TUNNELING METHOD FOR HAWKING RADIATION IN THE NARIAI CASE

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Abstract. We revisit the tunneling picture for the Hawking effect in light of the charged Nariai manifold, because this general relativistic solution, which displays two horizons, provides the bonus to allow the knowledge of exact solutions of the field equations. We first perform a revisitation of the tunneling ansatz in the framework of particle creation in external fields à la Nikishov, which corroborates the interpretation of the semiclassical emission rate $\Gamma_{\text{emission}}$ as the conditional probability rate for the creation of a couple of particles from the vacuum. Then, particle creation associated with the Hawking effect on the Nariai manifold is calculated in two ways. On the one hand, we apply the Hamilton-Jacobi formalism for tunneling, in the case of a charged scalar field on the given background. On the other hand, the knowledge of the exact solutions for the Klein-Gordon equations on Nariai manifold, and their analytic properties on the extended manifold, allow us a direct computation of the flux of particles leaving the horizon, and, as a consequence, we obtain a further corroboration of the semiclassical tunneling picture from the side of S-matrix formalism.

1. Introduction

Tunneling through the horizon is a longstanding approach to Hawking effect since the seminal papers by S.W.Hawking [1, 2]. Between the various methods corroborating the original calculations, the so-called tunneling method has been proposed. We limit ourselves herein to quote some seminal papers and a fine review [3-8]. The Hamilton-Jacobi (HJ henceforth) formalism for the calculation of the particle creation associated with the Hawking effect represents a semiclassical approach where the classical action of particles is computed along trajectories which pass through the horizon. A special version of the method is represented by the Parikh-Wilczek approach [9], where a tunneling through the horizon of a particle arises because of a quite unexpected mechanism, where the tunneling particle sets up the barrier by energy conservation, as nicely described by Parikh [9].

We revisit the HJ tunneling method for the Hawking effect by taking into account a the charged Nariai solution, in view of the fact that it allows to gain the knowledge of exact solutions of the field equations even in the inner black hole region. This is our basic reason for studying tunneling in this particular manifold, with the aim of considering it as a further benchmark for the tunneling method, which is of course a very useful and simple method for deriving the Hawking effect, but whose status is not so firmly grounded on the theoretical side [10]. With this aim, we first...
reinterpret the semiclassical emission rate

$$\Gamma_{\text{emission}} = \exp(-2\text{Im}S),$$

(1)

where $S$ is the classical action, as the conditional probability rate for the creation of a couple of particles from the vacuum (to be intended as the state of absence of particles). We remark that eq. (1) is the standard reference equation for the literature on the tunneling method, and e.g. in static backgrounds one may separate in the full action $S$ a temporal part $\omega t$, where $\omega$ is the particle energy, and a spatial one $S_0(x)$ which depends only on the spatial variables, and eq. (1) is often written as $\Gamma_{\text{emission}} = \exp(-2\text{Im}S_0) = \exp(-2\text{Im} \int p dq)$, where $p,q$ are canonically conjugate (see e.g. [7]). The latter expression, as well as (10), has the drawback to be not invariant under canonical transformations, as is remarked first in [11] and then in [7, 12]. Such an invariance is achieved by the expression $\Gamma_{\text{emission}} = \exp(-\text{Im} \oint p dq)$. See also sect. 5 for further discussion.

Then, we consider the field equations on the Nariai manifold, and set up a scattering picture for the tunneling process, whose key-ingredient is the requirement for analyticity of the exact solutions both in the inner region and in the outer one with respect to the black hole horizon. Explicit computation of the flux of particles through the horizon corroborates the standard tunneling ansatz, which is also taken into account.

The present analysis completes our previous studies concerning the quantum instability of the charged Nariai solutions [13, 14], where the Hawking effect was not derived. Moreover, we mention that in [15] an early study about the instability due to quantum matter of Nariai-like metrics appeared.

2. The HJ method for tunneling

The basic idea of the HJ method is very simple, and consists in adopting the WKB approximation and computing the tunneling probability for a straddling mode, to be intended as a mode whose wave function is regular on the horizon and also defined across the horizon itself [3, 5, 16]. Subtleties occur if singular coordinate systems on the horizon are adopted, as pointed out e.g. in [7, 12, 17, 18, 19, 20, 8]. Former calculations appear in [4], and further development are contained in [21, 22]. A thoroughful analysis and review is contained in [8], to which we refer the reader for a more complete list of references. We also refer to [4, 21, 22, 23].

As a basic ingredient of the approach, we have the classical action $S$ of particles (massless or not), to be computed along trajectories which pass through the horizon. The semiclassical emission rate is given by (1), whose right-hand side is easily realized to correspond to the standard form for the rate of emission associated with tunneling through a potential barrier in the WKB approximation. Still, the barrier is non-standard, being present a single turning point against the usual couple of turning points for standard barriers. The horizon plays the role of a unattainable limiting region for signals in the inner region of the black hole, much more than a real potential barrier. Moreover, a very non-trivial transition between a spacetime region with a time-dependent metric (black-hole region) to a static region (exterior of the black hole) is being occurring, so it is the case to remark that ‘standard interpretations’ are not so well grounded or, at the very least, free of misinterpretations. See also [10]. By following [20], it is interesting to write down the action
as follows:

\[ S = \int dS = \int \gamma (\partial_x, S) dx^i, \]  

(2)

where \( dS \) is the one-form corresponding to the differential of \( S \), and an integration along an oriented, null path is understood, and this is at the root of the so called null geodesic method [8]. In such a way, \( dS \) is written in terms of the differential of coordinates times the conjugate momenta \( p_i = (\partial_x, S) \) for \( i = 0, 1, 2, 3 \) (a change of sign in the 0-component can occur with respect to this definition).

3. TUNNELING METHOD AND A TRICK À LA NIKISHOV

Our ansatz herein is that the probability rate of pair creation near the black hole horizon \( \Gamma_{\text{emission}} \) can be interpreted as the conditional probability rate for the creation of a couple of particles from the vacuum (to be intended as the state of absence of particles). This interpretation is non-standard, and suggests that \( \Gamma_{\text{emission}} \) is just more the square of the relative weight between the outer part and the inner part of the straddling mode than the pair-creation rate itself. This argument is to be compared with the argument in [24], which is relative to the original picture by Damour and Ruffini [3].

For the following general picture, we refer to [25, 26]. We recall that the imaginary part of the effective action \( W \) is the signal of particle production. Indeed, the permanence of the vacuum has probability \( \mathbb{P} < 1 \): particle creation occurs with probability (per unit time)

\[ P_{0 \to 0} = \exp(-2\text{Im}W). \]  

(3)

One can notice the resemblance with the formula defining \( \Gamma_{\text{emission}} \), but the relation between \( \text{Im}W \) and \( \Gamma_{\text{emission}} \) is not so straightforward. Still, it exists and is found below.

Basically, the following idea is pursued. We proceed as in [26] for the general picture.

Let us introduce, for a diagonal scattering process,

\[ n_i^{IN} = R_i n_i^{OUT} + T_i p_i^{OUT}, \]  

(4)

where \( n_i \) stays for a negative energy mode and \( p_i \) for a positive energy one. In case an inner product different from the standard one for bosonic and fermionic fields occurs, ‘positive energy’ should be replaced by ‘positive norm’ (and analogously for negative energy). \( p_i^{IN}, n_i^{IN} \) form a scattering basis for the IN states, and \( p_i^{OUT}, n_i^{OUT} \) form a scattering basis for the OUT states. \( T_i \) is the transmission coefficient and \( R_i \) is the reflection one. It is evident that above we have written a Bogoliubov transformation between IN and OUT states, so the following identification is also true: \( R_i = \alpha_i, T_i = \beta_i \). Moreover, one defines as in [26]

\[ \eta_i := |T_i|^2, \]  

(5)

which can be shown to coincide with the mean number per unit time and unit volume of created particles. One has \( |R_i|^2 = 1 \mp \eta_i \), where, here and in the sequel, the upper sign holds for fermions and the lower one for bosons. By interpreting à la Stueckelberg the scattering process, one can also obtain \( n_i^{OUT} = R_i^{-1} n_i^{IN} - R_i^{-1} T_i p_i^{OUT} \), which is interpreted as the scattering of a negative mode incident from
the future and which is in part refracted in the past and in part reflected in the
future. The new reflection amplitude \( -R_i^{-1}T_i \) is such that the reflection coefficient
\[
|R_i^{-1}T_i|^2 = \frac{\eta_i}{1 \mp \eta_i} = \tilde{P}_i(1|0) \quad (6)
\]
can be interpreted as the conditional probability rate \( \tilde{P}_i(1|0) \) for the creation
of the pair \( n_{i,\text{out}}^{\text{out}} \cdot \tilde{p}_{i,\text{out}}^{\text{out}} \), starting from absence of particles in that state. The
conditional probability rate for \( n \) couples is \( \tilde{P}_i(n|0) = (\tilde{P}_i(1|0))^n \). Of course, in the
fermionic case only \( n = 1 \) is allowed. The probability rate for \( n \) couples is \( \tilde{P}_i(n) = \tilde{P}_i(n|0)\tilde{P}_i(0) \). \( \tilde{P}_i(0) \) represents the probability rate that no particles are created in
the given state \( i \). It can be calculated as follows:
\[
\sum_n \tilde{P}_i(n) = 1 = \tilde{P}_i(0) \sum_n \tilde{P}_i(n|0),
\]
and then
\[
\tilde{P}_i(0) = (1 \mp \eta_i)^{\pm 1}. \quad (7)
\]

As to the mean number of created couples, we have
\[
<n_i> = \sum_n n\tilde{P}_i(n) = \eta_i = |\beta_i|^2. \quad (9)
\]
As a consequence, in the above formulas we realize that \( \eta_i \rightarrow <n_i> \) is allowed.

Let us apply the above picture to our specific case. We interpret \( \Gamma_{\text{emission}} \) as
follows:
\[
\Gamma_{\text{emission}} = \tilde{P}_\omega(1|0), \quad (10)
\]
where \( \omega \) identifies the quantum state. In the present case, we get for bosons
\( \tilde{P}_\omega(0) = 1 - \exp(-\beta\omega) \), with \( \beta = \beta_H = 1/(k_{\text{boltzmann}}T_H) \) \( (T_H \) is the black hole
temperature). As a consequence, one gets \( \tilde{P}_\omega(n) = (1 - \exp(-\beta\omega))\exp(-\beta\omega n) \).
It is then easy to show that the mean number of created pairs in the state with
energy \( \omega \) is
\[
<n_\omega> = \sum_{n=0}^{\infty} n\exp(-\beta\omega n) = \frac{1}{\exp(\beta\omega) - 1}, \quad (11)
\]
which is the correct result. This argument is substantially equivalent to the one of
ref. \cite{27}. Note that a thermal particle distribution is obtained without recurring to
detailed balance arguments.

It is also worth noticing that it holds
\[
\text{Im}W = -\frac{1}{2} \int d\omega \log(1 - \exp(-\beta\omega)), \quad (12)
\]
which is the expected result\cite{27 28}. The calculation in the fermionic case is
analogous, and is based on the fact that the WKB approximation for the Dirac
equation coincides with the HJ equation. The only change is the statistics. We
point out again the substantial difference between the expressions for \( \Gamma_{\text{emission}} \) and
\( \exp(-2\text{Im}W) \) appearing in \cite{9}. As shown above, \( \Gamma_{\text{emission}} \) is the conditional probability
\cite{10} for the emission of a pair labeled by \( \omega \), whereas \( \exp(-2\text{Im}W) \) is the

\footnote{For the sake of completeness, one should write \( \text{Im}W = \frac{1}{2} \sum_{\omega,l,m} \log(1 + <n_{\omega,l,m}>) \), which takes into account the full dependence on quantum numbers, and one realizes that the label \( \omega \) introduced in \cite{10} is split, with some abuse of language, into \( \omega, l, m \), where \( \omega \) is the energy, and \( l, m \) are the usual quantum numbers for angular momentum.}
probability that, for any field mode with label \( \omega \), there is not a quantum instability in the field at hand, and then it involves a sum over all values of \( \omega \) (cf. (9)). Extensions to the cases where one takes into account also the backscattering (which is mandatory in 4D) are discussed in [21]. We also notice that the above interpretation concerning the meaning of \( \Gamma_{\text{emission}} \) hold true also for the Parikh-Wilczek approach.

4. **Charged Nariai manifold**

We describe herein the electrically charged Nariai solution. We shall consider Kruskal-like coordinates, that are introduced in the following for the black hole horizon \( \chi = \pi \), which is our main focus, and then for the cosmological horizon \( \chi = 0 \) (see below).

The manifold is described by the metric [29, 30, 31]

\[
ds^2 = \frac{1}{A}(- \sin^2(\chi) d\psi^2 + d\chi^2) + \frac{1}{B} (d\theta^2 + \sin^2(\theta) d\phi^2),
\]

with \( \psi \in \mathbb{R}, \chi \in (0, \pi) \), and the constants \( B = \frac{1}{2\sqrt{A}} \left( 1 - \sqrt{1 - 12 \frac{Q^2}{L^2}} \right) \), \( A = \frac{6}{L^2} - \frac{1}{B} \) are such that \( \frac{1}{B} < 1 \), and \( L^2 := \frac{3}{\Lambda} \). The black hole horizon occurs at \( \chi = \pi \).

This manifold has finite spatial section. In the Euclidean version, it corresponds to two spheres characterized by different radii. For the gauge potential we can choose \( A_i = -Q \frac{B}{A} \cos(\chi) \delta^0_0 \).

We consider the \( \psi-\chi \) part of the metric, and introduce a diffeomorphism \( \chi = \chi(r) \) such that \( \frac{1}{B}(- \sin^2(\chi) d\psi^2 + d\chi^2) = f(r) \frac{1}{B}(- d\psi^2 + dr^2) \). This can be obtained for \( f(r) = \sin^2(\chi) = \frac{1}{cosh^2(r)} \), and we can choose the branch \( \chi = 2 \arctan(\exp(-r)) \), which is useful because the tortoise-like coordinate \( r \) with this choice is such that \( r \to -\infty \) as \( \chi \to \pi^- \), as e.g. in the Schwarzschild case. Note that we get \( \cos(\chi) = \tanh(r) \).

We need to introduce Kruskal-like coordinates. Then, we first introduce null coordinates

\[
u = \frac{1}{\kappa}(\psi - r), \quad (14)
\]

\[
u = \frac{1}{\kappa}(\psi + r), \quad (15)
\]

where \( \kappa = \sqrt{A} \) is the surface gravity; then we can define the Kruskal-like coordinates adapted to the black hole horizon region \( \chi = \pi \):

\[
U = - \exp(-\kappa u), \quad (16)
\]

\[
V = \exp(\kappa v). \quad (17)
\]

Then we obtain \( ds^2 = -\frac{4}{A} \frac{1}{1-UV} dU dV + \frac{1}{B} (d\theta^2 + \sin^2(\theta) d\phi^2) \). We note that in our latter coordinate chart we get \( \partial_\psi = -U \partial_U + V \partial_V \), which will be useful in the following. We also need to introduce a gauge transformation in order to obtain a gauge potential which is regular on the horizon: \( A'_\mu = A_\mu - \partial_\mu G \), where \( G = -eQB \frac{\psi}{A} \).

Then we get \( eA'_U = \frac{1}{2U} eA_0 = -eQB \frac{V}{A} \frac{U}{1-UV} \). Analogously, we have \( eA'_V = \frac{1}{2V} eA_0 = eQB \frac{U}{A} \frac{V}{1-UV} \). The aforementioned gauge transformation is such that the following shift in the one-particle energy occurs:

\[
\omega \mapsto \omega + e\Phi_H, \quad (18)
\]
where $e^\Phi_H = \frac{e^{QB}}{A}$. For the cosmological horizon region we introduce further Kruskal coordinates $\bar{U} = \exp(\kappa u)$, $\bar{V} = -\exp(-\kappa v)$. The cosmological horizon $\chi = 0$ corresponds to $\bar{U} = 0$ and $\bar{V} = 0$, and analogous equations can be found. In particular, a further gauge transformation regularizing the potential on the cosmological horizon can be analogously given.

5. **Hawking effect on Nariai manifold in the HJ formalism**

In this section, we set up the HJ approach to tunneling for the Nariai charged solution in the case of a charged scalar field. It is worth mentioning that Hawking effect in the tunneling framework in de Sitter spacetime has been considered several times, in different situations which are mainly involved with the Schwarzschild-de Sitter solutions. See e.g. refs. [32, 33, 34, 35, 36, 37], where both the HJ approach and the Parikh-Wilczek one are considered.

We focus our attention on the HJ equation for the black hole Kruskal patch (we recall that the black hole horizon corresponds to $\chi = \pi$ in the original coordinates (13)): $$(19) \quad -(1 - UV)^2(k_U + eA_U)(k_V + eA_V) + \mu^2 = 0,$$
and then the action is automatically separated in its temporal part and in its spatial one. As a consequence, we have $\partial_e S = -Uk_U + Vk_V = -\frac{\dot{\chi}}{\kappa}$. The fundamental amplitude to be calculated is$$\Gamma = \exp(-2 \text{Im} \int dU (\partial_U S + eA'_U)). \quad (21)$$

We notice that the former expression (21) implicitly takes into account the temporal contribution to $\exp(-\text{Im} \oint p dq)$ one has to include in order to obtain a consistent implementation of the tunneling picture [17, 18, 19]. Explicitly, one has $\Gamma = \exp(-\text{Im} \int (\omega \Delta t_{\text{out}} + \omega \Delta t_{\text{in}} - (\int p_{\text{out}} dq - \int p_{\text{in}} dq)))$, where $\Delta t_{\text{out}}$ refers to the temporal contribution for outgoing particles, and analogously for $\Delta t_{\text{in}}$ [17]. See also [38] for further discussion. The point is that we are substantially implementing the null geodesic method discussed in [20, 8]. The result we obtain is thus correct, because, as shown in [8] (see in particular sect. 3 therein), the null geodesic method is covariant and invariant under canonical transformations, and equivalent to the HJ method. In order to perform the above integration, we must regularize the integral by choosing a suitable circuit in the complex plane. This amounts to a definition of the aforementioned integral, which is otherwise ill-defined. We obtain$$\int_\gamma dU K_U(U) = \int_{U_1}^\Gamma dU K_U(U) + \int_{C_\epsilon} dz K_U(z) + \int_{-\epsilon}^{U_2} dU K_U(U), \quad (22)$$

where $U_1 > 0, U_2 < 0, K_U := \partial_U S + eA'_U$, and $C_\epsilon$ is a semicircle which is oriented anti-clockwise centered in $U = 0$ and in the upper half $U$-plane, and then an antiparticle state is occurring, in agreement with the picture in [8] [5]. See also the following section. The only contribution to the imaginary part of the action
S is due to \( \int_{C_{\pm \epsilon}} dz \ K_U(z) \). Indeed, a simple application of the fractional residue theorem \([39], p. 209\) leads to

\[
\lim_{\epsilon \to 0^+} \int_{C_{\pm \epsilon}} dz \ K_U(z) = \pm i \pi \text{Res}[K_U(z), U = 0],
\]

where \( C_{\pm \epsilon} \) refers to a semicircle oriented clockwise for negative sign (\( C_{-\epsilon} \)) and anti-clockwise for positive sign (\( C_{+\epsilon} \)), which in the present case gives

\[
2 \text{ Im} \int dU (\partial U S + eA_U) = \frac{2 \pi}{\kappa} \left( \omega - \frac{eQB}{A} \right),
\]

which is the expected result. Absorption occurs instead with conditional probability equal to 1, which is both the classical result and also compatible with the detailed balance argument. In the case of the cosmological horizon, analogously one finds

\[
2 \text{ Im} \int d\bar{U} (\partial \bar{U} S + eA_{\bar{U}}) = \frac{2 \pi}{\kappa} \left( \omega + \frac{eQB}{A} \right).
\]

6. Hawking emission in S-matrix formalism

The semiclassical picture represented by the tunneling ansatz can also be corroborated by an S-matrix approach, due to the fact that we can calculate exact solutions of the Klein-Gordon equation and know their analyticity properties on the Nariai manifold. We consider the Hawking effect from the point of view of scattering theory. In order to set up a tunneling picture, we need to consider a rather unusual picture where the scattering takes place part inside the black hole horizon and part outside in the external region, with the black hole horizon playing the role of barrier. Needless to say, this is the original picture proposed by Hawking and then by Hartle and Hawking in a path integral formalism \([1, 2]\). Herein, we simply limit ourselves to propose this picture for the case at hand, in a non-dynamical situation.

We start from the Klein-Gordon equation of the Nariai background for the scalar field \( \Phi \). We use rescaled physical quantities, for example we have for particle energy \( \omega = \frac{\omega_{\text{phys}}}{\kappa} \). In the following, we shall indicate with \( \omega \) both the rescaled variable and the physical value, in order to avoid to make heavy the notation. Separation of variables \( \Phi = e^{-i\omega \psi} Y_{lm}(\Omega) \Phi(\chi) \) and a change of variable \( t = -\cos \chi \) lead to the following reduced ‘radial’ equation \([14]\)

\[
(1 - t^2)\Psi'' - 2t \Psi' + \left[ \frac{1}{1 - t^2} (\omega + eQB_A t) - \mu^2 \right] \Psi = 0,
\]

where the prime is the derivation w.r.t. \( t \). Note that this equation is invariant under \( \{ t \to -t, Q \to -Q \} \) so that we can look at the singularity in \( t = 1 \) only and obtain the properties of the singularity in \( t = -1 \) by \( Q \to -Q \). Now, the behaviour of the above equation near \( t = 1 \) suggests to set

\[
\Psi(t) = (1 - t)^{l_+} (1 + t)^{l_-} \Phi(t),
\]

where \( l_\pm = \frac{i}{2} |\omega \pm eE| \), and \( E := Q \frac{B_A}{A} \). The equation for the function \( \Phi \) is

\[
(1 - t^2)\Phi'' - 2(t - l_+(1 - t) + l_-(1 + t))\Phi' - d_t \Phi = 0,
\]
where \( d_1 := \mu_i^2 - \omega^2 + l_+ + l_- - (l_+ - l_-)^2 \). It is easy to infer that the general solution for \( \Phi \) is given by a hypergeometric function:

\[
\Phi(t) = C_+ F(a, b, c_+, t_+) + C_- F(a, b, c_-, t_-),
\]

where \( F(a, b; c; z) \) is the usual hypergeometric function and \( a := i e E + \frac{1}{2} + i \sqrt{\Delta} \), \( b := ie E + \frac{1}{2} - i \sqrt{\Delta} \), \( c_\pm := i(eE \pm \omega) + 1 \), \( t_\pm := \frac{1 \mp 1}{2} \), and \( \Delta := \mu_i^2 + (eE)^2 - \frac{1}{4} \). The above solution holds in the level-crossing region, i.e., for \(-eE < \omega < eE\), where also pair emission of charged particles occurs \[14\]. Still, that solution is easily shown to hold also outside the level crossing region. We then consider Kruskal-like coordinates as in the previous section and then we get \( \psi(U, V) \) and \( t(U, V) \) (which are not explicitly calculated). Of course, using \( U, V \) one is allowed to extend solutions inside the black hole. Analytic continuation is allowed, and we have to look about the branch singularities of the hypergeometric functions combined with the one associated to the factor \((1 - t)^l + (1 + t)^l\). We can easily deduce that \( \Psi(t) \) presents the same singularities as the function \( \tilde{\Psi}(t) := C_1 (1 - t)^l + (1 + t)^l - C_2(1 - t)^l + (1 + t)^l \). Thus, we have a logarithmic branch point at \( t = 1 \) (black hole horizon) and one at \( t = -1 \) (cosmological horizon). This implies the presence of a logarithmic branch cut (we choose the negative real axis, as usual) and the appearance of a suitable exponential factor as one passes the horizon. See below.

In a scattering picture, one sets up a so-called straddling mode \[5\]. This mode can be obtained by analytic continuation from the outgoing one as in the original picture by Damour and Ruffini \[3\]. See also \[40, 41\] for more recent applications of the Damour-Ruffini picture.

The most simple analysis can be performed in the case of a uncharged massless scalar field \( e = 0, m = 0 \) in the s-wave \( l = 0 \) (which is the leading contribution to the Hawking radiation). One obtains for the \( \psi - t \) part \( \eta(\psi, t) \) of the wave function in Kruskal-like coordinates

\[
\eta(U, V) = c_1 (-U)^{i\omega} + c_2 V^{-i\omega}
\]

which gives us an outgoing mode emerging from the black hole horizon for \( c_2 = 0 \). Then we find \( \eta(U, V)_{\text{outgoing}} = c_1 (-U)^{i\omega} \); a negative norm mode which straddles the horizon is obtained as follows \[3, 5\):

\[
\eta(U, V)_{\text{straddle}} = N_s \eta(-U + i \epsilon, V)_{\text{outgoing}},
\]

by analytic continuation. In passing the black hole horizon \( U = 0 \) \( (r = -\infty) \) a contribution from the logarithmic branch point arises, so that

\[
\eta(U, V)_{\text{straddle}} = N_s \int (-U)^{i\omega} e^{-\pi \omega} \frac{U}{U^\omega} - U > 0, \quad U > 0.
\]

As a consequence, one finds the usual temperature \( \beta = \frac{2\pi}{\kappa} \). Indeed, the following result is easily obtained from the above calculations: \( |N_s|^2 = \frac{1}{e^{\beta(\omega + c\Phi_H)} - 1} \), which also gives the mean number of created pairs.

In the general case, one proceeds as above, with the only difference that \[28\] is replaced by the more involved expression of the complete solution, where both the hypergeometric function and also the more involved factors \((t \pm 1)^l\) appear. Still, \[30\] remains true, and we get

\[
|N_s|^2 = \frac{1}{e^{\beta(\omega + c\Phi_H)} - 1}.
\]

\[\text{To be precise this is true only at finite, since they have different singularities at infinity.}\]
which is expected.

In order to corroborate the above scattering picture, we can proceed as follows. Let us consider the balance of fluxes by using the conserved current \( J_r := -\frac{1}{2} (\eta^*(\partial_r \eta) - (\partial_r \eta)^* \eta) \), in Kruskal-like coordinates, where for the region outside the black hole horizon we get \( \partial_r = U \partial_U + V \partial_V \). It is also interesting to point out that, in the inner part of the black hole, where \( U > 0, V > 0 \), by maintaining the same definition for \( u, v \), we have \( \partial_r = -U \partial_U + V \partial_V \), and also \( \partial_\psi = U \partial_U + V \partial_V \).

We get, as solution describing the Hawking radiation process for the Klein-Gordon equation both in the black hole region and in the external one, the following analytic continuation of the solution (29):

\[
\eta(U, V)_{\text{hawking}} = \begin{cases} 
  c_1 (U)^i \omega + c_2 V^{i \omega} & \text{for } U > 0, \\
  c_1 e^{-\pi \omega} (-U)^i \omega & \text{for } U < 0,
\end{cases}
\]

(33)

which represents a state composed by an ingoing negative norm state \( V^{i \omega} \) and a negative norm outgoing state in the black hole region \( U > 0 \) and an outgoing positive norm particle state \( e^{-\pi \omega} (-U)^i \omega \). As to the normalization, we get (cf. also [16])

\[
c_1 = \frac{1}{\sqrt{4\pi \omega \sqrt{1 - e^{-2\pi \omega}}}},
\]

(34)

\[
c_2 = \frac{1}{\sqrt{4\pi \omega}}.
\]

(35)

In particular, for the external region, we get external part of the straddling mode. Then we find the following transmission coefficient:

\[
|T|^2 = \frac{|J_r^{\text{transmitted}}|}{|J_r^{\text{incident}}|} = \frac{1}{e^{2\pi \omega} - 1}.
\]

(36)

which gives again a thermal spectrum for created pairs\(^3\)

The picture is not strictly the same as in the tunneling ansatz in the HJ formalism, because an ingoing internal mode appears too. In particular, we have an antiparticle (negative norm) state which, in a scattering picture, is composed by an ingoing negative energy state traveling backward in time towards the interior region (and then an antiparticle traveling forward in time towards the horizon) as initial state and a pair composed by a negative norm state traveling backward in time towards the horizon (and then an antiparticle traveling forward in time towards the interior region), and a particle state in the external region moving away from the horizon. The last two states are the same as in the original picture by Damour and Ruffini [3].

We stress that the bonus of the Nariai geometry consists in the fact that solutions inside the black hole do not suffer the problem to deal with a curvature singularity, and are exact.

7. Conclusions

We have discussed some aspects of the tunneling approach to Hawking radiation. In particular, we have shown that \( \Gamma_{\text{emission}} \) can be reinterpreted as a conditional probability of pair emission from vacuum, in the so-called transmission coefficient approach [25, 26], and that the current and expected decay rate for the vacuum is

\(^3\) One may wonder if the flux of \( J_r \), which is meaningful in the external region \( U < 0 \), is still meaningful also in the black hole region \( U > 0 \). We observe that the current involves substantially Wronskian relations also in the inner region, and so is conserved also there, even if its physical interpretation is not perspicuous.
obtained for the emission of thermal particles by the black hole. Then, we have studied Hawking emission in the Nariai case, with the aim of corroborating the tunneling ansatz by exploiting exact solutions of the field equations, which allow to set up an S-matrix approach even in the static situation for the Hawking process. A suitable use of fluxes allows to get the same result than in the semiclassical tunneling approach, which has been explored in regular (Kruskal-like) coordinate patches, both at the black hole horizon and also at the cosmological horizon and extended to the emission of charged particles.

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