ON THE BOUNDARY OF THE ATTAINABLE SET OF THE DIRICHLET SPECTRUM

LORENZO BRASCO, CARLO NITSCH, AND ALDO PRATELLI

Abstract. Denoting by $E \subseteq \mathbb{R}^2$ the set of the pairs $(\lambda_1(\Omega), \lambda_2(\Omega))$ for all the open sets $\Omega \subseteq \mathbb{R}^N$ with unit measure, and by $\Theta \subseteq \mathbb{R}^N$ the union of two disjoint balls of half measure, we give an elementary proof of the fact that $\partial E$ has horizontal tangent at its lowest point $(\lambda_1(\Theta), \lambda_2(\Theta))$.

1. Introduction

Given an open set $\Omega \subseteq \mathbb{R}^N$ with finite measure, its Dirichlet-Laplacian spectrum is given by the numbers $\lambda > 0$ such that the boundary value problem

$$-\Delta u = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,$$

has non trivial solutions. Such numbers $\lambda$ are called eigenvalues of the Dirichlet-Laplacian in $\Omega$, and form a discrete increasing sequence $0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \lambda_3(\Omega) \ldots$, diverging to $+\infty$ (see [4], for example). In this paper, we will work with the first two eigenvalues $\lambda_1$ and $\lambda_2$, for which we briefly recall the variational characterization: introducing the Rayleigh quotient as

$$R_\Omega(u) = \frac{\|\nabla u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2}, \quad u \in H^1(\Omega),$$

the first two eigenvalues of the Dirichlet-Laplacian satisfy

$$\lambda_1(\Omega) = \min \left\{ R_\Omega(u) : u \in H_0^1(\Omega) \setminus \{0\} \right\},$$

$$\lambda_2(\Omega) = \min \left\{ R_\Omega(u) : u \in H_0^1(\Omega) \setminus \{0\}, \int_\Omega u(x) u_1(x) \, dx = 0 \right\},$$

where $u_1$ is a first eigenfunction.

We are concerned about the attainable set of the first two eigenvalues $\lambda_1$ and $\lambda_2$, that is,

$$\mathcal{E} := \left\{ (\lambda_1(\Omega), \lambda_2(\Omega)) \in \mathbb{R}^2 : |\Omega| = \omega_N \right\},$$

where $\omega_N$ is the volume of the ball of unit radius in $\mathbb{R}^N$. Of course, the set $\mathcal{E}$ depends on the dimension $N$ of the ambient space. The set $\mathcal{E}$ has been deeply studied (see for instance [1, 3, 6]); an approximate plot is shown in Figure 1. Let us recall now some of the most important known facts. In what follows, we will always denote by $B$ a ball of unit radius (then, of volume $\omega_N$), and by $\Theta$ a disjoint union of two balls of volume $\omega_N/2$.

Basic properties of $\mathcal{E}$. The attainable set $\mathcal{E}$ has the following properties:

(i) for every $(\lambda_1, \lambda_2) \in \mathcal{E}$ and every $t \geq 1$, one has $(t \lambda_1, t \lambda_2) \in \mathcal{E}$;

(ii) $\mathcal{E} \subseteq \left\{ x \geq \lambda_1(B), y \geq \lambda_2(\Theta), 1 \leq \frac{y}{x} \leq \frac{\lambda_2(B)}{\lambda_1(B)} \right\}$. 

1
(iii) \( \mathcal{E} \) is horizontally and vertically convex, i.e., for every \( 0 \leq t \leq 1 \)

\[
(x_0, y), (x_1, y) \in \mathcal{E} \implies (1-t)x_0 + tx_1, y) \in \mathcal{E},
\]

\[
(x, y_0), (x, y_1) \in \mathcal{E} \implies (x, (1-t)y_0 + ty_1) \in \mathcal{E}.
\]

The first property is a simple consequence of the scaling property \( \lambda_i(t\Omega) = t^{-2}\lambda_i(\Omega) \), valid for any open set \( \Omega \subseteq \mathbb{R}^N \) and any \( t > 0 \). The second property is true because, for every open set \( \Omega \) of unit measure, the Faber–Krahn inequality ensures \( \lambda_1(\Omega) \geq \lambda_1(B) \), the Krahn–Szego inequality (see [5, 7, 8]) ensures \( \lambda_2(\Omega) \geq \lambda_2(\Theta) = \lambda_1(\Theta) \), and a celebrated result by Ashbaugh and Benguria (see [2]) ensures

\[
1 \leq \frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(B)}{\lambda_1(B)}.
\]

Finally, the third property is proven in [3]. It has been conjectured also that the set \( \mathcal{E} \) is convex, as it seems reasonable by a numerical plot, but a proof for this fact is still not known.

Thanks to the above listed properties, the set \( \mathcal{E} \) is completely known once one knows its “lower boundary”

\[
\mathcal{C} := \left\{ (\lambda_1, \lambda_2) \in \mathcal{E} : \forall t < 1, (t\lambda_1, t\lambda_2) \notin \mathcal{E} \right\},
\]

therefore studying \( \mathcal{E} \) is equivalent to study \( \mathcal{C} \). Notice in particular that \( \partial \mathcal{E} \) consists of the union of \( \mathcal{C} \) with the two half-lines

\[
\left\{ (t, t) : t \geq \lambda_1(\Theta) \right\} \quad \text{and} \quad \left\{ \left( t, \frac{\lambda_2(B)}{\lambda_1(B)} t \right) : t \geq \lambda_1(B) \right\}.
\]

Let us call for brevity \( P \) and \( Q \) the endpoints of \( \mathcal{C} \), that is, \( P \equiv (\lambda_1(\Theta), \lambda_2(\Theta)) \) and \( Q \equiv (\lambda_1(B), \lambda_2(B)) \).

The plot of the set \( \mathcal{E} \) seems to suggest that the curve \( \mathcal{C} \) reaches the point \( Q \) with vertical tangent, and the point \( P \) with horizontal tangent. In fact, Wolf and Keller in [6, Section 5] proved the first fact, and they also suggested that the second fact should be true, providing a numerical evidence. The aim of the present paper is to give a short proof of this fact.

**Theorem.** For every dimension \( N \geq 2 \), the curve \( \mathcal{C} \) reaches the point \( P \) with horizontal tangent.
The rest of the paper is devoted to prove this result: the proof will be achieved by exhibiting a suitable family \( \{ \tilde{\Omega}_\varepsilon \} \varepsilon > 0 \) of deformations of \( \Theta \) having measure \( \omega_N \) and such that

\[
\lim_{\varepsilon \to 0} \frac{\lambda_2(\tilde{\Omega}_\varepsilon) - \lambda_2(\Theta)}{\lambda_1(\Theta) - \lambda_1(\Omega_\varepsilon)} = 0. \tag{1.1}
\]

2. Proof of the Theorem

Throughout this section, for any given \( x = (x_1, ..., x_N) \in \mathbb{R}^N \), we will write \( x = (x_1, x') \) where \( x_1 \in \mathbb{R} \) and \( x' \in \mathbb{R}^{N-1} \).

We will make use of the sets \( \{ \Omega_\varepsilon \} \subseteq \mathbb{R}^N \), shown in Figure 2, defined by

\[
\Omega_\varepsilon := \left\{ (x_1, x') \in \mathbb{R}^+ \times \mathbb{R}^{N-1} : (x_1 - 1 + \varepsilon)^2 + |x'|^2 < 1 \right\} \\
\cup \left\{ (x_1, x') \in \mathbb{R}^+ \times \mathbb{R}^{N-1} : (x_1 + 1 - \varepsilon)^2 + |x'|^2 < 1 \right\}
= : \Omega_\varepsilon^+ \cup \Omega_\varepsilon^-.
\]

for every \( \varepsilon > 0 \) sufficiently small. The sets \( \tilde{\Omega}_\varepsilon \) for which we will eventually prove (1.1) will be rescaled copies of \( \Omega_\varepsilon \), in order to have measure \( \omega_N \).

To get our thesis, we need to provide an upper bound to \( \lambda_1(\Omega_\varepsilon) \) and an upper bound to \( \lambda_2(\Omega_\varepsilon) \); this will be the content of Lemmas 2.1 and 2.2 respectively.

**Figure 2.** The sets \( \Omega_\varepsilon = \Omega_\varepsilon^+ \cup \Omega_\varepsilon^- \)

**Lemma 2.1.** There exists a constant \( \gamma_1 > 0 \) such that for every \( \varepsilon \ll 1 \) it is

\[
\lambda_1(\Omega_\varepsilon) \leq \lambda_1(B) - \gamma_1 \varepsilon^{N/2}. \tag{2.1}
\]

**Proof.** Let \( B_\varepsilon \) be the ball of unit radius centered at \((1 - \varepsilon, 0)\), so that \( B_\varepsilon \subseteq \Omega_\varepsilon \) and in particular \( \Omega_\varepsilon^+ = B_\varepsilon \cap \{ x_1 > 0 \} \). Let also \( u \) be a first Dirichlet eigenfunction of \( B_\varepsilon \) with unit \( L^2 \) norm, and denote by \( T \) the region (shaded in Figure 3) bounded by the right circular conical surface \( \{ \sqrt{2\varepsilon - \varepsilon^2} - x_1 - |x'| = 0 \} \) and by the plane \( \{ x_1 = 0 \} \).

Since the normal derivative of \( u \) is constantly \( \kappa \) on \( \partial B_\varepsilon^+ \), we know that

\[
Du(x_1, x') = Du(0, x') + O(\sqrt{\varepsilon}) = (\kappa, 0) + O(\sqrt{\varepsilon}) \quad \text{on} \ T. \tag{2.2}
\]

Let us now define the function \( \tilde{u} : \Omega_\varepsilon^+ \to \mathbb{R} \) as

\[
\tilde{u}(x_1, x') := \begin{cases} 
    u(x_1, x') & \text{if } (x_1, x') \notin T, \\
    u(x_1, x') + \frac{\kappa}{2} \left( \sqrt{2\varepsilon - \varepsilon^2} - x_1 - |x'| \right) & \text{if } (x_1, x') \in T.
\end{cases}
\]
It is immediate to observe that $\tilde{u} = u$ on the surface $\left\{ \sqrt{2\varepsilon - \varepsilon^2} - x_1 - |x'| = 0 \right\} \cap \{ x_1 > 0 \}$, so that $\tilde{u} \in H^1(\Omega_\varepsilon^+)$.

Notice that $\tilde{u} \notin H^1_0(\Omega_\varepsilon^+)$ since $\tilde{u}$ does not vanish on $\{ x_1 = 0 \} \cap \partial \Omega_\varepsilon^+$. By construction and recalling (2.2),

$$D\tilde{u}(x_1, x') = Du(x_1, x') + \left( -\frac{\kappa}{2}, -\frac{\kappa}{2} \frac{x'}{|x'|} \right) = \left( \frac{\kappa}{2}, -\frac{\kappa}{2} \frac{x'}{|x'|} \right) + O(\sqrt{\varepsilon}) \quad \text{on } \mathcal{T}.$$  \hfill (2.3)

Since $\tilde{u} \geq u$ on $\Omega_\varepsilon^+$, and recalling that $u \in H^1_0(B_\varepsilon^+)$, one clearly has

$$\int_{\Omega_\varepsilon^+} \tilde{u}^2 dx \geq \int_{\Omega_\varepsilon^+} u^2 dx = \int_{B_\varepsilon^+} u^2 dx + O(\varepsilon^{(N+5)/2}) = 1 + O(\varepsilon^{(N+5)/2}),$$  \hfill (2.4)

since the small region $B_\varepsilon \setminus \Omega_\varepsilon^+$ has volume $O(\varepsilon^{(N+1)/2})$, and on this region $u = O(\varepsilon)$.

On the other hand, comparing (2.2) and (2.3), one has

$$|Du| = |D\tilde{u}| = |Du| - \frac{\kappa^2}{2} + O(\sqrt{\varepsilon}) \quad \text{on } \mathcal{T},$$

and since the volume of $\mathcal{T}$ is $\frac{\varepsilon^{N-1}}{N} (2\varepsilon - \varepsilon^2)^{N/2}$ we deduce

$$\int_{\Omega_\varepsilon^+} |D\tilde{u}|^2 dx = \int_{\Omega_\varepsilon^+} |Du|^2 dx - \frac{\varepsilon^{N-1}}{N} (2\varepsilon - \varepsilon^2)^{N/2} \left( \frac{\kappa^2}{2} + O(\sqrt{\varepsilon}) \right)$$

$$= \int_{\Omega_\varepsilon^+} |Du|^2 dx - \frac{\varepsilon^{N-1}}{N} \kappa^2 \varepsilon^{N/2} + O(\varepsilon^{(N+1)/2})$$

$$= \int_{B_\varepsilon^+} |Du|^2 dx - C_N \kappa^2 \varepsilon^{N/2} + O(\varepsilon^{(N+1)/2}),$$  \hfill (2.5)

where $C_N = \frac{\varepsilon^{N-1}}{N} 2^{(N/2)-1}$.

Therefore, by (2.4) and (2.5) we obtain

$$\mathcal{R}_{\Omega_\varepsilon^+}(\tilde{u}) = \frac{\int_{\Omega_\varepsilon^+} |D\tilde{u}|^2 dx}{\int_{\Omega_\varepsilon^+} \tilde{u}^2 dx} \leq \mathcal{R}_{B_\varepsilon^+}(u) - C_N \kappa^2 \varepsilon^{N/2} + O(\varepsilon^{(N+1)/2})$$

$$= \lambda_1(B) - C_N \kappa^2 \varepsilon^{N/2} + O(\varepsilon^{(N+1)/2}).$$

We can finally extend $\tilde{u}$ to the whole $\Omega_\varepsilon$, simply defining $\tilde{u}(x_1, x') = \tilde{u}(|x_1|, x')$ on $\Omega_\varepsilon^-$. By construction, $\tilde{u} \in H^1_0(\Omega_\varepsilon)$, and

$$\lambda_1(\Omega_\varepsilon) \leq \mathcal{R}_{\Omega_\varepsilon}(\tilde{u}) = \mathcal{R}_{\Omega_\varepsilon^+}(\tilde{u}) \leq \lambda_1(B) - C_N \kappa^2 \varepsilon^{N/2} + O(\varepsilon^{(N+1)/2}),$$
so that (2.1) follows and the proof is concluded.

Lemma 2.2. There exists a constant $\gamma_2 > 0$ such that for every $\varepsilon \ll 1$, it is

$$\lambda_2(\Omega_\varepsilon) \leq \lambda_1(B) + \gamma_2 \varepsilon^{(N+1)/2}.$$  \hfill (2.6)

Proof. First of all, we start underlining that

$$\lambda_2(\Omega_\varepsilon) \leq \lambda_1(\Omega^+_{\varepsilon})$$ \hfill (2.7)

in fact if we define

$$\tilde{u}(x_1, x') := \begin{cases} u_\varepsilon(x_1, x'), & \text{if } x_1 \in \Omega^+_{\varepsilon}, \\ -u_\varepsilon(-x_1, x'), & \text{if } x_1 \in \Omega^-_{\varepsilon}, \end{cases}$$

then by construction it readily follows that $-\Delta \tilde{u} = \lambda_1(\Omega^+_{\varepsilon})\tilde{u}$. As a consequence $\lambda_1(\Omega^+_{\varepsilon})$ is an eigenvalue of $\Omega_\varepsilon$, say $\lambda_1(\Omega^+_{\varepsilon}) = \lambda_\ell(\Omega_\varepsilon)$. Since $\Omega_\varepsilon$ is connected and $\tilde{u}$ changes sign, it is not possible $\ell = 1$, hence

$$\lambda_2(\Omega_\varepsilon) \leq \lambda_\ell(\Omega_\varepsilon) = \lambda_1(\Omega^+_{\varepsilon}).$$

It is then enough for us to estimate $\lambda_1(\Omega^+_{\varepsilon}).$ To this aim, define the set

$$\Omega_\varepsilon := \{(x_1, x') \in \Omega^+_{\varepsilon} : x_1 \geq \varepsilon\},$$

and take a Lipschitz cut-off function $\xi_\varepsilon \in W^{1, \infty}(\Omega^+_{\varepsilon})$ such that

$$0 \leq \xi_\varepsilon \leq 1 \text{ on } \Omega^+_{\varepsilon}, \quad \xi_\varepsilon \equiv 1 \text{ on } \Omega_\varepsilon, \quad \xi_\varepsilon \equiv 0 \text{ on } \partial \Omega^+_{\varepsilon} \cap \{x_1 = 0\}, \quad \|\nabla \xi_\varepsilon\|_{\infty} \leq L \varepsilon^{-1}.$$  

As in Lemma 2.1 let again $u$ be a first eigenfunction of the ball $B_{\varepsilon}$ of radius 1 centered at $(1 - \varepsilon, 0)$ having unit $L^2$ norm, and define on $\Omega_\varepsilon$ the function $\varphi = u \xi_\varepsilon.$ Since by construction $\varphi$ belongs to $H^1_0(\Omega_\varepsilon)$, we obtain

$$\lambda_1(\Omega^+_{\varepsilon}) \leq \mathcal{R}(\varphi, \Omega^+_{\varepsilon}) = \frac{\int_{\Omega^+_{\varepsilon}} |\nabla u|^2 \xi^2_\varepsilon dx + |\nabla \xi_\varepsilon|^2 u^2 + 2 u \xi_\varepsilon \langle \nabla u, \nabla \xi_\varepsilon \rangle}{\int_{\Omega^+_{\varepsilon}} u^2 \xi^2_\varepsilon dx}. \hfill (2.8)$$

We can start estimating the denominator very similarly to what already done in (2.4). Indeed, recalling that $|\Omega^+_{\varepsilon} \setminus \Omega_\varepsilon| = O(\varepsilon^{(N+1)/2})$ and that in that small region $u = O(\varepsilon)$, we have

$$\int_{\Omega^+_{\varepsilon}} u^2 \xi^2_\varepsilon dx = \int_{B_{\varepsilon}} u^2 dx - \int_{B_{\varepsilon} \setminus \Omega^+_{\varepsilon}} u^2 dx - \int_{\Omega^+_{\varepsilon} \setminus \Omega_\varepsilon} u^2 (1 - \xi^2_\varepsilon) dx = 1 + O(\varepsilon^{(N+5)/2}).$$

Let us pass to study the numerator: first of all, being $0 \leq \xi_\varepsilon \leq 1$ we have

$$\int_{\Omega^+_{\varepsilon}} |\nabla u|^2 \xi^2_\varepsilon dx \leq \int_{B_{\varepsilon}} |\nabla u|^2 dx = \lambda_1(B).$$

Moreover,

$$\int_{\Omega^+_{\varepsilon}} |\nabla \xi_\varepsilon|^2 u^2 dx = \int_{\Omega^+_{\varepsilon} \setminus \Omega_\varepsilon} |\nabla \xi_\varepsilon|^2 u^2 dx \leq \frac{L^2}{\varepsilon^2} \|u\|_{L^{\infty}(\Omega^+_{\varepsilon} \setminus \Omega_\varepsilon)}^2 = O(\varepsilon^{(N+1)/2}),$$

and in the same way

$$\int_{\Omega^+_{\varepsilon}} u \xi_\varepsilon \langle \nabla u, \nabla \xi_\varepsilon \rangle dx \leq \int_{\Omega^+_{\varepsilon} \setminus \Omega_\varepsilon} |u| |\nabla u| |\nabla \xi_\varepsilon| dx = O(\varepsilon^{(N+1)/2}).$$

Summarizing, by (2.8) we deduce

$$\lambda_1(\Omega^+_{\varepsilon}) \leq \lambda_1(B) + O(\varepsilon^{(N+1)/2}).$$
thus by (2.7) we get the thesis.

□

We are now ready to conclude the paper by giving the proof of the Theorem.

Proof of the Theorem. For any small $\varepsilon > 0$, we define $\tilde{\Omega}_\varepsilon = t_\varepsilon \Omega_\varepsilon$, where $t_\varepsilon = \sqrt[2]{\omega_N/|\tilde{\Omega}_\varepsilon|}$ so that $|\tilde{\Omega}_\varepsilon| = \omega_N$. Notice that $|\Omega_\varepsilon| = 2\omega_N + O(\varepsilon^{(N+1)/2})$, thus $t_\varepsilon = 2^{-1/N} + O(\varepsilon^{(N+1)/2})$. Recalling the trivial rescaling formula $\lambda_i(t\Omega) = t^{-2}\lambda_i(\Omega)$, valid for any natural $i$, any positive $t$ and any open set $\Omega$, we can then estimate by Lemma 2.1 and Lemma 2.2

$$
\lambda_1(\tilde{\Omega}_\varepsilon) = \left(\frac{|\Omega_\varepsilon|}{\omega_N}\right)^{2/N} \lambda_1(\Omega_\varepsilon) \leq 2^{2/N} \lambda_1(B) - 2^{2/N}\gamma_1 \varepsilon^{N/2} + O(\varepsilon^{(N+1)/2}),
$$

$$
\lambda_2(\tilde{\Omega}_\varepsilon) = \left(\frac{|\Omega_\varepsilon|}{\omega_N}\right)^{2/N} \lambda_2(\Omega_\varepsilon) \leq 2^{2/N} \lambda_1(B) + O(\varepsilon^{(N+1)/2}).
$$

Since $\lambda_1(\Theta) = \lambda_2(\Theta) = 2^{2/N}\lambda_1(B)$, the two above estimates give

$$
\lim_{\varepsilon \to 0} \frac{\lambda_2(\tilde{\Omega}_\varepsilon) - \lambda_2(\Theta)}{\lambda_1(\Theta) - \lambda_1(\tilde{\Omega}_\varepsilon)} = 0,
$$

which as already noticed in (1.1) implies the thesis.

□

Acknowledgements. The three authors have been supported by the ERC Starting Grant n. 258685; L. B. and A. P. have been supported also by the ERC Advanced Grant n. 226234.

References

[1] P. S. Antunes, A. Henrot, On the range of the first two Dirichlet and Neumann eigenvalues of the Laplacian, to appear in Proc. R. Soc. of Lond. A (2011), available at http://hal.inria.fr/hal-00511096/en
[2] M. S. Ashbaugh, R. D. Benguria, A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions, Ann. of Math. 135 (1992), 601–628.
[3] D. Bucur, G. Buttazzo, I. Figueiredo, The attainable eigenvalues of the Laplace operator, SIAM J. Math. Anal., 30 (1999), 527–536.
[4] A. Henrot, Extremum problems for eigenvalues of elliptic operators. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.
[5] I. Hong, On an inequality concerning the eigenvalue problem of membrane, Kôdai Math. Sem. Rep., 6 (1954), 113–114.
[6] J. B. Keller, S. A. Wolf, Range of the first two eigenvalues of the Laplacian, Proc. Roy. Soc. London Ser. A 447 (1994), 397–412.
[7] E. Krahn, Über Minimaleigenschaften der Krugel in drei un mehr Dimensionen, Acta Comm. Univ. Dorpat., A9 (1926), 1–44.
[8] G. Pólya, On the characteristic frequencies of a symmetric membrane, Math. Zeitschr., 63 (1955), 331–337.
