Calculation and assessment of measures of the residual operating life of non-restorable items

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Abstract. The widespread use of cutting-edge components in the composition of non-restorable items has made it difficult to assess their reliability at the stages of designing and production. Therefore, items with underestimated values of the assigned (warranted) operating life are put into operation. By now, the service life of such items has reached the assigned limits, the items still maintaining a fairly high dependability rate. Thus, there has been a problem of extending the service life beyond the originally assigned rates. As originally assigned rates are determined by such measures as the mean operating life and gamma-percentile operating life, the non-restorable item's life margin is determined as related to these rates by means of measures of the residual operating life, namely the mean residual operating life and gamma-percentile residual operating life. The mean residual operating life and gamma-percentile residual operating life are determined in this paper, to calculate the extendable service life. For these measures, we have derived calculation formulas and assessments which make it possible to determine the extended service life. In addition, for these measures, calculation formulas and assessments are obtained for non-restorable items whose margin allocation is distributed exponentially and uniformly. The paper examines the issues of attainability of the obtained assessments and the conditions of their attainability; also, the influence of failure rate monotonic change on the assessments is analyzed.

1. Mean residual operating life
Currently, there are different interpretations of the term "operating life", which is specially discussed in papers [1 – 4]. In this paper, according to GOST 27.002-2015, the term 'operating life' is only meant as the operating time of the item since the beginning of its operation to its transition to the limit state. Henceforth, the non-restorable item under study is assumed as operating without interruptions, i.e., its operating time is continuous.

Under the residual operating life over time \( \tau \) is assumed here as the item's operating time from the moment of time \( \tau \) until its transition to the limit state, under the established application modes and operating conditions. While solving the task of extending the service life in practice, the value \( \tau \) is usually understood as the assigned operating life [5].

If \( \zeta \) is the operating time of the item from the beginning of operation to its transition to the limit state, then the residual operating life \( \zeta_\tau \) after the time \( \tau \) is determined by formula

\[
\zeta_\tau = (\zeta - \tau)/(\zeta > \tau),
\]

here and further on, the slash will denote the condition for the event enclosed within the first parenthesis.
The mean residual operating life over time $\tau$ is assumed to be the value $R(\tau)$ which is determined by the formula [6]

$$R(\tau) = E(\zeta_{\tau}),$$

where $E(\cdot)$ is the mathematical expectation of a random variable which is inside the parentheses.

The measure $R(\tau)$ should not be confused with the mean residual operating life, which is equal to $r - \tau$, where $r$ is the mean (non-residual) operating life, since the value (1) and the value $\eta_r = \zeta - \tau$ are related as follows:

$$\zeta_{\tau} \geq \eta_r.$$  

Hence, taking into account (2), we obtain

$$R(\tau) \geq r - \tau,$$

where $r = E(\zeta)$ is the mean (non-residual) operating life.

For measure (2), the following formula is valid [8]:

$$R(\tau) = \frac{1}{P(\tau)} \int_0^\tau P(t)dt,$$  

where $P(\cdot)$ is the probability of failure during the time specified inside the parentheses.

For example, for the exponential law of item operating life allocation:

$$P(t) = e^{-\lambda t},$$  

where $\lambda > 0$ is constant value, the measure calculated by the formula (4), is equal to

$$R(\tau) = r,$$

here $r = 1/\lambda$ is mean residual operating life.

In the case of the item’s operating life being of a limited time interval $(0,l)$, the formula (4) uns as follows:

$$R(\tau) = \frac{1}{P(\tau)} \int_\zeta^\tau P(t)dt.$$  

So, for an item whose operating life allocation is uniformly distributed on the segment $(0,l)$:

$$P(t) = 1 - \frac{t}{l},$$  

according to the formula (7), we obtain

$$R(\tau) = \frac{1}{2}(l - \tau).$$

Since the mean operating life of the item $r$ is equal to [8]:

$$r = \int_0^l P(t)dt,$$
then, according to (4), we have

\[ R(0) = r . \quad (10) \]

For example, for the law (8), from (9) we obtain \( r = \frac{1}{2} . \)

In the tasks of extending the items' service life, information about the originally assigned values of dependability measure is important as a starting point of the residual operating life. Proceeding from this, let us prove the following.

Theorem 1. For the mean residual operating life of a non-restorable item over time \( \tau , \) the following formula is valid:

\[ R(\tau) = r - \tau + \int_0^\tau \lambda(u)R(u)du, \quad (11) \]

where \( \lambda(u) \) is the failure rate of the item.

Proof. While differentiating (4), we obtain

\[ R'(\tau) = \frac{1}{P^2(\tau)} \left[ -P^2(\tau) - P'(\tau) \int_\tau^\infty P(t)dt \right]. \]

Hence we have

\[ R'(\tau) = \lambda(u)R(u) - 1, \quad (12) \]

since [8]

\[ \lambda(u) = \frac{-P'(u)}{P(u)} \quad (13) \]

is the failure rate of the item.

While integrating the expression (12) in the range from zero to \( \tau , \) and taking into account (10), we get the required formula (11).

The meaning of formula (11) is that it determines the presence of the item's mean operating life margin at the time \( \tau \) with regard to the initially assigned rate \( r \) (or its absence). For example, for the allocation law of the item's operating life (5), there is not such a margin, since

\[ \lambda(u) \equiv \lambda , \quad R(u) = 1/\lambda \]

and then, taking into account (6), we have

\[ R(\tau) - r \equiv 0, \quad (\tau > 0). \]

Let us note that when

\[ \lambda(u) \equiv 0, \]

where \( u \in (0, \tau) , \) it follows from formula (11) that

\[ R(\tau) = r - \tau , \]

that is, the mean residual operating life is equal to the residual mean operating life, equal to \( r - \tau \). Hence, the assessment (3) is achievable.
2. Gamma-percentile residual operating life

For some items, high rates of failure-free operating are required for the extendable period of operation. In this case, the extendable period has to be determined depending on the specified rate of failure-free operating. It is obvious that the extendable service life, determined on the basis of the calculation and assessment of the mean residual operating life, does not have such properties. Let us determine one more measure of the residual operating life that has such properties [9].

Let the value be set $\gamma$, $(0 < \gamma < 1)$. Gamma-percentile residual operating life over time $\tau$ is assumed to be the time $T_\gamma(\tau)$ that is determined by the formula [9–14]:

$$
T_\gamma(\tau) = \max\left\{ t \big/ P_t(t) \geq \gamma \right\},
$$

(14)

where

$$
P_t(t) = \frac{P(\tau + t)}{P(\tau)},
$$

(15)

here $P(\cdot)$ is the probability of failure-free operation of the item during the time specified inside the parentheses.

Sometimes the level $\gamma$ is set not in fractions of a unit, but as a percentage (hence the name of this measure).

In other words, over a period of time $(\tau, \tau + T_\gamma(\tau))$, the conditional probability of the item's failure-free operation must be no less than the specified rate $\gamma$.

From the definition of measure (14), it follows that

$$
T_\gamma(\tau) > 0.
$$

The traditional measure "gamma-percentile (non-residual) operating life" $t_\gamma$ is the initially assigned level, since

$$
T_\gamma(0) = t_\gamma.
$$

(16)

If the probability of item's failure-free operation during time $t$ is a continuous and monotonically decreasing function, then the measure of operating life (14) can be determined from the equation

$$
P_t(t) = \gamma,
$$

(17)

as a solution with respect to time $t$, where the left part is defined by the formula (15).

For example, for a uniform distribution law (8), while solving equation (17), we find

$$
T_\gamma(\tau) = (1 - \tau)(1 - \gamma).
$$

It can be seen that

$$
T_\gamma(\tau) < t_\gamma = l(1 - \gamma).
$$

To find a solution for extending the item's service life, it is necessary to determine the operating life margin at the time $\tau$ with regard to the initial rate. Thereby, let us prove the following statement.

Theorem 2. For the gamma-percentile residual operating life of a non-restorable item over time $\tau$ the following formula is valid:
\[ T_y(\tau) = t_y - \tau + \int_0^\tau \frac{\lambda(u)du}{\lambda(u + T_y(u))}, \]  

(18)

where \( \lambda(\cdot) \) is the item’s failure rate at the time specified inside the parentheses.

Proof. Since, according to (17)

\[ P_\gamma(T_y(\tau)) = \gamma, \]  

(19)

then

\[ P(\tau + T_y(\tau)) = \gamma P(\tau). \]

Taking the derivative of both parts, we obtain

\[ P'(\tau + T_y(\tau)) \left( 1 + \frac{\partial T_y(\tau)}{\partial \tau} \right) = \gamma P'(\tau). \]

Hence we find

\[ 1 + \frac{\partial T_y(\tau)}{\partial \tau} = \gamma \frac{P'(\tau)}{P(\tau + T_y(\tau))}. \]

Since, according to (19)

\[ \gamma = \frac{P(\tau + T_y(\tau))}{P(\tau)}, \]

then we have

\[ 1 + \frac{\partial T_y(\tau)}{\partial \tau} = \frac{P'(\tau) P(\tau + T_y(\tau))}{P(\tau) P'(\tau + T_y(\tau))}. \]

Thereafter, by use of the formula (13), we obtain

\[ \frac{\partial T_y(u)}{\partial u} = \frac{\lambda(u)}{\lambda(u + T_y(u))} - 1. \]

While integrating the obtained expression in the range from zero to \( \tau \) and taking into consideration (16), we find the required formula (18), which proves the theorem.

The meaning of formula (18) is that it determines the presence or absence of the operating life margin at a time \( \tau \) using the value of the measure \( T_y(\tau) \) with regard to the initially assigned rate \( t_y \). For example, for items whose failure rate function is monotonically increasing, we have, from (18) at \( \tau > 0 \)

\[ T_y(\tau) < t_y, \]

that is, there is no margin of gamma-percentile residual operating life with regard to the originally assigned rate \( \gamma \). If the item’s failure rate function decreases monotonically, then it follows from (18) that when \( \tau > 0 \)

\[ T_y(\tau) > t_y, \]
that is, in this case, the item has a margin of gamma-percentile residual operating life at a time $\tau$ with regard to the originally assigned rate $t_\gamma$. Finally, for $\lambda(u) = \lambda$, where $\lambda > 0$ is a constant, from formula (18) we find

$$T_\gamma(\tau) = t_\gamma,$$

hence it follows that such an item does not have a margin of gamma-percentile residual operating life at a time $\tau$ with regard to the rate $t_\gamma$.

3. Conclusion
The calculation formulas and assessment of the residual operating life measures of non-restorable items are proved that make it possible to determine the extendable service life.

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