1. Introduction

This chapter describes widespread methods of model-based fuzzy control systems. The subject of this chapter is a systematic framework for the stability and design of nonlinear fuzzy control systems. We are trying to build a bridge between conventional fuzzy control and classic control theory. By building this bridge, the strong well developed tools of classic control could be used in model-based fuzzy control systems.

Model-based fuzzy control, with the possibility of guaranteeing the closed loop stability, is an attractive method for control of nonlinear systems. In recent years, many studies have been devoted to the stability analysis of continuous time or discrete time model based fuzzy control systems (Takagi & Sugeno, 1985; Rhee & Won, 2006; Chen et al., 1993; Wang et al., 1996; Zhao et al., 1996; Tanaka & Wang, 2001; Tanaka et al., 2001). Among such methods, the method of Takagi-Sugeno (Takagi & Sugeno, 1985) has found many applications for modelling complex nonlinear systems (Tanaka & Sano, 1994; Tanaka & Kosaki, 1997; Li et al., 1998). The concept of sector nonlinearity (Kawamoto et al., 1992) provided means for exact approximation of nonlinear systems by fuzzy blending of a few locally linearized subsystems. One important advantage of using such a method for control design is that the closed-loop stability analysis, using the Lyapunov method, becomes easier to apply. Various stability conditions have been proposed for such systems (Tanaka & Wang, 2001), (Ting, 2006), where the existence of a common solution to a set of Lyapunov equations is shown to be sufficient for guaranteeing the closed-loop stability. Some relaxed conditions are also proposed in (Kim & Lee, 2000; Ding et al, 2006; Fang et al., 2006, Tanaka & Ikeda, 1998). Parallel Distributed Compensator (PDC) is a generalization of the state feedback controller to the case of nonlinear systems, using the Takagi-Sugeno fuzzy model (Wang et al., 1996). This method is based on partitioning nonlinear system dynamics into a number of linear subsystems, for which state feedback gains are designed and blended in a fuzzy sense. Takagi-Sugeno model and parallel distributed compensation have been used in many applications successfully (Sugeno & Kang, 1986, Lee et al., 2006, Hong & Langari, 2000, Bonissone et al.,
1995). The Linear Matrix Inequality (LMI) technique offers a numerically tractable way to design a PDC controller with objectives such as stability (Wang et al., 1996; Ding et al., 2006; Fang et al., 2006; Tanaka & Sugeno, 1992), $H_\infty$ control (Lee et al., 2001), $H_2$ control (Lin & Lo, 2003), pole-placement (Jon et al., 1997; Kang & Lee, 1998), and others (Tanaka & Wang, 2001).

2. Takagi-Sugeno fuzzy model

The main idea of the Takagi-Sugeno fuzzy modeling method is to partition the nonlinear system dynamics into several locally linearized subsystems, so that the overall nonlinear behavior of the system can be captured by fuzzy blending of such subsystems. The fuzzy rule associated with the $i$-th linear subsystem for the continuous fuzzy system and the discrete fuzzy system, can then be defined as

Continuous fuzzy system

Rule i : IF $Z_i(t)$ is $M_{i1}$...and $Z_i(t)$ is $M_{il}$

THEN

$$\begin{align}
\dot{x}(t) &= A_ix(t) + B_iu(t) \\
y(t) &= C_ix(t)
\end{align}$$

(1)

Discrete Fuzzy System

Rule i : IF $Z_i(t)$ is $M_{i1}$...and $Z_i(t)$ is $M_{il}$

THEN

$$\begin{align}
x(t+1) &= A_ix(t) + B_iu(t) \\
y(t) &= C_ix(t)
\end{align}$$

(2)

where, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $A_i \in \mathbb{R}^{nxn}$, $B_i \in \mathbb{R}^{nxm}$, $C_i \in \mathbb{R}^{qn}$; $\{z_1(t), z_2(t), \ldots, z_p(t)\}$ are nonlinear functions of the state variables obtained from the original nonlinear equation, and $M_{ij}(z_i)$ are the degree of membership of $z_i(t)$ in a fuzzy set $M_{ij}$. Whenever there is no ambiguity, the time argument in $z(t)$ is dropped. The overall output, using the fuzzy blend of the linear subsystems, will then be as follows:

Continuous fuzzy system

$$\begin{align}
\dot{X} &= \sum_{i=1}^r w_i(z) \{ A_ix(t) + B_iu(t) \} \\
\dot{X} &= \sum_{i=1}^r h_i(z) \{ A_ix(t) + B_iu(t) \} \\
y(t) &= \frac{\sum_{i=1}^r w_i(z) C_ix(t)}{\sum_{i=1}^r w_i(z)} = \sum_{i=1}^r h_i(z) C_ix(t)
\end{align}$$

(3)
Discrete Fuzzy System

\[
x(t + 1) = \frac{\sum_{i=1}^{r} \omega_i(z(t)) [A_i x(t) + B_i u(t)]}{\sum_{i=1}^{r} \omega_i(z(t))}
\]

\[
= \sum_{i=1}^{r} h_i(z(t)) [A_i x(t) + B_i u(t)]
\]

\[
y(t) = \frac{\sum_{i=1}^{r} \omega_i(z(t)) C_i x(t)}{\sum_{i=1}^{r} \omega_i(z(t))}
\]

\[
= \sum_{i=1}^{r} h_i(z(t)) C_i x(t)
\]

Where

\[
w_1(z) = \prod_{j=1}^{i} M_{ij}(z_j)
\]

\[
h_i(z) = \frac{w_1(z)}{\sum_{i=1}^{r} w_1(z)}
\]

It is also true, for all \( t \), that

\[
\begin{cases}
\sum_{i=1}^{r} w_1(z) > 0, \\
w_1(z) \geq 0, i = 1, 2, \ldots, r
\end{cases}
\]

2.1. Building a fuzzy model

There are generally three approaches to build the fuzzy model: "sector nonlinearity," "local approximation," or a combination of the two.

2.1.1. Sector nonlinearity

Figure 1 illustrates the concept of global and local sector nonlinearity. Suppose the original nonlinear system satisfies the sector non-linearity condition (Kawamoto et al., 1992, as cited in Tanaka & Wang, 2001), i.e., the values of nonlinear terms in the state-space equation remain within a sector of hyper-planes passing through the origin. This model guarantees the stability of the original nonlinear system under the control law. A function \( \Phi: \mathbb{R} \rightarrow \mathbb{R} \) is said to be sector \([a,c]\) if for all \( x \in \mathbb{R} \), \( y = \Phi(x) \) lies between \( b_1 x \) and \( b_2 x \).
Example 1

The well-known nonlinear control benchmark, the ball-and-beam system is commonly used as an illustrative application of various control methods (Wang & Mendel, 1992) depicted in figure 2. Let $x_1(t)$ and $x_2(t)$ denote the position and the velocity of the ball and let $x_3(t)$ and $x_4(t)$ denote the angular position and the angular velocity of the beam. Then, the system dynamics can be described by the following state-space equation:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t)$$

Where

$$f(x) = \begin{bmatrix} x_2(t) \\ B(x_1(t)x_3^2(t) - G\sin(x_3(t))) \\ x_4(t) \\ 0 \end{bmatrix}$$

and

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Where $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{bmatrix}^T$ and $u(t)$ is torque.

$\sin(x_3)$ and $x_1x_4^2$ are nonlinear terms in the state-space equation. We define $z_1 = \sin(x_3)$ and $z_2 = x_1x_4^2$. Assume $x_3 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $x_1x_4 \in [-d, d]$ as the region within which the system will operate. Figure 3 shows that $z_1(t) = \sin(x_3(t))$ and its local sector operating...
region. The sector \([b_1, b_2]\) consists of two lines \(b_1x_1\) and \(b_2x_1\), where the slopes are \(b_1 = 1\) and \(b_2 = 2\). It follows that

\[
\left|\frac{2}{\pi}x\right| \leq |\sin(x)| \leq |x|,
\]

\[-dx_4 \leq x_1x_4^2 \leq dx_4.\]  \hspace{1cm} (7)

**Figure 3.** \(\sin(x_3(t))\) and its local sector

We present \(\sin(x_3(t))\) is represented as follows:

\[z_1 = \sin(x_3(t)) = \left(\sum_{i=1}^{2} M_i(z_1(t))b_i\right)x_3(t)\]  \hspace{1cm} (8)

From the property of membership functions \(\left[M_1(z_1(t)) + M_2(z_1(t)) = 1\right]\), we can obtain the membership functions

\[
M_1(z_1(t)) = \begin{cases} 
z_1(t) - \left(\frac{2}{\pi}\right)\sin^{-1}(z_1(t)) & z_1(t) \neq 0 \\
\left(1 - \frac{2}{\pi}\right)\sin^{-1}(z_1(t)) & \text{otherwise.} 
\end{cases}
\]

\[
M_2(z_1(t)) = \begin{cases} 
\sin^{-1}(z_1(t)) - z_1(t) & z_1(t) \neq 0 \\
\left(1 - \frac{2}{\pi}\right)\sin^{-1}(z_1(t)) & 0, \text{ otherwise.} 
\end{cases}
\]  \hspace{1cm} (9)

Similarly we obtain membership functions associated with \(z_2(t) = x_1(t)x_4(t)\). Assume \(\max(z_2(t)) = d = \alpha_1\) and \(\min(z_2(t)) = -d = \alpha_2\) we have:
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\[ z_2(t) = x_1(t)x_4(t) = \left( \sum_{i=1}^{2} N_i(z_2(t))b_i \right) \alpha_i \quad (10) \]

\[ N_1(z_2(t)) = \frac{z_2(t) - \alpha_2}{\alpha_1 - \alpha_2}, \]

\[ N_2(z_2(t)) = \frac{\alpha_1 - z_2(t)}{\alpha_1 - \alpha_2}, \quad (11) \]

The exact TS-fuzzy model-based dynamic system of the ball and beam system can be obtained as following:

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\dot{x}_3(t) \\
\dot{x}_4(t)
\end{bmatrix} = \sum_{i=1}^{2} \sum_{j=1}^{2} M_i(z_1(t))N_j(z_2(t)) \times
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & -G_b & \alpha_j \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t) \\
x_4(t)
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}u(t) \quad (12)
\]

The fuzzy model has the following 4 rules:

Rule 1: if \( z_1(t) \) is \( M_1 \) and \( z_2(t) \) is \( N_1 \)

Then \( \dot{x}(t) = A_1x(t) + B_1u(t) \),

Rule 2: if \( z_1(t) \) is \( M_1 \) and \( z_2(t) \) is \( N_2 \)

Then \( \dot{x}(t) = A_2x(t) + B_2u(t) \),

Rule 3: if \( z_1(t) \) is \( M_2 \) and \( z_2(t) \) is \( N_1 \)

Then \( \dot{x}(t) = A_3x(t) + B_3u(t) \),

Rule 4: if \( z_1(t) \) is \( M_2 \) and \( z_2(t) \) is \( N_2 \)

Then \( \dot{x}(t) = A_4x(t) + B_4u(t) \)

Where

\[ A_1 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & -G_b & \alpha_1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad A_1 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & -G_b & \alpha_2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad A_4 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & -G_b & \alpha_2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad A_4 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & -G_b & \alpha_2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} \]

\[ B_1 = B_2 = B_3 = B_4 = B = \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}, \quad z_1 = \sin(x_3) \text{ and } z_2 = x_1x_4 \]
2.1.2. Local approximation

The original system can be partitioned into subsystems by approximation of nonlinear terms about equilibrium points. This approach can have fewer rules and of course less complexity but it cannot guarantee the stability of the original system under the controller. Usually in this approach, construction of a fuzzy membership function requires knowledge of the behavior of the original system and of course different types of membership functions can be selected.

3. Parallel distributed compensation

Parallel distributed compensation (PDC) is a model-based design procedure introduced in (Wang et al., 1995). Using the Takagi-Sugeno fuzzy model, a fuzzy combination of the stabilizing state feedback gains, $F_i, i = 1,2,...,r$, associated with every linear subsystem is used as the overall state feedback controller. The general structure of the controller is then as

$$u = \sum_{i=1}^{r} \omega_i(z) F_i x(t)$$

If $z_1(t)$ is $M_{i1}$, and $z_2(t)$ is $M_{i2}$, ..., $z_m(t)$ is $M_{ip}$, then $u = -F_i x(t), i = 1,2,...,r$ (14)

The output of the controller is represented by

$$u = -\sum_{i=1}^{r} h_i(z) F_i x(t).$$

The Takagi-Sugeno model and the Parallel Distributed Compensation have the same number of fuzzy rules and use the same membership functions.

4. Stability conditions and control design

4.1. LMI

A variety of problems arising in system and control theory can be reduced to a few standard convex or quasi-convex optimization problems involving linear matrix inequalities (LMIs).

Lyapunov published his theory in 1890 and showed that $\frac{d}{dt}x(t) = Ax(t)$ is stable if and only if there exists a positive-definite matrix $P$ such that $A^T P + PA < 0$. The Lyapunov inequality, $P > 0$ and $A^T P + PA < 0$ is a form of an LMI.

An LMI has the form

$$F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i > 0,$$ (16)
Where $F_i \in \mathbb{R}^{n \times n}, i = 0, \ldots, m$ are the given symmetric matrices and $x \in \mathbb{R}^m$ is the variable and the inequality symbol shows that $F(x)$ is positive definite (Boyd, 1994).

### 4.2. Stability conditions

There are a large number of works on stability conditions and control design of fuzzy systems in the literature. A sufficient stability condition for ensuring stability of PDC was derived by Tanaka and Sugeno (Tanaka & Sugeno, 1990; 1992).

By substituting the controller output (15) into the TS model for the continuous fuzzy control (4), we have:

$$
\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t))h_j(z(t))\left[A_i - B_iF_j\right]x(t)
$$

or

$$
\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))h_i(z(t))G_{ii}x(t)
$$

$$
+ 2 \sum_{i=1}^{r} \sum_{i<j} h_i(z(t))h_j(z(t))\left[\frac{G_{ij} + G_{ji}}{2}\right]x(t)
$$

where $G_{ij} = A_i - B_iF_j$. Similarly for the discrete fuzzy system we have

$$
x(t+1) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t))h_j(z(t))\left[A_i - B_iF_j\right]x(t)
$$

or

$$
\dot{x}(t+1) = \sum_{i=1}^{r} h_i(z(t))h_i(z(t))G_{ii}x(t)
$$

$$
+ 2 \sum_{i=1}^{r} \sum_{i<j} h_i(z(t))h_j(z(t))\left[\frac{G_{ij} + G_{ji}}{2}\right]x(t)
$$

**Theorem 1:** The equilibrium of the continuous fuzzy system (3) with $u(t) = 0$ is globally asymptotically stable if there exists a common positive definite matrix $P$ such that

$$
A_i^TP + PA_i < 0, \quad i = 1, 2, \ldots, r
$$

that is, a common $P$ has to exist for all subsystems.

**Theorem 2:** The equilibrium of the discrete fuzzy system (4) with $u(t) = 0$ is globally asymptotically stable if there exists a common positive definite matrix $P$ such that

$$
A_i^TPA_i - P < 0, \quad i = 1, 2, \ldots, r
$$
that is, a common $P$ has to exist for all subsystems.

The stability of the closed loop system can be derived by using theorem 1 and 2.

Theorem 3: The equilibrium of the continuous fuzzy control system described by (18) is globally asymptotically stable if there exists a common positive definite matrix $P$ such that

$$G_{ii}^T P + P G_{ii} < 0,$$

$$\left( \frac{G_{ij} + G_{ji}}{2} \right)^T P + P \left( \frac{G_{ij} + G_{ji}}{2} \right) \leq 0,$$

$$i < j \text{ s.t. } h_i \cap h_j \neq \phi$$

(23)

(24)

Theorem 4: The equilibrium of the discrete fuzzy control system described by (20) is globally asymptotically stable if there exists a common positive definite matrix $P$ such that

$$G_{ii}^T P G_{ii} - P < 0,$$

$$\left( \frac{G_{ij} + G_{ji}}{2} \right)^T P \left( \frac{G_{ij} + G_{ji}}{2} \right) - P \leq 0,$$

$$i < j \text{ s.t. } h_i \cap h_j \neq \phi$$

(25)

(26)

4.3. Stable controller design

By using the following conditions, the solution of the LMI problem for continuous and discrete fuzzy systems gives us the state feedback gains $F_i$ and the matrix $P$ (if the problem is solvable).

Consider a new variable $X = P^{-1}$ then the stable fuzzy controller design problem is:

Continuous fuzzy system

Find $X > 0$ and $M_i$, $i = 1,2,\ldots,r$

$$-X A_i^T - A_i X + M_i^T B_i^T + B_i M_i > 0,$$

$$-X A_i^T - A_i X - X A_i^T - A_i X$$

$$+ M_i^T B_i^T + B_i M_i + M_i^T B_i^T + B_i M_i \geq 0.$$  

$$X = P^{-1} i < j \text{ s.t. } h_i \cap h_j \neq \phi$$

(27)

(28)

The conditions (27) and (28) gives us a positive definite matrix $X$ and $M_i$ (or that there is no solution). From the solution $X$ and $M_i$, a common $P$ and the feedback gains can be found as:
Similarly for a discrete fuzzy system, the design problem is

\[ X > 0 \text{ and } M_i, \ i = 1, 2, ..., r \]

\[
X - \left( A_i X - B_i M_i \right)^T X^{-1} \left( A_i X - B_i M_i \right) > 0,
\]

\[
X - \frac{1}{4} X \left( A_i X - B_i M_i + A_j X - B_j M_j \right)^T X^{-1}
\]

\[
\times \left( A_i X - B_i M_i + A_j X - B_j M_j \right) X \geq 0.
\]

4.4. Decay rate

Decay rate is associated with the speed of response. The decay rate fuzzy controller design helps to find feedback gains that provide better settling time (Tanaka et al., 1996; 1998a; 1998b).

Continuous fuzzy system: The condition that \( \dot{V}(x(t)) \leq -2\alpha V(x(t)) \) (Ichikawa et al., 1993, as cited in Tanaka & Wang, 2001) for all \( x(t) \) can be written as

\[
G_{ij}^T P + PG_{ij} + 2\alpha P < 0
\]

\[
\left( \frac{G_{ij} + G_{ji}}{2} \right)^T P + P \left( \frac{G_{ij} + G_{ji}}{2} \right) + 2\alpha P \leq 0
\]

Where

\[
G_{ij} = A_i - B_i F_i, \ \alpha > 0 \text{ and } i < j \text{ s.t. } h_i \cap h_j \neq \emptyset
\]

Therefore, by solving the following generalized eigenvalue minimization problem in \( X \), the largest lower bound on the decay rate that can be found by using a quadratic Lyapunov function:

\[ \text{maximize } \alpha \text{ subject to} \]

\[
X > 0,
\]

\[
-XA_i^T + A_i X + M_i^T B_i^T + B_i M_i - 2\alpha X > 0,
\]

\[
-XA_j^T - A_j X - XA_j^T - A_j X + M_j^T B_j^T + B_j M_j
\]

\[
\quad + M_i^T B_i^T + B_i M_i - 4\alpha X > 0,
\]

\[ i < j \text{ s.t. } h_i \cap h_j \neq \emptyset, \text{ where } X = P^{-1}, \ M_i = F_i X. \]
The condition that \( \Delta V(x(t)) \leq (\alpha^2 - 1)V(x(t)) \) (Ichikawa et al, 1993, as cited in Tanaka & Wang, 2001) for all \( x(t) \) can be written as

\[
G_{ij}^T P G_{ji} - \alpha^2 P < 0,
\]

\[
\left( \frac{G_{ij} + G_{ji}}{2} \right)^T P \left( \frac{G_{ij} + G_{ji}}{2} \right) - \alpha^2 P \leq 0
\]

\( i < j \) s.t. \( h_i \cap h_j \neq \emptyset \) and \( \alpha < 1 \) (35)

The generalized eigenvalue minimization can be found in (Tanaka & Wang, 2001).

4.5. Constraint on control

Theorem 5: Assume that the initial condition \( x(0) \) is known. The constraint \( \|u(t)\|_2 \leq \mu \) is satisfied at all times \( t \geq 0 \) if the LMIs

\[
\begin{bmatrix}
1 & x(0)^T \\
x(0) & X
\end{bmatrix} \geq 0
\]

\[
\begin{bmatrix}
X & M_i^T \\
M_i & \mu^2 I
\end{bmatrix} \geq 0
\]

(37)

Hold, where \( X = P^{-1} \) and \( M_i = F_i X \).

The above LMI design conditions depend on the initial states. Thus, if the initial states \( x(0) \) change, this means that the feedback gains \( F_i \) must be again determined. To overcome this disadvantage, modified LMI constraints on the control input have been developed, where \( x(0) \) is unknown but the upper bound \( \phi \) of \( \|x(t)\| \) is known, i.e., \( \|x(t)\| \leq \phi \).

Theorem 6: Assume that \( \|x(t)\| \leq \phi \), where \( x(0) \) is unknown but the upper bound \( \varphi \) is known. Then,

\[
x^T(0)X^{-1}x(0) \leq 1 \text{ if } \varphi^2 I \leq X,
\]

(38)

Where \( X = P^{-1} \)

Proofs of theorem 1 and 2 are given in (Tanaka & Wang, 2001)

4.6. Performance-oriented parallel distributed compensation

In the modified PDC proposed in (Seidi & Markazi, 2011), unlike the conventional PDC, state feedback gains associated with every linear subsystem, are not assumed fixed. Instead, based on some pre-specified performance criteria, several feedback gains are designed and
used for every subsystem. The overall gain associated with each of the subsystems, is then determined by a fuzzy blending of such gains, so that a better closed-loop performance can be achieved. The required membership functions are chosen based on some pre-specified performance indices, for example, a faster response or a smaller control input. In general, the rest of the method for calculating the overall state feedback gain remains similar to the conventional PDC method, as in (14) and (15). Figure 4, depicts the general framework for the proposed method, through which and depending on various performance criteria, different characteristics for the controller can be specified. For example, two different feedback gains could be designed for a typical subsystem; one providing a lower control input with a longer settling time response, and the other a faster response but with a larger control input. The idea is then to select the overall feedback gain for this subsystem as a weighted sum of such gains, where the weights are appropriately adjusted, in a fuzzy sense, during the time evolution of the system response, so that as a whole, a faster response with a lower control input can be achieved. For this purpose, when the magnitude of the control input becomes large, the relative weight of the first feedback gain is increased, so that the magnitude of the control input is kept within the permissible limits. On the other hand, when the control input is well below the permissible limit, the weight of the second feedback gain is increased, for a faster response. The dynamics of the resulting closed-loop control system can be analyzed as follows:

Consider the following Takagi–Sugeno model of the plant

\[
\dot{x} = \sum_{i=1}^{r} h_i(z) \left[ A_i x(t) + B_i u(t) \right]
\]  

(39)

The following structure is proposed for the fuzzy controller rules

\[i\text{th rule : If } Z_1(t) \text{ is } M_{i1} \text{ and } Z_2(t) \text{ is } M_{i2}, \ldots, Z_p(t) \text{ is } M_{ip}, J(t) \text{ is } H_{i1}, \ldots \text{ and } f(t) \text{ is } H_{iq}\]

then \[u_i(t) = \left\{ \sum_{n=1}^{q} m_{in}(J(t))K_{in} \right\} x(t)\]

(40)

Where \(i = 1, 2, \ldots, r\), \(q_i\) is the number of gain coefficients in the \(i\)th subsystem, \(m_{in}\) is the relevant membership degree for \(J(t)\), \(K_{in}\) is the \(n\)th state feedback gain associated with the \(i\)th subsystem, \(H_{iq}\) is the \(n\)th membership function for \(J(t)\), defined in the \(i\)th rule. Here \(J(t)\) is a term depicting a selected performance index, for instance, if one wants to limit the magnitude of the control signal \(u(t)\), then \(J(t) = \|u(t)\|\). Where the control input generated by the PDC controller is in the form of

\[u(t) = \sum_{i=1}^{r} h_i(z)u_i(t) = -\left\{ \sum_{n=1}^{r} h_i(z)K_i \right\} x(t)\]

(41)

\[K_i = \sum_{n=1}^{q} m_{in}(J(t))K_{in}\]
Lemma: The fuzzy control system (39), with the control strategy (41) is globally, asymptotically stable, if there exists a common positive definite matrix $P$ such that

$$G_{ijn}^T P + PG_{ijn} < 0$$

$$\left(\frac{G_{ijn} + G_{jin}}{2}\right)^T p + p \left(\frac{G_{ijn} + G_{jin}}{2}\right) \leq 0$$

(42)

where $i < j$, $h_i \cap h_j \neq \phi$, $G_{ijn} = A_i - B_i K_{jn}$.

Example 2

Consider a single link robot with flexible joint as in Figure 5. This benchmark problem is introduced in (Spong et al., 1987).

Figure 5. A single link robot with a flexible joint
The state space equations for the system of Figure 4 are

\[
\begin{align*}
\dot{x}_1 &= x_3(t) \\
\dot{x}_2 &= x_4(t) \\
\dot{x}_3 &= \frac{1}{I} (k(x_2(t) - x_1(t)) - mgL\sin(x_1(t))) \\
\dot{x}_4 &= \frac{1}{J} (u(t) - k(x_2(t) - x_1(t)))
\end{align*}
\tag{43}
\]

In order to apply the PDC methodology, the fuzzy Takagi-Sugeno Model is developed first (Seidi & Markazi, 2008). The nonlinear expression \( Z = \sin(x_1(t)) \), for \( x_1(t) \in [-\pi, \pi] \), can be expressed as

\[
z = \sin(x_1(t)) = \left( \sum_{i=1}^{2} M_i(z) b_i \right) x_1(t)
\tag{44}\]

Where, \( b_1 = 1, b_2 = 0 \) and, hence, the membership functions for \( z \) are obtained as

\[
M_1(z) = \begin{cases} 
    \frac{z}{\sin^{-1} z}, & z(t) \neq 0 \\
    1, & \text{Otherwise} 
\end{cases}
\]

\[
M_2(z) = \begin{cases} 
    \frac{\sin^{-1} z - z}{\sin^{-1} z}, & z(t) \neq 0 \\
    1, & \text{Otherwise} 
\end{cases}
\tag{45}\]

The resulting fuzzy model would then have the following fuzzy rules:

\begin{enumerate}
  \item Rule 1: If \( z(t) \) is \( M_1(z) \), then \( \dot{x}(t) = A_1x(t) + B_1u(t) \)
  \item Rule 2: If \( z(t) \) is \( M_2(z) \), then \( \dot{x}(t) = A_2x(t) + B_2u(t) \)
\end{enumerate}

Where,

\[
A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k - mgLb_1}{I} & \frac{k}{I} & 0 & 0 \\ \frac{k}{I} & \frac{-k}{I} & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k - mgLb_2}{I} & \frac{k}{I} & 0 & 0 \\ \frac{k}{I} & \frac{-k}{I} & 0 & 0 \end{bmatrix}, \tag{46}
\]

and

\[
B_1 = B_2 = B = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T.
\tag{47}\]
Assume $k = 100 \text{ Nm/rad}$, $g = 9.8 \text{ m/s}^2$ and other parameters are assumed unity then we have

$$
A_1 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-109.8 & 100 & 0 & 0 \\
100 & -100 & 0 & 0
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 100 & 0 & 0 \\
100 & -100 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix},
$$

Control Rule 1:
If $z(t)$ is $M_1(z)$, then $u(t) = -F_1 x(t)$

Control Rule 2:
If $z(t)$ is $M_2(z)$, then $u(t) = -F_2 x(t)$

The final output of the controller is

$$
u(t) = -\sum_{i=1}^{2} h_i F_i x(t) = h_1 F_1 x(t) + h_2 F_2 x(t)
$$

**Case 1: Stable controller design**

Using conditions (27) and (28) the stable controller can be obtained by solving below conditions

$$
X > 0
$$

$$
\begin{bmatrix}
-X A_1 - A_1 X + M_1^T B^T + B M_1 \\
-X A_2 - A_2 X + M_2^T B^T + B M_2 \\
-X A_1^T - A_1 X - X A_2^T - A_2 X + M_2^T B^T + B M_2 + M_1^T B^T + B M_1
\end{bmatrix} > 0,
$$

Using the MATLAB LMI Control Toolbox we obtain

$$
F_1 = [-495.76 \quad 668.96 \quad 14.112 \quad 47.388] \\
F_2 = [-497.23 \quad 671.34 \quad 14.356 \quad 47.552] \\
P = \begin{bmatrix}
42.1464 & -50.7108 & -1.5337 & -3.2007 \\
-50.7108 & 68.9721 & 2.4898 & 4.3456 \\
-1.5337 & 2.4898 & 0.2554 & 0.1719 \\
-3.2007 & 4.3456 & 0.1719 & 0.3527
\end{bmatrix}
$$

Figures 6 and 7 show the response of the system and control effort, respectively.

**Case 2: The decay rate**

Using conditions (31) and (32) the stable controller can be obtained by solving the conditions:
Figure 6. Response of flexible joint robots $x_1(t)$, case 1.

Figure 7. Control input for flexible joint robots, case 1.
\[ \begin{align*}
- X A_1^T & - A_1 X + 2X M_1 B^T + B M_1 - 2\alpha X > 0 \\
- X A_2^T & - A_2 X + 2X M_2 B^T + B M_2 - 2\alpha X > 0 \\
- X A_1^T & - A_1 X - X A_2^T - A_2 X + 2X M_2 B^T + B M_2 + 4\alpha X > 0
\end{align*} \] (52)

Considering \( \alpha = 10 \) and by using the MATLAB LMI Control Toolbox we obtain:

\[
F_1 = \begin{bmatrix} 4108.8 & 6545.2 & 1271.3 & 127.77 \end{bmatrix} \\
F_2 = \begin{bmatrix} 4066.9 & 6502.6 & 1261.7 & 127.1 \end{bmatrix} \\
\begin{bmatrix} 36.5087 & 24.0140 & 6.2135 & 0.3352 \\
24.0140 & 30.1341 & 6.3223 & 0.5013 \\
6.2135 & 6.3223 & 1.4260 & 0.0995 \\
0.3352 & 0.5013 & 0.0995 & 0.0099 \end{bmatrix} \\
P = \begin{bmatrix}
\end{bmatrix}
\] (53)

Figures 8 and 9 show the response of the system and control effort, respectively.

**Figure 8.** Response of flexible joint robots \( x_1(t) \), case 2.
Case 3: The decay rate with the constraint on the input

We design a stable fuzzy controller by considering the decay rate and the constraint on the control input. The design problem of the FJR is defined as follows:

Maximize $\alpha$

\[
X > 0 \\
\begin{bmatrix}
-XA_1^T - A_1X + M_1^TB^T + BM_1 - 2\alpha X \\
-XA_2^T - A_2X + M_2^TB^T + BM_2 - 2\alpha X \\
-XA_1^T - A_1X - XA_2^T - A_2X + M_1^TB^T + BM_2 + M_1^TB^T + BM_1 - 4\alpha X
\end{bmatrix} > 0
\]

\[
\begin{bmatrix}
X & M_1^T \\
M_1 & \mu^2I
\end{bmatrix} > 0
\]

\[
\begin{bmatrix}
X & M_2^T \\
M_2 & \mu^2I
\end{bmatrix} > 0
\]

\[
X - \phi^2I > 0
\]

(54)

Where $X = P^{-1}$, $M_1 = F_1X$, $\mu = 4600$, $\phi = 1$.

Using the MATLAB LMI toolbox to solve the LMI conditions (50), we can get the positive definite matrix and a set of gains (51), that make the system stable.

$\alpha = 0.072401$
\[
P = \begin{bmatrix}
0.7301 & 0.32486 & 0.096794 & 0.0034552 \\
0.32486 & 0.55483 & 0.10616 & 0.010209 \\
0.096794 & 0.10616 & 0.023049 & 0.0017139 \\
0.0034552 & 0.010209 & 0.0017139 & 0.00023565 \\
\end{bmatrix}
\]

\[
F_1 = [327.57 \quad 1745 \quad 261.86 \quad 57.475]
\]

\[
F_2 = [356.05 \quad 1739.2 \quad 259.77 \quad 57.5]
\]

Figures 10 and 11 show the response of the system and control effort, respectively.

\textbf{Figure 10.} System responses of the single-link flexible joint, case 3.

\textbf{Figure 11.} Control input for flexible joint robots, case 3.
Case 4: Performance-oriented parallel distributed compensation

The following stabilizing feedback gains are chosen using the pole placement method, so that $K_{11}$ and $K_{21}$ produce large magnitude inputs for subsystems 1 and 2, respectively, and $K_{22}$ and $K_{21}$ induce low magnitude inputs for those subsystems. In particular,

$$
K_{11} = \begin{bmatrix} 6667.2 & 4411.9 & 1052.4 & 92.6 \end{bmatrix} \\
K_{12} = \begin{bmatrix} -33.321 & 1413.7 & 191.63 & 51.2 \end{bmatrix} \\
K_{21} = \begin{bmatrix} 6658.7 & 4332.4 & 1025.4 & 91.1 \end{bmatrix} \\
K_{22} = \begin{bmatrix} 72.3 & 1389.8 & 189.6 & 50.6 \end{bmatrix}
$$

(56)

The required simple membership functions are selected as in Figure 12, so that, with a decrease in the corresponding plant input, in subsystems 1 and 2 respectively, the overall feedback gains come closer to $K_{11}$ and $K_{21}$, and with an increase in the corresponding control input respectively, the overall feedback gains come closer to $K_{21}$ and $K_{22}$. Now, the fuzzy rules for the controller are constructed as follows:

Rule 1: If $z(t)$ is $M_1(z)$ and $|u(t)|$ is "small" then $u(t) = K_{11} x(t)$

Rule 2: If $z(t)$ is $M_1(z)$ and $|u(t)|$ is "large" then $u(t) = K_{12} x(t)$

Rule 3: If $z(t)$ is $M_2(z)$ and $|u(t)|$ is "small" then $u(t) = K_{21} x(t)$

Rule 4: If $z(t)$ is $M_2(z)$ and $|u(t)|$ is "large" then $u(t) = K_{22} x(t)$

Figure 12. Membership functions for the control effort in the flexible joint robots.

A common positive definite matrix, $P$, satisfying the stability conditions (42) is obtained by solving the LMI problems:

$$
P = 10^4 \times \begin{bmatrix} 121710 & 15858 & 2558.5 & 63.525 \\ 15858 & 8624.4 & 1458.4 & 105.36 \\ 2558.5 & 1458.4 & 702.24 & 42.529 \\ 63.525 & 105.36 & 42.529 & 5.0962 \end{bmatrix}
$$
Applying a unit step reference signal for $x_1(t)$, the response history and the corresponding control input are shown in Figures (13) and (14), respectively. Simulation results are investigated for the following three controllers:

**Figure 13.** Response of flexible joint robot $x_1(t)$, case 4.

**Figure 14.** Control input for flexible joint robot, case 4.
1. A PDC controller with feedback gains $K_{11}$ and $K_{21}$ providing a high speed response, and with possible high control inputs (HPDC controller).

2. A PDC controller with feedback gains $K_{22}$ and $K_{21}$ providing a low speed response, and with a lower control input, as compared with the HPDC case (LPDC controller).

3. Proposed modified PDC controller, providing a fast response, yet with an acceptable level of control input (NPDC controller).

It is observed that the new controller provides a settling time similar to the HPDC case, with a much lower magnitude for the control input.

5. Conclusion

This chapter deals with approximation of the nonlinear system using Takagi-Sugeno (T-S) models with linear models as rule consequences and a construction procedure of T-S models. Also, the stability conditions and stabilizing control design of parallel distributed compensation (PDC) are discussed. It is seen that PDC a linear control method can be used to control the nonlinear system. Moreover, the stability analysis and control design problems for both continuous and discrete fuzz control systems can be transformed to linear matrix inequality (LMI) problems and they can be solved efficiently by convex programming techniques for LMIs. Design examples demonstrate the effectiveness of the LMI-based designs.

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6. References

Bonissone, P. P., Badami, V., Chaing, K.H., Khedkar, P.S., Marcelle, K. W. & Schutten, M. J. (1995). Industrial applications of fuzzy logic at general electric, Proceedings of IEEE, Vol. 83, No.3, (August 2002), p.p. 450-465, ISSN : 0018-9219.

Boyd, S., Ghaoui, L. E. & Feron, Eric & Balakrishnan V. (1994). Linear Matrix Inequalities in System and Control Theory, SIAM studies in applied mathematics, ISBN 0-89871-334X.

Chen, C. L., Chen, P. C. & Chen, C. K. ( 1993). Analysis and design of fuzzy control system, Fuzzy Sets and Systems, Vol. 57, No. 2, (July 1993) p.p. 125–140.

Ding, B. C., Sun, H. X. & Yang, P. (2006). Further study on LMI-based relaxed nonquadratic stabilization conditions for nonlinear systems in the Takagi–Sugeno’s form, Automatica, Vol. 42, No. 3, (March 2006) p.p. 503–508.

Fang, C. H., Liu, Y. S., Kau, S. W., Hong, L. & Lee, C. H. (2006). A new LMI-based approach to relaxed quadratic stabilization of T–S fuzzy control systems, IEEE Transactions on Fuzzy Systems, Vol. 14, No. 3, (June 2006), p.p. 386–397, ISSN : 1063-6706.

Hong, S.K. & Langari, R. (2000). Robust fuzzy control of a magnetic bearing system subject to harmonic disturbances, IEEE Transactions on Control System Technology, Vol. 8, No. 2, (August 2002), p.p. 366–371, ISSN : 1063-6536.
Ichikawa, A. et al. (1993). *Control Handbook*, Ohmu Publisher, Tokyo, in Japanese.

Joh, J., Langari, R., Jeung, E. & Chiuig, W. (1997). A new design method for continuous Takagi–Sugeno fuzzy controller with pole-placement constraints: an LMI approach, *Proceedings of IEEE International Conference on Systems, Man, and Cybernetics*, Vol. 3, p.p. 2969–2974, ISBN: 0-7803-4053-1, Orlando, Florida, USA, October 12-15, 1997.

Kawamoto, S., Tada, K., Ishigame A. & Taniguchi, T. (1992). An Approach to Stability Analysis of Second Order Fuzzy Systems, *Proceedings of First IEEE International Conference on Fuzzy Systems*, Vol. 1, pp. 1427-1434, ISBN: 0-7803-0236-2, San Diego, California, USA, March 8-12, 1992.

Kim, E., Lee, H. (2000). New approaches to relaxed quadratic stability conditions of fuzzy control systems, *IEEE Transactions on Fuzzy Systems*, Vol. 8, No. 5, p.p. 523–534, (August 2002), ISSN : 1063-6706.

Li, J., Niemann, D. & Wang, H. O. (1998). Robust tracking for high-rise/high-speed elevators, *Proceedings of American Control Conference*, Vol.6, p. 3445-3449, ISBN: 0-7803-4530-4, Philadelphia, Pennsylvania, USA, June 24-26, 1998.

Lee, H.J., Park, J.B. & Joo, Y.H. (2006). Robust load–frequency control for uncertain nonlinear power systems: a fuzzy logic approach, *Information Sciences*, Vol. 176, No.23, (December 2006), p.p. 3520–3537.

Lee, K.R., Jeung, E.T. & Park, H.B. (2001). Robust fuzzy H∞ control for uncertain nonlinear systems via state feedback: an LMI approach, *International Journal of Fuzzy Sets and Systems*, Vol. 120, No.1 (May 2001) p.p. 123–134.

Lin, Y.C., & Lo, J.C. (2003). Robust H∞ fuzzy control via dynamic output feedback for discrete-time systems, *Proceedings of IEEE International Conference on Fuzzy Systems*, Vol. 2, p.p. 1384–1388, ISBN: 0-7803-7810-5, May 25-28, 2003.

Kang, G. & Lee, W. (1998). Design of TSK fuzzy controller based on TSK fuzzy model using pole-placement, *Proceedings of IEEE International Conference on Fuzzy Systems*, Vol. 1, p.p. 246–251, ISSN : 1098-7584, Anchorage, Alaska, USA, May 4-9, 1998.

Rhee, B. J. & Won, S. (2006). A new Lyapunov function approach for a Takagi–Sugeno fuzzy control system design. *Journal of Fuzzy Sets System*, Vol. 157, No. 9, (May 2006), p.p. 1211–1228.

Seidi, M. & Markazi, A. H. D. (2008). Model-Based Fuzzy Control of Flexible Joint Manipulator: A LMI Approach, *Proceedings of the 5th International IEEE Symposium on Mechatronics and its Applications*, ISBN: 978-1-4244-2033-9, Amman, Jordan, May 27-29, 2008.

Seidi M. & Markazi, A. H. D. (2011). Performance-oriented parallel distributed compensation ,Vol. 348, No. 7, (September 2011), pp. 1231–1244.

Spong, M. W., Khorasani, K. & Kokotovic, P. V. (1987). " An integral manifold approach to the feedback control of flexible joint robots", *IEEE Journal of Robotics and Automation*, Vol.3, No.4, (January 2003), p.p. 291-300, ISSN : 0882-4967,.

Sugeno, M. & Kang , G. T. (1986). Fuzzy Modeling and Control of Multilayer Incinerator, *Journal of Fuzzy Sets Systems*, Vol. 18, No. 3, (April 1986), p.p. 329-345.

Takagi, T., & Sugeno, M. (1985). Fuzzy identification of systems and its applications to modeling and control, *IEEE Transactions on Systems Man and Cybernetics*, Vol. 15, NO.1, (February 1985), p.p.116–132.
Tanaka, K. & Sugeno, M. (1990). Stability Analysis of Fuzzy Systems Using Lyapunov's Direct Method, *Proceedings of the North America Fuzzy Information Processing Society NAFIPS’90*, Vol. 1, pp. 133-136, Toronto, Canada, June 1990.

Tanaka, K. & Sugeno M. (1992). Stability analysis and design of fuzzy control systems, *Journal of Fuzzy Sets and Systems*, Vol.45, No.2, (January 1992), p.p.135–156.

Tanaka, K. & Sano M. (1994). A robust stabilization problem of fuzzy control systems and its applications to backing up control of a truck trailer, *IEEE Transactions on Fuzzy Systems*, Vol. 2, No. 5, (August 2002), p.p. 119-134, ISSN : 1063-6706.

Tanaka, K., Ikeda, T. & Wang, H. O. (1996). Design of fuzzy control systems based on relaxed LMI stability conditions, *Proceedings of 35th IEEE Conference on Decision and Control*, Vol. 1, pp. 598-603, ISBN: 0-7803-3590-2, Kobe, Japan, December 11-13 1996.

Tanaka, K. & Kosaki, T. (1997). Design of a Stable Fuzzy Controller for an Articulated Vehicle, *IEEE Transactions Systems, Man and Cybernetics*, Vol. 27, No. 3, (August 2002), p.p. 552 – 558, ISSN : 1083-4419.

Tanaka, K., Ikeda, T. & Wang, H. O. (1998). Fuzzy regulator and fuzzy observer: Relaxed stability conditions and LMI-based designs, *IEEE Transactions on Fuzzy Systems*, Vol. 6, No. 2, (August 2002), p.p. 250-265, ISSN : 1063-6706.

Tanaka, K., Taniguchi, T. & Wang, H. O. (1998a). Model-Based Fuzzy Control of TORA System: Fuzzy Regulator and Fuzzy Observer Design via LMIs that Represent Decay Rate, Disturbance Rejection, Robustness, Optimality, *Proceedings of Seventh IEEE International Conference on Fuzzy Systems*, pp. 313-318, Alaska, USA, May 4-9 1998.

Tanaka, K., Ikeda, T., & Wang, H. O. (1998b). Fuzzy Regulators and Fuzzy Observers: relaxed stability conditions and LMI-based designs, *IEEE Transactions on Fuzzy Systems*, Vol. 6, No. 2, ( August 2002), pp. 250-265, ISSN : 1063-6706.

Tanaka, K. &Wang, H.O. (2001). Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach, 1st Edition, Wiley.

Tanaka, K., Hori, T. & Wang, H. O. (2003). A multiple Lyapunov function approach to stabilization of fuzzy control systems, *IEEE Transactions on Fuzzy Systems*, Vol. 11, No. 4, (August 2003), p.p. 582–589, ISSN : 1063-6706.

Ting, C.S. (2006). Stability analysis and design of Takagi–Sugeno fuzzy systems, *Journal of Information Sciences*, Vol. 176, No.19, (October 2006), p.p. 2817-2845.

Wang, H. O., Tanaka, K. & Griffin, M. F. (1995) "Parallel Distributed Compensation of Nonlinear Systems by Takagi-Sugeno Fuzzy Model, *Proceedings of International Joint Conference of the Fourth IEEE International Conference on Fuzzy Systems and The Second International Fuzzy Engineering Symposium*, p.p. 531-538, ISBN: 0-7803-2461-7, Yokohama, Japan, March 20-24, 1995.

Wang, H. O., Tanaka, K. & Griffin M. F. (1996). An approach to fuzzy control of nonlinear systems: Stability and the design issues. *IEEE Transactions on Fuzzy Systems*, Vol. 4, No. 1, (August 2002), p.p. 14–23, ISSN : 1063-6706.

Wang, L. X., & Mendel, J. M. (1992). Fuzzy basis functions, universal approximation, and orthogonal least-squares learning, *IEEE Transactions on Neural Networks*, Vol. 3, No. 5, (August 2002), p.p. 807–814, ISSN : 1045-9227.

Zhao, J., Wertz, V. & Gorez, R. (1996). Fuzzy gain scheduling controllers based on fuzzy models, *Proceedings of The 5th IEEE International Conference on Fuzzy Systems*, Vol. 3, No. 8, p.p. 1670–1676, ISBN: 0-7803-3645-3, New Orleans, Louisiana, USA, September 8-11, 1996.