Davenport constant for semigroups (II)

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Abstract

Let $S$ be a finite commutative semigroup. The Davenport constant of $S$, denoted $D(S)$, is defined to be the least positive integer $\ell$ such that every sequence $T$ of elements in $S$ of length at least $\ell$ contains a proper subsequence $T'$ ($T' \neq T$) with the sum of all terms from $T'$ equaling the sum of all terms from $T$. Let $q > 2$ be a prime power, and let $\mathbb{F}_q[x]$ be the ring of polynomials over the finite field $\mathbb{F}_q$. Let $R$ be a quotient ring of $\mathbb{F}_q[x]$ with $0 \neq R \neq \mathbb{F}_q[x]$. We prove that

$$D(S_R) = D(U(S_R)),$$

where $S_R$ denotes the multiplicative semigroup of the ring $R$, and $U(S_R)$ denotes the group of units in $S_R$.

Key Words: Davenport constant; Zero-sum; Finite commutative semigroups; Polynomial rings

1 Introduction

Let $G$ be an additive finite abelian group. A sequence $T$ of elements in $G$ is called a zero-sum sequence if the sum of all terms of $T$ equals to zero, the identity element of $G$. The Davenport constant $D(G)$ of $G$ is defined to be the smallest positive integer $\ell$ such that, every sequence $T$ of elements in $G$ of length at least $\ell$ contains a nonempty subsequence $T'$ with the sum of all terms of $T'$ equaling zero. H. Davenport [3] proposed the study of this constant in 1965. The Davenport constant together with the celebrated Erdős-Ginzburg-Ziv Theorem obtain by P. Erdős, A. Ginzburg and A. Ziv in 1961 were two pioneering researches for Zero-sum Theory (see [7] for a survey) which has been developed into a branch of Additive Group Theory.
Theorem A. [4] (Erdős-Ginzburg-Ziv Theorem) Every sequence of $2n - 1$ elements in an additive finite abelian group of order $n$ contains a zero-sum subsequence of length $n$.

During the past five decades, the Davenport constant and the Erdős-Ginzburg-Ziv Theorem together with a large of related problems have been studied extensively for the setting of groups (see [2, 5, 6, 8, 10, 11] for example). In 2008, the author of this paper and W.D. Gao formulated the definition of the Davenport constant for finite commutative semigroups which is stated as follows.

Definition B. [13] Let $S$ be a commutative semigroup (not necessary finite). Let $T$ be a sequence of elements in $S$. We call $T$ reducible if $T$ contains a proper subsequence $T'$ ($T' \neq T$) such that the sum of all terms of $T'$ equals the sum of all terms of $T$. Define the Davenport constant of the semigroup $S$, denoted $D(S)$, to be the smallest integer $\ell \in \mathbb{N} \cup \{\infty\}$ such that every sequence $T$ of length at least $\ell$ of elements in $S$ is reducible.

In fact, starting from the research of Factorization Theory in Algebra, A. Geroldinger and F. Halter-Koch in 2006 have formulated another closely related definition, $d(S)$, for any commutative semigroup $S$, which is called the small Davenport constant. For the completeness, their definition is also stated here.

Definition C. (Definition 2.8.12 in [9]) For a commutative semigroup $S$, let $d(S)$ denote the smallest $\ell \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:

For any $m \in \mathbb{N}$ and $a_1, \ldots, a_m \in S$ there exists a subset $I \subset [1, m]$ such that $|I| \leq \ell$ and

$$\sum_{i=1}^{m} a_i = \sum_{i \in I} a_i.$$ 

We have the following connection between the (large) Davenport constant $D(S)$ and the small Davenport constant $d(S)$ for any finite commutative semigroup $S$.

Proposition D. Let $S$ be a finite commutative semigroup. Then,

1. $d(S) < \infty$. (see Proposition 2.8.13 in [9])
2. $D(S) = d(S) + 1$. (see Proposition 1.2 in [1])

In 2014, the author together with S.D. Adhikari and W.D. Gao [1] also generalized the Erdős-Ginzburg-Ziv Theorem to finite commutative semigroups.
Very recently, H.L. Wang, L.Z. Zhang, Q.H. Wang and Y.K. Qu [14] made a study of the Davenport constant of the multiplicative semigroup of a quotient ring of \( F_p[x] \). In particular, they proved the following.

**Theorem E.** For any prime \( p > 2 \), let \( f(x) \) be a nonconstant polynomial of \( F_p[x] \) such that \( f(x) \) can be factorized into several irreducible polynomials which are not associated each other. Let \( R = \frac{F_p[x]}{(f(x))} \). Then

\[
D(S_R) = D(U(S_R)),
\]

where \( S_R \) denotes the multiplicative semigroup of the quotient ring \( \frac{F_p[x]}{(f(x))} \) and \( U(S_R) \) denotes the group of units in \( S_R \).

Moreover, they conjectured that \( D(S_R) = D(U(S_R)) \) holds true for all prime \( p > 2 \) and any nonconstant polynomial \( f(x) \in F_p[x] \).

In this paper, we obtained the following result for the quotient ring of the ring of polynomials over any finite field \( F_q \) where \( q > 2 \). As a special case, we affirmed their conjecture.

**Theorem 1.1.** Let \( q > 2 \) be a prime power, and let \( F_q[x] \) be the ring of polynomials over the finite field \( F_q \). Let \( R \) be a quotient ring of \( F_q[x] \) with \( 0 \neq R \neq F_q[x] \). Then

\[
D(S_R) = D(U(S_R)),
\]

where \( S_R \) denotes the multiplicative semigroup of the ring \( R \), and \( U(S_R) \) denotes the group of units in \( S_R \).

## 2 The proof of Theorem 1.1

We begin this section by giving some preliminaries.

Let \( S \) be a finite commutative semigroup. The operation on \( S \) is denoted by +. The identity element of \( S \), denoted \( 0_S \) (if exists), is the unique element \( e \) of \( S \) such that \( e + a = a \) for every \( a \in S \). If \( S \) has an identity element \( 0_S \), let

\[
U(S) = \{a \in S : a + a' = 0_S \text{ for some } a' \in S\}
\]

be the group of units of \( S \). For any element \( c \in S \) and any subset \( A \subseteq S \), let

\[
St_A(c) = \{a \in A : a + c = c\}
\]
denote the stabilizer of $c$ in $A$.

On a commutative semigroup $S$ the Green’s preorder, denoted $\leq_{H}$, is defined by

$$a \leq_{H} b \iff a = b \text{ or } a = b + c$$

for some $c \in S$. Green’s congruence, denoted $\mathcal{H}$, is a basic relation introduced by Green for semigroups which is defined by:

$$a \mathcal{H} b \iff a \leq_{H} b \text{ and } b \leq_{H} a.$$ For any element $a$ of $S$, let $H_a$ be the congruence class by $\mathcal{H}$ containing $a$. We write $a <_{\mathcal{H}} b$ to mean that $a \leq_{H} b$ but $H_a \neq H_b$. The following easy fact will be used later.

**Lemma 2.1.** (folklore) For any element $a \in S$, $U(S)$ acts on the congruence class $H_a$ and $\text{St}_{U(S)}(a)$ is a subgroup of $U(S)$.

In what follows, we also need some notations introduced by A. Geroldinger and F. Halter-Koch (see [9]), which are very helpful to dealing with the problems in zero-sum theory and factorization theory.

The sequence of elements in the semigroups $S$ is denoted by

$$T = a_1a_2\cdots a_\ell = \prod_{a \in S} a^{v_a(T)},$$

where $v_a(T)$ denotes the multiplicity of the element $a$ in the sequence $T$. By $\cdot$ we denote the operation to join sequences. Let $T_1, T_2$ be two sequences of elements in the semigroups $S$. We call $T_2$ a subsequence of $T_1$ if

$$v_a(T_2) \leq v_a(T_1)$$

for every element $a \in S$, denoted by

$$T_2 \mid T_1.$$

In particular, if $T_2 \neq T_1$, we call $T_2$ a proper subsequence of $T_1$, and write

$$T_3 = T_1T_2^{-1}$$

to mean the unique subsequence of $T_1$ with $T_2 \cdot T_3 = T_1$. Let

$$\sigma(T) = a_1 + a_2 + \cdots + a_\ell$$

be the sum of all terms in the sequence $T$. By $\lambda$ we denote the empty sequence. If $S$ has an identity element $0_S$, we allow $T = \lambda$ and adopt the convention that $\sigma(\lambda) = 0_S$. We say that $T$
is reducible if \( \sigma(T') = \sigma(T) \) for some proper subsequence \( T' \) of \( T \) (Note that, \( T' \) is probably the empty sequence \( \lambda \) if \( S \) has the identity element \( 0_S \) and \( \sigma(T) = 0_S \)). Otherwise, we call \( T \) irreducible. For more related terminology used in additive problems for semigroups, one is referred to [1, 12]. Here, the following two lemmas are necessary.

**Lemma 2.2.** ([9], Lemma 6.1.3) Let \( G \) be a finite abelian group, and let \( H \) be a subgroup of \( G \). Then, \( D(G) \geq D(G/H) + D(H) - 1 \).

**Lemma 2.3.** (see [13], Proposition 1.2) Let \( S \) be a finite commutative semigroup with identity. Then \( D(U(S)) \leq D(S) \).

Now we are in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** By Lemma 2.3, we need only to show that

\[ D(S_R) \leq D(U(S_R)). \]

Since the ring \( \mathbb{F}_q[x] \) is a principal ideal domain and \( 0 \neq R \neq \mathbb{F}_q[x] \), we have that \( R = \mathbb{F}_q[x]/(f(x)) \) for some nonconstant monic polynomial \( f(x) \in \mathbb{F}_q[x] \). Let

\[ f(x) = f_1(x)^{n_1} \ast f_2(x)^{n_2} \ast \cdots \ast f_r(x)^{n_r} \tag{1} \]

be the factorization of \( f(x) \) in \( \mathbb{F}_q[x] \), where \( r \geq 1, n_1, n_2, \ldots, n_r \geq 1, \) and \( f_1(x), f_2(x), \ldots, f_r(x) \) are pairwise non-associated irreducible monic polynomials of \( \mathbb{F}_q[x] \). To proceed, we need to introduce some notations.

Take an arbitrary element \( a \in S_R \). Let \( \theta_a(x) \in \mathbb{F}_q[x] \) be the unique polynomial corresponding to the element \( a \) with the least degree, i.e.,

\[ \overline{\theta_a(x)} = \theta_a(x) + (f(x)) \]

is the corresponding form of \( a \) in the quotient ring \( R \) with

\[ \deg(\theta_a(x)) \leq \deg(f(x)) - 1. \]

By gcd\((\theta_a(x), f(x))\) we denote the greatest common divisor of the two polynomials \( \theta_a(x) \) and \( f(x) \) in \( \mathbb{F}_q[x] \) (the unique monic polynomial with the greatest degree which divides both \( \theta_a(x) \) and \( f(x) \)), in particular, by (1),

\[ \gcd(\theta_a(x), f(x)) = f_1(x)^{\alpha_1} \ast f_2(x)^{\alpha_2} \ast \cdots \ast f_r(x)^{\alpha_r} \]
where \( \alpha_i \in [0, n_i] \) for \( i = 1, 2, \ldots, r \).

Now we show the following claim.

**Claim A.** Let \( a \) and \( b \) be two elements of \( S_R \). Then the following conclusions hold:

(i) If \( a \preceq_H b \) then \( \gcd(\theta_b(x), f(x)) \mid \gcd(\theta_a(x), f(x)) \) and \( \text{St}_{U(S_R)}(b) \subseteq \text{St}_{U(S_R)}(a) \);

(ii) \( a \not\preceq_H b \iff \gcd(\theta_b(x), f(x)) = \gcd(\theta_a(x), f(x)) \iff \text{St}_{U(S_R)}(b) = \text{St}_{U(S_R)}(a) \).

**Proof of Claim A.** Assume \( a \preceq_H b \). Since \( S_R \) has the identity element \( 0_{S_R} \), we have

\[ a = b + c \quad \text{for some} \quad c \in S_R. \]

It follows that

\[ \gcd(\theta_b(x), f(x)) \mid \gcd(\theta_a(x) + \theta_c(x), f(x)) = \gcd(\theta_a(x), f(x)). \]

For any element \( d \in \text{St}_{U(S_R)}(b) \), \( d + a = d + (b + c) = (d + b) + c = b + c = a \), and so \( d \in \text{St}_{U(S_R)}(a) \).

It follows that

\[ \text{St}_{U(S_R)}(b) \subseteq \text{St}_{U(S_R)}(a). \]

This proves Conclusion (i).

Now we prove Conclusion (ii).

Assume \( a \not\preceq_H b \). Then \( a \preceq_H b \) and \( b \preceq_H a \). It follows from Conclusions (i) that

\[ \gcd(\theta_b(x), f(x)) = \gcd(\theta_a(x), f(x)) \]

and

\[ \text{St}_{U(S_R)}(b) = \text{St}_{U(S_R)}(a). \]

Assume \( \gcd(\theta_b(x), f(x)) = \gcd(\theta_a(x), f(x)) \). Since

\[ \gcd\left( \frac{\theta_b(x)}{\gcd(\theta_a(x), f(x))}, \frac{f(x)}{\gcd(\theta_a(x), f(x))} \right) = 1_{\mathbb{F}_q} \]

and

\[ \gcd\left( \frac{\theta_b(x)}{\gcd(\theta_b(x), f(x))}, \frac{f(x)}{\gcd(\theta_b(x), f(x))} \right) = 1_{\mathbb{F}_q}, \]

it follows that there exist polynomials \( h(x), h'(x) \in \mathbb{F}_q[x] \) such that

\[ \frac{\theta_a(x)}{\gcd(\theta_a(x), f(x))} \ast h(x) \equiv \frac{\theta_b(x)}{\gcd(\theta_b(x), f(x))} \pmod{\frac{f(x)}{\gcd(\theta_b(x), f(x))}}. \]
and
\[
\frac{\theta_b(x)}{\gcd(\theta_b(x), f(x))} \ast h'(x) \equiv \frac{\theta_a(x)}{\gcd(\theta_a(x), f(x))} \pmod{\frac{f(x)}{\gcd(\theta_b(x), f(x))}},
\]
equivalently,
\[
\theta_a(x) \ast h(x) \equiv \theta_b(x) \pmod{f(x)}
\]
and
\[
\theta_b(x) \ast h'(x) \equiv \theta_a(x) \pmod{f(x)}.
\]
It follows that \( b \leq_H a \) and \( a \leq_H b \), i.e.,
\[
a \in H b.
\]

Assume \( \text{St}_{U(S_R)}(b) = \text{St}_{U(S_R)}(a) \). To prove \( a \in H b \), we suppose to the contrary that \( a \in H b \) does not hold. Then \( \gcd(\theta_b(x), f(x)) \neq \gcd(\theta_a(x), f(x)) \). We may suppose without loss of generality that there exist integers \( k \in [1, r] \) and \( m_k \in [1, n_k] \) such that
\[
f_k(x)^{m_k} \mid \gcd(\theta_a(x), f(x)) \quad (2)
\]
and
\[
f_k(x)^{m_k} \nmid \gcd(\theta_b(x), f(x)). \quad (3)
\]
Let
\[
h(x) = \frac{f(x)}{f_k(x)^{m_k}}. \quad (4)
\]
Take an element \( \xi \in \mathbb{F}_q \setminus \{0_{\mathbb{F}_q}, 1_{\mathbb{F}_q}\} \). Note that
\[
\gcd(h(x) + 1_{\mathbb{F}_q}, f(x)) = 1_{\mathbb{F}_q} \quad (5)
\]
or
\[
\gcd(\xi \ast h(x) + 1_{\mathbb{F}_q}, f(x)) = 1_{\mathbb{F}_q}. \quad (6)
\]
Take an element \( d \in S_R \) with
\[
\theta_d(x) \equiv h(x) + 1_{\mathbb{F}_q} \pmod{f(x)}
\]
or
\[
\theta_d(x) \equiv \xi \ast h(x) + 1_{\mathbb{F}_q} \pmod{f(x)}
\]
according to (5) or (6) holds respectively. It follows that
\[
d \in U(S_R),
\]
and follows from (2), (3) and (4) that

\[ \theta_a(x) \ast \theta_d(x) \equiv \theta_a(x) \pmod{f(x)} \]

and

\[ \theta_b(x) \ast \theta_d(x) \not\equiv \theta_b(x) \pmod{f(x)}. \]

That is, \( d \in \text{St}_{U(S_R)}(a) \setminus \text{St}_{U(S_R)}(b) \), a contradiction with \( \text{St}_{U(S_R)}(a) = \text{St}_{U(S_R)}(b) \). Hence, we have that

\[ a \trianglelefteq b. \]

This proves Claim A. \( \Box \)

Let \( T = a_1 a_2 \cdots a_\ell \) be an arbitrary sequence of elements in \( S_R \) of length

\[ \ell = U(S_R). \]

It suffices to show that \( T \) contains a proper subsequence \( T' \) with \( \sigma(T') = \sigma(T) \).

Take a shortest subsequence \( V \) of \( T \) such that

\[ \sigma(V) \trianglelefteq \sigma(T). \]  \hspace{1cm} (7)

We may assume without loss of generality that

\[ V = a_1 \cdot a_2 \cdot \cdots \cdot a_t \text{ where } t \in [0, \ell]. \]

By the minimality of \( |V| \), we derive that

\[ 0_{S_R} \trianglelefteq a_1 \trianglelefteq (a_1 + a_2) \trianglelefteq \cdots \trianglelefteq \sum_{i=1}^t a_i. \]

Denote

\[ K_0 = \{0_{S_R}\} \]

and

\[ K_i = \text{St}_{U(S_R)}(\sum_{j=1}^i a_j) \text{ for each } i \in [1, \ell]. \]

By Lemma 2.1, \( K_i \) is a subgroup of \( U(S_R) \) for each \( i \in [1, \ell] \). Moreover, since \( \text{St}_{U(S_R)}(0_{S_R}) = K_0 \), it follows from Claim A that

\[ K_0 \leq K_1 \leq K_2 \leq \cdots \leq K_\ell. \]
For $i \in [1, t]$, since $\frac{U(S_R)}{K_i} \cong \frac{U(S_R)}{K_{i-1}}$ and $D(K_i/K_{i-1}) \geq 2$, it follows from Lemma 2.2 that

$$
D(U(S_R)/K_i) = D\left(\frac{U(S_R)}{K_{i-1}}\right) \\
\leq D(U(S_R)/K_{i-1}) - (D(K_i/K_{i-1}) - 1) \\
\leq D(U(S_R)/K_{i-1}) - 1.
$$

It follows that

$$
1 \leq D(U(S_R)/K_i) \leq D(U(S_R)/K_{i-1}) - 1 \\
\vdots \\
\leq D(U(S_R)/K_0) - t \\
= D(U(S_R)) - t \\
= \ell - t \\
= |TV^{-1}|.
$$

By (7) and Conclusion (ii) of Claim A, we have

$$
gcd(\theta_{\sigma(V)}(x), f(x)) = gcd(\theta_{\sigma(T)}(x), f(x)).
$$

Let

$$
\mathcal{J} = \{ j \in [1, r] : f_j(x)^{n_j} | \theta_{\sigma(T)}(x) \}.
$$

By (9), we have that

$$
f_i(x) \nmid \theta_a(x) \quad \text{for each term } a \text{ of } TV^{-1} \text{ and each } i \in [1, r] \setminus \mathcal{J},
$$

and that

$$
f_j(x)^{n_j} | \theta_{\sigma(V)}(x) \quad \text{for each } j \in \mathcal{J}.
$$

For each term $a$ of $TV^{-1}$, let $\bar{a}$ be the element of $S_R$ such that

$$
\theta_\bar{a}(x) \equiv \theta_a(x) \pmod{f_i(x)^{n_i}} \quad \text{for each } i \in [1, r] \setminus \mathcal{J}
$$

and

$$
\theta_{\bar{a}}(x) \equiv 1_{\mathbb{Z}_q} \pmod{f_j(x)^{n_j}} \quad \text{for each } j \in \mathcal{J}.
$$

By (10), (12) and (13), we conclude that $gcd(\theta_a(x), f(x)) = 1_{\mathbb{Z}_q}$, i.e.,

$$
\bar{a} \in U(S_R) \quad \text{for each term } a \text{ of } TV^{-1}.
$$

By (11) and (12), we conclude that

$$
\sigma(V) + \bar{a} = \sigma(V) + a \quad \text{for each term } a \text{ of } TV^{-1}.
$$
By (8) and (14), we have that \( \prod_{\tilde{a}} \tilde{a} \) is a nonempty sequence of elements in \( U(S_R) \) of length \( | \prod_{\tilde{a}} \tilde{a} | = | TV^{-1} | \geq D(U(S_R) / K_t) \). It follows that there exists a nonempty subsequence \( W \mid TV^{-1} \), such that

\[
\sigma(\prod_{\tilde{a}} \tilde{a}) \in K_t,
\]

which implies

\[
\sigma(V) + \sigma(\prod_{\tilde{a}} \tilde{a}) = \sigma(V). \tag{16}
\]

By (15) and (16), we conclude that

\[
\sigma(T) = \sigma(TW^{-1}V^{-1}) + (\sigma(V) + \sigma(W)) = \sigma(TW^{-1}V^{-1}) + (\sigma(V) + \sigma(\prod_{\tilde{a}} \tilde{a})) = \sigma(TW^{-1}V^{-1}) + \sigma(V) = \sigma(TW^{-1}),
\]

and \( T' = TW^{-1} \) is the desired proper subsequence of \( T \). This completes the proof of the theorem. \( \square \)

3 Concluding remarks

We remark that if \( R \) is the quotient ring of \( F_2[x] \), the conclusion \( D(S_R) = D(U(S_R)) \) does not always hold true. For example, take \( f(x) = x \ast (x+1) \ast g(x) \in F_2[x] \) where \( \gcd(x \ast (x+1), g(x)) = 1_{F_2} \). Let \( R = F_2[x] / (f(x)) \). Take a sequence \( T = a_1 \cdot a_2 \cdot \ldots \cdot a_{\ell} \), where \( \theta_{a_1}(x) = x, \theta_{a_2}(x) = x+1, \) and \( a_3 \cdot \ldots \cdot a_{\ell} \) is a sequence of elements in \( U(S_R) \) of length \( \ell - 2 = D(U(S_R)) - 1 \) which contains no nonempty subsequence \( V \) with \( \sigma(V) = 0_{S_R} \). It is easy to verify that \( T \) is an irreducible sequence of length \( \ell = D(U(S_R)) + 1 \), which implies that \( D(S_R) \geq \ell + 1 = D(U(S_R)) + 2 \). Hence, we close this paper by proposing the following interesting problem.

**Problem.** Let \( R \) be a quotient ring of \( F_2[x] \) with \( 0 \neq R \neq F_2[x] \). Determine \( D(S_R) - D(U(S_R)) \).

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