Non-abelian tensor square of finite-by-nilpotent groups

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Abstract. Let $G$ be a group. We denote by $\nu(G)$ an extension of the non-abelian tensor square $G \otimes G$ by $G \times G$. We prove that if $G$ is finite-by-nilpotent, then the non-abelian tensor square $G \otimes G$ is finite-by-nilpotent. Moreover, $\nu(G)$ is nilpotent-by-finite (Theorem A). Also we characterize BFC-groups in terms of $\nu(G)$ among the groups $G$ in which the derived subgroup is finitely generated (Theorem B).

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1. Introduction. Let $G$ and $G^\varphi$ be groups, isomorphic via $\varphi : g \mapsto g^\varphi$ for all $g \in G$. Consider the group $\nu(G)$, introduced in [15] as

$$\nu(G) = \left\langle G \cup G^\varphi \mid [g, h^\varphi]^k = [g^k, (h^k)^\varphi] = [g, h^\varphi]^{k\varphi}, \forall g, h, k \in G \right\rangle. \quad (1.1)$$

It is a well-known fact (see [15]) that the subgroup $\Upsilon(G) = [G, G^\varphi]$ of $\nu(G)$ is canonically isomorphic with the non-abelian tensor square $G \otimes G$, as defined by R. Brown and J.-L. Loday in their seminal paper [2], the isomorphism being induced by $g \otimes h \mapsto [g, h^\varphi]$. The normality of $\Upsilon(G)$ in $\nu(G)$ gives the decomposition

$$\nu(G) = ([G, G^\varphi] \cdot G) \cdot G^\varphi, \quad (1.2)$$

where the dots mean (internal) semidirect products. With this in mind we shall write $\nu(G) = \Upsilon(G)(G^\varphi)$ and use $\Upsilon(G)$, $[G, G^\varphi]$, or $G \otimes G$ indistinctly to denote the non-abelian tensor square of $G$.

The group $\nu(G)$ inherits many properties of the argument $G$; for instance, if $G$ is a finite $\pi$-group ($\pi$ a set of primes), nilpotent, solvable, polycyclic-by-finite, or a locally finite group, then so is $\nu(G)$ (and hence also $G \otimes G$ and the

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non-abelian exterior square $G \wedge G$ [1, 3, 8, 9, 15]. For a deeper discussion of the commutator approach, we refer the reader to [7, 10].

In the present article we want to study the structure of $G \otimes G$ and $\nu(G)$ when $G$ is a finite-by-nilpotent group.

An important result in the context of finite-by-nilpotent groups, due to Baer [14, 14.5.1], states that if $Z_k(G)$ is a subgroup of finite index in $G$, for some positive integer $k$, then the subgroup $\gamma_{k+1}(G)$ is finite. The converse does not hold in general; however, in [5], Hall obtained that if $\gamma_{k+1}(G)$ is finite, for some positive integer $k$, then $Z_{2k}(G)$ is a subgroup of finite index in $G$. This theorem shows that if $G$ is finite-by-nilpotent, then $G$ is nilpotent-by-finite. In the present paper we establish the following related result.

**Theorem A.** Let $G$ be a finite-by-nilpotent group. Then

(a) The non-abelian tensor square $G \otimes G$ is finite-by-nilpotent;

(b) The group $\nu(G)$ is nilpotent-by-finite.

The converse of Theorem A (a) does not hold. In [1, Theorem 22] Blyth and Morse proved that $\Upsilon(D_\infty)$ is abelian, where $D_\infty = \langle a, b \mid a^2 = 1, a^b = a^{-1} \rangle$. More precisely, $\Upsilon(D_\infty)$ is isomorphic to $C_2 \times C_2 \times C_2 \times C_\infty$. On the other hand, $\gamma_k(D_\infty)$ is an infinite cyclic group, for every positive integer $k$. So, in particular, $D_\infty$ cannot be finite-by-nilpotent.

It is an immediate consequence of [3, Proposition 9] that if $G$ has a central subgroup $Z$ of finite index, then also $G \otimes G$ has a central subgroup of finite index. Such a group is called central-by-finite. In particular, in this case the center $Z(G)$ is a subgroup of finite index in $G$. I. Schur [14, 10.1.4] showed that if $G$ is a central-by-finite group, then the derived subgroup $G'$ is finite and $\exp(G')$ divides $|G/Z(G)|$. It is clear that every central-by-finite group is also a BFC-group. Recall that a group $G$ is called a BFC-group if there is a positive integer $d$ such that no element of $G$ has more than $d$ conjugates. B. H. Neumann improved Schur’s theorem in a certain way, showing that the group $G$ is a BFC-group if and only if the derived subgroup $G'$ is finite [11]. The following result gives us another characterization of BFC-groups in terms of $\nu(G)$ when $G$ has finitely generated derived subgroup.

**Theorem B.** Let $G$ be a group in which $G'$ is finitely generated. The following properties are equivalent:

(a) $G$ is a BFC-group;

(b) $\nu(G)'$ is central-by-finite;

(c) $\nu(G)''$ is finite.

In Theorem B, the hypothesis that the derived subgroup $G'$ is finitely generated is essential (see Remark 3.3 below). It is also interesting to note that when $G$ is a BFC-group, $\Upsilon(G)$ is central-by-finite (this follows from Corollary 3.4 below). Nonetheless, as seen above, $\Upsilon(D_\infty)$ is abelian and $D'_\infty$ is an infinite cyclic subgroup, while $D_\infty$ is not a BFC-group. This suggest the following:

**Question.** Let $G$ be a group in which $G'$ is generated by finitely many commutators of finite order and $\Upsilon(G)$ is central-by-finite. Is it true that $G$ is a BFC-group?
It is well-known that the converse of Schur’s theorem does not hold. Theorem B (b)–(c) could be used to obtain a converse of Schur’s theorem according to the following:

**Definition.** A group \(G\) is called a \(\nu\)-group if and only if there exists a group \(H\) with derived subgroup \(H'\) finitely generated such that \(G\) is isomorphic to \(\nu(H)'.\)

Thus, if \(G\) is a \(\nu\)-group with \(G'\) finite, then \(G\) is central-by-finite. For more details concerning groups satisfying the converse of Schur’s theorem, see [4,6,12,17].

In the next section we collect some results of the non-abelian tensor square and related constructions that are later used in the proofs of our main theorems. Section 3 contains the proofs of the main results.

2. The group \(\nu(G)\). The following basic properties are consequences of the defining relations of \(\nu(G)\) and the commutator rules (see [15, Section 2] for more details).

**Lemma 2.1.** The following relations hold in \(\nu(G)\), for all \(g, h, x, y \in G\).

(i) \([g,h^{\phi}, x^{\phi}] = [g,h^{\phi}, x] = [g^{\phi}, h, x] = [g^{\phi}, h^{\phi}, x] = [g^{\phi}, h, x].\)

(ii) \([g,h^{\phi}] \triangleleft \nu(G), [G, N^{\phi}] \triangleleft \nu(G)\).

In the notation of [16, Section 2], we have the derived map \(\rho' : \Upsilon(G) \to G'\) given by \([g, h^{\phi}] \mapsto [g, h]\). Let us denote by \(\mu(G)\) the kernel of \(\rho'\). In particular,

\[
\frac{\Upsilon(G)}{\mu(G)} \cong G'.
\]

**Remark 2.2.** Let \(N\) be a normal subgroup of \(G\). We set \(\overline{G}\) for the quotient group \(G/N\), and the canonical epimorphism \(\pi : G \to \overline{G}\) gives rise to an epimorphism \(\overline{\pi} : \nu(G) \to \nu(\overline{G})\) such that \(g \mapsto \overline{g}, g^{\phi} \mapsto \overline{g^{\phi}}\), where \(\overline{G^{\phi}} = G^{\phi}/N^{\phi}\) is identified with \(\overline{G^{\phi}}\).

In the following lemma we collect some results which can be found in [15] and [16], respectively.

**Lemma 2.3.** Let \(n\) be a positive integer and \(G\) be a group.

(i) ([15, Proposition 2.5]) With the above notation we have

(a) \([N, G^{\phi}] \triangleleft \nu(G), [G, N^{\phi}] \triangleleft \nu(G)\);

(b) \(Ker \overline{\pi} = \langle N, N^{\phi} \rangle [N, G^{\phi}] \cdot [G, N^{\phi}]\).

(ii) ([15, Theorem 3.3]) For \(i \geq 2\) the \(i\)-th term of the derived series of \(\nu(G)\) is given by

\[(\nu(G))^{(i)} = G^{(i)}(G^{\phi})^{(i)}[G^{(i-1)}, (G^{\phi})^{(i-1)}].\]

(iii) ([16, Proposition 2.7 (ii)]) \(\mu(G)\) is a central subgroup of \(\nu(G)\).

**Lemma 2.4.** Let \(n\) be a positive integer and \(G\) a group. Then

\([Z_n(G), G^{\phi}] [G, Z_n(G)^{\phi}] \leq Z_n(\nu(G)).\)
Proof. The case where \( n = 1 \) is covered by [15, Proposition 2.7]. So we will assume that \( n \geq 2 \). We first prove that \( [Z_n(G), G^\varphi] \leq Z_n(\nu(G)) \). Choose arbitrarily elements \( a \in Z_n(G) \) and \( b \in G \). Since \( \nu(G) = \Upsilon(G) G G^\varphi \), it is sufficient to show that

\[
[[a, b^\varphi], x_1^{\varphi}, \ldots, x_n^{\varphi}] = 1 = [[[a, b^\varphi], x_1, \ldots, x_n],]
\]

for all elements \( x_1, \ldots, x_n \in G \). Let \( x_1, \ldots, x_n \in G \). Repeated application of Lemma 2.1 (ii) enables us to write

\[
[[a, b^\varphi], x_1^{\varphi}, x_2^{\varphi}, \ldots, x_{n-1}^{\varphi}, x_n^{\varphi}] = [[[a, b, x_1, x_2, \ldots, x_{n-1}], x_n^{\varphi}], = 1
\]

and

\[
[[a, b^\varphi], x_1, x_2, \ldots, x_n] = [[[a, b, x_1, x_2, \ldots, x_{n-1}], x_n], = 1.
\]

Further, using only obvious modifications of the above argument, one can show that \( [G, Z_n(G)^\varphi] \leq Z_n(\nu(G)) \). This completes the proof. □

3. Proofs of the main results. The following lemma is well known; the equivalence follows directly from results of Hall [5] and Baer [14, 14.5.1].

Lemma 3.1. Let \( G \) be a group. The following properties are equivalent:

(i) \( G \) is finite-by-nilpotent;
(ii) There exists a positive integer \( k \) such that the subgroup \( Z_k(G) \) is a subgroup of finite index in \( G \).

Proof of Theorem A. (a) Since \( (G \otimes G)/\mu(G) \) is isomorphic to \( G' \) and \( \mu(G) \) is a central subgroup of \( \nu(G) \), it follows that the quotient \( (G \otimes G)/Z(G \otimes G) \) is isomorphic to a finite-by-nilpotent group. Therefore \( G \otimes G \) is finite-by-nilpotent.

(b) By definition, there exists a positive integer \( k \) such that \( \gamma_k(G) \) is finite. According to Hall’s result [5], \( Z_{2k}(G) \) is a subgroup of finite index in \( G \). Set \( \bar{G} = G/Z_{2k}(G) \). By Remark 2.2, there exists an epimorphism \( \bar{\pi} : \nu(G) \to \nu(\bar{G}) \). That \( \nu(\bar{G}) \) is finite follows from [15, Proposition 2.4].

Lemma 2.3 now shows that

\[
Ker \bar{\pi} = \langle Z_{2k}(G), Z_{2k}(G)^\varphi \rangle [Z_{2k}(G), G^\varphi][G, Z_{2k}(G)^\varphi].
\]

Since \( \nu(\bar{G}) \) is finite, it is sufficient to show that \( Ker \bar{\pi} \) is nilpotent. By Lemma 2.4, \( [Z_{2k}(G), G^\varphi][G, Z_{2k}(G)^\varphi] \leq Z_{2k}(\nu(G)) \). On the other hand, \( \nu(Z_{2k}(G)) \ni \langle Z_{2k}(G), Z_{2k}(G)^\varphi \rangle \) is nilpotent [15, Corollary 3.2]. Hence \( Ker \bar{\pi} \) is nilpotent. □

Now we will deal with Theorem B: Let \( G \) be a group in which \( G' \) is finitely generated. The following properties are equivalent:

(a) \( G \) is a BFC-group;
(b) \( \nu(G)' \) is central-by-finite;
(c) \( \nu(G)'' \) is finite.
It is well known that the finiteness of the non-abelian tensor square $G \otimes G$ does not imply that $G$ is a finite group. For instance, the Prüfer group $\mathbb{Z}(p^\infty)$ is an example of an infinite group such that $\mathbb{Z}(p^\infty) \otimes \mathbb{Z}(p^\infty) = 0$ (and so, finite). This is the case for all torsion, divisible abelian groups. A useful result, due to Niroomand and Parvizi [13], provides a sufficient condition for a group to be finite.

**Lemma 3.2.** Let $G$ be a finitely generated group in which the non-abelian tensor square $[G, G^\varphi]$ is finite. Then $G$ is finite.

We are now in a position to prove Theorem B.

**Proof of Theorem B.** (a) implies (b). By Lemma 2.3 (v), $\mu(G) \leq Z(\nu(G))$. Moreover, the quotient $\Upsilon(G)/\mu(G) \cong G'$. Since $G$ is a BFC-group, we have that $G'$ is finite [11]. Consequently, $\mu(G)$ is a subgroup of finite index in $\Upsilon(G)$. By Lemma 2.3 (iii), the derived subgroup of $\nu(G)$ is given by

$$\nu(G)' = \Upsilon(G) G' (G')^\varphi.$$ 

Since $G'$ is finite, we conclude that $\mu(G)$ is a central subgroup of finite index in $\nu(G)'$.

(b) implies (c). By Schur’s theorem [14, 10.1.4], $\nu(G)''$ is finite.

(c) implies (a). By Lemma 2.3 (iii), $\nu(G)'' = [G', (G')^\varphi]G''(G'')^\varphi$. Thus, $[G', (G')^\varphi]$ is finite. Since $G'$ is finitely generated, it follows that $G''$ is finite (Lemma 3.2). According to Neumann’s result $G$ is a BFC-group, which completes the proof.

**Remark 3.3.** In Theorem B, the hypothesis that the derived subgroup $G'$ is finitely generated is essential. Let $p \geq 3$ be a prime and consider the Prüfer group $A = \mathbb{Z}(p^\infty)$. We define the semi-direct product $D = A \cdot C_2$, where $C_2 = \langle c \rangle$ and

$$a^c = -a,$$

for every $a \in A$. Thus $D' \cong A$. By Lemma 2.3 (iii),

$$\nu(D)'' = \Upsilon(D') D''(D'')^\varphi.$$ 

Since $D'$ is a Prüfer group, it follows that the non-abelian tensor square $\Upsilon(D')$ is trivial. As $D$ is a metabelian group, we have $\nu(D)''$ is trivial (and so, finite). On the other hand, $D$ is not a BFC-group.

**Corollary 3.4.** Let $G$ be a BFC-group. Then $\Upsilon(G)$ is central-by-finite. Moreover, $\exp(\Upsilon(G)')$ divides $|G'|$.

**Proof.** Set $K = \Upsilon(G)$. Therefore $K$ is central-by-finite by Theorem B (b). That $\exp(K')$ divides $|K/Z(K)|$ follows from Schur’s theorem [14, 10.1.4]. Since $G$ is a BFC-group and $K/\mu(G) \cong G'$, we have $|K/Z(K)|$ divides $|G'|$, which completes the proof.
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