Self-Organized Percolation Model for Stock Market Fluctuations

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ABSTRACT: In the Cont-Bouchaud model \textsuperscript{1} of stock markets, percolation clusters act as buying or selling investors and their statistics controls that of the price variations. Rather than fixing the concentration controlling each cluster connectivity artificially at or close to the critical value, we propose that clusters shatter and aggregate continuously as the concentration evolves randomly, reflecting the incessant time evolution of groups of opinions and market moods. By the mechanism of “sweeping of an instability” [D. Sornette, Journal de Physique I 4, 209 (1994)], this market model spontaneously exhibits reasonable power law statistics for the distribution of price changes and accounts for the other important stylized facts of stock market price fluctuations.

Keywords: Clusters, activity, Monte Carlo, self-organized criticality, power laws, percolation

1 The Percolation model of stock market prices

A wealth of models \textsuperscript{1,2} (to our knowledge, the first stock market simulation was performed by the economist Stigler in 1964 \textsuperscript{3}), partially listed in \textsuperscript{4}, have been introduced in the financial and more recently in the physical community which attempt to capture the complex behavior of stock market prices and of market participants. Based on the competition between supply
and demand, the effort is to model the main observed stylized facts: absence
of two-point correlation of the returns \[1\], fat tail distribution of returns
(probabilities higher than Gaussian \[2\] \[3\] \[4\] \[5\]) and long-range volatility (stan-
dard deviation) correlations \[6\]. The goal is to have, on the one hand, the
simplest and most parsimonious description of the market and, on the other
hand, the most faithful representation of the observed market characteristics.

In this spirit, Cont and Bouchaud \[8\] introduced a percolation model
which assumes that investors can be classified into groups (clusters) of the
same opinion occurring with many different sizes. The simplest recipe to
aggregate interacting or inter-influencing traders into groups is to assume
that the connectivity between traders defining the groups can be seen as a
pure geometrical percolation problem with fixed occupancy on a given
network topology. Clusters are groups of neighboring occupied sites or investors.
Then, random percolation clusters make a decision to buy or sell on the stock
market, for all sites (corresponding to the individual investors and units of
wealth) in that cluster together. Thus, the individual investors are thought
to cluster together to form companies or groups of influence, which under
the guidance of a single manager buy (probability \(a\)), sell (probability \(a\)),
or refrain from trading (probability \(1 - 2a\)) within one time interval. The
traded amount is proportional to the number \(s\) of sites in the cluster, and

\[
\text{the logarithm of the price changes proportionally to}
\]

\[
\Delta = \text{the difference \Delta between demand and supply}.
\]

When the activity \(a\) is small, at most one cluster trades at a time. As
a consequence, the distribution \(P(R)\) of relative price changes or “returns”
\(R\) scales as the well-known \[7\] cluster size distribution \(n_s(p)\) of percolation
theory. In contrast, for large activity \(a\) and without an infinite cluster, the
relative price variation is the contribution (sum) of many clusters and the
central limit theorem implies that the distribution \(P(R)\) converges to the
Gaussian law for large systems (except exactly at the critical point \(p_c\)).

For low activity \(a\) and right at the site percolation threshold \(p = p_c\), when
the fraction \(p\) of lattice sites occupied by an investor in a \(d\)-dimensional lattice
of linear extent \(L\) barely suffices to form an “infinite” cluster stretching from
top to bottom, we observe power laws:

\[
n_s \propto s^{-\tau}, \quad P(R) \propto R^{-\tau} \quad \text{for } 1 \ll s \ll L^D \quad \text{where } D = d/(\tau - 1)
\]
is the fractal dimension of the percolating cluster.

For concentrations \( p \) different from \( p_c \), the cluster numbers decay as an exponential (resp. stretched exponential) for \( p < p_c \) (resp. for \( p > p_c \)). Their typical size \( s^* \) (not counting the “infinite” percolating cluster) is much smaller than at \( p = p_c \) where it is solely controlled by the system size (i.e. total number of traders). Since a cluster of size roughly comparable to the total system size appears at and above the percolation threshold \( p_c \), this value corresponds to a big crash in the market, and the region of \( p \) below \( p_c \) gives less volatile behavior.

Correlations of volatility in time are produced \cite{4} by letting the occupied sites (traders) diffuse slowly to empty neighbor sites on a lattice to reform or destroy new alliances. The volatility, i.e. the typical absolute value of the return, thus behaves similar to a mean cluster size and is correlated in time due to the slow diffusion \cite{4} with a typical decay slower than exponential.

This model at the percolation threshold thus agrees qualitatively (but not quantitatively especially on the exponents as discussed below) with the three stylized facts of real markets \cite{1,2}: the average return \( R \) is zero (if inflation and other regular trends are subtracted); the return distribution \( P(R) \) decays as a power law \( \propto R^{-\mu} \) for intermediate \( R \) with \( \mu \approx 4 \) (the probability \( P_>(R) \) of finding a change larger than \( R \) then varies as \( R^{1-\mu} \)), and the volatility

\[
V(t) = \langle R(t)^2 \rangle^{1/2}
\]  

clusters in time in the sense that its autocorrelation function

\[
C(\tau) = \langle \Delta V(t) \cdot \Delta V(t+\tau) \rangle , \quad {\text{with}} \quad \Delta V = V - \langle V \rangle , \quad (4)
\]

is positive and decays slowly to zero.

Two disadvantages of the model are:

- why should markets know the percolation threshold and work at \( p = p_c \)?

- How can a value of \( \mu = \tau \) be nearly 4 when \( \tau \) varies only from 2 to 2.5 if the dimensionality \( d \) increases from 2 to infinity ?

A mechanism for self-organized criticality like invasion percolation \cite{9} would only solve the first and not the second problem and would be difficult to justify form an economic view point.
2 A Simple Self-Organizing Market

2.1 Percolation connectivity evolving with time

We thus return to an alternative mechanism [10] which gives power laws without the need to tune $p$ to $p_c$ and which is very robust and simple. The new idea we propose in this context is that there is no reason a priori to expect that the parameter $p$ controlling the connectivity/influence between traders is fixed. The circle of professionals and colleagues to whom a trader is typically connected evolves as a function of time, not only in its structure at fixed average number of connections (corresponding to the diffusion effect discussed above) but also, in the average strength and number of interactions: at some times, traders are following strong herding behavior and the effective connectivity parameter $p$ is high; at other times, investors are more individualistic and smaller values of $p$ seem more reasonable. In order to take into account the complex dynamics of the network of interactions between traders, it thus seems reasonable to relax the hypothesis that $p$ is fixed at a given value but rather evolves with its own dynamics. The simplest version is to assume that $p$ is taken purely random at each time step. As a consequence, the distribution of relative price changes will be an average over those obtained for each sampled $p$’s. Averaging [11] over an interval in $p$ containing the percolation threshold $p_c$ will give its main contribution to the number of large clusters from a narrow region (width $\propto 1/s^\sigma$) about $p_c$, and thus lead to an integrated cluster numbers $\propto s^{\tau-\sigma}$, where $\sigma$ varies from 0.4 to 0.5 if $d$ varies from 2 to infinity. Now $\mu = \tau + \sigma$ varies from 2.45 to 3, closer to reality ($\mu \simeq 4$).

We stress that, as soon as $p$ samples an interval containing or close to $p_c$, the distribution of returns is a power law with no other truncation than given by the finite size of the total system. We thus obtain a robust critical behavior without any artificial adjustment of the connectivity parameter $p$. Notice that this mechanism in terms of a “sweeping of an instability” [11] is different from what is usually called self-organized criticality [11] which involves a dynamical feedback attracting the system dynamics to a dynamical critical point.

We thus [1] distribute randomly our investors on the $L \times L$ square lattice, with concentration $p$. We sum up all the results obtained by varying $p$ in steps of one percent, from 1 to 59 percent where the percolation threshold
$p_c = 0.593$ is reached. For each concentration, we make 1000 iterations where 
at each iteration one percent of the investors try to move to a randomly 
selected neighbor site. For each cluster configuration obtained in this way, 
we sum over 1000 different realizations of buying and selling decisions of the 
cluster, which thus allows much better averaging than in real markets where 
history cannot be repeated so easily. Many such simulations are averaged 
over to give smooth results.

Fama has argued [12] that the crash of Oct. 1987 on the US and other 
stock markets worldwide could be seen as the signature of an efficient re-
assessement of and convergence to the correct “fundamental” price after the 
long speculative bubble preceding it. In this spirit, we assume that, as $p$
reaches the crash concentration $p = p_c$, “everything” changes and the net-
work connectivity is afterwards reinitialized at a value $p < p_c$; therefore no 
data for $p > p_c$ are given here.

Fig.1 shows for the square lattice ($d = 2$) the distribution of returns $P(R)$.
The simulations confirm, for a range of about five orders of magnitude in $P$
similar to the range observed by Gopikrishnan et al [2], the predicted power 
law $P(R) \propto 1/R^\mu$ at intermediate $R$ with an exponent $\mu \approx 2.5$. For the 
largest $R$, finite size effects reduce $P(R)$ and, for small $R$, the probability 
is roughly constant. Increasing the linear lattice size $L$ from 31 via 101 to 
301 shifts the power-law region to larger $R$ without changing the effective 
exponent.

### 2.2 Size-dependent activity

The numerical deviation from the empirically observed $\mu = 4$ is thus large, 
and a different approach is needed. Instead of taking the activity $a$ as a free 
parameter between zero and $1/2$ (0.005 in Fig.1) and the same independently 
of the cluster size, we assume the following size dependence

$$a = 0.5/\sqrt{s},$$

thus getting rid of one free parameter. A priori, it is reasonable to consider 
that the big investors, such as the mutual and/or retirement funds with their 
prudent approach, their emphasis on low risk, and their enormous inertia due 
to the fact that large positions move the market unfavorably, have to and 
do trade less often than small professional investors who have to generate
their income from active trading rather than from sheer mass. In this spirit, recent works have documented that the growth dynamics of business firms \([13]\), the economies of countries \([14]\) and the university research activities \([15]\) depend on size, the smaller entities being the most active proportionally. Another not necessarily exclusive mechanism is that, within a large cluster, the \(s\) investors have to agree by some majority to buy and sell, and do not trade if no such majority is reached. A random decision process then could lead to the square-root behavior given by (4).

With this modification, the exponent \(\mu\) is predicted to be

\[
\mu = \tau + \sigma + 1/2 \simeq 3. 
\] (6)

In contrast, we measure in Fig.2 an effective exponent in the intermediate \(R\) range equal to 3.5, larger than the asymptotically expected value given by eq.(6) and close to the empirical value near 4. The volatility clustering is not destroyed by our change, and Fig.3 shows the volatility auto-correlations to decay slowly towards zero, also in agreement with empirical facts.

Since eq.(4) implies a zero probability of the “infinite” cluster to act for an “infinite” system above \(p_c\), we now can also integrate over the whole interval of \(p\) by summing from 1 to 99 percent. Fig.4 documents a new phenomenon, namely wings with rare price changes determined mainly by large clusters containing nearly the whole lattice and choosing randomly to buy, sell, or sleep. The larger the lattice is, the smaller is the overall absolute weight of these wings. However, relative to the power law represented by the dashed straight line, the larger the lattice, the larger is the relative weight of the wings. This means that these large price changes are more and more “outliers” of the power law statistics holding for intermediate price changes.

To understand this observation, recall first that the connectivity parameter \(p\) goes from a small number (1% in the simulations) to a larger number (99% in the simulations) above \(p_c\) and the observed distribution of log-price changes is mapped one-to-one onto the distribution \(P_{\text{sum}}\) of cluster sizes obtained by averaging over all the cluster size distributions obtained for \(p\) taken with equal weight between 1% and 99%. If the \(n_s\) of eq.(2) are the cluster numbers per lattice site, then \(L^2 n_s\) are those in the whole lattice and vary at \(p = p_c\) as \(L^2 s^{-\tau}\). The contribution \(P_{\text{sum}}(s)\) of \(s\)-clusters averaged over all \(p\), to the price changes reads

\[
P_{\text{sum}} \propto L^2 \frac{1}{s^{\tau+\sigma+1/2}}. 
\] (7)
The two additional terms $\sigma + 1/2$ in the exponent of (7) stem, as discussed above, from the two effects of “sweeping” of $p$ over $p_c$ and of the inverse square root dependence of the activity on the cluster size.

Now, extrapolating (7), we can estimate the probability $cL^2P_{\text{sum}}(cL^2)$ that this power law would predict for getting a cluster of size of order $L^2$ and compare it to the true probability of getting a cluster of this size. The factor $cL^2$ comes from the fact that one must count the large clusters around a typical size with proportional fluctuations, thus giving a true probability while $P_{\text{sum}}(cL^2)$ is the probability density to observe a cluster of size $cL^2$. From (7), we get

$$cL^2P_{\text{sum}}(cL^2) \propto (cL^2)^{L^2 \frac{1}{(cL^2)^{\tau+\sigma+1/2}}},$$

(8)

where $c$ is a number of order unity. Counting powers of $L$ in (8) and expressing the result for two dimensions gives

$$cL^2P_{\text{sum}} \propto L^{4-2\tau-2(1/2+\sigma)} = \frac{1}{L^{1.9}}.$$

(9)

In contrast, we know that there is exactly one infinite cluster in a quadratic lattice above $p_c$. Since the $p$'s are uniformly taken between 1% and 99%, this shows directly that the probability to get a cluster close to the maximum size $L^2$ is a constant fraction, independently of the lattice size $L$, and its contribution to a price change is multiplied by its activity $\propto 1/L$. (Indeed our data of Fig.4 give a value near $0.5/L$ for the fraction of returns larger than $L^2/2$.) This argument thus shows that the ratio of the true frequency to observe the large “outlier” (stemming from the infinite cluster truncated to the size of the lattice above $p_c$) is larger than the extrapolation of the power distribution of intermediate cluster size by a factor $L^{0.9}$ which increases with the system size $L$, in qualitative agreement with Fig.4.

We suggest that the large wings might correspond to the “outliers” in the stock market like the Wall Street crashes of 1929 and 1987 [16]. The normal autocorrelation functions of the volatility exhibiting long range dependence then apply to normal times on the stock market when no such outliers are relevant, and they are destroyed by the outliers. The wings vanish and the autocorrelations are restored if we follow [14] and omit the largest cluster from the market.
2.3 Nonlinear price change dependence

All our previous results derive from the assumption \(\text{(1)}\) that the change of (the logarithm of the) price is proportional to the difference between supply and demand. This assumption is often made and can be in fact derived rigorously \([17]\) from the two assumptions that it is not possible to make profits by repeatedly trading through a circuit and that the ratio of prices before and after a transaction is a function of the difference \(\Delta\) between demand and supply alone.

However, many recent empirical studies suggest that the relationship between the change of the logarithm of price and \(\Delta\) is highly nonlinear, especially for large orders \([18]\). Assuming that the time needed to complete a trade of size \(s\) is proportional to \(s\) and that the unobservable price fluctuations obey a diffusion process during that time, Zhang derives the relationship that the change of the logarithm of the price is proportional to the square root of the difference \(\Delta\) between demand and supply \([19]\), i.e. to the square root of \(s\) in our present formulation. This modifies all previous results as follows.

The result \((2)\) for the “pure” percolation model becomes

\[
n_s \propto s^{-\tau}, \quad P(R) \propto R^{-\mu} \quad \text{for} \quad 1 \ll s \ll L^D \quad \text{where} \quad R \propto \sqrt{s}, \tag{10}
\]

giving with \(ds/dR \propto R\) (with numerical estimates in two dimensions) the exponent \(\mu = 2\tau - 1\) around 3.1, still smaller than the empirical value close to 4.

The result obtained by the “sweeping” of the connectivity parameter \(p\) transforms \(\mu\) from \(\mu = \tau + \sigma\) into \(\mu = 2\tau - 1 + \sigma\), giving a value 3.5. Next, incorporating the size dependence \((5)\) of the activity leads to the prediction \(\mu = 2\tau + \sigma = 4.5\), in rough agreement with the empirical value 4.

We may even omit the size-dependent activity and use only this nonlinear price change dependence and \(0 < p < p_c\). Then the data of Fig.1 are transformed, without any additional simulations, into those of Fig.5 which give an effective \(\mu \simeq 3.9\) in better agreement with the real \(\mu \simeq 4\) than the theoretical prediction \(\mu = 2\tau - 1 + \sigma \simeq 3.5\).
3 Concluding Remarks

We have presented what we believe is probably the simplest and most robust model of stock market dynamics without tunable parameters that self-organizes into a regime where the most important empirical characteristics of stock market price dynamics are captured.

In this simplest version, we have chosen the most straightforward dynamics of the interaction/connectivity parameter $p$, i.e. a continuous increase up to the critical value $p_c$ followed by a reset to a low value and so on. Incorporating a size-dependence of the cluster activities has allowed us to let $p$ larger than $p_c$ for which we have documented the appearance of outliers corresponding to the infinite cluster truncated to the size of the lattice. This outlier might correspond to the large crashes observed in this century. A good agreement with empirical data is obtained alternatively by allowing for a nonlinear dependence of the change of (the logarithm of the) price as a function of the difference between supply and demand.

A random evolution of $p$, either pure white noise, or a random walk or with more correlation are interesting to investigate in the future, but will not change the most fundamental finding presented here of a power law distribution and long-range correlations of the volatility. More interesting presumably would be a dynamic of $p$ coupled to that of the price change, simulating the tendency to join a bullish market [2]. Note also that the present model is by construction up-down symmetric, which means that rallies appear as often statistically and in the same shape as crashes. There is not sharp peak versus flat trough asymmetry [20]. Such asymmetry can be easily incorporated by letting the trading activity be dependent on the function price(time), i.e. increasing prices causes more people to act than a decreasing price, but we have not persued this as this would imply adding novel ingredients in a model we have on purpose kept bare to its skeleton.

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Captions:

Fig.1: Return distribution at constant activity $a = 0.005$. The axis of relative price variations is scaled such that a buying cluster of $s$ investors produces an increase of the price by $s$.

Fig.2: Return distribution with activity decaying as $1/\sqrt{s}$.

Fig.3: Volatility autocorrelation function $C(T)$ versus time lag $T$ for the simulations of Fig.2.

Fig.4: As Fig.2, but with the cluster connectivity parameter $p$ varying from 1 to 99 percent.

Fig.5: Data from Fig.1 replotted by assuming a price change proportional to the square root of the (absolute value of the) difference $\Delta$ between demand and supply (and with sign opposite to that of $\Delta$).
a=0.005, summed from 1 to 59 %, L = 31 (2688 runs, +), 101 (320 runs, x), 201 (64 runs, stars)
Histogram, \( a = 0.5 / \sqrt{s} \), 64 lattices 101*101 summed from 10 to 59 \%; slope -3.5
Volatility clustering, $a=0.5/\sqrt{t}$, 64 lattices 101*101 summed from 10 to 59%
Sum from 0.01 to 0.99; 31*31 (x), 71*71 (+) and 100*100 (dots), 320, 64 and 64 samples; slope -3.5
Histogram of price changes, $a=0.005$, $64$ lattices $201 \times 201$ summed from 1 to 59 $\%$; slope $-3.9$