Conformal mechanics inspired by extremal black holes in $d = 4$

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Abstract
A canonical transformation which relates the model of a massive relativistic particle moving near the horizon of an extremal black hole in four dimensions and the conventional conformal mechanics is constructed in two different ways. The first approach makes use of the action–angle variables in the angular sector. The second scheme relies upon integrability of the system in the sense of Liouville.

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1 Introduction

In four dimensions, a stationary, rotating, asymptotically flat vacuum black hole with a single degenerate horizon is essentially unique and is given by the extremal Kerr solution [1]. The analog of the uniqueness theorem for charged black holes yields the extremal Kerr–Newman solution [1]. In the near horizon limit the solutions exhibit conformal invariance [2] which points at a possible dual description in terms of conformal field theory [3]. The Kerr/CFT correspondence provides an important pattern where a dual description of a black hole goes beyond string theory (for a review and further references see a recent work [4]).

In general, some important features of a black hole are adequately captured in the model of a relativistic particle moving on the curved background. A classic example of such a kind is the discovery of a quadratic first integral for a massive particle in the Kerr space–time [5] which preceded the construction of the second rank Killing tensor for the Kerr geometry [6]. In some instances the argument can be reversed. In Ref. [7] a massive charged particle moving near the horizon of the extremal Reissner-Nordström black hole was related to the conventional conformal mechanics in one dimension [8] by implementing a specific limit in which the black hole mass $M$ is large, the difference between the particle mass and the absolute value of its charge $(m - |e|)$ tends to zero with $M^2(m - |e|)$ fixed. The angular variables effectively decouple in the aforementioned limit and show up only in an indirect way via the effective coupling constant characterizing the conformal mechanics. In that setting the absence of a normalizable ground state in the conformal mechanics and the necessity to redefine its Hamiltonian [8] were given a new black hole interpretation [7].

In the near horizon limit the isometry group of the extremal black holes in four dimensions contains $SO(2,1)$ factor [9]. Because the latter is the conformal group in one dimension, the model of a relativistic particle propagating on such backgrounds is automatically conformal invariant. In principle, by applying a proper coordinate transformation its Hamiltonian can be put in the conventional conformal mechanics form

$$H = \frac{p_X^2}{2} + \frac{2f(u)}{X^2}, \quad \Omega = dp_X \wedge dX + \frac{1}{2} \omega_{\alpha\beta}(u) \, du^\alpha \wedge du^\beta.$$  

Here $X$ and $p_X$ are the effective radial coordinate and the momentum, and the dynamics of the angular variables $u^\alpha$ is governed by the effective coupling $f(u)$ which is given by the Casimir element of $so(2,1)$. The former and the latter descriptions are designated in the literature as the $AdS$ and conformal bases [10].

A salient feature of the conformal basis is that it gives rise to a reduced Hamiltonian system governed by the symplectic structure $\frac{1}{2} \omega_{\alpha\beta} du^\alpha \wedge du^\beta$ and the Hamiltonian in the angular sector $f(u)$ which is worthy of study by itself. A relation between the original conformal mechanics and its angular sector is far from trivial. For example, constants of motion of the initial conformal mechanics and those of its angular part are related in a rather intricate way [11, 12, 13]. One can also reverse the argument and construct new integrable conformal mechanical systems starting from known ones. For instance, the spherical part of the rational $A_N$–Calogero model leads to the superintegrable multi–center Higgs oscillator on an $(N-1)$–dimensional sphere [11]. Conformal basis also provides a useful means for supersymmetrization of conformal mechanics. Given $N = 4$ supersymmetric extension of the Hamiltonian system ($\frac{1}{2} \omega_{\alpha\beta} du^\alpha \wedge du^\beta, f(u)$), one can immediately construct its generalization which enjoys the full $D(1,2;\alpha)$ symmetry [12].

In the context of black hole physics, a coordinate transformation providing a link between the $AdS$ and conformal bases has been constructed only for a massive charged particle moving near the horizon the extremal Reissner-Nordström black hole. First the radial motion of the particle was related to $d = 1$ conformal mechanics of [8] at the off–shell Lagrangian level [10]. Then the analysis was extended to include the angular variables within the framework of the Hamiltonian formalism [14, 15]. The purpose of this work is to construct such a transformation for the near horizon extremal Kerr, Kerr-Newman, Kerr-Newman–AdS–dS, and rotating dilaton–axion black holes in four dimensions.

In the next section we demonstrate that the construction of a canonical transformation which relates the $AdS$ and conformal pictures is straightforward provided one resorts to the action–angle variables in the angular sector of the model. Because in some cases the explicit construction of the action–angle variables may prove to be technically involved, in Sect. 3 we discuss an alternative approach which relies upon integrability of the system in the sense of Liouville. In particular, it leads to explicit transformation rules for all the backgrounds mentioned above. We illustrate both the methods by the examples of the extremal Reissner-Nordström black
hole and the extremal rotating dilaton–axion black hole in Sect. 4. Concluding Sect. 5 contains the summary and the discussion of possible further developments.

2 Canonical transformation via action–angle variables

Conformal mechanics associated with the near horizon geometry of an extremal black hole in four dimensions is described by the triple

\[ H = r \left( \sqrt{(rp_r)^2 + L(\theta, p_\theta, p_\phi)} - q(p_\phi) \right), \quad D = tH + rp_r, \]

\[ K = \frac{1}{r} \left( \sqrt{(rp_r)^2 + L(\theta, p_\theta, p_\phi) + q(p_\phi)} \right) + t^2H + 2trp_r, \]

which involves the Hamiltonian \( H \), the generator of dilatations \( D \), and the generator of special conformal transformations \( K \). Under the Poisson bracket they form \( so(2,1) \) algebra

\[ \{H, D\} = H, \quad \{H, K\} = 2D, \quad \{D, K\} = K. \]

In general, the representation (1) is defined up to the rescaling

\[ H \to \lambda H, \quad K \to \frac{1}{\lambda} K. \]

The function \( L(\theta, p_\theta, p_\phi) \) entering Eq. (1) has the special structure

\[ L(\theta, p_\theta, p_\phi) = a(\theta) + b(\theta)p_\phi^2 + c(\theta)(p_\phi + f(\theta))^2, \]

where the coefficients \( a(\theta), b(\theta), c(\theta), f(\theta) \) and \( q(p_\phi) \) depend on the details of a particular black hole under consideration (see [14, 15] for the near horizon extremal Reissner-Nordström black hole, [16] for the rotating dilaton–axion black hole, [17] for the extremal Kerr throat solution, and [18] for the near horizon extremal Kerr-Newman and Kerr-Newman–AdS–dS black holes). Note that setting \( t = 0 \) in \( D \) and \( K \) above one does not spoil the algebra (2). In what follows we denote \( \tilde{D} = D|_{t=0}, \tilde{K} = K|_{t=0} \). It is worth mentioning that the Casimir element of \( so(2,1) \) is given in terms of \( L \) and \( q \)

\[ \mathcal{I} = HK - D^2 = L(\theta, p_\theta, p_\phi) - q(p_\phi)^2. \]

Along with the conformal generators, \( L(\theta, p_\theta, p_\phi) \) and \( p_\phi \) constitute the conserved charges of a particle which, in principle, allow one to integrate canonical equations of motion. The dynamics of the radial pair \( (r, p_r) \) is fixed algebraically from \( H \) and \( D \). \( p_\phi \) is a constant of the motion, while \( L \) determines \( p_\theta \). The remaining equations of motion for \( \theta \) and \( \phi \) can be integrated by quadratures.

Note that the angular sector of the system under consideration defined by the reduced Hamiltonian \( \mathcal{I}(\theta, p_\theta, p_\phi) \) and the symplectic structure \( \omega = dp_\theta \wedge d\theta + dp_\phi \wedge d\phi \) is an integrable system. Because canonical variables in the angular sector take finite values\(^1\), one can introduce the action–angle variables \( (I_a, \Phi^a) \) such that\(^2\)

\[ \mathcal{I} = L(I_1, I_2) - q(I_2)^2, \quad \omega = dI_a \wedge d\Phi^a, \quad \Phi^a \in [0, 2\pi), \quad a = 1, 2. \]

In these terms the conformal generators acquire the form

\[ H = r \left( \sqrt{(rp_r)^2 + L(I_1, I_2) - q(I_2)} \right), \quad \tilde{K} = \frac{1}{r} \left( \sqrt{(rp_r)^2 + L(I_1, I_2) + q(I_2)} \right), \quad \tilde{D} = rp_r, \]

\(^1\)The angular variables take finite values by definition. That the momenta are finite follows from the integrals of motion \( L \) and \( p_\phi \) which fix their dynamics.

\(^2\)Because \( L(\theta, p_\theta, p_\phi) \) is a conserved quantity, after passing to the action–angle variables it may depend on \( I_1 \) and \( I_2 \) only.
while the symplectic structure reads

$$\Omega = dp_r \wedge dr + dI_a \wedge d\Phi^a, \quad \Phi^a \in [0, 2\pi), \quad a = 1, 2.$$  \hfill (8)

At this point, the Hamiltonian can be put in the conventional conformal mechanics form by introducing the new radial coordinates as in [12] (for related earlier studies see [14, 15])

$$X = \sqrt{2K}, \quad P_X = -\frac{2D}{\sqrt{2K}}, \quad \{X, P_X\} = 1 \Rightarrow H = \frac{1}{2}P_X^2 + \frac{2I}{X^2},$$  \hfill (9)

with $I$ from (6). However, with respect to the Poisson bracket the new radial variables $(X, P_X)$ do not commute with $\Phi^a$. In order to split them, let us express $(r, p_r)$ via $(X, P_X, I_a)$

$$r = \frac{2}{X^2} \left( \sqrt{(XP_X)^2/4 + L(I_1, I_2) + q(I_2)} \right), \quad p_r = -\frac{P_X X^3}{4 \left( \sqrt{(XP_X)^2/4 + L(I_1, I_2) + q(I_2)} \right)}.$$  \hfill (10)

Then one can substitute these expressions into the symplectic two–form to get

$$\Omega = dP_X \wedge dX + dI_a \wedge d\Phi^a,$$  \hfill (11)

where

$$\tilde{\Phi}^a = \Phi^a + \frac{1}{2} \int d(XP_X) \frac{\partial \log \left( \sqrt{(XP_X)^2/4 + L(I_1, I_2) + q(I_2)} \right)}{\partial I_a}.$$  \hfill (12)

As a result, $(X, P_X)$ and $(\tilde{\Phi}^a, I_a)$ constitute canonical pairs.

Thus, we have demonstrated that the construction of a canonical transformation which provides a bridge between the model of a massive relativistic particle moving near the horizon of an extremal black hole in four dimensions and the conventional conformal mechanics is facilitated by resorting to the action–angle variables in the angular sector.

3 Alternative approach

In the previous section we presented the general scheme how to construct a canonical transformation which links the AdS and conformal bases. However, in some instances explicit derivation of the action–angle variables may constitute a non–trivial problem [19]. In particular, for the black hole configurations mentioned in the Introduction they are unknown in explicit form. In this section we discuss an alternative possibility which leads to explicit transformation rules. It relies upon the fact that the system at hand is integrable in the sense of Liouville.

Let us introduce new canonical momenta $F_1$ and $F_2$ which are related to the conserved charges $L$ and $p_0$\(^3\)

$$F_1 = \sqrt{L} = \sqrt{a(\theta) + b(\theta)p_0^2 + c(\theta)(p_\phi + f(\theta))^2}, \quad F_2 = p_\phi.$$  \hfill (13)

Transformation rules to new canonical coordinates $\Theta^1$ and $\Theta^2$ are then found by explicit integration of linear inhomogeneous partial differential equations which follow from the requirement that the full change of variables be canonical. The method of characteristics (see e.g. [20]) yields\(^4\)

$$\Theta^1 = \sqrt{a(\theta) + b(\theta)p_0^2 + c(\theta)(p_\phi + f(\theta))^2} \times \int_{\theta_0}^{\theta} \frac{d\tilde{\theta}}{\sqrt{b(\tilde{\theta})[L(\tilde{\theta}, p_\theta, p_\phi) - a(\tilde{\theta}) - c(\tilde{\theta})(p_\phi + f(\tilde{\theta}))^2]},}$$

\(^3\)For definiteness, we take $I_1$ to be positive. The case $I_1 = -\sqrt{a(\theta) + b(\theta)p_0^2 + c(\theta)(p_\phi + f(\theta))^2}$ can be considered likewise.

\(^4\)For the case at hand one has to assume $b(\theta)p_\theta > 0$. Another option $b(\theta)p_\theta < 0$ is realized for $F_1 = -\sqrt{a(\theta) + b(\theta)p_0^2 + c(\theta)(p_\phi + f(\theta))^2}.$
\[ \Theta^2 = \phi - \int_{\theta_0}^{\theta} \frac{c(\dot{\theta})(p_\phi + f(\dot{\theta}))d\dot{\theta}}{\sqrt{b(\dot{\theta})[L(\theta, p_\theta, p_\phi) - a(\dot{\theta}) - c(\dot{\theta})(p_\phi + f(\dot{\theta}))^2]}}. \]  

(14)

where \( \theta_0 \) is an arbitrary constant\(^5\). Given a black hole, one first identifies the coefficients \( a(\theta), b(\theta), c(\theta), f(\theta), q(p_\phi) \) and then computes the integrals. In general, both lines in (14) involve elliptic integrals. The inverse transformation from \( (\Theta^1, F_1), (\Theta^2, F_2) \) to \( (\theta, p_\theta), (\phi, p_\phi) \) is thus given in terms of elliptic functions. Among the black hole configurations mentioned in the Introduction, only the Reissner-Nordström black hole and the rotating dilaton–axion black hole yield (14) in terms of elementary functions. These two exceptional cases are considered in more detail in Sect. 4. Note that, because the new canonical coordinates \( \Theta^1 \) and \( \Theta^2 \) do not enter the transformed Hamiltonian, the explicit form of the transformation rules (14) does not really matter.

In the new canonical coordinates the conformal generators read

\[ H = r \left( \sqrt{(rp_r)^2 + F_1^2} - q(F_2) \right), \quad \tilde{K} = \frac{1}{r} \left( \sqrt{(rp_r)^2 + F_1^2 + q(F_2)} \right), \quad \tilde{D} = rp_r. \]  

(15)

At this point the passage to the conventional conformal mechanics can be realized as above which yields the effective coupling

\[ \mathcal{I} = F_1^2 - q(F_2)^2. \]  

(16)

It is amazing that in the conformal basis all the details of a particular black hole under consideration are encoded in a single function \( q(F_2) \) which enters the effective coupling. However, in contrast to the angular variables \( \Phi^a \) which parameterize a torus, \( \Phi^a \in [0, 2\pi] \), the domain of \( \Theta^2 \), in general, depends on the initial conditions and/or specific properties of a background. Thus, the price paid for the universality of the presented canonical transformation is its locality.

### 4 Examples

Let us illustrate the above considerations by the examples of the near horizon extremal Reissner-Nordström black hole and the near horizon extremal rotating dilaton–axion black hole for which the analysis can be done in elementary functions.

In the first case the conformal generators which characterize the particle probe read [15]

\[ H = \frac{r}{M^2} \left( \sqrt{(mM)^2 + (rp_r)^2 + p_\theta^2 + \sin^{-2}\theta p_\phi^2 + eq} \right), \quad \tilde{D} = rp_r, \]

\[ \tilde{K} = \frac{M^2}{r} \left( \sqrt{(mM)^2 + (rp_r)^2 + p_\theta^2 + \sin^{-2}\theta p_\phi^2 - eq} \right), \]  

(17)

where \( m \) and \( e \) are the mass and the electric charge of a particle while \( M \) and \( q \) are the mass and the electric charge of the black hole. The angular part is given by a free particle moving on a two-dimensional sphere

\[ \mathcal{I} = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} + (mM)^2 - (eq)^2, \quad \omega = dp_\theta \wedge d\theta + dp_\phi \wedge d\phi. \]  

(18)

In order to construct the action–angle variables, we introduce the generating function [19]

\[ S(\mathcal{E}, p_\phi|\theta, \phi) = p_\phi \phi + \int_{\mathcal{I} = \mathcal{E} = \text{const}} d\theta \sqrt{\mathcal{E} - (mM)^2 + (eq)^2 - \frac{p_\phi^2}{\sin^2\theta}}, \]  

(19)

\(^5\)By construction, there is a natural ambiguity in the definition of \( \Theta^1 \) and \( \Theta^2 \) which is related to the choice of the lower limit \( \theta_0 \) in the definite integrals (14). Note that the primitive integrals in (14) at the lower limit \( \theta_0 \) yield functions of \( L \) and \( p_\phi \) which, in our approach, are related to the new momenta \( F_1 \) and \( F_2 \).
which is then used to construct the desired canonical pairs

\[ \Phi^1 = -\arcsin \left( \frac{I_1 + I_2}{\sqrt{(I_1 + I_2)^2 - I_2^2}} \cos \theta \right), \quad I_1 = \sqrt{I - (mM)^2 + (eq)^2} - p_\phi, \]

\[ \Phi^2 = \varphi - \arcsin \left( \frac{p_\theta}{\sqrt{(I_1 + I_2)^2 - I_2^2}} \right) + \Phi^1, \quad I_2 = p_\phi. \]  

The effective coupling reads

\[ I = (I_1 + I_2)^2 + (mM)^2 - (eq)^2. \]  

Note that in the previous studies of the model [14, 15] the angular sector of the resulting conformal mechanics was described by a free particle moving on a two-dimensional sphere. The guiding principle behind the canonical transformation in [14] was to keep the explicit form of the so(3)–generators intact. In our approach no assumption is made on the form of the so(3)–generators and the resulting Hamiltonian in the angular sector reads simpler. As a further development of this work it would be interesting to construct the action–angle variables for the case of the near horizon extremal Kerr and Kerr-Newman black holes.

In order to illustrate the alternative method, let us consider the near horizon extremal rotating dilaton–axion black hole [21]. This solution to the Einstein–Maxwell–dilaton–axion theory has \[ SO(2, 1) \times U(1) \] symmetry group and can be viewed as interpolating between the near horizon extremal Reissner-Nordström black hole and the near horizon extremal Kerr black hole [21]. For a particle probe propagating on such a background the conformal generators read [16]

\[ H = r \left( \sqrt{m^2 + (rp_r)^2 + p_\theta^2 + \sin^{-2} \theta (p_\phi - e \cos \theta)^2} - p_\phi \right), \quad \tilde{D} = rp_r, \]

\[ \tilde{K} = \frac{1}{r} \left( \sqrt{m^2 + (rp_r)^2 + p_\theta^2 + \sin^{-2} \theta (p_\phi - e \cos \theta)^2} + p_\phi \right). \]  

Here \( m \) and \( e \) are the mass and the electric charge of a particle. Note that this realization of the conformal algebra looks very similar to that corresponding to the Bertotti–Robinson solution with a magnetic charge [15]. The only difference is in the last terms entering the Hamiltonian and the generator of special conformal transformations. These terms break \( SO(3) \) symmetry intrinsic to the Bertotti–Robinson solution down to \( U(1) \) in accord with the azimuthal symmetry of the dilaton–axion black hole.

Introducing new momenta in accord with (13)

\[ F_1 = \sqrt{m^2 + p_\theta^2 + \sin^{-2} \theta (p_\phi - e \cos \theta)^2}, \quad F_2 = p_\phi, \]  

one then derives from (14) (where for simplicity we choose \( \theta_0 = \pi/2 \)) the coordinates canonically conjugated to \( F_1 \) and \( F_2 \)

\[ \Theta^1 = -\frac{F_1}{\sqrt{F_1^2 - m^2 + e^2}} \left( \arcsin \left( \frac{(F_2^2 - m^2 + e^2) \cos \theta - eF_2}{\sqrt{(F_1^2 - F_2^2 - m^2)(F_1^2 - m^2 + e^2) + e^2F_2^2}} \right) + \arcsin \left( \frac{eF_2}{\sqrt{(F_1^2 - F_2^2 - m^2)(F_1^2 - m^2 + e^2) + e^2F_2^2}} \right) \right), \]

\[ \Theta^2 = \phi - \arctan \frac{(F_2p_\theta \sin \theta + [F_2 - e \cos \theta] \sqrt{F_1^2 - F_2^2 - m^2}) \cos \theta}{e \cos \theta (F_2 - e \cos \theta) - p_\theta^2 \sin^2 \theta - p_\phi^2 \sin^2 \theta \sqrt{F_1^2 - F_2^2 - m^2}}, \]  

(24)

To the best of our knowledge, the examples considered in this section together with the case of the near horizon extremal Reissner-Nordström black hole carrying a magnetic charge exhaust all the possibilities where a canonical transformation can be constructed in terms of elementary functions.
5 Conclusion

To summarize, in this work two related methods were proposed to construct a canonical transformation which links the model of a massive relativistic particle moving near the horizon of an extremal black hole in four dimensions and the conventional conformal mechanics. A key ingredient of the first scheme is the use of the action-angle variables in the angular sector of the system. Particular importance of this formulation is a precise indication regarding the (non)equivalence of various dynamical systems (cf. [22]). The second approach relies upon integrability of the model in the sense of Liouville. The advantage of the conformal basis is that it allows one to accumulate all specific information about the original model in a lower-dimensional integrable system governed by the effective coupling $I$. Moreover, with such a formulation at hand, one can reduce the problem of constructing an $\mathcal{N} = 4$ superconformal extension of the original system to that for the angular part [12].

Turning to further developments, it is tempting to investigate which geometries correspond to other conformal mechanics models known in the literature first by applying the inverse canonical transformation and then by reconstructing a metric tensor. Reduced integrable systems corresponding to higher dimensional near horizon extremal black holes are also worthy of study.

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