Nonlinear oscillation modes of double pendulum

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Abstract. The article investigates nonlinear oscillations of the double mathematical pendulum having identical parameters of its links and end loads. Approximate solutions in the nonlinear zone are constructed and analyzed on the basis of exact solution in the linear zone. These solutions clearly illustrate the drift of frequencies and modes of double pendulum with increasing oscillation amplitudes. The obtained analytical results are very useful for the synthesis of controlled movement modes of wide range of manipulators and various robotic structures, and they are also necessary for constructing optimal locomotion modes of walking machines and mechanisms. These solutions are also interesting as illustrative examples of nonlinear mechanics in the teaching and engineering practice.

1. Introduction

The study of movements of various pendulum structures has already been one of the main directions of research in analytical mechanics for a long time [1]. They are not only of fundamental theoretical importance, but also a large number of practical applications. Multi-link pendulums are of particular interest in this respect. They easily allow to increase the number of degrees of freedom, introducing and changing the inertial and dissipative parameters of individual links and also, if necessary, adding control actions in the articulated joints [2]. The double flat mathematical pendulum is the simplest of these pendulums [3]. This pendulum has found increasing application in recent decades in robotics as a manipulator and pedipulator, where it must perform working movements with both small and very large amplitudes [4].

The question arises, how such two-link link robot passes from the zone of small (i.e., linear) oscillations to the zone of weak nonlinearity. At the same time, regular modes with periodicity are the most valuable of all possible modes of motion of nonlinear systems, because they are advisable to use in practice. The study of such modes is closely related to the well-known problem of finding nonlinear vibration modes, which has recently attracted a large number of researchers [5]. We emphasize that double pendulum is a system with two degrees of freedom, for which the construction of analytical solutions in nonlinear zone turns out to be much more difficult than for systems with one degree of freedom. Therefore, nonlinear oscillations of the double pendulum are investigated only using numerical methods in most works on its dynamics. Nevertheless, the study of their qualitative nature on the basis of analytical dependences is also important in describing the regular movements of the system. These expressions would clearly demonstrate the drift of oscillations frequencies and modes with a gradual increasing the oscillation amplitudes of the links.
In this regard, the main attention in this article is focused precisely on the analytical study of nonlinear oscillations of double pendulum using specially developed asymptotic methods. The main purpose of this article is obtaining and interpreting formulas that make it possible to analyze in detail the nonlinear modes of oscillations of a double pendulum and establish their main qualitative features, and to accompany solutions with visual graphic illustrations as well.

2. Design scheme and selection of generalized coordinates
We consider the flat double mathematical pendulum with the same links of length \( l \) and identical end loads of mass \( m \) (figure 1). Such a system can be interpreted as the simplest nonlinear mechanical system with two degrees of freedom.

![Figure 1. Double mathematical pendulum.](image)

It is most convenient to choose the absolute angles \( \varphi_1 \) and \( \varphi_2 \) of deviations of the pendulum links from the vertical as generalized coordinates. We note that sometimes relative angles \( \theta_1 \) and \( \theta_2 \) of rotation in the joints of the pendulum are taken as generalized coordinates [6]. They are related to absolute angles by obvious relationships: \( \theta_1 = \varphi_1 \) and \( \theta_2 = \varphi_2 - \varphi_1 \). The angles \( \theta_1 \) and \( \theta_2 \) are often used in tasks of control motions because they are measurable. It should be emphasized that the choice of one or another generalized coordinates is conditioned by the specific research objectives. We will use absolute angles in this work, since in this case the equations of motion have the simplest form. Nevertheless, we will also demonstrate the behavior of the angles of rotation in the joints.

3. Derivation of nonlinear equations of motion
Proceeding to the derivation of motion equations of the double mathematical pendulum, we write its kinetic and potential energies [7]:

\[
T = \frac{1}{2} ml^2 \left[ 2\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2\cos(\varphi_2 - \varphi_1) \dot{\varphi}_1 \dot{\varphi}_2 \right] = \frac{1}{2} \phi^T A(\phi) \phi, \quad \Pi = mg l (3 - 2 \cos \varphi_1 - \cos \varphi_2), \tag{1}
\]

where \( \phi = [\varphi_1, \varphi_2]^T \) is the column of generalized coordinates and matrix \( A(\phi) \) of inertial coefficients has the form

\[
A(\phi) = ml^2 \begin{bmatrix}
2 & \cos(\varphi_2 - \varphi_1) \\
\cos(\varphi_2 - \varphi_1) & 1
\end{bmatrix}.	ag{2}
\]

Substituting expressions (1) into Lagrange equations of the second kind [8], we obtain the
nonlinear equation of system’s motion in the well-known matrix form:

$$\mathbf{A}(\phi)\dot{\phi} + \mathbf{B}(\phi, \phi) + \mathbf{C}(\phi) = 0,$$

(3)

where columns $\mathbf{B}(\phi, \phi)$ and $\mathbf{C}(\phi)$ are determined by expressions

$$\mathbf{B}(\phi, \phi) = ml^2 \sin(\phi_2 - \phi_1) \begin{bmatrix} -\phi_1^2 \\ \phi_1^2 \end{bmatrix}, \quad \mathbf{C}(\phi) = mgl \begin{bmatrix} 2\sin\phi_1 \\ \sin\phi_2 \end{bmatrix}.$$

(4)

The considered system is conservative, therefore there is the energy integral:

$$E = T + \Pi = \frac{1}{2}ml^2 \left[ 2\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\cos(\phi_2 - \phi_1)\dot{\phi}_1\dot{\phi}_2 \right] + mgl(3 - 2\cos\phi_1 - \cos\phi_2) = \text{const.}$$

(5)

Linearizing matrix equation (3), we obtain the equation that describes small oscillations:

$$\mathbf{A}_0\ddot{\phi} + \mathbf{C}_0\phi = 0,$$

(6)

where the constant matrices of inertial $\mathbf{A}_0$ and quasi-elastic $\mathbf{C}_0$ coefficients are [9]

$$\mathbf{A}_0 = ml^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{C}_0 = mgl \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

(7)

It is well-known, see [10], that frequencies and modes of linear model are defined by expressions

$$k_{s0} = \left(2 \pm \sqrt{2}\right)^{1/2} \cdot \left(\frac{g}{l}\right)^{1/2}, \quad \Phi_{(s)} = \begin{bmatrix} 1 \\ \mp\sqrt{2} \end{bmatrix}, \quad s = 1, 2.$$

(8)

Hereinafter, the lower sign corresponds to the first mode $(s = 1)$, and the upper sign corresponds to the second mode $(s = 2)$. We also need expressions for the norms of oscillations modes for subsequent constructions:

$$N_s = \Phi_{(s)\top} \mathbf{A}_0 \Phi_{(s)} = 2(2 \mp \sqrt{2})ml^2.$$

(9)

4. Construction of asymptotic solutions

We now turn to the problem of finding nonlinear oscillation modes. For this we will use asymptotic methods of nonlinear mechanics [11]. Let’s rewrite (3), so that on the left side there will be only linear terms corresponding to small oscillations (6), and all other terms will be on the right side. So we get:

$$\mathbf{A}_0\ddot{\phi} + \mathbf{C}_0\phi = \mathbf{A}(\phi)\dot{\phi} + \mathbf{B}(\phi, \phi) + \mathbf{C}(\phi) = \mathbf{Q}(\phi, \dot{\phi}, \ddot{\phi}),$$

(10)

where the matrices $\mathbf{A}_0$ and $\mathbf{C}_0$ are determined by expressions (7) and also new designations are made:

$$\mathbf{A}(\phi) = \mathbf{A}_0 - \mathbf{A}(\phi), \quad \mathbf{B}(\phi, \phi) = -\mathbf{B}(\phi, \phi), \quad \mathbf{C}(\phi) = \mathbf{C}_0\phi - \mathbf{C}(\phi).$$

(11)

Then column $\mathbf{Q}$, that is interpreted as column of disturbing forces acting in a linear system, takes the form:

$$\mathbf{Q} = ml^2 \sin(\phi_2 - \phi_1) \begin{bmatrix} \phi_1^2 \\ -\phi_1^2 \end{bmatrix} + mgl \begin{bmatrix} 2(\phi_1 - \sin\phi_1) \\ \phi_2 - \sin\phi_2 \end{bmatrix} + ml^2 (1 - \cos(\phi_2 - \phi_1)) \begin{bmatrix} \phi_2 \\ \phi_1 \end{bmatrix}.$$

(12)

We will be interested in the construction and detailed analysis of the formulas for the first approximation, i.e., for a weakly nonlinear model. To do this, we expand the trigonometric functions
in (12) in Taylor series so as to keep in the expression for $Q$ the terms no higher than the third order of smallness in generalized coordinates, velocities and accelerations:

$$Q = ml^2 (\varphi_z - \varphi_i) \left[ \frac{\dot{\varphi}_z^2}{-\dot{\varphi}_i^2} \right] + \frac{1}{6} mgl \left[ \frac{2\varphi_0^3}{\varphi_i^3} \right] + \frac{1}{2} ml^2 (\varphi_z - \varphi_i)^2 \left[ \frac{\ddot{\varphi}_z}{\ddot{\varphi}_i} \right]. \quad (13)$$

We will look for an approximate solution in the form of oscillations of constant amplitude according to one of the modes $\Phi_{(s)}$ of the linear system, but with some correction to the expression for the frequency:

$$\varphi = \Phi_{(s)} a \cos \varphi, \quad k_s = \dot{\varphi}_s = k_{i_0} (1 + \rho_s a^2), \quad a = \text{const.} \quad (14)$$

The correction factor $\rho_s$ follows from the harmonic balance equation for oscillation mode $\Phi_{(s)}$:

$$\int_0^{2\pi} \Phi_{(s)}^T (A_0 \ddot{\varphi} + C_0 \varphi - Q) \cos \psi d\psi = 0. \quad (15)$$

Calculating the columns of generalized velocities $\dot{\varphi}$ and accelerations $\ddot{\varphi}$ accurate to third-order values

$$\dot{\varphi} = -\Phi_{(s)} a \sin \psi k_{i_0} (1 + \rho_s a^2), \quad \ddot{\varphi} = -\Phi_{(s)} a \cos \psi k_{i_0} (1 + 2\rho_s a^2), \quad (16)$$

we define the column $Q$ with the same accuracy:

$$Q = (q_{i_1} \cos \psi + q_{i_2} \cos 3\psi) mgl a^3, \quad q_{i_1} = \left[ \begin{array}{c} 14 \pm 9\sqrt{2} \\ 4 \\ -22 \mp 17\sqrt{2} \\ 8 \end{array} \right], \quad q_{i_3} = \left[ \begin{array}{c} 46 \pm 33\sqrt{2} \\ 12 \\ -54 \mp 41\sqrt{2} \\ 24 \end{array} \right]. \quad (17)$$

Now we substitute expressions (16) and (17) into the harmonic balance equation (15) and take into account that oscillation modes satisfy equation $(C_0 - k_{i_0}^2 A_0) \Phi_{(s)} = 0$, and also use the notation (9) for the norms of these modes. Then we arrive at the following expressions for the correction factor $\rho_s$ and the frequency of nonlinear oscillations $k_s$ after transformations:

$$\rho_s = -\frac{mgl}{2N} \Phi_{(s)}^T q_{i_s} = -\frac{31 \pm 20\sqrt{2}}{32}, \quad k_s(a) = k_{i_0} \left(1 - \frac{31 \pm 20\sqrt{2}}{32} a^2\right). \quad (18)$$

It is easy to see that for the first frequency the correction factor is ($-0.0849$), while for the second one it is ($-1.8526$), i.e., the second frequency changes more significantly with increasing of amplitude.

However, the most interesting is not the frequency, but the nonlinear oscillations mode. We obtain a solution up to the third order of smallness to describe it. It is known from the theory of oscillations that if the harmonic force $Q_0 \cos \omega t$ acts in a linear system with two degrees of freedom, i.e.

$$A_0 \ddot{\varphi} + C_0 \varphi = Q_0 \cos \omega t, \quad (19)$$

then it excites the forced oscillations in the system:

$$\varphi_0 = \left[ \Phi_{(1)} \frac{\Phi_{(1)}^T Q_0}{N_1(k_{10}^2 - \omega^2)} + \Phi_{(2)} \frac{\Phi_{(2)}^T Q_0}{N_2(k_{20}^2 - \omega^2)} \right] \cos \omega t. \quad (20)$$

It is easy to understand based on this expression what forced oscillations each of the harmonic
components of expression (17) excites. When we construct the solution on the first mode, we should take into account only the second term for the first harmonic in expression (20), since this harmonic is already balanced on the first mode. On the contrary, only the first term should be taken into account when the solution is constructed on the second mode. Summing up the solutions corresponding to each harmonic of expression (17), we obtain the so-called "regularized oscillation" excited in the unperturbed system by generalized forces \( Q(\Phi_{\alpha,s}, \alpha \cos \psi, \ldots) \)

\[
\varphi_s(a, \psi) = \Phi_{(p)} \frac{\Phi_{(p)}^{\top} q_{1s} m g l a^3}{N_p(k_{p0}^2 - k_{10}^2)} \cos \psi + \left[ \Phi_{(1)} \frac{\Phi_{(1)}^{\top} q_{1s} m g l a^3}{N_1(k_{10}^2 - 9k_{10}^2)} + \Phi_{(2)} \frac{\Phi_{(2)}^{\top} q_{1s} m g l a^3}{N_2(k_{20}^2 - 9k_{10}^2)} \right] \cos 3\psi,
\]

where the additional index \( p \) is introduced, and \( p = 1 \) at \( s = 2 \) and vice versa \( p = 2 \) at \( s = 1 \). Adding (21) to the first expression (14), we obtain a solution up to the third order of smallness in the form

\[
\varphi = \Phi_{(s)} a \cos \psi + \Phi_{(s)} a \cos \psi + \Phi_{(s)} a \cos 3\psi,
\]

where columns \( a_{1s} \) and \( a_{3s} \) have the form

\[
a_{1s} = \begin{bmatrix} \pm \sqrt{2} + 1 \\ 2 \pm \sqrt{2} \\ 32 \end{bmatrix}, \quad a_{3s} = \begin{bmatrix} \mp 552 \sqrt{2} - 437 \\ 2688 \\ \pm 781 \sqrt{2} + 576 \\ 2688 \end{bmatrix}.
\]

We also can write the expression of the column of generalized velocities:

\[
\beta = -\left[ \left( \Phi_{(s)} a + \beta_{1s} a^3 \right) \sin \psi + \beta_{3s} a^3 \sin 3\psi \right] k_{s0},
\]

where columns \( \beta_{1s} \) and \( \beta_{3s} \) have the form

\[
\beta_{1s} = \begin{bmatrix} \mp 19 \sqrt{2} - 30 \\ 21 \pm 16 \sqrt{2} \\ 16 \end{bmatrix}, \quad \beta_{3s} = \begin{bmatrix} \mp 552 \sqrt{2} - 437 \\ 896 \\ \pm 781 \sqrt{2} + 576 \\ 896 \end{bmatrix}.
\]

It is possible to calculate the total mechanical energy of system (5) accurate to terms of the fourth smallness order as one of the checks of the obtained results and make sure that it is a constant for a given value of \( a \), i.e., it doesn’t depend on \( \psi \). Expanding the trigonometric functions included in (5) into series, one can obtain

\[
E = \frac{1}{2} m l^2 \left[ 2 \phi_1^2 + \phi_2^2 + 2 \phi_1 \phi_2 \left( 1 - \frac{\phi_2 - \phi_1}{2} \right) \right] + \frac{1}{2} m gl \left[ 2 \phi_1^2 + \phi_2^2 - \frac{\phi_2^4 - \phi_1^4}{6} - \frac{\phi_1^4}{12} \right].
\]

Substituting solutions (22) and (24) into this formula, we find the expression for the dimensionless energy level with the required accuracy:

\[
\gamma = \frac{E}{m gl} = 2a^2 + \frac{\mp 20 \sqrt{2} - 37}{32} a^4.
\]

It can be seen that there is no dependence on \( \psi \) in this formula which confirms the validity of the performed calculations.
5. Results and discussion

Now we turn to the interpretation of obtained results. We see, turning to the constructed solution (22) for oscillations modes in a weakly nonlinear model, that the motion on each mode is periodic, but no longer harmonic, and represents the sum of two harmonics. It is important to emphasize that if it is necessary to excite oscillations on one mode $\Phi_{(i)}$, in linear model then the columns $\Phi_0$ and $\Phi_0$ must be proportional to this mode, while in a nonlinear system the situation is much more complicated. Indeed, the ratio of the oscillation amplitudes of the deflection angles or angular velocities does not completely determine the nonlinear oscillations mode. Moreover, these ratios may be completely meaningless for sufficiently large oscillations, since generalized coordinates or velocities may have more than two extremes on the oscillations period.

We restrict ourselves to a detailed discussion of the first oscillation mode, i.e., the basic tone which is of main practical importance. Let’s turn to the construction of the system’s phase portrait in order to illustrate visually the process of the oscillation mode drift during the transition from the linear zone to the weakly nonlinear zone. It is clear that the phase space in our problem is four-dimensional. Therefore, it is convenient to construct the phase portrait for each generalized coordinate separately, that is on the planes $(\varphi_1, \dot{\varphi}_1/k_{10})$ and $(\varphi_2, \dot{\varphi}_2/k_{20})$, where the division of the generalized velocities by the value $k_{10}$ is done from dimensional considerations. We will construct a phase portrait using the analytical dependences (22) and (24) and compare it with the results that are obtained using numerical integration. We note that the same dimensionless energy level $\gamma$ must correspond to the approximate and exact phase trajectories for their adequate comparison. Let us assume for simplicity that $\dot{\varphi}_0 = 0$, i.e., the movement begins without an initial velocity. Then we have an exact expression for $\gamma$ according to (5):

$$\gamma = mg(3-2\cos \varphi_{10} - \cos \varphi_{20}).$$

It is necessary to select the values $\varphi_{10}$ and $\varphi_{20}$ fixing a certain value $\gamma$ and taking into account the expression (28), so that the movement of the system is periodic. Besides, the oscillation frequency of this movement should be close to the first frequency of the linear system. We use numerical methods for integrating the nonlinear matrix equation (3) to do this procedure.

Figures 2 and 3 demonstrate the phase portraits on the planes $(\varphi_1, \dot{\varphi}_1/k_{10})$ and $(\varphi_2, \dot{\varphi}_2/k_{20})$, where several phase trajectories corresponding to different values $\gamma$ are shown. The solid lines show here the phase trajectories constructed using numerical integration and the circles indicate the values obtained using formulas (22) and (24). There is a good agreement between the numerical and analytical results, although some differences appear with increase of the energy level. In addition, it can be seen that for small values $\gamma$ phase trajectories are concentric circles which corresponds to motion on the linear oscillations mode. These circles gradually begin to distort with increasing $\gamma$ and turn into other phase trajectories that are more complex in character.

It is also of interest to construct a phase portrait for the interlink angle $\theta_2 = \varphi_2 - \varphi_1$, i.e., on the plane $(\theta_2, \dot{\theta}_2/k_{10})$. It is shown in figure 4 and has a very unusual appearance. It is clearly seen here that the circles corresponding to analytical results taken at regular intervals are located on the phase trajectories more and more unevenly with increasing energy level.
Figure 2. Phase portrait for coordinate $\phi_1$.

Figure 3. Phase portrait for coordinate $\phi_2$. 
We emphasize here, that it is possible to obtain analytical solutions in the second approximation using the harmonic balance equation (15). Of course, they will quantitatively give more accurate results. However, such solutions will turn out to be rather cumbersome, while solutions (22) and (24) are compact and give an adequate qualitative picture of the double pendulum movements, clearly reflecting the influence of nonlinear factors.

Figure 5 shows the graphs of the dependence of the absolute angles $\phi_1$ and $\phi_2$, and also the interlink angle $\theta_2$ on time $t$ at one oscillation period. We note that these graphs correspond to the phase trajectory with the highest energy level of those plotted in figures 2-4. The numerical dependences are shown here as before by the solid line and the analytical values are represented by the circles. These graphs clearly demonstrate the non-trivial character of the double pendulum movements and there is also a good agreement between the numerical and analytical results. Similar graphs for angular velocities are shown in figure 6. It should be noted that the angle $\theta_2$, as well as the angular velocities $\dot{\phi}_1$ and $\dot{\theta}_2$, for the given case differ significantly from the sinusoidal functions and have six extremes on the oscillation period, and not two as in the small oscillations case. At the same time, the angles $\phi_1$ and $\phi_2$, as well as the angular velocity $\dot{\phi}_2$, have two extremes on the oscillation period as in the small oscillations case, but their form also differs significantly from sinusoidal functions.
6. Conclusion

The approximating analytical solutions for the oscillations frequencies and modes of double pendulum, which were constructed in this work, allow us to consider a smooth transition from its linear model to a nonlinear one and vice versa from the unified standpoint. These results are very necessary in the development and design of various devices of modern robotics, primarily androids and other mobile robots, and exactly this area is currently attracting great attention of specialists.

The main value of the obtained results lies in the fact that the role of nonlinear terms in the general picture of behavior sharply increases with the transition from small oscillations of such manipulators to the large ones. Therefore, it is necessary to know the qualitative character of their movements in order to synthesize the adequate control. The undoubted advantage of the above expressions is expressed in their simplicity and adequate correspondence to the results of numerical integration. This also refers to their visual interpretation in the form of phase portraits and dependence graphs of the deviation angles of the links on time.

Figure 5. Graphs of the dependence of angles $\varphi_1$, $\varphi_2$, and $\theta_2$ on time $t$.

Figure 6. Graphs of the dependence of angle velocities $\dot{\varphi}_1$, $\dot{\varphi}_2$, and $\dot{\theta}_2$ on time $t$. 
As final conclusion, we can note that described method of analytical search of frequencies and modes of nonlinear oscillations can be used not only for a two-link device, but also for three-link and four-link manipulators simulating the movements of the components of mobile robots.

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