Spatial cut-offs, Fermion Statistics, and Verlinde’s Conjecture

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December 20, 2018

Abstract

Verlinde conjectured eight years ago that gravitation might be an emergent entropic force. This rather surprising assertion was proved in [Physica A \textbf{505} (2018) 190] within a purely classical statistical context, and in [DOI: 10.13140/RG.2.2.34454.24640] for the case of bosons’ statistics. In the present work, we appeal to a quantum scenario involving fermions’ statistics. We consider also the classical limit of quantum (statistical) mechanics (QM). We encounter a lower bound to the distance $r$ between the two interacting masses, i.e., an $r$ cut-off. This is a new effect that exhibits some resemblance with the idea of space discretization proposed by recent gravitation theories.

Keywords: Gravitation, fermions, entropic force, emergent force, Verlides’s conjecture.
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1 Introduction

Eight years ago, Verlinde [1] proposed to establish a bridge linking gravity with an entropic force. The ensuing conjecture was proved recently I) in a purely classical environment [2] and II) in [3] for the Bose-Einstein statistics.

According to Verlinde, gravity should emerge as a result of information about the positions of material particles, connecting a thermal treatment of gravity to 't Hooft’s holographic principle. In this view, gravitation could be regarded as an emergent phenomenon. This interesting Verlinde’s notion was the focus of ample attention. See for instance [4, 5, 6]. A very good overview on the gravitation’s statistical mechanics is that of Padmanabhan [7], and references therein.

Verlinde’s idea motivated works on cosmology, the dark energy hypothesis, cosmological acceleration, cosmological inflation, and loop quantum gravity. The pertinent literature is abundant [3]. An important contribution was made by Guseo [8], who demonstrated that the local entropy function, related to a logistic distribution, is a catenary and vice versa, an invariance that can be interpreted in terms of Verlinde’s gravity’s origin conjecture. Guseo puts forward a new interpretation of the local entropy in a system [8]. Recapitulating: Verlinde’s conjecture has been proved:

- In a classical scenario in [2].
- In a quantum environment for bosons in ref. [3].

In this paper we wish to address the Verlinde-associated quantum fermionic case, that will be seen to yield rather surprising results.

2 Entropic force for an \( N \)-particles Fermi gas

We base our considerations on Chapter 6 of [9], where the reader is referred to for details. It is assumed that each fermion possesses an average energy \( E/N \). Such average energy approximation produces results that, while approximate, describe important features of the ideal Fermi gas [9].
2.1 Quantum entropic force

Our $N$-fermions gas’ entropy can be written in the fashion [see [9], Eq. (6.15)]

$$S = Nk_B \left[ \ln \left( \frac{n}{N} \right) - \left( \frac{n}{N} \right) \ln \left( 1 - \frac{N}{n} \right) \right], \quad (2.1)$$

where $n$ is related to the system’s energy $E$ according to [9]

$$n = V \left( \frac{E}{N} \right)^{\frac{3}{2}} \left( \frac{4\pi em}{2\hbar^2} \right)^{\frac{3}{2}}. \quad (2.2)$$

We can cast the volume as that of a sphere

$$V = \frac{4}{3} \pi r^3, \quad (2.3)$$

whose area is to be called $A$. Then we can recast (2.1) as

$$S = Nk_B \ln \left[ 1 - \frac{V}{N} \left( \frac{E}{N} \right)^{\frac{3}{2}} \left( \frac{4\pi em}{2\hbar^2} \right)^{\frac{3}{2}} \right] -$$

$$k_B V \left( \frac{E}{N} \right)^{\frac{3}{2}} \left( \frac{4\pi em}{2\hbar^2} \right)^{\frac{3}{2}} \ln \left[ \frac{N}{V} \left( \frac{E}{N} \right)^{\frac{3}{2}} \left( \frac{2\hbar^2}{4\pi em} \right)^{\frac{3}{2}} - 1 \right] \quad (2.4)$$

Now, according to [2], the entropic force $F_e$ is obtained via derivative with respect to $A$

$$F_e = -\lambda \frac{\partial S}{\partial A} =$$

$$\frac{\lambda 3k_B N}{8\pi r^2} \frac{1}{3N \left( \frac{E}{N} \right)^{\frac{3}{2}} \left( \frac{4\pi em}{2\hbar^2} \right)^{\frac{3}{2}} - 1} +$$

$$\frac{\lambda k_B}{2} \left( \frac{E}{N} \right)^{\frac{3}{2}} \left( \frac{4\pi em}{2\hbar^2} \right)^{\frac{3}{2}} r \ln \left[ \frac{3n}{4\pi r^3} \left( \frac{N}{E} \right)^{\frac{3}{2}} \left( \frac{3h^2}{4\pi em} \right)^{\frac{3}{2}} - 1 \right] -$$

$$\frac{\lambda k_B}{2} \left( \frac{E}{N} \right)^{\frac{3}{2}} \left( \frac{4\pi em}{2\hbar^2} \right)^{\frac{3}{2}} r \left[ \frac{4\pi r^3 \left( \frac{E}{N} \right)^{\frac{3}{2}} \left( \frac{4\pi em}{2\hbar^2} \right)^{\frac{3}{2}} - 1}{3N} \right], \quad (2.5)$$

where $\lambda$ is an arbitrary constant. We now recast the above equations as

$$F_e = \frac{12\lambda k_B N (\pi em)^{\frac{3}{2}} r}{32\pi r^3 (\pi em)^{\frac{3}{2}} - 3^{\frac{3}{2}} N \frac{\pi}{2} h^3} -$$
\[
\frac{4\pi \lambda k_B (\pi emE)^{\frac{3}{2}}}{(3N)^{\frac{3}{2}}h^3} r \left\{ \ln \left[ 32\pi r^3 (\pi emE)^{\frac{3}{2}} - (3N)^{\frac{3}{2}}h^3 \right] - \ln \left[ 32\pi r^3 (\pi emE)^{\frac{3}{2}} \right] \right\} - \\
\frac{12\lambda k_B N (\pi emE)^{\frac{3}{2}} r}{32\pi r^3 (\pi emE)^{\frac{3}{2}} + (3N)^{\frac{3}{2}}h^3}.
\]

Finally, we arrive at

\[
F_e = \frac{4\pi \lambda k_B (\pi emE)^{\frac{3}{2}}}{(3N)^{\frac{3}{2}}h^3} r \left\{ \ln \left[ 32\pi r^3 (\pi emE)^{\frac{3}{2}} - (3N)^{\frac{3}{2}}h^3 \right] - \ln \left[ 32\pi r^3 (\pi emE)^{\frac{3}{2}} \right] \right\},
\]

our central result here.

### 2.2 Fermionic entropic force in the classical limit (CL)

The CL obtains for \( n \ll 1 \),

\[
\frac{N}{n} \ll 1, \tag{2.8}
\]

and in this limit the entropy becomes \[ \]

\[
S = Nk_B \left[ 1 + \ln \left( \frac{n}{N} \right) \right], \tag{2.9}
\]

or

\[
S = \frac{5Nk_B}{2} + Nk_B \ln \left[ \frac{V}{N} \left( \frac{E}{N} \right)^{\frac{3}{2}} \left( \frac{4\pi m}{2h^2} \right)^{\frac{3}{2}} \right]. \tag{2.10}
\]

Now we have an entropic force of the form

\[
F_e = -\lambda \frac{\partial S}{\partial A} = -\frac{\lambda 3Nk_B}{8\pi r^2}, \tag{2.11}
\]

which is indeed of the Newton appearance, so that Verlinde’s conjecture gets proved in the classical limit. Note also that the entropic force (2.11) can be derived as well from (2.7) by taking \( r \) large enough. Using now the equality

\[
-\frac{\lambda 3Nk_B}{8\pi r^2} = -\frac{GmM}{r^2}, \tag{2.12}
\]

where \( G \) is the gravitational constant, we see that \( \lambda = \lambda(m, M, N) \) in the case \( r \) large enough.
2.3 Entropic Potential Energy

The entropic force is proportional to the derivative of the entropy with respect to the area $A$ of the sphere. It is interesting to calculate the corresponding potential energy $E_P$. For this we define the constants $a = (3N)^{1/2}h^{3}$ and $b = 32\pi (\pi emE)^{3/2}$. Using reference [11] we can compute the potential energy from the expression of the entropic force. The ensuing calculation is simple but lengthy. Its result reads

$$E_P(r) = -\frac{3Nk_B\lambda b}{8\pi a} \left\{ \frac{r^2}{2} \ln \left( 1 - \frac{a}{br^3} \right) - \frac{a^{3/2}}{2b^{3/2}} \left\{ \frac{1}{2} \ln \left[ \frac{r - \left( \frac{a}{b} \right)^{1/3}}{r^2 + \left( \frac{a}{b} \right)^{1/3} r + \left( \frac{a}{b} \right)^{2/3}} \right] \right\} + \sqrt{3} \left\{ \arctan \left[ \frac{2r + \left( \frac{a}{b} \right)^{1/3}}{\sqrt{3} \left( \frac{a}{b} \right)^{1/3}} - \frac{\pi}{2} \right] \right\} \right\}$$

(2.13)

where we have selected $E_P(r) = 0$ when $r \to \infty$. For $r$ large the potential energy adopts the appearance

$$E_P(r) = -\frac{\lambda 3Nk_B}{8\pi r}, \quad (2.14)$$

which is coherent with the result (2.11).

2.4 Results

In Fig. 1 we plot $F_e$ with $m=\text{uranium's atom mass}$, $E = \frac{Nmv^2}{2}$, $v = 1000 \text{ meter/second}$, and $N = 500$ yielding $r^3 > 1.525 \times 10^{-33}$.

The cut-off originates because $F_e$ takes now the simplified form

$$F_e = \frac{3N\lambda k_B}{8\pi} r \ln \left[ 1 - \frac{1.525 \times 10^{-33}}{r^3} \right]$$

(2.15)

and the plot is drawn for

$$\frac{F_e}{\left( \frac{3N\lambda k_B}{8\pi} \right) \left( 1.525 \times 10^{-33} \right)^{1/3}} = x \ln \left[ 1 - \frac{1}{x^3} \right].$$

(2.16)
3 Conclusions

We have considered, for fermions, Verlinde’s [entropic force - Gravitation] link, proved recently both in a classical context [2] and in a quantum bosonic scenario [3]. We have seen that Verlinde’s conjecture holds also for a Fermi environment. Further, the quantum emergent gravitation à la Verlinde does not diverge at the origin because a cut-off impedes reaching it. One finds, however, the emergent gravitation’s usual divergence-at-the-origin in the classical limit. One might perhaps wonder whether this divergence could be a classical artifact, since for bosons one does not have divergence at the origin neither [3]. Note that in two limits

- the classical limit of QM
- $r$ large enough.

the Newton $r$-dependence of the gravitation force is recovered.

Finally, we remark that we find an $r$-cut-off in the fermionic entropic force that somehow becomes reminiscent of the space-discretization ideas of loop gravity [10]. Our approach is not yet able to deal with geometric facets à la Einstein but might be coherent with granular space-time.

Acknowledgements

We are most grateful to Prof. A. R. Plastino for useful discussions.
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Figure 1: Here we plot $F_e/C$, $C = \frac{3Nk_B\lambda}{8\pi}$. Black curve $f$ (barely visible, very near the negative vertical axis): Fermion entropic force. Note the cut-off near the origin. Red curve $g$: approximate semi-classic Fermion-one. Both curves are almost-coincident for $r$—not too small. For better grasping details near the origin, see Fig.2.
Figure 2: Here we plot an amplified part, near the origin, of Fig. 1.