Influence of lacing bars on the buckling capacity of four-legged latticed columns considering geometric imperfections

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Abstract
The buckling behavior of latticed columns had been widely investigated based on the theory of Euler, Engesser and Timoshenko shear beam. Although these methods had been formulated and proved to be accurate in case of special assumptions, the influences of lacing bars on the buckling behavior of latticed columns were unclear. This paper modeled a general four-legged latticed column to study the influence of the cross-section characteristics of lacing bars along with their imperfections on the buckling capacity of latticed columns. Three loading conditions and four geometric imperfect models were built to testify the performance of lacing bars. To calculate the buckling load of latticed columns with imperfections accurately, advanced nonlinear analytical procedures using Newton-Raphson incremental-iterative method (ANAP-NR) and Risk arc-length incremental-iterative method (ANAP-Risk) were developed, and then validated by FE software ABAQUS. The current data in the paper show the maximum variation on the critical buckling load of latticed columns, caused by the cross-section area, the bending moment of inertia outer lacing plane, and the imperfections of lacing bars, could reach 68%, 30%, and 25%. The analytical results indicate the great importance of lacing bars on the buckling capacity of latticed columns.

Keywords
Latticed columns, buckling, geometric imperfections, nonlinear buckling analysis, FEA

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Introduction

Latticed columns have extensive applications in structural engineering, such as electricity industry, factory buildings, bridges, tower cranes, partly seen in Figures 1 and 2, providing economical solutions in case of large span and heavy load. Composed of legs (chords/flanges) and lacings, the latticed columns can be generally grouped into two main categories: laced and battened latticed columns decided by the type of lacings. The present work is to investigate the first one with four legs, whose lacings consist of transverse and diagonal bars (Figure 3).

Nearly all the present researches on the buckling capacity of latticed columns were based on the theory of Euler, Engesser or Timoshenko shear beam. Two main issues based on those theories were the buckling modes and the influence of nonlinear factors. The buckling modes included flexural buckling, torsional buckling, flexural-torsional bucking, and their interactions of global and local buckling modes. Razdolsky\(^1\)\(^{-7}\) proposed a statically indeterminate method to calculate Euler elastic stability with pure flexural and twisting buckling loads of latticed column. Different latticed columns with fir-shaped lattice, crosswise lattice, serpentine lattice and battened lattice were investigated and contrasted with the results of Eurocode 3 (EC3).\(^8\) These analytical results showed the design code EC3 was in need of revisions because of its higher values in forecasting the critical buckling load of latticed columns. Channel battened columns were studied by Ren et al.\(^9\) using piecewise cubic Hermit interpolation (PCHI) to express the rotational displacement function of column during flexural-torsional buckling deformation. It found that flexural-torsional buckling was prior to happen than pure flexural buckling for channel-section columns, which would be affected by the geometric characteristics and eccentricities.

Another issue is the influence of the nonlinear factors, such as shear deformations and initial geometric imperfections. Kalochairetis and Gantes\(^10\) investigated the interact influence of initial global and local geometric imperfections on the collapse buckling load of I-section laced columns. By replacing the bending rigidity with an effective value of the imperfections, the buckling calculation of latticed columns with imperfections becomes formulaic and convenient. Then transversely loads and arbitrary supports of laced columns were studied in Gantes and Kalochairetis.\(^11\) Approximate second-order analysis of imperfect Timoshenko member considering the above two issues was conducted on different boundary and load conditions of I-section laced columns. Li et al.\(^12\) deduced the strain energy expressions of three typical lacing bars systems, X-lacing, K-lacing, and E-lacing based on the potential energy functions for laced columns. The interactions of global and local imperfections were concerned to track the buckling equilibrium path of load-displacement curves.

In those theories of Euler, Engesser, and Timoshenko shear beam, a common assumption is that the bars or battens behave as a shear panel continuously connected to chords. Although it is simplified to consider lacings and chords of the latticed columns as an indivisible component, there exist two problems in the analytical process. One is the forecasting of the collapse buckling load was
inaccurate, usually in upper limit due to the assumption. The other is that the influence of geometrical characteristics of the individual bars or chords on the buckling capacity of the whole columns cannot be distinguished clearly.

Direct analysis of Finite Element Method and Advanced Nonlinear Analysis Method had proved more sophisticated without so many assumptions as traditional theories, but rarely seen in the buckling computation of latticed columns. In fact, both chords and lacing bars could behave as beam-column component between connected nodes in the direct analysis. Equilibrium path of bucking deformation could be acquired precisely with the iterations of nonlinear stiffness matrix using the numerical solution algorithms. Kim et al.\textsuperscript{13} and Nguyen and Kim\textsuperscript{14,15} presented an advanced analysis method of three-dimensional steel frame structure, which can consider the geometric and material nonlinearity of each component accounting for lateral-torsional buckling. Initial geometric imperfection with member and frame were considered in the advanced analysis.\textsuperscript{16,17} Steel frame with random geometric imperfections was integrated in the advanced second-order

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.jpg}
\caption{Applications in electricity engineering.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.jpg}
\caption{Applications in building engineering.}
\end{figure}
nonlinear analytical procedures. Fooladi\textsuperscript{18} proposed a new super-element of bars and chords components with twelve degrees of freedom to construct the finite element model of latticed columns. The characteristics of cross-section area, moments of inertia, shear coefficient, and torsional rigidity of lacing bars and chords could be counted entirely in the FE procedure.

**Figure 3.** Structural 3-D model for the four-legged latticed columns: (a) the cross-section profiles, (b) the expressions of the deformations of chord element section, and (c) the direction of bending deformation of lacing bars.
Note that many modeling experiments had validated the importance of lacing bars. Kalochairetis et al.,19 Bonab et al.,20 Yang et al.,21,22 and Chen and Ou23 conducted the experimental studies on the two-legged, three-legged and four-legged laced latticed columns, respectively. These testing results investigated the influence of columns’ characteristics on the buckling capacity of latticed columns, such as eccentricity, types of lacing bars, chords’ cross-section, panel’s length, material of specimen, diameter-to-thickness ratio of the tube section, etc. Mirtaheri et al.24,25 optimized the buckling capacity of the Buckling Restrained Brace (BRB) steel core, and derived the formula for the best length of BRB through the numerical method and experimental validation. Aghoury et al.26,27 experimentally investigated the buckling behavior of battened latticed columns composed of four equal slender angles. The ratio between geometrical characteristic of battens and chords were tested to play great roles on buckling behavior and collapse loads of whole columns. To validate the accuracy of experimental results, the geometric imperfections, loading deflection and residual stress were considered with finite element simulations of ABAQUS.

In this paper, both linear and nonlinear buckling analyses are conducted to investigate the influence of the characteristics of lacing bars along with their imperfections on the buckling capacity of latticed columns. Firstly, the linear buckling load was calculated considering three loading conditions, no eccentricity unidirectional eccentricity $e_x = 20$ mm, and bidirectional eccentricity $e_x = e_y = 20$ mm. The characteristics of lacing bars were parametric, including the cross-section area $A_d$, the bending moments of inertia $I_{d, in}$, $I_{d, out}$, inner and outer lacing plane. Then, advanced nonlinear analytical procedures ANAP for four imperfect models of lacing bars were developed to study their effect on the nonlinear buckling behavior of the latticed columns. Both the nonlinear solution algorithms of Newton-Raphson incremental-iterative method and Risk arc-length incremental-iterative method were integrated into ANAP. Finally, commercial nonlinear FE software ABAQUS and design code EC3 were introduced to validate the accuracy of ANAP.

**Structural model**

Typical 3-D structural model for the four-legged latticed columns consists of four identical legs (chords) and four-plane lacings (lacing bars), seen in Figure 3. The lacing bars were fixed (welded or bolted) regularly on the chord. The two ends of the columns were connected by the end plates, where one end was fixed on the ground with all freedom constrained, and the other was loaded by an eccentric compression load $P$ with offset $(e_x, e_y)$ in the global coordinate system $O-X-Y-Z$. To show the deformations of the chords under the compressive load $P$, Function $u(z), v(z)$ and $\theta(z)$ were used to express the lateral and twisting displacements of chord element in the local coordinate system $o-x-y-z$, where $o$, $o'$ and $s$, $s'$ represent the centroid and shear center of chord section, respectively (Figure 3(b)).
The cross section of chords is equilateral angel steel with dimension $a^*t$, whereas the lacing bars were set as arbitrary section with cross-section characteristics of area $A_d$, bending moment of inertia $I_{d, \text{in}}, I_{d, \text{out}}$ inner and outer lacing plane, twisting and warping moment of inertia $I_{d, r}, I_{d, w}$. The direction of bending deformation inner and outer lacing plane for lacing bars was plotted in Figure 3(c). Both chords and lacing bars are made of homogeneous and isotropic material with Yong’s elastic module $E$ and shears module $G$, respectively.

Based on the structural 3-D model for the four-legged latticed columns, a computational model considering the influence of the end plates was plotted in Figure 4. The computational model contains 24 nodes and 60 elements (4 end plate elements, 20 chord elements, 20 diagonal bar elements, and 20 transverse bar elements). The loading end plate was modeled with completely rigid elements with number 57, 58, 59, 60, and the constraint end plate was useless in the computational model because all the freedoms at nodes , , , and had been constrained. The eccentric compression load $P$ was distributed to the end nodes , , , and with compressive load $P_{21}, P_{22}, P_{23},$ and $P_{24}$, seen in equations (1)–(4).

$$P_{21} + P_{22} + P_{23} + P_{24} = P \quad (1)$$

$$\begin{align*}
(P_{21} + P_{22}) \times \left( \frac{d}{2} - \frac{a}{4} + e_x \right) &= (P_{23} + P_{24}) \times \left( \frac{d}{2} - \frac{a}{4} - e_x \right) \quad (2)
\end{align*}$$

$$\begin{align*}
(P_{21} + P_{24}) \times \left( \frac{d}{2} - \frac{a}{4} + e_y \right) &= (P_{22} + P_{23}) \times \left( \frac{d}{2} - \frac{a}{4} - e_y \right) \quad (3)
\end{align*}$$

$$\begin{align*}
P_{23} \left( \frac{\sqrt{2}}{2} d - \frac{\sqrt{2}}{4} a - \sqrt{e_x^2 + e_y^2} \cos \rho \right) - P_{21} \left( \frac{\sqrt{2}}{2} d - \frac{\sqrt{2}}{4} a + \sqrt{e_x^2 + e_y^2} \cos \rho \right) &= P \sqrt{e_x^2 + e_y^2} \cos \rho \quad (4)
\end{align*}$$

Where the equation (1) is the load equilibrium equation, and equation (2)–(4) are the bending moment equilibrium equations. $a$ and $d$ are the cross-section dimension of the chord and whole latticed columns in Figure 3, respectively. $\rho$ is the intersection angle between the vector $\overrightarrow{e_{xy}} = (e_x, e_y)$ and the diagonal edge of the cross section of the latticed columns.

$$\rho = \frac{\pi}{4} - acr \tan \left( \frac{e_y}{e_x} \right) \quad (5)$$

The necessary geometric dimensions used in Figures 3 and 4 were listed in Table 1. The cross-section characteristics of chords were calculated according to the dimension $a^*t = 36 \text{mm} \times 3 \text{mm}$, including cross-section area $A_c$, bending moment of inertia $I_{c,y}, I_{c,z}$ about $y$-axis and $z$-axis in local coordination system $o-x-y-z$, twisting and warping moment of inertia $I_{c,r}, I_{c,w}$, and the offsets $(y_0, z_0)$ between the centroid $o$ and shear center $s$. However, the cross-section
Table 1. Necessary geometric dimensions of the latticed columns.

| L (mm) | l (mm) | $\phi$ (°) | d (mm) | a (mm) | t (mm) | $y_0$ (mm) | $z_0$ (mm) |
|--------|--------|------------|--------|--------|--------|------------|------------|
| 1900   | 380    | 50.79      | 310    | 36     | 3      | 0          | 10.6       |

Figure 4. Structural computational model for the four-legged latticed columns.
characteristics of bars $A_d$, $I_{d,\text{in}}$ and $I_{d,\text{out}}$ were set as variable parameters, which refer to the chord section characteristics by multiplying a factor $m_1$, $m_2$ and $m_3$ ($0 < m_1, m_2, m_3 \leq 1$), and the cross-section characteristics $I_{d,t}$ and $I_{d,w}$ are same with the chord, seen in Figure 4.

**Linear buckling analysis of four-legged latticed columns**

*Computing method for linear critical buckling load*

Linear buckling analysis is used to express the structural buckling capacity under perfect conditions without considering initial geometric imperfections and plastic material behavior. Based on the computational model in Figure 4, the type of the chord element is same as the lacing bars but with different cross-section characteristics, so the chord element and lacing bars can be analyzed as a type of element. Figure 5 shows the forces and displacements of this element in system $o-x-y-z$, where $l_e$ is the length of chord or lacing bars element; $P_e$ is the axial force; $M_{yi}$, $M_{yj}$, $M_{zi}$, $M_{zj}$ are bending moment at the nodes $i$ and $j$, $V_i$, $V_j$, $Q_i$, $Q_j$ are the lateral force; $M_{xi}$, $M_{xj}$ are the torsional moment; and $M_{bi}$, $M_{bj}$ are the bimoments relating to the warping deformations. Then the force-displacement equilibrium equation for this element is

$$[K_{Le}]\{U_e\} + [K_{Ge}]\{U_e\} = \{F_e\} \quad (6)$$

Wherein, \(\{F_e\} = \{P_e, -P_e, M_{yi}, M_{yj}, V_i, V_j, M_{zi}, M_{zj}, Q_i, Q_j, M_{xi}, M_{xj}, M_{bi}, M_{bj}\}\) and \(\{U_e\} = \{w, -w, v', i', v', v_i, v_j, u', u' + u, i', u', i', \theta_i, \theta_j, \theta', \theta'\}\) are the nodal forces and displacements vector of the element. \([K_{Le}]\) and \([K_{Ge}]\) are the elastic stiffness matrix and geometric stiffness matrix of the element in local system $o-x-y-z$. Then the structural elastic stiffness matrix \([K_L]\) and geometric stiffness matrix \([K_G]\) of the whole columns in global coordination system $O-X-Y-Z$ could be assembled by the matrix \([K_{Le}]\) and \([K_{Ge}]\) of all the chord and bar elements. The process of derivation and assembly for \([K_{Le}]\) and \([K_{Ge}]\) had been explained in the researches of Chiorean.\(^{28,29}\) *Appendix* lists the expressions of the stiffness matrix \([K_{Le}]\) and \([K_{Ge}]\).

The structural equilibrium condition of the whole columns is:

$$[K_L]\{U\} + [K_G]\{U\} = \{F\} \quad (7)$$

Where \(\{U\}, \{F\}\) represent the global displacement vector and force vector of the columns nodes, respectively.

In the linear buckling analysis, the displacements of the elements, occurring in the pre-buckling state, were assumed to have no influence on the buckling loads. Meanwhile, the external compression load remains unchanged in the pre-buckling state. So the equation (7) in the state of buckling can be written as

$$([K_L] + \lambda [K_G])\{U_d\} = 0 \quad (8)$$
Where $\lambda$ is the buckling load factors, $\{U_d\}$ represents the global displacements in the state of buckling. In order to make the equation (8) has available solutions; its determinant should be zero.

$$[[K_L] + \lambda[K_G]] = 0$$  \hspace{1cm} (9)

In equation (9), the solutions of the load factors $\lambda$ represents the values of buckling loads corresponding to different bucking failure modes. The minimum value $\lambda_{\text{min}}$ corresponds to the critical linear buckling load of the columns.

**Influence of lacing bars on critical linear buckling load**

In previous researches Razdolsky\textsuperscript{1–7} and Kalochairetis and Gantes\textsuperscript{10,11} about the linear buckling analysis for latticed columns, the lacing bars are considered as
uniform distributed web connected with the chords. Only characteristic of cross-section area \( A_d \) is available. Bending moments of inertia \( I_{d,\text{in}}, I_{d,\text{out}} \) inner and outer lacing planes, twisting and warping moment of inertia \( I_{d,\tau}, I_{d,\psi} \) are not considered. Noting that in the studies of Razdolsky, the bending rigidity \( E^* I_{d,\text{out}} \) of lacing bars in two-legged laced columns was discussed, but thinking as infinitesimal characteristic compared to the axial rigidity \( E^* A_d \) without deeper verifications.

In the paper, the characterizes \( A_d, I_{d,\text{in}}, I_{d,\text{out}} \) of lacing bars are all set as variable parameters to investigate their influences deeply. Three different loading conditions of axial compression \( P \) with no eccentricity, unidirectional eccentricity \( e_x = 20 \text{ mm} \), and bidirectional eccentricity \( e_x = e_y = 20 \text{ mm} \) were considered to construct the relationship curves of the parameters \( A_d, I_{d,\text{in}}, I_{d,\text{out}} \) of lacing bars and the critical linear buckling load factor \( \lambda_{\min} \) of the whole columns.

Firstly, the parametric research focus on distinguishing the influence of parameters \( I_{d,\text{in}}, I_{d,\text{out}} \) of lacing bars on the global buckling load \( \lambda_{\min} \) of the latticed columns. The cross-section area \( A_d \) of lacing bars is set as \( A_d = 0.03 A_c (m_1 = 0.03) \) and \( A_d = 0.05 A_c (m_1 = 0.05) \), respectively. Figure 6(a)–(c) shows the relationships with X-axis \( m_2 = I_{d,\text{out}}/I_{c,y} \) (in the right), \( m_3 = I_{d,\text{in}}/I_{c,y} \) (in the left), and Y-axis \( \lambda_{\min}/10^4 \text{ N} \). From the curves of Figure 6, two conclusions can be clearly observed.

1. Compared the left and right figure of Figure 6(a), when the moment of inertia inner lacing planes \( m_2 \) varies from 0.1 to 0.5, the critical buckling load \( \lambda_{\min} \) of the latticed columns increase slightly about \( 0.2*10^4 \text{ N} \). However, when the moment of inertia outer lacing planes \( m_3 \) varies from 0.1 to 0.5, the corresponding \( \lambda_{\min} \) increase sharply about 30% with \( 1.8*10^4 \text{ N} \). The sensitivity of parameters \( I_{d,\text{out}} \) on the critical buckling load is about nine times larger than that of \( I_{d,\text{in}} \). This regular pattern is also suitable to the loading conditions of unidirectional and bidirectional eccentricity in Figure 6(b) and (c). The current data directly indicate the bars’ bending moment of inertia inner lacing plane \( EI_{d,\text{in}} \) has little influence on the buckling load of the latticed columns. However, the influence of bending stiffness \( EI_{d,\text{out}} \) of lacing bars outer lacing plane is heavy and should not be ignored.

2. Compared Figure 6(a)–(c), loading eccentricity would greatly reduce the buckling capacity of latticed columns. When the loading eccentricity \( (e_x, e_y) \) change from no eccentricity, unidirectional eccentricity to bidirectional eccentricity, the critical buckling load decreases from 81.4 kN, 72.6 kN to 65.7 kN. The eccentricity led to the declination of the buckling capacity of latticed columns with 10.8% and 19.3% according to the current data.

Then, to clearly show the influence of the cross-section area \( A_d \) on the critical buckling load, Figure 7 plots the relationship curves with X-axis \( m_1 = A_d/A_c \) ranging in (0, 0.45) and Y-axis \( \lambda_{\min}/10^4 \text{ N} \). From Figure 7(a)–(c), it can be concluded:
Figure 6. Influence of parameters $l_{d_{\text{in}}}$, $l_{d_{\text{out}}}$ of facing bars on the critical buckling load $\lambda_{\text{min}}$: (a) no eccentricity, (b) unidirectional eccentricity with $e_x = 20$ mm, and (c) bidirectional eccentricity with $e_x = e_y = 20$ mm.
1. The critical load $\lambda_{min}$ of latticed columns increase sharply along with the increment of bars’ cross-section area $A_d$, but the growth speed becomes slight abruptly after $A_d$ exceed a specific value nearly $A_d = 0.05A_e$. That means a threshold cross-section area $A_h = 0.05A_e$ exists for the lacing bars. Oversized cross-section area of lacing bars exceeding $A_h$ is meaningless to improve the buckling capacity of latticed columns in the practical structural engineering. The critical value of $A_d$ also coincides with the concept of the threshold-braced stiffness of bars in Zhang et al.’s research.

2. The loading eccentricity $e_x$, $e_y$, moment of inertia $I_{d_{\text{in}}}$, $I_{d_{\text{out}}}$ of lacing bars have almost no influence on the threshold cross-section area $A_h$.

Figure 7. Influence of parameter $A_d$ of lacing bars on the critical linear buckling load $\lambda_{min}$: (a) no eccentricity, (b) unidirectional eccentricity with $e_x = 20 \text{ mm}$, and (c) bidirectional eccentricity with $e_x = e_y = 20 \text{ mm}$.
Nonlinear buckling analysis of four-legged latticed columns

Initial geometric imperfection models

There always exist initial geometric imperfections for the latticed columns. In the design stage, The European code EC3 allowed the maximum amplitude $l_e/500$ with the imperfect type of single sine wave. However, in the stage of real structural engineering, the imperfect modes of latticed would be arbitrary, overlapping, and unpredictable. Imperfect types of double sine wave or maximum imperfections exceeding the amplitude $l_e/500$ are also commonly seen in the real latticed columns, seen in Figure 8(a) and (b).

To investigate the influence of the lacing bars imperfections on the critical buckling of latticed column, four typical imperfect models in structural engineering were designed in Figure 9, including single and double sine wave imperfections of the lacing bars. Table 2 lists the numerical expressions of each imperfect model. Where $v_0$ and $u_0$ represents the initial imperfect expressions of lacing bars inner and outer.

![Figure 8. (a) Imperfect type-1 of lacing bars and (b) imperfect type-2 of lacing bars.](image)

Table 2. Numerical expression for the initial geometric imperfections of the lacing bars.

| Imperfect models | Model_1 | Model_2 | Model_3 | Model_4 |
|------------------|---------|---------|---------|---------|
| Imperfect elements Expressions | 28 | 16, 40 | 16, 40 | 4, 6, 16, 18, 40, 42, 52, 54 |
| $u_0 = \frac{l_e}{N} \sin \left( \frac{\pi x}{L} \right)$ | $v_0 = \frac{l_e}{N} \sin \left( \frac{2 \pi x}{L} \right)$ | $u_0 = \frac{l_e}{N} \sin \left( \frac{\pi x}{L} \right)$ | $u_0 = \frac{l_e}{N} \sin \left( \frac{\pi x}{L} \right)$ | $u_0 = \frac{l_e}{N} \sin \left( \frac{\pi x}{L} \right)$ |

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the lacing plane, respectively. $l_e/N$ is the amplitude of the single and double sine wave imperfections ($N = 100$ in following calculations).

**Expresssions of the stiffness matrix of lacing bars with imperfections**

Figure 10 shows the deformed condition of lacing bars element and nodal forces with initial geometric imperfections $v_0(x)$ in x-z plane. Where $l_e$ is the length of lacing bars, $P_e$ is the axial force, $V_i$ and $V_j$ are the lateral force, $M_{yi}$, $M_{yj}$ are the bending moment of nodes $i$ and $j$, respectively.

Taking the unidirectional bending deformation $v(x)$ for example, the general expression in Table 2 of bending deformation imperfections $v_0(x)$ is

$$v_0(x) = v_{0m} \sin \left( \frac{n \pi x}{l_e} \right)$$

Where $v_{0m} = l_e/N$ is the amplitude of lacing bars imperfections; $n = 1$ or 2 represents the single or double sine wave of imperfect lacing bars, respectively.

The bending differential equations for the lacing bars’ displacement $v(x)$, considering the imperfections $v_0(x)$, is defined in equation (11).
Let \( g_2 = \frac{P_e}{EI_{d, in}} \) be the Yang's elastic modulus. Then the general solution of equation (11) is

\[
v(x) = b_1 \cos (\gamma_x x) + b_2 \sin (\gamma_x x) - \frac{M_{yi}}{P_e} + \frac{V_i}{P_e} + \beta \sin \left( \frac{n \pi x}{l_e} \right)
\]

In equation (12),

\[
\beta = \frac{l_e/N}{\left( \frac{n \pi}{l_e} \right)^2 + \gamma_y^2} \frac{P_e}{EI_{d, in}}
\]

Introduce the boundary conditions \( v(x = 0) = v_i \) and \( v(x = l) = v_j \) into equation (12), then coefficient \( b_1 \) and \( b_2 \) is

\[
b_1 = \frac{M_{yi}}{P_e}; \quad b_2 = \frac{1}{\sin(\gamma_x l_e)} \left( \frac{M_{yi}}{P_e} (1 - \cos(\gamma_x l_e)) - \frac{V_i l_e}{P_e} \right)
\]

Substituting \( b_1 \) and \( b_2 \) into equation (12), and the derivative of \( v(x) \) is

\[
v'(x) = - \frac{M_{yi} \gamma_y}{P_e} \sin (\gamma_x x) + \frac{\gamma_y}{\sin(\gamma_x l_e)} \left( \frac{M_{yi}}{P_e} (1 - \cos(\gamma_x l_e)) - \frac{V_i l_e}{P_e} \right) \cos (\gamma_x x)
\]

\[\]

\[
+ \frac{V_i}{P_e} - \beta n \pi \frac{1}{l_e} \cos \left( \frac{n \pi x}{l_e} \right)
\]

Combined equations (12), (14) with the mechanical equilibrium conditions between the two end points of elements \( V_i = -V_j = (M_{yi} + M_{yj})/l + P_e(v_j-v_i)/l_e \), the elastic stiffness matrix of lacing bars element with imperfections is obtained as following:
\[
\begin{bmatrix}
V_i \\
V_j \\
M_{xi} \\
M_{xj}
\end{bmatrix} = K_{xoz}^e \begin{bmatrix}
v_i \\
v_j \\
v'_i \\
v'_j \\
v_{0m}
\end{bmatrix} = \begin{bmatrix}
\frac{12i}{l_e}s_{z5} & -\frac{12i}{l_e}s_{z5} & \frac{6i}{l_e}s_{y2} & \frac{6i}{l_e}s_{y2} & 0 \\
\frac{6i}{l_e}s_{y2} & -\frac{6i}{l_e}s_{y2} & 4i_s s_{z3} & 2i_s s_{z4} & \frac{s_y}{l_e} \\
\frac{6i}{l_e}s_{y2} & -\frac{6i}{l_e}s_{y2} & 2i_s s_{z4} & 4i_s s_{z3} & -\frac{s_y}{l_e} \\
\frac{6i}{l_e}s_{y2} & -\frac{6i}{l_e}s_{y2} & 2i_s s_{z4} & 4i_s s_{z3} & -\frac{s_y}{l_e} \\
\frac{6i}{l_e}s_{y2} & -\frac{6i}{l_e}s_{y2} & 2i_s s_{z4} & 4i_s s_{z3} & -\frac{s_y}{l_e}
\end{bmatrix} \begin{bmatrix}
v_i \\
v_j \\
v'_i \\
v'_j \\
v_{0m}
\end{bmatrix}
\] (15)

In equation (15), \( K_{xoz}^e \) is the elastic stiffness matrix of bending deformation in \( x-z \) plane considering imperfections of \( v_0(x) \). \( i_y = EI_{d, in} l_e \) represents the stiffness resisting bending deformation inner lacing plane. The coefficients \( s_{z1} \sim s_{z6} \) are

\[
s_{z1} = \frac{\gamma y l_e}{2} \cot \left( \frac{\gamma y l_e}{2} \right); s_{z2} = \left( \frac{\gamma y l_e}{2} \right)^2 / (3 - 3s_{z1})
\]

\[
s_{z3} = (3s_{z2} + s_{z1})/4; s_{z4} = (3s_{z2} - s_{z1})/2
\]

\[
s_{z5} = s_{z1}s_{z2}; s_{z6} = \frac{n\pi(\gamma y l_e)^3}{(\gamma y l_e)^2 - (n\pi)^2} (s_{z3} - s_{z4})
\]

Similarly, the elastic stiffness matrix \( K_{xoy}^e \) of bending deformation in \( x-y \) plane considering imperfections of \( u_0(x) \) can be expressed as

\[
\begin{bmatrix}
Q_i \\
Q_j \\
M_{zi} \\
M_{zj}
\end{bmatrix} = K_{xoy}^e \begin{bmatrix}
u_i \\
u_j \\
u'_i \\
u'_j \\
u_{0m}
\end{bmatrix} = \begin{bmatrix}
\frac{12i}{l_e}s_{y5} & -\frac{12i}{l_e}s_{y5} & \frac{6i}{l_e}s_{y2} & \frac{6i}{l_e}s_{y2} & 0 \\
\frac{6i}{l_e}s_{y2} & -\frac{6i}{l_e}s_{y2} & 4i_z s_{y3} & 2i_z s_{y4} & \frac{s_y}{l_e} \\
\frac{6i}{l_e}s_{y2} & -\frac{6i}{l_e}s_{y2} & 2i_z s_{y4} & 4i_z s_{y3} & -\frac{s_y}{l_e} \\
\frac{6i}{l_e}s_{y2} & -\frac{6i}{l_e}s_{y2} & 2i_z s_{y4} & 4i_z s_{y3} & -\frac{s_y}{l_e} \\
\frac{6i}{l_e}s_{y2} & -\frac{6i}{l_e}s_{y2} & 2i_z s_{y4} & 4i_z s_{y3} & -\frac{s_y}{l_e}
\end{bmatrix} \begin{bmatrix}
u_i \\
u_j \\
u'_i \\
u'_j \\
u_{0m}
\end{bmatrix}
\] (16)

Where \( i_z = EI_{d, out} l_e \) represents the bending stiffness resisting bending deformation outer lacing plane. Let \( \gamma 2 z = P_c / EI_{d, out} \), the coefficients \( s_{y1} \sim s_{y6} \) are

\[
s_{y1} = \frac{\gamma y l_e}{2} \cot \left( \frac{\gamma y l_e}{2} \right); s_{y2} = \left( \frac{\gamma y l_e}{2} \right)^2 / (3 - 3s_{y1})
\]

\[
s_{y3} = (3s_{y2} + s_{y1})/4; s_{y4} = (3s_{y2} - s_{y1})/2
\]

\[
s_{y5} = s_{y1}s_{y2}; s_{y6} = \frac{n\pi(\gamma y l_e)^3}{(\gamma y l_e)^2 - (n\pi)^2} (s_{y3} - s_{y4})
\]

Then, the elastic stiffness matrix \([K_{Le}]\) of lacing bars element considering the initial geometric imperfections can be acquired by replacing the bending stiffness matrix \([K_{vr}]_{4\times4}, [K_{uu}]_{4\times4}\) in Appendix with the derived stiffness matrix \([K_{xoz}]\) and \([K_{xoy}]\).
Nonlinear solution algorithms

In order to trace the nonlinear equilibrium path of the four-legged latticed columns under compressive load, two kinds of nonlinear solution algorithm, Newton-Raphson load incremental-iterative method and Risk arc-length incremental-iterative method were introduced in Figure 11.

In the Newton-Raphson load incremental-iterative method (Figure 11(a)), the external force would be loaded in several steps with equal or unequal interval. Iterative process occurs in each load step to eliminate residual force between external force and internal nodal force. This method is sensitive to convergence failures and uncontrollable in snap of critical point of load-displacement curve. To meet the shortcomings of Newton-Raphson method, arc-length control method is developed in Risk constant arc-length approach (Figure 11(b)). The solution scheme was controlled by arc length consisting of nodal displacements and forces. This method could capture the whole equilibrium path efficiently and solute snaps back and snap through response successfully. Because of the superior performances, it had been integrated in most commercial nonlinear FE software ANSYS, ABAQUS, NASTRAN, etc.

Once the tangent matrix of bars and chords was assembled to the global stiffness of the latticed columns, the increment equilibrium equation can be written as

\[ [K_{Le}] = \begin{bmatrix} [k_{ww}]_{2 \times 2} & 0 & 0 & 0 \\ 0 & [K_{xoz}] & 0 & 0 \\ 0 & 0 & [K_{xoy}] & 0 \\ 0 & 0 & 0 & [k_{\theta\theta}]_{4 \times 4} \end{bmatrix} \]  

(17)
\[ ([K_L] + [K_G]_t) \Delta U_t = \lambda_t \{ F_{out} \} \]  

(18)

Wherein, \([K_G]_t\) is the geometric stiffness matrix in current load step \(t\). \(\{ \Delta U \}\) is the incremental displacement vector of structural nodes; \(\lambda_t\) is the incremental load factor for the external force \(\{ F_{out} \}\).

For Newton-Raphson increment approach, the unbalanced force in iterative step \(k\) of load step \(t\) is

\[ \{ \Psi \}_{t}^{k-1} = \lambda_t \{ F_{out} \} - \{ F_{in} \}_{t}^{k-1} \]  

(19)

\(\{ F_{in} \}\) is the inner nodal force vector of nodes. The convergence criteria in iterative process is

\[ \| \{ \Psi \}_{t}^{k-1} \| \leq \alpha_F \| \{ F_{out} \} \| \]  

(20)

For Risk constant arc-length approach, the constrained arc length in each load step is

\[ \Delta r^2 = \{ \Delta U \}_{t}^{k}^{T} \{ \Delta U \}_{t}^{k} + b(\lambda_t^k - \lambda_{t-1})^2 \{ F_{out} \}_{t}^{T} \{ F_{out} \} \]  

(21)

The convergence criteria in iterative process is

\[ \sqrt{\left[ \{ \Delta U \}_{t}^{k} - \{ \Delta U \}_{t}^{k-1} \right]^{T} \left[ \{ \Delta U \}_{t}^{k} - \{ \Delta U \}_{t}^{k-1} \right]} \leq \epsilon \]  

(22)

In equations (19)–(22), \(\{ \Psi \} = \) residual force vector between nodal force and external force. \(\Delta r = \) incremental arc length constraining the iterative process of force and displacement increment; \(b = \) type of arc length, sphere arc-length for \(b = 1\), cylinder arc length for \(b = 0\), ellipse arc length for \(b = \) other value; \(\alpha_F, \epsilon = \) convergence tolerance of iterative approaches Newton-Raphson and Risk Method, respectively.

**Procedure development with nonlinear solution algorithms**

Based on section 4.1–4.3, the advanced nonlinear analytical procedures ANAP considering the initial geometric imperfection of lacing bars were developed. Figure 12 showed the flowchart of the analytical procedures ANAP-NR and ANAP-Risk, developed with solution algorithms Newton-Raphson increment method and Risk arc-length increment method, respectively. As integrating the stiffness matrix of imperfect element, the procedure solvers could acquire the nonlinear buckling equilibrium path of the imperfect models in Figure 9.

The process of the analytical procedures in Figure 12 can described as the following steps:
1. Calculate the elastic stiffness matrix \( [K_{Le}] \) with imperfections and geometric stiffness matrix \( [K_{Ge}] \) for the chord and lacing bars element in local coordinate system \( o-x-y-z \);

2. Transfer the matrix \( [K_{Le}], [K_{Ge}] \) of the element from the local system \( o-x-y-z \) to the global system \( O-X-Y-Z \), and assemble the element stiffness matrix to the global stiffness matrix \( [K_L], [K_G] \) of the whole columns;

Figure 12. Advanced nonlinear analytical ANAP-NR and ANAP-risk.
3. Solve the incremental equilibrium equation \((\mathbf{K}_L + \mathbf{K}_G) \bullet \{\Delta \mathbf{U}\} = \lambda \bullet \{F_{out}\}\) in each load step, and acquire incremental displacement vector \(\{\Delta \mathbf{U}\}\);  
4. Apply nonlinear numerical iterative strategy of Newton-Raphson and Risk arc-length solution algorithms to eliminate residual force or displacement. The iterative principle of the two methods had been shown at Figure 11 in section 4.3.

For Newton-Raphson load increment iteration, the step length of every load step \([\lambda_1, \lambda_2, \ldots, \lambda_t, \ldots]\) could be equivalent distance and varied distance according to customized setting.

For Risk arc-length increment iteration, the load step were controlled by arc length \(r = [r_1, r_2, \ldots, r_t, \ldots]\) consisting of load and displacement vector, seen in equation (21). \(r_1\) is the initial arc length assigned at first, and then other arc length would update automatically according to equation (23).
\[ r_{t+1} = r_t^* \sqrt[n_d]{n_t} \]  

(23)

Where \( n_t \) is the number of iterations in the previous load step \( t \), \( n_d \) is the expected iterative numbers in the current load step.

5. After the convergence process of iteration completes in current load step, update the structural stiffness matrix of the latticed columns and prepare next load step. Print the equilibrium point consisting of nodal load vector \( \lambda_t^* \{ F_{out} \} \) and displacement vector \( \{ U \}_t \) in each end of load step. If all the load steps are done, plot the equilibrium path with load-displacements relationship curve in the test nodes.

Results and discussion

Four imperfect models in Figure 9 were solved by the advanced nonlinear analytical procedures (ANAP) in Figure 12. The purpose was to find the influence of lacing bars imperfections on the nonlinear buckling behavior of the four-legged latticed columns. To verify the solution accuracy of the analytical procedures ANAP, the nonlinear FE analytical software ABAQUS was introduced in section 5.2. Corresponding FE imperfect models were built, and then the nonlinear buckling analytical technique ABAQUS-Risk, having integrated Risk increment-iterative method already, was conducted to accurately track the nonlinear equilibrium path of the imperfect columns.

Effect of lacing bars imperfections

Figure 13(a)–(d) show the nonlinear equilibrium path with the load-displacement curves of the four imperfect models. Point_1–Point_4 were the test nodes, corresponding to the node , , , and in Figure 4. \( \bar{U} \) = the comprehensive displacement of \( U_x, U_y, \) and \( U_z \) of the node in global system \( O-X-Y-Z \), \( U = \sqrt{U_x^2 + U_y^2 + U_z^2} \). \( P \) = the applied compressive load with the bidirectional eccentricity \( e_x = e_y = 20 \) mm. In the procedure ANAP-NR, the incremental load parameter was set as 32 kN constantly. Meanwhile, in the procedure ANAP-Risk the arc length parameter was set with the initial arc length 0.01, minimum arc length \( 10^{-5} \), maximum arc length 10, and maximum iteration numbers 200, which all coincide with the nonlinear FE analytical technique ABAQUS-Risk.

From Figure 13, it can be seen the developed procedure of ANAP present a good performance in tracking the load-displacement equilibrium path of the imperfect latticed columns. Compared to the results of ABAQUS-Risk, method APAN-NR lead to the maximum error 2.01\% of the critical load from in solving Model_1, and the maximum error \(-6.25\%\) of the critical displacement in solving Model_2. Meanwhile, method ANAP-Risk just has errors with the critical loads ranging in 3.06\%–6.88\%, and the critical displacements ranging in \(-0.03\%\) to 7.18\%. However, method ANAP-Risk has the advantage of tracking the declining
stage of the equilibrium path, which cannot be achieved through ANAP-NR method.

What’s more, the critical buckling load $P_m$ and displacement $U_m$ from the four imperfect models in Figure 13 were plotted in Figure 14(a) and (b), respectively. Noting that the critical displacement in Figure 14(b) refers to the Point_1 (node $\odot$ in Figure 4). Comparing the critical nonlinear buckling loads and displacements of the four imperfect models, it can be concluded:

1. Different imperfect models of lacing bars have significant impact on the critical nonlinear buckling load and equilibrium displacements. It can be seen that the critical load of imperfect Model_4 is 327 kN (ABAQUS-Risk), which declined approximately 21.5% than that of Model_2 with the critical buckling load 417 kN. Imperfect model with many lacing bars’ imperfections would decrease more shapely the nonlinear buckling capacity of the latticed columns. The analytical results directly indicate the important influence of lacing bars imperfections on the critical buckling capacity of the four-legged latticed columns.

2. Compared the critical buckling loads of Model_2 and Model_3, the model with the imperfection of lacing bars outer lacing plane would bring more reduction for the buckling capacity of latticed columns. Under same imperfections of lacing bars with element number 16 and 40 in Figure 9, current data show the buckling load of Model_3 with imperfections outer lacing plane is about 17.03% (from 417 to 346 kN) less than that of Model_2 with imperfections inner lacing plane.
Verification with FE shell-beam models of ABAQUS

Most commercial FE software has integrated the module of geometric nonlinear buckling analysis. In this paper, Perfect four-legged latticed columns were modeled using nonlinear FE software ABAQUS (Figure 15) to validate the advanced
nonlinear analytical procedure ANAP. The chords and lacing bars were built with 4-node shell element S4R, 3-node shell element S3 and 2-node beam element B31, respectively. The meshing information of the FE model is listed in Table 3. The transverse and diagonal lacing bars were divided to four elements with meshing size 62.5 and 100.0 mm, respectively. The four imperfect models Model_1–Model_4 would be created automatically by modifying the nodal coordinates of the imperfect lacing bars with the imperfect nodal coordinates $R_1$–$R_3$, seen in Table 4.

Modeling techniques in ABAQUS were introduced and shown in Figure 16. The technique of master-slave nodes coupling technique was used to simulate the influence of the end plate in the loading and constraint plates. Meanwhile, the beam-shell stringer coupling technique was used to simulate the shared connected sections in chord and lacing bars.

In the nonlinear buckling analysis of ABAQUS platform, all the parameters setting were the same as the developed analytical procedure ANAP-NR and APAN-Risk. The nonlinear load-displacement equilibrium curves of method ANAP-NR, ANAP-Risk, and ABAQUS had been plotted in Figure 13. The corresponding critical buckling loads of the imperfect models were listed in Table 5. Meanwhile, Figure 17 showed the equilibrium path of the four imperfect models under the critical buckling conditions. It can be seen the imperfections of lacing bars have great influences on the bucking load and equilibrium path of latticed columns. The maximum computational errors of ANAP-NP and ANAP-Risk are within the variation of $[-2.09\%, 2.17\%]$ and $[2.97\%, 6.92\%]$, compared to the method of FE models in ABAQUS platform. The proposed methods ANAP have good computational accuracy.

**Comparison with the design code EC3**

According to the design code EC3-2005, the latticed built-up columns were smeared as continuous structure, where the lacing bars were considered as

| FE models Details | Items | Chords | Transverse lacings | Diagonal lacings |
|-------------------|-------|--------|-------------------|-----------------|
| Perfect model, Imperfect models (Model_1–Model_4) | Mesh size: mm | 10.0 × 9.0 | 62.5 (bars), 9.0 (stringer) | 100.0 (bars), 10.8 (stringer) |
| Number of nodes | 6758 | 208 | 248 (B31) |
| Number of elements | 5678 (S4R), 73 (S3) | 192 (B31) | 252 |

*Table 3. The meshing information of the FE models in ABAQUS.*
### Table 4. The imperfect nodal coordinates of lacing bars.

| Imperfect models | Imperfect elements | Imperfect types | Imperfect legends | Imperfect nodal coordinates |
|------------------|-------------------|----------------|------------------|----------------------------|
| Model_1          | 28                | Single sine wave outer lacing plane | 28 R₁ (62.5, 152.7, 981.9) R₂ (0, 153, 1060.1) R₃ (−62.5, 152.7, 1138.2) |
|                  |                   |                 |                  |                            |
| Model_2          | 16, 40            | Double sine wave inner lacing plane | 16 R₁ (−63.3, 152, 602.5) R₂ (0, 152, 680.1) R₃ (63.3, 152, 757.6) |
|                  |                   |                 |                  |                            |
| Model_3          | 16, 40            | Double sine wave outer lacing plane | 16 R₁ (−62.5, 151, 601.9) R₂ (0, 152, 680.1) R₃ (62.5, 153, 758.2) |
|                  |                   |                 |                  |                            |
| Model_4          | 4, 6, 16, 18 40, 42, 52, 54 | Single sine wave outer lacing plane | 4 R₁ (152.7, −62.5, 221.9) R₂ (153, 0, 300.1) R₃ (152.7, 62.5, 378.2) |

### Table 5. The comparison of the critical buckling loads of the latticed columns based on ABAQUS.

| Perfect model (linear buckling) | Imperfect models | Methods | Model_1 | Model_2 | Model_3 | Model_4 |
|--------------------------------|------------------|---------|---------|---------|---------|---------|
|                                |                  | Pₘ: kN | Pₘ: kN | Pₘ: kN | Pₘ: kN | Pₘ: kN |
| 657.83 kN                      | ANAP-NR          | 416    | 416    | 352     | 320     | −2.09   |
|                                | ANAP-risk        | 435.35 | 429.71 | 359.26  | 336.53  | 2.97    |
|                                | ABAQUS           | 407.17 | 416.95 | 346.27  | 326.82  | 0.0     |

ι% means the comparison with the results of FE software ABAQUS.
continuous web connected with the chords. The failure modes for the elastic buckling behavior of the latticed built-up columns include the global buckling of the whole columns and the local buckling of the chords.

Global elastic buckling of the whole columns considering the loading eccentricity and initial imperfections occurs at the load level $P_g$ in equation (24).

\[
P_g = P_{cr} \left[ \left( \frac{P_{ch, ed}}{P_{cr}} + \frac{w_0}{d} + \frac{1}{4} \right) - \sqrt{\left( \frac{P_{ch, ed}}{P_{cr}} + \frac{w_0}{d} + \frac{1}{4} \right)^2 - \frac{4P_{ch, ed}}{P_{cr}}} \right] (24)
\]

Where $P_{ch, ed}$ is the critical load of the most compressed chord. $w_0 = L/500$ is the amplitude of design imperfection of the whole columns. $P_{cr}$ is the effective critical force of the built-up members.

\[
P_{ch, ed} = \frac{P_g}{4} + \frac{w_0 + e_x + e_y}{1 - \frac{1}{S_v}} \frac{dA_c}{2I_{eff}} (25)
\]

\[
P_{cr} = \frac{\pi^2 E I_{eff}}{(\nu L)^2}, I_{eff} = d^2 A_c (26)
\]

$I_{eff}$ is the effective second moment of area of the built-up member. $A_c$ is the cross-section area of chord. $\nu$ is effective length factor relating to boundary conditions (equal to 2 for one end fixed and the other end free). $S_v$ is the shear stiffness of the lacing bars. For the bars contribution of the investigated model, the value of $S_v$ is:

\[
S_v = \frac{nEA_d a d^2}{\left( \sqrt{a^2 + d^2} \right)^3} (27)
\]
Local buckling is associated with the buckling of chord component between the adjacent lacing joints. The shape of the buckling mode was assumed to be sinusoidal along the length of chord.

Thus, the local elastic critical load $P_l$ is

$$P_l = \frac{4\pi^2 EI_{\text{min}}}{(\mu a)^2}$$

(28)

$\mu$ is effective length factor (equal to 1.52 for the investigated models). $I_{\text{min}} = I_{c,y}$ is the minimum second moment of the chord.

The critical buckling load $P_m$ considering the loading eccentricity and initial geometric imperfection is the minimum between the buckling loads $P_g$ and $P_l$. The critical load values from method ANAP-NR, ANAP-Risk, and EC3 were all listed in Table 6. As EC3 had ruled the types and values of the global and local imperfections of latticed built-up columns in the design stage, the critical buckling load $P_m$ from method EC3 is always 378.74 kN for the imperfect models. The maximum computational deviation between ANAP-NP, ANAP-Risk, and EC3 are within the variation of $[-15.15\%, 9.84\%]$ and $[-11.09\%, 14.95\%]$. The analytical results showed the shortage of method EC3 in solving the special imperfect models existing in the real latticed built-up columns.

**Summary and conclusions**

Linear and nonlinear buckling capacity of four-legged latticed columns was investigated in the study. The key purpose was to highlight the importance of lacing bars, including their cross-section characteristics of cross-section area $A_d$, bending moments of inertia $I_{d,\text{in}}$, $I_{d,\text{out}}$ inner and outer the lacing plane, and their initial

![Equilibrium path of the four imperfect models based on the geometric nonlinear analysis of ABAQUS.](image)

Figure 17. Equilibrium path of the four imperfect models based on the geometric nonlinear analysis of ABAQUS.
geometric imperfections. Three loading conditions with no eccentricity, unidirec-
tional eccentricity $e_x = 20$ mm, and bidirectional eccentricity $e_x = e_y = 20$ mm were
considered, and four models with lacing bars’ imperfections were introduced. To
implement the advanced nonlinear analysis procedure ANAP, elastic stiffness
matrix for the imperfect models of lacing bars was derived and the nonlinear solu-
tion algorithms of Newton-Raphson incremental iterations and Risk arc-length
incremental iterations were integrated into ANAP. Nonlinear FE software
ABAQUS was introduced to verify the accuracy of the analytical procedures.

Based on the outputs of the linear and nonlinear buckling analysis with method
ANAP-NR, ANAP-Risk, and ABAQUS-Risk, the following main conclusions
were drawn.

1. The assumptions in previous research of Razdolsky and Kalochairetis that
ignored the bending moment of inertia of lacing bars in calculating buck-
ling load of latticed columns need to be revised. The current analytical data
found the critical buckling load of latticed columns could improve sharply
with growth ratio 30%, when the bars’ bending moment of inertia outer
lacing plane $I_{d,\text{out}}$ increases. Compared with $I_{d,\text{out}}$, the moment of inertia
inner lacing plane $I_{d,\text{in}}$ was less important about 1/9 sensitivity than that of
$I_{d,\text{out}}$. At least the bars’ bending moment of inertia outer lacing plane $I_{d,\text{out}}$
should be counted.

2. There exists a threshold cross-section area $A_h$ for the lacing bars in improv-
ing the buckling load of latticed columns. When the cross-section area $A_d < A_h$, the critical buckling load of latticed columns increase sharply
along with the increment of area $A_d$. After $A_d$ exceed a specific value $A_h$
(nearly $A_h = 0.05A_c$ in this paper) the growth speed becomes slight abruptly.
Both loading eccentricity $e_x, e_y$, inertia parameters $I_{d,\text{in}}, I_{d,\text{out}}$ of lacing bars
cannot change the critical value of threshold cross-section area $A_h$.

3. The initial geometric imperfections of lacing bars would obviously decrease
the buckling capacity of latticed columns. In this paper, the different

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**Table 6.** The comparison of the critical loads of the latticed columns based on EC3.

| Methods      | Model_1 $P_m$: kN | $\varepsilon$% | Model_2 $P_m$: kN | $\varepsilon$% | Model_3 $P_m$: kN | $\varepsilon$% | Model_4 $P_m$: kN | $\varepsilon$% |
|--------------|------------------|----------------|------------------|----------------|------------------|----------------|------------------|----------------|
| ANAP-NR      | 416              | 9.84           | 416              | 9.84           | 352              | -7.06          | 320              | -15.51         |
| ANAP-Risk    | 435.35           | 14.95          | 429.71           | 13.46          | 359.26           | -5.14          | 336.53           | -11.09         |
| EC3          | 378.74           | 0.0            | 378.74           | 0.0            | 378.74           | 0.0            | 378.74           | 0.0            |

$\varepsilon$% means the comparison with the results of EC3.
imperfect model could cause the maximum declination of the critical buckling load with 21.5% from 416.8 kN (Model_2) to 327 kN (Model_4). What’s more, Imperfect models with lacing bars outer lacing plane or many lacing bars’ imperfections would decrease more sharply the buckling capacity of the four-legged latticed columns.

4. The analytical results validate the accuracy of ANAP developed in the study. When the parameters of nonlinear solution algorithm in each load step were set reasonably, the ANAP present a good performance in solving the nonlinear buckling capacity for the imperfect models. In this study, the maximum error of ANAP-NR and ANAP-Risk is 6.88% and 7.18%, compared with the results of FE validated method.

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Data availability statement
All the data, models, or codes that support the findings of this study are available from the corresponding author upon reasonable request.

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Appendix

The elastic stiffness matrix $[K_{Le}]$ of chord and lacing bar elements in the local coordinate system is in equation (29).

$$
[K_{Le}] = 
\begin{bmatrix}
[k_{ww}]_{2\times2} & 0 & 0 & 0 \\
0 & [k_{vv}]_{4\times4} & 0 & 0 \\
0 & 0 & [k_{uu}]_{4\times4} & 0 \\
0 & 0 & 0 & [k_{\theta\theta}]_{4\times4}
\end{bmatrix}
$$

(29)

$$
[k_{ww}] = \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix};
[k_{vv}]_{4\times4} = \frac{EI_y}{l_e^3} [K_1]_{4\times4};
[k_{uu}]_{4\times4} = \frac{EI_z}{l_e^3} [K_1]_{4\times4};
$$

$$
[k_{\theta\theta}]_{4\times4} = \frac{GI_w}{l_e} [K_2]_{4\times4} + \frac{EI_y}{l_e^3} [K_1]_{4\times4}
$$

$$
[K_1]_{4\times4} = 
\begin{bmatrix}
12 & 6l_e & -12 & 6l_e \\
6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\
-12 & -6l_e & 12 & -6l_e \\
6l_e & 2l_e^2 & -6l_e & 4l_e^2
\end{bmatrix};
[K_2]_{4\times4}
$$

$$
= 
\begin{bmatrix}
6/5 & l_e/10 & -6/5 & l_e/10 \\
l_e/10 & 2l_e^2/15 & -l_e/10 & -l_e^2/30 \\
-6/5 & -l_e/10 & 6/5 & -l_e/10 \\
l_e/10 & -l_e^2/30 & -l_e/10 & 2l_e^2/15
\end{bmatrix}
$$

The geometric stiffness matrix $[K_{Ge}]$ of chord and lacing bar elements in the local coordinate system is in equation (30).

$$
[K_{Le}] = 
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & [g_{vw}]_{4\times4} & 0 & [g_{v\theta}]_{4\times4} \\
0 & 0 & [g_{uw}]_{4\times4} & [g_{u\theta}]_{4\times4} \\
0 & 0 & 0 & [g_{\theta\theta}]_{4\times4}
\end{bmatrix}
$$

(30)

$$
[g_{vw}]_{4\times4} = [g_{uw}]_{4\times4} = \frac{P_e}{l_e} [K_2]_{4\times4};
$$

$$
[g_{v\theta}]_{4\times4} = \frac{W_{exi}}{l_e} [K_2]_{4\times4} + \frac{W_{exi} - W_{exj}}{l_e} [K_3]_{4\times4};
$$

$$
[g_{u\theta}]_{4\times4} = \left(\frac{P_e}{l_e} z_0 - \frac{M_{yi}}{l_e^3}\right) [K_2]_{4\times4} - \frac{M_{yi} - M_{yj}}{l_e} [K_3]_{4\times4} - V_i [K_4]_{4\times4} - (V_j - V_i) [K_5]_{4\times4}
$$
\[
\begin{align*}
[g_{uv}]_{4\times4} &= -\left(P_e \frac{y_0 - M_{zl}}{l_e} - M_{zl} - M_{zi} \right) [K_2]_{4\times4} - \frac{M_{zl} - M_{zi}}{l_e} [K_3]_{4\times4} \\
+ \frac{Q_i [K_4]_{4\times4} + (Q_j - Q_i) [K_5]_{4\times4}}{l_e} \\
[K_3]_{4\times4} &= \begin{bmatrix}
3/5 & l_e/10 & -3/5 & 0 \\
l_e/10 & l_e^2/30 & -l_e/10 & -l_e^2/60 \\
-3/5 & -l_e/10 & 3/5 & 0 \\
0 & -l_e^2/60 & 0 & l_e^2/30 \\
\end{bmatrix}; [K_4]_{4\times4} \\
[K_5]_{4\times4} &= \begin{bmatrix}
-13/70 & -3l_e/70 & -11/35 & 2l_e/35 \\
-l_e/105 & -l_e^2/210 & -31l_e/420 & l_e^2/84 \\
13/70 & 3l_e/70 & 11/35 & -2l_e/35 \\
-11l_e/420 & -l_e^2/210 & 23l_e/210 & -l_e^2/210 \\
\end{bmatrix}
\end{align*}
\]

Where \( A, I_y, I_z, I_t, I_w \) are the cross-section characteristics of the element. For chord element, \( \{A, I_y, I_z, I_t, I_w\} = \{A_c, I_{c_y}, I_{c_z}, I_{c_t}, I_{c_w}\} \), and for lacing bar element \( \{A, I_y, I_z, I_t, I_w\} = \{A_d, I_{c_{in}}, I_{c_{out}}, I_{d_t}, I_{d_w}\} \).