A Novel Approach to Quantum Gravity in the Presence of Matter without the Problem of Time

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Abstract

An approach to the quantization of gravity in the presence matter is examined which starts from the classical Einstein-Hilbert action and matter approximated by “point” particles minimally coupled to the metric. Upon quantization, the Hamilton constraint assumes the form of the Schrödinger equation: it contains the usual Wheeler-DeWitt term and the term with the time derivative of the wave function. In addition, the wave function also satisfies the Klein-Gordon equation, which arises as the quantum counterpart of the constraint among particles’ momenta. Comparison of the novel approach with the usual one in which matter is represented by scalar fields is performed, and shown that those approaches do not exclude, but complement each other. In final discussion it is pointed out that the classical matter could consist of superparticles or spinning particles, described by the commuting and anticommuting Grassmann coordinates, in which case spinor fields would occur after quantization.

Keywords: Canonical quantum gravity; problem of time; wave function of the universe; quantum field theory; configuration space

1 Introduction

Quantum gravity is an unfinished project. In one actively investigated approach, the state of the universe is represented by a wave functional that satisfies the Wheeler-DeWitt equation and comprises matter degrees of freedom as well as the gravitational field. Matter degrees of freedom are typically given in terms of scalar or spinor fields (see, e.g., Ref. [1]). However, when we consider the evolution of the universe, we usually talk about positions of objects, e.g., we say that galaxies are receding from us, that there is a black hole in the center of a galaxy, etc. The works such as those considered, e.g., in Refs. [2-7], inspired by the implications of the Wheeler-DeWitt equation, consider models in which the wave function of the universe is given in terms of positions of particles. Here we provide an explicit demonstration for the
first time of how a wave functional of, e.g., a scalar field and a gravitational field is related to a wave functional of many particle positions and a gravitational field. In the literature on quantum gravity the space of configurations of a scalar or whatever field is called configuration space, but the same name ‘configuration space’ is used in other branches of physics for the configuration space of one, two three, or many particle positions.

First we analyze the conventional field theory of a scalar field in the Schrödinger functional representation in which quantum fields are $c$-numbers, whilst the corresponding canonically conjugated variables are functional derivatives. A generic state, represented as a functional of the scalar field, can be expanded over the basis of the Fock space. In distinction to the usual approaches, in which a generic state as a superposition of multiparticle states in momentum space, we consider multiparticle states in position space. The scalar field QFT state functional can then be generalized to include a fixed (background) metric field $g_{ij}(x)$ on a space like 3D hypersurface $\Sigma$ in spacetime.

Next, the theory is formulated within a more general framework in which an unfixed, dynamical 4D metric is taken into account and is split according to the ADM prescription. The classical action contains a matter part, $I_m$, plus the gravitational part $I_G[g_{\mu\nu}]$. For the matter part it is customary to take a functional of some fields, such as a scalar, spinor or Maxwell field, etc. In the case of a scalar field, $\varphi$, the action is thus $I_m[\varphi, g]$.

In this paper we explore an alternative procedure in which we start directly from a classical action $I_m[X^\mu, g_{\mu\nu}]$ for a system of “point particles” coupled to a gravitational field. We point out that such a procedure makes sense, if the coordinates $X^\mu$ are not assumed to be associated with true point particles, but with effective positions of extended particles. In the Gupta-Bleuler quantization the classical constraints, obtained by variation of the Lagrange multipliers, become operator constraints on the quantum states. As usual, the Lagrange multipliers in $I_m[\varphi, g]$ are the lapse and shift functions $N, N_i, i = 1, 2, 3$, whereas in $I_m[X^\mu, g]$ we also have the Lagrange multiplier $\lambda$, associated with the $\tau$-reparametrization invariance, which gives the mass shell constraint, and, after quantization, the Klein-Gordon equation.

In the procedure, discussed in this paper, in which we start from the classical action $I_m[X^\mu, g] + I_G[g_{\mu\nu}]$, upon quantization the time $X^0$ does not disappear from the equations. On the contrary, in the usual procedure that starts from the action $I_m[\varphi, g] + I_G[g_{\mu\nu}]$, there is manifestly no time in the quantum equations, which is the notorious “problem of time”. In Ref. [8] are reviewed many different approaches to the resolution of this tough problem.

The paper is organized as follows. In Sec. 2 we first briefly review some basic facts about the Schrödinger functional representation for a scalar field and show how a fixed gravitational field can be included into the description. In Sec. 3 we consider
the dynamical gravity as well, first with the novel and then with the traditional approach to the matter term, and show how those distinct procedures are related to each other. In Sec. 4 we resume our findings, and discuss a larger framework into which they can be embedded. In particular, we point out, that the classical matter, consisting of particles, can be extended to superparticles or spinning particles that are described not only by the commuting coordinates \( X^\mu \), but also by anticommuting coordinates, in the literature often denoted as \( \xi^\mu \) or \( \theta^\mu \). Coupling of such a system to gravity and quantizing it along the lines discussed in this paper, would then bring spinor fields into the description.

2 Wave function in quantum field theory

To make our presentation self-consistent we will first review some known facts from quantum field theory, and then show a novel way of introducing the gravitational field into the description of quantum states. But first let us recall how in flat spacetime we arrive at a quantum field theory from a relativistic point particle. It is described by a “minimal” length action that can be cast into the following equivalent form:

\[
I[X^\mu(\tau)] = \frac{1}{2} \int d\tau \left( \frac{\dot{X}^\mu \dot{X}^\mu}{\lambda} + \lambda m^2 \right),
\]

where \( \lambda \) is a Lagrange multiplier, variation of which gives the constraint \((p^\mu p_\mu - m^2) = 0\). Upon quantization it becomes the Klein-Gordon equation

\[
(\partial_\mu \partial^\mu + m^2) \varphi(x) = 0.
\]

The latter can be derived from the action for a scalar field,

\[
I[\varphi(x)] = \frac{1}{2} \int d^4x \left( \partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2 \right),
\]

Here \( x \equiv x^\mu = (t, x), \mu = 0, 1, 2, 3, x \equiv x^i, i = 1, 2, 3, \) are spacetime coordinates. The canonically conjugated variables, \( \varphi(t, x) \) and \( \Pi(t, x) = \dot{\varphi}(t, x) \), where dot denotes the derivative with respect to the time \( t \), become upon quantization the operators \( \hat{\varphi}(t, x) \), \( \hat{\Pi}(t, x) \), satisfying the equal time commutation relations

\[
[\hat{\varphi}(t, x), \hat{\Pi}(t, x)] = i\delta^3(x - x'),
\]

\[
[\hat{\varphi}(t, x), \hat{\varphi}(t, x')] = 0, \quad [\hat{\Pi}(t, x), \hat{\Pi}(t, x')] = 0.
\]

A general solution of the Klein-Gordon equation \([2]\) can be expanded according to

\[
\varphi(t, x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_p}} \left( a(p)e^{-ipx} + a^\dagger(p)e^{ipx} \right),
\]

\footnote{We use the normalization as adopted, e.g., in the textbook by Peskin \[9\].}
where \( \omega_p = \sqrt{m^2 + p^2} \) and \( px \equiv k_\mu x^\mu \). The operators \( a(p), a^\dagger(p) \) are consistent with (1), (5) if they satisfy
\[
[a(p), a^\dagger(p')] = \delta^3(p - p'),
\]
\[
[a(p), a(p')] = 0, \quad [a^\dagger(p), a^\dagger(p')] = 0.
\] (7) (8)

In the \( \varphi(x) \) representation the field operator is represented by a \( c \)-number, time independent, field \( \varphi(x) \), whereas the conjugate momentum operator is represented by the functional derivative [10, 11]:
\[
\hat{\varphi}(x) \longrightarrow \varphi(x), \quad \hat{\Pi}(x) \longrightarrow -i \frac{\delta}{\delta \varphi(x)}.
\] (9)

A state \(|\Psi\rangle\) is represented by a time dependent wave functional \( \Psi[t, \varphi(x)] \equiv \langle \varphi(x)|\Psi \rangle \), which satisfies the Schrödinger functional equation
\[
i \frac{\partial \Psi[t, \varphi(x)]}{\partial t} = H \Psi[t, \varphi(x)].
\] (10)

The Hamilton operator is (see e.g.,Ref. [11])
\[
H = \frac{1}{2} \int d^3x \left( \hat{\Pi} \hat{\varphi} - \mathcal{L} \right) = \frac{1}{2} \int d^3x \left( -\frac{\delta^2}{\delta \varphi(x)^2} + \varphi(m^2 - \nabla^2)\varphi \right)
\] (11)

One could object that the second order functional derivatives give singularity. And yet, this is precisely what happens in quantum field theory in whatever representation. Namely, stationary solution of Eq. (11) satisfy
\[
H \Psi[\varphi(x)] = E \Psi[\varphi(x)].
\] (12)

An example of a solution is the vacuum functional
\[
\Psi_0[\varphi(x)] = \eta \exp \left[ -\frac{1}{2} \int d^3x \varphi(x) \sqrt{m^2 - \nabla^2} \varphi(x) \right],
\] (13)

which, when inserted into the stationary Schrödinger equation (12), gives
\[
\frac{1}{2} \int d^3x \left( \sqrt{m^2 - \nabla^2} \delta(0) \right) \Psi_0 = E \Psi_0.
\] (14)

This corresponds to the singular (“zero point”) energy of the vacuum.

However, in Ref. [12,13] (see also [14]) the following generalization of the Hamilton operator (11) was considered:
\[
H = \frac{1}{2} \left( -\rho^a(x)b(x') \partial_a(x) \partial_0(x') + \varphi^a(x) \omega_{a0}(x) b(x') \varphi^b(x') \right),
\] (15)
where $\rho^{ab}(x|x')$ is a metric in the space of fields $\varphi^a(x) \equiv \varphi^a(x)$. Then, in general, no singularity arises, despite that the expression contains the second order functional derivatives $\partial_a(x) \equiv \frac{\partial}{\partial \varphi^a(x)}$. As to the ordering ambiguity, one can extend the procedure of Ref. [15] by introducing the basis vectors $h_a(x)$, satisfying the Clifford algebra relations

$$h_a(x) \cdot h_b(x') \equiv \frac{1}{2} \left( h_a(x) h_b(x') + h_b(x') h_a(x) \right) = \rho_{a(b)}(x|x'),$$

(16)

and, instead of (15), define the following Hamilton operator, covariant in function space:

$$H = -\frac{1}{2} \left( (h^a(x) \partial_a(x)) \left( h^b(x') \partial_b(x') \right) + \varphi^a(x) \omega_{a(b)}(x|x') \varphi^b(x') \right).$$

(17)

Here $\omega_{a(b)}(x|x')$ is a coupling between the fields that generalizes $\sqrt{m^2 - \nabla^2}$. The momentum vector operator $\Pi = h^a(x) \Pi_a(x) = -ih^a(x) \partial_a(x)$ is then Hermitian [15].

We are not interested here into the mathematical intricacies concerning hermiticity of the momentum operator $\hat{\Pi}(x)$ that must be taken into account in the cases of generic function spaces. For us it is important that the definition (9) works well in the relevant physical calculations as nicely shown in Ref. [11]. The expectation value is

$$\langle \hat{\Pi}(x) \rangle = \int D\varphi(x) \Psi^*[\varphi(x)](-i) \frac{\delta}{\delta \varphi(x)} \Psi[\varphi(x)],$$

(18)

where for the measure we take

$$D\varphi(x) = \prod_x d\varphi(x),$$

(19)

which holds in the function space with the metric

$$\rho(x, x') = \delta^3(x - x'),$$

(20)

so that the squared distance element in the function space is

$$\int d^3x d^3x' \rho(x, x') d\varphi(x) d\varphi(x') = \int d^3x d^3x d\varphi(x)^2.$$

(21)

Taking the complex conjugate, we have

$$\langle \hat{\Pi}(x) \rangle^* = \int D\varphi(x) \Psi^*[\varphi(x)] i \frac{\delta}{\delta \varphi(x)} \Psi[\varphi(x)]$$

$$= \int D\varphi(x) \left[ \Psi^*[\varphi(x)](-i) \frac{\delta}{\delta \varphi(x)} \Psi[\varphi(x)] - i \frac{\delta}{\delta \varphi(x)} (\Psi^* \Psi) \right].$$

(22)

Let us assume that $\Psi^*[\varphi(x)] \Psi[\varphi(x)] \to 0$ if $\varphi \to \infty$, which means that the state is localized around a fixed, expected field configuration. Then the expectation value is real and the momentum operator is Hermitian. If we also assume that
\[ \int D\varphi(x) \Psi^*[\varphi(x)]\Psi[\varphi(x)] = 1, \] then \( \Psi[\varphi(x)] \) can be interpreted as the probability amplitude, and \( \Psi^*[\varphi(x)]\Psi[\varphi(x)] \) as the probability density for the field configuration \( \varphi(x) \), whose expectation value is

\[ \langle \varphi(x) \rangle = \int D\varphi(x) \Psi^*[\varphi(x)]\varphi(x)\Psi[\varphi(x)]. \]

(23)

Using (9), the operators \( a(p) \), \( a^\dagger(p) \) can be represented as follows \[10\]:

\[ a^\dagger(p) = \int d^3x e^{-ipx} \left( \frac{1}{\sqrt{(2\pi)^32\omega_p}} \varphi(x) - \frac{1}{\sqrt{(2\pi)^32\omega_p}} \delta \delta\varphi(x) \right), \]

(24)

\[ a(p) = \int d^3x e^{ipx} \left( \frac{1}{\sqrt{(2\pi)^32\omega_p}} \varphi(x) + \frac{1}{\sqrt{(2\pi)^32\omega_p}} \delta \delta\varphi(x) \right), \]

(25)

If we take the Fourier transform of the operators \( a^\dagger(p) \), \( a(p) \), namely

\[ a^\dagger(x) = \frac{1}{\sqrt{(2\pi)^3}} \int d^3p e^{ipx} a^\dagger(p) \]

(26)

\[ a(x) = \frac{1}{\sqrt{(2\pi)^3}} \int d^3p e^{-ipx} a(p) \]

(27)

we obtain

\[ [a(x), a^\dagger(x')] = \delta^3(x - x'), \]

(28)

\[ [a(x), a(x')] = 0, \quad [a^\dagger(x), a^\dagger(x')] = 0. \]

(29)

These are the commutation relations for creation (annihilation) operators that create (annihilate) a particle at \( x \). From (28), (29), by using (24), (25), we find the following explicit expressions:

\[ a^\dagger(x) = \sqrt{(2\pi)^32(m^2 - \nabla^2)^{1/4}} \varphi(x) - \frac{1}{\sqrt{(2\pi)^32(m^2 - \nabla^2)^{1/4}}} \delta \delta\varphi(x) \]

(30)

\[ a(x) = \sqrt{(2\pi)^32(m^2 - \nabla^2)^{1/4}} \varphi(x) + \frac{1}{\sqrt{(2\pi)^32(m^2 - \nabla^2)^{1/4}}} \delta \delta\varphi(x) \]

(31)

From the latter equations we obtain

\[ \varphi(x) = \frac{1}{\sqrt{(2\pi)^32(m^2 - \nabla^2)^{1/4}}} (a(x) + a^\dagger(x)), \]

(32)

\[ \frac{\delta}{\delta\varphi(x)} = \sqrt{(2\pi)^32(m^2 - \nabla^2)^{1/4}} (a(x) - a^\dagger(x)). \]

(33)
Using (32), (33), we can express the Hamiltonian (11) in terms of \(a(x)\) and \(a^\dagger(x)\): 
\[
H = \frac{1}{2} \int d^3x \left( a^\dagger(x)\sqrt{m^2 - \nabla^2} a(x) + \left( \sqrt{m^2 - \nabla^2} a(x) \right) a^\dagger(x) \right).
\] (34)

If expressed in terms of \(a(p)\), \(a^\dagger(p)\), the Hamiltonian has the usual form,
\[
H = \frac{1}{2} \int d^3p \omega_p (a^\dagger(p)a(p) + a(p)a^\dagger(p)).
\] (35)

A generic state is a superposition of many particle states:
\[
\Psi[t, \varphi(x)] = \sum_{r=0}^{\infty} \int d^3p_1 d^3p_2 \ldots d^3p_r \phi(t, p_1, p_2, \ldots, p_r) a^\dagger(p_1)a^\dagger(p_2) \ldots a^\dagger(p_r) \Psi_0[\varphi(p)],
\] (36)

where \(\phi(t, p_1, p_2, \ldots, p_r)\) is a complex valued wave packet profile for an \(r\)-particle state in momentum representation.

In terms of the Fourier transformed creation operators (26), (27), the same state reads:
\[
\Psi[t, \varphi(x)] = \sum_{r=0}^{\infty} \int d^3x_1 d^3x_2 \ldots d^3x_r \psi(t, x_1, x_2, \ldots, x_r) a^\dagger(x_1)a^\dagger(x_2) \ldots a^\dagger(x_r) \Psi_0[\varphi(x)],
\] (37)

where \(\psi(t, x_1, x_2, \ldots, x_r)\) is a complex valued wave packet profile for an \(r\)-particle state in position representation. Its absolute square \(|\psi|^2 = \psi^*\psi\) gives the probability density of observing at a time \(t\) a multi particle configuration at positions \(x_1, x_2, \ldots, x_r\), and as discussed in the next paragraph and the references cited therein, the apparent non covariance of \(|\psi(x)|^2\) is not problematic.

Within relativistic quantum field theories, the wave packet profiles in momentum representation are occasionally used in the literature when considering the \(S\)-matrix. In the textbook by Peskin and Schröder [9], a single particle wave packet is considered, and it is mentioned that \(\phi(p)\) is the Fourier transform of the spatial wavefunction. In general, a wave packet depends on time. In a given Lorentz reference frame one can thus consider either \(\phi(t, p)\) or \(\psi(t, x)\) (and their multiparticle extensions), the corresponding creation operators being \(a^\dagger(p)\) or \(a^\dagger(x)\). Although asymptotic states are usually taken to have definite momenta, there are situations, when they form wave packet profiles, in which case it is possible to measure the detection time, the time of flight, etc. Such wavepacket profiles, of course, can be observed from different Lorentz reference frames. When observed from another Lorentz frame, the wave packet, and hence the probability density, looks different, it is subjected to an

\[\text{A state with definite momentum, } a^\dagger(p)|0\rangle \equiv |p\rangle, \text{ is an approximation. In reality, there is always some spreading. A sharp momentum state is not in Hilbert space, it belongs to generalized states.}\]
appropriate transformation, but it is still spread around a centroid momentum, or equivalently, around a centroid position \[17\].

The vacuum state satisfies
\[a(x)\Psi_0[\varphi(x)] = 0.\] (38)

Using Eq. (31) in the latter equation, we find for the solution the vacuum functional (13) that was obtained directly from the stationary Schrödinger equation (12).

Usually authors do not work in the coordinate \((x)\), but in the momentum \((p)\) representation, which in many respects is more practical. In particular, in momentum representation it is straightforward to calculate the normalization constant for the vacuum functional. But for the purposes of our paper, the coordinate representation is useful, because once the expressions (30), (31) and (13) are obtained, we can generalize them to include the metric field \(q_{ij}(x)\) on a 3D hypersurface \(\Sigma\) by replacing \(\nabla^2\) with the covariant operator \(q_{ij}D_iD_j\), which when acting on a scalar field gives \(\frac{1}{\sqrt{q}}\partial_i(\sqrt{q}\partial^i\varphi(x))\), where \(q = \det q_{ij}\), i.e., the determinant of the 3-metric. To our knowledge such procedure has not yet been considered in the literature, and in the following we will show its usefulness in introducing a gravitational field into the description.

If we make the replacement
\[\nabla^2 \rightarrow q_{ij}D_iD_j = D_iD^i \equiv D^2,\] (39)
then the creation and annihilation operators (30), (31) become functionals of \(q_{ij}(x)\), and so does the vacuum functional (13):
\[a(x) \rightarrow a[x, q_{ij}(x)],\] (40)
\[a^\dagger(x) \rightarrow a^\dagger[x, q_{ij}(x)],\] (41)
\[\Psi_0[\varphi(x)] \rightarrow \Psi_0[\varphi(x), q_{ij}(x)].\] (42)

It turns out that they satisfy the same commutation relation (28), (29)

The so modified operator \(a^\dagger[x, q_{ij}(x)]\) creates a particle\(^3\) at position \(x\) in the gravitational field \(q_{ij}(x)\). One finds that the modified operator \(a[x, q_{ij}(x)]\) annihilates the modified vacuum functional \(\Psi_0[\varphi(x), q_{ij}(x)]\). A generic state can then be represented by the following functional:
\[\Psi[t, \varphi(t, x), q_{ij}(x)] = \sum_{r=1}^{\infty} \int d^3x_1 \cdots d^3x_r \psi[t, x_1, \ldots, x_r, q_{ij}(x)] \times a^\dagger[x_1, q_{ij}(x)] \cdots a^\dagger[x_r, q_{ij}(x)] \Psi_0[\varphi(x), q_{ij}(x)].\] (43)

\(^3\)It is usually stated that the notion of particle depends on the metric, and by “particle” it is understood an excitation with definite momentum. Here by “particles” we mean just the excitations, created or annihilated by the modified position dependent operators (30), (31).
Here $\psi[t, x_1, \ldots, x_r, q_{ij}(x)]$ is the amplitude for the probability that we will find matter particles at positions $x_1, x_2, \ldots, x_r$ in a gravitational field $q_{ij}(x)$. This is thus a wave functional that depends on particle positions and on the corresponding gravitational field at those positions. In the case when only one particle is created, its wave functional is $\psi[x_1, q_{ij}(x)]$. The state (43) is then given by

$$
\Psi[t, \varphi(x), q_{ij}(x)] = \int d^3x_1 \psi[t, x_1, q_{ij}(x)] a^\dagger[x_1, q_{ij}(x)] \Psi_0[\varphi(x), q_{ij}(x)].
$$

In Ref. [18] the wave functional of the form $\psi[X^\mu, q_{ij}(x)]$, which depends on a particle position and a gravitational field, was considered, and found to satisfy the Klein-Gordon and the Wheeler-DeWitt equation. This was an alternative to the usually considered state functional $\Psi[\varphi(x), q_{ij}(x)]$ that depends on a scalar field $\varphi(x)$ instead on $x$. In Eq. (44) we have a translation between those two possible descriptions, and in Eq. (43) we have a translation for the general case of many particles. In one description we have a functional $\Psi[t, \varphi(x), q_{ij}(x)]$, and in the other description we have a set of functionals $\psi[t, x_1, q_{ij}(x)], \psi[t, x_1, x_2, q_{ij}(x)], \ldots, \psi[t, x_1, \ldots, x_n, q_{ij}(x)]$, which are components of the expansion of the state functional in terms of the basis states $a^\dagger[x_1, q_{ij}(x)] \ldots a^\dagger[x_r, q_{ij}(x)] \Psi_0[\varphi(x), q_{ij}(x)], r = 0, 1, 2, \ldots, N$, where $N$ is arbitrary and can go to infinity.

**Schrödinger equation**

If we substitute the expression (37) for the state functional and (34) for the Hamiltonian into the Schrödinger equation (10), we obtain a set of equations for the multiparticle wavefunctions:

$$
i \frac{\partial \psi[(t, x_1, \ldots, x_r)]}{\partial t} = \sum_{\ell=1}^r \sqrt{m^2 - \nabla^2_{x_\ell}} \psi(t, x_1, \ldots, x_r), \quad r = 1, 2, \ldots, \infty,
$$

where

$$
\nabla^2_i \equiv -\frac{\partial^2}{\partial x_i^2}, \quad i = 1, 2, 3
$$

In Eq. (45) we have omitted the infinite zero point energy, because it cancels out in the expressions containing the probability density $\psi^* \psi$.

If we take into account also the gravitational field $q_{ij}(x)$, the Schrödinger equation (10) generalizes so that instead of $\Psi[t, \varphi(x)]$ we have $\Psi[t, \varphi(x), q_{ij}(x)]$ and in the Hamiltonian (34) the operator $\nabla^2$ generalizes according to (39):

$$
i \frac{\partial \Psi[t, \varphi(x), q_{ij}(x)]}{\partial t} = H \Psi[t, \varphi(x), q_{ij}(x)].
$$

Then, instead of (45) we obtain

$$
i \frac{\partial \psi[t, x_1, \ldots, x_r, q_{ij}(x)]}{\partial t} = \sum_{\ell=1}^r \sqrt{m^2 - D^2_{x_\ell}} \psi[t, x_1, \ldots, x_r, q_{ij}(x)], \quad r = 1, 2, \ldots, \infty.
$$
Equation (47), or equivalently, (48), describes the evolution of a many particle state in the presence of a background gravitational field $q_{ij}(x)$. At a certain time $t$, the probability density $|\psi[t, x_1, ..., x_r, q_{ij}(x)]|^2$ is centered around a configuration $x_1, ..., x_r$ and a 3-metric $q_{ij}(x)$. At another time $t$, it is centered around a different configuration and a different intrinsic metric. In other words, $\int_{\Omega} |\psi[t, x_1, ..., x_r, q_{ij}(x)]|^2 d^3x_1 d^3x_2 ... d^3x_r Dq_{ij}(x)$ is the probability that at time $t$ we find $r$ particles within a domain $\Omega$ around the positions $x_1, x_2, ..., x_r$ and in the gravitational field $q_{ij}(x)$. Because in the right hand side of Eq. (48) there is no operator term acting on $q_{ij}(x)$, the probability density remains at all times $t$ centered around the same 3-metric $q_{ij}(x)$. In particular, it can be $q_{ij}(x) = \delta_{ij}$, but in general it is a position dependent metric with a non vanishing 3-curvature. Equation (48) then describes evolution of a wave function in a fixed non trivial gravitational field. There is no dynamics of $q_{ij}(x)$ itself in such a formalism.

### 3 Dynamics of matter coupled to gravity

#### 3.1 A novel approach

In the previous section we started from a relativistic point particle in flat spacetime and arrived upon first quantization at the Klein-Gordon equation for a scalar field $\varphi(x)$. Then we quantized the field $\varphi(x)$ as well, and arrived at quantum field theory in which a generic state can be represented as a functional $\Psi[t, \varphi(x)]$, expanded in terms of many particle states with wave packet profiles (wave functions) $\phi(t, p_1, p_2, ..., p_r)$, or equivalently, $\psi(t, x_1, x_2, ..., x_r)$, satisfying the Schrödinger equation (45). We then discussed how such wavefunctions can be generalized to include a fixed gravitational field. We have thus arrived at the quantum evolution of a many particle wave function (48) in a fixed 3-metric field $q_{ij}(x)$ on a simultaneity hypersurface $\Sigma$. In order to include a dynamics of the gravitational field $g_{\mu \nu}(x)$, $x \equiv x^\mu$, $\mu = 0, 1, 2, 3$, and consequently of the induced metric $q_{ij}(x)$ on $\Sigma$, let us consider the classical action for a many particle system coupled to gravity:

$$I[X_n^\mu, g_{\mu \nu}(x), \lambda_n] = I_m[X_n^\mu, g_{\mu \nu}(x), \lambda_n] + I_G[g_{\mu \nu}(x)]$$

where

$$I_m[X_n^\mu, g_{\mu \nu}(x), \lambda_n] = \frac{1}{2} \sum_{n=1}^{N} \int d\tau \left( \frac{X_n^\mu X_n^\nu g_{\mu \nu}}{\lambda_n} + \lambda_n m_n^2 \right),$$

and

$$I_G = \kappa \int d^4x \sqrt{-g} R, \quad \kappa = (16\pi G)^{-1}.\footnote{The parameter $\tau$ is arbitrary and in principle different on each worldline. To simplify the notation we write $\tau$ instead of $\tau_n,}$
Here the functions \( X_\mu^n(\tau) \) of an arbitrary monotonically increasing parameter \( \tau \) describe the worldlines associated with positions of particles, e.g., their centers of mass. Realistic particles are not point like, they are extended beyond their Schwarzschild radii\(^5\), but in our description we take into account only the particle’s center of mass.

Despite that the definition of the center of mass is a subject of controversy \cite{19}, for our purpose here it is important that each of the objects whose motion is governed by the action (50) is extended, not point like, and that can be described by four coordinates only, so that the infinitely many degrees of freedom of an extended object are neglected. How this works for an extended object confined within a narrow tube in spacetime is shown in the derivation of the Papapetrou equation \cite{20}. The derivation is based on the moments of the stress-energy tensor around a chosen worldline within the tube. If only the first moment (“monopole”) is taken into account, one obtains the geodesic equation for such a worldline. In such sense one should also consider the worldlines occurring in the action (50). For further support of our argument, see Appendix A.

In the continuum limit of many densely packed worldlines the action (50) becomes the dust action. Coupling of dust to gravity was considered by Brown and Kuchar \cite{21} in order to resolve the problem of time in quantum gravity. In their approach dust is supposed to be present everywhere and its degrees of freedom incorporate time. We will show that instead of dust one can as well employ a system of “point” particles that even need not be present everywhere. It comes out that the generator of time translations is directly associated with the stress-energy tensor of such system of particles.

If one performs the ADM split of spacetime, then the action (49) can be written as a functional of the 3-metric \( q^{ij} \), and the lapse and shift functions, \( N \) and \( N^i \), \( i,j = 1,2,3 \). Rewriting the matter part of the action, (50), into the phase space form,

\[
I_m[X_\mu^n, p_{\mu n}, \lambda_n, g_{\mu \nu}] = \sum_n \int d\tau \left( p_{\mu n} \dot{X}_\mu^n - \frac{\lambda_n}{2} (g_{\mu \nu} p_{\nu n} p_{\mu n} - m_n^2) \right),
\]

and performing the ADM split, we obtain \cite{18}

\[
I_m[X_\mu^n, p_{\mu n}, \lambda_n, q_{ij}, N, N^i] = \sum_n \int d\tau \left( p_{\mu n} \dot{X}_\mu^n - \frac{\lambda_n}{2} \left[ \frac{1}{N^2} (p_{n0} - N^i p_{ni}) (p_{n0} - N^j p_{nj}) - q^{ij} p_{ni} p_{nj} - m_n^2 \right] \right)
\]

\(^5\)The action (50) can also describe a system of (mini) black holes with their positions being parametrized with \( X_\mu^\nu \) and tracing the worldlines \( X_\mu^\nu(\tau) \). Such a system, instead of being considered within a complicated detailed description involving the mutual dynamics of black holes’ gravitational field, could be as well approximately described in terms of their positions \( X_\mu^\nu \). The very fact that we talk about a black hole in the center of our galaxy, or in the center of another galaxy, means that we ascribe to a black hole a position, and parametrize it by a set of coordinates.
In the above action, momenta are the quantities, variation of which gives the relation $p^\mu_n = \dot{X}^\mu_n/\lambda_n$.

In order to have the matter action on the same footing as the gravitational action, we must insert $1 = \int \sqrt{-g} d^4x \delta^4(x - X_n(\tau))/\sqrt{-g} = \int d^4x \delta^4(x - X_n(\tau))$. But our particle is not exactly point like, it is extended, e.g., a ball, whose worldvolume is described by $X^\mu(\tau, \sigma)$, $\sigma \equiv \sigma^a = (R, \vartheta, \varphi)$, $0 \leq R \leq R_0$, $0 \leq \vartheta \leq \pi$, $0 \leq \varphi \leq 2\pi$, where $R_0$ is greater than the Schwarzschild radius. Therefore, $\delta^4(x - X_n(\tau))$ should be considered as an approximation to $\int d^3\sigma \sqrt{-\bar{f}} \delta^4(x - X_n(\tau, \sigma))$, where $\bar{f} \equiv \det \bar{f}_{ab}$, $\partial_a \equiv \partial/\partial \sigma^a$ (see also [22], where the so called “good” $\delta$-function is defined). Despite that such a $\delta$-function associated with a ball is not invariant, it is covariant. In Appendix A we consider the dynamics of a ball, modelled by a space filling brane, described by the action that is covariant with respect to general coordinate transformation of $x^\mu$ and $\xi^A = (\tau, \sigma^a)$.

For the gravitational part of the action we obtain [23]

$$I_G[q_{ij}, \pi^{ij}, N, N^i] = \int dt d^3x \left[ \pi^{ij} \dot{q}_{ij} - N \mathcal{H}_G(q_{ij}, \pi^{ij}) - N^i \mathcal{H}_G(q_{ij}, \pi^{ij}) \right],$$  

where

$$\mathcal{H}_G = -\frac{1}{\kappa} G_{ij,k\ell} \pi^{ij} \pi^{k\ell} + \kappa \sqrt{q} R^{(3)},$$  

$$\mathcal{H}_G^i = -2D_j \pi^{ij},$$

and

$$G_{ij,k\ell} = \frac{1}{2\sqrt{q}} (q_{ik}q_{j\ell} + q_{i\ell}q_{jk} - q_{ij}q_{k\ell}).$$

is the Wheeler-DeWitt metric.

Variation of $I = I_m + I_G$ with respect to $\lambda_n$, $N$, and $N^i$, which have the role of Lagrange multipliers, gives the constraints [18] that involve both the gravitational

\footnote{For instance, if we consider the case of Minkowski space, then in another Lorentz reference frame the $\delta$-function, corresponding to a ball-like extended object, looks different (no longer associated with a ball at rest, but a moving ellipsoid). However, in a new Lorentz frame one can have with respect to the new simultaneity 3-surface a ball-like extended object, described by $X_n(\tau, \sigma)$, the corresponding delta-function being exactly the same as our “original” delta-function. The concept of the delta-function, though not invariant, is covariant.}

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and matter degrees of freedom:

\[
\delta \lambda_n : \quad \frac{1}{N^2} (p_{n0} - N^i p_{ni})(p_{n0} - N^j p_{nj}) - q^{ij} p_{ni} p_{nj} - m_n^2, \quad (58)
\]

\[
\delta N : \quad \mathcal{H}_G = - \sum_n \int d\tau \frac{1}{N^3} \delta^4(x - X_n(\tau))(p_{n0} - N^i p_{ni})(p_{n0} - N^j p_{nj})
\]

\[
= - \sum_n \delta^3(x - X_n) \frac{1}{N} (p_{n0} - N^i p_{ni}) \quad (59)
\]

\[
\delta N^i : \quad \mathcal{H}_{G_i} = - \sum_n \int d\tau \frac{1}{N^2} \delta^4(x - X_n(\tau))p_{ni}(p_{n0} - N^j p_{nj}),
\]

\[
= - \sum_n \delta^3(x - X_n) p_{ni} \quad (60)
\]

In Eq. (59) we used \( p_n^0 = g^{0\nu} p_{n\nu} = (1/N^2)(p_{n0} - N^i p_{ni}) \), replaced \((p_n^0)^2\) with \( \dot{X}_n^0/\lambda_n \), and then proceeded as follows:

\[
\int d\tau \frac{1}{X_n^0} \delta^3(x - X_n^0(\tau)) \frac{1}{N} (p_{n0} - N^i p_{ni}) \dot{X}_n^0
\]

\[
= \int d\tau \frac{1}{|X_n^0|} \delta^3(x - X_n^0(\tau)) \frac{1}{N} (p_{n0} - N^i p_{ni}) X_n^0 |_{\tau = \tau_c} = \delta^3(x - X_n) \frac{1}{N} (p_{n0} - N^i p_{ni}), \quad (61)
\]

where \( \tau_c \) is the solution of the equation \( x^0 = X_n^0(\tau) \), i.e., \( \tau_c = (X_n^0)^{-1}(x^0) \), and where we have taken positive \( \dot{X}_n^0 \). Since this equation is valid at any \( x^0 \), we omit in the last step the subscript \( |_{\tau = \tau_c} \). Similarly we proceeded in Eq. (60).

Algebraic structure of the constraints (58)–(60) is independent of foliation of spacetime, determined by \( N, N^i \). In any foliation, the form of the equations remain the same. The constraints (58)–(60) are thus covariant under diffeomorphisms.

By taking a linear combination of those constraints, we obtain a Hamiltonian:

\[
H = \int d^3 \mathbf{x} \left( N \mathcal{H} + N^i \mathcal{H}_i \right) = 0. \quad (62)
\]

Here \( \mathcal{H} = \mathcal{H}_G + \mathcal{H}_m, \mathcal{H}_i = \mathcal{H}_{G_i} + \mathcal{H}_{m_i} \), where \( \mathcal{H}_m, \mathcal{H}_{m_i}, \mathcal{H}_G, \) and \( \mathcal{H}_{G_i} \) are given in Eqs. (50), (60), (55)–(57). The terms with \( N^i p_{ni} \) cancel out, and so we obtain

\[
H = \int d^3 \mathbf{x} (N \mathcal{H}_G + N^i \mathcal{H}_{G_i}) + \sum_n p_{n0} = 0. \quad (63)
\]

In the latter equation, which holds for arbitrary \( N, N^i \), the matter Hamiltonian is just the sum of particle’s momenta \( p_{n0} \), i.e., their energies.

Eq. (63) can also be obtained directly from the Einstein equations integrated over a space like hypersurface according to

\[
\frac{\kappa}{2} \int d\Sigma_{0\nu} \sqrt{-g} G_{0\nu} = - \int d\Sigma_{0\nu} \sqrt{-g} T_{0\nu} = -P_0. \quad (64)
\]

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If the stress-energy tensor is concentrated so that it effectively forms a discrete set of particles in the sense as previously described, then the total energy \( P_0 \) in Eq. (64) is just the sum of particle energies, occurring in Eqs. (64).

Instead of Eq. (64) we can take the corresponding covariant form

\[
\frac{\kappa}{2} \int d\Sigma^\nu \sqrt{-g} G_{\mu \nu} n^\mu = - \int d\Sigma^\nu \sqrt{-g} T_{\mu \nu} n^\mu = - P_\mu n^\mu, \tag{65}
\]

where \( n^\mu \) is a unit vector field. If \( T^{\mu \nu} = \rho u^\mu u^\nu \), i.e., a dust stress-energy tensor and \( u^\mu \) the 4-velocity, then it is convenient to take \( n^\mu = u^\mu \). A discrete version of the dust stress energy tensor is the one obtained from the system of "point particles", described by the action (52). Then for \( n^\mu \) we can take a vector field that at the locations of particles coincides with their 4-velocities \( u^\mu_n = p^\mu_n/m_n \).

Because the constraints (59)–(60) are in fact just the ADM split of Einstein’s equations, we see that as a consequence of the Bianchi identities they are conserved in time, or along any time like curve whose tangent is \( u^\nu \):

\[
u^\nu D_\nu (G^{\mu \nu} + 8\pi T^{\mu \nu}) = 0. \tag{66}
\]

The constraint (58), namely, \( g_{\mu \nu} p^\mu_n p^\nu_n - m_n^2 = 0 \), also is conserved,

\[
u^\alpha D_\alpha (g_{\mu \nu} p^\mu_n p^\nu_n - m_n^2) = 2 g_{\mu \nu} u^\alpha p^\mu_n D_\alpha p^\nu_n = 2 p_{\mu \nu} u^\alpha D_\alpha p^\nu_n = 0, \tag{67}
\]

where we now assume that \( u^\alpha \) is tangent along a geodesic, so that \( p^\alpha_n = m_n u^\alpha \). Then \( u^\alpha D_\alpha u^\nu = 0 \), and the equality (67) is satisfied.

When quantizing the theory, one has to fix a gauge. In our case this is achieved by choice of \( N, N^\alpha \). Because \( N \) and \( N^\alpha \) behave as Lagrange multipliers, they can be arbitrary functions of spacetime coordinates \( x^\mu \). We will use the choice \( N = 1 \), \( N^\alpha = 0 \). It is straightforward to show that such gauge fixing imposes no second class constraints. Upon quantization, in the Schrödinger representation in which \( X^\mu_n \) and \( q_{ij} \) are diagonal, the momentum operators are the covariant generalizations of \( \hat{p}_{\mu \nu} = -i \partial/\partial X^\mu_n \) and \( \hat{\pi}^{ij} = -i \partial/\partial q_{ij} \). If acting on a functional only once, then \( \hat{p}_{\mu \nu} \) behaves as partial derivative, and \( \hat{\pi}^{ij} \) as functional derivative, otherwise \( \hat{p}_{\mu \nu} \) is covariant derivative with respect to \( g_{\mu \nu} \), and \( \hat{\pi}^{ij} \) is covariant functional derivative with respect to the metric \( G_{ij \kappa \ell} \delta(x - x') \). The constraints (58)–(60) become conditions on a state represented as \( \tilde{\varphi}[T_1, T_2, ..., T_N, X^i_1, X^i_2, ..., X^i_N, q_{ij}(x)] = \tilde{\varphi}[T_n, X^i_n, q_{ij}(x)], n = 1, 2, ..., \infty \), where \( T_n \equiv X^\mu_n \).

\[
(\hat{p}^2_{00} - q^{ij\mu}_n \hat{p}_{\mu \nu} - m_n^2) \tilde{\varphi}[T_n, X^i_n, q_{ij}(x)] = 0, \tag{68}
\]

\[
(\mathcal{H}_G - \sum_n \delta^i(x - X^i_n) i \frac{\partial}{\partial T_n}) \tilde{\varphi}[T_n, X^i_n, q_{ij}(x)] = 0, \tag{69}
\]

\(^7\text{See explanation after Eq. (92).}\)
\[
\left( \mathcal{H}_G - \sum_n \delta^3(x - X_n) i \frac{\partial}{\partial X_n^i} \right) \tilde{\varphi}[T_n, X_n^i, q_{ij}(x)] = 0. \tag{70}
\]

Integrating the last two equations over \( x \), we obtain
\[
\left( H_G - \sum_n i \frac{\partial}{\partial T_n} \right) \tilde{\varphi}[T_n, X_n^i, q_{ij}(x)] = 0, \tag{71}
\]
\[
\left( H_{Gi} - \sum_n i \frac{\partial}{\partial X_n^i} \right) \tilde{\varphi}[T_n, X_n^i, q_{ij}(x)] = 0, \tag{72}
\]

where
\[
H_G = \int d^3x \mathcal{H}_G = \int d^3x \left( \frac{1}{\kappa} G_{ij} \frac{\delta^2}{\delta q_{ij} \delta q_{k\ell}} + \kappa \sqrt{q} R^{(3)} \right), \tag{73}
\]
\[
H_{Gi} = \int d^3x \mathcal{H}_{Gi} = -\int d^3x (-i) 2D_j \frac{\delta}{\delta q_{ij}(x)}. \tag{74}
\]

At this point let us recall that, as shown by Moncrief and Teitelboim [24], the momentum constraint (60) is a consequence of the conservation of the Hamilton constraint (59). Therefore, it is sufficient if we consider the Hamilton constraint (59) and its quantum versions (69) or (71) only. We then have
\[
(H_G + H_m) \tilde{\varphi}[T_n, X_n^i, q_{ij}(x)] = 0, \tag{75}
\]
where
\[
H_m = \sum_n \hat{p}_{n0} = -i \sum_n \frac{\partial}{\partial T_n}. \tag{76}
\]

In this description a quantum state is represented by \( \tilde{\varphi}[T_n, X_n^i, q_{ij}(x)] \), which is a function of the particles’ spacetime coordinates, \( X_n^\mu = (T_n, X_n^i) \), and a functional of the dynamical variables of gravity, \( q_{ij}(x) \). In the absence of horizons, which is the situation that we consider here, we can choose coordinates \( X_n^\mu \) so that all time coordinates \( X_n^0 = T_n \) on a given 3-surface \( \Sigma \) are equal: \( T_1 = T_2 = ... = T_r = T \). Then we can write \( \tilde{\varphi}[T_n, X_n^i, q_{ij}(x)] \) as a function of a single time coordinate \( T \):
\[
\tilde{\varphi}[T_n, X_n^i, q_{ij}(x)] = \phi[T, X_n^i, q_{ij}(x)], \quad n = 1, 2, ..., r, \tag{77}
\]
and
\[
\frac{d \tilde{\varphi}}{dT} = \sum_n \frac{\partial \tilde{\varphi}}{\partial T_n} \frac{\partial T_n}{\partial T} = \sum_n \frac{\partial \tilde{\varphi}}{\partial T_n} = \frac{\partial \phi}{\partial T}. \tag{78}\]

---

The ordering issue can be settled by replacing \( \delta / \delta q^{ij} \) with covariant functional derivatives with respect to the Wheeler-DeWitt metric, and proceeding à la Ref. [15]. A further development of this important point is beyond the scope of the present paper. As in so many other papers, also here \( \delta / \delta q^{ij} \) has only a symbolic meaning, unless considered as the covariant functional derivative.
Equation (75) then becomes

\[ H_G \phi[T, X^i_n, q_{ij}(x)] = i \frac{\partial \phi[T, X^i_n, q_{ij}(x)]}{\partial T} \]  \hspace{1cm} (79) \]

We see that if one starts from the classical action (49) in which the matter part is expressed in terms of the worldlines of individual particles, then upon quantization we arrive at the Wheeler-DeWitt equation (79) which contain time. The wave functional depends on coordinates of particles, \( X^\mu_n = (T_n, X^i_n) \), and the induced metric on a hypersurface \( \Sigma \), defined by \( X^0_n = T = \text{constant} \) (on which all particles have the same time coordinates \( T_n = T \)).

Besides the time dependent equation (79), the wave functional also satisfies the Klein-Gordon equation (68). Assuming real \( \tilde{\phi} \), the second order Klein-Gordon equation can be cast into the form of a first order equation for a complex wave function [25, 26]

\[ \psi = \psi_R + i \psi_I, \]  \hspace{1cm} (80) \]

where

\[ \psi_R = \phi(T, X_1, X_2, ..., X_r) = \tilde{\phi}(T_1, T_2, ..., T_r, X_1, X_2, ..., X_r), \]  \hspace{1cm} (81) \]

and

\[ \psi_I = i \Omega^{-1} \dot{\phi}(T, X_1, X_2, ..., X_r). \]  \hspace{1cm} (82) \]

So instead of (68) we obtain the equivalent equation (see Appendix B)

\[ i \frac{\partial \psi}{\partial T} = \Omega \psi. \]  \hspace{1cm} (83) \]

Here \( \Omega \) is a matrix, by means of which the above equation is just a compact form of Eq. (68) that we derived in Sec. 2 after expanding the Schrödinger wave functional \( \Phi[t, \phi(x)] \) in terms of multiparticle wave functions according to Eq. (37), and after generalizing it so to include the 3-surface induced metric \( q_{ij} \) as well.

The function \( \psi \), occurring in Eq. (83), is the true, complex valued, wave function (related to the probability density), whilst the function \( \tilde{\phi} \) satisfying the Klein-Gordon equation (68), is just a real field.

By using the expression (80) and Eq. (79), it is straightforward to derive that in addition to Eq. (79), valid for the real field \( \tilde{\phi} \), we also have the similar equation for the complex wave function \( \psi \):

\[ H_G \psi[T, X^i_n, q_{ij}(x)] = i \frac{\partial \psi[T, X^i_n, q_{ij}(x)]}{\partial T}. \]  \hspace{1cm} (84) \]

In the above treatment we started from the classical action (49) and the corresponding constraints which upon quantization became the conditions on states (68)–(70) that can be represented either as a wave functional \( \tilde{\phi}[T_n, X^i_n, q_{ij}(x)] \) or
φ[T, Xiem, qij(x)]. No further, i.e., a “second” quantization is performed here. The complex functionals ψ[T, Xiem, qij(x)], n = 1, 2, 3, ..., are obtained according to the prescriptions (80)–(82). The matter Hamiltonian $H_m$ plus the gravitational Hamiltonian $H_G$ acting together on $ψ[T, Xiem, qij(x)]$ give zero. Time automatically appears in the equations, such as (83) or (84).

3.2 Multiparticle states arising from the traditional approach, and their relation to the novel approach

In the previous subsections we discussed a novel approach in which we started from a classical matter represented as a multi particle system. In the usual treatments matter action is not $I_m[X^μ n, g_{μν}(x)], or equivalently, I_m[X^n μ, p_{νμ}, λ_n, g_{μν}] (Eqs. (52),(53)), but is a functional of fields, for instance scalar fields:

$$I_m[ϕa(x), g_{μν}(x)] = \frac{1}{2} \int d^4x \sqrt{-g} \left( g^{μν} \partial_μ ϕ^a \partial_ν ϕ_a - m^2 ϕ^a ϕ_a \right),$$

(85)

which after the ADM split reads:

$$I_m[ϕ^a, q_{ij}, N, N^i] = \frac{1}{2} \int d^3x N \sqrt{q} \left[ \frac{1}{N^2} (ϕ^a - N^i \partial_i ϕ^a)(ϕ_a - N^j \partial_j ϕ_a) 
- q^{ij} \partial_i ϕ^a \partial_j ϕ_a - m^2 ϕ^a ϕ_a \right].$$

(86)

Variation of the total action

$$I = I_m[ϕ^a, q_{ij}, N, N^i] + I_G[q_{ij}, N, N^i]$$

(87)

with respect to $N$ and $N^i$ gives the constraints $H = H_G + H_m$ and $H_i = H_{Gi} + H_{mi}$. Here the gravitational part of the constraints, $H_G$, $H_{Gi}$, are given in Eqs. (55),(56), whilst the matter parts $H_m$, $H_{mi}$ are now

$$H_m = \frac{\sqrt{q}}{2} \left( \frac{Π^a Π_a}{q} + q^{ij} \partial_i ϕ^a \partial_j ϕ_a + m^2 ϕ^a ϕ_a \right)$$

(88)

$$H_{mi} = \partial_i ϕ^a Π_a,$$

(89)

where

$$Π_a = \frac{∂L}{∂ϕ^a} = \frac{\sqrt{q}}{N} (ϕ_a - N^i \partial_i ϕ_a).$$

(90)

The Hamiltonian is a linear combination of the constraints

$$H = \int d^3x (N H + N^i H_i).$$

(91)
The Lagrange multipliers $N$ and $N^i$ are arbitrary, and, as before, we choose $N = 1$, $N^i = 0$, so that

$$H = \int d^3x \mathcal{H} = \int d^3x \left( \mathcal{H}_G + \mathcal{H}_m \right) = H_G + H_m.$$  \hspace{1cm} (92)

Such a choice brings no second class constraints, because it involves only Lagrange multipliers and their conjugate momenta $\pi$, $\pi_i$, and no other phase space variables. Namely, setting $\phi_1 = N - 1$, $\phi_2 = \pi$, $\phi_1^i = N^i$, $\phi_2^i = \pi_i$, and calculating the time derivative of the constraints according to $\dot{\phi} = \{\phi, H\}$, we obtain $\dot{\phi}_1 = 0$, $\dot{\phi}_2 = -H$, $\dot{\phi}_1^i = 0$, $\dot{\phi}_2^i = \mathcal{H}_i$, which are the original constraints.

Upon quantization we have

$$\langle H_G + H_m | \Phi \rangle = 0,$$  \hspace{1cm} (93)

which in the Schrödinger functional representation gives

$$\langle \varphi^a(x), q_{ij}(x) | (H_G + H_m) | \Phi \rangle = 0.$$  \hspace{1cm} (94)

Explicitly, the latter equations reads

$$\int d^3x \left[ \frac{1}{\kappa} G_{ijk\ell} \frac{\delta^2}{\delta q_{ij}\delta q_{k\ell}} + \sqrt{q} R^{(3)} \right] + \frac{\sqrt{q}}{2} \left( -\frac{\delta^2}{\delta \varphi^a\delta \varphi_a} + \varphi^a(-D^iD_i + m^2)\varphi_a \right) \Phi[\varphi^a(x), q_{ij}(x)] = 0,$$  \hspace{1cm} (95)

where $H_G$, $H_m$ are represented as matrices in the space of fields $\varphi^a(x)$, $q_{ij}(x)$, i.e., as functional differential operators. Concerning the factor ordering of the operators $\hat{\pi}^{ij}$ we can adopt and generalize the procedure of Ref. [15].

The functional $\Phi[\varphi^a(x), q_{ij}(x)]$ is “first quantized” with respect to the metric $q_{ij}(x)$, and “second quantized” with respect to particle position. This is a “hybrid” procedure, and there is manifestly no time in Eq. (95).

The matter Hamiltonian in Eq. (95),

$$H_m = \int d^3x \frac{\sqrt{q}}{2} \left( -\frac{\delta^2}{\delta \varphi^a\delta \varphi_a} + \varphi^a(-D_iD^i + m^2)\varphi_a \right),$$  \hspace{1cm} (96)

can be expressed in terms of the operators $a[x, q_{ij}(x)]$, $a^\dagger[x, q_{ij}(x)]$, defined in Sec. 2, so we have

$$H_m = \int d^3x \left( a^\dagger[x, q_{ij}(x)] \sqrt{m^2 - D^iD_i} a[x, q_{ij}(x)] + \text{z.p.} \right).$$  \hspace{1cm} (97)

This is an exact expression for $H_m$. 

Let us now take the following Ansatz:

\[ \Phi[\varphi(x), q_{ij}(x)] = \sum_n d^3x_1 \ldots d^3x_n \psi[x_1, \ldots, x_n, q_{ij}(x)] \tilde{\Phi}_n[\varphi(x), q_{ij}(x)], \tag{98} \]

where

\[ \tilde{\Phi}_n[\varphi(x), q_{ij}(x)] = a^\dagger[x_1, q_{ij}(x)] \ldots a^\dagger[x_n, q_{ij}(x)] \tilde{\Phi}_0[\varphi, q_{ij}(x)] \quad (99) \]

is obtained by the action of the creation operators on the vacuum \( \tilde{\Phi}_0[\varphi, q_{ij}(x)] \), which is now not the vacuum (42), valid in the case of a fixed background metric, but a suitably generalized expression accounting for the fact that now the metric is dynamical. Formally, let us set

\[ \tilde{\Phi}_0[\varphi, q_{ij}(x)] \propto \exp \left[ -\frac{1}{2} \int d^3x \sqrt{q(x)} \sqrt{m^2 - D_i^2} \varphi + Q[q_{ij}] \right], \tag{100} \]

which contains in the exponential an additional functional \( Q[q_{ij}] \), that could contain, among others, the expressions such as \( \int d^3x \sqrt{qR(3)} \).

Using the commutation relations\(^9\)

\[ [a[x, q_{ij}(x)], a^\dagger[x', q_{ij}(x)]] = \delta^3(x - x'), \tag{101} \]

\[ [a[x, q_{ij}(x)], a[x', q_{ij}(x)]] = 0, \quad [a^\dagger[x, q_{ij}(x)], a^\dagger[x', q_{ij}(x)]] = 0, \tag{102} \]

the expansion (98), and the matter Hamiltonian (96), we find

\[ H_m \Phi = \sum_{r=1}^{\infty} \int d^3x_1 d^3x_2 \ldots d^3x_n \sum_{n=1}^{r} \sqrt{m^2 - D^2_n} \psi[x_1, x_2, \ldots, x_r, q_{ij}(x)] \]

\[ \times a^\dagger[x_1, q_{ij}(x)] \ldots a^\dagger[x_r, q_{ij}(x)] \tilde{\Phi}_0[\varphi(x), q_{ij}(x)], \tag{103} \]

where

\[ D^2_n = q^{ij} D_{ni} D_{nj}, \quad D_{ni} = \frac{D}{DX_{ni}}. \tag{104} \]

If we postulate that the quantizations based on the actions (49) and (85) are equivalent, then the wave functional \( \psi[x_1, \ldots, x_n, q_{ij}(x)] \) is the same one as considered in Eqs. (68)–(72), depends on time \( T \) and satisfies (84). Here we have renamed \( X^i_n \equiv X_n \), occurring in Eqs. (68)–(72), into \( x_n, n = 1, 2, \ldots, n = \infty \). If time \( T \) occurs in \( \psi \), then it automatically also occurs in \( \Phi \).

Taking now into account Eq. (84), we obtain

\[ H_m \Phi = i \frac{\partial \Phi}{\partial T}, \tag{105} \]

\(^9\)As we pointed in Sec. 2, the metric and its determinant cancel out, so that they do not appear in the r.h.s. of the commutation relation.
We have thus reproduced the functional representation of the time dependent Schrödinger equation for the scalar field in the presence of a 3-metric field $q_{ij}(x)$.

Returning to Eq. (94) and using the latter result, namely that $H_m \Phi = i \partial \Phi / \partial T$, we obtain

$$H_G \Phi = -i \frac{\partial \Phi}{\partial T}.$$  \hspace{1cm} (106)

We have thus obtained a time dependent Schrödinger equation for the gravitational part of the Hamilton operator. Both equations, (105) and (106), together give the Wheeler-DeWitt equation (94). It thus turns out that though time does not manifestly take place in the Wheeler-DeWitt equation (94), it is hidden in the wave functional $\Phi$, if the quantization procedures discussed in Secs. (3.1) and (3.2) are equivalent in the sense that they lead to the same physics.

In the approach considered in this paper the problem of time does not exist, because we started from the classical action in which gravity is coupled to a multi-particle system that by definition contains time; upon quantization of such system, time does not disappear. This is different from the usual approaches in which the classical action to be quantized contains gravity coupled to a field, for instance a scalar field, and there is a problem of how to assign the role of time to certain degrees of freedom entering the quantum equations.

There are many different approaches to the problem of time in quantum gravity, extensively reviewed by Anderson [8]. In particular, Lapchinsky and Rubakov [27] showed that it is possible to introduce a Tomonaga bubble-time parameter $\sigma(x)$ denoting spatial hypersurfaces that spacetime is foliated to. In their approach, the gravity part of the Hamilton constraint gives in the case where $\sigma = \text{const} = t$, the time derivative of the matter state in the presence of a quasiclassical background gravitational field.

Such an approach that requires consideration of the quasiclassical approximation was criticized by Barvinsky [28], because it was not just a conventional calculational tool, but the only means to introduce the notion of time, probability, etc., in quantum cosmology. Therefore, Barvinsky developed a procedure that employed no (quasiclassical) approximation, but introduced spacetime foliation by imposing gauge conditions on the degrees of freedom of gravity and matter. In that approach, matter degrees of freedom were represented by scalar fields. But in principle an analogous procedure holds for the case in which matter degrees of freedom are represented by multi-particle spacetime coordinates $X^\mu_n$ occurring in the action (53). A peculiar feature of such approach is that the time-like coordinates $X^0_n \equiv T_n$ represent just the many-fingered time associated with a spacetime foliation. A particular choice of coordinates, such that $T_1 = T_2 = ... = T_n = T$ (see Eq. (77)) corresponds to $\sigma = \text{const} = t$, mentioned above. The many-fingered time thus occurs in the very construction of the matter action. Therefore, the gauge condition that reduces the
number of variables to the physical degrees of freedom can be $N = 1$, $N^i = 0$ and is straightforwardly associated with a particular spacetime foliation in the quasiclassical limit.

An alternative procedure was considered by Peres [29] who transformed a constrained system to an unconstrained one by a suitable canonical transformation of the phase space variables, such that one of the so obtained variables serves the role of time. No such a canonical transformation is necessary in our approach in which the coordinates $X_0^0 \equiv T_n$ already have the role of (many-fingered) time.

4 Discussion

When investigating the universe and behaviour of objects in it, we normally have their positions in mind. In quantum gravity the entire universe is envisaged to be describable by a wave function. However, in practical calculations only few degrees of freedom of the universe are taken into account, but we assume that in reality there exists a wave function(al) of the universe that comprises all degrees of freedom, including those of observers. We base our discussion of the universe on canonical gravity which we modified in the part that relates to matter. Usually matter is represented by fields, such as a scalar, spinor, gauge field, etc. In the model considered in this paper we started directly from the classical action for a multi particle system coupled to gravity, and arrived after quantization to the Schrödinger equation for a set of multi-particle wave functionals $\psi[t, x_1, x_2, ..., x_r, q_{ij}(x)], r = 1, 2, ...$. Thus, in our approach a state satisfying the Wheeler-DeWitt equation is represented as a functional of gravity and matter degrees of freedom, the latter being given in terms of multi particle configurations. It then turns out that time does not disappear from the quantum equations, it is given in terms of the time like coordinates of particles, associated with the clocks situated on particle’s worldlines. We have thus set up a theoretical framework for a quantum gravity description of the universe, which involves positions of particles, a concept in many respect closer to our intuition and observations than the concept of scalar field.

In this paper we have confined us to discussing the Wheeler-DeWitt equation whose semiclassical solutions are well-known to contain singularities. How to avoid singularities has been discussed, e.g., by Claus Kiefer and Barbara Sandhofer [30] (see also refs [31–35] cited therein) who conclude: ”Upon discussing the Wheeler-DeWitt equation, one finds that all normalizable solutions lead to a wave function that vanishes at the point of the classical singularity; this we interpret as singularity avoidance. An analogous situation of singularity avoidance is found in the loop quantum cosmology of this model [36].” Our classical action (49) of point particles coupled to gravity, if taken literally, would be impossible, because at the positions
of point particles there would be black hole singularities. We avoided singularity by postulating that the particles are extended and that $X^\mu$ are coordinates of their effective positions (analogous to the center of mass of a non-relativistic particle).

Our procedure could as well be upgraded into a promising direction: namely, by extending the classical point particle to the superparticle, by including, besides the commuting coordinates $X^\mu$, also the Grassmann anticommuting coordinates, and coupling such system to gravity. Thus, because of the presence of the additional, anticommuting coordinates, $\xi^\mu$, the problem of a point particle coupled to a gravitational field would be avoided, because the extra coordinates would bring into the game in an elegant way the particle’s effective extension. Upon quantization of such a model in which a superparticle is coupled to the metric (which now depends on $X^\mu$ and $\xi^\mu$), spinor fields would occur in the description. For the time being, in our current paper we describe a classical particle by four coordinates $X^\mu$ only, and assume that it is extended and held from collapsing into a black hole by the forces that are not included into the description. In our opinion this is legitimate, because a physical model necessarily involves a restricted set of variables, fields, etc., and neglects the rest.

However, we have to bear in mind that the action of Einstein’s gravity containing only the first order term of the curvature scalar, $R$, leads upon quantization to divergences that cannot be renormalized. In the present paper we have not addressed this problem. One possibility is to include into the gravity part of the action \( R^2 \), and higher order/derivative terms as well, and re-run the procedure considered in this paper, either with the same matter action consisting of point particles, or generalizing them to superparticles, as mentioned above. Inclusion of higher derivative terms would then solve the problem of renormalizability. But, as widely recognized, we would then have problems with ghosts and instabilities. There is a vast literature on how higher derivative gravity theories can be made physically viable (see, e.g., [42], and references therein.)

A higher derivative gravity action arises as an effective action of string theory. Another promising direction of research is loop quantum gravity. Both those theories have satisfactorily addressed the issue of quantum divergences. But string theory has run into serious problems, including the so-called “landscape”, while in loop quantum gravity it is not yet clear how spacetime manifold emerges from such a setup. I anticipate that the different theoretical structures, such as a higher derivative canonical gravity with superparticles as sources (that can be extended to superstrings), string theory, and loop quantum gravity, will at the end turn out to be revealing different aspects of an underlying more fundamental theory that is being crystalized in numerous works based on the powers of Clifford algebras.
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Appendix A: Coupling of “point-like” sources to the gravitational field

Because classical gravity contains black hole solutions, point particle sources are problematic\footnote{See the paper \cite{37}, where those problems are thoroughly analysed not only for point particles, but also for branes.}. However, if a particle is extended beyond its Schwarzschild radius, in principle there is no problem. When describing motion of an extended object, we may neglect its internal dynamics and consider only the motion of an effective worldline, a “center of mass”. In such a case the coupling of the object with the gravitational field may be given by Eq. (53), with understanding that $X^\mu(\tau)$ are center of mass coordinates, and that the range of the considered spacetime coordinates is outside the Schwarzschild radius.

As a model of extended object let us consider an open\footnote{This means that the brane does not fill the entire space, but only the volume inside a sphere, so that our brane is in fact a ball-filling brane.} space filling brane, described by coordinates $X^{\mu}(\xi^A)$, $\mu = 0, 1, 2, ..., D-1$, $A = 0, 1, 2, ... p$, where in our case of a space filling brane, $p = D - 1$. The action for such a system is

\[ I_m = \mu_B \int d^{p+1}\xi \left(-\det\partial_A X^\mu \partial_B X_\mu\right)^{1/2}, \quad f_{AB} \equiv \partial_A X^\mu \partial_B X_\mu, \quad (107) \]

where $\mu_B$ is the brane tension. When considered as a matter source of gravity the above action has to be rewritten so to include a $\delta$-function:

\[ I_m = \mu_B \int d^{p+1}\xi \left(-\det(\partial_A X^\mu \partial_B X_\mu)^{1/2}\delta^D(x - X(\xi))d^Dx. \quad (108) \]

This action is invariant under reparametrizations of the worldsheet parameters $\xi^A$, and under general coordinate transformations of spacetime coordinates $x^\mu$. Rewritten in terms of new $\xi^A$ and $x^\mu$, it retains the same form \footnote{This means that the brane does not fill the entire space, but only the volume inside a sphere, so that our brane is in fact a ball-filling brane.}, but with new functions $X^\mu(\xi^A)$, representing the same worldsheet.

Let us now split the derivative according to \cite{38,39}

\[ \partial_A = n_A \partial + \tilde{\partial}_A, \quad (109) \]

where $n_A$ is a vector in the direction of the hypersurface element $d\Sigma_A = d\Sigma n^A$, and $\partial$ is the derivative in the direction of $n_A$ (normal derivative), whilst $\tilde{\partial}_A$ is the tangential derivative, orthogonal to $n_A \partial$. Then we have

\[ f_{AB} = \partial_A X^\mu \partial_B X_\mu = n_A n_B \partial X^\mu \partial X_\mu + \tilde{\partial}_A X^\mu \tilde{\partial}_B X_\mu, \quad (110) \]

\[ \text{See the paper } \cite{37}, \text{ where those problems are thoroughly analysed not only for point particles, but also for branes.} \]
from which we obtain
\[ n^A n_A = \frac{1}{\partial X^\mu \partial X_\mu}, \]
\hspace{1cm} (111)
Inserting (110) into the definition of \( f = \det f_{AB} \), we obtain \[ 38, 39 \]
\[ f = \tilde{f}, \quad \tilde{f} = \frac{1}{p!} \epsilon^{A_1 \ldots A_p} \epsilon_{B_1 \ldots B_p} \tilde{f}_{A_1 B_1} \cdots \tilde{f}_{A_p B_p}, \quad \tilde{f}_{AB} \equiv \partial_A X^\mu \partial_B X_\mu. \] \hspace{1cm} (112)

In a gauge in which \( n_A = (1, 0, 0, \ldots) \), using the procedure of Ref. \[ 39 \], we have \( \xi^A = (\tau, \sigma^a), \quad a = 1, 2, \ldots, p, \partial X^\mu = \partial / \partial \tau \equiv \dot{X}^\mu, \quad \tilde{f} = \det f_{ab} \equiv \tilde{f}, \quad \partial_a = \partial / \partial \sigma^a, \) and \( d\Sigma = d^p \sigma \).

Using (109)–(112), the action (108) then reads
\[ I_m = \mu_B \int d\tau d^p \sigma \sqrt{-\tilde{f} \sqrt{\partial X^\mu \partial X_\mu} \delta^D (x - X(\tau, \sigma))} d^D x, \]
\hspace{1cm} (113)
which is a covariant expression, because it does not change its form under the transformations \( \xi^A \to \chi^A(\xi) \) and \( x^\mu \to x'^\mu = F^\mu(x) \). In a particular gauge (choice of parameters \( \xi^A \)), considered above, the action becomes
\[ I_m = \mu_B \int d\tau d^p \sigma \sqrt{-\dot{f} \sqrt{\dot{X}^\mu \dot{X}_\mu} \delta^D (x - X(\tau, \sigma))} d^D x, \]
\hspace{1cm} (114)

Let us now choose a line \( X_T^\mu(\tau) \) and write
\[ X^\mu(\tau, \sigma) = X_T^\mu(\tau) + w^\mu(\tau, \sigma). \] \hspace{1cm} (115)
Then
\[ I_m = \mu_B \int d\tau d^p \sigma \sqrt{-\tilde{f} \sqrt{\dot{X}_T^\mu \dot{X}_T^\mu} \delta^D (x - X_T(\tau) - w(\tau, \sigma))} d^D x, \]
\hspace{1cm} (116)
where we have now \( \tilde{f} = \det \partial_a w^\mu \partial_a w_\mu \).

Next, let us use the expansion\[ 12 \]
\[ \delta(x - X_T - w(\tau, \sigma)) = \delta(x - X_T(\tau)) \]
\[ + \int d^p \sigma w^\mu(\tau, \sigma) \frac{\delta \delta(x - X_T(\tau) - w(\tau, \sigma))}{\delta w^\mu(\tau, \sigma)} \bigg|_{w(\tau, \sigma)=0} + \ldots, \] \hspace{1cm} (117)
\[ \footnote{We can verify on a simpler example that such expansion indeed works:} \]
\[ \int dx F(x) \delta(x - a) = F(-a) \]
\[ \int dx F(x) \left( \delta(x) + \frac{\partial \delta(x - a)}{\partial a} \bigg|_{a=0} a + \ldots \right) = F(0) - F'(0)a + \ldots = F(-a) \]
and insert it into (116). Taking also a gauge such that the determinant $\bar{f}$ does not depend on $\tau$, the quantity $m = \mu_B \int d^p \sigma \sqrt{-\bar{f}}$ can be factored out. Assuming that the size of the brane is small and neglecting the terms with the powers of $w^{\mu}$, we obtain

$$I_m = m \int d\tau \sqrt{\dot{X}_T^\mu \dot{X}_T^\mu} d^D x \sqrt{-\bar{g}} \frac{\delta^D(x - X_T(\tau))}{\sqrt{-\bar{g}}}, \quad x \notin \Omega. \quad (118)$$

In the latter equation we have effectively approximated $X^\mu(\tau, \sigma)$ with $X_T(\tau)$, and taken only the region $x \notin \Omega$ of the spacetime outside the ball.

The action (118) implies that $X_T(\tau)$ is a geodesic. This is consistent with the brane equations of motions corresponding to the action (107), $D_A D^A X^\mu = 0$, which can be split as $D_\tau D^\tau X_T + D_\tau D^\tau w + D_a D^a w^\mu = 0$. The $w$-terms in the latter equation are due to the extension of the object and represent a deviation from the geodesic equation, like in the Papapetrou equation. If we neglect them, then we have the geodesic equation.

The above example of a ball, modelled `a la space filling brane, shows how an extended object can be approximately described as a point particle coupled to the gravitational field, with understanding that only the region outside the horizon is taken into account. The region inside horizon is not taken into account, because the object is actually not point-like, but extended. With the ball, we do not have a smeared $\delta$-function, but a “true” $\delta$ function, namely, $\delta^D(x - X(\tau, \sigma)) / \sqrt{-\bar{g}}$, i.e., an object which is covariant under general coordinate transformations of $x^\mu$, and also under reparametrization of $\xi^A = (\tau, \sigma^a)$.

The procedure explained above can be straightforwardly adapted to hold not only for a brane, filling a ball, but also for any brane, for instance, for a closed 2-brane, considered by Dirac as a model of electron.

In our procedure we in fact avoided the problem of defining the center of mass in special an general relativity. Namely, a far away observer cannot distinguish among arbitrarily chosen lines within the extended object whose size is negligible in comparison with the considered distances. Let us now nevertheless demonstrate how the center of mass could be defined, first in flat spacetime and then in a curved one.

Choosing a unit time like direction $n^\mu$ in Minkowski space, let us define the center of mass coordinates for a system of point particles according to

$$X^\mu_T = \frac{\sum_k p_k^\mu n_\mu X_k^\alpha N_\alpha^\mu}{\sum_k p_k^\mu n_\sigma}, \quad (119)$$

where $N_\alpha^\mu = \delta_\alpha^\mu - n_\alpha n^\mu$ is the projector onto the hypersurface $\Sigma_\mu$, orthogonal to $n_\mu$. The center of mass coordinates can thus be interpreted as being defined with respect to a chosen simultaneity surface $\Sigma_\mu$, associated with an observer.

The Poisson brackets between so defined center of mass coordinates and the total
momentum \( P^\nu = \sum_k P^\nu_k \) are
\[
\{ X^\mu_T, X^\nu_T \} = \frac{\partial X^\mu_T}{\partial X^\nu_T} \frac{\partial X^\nu_T}{\partial p_{k\beta}} - \frac{\partial X^\nu_T}{\partial X^\mu_T} \frac{\partial X^\mu_T}{\partial p_{k\beta}} = 0,
\]
(120)
\[
\{ X^\mu_T, P^\nu \} = N^\mu^\nu = \eta^\mu^\nu - n^\mu n^\nu.
\]
(121)

These equations are Lorentz covariant. In the particular case of \( n^\mu = (1, 0, 0, 0) \), we have
\[
\{ X^0_T, P^0 \} = \eta^{00} - 1 = 0,
\]
(122)
\[
\{ X^r_T, P^s \} = \eta^{rs}, \quad r, s = 1, 2, 3
\]
(123)
\[
\{ X^0_T, P^s \} = \eta^{0s} - n^0 n^s = 0,
\]
(124)
\[
X^\mu_T = \frac{\sum_k p^0_k (X^\mu_k - X^0_k n^\mu)}{\sum_k p^0_k} = \begin{cases} 0 & \text{if } \mu = 0, \\ \frac{\sum_k p^0_k X^\mu_k}{\sum_k p^0_k} & \text{if } \mu = r. \end{cases}
\]
(125)

We see that these are correct Poisson bracket relations.

For a generic stress-energy tensor we have
\[
X^\mu_T = \int d\Sigma_T T^{\nu\rho} n_\rho X^\alpha N^\alpha_{\mu}
\]
(126)
\[
\int d\Sigma_T T^{\nu\rho} n_\rho.
\]

How to define the center of mass in curved spacetime is much debated, with no unique generally accepted solution. Our tentative proposal is first to consider \( x^\alpha \) as a vector field which in given coordinates \([14]\) is \( a^\alpha(x) = x^\alpha \). Then we can generalize (126) to
\[
X^\mu_T = \int d\Sigma_T T^{\nu\rho} n_\rho a^\alpha(x) N^\alpha_{\mu}
\]
(127)
where \( \int_{[x_0]} \) denotes the covariant integral over a vector field, which in the above case is \( A^\mu(x) = a^\alpha N^\alpha_{\mu} \). This means that the vectors \( A^\mu(x) \) at different points \( x \) are paralellly transported along a geodesic from the point \( x \) to a chosen point \( x_0 \) (the “origin”), where they are summed (integrated). How precisely this works is shown in Refs. \([40, 41]\) and \([15]\).

For the purpose of the procedure adopted in this paper it is sufficient that the center of mass, or any other point that samples the motion of a finite size particle, does exist. The precise location of such point within the particle is not important for the validity of our procedure.
Appendix B: The connection between the multiparticle Schrödinger and Klein-Gordon equation

The multiparticle Schrödinger equation (48) can be cast into the following system of equation for the real and the imaginary part of the complex wave function $\psi = \psi_R + i\psi_I$:

$$\frac{\partial \psi_R}{\partial t} = \sum_{k=1}^{r} \omega_{x_k} \psi_I(t, x_1, x_2, \ldots, x_r), \quad (128)$$

$$\frac{\partial \psi_I}{\partial t} = -\sum_{k=1}^{r} \omega_{x_k} \psi_R(t, x_1, x_2, \ldots, x_r), \quad (129)$$

Introducing the compact notation

$$x_1, x_2, \ldots, x_r \equiv X_r, \quad \psi_R(x_1, x_2, \ldots, x_r) \equiv \psi_R(X_r) \equiv \psi^{(X_r)}_R,$$

we can rewrite Eqs. (128), (129) as

$$\dot{\psi}^{(X_r)}_R = \Omega^{(X_r)}(X_s) \psi^{(X_s)}_R, \quad (132)$$

$$\dot{\psi}^{(X_r)}_I = -\Omega^{(X_r)}(X_s) \psi^{(X_s)}_R, \quad (133)$$

Expressing $\psi^{(X_s)}_I$ in Eq. (132) in terms of $\dot{\psi}^{(X_r)}_R$,

$$\psi^{(X_s)}_I = \Omega^{-1}(X_s)(X_r) \dot{\psi}^{(X_r)}_R, \quad (134)$$

and inserting it into Eq. (133), we obtain the following second order equation:

$$\ddot{\psi}^{(X_r)}_R + \Omega^{(X_r)}(X_s) \Omega^{(X_s)}(X_k) \psi^{(X_k)}_R = 0. \quad (135)$$

Taking into account the explicit form (131) of the matrix $\Omega^{(X_r)}(X_s)$, the latter equation becomes

$$\ddot{\psi}_R(t, X_1, X_2, \ldots, X_r) + \sum_{m=1}^{r} \sum_{n=1}^{r} \omega_{x_m} \omega_{x_n} \psi_R(t, X_1, X_2, \ldots, X_r) = 0. \quad (136)$$

Introducing now the notation $\phi(t, X_1, X_2, \ldots, X_r) \equiv \psi_R(t, X_1, X_2, \ldots, X_r) \equiv \tilde{\phi}(t_1, t_2, \ldots, t_r, X_1, X_2, \ldots, X_r)$ and using the relation (78), which implies

$$\frac{d^2 \phi}{dt^2} = \sum_{m,n} \frac{\partial^2 \tilde{\phi}}{\partial t_m \partial t_n}, \quad (137)$$
Eq. (136) can be written in the form

\[ \sum_{M,n} \left( \frac{\partial^2 \tilde{\varphi}}{\partial t_m \partial t_n} + \omega_{x,n} \omega_{x,m} \tilde{\varphi} \right) = 0. \]  

(138)

In the latter equation is embraced the multiparticle Klein-Gordon equation (68)

\[ \frac{\partial^2 \tilde{\varphi}}{\partial t_n^2} + \omega_{x,n}^2 \tilde{\varphi} = 0. \]  

(139)

For illustration let us now consider a flat space solution of the above equation,

\[ \tilde{\varphi}(t_1, ..., t_r, x_1, ..., x_r) \]

\[ = \int d^3p_1 ... d^3p_r \left( c(p_1, ..., p_r) e^{-\sum_k p_k x_k} + c^*(p_1, ..., p_r) e^{\sum_k p_k x_k} \right). \]  

(140)

Taking successively the derivative with respect to \( t_n \) and \( t_m \), we obtain

\[ \frac{\partial^2 \tilde{\varphi}}{\partial t_m \partial t_n} = \int d^3p_1 ... d^3p_r \left( -\omega_{p_m} \omega_{p_n} \right) \left( c(p_1, ..., p_r) e^{-\sum_k p_k x_k} + c^*(p_1, ..., p_r) e^{\sum_k p_k x_k} \right). \]

(141)

where \( \omega_{p_m} = \sqrt{m^2 + p_n^2} \). Summing over \( m, n \) we obtain precisely Eq.(138) (apart from the fact that the above illustration is done for a flat space solution, while Eq. (138) holds in curved space as well).

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