Spin polarization-scaling quantum maps and channels

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We introduce a spin polarization-scaling map for spin-$j$ particles, whose physical meaning is the decrease of spin polarization along three mutually orthogonal axes. We find conditions on three scaling parameters under which the map is positive, completely positive, entanglement breaking, 2-tensor-stable positive, and 2-locally entanglement annihilating. The results are specified for maps on spin-1 particles. The difference from the case of spin-$\frac{1}{2}$ particles is emphasized.

Keywords: Spin polarization, qubit, qutrit, positive map, quantum channel, entanglement breaking, 2-tensor-stable positive, and 2-locally entanglement annihilating. The results are specified for maps.

I. INTRODUCTION

Quantum states of a spin-$j$ particle are described by $(2j+1)\times(2j+1)$ density matrices $\rho \in \mathcal{S}(\mathcal{H})$ satisfying the properties $\rho^\dagger = \rho$, $\text{tr}[\rho] = 1$, and $\langle \phi | \rho | \phi \rangle \geq 0$ for all $|\phi\rangle \in \mathcal{H}$, $\dim \mathcal{H} = 2j+1$. Taking into account the normalization condition, the density matrix $\rho$ is defined by $(2j+1)^2 - 1$ real parameters, which are usually treated as components of the generalized Bloch vector $\text{tr}[\rho]$, $\text{tr}[\rho^2]$, $\text{tr}[\rho^3]$, $\ldots$, $\text{tr}[(\rho^*)^3]$. However, many physical phenomena can be explained and visualized via a spin polarization vector $p \in \mathbb{R}$ with components $p_i = \text{tr}[\rho J_i]$, where $J_1, J_2, J_3$ are usual $(2j+1)$-dimensional representations of angular momentum operators (see, e.g., [34]). Angular momentum operators are Hermitian and satisfy the commutation relation $[J_i, J_j] = i\epsilon_{ijk} J_k$, where $\epsilon_{ijk}$ is the conventional Levi-Civita symbol and the summation over $m$ being assumed. Note that the spin-polarization vector $p$ does not contain the full information about the quantum state if $j \geq 1$. Despite this fact, it is of great use in quantum physics and chemistry as its components represent average spin projections onto three orthogonal axes and are experimentally measurable. Linear transformations of the spin polarization vector include rotations and scaling. Rotations are attributed to the unitary evolution, so we do not consider them in the present paper. Physically motivated scaling of the spin polarization vector is described by a map $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ of the following form:

$$\Phi[X] = \frac{1}{2j+1} \text{tr}[X] I + \frac{3}{j(j+1)(2j+1)} \sum_{i=1}^{3} \lambda_i \text{tr}[X J_i] J_i,$$

where $I \in \mathcal{B}(\mathcal{H})$ is the identity operator and $\lambda_i \in \mathbb{R}$. The factors take into account that $\text{tr}[I] = 2j+1$ and $\text{tr}[J_i J_j] = \frac{1}{2}j(j+1)(2j+1) \delta_{ij}$, $\delta_{ij}$ is the Kronecker delta. The map $[1]$ is trace-preserving and unital, i.e. $\text{tr}[\Phi[X]] = \text{tr}[X]$ and $\Phi[I] = I$. Note that the map $[1]$ differs in general from other classes of unital maps $[21, 29]$. Physical meaning of Eq. (1) is the transformation of the spin polarization

$$p_i \rightarrow \lambda_i p_i, \quad i = 1, 2, 3.$$

In case of spin-$\frac{1}{2}$ particles, formula (1) transforms into a well-known Pauli qubit map $\Phi[X] = \frac{1}{2} \left( \text{tr}[X] I + \sum_{i=1}^{3} \lambda_i \text{tr}[X \sigma_i] \sigma_i \right)$, where $(\sigma_1, \sigma_2, \sigma_3)$ is the conventional set of Pauli matrices (see, e.g., [11, 26]). The qubit ($j = \frac{1}{2}$) map $\Phi$ is known to be positive if and only if $|\lambda_i| \leq 1$, completely positive if and only if $1 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1$, entanglement breaking if and only if $|\lambda_1| + |\lambda_2| + |\lambda_3| \leq 1$, 2-local-entanglement-annihilating if and only if $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \leq 1$, 2-tensor-stable positive if and only if $1 \geq \lambda_1^2 \geq \lambda_2^3 \geq \lambda_3^2$. Similar characterization for higher spins ($j \geq 1$) is still missing, so the goal of the present paper is to analyze analogous properties of such maps and illustrate them for qutrits ($j = 1$).

The paper is organized as follows. In Sec. II, we analyze positivity of the spin polarization-scaling map (1). In Sec. III, we introduce a spin polarization-scaling quantum maps and illustrate them for qutrits ($j = 1$). In Sec. VI, brief conclusions are presented.

II. POSITIVITY

We will refer to an operator $R$ as positive-semidefinite and write $R \geq 0$ if $\langle \phi | R | \phi \rangle \geq 0$ for all $|\phi\rangle \in \mathcal{H}$. A map $\Phi$ is called positive if $\Phi[X] \geq 0$ for all $X \geq 0$ [33].

Let us now analyze positivity of the spin polarization-scaling map (1).

Since each $J_i$ is a spin projection operator with eigenvalues $j-j-1, \ldots, -j$, eigenvalues of the operator $a_1 J_1 + a_2 J_2 + a_3 J_3$ are $|a|\{j,-j-1, \ldots, -j\}$, where $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$. Therefore, the minimal eigenvalue of $\Phi[X]$ reads

$$\frac{1}{2j+1} \left( \text{tr}[X] - \frac{3}{j+1} \sum_{i=1}^{3} (\lambda_i \text{tr}[X J_i])^2 \right).$$

(3)

Suppose $X \geq 0$. As $|\text{tr}[X J_i]| \leq j |\text{tr}[X]|$, the minimal value of (3) is non-negative if $1 - \frac{3}{j+1} \sum_{i=1}^{3} \lambda_i^2 \geq 0$. Thus, we have found sufficient condition for positivity of the map $\Phi$.

**Proposition 1.** Spin-polarization-scaling map $\Phi$ is positive if $\sum_{i=1}^{3} \lambda_i^2 \leq \left( \frac{j+1}{j} \right)^2$.

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The necessary condition for positivity of the map (1) follows from the particular form of the positive-semidefinite operator
\[ X = x_0 I + \sum_{i=1}^{3} x_i J_i, \quad x_i \geq 0, \quad x_i \in \mathbb{R}, \quad x_0 = j \sqrt{\sum_{i=1}^{3} x_i^2}. \tag{4} \]

In fact, if \( X \) is given by formula (4), then \( \Phi^* X = x_0 I + \sum_{i=1}^{3} \lambda_i x_i J_i \) and \( \Phi^* [X] \not\geq 0 \) if and only if \( x_0 \not\geq j \sqrt{\sum_{i=1}^{3} \lambda_i^2 x_i^2} \). Suppose \( x_0 = j x_1 \) and \( x_2 = x_3 = 0 \), then \( \Phi^* [X] \not\geq 0 \) if \( |\lambda_1| \leq 1 \). Similarly, necessary conditions \( |\lambda_2| \leq 1 \) and \( |\lambda_3| \leq 1 \) appear for choices \( x_0 = j x_2 \), \( x_1 = x_3 = 0 \) and \( x_0 = j x_3 \), \( x_1 = x_2 = 0 \), respectively.

**Proposition 2.** Suppose the spin polarization-scaling map \( \Phi \) is positive, then \( |\lambda_i| \leq 1, i = 1, 2, 3. \)

### III. COMPLETE POSITIVITY

A linear map \( \Phi \) is called completely positive if \( \Phi \odot \text{Id}_k \) is positive for all \( k = 1, 2, \ldots \). Here \( \text{Id}_k \) is the identity transformation of \( k \)-dimensional operators \( \mathcal{B}(\mathcal{H}_k) \).

**Proposition 3.** The spin polarization-scaling map \( \Phi \) is completely positive if and only if
\[
I \otimes I + \frac{3}{j(j+1)} (\lambda_1 J_1 \otimes J_1 - \lambda_2 J_2 \otimes J_2 + \lambda_3 J_3 \otimes J_3) \geq 0. \tag{5}
\]

**Proof.** A linear map \( \Phi : \mathcal{B}(\mathcal{H}_d) \rightarrow \mathcal{B}(\mathcal{H}_d) \) is known to be completely positive if and only if its Choi matrix \( \Omega_\Phi = (\Phi \odot \text{Id}_2)[|\psi_\uparrow \rangle \langle \psi_\uparrow |] \) is positive-semidefinite (see also [17, 18, 22, 29]), where \( |\psi_\uparrow \rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i \rangle \otimes |i \rangle \) is the maximally entangled state and \( \{ |i \rangle \} \) is some orthonormal basis in \( \mathcal{H}_d \).

For our construction of the Choi operator let us choose eigenvectors of the operator \( J_3 \) as the basis, namely, \( J_3 |jm \rangle = m |jm \rangle \) and \( (jm^\dagger jm) = \delta_{m,m'} \). In this auxiliary basis \( J_\pm = J_1 \pm j J_2 \), then \( J_\pm |jm \rangle = \sqrt{(j \pm 1)(j \pm m \pm 1)} |jm \rangle \pm 1 \). Some algebra yields
\[
(2j+1)[(\Phi \odot \text{Id})]|\psi_\uparrow \rangle \langle \psi_\uparrow |] = \sum_{m,m'=-j}^{j} \Phi[|jm \rangle \langle jm'|] |jm \rangle \langle jm'| = \sum_{m=-j}^{j} \left( I + \frac{3m}{j(j+1)} \lambda_3 J_3 \right) |jm \rangle \langle jm | + \sum_{m=-j}^{j} \frac{3}{j(j+1)} \sqrt{(j-m)(j+m+1)} \times (\lambda_1 J_1 - i \lambda_2 J_2) |m \rangle \langle m | + \sum_{m=-j}^{j} \frac{3}{j(j+1)} \sqrt{(j+m)(j-m+1)} \times (\lambda_1 J_1 + i \lambda_2 J_2) |m \rangle \langle m | = I \otimes I + \frac{3}{j(j+1)} (\lambda_1 J_1 \otimes J_1 - \lambda_2 J_2 \otimes J_2 + \lambda_3 J_3 \otimes J_3). \]

Thus, \( \Phi \) is completely positive if and only if the operator (6) is positive-semidefinite.

**Example 1.** If \( j = \frac{1}{2} \), then angular momentum operators are given by matrices
\[
J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7}
\]
in the basis \( \{|\\frac{1}{\sqrt{2}} \rangle, |\frac{1}{2} - \frac{i}{2} \rangle \} \). The condition (5) reduces to \( 1 + \lambda_3 \geq |\lambda_1 | + |\lambda_2 | \). Geometrically, these inequalities correspond a tetrahedron with vertices \((1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)\) in the parameter space \((\lambda_1, \lambda_2, \lambda_3) \). [31]

**Example 2.** If \( j = 1 \), then angular momentum operators are given by matrices
\[
J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{8}
\]
in the basis \( \{|11 \rangle, |10 \rangle, |1-1 \rangle \} \). The condition (5) reduces to \( 4 - 9(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + 27 \lambda_1 \lambda_2 \lambda_3 \geq 0, |\lambda_i| \leq \frac{1}{2}, i = 1, 2, 3 \). Geometrical figure corresponding to such inequalities is depicted in Fig. 3(a).

### IV. ENTANGLEMENT BREAKING

A positive-semidefinite operator \( R \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \) is called separable (non-entangled) if there exist positive-semidefinite operators \( R_1^{(k)} \) and \( R_2^{(k)} \) such that \( R = \sum_{k} R_1^{(k)} \otimes R_2^{(k)} \). A linear map \( \Phi : \mathcal{H}_1 \rightarrow \mathcal{H}_1 \) is called entanglement breaking if \( \Phi \odot \text{Id}[|\varrho \rangle \langle \varrho |] \) is separable for all \( \varrho \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2) \) and identity transformation \( \text{Id} : \mathcal{H}_2 \rightarrow \mathcal{H}_2 \). The well-known result is that \( \Phi \) is entanglement breaking if and only if the Choi matrix \( \Omega_{\Phi} \) is separable.

The necessary condition for separability of \( \Omega_{\Phi} \) is that \( \Omega_{\Phi}^k \not\geq 0 \), where \( X^T = \sum_{j,j'} I \otimes |j \rangle \langle j'| (jX I \otimes |j \rangle \langle j'| (j \text{ is the partially transposed operator, } X \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \). Applying such a condition to the Choi matrix \( \Phi \) and taking into account that in conventional basis \(|jm \rangle \) the matrices \( J_x^T = J_x, J_y^T = -J_y, J_z^T = J_z \), we obtain the following result.

**Proposition 4.** Suppose the spin polarization-scaling map \( \Phi \) is completely positive and entanglement breaking, then
\[
I \otimes I + \frac{3}{j(j+1)} (\lambda_1 J_1 \otimes J_1 + \lambda_2 J_2 \otimes J_2 + \lambda_3 J_3 \otimes J_3) \geq 0. \tag{9}
\]

Note that the requirement (9) is sufficient for the channel \( \Phi \) to be entanglement binding [14].

**Example 3.** If \( j = \frac{1}{2} \), then Eq. (9) is equivalent to \( |\lambda_1 | + |\lambda_2 | + |\lambda_3 | \leq 1 \).

**Example 4.** If \( j = 1 \), then Eq. (9) is equivalent to \( 9(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + 27 \lambda_1 \lambda_2 \lambda_3 \geq 0 \).
V. 2-TENSOR-STABLE PROPERTIES

Some properties of linear maps do not change under tensoring the map with itself, for instance, \( \Phi \otimes \Phi \) is completely positive if and only if \( \Phi \) is completely positive. Similarly, \( \Phi \otimes \Phi \) is entanglement breaking if and only if \( \Phi \) is entanglement breaking (see, e.g., [9]). However, other properties of a map can change drastically under tensor power. For example, the map \( \Phi \otimes \Phi \) can be non-positive even if \( \Phi \) is positive [24].

A linear map \( \Phi \) is called 2-tensor-stable positive if \( \Phi \otimes \Phi \) is positive [3].

**Example 5.** It is shown in Ref. [3] that the spin polarization-scaling map \( \Phi \) given by Eq. (1) for qubits \( (j = \frac{1}{2}) \) is 2-tensor-stable positive if and only if \( \Phi^2 \) is completely positive, i.e. \( 1 + \lambda_3^2 \geq |\lambda_1^2 + \lambda_2^2| \).

For higher spins \( (j \geq 1) \) the result of Example 5 can be extended as follows.

**Proposition 5.** If the spin polarization-scaling map \( \Phi \) is 2-tensor-stable positive, then \( \Phi^2 \) is completely positive.

**Proof.** Consider a positive-semidefinite operator \( |\psi_+\rangle\langle\psi_+| \), where \( |\psi_+\rangle = (2j + 1)^{-1/2} \sum_{m=-j}^{j} |jm\rangle \otimes |jm\rangle \). The action of the positive map \( \Phi \otimes \Phi \) on such an operator reads

\[
0 \leq (2j + 1)^2 (\Phi \otimes \Phi) |\psi_+\rangle\langle\psi_+|
= (2j + 1)^2 (\text{Id} \otimes \Phi \otimes \Phi) |\psi_+\rangle\langle\psi_+|
= (\text{Id} \otimes \Phi) \left[ I \otimes I + \frac{3}{j(j+1)} \times (\lambda_1 J_1 \otimes J_1 - \lambda_2 J_2 \otimes J_2 + \lambda_3 J_3 \otimes J_3) \right]
= I \otimes I + \frac{3}{j(j+1)} \left( \lambda_1^2 J_1 \otimes J_1 - \lambda_2^2 J_2 \otimes J_2 + \lambda_3^2 J_3 \otimes J_3 \right)
= (2j + 1)^2 (\Phi^2 \otimes \text{Id}) |\psi_+\rangle\langle\psi_+|,
\]

i.e. the Choi matrix \( \Omega_{\Phi^2} \) is positive-semidefinite and \( \Phi^2 \) is completely positive.

In contrast to the case \( j = \frac{1}{2} \), for higher spins \( (j \geq 1) \) Proposition 5 provides the necessary condition only. For instance, it is not hard to see that for \( j = 1 \) there exists a spin polarization-scaling map \( \Phi \) such that \( \Phi^2 \) is completely positive but \( (\Phi \otimes \Phi) |\psi\rangle\langle\psi| \not\geq 0 \) for the Schmidt-rank-2 state \( |\psi\rangle = \frac{1}{\sqrt{2}} (|11\rangle + |1\rangle - |1\rangle) \).

In the case \( j = 1 \), the map \( \Phi^2 \) is completely positive if and only if \( 4 - 9(\lambda_1^4 + \lambda_2^4 + \lambda_3^4) + 27\lambda_1^2\lambda_2^2\lambda_3^2 \geq 0 \), which is depicted in Fig. 1(b).

A linear map \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) is called 2-locally entanglement annihilating if \( (\Phi \otimes \Phi)[|\psi\rangle\langle\psi|] \) is separable for all \( |\psi\rangle \in \mathcal{H} \otimes \mathcal{H} \). The same line of reasoning as for 2-tensor-stable positive maps leads to the following result.

**Proposition 6.** If the spin polarization-scaling map \( \Phi \) is 2-locally entanglement annihilating, then \( \Phi^2 \) is entanglement breaking.

VI. CONCLUSIONS

We have considered the physically motivated sets of operator maps for spin systems. The physical meaning of such maps is the degradation of spin polarization with scaling parameters \( \lambda_1, \lambda_2, \lambda_3 \) along the axes \( x, y, z \), respectively. We have found conditions (necessary, or sufficient, or both) under which the spin polarization-scaling map is positive, completely positive, entanglement breaking, 2-tensor-stable positive, 2-locally entanglement annihilating. These results can be of use in the analysis of data, where only spin polarization degrees of freedom are available. The crucial difference between the cases of spin-\( \frac{1}{2} \) and spin-1 particles is illustrated in a series of examples.

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