AN EXTENSION OF A CHARACTERIZATION OF THE
AUTOMORPHISMS OF HILBERT SPACE EFFECT
ALGEBRAS

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Abstract. The aim of this paper is to show that if an order preserving
bijective transformation of the Hilbert space effect algebra also preserves
the probability with respect to a fixed pair of mixed states, then it is
an ortho-order automorphism. A similar result for the orthomodular
lattice of all sharp effects (i.e., projections) is also presented.

Keywords: Hilbert space effect algebra, automorphisms, preservers.

Effect algebras play fundamental role in the theory of quantum measure-
ment [2] (also see [6] and [10]). In the Hilbert space setting, the so-called
Hilbert space effect algebra $E(H)$ is the set of all positive bounded linear
operators on the Hilbert space $H$ which are majorized by the identity $I$.
This set is usually equipped with certain operations and/or relations which
all have physical meaning. Therefore, there are different algebraic structures
on $E(H)$. In some respect, probably the most important such structure is
obtained when we equip $E(H)$ with the partial order $\leq$ (which is just the
usual order among self-adjoint operators restricted to $E(H)$) and a kind of
orthocomplementation, namely, $\perp: E \mapsto I - E$.

The study of the automorphisms of given algebraic structures is a very
important general problem in mathematics. As for effect algebras, the in-
vestigation of the so-called ortho-order automorphisms of $E(H)$ (that is, the

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automorphisms with respect to the order and orthocomplementation) was begun by Ludwig in [10]. In fact, in [10, Section V.5] he showed that, in case \( \dim H \geq 3 \), these automorphisms are implemented by unitary-antiunitary operators. However, his argument seemed to contain some gaps and the proof was recently clarified in [4] completely. (We mention that in our recent paper [14] we have shown that Ludwig’s result holds also in the 2-dimensional case and this answers a question that was open for quite some time.)

In our paper [11] we initiated the study of the automorphisms of effect algebras (or any other quantum structure) by means of their preserver properties. We expressed our belief there that, similarly to the case of linear preserver problems in matrix theory (concerning which we refer, for example, to the survey papers [8, 9]), such investigations may give important new information about the automorphisms in question and they may help to better understand the underlying algebraic structures. According to this, in [11] we presented some characterizations of the automorphisms of effect algebras via their preserver properties. This study was continued in [12, 13] where we obtained results of the same kind for the automorphisms of the Jordan algebra of all bounded observables.

Turning to the content of the present paper, we remark that the order is undoubtedly a very important relation on \( \mathcal{E}(H) \). One of the reasons is the following. As it turns out from [1] (and, in fact, was asserted already by Ludwig), the effects are determined by the weak atoms they majorize. As weak atoms can be defined by the order exclusively, it is obvious how essential the order is in the description of effects. We next refer to [15] to see how ”strong” this relation is. In spite of this, the order preserving property alone is not strong enough to characterize the ortho-order automorphisms in even some weak sense. In fact, if we consider the transformation

\[
E \mapsto \left( \frac{T^2}{2I - T^2} \right)^{-\frac{1}{2}} \left( (I - T^2 + T(I + E)^{-1}T)^{-1} - I \right) \left( \frac{T^2}{2I - T^2} \right)^{-\frac{1}{2}}
\]

where \( T \in \mathcal{E}(H) \) is fixed and invertible, one can easily check that this is a bijective map on \( \mathcal{E}(H) \) which preserves the order but has nothing to
do with the ortho-order automorphisms of $\mathcal{E}(H)$. (Here we mention that the situation is much different with the partially ordered set $B_s(H)$ of all bounded observables on $H$. It might be quite surprising that, as it turns out from our paper [12], if $\dim H > 1$ and $\phi$ is an order preserving bijective map on $B_s(H)$, then $\phi$ can be written in a nice explicit form which is closely related to the form of the automorphisms of $B_s(H)$ as a Jordan algebra (cf. [3]). So, it is clear that beside the order, the preservation of some other physical quantity is needed to characterize the automorphisms of $\mathcal{E}(H)$. The aim of this paper is to present such a result which is a significant generalization of one of our former results. Namely, in [11, Theorem 2] we proved that if $\dim H \geq 3$ and $\phi : \mathcal{E}(H) \to \mathcal{E}(H)$ is a bijective map which preserves the order and also preserves the probability with respect to a fixed pair of pure states, then $\phi$ is an ortho-order automorphism. It is now a natural question that what about the mixed states? In which follows we show that the same conclusion holds also for mixed states even in the case when $\dim H = 2$ (the 1-dimensional case is trivial). This generalizes and extends our result in [11]. Furthermore, we present a similar result concerning the orthomodular lattice of projections.

As for the notation, let $H$ be a complex Hilbert space and, just as before, let $\mathcal{E}(H)$ be the set of all self-adjoint bounded linear operators $E$ on $H$ for which $0 \leq E \leq I$. Denote by $\mathcal{P}(H)$ the set of all projections on $H$. The elements of $\mathcal{E}(H)$ are called effects (or, in other terminology, unsharp events), while the elements of $\mathcal{P}(H)$ are called sharp effects (or, in other terminology, sharp events). By a (mixed) state we mean a positive trace-class operator on $H$ with trace 1. The usual trace-functional is denoted by $\text{tr}$. The states form a convex set whose extreme points are exactly the rank-one projections which are called pure states. Finally, we emphasize that when we say that a transformation preserves a certain relation, we always mean that this relation is preserved in both directions.

Now, the main result of the paper reads as follows.
Theorem 1. Let $\phi : \mathcal{E}(H) \to \mathcal{E}(H)$ be a bijective map with the property that

$$E \leq F \iff \phi(E) \leq \phi(F) \quad (E, F \in \mathcal{E}(H))$$

and suppose that there are states $D, D'$ such that

$$\text{tr} (\phi(E)D') = \text{tr} (ED) \quad (E \in \mathcal{E}(H)).$$

Then there exists an either unitary or antiunitary operator $U$ on $H$ such that

$$\phi(E) = UEU^* \quad (E \in \mathcal{E}(H)).$$

Proof. Similarly to the proof of Theorem 1 in [11] we obtain that, by the order preserving property, $\phi$ preserves the projections and their ranks.

We show that $\phi(\lambda I) = \lambda I$ and $\phi(\lambda P) = \lambda \phi(P)$ holds for every $\lambda \in [0,1]$ and for every rank-one projection $P$. Let $\lambda \in [0,1]$ and pick a rank-one projection $P$ such that $\text{tr} \phi(P)D' = \text{tr} PD \neq 0$. As $\lambda P \leq P$, we obtain that $\phi(\lambda P) \leq \phi(P)$. Taking into account that $\phi(P)$ is of rank 1, it follows that $\phi(\lambda P) = \mu \phi(P)$ holds for some $\mu \in [0,1]$. Now, we compute on one hand that

$$\text{tr} \phi(\lambda P)D' = \text{tr} \mu \phi(P)D' = \mu \text{tr} PD$$

and on the other hand that

$$\text{tr} \phi(\lambda P)D' = \text{tr} (\lambda P)D = \lambda \text{tr} PD.$$

Comparing these two equalities and remembering that $\text{tr} PD \neq 0$, we deduce $\mu = \lambda$, that is, we get $\phi(\lambda P) = \lambda \phi(P)$.

At this point we need the useful concept of the strength of an effect $E$ along a rank-one projection $Q$ (or, equivalently, along the ray represented by the projection $Q$). According to [1], this number is, by definition, the supremum of the set of all $t \in [0,1]$ for which $tQ \leq E$.

Let $\lambda \in ]0,1]$. Pick a rank-one projection $P$ as above, that is, suppose that $\text{tr} \phi(P)D' = \text{tr} PD \neq 0$. The strength of $\lambda I$ along $P$ is obviously $\lambda$. By the order preserving property of $\phi$ and the homogeneity of $\phi$ on the set of effects of the form $tP$ (see the second paragraph of our proof), we infer that the strength of $\phi(\lambda I)$ along $\phi(P)$ is also $\lambda$. This holds for every
rank-one projection $\phi(P)$ for which $\text{tr} \phi(P)D' \neq 0$. It is clear that we have $\text{tr} \phi(P)D' \neq 0$ if and only if the range of $\phi(P)$ is not orthogonal to the range of $D'$. But the set of all such $\phi(P)$'s is easily seen to be dense in the set of all rank-one projections. This implies that the strength of $\phi(\lambda I)$ is equal to $\lambda$ along every member of a dense subset of rank-one projections. Lemma 5 in [11] tells us that in this case we have $\phi(\lambda I) = \lambda I$.

Now, pick an arbitrary rank-one projection $P$, let $\lambda \in [0, 1]$ be also arbitrary and take $\mu \in [0, 1]$ such that $\phi(\lambda P) = \mu \phi(P)$. We have on one hand that

$$\mu \phi(P) = \phi(\lambda P) \leq \phi(\lambda I) = \lambda I$$

implying that $\mu \leq \lambda$, and on the other hand that

$$\phi(\lambda P) = \mu \phi(P) \leq \mu I = \phi(\mu I)$$

implying that $\lambda P \leq \mu I$, i.e., $\lambda \leq \mu$. Therefore, we obtain that $\lambda = \mu$ which yields $\phi(\lambda P) = \lambda \phi(P)$ as we have claimed.

We next assert that $\phi$ preserves the orthogonality between rank-one projections. To see this, let $P, Q$ be mutually orthogonal rank-one projections. Denote $P^\perp = I - P$ the orthogonal complement of $P$. Pick numbers $0 < \lambda < \mu \leq 1$ and consider the effect $E = \lambda P + \mu P^\perp$. Clearly, we have $\lambda I \leq E \leq \mu I$, the strength of $E$ along $P$ is $\lambda$ and along $Q$ (which is a subprojection of $P^\perp$) is $\mu$. It follows from the order preserving property of $\phi$ and from the homogeneity of $\phi$ on the scalar multiples of rank-one projections that $\lambda I \leq \phi(E) \leq \mu I$, the strength of $\phi(E)$ along $\phi(P)$ is $\lambda$ and along $\phi(Q)$ is $\mu$. Now, Lemma 3 in [11] tells us that in this case the ranges of $\phi(P)$ and $\phi(Q)$ are subspaces of the eigenspaces of $\phi(E)$ corresponding to the eigenvalues $\lambda$ and $\mu$, respectively. This shows that the ranges of $\phi(P)$ and $\phi(Q)$ are orthogonal to each other.

Now we have to distinguish two cases. Suppose first that $\dim H = 2$. In that case it follows from what we have just seen (where we have made a reference to [11, Lemma 3]) that

$$\phi(\lambda P + \mu Q) = \lambda \phi(P) + \mu \phi(Q)$$
holds for every mutually orthogonal rank-one projections $P, Q$ and real numbers $\lambda, \mu \in [0, 1]$. Then it is trivial to check that $\phi$ is an ortho-order automorphism of $\mathcal{E}(H)$, that is, beside the order preserving property, $\phi$ also satisfies
\[
\phi(I - E) = I - \phi(E) \quad (E \in \mathcal{E}(H)).
\]
We can apply the result of our paper [14] to complete the proof in the case when $\dim H = 2$.

Suppose now that $\dim H \geq 3$. We know that $\phi$, when restricted onto the set of all rank-one projections, is a bijective transformation preserving orthogonality. It is a celebrated result of Uhlhorn [16] that in that case we have a unitary or antiunitary operator $U$ on $H$ such that
\[
\phi(P) = UPU^* \quad (P \in \mathcal{P}(H))
\]
holds for every rank-one projection $P$. Since $\phi(\lambda P) = \lambda \phi(P)$, it follows that
\[
\phi(\lambda P) = U(\lambda P)U^* \quad (\lambda \in [0, 1]).
\]
for every $\lambda \in [0, 1]$. The operators of the form $\lambda P$ are exactly the so-called weak atoms in $\mathcal{E}(H)$ and it is known from [1] that every effect is equal to the supremum of the set of all weak atoms which are majorized by the effect in question. As $\phi$ preserves the order, we thus obtain that
\[
\phi(E) = UEU^* \quad (E \in \mathcal{E}(H)).
\]
This completes the proof also in the present case.

In our second result which follows we formulate a similar statement concerning the orthomodular lattice $\mathcal{P}(H)$ of all projections on $H$. We recall that in the language of effects, the elements of $\mathcal{P}(H)$ are the so-called sharp effects. It should be emphasized that our result holds only in the case when $\dim H \geq 3$ (see Remark 1).

**Theorem 2.** Suppose that $\dim H \geq 3$. Let $\phi : \mathcal{P}(H) \to \mathcal{P}(H)$ be a bijective map with the property that
\[
P \leq Q \iff \phi(P) \leq \phi(Q) \quad (P, Q \in \mathcal{P}(H))
\]
and suppose that there are states $D$, $D'$ such that $D$ is not a scalar operator and

$$
(1) \quad \text{tr}(\phi(P)D') = \text{tr}(PD) \quad (P \in \mathcal{P}(H)).
$$

Then there exists an either unitary or antiunitary operator $U$ on $H$ such that $\phi$ is of the form

$$
\phi(P) = UPU^* \quad (P \in \mathcal{P}(H)).
$$

Proof. As the closed subspaces of $H$ and the projections on $H$ can be obviously identified, it is clear that our transformation $\phi$ gives rise to a so-called lattice automorphism of the lattice of all closed subspaces of $H$. The main result in [5] states that in the case when $\dim H \geq 3$, every such transformation is induced by a bijective semilinear map $A$ of $H$, and $A$ is bounded and either linear or conjugate-linear if $\dim H = \infty$. (In fact, the finite dimensional part of this result was not stated in the corresponding theorem in [5] as it is just a version of the fundamental theorem of projective geometry.) For our transformation $\phi$ this means that

$$
(2) \quad \phi(P_M) = P_{A(M)}
$$

holds for every closed subspace $M$ of $H$ ($P_M$ denotes the orthogonal projection onto $M$). The main point of our argument is to show that $A$ can be chosen to be an either unitary or antiunitary operator.

We first show that the semilinear map $A$ corresponding to our transformation $\phi$ is either linear or conjugate-linear also in the finite dimensional case. First recall the definition of semilinearity. This means that $A$ is additive and $A(\lambda x) = h(\lambda)A$ holds for every $x \in H$ and $\lambda \in \mathbb{C}$, where $h : \mathbb{C} \to \mathbb{C}$ is a given ring automorphism. By (2), for any nonzero $x \in H$ we have

$$
\phi \left( \frac{x \otimes x}{\|x\|^2} \right) = \frac{Ax \otimes Ax}{\|Ax\|^2}.
$$

Using (1) we obtain

$$
(3) \quad \frac{\text{tr}(Ax \otimes Ax \cdot D')}{\|Ax\|^2} = \frac{\text{tr}(x \otimes x \cdot D)}{\|x\|^2}
$$
which can be rewritten as
\[(4) \quad \frac{\|\sqrt{D}'Ax\|^2}{\|Ax\|^2} = \frac{\|\sqrt{D}x\|^2}{\|x\|^2}\]
for every nonzero \(x \in H\). Fix a vector \(y \in H\) such that \(\sqrt{D}'Ay \neq 0\). Replace \(x\) by \(x + \lambda y\) in (4) where \(x \notin \mathbb{C}y\) and \(\lambda \in \mathbb{C}\). Since \(A\) is semilinear, we obtain that
\[(5) \quad \frac{\|\sqrt{D}'(Ax + h(\lambda)Ay)\|^2}{\|Ax + h(\lambda)Ay\|^2} = \frac{\|\sqrt{D}(x + \lambda y)\|^2}{\|x + \lambda y\|^2}.
\]
We now prove that \(h\) is bounded in a neighbourhood of 0. In fact, if this is not the case, then there exists a sequence \((\lambda_n)\) in \(\mathbb{C}\) such that \(\lambda_n \to 0\) and \(|h(\lambda_n)| \to \infty\) as \(n \to \infty\). Putting these \(\lambda_n\)’s in the place of \(\lambda\) in (5) and then taking limit, it follows that
\[\frac{\|\sqrt{D}'Ay\|^2}{\|Ay\|^2} = \frac{\|\sqrt{D}x\|^2}{\|x\|^2}.
\]
Since \(y\) was fixed, the left hand side of this equation is constant. Hence, there is a positive real number \(c > 0\) such that
\[\|\sqrt{D}x\|^2 = c\|x\|^2
\]
holds for every \(x \notin \mathbb{C}y\). By continuity we have this equality for every \(x \in H\). This gives us that
\[\langle Dx, x \rangle = c\langle x, x \rangle \quad (x \in H)
\]
which yields \(D = cI\). But this case was excluded in the statement and thus we obtain that \(h\) is bounded in a neighbourhood of 0. It is known that even such a weak regularity property implies that \(h\) is either the identity or the conjugation on \(\mathbb{C}\) (one can find a detailed study of the ring automorphisms of \(\mathbb{C}\) in the book [7] including the mentioned result). Therefore, we get that \(A\) is either linear or conjugate-linear does not matter what the dimension of \(H\) is.

We next prove that \(A\) is a scalar multiple of a unitary or antiunitary operator. It follows from (3) that
\[\frac{\langle D'Ax, Ax \rangle}{\|Ax\|^2} = \frac{\langle Dx, x \rangle}{\|x\|^2}
\]
holds for every $0 \neq x \in H$. This implies that
\[
\langle D'Ax, Ax \rangle (x, x) = \langle Dx, x \rangle \langle Ax, Ax \rangle \quad (x \in H).
\]
Now, fixing $x, y \in H$ for a moment and replacing $x$ in (6) by $\lambda x + y$ where $\lambda$ runs through $\mathbb{C}$, we get polynomials of $\lambda$ and $\overline{\lambda}$ on both sides of (6) which are equal to each other. Therefore, the coefficients of the different powers of $\lambda$ and $\overline{\lambda}$ in those polynomials are also equal to each other. Comparing the coefficients of $\lambda^2$ we have
\[
\langle D'Ax, Ay \rangle (x, y) = \langle Dx, y \rangle \langle Ax, Ay \rangle
given A is linear and
\[
\langle D'Ay, Ax \rangle (x, y) = \langle Dx, y \rangle \langle Ay, Ax \rangle
given A is conjugate-linear. Consequently, in both cases we have the implication
\[
\langle x, y \rangle = 0 \implies \langle Dx, y \rangle \langle A^*Ax, y \rangle = 0.
\]
As $D$ is compact, according to the spectral theorem of compact self-adjoint operators, we can write $D = \sum \lambda_i P_i$, where the $\lambda_i$’s are nonnegative real numbers and the $P_i$’s are rank-one projections for which there exists an orthonormal basis $\{e_i\}$ in $H$ such that $P_i e_j = \delta_{ij} e_j$. Any $x \in H$ can be written as a sum $x = \sum_i c_i e_i$ with some coefficients $c_i \in \mathbb{C}$.

Suppose that $0 \neq x \in H$ is not an eigenvector of $D$. Then there exist $j, k$ such that $c_j \neq 0$, $c_k \neq 0$ and $\lambda_j \neq \lambda_k$. Set $z = \overline{c_k} e_j - \overline{c_j} e_k$. Then we have
\[
\langle x, z \rangle = \left\langle \sum_i c_i e_i, \overline{c_k} e_j - \overline{c_j} e_k \right\rangle = c_j c_k - c_k c_j = 0
\]
and
\[
\langle Dx, z \rangle = \langle x, Dz \rangle = \left\langle \sum_i c_i e_i, \lambda_j \overline{c_k} e_j - \lambda_k \overline{c_j} e_k \right\rangle = \lambda_j c_j c_k - \lambda_k c_k c_j \neq 0.
\]
These imply that for any $y \in H$ with $\langle x, y \rangle = 0$, $\langle Dx, y \rangle = 0$ we have
\[
\langle x, y + tz \rangle = 0 \quad \text{and} \quad \langle Dx, y + tz \rangle \neq 0
\]
whenever $t$ is a nonzero real number. Now, it follows from (7) that for such $y$ we get

$$\langle A^*Ax, y + tz \rangle = 0 \quad (t \neq 0).$$

Taking the limit $t \to 0$, we infer

$$\langle A^*Ax, y \rangle = 0.$$

By (7), we have this equality also in the case when $\langle x, y \rangle = 0$ and $\langle Dx, y \rangle \neq 0$. Therefore, we have proved that $\langle A^*Ax, y \rangle = 0$ holds for every $y \in H$ which is orthogonal to $x$. This readily implies that $A^*Ax$ is a scalar multiple of $x$. Remember that $x$ was an arbitrary nonzero vector in $H$ which is not an eigenvector of $D$. As the set of such vectors is dense in $H$ (this follows from the fact that $D$ is not scalar), by continuity we deduce that $A^*Ax$ is a scalar multiple of $x$ for every $x \in H$. This means that, so to say, the linear operators $A^*A$ and $I$ are locally linearly dependent.

Now, it is a folklore result whose proof requires only elementary linear algebra that for linear operators of rank at least 2, local linear dependence implies (global) linear dependence. Hence, we deduce that $A^*A = \mu I$ holds for some positive number $\mu$. Therefore, denoting $U = \frac{1}{\sqrt{\mu}}A$, we have an either unitary or antiunitary operator $U$ which clearly induces the same lattice automorphism on the collection of all closed subspaces of $H$ as $A$. It is easy to see that this implies

$$\phi(P) = UPU^* \quad (P \in \mathcal{P}(H))$$

and the proof is complete. $\square$

Remark 1. One might be interested whether the condition that $D$ is not a scalar operator can be omitted. We show that this can not be done if $\dim H \geq 2$. First we note that if $D$ is scalar (which can occur only in finite dimension), say $D = \lambda I$, then it is not hard to see that the assumptions in our statement imply that $D' = D$. In fact, the order preserving property of the transformation $\phi$ implies that it preserves the rank-one projections (more generally, the rank-$n$ projections) and then, considering the equation
(1) for such projections, we obtain that

$$\text{tr} \left( \phi(P)D' \right) = \text{tr} \left( PD \right) = \lambda \text{tr} P = \lambda \text{tr} \phi(P) = \text{tr} \left( \phi(P)(\lambda I) \right).$$

As $\phi(P)$ runs through the set of all rank-one projections, we deduce from this equality that $D' = \lambda I = D$. By the rank preserving property of $\phi$, it is now obvious that for such states (i.e., when $D' = D = \lambda I$) the equality (1) does not represent a proper condition, so all we know then is that $\phi$ is an order preserving bijection of $\mathcal{P}(H)$. Since, according to [5], such a transformation can be induced by any invertible semilinear operator, we see that in the treated case the conclusion in our theorem is no longer true.

We are in the same situation if we omit the condition $\dim H \geq 3$, that is, our statement does not remain valid when $\dim H = 2$. This is because on such a Hilbert space the only nontrivial projections are the rank-one projections, so in that case the order preserving property has no real content, it only implies that $\phi(0) = 0$ and $\phi(I) = I$. Otherwise, on the rank-one projections $\phi$ can behave arbitrarily. It is true that the equation (1) gives some restriction, but it is just an easy task to find such a transformation $\phi$ which fulfills the conditions but which does not preserve the orthogonality between the rank-one projections and hence can not be written in the form that appears in our theorem.

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