The collapsing sum
Gaussian blur on arbitrary matrices

Travis Dillon
8 December 2018

1 Introduction

Motivated by analysis of the Gaussian blur and an efficient algorithmic implementation of this image filter, we introduce a new operator on matrices called the collapsing sum. This operator takes \( m \times n \) matrices to \((m - 1) \times (n - 1)\) matrices by summing the entries of each \(2 \times 2\) block.

Section 2 gives an introduction to image filtering and the Gaussian blur in particular, and the collapsing sum is defined. In Section 3, we obtain some general results about the sum by reformulating the basic definition in terms of matrix multiplication; we also introduce the coefficient matrix associated to each sum.

Sections 4 and 5 examine the connections between the collapsing sum and Riordan arrays. [For those unfamiliar with Riordan arrays, the concept is introduced in Section 4 and further discussed in Appendix A.] The major result is Theorem 4.4, which reveals an intimate connection between the entries of the coefficient matrices and the entries of a particular set of Riordan arrays.

In Section 7, we revisit the Gaussian blur from the perspective of the collapsing sum. Theorem 7.1 shows that the collapsing sum operates on matrices identically to the Gaussian blur, which means that our work greatly extends results with the blur. In the remainder of the section, we exhibit parallels between extensions of the Gaussian blur and properties of the collapsing sum.

Sections 9 and 10 introduce and analyze a single-player game. Our previous work, including a generalization of the collapsing sum in Section 8, greatly assists with the analysis: Corollary 10.2 tells us that a special case of the generalized collapsing sum provides a test for equivalence of matrices.

The effects of the collapsing sum on the coefficient matrices motivates us to define an expanding sum in Section 11. Interestingly, Corollary 12.7 tells us that the expanding sum is invertible on the space of bi-infinite matrices, whereas the collapsing sum is not. Finally, in Section 13, we characterize the matrices fixed by the expanding and collapsing sums.

Throughout the paper, we will often need a matrix. Usually, this will be \( A \), and we will denote the entry in the \((i, j)\) position of \( A \) by \( a_{i,j} \). If we modify \( A \), we will modify its entries accordingly; for example, if we modify \( A \) to get
$A'$, the entries of $A'$ will be $a'_{i,j}$. At times, we will need other matrices; their entries will be denoted using the corresponding lowercase letter if no confusion will result.

2 Image filtering and the Gaussian blur

We begin with a description of image filtering. Grayscale images are stored as integer-valued matrices: shades of gray are stored as integers in a particular range (typically 0 to 255, although this range is by no means the only one possible), and each entry represents a pixel.

Now imagine that we want to apply a filter to a stored grayscale image. Filters are also be described by matrices and are applied to image matrices in a process called convolution. The matrix that represents the filter is called a kernel. Typically, kernels are square matrices of the form $(2r + 1) \times (2r + 1)$. The integer $r$ is called the radius of the filter. For simplicity in the convolution formula, kernels are indexed so that the central entry has coordinates $(0,0)$. Convolving the kernel $K = (k_{i,j})$ with an $m \times n$ matrix $A$ (which represents an $m \times n$ image) returns the $m \times n$ matrix $K \ast A$ with entries

$$(K \ast A)_{p,q} := \sum_{i,j=-r}^{r} k_{i,j} \cdot a_{p-i,q-j}.$$ 

We can picture this in the following way. Take the kernel matrix and flip it vertically and then horizontally. Then overlay the flipped kernel on the image such that the center of the kernel coincides with the pixel $(p,q)$. Take the product of the coinciding entries and sum these products. This gives the entry $(K \ast A)_{p,q}$. Note that we usually want $\text{sum}(K) = 1$ so that the average intensity of the image does not change.

This raises a question, though. What happens if, when we lay the flipped kernel on $A$, some of the kernel lays outside of $A$? This happens when $(p,q)$ is near an edge of the matrix. In terms of the formula, this would correspond to an “entry” of $A$ outside of the actual matrix, such as $a_{-1,0}$, which doesn’t exist. To fix this problem, we use what are called edge-handling techniques. There are two basic techniques: we can extend $A$ somehow to have values beyond its edges, or we can simply crop the pixels that would cause us to need values outside of $A$. For our purposes, we will suppose that we have chosen one of

---

1 Whether 0 represents black or white depends on the application; in printing, 0 represents white, whereas in computing, 0 represents black. We won’t need to pick between these conventions for our purposes.

2 We can also apply a filter to color images. The data for color images are stored as three separate values of red, green, and blue. To apply a filter, we can separate the data into three matrices by color type, apply the filter to each, and then recombine.

3 The filters we will work with are circularly symmetric, so the flipping steps will not be necessary here. In general, however, they are required.

4 A list of edge-handling techniques can be found at https://en.wikipedia.org/wiki/Kernel_(image_processing).
these edge-handling techniques, and we will let \( A^* \) denote the corresponding extension of \( A \). (If we choose the cropping edge-handling technique, then set \( A^* = A \).)

Blurring is one of the most common filters. These filters are also referred to as low-pass filters, since they are used to filter out high frequencies, or noise, in an image. Perhaps the simplest way to blur an image is to average nearby pixels. This is called a box blur, and the kernel \( B_{2r+1} \) for the box blur of radius \( r \) is the \((2r + 1) \times (2r + 1)\) all-ones matrix, normalized so that the entries sum to one. For example, we have

\[
B_3 = \frac{1}{3^2} \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}.
\]

As an example, suppose we have the following image.

The results of applying box blurs with radii of 1, 2, and 3, respectively, to this image are shown below.

One problem that becomes apparent with the box filter is that it weights pixels the same regardless of how distant they are from the pixel under consideration. Often, we want to weight closer pixels more heavily than distant pixels. The idea here is that pixels that are closer to each other will contain more information about each other than those that are farther away. Because of this, it is more common to use the Gaussian blur, which is often described as a “smoother” blur than the box blur. The Gaussian blur filter is based on the two-dimensional Gaussian curve

\[
f(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}.
\]

Sometimes values for the Gaussian blur are obtained directly from this function, but more often, the values are approximated using binomial coefficients. The \((2r + 1) \times (2r + 1)\) kernel \( G_{2r+1} \) of the approximate Gaussian blur has entries

\[
(G_{2r+1})_{i,j} = \frac{1}{(2r)^{i+r}} \binom{2r}{i+r} \binom{2r}{j+r}.
\]
[Recall that the central entry of a kernel is \((0, 0)\).] For example, the kernel for the \(5 \times 5\) approximate Gaussian blur is

\[
G_5 = \frac{1}{256} \begin{pmatrix}
1 & 4 & 6 & 4 & 1 \\
4 & 16 & 24 & 16 & 4 \\
6 & 24 & 36 & 24 & 6 \\
4 & 16 & 24 & 16 & 4 \\
1 & 4 & 6 & 4 & 1
\end{pmatrix}.
\]

Notice that the pixels near the center are weighted highest, and that the values, in some sense, smoothly taper off toward the edges. Applying the Gaussian blur of radii 1, 2, and 3, respectively, to our example image results in the images below.

![Example images](image1.png)

In the paper [4], Waltz and Miller develop an efficient algorithm for computing \(G_{2r+1} \ast A\) for a given matrix \(A\) and nonnegative integer \(r\). Their method is recursive and leverages the identity \(\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}\). In essence, the algorithm repeatedly adds adjacent entries of the image matrix to obtain the same effect as convolving with the kernel \(G_{2r+1}\). Their key insight is that all Gaussian blurs can be built up by adding adjacent entries. This motivates the following definition.

**Definition 2.1.** Let \(A\) be a real \(m \times n\) matrix with \(m, n \geq 2\). The \((m-1) \times (n-1)\) matrix \(\sigma(A)\) with entries \(\sigma(A)_{i,j} = a_{i,j} + a_{i,j+1} + a_{i+1,j} + a_{i+1,j+1}\) is called the **collapsing sum** of \(A\).

The remainder of this paper will focus on this operation. In Section 7, we will return to the Gaussian blur and examine the algorithm developed by Waltz and Miller in more depth to see how it connects with the collapsing sum. For now, we turn to examining the collapsing sum.

### 3 Basic properties of the collapsing sum

The definition of the collapsing sum should become intuitive with a few examples. Let’s take the simplest example: a \(2 \times 2\) matrix, say

\[
A = \begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}.
\]
Applying $\sigma$, we collapse $A$ down to the $1 \times 1$ matrix $\sigma(A) = (10)$. If we have a $3 \times 3$ matrix, say

$$B = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 3 \\ 0 & 2 & 1 \end{pmatrix},$$

then we apply $\sigma$ to get

$$\sigma(B) = \begin{pmatrix} 2 + 3 + 1 + 0 & 3 + 1 + 0 + 3 \\ 1 + 0 + 0 + 2 & 0 + 3 + 2 + 1 \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 3 & 6 \end{pmatrix}.$$ 

In short, each entry of $\sigma(A)$ is the sum of the entries of a $2 \times 2$ block of $A$.

The following properties of the collapsing sum can be proven directly from Definition 2.1.

**Proposition 3.1.** Let $A$ and $B$ be $m \times n$ matrices and $c$ any real number. Then

1. $\sigma(A + B) = \sigma(A) + \sigma(B)$,
2. $\sigma(cA) = c \cdot \sigma(A)$, and
3. $\sigma(A^T) = [\sigma(A)]^T$.

Much of the investigation will rely on repeatedly applying the collapsing sum.

**Definition 3.2.** Let $A$ be an $m \times n$ matrix. Define $\sigma^0(A) = A$, and for each positive integer $1 \leq s < \min\{m, n\}$, define $\sigma^s(A) = \sigma(\sigma^{s-1}(A))$ inductively.

With this new concept, it is natural to ask what a square matrix looks like in a “fully-collapsed” state. The following theorem answers this.

**Theorem 3.3.** Let $A$ be an $n \times n$ matrix. The value of the single entry of the matrix $\sigma^{n-1}(A)$ is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left((n-1)_{i-1} \left((n-1)_{j-1}\right) a_{i,j} \right).$$

Before proving this, we first develop better methods for analyzing the collapsing sum. Currently, we would only be able to prove this theorem directly from the definition, and this proof is essentially brute calculation. [For completeness, however, and as additional motivation for developing the following methods, the proof using only Definition 2.1 can be found in Appendix B.] As the first step in developing these methods, we define the following matrix.

**Definition 3.4.** Let $R_m$ be the $m \times (m+1)$ matrix with entries $(R_m)_{i,j} = \delta_{i,j} + \delta_{i+1,j}$.

Here and elsewhere throughout this paper, $\delta_{i,j}$ is the Kronecker delta, defined

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

The matrices $R_m$ have 1’s on the diagonal and superdiagonal and 0’s elsewhere. For example,

$$R_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$
We can use these matrices to give an equivalent and more workable definition of $\sigma$.

**Theorem 3.5.** Let $A$ be an $(m + 1) \times (n + 1)$ matrix with $m, n > 0$. Then $\sigma(A) = R_m A (R_n)^T$.

**Proof.** First note that $R_m A (R_n)^T$ is an $m \times n$ matrix. Let $S = R_m A (R_n)^T$ and choose any entry of $S$, say $s_{i,j}$. We have

$$
(R_m A)_{i,j} = \sum_{k=1}^{m+1} (R_m)_{i,k} a_{k,j}
$$

$$
= a_{i,j} + a_{i+1,j},
$$

since $(R_m)_{i,k} = 1$ when $k = i$ or $k = i + 1$ and is 0 otherwise. Thus

$$
s_{i,j} = [R_m A (R_n)^T]_{i,j}
$$

$$
= \sum_{k=1}^{n+1} (R_m A)_{i,k} (R_n)^T_{k,j}
$$

$$
= \sum_{k=1}^{n+1} [a_{i,k} + a_{i+1,k}] (R_n)_{j,k}
$$

$$
= [a_{i,j} + a_{i+1,j}] + [a_{i,j+1} + a_{i+1,j+1}]
$$

$$
= \sigma(A)_{i,j}.
$$

Since the entry of $S$ was arbitrary, $S = \sigma(A)$ and we are done.

From the proof, we can see that $R_m$ and $(R_n)^T$ essentially perform two separate functions: $R_m$ adds the rows of $A$ and $(R_n)^T$ adds the columns. The idea of separating the collapsing sum into row and column operators will form the basis of Section 6. For now, we extend the matrices $R_m$.

**Definition 3.6.** Let $m$ and $k \leq m$ be positive integers. Define $R^k_m$ as the product $R_{m-k+1} R_{m-k+2} \cdots R_m$, and let $(R_m)! = R_1 R_2 \cdots R_m = R^m_m$. Further, let $R^0_m = I_{m+1}$.

Repeated application of Theorem 3.5 shows that $\sigma^s(A) = (R^s_m) A (R^s_n)^T$ for any $(m + 1) \times (n + 1)$ matrix $A$, so it seems apt to investigate these matrices.

**Proposition 3.7.** Let $m$ be a positive integer and $s \leq m$ be a nonnegative integer. Then $R^s_m$ is an $(m-s+1) \times (m+1)$ matrix with entries $(R^s_m)_{i,j} = \binom{s}{j-i}$.

**Proof.** We proceed by induction. For $s = 0$, the theorem simplifies to the definition of $I_{m+1} = R^0_m$. Now suppose that the theorem holds for some nonnegative integer $k$. Then $R^k_m = R_{m-k} R^k_m$. Certainly this is an $(m-k) \times (m+1)$
matrix. Further, for any element \((R^k_m)_{i,j}\) we have
\[
(R^k_m)_{i,j} = \sum_{r=1}^{m-k+1} (R_{m-k})_{i,r} (R^k_m)_{r,j}
\]
\[
= \sum_{r=1}^{m-k+1} (R_{m-k})_{i,r} \binom{k}{j-r}
\]
\[
= \binom{k}{j-i} + \binom{k}{j-(i+1)}
\]
\[
= \binom{k+1}{j-i},
\]
so the formula holds by induction.

For example, we have
\[
R^3_5 = \begin{pmatrix}
1 & 3 & 3 & 1 & 0 & 0 \\
0 & 1 & 3 & 3 & 1 & 0 \\
0 & 0 & 1 & 3 & 3 & 1
\end{pmatrix}.
\]

The next corollary allows us to quickly prove Theorem 3.3.

**Corollary 3.8.** Let \(m\) be a positive integer. Then \((R_m)!\) is a \(1 \times (m+1)\) matrix with the entry \(\binom{m}{j-1}\) in column \(j\).

**Proof of Theorem 3.3** Let \(n\) be a positive integer and \(A\) be an \(n \times n\) matrix. Then \(\sigma^n(A) = [(R_{n-1})!]A[(R_{n-1})!]^T\) and the single entry of \(\sigma^n(A)\) is
\[
\sigma^n(A)_{1,1} = \sum_{i=1}^{n} \sum_{j=1}^{n} [(R_{n-1})!]_{i,j} a_{i,j} [(R_{n-1})!]_{j,1}^T
\]
\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} \binom{n-1}{i-1} \binom{n-1}{j-1} a_{i,j},
\]
which is what Theorem 3.3 states.

Compare this proof with the one in Appendix B. Introducing the matrices \(R_m\) significantly shortened and clarified the proof. As an example of Theorem 3.3 we take the identity matrices.

**Proposition 3.9.** The single entry in the fully-collapsed identity matrix \(\sigma^n(I_{n+1})\) is the central binomial coefficient \(\binom{2n}{n}\).

**Proof.** We note that the \((i,j)\)th entry of \(I_{n+1}\) is 1 if \(i = j\) and 0 otherwise.
Therefore we have

\[
\sigma^n(I_{n+1})_{1,1} = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (I_{n+1})_{i,j} \binom{n}{i-1} \binom{n}{j-1}
\]

\[
= \sum_{i=1}^{n+1} \binom{n}{i-1} \binom{n}{i-1}
\]

\[
= \sum_{i=1}^{n+1} \binom{n}{i-1} \binom{n}{n-i+1}
\]

\[
= \binom{2n}{n}.
\]

The last equality can be justified by the following counting argument. In choosing \(n\) items from \(2n\), we can first divide the \(2n\) items into two groups of \(n\) items. Then, for any \(i\) between 1 and \(n+1\), we can choose \(i-1\) items from the first group and \(n-i+1\) items from the second. Summing over all \(i\) from 1 to \(n+1\) gives the total number of ways to choose \(n\) items from \(2n\).

It is natural to next ask what the intermediate matrices look like. That is, for some \(m \times n\) matrix \(A\), we want to examine at the matrices \(\sigma^s(A)\) for each nonnegative integer \(s < \min\{m, n\}\). This is easily reducible to the previous problem, however, since each square block of order \(s+1\) is collapsed to one entry of \(\sigma^s(A)\), so we simply apply Theorem \[3.3\] to each \((s+1) \times (s+1)\) block of \(A\) to get the matrix \(\sigma^s(A)\). We can, however, still associate a single number with \(\sigma^s(A)\) by taking the sum over the elements of \(\sigma^s(A)\). This leads to a new problem and the following definition.

**Definition 3.10.** Let \(m\) and \(n\) be positive integers and \(s < \min\{m, n\}\) be a nonnegative integer. The **coefficient matrix** \(C^s_{m \times n}\) is the \(m \times n\) matrix with entries \(c_{i,j}\) such that for all \(m \times n\) matrices \(A\), we have \(\sum(\sigma^s(A)) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} a_{i,j}\).

If we have a particular \(m \times n\) matrix \(A\), we will sometimes write “the coefficient matrix for \(\sigma^s(A)\)” instead of \(C^s_{m \times n}\). We know that such a matrix exists and is unique, because we can obtain the values of its entries by simply running through calculations using the definition of the collapsing sum. To a certain extent, we can think of \(c_{i,j}\) as expressing how many times the entry \(a_{i,j}\) is represented in \(\sigma^s(A)\). If the context makes it clear, we may omit the superscript or subscript on \(C^s_{m \times n}\).

Finding a formula for \(\sum(\sigma^s(A))\) and finding the coefficient matrices are equivalent tasks. The reason for the restatement of this problem is that, by extracting the coefficients into a matrix, we can take advantage of the matrix methods we have developed thus far. To proceed, we will need to introduce the row sum and column sum vectors.

**Definition 3.11.** Let \(A\) be an \(m \times n\) matrix. The **column sum vector** \(\alpha\) of \(A\) has elements \(\alpha_j = \sum_{i=1}^{m} a_{i,j}\). The **row sum vector** \(\beta\) of \(A\) has elements \(\beta_i = \sum_{j=1}^{n} a_{i,j}\).
That is, the $j$th element of the column sum vector is the sum of the elements in the $j$th column of $A$. In this paper, we will take the row sum and column sum vectors to be column vectors. The next theorem uses these vectors to describe how the coefficient matrices relate to the matrices $R_m$.

**Lemma 3.12.** Let $A = (a_{i,j})$ be a matrix of indeterminates and $B$ and $C$ be matrices such that the product $BAC$ is defined. If $\beta$ is the column sum vector of $B$ and $\gamma$ is the row sum vector of $C$, then the coefficient of $a_{i,j}$ in the expression $\text{sum}(BAC)$ is $\beta_i \gamma_j$.

**Proof.** Choose any indeterminate $a_{p,q}$. We have

$$\text{sum}(BAC) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left[ \sum_{k=1}^{m} \sum_{r=1}^{n} b_{i,k} \cdot a_{k,r} \cdot c_{r,j} \right].$$

We obtain the coefficient on $a_{p,q}$ by setting $k = p$ and $r = q$. This coefficient is thus

$$\sum_{i=1}^{m} \sum_{j=1}^{n} B_{i,p} C_{q,j} = \left[ \sum_{i=1}^{m} b_{i,p} \right] \left[ \sum_{j=1}^{n} c_{q,j} \right].$$

But the left term in this product is just $\beta_p$, and the right term is just $\gamma_q$, so the coefficient on $a_{p,q}$ is $\beta_p \gamma_q$. Since $p$ and $q$ were arbitrary, we are done. \qed

**Theorem 3.13.** Let $\alpha$ be the column sum vector of $R_{m-1}^s$ and $\beta$ be the column sum vector of $R_{n-1}^s$. Then $C_{m \times n}^s = \alpha \beta^T$.

**Proof.** Let $\alpha$ and $\beta$ be as stated in the theorem, and let $A$ be any $m \times n$ matrix. Because $\sigma^s(A) = (R_{m-1}^s)A(R_{n-1}^s)^T$, we can apply Lemma 3.12. Note that the row sum vector of $(R_{m-1}^s)^T$ is simply the column sum vector of $R_{n-1}^s$. By Lemma 3.12 the $(i,j)$th entry of the coefficient matrix for $\sigma^s(A)$ is $\alpha_i \beta_j$. But $(\alpha \beta^T)_{i,j} = \alpha_i \beta_j$, and since the coefficient matrix is independent of the entries of $A$, this shows that $C_{m \times n}^s = \alpha \beta^T$. \qed

Note that Theorem 3.13 implicitly gives a formula for the coefficient matrix of $\sigma^s(A)$. The following corollary makes this formula explicit.

**Corollary 3.14.** Let $m$ and $n$ be positive integers, $s < \min\{m, n\}$ be a nonnegative integer, and $C_{m \times n}^s = (c_{i,j})$. Then $c_{i,j} = \left[ \sum_{\ell=1}^{m-s} \binom{s}{i-\ell} \right] \left[ \sum_{\ell=1}^{n-s} \binom{s}{j-\ell} \right]$.

And now, for just a bit of fun, since the coefficient matrices describe the sum of the entries of $\sigma^s(A)$, we examine the sum of the entries of $C_{m \times n}^s$.

**Lemma 3.15.** Let $m$ be a positive integer and $s < m$ be a nonnegative integer. Then $\text{sum}(R_{m}^s) = 2^s(m-s+1)$. 

9
Proof. We have

\[
\text{sum}(R^s_m) = \sum_{i=1}^{m-s+1} \sum_{j=1}^{m+1} \binom{s}{j-i} \\
= \sum_{i=1}^{m-s+1} \left[ \sum_{j=1}^{i-1} \binom{s}{j-i} + \sum_{j=1}^{i+s} \binom{s}{j-i} + \sum_{j=i+s+1}^{m+1} \binom{s}{j-i} \right] \\
= \sum_{i=1}^{m-s+1} \left[ 0 + 2^s + 0 \right] \\
= 2^s(m - s + 1).
\]

Note that \(1 \leq i \leq i + s \leq m + 1\), since \(1 \leq i \leq m - s + 1\) and \(s \geq 0\), so breaking the sum into three parts is justified. \(\square\)

**Proposition 3.16.** Let \(A\) be an \(m \times n\) matrix and \(s < \min\{m, n\}\) be a positive integer. Then \(\text{sum}(C^s_{m \times n}) = 4^s(m - s)(n - s)\).

**Proof.** Let \(A\) be \(m \times n\) and \(s < \min\{m, n\}\) be a nonnegative integer. Further, let \(\alpha\) be the column sum vector of \(R^s_{m-1}\) and \(\beta\) be the column sum vector of \(R^s_{n-1}\). We need to calculate \(\text{sum}(\alpha\beta^T)\). This is

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} (\alpha\beta^T)_{i,j} = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i\beta_j = \left[ \sum_{i=1}^{m} \alpha_i \right] \left[ \sum_{j=1}^{n} \beta_j \right].
\]

But the sum of the entries of a column sum vector is just the sum over all the entries of its corresponding matrix, so

\[
\text{sum}(\alpha\beta^T) = [\text{sum}(R^s_{m-1})] [\text{sum}(R^s_{n-1})] \\
= [2^s(m - s)] [2^s(n - s)] \\
= 4^s(m - s)(n - s),
\]

as claimed. \(\square\)

We close this section by exhibiting all coefficient matrices for a 5 \(\times\) 5 matrix.

[Note that \(C^4 = C^3\).]

\[
C^0 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix} \quad C^1 = \begin{pmatrix}
1 & 2 & 2 & 2 & 1 \\
2 & 4 & 4 & 4 & 2 \\
2 & 4 & 4 & 4 & 2 \\
2 & 4 & 4 & 4 & 2 \\
1 & 2 & 2 & 2 & 1
\end{pmatrix}
\]

\[
C^2 = \begin{pmatrix}
1 & 3 & 4 & 3 & 1 \\
3 & 9 & 12 & 9 & 3 \\
4 & 12 & 16 & 12 & 4 \\
3 & 9 & 12 & 9 & 3 \\
1 & 3 & 4 & 3 & 1
\end{pmatrix} \quad C^3 = \begin{pmatrix}
1 & 4 & 6 & 4 & 1 \\
4 & 16 & 24 & 16 & 4 \\
6 & 24 & 36 & 24 & 6 \\
4 & 16 & 24 & 16 & 4 \\
1 & 4 & 6 & 4 & 1
\end{pmatrix}
\]
4 The coefficients as Riordan arrays

Corollary 3.14 tells us that the entries of the coefficient matrices have the form
\[ \sum_{\ell=1}^{m-s} \binom{s}{i-\ell} \binom{n-s}{j-\ell}. \]
Therefore, we will study the sum \( \sum_{\ell=1}^{m-s} \binom{s}{i-\ell} \), since any information gained about this sum will translate directly into information about the entries of the coefficient matrices. To simplify the notation in this section, we will use \( t \) in place of \( m-s \). Specifically, we will examine what happens when we fix \( t \), so let us denote by \( C_t(s,i) \) the sum \( \sum_{\ell=1}^{t} \binom{s}{i-\ell} \).

When we fix \( t \) as a nonnegative integer, we can vary \( s \) and \( i \), and we represent this with a “triangular” array, as shown below, for the case \( t = 3 \).

\[
\begin{array}{cccccc}
0 & 1 & 1 & 1 & & \\
1 & 1 & 2 & 2 & 1 & \\
2 & 1 & 3 & 4 & 3 & 1 \\
3 & 1 & 4 & 7 & 7 & 4 & 1 \\
4 & 1 & 5 & 11 & 14 & 11 & 5 & 1 \\
5 & 1 & 6 & 16 & 25 & 25 & 16 & 6 & 1 \\
\end{array}
\]

Each fixed \( s \) in the array is a row, and each fixed \( i \) is a column. The entire array is denoted by \( C_t \). This figure, of course, is truncated at the level \( s = 5 \), but the array extends infinitely. Since for each fixed \( t \geq 1 \) we have one of these arrays, this gives us an infinite family of “triangular” arrays to explore. We begin by examining the rows of these arrays.

**Proposition 4.1.** Let \( C_t \) be given. Then \( C_t(s,i) \) is the coefficient of \( x^i \) after expanding \((x + x^2 + \cdots + x^t)(1 + x)^s\).

**Proof.** We proceed by induction on \( s \). In the first row of \( C_t \), we have \( s = 0 \). The entries of this row are \( \sum_{\ell=1}^{t} \binom{0}{i-\ell} = 1 \) for all \( 1 \leq i \leq t \) and 0 otherwise. Therefore the theorem holds when \( s = 0 \). Now assume that the theorem holds for some integer \( k \geq 0 \). We have

\[
C_t(k,i) + C_t(k,i+1) = \sum_{\ell=1}^{t} \binom{k}{i-\ell} + \sum_{\ell=1}^{t} \binom{k}{i+1-\ell} \\
= \sum_{\ell=1}^{t} \left[ \binom{k}{i-\ell} + \binom{k}{i+1-\ell} \right] \\
= \sum_{\ell=1}^{t} \binom{k+1}{i+1-\ell} \\
= C_t(k+1,i+1).
\]
Since \( C_t(k, i) \) and \( C_t(k, i + 1) \) are the coefficients on \( x^i \) and \( x^{i+1} \) respectively in the expression \((x + x^2 + \cdots + x^t)(1 + x)^k\), we see that \( C_t(k + 1, i + 1) \) is the coefficient on \( x^{i+1} \) in the expression

\[
(1 + x)[(x + x^2 + \cdots + x^t)(1 + x)^k] = (x + x^2 + \cdots + x^t)(1 + x)^{k+1},
\]

so we are done.

Thus fixing \( s \) and \( t \), expanding the polynomial \((x + x^2 + \cdots + x^t)(1 + x)^s\) and picking off the coefficients returns row \( s \) of \( C_t \).

We now introduce a concept that will give us an alternative way of describing these arrays.

**Definition 4.2.** Let \( f \) and \( g \) be ordinary power series generating functions with \( g(0) = 1 \), \( f(0) = 0 \), and \( f'(0) = 1 \). The Riordan array \((g, f)\) is an infinite matrix whose entries in column \( k \) are given by the generating function \( gf^k \). That is, the entry in the \((n, k)\)th position is \([x^n]g(x)[f(x)]^k\).

When no confusion will arise, we will denote the \((n, k)\) entry of \((g, f)\) by \( d_{n,k} \). In a Riordan array, the rows and columns are indexed starting at 0. The requirements on the functions \( g \) and \( f \) are not strictly necessary; the definition above describes so-called proper Riordan arrays, and some of the restrictions can be relaxed. The reasoning behind these restrictions, as well as Riordan arrays in general, is discussed in Appendix A.

Riordan arrays were first introduced as a generalization of Pascal’s triangle, and have found use in combinatorial sums; see, for example, the article \[3\]. They have since also been studied in their own right.

From the restrictions in Definition 4.2, the power series \( g \) and \( f \) look like

\[
g = 1 + a_1 x + a_2 x^2 + \cdots + a_n x^n \quad \text{and} \quad f = x + b_2 x^2 + \cdots + b_m x^m.
\]

Recalling that the generating function for column \( k \) of the array \((g, f)\) is \( gf^k \), we see that every Riordan array is lower triangular and has only 1’s on the main diagonal. As an example, Pascal’s triangle is the array \((\frac{1}{1-x}, \frac{x}{1-x})\). For a second example, take the array \((\frac{1}{1-x}, x)\). Both are shown below.

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & \cdots \\
1 & 2 & 1 & 0 & 0 & \cdots \\
1 & 3 & 3 & 1 & 0 & \cdots \\
1 & 4 & 6 & 4 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
2 & 1 & 0 & 0 & 0 & \cdots \\
3 & 2 & 1 & 0 & 0 & \cdots \\
4 & 3 & 2 & 1 & 0 & \cdots \\
5 & 4 & 3 & 2 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Associated with each Riordan array \((g, f) = (d_{n,k})\) is its corresponding \(A\)-sequence, a sequence of integers \((a_0, a_1, a_2, \ldots)\) such that \(a_0 \neq 0 \) and for each \( n, k \geq 0 \),

\[
d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots = \sum_{j=0}^{\infty} a_j d_{n,k+j}.
\]

12
Since \(d_{n,k} = 0\) if \(n > k\), the above sum is actually finite. It seems plausible that other infinite matrices besides Riordan arrays may have an associated A-sequence, that is, a sequence that satisfies the above equation. However, in [1], He and Sprugnoli proved the following result.

**Theorem 4.3.** An infinite lower triangular array is a Riordan array if and only if it has an A-sequence.

In the proof of Proposition 4.1, we showed that the arrays \(C_t\) satisfy the same recurrence relation as Pascal’s triangle; that is, for \(s, i \geq 0\) and \(t \geq 1\), we have

\[
C_t(s + 1, i + 1) = C_t(s, i) + C_t(s, i + 1).
\]

If the array \(C_t\) were an infinite lower triangular array, then this would mean that its A-sequence is \((1, 1, 0, 0, 0, \ldots)\). However, for \(t \neq 1\), the array \(C_t\) is not strictly a triangle. We can imagine adding rows above \(C_t\) to create a lower triangular matrix, and perhaps if we chose the entries of these new rows carefully, so that the A-sequence is maintained and the main diagonal contains only 1’s, we could build a Riordan array that contains \(C_t\). In fact we can, and it is unique. The following theorem gives the exact array.

**Theorem 4.4.** Let \(t\) and \(s\) be nonnegative integers. Further, let \(g_t(x) = \frac{(1-x)^t - x^t}{(1-x)(1-2x)}\) and \(f(x) = \frac{x}{1-x}\). Then \(C_t(s, i) = (g_t, f)_{s+t-1, i-1}\).

Instead of proving this through the extraction of coefficients by direct calculation, we instead establish a correspondence, for which we will need the following two lemmas. The proofs of these lemmas are somewhat tedious, so they can be found in Appendix B.

**Lemma 4.5.** Let \(t\) be a positive integer and \(g_t\) and \(f\) be as in Theorem 4.4. Then \((g_t, f)_{n,0} = 1\) for all \(n \geq t - 1\).

**Lemma 4.6.** Let \(t\) be a positive integer and \(g_t\) and \(f\) be as in Theorem 4.4. Then \((g_t, f)_{t-1, k} = 1\) for all \(0 \leq k < t\) and \((g_t, f)_{t-1, k} = 0\) for all \(k \geq t\).

We are now in a position to prove Theorem 4.4.

**Proof of Theorem 4.4.** Recall that \(C_t\) satisfies the Pascal recurrence relation. We will show that \((g_t, f) = (d_{n,k})\) also satisfies this recurrence. We have

\[
d_{n+1,k+1} = [x^{n+1}] \frac{x^k}{1-x} \frac{(1-x)^t - x^t}{(1-x)^k(1-2x)}
\]

\[
= [x^n] \frac{1}{1-x} \sum_{j=0}^{\infty} d_{j,k} x^j
\]

\[
= \sum_{j=0}^{n} d_{k,j}.
\]
Thus
\[d_{n,k} + d_{n,k+1} = d_{n,k} + \sum_{j=0}^{n-1} d_{k,j}\]
\[= \sum_{j=0}^{n} d_{k,j}\]
\[= d_{n+1,k+1}.
\]

Therefore the elements of \((g_t, f)\) satisfy the Pascal recurrence. From Lemmas 4.5 and 4.6 we know that row \(t-1\) of \((g_t, f)\) is identical to the first row of \(C_t\) [where \(s = 0\)], and column 0 of \((g_t, f)\), starting at row \(t-1\), corresponds to the first column of \(C_t\) [where \(i = 1\)]. Since, for each array, this row and column along with the Pascal recurrence are sufficient to determine the entire array, the two arrays must correspond. In general, row \(s\) of \(C_t\) corresponds to row \(s+t-1\) of \((g_t, f)\), and column \(i\) of \(C_t\) corresponds to column \(i-1\) of \((g_t, f)\). Therefore, for all \(i \geq 1\) and \(s \geq 0\), we have \(C_t(s,i) = (g_t, f)_{s+t-1,i-1}\), as desired.

It is also proved in [1] that the A-sequence of the Riordan array \((g,f)\) depends only on \(f\). In particular, if \(\bar{f}\) denotes the compositional inverse of \(f\), then the generating function of the A-sequence is
\[A(x) = \frac{x}{f(x)}.
\]

To make the proof self-contained, we did not reference this result. However, we can use it to more quickly prove that the Riordan array in Theorem 4.4 satisfies the Pascal recurrence, as follows. Let \(f = x^1 - x^2\). Then \(\bar{f} = \frac{x}{1+x}\) and the generating function for the A-sequence is
\[\frac{x}{1+x} = 1 + x.
\]
Therefore the array has A-sequence \((1,1,0,0,0,\ldots)\), so it satisfies the Pascal recurrence.

The important notion of Theorem 4.4 is that the arrays \(C_t\) are essentially Riordan arrays. In fact, the concept of deleting the upper rows from a Riordan array is not uncommon. We will see more of this in Section 5. After encountering a Riordan array, it is common to examine its row sums and diagonal sums, and we turn to that next.

**Proposition 4.7.** Let \(t\) and \(s\) be positive integers. The sum of the elements of row \(s\) in \(C_t\) is \(t2^s\).

**Proof.** By Proposition 4.1, the elements in row \(s\) of \(C_t\) are the coefficients of \((x + x^2 + \cdots + x^t)(1 + x)^s\) after expansion. We can find the sum of these coefficients by substituting \(x = 1\) to obtain \(t2^s\).

Before determining the diagonal sums, we first make the notion precise.
Definition 4.8. Let \( t \) and \( r \) be nonnegative integers. The \( r \)th diagonal sum of \( C_t \) is \( \sum_{i=0}^{r} C_t(r - i, i + 1) \).

The reader can check that this actually does correspond to the sum of the diagonal entries.

Proposition 4.9. Let \( t \) be a nonnegative integer and let \( F_k \) denote the \( k \)th Fibonacci number, with \( F_0 = F_1 = 1 \). If \( r \geq t - 2 \), then the \( r \)th diagonal sum of \( C_t \) is \( F_{r+2} - F_{r-t+2} \), and if \( r < t - 2 \), then the \( r \)th diagonal sum is \( F_{r+2} - 1 \).

Proof. The \( r \)th diagonal sum of \( C_t \) is given by

\[
\sum_{i=0}^{r} C_t(r - i, i + 1) = \sum_{i=0}^{r} \sum_{\ell=1}^{t} \binom{r - i - 1}{i + 1 - \ell}.
\]

We first switch the order and manipulate the limits of summation to yield

\[
\sum_{\ell=0}^{t-1} \sum_{i=0}^{r - \ell} \binom{r - i - \ell}{i},
\]

where we substituted \( \ell + 1 \) for \( \ell \) and then \( i + \ell \) for \( i \). Of course, the expression \( \binom{r - i - \ell}{i} \) is 0 when \( i < 0 \), so we can restrict the bounds to give

\[
\sum_{\ell=0}^{t-1} \sum_{i=0}^{r - \ell} \binom{r - \ell - i}{i}.
\]

The inner summation is just the \((r - \ell)\)th diagonal sum of Pascal’s Triangle, so it is \( F_{r-\ell} \), as long as \( r - \ell \geq -1 \).

Suppose \( r \geq t - 2 \). Then \( r \geq \ell - 1 \) for all values of \( \ell \), so the \( r \)th diagonal sum is

\[
\sum_{\ell=0}^{t-1} F_{r-\ell} = \sum_{k=0}^{r} F_k - \sum_{k=0}^{r-t} F_k = F_{r+2} - F_{r-t+2},
\]

as desired.

Finally, suppose that \( r < t - 2 \). Then the inner summation is empty for all \( \ell \geq r \), so the sum reduces to

\[
\sum_{\ell=0}^{r} F_{r-\ell} = \sum_{\ell=0}^{r} F_{\ell} = F_{r+2} - 1.
\]

Now, a theorem of Riordan arrays states that the generating function of the diagonal sums of array \((g, f)\) is \( \frac{g}{1 - xf} \). \( \Box \) [See Appendix A for a discussion of row \( g \).]
and diagonal sums of Riordan arrays.] If we apply that in our case, we get a diagonal sum generating function of
\[
g_t = \frac{(1-x)^t - x^t}{1-x} \frac{(1-2x)}{(1-x-x^2)}. \]

Extracting the coefficients from this generating functions would be no easier, and certainly no more enlightening, than the approach in the proof of Proposition 4.9, which explains why we did not use a Riordan array method.

Another way to view these arrays is as overlapped copies of Pascal’s Triangle. This is clear, of course, when \( t = 1 \), because \( C_1 \) is exactly Pascal’s Triangle. We rewrite the diagram given at the beginning of this section for \( t = 3 \) to make this clearer in general.

| s/i | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 1+0+0 | 0+1+0 | 0+0+1 |
| 1   | 1+0+0 | 1+1+0 | 0+1+1 | 0+0+1 |
| 2   | 1+0+0 | 2+1+0 | 1+2+1 | 0+1+2 | 0+0+1 |
| 3   | 1+0+0 | 3+1+0 | 3+3+1 | 1+3+3 | 0+1+3 | 0+0+1 |
| 4   | 1+0+0 | 4+1+0 | 6+4+1 | 4+6+4 | 1+4+6 | 0+1+4 | 0+0+1 |

Each “cell” contains the sum of three numbers and, if we consider only the first of these three numbers in each cell, what remains is Pascal’s Triangle. Similarly, considering only the second or third number, we again have with Pascal’s Triangle. This makes sense if we consider the recurrence relation on the array. We know that the arrays \( C_t \) follow the Pascal recurrence. If we consider each entry of the first row individually and use the recurrence relation on each entry, adding together entries that overlap in the array, we end up with the version of the array shown above. Therefore we can also construct \( C_t \) by marking off \( t \) positions in a row and “dropping” a copy of Pascal’s Triangle on each one, adding together the values of the entries from the various copies that share a cell. This gives some insight into the proof of Proposition 4.9. We essentially considered the diagonal sum of each of the copies of Pascal’s Triangle that comprise \( C_t \) and summed them to obtain our result.

In the next section, we analyze the effects of the collapsing sum on Riordan arrays.

5 Collapsing sums of Riordan arrays

As in Section 3, although we could approach this topic through direct calculation, a better approach will again be to develop matrix techniques. Therefore, we introduce the following infinite analogue of the \( R_n \) matrices.

**Definition 5.1.** The infinite matrix \( R_\infty \) has entries \((R_\infty)_{i,j} = \delta_{i,j} + \delta_{i+1,j}\) for all nonnegative integers \( i \) and \( j \).

One salient difference between \( R_m \) and \( R_\infty \) is that while in the former case we defined a sort of falling power, in this case we will be able to use ordinary
powers. Although our original definition of the collapsing sum only directly applies to finite matrices, it can be readily extended to infinite matrices, so we skip placing an extended definition here.

We will often need to multiply Riordan arrays in this section. It turns out that the arrays can be multiplied with reference only to the generating functions that determine them.

**Theorem 5.2.** Let \((g, f)\) and \((u, v)\) be Riordan arrays. Then \((g, f) \cdot (u, v) = (g \cdot u(f), v(f))\).

A proof of this can be found in [2]. We have one final preliminary. If \((g, f)\) is a Riordan array, often \((g, f)\) is used to denote infinite matrix defined by 
\[
\begin{align*}
(g, f)_{i+1,j} &= (g, f)_{i,j+1}. 
\end{align*}
\]
That is, removing the top row of \((g, f)\) results in the array \((g, f)\). However, we will need to extend this concept, and direct use of this notation would be cumbersome. Therefore we introduce the following matrix.

**Definition 5.3.** Let \(U\) be the infinite matrix with entries 
\[
U_{i,j} = \delta_{i+j+1, j}
\]
for all nonnegative integers \(i\) and \(j\).

Thus \(U\) is the infinite matrix with 1’s on the superdiagonal and 0’s elsewhere. Note that \((U(g, f)) = (g, f)\) and \((U(g, f)^2) = (g, f)^2\), and so on. We begin with the following lemma.

**Lemma 5.4.** The infinite matrix \((R_{\infty})^T\) is the Riordan array \((1 + x, x)\). Further, \(R_{\infty} = U(1 + x, x)\).

**Proof.** The matrix \((R_{\infty})^T\) has entries 
\[
[(R_{\infty})^T]_{i,j} = 1 \text{ when } i = j \text{ or when } i = j + 1 \text{ and } [(R_{\infty})^T]_{i,j} = 0 \text{ otherwise.}
\]
Thus the generating function \((1 + x)x^j\) represents the \(j\)th column of \((R_{\infty})^T\). By definition, we have \((R_{\infty})^T = (1 + x, x)\).

Now notice that
\[
(R_{\infty})_{i,j} = \delta_{i,j} + \delta_{i+1,j} = \delta_{j+1,i+1} + \delta_{j,i+1} = (R_{\infty})_{j,i+1} = [(R_{\infty})^T]_{i+1,j}.
\]
Therefore \(R_{\infty} = U(R_{\infty})^T = U(1 + x, x)\).

In a similar manner to Theorem 3.5, we have \(\sigma(g, f) = (R_{\infty})(g, f)(R_{\infty})^T\).

From Lemma 5.4 we can translate this to \(\sigma(g, f) = U(1 + x, x)(g, f)(1 + x, x)\). We extend this with the following theorem.

**Theorem 5.5.** Let \(s\) be a nonnegative integer and \((g, f)\) a Riordan array. Then \(\sigma^s(g, f) = U^s(g(1 + f))^s(1 + x)^s, f)\).

**Proof.** Induction using Theorem 5.2 shows that \((1 + x, x)^n = ((1 + x)^n, x)\). Thus, since the matrices \(U\) and \((1 + x, x)\) do not commute, we first show that 
\[
[U(1 + x, x)]^n = U^n((1 + x)^n, x).
\]
Because \(U(1 + x, x) = (1 + x, x)^T\), we have 
\[
[U(1 + x, x)]^n = [(1 + x, x)^T]^n = [(1 + x, x)^n] = ((1 + x)^n, x)^T.
\]
Now take any element
\[ [(1 + x)^n, x]^T]_{i,j} = ((1 + x)^n, x)_{j,i} = \binom{n}{j - i}. \]
The corresponding element in \( U^n((1 + x)^n, x) \) is
\[ [U^n((1 + x)^n, x)]_{i,j} = ((1 + x)^n, x)_{i+n,j} = \binom{n}{i + n - j}. \]
Since \( \binom{n}{j - i} = \binom{n}{i + n - j} \), it follows that
\[ [U((1 + x, x))^n] = ((1 + x)^n, x)^T = U^n((1 + x)^n, x). \]
With this in place, we can write
\[ \sigma^s(g, f) = [U((1 + x, x))^s(g, f)(1 + x, x)^s \]
\[ = U^s((1 + x)^s, x)(g, f)((1 + x)^s, x) \]
\[ = U^s((1 + x)^s, x)(g(1 + f)^s, f) \]
\[ = U^s(g(1 + f)^s(1 + x)^s, f), \]
which is what we needed.

We now construct the coefficient matrix for the collapsing sums of Riordan arrays. We can extend Definition 3.10 to the infinite case by finding the coefficient matrix for a bi-infinite matrix of distinct indeterminates. Summing over indeterminates is purely formal, so we do not have to worry about issues of convergence. The following theorem finds the coefficient matrix for bi-infinite matrices, of which Riordan arrays are a special case. Note that in Theorem 5.6 we index the matrix \( C \) beginning at 0.

**Theorem 5.6.** Let \( s \) be a nonnegative integer. The coefficient matrix \( C^s \) for any bi-infinite matrix has entries \( c_{i,j} = \sum_{\ell=0}^{i} \binom{s}{\ell} \sum_{\ell=0}^{j} \binom{s}{\ell} \) for all nonnegative integers \( i \) and \( j \).

**Proof.** Let \( \alpha \) be the column sum vector of \( R^s_\infty \). Then \( \alpha \) is also the row sum vector of \( (R^s_\infty)^T = ((1 + x)^s, x) \). Thus
\[ \alpha_i = \sum_{\ell=0}^{\infty} [x^\ell](1 + x)^s x^\ell = \sum_{\ell=0}^{\infty} \binom{s}{i - \ell} = \sum_{\ell=0}^{i} \binom{s}{\ell}. \]
From a straightforward application of Lemma 3.12 we see that the \((i, j)\)th entry of \( C^s \) is \( \alpha_i \alpha_j \), from which the conclusion follows swiftly.

This concludes our application of \( \sigma \) to Riordan arrays. However, this has again brought up the idea of “splitting” \( \sigma \). Looking back, we see that there are two distinct “parts” of applying \( \sigma \): the matrix \( U^s((1 + x)^s, x) \) and the Riordan array \( ((1 + x)^s, x) \), corresponding to adding the entries in the columns and rows, respectively, of the matrix to which we apply them. We also saw this with the matrices \( R_m \) and \( (R_m)^T \) earlier. In the next section, we examine this idea more closely.
6 Splitting $\sigma$

In this section, we study the decomposition of the collapsing sum into two operators, which our next definition introduces.

**Definition 6.1.** Let $A$ be an $m \times n$ matrix. Then $\sigma_\downarrow(A)$ is an $m \times (n-1)$ matrix with entries $\sigma_\downarrow(A)_{i,j} = a_{i,j} + a_{i,j+1}$. Also, $\sigma_\uparrow(A)$ is an $(m-1) \times n$ matrix with entries $\sigma_\uparrow(A)_{i,j} = a_{i,j} + a_{i+1,j}$.

We can see immediately that $\sigma(A) = \sigma_\downarrow(\sigma_\uparrow(A)) = \sigma_\uparrow(\sigma_\downarrow(A))$.

Further, if $A$ is $(m+1) \times (n+1)$, then $\sigma_\uparrow(A) = R_m A(1)$ and $\sigma_\downarrow(A) = A(R_n)^T$. It is straightforward to extend the coefficient matrices to this more general setting, and the following theorem is the corresponding extension of Corollary 3.14.

**Proposition 6.2.** Let $A$ be an $m \times n$ matrix and let $a$ and $b$ be positive integers with $a < m$ and $b < n$. Then the coefficient matrix for $\sigma_a \downarrow \sigma_b \uparrow(A)$ has entries $c_{i,j} = \sum_{\ell=1}^{m-a} \left( a_{i,j} - \ell \right) \sum_{\ell=1}^{n-b} \left( b_{j-\ell} \right)$.

**Proof.** We have $\sigma_a \downarrow \sigma_b \uparrow(A) = R_{m-1}^a A(R_{n-1}^b)^T$. Let $\alpha$ be the column sum vector of $R_{m-1}^a$ and $\beta$ be the column sum vector of $R_{n-1}^b$. From Lemma 3.12, the coefficient matrix for $\sigma_a \downarrow \sigma_b \uparrow(A)$ is $\alpha \beta^T$. Proposition 3.7 implies that $\alpha_j = \sum_{i=1}^{m-a} (R_{m-1}^a)_{i,j} = \sum_{i=1}^{m-a} \left( a_{j-i} \right)$, and a similar formula holds for $\beta_j$. The terms of the coefficient matrix are thus $(\alpha \beta^T)_{i,j} = \alpha_i \beta_j = \left[ \sum_{\ell=1}^{m-a} \left( a_{i-\ell} \right) \sum_{\ell=1}^{n-b} \left( b_{j-\ell} \right) \right]$, as stated. \qed

When working with the collapsing sum, we could associate a “maximally collapsed” $1 \times 1$ matrix only with square matrices. With these new operations, we can associate one with all matrices.

**Corollary 6.3.** Let $A$ be an $m \times n$ matrix. The single entry in the matrix $\sigma_m \downarrow \sigma_n \uparrow(A)$ is $\sum_{i=1}^{m} \sum_{j=1}^{n} (m)_{i-1} (n)_{j-1} a_{i,j}$.

As we did before, we can associate a number with each partial summation by adding the entries in the coefficient matrix. What follows is a direct extension of Proposition 3.16.

**Proposition 6.4.** Let $A$ be an $m \times n$ matrix and let $a$ and $b$ be nonnegative integers with $a < m$ and $b < n$. Let $C$ be the coefficient matrix of $\sigma_a \downarrow \sigma_b \uparrow(A)$. Then $\text{sum}(C) = 2^{a+b}(m-a)(n-b)$. 19
Proof. By similar manipulations as in the proof of Proposition 3.16 we let $\alpha$ be the column sum vector of $R_{m-1}^a$ and $\beta$ be the column sum vector of $R_{n-1}^b$.

$$\text{sum}(C) = \text{sum}(\alpha \beta^T)$$

$$= \left[ \sum_{i=1}^{m} \alpha_i \right] \left[ \sum_{j=1}^{n} \beta_j \right]$$

$$= \left[ \text{sum} (R_{m-1}^a) \right] \left[ \text{sum} (R_{n-1}^b) \right]$$

$$= [2^a(m-a)] [2^b(n-b)]$$

$$= 2^{a+b}(m-a)(n-b),$$

where the fourth equality follows from Lemma 3.15.

Splitting $\sigma$ gave us a more general pair of operations which allowed us to extend our previous results. These operations will allow us to re-examine the Gaussian blur and the efficient algorithm in [4] from the perspective of the collapsing sum.

7 Gaussian blur and the collapsing sum

If we look back at the matrix $G_5$ shown in Section 2, we see the coefficient matrix $C_{5 \times 5}$. Indeed, we can now see from the definition of the approximate Gaussian blur that the kernel $G$ is always a coefficient matrix for a fully-collapsed matrix, scaled by $4^{-2r}$. More precisely, for any $n \geq 1$, we have

$$G_n = C_{n \times n}^{n-1}.$$

This suggests a strong connection between the collapsing sum and the Gaussian blur, which was our hope in defining the collapsing sum. To make the connection explicit, we need only look at the effects of the Gaussian blur on a matrix. Let $A$ be any matrix, and suppose the edge-handling technique we use is cropping. Each entry of $G_{2r+1} * A$ corresponds to a block of $A$ of size $(2r+1) \times (2r+1)$. Convolving one of these blocks with $G_{2r+1}$ collapses it to a single entry, because the kernel $G_{2r+1}$ is proportional to the coefficient matrix for a fully-collapsed matrix of size $(2r+1) \times (2r+1)$. Since this holds for every entry, convolving with $G_{2r+1}$ is equivalent to applying $4^{-2r} \sigma^{2r}$.

We can extend this to other edge-handling techniques. Let $A^\bullet$ denote an extension of $A$. Then applying the Gaussian blur to $A$ is equivalent to applying the collapsing sum to $A^\bullet$. We record this in the following theorem.

**Theorem 7.1.** Suppose a matrix $A$ and an edge-handling technique that results in an extension $A^\bullet$ of $A$ are given. Then $G_{2r+1} * A = 4^{-2r} \sigma^{2r}(A^\bullet)$.

The efficient algorithm developed by Waltz and Miller takes advantage of two properties of the Gaussian blur. First, large blurs can be created by sequential applications of smaller blurs. The powers of the collapsing sum make...
this property evident, since $\frac{1}{2}\sigma$ effects a $2 \times 2$ blur. Second, Gaussian blurs can be separated into independent row and column operations. This is exactly what we noted in Section 6. Combining these two observations reduces the problem of computing a Gaussian blur to recursively calculating row and column sums. In terms of the collapsing sum, this is the statement

$$\sigma^*(A) = \sigma^*\sigma^*(A).$$

The main loop of Waltz and Miller’s code essentially consists of sequentially applying $\sigma_\rightarrow$, then sequentially applying $\sigma_\downarrow$, and finally normalizing by a factor of $4^{-s}$. Although this algorithm concentrates the normalization at the end, by distributing it amongst each step, a more straightforward interpretation of the algorithm becomes apparent: a Gaussian blur is built up from the operations $\frac{1}{2}\sigma_\rightarrow$ and $\frac{1}{2}\sigma_\downarrow$. In some sense, we might consider these operations as $1 \times 2$ and $2 \times 1$ blurs, respectively.

The authors address this in their paper through consideration of non-square blurs, that is, Gaussian-like blurs using non-square kernels. These can be defined in parallel to the square Gaussian blurs. If $G_{a \times b}$ denotes the kernel for the $a \times b$ Gaussian blur, then

$$(G_{a \times b})_{i,j} = 2^{-(a+b-2)} \binom{a-1}{i-1} \binom{b-1}{j-1}.$$  

Here, we index from $(1,1)$ in the top left corner of the matrix. This differs slightly from the definition in Section 2, since either $a$ or $b$ might be even, so there may be no center element to index from.

Note that the kernel $G_{a \times b}$ is proportional to the coefficient matrix for a fully collapsed $a \times b$ matrix. This extends Theorem 7.1, since convolving $G_{a \times b}$ with the matrix $A$ is equivalent to applying $2^{-(a+b-2)}\sigma_{a-1}\sigma_{b-1}$ to the extended matrix $A^\cdot$.  

Waltz and Miller discuss that all of these (possibly) non-square Gaussian blurs can be built through sequential convolution of $G_{1 \times 2}$ and $G_{2 \times 1}$. Of course, this is exactly what their algorithm does. In terms of the collapsing sum, this simply means that Gaussian blurs of any size can be built from $\frac{1}{2}\sigma_\rightarrow$ and $\frac{1}{2}\sigma_\downarrow$.

Finally, the authors venture into higher dimensions and discuss higher-dimensional blurs. We can easily transfer this idea to the language of the collapsing sum. Suppose we have an $n$-dimensional array. We can define $\sigma_i$, for $1 \leq i \leq n$, to be the operator that “collapses” the array in the $i$th direction, akin to the effects of $\sigma_\rightarrow$ and $\sigma_\downarrow$ in 2 dimensions. Define $\sigma_n := \sigma_\rightarrow \cdots \sigma_{n-1}$. Then powers of $2^{-n}\sigma_n$ give the higher-dimensional blur that Waltz and Miller describe. Of course, non-hypercube blurs are obtained by composing the operators $\frac{1}{2}\sigma_i$ for various values of $i$.

### 8 Generalizing $\sigma$

A simple way to extend the collapsing sum is to weight the entries before summing them. To make use of previous results, we want to maintain the separa-
bility of the row and column operations.

**Definition 8.1.** Let \( \varphi = (\varphi_1, \varphi_2)^T \) be a 2-element vector and \( A \) an \( m \times n \) matrix with \( m, n \geq 2 \). Then \( \sigma_\varphi(A) \) is an \( (m-1) \times n \) matrix with \( \sigma_\varphi(A)_{i,j} = \varphi_1 a_{i,j} + \varphi_2 a_{i+1,j} \). Similarly, \( \sigma_\varphi^T(A) \) is the \( m \times (n-1) \) matrix with \( \sigma_\varphi^T(A)_{i,j} = \varphi_1 a_{i,j} + \varphi_2 a_{i,j+1} \).

If \( \rho \) and \( \varphi \) are 2-element vectors, then \( \sigma_\rho(\sigma_\varphi^T(A)) = \sigma_{\rho \varphi^T}(\sigma_\rho(A)) \). This is an \( (m-1) \times (n-1) \) matrix constructed in a similar manner to \( \sigma(A) \). We will denote this matrix by \( \sigma_{\rho \varphi^T}(A) \). Note that \( \rho \varphi^T \) is a \( 2 \times 2 \) matrix that stores the weights used when computing entries in the new matrix. In particular, if \( \rho = \varphi = (1, 1)^T \), then \( \sigma_{\rho \varphi^T} = \sigma \). By maintaining separability, we can easily extend previous results.

**Definition 8.2.** Let \( \varphi = (\varphi_1, \varphi_2)^T \). The \( m \times (m+1) \) matrix \( R_\varphi^m \) has entries (\( R_\varphi^m \))_{i,j} = \varphi_1 \delta_{i,j} + \varphi_2 \delta_{i+1,j}.

Again, notice that if \( \varphi = (1, 1)^T \), then \( R_\varphi^m = R_m \). We define the falling powers of these matrices analogously to those of \( R_m \). A similar argument to the one in the proof of Proposition 3.7 shows the following result.

**Proposition 8.3.** Let \( m \) and \( s \leq m \) be positive integers and \( \varphi \) a two-element row vector. Then \( (R_\varphi^m)^s \) is an \( (m-s+1) \times (m+1) \) matrix with entries

\[
[(R_\varphi^m)^s]_{i,j} = \binom{s}{j-i} \varphi_1^{s-(j-i)} \varphi_2^{j-i}.
\]

We can also see that a generalized form of Theorem 3.13 holds, in that, for any \( m \times n \) matrix \( A \),

\[
\sigma_{\rho \varphi^T}^a \rho \varphi^T (A) = (R_\varphi^m)^s A [(R_\varphi^m)^s]^T.
\]

If we let \( C \) be the coefficient matrix for \( \sigma_{\rho \varphi^T}^a \rho \varphi^T (A) \), then \( C \) has entries

\[
c_{i,j} = \sum_{\ell=1}^{m-a} \binom{a}{i-\ell} \varphi_1^{a-(i-\ell)} \varphi_2^{j-\ell} \sum_{\ell=1}^{n-b} \binom{b}{j-\ell} \rho_1^{b-(j-\ell)} \rho_2^{j-\ell},
\]

which is a frightening thing to behold. This simplifies slightly when we look at the “maximal” sum \( \sigma_{\rho \varphi^T}^{m-1} \rho_1^{n-1} (A) \). Here we get the formula

\[
c_{i,j} = \binom{m-1}{i-1} \varphi_1^{m-i} \varphi_2^{j-1} \binom{n-1}{j-1} \rho_1^{n-j} \rho_2^{j-1}.
\]

Considering the sum of the elements in the coefficient matrix yields a nicer result. Using the same argument as in the proof of Proposition 3.16, we find that

\[
\text{sum}(C) = (\varphi_1 + \varphi_2)^a (\rho_1 + \rho_2)^b (m-a)(n-b).
\]

This is all, of course, a simple extension of previous results; as pointed out above, the special case \( \rho = \varphi = (1, 1)^T \) is the original collapsing sum. It turns out that another special case will be particularly relevant for the following two sections. If we take the vectors \( \rho = \varphi = (1, -1)^T \), we can form the operator \( \sigma_- := \sigma_{\rho \varphi^T} \). Note that since each generalized collapsing sum is linear, \( \sigma_- \) in particular is linear.
9 The switch game

In this section, we describe and begin to analyze a single-player game, and in the following section, we use the collapsing sum to obtain more significant results on this game. Suppose you have a rectangular board divided into \( m \) rows and \( n \) columns. On each square is a chip which is white on one side and black on the other. These chips are placed on the board in no particular order; some chips have their white side up, and others have their black side up. Each move, called a *row* or *column switch*, consists of choosing a row or a column and flipping every chip in that row or column, and the objective of the game is to make the board monochromatic through a sequence of moves. Our first question, then, is simple: For which starting positions is the game winnable?

We could begin analyzing the game immediately, but the analysis will become much easier by first rephrasing the game. We begin by substituting an \( m \times n \) matrix of 0's and 1's for the \( m \times n \) board: for each square on the board, the corresponding entry of the matrix is 0 if the chip on the square is white and 1 if it is black. Since we only require 0's and 1's, in this section, we will take all arithmetic operations modulo 2.

How, then, does a row or column switch transfer to this new setting?

**Definition 9.1.** Let \( A \) be an \( m \times n \) matrix, and let \( k \) be a positive integer. If \( k \leq m \), then the *row switch* applied to \( A \) at row \( k \) gives the \( m \times n \) matrix \( A' \) with entries \( a'_{i,j} = a_{i,j} + \delta_{i,k} \). If \( k \leq n \), then the *column switch* applied to \( A \) at column \( k \) gives the \( m \times n \) matrix \( A^* \) with entries \( a^*_{i,j} = a_{i,j} + \delta_{j,k} \).

By choosing a row or column and adding one to each entry, we effectively replace every 0 with a 1 and vice versa, since our arithmetic is modulo 2. Therefore, this models flipping the chips as described above. We now define an equivalence relation on our set which tells us when two matrices are “the same” in terms of playing the game.

**Definition 9.2.** Two matrices \( A \) and \( B \) are *switch-equivalent* if and only if there exists a sequence of row and column switches taking \( A \) to \( B \). In that case, we write \( A \sim B \). The set of all matrices equivalent to \( A \) is written \( [A] \).

The reader may check that \( \sim \) is an equivalence relation. We will later verify that it is in fact a congruence relation. We are particularly interested in one equivalence class: the class of winnable matrices.

**Definition 9.3.** A matrix \( S \) is *simple* if and only if it is equivalent to the zero matrix. The set of \( m \times n \) simple matrices will be denoted \( S_{m,n} \).

The simple matrices are exactly the winnable ones. [If we can perform switches to form the all-ones matrix, which represents a board with every chip black, then we can switch every row to get the zero matrix.] When the dimensions of the matrices are implied or unimportant, we will simply write \( S \). Our rephrased question is this: for what matrices \( A \in \mathbb{Z}_2^{m \times n} \) is \( A \in S \)?

For now, we postpone the answer to that question to build up some algebra. Suppose that \( S \) is a simple \( m \times n \) matrix and that it was formed from the
zero matrix by performing some set $X$ of switches. [Note that the order of
the switches is unimportant, so we can speak of a set of switches instead of a
sequence.] Notice that performing the switches in $X$ on any $m \times n$ matrix $A$
isan the same as taking the sum $A + S$. That is, we can represent the switches
of Definition 9.1 as the simple matrices of Definition 9.3 and two matrices $A$
and $B$ are equivalent exactly when there exists a simple matrix $S$ such that
$A + S = B$.

We can use this to show that $\sim$ respects addition. Suppose that $B \sim B'$.
Then there exists a simple matrix $S$ such that $B = B' + S$, and for any matrix
$A$, we have $A + B = A + B' + S$, so $A + B \sim A + B'$. Therefore $\sim$ is a congruence
relation. Importantly, the sum of any two simple matrices is simple: for any
two $S_1, S_2 \in S$, we have $S_1 \sim S_2 \sim 0$, so $S_1 + S_2 \sim 0$ and $S_1 + S_2 \in S$.

An interesting consequence of this is that each equivalence class is the same
size. Since $[A] = \{A + S : S \in S\}$, we see that the cardinality of $[A]$ is equal
to the cardinality of $S$. This size can be calculated using our current (quite
limited) methods, but a better proof will present itself when we develop this
subject more, so we will put this off until then [see Corollary 10.5]. We briefly
turn to equivalence in general.

**Proposition 9.4.** Two matrices $A$ and $B$ are equivalent if and only if $A - B$
is simple.

*Proof.* Two matrices $A$ and $B$ are equivalent if and only if there exists an
$S \in S$ such that $A = B + S$. This is true if and only if $A - B = S \in S$, as required. $\square$

Now we return to our previous question: what do simple matrices look like?
The next theorem provides an answer.

**Theorem 9.5.** The matrix $A$ is simple if and only if each row of $A$ is identical
up to row switches.

*Proof.* ($\Rightarrow$) Suppose $A$ is simple. Then there exists a sequence of row and
column switches that takes the zero matrix to $A$. Note that in the zero matrix,
each row is identical. Both row and column switches maintain this property, up
to row switches, so $A$ must have the row relationship described in the theorem.

($\Leftarrow$) Suppose that every row of $A$ is identical up to row switches. Perform
row switches to make each row identical to the first row. [It may be that some
rows do not require any row switches.] Call the resulting matrix $A'$. In $A'$, each
row is identical, so every column is either all 1’s or all 0’s. Perform a column
switch on each column of all 1’s to yield the zero matrix. Thus $A \sim A' \sim 0$, so
$A$ is simple. $\square$

Certainly, we can replace “row switches” with “column switches” in the
previous theorem and the result still holds. Theorem 9.5 provides a way to tell
whether a matrix is simple at a glance. For example, it is easy to tell that of
the following matrices, the left one is simple and the right one is not.

\[
\begin{pmatrix}
1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0
\end{pmatrix}
\quad
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0
\end{pmatrix}
\]

There is a second characterization of simple matrices, and this will prove more useful in later analysis.

**Theorem 9.6.** An \(m \times n\) matrix \(A\) with \(m, n \geq 2\) is simple if and only if each \(2 \times 2\) block of \(A\) is simple.

**Proof.** (\(\Rightarrow\)) Suppose \(A\) is simple and choose four elements \(a_{i,j}, a_{i+1,j}, a_{i,j+1},\) and \(a_{i+1,j+1}\). We know that row \(i + 1\) is a certain number of row switches from row \(i\). Therefore the rows of the \(2 \times 2\) block composed of the elements \(a_{i,j}, a_{i+1,j}, a_{i,j+1},\) and \(a_{i+1,j+1}\) are related by row switches, so by Theorem 9.5, the block is simple.

(\(\Leftarrow\)) Suppose every \(2 \times 2\) block of \(A\) is simple. Then each block obeys the row relationship of Theorem 9.5. Picking any two sequential rows of \(A\), we see that they must also obey the row relationship, because each \(2 \times 2\) block of \(A\) composed of elements from these rows obey the same row relationship. For example, say that \(a_{r,1} + k = a_{r+1,1}\), so that row \(r + 1\) is \(k\) row switches from row \(r\). Then \(a_{r,2} + k = a_{r+1,2}\), since \(a_{r,1}, a_{r+1,1}, a_{r,2}\), and \(a_{r+1,2}\) form a \(2 \times 2\) block of \(A\). We can iterate this argument across the elements of the two rows to show that the rows as a whole must be \(k\) row switches apart. Therefore any two consecutive rows are related by row switches. But then any two rows are related by row switches, so by Theorem 9.5, the matrix \(A\) is simple. 

Theorem 9.6 reduces the problem of finding simple matrices to finding all \(2 \times 2\) simple matrices. This is a simple calculation by hand, and it yields the following complete list of simple \(2 \times 2\) matrices:

\[
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\quad
\begin{pmatrix}
1 & 1 \\
0 & 0
\end{pmatrix}
\quad
\begin{pmatrix}
0 & 0 \\
1 & 1
\end{pmatrix}
\quad
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

10  **Simple matrices and \(\sigma_-\)**

In the previous section, we transferred the switch game on a rectangular board to a game on matrices with entries in \(\mathbb{Z}_2\). Upon doing that, an immediate generalization arises: in this section, we will let our matrices take values in one of \(\mathbb{Z}_k\) or \(\mathbb{Z}\) for any integer \(k \geq 1\). For notational convenience, we will sometimes write \(\mathbb{Z}_0\) for \(\mathbb{Z}\). We will write \(\mathbb{Z}_k^{m \times n}\) to mean the set of \(m \times n\) matrices whose entries are elements of \(\mathbb{Z}_k\). Now, to include the matrices \(\mathbb{Z}_k^{m \times n}\), we will allow switches to take a second form: a **negative row switch** applied at row \(k\) takes
A to \(A'\) with elements \(a'_{i,j} = a_{i,j} - \delta_{k,i}\). We similarly define negative column switches. Positive row and column switches are defined as in Definition 9.1. Note that in \(\mathbb{Z}_k\) for \(k > 0\), the distinction between positive and negative row and column switches does not appear: \(k - 1\) positive row switches has the same effect as a negative row switch. However, in \(\mathbb{Z}\), the distinction is necessary. From here on, we will simply write “switches” to mean positive or negative row or column switches.

Notice that Definitions 9.2 and 9.3 as well as Proposition 9.4 and Theorem 9.5 can be immediately extended to this more general setting. For example, in \(\mathbb{Z}_4^{2 \times 5}\), of the following matrices, the one on the left is simple, and the matrix on the right is not.

\[
\begin{pmatrix}
0 & 1 & 4 & 1 & 5 \\
2 & 3 & 6 & 3 & 0 \\
6 & 0 & 3 & 0 & 4 \\
3 & 4 & 0 & 4 & 1
\end{pmatrix}
\quad
\begin{pmatrix}
2 & 3 & 0 & 6 & 1 \\
4 & 1 & 4 & 1 & 4 \\
0 & 0 & 2 & 4 & 5 \\
3 & 2 & 5 & 0 & 6
\end{pmatrix}
\]

Checking these matrices is not nearly as quick as it is to check matrices with elements in \(\mathbb{Z}_2\). Additionally, this method is unsatisfying. Suppose we are given two \(734 \times 316\) matrices, and we wish to see whether they are equivalent. Having to visually examine their difference to see if it is simple would be a pain. The following theorem, which connects the collapsing sum to our generalized switch game, nicely answers the question of when a matrix is simple.

**Theorem 10.1.** Let \(A\) be an \(m \times n\) matrix with \(m,n \geq 2\). The matrix \(A\) is simple if and only if \(\sigma-(A)\) is the zero matrix.

Note that if one of either \(m\) or \(n\) is not greater than 2, then the matrix is automatically simple, so there is no real checking to be done. To prove this theorem, we will make use of Theorem 9.6. Note that the proof of this lemma holds in the generalized setting.

**Proof of Theorem 10.1**

\((\Rightarrow)\) Suppose \(A\) is simple. By Theorem 9.6, every \(2 \times 2\) block of \(A\) is simple, so we will examine the effect of \(\sigma-\) on simple \(2 \times 2\) matrices. Let \(B\) be such a matrix. Then \(B\) has entries \(b_{1,1}\) and \(b_{1,2}\) in the first row and \(b_{2,1} = b_{1,1} + k\) and \(b_{2,2} = b_{1,2} + k\), for some \(k \in \mathbb{Z}\), in the second row. Then

\[
\sigma-(B)_{1,1} = b_{1,1} - b_{1,2} - b_{2,1} + b_{2,2}
\]

\[
= b_{1,1} - b_{1,2} - (b_{1,1} + k) + (b_{1,2} + k)
\]

\[
= 0.
\]

Thus the effect of \(\sigma-\) on any simple \(2 \times 2\) matrix is to produce the \(1 \times 1\) zero matrix. Since every \(2 \times 2\) block of \(A\) is simple, each entry of \(\sigma-(A)\) is 0, so \(\sigma-(A)\) is the zero matrix.

\((\Leftarrow)\) In light of Theorem 9.6 if we can show that every \(2 \times 2\) block of \(A\) is simple, then we are done. Thus, let \(B\) be a \(2 \times 2\) matrix such that \(\sigma-(B)\) is the \(1 \times 1\) zero matrix. Then \(b_{1,1} - b_{1,2} - b_{2,1} + b_{2,2} = 0\), so \(b_{1,1} - b_{1,2} = b_{2,1} - b_{2,2}\). We know that \(b_{2,1} = b_{1,1} + k\) for some \(k \in \mathbb{Z}\). Substituting this in, we find that
Thus the two rows of $B$ are $k$ row switches apart, so $B$ is simple. Therefore, if $\sigma_.(A)$ is the zero matrix, every $2 \times 2$ block of $A$ is simple, so $A$ is simple.

**Corollary 10.2.** Two $m \times n$ matrices $A$ and $B$, with $m, n \geq 2$, are equivalent if and only if $\sigma_.(A) = \sigma_.(B)$.

**Proof.** From Proposition 9.4 we know that $A$ and $B$ are equivalent if and only if $A - B$ is simple. But $A - B$ is simple if and only if $\sigma_.(A - B) = 0$, which holds if and only if $\sigma_.(A) = \sigma_.(B)$.

Theorem 10.1 answers the first question of winnability from Section 9, and Corollary 10.2 answers the more general question of determining when two matrices are equivalent. In the case of the matrices have entries in $\mathbb{Z}_2$, the operators $\sigma_-$ and $\sigma$ are equivalent, so two matrices with entries in $\mathbb{Z}_2$ are equivalent if and only if their collapsing sums are equal.

We can view the set $\mathbb{Z}_k^{m \times n}$ as an additive group, of which $S_{m,n}$ is a subgroup. This perspective yields the following result, which will help answer the counting questions delayed in Section 9.

**Theorem 10.3.** Let $k \in \mathbb{N}$. Then $\mathbb{Z}_k^{(m-1) \times (n-1)} \cong \mathbb{Z}_k^{m \times n} / S_{m,n}$.

**Proof.** We claim that $\sigma_-$ provides a surjective homomorphism from $\mathbb{Z}_k^{m \times n}$ to $\mathbb{Z}_k^{(m-1) \times (n-1)}$. It is clearly a homomorphism, since $\sigma_-$ is linear. We now show that $\sigma_-$ is surjective. Thus, let $M \in \mathbb{Z}_k^{(m-1) \times (n-1)}$; we will construct a matrix $N$ such that $\sigma_.(N) = M$. Let $n_{i,j} = 0$ if $i = 1$ or $j = 1$. We can fill in the remaining entries of $N$ so that each $2 \times 2$ block of $N$ collapses to become the corresponding element of $M$. Begin with the $(2,2)$ entry and continue with that row. Then fill in the third row, and so on until the matrix is complete. Then $\sigma_.(N) = M$ by construction, so $\sigma_-$ is a surjective map. From Theorem 10.1 the kernel of $\sigma_-$ is exactly $S_{m,n}$. Thus $\mathbb{Z}_k^{(m-1) \times (n-1)} \cong \mathbb{Z}_k^{m \times n} / S_{m,n}$ by the Fundamental Homomorphism Theorem.

We now give an example of the “building” process described in the proof. Say that we want to build a matrix $N \in \mathbb{Z}_4^{3 \times 3}$ such that

$$\sigma_.(N) = M = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}. $$

We set the first row and column of $N$ to zeros. We know that

$$3 = m_{1,1} = n_{1,1} - n_{1,2} - n_{2,1} + n_{2,2} = 0 - 0 + n_{2,2}.$$

Therefore $n_{2,2} = 3$ and we have

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
Next,

\[2 = m_{1,2} = n_{1,2} - n_{1,3} - n_{2,2} + n_{2,3} = 0 - 0 - 3 + n_{2,3},\]

so \(n_{2,3} = 1\), and we fill in another entry. [Recall that the entries are in \(\mathbb{Z}_4\).] Continuing in this fashion, we get the matrix

\[
N = \begin{pmatrix}
0 & 0 & 0 \\
0 & 3 & 1 \\
0 & 3 & 2
\end{pmatrix}.
\]

After we chose the entries of first row and column, the rest of \(N\) was completely determined. This holds for matrices of any size, and it will be important in the proof of Theorem 10.6.

The answers to the counting questions that were alluded to above follow from Theorem 10.3.

**Corollary 10.4.** Let \(k > 0\). There are exactly \(k^{(m-1)(n-1)}\) equivalence classes in \(\mathbb{Z}_k^{m \times n}\).

**Proof.** Note that the equivalence classes of \(\mathbb{Z}_k^{m \times n}\) are exactly the cosets of \(S_{m,n}\) in \(\mathbb{Z}_k^{m \times n}\). By Theorem 10.3, each coset, and thus each equivalence class, of \(\mathbb{Z}_k^{m \times n}\) is associated with a unique element of \(\mathbb{Z}_k^{(m-1) \times (n-1)}\). Noting that \(\mathbb{Z}_k^{(m-1) \times (n-1)}\) contains \(k^{(m-1)(n-1)}\) elements completes the proof. \(\square\)

**Corollary 10.5.** Let \(k > 0\). Each equivalence class of \(\mathbb{Z}_k^{m \times n}\) has size \(k^{m+n-1}\).

**Proof.** There are \(k^{mn}\) matrices in \(\mathbb{Z}_k^{m \times n}\) and \(k^{(m-1)(n-1)}\) equivalence classes. Each equivalence class has the same size, so simple division yields that there are \(k^{m+n-1}\) elements in each equivalence class. \(\square\)

At this point, one might wonder what exactly these equivalence classes are. The next theorem characterizes them. First, though, we introduce some notation. For any \(m \times n\) matrix \(A\), let \(A^\circ\) be the block matrix

\[
\begin{pmatrix}
0_{1 \times 1} & 0_{1 \times n} \\
0_{m \times 1} & A
\end{pmatrix}.
\]

For example, if

\[
A = \begin{pmatrix}
2 & 3 & 4 \\
0 & 1 & 1 \\
3 & 0 & 4
\end{pmatrix},
\]

then

\[
A^\circ = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 2 & 3 & 4 \\
0 & 0 & 1 & 1 \\
0 & 3 & 0 & 4
\end{pmatrix}.
\]

With this, we can assign a unique representative for each equivalence class.
Theorem 10.6. Let $B \in \mathbb{Z}_k^{m \times n}$. For exactly one $A \in \mathbb{Z}_k^{(m-1) \times (n-1)}$, we have $B \in \left[A^\circ\right]$.

Proof. Let $B \in \mathbb{Z}_k^{m \times n}$. We first show that such an $A^\circ$ exists. We can construct one with the following process. From the definition, the first row and column of $A^\circ$ are filled with 0’s. We can then construct the remainder of the matrix $A^\circ$ such that $\sigma_-(A^\circ) = \sigma_-(B)$ in a process similar to that presented in the proof of Theorem 10.3. Therefore there exists a matrix $A \in \mathbb{Z}_k^{(m-1) \times (n-1)}$ such that $B \in \left[A^\circ\right]$.

Now we prove uniqueness. Suppose we have two matrices $A$ and $C$ such that $B \in \left[A^\circ\right]$ and $B \in \left[C^\circ\right]$. Then $A^\circ \sim B \sim C^\circ$, so $\sigma_-(A^\circ) = \sigma_-(C^\circ)$. Since we know the entries in the first row and column of both $A^\circ$ and $C^\circ$, we can reconstruct these matrices from their collapsing sums. But this process gives a unique matrix, and since their first rows and columns and their collapsing sums are the same, we must have $A^\circ = C^\circ$. Thus there exists exactly one matrix $A \in \mathbb{Z}_k^{(m-1) \times (n-1)}$ such that $B \in \left[A^\circ\right]$.

Now we do not only know the number of equivalence classes, but we can actually list them as $\{[A^\circ] : A \in \mathbb{Z}_k^{(m-1) \times (n-1)}\}$. If $k > 0$, we can provide an alternate proof of uniqueness using a counting argument. There are exactly $k^{(m-1)(n-1)}$ equivalence classes of matrices in $\mathbb{Z}_k^{m \times n}$ and also exactly $k^{(m-1)(n-1)}$ matrices in $\mathbb{Z}_k^{(m-1) \times (n-1)}$. Since each equivalence class has at least one representative of the form $[A^\circ]$, these counts show that there is at most one representative of this form for each equivalence class. Therefore this representation is unique.

Finally, note that we can generalize this section by replacing $\mathbb{Z}_k$ with an arbitrary abelian group $G$. We define a $g$-switch as entrywise multiplication of a row or column by the element $g \in G$. In this generalized setting, Theorems 30, 31, and 35 still hold, and if $|G| = k$, then the remaining assertions in this section hold as well.

11 An expanding sum

Up to this point, we’ve collapsed almost every matrix in sight. The notable exception to this is the coefficient matrices. With this in mind, let’s examine what happens when we apply $\sigma$ to a coefficient matrix for a fully-collapsed square matrix:

$$
\sigma \begin{pmatrix}
1 & 3 & 3 & 1 \\
3 & 9 & 9 & 3 \\
3 & 9 & 9 & 3 \\
1 & 3 & 3 & 1
\end{pmatrix} = 
\begin{pmatrix}
16 & 24 & 16 \\
24 & 36 & 24 \\
24 & 36 & 24 \\
16 & 24 & 16
\end{pmatrix}.
$$

Interestingly, when we collapse the coefficient matrix $C_4^{3 \times 4}$, we get the central region of the coefficient matrix for $C_5^4$. Computing this for other coefficient matrices, we see that the pattern continues: collapsing $C_n^{n-1}$ gives the central region of $C_{(n+1)}^{n \times (n+1)}$. This raises a question: can we define an operator that takes $C_n^{n-1}$ to $C_n^{n \times (n+1)}$ by expanding the matrix in some way? We can!
**Definition 11.1.** Let $A$ be an $m \times n$ matrix. The expanding sum of $A$, denoted $\tau(A)$, is defined

$$
\tau(A) = \sigma \left( \begin{array}{ccc}
0_{1 \times 1} & 0_{1 \times n} & 0_{1 \times 1} \\
0_{m \times 1} & A & 0_{m \times 1} \\
0_{1 \times n} & 0_{1 \times 1} & 0_{1 \times 1}
\end{array} \right).
$$

For example, we have

$$
\tau \left( \begin{array}{cccc}
1 & 3 & 3 & 1 \\
3 & 9 & 9 & 3 \\
3 & 9 & 9 & 3 \\
1 & 3 & 3 & 1
\end{array} \right) = \sigma \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 1 & 3 & 3 \\
0 & 3 & 9 & 9 \\
0 & 0 & 0 & 0
\end{array} \right) = \left( \begin{array}{cccc}
1 & 4 & 6 & 4 \\
4 & 16 & 24 & 16 \\
6 & 24 & 36 & 24 \\
1 & 4 & 6 & 4
\end{array} \right).
$$

It appears that we’ve defined an operator that does exactly what we wanted. [Corollary 11.7 confirms this.] Note that $\tau$ is not any kind of inverse of $\sigma$; indeed, we have

$$
\sigma \tau \left( \begin{array}{cccc}
1 & 0 \\
1 & 1
\end{array} \right) = \sigma \left( \begin{array}{ccc}
1 & 1 & 0 \\
2 & 3 & 1 \\
1 & 2 & 1
\end{array} \right) = \left( \begin{array}{cc}
7 & 5 \\
8 & 7
\end{array} \right).
$$

In fact, the two operators do not even commute:

$$
\tau \sigma \left( \begin{array}{cccc}
1 & 0 \\
1 & 1
\end{array} \right) = \tau \left( \begin{array}{c}
3 \\
3
\end{array} \right).
$$

With this in mind, we will study $\tau$ parallel to rather than together with $\sigma$. We first note that $\tau$ is linear. We will begin by giving $\tau$ a matrix form, as proved fruitful for $\sigma$ in Section 3. Before we do this, let us briefly reconsider the definition of $\tau$.

In this section and beyond, it will be convenient to let the entry in the $(i, j)$ position be 0 if $i$ or $j$ is outside the index range. Here, the lowest index is 1, but when we work with infinite matrices again, the lowest index will be 0. Let $B$ be the augmented version of $A$ in Definition 11.1 such that $\tau(A) = \sigma(B)$. Then

$$
\tau(A)_{i,j} = \sigma(B)_{i,j} = b_{i,j} + b_{i+1,j} + b_{i,j+1} + b_{i+1,j+1}.
$$

Since $B$ is augmented from $A$, we have $b_{i,j} = a_{i-1,j-1}$ [recalling our convention for entries outside the index range]. Therefore we have the following alternate definition of the expanding sum.

**Definition 11.2.** Let $A$ be an $m \times n$ matrix, and let $a_{i,j} = 0$ whenever $i$ or $j$ is outside its index range. Then $\tau(A)$ is an $(m+1) \times (n+1)$ matrix with entries

$$
\tau(A)_{i,j} = a_{i,j} + a_{i-1,j} + a_{i,j-1} + a_{i-1,j-1}.
$$

This definition will be convenient in future algebraic manipulation. We can give a third equivalent definition for $\tau$ using matrices. This will allow us to quickly analyze some aspects of $\tau$ analogously to that of our previous examination of $\sigma$. Recall the matrices $R_m$ from that Section 3.
Theorem 11.3. Let $A$ be an $m \times n$ matrix. Then $\tau(A) = (R_m)^TAR_n$.

Proof. First, note that both $\tau$ and $(R_m)^TAR_n$ are $(m+1) \times (n+1)$. We have

$$[AR_n]_{i,j} = \sum_{r=1}^{n} (R_n)_{i,r} a_{r,j}$$
$$= a_{i,j} + a_{i,j-1}.$$

Hence

$$[(R_m)^TAR_n]_{i,j} = \sum_{r=1}^{m} (R_m)_{r,i} (AR_n)_{r,j}$$
$$= \sum_{r=1}^{m} (R_m)_{r,i} [a_{i,j} + a_{i,j-1}]$$
$$= a_{i,j} + a_{i-1,j} + a_{i,j-1} + a_{i-1,j-1}$$
$$= \tau(A)_{i,j}.$$

Since these matrices agree in every position, we are done. \(\square\)

Theorem 11.3 gives us another reason to suspect that $\tau$ will be an interesting operator to study: the matrix representation of $\tau$ is very similar to that of $\sigma$. [Compare the previous theorem to Theorem 3.5.] The following definition continues this analogy, paralleling Definition 3.6.

Definition 11.4. Let $m$ and $k$ be positive integers. By $R^k_m$, we mean the product $R_m R_{m+1} \cdots R_{m+k-1}$. By convention we take $R^0_m = I_m$.

By repeatedly applying Theorem 11.3 we see that $\tau^k(A) = (R^k_m)^TAR^k_n$. With this form, we can easily find the entries of $\tau^k(A)$, but first we will need the following lemma.

Lemma 11.5. The matrix $R^k_m$ has size $m \times (m+k)$ and entries $(R^k_m)_{i,j} = \binom{k}{j-i}$.

Proof. We can rewrite the rising power as a falling power:

$$R^\pi_m = R_m R_{m+1} \cdots R_{m+k-1} = R^{k}_{m+k-1}.$$

From this, we simply apply Proposition 3.7 to obtain

$$(R^\pi_m)_{i,j} = (R^k_{m+k-1})_{i,j} = \binom{k}{j-i}.$$ \(\square\)

We can now easily obtain the entries of $\tau^k(A)$.

Proposition 11.6. Let $A$ be an $m \times n$ matrix and $k$ a nonnegative integer. Then $\tau^k(A)_{p,q} = \sum_{i=1}^{m} \sum_{j=1}^{n} \binom{k}{p-i} \binom{k}{q-j} a_{i,j}$.
Proof. We compute directly from the matrices:

\[ \tau^k(A)_{p,q} = \sum_{i=1}^{m} \sum_{j=1}^{n} (R^k_{m,j})^{T} a_{i,j} (R^k_{n,i})_{p,q} \]

\[ = \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \frac{k}{p-i} \right) \left( \frac{k}{q-j} \right) a_{i,j}. \]  

With this, we can prove a stronger version of the statement that motivated us to study \( \tau \).

Corollary 11.7. Let \( 1_{a \times b} \) be the \( a \times b \) all-ones matrix. Then for positive integers \( m \) and \( n \) and a nonnegative integer \( s < \min\{m,n\} \), we have \( C^s_{m \times n} = \tau^s(1_{(m-s) \times (n-s)}) \).

Proof. Let \( A = 1_{(m-s) \times (n-s)} \). From Proposition 11.6, we have

\[ \tau^s(A)_{p,q} = \sum_{i,j} \left( \frac{s}{p-i} \right) \left( \frac{s}{q-j} \right) a_{i,j} \]

\[ = \left[ \sum_{i=1}^{m-s} \left( \frac{s}{p-i} \right) \right] \left[ \sum_{j=1}^{n-s} \left( \frac{s}{q-j} \right) \right] \]

\[ = c_{i,j}. \]

Since both matrices are \( m \times n \), they must be equal, so we are done.

The conjecture at the beginning of this section is a special case of this corollary. Note that the matrix \( \sigma(A) \) is the center of the matrix \( \tau(A) \), since both operations sum \( 2 \times 2 \) blocks. Therefore \( \sigma(C^m_{n \times n}) = \sigma(\tau^{n-1}(1_{1 \times 1})) \) is the center region of \( \tau(\tau^{n-1}(1_{1 \times 1})) = C^m_{(n+1) \times (n+1)} \), which is exactly our conjecture.

In the next section, we turn to an aspect of \( \tau \) that differs quite significantly from \( \sigma \): reversing its effects.

12 Reversing \( \tau \)

If we think of \( \tau \) and \( \sigma \) as function on matrices, then \( \sigma \) maps to smaller matrices, while \( \tau \) maps to larger matrices. More to the point, \( \sigma \) is surjective, but not injective, and we will see that \( \tau \) is injective, but not surjective. This means that \( \tau \) has a left inverse, a function \( \hat{\tau} \) such that \( \hat{\tau}(A) = A \) for all matrices \( A \).

Definition 12.1. Let \( A \) be an \( m \times n \) matrix. Then \( \hat{\tau}(A) \) is the \( (m-1) \times (n-1) \) matrix with entries defined recursively by

\[ \hat{\tau}(A)_{i,j} = a_{i,j} - \hat{\tau}(A)_{i-1,j} - \hat{\tau}(A)_{i,j-1} - \hat{\tau}(A)_{i-1,j-1}. \]

\[ \text{The collapsing sum has a right inverse, and in fact we’ve already discussed it. The preimage construction used in Theorem 10.3 is a right inverse for \( \sigma \), and a similar construction gives a right inverse to \( \sigma \).} \]
It may seem as though there is no place to start the recursion, but recall that entries in position \((i,j)\) are 0 whenever \(i\) or \(j\) is outside outside the index range. With this, we can easily prove that \(\hat{\tau}\) is a left inverse of \(\tau\).

**Proposition 12.2.** Let \(A\) be any matrix. Then \(\hat{\tau}A = A\).

*Proof.* We will induct on both rows and columns. The edges [where one of \(i\) or \(j\) is less than the lowest index] provide our base cases, since in these cases \(\hat{\tau}A)_{i,j} = 0 = a_{i,j}\). Now suppose that \(\hat{\tau}A)_{i,j} = a_{i,j}\) whenever either \(i < p\) or both \(i = p\) and \(j < q\). Then

\[
\hat{\tau}A)_{p,q} = \tau(A)_{p,q} - \hat{\tau}A)_{p-1,q} - \hat{\tau}A)_{p,q-1} - \hat{\tau}A)_{p-1,q-1} \\
= \tau(A)_{p,q} - a_{p-1,q} - a_{p,q-1} - a_{p-1,q-1} \\
= a_{p,q},
\]

where the second equality comes from the induction assumption and the third from expanding \(\tau(A)_{p,q}\). This completes the induction. \(\square\)

A consequence of this proposition is that \(\tau\) is injective, as noted at the beginning of this section. [Suppose that \(\tau(A) = \tau(B)\). Then \(A = \hat{\tau}A = \hat{\tau}B\).] Note that in general, \(\hat{\tau}\) is not a right inverse of \(\tau\):

\[
\tau\hat{\tau} \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \tau \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

We noted previously that \(\tau\) is not surjective. Indeed, there is no \(1 \times 1\) matrix \(B\) such that

\[
\tau(B) = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}.
\]

A sensible question to ask, then, is the following: for what matrices \(A\) is there a matrix \(B\) such that \(A = \tau(B)\)? In other words, what matrices are in the image of \(\tau\)? This is an interesting question, and we address it in Theorem 12.5. First, we describe a matrix representation of \(\hat{\tau}\).

**Definition 12.3.** Let \(\hat{R}_m\) be the \(m \times (m + 1)\) matrix with entries

\[
(\hat{R}_m)_{i,j} = \begin{cases} (-1)^{i-j} & \text{if } i \geq j \\ 0 & \text{if } i < j. \end{cases}
\]

For example,

\[
\hat{R}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 \end{pmatrix}.
\]

The proof of the following theorem can be found in Appendix B.

**Proposition 12.4.** Let \(A\) be an \((m+1) \times (n+1)\) matrix. Then \(\hat{\tau}A = \hat{R}_m A \hat{R}^T_n\).
We now return to the question of determining the image of $\tau$. The following theorem provides a simple answer.

**Theorem 12.5.** The matrix $A$ is in the image of $\tau$ if and only if $\hat{\tau}(\hat{\tau}(A)) = A$.

**Proof.** ($\Rightarrow$) Suppose there exists a matrix $B$ such that $A = \tau(B)$. Then by Proposition 12.2 we have $\hat{\tau}(\hat{\tau}(A)) = \tau(B) = A$.

($\Leftarrow$) Conversely, if $\tau(A) = A$, then $A$ is the image of $\tau(A)$ under $\tau$. □

A stronger result holds for infinite matrices.

**Theorem 12.6.** Let $A$ be any infinite matrix. Then $A = \tau(\hat{\tau}(A))$, and therefore all infinite matrices are in the image of $\tau$.

**Proof.** We have

$$
\tau(\hat{\tau}(A))_{i,j} = \hat{\tau}(A)_{i,j} + \hat{\tau}(A)_{i-1,j} + \hat{\tau}(A)_{i,j-1} + \hat{\tau}(A)_{i-1,j-1}
$$

$$
= (a_{i,j} - \hat{\tau}(A)_{i-1,j} - \hat{\tau}(A)_{i,j-1} - \hat{\tau}(A)_{i-1,j-1})
$$

$$
+ \hat{\tau}(A)_{i-1,j} + \hat{\tau}(A)_{i,j-1} + \hat{\tau}(A)_{i-1,j-1}
$$

$$
= a_{i,j}.
$$

Since this holds for all $i$ and $j$, we are done. □

**Corollary 12.7.** The operator $\tau$ is invertible on the space of infinite matrices, and $\tau^{-1} = \hat{\tau}$.

Recall that Theorem 12.6 is not true for finite matrices. If $A$ is $m \times n$, then $\hat{\tau}(A)_{i,j}$ is defined by the recursive formula only when $i \leq m - 1$ and $j \leq n - 1$. Therefore the second equality above may not hold if $i = m$ or $j = n$, and this is where the argument fails for finite matrices.

### 13 Fixed matrices of $\sigma$ and $\tau$

In this section, we study the matrices fixed by $\sigma$ and $\tau$. If $A$ is a finite matrix, both $\sigma(A)$ and $\tau(A)$ have a different size than $A$. Therefore, in this section, we consider only bi-infinite matrices. Following the convention for Riordan arrays, we will index from 0, so these matrices will look like

$$
\begin{pmatrix}
    a_{0,0} & a_{0,1} & a_{0,2} & \cdots \\
    a_{1,0} & a_{1,1} & a_{1,2} & \cdots \\
    a_{2,0} & a_{2,1} & a_{2,2} & \cdots \\
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

We first examine $\tau$, because the characterization of the matrices it fixes is simple.

**Theorem 13.1.** Let $k$ be a positive integer. The only fixed matrix of $\tau^k$ is the all-zeros matrix.
Proof. Suppose that $\tau^k(A) = A$. We will show by induction on both rows and columns that all entries of $A$ must be 0. The base cases are the entries $a_{i,j}$ with one of $i$ or $j$ less than 0. For the inductive step, suppose that $a_{i,j} = 0$ when either $j < q$ or both $j = q$ and $i < p - 1$. Then by Proposition 11.6, we have

$$\tau^k(A)_{p,q} = \sum_{i,j} \binom{k}{p-i} \binom{k}{q-j} a_{i,j} = ka_{p-1,q} + a_{p,q}.$$ 

From the assumption that $A$ is a fixed matrix of $\tau^k$, we have $\tau^k(A)_{p,q} = a_{p,q}$. Therefore $a_{p-1,q} = 0$, which completes the induction.

Corollary 13.2. Let $k$ be a positive integer. The only fixed matrix of $\hat{\tau}^k$ is the all-zeros matrix.

Proof. A matrix is fixed under $\tau^k$ if and only if it is fixed under $\hat{\tau}^k$. To see this, note that we have $\tau^k(A) = A$ if and only if $\hat{\tau}^k \tau^k(A) = \hat{\tau}^k(A)$, that is, $A = \hat{\tau}^k(A)$. Since 0 is the only fixed matrix of $\tau^k$, it is also the only fixed matrix of $\hat{\tau}^k$. 

Now we turn to the fixed matrices of $\sigma$. We can see that this set is larger than the set of fixed matrices for $\tau$ by noting that

$$\begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is fixed by $\sigma$. Since $\sigma$ is linear, the set of fixed matrices is a subspace of the space of infinite matrices, so it is sufficient to find a generating set of this subspace. If we know that $\sigma(A) = A$ and we are given the entries $(a_{0,0}, a_{0,1}, a_{0,2}, \ldots) \cup (a_{1,0}, a_{2,0}, \ldots)$ of the first row and column, then we can determine every element of $A$ using a process similar to the method mentioned in the proof of Theorem 10.3. Put another way, there is a unique fixed matrix of $\sigma$ with a given first row and column.

Definition 13.3. Let $k$ be a nonnegative integer. Then $e_k$ is the fixed matrix of $\sigma$ that has a 1 in the $(0,k)$ position and 0’s everywhere else in the first row and column.

It may appear that we left out some of the matrices we need, since, for example, for any positive integer $k$, the fixed matrix with a 1 in the $(k,0)$ position and 0’s everywhere else in the first row and column seems to not be included in the definition. However, note that by Proposition 3.1, we have $\sigma(e_k^T) = \sigma(e_k)^T = e_k^T$, so $e_k^T$ is this matrix.

Theorem 13.4. The fixed matrices of $\sigma$ are exactly the (possibly infinite) linear combinations of the matrices in the set $\bigcup_{k \geq 0} \{e_k, e_k^T\}$. 

35
To prove this, we will need to examine the matrices $e_k$ more closely.

**Proposition 13.5.** Let $i$, $j$, and $k$ be positive integers. Then

$$
(e_k)_{i,j} = \begin{cases} 
0 & \text{if } j < k \\
\delta_{j,k} & \text{if } i = 0 \\
(-1)^{i+j-k}(i+j-k-1) & \text{if } i > 0 \text{ and } j \geq k.
\end{cases}
$$

The proof is given in Appendix B. To illustrate the proposition, we exhibit $e_1$ below.

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & -1 & 1 & -1 & 1 & -1 & \cdots \\
0 & 1 & -2 & 3 & -4 & 5 & \cdots \\
0 & -1 & 3 & -6 & 10 & -15 & \cdots \\
0 & 1 & -4 & 10 & -20 & 35 & \cdots \\
0 & -1 & 5 & -15 & 35 & -70 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

Now we can prove Theorem 13.4.

**Proof of Theorem 13.4.** To prove this theorem, we must prove two statements. First, that infinite linear combinations of elements of this set converge, and second, that these linear combinations are exactly the fixed matrices of $\sigma$.

First, consider the sum $\sum_{k=0}^{\infty}(\alpha_k e_k + \beta_k e_k^T)$. To find the entry in the $(i,j)$ position, we only need to compute the sum $\sum_{k=0}^{\max\{i,j\}}(\alpha_k e_k + \beta_k e_k^T)$, because if $k > \max\{i,j\}$, then $(e_k)_{i,j} = 0$ and $(e_k^T)_{i,j} = 0$ by Proposition 13.5. Since we need only compute a finite sum to determine each entry, the sum converges. Now suppose that $A$ is a fixed matrix of $\sigma$. By uniqueness we have $A = a_{0,0}e_0 + \sum_{k=1}^{\infty}(a_{0,k}e_k + a_{k,0}e_k^T)$. Conversely, every sum of this form must be a fixed matrix because the matrices $e_k$ are. \qed

**References**

[1] T.-X. He and R. Sprugnoli. Sequence characterization of Riordan arrays. *Discrete Mathematics* 309 (2009): 3962-3974.

[2] L. Shapiro, S. Getu, W. Woan, and L. Woodson. The Riordan group. *Discrete Applied Mathematics* 34 (1991): 229-239.

[3] R. Sprugnoli. Riordan arrays and combinatorial sums. *Discrete Mathematics* 132 (1994): 267-290.

[4] F. M. Waltz and J. W. V. Miller. An efficient algorithm for Gaussian blur using finite-state machines. *SPIE Conference on Machine Vision Systems for Inspection and Metrology VII*. November 1998.
Appendix A: Riordan arrays

Riordan arrays are commonly used to study integer sequences, and they also form a group. These two facts help explain the reasons for the restrictions in Definition 4.2. Formally, we consider the functions $g$ and $f$ as members of $\mathbb{Z}[[x]]$, the ring of formal power series over the integers. The identity element under the multiplication given in Theorem 5.2 is $(1, x)$. To make the Riordan arrays a group, we need to ensure that each element $(g, f)$ has an inverse. Now, a formal power series $g = a_0 + a_1 x + a_2 x^2 + \cdots$ has a multiplicative inverse if and only if $a_0$ has an inverse in the ring of coefficients. We will denote the multiplicative inverse of $g$ by $\frac{1}{g}$. Similarly, a formal power series $f = b_0 + b_1 x + b_2 x^2 + \cdots$ has a compositional inverse if and only if $b_0 = 0$ and $b_1$ has an inverse. We will denote the compositional inverse of $f$ by $\overline{f}$. Now, assuming that these inverses exist, we have

$$(g, f)^{-1} = \left(\frac{1}{g(f)}, \overline{f}\right).$$

Therefore, to make the Riordan arrays into a group, we need to place some restrictions on $g$ and $f$ to ensure that $(g, f)$ has an inverse. Since $-1$ and $1$ are the only integers whose inverses are integers, imposing the restrictions in Definition 4.2 makes the set of Riordan arrays into a group.

We can relax these restrictions if we extend the ring of coefficients. For example, for formal power series over the rational numbers, every nonzero coefficient is invertible. Therefore, in this case we need only require $g(0) \neq 0$, $f(0) = 0$, and $f'(0) \neq 0$ to ensure that the Riordan arrays form a group.

Row and diagonal sums

There is a theorem similar to Theorem 5.2 that describes the result of multiplying a Riordan array on the right by a column vector.

**Theorem.** Suppose $v$ is an infinite column vector with generating function $v(x)$. Then the generating function for $(g, f) \cdot v$ is $g(x) \cdot v(f(x))$.

This is known as the Fundamental Theorem of Riordan Arrays; its proof can be found in [1]. The row sums of the Riordan array $(g, f)$ are the entries in the column vector $(g, f) \cdot \mathbf{1}$. Since the generating function for the all-ones vector is $\frac{1}{1-x}$, the generating function for the row sums is

$$\frac{g}{1-f}.$$
an example with Pascal’s Triangle.

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 0 \\
1 & 3 & 3 & 1 & 0 & 0 \\
1 & 4 & 6 & 4 & 1 & 0 \\
1 & 5 & 10 & 10 & 5 & 1
\end{pmatrix}
\quad \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 \\
1 & 3 & 1 & 0 & 0 & 0 \\
1 & 4 & 3 & 0 & 0 & 0
\end{pmatrix}
\]

The Riordan array multiplication rules still apply to the new array \((g, xf)\). Thus the row sums of \((g, xf)\), and consequently the diagonal sums of \((g, f)\), have generating function

\[\frac{g}{1 - xf}.\]

**Appendix B: Omitted proofs**

This appendix contains the proofs that were omitted from the body of the paper.

**Theorem 3.3** Let \(A\) be an \(n \times n\) matrix. The value of the single entry of the matrix \(\sigma^{n-1}(A)\) is \(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} (n-1)_{i-1} (n-1)_{j-1} \).

**Proof.** We proceed by induction. If \(A\) is 1 \(\times\) 1, then \(\sigma^{0}(A) = A\) by definition. Thus the only entry of \(\sigma^{0}(A)\) is \(a_{1,1} = \sum_{i=1}^{1} \sum_{j=1}^{1} (0)_{i-1} (0)_{j-1} a_{i,j} \).

Now suppose that the theorem holds for some nonnegative integer \(k\), and let \(A\) be \((k+1) \times (k+1)\). We are interested in the single entry of \(\sigma^{k}(A)\). By the induction hypothesis, this entry is

\[\sigma^{k-1}(\sigma(A))_{1,1} = \sum_{i=1}^{k} \sum_{j=1}^{k} \sigma(A)_{i,j} \binom{k-1}{i-1} \binom{k-1}{j-1}.\]

Expanding the collapsing sum yields

\[\sigma^{k}(A)_{1,1} = \sum_{i=1}^{k} \sum_{j=1}^{k} a_{i,j} + a_{i,j+1} + a_{i+1,j} + a_{i+1,j+1} \binom{k-1}{i-1} \binom{k-1}{j-1}.\]

We can break this into four double sums. For example, we have

\[
\sum_{i=1}^{k} \sum_{j=1}^{k} a_{i+1,j} \binom{k-1}{i-1} \binom{k-1}{j-1} = \sum_{i=2}^{k} \sum_{j=1}^{k} a_{i,j} \binom{k-1}{i-2} \binom{k-1}{j-1} = \sum_{i=1}^{k+1} \sum_{j=1}^{k+1} a_{i,j} \binom{k-1}{i-2} \binom{k-1}{j-1},
\]
where the second equality holds because \( \binom{k-1}{i-2} \) and \( \binom{k-1}{i+1} \) are both 0. Using similar processes to change the limits of all four double sums yields

\[
\sigma^k(A)_{1,1} = \sum_{i=1}^{k+1} \sum_{j=1}^{k+1} a_{i,j} \left( \binom{k-1}{i-1} \binom{k-1}{j-1} \right) + \sum_{i=1}^{k+1} \sum_{j=1}^{k+1} a_{i,j} \left( \binom{k-1}{i-2} \binom{k-1}{j-1} \right)
\]

which simplifies to

\[
\sigma^k(A)_{1,1} = \sum_{i=1}^{k+1} \sum_{j=1}^{k+1} a_{i,j} \left[ \binom{k-1}{i-1} \binom{k-1}{j-1} \right] + \sum_{i=1}^{k+1} \sum_{j=1}^{k+1} a_{i,j} \left[ \binom{k-1}{i-2} \binom{k-1}{j-1} \right]
\]

This completes the induction. \(\square\)

**Lemma 4.5** Let \( t \) be a positive integer and \( g_t \) and \( f \) be as stated in Theorem 4.4. Then \( (g_t, f)_{n,0} = 1 \) for all \( n \geq t - 1 \).

**Proof.** Since \( \frac{1}{1-2x} \) is the generating function for the sequence \( \{2^r\} \), it follows that \( \frac{1}{(1-x)(1-2x)} \) is the generating function for the sequence \( \{\sum_{\ell=0}^{r} 2^\ell\} = \{2^{r+1} - 1\} \). We begin with the following preliminary calculation.

\[
[x^n] \frac{(1-x)^t}{(1-x)(1-2x)} = \sum_{\ell=0}^{n} (-1)^{n-\ell} \binom{t}{n-\ell} (2^{\ell+1} - 1)
\]

\[
= \sum_{\ell=n-t}^{n} (-1)^{n-\ell} \binom{t}{n-\ell} (2^{\ell+1} - 1) \quad (*)
\]

\[
= \sum_{\ell=0}^{t} (-1)^{t-\ell} \binom{t}{t-\ell} 2^{\ell+n-1} \quad (**)\]

\[
= 2^{n-1} \sum_{\ell=0}^{t} (-1)^{t-\ell} \binom{t}{\ell} 2^\ell
\]

\[
= 2^{n-t+1}(-1+2)^t
\]

\[
= 2^{n-t+1}.
\]
In (*), we simply eliminated values of $\ell$ where the binomial coefficient is 0; to get (**), we substituted $\ell + n - t$ for $\ell$ and noted that $\sum_{\ell=0}^{t}(-1)^{t-\ell}(\binom{\ell}{t-\ell})(-1) = -(1-1)^{t} = 0$. Thus

$$(gt,f)_{n,0} = [x^{n}]rac{(1-x)^{t} - x^{t}}{(1-x)(1-2x)}$$

$$= [x^{n}]\frac{(1-x)^{t}}{(1-x)(1-2x)} - [x^{n}]\frac{x^{t}}{(1-x)(1-2x)}$$

$$= 2^{n-t+1} - 1$$

$$= 2^{n-t+1} - 2^{n-t+1} - 1$$

$$= 1,$$

as claimed. To check that the above argument holds when $n = t - 1$, we need to show that

$$(x^{t-1})\frac{x^{t}}{(1-x)(1-2x)} = 2^{(t-1)-t+1} - 1,$$

but this clearly holds, since the coefficient on $x^{t-1}$ is 0 and $2^{(t-1)-t+1} - 1 = 2^{0} - 1 = 0$. 

Lemma 4.6. Let $t$ be a positive integer and $g_t$ and $f$ be as stated in Theorem 4.4. Then $(g_t,f)_{t-1,k} = 1$ for all $0 \leq k < t$ and $(g_t,f)_{t-1,k} = 0$ for all $k \geq t$.

Proof. Let $(g_t,f) = (d_{n,k})$. We will repeatedly use the recurrence

$$\frac{(1-x)^{t} - x^{t}}{(1-2x)} = (1-x)\frac{(1-x)^{t-1} - x^{t-1}}{(1-2x)} + x^{t-1},$$

which can be easily checked. For $0 \leq k < t$ we have

$$d_{t-1,k} = [x^{t-1}]x^{k}\frac{(1-x)^{t} - x^{t}}{(1-2x)(1-x)^{k+1}} = [x^{t-k-1}]\frac{(1-x)^{t} - x^{t}}{(1-2x)(1-x)^{k+1}}.$$ 

Now we make use of the recursion:

$$d_{t-1,k} = [x^{t-k-1}]\frac{1}{(1-x)^{k+1}} \left[ \frac{(1-x)((1-x)^{t-1} - x^{t-1})}{1-2x} + x^{t-1} \right].$$

However, since $t - k - 1 < t - 1$, we have

$$[x^{t-k-1}]\frac{x^{t-1}}{(1-x)^{k+1}} = 0,$$

and thus

$$d_{t-1,k} = [x^{t-k-1}]\frac{1}{(1-x)^{k}} \left[ \frac{(1-x)^{t-1} - x^{t-1}}{1-2x} \right].$$

[40]
We can iterate this process to yield

\[ d_{t-1,k} = \left[ x^{t-k-1} \right] \frac{1}{(1-x)^{k+1-q}} \left[ (1-x)^{t-q} - x^{t-q} \right], \]

so long as

\[ \left[ x^{t-k-1} \right] \frac{x^{t-q}}{(1-x)^{k+2-q}} = 0. \]

This holds when \( t - k - 1 < t - q \), that is, when \( q < k + 1 \). In particular, we can take \( q = k \) to get

\[ d_{t-1,k} = \left[ x^{t-k-1} \right] \frac{(1-x)^{t-k} - x^{t-k}}{(1-x)(1-2x)}. \]

From the proof of Lemma 4.5, we know that this is equal to 1.

Now we consider \( k \geq t \). Again, we have

\[ d_{t-1,k} = \left[ x^{t-1} \right] x^k \frac{(1-x)^{t} - x^t}{(1-2x)(1-x)^{k+1}} = \left[ x^{t-k-1} \right] \frac{(1-x)^{t} - x^t}{(1-2x)(1-x)^{k+1}}. \]

But since \( k \geq t \), there is no \( x^{t-k-1} \) term, so \( (g_t, f)_{t-1,k} = 0. \)

\[ \square \]

**Proposition 12.4.** Let \( A \) be an \((m+1) \times (n+1)\) matrix. Then \( \hat{\tau}(A) = \hat{R}_m A \hat{R}_n^T \).

**Proof.** We induct on the position of the entries of \( \hat{R}_m A \hat{R}_n^T \). The base cases are the positions \((i, j)\) with at least one of \( i \) or \( j \) less than 1. Now suppose that \( \hat{\tau}(A)_{i,j} = [\hat{R}_m A \hat{R}_n^T]_{i,j} \) whenever either \( i \leq p \) or both \( i = p \) and \( j \leq q \). Then

\[
[\hat{R}_m A \hat{R}_n^T]_{p,q} = \sum_{i=1}^{m+1} \sum_{j=1}^{n+1} (\hat{R}_m)_{p,i} a_{i,j} (\hat{R}_n^T)_{j,q}
\]

\[ = \sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{p-i}(-1)^{q-j} a_{i,j} \]

\[ = a_{p,q} + \sum_{i=1}^{p-1} \sum_{j=1}^{q} (-1)^{p-i}(-1)^{q-j} a_{i,j} + \sum_{i=1}^{p} \sum_{j=1}^{q-1} (-1)^{p-i}(-1)^{q-j} a_{i,j} \]

\[ - \sum_{i=1}^{p-1} \sum_{j=1}^{q} (-1)^{p-i}(-1)^{q-j} a_{i,j} \]

\[ = a_{p,q} - \hat{\tau}(A)_{p-1,q} - \hat{\tau}(A)_{p,q-1} + \hat{\tau}(A)_{p-1,q-1} \]

\[ = \hat{\tau}(A)_{p,q}. \]
where the penultimate equality follows from the induction hypothesis. Since all entries match, the matrices are equal.

\[ \text{Proposition 13.5} \]

Let \( i, j, \) and \( k \) be positive integers. Then

\[
(e_k)_{i,j} = \begin{cases} 
0 & \text{if } j < k \\
\delta_{j,k} & \text{if } i = 0 \\
(-1)^{i+j-k}(i+j-k-1) & \text{if } i > 0 \text{ and } j \geq k.
\end{cases}
\]

\[ \text{Proof.} \] Let \( k \) be given and \( A \) be the bi-infinite matrix with entries

\[
a_{i,j} = \begin{cases} 
0 & \text{if } j < k \\
\delta_{j,k} & \text{if } i = 0 \\
(-1)^{i+j-k}(i+j-k-1) & \text{if } i > 0 \text{ and } j \geq k.
\end{cases}
\]

Note that \( a_{0,k} = 1 \) and every other entry in the first row and column of \( A \) is 0. Therefore, if we show that \( \sigma(A) = A \), we are done by the uniqueness of a fixed matrix with a given first row and column. We first check the edge cases. Suppose that \( j < k - 1 \). Then

\[
\sigma(A)_{i,j} = a_{i,j} + a_{i+1,j} + a_{i,j+1} + a_{i+1,j+1} = 0 + 0 + 0 = a_{i,j}.
\]

If \( j = k - 1 \) and \( i > 0 \), we have

\[
\sigma(A)_{i,k-1} = 0 + 0 + (-1)^i \binom{i-1}{i-1} + (-1)^{i+1} \binom{i}{i} = 0 = a_{i,k-1}.
\]

If \( j = k - 1 \) and \( i = 0 \), then

\[
\sigma(A)_{0,k-1} = 0 + \delta_{k,k} + (-1)^1 \binom{0}{0} = a_{0,k-1}.
\]

If \( j \geq k \) and \( i = 0 \) then

\[
\sigma(A)_{0,j} = \delta_{j,k} + (-1)^{1+j-k} \begin{pmatrix} 1+j-k-1 \\ 0 \end{pmatrix} + \delta_{j+1,k} + (-1)^{1+j+1+k} \begin{pmatrix} 1+j-k \\ 0 \end{pmatrix}
\]

\[
= \delta_{j,k} \\
= a_{0,j}.
\]

Now we just need to check the case where \( i > 0 \) and \( j \geq k \). We have

\[
\sigma(A)_{i,j} = (-1)^{i+j-k} \left[ \begin{pmatrix} i+j+1-k \\ i \end{pmatrix} - \begin{pmatrix} i+j-k \\ i-1 \end{pmatrix} - \begin{pmatrix} i+j-k \end{pmatrix} - \begin{pmatrix} i+j-k-1 \end{pmatrix} \right]
\]

\[
= (-1)^{i+j-k} \begin{pmatrix} i+j-k \\ i \end{pmatrix} - \begin{pmatrix} i+j-k-1 \end{pmatrix}
\]

\[
= (-1)^{i+j-k} \begin{pmatrix} i+j+k-1 \\ i-1 \end{pmatrix}
\]

\[
= a_{i,j}.
\]

Thus \( \sigma(A) = A \), so by uniqueness, we have \( A = e_k \).