On the Minimal Displacement Vector of the Douglas-Rachford Operator

Goran Banjac

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Abstract
The Douglas-Rachford algorithm can be represented as the fixed point iteration of a firmly nonexpansive operator, which converges to a fixed point, provided it exists. When the operator has no fixed points, the algorithm’s iterates diverge, but the difference between consecutive iterates converges to the minimal displacement vector, which can be used to certify infeasibility of an optimization problem. In this paper, we establish new properties of the minimal displacement vector, which allow us to generalize some existing results.

1 Introduction

The Douglas-Rachford algorithm is a powerful method for minimizing the sum of two convex functions and its asymptotic behavior is well-understood when the problem has a solution. While there exist some results studying feasibility problems involving two convex sets that do not intersect [BDM16, BM16, BM17], some recent works also study a more general setting in which the asymptotic behavior of the algorithm is characterized via the so-called minimal displacement vector. The authors in [BHM16] characterize this vector in terms of the domains of the functions, whose sum is to be minimized, and their Fenchel conjugates. This characterization is used in [RLY19] to show that a nonzero minimal displacement vector implies either primal or dual infeasibility of the problem, but there is an additional assumption imposed, which excludes the case of simultaneous primal and dual infeasibility. The authors in [BM19] derive a new convergence result on the algorithm applied to the problem of minimizing a convex function subject to a linear constraint, but they assume that the Fenchel dual problem is feasible. The analysis in [BGSB19, BL20] covers the case of simultaneous primal and dual infeasibility for a restricted class of problems and shows that the minimal displacement vector can be decomposed as the sum of two orthogonal vectors, one of which is a certificate of primal, and the other of dual infeasibility.

In this paper, we show that the orthogonal decomposition of the minimal displacement vector of the Douglas-Rachford operator established in [BGSB19, BL20] holds in the general case as well. We then consider a class of problems of minimizing a convex function
subject to a convex constraint and show that the algorithm generates certificates of both primal and dual strong infeasibility. This allows us to recover the results reported in \cite{BGSB19, BL20} as a special case of our analysis.

The paper is organized as follows. We introduce some definitions and notation in the sequel of Section 1, and some known results on the Douglas-Rachford algorithm in Section 2. Section 3 presents a decomposition of the minimal displacement vector and a new convergence result for a class of constrained convex minimization problems. Section 4 applies these new results to the problem of minimizing a convex quadratic function subject to a convex constraint. Finally, Section 5 concludes the paper.

1.1 Notation

All definitions introduced here are standard and can be found in \cite{BC17}, to which we also refer for basic results on convex analysis and monotone operator theory.

Let $\mathbb{N}$ denote the set of nonnegative integers, $\mathbb{R}$ the set of real numbers, and $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$ be real Hilbert spaces with inner products $\langle \cdot | \cdot \rangle$, induced norms $\| \cdot \|$, and identity operators $\text{Id}$. The power set of $\mathcal{H}$ is denoted by $2^\mathcal{H}$. Let $D$ be a nonempty subset of $\mathcal{H}$ with $D$ being its closure. We denote the range of operator $T : D \to \mathcal{H}$ by $\text{ran} \ T$ and the kernel of a linear operator $A$ by $\ker \ A$. For a proper lower semicontinuous convex function $f : \mathcal{H} \to ]-\infty, +\infty]$, we define its:

- **domain**: $\text{dom} \ f = \{ x \in \mathcal{H} \mid f(x) < +\infty \}$,
- **Fenchel conjugate**: $f^* : \mathcal{H} \to ]-\infty, +\infty] : u \mapsto \sup_{x \in \mathcal{H}} \langle x \mid u \rangle - f(x)$,
- **recession function**: $\text{rec} \ f : \mathcal{H} \to ]-\infty, +\infty] : y \mapsto \sup_{x \in \text{dom} f} (f(x+y) - f(x))$,
- **proximity operator**: $\text{Prox}_f : \mathcal{H} \to \mathcal{H} : x \mapsto \argmin_{y \in \mathcal{H}} (f(y) + \frac{1}{2}\|y-x\|^2)$,
- **subdifferential**: $\partial f : \mathcal{H} \to 2^{\mathcal{H}} : x \mapsto \{ u \in \mathcal{H} \mid \forall y \in \mathcal{H} \langle y-x \mid u \rangle \leq f(x) \}$.

For a nonempty closed convex set $C \subseteq \mathcal{H}$, we define its:

- **polar cone**: $C^\circ = \left\{ u \in \mathcal{H} \mid \sup_{x \in C} \langle x \mid u \rangle \leq 0 \right\}$,
- **recession cone**: $\text{rec} C = \{ x \in \mathcal{H} \mid (\forall y \in C) \langle x+y \mid x \rangle \leq 0 \}$,
- **indicator function**: $\iota_C : \mathcal{H} \to [0, +\infty] : x \mapsto \begin{cases} 0 & x \in C \\ +\infty & \text{otherwise} \end{cases}$,
- **support function**: $\sigma_C : \mathcal{H} \to ]-\infty, +\infty] : u \mapsto \sup_{x \in C} \langle x \mid u \rangle$,
- **projection operator**: $P_C : \mathcal{H} \to \mathcal{H} : x \mapsto \argmin_{y \in \mathcal{H}} \|y-x\|$,
- **normal cone operator**: $N_C : \mathcal{H} \to 2^{\mathcal{H}} : x \mapsto \begin{cases} \{ u \in \mathcal{H} \mid \sup_{y \in C} \langle y-x \mid u \rangle \leq 0 \} & x \in C \\ \emptyset & x \notin C \end{cases}$. 

2
2 Douglas-Rachford Algorithm

The Douglas-Rachford algorithm can be used to solve composite minimization problems of the form

$$\min_{x \in \mathcal{H}} f(x) + g(x), \quad (P)$$

where $f : \mathcal{H} \to ]-\infty, +\infty]$ and $g : \mathcal{H} \to ]-\infty, +\infty]$ are proper lower semicontinuous convex functions. Observe that $(P)$ is feasible if $0 \in \text{dom } f - \text{dom } g$ and strongly infeasible if $0 \not\in \text{dom } f - \text{dom } g$. The Fenchel dual of $(P)$ can be written as

$$\min_{\nu \in \mathcal{H}} f^*(\nu) + g^*(-\nu). \quad (D)$$

Starting from some $s_0 \in \mathcal{H}$, the Douglas-Rachford algorithm applied to $(P)$ generates the following iterates:

$$x_n = \text{Prox}_f s_n \quad (1a)$$
$$\nu_n = s_n - x_n \quad (1b)$$
$$\tilde{x}_n = \text{Prox}_g (2x_n - s_n) \quad (1c)$$
$$s_{n+1} = s_n + \tilde{x}_n - x_n, \quad (1d)$$

which can be written compactly as $s_n = T^n s_0$, where

$$T = \frac{1}{2} \text{Id} + \frac{1}{2} (2 \text{Prox}_g - \text{Id})(2 \text{Prox}_f - \text{Id})$$

is a firmly nonexpansive operator [LM79]. It is easy to show from (1) that

$$s_n - Ts_n \in (\text{dom } f - \text{dom } g) \cap (\text{dom } f^* + \text{dom } g^*).$$

Note that $T$ has a fixed point if and only if $0 \in \text{ran} (\text{Id} - T)$. To deal with the potential lack of a fixed point of $T$, we define its minimal displacement vector as

$$v = P_{\text{ran}(\text{Id} - T)}(0).$$

Since the set $\text{ran}(\text{Id} - T)$ is nonempty closed convex [Paz71, Lem. 4], the projection above is unique. We next show some useful relations among vector $v$, problem $(P)$, and the Douglas-Rachford iterates, which hold regardless of the existence of a fixed point of $T$.

**Fact 2.1.** Let $s_0 \in \mathcal{H}$ and $s_n = T^n s_0$. Then

(i) $v = P_{\text{dom } f - \text{dom } g^* + \text{dom } f^* + \text{dom } g^*}(0).$

(ii) $\frac{1}{n} s_n \to -v.$

(iii) $s_n - s_{n+1} \to v.$

**Proof.** The first result is [BHM16, Cor. 6.5], the second is [Paz71, Cor. 3], and the third is [BBR78, Cor. 2.3].
3 Minimal Displacement Vector

Motivated by the characterization of the minimal displacement vector given in Fact 2.1(i) and [BM19, Prop. 2.3], we define vectors
\[ v_P = P_{\text{dom } f - \text{dom } g}(0) \quad \text{and} \quad v_D = T_{\text{dom } f^* + \text{dom } g^*}(0). \]

3.1 Static Results

Although it is obvious that nonzero \( v_P \) and \( v_D \) imply strong infeasibility of \((P)\) and \((D)\), respectively, we next provide some useful identities.

Proposition 3.1. Vectors \( v_P \) and \( v_D \) satisfy the following equalities:
\[
\begin{align*}
\text{rec } f^*(-v_P) + \text{rec } g^*(v_P) &= -\|v_P\|^2, \\
\text{rec } f(-v_P) + \text{rec } g(-v_P) &= -\|v_P\|^2.
\end{align*}
\]

Proof. Since the proofs of both equalities follow very similar arguments, we only provide a proof for the first. Using the definition of \( v_P \) and [BC17, Prop. 6.47], we have
\[ -v_P \in N_{\text{dom } f - \text{dom } g}(v_P). \]
Using [BC17, Thm. 16.29] and the facts that \( \iota_D = \sigma_D \) and \( \partial \iota_D = N_D \), the inclusion above is equivalent to
\[ -\|v_P\|^2 = \sigma_{\text{dom } f - \text{dom } g}(-v_P) = \sigma_{\text{dom } f}(-v_P) + \sigma_{\text{dom } g}(v_P) = \text{rec } f^*(-v_P) + \text{rec } g^*(v_P), \]
where the second equality follows from \( \sigma_{C+D} = \sigma_{C+D} = \sigma_C + \sigma_D \) and \( \sigma_{-C} = \sigma_C \circ (-\text{Id}) \), and the third from [BC17, Prop. 13.49].

Proposition 3.2. The following relations hold between vectors \( v_P \), \( v_D \), and \( v \):
\[
\begin{align*}
\text{(i)} & \quad -v_P \in \left( \text{rec } (\text{dom } f) \right)^\oplus \cap \left( \text{rec } (\text{dom } g) \right)^\ominus, \\
\text{(ii)} & \quad -v_D \in \left( \text{rec } (\text{dom } f^*) \right)^\oplus \cap \left( \text{rec } (\text{dom } g^*) \right)^\ominus, \\
\text{(iii)} & \quad -v_P \in \left( \text{rec } (\text{dom } f^*) \right)^\ominus \cap \left( \text{rec } (\text{dom } g^*) \right)^\oplus, \\
\text{(iv)} & \quad -v_D \in \left( \text{rec } (\text{dom } f) \right)^\ominus \cap \left( \text{rec } (\text{dom } g) \right)^\oplus, \\
\text{(v)} & \quad \langle v_P \mid v_D \rangle = 0, \\
\text{(vi)} & \quad v_P + v_D \in \text{dom } f - \text{dom } g \cap \text{dom } f^* + \text{dom } g^*, \\
\text{(vii)} & \quad v = v_P + v_D.
\end{align*}
\]

Proof. (i)&(ii): Follow from [BCL04, Cor. 2.7] and the definitions of \( v_P \) and \( v_D \).
(iii)&(iv): Follow from parts (i)&(ii) and Lem. A.1.
(v): Since \( -v_P \in \left( \text{rec } (\text{dom } f) \right)^\oplus \) and \( -v_D \in \text{rec } (\text{dom } f) \), we have \( \langle v_P \mid v_D \rangle \leq 0 \). Also, since \( -v_P \in \left( \text{rec } (\text{dom } g) \right)^\ominus \) and \( -v_D \in \text{rec } (\text{dom } g) \), we have \( \langle v_P \mid v_D \rangle \geq 0 \). Therefore, it must be that \( \langle v_P \mid v_D \rangle = 0 \).
(vi): By (iv), we have \(-v_D \in \text{rec}(\text{dom } g)\), hence
\[ v_P + v_D \in \text{dom } f - \text{dom } g + v_D = \text{dom } f - (\text{dom } g - v_D) \subseteq \text{dom } f - \text{dom } g. \]

Similarly, by (iii) we have \(v_P \in \text{rec}(\text{dom } g^*)\), hence
\[ v_P + v_D \in \text{dom } f^* + \text{dom } g^* = \text{dom } f^* + (\text{dom } g^* + v_P) \subseteq \text{dom } f^* + \text{dom } g^*. \]

(vii): Assuming that \(v_P + v_D = 0\), the identity follows from Fact 2.1(i) and part (vi). We next assume that \(v_P + v_D \neq 0\). Using [BC17, Thm. 3.16] together with the definitions of \(v_P\), \(v_D\), and \(v\), we have
\[
\langle v - v_P \mid -v_P \rangle \leq 0 \iff \|v_P\|^2 \leq \langle v \mid v_P \rangle
\]
\[
\langle v - v_D \mid -v_D \rangle \leq 0 \iff \|v_D\|^2 \leq \langle v \mid v_D \rangle,
\]
which together with part (v) implies
\[
\|v_P + v_D\|^2 = \|v_P\|^2 + \|v_D\|^2 \leq \langle v \mid v_P + v_D \rangle \leq \|v\|\|v_P + v_D\|.
\]
Dividing the inequality by \(\|v_P + v_D\| \neq 0\), we get \(\|v_P + v_D\| \leq \|v\|\). Combining this with Fact 2.1(i) and part (vi), we obtain the result.

**Corollary 3.3.** The following relations hold between vectors \(v\), \(v_P\), and \(v_D\):

1. \(-v_P = P_{\text{rec}(\text{dom } f)}(-v)\).
2. \(-v_D = P_{\text{rec}(\text{dom } f)}(-v)\).

**Proof.** Follows directly from Prop. 3.2 and [BC17, Cor. 6.31].

The authors in [RLY19] have also established connections between recession functions and the minimal displacement vector, but the equalities in Prop. 3.1 provide a tight characterization of the left-hand sides and improve the bounds given in [RLY19]. Also, if problem \((P)\) is feasible, then \(0 \in \text{dom } f - \text{dom } g\) and \(v_P = 0\), which according to Prop. 3.2(vii) implies \(v = v_P\); similarly, if problem \((D)\) is feasible, then \(v = v_D\). Although these identities were established in [RLY19], they follow as a special case of our analysis, which is also applicable when both \((P)\) and \((D)\) are infeasible.

### 3.2 Dynamic Results

**Proposition 3.4.** Let \((c_n)_{n \in \mathbb{N}}\) be a sequence in \(H\) satisfying \(\frac{1}{n}c_n \to c\) and \(D \subseteq H\) a nonempty closed convex set. Then \(\frac{1}{n}P_{Dc_n} \to P_{\text{rec } Dc}\).

**Proof.** A related result is shown in [FP03, Lem. 6.3.13] and [GS19, Prop. 2.2] in a finite-dimensional setting. Using similar arguments here, together with those in [SL15, Lem. 4.3], we can only establish the weak convergence, i.e., \(\frac{1}{n}a_n := \frac{1}{n}P_{Dc_n} \to P_{\text{rec } Dc} = a\).

Using Moreau’s decomposition [BC17, Thm. 6.30], it follows that \(\frac{1}{n}a_n := \frac{1}{n}(\text{Id} - P_D)c_n \to \frac{1}{n}(\text{Id} - P_D)c_n \to \ldots\)
\( P_{(rec \, D)^\circ} c = b \) and \( \|c\|^2 = \|a\|^2 + \|b\|^2 \). For an arbitrary vector \( z \in D \), [BC17, Thm. 3.16] yields

\[
\|c_n - z\|^2 \geq \|a_n - z\|^2 + \|b_n\|^2, \quad \forall n \in \mathbb{N}.
\]

Dividing the inequality by \( n^2 \) and taking the limit superior, we get

\[
\lim \|\frac{1}{n}c_n\|^2 \geq \lim \|\frac{1}{n}a_n\|^2 + \lim \|\frac{1}{n}b_n\|^2,
\]

and thus

\[
\lim \|\frac{1}{n}a_n\|^2 \leq \lim \|\frac{1}{n}c_n\|^2 - \lim \|\frac{1}{n}b_n\|^2 \leq \|c\|^2 - \|b\|^2 = \|a\|^2,
\]

where the second inequality follows from [BC17, Lem. 2.42]. The inequality above yields

\[
\lim \|\frac{1}{n}a_n\|^2 \leq \|a\|^2,
\]

which due to [BC17, Lem. 2.51] implies \( \frac{1}{n}a_n \to a \).

The proposition above generalizes [BL20, Prop. 3.2(iii)], in which a similar result is shown, but with an additional assumption that \( \frac{1}{n}P_Dc_n \) exists.

We next consider a restricted class of problem \((P)\) in which \( f \) is the indicator function of a nonempty closed convex set.

**Corollary 3.5.** Let \( s_0 \in \mathcal{H} \) and \((x_n, \tilde{x}_n, \nu_n)_{n \in \mathbb{N}}\) be the sequences generated by (1) with \( f = \iota_D \), where \( D \subseteq \mathcal{H} \) is a nonempty closed convex set. Then

(i) \( \frac{1}{n}x_n \to P_{rec \, D}(-v) = -v_D \).

(ii) \( \frac{1}{n}\tilde{x}_n \to P_{rec \, D}(-v) = -v_D \).

(iii) \( \frac{1}{n}\nu_n \to P_{(rec \, D)^\circ}(-v) = -v_P \).

**Proof.** (i): Follows from (1a), Fact 2.1(ii), Prop. 3.4, and Cor. 3.3.

(ii): Due to (1d), Fact 2.1(iii), and part (i), we have

\[
\frac{1}{n}\tilde{x}_n = \frac{1}{n}(s_{n+1} - s_n + x_n) \to P_{rec \, D}(-v).
\]

(iii): Due to (1b), Fact 2.1(ii), part (i), and Moreau’s decomposition [BC17, Thm. 6.30], we have

\[
\frac{1}{n}\nu_n = \frac{1}{n}(s_n - x_n) \to (-v) - P_{rec \, D}(-v) = P_{(rec \, D)^\circ}(-v) = -v_P,
\]

where the last equality follows from Cor. 3.3.

### 4 Constrained Minimization of a Quadratic Function

Consider the following convex optimization problem:

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \langle z | Qz \rangle + \langle q | z \rangle \\
\text{subject to} & \quad Az \in C,
\end{align*}
\]

with \( Q : \mathcal{H}_1 \to \mathcal{H}_1 \) a monotone self-adjoint bounded linear operator, \( q \in \mathcal{H}_1 \), \( A : \mathcal{H}_1 \to \mathcal{H}_2 \) a bounded linear operator, and \( C \) a nonempty closed convex subset of \( \mathcal{H}_2 \); we assume that \( \text{ran} \, Q \) and \( \text{ran} \, A \) are closed. The objective function of the problem is convex, continuous, and Fréchet differentiable [BC17, Prop. 17.36(i)].
Proposition 4.1 ([BGSB19, Prop. 3.1]).

(i) If there exists $\bar{\mu} \in (\text{rec } C)^\circ$ such that $A^*\bar{\mu} = 0$ and $\sigma_C(\bar{\mu}) < 0$, then problem (2) is strongly infeasible.

(ii) If there exists $\bar{z} \in \mathcal{H}_1$ such that $Q\bar{z} = 0$, $A\bar{z} \in \text{rec } C$, and $\langle q \mid \bar{z} \rangle < 0$, then the dual of problem (2) is strongly infeasible.

Observe that (2) is an instance of problem (P) with $f: \mathcal{H}_1 \times \mathcal{H}_2 \to ]-\infty, +\infty]$ and $g: \mathcal{H}_1 \times \mathcal{H}_2 \to ]-\infty, +\infty]$ given by

\[
\begin{align*}
 f(z, y) &= \iota_C(y) \\
g(z, y) &= \frac{1}{2} \langle z \mid Qz \rangle + \langle q \mid z \rangle + \iota_{Az=y}(z, y),
\end{align*}
\]

where $\iota_{Az=y}$ denotes the indicator function of the set $\{(z, y) \in \mathcal{H}_1 \times \mathcal{H}_2 \mid Az = y\}$. Due to Lem. A.2, $f^*: \mathcal{H}_1 \times \mathcal{H}_2 \to ]-\infty, +\infty]$ and $g^*: \mathcal{H}_1 \times \mathcal{H}_2 \to ]-\infty, +\infty]$ are given by

\[
\begin{align*}
 f^*(\lambda, \mu) &= \iota_{\{0\}}(\lambda) + \sigma_C(\mu) \\
g^*(\lambda, \mu) &= \frac{1}{2} \langle \lambda + A^*\mu - q \mid Q^\dagger(\lambda + A^*\mu - q) \rangle + \iota_{\text{ran } Q}(\lambda + A^*\mu - q).
\end{align*}
\]

We next consider iteration (1) applied to the problem of minimizing the sum of the functions given in (3).

When $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite-dimensional Euclidean spaces and $C$ has some additional structure, problem (2) reduces to the one considered in [BGSB19], where the Douglas-Rachford algorithm (which is equivalent to the alternating direction method of multipliers) was shown to generate certificates of primal and dual strong infeasibility. This result was generalized in [BL20] to the case where $\mathcal{H}_1$ and $\mathcal{H}_2$ are real Hilbert spaces and $C$ is a general nonempty closed convex set.

The following proposition was first proven in [BGSB19] and then extended in [BL20] to a more general setting. We next show that the same result is a direct consequence of our analysis presented in Section 3. We use the notation

\[
v = (v', v''), \quad v_p = (v'_p, v''_p), \quad v_D = (v'_D, v''_D),
\]

where the first and second components are elements of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Proposition 4.2. Let $f: \mathcal{H}_1 \times \mathcal{H}_2 \to ]-\infty, +\infty]$ and $g: \mathcal{H}_1 \times \mathcal{H}_2 \to ]-\infty, +\infty]$ be given by (3), and $(z_n, y_n)$ and $(\lambda_n, \mu_n)$ be the Douglas-Rachford iterates corresponding to $x_n$ and $\nu_n$ in (1), respectively. Then

\[
\begin{align*}
 (i) \quad &\lambda_n = 0 \text{ for all } n \in \mathbb{N}. \\
 (ii) \quad &\frac{1}{2}(z_n, y_n, \mu_n) \to -(v'_D, v''_D, v''_D). \\
 (iii) \quad &(-v'_D, -v''_D) = (-v', P_{\text{rec } C}(v'')). \\
 (iv) \quad &(v''_p, -v''_D) = (0, P_{\text{rec } C}(v'')) \\
 (v) \quad &Qv'_D = 0. \\
 (vi) \quad &A\nu'_D = v''_D. \\
 (vii) \quad &\langle q \mid -v'_D \rangle = -\|v'_D\|^2. \\
 (viii) \quad &A^*v''_p = 0. \\
 (ix) \quad &\sigma_C(-v''_p) = -\|v''_p\|^2.
\end{align*}
\]
Proof. Let \((p_n, r_n)\) be the Douglas-Rachford iterates corresponding to \(s_n\) in (1) so that \((p_{n+1}, r_{n+1}) = T(p_n, r_n)\). As \(\text{Prox}_f = P_D\) with \(D = \mathcal{H}_1 \times C\), we have
\[
(z_n, y_n) = P_D(p_n, r_n) = (p_n, P_C r_n).
\] (5)

(i): From (1b) and (5), we have \(\lambda_n = p_n - z_n = 0\).

(ii)&(iii)&(iv): Follow from Cor. 3.5 with \(D = \mathcal{H}_1 \times C\).

(v)&(vi)&(vii): Using the identity \(\text{rec } f = \sigma_{\text{dom } f}\) [BC17, Prop. 13.49], it is easy to show that the recession functions of those in (3) are given by
\[
\text{rec } f(\bar{z}, \bar{y}) = \nu_{\text{rec } C}(\bar{y}) \\
\text{rec } g(\bar{z}, \bar{y}) = \langle q \mid \bar{z} \rangle + \nu_{\text{ker } Q}(\bar{z}) + \nu_{A z = y}(\bar{z}, \bar{y}).
\]

Due to Prop. 3.1, we obtain
\[
\nu_{\text{rec } C}(-v_D') + \langle q \mid -v_D' \rangle + \nu_{\text{ker } Q}(-v_D') + \nu_{Az = y}(-v_D', -v''_D) = -\|v_D\|^2,
\]
which implies
\[
Qv_D' = 0, \quad Av_D' = v''_D, \quad \langle q \mid -v_D' \rangle = -\|v_D\|^2.
\]

(viii)&(ix): Using the identity \(\text{rec } f^* = \sigma_{\text{dom } f}\), it is easy to show that the recession functions of those in (4) are given by
\[
\text{rec } f^*(\bar{\lambda}, \bar{\mu}) = \lambda_{\{0\}}(\bar{\lambda}) + \sigma_C(\bar{\mu}) \\
\text{rec } g^*(\bar{\lambda}, \bar{\mu}) = \lambda_{\{0\}}(\bar{\lambda} + A^*\bar{\mu}).
\]

Due to Prop. 3.1, we obtain
\[
\lambda_{\{0\}}(-v_D') + \sigma_C(-v''_D) + \lambda_{\{0\}}(v'_D + A^*v''_D) = -\|v_p\|^2,
\]
which implies
\[
A^*v''_D = 0, \quad \sigma_C(-v''_D) = -\|v_p\|^2. \square
\]

Prop. 4.1 and Prop. 4.2 imply that, if \(-v''_p\) is nonzero, then problem (2) is strongly infeasible, and similarly, if \(-v'_p\) is nonzero, then its dual is strongly infeasible. Moreover, these infeasibility certificates are limits of the sequences \((\frac{1}{n}\mu_n)_{n \in \mathbb{N}}\) and \((\frac{1}{n}z_n)_{n \in \mathbb{N}}\).

Remark 4.3. Using Fact 2.1(iii), the identity in (5), and the structure of \(g\) in (3b), it is easy to show that \((z_n - z_{n+1}, y_n - y_{n+1}, \mu_n - \mu_{n+1}) \to (v'_D, v''_D, v''_p)\). We do not know whether \(x_n - x_{n+1} \to v_D\) and \(\nu_n - \nu_{n+1} \to v_p\) hold in a more general setting.

5 Conclusions

We have presented some useful properties of the minimal displacement vector of the Douglas-Rachford operator applied to the problem of minimizing the sum of two convex
functions. In particular, we showed that the minimal displacement vector can be decomposed as the sum of two orthogonal vectors, one of which is a certificate of primal, and the other of dual strong infeasibility of the problem. Moreover, we showed that these infeasibility certificates can be obtained as the limits of sequences constructed from the Douglas-Rachford iterates, which allowed us to recover and generalize some existing results. It would be interesting to explore whether these convergence results hold in a more general case in which one of the functions is not necessarily the indicator function of a convex set.

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Appendix A Supporting Results

Lemma A.1. Let \( f : \mathcal{H} \to ]-\infty, +\infty] \) be a proper lower semicontinuous convex function. Then
\[
\text{dom} (\text{rec} (\text{dom} f))^\circ = \text{dom} \sigma_{\text{dom} f} = \text{dom} (\text{rec} f^*) \subseteq \text{rec} (\text{dom} f^*).
\]

Proof. The first equality can be found in [AET04] and the second is [BC17, Prop. 13.49]. To show the last inclusion, let \( d \in \text{dom}(\text{rec} f^*) \). Then \( \text{rec} f^*(d) < +\infty \), which implies
\[
(\forall y \in \text{dom} f^*) f^*(y + d) < +\infty \iff (\forall y \in \text{dom} f^*) y + d \in \text{dom} f^* \\
\iff d \in \text{rec} (\text{dom} f^*),
\]
and thus \( \text{dom}(\text{rec} f^*) \subseteq \text{rec} (\text{dom} f^*) \). Moreover, since \( \text{rec} (\text{dom} f^*) \) is always closed, we have \( \text{dom} (\text{rec} f^*) \subseteq \text{rec} (\text{dom} f^*) \).

Lemma A.2. Let \( g : \mathcal{H}_1 \times \mathcal{H}_2 \to ]-\infty, +\infty] \) be given by (3b). Its Fenchel conjugate \( g^* : \mathcal{H}_1 \times \mathcal{H}_2 \to ]-\infty, +\infty] \) is given by
\[
g^*(\lambda, \mu) = \frac{1}{2} \langle \lambda + A^* \mu - q | Q^\dagger (\lambda + A^* \mu - q) \rangle + \iota_{\text{ran} Q}(\lambda + A^* \mu - q).
\]

where \( Q^\dagger \) is the Moore-Penrose inverse of \( Q \).

Proof. For the function \( h : \mathcal{H}_1 \to \mathbb{R} : z \mapsto \frac{1}{2} \langle z | Qz \rangle + \langle q | z \rangle \), its Fenchel conjugate is given by
\[
h^*(\lambda) = \sup_{z \in \mathcal{H}_1} \left( \langle \lambda | z \rangle - \frac{1}{2} \langle z | Qz \rangle - \langle q | z \rangle \right) = \frac{1}{2} \langle \lambda - q | Q^\dagger (\lambda - q) \rangle + \iota_{\text{ran} Q}(\lambda - q),
\]
which follows directly from [BC17, Prop. 13.23(iii) & Prop. 17.36(iii)]. Thus, the Fenchel conjugate of $g$ is given by

$$g^*(\lambda, \mu) = \sup_{(z,y) \in H_1 \times H_2} \left( \langle \lambda \mid z \rangle + \langle \mu \mid y \rangle - \frac{1}{2} \langle z \mid Qz \rangle - \langle q \mid z \rangle - \lambda_{Az=y}(z,y) \right)$$

$$= \sup_{z \in H_1} \left( \langle \lambda + A^*\mu \mid z \rangle - \frac{1}{2} \langle z \mid Qz \rangle - \langle q \mid z \rangle \right)$$

$$= h^*(\lambda + A^*\mu).$$

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