A COUNTEREXAMPLE TO A MULTILINEAR ENDPOINT QUESTION OF CHRIST AND KISELEV

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Abstract. Christ and Kiselev [2],[3] have established that the generalized eigenfunctions of one-dimensional Dirac operators with $L^p$ potential $F$ are bounded for almost all energies for $p < 2$. Roughly speaking, the proof involved writing these eigenfunctions as a multilinear series $\sum_n T_n(F,\ldots,F)$ and carefully bounding each term $T_n(F,\ldots,F)$. It is conjectured that the results in [3] also hold for $L^2$ potentials $F$. However in this note we show that the bilinear term $T_2(F,F)$ and the trilinear term $T_3(F,F,F)$ are badly behaved on $L^2$, which seems to indicate that multilinear expansions are not the right tool for tackling this endpoint case.

1. Introduction

Let $F(x)$ be a real potential on $\mathbb{R}$. For each energy $k^2 > 0$ we can consider the Dirac generalized eigenfunction equation

$$(\frac{d}{dx} + F)(-\frac{d}{dx} + F)\phi(x) = k^2 \phi(x)$$

on $\mathbb{R}$. This Dirac equation can be thought of as a Schrödinger equation with potential $V = F'' + F^2$. For each $k$ there are two linearly independent eigenfunctions $\phi = \phi_k$. A natural question from spectral theory is to ask whether these eigenfunctions are bounded (i.e. are in $L^\infty$) for almost every real $k$. In [3] Christ and Kiselev showed among other things that this was true when $F \in L^p_x$ for any $1 \leq p < 2$. It is well known (see e.g. [12]) that the statement fails when $p > 2$, but the $p = 2$ case remains open. In [5] it is shown that for $L^2$ potentials one has absolutely continuous spectrum on $[0,\infty)$, but this is a slightly weaker statement.

We briefly outline the arguments in [2],[3]. The method of variation of constants suggests the ansatz

$$\phi(x) = a(x)e^{ikx} + b(x)e^{-ikx}$$

$$(\frac{d}{dx} + F)\phi(x) = -ika(x)e^{ikx} + ikb(x)e^{-ikx}.$$

Substituting this into the previous and simplifying, we reduce to the first-order system

$$a'(x) = F(x)e^{-2ikx}b(x)$$
$$b'(x) = F(x)e^{2ikx}a(x).$$

For simplicity we may assume $F$ is supported on the positive half axis. If we set initial conditions $a(-\infty) = 1$, $b(-\infty) = 0$ for instance, and then solve this system by iteration,
we thus obtain the formal multilinear expansions
\[ a = 1 + \sum_{n \geq 2, \text{even}} T_n(F_1, \ldots, F); \quad b = \sum_{n \geq 1, \text{odd}} T_n(F_1, \ldots, F) \]
where for each \( n \geq 1 \), \( T_n \) is the \( n \)-linear operator
\[ T_n(F_1, \ldots, F_n)(k, x) := \int_{x_1 < \cdots < x_n < x} e^{-2ik\sum_{j=1}^{n}(-1)^jx_j} F_1(x_1) \cdots F_n(x_n) \, dx_1 \cdots dx_n. \]

For integrable \( F_j \) we can define the \( n \)-linear operators
\[ T_n(F_1, \ldots, F_n)(k, +\infty) := \int_{x_1 < \cdots < x_n} e^{-2ik\sum_{j=1}^{n}(-1)^jx_j} F_1(x_1) \cdots F_n(x_n) \, dx_1 \cdots dx_n. \]

The strategy of Christ and Kiselev was then to control each individual expression \( T_n \) on \( L^p \). Specifically, they showed the estimate
\[ \| \sup_x |T_n(F, \ldots, F)(k, x)| \|_{L^p_k/L^{n,\infty}} \leq C_{p,n} \| F \|_{L^p_k}^n \]
(1)
for all \( n \geq 1 \) and \( 1 \leq p < 2 \), where \( C_{p,n} \) was a constant which decayed rapidly in \( n \) and \( 1/p + 1/p' = 1 \). In particular one has the non-maximal variant
\[ \| T_n(F_1, \ldots, F_n)(k, +\infty) \|_{L^{p'}_k/L^{n,\infty}} \leq C_{p,n} \| F \|_{L^p_k}^n. \]
(2)

The boundedness of eigenfunctions for almost every \( k \) then follows by summing these bounds carefully.

It is tempting to try this approach for the endpoint \( p = 2 \). For \( n = 1 \) we see that \( T_1(F)(k, +\infty) \) is essentially the Fourier co-efficient \( \hat{F}(k) \), while \( \sup_k |T_1(F)(k, x)| \) is essentially the Carleson maximal operator \( CF(k) \). The estimates (2), (1) for \( p = 2 \) then follow from Plancherel’s theorem and the Carleson-Hunt theorem \([4, 5]\) respectively.

For \( n = 2 \) the expression \( T_2(F,F)(k, +\infty) \) is essentially \( H_{-\infty}(\hat{F})^2(k) \), where \( H_{-\infty} \) is the Riesz projection
\[ \overline{H_{-\infty}F} := \chi_{(-\infty,0]} \hat{F}, \]
and so (2) follows for \( p = 2 \) by Hölder’s inequality and the weak-type \((1,1)\) of the Riesz projections. We also remark that if the phase function \( x_1 - x_2 \) in the definition of \( T_2 \) were replaced by \( \alpha_1 x_1 + \alpha_2 x_2 \) for generic numbers \( \alpha_1, \alpha_2 \) then the operator is essentially a bilinear Hilbert transform and one still has boundedness from the results in \([7, 8, 13]\).

It may thus appear encouraging to try to estimate the higher order multilinear operators for \( L^2 \) potentials \( F \). However, in this note we show

**Theorem 1.1.** When \( p = 2 \) and \( n = 2 \), the estimate (2) fails. When \( p = 2 \) and \( n = 3 \), the estimate (2) fails.

Because of this, we believe that it is not possible to prove the almost everywhere boundedness of eigenfunctions for Dirac or Schrödinger operators with \( L^2 \) potential purely by multilinear expansions; we discuss this further in the remarks section.

The counterexample has a logarithmic divergence, and essentially relies on the fact that while convolution with the Hilbert kernel \( p.v. \frac{1}{x} \) is bounded, convolution with \( \frac{\text{sgn}(x)}{x} \) or \( \frac{\chi_{(-\infty,0]}(x)}{x} \) is not. It may be viewed as an assertion that \( L^2 \) potentials create significant long-range interaction effects which are not present for more rapidly decaying potentials.
Interestingly, our counterexamples rely strongly on a certain degeneracy in the phase function \( \sum_j (-1)^j x_j \) on the boundary of the simplex \( x_1 < \ldots < x_n \). If one replaced this phase by \( \sum_j x_j \), then we have shown in [9], [10] that the bound (2) in fact holds when \( p = 2 \) and \( n = 3 \). Indeed this statement is true for generic phases of the form \( \sum_j \alpha_j x_j \). A similar statement holds for (1) when \( p = 2 \) and \( n = 2 \) and will appear elsewhere.

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2. Proof of Theorem 1.1

The letter \( C \) may denote different large constants in the sequel. To be consistent with the previous notation we shall define the Fourier transform as

\[
\hat{F}(k) := \int e^{-2ikx} F(x) \, dx.
\]

We let \( N \gg 1 \) be a large integer parameter, which we shall take to be a square number, and test (1), (2) with the real-valued potential

\[
F(x) := \sum_{j=N}^{2N} F_j(x)
\]

where the \( F_j \) are given by

\[
F_j(x) := N^{-1} \cos(2A_j x) \phi(\frac{x}{N} - j),
\]

\( \phi \) is a smooth real valued function supported in \([-\frac{1}{4}, \frac{1}{4}]\) with total mass \( \int \phi = 1 \) such that \( \hat{\phi} \) stays away from 0 in \([-1, 1]\), and \( A \) is a sufficiently large absolute constant whose purpose is to ensure that

\[
4 \sum_{j \in \mathbb{Z} \setminus \{0\}} \left| \hat{\phi}(\xi - A j) \right| \leq |\hat{\phi}(\xi)|
\]

for \( \xi \in [-1, 1] \). Informally, \( F \) is a “chirp” which is localized in phase space to the region

\[
\{(k, x) : k = \pm \frac{A_j}{N} + O(\frac{1}{N}); x = N j + O(N), N \leq j \leq 2N\}.
\]

We may compute the Fourier transform of the \( F_j \) using the rapid decay of \( \hat{\phi} \) as

\[
\hat{F}_j(k) = \frac{1}{2} e^{-2i(Nk - A j)j} \hat{\phi}(Nk - A j) + O(N^{-200}) \tag{3}
\]

in the region \( \frac{A}{2} < k < 3A \). We remark that the error term \( O(N^{-200}) \) has a gradient which is also \( O(N^{-200}) \).

Clearly we have \( \|F_j\|_2 = O(N^{-1/2}) \), and hence that \( \|F\|_2 = O(1) \).
We now compute

\[ T_2(F, F)(k, x) = \int_{x_1 < x_2 < x} e^{2ik(x_1 - x_2)} F(x_1)F(x_2) \, dx_1 dx_2 \]  

in the region

\[ |Nk - Aj_0| \leq 1; \quad x = N(j_0 - \sqrt{N} + \frac{1}{2}) \]  

for some integer \( \frac{3N}{2} < j_0 < 2N \). In this region we show that

\[ |T_2(F, F)(k, x)| \geq C^{-1} \log N, \]  

which will imply that

\[ \| \sup_x |T_2(F, F)(k, x)| \|_{L^2, \infty} \geq C^{-1} \log N \]  

and thus contradict (1) for \( n = 2 \) and \( p = 2 \) by letting \( N \) go to infinity.

We now prove (6). Fix \( k, j_0, x \). Observe from (4) that \( T_2(F_j, F_{j'})(k, x) \) vanishes unless \( j \leq j' \leq j_0 - \sqrt{N} \). Thus we may expand

\[ T_2(F, F)(k, x) = \sum_{N \leq j \leq j_0 - \sqrt{N}} T_2(F_j, F_j)(k, x) + \sum_{N \leq j < j' \leq j_0 - \sqrt{N}} T(F_j, F_{j'})(k, x). \]  

We first dispose of the error term (8). In the region \( j < j' \leq j_0 - \sqrt{N} \), the conditions \( x_1 < x_2 < x \) in (4) become superfluous, so we may factor

\[ T_2(F_j, F_{j'})(k, x) = \hat{F}_j(k) \hat{F}_{j'}(k). \]

However, since \( \hat{\phi} \) is rapidly decreasing and \( |j - j_0|, |j' - j_0| \geq \sqrt{N} \), we see from (3) that

\[ |\hat{F}_j(k)|, |\hat{F}_{j'}(k)| \leq CN^{-100}. \]

Summing this, we see that the total contribution of (8) is \( O(N^{-198}) \).

Now we consider the contribution of (7). We use the identity

\[ T_2(F_j, F_j)(k, x) = T_2(F_j, F_j)(k, +\infty) = H_-(|\hat{F}_j|^2)(k) \]  

combined with (3). The operator \( H_- \) is a non-trivial linear combination of the identity and the Hilbert transform, while \( |\hat{F}_j|^2 \) is essentially a non-negative bump function rapidly decreasing away from the interval \( [jA/N - O(1/N), jA/N + O(1/N)] \). Because of this we see that for \( j \neq j_0 \) we have

\[ H_-(|\hat{F}_j|^2)(k) = \frac{c}{j - j_0} + O(|j - j_0|^{-2}) \]  

where \( c \) is a non-zero absolute constant. Summing this over all \( j \leq j_0 - \sqrt{N} \) and observing that \( j - j_0 \) has a consistent sign we see that the contribution of (7) has magnitude at least \( C^{-1} \log N \), and (6) follows.

We now compute \( T_3(F, F, F)(k, +\infty) \) in the region

\[ |Nk - Aj_0| \leq 1; \quad 1.4N < j_0 < 1.6N. \]
We will show that
\[ |T_3(F, F, F)(k, +\infty)| \geq C^{-1} \log N \] (12)
in this region, which will disprove (4) for \( n = 3 \) and \( p = 2 \) similarly to before.

It remains to prove (12). Fix \( j_0 \). Observe that \( T_3(F_j, F_{j'}, F_{j''})(k, +\infty) \) vanishes unless \( j \leq j' \leq j'' \). Thus we can split
\[ T_3(F, F, F)(k, +\infty) = \sum_{N \leq j \leq 2N} T_3(F_j, F_j, F_j)(k, +\infty) \] (13)
\[ + \sum_{N \leq j < j' \leq 2N} T_3(F_j, F_j, F_{j'})(k, +\infty) \] (14)
\[ + \sum_{N \leq j' < j \leq 2N} T_3(F_{j'}, F_j, F_j)(k, +\infty) \] (15)
\[ + \sum_{N \leq j < j' < j'' \leq 2N} T_3(F_j, F_{j'}, F_{j''})(k, +\infty). \] (16)

We first consider (13). We expand
\[ T_3(F_j, F_j, F_j)(k, +\infty) = \int_{x_1 < x_2 < x_3} e^{2ik(x_1-x_2+x_3)}F_j(x_1)F_j(x_2)F_j(x_3) \, dx_1dx_2dx_3. \]
This is a linear combination of eight terms of the form
\[ N^{-3} \int_{x_1 < x_2 < x_3} e^{2ik(x_1-x_2+x_3)}e^{2i\frac{\lambda}{N}(\pm x_1 \pm x_2 \pm x_3)} \phi(x_1-j)\phi(x_2-j)\phi(x_3-j) \, dx_1dx_2dx_3; \]
making the substitutions \( y_s := \frac{x_s}{N} - j \) for \( s = 1, 2, 3 \), this becomes
\[ e^{i\theta} \int_{y_1 < y_2 < y_3} e^{2ikN(y_1-y_2+y_3)}e^{2iA_j(y_1+y_2+y_3)} \phi(y_1)\phi(y_2)\phi(y_3) \, dy_1dy_2dy_3 \]
for some phase \( e^{i\theta} \) depending on all the above variables.

We shall only consider the choice of signs \((-y_1 + y_2 - y_3)\); the reader may easily verify that the other choices of signs are much smaller thanks to stationary phase. In this case we can write the above as
\[ e^{i\theta} \int_{y_1 < y_2 < y_3} e^{2i(kN-A_j)(y_1-y_2+y_3)} \phi(y_1)\phi(y_2)\phi(y_3) \, dy_1dy_2dy_3. \]
If \( kN - A_j = O(1) \) we estimate this crudely by \( O(1) \). Otherwise we can perform the \( y_1 \) integral using stationary phase to obtain
\[ e^{i\theta} \frac{1}{2i(kN-A_j)} \int_{y_2 < y_3} e^{2i(kN-A_j)y_3} \phi(y_2)\phi(y_3) \, dy_2dy_3 + O(|kN-A_j|^{-2}). \]
Performing another stationary phase we see that this quantity is \( O(|kN-A_j|^{-2}) \). Summing over all \( j \) we see that (13) is \( O(1) \).

Let us now consider (16). When \( j < j' < j'' \), the constraints \( x_1 < x_2 < x_3 \) in the definition of \( T_3 \) are redundant, and we can factorize
\[ T_3(F_j, F_{j'}, F_{j''})(k, +\infty) = \overline{F_j(k)}\overline{F_{j'}(k)}\overline{F_{j''}(k)}. \]
Applying (3) and using the rapid decay of \( \phi \) we see that

\[
|T_3(F_j, F_{j'}, F_{j''})(k, +\infty)| \leq C(1 + |j - j_0| + |j' - j_0| + |j'' - j_0|)^{-10} + CN^{-100}.
\]

Summing over all \( j, j', j'' \) we see that (16) is \( O(1) \).

It remains to control (15) + (14). First we consider (14). For this term the condition \( x_2 < x_3 \) is redundant, so we can factorize

\[
T_3(F_j, F_{j'}, F_{j'})(k, +\infty) = T_2(F_j, F_j)(k, +\infty)\hat{F}_{j'}(k).
\]

Now consider (15). For this term the condition \( x_1 < x_2 \) is the dominant contribution to (17). Using (3) as in (16) we see that (15) is

\[
|T_3(F_j, F_j, F_j)(k, +\infty) - T_2(F_j, F_j)(k, +\infty)\hat{F}_{j'}(k)| \leq CN^{-100}.
\]

We claim these terms are the dominant contribution to (17). We consider the terms with \( j = j_0 \). We claim these terms are the dominant contribution. From (3), (10) we conclude

\[
\sum_{N \leq j, j' \leq 2N} \sgn(j' - j)T_2(F_j, F_j)(k, +\infty)\hat{F}_{j'}(k) + \sum_{N \leq j' < j \leq 2N} \hat{F}_{j'}(k)|\hat{F}_j(k)|^2.
\]

Using (3) as in (16) we see that (16) is \( O(1) \), so to prove (12) it will suffice to show

\[
\sum_{N \leq j, j' \leq 2N, j \neq j'} \sgn(j' - j)T_2(F_j, F_j)(k, +\infty)\hat{F}_{j'}(k) \geq C^{-1} \log N. \tag{17}
\]

We first consider the terms with \( j' = j_0 \). We claim these terms are the dominant contribution. From (3), (10) we conclude

\[
\sum_{N \leq j \leq 2N, j \neq j_0} \sgn(j_0 - j)T_2(F_j, F_j)(k, +\infty)\hat{F}_{j_0}(k)
\]

\[
= \sum_{N \leq j \leq 2N, j \neq j_0} c \frac{\sgn(j_0 - j)}{j_0 - j} \hat{F}_{j_0}(k) + O(1). \tag{18}
\]

Here \( c \) is the same non-zero constant as in (10), and \( \hat{F}_{j_0}(k) \) is bounded away from 0 by choice of \( \phi \). Thus the first term is greater than \( C^{-1} \log N \), so it suffices indeed to show that this term is the dominant contribution to (17).

We consider the terms with \( j = j_0 \). Using that \( |T_2(F_j, F_j)(k, +\infty)| \leq C \) we obtain

\[
\sum_{N \leq j' \leq 2N, j_0 \neq j'} |T_2(F_{j_0}, F_{j_0})(k, +\infty)\hat{F}_{j'}(k)| \leq C.
\]

This term is therefore negligible.

Finally, we have to consider the terms with \( j, j' \neq j_0 \). We have by the choice of \( A \),

\[
\sum_{N \leq j, j' \leq 2N, j, j' \neq j_0} |T_2(F_j, F_j)(k, +\infty)|\hat{F}_{j'}(k)| \leq \frac{1}{2} \sum_{N \leq j \leq 2N, j \neq j_0} \frac{c}{|j - j_0|}|\hat{F}_{j_0}(k)| + C
\]
3. Remarks

- The counterexample can easily be extended to larger $n$ (e.g. by appending some bump functions to the left or right of $F$).
- The counterexample above involved a potential $F$ which was bounded in $L^2$, but for which $\sup_x |T_2(F, F)(k, x)|$ and $|T_3(F, F, F)(k, +\infty)|$ were large (about $\log N$) on a large subset of $[A, 2A]$. By letting $N$ vary and taking suitable linear combinations of such variants of the above counterexample, one can in fact generate a potential $F$ bounded in $L^2$ for which $\sup_x |T_2(F, F)(k, x)|$ and $|T_3(F, F, F)(k, x)|$ accumulate at $\infty$ for $x \to \infty$ for all $k$ in a set of positive measure (one can even achieve blow-up almost everywhere). Thus it is not possible to estimate these multilinear expansions in any reasonable norm if one only assumes the potential to be in $L^2$.
- Similarly if $F$ had a derivative in $L^2$; it is the decay of $F$ which is relevant here, not the regularity.
- The unboundedness of $T_3$ on $L^2$ can be interpreted as stating that the (non-linear) scattering map $F \mapsto b_k(+\infty)$ from potentials to reflection coefficients is not $C^3$ on the domain of $L^2$ potentials. Similarly the map $F \mapsto a_k(+\infty)$ from potentials to transmission coefficients is not $C^4$ on the domain of $L^2$ potentials. In particular these scattering maps are not analytic.
- Despite the bad behavior of the individual terms $T_k(F, \ldots, F)$, the transmission and reflection coefficients $a_k(x), b_k(x)$ are still bounded for the counterexample given above. This phenomenon is similar to the observation that the function $e^{ix} = 1 + ix - x^2/2 - \ldots$ is bounded for arbitrarily large real $x$, even if the individual terms $(ix^n/n!$ are not.

We now sketch the proof of boundedness of $a_k, b_k$. Suppose that $k = A j_0/N + O(1/N)$ for some $N \leq j_0 \leq 2N$; we now fix $j_0$ and $k$. We can write

\[
\begin{pmatrix}
a_k(x) \\
b_k(x)
\end{pmatrix} = G(x) \begin{pmatrix}1 \\ 0\end{pmatrix}
\]

where $G$ is the $2 \times 2$ matrix solving the ODE

\[
G'(x) = \begin{pmatrix}0 & F(x) e^{-2ikx} \\ F(x) e^{2ikx} & 0\end{pmatrix} G(x); \quad G(-\infty) = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}.
\]

We define the matrices $G_j$ similarly by

\[
G'_j(x) = \begin{pmatrix}0 & F_j(x) e^{-2ikx} \\ F_j(x) e^{2ikx} & 0\end{pmatrix} G_j(x); \quad G_j(-\infty) = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}.
\]

We observe the identity

\[
G(x) = G_{j_1}(+\infty) G_{j_1-1}(+\infty) \ldots G_N(+\infty)
\]

whenever $N \leq j_1 \leq 2N$ and $x = N(j_1 + \frac{1}{2})$; this can be proven by an easy induction on $j_1$ and the observation that the above ODE are invariant under right-multiplication.
One can compute the $G_j(+\infty)$ using multilinear expansions (or using Gronwall’s inequality), eventually obtaining
\[
G_j(+\infty) = \left( 1 + \frac{ic}{j-j_0} \begin{array}{c} 0 \\ 1 - \frac{ic}{j-j_0} \end{array} \right) + O(|j-j_0|^{-2})
\]
for all $j \neq j_0$, where $C$ is a non-zero real constant. Because of the crucial factor of $i$ in the diagonal entries we see that the operator norm $\|G_j(+\infty)\|$ of $G_j$ is
\[
\|G_j(+\infty)\| = 1 + O(|j-j_0|^{-2}).
\]
This allows one to multiply the $G_j(+\infty)$ together and obtain boundedness of $G(x)$ and hence $a_k(x), b_k(x)$.

In analogy with the observation concerning $e^{ix}$, one may need to use the fact that $F$ is real in order to obtain boundedness of eigenfunctions in the $L^2$ case. When $F$ is real there are additional estimates available, such as the scattering identity
\[
\int \log |a_k(+\infty)| \, dk = C \int |F(x)|^2 \, dx
\]
for some absolute constant $C$; see for instance [3].

We do not yet know how to obtain boundedness of eigenfunctions for $L^2$ potentials $F$. However we have been able to achieve this for a model problem in which the Fourier phases $e^{2i(k,z)}$ are replaced by a dyadic Walsh variant $e(k,x)$. See [1].

- One can modify the counterexample to provide similar counterexamples for Schrödinger operators $-\frac{d^2}{dx^2} + V$ with $V \in L^2$, either by using the Miura transform $V = F' + F^2$ mentioned in the introduction, or by inserting the standard WKB phase modification to the operators $T_k$ as in [3]. We omit the details.
- The multilinear expansion of $a$ leads to an expansion of $|a|^2$, whose quadratic term is equal to
\[
2Re(T_2(F,F)) = 2Re(H_-(|\hat{F}|^2)) = |\hat{F}|^2
\]
This term is in $L^1$, which is better than the term $T_2(F,F)$, which is in general only in the Lorentz space $L^{1,\infty}$. The higher order terms of the expansion of $|a|^2$ are however unbounded again. Using the identity $|a|^2 = 1 + |b|^2$ we see that the fourth order term of $|a|^2$ is equal to
\[
2Re(T_1(F)T_3(F,F,F))
\]
We now define the modified potential
\[
G(x) = F(x) + G_0(x)
\]
where $F$ is as in the proof of Theorem [1] and $G_0(x) = \phi(x - N^3)$. Expanding the fourth order term by multilinearity, one observes that all terms can be estimated from above nicely with the exception of
\[
2Re(T_1(G)T_3(F,F,F))
\]
Since $T_1(G) = \hat{G}$ has more rapidly changing phase than $T_3(F,F,F)$, the real part and the modulus $\overline{T_1(G)}T_3(F,F,F)$ are of comparable size on a large set, and so this term is of the order $\log(N)$ on a large set just like $T_3(F,F,F)$ itself.
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