Yang–Baxter systems, solutions and applications

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Abstract
Two types of Yang–Baxter systems play roles in the theoretical physics – constant and colour dependent. The constant systems are used mainly for construction of special Hopf algebra while the colour or spectral dependent for construction of quantum integrable models. Examples of both types together with their particular solutions are presented. The complete solution is known only for the constant system related to the quantized braided groups in the dimension two. The strategy for solution of the system related to quantum doubles is suggested and partial results are presented.

1 Introduction
The Yang–Baxter equations proved to be an important tool for various branches of theoretical physics. There are several types of the Yang–Baxter equations.

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The simplest are the constant Yang–Baxter equations. They represent a system of \( N^6 \) cubic equations for elements of \( N^2 \times N^2 \) matrix \( R \) and can be written in the well known form
\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{1}
\]
A more complicated version of the Yang–Baxter equations is the case where the matrix \( R \) depends on one or two parameters and the equations acquire the forms
\[
R_{12}(u)R_{13}(u + v)R_{23}(v) = R_{23}(v)R_{13}(u + v)R_{12}(u) \tag{2}
\]
or
\[
R_{12}(u_1, u_2)R_{13}(u_1, u_3)R_{23}(u_2, u_3) = R_{23}(u_2, u_3)R_{13}(u_1, u_3)R_{12}(u_1, u_2). \tag{3}
\]
They are called spectral and colour dependent Yang–Baxter equations. It is easy to check that (1) and (2) are special cases of (3) for \( R(u, v) = R(u - v) \) and \( R(u, v) = R \) respectively.

Even though many solutions are known for all types of the Yang–Baxter equations in various dimensions [1, 2, 3, 4], the complete solution is known only for the constant Yang–Baxter equations in the dimension two, i.e. matrices \( 4 \times 4 \), until now [5].

Few years ago, extensions of the Yang–Baxter equations for several matrices, called Yang–Baxter systems, appeared in literature. The goal of the present paper is to give a review of such systems and show results on classification of solutions of some constant systems.

2 Review of Yang–Baxter systems

As the Yang–Baxter systems usually contain several Yang–Baxter–type equations it is convenient to introduce the following notation: Constant Yang–Baxter commutator \([R, S, T]\) of (constant) \( N^2 \times N^2 \) matrices \( R, S, T \) is \( N^3 \times N^3 \) matrix
\[
[R, S, T] := R_{12}S_{13}T_{23} - T_{23}S_{13}R_{12} \tag{4}
\]
and spectral or colour dependent Yang–Baxter commutator \([[[R, S, T]]]\) of (spectral or colour dependent) \( N^2 \times N^2 \) matrices \( R, S, T \) is \( N^3 \times N^3 \) matrix
\[
[[[R, S, T]]] = [[[R, S, T]]](u_1, u_2, u_3) :=
\]
\[ R_{12}(u_1, u_2)S_{13}(u_1, u_3)T_{23}(u_2, u_3) - T_{23}(u_2, u_3)S_{13}(u_1, u_3)R_{12}(u_1, u_2) \]  

In this notation, the constant and spectral or colour dependent Yang–Baxter equations read
\[ [R, R, R] = 0, \quad [[R, R, R]] = 0. \]  

2.1 Constant Yang–Baxter systems

2.1.1 System for quantized braided groups

The quantized braided groups were introduced recently in [6] combining Majid’s concept of braided groups [7] and the FRT formulation of quantum supergroups [8]. The generators \( T_{ij}, \ i, j \in \{1, \ldots, d = \text{dim} V\} \) of quantized braided groups satisfy the algebraic and braid relations
\[ Q_{12}R_{12}^{-1}T_1R_{12}T_2 = R_{21}^{-1}T_2R_{21}T_1Q_{12} \]  

\[ \psi(T_1 \otimes R_{12}T_2) = R_{12}T_2 \otimes R_{12}^{-1}T_1R_{12} \]  

where the numerical matrices \( Q, \ R \) satisfy the system of Yang–Baxter–type equations
\[ [Q, Q, Q] = 0, \quad [R, R, R] = 0, \]  
\[ [Q, R, R] = 0, \quad [R, R, Q] = 0. \]  

For classification of the quantized braided groups in the dimension two we have to find all solutions of this Yang–Baxter system that are both invertible and have the so called second inversion \( (R^{11})^{-1} \).

The system (9) – (10) is invariant under
\[ Q' = \lambda(S \otimes S)Q(S \otimes S)^{-1}, \quad R' = \kappa(S \otimes S)R(S \otimes S)^{-1}, \quad \lambda, \kappa \in \mathbb{C}, \ S \in SL(2, \mathbb{C}) \]  

and
\[ Q'' = Q^+ = PQP, \quad R'' = R^+ = PRP. \]  

These symmetries are important for classification of solutions.

The complete set of invertible solutions in the dimension two, i.e. for matrices \( 4 \times 4 \), is given in [9]. Simple solutions are \((R, Q = R)\) or \((R, Q = PR^{-1}P)\), \( R \) being solution of the Yang–Baxter equation (11). On the other hand these solutions are the only ones that solve (11) – (14) for all but four classes of invertible Yang–Baxter solutions \( R \) of the Hietarinta’s list [9].
The exceptional cases of $R$, for which other solutions exist are

$$
R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & -1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & t & 0 \\
0 & t & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
x & 1 & 0 & 0 \\
y & 0 & 1 & 0 \\
z & y & x & 1
\end{pmatrix}, \quad (13)
$$

and diagonal matrix. The corresponding solutions of (9)–(10) were found in [9].

2.1.2 System for quantum doubles

Quantum doubles are special quasitriangular Hopf algebras constructed from the tensor product of Hopf algebras by defining a pairing between them. In the paper [10] a method of obtaining the quantum doubles for pairs of FRT quantum groups is presented. Let two quantum groups are given by relations [11]

$$
W_{12}U_1U_2 = U_2U_1W_{12}
$$
$$
Z_{12}T_1T_2 = T_2T_1Z_{12}
$$

where $W$ and $Z$ are matrices satisfying the Yang–Baxter equations

$$
[W, W, W] = 0, \quad [Z, Z, Z] = 0,
$$

(14)

and suppose that there is a matrix $X$ that satisfies the equations

$$
[W, X, X] = 0, \quad [X, X, Z] = 0.
$$

(15)

Then the relations

$$
X_{12}U_1T_2 = T_2U_1X_{12}
$$

(16)

define quantum double with the pairing

$$
<U_1, T_2> = X_{12}.
$$

(17)

The problem of solution of the system (14), (13) will be attacked in the section [10].
2.1.3 System for generalized reflection algebras

The system (14), (15) is a special case of a more general Yang–Baxter system

\[ [A, A, A] = 0, \quad [D, D, D] = 0, \]
\[ [A, C, C] = 0, \quad [D, B, B] = 0, \]
\[ [A, B^+, B^+] = 0, \quad [D, C^+, C^+] = 0, \]
\[ [A, C, B^+] = 0, \quad [D, B, C^+] = 0, \]

where the superscript \( X^+ \) means \( PXP \), and \( P \) is the permutation matrix \( P_{ij}^{kl} = \delta_i^l \delta_j^k \).

This system has appeared as the consistency conditions for the algebra generated by elements \( L_{jk}^i, j, k \in \{1, 2, \ldots, N\} \) satisfying quadratic relations

\[ A_{12}L_{12}L_{2} = L_{2}C_{12}L_{1}D_{12} \]  \hspace{1cm} (19)

where \( L = \{L_{jk}^i\}_{j,k=1}^M \) and \( A, B, C, D \) are numerical matrices \( N^2 \times N^2 \) (i.e. \( A_{12} = \{(A)_{i_1j_2}^{i_2j_1} \in C\}_{i_1,j_1,j_2=1}^N \) and similarly for \( B, C, D \)). These algebras were considered in [12] and include (algebras of functions on) quantum groups, quantum supergroups, braided groups, quantized braided groups, reflection algebras and others.

2.2 Spectral or colour dependent Yang–Baxter systems

In quantized ultralocal models, i.e. such that the Poisson brackets of the fields is proportional to \( \delta \)–function, commutation relations for elements of the Lax operator can be written as

\[ R_{12}(\lambda - \mu)L_1(\lambda)L_2(\mu) = L_2(\mu)L_1(\lambda)R_{12}(\lambda - \mu). \]  \hspace{1cm} (20)

where the matrix \( R(\lambda - \mu) \) satisfies the spectrally dependent Yang–Baxter equation (2).

The consistency conditions for the algebras used for quantization of nonultralocal models yield spectral or colour dependent Yang–Baxter systems.
2.2.1 Quadratic algebras for nonultralocal models

In the paper [13], Freidel and Maillet suggested the commutation relations for the elements of the quantized Lax operator $L(\lambda)$ in the form

$$A_{12}(u_1, u_2)L_1(u_1)B_{12}(u_1, u_2)L_2(u_2) = L_2(u_2)C_{12}(u_1, u_2)L_1(u_1)D_{12}(u_1, u_2)$$  \hspace{1cm} (21)

where $A, B, C, D$ are spectral or colour dependent $N^2 \times N^2$ matrices that satisfy the equations of the Yang–Baxter system

$$[[A, A, A]] = 0, \quad [[D, D, D]] = 0, \quad [[A, C, C]] = 0, \quad [[D, B, B]] = 0, \quad [[A, B^\dagger, B^\dagger]] = 0, \quad [[D, C^\dagger, C^\dagger]] = 0,$$

$$[[A, C, B^\dagger]] = 0, \quad [[D, B, C^\dagger]] = 0.$$  \hspace{1cm} (22)

The superscript $X^\dagger$ is defined by $X^\dagger(u, v) := PX(v, u)P$. This algebra with $C = B^\dagger$ was introduced also in [14, 15].

Several particular solutions are known, e.g.

$$A(u, v) = (u - v)1 + P$$

$$B^\dagger(u, v) = C(u, v) = v1 + \sigma_- \otimes \sigma_+$$

$$D(u, v) = u - v + (1 - u/v)\sigma_- \otimes \sigma_+(1 - v/u)\sigma_+ \otimes \sigma_-$$

where

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$  \hspace{1cm} (23)

2.2.2 Multiple braided product of quadratic algebras

The crucial object for construction of quantum integrable models is the monodromy matrix [16]. In the quantized models it is a representation of the multiple matrix coproduct $\Delta^N(L)$ for a quadratic algebra generated by $L$. The usual matrix coproduct cannot be used for nonultralocal models unless we introduce the braiding structure to the tensor products of the algebras [14]. The structure can be expressed as the algebra generated by $N \times M^2$ generators

$$(L^I)^k_j(u), \quad I \in \{1, 2, \ldots, N\}, \quad j, k \in \{1, 2, \ldots, M\}$$  \hspace{1cm} (24)
satisfying quadratic relations
\begin{equation}
L^I_1(u_1)X^{JK}_{12}(u_1, u_2) L^K_2(u_2) = W^{JK}_{12}(u_1, u_2) L^K_2(u_2) Y^{JK}_{12}(u_1, u_2) L^I_1(u_1) Z^{JK}_{12}(u_1, u_2)
\end{equation}
(25)
where \(X^{JK}, W^{JK}, Y^{JK}, Z^{JK}\) for fixed \(J, K \in \{1, 2, \ldots, N\}\) are numerical invertible colour dependent \(M^2 \times M^2\) matrices. No summation over indices \(I, J, K, \ldots\) is assumed.

The importance of these algebra consists in the fact that we can find a commuting subalgebra that can be used for construction of quantum Hamiltonian of a model together with conserved quantities.

The quadratic algebras defined by relations of the form (25) must satisfy consistency conditions of the Yang–Baxter–type that follow from the requirement that no supplementary higher degree relations are necessary for unique transpositions of three and more elements [11]. For relations of the form (25) the conditions read (cf. (22))

\begin{align}
\{[Z, Z, Z]\} &= 0, \quad \{[W, W, W]\} = 0 \quad (26) \\
\{[Z, X, X]\} &= 0, \quad \{[X, X, W]\} = 0 \quad (27) \\
\{[Z, Y^{\dagger}, Y^{\dagger}]\} &= 0, \quad \{[Y^{\dagger}, Y^{\dagger}, W]\} = 0 \quad (28) \\
\{[Z, X, Y^{\dagger}]\} &= 0, \quad \{[Y^{\dagger}, X, W]\} = 0 \quad (29)
\end{align}

where by \(\{[R, S, T]\} = 0\) we mean that

\begin{align}
[[R^{J_1J_2} S^{J_3J_4} T^{J_5J_6}]] &= R_{12}^{J_1J_2}(u_1, u_2) S_{13}^{J_3J_4}(u_1, u_3) T_{23}^{J_5J_6}(u_2, u_3) \\
&\quad - T_{23}^{J_5J_6}(u_2, u_3) S_{13}^{J_3J_4}(u_1, u_3) R_{12}^{J_1J_2}(u_1, u_2) = 0
\end{align}
(30)

for all \(J_1, J_2, J_3 \in \{1, \ldots, N\}\) and

\[(Y^{\dagger})^{JK}(u, v) := (Y^{KJ}(v, u))^+ = PY^{KJ}(v, u)P.\]

Several particular solutions of (26) – (29) are given in [12].

3 Solving the Yang–Baxter system for the quantum double

Our longtime goal is solution of the spectral dependent Yang–Baxter systems for nonultralocal models presented in the previous section. Nevertheless to
get a deeper experience with such complicated systems we decided to solve their constant versions first. One can easily check that the constant version of the system (22) where $B = C^\dagger$ can be written as the system for the quantum doubles

$$[W, W, W] = 0, \quad (31)$$

$$[W, X, X] = 0, \quad (32)$$

$$[X, X, Z] = 0, \quad (33)$$

$$[Z, Z, Z] = 0, \quad (34)$$

We shall look for solutions of this system in the lowest nontrivial dimension two (i.e. matrices $4 \times 4$).

Solution of the system is essentially facilitated by knowledge of symmetries of the system (31)–(34). The space of solutions is invariant under both continuous transformations

$$W' = \omega (T \otimes T) W (T \otimes T)^{-1} \quad (35)$$

$$X' = \xi (T \otimes S) X (T \otimes S)^{-1} \quad (36)$$

$$Z' = \zeta (S \otimes S) Z (S \otimes S)^{-1} \quad (37)$$

where

$$\omega, \xi, \zeta \in \mathbb{C}, \quad T, S \in SL(2, \mathbb{C})$$

and the discrete transformations

$$(W', X', Z') = (W^t, X^t, Z^t) \quad (38)$$

$$(W', X^t, Z') = (W^a, X, Z^b), \quad a = id, \#; \quad b = id, \# \quad (39)$$

$$(W', X^t, Z') = (W^c, X^-, Z^d), \quad c = +, -, \quad d = +, - \quad (40)$$

$$(W', X^t, Z') = (Z^c, X^+, W^d), \quad c = +, -, \quad d = +, - \quad (41)$$

where $Y^t$ is transpose of $Y$, $Y^+ := PYP$, $Y^- := Y^{-1}$, $Y^id := Y$, $Y^\# := (Y^+)^{-1} = (Y^-)^+$. Beside that one can guess several simple solutions of the system, (31)–(33), namely

$$W = X = Z = R, \quad \text{where} \quad [R, R, R] = 0, \quad (42)$$

$$X = 1, \quad W, Z \text{ arbitrary solutions of } [W, W, W] = 0, \quad [Z, Z, Z] = 0, \quad (43)$$
and
\[ W = Z = P, \quad X \text{ arbitrary matrix}, \]
where \( P \) is the permutation matrix and \( X \) is arbitrary.

The last two solutions give positive answers to the following questions. Is there a matrix \( X \) such that for any pair of matrices \( W, Z \) that solve the Yang–Baxter equations the triple \((W, X, Z)\) solves the system (31)–(34)? Is there for any matrix \( X \) a pair of matrices \( W, Z \) such that the triple \((W, X, Z)\) solves the system (31)–(34)?

Besides the above mentioned, one can find solutions of the system (31)–(34) from the knowledge of solutions of the system (9–10). Namely, if \((Q, R)\) is a solution of the system of Yang–Baxter type equations (9–10) then \((W = Q, X = R, Z = R + QR -)\) is a solution of the system (31)–(34).

The question is if there are other solutions than those given above. The answer is also positive and the author believes that the complete solution of the system (31)–(34) in the dimension two can be found.

### 3.1 Strategies of solution

Even for the lowest nontrivial dimension two, the system (31)–(34) represents a tremendous task - solving 256 cubic equations for 48 unknowns. That’s why it is understandable that the assistance of computer programs for symbolic calculations is essential in the following\(^1\). On the other hand it does not mean that one can find the solutions by pure brute force, namely applying a procedure SOLVE to the system of the 256 equations.

There are several strategies for solution of the above given problem. All of them are based on the knowledge of the symmetries of the system and the knowledge of the complete set of solutions of the Yang–Baxter equation in the dimension two.

One possible (and obvious) strategy is solving the equations (32), (33) for all pairs of matrices \( W, Z \) in the Hietarinta’s list of solutions of the Yang–Baxter equation [5]. By this way we reduce the problem to 128 quadratic equations for 16-22 unknowns (depending on the number of parameters in the solutions of the Yang–Baxter equation).

Another strategy is to use the symmetry (36) to simplify the matrix \( X \) as much as possible, then solve the linear equations (32), (33) for \( W \) and \( Z \)

\(^1\)Reduce 3.6 was used
and finally solve the Yang–Baxter equations (31),(34). This is an analogue of the strategy accepted in [9] for solving the system (9)–(10).

Both the mentioned strategies yield sets of equations and unknowns that are still too large to be solved by the computer program. The strategy that seems to be working is a combination of the two previous. We start with a solution of the Yang–Baxter equation $W$ and solve $X$ from the equation (32) using the symmetries (35)–(41) that leave $W$ form–invariant (i.e. changing only values of its parameters). Then we solve the equation (33) (linear in elements of $Z$) and determine $Z$ from (34) using the results of the previous step. The results of this strategy for the so called (non)standard solution of the Yang–Baxter equation are given in the next subsection.

3.2 Results for the standard and nonstandard solution of the Yang–Baxter equation

Let

$$W = W_{q,s} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & s^{-1} & 0 & 0 \\ 0 & q - q^{-1} & s & 0 \\ 0 & 0 & 0 & t \end{pmatrix}, \quad t = q \text{ or } t = -q^{-1}, \quad q, s \in \mathbb{C}\{0\}, \quad q^2 \neq 1.$$  \hspace{1cm} (45)

This matrix is called standard, for $t = q$, or nonstandard, for $t = -q^{-1}$, solution of the Yang–Baxter equation. Up to the transformations (35)–(41) we get the following list of solutions of the equation (32) for any (invertible) $q, s \in \mathbb{C}$.

$$X_1 = \begin{pmatrix} a & 0 & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & d & b \end{pmatrix}, \quad X_2 = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & s^{-1} & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & 0 & \frac{b}{zt} \end{pmatrix},$$  \hspace{1cm} (46)

$$X_3 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}.$$  \hspace{1cm} (47)
Beside that we get special solutions

\[ X_4 = \begin{pmatrix} q & 0 & 0 & c \\ 0 & s^{-1} & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & 0 & \frac{bt}{s} \end{pmatrix}, \quad \text{for } s^2 = 1. \] (48)

\[ X_5 = \begin{pmatrix} a & 0 & 0 & b \\ 0 & -a & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix}, \quad \text{for } q = -s = i. \] (49)

\[ X_6 = \begin{pmatrix} 0 & 0 & ia & 0 \\ 2iab/c & 0 & 0 & a \\ ib & 0 & 0 & c \\ 0 & b & 0 & 0 \end{pmatrix}, \quad \text{for } q = i, s = 1. \] (50)

respectively.

Each of these solutions of the equation (32) can be extended to the solution of the system (31)–(34) by

\[ Z = P \]

where

\[ P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \] (51)

For \( X_1 \sim X_5 \) there are still other solutions.

Solutions of the equations (33)–(34) where \( X = X_1 \) are

\[ Z_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & y & x & 1 \end{pmatrix}, \quad Z_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ -x & 0 & 1 & 0 \\ -xy & -y & -y & y & 1 \end{pmatrix}, \] (52)

so that the triples \((W_{q,s}, X_1, Z_{11})\) and \((W_{q,s}, X_1, Z_{12})\) are solutions to the system (31)–(34).

Similarly, for \( X_2 \) we get the solutions of the equations (33)–(34)

\[ Z_{20} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & b^{-1} & 0 & 0 \\ 0 & q - g^{-1} & b & 0 \\ 0 & 0 & 0 & t \end{pmatrix}. \] (53)
and if \( q^2 = -1 \) then moreover
\[
Z_{21} = \begin{pmatrix} q & 0 & 0 & \delta \\ 0 & r & 0 & 0 \\ 0 & q - rbq^{-1} & b & 0 \\ 0 & 0 & 0 & -rbq^{-1} \end{pmatrix},
\] (54)
where \( \delta = 0 \) if \( b^2 \neq -1 \).

For the diagonal \( X_3 \) we find that any "six-vertex" matrix \( Z \) solves (33) so that the solutions of (33)–(34) are
\[
Z_{30} = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & y & 0 \end{pmatrix},
\](55)
\[
Z_{31} = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & r^{-1} & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & p \end{pmatrix},
\]
\[
Z_{32} = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & r^{-1} & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & -p^{-1} \end{pmatrix}.
\](56)

If \( a = b, c = -d \) in \( X_3 \) then any "eight-vertex" matrix \( Z \) solves (33). To obtain all invertible solutions of the system (33)–(34) in this case one must perform the classification of the "eight-vertex" solutions of the Yang–Baxter equation(34) up to the symmetries (37) where \( S \) is the (anti)diagonal matrix because only transformations (36) with (anti)diagonal \( S \) leave \( X_3 \) form–invariant. This classification was done in \[9\] and beside the "eight-vertex" solutions given in \[2\] contains the matrices
\[
Z = \begin{pmatrix} x & 0 & 0 & y \\ 0 & \pm x & y & 0 \\ 0 & y & \pm x & 0 \\ y & 0 & 0 & x \end{pmatrix}, \quad x, y \neq 0
\] (57)

If \( a = b, c = d \) then any triple \((W_{q,s}, X_3, Z)\) where \( Z \) solves the Yang–Baxter equation, is solution to the system (31)–(34). The list of these solutions is then equivalent to the list of solutions in \[4\].

The only matrix \( Z \) that solves the equations (33) for \( X = X_4 \) where \( q^2 \neq -1 \) or \( b^2 \neq 1 \) is \( Z = P \). If \( q^2 = -b^2 = -1 \) then there is another solution,
namely

\[
Z_{41} = \begin{pmatrix}
p & 0 & 0 & \frac{ac}{2}(p + p^{-1}) \\
0 & bp^{-1} & 0 & 0 \\
p - p^{-1} & bp & 0 \\
0 & 0 & 0 & -p^{-1}
\end{pmatrix}.
\]  (58)

Finally for \( q = -s = i \) we get solutions \((W_{i,-i}, X_5, Z)\) where

\[
Z = Z_{51} = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & \epsilon & 1 & 0 \\
0 & 1 & -\epsilon & 0 \\
-1 & 0 & 0 & 1
\end{pmatrix},
\]  (59)

or

\[
Z = Z_{52} = \begin{pmatrix}
k - k^{-1} + 2 & 0 & 0 & k - k^{-1} \\
0 & k + k^{-1} & k - k^{-1} & 0 \\
0 & k - k^{-1} & k + k^{-1} & 0 \\
k - k^{-1} & 0 & 0 & k - k^{-1} - 2
\end{pmatrix},
\]  (60)

or

\[
Z = Z_{53} = \begin{pmatrix}
k & 0 & 0 & 0 \\
0 & \epsilon k & 0 & 0 \\
0 & k - 1 & 1 & 0 \\
\epsilon (k - 1) & 0 & 0 & -1
\end{pmatrix},
\]  (61)

or

\[
Z = Z_{54} = \begin{pmatrix}
k & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & k - k^{-1} & 1 & 0 \\
0 & 0 & 0 & -k^{-1}
\end{pmatrix},
\]  (62)

where \( \epsilon^2 = 1, \ k = c/a. \)

### 4 Conclusions

Recently two types of Yang–Baxter systems have appeared in the theoretical physics – constant and colour dependent. The constant systems are used mainly for construction of special Hopf algebras while the colour or spectral dependent for construction of quantum integrable models. We have presented
examples of both types together with their particular solutions. The complete solution is known only for the constant system (10)–(11) in the dimension two.

The strategy for solution of the quantum double system (14), (15) was suggested and partial results were presented. Namely, if $W$ is the standard or nonstandard solution (45) of the Yang–Baxter equation then the complete set of invertible solutions of this system up to symmetry transformations (35)–(41) is formed by the triples

$$(W_{q,s}, X, P) \text{ where } X = X_1, X_2, X_3$$

$$(W_{q,s}, X_1, Z_{11}), (W_{q,s}, X_1, Z_{12})$$

$$(W_{q,s}, X_2, Z_{20}), (W_{i,s}, X_2(q = i), Z_{21})$$

$$(W_{q,s}, X_3, Z_{30}), (W_{q,s}, X_3, Z_{31}), (W_{q,s}, X_3, Z_{32}), (W_{q,s}, \text{diag}(a, -a, b, b), Z_{8V}),$$

where $Z_{8V}$ is invertible "eight-vertex" solution of the Yang–Baxter equation (for the classification see [2, 4]),

$$(W_{q,s}, \text{diag}(a, a, b, b), Z),$$

where $Z$ is arbitrary invertible solution of the Yang–Baxter equation (for the classification see [3]),

$$(W_{q,\pm 1}, X_4, P), (W_{i,\pm 1}, X_4(q = i), Z_{41}),$$

$$(W_{i,-i}, X_5, P), (W_{i,-i}, X_5, Z_{51}), (W_{i,-i}, X_5, Z_{52}), (W_{i,-i}, X_5, Z_{53}), (W_{i,-i}, X_5, Z_{54}),$$

and $(W_{i,1}, X_6, P)$. The forms of the matrices $P, X_i$ and $Z_{ij}$ are given in the previous section.

From this result it is clear that quite distinct types of FRT quantum groups can be combined into the quantum doubles.

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