MAYER-VIETORIS PROPERTY FOR RELATIVE SYMPLECTIC COHOMOLOGY

UMUT VAROLGUNES

Abstract. In this paper, we construct a Hamiltonian Floer theory based invariant called relative symplectic cohomology, which assigns a module over the Novikov ring to compact subsets of closed symplectic manifolds. We show the existence of restriction maps, and prove some basic properties. Our main contribution is to identify a natural geometric situation in which relative symplectic cohomology of two subsets satisfy the Mayer-Vietoris property. This is tailored to work under certain integrability assumptions, the weakest of which introduces a new geometric object called a barrier - roughly, a one parameter family of rank 2 coisotropic submanifolds. The proof uses a deformation argument in which the topological energy zero (i.e. constant) Floer solutions are the main actors.

1. Introduction

Denote the Novikov ring and field over \( \mathbb{Q} \), by \( \Lambda_{\geq 0} \) and \( \Lambda \), respectively. Let \( M \) be a closed symplectic manifold. Relative symplectic cohomology \( SH_M(K) \) is a \( \mathbb{Z}_2 \)-graded \( \Lambda_{\geq 0} \)-module assigned to each compact \( K \subset M \).

Relative symplectic cohomology satisfies the following properties.

- (coordinate independence) Let \( \phi : M \to M \) be a symplectomorphism, then there exists a canonical relabeling isomorphism \( SH_M(K) \to SH_M(\phi(K)) \).
- (global sections) \( SH_M(M) = H(M, \mathbb{Z}) \otimes \Lambda_{>0} \) as \( \mathbb{Z}_2 \)-graded \( \Lambda_{\geq 0} \)-modules, where \( \Lambda_{>0} \) is the maximal ideal of \( \Lambda_{\geq 0} \).
- (empty set) \( SH_M(\emptyset) = 0 \).
- (restriction maps) For any \( K' \subset K \), there are canonical graded module maps, called restriction maps:

\[
SH_M(K) \to SH_M(K').
\]

Moreover, if \( K'' \subset K' \subset K \), the map \( SH_M(K) \to SH_M(K'') \) is equal to the composition \( SH_M(K) \to SH_M(K') \to SH_M(K'') \).

We construct \( SH_M(K) \) and prove the properties above in this paper. For further properties (and their proofs), including:

- (Hamiltonian isotopy invariance of restriction maps) Let \( \phi_t : M \to M, \ t \in [0,1] \), be a Hamiltonian isotopy such that \( \phi_t(K) \subset K' \) for all \( t \). We have a commutative diagram

\[
\begin{array}{ccc}
SH_M(K') & \longrightarrow & SH_M(\phi_1(K)) \\
\downarrow & & \downarrow \phi_1^{-1} \\
& SH_M(K). & \\
\end{array}
\]
Figure 1. On the left there are two subsets that cannot satisfy Mayer-Vietoris, and on the right are two that do. The thick circle on the left divides the sphere into equal areas.

- (displaceability condition) Let $K \subset M$ be displaceable by a Hamiltonian diffeomorphism, then $SH_M(K) \otimes_{\Lambda_{>0}} \Lambda = 0$;

as well as a lengthy motivational and historical discussion we refer the reader to author’s thesis [15]. Similar, but a priori different, invariants with similar properties have independently appeared in the literature ([5], [16], [8], [2]) and the reader will find a comprehensive list of references along with the appropriate comparisons in the aforementioned thesis.

1.1. Mayer-Vietoris property. The main task of this paper is to analyze the question: does $SH_M(\cdot)$ satisfy the Mayer-Vietoris property, i.e. for $K_1, K_2$ compact subsets of $M$, is there an exact sequence

$$(1.1.0.1) \quad \begin{array}{c} SH_M(K_1 \cup K_2) \to SH_M(K_1) \oplus SH_M(K_2) \to \end{array}$$

where the degree preserving maps are the restriction maps (up to sign)?

A Mayer-Vietoris sequence for their version of symplectic homology, when $K_1$ and $K_2$ are Liouville cobordisms inside a Liouville domain $M$ satisfying a number of conditions (one of them being that their union and intersection is also a Liouville cobordism) was established by Cieliebak-Oancea in Theorem 7.17 of [2]. The most rudimentary version of our results Theorem [1.3.3] can be seen as a generalization of theirs. As far as we know this is the first investigation of a symplectic Mayer-Vietoris property where the boundaries of the domains under question intersect non-trivially.

Mayer-Vietoris property does not hold in general. In Figure 1, we see examples of pairs of subsets inside the two sphere that does and cannot satisfy Mayer-Vietoris property.
Remark 1.1.1. Using the displaceability and global sections properties, and the fact that any symplectic manifold can be covered by displaceable subsets, we get a conceptual counterexample to any possible notion of locality in the manifold.

One piece of good news is that we can measure the failure of the Mayer-Vietoris property to hold. Slightly generalizing the situation, let \( K_1, \ldots, K_n \) be compact subsets of \( M \). Using the full package of Hamiltonian Floer theory, we can construct a chain complex \( SC_M(K_1, \ldots, K_n) \): an explicit deformation of the chain complex \( \bigoplus_{I \in [n]} SC_M(\bigcap_{i \in I} K_i) \), w.r.t the \( |I| \)-filtration (i.e. the full differential is lower triangular, and the diagonal entries are the differentials from before). Here \( I \) being the empty set means taking the union of \( K_i \)'s.

More specifically, in this deformation the part of the differential that increases \( |I| \)-filtration by 1 are given by restriction maps, the ones that increase by 2 are chain homotopies between compositions of restriction maps in different directions and so on. The data of \( SC_M(K_1, \ldots, K_n) \) should be visualized in the following way. The modules underlying the summands of \( \bigoplus_{I \in [n]} SC_M(\bigcap_{i \in I} K_i) \) are placed on the vertices of an \( n \)-dimensional cube (with an ordering of its coordinates), and the differential is the direct sum of maps indexed by the faces (including the vertices) of the cube, going between the initial and terminal vertices of that face. Such diagrams will be called cubical diagrams, or \( n \)-cubes (see \[2.1.1\]).

The homology of \( SC_M(K_1, \ldots, K_n) \) only depends on \( K_1, K_2, \ldots, K_n \), therefore the following definition makes sense.

**Definition 1.** \( K_1, K_2, \ldots, K_n \) satisfies descent, if \( SC_M(K_1, \ldots, K_n) \) is acyclic.

Satisfying descent implies the existence of a convergent spectral sequence:

\[
\bigoplus_{0 \neq I \subseteq [n]} SH_M(\bigcap_{i \in I} K_i) \Rightarrow SH_M(\bigcup_{i=1}^{n} K_i),
\]

which produces a Mayer-Vietoris sequence for \( n = 2 \) as in Equation \[1.1.0.1\] above.

**Definition 2.** Let \( Z^{2n-2} \) be a closed manifold. We define a barrier to be an embedding \( Z \times [-\epsilon, \epsilon] \to M^{2n} \), for some \( \epsilon > 0 \), where \( Z \times \{a\} \to M \) is a coisotropic for all \( a \in [-\epsilon, \epsilon] \). We call the image of \( Z \times \{0\} \) the center of the barrier, and the vector field obtained by pushing forward \( \partial_{\epsilon} \in \Gamma(\mathbb{R} \times \{0\}, T(Z \times (-\epsilon, \epsilon)) |_{Z \times \{0\}}) \) to \( M \) the direction of the barrier.

**Theorem 1.1.2.** (Mayer-Vietoris sequence) Let \( K_1, K_2 \subset M \) be compact domains. Assume that \( \partial K_1 \) and \( \partial K_2 \) transversally intersect along a rank 2 coisotropic which, if non-empty, is the center of a barrier whose direction points out of \( K_1 \) and \( K_2 \). Then, \( K_1 \) and \( K_2 \) satisfy descent. Therefore, we have an exact sequence:

\[
\begin{array}{c}
SH_M(K_1 \cup K_2) \longrightarrow SH_M(K_1) \oplus SH_M(K_2), \\
[1] \downarrow \quad \downarrow \\
SH_M(K_1 \cap K_2)
\end{array}
\]

where the degree preserving maps are the restriction maps (up to signs).

We made the assumption that \( K_1, K_2 \subset M \) are domains purely for the sake of keeping the statement simple. For the actual statement see Theorem \[4.6.1\]. Note that in dimension 2, the condition is equivalent to boundaries not intersecting, as a
point in a surface can never be coisotropic (see Figure 1). In dimension 4, it implies that the intersection is a disjoint union of Lagrangian tori, but unfortunately being outward pointing is an extra condition in this case, see Corollary 4.7.4.

Coming closer to our starting point of integrable systems, we make the following definition.

**Definition 3.** An involutive map is a smooth map \( \pi : M \to B \) to a smooth manifold \( B \), such that for any \( f, g \in C^\infty(B) \), we have \( \{ f \circ \pi, g \circ \pi \} = 0 \)

**Remark 1.1.3.** The most studied examples of involutive maps are Lagrangian fibrations. These correspond to the case where the image of \( \pi \) has half the dimension of \( M \) (which is the most it can be).

**Theorem 1.1.4.** Let \( \pi : M \to B \) be an involutive map, and \( X_1, \ldots X_n \) be closed subsets of \( B \). Then \( \pi^{-1}(X_1), \ldots \pi^{-1}(X_n) \) satisfy descent.

**Remark 1.1.5.** A fancy way of saying the same thing is that \( SC_M(\pi^{-1}(\cdot)) \) gives a homotopy sheaf over the Grothendieck topology of compact subsets on \( B \). We obtain Theorem 1.1.4 as a corollary of Theorem 4.8.1.

The following corollary of Theorem 1.1.4 (generally referred to as the Stem theorem) was first proven by Entov-Polterovich using a completely different set of tools [3].

**Theorem 1.1.6.** Any involutive map admits at least one fiber that is not displaceable by Hamiltonian isotopy.

**Proof.** Let \( \bigcup C_i \) be any finite cover of the image of \( M \) inside \( B \) by compact subsets. Theorem 1.1.4 and the global sections property shows that \( SH_M(\pi^{-1}(\bigcup C_i)) \otimes \Lambda \neq 0 \), for some non-empty \( J \subset [n] \), by a spectral sequence argument. Hence, by the displaceability property, \( C_i \) is not displaceable for some \( i \). Now assuming that each fiber is displaceable easily leads to a contradiction. \( \square \)

**Remark 1.1.7.** Even though the tools are different, the logic of our proof is similar to [3] as the experts will notice. We also refer the reader to [3] for a more detailed exposition of the corollary above including many interesting examples.

1.2. A remark on relative open string invariants. Let \( L \subset M \) be a closed aspherical Lagrangian (one can be a lot less restrictive, but we choose to be brief here). Replacing Hamiltonian Floer theory of closed orbits with Lagrangian Floer theory of chords with endpoints on \( L \), we immediately obtain a relative invariant \( FH_L(K) \), for any compact subset \( K \). We leave the discussion of this invariant to an upcoming paper, but we would like to advertise one result:

**Theorem 1.2.1.** Any involutive map admits at least one fiber that is not displaceable from \( L \) by Hamiltonian isotopy.

This open string version of the Stem theorem seems to be new. Its proof only notationally differs from the one of Theorem 1.1.6.

1.3. Outline of the thesis. In Section 2 we collect some algebraic facts together (none of the results are new). In 2.1 we discuss the homotopical algebra of cubical diagrams. In the sequel 2.2 we consider the relationship between colimits and homotopy colimits of linear diagrams of chain complexes. In 2.3 we recall the notions of completion and completeness for modules over the Novikov ring, and
discuss their interaction with taking homology of chain complexes. We end with a short summary in 2.4.

In Section 3, first, we list our conventions for Hamiltonian Floer homology in 3.1 and review Hamiltonian Floer theory in 3.2. In 3.3 we define relative symplectic cohomology, and show its basic properties as listed in the Introduction. In the last subsection 3.4, we introduce relative symplectic cohomology of multiple compact subsets.

Section 4 is where we discuss the Mayer-Vietoris/descent properties. We focus on the homology level statement for two subsets (i.e. the Mayer-Vietoris sequence) until the last subsection for better readability. In 4.1 we reduce the problem to showing the existence of a sequence of (pairs of) Hamiltonians that can be used as acceleration data for our subsets, which satisfy a dynamical property. In 4.2 we explain a controlled way of choosing acceleration data, and immediately show the Mayer-Vietoris property for two domains with non-intersecting boundary in 4.3. Subsections 4.4 and 4.5 introduce and motivate barriers and some relevant notions. In 4.6 we prove our main theorem (Theorem 4.6.1). In 4.7 we give examples of barriers. In the last subsection 4.8, after generalizing the main theorem slightly (Theorem 4.8.1), we show the descent result for multiple subsets that are preimages of involutive maps.

In Appendix A we establish the easy translation from Pardon’s simplicial diagrams to our cubical ones. Finally, in Appendix B we reduce the descent statement for $n > 2$ subsets to a bunch of others but each involving only 2 subsets.

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2. Algebra preparations

2.1. Homotopical constructions. In this subsection we assume that all our chain complexes are $\mathbb{Z}/2$-graded. However, whenever there is a $\mathbb{Z}/N$ or $\mathbb{Z}$-grading statements can be modified to take into account those gradings without a problem.

2.1.1. Cubes. Consider the standard unit cube $\text{Cube}^n := \{(x_1, \ldots, x_n) \mid x_j \in [0, 1]\} \subset \mathbb{R}^n$. Note that the ordering of the coordinates will be part of the data. For $0 \leq k \leq n$, a $k$-dimensional face of $\text{Cube}^n$ is any subset of $\text{Cube}^n$ given by setting $n - k$ of the coordinates to either 0 or 1. Let us call a vertex of a $k$-face the initial vertex if it has the maximum number of zeros and terminal if it has the maximum number of ones.

Let us call two faces $F'$ and $F''$ adjacent if the terminal vertex of $F'$ equals the initial vertex of $F''$. We denote this relationship by $F' > F''$. We say that two adjacent faces form a boundary of a face $F$ if $F$ is the smallest face that contains both $F'$ and $F''$.

Let $R$ be a commutative ring. We define an $n$-cube of chain complexes over $R$ in the following way. To 0-dimensional faces (i.e. vertices) of $\text{Cube}^n$ we associate an $R$-module $C^\nu$, and for any $k$-dimensional face (including $k = 0$) $F$ we give
maps $f_F : C^\nu \rightarrow C^{\nu'}$ from its initial vertex to its terminal vertex, of degree $\dim(F) + 1$.

These maps are required to satisfy the following relations. For each face $F$ we have:

\[(2.1.1.1) \sum_{F' \text{ is a bdry of } F} (-1)^{\ast F'} f_{F'} f_F = 0,\]

where $\ast F' = \#_v 1 + \#_v 01$ for $v = \nu_{\text{ter}} F' - \nu_{\text{in}} F'$ considered as a vector inside $F$.

Let us explain this notation a little bit. The coordinates of $v$ is a sequence of 0's and 1's of length $\dim(F)$. If $abc..$ is a word of 0's and 1's, $\#_v abc..$ is the number of subsequences of the coordinates of $v$ that is equal to $abc..$ It's clear how this definition would extend to an alphabet with more letters.

In Figure 2.1.1 we present a 3-cube to illustrate the definition. At the corners there are chain complexes, at the edges chain maps, at the square faces homotopies between the two different ways of going between the initial and terminal vertices of that square, and lastly at the codimension 0 face we have one map $H$ that satisfies:

\[(2.1.1.2) -g^{100} + g^{010} - g^{001} - g^{101} + g^{110} - dH + H d = 0,\]

where $g^{100}$ is the composition $C^{000} \rightarrow C^{100} \rightarrow C^{111}$ (the second map is the homotopy) etc.

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**Figure 2.** A 3-cube

2.1.2. Maps between $n$-cubes. A partially defined $n$-cube is one where we have chain complexes at the vertices of $\text{Cube}_n$, and maps for some of the faces specified so that whenever it makes sense Equation $2.1.1.1$ is satisfied. If this data is extended to a full $n$-cube, we call the extension a filling.

We define a map between two $n$-cubes to be a filling of an $(n+1)$-cube where the two opposite faces $\{x_{n+1} = 0\}$ and $\{x_{n+1} = 1\}$ are the given $n$-cubes.

\[(2.1.2.1) C \rightarrow C'\]

An example of a map of $n$-cubes is the $id$ map, where the two $n$-cubes are connected to each other with identity maps between the chain complexes at the edges and all the homotopies are zero. Note that even if we added the new coordinate in a different place than the last in the ordering, we would get an $(n+1)$-cube this way.
A homotopy of two maps of \( n \)-cubes is a filling of an \((n+2)\)-cube where two opposite faces \( \{x_{n+2} = 0\} \) and \( \{x_{n+2} = 1\} \) are the given maps of \( n \)-cubes and the copies of each given \( n \)-cube at those opposite faces are connected to each other with identity maps.

\[(2.1.2.2)\]

\[
\begin{array}{ccc}
C & \longrightarrow & C' \\
| id | & | id | \\
\downarrow & \downarrow \\
C' & \longrightarrow & C''
\end{array}
\]

Let us call an \((n+2)\)-cube of such shape an \((n+2)\)-slit.

A triangle of maps of \( n \)-cubes is a triple of \( n \)-cubes and maps between them placed in a partial \((n+2)\)-cube in the following manner, and of course a filling of that cube. We require that the coordinate with the axis parallel to the edges where we inserted the \( id \) map is the last of the \( n+2 \) coordinates.

\[(2.1.2.3)\]

\[
\begin{array}{ccc}
C & \longrightarrow & C'' \\
| id | & | id | \\
\downarrow & \downarrow \\
C' & \longrightarrow & C''
\end{array}
\]

Let us call an \((n+2)\)-cube of such shape an \((n+2)\)-triangle.

We now give examples of these definitions in low dimensions.

The data of \( f_0, f_1, h \) below:

\[(2.1.2.4)\]

\[
\begin{array}{ccc}
C_0 & \longrightarrow & C_1 \\
| f_0 | & | h | \\
\downarrow & \downarrow \\
C'_0 & \longrightarrow & C'_1
\end{array}
\]

such that \( f_0, f_1 \) are chain maps, and \( c'f_0 - f_1c + hd + dh = 0 \) a map of 1-cubes.

Let \( f'_0, f'_1, h' \) be another such map. Then a homotopy from the first triple to the second one (primed ones) would be given by \( F_0 \), a chain homotopy between \( f_0 \) and \( f'_0 \), similarly \( F_1 \), and also an \( H \) that satisfies the equation that is associated to the maximal face of the 3-cube:

\[(2.1.2.5)\]

\[c'F_0 - F_1c + h' - h - dH + Hd = 0,\]

as a special case of Equation 2.1.1.2.

Finally consider the following homotopy commutative triangle:

\[(2.1.2.6)\]

\[
\begin{array}{ccc}
C_0 & \longrightarrow & C_1 \\
| g | & | f_1 | \\
\downarrow & \downarrow \\
C_2
\end{array}
\]

and another map that fills the triangle \( h : C_0 \to C_2 \) such that,

\[(2.1.2.7)\]

\[f_1f_0 - g + dh + hd = 0.\]
We are thinking of this data as the following 2-cube (with $C_1$ sitting at the vertex with coordinates $(0,1)$):

$$
\begin{array}{c}
C_0 \xrightarrow{f_0} C_1 \\
\downarrow g & \downarrow h \\
C_2 \xrightarrow{id} C_2.
\end{array}
$$

2.1.3. $n$-cubes with positive signs. There is a slightly different definition of $n$-cubes where the signs are not present. We define an $n$-cube with positive signs to be one where the signs in the Equation 2.1.1 are all +1, in other words Lemma 2.1.1.

Faces of $\text{Cube}_n$ are in one-to-one correspondence with

$$
\{(i_1, \ldots, i_n) \mid i_k \in \{0,1,-\}\},
$$

where $-$ represents the coordinates that vary in the face. Let us denote this assignment by $F \mapsto \mu(F)$.

Lemma 2.1.1. Let $(C_v, f_F)$ be an $n$-cube ($f_{\{v\}}$ are the differentials). There exists a canonical way of changing the signs of $f_F \mapsto (-1)^{\mu(F)}f_F$ so that $(C_v, (-1)^{\mu(F)}f_F)$ is an $n$-cube with positive signs.

Proof. We define $\mu(F) := \#_{\mu(F)}(0-) + \#_{\mu(F)}0$.

Let $F' > F''$ be a boundary of $F$. Let $S \subset [n]$ be the entries of $\mu(F)$ that are equal to $-$. Then, there is a subset $S' \subset S$ such that $\mu(F')$ is obtained by changing the entries of $S$ corresponding to $S'$ to 0, and $\mu(F'')$ is obtained by changing the entries corresponding to $S - S'$ to 1.

We claim that $\#_{\mu(F)}(0-) + \mu(F') + \mu(F'') + *_{F',F} \text{ is even.}$ The parity of $\#_{\mu(F)}(0-) + \#_{\mu(F')}(0-) + \#_{\mu(F'')}0$ is equal to the one of the number of 01’s in $\nu_{\text{ter}}F' - \nu_{\text{in}}F'$ considered as a vector inside $F$. Moreover the one of the number of 1’s in $\nu_{\text{ter}}F' - \nu_{\text{in}}F'$ considered as a vector inside $F$ plus $\#_{\mu(F')}0 + \#_{\mu(F'')}0$ has the same parity as $\dim F$. This proves the claim because the overall factor $(-1)^{\#_{\mu(F)}(0-)+\dim F}$ can be canceled from the equation. \hfill \qed

2.1.4. Cones of $n$-cubes. Recall that the usual cone operation takes a chain map (i.e. a 1-cube) between two chain complexes, and splits out a single chain complex (a 0 cube):

$$
(\text{C},d) \xrightarrow{\nu} (\text{C}',d') \rightarrow \left[ \text{C}[1] \oplus \text{C}', \begin{pmatrix} -d & 0 \\ f & d' \end{pmatrix} \right].
$$

This can be generalized to all cubes. First, given an $n$-cube with positive signs and one of the $n$ directions, we explain how to construct an $(n-1)$-cube with positive signs with the cone construction.

Let $(C_v, f_F)$ be an $n$-cube with positive signs, and $1 \leq i \leq n$ an integer. If $w$ is a sequence of length $n-1$, we let $(w, i, a)$ be the sequence of length $n$ with $a$ added as the $i$th entry to $w$. Recall also that we can identify a face $F$ with $\mu(F)$ as defined in the previous subsection.

The cone of $(C_v, f_F)$ in direction $i$ is defined by:

$$
C_w = C_{(w,i,0)}[1] \oplus C_{(w,i,1)},
$$

2.1.4.1
and \( f_F : C_{in(F)} \to C_{ter(F)} \) is given by the matrix,
\[
\begin{pmatrix}
\int_{(F)_{i,0}} & 0 \\
\int_{(F)_{i,-}} & \int_{(F)_{i,1}}.
\end{pmatrix}
\]
(2.1.4.3)

It is readily seen that this defines an \((n-1)\)-cube with positive signs.

Now, we define cones on \( n \)-cubes by
(2.1.4.4)

\[ n \text{-cube} \rightarrow n \text{-cube with p. signs} \rightarrow (n-1) \text{-cube with p. signs} \rightarrow (n-1) \text{-cube} \]

Note that the signs in the formulas will be different for different directions. We will call the fact that the cone operation turns an \( n \)-cube into an \((n-1)\)-cube, the functoriality of the cone operation.

Lemma 2.1.2. (1) Cone operation in two different directions commute.

(2) The cone operation in a direction \( d \) other than the last one sends the map \( C \xrightarrow{id} C \) to \( \text{cone}^d(C) \xrightarrow{id} \text{cone}^d(C) \)

(3) Cones in directions except the last one send \((n+1)\)-slits to \( n \)-slits, and except the last two send \((n+1)\)-triangles to \( n \)-triangles.

Proof. (1) The sign change is easily seen to not depend on the order.

(2) The identity maps do not change sign because if the \( d \)th entry is 0 then they get negated twice, and if it is 1 not at all. The two opposite faces (connected by \( id \)) get the same sign changes because their last coordinates being 0 or 1 do not affect the sign change.

(3) Follows from (2).

We explain this on 2-cubes. There are two cones of the 2-cube in Diagram 2.1.2.4 (called \( C \)): one that contracts the direction parallel to \( f \)’s \( \text{cone}^f(C) \), and the one that contracts \( c \)’s \( \text{cone}^c(C) \). Let us write them down explicitly.

(2.1.4.5)

\[
\text{cone}^f(C) = \left[ C_0[1] \oplus C'_0, \begin{pmatrix} -d & 0 \\ -f_1 & d \end{pmatrix} \right] \xrightarrow{\begin{pmatrix} -c & 0 \\ h & c' \end{pmatrix}} \left[ C_1[1] \oplus C'_1, \begin{pmatrix} -d & 0 \\ f_2 & d \end{pmatrix} \right]
\]

(2.1.4.6)

\[
\text{cone}^c(C) = \left[ C_0[1] \oplus C_1, \begin{pmatrix} -d & 0 \\ c & d \end{pmatrix} \right] \xrightarrow{\begin{pmatrix} f_1 & 0 \\ h & f_2 \end{pmatrix}} \left[ C'_0[1] \oplus C_1', \begin{pmatrix} -d & 0 \\ c & d \end{pmatrix} \right]
\]

In both cases, taking the cone in the remaining direction results in \( C_0 \oplus (C_1[1] \oplus C'_0[1]) \oplus C'_1 \) with differential:

(2.1.4.7)

\[
\begin{pmatrix}
  d & 0 & 0 & 0 \\
  -c & -d & 0 & 0 \\
  f_1 & 0 & -d & 0 \\
  h & c' & f & d
\end{pmatrix}.
\]
2.1.5. **Composing n-cubes.** The composition of two chain maps is a chain map. We generalize this construction to higher dimensional cubes.

Let us start with 2-cubes. Let the two squares below be commutative up to the given homotopies.

\[
\begin{array}{ccc}
C_0 & \rightarrow & C_1 \\
\downarrow & & \downarrow \\
D_0 & \rightarrow & D_1
\end{array} \quad \begin{array}{ccc}
C_1 & \rightarrow & C_2 \\
\downarrow & & \downarrow \\
D_1 & \rightarrow & D_2
\end{array}
\]

In this case, we say that the two 2-cubes are glued along \( f_1 \) a 1-cube.

We can define the composite 2-cube:

\[
\begin{array}{ccc}
C_0 & \rightarrow & C_2 \\
\downarrow & & \downarrow \\
D_0 & \rightarrow & D_2
\end{array}
\]

where \( G = g_1 c_0 + g_0 c_1 \).

In general, if we are given two n-cubes glued along an \((n-1)\)-cube, we can define an n-cube in a similar fashion. This operation also depends on the ordering, more precisely, on the place of the special direction in the ordering.

Note that any iterated cone of an n-cube (remembering its direct sum decomposition) has the same information as the cone itself. Only some signs are different but we know exactly how the signs change.

Let \( C \rightarrow C' \rightarrow C'' \) be two n-cubes \( C \rightarrow C' \) and \( C' \rightarrow C'' \) glued along \( C' \). By taking the \((n-1)\) times iterated cone we get two chain maps glued along a chain complex \( \text{cone}^{n-1}(C) \rightarrow \text{cone}^{n-1}(C') \rightarrow \text{cone}^{n-1}(C'') \). We of course know how to compose these two maps, and all we need to do is to de-cone this as described in the previous paragraph. We omit the explicit formulas. The following is immediate by definition.

**Lemma 2.1.3.**

- The composition operation is associative. Namely if we have three cubes glued along linearly, then the final composition is independent of the order in which we performed the compositions.
- Composition commutes with the cone operation done in a direction parallel to the glued face.

2.1.6. **Rays.** We call an infinite sequence of n-cubes \( D_1, D_2, \ldots \) an n-ray if they are glued together to form a half-infinite box, more precisely an \((n-1)\)-dimensional face of \( D^1 \) is the same as one of \( D_2 \), the opposite face of \( D_2 \) is the same as one of \( D_3 \), etc. Below is a 1-ray:

\[
C_0 \rightarrow C_1 \rightarrow C_2 \rightarrow \ldots
\]

And a 2-ray:

\[
C_0 \rightarrow C_1 \rightarrow C_2 \rightarrow \ldots
\]
For an \( n \)-ray, there are \( n - 1 \) finite, and 1 infinite directions. We always think of the infinite direction as the first in order. We call the faces of the \( n \)-cubes forming an \( n \)-ray that are perpendicular to the infinite direction the slices of the ray. Slices are \( (n - 1) \)-cubes. We will generally present an \( n \)-ray as

\[
C_0 \to C_1 \to C_2 \to \ldots,
\]

where \( C_i \) are the slices.

We define a map between two \( n \)-rays to be an \((n + 1)\)-ray filling the two \( n \)-rays, in other words, \((n + 1)\)-cubes filling the two infinite sequence of \( n \)-cubes which glue together. The 2-ray above is map between the upper and lower 1-rays.

A homotopy between two maps of \( n \)-rays is again given by a sequence of homotopies for the given maps of \( n \)-cubes that glue together. A triangle of maps is defined in the same way.

2.1.7. Cones and telescopes of \( n \)-rays. Note that, using functoriality of cones, along the \( n - 1 \) finite directions of an \( n \)-ray we can take cones and end up with an \((n - 1)\)-ray.

Let \( C = C_0 \to C_1 \to C_2 \to \ldots \) be a 1-ray. The telescope \( tel(C) \) of such a diagram is defined to be the chain complex with the underlying \( R \)-module \( \bigoplus_{i \in \mathbb{N}} C_i[1] \oplus C_i \) and the differential as depicted below:

\[
\begin{array}{ccc}
C_0 & \xrightarrow{d} & C_1 \\
\downarrow{id} & & \downarrow{id} \\
C_0[1] & \xrightarrow{-d} & C_1[1]
\end{array}
\begin{array}{ccc}
C_1 & \xrightarrow{d} & C_2 \\
\downarrow{id} & & \downarrow{id} \\
C_1[1] & \xrightarrow{-d} & C_2[1]
\end{array}
\begin{array}{ccc}
C_2 & \xrightarrow{d} & \ldots \\
\downarrow{id} & & \downarrow{id} \\
C_2[1] & \xrightarrow{-d} & \ldots
\end{array}
\]

More generally, the telescope \( tel(C) \) of an \( n \)-ray \( C \) is an \((n - 1)\)-cube. Let \( C \) be the \( n \)-ray \( C_0 \to C_1 \to C_2 \to \ldots \). Now define the \( R \)-modules at the vertices of \( tel(C) \) as the entrywise direct sum \( \bigoplus_{i \in \mathbb{N}} C_i[1] \oplus C_i \). The maps in the \((n - 1)\)-cube structure are depicted in:

\[
\begin{array}{ccc}
C_0 & \xrightarrow{id} & C_1 \\
\downarrow{id} & & \downarrow{id} \\
\pm C_0[1] & \xrightarrow{-} & \pm C_1[1]
\end{array}
\begin{array}{ccc}
C_1 & \xrightarrow{id} & C_2 \\
\downarrow{id} & & \downarrow{id} \\
\pm C_1[1] & \xrightarrow{-} & \pm C_2[1]
\end{array}
\begin{array}{ccc}
C_2 & \xrightarrow{id} & \ldots \\
\downarrow{id} & & \downarrow{id} \\
\pm C_2[1] & \xrightarrow{-} & \ldots
\end{array}
\]

Note that the \( C_i \)'s (and the shifted copies) have internal structure that make them an \((n - 1)\)-cube that is taken into account here, and the \( \pm \) in front means that some those maps are negated (as described in the next sentence). The pieces formed by diagonal arrows are the cones of \( D_i = C_{i-1} \to C_i \)'s in the infinite direction, and the vertical arrows are the cones of \( C_i \xrightarrow{id} C_i \), where \( id \) is put as the first coordinate. In particular, the fact that this is an \((n - 1)\)-cube follows from the functoriality of cones.

**Lemma 2.1.4.** \( \bullet \) We get a canonical 1-ray from any \( n \)-ray by an \((n - 1)\) times iterated cone. This commutes with the telescope.
Telescopes are functorial in the sense that (1) a map of $n$-rays canonically give a map of the telescopes (which are $(n-1)$-cubes), (2) a homotopy between two maps give a homotopy, (3) a triangle of maps gives a triangle of maps.

2.2. 1-rays and quasi-isomorphisms. In this subsection we give a low level discussion of the fact that the telescope of a 1-ray is the homotopy colimit of the diagram in the appropriate category of chain complexes. Let $\mathcal{C} = C_0 \to C_1 \to C_2 \to \ldots$ be a 1-ray.

**Lemma 2.2.1.** There is a canonical quasi-isomorphism

\[
\text{tel}(\mathcal{C}) \to \lim_{\to}(C_1 \to C_2 \to \ldots)
\]  

**Proof.** Define $F^n(\text{tel}(\mathcal{C}))$ to be $(\bigoplus_{i \in [1,n-1]} C_i[1] \oplus C_i) \oplus C_n$. Notice that $\text{tel}(\mathcal{C})$ is the usual direct limit of $F^n(\text{tel}(\mathcal{C}))$. Moreover, there are canonical quasi-isomorphisms $F^n(\mathcal{C}) \to C_n$ induced by the given maps $C_i \to C_n$, $i \in [1,n]$ and the zero maps $C_i[1] \to C_n$, $i \in [1,n-1]$, which makes the diagrams

\[
\begin{array}{ccc}
F^n(\text{tel}(\mathcal{C})) & \to & F^{n+1}(\text{tel}(\mathcal{C})) \\
\downarrow & & \downarrow \\
C_n & \to & C_{n+1}
\end{array}
\]

commutative. The induced map $\text{tel}(\mathcal{C}) \to \lim_{\to} C_i$ is also a quasi-isomorphism, since direct limits commute with homology. □

Let $i(0) < i(1) < i(2) < \ldots$ be an infinite subsequence of $\mathbb{Z}_{\geq 0}$. Note that by composing maps we get a unique map $C_n \to C_m$ for all $m \geq n$. Then we canonically obtain a 1-ray $\mathcal{C}^i = C_{i(1)} \to C_{i(2)} \to \ldots$. Let us call this a subray. Let us call the canonical map of 1-rays $\mathcal{C} \to \mathcal{C}^i$ a compression map:

\[
\begin{array}{ccc}
C_1 & \to & C_2 \\
\downarrow & & \downarrow \\
C_{i(1)} & \to & C_{i(2)}
\end{array}
\]

**Lemma 2.2.2.** The compression map induces a quasi-isomorphism: $\text{tel}(\mathcal{C}) \to \text{tel}(\mathcal{C}^i)$.

**Proof.** This follows from the commutativity of the diagram:

\[
\begin{array}{ccc}
\text{tel}(\mathcal{C}) & \to & \text{tel}(\mathcal{C}^i) \\
\downarrow & & \downarrow \\
\lim_{\to}(C_1 \to C_2 \to \ldots) & \to & \lim_{\to}(C_{i(1)} \to C_{i(2)} \to \ldots)
\end{array}
\]

since bottom horizontal map is a quasi-isomorphism, using that the homology commutes with direct limits and that $i$ is a cofinal subsequence of natural numbers. □

This generalizes to higher dimensional rays too. Using the composition operation as in the subsubsection 2.1.5 we can define the notion of subrays, and compression...
morphisms in exactly the same way. The lemma above holds with \(\text{tel} \circ \text{cone}^{n-1}\).

2.3. **Completion of modules and chain complexes over the Novikov ring.**

Let us start by writing down our conventions for the Novikov field:

\[
\Lambda = \{ \sum_{i \in \mathbb{N}} a_i T^{\alpha_i} \mid a_i \in \mathbb{Q}, \alpha_i \in \mathbb{R}, \text{ and for any } R \in \mathbb{R}, \}
\]

There is a valuation map \(\text{val} : \Lambda \to \mathbb{R} \cup \{+\infty\}\) given by \(\text{val}(\sum_{i \in \mathbb{N}} a_i T^{\alpha_i}) = \min_i(\alpha_i \mid a_i \neq 0)\) for non zero elements, and \(\text{val}(0) = +\infty\). We define \(\Lambda_{\geq r} := \text{val}^{-1}([r, \infty[)\) and \(\Lambda_{> r} := \text{val}^{-1}(r, \infty)\). \(\Lambda_{\geq 0}\) is called the Novikov ring. The valuation we described makes \(\Lambda_{\geq 0}\) a complete valuation ring with real numbers as the value group.

**Lemma 2.3.1.** Let \(A\) be a \(\Lambda_{\geq 0}\)-module. Then, \(A\) is flat if and only if it is torsion free.

**Proof.** This is true for any valuation ring [12, Tag 0539]. \(\square\)

**Corollary 2.3.2.** Let \(C\) be an acyclic chain complex over \(\Lambda_{\geq 0}\) with a torsion free underlying module. Then, \(C \otimes_{\Lambda_{\geq 0}} \Lambda_{\geq 0}/\Lambda_{> r}, r > 0,\) and \(C \otimes_{\Lambda_{\geq 0}} \Lambda_{\geq 0}/\Lambda_{\geq r},\) for \(r \geq 0,\) are also acyclic.

Completion is a functor \(\text{Mod}(\Lambda_{\geq 0}) \to \text{Mod}(\Lambda_{\geq 0})\) defined by

\[
A \mapsto \hat{A} : \lim_{r \geq 0} A \otimes_{\Lambda_{\geq 0}} \Lambda_{\geq 0}/\Lambda_{\geq r},
\]

and by functoriality of inverse limits on the morphisms. There is a natural map of modules \(A \to \hat{A}\).

One can construct the completion in the following way. Let us say that a sequence \((a_1, a_2, \ldots)\) of elements of \(A\)

- is a **Cauchy sequence**, if for every \(r \geq 0\) there exists a positive integer \(N\) such that for every \(n, n' > N, a_n - a_{n'} \in T^r A,\)
- converges to \(a \in A,\) if for every \(r \geq 0\) there exists a positive integer \(N\) such that for every \(n > N, a - a_n \in T^r A.\)

Then, we have that \(\hat{A}\) is isomorphic to all Cauchy sequences in \(M\) (with its natural \(\Lambda_{\geq 0}\)-module structure) modulo the ones that converge to 0.

In case \(A\) is free, this description becomes simpler. Choose a basis \(\{v_i\}, i \in I.\) Then, \(\hat{A}\) is isomorphic to

\[
\{ \sum_{i \in I} \beta_i v_i \mid \beta_i \in \Lambda_{\geq 0}, \text{ and for every } R \geq 0, \text{ there is only finitely many } i \in I \text{ s.t. } \text{val}(\beta_i) < R \}.
\]

The following lemma is immediate from this description.

**Lemma 2.3.3.** Let \(A\) be a free \(\Lambda_{\geq 0}\)-module. Then

- \(\hat{A}\) is torsion free.
- The map \(A \otimes_{\Lambda_{\geq 0}} \Lambda_{\geq 0}/\Lambda_{\geq r} \to \hat{A} \otimes_{\Lambda_{\geq 0}} \Lambda_{\geq 0}/\Lambda_{\geq r}\) is an isomorphism for all \(r \geq 0.\)

The completion functor automatically extends to a functor \(\text{Ch}(\Lambda_{\geq 0}) \to \text{Ch}(\Lambda_{\geq 0}).\)
Lemma 2.3.4. Let $C$ be a chain complex over $\Lambda_{\geq 0}$, and $r > 0$. If the underlying module of $C$ is torsion-free and complete (meaning that every Cauchy sequence converges), then $C \otimes_{\Lambda_{\geq 0}} \Lambda_{\geq 0}/\Lambda_{> r}$ is acyclic only if $C$ is acyclic.

Proof. Let $\alpha \in C$, and $d\alpha = 0$. We need to show that $\alpha$ is exact. Our assumption implies that there exists $a, b \in C$ such that $\alpha = d b + T^r a$.

We have that $d(T^r a) = T^r d a = 0$, which implies that $d a = 0$ by torsion-freeness. Now we repeat the previous step for $a$, and keep going. Because of our completeness assumption this defines a primitive of $\alpha$. □

Corollary 2.3.5. (1) Assume that $C$ is finitely generated free as a module, then if $C \otimes_{\Lambda_{\geq 0}} \Lambda_{\geq 0}/\Lambda_{> r}$ is acyclic then so is $C$.

(2) Assume that $C$ is free as a module, then $C$ acyclic implies $\hat{C}$ acyclic.

(3) Let $f : C \to C'$ be a chain map. Assume that the underlying modules of $C$ and $C'$ are free. Then $\hat{f} : \hat{C} \to \hat{C'}$ is a quasi-isomorphism if $f$ is one.

Proof. For (1), choose a basis for $C$ and write $d$ as a matrix. There exists a smallest positive number $r$ such that $T^r$ has a non-zero coefficient in a matrix entry. Then our assumption actually implies that $C \otimes_{\Lambda_{\geq 0}} \Lambda_{\geq 0}/\Lambda_{> r}$ is acyclic.

For (2), combine the previous two lemmas (noting that the completion of a module is complete), and for (3) use the fact a chain map is a quasi-isomorphism if its cone is acyclic. □

Even though taking homotopy colimits are better suited for general constructions, sometimes usual direct limits are better for computations. To this end we show that Lemma 2.2.1 still holds after completions.

Lemma 2.3.6. Let $C = C_0 \to C_1 \to C_2 \to \ldots$ be a $1$-ray. There is a canonical quasi-isomorphism

$$\widehat{\text{tel}}(C) \to \lim \hat{C}.$$ (2.3.0.6)

Proof. We have canonical quasi-isomorphisms

$$f_r : \text{tel}(C) \otimes_{\Lambda_{\geq 0}} \Lambda_{\geq 0}/\Lambda_{> r} \to \lim \hat{C} \otimes_{\Lambda_{\geq 0}} \Lambda_{\geq 0}/\Lambda_{> r},$$ (2.3.0.7)

that are compatible with each other, using Lemma 2.2.1 and that tensor product commutes with telescopes and direct limit. We claim that the inverse limit over $r$ of these maps give the desired map.

We show that the inverse limit of $\text{cone}(f_r)$ is acyclic, which is clearly enough. Note that the maps in this inverse system are all surjective. Therefore we have a Milnor short exact sequence (see Theorem 3.5.8 in [17]), and the fact that $\text{cone}(f_r)$‘s are acyclic implies the desired acyclicity. □

2.4. Acyclic cubes and an exact sequence. Starting from an $n$-ray we can obtain a $(n - 1)$-cube by applying telescope. We can then apply completion functor to the result. Hence, we obtain an assignment $\hat{\text{tel}} : (n - 1) - \text{rays} \to ((n - 1) - \text{cubes})$. This trivially extends to morphisms, and respects homotopies. It is functorial in the sense that it also preserves triangles. We can also apply the maximally iterated cone functor to obtain a chain complex. In fact we could have applied it before the other two operations and the result would not change: $\text{cone}^{n-1} \circ \text{tel} = \text{cone}^{n-1} \circ \text{tel} = \text{tel} \circ \text{cone}^{n-1}$. Note that completion is always applied after telescope.
Let us call an \( n \)-cube **acyclic** if its maximally iterated cone is an acyclic chain complex. Note that by Corollary 2.3.5 Part (1), if the modules in this cube are finitely generated free, then this acyclicity is equivalent to acyclicity after tensoring with the residue field.

**Lemma 2.4.1.** Let \( \mathcal{C} \) be a \( n \)-ray where the underlying modules are free. Assume that all the slices are acyclic \((n-1)\)-cubes, then \( \text{tel}(\mathcal{C}) \) is acyclic, and hence \( \tilde{\text{tel}}(\mathcal{C}) \) is also acyclic.

**Proof.** The first follows because the maximally iterated cone commutes with the telescope functor, and Lemma 2.2.1. The second part is Lemma 2.3.5 Part (2). □

**Lemma 2.4.2.** An acyclic 2-cube

\[
\begin{array}{c}
C_{00} \rightarrow C_{10} \\
| & | \\
C_{01} \rightarrow C_{11}
\end{array}
\]

(2.4.0.1)

\[
\begin{array}{c}
H(C_{00}) \rightarrow H(C_{10}) \oplus H(C_{01}) \\
\text{[1]} \\
H(C_{11})
\end{array}
\]

(2.4.0.2)

where the degree preserving arrows are induced from the ones in the 2-cube.

**Proof.** The acyclicity implies that \( C_{00} \rightarrow \text{cone}(C_{10} \oplus C_{01} \rightarrow C_{11}) \) is a quasi-isomorphism. Then the long exact sequence of homology associated to the cone finishes the proof. □

### 3. Definition and Basic properties

In this section, we assume familiarity with Hamiltonian Floer theory at the level of Pardon [10], Section 10. We also freely use notations and results of the previous section.

#### 3.1. Conventions.** In this short subsection, we put together our conventions in setting up Hamiltonian Floer theory.

1. \( \omega(X_H, \cdot) = dH \).
2. \( \omega(\cdot, J \cdot) = g \), hence \( JX_H = \text{grad}_g H \).
3. Floer equation: \( J \frac{\partial u}{\partial s} = \frac{\partial u}{\partial t} - X_H \).
4. Topological energy of (arbitrary) \( u : S^1 \times \mathbb{R} \rightarrow M \) for a given Hamiltonian \( S^1 \times \mathbb{R} \times M \rightarrow \mathbb{R} \):

\[
\int \omega + \int \partial_s (H(s, t, u(s, t))) ds dt
\]

(3.1.0.1)

\[
= \int \omega \left( \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} \right) ds dt + \int \left[ (\partial_s H_s) + d_{u(s, t)} H_{s, t} \left( \frac{\partial u}{\partial s} \right) \right] ds dt
\]

(3.1.0.2)

\[
= \int \omega \left( \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} - X_H \right) ds dt + \int (\partial_s H_s) ds dt
\]

(3.1.0.3)
(5) Homomorphisms defined by moduli problems always send the generator of the Floer complex at the negative punctures to the one at the positive puncture.

(6) We consider all orbits, not just contractible ones.

(7) We always work over $\Lambda \geq 0$. The generators have no action but the solutions of Floer equations are weighted by their topological energy.

(8) We use Floer-Hofer’s coherent orientations to fix the signs.

(9) The generators have the $\mathbb{Z}/2$ grading given by the Lefschetz sign, and the Novikov parameter has degree 0.

Remark 3.1.1. Assume that the minimal Chern number of $M$ is $k$. Then, all our cochain complexes can be given a $\mathbb{Z}/2k\mathbb{Z}$-grading (a $\mathbb{Z}$-grading, if $k = \infty$). All the statements that we prove can be extended to take into account this grading with no extra work.

3.2. Hamiltonian Floer theory. Let $M$ be a closed symplectic manifold. Take a one periodic time-dependent Hamiltonian $H : M \times S^1 \to \mathbb{R}$ with non-degenerate one-periodic orbits $\mathcal{P}(H)$. Then, there exists choices of a compatible almost complex structure $J$, extra Pardon data $P$ (as in Definition 7.5.3 in [10]), and coherent orientations (as in Appendix C of [10]) so we can define a chain complex over $\Lambda \geq 0$ as follows:

- As a $\mathbb{Z}_2$-graded module:
  \[
  CF(H, J, P) = \bigoplus_{\gamma \in \mathcal{P}(H)} \Lambda_{\geq 0} \cdot \gamma,
  \]
  i.e. $CF(H, J, P)$ is freely generated over $\Lambda_{\geq 0}$ by the elements of $\mathcal{P}(H)$. The grading is given by the Lefschetz sign of the fixed point associated to each periodic orbit.

- We define the differential by the formula:
  \[
  d\gamma = \sum_{\gamma', A \in \pi_2(\gamma, \gamma')} \#_{\text{vir}} M(\gamma, \gamma', A, H, J, P) T^{\omega(A) + \int_{S^1} \gamma'^* H dt - \int_{S^1} \gamma^* H dt} \gamma',
  \]
  and extend it $\Lambda_{\geq 0}$-linearly. Here $\pi_2(\gamma, \gamma')$ denotes homotopy classes of maps $S^1 \times I \to M$, such that $S^1 \times \{0\} \to M$ and $S^1 \times \{1\} \to M$ are the defining parametrizations of $\gamma$ and $\gamma'$. $\#_{\text{vir}} M(\gamma, \gamma', A, H, J, P) \in \mathbb{Q}$ are virtual numbers defined as in Pardon. These are virtual counts of genus 0 nodal curves with two ordered punctures in total, where both punctures are at the same component, mapping into $M$. The component with punctures is a cylinder and the restriction of the map to it $u : \mathbb{R} \times S^1 \to M$ satisfies the equation:
  \[
  J \frac{\partial u}{\partial s} = \frac{\partial u}{\partial t} - X_H,
  \]
  with the asymptotic conditions
  \[
  u(t, s) \to \begin{cases} \gamma(t), & s \to -\infty \\ \gamma'(t), & s \to \infty \end{cases}
  \]
  The other components of the curve are $J$-holomorphic spheres. The homotopy class of the map is given by $A$. 

A square family of Hamiltonians as depicted on the left gives rise to a 2-cube of chain complexes as below. Note that the homotopy is defined by counting the accidental solutions in the one parameter family of continuation map equations.

\[
\begin{array}{c}
\text{CH}(H_{00}) \to \text{CH}(H_{10}) \\
\downarrow \downarrow \downarrow \downarrow \\
\text{CH}(H_{01}) \to \text{CH}(H_{11})
\end{array}
\]

The exponent of \( T \) in the formula, \( \omega(A) + \int_{S^1} \gamma^* H dt - \int_{S^1} \gamma^* H dt \), is the topological energy (as in 4. in subsection 3.1) of \( u \) plus the integral of \( \omega \) along the sphere components. It follows from the well-known computation presented there that each of these terms, and hence \( \omega(A) + \int_{S^1} \gamma^* H dt - \int_{S^1} \gamma^* H dt \), is always non-negative whenever \( \#^{\text{vir}} M(\gamma, \gamma', A, H, J, P) \neq 0 \).

For a more careful description of the moduli spaces involved see Definition 10.2.2 for \( n = 0 \) in \([10]\). This makes \( d \) a degree one \( \Lambda_{\geq 0} \)-module map that squares to zero.

Continuing to follow Pardon, we outline what Hamiltonian Floer theory gives us for higher dimensional families of Hamiltonians. It will be more convenient to use cubes, so we give the theory in that framework, instead of the simplices as Pardon does.

Let \( \text{Cube}_n = [0,1]^n \subset \mathbb{R}^n \). Let us consider the Morse function

\[
f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \cos(\pi x_i).
\]

Critical points of \( f \) are precisely the vertices of the cube, and its gradient vector field is tangent to all the strata of the cube.

By an \( n \)-cube of Hamiltonians, we mean a smooth map \( H : \text{Cube}_n \to C^\infty(M \times S^1, \mathbb{R}) \), which is constant on an open neighborhood of each of the vertices, and also so that the Hamiltonians at the vertices are non-degenerate. We also choose a \( \text{Cube}_n \)-family of compatible almost complex structures \( J \), Pardon data \( P \), and coherent orientations. Now, for each face \( F \) of the cube we can consider virtual counts \( \#^{\text{vir}} M(\gamma, \gamma', A, H, J, P, F) \) of Floer trajectories associated to that face, intuitively counting rigid buildings of bubbled solutions of Equation 3.2.0.3 with \((s,t)\)-dependent \( H \) and \( J \) prescribed by the gradient flow lines of \( f \) (see Figure 3 for a picture, and Definition 10.2.2 in \([10]\) for a precise definition). We again weight these counts by their topological energy.

We want to make three remarks about these virtual counts:

- If the compactified moduli space of stable Floer trajectories \( \overline{M}(\gamma, \gamma', A, H, J, F) \) (as in Definition 10.2.3 iv. of \([10]\)) is empty for some homotopy class \( A \), then the virtual count is necessarily zero. In particular, if \( \#^{\text{vir}} M(\gamma, \gamma', A, H, J, P, F) \neq 0 \)...
0, then, by the computation shown in the bullet point numbered 4. of subsection 3.1,

\[ \omega(A) + \int_{S^1} \gamma^* H' - \int_{S^1} \gamma^* H \geq \int |\frac{\partial u}{\partial s}|^2 dsdt + \int (\partial_s H_s) dsdt, \]

such that there exists a broken flow line of \( f \) in \( Cube_n \) with intermediate vertices \( v_1, \ldots, v_i \) (possibly equal to each other, \( v_{in(F)} \) or \( v_{ter(F)} \)), and \( u : \mathbb{R} \times S^1 \sqcup \ldots \sqcup \mathbb{R} \times S^1 \to M \) is a building of solutions of continuation map equations (as dictated by the broken flow line) from \( \gamma \) to \( \gamma_1 \), \( \gamma_1 \) to \( \gamma_2 \), \ldots, \( \gamma_i \) to \( \gamma' \), for some \( \gamma_i \), a one-periodic orbit of the Hamiltonian at \( v_i \). Note that we only have an inequality because we are not considering the geometric energy of the bubbles on the right hand side. We call this the **energy inequality**. We have already alluded to a special case of this inequality once in the discussion of the differential, where the second term on the right is zero.

- If a compactified moduli space of stable Floer trajectories consists of one point and that point is regular, then the virtual number associated to it is non-zero. This is a consequence of Lemma 5.2.6 of [10].
- If the virtual dimension of a moduli space is not equal to zero, then the virtual counts necessarily give zero.

The upshot for us is that these (weighted) counts fit together to give an \( n \)-cube: the chain complexes at the vertices are the Hamiltonian Floer cochain complexes; at the edges we have what is known as continuation maps; and higher dimensional faces give a hierarchy of homotopies as in the definition of an \( n \)-cube. Instead of showing this from scratch, we deduce it from Pardon’s results for simplex families in Appendix A.

![Figure 3. Picture of Floer trajectory (taken from Salamon [11])](image)

**Remark 3.2.1.** Whenever we pass from a family of Hamiltonians to a diagram of chain complexes we have to make choices of almost complex structures, Pardon data, and coherent orientations. Our final statements do not depend on these choices. In proofs and constructions all we need is their existence. We can handle these choices in two different ways (1) make a universal choice once and for all, or (2) make the choices inductively whenever you need one as in Pardon [9]. We generally suppress these choices and omit them from the labeling of the diagrams.
3.2.1. Monotone families.

Definition 4. We call an $n$-cube family of Hamiltonians monotone if the Hamiltonians are non-decreasing along all of the flow lines of $f$ (as defined in (3.2.0.5)). By the energy inequality (3.2.0.7), a monotone $n$-cube of Hamiltonians gives rise to an $n$-cube defined over $\Lambda_{\geq 0}$.

We will also use two other shapes $\text{Triangle}_n$ and $\text{Slit}_n$ which are subsets of $\text{Cube}_n$. These are used to define $n$-triangle and $n$-slit families of Hamiltonians.

We define $\text{Triangle}_2 := \{ x_1 \geq x_2 \} \subset \text{Cube}_2$ and $\text{Slit}_2$ to be the closed region that lies between the flow lines of $f$ that pass through the points $(1/3, 2/3)$ and $(2/3, 1/3)$, see Figure 4. Then we define $\text{Triangle}_n$ and $\text{Slit}_n$ by taking cartesian product with $\text{Cube}_{n-2}$. The gradient flow of $f$ is tangent to $\text{Triangle}_n$ and $\text{Slit}_n$ as well. The notion of monotonicity is defined in the same way as we did for the cube. Families of Hamiltonians parametrized by these shapes give rise to special $n$-cubes as in subsubsection 2.1.2:

- $\text{Slit}_n$ gives two $(n-2)$-cubes, two maps between them, and a homotopy between the two maps, i.e. an $n$-slit.
- $\text{Triangle}_n$ gives three $(n-2)$-cubes, three maps between them as dictated by the connections in the triangle, and a filling of the remainder of the diagram, i.e. an $n$-triangle.

3.2.2. Contractibility.

Definition 5. We define a homotopy of Hamiltonians with stations to be a map $h : I \times M \times S^1 \to \mathbb{R}$, and numbers $s_0 = 0 < s_1 < \ldots < s_k < s_{k+1} = 1$ such that the Hamiltonians $H|_{s_a}$, for $a \in \{0, s_1, \ldots, s_k, 1\} \subset I$, are non-degenerate. We say $h$ is monotone if it is increasing in the $I$-direction.

We choose non-decreasing functions $l_i : \mathbb{R} \to [s_i, s_{i+1}]$ which are equal to $s_i$, and $s_{i+1}$ near $-\infty$, and $+\infty$, respectively, for every $i$. After choosing almost complex structures this lets us write down a moduli problem for $u : \bigsqcup_{i=0}^{k} \mathbb{R} \times S^1 \to M$, where the equation corresponding to the $i$th component is the continuation map equation with Hamiltonian term given by $H_{l_i(s)}$. Therefore, a homotopy of Hamiltonians with stations (plus extra auxiliary choices as usual) define a map $CF(H|_0) \to CF(H|_1)$.
by composing $CF(H \mid 0) \to CF(H \mid s_1) \to \cdots \to CF(H \mid s_k) \to CF(H \mid 1)$. If the homotopy is monotone, the map is defined over $\Lambda_{\geq 0}$.

In the following, by a face of a simplex $\Delta^n = \{(x_1, \ldots, x_{n+1} \mid x_i \geq 0, x_1 + \cdots + x_{n+1} = 1) \subset \mathbb{R}^{n+1}$ we mean any of its subsets that can be obtained by setting a subset (possibly empty) of the coordinates to 0. A function on a closed subset of $\Delta^n$ being smooth means that it can be extended to a smooth function on a neighborhood of it inside $\mathbb{R}^{n+1}$.

**Definition 6.** We define an $n$-simplex family of homotopy of Hamiltonians with stations between $H_0$ and $H_1$ as a smooth map $H : \Delta^n \times I \times M \times S^1 \to \mathbb{R}$ such that $\{a\} \times \{0\} \times M \times S^1 \to \mathbb{R}$ is $H_0$ for all $a \in \Delta^n$, and $\{a\} \times \{1\} \times M \times S^1 \to \mathbb{R}$ is $H_1$ for all $a \in \Delta^n$. Moreover, we are given a subset $S \subset \Delta \times I$ (the stations) satisfying the conditions:

- There exists numbers $0 < s_1 \leq \ldots \leq s_k < 1$ and faces $F_1, \ldots, F_k$ of $\Delta^n$ such that $S = \bigcup_{i=1}^k F_i \times \{s_i\}$.
- There exists a neighborhood $U$ of $S \cup \Delta \times \{0\} \cup \Delta \times \{1\}$ in $\Delta^n \times I$ such that for every $x \in M$ and $t \in S^1$, $H \mid U \times \{x\} \times \{t\}$ is locally constant.

We say such family is **monotone** if it is increasing in the $I$-direction.

**Remark 3.2.2.** Note that there is a cosmetic difference between a 0-simplex family of homotopy of Hamiltonians and a homotopy of Hamiltonians (as in the first definition of this subsubsection, which we gave as a warm-up) related to how we choose to turn the data into a form that lets us write down the corresponding Floer equation (the next paragraph versus the paragraph right after Definition 5). Definition 0 is the one we use in practice.

Let us denote the coordinate in the $I$-direction by $r$. Given $n > 0$ integer and an $S$ as above, we fix a function $g_{n,S} : \Delta_n \times I \to \mathbb{R}$ such that:

- $g_{n,S} \geq 0$,
- $g_{n,S}$ vanishes precisely along $S$,
- all the integral curves of the vector field $g\theta_r$ are defined for all times $(-\infty, \infty)$.

We also want these functions to be compatible in the sense that if $F$ is a face of $\Delta_n$, $g_{n,S} \mid F = g_{\dim F, S \cap F}$. It is possible to make such choices (see the proof of Lemma 3.2.4).

Families of homotopies of Hamiltonians are then used to define homotopy coherent diagrams of maps from $CF(H \mid 0)$ to $CF(H \mid 1)$, which are defined over $\Lambda_{\geq 0}$, if the family is monotone. This follows from the gluing results of [10]. See Figure 5 for an example.

**Remark 3.2.3.** Succinctly, we defined an $(\infty,1)$ category where the objects are non-degenerate Hamiltonians, and the Hom sets are Kan complexes given by the simplex families as above. In the monotone version, Hom sets are the monotone simplex families (which might be empty of course). The following lemma says that the non-empty Hom sets are contractible in either case. Floer theory constructs an $\infty-$ functor from these categories to the $\infty$ category of chain complexes (over $\Lambda_{\geq 0}$ in the monotone case).

A family of homotopy of Hamiltonians with stations on the boundary of $\Delta_n$ is a $\Delta_{n-1}$-family of homotopy of Hamiltonians with stations defined on each $n-1$.
Figure 5. A family of Hamiltonians with stations as on the left gives rise to a diagram as on the right. Note that in the right picture there is a face in the back, and by a double edge we mean the identity map. Moreover, all faces carry homotopies, in particular the maximal dimensional face. We omit writing down the equations.

dimensional face of $\Delta_{n-1}$ so that the Hamiltonians glue together to a continuous function $\partial \Delta^n \times I \times M \times S^1 \to \mathbb{R}$ (we also have stations but no conditions on them). Note that this implies that $\partial \Delta^n \times I \times M \times S^1 \to \mathbb{R}$ is in fact smooth (this is not hard, see Lemma 16.8 of [7] for example).

Lemma 3.2.4. Any family of homotopy of Hamiltonians with stations that is defined on the boundary of a simplex can be extended to the interior of the simplex. Crucially, if the initial family is monotone, the extension can be made monotone.

Proof. First we note that we do not add more stations, so the new $S$ is simply the union of old ones considered as a subset of $\Delta_n$.

This is an application of Whitney Extension theorem [17], more accurately of the construction that is involved in proving it. We refer to Subsections VI.2.2 and VI.2.3 in [13] for the construction (i.e. Equation (8) in [13]) and its properties. The only property that the construction does not immediately satisfy is constancy near the stations. This is easy to fix. Let us call the extension so far $\tilde{F}$. We first define a function $C$ in a neighborhood $N$ of $S$ via extending by constants. Then, we take a positive partitions of unity $f_1 + f_2 = 1$, where $f_1$ is supported inside $N$, and is equal to 1 in a neighborhood of $S$. We then define our final extension to be $F = f_1 \tilde{F} + f_2 C$. It is easy to see that $F$ satisfies all the properties, including monotonicity. \[\Box\]

The upshot of this discussion informally is that any partially defined homotopy coherent diagram of maps $CF(H)$ to $CF(H')$ can be filled, even over $\Lambda_{\geq 0}$. Instead of trying explain this more rigourously, we give an example.

Example. Assume that we have Hamiltonians defined on the boundary of the $Cube_3$, which are also increasing along the vector field that we use to define monotonicity. This almost defines a 3-cube of chain complexes over $\Lambda_{\geq 0}$ except that we
do not yet have the map associated to the top dimensional face. Note that what we are given can be repackaged as a family of homotopies of Hamiltonians with stations that is defined on the boundary of a hexagon. We can now triangulate our hexagon and fill in the inside (we could directly fill the hexagon too, but we say it in this way to be consistent with the general framework). Floer theory then gives us the desired map to complete our initial diagram to a 3-cube.

3.3. Construction of the invariant.

3.3.1. Cofinality. Let $X$ be a closed smooth manifold, and $A \subset X$ be a compact subset. We define $C^\infty_{A \subset X} := \{ H \in C^\infty(X, \mathbb{R}) | H |_K < 0 \}$. Note that $C^\infty_{A \subset X}$ is a directed set, with the relation $H \geq H'$ if $H(x) \geq H'(x)$ for all $x \in X$.

**Lemma 3.3.1.** Let $H_1 \leq H_2 \leq \ldots$ be elements of $C^\infty_{A \subset X}$. They form a cofinal family if and only if $H_i(x) \to 0$, for $x \in A$, and $H_i(x) \to \infty$, for $x \in X - A$, as $i \to \infty$.

**Proof.** The only if direction is trivial, we prove the if direction. Take any $f \in C^\infty_{A \subset X}$, we need to show that there exists an $i > 0$ such that $f \leq H_i$.

By compactness (and Dini’s theorem), there is a $j > 0$ such that $f < H_j$ on $A$. But, then there has to be a neighborhood $U$ of $A$ such that $f < H_j$ on $U$.

Again, by compactness, there is a $j' > 0$ such that $f < H_{j'}$ on $X - U$. Choosing, $i = \max(j, j')$ finishes the proof.

3.3.2. Definition and basic properties. Let $M$ be a closed symplectic manifold, $K \subset M$ be a compact subset. We call the following data an acceleration data for $K$:

- $H_1 \leq H_2 \leq \ldots$ a cofinal family in $C^\infty_{K \times S^1 \subset M \times S^1}$, where $H_i$ are non-degenerate for all $i \geq 1$.

- Monotone 1-cube of Hamiltonians $\{H_s\}_{s \in [i, i+1]}$, for all $i$.

Note that acceleration data gives one $\mathbb{R}_{\geq 0}$ family of Hamiltonians, which we will denote by $H_s$. From an acceleration data, we obtain a 1-ray of chain complexes over $\Lambda_{\geq 0}$: $C(H_s) := CF(H_1) \to CF(H_2) \to \ldots$.

We define $SC_M(K, H_s) := \text{tel}(C(H_s))$.

If $H_s$ and $H'_s$ are two acceleration data for $K$ such that $H_n \geq H'_n$ for all $n \in \mathbb{N}$, we can produce a map of 1-rays $C(H'_s) \to C(H_s)$ by filling in the 2-cubes.

\[
\begin{align*}
CF(H'_1) & \to CF(H'_2) & \to CF(H'_3) & \to \ldots \\
| & \phantom{\ldots} \downarrow & \phantom{\ldots} \downarrow & \phantom{\ldots} \downarrow \phantom{\ldots} \\
CF(H_1) & \to CF(H_2) & \to CF(H_3) & \to \ldots
\end{align*}
\]

This map is unique up to homotopy of maps of 1-rays by filling in the 3-slits. Therefore we get a canonical map:

\[
H(SC_M(K, H'_s)) \to H(SC_M(K, H_s)).
\]

Moreover, if we have $H_n \geq H'_n \geq H''_n$, the canonical triangle is commutative, this time by filling in the 3-triangles.

Recall that for a 1-ray, we had the notion of a compression morphism, which induced an isomorphism after applying $H(\text{tel}(\cdot))$ (as in Section 2.2). A priori this isomorphism is not induced by Floer theory, so we need to remedy that.
Let $\mathcal{C}(H_s)^n = CF(H_{n(1)}) \to CF(H_{n(2)}) \to \ldots$ be subray. We can also think of $H_{n(1)} < H_{n(2)} < \ldots$ as part of another acceleration data $H'_s$.

**Lemma 3.3.2.**

- There is a canonical isomorphism $H(\widehat{\text{tel}}(\mathcal{C}(H_s)^n)) \to H(\widehat{\text{tel}}(\mathcal{C}(H_s)^n))$.

- The diagram commutes:

$$
\begin{array}{c}
H(\widehat{\text{tel}}(\mathcal{C}(H_s)^n)) \\
\downarrow \\
H(\widehat{\text{tel}}(\mathcal{C}(H_s)^n))
\end{array}
\xrightarrow[\text{commutes}]{H(\widehat{\text{tel}}(\mathcal{C}(H_s)^n))}
\begin{array}{c}
H(\widehat{\text{tel}}(\mathcal{C}(H'_s))) \\
\downarrow \\
H(\widehat{\text{tel}}(\mathcal{C}(H'_s)))
\end{array}
$$

**Proof.** These follow from the results of the Contractibility subsection. We omit the details.

**Proposition 3.3.3.** The comparison maps (as defined in (3.3.2.2)) $H(SC_M(K, H'_s)) \to H(SC_M(K, H_s))$ are isomorphisms.

**Proof.** We can find subsequences $n(i)$ and $m(i)$ of natural numbers such that $H'_1 < H_i < H'_n(i) < H_m(i)$. We then get three 2-rays glued to each other.

$$
\begin{array}{c}
CF(H'_1) \\
\downarrow \\
CF(H'_1)
\end{array}
\xrightarrow[\text{commutes}]{CF(H'_1)}
\begin{array}{c}
CF(H'_2) \\
\downarrow \\
CF(H'_2)
\end{array}
\xrightarrow[\text{commutes}]{CF(H'_2)}
\begin{array}{c}
CF(H'_3) \\
\downarrow \\
CF(H'_3)
\end{array}
\ldots
\begin{array}{c}
CF(H'_{n(1)}) \\
\downarrow \\
CF(H'_{n(1)})
\end{array}
\xrightarrow[\text{commutes}]{CF(H'_{n(1)})}
\begin{array}{c}
CF(H'_{n(2)}) \\
\downarrow \\
CF(H'_{n(2)})
\end{array}
\xrightarrow[\text{commutes}]{CF(H'_{n(2)})}
\begin{array}{c}
CF(H'_{n(3)}) \\
\downarrow \\
CF(H'_{n(3)})
\end{array}
\ldots
\begin{array}{c}
CF(H_{m(1)}) \\
\downarrow \\
CF(H_{m(1)})
\end{array}
\xrightarrow[\text{commutes}]{CF(H_{m(1)})}
\begin{array}{c}
CF(H_{m(2)}) \\
\downarrow \\
CF(H_{m(2)})
\end{array}
\xrightarrow[\text{commutes}]{CF(H_{m(2)})}
\begin{array}{c}
CF(H_{m(3)}) \\
\downarrow \\
CF(H_{m(3)})
\end{array}
\ldots
\end{array}
$$

Now we apply $\widehat{\text{tel}}$ to this diagram. By the previous lemma, the composition of the first and last two maps is a quasi-isomorphism. The same is true for the second and third maps for the same reason. Hence all three maps are quasi-isomorphisms.

**Proposition 3.3.4.**

1. Let $H_s$ and $H'_s$ be two different acceleration data, then $H(SC_M(K, H_s)) = H(SC_M(K, H'_s))$ canonically. Therefore we simply denote the invariant by $SH_M(K)$.

2. For $K \subset K'$, there exists canonical restriction maps $SH_M(K') \to SH_M(K)$. This satisfies the presheaf property.

3. Let $\phi : M \to M$ be a symplectomorphism. There exists a canonical isomorphism $SH_M(K) = SH_M(\phi(K))$ by relabeling an acceleration data by $\phi$.

**Proof.** To construct the maps in (1), we take acceleration data that dominates both cofinal sequence in question, and use the roof that it gives. The maps in (2), are defined exactly as the maps (3.3.2) were defined. The canonicality of maps in (1) and (2) are applications of contractibility. (3) is easy as we can relabel all choices by the symplectomorphism.
3.3.3. Computing $SH_M(M)$ and $SH_M(\emptyset)$. When $K = M$, take a $C^2$-small non-degenerate $H$ with no non-constant time-1 orbits, which is negative everywhere (see Lemma 4.2.1). We define $H_s = s^{-1}H$, for $s > 1$, as the acceleration data.

Let $CM(H)$ be the Morse complex of $H$ with $\mathbb{Z}$-coefficients. On the other hand we denote by $CM(H, \Lambda_{\geq 0})$ the complex freely generated over $\Lambda_{\geq 0}$ by the critical points, but with the terms in the differential weighed by $T^H(p_+) - T^H(p_-)$.

By Pardon [10] Theorem 10.7.1, we see that the associated diagram for this acceleration data looks like

\begin{equation}
\ldots CM(H_n, \Lambda_{\geq 0}) \to CM(H_{n+1}, \Lambda_{\geq 0}) \ldots,
\end{equation}

where a generator $p$ in the $n$th level is sent to $T^H(p)$ by the continuation map, using $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

It is easy to see that the direct limit of this diagram of chain complexes is $CM(H) \otimes_\mathbb{Z} \Lambda_{\geq 0}$ with maps

\begin{equation}
CM(H_n, \Lambda_{\geq 0}) \to CM(H) \otimes_\mathbb{Z} \Lambda_{\geq 0},
\end{equation}

sending $p$ to $T^{-H(p)}p$. Completion does nothing to $CM(H) \otimes_\mathbb{Z} \Lambda_{\geq 0}$. Using that $\Lambda_{\geq 0}$ is flat, we get the result that was stated in the Introduction:

\begin{equation}
SH_M(M) = H(M, \mathbb{Z}) \otimes_\mathbb{Z} \Lambda_{\geq 0}.
\end{equation}

For $K = \emptyset$, we start with any non-degenerate $H$, and define $H_s = H + s$. The linear diagram for this acceleration data looks like $\ldots C \to C \to \ldots$ for some chain complex $C$, where all the maps are also the same. By the energy inequality, this map sends $C$ to $TC$. Now looking at the definition of completed direct limit (using Lemma 2.3.6), we see that we are computing the inverse limit of 0-modules, which is also 0.

3.4. Multiple subsets. Let $K_1, \ldots, K_n$ be compact subsets of $M$. For every $I \subseteq [n]$, choose a cofinal sequence $H^C_k$ for $C = \bigcap_{i \in I} K_i$, such that $H^C_k \geq H^C_{k'}$ whenever $C \subset C'$. Here $I$ being the empty set means taking the union of $K_i$’s. For each $k$, extend $Cube_{n+1}$, where the vertices are given by $H^C_k$ and $H^C_{k+1}$, to a monotone $(n + 1)$-cube family of Hamiltonians. Let us call this entire $n + 1$ dimensional family of Hamiltonians $\mathcal{H}$. Using Hamiltonian Floer theory, this data gives us an $(n + 1)$-ray. The ordering of the coordinates of the slices is given by the ordering of the subsets. Here is a diagram for how this looks like for $n = 2$:

\begin{verbatim}
... \longrightarrow CH(H^K_1 \cup K_2) \longrightarrow CH(H^K_{n+1} \cup K_2) \longrightarrow ...
\end{verbatim}

\begin{verbatim}
... \longrightarrow CH(H^K_1) \longrightarrow CH(H^K_{n+1}) \longrightarrow ...
\end{verbatim}

\begin{verbatim}
... \longrightarrow CH(H^K_1 \cap K_2) \longrightarrow CH(H^K_{n+1} \cap K_2) \longrightarrow ...
\end{verbatim}

Applying $tel \circ cone^n$ to this $(n + 1)$-ray, we construct a chain complex $SC_M(K_1, \ldots, K_n, \mathcal{H})$. Note that $SC_M(K_1, \ldots, K_n, \mathcal{H})$ depends on the ordering of the subsets.
The methods of the previous subsection with higher dimensional families of cubes, triangles, and slits etc. let us show that \( SH_M(K_1, \ldots, K_n) := H(SC_M(K_1, \ldots, K_n, \mathcal{H})) \) is well defined. We omit more details.

4. Mayer Vietoris property

4.1. Zero energy solutions. Let \( X, Y \subset M \) be two compact subsets. We are going to show under certain assumptions that \( SH_M(X, Y) = 0 \), which implies a Mayer-Vietoris sequence by Lemma 2.4.2. We will do this by making a special choice of \( \mathcal{H} \) for defining \( SC_M(X, Y, \mathcal{H}) \). We explain all this from scratch in this subsection, sometimes at the expense of repeating ourselves.

We can choose acceleration data \( H_{A^s} \), for \( A = X \cap Y, X, Y, X \cup Y \), so that \( H_{A^s} \geq H_{B^s} \) whenever \( A \subset B \). We can then construct a 3-ray with \( \mathcal{C}(H_{A^s}) \) at the four infinite edges:

\[
\ldots \quad \rightarrow \quad CH(H_{n+1}^{X \cup Y}) \quad \rightarrow \quad CH(H_{n+1}^{X}) \quad \rightarrow \quad \ldots \\
\ldots \quad \rightarrow \quad CH(H_n^{X}) \quad \rightarrow \quad CH(H_{n+1}^{X}) \quad \rightarrow \quad \ldots \\
\ldots \quad \rightarrow \quad CH(H_n^{Y}) \quad \rightarrow \quad CH(H_{n+1}^{Y}) \quad \rightarrow \quad \ldots \\
\ldots \quad \rightarrow \quad CH(H_n^{X \cap Y}) \quad \rightarrow \quad CH(H_{n+1}^{X \cap Y}) \quad \rightarrow \quad \ldots 
\]

The 2-cube slices of this 3-ray look like:

\[
CH(H_n^{X \cup Y}) \quad \rightarrow \quad CH(H_n^{X})
\]

\[
CH(H_n^{Y}) \quad \rightarrow \quad CH(H_n^{X \cap Y})
\]

We want to show that we can set-up our 3-ray in such a way that all of these slices are acyclic 2-cubes. This implies that \( SH_M(X, Y) = 0 \) by Lemma 2.4.1.

Let \( h_0 \leq h_1 \) be non-degenerate Hamiltonians with a monotone homotopy \( h_s \) between them. Let \( \gamma_0 \) and \( \gamma_1 \) be one-periodic orbits of \( h_0 \) and \( h_1 \) respectively. We make more choices and define the continuation map \( con : CF(h_0) \to CF(h_1) \). We consider the matrix coefficient \( \alpha := < con(\gamma_0), \gamma_1 > \in \mathbb{A}_{\geq 0} \).

Lemma 4.1.1.

- If \( val(\alpha) = 0 \), then \( \gamma_0 = \gamma_1 \), and \( h_s \circ \gamma_0 : S^1 \to \mathbb{R} \) is independent of \( s \).
- Assume that \( h_s \) is \( C^\infty \)-wise constant along \( \gamma_0 \). Then \( \gamma_0 \) is a non-degenerate one-periodic orbit for \( h_1 \), and if we take \( \gamma_1 \) to be that, \( val(\alpha) = 0 \).

Proof. The first statement immediately follows from the energy identity (Equation (3.2.0.7) in subsubsection 3.2). For the second statement, note that \( u(s, t) = \gamma_0(t) \) satisfies the Floer equation. This solution is regular. By the energy identity, it is the only solution with zero topological energy. Moreover, it is easy to see that the compactified moduli space of (possibly bubbled and broken) stable Floer trajectories in the homotopy class of the constant solution also consists only of this solution. This proves the statement by Lemma 5.2.6 of [10].
Let $f$ and $g$ be two non-degenerate Hamiltonians. We define $U = \{ f < g \} \subset M \times S^1$ and $V = \{ f > g \} \subset M \times S^1$. The graph of $\gamma : S^1 \to M$ is the image of the map $\gamma \times id : S^1 \to M \times S^1$.

**Proposition 4.1.2.** Assume that $\bar{U}$ and $\bar{V}$ are disjoint. Then, $\max(f,g)$ and $\min(f,g)$ are smooth functions.

Moreover, assume that no one-periodic orbit of $X_f, X_g, X_{\min(f,g)}$ or $X_{\max(f,g)}$ has a graph that intersects both $U$ and $V$ (see Figure [6]). Then, $\max(f,g)$ and $\min(f,g)$ are non-degenerate, and,

$$
\begin{align*}
\text{CF}(\min(f,g)) & \to \text{CF}(f) \\
\text{CF}(g) & \to \text{CF}(\max(f,g))
\end{align*}
$$

is acyclic, for any choice of monotone 2-cube family of Hamiltonians and extra data necessary to define the maps.

**Proof.** Note that if $h = h'$ on an open set $S$, and $h_s$ is a monotone homotopy from $h$ to $h'$, then $h_s = h$ on $\bar{S}$ with all derivatives.

The first statement is elementary. Non-degeneracy of $\max(f,g)$ and $\min(f,g)$ follow from the fact that their one periodic orbits all occur also as orbits of $X_f$ or $X_g$ with the same return map, by our assumption.

The one periodic orbits of the 4 Hamiltonians in question fall under 3 groups: the ones whose graph

1. intersects $U$
2. intersects $V$
3. lies in $M \times S^1 - (U \cup V) = \{ f = g \}$

The group 3 is common to all of them. 1 of $f$ is the 1 of $\min(f,g)$; 2 of $f$ is the 2 of $\max(f,g)$; 1 of $g$ is the 1 of $\max(f,g)$ and 2 of $g$ is the 1 of $\min(f,g)$.

Now set $T = 0$ and use the previous lemma. Note that the homotopy map is necessarily zero (after $T = 0$). This is because the family of topological energy zero solutions have virtual dimension 1 and hence the corresponding virtual count is 0. One can also use the mod 2 grading to reach to the same conclusion since homotopy map is supposed to have degree 1.

This finishes the proof by Corollary 2.3.5 Part (1). \qed

**Remark 4.1.3.** In the applications below $U$ and $V$ will be of the form $\bar{U} \times S^1$ and $\bar{V} \times S^1$. Note that in that case, the condition of not intersecting both $U$ and $V$ is empty for constant orbits.

### 4.2. Boundary accelerators.

In this subsection we explain how to choose an acceleration data so that the interesting Hamiltonian dynamics concentrates near hypersurfaces that tightly envelop the compact subset in question.

**Definition 7.** Let $K$ be a subset, we say that a sequence of compact domains $D^1_K, D^2_K, \ldots$ approximate $K$ if

- $\bigcap D^i_K = K$
- $D^i_K \subset \text{int}(D^i_K)$

Note that every compact subset can be approximated by compact domains.
Definition 8. A boundary accelerator consists of three pieces of data:

1. A strictly increasing sequence of positive numbers $\Delta_i$ which converge to infinity as $i \to \infty$.
2. A sequence of triplets of compact domains $\{(\text{fill}(\partial N_i^{-}), N_i^+, \text{fill}(\partial N_i^{+}))\}_{i \in \mathbb{Z}^+}$ such that
   - $\text{fill}(\partial N_i^{-}) \cup N_i^+ \cup \text{fill}(\partial N_i^{+}) = M$.
   - The interiors of $\text{fill}(\partial N_i^{-}), N_i^+, \text{fill}(\partial N_i^{+})$ are pairwise disjoint.
   - $\partial N_i^+ = \partial N_i^{-} \cup \partial N_i^{+}$, $\text{fill}(\partial N_i^{-}) = \partial N_i^{+}$, $\text{fill}(\partial N_i^{+}) = \partial N_i^{-}$.
   - $N_i^+$ is contained in the interior of $\text{fill}(\partial N_i^{+})$.
   - $N_i^+$ is contained in the interior of $\text{fill}(\partial N_i^{+})$.

We call $N_i^+$ the mixing regions, $\text{fill}(\partial N_i^{+})$ the inner fillers, and $\text{fill}(\partial N_i^{-})$ the outer fillers.

3. Smooth functions $f_i : N_i^+ \to [0, \Delta_i]$ such that
   - $f_i$ has no critical points along $\partial N_i^+$.
   - $f_i^{-1}(0) = \partial N_i^+$ and $f_i^{-1}(\Delta_i) = \partial N_i^{+}$.

We call these the excitation functions.

See Figure 7 for a cartoon depicting the situation. We will generally drop the fillers from notation, but they are always there.

Now we explain how we get a valid acceleration data starting from a boundary accelerator. An extra property we want is to restrict the points that a non-constant periodic orbit can pass through to the mixing regions. The following lemma is our main tool in that respect.

Lemma 4.2.1. Let $H : X \to \mathbb{R}$ be a smooth function, where $X$ is a manifold with boundary and $H$ is constant along the boundary. For small enough $\epsilon > 0$, all time-1 orbits of $X_{\epsilon H}$ are constant.
Figure 7. One mixing region in a boundary accelerator, and the relevant notation.

Proof. This follows from the more general theorem of Yorke [18].

Moreover, we will need to perturb the excitation functions to have non-degenerate orbits, but we will have to perturb in a very controlled fashion. We start with a preparatory lemma.

Lemma 4.2.2. Let $F : M \times [0, 1] \rightarrow \mathbb{R}$ be a Hamiltonian, and $\gamma : [0, 1] \rightarrow M$ a flow line of $X_F$. Then, for any $t_0 \in (0, 1)$, and neighborhood $V$ of $\gamma(t_0)$; we have that for every $v \in T_{\gamma(1)}M$, there is a smooth one parameter family of functions $f_s : M \times [0, 1] \rightarrow \mathbb{R}$, $s \in [0, \tau)$, for some $\tau > 0$, such that

- $f_0 = 0$
- for some $\epsilon > 0$, $f_s(x, t) = 0$, for $t < \epsilon$ and $t > 1 - \epsilon$ and all $s$
- $f_s \geq 0$, for all $s$
- $\text{supp}(f_s) \subset V$, for all $s$
- The tangent vector to the curve $s \mapsto \phi_{F+f_s}^{1}(\gamma(0))$ at $s = 0$ is equal to $v$.

Proof. We can easily reduce to the case $\gamma([0, t_0]) \subset V$. Moreover, because $X_{F_t}$ induces a one parameter family of diffeomorphisms, we can in fact assume that the entirety of $\gamma$ is contained in $V$. Using the formula for the Hamiltonian function inducing the composition of two Hamiltonian functions, we can moreover assume that $\gamma$ is a constant orbit, and the flow of $X_{F_t}$ is identity in a neighborhood of it.

Finally, we can assume that $M = \mathbb{C}^n$, $F_t = 0$ everywhere, and $V$ is an open ball around the origin. Consider a linear function $-Jv \cdot x + c$, which is positive on $V$. By changing its support in the time domain by reparametrizing we can make it supported away from 0 and 1 while keeping it positive. Call the resulting
Hamiltonian $f$ and define $f_s = sf$. For any positive cutoff function $\beta$ supported on $V$ and equal to 1 near the origin, $\beta f_s$ does the job.

\[ \square \]

**Lemma 4.2.3.** Let $H : M \times S^1 \to \mathbb{R}$ be a Hamiltonian, $n$ be a positive integer, and $\delta > 0$. Let $U \subset M$ be an open subset, and $f$ be any strictly positive smooth function on $U$. Then there exists a $H' : M \times S^1 \to \mathbb{R}$ such that

1. $\text{supp}(H - H') \subset U \times S^1$
2. $|H' - H|_{C^n} < \delta$
3. $|H' - H| < f$
4. $H \geq H'$
5. All one-periodic orbits of $X_{H'}$ which intersect $U$ are non-degenerate.

**Proof.** Let us consider the space $F$ all nonnegative functions $M \times S^1 \to \mathbb{R}$ that are supported in $U \times S^1$, and is less than $f$, with the $C^n$ norm. This is a convex subset of a Banach space.

Consider the open subset $\mathcal{V}$ of $F \times M$ given by $(h, x)$, where the Hamiltonian flow $\phi_t^{h+H}$ of $h + H$, starting at $x$, intersects $U$. We have the map $\mathcal{V} \to M \times M$ given by $(h, x) \mapsto (x, \phi_1^{h+H}(x))$.

The Sard-Smale theory of transversality extends to this setting [1]. The previous lemma shows that this map is transverse to the diagonal in $M \times M$ in the appropriate sense, and finishes the proof. \[ \square \]

Whenever we say we make perturbations, or apply perturbation lemma, we will mean that we are using this lemma. If we want to stress that we are using the fourth bullet point, we will say that we are making monotone perturbations.

**Proposition 4.2.4.** Let $K$ be a compact subset of a closed symplectic manifold $M$, then we can find functions $h_i : M \to \mathbb{R}$, $i \in \mathbb{Z}_{>0}$ such that

1. There exists mixing regions $N^i_K$ (with fillers) and a sequence of numbers $\Delta_i$ so that $\{(h_i |_{N^i_K}, N^i_K, \Delta_i)\}$ is a boundary accelerator.
2. The critical points of $h_i$ inside the fillers are non-degenerate as time-1 orbits of $X_{h_i}$
3. All non-constant one-periodic orbits of $X_{h_i}$ are contained outside of a neighborhood of the fillers of $N^i_K$.
4. There exists a sequence of positive numbers $\delta_i \to 0$ such that $-\delta_i < h_i |_{\text{fill}(\partial N^i_K)} \leq 0$, with equality only the boundary, and for $x \in \text{fill}(\partial N^i_K)$, $h_i(x) \leq h_i(x)$.
5. $\Delta_i \leq h_i |_{\text{fill}(\partial N^i_K)} < \Delta_{i+1}$, with equality only on the boundary.

**Proof.** Using compactness, we first find approximating domains for $K$. Then using tubular neighborhood theorem, we construct boundary accelerators. Lastly, we extend the excitation functions to the fillers step by step.

1. We extend the excitation functions to the fillers so that the extension is negative in the interior of the inner filler, and it is bigger than $\Delta_i$ in the interior of the outer filler.
2. By making compactly supported perturbations in the interior of the fillers we can make the functions Morse on the fillers. Note that our perturbation theorem does not apply to this situation as we used time dependent Hamiltonians there. Nevertheless, this is standard, and we omit more details.
(3) Momentarily denote the function restricted to a small neighborhood of the inner filler by \( f \). Let \( \tilde{w} : [0, \Delta_i] \to [\epsilon, 1] \) be a non-decreasing function which is equal to \( \epsilon \) in a neighborhood of 0, and to 1 in a neighborhood of \( \Delta_i \). We then extend \( \tilde{w} \circ h_i \) to a function \( w \) on \( M \) by constants. If we multiply the function we had constructed in (2) by \( w \), it still satisfies all the previous properties, but now \( f \to \epsilon f \). By choosing \( \epsilon \) small enough we can make sure that there are no non-constant orbits contained in a neighborhood of the inner filler. We do the same for the outer filler, but this time we have to think of \( \Delta_i \) as the zero level, and hence the rescaling results in \( f \to \Delta_i + \epsilon(f - \Delta_i) \). Finally notice that choosing \( \epsilon \) small enough also achieves the extra non-degeneracy condition on the Morse critical points inside the fillers, as well as the last two conditions from the statement of the proposition.

□

Proposition 4.2.5. Let \( h_i \) be as in Proposition 4.2.4. We also fix \( n > 0 \) an integer, and \( \tau > 0 \) a real number.

We can find \( H_i : M \times S^1 \to \mathbb{R} \) such that

- \( H_i = h_i \) on the fillers.
- \( H_i(\text{int}(N_K^i)) \subset (0, \Delta_i) \).
- \( |H_i - h_i|_{C^\alpha} < \tau \).
- All one-periodic orbits of \( X_{H_i} \) are non-degenerate.

In particular, \( H_i \)'s form a non-degenerate cofinal sequence for \( K \).

Proof. We apply the perturbation lemma with \( U \) being the interior of \( N_K^i \)'s. □

See Figure 8 for a summary of this procedure that constructs a cofinal sequence (and by linear interpolation acceleration data) from boundary accelerators.

Remark 4.2.6. The main gain from this construction is that we obtained an acceleration data with no non-constant orbits outside of the mixing region while inside the mixing regions changing the excitation functions only in very controlled ways from what they were originally (for example we have not changed the level sets of the functions inside the mixing regions until the very last step). In the remaining sections, we will have to go through this construction again, trying to do it for two subsets simultaneously, while satisfying certain extra conditions related to Proposition 4.1.2. Roughly speaking, the excitation functions will satisfy these extra conditions by the assumptions, and our goal will be to not ruin it.

4.3. Non-intersecting boundaries. In this subsection we investigate the case when \( X \) and \( Y \) are two compact domains with disjoint boundaries.

Definition 9. We say that boundary accelerators \( (f_i^X, N_i^X, \Delta_i^X) \) and \( (f_i^Y, N_i^Y, \Delta_i^Y) \) are compatible if

- \( N_i^X \) and \( N_i^Y \) are disjoint
- \( \Delta_i^X = \Delta_i^Y \)

Let us define a standard neighborhood of a compact domain \( D \) to be a subset of the form \( D \cup V \) where \( V \) is a product neighborhood of \( \partial D \).

Proposition 4.3.1. We can find \( h_i^X \) and \( h_i^Y \) as in Proposition 4.2.4 such that
Figure 8. This is a summary of the construction of a cofinal sequence for $K$ via boundary accelerators. 1) Boundary accelerators, 2) Extending excitation functions to smooth functions on the entire manifold, 3) Morsifying inside the fillers without changing the function along the mixing regions, 4) Scaling the functions in a neighborhood of the fillers, so that the non-constant one-periodic orbits are forced to lie inside the mixing region, 5) Making the non-constant orbits non-degenerate (note that in reality we start using time dependent Hamiltonians at this step), 6) Two Hamiltonians constructed in this way for $K$ to illustrate how the cofinal family looks.

- The corresponding boundary accelerators are compatible
- $h^X_i = h^Y_i$ is satisfied in a compact domain.
- $\min(h^X_i, h^Y_i) \leq \min(h^X_{i+1}, h^Y_{i+1})$ on $\text{fill}(N^{(i+1)-}_X) \cap \text{fill}(N^{(i+1)-}_Y)$.

Proof. We start with any pair of compatible boundary accelerators. We extend the excitation functions to smooth functions as in Step (1) of the proof of Proposition 4.2.4 so that the extensions are the same along a compact domain $D$, which is the complement of standard neighborhood of $N^i_X \cup N^i_Y$. We now want to perturb these
to achieve Morseness (Step (2)). First, we make some common perturbation inside $D$. And then we take a smaller compact domain and separately apply monotone perturbations outside of it. We repeat this for outer fillers. All the perturbations are compactly supported and are away from the mixing regions of the boundary accelerators. Finally we make the functions very flat (Step (3)) compatibly.

As the final step, we independently apply the perturbation lemma to obtain $H^X_i$ and $H^Y_i$, again using that the mixing regions are disjoint.

**Proposition 4.3.2.**

1. $\min(H^X_i, H^Y_i)$ form a non-degenerate cofinal family for $X \cup Y$. Similarly with $\max$ for the intersection.

2. These functions can be filled into a 3-ray, which satisfies the conditions of Proposition 4.1.2.

**Proof.** We have $\min(H^X_i, H^Y_i) \leq \min(H^X_{i+1}, H^Y_{i+1})$ by construction, and cofinality follows from subsubsection 3.3.1 (same holds for $\max$). We arranged our functions so that the region of equality is of the form $D \times S$ for some domain $D$ (as in Remark 4.1.3). Notice that the mixing regions, which contain all the non-constant orbits, are disjoint from $D$. It follows that the conditions of Proposition 4.1.2 are satisfied for $H^X_i$ and $H^Y_i$. □

Therefore, we proved:

**Theorem 4.3.3.** Let $X$ and $Y$ be two compact domains such that $\partial X \cap \partial Y$ is empty. Then, we have an exact sequence:

\[ SH_M(X \cup Y) \xrightarrow{[1]} SH_M(X) \oplus SH_M(Y) \xrightarrow{\delta} SH_M(X \cap Y), \]

where the degree preserving maps are the restriction maps (up to sign).

4.4. **Barriers.** We start with an informal discussion. Let us consider the simplest example with the boundaries of two domains intersecting to explain what goes wrong for our strategy in general. Take two small disks inside a surface intersecting in the minimal way in an eye-shaped region. Now the Hamiltonians in the acceleration data coming from boundary accelerators will have periodic orbits that make circles around the boundary for all 4 subsets in question. It is clear that in this case no continuation map equation can have topological energy 0 solutions.

Continuing the informal discussion, we now motivate the definition to come in a slightly simplified setup. Let $N = Y \times [0, 1]$ be a symplectic manifold with boundary, and $f : N \to [0, 1]$ be any Hamiltonian such that $f^{-1}(0) = Y \times \{0\}$ and $f^{-1}(1) = Y \times \{1\}$. Let $D \subset Y$ be a compact domain, and consider the subset $S := D \times [0, 1] \subset N$. The boundary of $S$ has two portions: the horizontal one that overlaps with the boundary of $N$, and the vertical one coming from the boundary of $D$. We want to come up with a way to guarantee that if an orbit of $X_f$ intersects $S$ then it is contained in it. It appears as though the only feasible way to guarantee this is to assume that $X_f$ has some directionality along the vertical boundary of $S$, more precisely, that $X_f$ cannot be (strictly) inward pointing and (strictly) outward pointing at different points along the vertical boundary of $S$. Let us assume that it is never strictly outward pointing. See Figure 9 for a depiction of the situation.
Using energy conversation at the horizontal boundary, this shows that the flow of $X_f$ moves $S$ into itself. But, since Hamiltonian flows preserve volume, the only way for this happen is that $X_f$ should be everywhere tangent to the vertical boundary as well.

Note that this is a very non-generic situation. Energy levels of $f$ will generically be transverse to the vertical boundary. Elementary symplectic geometry shows that intersections of these level sets with the vertical boundary then have to be coisotropic manifolds of rank 2 (set $X$ = level set, and $Z = X \cap \partial S$ in the following lemma).

**Lemma 4.4.1.** Let $X \subset M$ be a hypersurface. Take another hypersurface $Z \subset H$. Then, the characteristic line field of $X$ is tangent to $Z$ if and only if $Z$ is a coisotropic (rank 2).

**Proof.** If the characteristic line field is tangent to $Z$, then the kernel of $\omega |_Z$ is at least one dimensional. By the classification of skew-symmetric bilinear forms this means that the kernel in question is actually at least two dimensional. By the non-degeneracy of the symplectic form on $M$, we get that $Z$ is a coisotropic.

Conversely, if $Z$ is a coisotropic, then its symplectic orthogonal distribution needs to contain the characteristic line field of $X$. This is because a linear map from a two dimensional vector space to a one dimensional one has at least one dimensional kernel. □

We repeat the definition of a barrier from the introduction in light of this discussion.

**Definition 10.** Let $Z^{2n-2}$ be a closed manifold. We define a **barrier** to be an embedding $Z \times [-\epsilon, \epsilon] \to M^{2n}$, for some $\epsilon > 0$, where $Z \times \{a\} \to M$ is a coisotropic for all $a \in [-\epsilon, \epsilon]$. We call the vector field obtained by pushing forward $\partial_\epsilon \in \Gamma(Z \times \{0\}, T(Z \times (-\epsilon, \epsilon)) |_{Z \times \{0\}})$ to $M$ the **direction** of the barrier.
Remark 4.4.2. The reader will notice that we lost some generality here. All we need from the barrier is that the Hamiltonian flow of certain functions, of which the barrier is not a level set, are tangent to it. The product decomposition into coisotropics is not necessary but has a more geometric flavor, which we find appealing. We will come back to the more general statement, which uses a more functional language, in subsection 4.8 (Theorem 4.8.1), and the proofs are all written so that no extra work is necessary for the generalization.

Now, we go back to the formal discussion.

Definition 11. We say that a Hamiltonian $f : M \to \mathbb{R}$ is compatible with a hypersurface $Y$ (possibly with boundary) if the Hamiltonian vector field $X_f$ is tangent to $Y$ and $\partial Y$.

Let $B$ be the image of a barrier $Z \times [-\epsilon, \epsilon] \to M$.

Lemma 4.4.3. Let $h : M \to \mathbb{R}$ be a Hamiltonian. If $h$ is constant along $Z \times \{a\}$ for all $a \in [-\epsilon, \epsilon]$, then $h$ is compatible with $B$.

Proof. Let $z \in Z \times \{a\}$. We know that for any vector $v$ at $z$ tangent to $Z \times \{a\}$, the directional derivative of $h$ along $v$ is zero. This is equivalent to $\omega(v, X_h(z)) = 0$. By coisotropicity, $X_h(z)$ is tangent to $Z \times \{a\}$, finishing the proof. □

In the following $B$ could be any hypersurface with boundary, but we will apply it when it is the image of a barrier, so we state it in that situation.

Lemma 4.4.4. Take an embedding $B \times [-\delta, \delta] \to M$ extending $B = \text{im}(Z \times [-\epsilon, \epsilon]) \subset M$. Let $h : M \times [0, 1] \to \mathbb{R}$ be a function, and let $\phi_h$ denote its Hamiltonian flow. There exists a $\tau > 0$ such that, if for some $B$-compatible $\tilde{h}$, $|h_t - \tilde{h}|_{C^2} < \tau$ for all $t \in [0, 1]$, then no trajectory of $\phi_h$ starting at a point on $B \times \{\pm \delta\}$ can intersect $B$ within time 1.

Proof. This is a simple application of Arzela-Ascoli theorem. Assuming the contrary, we find a contradiction to the fact the Hamiltonian flow of a $B$-compatible function is tangent to the barrier and its boundary. □

4.5. Non-degeneracy. This subsection is a long remark on why we cannot restrict ourselves to barrier compatible Hamiltonians, and can be skipped on first reading. We start with an elementary lemma.

Lemma 4.5.1. $h$ is compatible with a hypersurface $X$, if and only if it is constant along the characteristic leaves of $X$.

Proof. Let $X = \{f = 0\}$ for some function $f$ with $df \neq 0$ along $X$. $h$ is compatible with $X$ iff $df(X_h) = 0$ along $X$. Moreover, $df(X_h) = \{f, h\} = -dh(X_f) = -X_f(h)$. The claim follows. □

Using the embedding of the barrier, we can construct compatible Hamiltonians with the stronger property that they are constant along the rank 2 coisotropics making up the barrier (as in Lemma 4.4.3). The definition of compatibility does not impose this a priori, but the lemma above shows that we may be forced to it nevertheless as there might be characteristic lines of $B$, which are dense inside $Z \times \{a\}$ for almost all $a$'s.

The upshot for us is that we may not have a single compatible $M \times S^1 \to \mathbb{R}$ with non-degenerate periodic orbits. The problematic orbits are the ones that lie inside...
the barrier. In fact if \( \dim(M) \geq 8 \), one can show by a Jacobian computation that in the scenario described above with the dense characteristic lines we can never make those orbits non-degenerate for barrier compatible Hamiltonians. Fortunately, we can be a little more flexible as in Lemma 4.4.4.

4.6. The proof of the main theorem.

Definition 12. We say that a sequence of approximating domains \( D^i_X \) and \( D^i_Y \) have barriers if there are barriers \( Z^i \times [-\epsilon_i, \epsilon_i] \to M \) such that

- \( \partial D^i_X \cap \partial D^i_Y = Z^i \times \{0\} \)
- The direction of the barrier points strictly outside of \( D^i_X \) and \( D^i_Y \)

Theorem 4.6.1. Assume that \( X \) and \( Y \) admit a sequence of approximating domains with barriers. Then, we have the following exact sequence:

\[
\begin{align*}
SH_M(X \cup Y) & \longrightarrow SH_M(X) \oplus SH_M(Y), \\
\uparrow & \\
SH_M(X \cap Y) & \longrightarrow
\end{align*}
\]

where the degree preserving maps are the restriction maps (up to sign).

Our strategy is exactly the same as in the proof of Theorem 4.3.3. We will construct a cofinal sequence \( H^i_X \) and \( H^i_Y \) satisfying the conditions of Proposition 4.1.2. Of course now we have to deal with the intersection of the mixing regions using the barrier.

4.6.1. Neighborhoods of intersections of the boundary. Let \( K_1 \) and \( K_2 \) be two domains such that \( \partial K_1 \cap \partial K_2 = Z \). Let \( F : D \times Z \to M \) to be an embedding, where \( D \) is an open disk, and the map is identity at the zero subsection. We can assume that \( D \times \{z\} \) is transverse to both \( \partial K_1 \) and \( \partial K_2 \) for all \( z \in Z \), by restricting the domain of \( F \).

Making compactly supported modifications to a domain \( K \) inside \( F \) means that we find another domain \( K' \) such that outside of a compact subset of \( \text{im}(P) \) we have \( K = K' \). We will be able to apply this operation as we wish in what is to come.

We can picture \( P^{-1}(K \cap \text{im}(P)) \) as the union (over \( z \in Z \)) of regions inside \( D \times \{z\} \). Possibly restricting \( P \) to a smaller disk neighborhood we can assume that all these regions look like one side of a curve, passing through the origin, properly embedded inside the disk. By making compactly supported modifications and then restricting to a smaller neighborhood we can make all those curves linear.

Assume that our \( F \) makes the curves in the disks linear for both \( K_1 \) and \( K_2 \). Note that these lines in the disk are all oriented. We can also make the lines perpendicular to each other for the standard metric on the disk. Note that oriented lines of \( K_2 \) are obtained by making a 90 degrees rotation to the ones of \( K_1 \) along the quadrant which does not belong to either of the subsets in question. On each connected component of \( Z \), this rotation is either always positive or always negative. Hence the data of the portion of the sets inside \( F \) is equivalent to a map \( Z \to S^1 \), and a sign assigned to each connected component of \( Z \). The sign does not play a role in the following discussion.

By making compactly supported modifications and restricting domains, we can make this map \( Z \to S^1 \) any other one that is homotopic to it. Moreover, if we want
to, by reparametrizing $F$ with a fibrewise rotation diffeomorphism of $D \times Z$, we can make it nullhomotopic. Let us call such an $F$ an intersection framing.

### 4.6.2. Tangentialization

The last ingredient in the proof is a procedure we call tangentialization. See Figure 10 for a simple cartoon - we will have to be a lot more careful. We want to construct mixing regions for $X$ and $Y$ which can be rearranged to mixing regions for $X \cap Y$ and $X \cup Y$ (note though that in the end what matters is the cofinal functions we constructed and that they satisfy Proposition 4.1.2).

**Definition 13.** Let $Z \times [-\epsilon, \epsilon] \to M$ be a barrier. We call an embedding $Z \times [-\epsilon', \epsilon'] \times [-\delta, \delta] \to M$ with $\delta > 0$ and $\epsilon' > \epsilon$ extending the barrier a thickening of the barrier. Let us call the image of the barrier $B$, and the image of the thickened barrier $P$. A subset $A$ of $M$ is called barrier-friendly if the preimage of $A \cap P$ in $Z \times [-\epsilon', \epsilon'] \times [-\delta, \delta]$ is of the form $Z \times S$, where $S$ is a subset of $[-\epsilon', \epsilon'] \times [-\delta, \delta]$ for some thickening.

Let $D^X_i$ and $D^Y_i$ be a sequence of approximating domains with barriers. The upshot of the discussion in the previous subsection is that, for their defining barrier, we can assume that $D^X_i$ and $D^Y_i$ are barrier friendly and the subset of the square look as in the left picture of Figure 11 because of the outward pointing condition.

**Definition 14.** We say that the boundary accelerators $(f^X_i, N^X_i, \Delta^X_i)$ and $(f^Y_i, N^Y_i, \Delta^Y_i)$ are compatible with barriers if:

- $\Delta^X_i = \Delta^Y_i$
- $N^X_i$ and $N^Y_i$ are barrier-friendly (for the same thickening), with the subsets of the square as described in the right picture of Figure 17

To elaborate, we take a curve in $[-\epsilon', \epsilon'] \times [-\delta, \delta]$ that is the graph of a non-decreasing smooth function of $\delta$ that is equal to 0 exactly for $[-\delta/10, \delta/10]$. We take one of the subsets as the $\kappa$-neighborhood (for the standard metric) of this curve, and the other subset is obtained by reflecting along the $\epsilon'$-axis. Let us call the barrier friendly subset obtained from the rectangle that is the product of $[-\delta/10, \delta/10]$ on the $\epsilon$-axis and $[-\kappa, \kappa] \subset [-\epsilon', \epsilon']$ the plaster. Finally, $N^X_i$ and $N^Y_i$ should not intersect elsewhere.

- $f^X_i = f^Y_i$ along the plaster, and $f^X_i \neq f^Y_i$ anywhere else on $N^X_i \cap N^Y_i$.
- $f^X_i$ is compatible with the barrier $B_i$.

**Proposition 4.6.2.** We can find $h^X_i$ and $h^Y_i$ as in Proposition 4.2.4 such that:

- The corresponding boundary accelerators are compatible with barriers.
- $\min(h^X_i, h^Y_i) \leq \min(h^X_{i+1}, h^Y_{i+1})$ on $\text{fill}(N^X_{i+1}) \cap \text{fill}(N^Y_{i+1})$.
- The region where $h^X_i = h^Y_i$ contains a subset that looks like the black region from Figure 12. Let us be more precise. We push $\text{fill}(\partial N^X_i) \cap \text{fill}(\partial N^Y_i)$ inwards, hence obtaining a domain with $\text{fill}(\partial N^X_i) \cap \text{fill}(\partial N^Y_i)$ as a standard neighborhood, and similarly $\text{fill}(\partial N^X_i) \cap \text{fill}(\partial N^Y_i)$ outwards a little (so that they both still intersect the barrier). We also take a (thinner) thickening of the barrier, which in particular intersects $N^X_i$ and $N^Y_i$ only along the plaster. The union of these three regions is what the black region represents. We refer to the new (thinner) thickening as the bridge.

- The connected components of the complement of the black region fall into two groups: the ones that contain $\text{fill}(\partial N^X_i) - \text{fill}(\partial N^Y_i)$ (X-dominated), and the ones that contain $\text{fill}(\partial N^X_i) - \text{fill}(\partial N^Y_i)$ (Y-dominated). We
Figure 10. A cartoon of the tangentialization process. On the left we see a member of the approximating domains with their barrier, and on the right the mixing regions that are compatible with the barrier. Note that all the labels have an $i$ superscript which we dropped from the picture.

Figure 11. Zooming in near the barrier for barrier friendly regions at a slice in the thickening. The left picture shows the barrier compatible approximating domains after making the original ones barrier friendly by compactly supported perturbations. The right one shows the barrier compatible mixing regions and the barrier. The plaster is seen as the rectangle in black contained in the intersection of the slices of the mixing regions.

require that $h^X_i \geq h^Y_i$ on $X$ dominated components, and $h^Y_i \geq h^X_i$ on the $Y$-dominated ones.
Figure 12. The black region is a subset of the region of equivalence for the two functions we construct. One can also see the \( X \) and \( Y \)-dominated regions. Notice how the conditions of Proposition 4.1.2 are going to hold by way of restricting the non-constant orbits to lie on the mixing regions and using almost barrier compatible functions.

\textbf{Proof.} We first construct the boundary accelerators that are compatible with the barriers. We do compactly supported modifications to barrier friendly neighborhoods of \( \partial D^X_i \) and \( \partial D^Y_i \) inside the thickening, and get the mixing regions of the desired shape. We construct the excitation functions so that inside the thickening they are lifts of functions on the square.

We then extend the excitation functions to smooth functions on \( M \) as in the first bullet point of the proof of the Proposition 4.2.4 so that their regions of equivalence is a smoothing of a given black region, and moreover the domination property is satisfied. Then, we use compactly supported (monotone) Morsifications outside of the mixing regions (the black region might get slightly smaller at this step) and a compatible flating procedure to achieve what we want as before. \( \square \)

Final step is to make the Hamiltonians non-degenerate. Let us call the intersection of the bridge with the plaster \( T \), and let us also fix a slightly thinner one, and call the intersection \( T' \).

\textbf{Proposition 4.6.3.} Let \( h^X_i \) and \( h^Y_i \) be as above. We can find \( H^X_i : M \times S^1 \to \mathbb{R} \) and \( H^Y_i : M \times S^1 \to \mathbb{R} \) such that

\begin{itemize}
  \item They satisfy the conditions in Proposition 4.2.5 (with \( n = 3 \) and \( a \tau > 0 \)).
  \item \( H^X_i = H^Y_i \) along \( T' \), and the \( X \) and \( Y \) domination property still holds, outside of the new black region where \( T \) is replaced by \( T' \).
\end{itemize}

\textbf{Proof.} As before we only do perturbations that are compactly supported in the corresponding mixing regions. First make a perturbation inside \( T \) to both functions. Then do monotone perturbations separately in the complement of \( T' \) ensuring that the domination property continues to hold. \( \square \)
Finally, choose the $\tau$ so that the Lemma 4.4.4 applies with the thickening there being $T'$. This finishes the proof of Theorem 4.6.1 by Proposition 4.1.2 as before, because no periodic orbit can pass from the $X$-dominated region to the $Y$-dominated one (and vice versa).

4.7. Instances of barriers. As we have mentioned before, the outward pointing condition can be relaxed to a more cohomological condition. Namely:

**Proposition 4.7.1.** Assume that we have a sequence of approximating domains $D^i_X$ and $D^i_Y$, and barriers $Z_i \times [-\epsilon_i, \epsilon_i] \to M$ such that

- $\partial D^i_X \cap \partial D^i_Y = Z_i \times \{0\}$
- The vector field $\partial \epsilon_i$ has winding number 0 with respect to the homotopy class of of trivializations of the normal bundle of $Z_i$ induced by $D^i_X$ and $D^i_Y$.

Then there exists a sequence of approximating domains with barriers for $X$ and $Y$.

When $\text{dim}(M) = 2$ the barrier condition can be satisfied only when the boundaries of the approximating domains do not intersect. For $\text{dim}(M) = 4$, the cohomological condition becomes of importance.

**Lemma 4.7.2.** Consider the standard neighborhood of a Lagrangian torus $T^2 \times \mathbb{R}^2, \omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$, where we think of $T^2 = \mathbb{R}^2/\sim$. If $T^2 \to T^2 \times \mathbb{R}^2$ is a Lagrangian subsection, which is nowhere zero, then the map $T^2 \to \mathbb{R}^2 - \{0\} \to S^1$ is nullhomotopic.

**Proof.** Such Lagrangian sections correspond to closed 1-forms on $T^2$. Any nowhere vanishing 1-form $\alpha$ on $T^2$ would define a map $T^2 \to \mathbb{R}^2 - \{0\} \to S^1$, and we can talk about its homotopy class $h_\alpha$. Notice that $h_\alpha$ only depends on the cooriented foliation given by $\alpha$. More precisely, we fix an orientation of $T^2$ and hence a coorientation of the foliation induces an orientation. We also fix a trivialization of $TT^2$ given by the coordinates we used in the statement of the Lemma. Then to any embedded loop $S^1 \to T^2$ we can assign a number that is the winding of the oriented line field given by the foliation w.r.t to the trivialization of the tangent bundle. This number is the same for homotopic loops, and the assignment determines the homotopy class $h_\alpha$ in question. In particular, if we can show that, for $d\alpha = 0$, the number associated to two non-homotopic non-contractible embedded loops are 0, we will be done.

By an elementary result of Tischler ([6] Theorem 29, which follows from the proof of [13] Theorem 1), we can find a submersion $\theta : X \to S^1$ such that the foliation given by the fibers is arbitrarily close to the foliation defined by $\alpha$. Hence, we are reduced to showing the statement for $\alpha = d\theta$. Notice that we can find an embedded loop that is transverse to all the fibers of $\theta$. If we can show that the winding number of the fiber loops and the transverse loop are both zero, we will be done.

First note that any homotopically non-trivial embedded loop on our torus can be isotoped through embedded loops into a linear loop. This can be shown by unfolding the given loop to $\mathbb{R}^2$. We draw the straight line between its endpoints, and by a small isotopy make our curve transverse to the straight line. Then we cancel intersections between the two curves by isotoping our (curvy) curve along ribbons, using the Schoenflies theorem. We finish by Schoenflies theorem again.
This shows that the winding number of the tangent lines of any homotopically non-trivial embedded loop is zero. Applying this statement to the fiber and transverse loops finishes the proof. □

**Remark 4.7.3.** Note that if the subsection is not required to be Lagrangian, meaning that \( \alpha \) is not necessarily closed, we can realize all homotopy classes of maps \( T^2 \to S^1 \) by inserting Reeb components. Our proof above is basically showing that when \( \alpha \) is closed there can be no Reeb components in the foliation.

**Corollary 4.7.4.** Let \( D \) and \( D' \) be two domains with transversely intersecting boundaries along a disjoint union of Lagrangian tori \( L \). Then, \( L \) can be extended to an outward pointing barrier if and only if the intersection (as in subsubsection 4.6.1) and Lagrangian (see [4] for the simple definition) framings of \( L \) agree.

4.8. **Involutive systems.** Recall the following definition from the introduction.

**Definition 15.** We say that compact subsets \( K_1, K_2, \ldots, K_n \) of \( M \) satisfy descent, if \( SC_M(K_1, \ldots, K_n) \) is acyclic.

4.8.1. **A slight generalization of the main theorem.**

**Theorem 4.8.1.** Let \( f^X_i : M \to \mathbb{R} \) and \( f^Y_i : M \to \mathbb{R} \) be smooth functions such that

1. \((f^X_i)^{-1}((\infty, 0])\) and \((f^Y_i)^{-1}((\infty, 0])\) approximate \( X \) and \( Y \) respectively
2. Let \( f := (f^X_i, f^Y_i) : M \to \mathbb{R}^2 \). There exists a smooth curve \( C \) passing through the origin once and intersecting only the first and third quadrants such that

\[
\{f^X_i, f^Y_i\} \mid_{f^{-1}(C)} = 0.
\]

Note that this condition is automatically satisfied if \( f^{-1}(C) = \emptyset \). Then \( X \) and \( Y \) satisfy descent.

The proof of this version is absolutely the same. \( f^{-1}(C) \) plays the role of a barrier. It is a little more general in that it admits a map \( f^{-1}(C) \to \mathbb{R} \) with coisotropic fibres, but the fibres are possibly singular. We draw the pictures that we were drawing in the \( \partial \phi \) plane before, for example near the barrier, in the target plane of the map \( f \) near the origin (Figure 13). In this framework, we can see the entirety of \( M \) and the subsets in our pictures, which is nice. We make the subsets tangential by making the subsets inside \( \mathbb{R}^2 \) tangential near the origin, tangent direction being transverse to \( C \). We construct the excitation functions as functions of \( f^X_i \) and \( f^Y_i \). Such functions are all compatible with \( f^{-1}(C) \), because of the following lemma (we are using \( k = 2 \) only here).

**Lemma 4.8.2.** Let \( f_1, \ldots, f_k : M \to \mathbb{R} \), and \( g_1, g_2 : \mathbb{R}^k \to \mathbb{R} \) be smooth functions. Assume that \( \{f_i, f_j\} = 0 \) at \( x \in M \), for all \( i, j \). Then the functions \( G_i : M \to \mathbb{R} \), \( i = 1, 2 \), defined by \( x \mapsto g_i(f_1(x), \ldots, f_k(x)) \) also satisfy \( \{G_1, G_2\} = 0 \) at \( x \in M \).

**Proof.** We have that \( \{h, h'\} = \omega(X_h, X_{h'}) \). Moreover, \( dG_i \) is a \( C^\infty \) linear combination of \( df_1, \ldots, df_k \), and hence \( X_{G_i} \) is the same linear combination of \( X_{f_1}, \ldots, X_{f_k} \). This finishes the proof. □

We are able to satisfy the regularity conditions that are required from the excitation functions at the boundary of mixing regions by Sard’s lemma. The construction proceeds as before.
4.8.2. Descent for symplectic manifolds with involutive structure.

Definition 16. An involutive map is a smooth map $\pi : M \to B$ to a smooth manifold $B$, such that for any $f, g \in C^\infty(B)$, we have $\{f \circ \pi, g \circ \pi\} = 0$.

Theorem 4.8.3. Let $X_1, \ldots, X_n$ be closed subsets of $B$. Then $\pi^{-1}(X_1), \ldots, \pi^{-1}(X_n)$ satisfy descent.

Proof. It suffices to show this for $n = 2$ (see Appendix B for the easy inductive argument). In that case, we have already proved a stronger version in Theorem 4.8.1, as we can use functions on $B$ to get the sequences of functions in Theorem 4.8.1.

Remark 4.8.4. For multiple subsets, there is a more optimal theorem we could have proved. First of all, note that for $n > 2$, domains being pairwise equipped with barriers (generalized or not) is not enough to conclude that the $n$ subsets satisfy descent. Let us stick to $n = 3$ for simplicity. Having a barrier between $D_1$ and $D_2$, and $D_1$ and $D_3$ does not imply a priori that there is a barrier between $D_1$ and $D_2 \cup D_3$. Apart from the non-matching problem at the triple intersection at the boundary, there can also be no guaranteed way of gluing the barriers together. This is because of the outward pointing condition near the triple intersection that is essential. In this case, it would be enough to assume that the three functions in question all pairwise commute in a neighborhood of the triple intersection of the boundaries of the domains. Currently, such generalizations seem to be useless.
APPENDIX A. CUBICAL DIAGRAMS FROM SIMPLICIAL ONES

We show that $n$-cube families of Hamiltonians give $n$-cubes using Pardon’s results on simplex families. The main challenge here is to show that the signs work out correctly.

Let $Cube = [0, 1]^n$, with an ordering of its coordinates. We can cover it by $n!$ simplices, one for each permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$. We can think of such a permutation as a path that starts at $(0, \ldots, 0)$ and takes a unit step in the positive $i_k$ direction at time $k = 1, \ldots, n$, and ends up at $(1, \ldots, 1)$. The corresponding simplex $\Delta^n \to Cube$ is the linear map that sends the $i$th vertex of the simplex to the $i$th vertex we encounter on this path. Note also that the function $f$ (as in 3.2.0.5) is tangent to all the faces of all these simplices.

Now let $H$ be a $Cube$ family of Hamiltonians. Let $F$ be a $k$-dimensional face of the $Cube$. $F$ itself is a cube with an induced ordering of its coordinates. By the above procedure it can be covered by $k!$ simplices. We get a map $C_{\nu, n} \to C_{\nu, ter}$ for each of these simplices by restricting the family of Hamiltonians. We define

$$f_F = \sum_{k! \text{ simplices}} (-1)^{\text{sign}} f_{(j_1, \ldots, j_k)},$$

where $(j_1, \ldots, j_k)$ is the permutation corresponding to the simplex, and sign is given by its signature. We claim that these define a cubical diagram. We need to show that the Equation (2.1.1.1) from subsubsection 2.1.1:

$$\sum_{F' \neq F'' \text{ is a bdry of } F} (-1)^{*_{F', F}} f_{F'} f_{F''} = 0,$$

is satisfied for each face $F$. Recall that $*_{F', F} = \#_v 1 + \#_v 01$ for $v = \nu_{\text{ter}} F' - \nu_{\text{in}} F'$ considered as a vector inside $F$. Without loss of generality, we will show the one for the top dimensional face.

By Pardon (Equation (7.6.5)), we get $n!$ equations of the form below for the top dimensional face of each of the simplices in the cover.

$$\sum_k (-1)^{k+1} g_{(1,\ldots,k)} g_{(k,\ldots,n)} + \sum (-1)^k g_{(1,\ldots,k,\ldots,n)} = 0$$

We add all of these equations up after multiplying them with $(-1)^{\text{sign}}$, where sign is again the signature of the permutation corresponding to the simplex. The second group of terms cancel out because the signature of a permutation changes after one transposition. Using the description of the signature of a permutation via the number of inversions, we see that we get exactly the equation we wanted from the first group of terms.
Let $K_1, \ldots, K_n$ be compact subsets of $M$. Let $\mathcal{K}$ be smallest set of subsets of $M$, which is closed under intersection and union, and contains $K_1, \ldots, K_n$.

Assume that for any $X, Y \in \mathcal{K}$, $SH_M(X, Y) = 0$. Then, we want to show that for any $X_1, \ldots, X_l \in \mathcal{K}$, $SH_M(X_1, \ldots, X_l) = 0$. We do this by induction. Assume that it holds for $l - 1 \geq 2$.

By the descent for two subsets we have that the natural map

$$\text{cone}(SC_M(X_1) \oplus SC_M(X_2 \cup \ldots \cup X_l) \to SC_M((X_2 \cap X_1) \cup \ldots \cup (X_l \cap X_1)))$$

is a quasi-isomorphism.

We also have homotopy commutative diagrams:

$$\begin{align*}
SC_M(X_2 \cup \ldots \cup X_l) & \longrightarrow SC_M((X_2 \cap X_1) \cup \ldots \cup (X_l \cap X_1)) \\
\bigoplus_{0 \neq I \subset \{2, \ldots, l\}} SC_M(\bigcap_{i \in I} X_i) & \longrightarrow \bigoplus_{0 \neq I \subset \{2, \ldots, l\}} SC_M(\bigcap_{i \in I} (X_i \cap X_1)),
\end{align*}$$

and

$$\begin{align*}
SC_M(X_1) & \longrightarrow SC_M((X_2 \cap X_1) \cup \ldots \cup (X_l \cap X_1)) \\
\bigoplus_{0 \neq I \subset \{2, \ldots, l\}} SC_M(\bigcap_{i \in I} (X_i \cap X_1)) & \longrightarrow \bigoplus_{0 \neq I \subset \{2, \ldots, l\}} SC_M(\bigcap_{i \in I} X_i),
\end{align*}$$

In these two diagrams, by the direct sum we mean the homotopy colimit of the corresponding homotopy coherent diagram. By the induction hypothesis all the vertical arrows are quasi-isomorphisms.

By piecing together these diagrams, we see that the cone in (B.0.0.1) is quasi-isomorphic to $\bigoplus_{0 \neq I \subset \{1, \ldots, l\}} SC_M\left(\bigcap_{i \in I} X_i\right)$ in a way that is compatible with the maps that they receive from $SC_M(X_1 \cup \ldots \cup X_l)$. This finishes the proof.

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