LINE GRAPHS AND THE TRANSPLANTATION METHOD

PETER HERBRICH

Abstract. We study isospectrality for mixed Dirichlet-Neumann boundary conditions, and extend the previously derived graph-theoretic formulation of the transplantation method. Led by the theory of Brownian motion, we introduce vertex-colored and edge-colored line graphs that give rise to block diagonal transplantation matrices. In particular, we rephrase the transplantation method in terms of representations of free semigroups, and provide a method for generating adjacency cospectral weighted directed graphs.

1. Introduction

Inverse spectral geometry studies the extent to which a geometric object, e.g., a Euclidean domain, is determined by the spectral data of an associated operator, e.g., the eigenvalues of the Laplace operator with suitable boundary conditions. This objective is beautifully summarized by Kac’s influential question “Can one hear the shape of a drum?” [Kac66]. Recently, the author [Her14] studied broken drums each of which is modeled as a compact flat manifold \( M \) with boundary \( \partial M = \partial_D M \cup \partial_N M \), where \( \partial_D M \) and \( \partial_N M \) represent the attached and unattached parts of the drumhead, respectively. The audible frequencies of such a broken drum are determined by the eigenvalues of the Laplace-Beltrami operator \( \Delta M \) of \( M \) with Dirichlet and Neumann boundary conditions along \( \partial_D M \) and \( \partial_N M \), respectively. Provided that \( \partial M \) is sufficiently smooth, this operator has discrete spectrum given by an unbounded non-decreasing sequence of non-negative eigenvalues.

Using number-theoretic ideas, Sunada [Sun85] developed a celebrated method involving group actions to construct isospectral manifolds, i.e., manifolds whose spectra coincide. It ultimately allowed Gordon et al. [GWW92] to answer Kac’s question in the negative. Buser [Bus86] distilled the combinatorial core of Sunada’s method into the transplantation method, which involves tiled manifolds that are composed of identical building blocks, e.g., \( M \) and \( M' \) in Figure 1(A). If \( \varphi \) is an eigenfunction on \( M \), then its restrictions \( (\varphi_i)_{i=1}^4 \) to the blocks of \( M \) are superposed linearly as \( (\sum_{i=1}^4 T_{ij} \varphi_i)_{j=1}^4 \) on the blocks of \( M' \) such that the result is an eigenfunction on \( M' \), and vice versa. All known pairs of isospectral planar domains with Dirichlet boundary conditions arise in this way [BCDS94], i.e., they are transplantable.

Following [Her14], we encode each tiled manifold with mixed Dirichlet-Neumann boundary conditions by an edge-colored graph with signed loops that encode boundary conditions, e.g., \( G \) and \( G' \) in Figure 1(C). By definition, every vertex of an edge-colored loop-signed graph \( G \) has one incident edge of each color, either as a link to another vertex or as a signed loop. If \( G \) has \( k \) edge colors, then it is determined by its \( k \)-tuple of adjacency matrices \( (A_G^c)_{c=1}^k \), which are diagonally-signed permutation matrices with \( A_G^c = (A_G^c)^T = (A_G^c)^{-1} \) given by

\[
[A_G^c]_{v\bar{v}} = \begin{cases} 
1 & \text{if } v \neq \bar{v} \text{ and there is a } c-\text{colored link between vertices } v \text{ and } \bar{v}, \\
\pm 1 & \text{if } v = \bar{v} \text{ and there is a } c-\text{colored } N- \text{ or } D-\text{loop at vertex } v, \text{ respectively,} \\
0 & \text{otherwise.}
\end{cases}
\]

Acknowledgments. I am indebted to Peter Doyle for his indispensable contributions.
Definition 1. Let $G$ and $G'$ be edge-colored or vertex-colored graphs given by $k$-tuples of $n \times n$ adjacency matrices $(A_G^c)^{k}_{c=1}$ and $(A_{G'}^c)^{k}_{c=1}$, respectively. Then $G$ and $G'$ are said to be

1. transplantable if there exists an invertible transplantation matrix $T \in \mathbb{R}^{n \times n}$ such that $A_G^c = T A_G^c T^{-1}$ for every $c \in \{1, 2, \ldots, k\}$,
2. cycle equivalent if for every finite sequence of colors $c_1, c_2, \ldots, c_t \in \{1, 2, \ldots, k\}$
   \[
   \text{tr}(A_G^{c_1} A_G^{c_2} \cdots A_G^{c_t}) = \text{tr}(A_{G'}^{c_1} A_{G'}^{c_2} \cdots A_{G'}^{c_t}).
   \]

Note that transplantable graphs are cycle equivalent. The following characterization of transplantable tiled manifolds says that the converse holds for edge-colored loop-signed graphs.

Theorem 2. [Her14] Let $G$ and $G'$ be edge-colored loop-signed graphs with the same numbers of vertices and colors. Let $M$ and $M'$ be tiled manifolds with mixed Dirichlet-Neumann boundary conditions obtained by choosing a building block. Then the following are equivalent:

1. $M$ and $M'$ are transplantable (and therefore isospectral),
2. $G$ and $G'$ are transplantable,
3. $G$ and $G'$ are cycle equivalent.

The equivalence of (1) and (2) is shown using regularity and continuation theorems for elliptic operators, and the equivalence of (2) and (3) is shown using representation theory. In the following, we derive further characterizations of transplantability. As is well-known, Brownian motion on a manifold $M$ has $\frac{1}{2} \Delta_M$ as its infinitesimal generator, rendering it a natural object of study for spectral questions. Consider a particle moving on the tiled manifold $M$ in Figure 1(A). Each time the particle hits $\partial_N M$, it is reflected back, whereas contact with $\partial_D M$ destroys the particle. Since the 4 triangular building blocks of $M$ are isometric, we only keep track of the triangle sides visited, which corresponds to a walk on the colored vertices of the associated directed line graph $L^c(G)$ shown in Figure 1(E). Each $N$-loop of $G$ in Figure 1(C) contributes 3 directed edges of equal weight to $L^c(G)$ in Figure 1(E) whereas $D$-loops of $G$ do not contribute at all. The edge-colored directed line graph $L^c(G)$ in Figure 1(G) is obtained by coloring edges instead of vertices. In Section 2 we define $L^c(G)$ and $L^c(G')$ rigorously, and prove the following main theorem.

Theorem 3. Let $G$ and $G'$ be edge-colored loop-signed graphs with $n$ vertices and $k$ colors. If $\text{tr}(A_G^c) = \text{tr}(A_{G'}^c)$ for every $c \in \{1, 2, \ldots, k\}$, then the following are equivalent:

1. $G$ and $G'$ are transplantable,
2. $G$ and $G'$ are cycle equivalent,
3. $L^c(G)$ and $L^c(G')$ are transplantable,
4. $L^c(G)$ and $L^c(G')$ are cycle equivalent,
5. $L^c(G)$ and $L^c(G')$ are transplantable,
6. $L^c(G)$ and $L^c(G')$ are cycle equivalent.

If any of the above conditions holds, then there exists an invertible transplantation matrix for both (3) and (2) that is the direct sum of square matrices of sizes $(\text{tr}(I_n + A_G^c)/2)^{k}_{c=1}$.

Theorem 3 has the following representation-theoretic interpretation. For $K = \{1, 2, \ldots, k\}$, we denote the free group on $K$ by $F(K)$, and the free semigroup on $K$ by $K^+$. The graphs $G$ and $G'$ give rise to a pair of representations of $F(K)$ by virtue of $c^{\pm 1} \mapsto A_G^c$ and $c^{\pm 1} \mapsto A_{G'}^c$, respectively. Similarly, $L^c(G)$ and $L^c(G')$, as well as $L^c(G)$ and $L^c(G')$, give rise to pairs of representations of $K^+$. If the assumptions of Theorem 3 are satisfied, and the representations of one pair are equivalent or have equal characters, then both are true for all pairs.
Theorem 3 that have 4 planar domains with pure Dirichlet boundary conditions [BCDS94], and introduce their line graphs in [MM03], McDonald and Meyers consider the finitely many known pairs of transplantable graphs. For each of the pairs in [BCDS94], they verify that the associated edge-colored line graphs are isomorphic if and only if their line graphs are isomorphic, with the exception of theorem [Whi32] which states that two uncolored connected graphs without loops or parallel edges are isomorphic if and only if their line graphs are isomorphic, with the exception of the classical Whitney graph isomorphism theorem. It is worth mentioning that [Her14] gives examples of non-isomorphic graphs $G$ and $G’$ as in Theorem 3 that have 4 vertices, no N-loops, and isomorphic line graphs $L^{ec}(G)$ and $L^{ec}(G’)$.

These pairs closely resemble the single exception of the classical Whitney graph isomorphism theorem [Whi32] which states that two uncolored connected graphs without loops or parallel edges are isomorphic if and only if their line graphs are isomorphic, with the exception of the triangle graph $K_3$ and the star graph $S_3 = K_{1,3}$, which both have $K_3$ as their line graph.

We want to point out the results in [MM03, OB12], which initiated our investigations. In [MM03], McDonald and Meyers consider the finitely many known pairs of transplantable planar domains with pure Dirichlet boundary conditions [BCDS94], and introduce their line graph construction, which, in our notation, corresponds to the assignment $M \mapsto L^{ec}(G)$. For each of the pairs in [BCDS94], they verify that the associated edge-colored line graphs...
are cospectral with respect to a certain discrete Laplace operator. In [OB12], Oren and Band note that these graphs are also cospectral with respect to their weighted adjacency matrices. However, the line graph construction was neither known to always produce cospectral graphs, nor could it deal with Neumann boundary conditions, and it had not been noticed that there exist canonical transplantations as in the second part of Theorem 3.

2. Colored directed line graphs

Let \( G \) be an edge-colored loop-signed graph with \( n \) vertices and adjacency matrices \( (A_c^\epsilon)^k_{c=1} \). In particular, \( \text{tr}(I_n + A_{\epsilon}^c)/2 \) equals the number of \( c \)-colored links and \( N \)-loops of \( G \).

**Definition 4.** Let \( G^* \) be the graph obtained from \( G \) by removing all \( D \)-loops. Let

\[
n_L = \sum_{c=1}^{k} \frac{\text{tr}(I_n + A_c^\epsilon)}{2}
\]

denote the number of edges of \( G^* \). The vertex-colored directed line graph \( L^{vc}(G) \) of \( G \) has one \( c \)-colored vertex for each \( c \)-colored edge of \( G^* \), and two vertices of \( L^{vc}(G) \) are connected if and only if the corresponding edges in \( G^* \) are incident. More precisely, \( L^{vc}(G) \) is defined by its \( k \)-tuple of \( n_L \times n_L \) adjacency matrices \( (A_c^{vc}(G))^k_{c=1} \) given by

\[
[A^{vc(G)}_{c=1}]_{\epsilon \bar{\epsilon}} = \begin{cases} 
2 & \text{if edge } \epsilon \text{ of } G^* \text{ is } c\text{-colored and shares all of its vertices with edge } \bar{\epsilon} \neq \epsilon, \\
1 & \text{if edge } \epsilon \text{ of } G^* \text{ is a } c\text{-colored link and shares one vertex with edge } \bar{\epsilon} \neq \epsilon, \\
0 & \text{otherwise.}
\end{cases}
\]

The edge-colored directed line graph \( L^{ec}(G) \) has \( n_L \) vertices, colors \( \{\{c, \bar{c}\} \mid 1 \leq c < \bar{c} \leq k\} \), and is obtained from \( L^{vc}(G) \) by coloring its edges with the colors of their incident vertices. More precisely, \( L^{ec}(G) \) is defined by its \( \binom{k}{2} \) adjacency matrices \( (A_{L^{ec}(G)})_{1 \leq c < \bar{c} \leq k} \) given by

\[
[A^{ec(G)}_{1 \leq c < \bar{c} \leq k}]_{\epsilon \bar{\epsilon}} = \begin{cases} 
[A^{vc(G)}_{c=1}]_{\epsilon \bar{\epsilon}} + [A^{\bar{c}}_{vc(G)}]_{\epsilon \bar{\epsilon}} & \text{if the set of colors of edges } \epsilon \text{ and } \bar{\epsilon} \text{ of } G^* \text{ is } \{c, \bar{c}\}, \\
0 & \text{otherwise.}
\end{cases}
\]

We note that if \( G \) has no \( N \)-loops or parallel links, then \( L^{ec}(G) \) is a simple edge-colored undirected graph, meaning it has symmetric \( \{0,1\} \)-adjacency matrices with zero diagonal.

**Definition 5.** Let \( B_w \in \{0,1,w\}^{n \times n_L} \) be the weighted incidence matrix of \( G^* \) given by

\[
[B_w]_{ve} = \begin{cases} 
w & \text{if edge } \epsilon \text{ of } G^* \text{ is an } N\text{-loop incident to vertex } v, \\
1 & \text{if edge } \epsilon \text{ of } G^* \text{ is a link incident to vertex } v, \\
0 & \text{otherwise.}
\end{cases}
\]

For each \( c \in \{1,2,\ldots,k\} \), let \( C^c \in \{0,1\}^{n_L \times n_L} \) be the diagonal matrix given by

\[
[C^c]_{ee} = \begin{cases} 
1 & \text{if edge } \epsilon \text{ of } G^* \text{ has color } c, \\
0 & \text{otherwise.}
\end{cases}
\]

**Lemma 6.** For every \( c \in \{1,2,\ldots,k\} \),

\[
A_G^c = B_1 C^c B_2^T - I_n \quad \text{and} \quad A^{vc(G)}_{L^{ec}(G)} = C^c (B_2^T B_1 - 2I_{n_L}).
\]
Proof. We show the matrix equalities row by row. If \( M \) is a matrix, let \([M]_m\) denote its row \( m \). Let \( v \in \{1, 2, \ldots, n\} \). Recall that vertex \( v \) of \( G \) has exactly one \( c \)-colored incident edge \( e \), either as a link, \( N \)-loop, or \( D \)-loop, respectively. If \( e \) is a link to vertex \( \tilde{v} \neq v \), then \([A^e_G]_{v\tilde{v}} = 1\) and \([B_1C^e]_{v\tilde{v}} = 1\) are the only non-vanishing entries in \([A^e_G]_v\) and \([B_1C^e]_v\), respectively. In particular, \([B_1C^eB^2_G]_e\), equals \([B^2_G]_e\) whose non-zero entries are \([B^2_G]_{e\tilde{v}} = [B^2_G]_{e\tilde{v}} = 1\). Similarly, if \( e \) is a \( D \)-link, then \([A^e_G]_{v\tilde{v}} = -1\) and \([B_1C^eB^2_G]_e = 0\). Finally, if \( e \) is an \( N \)-link, then \([A^e_G]_{v\tilde{v}} = 1\) and \([B_1C^e]_{v\tilde{v}} = 1\) are the only non-vanishing entries in their respective rows. In particular, the same is true for \([B_1C^eB^2_G]_{v\tilde{v}} = [B^2_G]_{v\tilde{v}} = 2\), which shows the first equality.

Let \( e \in \{1, 2, \ldots, n_L\} \). If edge \( e \) of \( G^* \) is not \( c \)-colored, then \([A^e_G]_{v\tilde{v}} = 0\) and \([C^e]_e = 0\). We therefore assume that \( e \) is \( c \)-colored, in which case \([C^e(B^2_GB_1^2 - 2I_{n_L})]_e = [B^2_GB_1^2 - 2I_{n_L}]_e\).

If \( e \) is a link between vertices \( v \) and \( \tilde{v} \) of \( G \), then \([B^2_G]_{v\tilde{v}} = [B^2_G]_{v\tilde{v}} = 1\) are the only non-zero entries in \([B^2_G]_e\). In particular, \([B^2_GB_1]_e = [B_1]_v + [B_1]_{\tilde{v}}\), which coincides with \([A^e_G]_{v\tilde{v}} + 2I_{n_L}]_e\). Similarly, if \( e \) is an \( N \)-loop at \( v \), then \([B^2_G]_{v\tilde{v}} = 2\) is the only non-zero entry in \([B^2_G]_e\) which gives \([B^2_GB_1]_e = 2[B_1]_v = [A^e_G]_{v\tilde{v}} + 2I_{n_L}]_e\).

\[\square\]

Lemma 7. In Theorem 5 the statements (2), (4), and (6) are equivalent.

Proof. We start by showing that (2) and (4) are equivalent, which amounts to showing that the traces of products of adjacency matrices of \( G \) determine those of \( L^\infty(G) \), and vice versa. Since \( \text{tr}(I_n) = n \) and \( (\text{tr}(A^c_G))_{c=1}^k \) are given by assumption, \( \text{tr}(I_n) = \sum_{c=1}^k (\text{tr}(I_n + A^c_G))/2 \) can be assumed as given as well. For \( c \in \{1, 2, \ldots, k\} \), we have \( \text{tr}(A^c_{L^\infty(G)}) = 0 \), \( (A^c_G)^2 = I_n \), and \( (A^c_{L^\infty(G)})^2 = 0 \). It therefore suffices to consider products of adjacency matrices with cyclically square-free color sequences, i.e., sequences of the form \( c_1, c_2, \ldots, c_l \in \{1, 2, \ldots, k\} \) with \( c_1 \neq c_l \) and \( c_i \neq c_{i+1} \) for \( i \in \{1, 2, \ldots, l-1\} \). As \( C^{c_1}C^{c_2} = C^{c_2}C^{c_1} = 0 \), Lemma 6 yields

\[A^c_{L^\infty(G)}A^c_{L^\infty(G)} = (C^{c_1}B_1^2B_1 - 2C^{c_1})(C^{c_2}B_2^2B_1 - 2C^{c_2}) = C^{c_1}B_1^2B_1C^{c_2}B_2^2B_1 - 2C^{c_1}B_2^2B_1C^{c_2},\]

which has trace

\[\text{tr}(A^c_{L^\infty(G)}A^c_{L^\infty(G)}) = \text{tr}(B_1C^{c_1}B_2^2B_1C^{c_2}B_2^2B_1 - 2C^{c_2}) = \text{tr}(C^{c_1}B_1^2B_1C^{c_2}B_2^2B_1 - 2C^{c_1}B_1^2B_1C^{c_2},\]

which gives the desired statement by induction on \( l \).

We finish by showing that (4) and (5) are equivalent, which is essentially due to the fact that \( L^\infty(G) \) and \( L^\infty(G) \) have the same set of cycles, i.e., closed walks on their vertices. Proceeding as above, we note that \( \text{tr}(A^{[c_1,c_\tilde{c}]}_{L^\infty(G)}) = 0 \) for every \( c, \tilde{c} \in \{1, 2, \ldots, k\} \) with \( c \neq \tilde{c} \). Also, \( A^{[c_1,c_\tilde{c}]}_{L^\infty(G)}A^{[c_2,c_\tilde{c}]}_{L^\infty(G)} = 0 \) whenever \( \{c_1,c_\tilde{c}\} \cap \{c_2,c_\tilde{c}\} = \emptyset \). Thus,

\[\text{tr}(A^{[c_1,c_\tilde{c}]}_{L^\infty(G)}A^{[c_2,c_\tilde{c}]}_{L^\infty(G)} \cdots A^{[c_1,c_\tilde{c}]}_{L^\infty(G)}) = 0\]

unless \( \{c_1,c_\tilde{c}\} \cap \{c_2,c_\tilde{c}\} \neq \emptyset \) and \( \{c_2,c_\tilde{c}\} \cap \{c_{i+1},c_\tilde{c}_{i+1}\} \neq \emptyset \) for \( i \in \{1, 2, \ldots, l-1\} \). Due to the cyclic invariance of the trace, every possibly non-zero trace is of the form

\[\text{tr}(A^{[c_1,c_\tilde{c}]}_{L^\infty(G)}A^{[c_2,c_\tilde{c}]}_{L^\infty(G)} \cdots A^{[c_1,c_\tilde{c}]}_{L^\infty(G)}) = \text{tr}(A^{[c_1,c_\tilde{c}]}_{L^\infty(G)}A^{[c_2,c_\tilde{c}]}_{L^\infty(G)} \cdots A^{[c_1,c_\tilde{c}]}_{L^\infty(G)}),\]

where \( c_1 \neq c_3 \).
The second equation follows in a similar fashion since
\[ [(A_{\text{L}^{c}(G)^{2}})_{ee}]_{ee} \neq 0 \text{ only if vertex } e \text{ of } L^{c}(G) \text{ has color } c, \text{ in which case it equals the number of closed walks that start at } e \text{ and have color sequence } c, \bar{c}, c, \ldots, \bar{c}, \text{ likewise for } [(A_{\text{L}^{c}(G)^{2}})_{ee}]_{ee}. \]

Thus,
\[ [(A_{\text{L}^{c}(G)^{2}})_{ee}]_{ee} = [(A_{\text{L}^{c}(G)^{2}})_{ee}]_{ee} + [(A_{\text{L}^{c}(G)^{2}})_{ee}]_{ee}. \]

The second equation follows in a similar fashion since \([A_{L^{c}(G)^{2}}]_{ee} \neq 0 \text{ only if } e \text{ of } L^{c}(G) \text{ has color } c_{1}. \]

Theorem 2 and Lemma 7 show that (1), (2), (4), and (6) in Theorem 3 are equivalent. Since transplantablility of graphs implies their cycle equivalence, e.g., \((3) \Rightarrow (1) \text{ and (5) } \Rightarrow (6)\), Theorem 3 will be proven once we have shown that (1) implies the existence of a block diagonal transplantation matrix for both (3) and (5) as claimed, i.e., \((1) \Rightarrow (3) \text{ and (1) } \Rightarrow (5)\).

In the following, let \(G \) and \(G' \) be transplanatable edge-colored loop-signed graphs with \(n \times n\) adjacency matrices \((A_{G})_{c=1}^{k}\) and \((A_{G'})_{c=1}^{k}\), respectively. Let \(T \in \mathbb{R}^{n \times n}\) be an invertible transplantation matrix satisfying \(A_{G}T = TA_{G'}\) for every \(c \in \{1, 2, \ldots, k\}\). In particular,
\[ n_{L}^{c} = \frac{\text{tr}(I_{n} + A_{G})}{2} = \frac{\text{tr}(I_{n} + A_{G'})}{2}, \]

which equals the number of \(c\)-colored links and \(N\)-loops of \(G\) or \(G'\), respectively. Each of the graphs \(L^{c}(G)\) and \(L^{c}(G')\) has \(n_{L}^{c} = n_{L}^{c} + n_{L}^{c} + \ldots + n_{L}^{c}\) vertices, which we number accordingly, i.e., the respective first \(n_{L}^{c}\) vertices have color 1, followed by \(n_{L}^{c}\) vertices of color 2, and so on. Let \(e, e' \in \{1, 2, \ldots, n_{L}\}\). We denote the color of edge \(e\) of \(G'\) by \(c\), and its incident vertices by \(v\) and \(\bar{v}\), where \(v = \bar{v}\) if it is an \(N\)-loop. Analogously, we let edge \(e'\) of \((G')^{*}\) have color \(c'\) and possibly identical incident vertices \(v'\) and \(\bar{v}'\). Then,
\[ [A_{G}]_{v\bar{v}} = [A_{G'}]_{v'\bar{v}'} = [A_{G'}]_{v'\bar{v}'} = 1 \]

are the only non-vanishing entries in their respective row and column. In particular,
\[ [T]_{v'\bar{v}'} = [A_{G}T]_{v'\bar{v}'} = [T]_{v'\bar{v}'} \quad \text{and} \quad [T]_{v\bar{v}'} = [A_{G'}T]_{v\bar{v}'} = [T]_{v\bar{v}'} \]

Hence,
\[ [T]_{v'\bar{v}'} + [T]_{v'\bar{v}'} = [T]_{v'\bar{v}'} + [T]_{v\bar{v}'} = [T]_{v'\bar{v}'} + [T]_{v\bar{v}'} + [T]_{v\bar{v}'} = [T]_{v'\bar{v}'} + [T]_{v\bar{v}'} = [T]_{v'\bar{v}'} + [T]_{v\bar{v}'} \]

which allows to define a transplantation matrix for the line graphs associated with \(G\) and \(G'\).

**Definition 8.** The line graph transplantation matrix \(T_{L} \in \mathbb{R}^{n_{L} \times n_{L}}\) coming from \(T\) is given by
\[ [T_{L}]_{ee'} = \begin{cases} [T]_{v'\bar{v}'} + [T]_{v\bar{v}'} = [T]_{v'\bar{v}'} + [T]_{v\bar{v}'} & \text{if } c = c' \text{ and } v' \neq \bar{v}' , \\ [T]_{v'\bar{v}'} = [T]_{v'\bar{v}'} = [T]_{v'\bar{v}'} = [T]_{v'\bar{v}'} & \text{if } c = c' \text{ and } v' = \bar{v}' , \\ 0 & \text{otherwise.} \end{cases} \]

For later reference, we note that if \(e\) was a \(c\)-colored \(D\)-loop, i.e., if \([A_{G}]_{v\bar{v}} = -1\), then
\[ [T]_{v\bar{v}} = -[A_{G}T]_{v\bar{v}} = -[T]_{v\bar{v}} \quad \text{and} \quad [T]_{v'\bar{v}'} = -[A_{G'}]_{v'\bar{v}'} = -[T]_{v'\bar{v}'} , \]

so that \([T]_{v\bar{v}} + [T]_{v\bar{v}} = 0\). Similarly, if \(e'\) was a \(c\)-colored \(D\)-loop, i.e., if \([A_{G'}]_{v'\bar{v}'} = -1\), then
\[ [T]_{v\bar{v}} = [A_{G'}T]_{v'\bar{v}'} = [T]_{v'\bar{v}'} = -[T]_{v'\bar{v}'} , \]

so that \([T]_{v\bar{v}} + [T]_{v\bar{v}} = 0\).

**Lemma 9.** The line graph transplantation matrix \(T_{L}\) is invertible and satisfies
\[ A_{L^{c}(G)}T_{L} = T_{L}A_{L^{c}(G')} \quad \text{for every } c \in \{1, 2, \ldots, k\} , \]

as well as
\[ A_{L^{c}(G)}T_{L} = T_{L}A_{L^{c}(G')} \quad \text{for every } c, \bar{c} \in \{1, 2, \ldots, k\} \text{ with } c \neq \bar{c} . \]
Proof. For each \( c \in \{1, 2, \ldots, k\} \), let \( E^c = \{ n_1^c + n_2^c + \ldots + n_k^c + e^c \mid e^c = 1, 2, \ldots, n_k^c \} \), which corresponds to the \( c \)-colored edges of \( G^* \) as well as to the \( c \)-colored edges of \( (G')^* \), which in turn correspond to the \( c \)-colored vertices of \( L^c(G) \) and \( L^c(G') \), respectively. In order to show that the block diagonal matrix \( T_L \) is invertible, it suffices to show that each of its \( k \) diagonal blocks has linearly independent rows. Let \( c \in \{1, 2, \ldots, k\} \), and assume that \( (a_e)e \in E^c \in \mathbb{R}^{n_k^c} \) satisfies

\[
0 = \sum_{e \in E^c} a_e[T_L]_{ee'} \quad \text{for every } e' \in E^c.
\]

As before, we let edge \( e' \in E^c \) of \( (G')^* \) have possibly identical incident vertices \( v' \) and \( \tilde{v}' \). We first consider the case \( v' = \tilde{v}' \). Recall that every vertex \( v \) of \( G \) has exactly one incident \( c \)-colored edge, which we denote by \( e(v, c) \). Let

\[
\tilde{a}_{e(v, c)} = \begin{cases} 
\frac{1}{2} a_{e(v, c)} & \text{if } e(v, c) \text{ is a link}, \\
a_{e(v, c)} & \text{if } e(v, c) \text{ is an } N\text{-loop}, \\
0 & \text{if } e(v, c) \text{ is a } D\text{-loop}.
\end{cases}
\]

If \( e = e(v, c) \) is an \( N \)-loop at \( v \), then

\[
a_e[T_L]_{ee'} = \tilde{a}_{e(v, c)}([T]_v + [T]_{v_0}),
\]

whereas if \( e = e(v, c) = e(\tilde{v}, c) \) is a link between \( v \) and \( \tilde{v} \), then

\[
a_e[T_L]_{ee'} = 2\tilde{a}_{e(v, c)}([T]_v + [T]_{v_0}) + \tilde{a}_{e(\tilde{v}, c)}([T]_{\tilde{v}}).\]

Hence,

\[
0 = \sum_{v=1}^{n} \tilde{a}_{e(v, c)}([T]_v + [T]_{v_0}) = \sum_{v=1}^{n} \tilde{a}_{e(v, c)}[T]_v + \sum_{\tilde{v}=1}^{n} \tilde{a}_{e(\tilde{v}, c)}[T]_{\tilde{v}} = 2 \sum_{v=1}^{n} \tilde{a}_{e(v, c)}[T]_v,
\]

where in the last equality we used that each vertex \( \tilde{v} \) is the unique \( c \)-neighbor of some vertex \( v \), meaning \( [A^c_G]_{\tilde{v}v} \neq 0 \), in which case \( \tilde{a}_{e(\tilde{v}, c)}[T]_{\tilde{v}v} = \tilde{a}_{e(v, c)}[T]_{v\tilde{v}} \). The same arguments apply if \( v' = \tilde{v}' \), except for the summands involving \( \tilde{v}' \) which disappear. Since the rows of \( T \) are linearly independent, we deduce that \( \tilde{a}_{e(v, c)} = 0 \) for every \( v \in \{1, 2, \ldots, n\} \). In other words, \( (a_e)e \in E^c = 0 \), which proves that \( T_L \) is invertible.

Next, we show that for every \( c \in \{1, 2, \ldots, k\} \) and \( e, e' \in \{1, 2, \ldots, n_k^c\} \)

\[
[A^c_{L^c(G)}T_L]_{ee'} = [T_LA^c_{L^c(G')}_{ee'}].
\]

Let \( e \in E^c \) and \( e' \in E^{c'} \). Since \( T_L \) is block diagonal, we have

\[
[A^c_{L^c(G)}T_L]_{ee'} = \sum_{\tilde{e} \in E^{c'}} [A^c_{L^c(G)}]_{\tilde{e}e}[T_L]_{\tilde{e}e'} \quad \text{and} \quad [T_LA^c_{L^c(G')}_{ee'}] = \sum_{\tilde{e} \in E^{c'}} [T_L]_{e\tilde{e}}[A^c_{L^c(G')}_{\tilde{e}e'}].
\]

If \( \tilde{c} \neq c \) or \( \tilde{c} = c = c' \), then \( [A^c_{L^c(G)}]_{\tilde{e}e} = 0 \) for all \( \tilde{e} \in E^{c'} \), and \( [A^c_{L^c(G')}_{\tilde{e}e'} = 0 \) for all \( \tilde{e} \in E^{c'} \). It therefore suffices to consider the case \( \tilde{c} = c \neq c' \). As before, we let edge \( e \) of \( G^* \) and edge \( e' \) of \( (G')^* \) have incident vertices \( \{v, \tilde{v}\} \) and \( \{v', \tilde{v}'\} \), respectively. Note that each of the sums above has at most 2 non-vanishing summands, which correspond to the \( c \)-colored edges at \( v \) and \( \tilde{v} \), as well as the \( c \)-colored edges at \( v' \) and \( \tilde{v}' \), respectively. Regardless of whether these edges are links, \( N \)-loops, or \( D \)-loops, if \( v' \neq \tilde{v}' \), then

\[
[A^c_{L^c(G)}T_L]_{ee'} = [T]_{v\tilde{v}} + [T]_{v\tilde{v'}} + [T]_{\tilde{v}v'} + [T]_{\tilde{v}v'} = [T_LA^c_{L^c(G')}_{ee'}],
\]

whereas if \( v' = \tilde{v}' \), then

\[
[A^c_{L^c(G)}T_L]_{ee'} = [T]_{v\tilde{v}} + [T]_{\tilde{v}v'} = [T_LA^c_{L^c(G')}_{ee'}].
\]
Finally, we show that for every \(c, \tilde{c} \in \{1, 2, \ldots, k\}\) with \(c \neq \tilde{c}\) and \(e \in \{1, 2, \ldots, n_L\}\)

\[
[A_{L^{(c,\tilde{c})}} T_L]_e = [T_L A_{L^{(c,\tilde{c})}}]_e,
\]

where we reused the notation \([M]_m\) for row \(m\) of the matrix \(M\). We use the same idea as above and note that if \(e \notin E^c \cup E^{\tilde{c}}\), then \(A_{L^{(c,\tilde{c})}} T_L = 0\), i.e., \([A_{L^{(c,\tilde{c})}} T_L]_e = 0\), and \([T_L]_{e}\tilde{e} \neq 0\) only if \(\tilde{e} \notin E^c \cup E^{\tilde{c}}\), i.e., \([T_L A_{L^{(c,\tilde{c})}}]_e = 0\). If \(e \in E^c\), then \([A_{L^{(c,\tilde{c})}} T_L]_e = [A_{L^{(c,\tilde{c})}}]_e\), and \([T_L]_{e}\tilde{e} \neq 0\) only if \(\tilde{e} \in E^c\) in which case we have \([A_{L^{(c,\tilde{c})}}]_{\tilde{e}} = [A_{L^{(c,\tilde{c})}}]_{\tilde{e}}\), i.e., \([T_L A_{L^{(c,\tilde{c})}}]_e = [T_L A_{L^{(c,\tilde{c})}}]_e\). Similarly, if \(e \in E^{\tilde{c}}\), then

\[
[A_{L^{(c,\tilde{c})}} T_L]_e = [A_{L^{(c,\tilde{c})}}]_e = [T_L A_{L^{(c,\tilde{c})}}]_e = [T_L A_{L^{(c,\tilde{c})}}]_e.
\]

\[\Box\]

References

[BCDS94] Peter Buser, John Conway, Peter Doyle, and Klaus-Dieter Semmler, Some planar isospectral domains, Internat. Math. Res. Notices (1994), no. 9, 391ff., approx. 9 pp. (electronic).

[Bus86] Peter Buser, Isospectral Riemann surfaces, Ann. Inst. Fourier (Grenoble) 36 (1986), no. 2, 167–192.

[GWW92] C. Gordon, D. Webb, and S. Wolpert, Isospectral plane domains and surfaces via Riemannian orbifolds, Invent. Math. 110 (1992), no. 1, 1–22.

[Her14] Peter Herbrich, On inaudible properties of broken drums – Isospectrality with mixed Dirichlet-Neumann boundary conditions, arXiv:1111.6789v3 (2014).

[Kac66] Mark Kac, Can one hear the shape of a drum?, Amer. Math. Monthly 73 (1966), no. 4, part II, 1–23.

[MM03] Patrick McDonald and Robert Meyers, Isospectral polygons, planar graphs and heat content, Proc. Amer. Math. Soc. 131 (2003), no. 11, 3589–3599 (electronic).

[OB12] Idan Oren and Ram Band, Isospectral graphs with identical nodal counts, J. Phys. A 45 (2012), no. 17, 135203, 12.

[Sun85] Toshiyuki Sunada, Riemannian coverings and isospectral manifolds, Ann. of Math. (2) 121 (1985), no. 1, 169–186.

[Whi32] Hassler Whitney, Congruent Graphs and the Connectivity of Graphs, Amer. J. Math. 54 (1932), no. 1, 150–168.

Department of Mathematics, Dartmouth College, Hanover, NH, USA

E-mail address: peter.herbrich@dartmouth.edu