ORNSTEIN-UHLENBECK PROCESSES ON LIE GROUPS

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ABSTRACT. We consider Ornstein-Uhlenbeck processes (OU-processes) related to hypoelliptic diffusion on finite-dimensional Lie groups: let $L$ be a hypoelliptic, right invariant “sum of the squares”-operator on a Lie group $G$ with associated Markov process $X$, then we construct OU-type processes by adding horizontal gradient drifts of functions $U$. In the natural case $U = -\log p(1, x)$, where $p(1, x)$ is the density of the law of the $X_1$ starting at the identity $e$ with respect to the left-invariant Haar measure on $G$, we show the Poincaré inequality by applying the Driver-Melcher inequality for “sum of the squares” operators on Lie groups.

The Markov process associated to $-\log p(1, x)$ is called the OU-process related to the given hypoelliptic diffusion on $G$. We prove the global strong existence of this OU-process on $G$. The Poincaré inequality for a large class of potentials $U$ is then shown by perturbation methods and used to obtain a hypoelliptic equivalent of the standard result on cooling schedules for simulated annealing. The relation between local results on $L$ and global results for the constructed OU-process is widely used in this study.

1. Preparations from functional analysis

We consider a finite-dimensional Lie group $G$ with Lie algebra $\mathfrak{g}$, its right-invariant Haar measure $\mu$ and a family of left-invariant vector fields $V_1, \ldots, V_d \in \mathfrak{g}$. We assume that Hörmander’s condition holds, i.e. the sub-algebra generated by $V_1, \ldots, V_d$ coincides with $\mathfrak{g}$.

We consider furthermore a stochastic basis $(\Omega, \mathcal{F}, P)$ with a $d$-dimensional Brownian motion $B$ and the Lie group valued process

$$dX^x_t = \sum_{i=1}^d V_i(X^x_t) \circ dB^i_t, \quad x \in G.$$ (1.1)

The generator of this process is denoted by $L$ and we have

$$L = \frac{1}{2} \sum_{i=1}^d V_i^2,$$

where we interpret the vector fields as first order differential operators on $C^\infty(G, \mathbb{R})$. Furthermore, we define a semigroup $P_t$ acting on bounded measurable functions $f : G \to \mathbb{R}$ by

$$P_t f(x) = \mathbb{E}(f(X^x_t)).$$

This semigroup can be extended to a strongly continuous semigroup on $L^2(G, \mu)$, which we will denote by the same letter $P_t$. The carré du champ operator $\Gamma$ is defined for functions $f$, where it makes sense, by

$$\Gamma(f, g) = L(fg) - fLg - gLf.$$

(1.2)
In our particular case, we obtain immediately

$$\Gamma(f, f) = \sum_{i=1}^{d} (V_i f)^2 .$$

Notice that the carré du champ operator does not change if we add a drift to the generator $L$.

Due to the right invariance of the Haar measure $\mu$ and the left-invariance of the vector fields $V_i$, the operator $L$ is symmetric (reversible) with respect to $\mu$ and therefore $\mu$ is an infinitesimal invariant measure in the sense that $\int L f(x) \mu(dx) = 0$ for all smooth compactly supported functions $f$. Furthermore, due to the symmetry of $L$ and the invariance of $\mu$, we have from (1.2) the relation

$$2 \int f L g \mu = - \int \Gamma(f, g) \mu$$

(1.3) for all $f, g \in C_0^\infty(G)$.

Let now $U : G \to \mathbb{R}$ be an arbitrary smooth function such that

$$\int_G \exp(-U(x)) \mu(dx) < \infty ,$$

and consider the modified generator

$$L^U := L - \frac{1}{2} \Gamma(U, \cdot) .$$

Notice that $\mu^U = \exp(-U)\mu$ is an invariant (finite) measure for $L^U$, since, by (1.3),

$$\int L^U f \mu^U = \int (L f) \exp(-U) \mu - \frac{1}{2} \int \Gamma(U, f) \exp(-U) \mu$$

$$= -\frac{1}{2} \int \Gamma(f, \exp(-U)) \mu - \frac{1}{2} \int \Gamma(f, U) \exp(-U) \mu = 0 .$$

Here, the last step uses the fact that $\Gamma(f, \cdot)$ is a derivation.

We have the following observation on the existence of a spectral gap at 0:

**Proposition 1.1.** The operator $L^U$ has a spectral gap at 0 of size $a > 0$ if and only if

$$\int_G \Gamma(f, f)(x) \mu^U(dx) \geq 2a \int_G f(x)^2 \mu^U(dx) ,$$

for all compactly supported smooth functions $f$ on $G$ satisfying

$$\int_G f(x) \mu^U(dx) = 0 .$$

**Proof.** $L^U$ has a spectral gap of size $a > 0$ at 0 if

$$\int_G f(x) L^U f(x) \mu^U(dx) \leq -a \int_G f(x)^2 \mu^U(dx)$$

for test functions with $\int_G f(x) \mu^U(dx) = 0$. Now, by construction, the integral on the left hand side can be partially integrated, hence

$$-\frac{1}{2} \int_G \Gamma(f, f)(x) \mu^U(dx) \leq -a \int_G f(x)^2 \mu^U(dx)$$

for any test function satisfying the integral constraint. This proves the desired inequality. \qed
Remark 1.2. Assume that there is a spectral gap. Then the largest $a > 0$ in the previous inequality is the modulus of the smallest non-zero spectral value of $L^U$.

Remark 1.3. If we want to write an inequality for all test functions $f$, it reads like

$$\int_G \Gamma(f,f)(x)\mu^U(dx) \geq 2a\left(\int_G f(x)^2\mu^U(dx) \int_G \mu^U(dx) - \left(\int_G f(x)\mu^U(dx)\right)^2\right)$$

for all test functions $f \in C^\infty_0(G)$.

2. Strong existence of OU-processes with values in Lie groups

We consider now the special case where we take as our 'potential' $W_t(x) = -\log p(t,x)$, $t > 0$, where $p(t,x)$ is the density of the law of $X_t$ with respect to $\mu$. By Hörmander’s Theorem [Hör67, Hör07], the function $(t,x) \mapsto p(t,x)$ is a positive and smooth function, hence the potential $W_t$ is as in the previous section. We write for short $L_t$ instead of $L^U_{W_t}$ and we call the associated Markov process the **OU-process on $G$**. We show that we have in fact global strong solutions for the corresponding Stratonovich SDE with values in $G$. The next proposition is slightly more general.

**Proposition 2.1.** Consider a smooth potential $U : G \to \mathbb{R}$ such that

$$\int \exp(-U(x))\mu(dx) < \infty.$$ 

Consider the following Stratonovich SDE with values in $G$:

$$dY^y_t = V_0(Y^y_t)dt + \sum_{i=1}^d V_i(Y^y_t) \circ dB^i_t, \quad Y^y_0 = y \in G,$$

where $V_0f = -\frac{1}{2}\Gamma(U,f)$ for smooth test functions $f$. Then there is a global strong solution to (2.1) for all initial values $y \in G$.

**Proof.** Since the coefficients defining (2.1) are smooth by assumption, there exists a unique strong solution up to the explosion time

$$\zeta_y = \inf\{t : \lim_{\tau \to t} Y^y_\tau = \infty\}.$$

We then define a semigroup $P_t$ on $L^2(G,\mu^U)$ by

$$(P_t f)(y) = \mathbb{E}\{f(Y^y_t) 1_{\zeta_t < \infty}\}.$$  

(2.2)

It can be shown in the exact same way as in [Che73, Li92] that $P_t$ is a strongly continuous contraction semigroup and that its generator $A$ coincides with $L^U$ on the set $C^\infty_0(G)$ of compactly supported smooth functions.

On the other hand, setting $D(L^U) = C^\infty_0(G)$, one can show as in [Che73, Li92] that $L^U$ is essentially self-adjoint, so that one must have $A = L^U = (L^U)^*$. In particular, since the constant function $1$ belongs to $L^2(G,\mu^U)$ by the integrability of $\exp(-U)$ and since $\int (L^U \psi)(x)\mu^U(dx) = 0$ for any test function $\psi \in C^\infty_0(G)$, $1$ belongs to the domain of $(L^U)^*$ and therefore also to the domain of $A$. This then implies that $P_t 1 = 1$ by the same argument as in [Li92]. In particular, coming back to the definition (2.2) of $P_t$, we see that $P(\zeta_y = \infty) = 1$ for every $y$, which is precisely the non-explosion result that we were looking for.  

□
Remark 2.2. While this argument shows that, given a fixed initial condition $y$, there exists a unique global strong solution $Y^y_t$ to (2.1), it does not prevent more subtle kinds of explosions, see for example [Elw78].

By Proposition 2.1 and since $p(t,x)$ is smooth and integrable, it follows immediately that the OU-process exists globally in a strong sense.

Corollary 2.3. For any given $\tau > 0$, the process

$$dY^y_t = V_0(Y^y_t)dt + \sum_{i=1}^{d} V_i(Y^y_t) \circ dB^i_t, \quad Y^y_0 = y \in G,$$

with $V_0f = -\frac{1}{2} \Gamma(W_t, f)$ has a global strong solution.

Remark 2.4. More traditional Lyapunov-function based techniques seem to be highly non-trivial to apply to this situation, due to the lack of information on the behaviour of $U(y)$ at large $y$. In view of [BA88, Léa87b, Léa87a], it is tempting to conjecture that one has the asymptotic

$$\lim_{t \to 0} t^2 \log \partial_t p(t,y) = -d^2(e,y), \quad (2.3)$$

and that the limit holds uniformly over compact sets $K$ that do not contain the origin $e$. (Note that it follows from [BA88] that this is true provided that $K$ does not intersect the cut-locus.) If it were the case that (2.3) holds, the space-time scaling properties of $p(t,x)$ would imply that, for every $t > 0$, there exists a compact set $K$ such that $Lp(t,x) = \partial_t p(t,x) > 0$ for $x \notin K$. On the other hand, one has

$$\mathcal{L}^t W_t = -\frac{1}{2} \Gamma(W_t, W_t) + \mathcal{L} W_t = -\frac{Lp(t,\cdot)}{p(t,\cdot)},$$

so that this would imply that $W_t$ is a Lyapunov function for the corresponding OU-process.

3. Spectral Gaps for OU-processes

Next we consider the question if $\mathcal{L}^t$ admits a spectral gap, which turns out to be a consequence of the Driver-Melcher inequality.

Theorem 3.1. The following assertions are equivalent:

- The operator $\mathcal{L}^t$ has a spectral gap of size $a_t \geq 0$ for $t > 0$ and a positive $H^1$-function $a : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$.

- The local estimate

$$P_t(\Gamma(f, f))(g) \geq 2a_t(\langle P_t f \rangle^2(g) - \langle (P_t f) \rangle^2)$$

holds true for all test functions $f : G \to \mathbb{R}$, $t > 0$ and a positive $H^1$-function $a : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ at one (and therefore all) point $g \in G$.

Furthermore, if we know that

$$\Gamma(P_t f, P_t f)(e) \leq \varphi(t) P_t(\Gamma(f, f))(e)$$

holds true for all test functions $f \in C_0^\infty(G)$, all $t \geq 0$, and a strictly positive locally integrable function $\varphi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, then we can choose $a_t$ by

$$a_t \int_0^t \varphi(t-s)ds = \frac{1}{2},$$

for $t > 0$ and the two equivalent assertions hold true.
Proof. Since $\mu^W_t$ is equal to the law of $X_t^e$, one has $\int f \mu^W_t = P_t f(e)$ for every $f \in C_0^\infty(G)$. The equivalence of the first two statements then follows from (1.4) and the fact that the translation invariance of (1.1) implies that if the bound holds at some $g$, it must hold for all $g \in G$. We fix a test function $f : G \to \mathbb{R}$ as well as $t > 0$ and consider

$$H(s) = P_s((P_{t-s}f)^2)$$

for $0 \leq s \leq t$. The derivative of this function equals

$$H'(s) = P_s(\Gamma(P_{t-s}f, P_{t-s}f))$$

and therefore – assuming the third statement – we obtain

$$H'(s) \leq \varphi(t-s)P_t(\Gamma(f, f)).$$

Whence we can conclude

$$H(t) - H(0) \leq \int_0^t \varphi(t-s)ds P_t(\Gamma(f, f)),$$

which is the second of the two equivalent assertions for an appropriately chosen $a$. □

Remark 3.2. We can replace the Lie group $G$ by a general manifold $M$, on which we are given a hypo-elliptic, reversible diffusion $X$ with “sum of the squares” generator $L$. Then the completely analogous statement holds on $M$, in particular local Poincaré inequalities on $M$ for $L$ lead to a spectral gap for the OU-type process $L^t$ with $t > 0$.

Corollary 3.3. Let $G$ be a free, nilpotent Lie group with $d$ generators $e_1, \ldots, e_d$ and denote by $X$ the canonical diffusion process on $G$, i.e.

$$dX_t = \sum_{i=1}^d X_t e_i \circ dB_t^i.$$

Then the operator $L^t$ has a spectral gap of size $a_t = \frac{1}{2Kt}$ for some constant $K > 0$.

Proof. Due to the very interesting thesis [DM07], there is a constant $K$ such that the bound

$$\Gamma(P_t f, P_t f)(e) \leq K P_t(\Gamma(f, f))(e)$$

holds true for all test functions $f \in C_0^\infty(G)$ and for all times $t \geq 0$. Furthermore, due to the scaling properties of $P_t$, there exists a best constant $K$ such that this bound is sharp for all $t \geq 0$. This shows that $a_t Kt = \frac{1}{2}$, due to the assertions of Theorem 3.1. □

3.1. Generalisation to homogeneous spaces. Consider now $M$ a homogeneous space with respect to the Lie group $G$, i.e. we have a (right) transitive action $\hat{\pi} : G \times M \to M$ of $G$ on $M$. We assume that there exists a measure $\mu^M$ on $M$ which is invariant with respect to this action. We also assume that we are given a family $V_1, \ldots, V_d$ of left-invariant vector fields on $G$ that generate its entire Lie algebra $\mathfrak{g}$ as before. These vector fields induce vector fields $V_i^M$ on $M$ by means of the action $\hat{\pi}$. 

By choosing an ‘origin’ $o \in M$, we obtain a surjection $\pi : G \to M$ by $\pi(g) = \hat{\pi}(g, o)$. Due to the invariance of $\mu^M$ with respect to the action $\hat{\pi}$, the vector fields $V^M_i$ are anti-symmetric operators on $L^2(\mu^M)$ and the generator
\[ L^M = \frac{1}{2} \sum_{i=1}^{d} (V^M_i)^2 \]
is consequently symmetric on $L^2(\mu^M)$. In particular we have
\[ (V^M_i f) \circ \pi = V_i (f \circ \pi) \]
for $i = 1, \ldots, d$. The local Driver-Melcher inequality on $G$ translates to the same inequality on $M$ by means of
\[ P^M_t(f) \circ \pi = P_t(f \circ \pi) \]
for test functions $f : M \to \mathbb{R}$, hence we obtain the corresponding Driver-Melcher inequality on $M$.

4. A SIMPLE RESULT ON SIMULATED ANNEALING

For applications to simulated annealing, our main tool will be the following simple Theorem:

**Theorem 4.1.** Let $U : G \to \mathbb{R}$ be a potential $U$ with
\[ |U + \log p(\varepsilon, \cdot)| \leq D \varepsilon \]
for some $\varepsilon > 0$ and some constant $D > 0$. Assume furthermore that a Poincaré inequality holds for $L^2$, i.e.
\[ P_\varepsilon(f^2)(e) \leq K_\varepsilon P_\varepsilon(\Gamma(f, f))(e) \]
for test functions $f \in C_0^\infty(G)$ with $P_\varepsilon f(e) = 0$ and some constant $K > 0$. Then one has $\exp(-U) \in L^1(\mu(dx))$ and the Poincaré inequality
\[ \int f^2(x) \exp(-U(x)) \mu(dx) \leq C \int \Gamma(f, f)(x) \exp(-U(x)) \mu(dx) \]
holds for all test functions $f \in C_0^\infty(G)$ with $\int f(x) \exp(-U(x)) \mu(dx) = 0$ and some constant $C = K_\varepsilon \exp(2D \varepsilon) > 0$. In particular, this leads to a spectral gap of size at least $\frac{1}{C_\varepsilon}$ for $L^U$.

**Proof.** It follows immediately from the inequality
\[ p(\varepsilon, x) = \exp(-U(x)) \exp(U(x)) p(\varepsilon, x) \geq \exp(-D) \exp(-U(x)) \]
that $\exp(-U) \in L^1(\mu)$. Furthermore,
\[ \exp(-U(x)) = \frac{1}{p(\varepsilon, x) \exp(U(x))} p(\varepsilon, x) \geq \exp(-D) p(\varepsilon, x) \]
for all $x \in G$ by assumption. Hence we deduce (4.2) with $C_\varepsilon = K_\varepsilon \exp(2D \varepsilon)$ from (4.1).

**Remark 4.2.** See [BLW07] for results on unbounded perturbations, where one can hope for similar conclusions.
Throughout the remainder of this section we assume that $M$ is a nilmanifold, that is a homogeneous space with respect to a nilpotent Lie group $G$. We consider the same structures as in Section 3.1 on $M$, but we omit the index $M$ in the vector fields and measures in order to improve readability. We shall furthermore assume that $M$ satisfies the following global estimate:

**Assumption 4.3.** There is a constant $	ilde{D}$ such that

$$|d(x_0, x)^2 + t \log p(t, x_0, x)| \leq \tilde{D} \quad (4.3)$$

for all $0 < t < 1$ and all $x \in M$ and some (and therefore all by translation invariance) $x_0 \in M$. Here, $d$ denotes the lift of the Carnot-Carathéodory distance from $G$ to $M$.

**Remark 4.4.** If $M$ is a compact nilmanifold, we can apply Léandre’s beautiful result [Léa87b, Léa87a] that

$$\lim_{t \to 0} t \log p(t, x_0, x) = -d(x_0, x)^2$$

holds true uniformly on $M$, which implies Assumption 4.3.

A non-compact example of a nilmanifold, where this estimate still holds true is given by the Heisenberg group $G^2_d$. Notice that this is an example of dimension $d + \frac{d(d-1)}{2}$.

We prove a quantitative simulated annealing result under the previous assumption on the nilmanifold $M$. The idea is to introduce a parameter $\varepsilon$ in the operators,

$$\mathcal{L}^U, \varepsilon = \mathcal{L} - \frac{1}{2} \Gamma \left( \frac{U}{\varepsilon^2}, \cdot \right),$$

such that the associated invariant measure $\exp(-\frac{U}{\varepsilon^2})\mu$ concentrates around the minima of $U$. Notice that the previous operator satisfies

$$\varepsilon^2 \mathcal{L}^U, \varepsilon = \varepsilon^2 \mathcal{L} - \frac{1}{2} \Gamma(U, \cdot),$$

hence a spectral gap for $\varepsilon^2 \mathcal{L}^U, \varepsilon$ leads to a spectral gap for the diffusion process

$$dY^\mu_t = V_0(Y^\mu_t)dt + \sum_{i=1}^d \varepsilon V_i(Y^\mu_t) \circ dB^i_t, \quad Y^\mu_0 = y \in G,$$

with $V_0f = -\frac{1}{2} \Gamma(U, f)$. Consequently we know – given strong existence – that the law of $Y^\mu_t$ converges to $\exp(-\frac{U}{\varepsilon^2})\mu$ and concentrates a posteriori around the minima of $U$.

In the following theorem we try to quantify this behaviour. We denote by $\mu^{U, \varepsilon}$ the probability measure invariant for $\mathcal{L}^{U, \varepsilon}$ and we use the notation

$$\var_{\varepsilon}(f) = \langle (f - \langle f \rangle_{\varepsilon})^2 \rangle_{\varepsilon}$$

with $\langle f \rangle_{\varepsilon} = \int_M f(g) \mu^{U, \varepsilon}(dg)$.

**Theorem 4.5.** Let $U : M \to \mathbb{R}$ be a smooth function such that there exist a constant $D$ and a point $x_0 \in M$ such that

$$|U(x) - d(x_0, x)^2| \leq D,$$

for all $x \in M$. Then there exist constants $R, c > 0$ such that for $\varepsilon(t) = \frac{\varepsilon}{\sqrt{\log(R+t)}}$

and

$$\var_{\varepsilon(t)}(f) \leq K(R + t)\langle \Gamma(f, f) \rangle_{\varepsilon(t)},$$
for all test functions \( f \in C_0^\infty(M) \) and \( t \geq 0 \).

**Proof.** We can start to collect results. Combining Corollary 3.3 and Assumption 4.3 with Theorem 4.1, we obtain that spectral gap for the operator \( \mathcal{L}^{U,\varepsilon} \) has size at least

\[
\frac{1}{K\varepsilon^2} \exp\left(-\frac{2(D + \tilde{D})}{\varepsilon^2}\right)
\]

for \( 0 < \varepsilon < 1 \), so that \( \varepsilon^2 \mathcal{L}^{U,\varepsilon} \) has a spectral gap of size at least

\[
\frac{1}{K} \exp\left(-\frac{2(D + \tilde{D})}{\varepsilon^2}\right).
\]

We choose \( \varepsilon^2 = 2(D + \tilde{D}) \) and \( R \) sufficiently big so that \( \varepsilon(t) < 1 \) for \( t \geq 0 \), and we conclude that

\[
K \exp\left(\frac{2(D + \tilde{D})}{\varepsilon(t)^2}\right) \leq K(R + t)
\]

for all \( t \geq 0 \), which is the desired result. \( \square \)

We denote by \( Z \) the process with cooling schedule \( t \mapsto \varepsilon(t) \) as in the previous theorem,

\[
dZ_t = V_0(Z_t)dt + \sum_{i=1}^d \varepsilon(t)V_i(Z_t) \circ dB^i_t.
\]

Then the previous conclusion leads to the following Proposition.

**Proposition 4.6.** Let \( f_t \) denote the Radon-Nikodym derivative of the law of \( Z_t^\varepsilon \) with respect to \( \mu_t = \mu^{U,\varepsilon(t)} \) and let

\[
u(t) := \|f_t - 1\|^2_{L^2(\mu_t)}
\]

denote the distance in \( L^2(\mu_t) \), then

\[
u'(t) \leq -\frac{1}{K(R + t)} \nu(t) + \frac{1}{c^2(R + t)} \nu(t) + \frac{1}{c^2(R + t)} \sqrt{\nu(t)}
\]

for the constants \( R, c, K \) from Theorem 4.5.

**Remark 4.7.** If we choose \( c^2 > K \), which is always possible, we conclude that \( \sup_{t \geq 0} \nu(t) \) is bounded from above by a constant depending on \( f_0, c, R \) and \( K \).

**Proof.** The proof follows closely the lines of [HS88]. By assumption we know that \( \nu(t) = \|f_t\|^2_{L^2(\mu_t)} - 1 \) and hence with the notation \( \beta(t) = \frac{1}{\varepsilon(t)^2} \),

\[
u'(t) \leq -2(\Gamma(f_t, f_t))_{\varepsilon(t)} - \beta'(t) \int (U - \langle U \rangle_{\varepsilon(t)}) f_t^2 \mu_t
\]

\[
= -2(\Gamma(f_t, f_t))_{\varepsilon(t)} - \beta'(t) \int (U - \langle U \rangle_{\varepsilon(t)})(f_t - 1)^2 \mu_t
\]

\[
- 2\beta'(t) \int (U - \langle U \rangle_{\varepsilon(t)})(f_t - 1) \mu_t
\]

\[
\leq -\frac{1}{K(R + t)} \nu(t) + \frac{1}{c^2(R + t)} \nu(t) + \frac{1}{c^2(R + t)} \sqrt{\nu(t)}.
\]

Here, we used the Cauchy-Schwarz inequality and the conclusions of the previous Theorem 4.5. \( \square \)
Theorem 4.8. Assume that we are in the previous settings with $c^2 > K$, so that $\sup_{t \geq 0} \|f_t\|_{L^2(\mu_t)} \leq M < \infty$. Define $U_0 = \inf_{x \in M} U(x)$ and, for every $\delta > 0$, denote by $A_\delta$ the set $A_\delta = \{ x \in M \mid U(x) \geq U_0 + \delta \}$. Then we can conclude that

$$P(Z_t^i \in A_\delta) \leq M \sqrt{\mu_t(A_\delta)}$$

for every $t > 0$ and every $\delta \geq 0$.

Proof. It follows from the Cauchy-Schwarz inequality that

$$P(Z_t^i \in A_\delta) = \int_{A_\delta} f_t \mu_t \leq M \sqrt{\mu_t(A_\delta)},$$

as required. □

Remark 4.9. Since $\lim_{\varepsilon \to 0} \mu_{U,\varepsilon}(A_\delta) = 1$ for every $\delta > 0$, we obtain that for all continuous bounded test functions $f$, we have

$$E(f(Z_t^i)) \to f(x_0),$$

provided that there is only one element $x_0 \in M$ such that $U(x_0) = U_0$.

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