Demazure Slices of Type $A_{2l}^{(2)}$

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Abstract
We consider a Demazure slice of type $A_{2l}^{(2)}$, that is an associated graded piece of an infinite-dimensional version of a Demazure module. We show that a global Weyl module of a hyperspecial current algebra of type $A_{2l}^{(2)}$ is filtered by Demazure slices. We calculate extensions between a Demazure slice and a usual Demazure module and prove that a graded character of a Demazure slice is equal to a nonsymmetric Macdonald-Koornwinder polynomial divided by its square norm. In the last section, we prove that a global Weyl module of the special current algebra of type $A_{2l}^{(2)}$ is a free module over the polynomial ring arising as the endomorphism ring of itself.

Keywords
Affine Lie algebra · Demazure modules · Macdonald-Koornwinder polynomials · Weyl modules

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1 Introduction

A Demazure module in a highest weight module $L(\lambda)$ of a Kac-Moody Lie algebra $\mathfrak{g}$ is studied for a long time. For an affine Lie algebra $\mathfrak{g}$, there are two types of Demazure modules in the literature [17, 20]. One is a thin Demazure module, that is usual Demazure module. The other is a thick Demazure module, that is an infinite-dimensional version of a thin Demazure module. Consider an affine Lie algebra of type $X_{l}^{(r)}$ ($X = A, D, E$) and $r = 1, 2, 3$ that is called type I in [2]. Its level one thin Demazure module has special features. Sanderson [25] and Ion [13] showed that its graded character is equal to a nonsymmetric Macdonald polynomial specialized at $t = 0$ in $X_{l}^{(r)} \neq A_{2l}^{(2)}$-case and equal to a nonsymmetric Macdonald-Koornwinder polynomial specialized at $t = 0$ in $A_{2l}^{(2)}$-case. Another special feature is the connection with a local Weyl module of a current algebra.

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that is a hyperspecial maximal parabolic subalgebra of $\mathfrak{g}$ [6]. Chari-Loktev [3], Fourier-Littelmann [3], Fourier-Kus [10] and Chari-Ion-Kus [6] showed that a $\mathfrak{c}_\mathfrak{g}$-stable level one thin Demazure module is isomorphic to a local Weyl module as a $\mathfrak{c}_\mathfrak{g}$-module.

Less is known about a thick Demazure module compared to a thin Demazure module. A thick Demazure module is a module of a lower Borel subalgebra that is generated from an extremal weight vector of $L(\Lambda)$. Cherednik and Kato [7] recently studied a Demazure slice that is defined as a quotient module of a thick Demazure module. In type I but not of type $A(2)_2$, they showed that a global Weyl module of $\mathfrak{c}_\mathfrak{g}$ have a filtration by level 1 Demazure slices. Moreover they calculated extensions between a level one Demazure slice and a level one thin Demazure module. As a result, they showed graded characters of a level one Demazure slice and a thin Demazure module are orthogonal to each other with respect to the Euler-Poincaré-pairing. In particular, the graded character of a Demazure slice is equal to a nonsymmetric Macdonald polynomial specialized at $t = \infty$ divided by its square norm.

In this paper, we provide analogues of these results in [7] for $A(2)_2$. Let $\mathfrak{g}$ be an affine Kac-Moody Lie algebra of type $A(2)_2$ and $\mathfrak{g}$ be a simple Lie algebra of type $\mathfrak{c}_\mathfrak{g}$ contained in $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{p}$ be the integral weight lattice of $\mathfrak{g}$ and $\mathfrak{p}^+$ be the set of dominant integral weights of $\mathfrak{g}$. For each $\lambda \in \mathfrak{p}^+$, we have a $\mathfrak{c}_\mathfrak{g}$-module $W(\lambda)$, that is called a global Weyl module. Level one Demazure slices and thin Demazure modules are parametrized by $\lambda \in \mathfrak{p}^+$ as $D_\lambda$ and $D_{\lambda}$, respectively. Let $\Lambda_0$ be the unique level one dominant integral weight of $\mathfrak{g}$ and let $\delta$ be the simple imaginary root of $\mathfrak{g}$. Let $\tilde{W}$ be the Weyl group of $\mathfrak{g}$. Let $b_-$ be a lower-triangular Borel subalgebra of $\mathfrak{g}$.

**Theorem A** (Theorem 3.28) For each $\lambda \in \mathfrak{p}^+$, the global Weyl module $W(\lambda) \otimes \mathbb{C}_{\Lambda_0}$ has a filtration by Demazure slices as $b_-$-module and each $D_\mu$ ($\mu \in \tilde{W} \lambda$) appears exactly once.

Let $\mathcal{B}$ be a full subcategory of the category of $U(b_-)$-modules and $\langle -, - \rangle_{\text{Ext}}$ be the Euler-Poincaré-pairing associated to $\text{Ext}^\mathcal{B}$ (see Section 1 for their precise definitions).

**Theorem B** (Theorem 3.42) For each $\lambda, \mu \in \mathfrak{p}$, $m \in \frac{1}{2}\mathbb{Z}$ and $k \in \mathbb{Z}$, we have

$$\dim_{\mathbb{C}} \text{Ext}^m_{\mathcal{B}}(\mathbb{D}^\lambda \otimes_{\mathbb{C}} \mathcal{C}_{m\delta+k\Lambda_0}, D^\mu_\nu) = \delta_{n,0}\delta_{m,0}\delta_{k,0}\delta_{\lambda,\mu} \quad n \in \mathbb{Z}^+,$$

where $\nu$ means the restricted dual.

For each $\lambda \in \mathfrak{p}$, let $\tilde{E}_\lambda(x_1, ..., x_l, q)$ and $E^\dagger_\lambda(x_1, ..., x_l, q)$ be nonsymmetric Macdonald polynomials specialized at $t = 0, \infty$ respectively. Let $(-, -)$ be the Weyl group invariant inner product on the dual of a Cartan subalgebra $\mathfrak{h}^*$ normalized so that the square length of the shortest roots of $\mathfrak{g}$ with respect to $(-, -)$ is 1. Let $\text{gch} M$ be a graded character of $M$ (see Section 2.6 for the definition). As a corollary of Theorem B, we have

**Theorem C** (Corollary 3.44) For each $\lambda \in \mathfrak{p}$, we have

$$\text{gch} \mathbb{D}^\lambda = q^{\langle b \mid b \rangle} E^\dagger_\lambda(x_1^{-1}, ..., x_l^{-1}, q^{-1})/(\tilde{E}_\lambda, E^\dagger_\lambda)_{\text{Ext}}.$$

In this paper, we refer to a maximal parabolic subalgebra of affine Lie algebra that contains a finite dimensional simple Lie algebra as a current algebra. For an affine Lie algebra
of type \( A_{2/l}^{(2)} \), two kind of current algebras are studied in the literature. They contain simple Lie algebras of type \( C_l \) and \( B_l \), respectively. The former is called a hyperspecial current algebra. A dimension formula of a local Weyl module of a hyperspecial current algebra and freeness of a global Weyl module over its endomorphism ring are proved in [6]. The latter is called a special current algebra and a dimension formula of a local Weyl module of a special current algebra is proved in [10] and [8]. Let \( \mathfrak{C} \hat{g}^\dagger \) be a special current algebra of \( g \). Then \( \mathfrak{C} \hat{g}^\dagger \) contains a simple Lie algebra \( \hat{g}^\dagger \) of type \( B_l \). Let \( W(\lambda)^\dagger \) be a global Weyl module of \( \mathfrak{C} \hat{g}^\dagger \). In the last section, we prove the following theorem.

**Theorem D** (=Theorem 4.15+Theorem 4.16) Let \( \lambda \) be a dominant integral weight of \( \hat{g}^\dagger \). The endomorphism ring \( \text{End}_{\mathfrak{C} \hat{g}^\dagger}(W(\lambda)^\dagger) \) is a polynomial ring and \( W(\lambda)^\dagger \) is free over \( \text{End}_{\mathfrak{C} \hat{g}^\dagger}(W(\lambda)^\dagger) \).

The organization of the paper is as follows: In section one, we prepare basic notation and definitions. Section two is about a Demazure slice. Main contents of section two are the relation between a global Weyl module and a Demazure slice (Theorem A), and calculation of extensions between a Demazure slice and a thin Demazure module (Theorem B). As a corollary, we prove a character formula of a Demazure slice (Theorem C). In section three, we study a global Weyl module of a special current algebra of type \( A_{2/l}^{(2)} \). We prove the endomorphism ring of a global Weyl module is isomorphic to a polynomial ring and a global Weyl module is free over its endomorphism ring (Theorem D).

### 2 Preliminaries

We refer to [24], [16, Chapter 6] and [2] for general terminologies throughout this section. Mainly we refer to [16] for Sections 2.2 and 2.4 and refer to [2] for the Section 2.3.

#### 2.1 Notations

We denote the set of complex numbers by \( \mathbb{C} \), the set of integers by \( \mathbb{Z} \), the set of nonnegative integers by \( \mathbb{Z}_+ \), the set of rational numbers by \( \mathbb{Q} \), and the set of natural numbers by \( \mathbb{N} \). We work over the field of complex numbers. In particular, a vector space is a \( \mathbb{C} \)-vector space. For each \( x \in \mathbb{Q} \), we set \( \lfloor x \rfloor := \max\{z \in \mathbb{Z} | x \geq z\} \). We set \( x(r) := x^r / r! \) for an element \( x \) of a \( \mathbb{C} \)-algebra.

#### 2.2 Affine Kac-Moody Algebra of Type \( A_{2/l}^{(2)} \)

Let \( \mathfrak{g} \) be an affine Kac-Moody algebra of type \( A_{2/l}^{(2)} \) and \( \mathfrak{h} \) be its Cartan subalgebra. We denote the set of roots of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \) by \( \Delta \) and fix a set of simple roots \( \{\alpha_0, \alpha_1, \ldots, \alpha_l\} \), where \( \alpha_0 \) is the shortest simple root of \( \mathfrak{g} \). Let \( \Delta_+ \) and \( \Delta_- \) be the set of positive and negative roots, respectively. We set the simple imaginary root as \( \delta := 2\alpha_0 + \alpha_1 + \cdots + \alpha_l \), the set of imaginary roots as \( \Delta_{im} := \mathbb{Z}\delta \), and the set of real roots \( \Delta_{re} := \Delta \setminus \Delta_{im} \). We set \( \mathcal{Q} := \bigoplus_{i=0}^{l} \mathbb{Z}\alpha_i \), \( \hat{\mathcal{Q}} := \bigoplus_{i=1}^{l} \mathbb{Z}\alpha_i \), and \( \hat{\mathcal{Q}}^\dagger := \bigoplus_{i=0}^{l-1} \mathbb{Z}\alpha_i \). We set \( \mathcal{Q}_+ := \bigoplus_{i=0}^{l} \mathbb{Z}_+\alpha_i \), \( \hat{\mathcal{Q}}_+ := \bigoplus_{i=1}^{l} \mathbb{Z}_+\alpha_i \), and \( \hat{\mathcal{Q}}^\dagger_+ := \bigoplus_{i=0}^{l-1} \mathbb{Z}_+\alpha_i \). Let \( \hat{\Delta} = \Delta \cap \hat{\mathcal{Q}} \). The set \( \hat{\Delta} \) is a root system of type \( C_l \). Using the standard basis \( \varepsilon_1, \ldots, \varepsilon_l \) of \( \mathbb{R}^l \), we have:

\[
\hat{\Delta} = \{ (\pm \varepsilon_i \pm \varepsilon_j), \pm 2\varepsilon_i | i, j = 1, \ldots, l \}.
\]
We denote the set of short roots of \( \hat{\mathfrak{g}} \) by \( \hat{\Delta}_s \) and the set of long roots of \( \hat{\mathfrak{g}} \) by \( \hat{\Delta}_l \). We have
\[
\Delta_{re} = (\hat{\Delta}_s + \mathbb{Z}\delta) \cup (\hat{\Delta}_l + 2\mathbb{Z}\delta) \cup \frac{1}{2}(\hat{\Delta}_l + (2\mathbb{Z} + 1)\delta)
\]
and
\[
\alpha_0 = \frac{\delta}{2} + \varepsilon_1, \quad \alpha_1 = -\varepsilon_1 + \varepsilon_2, \ldots, \alpha_{l-1} = -\varepsilon_{l-1} + \varepsilon_l, \quad \alpha_l = -2\varepsilon_l.
\]
We set \( \Delta_{i\pm} := \Delta_\pm \cap \Delta_i, \Delta_{s\pm} := \Delta_\pm \cap \Delta_s \) and \( \hat{\Delta}_\pm := \Delta_\pm \cap \hat{\Delta} \). For each \( \alpha \in \Delta_{re} \), let \( \check{\alpha} \in \mathfrak{h} \) be the corresponding coroot of \( \mathfrak{g} \). Let \( \theta \) be the highest root of \( \hat{\Delta} \). Let \( d \in \mathfrak{h} \) be the scaling element that satisfies \( \alpha_i(d) = \delta_i,0 \). We denote a central element of \( \mathfrak{g} \) by \( K = \hat{\alpha}_0 + 2\hat{\alpha}_1 + \cdots + 2\hat{\alpha}_l \). For each \( \alpha \in \Delta \), we denote the root space corresponding to \( \alpha \) by \( \mathfrak{g}_\alpha \). For each \( \alpha \in \Delta_{re} \), the root space \( \mathfrak{g}_\alpha \) is one dimensional and we denote a nonzero vector in \( \mathfrak{g}_\alpha \) by \( e_\alpha \). A Borel subalgebra \( \mathfrak{b}_\pm \) and a maximal nilpotent subalgebra \( \mathfrak{n}_\pm \) of \( \mathfrak{g} \) are
\[
\mathfrak{b}_+ = \mathfrak{h} \oplus n_+, \quad n_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha, \quad \mathfrak{b}_- = \mathfrak{h} \oplus n_-, \quad \text{and} \quad n_- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha.
\]
For each \( i \in \{0, 1, \ldots, l\} \), we define \( \Lambda_i \in \mathfrak{h}^\ast \) by
\[
\Lambda_i(\check{\alpha}_j) = \delta_{i,j}, \quad \Lambda_i(d) = 0.
\]
We set
\[
P := \mathbb{Z}\Lambda_0 \oplus \cdots \oplus \mathbb{Z}\Lambda_l \oplus \mathbb{Z}\frac{\delta}{2}, \quad \text{and} \quad P_+ := \mathbb{Z}_+\Lambda_0 \oplus \cdots \oplus \mathbb{Z}_+\Lambda_l \oplus \mathbb{Z}_+\frac{\delta}{2}.
\]
We set \( \sigma_i := \Lambda_i - 2\Lambda_0 \) (\( i \in \{1, \ldots, l\} \)),
\[
\hat{P} = \mathbb{Z}\sigma_1 \oplus \cdots \oplus \mathbb{Z}\sigma_l \quad \text{and} \quad \hat{P}_+ = \mathbb{Z}_+\sigma_1 \oplus \cdots \oplus \mathbb{Z}_+\sigma_l.
\]
We set \( \hat{Q}' := \hat{Q} + \frac{1}{2}\mathbb{Z}\hat{\Delta}_l \) and \( \hat{Q}'_+ := \hat{Q}_+ + \frac{1}{2}\mathbb{Z}_+\hat{\Delta}_l \).

### 2.3 Hyperspecial Current Algebra of \( A_{2l}^{(2)} \)

We set \( \hat{\mathfrak{h}} := \bigoplus_{i=1}^l \mathbb{C}\alpha_i, \quad \hat{\mathfrak{g}} := \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \oplus \hat{\mathfrak{h}}, \) and \( \hat{\mathfrak{b}}_+ := \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \). Then \( \hat{\mathfrak{g}} \) is a finite dimensional simple Lie algebra of type \( \mathfrak{c}_l \), the Lie subalgebra \( \hat{\mathfrak{h}} \) is a Cartan subalgebra of \( \hat{\mathfrak{g}} \), the Lie subalgebra \( \hat{\mathfrak{b}}_+ \) is a Borel subalgebra of \( \hat{\mathfrak{g}} \), and \( \hat{\Delta} \) is the set of roots of \( \hat{\mathfrak{g}} \) with respect to \( \hat{\mathfrak{h}} \). The lattice \( \hat{P} \) is the integral weight lattice of \( \hat{\mathfrak{g}} \), and \( \hat{P}_+ \) is the set of dominant integral weight of \( \hat{\mathfrak{g}} \). A hyperspecial current algebra \( \mathfrak{c}_l \) is a maximal parabolic subalgebra of \( \mathfrak{g} \) that contains \( \hat{\mathfrak{g}} \), i.e.,
\[
\mathfrak{c}_l := \hat{\mathfrak{g}} + \mathfrak{b}_-.
\]
We set \( \mathfrak{c}_l' := \{\mathfrak{c}_l, \mathfrak{c}_l\} \).

**Remark 2.1** Usually \( \mathfrak{c}_l' \) is called current algebra in the literature. We have \( \mathfrak{c}_l = \mathfrak{c}_l' \oplus \mathfrak{g} \oplus \mathfrak{K} \).

We define a subalgebra \( \mathfrak{c}_{\mathfrak{g}_{im}} \) of \( \mathfrak{c}_l \) by
\[
\mathfrak{c}_{\mathfrak{g}_{im}} := \bigoplus_{n \in \mathbb{N}} \mathfrak{g}_{n\delta},
\]
and define a subalgebra \( \mathfrak{n}_+ \) of \( \mathfrak{c}_l \) by
\[
\mathfrak{n}_+ := \bigoplus_{\alpha \in (\hat{\Delta}_+ - \mathbb{Z}\delta) \cup (\hat{\Delta}_+ - 2\mathbb{Z}\delta) \cap \frac{1}{2}(\hat{\Delta}_+ - (2\mathbb{Z} + 1)\delta)} \mathfrak{g}_\alpha.
\]
2.4 Weyl Group

Let \( s_\alpha \in \text{Aut}(\mathfrak{h}^*) \) be the simple reflection corresponding to \( \alpha \in \Delta_{re} \). We have
\[
s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^* \rangle \alpha, \quad \text{for } \lambda \in \mathfrak{h}^*.
\]

We set \( W \) as the subgroup of \( \text{Aut}(\mathfrak{h}^*) \) generated by \( s_\alpha \) \((\alpha \in \Delta_{re})\), and \( \hat{W} \) as the subgroup generated by \( s_\alpha \) \((\alpha \in \Delta)\). For each \( i = 0, \ldots, l \), let \( s_i := s_{\alpha_i} \). Then \( W \) is generated by \( s_i \) \((i = 0, \ldots, l)\), and \( \hat{W} \) is generated by \( s_i \) \((i = 1, \ldots, l)\). Let \((-|-)\) be a \( W \)-invariant bilinear form on \( \mathfrak{h}^* \) normalized so that \( (\alpha_0|\alpha_0) = 1 \). For each \( \mu \in \hat{P} \), we define \( t_\mu \in \text{Aut}(\mathfrak{h}^*) \) by
\[
t_\mu(\lambda) = \lambda + \langle \lambda, K \rangle \mu - \left( \frac{1}{2} \langle \mu | \mu \rangle \langle \lambda, K \rangle \right) \delta.
\]
We have \( t_\mu \in W \) and
\[
W = \hat{W} \ltimes \hat{P}. \tag{2.1}
\]

For each \( \lambda \in \hat{P} \), we denote the unique element of \( \hat{W} \lambda \cap \pm P_+ \) by \( \lambda_+ \), respectively. We set \( \rho := \frac{1}{2} \sum_{\alpha \in \Delta_{re}} \alpha \). For each \( w \in \hat{W} \lambda \) and \( \lambda \in \hat{P} \), we define \( w \circ \lambda := w(\lambda + \rho) - \rho \). For each \( \Lambda \in P \), we set \( W^\Lambda := \{ w \in W \mid w \Lambda = \Lambda \} \). We denote the set of minimal coset representatives of \( \hat{W} \lambda \) by \( W \). 

**Definition 2.2** (Reduced expression) Each \( w \in W \) can be written as a product \( w = s_{i_1} s_{i_2} \cdots s_{i_n} \) \((i_j \in \{0, \ldots, l\})\). If \( n \) is minimal among such expressions, then \( s_{i_1} s_{i_2} \cdots s_{i_n} \) is called a reduced expression of \( w \) and \( n \) is called the length of \( w \) (written as \( l(w) \)).

**Definition 2.3** (Left weak Bruhat order) Let \( w \in W \) and \( i = 0, \ldots, l \). We write \( s_i w > w \) if \( l(s_i w) > l(w) \) holds. Left weak Bruhat order is the partial order on \( W \) generated by \( > \).

**Definition 2.4** (Macdonald order) We write \( \mu \succeq \lambda \) if and only if one of the following two conditions holds:

1. \( \mu - \lambda \in \hat{Q}_+ \) if \( \mu \in \hat{W} \lambda \);
2. \( \lambda_+ - \mu_+ \in \hat{Q}_+ \) if \( \mu_+ \neq \lambda_+ \).

For \( w \in W \) and \( \mu \in \hat{P} \), let \( w(\mu) \in \hat{P} \) be the restriction of \( w(\mu + \Lambda_0) \) to \( \mathfrak{h} \). For each \( \lambda \in \hat{P} \), let \( \pi_\lambda \in W \) be a minimal length element such that \( \pi_\lambda(00) = \lambda \). For each \( \mu \in \hat{P} \), we denote the convex hull of \( \hat{W} \mu \) by \( C(\mu) \).

**Lemma 2.5** ([21] Proposition 2.6.2) If \( \mu \in \hat{P} \), then \( C(\mu) \cap (\mu + \hat{Q}') \subseteq \bigcap_{w \in \hat{W}} w(\mu - \hat{Q}_+'). \)

**Proof** The set \( w(\mu_+ - \hat{Q}_+') \) is the intersection of \( \mu_+ + \hat{Q}' \) with the convex hull of \( w(\mu_+ - \hat{Q}_+) \). The set \( \hat{W} \mu \) is contained in \( \bigcap_{w \in \hat{W}} w(\mu_+ - \hat{Q}_+) \). Hence we have \( C(\mu) \cap (\mu + \hat{Q}') \subseteq \bigcap_{w \in \hat{W}} w(\mu_+ - \hat{Q}_+) \). \( \square \)

**Lemma 2.6**

1. If \( w > v \in W \), then \( v(0) \succeq w(0) \);
2. Let \( b, c \in \hat{P} \) satisfy \( b = s_i(\langle c \rangle) \) for some \( i = 0, \ldots, l \). Then
\[
c \succ b \iff c = \pi_{s_i} \pi_c > \pi_c.
\]

**Proof** First, we prove (1). It is enough to prove the assertion for \( w = s_i v \). Since \( w > v \), we have \( \langle v \Lambda_0, \alpha_i \rangle \geq 0 \). This implies \( v \Lambda_0 - w \Lambda_0 \in Q_+ \). Hence we have \( v(0) \succeq w(0) \).
$w(0)$ if $i \neq 0$. If $i = 0$, then we have $w(0) - v(0) = \langle v \Lambda_0, \delta_0 \rangle \theta/2$. We set $N = \langle v \Lambda_0, \delta_0 \rangle$. We have $\langle w(0), \Theta \rangle = (N + 1)/2$. Hence $s_0(w(0)) = w(0) - N+1/2$ and $v(0) = N + 1 w(0)$. Therefore, $v(0) \in C(w(0)) \cap (w(0) + \tilde{Q}')$. By Lemma 2.5, $w(0)+ - v(0)+ = 0$. Hence $v(0) \geq w(0)$.

Next, we prove (2). We already proved ($\Leftarrow$). So we prove ($\Rightarrow$). By Definition 2.3, we have $s_i c > \pi c$ or $s_i c < \pi c$. From $c > b$ and (1), we have $s_i c > \pi c$ and $\pi b > s_i \pi b$. We have $(s_i c)(0) = b$ thanks to $b = s_i (c)$. We show that $\pi b = s_i c$. If $\pi b \neq s_i c$, then we have $l(s_i c) > l(\pi b)$ by the minimality of $l(\pi b)$. Since $l(\pi b) = l(s_i c) + 1$, $l(s_i c) = l(\pi c) + 1$ and $l(s_i c) > l(\pi c)$, we get $l(\pi c) > l(s_i c)$. This contradicts the minimality of $l(\pi c)$. Hence the assertion follows.

2.5 Macdonald-Koornwinder Polynomials

In this subsection, we recall materials presented in [24, §3] and [13], and we specialize parameters $t$, $t_0$, $u_0$, $u_l$ in [24] as $t_0 = t_l = u_0 = t$ and $u_l = 1$ [13].

2.5.1 Nonsymmetric Case

We set $\mathbb{F} := \mathbb{Q}(t, q^{1/2})$. Let $\mathbb{F}[\hat{P}]$ be a group ring of $\hat{P}$ over $\mathbb{F}$ and $X^\lambda$ be an element of $\mathbb{F}[\hat{P}]$ corresponding to $\lambda \in \hat{P}$. We identify $\mathbb{F}[X_1^{\pm 1}, ..., X_l^{\pm 1}]$ with $\mathbb{F}[\hat{P}]$ by $x_i = X^{\varepsilon_i}$ for each $i \in \{1, ..., l\}$. We define

$$\Delta(x) := \Delta(x)_+ \Delta(x)^{-1}_+ \prod_{n \in \mathbb{N}} (1 - q^n) \in \mathbb{F}[X_1^{\pm 1}, ..., X_l^{\pm 1}]$$

by

$$\Delta(x)_+ := \prod_{i=1, \ldots, l} \frac{(x_i)_\infty (-x_i)_\infty (q^{1/2} x_i)_\infty}{(tx_i)_\infty (-tx_i)_\infty (q^{1/2} x_i)_\infty} \prod_{1 \leq i < j \leq l} \frac{(x_i x_j)_\infty (x_i x_j^{-1})_\infty}{(tx_i x_j)_\infty (tx_i x_j^{-1})_\infty}.$$ 

Here $(u)_\infty = \prod_{n \in \mathbb{Z}_+} (1 - q^n u)$. We define

$$\varphi(x) := \prod_{i=1, \ldots, l} \frac{(x_i - t)(x_i + t)}{x_i - 1} \prod_{1 \leq i < j \leq l} \frac{(x_i x_j - t)(x_i x_j^{-1} - t)}{(x_i x_j - 1)(x_i x_j^{-1} - 1)}$$

and $C(x) := \Delta(x) \varphi(x)$. We have

$$\Delta(x)_+|_{t=0} = \prod_{i=1, \ldots, l} \frac{1}{(x_i)_\infty (-x_i)_\infty (q^{1/2} x_i)_\infty} \prod_{1 \leq i < j \leq l} \frac{(x_i x_j)_\infty (x_i x_j^{-1})_\infty}{(tx_i x_j)_\infty (tx_i x_j^{-1})_\infty}$$

and

$$\varphi(x)|_{t=0} = \prod_{i=1, \ldots, l} \frac{1}{1 - x_i^{-2}} \prod_{1 \leq i < j \leq l} \frac{1}{(1 - x_i^{-1} x_j^{-1})(1 - x_i^{-1} x_j)}.$$ 

Under the identification $x_i = X^{\varepsilon_i}$, we have

$$\Delta(x)|_{t=0} = \prod_{\alpha \in \Delta \text{ and } \alpha(d) \leq 0} (1 - X^\alpha)^{\dim g_\alpha} \text{ and } \varphi(x)|_{t=0} = \prod_{\alpha \in \Delta_+} \frac{1}{1 - X^\alpha}.$$ 

Hence we have

$$C|_{t=0} = \prod_{\alpha \in \Delta_-} \frac{1}{1 - X^\alpha}.$$
Definition 2.7 We define an inner product on \( \mathbb{F}[x_1^{\pm 1}, \ldots, x_l^{\pm 1}] \) by
\[
\langle f, g \rangle'_{\text{nonsym}} := \text{the constant term of } fg^* C \in \mathbb{F}.
\]
Here \( \star \) is the involution on \( \mathbb{F}[x_1^{\pm 1}, \ldots, x_l^{\pm 1}] \) such that \( q^\star = q^{-1}, x_i^\star = x_i^{-1} \) and \( t^\star = t^{-1} \).

Definition 2.8 The set of nonsymmetric Macdonald-Koornwinder polynomials \( \{ E_\lambda(x, q, t) \}_{\lambda \in \hat{P}} \) is a collection of elements in \( \mathbb{F}[\hat{P}] \) indexed by \( \hat{P} \) with the following properties:

1. \( \langle E_\lambda, E_\mu \rangle'_{\text{nonsym}} = 0 \) if \( \lambda \neq \mu \);
2. \( E_\lambda = X^\lambda + \sum_{\mu > \lambda} c_\mu X^\mu \).

As in [13, §3.2], we set \( \bar{E}_\lambda := \lim_{t \to 0} E_\lambda \) and \( E^\dagger := \lim_{t \to 0} E_\lambda^* \).

Let \( \langle -, - \rangle_{\text{nonsym}} \) be a specialization of \( \langle -, - \rangle'_{\text{nonsym}} \) at \( t = 0 \).

2.5.2 Symmetric Case

The Weyl group \( \hat{W} \) acts linearly on \( \mathbb{F}[\hat{P}] \) by \( w(e^{\lambda}) = e^{\lambda} \) for each \( w \in \hat{W} \) and \( \lambda \in \hat{P} \).

Definition 2.9 We define an inner product on \( \mathbb{F}[x_1^{\pm 1}, \ldots, x_l^{\pm 1}] \) by
\[
\langle f, g \rangle_{\text{sym}} := \text{the constant term of } fg \Delta(x) \in \mathbb{F}.
\]

We set \( \bar{E}_\lambda \) and \( E^\dagger \) as above.

Definition 2.10 The set of symmetric Macdonald-Koornwinder polynomials \( \{ P_\lambda(x, q, t) \}_{\lambda \in \hat{P}} \) is a collection of elements in \( \mathbb{F}[\hat{P}]^\hat{W} \) indexed by \( \hat{P}_+ \) with the following properties:

1. \( \langle P_\lambda, P_\mu \rangle_{\text{sym}} = 0 \) if \( \lambda \neq \mu \);
2. \( P_\lambda = X^\lambda + \sum_{\mu > \lambda} c_\mu X^\mu \).

We set \( \bar{P}_\lambda := \lim_{t \to 0} P_\lambda \).

Let \( \langle -, - \rangle_{\text{sym}} \) be a specialization of \( \langle -, - \rangle'_{\text{sym}} \) at \( t = 0 \). We abbreviate \( \bar{E}_\lambda(x_1, \ldots, x_l, q) \), \( E^\dagger(x_1, \ldots, x_l, q) \), \( \bar{P}_\lambda(x_1, \ldots, x_l, q) \) and \( P^\dagger(x_1, \ldots, x_l, q) \) as \( \bar{E}_\lambda(X, q) \), \( E^\dagger(X, q) \), \( \bar{P}_\lambda(X, q) \) and \( P^\dagger(X, q) \), respectively.

Proposition 2.11 ([13] Theorem 4.2) For each \( \lambda \in \hat{P}_+ \), we have
\[
\bar{P}_\lambda(X^{-1}, q^{-1}) = \bar{E}_\lambda(X^{-1}, q^{-1}).
\]

2.6 Representation of \( b_- \) and \( \mathfrak{g} \) and Their Euler-Poincaré-Pairing

2.6.1 Representations of \( b_- \)

For each \( b_- \)-module \( M \) and \( \lambda \in \hat{P} \), we set \( M_\lambda := \{ m \in M | hm = \lambda(h)m \text{ for } h \in h \} \). Let \( \mathfrak{B} \) be the full subcategory of the category of \( U(b_-) \)-module such that a \( b_- \)-module \( M \) is an
object of \( \mathcal{B} \) if and only if \( M \) has a weight decomposition
\[
M = \bigoplus_{\lambda \in P} M_{\lambda},
\]
where \( M_{\lambda} \) has at most countable dimension for all \( \lambda \in P \). We set \( \text{wt } M := \{ \lambda \in P \mid M_{\lambda} \neq \{0\} \} \). Let \( \mathcal{B}' \) be the full subcategory of \( \mathcal{B} \) such that \( M \in \mathcal{B} \) is an object of \( \mathcal{B}' \) if and only if \( M \) is a \( b_- \)-module such that the set of weights \( \text{wt } M \) is contained in \( \bigcup_{i=1,\ldots,k} (\mu_i - Q_+) \) for some \( \mu_i \in P \), and every weight space is finite dimensional. Let \( \mathcal{B}_0 \) be the full subcategory of \( \mathcal{B}' \) consisting of finite dimensional \( b_- \)-modules. For each \( M \in \mathcal{B}' \), we define a graded character of \( M \) by the following formal sum
\[
\text{gch } M := \sum_{\lambda - m\delta \in P \oplus \frac{1}{2} \mathbb{Z}\delta} q^m X^\lambda \dim \mathbb{C} \text{Hom}_{\mathfrak{h} \oplus \mathfrak{d}} (\mathbb{C}_{\lambda - m\delta}, M),
\]
where \( \mathbb{C}_{\lambda - m\delta} \) is a 1-dimensional \( \mathbb{h} \oplus \mathbb{d} \)-module with its weight \( \lambda - m\delta \). For each \( \Lambda \in P \), let \( \mathbb{C}'_{\Lambda} \) be the 1-dimensional \( \mathbb{h} \)-module with its weight \( \Lambda \), and \( \mathbb{C}_{\Lambda} \) be the 1-dimensional simple module of \( b_- \) with its weight \( \Lambda \). For each \( \Lambda \in P \), we set \( P(\Lambda) := U(b_-) \otimes_{U(\mathfrak{h})} \mathbb{C}'_{\Lambda} \) and \( N(\Lambda) := \sum_{\mu \in P\setminus\{\Lambda\}} P(\Lambda)_{\mu} \). Then \( N(\Lambda) \) is a \( b_- \)-submodule of \( P(\Lambda) \) and \( \mathbb{C}_{\Lambda} \cong P(\Lambda)/N(\Lambda) \).

**Proposition 2.12** For each \( \Lambda \in P \), the \( b_- \)-module \( P(\Lambda) = U(b_-) \otimes_{U(\mathfrak{h})} \mathbb{C}'_{\Lambda} \) is a projective cover of \( \mathbb{C}_{\Lambda} \) in \( \mathcal{B} \).

**Proof** For each \( M \in \mathcal{B} \), we have \( \text{Hom}_{\mathcal{B}} (P(\Lambda), M) = \text{Hom}_{\mathfrak{h}} (\mathbb{C}_{\Lambda}, M) \). Hence, \( P(\Lambda) \) is a projective cover of \( \mathbb{C}_{\Lambda} \) in \( \mathcal{B} \).

**Proposition 2.13** ([9] Lemma 5.2) The category \( \mathcal{B} \) has enough projectives.

**Definition 2.14** Let \( M \) be a \( b_- \)-module with \( \mathfrak{h} \)-weight decomposition \( M = \bigoplus_{\mu \in \mathfrak{h}^*} M_{\mu} \). Then \( M^\vee := \bigoplus_{\mu \in \mathfrak{h}^*} M^*_{\mu} \) is a \( b_- \)-module with a \( b_- \)-action defined by
\[
X f (v) := - f (X v) \text{ for } X \in b_-, \ f \in M^\vee \text{ and } v \in M.
\]

**Definition 2.15** For each \( M \in \mathcal{B}' \) and \( N \in \mathcal{B}_0 \), we define the Euler-Poincaré-pairing \( \langle M, N \rangle_{\text{Ext}} \) as a formal sum by
\[
\langle M, N \rangle_{\text{Ext}} := \sum_{p \in \mathbb{Z}^+, m \in \frac{1}{2} \mathbb{Z}} (-1)^p q^m \dim \mathbb{C} \text{Ext}_{\mathcal{B}}^p (M \otimes_{\mathbb{C}} \mathbb{C}_{m\delta}, N^\vee).
\]

**Proposition 2.16** For each \( M \in \mathcal{B}' \) and \( N \in \mathcal{B}_0 \), the following hold:

1. The pairing \( \langle M, N \rangle_{\text{Ext}} \) is an element of \( \mathbb{C}(q^{1/2}) \);
2. This pairing depends only on the graded characters of \( M \) and \( N \).

**Proof** First, we prove (1). Let \( S \) be the set of highest weight vectors of \( M \). Since \( \text{wt } M \) is bounded from above, we have a surjection \( \phi^0 : P^0 := \bigoplus_{v \in S} P(\text{wt}(v)) \to M \), where \( \text{wt}(v) \) is the \( \mathfrak{h} \)-weight of \( v \). If \( v \in S \) such that \( (\text{wt}(v) + Q_+ \setminus \{0\}) \cap \text{wt } M = \emptyset \), then the vector \( v \) is
not an element of \( \ker \varphi^0 \). Hence the set \( \ker \varphi^0 \) is a proper subset of \( \ker P^0 \). For \( \ker \varphi^0 \), we define \( \varphi^1 : P^1 \to \ker \varphi^0 \) in the same way. Repeating this procedure, we get a projective resolution \( \cdots \to P^1 \to P^0 \to M \to 0 \) such that \( \ker P^{k+1} \) is a proper subset of \( \ker P^k \) for all \( k \in \mathbb{Z}_+ \). The complex \( P^* \otimes_C C_{m\delta} \) is a projective resolution of \( M \otimes_C C_{m\delta} \). For each \( m \in \frac{1}{2} \mathbb{Z} \), we have \( (P^k \otimes_C C_{m\delta}) \cap \ker P^0 = 0 \) for all \( k \gg 0 \) since \( N \) and every weight space of \( M \) are finite dimensional. This implies \( \text{Ext}_B^k(M \otimes_C C_{m\delta}, N^\vee) = \{0\} \) for all \( k \gg 0 \). Hence \( \sum_{k \in \mathbb{Z}_+} (-1)^k q^m \text{dim}_C \text{Ext}_B^k(M \otimes_C C_{m\delta}, N^\vee) \) is well-defined. Since \( b_- \)-action on \( P^0 \) does not increase \( d \)-eigenvalues, and the set of weights of an object of \( \mathcal{B}' \) is bounded from above, the intersection of the set of \( d \)-eigenvalues of \( N^\vee \) and \( P^0 \otimes_C C_{m\delta} \) is empty for all \( m \ll 0 \). This implies the assertion.

Next, we prove (2). Let \( N' \) be an object of \( \mathcal{B}_0 \) such that \( \text{gch} \ N = \text{gch} \ N' \). The sets of composition factors of \( N \) and \( N' \) are the same. We denote the set of composition factors by \( S \). For each exact sequence \( 0 \to L_1 \to L_2 \to L_3 \to 0 \), we have \( (M, L_2)_{\text{Ext}} = (M, L_1)_{\text{Ext}} + (M, L_3)_{\text{Ext}} \). This implies \( (M, N)_{\text{Ext}} = \sum_{\lambda \in S} (M, C_{\lambda})_{\text{Ext}} = (M, N')_{\text{Ext}} \). Hence the assertion for the second argument follows. Let \( K^0 := \bigoplus_{v \in S} N(\text{wt}(v)) \) be a \( b_- \)-submodule of \( P^0 \). We set \( N^0 := M \) and \( N^1 := \varphi^0(K^0) \). We define a \( b_- \)-submodule \( N^2 \) of \( N^1 \) instead of \( M \). Repeating this, we get a sequence of \( b_- \)-submodules \( M = N^0 \supset N^1 \supset N^2 \supset \cdots \). Since every weight space of \( M \) is finite dimensional, for each \( \mu \in P \), we have \( \text{wt}^s(\mu) = \{0\} \) for \( s \gg 0 \) by construction. We can take a composition series \( M = M^0 \supset \cdots \supset M^s \supset M^{s+1} \supset \cdots \) of \( M \) as a refinement of the above sequence of \( b_- \)-submodules. Since \( N \) is finite dimensional, for \( s \gg 0 \), we have \( \text{wt}(v) - \text{wt}(u) \notin \mathbb{Q}_+ \) for each \( v \in M^s \) and \( u \in N \). By taking a projective resolution of \( M^s \) as in the proof of (1), we have \( \text{Ext}_B^k(M^s \otimes_C C_{m\delta}, N^\vee) = \{0\} \) for \( s \gg 0 \). Using this composition series, we can prove the assertion for the first argument in the same way.

Thanks to Proposition 2.16, we get a bilinear map from \( \mathbb{C}[[q^{1/2}]] \langle \hat{P} \rangle \times \mathbb{C}[[q^{1/2}]] \langle \hat{P} \rangle \) to \( \mathbb{C}[[q^{1/2}]] \langle \hat{P} \rangle \), that we denote also \( \langle -,- \rangle_{\text{Ext}} \)

**Proposition 2.17** For each \( M \in \mathcal{B} \) and \( N \in \mathcal{B}_0 \), we have \( \langle \text{gch} \ M, \text{gch} \ N \rangle_{\text{Ext}} = \langle \text{gch} \ M, \text{gch} \ N \rangle_{\text{nsym}} \).

**Proof** \( \langle \text{gch} \ C_{\lambda} \rangle_{\text{Ext}} \) and \( \langle \text{gch} \ P(\Lambda) \rangle_{\text{Ext}} \) are \( \mathbb{C}[[q^{1/2}]] \)-basis of \( \mathbb{C}[[q^{1/2}]] \langle \hat{P} \rangle \). Therefore, it suffices to check the assertion for \( M = C_{\lambda} \) and \( N = P(\Lambda) \). By the PBW theorem, we have \( \text{ch} \ P(\Lambda) = X^\lambda / \prod_{\alpha \in A_+} (1 - X^\alpha)^{\text{dim}_C \varphi_\alpha} \). Hence we have \( \text{ch} \ P(\Lambda) = X^\lambda / C_{t=0} \). Hence we get

\[
\langle \text{gch} \ P(\Lambda), \text{gch} \ C_{\lambda} \rangle_{\text{Ext}} = 1 = \langle \text{gch} \ P(\Lambda), \text{gch} \ C_{\lambda} \rangle_{\text{nsym}}.
\]

The assertion follows.

### 2.6.2 Representations of \( \mathfrak{g}_G \)

Let \( \mathfrak{g}_G \)-mod_{wt} be the full subcategory of the category of \( \mathfrak{g}_G \)-modules such that \( M \) is an object of \( \mathfrak{g}_G \)-mod_{wt} if and only if \( M \) is a \( \mathfrak{g}_G \)-module which has a weight decomposition

\[
M = \bigoplus_{\Lambda \in P} M_\Lambda
\]

such that every weight space has at most countable dimension. Let \( \mathfrak{g}_G \)-mod_{int} be the full subcategory of the category \( \mathfrak{g}_G \)-mod_{wt} such that an object \( M \) of \( \mathfrak{g}_G \)-mod_{wt} is an object of \( \mathfrak{g}_G \)-mod_{int} if and only if \( M \) is an integrable \( \hat{g} \)-module and the set of weights \( \text{wt} \ M = \{ \lambda \in \mathfrak{g}_G \})
\[ P \mid M_\lambda \neq \{0\} \] is contained in \( \bigcup_{i=1,...,k} (\mu_i - Q_+) \) for some \( \mu_i \in P \) and every weight space is finite dimensional. For each \( \lambda \in \hat{P}_+, \mu \in \hat{P} \) and \( n, 2m \in \mathbb{Z} \), we set
\[
P(\lambda + n\Lambda_0 + m\delta)_{\text{int}} := U(\mathcal{C}_g) \otimes_{U(\hat{\mathfrak{g}} + \mathfrak{h})} V(\lambda + n\Lambda_0 + m\delta)
\]
and
\[
P(\mu + n\Lambda_0 + m\delta)_{\text{wt}} := U(\mathcal{C}_g) \otimes_{U(\mathfrak{h})} \mathcal{C}_{\mu + n\Lambda_0 + m\delta},
\]
where \( V(\lambda + n\Lambda_0 + m\delta) \) is the highest weight simple module of \( \hat{\mathfrak{g}} + \mathfrak{h} \) with its highest weight \( \lambda + n\Lambda_0 + m\delta \) and \( \mathcal{C}_{\mu + n\Lambda_0 + m\delta} \) is the 1-dimensional module of \( \mathfrak{h} \) with its weight \( \mu + n\Lambda_0 + m\delta \).

Let \( \pi : \mathcal{C}_g \to \hat{\mathfrak{g}} \) be a homomorphism of Lie algebras defined by
\[
\pi |_{\hat{\mathfrak{g}}} = \text{id}_{\hat{\mathfrak{g}}}, \quad \pi (\mathcal{C}_g \neq 0) = \{0\},
\]
where \( \mathcal{C}_g \neq 0 := \{X \in \mathcal{C}_g \mid [d, X] \neq 0\} \). We can prove the following proposition in the same way as Proposition 2.12, and we omit its proof.

**Proposition 2.18** For each \( \mu \in \hat{P} \) and \( n, 2m \in \mathbb{Z} \), the \( \mathcal{C}_g \)-module \( P(\mu + n\Lambda_0 + m\delta)_{\text{wt}} \) is a projective module.

**Proposition 2.19** ([2] Proposition 2.3) Let \( \lambda \in \hat{P}_+ \) and \( n, 2m \in \mathbb{Z} \).

1. \( \pi^* V(\lambda + n\Lambda_0 + m\delta) \) is a simple object in \( \mathcal{C}_g \)-mod\text{int}.
2. \( P(\lambda + n\Lambda_0 + m\delta)_{\text{int}} \) is a projective cover of its unique simple quotient \( \pi^* V(\lambda + n\Lambda_0 + m\delta) \) in \( \mathcal{C}_g \)-mod\text{int}.

**Proposition 2.20** The categories \( \mathcal{C}_g \)-mod\text{wt} and \( \mathcal{C}_g \)-mod\text{int} have enough projectives.

**Proof** We can prove that \( \mathcal{C}_g \)-mod\text{wt} has enough projectives in the same way as Proposition 2.13. Let \( M \) be an object of \( \mathcal{C}_g \)-mod\text{int}. Since \( M \) is an integrable \( \hat{\mathfrak{g}} \)-module, for each \( \hat{\mathfrak{g}} \)-highest weight vector \( v \in M \) with its weight \( \Lambda \), we have a morphism of \( \mathcal{C}_g \)-module \( P(\Lambda)_{\text{int}} \to M \). Collecting them for all \( \hat{\mathfrak{g}} \)-highest weight vector, we obtain a surjection from a projective module to \( M \). \( \square \)

**Definition 2.21** For each \( M, N \in \mathcal{C}_g \)-mod\text{int} such that \( N \) is finite dimensional, we define the Euler-Poincaré-pairing \( \langle M, N \rangle_{\text{int}} \) as a formal sum by
\[
\langle M, N \rangle_{\text{int}} := \sum_{p \in \mathbb{Z}_+, m \in \frac{1}{2} \mathbb{Z}} (-1)^p q^m \text{dim}_\mathbb{C} \text{Ext}_{\mathcal{C}_g \text{-mod\text{int}}}^p (M \otimes_{\mathbb{C}} \mathcal{C}_{m\delta}, N).\]

We can prove the following proposition in the same way as Proposition 2.16, and we omit its proof.

**Proposition 2.22** For each \( M, N \in \mathcal{C}_g \)-mod\text{int} such that \( N \) is finite dimensional, the following hold:

1. The pairing \( \langle M, N \rangle_{\text{int}} \) is an element of \( \mathbb{C}((q^{1/2})) \);
2. This pairing depends only on the graded characters of \( M \) and \( N \).
3 Demazure Modules

We continue to work in the setting of the previous section.

3.1 Representations of \( g \)

3.1.1 Highest Weight Simple Module

**Definition 3.1** Let \( \Lambda \in P \) and let \( C_{\Lambda} \) be the corresponding 1-dimensional module of \( b_+ \). The Verma module \( M(\Lambda) \) of highest weight \( \Lambda \) is a \( g \)-module defined by

\[
M(\Lambda) := U(g) \otimes_{U(b_+)} C_{\Lambda}.
\]

The Verma module \( M(\Lambda) \) has a unique simple quotient (see [16] Proposition 9.2). We denote it by \( L(\Lambda) \).

**Theorem 3.2** (see [16] Proposition 3.7, Lemma 10.1 and §9.2) For each \( \Lambda \in P \), the following hold.

1. \( L(\Lambda) \) is an integrable \( g \)-module if and only if \( \Lambda \in P_+ \);
2. For each \( \Lambda \in P_+ \) and \( w \in W \), we have \( \dim_{\mathbb{C}} L(\Lambda)_w \Lambda = 1 \);
3. \( L(\Lambda) \) has a \( h \)-weight decomposition

\[
L(\Lambda) = \bigoplus_{\mu \in P} L(\Lambda)_\mu
\]

and \( L(\Lambda)_\mu \) is finite-dimensional for all \( \mu \in P \).

We remark that \( g_{\text{ch}} L(\Lambda) \) is well-defined thanks to Theorem 3.2 (3).

3.1.2 Realization of \( L(\Lambda_0) \)

**Definition 3.3** (Heisenberg algebra) For each \( l \in \mathbb{N} \), let \( S_l \) be a unital \( \mathbb{C} \)-algebra generated by \( x_{i,n} (i = 1, \ldots, l, 0 \neq n \in \mathbb{Z}) \) and \( K \) which satisfy the following conditions:

1. \([x_{i,n}, x_{j,m}] = n\delta_{i,j} \delta_{n,-m} K\);
2. \([K, S_l] = 0\).

We set \( R = \mathbb{C}[y_{i,n} \mid i \in \{1, \ldots, l\}, \ n \in \mathbb{N}] \). We define a representation \( p : S_l \to \text{End}_{\mathbb{C}}(R) \) by

\[
p(x_{i,-n}) = y_{i,n}, \quad p(x_{i,n}) = n \frac{\partial}{\partial y_{i,n}}, \quad p(K) = \text{id}_R \ (n > 0).
\]

Let \( g_{im} := \bigoplus_{n \in \mathbb{Z} \setminus \{0\}} g_{n\delta} \). The algebra \( S_l \) is a \( \mathbb{Z} \)-graded algebra by setting \( \deg x_{i,n} = n \) and \( \deg K = 0 \), and \( U(g_{im} \oplus \mathbb{C} K) \) is a \( \mathbb{Z} \)-graded algebra by the \( \mathbb{Z} \)-grading induced from the adjoint action of the scaling element \( d \). For \( g \) of type \( A_{2l}^{(2)} \), we have \( \dim_{\mathbb{C}} g_{n\delta} = l \) for \( n \in \mathbb{Z} \), and we have the following.

**Proposition 3.4** (see [16] Proposition 8.4) The algebras \( U(g_{im} \oplus \mathbb{C} K) \) and \( S_l \) are isomorphic as \( \mathbb{Z} \)-graded algebras.
By Proposition 3.4, we identify $S_l$ with $U(\mathfrak{gl}_m \oplus \mathbb{C} K)$. Since $\mathfrak{h}$ and $U(\mathfrak{gl}_m \oplus \mathfrak{h} \oplus \mathbb{C} K)$ are mutually commutative, the following $\mathbb{C}$-algebra homomorphism $p_\lambda : U(\mathfrak{gl}_m \oplus \mathfrak{h} \oplus \mathbb{C} K) \to \operatorname{End}_\mathbb{C} (R) (\lambda \in \hat{P})$ is well-defined

$$p_\lambda|_{S_l} = p \quad \text{and} \quad p_\lambda(h) = \lambda(h) \text{id}_R \quad \text{for} \quad h \in \mathfrak{h}.$$  

We denote this $U(\mathfrak{gl}_m \oplus \mathfrak{h} \oplus \mathbb{C} K)$-module by $\mathbb{R}_\lambda$.

Theorem 3.5 ([27] Theorem 6.4) We put $\tilde{p} := \prod_{\lambda \in \hat{P}} p_\lambda : U(\mathfrak{gl}_m \oplus \mathfrak{h} \oplus \mathbb{C} K) \to \operatorname{End}_\mathbb{C} \left( \bigoplus_{\lambda \in \hat{P}} R_\lambda \right)$. Then $\tilde{p}$ extends to an algebra homomorphism $U(\mathfrak{g}) \to \operatorname{End}_\mathbb{C} \left( \bigoplus_{\lambda \in \hat{P}} R_\lambda \right)$ and $\bigoplus_{\lambda \in \hat{P}} R_\lambda$ is isomorphic to $L(\Lambda_0)$ as a $\mathfrak{g}$-module.

### 3.2 Thin and Thick Demazure Modules

**Definition 3.6** For each $w \in W$ and $\Lambda \in P_+$, we define $\mathfrak{b}_-$-modules

$$D_{w\Lambda} := U(\mathfrak{b}_-) v_{w\Lambda}^* \subset L(\Lambda)^\vee \quad \text{and} \quad D^{w\Lambda} := U(\mathfrak{b}_-) v_{w\Lambda} \subset L(\Lambda).$$

Here $v_{w\Lambda} \in L(\Lambda)_{w\Lambda}$ and $v_{w\Lambda}^* \in (L(\Lambda)_{w\Lambda})^*$ are nonzero vectors. By Theorem 3.2 (3), these vectors are unique up to scalars. Hence $D_{w\Lambda}$ and $D^{w\Lambda}$ are well-defined. We call $D_{w\Lambda}$ a thin Demazure module and $D^{w\Lambda}$ a thick Demazure module.

**Remark 3.7** A Demazure module in [20] means the thin Demazure module $D_{w\Lambda}$. 

**Lemma 3.8** ([16] Proposition 3.6) For each $w \in W$, $\Lambda \in P_+$ and $\alpha \in \Delta_+$, we have

$$v_{\alpha w\Lambda} \in \begin{cases} \mathfrak{g}_{\alpha}^{(w\Lambda, \alpha)} v_{w\Lambda} & (\langle w\Lambda, \alpha \rangle > 0) \\ \mathfrak{g}_{-\alpha}^{-(w\Lambda, \alpha)} v_{w\Lambda} & (\langle w\Lambda, \alpha \rangle < 0) \\ \mathbb{C} v_{w\Lambda} & (\langle w\Lambda, \alpha \rangle = 0) \end{cases},$$

where $\mathfrak{g}_\alpha^m = \{X_1 X_2 \cdots X_m \in U(\mathfrak{g}) \mid X_i \in \mathfrak{g}_\alpha\}$.

Lemma 3.9 and Corollary 3.12 in the below are proved in [7] for the dual of the untwisted affine Lie algebra. The proofs in [7] are also valid for type $A_{2l}^{(2)}$.

**Lemma 3.9** ([7] Corollary 4.2) For each $w, v \in W$ and $\Lambda \in P_+$, we have the following:

1. If $w \leq v$, then $D^v \subseteq D^w$.
2. If $w$ and $v$ are minimal representatives of cosets in $W/W^\Lambda$ and $D^v \subseteq D^w$, then $w \leq v$.

Lemma 3.9 allows us to define as follows:

**Definition 3.10** For each $w \in W$ and $\Lambda \in P_+$, we define a $U(\mathfrak{b}_-)$-module $\mathbb{D}^{w\Lambda}$ as

$$\mathbb{D}^{w\Lambda} := D^{w\Lambda} / \sum_{w < v} D^v.$$  

We call this module a Demazure slice.
Proposition 3.11 ([18] Corollary 2.22) For each \( \Lambda \in P_+ \) and \( S \subset W \), there exists \( S' \subset W \) such that

\[
\bigcap_{w \in S} D^{w\lambda} = \sum_{w \in S'} D^{w\Lambda}.
\]

Corollary 3.12 ([7] Corollary 4.4) For each \( w, v \in W \) and \( \Lambda \in P_+ \), we have

\[
(D^{w\Lambda} \cap D^{v\Lambda})/\left(D^{v\Lambda} \cap \sum_{u > w} D^{u\Lambda}\right) = D^{w\Lambda} \text{ or } \{0\}.
\]

### 3.3 Level One Case

In this subsection, we consider level one Demazure modules. The unique level one dominant integral weight of \( A(2)_2 \) is \( \Lambda_0 \). From Eq. 2.1,

\[
\hat{P} \ni \lambda \mapsto \lambda + \Lambda_0 + \frac{\langle \lambda, \lambda \rangle}{2} \delta \in W\Lambda_0
\]

is a bijection. For each \( \lambda \in \hat{P} \), we set

\[
D_{\lambda} := D_{\pi_{\lambda}}, \quad D^\lambda := D^{\pi_{\lambda}}, \quad D_{\lambda} := D^{\pi_{\lambda}}.
\]

Lemma 3.13 For each \( \lambda, \mu \in \hat{P} \), we have \( D^\lambda \subsetneq D^\mu \) if and only if \( \mu \succ \lambda \).

Proof If \( D^\lambda \subsetneq D^\mu \), then we have \( \pi_{\mu} < \pi_{\lambda} \) by Lemma 3.9. Then, Lemma 2.6 (1) implies \( \mu \succ \lambda \). Conversely, we assume that \( \mu \succ \lambda \). There exists \( w \in W \) such that \( \mu > \lambda = w(\mu) \). Let \( w = s_{i_1} \cdots s_{i_n} \) be a reduced expression of \( w \) such that \( (s_{i_k+1} \cdots s_{i_1})(\mu) > (s_{i_k} s_{i_k+1} \cdots s_{i_1})(\mu) \) for all \( k \). If \( n = 1 \), then Lemma 2.6 (2) implies \( \pi_{\mu} < \pi_{\lambda} \). Hence, we have \( D^\lambda \subsetneq D^\mu \). If \( n > 1 \), then we have \( D^\lambda \subsetneq D^{(s_{i_2} \cdots s_{i_n})(\mu)} \subsetneq \cdots \subsetneq D^\mu \) inductively. \( \square \)

Theorem 3.14 ([13] Theorem 1) For each \( \lambda \in \hat{P} \), we have

\[
gch D_{\lambda} = q^{(b|b)} E_{\lambda}(X^{-1}, q^{-1}).
\]

### 3.4 Weyl Modules

Definition 3.15 ([6] §3.3) For each \( \lambda \in \hat{P}_+ \), the global Weyl module is a cyclic \( \mathbb{C}g \)-module \( W(\lambda) \) generated by a vector \( v_{\lambda} \) that satisfies following relations:

1. \( h v_{\lambda} = \lambda(h) v_{\lambda} \) for each \( h \in \mathfrak{h} \);
2. \( e_{-a}^{(\lambda, \lambda)+1} v_{\lambda} = 0 \) for each \( a \in \Delta_+ \);
3. \( \mathfrak{e}_{\pi_{\lambda}} v_{\lambda} = 0 \).

Definition 3.16 ([6] §3.5 and §7.2) For each \( \lambda \in \hat{P} \), the local Weyl module is a cyclic \( \mathbb{C}g \)-module \( W(\lambda)_{loc} \) generated by a vector \( v_{\lambda} \) that satisfies relations (1), (2), (3) of Definition 3.15 and

4. \( X v_{\lambda} = 0 \) for \( X \in \mathbb{C}g_{im} \).
**Theorem 3.17** ([6] Theorem 2) *Let* \( \lambda \in \hat{P}_+ \). *Then* \( D_\lambda \otimes_\mathbb{C} \mathbb{C}_{(\lambda|\lambda)\delta/2-\Lambda_0} \) *is isomorphic to* \( W(\lambda)_{\text{loc}} \) *as* \( \mathfrak{g} \)-*module, where \( \mathbb{C}_{(\lambda|\lambda)\delta/2-\Lambda_0} \) *is the 1-dimensional module with its h-weight* \( (\lambda|\lambda)\delta/2-\Lambda_0 \).

**Corollary 3.18** *For each* \( \lambda \in \hat{P}_+ \), *we have*  
\[
gch W(\lambda)_{\text{loc}} = q^{-\frac{\langle \lambda, \lambda \rangle}{2}} \bar{P}_\lambda(X^{-1}, q^{-1}).
\]

**Proof** *By Theorem 3.17, we have*  
\[
gch W(\lambda)_{\text{loc}} = gch D_\lambda.
\]

*By Proposition 2.11 and Theorem 3.14, the assertion follows.*

**Theorem 3.19** ([2] Theorem 2.5 (3), Theorem 4.7 and [19] Theorem 7.21) *For each* \( \lambda, \mu \in \hat{P}_+ \) *and* \( m \in \frac{1}{2}\mathbb{Z} \), *we have*  
\[
\dim_{\mathbb{C}} \text{Ext}^n_{\mathfrak{g}}(W(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_{m\delta}, W(\mu)^\vee_{\text{loc}}) = \delta_{m,0}\delta_{0,n}\delta_{\lambda,\mu}.
\]

**Corollary 3.20** *For each* \( \lambda, \mu \in \hat{P}_+ \), *we have*  
\[
gch W(\lambda), gch W(\mu)_{\text{loc}} \rangle_{\text{int}} = \delta_{\lambda,\mu}.
\]

**Proof** *The assertion follows from Definition 2.21 and Theorem 3.19.*

### 3.4.1 Extensions Between Weyl Modules in \( \mathcal{B} \)

In this subsection, we prove the following corollary of Theorem 3.19.

**Theorem 3.21** *For each* \( \lambda, \mu \in \hat{P}_+ \), *and* \( m \in \mathbb{Z}, n \in \mathbb{Z}_+ \), *we have*  
\[
\dim_{\mathbb{C}} \text{Ext}^n_{\mathcal{B}}(W(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_{m\delta}, W(\mu)^\vee_{\text{loc}}) = \delta_{m,0}\delta_{0,n}\delta_{\lambda,\mu}.
\]

**Definition 3.22** ([11] §2.1) *Let* \( \mathcal{C}, \mathcal{D} \) *be abelian categories. A contravariant* \( \delta \)-*functor from* \( \mathcal{C} \) *to* \( \mathcal{D} \) *consists of the following data:

1. A collection \( T = \{T^i\}_{i \in \mathbb{Z}_+} \) of contravariant additive functors from \( \mathcal{C} \) to \( \mathcal{D} \);
2. For each exact sequence \( 0 \rightarrow M'' \rightarrow M \rightarrow M' \rightarrow 0 \), a collection of morphisms \( \{\delta^n : T^n(M'') \rightarrow T^{n+1}(M')\}_{n \in \mathbb{Z}_+} \) with the following conditions:
   1. For each exact sequence \( 0 \rightarrow M'' \rightarrow M \rightarrow M' \rightarrow 0 \), there is a long exact sequence  
      \[
      0 \rightarrow T^0(M') \rightarrow T^0(M) \rightarrow T^0(M'') \delta^0 \rightarrow T^1(M') \rightarrow \cdots \rightarrow T^{n-1}(M'') \delta^{n-1} \rightarrow T^n(M') \rightarrow T^n(M) \rightarrow T^n(M'') \delta^n \rightarrow \cdots; 
      \]
   2. For each morphism of short exact sequence  
      \[
      0 \longrightarrow M'' \longrightarrow M \longrightarrow M' \longrightarrow 0 \quad \text{with} \quad 0 \longrightarrow N'' \longrightarrow N \longrightarrow N' \longrightarrow 0,
      \]

we have the following commutative diagram

\[
\begin{array}{ccc}
T^{n-1}(N'') & \xrightarrow{\delta^{n-1}} & T^n(N') \\
\downarrow & & \downarrow \\
T^{n-1}(M'') & \xrightarrow{\delta^{n-1}} & T^n(M')
\end{array}
\]

**Definition 3.23** ([11] §2.1) For each contravariant $\delta$-functors $T = \{T^i\}_{i \in \mathbb{Z}_+}$ and $S = \{S^i\}_{i \in \mathbb{Z}_+}$, a morphism of $\delta$-functor from $T = \{T^i\}_{i \in \mathbb{Z}_+}$ to $S = \{S^i\}_{i \in \mathbb{Z}_+}$ is a collection of natural transformations $F = \{F^i : T^i \rightarrow S^i\}_{i \in \mathbb{Z}_+}$ with the following condition:

($\ast$) For each exact sequence $0 \rightarrow M'' \rightarrow M \rightarrow M' \rightarrow 0$, the following diagram is commutative

\[
\begin{array}{ccc}
T^{n-1}(M'') & \xrightarrow{\delta^{n-1}} & T^n(M') \\
F^{n-1}(M'') & \downarrow & F^n(M') \\
S^{n-1}(M'') & \xrightarrow{\delta^{n-1}} & S^n(M')
\end{array}
\]

**Definition 3.24** ([11] §2.2) A contravariant $\delta$-functor $T = \{T^i\}_{i \in \mathbb{Z}_+}$ is called a universal $\delta$-functor if for each $\delta$-functor $S = \{S^i\}_{i \in \mathbb{Z}_+}$ and for each natural transformation $F^0 : T^0 \rightarrow S^0$, there exists a unique morphism of $\delta$-functor $\{F^i : T^i \rightarrow S^i\}_{i \in \mathbb{Z}_+}$.

**Definition 3.25** ([11] §2.2) An additive functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called coeffaceable if for each object $M$ of $\mathcal{C}$, there is a epimorphism $P \rightarrow M$ such that $F(P) = 0$.

**Theorem 3.26** ([11] Proposition 2.2.1) For each $\mathcal{C}$, $\mathcal{D}$ be abelian categories and let $T = \{T^i\}_{i \in \mathbb{Z}_+}$ be a contravariant $\delta$-functor from $\mathcal{C}$ to $\mathcal{D}$. If $T^i$ is coeffaceable for $i > 0$, then $T$ is universal.

**Lemma 3.27** (Shapiro’s lemma) For each $M \in \mathcal{B}$, $N \in \mathcal{C}_g$-$\text{mod}_{\text{wt}}$ and $n \in \mathbb{Z}_+$, we have

\[
\text{Ext}_\mathcal{B}^n(M, N) = \text{Ext}_{\mathcal{C}_g}$-$\text{mod}_{\text{wt}}^n\left(U(\mathcal{C}_g) \otimes_{U(\mathcal{B})} M, N\right).
\]

**Proof** Let $P^* \rightarrow M \rightarrow 0$ be a projective resolution of $M$ in $\mathcal{B}$. Since $U(\mathcal{C}_g)$ is free over $U(\mathcal{B})$, the complex $U(\mathcal{C}_g) \otimes_{U(\mathcal{B})} P^*$ is a projective resolution of $U(\mathcal{C}_g) \otimes_{U(\mathcal{B})} M$ in $\mathcal{C}_g$-$\text{mod}_{\text{wt}}$. The assertion follows from the Frobenius reciprocity. \hfill \Box

**Lemma 3.28** We have the following:

1. For each $M$, $N \in \mathcal{C}_g$-$\text{mod}^{\text{int}}$, we have

\[
\text{Ext}_{\mathcal{C}_g}$-$\text{mod}_{\text{wt}}^k(M, N^\vee) = \text{Ext}_{\mathcal{C}_g}$-$\text{mod}_{\text{wt}}^k(M, N^\vee) \ k \in \mathbb{Z}_+;
\]

2. For each $N \in \mathcal{C}_g$-$\text{mod}^{\text{int}}$, we have

\[
\text{Ext}_{\mathcal{C}_g}$-$\text{mod}_{\text{wt}}^k\left(U(\mathcal{C}_g) \otimes_{U(\mathcal{B})} \mathbb{C}_0, N^\vee\right) = \text{Ext}_{\mathcal{C}_g}$-$\text{mod}_{\text{wt}}^k(\mathbb{C}_0, N^\vee) \ k \in \mathbb{Z}_+.
\]
Proof First, we prove the first assertion. The sets of functors \( \{ \text{Ext}^k_{\mathcal{C} \text{-mod}_{\text{wt}}}(-, N^\vee) \}_{k \in \mathbb{Z}_+} \) and \( \{ \text{Ext}^k_{\mathcal{C} \text{-mod}^{\text{int}}(-, N^\vee) \}_{k \in \mathbb{Z}_+} \) are contravariant \( \delta \)-functors from \( \mathcal{C} \text{-mod}_{\text{wt}} \) to the category of vector spaces. From Theorem 3.26, \( \{ \text{Ext}^k_{\mathcal{C} \text{-mod}_{\text{wt}}}(-, N^\vee) \}_{k \in \mathbb{Z}_+} \) is a universal \( \delta \)-functor.

We prove \( \{ \text{Ext}^k_{\mathcal{C} \text{-mod}_{\text{wt}}}(-, N^\vee) \}_{k \in \mathbb{Z}_+} \) is also a universal \( \delta \)-functor. From Theorem 3.26, it is sufficient to show \( \text{Ext}^l_{\mathcal{C} \text{-mod}_{\text{wt}}}(P(\lambda + n\Lambda_0 + m\delta)_{\text{int}}, N^\vee) = \{0\} \) for \( \lambda \in \hat{P}_+, n, 2m \in \mathbb{Z} \) and \( l > 0 \). From the BGG-resolution, we have an exact sequence

\[
\cdots \to \bigoplus_{w \in \hat{W}, l(w) = n+1} \tilde{M}(w \circ \lambda + n\Lambda_0 + m\delta) \to \bigoplus_{w \in \hat{W}, l(w) = n} \tilde{M}(w \circ \lambda + n\Lambda_0 + m\delta) \to \cdots
\]

\[
\cdots \to \bigoplus_{w \in \hat{W}, l(w) = 1} \tilde{M}(w \circ \lambda + n\Lambda_0 + m\delta) \to \tilde{M}(\lambda + n\Lambda_0 + m\delta) \to V(\lambda + n\Lambda_0 + m\delta) \to 0,
\]

where

\[
\tilde{M}(\mu) := U(\hat{g} + h) \otimes_{U(\hat{b}_+ + h)} \mathbb{C}_\mu.
\]

Since \( U(\mathcal{C} \mathfrak{g}) \) is free over \( U(\hat{g} + h) \), by tensoring \( U(\mathcal{C} \mathfrak{g}) \), we obtain a projective resolution \( P^* \to P(\lambda + n\Lambda_0 + m\delta)_{\text{int}} \to 0 \) of \( P(\lambda + n\Lambda_0 + m\delta)_{\text{int}} \) in \( \mathcal{C} \text{-mod}_{\text{wt}} \) such that \( P^n = \bigoplus_{w \in \hat{W}, l(w) = n} U(\mathcal{C} \mathfrak{g}) \otimes_{U(\hat{g} + h)} \tilde{M}(w \circ \lambda + n\Lambda_0 + m\delta) \). For each \( l(w) > 0 \),

\[
\text{Hom}_{\mathcal{C} \text{-mod}_{\text{wt}}}(U(\mathcal{C} \mathfrak{g}) \otimes_{U(\hat{g} + h)} \tilde{M}(w \circ \lambda + n\Lambda_0 + m\delta), N^\vee) = \{0\}.
\]

This implies \( \text{Ext}^l_{\mathcal{C} \text{-mod}_{\text{wt}}}(P(\lambda + n\Lambda_0 + m\delta)_{\text{int}}, N^\vee) = \{0\} \) for \( l > 0 \). Hence \( \{ \text{Ext}^k_{\mathcal{C} \text{-mod}_{\text{wt}}}(-, N^\vee) \}_{k \in \mathbb{Z}_+} \) is a universal \( \delta \)-functor by Theorem 3.26. Since \( \text{Ext}^0_{\mathcal{C} \text{-mod}_{\text{wt}}}(-, N^\vee) = \text{Ext}^0_{\mathcal{C} \text{-mod}^{\text{int}}}(-, N^\vee) \), the assertion follows.

Next, we prove the second assertion. Two sets of functors \( \{ \text{Ext}^k_{\mathcal{C} \text{-mod}_{\text{wt}}}(U(\mathcal{C} \mathfrak{g}) \otimes_{U(\hat{b}_-)} \mathbb{C}_0, (-)^\vee) \}_{k \in \mathbb{Z}_+} \) and \( \{ \text{Ext}^k_{\mathcal{C} \text{-mod}^{\text{int}}}(\mathbb{C}_0, (-)^\vee) \}_{k \in \mathbb{Z}_+} \) are contravariant \( \delta \)-functors from \( \mathcal{C} \text{-mod}^{\text{int}} \) to the category of vector spaces. Since \( \mathbb{C}_0 \) is an object of \( \mathcal{C} \text{-mod}^{\text{int}} \), we can prove that the latter is a universal \( \delta \)-functor by the same argument as in the proof of (1). We show that \( \text{Ext}^l_{\mathcal{C} \text{-mod}_{\text{wt}}}(U(\mathcal{C} \mathfrak{g}) \otimes_{U(\hat{b}_-)} \mathbb{C}_0, P(\lambda + n\Lambda_0 + m\delta)_{\text{int}}) = \{0\} \) for each \( l > 0 \). For each \( w \in \hat{W} \), by the PBW theorem and the Frobenius reciprocity, we have

\[
\text{Hom}_{\mathcal{C} \mathfrak{g}}(U(\mathcal{C} \mathfrak{g}) \otimes_{U(\hat{b}_-)} \mathbb{C}_0, (U(\mathcal{C} \mathfrak{g}) \otimes_{U(\hat{g} + h)} \tilde{M}(w \circ \lambda + n\Lambda_0 + m\delta))^\vee)
\]

\[
= \text{Hom}_{\mathcal{C} \mathfrak{g}}(U(\mathcal{C} \mathfrak{g}) \otimes_{U(\hat{g} + h)} \tilde{M}(w \circ \lambda + n\Lambda_0 + m\delta), (U(\mathcal{C} \mathfrak{g}) \otimes_{U(\hat{b}_-)} \mathbb{C}_0)^\vee)
\]

\[
= \text{Hom}_{\hat{b}_+ + h}(\mathbb{C}_{w \circ \lambda + n\Lambda_0 + m\delta}, (U(\mathcal{C} \mathfrak{g}) \otimes_{U(\hat{b}_-)} \mathbb{C}_0)^\vee)
\]

\[
= \text{Hom}_{\hat{b}_+ + h}(\mathbb{C}_{w \circ \lambda + n\Lambda_0 + m\delta}, (U(\hat{b}_+ + h) \otimes_{U(\hat{b}_-)} \mathbb{C}_0)^\vee)
\]

\[
= \text{Hom}_{\hat{b}_+ + h}(U(\hat{b}_+ + h) \otimes_{U(\hat{b}_-)} \mathbb{C}_0, \mathbb{C}_{-w \circ \lambda - n\Lambda_0 - m\delta})
\]

\[
= \text{Hom}_{\hat{b}_+ + h}(\mathbb{C}_0, \mathbb{C}_{-w \circ \lambda - n\Lambda_0 - m\delta}).
\]

If \( l(w) > 0 \), then \( \text{Hom}_{\hat{b}_+ + h}(\mathbb{C}_{w \circ \lambda + n\Lambda_0 + m\delta}, (U(\hat{b}_+ + h) \otimes \mathbb{C}_0)^\vee) \) = \{0\}. Using the projective resolution of \( P(\lambda + n\Lambda_0 + m\delta)_{\text{int}} \) considered in the proof of (1), this
implies $\text{Ext}_{\mathcal{C}_g\text{-mod}^{\operatorname{int}}}^l(U(\mathcal{C}_g) \otimes_{U(b_{-})} \mathcal{C}_0, P(\lambda + n \Lambda_0 + m \delta)_{\operatorname{int}}) = \{0\}$ for each $l > 0$. Hence $\text{Ext}_{\mathcal{C}_g\text{-mod}^{\operatorname{int}}}^k(U(\mathcal{C}_g) \otimes_{U(b_{-})} \mathcal{C}_0, (-)_{\mathcal{C}}) = \{0\}$ is a universal $\delta$-functor. Since $\text{Ext}_{\mathcal{C}_g\text{-mod}^{\operatorname{int}}}^0(U(\mathcal{C}_g) \otimes_{U(b_{-})} \mathcal{C}_0, N_{\mathcal{C}}) = \text{Ext}_{\mathcal{C}_g\text{-mod}^{\operatorname{int}}}^0(\mathcal{C}_0, N_{\mathcal{C}})$, the assertion follows.

**Lemma 3.29** For each $M, N \in \mathcal{B}$, we have

$$\text{Ext}_{\mathcal{B}}^n(M, N_{\mathcal{C}}) = \text{Ext}_{\mathcal{C}_g\text{-mod}^{\operatorname{int}}}^n(\mathcal{C}_0, M_{\mathcal{C}} \otimes_{\mathcal{C}} N_{\mathcal{C}}) \text{ for } n \in \mathbb{Z}_+.$$ 

**Proof** We show that $\{\text{Ext}_{\mathcal{B}}^n(\mathcal{C}_0, (-)_{\mathcal{C}} \otimes_{\mathcal{C}} N_{\mathcal{C}})\}_{n \in \mathbb{Z}_+}$ is a universal $\delta$-functor. For each injective object $I \in \mathcal{B}$, the object $I \otimes_{\mathcal{C}} N_{\mathcal{C}}$ is an injective object in $\mathcal{B}$. Hence we have $\text{Ext}_{\mathcal{B}}^n(\mathcal{C}_0, P_{\mathcal{C}} \otimes_{\mathcal{C}} N_{\mathcal{C}}) = \{0\}$ for each projective object $P \in \mathcal{B}$ and $k \in \mathbb{N}$. From Theorem 3.26, this implies $\text{Ext}_{\mathcal{B}}^n(\mathcal{C}_0, (-)_{\mathcal{C}} \otimes_{\mathcal{C}} N_{\mathcal{C}})_{n \in \mathbb{Z}_+}$ is a universal $\delta$-functor. For each $R \in \mathcal{B}$, we have $\text{Hom}_{\mathcal{B}}(R, N_{\mathcal{C}}) = \text{Hom}_{\mathcal{B}}(\mathcal{C}_0, R_{\mathcal{C}} \otimes_{\mathcal{C}} N_{\mathcal{C}})$. Since $\{\text{Ext}_{\mathcal{B}}^n(-, N_{\mathcal{C}})\}_{n \in \mathbb{Z}_+}$ is a universal $\delta$-functor, this implies $\{\text{Ext}_{\mathcal{B}}^n(-, N_{\mathcal{C}})\}_{n \in \mathbb{Z}_+} \cong \{\text{Ext}_{\mathcal{B}}^n(\mathcal{C}_0, (-)_{\mathcal{C}} \otimes_{\mathcal{C}} N_{\mathcal{C}})\}_{n \in \mathbb{Z}_+}$. Hence the assertion follows.

**Remark 3.30** The conclusion of Lemma 3.29 remains valid if we replace $\text{Ext}_{\mathcal{B}}^n$ with $\text{Ext}_{\mathcal{C}_g\text{-mod}^{\operatorname{int}}}^n$ by the same argument.

**Theorem 3.31** For $M, N \in \mathcal{C}_g\text{-mod}^{\operatorname{int}}$, we have

$$\text{Ext}_{\mathcal{B}}^n(M, N_{\mathcal{C}}) = \text{Ext}_{\mathcal{C}_g\text{-mod}^{\operatorname{int}}}^n(M, N_{\mathcal{C}}).$$

**Proof** We have

$$\text{Ext}_{\mathcal{B}}^n(M, N_{\mathcal{C}}) = \text{Ext}_{\mathcal{C}_g\text{-mod}^{\operatorname{int}}}^n(\mathcal{C}_0, M_{\mathcal{C}} \otimes_{\mathcal{C}} N_{\mathcal{C}}) \text{ from Lemma 3.29}$$

$$= \text{Ext}_{\mathcal{C}_g\text{-mod}^{\operatorname{int}}}^n(U(\mathcal{C}_g) \otimes_{U(b_{-})} \mathcal{C}_0, M_{\mathcal{C}} \otimes_{\mathcal{C}} N_{\mathcal{C}}) \text{ from Lemma 3.27}$$

$$= \text{Ext}_{\mathcal{C}_g\text{-mod}^{\operatorname{int}}}^n(\mathcal{C}_0, M_{\mathcal{C}} \otimes_{\mathcal{C}} N_{\mathcal{C}}) \text{ from Lemma 3.28 (2)}$$

$$= \text{Ext}_{\mathcal{C}_g\text{-mod}^{\operatorname{int}}}^n(\mathcal{C}_0, M_{\mathcal{C}} \otimes_{\mathcal{C}} N_{\mathcal{C}}) \text{ from Lemma 3.28 (1)}$$

$$= \text{Ext}_{\mathcal{C}_g\text{-mod}^{\operatorname{int}}}^n(M, N_{\mathcal{C}}) \text{ from Remark 3.30}.$$

**Proof of Theorem 3.31** If we set $M = W(\lambda) \otimes_{\mathcal{C}} C_{m \delta}$ and $N = W(\mu)_{\operatorname{loc}}$ in Theorem 3.31, then we obtain Theorem 3.31.

**Corollary 3.32** For each $f, g \in \mathbb{C}(\langle q^{1/2} \rangle)[\hat{P}]^\mathbb{W}$, we have

$$\langle f, g \rangle_{\text{int}} = \langle f, g \rangle_{\text{Ext}}.$$ 

**Proof** From Theorem 3.31, we have

$$\langle \text{gch} W(\lambda), \text{gch} W(\mu)_{\operatorname{loc}} \rangle_{\mathcal{C}} = \langle \text{gch} W(\lambda), \text{gch} W(\mu)_{\operatorname{loc}} \rangle_{\mathcal{C}}_{\mathbb{W}}$$

for each $\lambda, \mu \in \hat{P}_+$. Since $\{\text{gch} W(\lambda)\}_{\lambda \in \hat{P}_+}$ and $\{\text{gch} W(\lambda)_{\operatorname{loc}}\}_{\lambda \in \hat{P}_+}$ are $\mathbb{C}(\langle q^{1/2} \rangle)$-basis of $\mathbb{C}(\langle q^{1/2} \rangle)[\hat{P}]^\mathbb{W}$, we obtain the assertion.
3.4.2 Demazure-Joseph Functors

For each \( i = 0, \ldots, l \), let \( \mathfrak{sl}(2, i) \) be a Lie subalgebra of \( \mathfrak{g} \) isomorphic to \( \pm \alpha_i \) and \( p_i := \mathfrak{b}_- + \mathfrak{sl}(2, i) \). For each \( i = 0, \ldots, l \) and a \( \mathfrak{b}_- \)-module \( M \) with semisimple \( \mathfrak{h} \)-action, \( D_i(M) \) is the unique maximal \( \mathfrak{sl}(2, i) \)-integrable quotient of \( U(p_i) \otimes M \). Then \( D_i \) defines a functor called Demazure-Joseph functor [14].

**Theorem 3.33 ([14])** For each \( i = 0, \ldots, l \) and a \( \mathfrak{b}_- \)-semisimple \( \mathfrak{h} \)-module \( M \), the following hold:

1. The functors \( \{ D_i \}_{i=0, \ldots, l} \) satisfy braid relations of \( \mathcal{W} \);
2. There is a natural transformation \( \text{Id} \to D_i \);
3. If \( M \) is an \( \mathfrak{sl}(2, i) \)-integrable \( p_i \)-module then \( D_i(M) \cong M \);
4. If \( N \) is an \( \mathfrak{sl}(2, i) \)-integrable \( p_i \)-module then \( D_i(M \otimes N) \cong D_i(M) \otimes N \);
5. The functor \( D_i \) is right exact.

For a reduced expression \( w = s_{i_1} s_{i_2} \cdots s_{i_n} \in \mathcal{W} \), we define

\[
D_w := D_{i_1} \circ D_{i_2} \circ \cdots \circ D_{i_n}.
\]

This is well-defined by Theorem 3.33 (1).

**Theorem 3.34** For each \( \Lambda \in P_+ \), \( w \in \mathcal{W} \) and \( i \in \{ 0, \ldots, l \} \), we have

\[
D_i(D_w \Lambda) = \begin{cases} D_{w \Lambda} & (w \geq s_i w) \\ D_{s_i w \Lambda} & (w < s_i w). \end{cases}
\]

**Proof** By Lemma 3.8 and the PBW theorem, \( D_{w \Lambda} \) has an integrable \( \mathfrak{sl}(2, i) \)-action if \( w \geq s_i w \). Hence Theorem 3.33 (3) implies \( D_i(D_w \Lambda) = D_{w \Lambda} \) if \( w \geq s_i w \). If \( w < s_i w \), then \( D_{s_i w \Lambda} \) is a \( p_i \)-module with an integrable \( \mathfrak{sl}(2, i) \)-action by Lemma 3.8 and the PBW theorem, and we have an inclusion \( D_{w \Lambda} \to D_{s_i w \Lambda} \). Hence we have a morphism of \( p_i \)-module \( U(p_i) \otimes D_{w \Lambda} \to D_{s_i w \Lambda} \). This morphism is surjective since \( D_{s_i w \Lambda} \) is generated by a vector with its weight \( w \Lambda \) as \( p_i \)-module. Therefore we obtain a surjection \( D_i(D_w \Lambda) \to D_{s_i w \Lambda} \) by taking a maximal \( \mathfrak{sl}(2, i) \)-integrable quotient. By [17, Proposition 3.3.4], we have \( \text{gch} \ D_i(D_w \Lambda) = \text{gch} \ D_{s_i w \Lambda} \). Hence the above surjection is an isomorphism. \( \square \)

We set \( D_i^\# := \vee \circ D_i \circ \vee \).

**Proposition 3.35 ([9] Proposition 5.7)** For each \( i = 0, 1, \ldots, l, n \in \mathbb{Z}_+ \), \( M \in \mathcal{B}' \), \( N \in \mathcal{B}_0 \), we have

\[
\text{Ext}_n^{\mathfrak{g}}(D_i(M), N) \cong \text{Ext}_n^{\mathfrak{g}}(M, D_i^\#(N)) \quad n \in \mathbb{Z}_+.
\]

3.4.3 Realization of Global Weyl Modules

For each \( \lambda \in \tilde{P}_+ \), we define

\[
\text{Gr}^\lambda D := D^\lambda / \sum_{\lambda > \mu, \mu \notin \tilde{W} \lambda} D^\mu.
\]

\( \text{Springer} \)
From the PBW theorem and Lemma 3.8, $D^\lambda$ and $\sum_{\lambda \succ \mu, \mu \notin \hat{W}_\lambda} D^\mu$ are stable under the action of $\mathfrak{g}$. Hence $\operatorname{Gr}^\lambda D$ admits a $\mathfrak{g}$-module structure.

**Proposition 3.36** Let $\lambda \in \hat{P}_+$. Then $\operatorname{Gr}^\lambda D$ has a filtration of $\mathfrak{b}_-$-submodules

$$\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{N-1} \subset F_N = \operatorname{Gr}^\lambda D$$

such that

$$\{F_i/F_{i-1}\}_{i=1,\ldots,N} = \{D^\mu\}_{\mu \in \hat{W}_\lambda}.$$

**Proof** Let $\geq'$ be a total order on $W$ such that if $w \geq v$ then $w \geq' v$. For each $w \geq \pi \lambda$, define

$$F_w := \left( \sum_{v \geq' w} D^w_{\lambda_0} + \sum_{\lambda \succ \mu, \mu \notin \hat{W}_\lambda} D^\mu \right) / \sum_{\lambda \succ \mu, \mu \notin \hat{W}_\lambda} D^\mu.$$

This is a $\mathfrak{b}_-$-submodule of $\operatorname{Gr}^\lambda D$ and $F_w \subseteq F_v$ if $w \geq' v$.

By Corollary 3.12, $\{F_w\}_{w \in W}$ gives the assertion. $\square$

**Lemma 3.37** We have the following equality of graded characters.

$$\operatorname{gch} L(\Lambda_0) = \sum_{\lambda \in \hat{P}_+} q^{(\lambda, \lambda)/2} \operatorname{gch} W(\lambda).$$

**Proof** Let $\lambda \in \hat{P}_+$ and $k \in \mathbb{Z}_+$. By Theorem 3.17,

$$\operatorname{Ext}^k_{\mathfrak{g} \text{-mod int}}(L(\Lambda_0), (\mathbb{C}_{-(\lambda, \lambda)/2} \otimes \mathbb{C} W(\lambda)_{\text{loc}})^\vee) = \operatorname{Ext}^k_{\mathfrak{g} \text{-mod int}}(L(\Lambda_0), D^{\lambda \vee}_\lambda).$$

Applying Theorem 3.34 and Proposition 3.35 repeatedly, we have

$$\operatorname{Ext}^k_{\mathfrak{g} \text{-mod int}}(L(\Lambda_0), D^{\lambda \vee}_\lambda) = \operatorname{Ext}^k_{\mathfrak{g} \text{-mod int}}(L(\Lambda_0), D^0_0)^\vee$$

$k \in \mathbb{Z}_+$.

Here $D_0$ is isomorphic to the trivial $\mathfrak{g}$-module $\mathbb{C}_{\Lambda_0}$ with its weight $\Lambda_0$. By [12, Theorem 3.6], We have a projective resolution of a $\mathfrak{g}$-module

$$\cdots \rightarrow P^1 \rightarrow P^0 \rightarrow \mathbb{C}_{\Lambda_0} \rightarrow 0,$$

where $P^n = \bigoplus_{w \in W^0, l(w)=n} P(w \circ 0 + \Lambda_0)_{\text{int}}$. Since $\dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}}(P^n, \mathbb{C}_{\Lambda_0}) = \delta_{0,n}$, we obtain

$$\dim_{\mathbb{C}} \operatorname{Ext}^k_{\mathfrak{g} \text{-mod int}}(L(\Lambda_0), (\mathbb{C}_{-(\lambda, \lambda)/2} \otimes \mathbb{C} W(\lambda)_{\text{loc}})^\vee) = \delta_{0,k} \quad k \in \mathbb{Z}_+.$$

Therefore, we have

$$\langle \operatorname{gch} L(\Lambda_0), \operatorname{gch} (\mathbb{C}_{-(\lambda, \lambda)/2} \otimes \mathbb{C} W(\lambda)_{\text{loc}})^\vee \rangle_{\text{int}} = 1.$$

By Corollary 3.18, the set of graded characters $\{\operatorname{gch} W(\lambda)_{\text{loc}}\}_{\lambda \in \hat{P}_+}$ is an orthogonal $\mathbb{C}((q^{1/2}))$-basis of $\mathbb{C}((q^{1/2}))[\hat{P}]$. Hence Corollary 3.20 implies the assertion. $\square$

If a $\mathfrak{b}_-$-module $M$ admits a finite sequence of $\mathfrak{b}_-$-submodules such that every successive quotient is isomorphic to some $D^{\mu}$ ($\mu \in \hat{P}$), then we say $M$ is filtered by Demazure slices. Let $f, g \in \mathbb{C}((q^{1/2}))[\hat{P}]$. Here we make a convention that $f \geq g$ means all the coefficients of $f$ and $g$ belong to $\mathbb{Z}_+$. 

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Theorem 3.38 For each $\lambda \in \check{P}_+$, the global Weyl module $W(\lambda) \otimes_{C} C_{\check{A}_0}$ is isomorphic to $\text{Gr}^D \check{D}$ as $C_{\check{g}}$-module. In particular, $W(\lambda) \otimes_{C} C_{\check{A}_0}$ is filtered by Demazure slices and each $D_{\mu}$ ($\mu \in \check{W}_{\lambda}$) appears exactly once as a successive quotient.

Proof First, we show that there exists a surjection $W(\lambda) \otimes_{C} C_{\check{A}_0} \rightarrow \text{Gr}^D \check{D}$. Let $v_{\lambda} \in \text{Gr}^D \check{D}$ be the nonzero cyclic vector with its weight $\lambda + \check{A}_0 - (\lambda | \lambda) / 2 \delta$. We check $v_{\lambda}$ satisfies Definition 3.15 (1), (2), (3). The condition (1) is trivial from the definition of $v_{\lambda}$. Since $L(\check{A}_0)$ is an integrable $g$-module, $v_{\lambda}$ is an extremal weight vector. This implies the condition (2). We check the condition (3) in the sequel. Since $\langle \lambda, \check{\alpha} \rangle \geq 0$ and $v_{\lambda}$ is an extremal weight vector, we have $e_{\alpha} v_{\lambda} = 0$ for $\alpha \in \check{\Delta}_+$. For each $\mu \in \check{P}$, we set $|\mu\rangle := 1 \in R_{\mu}$. For each $\beta = \alpha + n\delta \in \check{\Delta}$ with $n \in -\mathbb{Z}_+/2$ and $\alpha \in \check{\Delta}_+ \cup \frac{1}{2} \check{\Delta}_{l+}$, we have $v := e_{\alpha + n\delta} \langle \lambda \rangle \in U(C_{\check{g}}) |\lambda + \alpha\rangle$ by Theorem 3.5. Since $U(\check{g}) v$ is finite dimensional, $U(\check{g}) v$ has a highest weight vector whose weight is $v$. Then, $v \in U(C_{\check{g}}(m)) U(\check{n}_-) |v\rangle \subset D^v$. Hence $v - \lambda \in \check{Q}_+$. Since $\lambda$ and $v$ is dominant, we have $\lambda > v$. Therefore $D^v$ is 0 in $\text{Gr}^D \check{D}$ as $|v\rangle \in D^v$. This implies $v = 0$ and we have the desired surjection. In particular, we have an inequality $q^{(\lambda, \lambda)/2} \text{gch} W(\lambda) \geq \text{gch} \text{Gr}^D \check{D}$.

On the other hand, we have,

$$\text{gch} L(\check{A}_0) = \sum_{\lambda \in \check{P}_+} \text{gch} \text{Gr}^D \check{D}$$

and

$$\text{gch} L(\check{A}_0) = \sum_{\lambda \in \check{P}_+} q^{(\lambda, \lambda)/2} \text{gch} W(\lambda)$$

by Lemma 3.37. Thus the above inequality is actually an equality and the assertion follows.

3.5 Extensions Between $D^\lambda$ and $D_\mu$

3.5.1 Demazure-Joseph Functor and Demazure Slices

Theorem 3.39 For each $w \in \check{W}$ and $i \in \{0, \ldots, l\}$, we have the following:

$$D_i(D^w) = \begin{cases} D^{s_i w} & \text{if } s_i w < w \\ D^w & \text{if } s_i w \geq w. \end{cases}$$

Proof The proof is the same as proof of Theorem 3.34 using the analog of [17, Proposition 3.3.4] for thick Demazure modules (cf. [17, §4]).
Proof We set $M_c := \sum_{c \succ a} D^a$. We have a short exact sequence

$$0 \to M_c \to D^c + M_c \to W(c) \to 0.$$ 

The module $M_c$ is invariant under $D_i$ by Theorem 3.39, and hence we obtain a following exact sequence

$$\mathbb{L}^{-1} D_i(W(c)) \to M_c \to D^c + M_c \to D_i(W(c)) \to 0.$$ 

Here

$$c' = \begin{cases} s_i c & (s_i c \succeq c) \\ c & (s_i c \prec c) \end{cases}$$

and $\mathbb{L} \cdot D_i$ is the left derived functor of $D_i$. By Theorem 3.33 (2), we have the following commutative diagram

$$\begin{array}{ccc} M_c & \longrightarrow & D^c + M_c \\
\| & & \| \\
M_c & \longrightarrow & D^{c'} + M_c \end{array}.$$ 

Since $L(\Lambda_0)$ is completely reducible as a $\mathfrak{sl}(2, i)$-module and $D^c + M_c$ is a $b$-submodule of $L(\Lambda_0)$, the above morphism $D^c + M_c \to D^c + M_c$ is injective by [14, Lemma 2.8 (1)]. Hence $M_c \to D^{c'} + M_c$ is injective. Therefore we obtain $D_i(W(c)) \cong (D^{c'} + M_c)/M_c$ from the above exact sequence.

Proposition 3.41 ([7] Corollary 4.15) Let $i \in \{0, 1, ..., l\}$ and $c \in \hat{P}$. If $s_i c > c$, then we have an exact sequence

$$0 \to \mathbb{D}^c \to D_i(\mathbb{D}^c) \to \mathbb{D}^{s_i c} \to 0$$

and $D_i(\mathbb{D}^{s_i c}) = \{0\}$.

Proof We set $S := \{ w \in W | w \not\leq \pi_c, s_i w \not\leq \pi_c \}$ and $M := \sum_{w \in S} D^w$. Then we have $D^c \cap M = \sum_{\pi_c < w} D^w$. Hence we have an exact sequence

$$0 \to M \to D^c + M \to \mathbb{D}^c \to 0.$$ 

As $s_i(S) \subset S$, we have $D_i(M) \cong M$. By the same argument as in the proof of Proposition 3.40, applying $D_i$, we obtain

$$0 \to M \to D^{s_i c} + M \to D_i(\mathbb{D}^c) \to 0.$$ 

In particular, we have

$$\mathbb{D}^c \cong (D^c + M)/M \quad \text{and} \quad D_i(\mathbb{D}^c) \cong (D^{s_i c} + M)/M.$$ 

Hence we have

$$0 \to \mathbb{D}^c \to D_i(\mathbb{D}^c) \to (D^{s_i c} + M)/(D^c + M) \to 0.$$ 

Here $(D^{s_i c} + M)/(D^c + M) \cong D^{s_i c}/(D^{s_i c} \cap (D^c + M))$ is isomorphic to $\mathbb{D}^{s_i c}$ since $D^{s_i c} \cap (D^c + M) = \sum_{w > s_i \pi_c} D^w$. Hence the first assertion follows. Applying $D_i$ to the last exact sequence, from right exactness of $D_i$, we have an exact sequence

$$D_i(\mathbb{D}^c) \to D_i^2(\mathbb{D}^c) \to D_i(\mathbb{D}^{s_i c}) \to 0.$$
From Theorem 3.33 (3), the above homomorphism $D_t(D^\ast) \to D^2_t(D^\ast)$ is an isomorphism. This implies the second assertion.

### 3.5.2 Calculation of $\text{Ext}^n_{\mathcal{B}}(\mathbb{D}^\lambda \otimes_{\mathcal{C}} C_{m\delta+k\Lambda_0}, D^\gamma_\mu)$

The following theorem is an $A_{2\mathcal{B}}^{(2)}$ version of [7, Theorem 4.18].

**Theorem 3.42** For each $\lambda$, $\mu \in \hat{P}$, $m \in \frac{1}{2}\mathbb{Z}$ and $k \in \mathbb{Z}$, we have

$$\dim_{\mathcal{C}} \text{Ext}^n_{\mathcal{B}}(\mathbb{D}^\lambda \otimes_{\mathcal{C}} C_{m\delta+k\Lambda_0}, D^\gamma_\mu) = \delta_{n,0}\delta_{m,0}\delta_{k,0}\delta_{\lambda,\mu} \quad n \in \mathbb{Z}_+.$$

**Proof** By comparing the level, the extension vanishes if $k \neq 0$. We prove the assertion by induction on $\mu$ with respect to $\triangleright$. By Theorem 3.33 (3), we have $D_w(D_0) = D_0$ for all $w \in \hat{W}$. If $\lambda$ is not anti-dominant, then there exists $i \in \{1, ..., l\}$ such that $s_i \lambda > \lambda$. Hence

$$\text{Ext}^n_{\mathcal{B}}(\mathbb{D}^\lambda \otimes_{\mathcal{C}} C_{m\delta+k\Lambda_0}, D^\gamma_\mu) = \text{Ext}^n_{\mathcal{B}}(\mathbb{D}^\lambda \otimes_{\mathcal{C}} C_{m\delta+k\Lambda_0}, D^\gamma_i(D^\gamma_0)) = \text{Ext}^n_{\mathcal{B}}(D_t(\mathbb{D}^\lambda \otimes_{\mathcal{C}} C_{m\delta+k\Lambda_0}), D^\gamma_0) = \{0\}.$$

Here we used Proposition 3.35 in the second equality and Proposition 3.41 in the third equality. If $\lambda$ is anti-dominant, then we have $D_{w_0}(\mathbb{D}^\lambda) = W(\lambda_+) \otimes_{\mathcal{C}} C_{\Lambda_0}$ for the longest element $w_0$ of $\hat{W}$ by Proposition 3.40. Hence we have $\text{Ext}^n_{\mathcal{B}}(\mathbb{D}^\lambda \otimes_{\mathcal{C}} C_{m\delta+k\Lambda_0}, D^\gamma_0) = \text{Ext}^n_{\mathcal{B}}(W(\lambda_+) \otimes_{\mathcal{C}} C_{m\delta+k\Lambda_0}, W(0)^{\text{loc}})_{\lambda}$ by Theorem 3.17. From Theorem 3.21, the assertion follows in this case.

Let $s_i \mu > \mu$. We set $\mathbb{D}^\gamma_\lambda := \mathbb{D}^\lambda \otimes_{\mathcal{C}} C_{m\delta+k\Lambda_0}$ for $\lambda \in \hat{P}$. By Proposition 3.41, we have the following exact sequence

$$0 \to \text{Ext}^0_{\mathcal{B}}(\mathbb{D}^\gamma_{s_i \lambda}, D^\gamma_\mu) \to \text{Ext}^0_{\mathcal{B}}(D_t(\mathbb{D}^\gamma_\lambda), D^\gamma_\mu) \to \text{Ext}^0_{\mathcal{B}}(\mathbb{D}^\gamma_\lambda, D^\gamma_\mu) \to \cdots \to \text{Ext}^0_{\mathcal{B}}(\mathbb{D}^\gamma_{s_i \lambda}, D^\gamma_\mu) \to \text{Ext}^0_{\mathcal{B}}(D_t(\mathbb{D}^\gamma_\lambda), D^\gamma_\mu) \to \text{Ext}^0_{\mathcal{B}}(\mathbb{D}^\gamma_\lambda, D^\gamma_\mu) \to \cdots.$$

From Theorem 3.34 and Proposition 3.35, we have

$$\text{Ext}^n_{\mathcal{B}}(D_t(\mathbb{D}^\gamma_{s_i \lambda} \otimes_{\mathcal{C}} C_{m\delta+k\Lambda_0}), D^\gamma_\mu) \cong \text{Ext}^n_{\mathcal{B}}(\mathbb{D}^\gamma_{s_i \lambda} \otimes_{\mathcal{C}} C_{m\delta+k\Lambda_0}, D^\gamma_\mu) \cong \text{Ext}^n_{\mathcal{B}}(\mathbb{D}^\gamma_{s_i \lambda} \otimes_{\mathcal{C}} C_{m\delta+k\Lambda_0}, D^\gamma_\mu).$$

Therefore, the assertion follows from the induction hypothesis and the long exact sequence.

**Corollary 3.43** For each $\lambda$, $\mu \in \hat{P}$, we have

$$\langle \text{gch} \mathbb{D}^\lambda, q^{-\frac{1}{2}}E_{\mu}(X^{-1}, q^{-1}) \rangle_{\text{Ext}} = \delta_{\lambda,\mu}.$$

**Proof** By Theorem 3.42, we have

$$\langle \text{gch} \mathbb{D}^\lambda, \text{gch} D_\mu \rangle_{\text{Ext}} = \delta_{\lambda,\mu}.$$

Using Theorem 3.14, we obtain the assertion.

**Corollary 3.44** For each $\lambda \in \hat{P}$, we have

$$\text{gch} \mathbb{D}^\lambda = q^{\frac{1}{2}}E_{\lambda}(X^{-1}, q^{-1})/\langle \tilde{E}_{\lambda}, E_{\lambda}^\dagger \rangle_{\text{Ext}}.$$
\textbf{Proof} Since \( \{ E_{\lambda}(X^{-1}, q^{-1})/\langle \bar{E}_{\lambda}, E_{\lambda}^\dagger\rangle_{\text{Ext}} \}_{\mu \in \hat{P}} \) is a \( \mathbb{C}(q^{1/2}) \)-basis of \( \mathbb{C}(q^{1/2})[\hat{P}] \), we have
\[
gch \mathbb{D}^\lambda = \sum_{\mu \in \hat{P}} a_\mu E_{\lambda}(X^{-1}, q^{-1})/\langle \bar{E}_{\lambda}, E_{\lambda}^\dagger\rangle_{\text{Ext}}
\]
for some \( a_\mu \in \mathbb{C}(q^{1/2}) \). Since \( \langle - , - \rangle_{\text{nonsym}} \mid_{\lambda=0} = \langle - , - \rangle_{\text{Ext}} \), and \( \{ E_{\lambda}(X, q) \}_{\lambda \in \hat{P}} \) are orthogonal with respect to \( \langle - , - \rangle_{\text{nonsym}} \) each other, we have
\[
\langle \bar{E}_{\lambda}(X^{-1}, q^{-1}), E_{\mu}(X^{-1}, q^{-1}) \rangle_{\text{Ext}}/\langle \bar{E}_{\lambda}, E_{\lambda}^\dagger \rangle_{\text{Ext}} = \delta_{\lambda, \mu}.
\]
Hence we have \( \langle \text{gch } \mathbb{D}^\lambda, \text{gch } D_\mu \rangle_{\text{Ext}} = a_\mu \) by Theorem 3.14. Therefore the assertion follows from Corollary 3.43.

\[\square\]

\section{Weyl Modules for Special Current Algebra of \( A^{(2)}_{2l} \)}

We continue to work in the setting of the previous section.

\subsection{Special Current Algebra of \( A^{(2)}_{2l} \)}

In this subsection, we refer for general terminologies to [10, Chapter 2], [8, §2.2] and [1, Appendix]. We set
\[
\mathfrak{h}^\dagger := \bigoplus_{i=0}^{l-1} \mathbb{C} \alpha_i, \quad \Delta^\dagger := \Delta \cap \hat{Q}^\dagger \quad \text{and} \quad \mathfrak{g}^\dagger := \left( \bigoplus_{\alpha \in \Delta^\dagger} \mathfrak{g}_\alpha \right) \oplus \mathfrak{h}^\dagger.
\]
Then \( \mathfrak{g}^\dagger \) is a finite dimensional simple Lie algebra of type \( B_l \). The subalgebra \( \mathfrak{h}^\dagger \) is a Cartan subalgebra of \( \mathfrak{g}^\dagger \), and \( \Delta^\dagger \) is the set of roots of \( \mathfrak{g}^\dagger \) with respect to \( \mathfrak{h}^\dagger \). Using the standard basis \( v_1, ..., v_l \) of \( \mathbb{R}^l \), we have:
\[
\hat{\Delta}^\dagger = \{ \pm (v_i \pm v_j) \mid \pm v_i \mid 1 \leq i \neq j \leq l \}.
\]
We denote the set of short roots of \( \mathfrak{g}^\dagger \) by \( \hat{\Delta}_s^\dagger \) and the set of long roots of \( \mathfrak{g}^\dagger \) by \( \hat{\Delta}_l^\dagger \). We have
\[
\hat{\Delta}_s^\dagger = \{ \pm v_i \mid 1 \leq i \leq l \}
\]
and
\[
\hat{\Delta}_l^\dagger = \{ \pm (v_i \pm v_j) \mid 1 \leq i \neq j \leq l \}.
\]
Let \( \{ \alpha_1^\dagger, ..., \alpha_l^\dagger \} \) be a set of simple roots of \( \mathfrak{g}^\dagger \). We have \( \alpha_i^\dagger = -v_i + v_{i+1} \) if \( i \neq l \) and \( \alpha_l^\dagger = -v_l \). In the notation of §1.2, we have \( \alpha_i^\dagger = \alpha_{l-i} \). We set \( \hat{Q}_+^\dagger := \bigoplus_{i=1, ..., l} \mathbb{Z} \alpha_i^\dagger \). We have
\[
\Delta_{re} = (\hat{\Delta}^\dagger + \mathbb{Z} \delta) \cup (2\hat{\Delta}^\dagger + (2\mathbb{Z} + 1)\delta).
\]
The special current algebra \( \mathfrak{c}_\mathfrak{g}^\dagger \) is the maximal parabolic subalgebra of \( \mathfrak{g} \) that contains \( \hat{\mathfrak{g}}^\dagger \). We have \( \mathfrak{c}_\mathfrak{g}^\dagger = \hat{\mathfrak{g}}^\dagger + \mathfrak{b}_- \). We set
\[
\mathfrak{c}_\mathfrak{g}^\dagger_{im} := \mathfrak{c}_\mathfrak{g}^\dagger_{im}, \quad \mathfrak{c}_\mathfrak{g}^\dagger' := [\mathfrak{c}_\mathfrak{g}^\dagger, \mathfrak{c}_\mathfrak{g}^\dagger]
\]
and
\[
\mathfrak{c}_n^\dagger_+ := \bigoplus_{\alpha \in (\hat{\Delta}^\dagger_{-} - \mathbb{Z} \delta) \cup (2\hat{\Delta}^\dagger_{+} - (2\mathbb{Z} + 1)\delta)} \mathfrak{g}_\alpha
\]
Let \( \hat{\mathbb{P}}^\dagger \) be the integral weight lattice of \( \hat{\mathfrak{g}}^\dagger \) and \( \hat{\mathbb{P}}^\dagger_+ \) be the set of dominant integral weights of \( \hat{\mathfrak{g}}^\dagger \). Let \( \mathfrak{c}_s^\dagger \) (\( i = 1, ..., l \)) be the fundamental weights of \( \hat{\mathfrak{g}}^\dagger \). We identify \( \hat{\mathbb{P}}^\dagger \) and \( \mathbb{Z} \mathfrak{c}^\dagger_{1} \oplus \mathbb{Z} \mathfrak{c}^\dagger_{2} \oplus \mathbb{Z} \mathfrak{c}^\dagger_{3} \oplus \mathbb{Z} \mathfrak{c}^\dagger_{4} \).
\[ \cdots \oplus \mathbb{Z} \omega_i^+ \oplus \mathbb{Z} \omega_i^+ \] by \( \omega_i^+ = \Lambda_{i-1} - \Lambda_i \) for \( i \neq 1 \) and \( \omega_1^+ = \Lambda_0 - \Lambda_1 / 2 \). Let \( W^+ \) be the subgroup of \( W \) generated by \( \{ s_\alpha \}_{\alpha \in \Delta^+} \).

### 4.2 Realization of \( \mathcal{C}_g^\dagger \)

We refer to [6, §4.6] in this subsection. Let \( X_i, j \) be a \((2l + 1) \times (2l + 1)\) matrix unit whose \( i j \)-entry is one. We set \( H_i = X_{i, i} - X_{i+1,i+1} \) \((i = 1, \ldots, 2l)\). The Lie algebra \( \mathfrak{sl}_{2l+1} \) is spanned by \( X_{i, j} \) \((i \neq j)\) and \( H_i \) \((i = 1, \ldots, 2l)\). The assignment

\[
X_{i, i+1} \rightarrow X_{2l+1-i, 2l+2-i}, \quad X_{i+1, i} \rightarrow X_{2l+2-i, 2l+1-i}
\]

extends on \( \mathfrak{sl}_{2l+1} \) as a Lie algebra automorphism. We write this automorphism by \( \sigma \). Let \( L(\mathfrak{sl}_{2l+1}) = \mathfrak{sl}_{2l+1} \otimes \mathbb{C}[t^{\pm 1}] \) be the loop algebra corresponding to \( \mathfrak{sl}_{2l+1} \) and extend \( \sigma \) on \( L(\mathfrak{sl}_{2l+1}) \) by \( \sigma(X \otimes f(t)) = \sigma(X) \otimes f(-t) \). We denote the fixed point of \( \sigma \) in \( \mathfrak{sl}_{2l+1} \otimes \mathbb{C}[t] \) by \( (\mathfrak{sl}_{2l+1} \otimes \mathbb{C}[t])^\sigma \).

**Proposition 4.1** (see [16] Theorem 8.3) *The Lie algebra \( (\mathfrak{sl}_{2l+1} \otimes \mathbb{C}[t])^\sigma \) is isomorphic to \( \mathcal{C}_g^\dagger \).*

### 4.3 Weyl Modules for \( \mathcal{C}_g^\dagger \)

**Definition 4.2** For each \( \lambda \in \hat{P}_+ \), the global Weyl module is a cyclic \( \mathcal{C}_g^\dagger \)-module \( W(\lambda)^\dagger \) generated by \( v_\lambda \) that satisfies the following relations:

1. \( h v_\lambda = \lambda(h) v_\lambda \) for each \( h \in \mathfrak{h}; \)
2. \( e_\alpha^{(\lambda, \alpha)+1} v_\lambda = 0 \) for each \( \alpha \in \Delta_+^\dagger; \)
3. \( \mathcal{C}_g^\dagger v_\lambda = 0. \)

**Definition 4.3** For each \( \lambda \in \hat{P} \), the local Weyl module is a cyclic \( \mathcal{C}_g^\dagger \)-module \( W(\lambda)_{loc}^\dagger \) generated by \( v_\lambda \) satisfies relations (1), (2), (3) of Definition 4.2 and

4. \( X v_\lambda = 0 \) for each \( X \in \mathcal{C}_g^\dagger_{im}. \)

**Theorem 4.4** ([10] Corollary 6.0.1 and [8] Corollary 2.19) *For each \( \lambda \in \hat{P}_+^\dagger \), we have

1. If \( \lambda = \sum_{i=1}^{l-1} m_i \sigma_i^\dagger + (2k - 1) \sigma_l^\dagger \), then
   \[
   \dim_{\mathbb{C}} W(\lambda)^\dagger_{loc} = \left( \prod_{i=1}^{l-1} \binom{2l + 1}{i} \right) \binom{2l + 1}{k-1} \binom{2l}{l};
   \]
2. If \( \lambda = \sum_{i=1}^{l-1} m_i \sigma_i^\dagger + 2m_l \sigma_l^\dagger \), then
   \[
   \dim_{\mathbb{C}} W(\lambda)^\dagger_{loc} = \prod_{i=1}^l \binom{2l + 1}{i}^{m_i}.
   \]

### 4.4 The Algebra \( \mathcal{A}_\lambda \)

Let \( \lambda \in \hat{P}_+^\dagger \). We set

\[ \text{Ann}(v_\lambda) := \{ X \in \mathcal{C}_g^\dagger_{im} \mid X v_\lambda = 0 \} \quad \text{and} \quad \mathcal{A}_\lambda := \mathcal{C}_g^\dagger_{im} / \text{Ann}(v_\lambda), \]
where $v_\lambda$ is the cyclic vector of $W(\lambda)_{loc}^\dagger$ in Definition 4.2.

**Proposition 4.5** ([6] §7.2) For each $\lambda \in \hat{P}_+^\dagger$, the algebra $A_\lambda$ acts on $W(\lambda)^\dagger$ by

\[ X.Yv_\lambda := YXv_\lambda \text{ for } X \in A_\lambda \text{ and } Y \in U(C \xi^\dagger). \]

### 4.4.1 Generators of $A_\lambda$

For $i = 1, ..., l - 1$, we set

\[
\begin{align*}
    h_{i,0} &:= H_i + H_{2l+1-i}, \quad h_{i,1} := H_i - H_{2l+1-i}, \\
    x_{i,0} &:= X_{i,i+1} + X_{2l+1-i,2l+2-i}, \quad x_{i,1} := X_{i,i+1} - X_{2l+1-i,2l+2-i}, \\
    y_{i,0} &:= X_{i,i+1} + X_{2l+2-i,2l+1-i}, \quad y_{i,1} := X_{i,i+1} - X_{2l+2-i,2l+1-i}
\end{align*}
\]

and

\[
\begin{align*}
    h_{l,0} &:= 2(H_l + H_{l+1}), \quad h_{l,1} := H_l - H_{l+1}, \\
    x_{l,0} &:= \sqrt{2}(X_{l,l+1} + X_{l+1,l+2}), \quad x_{l,1} := -\sqrt{2}(X_{l,l+1} - X_{l+1,l+2}), \\
    y_{l,0} &:= \sqrt{2}(X_{l+1,l} + X_{l+2,l+1}), \quad y_{l,1} := -\sqrt{2}(X_{l+1,l} - X_{l+2,l+1}).
\end{align*}
\]

The Lie algebra generated by $\{x_{i,0}, y_{i,0}, h_{i,0}\}_{i=1, ..., l}$ is isomorphic to the simple Lie algebra of type $B_l$ and $\{h_{i,0}\}_{i=1, ..., l}$ is the set of its simple coroots [1, Theorem 9.19]. We set $z_{l,1} := \frac{1}{\sqrt{2}}[y_{l,0}, y_{l,1}]$. As in [5, §3.3], we define $p_{i,r} \in U(C \xi^\dagger_{im})$ ($i = 1, ..., l$ and $r \in \mathbb{Z}_+$) by

\[
\sum_{r \in \mathbb{Z}_+} p_{i,r}z^r := \exp \left( -\sum_{k=1}^{\infty} \frac{h_{i,0}}{k} \otimes t^{-2k} - \frac{h_{i,1}}{2k - 1} z^{2k} \right)
\]

for $i \neq l$ and

\[
\sum_{r \in \mathbb{Z}_+} p_{i,r}z^r := \exp \left( -\sum_{k=1}^{\infty} \frac{h_{i,0}/2}{2k} \otimes t^{-2k} + \frac{h_{i,1}}{2k - 1} z^{2k} \right)
\]

**Proposition 4.6** The algebra $U(C \xi^\dagger_{im})$ is isomorphic to the polynomial ring $\mathbb{C}[p_{i,r} | i = 1, ..., l, r \in \mathbb{Z}_+]$.

**Proof** We have $\mathbb{C}[p_{i,r} | i = 1, ..., l, r \in \mathbb{Z}_+] \subset U(C \xi^\dagger_{im})$. The set of generators of $U(C \xi^\dagger_{im})$ is $\{h_{n,\xi} \otimes t^{-2k+\xi} | n \in \{1, ..., l\}, k \in \mathbb{N} \text{ and } \xi \in \{0, 1\}\}$. It suffices to see that $h_{n,\xi} \otimes t^{-2k+\xi} \in \mathbb{C}[p_{i,r} | i = 1, ..., l, r \in \mathbb{Z}_+]$ for each $i \in \{1, ..., l\}$, $k \in \mathbb{N}$ and $\xi \in \{0, 1\}$. We have $h_{l,\xi} \otimes t^{-1} = p_{l,1}$ up to a constant multiple. By definition, $p_{l,2k-\xi} + (h_{l,\xi} \otimes t^{-2k+\xi})/(2k - \xi)$ is an element of $\mathbb{Q}[h_{l,s} | s < 2k - \xi]$ if $i \neq l$, and $p_{l,2k-\xi} - (1)^{\xi+1}h_{l,s}z^{1-\xi} \otimes t^{-2k+\xi} / (2k - \xi)$ is an element of $\mathbb{Q}[h_{l,s} | s < 2k - \xi]$. The assertion follows by induction on $2k - \xi$.

**Lemma 4.7** ([5] Lemma 3.2, Lemma 3.3 (iii) (b) and [4] Lemma 1.3 (iii)) Let $V$ be a $C \xi^\dagger$-module and $v \in V$ be a nonzero vector such that $C \xi_{im}v = 0$. We have the following:

1. For $i \neq l$, we have $(x_{i,1} \otimes t^{-1})(y_{i,0})(r)v = (-1)^{r} p_{i,r}v$ for $r \in \mathbb{N}$;
2. We have $(x_{l,0})(2r)(z_{l,1} \otimes t^{-1})(r)v = (-1)^{r} p_{l,r}v$ for $r \in \mathbb{N}$.

**Proposition 4.8** Let $\lambda \in \hat{P}_+^\dagger$, $i \in \{1, ..., l - 1\}$ and $v_\lambda$ be cyclic vector of $W(\lambda)^\dagger$ with its weight $\lambda$. We have $p_{i,r}v_\lambda = 0$ for $r > \langle \lambda, \alpha_i^\dagger \rangle$, and $p_{l,r}v_\lambda = 0$ for $r > |\langle \lambda, \alpha_l^\dagger \rangle|$.\[\square\]
Proof Definition 4.2 (3) implies the set of \( \hat{h}^{\dagger} \)-weights of \( W(\lambda)^{\dagger} \) is the subset of \( \lambda - \hat{Q}^{\dagger} \).

From Definition 4.2 (2) and Lemma 4.7 (1), we get \( p_{l,r} v_{\lambda} = 0 \) for \( r > \langle \lambda, \hat{\alpha}_{l}^{\dagger} \rangle \). By Definition 4.2 (2), \( W(\lambda)^{\dagger} \) is an \( \hat{g}^{\dagger} \)-integrable module. Since the set of \( \hat{h}^{\dagger} \)-weights of \( W(\lambda)^{\dagger} \) is contained in \( \lambda - \hat{Q}^{\dagger} \), this implies \( \lambda - k\alpha_{l}^{\dagger} \) for \( k > \langle \lambda, \hat{\alpha}_{l}^{\dagger} \rangle \) is not a weight of a vector of \( W(\lambda)^{\dagger} \). Since \( (z_{l,1} \otimes t) \) is a root vector corresponding to \( 2\alpha_{l}^{\dagger} - \delta \), we obtain \( p_{l,r} v_{\lambda} = 0 \) for \( r > \left\lfloor \frac{\langle \lambda, \hat{\alpha}_{l}^{\dagger} \rangle}{2} \right\rfloor \).

We set \( A^{\prime}_{\lambda} := \mathbb{C}[p_{i,r} \mid 1 \leq r \leq \langle \lambda, \hat{\alpha}_{l}^{\dagger} \rangle \text{ for } i \neq l, \ 1 \leq r \leq \left\lfloor \frac{\langle \lambda, \hat{\alpha}_{l}^{\dagger} \rangle}{2} \right\rfloor \text{ for } i = l] \).

Corollary 4.9 For each \( \lambda \in \hat{P}^{\dagger}_{+} \), there exists a \( \mathbb{C} \)-algebra surjection \( A^{\prime}_{\lambda} \rightarrow A_{\lambda} \).

Proof By Proposition 4.8, we have \( p_{i,r}, p_{l,k} \in \text{Ann}(v_{\lambda}) \) for each \( r > \langle \lambda, \hat{\alpha}_{l}^{\dagger} \rangle \) (\( i \neq l \)) and each \( k > \left\lfloor \frac{\langle \lambda, \hat{\alpha}_{l}^{\dagger} \rangle}{2} \right\rfloor \). Hence we have a surjection \( A^{\prime}_{\lambda} \rightarrow A_{\lambda} \) by Proposition 4.6.

We set \( \hat{P}^{\dagger}_{+} := \{ \lambda \in \hat{P}^{\dagger}_{+} \mid \langle \lambda, \hat{\alpha}_{l}^{\dagger} \rangle \in 2\mathbb{Z}_{+} \} \).

Theorem 4.10 ([6] §5.6 and Theorem 1) For each \( \lambda \in \hat{P}^{\dagger}_{+} \) and nonzero element \( f \in A^{\prime}_{\lambda} \), there exists a quotient of \( W(\lambda)^{\dagger} \) such that \( f \) acts nontirivially on the image of the cyclic vector \( v_{\lambda} \) of \( W(\lambda)^{\dagger} \).

In particular \( A_{\lambda} \cong A^{\prime}_{\lambda} \).

Lemma 4.11 ([6] Lemma 5.4) For each \( 1 \leq s \leq k \), let \( V_{s} \) be representations of \( \mathbb{C}^{g^{\dagger}} \) and let \( v_{s} \) be vectors of \( V_{s} \) such that \( C^{n_{l}} v_{s} = 0 \). We have

\[ p_{i,r}(v_{1} \otimes \cdots \otimes v_{k}) = \sum_{r=j_{1}+\cdots+j_{k}, \ j_{i} \geq 0} p_{i,j_{i}} v_{1} \otimes \cdots \otimes p_{i,j_{k}} v_{k} \]

for all \( 1 \leq i \leq l \) and \( r \in \mathbb{Z}_{+} \).

4.4.2 Dimension Inequalities

For each maximal ideal \( I \) of \( A_{\lambda} \), we define

\[ W(\lambda, I)^{\dagger} := (A_{\lambda}/I) \otimes W(\lambda)^{\dagger} \]

Let \( U(\mathbb{C}^{g_{im}})_{+} \) be the augmentation ideal of \( U(\mathbb{C}^{g_{im}}) \) and \( I_{\lambda,0} \) be a maximal ideal of \( A_{\lambda} \) defined by \( (U(\mathbb{C}^{g_{im}})_{+} + \text{Ann}(v_{\lambda}))/\text{Ann}(v_{\lambda}) \).

Proposition 4.12 For each \( \lambda \in \hat{P}^{\dagger}_{+} \), we have \( W(\lambda)^{\dagger}_{loc} \cong W(\lambda, I_{\lambda,0})^{\dagger} \).

Proof The assertion follows from Definition 4.3 (4).

Proposition 4.13 ([6] Proposition 6.4 and 6.5) Let \( \lambda \in \hat{P}^{\dagger}_{+} \) and let \( I \) be a maximal ideal of \( A_{\lambda} \).

\[ \Box \text{ Springer} \]
If \( \mu \in \hat{P}_+ \) satisfies \( \lambda - \mu \in \hat{P}_+^\vee \), then we have
\[
\dim_{\mathbb{C}} W(\lambda, I)^\dagger \geq \dim_{\mathbb{C}} W(\mu)_{\text{loc}}^{\dagger} \left( \prod_{i=1}^{l-1} \binom{2l + 1}{i} \right)^{(\lambda - \mu)(\alpha_i^\vee) (l+1) \binom{(l-\mu)(\alpha_i^\vee)/2}}.
\]

We have
\[
\dim_{\mathbb{C}} W(\lambda)^\dagger \geq \dim_{\mathbb{C}} W(\lambda, I)^\dagger.
\]

**Corollary 4.14** ([6] Theorem 10 when \( \lambda \in \hat{P}_+^\vee \)) For each \( \lambda \in \hat{P}_+^\vee \) and each maximal ideal \( I \) of \( A_\lambda \), the dimension \( \dim_{\mathbb{C}} W(\lambda, I)^\dagger \) does not depend on \( I \) and is given by Theorem 4.4.

**Proof** If \( \lambda = \sum_{i=1}^{l-1} m_i \sigma_i^\vee + 2m_l \sigma_l^\vee \), then we have
\[
\dim_{\mathbb{C}} W(\lambda)^\dagger \geq \dim_{\mathbb{C}} W(\lambda, I)^\dagger \geq \prod_{i=1}^{l-1} \binom{2l + 1}{i}^{m_i} \binom{2l + 1}{l}^{k-1}
\]

by Proposition 4.13. From Theorem 4.4 (2), this inequality is actually equality. If \( \lambda = \sum_{i=1}^{l-1} m_i \sigma_i^\vee + (2k - 1) \sigma_l^\vee \), then we have
\[
\dim_{\mathbb{C}} W(\lambda)^\dagger \geq \dim_{\mathbb{C}} W(\lambda, I)^\dagger \geq \dim_{\mathbb{C}} W(\sigma_l^\vee)_{\text{loc}}^{\dagger} \left( \prod_{i=1}^{l-1} \binom{2l + 1}{i}^{m_i} \binom{2l + 1}{l}^{k-1}
\]

by Proposition 4.13. From Theorem 4.4 (1), this inequality is actually equality. Hence the assertion follows.

### 4.5 Freeness of \( W(\lambda)^\dagger \) over \( A_\lambda \)

In this subsection, we prove the following theorem

**Theorem 4.15** For each \( \lambda \in \hat{P}_+^\vee \), the global Weyl module \( W(\lambda)^\dagger \) is free over \( A_\lambda \).

To prove this theorem, we need the following preparatory result:

**Theorem 4.16** For each \( \lambda \in \hat{P}_+^\vee \), the algebra \( A_\lambda \) is isomorphic to \( A_\mu^\prime \).

Theorem 4.16 and Corollary 4.14 imply Theorem 4.15 by [23, 26]. We prove Theorem 4.15 after proving Theorem 4.16.

**Proof of Theorem 4.16** We show that the surjection \( A_\lambda^\prime \to A_\lambda \) is the isomorphism. We have \( \dim_{\mathbb{C}} A_\lambda^\prime = 1 \). Since \( \dim_{\mathbb{C}} A_{\sigma_l^\vee}^\prime \geq 1 \). Hence \( A_\lambda^\prime \to A_{\sigma_l^\vee}^\prime \) is the isomorphism. If \( \lambda \in \hat{P}_+^\vee \), then the assertion is Theorem 4.10. We prove the assertion for \( \lambda = \sum_{i=1}^{l-1} m_i \sigma_i^\vee + (2m + 1) \sigma_l^\vee \). Let \( f \in A_\lambda^\prime \) be a nonzero element. It is suffice to show that there exists a quotient of \( W(\lambda)^\dagger \) such that \( f \) acts nontrivially on the image of the cyclic vector \( v_\lambda \) of \( W(\lambda)^\dagger \). Let \( \mu := \lambda - \sigma_l^\vee \). We have \( A_\lambda^\prime \cong A_\mu^\prime \). By checking the defining relations, we have a homomorphism of \( \mathbb{C}g^\vee \)-module
\[
W(\lambda)^\dagger \to W(\sigma_l^\vee)^\dagger \otimes_{\mathbb{C}} W(\mu)^\dagger
\]
which maps \( v_\lambda \) to \( v_{\sigma_i^j} \otimes v_\mu \). By Theorem 4.10, we have a quotient module \( V \) of \( W(\mu)^\dagger \) such that \( f \) acts nontrivially on the image of \( v_\mu \in W(\mu)^\dagger \). We have a homomorphism

\[
W(\lambda)^\dagger \to W(\sigma_i^j)^\dagger \otimes_C V \to W(\sigma_i^j)^{\dagger \text{loc}} \otimes_C V.
\]

Let \( v \in V \) and \( w_{\sigma_i^j}^\dagger \in W(\sigma_i^j)^{\dagger \text{loc}} \) be the image of \( v_\mu \) in \( V \) and the image of \( v_{\sigma_i^j} \) in \( W(\sigma_i^j)^{\dagger \text{loc}} \), respectively. By Lemma 4.11, we have \( p_{i,r}(w_{\sigma_i^j}^\dagger \otimes v) = w_{\sigma_i^j}^\dagger \otimes p_{i,r}(v) \) for each \( i \in \{1, \ldots, l\} \) and \( r \in \mathbb{Z}_+ \). Therefore, \( f \) acts nontrivially on the highest weight vector \( w_{\sigma_i^j}^\dagger \otimes v \) of \( W(\sigma_i^j)^{\dagger \text{loc}} \otimes V \). Hence \( f v_\lambda \neq 0 \). Hence the assertion follows.

\[\square\]

**Proof of Theorem 4.15** We set \( N := \dim W(\lambda)^{\dagger \text{loc}} \). Let \( \mathfrak{m} \) be a maximal ideal of \( A_\lambda \). By Nakayama's lemma [22, Lemma 1.M], there exists \( f \not\in \mathfrak{m} \) such that \((W(\lambda)^\dagger)_f \) is generated by \( N \) elements as \((A_\lambda)_f\)-module, where \((W(\lambda)^\dagger)_f \) and \((A_\lambda)_f \) are the localization of \((W(\lambda)^\dagger) \) and \((A_\lambda) \) by \( f \), respectively. Since \((A_\lambda)_f \) is Noetherian, we have an exact sequence

\[
(A_\lambda)_f \xrightarrow{\phi} (A_\lambda)_f^{\oplus N} \xrightarrow{\psi} (W(\lambda)^\dagger)_f \to 0.
\]

For any maximal ideal \( \mathfrak{n} \) such that \( f \not\in \mathfrak{n} \), the induced morphism \( \overline{\psi} : (A_\lambda)_f^{\oplus N} / \mathfrak{n}(A_\lambda)_f^{\oplus N} \to (W(\lambda)^\dagger)_f / \mathfrak{n}(W(\lambda)^\dagger)_f \) is an isomorphism by Corollary 4.14. This implies the matrix coefficients of \( \phi \) are contained in the Jacobson radical of \((A_\lambda)_f \). Since \((A_\lambda)_f \) is an integral domain and finitely generated over \( C \), we deduce \( \phi = 0 \). It follows that \((W(\lambda)^\dagger) \) is flat over \( A_\lambda \) by [15]. Since \( A_\lambda \) is a polynomial ring, \((W(\lambda)^\dagger) \) is a projective \( A_\lambda \)-module. From [23, 26], a finitely generated projective module over a polynomial ring is free. Hence the assertion follows.

\[\square\]

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**References**

1. Carter, R.: Lie Algebras of Finite and Affine Type. Cambridge University Press, Cambridge (2010)
2. Chari, V., Ion, B.: BGG reciprocity for current algebras. Compos. Math. 151, 7 (2015)
3. Chari, V., Loktev, S.: Weyl, Demazure and fusion modules for the current algebra of \( \mathfrak{sl}_2 \). Int. Math. Res. Not. 5038 (2008)
4. Chari, V., Ion, B.: BGG reciprocity for current algebras. Compos. Math.
5. Carter, R.: Lie Algebras of Finite and Affine Type. Cambridge University Press, Cambridge (2010)
6. Heckenberger, I., Kolb, S.: On the Bernstein-Gelfand-Gelfand resolution for Kac-Moody algebras and quantized enveloping algebras. Transform. Groups 12(4), 647–655 (2007)
7. Joseph, A.: On the Demazure character formula. Ann. Sci. l’E.N.S. 389–419 (1985)
8. Feigin, E., Kato, S., Makedonskyi, I.: Representation theoretic realization of non-symmetric Macdonald polynomials at infinity, Preprint: arXiv:1703.04108. J. Reine Angew. Math. to appear (2017)
9. Joseph, A.: On the Demazure character formula. Ann. Sci. l’E.N.S. 389–419 (1985)
15. Jothilingam, P.: When is a flat module projective, Indian. J. pure appl. Math. 15(1), 65–66 (1984)
16. Kac, V.G.: Infinite-Dimensional Lie Algebras, 3rd edn. Cambridge University Press, Cambridge (1990)
17. Kashiwara, M.: The crystal base and Littelmann’s refined Demazure character formula. Duke Math. J. 71, 839–858 (1993)
18. Kato, S.: Frobenius splitting of thick flag manifolds of Kac-Moody algebras. Int. Math. Res. Not. my174 July (2018)
19. Kleshchev, A.: Affine highest weight categories and affine quasi-hereditary algebras. Proc. Lond. Math. Soc. 110(4), 841–882 (2015)
20. Kumar, S.: Kac-Moody Groups, Their Flag Varieties and Representation Theory, volume 204 of Progress in Mathematics. Birkhäuser Boston, Inc, Boston (2002)
21. Macdonald, I.G.: Affine Hecke Algebras and Orthogonal Polynomials Cambridge Tracts in Mathematics, vol. 157. Cambridge University Press, Cambridge (2003)
22. Matsumura, H.: Commutative Algebra. Benjamin/Cummings (1980)
23. Quillen, D.: Projective modules over polynomial rings. Invent. Math. 36, 167–171 (1976)
24. Sahi, S.: Some Properties of Koornwinder Polynomials. q-Series from a Contemporary Perspective (South Hadley, MA, 1998), Contemp. Math. 254. AMS, Providence (2000)
25. Sanderson, Y.B.: On the connection between Macdonald polynomials and Demazure characters. J. Algebraic Combin. 11, 269 (2000). https://doi.org/10.1023/A:1008786420650
26. Suslin, A.A.: Projective modules over polynomial rings are free. Dokl. Akad. Nauk SSSR 229(5), 1063–1066 (1976)
27. Xia, L.-M., Hu, N., Bai, X.: Vertex representations for twisted affine Lie algebra of type $A_{2l}^{(2)}$, arXiv:0811.0215 (2008)

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