ON A P-LAPLACIAN EIGENVALUE PROBLEM WITH SUPERCRITICAL EXponent

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Abstract. In this paper, we prove the existence of the positive and negative solutions to p-Laplacian eigenvalue problems with supercritical exponent. This extends previous results on the problems with subcritical and critical exponents.

1. Introduction. A new framework (established recently in [21]) is used to deal with the following p-Laplacian eigenvalue problem: For any given $\alpha \in \mathbb{R}^+$, find a solution $u$ to the following equation

$$\begin{cases}
-\text{div}(|\nabla u|^{p-2}\nabla u) + |u|^{q-2}u = \lambda g(x, u), & x \in \Omega, \\
\frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{1}{q} \int_{\Omega} |u|^q dx = \alpha.
\end{cases}
$$

Here $\Omega \subset \mathbb{R}^n(n > p \geq 2)$ is a bounded domain with smooth boundary $\partial \Omega$ and $q > p^* \triangleq \frac{np}{n-p}$.

The force term $g(x, u)$ satisfies the following assumptions:

(*) $g \in C^0(\Omega \times \mathbb{R}, \mathbb{R})$, $g(x, t)t \geq 0$ and there is a neighborhood $U$ of zero such that $g(x, t)t > 0$, for any $t \in U \setminus \{0\}$ and $x \in \Omega$;

(**) $|g(x, t)| \leq c_1 |t|^\ell + c_2$, $c_1, c_2 \in \mathbb{R}$, $1 \leq \ell < q - 1$.

There are many papers discussing the following eigenvalue problem

$$\begin{cases}
-\text{div}(|\nabla u|^{p-2}\nabla u) = \lambda g(x, u), & x \in \Omega, \\
\frac{1}{p} \int_{\Omega} |\nabla u|^p dx = \alpha.
\end{cases}
$$

For the case $p = 2$, the authors in [1, 15, 19] proved the existence of multiple solutions to (I) when $g$ is odd, later the existence of three solutions was obtained in [9] and Y.Q. Li and Z.L. Liu in [10] applied the descent flow to obtain the multiple and sign-changing solutions when $g$ is non odd. Recently, we studied the existence of positive and negative solutions to (I) for the supercritical exponent (i.e., $q > 2^*$) by establishing a new framework in [21].

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Lemma 1.2. For any space, the following fundamental lemma holds (see also Lemma 1.1 in [21]).

The coefficient \( \langle \Psi, \Phi \rangle \) given above complicates checking the Palais-Smale condition when the nonlinear term is supercritical. So we simplify the coefficient \( \langle \Psi', u \rangle_{X^*, X} \) in (1.4) by giving a new form for \( \tilde{\Psi}'(u) \).

Due to the Best Approximation Theorem in a strictly convex reflexive Banach space, the following fundamental lemma holds (see also Lemma 1.1 in [21]).

**Lemma 1.2.** For any \( u \in N_\alpha \), there exists a unique element \( e(u) \in X \), such that

\[
\langle \Phi'(u), e(u) \rangle = 1 \quad \text{and} \quad \|e(u)\| = \inf_{v \in X} \{\|v\| : \langle \Phi'(u), v \rangle = 1\},
\]
where $\langle \cdot, \cdot \rangle$ denotes the dual pair between $X$ and $X^*$, the element $e(u) \in X$ dependent on $u$.

Furthermore, there exist two positive constants $M_1, M_2$ independent on $u$, such that

$$0 < M_1 \leq \|e(u)\| \leq M_2, \quad \forall \ u \in N_\alpha. \quad (1.6)$$

Then, let $\ker(\Phi'(u)) = \{ w \in X : \langle \Phi'(u), w \rangle = 0 \}$ and $\ker(e(u)) = \{ w^* \in X^* : \langle w^*, e(u) \rangle = 0 \}$. We give our new form for $\tilde{\Psi}'(u)$ with the best approximation $e(u)$ given above (see also Theorem 1.1 in [21]).

**Lemma 1.3.** Under the assumption of Lemma 1.2, it holds that

$$\tilde{\Psi}'(u) = \Psi'(u) - \langle \Psi'(u), e(u) \rangle \Phi'(u). \quad (1.7)$$

Meanwhile, the tangent (or cotangent) space of $N_\alpha$ at $u$: $T_u(N_\alpha)$ (or $T_u^*(N_\alpha)$) is respectively given by

$$T_u(N_\alpha) = \ker(\Phi'(u)), \quad \tilde{\Psi}'(u) \in T_u^*(N_\alpha) = \ker(e(u)), \quad (1.8)$$

Moreover, provided $X$ is a Hilbert space, $\tilde{\Psi}'(u)$ is just a tangent vector (in the sense of isomorphism).

From (1.7), the existence of solutions for the eigenvalue problem (I) is shown in the following Theorem:

**Theorem 1.4.** Under the assumptions $(\ast)$, $(\ast\ast)$ on $g$, there exist at least two solutions to problem (I), one of which is positive and the other is negative.

This article is organized as follows: In Section 2, some preliminaries are given; In Section 3, we check the Palais-Smale condition for the formula (1.7); In Section 4, we show Theorem 1.4. Throughout the rest of the article, $C, C_i, c, c_i, i = 1, 2, \ldots$ denote constants and may be different in different places; “$\rightarrow$” and “$\rightharpoonup$” represent strong and weak convergence in related function spaces, respectively.

2. Preliminaries. We begin with the following lemma about the convergence property.

**Lemma 2.1.** If $u_n \rightharpoonup v$ in $X$ as $n \to \infty$, then

$$u_n \rightarrow v \text{ in } W^{1,p}_0(\Omega), \quad u_n \rightarrow v \text{ in } L^q(\Omega), \quad \text{as } \ n \to \infty, \quad (2.1)$$

$$u_n \rightarrow v \text{ in } L^t(\Omega), \quad \forall \ 2 \leq t < q, \quad \text{as } \ n \to \infty, \quad (2.2)$$

$$u_n(x) \rightarrow v(x), \quad \text{a.e. } x \in \Omega, \quad \text{as } \ n \to \infty. \quad (2.3)$$

Furthermore,

$$|u_n|^{q-2} u_n \rightharpoonup |v|^{q-2} v \text{ in } L^q(\Omega) \text{ with } \frac{1}{q} + \frac{1}{q'} = 1, \quad \text{as } \ n \to \infty. \quad (2.4)$$

Since the proof is similar to that of Lemma 2.1 in [21], we omit it here.

Next is to give the differential of functionals $\Phi(u)$, $\Psi(u)$.

**Lemma 2.2.** Let the functionals $\Phi(u)$, $\Psi(u)$ be defined in (1.1), (1.2), then $\Phi(u)$, $\Psi(u)$ is $C^1$ on $X$ with

$$\Phi'(u) \triangleq -\text{div}(\nabla u |^{p-2} \nabla u) + |u|^{q-2} u, \quad (2.5)$$

$$\Psi'(u) \triangleq g(x, u). \quad (2.6)$$
By the mean value theorem and the Lebesgue dominant convergence, it is easy to prove Lemma 2.2. We shall not develop this here and refer the interested reader to Lemma 2.2 in [21].

Before considering the formula of \( \tilde{\Psi}'(u) \), we show the smoothness of the manifold \( N_\alpha \).

**Lemma 2.3.** Let \( N_\alpha \) be defined in (1.1), then \( N_\alpha \) is a smooth manifold for each \( \alpha \neq 0 \).

**Proof.** It suffices to show that \( \alpha(\neq 0) \) is a regular value of \( \Phi \). That is to say, for any \( u \in N_\alpha \) with \( \alpha \neq 0 \), \( \Phi'(u) \neq 0 \). In fact, if there exists a \( u \in N_\alpha \) such that \( \Phi'(u) = 0 \), namely,

\[
\Phi'(u) = -\text{div}(|\nabla u|^{p-2}\nabla u) + |u|^{q-2}u = 0 \quad \text{in} \quad X^*,
\]

then,

\[
0 = \langle \Phi'(u), u \rangle = \langle -\text{div}(|\nabla u|^{p-2}\nabla u) + |u|^{q-2}u, u \rangle = \int_{\Omega} |\nabla u|^p dx + \int_{\Omega} |u|^q dx.
\]

Hence \( u = 0 \), which contradicts with the definition of \( N_\alpha \) for \( \alpha \neq 0 \) (see (1.1)). Thus, Lemma 2.3 holds. □

3. **Palais-Smale condition for \( \tilde{\Psi}'(u) \) defined in (1.7).** Recall that there is a one-to-one correspondence between critical points of \( \tilde{\Psi} \) and weak solutions to problem (I). Therefore, (1.7) implies that there is a one-to-one correspondence between critical points of \( \tilde{\Psi} \) and weak solutions to problem (I) with \( \lambda = \frac{1}{\langle \Phi'(u), e(u) \rangle_{X^*,X}} \), where \( \langle \Phi'(u), e(u) \rangle_{X^*,X} \neq 0 \) follows from the subsequent remark.

**Remark 1.** When \( \tilde{\Psi}'(u) = 0 \), \( \langle \Phi'(u), e(u) \rangle = 0 \) if and only if \( \tilde{\Psi}'(u) = 0 \), which is contradictory to the assumption \((\ast)\) on \( g \) with the fact that if \( u \in W^{1,p}_0(\Omega) \setminus \{0\} \) then for any \( \epsilon > 0 \), \( \text{mes}\{x : |u(x)| \leq \epsilon\} > 0 \) (see the details in [8]).

Denote

\[
\mathbb{K} = \{ u \in N_\alpha : \tilde{\Psi}'(u) = 0 \}, \quad \mathbb{K}_c = \{ u \in \mathbb{K} : \tilde{\Psi}(u) = c \} \quad (3.1)
\]

and

\[
\tilde{\mathbb{K}}_c = \{ u \in N_\alpha : \tilde{\Psi}(u) \leq c \}. \quad (3.2)
\]

Similarly as Lemma 2.6 in [21], we have

**Lemma 3.1.** \( \tilde{\Psi}' \), \( \Psi \) and \( \tilde{\Psi} \) are completely continuous, namely, it maps weakly convergent sequences to strongly convergent ones.

Now we are in position to testify the Palais-Smale condition for \( \tilde{\Psi}'(u) \) given in (1.7).

**Lemma 3.2.** Assume that a sequence \( u_n \in N_\alpha \) satisfies \( \tilde{\Psi}(u_n) \to c(\neq 0) \) and \( \tilde{\Psi}'(u_n) \to 0 \), then it has a convergent subsequence in \( X \).

**Proof.** From the definition of \( N_\alpha \), \( u_n \) is a bounded sequence in \( X \). Thus, \( u_n \) has a weak convergent subsequence (also denoted by \( u_n \rightharpoonup u \) in \( X \) for convenience). Lemma 3.1 shows that

\[
\tilde{\Psi}(u_n) \to \tilde{\Psi}(u), \quad \text{i.e.,} \quad \tilde{\Psi}(u) = c. \quad (3.3)
\]

Since \( c \neq 0 \), \( u \neq 0 \). Owing to \( \tilde{\Psi}'(u_n) \to 0 \), (1.7) indicates that

\[
\tilde{\Psi}'(u_n) = \Psi'(u_n) - \langle \Psi'(u_n), e(u_n) \rangle \Psi'(u_n) \to 0. \quad (3.4)
\]
Meanwhile, Lemma 3.1 and $u_n \to u$ in $X$ implies that

$$\Psi'(u_n) \to \Psi'(u) \text{ in } X^*. \quad (3.5)$$

Denote $\beta_n = \langle \Psi'(u_n), e(u_n) \rangle$ in (3.4). Since $\|e(u_n)\| < M_2$ (see (1.6)) and $\Psi'(u_n) \to \Psi'(u)$, the sequence $\beta_n$ is bounded. The standard analysis yields that there exists a subsequence $\beta_{n_k} \to \beta_0$. Thus, (3.4) and (3.5) lead to

$$0 = \lim_{k \to \infty} \{ \Psi'(u_{n_k}) - \langle \Psi'(u_{n_k}), e(u_{n_k}) \rangle \Psi'(u_{n_k}) \}$$

$$= \lim_{k \to \infty} \Psi'(u_{n_k}) - \lim_{k \to \infty} \beta_{n_k} \Phi'(u_{n_k})$$

$$= \Psi'(u) - \beta_0 \lim_{k \to \infty} \Phi'(u_{n_k}). \quad (3.6)$$

We claim that

**Claim 1**: $\beta_0 \neq 0$.

If $\beta_0 = 0$, combining with $\Phi'(u_{n_k}) \to \Phi'(u)$ in $X^*$ (see following Claim 2), (3.6) yields $\Psi'(u) = 0$. By a similar argument used in Remark 1, it contradicts with $u \neq 0$ (due to $c \neq 0$ in (3.3)). The proof of Claim 1 is done.

Therefore, (3.6) implies

$$\lim_{k \to \infty} \Phi'(u_{n_k}) = \lim_{k \to \infty} \left( \frac{\Psi'(u_{n_k})}{\beta_{n_k}} + o(1) \right) = \frac{\Psi'(u)}{\beta_0}. \quad (3.7)$$

**Claim 2**: $\Phi'(u_{n_k}) \to \Phi'(u)$ in $X^*$.

The proof of Claim 2 will be given later.

Claim 2 and (3.7) state that

$$\lim_{k \to \infty} \Phi'(u_{n_k}) = \frac{\Psi'(u)}{\beta_0} = \Phi'(u) \text{ in } X^*. \quad (3.8)$$

Then (2.5) and (3.8) lead to

$$-\text{div}(|\nabla u_{n_k}|^{p-2}\nabla u_{n_k}) + |u_{n_k}|^{q-2}u_{n_k} \to -\text{div}(|\nabla u|^{p-2}\nabla u) + |u|^{q-2}u \text{ in } X^*, \quad (3.9)$$

i.e.,

$$-\text{div}(|\nabla u_{n_k}|^{p-2}\nabla u_{n_k}) - (-\text{div}(|\nabla u|^{p-2}\nabla u)) + |u_{n_k}|^{q-2}u_{n_k} - |u|^{q-2}u \to 0 \text{ in } X^*. \quad (3.10)$$

Multiplying (3.10) by $u_{n_k} - u$, we can get

$$0 = \lim_{k \to \infty} \left( -\text{div}(|\nabla u_{n_k}|^{p-2}\nabla u_{n_k}) - (-\text{div}(|\nabla u|^{p-2}\nabla u)) + |u_{n_k}|^{q-2}u_{n_k} - |u|^{q-2}u, u_{n_k} - u \right)$$

$$\geq \frac{1}{2p} \int_\Omega |\nabla u_{n_k} - \nabla u|^p dx + \frac{1}{2q} \int_\Omega |u_{n_k} - u|^q dx \geq 0. \quad (3.11)$$

This implies that $\lim_{k \to \infty} \|u_{n_k} - u\|_{W^{1,p}(\Omega)} = 0$ and $\lim_{k \to \infty} \|u_{n_k} - u\|_{L^q(\Omega)} = 0$, namely,

$$u_{n_k} \to u \text{ in } X. \quad (3.12)$$

Furthermore, combining with $u_{n_k} \in N_\alpha$, we have $u \in N_\alpha$. Thus, Lemma 3.2 holds.

It remains to prove Claim 2.

**Proof**. It needs to testify

$$-\text{div}(|\nabla u_{n_k}|^{p-2}\nabla u_{n_k}) + |u_{n_k}|^{q-2}u_{n_k} \to -\text{div}(|\nabla u|^{p-2}\nabla u) + |u|^{q-2}u \text{ in } X^*. \quad (3.13)$$

Owing to (2.4), the crucial difficulty is to show

$$-\text{div}(|\nabla u_{n_k}|^{p-2}\nabla u_{n_k}) \to -\text{div}(|\nabla u|^{p-2}\nabla u) \text{ in } W^{-1,p'}(\Omega). \quad (3.14)$$
From (2.1), it is equivalent to showing
\[ \nabla u_{nk}(x) \to \nabla u(x), \text{ a.e. } x \in \Omega. \]
i.e.,
\[ \int_{\Omega} (|\nabla u_{nk}|^{p-2}\nabla u_{nk} - |\nabla u|^{p-2}\nabla u)(\nabla u_{nk} - \nabla u)dx \to 0. \quad (3.15) \]

For this aim, we rewrite the equation (3.4) as
\[ \Phi'(u_{nk}) = \Psi'(u_{nk}) \beta_{nk} + o(1) \quad \text{with } \beta_{nk} = \langle \Psi'(u_{nk}), e(u_{nk}) \rangle. \quad (3.16) \]

Multiplying (3.16) with \( u_{nk} \) and integrating on \( \Omega \)(noting that \( \{u_{nk}\} \) is bounded in \( W^{1,p}_0(\Omega) \cap L^q(\Omega) \)), we conclude that
\[ \int_{\Omega} |\nabla u_{nk}|^p dx + \int_{\Omega} |u_{nk}|^q dx = \int_{\Omega} \frac{\Psi'(u_{nk})}{\beta_{nk}} u_{nk} dx + o(1). \quad (3.17) \]
Recall that \( 1 \leq \ell < q - 1 \) and \( q > p^* \). Owing to the Strauss Lemma and (2.2), the right hand side of (3.17) yields
\[ \lim_{k \to \infty} \int_{\Omega} \frac{\Psi'(u_{nk})}{\beta_{nk}} u_{nk} dx = \int_{\Omega} \frac{\Psi'(u)}{\beta_0} u dx. \quad (3.18) \]

From (3.17) and (3.18), we obtain
\[ \lim_{k \to \infty} \left\{ \int_{\Omega} |\nabla u_{nk}|^p dx + \int_{\Omega} |u_{nk}|^q dx \right\} = \int_{\Omega} \frac{\Psi'(u)}{\beta_0} u dx + o(1). \quad (3.19) \]

Similarly, multiplying (3.16) by \( u \) and integrating over \( \Omega \), as \( k \to \infty \), we get that
\[ \int_{\Omega} |\nabla u_{nk}|^{p-2}\nabla u_{nk} \cdot \nabla u_{nk} dx + \int_{\Omega} |u_{nk}|^{q-2} u_{nk} u dx = \int_{\Omega} \frac{\Psi'(u_{nk})}{\beta_{nk}} u_{nk} dx + o(1). \quad (3.20) \]
Clearly, the right hand side of (3.20) leads to
\[ \lim_{k \to \infty} \int_{\Omega} \frac{\Psi'(u_{nk})}{\beta_{nk}} u_{nk} dx + o(1) = \int_{\Omega} \frac{\Psi'(u)}{\beta_0} u dx. \quad (3.21) \]
Using the Strauss Lemma together with (2.4), (3.20) and (3.21), we deduce that
\[ \lim_{k \to \infty} \int_{\Omega} |\nabla u_{nk}|^{p-2}\nabla u_{nk} \cdot \nabla u_{nk} dx + \int_{\Omega} |u_{nk}|^{q-2} u_{nk} u dx \\
= \lim_{k \to \infty} \int_{\Omega} |\nabla u_{nk}|^{p-2}\nabla u_{nk} \cdot \nabla u_{nk} dx + \int_{\Omega} |u|^{q} dx \\
= \int_{\Omega} \frac{\Psi'(u)}{\beta_0} u dx, \quad (3.22) \]
where the term \( \lim_{k \to \infty} \int_{\Omega} |\nabla u_{nk}|^{p-2}\nabla u_{nk} \cdot \nabla u_{nk} dx \) exists because \( \{\nabla u_{nk}\} \) is bounded in \( L^{\frac{p}{p-1}}(\Omega) \) and by choosing a subsequence, \( \{\nabla u_{nk}\} \) is weakly convergent in \( L^{\frac{p}{p-1}}(\Omega) \).
Due to $u_{n_k} \to u$ in $W^{1,p}_0(\Omega)$, taking the difference of (3.19) and (3.22), we derive that
\[
\lim_{k \to \infty} \int_{\Omega} \left( (|\nabla u_{n_k}|^{p-2} \nabla u_{n_k} - |\nabla u|^{p-2} \nabla u) (\nabla u_{n_k} - \nabla u) \right) dx
\]
\[
= \lim_{k \to \infty} \int_{\Omega} \left( (|\nabla u_{n_k}|^p - |\nabla u_{n_k}|^{p-2} \nabla u_{n_k} \nabla u) dx - \lim_{k \to \infty} \int_{\Omega} \left( (|\nabla u|^{p-2} \nabla u (\nabla u_{n_k} - \nabla u) \right) dx
\]
\[
= \lim_{k \to \infty} \int_{\Omega} \left( (|\nabla u_{n_k}|^p - |\nabla u_{n_k}|^{p-2} \nabla u_{n_k} \nabla u) dx
\]
\[
= \left( \int_{\Omega} \frac{\Psi'(u)}{\beta_0} u dx - \lim_{k \to \infty} \int_{\Omega} |u_{n_k}|^q dx \right) - \left( \int_{\Omega} \frac{\Psi'(u)}{\beta_0} u dx - \int_{\Omega} |u|^q dx \right)
\]
\[
= \int_{\Omega} |u|^q dx - \lim_{k \to \infty} \int_{\Omega} |u_{n_k}|^q dx
\leq 0. \tag{3.23}
\]
Here the final inequality in (3.23) follows from the Fatou’s lemma with the fact that $u_n \to u$ almost everywhere in $\Omega$.

Nevertheless, one has
\[
(|\nabla u_{n_k}|^{p-2} \nabla u_{n_k} - |\nabla u|^{p-2} \nabla u) (\nabla u_{n_k} - \nabla u) \geq 0. \tag{3.24}
\]

Hence, (3.23) and (3.24) yield that
\[
\lim_{k \to \infty} \int_{\Omega} \left( (|\nabla u_{n_k}|^{p-2} \nabla u_{n_k} - |\nabla u|^{p-2} \nabla u) (\nabla u_{n_k} - \nabla u) \right) dx = 0. \tag{3.25}
\]

Then standard arguments lead to
\[
\nabla u_{n_k}(x) \to \nabla u(x), \text{ a.e. } x \in \Omega. \tag{3.26}
\]

Recall that $\{|\nabla u_{n_k}|^{p-2} \nabla u_{n_k}\}$ is bounded in $L^{\frac{2}{p-1}}(\Omega)$. Choosing a subsequence with (3.26), we have
\[
|\nabla u_{n_k}|^{p-2} \nabla u_{n_k} \to |\nabla u|^{p-2} \nabla u \text{ in } L^{\frac{2}{p-1}}(\Omega).
\]

Combining with (2.4), one has $\Phi'(u_n) \to \Phi'(u)$ in $X^*$. The proof of Lemma 3.2 is completely finished. \qed

By applying the similar argument as Theorem 2.1 in [21], we have the convergence of sequences $\{e(u_n)\}$ defined in (3.16) as $u_n \to u$ in $X$. This plays an important role in our formula $\tilde{\Psi}'(u_n_k) = \Psi'(u_{n_k}) - (\Psi'(u_{n_k}), e(u_{n_k})) \Phi'(u_{n_k})$.

**Theorem 3.3.** Under the assumptions of Lemma 3.2, there exists a subsequence (also denoted by $\{e(u_{n_k})\}$) such that $\lim_{k \to \infty} e(u_{n_k}) = e(u)$ in $X$ and
\[
0 = \tilde{\Psi}'(u) = \Psi'(u) - \langle \Psi'(u), e(u) \rangle \Phi'(u). \tag{3.27}
\]

4. **Existence of positive and negative solutions.** Similarly as in [21], we set,
\[
\Omega(x,t) = \begin{cases} \frac{g(x,t)}{t}, & \text{if } t \geq 0, \\ 0, & \text{otherwise,} \end{cases} \tag{4.1}
\]
\[
\Omega(x,u) = \int_0^u \Omega(x,t) dt, \quad J(u) = -\int_{\Omega} \Omega(x,u(x)) dx, \tag{4.2}
\]
c = \inf_{u \in N_\alpha} J(u). \quad (4.3)

From the assumptions (\ast), (\ast\ast) on g, we know that \( J(u) < 0 \) if \( u \geq 0 \) and \( u \neq 0 \).

We claim that

**Claim 3:** If \( u \in N_\alpha \), then \(|u| \in N_\alpha\) and

\[ J(|u|) \leq J(u). \quad (4.4) \]

**Proof.** In fact, for any \( u \in N_\alpha \), then \( \Omega \) can be decomposed into \( \Omega = \Omega_1 + \Omega_2 + \Omega_3 \) with

\[
\Omega_1 = \{ x : u(x) > 0 \}, \quad \Omega_2 = \{ x : u(x) < 0 \}, \quad \Omega_3 = \{ x : u(x) = 0 \}
\]

and

\[
\frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx + \frac{1}{q} \int_{\Omega} |u|^q \, dx = \alpha. \quad (4.5)
\]

Based on the Theorem 6.17, p. 152, of [13], we know that

\[
(\nabla |u|)(x) = \begin{cases} 
\nabla u(x), & \text{if } u(x) > 0, \\
-\nabla u(x), & \text{if } u(x) < 0, \\
0, & \text{if } u(x) = 0.
\end{cases} \quad (4.6)
\]

One can easily deduce that \(|u| \in N_\alpha\) with \(|u| \geq 0\) and \(|u| \neq 0\). Thus (4.4) follows from the assumption (\ast) on \( g \) and (4.2).

Claim 3 asserts that the critical value \( c < 0 \) in (4.3) and meanwhile if there exists a critical point for the value \( c \), then the critical point is nonnegative. Due to Lemma 3.2, the Ekeland variational principal indicates that there exists a nonnegative solution \( u \in N_\alpha \) such that \( J(u) = c \) and

\[
\begin{cases}
-\text{div}(|\nabla u|^{p-2}\nabla u) + |u|^{q-2}u = \lambda \overline{g}(x, u), & \text{if } x \in \Omega, \\
u(x) \geq 0, & \text{if } x \in \Omega, \\
u(x) = 0, & \text{if } x \in \partial \Omega.
\end{cases} \quad (4.7)
\]

Here \( \frac{1}{\lambda} = (\overline{g}(x, u), c(u)) \) (see (3.27)).

Multiplying (4.7) by \( u \) and integrating in \( \Omega \), we have

\[
\int_{\Omega} |\nabla u|^p \, dx + \int_{\Omega} |u|^q \, dx = \lambda \int_{\Omega} \overline{g}(x, u)u \, dx. \quad (4.8)
\]

From the assumption (\ast) on \( g \) and the fact that \( u \geq 0 \) in \( \Omega \), we obtain \( \lambda > 0 \).

Using the regularity results for quasilinear elliptic equations (see Theorem 2.3, p. 352 of [18]), we know that \( u \) is a classical solution. In particular,

\[ u \in C^1_0(\Omega). \quad (4.9) \]

Furthermore, (4.7) shows

\[
\text{div}(|\nabla u|^{p-2}\nabla u) + (-|u|^{p-2})u = -\lambda \overline{g}(x, u).
\]

On account of \( \lambda > 0 \) and \( \overline{g}(x, u) > 0 \), then

\[
\text{div}(|\nabla u|^{p-2}\nabla u) + c(x)u \leq 0.
\]

where \( c(x) \triangleq (-|u(x)|^{p-2}) \leq 0 \) (by (4.9)).

Applying the strong maximum principle (see [16]) to (4.11), we have

\[ u(x) > 0, \forall x \in \Omega \quad \text{and} \quad \frac{\partial u(x)}{\partial \nu(x)} < 0, \forall x \in \partial \Omega, \quad (4.12) \]
where $\nu(x)$ is the outward pointing unit normal to $\partial \Omega$. On the basis of the definition of $\overline{g}$, we obtain a positive solution (denoted by $u_1$) to problem (I).

If we set

$$g(x,t) = \begin{cases} g(x,t), & \text{if } t \leq 0, \\ 0, & \text{if } t > 0. \end{cases}$$

(4.13)

By a similar argument, we obtain a negative solution (denoted by $u_2$) to problem (I), that is the conclusion of Theorem 1.4.

As an auxiliary result, the following holds:

**Proposition 1.** Both the positive and the negative solutions are local minima of $\hat{\Psi}$ in the $C^1_0(\Omega)$ topology.

For the readers’ convenience, we provide its proof below (see also [21]).

**Proof.** Since $\partial \Omega$ is compact, (4.12) implies that there exists an $\epsilon > 0$ such that

$$\frac{\partial u_1(x)}{\partial \nu(x)} < -\epsilon, \quad \forall x \in \partial \Omega.$$  

(4.14)

Note that

$$\frac{\partial v(x)}{\partial \nu(x)} \leq \frac{\partial u_1(x)}{\partial \nu(x)} + n \|v - u_1\|_{C^1_0(\Omega)}, \quad \forall v \in C^1_0(\Omega).$$  

(4.15)

Therefore, if $\|v - u_1\|_{C^1_0(\Omega)} < \frac{\epsilon}{n}$, then $\frac{\partial v}{\partial \nu} < 0$ on $\partial \Omega$. It implies that $v > 0$ in $\Omega$ whenever $\|v - u_1\|_{C^1_0(\Omega)}$ is small enough, so $\hat{\Psi}(v) = \tilde{\Psi}(v) \geq \hat{\Psi}(u_1) = \tilde{\Psi}(u_1)$. Thus, $u_1$ is a local minimum of $\hat{\Psi}$ (where $\hat{\Psi}$ is given by (1.3)). The same argument shows that $u_2$ is a local minimum of $\tilde{\Psi}$ as well. Then we are done. 

**Remark 2.** Our method also works for the subcritical or critical case.

REFERENCES

[1] H. Amann, *Lusternik-Schnirelman theory and nonlinear eigenvalue problems*, Math. Ann., 199 (1972), 55–72.

[2] J. Benedikt and P. Drábek, *Asymptotics for the principal eigenvalue of the p-Laplacian on the ball as p approaches 1*, Nonlinear Anal. TMA, 93 (2013), 23–29.

[3] J. Q. Chen, S. W. Chen and Y. Q. Li, *On a quasilinear elliptic eigenvalue problem with constraint*, Sci. China, Ser. A: Math., 47 (2004), 523–537.

[4] D. G. De Figueiredo and J. P. Gossez, *P. Ubilla, Local “superlinearity” and “sublinearity” for the p-Laplacian*, J. Funct. Anal., 257 (2009), 721–752.

[5] J. Fleckinger, E. M. Harrell II and F. de Thélin, *On the fundamental eigenvalue ratio of the p-Laplacian*, Bull. Sci. Math., 131 (2007), 613–619.

[6] B. L. Guo, Q. X. Li and Y. Q. Li, *Sign-changing solutions of a p-Laplacian elliptic problem with constraint in $\mathbb{R}^N$*, J. Math. Anal. Appl., 451 (2017), 604–622.

[7] S. C. Hu and N. S. Papageorgiou, *Multiple positive solutions for nonlinear eigenvalue problems with the p-Laplacian*, Nonlinear Anal. TMA, 69 (2008), 4286–4300.

[8] Y. Q. Li, *On a nonlinear elliptic eigenvalue problem*, J. Differ. Equ., 117 (1995), 151–164.

[9] Y. Q. Li, *Three solutions of a semilinear elliptic eigenvalue problem*, Acta Math. Sin., New Ser., 11 (1995), 142–152.

[10] Y. Q. Li and Z. L. Liu, *Multiple and sign-changing solutions of an elliptic eigenvalue problem with constraint*, Sci. China, Ser. A., 44 (2001), 48–57.

[11] A. Lé, *Eigenvalue problems for the p-Laplacian*, Nonlinear Anal. TMA, 64 (2006), 1057–1099.

[12] J. Q. Liu and X. Q. Liu, *On the eigenvalue problem for the p-Laplacian operator in $\mathbb{R}^N$*, J. Math. Anal. Appl., 379 (2011), 861–869.

[13] E. H. Lieb and M. Loss, *Analysis*, second edition, Americal Mathematical socirty, provedince Rhode Island, 2001.

[14] R. E. Megginson, *An introduction to Banach Space Theory*, Springer, 1998.
[15] A. Szulkin, Ljusternik-Schnirelman Theory on $C^1$-manifolds, *Ann. Inst. Henri Poincaré*, 5 (1988), 119–139.

[16] S. Sakaguchi, Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet Problems, *Ann. Scuola Normale Sup. di Pisa Serie 4*, 14 (1987), 403–421.

[17] D. Valtorta, Sharp estimate on the first eigenvalue of the p-Laplacian, *Nonlinear Anal.*, 75 (2012), 4974–4994.

[18] M. Xu and X. P. Yang, Remark on solvability of p-laplacian equations in large dimension, *Israel J. Math.*, 172 (2009), 349–356.

[19] E. Zeidler, Ljusternik-Schnirelman theory on general level sets, *Math. Nachr.*, 129 (1986), 235–259.

[20] E. Zeidler, *Nonlinear Functional Analysis and Its Applications III*, New-York: Springer-Verlag, 1985.

[21] Y. S. Zhong and Y. Q. Li, A new form for the differential of the constraint functional in strictly convex reflexive Banach spaces, *J. Math. Anal. Appl.*, 455 (2017), 1783–1800.

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