Proper resolutions and Gorensteinness in extriangulated categories

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Abstract Let $(\mathcal{C}, \mathcal{E}, \mathcal{S})$ be an extriangulated category with a proper class $\xi$ of $\mathcal{E}$-triangles, and $\mathcal{W}$ an additive full subcategory of $(\mathcal{C}, \mathcal{E}, \mathcal{S})$. We provide a method for constructing a proper $\mathcal{W}(\xi)$-resolution (resp., coproper $\mathcal{W}(\xi)$-coresolution) of one term in an $\mathcal{E}$-triangle in $\xi$ from that of the other two terms. By using this way, we establish the stability of the Gorenstein category $\mathcal{G}(\mathcal{W}(\xi))$ in extriangulated categories. These results generalize the work of Z. Y. Huang [J. Algebra, 2013, 393: 142–169] and X. Y. Yang and Z. C. Wang [Rocky Mountain J. Math., 2017, 47: 1013–1053], but the proof is not too far from their case. Finally, we give some applications about our main results.

Keywords Proper resolution, coproper coresolution, extriangulated categories, Gorenstein categories

MSC\textsuperscript{2020} 18G80, 18E10, 18G25, 18G10

1 Introduction

Let $\mathcal{A}$ be an abelian category and $\mathcal{W}$ an additive full subcategory of $\mathcal{A}$. Huang [3] provided a method for constructing a proper $\mathcal{W}$-resolution (resp., coproper $\mathcal{W}$-coresolution) of one term in a short exact sequence in $\mathcal{A}$ from those of the other two terms. By using these, he affirmatively answered an open question on the stability of the Gorenstein category $\mathcal{G}(\mathcal{W})$ posed by Sather-Wagstaff et al. [7] and also proved that $\mathcal{G}(\mathcal{W})$ is closed under direct summands. Later, Yang-Wang [11] extended Huang’s results to triangulated categories in parallel. Some further investigations of proper resolutions (resp., coproper coresolutions) and Gorenstein categories for abelian categories or triangulated categories can be seen in [3,4,6,8–10].

The notion of extriangulated categories was introduced by Nakaoka and...
Palu [5] as a simultaneous generalization of exact categories and triangulated categories. Exact categories and extension closed subcategories of an extriangulated category are extriangulated categories, while there exist some other examples of extriangulated categories which are neither exact nor triangulated (see [2,5,12]). Hence, many results hold on exact categories and triangulated categories can be unified in the same framework. Based on this idea, we will unify the results of Huang and Yang-Wang in the framework of extriangulated categories.

Let \((\mathcal{C}, \mathcal{E}, s)\) be an extriangulated category with a proper class \(\xi\) of \(\mathcal{E}\)-triangles. The authors [2] studied a relative homological algebra in \(\mathcal{C}\) which parallels the relative homological algebra in a triangulated category. By specifying a class of \(\mathcal{E}\)-triangles, which is called a proper class \(\xi\) of \(\mathcal{E}\)-triangles, the authors introduced \(\xi\)-projective dimensions and \(\xi\)-injective dimensions, and discussed their properties. Inspired by Huang and Yang-Wang’s work, in this paper, we introduce and study the Gorenstein category in extriangulated categories and demonstrate that this category shares some basic properties with the Gorenstein category in abelian categories or in triangulated categories.

This paper is organized as follows. Section 2 gives some preliminaries and basic facts about extriangulated categories which will be used throughout the paper. Section 3 provides a method for constructing a proper resolution (resp., coproper coresolution) of one term in an \(\mathcal{E}\)-triangle in \(\xi\) from those of the other two terms, which generalizes Huang’s results on an abelian category and Yang-Wang’s results on a triangulated category and is new for an exact category case (see Theorems 1–4 and Remark 2 below). Section 4 is devoted to studying the Gorenstein category in extriangulated categories. More precisely, we prove that this Gorenstein category is closed under direct summands and the stability of the Gorenstein category is also established in extriangulated categories, which refines a result of Yang and Wang (see Theorem 5 and Remark 4 below).

2 Preliminaries

Throughout this paper, we always assume that \(\mathcal{C} = (\mathcal{C}, \mathcal{E}, s)\) is an extriangulated category with enough \(\xi\)-projectives and enough \(\xi\)-injectives, and it satisfies Condition (WIC) (for details, see Condition 1 below). We also assume that \(\xi\) is a proper class of \(\mathcal{E}\)-triangles in \((\mathcal{C}, \mathcal{E}, s)\).

Let us briefly recall some definitions and basic properties of extriangulated categories from [5], and we omit some details here.

Let \(\mathcal{C}\) be an additive category equipped with an additive bifunctor

\[ \mathcal{E}: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Ab}, \]

where \text{Ab} is the category of abelian groups. For any objects \(A, C \in \mathcal{C}\), an element \(\delta \in \mathcal{E}(C, A)\) is called an \(\mathcal{E}\)-extension. Let \(s\) be a correspondence which associates an equivalence class
\( s(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C] \)

to any \( E \)-extension \( \delta \in E(C, A) \). This \( s \) is called a realization of \( E \), if it makes the diagrams in [5, Definition 2.9] commutative. A triplet \((\mathcal{C}, E, s)\) is called an extriangulated category if it satisfies the following conditions:

1. \( E: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Ab} \) is an additive bifunctor;
2. \( s \) is an additive realization of \( E \);
3. \( E \) and \( s \) satisfy the compatibility conditions in [5, Definition 2.12].

**Remark 1** Note that both exact categories and triangulated categories are extriangulated categories (see [5, Example 2.13]) and extension closed subcategories of extriangulated categories are again extriangulated (see [5, Remark 2.18]). Moreover, there exist extriangulated categories which are neither exact categories nor triangulated categories (see [5, Proposition 3.30], [12, Remark 4.13], and [2, Remark 3.3]).

**Lemma 1** [5, Proposition 3.15] Assume that \((\mathcal{C}, E, s)\) is an extriangulated category. Let \( C \) be any object, and let

\[
\begin{array}{ccc}
A_1 & \xrightarrow{x_1} & B_1 & \xrightarrow{y_1} & C \\
& \delta_1 & & \downarrow \\
A_2 & \xrightarrow{x_2} & B_2 & \xrightarrow{y_2} & C \\
\end{array}
\]

be any pair of \( E \)-triangles. Then there is a commutative diagram in \( \mathcal{C} \)

\[
\begin{array}{ccc}
A_2 & \xrightarrow{m_2} & A_2 \\
\downarrow & & \downarrow \\
A_1 & \xrightarrow{m_1} & M & \xrightarrow{e_1} & B_2 \\
& \downarrow & \downarrow \\
A_1 & \xrightarrow{x_1} & B_1 & \xrightarrow{y_1} & C \\
\end{array}
\]

which satisfies

\[
s(y_2^*\delta_1) = [A_1 \xrightarrow{m_1} M \xrightarrow{e_1} B_2], \quad s(y_1^*\delta_2) = [A_2 \xrightarrow{m_2} M \xrightarrow{e_2} B_1].\]

A class of \( E \)-triangles \( \xi \) is called saturated if, in the situation of Lemma 1, whenever the \( E \)-triangles

\[
\begin{array}{ccc}
A_1 & \xrightarrow{m_1} & M & \xrightarrow{e_1} & B_2 \\
\downarrow & & \downarrow \\
A_1 & \xrightarrow{x_1} & B_1 & \xrightarrow{y_1} & C \\
\end{array}
\]

belong to \( \xi \), the \( E \)-triangle

\[
\begin{array}{ccc}
A_2 & \xrightarrow{x_2} & B_2 & \xrightarrow{y_2} & C \\
\end{array}
\]

belongs to \( \xi \).
An \( E \)-triangle \( A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \) is called split if \( \delta = 0 \). By [5, Corollary 3.5], we know that it is split if and only if \( x \) is section, if and only if \( y \) is retraction. The full subcategory consisting of the split \( E \)-triangles will be denoted by \( \Delta_0 \).

A class of \( E \)-triangles \( \xi \) is closed under base change if for any \( E \)-triangle \( A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \) belongs to \( \xi \) and any morphism \( c: C' \rightarrow C \), any \( E \) -triangle \( A \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{c \circ \delta} \) belongs to \( \xi \).

Dually, a class of \( E \)-triangles \( \xi \) is closed under cobase change if for any \( E \)-triangle \( A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \) belongs to \( \xi \) and any morphism \( a: A \rightarrow A' \), any \( E \) -triangle \( A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{a \circ \delta} \) belongs to \( \xi \).

**Definition 1** [2, Definition 3.1] Let \( \xi \) be a class of \( E \)-triangles which is closed under isomorphisms. \( \xi \) is called a proper class of \( E \)-triangles if the following conditions hold:

1. \( \xi \) is closed under finite coproducts and \( \Delta_0 \subseteq \xi \);
2. \( \xi \) is closed under base change and cobase change;
3. \( \xi \) is saturated.

**Definition 2** [2, Definition 4.1] An object \( P \in C \) is called \( \xi \)-projective if for any \( E \)-triangle \( A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \) belongs to \( \xi \), the induced sequence of abelian groups

\[
0 \rightarrow \mathcal{C}(P, A) \rightarrow \mathcal{C}(P, B) \rightarrow \mathcal{C}(P, C) \rightarrow 0
\]

is exact. Dually, we have the definition of \( \xi \)-injective.

We denote by \( \mathcal{P}(\xi) \) (resp., \( \mathcal{I}(\xi) \)) the class of \( \xi \)-projective (resp., \( \xi \)-injective) objects of \( \mathcal{C} \). It follows from the definition that these subcategories \( \mathcal{P}(\xi) \) and \( \mathcal{I}(\xi) \) are full, additive, closed under isomorphisms and direct summands.

An extriangulated category \((\mathcal{C}, \mathcal{E}, s)\) is said to have enough \( \xi \)-projectives (resp., enough \( \xi \)-injectives) provided that for each object \( A \), there exists an \( E \)-triangle \( K \rightarrow P \rightarrow A \rightarrow \) (resp., \( A \rightarrow I \rightarrow K \rightarrow \) ) in \( \xi \) with \( P \in \mathcal{P}(\xi) \) (resp., \( I \in \mathcal{I}(\xi) \)).

Let \( A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \) be an \( E \)-triangle (in \( \xi \)). The morphism \( x: A \rightarrow B \) is called \( (\xi) \)-inflation, and \( y: B \rightarrow C \) is called \( (\xi) \)-deflation; \( x \) is called the hokernel of \( y \) and \( y \) is called the hocokernel of \( x \). Let \( \mathcal{W} \) be a class of objects in \( \mathcal{C} \). We say that \( \mathcal{W} \) is closed under hokernels of \( \xi \)-deflation if, whenever the \( E \)-triangle \( A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \) in \( \xi \) with \( B, C \in \mathcal{W}, A \in \mathcal{W} \). Dually, we
say that $\mathcal{W}$ is closed under hocokernels of $\xi$-inflation if, whenever the $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{-\delta} \in \xi$ with $A, B \in \mathcal{W}$, $C \in \mathcal{W}$.

In addition, we assume the following condition for the rest of the paper (see [5, Condition 5.8]).

**Condition 1** (Condition (WIC)) Consider the following conditions.

(1) Let $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$ be any composable pair of morphisms. If $gf$ is an inflation, then so is $f$.

(2) Let $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$ be any composable pair of morphisms. If $gf$ is a deflation, then so is $g$.

**Fact 1** (1) The class of $\xi$-inflations (resp., $\xi$-deflations) is closed under compositions (see [2, Corollary 3.5]).

(2) Let $x: A \rightarrow B$ and $y: B \rightarrow C$ be a composable pair of morphisms. Then $x$ is a $\xi$-inflation (resp., $y$ is a $\xi$-deflation) whenever $yx$ is a $\xi$-inflation (resp., $\xi$-deflation) (see [2, Proposition 4.13]).

Similar to the proof of [2, Lemma 4.15], we have the following result.

**Proposition 1** Let $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{-\delta} \in \xi$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{-\delta'} \in \xi$ be $\mathbb{E}$-triangles in $\xi$. If $(a, c): \delta \rightarrow \delta'$ is a morphism of $\mathbb{E}$-triangles, where $a$ and $c$ are $\xi$-inflations (resp., $\xi$-deflations), then there is a $\xi$-inflation (resp., $\xi$-deflation) $b: B \rightarrow B'$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
A & \xrightarrow{x} & B \\
\downarrow a & & \downarrow b \\
A' & \xrightarrow{x'} & B'
\end{array}
\quad \begin{array}{ccc}
& & \\
& & \\
& & \\
& & \\
C & \xrightarrow{-\delta} & \\
\downarrow c & & \downarrow c \\
C' & \xrightarrow{-\delta'} & 
\end{array}
\]

**Definition 3** [2, Definition 4.4] A complex

\[
\cdots \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0 \xrightarrow{d_{-1}} X_{-1} \xrightarrow{d_{-2}} \cdots
\]

in $\mathcal{C}$ is called $\xi$-exact complex if for each integer $n$, there exists an $\mathbb{E}$-triangle

\[
K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \xrightarrow{-\delta_n} \in \xi
\]

in $\xi$ and $d_n = g_{n-1}f_n$.

**Definition 4** [2, Definition 4.5] Let $\mathcal{W}$ be a class of objects in $\mathcal{C}$. An $\mathbb{E}$-triangle

\[
A \xrightarrow{} B \xrightarrow{} C \xrightarrow{}
\]

in $\xi$ is called to be $\mathcal{C}(\cdot, \mathcal{W})$-exact (resp., $\mathcal{C}(\mathcal{W}, \cdot)$-exact) if for any $W \in \mathcal{W}$, the induced sequence of abelian group

\[
0 \rightarrow \mathcal{C}(C, W) \rightarrow \mathcal{C}(B, W) \rightarrow \mathcal{C}(A, W) \rightarrow 0
\]

(resp., $0 \rightarrow \mathcal{C}(W, A) \rightarrow \mathcal{C}(W, B) \rightarrow \mathcal{C}(W, C) \rightarrow 0$)
is exact in $\text{Ab}$.  

**Definition 5** [2, Definition 4.6] Let $\mathcal{W}$ be a class of objects in $\mathcal{C}$. A complex $X$ is called $\mathcal{C}(-, \mathcal{W})$-exact (resp., $\mathcal{C}(\mathcal{W}, -)$-exact) if it is a $\xi$-exact complex

$$
\cdots \longrightarrow X_1 \overset{d_1}{\longrightarrow} X_0 \overset{d_0}{\longrightarrow} X_{-1} \longrightarrow \cdots
$$

in $\mathcal{C}$ such that there exists a $\mathcal{C}(-, \mathcal{W})$-exact (resp., $\mathcal{C}(\mathcal{W}, -)$-exact) $\mathcal{E}$-triangle

$$
K_{n+1} \overset{g_n}{\longrightarrow} X_n \overset{f_n}{\longrightarrow} K_n \overset{\delta_n}{\longrightarrow}
$$
in $\xi$ and $d_n = g_{n-1} f_n$ for each integer $n$.

**Definition 6** Let $M$ be an object in $\mathcal{C}$. A $\mathcal{W}(\xi)$-resolution of $M$ is a $\xi$-exact complex

$$
\cdots \rightarrow W_i \rightarrow \cdots \rightarrow W_1 \rightarrow W_0 \rightarrow M \rightarrow 0
$$
in $\mathcal{C}$ with all $W_i \in \mathcal{W}$. A $\mathcal{W}(\xi)$-resolution of $M$ is called proper $\mathcal{W}(\xi)$-resolution if it is $\mathcal{C}(\mathcal{W}, -)$-exact. Dually, one can define the notion of a (coproper) $\mathcal{W}(\xi)$-coresolution.

### 3 Proper resolutions and coproper coresolutions

In this section, we provide a method for constructing a proper $\mathcal{W}(\xi)$-resolution (resp., coproper $\mathcal{W}(\xi)$-coresolution) of one term in an $\mathcal{E}$-triangle in $\xi$ from those of the other two terms. At first, we need the following easy observations.

**Lemma 2** Let $\mathcal{W}$ be a class of objects in $\mathcal{C}$.

1. Consider the following commutative diagram of $\mathcal{E}$-triangles in $\xi$:

$$
\begin{array}{ccc}
A & \overset{x}{\longrightarrow} & B \\
\downarrow & & \downarrow \\
A' & \overset{x'}{\longrightarrow} & B'
\end{array}
\quad
\begin{array}{ccc}
B & \overset{y}{\longrightarrow} & C \\
\downarrow b & & \downarrow c \\
B' & \overset{y'}{\longrightarrow} & C'
\end{array}
\quad
\begin{array}{ccc}
C & \overset{c^* \delta'}{\longrightarrow} & \cdots \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & C
\end{array}
$$

If the second row is $\mathcal{C}(\mathcal{W}, -)$-exact, then so is the first row.

2. Consider the following commutative diagram of $\mathcal{E}$-triangles in $\xi$:

$$
\begin{array}{ccc}
A & \overset{x}{\longrightarrow} & B \\
\downarrow a & & \downarrow b \\
A' & \overset{x'}{\longrightarrow} & B'
\end{array}
\quad
\begin{array}{ccc}
B & \overset{y}{\longrightarrow} & C \\
\downarrow & & \downarrow \\
B' & \overset{y'}{\longrightarrow} & C'
\end{array}
\quad
\begin{array}{ccc}
C & \overset{a_* \delta}{\longrightarrow} & \cdots \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & C
\end{array}
$$

If the first row is $\mathcal{C}(-, \mathcal{W})$-exact, then so is the second row.
Proof. We only prove (1), the proof of (2) is similar. Let $W$ be any object in $\mathcal{C}$. By [5, Corollary 3.12], we have the following exact sequence:

$$\mathcal{C}(W, A) \xrightarrow{\mathcal{C}(W,x)} \mathcal{C}(W, B) \xrightarrow{\mathcal{C}(W,y)} \mathcal{C}(W, C) \xrightarrow{(c^*\delta')_2} \mathbb{E}(W, A),$$

where

$$(c^*\delta')_2(f) = f^*c^*\delta', \quad \forall f \in \mathcal{C}(W, C).$$

Note that the sequence

$$0 \longrightarrow \mathcal{C}(W, A) \xrightarrow{\mathcal{C}(W,x')} \mathcal{C}(W, B') \xrightarrow{\mathcal{C}(W,y')} \mathcal{C}(W, C') \longrightarrow 0$$

is exact in Ab by hypothesis. Therefore, for any morphism $f \in \mathcal{C}(W, C)$, there exists a morphism $g \in \mathcal{C}(W, B')$ such that $cf = y'g$. Thus,

$$(c^*\delta')_2(f) = f^*c^*\delta' = g^*y'^*\delta' = 0,$$

and hence, $\mathcal{C}(W, y)$ is epic. Since $\mathcal{C}(W, x') = \mathcal{C}(W, b)\mathcal{C}(W, x)$ is monic, so is $\mathcal{C}(W, x)$. This implies that

$$0 \longrightarrow \mathcal{C}(W, A) \xrightarrow{\mathcal{C}(W,x)} \mathcal{C}(W, B) \xrightarrow{\mathcal{C}(W,y)} \mathcal{C}(W, C) \longrightarrow 0$$

is exact in Ab, as desired. $\square$

Similar to the proof of [3, Lemma 2.5] and its dual, we have the following result.

**Lemma 3** Consider the morphism of $\mathbb{E}$-triangles

$$\begin{array}{ccc}
A & \xrightarrow{x} & B \\
\downarrow a & & \downarrow b \\
A' & \xrightarrow{x'} & B'
\end{array}
\quad
\begin{array}{ccc}
& & C \\
& \downarrow c & \\
& & C'
\end{array}
\quad
\begin{array}{ccc}
\delta & \longrightarrow & \\
\delta' & \longrightarrow & \\
\end{array}$$

Then the following statements hold.

1. If all morphisms of $\mathcal{C}(a, W)$, $\mathcal{C}(c, W)$, and $\mathcal{C}(x', W)$ are epic for an object $W$ in $\mathcal{C}$, then so is $\mathcal{C}(b, W)$.
2. If all morphisms of $\mathcal{C}(a, W)$, $\mathcal{C}(c, W)$, and $\mathcal{C}(y, W)$ are monic for an object $W$ in $\mathcal{C}$, then so is $\mathcal{C}(b, W)$.
3. If all morphisms of $\mathcal{C}(W, a)$, $\mathcal{C}(W, c)$, and $\mathcal{C}(W, y)$ are epic for an object $W$ in $\mathcal{C}$, then so is $\mathcal{C}(W, b)$.
4. If all morphisms of $\mathcal{C}(W, a)$, $\mathcal{C}(W, c)$, and $\mathcal{C}(W, x')$ are monic for an object $W$ in $\mathcal{C}$, then so is $\mathcal{C}(W, b)$.

**Proposition 2** (1) If

$$X \xrightarrow{f} Y \xrightarrow{f'} Z \xrightarrow{\delta} K \xrightarrow{g} L \xrightarrow{g'} Y \xrightarrow{\delta'} L,$$

then

$$\text{(3.1)}$$
are both \( C(\mathcal{W},-)-\)exact and \( C(-,\mathcal{W})-\)exact \( \mathcal{E} \)-triangles in \( \xi \), then we have the following commutative diagram:

\[
\begin{array}{ccc}
K & \xrightarrow{d} & K \\
\downarrow{h} & & \downarrow{g} \\
N & \xrightarrow{f} & L & \xrightarrow{h'} & Z & \xrightarrow{\delta''} & \to \\
\downarrow{e} & & \downarrow{g'} & & \downarrow{\delta} & & \;
\end{array}
\]

where all rows and columns are both \( C(\mathcal{W},-)-\)exact and \( C(-,\mathcal{W})-\)exact \( \mathcal{E} \)-triangles in \( \xi \).

(2) If \( X \xrightarrow{f} Y \xrightarrow{f'} Z \xrightarrow{\delta} \) and \( Y \xrightarrow{g} L \xrightarrow{g'} K \xrightarrow{\delta'} \) are both \( C(\mathcal{W},-)-\)exact and \( C(-,\mathcal{W})-\)exact \( \mathcal{E} \)-triangles in \( \xi \), then we have the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y & \xrightarrow{f'} & Z & \xrightarrow{\delta} & \to \\
\downarrow{h} & & \downarrow{g} & & \downarrow{d} & & \;
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{h} & L & \xrightarrow{h'} & N & \xrightarrow{\delta''} & \to \\
\downarrow{g'} & & \downarrow{e} & & \;
\end{array}
\]

\[
\begin{array}{ccc}
K & \xrightarrow{\delta'} & K \\
\downarrow{\delta} & & \downarrow{f'\delta'} \\
& & \\
\end{array}
\]

Proof (1) It follows from [2, Theorem 3.2] that we have the desired commutative diagram where all rows and columns are \( \mathcal{E} \)-triangles in \( \xi \). Since the \( \mathcal{E} \)-triangle \( K \xrightarrow{g} L \xrightarrow{g'} Y \xrightarrow{\delta'} \) is a \( C(\mathcal{W},-)-\)exact in \( \xi \), the first column in this diagram is \( C(\mathcal{W},-)-\)exact by Lemma 2 (1). It is easy to check that the second row is \( C(\mathcal{W},-)-\)exact by (3) and (4) in Lemma 3. Then all rows and columns in this diagram are \( C(\mathcal{W},-)-\)exact. Note that \( \mathcal{E} \)-triangles (3.1) are \( C(-,\mathcal{W})-\)exact, it is easy to check that \( C(h,W) \) is epic by Lemma 3 (1) for any object \( W \in \mathcal{W} \), and \( C(h',W) \) is monic since

\[
C(h',W) = C(g',W)C(f',W).
\]

This implies that the second row is \( C(-,\mathcal{W})-\)exact. It is easy to check that the first column is \( C(-,\mathcal{W})-\)exact by \( 3 \times 3 \)-Lemma.
(2) The proof is dual to that of (1).

**Lemma 4** Let

$$A \xrightarrow{x} B \xrightarrow{y} C -\delta-$$  \hspace{1cm} (3.2)

be a $\mathcal{C}(\mathcal{W},-)\text{-exact } \mathbb{E}\text{-triangle in } \xi$. If both

$$K_1^A \xrightarrow{g_0^A} W_0^A \xrightarrow{f_0^A} A -\delta- \rightarrow, \hspace{1cm} K_1^C \xrightarrow{g_0^C} W_0^C \xrightarrow{f_0^C} C -\delta- \rightarrow,$$

are $\mathbb{E}\text{-triangles in } \xi$ with $W_0^C \in \mathcal{W}$, then we have the following commutative diagram:

$$
\begin{array}{ccccccc}
K_1^A & \xrightarrow{x_1} & K_1^B & \xrightarrow{y_1} & K_1^C & \xrightarrow{\delta_1} & \\
\downarrow{g_0^A} & & \downarrow{g_0^B} & & \downarrow{g_0^C} & & \\
W_0^A & \xrightarrow{(1_0)} & W_0^A \oplus W_0^C & \xrightarrow{(0_1)} & W_0^C & \xrightarrow{-} & \\
\downarrow{f_0^A} & & \downarrow{f_0^B} & & \downarrow{f_0^C} & & \\
A & \xrightarrow{\delta_0^A} & B & \xrightarrow{\delta_0^B} & C & \xrightarrow{\delta_0^C} & \\
\end{array}
$$

where all rows and columns are $\mathbb{E}\text{-triangles in } \xi$. Moreover,

(1) if the first and third columns in this diagram are $\mathcal{C}(\mathcal{W},-)\text{-exact},$ then so are all $\mathbb{E}\text{-triangles in this diagram};$

(2) if the third row and the first and third columns in this diagram are $\mathcal{C}(\mathcal{-},\mathcal{W})\text{-exact},$ then so are all $\mathbb{E}\text{-triangles in this diagram.}$

**Proof** Since (3.2) is a $\mathcal{C}(\mathcal{W},-)\text{-exact},$ there exists $z \in \mathcal{C}(W_0^C, B)$ such that $f_0^C = yz.$ So

$$(f_0^C)^*\delta = z^*y^*\delta = 0,$$

and there exists a $\xi\text{-deflation } f_0^B \in \mathcal{C}(W_0^A \oplus W_0^C, B)$ by Proposition 1, which makes the following diagram commutative:

$$
\begin{array}{ccccccc}
W_0^A & \xrightarrow{(1_0)} & W_0^A \oplus W_0^C & \xrightarrow{(0_1)} & W_0^C & \xrightarrow{-} & \\
\downarrow{f_0^A} & & \downarrow{f_0^B} & & \downarrow{f_0^C} & & \\
A & \xrightarrow{x} & B & \xrightarrow{y} & C & \xrightarrow{\delta} & \\
\end{array}
$$

Assume that

$$K_1^B \xrightarrow{g_0^B} W_0^A \oplus W_0^C \xrightarrow{f_0^B} B -\delta- \rightarrow$$  \hspace{1cm} (3.3)
is an $E$-triangle in $\xi$. Then we have the following commutative diagram:

\[
\begin{array}{c}
K_1^A \dashrightarrow y_1 \rightarrow K_1^B \dashrightarrow y_1 \rightarrow K_1^C \dashrightarrow \\
g_0^A \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad 
Theorem 1 Let

\[ X \xrightarrow{x} X^0 \xrightarrow{y} X^1 -\to \]  

(3.4)

be an $\mathcal{E}$-triangle in $\xi$. Assume that $\mathcal{W}$ is closed under finite direct sums and hokernel of $\xi$-deflations, and let

\[ \cdots \to W_i^0 \xrightarrow{d_i^0} \cdots \to W_1^0 \xrightarrow{d_1^0} W_0^0 \xrightarrow{d_0^0} X^0 \to 0 \]  

(3.5)

and

\[ \cdots \to W_i^1 \xrightarrow{d_i^1} \cdots \to W_1^1 \xrightarrow{d_1^1} W_0^1 \xrightarrow{d_0^1} X^1 \to 0 \]  

(3.6)

be proper $\mathcal{W}(\xi)$-resolutions of $X^0$ and $X^1$, respectively.

(1) We have a proper $\mathcal{W}(\xi)$-resolution of $X$,

\[ \cdots \to W_{i+1}^1 \oplus W_i^0 \to \cdots \to W_2^1 \oplus W_1^0 \to W \to X \to 0, \]  

(3.7)

and an $\mathcal{E}$-triangle

\[ W \to W_1^1 \oplus W_0^0 \to W_0^1 -\to \]  

(3.8)

in $\xi$.

(2) If the $\xi$-exact complexes (3.5), (3.6), and the $\mathcal{E}$-triangle (3.4) are $\mathcal{C}(\mathcal{W}, -)$-exact, then so is (3.7).

Proof (1) Since (3.5) and (3.6) are proper $\mathcal{W}(\xi)$-resolutions of $X^0$ and $X^1$, respectively, there exist $\mathcal{C}(\mathcal{W}, -)$-exact $\mathcal{E}$-triangles

\[ K_{i+1}^0 \xrightarrow{g_{i+1}^0} W_i^0 \xrightarrow{f_i^0} K_i^0 -\to, \quad K_{i+1}^1 \xrightarrow{g_{i+1}^1} W_i^1 \xrightarrow{f_i^1} K_i^1 -\to, \]

in $\xi$ with $W_i^0, W_i^1 \in \mathcal{W}$ for each integer $i \geq 0$, where $K_0^0 = X^0$ and $K_0^1 = X^1$. It follows from [2, Theorem 3.2] that there exists the following commutative diagram:

\[ \begin{array}{ccccccccc}
K_1^1 & \to & K_1^1 \\
\downarrow & & \downarrow \\
X & \xrightarrow{m_2} & M & \xrightarrow{e_2} & W_1^0 & \xrightarrow{\eta_2} & \to \\
\parallel & & \parallel & & \parallel & & \parallel \\
X & \xrightarrow{m_1} & X^0 & \xrightarrow{\eta} & X^1 & \xrightarrow{\delta_1} & \to \\
\parallel & & \parallel & & \parallel & & \parallel \\
& & & & & & \\
& & & & & & \\
\end{array} \]  

(3.9)

where all rows and columns are $\mathcal{E}$-triangles in $\xi$. Because the third column in diagram (3.9) is $\mathcal{C}(\mathcal{W}, -)$-exact, so is the second column by Lemma 2 (1). Thus,
by Lemma 4 (1), we get the following commutative diagram:

\[
\begin{array}{ccc}
K_1^1 & \rightarrow & L_1 \\
\downarrow & & \downarrow \\
W_1^1 & \rightarrow & W_1^1 \oplus W_0^0 \\
\downarrow & & \downarrow \\
K_1^0 & \rightarrow & E_1^0 \\
\end{array}
\]

where all rows and columns are \( \mathcal{C}(\mathcal{W},-) \)-exact \( \mathcal{E} \)-triangles in \( \xi \) and the second row is split. It is clear that \( W_1^1 \oplus W_0^0 \in \mathcal{W} \) by assumption.

On the other hand, we have the following commutative diagram:

\[
\begin{array}{ccc}
L_1 & \rightarrow & L_1 \\
\downarrow & & \downarrow \\
W & \rightarrow & W_1^1 \oplus W_0^0 \\
\downarrow & & \downarrow \\
X & \rightarrow & M \\
\downarrow & & \downarrow \\
& & \rightarrow W_0^1 \\
\end{array}
\]

where all rows and columns are \( \mathcal{E} \)-triangles in \( \xi \) by [2, Theorem 3.2] and the second row is the desired \( \mathcal{E} \)-triangle (3.8). It is easy to see that \( W \in \mathcal{W} \) because \( \mathcal{W} \) is closed under hokernel of \( \xi \)-deflations. Since the second column in diagram (3.11) is \( \mathcal{C}(\mathcal{W},-) \)-exact, so is the first column by Lemma 2 (1).

By Lemma 4 (1) again, we get the following commutative diagram:

\[
\begin{array}{ccc}
K_1^1 & \rightarrow & L_1 \\
\downarrow & & \downarrow \\
W_1^1 & \rightarrow & W_1^1 \oplus W_0^0 \\
\downarrow & & \downarrow \\
K_1^0 & \rightarrow & E_1^0 \\
\end{array}
\]
where all rows and columns are $\mathcal{C}(\mathcal{W}, -)$-exact $\mathcal{E}$-triangles in $\xi$. Continuing in this process, we get the desired $\xi$-exact complex (3.7), where

$$L_{i+1} \longrightarrow W_{i+1}^1 \oplus W_i^0 \longrightarrow L_i \longrightarrow \cdots, \quad \forall i \geq 1,$$

and $L_1 \longrightarrow W \longrightarrow X \longrightarrow$ are $\mathcal{C}(\mathcal{W}, -)$-exact $\mathcal{E}$-triangles in $\xi$.

(2) Note that both the third row and the third column in diagram (3.9) are $\mathcal{C}(\mathcal{W}, -)$-exact, so the second row and the second column in this diagram are also $\mathcal{C}(\mathcal{W}, -)$-exact. Since both the first and third columns in diagram (3.10) are $\mathcal{C}(\mathcal{W}, -)$-exact by assumption, both the first row and the second column in this diagram are also $\mathcal{C}(\mathcal{W}, -)$-exact by Lemma 4 (2). It is easy to check that the first column in diagram (3.11) is $\mathcal{C}(\mathcal{W}, -)$-exact because the third row and the second column in this diagram are $\mathcal{C}(\mathcal{W}, -)$-exact. Note that the third row and the first and third columns in diagram (3.12) are $\mathcal{C}(\mathcal{W}, -)$-exact, so is the second column in this diagram by Lemma 4 (2). Finally, we deduce that the $\xi$-exact complex (3.7) is $\mathcal{C}(\mathcal{W}, -)$-exact. □

Dual to Theorem 1, we have the following result which provides a method for constructing a coproper $\mathcal{W}(\xi)$-coresolution of the last term in an $\mathcal{E}$-triangle in $\xi$ from those of the first two terms.

**Theorem 2** Let

$$Y_1 \longrightarrow Y_0 \longrightarrow Y \longrightarrow$$

be an $\mathcal{E}$-triangle in $\xi$. Assume that $\mathcal{W}$ is closed under finite direct sums and hocokernel of $\xi$-inflations, and let

$$0 \rightarrow Y_0 \rightarrow W_0^0 \rightarrow W_0^1 \rightarrow \cdots \rightarrow W_i^0 \rightarrow \cdots \quad (3.14)$$

and

$$0 \rightarrow Y_1 \rightarrow W_1^0 \rightarrow W_1^1 \rightarrow \cdots \rightarrow W_i^1 \rightarrow \cdots \quad (3.15)$$

be coproper $\mathcal{W}(\xi)$-coresolutions of $Y_0$ and $Y_1$, respectively.

(1) We have a coproper $\mathcal{W}(\xi)$-coresolution of $Y$,

$$0 \rightarrow Y \rightarrow W \rightarrow W_0^1 \oplus W_1^2 \rightarrow \cdots \rightarrow W_i^0 \oplus W_i^{i+1} \rightarrow \cdots, \quad (3.16)$$

and an $\mathcal{E}$-triangle

$$W_0^0 \longrightarrow W_0^0 \oplus W_1^1 \longrightarrow W \longrightarrow$$

in $\xi$.

(2) If the $\xi$-exact complexes (3.14), (3.15), and the $\mathcal{E}$-triangle (3.13) are $\mathcal{C}(\mathcal{W}, -)$-exact, then so is (3.16).

**Proof** The proof is dual to that of Theorem 1, so we omit it here. □

The next result provides a method for constructing a proper $\mathcal{W}(\xi)$-resolution of the last term in an $\mathcal{E}$-triangle in $\xi$ from those of the first two terms.
**Theorem 3** Let

\[ X_1 \longrightarrow X_0 \longrightarrow X \longrightarrow \cdots \]  

(3.18)

be a \( C(\mathcal{W}, -) \)-exact \( E \)-triangle in \( \xi \). Assume that \( \mathcal{W} \) is closed under finite direct sums, and assume that

\[ W^n \rightarrow \cdots \rightarrow W_0^1 \rightarrow W_0^0 \rightarrow X_0 \rightarrow 0 \]  

(3.19)

and

\[ W_1^{n-1} \rightarrow \cdots \rightarrow W_1^1 \rightarrow W_1^0 \rightarrow X_1 \rightarrow 0 \]  

(3.20)

are proper \( \mathcal{W}(\xi) \)-resolutions of \( X_0 \) and \( X_1 \), respectively.

1. We have a proper \( \mathcal{W}(\xi) \)-resolution of \( X \),

\[ W^n_0 \oplus W_1^{n-1} \rightarrow \cdots \rightarrow W^n_0 \oplus W_1^1 \rightarrow W^n_0 \oplus W_1^0 \rightarrow W_1^0 \rightarrow X \rightarrow 0. \]  

(3.21)

2. If the \( \xi \)-exact complexes (3.19), (3.20), and the \( E \)-triangle (3.18) are \( C(\mathcal{W}, -, \mathcal{W}) \)-exact, then so is the \( \xi \)-exact complex (3.21).

**Proof**  
(1) Since (3.19) and (3.20) are proper \( \mathcal{W}(\xi) \)-resolutions of \( X_0 \) and \( X_1 \) respectively, there exist \( C(\mathcal{W}, -) \)-exact \( E \)-triangles

\[ K_{i+1}^0 \longrightarrow W_i^0 \longrightarrow K_i^0 \overset{\delta_i}{\longrightarrow}, \quad K_{i+1}^1 \longrightarrow W_i^1 \longrightarrow K_i^1 \overset{\delta_i}{\longrightarrow}, \]

in \( \xi \) with \( W_i^0, W_i^1 \in \mathcal{W} \) for each integer \( i \geq 0 \), where \( K_0^0 = X_0 \) and \( K_1^0 = X_1 \).

It follows from [2, Theorem 3.2] that there exists the following commutative diagram:

\[ \begin{array}{ccc}
K_0^1 & \longrightarrow & K_0^1 \\
\downarrow & & \downarrow \\
L_1 & \longrightarrow & W_0^0 \longrightarrow X \longrightarrow \\
\downarrow & & \downarrow \\
X_1 & \longrightarrow & X_0 \longrightarrow X \longrightarrow \\
\end{array} \]  

(3.22)

where all rows and columns are \( E \)-triangle in \( \xi \). Note that the second column in diagram (3.22) is \( C(\mathcal{W}, -) \)-exact, so is the first column by Lemma 2 (1). Since the third row is \( C(\mathcal{W}, -) \)-exact by assumption, it is easy to check that the second row is \( C(\mathcal{W}, -) \)-exact. By Lemma 4 (1), we get the following
commutative diagram:

\[
\begin{array}{c}
K^2_0 \rightarrow L_2 \rightarrow K^1_1 \rightarrow \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
W^1_0 \rightarrow W^1_0 \oplus W^0_1 \rightarrow W^0_1 \rightarrow \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
K^1_0 \rightarrow L_1 \rightarrow X_1 \rightarrow
\end{array}
\]

(3.23)

where all rows and columns are \( C(\mathcal{W},-)-\)exact \( \mathcal{E} \)-triangles in \( \xi \). It is clear that \( W^1_0 \oplus W^0_1 \in \mathcal{W} \) by assumption. Then by using Lemma 4 (1) iteratively, we get the desired proper \( \mathcal{W}(\xi) \)-resolution (3.21) of \( X \), which is spliced by \( C(\mathcal{W},-)-\)exact \( \mathcal{E} \)-triangles

\[
L_1 \rightarrow W^0_0 \rightarrow X \rightarrow , \quad L_{i+1} \rightarrow W^0_i \oplus W^1_{i-1} \rightarrow L_i \rightarrow , \quad \forall \ i \geq 1.
\]

(2) It follows from Proposition 2 (1) that all \( \mathcal{E} \)-triangles in diagram (3.22) are both \( C(\mathcal{W},-) \)-exact and \( C(-,\mathcal{W}) \)-exact. Both the first and third columns in diagram (3.23) are \( C(-,\mathcal{W}) \)-exact by assumption, so the second column in this diagram is also \( C(-,\mathcal{W}) \)-exact by Lemma 4 (2). Finally, we deduce that the \( \xi \)-exact complex (3.21) is \( C(-,\mathcal{W}) \)-exact.

The next result, which is dual to Theorem 3, provides a method for constructing a coproper \( \mathcal{W}(\xi) \)-coresolution of the first term in an \( \mathcal{E} \)-triangle in \( \xi \) from those of the last two terms.

**Theorem 4** Let

\[
Y \rightarrow Y^0 \rightarrow Y^1 \rightarrow
\]

(3.24)

be a \( C(-,\mathcal{W}) \)-exact \( \mathcal{E} \)-triangle in \( \xi \). Assume that \( \mathcal{W} \) is closed under finite direct sums, and let

\[
0 \rightarrow Y^0 \rightarrow W^0_0 \rightarrow W^0_1 \rightarrow \cdots \rightarrow W^0_n
\]

(3.25)

and

\[
0 \rightarrow Y^1 \rightarrow W^1_0 \rightarrow W^1_1 \rightarrow \cdots \rightarrow W^1_{n-1}
\]

(3.26)

be coproper \( \mathcal{W}(\xi) \)-coresolutions of \( Y^0 \) and \( Y^1 \), respectively.

(1) We have a coproper \( \mathcal{W}(\xi) \)-coresolution of \( Y \),

\[
0 \rightarrow Y \rightarrow W^0_0 \rightarrow W^1_0 \oplus W^0_1 \rightarrow W^1_1 \oplus W^0_2 \rightarrow \cdots \rightarrow W^1_{n-1} \oplus W^0_n.
\]

(3.27)

(2) If the \( \xi \)-exact complexes (3.25), (3.26), and the \( \mathcal{E} \)-triangle (3.24) are \( C(\mathcal{W},-) \)-exact, then so is the \( \xi \)-exact complex (3.27).

**Proof** The proof is dual to that of Theorem 3, so we omit it here. \( \square \)
Remark 2 Note that the extriangulated categories are a simultaneous generalization of abelian categories and triangulated categories. It follows that Theorems 1–4 here unify [3, Theorems 3.2, 3.4, 3.6, 3.8] in abelian categories, and [11, Theorems 2.3, 2.5, 2.7 2.9] in triangulated categories. It should be noted that our results here are new for an exact category case.

4 Gorensteinness in extriangulated categories

In this section, we give some applications of the results in Section 3. In the following, we always assume that \( \mathcal{W} \) is a class of objects in \( \mathcal{C} \) which is closed under isomorphisms and finite direct sums. We introduce the Gorenstein category \( \mathcal{GW}(\xi) \) in extriangulated categories and demonstrate that this category shares some basic properties with the Gorenstein category in abelian categories or triangulated categories.

Definition 7 Let \( X \) be an object of \( \mathcal{C} \). A complete \( \mathcal{W}(\xi) \)-resolution of \( X \) is both \( \mathcal{C}(\mathcal{W},-) \)-exact and \( \mathcal{C}(-,\mathcal{W}) \)-exact \( \xi \)-exact complex

\[
\cdots \to W_1 \to W_0 \to W^0 \to W^1 \to \cdots
\]

with each term in \( \mathcal{W} \) such that

\[
X_1 \to W_0 \to X \to \to
\]

and

\[
X \to W^0 \to X^1 \to \to
\]

are corresponding \( \mathcal{E} \)-triangles in \( \xi \).

The Gorenstein subcategory \( \mathcal{GW}(\xi) \) of \( \mathcal{C} \) is defined as

\[
\mathcal{GW}(\xi) = \{ X \in \mathcal{C} \mid X \text{ admits a complete } \mathcal{W}(\xi) \text{-resolution} \}.
\]

Set

\[
\mathcal{GW}^1(\xi) = \mathcal{GW}(\xi), \quad \mathcal{GW}^{n+1}(\xi) = \mathcal{G}(\mathcal{GW}^n(\xi)), \quad \forall n \geq 1.
\]

Remark 3 (1) Assume that \( \mathcal{C} \) is an abelian category. If \( \xi \) is the class of exact sequences and \( \mathcal{W} \) is a full additive subcategory of \( \mathcal{C} \) that is closed under isomorphisms, then Gorenstein subcategory \( \mathcal{GW}(\xi) \) defined here coincides with the earlier one given by Sather-Wagstaff et al. [7].

(2) Assume that \( (\mathcal{T}, \Sigma, \Delta) \) is a triangulated category and \( \xi \) is a proper class of triangles which is closed under suspension (see [1, Section 2.2]). If \( \mathcal{W} \) is an additive full subcategory of \( \mathcal{T} \) closed under isomorphisms and \( \Sigma \)-stable, i.e., \( \Sigma(\mathcal{W}) = \mathcal{W} \), then Gorenstein subcategory \( \mathcal{GW}(\xi) \) defined here coincides with the earlier one given by Yang and Wang [11].

(3) Assume that the extriangulated category \( (\mathcal{C}, \mathcal{E}, \mathcal{S}) \) has enough \( \xi \)-projectives and enough \( \xi \)-injectives. If \( \mathcal{W} \) is the class of \( \xi \)-projectives (resp., \( \xi \)-injectives) objects, then Gorenstein subcategory \( \mathcal{GW}(\xi) \) defined here coincides
with the subcategory consisting of $\xi$-$\mathcal{G}$-projective (resp., $\xi$-$\mathcal{G}$-injective) objects in [2].

**Lemma 6**  Given both $\mathcal{C}(\mathcal{W}, -)$-exact and $\mathcal{C}(-, \mathcal{W})$-exact $\mathbb{E}$-triangles

$$
X \longrightarrow Y \longrightarrow Z \longrightarrow
$$
in $\xi$, if $X, Z \in \mathcal{GW}(\xi)$, then so is $Y$.

**Proof** Assume $X, Z \in \mathcal{GW}(\xi)$. Then there exist complete $\mathcal{W}(\xi)$-resolutions

$$
\cdots \rightarrow W_1 \rightarrow W_0 \rightarrow W^0 \rightarrow W^1 \rightarrow \cdots
$$

and

$$
\cdots \rightarrow V_1 \rightarrow V_0 \rightarrow V^0 \rightarrow V^1 \rightarrow \cdots
$$
of $X$ and $Z$, respectively. It is straightforward to show that

$$
\cdots \rightarrow W_1 \oplus V_1 \rightarrow W_0 \oplus V_0 \rightarrow W^0 \oplus V^0 \rightarrow W^1 \oplus V^1 \rightarrow \cdots
$$
is a complete $\mathcal{W}(\xi)$-resolution of $Y$ by repeating application of Lemmas 4 and 5. □

As a main application of the results in Section 3, we have the main result of this section.

**Theorem 5**  The following are true for any extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$:

1. $\mathcal{GW}^n(\xi) = \mathcal{GW}(\xi)$ for any $n \geq 1$;
2. $\mathcal{GW}(\xi)$ is closed under direct summands.

**Proof**  (1) It suffices to show that $\mathcal{GW}^2(\xi) = \mathcal{GW}(\xi)$. Let $G \in \mathcal{GW}(\xi)$. Note that

$$
G \xrightarrow{1} G \longrightarrow 0 \xrightarrow{-} , \quad 0 \longrightarrow G \xrightarrow{1} G \longrightarrow ,
$$

are $\mathbb{E}$-triangles in $\xi$. It is easy to check that

$$
\cdots \rightarrow 0 \rightarrow G \xrightarrow{1} G \rightarrow 0 \rightarrow \cdots
$$
is a complete $\mathcal{GW}(\xi)$-resolution of $G$, and then $G \in \mathcal{GW}^2(\xi)$, which implies that $\mathcal{GW}(\xi) \subseteq \mathcal{GW}^2(\xi)$.

Let $X$ be an object in $\mathcal{GW}^2(\xi)$ and

$$
\cdots \rightarrow G_1 \rightarrow G_0 \rightarrow G^0 \rightarrow G^1 \rightarrow \cdots
$$
a complete $\mathcal{GW}(\xi)$-resolution of $X$. That is, there exist both $\mathcal{C}(\mathcal{GW}(\xi), -)$-exact and $\mathcal{C}(-, \mathcal{GW}(\xi))$-exact $\mathbb{E}$-triangles

$$
K_{i+1} \longrightarrow G_i \longrightarrow K_i \xrightarrow{-} , \quad K^i \longrightarrow G^i \longrightarrow K^{i+1} \xrightarrow{-}
$$
in $\xi$ with $G_i, G^i \in \mathcal{GW}(\xi)$ for any $i \geq 0$, where $K_0 = K^0 = X$. Since $G_0 \in \mathcal{GW}(\xi)$, there exists a both $\mathcal{C}(\mathcal{W}, -)$-exact and $\mathcal{C}(-, \mathcal{W})$-exact $\mathbb{E}$-triangle
\[L_1 \longrightarrow W_0 \longrightarrow G_0 \longrightarrow X \longrightarrow \] in \(\xi\) with \(W_0 \in \mathcal{W}\) and \(L_1 \in \mathcal{GW}(\xi)\). Note that the \(\mathbb{E}\)-triangle \(K_1 \longrightarrow G_0 \longrightarrow X \longrightarrow \) in \(\xi\) is both \(C(\mathcal{W}, -)\)-exact and \(C(-, \mathcal{W})\)-exact, so we have the following commutative diagram:

\[
\begin{array}{c}
L_1 \\
\downarrow \\
M_1 \\
\downarrow \\
K_1
\end{array}
\begin{array}{c}
\longrightarrow W_0 \\
\longrightarrow X \\
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
\longrightarrow G_0 \\
\longrightarrow X \\
\rightarrow \\
\rightarrow
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array}
\]

where all rows and columns are both \(C(\mathcal{W}, -)\)-exact and \(C(-, \mathcal{W})\)-exact \(\mathbb{E}\)-triangles in \(\xi\) by Proposition 2 (1). It follows from [2, Theorem 3.2] that we have the following commutative diagram:

\[
\begin{array}{c}
K_2 \\
\downarrow \\
L_1
\end{array}
\begin{array}{c}
\longrightarrow N_1 \\
\rightarrow G_1 \\
\rightarrow
\end{array}
\begin{array}{c}
\longrightarrow M_1 \\
\rightarrow K_1 \\
\rightarrow
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array}
\]

where all rows and columns are \(\mathbb{E}\)-triangles in \(\xi\). Moreover, all \(\mathbb{E}\)-triangles in diagram (4.1) are \(C(\mathcal{W}, -)\)-exact by Lemma 2 (1) and \(3 \times 3\)-Lemma.

Next, we claim that the \(\mathbb{E}\)-triangle \(L_1 \longrightarrow N_1 \longrightarrow G_1 \longrightarrow \) in \(\xi\) is \(C(-, \mathcal{W})\)-exact. In fact, there exists a both \(C(\mathcal{W}, -)\)-exact and \(C(-, \mathcal{W})\)-exact \(\mathbb{E}\)-triangle \(H \longrightarrow W \longrightarrow G_1 \longrightarrow \) in \(\xi\) with \(W \in \mathcal{W}\) and \(H \in \mathcal{GW}(\xi)\) since \(G_1 \in \mathcal{GW}(\xi)\). So we have the following commutative diagram:

\[
\begin{array}{c}
H \\
\downarrow \\
L_1
\end{array}
\begin{array}{c}
\longrightarrow Z \\
\rightarrow W \\
\rightarrow
\end{array}
\begin{array}{c}
\longrightarrow N_1 \\
\rightarrow G_1 \\
\rightarrow
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array}
\]
where all rows and columns are $E$-triangles in $\xi$. Moreover, the second row in diagram (4.2) is $C(\mathcal{W},-)\text{-exact}$ by Lemma 2 (1), and it is split as $W \in \mathcal{W}$. Hence, the second row in diagram (4.2) is $C(-,\mathcal{W})\text{-exact}$, which follows from [2, Lemma 4.10 (1)] that the $E$-triangle $L_1 \rightarrow N_1 \rightarrow G_1$ in $\xi$ is $C(-,\mathcal{W})\text{-exact}$ because the third column in diagram (4.2) is $C(-,\mathcal{W})\text{-exact}$. Since $G_1, L_1 \in \mathcal{GW}(\xi)$, we have $N_1 \in \mathcal{GW}(\xi)$ by Lemma 6. It is easy to show that all $E$-triangles in diagram (4.1) are $C(\mathcal{W},-)\text{-exact}$ and $C(-,\mathcal{W})\text{-exact}$ $E$-triangles in $\xi$ by Proposition 2 (1). Proceeding in this manner, we can get both $C(\mathcal{W},-)\text{-exact}$ and $C(-,\mathcal{W})\text{-exact}$ $E$-triangles $M_{i+1} \rightarrow W_i \rightarrow M_i$ in $\xi$ with $W_i \in \mathcal{W}$ and $L_2 \in \mathcal{GW}(\xi)$. So we have the following commutative diagram:

\[
\begin{array}{c}
\text{L}_2 \quad \text{M}_2 \quad \text{K}_2 \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{W}_1 \quad \text{M}_1 \quad \text{N}_1 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \quad 0 \quad 0 \\
\end{array}
\]

where all rows and columns are both $C(\mathcal{W},-)\text{-exact}$ and $C(-,\mathcal{W})\text{-exact}$ $E$-triangles in $\xi$ by Proposition 2 (1). Proceeding in this manner, we can get both $C(\mathcal{W},-)\text{-exact}$ and $C(-,\mathcal{W})\text{-exact}$ $E$-triangles $M_{i+1} \rightarrow W_i \rightarrow M_i$ in $\xi$ with $W_i \in \mathcal{W}$. Spliced these $E$-triangles together, we obtain a both $C(\mathcal{W},-)\text{-exact}$ and $C(-,\mathcal{W})\text{-exact}$ $E$-triangle $\cdots \rightarrow W_1 \rightarrow W_0 \rightarrow X \rightarrow 0$ with $W_i \in \mathcal{W}$ for any $i \geq 0$.

Dually, we can obtain a both $C(\mathcal{W},-)\text{-exact}$ and $C(-,\mathcal{W})\text{-exact}$ $E$-triangle $\cdots \rightarrow X \rightarrow W^0 \rightarrow W^1 \rightarrow \cdots$ with $W^i \in \mathcal{W}$ for any $i \geq 0$. Hence, $X \in \mathcal{GW}(\xi)$, as desired.

(2) Assume that $G \in \mathcal{GW}(\xi)$ and $H$ is a direct summand of $G$. Then there exists $H' \in \mathcal{C}$ such that $G = H \oplus H'$. Therefore, there exist two split $E$-triangles

\[
\begin{array}{c}
H \xrightarrow{(0)} G \xrightarrow{(1)} H' \\
\end{array}
\]

\[
\begin{array}{c}
H' \xrightarrow{(0)} G \xrightarrow{(1)} H \\
\end{array}
\]

in $\xi$. Since $G \in \mathcal{GW}(\xi)$, there exists a both $C(\mathcal{W},-)\text{-exact}$ and $C(-,\mathcal{W})\text{-exact}$ $E$-triangle $G \xrightarrow{\alpha} W_{-1} \xrightarrow{\beta} K_{-1}$ in $\xi$ with $W_{-1} \in \mathcal{W}$ and $K_{-1} \in \mathcal{GW}(\xi)$.
It follows from Proposition 2 (2) that we have the following commutative diagram:

\[
\begin{array}{cccccccc}
\alpha & \beta & \delta & \\
H & \rightarrow & G & \rightarrow & H' & \rightarrow & 0 & \rightarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
H & \rightarrow & W & \rightarrow & X & \rightarrow & 0 & \rightarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
K & \rightarrow & K & \rightarrow & & & & \\
\end{array}
\]

where all rows and columns are both \(\mathcal{C}(\mathcal{W}, -)\)-exact and \(\mathcal{C}(-, \mathcal{W})\)-exact E-triangles in \(\xi\). Note that

\[
\begin{array}{cccccccc}
\alpha' & \beta' & \delta' & \\
H' & \rightarrow & X & \rightarrow & K & \rightarrow & 0 & \rightarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
H & \rightarrow & G & \rightarrow & H & \rightarrow & 0 & \rightarrow \\
\end{array}
\]

are E-triangles in \(\xi\). Then there exists a commutative diagram

\[
\begin{array}{cccccccc}
\alpha & \beta & \delta & \\
H' & \rightarrow & X & \rightarrow & K & \rightarrow & 0 & \rightarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
H & \rightarrow & G & \rightarrow & H & \rightarrow & 0 & \rightarrow \\
\end{array}
\]

where \(G \rightarrow G \rightarrow G \rightarrow K \rightarrow 0 \rightarrow \)

are E-triangles in \(\xi\) because \(\xi\) is closed under cobase change. It is easy to check that the second row is both \(\mathcal{C}(\mathcal{W}, -)\)-exact and \(\mathcal{C}(-, \mathcal{W})\)-exact E-triangle in \(\xi\) because the first row and the first and second columns are both \(\mathcal{C}(\mathcal{W}, -)\)-exact and \(\mathcal{C}(-, \mathcal{W})\)-exact. It follows from Lemma 6 that \(G_{-1} \in \mathcal{GW}(\xi)\) since \(G, K_{-1} \in \mathcal{GW}(\xi)\). Hence, there exists a both \(\mathcal{C}(\mathcal{W}, -)\)-exact and \(\mathcal{C}(-, \mathcal{W})\)-exact E-triangle \(G_{-2} \rightarrow W_{-2} \rightarrow K_{-2} \rightarrow 0 \rightarrow \) in \(\xi\) with \(W_{-2} \in \mathcal{W}\) and \(K_{-2} \in \mathcal{W}\).
\(GW(\xi)\). So there exists a commutative diagram

\[
\begin{array}{ccccccc}
X & \xrightarrow{g'_{-1}} & G_{-1} & \xrightarrow{f'_{-1}} & H & \rightarrow & 0 \\
X & \xrightarrow{g_{-2}} & W_{-2} & \xrightarrow{f_{-2}} & Y & \xrightarrow{\rho_{-2}} & 0 \\
K_{-2} & \xrightarrow{\delta_{-2}} & K_{-2} & \xrightarrow{(f'_{-1}) \cdot \delta_{-2}} & K_{-2}
\end{array}
\]

where all rows and columns are both \(\mathcal{C}(\mathcal{W}, -)\)-exact and \(\mathcal{C}(-, \mathcal{W})\)-exact \(E\)-triangles in \(\xi\) by Proposition 2 (2). Proceeding this manner, one can get a both \(\mathcal{C}(\mathcal{W}, -)\)-exact and \(\mathcal{C}(-, \mathcal{W})\)-exact \(\xi\)-exact complex

\[
0 \rightarrow H \rightarrow W_{-1} \rightarrow W_{-2} \rightarrow \cdots
\]

with each \(W_{-i} \in \mathcal{W}\) for \(i \geq 1\). Dually, we can get the following both \(\mathcal{C}(\mathcal{W}, -)\)-exact and \(\mathcal{C}(-, \mathcal{W})\)-exact \(\xi\)-exact complex:

\[
\cdots \rightarrow W_{1} \rightarrow W_{0} \rightarrow H \rightarrow 0
\]

with \(W_{i} \in \mathcal{W}\) for any \(i \geq 0\). Hence, \(H \in \mathcal{GW}(\xi)\), as desired. \(\square\)

By Remark 3 (1) and Theorem 5, we have the following corollary.

Corollary 1 [3, Theorems 4.1, 4.6 (2)] Assume that \(\mathcal{C}\) is an abelian category and \(\xi\) is the class of exact sequences. If \(\mathcal{W}\) is a full additive subcategory of \(\mathcal{C}\), which is closed under isomorphisms, then \(\mathcal{GW}^{n}(\xi) = \mathcal{GW}(\xi)\) for any \(n \geq 1\) and \(\mathcal{GW}(\xi)\) is closed under direct summands.

As a consequence of Remark 3 (2) and Theorem 5, we have the following corollary.

Corollary 2 Assume that \(\mathcal{C}\) is a triangulated category and \(\xi\) is a proper class of triangles. If \(\mathcal{W}\) is an additive full subcategory of \(\mathcal{F}\), which is closed under isomorphisms and \(\Sigma\)-stable, then \(\mathcal{GW}^{n}(\xi) = \mathcal{GW}(\xi)\) for any \(n \geq 1\) and \(\mathcal{GW}(\xi)\) is closed under direct summands.

Remark 4 We note that Corollary 2 refines one of the main results of Yang and Wang [11]. In their paper, they showed that for any triangulated category with countable coproducts, if the class \(\mathcal{W}\) is closed under countable coproducts, then \(\mathcal{GW}^{n}(\xi) = \mathcal{GW}(\xi)\) for any \(n \geq 1\) and \(\mathcal{GW}(\xi)\) is closed under direct summands (see [11, Theorem 3.3]).

Corollary 3 For given both \(\mathcal{C}(\mathcal{W}, -)\)-exact and \(\mathcal{C}(-, \mathcal{W})\)-exact \(E\)-triangle

\[
X \longrightarrow Y \longrightarrow Z \longrightarrow
\]
in $\xi$ with $Y \in \mathcal{GW}(\xi)$, $X \in \mathcal{GW}(\xi)$ if and only if $Z \in \mathcal{GW}(\xi)$.

Proof Assume $Y, Z \in \mathcal{GW}(\xi)$. Then Theorem 4 implies that $X$ has a coproper $\mathcal{W}(\xi)$-coresolution which is $C(\mathcal{W},-)\text{-exact}$. Consider a both $C(\mathcal{W},-)\text{-exact}$ and $C(-,\mathcal{W})\text{-exact} \mathcal{E}\text{-triangle}$ $L \rightarrow W \rightarrow Y \rightarrow$ in $\xi$ with $W \in \mathcal{W}$ and $L \in \mathcal{GW}(\xi)$. Then we have the following commutative diagram:

$$
\begin{array}{cccc}
L & \rightarrow & L \\
\downarrow & & \downarrow \\
Z' & \rightarrow & W & \rightarrow & Z \rightarrow \rightarrow \\
\downarrow & & \downarrow & & \downarrow \\
X & \rightarrow & Y & \rightarrow & Z \rightarrow \rightarrow
\end{array}
$$

where all rows and columns are both $C(\mathcal{W},-)\text{-exact}$ and $C(-,\mathcal{W})\text{-exact} \mathcal{E}\text{-triangles}$ in $\xi$ by Proposition 2 (1). Note that $Z \in \mathcal{GW}(\xi)$. Then there exists a both $C(\mathcal{W},-)\text{-exact}$ and $C(-,\mathcal{W})\text{-exact} \mathcal{E}\text{-triangle}$ $K \rightarrow V \rightarrow Z \rightarrow \rightarrow$ in $\xi$ with $V \in \mathcal{W}$ and $K \in \mathcal{GW}(\xi)$. So we have the following commutative diagram:

$$
\begin{array}{cccc}
K & \rightarrow & K \\
\downarrow & & \downarrow \\
Z' & \rightarrow & N & \rightarrow & V \rightarrow \rightarrow \\
\downarrow & & \downarrow & & \downarrow \\
Z' & \rightarrow & W & \rightarrow & Z \rightarrow \rightarrow
\end{array}
$$

where all rows and columns are $\mathcal{E}\text{-triangles}$ in $\xi$. Moveover, the second row and column are $C(\mathcal{W},-)\text{-exact}$ by Lemma 2 (1). Hence, $K \oplus W \cong Z' \oplus V$, which implies that $Z' \in \mathcal{GW}(\xi)$ because $\mathcal{GW}(\xi)$ is closed under direct summands by Theorem 5 (2). Hence, $X$ has a proper $\mathcal{W}(\xi)$-resolution which is $C(-,\mathcal{W})\text{-exact}$ by Theorem 3. Therefore, $X \in \mathcal{GW}(\xi)$.

Dually, we can prove that $Z \in \mathcal{GW}(\xi)$ whenever $X, Y \in \mathcal{GW}(\xi)$.

\[\square\]

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