Intrinsic nonlinearity of a PN-junction diode and higher order harmonic generation

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Abstract

Voltage current characteristics of a PN-junction diode are intrinsically nonlinear in nature. It is shown in this paper that a mathematical form of nonlinearity of a PN-junction diode resembles the nonlinear response of electric polarization of a dielectric medium to the electric field. Nonlinearity of a PN-junction can be expressed in a series of successively increasing orders of the nonlinearity. For a PN-junction diode, higher order nonlinear terms become significant as a voltage across the diode is increased. In this paper, a gradual emergence of a nonlinear regime with the amplitude of a sinusoidal voltage is presented. Higher order harmonics are produced by utilizing the nonlinearity of a single PN-junction diode. An experimental realization of a frequency comb with the highest frequency up to the twentieth harmonics is also presented. In addition, in the same circuit by making the nonlinearity significant up to the second order, an experiment on generation of the sum and difference of frequencies is realized.

1 Introduction

Nonlinearity is present in most of the natural processes. A linear behaviour of a system is an approximation of a generalised nonlinear response [non (2015)]. Whether it is a simple pendulum or interaction of light with matter, a large amplitude of oscillation leads to a breakdown of a linear approximation and brings out the nonlinear response of a system. Nonlinear physics is a well established field of research. Some of the prominent concepts such as the emergence of chaos [Gleick (1988)] and in nonlinear optics where light fields interact with each other [Shen (2003); Boyd (2008); Remoissenet (2003)] have been of great interest. In this paper, an experiment based on a PN-junction diode is introduced which utilizes the intrinsic nonlinearity of a PN-junction diode to experimentally realize the effects analogous to those observed in classical nonlinear optics. The voltage current characteristics of a PN-junction diode can be expressed in the form of an infinite series.
of linear and nonlinear terms. With an increase in the amplitude of a sinusoidal voltage applied across a PN-junction diode the higher order nonlinearities become important. As the amplitude of a sinusoidal voltage is increased, new components of frequency appears in current passing through the PN-junction diode. A gradual generation of new frequencies and generation of a frequency comb [Hansch (2006)] with the highest frequency up to the twentieth harmonics are clearly demonstrated in the experiment. Further, by using a source of two different frequency sinusoidal voltage waveforms (a sinusoidal voltage has a wave like form in the time domain), the sum and difference frequency generation have been demonstrated experimentally.

2 Nonlinear dielectric medium: Introduction

In a linear dielectric medium in presence of an electric field, the displacement of bound charges from their equilibrium position remains linearly proportional to the applied force (Hooke’s law). As a consequence, the induced electric polarization ($P$) remains linearly proportional to the net electric field ($E$) such that $P = \varepsilon_0 \chi_e E$, where $\varepsilon_0$ is the vacuum permittivity and $\chi_e$ is the electric susceptibility, which is a measure of the ability of a medium to get polarized. Therefore, when light propagates in a ‘linear’ dielectric medium, the frequency of the propagating light wave and the refractive index of the medium remain unaltered. As a consequence, different electromagnetic fields do not interact with each other. However, a linear response is an approximation and as the intensity of light (electric field) is increased, the displacement of bound charges increases. Hooke’s law does not remain valid for a large displacement of bound charges and hence the induced polarization does not remain linearly proportional to the electric field. The dependence of the induced electric polarization on the electric field becomes nonlinear which can be expressed as [Shen (2003); Saleh and Teich (2007); Boyd (2008)]

$$P = \varepsilon_0 (\chi_e^{(1)} E + \chi_e^{(2)} E^2 + \chi_e^{(3)} E^3 + \cdots) \tag{1}$$

Where $\chi_e^{(1)}$, $\chi_e^{(2)}$ and $\chi_e^{(3)}$ are the first, second and third order electric susceptibilities respectively, the order of nonlinearity is denoted by a superscript. Equation (1) can be rewritten as the sum of a linear and a nonlinear term such that $P = \varepsilon_0 \chi_e^{(1)} E + P_{NL}$, where $P_{NL}$ is the nonlinear component of the electric polarization. At a low light intensity, only the first term is significant and all the nonlinear terms are negligible. For a large intensity of light, the $P_{NL}$ term becomes pronounced. The relative magnitude of the nonlinear susceptibilities in Equation (1) depends on the crystal symmetry [Haueisen (1973)]. For non centro-symmetric crystals such as Lithium Niobate (LiNbO$_3$), Potassium Titanyl Phosphate (KTP) and Beta Barium Borate (BBO) the second order nonlinearity $\chi_e^{(2)}$ is nonzero. A nonlinear medium
having only the second order nonzero nonlinearity is known as a second order non-linear medium [Shen (2003); Saleh and Teich (2007); Boyd (2008)]. One of the important processes resulting from the second order nonlinearity ($\chi^{(2)}_e$ process) is the second harmonic generation [Franken et al. (1961)], where frequency of light is doubled. According to the quantum description of second harmonic generation, two photons of frequency $\omega$ interact with each other via the second order interaction and merge together as a single photon of frequency $2\omega$. In a reverse process which is known as frequency down conversion, a photon can split into two photons of lower frequencies. Other examples of second order nonlinear processes are the sum and difference frequency generation [Bass et al. (1962); Shen (1976, 2003); Saleh and Teich (2007); Boyd (2008); Franken and Ward (1963); Hansch (2006)]. For a medium such as the silica optical fiber, $\chi^{(3)}_e$ is significant which results in third order nonlinear effects [Ghatak and Thyagarajan (2000); Agarwal (2013)], where the refractive index becomes intensity dependent - a phenomenon known as the optical Kerr effect. Optical Kerr effect can be utilised to modulate the phase of a wave by modulating the intensity of another wave. A few important examples of third order nonlinear processes [Saleh and Teich (2007); Agarwal (2013)] are the third harmonic generation [Young et al. (1971)], self phase modulation, cross phase modulation, four wave mixing, supercontinuum generation and optical phase conjugation. These nontrivial effects have revolutionized the field of nonlinear optics over the decades and have played a significant role to produce quantum entangled photons and in experiments on quantum information processing.

3 Nonlinearity of a PN-junction diode

An ideal diode conducts current in the forward bias and completely blocks the flow of current in the reverse bias. A PN-junction is formed, if a P-type semiconductor is joined to a N-type semiconductor. Immediately after joining them, electrons which are majority carriers of N-type region start diffusing into P-type region due to their concentration gradient across the contact of two semiconductors. Diffused electrons recombine with holes which are majority carriers of P-type region. Flow of charge carries through the contact due to concentration gradient generates a diffusion current. However, this process is gradually stopped by an opposing force. As electrons diffuse into the P-type region, a net positive charge develops near the contact in N-type region and a net negative charge is formed near the contact in P-type region. This charge distribution produces an electric field in the PN-junction pointing from N-type region towards P-type region. Electric field generates an electric potential barrier at the junction and an opposing force for the flow due to diffusion of ma-
majority carriers. However, minority carriers which are produced by thermal agitation, are accelerated by the junction electric field. This process produces current named as the drift current. In equilibrium condition, if a PN-junction is unbiased, the diffusion current and drift current flow in opposite direction. Therefore, a net flow of current is zero through a PN-junction i.e. majority carriers climb the potential barrier and minority carriers descends the potential barrier. Under the forward bias, P-type region is kept at a higher potential as compared to N-type region by connecting to a battery. In the forward bias, the net electric field and height of potential barrier across the junction is decreased. Therefore, probability of carriers to climb the potential barrier increases in the forward bias. This process increases electron concentration in P-type region at the junction edge and hole concentration in N-type region at the junction edge. Such an excess carrier concentration increases exponentially with an applied bias voltage across the junction. Therefore, in the forward bias, the diffusion current increases exponentially with applied bias voltage while the drift current remains negligibly small. In the reverse bias, P-type region is kept at a lower potential as compared to N-type region. Therefore, potential barrier height increases and probability to cross the potential barrier decreases. However, the flow of drift current saturates with the increase in the reverse bias voltage due to an extremely low carrier concentration of thermally generated minority carriers. The flow of net current through a PN-junction is a nonlinear function of the bias voltage. However, in case of a nonlinear dielectric medium, it is the displacement of charge from the equilibrium position which depends nonlinearly on the applied force. Nonlinear dielectric medium and a PN-junction diode are analogous to each other in the context of a mathematical form of their nonlinearity, which is the central idea of this paper.

In a realistic PN-junction diode, for a bias voltage $V$ across a junction the net current $I$ passing through the junction is [Hayes and Horowitz (2002)]

$$I = I_0(e^{V/\eta V_{th}} - 1)$$

(2)

Where $I_0$ is the reverse saturation current, $\eta$ is the ideality factor which is constant (value between 1 to 2) and $V_{th} = \frac{k_B T}{q}$ is the thermal voltage, $k_B$ is the Boltzmann constant, $T$ is the absolute temperature and $q$ is the charge on electron. A typical numerical value of the thermal voltage is 25.85 mV at 300 K. According to Equation (2) a PN-junction diode current increases exponentially when a forward bias voltage $V$ is increased, while under a reverse bias the current saturates to $I_0$.

Taylor series expansion of Equation (2) can be written as

$$I = I_0\left(\frac{V}{\eta V_{th}} + \frac{1}{2! \eta^2 V_{th}^2} + \frac{1}{3! \eta^3 V_{th}^3} + \frac{1}{4! \eta^4 V_{th}^4} + \frac{1}{5! \eta^5 V_{th}^5} + \ldots \right)$$

(3)
Figure 1: A schematic showing an exponential dependence of current on the forward bias voltage across a $PN$-junction diode. A variation of diode current for a sinusoidal bias voltage of amplitude $V_o$ with a constant dc-offset voltage $V_{dc}$ is also shown.

Which can be expressed as

$$I = \chi_v^{(1)} V + \chi_v^{(2)} V^2 + \chi_v^{(3)} V^3 + \chi_v^{(4)} V^4 + \chi_v^{(5)} V^5 + \ldots$$

(4)

Where $\chi_v^{(1)} = \frac{I_0}{\eta V_{th}}$, $\chi_v^{(2)} = \frac{I_0}{2! \eta^2 V_{th}^2}$, $\chi_v^{(3)} = \frac{I_0}{3! \eta^3 V_{th}^3}$, and so on. Where a superscript denotes the order of nonlinearity. The first term of Equation 4 is linear and all the remaining terms represent a nonlinear dependence of current on voltage. Equation 4 is analogous to Equation 1 where the voltage is analogous to the electric field and the current is analogous to the electric polarization.

The voltage current characteristics of a $PN$-junction diode are qualitatively presented in Figure 1 where an exponential rise of current with the bias voltage is shown. A sinusoidal voltage of amplitude $V_o$ with a constant dc-offset voltage $V_{dc}$ is applied across the diode. Because of the nonlinear characteristics of the $PN$-junction diode the current is nonsinusoidal. For a positive dc-offset voltage such that $V_{dc} > V_o$, the diode is always forward biased. However, for $V_{dc} = 0$ the diode is reverse biased during the negative half cycle and consequently a half wave rectification occurs.

A circuit designed to study the nonlinearity of a $PN$-junction diode is shown in Figure 2. This circuit produces a voltage output which is linearly proportional to current passing through the diode. Input voltage of the circuit is applied parallel to the diode. Circuit consists of two 741 general purpose operational amplifiers and a $PN$-junction diode IN4007 which is a nonlinear element in the circuit. The first operational amplifier is configured in an inverting voltage adder configuration whose output voltage is a summation of the input voltages with reversed polarity $-(V_s + V_{dc})$. The output of the voltage adder is connected to the N-type terminal of a $PN$-junction diode (IN4007) while the P-type terminal is connected to the inverting terminal of a second operational amplifier. A typical numerical value of reverse saturation current $I_o$ of the $PN$-junction diode-IN4007 is 5.0 $\mu$A. Since an inverting terminal of the second operational amplifier is biased at the virtual
ground potential, therefore, the PN-junction diode becomes forward biased if the output of the voltage adder is negative. The second operational amplifier is operating in an inverting current-to-voltage converter configuration and its output voltage is $V_t = R_F I$, where $I$ is the current flowing through the PN-junction diode. Therefore, the output voltage of the current-to-voltage converter is $V_t = R_F I_0(e^{(V_s + V_{dc})/V_{th}} - 1)$. Final output voltage $V_t$ of the circuit and voltage across the PN-Junction diode (potential at the N-type terminal where the P-type terminal is connected to a virtual ground) is shown in Figure 3, where $V_{dc} = 360\text{mV}$ and $V_o = 180\text{mV}$.

Nonlinear response of the circuit is clearly shown in the upper plot of Figure 3 where the current passing through the diode (converted to voltage by an operational amplifier) is non sinusoidal for a sinusoidal input voltage. The output voltage waveform is non-sinusoidal therefore, it shows the presence of more than one frequency components.

4 Phase Matching

In nonlinear wave mixing processes, conditions of energy and momentum conservation should be fulfilled. The first condition implies that the total energy of interacting fields must be same before and after the nonlinear interaction. If all the waves at different spatial points at an instant of time in an extended medium are in phase with each other then a constructive interference occurs. This results in a high efficiency of harmonic generation. Any deviation from an exact phase matching results in a decrement of efficiency of harmonic generation. A phase matching condition implies conservation of momentum in an extended nonlinear medium. In case of a PN-junction diode, the nonlinearity is not extended however, its source of origin is localized at the junction. For a typical frequency of oscillating voltage waveform the wavelength of oscillating voltage waveform in circuit is much larger than the extension of the circuit. A nonlinear PN-junction is equivalent to a single nonlinear dipole and in this case the condition of phase matching implies resonance of driving field with the oscillator modes. In case of an extended nonlinear medium the nonlinear-
Figure 3: A plot showing an applied sinusoidal voltage across a $PN$-junction diode (lower plot). Output voltage $V_t$ (upper plot) is linearly proportional to current passing through the diode. It is clearly shown that a sinusoidal input voltage gives rise to a non-sinusoidal output voltage due to the intrinsic nonlinearity of the diode.

### 5 Emergence of Nonlinear Regime

We now move to experiment to demonstrate a generation of new frequencies from the nonlinear response of the $PN$-junction diode. The output voltage $V_t$ of the circuit shown in Figure. 2 can be written as

$$V_t = R_F(\chi_v^{(1)} V + \chi_v^{(2)} V^2 + \chi_v^{(3)} V^3 + \chi_v^{(4)} V^4 + \chi_v^{(5)} V^5 + \ldots)$$

(5)

Where a voltage applied at input of circuit is $V = V_{dc} + V_s$ and $V_s = V_o \cos(\omega_o t)$ is a sinusoidal voltage corresponding to a fundamental harmonic of angular frequency $\omega_o$. For a low input voltage $V_o + V_{dc}$ the linear term is the only significant term, therefore, the output voltage $V_t$ is $R_F \chi_v^{(1)} (V_{dc} + V_o \cos(\omega_o t))$, which has the same frequency spectrum as the spectrum of the input waveform, namely the presence of a single frequency $\omega_o$. If the amplitude of the input sinusoidal voltage $V_o$ is increased such that only the second order term of Equation. 5 becomes significant and all the remaining nonlinear terms are negligible then the total output voltage due to the first two terms of Equation. 5 can be written as

$$V_t = R_F \left( \chi_v^{(1)} V_{dc} + \chi_v^{(2)} V_{dc}^2 + \frac{V_o^2}{2} \right) +$$

$$\left( \chi_v^{(1)} + 2 \chi_v^{(2)} V_{dc} \right) V_o \cos(\omega_o t) +$$

$$\chi_v^{(2)} \frac{V_o^2}{2} \cos(2 \omega_o t)$$

(6)

The voltage waveform $V_t$ which is proportional to the current passing through the $PN$-junction diode contains a new frequency component $(2\omega_o)$ of frequency twice the frequency $(\omega_o)$ of the input sinusoidal voltage $V_s$. The generation of new frequency component corresponds to the second harmonic generation. A further increase of amplitude $V_o$ reinforces the third order nonlinear term to be significant in addition to the...
second order term. The contribution to the output voltage $V_t$ due to the third term of Equation is written as

$$R_F \chi^{(3)}_o \left( V_{dc}^3 + \frac{3}{2} V_{dc} V_o^2 + 3 \left( V_{dc}^2 V_o + \frac{V_o^3}{4} \right) \cos(\omega_o t) \right) + \left( \frac{3}{2} V_{dc}^2 \cos(2\omega_o t) + \frac{V_o^3}{4} \cos(3\omega_o t) \right)$$ (7)

which has frequency components at zero, $\omega_o$, $2\omega_o$ and $3\omega_o$. The harmonics of frequency $3\omega_o$ corresponds to a third harmonic generation. The total output voltage $V_t$ is a summation of contribution from the first three terms of Equation. Similarly, the contribution from the fourth order nonlinear term of Equation to the output voltage $V_t$ is written as

$$R_F \chi^{(4)}_o \left( V_{dc}^4 + 3 V_{dc}^2 V_o^2 + \frac{3 V_o^4}{8} + \left( 4 V_{dc}^3 V_o + 3 V_{dc} V_o^3 \right) \cos(\omega_o t) + \left( 3 V_{dc}^2 V_o^2 + \frac{V_o^4}{2} \right) \cos(2\omega_o t) + V_{dc} V_o^3 \cos(3\omega_o t) + \frac{V_o^4}{8} \cos(4\omega_o t) \right)$$ (8)

which has frequency components at zero, $\omega_o$, $2\omega_o$, $3\omega_o$ and $4\omega_o$. The frequency component at $4\omega_o$ signifies the fourth harmonic generation. The total output voltage $V_t$ is again an addition of contributions from all the significant terms in Equation. A gradual emergence of a nonlinear regime is observed by generation of higher order harmonics as the amplitude of the sinusoidal input voltage is increased. The fundamental frequency of the input sinusoidal voltage waveform is chosen to be $f_o = \omega_o/2\pi = 5kHz$ and the offset voltage is $V_{dc} = 360$ mV in all the plots. Figure (a) corresponds to a linear regime, where the amplitude of higher harmonics is negligibly small. A plot in Figure (b) shows the emergence of the nonlinear regime, where the second order nonlinearity becomes significant, and a second harmonics at frequency $10 kHz$ is observed. Another plot in Figure (c) shows a regime where the nonlinearity up to the third order is pronounced as shown by a third harmonic generation at frequency $15 kHz$. A plot in Figure (d) shows a regime where the nonlinearity up to the fourth order is pronounced as shown by a fourth harmonic generation at frequency $20 kHz$. Therefore, higher orders of nonlinearity can be addressed by increasing voltage across a PN-junction diode.

5.1 Frequency comb generation

A frequency comb is a series of equally spaced discrete harmonics. In this experiment a frequency comb has been generated by utilizing the nonlinear response of a PN-junction diode. A sinusoidal input voltage with a nonzero voltage offset is applied at the input of the circuit as shown in Fig-
Figure 4: A series of plots showing a gradual emergence of a nonlinear regime of higher orders as the amplitude of the input voltage waveform is increased. The amplitude of the input sinusoidal voltage is shown on each spectrum. The dc-offset is constant and frequency of the input sinusoidal voltage is 5 kHz in all plots. Plot (a) shows a linear regime where no new frequency is generated. (b) Represents an emergence of the nonlinear regime where the second harmonic generation is observed. (c) A third order nonlinearity is also pronounced and frequency components up to the third harmonics are observed. (d) A fourth order nonlinearity is gradually pronounced and a frequency component of fourth harmonics is observed.
ure. The output voltage is observed in time domain on a digital oscilloscope. The amplitude of the input sinusoidal voltage of frequency 5 kHz is gradually increased until the output similar to the output voltage shown in Figure is observed. A frequency spectrum analyzer is connected to the output of the circuit to measure a frequency spectrum of the output voltage. For a large amplitude the higher order nonlinear terms are pronounced and a comb of frequencies with the highest frequency up to the twentieth harmonics of frequency 100 kHz is observed, where the frequency of input sinusoidal voltage is 5 kHz. An electrical power spectrum of the output voltage is shown in Figure, where the measured electrical power is indicated on a logarithmic scale (1 dBm = 10 log₁₀ P/1 mW and P is measured in milli-Watt (mW)). This is essential to keep the amplitudes corresponding to different frequency components within the scale margins. Frequency combs are useful in many areas of science and this demonstration introduces the idea in a simple system.

5.2 Sum and difference frequency generation

The sum and difference of frequencies of different sinusoidal voltage waveforms can be produced from the second order nonlinearity. Two independent sinusoidal voltage waveforms $V_{s1}$ and $V_{s2}$ with a constant dc voltage offset are applied at the inputs of the circuit shown in Figure. A second sinusoidal voltage waveform $V_{s2}$ is connected to an inverting terminal of the first operational amplifier through an additional 10 kΩ resistance (not shown in the circuit diagram). In this case the total input voltage is $V = V_{dc} + V_{s1} + V_{s2}$ where, $V_{s1} = V_{o1} \cos(\omega_1 t)$ and $V_{s2} = V_{o2} \cos(\omega_2 t)$. The output voltage is given by Equation. Consider the waveform amplitudes $V_{o1}$ and $V_{o2}$ are such that only first two terms of the Equation are significant, i.e., only the second order nonlinearity is significant. Therefore, the total

![Figure 5: A frequency spectrum of a frequency comb. A highest frequency 100 kHz corresponds to the twentieth harmonics and frequency of input voltage waveform is $f_o = 5$ kHz.](image)
Figure 6: A frequency spectrum of the output voltage waveform shows the sum \((f_1 + f_2)\) and difference \((f_1 - f_2)\) frequency generation. Second harmonics of frequencies 2\(f_1\) and 2\(f_2\) corresponding to each input voltage waveforms are also produced.

The output voltage of the circuit is written as

\[
V_t = R_F I_0 \left( \frac{\chi_v^{(2)}}{2} (V_{o1}^2 + V_{o2}^2 + 2V_{dc}^2) + \chi_v^{(1)} V_{dc} + (\chi_v^{(1)} V_{o1} + 2\chi_v^{(2)} V_{o1} V_{dc}) \cos(\omega_1 t) + \frac{\chi_v^{(2)}}{2} V_{o2}^2 \cos(2\omega_1 t) + \chi_v^{(2)} V_{o1} V_{o2} \cos(\omega_1 - \omega_2) t + \chi_v^{(2)} V_{o1} V_{o2} \cos(\omega_1 + \omega_2) t \right)
\]

\(9\)

The output voltage waveform consists of harmonics of frequencies equal to the sum \((\omega_1 + \omega_2)\) and the difference \((\omega_1 - \omega_2)\) of frequencies of the input voltage waveforms. In addition, there are frequency components at 2\(\omega_1\) and 2\(\omega_2\) which correspond to second harmonic generation corresponding to the individual voltage waveforms \(V_{o1}\) and \(V_{o2}\), respectively. A measured frequency spectrum of the output voltage waveform \(V_t\) is shown in Figure.\([6]\) for \(f_1 = \omega_1 / 2\pi = 5\ kHz\) and \(f_2 = \omega_2 / 2\pi = 3\ kHz\) where, \(V_{o1} = 21\ mV\) and \(V_{o2} = 26\ mV\) are chosen such that only the second order nonlinearity is significant. A measured spectrum explicitly shows the sum of frequencies \((f_1 + f_2 = 8\ kHz)\) and the difference of frequencies \((f_1 - f_2 = 2\ kHz)\). In addition, the measured spectrum contains frequency components of second harmonic generation at 2\(f_1 = 10\ kHz\) and 2\(f_2 = 6\ kHz\) corresponding to the individual voltage waveforms \(V_{o1}\) and \(V_{o2}\), respectively.

6 Conclusions

In summary, we have explored the nonlinearity of voltage current characteristics of a \(PN\)-junction diode and experimentally demonstrated various concepts related to nonlinear physics in particular nonlinear optics. Taylor expansion of the voltage current characteristics of a \(PN\)-junction diode resembles the nonlinear dependence of electric polarization of a dielectric medium on the electric field which allows an analogy between the two systems.

A gradual emergence of the nonlin-
ear regime of successively increasing orders has been shown experimentally. A frequency comb with the highest frequency up to the twentieth harmonics has also been produced. In addition, an experiment to create the sum and difference of frequencies of two independent sinusoidal voltage waveforms is presented. It may also be possible to observe certain other aspects of nonlinear physics using the PN-junction diode experiment and these aspects will be presented elsewhere.

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