Statistical properties of physical-like measures

Shaobo Gan¹, Fan Yang²,∗∗, Jiagang Yang³ and Rusong Zheng⁴

¹ School of Mathematical Sciences, Peking University, Beijing 100871, People’s Republic of China
² Department of Mathematics, Michigan State University, East Lansing, MI, United States of America
³ Departamento de Geometria, Instituto de Matemática e Estatística, Universidade Federal Fluminense, Niterói, Brazil
⁴ Southern University of Science and Technology, Shenzhen 518055, People’s Republic of China

E-mail: gansb@pku.edu.cn, yangfa31@msu.edu, yangjg@impa.br and zhengrs@sustech.edu.cn

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Abstract
In this paper we consider the semi-continuity of the physical-like measures for diffeomorphisms with dominated splittings. We prove that any weak-* limit of physical-like measures along a sequence of $C^1$ diffeomorphisms $\{f_n\}$ must be a Gibbs $F$-state for the limiting map $f$. As a consequence, we establish the statistical stability for the $C^1$ perturbation of the time-one map of three-dimensional Lorenz attractors, and the continuity of the physical measure for the diffeomorphisms constructed by Bonatti and Viana.

Keywords: physical-like measures, Gibbs $F$-states, dominated splitting, statistical stability
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1. Introduction

Let $f : M \to M$ be a diffeomorphism on a compact Riemannian manifold $M$. An $f$-invariant probability measure $\mu$ is a physical measure if the set of points $x \in M$ for which the empirical measures $\delta_{f^n}^{\mu}$ satisfy

$$\delta_{f^n}^{\mu} := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^{i+1}} \to \mu \text{ (in the weak-* sense)} \quad (1)$$

has positive volume. This set is called the basin of $\mu$ and is denoted by $B(\mu)$. Following the pioneer work of Catsigeras and Enrich [13], we say that an invariant probability $\mu$ is physical-like if: for any small neighborhood $U$ of $\mu$ inside the space of probabilities $P(M)$ (not necessarily invariant under $f$) with respect to the weak-* topology, the set

$$\{ x \in M : \text{there are infinitely many } n, k \in \mathbb{N} \text{ such that } \delta_{k}^{f^{n}} \in U \}$$

has positive volume. The set of physical-like measures of $f$ is denoted by $\text{PhL}(f)$. Even though $f$ may not have any physical measure, $\text{PhL}(f)$ is always nonempty, and any physical measure is physical-like (see [13]).

In this paper, we investigate the properties of physical-like measures under the setting of diffeomorphisms with dominated splitting. More precisely, we assume that there exists a splitting $TM = E \oplus F$ of the tangent bundle that is invariant under the tangent map $Df$, and satisfies

$$\| (Df|_{F(x)})^{-1} \| \| Df|_{E(x)} \| < 1 \text{ at every } x \in M. \quad (2)$$

In other words, the bundle $F$ dominates the bundle $E$.

A program for investigating the physical measures of partially hyperbolic diffeomorphisms or diffeomorphisms with dominated splitting was initiated by Alves et al in [1, 9]. Since then, there have been great progress in many directions, including the existence, finiteness and statistical stability of physical measures for diffeomorphisms with mostly contracting center [19, 20] and with mostly expanding centers [2–4, 36, 40], and the relation between physical measures and the partial entropy along the unstable foliations [18, 23, 39], to name but a few.

All these works rely on the Pesin theory, in particular, the absolute continuity of the Pesin stable lamination. As a result, $f$ has been assumed to be at least $C^{1+\alpha}$.

Below we will introduce a different method to describe the physical and physical-like measures, which focuses more on the Ruelle’s inequality and Pesin’s entropy formula, and works even for $C^1$ systems.

**Definition 1.** We say that a probability measure $\mu$ is a Gibbs $F$-state of $f$ if

$$h_{\mu}(f) \geq \int \log | \det(Df|_{F(x)})| \, d\mu(x), \quad (3)$$

where $h_{\mu}(f)$ is the measure-theoretic entropy of $\mu$. We denote the space of Gibbs $F$-states by $\text{Gibbs}^F(f)$.

When all the Lyapunov exponents of $\mu$ along the $F$ bundle at almost every point are positive, and when the other exponents are non-positive, then combined with Ruelles’s inequality [33], it is easy to see that the previous inequality is indeed an equality, which is known as Pesin’s entropy formula.
The relation between physical-like measures and Gibbs $F$-states is established by Catsigeras, Cerminara, and Enrichin [14]:

**Proposition 1.1.** Let $f$ be a $C^1$ diffeomorphism on a compact manifold $M$ which admits a dominated splitting $TM = E \oplus F$. Then there exists a full volume subset $\Gamma \subset M$ such that, for any $x \in \Gamma$, any limit point $\mu$ of the sequence $\{\delta_{f^n}\}$ belongs to $\text{Gibbs}^F(f)$. Moreover, we have $\text{PhL}(f) \subset \text{Gibbs}^F(f)$.

In particular, their result shows that $\text{Gibbs}^F(f)$ is always non-empty; furthermore, if $f$ admits a unique Gibbs $F$-state $\mu$, then $\mu$ is a physical measure whose basin has full volume.

The relation between physical-like measures and the inequality (3) was first discovered by Keller [24, theorem 6.1.8] for one-dimensional $C^1$ map with Markov partition. Campbell and Quas used Keller’s result in [15] to show that every $C^1$ generic circle expanding map admits a unique physical measure, whose basin has full volume. The key point of Campbell and Quas’ proof is to show that generic expanding map admits a unique Gibbs $F$-state. Later, Qiu ([32]) built the same result for uniformly hyperbolic attractors, and proved that $C^1$ generic hyperbolic attractor admits a unique physical measure.

Despite that all the works mentioned above are under $C^1$ generic context, we would like to point out that proposition 1.1 works beyond $C^1$ generic setting. It can be applied to a given diffeomorphism with higher regularity ($C^{1+\alpha}$) where it is used in combination with Pesin’s theory. Examples of such applications include partially hyperbolic diffeomorphisms with mostly contracting center or with mostly expanding center, see for instance [22, 23, 39, 40].

The table below summarizes the various properties of $\text{PhL}(f)$ and $\text{Gibbs}^F(f)$. For the precise statements and proofs, see proposition 2.1 in the next section.

|                      | $\text{PhL}(f)$ | $\text{Gibbs}^F(f)$ |
|----------------------|-----------------|---------------------|
| **Existence**        | True            | True                |
| **Convexity**        | False           | True                |
| **Compactness**      | True            | If $h(f)$ is upper semi-continuous |
| **Semi-continuity**  | Theorem A below | If $h_\mu(\cdot)$ is upper semi-continuous |

Here note that the compactness and semi-continuity of $\text{Gibbs}^F(f)$ largely depend on the continuity of the metric entropy as a function of the invariant measure and of the diffeomorphism. Examples of such continuity include $C^\infty$ maps (by Buzzi [10] and Yomdin [41]), diffeomorphisms away from tangencies (by Liao et al [26]), time-one map of Lorenz-like flows (by Pacifico et al [30]) and many more. However, there are also many counterexamples where the metric entropy function fails to be upper semi-continuous (see [11, 28, 29]).

In this paper, we are going to reveal further connections between the two spaces of measures, under the context of $C^1$ perturbation theory. More precisely, we prove:

**Theorem A.** Suppose that $f$ is a $C^1$ diffeomorphism which admits a dominated splitting $E \oplus F$, and $\{f_n\}$ is a sequence of diffeomorphisms converging to $f$ in the $C^1$ topology. Then any weak-* limit of any sequence of physical-like measures $\mu_n$ of $f_n$ is a Gibbs $F$-state of $f$.

As a result, if a diffeomorphism $f$ admits a dominated splitting and has a unique Gibbs $F$-state $\mu$, then for any $C^1$ nearby diffeomorphism $g$, and for any point $x$ in a full volume
subset, any weak-* limit of the empirical measures \( \{ \delta_{g^n(x)} \} \) must be close to \( \mu \). In particular, any physical measure of \( g \) (when exists) must be close to \( \mu \). This means that the existence of a unique Gibbs \( F \)-state is an intrinsic statistically stable property. It is worth noting that we do not require any extra hypothesis such as \( h \)-expansiveness.

As an application of theorem A, we will show in section 4 that the time-one map of three-dimensional singular hyperbolic attractors and the example of Bonatti and Viana on \( T^4 \) are statistically stable. More precisely, we will show that for the diffeomorphism of Bonatti and Viana, the unique physical measure varies continuously in \( C^1 \) and weak-* topology. We would like to remark that this result could also be obtained from the general criterion of [34, theorem E] combined with the careful study of the entropy structure for the Bonatti–Viana maps in [17], which already shows that the unique physical measure is “almost expansive” (for the precise definition, see [17, definition 2.3]). However, the method in this paper does not rely on this fact.

Remark 1.2. Even though we assume that the dominated splitting is defined on the entire manifold \( M \), it is straightforward to check that the theorem A remains true when the dominated splitting is defined on a compact invariant set \( \Lambda \). In this case the dominated splitting on \( \Lambda \), together with the invariant cones, can be extended to a small neighborhood of \( \Lambda \), see [8, appendix B] for more detail. Then theorem A can be applied to a sequence of physical-like measures supported in this neighborhood. This allows one to obtain the statistical stability for a large family of diffeomorphisms that support finitely many Gibbs \( F \)-states whose supports are mutually disjoint.

2. Properties of physical-like measures and Gibbs \( F \)-states

In this section we collect some properties of \( \text{PhL}(f) \) and \( \text{Gibbs}^F(f) \).

Proposition 2.1. Let \( f \) be a \( C^1 \) diffeomorphism with dominated splitting \( E \oplus F \), then

(a) \( \text{PhL}(f) \subset \text{Gibbs}^F(f) \).
(b) \( \text{PhL}(f) \) is non-empty and compact.
(c) \( \text{Gibbs}^F(f) \) is non-empty and convex.
(d) \( \text{Gibbs}^F(f) \) is compact if \( h_\mu(f) \) is upper semi-continuous w.r.t. \( \mu \).
(e) \( \text{Gibbs}^F(f) \) varies upper semi-continuously w.r.t. \( f \) if \( h_\mu(f) \) varies upper semi-continuously w.r.t. both \( f \) and \( \mu \). To be more precise, if \( f_n \to f \) (in, say, \( C^1 \) topology) and \( \mu_n \in \text{Gibbs}^F(f_n) \) with \( \mu_n \to \mu \) and satisfy

\[
\limsup_n h_{\mu_n}(f_n) \leq h_\mu(f),
\]

then \( \mu \in \text{Gibbs}^F(f) \).

Proof. (a) Follows from proposition 1.1.

For (b), the non-emptiness and compactness follows immediately from the definition of \( \text{PhL}(f) \) and the fact that \( \mathcal{P}(M) \) is compact. See also [13] for more detail.

(c) \( \text{Gibbs}^F(f) \) is non-empty because of (a) and (b). It is convex since \( h_\mu(f) \) is affine in \( \mu \). To prove (d), take \( \mu_n \in \text{Gibbs}^F(f) \) and assume that \( \mu_n \to \mu \) in weak-* topology. If \( h_\mu(f) \), as a function of \( \mu \), is upper semi-continuous, we obtain

\[
h_\mu(f) \geq \limsup_n h_{\mu_n}(f) \geq \limsup_n \int \log |\det(Df|_{F(x)})|d\mu_n(x) = \int \log |\det(Df|_{F(x)})|d\mu(x).
\]
So $\mu \in \text{Gibbs}^f(f)$.

For (e) the proof is similar to (d) and omitted. Note that dominated splitting is persistent under $C^1$ topology: if $f_n \to f$ then $f_n$ has dominated splitting $E_n \oplus F_n$ such that $E_n \to E$ and $F_n \to F$ in the Grassmannian.

It is also worthwhile to note that $\text{PhL}(f)$ may not be convex, and there exist examples for which $\text{PhL}(f) \subsetneq \text{Gibbs}^F(f)$. To see such an example, consider a uniformly hyperbolic diffeomorphism $f$ with two disjoint transitive attractors $\Lambda_1$ and $\Lambda_2$, each of which supports an ergodic physical measure $\mu_i$, $i = 1, 2$. Then $\text{PhL}(f) = \{\mu_1, \mu_2\}$ is not convex. Meanwhile, $\text{Gibbs}^F(f) = \{a\mu_1 + (1-a)\mu_2 : a \in [0, 1]\}$, so $\text{PhL}(f) \subsetneq \text{Gibbs}^F(f)$. See also [13] and the discussion following [14, corollary 2].

Finally we would like to point out that neither $\text{PhL}(f)$ nor $\text{Gibbs}^F(f)$ behave well under ergodic decomposition. There are examples (see [13]) such that $\text{PhL}(f)$ consists of a single measure which is not ergodic. Meanwhile it is easy to construct examples such that typical ergodic components of a measure $\mu \in \text{Gibbs}^F(f)$ are no longer in $\text{Gibbs}^F(f)$.

3. Proof of the main theorem

In this section we prove theorem A. From now on, $\{f_n\}$ is a sequence of $C^1$ diffeomorphisms with $f_n \to f$. For convenience we will write $f_0 = f$. Denote by $E_n \oplus F_n$ the dominated splitting for $f_n$. We will take $\mu_n \in \text{PhL}(f_n)$ with $\mu_n \rightharpoonup^\star \mu$. Then $\mu$ is an invariant probability of $f$.

Let us briefly explain the structure of the proof. Proving by contradiction, we will assume that the limiting measure $\mu$ is not a $\text{Gibbs} F$-state. As a result, the metric pressure of $\mu$:

$$P_\mu(f) := h_\mu(f) - \int \log |\det(Df|_{F(x)})|d\mu(x)$$

must be negative. Then, for a proper finite partition $\mathcal{A}$ the metric pressure of $f_n$ with respect to the $n$th join of $\mathcal{A}$ (and note that such join depends on the map $f_n$) is negative, uniformly in $n$: there exists $b > 0$, $N > 0$ such that for all $n$ large enough:

$$\frac{1}{N} H_{\mu_n} \left( \bigvee_{i=0}^{N-1} f_n^{-i}(\mathcal{A}) \right) - \int \log |\det(Df_n|_{F_n(x,i)})|d\mu_n < -b < 0.$$ 

This step is carried out in sections 3.1 and 3.2.

From here the proof largely follows the idea of the variational principle [37]. We will consider the following good set:

$$G_m^n = \{x : \delta_{f^{n-m}} \in U_n\},$$

where $U_n$ is a small neighborhood of $\mu_n$ in the space of probability measures. Using the pressure estimate above, we will show in section 3.3 that the volume of $G_m^n$, when restricted to any disk tangent to local $F_n$-cone with dimension equal to $\dim F$, is of order $e^{-bm}$; furthermore, this estimate can be made uniform in $n$. Then it follows that for Lebesgue almost every point, the empirical measures $\delta_{f^{n-m}}$ can only be in $U_n$ for finitely many $m$’s, contradicting with the choice of $\mu_n \in \text{PhL}(f_n)$. 

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To this end, we will assume from now on that $\mu \notin \text{Gibbs}^F(f)$. To simplify notation we write

$$\phi_f^\mu(x) = -\log |\det(Df|_{F(x)})|,$$

then there exists $a > 0$ such that

$$h_\mu(f) + \int \phi_f^\mu(x) d\mu(x) \leq -a < 0. \quad (4)$$

Also note that the definition of the function $\phi_f^\mu(x)$ can be extended to any subspace $\bar{F}(x) \subset T_xM$.

### 3.1. A $C^1$ neighborhood of $f$

We denote by $d^G$ the distance in the Grassmannian manifold. By the continuity of $\phi_f^\mu$ and the compactness of the Grassmannian, we can choose a $C^1$ neighborhood $U$ of $f$ and $\delta_0 > 0$ small enough, with the following property:

For any $g \in U$, $x, y \in M$ with $d(x, y) < \delta_0$, and any subspaces $\bar{F}(x) \subset T_xM, \bar{F}(y) \subset T_yM$ with $\dim \bar{F}(x) = \dim \bar{F}(y) = \dim F_0$ and $d^G(F_g(x), \bar{F}(x)) < \delta_0, \quad d^G(F_g(y), \bar{F}(y)) < \delta_0,$

where $E_g \oplus F_g$ is the dominated splitting of $g$ which is the continuation of $E \oplus F$, one has

$$|\phi_{\bar{F}}^\mu_g(x) - \phi_{\bar{F}}^\mu_g(y)| < \frac{a}{1000}. \quad (5)$$

We further assume that $\delta_0$ is small enough, such that for every point $x \in M$ the exponential map $\exp_x : T_xM \rightarrow M$ sends the $\delta_0$-ball $B_{\delta_0}(0_x) \subset T_xM$ diffeomorphically onto its image.

For any $\delta > 0$, we denote by $C_\delta(f_g(x)) \subset T_xM$ the $\delta$-cone around $F_g(x)$:

$$C_\delta(F_g(x)) = \{ v \in T_xM : |v_{E_g}| \leq \delta |v_{F_g}| \text{ for } v = v_{E_g} + v_{F_g} \in E_g \oplus F_g \}.$$

By the dominated assumption, the cone field $C_\delta(f_g)$ is invariant under the iteration of $Dg$, i.e., there is $0 < \lambda < 1$ independent of $\delta$ such that

$$Dg(C_\delta(F_g(x))) \subset C_{\lambda \delta}(F_g(g(x))).$$

For $\delta_0$ satisfying (5) above, we will refer to

$$C_{\delta_0}(F_g(x)) = \exp_x \left( C_{\delta_0}(F_g(x)) \cap B_{\delta_0}(0_x) \right)$$

as the local $F_g$ cone on the underlying manifold $M$. Note that the local $F_g$ cones are invariant in the following sense: there exists $\delta_1 \in (0, \delta_0)$ small enough such that for all $x \in M$ one has

$$g \left( C_{\delta_0}(F_g(x)) \cap B_{\delta_1} \left( \frac{x}{\|x\|_1} \right) \right) \subset C_{\delta_0}(F_g(g(x))) \cap B_{\delta_1}(g(x)).$$

From now on, $\delta_0$ and $\delta_1$ will be fixed.

**Definition 2.** Given $g \in U$, an embedded submanifold $K \subset M$ is said to be **tangent to local $F_g$ cone**, if for any $x \in K$ one has
\[ K \subset C_{\delta_0}(F_g(x)) \cap B_{\delta_1}(x). \]

The following simple lemma is taken from [30, lemma 2.3].

**Lemma 3.1.** There is a constant \( L > 0 \) such that for any \( g \in \mathcal{U} \) and for every \( x \in M \) and any disk \( D \) tangent to local \( F_g \) cone, we have \( \text{vol}(D) < L \).

Writing

\[ B_{\delta_n}(x, g) := \{ y \in M : d(g^i(x), g^i(y)) < \delta, \quad i = 0, \ldots, n-1 \} \]

for the \((\delta, n)\)-Bowen ball around \( x \). By the contraction of the cone filed on the tangent space, it follows that

**Lemma 3.2.** If \( K \subset M \) is a disk tangent to local \( F_g \) cone with dimension \( \dim(f_g) \), then for any \( x \in K \) and \( n \geq 0 \),

\[ g^n \left( K \cap B_{\frac{\delta}{\|g\|_{C^1}}, \delta}(x, g) \right) \]

is still tangent to local \( F_g \) cone. Moreover,

\[ \text{vol}_{g^n(K)} \left( g^n \left( K \cap B_{\frac{\delta}{\|g\|_{C^1}}, \delta}(x, g) \right) \right) \leq L. \] (6)

**Proof.** The first part of the lemma follows from the forward invariance of the local cone and induction. The second part is a consequence of lemma 3.1. \( \square \)

From now on, we take \( \delta_2 = \frac{\delta}{\sup_{g \in \mathcal{U}} \|g\|_{C^1}} \) which will be the size of the Bowen balls and separated sets.

### 3.2. A finite partition

The goal of this section is to rewrite (4) in terms of the information entropy \( H_\nu \) of the finite join (under the iteration of the perturbed maps \( f_n \)) of a finite partition; here \( \nu \) is a probability measure (not necessarily invariant under \( f \) or \( f_n \)) that is close to some \( \mu_\nu \). To this end, let \( \delta_0 \) and \( \mathcal{U} \) be given in the previous section, and recall that \( \mu_\nu \to \mu \) in weak-* topology. We also write \( \mu_0 = \mu \). Fix \( \mathcal{A} \) a finite, measurable partition of \( M \) with \( \text{diam}(\mathcal{A}) < \delta_2 \), such that \( \mu_i(\partial A) = 0 \) for all \( i = 0, 1, \ldots \). The existence of such a partition follows from the fact that there are at most countable disjoint sets with positive \( \mu_i \) measure for each \( i \), thus for any point \( x \), there is a ball with radius \( r_x \) arbitrarily small such that the boundary of this ball has vanishing \( \mu_i \) measure for any \( i \). Each ball and its complement form a partition, we can take \( \mathcal{A} \) as the refinement of finitely many such partitions.

Moreover, we can take \( \mathcal{A} \) to be fine enough, such that:

\[ h_\mu(f, \mathcal{A}) + \int \phi_f^* \, d\mu < \frac{999}{1000} \sigma. \] (7)

\(^7\)We slightly abuse notation and use \( B_{\delta_0}(\cdot) \) both for balls in \( T_xM \) and in \( M \); one could easily tell the difference by looking at the center.
Then there is $N$ large enough, such that
\[
\frac{1}{N} H_\mu \left( \bigcup_{i=0}^{N-1} f^{-i}(A) \right) + \int \phi_{f^n} \, d\mu < -\frac{998}{1000} a. \quad (8)
\]

We would like to replace $\frac{1}{N} H_\mu \left( \bigcup_{i=0}^{N-1} f^{-i}(A) \right)$ by $\frac{1}{N} H_\mu \left( \bigvee_{i=0}^{N-1} f^{-i}(A) \right)$. This creates an extra difficulty since the partition in question depends on $f_n$. To solve this issue, we introduce the following lemma, whose proof is standard in the measure theory and is thus omitted.

**Lemma 3.3.** Let $\mu$, $\mu_i$, $i = 1, 2, \ldots$ be probability measures such that $\mu_i \xrightarrow{\text{weak}^*} \mu$. Let $A_i, i = 1, 2, \ldots$ be a sequence of measurable sets with the following properties:

(a) $\tilde{A} := \text{int}(A)$, $\tilde{A}_n := \text{int}(A_n)$ satisfy that for every compact set $K \subset \tilde{A}$, there exists $N_K > 0$ such that $K \subset \tilde{A}_n$ for all $n > N_K$;

(b) the above property holds with $A$ and $A_n$ replaced by $A'$ and $A'_n$;

(c) $\mu(\partial A) = \mu_i(\partial A_i) = 0$, $i = 1, 2, \ldots$

Then we have $\lim_{n \to \infty} \mu_i(A_n) = \mu(A)$.

As an immediate application, we have:

**Lemma 3.4.** For $N > 0$ fixed, we have
\[
\lim_{n \to \infty} \frac{1}{N} H_{\mu_n} \left( \bigvee_{i=0}^{N-1} f^{-i}(A) \right) = \frac{1}{N} H_\mu \left( \bigvee_{i=0}^{N-1} f^{-i}(A) \right).
\]

**Proof.** Elements of $\bigvee_{i=0}^{N-1} f^{-i}(A)$ have the form:
\[
A = \bigcap_{i=0}^{N-1} f^{-i}(B_i)
\]
for some sequence $\{B_i \in A\}_{i=0}^{N-1}$. Given such a sequence, we denote by
\[
A_n = \bigcap_{i=0}^{N-1} f^{-i}(B_i).
\]
Since $f_n$ converges to $f$ in $C^1$ topology and elements of $\mathcal{A}$ are finite intersections of open balls $B_{r_{\mathcal{A}}}(x_k)$ and their complements, (a) and (b) of Lemma 3.3 are satisfied by the sets $A$ and $A_n$. For (c) of Lemma 3.3, observe that $\mu_n(\partial A) = 0$ implies that $\mu_n \left( \bigcup_{i=0}^{N-1} f^{-i}(\partial A) \right) = 0$ for all $n$, and the same holds for $\mu$ and $f$. It follows that $\mu(\partial A) = \mu_n(\partial A_n) = 0$.

Now we can apply the previous lemma to get $\lim_n \mu_n(A_n) = \mu(A)$. In particular,
\[
-\mu_n(A_n) \log \mu_n(A_n) \to -\mu(A) \log \mu(A).
\]
Summing over all elements of $\bigvee_{i=0}^{N-1} f^{-i}(A)$ (and keep in mind that this is a finite partition for fixed $N$) and divide by $N$, we obtain the desired result. \[\square\]

Combine Lemma 3.4 with (8) and use the continuity of $\phi_{f^n}$, we conclude that there exists $N_1 \in \mathbb{N}$ such that
\[
\frac{1}{N} H_{\mu_n} \left( \bigvee_{i=0}^{N-1} f^{-i}(A) \right) + \int \phi_{f^n} \, d\mu_n < -\frac{996}{1000} a, \quad (9)
\]
for all \( n > N_1 \).

Note that for each \( n \) and \( N \), the set \( \bigcup_{i=0}^{N-1} f_n^{-i}(\partial A) \) is closed, so its measure varies upper semi-continuously with respect to probability measures (not necessarily invariant by any of \( f_n \)). Fix \( \varepsilon > 0 \) small enough. For each \( n > N_1 \), we can take a small convex neighborhood \( U_n \subset \mathcal{P}(M) \) of \( \mu_n \) such that for any \( \nu \in U_n \),

\[
\nu \left( \bigcup_{i=0}^{N-1} f_n^{-i}(\partial A) \right) < \varepsilon, \text{ and consequently }
\]

\[
\frac{1}{N} H_\nu \left( \bigcap_{i=0}^{N-1} f_n^{-i}(A) \right) \leq \frac{1}{N} H_{\mu_n} \left( \bigcap_{i=0}^{N-1} f_n^{-i}(A) \right) + \frac{1}{1000} a. \tag{10}
\]

By the continuity of \( \phi_{fn}^p \), we can shrink \( \varepsilon \) and finally obtain

\[
\frac{1}{N} H_\nu \left( \bigcap_{i=0}^{N-1} f_n^{-i}(A) \right) + \int \phi_{fn}^p \, d\nu < \frac{-994}{1000} a, \text{ for all } \nu \in U_n. \tag{11}
\]

### 3.3. From pressure to the measure of the good set

In this subsection we assume that \( I \) is a smooth disk with dimension \( \text{dim } F \) that is tangent to local \( F \) cone (to simplify notation we write \( F_n = F_{f_n} \), for some \( n = 0, 1, 2, \ldots \)). Recall that every \( \mu_n \) is a physical-like measure of \( f_n \). As a result, there is a positive volume subset \( \Lambda_n \) such that for every \( x \in \Lambda_n \), there is a sequence \( \{i_k\} \) such that the empirical measures satisfy

\[
\delta_{f_n^{i_k}} \in U_n \tag{12}
\]

for any \( k \), where \( U_n \) is the neighborhood of \( \mu_n \) in \( \mathcal{P}(M) \) that we chose in the previous subsection such that (10) and (11) hold.

To obtain a contradiction, for \( m \in \mathbb{N} \), we define

\[
\mathcal{G}^{I,n}_m = \{ x \in I \cap \Lambda_n : \delta_{f_n^{i_k}} \in U_n \}.
\]

The goal is to show that \( \mathcal{G}^{I,n}_m \) have small measure with respect to the Riemannian volume on \( I \), uniformly in \( n \).

For this purpose, we let \( E^{I,n}_m \) be a maximal \((\delta_2, m)\)-separated set of \( \mathcal{G}^{I,n}_m \) w.r.t. the map \( f_n \). Here we drop the dependence of \( E^{I,n}_m \) on \( \delta_2 \) since it is fixed throughout the paper. Consider the following probability measure \( \sigma_{nm}^m \) supported on \( E^{I,n}_m \):

\[
\sigma_{nm}^m = \sum_{i \in E^{I,n}_m} e^{S_{f_n}^i \phi_{f_n}^p (\cdot)} \delta_{f_n^{i_k}} \sum_{z \in E^{I,n}_m} e^{S_{f_n}^i \phi_{f_n}^p (z)}.
\]

Here \( S_{f_n}^i \phi \) is the Birkhoff sum of \( \phi \) w.r.t. the map \( f_n \), i.e., \( S_{f_n}^i \phi = \sum_{j=0}^{i-1} \phi \circ f_n^j \).

By the convexity of the neighborhoods \( U_n \) of \( \mu_n \) and the definition of \( \mathcal{G}^{I,n}_m \), the measure

\[
\mu_{nm}^m = \frac{1}{m} \sum_{i=0}^{m-1} (f_n^i)_* \sigma_{nm}^m
\]
is a convex combination of $\delta f_n^{m}$ for $x \in G_m^{f_n}$ and thus is contained in $U_m$. Denote by

$$P_m^a = \frac{1}{m} \log \sum_{x \in F_m^a} e^{\delta f_n^{m} \phi f_n^{m}(x)}$$

the pressure of the separated set $E_m^a$ and $P_m^a = \lim \sup_m P_m^a$ the limiting pressure for each $f_n$.

**Lemma 3.5.** For $n > N_1$, we have

$$P_m^a < -\frac{994}{1000} a.$$

**Proof.** The proof is motivated by the proof of variational principle [37].

For any $l \geq 2N$, write $a(j) = \lfloor \frac{l}{N} \rfloor$ for $0 \leq j \leq N - 1$. Here $N > 0$ is the integer chosen in the previous section such that (8) to (11) holds. Then for each $j$ we have

$$\int f_n^{-i} A = \sum_{i=0}^{a(j)-1} f_n^{-i} \left( \bigvee_{i=0}^{N-1} f_n^{-i} A \right) \vee \bigvee_{i \in S} f_n^{-i} A,$$

where $S$ is a subset of $\{0, 1, \ldots, l - 1\}$ with $\#S \leq 2N$. Therefore

$$mP_m^a = \log \sum_{x \in G_m^{f_n}} e^{\delta f_n^{m} \phi f_n^{m}(x)} = H_{\sigma_m}^{a} \left( \bigvee_{i=0}^{m-1} f_n^{-i} A \right) + \int S_m^f \phi^{f_n^m} d\sigma_m^a$$

$$\leq \sum_{r=0}^{a(j)-1} H_{\sigma_m}^{a} f_n^{-r} \left( \bigvee_{i=0}^{N-1} f_n^{-i} A \right) + H_{\sigma_m}^{a} \left( \bigvee_{i \in S} f_n^{-i} A \right) + \int S_m^f \phi^{f_n^m} d\sigma_m^a$$

$$\leq \sum_{r=0}^{a(j)-1} H_{\sigma_m}^{a} f_n^{-r} \left( \bigvee_{i=0}^{N-1} f_n^{-i} A \right) + 2N \log \#A + \int S_m^f \phi^{f_n^m} d\sigma_m^a.$$

Summing over $j$ from 0 to $N - 1$:

$$NmP_m^a \leq \sum_{r=0}^{m-1} \sum_{j=0}^{a(j)-1} H_{\sigma_m}^{a} f_n^{-r} \left( \bigvee_{i=0}^{N-1} f_n^{-i} A \right) + 2N \log \#A + N \int S_m^f \phi^{f_n^m} d\sigma_m^a.$$

Dividing by $mN$ yields

$$P_m^a \leq \frac{1}{N} H_{\sigma_m}^{a} \left( \bigvee_{i=0}^{N-1} f_n^{-i} A \right) + \frac{2N}{m} \log \#A + \int \phi^{f_n^m} d\mu_m^a$$

$$\leq \frac{994}{1000} a + \frac{2N}{m} \log \#A,$$

where the second inequality follows from (11).

Sending $m$ to infinity, we conclude the proof.

The main result of this section is the following lemma:

**Lemma 3.6.** For every $n > N_1$ and every ($\dim F$)-dimensional disk $I$ that is tangent to local $F_n$ cone, we have

$$\lim \sup_{m \to \infty} \frac{1}{m} \log \text{vol}(G_m^{f_n}) < \mathcal{P}_n^a + \frac{1}{1000} a < -\frac{993}{1000} a.$$
Proof. By the choice of $\delta_0, \delta_2$ and (5), we obtain
\[
\text{vol}(G_{m}^{L_n}) \leq \sum_{z \in E_n} \text{vol}(B_{\delta_0,n}(z, f_n)) \\
\leq \sum_{z \in E_n} \text{vol}_{f_n^{-m}(f_n(z))} \left( f_n^{-m} \right) (e^{\frac{m}{1000}})^m.
\]

By lemma 3.2, the previous inequality is bounded by
\[
\text{vol}(G_{m}^{L_n}) \leq L \sum_{z \in E_n} \left| \det(Df_n^{-m}|_{f_n^{-m}(z)}) \right| (e^{\frac{m}{1000}})^m.
\]

Thus
\[
\frac{1}{m} \log \text{vol}(G_{m}^{L_n}) \leq \frac{1}{m} \log L + P_m + \frac{a}{1000},
\]
and
\[
\limsup \frac{1}{m} \log \text{vol}(G_{m}^{L_n}) \leq \limsup \frac{a}{1000} = \frac{a}{1000} < -\frac{993}{1000}a.
\]

3.4. Proof of theorem A

Fix any $n > N_1$. Recall that $\Lambda_n$ is a positive volume subset such that (12) holds. We take a smooth foliation box $B : f_{\dim E} \times f_{\dim F} \to M$ such that $\text{vol}(B \cap \Lambda_n) > 0$ and for any $a \in f_{\dim E}$, $B(a, \cdot)$ maps $\{a\} \times f_{\dim F}$ to a disk $I_a$ that is tangent to local $F_a$ cone. Since the foliation chart is smooth, by Fubini theorem, there is $a_n \in f_{\dim E}$ such that the corresponding disk $I_{a_n}$ satisfies $\text{vol}_{I_a}(\Lambda_{a_n}) > 0$.

On the other hand, lemma 3.6 applied to $I_{a_n}$ yields
\[
\limsup \frac{1}{m} \log \text{vol}_{I_a}(G_{m}^{L_n}) < -\frac{993}{1000}a < 0.
\]

In particular, this means that
\[
\sum_{m=1}^{\infty} \text{vol}_{I_a}(G_{m}^{L_n}) < \infty.
\]

By the Borel–Contelli lemma, we have
\[
\text{vol}_{I_a}\{x \in I_a : \delta_n^{f_{n,m}} \in U_n \text{ infinitely often} \} = 0.
\]

However, this contradicts with the choice of $a_n$ such that $\text{vol}_{I_a}(\Lambda_{a_n}) > 0$. We conclude the proof of theorem A.

4. Examples of application

In this section, we will provide examples where the metric entropy function is not upper semi-continuous, yet our main result still applies. Observe that in these cases, one cannot expect to obtain the continuity of Gibbs $F$-states only from its definition.
4.1. Statistical stability of singular hyperbolic attractors

We consider $C^1$ perturbations for the time-one map of singular hyperbolic attractors on three-dimensional manifolds. Let $X$ be a $C^2$ vector field on a compact boundaryless three-manifold $M$ and $\phi_t$ be the flow induced by $X$. An attractor $\Lambda$ is called singular hyperbolic, if all the singularities in $\Lambda$ are hyperbolic, and if there is a dominated splitting for $\phi_t$:

$$T_\Lambda M = E^s \oplus F^{cu}$$

with $\dim E^s = 1$, such that $D\phi_t|_{E^s}$ is uniformly contracting, and $D\phi_t|_{F^{cu}}$ is volume expanding: there exist $C > 0$ and $\lambda > 1$ such that

$$|\det D\phi_t|_{F^{cu}}(x)| \geq C\lambda^t$$

for all $x \in \Lambda$ and $t > 0$. Note that this condition prevents trivial measures (i.e. Dirac measure of a singularity or trivial measures on a periodic orbit) to be Gibbs $F^{cu}$-states. Examples of singular hyperbolic attractors include the famous Lorenz attractor. We invite the reader to the book [6] for a comprehensive study on this topic.

It is proven in [7, theorems B and C, corollary 2] that every singular hyperbolic attractor has a unique physical measure $\mu$. Moreover, $\mu$ is ergodic, hyperbolic (meaning that $\mu$ has a unique zero Lyapunov exponent which is given by the flow direction), fully supported on $\Lambda$, has absolutely continuous conditional measures on the center-unstable manifolds (these are the images of the Pesin strong unstable manifolds under the flow), and satisfies the entropy formula:

$$h_\mu(\phi_1) = \int \log |\det D\phi_1|_{F^{cu}}|d\mu.$$ 

Note that $\phi_1$ is the time-one map of the flow. Denote by $U$ the attracting neighborhood of $\Lambda$. Since $\phi_1$ is $C^2$, by Ledrappier–Young formula [25], we see that $\mu$ is the unique Gibbs $F^{cu}$-state:

$$Gibbs^{F^{cu}}(\phi_1|_U) = \{\mu\}.$$ 

By theorem A, we obtain the statistical stability for $C^1$ perturbations of the time-one map:

**Theorem 4.1.** Assume that $\Lambda$ is a singular hyperbolic attractor for a three-dimensional flow $\phi_t$ with attracting neighborhood $U$. Let $\{f_n\}$ be a sequence of $C^1$ diffeomorphisms converging to $\phi_1$ in $C^1$ topology, and $\mu_n$ be a physical-like measure of $f_n$ supported in $U$. Then we have $\mu_n \rightharpoonup^* \mu$, where $\mu$ is the unique physical measure of $\phi_1$ on $\Lambda$.

In a recent work [5], Araújo established the finiteness and statistical stability for physical measures of sectional hyperbolic attractors among $C^{1+}$ vector fields under $C^1$ topology. Here, we remark that $f_n$ in theorem 4.1 need not be the time-one map of a vector field $X_n$ that is $C^1$ close to $X$. In this case, since the bundle $F^{cu}$ admits no further domination, it is possible to create homoclinic tangency after $C^1$ perturbation, see Gourmelon [21, theorem 3.1], Pujals, Sambarino [31] and Wen [38] for previous results along this direction. Following the work of Newhouse [29] (see also [11] for a refined construction) local horseshoes with large entropy can be created, which prevents the metric entropy from being upper semi-continuous\(^8\).

\(^8\)Note that the robust $h$-expansiveness (which implies the upper semi-continuity of $h_\mu(X)$) proven in [30] applies only to nearby flows.
4.2. The example of Bonatti and Viana on $\mathbb{T}^4$

Following Mañé’s study [27] of derived from Anosov diffeomorphisms on $\mathbb{T}^3$, Bonatti and Viana constructed in [9] (see also [35]) a family of robustly transitive diffeomorphisms without any hyperbolic direction. Their examples are obtained by perturbing a linear Anosov diffeomorphism $A : \mathbb{T}^4 \rightarrow$ near two fixed points $p$ and $q$.

Let us briefly recall the construction of the example. Let $A \in \text{SL}(4, \mathbb{Z})$ be a linear Anosov diffeomorphism with four distinct real eigenvalues:

$$0 < \lambda_1 < \lambda_2 < 1/3 < \lambda_3 < \lambda_4,$$

and let $p, q$ be two fixed points of $A$. We fix some $r > 0$ small enough and consider the following perturbation of $A$, and note that such perturbations are $C^0$ small but $C^1$ large:

(a) outside $B_r(p)$ and $B_r(q)$ the map is untouched;
(b) in $B_r(p)$ the fixed point $p$ undergoes a pitchfork bifurcation in the direction corresponding to $\lambda_2$; this changes the stable index of $p$ from 2 to 1 and creates two new fixed points inside $B_r(p)$ with stable index 2, which we denote by $p_1$ and $p_2$;
(c) a small perturbation near $B_r(p_1) \subset B_r(p)$ makes the contracting eigenvalues complex;
(d) repeat steps (b) and (c) in $B_r(q)$ for $A^{-1}$.

Write

$$\lambda_{cs} = \sup \{ \log \| Df_{BV}[E_{cs}(x)] \| : x \in B_r(p) \} > 0,$$
$$\lambda_{cu} = \sup \{ \log \| Df_{BV}^{-1}[E_{cu}(x)] \| : x \in B_r(q) \} > 0,$$
$$\lambda = \max\{\lambda_{cs}, \lambda_{cu}\}.$$

Choosing $r, \lambda > 0$ small enough (we refer to [9] for full detail and [12] for a refined construction), we obtain a diffeomorphism which we denote by $f_{BV}$, such that:

- there exists an open neighborhood $U_{BV} \subset \text{Diff}^1(\mathbb{T}^4)$ of $f_{BV}$ such that every $g \in U_{BV}$ is transitive;
- $g \in U_{BV}$ admits a dominated splitting $T\mathbb{T}^4 = E^{cs} \oplus E^{cu}$ with $\dim E^{cs} = \dim E^{cu} = 2$; moreover, $E^{cs}$ and $E^{cu}$ cannot be further split into one-dimensional invariant subbundles;
- $E^{cs}$ and $E^{cu}$ are integrable (this requires the refined construction in [12]).

The following theorem is proven in [17].

**Theorem 4.2 [17, theorem B].** For $r, \lambda > 0$ small enough, there exists a $C^1$ neighborhood $U$ of $f_{BV}$ such that every $g \in U \cap \text{Diff}^2(\mathbb{T}^4)$ has a unique physical measure $\mu_g$ which is the unique equilibrium state for the potential $\varphi_g = -\log \det(Dg|_{E^{cs}})$, and satisfies

$$P(\varphi_g, g) = h_{\mu_g}(g) + \int \varphi_g \, d\mu_g = 0.$$

Combining with theorem A, we obtain the continuity of the physical measures for the example of Bonatti and Viana:

**Theorem 4.3.** Let $U$ be the neighborhood of $f_{BV}$ given by theorem 4.2. Then restricted to $U \cap \text{Diff}^2(\mathbb{T}^4)$, it satisfies that $\mu_g$ varies continuously (in the weak-* topology) with respect to $g$ in the $C^1$ topology.
Proof. The previous theorem states that
\[ \text{Gibbs}^{E^u}(g) = \{ \mu_g \} \]
for \( g \in \mathcal{U} \cap \text{Diff}^2(T^d) \). Let \( g_n, g \) be \( C^2 \) diffeomorphisms in \( \mathcal{U} \), with \( g_n C^1 \xrightarrow{\text{weak}^*} g \). Then theorem A shows that \( \mu_{g_n} \xrightarrow{\text{weak}^*} \mu_g \), as desired. \( \square \)

Similar to the previous example, since the bundles \( E^{cs} \) and \( E^{cu} \) admit no further domination, it is possible to create homoclinic tangency after \( C^1 \) perturbation and therefore the metric entropy is not upper semi-continuous. See also [17], particularly lemma 6.11, for the characterization on the refined entropy structure for \( g \in \mathcal{U}_{BV} \).

4.3. The open examples in [1], including the example of Mañé [27]

We conclude our paper with a brief discussion on the \( C^1 \) open family of diffeomorphisms constructed in [1], which generalizes both the example of Mañé [27] and Bonatti–Viana [9].

The construction in [27] calls for a linear Anosov diffeomorphism \( f_0 \) on \( T^d \), \( d \geq 2 \) and a small closed domain \( V \subset T^d \). Then one considers diffeomorphisms \( f \) on \( T^d \) such that:

- \( f \) admits invariant cone fields \( C^{\alpha} \) and \( C^{\alpha} \) containing the stable and the unstable bundle of \( f_0 \), respectively;
- \( Df \) is volume expanding on disks tangent to \( C^{\alpha} \), and volume contracting on disks tangent to \( C^{\alpha} \), for all \( x \in M \);
- outside \( V \), \( f \) is \( C^1 \) close to \( f_0 \);
- inside \( V \), \( \|Df^{-1}\|_{TD^u} \| < (1 + \delta_0) \) and \( \|Df\|_{TD^c} \| < (1 + \delta_0) \) for some \( \delta_0 > 0 \) small enough and for all disks \( D^c, D^u \) tangent to \( C^c, C^u \) respectively.

It is proven that any map \( f \) satisfying the assumptions above are non-uniformly expanding along the centre-unstable direction (for the precise definition, see [1, equation (2)]). In particular, such maps support ergodic SRB measures which are Gibbs \( F \)-states. Our main result could be applied to such family of diffeomorphisms, provided that one follows closely the argument in [17] to obtain the uniqueness of the Gibbs \( F \)-state.

A special case is Mañé’s derived-from-Anosov diffeomorphisms [27]. Here \( f_0 \) is a linear Anosov diffeomorphism on \( T^3 \). We further consider the following situations:

Case 1. The eigenvalues of \( f_0 \) are \( 0 < \lambda_1 < \lambda_2 < 1 < \lambda_3 \).

In this case we take \( E = E^a \), and \( F = E^u \) to be the unstable bundle; \( f \) constructed above has mostly contracting center [9, 19], in the sense that every Gibbs \( u \)-state has negative center Lyapunov exponent. The uniqueness of the Gibbs \( F \)-state (in this case, Gibbs \( u \)-state) has been established in [16, 20]. With theorem A, we obtain the statistical stability for the physical measure.

Case 2. The eigenvalues of \( f_0 \) are \( 0 < \lambda_1 < 1 < \lambda_2 < \lambda_3 \).

In this case \( E = E^c \) is the stable bundle, and \( F = E^{cs} \); following [1], \( f \) has mostly expanding center. The finiteness and the statistical stability of the physical measure have been established under various situations [3, 4, 36, 40]. Alternatively, the argument in [17] can be easily adapted to prove the uniqueness of the Gibbs \( F \)-state, and the statistical stability follows from theorem A.

ORCID iDs

Fan Yang https://orcid.org/0000-0002-4954-9681
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