Unsteady stagnation-point flow and heat transfer over an exponential stretching sheet in Copper-water nanofluid with slip velocity effect

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Abstract. This paper is performed in order to study the behavior of unsteady stagnation-point flow and the heat transfer in nanofluid with the presence of velocity slip over an exponential stretching sheet. The nanofluid model namely Tiwari-Das model is considered where the influence of volume of nanoparticle in the base fluid is also part of the focus in this study. Copper is the nanoparticle and water is chosen as the base fluid. The transformation of governing equation in partial differential equations form to the system of ordinary differential equations is carried out using suitable similarity variables. The system is solved using bvp4c program in Matlab software where the numerical results for dual solutions are presented graphically for the skin friction coefficient, the local Nusselt number and the velocity as well as the temperature profiles are obtained for different parameters namely, velocity slip, nanoparticle volume fraction, suction/ injection and unsteadiness parameters.

1. Introduction
The stagnation-point flow is a flow that describes the behaviour or the characteristic of motion of the fluid near the stagnation region. This kind of flow has been used in various fields for instance Yang et al. [1] considered this flow in dentistry that showed the deposition of bacteria in a stagnation-point flow has larger influence compared to parallel plate flow. Besides, stagnation-point flow also is applied in air purification as reported by Montecchio et al. [2]. The study involves the flow over an exponential stretching/ shrinking sheet compared to an ordinary stretching/shrinking sheet has been considered by some authors which can be found from the works by Wei et al. [3], Suali et al. [4], Bhattacharyya and Vajravelu [5], Bachok et al. [6], Bhattacharyya [7] and Zaimi and Ishak [8].

Nanofluid which is introduced by Choi in 1995 has become one of the medium that can be considered for the purpose of heat transfer enhancement. This fluid has been applied in numerous applications for example in energy saving as reported by Firouzfar et al. [9] , cell biology (Wang and Wang* [10]), solar energy system (Verma and Tiwari [11]), and so on. Among nanofluid models, there are two models that have been applied to solve the boundary layer system namely Buongiorno model proposed by Buongiorno [12] that measures the effects of thermophoresis and Brownian motion while Tiwari-Das [13] model which study the influence of nanoparticle volume fraction towards the flow behaviour. The movement of nanoparticle in the base fluid randomly has caused the
slip velocity. Hence, the classical no-slip condition can be replaced with the slip velocity condition. Some studies have applied slip velocity condition in their research as can be found in Abbas et al. [14], Pandey and Kumar [15], Najib et al. [16] and so on.

The present study is extended from the work done by Bachok et al. [17], Zaib et al. [18] and Bhattacharyya et al. [19] to investigate the characteristics of unsteady stagnation-point flow and heat transfer rate over an exponentially stretching sheet in Copper-water nanofluid where the presence of velocity slip between surface and the fluid with mass suction/injection effects are taken into account. The Tiwari and Das model is considered in the system to investigate the influence of volume of nanoparticle in the base fluid. Besides, the effects of some other parameters namely unsteadiness, slip, suction/injection are also considered to study the behaviour of the flow and heat transfer rate at the surface. The bvp4c program in Matlab software is used to find the numerical solutions and the results are presented graphically.

### 2. Mathematical Formulation

The system of two-dimensional laminar incompressible stagnation flow (density is constant) where the unsteady free stream velocity is \( U_\infty = \left( a / (1 - ct) \right) e^{x/t} \) which \( a \) is the stagnation length and \( c \) is the positive constant are considered. Assume that the plate velocity is \( u_\infty = \varepsilon (a / (1 - ct)) e^{x/t} \) where \( \varepsilon > 0 \) for the stretching velocity rate and \( l \) is the characteristic length. Let the slip occurs between fluid and surface which represents as \( u = L \partial u / \partial y \) where \( L \) is the length of the slip (see Bhattacharyya et al. [19]). The reference temperature, ambient temperature and temperature distribution near the surface are \( T_0, T_\infty \) and \( T_\infty = T_0 + (T_0 / (1 - ct)) e^{x/t} \) where both \( T_0 \) and \( T_\infty \) are constants. \( x \)-axis is parallel to the stream direction while \( y \)-axis is perpendicular to it. The governing equations below are referred from Bachok et al. [17] and Zaib et al. [18]:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial U_\infty}{\partial t} + U_\infty \frac{\partial u}{\partial x} + \frac{\mu_{nf}}{\rho_{nf}} \frac{\partial^2 u}{\partial y^2}, \quad (2)
\]

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha_{nf} \frac{\partial^2 T}{\partial y^2}, \quad (3)
\]

subject to boundary condition

\[
v = -v_\infty, \quad u = \varepsilon (L \partial u / \partial y), \quad T = T_\infty \quad \text{at} \quad y = 0, \quad (4)
\]

\[
u \to U_\infty, \quad \text{as} \quad y \to \infty,
\]

where \( u \) and \( v \) are the velocity components in \( x \) and \( y \) axes, \( t \) denotes the time, \( T \) is the nanofluid temperature, \( \mu_{nf}, \alpha_{nf} \) and \( k_{sf} \) represent the viscosity, thermal diffusivity and density of the nanofluid, respectively. Equation (5) is given by Oztop and Abu-Nada [20]:

\[
\mu_{nf} = \frac{\mu_f}{(1 - \phi)^{2.5}}, \quad \alpha_{nf} = \frac{k_{sf}}{(\rho c_p)_s}, \quad k_{sf} = \frac{k_f + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)},
\]

\[
(\rho c_p)_s = (\rho c_p)_f \left[ 1 - \phi + \phi \left( (\rho c_p)_s / (\rho c_p)_f \right) \right], \quad \rho_{sf} = \rho_f \left[ 1 - \phi + \phi (\rho_s / \rho_f) \right]. \quad (5)
\]
where $\phi$ denotes the volume of the nanoparticle, $\mu_f$ is the viscosity of the fluid, $(\rho C_p)_s f$ and $(\rho C_p)_n f$ are the heat capacity of the nanofluid, nanoparticle and fluid, $k_{nf}$, $k_s$ and $k_f$ are the thermal conductivities of the nanofluid, nanoparticle and fluid, respectively, $\rho_f$ and $\rho_s$ are the densities of fluid and nanoparticle.

The system of partial differential equations (1) – (3) subjected to (4) is transformed to the simplest form using the similarity variables which is given by:

$$
\eta = \left(\frac{a}{2\nu_f(1-ct)}\right)^{\frac{1}{2}} e^{\nu_f^2 \eta}, \quad \psi = \left(\frac{2\nu_f a}{1-ct}\right)^{\frac{1}{2}} e^{\nu_f^2 \eta} f(\eta), \quad \theta = \frac{T - T_\infty}{T_w - T_\infty},
$$

where $\psi$ is the stream function for which

$$
u = \frac{a}{1-ct} e^{\nu_f f'(\eta)} \text{ and } v = -\frac{a}{1-ct} e^{\nu_f f'(\eta)}.$$

Using equation (6), one can show that equation (1) is satisfied while equations (2) and (3) subjected to the boundary conditions (4) are transformed into the following equations (7) – (9)

$$
\frac{1}{(1-\phi)^{2.5} \left(1-\phi+\phi\left(\rho_s / \rho_f\right)\right)} f'''' + f'' + 2f'' + 2 - A(2f'' + \eta f' - 2) = 0
$$

$$
\frac{k_{nf}}{Pr} \left(1-\phi+\phi\left(\rho C_p_s / (\rho C_p_f)\right)\right) \theta''' + f' \theta'' - f' \theta - A(2\theta + \eta \theta') = 0.
$$

subject to boundary conditions

$$
f(0) = s, \quad f'(0) = \sigma + \sigma f''(0), \quad \theta(0) = 1 \quad \text{at} \quad \eta = 0,
$$

$$
f' (\eta) \rightarrow 1, \quad \theta (\eta) \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty,
$$

where the differentiation is with respect to $\eta$, $\sigma = L(a / 2\nu_f(1-ct))^{\frac{1}{2}} e^{\nu_f^2}$ is the slip velocity parameter, $Pr = \nu_f / \alpha_f$ is the Prandtl number and $A = \alpha_f / \omega^{\nu_f^2}$ is the dependent unsteadiness parameter, $s = v_0 \left(2/\omega \nu_f\right)^{\frac{1}{2}} > 0$ represents the suction parameter and $s = v_0 \left(2/\omega \nu_f\right)^{\frac{1}{2}} < 0$ is the injection parameter.

The interest of physical quantities namely the skin friction coefficient $C_f$ and the local Nusselt number $Nu_s$ is given by equation (10) (see Rao et al. [21]),

$$
C_f = \frac{\tau_w}{\rho_f U_\infty^2}, \quad Nu_s = \frac{q_w}{k_f(T_w - T_\infty)},
$$

where $\tau_w$ is the shear stress and $q_w$ is the heat flux at the surface which represented by equation (11)
\[
\tau_w = \mu_W \left( \frac{\partial u}{\partial y} \right)_{y=0}, \quad q_w = -k_W \left( \frac{\partial T}{\partial y} \right)_{y=0}, \quad (11)
\]

Using (6) and (10) and (11), \( C_f \left( 2 \text{Re}_x \right)^{1/2} \) and \( \text{Nu}_x \left( 2 / \text{Re}_x \right)^{1/2} \) are given by equations (12), respectively.

\[
C_f \left( 2 \text{Re}_x \right)^{1/2} = f'(0) / (1 - \phi)^{2.5}, \quad \text{Nu}_x \left( 2 / \text{Re}_x \right)^{1/2} = -\left( k_W / k_f \right) \theta'(0) \quad (12)
\]

where \( \text{Re}_x = U_x l / v_f \) denotes the local Reynolds number.

3. Results and discussion

The ordinary differential equations (7) and (8) subjected to the boundary conditions (9) are solved by bvp4c program build in Matlab software. Dual solutions for some values of slip \( \sigma \) and nanoparticle volume fraction parameter \( \phi \) with suction/ injection parameter \( s \) are obtained for the skin friction coefficient \( f'(0) \) which represents the skin friction coefficient and the local Nusselt number \( -\theta'(0) \) that shows the heat transfer rate at the surface. Besides, the velocity and temperature profiles are also presented in order to depict that the far-field boundary condition is asymptotically fulfilled and support the obtained results of dual solutions graphically. Following Tiwari-Das model, the water with Prandtl number 6.2 (\( \text{Pr} = 6.2 \)) is chosen as the base fluid while the Copper is the nanoparticle which is dispersed in the base fluid. Based on Oztop and Abu-Nada [20] the thermo physical properties for both water and Copper is presented in Table 1 and the value of nanoparticle volume fraction is taken between 0 to 0.2 (\( 0 \leq \phi \leq 0.2 \)) which \( \phi = 0 \) indicates the nanoparticle is absent in the base fluid. Table 2 tabulates the comparison results from Bachok et al. [6] and the present which shows an excellent agreement. The unsteadiness parameter value is fixed at 0.1 (\( A = 0.1 \)) and the value of the stretching sheet parameter is 0.5 (\( \varepsilon = 0.5 \)).

| Substance | Density, \( \rho \) (kg/m\(^3\)) | Specific heat, \( C_\rho \) (J/kg\(-k\)) | Conductivity, \( K \) (W/m\(-K\)) |
|-----------|-------------------------------|---------------------------------|-----------------------------|
| Water     | 997.1                         | 4179                            | 0.613                       |
| Copper    | 8933                          | 385                             | 400                         |

| \( \varepsilon \) | Bachok et al. [6] | Present |
|-------------------|-------------------|---------|
| \( f'(0) \)       | 2.1182            | 2.118168665 |
| \( -\theta'(0) \) | 0.6870            | 0.687002248 |
| \( f'(0) \)       | 1.6872            | 1.687218164 |
| \( -\theta'(0) \) | 1.7148            | 1.714771539 |
| \( f'(0) \)       | 2.4874            | 2.487418731 |
| \( -\theta'(0) \) | 0.9604            | 0.960416075 |

The effects of some values of nanoparticle volume fraction \( \phi \) with suction/ injection parameter of the skin friction coefficient and local Nusselt number are displayed in figures 1 and 2 and the dual solutions are obtained in both figures. Figure 1 shows that the first solution of the skin friction coefficient increases as the volume of nanoparticle of Copper in the water increases. This indicates that the shear stress at the surface increases as the volume of Copper nanoparticle in the fluid increases. While the second solution of skin friction coefficient shows the opposite trend compared to the first solution. As for the local Nusselt number which show the heat transfer rate at the surface both
solutions are seen decreases when the concentration of nanoparticle in the water increases as displayed in figure 2. Besides that, the increasing value of suction parameter causes the increment in the shear stress and the heat transfer rate at the surface for all tested values of nanoparticle volume fraction. Furthermore, the greater value of $S$ makes the second solution can be found easier since the value obtained between the first and second solutions are significantly different compared to the solutions with smaller value of $s$. This situation also shows that the shear stress and heat transfer rate for suction effect $s > 0$ is greater than injection effect $s < 0$ as well as for the impermeable surface $s = 0$.

Figures 3 and 4 present the skin friction coefficient and local Nusselt number for some values of slip velocity where this slip occurs between the fluid and the surface with suction/injection parameter. Figure 3 displays the increasing effect of slip parameter decreases the skin friction coefficient for both solutions and leads to increase the velocity boundary layer thickness as reported by Zaimi and Ishak [8]. Figure 4 depicts the local Nusselt number increases for first solution but the second solution shows the decreasing trend for the increasing value of slip parameter. This implies that the heat transfer rate at the surface increases as the slip effect increases for the first solution but it decreases for the second solution. Apart from that the values of local Nusselt number are the positive values which indicate that the transfer of the heat is from the heated surface to the cooler nanofluid. In addition, for some values of slip parameters, the escalation of suction value shows the rise of shear stress and heat transfer rate at the surface.
The variations of the skin friction coefficient and local Nusselt number for different values of unsteadiness parameter $A$ from equations (12) with nanoparticle volume fraction $\varphi$ are shown in Figures 5 and 6. Both figures show that accelerating the unsteadiness flow parameter $A$ increases the shear stress and the rate of heat transfer at the surface. Meanwhile, increasing the volume fraction of Copper in the water has also affected the increment in both of the skin friction as well as the heat transfer rate which denotes the cooling process is faster with the presence of higher dispersion of Copper in the base fluid enhances the thermal conductivity of the nanofluid. Figures 7 and 8 depicted the effect of suction/ injection parameter $s$ towards the skin friction and the local Nusselt number. It is notable that the stronger effect of suction (or injection becomes weaker) has caused the shear stress and the heat transfer rate at the surface are increased. Furthermore increasing the nanoparticle volume fraction $\varphi$ of Copper in the water has enhanced the transportation rate of heat at the surface which speeding up the cooling process.

![Figure 5](image1.png)  
*Figure 5. The $C_f \left( 2 Re \right)^{1/2}$ for different $A$ in Cu-water with $\varphi$ when $Pr = 6.2$, $\epsilon = 0.5$, $s = 2$ and $\sigma = 0.1$.  

![Figure 6](image2.png)  
*Figure 6. The $Nu_\lambda \left( 2 / Re \right)^{1/2}$ for different $A$ in Cu-water with $\varphi$ when $Pr = 6.2$, $\epsilon = 0.5$, $s = 2$ and $\sigma = 0.1$. 

![Figure 7](image3.png)  
*Figure 7. The $C_f \left( 2 Re \right)^{1/2}$ for different $s$ in Cu-water with $\varphi$ when $Pr = 6.2$, $\epsilon = 0.5$, $A = 0.1$ and $\sigma = 0.1$.  

![Figure 8](image4.png)  
*Figure 8. The $Nu_\lambda \left( 2 / Re \right)^{1/2}$ for different $s$ in Cu-water with $\varphi$ when $Pr = 6.2$, $\epsilon = 0.5$, $A = 0.1$ and $\sigma = 0.1$. 

The velocity and temperature profiles are shown in figures 9 and 10 for some values of slip parameters over an exponential stretching sheet in Copper water nanofluid respectively. Figure 9 presents the velocity profile for different values of slip parameter. This figure displays that the enhancement of slip effect causes the increasing in the velocity boundary layer thickness for the first solution and the opposite behaviour for the second solution. While the temperature boundary layer thickness shows the increasing trend for the first solution and decreasing temperature for the second solution as the value of slip parameter becomes more significant as can be seen in figure 10.
Figure 9. Velocity profile for different $\sigma$ in Cu-water when $\Pr = 6.2$, $\varepsilon = 0.5$, $\varphi = 0.1$, $s = 2$ and $A = 0.1$.

Figure 10. Temperature profile for different $\sigma$ in Cu-water when $\Pr = 6.2$, $\varepsilon = 0.5$, $\varphi = 0.1$, $s = 2$ and $A = 0.1$.

In the meantime, figure 11 illustrates the effect of nanoparticle volume fraction towards the velocity profile where the increment of nanoparticle in the base fluid causes the increasing in the velocity profiles for first solution but decreases for second solution. Figure 12 implies that the temperature profile for different values of nanoparticle volume fraction where the increasing amount of nanoparticle disperses in the fluid increases the temperature of the fluid for both first and second solutions. Apart of that, all of the velocity and the temperature profiles above have shown the far field boundary conditions are fulfilled and they supported the existence of dual solutions as displayed in figures 1 – 4. Besides that, the velocity and temperature boundary layer thicknesses for the first solution are found thinner than the second solution for all cases.

Figure 11. Velocity profile for different $\varphi$ in Copper-water nanofluid when $\Pr = 6.2$, $\varepsilon = 0.5$, $\sigma = 0.1$, $s = 2$ and $A = 0.1$.

Figure 12. Temperature profile for different $\varphi$ in Copper-water nanofluid when $\Pr = 6.2$, $\varepsilon = 0.5$, $\sigma = 0.1$, $s = 2$ and $A = 0.1$.

4. Conclusion
The unsteady stagnation-point flow over an exponentially stretching sheet in Copper-water nanofluid with the presence of slip effect is numerically solved using the similarity transformation by introducing the appropriate similarity variables. The Tiwari-Das model is considered in the system where water is chosen as the base fluid and Copper as the chosen nanoparticle. The dual solutions are obtained by bvp4c program in Matlab software for the values of the skin friction coefficient and also local Nusselts number that represent the shear stress and heat transfer rate at the surface, respectively
for some values of nanoparticle volume fraction, slip parameter and mass suction/injection parameter. The far-field boundary condition is fulfilled for the tested parameter and the results are displayed graphically. Hence, the conclusions can be made from the present work are:

- The dual solutions are obtained for any value of $s$ but the significant difference between the first and second solutions can be found for suction effect $s > 0$ compared to injection effect $s < 0$ which indicates that the second solution is easier to be found for suction case.
- The shear stress at the surface increases as the nanoparticle volume fraction, unsteadiness and mass suction parameter is increasing while the decreasing behaviour occurs when the slip parameter increases.
- The heat transfer rate at the surface increases when the effects of slip and suction are higher and decreases as the volume of nanoparticle and unsteadiness parameter become significant.
- The velocity of the fluid increases with the presence of the nanoparticle in the base fluid and the slip between surface and the fluid.
- The temperature of the fluid increases when nanoparticle is dispersed in the water and decreases as the effect of the slip becomes stronger.

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References
[1] Yang J, Bos R, Belder G F, Engel J and Busscher H J 1999 Deposition of Oral Bacteria and Polystyrene Particles to Quartz and Dental Enamel in a Parallel Plate and Stagnation Point Flow Chamber Journal of Colloid and Interface Science 220 410–418
[2] Montecchio F, Persson H, Engvall K, Delin J and Lanza R 2016 Development of a stagnation point flow system to screen and test TiO 2-based photocatalysts in air purification applications Chemical Engineering Journal 306 734–744
[3] Wei S, Omar A and Ishak A 2011 Stagnation-Point Flow over an Exponentially Shrinking / Stretching Sheet Zeitschrift fur Naturforschung - Section A Journal of Physical Sciences 66 705-711
[4] Suali M, Long N M A N, and Ariffin N M 2012 Unsteady Stagnation Point Flow and Heat Transfer over a Stretching / Shrinking Sheet with Suction or Injection Journal of Applied Mathematics 2012 1-12
[5] Bhattacharyya K, and Vajravelu K 2012 Stagnation-point flow and heat transfer over an exponentially shrinking sheet Communications in Nonlinear Science and Numerical Simulation 17(7) 2728–2734
[6] Bachok N, Ishak A, and Pop I 2012a Boundary layer stagnation-point flow and heat transfer over an exponentially stretching / shrinking sheet in a nanofluid International Journal of Heat and Mass Transfer 55(25–26) 8122–8128
[7] Bhattacharyya K 2013 Heat transfer analysis in unsteady boundary layer stagnation-point flow towards a shrinking/stretching sheet Ain Shams Engineering Journal 4(2) 259–264
[8] Zaimi K, and Ishak A 2016 Stagnation-Point Flow towards a Stretching Vertical Sheet with Slip Effects Mathematics 4(2) 27
[9] Firouzfar E, Soltanieh M, Noie S H, and Saidi S H 2011 Energy saving in HVAC systems using nano fluid Applied Thermal Engineering 31(8–9) 1543–1545
[10] Wang E C and Wang A Z 2014 Nanoparticles and their applications in cell and molecular biology Integrative Biology 6(1) 9–26
[11] Verma S K, and Tiwari A K 2015 Progress of nanofluid application in solar collectors : A review Energy Conversion and Management 100 324–346
[12] Buongiorno J 2006 Convective Transport in Nanofluids Journal Of Heat Transfer 128 240-250
[13] Tiwari R K, and Das M K 2007 Heat transfer augmentation in a two-sided lid-driven differentially heated square cavity utilizing nanofluids International Journal of Heat and Mass Transfer 50(9–10) 2002–2018
[14] Abbas Z, Perveen R, Sheikh M and Pop I 2016 Thermophoretic diffusion and nonlinear radiative heat transfer due to a contracting cylinder in a nanofluid with generalized slip condition Results in Physics 6 1080–1087
[15] Pandey A K and Kumar M 2017 Boundary layer flow and heat transfer analysis on Cu-water nanofluid flow over a stretching cylinder with slip Alexandria Engineering Journal Article in Press
[16] Najib N, Bachok N, Arifin N, and Senu N 2017 Boundary Layer Flow and Heat Transfer of Nanofluids over a Moving Plate with Partial Slip and Thermal Convective Boundary Condition: Stability Analysis International Journal of Mechanics 11(1) 18-24
[17] Bachok N, Ishak A, and Pop I 2012b The boundary layers of an unsteady stagnation-point flow in a nanofluid International Journal of Heat and Mass Transfer 55(23–24) 6499–6505
[18] Zaib A, Bhattacharyya K and Shafie S 2015 Unsteady boundary layer flow and heat transfer over an exponentially shrinking sheet with suction in a copper-water nanofluid Journal of Central South University 22 4856–4863
[19] Bhattacharyya K, Mukhopadhyay S, and Layek G C 2011 Slip effects on boundary layer stagnation-point flow and heat transfer towards a shrinking sheet International Journal of Heat and Mass Transfer 54 308–313
[20] Oztop H F, and Abu-Nada E 2008 Numerical study of natural convection in partially heated rectangular enclosures filled with nanofluids International Journal of Heat and Fluid Flow 29 1326–1336
[21] Rao J A, Vasumathi G, and Mounica J 2015 Joule Heating and Thermal Radiation Effects on MHD Boundary Layer Flow of a Nanofluid over an Exponentially Stretching Sheet in a Porous Medium World Journal of Mechanics 5 151–164