COINCIDING MEAN OF THE TWO SYMMETRIES ON THE SET OF MEAN FUNCTIONS

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Abstract. On the set $\mathcal{M}$ of mean functions the symmetric mean of $M$ with respect to mean $M_0$ can be defined in several ways. The first one is related to the group structure on $\mathcal{M}$ and the second one is defined trough Gauss’ functional equation. In this paper we provide an answer to the open question formulated by B. Farhi about the matching of these two different mappings called symmetries on the set of mean functions. Using techniques of asymptotic expansions developed by T. Burič, N. Elezović and L. Mihoković (Vukišić) we discuss some properties of such symmetries through connection with asymptotic expansions of means involved. As a result of coefficient comparison, new class of means was discovered which interpolates between harmonic, geometric and arithmetic mean.

1. Introduction

Function $M: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is called a mean if for all $s, t \in \mathbb{R}^+$
\[ \min(s, t) \leq M(s, t) \leq \max(s, t). \] (1)

Mean $M$ is symmetric if for all $s, t \in \mathbb{R}^+$
\[ M(s, t) = M(t, s) \]
and homogeneous (of degree 1) if for all $\lambda, s, t \in \mathbb{R}^+$
\[ M(\lambda s, \lambda t) = \lambda M(s, t) \]

This paper was motivated by the problem of matching two different mappings on the set of mean functions formulated in paper [9] in which author introduced algebraic and topological structures on the set $\mathcal{M}_D$ of symmetric means on a symmetric domain $D$ with additional property
\[ M(s, t) = s \Rightarrow s = t, \quad \forall (s, t) \in D. \]

The first mapping is related to the group structure and the second one is defined trough Gauss’ functional equation. It was found that those mappings coincide for arithmetic, geometric and harmonic mean but the question of the existence of other solutions remained open. We shall take $D = \mathbb{R}^+ \times \mathbb{R}^+$.

First, let $\mathcal{A}_D$ be set of all functions $f: D \rightarrow \mathbb{R}$ such that
\[ (\forall (x, y) \in D) \quad f(x, y) = -f(y, x). \]
\((A_D, +)\) is an abelian group with the neutral element 0. Function \(\varphi: \mathcal{M}_D \to \mathcal{A}_D\) defined by
\[
\varphi(M)(x, y) := \begin{cases} 
\log \left( \frac{M(x, y) - x}{M(x, y) - y} \right), & x \neq y, \\
0, & x = y,
\end{cases}
\]
is a bijection. The composition law \(*: \mathcal{M}_D \times \mathcal{M}_D \to \mathcal{M}_D\) is defined by
\[
M_1 * M_2 = \varphi^{-1}(\varphi(M_1) + \varphi(M_2)).
\]
Thus \((\mathcal{M}_D, \ast)\) is an abelian group with the neutral element \(\varphi^{-1}(0) = A\). It can also easily be shown that the explicit formula for the composition law \(\ast\) holds:
\[
(M_1 * M_2)(x, y) = \begin{cases} 
\frac{x(M_1 - y)(M_2 - y) + y(M_1 - x)(M_2 - x)}{(M_1 - x)(M_2 - y) + (M_1 - y)(M_2 - x)}, & x \neq y, \\
x, & x = y.
\end{cases}
\]

Based on this operation, the first type of the symmetry was defined.

**Definition 1** ([9]). The symmetric mean \(M_2\) to a mean \(M_1\) with respect to mean \(M_0\) via the group structure \((\mathcal{M}_D, \ast)\) is defined with the expression
\[
S_{M_0}(M_1) = M_2 \Leftrightarrow M_1 * M_2 = M_0 * M_0.
\]

Combining (3) with (2) the explicit formula for symmetric mean of mean \(M_1\) with respect to \(M_0\) can easily be calculated:
\[
S_{M_0}(M_1) = \frac{x(M_1 - x)(M_0 - y)^2 - y(M_0 - x)^2(M_1 - y)}{(M_1 - x)(M_0 - y)^2 - (M_0 - x)^2(M_1 - y)}. \tag{4}
\]

We shall see the behavior of \(S_{M_0}\) for some basic well known means \(M_0\). For \((s, t) \in \mathcal{D} = \mathbb{R}^+ \times \mathbb{R}^+\) let
\[
A(s, t) = \frac{s + t}{2}, \quad G(s, t) = \sqrt{st}, \quad H(s, t) = \frac{2st}{s + t}.
\]
be the arithmetic, geometric and harmonic means respectively.

**Example 1** ([9]). For any mean \(M \in \mathcal{M}_D\), we have:
\[
\begin{align*}
(1) \quad S_A(M) &= 2A - M, \\
(2) \quad S_G(M) &= \frac{G^2}{M}, \\
(3) \quad S_H(M) &= \frac{H(M)}{2M - H(M)}.
\end{align*}
\]

Another type of symmetry, independent of the group structure \((\mathcal{M}_D, \ast)\), can also be defined.

**Definition 2** ([9]). \(M_2\) is said to be functional symmetric mean of \(M_1\) with respect to \(M_0\) if the following functional equation is satisfied:
\[
\sigma_{M_0}(M_1) = M_2 \Leftrightarrow M_0(M_1, M_2) = M_0. \tag{5}
\]

We can also say that mean \(M_0\) is the functional middle of \(M_1\) and \(M_2\). Defining equation on the right side of the equivalence relation (5) is known as the Gauss functional equation. Some authors refer to means \(M_1\) and \(M_2\) as a pair of \(M_0\)-complementary means. Mean \(M_0\) is also said to be \((M_1, M_2)\)-invariant. For recent related results see [13, 14, 15] and also survey article on invariance of means [12] and references therein. Furthermore, functional symmetric mean exists and it is unique.

With respect to the same means as in the latter example we may calculate the symmetric means.
Example 2 \([\text{[9]}]\). For any mean \(M \in \mathcal{M}_D\), we have:

1. \(\sigma_A(M) = 2A - M\),
2. \(\sigma_G(M) = \frac{G^2}{M}\),
3. \(\sigma_H(M) = \frac{HM}{2M-H}\).

Taking into account Examples \([1]\) and \([2]\) in which the same mappings appear with respect to arithmetic, geometric and harmonic mean appear, author in \([9]\) states the following.

**Open question.** For which mean functions \(M_0\) on \(\mathcal{D} = \mathbb{R}^+ \times \mathbb{R}^+\) the two symmetries, \(S\) and \(\sigma\), with respect to \(M_0\) coincide?

The goal of this paper is to analyze the open question and offer the answer in the setting of symmetric homogeneous means which possess the asymptotic expansion.

Techniques of asymptotic expansions were developed in \([2, 5, 6]\) and appeared to be very useful in comparison and finding inequalities for bivariate means \([3, 4]\), comparison of bivariate parameter means \([6]\), finding optimal parameters in convex combinations of means \([3, 16]\) and solving the functional equations of the form \(B(A(x)) = C(x)\), where asymptotic expansions of \(B\) and \(C\) are known, in order to obtain the asymptotic expansion of integral means \([7]\).

Techniques and results applied in this paper were described in Section 2. In the next step we obtained the algorithm for calculating the coefficients in the asymptotic expansions of means \(M_2^S = S_{M_0}(M_1)\) and \(M_2^\sigma = \sigma_{M_0}(M_1)\). Comparing the first few obtained coefficients we anticipated the general form of the coefficients in the asymptotic expansion of mean \(M_0\) for which \(M_2^S = M_2^\sigma\).

At the beginning of Section 3 we found closed formula and proved that proposed function represents the well defined one parameter class of means. Later we have shown that it also covers, as the special cases, means from Examples \([1]\) and \([2]\). Other properties were also explored such as limit behavior and monotonicity with respect to the parameter.

Lastly, in Section 4 we have proved that this class of means answered the open question and stated the hypothesis that there weren’t any other solutions in the context of homogeneous symmetric means which possess asymptotic power series expansions.

In addition, methods presented in this paper may be useful with similar problems regarding functional equations, especially in case when the explicit formula for included function was not known.

2. Asymptotic expansions

Recall the definition of an asymptotic power series expansion as \(x \to \infty\).

**Definition 3.** The series \(\sum_{n=0}^{\infty} c_n x^{-n}\) is said to be an asymptotic expansion of a function \(f(x)\) as \(x \to \infty\) if for each \(N \in \mathbb{N}\)

\[
f(x) = \sum_{n=0}^{N} c_n x^{-n} + o(x^{-N}).
\]

Main properties of asymptotic series and asymptotic expansions can be found in \([8]\). Taylor series expansion can also be seen as an asymptotic expansion but the converse is not generally true and the asymptotic series may also be divergent. The main characteristic of asymptotic expansion is that it provides good approximation using finite number of terms while letting \(x \to \infty\).
Because of the intrinsicity (1), mean $M$ would possess the asymptotic power series as $x \to \infty$ of the form

$$M(x + s, x + t) = \sum_{n=0}^{\infty} c_n(s, t)x^{-n+1}$$

with $c_0(s, t) = 1$. For a homogeneous symmetric mean the coefficients $c_n(s, t)$ are also homogeneous symmetric polynomials of degree $n$ in variables $s$ and $t$ and for $s = -t$ they have simpler form. Let the means included possess the asymptotic expansions as $x \to \infty$ of the form

$$M_0(x - t, x + t) = \sum_{n=0}^{\infty} a_n t^{2n} x^{-2n+1},$$

$$M_1(x - t, x + t) = \sum_{n=0}^{\infty} b_n t^{2n} x^{-2n+1},$$

$$M_2(x - t, x + t) = \sum_{n=0}^{\infty} c_n t^{2n} x^{-2n+1}.$$  

Conversely, it can also be shown that the expansion in variables $(x - t, x + t)$ is sufficient to obtain so called two variable expansion, i.e. the expansion in variables $(x + s, x + t)$. Furthermore, note that

$$a_0 = b_0 = c_0 = 1.$$ 

In this section we will find the asymptotic expansions of means $M^S_2 = S_{M_0}(M_1)$ and $M_2^\sigma = \sigma_{M_0}(M_1)$.

2.1. Symmetry $S_{M_0}$. Recall the recently developed results for transformations of asymptotic series, i.e. the complete asymptotic expansions of the quotient and the power of asymptotic series.

Lemma 1 ([6], Lemma 1.1.). Let function $f(x)$ and $g(x)$ have following asymptotic expansions ($a_0 \neq 0, b_0 \neq 0$) as $x \to \infty$:

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}, \quad g(x) \sim \sum_{n=0}^{\infty} b_n x^{-n}.$$ 

Then asymptotic expansion of their quotient $f(x)/g(x)$ reads as

$$\frac{f(x)}{g(x)} \sim \sum_{n=0}^{\infty} c_n x^{-n},$$

where coefficients $c_n$ are defined by

$$c_n = \frac{1}{b_0} \left( a_n - \sum_{k=0}^{n-1} b_{n-k} c_k \right).$$

Lemma 2 ([2][10]). Let $m(x)$ be a function with asymptotic expansion ($c_0 \neq 0$):

$$m(x) \sim \sum_{n=0}^{\infty} c_n x^{-n}, \quad (x \to \infty).$$
Then for all real \( r \) it holds
\[
[m(x)]^r \sim \sum_{n=0}^{\infty} P[n, r, (c_j)_{j \in \mathbb{N}_0}] x^{-n}
\]
where
\[
P[0, r, (c_j)_{j \in \mathbb{N}_0}] = c_0,
\]
\[
P[n, r, (c_j)_{j \in \mathbb{N}_0}] = \frac{1}{nc_0} \sum_{k=1}^{n} [k(1 + r) - n]c_k P[n - k, r, (c_j)_{j \in \mathbb{N}_0}]. \tag{8}
\]

Symmetric mean with respect to mean \( M_0 \) of mean \( M_1 \) via the group structure \((\mathcal{M}_D, \ast)\) as a consequence of (1) can be expressed as
\[
M^S_2(x - t, x + t) = S_{M_0}(M_1)(x - t, x + t)
\]
\[
= \frac{(x - t)(M_1 - x + t)(M_0 - x - t)^2 - (x + t)(M_0 - x + t)^2(M_1 - x - t)}{(M_1 - x + t)(M_0 - x - t)^2 - (M_0 - x + t)^2(M_1 - x - t)}
\]
\[
= \frac{(x - t)(M_1 + t)(M_0 - t)^2 - (x + t)(M_0 + t)^2(M_1 - t)}{(M_1 + t)(M_0 - t)^2 - (M_0 + t)^2(M_1 - t)}
\]
\[
= x + \frac{2tM_0 - t^2M_1 - M_0^2 M_1}{t^2 + M_0^2 - 2M_0 M_1},
\]
where \( M_i, i = 1, 2, 3, \) stands for \( M_i - x \). The variables \( (x - t, x + t) \) were omitted for the sake of simplicity. Further calculations reveal that
\[
M^S_2(x - t, x + t) = x + t^2 x^{-1} \left[ (2c_1 - a_1) + \right.
\]
\[
+ \sum_{n=0}^{\infty} \left( 2c_{n+2} - a_{n+2} + \sum_{k=0}^{n} \left( \sum_{j=0}^{k} (c_{j+1} c_{k-j+1}) a_{n+1-k} \right) \right) t^{2n+2} x^{-2n-2} \left. \right] \times
\]
\[
\left[ 1 + \sum_{n=0}^{\infty} \sum_{k=0}^{n} c_{k+1} (c_{n-k+1} - 2a_{n-k+1}) t^{2n+2} x^{-2n-2} \right]^{-1}.
\]

Coefficients \( b^S_n \) for \( n \geq 1 \) are obtained using Lemma (1) for division of asymptotic series. Hence, we have the following:
\[
b^S_0 = 1,
\]
\[
b^S_n = num_n - \sum_{k=0}^{n-2} den_{n-1-k} b^S_{k+1}, \quad n \geq 1,
\]
where \((num_n)_{n \in \mathbb{N}_0}\) and \((den_n)_{n \in \mathbb{N}_0}\) denote auxiliary sequences which appear in numerator and denominator:
\[
um_0 = 2c_1 - a_1,
\]
\[
um_n = 2c_{n+1} - a_{n+1} + \sum_{k=0}^{n-1} \left( \sum_{j=0}^{k} (c_{j+1} c_{k-j+1}) a_{n-k} \right), \quad n \geq 1,
\]
and
\[
den_0 = 1,
\]
\[
den_n = \sum_{k=0}^{n-1} c_{k+1} (c_{n-k} - 2a_{n-k}), \quad n \geq 1.
\]
We shall calculate the first few coefficients:

\[ b_0^S = 1, \]
\[ b_1^S = 2c_1 - a_1, \]
\[ b_2^S = 2c_2 - a_2 - 2c_1(a_1 - c_1)^2, \]
\[ b_3^S = 2c_3 - a_3 - 2(a_1 - c_1)(2a_2c_1 + c_1^2)(2a_1^2 - 3a_1c_1 + c_1^2) + (a_1 - 3c_1)c_2, \]
\[ b_4^S = 2c_4 - a_4 - 2(a_2^2c_1 + 4a_1^2c_1^3 + 4a_3^3c_1(-3c_1^3 + c_2)
\quad + 2a_2((3a_1 - 2c_1)(a_1 - c_1)c_1^2 + (a_1 - 2c_1)c_2)
\quad + a_1^2(13c_1^5 - 15c_1^3c_2 + c_3) + 2a_1(a_3c_1 - 3c_1^6 + 8c_1^3c_2 - c_2^3 - 2c_1c_3)
\quad + c_1(-2a_3c_1 + c_1^6 - 5c_3^3c_2 + 3c_2^2 + 3c_1c_3)), \]
\[ b_5^S = 2c_5 - a_5 - 2(-2a_4c_1^2 + 8a_1^5c_1^4 + 4a_3c_1^3 - c_1^5 - 4a_3c_1c_2 + 7c_1^5c_2 - 10c_1^3c_2^2
\quad + c_2^3 + a_2^2(6a_1^2c_1^2 - 5c_1^5 + c_2) + 4a_1^4c_1^2(-7c_1^3 + 3c_2) - 5c_1^6c_3 + 6c_1c_2c_3
\quad + 2a_3^2(19c_1^6 - 24c_1^3c_2 + c_2^3 + 2c_1c_3) + 2a_2(a_3c_1 + 8a_1^3c_1^3 - 3c_1^6 + 8c_1^3c_2
\quad - c_2^3 + 6a_1^2c_1(-3c_1^3 + c_2) - 2c_1c_3 + a_1(13c_1^5 - 15c_1^3c_2 + c_3)) + 3c_1^2c_4
\quad + c_1^2(6a_3c_1^2 - 5c_1^5 - 13c_1^3c_2 + 3c_2^3 + 3c_1c_3) + c_4) + 2a_1(a_4c_1
\quad + a_3(-5c_1^5 + c_2) + 2(2c_1^3 - 9c_1^2c_2 + 6c_1^2c_2^2 + 4c_1^3c_3 - c_2c_3 - c_1c_4)). \]

2.2. Symmetry \( \sigma_{M_0} \). The problem of functional symmetric mean corresponds to the functional equation

\[ M_0(x - t, x + t) = M_0(M_1(x - t, x + t), M_2(x - t, x + t)) \]

which we will solve in terms of asymptotic series. To this end, we shall use the following result from Burić and Elezović about the asymptotic expansion of the composition of means.

**Theorem 3 (I, Theorem 2.2.).** Let \( M \) and \( N \) be given homogeneous symmetric means with asymptotic expansions

\[ M(x - t, x + t) = \sum_{k=0}^{\infty} a_k t^{2k} x^{-2k+1}, \quad N(x - t, x + t) = \sum_{k=0}^{\infty} b_k t^{2k} x^{-2k+1}, \]

and let \( F \) be homogeneous symmetric mean with expansion

\[ F(x - t, x + t) = \sum_{k=0}^{\infty} \gamma_k t^{2k} x^{-2k+1}. \]

Then the composition \( H = F(M, N) \) has asymptotic expansion

\[ H(x - t, x + t) = \sum_{k=0}^{\infty} h_n t^{2n} x^{-2n+1}, \]

where coefficients \( (h_n) \) are calculated by

\[ h_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2sk} P[j, 2k, (d_m)_{m \in \mathbb{N}_0}] P[n - 2zk - j, -2k + 1, (e_m)_{m \in \mathbb{N}_0}]. \]
Sequences \((c_n)\) and \((d_n)\) are defined by
\[
c_n = \frac{1}{2}(a_n + b_n), \quad d_n = \frac{1}{2}(a_{n+z} - b_{n+z}), \quad n \geq 0,
\]
where \(z\) is the smallest number such that \(d_n \neq 0\).

Applying Theorem 3 on \(M = M_1, N = M_2\) (or equivalently \(M = M_2, N = M_1\)) and \(F = M_0\) we obtain the asymptotic expansion of the composition \(M_0(M_1 M_2)\). Since the equation \(M_0 = M_0(M_1 M_2)\) holds, on the other side in Theorem 3 we also have \(H = M_0\). The coefficients in the asymptotic expansion of the composition \(M_0(M_1 M_2)\) equal the coefficients \(c_n\) in the asymptotic expansion of mean \(M_0\). In the end, we obtain the recursive algorithm for coefficients \(c_n\):
\[
c_0 = 1;
\]
\[
c_n = \sum_{k=0}^{n-2}\sum_{j=0}^{n-2k} c_k P[j, 2k, (\frac{1}{2}(a_m - b_m^2))_{m \geq z}] \times P[n - 2zk - j - 2k + 1, (\frac{1}{2}(a_m + b_m^2))_{m \in N_0}], \quad n \geq 1,
\]
where \(P[n, r, (c_m)_{m \in N_0}], n \in N_0\), denotes the \(n\)-th coefficient in the asymptotic expansion of \(r\)-th power of the asymptotic series with coefficients \((c_m)_{m \in N_0}\), as it was defined in (8). Because of (7), \(z\) is always greater or equal to 1.

For \(z = 1\) we calculate the first few coefficients:
\[
c_0 = 1,
\]
\[
c_1 = \frac{1}{2}(a_1 + b_1^2),
\]
\[
c_2 = \frac{1}{2}(a_2 + b_2^2) + \frac{1}{4}(a_1 - b_1^2)^2 c_1,
\]
\[
c_3 = \frac{1}{2}(a_3 + b_3^2) - \frac{1}{8}(a_1 - b_1^2)(a_1^2 - 4a_2 - (b_1^2)^2 + 4b_2^2) c_1,
\]
\[
c_4 = \frac{1}{2}(a_4 + b_4^2) + \frac{1}{16}((a_1^4 + 4a_2^2 - 8a_3b_1^2 + (b_1^2)^4 + 2a_2(b_1^2)^2 - 4b_2^2)
\]
\[\quad - 2a_1^2(3a_1 + (b_1^2)^2 - b_2^2) - 6(b_1^2)^2b_2^2 + 4(b_2^2)^2
\]
\[\quad + 4a_1(2a_3 + b_1^2(a_2 + b_2^2) - 2b_3^2) + 8b_1^2b_2^2) c_1 + (a_1 - b_1^2)^4 c_2),
\]
\[
c_5 = \frac{1}{2}(a_5 + b_5^2) - \frac{1}{32}((a_1^2 + a_1^4b_1^2 - 4a_2^2b_1^2 + 16a_4b_1^2 - 4a_3(b_1^2)^2 + (b_1^2)^5
\]
\[\quad - 2a_1^2(4a_2 + (b_1^2)^2) + 16a_3b_2^2 - 8(b_1^2)^3b_2^2 + 12b_1^2(b_2^2)^2
\]
\[\quad - 8a_2(3a_3 + b_1^2b_2^2 - 2b_3^2) + 2a_1^2(6a_3 - (b_1^2)^3 + 4b_1^2b_2^2 - 2b_3^2) + 12(b_1^2)^2b_2^2
\]
\[\quad - 16b_1^2b_3^2 - 16b_1^2b_2^2 + a_1(12a_2 - 16a_4 - 8a_3b_1^2 + (b_1^2)^4 + 8a_2((b_1^2)^2 - b_2^2)
\]
\[\quad - 4(b_2^2)^2 - 8b_1^2b_3^2 + 16b_1^2) c_1 - (a_1 - b_1^2)^3(3a_1^2 - 8a_2 - 3(b_1^2)^2 + 8b_1^2) c_2).
\]

The connection between \(b_n^m\) and \(c_m\) with the highest index \(n\) in each equation is linear. In the expression (9) \(b_n^m\) appears only in the second part
\[
P[n - 2zk - j - 2k + 1, (\frac{1}{2}(a_m + b_m^2))_{m \in N_0}],
\]
when \(k = j = 0\). Then (10) becomes \(P[n, 1, (\frac{1}{2}(a_m + b_m^2))_{m \in N_0}]\) which represents the \(n\)-th coefficient in the \(\sum_{n=0}^{\infty} \frac{1}{2}(a_n + b_n^2)t^{2n}x^{-2n+1}\) to the power of 1 which equals \(\frac{1}{2}(a_n + b_n^2)\). So we can easily extract \(b_n^m\). The first few coefficients \(b_n^m\) are:
\[
b_0^m = 1,
\]
**Comparison of coefficients.** Sequences \((b_n^S)_{n \in \mathbb{N}_0}\) and \((b_n^T)_{n \in \mathbb{N}_0}\) represent the coefficients in asymptotic expansions of means which are results of mappings \(S_{M_0}(M_1)\) and \(\sigma_{M_0}(M_1)\) respectively. Since we are looking for a mean \(M_1\) such those mappings coincide, \(b_n^S\) and \(b_n^T\) need to be equal. Since the equality must hold for any mean \(M_1\) we may suppose that \(z = 1\) which is equivalent with \(a_1 \neq c_1\). Equating \(b_n^S\) with \(b_n^T\) and \(b_1^S\) with \(b_1^T\) doesn’t provide any new information except

\[b_0 = b_0^S = b_0^T = 1 \text{ and } b_1 = b_1^S = b_1^T = 2c_1 - a_1.\]

With such \(b_1^T\) we may express \(b_2^T\) as

\[b_2^T = 2c_2 - a_2 - 2c_1(a_1 - c_1)^2,\]

which is already equal to \(b_2^S\). Now we can substitute

\[b_2 = b_2^S = 2c_2 - a_2 - 2c_1(a_1 - c_1)^2,\]

in \(b_3^T\) to obtain

\[b_3^T = 2c_3 - a_3 - 2c_1(a_1 - c_1)(2a_2 + 2c_1(a_1 - c_1)^2 + c_1^2 - a_1c_1 - 2c_2),\]
which after equating with $b_3^S$ gives the following condition

$$(a_1 - c_1)^2(c_1^2 + c_1^3 + c_2) = 0.$$ 

Since we assumed that $a_1$ and $c_1$ are not equal, it is necessarily

$$c_2 = -c_1^2(1 + c_1).$$

Now we have

$$b_3 = b_3^S = b_3^S = 2c_3 - a_3 - 2c_3(a_1 - c_1) \left( (3 - 4a_1)c_1^2 + a_1(2a_1 - 1)c_1 + 2a_2 + 4c_1^3 \right).$$

After substitutions we observe the next coefficient

$$b_4^S = 2c_4 - a_4 - 2c_1 \left( 2a_2c_1(-a_3(6c_1 + 1) + 3a_1^2 + 2c_1(2c_1 + 1)) \right)$$

$$+ c_1 \left( c_1(-4a_1^3(4c_1 + 1) + a_1^2(2c_1 + 1)(15c_1 + 2) - 2a_1c_1(14c_1(c_1 + 1) + 3) \right)$$

$$+ 4a_1^2 + c_1^2(c_1(11c_1 + 15) + 5)) - 2a_3) + 2c_3(c_1 - a_1) + a_2^2 + 2a_1a_3$$

which after equating with $b_4^S$ gives the following condition:

$$(a_1 - c_1)^2 \left( 2c_1^3(c_1 + 1)^2 - c_3 \right) = 0,$$

and we conclude that it must be

$$c_3 = 2c_1^3(1 + c_1)^2.$$

We continue with this procedure as it was described above. Further calculations reveal that the first few coefficients $c_n$ have the following form:

$$c_0 = 1,$$

$$c_1 = c,$$

$$c_2 = -c^2(1 + c),$$

$$c_3 = 2c^3(1 + c)^2,$$

$$c_4 = -5c^4(1 + c)^3,$$

$$c_5 = 14c^5(1 + c)^4,$$

$$c_6 = -42c^6(1 + c)^5.$$

After these first steps it is natural to state the following hypothesis about the general formula for the coefficients in the asymptotic expansion of mean $M_0$:

$$c_0 = 1,$$

$$c_n = (-1)^{n-1}C_{n-1}c^n(1 + c)^{n-1}, \quad n \geq 1,$$

where $C_n$ denotes the $n$-th Catalan number. Catalan numbers appear in many occasions and their behavior has been widely explored. Here we mention only a few properties which we will use in sequel. Catalan numbers are defined by

$$C_n = \frac{1}{n+1} \binom{2n}{n}, \quad n \in \mathbb{N}_0$$

and they satisfy the recursive relation

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad n \in \mathbb{N}_0.$$
Based on this recursive relation the generating function for Catalan numbers can be obtained (11):

$$
\sum_{n=0}^{\infty} C_n y^n = \frac{1 - \sqrt{1 - 4y}}{2y}.
$$

(12)

3. New mean function

In this section we shall find closed a form for a mean whose coefficients are given in (11). We start from asymptotic expansion (6):

$$
M_0(x - t, x + t) = x + \sum_{n=1}^{\infty} (-1)^{n-1} C_{n-1} (1 + c)^n t^{2n} x^{1-2n}.
$$

(13)

Introducing the substitution

$$
y = -\frac{c(1+c)t^2}{x^2},
$$

yields

$$
M_0(x - t, x + t) = x + ct^2 x^{-1} \sum_{n=0}^{\infty} C_n y^n,
$$

and then according to the formula (12), for $c + 1 \neq 0$, we obtain

$$
M_0(x - t, x + t) = x + ct^2 x^{-1} \sum_{n=0}^{\infty} C_n y^n = x + \frac{1 + 2c}{2(1 + c)} x^2 + \frac{1}{2(1 + c)} \sqrt{x^2 + 4c(1 + c)t^2}.
$$

(14)

With substitution

$$
x = \frac{a + b}{2}, \quad t = \frac{b - a}{2},
$$

in (14), we obtain the expression for $M_0$ in terms of variables $a$ and $b$. For $c \in \mathbb{R} \setminus \{-1\}$ and $a, b > 0$ we define function $L_c: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$

$$
L_c(a, b) = \frac{a + b}{2} + \frac{1 + 2c}{2(1 + c)} \sqrt{\left(\frac{a + b}{2}\right)^2 + 4c(1 + c) \left(\frac{b - a}{2}\right)^2}.
$$

(15)

Theorem 4. For $c \in \langle -1, +\infty \rangle$ function $L_c$ is a mean.

Remark 1. For $c \in (-1, +\infty)$ function $L_c$ corresponds to the harmonic mean.

Proof. We shall divide proof into the several parts.

1. Function $L_c$ is well defined for all $(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+$. We can rearrange terms under the square root

$$
\left(\frac{a + b}{2}\right)^2 + 4c(1 + c) \left(\frac{b - a}{2}\right)^2 = \frac{1}{4} ((a + b)^2 + 4c(1 + c)(b - a)^2)
$$

$$
= \frac{1}{4}(1 + 2c)^2(a - b)^2 + 4ab > 0.
$$
2. Function $L_c$ is a mean. $L_c$ is obviously a symmetric function. Without the loss of generality we may suppose that $a < b$. Now the condition
$$\min(a, b) < L_c(a, b) < \max(a, b)$$
is equivalent to
$$a < \frac{a + b}{2} + \frac{1}{2(1 + c)} \sqrt{\left(\frac{a + b}{2}\right)^2 + 4c(1 + c)\left(\frac{b - a}{2}\right)^2} < b.$$
Let $s = \frac{a}{b}$. Then $0 < s < 1$ and previous expression becomes
$$s < \frac{s + 1}{2} + \frac{1}{4(1 + c)} \sqrt{(1 + s)^2 + 4c(1 + c)(1 - s)^2} < 1,$$
or in other words
$$\frac{1}{4(1 + c)}[(2c + 3)s - (2c + 1)] < \frac{1}{4(1 + c)}[\sqrt{(1 + s)^2 + 4c(1 + c)(1 - s)^2} < \frac{1}{4(1 + c)}[-(1 + 2c)s + (2c + 3)].$$
Denote
$$I_1(s) = (2c + 3)s - (2c + 1), \quad I_2(s) = -(1 + 2c)s + (2c + 3).$$
Suppose $c + 1 > 0$. We need to prove the following inequalities
$$I_1(s) < \frac{1}{4(1 + c)}\sqrt{(1 + s)^2 + 4c(1 + c)(1 - s)^2} < I_2(s). \quad (16)$$
If $I_1(s) \geq 0$, squaring the left side inequality yields
$$(2c + 3)^2s^2 - 2s(2c + 3)(2c + 1) + (2c + 1)^2 < (1 + s)^2 + 4c(1 + c)(1 - s)^2,$$which reduces to the condition
$$8(s - 1)(c + 1) < 0,$$which is fulfilled. If $I_1(s) < 0$ then the left side inequality in (16) is trivially satisfied.

On the other side,
$$I_2(s) \geq 0 \Leftrightarrow \left(c < \frac{1}{2} \text{ and } s \geq \frac{2c + 3}{2c + 1}\right) \text{ or } \left(c > \frac{1}{2} \text{ and } s \leq \frac{2c + 3}{2c + 1}\right) \text{ or } c = -\frac{1}{2}.$$
Since for $c \in (-1, -\frac{1}{2})$ we have $\frac{2c + 3}{2c + 1} < 0$ and for $c \in (-\frac{1}{2}, +\infty)$ $\frac{2c + 3}{2c + 1} > 1$, it holds that $I_2(s) \geq 0$. Squaring the right side inequality in (16) then yields
$$(1 + s)^2 + 4c(1 + c)(1 - s)^2 < (1 + 2c)^2s^2 - 2s(1 + 2c)(2c + 3) + (2c + 3)^2$$which reduces to obviously fulfilled condition
$$8(s - 1)(c + 1) < 0.$$
Proof of the theorem is complete. $\square$

Remark 2. Notice that we proved that $L_c$ is a strict mean, i.e. for $s \neq t$ strict inequalities hold:
$$\min(s, t) < M(s, t) < \max(s, t).$$
3.1. Special cases. Before we continue further, let us see what happens with some of the special cases of parameter $c$.

**Example 3.**  
(1) $c = -1$. Then mean has two non-zero coefficients:

\[ c_0 = 1, \quad c_1 = c, \quad c_n = 0, \quad n \geq 2. \]

Corresponding asymptotic expansion is finite. From (13) we obtain

\[ L_c(x - t, x + t) = x + ct x^{-1}, \]

which, after substitution $x = \frac{a+b}{2}$, $t = \frac{b-a}{2}$, becomes

\[ L_c(a, b) = \frac{a+b}{2} + c \cdot \frac{(b-a)^2}{4} \cdot \frac{2}{a+b} = \frac{2ab}{a+b} = H(a, b), \]

where $H$ is the harmonic mean.

(2) $c = 0$. All coefficients except $c_0$ equal zero. Then either from the (13) or (14) we obtain

\[ L_c(x - t, x + t) = x, \]

and after the substitution

\[ L_c(a, b) = \frac{a+b}{2} = A(a, b), \]

where $A$ is the arithmetic mean.

(3) $c = -\frac{1}{2}$. The coefficients are

\[ c_0 = 1, \quad c_n = -\frac{1}{2^{2n-1}} C_{n-1}, \quad n \geq 1. \quad (17) \]

Coefficients (17) correspond to the coefficients in asymptotic expansion of geometric mean obtained in [5] for $\alpha = 0$ and $\beta = t$, and also to coefficients of power mean $M_p$ with $p = 0$ obtained in [6]. On the other side, from the formula (14) we obtain

\[ L_c(x - t, x + t) = \sqrt{x^2 - t^2}, \]

and after substitution

\[ L_c(a, b) = \sqrt{ab} = G(a, b), \]

where $G$ is the geometric mean.

From the example above we see that we covered the cases of means for which in [9] was stated that symmetries $S$ and $\sigma$ coincide.

3.2. Limit cases and monotonicity. In this subsection we study properties of $L_c$ with respect to parameter $c$. First, we state the following proposition which can be proved using basic methods of mathematical analysis.

**Proposition 5.** For a fixed pair $(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+$ and function $L_c$ holds

(1) $\lim_{c \to -\infty} L_c(a, b) = \lim_{c \to -\infty} L_c(a, b) = \max(a, b)$,

(2) $\lim_{c \to -1^-} L_c(a, b) = +\infty$,

(3) $\lim_{c \to -1^+} L_c(a, b) = \frac{2ab}{a+b} = H(a, b)$. 


It is well known that the following double inequality hold

\[ H < A < G. \]

Also, \( H = L_c \) for \( c \to -1 \), \( G = L_c \) for \( c = -\frac{1}{2} \) and \( A = L_c \) for \( c = 0 \). In the next Theorem we explore the ordering of means \( L_c \) with respect to parameter \( c \).

**Theorem 6.** For a fixed pair \((a, b) \in \mathbb{R}^+ \times \mathbb{R}^+\), \( a \neq b \), function \( f : \mathbb{R} \setminus \{-1\} \to \mathbb{R} \),

\[ f(c) = L_c(a, b) \]

is strictly increasing on intervals \((-\infty, -1)\) and \((-1, +\infty)\). In addition, for \( c_1 \in (-\infty, -1) \) and \( c_2 \in (-1, +\infty) \) it holds \( f(c_1) \geq f(c_2) \).

**Proof.** From the definition of \( L_c \) it follows

\[ f(c) = (a + b)g(c) + \sqrt{(a + b)^2g(c)^2 - 4abg(c) + ab}, \]

where

\[ g(c) = \frac{1 + 2c}{4(1 + c)}. \]

The first derivative of function \( f \) equals

\[ f'(c) = g'(c) \left[ (a + b) + \frac{(a + b)^2g(c) - 2ab}{\sqrt{(a + b)^2g(c)^2 - 4abg(c) + ab}} \right]. \]

We easily see that

\[ g'(c) = \frac{1}{4} \left( \frac{1 + 2c}{1 + c} \right)' = \frac{1}{4} \cdot \frac{1}{(1 + c)^2} > 0. \]

So the condition \( f'(c) > 0 \) is equivalent to

\[ g(c) > (a + b) + \frac{(a + b)^2g(c) - 2ab}{\sqrt{(a + b)^2g(c)^2 - 4abg(c) + ab}} > 0 \]

or

\[ (a + b)\sqrt{(a + b)^2g(c)^2 - 4abg(c) + ab} > 2ab - (a + b)^2g(c), \quad (18) \]

which is satisfied when the right side is less than zero. On the other side, when

\[ 2ab - (a + b)^2g(c) \geq 0, \]

condition (18) is equivalent to

\[ (a + b)^2 \left[ (a + b)^2g(c)^2 - 4abg(c) + ab \right] > 4a^2b^2 + (a + b)^2 \left[ (a + b)^2g(c)^2 - 4abg(c) \right] \]

which is true for \( a \neq b \).

The second part of the theorem follows from the first part and the Proposition \( \Box \).
4. Answer to the open question

**Theorem 7.** For mean \( L_c, c \in [-1, +\infty) \), defined in (15) symmetries \( S_{L_c} \) and \( \sigma_{L_c} \) coincide.

**Proof.** Let us rewrite mean \( L_c \) in the following manner:

\[
L_c(a, b) = \frac{1}{4(1+c)} \left[ (1 + 2c)(a + b) + \sqrt{(a + b)^2 + 4c(1+c)(b - a)^2} \right].
\]

For \( M_0 = L_c \) and variable mean \( M = M \), there exists symmetric mean \( \sigma = \sigma_{L_c}(M) \), i.e. the condition \( L_c(M, \sigma) = L_c \) holds, which yields (for the sake of brevity, the variables will be omitted):

\[
\frac{1}{4(1+c)} \left[ (1 + 2c)(M + \sigma) + \sqrt{(M + \sigma)^2 + 4c(1+c)(M - \sigma)^2} \right] = L_c,
\]

or equivalently

\[
\sqrt{(M + \sigma)^2 + 4c(1+c)(M - \sigma)^2} = 4(1+c)L_c - (1 + 2c)(M + \sigma).
\]

We rearrange the terms and because of the existence of mean \( \sigma = \sigma_{L_c}(M) \), we may square the latter expression:

\[
M^2(1 + 2c)^2 + 2M\sigma(1 - 4c - 4\sigma^2) + \sigma^2(1 + 2c)^2
\]

\[
= [4(1+c)L_c - (1 + 2c)M]^2 - 2 [4(1+c)L_c - (1 + 2c)M] + \sigma^2(1 - 2c)^2.
\]

The terms \( \sigma^2(1 - 2c)^2 \) cancel from both sides. Further calculation gives

\[
2M(1 - 4c - 4\sigma^2)\sigma + 2(4(1+c)L_c - (1 + 2c)M)(1 + 2c)\sigma
\]

\[
= -M^2(1 + 2c)^2 + (4(1+c)L_c - (1 + 2c)M)^2,
\]

and finally

\[
\sigma = \frac{L_c((1 + 2c)M - 2(1 + c)L_c)}{2cM - (1 + 2c)L_c} \quad (19)
\]

Thus we obtained the explicit expression for mean \( \sigma = \sigma_{L_c}(M) \) in terms of \( M \) and \( L_c \).

On the other side, from (19) we know that

\[
S_{L_c}(M) = \frac{a(M-a)(L_c-b)^2 - b(L_c-a)^2(M-b)}{(M-a)(L_c-b)^2 - (L_c-a)^2(M-b)},
\]

which may be written as

\[
S_{L_c}(M) = \frac{K_1 M - K_2}{K_0 M - K_1} \quad (20)
\]

where

\[
K_0 = (L_c - b)^2 - (L_c - a)^2,
\]

\[
K_1 = a(L_c - b)^2 - b(L_c - a)^2,
\]

\[
K_2 = a^2(L_c - b)^2 - b^2(L_c - a)^2.
\]

By equating the results of mappings \( \sigma \) and \( S \) with respect to mean \( L_c \) of a mean \( M \) and employing formulas (19) and (20), we obtain

\[
\frac{L_c((1 + 2c)M - 2(1 + c)L_c)}{2cM - (1 + 2c)L_c} = \frac{K_1 M - K_2}{K_0 M - K_1}
\]
which needs to be proved. We calculate

\[ \frac{2(1 + c)L_c - (1 + 2c)M}{K_0M - K_1} = [(1 + 2c)L_c - 2cM] (K_1M - K_2). \]

Grouping by the powers of \( M \) yields

\[ [M_0(1 + 2c)K_0 - 2cK_1] M^2 + 2 [K_2c - (1 + c)L_c^2K_0] M + L_c [2(1 + c)L_c K_1 - (1 + 2c)K_2] = 0. \]  

Now we simplify each coefficient by the powers of \( M \). First,

\[ M_0(1 + 2c)K_0 - 2cK_1 = \]

\[ = M_0(1 + 2c) [(L_c - b)^2 - (L_c - a)^2] - 2c [a(L_c - b)^2 - b(L_c - a)^2] \]

\[ = (a - b) [2(1 + c)L_c^2 - (a + b)(1 + 2c)L_c + 2abc], \]

second,

\[ cK_2 - (1 + c)L_c^2K_0 = \]

\[ = c [a^2(L_c - b)^2 - b^2(L_c - a)^2] - (1 + c)L_c^2 [(L_c - b)^2 - (L_c - a)^2] \]

\[ = -(a - b)L_c [2(1 + c)L_c^2 - (a + b)(1 + 2c)L_c + 2abc], \]

and third

\[ 2(1 + c)L_c K_1 - (1 + 2c)K_2 = \]

\[ = 2(1 + c)L_c [a(L_c - b)^2 - b(L_c - a)^2] - (1 + 2c) [a^2(L_c - b)^2 - b^2(L_c - a)^2] \]

\[ = (a - b)L_c [2(1 + c)L_c^2 - (a + b)(1 + 2c)L_c + 2abc]. \]

Hence, the equation \( 21 \) factorizes as

\[ (a - b) [2(1 + c)L_c^2 - (a + b)(1 + 2c)L_c + 2abc] \] \( (M^2 - 2L_cM + L_c^2) = 0. \]  

Notice that the mean \( L_c \) defined in \( 15 \) is one of the solutions of quadratic equation

\[ 2(1 + c)L_c^2 - (a + b)(1 + 2c)L_c + 2abc, \]

and the condition \( 22 \) is fulfilled which proves the theorem. \( \square \)

We will close this section with a conjecture. Based on the analysis in this paper we may conclude the following.

**Conjecture 1.** Symmetric homogeneous mean which has the asymptotic power series expansion and fulfills the requirements of the Open question from \( 9 \) necessarily has the same coefficients as mean \( L_c, c \in [-1, +\infty). \)

5. **Concluding remarks**

Using techniques of asymptotic expansions we were able to compare two symmetries of different origins on the set of mean functions. Finding asymptotic series expansion for both of them, in terms of recursive algorithm for their coefficients, enabled us to carry out the coefficient comparison which resulted with obtaining class of means which interpolates between harmonic, geometric and arithmetic mean. Furthermore, various extensions of \( L_c, c \in [-1, +\infty), \) could be observed, such that for all \( c \in \mathbb{R} \) function \( L_c \) would be a mean.

Methods presented in this paper may be useful with various problems regarding bivariate means and further. For example, in case of dual means, generalized inverses of means and similar problems where some functional connection is given
and especially when the explicit formula for some of the means involved was not known.

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