INFEINCE IN HIGH-DIMENSIONAL REGRESSION MODELS WITHOUT THE EXACT OR $L^p$ SPARSITY

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ABSTRACT. We propose a new inference method in high-dimensional regression models and high-dimensional IV regression models. The method is shown to be valid without requiring the exact sparsity or $L^p$ sparsity conditions. Simulation studies demonstrate superior performance of this proposed method over those based on the LASSO or the random forest, especially under less sparse models. We illustrate an application to production analysis with a panel of Chilean firms. Our results are closer to the benchmark conventional estimates than the estimates by other machine learning methods.

Keywords: double/debiased machine learning, high-dimensional Akaike information criterion, orthogonal greedy algorithm, production function.

1. INTRODUCTION

The advent of modern machine learning techniques has significantly widened the class of analyzable regression models that include models with high-dimensional controls and/or models with flexible nonlinearity. Perhaps the most popular and important machine learning approaches to estimation and inference in high-dimensional regression models today are those based on shrinkage and regularization, such as the least absolute shrinkage and selection

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1We use the phrase “machine learning” following the recent related literature, but it is worthy to remark that it is a synonym of semiparametric or high-dimensional ‘estimation.’
operation (LASSO). While they have practically appealing properties, these popular machine learning methods yet rely on a list of assumptions that may not be necessarily mild under certain applications. In particular, the assumptions of the exact sparsity and the $L^p$ sparsity required for these methods are sometimes controversial and perceived to be strong for some applications. As such, there still remains room in the literature for further widening the class of analyzable high-dimensional regression models if these assumption can be relaxed by a new machine learning method.

This paper proposes a method of inference in high-dimensional regression models without requiring the exact sparsity or the $L^p$ sparsity. We set a low-dimensional parameter vector as the object of interest, and treat the remaining high-dimensional parameter vector as a nuisance component. Under this common setting, our proposed method of inference works as follows. First, use the orthogonal greedy algorithm (OGA; Temlyakov, 2000) to order the high-dimensional regressors in a descending order of explanatory power. Second, use the high-dimensional Akaike information criterion (HDAIC; Ing, 2020) to select a model among the ordered list of models constructed in the first step. Third, estimate the models selected in the second step. Fourth, plug the estimated selected models in a Neyman orthogonal score and estimate the low-dimensional parameter vector of interest.

We take advantage of a number of recent methodological and theoretical developments, namely OGA, HDAIC, and DML, to derive asymptotic statistical properties of this new method of inference. Ing (2020) investigates convergence rate properties of an estimator of high-dimensional regression models based on OGA and HDAIC (hereafter referred to as OGA+HDAIC). Importantly, the setting imposes assumptions on the high-dimensional parameter vector that are weaker than the exact sparsity and the $L^p$ sparsity. Furthermore, this approach does not require to impose high-level conditions on the sample Gram matrix, such
as a restricted eigenvalue type condition, that are often required in the literature. Even under these weaker conditions, it is still possible to obtain similar rates of convergence to those based on existing machine learning techniques such as the LASSO. Given the adequately fast convergence rates of the preliminary nuisance parameter estimators based on OGA+HDAIC, we can apply the post-double-selection approach (Belloni, Chernozhukov, and Hansen, 2013) or the double/debiased machine learning (DML) framework (Chernozhukov, Chetverikov, Demirer, Duflo, Hansen, Newey, and Robins, 2018) to in turn obtain a root-$N$ convergence of the low-dimensional parameter vector of interest with a limit normal distribution. Simulation studies demonstrate that this proposed method performs significantly better than those based on the LASSO or the random forest, especially when the data generating model becomes less sparse. We apply the proposed method to an analysis of production functions using a panel of Chilean firms. Our results are closer to the benchmark conventional estimates (Levinsohn and Petrin, 2003) than the estimates based on other machine learning methods, namely the LASSO and the random forest.

**Relation to the literature:** This paper is related to several branches of the econometrics and statistics literature. First, it is closely related to the literature on inference in high-dimensional regression models and high-dimensional IV regression models, e.g., Belloni, Chen, Chernozhukov, and Hansen (2012), Belloni et al. (2013), Javanmard and Montanari (2014), van de Geer, Bühlmann, Ritov, and Dezeure (2014), Zhang and Zhang (2014), Caner and Kock (2018a, 2018b), Belloni, Chernozhukov, Chetverikov, Hansen, and Kato (2018a), Galbraith and Zinde-Walsh (2020), Gold, Lederer, and Tao (2020), Kueck, Luo, Spindler, and Wang (2021) to list a few. The majority of these papers focus on utilizing LASSO (Tibshirani, 1996) and its variants for estimation, and thus rely crucially on the exact sparsity, approximate sparsity or $L^p$ sparsity. We contribute to this literature, as emphasized above, by relaxing the

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2The approximate sparsity is closely related to the exact sparsity.
conventional assumptions of these notions of sparsity. Second, also related is the literature on model selection methods for high-dimensional models. For extensive reviews of this vast literature, we refer readers to the monographs of Bühlmann and van de Geer (2011), Giraud (2015), and Hastie, Tibshirani, and Wainwright (2019). In particular, we take advantage of the theoretical results of OGA+HDAIC by Ing (2020) as one of the main auxiliary steps to our goal as emphasized earlier. Methodologically, our paper benefits from the theoretical studies of various greedy algorithms in Temlyakov (2000), Tropp (2004), Tropp and Gilbert (2007), Ing and Lai (2011), and share ties with other iterated model selection methods such as the least absolute angle regression of Efron, Hastie, Johnstone, Tibshirani et al. (2004), the $L_2$-boosting of Bühlmann and Yu (2003), the test-based forward model selection of Kozbur (2017, 2020), and so forth. Third, this paper is related to the literature on Neyman orthogonal scores or locally robust scores, e.g., Belloni, Chernozhukov, and Kato (2015), Chernozhukov, Escanciano, Ichimura, Newey, and Robins (2016), Belloni, Chernozhukov, Chetverikov, and Demirer (2018b), and, in particular, the DML (Chernozhukov, Chetverikov, Demirer, Duflo, Hansen, Newey, and Robins, 2018). We contribute to this literature by proposing to add OGA+HDAIC to the library of the list of preliminary estimators. Fourth, the conditions that we impose in place of the exact sparsity and the $L^p$ sparsity concern the speed at which the absolute size of the regression parameters decays when descendingly ordered. These conditions are analogous to those that are used for model selection problems in autoregressive time series models (e.g., Shibata, 1980; Ing, 2007) as well as the ordinary- and super-smoothness on probability density functions that are used in the deconvolution literature (e.g., Fan, 1991; Fan and Truong, 1993).
2. High-Dimensional Linear Regression Models

2.1. The Model. Consider the linear regression model

\[ Y = D\theta_0 + X'\Lambda_0 + U, \quad E[U|X, D] = 0, \]  

(2.1)

where \( Y \) denotes an outcome variable, \( D \) denotes a treatment variable, \( X \) denotes a \( p \)-dimensional vector of controls, and \( U \) denotes unobserved factors. We allow for a high dimensionality in the sense that \( p \) can be increasing in \( N \) and may be even larger than \( N \) – more details will follow. In this framework, we are interested in the partial effect \( \theta_0 \) of \( D \) on \( Y \). Also write the linear projection of \( D \) on \( X \):

\[ D = X'\beta_0 + V, \quad E[V|X] = 0, \]  

(2.2)

In Section 4.1, we consider an extended model in which we introduce approximation errors in (2.1)–(2.2).

To construct a moment restriction under (2.1)–(2.2), consider the orthogonal score function from Robinson (1988):

\[ \psi(Y, D, X; \theta, \eta) := \{Y - X'\gamma - \theta(D - X'\beta)\} (D - X'\beta), \]  

(2.3)

where \( X'\gamma_0 = E[Y|X] \) and \( \eta = (\gamma, \beta) \). Note also that \( X'\beta_0 = E[D|X] \) follows by construction from (2.2).

Notations. To proceed, we first fix basic notations. We use subscripts \( i \) and \( j \) to denote indices of observations and coordinates, respectively. Define \( X_{IJ} = (X_{ij}, i \in I) \) as a \(|I| \times 1\) vector, \( X_{iJ} = (X_{ij}, j \in J) \) as a \(|J| \times 1\) vector, and \( X_{IJ} = (X_{ij}, i \in I)' \) as a \(|I| \times |J|\) matrix, where \( I \) is a subset of observation indices \( \{1, 2, ..., N\} \), \( J \) is a subset of coordinate indices \( \mathcal{P} \equiv \{1, 2, ..., p\} \), and \(|.|\) denotes the set cardinality. For any vector, \( \|.| \) refers to the Euclidean norm. The \( L^q \) norm is defined by \( \|\xi\|_q = (\sum_{j=1}^{p} \xi_j^q)^{1/q} \) for \( q < \infty \) and \( \|\xi\|_{\infty} = \max_{1 \leq j \leq p} |\xi_j| \).
2.2. The Method. This section provides an overview of the method. We propose the following procedure for a root-$N$ consistent estimation and inference about the partial effect $\theta_0$ without assuming sparsity on the high-dimensional parameters, $\beta_0$ or $\gamma_0$.

**Algorithm 1** (OGA+HDAIC with DML for high-dimensional linear models).

**Step 1.** Randomly split the sample indices $\{1, ..., N\}$ into $K$ folds $(I_k)_{k=1}^K$. For simplicity, let the size of each fold be $n = N/K$ and the size of $I_k^c$ be $n^c$.

**Step 2.** For each fold $k \in \{1, ..., K\}$, perform following procedure using \{(X', D)\}_{i \in I_k^c} to get $\hat{\beta}_k$.

(a) Compute $\hat{\mu}_{0,j} = X'_{I_k,j} D_{I_k} / \sqrt{n^c} \|X_{I_k,j}\|$. Select the coordinate $\hat{j}_1 = \arg\max_{1 \leq j \leq p} |\hat{\mu}_{0,j}|$. Define $\hat{J}_1 = \{\hat{j}_1\}$.

(b) Compute $\hat{\mu}_{1,j} = X'_{I_k,j} (I_{n^c} - H_1) D_{I_k} / \sqrt{n^c} \|X_{I_k,j}\|$, where $H_1 = X'_{I_k,\hat{J}_1} (X'_{I_k,\hat{J}_1} X_{I_k,\hat{J}_1})^{-1} X'_{I_k,\hat{J}_1}$. Select the coordinate $\hat{j}_2 = \arg\max_{1 \leq j \leq p, j \notin \hat{J}_1} |\hat{\mu}_{1,j}|$. Update $\hat{J}_2 = \hat{J}_1 \cup \{\hat{j}_2\}$.

(c) Given $m-1$ coordinates $\hat{J}_{m-1}$ that have been obtained, compute $\hat{\mu}_{m-1,j} = X'_{I_k,j} (I_{n^c} - H_{m-1}) D_{I_k} / \sqrt{n^c} \|X_{I_k,j}\|$, where $H_{m-1} = X'_{I_k,\hat{J}_{m-1}} (X'_{I_k,\hat{J}_{m-1}} X_{I_k,\hat{J}_{m-1}})^{-1} X'_{I_k,\hat{J}_{m-1}}$. Select the coordinate $\hat{j}_m = \arg\max_{1 \leq j \leq p, j \notin \hat{J}_{m-1}} |\hat{\mu}_{m,j}|$. Iteractively update $\hat{J}_m = \hat{J}_{m-1} \cup \{\hat{j}_m\}$.

(d) Compute HDAIC ($\hat{J}_m = (1 + C^*|\hat{J}_m| \log p/n^c)\hat{\sigma}_m^2$) for each $m$, where $C^*$ is from (D.2) in Appendix D.1 and $\hat{\sigma}_m^2 = 1/n^c D'_{I_k} (I - H_m) D_{I_k}$. Choose $\hat{m} = \arg\min_{1 \leq m \leq M^*_n} \text{HDAIC}(\hat{J}_m)$, where $M^*_n$ is defined in (D.1) in Appendix D.1.

(e) With coordinates $\hat{J}_{\hat{m}}$ run OLS of $D_i$ on $X_{i,\hat{J}_{\hat{m}}}$ to get $\hat{\beta}_k$.

**Step 3.** Repeat Step 2 with \{(X', Y')\}_{i \in I_k^c} instead of \{(X', D)\}_{i \in I_k^c}, to get $\hat{\gamma}_k$ for each fold $k \in \{1, ..., K\}$.

**Step 4.** Obtain $\tilde{\theta}$ as a solution to $1/K \sum_{k=1}^K 1/n \sum_{i \in I_k} \psi(Y_i, D_i; X_i; \hat{\gamma}_k) = 0$ where $\hat{\eta}_k = (\hat{\gamma}_k, \hat{\beta}_k)$ and $\psi$ is defined in (2.3).
Step 5. Compute $\hat{M} = -1/K \sum_{k=1}^{K} 1/n \sum_{i \in I_k} (D_i - X^\prime_i \hat{\beta})^2$. Obtain a variance estimator of $\hat{\theta}$ as $\hat{\Omega} = \hat{M}^{-1} \frac{1}{K} \sum_{k=1}^{K} \frac{1}{n} \sum_{i \in I_k} [\psi(Y, D, X; \hat{\theta}, \hat{\eta}_k) \psi(Y, D, X; \hat{\theta}, \hat{\eta}_k)'] (\hat{M}^{-1})'$.

We highlight three notable elements of this algorithm. First, the overall procedure (Steps 1–4) uses the cross fitting to remove an over-fitting bias. Specifically, by using complementary sub-sample $I^c_k$ to estimate the nuisance parameters $\hat{\eta}_k = (\hat{\gamma}_k, \hat{\beta}_k)$ that are in turn evaluated in the $I_k$-mean of the score, we can circumvent a bias that arises from products of dependent factors in the score. Our combined use of the orthogonal score (2.3) and this cross-fitting method allows for the high-level theory of the double/debiased machine learning (DML, Chernozhukov et al., 2018) to be applicable. Section 4.2 discusses an alternative algorithm that does not rely on the cross fitting at the cost of an additional assumption. Second, the coordinates $\{\hat{j}_1, \ldots, \hat{j}_p\}$ are ranked in Step 2 (a)–(c) in the order of decreasing importance after successive orthogonalization using OGA as in Ing (2020). Third, a subset $\hat{J}_m = \{\hat{j}_{1}, \ldots, \hat{j}_{m}\}$ of the ordered set $\{\hat{j}_1, \ldots, \hat{j}_p\}$ is selected in Step 2 (d) using HDAIC as in Ing (2020). Our combined use of these three elements (DML, OGA, and HDAIC) together allows for a novel root $N$ consistent estimation of $\theta_0$ without assuming traditional functional class restrictions (e.g., the sparsity) required by existing popular estimators (e.g., LASSO). In Section 2.3, we formally present theoretical arguments in support of this claim.

While Algorithm provides nearly full details of the proposed method, it omits a couple of details. Specifically, Step 2 (d) on HDAIC uses two tuning parameters, $C^*$ and $M^*_n$. We present details about these aspects of the algorithm in Appendix D.1.

2.3. The Theory. This section proposes and discusses assumptions under which one can conduct an inference about $\hat{\theta}$ based on root-$N$ asymptotic normality using the method described in Algorithm. We use the notations $c, C, \overline{C}, \overline{\tau}$ and $q$ for strictly positive constants such that their values can differ depending on the location. Let $q > 4$ be a positive integer, $c_q, C_q, \lambda_1$ be some positive constants and $K_{N,q}$ be a positive sequence of constants such that
\( K_{N,q} \geq E[\max_{1 \leq j \leq p} |X_{ij}|^q] \). Wherever there is no risk of confusion, we also use the generic notation \( \xi \) to refer to both \( \beta \) and \( \gamma \) to avoid repetitions. All the random variables and parameter vectors are \( N \)-dependent unless otherwise specified. We abbreviate the \( N \) index for brevity.

**Assumption 1.** For each \( N \in \mathbb{N} \), it holds that

(a) \( (Y_i, D_i, X_i') \) are i.i.d. copies of \( (Y, D, X') \).
(b) (2.1) and (2.2) hold.
(c) \( E[|Y|^q] + E[|D|^q] \leq C_q \).
(d) \( E[|UV|^2] \geq c_q^2 \) and \( E[V^2] \geq c_q \).
(e) \( \max_{1 \leq j \leq p} E[|X_{ij}|^q] \leq C_q \), \( E[|V|^q] \leq C_q \), and \( E[|U|^q] \leq C_q \).

Furthermore, it holds asymptotically that (f) \( K_{N,q}^2 \log p/N^{1-2/q} = o(1) \).

Assumption 1 (a) requires a random sampling of data. Assumption 1 (b) requires that the correct model is given by (2.1) and (2.2). Assumption 1 (c)–(e) requires bounded moments of various variables. Assumption 1 (f) requires constraints on the speed at which the dimensionality as well as the maximal of the covariate vector can grow. Vectors consist of independent subgaussian random variables with bounded variances, for example, are special cases satisfying this restriction, as the expectation of the maximum of \( X_{ij} \) is bounded by a factor of \( \sqrt{\log p} \). We emphasize that these assumptions are mild in comparison with the counterpart assumptions made in the high-dimensional regression literature.

**Assumption 2.** It holds over \( N \in \mathbb{N} \) that

(a) \( \lambda_{\min}(\Gamma) \geq \lambda_1 > 0 \) and \( \lambda_{\max}(\Gamma) \leq C_q \), where \( \Gamma = E[XX'] \).
(b) Define \( \Gamma(J) = E[X_{i,J}X_i'] \) and \( d_\ell(J) = E[X_{i,J}X_{i,J}] \) for a set of coordinate indices \( J \subseteq \mathcal{P} \). Then

\[
\max_{1 \leq |J| \leq C(N/\log p)^{1/2}, \ell \notin J} |\Gamma^{-1}(J)d_\ell(J)| < C_q.
\]
Assumption 2(a) requires that the minimum eigenvalue $\lambda_{\min}(\Gamma)$ of the Gram matrix $\Gamma$ to be positive, and it is a very common restriction. Assumption 2(b) is a restriction on the covariance structure of $X_{iJ}$. Observe that $\Gamma^{-1}(J)d_{\ell}(J)$ takes the form of regression coefficient of $X_{i\ell}$ on $X_{iJ}$, and so Assumption 2(b) means that $X_{i\ell}$ cannot be strongly correlated with $X_{iJ}$ for $\ell \notin J$. We remark that these conditions are imposed at the population level; unlike in the LASSO or Dantzig selector (Candes and Tao, 2007), a restricted eigenvalue type condition for the sample Gram matrix (see e.g., Bickel, Ritov, and Tsybakov, 2009) is not required here – see also the discussion in Ing (2020, Sec. 3.2).

The following assumption imposes restrictions on the function classes in terms the parameters $\beta_0$ and $\gamma_0$. We will use the generic notation $\xi_0$ to refer to $\beta_0$ and $\gamma_0$. Note that $\xi_0 \in \mathbb{R}^p$ in both cases. Define $\xi(J) = (\xi_j)_{j \in J}$ to be a $|J| \times 1$ vector, where recall that $\xi$ is a generic notation to refer to $\beta_0$ and $\gamma_0$.

**Assumption 3.** It holds over $N \in \mathbb{N}$ that for each of $\xi_0 = \beta_0$ and $\gamma_0$, $\xi_0$ follows either (a) or (b) described below.

(a) Polynomial decay: $\log p = o(N^{1-2/q})$. Each $\xi_0$ is such that $\|\xi_0\|_2^2 \leq C_0$ for some $C_0 > 0$ and there exist $\alpha > 1$ such that for any $J \subseteq \mathcal{P}$,

$$\|\xi_0(J)\|_1 \leq C \left(\|\xi_0(J)\|_2^2\right)^{(\alpha-1)/(2\alpha-1)}.$$  

(b) Exponential decay: $\log p = o(N^{1/4})$. Each $\xi_0$ is such that $\|\xi_0\|_{\infty} \leq C_0$ for some $C_0 > 0$ and there exists $C_1 > 1$ such that for any $J \subseteq \mathcal{P}$,

$$\|\xi_0(J)\|_1 \leq C_1 \|\xi_0(J)\|_\infty.$$  

This is a key assumption in this paper, and defines admissible function classes for the high-dimensional linear models. While the literature on LASSO requires the exact sparsity and the $L^p$ sparsity (including approximate sparsity) conditions, Assumption 3 does not
impose such conditions. We remark that, if we rearrange the components of the parameter vector $\xi_0$ by their absolute values in a descending order (denote it again as $\xi_0$ with an abuse of notation), then Condition (a) contains special cases such as the conventional polynomial decay condition

$$L_j^{-\alpha} \leq |\xi_{0j}| \leq U_j^{-\alpha}, \quad 0 < L \leq U < \infty,$$

as well as the polynomial summability condition

$$\sum_{j=1}^{p} |\xi_{0j}|^{1/\alpha} < M, \quad M \in (0, \infty),$$

following the discussion in Ing (2020, pp. 1962). On the other hand, Condition (b) implies the conventional exponential decay condition that, for some $\alpha' > 0$,

$$L' \exp(-\alpha' j) \leq |\xi_{0j}| \leq U' \exp(-\alpha' j), \quad 0 < L' \leq U' < \infty,$$

so long as the regressors have bounded second moments. Hence throughout the paper, Conditions (a) and (b) are referred to as the polynomial decay condition and the exponential decay condition, respectively, albeit their extra generality. Clearly, the case of polynomial decay accommodates a larger function class, but we remark that there is a tradeoff in terms of how fast the dimension $p$ can diverge as the sample size $N$ increases.

Following Ing (2020, pp. 1960), we now present a concrete example where our Assumption 3 holds but the sparsity does not. Suppose that

$$L_j^{-\alpha} \leq |\Lambda_{0(j)} \sigma_{(j)}| \leq U_j^{-\alpha}, \quad j = 1, \ldots, p,$$

for some $\alpha > 1$, where $0 < L \leq U < \infty$, and $|\Lambda_{0(1)} \sigma_{(1)}| \geq |\Lambda_{0(2)} \sigma_{(2)}| \geq \cdots \geq |\Lambda_{0(p)} \sigma_{(p)}|$ is a descending reordering of $\{\Lambda_{0,j} \sigma_j\}$ with $\sigma_j^2 = E[X_{ij}^2]$. In this setting, our Assumption 3 holds for the same $\alpha$, but $\sum_{j=1}^{p} |\Lambda_{0,j} \sigma_j|^{1/\gamma}$ is now unbounded as $L(1 + \log p) \leq \sum_{j=1}^{p} |\Lambda_{0,j} \sigma_j|^{1/\gamma} \leq U(1 + \log p)$. 

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Theorem 1. Let \((\mathcal{P}_N)_{N \in \mathbb{N}}\) be a sequence of sets of DGPs such that Assumptions \(1\)–\(3\) are satisfied on the model \((2.1)\)–\((2.2)\). Then, the estimator \(\hat{\theta}\) satisfies
\[
\sqrt{N} (\hat{\theta} - \theta_0) \xrightarrow{d} N(0, \Omega),
\]
where \(\Omega = (E[V^2])^{-1}E[V^2U^2](E[V^2])^{-1} \). Define \(\hat{M} := -1/K \sum_{k=1}^{K} 1/n \sum_{i \in I_k} (D_i - X_i'\hat{\theta})^2\). Then, we can define the variance estimator
\[
\hat{\Omega} = \hat{M}^{-1} \frac{1}{K} \sum_{k=1}^{K} \frac{1}{n} \sum_{i \in I_k} [\psi(Y, D, X; \hat{\theta}, \hat{\eta}_k)\psi(Y, D, X; \hat{\theta}, \hat{\eta}_k)'](\hat{M}^{-1})'
\]
and the confidence regions with significance level \(a \in (0, 1)\) have uniform asymptotic validity:
\[
\sup_{P \in \mathcal{P}_N} \left| P \left( \theta_0 \in \left[ \hat{\theta} \pm \Phi^{-1}(1-a/2)\sqrt{\hat{\Omega}/N} \right] - (1-a) \right) = o(1). \]

A proof is provided in Appendix A.1. This theorem guarantees that the estimator \(\hat{\theta}\) of \(\theta_0\) provided by Algorithm II converges at the rate of \(\sqrt{N}\) and asymptotically follows the normal distribution under Assumption II.3. Furthermore, the sample-counterpart asymptotic variance estimator constructs an asymptotically valid confidence interval. We emphasize that this result does not rely on the sparsity assumption which is used in the literature on high-dimensional linear models.

3. Simulation Studies

In this section, we investigate the finite sample properties of our proposed estimator \(\hat{\theta}\) and compare them with those of two existing estimators, namely the LASSO-based DML and random-forest-based DML.\(^3\)

We follow Belloni et al. (2013) in developing baseline data generating processes (DGPs). The linear regression model is specified by
\[
Y = D\theta_0 + X'\Lambda_0 + U,
\]

\(^3\)For these two existing methods, we use the R package “DoubleML : Double Machine Learning in R.”
where $\theta_0 = 0.5$ and $p = dim(X) = 500$. Consistently with this specification, data are generated by the system

$$Y = \theta_0(D - X'\beta_0) + X'\gamma_0 + U, \quad U \sim N(0, 1),$$

$$D = X'\beta_0 + V, \quad V \sim N(0, 1),$$

where the covariates are in turn generated by $X \sim N(0, \Sigma)$ with $\Sigma_{jk} = (0.5)^{|k-j|}$.

For the high-dimensional nuisance parameters, $\eta_0 = (\gamma_0, \beta_0)$, we set $p = 500$ throughout and consider a couple of alternative designs. In the first design, each of $\beta_0$ and $\gamma_0$ has ten coordinates taking the value of 1 and $p - 10$ coordinates taking the value of zero, i.e., sparse design. In the second design, both $\beta_0$ and $\gamma_0$ decay exponentially. Specifically, the $j$-th coordinate of each of $\beta_0$ and $\gamma_0$ is set to $e^{-j}$. The third design has both $\beta_0$ and $\gamma_0$ decaying at polynomial rates. Specifically, the $j$-th coordinate of each of $\beta_0$ and $\gamma_0$ is set to $j^{-2}$, $j^{-1.75}$, $j^{-1.5}$, $j^{-1.25}$ and $j^{-1}$ for five sets of simulations. For each of these sets of simulations, we experiment with the two sample sizes $N \in \{500, 1000\}$.

Table I summarizes simulation results. Displayed are four Monte Carlo simulation statistics for each set of simulations, including the bias, standard deviation (SD), root mean square error (RMSE), and 95% coverage frequency. In the first row group of the table displaying the results the sparse design, both LASSO-based method and our proposed method based on the OGA and HDAIC work well, while that based on Random Forest significantly underperforms. In the second row group of the table displaying the results under the exponential decay, all the three machine learning methods yield desired results both in terms of all the displayed statistics. There are no significant differences across the three methods under this sparse model. In the subsequent row groups of the table displaying the results for the cases of the polynomial decays, however, observe that the performance varies across the three machine learning methods. While our proposed method based on the OGA and
| \( \beta_{0,i}, \gamma_{0,i} \) | N | p | Method of Preliminary Estimation | Bias | SD | RMSE | 95% |
|---|---|---|---|---|---|---|---|
| Sparse | 500 | 500 | LASSO | 0.020 | 0.044 | 0.049 | 0.937 |
| | | | Random Forest | -0.481 | 0.000 | 0.498 | 0.000 |
| | | | OGA+HDAIC | -0.003 | 0.045 | 0.046 | 0.943 |
| | 1000 | 500 | LASSO | 0.012 | 0.031 | 0.033 | 0.929 |
| | | | Random Forest | -0.347 | 0.003 | 0.485 | 0.000 |
| | | | OGA+HDAIC | 0.000 | 0.032 | 0.032 | 0.947 |
| | 500 | 500 | LASSO | 0.006 | 0.044 | 0.045 | 0.934 |
| | | | Random Forest | 0.008 | 0.044 | 0.045 | 0.940 |
| | | | OGA+HDAIC | 0.000 | 0.045 | 0.045 | 0.941 |
| | 1000 | 500 | LASSO | 0.007 | 0.031 | 0.032 | 0.942 |
| | | | Random Forest | 0.006 | 0.031 | 0.031 | 0.950 |
| | | | OGA+HDAIC | 0.000 | 0.032 | 0.032 | 0.950 |
| | 500 | 500 | LASSO | 0.010 | 0.044 | 0.046 | 0.934 |
| | | | Random Forest | 0.033 | 0.043 | 0.055 | 0.878 |
| | | | OGA+HDAIC | -0.002 | 0.045 | 0.046 | 0.938 |
| | 1000 | 500 | LASSO | 0.009 | 0.031 | 0.032 | 0.939 |
| | | | Random Forest | 0.025 | 0.031 | 0.039 | 0.882 |
| | | | OGA+HDAIC | 0.001 | 0.032 | 0.032 | 0.945 |
| | 500 | 500 | LASSO | 0.016 | 0.044 | 0.047 | 0.931 |
| | | | Random Forest | 0.046 | 0.043 | 0.063 | 0.818 |
| | | | OGA+HDAIC | -0.001 | 0.045 | 0.047 | 0.930 |
| | 1000 | 500 | LASSO | 0.011 | 0.031 | 0.033 | 0.932 |
| | | | Random Forest | 0.035 | 0.031 | 0.046 | 0.805 |
| | | | OGA+HDAIC | 0.001 | 0.032 | 0.032 | 0.951 |
| | 500 | 500 | LASSO | 0.020 | 0.044 | 0.049 | 0.931 |
| | | | Random Forest | 0.066 | 0.042 | 0.079 | 0.651 |
| | | | OGA+HDAIC | 0.001 | 0.045 | 0.046 | 0.936 |
| | 1000 | 500 | LASSO | 0.013 | 0.031 | 0.034 | 0.928 |
| | | | Random Forest | 0.053 | 0.030 | 0.060 | 0.576 |
| | | | OGA+HDAIC | 0.002 | 0.031 | 0.033 | 0.938 |
| | 500 | 500 | LASSO | 0.028 | 0.044 | 0.053 | 0.904 |
| | | | Random Forest | 0.108 | 0.041 | 0.115 | 0.245 |
| | | | OGA+HDAIC | 0.006 | 0.044 | 0.047 | 0.933 |
| | 1000 | 500 | LASSO | 0.018 | 0.031 | 0.036 | 0.909 |
| | | | Random Forest | 0.096 | 0.029 | 0.100 | 0.083 |
| | | | OGA+HDAIC | 0.004 | 0.031 | 0.034 | 0.923 |
| | 500 | 500 | LASSO | 0.038 | 0.043 | 0.061 | 0.827 |
| | | | Random Forest | 0.193 | 0.037 | 0.196 | 0.001 |
| | | | OGA+HDAIC | 0.022 | 0.043 | 0.053 | 0.893 |
| | 1000 | 500 | LASSO | 0.026 | 0.031 | 0.042 | 0.838 |
| | | | Random Forest | 0.179 | 0.026 | 0.181 | 0.000 |
| | | | OGA+HDAIC | 0.014 | 0.031 | 0.037 | 0.901 |

Table 1. Monte Carlo simulation results. Displayed are Monte Carlo simulation statistics including the bias, standard deviation (SD), root mean square error (RMSE), and 95% coverage frequency.
HDAIC continues to perform well in terms of all the displayed statistics, the LASSO-based method slightly underperforms and the random-forest-based method significantly underperforms. In particular, these differences in the finite-sample performance widen as the degree of polynomial decay becomes smaller, i.e., as the model becomes less sparse. These results demonstrate the relative robustness of the method proposed in this paper under less sparse high-dimensional regression models.

We ran many other sets of simulations and present their results in Appendix E. In particular, Appendix E.1 presents simulation results with various values of the tuning parameters, and demonstrate the robustness of the qualitative patterns observed above. From these results, we recommend to use the method based on the OGA and HDAIC over the two alternative methods for its robust performance across various designs even including the sparse design.

4. Extensions

4.1. Models with Approximation Errors. Extending the baseline model (2.1)–(2.2), consider the following partially linear model motivated by Belloni et al. (2013):

\[
Y = D\theta_0 + f(X) + U, \quad E[U|X] = 0, \quad (4.1)
\]
\[
D = g(X) + V, \quad E[V|X] = 0, \quad (4.2)
\]

where \(Y\) denotes an outcome variable, \(D\) denotes a treatment variable, \(X\) denotes a \(p\)-dimensional vector of controls, and \(U\) and \(V\) denotes unobserved factors. We do not directly impose any parametric restriction on \(f\) or \(g\) unlike the baseline model presented in Section 2. This extension is useful in certain applications, such as the one we present in Section 5. In this semi-parametric framework, we are interested in the partial effect \(\theta_0\) of \(D\) on \(Y\).
Now consider the following reduced form regressions for (4.1)–(4.2):

\[
Y = X'\gamma_0 + r_Y(X) + \mathcal{E}, \quad E[\mathcal{E} | X] = 0, \quad (4.3)
\]

\[
D = X'\beta_0 + r_D(X) + V, \quad E[V | X] = 0, \quad (4.4)
\]

where \(X'\gamma_0\) and \(X'\beta_0\) are approximations to \(E[Y | X]\) and \(E[D | X]\), and \(r_Y(X)\) and \(r_D(X)\) are approximation errors. The functions \(r_Y\) and \(r_D\) are nonparametric as are \(f\) and \(g\). We will impose conditions on the magnitudes of \(r_Y\) and \(r_D\) below. Models under these conditions, along with certain sparsity conditions imposed on \(\beta_0\) and \(\gamma_0\), are said to be “approximate sparse” in Belloni et al. (2012, 2013).

Recall the orthogonal score \(\psi(Y,D,X;\theta,\eta)\) defined in (2.3). With this orthogonal score, we propose to obtain \(\hat{\theta}\) and \(\hat{\Omega}\) via Algorithm 1 presented in Section 2 even under the current extended setting with approximation errors.

With the extended model (4.3)–(4.4), a different set of assumptions are imposed from those in the baseline model. First, we slightly modify Assumption 1 as follows.

**Assumption 4.** For each \(N \in \mathbb{N}\), it holds that

(a) \((Y_i, D_i, X_i')_{i=1}^N\) are i.i.d. copies of \((Y, D, X')\).

(b) (4.3) and (4.4) hold.

(c) \(E[|Y|^q] + E[|D|^q] \leq C_q\).

(d) \(E[|UV|^2] \geq c_q^2\) and \(E[V^2|(Y,D,X')] \geq c_q\).

(e) \(\max_{1 \leq j \leq p} E[|X_{ij}|^q] \leq C_q, E[|V|^q] \leq C_q,\) and \(E[|\mathcal{E}|^q] \leq C_q\).

Furthermore, it holds asymptotically that

(f) \(K_{N,q}^2 C \log p/N^{1-2/q} = o(1)\).

In part (d), we require the conditional variance of \(V\) given \((Y, D, X')\) to be bounded away from zero whereas the counterpart in the baseline model assumed the unconditional variance to be bounded away from zero.
We continue to use Assumption 2 from the baseline model. However, it should be stressed that we now impose Assumption 2 on (4.3)–(4.4) rather than (2.1)–(2.2). With the approximation errors introduced in the current extended model, we make the following assumption on the approximation error functions $r_Y$ and $r_D$.

**Assumption 5.** For $r(X) = r_Y(X)$ and $r_D(X)$, it holds that

(a) $E[r^4(X)] \leq C$.
(b) $E[r^2(X)] \leq C \log p/N$.
(c) $\max_{1 \leq j \leq p} |E[r(X)X_{ij}]| \leq C_{p,1} \sqrt{\log p/N^{1/4}}$.

Assumption 5(a) requires the fourth moment of the approximation error to be bounded, (b) assumes the second moment to be of order $\log p/N$, and (c) bounds the maximum cross moment of the approximation and the covariates.

Finally, we focus on the more difficult case, namely the polynomial decay case, for brevity in this section.

**Assumption 6.** It holds over $N \in \mathbb{N}$ that for each of $\xi_0 = \beta_0$ and $\gamma_0$, $\xi_0$ follows polynomial decay, i.e., $\log p = o(N^{1-2/q})$. Each $\xi_0$ is such that $\|\xi_0\|_2^2 \leq C_0$ for some $C_0 > 0$ and there exist $\alpha > 1$ and $C_\alpha > 0$ such that for any $J \subseteq P$, 

$$\|\xi_0(J)\|_1 \leq C_\alpha \left(\|\xi_0(J)\|_2^2\right)^{(\alpha-1)/(2\alpha-1)}.$$

The following theorem establishes the asymptotic normality of $\hat{\theta}$ along with the asymptotic validity of inference under the extended model with approximation errors.

**Theorem 2.** Let $(P_N)_{N \in \mathbb{N}}$ be a sequence of sets of DGPs such that Assumptions 2 and 4–7 are satisfied on the model (4.1)–(4.2) entailing the reduced forms (4.3)–(4.4). Then, the estimator $\hat{\theta}$ defined in Algorithm 4 satisfies

$$\sqrt{N} (\hat{\theta} - \theta_0) \xrightarrow{d} N(0, \Omega),$$
where \( \Omega = (E[V^2])^{-1}E[V^2U^2](E[V^2])^{-1} \). The confidence regions with significance level \( a \in (0, 1) \) have uniform asymptotic validity:

\[
\sup_{P \in \mathcal{P}_N} \left| P \left( \theta_0 \in \left[ \hat{\theta} \pm \Phi^{-1}(1 - a/2)\sqrt{\hat{\Omega}/N} \right] \right) - (1 - a) \right| = o(1),
\]

where \( \hat{\Omega} \) is defined in Section 2.

A proof is presented in Appendix C.1. The same remarks as those presented below the statement of Theorem 1 apply here.

We want to stress that Theorem 2 is not an immediate consequence given Theorem 1 because the original proofs for convergence rates of OGA+HDAIC in Ing (2020) do not permit approximately sparse models. In order to show Theorem 2 we establish convergence rates for the OGA+HDAIC in approximately sparse regression models.

4.2. Estimation and Inference without Cross Fitting. Thus far, our proposed procedures of estimation and inference are based on cross fitting. A drawback of using the cross fitting is the randomness of estimates given the data. To overcome this drawback, we provide an alternative procedure of estimation and inference without relying on cross fitting in this section. However, we stress that this benefit comes with costs in some assumptions as discussed below.

We continue from Section 4.1 to consider the partial linear model (4.1)–(4.2) entailing the reduced forms (4.3)–(4.4). Furthermore, we continue to use the same orthogonal score \( \psi(Y, D, X; \theta, \eta) \) defined in (2.3). However, we now replace Algorithm 1 by the following algorithm which does not involve the cross-fitting procedure. Let \([N] = \{1, \ldots, N\}\), so that \(X_{[N]j} = \{X_{ij}, i \in [N]\}\) and \(D_{[N]} = (D_1, \ldots, D_N)'\).

Algorithm 2 (OGA+HDAIC with DML for high-dimensional linear models without cross fitting).

\emph{Step 1.} Perform following procedure using \( \{(X_i', D_i')\}_{i=1}^N \) to get \( \hat{\beta} \).
Step 3. Obtain 

(a) Compute \( \hat{\mu}_{0,j} = X'_{[N]} D_{[N]} / \sqrt{N} \| X_{[N]} \| \). Select the coordinate \( \hat{J}_1 = \text{argmax}_{1 \leq j \leq p} |\hat{\mu}_{0,j}| \).

Define \( \hat{J}_1 = \{ \hat{j}_1 \} \).

(b) Compute \( \hat{\mu}_{1,j} = X'_{[N]} (I_N - H_1) D_{[N]} / \sqrt{N} \| X_{[N]} \| \), where \( H_1 = X_{[N]} \hat{\mu}_{1,j} (X'_{[N]} \hat{\mu}_{1,j} X_{[N]} \hat{\mu}_{1,j})^{-1} X'_{[N]} \hat{\mu}_{1,j} \). Select the coordinate \( \hat{j}_2 = \text{argmax}_{1 \leq j \leq p,j \notin \hat{j}_1} |\hat{\mu}_{1,j}| \). Update \( \hat{J}_2 = \hat{J}_1 \cup \{ \hat{j}_2 \} \).

(c) Given \( m-1 \) coordinates \( \hat{J}_{m-1} \) that have been obtained, compute \( \hat{\mu}_{m-1,j} = X'_{[N]} (I_N - H_{m-1}) D_{[N]} / \sqrt{N} \| X_{[N]} \| \), where \( H_{m-1} = X_{[N]} \hat{\mu}_{m-1,j} (X'_{[N]} \hat{\mu}_{m-1,j} X_{[N]} \hat{\mu}_{m-1,j})^{-1} X'_{[N]} \hat{\mu}_{m-1,j} \). Select the coordinate \( \hat{j}_m = \text{argmax}_{1 \leq j \leq p,j \notin \hat{j}_{m-1}} |\hat{\mu}_{m,j}| \). Iteractively update \( \hat{J}_m = \hat{J}_{m-1} \cup \{ \hat{j}_m \} \).

(d) Compute HDAIC (\( \hat{J}_m \)) = \( 1 + C^* |\hat{J}_m| \log p/N \hat{\sigma}_m^2 \) for each \( m \) and \( \hat{\sigma}_m^2 = 1/N D' (I - H_m) D \). Choose \( \hat{m} = \text{argmin}_{1 \leq m \leq M_n^*} \text{HDAIC}(\hat{J}_m) \), where \( C^* \) and \( M_n^* \) are defined in (D.2) and (D.1) in Appendix D.1.

(e) With coordinates \( \hat{J}_m \), run OLS of \( D_i \) on \( X_i \hat{J}_m \) to get \( \hat{\beta} \).

Step 2. Repeat Step 2 with \( \{ (X'_i, Y_i)' \}_{i=1}^N \) instead of \( \{ (X'_i, D_i)' \}_{i=1}^N \), to get \( \hat{\gamma} \).

Step 3. Obtain \( \hat{\theta} \) as a solution to \( 1/N \sum_{i=1}^N \psi(Y_i, D_i, X_i; \hat{\theta}, \hat{\eta}) = 0 \) where \( \hat{\eta} = (\hat{\gamma}, \hat{\beta}) \) and \( \psi \) is defined in (2.3).

To establish asymptotic properties for this new estimator \( \hat{\theta} \), we continue to impose Assumptions 2, 4, and 5. As in Section 4.1 we focus on the more difficult case, namely the polynomial decay case, for brevity.

**Assumption 7.** It holds over \( N \in \mathbb{N} \) that \( |\theta_0| \leq C \), and for each \( \xi_0 = \beta_0 \) and \( \gamma_0 \), \( \xi_0 \) follows polynomial decay, i.e., each \( \xi_0 \) is such that \( \| \xi_0 \|_2^2 \leq C_0 \) for some \( C_0 > 0 \) and there exist \( \alpha > 1 \) such that \( \log p = o(N^{(\alpha - 1)/(3\alpha - 1)}) \) and for any \( J \subseteq \mathcal{P} \),

\[
\| \xi_0(J) \|_1 \leq C \left( \| \xi_0(J) \|_2^2 \right)^{(\alpha - 1)/(2\alpha - 1)}.
\]

Unlike the previous sections, however, we now require \( \log p = o(N^{(\alpha - 1)/(3\alpha - 1)}) \) for \( \alpha > 1 \).
The following theorem establishes the asymptotic normality of $\tilde{\theta}$ defined without the cross-fitting procedure.

**Theorem 3.** Let $(\mathcal{P}_N)_{N \in \mathbb{N}}$ be a sequence of sets of DGPs such that Assumptions 2, 4, 5, and 7 are satisfied on the model $(4.1)-(4.2)$ entailing the reduced forms $(4.3)-(4.4)$. Then, the estimator $\tilde{\theta}$ satisfies

$$\sqrt{N} \left( \tilde{\theta} - \theta_0 \right) \overset{d}{\rightarrow} N(0, \Omega),$$

where $\Omega = (E[V^2])^{-1}E[V^2U^2](E[V^2])^{-1}$.

A proof is given in Appendix C.2.

We stress that, although the proof builds on that of Theorem 1 in Belloni et al. (2013), it is far from being trivial as the lack of cross-fitting and $L^p$-sparsity creates extra challenges. Specifically, a key intermediate step is to control the $L^1$ distances between $\beta_0$ and $\tilde{\beta}(\tilde{J})$, an oracle regression estimator defined in the proof of Theorem 3. Due to the lack of exact or approximate sparsity of $\beta_0$, this is shown via different strategies from those employed in Belloni et al. (2013).

As emphasized at the beginning of the current subsection, the main advantage of the estimation procedure without cross-fitting is that the estimate is now non-random given data. Besides, this framework without cross-fitting offers an additional advantage. Recall that our main motivation to use the OGA+HDAIC is to weaken the sparsity assumptions required for conventional high-dimensional methods such as the LASSO. In the current framework without cross-fitting, there is another motivation to use the OGA+HDAIC. Namely, it selects regressors based on their strength in explanatory power by the algorithm. Hence, we have better interpretations of the model selected by the OGA+HDAIC than the conventional high-dimensional methods under the current framework without cross-fitting. We highlight this additional advantage of our proposed method.
Appendix E.2 presents simulation results with this modified method without cross fitting. The results are similar to those obtained for the baseline model presented in Section 3.

5. AN EMPIRICAL APPLICATION

In this section, we demonstrate an application of the proposed method to estimation of production functions. The main challenge in the econometrics of production functions is the simultaneity in the choice of input firms (Marschak and Andrews, 1944). While early studies of production functions address this simultaneity problem by explicitly modeling rational choice structures of firms, Olley and Pakes (1996) more recently propose a novel idea to use the inverse of the reduced-form investment choice function as a control function. Levinsohn and Petrin (2003) propose to use intermediate input, instead of investment, as a control variable for a number of advantages.

The use of the control function a la Levinsohn and Petrin (2003) entails the partial linear estimating equation for the labor elasticity of the form

\[ y_{it} = \ell_{it}\theta + f(k_{it}, m_{it}) + u_{it}, \]  

(5.1)

where \( y_{it} \) denotes the logarithm of output, \( \ell_{it} \) denotes the logarithm of labor input, \( k_{it} \) denotes the logarithm of capital input, \( m_{it} \) denotes the logarithm of intermediate input, \( g \) is a nonparametric function that subsumes a part of the production function and the control function, and \( u_{it} \) denotes a mean-orthogonal reduced-form composite error. See Olley and Pakes (1996) and Levinsohn and Petrin (2003) for details.

In light of the partial linear form (5.1), Olley and Pakes (1996) and Levinsohn and Petrin (2003) propose to use the estimator of Robinson (1988) which is semiparametric root-\( n \) consistent for \( \theta \). Following these seminal papers, numerous researchers have estimated production functions. That said, many of these subsequent studies follow the Stata command (Petrin, Poi, and Levinsohn, 2004) which implements estimation of (5.1) via the parametric
third-degree polynomial approximation

\[ y_{it} = \ell_{it} \theta + \sum_{\rho_1=0}^{3} \sum_{\rho_2=0}^{3-\rho_1} \delta_{\rho_1\rho_2} k_{it}^{\rho_1} m_{it}^{\rho_2} + u_{it}. \]  

(5.2)

See Petrin et al. (2004, pages 116–118).

To mitigate the approximation bias asymptotically, we consider a higher-dimensional approximation

\[ y_{it} = \ell_{it} \theta + \sum_{j=1}^{p} \delta_j \phi_j(k_{it}, m_{it}) + r_p(k_{it}, m_{it}) + u_{it} \]  

(5.3)

with an error \( r_p(k_{it}, m_{it}) \) in approximation, where \( \phi = (\phi_1, \phi_2, \phi_3, \ldots) \) is a basis and \( p \) can be large and increasing with the sample size. The basis \( \phi \) could be defined as the Cartesian product of polynomials, i.e., \( (\phi_1(k, m), \phi_2(k, m), \phi_3(k, m), \ldots) = (1, k, m, k^2, m^2, km, \ldots) \), as a generalization of the popular estimating equation (5.2) in the Stata command. More generally, we can define the basis \( \phi \) as the tensor product of orthonomal bases. We employ the tensor product of Hermite bases (Gallant and Nychka, 1987; Chen, 2007) for our basis \( \phi \), and apply our proposed method to (5.3) to get an estimate of \( \theta \) and its standard error.

Following Levinsohn and Petrin (2003), we use a plant-level panel of Chilean firms from 1979 to 1986. See Liu (1991) for details about the construction of the data. Among others, we focus on the 3-digit level industry of food products (311) because of its large sample size compared to other industries. We are interested in the elasticity with respect to unskilled labor input \( \ell_{it}^u \) and skilled labor input \( \ell_{it}^s \). The intermediate input variables include electricity \( m_{it}^e \), fuels \( m_{it}^f \), and materials \( m_{it}^m \). To estimate the elasticity with respect to unskilled labor input \( \ell_{it}^u \) using \( m_{it}^m \) as a proxy following Levinsohn and Petrin (2003), we consider the estimating equation of the form

\[ \begin{array}{c}
  y_{it} \seteq \ell_{it}^u \theta_u + \ell_{it}^s \theta_s + m_{it}^e \theta^e + m_{it}^f \theta^f + \sum_{j=1}^{p} \delta_j \phi_j(k_{it}, m_{it}^m) + \tau_t + r_p(k_{it}, m_{it}^m) + u_{it} \\
  \end{array} \]  

(5.4)
as in (4.1). To estimate the elasticity with respect to skilled labor input \( \ell_{it}^s \), we swap \( \ell_{it}^u \theta^u \) and \( \ell_{it}^s \theta^s \) in the above estimating equation:

\[
y_{it} = \ell_{it}^s \theta^s + \ell_{it}^u \theta^u + m_{it}^e \theta^e + m_{it}^f \theta^f + \sum_{j=1}^{p} \delta_j \phi_j (k_{it}, m_{it}^m) + \tau_t + r_p (k_{it}, m_{it}^m) + u_{it}
\]

as in (4.1).

The term \( \tau_t \) represents time effects. Following Levinsohn and Petrin (2003), we include the indicator for year groups 1979–1981, 1982–1983, and 1984–1986.

For estimation of (5.4) using a polynomial basis, we let \( X \) consist of (i) \( \ell_{it}^l \) (ii) \( m_{it}^e \), (iii) \( m_{it}^f \), (iv) \( k_{it} \), \ldots, \( k_{it}^{10} \), (v) \( m_{it}^m \), \ldots, \( (m_{it}^m)^{10} \), (vi) dummy for 1979–1981, (vii) dummy for 1982–1983, and (viii) interactions of the terms in (iv) and (v). We also consider an estimation of (5.4) using a Hermite basis (\( \psi_0, \ldots, \psi_9 \)), we let \( X \) consist of (i) \( \ell_{it}^u \) (ii) \( m_{it}^e \), (iii) \( m_{it}^f \), (iv) \( \psi_0 (k_{it}) \), \ldots, \( \psi_9 (k_{it}) \), (v) \( \psi_0 (m_{it}^m) \), \ldots, \( \psi_9 (m_{it}^m) \), and (vi) dummy for 1979–1981, (vii) dummy for 1982–1983, and (vii) interactions of the terms in (iv) and (v). We use a finite-sample adjusted version of the DML estimates following Chernozhukov et al. (2018, Sec. 3.4) – see Appendix D.2 for details. See Appendix D.3 for details about the Hermite basis. We repeat analogous estimation procedures for (5.4).

Table 2 summarizes estimation results. Row (I) copies estimates from Levinsohn and Petrin (2003). Rows (II), (III), and (IV) report results based on the DML with LASSO, DML with random forest, and DML with the OGA and HDAIC (the estimator proposed in this paper), respectively. The first two columns show results based on the tensor product of polynomial bases, while the last two columns show results based on the tensor product of Hermite bases.

---

<sup>4</sup>For (II) and (III), we use the R package “DoubleML: Double Machine Learning in R.” We set the parameters as folds = 10, num.trees = 100, min.node.size = 2, max.depth = 5, and the number of repetitions = 20.
First, observe that all the three machine learning estimates, (II), (III), and (IV), based on the polynomial basis yield larger point estimates than the low-dimensional estimates (I). This may indicate a potential bias of the conventional estimator based on a low-dimensional polynomial approximation. However, it is also worthy of remarking that polynomial bases (including the Legendre bases) are not suitable to approximating functions of variables that have unbounded supports. Hermite bases, on the other hand, are capable of approximating functions of variables with unbounded supports. We therefore focus on the results based on

\[\text{Table 2. Estimates of labor elasticities in the 3-digit level industry of food products (311) in Chile.}\]

|                  | Polynomial Basis | Hermite Basis |
|------------------|------------------|---------------|
|                  | Unskilled Labor  | Skilled Labor |
|                  | Unskilled Labor  | Skilled Labor |
| (I) Levinsohn and Petrin (2003) | 0.139 (0.010) | 0.051 (0.009) | — | — |
| (II) Double Machine Learning with LASSO Preliminary Estimation | 0.170 (0.011) | 0.063 (0.008) | 0.196 (0.011) | 0.085 (0.010) |
| (III) Double Machine Learning with Random Forest Preliminary Estimation | 0.185 (0.013) | 0.061 (0.010) | 0.189 (0.013) | 0.062 (0.011) |
| (IV) Double Machine Learning with OGA+HDAIC Preliminary Estimation | 0.279 (0.017) | 0.161 (0.012) | 0.165 (0.011) | 0.038 (0.010) |

\[\text{Namely, the set of functions spanned by the standard polynomials is dense in the set } C^0(K) \text{ of continuous functions (or } L^2(K) \text{ of square integrable functions) defined only on a compact support } K \subset \mathbb{R}, \text{ whereas the set of functions spanned by the Hermite polynomials is dense in the set } L^1(\mathbb{R}) \text{ of integrable functions defined on the entire real line } \mathbb{R}. \text{ As such, when the regressor(s) are infinitely supported, as is likely the case in the current application, the approximation by the standard polynomial basis is not credible but that by the Hermite polynomial basis is credible.}\]
For the unskilled labor coefficient, all the three machine learning methods, (II), (III), and (IV), still yield larger point estimates than that of the low-dimensional method (I), but the estimate based on our proposed method (IV) is relatively smaller and closer to that of (I). For the skilled labor coefficient, the two machine learning methods, (II) and (III), yield slightly larger estimates than that of (I), while our proposed machine learning method (IV) yields a slightly smaller estimate than that of (I). In summary, the results of the DML based on the LASSO or the random forest significantly differ from the result of the conventional low-dimensional method, but our proposed method also yields slightly different results from those of the DML based on the LASSO or the random forest, as is also the case in our simulation studies presented in Section 3.

We ran several other estimates for robustness checks. Appendix F presents estimation results based on alternative values of the tuning parameters. Appendix F also presents results based on the method without cross fitting introduced in Section 4.2. It turns out that the qualitative pattern of the results summarized above remain robust.

Finally, we conclude this section with a few remarks about the validity of the estimation approach employed in this empirical application following Olley and Pakes (1996) and Levinsohn and Petrin (2003). It is well known today that the estimating equation (5.1) fails to identify the parameter \( \theta \) in general, as first pointed out by Ackerberg, Caves, and Frazer (2015). That said, they also suggest that \( \theta \) can be correctly identified by (5.1) under certain DGPs. They include DGPs with: (1) i.i.d. optimization error in \( \ell_{it} \) and not in \( m_{it} \); or (2) i.i.d. shocks to the price of labor or output after \( m_{it} \); for instance (Ackerberg et al., 2015, Sec. 3.1). As such, we stress that the validity of the estimation method present above is contingent on these assumptions about the underlying DGPs.

\(^6\)In addition to this difference in the approximation theoretic properties between the polynomial and Hermite bases, we also remark that the difference may be due to the fact that the proposed method is not invariant to an invertible linear transformation of the regressors like the LASSO.
6. Summary and Discussions

In this paper, we propose a new method of inference in high-dimensional regression models. The estimation procedure is based on a combined use of the OGA, HDAIC, and DML. The method of inference about any low-dimensional subvector of high-dimensional parameters is based on a root-$N$ asymptotic normality, which does not require the exact sparsity condition or the $L^p$ sparsity condition. Instead imposed are conditions on the rate at which the absolute size of parameters decays when descendingly ordered. We demonstrate through simulation studies superior finite sample performance of this proposed method over those based on two popular alternatives, namely the LASSO and the random forest. The extent of this outperformance is more prominent under less sparse models characterized by slower polynomial decays. Finally, we illustrate an application of the method to production analysis using a panel of Chilean firms. Using the tensor product of Hermite basis as high-dimensional controls, we find that estimates based on our proposed method differ from those based on the LASSO and random forest, similarly to what we observe in the simulation studies.

We close this paper with discussions of limitations, omitted extensions and potential directions for future research. First, unlike regressions and like the LASSO, the method is not invariant to invertible linear transformation of the regressors $X$. Practitioners should be aware of this drawback in our proposed method. Second, as is the case with other DML methods, our proposed method based on DML is subject to random estimates. To overcome this problem, we present an alternative procedure without cross fitting in Section 4.2, but this comes at the expense of an alternative set of assumptions. Again, practitioners should be aware of these tradeoffs in choosing an appropriate method. Third, we focus on high-dimensional linear regression models with exogenous regressors throughout the main text. We provide an extension to high-dimensional linear IV models in Section B. Extensions to other important models are left for future research.
This appendix section collects a proof of the main theorem and an auxiliary lemma.

A.1. Proof of Theorem 1

Proof. In this proof, we first show that the convergence rates from Theorem 3.1. of Ing (2020) can be guaranteed under our Assumptions 1-3. It then suffices to demonstrate that Theorem 4.1. in Chernozhukov et al. (2018) can be applied under our assumptions and with such convergence rates. Following Corollary 1 in Belloni et al. (2013), we fix a sequence of DGPs \((P_N)_{N \in \mathbb{N}}, P_N \in \mathcal{P}_N\) and establish the asymptotic statements in order to show uniformity over the sequence of sets of DGPs.

Step 1. We shall verify all the conditions for Theorem 3.1. in Ing (2020). Let us first examine each of the regularity conditions (A1)-(A5) for Theorem 3 in Ing (2020). Note that Assumption 3 (a) and (b), correspond to the conditions (A3) and (A4) in Ing (2020), respectively; Assumption 2 (b) corresponds to the condition (A5) in Ing (2020). Assumption 2 (a) is the additional conditions listed in Theorem 2.1, equations (2.19)-(2.21) in Ing (2020).

Now, we show that Assumption 1(e) and Assumption 3 imply (A1) in Ing (2020). We shall verify that there exists a strictly positive constant \(c_1\) such that

\[
P \left( \max_{1 \leq j \leq p} \frac{1}{N} \sum_{i=1}^{N} |X_{ij}V_i| \geq c_1 \sqrt{\frac{\log p}{N}} \right) = o(1). \tag{A.1}
\]

uniformly over \(P \in \mathcal{P}_N\). Set to \(Z_{ij} = X_{ij}V_i\), \(F = \max_{1 \leq i \leq N} \max_{1 \leq j \leq p} X_{ij}V_i\), and

\[
\sigma^2 = \max_{1 \leq j \leq p} E[X_{ij}^2V_i^2] \leq \max_{1 \leq j \leq p} \sqrt{E[X_{ij}^4]E[V_i^4]} \leq C_q
\]

by Assumption 3(e). Then Jensen’s inequality and some calculations yield

\[
\sqrt{E[F^2]} = \sqrt{E \left[ \max_{1 \leq i \leq N} \max_{1 \leq j \leq p} X_{ij}^2V_i^2 \right]} \leq \left( E \left[ \max_{1 \leq i \leq N} \max_{1 \leq j \leq p} |X_{ij}^{q/2}V_i^{q/2}|^2 \right] \right)^{2/q} \\
\leq \left( \sum_{i=1}^{N} E \left[ \max_{1 \leq j \leq p} |X_{ij}^{q/2}V_i^{q/2}|^2 \right] \right)^{2/q} \leq \left( \sum_{i=1}^{N} \left( E \left[ \max_{1 \leq j \leq p} |X_{ij}|^q E[|V_i|^q] \right] \right)^{1/2} \right)^{2/q} \tag{26}
\]
\[ \leq N^{2/q} \left( E \left[ \max_{1 \leq j \leq p} |X_{ij}|^q \right] E \left[ |V_i|^q \right] \right)^{1/q}. \]

By applying Lemma 1, we have
\[ E \left[ \max_{1 \leq j \leq p} \left| \frac{1}{N} \sum_{i=1}^{N} X_{ij} V_i \right| \right] \leq C \left\{ \sqrt{\frac{\log p}{N}} + K^{1/q} \frac{\log p}{N} \right\}. \]

Note that the constant \( C \) is universal and the bound depends on the DGP \( P \in \mathcal{P}_N \) only via \( q, N, p, C_q, \) and \( K_{N,1/q} \). Thus the right hand side of the bound is \( o(1) \) uniformly as \( N \to \infty \) by Assumption 11. A similar argument applies to the case of estimating \( \gamma_0 \), where we replace \( V \) with \( E \). Therefore, Assumption 1(e), Assumption 3, and Markov’s inequality imply (A.1).

Next, we show that Assumption 1(e) and Assumption 3 imply (A2) in Ing (2020). We shall illustrate that there exists a strictly positive constant \( c_2 \) such that
\[ P \left( \max_{1 \leq j, \ell \leq p} \left| \frac{1}{N} \sum_{i=1}^{N} X_{ij} X_{i\ell} - E[X_{1j} X_{1\ell}] \right| \geq c_2 \sqrt{\frac{\log p}{N}} \right) = o(1). \] (A.2)

We apply Lemma 1 with \( Z_{ij\ell} = X_{ij} X_{i\ell}, F = \max_{1 \leq i \leq N} \max_{1 \leq j, \ell \leq p} |Z_{ij\ell}|, \) and \( \sigma^2 = \max_{1 \leq j, \ell \leq p} E[X_{ij}^2 X_{i\ell}^2] \leq \max_{1 \leq j \leq p} E[X_{ij}^4] \leq C_q \) by Assumption 1(e). Also note that
\[ \sqrt{E[F^2]} = \sqrt{E \left[ \max_{1 \leq i \leq N} \max_{1 \leq j, \ell \leq p} X_{ij}^2 X_{i\ell}^2 \right]} \leq \left( E \left[ \max_{1 \leq i \leq N} \max_{1 \leq j, \ell \leq p} X_{ij}^{q/2} X_{i\ell}^{q/2} \right] \right)^{2/q} \]
\[ \leq \left( \sum_{i=1}^{N} E \left[ \max_{1 \leq j, \ell \leq p} X_{ij}^{q/2} X_{i\ell}^{q/2} \right] \right)^{2/q} \leq \left( \sum_{i=1}^{N} \left( E \left[ \max_{1 \leq j \leq p} |X_{ij}|^q \right] E \left[ \max_{1 \leq \ell \leq p} |X_{i\ell}|^q \right] \right)^{1/2} \right)^{2/q} \]
\[ \leq N^{2/q} \left( E \left[ \max_{1 \leq j \leq p} |X_{ij}|^q \right] \right)^{2/q}. \]

An application of Lemma 1 yields that
\[ E \left[ \max_{1 \leq j, \ell \leq p} \left| \frac{1}{N} \sum_{i=1}^{N} X_{ij} X_{i\ell} - E[X_{1j} X_{1\ell}] \right| \right] \leq C \left\{ \sqrt{\frac{\log p}{N}} + K^{2/q} \frac{\log p}{N} \right\}, \]

where the last equality is \( o(1) \) uniformly as \( N \to \infty \) following Assumptions 14 and 3. By Markov’s inequality, we conclude that Assumption 1(e) and Assumption 3 imply (A.2).
Now, by invoking Theorem 3.1. in Ing (2020) and Equation (3.16), we obtain the convergence rates
\[
\| \hat{\beta} - \beta_0 \|^2 = O_p \left( \left( \frac{\log p}{N} \right)^{1-1/2\alpha} \right) \tag{A.3}
\]
for the polynomial decay case, and
\[
\| \hat{\beta} - \beta_0 \|^2 = O_p \left( \frac{\log p \log N}{N} \right) \tag{A.4}
\]
for the exponential decay case. Note that we use \( N \) instead of \( |I_k| \) since the number of folds \( K \) is fixed and the fold size is proportional to \( N \).

**Step 2.** Given the convergence rates obtained in Step 1, we shall next show that Theorem 4.1. in Chernozhukov et al. (2018) can be applied. To this end, it suffices to verify Assumption 4.1. in Chernozhukov et al. (2018). First, note that Assumption II (b)-(e) corresponds to Assumption 4.1 (a)-(d) in Chernozhukov et al. (2018).

Let us now vindicate that Assumption 4.1. (e) in Chernozhukov et al. (2018) is implied by our (A.3) and (A.4), obtained in Step 1. Observe that (A.3) implies that there exists a sequence of events \((A_N)_N\) with probability \( P(A_N) = 1 - o(1) \) such that conditionally on \( A_N \), we have
\[
\| \hat{\beta} - \beta_0 \|^2 \lesssim \left( \frac{\log p}{N} \right)^{1-1/2\alpha}.
\]
Therefore, for \( X \), an independent copy of the regressor vector \( X_i \), we have
\[
E \left[ \left\| X'(\hat{\beta} - \beta_0) \right\|^2 \mid A_N \right] = E\left[ E[(X'(\hat{\beta} - \beta_0))^2 \mid \hat{\beta}, A_N] \mid A_N \right] \tag{A.5}
\]
\[
= E[(\hat{\beta} - \beta_0)'E[XX' \mid \hat{\beta}, A_N](\hat{\beta} - \beta_0) \mid A_N] \tag{A.6}
\]
\[
= E[(\hat{\beta} - \beta_0)'E[XX'](\hat{\beta} - \beta_0) \mid A_N] \tag{A.7}
\]
\[
= \lambda_{\max}(E[XX'])\| \hat{\beta} - \beta_0 \|^2, \tag{A.8}
\]
where the third equality follows from the fact that the sigma-algebra generated by \( X \) is independent of \( A_N \). Observe that (A.8) is bounded by Assumption (A.9), (A.3), and (A.4). Thus the above bound holds with probability at least \( 1 - \Delta_N \) for some \( \Delta_N = o(1) \).

Define \( \| \cdot \|_{P,q} \) as the \( L^q(P) \) norm, where \( \| f \|_{P,q} = (\int |f(w)|^q dP(w))^{1/q} \) with \( P \) being the law with respect to \((Y, D, X)\). We shall now establish that with \( P \)-probability no less than \( 1 - \Delta_N \),

\[
\| \hat{\eta} - \eta_0 \|_{P,q} \leq C, \| \hat{\eta} - \eta_0 \|_{P,2} \leq \delta_N, \text{ and } \| \hat{\beta} - \beta_0 \|_{P,2} \times (\| \hat{\beta} - \beta_0 \|_{P,2} + \| \hat{\gamma} - \gamma_0 \|_{P,2}) \leq \delta_N N^{-1/2}
\]

is satisfied for some \( \Delta_N, \delta_N \) such that both are sequences of strictly positive constants converging to zero.

From a similar argument as in (A.9)–(A.8),

\[
\| \hat{\eta} - \eta_0 \|_{P,q} = \| \hat{\beta} - \beta_0 \|_{P,q} \vee \| \hat{\gamma} - \gamma_0 \|_{P,q} = \left( E \left[ \left\| X'(\hat{\beta} - \beta_0) \right\| q \mid A_N \right] \right)^{1/q} \\
\leq \left( E \left[ \left( (\hat{\beta} - \beta_0)'E[XX'](\hat{\beta} - \beta_0) \right)^{q/2} \mid A_N \right] \right)^{1/q} = \left( \lambda_{\max}(E[XX']) \| \hat{\beta} - \beta_0 \|^{q/2} \right)^{1/q} \\
= \left( \lambda_{\max}(E[XX']) \| \hat{\beta} - \beta_0 \|^{2} \right)^{1/2},
\]

where it is bounded by Assumption (A.3), (A.8), and (A.4). Thus, \( \| \hat{\eta} - \eta_0 \|_{P,q} \leq C \) holds with probability at least \( 1 - \Delta_N \) for some \( \Delta_N = o(1) \).

Since we use the identical procedure, namely the OGA and HDAIC, in estimating both of \( \beta_0 \) and \( \gamma_0 \), the convergence rates apply to both. We consider two cases where both parameters follow the polynomial decay case or the exponential decay case.

Case 1. For the polynomial decay case, let \( \delta_N = (\log p)^{1-1/2\alpha} N^{1/2\alpha - 1/2} \). Then \( \delta_N = o(1) \) since \( \alpha \) is assumed to be strictly larger than 1 in Assumption (3) (a). We have

\[
\| \hat{\eta} - \eta_0 \|_{P,2} = \| \hat{\beta} - \beta_0 \|_{P,2} \vee \| \hat{\gamma} - \gamma_0 \|_{P,2} \lesssim \left( \frac{\log p}{N} \right)^{1/2-1/4\alpha} \leq \delta_N,
\]

where the last inequality holds since \( (\log p/N)^{1/4\alpha} \leq \log p \) holds with \( \alpha > 1 \). Also,

\[
\| \hat{\beta} - \beta_0 \|_{P,2} \times (\| \hat{\beta} - \beta_0 \|_{P,2} + \| \hat{\gamma} - \gamma_0 \|_{P,2}) \lesssim \left( \frac{\log p}{N} \right)^{1-1/2\alpha} = \delta_N N^{-1/2}
\]
holds.

Case 2. For the exponential decay case, let $\delta_N = N^{-1/2} \log p \log N$. By the assumption $\log p = o(N^{1/4})$, $\delta_N = o(1)$. We have

$$\|\hat{\eta} - \eta_0\|_{P,2} \lesssim \sqrt{\frac{\log p \log N}{N}} < \delta_N$$

and

$$\|\hat{\beta} - \beta_0\|_{P,2} \times \left(\|\hat{\beta} - \beta_0\|_{P,2} + \|\hat{\gamma} - \gamma_0\|_{P,2}\right) \lesssim \frac{\log p \log N}{N} = \delta_N N^{-1/2}.$$ 

A similar argument applies to the cross cases.

We have shown that Assumption 4.1 in Chernozhukov et al. (2018) holds for both sparsity assumptions. Therefore, applying Theorem 4.1. in Chernozhukov et al. (2018), we get the desired results. \qed

### Appendix B. High-Dimensional Linear IV Regression Models

Section 2 in the main text presented the method for high-dimensional linear regression models. In this section, we extend the method by accommodating high-dimensional linear IV regression models.
B.1. **The Model.** Consider the high-dimensional linear IV model

\[
Y = D\theta_0 + X'\Lambda_0 + U, \quad E[U|X, Z] = 0, \quad (B.1)
\]

\[
Z = X'\beta_0 + V, \quad E[V|X] = 0, \quad (B.2)
\]

where \(Z\) denotes an instrumental variable and the parameter of interest is the partial effect \(\theta_0\) of the endogenous treatment variable \(D\) on the outcome variable \(Y\). To construct a moment restriction under \((B.1)–(B.2)\), consider the orthogonal score function

\[
\psi(Y, D, X, Z; \theta, \eta) := \{Y - X'\gamma - \theta(D - X'\zeta)\}(Z - X'\beta), \quad (B.3)
\]

where \(X'\gamma_0 = E[Y|X], X'\zeta_0 = E[D|X]\) and \(\eta = (\gamma, \zeta, \beta)\).

B.2. **The Method.** This section describes the algorithm for estimation and inference about \(\theta_0\) in the high-dimensional linear IV regression model \((B.1)–(B.2)\).

**Algorithm 3** (OGA+HDAIC with DML for high-dimensional linear IV models).

**Step 1.** Randomly split the sample indices \(\{1, ..., N\}\) into \(K\) folds \((I_k)_{k=1}^K\). For simplicity, let the size of each fold be \(n = N/K\) and the size of \(I_k\) be \(n^c\).

**Step 2.** For each fold \(k \in \{1, ..., K\}\), perform following procedure using \(\{(X'_i, Z_i)\}_{i \in I_k}\) to get \(\hat{\beta}_k\).

(a) Compute \(\hat{\mu}_{0,j} = X'_{I_k^{\hat{j}}}Z_{I_k^{\hat{j}}}/\sqrt{n^c}\|X_{I_k^{\hat{j}}}\|. \) Select the coordinate \(\hat{j}_1 = \text{argmax}_{1 \leq j \leq p} |\hat{\mu}_{0,j}|. \) Define \(\hat{J}_1 = \{\hat{j}_1\}. \)

(b) Compute \(\hat{\mu}_{1,j} = X'_{I_k^{\hat{j}}} (I_{n^c} - H_1) Z_{I_k^{\hat{j}}}/\sqrt{n^c}\|X_{I_k^{\hat{j}}}\|, \) where \(H_1 = X_{I_k^{\hat{j}1}} (X'_{I_k^{\hat{j}1}} X_{I_k^{\hat{j}1}})^{-1} X'_{I_k^{\hat{j}1}}. \) Select the coordinate \(\hat{j}_2 = \text{argmax}_{1 \leq j \leq p, j \notin \hat{J}_1} |\hat{\mu}_{1,j}|. \) Update \(\hat{J}_2 = \hat{J}_1 \cup \{\hat{j}_2\}. \)

(c) Given \(m-1\) coordinates \(\hat{J}_{m-1}\) that have been obtained, compute \(\hat{\mu}_{m-1,j} = X'_{I_k^{\hat{j}}} (I_{n^c} - H_{m-1}) Z_{I_k^{\hat{j}}}/\sqrt{n^c}\|X_{I_k^{\hat{j}}}\|, \) where \(H_{m-1} = X_{I_k^{\hat{j}m-1}} (X'_{I_k^{\hat{j}m-1}} X_{I_k^{\hat{j}m-1}})^{-1} X'_{I_k^{\hat{j}m-1}}. \) Select the coordinate \(\hat{j}_m = \text{argmax}_{1 \leq j \leq p, j \notin \hat{J}_{m-1}} |\hat{\mu}_{m,j}|. \) Iteratively update \(\hat{J}_m = \hat{J}_{m-1} \cup \{\hat{j}_m\}. \)
(d) Compute $\text{HDAIC}(\hat{J}_m) = (1 + C^* |\hat{J}_m| \log p/n)\hat{\sigma}_m^2$ for each $m$, where $C^*$ is from (D.3) in Appendix D.1 and $\hat{\sigma}_m^2 = 1/nZ_k'(I-H_m)Z_k'$. Choose $\hat{m} = \text{argmin}_{1 \leq m \leq M^*_n}$ HDAIC$(\hat{J}_m)$, where $M^*_n$ is defined in (D.1) in Appendix D.1.

(e) With coordinates $\hat{J}_m$, run OLS of $Z_i$ on $X_i\hat{J}_m$ to get $\hat{\beta}_k$.

Step 3. Repeat Step 2 with $\{(X'_i,D_i)\}_{i \in I_k^c}$ in place of $\{(X'_i,Z_i)\}_{i \in I_k}$, to get $\hat{\zeta}_k$ for each fold $k \in \{1, \ldots, K\}$.

Step 4. Repeat Step 2 using $\{(X'_i,Y_i)\}_{i \in I_k^c}$ to get $\hat{\gamma}_k$ for each fold $k \in \{1, \ldots, K\}$.

Step 5. Obtain $\hat{\theta}$ as a solution to $1/K \sum_{k=1}^{K} 1/n \sum_{i \in I_k} \psi(Y_i, D_i, X_i, Z_i; \hat{\theta}, \hat{\eta}_k) = 0$ where $\hat{\eta}_k = (\hat{\gamma}_k, \hat{\beta}_k, \hat{\zeta}_k)$ and $\psi$ is defined in (B.3).

Step 6. Compute $\hat{M} = -1/K \sum_{k=1}^{K} 1/n \sum_{i \in I_k} (D_i - X'_i\hat{\gamma})(Z_i - X'_i\hat{\beta})$. Obtain a variance estimator of $\hat{\theta}$ as $\hat{\Omega} = \hat{M}^{-1} \frac{1}{K} \sum_{k=1}^{K} \frac{1}{n} \sum_{i \in I_k} [\psi(Y, D, X, Z; \hat{\theta}, \hat{\eta}_k)\psi(Y, D, X, Z; \hat{\theta}, \hat{\eta}_k)](\hat{M}^{-1})'$.

Observe that this algorithm parallels Algorithm 1 and hence similar remarks are in order. First, the procedure (Steps 1–4) uses the cross fitting to remove an over-fitting bias. Second, the coordinates $\{\hat{j}_1, \ldots, \hat{j}_p\}$ are ranked in Step 2 (a)–(c) in the order of decreasing importance after successive orthogonalization using OGA as in Ing (2020). Third, a subset $\hat{J}_m = \{\hat{j}_1, \ldots, \hat{j}_\hat{m}\}$ of the ordered set $\{\hat{j}_1, \ldots, \hat{j}_p\}$ is selected in Step 2 (d) using HDAIC as in Ing (2020). The combined use of these three elements (DML, OGA, and HDAIC) together allows for a novel root $N$ consistent estimation of $\theta_0$ without assuming traditional functional class restrictions (e.g., the sparsity) required by existing popular estimators (e.g., LASSO). Section B.3 formally presents theoretical arguments in support of this claim.

B.3. The Theory. This section follows as a corollary to the main theory in section 2. Again, we use a generic notation $\mathcal{E}$ to refer to $\mathcal{E}_D = D - X'\zeta_0$ and $\mathcal{E}_Y = Y - X'\gamma_0$.

Assumption 8. For each $N \in \mathbb{N}$, it holds that

(a) $(Y_i, D_i, X'_i, Z_i)_{i=1}^{N}$ are i.i.d. copies of $(Y, D, X', Z)$.
(b) (B.1) and (B.2) hold.

(c) \( E[|Y|^q] + E[|D|^q] + E[|Z|^q] \leq C_q. \)

(d) \( E[|UV|^2] \geq c_q^2 \) and \( E[DV] \geq c_q. \)

(e) \( \max_{1 \leq j \leq p} E[|X_{ij}|^q] \leq C_q, \ E[|V|^q] \leq C_q, \) and \( E[|E|^q] \leq C_q. \)

Furthermore, it holds asymptotically that (f) \( K_{N,q}^2 \log p / N^{1-2/q} = o(1). \)

**Assumption 9.** It holds over \( N \in \mathbb{N} \) that

(a) \( \lambda_{\min}(\Gamma) \geq \lambda_1 > 0 \) and \( \lambda_{\max}(\Gamma) \leq C_q, \) where \( \Gamma = E[XX']. \)

(b) Define \( \Gamma(J) = E[X_{ij}X'_{iJ}] \) and \( d_\ell(J) = E[X_{i\ell}X_{iJ}] \) for the set of coordinate indices \( J. \)

\[
\max_{1 \leq |J| \leq C(N/\log p)^{1/2}, \ell \not\in J} \left| \Gamma^{-1}(J)d_\ell(J) \right| < C_q.
\]

**Assumption 10.** For each of \( \xi_0 = \beta_0, \zeta_0, \xi_0 \) follows either (a) or (b) described below.

(a) Polynomial decay: \( \log p = o(N^{1-2/q}). \) Each \( \xi_0 \) is such that \( \|\xi_0\|_2^2 \leq C_0 \) for some \( C_0 > 0, \) there exist \( \alpha > 1 \) such that for any \( J \subseteq \mathfrak{P}, \)

\[
\|\xi_0(J)\|_1 \leq C \left( \|\xi_0(J)\|_2^2 \right) \left( \alpha^{-1}/(2\alpha - 1) \right).
\]

(b) Exponential decay: \( \log p = o(N^{1/4}). \) Each \( \xi_0 \) is such that \( \|\xi_0\|_\infty \leq C_0 \) for some \( C_0 > 0 \) and there exists \( C_1 > 1 \) such that for any \( J \subseteq \mathfrak{P}, \)

\[
\|\xi_0(J)\|_1 \leq C_1 \|\xi_0(J)\|_\infty.
\]

Assumptions 8–10 closely parallel Assumptions 1–3, and thus similar remarks apply here. The following theorem supports the estimation and inference procedure presented in Algorithm 3.
Theorem 4. Let \((P_N)_{N \in \mathbb{N}}\) be a sequence of sets of DGPs such that Assumptions 8–10 are satisfied on the model (B.1)–(B.2). Then, the estimator \(\hat{\theta}\) follows

\[
\sqrt{N} \left( \hat{\theta} - \theta_0 \right) \xrightarrow{d} N(0, \Omega),
\]

where \(\Omega = (E[DV])^{-1}E[V^2U^2](E[DV])^{-1}\). Define \(\tilde{M} := -1/K \sum_{k=1}^{K} 1/n \sum_{i \in I_k} (D_i - X_i'\hat{\zeta})(Z_i - X_i'\hat{\beta})\). Then, we can define the variance estimator

\[
\hat{\Omega} = \tilde{M}^{-1} \frac{1}{K} \sum_{k=1}^{K} 1/n \sum_{i \in I_k} [\psi(Y,D,X,Z; \theta, \hat{\eta}_k)\psi(Y,D,X,Z; \theta, \hat{\eta}_k)'](\tilde{M}^{-1})'
\]

and the confidence regions with significance level \(a \in (0,1)\) have uniform asymptotic validity:

\[
\sup_{P \in P_N} \left| P \left( \theta_0 \in \left[ \hat{\theta} \pm \Phi^{-1}(1-a/2)\sqrt{\hat{\Omega}/N} \right] - (1-a) \right) = o(1). \right.
\]

A proof is provided in Appendix C.3. As in the case of the baseline regression model, we once again emphasize that this result does not rely on the sparsity assumption which is used in the literature on high-dimensional linear models.

Appendix E.3 presents simulation designs and results for high-dimensional IV regression models studied in the current appendix section. The results are similar to those obtained for the baseline model presented in Section 3. Namely, while our proposed method based on the OGA and HDAIC perform well in terms of all the simulation statistics, the LASSO-based method slightly underperforms and the random-forest-based method significantly underperforms. These differences in the finite-sample performance widen as the degree of polynomial decay becomes smaller.

APPENDIX C. PROOFS FOR THE EXTENSIONS

C.1. Proof of Theorem 2

Proof. Throughout this proof, write \(\|v\|_A^2 = v^\top Av\) for a vector \(v\) and nonnegative definite matrix \(A\). In this proof, we show the prediction norm rates of the nuisance parameters, that
are subsequently used in the proof of Theorem 1 to be attainable with the reduced form models (4.3) and (4.4). The proof consists of two parts: first we establish the convergence rates of OGA under the setting with approximation errors and, in the second part, the convergence rates of OGA coupled with HDAIC. These two parts correspond to Theorems 2.1 and 3.1 in [Ing (2020)], respectively. Hence the proof strategies follow closely the proofs of these two results with appropriate modifications.

We consider the estimation of $\beta_0$ from (4.4) within a partition with sample size $n$. The same logic applies to $\gamma_0$ in (4.3).

**Step 1.** (OGA part 1: Definitions) In this step, we show how to modify the definitions of some objects and sets of events in [Ing (2020)] to accommodate the presence of the extra approximation errors. Define $X_j$ as the $j$–th coordinate of $X$. Consider the model (4.4) and define

$$D(X) = \sum_{j=1}^{p} \beta_j X_j, \quad D_J(X) = \sum_{j \in J} \beta_j X_j,$$

$$\hat{D}_m(X) \equiv \hat{D}_{\hat{J}_m}(X) = \sum_{j \in \hat{J}_m} \hat{\beta}_j X_j, \quad \hat{D}_{i,\hat{J}_m} = \sum_{j \in \hat{J}_m} \hat{\beta}_j X_{ij},$$

$$\mu_{J,k} = \text{E}[(D(X) - D_J(X))X_{ik}] / \sigma_k, \quad \sigma_k = \sqrt{\text{E}[X_{ik}^2]},$$

$$\hat{\mu}_{J,k} = \frac{1}{n} \sum_{i=1}^{n} (D_i - \hat{D}_{i,\hat{J}_m})X_{ik} \left( \frac{1}{n} \sum_{i=1}^{n} X_{ik}^2 \right)^{1/2},$$

and the collections of events

$$A_n(m) = \{ \max_{(J,k):|J| \leq m-1,k \notin J} |\hat{\mu}_{J,k} - \mu_{J,k}| \leq C(\log p/n)^{1/2} \}, \quad (C.1)$$

$$B_n(m) = \{ \min_{0 \leq j \leq m-1} \max_{1 \leq k \leq p} |\mu_{\hat{J},k} - \hat{\mu}_{\hat{J},k}| > \bar{\xi}C(\log p/n)^{1/2} \}, \quad (C.2)$$

where $\bar{\xi}, C > 0$ are some large constants.
Now, define the corresponding variables with the approximation errors:

\[ D_i^r = \sum_{j=1}^{p} \beta_j X_{ij} + r_D(X_i) + V_i, \quad D'(X) = \sum_{j=1}^{p} \beta_j X_{j} + r_D(X), \]

\[ \mu_{r,j,k}^* = E[(D'(X) - D_J(X))X_{j,k}] / \sigma_k = \mu_{j,k} + E[r(X)X_{j,k}] / \sigma_k, \]

\[ \tilde{\mu}_{r,j,k}^* = \frac{1}{n} \sum_{i=1}^{n} (D_i^r - \tilde{D}_{i,j})X_{ik} = \tilde{\mu}_{j,k} + \frac{1}{n} \sum_{i=1}^{n} r(X_i)X_{ik} / \sigma_k, \]

and

\[ A_r^*(m) = \{ \max_{(j,k) : |j| \leq m - 1, k \neq j} \left| \tilde{\mu}_{r,j,k}^* - \mu_{r,j,k}^* \right| \leq C(\log p/n)^{1/2} \}, \quad \text{and} \quad (C.3) \]

\[ B_r^*(m) = \{ \min_{0 \leq j \leq m - 1} \max_{1 \leq k \leq p} \left| \mu_{r,j,k}^* \right| > \tilde{\xi} C(\log p/n)^{1/2}, \quad \text{and} \quad (C.4) \]

where \( \tilde{\xi} = 2/(1 - \xi) \) for some \( 0 < \xi < 1 \).

We will show that \((C.1), (C.2)\) and Assumption \(5\) imply \((C.3)\) and \((C.4)\) with appropriate choices on the constants.

\[ |\tilde{\mu}_{r,j,k}^* - \mu_{r,j,k}^*| = |\tilde{\mu}_{j,k} - \mu_{j,k}| + \frac{1}{n} \sum_{i=1}^{n} r(X_i)X_{ik} / (1/n \sum_{i=1}^{n} X_{ik}^2)^{1/2} - E[r(X)X_{j,k}] / \sigma_k \]

\[ \leq |\tilde{\mu}_{j,k} - \mu_{j,k}| + \left| \frac{1}{n} \sum_{i=1}^{n} r(X_i)X_{ik} / (1/n \sum_{i=1}^{n} X_{ik}^2)^{1/2} - E[r(X)X_{j,k}] / \sigma_k \right| \]

\[ \leq |\tilde{\mu}_{j,k} - \mu_{j,k}| + R_{p,3}, \]

where \( R_{p,3} \equiv \max_{1 \leq k \leq p} \left| \frac{1}{n} \sum_{i=1}^{n} r(X_i)X_{ik} / (1/n \sum_{i=1}^{n} X_{ik}^2)^{1/2} - E[r(X)X_{j,k}] / \sigma_k \right| \) and \( R_{p,3} = o_p(1) \) by Assumption \(5(a), (c)\), and Lemma \(1\) in Appendix A.2.

Conditional on the events \( B_r(m) \),

\[ |\mu_{r,j,k}^*| = |\mu_{j,k} + E[r(X)X_{j,k}] / \sigma_k| \]

\[ \leq |\mu_{j,k}| + |E[r(X)X_{j,k}] / \sigma_k| \]

\[ \leq \tilde{\xi} C(\log p/n)^{1/2} + C(\log p/n)^{1/2}, \]

where \( C > 0 \) and \( \tilde{\xi} \) is so that \( \tilde{\xi} \equiv \tilde{\xi} + 1 = 2/(1 - \xi) \).
Using the above definitions, it holds for all $1 \leq q \leq m$ on $A^r_n(m) \cap B^r_n(m)$,
\[
\left| \mu^r_{\hat{j}_{q-1},j_q} \right| \geq - \left| \tilde{\mu}^r_{\hat{j}_{q-1},\hat{j}_q} - \mu^r_{\hat{j}_{q-1},j_q} \right| + \left| \tilde{\mu}^r_{\hat{j}_{q-1},\hat{j}_q} \right| \\
\geq - \max_{(j,i) \in (J,i) \leq m-1} \left| \mu^r_{\hat{j}_{q-1},\hat{j}_q} - \mu^r_{\hat{j}_{q-1},j_q} \right| \\
\geq - C (\log p_n/n)^{1/2} + \max_{1 \leq j \leq p_n} \left| \mu^r_{\hat{j}_{q-1},j} \right| \\
\geq - 2C (\log p_n/n)^{1/2} + \max_{1 \leq j \leq p_n} \left| \mu^r_{\hat{j}_{q-1},j} \right| \\
> \xi \max_{1 \leq j \leq p_n} \left| \mu^r_{\hat{j}_{q-1},j} \right|,
\]
where the first inequality comes from the triangle inequality, the second from taking the maximum, the third from (C.3) and since $\left| \tilde{\mu}^r_{\hat{j}_{q-1},\hat{j}_q} \right| = \max_{1 \leq j \leq p_n} \left| \tilde{\mu}^r_{\hat{j}_{q-1},j} \right|$, the fourth from the triangle inequality and (C.3), and the last from $2C (\log p_n/n)^{1/2} < (2/\tilde{\xi}) \max_{1 \leq j \leq p_n} \left| \mu^r_{\hat{j}_{q-1},j} \right|$ on $B^r_n(m)$ and $1 - \xi = 2/\tilde{\xi}$.

Hence, with Assumption 5 and $\tilde{\xi} = \xi + 1$, $A_n(m)$ implies $A^r_n(m)$ and thus $\lim_{n \to \infty} P(A_n(m)) = 1$ implies $\lim_{n \to \infty} P(A^r_n(m)) = 1$. In Step 3, we derive the bounds on $A^r_n(m) \cap B^r_n(m)$ and $A^r_n(m) \cap (B^r_n(m))^c$, respectively, and use the fact that $\lim_{n \to \infty} P(A_n(m)) = 1$ to show that it is always the case that either $A^r_n(m) \cap B^r_n(m)$ or $A^r_n(m) \cap (B^r_n(m))^c$ holds.

**Step 2.** (OGA part 2: Lemma A.1. from Ing (2020)) We now establish error bounds for the population OGA under some high-level conditions (for polynomial decay). Recursively define $J_{\xi,m} = J_{\xi,m-1} \cup \{j_{\xi,m}\}$, with $J_{\xi,0} = \emptyset$ and $j_{\xi,m}$ any element $\ell \in \{1, \ldots, p\}$ satisfying
\[
|E[V_{m-1}X_{\ell}]| \geq \xi \max_{1 \leq j \leq p} |E[V_{m-1}X_j]|. \tag{C.5}
\]
Denote $V_m = D^r(X) - D_{J_{\xi,m}}(X)$, then
\[
E[V_m^2] \leq E \left[ (D^r(X) - D_{J_{\xi,m}}(X)) \sum_{j=1}^p \beta_j X_j \right] \\
\leq \max_{1 \leq j \leq p} \left| \mu^r_{J_{\xi,m},j} \right| \sum_{j=1,j \notin J_{\xi,m}}^p |\beta_j|, \tag{C.6}
\]
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Using (C.9) and Lemma 1 of Gao, Ing, and Yang (2013), we obtain the following bound for

\[ E[V_m^2] \geq \lambda_1 \sum_{j=1,j \notin J_{\xi,m}}^p \beta_j^2. \]  

(C.7)

Combining (C.6) and (C.7), we obtain

\[ \leq C\lambda_1^{-\alpha/(2a-1)} \sum_{1 \leq j \leq p} \left| \mu^r_{J_{\xi,m,j}} \right| \{ E[V_m^2]\}^{\alpha/(2a-1)}. \]

(C.8)

Note that \( \beta(J) = \Gamma^{-1}(J)E[X(J)'D] = \argmin_{c \in \mathbb{R}^n} E[(D - X'c)^2]. \) We now have

\[ E[V_{m+1}^2] = E \left[ (D'(X) - \sum_{j \in J_{\xi,m}} \beta_j (J_{\xi,m}+1)X_j - \beta_{j_{m+1}} (J_{\xi,m}+1)X_{j_{m+1}})^2 \right] \]

\[ \leq E \left[ (D'(X) - \sum_{j \in J_{\xi,m}} \beta_j (J_{\xi,m}+1)X_j - \mu^r_{J_{\xi,m},j}X_{j_{m+1}})^2 \right] \]

\[ = E \left[ (V_m - \mu^r_{J_{\xi,m},j}X_{j_{m+1}})^2 \right] \]

\[ \leq E[V_m^2] - \xi^2 \sum_{1 \leq j \leq p} \left| \mu^r_{J_{\xi,m,j}} \right|^2 \]

\[ \leq E[V_m^2] - \xi^2 \lambda_1^{2/(2a-1)} C_\gamma^{-2} \left[ E(V_m^2) \right]^{2/(2a-1)} \]

\[ = E(V_m^2) \left\{ 1 - \xi^2 \lambda_1^{2/(2a-1)} C_\gamma^{-2} \left[ E(V_m^2) \right]^{1/(2a-1)} \right\}, \]

where the second inequality comes from (C.5) and the third inequality comes from (C.8).

Using (C.9) and Lemma 1 of Gao, Ing, and Yang (2013), we obtain the following bound for \( G_1 > 0: \)

\[ E[V_m^2] \leq G_1 m^{-2a+1}. \]  

(C.10)

**Step 3.** (OGA part 3: Combining Steps 1 and 2) Define the shorthand notations \( W_1^N = (Y_i, D_i, X_i)_i=1^N \) and \( E_{W_1^N} = E[|W_1^N|]. \) Combining Step 1 and Step 2, we obtain the bound

\[ E_{W_1^N}[(D'(X) - D_{j_{m}'}(X))^2] \leq G_1 m^{-2a+1} \text{ on } A_\eta^r(m) \cap B_\eta^r(m). \]  

(C.11)
Using Assumption \textbf{[6](a)} and (\textbf{C.7}), we have for any $0 \leq l \leq m - 1$,

$$E_{W^N_i}[(D^r(X) - D_{\hat{j}_l}(X))^2] \leq \left(C_\alpha \max_{1 \leq j \leq p} \left| \mu_{\tilde{j}_l,j}^r \right| \right)^{2-1/\alpha} \lambda_l^{-1+1/\alpha}. \tag{C.12}$$

By (C.12), we have

$$E_{W^N_i}[(D^r(X) - D_{\tilde{j}_m}(X))^2] \leq \min_{0 \leq l \leq m-1} E_{W^N_i}[(D^r(X) - D_{\hat{j}_l}(X))^2] \leq C_{\alpha}^{2-1/\alpha} \lambda_l^{-1+1/\alpha} \left( \min_{0 \leq l \leq m-1} \max_{1 \leq j \leq p} \left| \mu_{\tilde{j}_l,j}^r \right| \right)^{2-1/\alpha} \tag{C.13}$$

where the last inequality holds conditioning on $(B_n^r(m))^c$.

Combining (C.11) and (C.13), for all $1 \leq m \leq K_n$ and $C > 0$, we have

$$E_{W^N_i}[(D^r(X) - D_{\tilde{j}_m}(X))^2] I_{A_n^c(K_n)} \leq C \max \left\{ m^{-2\alpha+1}, \{ \log p/n \}^{1-1/2\alpha} \right\}. \tag{C.14}$$

Under Assumption \textbf{5} $\lim_{n \to \infty} P(A_n(m)) = 1$ as shown in Section S1 of supplementary material of Ing (2020), we then have $\lim_{n \to \infty} P(A_n^c(m)) = 1$ following the conclusion of Step 1. With (C.14) we achieve

$$\max_{1 \leq m \leq K_n} \frac{E_{W^N_i}[(D^r(X) - D_{\tilde{j}_m}(X))^2]}{\max \left\{ m^{-2\alpha+1}, \{ \log p/n \}^{1-1/2\alpha} \right\}} \preceq_p C. \tag{C.15}$$

Note that we are interested in the conditional mean squared prediction error, $E_{W^N_i}[(D^r(X) - \tilde{D}_m(X))^2] = E_{W^N_i}[(D^r(X) - D_{\tilde{j}_m}(X))^2] + E_{W^N_i}[(D_{\tilde{j}_m}(X) - \tilde{D}_m(X))^2]$. The convergence rate for the latter term is

$$\max_{1 \leq m \leq K_n} \frac{E_{W^N_i}[(D_{\tilde{j}_m}(X) - \tilde{D}_m(X))^2]}{m \log p/n} \preceq_p C, \tag{C.16}$$

where the proof follows exactly the same arguments as in Section S1 of supplementary material in Ing (2020) under our current setting. Combining (C.14) and (C.16), we obtain

$$\max_{1 \leq m \leq K_n} \frac{E_{W^N_i}[(D^r(X) - \tilde{D}_m(X))^2]}{m^{-2\alpha+1} + m \log p/n} \preceq_p C. \tag{C.17}$$
Step 4. (OGA+HDAIC) Using the results in the previous steps, we now replace \( m \) with \( \hat{k}_n \) obtained from HDAIC and establish the convergence rate under such setting. Define

\[
V(J) = D(X) - X(J)'\beta(J),
\]

\[
V^r(J) = V(J) + r(X) = D(X) - X(J)'\beta(J) + r(X),
\]

\[
V_i(J) = D_i - V_i - X_i(J)'\beta(J), \quad \text{and}
\]

\[
V^r_i(J) = V_i(J) + r(X_i) = D_i - V_i - \sum_{j \in J} \beta_j X_{ij} + r(X_i).
\]

We will establish the following four inequalities for any \( 1 \leq m \leq K_n \)

\[
\left| \frac{1}{n} \sum_{i=1}^{n} (V^r_i(\hat{J}_m))^2 - E_W^N [(V^r(\hat{J}_m))^2] \right| \leq CR_{1,p} \left\{ E_W^N [V^2(\hat{J}_m)] \right\}^{(\alpha-1)/(2\alpha-1)}, \tag{C.18}
\]

\[
\left| \frac{1}{n} \sum_{i=1}^{n} V_i V_i^r(\hat{J}_m) \right| \leq CR_{2,p} \left\{ E_W^N [V^2(\hat{J}_m)] \right\}^{(\alpha-1)/(2\alpha-1)}, \tag{C.19}
\]

\[
\max_{1 \leq m \leq K_n} \left\| \frac{1}{n} \sum_{i=1}^{n} X_i(\hat{J}_m)V_i^r(\hat{J}_m) \right\|_{\hat{\Gamma}^{-1}(\hat{J}_m)}^2 \leq \left\| \hat{\Gamma}^{-1}(K_n) \right\| CR_{1,p}^2, \tag{C.20}
\]

\[
\max_{1 \leq m \leq K_n} \left\| \frac{1}{n} \sum_{i=1}^{n} X_i(\hat{J}_m) V_i \right\|_{\hat{\Gamma}^{-1}(\hat{J}_m)}^2 \leq \left\| \hat{\Gamma}^{-1}(K_n) \right\| R_{2,p}^2, \tag{C.21}
\]

where \( R_{r,1} \equiv |1/n \sum_{i=1}^{n} r^2(X_i) - E[r^2(X)]| = o_p(1) \) by Assumption 4 and the LLN, \( R_{2,p} \equiv \max_{1 \leq j \leq p} |1/n \sum_{i=1}^{n} X_{ij} V_i| \lesssim_p (\log p/n)^{1/2} \) from (A.1), and recall that \( \|v\|_A^2 = v^\top A v \) for a vector \( v \) and nonnegative definite matrix \( A \).

Among the above, we prove (C.18)–(C.20) since they include the variables with the approximation errors as (C.21) does not depend on the newly introduced approximation error.
in the current result.

\[
\frac{1}{n} \sum_{i=1}^{n} (V^r_{i} (\hat{J}_m))^2 - E_{W_i}^N [(V^r (\hat{J}_m))^2]
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} V^2_{i} (\hat{J}_m) + \frac{2}{n} \sum_{i=1}^{n} V_i (\hat{J}_m) r(X_i) + \frac{1}{n} \sum_{i=1}^{n} r^2(X_i) - E_{W_i}^N [V^2 (\hat{J}_m)] - E_{W_i}^N [r^2(X)]
\]
\[
\leq \frac{1}{n} \sum_{i=1}^{n} V^2_{i} (\hat{J}_m) - E_{W_i}^N [V^2 (\hat{J}_m)] + 2 \frac{1}{n} \sum_{i=1}^{n} V_i (\hat{J}_m) r(X_i) + \frac{1}{n} \sum_{i=1}^{n} r^2(X_i) - E_{W_i}^N [r^2(X)]
\]
\[
\leq \frac{1}{n} \sum_{i=1}^{n} V^2_{i} (\hat{J}_m) - E_{W_i}^N [V^2 (\hat{J}_m)] \quad (a) + 2 \sqrt{\frac{1}{n} \sum_{i=1}^{n} V^2_{i} (\hat{J}_m)} \sqrt{\frac{1}{n} \sum_{i=1}^{n} r^2(X_i)} + R_{r,1},
\]

where we want to show that \((b) \lesssim_P (a)\). Note that from Assumption [6][a] and (C.7) as shown in Section S2 of supplementary material of Ing (2020), we have

\[
\left| \frac{1}{n} \sum_{i=1}^{n} V^2_{i} (\hat{J}_m) \right| \lesssim_P E_{W_i}^N [V^2 (\hat{J}_m)] + C R_{1, p} \{E_{W_i}^N [V^2 (\hat{J}_m)]\}^{(a-1)/(2a-1)},
\]
\[
\lesssim_P E_{W_i}^N [V^2 (\hat{J}_m)],
\]

where the second inequality comes from Assumption [4][d] and hence

\[
\sqrt{\frac{1}{n} \sum_{i=1}^{n} V^2_{i} (\hat{J}_m)} \sqrt{\frac{1}{n} \sum_{i=1}^{n} r^2(X_i)} \lesssim_P E_{W_i}^N [V^2 (\hat{J}_m)]^{1/2} (\log p/n)^{1/2} \lesssim_P (a),
\]

where the first bound comes from Assumption [3][b] and the last comes from Assumption [4] [d]. Since \((b) \lesssim_P (a)\) and \((c) = o_P(1)\) by Assumption [3][a] we obtain (C.18).
Now, note that

\[
\left| \frac{1}{n} \sum_{i=1}^{n} V_i V_i^r(\hat{J}_m) \right| = \left| \frac{1}{n} \sum_{i=1}^{n} V_i V_i(\hat{J}_m) + \frac{1}{n} \sum_{i=1}^{n} V_i r(X_i) \right|
\]

\[
\leq \left| \frac{1}{n} \sum_{i=1}^{n} V_i V_i(\hat{J}_m) \right| + \left| \frac{1}{n} \sum_{i=1}^{n} V_i r(X_i) \right|
\]

\[
\leq \left( \frac{1}{n} \sum_{i=1}^{n} V_i V_i(\hat{J}_m) \right) + \left( \frac{1}{n} \sum_{i=1}^{n} V_i^2 r(X_i) \right).
\]

By an argument similar to deriving (C.18) and since term (e) above is bounded by a constant from Assumption 4(e) and 5(a), we obtain (C.19).

Next, observe that

\[
\left| \frac{1}{n} \sum_{i=1}^{n} X_i(\hat{J}_m)V_i^r(\hat{J}_m) \right|_\hat{\Gamma}^{-1}(m)
\]

\[
\leq \left\| \hat{\Gamma}^{-1}(K_n) \right\|^{1/2} \left| \frac{1}{n} \sum_{i=1}^{n} X_i(\hat{J}_m)V_i(\hat{J}_m) \right|_\hat{\Gamma}^{-1}(m) + \left| \frac{1}{n} \sum_{i=1}^{n} X_i(\hat{J}_m)r(X_i) \right|_\hat{\Gamma}^{-1}(K_n)^{1/2} \left| \frac{1}{n} \sum_{i=1}^{n} X_i(\hat{J}_m)r(X_i) \right|.
\]

By a similar manipulations as in (C.18) and since

\[
\left| \frac{1}{n} \sum_{i=1}^{n} X_i(\hat{J}_m)r(X_i) \right| \leq \left| \frac{1}{n} \sum_{i=1}^{n} X_i(\hat{J}_m)r(X_i) \right|_1 \lesssim_p (\log p/n)^{1/2},
\]

where the last inequality holds from Assumption 5(a), (c), and Lemma 4 in Appendix A.2., we achieve (C.20).

Recall \( M_n^* = \min \{(n/\log p)^{1/2\alpha}, \tilde{c}(n/\log p)^{1/2}\} \) and let \( \tilde{\kappa}_n = \min_{1 \leq k \leq K_n} \{E_{\hat{\mathcal{B}}_n^k}(V^2(\hat{J}_k)) \leq GM_n^{*2\alpha+1} \} \) where \( G > C \) is an appropriate constant that is large enough, where \( C \) is defined in (C.14).
Using (C.18)–(C.21), it follows exactly the same the proof shown in [20] under our current setting that

\[
\lim_{n \to \infty} P(\hat{k}_n \leq \tilde{k}_n - 1) = 0, \quad (C.22)
\]

\[
\lim_{n \to \infty} P(\hat{k}_n \geq CM^*_n) = 0, \quad (C.23)
\]

hold and hence we have the following results:

\[
E_{\mathcal{W}_1^N}[(D^r(X) - \hat{D}_{\hat{k}_n}(X))^2]I_{\{\tilde{k}_n \leq \hat{k}_n < CM^*_n\}} = O_p \left( M^*_{n}^{r - 2\gamma + 1} \right), \quad (C.24)
\]

and the desired result follows:

\[
E_{\mathcal{W}_1^N}[(D^r(X) - \hat{D}_{\hat{k}_n}(X))^2] \lesssim P \left( \log p/N \right)^{1-1/2\alpha}. \quad (C.25)
\]

Following the same argument as in the proof of Theorem 1, the conditions for Assumption 4.1. in Chernozhukov et al. (2018) is satisfied and thus by applying Theorem 4.1. in Chernozhukov et al. (2018) we obtain the desired asymptotic normality results.

C.2. Proof of Theorem 3

Proof. For notational simplicity, we write \(\mathcal{Y} = [Y_1, \ldots, Y_N]'\), \(\mathcal{X} = [X_1, \ldots, X_N]'\), \(\mathcal{D} = [D_1, \ldots, D_N]'\), \(\mathcal{V} = [V_1, \ldots, V_N]'\), \(\mathcal{U} = [U_1, \ldots, U_N]'\), \(\mathcal{R}_{\theta} = [r_{\theta}(X_1), \ldots, r_{\theta}(X_N)]'\), \(\mathcal{R}_D = [r_D(X_1), \ldots, r_D(X_N)]'\), and \(f = [f(X_1), \ldots, f(X_N)]'\) and \(g = [g(X_1), \ldots, g(X_N)]'\) with

\[
f(X) = X'\Lambda_0 + r_{\theta}(X), \quad g(X) = X'\beta_0 + r_D(X).
\]

Define for a non-empty set of coordinate indices \(J \subseteq \mathfrak{P}\) that \(X_{[N],J} = \{X_{ij}, i \in \{1, \ldots, N\}, j \in J\}\), \(P_J = X_{[N],J}(X_{[N],J}^r X_{[N],J})^{-1} X_{[N],J}^r\), and \(M_J = I_N - P_J\), where \(I_N\) is an \(N\)-dimensional identity matrix. Define the indices of chosen coordinates from Algorithm 2 using \((\mathcal{X}, \mathcal{D})\) as \(\tilde{J}_n^P\), \((\mathcal{X}, \mathcal{Y})\) as \(\tilde{J}_n^Y\), and let \(\tilde{J} = \tilde{J}_n^P \cup \tilde{J}_n^Y\).
First note that
\[
\tilde{\theta} = \left( \frac{1}{N} D'M_j D \right)^{-1} \left( \frac{1}{N} D'M_j Y \right), \quad \text{and}
\]
\[
\sqrt{N} \left( \tilde{\theta} - \theta_0 \right) = \left( \frac{1}{N} D'M_j D \right)^{-1} \left( \frac{1}{\sqrt{N}} D'M_j (f + U) \right) = A^{-1} = B.
\]

Thus, if we show
\[
A = \frac{1}{N} Y'V + o_p(1) \quad \text{and} \quad B = \frac{1}{\sqrt{N}} Y'U + o_p(1), \quad (C.26)
\]
then an application of CLT yields the desired conclusion. In Step 1 we derive component-wise bounds that will be used in the following steps. We show (C.26) in Step 2 and in Step 3 we conclude.

Step 1. (Component-wise bounds) Note we have \( R_{2,p} = \max_{1 \leq j \leq p} \left| 1/N \sum_{i=1}^{N} X_{ij} V_i \right| \lesssim P (\log p/N)^{1/2} \) from (A.1). First we derive bounds for \( \left\| \frac{1}{N} X'V \right\| \infty \):
\[
\left\| \frac{1}{N} X'V \right\| \infty \leq R_{2,p} \lesssim P (\log p/N)^{1/2}, \quad (C.27)
\]
and similarly \( \left\| X'U/N \right\| \infty \lesssim P (\log p/N)^{1/2} \).

Note \( \tilde{\beta}(J) = (X'_{[N]}X_{[N]})^{-1} X'_{[N]}J \). From the convergence rates from Ing (2020), we have
\[
\left\| \tilde{\beta}(\tilde{J}) - \beta_0 \right\| \lesssim P \left\| \tilde{\beta}(\tilde{J}^D) - \beta_0 \right\|
\leq \left( \lambda_1^{-1} E_{W_1} \left[ (D'(X) - \hat{D}_m(X))^2 \right] \right)^{1/2} \quad (C.28)
\lesssim P (\log p/N)^{(2\alpha - 1)/4\alpha},
\]
where the first inequality comes from \( \tilde{J}^D_m \subseteq \tilde{J} \), the second from Assumption 2 (a) and Equation (3.16) of Ing (2020), and the last from the results in Theorem 3.1. of Ing (2020).

In what followings, we establish a bound for \( \| \tilde{\beta}(\tilde{J}) - \beta_0 \|_1 \). Recall \( \beta_0(J) = (\beta_{0j}, j = 1; \ldots, p) \), where \( \beta_{0j} = 0 \) for \( j \notin J \). Notice that
\[
\| \tilde{\beta}(\tilde{J}) - \beta_0 \|_1 = \| \tilde{\beta}(\tilde{J}) - \beta_0(\tilde{J}) \|_1 + \| \beta_0(\tilde{J}^c) \|_1 = \| \tilde{\beta}(\tilde{J}) - \beta_0(\tilde{J}) \|_1 + \| \beta_0 - \beta_0(\tilde{J}) \|_1.
\]
The first term on RHS is bounded by
\[
\left\| \tilde{\beta}(J) - \beta_0(J) \right\|_1 \leq \sqrt{\tilde{m}} \left\| \tilde{\beta}(\tilde{J}) - \beta_0(\tilde{J}) \right\|
\leq \sqrt{\hat{m}^D + \hat{m}^Y} \left\| \tilde{\beta}(\tilde{J}^D_m) - \beta_0(\tilde{J}^D_m) \right\|
\lesssim_P (M^*_N)^{1/2} (\log p/N)^{(2\alpha - 1)/4\alpha}
\leq_P (\log p/N)^{-1/4\alpha} (\log p/N)^{(2\alpha - 1)/4\alpha}
\leq_P (\log p/N)^{(\alpha - 1)/2\alpha},
\]
where the second inequality comes from \(\tilde{J} = \tilde{J}^D_m \cup \tilde{J}^Y_m\), and the third from (C.23). The fourth comes from the definition of \(M^*_N\) defined in (D.1), and the last follows. On the other hand, the second term on RHS can be controlled by
\[
\left\| \beta_0 - \beta_0(\tilde{J}) \right\|_1 \leq C \sum_{j \notin \tilde{J}} |\beta_{0j}|
\leq CC\alpha \left( \sum_{j \notin \tilde{J}} \beta_{0j}^2 \right)^{(\alpha - 1)/(2\alpha - 1)}
\leq CC\alpha \lambda_1^{(-\alpha + 1)/(2\alpha - 1)} \left( E[V(J)^2] \right)^{(\alpha - 1)/(2\alpha - 1)}
\leq CC\alpha \lambda_1^{(-\alpha + 1)/(2\alpha - 1)} G_1 |\tilde{J}|^{-2\alpha + 1}
\leq CC\alpha \lambda_1^{(-\alpha + 1)/(2\alpha - 1)} G_1 (2M^*_N)^{-2\alpha + 1}
\leq CC\alpha \lambda_1^{(-\alpha + 1)/(2\alpha - 1)} G_1 (2)^{-2\alpha + 1} (\log p/N)^{(2\alpha - 1)/2\alpha}
\lesssim_P (\log p/N)^{(\alpha - 1)/2\alpha}
\]
The first inequality comes from Assumption 2(b), where it holds for all \(J \subseteq \Phi\) such that \(|J| \leq C(N/\log p)^{1/2}\), as shown in  
Ing (2020) Equation (2.16) and the following equation. The second inequality comes from Assumption 7. The third comes from (C.7), which holds under Assumption 2(a). The fourth comes from (C.10) in the previous section’s proof. The
fifth comes from $|\tilde{J}| \leq |\tilde{J}^D_m| + |\tilde{J}^N_m|$, where $|\tilde{J}^N_m| \leq M^*_N$. The sixth comes from the definition of $M^*_N$ given in (D.1), and the last follows. By combining the bounds, we conclude that

$$
\|\tilde{\beta}(\tilde{J}) - \beta_0\|_1 \lesssim_P (\log p/N)^{(\alpha-1)/4\alpha} + (\log p/N)^{(2\alpha-1)/2\alpha} \lesssim_P (\log p/N)^{(\alpha-1)/4\alpha}.
$$

(C.29)

It also holds that

$$
\left\| \frac{1}{\sqrt{N}}M_{\tilde{j}g} \right\| \leq \left\| \frac{1}{\sqrt{N}}M_{\tilde{j}^D_p g} \right\| \\
\leq \left\| \frac{1}{\sqrt{N}} \left( X\tilde{\beta}(\tilde{J}^D_m) - g \right) \right\| \\
\leq \left\| \frac{1}{\sqrt{N}}X\left( \tilde{\beta}(\tilde{J}^D_m) - \beta_0 \right) \right\| + \left\| \frac{1}{\sqrt{N}}R_D \right\| \\
\lesssim_P (\log p/N)^{(2\alpha-1)/4\alpha} + \sqrt{\frac{1}{N} \sum_{i=1}^N p_i^2(X_i)} \\
\lesssim_P (\log p/N)^{(2\alpha-1)/4\alpha} + \sqrt{\log p/N} \\
\lesssim_P (\log p/N)^{(2\alpha-1)/4\alpha},
$$

(C.30)

where the fourth comes from the convergence rates from (C.25), and the fifth from Assumption 5(b) and the remainder follows.

Similarly, note the convergence rate for $\|\tilde{\gamma}(\tilde{J}^Y_m) - \gamma_0\|$ is of the same order as in (C.28), hence we have

$$
\left\| \frac{1}{\sqrt{N}}M_{\tilde{j}}(\theta_0 g + f) \right\| \leq \left\| \frac{1}{\sqrt{N}} \left( X\tilde{\gamma}(\tilde{J}) - (\theta_0 g + f) \right) \right\| \\
\leq \left\| \frac{1}{\sqrt{N}} \left( X\tilde{\gamma}(\tilde{J}^Y_m) - (\theta_0 g + f) \right) \right\| \\
\leq \left\| \frac{1}{\sqrt{N}}X\left( \tilde{\gamma}(\tilde{J}^Y_m) - \gamma_0 \right) \right\| + \left\| \frac{1}{\sqrt{N}}R_Y \right\| \\
\lesssim_P (\log p/N)^{(2\alpha-1)/4\alpha},
$$

where the second inequality comes from $\tilde{J}^Y_m \subseteq \tilde{J}$, the third from triangle inequality, and the last follows Assumption 5(b) and the convergence rate of $\|\tilde{\gamma}(\tilde{J}^Y_m) - \gamma_0\|$. Using triangle
inequality,
\[ \left\| M_j \theta_0 g \right\| - \left\| M_j f \right\| \leq \left\| M_j (\theta_0 g + f) \right\| , \]

where \( \left\| M_j \theta_0 g / \sqrt{N} \right\| = \left\| \theta_0 \right\| \chi(\tilde{J}) - \beta_0) / \sqrt{N} \right\| \lesssim_P (\log p/N)^{(2\alpha-1)/4\alpha} \) by the assumption on bounded \( \| \theta_0 \| \) in Assumption 7. Therefore, \( \left\| M_j f \right\| \) is bounded by the same bound and

\[ \left\| \tilde{\Lambda}(\tilde{J}) - \Lambda_0 \right\| \lesssim_P \left\| \frac{1}{\sqrt{N}} \chi \left( \tilde{\Lambda}(\tilde{J}) - \Lambda_0 \right) \right\| \lesssim_P \left\| \frac{1}{\sqrt{N}} M_j f \right\| \lesssim_P (\log p/N)^{(2\alpha-1)/4\alpha} \quad \text{(C.31)} \]

by a similar argument as in (C.30).

Let \( \tilde{\beta}_V(J) = (\chi'(J) \chi(J))^{-1} \chi'(J) \Psi. \)

\[
\left\| \tilde{\beta}_V(\tilde{J}) \right\|_1 \leq \sqrt{\tilde{J}} \left\| \tilde{\beta}_V(\tilde{J}) \right\|_2 \\
\leq \sqrt{\tilde{J}} \left\| \tilde{\Gamma}^{-1}(\tilde{J}) \right\|_2 \left\| \frac{1}{\sqrt{N}} \chi(\tilde{J}) \Psi \right\|_2 \\
\leq \tilde{J} \left\| \tilde{\Gamma}^{-1}(\tilde{J}) \right\|_2 \left\| \frac{1}{\sqrt{N}} \chi(\tilde{J}) \Psi \right\|_\infty \\
\lesssim_P (\log p/N)^{(\alpha-1)/2\alpha} .
\]

The last inequality comes from \( \lim_{N \to \infty} P(\| \tilde{\Gamma}^{-1}(\tilde{J}_{K_n}) \| \leq \overline{B}) = 1 \) as shown in Section S1 of the Supplementary Material of Ing (2020), which holds under current setting and \( \overline{B} \) is some large constant defined in Theorem 2.1. of Ing (2020).

The last component is

\[ \left| \frac{1}{\sqrt{N}} R_p' \mathcal{U} \right| \lesssim_P \sqrt{E \left[ \frac{1}{N} \sum_{i=1}^N r_D^2(X_i) \right]} \lesssim_P (\log p/N)^{1/2}, \quad \text{(C.33)} \]

where the first inequality comes from Chebyshev and the last from \( \delta(b) \). The same logic applies to \( \left| R_{Y_0}' \Psi / \sqrt{N} \right| \).
Step 2. (Bounding $A$ and $B$) Decompose the two objects in \((C.26)\) into

\[
A = \frac{1}{N} V' V + \frac{1}{N} f'M_j f + \frac{2}{N} f'M_j V - \frac{1}{N} V' P_j V,
\]
\[
B = \frac{1}{\sqrt{N}} V' U + \frac{1}{\sqrt{N}} g'M_j f + \frac{1}{\sqrt{N}} g'M_j U + \frac{1}{\sqrt{N}} V' f_j f - \frac{1}{\sqrt{N}} V' P_j U,
\]

where the components (a)-(g) can be further controlled by

\[
| (a) | \leq \left\| \frac{1}{\sqrt{N}} M_j f \right\|^2 \lesssim_P (\log p/N)^{(2\alpha-1)/2},
\]
\[
| (b) | \leq 2 \left\| \frac{1}{N} \mathcal{R}_{\Lambda_0} V \right\| + 2 \left\| \left( \tilde{\Lambda}(\tilde{J}) - \Lambda_0 \right)' \frac{1}{N} \mathcal{X}' V \right\|
\leq 2 \left\| \frac{1}{N} \mathcal{R}_{\Lambda_0} V \right\| + 2 \left\| \tilde{\Lambda}(\tilde{J}) - \Lambda_0 \right\|_1 \left\| \frac{1}{N} \mathcal{X}' V \right\|_\infty
\lesssim_P \sqrt{N}^{-1} (\log p/N)^{1/2} + (\log p/N)^{(\alpha-1)/4} (\log p/N)^{1/2},
\]
\[
| (c) | \leq \left\| \tilde{\beta}_V (\tilde{J})' \frac{1}{N} \mathcal{X}' V \right\| \leq \left\| \tilde{\beta}_V (\tilde{J}) \right\|_1 \left\| \frac{1}{N} \mathcal{X}' V \right\|_\infty
\lesssim_P (\log p/N)^{(\alpha-1)/2} (\log p/N)^{1/2},
\]

(C.34)
and

\[ |(d)| \leq \sqrt{N} \left\| \frac{1}{\sqrt{N}} M_j f \right\| \cdot \left\| \frac{1}{\sqrt{N}} M_j g \right\| \lesssim_P \sqrt{N} \left( \log p / N \right)^{(2\alpha - 1)/2}, \]

\[ |(e)| \leq \left\| \frac{1}{\sqrt{N}} R'_D \mathcal{U} \right\| + \left\| (\tilde{\beta}(\tilde{J}) - \beta_0)' \frac{1}{\sqrt{N}} \mathcal{V}' \mathcal{U} \right\| \lesssim_P \sqrt{N} \left( \log p / N \right)^{(2\alpha - 1)/2}, \]

\[ |(f)| \leq \left\| \frac{1}{\sqrt{N}} R'_Y \mathcal{V} \right\| + \left\| (\tilde{\Lambda}(\tilde{J}) - \Lambda_0)' \frac{1}{\sqrt{N}} \mathcal{V}' \mathcal{V} \right\| \lesssim_P \sqrt{N} \left( \log p / N \right)^{(2\alpha - 1)/2}, \]

\[ |(g)| \leq \left\| \tilde{\beta}_V(\tilde{J})' \frac{1}{\sqrt{N}} \mathcal{V}' \mathcal{U} \right\| \lesssim_P \sqrt{N} \left( \log p / N \right)^{(2\alpha - 1)/2}. \]

Now, we show that each part in (C.34) and (C.35) is \( o_p(1) \). Note that \( |(a)|, |(b)|, \) and \( |(c)| \) are \( o(1) \) if \( |(d)|, |(e)|, \) and \( |(g)| \) are all \( o(1) \). Since

\[ |(d)| \lesssim_P \sqrt{N} \left( \log p / N \right) = o(1), \]

\[ |(e)| \lesssim_P \sqrt{N} \left( \log p / N \right)^{(a - 1)/4} \left( \log p / N \right)^{1/2} = o(1), \]

\[ |(g)| \lesssim_P \left( \log p / N \right)^{(a - 1)/2} \left( \log p \right)^{1/2} = o(1), \]

and as \( (e) \) and \( (f) \) both share the same upper bound, \( \log p = o(N^{(a-1)/(3\alpha-1)}) \) is a sufficient condition for all the components in (C.34) and (C.35) to be \( o_p(1) \), given any \( \alpha > 1 \) for Assumption 7(a). Therefore we achieve (C.26).
Step 3. (CLT) From Assumption 4 on the error terms, (C.26) implies
\[
\sqrt{N} \left( \tilde{\theta} - \theta_0 \right) = \left( E[V^2]^{-1} + o_p(1) \right) \left( \frac{1}{\sqrt{N}} V' U + o_p(1) \right)
\]
\[
\overset{d}{\rightarrow} E[V^2]^{-1} N(0, E[V^2 U^2])
\]
following Lindeberg–Lévy CLT, which concludes the proof. \qed

C.3. Proof of Theorem 4

Proof. In proof of Theorem 1 we have shown that the convergence rates from Theorem 3.1. of Ing hold under our assumptions. Hence it is enough to show that Assumption 4.2. in CCDDHNR holds. Note again that Assumption 8 (b)–(e) corresponds to Assumption 4.2. (a)–(d) in Chernozhukov et al. (2018).

We shall thus verify condition (e) in Chernozhukov et al. (2018) is implied by the convergence rates (A.3) and (A.4). Recall that by (A.3) and an argument similar to (A.5)–(A.8),
\[
\left\| \hat{\xi} - \xi_0 \right\|^2 \lesssim \left( \frac{\log p}{N} \right)^{1-1/2\alpha} \quad (C.36)
\]
holds with probability at least 1 – \( \Delta_N \) for some \( \Delta_N = o(1) \) and \( \xi = \gamma, \zeta, \) and \( \beta \).

Now we will show that with \( P \)-probability no less than 1 – \( \Delta_N \), \( \left\| \hat{\eta} - \eta_0 \right\|_{P,q} \leq C \), \( \left\| \hat{\eta} - \eta_0 \right\|_{P,2} \leq \delta_N \), and \( \left\| \hat{\beta} - \beta_0 \right\|_{P,2} \leq \delta_N \) hold that for \( \Delta_N, \delta_N \) such that both are sequences of strictly positive constants converging to zero, where \( \eta = (\beta, \gamma, \zeta) \).

From (C.36), we have
\[
\left\| \hat{\eta} - \eta_0 \right\|_{P,q} = \left\| \hat{\beta} - \beta_0 \right\|_{P,q} \vee \left\| \hat{\gamma} - \gamma_0 \right\|_{P,q} \vee \left\| \hat{\zeta} - \zeta_0 \right\|_{P,q} = \left( E \left[ \left\| X'(\hat{\beta} - \beta_0) \right\| \right] A_N \right)^{1/q}
\]
\[
\leq \left( E \left[ (\hat{\beta} - \beta_0)' E[XX'] (\hat{\beta} - \beta_0) \right]^{q/2} A_N \right)^{1/q} = \left( \lambda_{\max}(E[XX']) \left\| \hat{\beta} - \beta_0 \right\|^2 \right)^{1/q},
\]
where it is bounded by Assumption 9(a), (A.3), and (A.4).

Thus, \( \left\| \hat{\eta} - \eta_0 \right\|_{P,q} \leq C \) holds with probability at least 1 – \( \Delta_N \) for some \( \Delta_N = o(1) \).
Since we use the identical procedure, namely the OGA and HDAIC, in estimating $\beta_0$, $\gamma_0$, and $\eta_0$, the convergence rates apply to all the nuisance parameters. Like we did in the proof of Theorem 1, we consider two cases where all the parameters follow the polynomial decay case or the exponential decay case.

Case 1. For the polynomial decay case, let $\delta_N = (\log p)^{1-1/2\alpha}N^{1/2\alpha-1/2}$. Then $\delta_N = o(1)$ since $\alpha$ is assumed to be strictly larger than 1 in Assumption 10 (a). We have

$$\|\hat{\eta} - \eta_0\|_{P,2} = \|\hat{\beta} - \beta_0\|_{P,q} \vee \|\hat{\gamma} - \gamma_0\|_{P,q} \vee \|\hat{\zeta} - \zeta_0\|_{P,q} \lesssim \left(\frac{\log p}{N}\right)^{1/2-1/4\alpha} \leq \delta_N,$$

where the last inequality holds since $(\log p/N)^{1/4\alpha} \leq \log p$ holds with $\alpha > 1$. Also,

$$\|\hat{\beta} - \beta_0\|_{P,2} \times \left(\|\hat{\zeta} - \zeta_0\|_{P,2} + \|\hat{\gamma} - \gamma_0\|_{P,2}\right) \lesssim \left(\frac{\log p}{N}\right)^{1-1/2\alpha} = \delta_N N^{-1/2}$$

holds.

Case 2. For the exponential decay case, let $\delta_N = N^{-1/2} \log p \log N$. By the assumption $\log p = o(N^{1/4})$, $\delta_N = o(1)$. We have

$$\|\hat{\eta} - \eta_0\|_{P,2} \lesssim \sqrt{\frac{\log p \log N}{N}} < \delta_N$$

and

$$\|\hat{\beta} - \beta_0\|_{P,2} \times \left(\|\hat{\zeta} - \zeta_0\|_{P,2} + \|\hat{\gamma} - \gamma_0\|_{P,2}\right) \lesssim \frac{\log p \log N}{N} = \delta_N N^{-1/2}.$$

A similar argument applies to the cross cases.

We have shown that Assumption 4.2 in Chernozhukov et al. (2018) holds for both sparsity assumptions. Therefore, applying Theorem 4.2. in Chernozhukov et al. (2018), we get the desired results.

□

Appendix D. Omitted Details

This appendix section presents details that are omitted from the main text.
D.1. Details of the Method. This section provide details about two tuning parameters, $C^*$ and $M^*_n$, used in Algorithm 1. Let $c_1, c_2$ be sufficiently large positive constants that satisfy

$$
P\left( \max_{1 \leq j \leq p} \left| \frac{1}{N} \sum_{i=1}^{N} X_{ij} \epsilon_i \right| \geq c_1 \sqrt{\frac{\log p}{N}} \right) = o(1)
$$

and

$$
P\left( \max_{1 \leq j, \ell \leq p} \left| \frac{1}{N} \sum_{i=1}^{N} X_{ij} X_{i\ell} - E[X_{ij} X_{i\ell}] \right| \geq c_2 \sqrt{\frac{\log p}{N}} \right) = o(1),
$$

for each of $\epsilon = V$ and $\epsilon = Y - X'\gamma_0$. We require $c_1, c_2$ to satisfy the above restrictions with $\epsilon = V$ for the estimation of the part $\beta_0$ of the nuisance parameters and with $\epsilon = Y - X'\gamma_0$ for the estimation of the part $\gamma_0$ of the nuisance parameters.

Define $\Gamma(J) = E[X_{iJ} X_{iJ}']$,

$$
M^*_N = \min \left\{ \left( \frac{N}{\log p} \right)^{1/2}, \bar{\delta} \left( \frac{N}{\log p} \right)^{1/2} \right\},
$$

and

$$
\tau = \sup \left\{ \tau : \tau > 0, \limsup_{N \to \infty} \frac{\tau c_2}{\min_{|J| \leq \tau(N/\log p)^{1/2} \lambda_{\min}(\Gamma(J))} \lambda_{\min}(\Gamma(J))} \leq 1 \right\},
$$

where $0 < \bar{\delta} < \min\{\bar{\tau}, C\}$ with an arbitrary strictly positive constant $\bar{C}$ restricted in Assumption 2(b). In our simulation and real data analysis, we set $\bar{\delta} = 5$ following Ing (2020).

Let $\bar{B}$ be a positive constant satisfying

$$
\frac{1}{\liminf_{N \to \infty} \min_{|J| \leq \tau(N/\log p)^{1/2} \lambda_{\min}(\Gamma(J))} \lambda_{\min}(\Gamma(J)) - c_2 \bar{\delta}}
$$

Define a sufficiently large positive constant $C^*$ satisfying

$$
C^* > \frac{2\bar{B}(c_1^2 + c_2^2)}{\sigma^2},
$$

for each of $\sigma^2 = E[V^2]$ and $\sigma^2 = E[(Y - X'\gamma_0)^2]$. In our simulation and real data analysis, we set $C^* = 2$ following Ing (2020). We require $C^*$ to satisfy this restriction with $\sigma^2 = E[V^2]$. 

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for estimation of the part $\beta_0$ of the nuisance parameters and with $\sigma^2 = E[(Y - X'\gamma_0)^2]$ for estimation of the part $\gamma_0$ of the nuisance parameters.

D.2. Finite Sample Adjustment. The DML estimator is random even conditionally on data, because of the random splitting of the sample for cross fitting. To mitigate the effects of this randomness of the DML, Chernozhukov et al. (2018, Sec. 3.4) propose procedures of finite-sample adjustments. In this section, we present one of these procedures for completeness.

Suppose that we repeat the DML estimation $S$ times to obtain $\{\hat{\theta}^s\}_{s=1}^S$. A robust estimator that incorporates the impact of sample splitting is defined by

$$\tilde{\theta}^{Med} = \text{Median} \{\hat{\theta}^s\}_{s=1}^S.$$

Chernozhukov et al. (cf. 2018, Definition 3.3). Its associated variance estimator is given by

$$\tilde{\Omega}^{Med} = \text{Median} \left\{\tilde{\Omega}^s + (\hat{\theta}^s - \tilde{\theta}^{Med})(\hat{\theta}^s - \tilde{\theta}^{Med})'\right\}_{s=1}^S.$$

Chernozhukov et al. (cf. 2018, Equation 3.14). We use these estimators with $S = 20$ in reporting the estimation results in Section 5.

D.3. The Hermite Basis. In Section 5 we use the Hermite basis ($\psi_0, \ldots, \psi_9$). The $k$-th basis element $\psi_j$ of the Hermite basis is defined by

$$\psi_k(x) = e^{-x^2/2}H_k(x),$$

where $H_k$ is the Hermite polynomial defined by

$$H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}$$

for each $k \in \mathbb{Z}_+$. The the Hermite polynomial can be also written as

$$H_k(x) = k! \sum_{\kappa=0}^{\lfloor k/2 \rfloor} \frac{(-1)^\kappa}{\kappa!(k - 2\kappa)!} \frac{x^{k-2\kappa}}{2^{\kappa}}$$
in a closed form, where $\lfloor \cdot \rfloor$ denotes the floor function.

APPENDIX E. ADDITIONAL SIMULATIONS

E.1. Alternative Values of the Tuning Parameters. In Section 3 in the main text, we present simulation results using the choice $C^* = 2$ of the tuning parameter following Ing (2020) – see Appendix D.1 for the implementation details. In the current appendix section, we show additional simulation results that we obtain by varying the value of $C^*$ up and down by ten percent and twenty percent, and report the sensitivity of the results to this variation.

Table 3 shows the results under $C^* = 1.6, 1.8, 2.2, \text{ and } 2.4$ along with the baseline value of $C^* = 2.0$. We focus on one DGP, namely the case of $\beta_{0,j} = \gamma_{0,j} = j^{-1.5}$, for simplicity. Observe that the results are fairly insensitive to variations in the values of $C^*$.

E.2. Estimation and Inference without Cross Fitting. In Section 3 in the main text, we present simulation results for Algorithm 1 which uses sample splitting for cross fitting. In the current appendix section, we present simulation results for Algorithm 2 based on the
full sample without cross fitting. We use the the same DGPs as in Section 3 in the main text.

The results presented in Table 4 for Algorithm 2 are almost the same as those presented in Table 1 in the main text. In other words, Algorithm 2 behaves similarly to Algorithm 1 under DGPs, while the latter slightly outperforms especially under less sparse designs.

| $\beta_{0,j}, \gamma_{0,j}$ | $N$ | $p$ | Method of Preliminary Estimation | Bias  | SD    | RMSE  | 95%   |
|---------------------------|-----|-----|----------------------------------|-------|-------|-------|-------|
| Sparse                    | 500 | 500 | OGA+HDAIC                        | -0.002| 0.045 | 0.045 | 0.943 |
|                           | 1000| 500 | OGA+HDAIC                        | 0.000 | 0.032 | 0.032 | 0.943 |
| $e^{-j}$                  | 500 | 500 | OGA+HDAIC                        | 0.001 | 0.045 | 0.045 | 0.937 |
|                           | 1000| 500 | OGA+HDAIC                        | 0.000 | 0.032 | 0.032 | 0.946 |
| $j^{-2}$                  | 500 | 500 | OGA+HDAIC                        | -0.001| 0.045 | 0.046 | 0.939 |
|                           | 1000| 500 | OGA+HDAIC                        | 0.001 | 0.032 | 0.032 | 0.947 |
| $j^{-1.75}$               | 500 | 500 | OGA+HDAIC                        | 0.000 | 0.045 | 0.046 | 0.942 |
|                           | 1000| 500 | OGA+HDAIC                        | 0.001 | 0.032 | 0.032 | 0.946 |
| $j^{-1.5}$                | 500 | 500 | OGA+HDAIC                        | 0.002 | 0.045 | 0.046 | 0.935 |
|                           | 1000| 500 | OGA+HDAIC                        | 0.003 | 0.032 | 0.033 | 0.936 |
| $j^{-1.25}$               | 500 | 500 | OGA+HDAIC                        | 0.007 | 0.044 | 0.048 | 0.924 |
|                           | 1000| 500 | OGA+HDAIC                        | 0.005 | 0.031 | 0.034 | 0.923 |
| $j^{-1}$                  | 500 | 500 | OGA+HDAIC                        | 0.022 | 0.044 | 0.056 | 0.872 |
|                           | 1000| 500 | OGA+HDAIC                        | 0.015 | 0.031 | 0.037 | 0.898 |

Table 4. Monte Carlo simulation results without cross fitting. Displayed are Monte Carlo simulation statistics including the bias, standard deviation (SD), root mean square error (RMSE), and 95% coverage frequency.

E.3. High-Dimensional IV Regression. We consider the simple setting where the data are generated by the system

$$Y = \theta_0(D - X'\zeta_0) + X'\gamma_0 + U,$$

$$D = \mu Z + X'\zeta_0 + E,$$

$$Z \sim N(0, 1),$$
\[ \gamma_{0,j}, \zeta_{0,j} \] 

| Method of Preliminary Estimation | Bias  | SD    | RMSE | 95%  |
|----------------------------------|-------|-------|------|------|
| \text{Sparse} 500 500 OGA+HDAIC  | -0.001 | 0.046 | 0.047 | 0.949 |
| 1000 500                                      | 0.002  | 0.032 | 0.033 | 0.951 |
| \text{e}^{-j} 500 500 OGA+HDAIC  | -0.001 | 0.046 | 0.046 | 0.951 |
| 1000 500                                      | 0.002  | 0.032 | 0.032 | 0.947 |
| \text{j}^{-2} 500 500 OGA+HDAIC  | -0.001 | 0.046 | 0.047 | 0.943 |
| 1000 500                                      | 0.002  | 0.032 | 0.033 | 0.951 |
| \text{j}^{-1.75} 500 500 OGA+HDAIC | -0.001 | 0.047 | 0.047 | 0.942 |
| 1000 500                                      | 0.002  | 0.032 | 0.033 | 0.948 |
| \text{j}^{-1.5} 500 500 OGA+HDAIC | -0.001 | 0.047 | 0.047 | 0.942 |
| 1000 500                                      | 0.001  | 0.033 | 0.033 | 0.952 |
| \text{j}^{-1.25} 500 500 OGA+HDAIC | -0.002 | 0.048 | 0.049 | 0.943 |
| 1000 500                                      | 0.002  | 0.033 | 0.033 | 0.934 |
| \text{j}^{-1} 500 500 OGA+HDAIC               | -0.001 | 0.051 | 0.051 | 0.951 |
| 1000 500                                      | 0.002  | 0.034 | 0.035 | 0.946 |

Table 5. Monte Carlo simulation results for IV model. Displayed are Monte Carlo simulation statistics including the bias, standard deviation (SD), root mean square error (RMSE), and 95% coverage frequency.

\[
(U, E) \sim N \left( 0, \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix} \right).
\]

For our method, we use Algorithm 3 for the estimation. For DML methods, we used the R package DoubleML’s partially linear IV setting. The results are shown in Table 5.

**APPENDIX F. ADDITIONAL EMPirical RESULTS**

In the current appendix section, we provide additional empirical estimates following Section 5 in the main text. We provide the following two types of alternative estimates. First, we vary the value of the tuning parameter \( C^* \). Second, we use Algorithm 2 without cross fitting, instead of Algorithm 1. Table 6 gives the baseline results using our method in row (I). Rows (II) and (III) show the estimates under the choices \( C^* = 1.8 \) and \( 2.2 \), respectively.
Table 6. Estimates of labor elasticities in the 3-digit level industry of food products (311) in Chile based on four alternative methods.

Row (IV) shows the estimates using the whole sample without cross fitting. Observe that the results are mostly robust, and our empirical findings qualitatively remain the same as those discussed in the main text.

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