Cosmological Perturbations in the “Healthy Extension” of Hořava-Lifshitz gravity

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We study linear cosmological perturbations in the “healthy extension” of Hořava-Lifshitz gravity which has recently been analyzed [1]. We find that there are two degrees of freedom for scalar metric fluctuations, but that one of them decouples in the infrared limit. Also, for appropriate choices of the parameters defining the Lagrangian, the extra mode can be made well-behaved even in the ultraviolet.

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I. INTRODUCTION

Following Hořava’s proposal [2] of a power-counting renormalizable Lagrangian for gravity based on anisotropic scaling between space and time \(^1\), several conceptual challenges to the theory were raised (see e.g. \(^2\)). One of the key challenges concerns the extra scalar metric degree of freedom which arises \(^2, 10, 11\) since there are the same number of variables appearing in the action, but there is less gauge symmetry. This extra degree of freedom appears when expanding about flat space-time. It has been shown \(^12\) that in a particular “non-projectable” version \(^2\) of the theory, the extra mode is non-dynamical in linear cosmological perturbation theory about an isotropic background (also in the presence of spatial curvature \(^13\)), although it is expected that the mode will become dynamical when going beyond linear perturbation theory \(^7\). In recent work, we have shown \(^14\) that the extra scalar gravitational mode survives in the “projectable” version of the theory, and that it has either ghost-like or tachyonic properties \(^3\).

Fluctuations in Hořava-Lifshitz gravity beyond linear order are also plagued by a “strong coupling” problem \(^5, 7\). This problem is absent in the “healthy” extension of the theory \(^14, 5\). In recent work \(^15\) it has been shown that the fluctuations about Minkowski space-time in the healthy extension of Hořava-Lifshitz (HL) gravity can be made well-behaved (i.e. neither ghosts nor tachyons) by appropriate choices of the parameters which enter into the Lagrangian, and that the extra mode does not lead to any phenomenological inconsistencies. In this note we show that similar conclusions hold for linear fluctuations about an isotropic cosmological background: there is a dynamical extra scalar mode, but it can be made ghost-free and non-tachyonic. In addition, it decouples in the infrared limit (a limit which cannot be seen when expanding about Minkowski space-time but a limit which is crucial for actual cosmological perturbations) \(^6\).

The outline of this short paper is as follows: We first give a very brief review of HL gravity, before writing down the specific Lagrangian of the healthy extension (Section 2) which will be studied in the following. In Section 3 we define the linearized cosmological fluctuations. Sections 4 and 5 are the key ones of this work: in Section 4 we derive the constraint equations and use them to solve for two of the fluctuation variables. In Section 5 we then derive the quadratic action for the remaining two cosmological fluctuation variables, making use of the constraint equations in

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\(^1\) See e.g. \(^3, 4\) for general overviews.

\(^2\) A version in which only the terms in the Lagrangian which appear in the “projectable” version are kept, but in which the lapse function is allowed to depend on space and time.

\(^3\) In pure de Sitter space it appears that this problem is absent \(^13\).

\(^4\) The version of the theory which contains all the terms in the Lagrangian consistent with the symmetries, taking into account that the lapse function now depends on space and time.

\(^5\) In the far ultraviolet the strong coupling instability may reappear \(^16\), but at the scale where this appears higher derivative operators become important and may prevent these instabilities \(^17\).

\(^6\) We are not the first to consider cosmological fluctuations in the healthy extension of HL gravity. In \(^18\) the evolution of super- and sub-Hubble curvature fluctuations was studied both analytically and numerically. The emphasis in our work is on the general properties of the perturbation modes rather than on the specific form of the solutions of the equations of motion.
the derivation. Working in Fourier space, we then analyze the coefficient matrix of the quadratic form of the kinetic terms in the action. This yields the conditions for the absence of ghosts. We study the infrared (IR) and ultraviolet (UV) limits of the expressions for the eigenvalues of the kinetic coefficient matrix. We find that in the IR limit one of the eigenvalues tends to zero. Since the coefficients of the mass matrix do not tend to zero in the IR limit, the previous result implies that the extra scalar metric degree of freedom decouples in the IR limit. We discuss the implications of this result in the concluding section of the paper. In the penultimate section we make contact with the Minkowski space-time limit. This is a followup paper to our previous work \[14\] to which the reader is referred for a more detailed introduction and description of the notation, and for more references to previous works.

II. MODEL

As in Einstein gravity, in the HL theory the basic variables are the metric tensor components in four dimensional space-time. However, instead of demanding full space-time diffeomorphism invariance, HL gravity has a reduced set of symmetries - only time-dependent spatial diffeomorphisms and space-independent time reparametrizations. Due to the reduced symmetry there is an extra scalar gravitational degree of freedom.

It is convenient to work in terms of the ADM metric variables, i.e. the spatial metric $g_{ij}$, the lapse function $N$ and the shift vector $N^i$ (latin indices run over space components). There are two versions of HL gravity depending on what coordinates $N$ is allowed to depend on. In the projectable version $N$ is a function of time $t$ only, in the non-projectable version it depends on both space and time. The “healthy” extension studied in [1] is the version of the non-projectable model in which all terms consistent with the residual symmetries are kept, not only those appearing in the Lagrangian of the projectable version.

The action for HL gravity is based on demanding power-counting renormalizability under anisotropic scaling between space and time. We write the action for the healthy extension of HL gravity in the following form (merging the notations of [20, 21] and [19]):

$$S = \chi^2 \int dt d^3x N \sqrt{g} \left( \mathcal{L}_K - \mathcal{L}_V - \mathcal{L}_E + \chi^{-2} \mathcal{L}_M \right)$$

(2.1)

where $g \equiv \det(g_{ij})$ and $\chi^2 \equiv 1/(16\pi G)$. The first two terms in the Lagrangian are present in the initial version of the HL model. They are the kinetic and potential Lagrangians, respectively. The final term is the matter Lagrangian. The third term is the new term specific to the healthy extension [1] of HL gravity. The four terms are given by

$$\mathcal{L}_K = K_{ij} K^{ij} - \lambda K^2$$

(2.2a)

$$\mathcal{L}_V = 2\Lambda - R + \frac{1}{\chi} \left( g_2 R^2 + g_3 R_{ij} R^{ij} \right) + \frac{1}{\chi} \left( g_4 R^3 + g_5 R R_{ij} R^{ij} + g_6 R^i R^j R^k R^l \right) +$$

$$+ \frac{1}{\chi^4} \left[ g_7 R \nabla^2 R + g_8 (\nabla_i R_{jk}) (\nabla^i R^{jk}) \right]$$

(2.2b)

$$\mathcal{L}_E = -\eta a_i a^i + \frac{1}{\chi} \left( \eta_2 a_i \Delta a^i + \eta_3 \nabla^2 a^i \right) + \frac{1}{\chi} \left( \eta_4 a_i \Delta^2 a^i + \eta_5 \Delta R \nabla_i a^i + \eta_6 \nabla^2 \nabla_i a^i \right) + \ldots$$

(2.2c)

$$\mathcal{L}_M = \frac{1}{2N^2} \left( \dot{\phi} - N^i \nabla_i \phi \right)^2 - V(g_{ij}, \mathcal{P}_n, \varphi)$$

(2.2d)

where $K_{ij}$ is the extrinsic curvature tensor of the constant time hypersurfaces, $\Lambda$ is the cosmological constant, $a_i \equiv \frac{\partial_i N}{N}$, and where for simplicity we have taken matter to be a single scalar field $\varphi$. The constants $\eta, \eta_2, \eta_3, \eta_4, \eta_5$ and $\eta_6$ are the parameters that characterize the specific healthy extension. Note that the constant $\lambda$ equals 1 in General Relativity. In HL gravity, $\lambda$ is expected to flow to 1 in the IR limit.

For consistency, the matter potential energy should also contain all terms compatible with the anisotropic scaling symmetry and power-counting renormalizability. Thus [22]

$$V = V_0(\varphi) + V_1(\varphi) P_0 + V_2(\varphi) P_1^2 + V_3(\varphi) P_2^2 + V_4(\varphi) P_1 P_2 + V_5(\varphi) P_1 P_0 P_2 + V_6(\varphi) P_1 P_2$$

(2.4)

\footnote{We have not written down the full list of extra terms (see [1] for this list). However, the terms we have omitted all vanish to quadratic order when expanding about a homogeneous and spatially flat cosmological background.}
where
\[ \mathcal{P}_0 \equiv (\nabla \varphi)^2, \quad \mathcal{P}_i \equiv \Delta^i \varphi, \quad \Delta \equiv g^{ij} \nabla_i \nabla_j. \] (2.5)

Note that the only term which may receive contributions from the second order \( a_i \) is the term \( R \nabla_i a^i \). For some other applications of the theory of cosmological perturbations to HL gravity, we refer to Ref. [20, 21, 24].

The scalar metric perturbations can be written as follows (see e.g. [23] for an extensive review article on cosmological fluctuations, and [21, 22, 24] for some other applications of the theory of cosmological perturbations to HL gravity):
\[ \delta N(t, x^k) = \nu(t, x^k) \] (3.1a)
\[ \delta N_i(t, x^k) = \partial_i B(t, x^k) \] (3.1b)
\[ \delta g_{ij}(t, x^k) = a^2(t) [ -2 \psi(t, x^k) \delta_{ij} + 2 E(t, x^k) ] \] (3.1c)

where the subscript \( |ij| \) denotes the covariant derivative. Correspondingly, also matter fluctuations must be taken into account:
\[ \varphi(t, x^k) = \varphi_0(t) + \delta \varphi(t, x^k) \] (3.2)

We can use the freedom under spatial diffeomorphisms to set
\[ E = 0. \] (3.3)

Note that in Ref. [19] the authors use the variable \( \beta \equiv B/a^2 \) instead of \( B \).

**III. PERTURBATIONS**

The constraint equations are important since they allow us to reduce the number of independent dynamical degrees of freedom. We discuss the Hamiltonian and momentum constraints in turn.

**A. Hamiltonian constraint**

The Hamiltonian constraint in the healthy extension of HL gravity takes the form
\[ \mathcal{L}_H + (\mathcal{L}_V + \mathcal{L}_E) + N \frac{\delta(\mathcal{L}_V + \mathcal{L}_E)}{\delta N} = \frac{1}{2 \chi^2} J^t \] (4.1)

where
\[ J^t = 2 \left( N \frac{\delta \mathcal{L}_M}{\delta N} + \mathcal{L}_M \right). \] (4.2)

Note that the term
\[ N \frac{\delta(\mathcal{L}_V + \mathcal{L}_E)}{\delta N} = N \frac{\delta \mathcal{L}_E}{\delta N} = 2 \eta \nabla_i a^i + \frac{2 \eta_2}{\chi^2} \Delta \nabla_i a^i + \frac{\eta_3}{\chi^2} \Delta R - \frac{2 \eta_4}{\chi^4} \Delta^2 \nabla_i a^i + \frac{\eta_5}{\chi^6} \Delta^2 \nabla_i a^i + \frac{\eta_6}{\chi^8} \Delta R^2 + \ldots \] (4.3)

is absent in the “unhealthy” HL gravity. Expanding the Hamiltonian constraint to first order we obtain the following:
\[ 3H(3\lambda - 1) (\dot{\psi} + H \nu) + (3\lambda - 1) H \Delta B - 2 \Delta \psi + \frac{\delta \rho_M}{2 \chi^2} + \eta \Delta \nu - \frac{\eta_2}{\chi^2} \Delta^2 \nu + 2 \frac{\eta_3}{\chi^4} \Delta^2 \psi - \frac{\eta_4}{\chi^4} \Delta^3 \nu + 2 \frac{\eta_5}{\chi^6} \Delta^3 \psi + 3 \frac{\mathcal{K}}{a^2} \left( \frac{\eta_3}{\chi^2} + 6 \frac{\mathcal{K} \eta_6}{a^2 \chi^4} \right) \Delta \nu - 6 \frac{\mathcal{K}}{a^2} \left( 1 - 4 \frac{\mathcal{K}}{a^2} \beta_1 - 12 \frac{\mathcal{K}^2}{a^4} \beta_2 \right) \psi + 6 \frac{\mathcal{K}}{a^2} (\eta_5 + 4 \eta_6 + 2g_7) \frac{\Delta^2 \psi}{\chi^4} + 2 \frac{\mathcal{K}}{a^2} \left( 3 \frac{\eta_3}{\chi^2} + 36 \frac{\mathcal{K} \eta_6}{a^2 \chi^4} + 4 \beta_1 + 18 \frac{\mathcal{K} g_7}{a^2 \chi^4} + 12 \frac{\mathcal{K}}{a^2} \beta_2 \right) \Delta \psi = 0 , \] (4.4)
where $\mathcal{K}$ is the spatial curvature constant \(^8\),

\[
\beta_1 \equiv \frac{3g_2 + g_3}{\chi^2}, \quad \beta_2 \equiv \frac{9g_4 + 3g_5 + g_6}{\chi^4}
\]

and

\[
\delta \rho_M = \dot{\varphi}_0 \dot{\varphi} - \nu \dot{\varphi}_0^2 + V_{0,\varphi}(\varphi_0) \delta \varphi + V_4(\varphi_0) \Delta^2 \delta \varphi.
\]

Note that the coupling constants $\eta$ newly introduced in the healthy extension of HL gravity multiply higher order derivatives of the gravitational degrees of freedom, even in the case of a spatially flat background.

### B. Momentum constraint

To first order the momentum constraint in the healthy extension of HL gravity takes the form

\[
\partial_j \left[ (\lambda - 1) \Delta B + (3\lambda - 1) \left( \dot{\varphi} + H \nu \right) - \frac{2\mathcal{K}}{a^2} B - \frac{1}{2\chi^2} q_M \right] = 0,
\]

where

\[
q_M = \dot{\varphi}_0 \delta \varphi.
\]

### C. Solving the constraints in a spatially flat background

Now we specialize to the case of a spatially flat background. Based on the lessons which can be drawn by comparing cosmological perturbation theory in the regular non-projectable version of HL gravity in the cases with \([13]\) and without \([12]\) spatial curvature, we do not expect any differences with respect to the number and general properties of dynamical degrees of freedom.

In analogy of what is done in the theory of cosmological perturbations in the regular non-projectable version of HL gravity \([12]\), we will use the constraints to solve for two of the metric degrees of freedom, namely $\nu$ and $B$. The expressions for $\nu$ and $B$ can be obtained by combining the two constraint equations. It is instructive to express the results in terms of the physical momentum $\bar{k} \equiv k/a$. The results can be written in a compact form if we introduce the notation

\[
f_1(\bar{k}) \equiv -\eta + \eta_2 \frac{\bar{k}^2}{\chi^2} + \eta_4 \frac{\bar{k}^4}{\chi^4},
\]

\[
f_2(\bar{k}) \equiv -1 + \eta_3 \frac{\bar{k}^2}{\chi^2} + \eta_5 \frac{\bar{k}^4}{\chi^4}
\]

and

\[
d(\bar{k}) \equiv 4(3\lambda - 1)H^2 + (\lambda - 1) \left[ \frac{\dot{\varphi}_0^2}{\chi^2} + 2f_1(\bar{k}) \bar{k}^2 \right].
\]

\(^8\) We recall that in the presence of spatial curvature HL gravity can lead to a non-singular bouncing cosmology \([22]\) which is an interesting property of the theory from the point of view of cosmology.
We then obtain
\[ d(\vec{k}) B_k(t) = - (3\lambda - 1) \left[ \frac{\dot{\varphi}_0^2}{\chi^2 k^2} + 2 f_1(\vec{k}) \right] \dot{\psi}_k(t) - (3\lambda - 1) \frac{H \dot{\varphi}_0}{\chi^2 k^2} \delta \varphi_k(t) \]
\[ - 4(3\lambda - 1) f_2(\vec{k}) H \psi_k(t) - \left\{ (3\lambda - 1) [V_0(\varphi_0) + V_4(\varphi_0) \vec{k}^4] H + \right. \]
\[ + 3(3\lambda - 1) \dot{\varphi}_0 H^2 - \frac{\dot{\varphi}_0^3}{2\chi^2} - \dot{\varphi}_0 f_1(\vec{k}) k^2 \left\} \frac{\delta \varphi_k(t)}{\chi^2 k^2} \right] \]
\[ d(\vec{k}) \nu_k(t) = (\lambda - 1) \frac{\dot{\varphi}_0}{\chi^2} \dot{\psi}_k(t) - 4(3\lambda - 1) H \dot{\psi}_k(t) + \]
\[ + \left\{ (3\lambda - 1) \dot{\varphi}_0 H + (\lambda - 1) [V_0(\varphi_0) + V_4(\varphi_0) \vec{k}^4] \right\} \frac{\delta \varphi_k(t)}{\chi^2} + \]
\[ + 4(\lambda - 1) f_2(\vec{k}) k^2 \psi_k(t). \]

Given the form of the common denominator (4.11), the solutions for \( B_k(t) \) and \( \nu_k(t) \) are regular in the limit \( \lambda \to 1 \) whenever \( H \neq 0 \).

In order to better understand how to interpret the low-momentum and the high-momentum limits, it is useful to rewrite the coefficient function \( d(\vec{k}) \) of Eq. (4.11) as follows (valid for \( H \neq 0 \) and \( \lambda \neq 1/3 \)):
\[ d(\vec{k}) = 4(3\lambda - 1) H^2 \left[ 1 + \frac{\lambda - 1}{2(3\lambda - 1)} \frac{\dot{\varphi}_0^2}{\chi^2 H^2} + \frac{\lambda - 1}{2(3\lambda - 1)} \left( -\eta + \frac{k^2}{\chi^2} + \frac{\eta^2}{\chi^2} \right) \frac{k^2}{H^2} \right]. \]

From this form of the expression, we see that the value of \( \vec{k} = k/a \) which separates the low momentum from the high momentum region is the Hubble momentum \( H \). This is not surprising since we expect fluctuations to behave differently for wavelengths larger and smaller than the Hubble radius. Note that in the short wavelength region \( k > aH \), the next-to-leading order terms in the expression for \( d(\vec{k}) \) are controlled by the ratio \( k/\chi \) (as long as \( H < \chi \) which will hold in the region of validity of the effective field theory).

In the long wavelength (IR) limit, the expression for \( d(\vec{k}) \) becomes
\[ d(\vec{k}) \sim 4(3\lambda - 1) H^2 + (\lambda - 1) \dot{\varphi}_0^2 / \chi^2 \]
\[ = \frac{3\lambda - 1}{3} \frac{\dot{\varphi}_0^2}{\chi^2} + \frac{4 V_0(\varphi_0)}{3} + \frac{8 \Lambda}{3}. \]

We see that the sign of the first term changes when \( \lambda \) crosses the critical value \( \lambda = 1/3 \). In the IR limit the first term dominates and hence \( d(\vec{k}) \) is positive. More generally, sufficient conditions for positivity of \( d(\vec{k}) \) are \( \lambda > 1/3, V_0(\varphi_0) > 0 \) and \( \Lambda > 0 \).

In the short wavelength (UV) limit, the expression for \( d(\vec{k}) \) becomes
\[ d(\vec{k}) \xrightarrow{k \to \infty} 2(\lambda - 1) f_1(\vec{k}) k^2, \]
which changes sign as \( \lambda \) crosses the value \( \lambda = 1 \).

V. SECOND ORDER ACTION

A. The action

We are now ready to discuss the second order action for cosmological fluctuations. We insert the metric ansatz including fluctuations into the full action, make use of the constraint equations to eliminate the variables \( \nu \) and \( B \), and expand. Working in Fourier space, after a lot of algebra the terms in the total action which are second order in the perturbation variables are

\[ \delta_2 S^{(\nu)} = \chi^2 \int dt \int \frac{d^3k}{(2\pi)^3} a^3 \left\{ c_{\varphi \dot{\varphi}^2} + c_{\psi \dot{\psi}^2} + c_{\psi \dot{\psi} \delta \varphi} + f_{\varphi \delta \varphi \dot{\varphi}} + f_{\psi \dot{\psi} \delta \varphi} + f_{\varphi \dot{\varphi} \psi \dot{\psi}} + \right. \]
\[ + f_{\varphi \psi \dot{\varphi} \delta \varphi} + \left. f_{\psi \dot{\phi} \delta \varphi} - m_{\varphi}^2 \phi_k^2 - m_{\psi}^2 \psi_k^2 - m_{\varphi \psi}^2 \psi_k \delta \varphi_k \right\}. \]
We first focus on the coefficients of the kinetic terms since they will tell us how many dynamical degrees of freedom survive and whether there are ghosts. These coefficients are given by

\[ d(k) c_\phi = 2(3\lambda - 1) \frac{H^2}{\chi^2} + (\lambda - 1) f_1(k) \frac{k^2}{\chi^2} \]  \tag{5.2}

\[ d(k) c_\phi = 2(3\lambda - 1) \left[ \frac{\phi_0^2}{\chi^2} + 2f_1(k)k^2 \right] \]  \tag{5.3}

\[ d(k) c_{\phi\phi} = 4(3\lambda - 1) \frac{H\phi_0}{\chi^2} \]  \tag{5.4}

We will discuss these terms below. First, we give the expressions for the coefficients of the terms involving one time derivative of a dynamical variable. They are

\[ d(k) f_\phi = -(3\lambda - 1) \frac{\phi_0^2 H}{\chi^4} - (\lambda - 1) \left[ V_{0,\phi}(\phi_0) + V_4(\phi_0)k^4 \right] \frac{\phi_0}{\chi^2} \]  \tag{5.5}

\[ d(k) f_\phi = -24(3\lambda - 1)H \Lambda + 12(3\lambda - 1)^2 H^2 \]  \tag{5.6}

\[ d(k) f_{\phi\phi} = 6(\lambda - 1) \frac{\phi_0 H}{\chi^3} - 3(\lambda - 1)(3\lambda + 1) \frac{\phi_0 H^2}{\chi^2} - \frac{3}{2}(\lambda - 1) \frac{\phi_0^3}{\chi^4} \]  \tag{5.7}

\[ d(k) f_{\phi\phi} = 4(3\lambda - 1) \frac{V_{0,\phi}(\phi_0)H}{\chi^2} - (3\lambda - 1) \frac{\phi_0^2}{\chi^4} \]  \tag{5.8}

Finally, we also give the expressions for the coefficients of the mass matrix. They are

\[ d(k) m^2_{\phi} = 2(3\lambda - 1) \frac{V_{0,\phi}(\phi_0)H^2}{\chi^2} + \frac{3}{2}(3\lambda - 1) \frac{\phi_0^2 H^2}{\chi^4} + (3\lambda - 1) \frac{V_{0,\phi}(\phi_0)\phi_0 H}{\chi^2} + \frac{1}{2}(\lambda - 1) \frac{V_{0,\phi}(\phi_0)\phi_0^2}{\chi^4} + \]  

\[ + \frac{1}{2}(\lambda - 1) \frac{V_{0,\phi}(\phi_0)^2}{\chi^6} - \frac{1}{4} \frac{\phi_0^4}{\chi^6} \left\{ 4(3\lambda - 1) \frac{V_{1}(\phi_0)H^2}{\chi^2} - (\lambda - 1) f_1(k) \frac{V_{0,\phi}(\phi_0)}{\chi^2} + \right\} \]  

\[ + \frac{1}{2} \left[ f_1(k) + 2(\lambda - 1) \right] \frac{\phi_0^2}{\chi^4} k^2 + \left\{ 4(3\lambda - 1) [V_{4,\phi}(\phi_0) + V_2(\phi_0)] H^2 - 2(\lambda - 1) f_1(k) V_1(\phi_0) + \right\} \]  

\[ + (3\lambda - 1) \frac{V_{1}(\phi_0)\phi_0 H}{\chi^2} + (\lambda - 1) [V_{4,\phi}(\phi_0) + V_2(\phi_0)] \frac{\phi_0^2}{\chi^4} \right\} \frac{k^4}{\chi^6} + \]  

\[ + \left\{ 4(3\lambda - 1) V_0(\phi_0)H^2 \chi^2 + 2(\lambda - 1) f_1(k) [V_2(\phi_0) + V_{4,\phi}(\phi_0)] \chi^2 + (\lambda - 1) V_0(\phi_0) \phi_0^2 \right\} \frac{k^6}{\chi^6} + \]  

\[ + (4\lambda - 1) f_1(k) V_0(\phi_0) \chi^4 + \frac{1}{2}(\lambda - 1) V_4(\phi_0) \phi_0^2 \chi^2 + \frac{k^8}{\chi^6} \]  \tag{5.9}

\[ d(k) m^2_{\phi\phi} = \frac{9}{2} (3\lambda - 1)^2 \frac{\phi_0 H^3}{\chi^2} + \frac{3}{2} (3\lambda - 1)(3\lambda - 7) \frac{V_{0,\phi}(\phi_0)H^2}{\chi^4} - \frac{3}{4} (3\lambda - 1) \left[ \phi_0^2 + 2V_0(\phi_0) \right] \frac{\phi_0 H}{\chi^4} + \]  

\[ + \frac{3}{2}(\lambda - 1) \left[ \frac{3}{2} \phi_0^2 + V_0(\phi_0) \right] \frac{V_{0,\phi}(\phi_0)}{\chi^2} + \frac{3}{2} (3\lambda - 1) \frac{\phi_0 H \Lambda}{\chi^2} - 3(\lambda - 1) \frac{V_{0,\phi}(\phi_0)\Lambda}{\chi^2} + \]  

\[ + \left\{ 2(3\lambda - 1) f_2(\bar{k}) \frac{\phi_0 H}{\chi^2} - (\lambda - 1) [3f_1(\bar{k}) - 2f_2(\bar{k})] \frac{V_{0,\phi}(\phi_0)}{\chi^2} \right\} \frac{\bar{k}^2}{\chi^4} + \left\{ \frac{3}{2} V_4(\phi_0) \Lambda^2 \right\} \frac{\bar{k}^6}{\chi^6} + \]  

\[ - \frac{9}{2} (3\lambda - 1) \frac{V_4(\phi_0)H^2}{\chi^2} + \frac{3}{4} \left[ \phi_0^2 + 2V_0(\phi_0) \right] \frac{V_4(\phi_0)}{\chi^4} + \frac{1}{2} (\lambda - 1) \bar{k}^4 + 2(\lambda - 1) f_2(\bar{k}) V_4(\phi_0) \frac{\bar{k}^6}{\chi^6} \]  \tag{5.10}
\[ d(\bar{k}) m_{\bar{k}}^2 = 3\Lambda \left[ 4(3\lambda - 1)H^2 + (\lambda - 1)\frac{\dot{\phi}_0^2}{\chi^2} \right] - \left[ \frac{3}{2}(13\lambda - 11)\frac{\dot{\phi}_0^2}{\chi^2} - 6V_0(\varphi_0) \right](3\lambda - 1)\frac{H^2}{\chi^2} + \\
-78(3\lambda - 1)^2 H^4 - 3(\lambda - 1)\frac{\dot{\phi}_0^2}{\chi^2} + \frac{3}{2}(\lambda - 1)V_0(\varphi_0)\frac{\dot{\phi}_0^2}{\chi^2} + \left\{ 6[f_1(\bar{k}) - 4f_2(\bar{k})] \Lambda + \\
-3(\lambda - 1)[13f_1(\bar{k}) - 12f_2(\bar{k})]H^2 - \frac{1}{2}[3f_1(\bar{k}) + 12f_2(\bar{k}) - 4\frac{\dot{\phi}_0^2}{\chi^2} + \\
+3[f_1(\bar{k}) - 4f_2(\bar{k})]V_0(\varphi_0) \right\}(\lambda - 1)k^2 + \left\{ 4(\lambda - 1)[f_1(\bar{k}) + 2f_2(\bar{k})] + \\
+8(3\lambda - 1)(8g2 + 3g3)\frac{H^2}{\chi^2} + 2(\lambda - 1)(8g2 + 3g3)\frac{\dot{\phi}_0^2}{\chi^2} \right\}\dot{k}^4 + \left\{ 4(\lambda - 1)f_1(\bar{k})(8g2 + 3g3) + \\
+8(3\lambda - 1)(8g7 - 3g8)\frac{H^2}{\chi^2} + 2(\lambda - 1)(8g7 - 3g8)\frac{\dot{\phi}_0^2}{\chi^2} \right\}\dot{\chi}^6 + 4(\lambda - 1)f_1(\bar{k})(8g7 - 3g8)\frac{\dot{\phi}_0^2}{\chi^2} \right\}\dot{\chi}^6 \right\} \cdot \right. \\
\times \left. \left( \frac{8}{\chi^2} \right)^2 \right\} \ . \quad (5.11) \]

Note that the coefficients of the mass matrix remain finite in the limit \( \bar{k} \to 0 \).

**B. Observations**

Recall that in the case of the original non-projectable version of HL gravity, only one of the two metric degrees of freedom was dynamical \cite{12}. Will this also be the case here? Note that, on setting \( \bar{k} = 0 \), the kinetic part of the Lagrangian becomes

\[ c_\varphi c_\psi^2 + c_\psi c_\varphi^2 + c_\varphi c_\psi \dot{\varphi}_k \delta \varphi_k \propto \left( \frac{H}{\dot{\varphi}_0} \delta \varphi_k + \dot{\psi}_k \right)^2 . \quad (5.12) \]

This suggests that the introduction of the Sasaki-Mukhanov variable \( \zeta \) defined as \cite{26 27}

\[ - \zeta \equiv \psi + \frac{H}{\dot{\varphi}_0} \delta \varphi \quad (5.13) \]

may reduce to one the actual number of dynamical degrees of freedom.

In terms of \( \zeta \) and \( \psi \), the kinetic part of the Lagrangian is a quadratic form with coefficients

\[ d(\bar{k}) c_\zeta = \left[ 2(3\lambda - 1) + (\lambda - 1)f_1(\bar{k}) \frac{\dot{\bar{k}}^2}{H^2} \right] \frac{\dot{\phi}_0^2}{\chi^2} \quad (5.14) \]

\[ d(\bar{k}) c_\psi = \left[ 4(3\lambda - 1) + (\lambda - 1)\frac{\dot{\phi}_0^2}{\chi^2 H^2} \right] f_1(\bar{k})\dot{k}^2 \quad (5.15) \]

\[ d(\bar{k}) c_\zeta = 2(\lambda - 1)\frac{\dot{\phi}_0^2}{\chi^2 H^2} f_1(\bar{k})\dot{k}^2 . \quad (5.16) \]

Observe that \( c_\psi^2 \) and \( c_\zeta \psi \) tend to zero as \( \bar{k} \to 0 \), whereas \( c_\zeta \) is non-trivial as long as the matter field is present. However, we see how the presence of the term \( f_1(\bar{k}) \neq 0 \) is present only in the healthy extension of HL gravity - alters the findings of Ref. \cite{12}: Both metric degrees of freedom survive as dynamical variables.

To exclude the possibility that there is a single dynamical metric fluctuation variable (which would by the above analysis have to be different from \( \zeta \)), we need to find the eigenvalues of the coefficient matrix of the kinetic part of the Lagrangian. Returning to the original variables \( \varphi, \psi \), we consider the kinetic matrix \cite{9}

\[ \begin{pmatrix} \chi^2 c_\varphi & \chi c_\varphi c_\psi \\ \frac{\chi^2 c_\psi}{2} & c_\psi \end{pmatrix} \ . \quad (5.17) \]

\cite{9} We have multiplied the \( c \)’s by proper powers of \( \chi \) in order to make the matrix dimensionally homogeneous. Such a rescaling is equivalent to considering \( \delta \varphi_k / \chi \) and \( \psi \) as the two dynamical variables.
which has the following eigenvalues:

\[
d(k)c_{1,2} = (3\lambda - 1) \left( \frac{\dot{\phi}^2}{\chi^2} + H^2 \right) + \frac{13\lambda - 5}{2} f_1(\bar{k})k^2 \\
\pm \sqrt{(3\lambda - 1)^2 \left( \frac{\dot{\phi}^2}{\chi^2} + H^2 \right)^2 + (11\lambda - 3)(3\lambda - 1) \left( \frac{\dot{\phi}^2}{\chi^2} - H^2 \right) f_1(\bar{k})k^2 + \left( \frac{11\lambda - 3}{2} \right)^2 f_1(\bar{k})^2k^4}
\]

(5.18)

It is easy to see that for \( f_1(\bar{k}) = 0 \) one eigenvalue is exactly zero. However, in the healthy extension of HL gravity both eigenvalues are non-vanishing and hence both degrees of freedom are dynamical. This sounds like bad news for the model. However, we shall now show that in the infrared limit one of the modes decouples (its mass tends to infinity).

First, however, let us consider under which conditions the linear cosmological perturbations are ghost-free. We first realize that if

\[
\frac{1}{d(k)} \left[ (3\lambda - 1) \left( \frac{\dot{\phi}^2}{\chi^2} + H^2 \right) + \frac{13\lambda - 5}{2} f_1(\bar{k})k^2 \right] < 0,
\]

(5.19)

then we will for sure have one negative eigenvalue, i.e. the extra dynamical degree of freedom will be ghost-like. In the opposite case, in which the expression on the left-hand side of Eq. (5.19) is positive, one needs to look more carefully at the expressions. We rewrite \( c_{1,2} \) as

\[
c_{1,2} = A \pm \sqrt{B}
\]

(5.20)

and then need to check whether \( A^2 - B \) is positive or negative in the case \( A > 0 \). The difference is

\[
A^2 - B = 2(3\lambda - 1)f_1(\bar{k})\bar{k}^2.
\]

(5.21)

In the IR limit, \( f_1(\bar{k}) \) tends to \(-\eta\), and thus the condition for ghost freeness is simply

\[
\eta < 0.
\]

(5.22)

For larger values of \( \bar{k} \) is not so easy to estimate the sign of the difference in (5.21), since we are dealing with several parameters - \( \lambda, \eta, \eta_2, \eta_4 \) - which can have arbitrary signs in the most general case. The same difficulty arises when wishing to determine the region in parameter space where Eq. (5.19) is satisfied.

In the following we will assume that the inequality in Eq. (5.19) is reversed, and \( \eta < 0 \) so that the theory is ghost-free in the infrared. First let us consider the IR limit of the expressions for the eigenvalues \( c_{1,2} \). We have\(^{10}\)

\[
c_1 \simeq \frac{1}{2},
\]

(5.23)

\[
c_2 \simeq -\eta \left( \frac{\bar{k}}{H} \right)^2.
\]

(5.24)

Note that the eigenvalue of the extra degree of freedom goes to zero in the IR limit. From the expressions for the mass matrix coefficients we see that they do not tend to zero in the IR limit. Hence, if we re-scale the new scalar gravitational degree of freedom such that it has canonical kinetic normalization in the IR, we see that the effective mass of this degree of freedom will diverge as \( H/\bar{k} \). Thus, we see that the extra scalar degree of freedom in the healthy extension of HL gravity decouples in the IR limit. Hence, at late times, it will not contribute to cosmological perturbations on scales relevant to current cosmological observations. This also explains the results of \(^{19}\) who find that the cosmological fluctuations in the healthy extension of HL gravity agree quite well with those in Einstein gravity\(^{11}\). On UV scales, however, the extra scalar gravitational mode will play a role. This may effect the early evolution of

\(^{10}\) We use the approximation \( H^2 \gg \dot{\phi}^2/\chi^2 \) which is justified since the right hand side is one several positive terms which appear in the expression for \( H^2 \).

\(^{11}\) There is a potential worry: in terms of the re-scaled variable, it appears that the theory could be strongly coupled since the coefficients of the interaction terms will diverge more strongly than the mass (we thank Omid Saremi for raising this concern). This issue deserves further study. We believe, however, that there should be no problem since in terms of the original variables the interaction terms are Planck-suppressed.
fluctuations in inflationary cosmology on sub-Hubble scales and hence be relevant to the “trans-Planckian” problem of the inflationary universe scenario.

In the UV limit we have
\[ d(\bar{k}) \simeq 2(\lambda - 1)f_1(\bar{k})\bar{k}^2, \]
and from this we can easily show that the eigenvalues are
\[ c_1 \simeq \frac{1}{2}, \]
\[ c_2 \simeq \frac{3\lambda - 1}{\lambda - 1}, \]
which is negative for all values of $\lambda$ between $1/3$ and $1$. In this range of values of $\lambda$ the extra scalar degree of freedom will be a ghost.

The transition scale between the IR and UV is, as expected, at $\bar{k} = H$. Thus, for applications to cosmology the theory should be ghost-free both in the IR and UV. Hence, we see that the requirements for a “good” behavior of the extra degree of freedom are $\eta < 0$ and $\lambda > 1$. Note that the limit $\lambda \rightarrow 1$ is smooth as long as $H \neq 0$.

VI. MINKOWSKI LIMIT

Let us finally discuss the Minkowski limit of our analysis which is obtained if we drop the matter contribution to the action. In this case, we only have one dynamical degree of freedom for scalar metric fluctuations, namely $\psi$. We can observe that $m_\psi^2 = 0$ in the absence of matter and of the cosmological constant.

The constraint equations can easily be solved and yield:
\[ \bar{k}^2 B_k(t) = -\frac{3\lambda - 1}{\lambda - 1} \dot{\psi}_k(t) \]
and
\[ \nu_k(t) = \frac{2f_2(\bar{k})}{f_1(\bar{k})} \psi_k(t). \]

Inserting these expressions into the action for fluctuations and dropping the matter terms gives us
\[ \delta_2 S^{(s)} = 2\lambda^2 \int dt \frac{d^3k}{(2\pi)^3} \left\{ \frac{3\lambda - 1}{\lambda - 1} \dot{\psi}_k^2 - \left[ \left( 1 + \frac{2f_2(\bar{k})^2}{f_1(\bar{k})} \right) \bar{k}^2 + (8g_2 + 3g_3)\frac{\bar{k}^4}{\lambda^2} + (8g_7 - 3g_8)\frac{\bar{k}^6}{\lambda^4} \right] \psi_k(t)^2 \right\} \]
(6.3)

In the IR limit, the equation of motion becomes
\[ \frac{3\lambda - 1}{\lambda - 1} \dot{\psi}_k - \frac{2 - \eta}{\eta} \bar{k}^2 \psi_k = 0. \]
(6.4)

We can make the following observations. First, there is ghost instability for $1/3 < \lambda < 1$. Secondly, there is a classical instability unless $0 < \eta < 2$. These conclusions are in perfect agreement with those of [19].

VII. DISCUSSION AND CONCLUSIONS

We have studied cosmological perturbations in the “healthy” extension of HL gravity proposed in [1], assuming the simplest form of matter, namely a scalar matter field. Starting from the second order action for the fluctuations, we studied the general properties of the scalar modes. We find that there are two dynamical degrees of freedom, unlike in Einstein gravity and unlike what happens in the non-projectable version of the original HL model [12]. We identified the conditions under which the extra degree of freedom is well behaved (i.e. not a ghost). In the infrared (IR) limit, the condition is $\eta < 0$, where $\eta$ is one of the constant coefficients defining the extension of the Lagrangian. In the ultraviolet regime the condition is $\lambda > 1$ or $\lambda < 1/3$. For the theory to be healthy overall, both the condition on $\lambda$ and that on $\eta$ have to be satisfied.

Our second main result is that in the IR limit, the mass of the canonically normalized extra scalar gravitational degree of freedom tends to infinity. Thus, the extra mode decouples from low energy physics. Hence, when applied to
cosmology, we find that the extra gravitational degree of freedom is harmless for late-time cosmological perturbations. It may, however, have interesting consequences for early universe cosmology.

We note that another “healthy” extension of HL gravity has recently been proposed [29]. In that model, the Lagrangian is constructed so that there is no extra scalar gravitational mode. It would be interesting to study cosmological fluctuations in that model, as well.

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