On perfect fluids and black holes in static equilibrium

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Abstract. Proofs of spherical symmetry of static black holes and of spherical symmetry of static perfect fluids normally require, a priori, “black holes only” or “fluid only”. In a recent paper Shiromizu, Yamada and Yoshino \cite{shiromizu05} admit a priori (and exclude) coexistence of fluids and holes. This work assumes connectedness of the fluid region and the same assumptions on the equation of state as earlier papers on the “fluid only” case, and requires in addition an upper bound for the fluid mass in terms of the black holes masses. We discuss this paper. As a new result we show that there cannot exist static fluid shells (i.e. fluid regions of the topology of an annulus) even if one a priori admits, inside and outside the shell, any arrangement of black holes or additional matter which satisfies the energy condition.

1. Introduction
Field configurations in vacuum or with isotropic energy momentum tensor cannot resist tangential stresses. This leads, in particular, to the conjecture that static black holes as well as static perfect fluids take on a spherically symmetric form. The “no-hair” theorem for static, asymptotically flat vacuum solutions of Einstein’s equations which a priori admits several non-degenerate black holes components was proven by Bunting and Masood-ul-Alam \cite{bunting83}. The crucial steps of this proof are the “doubling” of the manifold along the horizons, the application of a conformal rescaling which preserves asymptotic flatness, keeps the curvature non-negative but removes the mass, and the use of the rigidity case of the positive mass theorem. On the other hand, the “fluid ball problem”, has a long history in Newtonian theory already \cite{teens}. In Relativity, spherical symmetry is expected to hold under analogous conditions, namely for static, asymptotically flat solutions of Einstein’s equations with a sufficiently smooth equation of state (eqos) $\rho = \rho(p)$ such that $\rho \geq 0$ for $p \geq 0$ (called “the positive eqos” in what follows). For such a fluid, it is only known that there are no non-spherical static perturbations around the spherically symmetric solutions \cite{wetterich93}. The non-perturbative arguments \cite{hawking79, iyer85, dain87} proceed along the same lines as the black hole proof after doubling. However, the procedure of finding suitable conformal rescalings is here quite an intricate business which has only been achieved under additional restrictions, both on the equation of state as well as on the connectedness of the fluid region. In particular, for connected fluids of spherical topology, the more recent results \cite{dain87, dain00} require the restriction $I \leq 0$ where

$$I \equiv (\rho + p) \frac{d}{dp} \kappa + 2\kappa + \frac{1}{5} \kappa^2 \quad \text{with} \quad \kappa = \frac{d}{dp} \frac{(\rho + p)}{(\rho + 3p)}. \quad (1)$$

In a recent paper Shiromizu, Yamada and Yoshino \cite{shiromizu05} consider the situation where fluids and black holes are a priori allowed to coexist in static asymptotically flat spacetimes. Admitting an
arbitrary number of black holes, but as above a connected fluid satisfying $I \leq 0$, these authors obtain spherical symmetry under an additional technical assumption which amounts to requiring that the fluid mass is sufficiently small compared to the black hole masses. In this paper we present, after the technical section 2, in Sect. 3 a related result (Theorem 1) which excludes fluid regions of annular shape. This result is quite simple in the sense that it uses little more than the maximum principle and the boundary point lemma applied to the field equations for the gravitational potential in the respective regions and on the inner fluid boundary. On the other hand, the result is surprisingly general as it admits a priori any, possibly disconnected, static arrangement of black holes and arbitrary matter as long as the latter satisfies an energy condition and has at least one fluid component of “annular” shape. In the same manner we can exclude inner boundaries for fluids extending to infinity, consistent with asymptotic flatness. The resulting spacetimes are either Schwarzschild if there is an exterior vacuum, or one of the Buchdahl solutions [9] otherwise. The second part of Sect. 3 contains a review of the result [1]. Lindblom and Masood-ul-Alam have also shown spherical symmetry of connected fluids under other conditions than $I \leq 0$, namely by imposing a weaker restriction on the equation of state but in addition a bound on the surface potential [8]. This set of conditions can also substitute $I \leq 0$ in the result [1]. In a final remark we point out that there are no examples known for which these two conditions are consistent. After this work was completed and presented at the ERE conference, one of us (W.S) received a preprint by Masood-ul-Alam in which he claims to prove uniqueness of fluid solutions for all positive eqos, also allowing disconnected fluid regions. This work employs a subtle refinement of the spinorial positive mass theorem admitting curvatures which are negative on sufficiently small sets. It seems possible to combine the techniques of Masood-ul-Alam with the ones of the black hole uniqueness proofs, along similar lines as sketched below. However, as the new positive mass theorem is quite subtle and not manifestly compatible with horizon boundary conditions, it would be undue to make a statement on this point here.

2. Basics

2.1. The field equations and the boundary conditions

We consider a static spacetime represented by a manifold $M$ with Lorentzian metric and timelike Killing field $\xi^\mu$, $(\mu = 0, 1, 2, 3)$, and call its norm $V$. The surfaces orthogonal to $\xi^\mu$ with induced metric are denoted by $(\Sigma, g)$. We assume that all quantities are smooth except at certain smooth boundaries, see Sect. 2.2. We next impose standard conditions for asymptotic flatness, namely that outside a compact set of $\Sigma$ the function $V$ the metric $g_{ij}$ $(i, j = 1, 2, 3)$ and their partial derivatives take the form

\begin{align}
V &= 1 + O(1/r), \\
\partial_i V &= O(1/r^2), \\
\partial_i \partial_j V &= O(1/r^3),
\end{align}

in suitable coordinates, where $r = (x^ix^j\delta_{ij})^{1/2}$. These conditions guarantee that the Komar mass and the ADM mass exist and coincide [10]. As to the black holes, we recall that in a static spacetime an event horizon is a Killing horizon. If the horizon is non-degenerate, it has a bifurcation surface given on $\Sigma$ by $V = 0$ (see e.g. [11]). We remove the possible black hole region from $\Sigma$, i.e. we take the latter to be of topology $\mathbb{R}^3 \setminus B_i$, where $B_i$ are finitely many balls given by $V = 0$. Einstein’s equations with a perfect fluid lead to the equations

\begin{align}
\Delta V &= 4\pi V (\rho + 3p), \\
R_{ij} &= V^{-1}\nabla_i \nabla_j V + 4\pi (\rho - p),
\end{align}
where $\rho$ and $p$ are the density and the pressure of the fluid, and $\nabla_i$ and the Ricci tensor $R_{ij}$ refer to $g_{ij}$. The corresponding Euler or Bianchi identities read

$$\nabla_i p = -V^{-1}(\rho + p) \nabla_i V.$$  

(7)

We consider fluids satisfying a “positive eqos”, i.e. $\rho = \rho(p)$ with $\rho \geq 0$ and $p \geq 0$. If the fluid has a boundary, the latter is given by $p = 0$. Otherwise, we require that $p \to 0$ at infinity in a way compatible with (2) and the field equations. These assumptions imply (see e.g. [7]) that $p$ depends uniquely and monotonically on $V$, that $V = \text{constant}$ wherever $p = 0$, and that

$$V = V_s \exp \left(-\int_0^p \frac{dp'}{\rho(p') + p'} \right).$$  

(8)

In Thm. 1 of Sect. 2, we admit any matter which satisfies the weak energy condition. In this case the potential equation (5) generalizes to

$$\Delta V = 4\pi V R_{\mu\nu} \xi^\mu \xi^\nu$$  

(9)

where $R_{\mu\nu}$ is the Ricci tensor, with $R_{\mu\nu} \xi^\mu \xi^\nu \geq 0$ on $\Sigma$.

2.2. The maximum principle

For our main result (Thm. 1) we need to apply the maximum principle to Eqs.(5) and (9). There are weak maximum principles which can handle these equations under physically reasonable boundary conditions for the matter variables and matching conditions for the field variables. To simplify the presentation, we will instead assume here that the metric and the matter variables are smooth, except at smooth boundaries. We can then apply the strong maximum principle on the domains where the matter is smooth, and the boundary point lemma at the interfaces. We recall these theorems below. Together they imply that $V$ will always take on its maximum $V_s = 1$ at infinity, no matter what is inside. On the other hand, in Thm. 2 we have to deal with more complicated elliptic equations containing higher derivatives of $V$ and singular points. Like the results in the “fluid only” case [6, 7] the complete proof of this theorem requires the more general maximum principle of Alexandrov (Thm. 9.1 of [12]) to deal in particular with potential critical points of $V$ inside the fluid region. We refer to [6, 7] for these technicalities.

**Uniformly Elliptic Operators:** Let $\Omega$ be a smooth domain of $\mathbb{R}^3$, $u \in C^\infty(\Omega)$, and $L$ be the operator

$$Lu = a^{ij}(x)\partial_i \partial_j u + b^i(x)\partial_i u + c(x)u, \quad a^{ij} = a^{ji}.$$  

(10)

whose coefficients are smooth as well. $L$ is uniformly elliptic when $\lambda \delta^{ij} \leq a^{ij}(x) \leq \Lambda \delta^{ij}$ for positive constants $\lambda, \Lambda$.

**The Boundary Point Lemma:** Suppose that $L$ is uniformly elliptic, $c = 0$ and $Lu \geq 0$ in $\Omega$. Let $\partial \Omega$ be smooth and assume that $u$ extends smoothly to the boundary and achieves a maximum at $x_0 \in \partial \Omega$. Then the outer normal derivative of $u$ at $x_0$ is positive, i.e.

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$  

(11)

**The Strong Maximum Principle:** Let $L$ be uniformly elliptic, $c = 0$ and $Lu \geq 0$ in a domain $\Omega$. Then $u$ achieves its maximum in the interior of $\Omega$ if and only if $u$ is constant. If $c \leq 0$ and $c/\lambda$ is bounded, then $u$ cannot achieve a non-negative maximum in the interior of $\Omega$ unless it is constant.
3. Uniqueness
This section contains the two main theorems and a final remark. As a preliminary remark we note that, to avoid singularities, a fluid can never intersect a black hole horizon in a static spacetime. We shall say that a fluid component \( F \) has an inner boundary with respect to the asymptotically flat end when a connected component \( (\partial F)_i \) of its boundary separates the manifold into two regions, one compact and one non-compact, and the matter content is vacuum in some neighbourhood of \( (\partial F)_i \) within the compact region.

**Theorem 1.** Let \((\Sigma, g)\) be an asymptotically flat static solution of Einstein’s equations, with \( \Sigma \) of topology \( R \setminus B_i \). We assume that the matter satisfies the weak energy condition and that all matter variables are smooth on finitely many domains on \( \Sigma \) separated by smooth boundaries. Then none of the matter components can be a fluid with inner boundary with respect to the asymptotically flat end. In particular,

(i) \((\Sigma, g)\) cannot contain a fluid component \( F \subset \Sigma \) extending to infinity but have an inner boundary \( \partial F \) with vacuum in an interior neighbourhood, and

(ii) \((\Sigma, g)\) cannot contain any fluid component \( F \) of annular shape with vacuum everywhere in some neighbourhood.

**Proof.** We assume that the inner boundary is present and arrive at a contradiction. Let \( \Sigma = \mathcal{E} \cup \mathcal{F} \cup \mathcal{I} \) where \( \mathcal{I} \) is the compact interior region and \( \mathcal{E} \) is the exterior region (which is empty if the fluid extends to infinity). As \( p = p(V) \), \( V \) takes the same positive value \( V_s \) at the outer and the inner boundary (if \( \mathcal{E} = \emptyset \), \( V_s = 1 \)). From the maximum principle and the boundary point lemma applied to (5) on \( \mathcal{F} \), \( V_s \) must be a maximum with gradient \( \nabla_i V \) pointing outside \( \mathcal{F} \) everywhere on \( \partial \mathcal{F} \). On the other hand, from the maximum principle applied to (5) on \( \mathcal{I} \), (c.f. Sect. 2.2) \( V \) takes on its maximum \( V_s > 0 \) on the boundary component \( \partial \mathcal{I} \cap \partial \mathcal{F} \), as \( V = 0 \) on the black hole boundaries (if present). Applying again the boundary point lemma we find that \( \nabla_i V \) points outside \( \mathcal{I} \) on \( \partial \mathcal{I} \cap \partial \mathcal{F} \) which gives the required contradiction. □

The following Theorem is a combination of the Theorem of Shiromizu, Yamada and Yoshino [1], with a result on finite extent for fluids which satisfy \( I \leq 0 \) [13, 14].

**Theorem 2** Assume \((\Sigma, g)\) contains a connected perfect fluid \( F \) whose equation of state satisfies \( I \leq 0 \). If the fluid region is finite (i.e., if \( V_s < 1 \)), assume further that

\[
\kappa_i \leq (1 - V_s^2)^{-2} \kappa_s \quad (12)
\]

holds for all black holes \( B_i \), where \( V_s \) is the surface potential of the fluid, and \( \kappa_s \) and \( \kappa_i \) are the surface gravity of the fluid and of the black holes, respectively. Then \((\Sigma, g)\) is spherically symmetric and either Schwarzschild in the exterior, or the Buchdahl solution (which extends to infinity).

**Proof.** As mentioned in the introduction, the proof rests on the ideas of the “black hole only” and the “fluid only” cases. If the fluid is finite and the exterior contains black holes, one uses the “doubling argument” by Bunting and Masood-ul-Alam together with a suitable conformal rescaling. This produces, out of two copies \( \Sigma^+ \) and \( \Sigma^- \) of \( \Sigma \), an asymptotically flat manifold \( \tilde{\Sigma} = \Sigma^+ \cup \Sigma^- \) of topology \( \mathbb{R}^3 \), with non-negative Ricci scalar and vanishing mass. Then the rigidity part of the positive mass theorem gives flatness of \( \Sigma \), which leads to spherical symmetry by standard arguments. The required conformal factors are given by

\[
\Omega_+ = \begin{cases} 
(1 + V)/2 & \text{outside the fluid}, \\
\psi(V) & \text{inside the fluid}
\end{cases} \quad (13)
\]

\[
\Omega_- = (1 - V)/2; \quad (14)
\]
where in particular $\psi(V)$ is the function used in [7, 8]. The Ricci scalars take the forms

$$R(g_+) = -G(V) [W - W_\mu(V)],$$
$$R(g_-) = 8\pi\Omega^{-5}_{-5}[(1 + V)p + 6Vp] \geq 0;$$  (15)

where $W = |\nabla V|^2$ and $W_\mu(V)$ and $G(V)$ are smooth functions of $V$. While $R(g_-)$ is thus manifestly positive, the non-negativity of $R(g_+)$ follows by showing non-negativity of each factor in (15), each of which requires the condition $I \leq 0$. Showing $G(V) \geq 0$ is the easy part. For $W - W_\mu$ one derives equations of the form

$$L[H(V)(W - W_\mu(V))] \geq 0$$  (17)

inside and outside the fluid. Here $H(V)$ is a function of $V$ and the operator $L$ is elliptic everywhere, with $c \leq 0$. However, as $H(V)$ is singular at $V = 0$, $L$ is uniformly elliptic only in domains whose closure does not intersect any horizon components. Accordingly, the maximum principle and the boundary point lemma exclude a positive maximum of $W - W_\mu(V)$ everywhere except on the horizon. The latter possibility is now excluded by the extra assumption (12) with $\kappa_s = \sqrt{W_s}$ and $\kappa_i = \sqrt{W_i}$, since $[W - W_\mu(V)]_s \leq 0$ and $(1 - V_s^2)^{-4}W_\mu(V)|_s = W_\mu(V)|_i = const.$ for all black holes $B_i$. Finally, we recall from [7, 15] that the fluids extending to an asymptotically flat infinity which satisfy $I \leq 0$ automatically satisfy $I = 0$ and are given by the 1-parameter family

$$p = \frac{1}{n+1} \rho^{n+1} \left( \frac{1}{\rho_0^n} - \frac{1}{\rho_s^n} \right)^{-1}$$  (18)

of the “Buchdahl” equations of state [9], where $\rho_0$ is the parameter. For each member of this family there exists a 1-parameter family of solutions for which the central pressure goes from zero to infinity. The family is spherically symmetric and can be written in a simple way in terms of the coordinate $V$ as the metric $\Omega^2 \delta_{ij}$ in (13) has to be flat.

**Alternative conditions:** We finally comment on the conditions

$$\kappa \leq \frac{5\rho^2}{6\rho(\rho + 3p)}$$  (19)
$$V_s < \sqrt{\frac{\kappa}{\kappa + 10}} \exp \left( -\int_0^p \frac{dp'}{\rho(p') + \rho} \right)$$  (20)

given by Lindblom and Masood-ul-Alam [8], which can be used instead of $I \leq 0$ to show uniqueness, also if black holes are present a priori [1]. Equ. (19) is a condition on the equation of state closely related to $I \leq 0$ in the sense that $I \leq 0$ implies (19), while $I \equiv 0$ implies either equality in (19) or $\rho = const.$ [7, 14]. However, (20) is an upper bound on the surface potential which is in danger of becoming inconsistent with (19). This is already indicated by the well known Buchdahl limit $V_s \geq 1/3$ [16] for positive eqos which also satisfy $dp/dp \geq 0$, but also by other limits on $V_s$ in terms of the equation of state which have been given explicitly or follow easily from the results of [13, 14, 17, 18, 19]. To get confidence in the consistency of the set (19) with (20), an example would therefore be useful. Contrary to the claim in [8], no equation of state satisfying $I \equiv 0$ can serve this purpose: The latter equation is solved by $\rho = const.$ and by a two-parameter family of equations of state, whose spherically symmetric, asymptotically flat solutions have been determined explicitly [20]. All of them violate (20).

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