Rényi entropy and $C_T$ for 
$p$–forms on even spheres

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Rényi entropy and central charge, $C_T$, are calculated for a coexact $p$–form on an even sphere with particular reference to the conformally invariant case. It is shown, for example, that the entanglement entropy is minus the standard conformal anomaly with no ‘shift’ being required.

The shift necessary for a conformal $p$–form, when using a hyperbolic technique, is predicted, on a numerical basis, to be (minus) the entanglement entropy of a conformal $(p-1)$–form.

The central charges agree numerically with a general formula of Buchel et al.

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1. Introduction

I give a further calculation of Rényi entropy as a partial extension of earlier work, [1], to which I have to refer for basic explanation and definitions. For this reason I proceed somewhat rapidly.

The field under consideration is a $p$–form propagated by the de Rham Laplacian, $\mathcal{O} = -d\delta - \delta d$. I am not aware of any simple general higher derivative product operator like the GJMS one that involves the de Rham Laplacian.\(^2\) Specific cases do factorise (see [3] and references there) but on a rather ad hoc basis. Osborn and Stegiou, [4], exhibit a four–derivative conformal form theory.

The conically deformed manifold used to define the Rényi entropy is the $d$–dimensional periodic spherical $q$–lune described in [1].

One motivation for the calculation is that it can be carried out in fair generality and that it reinforces some general points appearing in [1]. It also lends concrete credence to the suggestions of Donnelly, Michel and Wall, [5], on the role of edge modes in entanglement entropy. (See also, [6]).

The hyperbolic cylinder technique was applied by Nian and Zhou, [7], in order to determine $p$–form Rényi entropies, A general procedure was outlined and the case of $p = 2$ treated in detail. (See later in section 7.)

2. Rényi entropy

The Rényi entropy, $S_n$, is defined by,

$$S_n = \frac{nW(1) - W(1/n)}{1 - n},$$

where $W(q)$ here is the effective action on the periodic $q$–lune. $n = 1/q$ is the Rényi, or replica, index. $S_1$ is the entanglement entropy and $S'_1$ determines the central charge, $C_T$, [8].

In even dimensions, the universal component of $S_n$, denoted by $\mathfrak{S}_q$, is obtained by substituting the value $\zeta(0)$ for the effective action, $W$, in (1). $\zeta(s)$ is the spectral $\zeta$–function of the propagating operator, $\mathcal{O}$, on the conically deformed manifold. I will refer to $\zeta(0)$ as the (universal part of the) ‘free energy’ and sometimes, imprecisely, as the conformal anomaly, even for any $p$ and $d$. For conformal gauge fields, $p = d/2 - 1$. Their quantisation is well known. Because of $p$–form isomorphisms, it is sufficient to consider coexact forms and make the full reconstruction later.

\(^2\) Wünsch, [2], gives a useful treatment of standard conformally invariant operators referring to $\delta d$ as the Maxwell operator.
3. The coexact \( \zeta \)–function at 0

The \( \zeta \)–function is determined by the coexact spectrum of the de Rham Laplacian on the \( q \)–lune, \( S_q \). The eigenvalues are the same as on the full sphere and standard. They are \( \lambda(p, m, a) \) where

\[
\lambda(p, m, a) = (a + 1 + m)^2 - \alpha^2(a, p), \quad m = 0, 1, 2, \ldots, \tag{2}
\]

setting \( a = (d - 1)/2 \) and \( \alpha(a, p) \equiv a - p \). Note that there is a zero mode at \( m = 0 \) when \( p = d \).

The degeneracies are not the same as on the round sphere, of course. They are given below.

According to (2) I can formally factorise the coexact de Rham Laplacian as follows

\[
-d\delta - \delta d \equiv -\delta d = (\hat{B} - \alpha)(\hat{B} + \alpha)
\]

where

\[
\hat{B}(a, p) = \sqrt{-\delta d + \alpha^2(a, p)},
\]

has eigenvalues

\[
a + 1 + m, \quad m = 0, 1, \ldots,
\]

and \( \zeta \)–function,

\[
\zeta(s, a) = \sum_{m=0}^{\infty} \frac{d(m)}{(a + 1 + m)^s}.
\tag{3}
\]

\( d(m) \) is the level degeneracy, still to be determined.

This ‘simple’ \( \zeta \)–function can be found in terms of the Barnes \( \zeta \)–function as I show below. This is computationally advantageous because it can be shown, [9], that \( \zeta(0) \) of \( -\delta d \) (i.e. the universal part of the free energy) is the average of the \( \zeta \)–functions evaluated at 0 of each linear factor, written,

\[
\mathfrak{F}(p, d, q) = \frac{1}{2} \left( \zeta(0, a + \alpha) + \zeta(0, a - \alpha) \right).
\tag{4}
\]
4. The degeneracy

To compute the coexact free energy one needs the degeneracy, $d(m)$. The solution to the $p$–form spectral problem on regular sphere orbifolds, $S^d/\Gamma$, has been given in [10,11]. In the case when the deck group $\Gamma$ is the extended dihedral action, the fundamental domain (the orbifold) is a single lune of angle $\pi/q$. Geometrically, the periodic lune, referred to above, is obtained by combining such a single lune with its contiguous reflection. The spectrum on the periodic lune is (or can be) obtained by uniting the absolute (a) and relative (r) $p$–spectra on a single lune. I find this convenient and adopt it here.

I should mention that the vector, $p = 1$, spectrum and $\zeta(0)$ on the $d$–dimensional, $q$–sphere, was early directly determined by De Nardo, Fursaev and Miele, [12].

The degeneracies are of course altered by the factoring. As often, it is best to organise them into a generating function,

$$d_b(p, \sigma) = \sum_{m=0}^{\infty} d_b(p, m) \sigma^m,$$

where I have now explicitly indicated the dependence on the form order. The suffix $b$ is the condition that the $p$–form satisfies on the boundary of the fundamental domain of the action of $\Gamma$, i.e. either $b = a$ (absolute) or $b = r$ (relative). These are dual is the sense that, for coexact forms,

$$d_b(p, \sigma) = d_{a\ast}(d-1-p, \sigma), \quad *a = r, ** = id.$$

Note that there are no coexact $d$–forms, $d_b(d, \sigma) = 0$, although there is a zero mode.

Molien’s theorem and invariant theory produce an expression for $d_b(p, \sigma)$ in terms of the algebraic (integer) degrees, $\omega_i$, $i = 1, \ldots, d$, which define the polytope symmetry group, $\Gamma$. In the simple case of a dihedral action all the $\omega_i$ are unity except for one, which equals $q$. (It is helpful to note that if $q = 1$, the fundamental domain is a hemisphere).

The corresponding coexact generating functions are functions of $q$ and are given, [11], by (in even dimensions),

$$d_a(p, \sigma, q) = \frac{(-1)^{p+1}}{\sigma^{p+1}(1-\sigma)^{d-1}(1-\sigma q)} \sum_{r=p+1}^{d} (-1)^r e_r(\sigma^q, \sigma, \ldots, \sigma), \quad (5)$$

3 A finite, ‘fermionic’ Poincaré series on the form orders, $p$, can also be introduced, but I will not use this here. $\sigma$ is often written as $q$ but here this stands for the orbifold order.
in terms of elementary symmetric functions, \( e_r \), on \( d \) arguments. Explicitly

\[
e_r(\sigma^q, \sigma, \ldots, \sigma) = \left( \frac{d-1}{r} \right) \sigma^r + \left( \frac{d-1}{r-1} \right) \sigma^{r-1+q},
\]

(6)

which allows the simple \( \zeta \)–function, (3), to be obtained. At this point, to make sense, \( q \) is an integer. After explicit calculation it can be continued into the reals.

5. The cylinder kernel and the \( \zeta \)–function by Mellin transform

The form of the eigenvalues suggests the construction of the quantity,

\[
T_b(p, \sigma, q, a) = \sigma^{a+1} d_b(p, \sigma, q),
\]

(7)

which can be interpreted as the (traced) cylinder kernel\(^4\) for the pseudo operator \( \hat{B} \), on putting \( \sigma = e^{-\tau} \), where \( \tau \) is a propagation ‘time’.

The simple coexact spectral \( \zeta \)–function, (3), then follows immediately as the Mellin transform\(^5\) (I give the absolute expression),

\[
\zeta_a(s, a, p, q) = i \frac{\Gamma(1-s)}{2\pi} \int_{C_0} d\tau (-\tau)^{s-1} T_a(p, e^{-\tau}, q, a)
\]

\[
= (-1)^p+1 i \frac{\Gamma(1-s)}{2\pi} \int_{C_0} d\tau (-\tau)^{s-1} \frac{e^{-(a+1)p\tau}}{(1-e^{-\tau})^{d-1}(1-e^{-q\tau})} \times 
\]

\[
\sum_{r=p+1}^{d} (-1)^r \left[ \left( \frac{d-1}{r} \right) e^{-r\tau} + \left( \frac{d-1}{r-1} \right) e^{-(r-1+q)\tau} \right]
\]

\[
= (-1)^p+1 \sum_{r=p+1}^{d} (-1)^r \left[ \left( \frac{d-1}{r} \right) \zeta_B(s, a-p+r \mid \omega) 
\right.
\]

\[
\left. + \left( \frac{d-1}{r-1} \right) \zeta_B(s, a-p+r+q-1 \mid \omega) \right].
\]

(8)

The \( \zeta_B \) are Barnes \( \zeta \)–functions and the vector, \( \omega \), stands for the \( d \)–dimensional set \( \omega = (q, 1, \ldots, 1) = (q, 1_{d-1}) \).

Duality gives the relative \( \zeta_r(s, a, p, q) = \zeta_a(s, a, d-1-p, q) \) which has to be added to the absolute expression in order to get the coexact \( p \)–form value on the periodic (double) lune.

\(^4\) Other names are ‘wave kernel’, ‘Poisson kernel’, ‘single particle partition function’, depending on the interpretation of the parameter, \( \tau \).

\(^5\) This approach has been around for a considerable time. See [13] where the inversion behaviour under \( \sigma \to 1/\sigma \) was also brought into play.
This \( \zeta \)-function is now substituted into (4) to give the required (coexact) free energy. One sees that the four arguments of the pair of Barnes functions in (8) have the values \( (\alpha + r \pm \alpha, \alpha + r + q - 1 \pm \alpha) \).

The Barnes function at \( s = 0 \) is a generalised Bernoulli polynomial and the square bracket in (8) equals,

\[
\frac{1}{d!q} \left[ \binom{d-1}{r} B_d^{(d)}(\alpha + r \pm \alpha | \omega) + \binom{d-1}{r-1} B_d^{(d)}(\alpha + r + q - 1 \pm \alpha | \omega) \right].
\]

(9)

The \( q \)-dependence can be simplified by using a symmetry property of the Bernoulli functions. This produces for (9),

\[
\frac{1}{d!q} \left[ \binom{d-1}{r} B_d^{(d)}(\alpha + r \pm \alpha | \omega) + (-1)^d \binom{d-1}{r-1} B_d^{(d)}(d - \alpha - r \mp \alpha | \omega) \right].
\]

(10)

Written out, this equals

\[
\frac{1}{d!q} \left[ \binom{d-1}{r} B_d^{(d)}(2\alpha + r | \omega) + (-1)^d \binom{d-1}{r-1} B_d^{(d)}(d - 2\alpha - r | \omega) \\
+ \binom{d-1}{r} B_d^{(d)}(r | \omega) + (-1)^d \binom{d-1}{r-1} B_d^{(d)}(d - r | \omega) \right].
\]

(11)

For given form order, \( p \), and dimension, \( d \), all quantities can be evaluated easily by machine and lead to an expression for the (absolute) coexact free energy, \( \mathfrak{F}_a(p, d, q) \), as a rational function of the lune parameter, \( q \), which can now be taken as a real number.

6. The complete field theory \( \zeta(0) \). Free energy and Rényi entropy

Having the single (coexact) \( p \)-form quantity, it is necessary next to assemble the ghosts–for–ghosts sum to get the complete \( p \)-form free energy, \( \mathcal{F}(p, d, q) \). This is a standard construction (e.g. [14]) and yields, in particular, on the single lune,

\[
\mathcal{F}_b(p, d, q) = \sum_{l=0}^{p} (-1)^{p+l} \mathfrak{F}_b(l, d, q) + (-1)^p (p + 1) \delta_{br}.
\]

(12)

The last term is a zero mode effect which exists only for relative conditions. The end result is again a rational function of \( q \) for given \( p \) and \( d \). I give a few absolute examples,

\[
\begin{align*}
\mathcal{F}_a(1, 4, q) &= -\frac{q^4 + 30 q^2 - 660 q + 33}{360 q} \\
\mathcal{F}_a(2, 4, q) &= -\frac{q^4 - 60 q^2 + 1440 q - 57}{720 q} \\
\mathcal{F}_a(3, 6, q) &= \frac{2 q^6 - 35 q^4 - 1260 q^2 + 42924 q - 1355}{15120 q}.
\end{align*}
\]
The relative expressions can be most easily found from the difference,

$$\mathcal{F}_a(p, d, q) - \mathcal{F}_r(p, d, q) = (-1)^p 2(p + 1),$$
or from duality.

As a final step, adding the absolute and relative quantities gives those on the periodic lune (the \(q\)-deformed sphere). A few examples are,

\[
\begin{align*}
\mathcal{F}(0, 2, q) &= \frac{q}{6} + \frac{1}{6q} \\
\mathcal{F}(1, 4, q) &= -\frac{q(q^2 + 30)}{180} - \frac{1}{3} - \frac{11}{60q} \\
\mathcal{F}(2, 4, q) &= -\frac{q(q^2 - 60)}{360} + 2 + \frac{57}{360q} \\
\mathcal{F}(1, 6, q) &= \frac{q(2q^4 - 35q^2 - 1260)}{7560} - \frac{29}{90} - \frac{271}{1512q} \\
\mathcal{F}(2, 6, q) &= \frac{q(2q^4 + 35q^4 + 840)}{5040} + \frac{31}{45} + \frac{191}{1008q} \\
\mathcal{F}(3, 6, q) &= \frac{q(2q^4 - 35q^2 - 1260)}{7560} - \frac{209}{90} - \frac{271}{1512q} \\
\mathcal{F}(3, 8, q) &= -\frac{q(3q^6 + 56q^4 + 686q^2 + 15120)}{90720} - \frac{221}{210} - \frac{2497}{12960q}.
\end{align*}
\]

As check, evaluation at \(q = 1\), the round sphere, provides agreement with the values obtained by Raj, [15].

Samples of the resulting Rényi entropies are,

\[
\begin{align*}
\mathcal{S}_q(1, 4) &= \frac{(q + 1)(q^2 + 31)}{180} + \frac{1}{3} \\
\mathcal{S}_q(2, 4) &= \frac{(q + 1)(q^2 - 59)}{360} - 2 \\
\mathcal{S}_q(2, 6) &= -\frac{(q + 1)(2q^4 + 37q^2 + 877)}{5040} - \frac{31}{45} \\
\mathcal{S}_q(3, 6) &= -\frac{(q + 1)(2q^4 - 33q^2 - 1293)}{7560} + \frac{209}{90} \\
\mathcal{S}_q(3, 8) &= \frac{(q + 1)(3q^6 + 59q^4 + 745q^2 + 15865)}{90720} + \frac{221}{210}.
\end{align*}
\]

\(^{6}\) When calculating the heat–kernel coefficient, relevant for any conformal anomaly, one should either use the \(\zeta\)-functions with zero modes included, or add these in separately. This point arises for the \(p = d\) form for which there is a coexact zero mode. Excluding this, the coexact \(d\)-form vanishes.
I have arranged the expressions in a way helpful for the remarks in the next section.

The significance of the non–conformal entropy is not clear to me.

Evaluation at \( q = 1 \) gives the entanglement entropy and one finds \( \mathcal{S}_1(p,d) = -\mathcal{F}(p,d,1) \) essentially as a consequence of the fact that the free energy is an extremum at the round sphere, \( q = 1 \).\(^7\) In the conformal case, this says that the entanglement entropy is minus the conformal anomaly (in my sign conventions) on the round sphere. For example, for one–forms in four–space this equals \( 31/45 \), the standard value. The higher–dimensional values were calculated by Capelli and D’Apollonio, [14], long ago using the same spectral data as here but organised differently.

7. The shift and edge modes. Comparison with hyperbolic

In [3], [7] and elsewhere, in order to regain the standard conformal anomaly on the round sphere, a ‘shift’ was made to the value obtained from the Rényi entropy found there on a hyperbolic cylinder approach.

Here, this is not necessary and I have written the obtained entropies in (14) so as to make comparisons easier. The first part on the right–hand side is the hyperbolic result, agreeing with the expressions in [7] for \( p = 1, 2 \). The second part is the shift, applied in [7] to get the ‘correct’ entropy. See also [6,3] for \( p = 1 \). The expression for the gauge boson \( (p = 1) \) free energy is given by Fursaev, [16].

The numerics\(^8\) thus reveal that the constant term in the conformal \( p \)–form free energy is just minus the entanglement entropy of a conformal \( p+1 \) form. These are the shifts that would have to be applied when computing the entanglement entropy for a conformal gauge theory by any hyperbolic method. It confirms the expectation that the shift is an entangling surface, ghost edge mode effect, in particular the specific \( p \)–form gauge theory suggestion by Donnelly, Michel and Wall, [5]. The \( p = 1 \) case was earlier considered by Huang, [6].

I note that the conformal Rényi entropy evaluated at \( q = -1 \) equals the constant part of the free energy. This is expressed as,

\[
\mathcal{S}_{-1}(p,2p + 2) = \mathcal{S}_1(p-1,2p),
\]

\(^7\)Unfortunately, I have not been able to prove this in general, but only case by case, the derivatives all having a factor of \( (q^2 - 1) \), even for any \( p \) and \( d \).

\(^8\)An analyticall proof may be provided later.
which is in accord with the hyperbolic expressions for the entropy as they vanish when \( q = -1 \).

A comparison of the free energies here and those in \([7]\) and \([3]\) show that they differ by the shift constant, and also by a term proportional to \( 1/q \) which numerically equals minus twice the (Maxwell) Casimir energy on \( R \times S^{d-1} \), \([17]\), \([18]\), as was the case also in \([1]\).\(^9\) The hyperbolic expressions are given as the first terms on the right–hand side of (13).

8. Derivatives and central charge

Beccaria and Tseytlin, \([3]\), employ Perlmutter’s relation, \([8]\), to find the central charge, \( C_T \), from the Rényi entropy. I followed the same route, in a different, compact geometry, in \([1]\) for scalars and spinors and now extend these to standard gauge \( p \)–forms (\( cf \) Nian and Zhou, \([7]\)).

I can, at the moment, proceed only dimension by dimension and form by form. Also, to give the central charge meaning, I have to limit myself to the conformal results, \( (p = d/2 - 1) \). Then \( C_T(d) \) computes to \( [2, 16, 108, 640, 3500, \ldots] \) for \( d = 2, 4, \ldots, 10, \ldots \).

In fact I have added nothing numerically to the general formula of Buchel \textit{et al}, \([20]\),

\[
C_T(d) = \frac{d^2 (d - 2)!}{2((d/2 - 1)!)^2}
\]  

(15)
derived, after lengthy calculation, directly from the two point function of the energy–momentum tensor in flat space. I have thus tested their formula by a quite different, spectral method.

9. Comments

In \([1]\) I derived general formulae for \( C_T \) like (15). Without further serious simplification of the expressions used here, it is difficult to see how one might obtain this simple result.

Although the organisation of the spectral data by using Barnes \( \zeta \)–functions is very efficient at producing the answers, it generally gives no indication of any

\(^9\)This can probably be proved analytically but the calculation is complicated by the ghost sum. This actually collapses, up to zero modes, on the Einstein cylinder, \([19]\). A similar collapse should occur on the \( q \)–sphere as \( q \to 0 \).
underlying reason for a particular result which is often a consequence of machine evaluation.

I suggest that a GJMS–type product for the Maxwell operator, $-\delta d$, exists.

A more systematic hyperbolic calculation should be undertaken for higher $p$–forms in order to confirm the nature of the shifts.

The $q \to 0$ limit should be carefully investigated.

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