{R \rho_\omega(\bar{Y})} \left[ \sum_{x \in \partial B_2} \int_0^{\bar{s}} \rho_\omega(\hat{x}) v(\hat{x})dt + \sum_{x \in B_2} \rho_\omega(x, 0) \text{dist}(x, \partial B_2)v(x, 0) \right]. \tag{38}
\]
Similarly, by Proposition 31 and (37), we have
\[
v(Y) \leq C \frac{u(0, \bar{s} - R^2)}{R \rho_\omega(Y)} \left[ \sum_{x \in \partial B_{2R}} \int_0^{\bar{s}} \rho_\omega(\hat{x}) v(\hat{x}) \, dt \right. \\
+ \left. \sum_{x \in B_{2R}} \rho_\omega(x, 0) \text{dist}(x, \partial B_{2R}) v(x, 0) \right]. \tag{39}
\]

Combining (38) and (39), we get
\[
v(\bar{Y}) v(Y) \geq C \frac{\bar{u}(0, \bar{s} - R^2)/\rho_\omega(\bar{Y})}{\bar{u}(0, s - R^2)/\rho_\omega(Y)}.
\]

Next, applying Proposition 31 to the constant function 1 and using (37),
\[
1 = \sum_{x \in \partial B_{2R}, z \in B_{2R}, z \sim x} \int_0^\bar{s} \frac{\rho_\omega(\hat{x})}{\rho_\omega(Y)} \omega_t(z, x) \tilde{u}(z, t) \, dt + \sum_{x \in B_{2R}} \frac{\rho_\omega(x, 0)}{\rho_\omega(Y)} \tilde{u}(x, 0) \\
\leq C \frac{\tilde{u}(0, \bar{s} - R^2)}{R \rho_\omega(Y)} \left[ \sum_{x \in \partial B_{2R}} \int_0^\bar{s} \rho_\omega(\hat{x}) \, dt + \sum_{x \in B_{2R}} \rho_\omega(x, 0) \text{dist}(x, \partial B_{2R}) \right].
\]

Similarly, by Proposition 31 and (36),
\[
1 \geq C \frac{u(0, \bar{s} - R^2)}{R \rho_\omega(Y)} \left[ \sum_{x \in \partial B_{2R}} \int_0^{\bar{s}/2} \rho_\omega(\hat{x}) \, dt + \sum_{x \in B_{2R}} \rho_\omega(x, 0) \text{dist}(x, \partial B_{2R}) \right].
\]

These inequalities, together with (40), yield the lemma. \(\square\)

**Remark 33** It is clear that for static environments, the adjoint Harnack inequality (Theorem 6) follows immediately from Lemma 32. However, in time-dependent case, we need the parabolic volume-doubling property of \(\rho_\omega\).

**Proof of Theorem 6** First, we will show that for all \(R > 0\),
\[
\int_0^s \rho_\omega(\partial B_R, t) \, dt + \sum_{x \in B_R} \rho_\omega(x, 0) \text{dist}(x, \partial B_R) \\
\asymp \frac{1}{R} \int_0^s \rho_\omega(B_R, t) \, dt + \sum_{x \in B_R} \rho_\omega(x, s) \text{dist}(x, \partial B_R). \tag{41}
\]
Recall \( \tau_R \) at (34) and set \( g(x, t) = E_\omega^{x,t}[\tau_R(\hat{X})] \). Note that \( g(x, \cdot) = 0 \) for \( x \notin B_R \) and \( \mathcal{L}_\omega g(x, t) = -1 \) if \( x \in B_R \). By (3), for any \( s > 0 \),

\[
0 = \sum_{x \in \mathbb{Z}^d} \int_0^s g(x, t) \left[ \sum_y \rho_\omega(y, t) \omega_t(y, x) - \partial_t \rho_\omega(x, t) \right] dt
\]

\[
= \sum_{x \in \partial B_R} \int_0^s \rho(x, t) \omega_t(x, y) g(y, t) dt + \sum_{x \in B_R} g(x, 0) \rho(x, 0) - \sum_{x \in B_R} \int_0^s \rho(x, t) dt - \sum_{x \in B_R} g(x, s) \rho(x, s).
\]

Moreover, since \( |X_t|^2 - \frac{d}{\kappa} t \) and \( |X_t|^2 - \kappa t \) are super- and sub- martingales,

\[
g(x, t) \asymp E_\omega^{x,t}[|\tau_R|^2 - |x|^2] \asymp R \text{dist}(x, \partial B_R) \quad \forall (x, t) \in B_R \times \mathbb{R}
\]

by the optional-stopping theorem. Display (41) then follows.

Combining (41) and Lemma 32, we obtain

\[
\frac{v(\bar{Y})}{v(Y)} \geq C \int_0^{R^2} \rho(B_{2R}, t) dt + R \sum_{x \in \partial B_{2R}} \rho(x, R^2) \text{dist}(x, \partial B_{2R})
\]

\[
\int_0^{4R^2} \rho(B_{2R}, t) dt + R \sum_{x \in B_{2R}} \rho(x, 4R^2) \text{dist}(x, \partial B_{2R}).
\]

Finally, Theorem 6 follows by Theorem 8 and the above inequality. \( \square \)

### 6 Proof of Theorems 4, 11, and Corollary 12

The goal of this section is to prove the LLT (Theorem 4), the HKE (Theorem 11), and Corollary 12. With the QCLT and the Hölder regularity for \( \omega^* \)-caloric functions, the LLT, quenched heat kernel bound (6), and the Green’s function asymptotics Corollary 12(ii)(iii) all follow from rather standard arguments, which have been successfully implemented for random conductance models, e.g., [2,3,5,7]. Our main novelty in this section is the bounds (7)(8) of positive and negative moments for the heat kernel.

#### 6.1 Proof of Theorem 11

**Proof** First, using Theorem 6 and standard arguments, we will prove (6). Recall that \( v(\hat{x}) := q_0^\omega(\hat{0}, \hat{x}) \) satisfies \( \mathcal{L}_\omega^* v = 0 \) in \( \mathbb{Z}^d \times (0, \infty) \). By Theorem 6, for \( \hat{x} = (x, t) \in \mathbb{Z}^d \times (0, \infty) \), we have \( v(\hat{x}) \leq C \min_{y \in B_{\sqrt{t}}(x)} v(y, 3t) \) and so

\[
v(\hat{x}) \leq \frac{C}{\rho(B_{\sqrt{t}}(x), 3t)} \sum_{y \in B_{\sqrt{t}}(x)} \rho(y, 3t) v(y, 3t)
\]

\[
= \frac{C}{\rho(B_{\sqrt{t}}(x), 3t)} P^{0,0}_\omega(X_{3t} \in B_{\sqrt{t}}(x)) \leq C \exp[-c \Phi(|x|, t)],
\]

\[\square\] Springer
where Corollary 14 is used in the last inequality. Moreover, for any \( s \in [0, t], |y| \leq |x| + c\sqrt{t} \), by Theorem 8 and iteration,

\[
\rho(B_{\sqrt{t}}(x), 3t) \geq C \rho(B_{\sqrt{t/4}}(x), s) \geq C \left( \frac{|x|}{\sqrt{t/4}} + 1 \right)^{-c} \rho(B_{\sqrt{t/4}}(y), s).
\]

Since \( \frac{|x|}{\sqrt{t/4}} + 1 \leq C \epsilon e^{\epsilon f(|x|, t)} \) for any \( \epsilon > 0 \), the upper bound in (6) follows.

To obtain the lower bound in (6), by similar argument as above and Theorem 6, \( v(\hat{x}) \geq C \max_{y \in B_{\sqrt{t/2}}(x)} v(y, t/4) \) for \( \hat{x} \in \mathbb{Z}^d \times (0, \infty) \), and so

\[
v(\hat{x}) \geq \frac{C}{\rho(B_{\sqrt{t/2}}(x), t/4)} P_{\omega}^{0,0}(X_{t/4} \in B_{\sqrt{t/2}}(x)). \tag{42}
\]

We claim that for any \( (y, s) \in \mathbb{Z}^d \times (0, \infty) \),

\[
P_{\omega}^{x,0}(X_s \in B_{\sqrt{s}}) \geq C e^{-c|y|^2/s} \tag{43}
\]

Indeed, the case \( |y|/\sqrt{s} \leq 3 \) follows from Lemma 15. When \( |y|/\sqrt{s} > 3 \), let

\[
n = \left\lfloor \frac{2|y|^2}{s} \right\rfloor.
\]

Set \( u(x, t) := p^{\omega}(x, t; B_{\sqrt{t}}) \). Then \( u \) is \( \omega \)-caloric on \( \mathbb{Z}^d \times (-\infty, s) \). Taking a sequence of points \( (y_i)_{i=1}^n \) such that \( y_0 = y, y_n = 0 \) and \( |y_i - y_{i+1}| \leq |y|/n \), for \( i = 0, \ldots, n-1 \),

\[
\min_{x \in B_{|y|/\sqrt{n}}(y_i)} u(x, \frac{i|y|^2}{n^2}) \geq \min_{x \in B_{|y|/\sqrt{n}}(y_i)} p^{\omega}(z, \frac{i|y|^2}{n^2}; B_{|y|/\sqrt{n}}(y_{i+1}), \frac{(i+1)|y|^2}{n^2}) \min_{x \in B_{|y|/\sqrt{n}}(y_{i+1})} u(x, \frac{(i+1)|y|^2}{n^2}).
\]

Lemma 15

\[
C \min_{x \in B_{|y|/\sqrt{n}}(y_{i+1})} u(x, \frac{(i+1)|y|^2}{n^2}).
\]

Iteration then yields \( u(y, 0) \geq C^{n-1} \min_{x \in B_{|y|/\sqrt{n}}} u(x, \frac{|y|^2}{n}) \) \( \geq C^n \). Inequality (43) is proved. Then, by (42),

\[
v(x, t) \geq \frac{C}{\rho(B_{\sqrt{t/2}}(x), t/4)} e^{-c|x|^2/t}.
\]

Moreover, by Theorem 8, we have for any \( s \in [0, t], |y| \leq |x| \),

\[
\rho(B_{\sqrt{t/2}}(x), t/4) \leq C \rho(B_{\sqrt{t/2}}(x), s) \leq C \left( \frac{|x|}{\sqrt{t}} + 1 \right)^c \rho(B_{\sqrt{t}}(y), s).
\]

The lower bound in (6) is proved.
Next, we will prove the moment bounds (7) and (8), which, by (6) and (22), are equivalent to showing that, for $r := \sqrt{t} > 0$,

$$
\|\frac{\rho(\hat{0})}{\rho(Q_r)}\|_{L^{d+1/d}(P)} \leq C r^2 (r \vee 1)^{-d}
$$

and

$$
\|\frac{\rho(\hat{0})}{\rho(Q_r)}\|_{L^{-p}(P)} \geq C r^2 (r \vee 1)^{-d},
$$

where $Q_r$ is as defined in (18). Indeed, using the translation-invariance of $P$ and the volume-doubling property of $\rho$, for $q := (d+1)/d$,

$$
\|\frac{\rho(\hat{0})}{\rho(Q_r)}\|_{L^q(P)} \leq C |Q_r|^{1/q},
$$

where we used the Reverse Hölder inequality (Corollary 24(i)) in the last inequality. Recalling (24), inequality (7) then follows from the fact that $|Q_r| = r^2 \sum_{x \in B_r} 1 \times r^2 (r \vee 1)^{-d}$.

To obtain (8), note that by translation invariance and $P$ and the volume-doubling property of $\rho$, taking $\epsilon \in (0, 1/(p-1))$,

$$
\|\frac{\rho(Q_r)}{\rho(\hat{0})}\|_{L^{-\epsilon}(P)} \leq C |Q_r|^{-\epsilon},
$$

where we used the $A_p$ inequality (Theorem 26(a)) of $\rho$ in the last inequality. Therefore $\|\rho(\hat{0})/\rho(Q_r)\|_{L^{-p}(P)} \geq C r^2 (r \vee 1)^{-d}$ and (8) is proved.

Display (9) follows from (7), (8), and Minkowski’s integral inequality.

6.2 Proof of Theorem 4 and Corollary 12

Recall $q^\omega(\hat{y}, \hat{x})$ in (4). For any $\hat{x} = (x, t) \in \mathbb{R}^d \times \mathbb{R}$, set

$$
v(\hat{x}) := q^\omega(\hat{0}; \lfloor x \rfloor, t),
$$

where $\lfloor x \rfloor$ is as in Theorem 4. Note that $L^\omega v = 0$ in $\mathbb{Z}^d \times (0, \infty)$. By Corollary 7 and Theorem 11, for any $\hat{y} = (y, s) \in B_{\sqrt{t}}(x) \times (\frac{t}{2}, t)$,

$$
|v(\hat{x}) - v(\hat{y})| \leq C \left( \frac{|x - y| + \sqrt{t - s}}{\sqrt{t}} \right)^Y \sup_{B_{\sqrt{t}}(x) \times (\frac{t}{2}, t)} v \\
\leq C \left( \frac{|x - y| + \sqrt{t - s}}{\sqrt{t}} \right)^Y t^{-d/2}
$$

when $t > t_0(\omega)$ is big enough. Here in the last inequality we used Corollary 12(i) which is an immediate consequence of Theorem 11 and (23).

Recall $O_r$ in (27). For $\hat{x} = (x, t) \in \mathbb{R}^d \times \mathbb{R}$, write

$$
\hat{x}^n := (\lfloor nx \rfloor, n^2 t).
$$
To prove Theorem 4, it suffices to show that for any $K > T$,

$$\lim_{n \to \infty} \sup_{\hat{x} \in O_K \times [T, K]} |n^d v(\hat{x}^n) - p_T^\Sigma (0, x)| = 0. \quad (45)$$

Indeed, for any $\epsilon > 0$, there exists $K = K(T, \epsilon, d, \kappa) > 0$ such that, writing $\mathcal{D} := (\mathbb{R}^d \times [T, \infty)) \setminus (O_K \times [T, K])$, we have

$$\lim_{n \to \infty} \sup_{\mathcal{D}} n^d v(\hat{x}^n) + p_T^\Sigma (0, x) \leq C \sup_{\mathcal{D}} t^{-d/2} e^{-c|x|^2/t} \leq \epsilon.$$ 

Hence Theorem 4 follows provided that (45) is proved.

**Proof of Theorem 4** As we discussed in the above, it suffices to prove (45). For any $\epsilon > 0$,

$$\left| n^d v(\hat{x}^n) - p_T^\Sigma (0, x) \right| \leq A(\hat{x}, \epsilon) + B_n(\hat{x}, \epsilon) + C_n(\hat{x}, \epsilon), \quad (46)$$

where $A(\hat{x}, \epsilon) = \left| \int_t^{t+\epsilon^2} \frac{p_T^\Sigma (0, nO_\epsilon (x))}{\epsilon^2 |O_\epsilon|} ds - p_T^\Sigma (0, x) \right|$, $B_n(\hat{x}, \epsilon) = \int_t^{t+\epsilon^2} \frac{P_\omega (X_{n^2 s} \in nO_\epsilon (x)) - p_T^\Sigma (0, nO_\epsilon (x))}{\epsilon^2 |O_\epsilon|} ds$, and $C_n(\hat{x}, \epsilon) = \left| n^d v(\hat{x}^n) - \int_t^{t+\epsilon^2} \frac{p_\omega (X_{n^2 s} \in nO_\epsilon (x))}{\epsilon^2 |O_\epsilon|} ds \right|$.

First, we will show that

$$\lim_{n \to \infty} \sup_{\hat{x} \in O_K \times [T, K]} C_n(\hat{x}, \epsilon) = O(\epsilon^\gamma). \quad (47)$$

To this end, note that there exists $N = N(T, \omega, d, \kappa)$ such that for $n \geq N$,

$$C_n(\hat{x}, \epsilon) \leq n^d v(\hat{x}^n) \left| 1 - \int_t^{t+\epsilon^2} \frac{\rho(nO_\epsilon (x), n^2 s) ds}{\epsilon^2 nO_\epsilon} \right| + \sum_{y \in nO_\epsilon (x)} \int_t^{t+\epsilon^2} |v(y, n^2 s) - v(\hat{x}^n)| \rho(y, n^2 s) ds / (\epsilon^2 |O_\epsilon|)$$

$$\leq C T^{-d/2} \left| 1 - \int_t^{t+\epsilon^2} \frac{\rho(nO_\epsilon (x), n^2 s) ds}{\epsilon^2 nO_\epsilon} \right| + C T^{-d/2} \epsilon^\gamma \int_t^{t+\epsilon^2} \rho(nO_\epsilon (x), n^2 s) ds / (\epsilon^2 nO_\epsilon),$$
where in the second inequality we used Corollary 12(i) and (44). Further, by an ergodic theorem of Krengel and Pyke [21, Theorem 1] and (2),

$$\lim_{n \to 0} \sup_{\hat{x} \in \mathcal{O}_K \times [T, K]} \left| 1 - \int_{t}^{t+2} \frac{\rho(n\mathcal{O}_e(x), n^2 s) ds}{\epsilon^2 |n\mathcal{O}_e|} \right| = 0.$$  \hspace{1cm} (48)

Display (47) follows.

Next, for $\hat{x} = (x, t)$, by writing $B_n(\hat{x}, \epsilon)$ as

$$\left| \int_{t}^{t+2} \frac{\rho^0(X_n s \in n\mathcal{O}_e(x)) ds}{\epsilon^2 |\mathcal{O}_e|} - \int_{t}^{t+2} \frac{\Sigma(0, \mathcal{O}_e(x)) ds}{\epsilon^2 |\mathcal{O}_e|} \right| =: |B_n^1(\hat{x}, \epsilon) - B^2(\hat{x}, \epsilon)|,$$

we will show that

$$\lim_{n \to \infty} \sup_{\hat{x} \in \mathcal{O}_K \times [T, K]} B_n(\hat{x}, \epsilon) = O(\epsilon^{\gamma}).$$  \hspace{1cm} (49)

We claim that $B_n(\hat{x}, \epsilon)$ is approximately equicontinuous (with order $\epsilon^{\gamma}$). That is, there exist $N, \delta$ depending on $(\epsilon, \omega, d, \kappa, T, K)$ such that whenever $n \geq N$ and $\hat{x}_1 = (x_1, t_1), \hat{x}_2 = (x_2, t_2) \in \mathcal{O}_K \times [T, K]$ satisfy $|\hat{x}_1 - \hat{x}_2| := |x_1 - x_2| + |t_1 - t_2| < \delta$, we have

$$|B_n(\hat{x}_1, \epsilon) - B_n(\hat{x}_2, \epsilon)| < C\epsilon^{\gamma}.$$  

It suffices to show that $B_n^1(\hat{x}, \epsilon)$ is approximately equicontinuous. Indeed, by (47) and (44), when $n \geq N$ is large and $\hat{x}_1, \hat{x}_2 \in \mathcal{O}_K \times [T, K]$,

$$|B_n^1(\hat{x}_1, \epsilon) - B_n^1(\hat{x}_2, \epsilon)| \leq C_n(\hat{x}_1, \epsilon) + C_n(\hat{x}_2, \epsilon) + n^d |v(\hat{x}_1^n) - v(\hat{x}_2^n)|$$

$$\leq C\epsilon^{\gamma} + C(|x_1 - x_2| + \sqrt{|t_1 - t_2|})^{\gamma}.$$  

The approximate equicontinuity of $B_n^1(\hat{x}, \epsilon)$ follows. To prove (49), we choose a finite sequence $\{\hat{x}_i\}_{i=1}^M$ such that $\min_{1 \leq i \leq M} |\hat{x} - \hat{x}_i| < \delta$ for all $\hat{x} \in \mathcal{O}_K \times [T, K]$. Since $\lim_{n \to \infty} \max_{1 \leq i \leq M} B_n(\hat{x}_i) = 0$ by the QCLT (Theorem 1), display (49) follows by the approximate equicontinuity.

Clearly, $\lim_{\epsilon \to 0} \sup_{\hat{x} \in \mathcal{O}_K \times [T, K] \\setminus \{\hat{x}_i\}} A(\hat{x}, \epsilon) = 0$. This, together with (47) and (49), yields the uniform convergence of (46) by sending first $n \to \infty$ and then $\epsilon \to 0$. Our proof of Theorem 4 is complete. \hspace{1cm} \square

Proof of Corollary 12: (i) follows from Theorem 11 and (23). (ii) and (iii) are consequences of Theorems 4 and 11. Their proofs, which are similar to [2, Theorem 1.14] and [5, Theorem 1.4], can be found in Sect. A.6. \hspace{1cm} \square

Acknowledgements We thank two anonymous referees for their careful reading of our paper and their valuable comments. XG did the main part of his work while at the University of Wisconsin-Madison. He thanks Professors Timo Sepäläinen, Hung Vinh Tran, and other colleagues at the UW for their hospitality and supportive environment they created.
A Appendix

In Sect. A.1 we will show properties (i)-(iii) of $\rho_{\omega}$ in Remark 3. In Sect. A.2, we prove the Hölder estimate (Corollary 7) for $\omega^*$-caloric functions. Sections A.3 and A.4 are devoted to the maximum principle (which is used in various places in the paper) and the mean value inequality (used in the proof of the PHI Proposition 5) for $L_{\omega}$, respectively. Section A.5 contains the proof of Lemma 25. Corollary 12(ii)(iii) are proved in Sect. A.6.

A.1 Properties (i)--(iii) in Remark 3

Proof (i) Since $\mathbb{Q}$ is an invariant measure for $(\tilde{\omega}_t)$, we have for any bounded measurable function $f$ on $\Omega$, $y \in \mathbb{Z}^d$, $\hat{x} = (x, t)$, and $s < t$,

$$0 = E_{\mathbb{Q}} E_{\omega}^{0,0} [f(\theta_{-\xi}(\tilde{\omega}_t-s)) - f(\theta_{-\xi}\omega)]$$

$$= E_{\mathbb{P}} \left[ \rho(\omega) \sum_{y \in \mathbb{Z}^d} p^\omega(0, 0; x - y, t - s)[f(\theta_{-\xi}(\tilde{\omega}_t-s)\omega) - f(\theta_{-\xi}\omega)] \right]$$

$$= E_{\mathbb{P}} \left[ f(\omega) \left[ \sum_{y \in \mathbb{Z}^d} \rho(\theta_{\hat{y}\omega}) p^\omega(\hat{y}, \hat{x}) - \rho(\theta_{\hat{x}\omega}) \right] \right], (50)$$

where $\hat{y} = (y, s)$ and we used the translation-invariance of $\mathbb{P}$ in the last equality. Moreover, by Fubini’s theorem, for any bounded compactly-supported continuous function $\phi: \mathbb{R} \to \mathbb{R}$,

$$E_{\mathbb{P}} \left[ f(\omega) \int_{-\infty}^t \phi(s) \left[ \sum_{y \in \mathbb{Z}^d} \rho(\theta_{\hat{y}\omega}) p^\omega(\hat{y}, \hat{x}) - \rho(\theta_{\hat{x}\omega}) \right] ds \right] = 0$$

Thus we have that $\mathbb{P}$-almost surely, for any such test function $\phi$ on $\mathbb{R}$,

$$\int_{-\infty}^t \phi(s) \left[ \sum_{y \in \mathbb{Z}^d} \rho(\hat{y}) p^\omega(\hat{y}, \hat{x}) - \rho(\hat{x}) \right] ds = 0.$$
which (together with the translation-invariance of $\mathbb{P}$) implies that $\mathbb{P}$-almost surely, $\rho_\omega(x, t)\delta_x dt$ is an invariant measure for the process $(\hat{X}_t)_{t \geq 0}$.

(ii) We have $\rho_\omega > 0$ since the measures $Q$ and $\mathbb{P}$ are equivalent. The uniqueness follows from the uniqueness of $Q$ in [14, Theorem 2.1(iii)].

(iii) By (50) and Fubini’s theorem, we also have that $\mathbb{P}$-almost surely, for any test function $\phi(t)$ as in (i) and any $h > 0$, $x \in \mathbb{Z}^d$,

$$\int_{-\infty}^{\infty} \phi(t) \left[ \sum_{y \in \mathbb{Z}^d} \rho_\omega(y, t) (p^\omega(y, t; x, t + h) - \delta_x(y)) - (\rho_\omega(x, t + h) - \rho_\omega(\hat{x})) \right] dt = 0.$$ 

Dividing both sides by $h$ and letting $h \to 0$, we obtain (3) with $\delta_\rho_\omega$ replaced by the weak derivative. Note that the weak differentiability of $\rho_\omega$ in $t$ implies that it has an absolutely continuous (in $t$) version. Since $\rho_\omega$ is only used as a density, we may always assume that $\mathbb{P}$-almost surely, $\rho_\omega(x, \cdot)$ is continuous and almost-everywhere differentiable in $t$. \hfill $\square$

A.2 Proof of Corollary 7

**Proof** Assume $(x_0, t_0) = (0, 0)$ and fix $R > 0$. Let $R_k = 2^{-k} R$ and $Q^k = B_{R_k} \times (-R_k^2, 0]$. Note that $Q^{k+1} \subset Q^k$. For any bounded subset $E \subset \mathbb{Z}^d \times \mathbb{R}$, denote $osc_E u := \sup_E u - \inf_E u$. Set

$$v_k := (u - \inf_{Q^k} u) / osc_{Q^k} u.$$ 

Notice that $\inf_{Q^k} v_k = 0$, $\sup_{Q^k} v_k = 1$ and

$$osc_{Q^{k+1}} u = osc_{Q^{k+1}} v_k \cdot osc_{Q^k} u.$$ 

We claim that $osc_{Q^{k+1}} v_k \leq 1 - \delta$ for some $\delta = \delta(d, \kappa) \in (0, 1)$. Indeed, replacing $v_k$ by $1 - v_k$ if necessary, we can assume $\sup_{B_{R_{k+1}} \times (-\frac{3}{4} R_{k+1}^2, -\frac{1}{2} R_k^2)} v_k \geq 1/2$. By the PHI for $L^*_{\omega}$ (Theorem 6),

$$\inf_{Q^{k+1}} v_k \geq c \sup_{B_{R_{k+1}} \times (-\frac{3}{4} R_{k+1}^2, \frac{1}{2} R_k^2)} v_k \geq \frac{c}{2} := \delta \in (0, 1)$$ 

and so $osc_{Q^{k+1}} v_k \leq \sup_{Q^k} v_k - \inf_{Q^{k+1}} v_k \leq 1 - \delta$. The claim is proved. So

$$osc_{Q^{k+1}} u \leq (1 - \delta) osc_{Q^k} u.$$ 

If $r > R/2$, the Corollary is trivial. If $r \leq R/2$, we iterate the above inequality $k = \lfloor \log_2 (R/r) \rfloor$ times (so that $Q^{k+1} \subset B_r \times (-r^2, 0] \subset Q^k$) to obtain

$$osc_{B_r \times (-r^2, 0]} u \leq osc_{Q^k} u \leq (1 - \delta)^k osc_{Q^0} u \leq (1 - \delta)^{-1} (r/R)^\gamma osc_{Q^0} u.$$ 

where $\gamma = -\log_2 (1 - \delta)$. Our proof is complete. \hfill $\square$
A.3 Parabolic maximum principle

In what follows we will prove a maximum principle for parabolic difference operators under the discrete space and continuous time setting. Its statement is a tiny modification from [14, Theorem 5.1] where its proof, which follows verbatim the lines of [14, Theorem 2.2], was omitted. For the purpose of completeness we will include a full proof in the below.

For any \( D \subset B_R \times (0, T) \), define \( \hat{x} := (x, t) \in D \) and \( u : D \cup \partial^D D \to \mathbb{R} \), define

\[
I_u(\hat{x}) := \{ p \in \mathbb{R}^d : u(\hat{x}) - u(y, s) \geq p \cdot (x - y), \forall (y, s) \in D \cup \partial^D D \text{ with } s > t \},
\]

\( \Gamma = \Gamma(u, D) := \{ (x, t) \in D : I_u(x, t) \neq \emptyset \} \),

\( \Gamma^+ = \Gamma^+(u, D) = \{ \hat{x} \in \Gamma : R|p| < u(\hat{x}) - p \cdot x \text{ for some } p \in I_u(\hat{x}) \}. \) (51)

**Theorem A.3.1** (Maximum principle) Let \( \omega \in \Omega_{\kappa} \). Recall \( \int_{\mathcal{D}} \) in (24). Assume that \( D \subset B_R \times (0, T) \) is an open subset of \( \mathbb{Z}^d \times \mathbb{R} \) for some \( R, T > 0 \). Let \( f \) be a measurable function on \( D \). For any function \( u : D \cup \partial^D D \to \mathbb{R} \) that solves \( \mathcal{L}_a u \geq -f \text{ in } D \), we have

\[
\sup_D u \leq \sup_{\partial^D D} u + CR^d/(d+1) \left( \int_{\Gamma^+} |f|^{d+1} \right)^{1/(d+1)}.
\]

**Proof** Without loss of generality, assume \( f \geq 0 \), \( \sup_{\partial^D D} u = 0 \), and

\[
\sup_D u := M > 0.
\]

Let \( \Lambda = \{ (\xi, h) \in \mathbb{R}^{d+1} : R|\xi| < h < M/2 \} \). For \( (x, t) \in D \), define a set

\[
\chi(x, t) = \{ (p, u(x, t) - x \cdot p) : p \in I_u(x, t) \} \subset \mathbb{R}^{d+1}.
\]

First, we claim that

\[
\Lambda \subset \chi(\Gamma^+) := \bigcup_{(x, t) \in \Gamma^+} \chi(x, t). \quad (52)
\]

This will be proved by showing that for any \( (\xi, h) \in \Lambda \), we have \( (\xi, h) \in \chi(x_1, t_1) \) for some \( (x_1, t_1) \in \Gamma^+ \). Indeed, fix \( (\xi, h) \in \Lambda \) and define

\[
\phi(x, t) := u(x, t) - \xi \cdot x - h.
\]

Since \( \sup_{\mathcal{D}} \phi \geq M - |\xi| R - h > 0 \), there exists \( (x_0, t_0) \in \mathcal{D} \) with \( \phi(x_0, t_0) > 0 \). Now for any \( x \in \mathbb{Z}^d \), set (with the convention \( \sup \emptyset = -\infty \))

\[
N_x = \sup\{ t : (x, t) \in \mathcal{D} \text{ and } \phi(x, t) \geq 0 \}.
\]
and let \((x_1, t_1)\) be such that

\[
t_1 = N_{x_1} = \max_{x \in B_R} N_x \geq N_{x_0} \geq t_0.
\]

By the continuity of \(\phi\), we get \(\phi(x_1, t_1) \geq 0\) and \((x_1, t_1) \in D \cup \partial^p D\). Since \(\phi|_{\partial^p D} < 0\), we have \((x_1, t_1) \in D\). Moreover, since \(D\) is an open set, we can conclude that \(\phi(x_1, t_1) = 0\) and \(\phi(x_1, s) < 0\) for all \(s > t_1\) with \((x_1, s) \in D \cup \partial^p D\). Hence \(\xi \in I_u(x_1, t_1)\) and \(u(x_1, t_1) - \xi \cdot x_1 = h > R|\xi|\), which implies that \((\xi, h) \in \chi(x_1, t_1)\) and \((x_1, t_1) \in \Gamma^+\). Display (52) is proved.

Next, setting

\[
\chi(\Gamma^+, x) := \bigcup_{s: (x, s) \in \Gamma^+} \chi(x, s),
\]

we will show that

\[
\text{Vol}_{d+1}(\chi(\Gamma^+, x)) \leq C \int_0^T (f(x, t)/\epsilon)^{d+1} 1_{(x,t) \in \Gamma^+} dt,
\]

where \(\text{Vol}_{d+1}\) is the volume in \(\mathbb{R}^{d+1}\). To this end, let \(\tilde{\chi}(x, t) = I_u(x, t) \times \{u(x, t)\} \subset \mathbb{R}^{d+1}\). Noting that, for fixed \(x\), the map \((y, s) \mapsto (y, s + y \cdot x)\) is volume preserving, we then have

\[
\text{Vol}_{d+1}(\chi(\Gamma^+, x)) = \text{Vol}_{d+1}(\tilde{\chi}(\Gamma^+, x)) = \int_0^T (-\partial_t u) \text{Vol}_d(I_u(x, t)) 1_{(x,t) \in \Gamma^+} dt.
\]

For any fixed \(p \in I(x, t), (x, t) \in \Gamma^+\), set

\[
w(y, s) = u(y, s) - p \cdot y.
\]

Then \(I_w(x, t) = I_u(x, t) + p\). Since \(w(x, t) - w(x \pm e_i, t) \geq \mp q_i\) for any \(q \in I_w(x, t), i = 1, \ldots d\), we have

\[
\text{Vol}_d(I_u(x, t)) = \text{Vol}_d(I_w(x, t)) \leq \prod_{i=1}^d [2u(x, t) - u(x + e_i, t) - u(x - e_i, t)].
\]

This inequality, together with (54), yields

\[
\text{Vol}_{d+1}(\chi(\Gamma^+, x)) \leq -C \int_0^T \partial_t u \prod_{i=1}^d a_i(x, x + e_i)[2u(x, t) - u(x + e_i, t) - u(x - e_i, t)] 1_{(x,t) \in \Gamma^+} dt
\]

\spacefactor=1000
\[ \leq C \int_0^T [-L_a u(x,t)]^{d+1} 1_{(x,t) \in \Gamma^+} \, dt. \]

Display (53) is proved. Finally, by (52), (53) and \( \text{Vol}_{d+1}(\Lambda) = CM^{d+1}/R^d \), we conclude that \( M^{d+1}/R^d \leq C \int_{\Gamma^+} f^{d+1} \). The theorem follows. \( \square \)

A.4 Mean value inequality

**Theorem A.4.1** Let \( \theta > 0 \) and \( a \in \Omega_{\kappa} \). Recall \( \| \cdot \|_{\mathcal{D},p} \) in (25). For any \( \theta_1 \in (0, \theta) \), \( \rho \in (0, 1) \) and \( p > 0 \), there exists \( C = C(\kappa, d, p, \theta, \theta_1, \rho) \) such that for any function \( u \) that solves \( L_a u \geq 0 \) in \( \mathcal{D} = B_R \times [0, \theta R^2) \), we have

\[ \sup_{B_{\rho R} \times [0, \theta_1 R^2]} u \leq C \| u^+ \|_{\mathcal{D}, p}. \]

**Proof** Since \( \| u^+ \|_{\mathcal{D}, p} \) is increasing in \( p > 0 \), it suffices to consider \( p \in (0, 1) \). Let \( \beta \geq 2 \) be a constant to be determined, and set

\[ \eta(x) := (1 - |x|^2/R^2)^{\beta} 1_{x \in B_R}, \quad \zeta(t) := (1 - t/\theta R^2)^{\beta} 1_{0 \leq t < \theta R^2}, \]

and set \( v = \eta u^+, \bar{v} = v \xi \). Define an elliptic operator \( L_a^E \) to be

\[ L_a^E f(x,t) = \sum_{y : y \sim x} a_t(x,y)(f(y,t) - f(x,t)), \]

so that \( L_a = L_a^E + \partial_t \). Note that \( \bar{v} \big|_{\partial^+ \mathcal{D}} = 0 \) and \( \bar{v}(\hat{x}) > 0 \) for \( \hat{x} \in \Gamma^+(\bar{v}, \mathcal{D}) \). (Recall the notation \( \Gamma^+ \) above Theorem A.3.1.) By the same argument as in [19, displays (27),(28) and (29)], we have that on \( \Gamma^+(\bar{v}, \mathcal{D}) \), \( u^+ = u \) and

\[ L_a^E v \geq \eta L_a^E u - C_\beta \eta^{1-2/\beta} R^{-2} u^+. \]

Hence, for \( X = (x,t) \in \Gamma^+(\bar{v}, \mathcal{D}) \),

\[ L_a \bar{v} = \zeta L_a^E v + \partial_t(\zeta \eta u) \]

\[ \geq \zeta \eta L_a^E u - C_\beta \zeta \eta^{1-2/\beta} R^{-2} u^+ + \eta \mu \partial_t \zeta + \zeta \eta \partial_t u \]

\[ = \zeta \eta L_a u - C_\beta \zeta \eta^{1-2/\beta} R^{-2} u^+ + \eta \mu \partial_t \zeta \]

\[ \geq -C_\beta \zeta \eta^{1-2/\beta} R^{-2} u^+ + \eta \mu \partial_t \zeta. \]

Noting that in \( \mathcal{D} \), \( \partial_t \zeta \geq -C_\beta R^{-2} \zeta^{1-1/\theta} / \theta \) and \( \zeta, \eta \in [0, 1] \), we have

\[ L_a \bar{v} \geq -C(\eta \zeta)^{1-2/\beta} R^{-2} u^+ \quad \text{in } \Gamma^+(\bar{v}, \mathcal{D}). \]
Applying Theorem A.3.1 to $\tilde{v}$ and taking $\beta = 2(d + 1)/p$, 

$$
\sup_{\mathscr{D}} \tilde{v} \leq C ((\eta \xi)^{1-2/\beta} u^+ / \varepsilon)_{\mathscr{D}, d+1}
\leq C \left( \sup_{\mathscr{D}} \tilde{v} \right)^{(1-p/(d+1))} \left( (u^+)_{\mathscr{D}, d+1} / \varepsilon \right).$

Since $\sup_{B_{\rho R \times [0, \theta_1 R^2]} u} \leq C \sup_{\mathscr{D}} \tilde{v}$, the theorem follows. \hfill $\Box$

### A.5 Proof of Lemma 25

Recall $|\mathcal{D}|$, $\int_{\mathcal{D}}$, $\|\|_{\mathcal{D}, p}$, and the parabolic cubes in (24), (25) and (26).

**Proof** First, we claim that there exist constants $\gamma, \delta \in (0, 1)$ such that $w(E) > \gamma w(K)$ implies $|E| > \delta |K|$ for all $E \subset K$ where $K \neq \emptyset$ is a subcube of $K^0$. Indeed, this is a simple consequence of Hölder’s inequality:

$$
\frac{1}{|K|} w(E) = \frac{1}{|K|} \int_K w \mathbb{1}_E \leq \left( \frac{|E|}{|K|} \right)^{1/q'} w_{K, q}^{(RH_q)} \leq C \left( \frac{|E|}{|K|} \right)^{1/q'} \frac{w(K)}{|K|},
$$

where $q' = q/(q - 1)$ denotes the conjugate of $q$.

Assume $K^0 = K_r$. Let $M_k(K_r)$, $k > 1$ be the family of nonempty subcubes of $K_r$ of the form

$$
\left( \prod_{i=1}^d \left( \frac{m_i}{2^k}, \frac{1 + m_i}{2^k} \right) \cap \mathbb{Z}^d \right) \times \left[ \frac{n_i}{4^k} r^2, \frac{1 + n_i}{4^k} r^2 \right)
$$

where $m_i, n_i$’s are integers. Elements in $M_k(K_r)$ are called $k$-level dyadic subcubes of $K_r$. Note that every $k$-level cube $K$ is contained in a unique $(k - 1)$-level "parent" denoted by $K^{-1}$. Since the class $A_p$ is invariant under constant multiplication, we may assume that $w(K^0)/|K^0| = 1$.

Let $f := w^{-1} \mathbb{1}_{K^0}$ and define a maximal function

$$
M_f(x) := \sup_{K \ni x} \frac{1}{w(K)} \sum_K |f| w,
$$

where the supremum is taken over all dyadic subcubes $K$ of $K_0$. Consider the level sets

$$
E_k = \{ x \in K^0 : M_f(x) > 2^{Nk} \}, \quad k = 0, 1, 2, \ldots
$$

where $N$ is a big constant to be determined. Notice that by assumption, $E_0$ is comprised of dyadic subcubes strictly smaller than $K_0$. Since $w$ is volume-doubling, there exists a constant $c_0 > 0$ such that for any maximal dyadic subcube $K$ of $E_{k-1}$,

$$
\int_K f w \leq \int_{K^{-1}} f w \leq 2^{N(k-1)} w(K^{-1}) \leq 2^{N(k-1)+c_0} w(K).
$$
Moreover, for the same $K$, we have $2^{N_k} w(E_k \cap K) \leq \int_K f w$ and so, by the inequality above, $w(E_k \cap K) \leq 2^{c_0-N} w(K)$. We now take $N$ to be large enough that $w(E_k \cap K) \leq (1-\gamma) w(K)$ which implies $|E_k \cap K| \leq (1-\delta)|K|$. Summing over all such $K$’s, we have $|E_k| \leq (1-\delta)|E_{k-1}|$, $k \geq 1$. Thus

$$|E_k| \leq \delta^k |E_0| \leq \delta^k |K^0|, \quad k = 0, \ldots$$

and so, for $p > 1$ chosen so that $p' = p/(p-1)$ is sufficiently close to 1,

$$\int_{K^0} f^{p'-1} = \int_{K^0 \cap \{x: M_f \leq 1\}} f^{p'-1} + \sum_{k=0}^{\infty} \int_{E_k \setminus E_{k+1}} f^{p'-1} \leq |K^0| + \sum_{k=0}^{\infty} 2^{(p'-1)(k+1)} \delta^k |K^0| \leq C |K^0|.$$  

(i) is proved. (ii) then follows from Hölder’s inequality

$$\frac{1}{w(K)} \int_E w^{-1} w \leq \left( \frac{1}{w(K)} \int_K w^{-p'} w \right)^{1/p'} \left( \frac{1}{w(K)} \int_K 1_E w \right)^{1/p}$$

and the $A_p$ inequality. \hfill $\square$

**A.6 Proof of Corollary 12(ii)(iii)**

**Proof** (ii) For any $\hat{x} = (x, t) \in \mathbb{R}^d \times [0, \infty)$ and $\omega \in \Omega_\kappa$, set

$$v(\hat{x}) = q^\omega (\hat{0}; [x], t) \quad \text{and} \quad a^\omega (x) := \int_0^\infty (v(0, t) - v(x, t)) dt.$$  

When $d = 2$, it suffices to consider $x \in \mathbb{B}_1 \setminus \{0\}$. We fix a small number $\epsilon \in (0, 1)$ and split the integral $a^\omega (nx)$ into four parts:

$$a^\omega (nx) = \int_0^{n^\epsilon} + \int_{n^\epsilon}^{n^2} + \int_{n^2}^\infty =: I + II + III,$$

where it is understood that the integrand is $(v(0, t) - v(x, t)) dt$.

First, we will show that $P$-almost surely,

$$\lim_{n \to \infty} |I|/ \log n \leq \epsilon. \quad (55)$$
By Theorem 11, for any \( t \in (0, n^\epsilon) \), \( x \in \mathbb{Z}^2 \setminus \{0\} \) and all \( n \) large enough, \( v(nx, t) \leq Ce^{-cn|x|}/\rho_\omega(B_{\sqrt{t}}, 0) \). Thus

\[
\int_0^{n^\epsilon} v(nx, t) dt \leq \frac{n^\epsilon}{\rho_\omega(0)} e^{-cn|x|}.
\]

By (i), there exists \( t_0(\omega) > 0 \) such that for \( n \) big enough with \( n^\epsilon > t_0 \),

\[
\int_0^{n^\epsilon} v(0, t) dt \leq \frac{Ct_0}{\rho_\omega(0)} + \int_{t_0}^{n^\epsilon} \frac{C}{t} dt \leq \frac{Ct_0}{\rho_\omega(0)} + C \epsilon \log n.
\]

Display (55) follows immediately.

In the second step, we will show that (note that \( 2p_1^\Sigma(0, 0) = 1/\pi \sqrt{\det \Sigma} \))

\[
\limsup_{n \to \infty} |II - 2p_1^\Sigma(0, 0) \log n| / \log n \leq C \epsilon, \quad \mathbb{P}\text{-a.s.}
\] (56)

Indeed, by Theorem 4, there exists \( C(\omega, \epsilon) > 0 \) such that \( |tv(0, t) - p_1^\Sigma(0, 0)| \leq \epsilon \) whenever \( t \geq C(\omega, \epsilon) \). Now, taking \( n \) large enough such that \( n^\epsilon > C(\omega, \epsilon) \),

\[
\left| \int_{n^\epsilon}^{n^2} v(0, t) dt - (2 - \epsilon) p_1^\Sigma(0, 0) \log n \right|
\leq \int_{n^\epsilon}^{n^2} \frac{|tv(0, t) - p_1^\Sigma(0, 0)|}{t} dt
\leq \epsilon \int_{n^\epsilon}^{n^2} \frac{dt}{t} < 2 \epsilon \log n.
\] (57)

On the other hand, for \( t \geq n^\epsilon > t_0(\omega) \), by (i), \( v(nx, t) \leq \frac{C}{t} (e^{-cn|x|} + e^{-cn^2|x|^2/t}) \). Thus

\[
\int_{n^\epsilon}^{n^2} v(nx, t) dt \leq \int_{n^\epsilon}^{n^2} \frac{C}{t} e^{-cn^\epsilon|x|^2} dt + \int_{n^\epsilon}^{n^2} \frac{C}{t} dt \leq C \epsilon \log n.
\] (58)

Displays (57) and (58) imply (56).

Finally, we will prove that for \( \mathbb{P} \)-almost every \( \omega \),

\[
\limsup_{n \to \infty} |III| / \log n = 0.
\] (59)

Since \( |x| < 1 \), by (44), for any \( t \geq n^2 \geq t_0(\omega) \),

\[
|v(0, t) - v(nx, t)| \leq C \left( \frac{n}{\sqrt{t}} \right)^Y t^{-1}.
\]
Therefore, $\mathbb{P}$-almost surely, when $n^2 > t_0(\omega)$,

$$\left| \int_{n^2}^{\infty} v(0, t) - v(nx, t) \, dt \right| \leq C n^r \int_{n^2}^{\infty} \frac{1}{t^{r/2+1}} \, dt \leq C.$$

Display (59) follows. Combining (55), (56) and (59), we have for $d = 2$,

$$\lim_{n \to \infty} \left| \frac{a^\omega(nx)}{\log n} - 2p^\omega(0, 0) \right| \leq C \epsilon,$$

Noting that $\epsilon > 0$ is arbitrary, we obtain Corollary 12(ii).

(iii) We fix a small constant $\epsilon \in (0, 1)$. Note that

$$nd - \int_0^{\infty} q^\omega(\hat{0}; [nx], t) \, dt = \int_0^{\infty} nd v(nx, n^2 s) \, ds.$$

For any fixed $x \in \mathbb{R}^d$, write

$$\int_0^{\infty} nd v(nx, n^2 s) \, ds = \int_0^{\epsilon} + \int_{\epsilon}^{\epsilon^2} + \int_{\epsilon^2}^{1/\sqrt{\epsilon}} + \int_{1/\sqrt{\epsilon}}^{\infty} =: I + II + III + IV.$$

First, by Theorem 11, for $s \in (0, n^{-\epsilon})$, we have $v(nx, n^2 s) \leq Ce^{-cn^2|x|^2/\rho_\omega(\hat{0})}$, hence

$$\lim_{n \to \infty} I \leq C \lim_{n \to \infty} nd - e^{-cn^2|x|^2/\rho_\omega(\hat{0})} = 0.$$

Second, by (i), when $n$ is large enough, then for all $t \geq n^{2-\epsilon}$, we have $v(nx, t) \leq Ct^{-d/2}e^{-cn^2|x|^2/t}$. Hence

$$\lim_{n \to \infty} II \leq \lim_{n \to \infty} C n^d \int_{n^{-\epsilon}}^{\epsilon} (n^2 s)^{-d/2} e^{-c|x|^2/s} \, ds \leq C \epsilon.$$

Moreover, by Theorem 4, there exists $N(\omega, \epsilon)$ such that for $n \geq N(\omega, \epsilon)$, we have $\sup_{|s| \geq \epsilon} |v(nx, n^2 s) - p_s^\omega(0, x)| \leq \epsilon$. Hence

$$\lim_{n \to \infty} \left| III - \int_\epsilon^{1/\sqrt{\epsilon}} p_s^\omega(0, x) \, ds \right| \leq \sqrt{\epsilon}.$$

Further, by (i), for $d \geq 3$,

$$\lim_{n \to \infty} IV \leq C \int_{1/\sqrt{\epsilon}}^{\infty} \frac{n^d}{(n^2 s)^{d/2}} \, ds = Ce^{(d-2)/4}.$$
Finally, combining (60),(61), (62) and (63), we get

$$\lim_{n \to \infty} \left| \int_0^\infty n^d v(nx, n^2 s) \, ds - \int_{\epsilon}^{1/\sqrt{\epsilon}} p_\Sigma^\epsilon(0, x) \, ds \right| \leq C \epsilon^{1/4}.$$

Letting $\epsilon \to 0$, (iii) is proved. □

References

1. Andres, S.: Invariance principle for the random conductance model with dynamic bounded conductances. Ann. Inst. Henri Poincaré Probab. Stat. 50(2), 352–374 (2014)
2. Andres, S., Chiarini, A., Deuschel, J.-D., Slowik, M.: Quenched invariance principle for random walks with time-dependent ergodic degenerate weights. Ann. Probab. 46(1), 302–336 (2018)
3. Andres, S., Chiarini, A., Slowik, M.: Quenched local limit theorem for random walks among time-dependent ergodic degenerate weights. Probab. Theory Related Fields 179, 1145–1181 (2021)
4. Andres, S., Deuschel, J.-D., Slowik, M.: Harnack inequalities on weighted graphs and some applications to the random conductance model. Probab. Theory Related Fields 164(3–4), 931–977 (2016)
5. Andres, S., Deuschel, J.-D., Slowik, M.: Green kernel asymptotics for two-dimensional random walks under random conductances. Electron. Commun. Probab. 25(58), 1–14 (2020)
6. Bauman, P.: Positive solutions of elliptic equations in nondivergence form and their adjoints. Ark. Mat. 22(2), 153–173 (1984)
7. Barlow, M., Hambly, B.: Parabolic Harnack inequality and local limit theorem for percolation clusters. Electron. J. Probab. 14(1), 1–27 (2009)
8. Berger, N., Criens, D.: A parabolic Harnack inequality for balanced random environments. Work in progress
9. Berger, N., Deuschel, J.-D.: A quenched invariance principle for non-elliptic random walk in i.i.d. balanced random environment. Probab. Theory Related Fields 158(1–2), 91–126 (2014)
10. Coifman, R., Fefferman, C.: Weighted norm inequalities for maximal functions and singular integrals. Studia Math. 51, 241–250 (1974)
11. Criens, D.: Essays on Stochastic Processes and their Applications. PhD Thesis, (2020)
12. Delmotte, T.: Parabolic Harnack inequality and estimates of Markov chains on graphs. Rev. Mat. Iberoam. 15, 181–232 (1999)
13. Delmotte, T., Deuschel, J.-D.: On estimating the derivatives of symmetric diffusions in stationary random environment, with applications to $\nabla \phi$ interface model. Probab. Theory Relat. Fields 133(3), 358–390 (2005)
14. Deuschel, J.-D., Guo, X., Ramirez, A.: Quenched invariance principle for random walk in time-dependent balanced random environment. Ann. Inst. Henri Poincaré Probab. Stat. 54(1), 363–384 (2018)
15. Escauriaza, L.: Bounds for the fundamental solutions of elliptic and parabolic equations: In memory of Eugene Fabes. Commun. Part. Differ. Equ. 25(5–6), 821–845 (2000)
16. Fabes, E., Safonov, M., Yuan, Y.: Behavior near the boundary of positive solutions of second order parabolic equations. II. Trans. Am. Math. Soc. 351(12), 4947–4961 (1999)
17. Fabes, E., Stroock, D.: The $L^p$-integrability of Green’s functions and fundamental solutions for elliptic and parabolic equations. Duke Math. J. 51(4), 997–1016 (1984)
18. Garofalo, N.: Second order parabolic equations in nonvariational form: boundary Harnack principle and comparison theorems for nonnegative solutions. Annali di matematica pura ed applicata 138(1), 267–296 (1984)
19. Guo, X., Zeitouni, O.: Quenched invariance principle for random walks in balanced random environment. Probab. Theory Related Fields 152, 207–230 (2012)
20. Huang, R., Kumagai, T.: Stability and instability of Gaussian heat kernel estimates for random walks among time-dependent conductances, Electron. Commun. Probab. 21, paper no. 5 (2016)
21. Krengel, U., Pyke, R.: Uniform pointwise ergodic theorems for classes of averaging sets and multiparameter subadditive processes. Stochastic Process. Appl. 26(2), 289–296 (1987)
22. Krylov, N., Safonov, M.: A property of the solutions of parabolic equations with measurable coefficients. Izv. Akad. Nauk SSSR Ser. Mat. 44(1), 161–239 (1980)
23. Kuo, H.-J., Trudinger, N.: Evolving monotone difference operators on general space-time meshes. Duke Math. J. 91(3), 587–607 (1998)
24. Lawler, G.: Weak convergence of a random walk in a random environment. Comm. Math. Phys. 87(1), 81–87 (1982/83)
25. Lawler, G.: Estimates for differences and Harnack inequality for difference operators coming from random walks with symmetric, spatially inhomogeneous increments. Proc. Lond. Math. Soc. 63(3), 552–568 (1991)
26. Mustapha, S.: Gaussian estimates for spatially inhomogeneous random walks on $\mathbb{Z}^d$. Ann. Probab. 34(1), 264–283 (2006)
27. Papanicolaou, G., Varadhan, S.R.S.: Diffusions with random coefficients. Statistics and probability: essays in honor of C. R. Rao, pp. 547–552, North-Holland, Amsterdam, (1982)
28. Safonov, M., Yuan, Y.: Doubling properties of second order parabolic equations. Ann. Math. Second Ser. 150(1), 313–328 (1999)
29. Stein, E.: Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ (1993)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.