On–Off Random Access Channels: A Compressed Sensing Framework

Alyson K. Fletcher, Sundeep Rangan, and Vivek K Goyal

Abstract—This paper considers a simple on–off random multiple access channel, where $n$ users communicate simultaneously to a single receiver over $m$ degrees of freedom. Each user transmits with probability $\lambda$, where typically $\lambda n < m \ll n$, and the receiver must detect which users transmitted. We show that when the codebook has i.i.d. Gaussian entries, detecting which users transmitted is mathematically equivalent to a certain sparsity detection problem considered in compressed sensing. Using recent sparsity results, we derive upper and lower bounds on the capacities of these channels. We show that common sparsity detection algorithms, such as lasso and orthogonal matching pursuit (OMP), can be used as tractable multiuser detection schemes and have significantly better performance than single-user detection. These methods do achieve some near–far resistance but—at high signal-to-noise ratios (SNRs)—may achieve capacities far below optimum. Maximum likelihood detection. We then present a new algorithm, called sequential OMP, that illustrates that iterative detection combined with power ordering or power shaping can significantly improve the high SNR performance. Sequential OMP is analogous to successive interference cancellation in the classic multiple access channel. Our results thereby provide insight into the roles of power control and multiuser detection on random-access signalling.

Index Terms—compressed sensing, convex optimization, lasso, maximum likelihood estimation, multiple access channel, multiuser detection, orthogonal matching pursuit, power control, random matrices, single-user detection, sparsity, thresholding

I. INTRODUCTION

In wireless systems, random access refers to any multiple access communication protocol where the users autonomously decide whether or not to transmit depending on their own traffic requirements and estimates of the network load. While random access is best known for its use in packet data communication in wireless local area networks (LANs) [1], this paper considers random access for simple on–off messaging. On-off random access signaling can be used for a variety of control tasks in wireless networks such as user presence indication, initial access, scheduling requests and paging. Random on–off signaling is already used for some of these tasks in current cellular systems [2], [3]

The limits of on–off random access signaling with multiple users are not fully understood. To this end, we consider a simple random multiple access channel where $n$ users transmit to a single receiver. Each user is assigned a single codeword which it transmits with probability $\lambda$. We wish to understand the capacity of these channels, by which we mean the total number of degrees of freedom $m$ needed to reliably detect which users transmit as a function of $n$, $\lambda$, and the channel conditions. We also wish to establish performance bounds for specific decoding algorithms.

This on–off random access channel is related to the classic multiple access channel (MAC) in network information theory [4], [5]. The theory of the MAC channel is well understood [6]–[9] and has been applied in commercial CDMA systems [10]. Unfortunately, it is difficult to apply the classic MAC channel analysis directly to the on–off random access channel under consideration here.

In the traditional analysis of the MAC channel, the number of users remains constant, while the number of degrees of freedom of the channel goes to infinity. As a result, each user can employ a capacity-achieving code with an infinite block length. However, in the on–off random access channel considered here, as the number of degrees of freedom of the channel is increased, the goal is not to scale the number of bits per user, but rather the total number of users. Since each user only transmits at most one bit of information, channel coding cannot be used for reliability, and the classic MAC capacity results do not apply.

Our analysis is instead based on identifying a connection between the on–off random access channel and the recovery of the sparsity pattern of a signal from noisy random linear measurements. The feasibility of recovering sparse, approximately sparse, or compressible signals from a relatively small number of random linear measurements has recently been termed compressed sensing [11]–[13]. When the users in the on–off random access channel employ certain large random codebooks, we show that the problem at the receiver of detecting the active users is precisely the sparsity detection problem addressed in several recent works in the compressed sensing literature [14]–[17].

Results in compressed sensing generally provide bounds on the $\ell^2$ estimation error of a signal as a function of the number of measurements, the signal sparsity and other factors. However, what is relevant for the random on–off multiple access channel is detecting the positions of the nonzero entries. This problem arises in subset selection in linear regression [18].

By exploiting recent compressed sensing results and providing an analysis of a new algorithm, we are able to provide
a number of insights:

- **Performance bounds with ML detection:** Recent results in [14], [15], [19] provide simple upper and lower bounds on the number of measurements required to detect the users reliably assuming maximum likelihood (ML) detection. One of the consequences of these bounds is that, unlike the classic MAC channel, the sum rate achievable with random access signaling can be strictly less than the rate achievable with coordinated transmissions with the same total power.

- **Potential gains over single-user detection:** ML detection can be considered as a type of multiuser detection. Current commercial designs, however, almost universally use simple single-user detection (see, for example [20] for a typical WCDMA design). The single-user detection performance can be estimated by bounds given in [15], [21]. The bounds show that ML detection offers a potentially large gain over single-user detection, particularly at high SNRs. The gap at high SNRs can be explained by a certain self-noise limit experienced by single-user detection.

- **Lasso- and OMP-based multiuser detection and near–far resistance:** ML sparsity detection is a well-known NP-hard problem [22]. However, there are practical, but suboptimal, algorithms such as the orthogonal matching pursuit (OMP) [23]–[26] and “lasso” [27] methods in sparse estimation that can be used for multiuser detection methods for the on–off random access channel. In comparison to single-user detection, we show that these methods can offer improved performance when the dynamic range in received power levels is large. This near–far resistance feature is similar to that of standard MMSE multiuser detection in CDMA systems [28].

- **Improved high SNR performance with power shaping:** While both lasso and OMP offer improvements over single-user detection, there is still a large gap in the performance of these algorithms in comparison to ML detection at high SNRs. Specifically, at high SNRs, ML achieves a fundamentally different scaling in the number of measurements required for reliable detection than that required by lasso, OMP and single-user detection.

We show, however, that when accurate power control is available, the ML scaling can be theoretically achieved with a simplified version of OMP, which we call sequential OMP (SeqOMP). The method is analogous to the classic successive interference cancellation (SIC) method for the MAC channel. Specifically, users are deliberately targeted at different received power levels and then detected and cancelled out in descending order of power. While SeqOMP shows significant gains over single-user detection, for most practical problem sizes it does worse than standard OMP, even without power shaping. However, we show, at least by simulation, that power shaping can improve the performance of OMP as well.

The connection between sparsity detection methods such as OMP and the SIC technique for the MAC channel has also been observed in the recent work of Jin and Rao [29]. A related work by Wipf and Rao [30] also gave some empirical evidence for the benefit of power shaping when used in conjunction with sparse Bayesian learning algorithms. Both the works [29] and [30] are discussed in more detail below. The results in this work make the connections between sparsity detection and the random access MAC channel more precise by giving concrete conditions on the detectability of the sparsity pattern, characterizing the optimal power shaping distribution, and contrasting the classic MAC and on–off random access MAC capacities.

The remainder of the paper is organized as follows. The setting is formalized in Section II. In particular, we define all the key problem parameters. Results that can be derived from existing necessary and sufficient conditions for sparsity pattern recovery are then presented in Section III. We will see that there is a potentially-large performance gap between single-user detection and the optimal ML detection. Existing “practical” multiuser detection techniques perform significantly better than single-user detection in that they are near–far resistant. However, their performance saturates at high SNRs, falling well short of ML detection. Section IV presents a new detection algorithm, sequential orthogonal matching pursuit (SeqOMP), that has near–far resistance under certain assumptions on power control. Furthermore, with optimal power shaping, it does not suffer from saturation at high SNRs. Numerical experiments are reported in Section V. Connection to MAC capacity are discussed in Section VI. Conclusions are given in Section VII and proofs are relegated to the Appendix.

II. ON-OFF RANDOM ACCESS CHANNEL MODEL

A. Problem Formulation

Assume that there are $n$ transmitters sharing a wireless channel to a single receiver. Each user $j$ is assigned a unique, dedicated codeword represented as an $m$-dimensional vector $\mathbf{a}_j \in \mathbb{C}^m$, where $m$ is the total number of degrees of freedom in the channel. By degrees of freedom we simply mean the dimension of the received vector, which represents the number of samples in time or frequency depending on the modulation. In any channel use, only some fraction of the users, $\lambda \in (0, 1)$, transmit their codeword. The fraction $\lambda$ will be called the activity ratio and any user that transmits will be called active.

The signal at the receiver from each user $j$ is modeled as $x_j \mathbf{a}_j$ where $x_j$ is a complex scalar. If the user is not active, $x_j = 0$. If the user is active, $x_j$ would represent the product of the transmitted symbol and channel gain. The total signal at the receiver is given by

$$\mathbf{y} = \sum_{j=1}^{n} x_j \mathbf{a}_j + \mathbf{w} = \mathbf{A}\mathbf{x} + \mathbf{w},$$

where $\mathbf{w} \in \mathbb{C}^m$ represents noise. The matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$ is formed by codewords $\mathbf{a}_j$,

$$\mathbf{A} = [\mathbf{a}_1 \cdots \mathbf{a}_n],$$

and will be called the codebook. The vector $\mathbf{x} = [x_1 \cdots x_n]^T$ will be called the modulation vector, and its components $\{x_j\}_{j=1}^{n}$ are referred to as the received modulation symbols.
Given a modulation vector $x$, define the active user set as

$$I_{true} = \{ j : x_j \neq 0 \},$$

(2)

which is the “true” set of active users. The size of the active user set is related to the activity ratio through

$$\lambda = \frac{1}{n} |I_{true}|.$$  \hspace{1cm} (3)

The goal of the receiver is to determine an estimate $\hat{I} = \hat{I}(y)$ of $I_{true}$ based on the received noisy vector $y$.

For the most part, we will be interested in estimators that exploit minimal prior knowledge of the modulation vector $x$ other than it being sparse. In particular, we will limit our attention to estimators that do not explicitly require a priori knowledge of the complex modulation symbols $x_j$. This assumption is required since the channel gain is typically unknown at the receiver in random access channels, since users conducting random access communication would be unlikely to be sending any other persistent pilot reference.

We consider large random codebooks where the entries of $A$ are i.i.d. $CN(0, 1/m)$. We assume the noise vector is also Gaussian: $w \sim CN(0, (1/m)I_m)$. Given an estimator, $\hat{I} = \hat{I}(y)$, the probability of error,

$$p_{err} = Pr \left( \hat{I} \neq I_{true} \right),$$

(4)

is taken with respect to random codebook $A$, the noise vector $w$, and the statistical distribution of the modulation vector $x$. We want to find estimators $\hat{I}$ that bring $p_{err}$ close to zero.

We will see that two key factors influence the ability to detect the active user set. The first is the total SNR defined as

$$\text{SNR} = \frac{\mathbb{E}[\|Ax\|^2]}{\mathbb{E}[\|w\|^2]}.$$  \hspace{1cm} (5)

Since the components of the matrix $A$ and noise vector $w$ are i.i.d. $CN(0, 1/m)$, it can be verified that, for deterministic $x$,

$$\text{SNR} = \|x\|^2.$$  \hspace{1cm} (6)

In the case of random $x$, this expression is the conditional SNR given $x$; we will have both deterministic and random formulations.

The second term is what we will call the minimum-to-average ratio

$$\text{MAR} = \frac{\min_{j \in I_{true}} |x_j|^2}{\|x\|^2 / \lambda n}.$$  \hspace{1cm} (7)

Since $I_{true}$ has $\lambda n$ elements, $\|x\|^2 / \lambda n$ is the average of $\{|x_j|^2 | j \in I_{true}\}$. Therefore, $\text{MAR} \in (0, 1]$ with the upper limit occurring when all the nonzero entries of $x$ have the same magnitude. $\text{MAR}$ is a deterministic quantity when $x$ is deterministic and a random variable otherwise.

One final value that will be important is the minimum component SNR, which, for a given $x$, is given by

$$\text{SNR}_{min} = \frac{1}{\mathbb{E}[\|w\|^2]} \min_{j \in I_{true}} \mathbb{E}[\|a_j x_j\|^2],$$  \hspace{1cm} (8)

where $a_j$ is the $j$th column of $A$. The quantity $\text{SNR}_{min}$ has a natural interpretation: The numerator, $\min_{j \in I_{true}} \mathbb{E}[\|a_j x_j\|^2]$ is the signal power due to the smallest nonzero component in $x$, while the denominator, $\mathbb{E}[\|w\|^2]$, is the total noise power. The ratio $\text{SNR}_{min}$ thus represents the contribution to the SNR from the smallest nonzero component of the unknown vector $x$.

The final equality in (8) is a consequence of the fact that $\mathbb{E}[\|a_j\|^2] = \mathbb{E}[\|w\|^2] = 1$. Observe that (6) and (7) show

$$\text{SNR}_{min} = \frac{1}{\lambda n} \text{SNR} \cdot \text{MAR}.$$  \hspace{1cm} (9)

### B. MAR and Power Control

For wireless systems, the factor MAR in (7) has an important interpretation as a measure of the dynamic range of received power levels. With accurate power control, all users can be controlled to arrive at the same power. In this case, $\text{MAR} = 1$. However, if power control is difficult due to fading or lack of power control feedback, there can be a considerable dynamic range in the received powers from different users. In this case, some users could arrive at powers much below the average making $\text{MAR}$ closer to zero.

One of the results in this paper is a precise quantification of the effect of $\text{MAR}$ on the detectability of the active user set. Specifically, we will show that low $\text{MAR}$ can make reliable detection significantly more difficult for certain algorithms. The problem is analogous to the well-known near-far effect in CDMA systems [28], where users with weak signals can be dominated by higher-power signals.

### C. Synchronization and Multi-Path

It is important to recognize that an implicit assumption in the above model is that the transmissions from different users are perfectly synchronized. At a minimum, the timing offsets from the users are exactly known at the receiver and there is no multipath.

Of course, in many wireless applications, exact synchronization is not possible and the receiver must estimate the timing delay of the transmission as part of the detection process. In most practical receivers, timing offsets are estimated by discretizing the delay search space, typically to a quarter or half-chip resolution. The receiver then searches over a finite set of delay hypotheses depending on the range of timing uncertainty. In the presence of multipath, the receiver could detect multiple delay hypotheses.

To model this search in the theoretical framework of this paper, we would need to model each timing shift of the codeword as a different codeword. The total number of codewords would then grow to the number of users times the number of delay hypotheses per user. While the algorithms we will present can be applied in this manner to deal with the asynchronous case, there are several theoretical issues with extending the analysis. In particular, this extended codebook would lack the independence of codewords that the simpler model has by construction. We will thus just consider only the synchronous case for the remainder of this paper.

### III. PERFORMANCE WITH CURRENT SPARSITY DETECTION METHODS

The problem of detecting the active user set is precisely equivalent to a sparsity pattern recovery problem. To see this,
note that the modulation vector $x$ is sparse, with nonzero components only in positions corresponding to the active users. The problem at the receiver is to detect these nonzero positions in $x$ from noisy linear observations $y$ in (1).

In this section, we develop asymptotic analyses for detection of the active users based on previous results on sparsity pattern recovery. We model $x$ as deterministic, so the quantities $\lambda$, $\text{SNR}$, $\text{MAR}$ and $\text{SNR}_{\text{min}}$ are also deterministic. Since our formulation allows simple translation of results from [14]–[17], we state these translations without detailed justifications. Several results are here adjusted by a factor of two because we have complex, rather than real, measurements.

Our results are expressed as scaling laws on the number of measurements for asymptotic reliable detection of the active user set. We define this as follows:

**Definition 1:** Suppose that we are given deterministic sequences $m = m(n)$ and $x = x(n) \in \mathbb{C}^n$ that vary with $n$. For a given detection algorithm $\hat{I} = \hat{I}(y)$, we then define the probability of error $p_{\text{err}}$ in (4) where the probability is taken over the randomness of the codebook $A$ and the noise vector $w$. Given the number of measurements $m(n)$ and modulation vector $x(n)$, the probability of error will then simply be a function of $n$. We say that the detection algorithm achieves asymptotic reliable detection when $p_{\text{err}}(n) \to 0$.

Table [I] summarizes the results from this section and previews results from Section [IV].

### A. Optimal Detection With No Noise

To understand the limits of detection, it is useful to first consider the minimum number of measurements when there is no noise. Since the activity ratio is $\lambda$, $x$ will have $k = \lambda n$ nonzero components. For a lower bound on the minimum number of measurements needed for reliable detection, suppose that the receiver knows the number of active users $k$ as side information.

With no noise, the received vector is $y = Ax$, which will belong to one of $J = \binom{k}{0}$ subspaces spanned by $k$ columns of $A$. If $m > k$, then these subspaces will be distinct with probability 1. Thus, an exhaustive search through the subspaces will reveal which subspace $y$ belongs to and thus determine the active user set. This shows that with no noise and no computational limits, the scaling in measurements of

$$m > \lambda n$$

is sufficient for asymptotic reliable detection.

Conversely, if no prior information is known at the receiver other than $x$ being $k$-sparse, then the condition (10) is also necessary. If $m \leq k = \lambda n$, then for almost all codebooks $A$, any $k$ columns of $A$ span $\mathbb{C}^n$. Consequently, any received vector $y = Ax$ is consistent with any $k$ users transmitting. Thus, the active user set cannot be determined without further prior information on the modulation vector $x$.

### B. ML Detection with Noise

Now suppose there is noise. Since $x$ is an unknown deterministic quantity, the probability of error in detecting the active user set is minimized by maximum likelihood (ML) detection. Since the noise $w$ is Gaussian, the ML detector finds the $k$-dimensional subspace spanned by $k$ columns of $A$ containing the maximum energy of $y$.

The ML estimator was first analyzed by Wainwright [14]. The results in that work, along with the fact that $k = \lambda n$, show that there exists a constant $C > 0$ such that if

$$m \geq C\max\left\{ \frac{1}{\text{MAR} \cdot \text{SNR}} \lambda n \log(n(1-\lambda)), \lambda n \log(1/\lambda) \right\}$$

$$= C\max\left\{ \frac{1}{\text{SNR}_{\text{min}}} \log(n(1-\lambda)), \lambda n \log(1/\lambda) \right\}$$

(11) then ML will asymptotically detect the correct active user set. The equivalence of the two expressions in (11) is due to (9). Also, [15, Thm. 1] (generalized in [19, Thm. 1]) shows that, for any $\delta > 0$, the condition

$$m \geq \frac{1 - \delta}{\text{MAR} \cdot \text{SNR}} \lambda n \log(n(1-\lambda)) + \lambda n,$$

$$= \frac{1 - \delta}{\text{SNR}_{\text{min}}} \log(n(1-\lambda)) + \lambda n,$$

(12) is necessary. Observe that when $\text{SNR} \cdot \text{MAR} \to \infty$, the lower bound (12) approaches $m \geq \lambda n$, matching the noise free case (10) as expected.

These necessary and sufficient conditions for ML appear in Table [I] with smaller terms and the infinitesimal $\delta$ omitted for simplicity.

### C. Single User Detection

The most common and simple method to detect the active user set is a single-user detection estimator of the form,

$$\hat{I}_{\text{SUD}} = \{ j : \rho(j) > \mu \},$$

(13)

where $\mu > 0$ is a threshold parameter and $\rho(j)$ is the correlation coefficient,

$$\rho(j) = \frac{|a_j^\ast|^2}{\|a_j\|^2 \|y\|^2}.$$

(14)

Single-user detection has been analyzed in the compressed sensing context in [15], [21], [31]. A small modification of [15] shows the following result: Suppose,

$$m(n) > \frac{(1 + \delta) L(\lambda, n)(1 + \text{SNR})}{\text{SNR} \cdot \text{MAR}} \lambda n,$$

$$= \frac{(1 + \delta) L(\lambda, n)(1 + \text{SNR})}{\text{SNR}_{\text{min}}} \lambda n,$$

(15)

where $\delta > 0$ and

$$L(\lambda, n) = \left[ \sqrt{\log(n(1-\lambda))} + \sqrt{\log(n\lambda)} \right]^2.$$

(16)

Then there exists a sequence of detection thresholds $\mu = \mu(n)$ such that single-user detection achieves asymptotic reliable detection of the active user set. As before, the equivalence of the two expressions in (15) is due to (9).

Comparing the sufficient condition (15) for single-user detection with the necessary condition (12), we see two distinct problems in single-user detection:
### Constant offset
The scaling [15] for single-user detection shows a factor \( L(\lambda, n) \) instead of \( \log((1 - \lambda)n) \) in (12). It is easily verified that, for \( \lambda \in (0, 1/2) \),

\[
\log((1 - \lambda)n) < L(\lambda, n) < 4 \log((1 - \lambda)n),
\]

so this difference in factors alone could require that single-user detection use up to four times more measurements than ML for asymptotic reliable detection. Combining the inequality (17) with (15), we see that the more stringent, but simpler, condition

\[
m(n) > \frac{(1 + \delta)(1 + \text{SNR})}{\text{MAR}} \lambda n \log((1 - \lambda)n)
\]

is also sufficient for asymptotic reliable detection with single-user detection. This simpler condition is shown in Table I where we have omitted the infinitesimal \( \delta \) quantity to simplify the table entry.

### Self noise limit
In addition to the \( L(\lambda, n) / \log(n(1 - \lambda)) \) offset, single-user detection also requires a factor of \( 1 + \text{SNR} \) more measurements than ML. This \( 1 + \text{SNR} \) factor has a natural interpretation as self-noise: When detecting any one component of the vector \( x \), single-user detection sees the energy from the other \( n - 1 \) components of the signal as interference. We can think of this additional noise as self-noise, by which we mean the interference caused from different components of the signal \( x \) interfering with one another in the observed signal \( y \) through the measurement matrix \( A \). This self-noise is distinct from the additive noise \( w \). This self-noise increases the effective noise by a factor of \( 1 + \text{SNR} \), which results in a proportional increase in the minimum number of measurements.

This self-noise results in a large performance gap at high SNRs. In particular, as \( \text{SNR} \to \infty \), (15) reduces to

\[
m(n) > \frac{(1 + \delta)L(\lambda, n)}{\text{MAR}} \lambda n \log((1 - \lambda)n).
\]

In contrast, ML may be able to succeed obtain with a scaling \( m = O(\lambda n) \) for high SNRs, which is fundamentally better than the \( m = \Omega(\lambda n \log((1 - \lambda)n)) \) required by single-user detection.

### D. Lasso and OMP Estimation
While ML has clear advantages over single-user detection, it is not computationally feasible. However, one practical method used in sparse signal estimation is the lasso estimator [27], also called basis pursuit denoising [32]. In the context of the random access channel, the lasso estimator would first estimate the modulation vector \( x \) by solving the convex minimization

\[
\hat{x} = \arg \min_{x} \left( \|y - Ax\|_2^2 + \mu \|x\|_1 \right),
\]

where \( \mu > 0 \) is an algorithm parameter that “encourages” sparsity in the solution \( \hat{x} \). The nonzero components of \( \hat{x} \) can then be used as an estimate of the active user set.

The exact performance of lasso is not known at finite SNR. However, Wainwright [16] has exactly characterized the conditions for lasso to work in the high SNR regime. Specifically, if \( m, n \) and \( \lambda n \to \infty \), with \( \text{SNR} \cdot \text{MAR} \to \infty \), the scaling

\[
m > \lambda n \log(n(1 - \lambda)) + \lambda n + 1,
\]

is both necessary and sufficient for asymptotic sparsity recovery.

Another common approach to sparsity pattern detection is the greedy OMP algorithm [23], [25], [26]. This has been analyzed by Tropp and Gilbert [17] in a setting with no noise. They show that, when \( A \) has Gaussian entries, a sufficient condition for asymptotic reliable recovery is

\[
m > 2\lambda n \log(n) + C\lambda n,
\]

where \( C > 0 \) is a constant. Numerical experiments reported in [17] suggest that the constant factor 2 may be removed.

### TABLE I
Summary of results on measurement scalings for asymptotic reliable detection for various detection algorithms.

Only leading terms are shown. See body for definitions and additional technical limitations.

| Necessary for ML | \( m > \frac{1}{\text{MAR} \cdot \text{SNR}} \lambda n \log((1 - \lambda)n) \) | \( m > \lambda n \) |
|------------------|--------------------------|------------------|
| Fletcher et al. [15, Thm. 1] | (elementary) | |
| Sufficient for ML | \( m > \frac{C}{\text{MAR} \cdot \text{SNR}} \lambda n \log((1 - \lambda)n) \) | \( m > \lambda n \) |
| Wainwright [14] | (elementary) | |
| Sufficient for sequential OMP with power shaping | \( m > \frac{1}{\log(1 + \text{SNR})} \lambda n \log((1 - \lambda)n) \) | \( m > 5\lambda n \) |
| From Theorem I (Section IV-E) | From Theorem I (Section IV-F) |
| Necessary and sufficient for lasso detection | unknown (expression to the right is necessary) | \( m > \lambda n \log(n(1 - \lambda)) \) |
| Wainwright [16] | |
| Sufficient for OMP | unknown | \( m > 2\lambda n \log(n) \) |
| Tropp and Gilbert [17] | |
| Sufficient for single user detection [13] | \( m > \frac{4(1 + \text{SNR})}{\text{MAR} \cdot \text{SNR}} \lambda n \log((1 - \lambda)n) \) | \( m > \frac{4}{\text{MAR}} \lambda n \log(n(1 - \lambda)) \) |
| Fletcher et al. [15, Thm. 2] | |
although this has not be proven. In any case, OMP with no noise has a similar scaling in the sufficient number of measurements as lasso.

The conditions (21) and (22) are both shown in Table I. As usual, the table entries are simplified by including only the leading terms.

The lasso and OMP scaling laws, (21) and (22), can be compared with the high SNR limit for the single-user detection scaling law in (19). This comparison shows the following:

- **Removal of the constant offset:** The $L(\lambda, n)$ term in the single-user detection expression (19) is replaced by a $\log(n(1 - \lambda))$ term in the lasso scaling law (21) and $2\log(n)$ for the OMP scaling law (22). Similar to the discussion above, this implies that lasso could require up to 4 times fewer measurements than single-user detection. OMP could require 2 times fewer.

- **Near-far resistance:** In addition, both the lasso and OMP methods do not have a dependence on MAR; thus, in the high SNR regime, they have a near–far resistance that single-user detection does not. This gain can be large when there are users whose received powers are much below the average (low MAR). The near–far resistance of lasso and OMP is analogous to that of MMSE multiuser detection in CDMA systems [28]. In that case, when the number of degrees of freedom $m$ exceeds the number of users $n$, a decorrelating detector can null out strong users while recovering weak ones. An interesting property that we see in the random access case is that near–far resistance may be possible when $m < n$, provided that $m$ is sufficiently greater than the number of active users, $\lambda n$.

- **Limits at high SNR:** We also see from (21) and (22) that both lasso and OMP are unable to achieve the scaling $m = O(\lambda n)$ that may be achievable with ML at high SNR. Instead, both lasso and OMP have the scaling, $m = O(\lambda n \log((1 - \lambda)n))$, similar to the minimum scaling possible with single-user detection, which suffers from a self-noise limit.

### IV. Sequential Orthogonal Matching Pursuit

The analyses in the previous section suggest that ML detection may offer significant gains over the provable performance of current “practical” algorithms such as single-user detection, lasso and OMP, when the SNR is high. Specifically, as the SNR increases, the performance of these practical methods saturates at a scaling in the number of measurements that can be significantly higher than that for ML.

In this section, we show that if accurate power control is available, an OMP-like algorithm, which we call sequential orthogonal matching pursuit or SeqOMP, can break this barrier. Specifically, the performance of SeqOMP does not saturate at high SNR.

#### A. Algorithm

**Algorithm 1 (SeqOMP):** Given a received vector $y$ and threshold level $\mu > 0$, the algorithm produces an estimate $\hat{I}_{\text{OMP}}$ of the active user set with the following steps:

1. Initialize the counter $j = 1$ and set the initial active user set estimate to empty: $\hat{I}(0) = \emptyset$.
2. Compute $P(j)\alpha_j$ where $P(j)$ is the projection operator onto the orthogonal complement of the span of $\{\alpha_\ell, \ell \in I(j - 1)\}$.
3. Compute the correlation, 
   \[
   \rho(j) = \frac{|\alpha_j' P(j)y|^2}{\|P(j)\alpha_j\|^2\|P(j)y\|^2}.
   \]
4. If $\rho(j) > \mu$, add the index $j$ to $\hat{I}(j - 1)$. That is, $\hat{I}(j) = \hat{I}(j - 1) \cup \{j\}$. Otherwise, set $\hat{I}(j) = \hat{I}(j - 1)$.
5. Increment $j = j + 1$. If $j \leq n$ return to step 2.
6. The final estimate of the active user set is $\hat{I}_{\text{OMP}} = \hat{I}(n)$.

The SeqOMP algorithm can be thought of as an iterative version of single-user detection with the difference that, after an active user is detected, subsequent correlations are performed only in the orthogonal complement to the detected codeword. The method is identical to the standard OMP algorithm of [23], [25], [26], except that SeqOMP passes through the data only once. For this reason, SeqOMP is actually computationally simpler than standard OMP.

As simulations will illustrate later, SeqOMP generally has much worse performance than standard OMP. It is not intended as a competitive practical alternative. Our interest in the algorithm lies in the fact that we can prove positive results for SeqOMP. Specifically, we will be able to show that this relatively poor algorithm, when used in conjunction with power shaping, can achieve a fundamentally better scaling at high SNRs than what has been proven is achievable with methods such as OMP. We will also provide some simulation evidence that OMP can also benefit somewhat from power shaping, although we will not be able to prove this here.

#### B. Sequential OMP Performance

The analysis in Section III was based on deterministic vectors $x$. To characterize the SeqOMP performance, it is simpler to use a partially-random model where the active user set
is random while the received modulation signal power $|x_j|^2$, conditioned on user $j$ being active, remains deterministic. We reuse the notation $\lambda$ because its meaning remains almost the same.

We assume that each user is active with some probability $\lambda \in (0, 1)$, which we now call the activity probability. The activities of different users are assumed to be independent. Thus, unlike in Section III, $\lambda_n$ represents the average number of users that are active, as opposed to the actual number.

Let $p_j$ denote the received modulation symbol power

$$p_j = |x_j|^2,$$  

(24)

conditional that user $j$ is active. We will call the set $\{p_j\}_{j=1}^n$ the power profile, which we will treat as a deterministic quantity. Since each user transmits with a probability $\lambda$, the total average SNR is given by

$$\text{SNR} = \lambda \sum_{\ell=1}^n p_j.$$  

(25)

This factor is also deterministic.

Given a power profile, we will see that a key parameter in estimating the performance of the SeqOMP algorithm is what we will call the minimum signal-to-interference and noise ratio (SINR) defined as

$$\gamma = \min_{\ell=1,\ldots,n} p_j / \tilde{\sigma}^2(\ell),$$  

(26)

where $\tilde{\sigma}^2(\ell)$ is given by

$$\tilde{\sigma}^2(\ell) = 1 + \lambda \sum_{j=\ell+1}^n p_j.$$  

(27)

The parameters $\gamma$ and $\tilde{\sigma}^2(\ell)$ have simple interpretations: Suppose that the SeqOMP algorithm has correctly decoded all the users for $j < \ell$. Then, in detecting the $\ell$th user, the receiver sees the noise $w$ with power $\mathbb{E}[|w|^2] = 1$ and, for each user $j > \ell$, an interference power $p_j$ with probability $\lambda$. Hence, $\tilde{\sigma}^2(\ell)$ is the total average interference power seen when detecting $\ell$th user, assuming perfect cancellation. Since user $\ell$ arrives at a power $p_\ell$, the ratio $p_\ell / \tilde{\sigma}^2(\ell)$ in (26) represents the average SINR seen by user $\ell$. The value $\gamma$ is the minimum SINR over all $n$ users.

**Theorem 1:** Let $\lambda = \lambda(n)$, $m = m(n)$ and the power profile $\{p_j\}_{j=1}^n = \{p_j(n)\}_{j=1}^n$, be deterministic quantities that all vary with $n$ satisfying the limits $m - \lambda n$, $\lambda n$ and $(1 - \lambda)n \to \infty$, and $\gamma \to 0$. Also, assume the sequence of power profiles satisfies the limit

$$\lim_{n \to \infty} \max_{i=1,\ldots,n-1} \log(n)\tilde{\sigma}^{-4}(i) \sum_{j=i+1}^n p_j^2 = 0.$$  

(28)

Finally, assume that for all $n$,

$$m \geq \frac{(1 + \delta) L(n, \lambda)}{\gamma} + \lambda n,$$  

(29)

for some $\delta > 0$ and $L(n, \lambda)$ defined in (16). Then, there exists a sequence of thresholds, $\mu = \mu(n)$, such that SeqOMP will achieve asymptotic reliable detection of the active user set in that

$$p_{err} = \Pr \left( \hat{I}_{\text{OMP}} \neq I_{\text{true}} \right) \to 0,$$

where the probability is taken over the randomness in the activities of the users, the codebook $A$, and the noise $w$. The sequence of threshold levels can be selected independent of the sequence of power profiles.

**Proof:** See Appendix A.

The theorem provides a simple sufficient condition on the number of measurements as a function of the SINR $\gamma$, activity probability $\lambda$ and number of users $n$. The condition (25) is somewhat technical, but is satisfied in the cases that interest us. The remainder of this section will discuss some of the implications of this theorem.

**C. Near–Far Resistance with Known Power Ordering**

First, suppose that the power ordering $p_j$ is known at the receiver so the receiver can detect the users in order of decreasing power. If, in addition, the SNRs of all the users go to infinity so that $p_j \to \infty$ for all $j$, then it can be verified that $\gamma > 1/(\lambda n)$. In this case, the sufficiency of the scaling (29) shows that

$$m \geq (1 + \delta) \lambda n L(n, \lambda) + \lambda n$$

is sufficient for asymptotic reliable detection. This is identical to the lasso performance except for the factor $L(\lambda, n) / \log((1 - \lambda) n)$, which lies in $(0, 4)$ for $\lambda \in (0, 1/2)$. In particular, the minimum number of measurements does not depend on MAR; therefore, similar to lasso and OMP, SeqOMP can theoretically detect users even when they are much below the average power.

With SeqOMP, simply knowing the order of powers is sufficient to achieve near–far resistance when the SNR is sufficiently high. Unlike for single-user detection, unequal received powers do not hurt the performance of SeqOMP, as long as the order of the powers are known at the receiver.

The feasibility of knowing the power ordering is addressed in Section IV below. We will now look at the effect of the power profile on the performance.

**D. Performance with Constant Power**

Consider the case when all the powers $p_j$ are equal. To satisfy the constraint (25), the constant power level must be $p_j = \text{SNR}/(\lambda n)$. From (26), the minimum SINR is $\gamma = \gamma_{\text{const}}$, where

$$\gamma_{\text{const}} = \frac{\text{SNR}}{\lambda (n + (n - 1) \text{SNR})} \approx \frac{\text{SNR}}{\lambda n (1 + \text{SNR})},$$  

(30)

and the approximation holds for large $n$.

It can be verified that the constant power profile satisfies the technical condition (28) provided $\lambda$ is bounded away from zero and the SNR does not grow “too fast”. Specifically, the SNR must satisfy $\text{SNR} = o(n / \log(n))$. In this case, we can substitute $\gamma = \gamma_{\text{const}}$ in (29) to obtain the condition

$$m > \frac{(1 + \delta)(1 + \text{SNR}) L(n, \lambda)}{\text{SNR}} \lambda n + \lambda n$$
for asymptotic reliable detection. The condition is precisely the condition for single-user detection in [13] with \( \text{MAR} = 1 \) and an additional \( \lambda n \) term.

Thus, for a constant power profile, Theorem 1 does not show any benefit in using SeqOMP.

E. Optimal Power Shaping

The constant power profile, however, is not optimal. Suppose that accurate power control is feasible so that the receive power levels \( p_j \) can be set by the receiver. In this case, we can maximize the SINR \( \gamma \) in (26) for a given total SNR constraint (25). It is easily verified that any power profile \( p_j \) maximizing the SINR \( \gamma \) in (26) will satisfy

\[
p_\ell = \gamma \left( 1 + \lambda \sum_{j=\ell+1}^n p_j \right)
\]

for all \( \ell = 1, \ldots, n \). The solution to (31) and (25) is given by

\[
p_\ell = \gamma (1 + \gamma) n - \ell,
\]

where \( \gamma = \gamma_{\text{opt}} \) is the SINR,

\[
\gamma_{\text{opt}} = \frac{1}{\lambda} \left[ (1 + \text{SNR})^{1/n} - 1 \right] \approx \frac{1}{\lambda n} \log(1 + \text{SNR}).
\]

Here, the approximation holds for large \( n \). Again, some algebra shows that, when \( \lambda \) is bounded away from zero, the power profile \( p_j \) in (32) will satisfy the technical condition (28) when \( \log(1 + \text{SNR}) = o(n / \log(n)) \).

The power profile (32) is exponentially decreasing in the index order \( \ell \). Thus, users early in the detection sequence are allocated exponentially higher power than users later in the sequence. This allocation insures that early users have sufficient power to overcome the interference from all the users in the sequence. This allocation insures that early users have sufficient power to overcome the interference from all the users later in the detection sequence that are not yet cancelled. This power shaping is analogous to the optimal power allocations in the classic MAC channel when using a SIC receiver [5].

The ratio of the optimal SINR \( \gamma_{\text{opt}} \) in (33) to the SINR with a constant power profile, \( \gamma_{\text{const}} \) in (30), is given by

\[
\frac{\gamma_{\text{opt}}}{\gamma_{\text{const}}} = \frac{(1 + \text{SNR}) \log(1 + \text{SNR})}{\text{SNR}}.
\]

This ratio represents the potential increase in SINR with exponential power shaping relative to the SINR with equal power for all users. The ratio increases with SNR and can be large when the SNR is high. For example, when \( \text{SNR} = 10 \text{ dB} \), \( \gamma_{\text{opt}} / \gamma_{\text{const}} \approx 2.6 \). When \( \text{SNR} = 20 \text{ dB} \), the gain is even higher at \( \gamma_{\text{opt}} / \gamma_{\text{const}} \approx 4.7 \).

Based on Theorem 1, this gain in SINR will result in a proportional decrease in the minimum number of measurements. Specifically, if we substitute the SINR \( \gamma_{\text{opt}} \) in (33) into (29), we see that the condition

\[
m \geq \frac{1 + \delta}{\log(1 + \text{SNR})} L(n, \lambda) \lambda n + \lambda n
\]

is sufficient for SeqOMP to achieve asymptotic reliable detection of the active users, when the users use exponential power shaping (32).

As before, if \( \lambda < 1/2 \), we can bound \( L(n, \lambda) < 4 \log(n(1 - \lambda)) \) and the sufficient condition (34) can be simplified to

\[
m \geq \frac{4(1 + \delta) \log(n(1 - \lambda))}{\log(1 + \text{SNR})} \lambda n + \lambda n
\]

the leading term of which appears in Table II with the \( \delta \) omitted.

F. SNR Saturation

As discussed earlier, a major problem with both single-user detection and lasso multiuser detection was that their performance “saturates” with high SNR. That is, even as the SNR scales to infinity, the minimum number of measurements scales as \( m = O(\lambda n \log((1 - \lambda) n)) \). In contrast, optimal ML detection can achieve a scaling \( m = O(\lambda n) \), when the SNR is sufficiently high.

An important consequence of (34) is that SeqOMP with exponential power shaping can overcome this bound. Specifically, if we take the scaling of \( \text{SNR} = \Theta(\lambda n) \) in (34) and assume that \( \lambda \) is bounded away from zero we see that asymptotically, SeqOMP requires only

\[
m \geq 5 \lambda n
\]

measurements. In this way, unlike single-user and lasso detection, SeqOMP is able to obtain the scaling \( m = O(\lambda n) \) when the SNR \( \to \infty \).

G. Power Shaping with Sparse Bayesian Learning

The fact that power shaping can provide benefits when combined with certain iterative detection algorithms confirms the observations in the work of Wipf and Rao [30]. That work considers signal detection with a certain sparse Bayesian learning (SBL) algorithm. They show the following result: Suppose \( x \) has \( k \) non-zero components and \( p_i, i = 1, 2, \ldots, k \), is the power of the \( i \)th largest component. Then, for a given measurement matrix \( A \), there exist constants \( \nu_i > 1 \) such that if

\[
p_i \geq \nu_i p_{i-1},
\]

the SBL algorithm will correctly detect the sparsity pattern of \( x \).

The condition (37) shows that a certain growth in the powers can guarantee correct detection. The parameters \( \nu_i \) however depend in some complex manner on the matrix \( A \), so the appropriate growth is difficult to compute. They also provide strong empirical evidence that shaping the power with certain profiles can greatly reduce the number of measurements needed.

The results in this paper add to Wipf and Rao’s observations showing that growth in the powers can also assist sequential OMP. Moreover, for the SeqOMP case, we can explicitly derive the optimal power profile for certain large random matrices.

This is not to say that SeqOMP is better than SBL. In fact, empirical results in [34] suggest that SBL will outperform OMP, which will in turn do better than SeqOMP. As we have stressed before, the point here of analyzing SeqOMP is that we can easily derive concrete analytic results. These results may provide guidance for more sophisticated algorithms.
H. Robust Power Shaping

The above analysis shows certain benefits of SeqOMP used in conjunction with power shaping. However, these gains are theoretically only possible at infinite block lengths. Unfortunately, when the block length is finite, power shaping can actually reduce the performance.

The problem is that when an active user is not detected in SeqOMP, the user’s energy is not cancelled out and remains as interference for all subsequent users in the detection sequence. With power shaping, users early in the detection sequence have much higher power than users later in the sequence, so missing an early user can make the detection of subsequent users difficult. At infinite block lengths, the probability of missing an active user can be driven to zero. But, at finite block lengths, the probability of missing an active user early in the sequence will always be nonzero, and therefore a potential problem with power shaping.

The work [36] observed a similar problem when SIC is used in the CDMA uplink. To mitigate the problem, [36] proposed to adjust the power allocations to make them more robust to decoding errors early in the decoding sequence. The same technique, which we will call robust power shaping, can be applied to the SeqOMP as follows.

In the condition (31), it is assumed that all the energy of users with index \( j < \ell \) have been correctly detected and subtracted. But, following [36], suppose that on average some fraction \( \theta \in [0, 1] \) of the energy of users early in the detection sequence is not cancelled out due to missed detections. We will call \( \theta \) the leakage fraction. With nonzero leakage, the condition (31) would be replaced by

\[
p_{\ell} = \gamma \left( 1 + \theta \lambda \sum_{j=1}^{\ell-1} p_j + \lambda \sum_{j=\ell+1}^{n} p_j \right). \tag{38}
\]

For given SNR, \( \theta \) and \( \lambda \), the linear equations (25) and (38) can be solved to obtain the optimal power profile, given by

\[
p_j = \frac{(1 - \theta)\gamma}{1 + \lambda \theta \gamma} \left( \frac{1 + \lambda \gamma}{1 + \lambda \theta \gamma} \right)^{n-j}, \tag{39}
\]

where \( \gamma = \gamma(\theta) \), the optimal SINR

\[
\gamma(\theta) = \frac{1}{\lambda(1 - \theta)} \left[ \left( \frac{1 + \text{SNR}}{1 + \theta \text{SNR}} \right)^{1/n} - 1 \right]
\]

\[
\approx \frac{1}{\lambda n (1 - \theta)} \log \left( \frac{1 + \text{SNR}}{1 + \theta \text{SNR}} \right). \tag{40}
\]

The approximation here is valid for large \( n \).

Fig. 1 plots the SINR, \( \gamma(\theta) \), as a function of the leakage fraction \( \theta \). The SINR is plotted relative to \( \gamma_{\text{const}} \) in (30), which is the SINR that one obtains with a constant power profile. The increase in SINR is maximized when the leakage fraction, \( \theta \rightarrow 0 \). When \( \theta = 0 \), \( \gamma(\theta) = \gamma_{\text{exp}} \), the SINR (33) for the exponential power shaping. This is the optimal SINR, but assumes that there are no missed detections.

As the leakage fraction \( \theta \) is increased, the SINR, \( \gamma(\theta) \), decreases, which is price for the robustness to missed detections. In the limit as \( \theta \rightarrow 1 \), the optimal power profile, \( p_j \) in (39) approaches a constant and the corresponding SINR, \( \gamma(\theta) \), converges to \( \gamma_{\text{const}} \). However, even at a reasonable leakage fraction, say \( \theta = 0.1 \), the SINR \( \gamma(\theta) \) can still be significantly larger than \( \gamma_{\text{const}} \).

It is illustrative to actually look at the optimal power profiles as a function of \( \theta \). Fig. 2 plots the optimal power profile, \( p_j \) in (39), for leakage values of \( \theta = 0, 0.1 \) and 1. In the plot, \( n = 100 \), SNR = 20 dB, and activity probability is \( \lambda = 0.1 \).

![Fig. 1. Increase in SINR. \( \gamma(\theta)/\gamma_{\text{const}} = \gamma(\theta)/\gamma(1) \), as a function of the leakage fraction \( \theta \).](image1)

![Fig. 2. Robust power shaping: Power profiles for various leakage values \( \theta \). For all the curves, the number of users is \( n = 100 \), SNR = 20 dB, and activity probability is \( \lambda = 0.1 \).](image2)
I. Practical Power Control Considerations

In the original description of the problem in Section II, we said that we would restrict our attention to estimators that do not require a priori knowledge of the modulation vector $\mathbf{x}$. However, although SeqOMP does not require knowledge at the receiver of the channel phases, the above analysis shows that knowledge of the order of the conditional received powers is necessary to achieve near-far resistance. Additionally, eliminating the self-noise limit requires that powers are explicitly targeted to a certain profile.

The use of power control for on-off random access communication requires some justification. On-off random access signaling is most likely to be used when the users do not already have some ongoing communication. For example, in cellular systems, it is used for initial access or requests to transmit. If the users were already transmitting, the one bit could be embedded in the other communication and on-off random access signaling would not be needed. Consequently, fast feedback power control would likely not be available for such on-off random access transmissions since the users are not likely to have a continuous transmission to measure the received power.

Thus, in practice, power control is likely achievable only by open-loop methods. Open-loop power control is used for example in cellular systems where each mobile estimates the path loss in the downlink and adjusts its access power appropriately in the uplink. Open-loop power control is most accurate when the uplink and downlink are time-division duplexed (TDD) in the same band.

V. Numerical Simulation

A. Threshold Settings

The performance of the single-user detection and SeqOMP algorithms depend on the setting of the threshold level $\mu$. In the theoretical analysis of Theorem 1, an ideal threshold is calculated assuming infinite block lengths that guarantees perfect detection of the active user set. However, in simulations with finite block lengths, it is more reasonable to set the threshold based on a desired false alarm probability. A false alarm is the event when the algorithm falsely detects that a user is active when it is not. For the single-user detection algorithm in Section IV-C or the SeqOMP algorithm in Section IV-A, the false alarm probability is

$$p_{FA} = \Pr\left(j \in \hat{I} \mid j \notin I_{true}\right) = \Pr\left(\rho(j) > \mu \mid j \notin I_{true}\right),$$

which is the probability that the correlation $\rho(j)$ exceeds the threshold $\mu$ when the user $j$ is not active.

It is shown in the proof of Theorem 1 that, when $j \notin I_{true}$, $\rho(j)$ follows a Beta $B(2, 2(m - 1))$ distribution. When $m$ is large, this beta distribution is approximately Rayleigh and the false alarm probability is given by

$$p_{FA} \approx \exp(-\mu m).$$

Thus, the threshold level $\mu$ can be set to

$$\mu = -\log(p_{FA})/m$$

for a given desired false alarm probability.

In the simulations below, we will run the algorithms with a fixed false alarm probability (typically $p_{FA} = 10^{-3}$), and measure the missed detection rate given by

$$p_{MD} = \Pr\left(j \notin \hat{I} \mid j \in I_{true}\right).$$

The missed detection rate will be averaged over all $j \in I_{true}$.

B. Evaluation of Bounds

We first compare the actual performance of the SeqOMP algorithm with the bound in Theorem 1. Fig. 3 plots the simulated missed detection probability for using SeqOMP at various SNRs, activity probabilities $\lambda$, and numbers of measurements $m$. In all simulations, the number of users was fixed to $n = 100$ and the users arrived at equal power ($\text{MAR} = 1$). The false alarm probability was set to $p_{FA} = 10^{-3}$. The robust power profile of Section IV-H is used with a leakage fraction $\theta = 0.1$.

The dark line in Fig. 3 represents the number of measurements $m$ for which Theorem 1 would theoretically guarantee reliable detection of the active user set at infinite block lengths. To apply the theorem, we used the SINR $\gamma = \gamma(\theta)$ in (40). At the block lengths considered in this simulation, the missed detection probability at the theoretical sufficient condition is small, typically between 2 and 10%. Thus, even at moderate block lengths, the theoretical bound in Theorem 1 can provide a good estimate for the number of measurements for reliable detection.

C. SeqOMP vs. Single User Detection

Fig. 4 shows a more direct comparison of the performance of single-user detection and SeqOMP with power shaping. In the simulation, there are $n = 100$ users, the activity probability is $\lambda = 0.1$, and the total SNR is $20$ dB. The number of measurements $m$ was varied, and for each $m$, the missed detection probability was estimated with 1000 Monte Carlo trials.

As expected, single-user detection requires the most number of measurements. For a missed detection rate of 1%, Fig. 4 shows that single-user detection requires approximately $m \approx 210$ measurements. In this simulation of single-user detection, all users arrived at the same power. Employing SeqOMP, but keeping the power profile of the users constant, decreases the number of measurements somewhat to $m \approx 170$ for a 1% missed detection rate. However, using SeqOMP with power shaping decreases the number of measurements by more than a factor of two to $m \approx 95$. Thus, at least at high SNRs, SeqOMP may provide significant gains over simple single-user detection.

D. OMP with Power Shaping

As discussed earlier, although SeqOMP can provide gains over single-user detection, its performance is typically worse than OMP, even if SeqOMP is used with power shaping. Our interest in the algorithm is that it is simple to analyze.
However, we can in principle use power shaping with the better OMP algorithm as well.

While we do not have any analytical result, the simulation in Fig. 5 shows that power shaping provides some gains with OMP as well. Specifically, when the users are targeted at equal receive power, \( m \approx 85 \) measurements are needed for a missed detection probability of 1%. This number is slightly lower than that required by SeqOMP, even when SeqOMP uses power shaping. When OMP is used with power shaping, the number of measurements decreases to about \( m \approx 65 \).

VI. RELATIONS TO MAC CAPACITY

As discussed in the introduction, the random access channel is a special case of a multiple access channel (MAC). One of the fundamental results in network information theory [4], [5] is that, under certain assumptions, the sum rate with multiple users transmitting to a single receiver without coordination can equal the capacity with coordination. However, one of the key assumptions in this classic result is that the users employ capacity-achieving block codes. In the on–off random access channel considered here, users transmit on a single codeword and therefore cannot benefit from channel coding. Thus, unlike the classic MAC channel, the random access channel may incur a loss in capacity due to the lack of coordination amongst users.

To evaluate this possibility, let us first compute the effective “sum rate” transmitted in the on–off random access channel. Each user transmits with a probability \( \lambda \), so the information conveyed in detecting the user’s activity is \( h(\lambda) \), where \( h(\lambda) \) is the binary entropy,

\[
h(\lambda) = -\lambda \log(\lambda) - (1 - \lambda) \log(1 - \lambda) \text{ nats.}
\]

Since there are \( n \) users, if all users can be reliably detected, the total information rate is

\[
R = nh(\lambda).
\]

We can compare this rate with the Shannon capacity of the channel. If all the users coordinate their transmissions, the capacity would be identical to a single user transmitting with the same total power. Since the channel is AWGN with \( m \) channel uses, the capacity of the channel with a single coordinated transmission would be \( C = m \log(1 + \text{SNR}) \).

If the number of measurements \( m \) is selected for reliable detection, the necessary condition (12) shows that the capacity is bounded below by

\[
C = m \log(1 + \text{SNR}) \geq \frac{\log(1 + \text{SNR})}{\text{SNR}} \lambda n \log((1 - \lambda)n).
\]

Thus, the ratio of the sum rate to capacity is bounded above by

\[
\frac{R}{C} \leq \frac{\log(1 + \text{SNR})h(\lambda)}{\text{SNR} \log((1 - \lambda)n)}.
\]

This ratio represents a bound on the maximum rate without coordination amongst the users to the maximum rate possible...
with coordination. If $\lambda$ and the SNR are fixed and $n \to \infty$, the ratio $R/C \to 0$. Thus, the sum rate of the random access channel has a fundamentally lower scaling than the standard AWGN channel.

There is, however, one case where the random access channel’s sum rate achieves the single-user Shannon capacity. Suppose that the SeqOMP algorithm is used with exponential power shaping. The sufficient condition (34) shows that the Shannon capacity is

$$C = m \log(1 + \text{SNR}) \approx \lambda n L(\lambda, n),$$

where, in the approximation, we have ignored the infinitesimal $\delta$, and the $\lambda n$ term. In this case, the ratio of the sum rate to capacity is

$$\frac{R}{C} \approx \frac{h(\lambda)}{\lambda L(\lambda, n)}.$$

Now suppose the expected number of active users is fixed to some value $k$, and we let the activity probability scale as $\lambda(n) = k/n$. It is easily checked that $R/C \to 1$ as $n \to \infty$. Therefore, with a fixed expected number of active users, the sum rate of the random access channel matches the Shannon capacity as the number of user $n \to \infty$. Moreover, the random access capacity can be achieved with the SeqOMP method with exponential power shaping.

In a way, this result is perhaps not surprising. When the expected number of used is fixed to some value $k$, and the block length $m$ scales to infinity, the random access channel becomes identical to a standard MAC channel with $k$ users, each transmitting on a random codebook of size $n/k$. Moreover, the SeqOMP algorithm is precisely equivalent to the classic SIC used in conjunction with ML detection for each user. SIC combined with optimal decoding for each user is known to achieve the sum rate.

The connection between the MAC channel and sparsity detection has also been observed by Jin and Rao [29]. Specifically, they show that OMP is clearly an analogue to the classic SIC method. Moreover, they argue, at least heuristically, that if $\lambda \ll 1$, the sum-rate $R$ achievable by OMP should approach the capacity $C$.

Our analysis of SeqOMP provides analytic evidence for these claims by showing a specific regime where $R/C \to 1$. However, it also shows when this intuition fails by showing that when the SNR and activity probability $\lambda$ are fixed, then $R/C \to 0$. In this case, there is a potentially-large gap between the MAC capacity and the sum rate in the random on–off channel.

VII. CONCLUSIONS

Sparse signal detection is a valuable framework for understanding multiple access on–off random signaling. Results can provide simple capacity estimates and clarify the role of power control and multiuser detection. Methods such as OMP and lasso, which are widely used in sparse detection problems, can be applied as multiuser detection methods for on–off random access channels. Analysis shows that these methods may offer improved near–far resistance over single-user detection in high SNRs. Optimal ML detection may theoretically offer further gains in the high SNR regime, but is not computationally possible. However, some gains at high SNR may be practically achievable through power shaping and SIC-like techniques such as OMP.

APPENDIX

PROOF OF THEOREM 1

A. Proof Outline

At a high level, the proof of Theorem 1 is similar to the proof of [15, Thm. 2], the single-user detection condition (18). One of the difficulties in the proof is to handle the relationships...
between random events at different iterations of the SeqOMP algorithm. To avoid this difficulty, we first show an equivalence between the success of SeqOMP and an alternative sequence of events that is easier to analyze. After this simplification, small modifications handle the cancellations of detected vectors.

Fix \( n \) and define
\[
I_{\text{true}}(j) = \{ \ell : \ell \in I_{\text{true}}, \ell \leq j \},
\]
which is the set of elements of the active set with indices \( \ell \leq j \). Observe that \( I_{\text{true}}(0) = \{ \emptyset \} \) and \( I_{\text{true}}(n) = I_{\text{true}} \).

Let \( P_{\text{true}}(j) \) be the projection operator onto the orthogonal complement of \( \{ a_\ell, \ell \in I_{\text{true}}(j-1) \} \), and define
\[
\rho_{\text{true}}(j) = \frac{|a_j^T P_{\text{true}}(j)y|^2}{\|P_{\text{true}}(j)a_j\|^2 \|P_{\text{true}}(j)y\|^2}.
\]
A simple induction argument shows that Algorithm 1 correctly detects the elements in the active set if and only if, at each iteration \( j \), the variables \( I(j) \), \( P(j) \) and \( \rho(j) \) defined in the algorithm are equal to \( I_{\text{true}}(j) \), \( P_{\text{true}}(j) \) and \( \rho_{\text{true}}(j) \), respectively. Therefore, if we define
\[
\hat{I} = \{ j : \rho_{\text{true}}(j) > \mu \}
\]
then Algorithm 1 correctly detects all users if and only if \( I = I_{\text{true}} \). In particular,
\[
p_{\text{err}}(n) = \Pr \left( \hat{I} \neq I_{\text{true}} \right).
\]

To prove that \( p_{\text{err}}(n) \to 0 \) it suffices to show that there exists a sequence of threshold levels \( \mu(n) \) such that the following two limits
\[
\lim_{n \to \infty} \inf_{j \in I_{\text{true}}(n)} \frac{\rho_{\text{true}}(j)}{\mu} > 1, \quad (43)
\]
\[
\lim_{n \to \infty} \sup_{j \notin I_{\text{true}}(n)} \frac{\rho_{\text{true}}(j)}{\mu} < 1, \quad (44)
\]
hold in probability. The first limit ensures that all the components in the active set will not be missed and will be called the zero missed detection condition. The second limit ensures that all the components not in the active set will not be falsely detected and will be called the zero false alarm condition.

Set the sequence of threshold levels as follows. Since \( \delta > 0 \), we can find an \( \epsilon > 0 \) such that
\[
1 + \delta \geq (1 + \epsilon)^2.
\]
For each \( n \), let the threshold level be
\[
\mu = (1 + \epsilon) \frac{\log(n(1 - \lambda))}{m - \lambda n}.
\]
The asymptotic lack of missed detections and false alarms with these thresholds are proven in Appendices D and E respectively. In preparation for these sections, Appendix B reviews some facts concerning tail bounds on Chi-squared and Beta random variables and Appendix C performs some preliminary computations.

B. Chi-Squared and Beta Random Variables

The proof requires a number of simple facts concerning chi-squared and beta random variables. These variables are reviewed in [37]. We will omit or just provide some sketches of the proofs of the results in this section since they are all standard.

A random variable \( u \) has a chi-squared distribution with \( r \) degrees of freedom if it can be written as \( u = \sum_{i=1}^r z_i^2 \), where \( z_i \) are i.i.d. \( \mathcal{N}(0, 1) \). If \( u \) is a chi-squared with two degrees of freedom, the random variable \( v = u/2 \) has a Rayleigh distribution. For this work, chi-squared and Rayleigh distributed random variables arise in two important instances.

Lemma 1: Suppose \( x \in \mathbb{C}^r \) has a complex Gaussian distribution \( \mathcal{CN}(0, \sigma^2 I_r) \). Then:
(a) \( 2\|x\|^2/\sigma^2 \) is chi-squared with \( 2r \) degrees of freedom; and
(b) if \( y \) is any other \( r \)-dimensional random vector that is nonzero with probability one and independent of \( x \), then the variable
\[
u = \frac{|x^T y|^2}{\sigma^2 \|y\|^2}
\]
has a Rayleigh distribution.

Proof: Part (a) follows from the fact that \( 2\|x\|^2/\sigma^2 \) is a sum of squares of \( 2r \) unit-variance Gaussian random variables, one for each component of \( \sqrt{2/\sigma} x \). Part (b) follows from the fact that \( x^T y/(\|y\|\sigma) \) is a unit-variance complex Gaussian random variable.

The following two lemmas provide standard tail bounds.

Lemma 2: Suppose that for each \( n \), \( \{x_j^{(n)}\}_{j=1}^n \) is a set of complex Gaussian random vectors with each \( x_j^{(n)} \) spherically symmetric in an \( m_j(n) \)-dimensional space. The variables may be dependent. Suppose also that \( E\|x_j^{(n)}\|^2 = 1 \) and
\[
\lim_{n \to \infty} \log(n)/m_{\min}(n) = 0
\]
where
\[
m_{\min}(n) = \min_{j=1,\ldots,n} m_j(n).
\]
Then the limits
\[
\lim_{n \to \infty} \max_{j=1,\ldots,n} \|x_j^{(n)}\|^2 = \lim_{n \to \infty} \min_{j=1,\ldots,n} \|x_j^{(n)}\|^2 = 1
\]
hold in probability.

Proof: From Lemma 1 for every \( j \) and \( n \), the norms
\[
z(j, n) = 2m_j(n)\|x_j^{(n)}\|^2;
\]
are chi-squared random variables with \( 2m_j(n) \) degrees of freedom. A standard tail bound (see, for example [14]), shows that for any \( \epsilon > 0 \),
\[
\Pr \left( \frac{z(j, n)}{2m_j(n)} > 1 + \epsilon \right) \leq \exp(-2\epsilon m_j(n))
\]
where the last step is due to the fact that \( m_j(n) \geq m_{\min}(n) \).
So, using the union bound,
\[
\Pr \left( \max_{j=1, \ldots, n} \| x_j^{(n)} \|^2 > 1 + \epsilon \right)
\]
\[=
\Pr \left( \max_{j=1, \ldots, n} \frac{z(j, n)}{2m_j(n)} > 1 + \epsilon \right)
\]
\[\leq \sum_{j=1, \ldots, n} \Pr \left( \frac{z(j, n)}{2m_j(n)} > 1 + \epsilon \right)
\]
\[\leq n \max_{j=1, \ldots, n} \Pr \left( \frac{z(j, n)}{2m_j(n)} > 1 + \epsilon \right)
\]
\[= n \exp(-2\epsilon m_{\min}(n))
\]
\[= \exp(-2\epsilon m_{\min}(n) + \log(n)) \rightarrow 0,
\]
where the last step is due to the fact that \( \log(n)/m_{\min}(n) \rightarrow 0 \).

This shows that
\[
\limsup_{n \to \infty} \max_{j=1, \ldots, n} \| x_j^{(n)} \|^2 \leq 1
\]
in probability.

Similarly, using the tail bound that
\[
\Pr \left( \frac{z(j, n)}{2m_j(n)} < 1 - \epsilon \right) \leq \exp(-\epsilon^2 m_j(n)),
\]
one can show that
\[
\liminf_{n \to \infty} \min_{j=1, \ldots, n} \| x_j^{(n)} \|^2 \geq 1
\]
in probability, and this proves the lemma.

**Lemma 3:** Suppose that for each \( n \), \( \{ u_j^{(n)} \}_{j=1}^n \) is a set of Rayleigh random variables. The variables may be dependent. Then
\[
\limsup_{n \to \infty} \max_{j=1, \ldots, n} \frac{u_j^{(n)}}{\log(n)} \leq 1,
\]
where the limit is in probability.

**Proof:** Since each \( u_j^{(n)} \) is Rayleigh, for any \( \mu > 0 \),
\[
\Pr(u_j^{(n)} > \mu) = e^{-\mu}.
\]
Combining this with the union bound, we see that for any \( \epsilon > 0 \),
\[
\Pr \left( \max_{j=1, \ldots, n} \frac{u_j^{(n)}}{\log(n)} > (1 + \epsilon) \right)
\]
\[\leq n \exp(-(1 + \epsilon) \log(n)) = n^{-\epsilon} \rightarrow 0.
\]
This proves the limit (47).

The final two lemmas concern certain beta distributed random variables. A real-valued scalar random variable \( w \) follows a Beta(\( r, s \)) distribution if it can be written as \( w = u_r/(u_r + v_s) \), where the variables \( u_r \) and \( v_s \) are independent chi-squared random variables with \( r \) and \( s \) degrees of freedom, respectively. The importance of the beta distribution is given by the following lemma.

**Lemma 4:** Suppose \( x \) and \( y \) are independent random \( r \)-dimensional complex random vectors with \( x \) being spherically-symmetrically distributed in \( \mathbb{C}^r \) and \( y \) having any distribution that is nonzero with probability one. Then the random variable
\[
w = \frac{|x^*y|^2}{\|x\|^2\|y\|^2}
\]
is independent of \( x \) and follows a Beta(\( 2, r - 2 \)) distribution.

**Proof:** This can be proven along the lines of the arguments in [38].

The following lemma provides a simple expression for the maxima of certain beta distributed variables.

**Lemma 5:** For each \( n \), suppose \( \{ w_j^{(n)} \}_{j=1}^n \) is a set of random variables with \( w_j^{(n)} \) having a Beta(\( 2, m_j(n) - 2 \)) distribution. Suppose that
\[
\lim_{n \to \infty} \log(n)/m_{\min}(n) = 0, \quad \lim_{n \to \infty} m_{\min}(n) = \infty \tag{48}
\]
where
\[
m_{\min}(n) = \min_{j=1, \ldots, n} m_j(n).
\]
Then,
\[
\limsup_{n \to \infty} \max_{j=1, \ldots, n} \frac{m_j(n)}{\log(n)} w_j^{(n)} \leq 1
\]
in probability.

**Proof:** We can write \( w_j^{(n)} = u_j^{(n)}/(u_j^{(n)} + v_j^{(n)}) \) where \( u_j^{(n)} \) and \( v_j^{(n)} \) are independent chi-squared random variables with 2 and \( m_j(n) - 2 \) degrees of freedom, respectively. Let
\[
U_n = \frac{1}{2\log(n)} \max_{j=1, \ldots, n} u_j^{(n)},
\]
\[
V_n = \min_{j=1, \ldots, n} \frac{1}{2m_j(n) - 2} v_j^{(n)},
\]
\[
T_n = \max_{j=1, \ldots, n} \frac{m_j(n)u_j^{(n)}}{\log(n)}.
\]
The condition (48) and an argument similar to the proof of Lemma 4 shows that \( V_n \to 1 \) in probability. Also, \( U_n/2 \) is Rayleigh distributed so Lemma 3 shows that
\[
\limsup_{n \to \infty} U_n \leq 1
\]
in probability. Using these two limits along with (48) shows that
\[
\limsup_{n \to \infty} T_n = \limsup_{n \to \infty} \max_{j=1, \ldots, n} \frac{m_j(n)w_j^{(n)}}{\log(n)}
\]
\[= \limsup_{n \to \infty} \max_{j=1, \ldots, n} \frac{m_j(n)}{\log(n)} \frac{u_j^{(n)}}{u_j^{(n)} + v_j^{(n)}}
\]
\[= \limsup_{n \to \infty} \frac{U_n}{V_n + U_n/2} \leq \frac{1}{1 + (1/2)} = 1,
\]
where the limit is in probability.

**C. Preliminary Computations and Technical Lemmas**

We first need to prove a number of simple but technical bounds. We begin by considering the dimension \( m_i \) defined as
\[
m_i = \dim(\text{range}(P_{\text{true}}(i))). \tag{49}
\]
Our first lemma computes the limit of this dimension.

**Lemma 6:** The following limit
\[
\lim_{n \to \infty} \min_{i=1, \ldots, n} \frac{m_i}{m - \lambda n} = 1 \tag{50}
\]
holds in probability and almost surely. The deterministic limits
\[
\lim_{n \to \infty} \frac{\log(\lambda n)}{m - \lambda n} = \lim_{n \to \infty} \frac{\log((1 - \lambda)n)}{m - \lambda n} = 0 \tag{51}
\]
also hold.

Proof: Recall that \( P_{\text{true}}(i) \) is the projection onto the orthogonal complement of the vectors \( a_j \) with \( j \in I_{\text{true}}(i-1) \). With probability one, these vectors will be linearly independent, so \( P_{\text{true}}(i) \) will have dimension \( m - |I_{\text{true}}(i-1)| \). Since \( I_{\text{true}}(i) \) is increasing with \( i \),
\[
\min_{i = 1, \ldots, n} m_i = m - \max_{i = 1, \ldots, n} |I_{\text{true}}(i-1)| = m - |I_{\text{true}}(n-1)|. \tag{52}
\]
Since each user is active with probability \( \lambda \) and the activities of the users are independent, the law of large numbers shows that
\[
\lim_{n \to \infty} \frac{|I_{\text{true}}(n-1)|}{\lambda(n-1)} = 1
\]
in probability and almost surely. Combining this with (52) shows (51).

We next show (51). Since the hypothesis of the theorem requires that \( \lambda n, (1 - \lambda)n \) and \( m - \lambda n \) all approach infinity, the fractions in (51) are eventually positive. Also, from (16), \( L(\lambda, n) < \max\{\log(\lambda n), \log((1 - \lambda)n)\} \). Therefore, from (29),
\[
\frac{1}{m - \lambda n} \max\{\log(\lambda n), \log((1 - \lambda)n)\} \leq \frac{\gamma}{L(\lambda, n)} \max\{\log(\lambda n), \log((1 - \lambda)n)\} \leq \gamma \to 0,
\]
where the last step is from the hypothesis of the theorem.

Next, for each \( i = 1, \ldots, n \), define the residual vector,
\[
e_i = P_{\text{true}}(i)(y - a_i x_i). \tag{53}
\]
Observe that
\[
e_i \overset{(a)}{=} P_{\text{true}}(i)(w + \sum_{j \neq i} a_j x_j) \overset{(b)}{=} P_{\text{true}}(i)(w + \sum_{j > i} a_j x_j), \tag{54}
\]
where (a) follows from (11) and (b) follows from the fact that \( P_{\text{true}}(i) \) is the projection onto the orthogonal complement of the span of all vectors \( a_j \) with \( j < i \) and \( x_j \neq 0 \).

The next lemma shows that the power of the residual vector is described by the random variable
\[
\sigma^2(i) = 1 + \sum_{j = i+1}^n |x_j|^2. \tag{55}
\]

Lemma 7: For all \( i = 1, \ldots, n \), the residual vector \( e_i \), conditioned on the modulation vector \( x \) and projection \( P_{\text{true}}(i) \), is a spherically symmetric Gaussian in the range space of \( P_{\text{true}}(i) \) with total variance
\[
E(\|e_i\|^2 | x) = \frac{m_i}{m} \sigma^2(i), \tag{56}
\]
where \( m_i \) and \( \sigma^2(i) \) are defined in (49) and (55), respectively.

Proof: Let \( v_i = w + \sum_{j > i} a_j x_j \), so that \( e_i = P_{\text{true}}(i)v_i \). Since the vectors \( a_j \) and \( w \) have Gaussian \( \mathcal{CN}(0, 1/mI_m) \) distributions, for a given modulation vector \( x \), \( v_i \) must be a zero-mean white Gaussian vector with total variance \( E(\|v_i\|^2) = \sigma^2(i) \). Also, since the operator \( P_{\text{true}}(i) \) is a function of the components \( x_\ell \) and vectors \( a_j \), for \( \ell < i \), \( P_{\text{true}}(i) \) is independent of the vectors \( w \) and \( a_j \), \( j > i \), and therefore independent of \( v_i \). Since \( P_{\text{true}}(i) \) is a projection from an \( m \)-dimensional space to an \( m_i \)-dimensional space, \( e_i \), conditioned on the modulation vector \( x \), must be spherically symmetric Gaussian in the range space of \( P_{\text{true}}(i) \) with total variance satisfying (56).

Our next lemma requires the following version of the well-known Hoeffding’s inequality.

Lemma 8 (Hoeffding’s Inequality): Suppose \( z \) is the sum
\[
z = z_0 + \sum_{i=1}^r z_i
\]
where \( z_0 \) is a constant and the variables \( z_i \) are independent random variables that are almost surely bounded in some interval \( z_i \in [a_i, b_i] \). Then, for all \( \epsilon > 0 \),
\[
\Pr(z - E(z) \geq \epsilon) \leq \exp\left(-\frac{2\epsilon^2}{C}\right),
\]
where
\[
C = \sum_{i=1}^r (b_i - a_i)^2.
\]

Proof: See [39].

Lemma 9: Under the assumptions of Theorem 1, the limit
\[
\limsup_{n \to \infty} \max_{i = 1, \ldots, n} \frac{\sigma^2(i)}{\sigma^2(i)} \leq 1
\]
holds in probability.

Proof: Let \( z(i) = \frac{\sigma^2(i)}{\sigma^2(i)} \). From the definition of \( \sigma^2(i) \) in (55), we can write
\[
z(i) = \frac{1}{\sigma^2(i)} + \sum_{j=i+1}^n z(i, j),
\]
where \( z(i, j) = |x_j|^2/\sigma^2(i) \) for \( j > i \).

Now recall that in the problem formulation, each user is active with probability \( \lambda \), with power \( |x_j|^2 = p_j \) conditioned on when the user being active. Also, the activities of different users are independent, and the conditional powers \( p_j \) are treated as deterministic quantities. Therefore, the variables \( z(i, j) \) are independent with
\[
z(i, j) = \begin{cases} p_j/\sigma^2(i), & \text{with probability } \lambda; \\ 0, & \text{with probability } 1 - \lambda, \end{cases}
\]
for \( j > i \). Combining this with the definition of \( \sigma^2(i) \) in (27), we see that
\[
E(z(i)) = \frac{1}{\sigma^2(i)} \left(1 + \lambda \sum_{j=i+1}^n p_j \right) = 1.
\]
Also, for each \( j > i \), we have the bound
\[
z(i, j) \in [0, \frac{1}{\Delta_j^2(i)}].
\]
So for use in Hoeffding’s Inequality (Lemma 5), define
\[
C = C(i, n) = \bar{\sigma}^{-1}(i) \sum_{j=i+1}^n p_j^2,
\]
where dependence of the power profile and \( \bar{\sigma}(i) \) on \( n \) is implicit. Now define
\[
c_n = \max_{i=1, \ldots, n} \log(n) C(i, n),
\]
so that \( C(i, n) \leq c_n/\log(n) \) for all \( i \). Hoeffding’s Inequality (Lemma 8) now shows that for all \( i < n \),
\[
\Pr(z(i) \geq 1 + \epsilon) \leq \exp\left(-2\epsilon^2/C(i, n)\right) \leq \exp\left(-2\epsilon^2 \log(n)/c_n\right).
\]
Using the union bound,
\[
\lim_{n \to \infty} \Pr\left(\max_{j=1, \ldots, n} z(i) > 1 + \epsilon\right) \leq \lim_{n \to \infty} n \exp\left(-2\epsilon^2 \log(n)/c_n\right) = \lim_{n \to \infty} n^{1-2\epsilon^2/c_n} = 0.
\]
The final step is due to the fact that the technical condition \( \Delta \) in the theorem implies \( c_n \to 0 \). This proves the lemma.

**D. Missed Detection Probability**

Consider any \( j \in I_{\text{true}} \). Using (53) to rewrite (41) along with some algebra shows
\[
\rho_{\text{true}}(j) = \frac{|a_j^TP_{\text{true}}(j)y_j|^2}{|P_{\text{true}}(j)a_j|^2|P_{\text{true}}(j)y_j|^2} = \frac{|P_{\text{true}}(j)a_j|^2|x_j P_{\text{true}}(j)a_j + e_j|^2}{s_j - 2\sqrt{s_j} z_j + s_j} \geq \frac{s_j - 2\sqrt{s_j} z_j + s_j}{s_j + 2\sqrt{s_j} z_j + 1},
\]
where
\[
s_j = \frac{|x_j|^2|P_{\text{true}}(j)a_j|^2}{|e_j|^2}, \quad z_j = \frac{|a_j^TP_{\text{true}}(j)e_j|^2}{|P_{\text{true}}(j)a_j|^2|e_j|^2}.
\]
Define
\[
s_{\text{min}} = \min_{j \in I_{\text{true}}} s_j, \quad s_{\text{max}} = \max_{j \in I_{\text{true}}} z_j.
\]
We will now bound \( s_{\text{min}} \) from below and \( s_{\text{max}} \) from above.

We first start with \( s_{\text{min}} \). Conditional on \( x \) and \( P_{\text{true}}(j) \), Lemma 2 shows that each \( e_j \) is a spherically-symmetrically distributed Gaussian on the \( m_j \)-dimensional range space of \( P_{\text{true}}(j) \). Since there are asymptotically \( \lambda n \) elements in \( I_{\text{true}} \), Lemma 3 along with (51) show that
\[
\lim_{n \to \infty} \max_{j \in I_{\text{true}}} \frac{|e_j|^2}{m_j \sigma(j)^2} = 1,
\]
where the limit is in probability. Similarly, \( P_{\text{true}}(j)a_j \) is also a spherically-symmetrically distributed Gaussian in the range space of \( P_{\text{true}}(j) \). Since \( P_{\text{true}}(j) \) is a projection from an \( m \)-dimensional space to a \( m_j \)-dimensional space and \( E|a_j|^2 = 1 \), we have that \( E|P_{\text{true}}(j)a_j|^2 = m_j/m \). Therefore, Lemma 2 along with (51) show that
\[
\lim_{n \to \infty} \min_{j \in I_{\text{true}}} m_j \sigma(j)^2 = 1.
\]

Taking the limit (in probability) of \( s_{\text{min}} \),
\[
\text{lim inf}_{n \to \infty} \frac{s_{\text{min}}}{\gamma} = \text{lim inf}_{n \to \infty} \min_{j \in I_{\text{true}}} \gamma \|P_{\text{true}}(j)a_j\|^2
\]
\[
= \text{lim inf}_{n \to \infty} \min_{j \in I_{\text{true}}} \frac{|x_j|^2|P_{\text{true}}(j)a_j|^2}{\gamma |e_j|^2}
\]
\[
\geq \text{lim inf}_{n \to \infty} \min_{j \in I_{\text{true}}} \frac{p_j}{\gamma |e_j|^2} \geq 1,
\]
where (a) follows from (58); (b) follows from (60) and (61); (c) follows from (54); (d) follows from Lemma 9 and (e) follows from (26).

We next consider \( s_{\text{max}} \). Conditional on \( P_{\text{true}}(j) \), the vectors \( P_{\text{true}}(j)a_j \) and \( e_j \) are independent spherically-symmetric complex Gaussians in the range space of \( P_{\text{true}}(j) \). It follows from Lemma 2 that each \( z_j \) is a Beta(2, \( m_j - 2 \)) random variable. Since there are asymptotically \( \lambda n \) elements in \( I_{\text{true}} \), Lemma 3 along with (50) and (51) show that
\[
\text{lim sup}_{n \to \infty} \frac{m - \lambda n}{\log(\lambda n)} s_{\text{max}} = \text{lim sup}_{n \to \infty} \frac{m - \lambda n}{\log(\lambda n)} \max_{j \in I_{\text{true}}} z_j \leq 1.
\]
The above analysis shows that for any \( j \in I_{\text{true}} \),
\[
\text{lim inf}_{n \to \infty} \min_{j \in I_{\text{true}}} \frac{1}{\mu} \left( \sqrt{s_j} - \sqrt{z_j} \right)
\]
\[
\geq \text{lim inf}_{n \to \infty} \frac{1}{\mu} \left( s_{\text{min}} - s_{\text{max}} \right)
\]
\[
\geq \text{lim inf}_{n \to \infty} \frac{1}{\mu} \left( \sqrt{\gamma} - \sqrt{\log(\lambda n)/m - \lambda n} \right)
\]
\[
\geq \text{lim inf}_{n \to \infty} \sqrt{1 + \delta} \left( \sqrt{1 + \delta} - \sqrt{\log(\lambda n)/m - \lambda n} \right)
\]
\[
\geq \text{lim inf}_{n \to \infty} \frac{1 + \delta}{m - \lambda n} \left( \sqrt{L(\lambda n, \mu)} - \sqrt{\log(\lambda n)} \right)
\]
\[
\geq \text{lim inf}_{n \to \infty} \frac{1 + \delta}{m - \lambda n} \left( \frac{(1 + \delta) \log(n(1 - \lambda))}{\log(\lambda n) \mu} \right)
\]
\[
\geq \text{lim inf}_{n \to \infty} \frac{1 + \delta}{1 + \epsilon}
\]
\[
\geq \sqrt{1 + \epsilon}
\]
where (a) follows from the definitions of \( s_{\min} \) and \( s_{\max} \); (b) follows from (62) and (63); (c) follows from (50); (d) follows from (65); (e) follows from (46); and (f) follows from (45). Therefore, starting with (57),

\[
\liminf_{n \to \infty} \min_{j \in I_{\text{true}}} \frac{\rho(j)}{\mu} \geq \liminf_{n \to \infty} \min_{j \in I_{\text{true}}} \frac{1}{\mu} \left\{ \frac{s_j - 2\sqrt{s_j s_j} + s_j}{s_j + 2\sqrt{s_j s_j} + 1} \right\} = \liminf_{n \to \infty} \min_{j \in I_{\text{true}}} \frac{1}{\mu} \left\{ \frac{1}{\left(\sqrt{s_j} - \sqrt{s_j} \right)^2} \right\}
\]

\[
\liminf_{n \to \infty} \min_{j \in I_{\text{true}}} \frac{1 + \epsilon}{\mu} \left\{ \frac{s_j + 2\sqrt{s_j s_j} + 1}{s_j + 2\sqrt{s_j s_j} + 1} \right\} \geq \liminf_{n \to \infty} \min_{j \in I_{\text{true}}} \frac{1 + \epsilon}{\mu} \left\{ \frac{s_{\min} + 2\sqrt{s_{\min} s_j}}{s_{\min} + 2\sqrt{s_{\min} s_j}} \right\} \geq \liminf_{n \to \infty} \min_{j \in I_{\text{true}}} \frac{1 + \epsilon}{\mu} \left\{ \frac{(\sqrt{s_j} + 1)^2}{1 + \epsilon} \right\} \geq 1 + \epsilon,
\]

where (a) follows from (57); (b) follows from (64); (c) follows from the fact that \( z_j \in [0, 1] \) (it is a Beta distributed random variable); (d) follows from (62); and (e) follows from the condition of the hypothesis of the theorem that \( \gamma \to 0 \). This proves the first requirement, condition (43).

**E. False Alarm Probability**

Now consider any index \( j \notin I_{\text{true}} \). This implies that \( x_j = 0 \) and therefore (53) shows that

\[
P_{\text{true}}(j)y = e_j.
\]

Hence from (41),

\[
\rho_{\text{true}}(j) = \frac{|a_j^e|^2}{||P_{\text{true}}(j)a||^2 ||e_j||^2} = z_j
\]

where \( z_j \) is defined in (59). From the discussion above, each \( z_j \) has the Beta(2, \( m_j - 2 \)) distribution. Since there are asymptotically \( (1 - \lambda)n \) elements in \( I_{\text{true}} \), the conditions (50) and (51) along with Lemma 5 show that the limit

\[
\limsup_{n \to \infty} \max_{j \notin I_{\text{true}}} \frac{m - \lambda n}{\log(n(1 - \lambda n))} z_j \leq 1
\]

holds in probability. Therefore,

\[
\limsup_{n \to \infty} \max_{j \notin I_{\text{true}}} \frac{1}{\mu} \rho_{\text{true}}(j) \geq \limsup_{n \to \infty} \max_{j \notin I_{\text{true}}} \frac{1}{\mu} z_j \geq \limsup_{n \to \infty} \max_{j \notin I_{\text{true}}} \frac{m - \lambda n}{\log(n(1 - \lambda n))} z_j \leq \frac{1}{1 + \epsilon}
\]

where (a) follows from (57); (b) follows from (46); and (c) follows from (66). This proves (44) and thus completes the proof of the theorem.

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