A Note on Non-Isolated Real Singularities and Links

Lars Andersen

June 2021

Abstract

For analytic map germs $f: (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ having an isolated critical value in the origin with $\dim V(f) > 0$ and satisfying the transversality property of D.B. Massey we show that for $c > 0$ a large enough constant, and $k \in \mathbb{N}$ a large enough natural number the local Milnor-Lê fibers of $f$ and of the isolated singularity $g = f - c(\sum_{i=1}^{n} x_i^2)^k$ satisfies the following.

There exists a homotopy equivalence between the negative Milnor-Lê fiber of $f$, to which a cobordism between its boundary and the link of $f$ has been adjoined, and the Milnor-Lê fiber of $g$.

1 Introduction

1.1 Introduction

The topological nature of analytic map germs has grown to this day to become an immense field of research. For real analytic maps, and more particularly singular such maps, we can for the purpose of this paper mention the works of H. Hamm (e.g [4]), Z. Szafraniec ([7]) and more recently the paper of Séade, Cisneros-Molina and Snoussi ([1]) and the preprint ([3]) of N. Dutertre. The result presented in following paper is in fact a mere consequence of the work of the latter authors, and should ideally be called a corollary of [6, Theorem 13.5] and [7, Lemma 1].

The result in this paper is a topological relation between the link of a singularity of real analytic function germs $f: (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, its Milnor-Lê fiber, and the Milnor fiber of an isolated singularity of the form $f - c\omega^k$ where $\omega$ is the sum of the squares of the coordinate functions on $\mathbb{R}^n$, $c > 0$ a constant, and $k \in \mathbb{N}$ a sufficiently large natural number.

1.2 The Local Milnor-Lê Fibration

Suppose that the real analytic function germ $f: (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ has an isolated critical value in the origin, satisfies the transversality property of D.B. Massey ([5, Definition 13.4]) and that $\dim V(f) > 0$. Then the following theorem holds.
Theorem 1 ([E] Theorem 13.5]). The germ $f$ has a local Milnor-Lé fibration

$$f : N_f(\epsilon, \delta) \to \mathbb{R} \setminus \{0\},$$

where $N_f(\epsilon, \delta) = f^{-1}((-\epsilon, \epsilon) \setminus \{0\}) \cap B_\delta$ for some ball $B_\delta \subset \mathbb{R}^n$ and for $0 < \epsilon << \delta$. This determines an equivalent fiber bundle

$$\phi : S_\delta \setminus K \to S^1$$

where $K = f^{-1}(0) \cap S_\delta$ is the link of the map germ $f$, and where the projection map $\phi$ is $f/\|f\|$ when restricted to $N_f(\epsilon, \delta)$.

Let $p > 0$ be a positive integer. Then the transversality property is in fact equivalent to the existence of the first fibration for locally surjective map germs

$$g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0), \quad \dim V(g) > 0$$

with an isolated critical value in the origin.

In our case ($p = 1$) the transversality property implies the equivalence of the bundles. This is done, following Milnor, by integrating an appropriate vector field to "inflate" the Milnor tube $N_f(\epsilon, \delta)$ and push the fibers of the first fibration outwards towards the sphere. For $p > 1$, the equivalence of the bundles is more delicate yet has recently been shown ([I]) by J. L. Cibernos-Molina, J. Séade and J. Snoussi that the notion of $d$-regularity which they introduced, plays an essential rôle.

1.3 A Result of Szafraniec

Let us now recall a result of Szafraniec.

Lemma 1.0.1 ([7] Lemma 1]). Let $f : (U, 0) \to (\mathbb{R}, 0)$ be a real analytic function defined in an open subset of $\mathbb{R}^n$. Then there exist constants $C > 0, \alpha > 0$ such that:

If $c \in (0, C), k \geq \alpha$ is an integer, $r \not= 0$ is sufficiently close to the origin and

$$g : U \subset \mathbb{R}^n \to \mathbb{R}, \quad g = f - c(x_1^2 + \cdots + x_n^2)^k$$

then $\{0\} \subset \mathbb{R}$ is a regular value of $g|_{S_r}$. In particular $g$ has an isolated critical point in the origin.

In particular it follows from the proof of the lemma that if $r = r(c, k)$ is chosen sufficiently small and if

$$N_f^-(r) := \{x \in S_r : f(x) \leq 0\}$$

$$N_g^-(r) := \{x \in S_r : g(x) \leq 0\}$$

then $N_f(r) \subset \text{int}N_g(r)$ and $N_f(r)$ is a deformation retract of $N_g(r)$. 
1.4 The Result

Let us from now on assume that $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ satisfies the hypotheses of Theorem 1. We then apply the theorem to $-f$ and consequently let $N_f^-(\epsilon, \delta)$ be the negative Milnor tube of $f$, so that

$$f : N_f^-(\epsilon, \delta) \to (-\infty, 0)$$

is the projection of a trivial fibre bundle. Let

$$F^-(f) = f^{-1}(\epsilon) \cap B_\delta.$$ 

By inflating the Milnor tube as in the proof of Theorem 1 one obtains a homeomorphism

$$F^-(f) \cong \{x \in S_\delta : f(x) \leq -\epsilon\}.$$ 

As a consequence,

$$F^-(f) \cup_{\partial F^-(f)} (N_f^- (\epsilon, \delta) \cap S_\delta) \cong N_f(\delta)$$

where

$$N_f^- (\epsilon, \delta) \cap S_\delta = f^{-1}([-\epsilon, 0]) \cap S_\delta.$$ 

On the other hand, by Lemma 1.0.1 $N_f(\delta) \hookrightarrow N_g(\delta)$ is a deformation retract so

$$F^-(f) \cup_{\partial F^-(f)} (N_f^- (\epsilon, \delta) \cap S_\delta) \hookrightarrow N_g^- (\delta)$$

is a homotopy equivalence. Note that by Lemma 1.0.1 $g$ has an isolated critical point in the origin.

Let us now recall a result due to A. Durfee. Suppose given an algebraic set $M$ in the affine space $\mathbb{R}^n$ and let $X \subset M$ be a compact algebraic subset. Suppose that $M \setminus X$ is nonsingular. Recall that a subset $T \subset M$ is an algebraic neighborhood of $X$ if:

there exists a nonnegative proper polynomial map $\alpha : M \to \mathbb{R}$ and a positive real number $\gamma$ smaller than any critical value of $\alpha$ such that

$$X = \alpha^{-1}(0), \quad T = \alpha^{-1}([0, \gamma])$$

In this situation Durfee (2) proved

**Lemma 1.0.2 (2 Proposition 1.6).** If $T$ is an algebraic neighborhood of $X$ then the inclusion $X \hookrightarrow T$ is a homotopy equivalence.

**Remark 1.0.1.** The main point of the proof is to pick a well-chosen strictly increasing neighborhood basis of $X$ and then use the vector field grad $\alpha$ to trivialise these.

We now apply this with

$$X = K_f = g^{-1}(0) \cap S_\delta, \quad T = g^{-1}([-\epsilon', \epsilon']) \cap S_\delta$$

for $\epsilon' << \delta$, and assume that

$$g^{-1}([-\epsilon', 0)) \cap S_\delta \neq \emptyset.$$ 

Of course this is always the case whenever the boundary of the negative Milnor fiber $F^-(g)$ of $g$ is nonempty. Then Lemma 1.0.1 together with Theorem 1 applied to the isolated singularity $g$ give

$$N_g^- (\delta) \sim \{g \leq -\epsilon'\} \cap S_\delta \cong F^-_{q'}(g).$$
We have proven

**Theorem 2.** Suppose that \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0) \) has an isolated critical value in the origin, satisfies the transversality property and that \( \dim V(f) > 0 \). Let \( F_\epsilon^-(f) \) denote the local negative Milnor-Lê fiber of \( f \), where \( 0 < \epsilon << \delta \). There exist constants \( C > 0, \alpha > 0 \) such that:

if \( c \in (0, C) \), if \( k \geq \alpha \) is an integer and if \( \delta \) is chosen so small that

\[
g : \mathbb{B}_\delta \to \mathbb{R}, \quad g = f - c(x_1^2 + \cdots + x_n^2)^k
\]

has no critical points except an isolated critical point in the origin then there exists a homotopy equivalence

\[
F^-_\epsilon(f) \cup_{\partial F^-_\epsilon(f)} (N^-_f(\epsilon, \delta) \cap S_\delta) \sim F^-_\epsilon(g)
\]

whenever the negative Milnor-Lê fiber \( F^-_\epsilon(g) \) of \( g \) at the origin is non-empty.

**Corollary 1.0.1.** Under the same assumptions as in [Theorem 2] there is a long exact sequence in homology

\[
\rightarrow H_n(\partial F^-_\epsilon(f)) \rightarrow H_n(F^-_\epsilon(f)) \oplus H_n(N^-_f(\partial F^-_\epsilon(f))) \rightarrow H_n(F^-_\epsilon(g)) \rightarrow
\]

**References**

[1] Jose Cisneros, Jose Seade, and J. Snoussi. Milnor fibrations and d-regularity for real analytic singularities. *International Journal of Mathematics*, 21, 01 2012.

[2] Alan H. Durfee. Neighborhoods of Algebraic Sets. *Transactions of the American Mathematical Society*, 276(2):517–530, 1983.

[3] Nicolas Dutertre. On the topology of non-isolated real singularities, 2019.

[4] Helmut Hamm. On the euler characteristic of real milnor fibres. *Journal of Singularities*, 10, 01 2014.

[5] Jose Seade. *On the Topology of Isolated Singularities in Analytic Spaces*. Birkhäuser Verlag, Basel, 2006.

[6] Jose Seade. On Milnor’s fibration theorem and its offspring after 50 years. *Bulletin of the American Mathematical Society*, 56:1, 11 2018.

[7] Zbigniew Szafraniec. On the euler characteristic of analytic and algebraic sets. *Topology*, 25:411–414, 1986.