Flexible multibody system modelling of an aerial rescue ladder using Lagrange’s equations

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1. Introduction

Fire departments throughout the world use aerial rescue ladders, for example shown in Figure 1, as an important tool in fulfilling their duties.

A rescue ladder typically consists of a telescopic boom mounted on a truck chassis such that its inclination and orientation about the vertical axis can be changed. A rescue cage attached to the tip of the boom serves as working platform. Some types of aerial ladders feature an additional articulated arm between boom and rescue cage which can be rotated downwards, in order to reach a target behind a roof or any other obstacle. Unlike other similar apparatus, such as cherry picker working platforms or truck mounted cranes, an aerial rescue ladder consists of u-shaped truss structure segments providing an ascendable path from base to tip or rescue cage. To increase the reach and to comply with the legal weight limitation of trucks, a lightweight boom construction is realized. Like all large-scale manipulators, an aerial ladder is susceptible to structural oscillations induced by external disturbances, e.g. wind forces, or due to motion of the manipulator itself. To increase precision, speed and comfort during operation, active oscillation damping techniques are used [1,2] which require a dynamic model of the system. An accurate model highly simplifies the controller design especially for diverse ladder
portfolios. Additionally, an exact model of the ladder dynamics can be used as one module for a digital twin of the complete aerial ladder truck.

Modelling of large-scale flexible robots is in active research since several decades. A brief overview of present literature and approaches is given by [3]. Rigid body systems with virtual joints are used by [1,4] to model the flexible structure of an aerial ladder or other large-scale manipulators [5,6]. The advantage of this approach is a resulting low dimensional system model which can directly be used for controller design. Strain gauges and a gyroscope are used as sensors for the structural dynamics of the ladder. The rigid system model with virtual joints simplifies this dynamics. Due to this simplification, a physical motivated sensor equation based on the system states is not straightforward or even impossible to formulate without experimental data.

A direct formulation of sensor output equations based on system states is possible using distributed parameter models as done by [2,7,8] for aerial ladders. Articulated arm and rescue cage are thereby seen as rigid bodies. The telescopic boom is modelled as one single Euler-Bernoulli beam with piecewise constant parameters. A modal transformation leads to a finite dimensional system approximation. This approach implies a rigid connection between all overlapping ladder segments, which is not true for the real world manipulator. While these modelling approaches are appealing in the sense of simplicity or the available academic tools for analysis, they do not accurately represent the real

Figure 1. A Magirus aerial ladder used as application example. The ladder consists of a telescopic main boom and an articulated arm.
world dynamics. This can be seen best for the example of a fully retracted ladder as displayed in Figure 2. The stepped beam model becomes a single Clamped-Pinned beam with very high stiffness $EI$ which is the sum of the stiffness of all ladder segments resulting in high eigenfrequencies. But one can observe during operation or easily imagine that the very inner ladder segment of the real manipulator oscillates similar to a Pinned-Pinned beam, as shown in Figure 2a, with the comparable lower stiffness $^1EI$ of the inner ladder segment resulting in a much lower eigenfrequency. Thus, the stepped beam model is not valid especially for short ladder lengths.

This contribution aims to increase the modelling accuracy by a more precise representation of the interconnection between the telescopic segments. Therefore, the aerial ladder will be modelled as flexible multibody system (FMBS). There is an extensive body on literature on FMBS. The Newton-Euler approach to compute the system dynamics is used by [9–12] and many others. While being computationally efficient, the downside is a comparably complex formulation of the FMBS [13].

A more simple and systematic formulation is possible using the Lagrange approach as done by [14–19]. An advantage is the description of the system solely by its position and orientation kinematics. However, one disadvantage is that the computational effort needed to solve the Lagrange equation increases exponentially and limits the number of used flexible elements. Due to the computational effort, this approach has only been applied to manipulators with few flexible links or a limited number of eigenmodes [14–17, 20].

A quantitatively exact model of the dynamics of the turntable rescue ladder using the Lagrange approach and mainly based on CAD parameters is presented in this publication. Additionally, the computational effort is drastically decreased. As a proof of concept and to present the main ideas of the modelling algorithm, the vertical dynamics of the aerial rescue ladder are modelled using the Lagrange approach. Even though only vertical dynamics are considered, the manipulator is modelled in 3-dimensional space to allow straightforward modelling of the horizontal dynamics in future applications. The resulting example model is consisting of five flexible segments each using an arbitrary number of flexible modes. Interconnections are formulated as a physically motivated connection at two discrete points which is a more exact representation of the real world interconnection than the currently used stepped beam approach. Kinematic loops occurring due to two discrete points of interconnection between each telescopic element are implicitly solved by the chosen assumed modes. The modelling algorithm is split into two parts: Integrals occurring during the formulation of the system are solved for a general flexible segment by using a computer algebra system. Subsequently, the Lagrange equation is used to derive the system dynamics with automatic
differentiation software [21]. Since the manipulator is described only by its position and orientation kinematics and main mechanical properties like mass or stiffness, adaption to a different apparatus is easy and straightforward. Linearized system dynamics are extracted directly and efficiently and can be used for control design. Due to the efficient computing structure, derivation of the linearized system dynamics for different working points and poses is even possible online on embedded systems. Validation with real world system experimental data confirms the high model accuracy.

The paper is organized as follows: Section 2 introduces the kinematics of the used manipulator and of the flexible elements. The Lagrange’s equation is shortly recapitulated in Section 3 and the corresponding energy expressions are given in Section 4. Damping effects are modelled in Section 5 using the Rayleigh damping model. To be able to compare the resulting model with real world testing data in Section 8, the sensor equations are given in Section 6 and the modal transformation is briefly recapitulated in Section 7. A concluding discussion is given in Section 9.

2. Kinematic model

For the chosen modelling approach, position and orientation kinematics mainly characterize the used manipulator and determine the position and orientation of each manipulator mass element. As shown in Figure 3, each ladder segment i features a local right-handed coordinate system $\Sigma_i$. The inertial frame is denoted by $\Sigma_i^I$ with its origin in vertical rotational axis of the turntable on the ground surface. Note that the vertical axis of each frame is the z-axis to comply with the coordinate system for the inertial frame used by the manufacturer of the manipulator. This results in actuated angular coordinates $\theta_2$ and $\theta_3$ which are defined in negative direction of the right hand side coordinate system. Throughout this paper, the notation follows the Nomenclature/Notation described in the appendix. Results obtained in this section are used in the following to formulate the energy expressions.

2.1. General structure

The considered aerial ladder manipulator is shown in Figure 3. It consists of five truss segments of which the first four build a telescopic structure. Actuated coordinates $\theta$ are the extraction length $\theta_1$, the elevation angle $\theta_2$ of the main boom and the angle of the articulated arm $\theta_3$. A rotation about the vertical $z^I$-axis is possible for the real manipulator but not in the scope of this paper.

All actuated coordinates are driven by load-sensing hydraulics. Hydraulic systems with load-sensing include an internal control loop for the hydraulic flow to the actuators [22]. It is assumed, that the load-sensing system dynamics are much faster than the manipulator dynamics. Because of this assumption any load-sensing system dynamics are neglected. For the used hydraulic actors there is a direct relationship between the hydraulic flow and the actor velocity.

The actuator acceleration $\ddot{\theta}$ is seen as model input signal. To transform the actuator acceleration signal to a valid hydraulic input signal for the real world manipulator, the acceleration is integrated over time to get the resulting velocity. Using the relationship between velocity and hydraulic flow, the hydraulic setpoint can be computed.
In fully retracted pose, the articulated arm, segment five, is completely inside the telescopic structure and thus $\theta_3$, as shown in Figure 4, has to be zero. To be able to articulate the last segment as soon as possible, telescoping extracts segment four at first. If segment four cannot be extracted anymore, segments two and three start proportional extraction. Therefore,

$$l_{\text{ext}} = \begin{bmatrix} l_{\text{ext}}^2 \\ l_{\text{ext}}^3 \\ l_{\text{ext}}^4 \end{bmatrix} = \begin{bmatrix} \max(0, (\theta_1 - l_{\text{ext, max}}^4)/2) \\ \max(0, (\theta_1 - l_{\text{ext, max}}^4)/2) \\ \min(l_{\text{ext, max}}^4, \theta_1) \end{bmatrix} \tag{1}$$

with $l_{\text{ext}}^i$ the extraction of the $i^{\text{th}}$ segment and $l_{\text{ext, max}}^4$ the maximum possible extraction length of the fourth segment.

The rescue cage at the tip and the cable winch at the lower end, which is used for extraction and retraction of the telescopic ladder segments, are modelled as a rigid body with a discrete mass $m_W$, $m_C$ and moment of inertia $J_C$. Segments two to five are modelled as flexible elements. The first segment is a series connection of a rigid beam, displayed as the thicker lower end in Figure 4, and a flexible beam element. The lower end is seen as rigid since this part is restrained to the rigid turntable.

The vehicle chassis and hydraulic cylinder dynamics are approximated by a spring-damper system.

Figure 3. Manipulator of aerial ladder with used frames $\Sigma$ and actuated coordinates $\theta$. $l_{\text{ext}}^2$ to $l_{\text{ext}}^4$ are the extractions lengths for segments two to four, summing up to the overall extraction length $\theta_1$. Articulation $\theta_3$ is possible in downwards direction between $0^\circ$ and $-75^\circ$. The ladder segments form an upward curved arc of a very large circle in unencumbered condition. Segment three is analogue to segment two and thus omitted in this figure for the sake of simplicity.
2.2. Segment shape and deformation

By design, the ladder segments form an upward curved arc of a very large circle in unencumbered condition. With increasing payload, the segments get straightened, but the direction of static bending does not change. Due to this shape, passengers standing in the rescue cage are always able to see the complete ladder set which is psychologically advantageous.

With any deformation \( w^i(x, t) \), the position of mass elements along the segment \( i \) can be expressed as

\[
{^i}\mathbf{r}(x) = \begin{bmatrix} x \\ 0 \\ (x^2)/2R \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ w^i(x, t) \end{bmatrix}
\]

where \( R \) is the radius of the used circle, \( (x^2)/2R \) an approximation of the circle segment and \( x \) the material point \( x \)-axis position in \( \Sigma^j \) without deformations. \( (x^2)/2R \) results from the Taylor series expansion of the expression for a quarter circle

\[
z(x) = -\sqrt{R^2 - (x)^2} + R = 0 + \frac{(x^2)}{2R} + O((x)^4)
\]

neglecting higher order terms.

In order to obtain a finite approximation of the deformation \( w^i(x, t) \), the method of Ritz series expansion is used [23]. The deformation is approximated by a finite series expansion of the form

\[
w^i(x, t) = \sum_{j=1}^{N} \psi^i_j(x)q^i_j(t) = \psi^i(x)^Tq^i(t), \quad N \in \mathbb{N}
\]
with basis functions $\psi^i(x)$, $N$ the number of used basis functions and coordinates $q^i(t)$. The basis functions must be linearly independent and comply with the essential boundary conditions of the considered system. Additionally, for $N \to \infty$, $\psi^i(x)$ must form a complete set of continuous functions. With increasing number $N$ of used basis functions, the approximation will improve. Combining Equation (2) and Equation (4) yields

$$i\mathbf{r}^i(x) = \begin{bmatrix} x \\ 0 \\ \frac{z^2}{2R} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ q^i(t)^T \end{bmatrix} \psi^i(x)$$

$$= b^i(x) + Q^i(t)\psi^i(x). \quad (5)$$

In this paper, the eigenmodes of Euler-Bernoulli beams are used as basis functions $\psi^i(x)$. The main assumptions of Euler-Bernoulli beam theory are small deflection, always orthogonal cross-sections in relation to the neutral axis of the beam and a high length-to-thickness ratio [24]. Since the ladder segments have a slender shape and the deflection during normal operation is relatively small, this beam model is seen as valid. The eigenmodes comply with the above stated requirements and are orthonormal. For the upper part of segment 1, Clamped-Free eigenmodes [9] as shown in Figure 5 are used.

Pinned-Pinned-Free eigenmodes [25,26], an example is given in Figure 6, are chosen for Segments 2 to 4 due to the type of interconnection with the other segments. Beams in Pinned-Pinned-Free configuration have an additional inner pinned support which prevents lateral movement. The position for the inner pinned support is chosen such that it coincides with the connection point to the underlying beam. Segment 5 is modelled by Clamped-Mass eigenmodes since the cage mass and inertia are known.

Clamped-Free and Clamped-Mass basis functions are of the form

$$\psi^i_j(x) = p_{1,j} \sin(\kappa_j x) + p_{2,j} \cos(\kappa_j x) + p_{3,j} \sinh(\kappa_j x) + p_{4,j} \cosh(\kappa_j x) \quad (7)$$

while Pinned-Pinned-Free modes are of the form

$$\psi^i_j(x) = \begin{cases} p_{1,j} \sin(\kappa_i \bar{x}) + p_{2,j} \cos(\kappa_i \bar{x}) + p_{3,j} \sinh(\kappa_i \bar{x}) + p_{4,j} \cosh(\kappa_i \bar{x}), & x \in [0, x_p] \\ p_{5,j} \sin(\kappa_i \bar{x}) + p_{6,j} \cos(\kappa_i \bar{x}) + p_{7,j} \sinh(\kappa_i \bar{x}) + p_{8,j} \cosh(\kappa_i \bar{x}), & x \in (x_p, L) \end{cases} \quad (8)$$

\[ \text{Figure 5. First three eigenmodes for a Clamped-Free normalized length Euler-Bernoulli beam.} \]
with the eigenvalue $\kappa_i \in \mathbb{R}$, parameters $p_{i,j} \in \mathbb{R}$, $x = \tilde{x} - x_p$ and $x_p$ the local position of the inner pinned constraint. The eigenvalue $\kappa_i$ depends on the mechanical characteristics and the position of the inner pinned constraint. It is the numerical determined root of the characteristic expression of the beam. With a known $\kappa_i$, the parameters $p_{i,j}$ are explicitly defined.

The later introduced energy and dissipation expressions for the Lagrange approach include integrals over the position $i\mathbf{r}^i$ and its time derivatives. The integrals are of the form

$$
\int_0^{L_i} i\mathbf{r}^i(x)\mathbf{r}^i(x)^\top \, dx
= Q^i(t) \left( \int_0^{L_i} \psi^i(x)\psi^i(x)^\top \, dx \right) Q^i(t)^\top \ldots
+ Q^i(t) \left( \int_0^{L_i} \mathbf{b}^i(x)\mathbf{b}^i(x)^\top \, dx \right) \mathbf{Q}^i(t)^\top \ldots
+ \int_0^{L_i} \mathbf{b}^i(x)\mathbf{b}^i(x)^\top \, dx
\quad \text{(9)}
$$

and

$$
\int_0^{L_i} \frac{d^k}{dt^k} i\mathbf{r}^i(x)\mathbf{r}^i(x)^\top \, dx, \quad k \in \mathbb{N}
= \frac{d^k}{dt^k} Q^i(t) \left( \int_0^{L_i} \psi^i(x)\psi^i(x)^\top \, dx \right) Q^i(t)^\top + \frac{d^k}{dt^k} \left( \int_0^{L_i} \psi^i(x)\mathbf{b}^i(x)^\top \, dx \right). \quad \text{(10)}
$$

Due to the orthonormal basis functions $\psi^i(x)$, further simplification is achieved by

$$
\int_0^{L_i} \psi^i(x)\psi^i(x)^\top dx = I
\quad \text{(11)}
$$
with \( I \) denoting the identity matrix.

Similar holds for integrals over spatial derivatives of the eigenmodes \( \psi^j \) like

\[
\int_0^{L^j} \frac{\partial^k \psi^j(x)}{\partial x^k} \left( \frac{\partial^k \psi^j(x)}{\partial x^k} \right)^T \, dx = \begin{bmatrix} *_1 & 0 \\ \vdots & \ddots \\ 0 & *_n \end{bmatrix}
\]

(12)

\[
*_{n} = \int_0^{L^j} \left( \frac{\partial^k \psi^j_n(x)}{\partial x^k} \right)^2 \, dx.
\]

(13)

In summary, due to the chosen kind of eigenmodes and structure of expression (6), all of the above integrals can be simplified to scalar integrals, which are rapidly solved by computer algebra.

### 2.3. Interconnections

As shown in Figure 4, the segments forming the telescopic part of the manipulator are interconnected by two comparably small points. Photographs of the connection points are displayed in Figure 7: a slide bearing at the lower end of the segment and roller bearing at the upper end of the underlying segment. In standard modelling, this kind of connection results in a kinematic loop that needs to be solved. The chosen Eigenmodes implicitly incorporate this loop.

To compute the angle \( \phi^{i+1} \) around the \( y^i \)-axis between frame \( \Sigma^i \) and \( \Sigma^{i+1} \) for \( i \in [1, 3] \) one can exploit the fact that all segments have the same shape in unencumbered condition and the segment \( i+1 \) does not deflect in any direction at the interconnection point. The connection is illustrated for segment one to two in Figure 8 as an example. Without any deformation, i.e. \( q = 0 \), the direction \( \frac{\partial r^i(x)}{\partial x} \) of segment one at \( l^2_{\text{ext}} \) must be identical to the direction \( \frac{\partial r^2(x)}{\partial x} \) of segment two at \( x = 0 \) with \( x \) in \( \Sigma^2 \) coordinates. Mathematically speaking,
\[
\begin{aligned}
\frac{\partial i \dot{r}^i(x)}{\partial x} \bigg|_{q=0}^{x=i+1} &= \frac{1}{c} i R^{i+1}(i, \phi^{i+1}) \bigg|_{q=0}^{x=i+1} = c \begin{bmatrix}
\cos(i, \phi^{i+1}) \\
0 \\
-\sin(i, \phi^{i+1})
\end{bmatrix}, c \in \mathbb{R},
\end{aligned}
\]

Adding the angle arising from deflection \( w^i \), the resulting expression for the angle \( i \phi^{i+1} \) is
\[
\begin{aligned}
i \phi^{i+1} &= -\arctan \left( \frac{l_{i+1}^{\text{ext}}}{R} \right) - \arctan \left( \frac{w^i(L^i, t) - w^i(L^{i+1}, t)}{L^i - l_{i+1}^{\text{ext}}} \right).
\end{aligned}
\]

Segment 5 is connected to the tip of segment 4. The angle around the \( y^4 \)-axis between both systems is given by
\[
\begin{aligned}
4 \phi^5 &= -\arctan \left( \frac{\partial 4 r^4_2(x)}{\partial x} \bigg|_{x=L^4} \right) - \theta_3.
\end{aligned}
\]

### 2.4. Transformations

Positions can be transformed between all frames by the use of homogeneous transformation matrices that include rotational and translational transformation [27]. A position \( i \dot{r} \) given in coordinates of system \( \Sigma^i \) can be transformed to coordinates of the inertial frame \( \Sigma^I \) by
\[
\begin{aligned}
i \dot{r} &= \begin{bmatrix}
i R^i(q, \theta) \\
i r_0^i(q, \theta) \\
1
\end{bmatrix}, i \dot{r} = i A_r^i(q, \theta)^i \dot{r}
\end{aligned}
\]

with the rotational matrix \( i R^i \) and the translational offset \( i r_0^i \) between the frames. The position vector is expanded to \( \dot{r} = [x \ y \ z \ 1]^T \in \mathbb{R}^{4 \times 1} \) and will be used in this form in the following. Rotational velocities are accordingly expanded to \( \omega = [\omega_x \ \omega_y \ \omega_z \ 1]^T \in \mathbb{R}^{4 \times 1} \) and transformed by

![Figure 8](image.png)

**Figure 8.** Simplified drawing of the interconnection between segment one and two given in \( \Sigma^1 \) frame. The undeformed state is plotted in black. The grey lines show an exemplary deformed configuration.
\[
I \omega = \begin{bmatrix}
I \mathbf{R}^i(q, \theta) & I \omega_b^i(q, \theta) \\
0 & 1
\end{bmatrix} \quad I \omega = I \mathbf{A}^i(q, \theta) I \omega
\]

with \( I \omega_b^i \) the angular velocity of the local frame \( i \) in target coordinates \( I \). For the sake of short notation, arguments \( q \) and \( \theta \) are omitted in the following.

### 2.5. Angular velocity

The angular velocity vector \( \omega^i \) of frame \( \Sigma^i \) is given by [28]

\[
\dot{\mathbf{R}}^i = \tilde{\omega}^i \mathbf{R}^i
\]

with the rotational transformation matrix \( \mathbf{R} \) and the tilde operator

\[
\tilde{\omega}^i = \begin{bmatrix}
0 & -\omega_z^i & \omega_y^i \\
\omega_z^i & 0 & -\omega_x^i \\
-\omega_y^i & \omega_x^i & 0
\end{bmatrix}, \quad \omega \in \mathbb{R}^{4 \times 1}.
\]

Usually, Equation (19) is used to compute \( \dot{\mathbf{R}}^i \). Since the approach used in this publication utilizes automatic differentiation resulting in very efficient computation of \( \dot{\mathbf{R}}^i \) solely based on \( \mathbf{R}^i \), the relation can be used to compute the angular velocity of frame \( \Sigma^i \). The inverse tilde operator (\( \tilde{\cdot} \)) is defined by

\[
A^i = \frac{1}{2} \begin{bmatrix}
a_{3,2} - a_{2,3} \\
a_{1,3} - a_{3,1} \\
a_{2,1} - a_{1,2} \\
1
\end{bmatrix}, \quad A \in \mathbb{R}^{3 \times 3}
\]

and the angular velocity \( \omega^i \) of system \( \Sigma^i \) can be computed by

\[
\omega^i = \left( \tilde{\mathbf{T}}^i \mathbf{R}^i \right).
\]

This relationship allows the extraction of the frame angular velocity by only using the position and orientation kinematics.

### 3. Lagrange’s equation

Variational methods in analytical mechanics, especially the application of Lagrange’s equation of the second kind, allow the derivation of equations of motion from expressions of the kinetic and potential energy of the modelled system. The Lagrange’s equation of the second kind is given by

\[
\frac{d}{dt} \left( \frac{\partial (T - V)}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial (T - V)}{\partial \mathbf{q}} - \mathbf{\tau} = 0
\]

with the generalized coordinates \( \mathbf{q} \), the kinetic energy \( T \), the potential energy \( V \) and the sum of all non-conservative forces \( \mathbf{\tau} \) including damping. The input of the actors used in the application example directly set the actor velocity \( \dot{\theta} \). Thus, \( \dot{\theta} \) is seen as the system
input and transformed to \( \dot{\theta} \) by integration. Actor coordinates can be replaced, the non-conservative forces can be rewritten and Equation (23) becomes

\[
\frac{\partial}{\partial[q^T \theta^T \dot{\theta}^T]^T} \left( \frac{\partial(T - V)}{\partial \dot{q}} \right) \begin{bmatrix} \dot{q} \\ \dot{\theta} \end{bmatrix} - \frac{\partial(T - V)}{\partial q} - \tau = 0
\]  

with the remaining generalized coordinates \( q \) and the non conservative forces \( \tau \).

Linearization and extraction of all needed system matrices for mechanical systems of the form

\[
MA\dot{q} = D\Delta \dot{q} + K\Delta q + B\Delta \ddot{\theta} + r
\]  

with \( \Delta \dot{q} = \dot{q} - \dot{q}_0, \Delta q = q - q_0 \) and \( \Delta \ddot{\theta} = \ddot{\theta} - \ddot{\theta}_0 \), is done using

\[
M = \left| \frac{\partial LF}{\partial \dot{q}} \right|_{q=q_0, \ \dot{q}=\dot{q}_0, \ \ddot{q}=\ddot{q}_0, \ \theta=\theta_0} \quad D = \left| \frac{\partial LF}{\partial \dot{q}} \right|_{q=q_0, \ \dot{q}=\dot{q}_0, \ \ddot{q}=\ddot{q}_0, \ \theta=\theta_0} \quad K = \left| \frac{\partial LF}{\partial \dot{q}} \right|_{q=q_0, \ \dot{q}=\dot{q}_0, \ \ddot{q}=\ddot{q}_0, \ \theta=\theta_0} \quad B = -\left| \frac{\partial LF}{\partial \theta} \right|_{q=q_0, \ \dot{q}=\dot{q}_0, \ \ddot{q}=\ddot{q}_0, \ \theta=\theta_0} \quad r = -LF
\]  

with the arbitrary working point \( q_0, \ \dot{q}_0, \ \ddot{q}_0, \ \theta_0 \). The resulting matrices and vector make it possible to use all standard techniques of linear control theory.

### 4. Energy expressions

In the following, the energy expressions will be derived in a form, which can be efficiently used for implementation. The efficient formulation of the kinetic energy is presented in more detail. Using the same approach for formulation, the potential energy and the dissipation model are given in a shortened version.

#### 4.1. Kinetic energy

The kinetic energy \( T^i \) of a flexible segment \( i \) using the Euler-Bernoulli assumptions can be expressed as

\[
T^i = \int_0^L \frac{\mu}{2} \left( I^i \dot{r}^i(x) \right)^T I^i \dot{r}^i(x) \ dx
\]  

with \( \mu \) the mass per length. To be able to easily solve the integral, the aim is to shift the scope of the integrals from the inertial frame \( \Sigma^I \) to the local frame \( \Sigma^i \). Integrals over the local frame are already solved in Section 2.2 deformation and identical for all used beam elements.

For positions given in local coordinates, i.e.
\[ \dot{\mathbf{r}}(x) = \dot{\mathbf{A}}_r(x) \dot{\mathbf{r}}(x) + \dot{\mathbf{A}}_r(x) \dot{\mathbf{r}}(x) \] (28)

the expression (27) is reformulated to

\[ T^i = \int_0^L \frac{\mu}{2} \left( (\dot{\mathbf{r}}^i(x))^T \dot{\mathbf{A}}_r(x) \dot{\mathbf{r}}^i(x) + (\dot{\mathbf{r}}^i(x))^T \dot{\mathbf{A}}_r(x) \dot{\mathbf{r}}^i(x) + \ldots \right) \]

\[ \ldots + (\dot{\mathbf{r}}^i(x))^T \dot{\mathbf{A}}_r(x) \dot{\mathbf{r}}^i(x) + (\dot{\mathbf{r}}^i(x))^T \dot{\mathbf{A}}_r(x) \dot{\mathbf{r}}^i(x) \right) \, dx. \] (29)

Using the fact, that

\[ \mathbf{v}^T \mathbf{v} = \text{tr}(\mathbf{vv}^T), \quad \mathbf{v} \in \mathbb{R}^{n \times 1}, \] (30)

where \( \text{tr}(\cdot) \) is the trace function, the transformation matrices can be placed outside of the integral. The resulting expression yields

\[ T^i = \frac{\mu}{2} \text{tr} \left( \mathbf{A}_r^i \int_0^L \dot{\mathbf{r}}^i(x) \dot{\mathbf{r}}^i(x)^T \, dx + \dot{\mathbf{A}}_r^i \int_0^L \dot{\mathbf{r}}^i(x) \dot{\mathbf{r}}^i(x)^T \, dx \right) \]

\[ \ldots + 2 \mathbf{A}_r^i \int_0^L \dot{\mathbf{r}}^i(x) \dot{\mathbf{r}}^i(x)^T \, dx \mathbf{A}_r^i \] (31)

with all integrals common for all used beam elements. The remaining integrals in local coordinates even further simplify using Equations (9) to (11).

For discrete masses, the kinetic energy is

\[ T_d = \sum_k \frac{m_k}{2} (\dot{\mathbf{r}}_k + \dot{\mathbf{r}}_k)^2 + \sum_k \frac{J_k}{2} (\dot{\mathbf{\omega}}_k + \dot{\mathbf{\omega}}_k)^2 \] (32)

resulting in the overall kinetic energy

\[ T = \sum_i T^i + T_d \] (33)

for the complete manipulator.

### 4.2. Potential energy

The potential energy of a flexible segment \( i \) can be expressed as

\[ V^i = \mu^i \mathbf{g}^T \mathbf{A}_r^i \int_0^L \dot{\mathbf{r}}^i(x) \, dx + E I^i \left( \mathbf{q}^i \right)^T \int_0^L \frac{\partial^2 \psi^i(x)}{\partial x^2} \left( \frac{\partial^2 \psi^i(x)}{\partial x^2} \right)^T \, dx \mathbf{q}^i \] (34)

with the gravity vector

\[ \mathbf{g} = [0 \quad 0 \quad g \quad 0]^T, \quad g \in \mathbb{R} \] (35)

and the beam rigidity \( E I \). The first part of Equation (34) represents the gravitational energy of each mass element. According to the Euler-Bernoulli assumptions, the second
part represents the internal potential energy of a beam due to elastic deformations. For discrete masses and springs the potential energy is

$$V_d = \sum_k m_k g^T r_k + \sum_i \frac{K_i}{2} (l^i \Delta r_i^T \Delta r_i)$$

(36)

where $l^i \Delta r_i$ is the current spring elongation. The overall potential energy results in

$$V = \sum_i V^i + V_d$$

(37)

for the complete manipulator.

5. Dissipation model

Dissipative effects will be modelled according to the Rayleigh damping model [29,30].

The viscous damping part is assumed to be proportional to the mass resulting in discrete element generalized damping forces $f_v$ of

$$f_v = \left( \frac{\partial^i r}{\partial q} \right)^T d_v m^i \dot{r}$$

(38)

with damping factor $d_v$ and mass $m$ for a generic mass element. Applying this to the distributed parameter segment $i$ and shifting integrals to local coordinates leads to

$$f^i_v = d_v m^i \int_0^{l^i} \frac{\partial^i r^i(x)}{\partial q} l^i \dot{r}^i(x)^T \, dx$$

(39)

$$= d_v m^i \left. \frac{\partial}{\partial q} \right|_{q=q} \text{tr} \left( \ldots l^i A^i_r(x, \bar{q}) \int_0^{l^i} r^i(x, \bar{q}) r^i(x, q)^T \, dx \right. \ldots$$

$$\left. \ldots + l^i A^i_r(x, \bar{q}) \int_0^{l^i} r^i(x, \bar{q}) r^i(x, q)^T \, dx \right) \ldots$$

(40)

The structural damping of a flexible beam element is modelled according to the Kelvin-Voigt approach as proportional to the fourth spatial derivative of the deformation $w^i(x, t)$ [30]. The resulting damping force along the beam is given by

$$F^i(x, t) = d_s E^i \frac{\partial^4 w^i(x, t)}{\partial t \partial x^4}$$

(41)
with structural damping factor \( d_s \). A modal transformation of this force results in the damping force for each flexible coordinate \( \dot{q}_k \) given by

\[
f_{i,k}^i = \int_0^{L_i} \psi_k^i(x)d_sEI^i \frac{\partial^3 w^i(x,t)}{\partial t \partial x^4} \, dx.
\]  

(42)

Using Equations (4) and (11) and the dependence

\[
(k_k^i)^4 = \left( \frac{\omega_k^i}{\mu^i} \right)^2 \frac{\mu^i}{EI} \]

(43)

between the eigenvalue \( k_k^i \) and the eigenfrequency \( \omega_k^i \) of Euler-Bernoulli beams, the expression becomes

\[
f_{i,k}^i = d_sEI^i \sum_l \left( \int_0^{L_i} \psi_k^i(x) \frac{\partial^4 \psi_l^i(x)}{\partial x^4} \, dx \right) \dot{q}_k
\]

\[= d_sEI^i (k_k^i)^4 \dot{q}_k = d_s\mu^i (\omega_k^i)^2 \dot{q}_k.\]

(44)

Equation (44) simplifies to the squared eigenfrequency \( \omega^i \) of the segment \( i \) since orthogonal and normalized eigenmodes are used as basis functions.

Forces \( f_d \) generated by discrete dampers are modelled by

\[
f_d = d_d \left( \frac{\partial^4 \Delta r}{\partial q} \right)^T \Delta \dot{r}
\]

(46)

with the damping coefficient \( d_d \) and \( \Delta \dot{r} \) the damper elongation.

6. Sensor equations

As shown in Figure 4, a strain gauge and a gyroscope are used as sensors. The sensor signal of the strain gauge is proportional to the inner beam torque. The sensor equation

\[
y_{sg}(t) = c_{sg} (q^1(t)) \frac{T \partial^2 \psi^i(x)}{\partial x^2} \bigg|_{x=x_{sg}} + y_{sg,0}
\]

(47)

is used with \( x_{sg} \) the local strain gauge position, \( c_{sg} \) a proportionality constant and \( y_{sg,0} \) a constant sensor offset. The angular velocity about the local \( y \)-axis measured by the gyroscope can be expressed as

\[
y_{gyro}(t) = \left[ \begin{array}{ccc} 0 & 0 & -1 \end{array} \right] \frac{d}{dt} \frac{\partial (4^4(x,t))}{\partial x} \bigg|_{x=x_{gyro}} + \left[ \begin{array}{ccc} 0 & 0 & 1 \end{array} \right] R^4 T A \omega^4
\]

(48)

with \( x_{gyro} \) the local position of the gyroscope. The first part of the expression represents the angular velocity about the \( y \)-axis caused by the deformation of the segment and the second part the velocity of the frame \( \Sigma^4 \) in local coordinates. Linearization leads to the output expression in the form
\[ y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + r_y(\theta). \quad (49) \]

### 7. Modal transformation

Modal transformation [31] of a system as given in Equation (25) allows the analysis of eigenfrequencies and modes of the overall system. The eigenfrequencies \( \omega_i \) and eigenmodes \( \nu_i \) of the undamped system can be found as the solutions of the eigenvalue problem

\[ (M\omega_i^2 - K)\nu_i = 0. \quad (50) \]

Using the eigenmodes \( \nu_i \), the system dimension can be reduced such that only eigenfrequencies of interest are considered. For mechanical systems, the eigenmodes of interest are typically the ones with the lowest eigenfrequencies.

### 8. Numerical results and experimental validation

As a proof of concept, the flexible multibody system is implemented for a Magirus M32L-AS turntable ladder. For each flexible element, the first three eigenmodes are used as basis functions for flexible movement. Measurements with a payload of 430 kg are used for validation.

#### 8.1. Parameters

Table 1 gives an overview of all used parameters and theirs sources. All beam elements need a rigidity parameter \( EI \). FE computer simulations of the truss structure forming a ladder part with a static load at the tip of the segment lead to a displacement and the rigidity immediately follows.

The spring coefficient \( K \) modelling the vehicle chassis and hydraulic cylinder dynamics and the strain gauge proportionality constant \( c_g \) are chosen such that for the fully retracted ladder, the output and lowest eigenfrequency of the resulting model matches with the measured output and eigenfrequency of the system with a 500 kg payload. Damping coefficients \( d_v, d_s \) and \( d_d \) are chosen empirically to match the time domain behaviour for this payload. The chosen payload of 500 kg for parameter identification is different to the chosen payload for model validation to reduce the risk of model overfitting.

The working point for the linearization in Equation (26) is chosen as constant \( \mathbf{q}_0 = \dot{\mathbf{q}}_0 = \ddot{\mathbf{q}}_0 = \mathbf{0} \) and \( \theta = \mathbf{0} \). Since the algorithm can rapidly compute the matrices for an arbitrary working point, the rest position would be another common choice. But for the sake of simplicity, the zero working point is chosen.

#### 8.2. Software framework

The FMBS model is implemented using Matlab Symbolic Toolbox and CasADi automatic differentiation software [32]. Figure 9 gives a brief overview of the resulting software structure. The Matlab computer algebra system is used to solve the small set of remaining integrals. Afterwards, the derivatives are computed very efficiently by automatic differentiation (AD).
The manipulator is constructed simply by defining its position and orientation kinematics and sensor equations. Velocities are rapidly derived from the position and orientation kinematics by means of AD. The linearized model, as shown in Equation (26), is consisting of five flexible elements, each utilizing three basis functions. A full linearized model is computed in less than 1 ms on a standard PC. Usage in embedded systems by CasADi C-function export enables an online derivation of the linearized model based on the current working point and manipulator pose, i.e. for the current \( q, \dot{q}, \theta \) and \( \dot{\theta} \). This opens up a multitude of possibilities, e.g. for controller and observer design. Simulation of a different manipulator with seven flexible elements and 15 basis functions per segment was possible in a comparable computation time.

### 8.3. Comparison of modal system parameters

For the considered ladder type, an active oscillation damping functionality already exists for the vertical direction which utilizes a system model of the form

![Diagram](image)

**Figure 9.** Overview of resulting software structure. The user input consists of the mechanical parameters, position and orientation kinematics and sensor equations.

| Parameter | Description | Source |
|-----------|-------------|--------|
| \( q \) | position and orientation kinematics | CAD |
| \( EI' \) | beam rigidity | CAD |
| \( \mu' \) | beam mass per length | CAD |
| \( m_w \) | mass winch | CAD |
| \( m_c \) | mass cage | CAD |
| \( J_c \) | moment of inertia cage | CAD |
| \( c_{eg} \) | strain gauge proportionality | single pose model identification |
| \( K \) | spring for chassis & hydraulics | single pose model identification |
| \( d_d \) | damper for chassis & hydraulics | single pose model identification |
| \( d_v \) | viscous damping factor | model identification |
| \( d_s \) | structural damping factor | model identification |

Table 1. Source of chosen parameter values. Damping factors are printed in grey. Single pose model identification means, that the experiment for parameter identification is conducted in one single manipulator pose and a payload of 500 kg.
\[
\begin{align*}
\dot{x} &= \begin{bmatrix} 0 & 1 \\
-\omega_1^2 & -2d_1\omega_1 \end{bmatrix} x + \begin{bmatrix} 0 \\
b_1 \end{bmatrix} \dot{\theta} \\
y &= \begin{bmatrix} y_{\text{sg}} \\ y_{\text{gyro}} \end{bmatrix} = \begin{bmatrix} c_1 & 0 \\
0 & 1 \end{bmatrix} x
\end{align*}
\]

with \(x(0) = x_0\) [1,2]. The system parameters \(\omega_1, b_1\) and \(c_1\) depend on the extraction length \(\theta_1\) and the angle \(\theta_3\) of the articulated arm. All parameters are known by identification measurements for eleven different extraction lengths \(\theta_1\) and two different angles \(\theta_2\). These known parameters will be used for validation of the derived model.

For comparison, the considered flexible multibody system model is linearized, see Section 3, reduced by modal transformation to the lowest eigenfrequency, see Section 7, and transformed to the same system structure. Figure 10 shows the mean value of the eigenfrequencies for \(\theta_3 = 0^\circ\) and \(\theta_3 = -75^\circ\) of the derived model and the identified eigenfrequencies of the benchmark from real world testing. The three different kinds of payloads correspond to a turntable ladder without a rescue cage and two different cage types. For high extraction length and high payload, the model fits with the identified benchmark data very well. Without payload and for short extraction length, the benchmark eigenfrequency is lower than the modelled frequency. For these data points, the signal-to-noise ratio of the sensor signals is low and bearing play resonances occur. This results in an uncertain estimation of the benchmark eigenfrequency parameter. Additionally, due to the missing payload, small differences between real masses for each ladder segment and the modelled masses have a high influence on the resulting eigenfrequency.

A comparison of the input and output parameters in the 430 kg case of payload is shown in Figures 11 and 12. The output parameter \(c_1\) matches very well which is demonstrating a high conformity of the derived and empirically identified parameters. Analysing \(b_1\), the shape and range of the model derived parameters match but there is a small quantitative deviation for low and high extracting lengths. Due to the structure of the system in Equation (52), the input parameter \(b_1\) additionally influences the sensitivity of the gyroscope output. The model predicts slightly smaller output values for the gyroscope than the identified parameters. Comparison of the parameters for the two remaining payloads leads to similar positive results and supports the quality of the derived model.

![Figure 10](image-url)

**Figure 10.** Lowest eigenfrequency, mean for \(\theta_3 = 0^\circ\) and \(\theta_3 = -75^\circ\), of the derived model without payload (○), with 430 kg payload (+) and with 500 kg payload (*) compared to eigenfrequencies identified by measurements in grey (—).
The derived model is used to replicate experimental data for the manipulator with 430 kg payload. During the real world experiment, the actor \( \theta_2 \) is used to excite the manipulator. For several different extraction lengths \( \theta_1 \) and angles \( \theta_3 \), the hydraulic set point of \( \theta_2 \) is controlled at maximum speed. Maximum speed is set for a duration of 0.5 s in elevation or 1.5 s in depression direction before returning to stand still. To replicate this excitation for the simulation, the recorded quantized actor position sensor signal is filtered and numerically differentiated to obtain the first and second derivative.

Figure 13 shows the used excitation trajectory for an extraction length \( \theta_1 \) of 20.7 m, articulated arm angle \( \theta_3 = 0^\circ \) and 430 kg payload. The resulting shape of trajectory for depression direction in Figure 13 around the time of 113 s is due to the delayed opening of the used hydraulic load-holding valve. Resulting sensor values are displayed in Figures 14 and 15. While the modal analysis in Section 8.3 only confirms the compliance of the model for the first eigenfrequency, the gyroscope signal in Figure 14 between 80 s and 90 s shows a strong resemblance even for the second eigenfrequency. Especially right after the excitation, high-
frequency oscillation have almost identical frequency and phase in simulation and measurement. With increasing time, a slight mismatch of the eigenfrequencies accumulates to a growing phase difference. The strain gauge signal shown in Figure 15 confirms the match of eigenfrequencies. The difference in measured and simulated signals at 80 s is caused by a slight mismatch of the phase of oscillation between real world manipulator and simulation right before the excitation.

The frequency spectrum shown in Figure 16 confirms the observation of well-matching first and second eigenfrequency.

The same analysis is repeated for an excitation at the extraction length $\theta_1$ of 6.7 m as shown in Figure 17. In the case of this very short manipulator, the first eigenfrequency is comparable high and thus the second is not visible in the simulation data shown in Figures 18 and 19. The high oscillation in the measurement data of the gyroscope results from small play in the slide bearing between the segments.

The frequency spectrum shown in Figure 20 confirms the observation of well matching first eigenfrequency.

In summary, even though the simulated poses and used payload do not correspond to the pose and payload used to adjust the very few free parameters, the time domain behaviour

![Figure 13](image13.png)

**Figure 13.** Used excitation for actor $\theta_2$ for an extraction length $\theta_1$ of 20.7 m, articulated arm angle $\theta_3 = 0^\circ$ and 430 kg payload.

![Figure 14](image14.png)

**Figure 14.** Signals of simulated gyroscope (--) compared to measured signal values in grey (--) for an extraction length $\theta_1$ of 20.7 m, articulated arm angle $\theta_3 = 0^\circ$ and 430 kg payload.
matches very well with the measured data. This is remarkable considering the fact that only very few parameters are freely chosen and all others are extracted from CAD.

9. Conclusion

The derived vertical dynamics model for the aerial rescue ladder treats each ladder segment as flexible element with distributed parameters and uses Ritz series expansion to obtain a finite approximation. The basis functions are chosen as orthonormal modes of Euler-Bernoulli beams satisfying the essential boundary conditions of each segment. Due to the type of boundary conditions, kinematic loops occurring between the segment connections are solved implicitly. A linearized expression for the manipulator dynamics is computed very efficiently and can be directly used for control development. Since the
Figure 17. Used excitation for actor $\theta_2$ for an extraction length $\theta_1$ of 6.5 m, articulated arm angle $\theta_3 = 0^\circ$ and 430 kg payload.

Figure 18. Signals of simulated gyroscope (--) compared to measured signal values in grey (--) for an extraction length $\theta_1$ of 6.5 m, articulated arm angle $\theta_3 = 0^\circ$ and 430 kg payload.

Figure 19. Signals of simulated strain gauges (--) compared to measured signal values in grey (--) for an extraction length $\theta_1$ of 6.5 m, articulated arm angle $\theta_3 = 0^\circ$ and 430 kg payload.
resulting model is very efficient and can be exported to C-functions easily, an online computation of the linearized model according to the current working point is possible on embedded systems. Validating the model against measurement data confirms the very high accuracy. System parameters extracted from the model match with parameters identified by real world testing for any payload and pose. It is shown, that measured time domain sensor data can be reproduced exceptionally well by the derived model.

**Note**

1. For a ladder consisting of five segments, the stiffness of the inner ladder segments is lower than $\frac{1}{5}$ of the stiffness of the stepped beam model beam.

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**Disclosure statement**

No potential conflict of interest was reported by the author(s).

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Appendix

Nomenclature/Notation

Vectors are named in lower case bold symbols while matrices are upper case bold Latin, e.g. ‘\( \boldsymbol{v} \)’ and ‘\( \boldsymbol{M} \)’.

Symbols and functions depending on a coordinate system or affiliated to a specific element are named according to the scheme

\[ a^b_c(x) \]  \hspace{1cm} (53)

with \( a \) indicating the frame of resolution and \( b \) showing affiliation of this function to the element \( b \). The subscript \( c \) can be used for further specification, e.g. indicating the \( c \)th element of a vector. \( a \) or \( b \) are omitted if \( r \) is valid for all frames or affiliations. Arguments of \( r \) which depend on a coordinate system are always given in the affiliated coordinate system. In this example, \( x \) is the \( x \) used in the \( b \) frame.

Rotational matrices \( \boldsymbol{R} \) and offsets between two coordinate systems \( r_0 \) are named according to

\[ a^b_R \quad \text{and} \quad a^b_r_0 \]  \hspace{1cm} (54)

with \( a \) indicating the target frame and \( b \) the current coordinate system. Homogeneous transformations are written in a similar manner as

\[ a^b_A \]  \hspace{1cm} (55)

with \( r \) indicating that the matrix can be used for transformation of positions. \( a^b_A \) is accordingly the transformation matrix for angular velocity vectors.