Linear Differential Equations on Some Classes of Weighted Function Spaces

Ahmed El-Sayed Ahmed 1,* and Amnah E. Shammaky 2

1 Mathematics Department, Faculty of Science, Taif University, Taif 21944, Saudi Arabia
2 Department of Mathematics, Faculty of Science, Jazan University, Jazan 45142, Saudi Arabia; aeshamaki@jazanu.edu.sa
* Correspondence: a.elsayed@tu.edu.sa

Abstract: Some weighted-type classes of holomorphic function spaces were introduced in the current study. Moreover, as an application of the new defined classes, the specific growth of certain entire-solutions of a linear-type differential equation by the use of concerned coefficients of certain analytic-type functions, that is the equation \( h^{(k)} + K_{k-1}(v)h^{(k-1)} + \ldots + K_1(v)h' + K_0(v)h = 0 \), will be discussed in this current research, whereas the considered coefficients \( K_0(v), \ldots, K_{k-1}(v) \) are holomorphic in the disc \( \Gamma_R = \{ v \in \mathbb{C} : |v| < R \}, \ 0 < R \leq \infty \). In addition, some non-trivial specific examples are illustrated to clear the roles of the obtained results with some sharpness sense. Hence, the obtained results are strengthen to some previous interesting results from the literature.

Keywords: complex linear differential equation; analytic functions; growth order

PACS: 30B10; 30B50; 46E15

1. Introduction

The known theory of Function spaces is one of the most interesting and active research areas with various crucial applications in many branches of mathematics. This important theory has a major role in both mathematical analysis (e.g., approximation theory, theory differential and integral equations, measure theory, operator theory) and engineering sciences (e.g., recent computer-aided geometric design, the known image processing). This joyful theory has numerous generalizations including, among the others, \( \mathbb{C}^n \) extensions as well as Clifford analysis generalizations. Recently, the theory has been intensively researched in differential equations. Some radial growth of certain types linear differential equations have been introduced and studied with the help of some certain classes of analytic function spaces.

Suppose that \( \Gamma = \Gamma_v = \{ v : |v| < 1 \} \) stands for the open disc in \( \mathbb{C} \). Let \( H(\Gamma_v) \) denote the class of all holomorphic functions on \( \Gamma_v \). For \( a \in \Gamma_v \), the known Möbius transformations \( \varphi_a(v) \) is given by

\[
\varphi_a(v) = \frac{a - v}{1 - \bar{a}v}, \text{ for } v \in \Gamma_v.
\]

For a point \( a \in \Gamma_v \) and \( r \in (0,1) \), the specific pseudo-hyperbolic concerned disk \( \Gamma(a,r) \) with specific pseudo-hyperbolic concerned center \( a \) and specific pseudo-hyperbolic radius \( r \) is defined by \( \Gamma(a,r) = \varphi_a(\Omega_r) \).

This type of pseudo-concerned disk \( \Gamma(a,r) \) is a considered Euclidean-type disk, for which its Euclidean concerned supposed center, as well as its concerned Euclidean-type radius are \( \frac{1 - r^2}{1 - |a|^2} \) and \( \frac{(1 - |a|^2)r}{1 - r^2} \), respectively (see [1]). The considered normalized Lebesgue-type area measure on \( \Gamma_v \) will be symbolized here by \( dm(v) \). The following known identity can be stated as follows (see [1]):
\[ 1 - |\varphi_a(v)|^2 = \frac{(1 - |a|^2)(1 - |v|^2)}{|1 - \overline{a}v|^2} = (1 - |v|^2)|\varphi_a'(v)|. \]

For \( a \in \Gamma_v \), by the assumption \( v = \varphi_a(w) \), we easily have \( dm(w) = |\varphi_a'(v)|^2 dm(v) \). In addition, we have \( \varphi_a(\varphi_a(v)) = v \), then \( \varphi_a^{-1}(v) = \varphi_a(v) \). For \( a, v \in \Gamma_v \) and \( r \in (0, 1) \), the specific supposed pseudo-disc \( \Gamma(a, r) \) is given by \( \Gamma(a, r) = \{ v \in \Gamma_v : |\varphi_a(v)| < r \} \). Further, the known analytic Green’s function on the disc \( \Gamma_v \) with a specific singularity at the point \( a \), is defined by
\[ g(v, a) = \log \frac{|1 - \overline{a}v|}{|v - a|} = \log \frac{1}{|\varphi_a(v)|}. \]

For two specific quantities \( K_h \) and \( C_{h,t} \), these specific quantities are depending on the holomorphic-type function \( h \) on \( \Gamma_v \), we say that these quantities are equivalent and write \( K_h \approx C_{h,t} \), when we can obtain a constant \( N^* \) that is finite, positive and not depend on the holomorphic function \( h \), for which
\[ \frac{1}{N^*} C_{h,t} \leq K_h \leq N^* C_{h,t}. \]

In the case of equivalence between \( K_h \) and \( C_{h,t} \), thus we obtain that
\[ K_h < \infty \iff C_{h,t} < \infty. \]

It should be noted:

The symbol \( F_1 \lesssim F_2 \) means that, we can find a specific constant \( \gamma > 0 \), for which \( F_1 \leq \gamma F_2 \), whereas \( F_1 \) and \( F_2 \) are two holomorphic functions on \( \Gamma_v \).

The authors in [2] have introduced an interesting active class of holomorphic functions, which is denoted by holomorphic \( Q_p \)-spaces as follows:
\[ Q_p = \{ h : h \text{ holomorphic in } \Gamma_v \text{ and } \sup_{a \in \Gamma_v} \int_{\Gamma_v} |h'(v)|^2 g^n(v, a) dm(v) < \infty \}, \]
where \( dm(v) \) is the known considered Euclidean area element on \( \Gamma_v \) and \( p \in (0, \infty) \). Here the considered weighted-type function \( g(v, a) = \log \frac{|1 - \overline{a}v|}{|v - a|} \) is introduced and studied as the well defined composition of the known Möbius-type transformation \( \varphi_a(v) \) with the essential concerned solution of the two dimensional-type Laplacian. Also, the considered weighted-type holomorphic function \( g(v, a) \) is an actually the holomorphic Green’s-type function in \( \Gamma_v \), that has a specific pole at the point \( a \in \Gamma_v \).

From now on, we will let the function \( \varphi : \Gamma_v \to \mathbb{R}^+ \), to be bounded and continuous function. With the help of this function, we introduce the following new holomorphic function classes:

**Definition 1.** Let \( h \in H(\Gamma_v) \), for \( \alpha \in (0, \infty) \), we say that \( h \) is in the exponential \( \varphi \)-weighted Bloch space \( B^{\varphi \alpha}_{\alpha} \), when
\[ \|h\|_{B^{\varphi \alpha}_{\alpha}} = \sup_{v \in B} (1 - |v|^2)^\alpha e^{\varphi(1-|v|^2)} |h^{(n)}(v)| < \infty; \quad n \in \mathbb{N}. \]

**Definition 2.** Let \( h \in H(\Gamma_v) \), for \( \alpha \in (0, \infty) \), we say that \( h \) is in the little exponential \( \varphi \)-Bloch space \( B^{\varphi \alpha}_{\alpha} \), when
\[ \|h\|_{B^{\varphi \alpha}_{\alpha}} = \lim_{|v| \to 1} (1 - |v|^2)^\alpha e^{\varphi(1-|v|^2)} |h^{(n)}(v)| = 0; \quad n \in \mathbb{N}. \]
Remark 1. It should be noted here that the exponential $\omega$-weighted Bloch space $\mathcal{B}^\omega_{\alpha,p}$, is more general than the holomorphic Bloch space from algebraic point of view but for geometric meaning it needs more study on connections of Bergman metric and the exponential $\omega$-weighted Bloch space $\mathcal{B}^\omega_{\alpha,p}$. When $\omega(v) \equiv 1$ and $n = 1$, thus the definition of known $\omega$-Bloch is obtained (see [3]).

The following general analytic function spaces can be introduced as follows:

Definition 3. Suppose that $0 < p < \infty$. The holomorphic function $h \in H(\Gamma_v)$ is said to belong to the space $P_{\omega}^{\alpha,n}(\varphi)$, when
\[
\|h\|^2_{\omega,\alpha,n} = \sup_{a \in \Gamma_v} \int_{\Gamma_v} |h^{(n)}(v)|^2 (1 - |v|)^{p-2} e^{2\omega(1-|v|^2)} (1 - |\varphi_a(v)|)^p \, dm(v) < \infty; \quad n \in \mathbb{N}
\]

Definition 4. Suppose that $0 < p < \infty$. The holomorphic function $h \in H(\Gamma_v)$ is said to belong to the space $P_{\omega}^{2\alpha,n}(g)$, when
\[
\|h\|^2_{\omega,2\alpha,n} = \sup_{a \in \Gamma_v} \int_{\Gamma_v} |h^{(n)}(v)|^2 (1 - |v|)^{p-2} e^{2\omega(1-|v|^2)} g(v,a)^p \, dm(v) < \infty; \quad n \in \mathbb{N}
\]

Definition 5. Suppose that $0 < p < \infty$. The holomorphic function $h \in H(\Gamma_v)$ is said to belong to the exponential Hardy space $H^p(e^{\omega})$.
\[
\|h\|^p_{\omega} = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |h(re^i\theta)|^p e^{p\omega(1-|re^i\theta|^2)} \, d\theta < \infty,
\]
when $\|h\|_{\infty} = \sup_{v \in \Gamma_v} |h(v)| e^{\omega(1-|v|^2)} < \infty$, thus the function $h$ belongs to the exponential Hardy space $H^\omega(e^{\omega})$.

Furthermore, $h \in H^2(e^{\omega})$ if and only if
\[
\int_{\Gamma_v} |h'(v)|^2 (1 - |v|^2)^2 e^{2\omega(1-|v|^2)} \, dm(v) < \infty.
\]

Definition 6 ([1,4]). Let $h \in H(\Gamma_v)$ be holomorphic-type function in $\Gamma_v$ and let $p \in (1, \infty)$. Then,
\[
\|h\|^p_2 = \sup_{v \in \Gamma_v} \int_{\Gamma_v} |h'(v)|^p (1 - |v|^2)^{p-2} \, dm(v) < \infty,
\]
thus $h$ belongs to the holomorphic-type Besov space $B^p_v$.

Because Möbius-type mapping $\varphi$ can be expressed as $\varphi(v) = e^{i\theta} \varphi_a(v)$, whereas $\theta$ is real.

For a concerned sub-arc $\ell \subset \partial \Gamma_v$, let
\[
S(\ell) = \{ r\zeta \in \Gamma_v : 1 - |r\zeta| < 1, \zeta \in \ell \}.
\]

When $|\ell| \geq 1$ then we set $S(\ell) = \Gamma_v$. Assume that $p \in (0, \infty)$, thus the specific positive measure $dF$ is to be a concerned $p$-Carleson-type measure on $\Gamma_v$ when
\[
\sup_{\ell \subset \partial \Gamma_v} \frac{F(S(\ell))}{|\ell|^p} < \infty.
\]

Here and henceforth $\sup_{\ell \subset \partial \Gamma_v}$ indicates the supremum taken over all specific sub-arcs $\ell$ of $\partial \Gamma_v$. 
Note that $p = 1$ results the classical known Carleson-type measure (cf. [5]).

2. Linear-type Complex Differential Equations

The specific growth of the solutions of the differential-type complex equations of the form:

$$h^{(k)} + K_{k-1}(v)h^{(k-1)} + \ldots + K_1(v)h' + K_0(v)h = 0 \quad (1)$$

will be studied in this current research. Meanwhile, the considered coefficients $K_0(v), \ldots, K_{k-1}(v)$ are holomorphic in the disc $\Gamma_R = \{ v \in \mathbb{C} : |v| < R \}$, $0 < R \leq \infty$. For further details on the numerous studies of the theory of complex linear differential-type equations on some various classes of complex function spaces, the following citations can be used [6–16].

Pommerenke [16] studied the second-type order differential complex equation, that given by:

$$h'' + B(v)h = 0, \quad (2)$$

whereas $B(v)$ is a holomorphic function in $\Gamma_v$.

Heittokangas [10] improved the research due to Pommerenke by using the equation of the form:

$$h^{(k)} + B(v)h = 0, \quad (3)$$

whereas $B(v)$ is a holomorphic-type function in $\Gamma_v$ and $k \in \mathbb{N}$.

This study followed by an interesting research in [12], which considered a joyful research with the help of the complex linear differential equation of the form (1) with all obtained solutions are belonging to the classes $H^\infty(\omega)$-space or the Dirichlet $D^p(\omega_0)$-space.

Recall now some equivalent statements for holomorphic-type functions to be in the holomorphic exponential $\omega$-Bloch spaces $B^a(\omega, n)$ and the little holomorphic exponential little $\omega$-Bloch spaces $B^a_{\omega}(\omega, n, 0)$.

**Proposition 1** ([3]). Suppose that $a > -1$ and $v \in \Gamma_v$, then

$$h(v) = (a + 1) \int_{\Gamma} \frac{(1 - |\zeta|^2) h(\zeta)}{(1 - v|\zeta|^2)^{a+1}} \, dm(\zeta)$$

if $h$ is an analytic function on $\Gamma$ with

$$\int_{\Gamma} (1 - |v|^2)^a |h(v)| \, dm(v) < +\infty.$$

**Lemma 1** (see [17]). Let $F$ be a positive measure on $\Gamma_v$, and assume that $t \in (0, 1)$, $0 < p < \infty$. Hence,

$$\sup_{|\ell| \leq 1-t} \frac{F(S(\ell))}{|\ell|^p} \leq C \sup_{|r| \geq 1} \int_{\Gamma_v} |q^p_r(v)|^p \, dF(v).$$

**Lemma 2** ([18]). Let $h$ be an analytic function in $\Gamma_v$, and let $0 < p < \infty$, $0 < q < \infty$, $0 < r < 1$ and $a \in \Gamma_v$. Then there exists constant $C > 0$, such that

$$|h(a)|^p (1 - |a|^2)^q \leq C \int_{\Gamma(a, r)} |h(v)|^p (1 - |v|^2)^{q-2} \, dm(v).$$
Lemma 3 ([19]). Let $f$ be an analytic function in $\Gamma_v$ and let $0 < p < \infty$, $-1 < q < \infty$. Then there exist two positive constant $C_1$ and $C_2$, depending only on $p$ and $q$, such that

$$C_1 \left( |h(0)|^p + \int_{\Gamma_v} |h'(v)|(1 - |v|^2)^p |q_{a}(v)|^q dv \right) \leq \int_{\Gamma_v} |h(v)|(1 - |v|^2)^p |q_{a}(v)|^q dv \leq C_2 \left( |h(0)|^p + \int_{\Gamma_v} |h'(v)|(1 - |v|^2)^p |q_{a}(v)|^q dv \right).$$

Theorem 1. For a bounded continuous function $\omega : \Gamma_v \to (0, \infty)$, let $h$ be an analytic function in $\Gamma_v$. Let $n \in \mathbb{N}$ and $p > 0$. Let $dF(v) = |h^{(n)}(v)|^2 (1 - |v|^2)^{2n-2} e^{2\omega(1-|v|^2)} dm(v)$ Then the following quantities are comparable:

(i) $\|h\|_{P_{\omega, p, r}(g)}^2 + |h(0)|^2$,

(ii) $\sup_{a \in \Gamma_v} \int_{\Gamma_v} |h^{(n)}(v)|^2 (1 - |v|^2)^p e^{2\omega(1-|v|^2)} (1 - |q_{a}(v)|^2)^q dm(v) \leq \sum_{j=1}^{k-1} |h^{(j)}(0)|^2$,

(iii) $\sup_{a \in \Gamma_v} \int_{\Gamma_v} |h^{(n)}(v)|^2 (1 - |v|^2)^p e^{2\omega(1-|v|^2)} g^p(v, a) dm(v) \leq \sum_{j=1}^{k-1} |h^{(j)}(0)|^2$,

(iv) $\sup_{\ell} \frac{F(S(\ell))}{|\ell|^p} \leq \sum_{j=1}^{k-1} |h^{(j)}(0)|^2$.

Proof. (i) $\Rightarrow$ (ii). We can prove this assertion by the known induction method. Now, consider the following:

When $h \in P_{\omega, p, r}(g)$, thus as in [20], $h \in B_{\omega}$. In view of the inequality $1 - |q_{a}(v)|^2 \leq 2g(v, a)$, and Proposition 3, we deduce that (i) $\Rightarrow$ (ii). For $n = 1$. Suppose that (i) $\Rightarrow$ (ii) for some specific concerned fixed natural number $n > 1$. Using Lemma 3 to the specific function $(g \circ q_{a})(rv)$, $0 < r < 1$, we get

$$\frac{1}{\Gamma_{(a, r)}} \int_{\Gamma_{(a, r)}} |h'(v)|^2 (1 - |v|^2)^p dv \leq C \frac{1}{\Gamma_{(a, r)}} \int_{\Gamma_{(a, r)}} |h^{(n)}(v)|^2 (1 - |v|^2)^p dm(v),$$

whereas the positive specific constant $C$ depends only on the point $p$ as well as on $r$, the holomorphic function $h(v) = h^{(n)}(v)$, whereas $a \in \Gamma_v$, thus

$$\frac{1}{\Gamma_{(a, r)}} \int_{\Gamma_{(a, r)}} |h^{(n)}(v)|^2 dm(v) \leq C \frac{1}{\Gamma_{(a, r)}} \int_{\Gamma_{(a, r)}} |h^{(n+1)}(v)|^2 (1 - |v|^2)^2 dm(v) \leq \frac{C}{(1 - |v|^2)^{2n-4}} \frac{1}{\Gamma_{(a, r)}} \int_{\Gamma_{(a, r)}} |h^{(n+1)}(v)|^2 (1 - |v|^2)^{2n-2} dm(v) \leq \frac{C}{(1 - |v|^2)^{2n-4}} \frac{1}{\Gamma_{(a, r)}} \int_{\Gamma_{(a, r)}} |h^{(n)}(v)|^2 (1 - |v|^2)^{2n-2} dm(v) \leq \frac{1}{(1 - |v|^2)^{2n-4}} \frac{1}{\Gamma_{(a, r)}} \int_{\Gamma_{(a, r)}} |h^{(n)}(v)|^2 (1 - |v|^2)^{2n-2} dm(v).$$

Hence, (ii) is obtained with the specific index $n + 1$, thus the implication (i) $\Rightarrow$ (ii) has been deduced. For the implication (ii) $\Rightarrow$ (iii). Suppose that
whereas $\Gamma(a, \frac{1}{2}) = \{ v \in \Gamma_v : |\varphi_a(v)| < \frac{1}{4} \}$. Because $g(v, a) \geq \log 4 > 1$, for $v \in \Gamma(a, \frac{1}{2})$, and

$$g(v, a) = \log \frac{1}{|\varphi_a(v)|} \leq 4(1 - |\varphi_a(v)|^2),$$

for $v \in \Gamma_v \setminus \Gamma(a, \frac{1}{2})$, we deduce that

$$I_1(v) \leq \int_{\Gamma(a, \frac{1}{2})} |h^n(v)|^2 (1 - |v|^2)^{2n-2} e^{\alpha(1-|v|^2)} g^p(v, a) \, dm(v),$$

and

$$I_2(v) \leq \int_{\Gamma_v \setminus \Gamma(a, \frac{1}{2})} |h^n(v)|^2 (1 - |v|^2)^{2n-2} e^{\alpha(1-|v|^2)} g^p(v, a) \, dm(v).$$

For $h \in \mathcal{B}_{\alpha}$, using (ii) and considering the equivalence of (i) and (ii), thus for all $a \in \Gamma_v$, we deduce that

$$I_1(v) \leq \left( M^p_{h,1} \right)^2 \int_{\Gamma(a, \frac{1}{2})} (1 - |v|^2)^{-2} e^{\alpha(1-|v|^2)} g^p(v, a) \, dm(v).$$

Therefore, $\sup_{a \in \Gamma_v} I_1(a) < \infty$. Thus, by (ii), we obtain that $\sup_{a \in \Gamma_v} I_2(a) < \infty$, and so $\sup_{a \in \Gamma_v} I(a) < \infty$, hence (iii) is obtained. Next, we will show that $(4) \Rightarrow (1)$, since

$$F(S(\ell)) < A |\ell|^p.$$

Then,

$$\frac{F(S(\ell))}{|\ell|^p} \leq \int_{S(\ell)} \left( \frac{1 - |a|^2}{1 - a\bar{v}} \right)^2 \, dF(v)$$

$$\leq \int_{\Gamma_v} \left( \frac{1 - |a|^2}{1 - a\bar{v}} \right)^2 \, dF(v) = \sup_{a \in \Gamma_v} \int_{\Gamma_v} |\varphi_a'(v)|^2 \, dF(v)$$

$$\leq C \sup_{a \in \Gamma_v} \int_{\Gamma_v} |\varphi_a'(v)|^2 g^p(v, a) e^{\alpha(1-|v|^2)} \, dm(v)$$

$$\leq C \|\varphi_a\|_{p,w,1(\ell)}^2.$$

This can be verified for $f(v) = \varphi_a(v) = \frac{a \bar{v}}{1 - a\bar{v}}$. Thus

$$\frac{F(S(\ell))}{|\ell|^p} \leq C \|h\|_{p,w,1(\ell)}^2.$$

The the proof is therefore obviously established. \( \square \)
Lemma 4. For a bounded continuous nondecreasing function \( \omega : \Gamma \to (0, \infty) \), let \( h \in H(\Gamma) \) and assume that \( 0 < p \leq 2 \). Thus, we can find a specific positive constant \( C > 0 \), depending on \( p \), for which

\[
\|h\|_{H^p(\omega)} \leq C\left(\|h\|_{P^p_{\omega}} + |h(0)|\right).
\]

When \( p \geq 2 \), thus

\[
\|h\|_{P^p_{\omega}} \leq C\|h\|_{H^p(\omega)}.
\]

Proof. The proof can be obtained simply as given in Lemma 4.1 (see [21]), thus it will be removed. \( \square \)

Theorem 2. For \( p \in (0, \infty) \). There are equivalence between the next specific conditions:

(i) \( F \) is a \( (P_{\omega;0,1}(g)) \) – Carleson measure,

(ii) there is a constant \( A \) such that \( F(S(\ell, \theta)) \leq A\ell^p, \forall \ell \in (0,1), \) and all \( \theta \in [0, 2\pi) \),

(iii) there is a constant \( C \) such that

\[
\sup_{a \in \Gamma} \int \left| \phi'(v) \right|^p dF(v) \leq C \quad \text{for all} \quad a \in \Gamma.
\]

Proof. First, assume that (i) holds. Thus, we obtain

\[
\frac{1}{|a|^2} \int_{\Gamma_v} \left| h'(v) \right|^2 dF(v) \leq C \int_{\Gamma_v} \left| h'(v) \right|^2 e^{2\omega(1-|v|^2)} g^p(v,a) dm(v),
\]

for all \( h \in P_{\omega;0,1}(g) \). In a special case this can be obtained for \( h(z) = \phi_a(v) = \frac{e^{|v|} - 1}{1 - e^{|v|}} \). Therefore,

\[
\sup_{a \in \Gamma_v} \int \left| \phi'_a(v) \right|^2 dF(v) \leq C \sup_{a \in \Gamma_v} \int \left| \phi'_a(v) \right|^2 e^{2\omega(1-|v|^2)} g^p(v,a) dm(v)
\]

\[
\leq C \| \phi'_a \|^2_{P^p_{\omega;0,1}} \leq C \lambda, \quad \forall \ a \in \Gamma
\]

Thus (iii) follows.

Next assume that the assertion (iii) holds, we want to prove that (ii) is correct. Then we have the following,

\[
C \geq \int_{\Gamma_v} \left( \frac{1 - |v|^2}{1 - |a|^2} \right)^2 dF(v) \geq \int_{S(\ell)} \left( \frac{1 - |v|^2}{1 - |a|^2} \right)^p dF(v),
\]

\[
\geq \frac{\mu(S(\ell))}{|\ell|^p} \geq \frac{\lambda}{|\ell|^p} \mu(S(\ell)),
\]

which implies that

\[
\mu(S(\ell)) < A|\ell|^p
\]

Hence, (ii) is obtained.

Assume that (ii) holds, we will clear that (i) holds. For \( v = \rho e^{i\theta} \), we suppose that

\[
\Omega_1(v) = \left\{ \zeta : |\zeta - v| < \frac{1 - |v|}{2} \right\},
\]

\[
\Omega_2(v) = \left\{ \zeta : |\zeta - v| < 1 - |v| \right\}.
\]

Hence,

\[
\Omega_1(v) \subseteq \Omega_2(v) \subseteq S(2(1 - |v|), \theta).
\]
Moreover, when $\zeta \in \Omega_1(v)$, we deduce that
\[
\frac{1}{2} \leq \frac{1 - |\zeta|}{1 - |v|} \leq \frac{3}{2}.
\]

Let $h \in P_{\alpha, \rho \pi}$; since $h$ is holomorphic, we get that
\[
h'(z) = \frac{4}{\pi(1 - |v|)^2} \int_{\Omega_1(v)} h'(\zeta) dm(\zeta).
\]

Thus, using the known inequality of Jensen, we obtain that
\[
|h'(v)|^2 \leq \frac{4}{\pi(1 - |v|)^2} \int_{\Omega_1(v)} |h'(\zeta)|^2 dm(\zeta).
\]

Then,
\[
\int_{\Gamma_v} |h'(v)|^2 dF(v) \leq \int_{\Gamma_v} \frac{4}{\pi(1 - |v|)^2} \left( \int_{\Omega_1(v)} |h'(\zeta)|^2 dm(\zeta) \right) dF(v)
\]
\[
\leq \frac{4}{\pi} \int_{\Gamma_v} \left( \int_{\Omega_1(v)} |h'(\zeta)|^2 \left( \frac{3}{2(1 - |\zeta|)^2} dm(\zeta) \right) \right) dF(v)
\]
\[
\leq \frac{9}{\pi} \int_{\Gamma_v} \left( \int_{\Omega_1(v)} |h'(\zeta)|^2 \chi_{\Omega_1(v)}(\zeta)(1 - |\zeta|)^{-2} dm(\zeta) \right) dF(v)
\]
\[
\leq \frac{9}{\pi} \int_{\Gamma_v} \left( \int_{\Omega_1(v)} |h'(\zeta)|^2 (1 - |\zeta|)^{-2} \chi_{\Omega_1(v)}(\zeta) dm(\zeta) \right) dF(v).
\]

Although,
\[
\chi_{\Omega_1(v)}(\zeta) \leq \chi_{S(2(1 - |\zeta|), \theta)}(\zeta),
\]
whereas $v = |v|e^{i\theta}$ and $\zeta \in \Omega_1(v)$, we infer that
\[
|\zeta - e^{i\theta}| < 2(1 - |\zeta|).
\]

In view of (ii), the next specific inequality can be obtained
\[
\int_{\Gamma_v} \chi_{\Omega_1(v)} dF(v) \leq \mu(S(2(1 - |\zeta|), \theta)) \leq A^2(1 - |\zeta|)^p.
\]

Thus,
\[
\int_{\Gamma_v} |h'(v)|^2 dF(v) \leq \frac{9}{\pi} A^2 \int_{\Gamma_v} \left( \frac{|h'(\zeta)|^2 (1 - |\zeta|)^{-p - 2} e^{2\alpha(1 - |\zeta|^2)} g^p(\zeta, a) dm(\zeta)}{(1 - |\zeta|)^p} \right)
\]
\[
\leq c \int_{\Gamma_v} \left( \frac{|h'(\zeta)|^2 (1 - |\zeta|)^{-p - 2} e^{2\alpha(1 - |\zeta|^2)} g^p(\zeta, a) dm(\zeta)}{(1 - |\zeta|)^p} \right),
\]

whereas $c$ is a specific positive constant. Hence using Theorem 2.1, we obtain
\[
\int_{\Gamma_v} |h'(v)|^2 dF(v) \leq c \|h\|_{P_{\alpha, \rho \pi}(g)}^2,
\]
which gives (i). This completes the proof of Theorem 2. \(\Box\)
A collocation of interesting approach for solving a class of certain complex linear-type differential equations in the unit disc will be studied in the next section. By using a concerned collocation of holomorphic functions defined in some general weighted spaces of the analytic-type, some joyful solutions of a complex linear differential equations can be investigated. Such solutions transforms the considered linear complex differential equations into certain weighted holomorphic function spaces. For some certain classes of holomorphic function spaces, some concerned coefficients have important roles in the obtained results with the help of a specific system of complex linear equations. The used method results the analytic solution if the exact solutions are in certain weighted holomorphic functions. An interesting example is also illustrated to investigate the validity and applicability of the given technique and the comparisons which made with some obtained results. The given examples have reflexed and demonstrated the importance of the current research.

3. Entire Solutions

Some few decades ago, many interesting techniques have been developed and evolved for the solutions of the concerned complex-type differential Equations (1). While quite a good number portion of the solutions is useful for certain research purposes, there are special some which are so important by the complex function spaces solutions. In this current manuscript, very general classes of complex function spaces for solving complex-type differential equations are considered and deeply discussed. After that, the research results are established with two basic methods commonly proved by the help of holomorphic norms with the defined spaces and by introducing certain entire solutions for the complex-type linear differential equations. The obtained results are pursued by the corresponding results in literature.

Proposition 2. Suppose that $\alpha > 1$ and $v \in \Gamma_v$, then

$$h \in B^\alpha(e^\omega, 1) \iff (1 - |v|^2)^{a-1} e^{\alpha(1 - |v|^2)} h(v) < +\infty.$$  

Moreover,

$$h \in B^\alpha(e^\omega, 1, 0) \iff (1 - |v|^2)^{a-1} e^{\alpha(1 - |v|^2)} h(v) \rightarrow 0 \text{ when } |v| \rightarrow 0.$$  

Proof. From [3], we have that

$$h(v) = h(0) + \int_\Gamma \frac{(1 - |\zeta|^2)^a h(\zeta)}{(1 - v\bar{\zeta})^{1+a}} dm(\zeta).$$

Thus,

$$|h(v) - h(0)| \leq \|h\|_{B^\alpha(\ed, 1)} \int_\Gamma \frac{dm(\zeta)}{|\zeta|^a (1 - v\bar{\zeta})^{1+a}} \leq \frac{1}{e^{\alpha(1 - |v|^2)}} \|h\|_{B^\alpha(\ed, 1)} \int_\Gamma \frac{dm(\zeta)}{|\zeta|^a (1 - v\bar{\zeta})^{1+a}}.$$  

because [3]

$$\int_\Gamma \frac{dm(\zeta)}{|\zeta|^a (1 - v\bar{\zeta})^{1+a}} \sim (1 - |v|^2)^{-(a-1)} \text{ (see proposition 6 in ).}$$

Therefore,

$$(1 - |v|^2)^{a-1} e^{\alpha(1 - |v|^2)} h(v) < +\infty.$$  

For the converse, suppose that $(1 - |v|^2)^{a-1} e^{\alpha(1 - |v|^2)} h(v) < \eta$, where $\eta > 0$. 

Now, using Proposition 6 in [3], we can obtain that
\[
h'(v) = \alpha (\alpha + 1) \int \frac{\beta e^{\omega (1-|v|^2)} (1-\zeta)^{a-1} dm(\zeta)}{|\zeta| e^{\omega (1-|v|^2)} (1-v_2)^2 + \alpha},
\]
which implies that,
\[
|h'(v)| \leq \alpha (\alpha + 1) \eta \int \frac{dm(\zeta)}{|\zeta| e^{\omega (1-|v|^2)} (1-\zeta)^{2 + \alpha}} \leq \alpha (\alpha + 1) \eta \int \frac{dm(\zeta)}{|1-\zeta|^{2 + \alpha}} \leq \frac{\eta (1-|v|^2)^{-\alpha}}{e^{\omega (1-|v|^2)}},
\]
then \( h \in B^a(\omega, 1). \)

**Remark 2.** From Proposition 2, we can deduce that the norm of the exponential Bloch-type space is equivalent to the following norm
\[
\|h\|_{\omega, a, n} = \sup \{ (1-|v|^2)^{a-1} e^{\omega (1-|v|^2)} |h(v)| : \ v \in \Gamma \}.
\]

Using Proposition 3.1 and mathematical induction, the next result can be deduced easily.

**Proposition 3.** Let \( h \in H(\Gamma, v) \) and let either \( \alpha < 0, \infty \) for \( n \in \mathbb{N} \) or \( \alpha > 0, n = 0 \). Thus
\[
h \in B^a(\omega, n) \iff M_{H_n, \omega}(\omega) = \sup_{v \in \Gamma, n} (1-|v|^2)^{a+n-1} e^{\omega (1-|v|^2)} |h^{(n)}(v)| < \infty.
\]

Further,
\[
h \in B^a(\omega, n, 0) \iff \lim_{|v| \to 1} (1-|v|^2)^{a+n-1} e^{\omega (1-|v|^2)} |h^{(n)}(v)| = 0.
\]

For the \( n \)-th derivatives of holomorphic-type functions in \( H_q^b(\omega) \) and \( P_{p, \rho, \omega, n} \), the next interesting proposition can be deduced.

**Proposition 4.** Let \( \omega : \Gamma, v \to (0, \infty) \), be a bounded continuous function and let \( h \in H(\Gamma, v) \), \( 1 < \alpha < \infty \) and \( n \in \mathbb{N} \). Hence, the next concerned specific statements are comparable:
\[
(a_1) \quad L(\mathcal{B}^a(\omega, n)) = L(\mathcal{B}^a(\omega, n, 0)) = \|h\|_{H_n^{\omega}, \omega}.
\]
\[
(a_2) \quad \|h\|_{\rho_{\omega} + |h(0)|},
\]
\[
(a_3) \quad \sup_{v \in \mathbb{B}} |h^{(n)}(v)|(1-|v|^2)^{a+n-1} e^{\omega (1-|v|^2)} + \sum_{j=0}^{n-1} |h^{(j)}(0)|.
\]

**Proof.** Let \( Y \) be a specific Banach space of holomorphic functions on \( \Gamma \). Assume that \( L(Y) \) denotes the space of all known defined point-wise multipliers of \( Y \). Since we have the fact, \( L(Y) \subset H_q^\infty \), the proof of \((a_1)\) can be obtained directly from Propositions 2, 3 and this fact. For the proof of \((a_2)\), we can use the fact that \( L(Y) \subset Y \). Proof of \((a_3)\) is clearly from the definition of the exponential Bloch functions. \( \square \)

Now, we give the following interesting result, which gives some solutions of the complex differential Equation (1).

**Theorem 3.** Let \( \omega : \Gamma, v \to (0, \infty) \), be a bounded continuous nondecreasing function. For every \( q > 0 \), thus we can find a specific constant \( \beta = \beta(q, k) > 0 \), for which that when the specific coefficients \( K_j(v) \) of (1) hold
\[
\|K_j\|_{H_q^b(\omega)} = \sup_{|v| \geq \delta} |K_j(v)|(1-|v|^2)^{k-j} e^{\omega (1-|v|^2)} \leq \beta, \quad j = 0, \ldots, k-1,
\]
thus all specific solutions of (1) are belonging to the holomorphic space \( H_q^b(\omega) \).
Theorem 4. Let there therefore all entire-type solutions of (1) are belonging to the holomorphic space $H^\infty_q(e^\omega)$ depending only some initial values of the analytic function $h$ which that when the analytic coefficients $K_j(\rho)$ then all entire-type solutions of (1) are belonging to the holomorphic space $H^\infty_q(e^\omega)$.

Corollary 1. Let $0 < \delta < 1$. For every $q > 0$ there exists a constant $\beta = \beta(q, k) > 0$, for which the analytic coefficients $K_j(v)$ of (1) satisfy

$$\|K_j\|_{H^\infty_q(e^\omega)} = e \sup_{v \in D} |K_j(v)| (1 - |v|^2)^{(k-j)} \leq \beta, \quad j = 0, \ldots, k-1,$$

then all entire-type solutions of (1) are belonging to the holomorphic space $H^\infty_q(e^\omega)$.

Proof. The proof can be followed by setting $\omega \equiv 1$ in Theorem 3.

Theorem 4. Let $\delta \in (0, 1)$. Then, For every $q > 0$, we can find a constant $\beta = \beta(q, k) > 0$, for which that when the analytic coefficients $K_j(v)$ of (1) verify

$$\|K_j\|_{H^\infty_q(e^\omega)} = \sup_{|v| \geq \delta} |K_j(v)| (1 - |v|^2)^{(k-j)} e^{\omega(1 - |v|^2)} \leq \beta, \quad j = 0, \ldots, k-1,$$

(4)

therefore, all entire-type solutions of (1) are belonging to the holomorphic $H^\infty_q(e^\omega)$.

Proof. Suppose that $\beta \leq 1$ and $r \in (\delta, 1)$. As in [11], we can find a specific constant $\lambda_1 > 0$, depending only some initial values of the analytic function $h$, for which

$$|h(re^{i\theta})| \leq \lambda_1 \exp \left( k \int_0^r \sum_{j=0}^{k-1} |K_j(se^{i\theta})|^2 \, ds \right)$$

for all $0 < \theta < 2\pi$, then the specific assumption (4) gives
|h(re^{i\theta})| \leq \lambda_1 \exp \left( k \int_0^\delta \sum_{j=0}^{k-1} |K_j(se^{i\theta})| |\bar{s}^j ds + k \int_\delta^r \sum_{j=0}^{k-1} |K_j(se^{i\theta})| |\bar{s}^j ds \right) \\
= \lambda_2 \exp \left( k \int_\delta^r \sum_{j=0}^{k-1} |K_j(se^{i\theta})| |\bar{s}^j ds \right) \\
\leq \lambda_2 \exp \left( \beta^k k^2 \int_\delta^r ds \frac{1}{1-s} \right) \leq \frac{\lambda_2}{(1-r)^k k^2}.

Choosing \( \beta = \left( \frac{q}{\bar{q}} \right)^k \), we deduce that the proof is completely established. \( \square \)

Example 1. The following complex-valued functions (see citeHkr)

\[ h_n(v) = (1 - v)^\frac{1}{2}(-a_1 + 1 + (-1)^n \sqrt{(a_1 - 1)^2 + 4a_0}), \quad n = 1, 2 \]

are linearly independent entire-solutions of the next complex-type differential equation:

\[ h'' - \frac{a_1}{1-v} h' - \frac{a_0}{(1-v)^2} h = 0, \quad \text{with} \ \omega \equiv 1 \]

whereas \( a_0, a_1 \in \mathbb{R} \) such that \((a_1 - 1)^2 + a_0 > 0\). So, when \( a_1 > q + 1 - \frac{a_0}{q} \), then \( h_1 \not\in H_k^q(e) \), and thus the specific constant \( \beta \) in Theorems 3 and 4 gives

\[ \beta \leq 2 \min_{a_0 \in \mathbb{R}} \max \{|q + 1 - \frac{a_0}{q}, 2|a_0|\} = \frac{4q(q + 1)}{2q + 1} \]

When \( k = 2 \). This gives us that \( \beta \to 0 \) as \( q \to 0 \).

The concerned functions \( h_1(v), h_2(v) \) and \( h_3(v) = (1 - v)^2 \) are in fact linear independent specify entire-solutions of the following equation:

\[ h'' - \frac{a_1}{1-v} h' - \frac{a_0 + a_1}{(1-v)^2} h' - \frac{2a_0}{(1-v)^2} h = 0, \quad \text{with} \ \omega \equiv 1 \]

whereas \( a_0, a_1 \in \mathbb{R} \), then the specific constant \( \beta \) in Theorems 3 and 4 gives

\[ \beta \leq 2 \min_{a_0 \in \mathbb{R}} \max \{|q + 1 - \frac{a_0}{q}, 2|q + 1 + a_0(1 - \frac{1}{q}), 8|a_0|\} = C(q) \]

where \( C(q) = \frac{16q(q+1)}{3q+1}, \quad 0 < q \leq 1 \) and \( C(q) = \frac{16q(q+1)}{3q-1}, \quad 1 \leq q < \infty \) in the case \( k = 3 \). Thus, we deduce that \( \beta \to 0 \) as \( q \to 0 \).

Theorem 5. Let \( \omega : \Gamma_v \to (0, \infty) \), be a bounded continuous function and \( \delta \in (0, 1) \). For every \( p \in (0, \infty) \), we can find a positive constant \( \beta = \beta(p, k) > 0 \), for which that if the analytic coefficients \( K_j(v) \) of (1) verify

\[
\sup_{0 < |\ell| \leq \delta} \frac{1}{|\ell|} \int_{S(\ell)} |K_j^{(n)}(u)|^2 (1 - |v|^2)^{2(k+n-j)-1} \frac{1 - |a|^2}{|1 - \bar{a}v|^2} \mathbb{R}^p(v, a) e^{\omega(1 - |v|^2)} dm(v) \leq \beta, \quad j = 0, \ldots, k - 1,
\]

for all \( n = 0, \ldots, k - 1 \), then all solutions of (1) belong to \( D_\alpha^p \cap H_k^q(e^{\omega}) \).
\[
|K_j(a)|^2 (1 - |a|^2)^{2(k-j)}
\]

\[
\leq \frac{C_1}{1 - |a|} \int_{\Gamma(a,r)} |K_j(v)|^2 (1 - |v|^2)^{2(k-j)-1} dm(v)
\]

\[
= \frac{C_1}{1 - |a|} \int_{\Gamma(a,r)} |K_j(v)|^2 (1 - |v|^2)^{2(k-j)-1} \frac{e^{\omega(1-|v|^2)}}{\omega^{(1-|v|^2)}} dm(v)
\]

\[
\leq \frac{\gamma_1}{(1 - |a|)^{2\omega(1-r^2)} (\log(\frac{1}{r}))^p} \int_{\Gamma(a,\frac{1}{2})} |K_j(v)|^2 (1 - |v|^2)^{2(k-j)-1} \frac{e^{\omega(1-|v|^2)}}{\omega^{(1-|v|^2)}} dm(v),
\]

whereas \(C_1\) is a positive constant that depends on both values of \(k\) and \(p\). Let \(D\) define a known arc that lies on the boundary of \(\Gamma_v\) with a determined center at the point \(e^{\arg a}\) and with the determined length \(\frac{2}{3} (1 - |a|)/(1 - \frac{|a|}{2})\). Then \(1 - |a| \geq \frac{1}{3} |\ell_a|\), the specific disc \(\Gamma(a, \frac{1}{2})\) defines a subset of \(S(\ell_a)\). Thus, using (6), we have

\[
\sup_{|z| \leq \frac{2}{3}|a|} |K_j(a)|^2 (1 - |a|^2)^{2(k-j)}
\]

\[
\leq \sup_{|z| \leq \frac{2}{3}|a|} \frac{C_2}{|\ell_a| |e^{2\omega(1-r^2)} (\log(\frac{1}{r}))^p} \int_{S(\ell)} |K_j(v)|^2 (1 - |v|^2)^{2(k-j)-1} \frac{e^{\omega(1-|v|^2)}}{\omega^{(1-|v|^2)}} dm(v),
\]

\[
= \sup_{|\ell| \leq \delta} \frac{C_2}{|D(\ell_a)| |e^{2\omega(1-r^2)} (\log(\frac{1}{r}))^p} \int_{D(\ell)} |K_j(v)|^2 (1 - |v|^2)^{2(k-j)-1} \frac{e^{\omega(1-|v|^2)}}{\omega^{(1-|v|^2)}} dm(v),
\]

hence, for choosing \(\beta\) enough small in the inequality (6), in view of Theorem 4 we get that all entire-solutions of (1) are in the space \(L^p(\mathbb{C}^\delta)\).

To prove that all entire-solutions are in the space \(P_{\rho,\omega,n}(S)\), assume that \(\frac{1}{2} \leq \rho < 1\), \(r = 1 - \delta\), and let \(h\) be an entire-solution of (1). Thus, using both Theorem 1 and the known Leibniz formula, we obtain

\[
\|h_p\|_{L^p_{\omega,v}} \leq C_1 \left( \int_{\Gamma_v} |h^{(k)}(\rho v)|^2 (1 - |v|^2)^{2k-1} dm(v) + \sum_{j=1}^{k-1} |h^{(j)}(0)|^2 \right)
\]

\[
\leq C_2 \left( \sum_{j=0}^{k-1} \sum_{n=0}^{j} \int_{\Gamma_v} |K_j^{(n)}(\rho v)| h^{(j-n)}(\rho v) |(j-n)|^2 (1 - |v|^2)^{2k-1} dm(v) + \sum_{j=1}^{k-1} |h^{(j)}(0)|^2 \right)
\]

\[
\leq C_3 \left( \sum_{j=0}^{k-1} \sum_{n=0}^{j} \int_{\Gamma_v \setminus D(\ell_a)} |(\rho v)^2 |K_j^{(n)}(\rho v) |(j-n)|^2 (1 - |v|^2)^{2(k-n-j)-1} dm(v) + C_4 \right)
\]

whereas

\[
C_4 = C_3 \left( \sum_{j=0}^{k-1} \sum_{n=0}^{j} \int_{D(\ell_a)} |(\rho v)^2 |K_j^{(n)}(\rho v) |(j-n)|^2 (1 - |v|^2)^{2(k-n-j)-1} dm(v) + \sum_{j=1}^{k-1} |h^{(j)}(0)|^2 \right)
\]

and the constant \(C_3\) is a positive constant which depending only on the values of \(p\) and \(k\). In view of our assumption and Theorem 1, with the positive measure \(dF_{\omega,n,j} = |K_j^{(n)}(\rho v)|^2 (1 - |v|^2)^{2(k+n-j)-1} \omega^{(1-|v|^2)} dm(v)\) is a known as bounded Carleson-type measure \(\forall \frac{1}{2} \leq \rho \leq 1, j = 0, \ldots, k-1\) and \(n = 0, \ldots, k-1\). By Theorem 3.2 and Lemma 3.1, we deduce that
\[
\int_{\mathbb{D}_r \setminus \{0\}} |h(\rho \nu)|^2 d\xi_{\varphi,n,j}(v) \leq C(\varphi, n, j) \|h\|_{L_p(\varphi)}^2 \\
\leq C(\varphi, n, j) (\|h\|_{T_{\varphi}}^2 + |h(0)|^2).
\] (7)

Hence,
\[
C(\varphi, n, j) = \sup_{|t| \leq 1-r} \frac{1}{|t|} \int_{S(t)} d\xi_{\varphi,n,j}(v) \\
= \sup_{|t| \leq \delta} \frac{1}{|t|} \int_{S(t)} |K_j^n(\rho \nu)|^2 (1-|v|^2)^{2(k+n-j)-\beta} \psi^p(v, \varphi) e^{\omega(1-|v|^2)} \ dm(v).
\] (8)

Therefore, when \( \rho \to 1^- \), the assumption (8) yields
\[
\|h\|_{L_p}^2 K_p(1 - C_3 \beta^4) \leq C_3 |h(0)|^{\beta^4} + C_4.
\]

where \( C_3 = C(p, k) \). The needed assertion is therefore proved completely. \( \square \)

**Remark 3.** Since some years ago, there are some interesting generalizations for classes of hypercomplex function spaces (see [22–25]). The following interesting question can be emerged. How about the discussion and studying of Equation (1) in hypercomplex (or quaternion) sense?

4. Conclusions

Both theories of function spaces and differential equations play very crucial roles in the recent research area of mathematical sciences, as well as mathematical physics. The current research is an interesting combination between both important fields.

In this manuscript, certain interesting properties of entire-solutions of a specific linear complex differential equation with the use of analytic-type functions in the defined unit disc \( \mathbb{D}_r \) are clearly established. Furthermore, some relevant conditions on the coefficient \( K_j(v) \) which give guarantee that all concerned normal analytic solutions \( h \) are in the union of the weighted holomorphic Hardy spaces or that the type of analytic zero-sequence of every specific entire-solution \( h \neq 0 \). In addition, the relevant conditions on the coefficients are illustrated regarding the Carleson-type measures.

**Author Contributions:** Both authors have equally contributed in preparing the manuscript. All authors have read and agreed to the published version of the manuscript.

**Funding:** Taif University supported this current research work under project number (TURSP-2020/159).

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** The authors would like to appreciate their thanks to Taif University Researchers for supporting Project number (TURSP-2020/159), Taif University—Saudi Arabia.

**Conflicts of Interest:** The authors completely declare that they have no any competing-type of interests.

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