ABELIAN AUTOMORPHISM GROUPS
OF 3-FOLDS OF GENERAL TYPE

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Abstract. This paper is devoted to the study of abelian automorphism groups of surfaces and 3-folds of general type over complex number field $\mathbb{C}$. We obtain a linear bound in $K^3$ for abelian automorphism groups of 3-folds of general type whose canonical divisor $K$ is numerically effective, and we improve on Xiao’s results on abelian automorphism groups of minimal smooth projective surfaces of general type. More precisely, the main results in this paper are the following.

Theorem 3.0. Let $X$ be a smooth 3-fold of general type over the complex number field, and $K_X$ the canonical divisor of $X$. Let $G$ be an abelian group of automorphisms of $X$ (i.e. $G \subset \text{Aut}(X)$). Suppose $K$ is nef. Then there exists a universal constant coefficient $c$ such that

$$\# G \leq cK_X^3.$$  

Theorems 6.5 and 7.1. Let $S$ be a minimal smooth surface of general type over the complex number field, and $K$ the canonical divisor of $S$. Let $G$ be an abelian group of automorphisms of $S$ (i.e. $G \subset \text{Aut}(S)$). Suppose that $\chi(O_S) \geq 8$. Then

$$\# G \leq 36K^2 + 24.$$  

Moreover, suppose that $K^2 \geq 181$, and that either $S$ has no pencil of curves of genus $g$, $3 \leq g \leq 5$, or $K^2_S \geq \frac{12(g-1)}{g+5} \chi(O_S)$ when $S$ has a pencil of curves of genus $g$, $3 \leq g \leq 5$. Then

$$\# G \leq 24K^2 + 256.$$  

Introduction

Let $V$ be a smooth projective variety of general type over the complex number field, $K_V$ the canonical divisor of $V$. It is well-known that the automorphism group $\text{Aut}(V)$ of $V$ is finite. When $d := \text{dim } V = 1$, a classical theorem of Hurwitz says that $\# \text{Aut}(V) \leq 42 \text{deg } K_V$. When $d = 2$, the automorphism groups for a complex smooth projective surface of general type have been thoroughly studied by many authors. In 1950, Andreotti showed that the automorphism group $\text{Aut}(S)$ of a complex smooth projective surface $S$ of general type is finite and bounded by an exponential function of $K^2$ [A]. Then Corti, Huckleberry and Saner independently...
proved that $|\text{Aut}(S)|$ is bounded by a polynomial function of small degree in $K^2$ [C], [HS]. Recently, progress has been achieved by Xiao, who has obtained a linear bound for $\text{Aut}(S)$, in good analogy with the case of curves [X2], [X3]:

Xiao’s theorem 1. Let $S$ be a minimal smooth surface of general type over the complex number field, $K$ the canonical divisor of $S$. Then

$$|\text{Aut}(S)| \leq 42^2K^2,$$

and this bound is the best.

Soon thereafter Chen has completely studied the automorphism groups for a surface of general type with a pencil of genus 2 [Ch1]:

Chen’s theorem 1. Let $S$ be a minimal smooth surface of general type with a minimal genus 2 fibration, and $K$ as above. Suppose $K^2_S \geq 140$. Then (i) $|\text{Aut}(S)| \leq 504K^2$; moreover if $f$ is not locally trivial, then $|\text{Aut}(S)| \leq 288K^2$. And these are all the best bounds.

When $d \geq 3$, Xiao conjectured that under the assumption that $K_V$ is nef (i.e., numerically effective), then there is a number $\Delta_d$, depending only on $d$, such that

$$\# \text{Aut}(V) \leq \Delta_dK^d_V.$$

It is an intriguing problem to generalize these bounds to higher dimensions. In the attempts to proving such a conjecture, the order of abelian subgroups has a special importance (see [X1] for a discuss).

Let $G$ be an abelian subgroup of $\text{Aut}(V)$. One has that $\#G \leq 2 \deg K_V + 8$ when $d = 1$ [N, W]. For two dimensional case, Howard and Sommese proved that $|G|$ is bounded by the square of $K^2$ times a constant (among other things). Soon before finishing the great work mentioned above, Xiao considered the induced linear representation of $G$ on the space of global sections of pluricanonical sheaves and obtained a linear bound for $\#G$ in terms of $K^2_V$ [X1]:

Xiao’s theorem 2. Let $S$, $K$ and $G$ be as above. Suppose $K^2_S \geq 140$. Then $|G| \leq 52K^2_S + 32$.

For abelian automorphism groups of surfaces of general type with a pencil of genus 2, Chen obtained a best bound [Ch1]:

Chen’s theorem 2. Let $S$ be a minimal smooth surface of general type with a minimal genus 2 fibration, and $G$ be as above. Then $|G| \leq 12.5K^2_S + 100$, if $K^2 \geq 9$.

The purpose of this paper is to study the abelian automorphism groups of surfaces and 3-folds of general type. Of the two topics which we will concentrate on, one is to give a 3-dimensional generalization of Xiao’s results in [X1] and the other is to improve on Xiao’s Theorem 2. Our main results are the following.

Theorem 3.0. Let $X$ be a smooth 3-fold of general type over the complex number field, and $K_X$ the canonical divisor of $X$. Let $G$ be an abelian group of automorphisms of $X$ (i.e., $G \subseteq \text{Aut}(X)$). Suppose $K$ is nef. Then there exists a universal
constant coefficient $c$ such that

$$
\#G \leq cK^3_X.
$$

Theorems 6.5 and 7.1. Let $S$ be a minimal smooth surface of general type over
the complex number field, and $K$ the canonical divisor of $S$. Let $G$ be an abelian
group of automorphisms of $S$ (i.e. $G \subset \text{Aut}(S)$). Suppose that $\chi(O_S) \geq 8$. Then

$$
\#G \leq 36K^2 + 24.
$$

Moreover, suppose that $K^2 \geq 181$, and that either $S$ has no pencil of curves of
genus $g$, $3 \leq g \leq 5$ or $K^2_S \geq \frac{12(g-1)}{g+5}\chi(O_S)$ when $S$ has a pencil of curves of genus $g$, $3 \leq g \leq 5$. Then

$$
\#G \leq 24K^2 + 256.
$$

The arguments here are inspired by the work of Xiao [X1]. We look at the induced
linear representation of $G$ on the space $H_n$ of global sections of pluricanonical
sheaves $\omega^n_V$. Consider the natural map

$$
H_{n-t} \otimes H_{n+t} \oplus H_n \otimes H_n \to H_{2n},
$$

instead of $H_n \otimes H_n \to H_{2n}$ as in [X1], we have that under some weak conditions,
there are at least two $G$-semi-invariants in $H_{2n}$ corresponding to a same character
of $G$. Then the corresponding divisors in $|2nK_S|$. generate a pencil $\Lambda$ whose
general fiber $F$ is fixed by $G$. Therefore $\#G$ is limited by the order of the group of
automorphisms of the normalization $\tilde{F}$ of $F$ as a smooth curve when $\dim(V) = 1$ or
that of the minimal model $\tilde{F}_0$ of the desingularization $\tilde{F}$ of $F$ as a minimal smooth
surface of general type when $\dim(V) = 2$. Theorems 3.0, 6.5 and 7.1 are obtained
in this way.

I am convinced that essentially the same methods used to prove Theorem 3.0
can be used to prove the statement for higher dimensional algebraic varieties of
general type.

I hope to return to this subject in a later occasion.

Chapter I Abelian automorphism groups of 3-folds: a linear bound

§1 Preliminaries on 3-folds

We work throughout over the complex number $\mathbb{C}$.

For the reader’s convenience, we collect some Preliminaries on 3-folds of general
type in this section. For Preliminaries on surfaces we refer the reader to the standard
text, e.g. [BPV]. We use the standard notation as in [BPV] unless otherwise stated.
Let $X$ be a nonsingular projective 3-fold over $\mathbb{C}$, and $D \in \text{Div}(X)$, where $\text{Div}(X)$ is a free abelian group generated by Weil divisors on $X$. We say that $D$ is nef if $D.C \geq 0$ for any curve $C$ on $X$, and that $D$ is big if the Iitaka dimension $\kappa(D, X) = 3$ [Ii]. If $D$ is nef, then for any integer $0 \leq i \leq 3$ we know that $D^i.W \geq 0$ for every codimension $i$ subvariety $W$ of $X$ (cf. [Ha, p.34]). We say that $X$ is of general type if $K_X$ is big. If $K_X$ is nef, then $K_X$ is big iff $K_X^3 > 0$ (Sommese, cf. [Ka, Lemma 3]).

We denote the linear equivalence and the numerical equivalence by $\equiv$ and $\sim$ respectively. Denote by $\Phi_n$ the rational map associated with the complete linear system $|nK_X|$. By Hirzebruch-Riemann-Roch theorem we have

$$\chi(\mathcal{O}_X(nK_X)) = \frac{1}{12}(2n - 1)n(n - 1)K_X^3 + (1 - 2n)\chi(\mathcal{O}_X).$$

**Fact 1.2.** (cf. [Ka, Theorem 2], [V, Theorem 1]) If $L$ is an nef and big line bundle on $X$, then $H^i(X, L \otimes \omega_X) = 0$ for $i > 0$.

**Corollary 1.3.** If $X$ is of general type and $K_X$ is nef, then

(i) $p_n := h^0(nK_X) = \frac{1}{12}(2n - 1)n(n - 1)K_X^3 + (1 - 2n)\chi(\mathcal{O}_X)$ for $n \geq 2$;

(ii) $\chi(\mathcal{O}_X) \leq K_X^3/6$, and $K_X^3$ is even;

(iii) $p_n \geq 5$ for $n \geq 3$.

**Fact 1.4.** Let $X$ be a smooth 3-fold of general type and $f: X \to B$ be a fiber space, where $B$ is a smooth projective curve. By the easy addition formula (cf. [U, Chapter 6]), the general fiber of $f$ is a smooth projective surface of general type.

**Fact 1.5.** (cf. [Ma, Theorem 8]) Let $X$ be as in Corollary 1.3. Then $\Phi_n$ is birational for $n \geq 8$.

Finally we need a rough upper estimate of $-\chi(\mathcal{O}_X)$ in terms of $K_X^3$.

**Proposition 1.6.** Let $X$ be as in Corollary 1.3. Then $-\chi(\mathcal{O}_X) \leq \frac{5}{2}K_X^3 + 1$.

**Proof.** We can suppose $-\chi(\mathcal{O}_X) > 0$; consequently $p_2 \geq 2$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
X' & \xrightarrow{f} & C \\
\downarrow \pi & & \downarrow h \\
X \cdots \Phi_2 & \rightarrow & W := \Phi_2(X) \subset \mathbb{P}^{p_2-1}
\end{array}
$$

where $\pi$ is a succession of blowing-ups with nonsingular centers such that $g := \Phi_2 \circ \pi$ is a morphism, and $\alpha = h \circ f$ is the stein factorization. Let $\alpha = \deg h$, and $E$ be a

general fiber of \( f \). Let \( H \) be a hyperplane section of \( W \) in \( \mathbb{P}^{p_2-1} \), and \( \beta \) the degree of \( W \) in \( \mathbb{P}^{p_2-1} \). We have
\[
\pi^*(2K_X) \equiv f^*h^*(H) + Z,
\]
where \( Z \) is the fixed part of \( |\pi^*(2K_X)| \).

If \( \dim W = 1 \), then \( \pi^*(2K_X) \sim \alpha \beta F + Z \). Multiplying this equality by \( \pi^*(K_X)^2 \), we have
\[
2K_X^3 = \alpha \beta \pi^*(K_X)^2.F + \pi^*(K_X)^2.Z.
\]
Since \( \pi^*(K_X) \) is nef and big, and \( F \) is nef and \( F \not\sim 0 \), we have \( \pi^*(K_X)^2.F \geq 1 \) and \( \pi^*(K_X)^2.Z \geq 0 \). Hence \( 2K_X^3 \geq \beta \).

If \( \dim W = 2 \), then \( \pi^*(K_X)^2 \sim \beta F + \pi^*(2K_X).Z + (f^*h^*H).Z \). Multiplying this equality by \( \pi^*(K_X) \), for the same reason as above we have
\[
4K_X^3 = \beta \pi^*(K_X).F + 2\pi^*(K_X)^2.Z + \pi^*(K_X).f^*h^*H.Z \geq \beta.
\]
If \( \dim W = 3 \), for the same reason as above we have
\[
\pi^*(K_X)^3 - (f^*h^*H)^3 = Z(\pi^*(2K_X)^2 + \pi^*2K_X.f^*h^*H + (f^*h^*H)^2) \geq 0.
\]
Thus \( 8K_X^3 \geq \beta \).

Summing up the above inequalities, we have \( 8K_X^3 \geq \beta \).

Now the result follows from the well-known fact that \( \beta \geq \text{codim} W + 1 \) (cf. e.g. [Mu, p. 77]). \( \square \)

§2 Mid-points of finite lattice points: some technical lemmas

Our goal in the next section is to find a linear bound for the order of abelian automorphism groups of 3-folds of general type. Our technique is to reduce the problem to that of estimating the number of mid-points of some finite lattice points in a linear space.

**Definition 2.1** ([X1]). Let \( \mathcal{A} \) and \( \mathcal{B} \) be finite sets of points in a linear space \( P \).
We define \( \mathcal{A.B} \) the set of mid-points \( \frac{1}{2}(p + q) \) of two points \( p \) in \( \mathcal{A} \) and \( q \) in \( \mathcal{B} \) (\( p \) and \( q \) may be the same point if \( \mathcal{A} \cap \mathcal{B} \neq \emptyset \); so \( \mathcal{A.B} \subset \mathcal{A.B} \)).

We define the **dimension** of \( \mathcal{A} \) to be the dimension of the (affine) space generated by \( \mathcal{A} \). Let \( \mathcal{A} \) be a finite set of points in \( P \), and \( \mathcal{B} \) a subset of \( \mathcal{A} \). The set \( \mathcal{B} \) is said to be **relatively convex** in \( \mathcal{A} \), if no point of \( \mathcal{A} - \mathcal{B} \) is contained in the convex hull of \( \mathcal{B} \). The set \( \mathcal{B} \) is called **integrally convex** if it is relatively convex in some lattice \( \mathcal{A} \) generating \( P \). With such a lattice \( \mathcal{A} \) fixed, we will call the points in \( \mathcal{A} \) integral points.

A **chain** in a set \( \mathcal{B} \) is by definition a series of points \( p_1, \ldots, p_n \) in \( \mathcal{B} \) such that the vectors \( p_i - p_{i-1} \) \((i = 2, \ldots, n)\) are equal. In this case, \( n \) is called the **length** of the chain. If \( \mathcal{B} \) is integrally convex (in a fixed lattice \( \mathcal{A} \)) and \( p, q \) are two points in \( \mathcal{B} \), then the integral points on the line segment joining \( p \) and \( q \) form a chain in \( \mathcal{B} \) in an obvious way.
Lemma 2.2. Let $A_i$ ( $i = 1, 2, 3$) be finite integral sets in a $\mathbb{Q}$-linear space with $A_1 \subset A_2 \subset A_3$. Let $P_i$ be the enveloping space of $A_i$ for $i = 1, 2, 3$. Assume $\dim P_2 < \dim P$. Then there exists an integral linear map $\varphi$ (i.e., $\varphi$ maps integral points to integral points) from $P_3$ to $P_2$ such that

(i) $\varphi|_{A_3}$ is injective, and $\varphi|_{P_2}$ is identity;

(ii) $(A_1 \cup A_3 \cup A_2) \geq (A_1 \cup A_3 \cup A_2 \cup A_2)$.

Proof. Let $x_1, \ldots, x_n$ be a (integral) basis of $P$. We can suppose $\dim P_2 = \dim P - 1$, and $P_2$ is the hyperplane $x_n = 0$. Since $A_3$ is a finite set, we can choose an integer $t$ such that $|a_i| + |b_i| < t$ ($i = 1, \ldots, n$) for any two points $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$ in $A_3$. We define

$$\varphi: P \to P_2$$

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1 + tx_n, x_2 + tx_n, \ldots, x_{n-1} + tx_n, 0).$$

Clearly, $\varphi$ satisfies (i) by the choice of $t$. Now the mid-point of two points $p$ in $A_1$ and $q$ in $\varphi(A_3)$ is the image of the mid-point of two points $p$ in $A_1$ and $\varphi^{-1}(q)$ in $A_3$, and if $\frac{1}{2}(p_1 + q_1) = \frac{1}{2}(p_2 + q_2)$ for $p_i$ in $A_1$ and $q_i$ in $A_3$, then

$$\frac{1}{2}(p_1 + \varphi(q_1)) = \frac{1}{2}(p_2 + \varphi(q_2)).$$

Hence $\varphi$ satisfies (ii). \qed

Similarly, we have the following two lemmas.

Lemma 2.2.1. Let $A_1$, $P_1$ be as in Lemma 2.2. Assume $\dim P_1 < \dim P_2$, and $P_2 = P_3$. Then there exists an integral linear map $\varphi$ (i.e., $\varphi$ maps integral points to integral points) from $P_2$ to $P_1$ such that

(i) $\varphi|_{A_3}$ is injective, and $\varphi|_{P_2}$ is identity;

(ii) $(A_1 \cup A_3 \cup A_2) \geq (A_1 \cup A_3 \cup A_2 \cup A_2)$.

Lemma 2.2.2. Let $A_i$, $P_i$ be as in Lemma 2.2. Assume $P := P_1 = P_2 = P_3$, and the dimension of $P$ is $d \geq 2$. Let $x_1, \ldots, x_n$ be a (integral) basis of $P$. Then there exists an integral linear map $\varphi$ (i.e., $\varphi$ maps integral points to integral points) from $P$ to the hyperplane $P': x_n = 0$, such that. Then there exists an integral linear map $\varphi$ (i.e., $\varphi$ maps integral points to integral points) from $P$ to $P'$ such that

(i) $\varphi|_{A_3}$ is injective; and

(ii) $(A_1 \cup A_3 \cup A_2) \geq (\varphi(A_1) \cup \varphi(A_3) \cup \varphi(A_2)).$

Definition 2.3 ([X1]). Let $A$ be a finite integral set in a space $P$ with a basis $x_1, \ldots, x_n$. The arrangement of $A$ with respect to $x_i$ is a new integral set $A'$ such that a point $(a_1, \ldots, a_n)$ is contained in $A'$ iff

(i) $a_i \geq 0$, and

(ii) there are at least $a_i + 1$ points $(b_1, \ldots, b_n)$ in $A$ such that $a_j = b_j$ for all $j \neq i$. 

It is immediate that \( \mathcal{A}' \) is integral, and has the same cardinality and dimension as that of \( \mathcal{A} \). But if \( \mathcal{A} \) is convex, \( \mathcal{A}' \) need not be convex. Clearly, if \( \mathcal{A}_i \) \( (i = 1, 2, 3) \) are finite integral sets in \( P \) with \( \mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \), then by the definition we have \( \mathcal{A}'_1 \subset \mathcal{A}'_2 \subset \mathcal{A}'_3 \).

**Lemma 2.4.** Let \( \mathcal{A}_i \) \( (i = 1, 2, 3) \) be finite integral sets of dimension \( d \geq 3 \) with \( \mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \), and let \( \mathcal{A}'_i \) be the arrangement of \( \mathcal{A}_i \) with respect to a coordinate axis. Then

\[
\#(\mathcal{A}_1 \cdot \mathcal{A}_3 \cup \mathcal{A}_2 \cdot \mathcal{A}_3) \geq \#(\mathcal{A}'_1 \cdot \mathcal{A}'_3 \cup \mathcal{A}'_2 \cdot \mathcal{A}'_3).
\]

**Proof.** We prove the lemma for the case \( d = 3 \); for the case \( d > 3 \), the proof is similar, and left to the reader.

For the sake of simplicity, let \( i = 1 \), and use \( x, y, z \) instead of \( x_1, x_2, x_3 \). Take any two integers \( y_m, z_m \), and assume that there are \( k \) points in \( \mathcal{A}'_1, \mathcal{A}'_3 \) with \( y = y_m, z = z_m \). Then there exist two points \( p = (x_1, y_1, z_1) \) and \( q = (x_2, y_2, z_2) \) in \( \mathcal{A}_3 \) whose mid-point is \((\frac{1}{2}(k-1), y_m, z_m)\). Now \((\frac{1}{2}(k-1), y_m, z_m)\) is either in \( \mathcal{A}'_1, \mathcal{A}'_3 \) or in \( \mathcal{A}_2, \mathcal{A}'_2 \). Hence we have either \( p \in \mathcal{A}'_1, q \in \mathcal{A}'_3 \) or \( p, q \in \mathcal{A}_2 \). Now we suppose \( p \in \mathcal{A}'_1, q \in \mathcal{A}'_3 \). (for the latter case, the proof is similar.) By the definition of arrangement, this means that there are at least \( x_1 + 1 \) points with \( y = y_1, z = z_1 \) in \( \mathcal{A}_1 \) and at least \( x_2 + 1 \) points with \( y = y_2, z = z_2 \) in \( \mathcal{A}_3 \). Because \( x_1 + x_2 + 1 = k \), we see that the points in \( \mathcal{A}_1 \) with \( y = y_1, z = z_1 \) and the points in \( \mathcal{A}_3 \) with \( y = y_2, z = z_2 \) produce at least \( k \) mid-points with \( y = y_m, z = z_m \) in \( \mathcal{A}_1, \mathcal{A}_3 \). In fact, let the \( x_1 + 1 \) (resp. \( x_2 + 1 \)) points in the first (resp. second) row of \( \mathcal{A}_1 \) (resp. \( \mathcal{A}_3 \)) be \( p_1, \ldots, p_{x_1+1} \) (resp. \( q_1, \ldots, q_{x_2+1} \)) such that if \( i < j \) then the \( x \)-coordinate of \( p_i \) (resp. \( q_i \)) is less than \( p_j \) (resp. \( q_j \)). Then the mid-points

\[
\frac{1}{2}(p_1 + q_1), \frac{1}{2}(p_1 + q_2), \ldots, \frac{1}{2}(p_1 + q_{x_2+1}), \frac{1}{2}(p_2 + q_{x_2+1}), \ldots, \frac{1}{2}(p_{x_1+1} + q_{x_2+1})
\]

form the desired subset of \( \mathcal{A}_1, \mathcal{A}_3 \). \( \square \)

**Lemma 2.5.** Let \( \mathcal{A}_i \) \( (i = 1, 2, 3) \) be finite integrally convex sets with \( \mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \), and the dimension of \( \mathcal{A}_1 \) is \( 3 \). Suppose that the length of the longest chain in \( \mathcal{A}_3 \) (resp. \( \mathcal{A}_2 \)) is less than \( \frac{1}{6} \# \mathcal{A}_3 \) (resp. \( \frac{1}{15} \# \mathcal{A}_2 \)). And suppose

\[
\# \mathcal{A}_2 \geq 21, \quad \# \mathcal{A}_3 \leq 2 \# \mathcal{A}_2.
\]

Then

\[
\#(\mathcal{A}_1 \cdot \mathcal{A}_3 \cup \mathcal{A}_2 \cdot \mathcal{A}_3) \geq \min \left\{ \# \mathcal{A}_3 + 3 \# \mathcal{A}_2 - 23; \right. \left. \frac{5}{6} \# \mathcal{A}_3 + \frac{10}{3} \# \mathcal{A}_2 - 10; \right. \left. \frac{5}{6} \# \mathcal{A}_3 + \frac{13}{4} \# \mathcal{A}_2 - 2; \right. \left. \frac{7}{12} \# \mathcal{A}_3 + \frac{15}{4} \# \mathcal{A}_2 - 6; \right. \left. \frac{1}{2} \# \mathcal{A}_3 + 4 \# \mathcal{A}_2 - 4; \right. \left. 5 \# \mathcal{A}_3 - 91 \right\}
\]
Proof. By Lemmas 2.2, 2.2.1, 2.2.2, we can suppose the dimension of the enveloping space of $A_3$ is 3. Let $p_1, \ldots, p_l$ be a longest chain in $A_2$. Assume $p_1$ is the origin of the enveloping space, and that $p_2 = (1, \ 0, \ 0)$. Let $x$, $y$, $z$ be this basis. Then

$$3 \sqrt{\#A_2} \leq l \leq \frac{1}{4} \#A_2.$$  

Arrange $A_n$ ($i = 1, \ldots, 3$) with respect to $x$-axis, then with respect to $y$, and then to $z$. By Lemma 4, we have only to count the number of points in the set of mid-points of the new set $(A_1', A_3' \cup A_2' \cup A_3')$ thus produced.

Let $m_{xi}$ (resp. $m_{yi}$, $m_{zi}$) be the number of points of $A_i'$ on the $x$ (resp. $y$, $z$) axis, $i = 1, \ 2, \ 3$. We have $m_{x2} = l \leq \frac{1}{4} \#A_2$ (cf. [5, p. 625]) and $m_{x3} \leq \frac{3}{6} \#A_3$, $m_{x1} \leq m_{x2} \leq m_{x3}$, etc. If the basis elements $y$ and $z$ are chosen carefully, $A_i'$ will also be of dimension 3. We may assume that the points of $A_3'$ have either $z = 0$, or $z = 1$, $y = 0$ (cf. [5, p. 626]).

Now for simplicity of the notation, we replace $A_i'$ by $A_i$. Remark that $(0, \ 0, \ 1)$ is in $A_1$, and although $A_i$ is not convex now, it has the property that if a point $(a, b, c)$ is in $A_i$, then all integral points $(a', b', c')$ such that

$$0 \leq a' \leq a, \ 0 \leq b' \leq b, \ 0 \leq c' \leq c$$

are in $A_i$ too.

Let $A_{i0}$ be just the subset of points of $A_i$ in the plane $z = 0$. Let $t_i$ ($i = 1, \ 2, \ 3$) be the number of points in $A_i$ with $z = 1$. Clearly,

$$1 \leq t_1 \leq t_2 \leq t_3 \leq m_{x3} \leq \frac{1}{6} \#A_3.$$  

In what follows we denote by $S$ the set $A_1.A_3 \cup A_2.A_2$. Consider the mid-points $\frac{1}{2}(r + s)$ with $r, s \in A_2$ and $r$ on the $x$-axis and $s$ on the $y$-axis. Two of such points $\frac{1}{2}(r_1 + s_1)$ and $\frac{1}{2}(r_2 + s_2)$ are different if $r_1 \neq r_2$ or $s_1 \neq s_2$. Consider the mid-points $\frac{1}{2}(p + q)$ with $p = (0, \ 0, \ 1) \in A_1$ and $q$ is in the plane $z = 0$. Therefore $S$ contains at least $m_{x2}m_{y2} + \#A_{30}$ points. Hence we can assume $m_{x2}m_{y2} \leq \frac{13}{3} \#A_2$. Consequently,

$$m_{x2} + m_{y2} \leq \frac{1}{4} \#A_2 + 12,$$

as we have $3 \sqrt{\#A_2} \leq m_{x2} \leq \frac{1}{4} \#A_2$, $m_{y2} \leq \frac{1}{4} \#A_2$.

First, we have

$$\#(A_{20}.A_{20}) = 4 \#A_{20} - 2(m_{x2} + m_{y2}) + 1,$$

$$\#S \geq \#(A_1.A_3) = (t_1 - 1)(m_{x3} + 3) + 5 \#A_3 - 2(m_{x2} + m_{y2}) - 1.$$
See [X1, Lemma 6] for proofs of (2.5.2) and (2.5.3).

Second, we estimate \( \#S \) by considering not only the set \( A_2.A_2 \) but also the set \( A_1.A_3 \).

Since \( p := (0, \, 0, \, 1) \) is in \( A_1 \), the mid-points in
\[
B_1 := \left\{ \frac{1}{2}(p + q), q \text{ is in } A_{30} \right\}
\]
are in \( A_1.A_3 \); clearly, the mid-points in \( B_2 := A_{30}.A_{30} \) and
\[
B_3 := \left\{ \frac{1}{2}((a, \, 0, \, 1) + (b, \, 0, \, 1)), 0 \leq a, \, b \leq t_2 - 1, a, \, b \in \mathbb{Z} \right\}
\]
are in \( A_2.A_2 \). It is easy to see that \( B_i \cap B_j = \emptyset \) for \( i \neq j \). Let \( B = B_1 \cup B_2 \cup B_3 \).

Then
\[
\#B = \#A_{30} + \#(A_{20}.A_{20}) + 2t_2 - 1.
\]

By (2.5.1) and (2.5.2), we have
\[
\begin{align*}
\#S & \geq \#B = \#A_3 + 4\#A_2 - 2(m_{x_2} + m_{y_2}) - t_3 - t_2 \\
& \geq \#A_3 + \frac{7}{2}\#A_2 - t_3 - 2t_2 - 24.
\end{align*}
\]  

Now we consider the following cases separately.

Case II. \( \frac{1}{2}t_3 < t_2 \leq \frac{1}{8}\#A_2 \). By (2.5.5), we have
\[
\#S \geq \#A_3 + \frac{7}{2}\#A_2 - 4t_2 - 23 \geq \#A_3 + 3\#A_2 - 23.
\]

Case II. \( t_2 \leq \min\{\frac{1}{2}t_3, \frac{1}{8}\#A_2\} \). Let \( a = t_3 - 2t_2 + 1 \). Since \( p := (0, \, 0, \, 1) \) is in \( A_1 \), the mid-points in
\[
C := \left\{ \frac{1}{2}(p + (t_3 - 1, \, 0, \, 1)), \frac{1}{2}(p + (t_3 - 2, \, 0, \, 1)), ..., \frac{1}{2}(p + (t_3 - a, \, 0, \, 1)) \right\}
\]
are in \( S \); it is clear that \( B \cap C = \emptyset \), and \( \#C = a = t_3 - 2t_2 + 1 \). Now by (2.5.5), we have
\[
\#S \geq \#B + \#C \geq \#A_3 + 3\#A_2 - 23.
\]

Case III. \( t_2 \geq \frac{1}{8}\#A_2 \) and \( m_{y_2} \geq 7 \). By (2.5.1) and (2.5.3), we have
\[
\#S \geq 4(t_2 - 1) + \frac{9}{2}\#A_2 - 27 \geq 5\#A_2 - 31.
\]

Case II. \( t_2 \geq \frac{1}{8}\#A_2 \) and \( m_{y_2} \leq 6 \). Note that we have \( m_{y_2} \geq 3 \) since \( m_{x_2} \leq \frac{1}{8}\#A_2 \). Let \( e = m_{y_2}m_{x_2} + t_2 - \#A_2 \geq 0 \), and let \( n_i \) be the number of points of \( A_2 \) with \( y = i, \, z = 0 \). Clearly, \( m_{x_2} = n_0 \). The mid-points in
\[
E := \left\{ \frac{1}{2}((n_i - 1, \, i, \, 0) + (n_{i+1} - 1, \, i + 1, \, 0)), ..., \frac{1}{2}((n_{i+1}, \, i, \, 0) + (n_{i+1} - 1, \, i + 1, \, 0)), \right\}
\]
provided \( n_i > n_{i+1}, \, i = 0, ..., 3 \).
are in $S$, $B \cap E = \emptyset$, and $\#E = e$. By (2.5.4), we have
\begin{equation}
\#S \geq \#B + \#E \geq \#A_3 + 4\#A_2 - 2(m_{x2} + m_{y2}) - t_3 - 2t_2 + e. 
\end{equation}

**Case II2.1.** $m_{y2} = 5$ or $6$. If $m_{y2} = 6$, then $m_{x2} \geq \frac{1}{7}(\#A_2 + e)$. By (2.5.6), we have
\[\#S \geq \#A_3 + 2\#A_2 + 10m_{x2} - t_3 - e - 12 \geq \frac{5}{6}\#A_3 + \frac{24}{7}\#A_2 - 12 \geq \frac{5}{6}\#A_3 + \frac{10}{3}\#A_2 - 10.\]
Similarly, if $m_{y2} = 5$, then we have $\#S \geq \frac{5}{6}\#A_3 + \frac{10}{3}\#A_2 - 10$.

**Case II2.2.** $m_{y2} = 4$. By (2.5.6), we have
\[\#S \geq \frac{5}{6}\#A_3 + \frac{13}{4}\#A_2 - 2 + e,\]
provided $m_{x2} \geq \frac{5}{24}(\#A_2 + e) + 1$. This allows us to assume
\[\frac{1}{5}(\#A_2 + e) \leq m_{x2} \leq \frac{5}{24}(\#A_2 + e).\]
Therefore, $t_2 \geq \frac{1}{6}(\#A_2 + e)$. Let $w$ be the integer part of $\frac{1}{12}\#A_3$. Then the point $p = (w-1, 0, 1)$ is in $A_2$ since $\#A_3 \leq 2\#A_2$ by the hypothesis, and the mid-points in
\[F := \left\{ \frac{1}{2}(p + (a, b, 0)), \ a \in \mathbb{Z}, \ a \geq w - 1, \ b = 0, \ldots, 3, (a, b, 0) \text{ is in } A_{20} \right\}\]
are in $S$, $B \cap F = \emptyset$, and
\[\#F = 4(m_{x2} - w + 1) - e \geq 4(m_{x2} - \frac{1}{12}\#A_3 + 1) - e.\]
Now by (2.5.6), we have
\[\#S \geq \#A_3 + 4\#A_2 - 2(m_{x2} + m_{y2}) - t_3 - 2t_2 + e + \#F.\]
\[\geq \frac{1}{2}\#A_3 + 2\#A_2 + 10m_{x2} - 2e - 4 \geq \frac{1}{2}\#A_3 + 4\#A_2 - 4.\]

**Case II2.3** $m_{y2} = 3$. In this case $m_{x2} = t_2 = \frac{1}{4}\#A_2$. The mid-points in
\[G := \left\{ \frac{1}{2}((a, 0, 1) + (a, b, 0)), \ \text{where } \frac{1}{12}\#A_3 \leq a \leq \frac{1}{4}\#A_2, \ a \in \mathbb{Z}, \ b = 0, 1, 2 \right\}\]
are in $S$, $B \cap G = \emptyset$ since $2a > m_{x2}$, and $\#G = 3(\frac{1}{4}\#A_2 - \frac{1}{12}\#A_3)$. Then by (2.5.5), we have
\[\#S \geq \#B + \#G \geq \frac{7}{12}\#A_3 + \frac{15}{4}\#A_2 - 6.\]
Summing up above inequalities, we get what we wanted. □
Lemma 2.6. Let \( A_i (i = 1, 2, 3) \) be finite integrally convex sets with \( A_1 \subset A_2 \subset A_3 \), and the dimension of \( A_1 \) is at least 4. Set \( \epsilon = \frac{1}{530} \). Suppose that the length of the longest chain in \( A_3 \) is less then \( \epsilon \# A_3 \), and \( 4 \# (A_2 \geq A_3) \). Then

\[
\#(A_1 \cdot A_3 \cup A_2 \cdot A_2) \geq (1 - \epsilon) \# A_3 + \frac{14(1 - 4\epsilon)}{3} \# A_2 - 57.
\]

Proof. By Lemmas 2.2, 2.2.1, 2.2.2, we can suppose the dimension of the enveloping space of \( A_i \) is 4 for \( i = 1, 2, 3 \). Let \( p_1, ..., p_l \) be a longest chain in \( A_1 \). Assume \( p_1 \) is the origin of the enveloping space, and that \( p_2 = (1, 0, 0, 0) \). Let \( x, y, z, u \) be this basis. Arrange \( A_i (i = 1, ..., 3) \) with respect to \( x \)-axis, then with respect to \( y \), to \( z \), and then to \( u \). By lemma 2.4, we have only to count the number of points in the set of mid-points of the new set \((A'_1 \cdot A'_3 \cup A'_2 \cdot A'_2)\) thus produced.

Let \( m_{xi} \) be the number of points of \( A'_i \) on the \( x \)-axis, \( i = 1, 2, 3 \). We have \( m_{xi} \leq \epsilon \# A_3 \).

If the basis elements \( y, z \) and \( u \) are chosen carefully, then \( A'_1 \) will also be of dimension 4. We may assume that the points of \( A'_3 \) have either \( u = 0 \), or \( u = 1 \), \( y = 0 \), \( z = 0 \) as in the proof of [X1, Lemma 6].

Now for the simplicity of the notations, we replace \( A'_i \) by \( A_i \). Remark that although \( A_i \) is not convex now, it has the property that if a point \((a, b, c, d)\) is in \( A_i \), then all integral points \((a', b', c', d')\) such that

\[ 0 \leq a' \leq a, \quad 0 \leq b' \leq b, \quad 0 \leq c' \leq c, \quad 0 \leq d' \leq d \]

are in \( A_i \) too.

Let \( A_{i0} \) be the subset of points of \( A_i \) in the plane \( u = 0 \). Let \( t_i (i = 1, 2, 3) \) be the number of points in \( A_i \) with \( u = 1 \). Clearly,

\[ 1 \leq t_1 \leq t_2 \leq t_3 \leq m_{x3} \leq \epsilon \# A_3. \]

In what follows we denote by \( S \) the set \( A_1 \cdot A_3 \cup A_2 \cdot A_2 \).

Consider the subset \( A_{20} \). Clearly it is of dimension 3. Since \( t_2 \leq \epsilon \# A_3 \leq 4\epsilon \# A_2 \), we have \( \# A_{20} = \# A_2 - t_2 \geq (1 - 4\epsilon) \# A_2 \), i.e., \( \# A_2 \leq \frac{1}{1 - 4\epsilon} \# A_{20} \).

Let \( k \) be the length of the longest chain in \( A_{20} \), then

\[ k \leq \epsilon \# A_3 \leq 4\epsilon \# A_2 \leq \frac{4\epsilon}{1 - 4\epsilon} \# A_{20} < \frac{1}{6} \# A_{20}. \]

Hence the condition of [X1, Lemma 6] is satisfied for \( A_{20} \); one has that

\[ (2.6.1) \quad \#(A_{20} \cdot A_{20}) \geq \frac{14}{3} \# A_{20} - 57. \]

Second, we estimate \( \# S \) by considering not only the set \( A_2 \cdot A_2 \) but also the set \( A_1 \cdot A_3 \).
Since $r := (0, 0, 0, 1)$ is in $A_1$, the mid-points in

$$B_1 := \left\{ \frac{1}{2}(r + q), q \text{ is in } A_{30} \right\}$$

are in $A_1 \cdot A_3$; clearly, the mid-points in $B_2 := A_{20} \cdot A_{20}$ are in $A_2 \cdot A_2$. It is easy to see that $B_1 \cap B_2 = \emptyset$. Let $B = B_1 \cup B_2$. Then by (2.6.1), we have

$$\#S \geq \#B$$
$$\geq \#A_3 + \frac{14}{3} \#A_{20} - t_3 - 57$$
$$\geq (1 - \epsilon) \#A_3 + \frac{14(1 - 4\epsilon)}{3} \#A_2 - 57. \quad \square$$

Similarly, we have

**Lemma 2.7.** Let $A_i$ ($i = 1, 2, 3$) be finite integrally convex sets with $A_1 \subset A_2 \subset A_3$, and the dimension of $A_1$ (resp. $A_2$) is $\geq 2$ (resp. $\geq 3$). Suppose that the length of the longest chain in $A_3$ (resp. $A_2$) is less then $\frac{1}{10} \#A_3$ (resp. $\frac{1}{5} \#A_2$). Then

$$\#(A_1 \cdot A_3 \cup A_2 \cdot A_2) \geq \frac{9}{10} \#A_3 + \frac{16}{5} \#A_2 - 30.$$

§3. A linear bound of abelian automorphism groups of 3-folds of general type

In this section we prove the following

**Theorem 3.0.** Let $X$ be a smooth 3-fold of general type over the complex number field, and $K_X$ the canonical divisor of $X$. Let $G$ be an abelian group of automorphisms of $X$ (i.e. $G \subset \text{Aut}(X)$). Suppose $K$ is nef. Then there exists a universal constant coefficient $c$ such that

$$\#G \leq cK_X^3.$$

**Remark.** It is easy to construct examples with $\#G$ increasing linearly with $K^3$: let $S$ be a surface of general type with an abelian automorphism group $G_1$ of order $12.5K^2_S + 100$ [Ch], and let $C$ be a curve of genus 2 with an abelian automorphism group $G_2$ of order 12 [N]. Let $X = S \times C$. Let $p_1, p_2$ be the projections of $X$ onto the two factors. The group $G$ generated by $p_1^*G_1$ and $p_2^*G_2$ is an abelian automorphism group of $X$ of order $25K^3_X + 1200$.

We can give explicit estimates in Theorem 3.0 but these are far from being the best possible. An interesting question is, roughly speaking: what is asymptotically the best upper bound for $G^3$?
The arguments here are inspired by Xiao’s work on abelian automorphism groups of surfaces of general type [X1]. We consider the natural action of the abelian group $G$ on the space $H_n = H^0(X, nK_X)$, for a fixed positive integer $n$. Because $G$ is finite abelian, such an action is diagonalisable, in other words $H_n$ has a basis consisting of semi-invariant vectors. Consider two such semi-invariants $v_1, v_2$ in $H_n$, with

$$
\sigma(v_i) = \alpha_i(\sigma)v_i \text{ for } \sigma \in G,
$$

where $\alpha_i$ are the corresponding characters of $G$. Suppose that the two semi-invariants $v_1, v_2$ corresponding to a same character of $G$ (i.e., $\alpha_1 = \alpha_2$), and let $D_1$ and $D_2$ be the corresponding divisors in $|nK_X|$. Then $D_1$ and $D_2$ generate a pencil $\Lambda$ whose general fiber $F$ is fixed by $G$. Therefore $\#G$ is limited by the order of the group of automorphisms of the minimal model $\tilde{F}_0$ of the desingularization $\tilde{F}$ of $F$ as a minimal smooth surface of general type. But $\# \text{Aut}(\tilde{F}_0)$ increases proportionally with $K^3_X$, as $p_g(\tilde{F})$ so does.

We consider the natural map

$$
H_n \otimes H_{3n} \oplus H_{2n} \otimes H_{2n} \to H_{4n},
$$

which is compatible with the above actions of $G$: i.e., if $v_1 \in H_n$, $v_2 \in H_{3n}$ (resp. $w_i \in H_{2n}$ $i = 1, 2$) are two semi-invariants, then $v_1 \otimes v_2$ (resp. $w_1 \otimes w_2$) is semi-invariant in $H_{4n}$. If there are more than dim $H_i$ semi-invariants in $H_i$ for some $i \leq 3n$, then there are semi-invariants in $H_i$ (therefore in $H_{4n}$) with the same character, and we are done. So we may assume that there are exactly $\delta_i = \text{dim } H_i$ ($i \leq 3n$) semi-invariants $v_j^i$ ($j = 1, \ldots, \delta_i$) in $H_i$, corresponding to mutually different characters. Each vector $v_j^i$ corresponds to a unique divisor $D_j^i$ in $|iK|$. The relation $v_j^i \otimes v_k^l = cv_m^r \otimes v_n^s$ (where $c$ is a constant, and $i + l = r + s = 4n$) in $H_{4n}$ translates to a relation

\begin{equation}
(*) \quad D_j^i + D_k^l = D_m^r + D_n^s
\end{equation}

between these divisors.

Fix a semi-invariant $u \in H_n$ when $H_n \neq 0$. Then $u$ corresponds to a unique divisor $U$ in $|nK|$ which is fixed by $G$. We can consider the finite set $\Sigma_i$ of points corresponding to $D_j^i$ in a certain divisorial space $P_i$ defined in [X1, §1], and there are natural embeddings:

$$
\Sigma_n \to \Sigma_{2n} \to \Sigma_{3n}, \quad \text{and} \quad P_n \to P_{2n} \to P_{3n}
$$

defined by $U$. In such a setting, a semi-invariant in $H_{4n}$ of the form $v_j^i \otimes v_k^l$ corresponds naturally to the mid-point of two points in $\Sigma_{3n}$ corresponding to $D_j^i$.
and $D_k^l$, and a relation of the form (*) means that the corresponding mid-points coincide.

Denote by $S_1$ the set of mid-points of two points $p, q$ in $\Sigma_{3n}$ such that either $p$ is in $\Sigma_n$ and $q$ is in $\Sigma_{3n}$ or $p$ and $q$ are in $\Sigma_{2n}$. Now the problem has been reduced to that of comparing the number of points in $S_1$ and the dimension of $H_{4n}$. In this way we show that for $n$ large enough the number of points in $S_1$ is larger than the dimension of $H_{4n}$.

We fix a smooth complex projective 3-fold of general type $X$ in the future, and let $K$ be the canonical divisor of $X$, $H_n = H^0(X, nK)$, and $\chi = \chi(O_X)$. We also fix an abelian group $G$ of automorphisms of $X$.

For the reader’s convenience, we recall some notation defined in [X1].

**Definition ([X1]).** Let $v_1, \ldots, v_{\delta_n}$ be a basis of $H_n$ consisting of semi-invariants for the natural action of $G$ on $H_n$, and $D_1, \ldots, D_{\delta_n}$ the divisors in $|nK|$ corresponding to these vectors, where $\delta_n = \dim H_n$. We say that $H_n$ is *uniquely decomposable* (under the action of $G$) if the set $\{D_i\}$ is uniquely determined, or equivalently if there are exactly $\delta_n$ different characters for the natural action of $G$ on $H_n$.

Fix the divisor $D_1$. Denote by $P'_n$ the set

$$\{\mathbb{Q}\text{-divisors } D \text{ on } X \mid \text{there is an } m \in \mathbb{Z}^+ \text{ such that } mD \text{ is linearly equivalent to } mD_1\}.$$  

Denote by $[D]$ the element of $P'_n$ corresponding to the $\mathbb{Q}$-divisor $D$. We define addition and scalar multiplication as follows:

$$[D] + [D'] = [D + D' - D_1],$$

$$c[D] = [cD + (1 - c)D_1], \quad c \in \mathbb{Q}.$$  

Then $P'_n$ is a generally infinite dimensional linear space, with $[D_1]$ as the origin.

The subset $I$ in $P'_n$ of points corresponding to integral divisors linearly equivalent to $D_1$ is an additive subgroup, and there is a set of generators of $I$ which form a basis of $P'_n$. Under such a basis, $I$ is a subset of points with integral coordinates.

Denote by $P_n$ the finite dimensional subspace generated by the set

$$\{[D_1], [D_2], \ldots, [D_{\delta_n}]\}.$$  

Let $\Sigma_n$ be the finite set in $P_n$ consisting of points corresponding to effective divisors in $|nK|$ fixed by $G$. Then $P_n$, therefore $\Sigma_n$, is uniquely determined up to choices of $D_1$ only if $H_n$ is uniquely decomposable. We will call the set $\Sigma_n$ a *basic set* in $P_n$.

Clearly, $P_n$ depends on the choice of $D_1$; but if we replace $D_1$ by another divisor, say $D_i$, $P_n$ differs only by an integral translation. Because $\# \Sigma_n$ and the number
of middle points of $\Sigma_n$, which are all the properties about $\Sigma_n$ we use later, are integral translation invariants, it dose not matter which such $D_i$ is selected. Also, $\Sigma_n$ is determined up to integral translations as above iff $H_n$ has exactly $\delta_n$ semi-invariants, and thus iff there are $\delta_n$ different characters for the action of $G$ on $H_n$.

Fix a semi-invariant $u \in H_n$ for the natural action of $G$ on $H_n$. Let $U$ be the divisor in $|nK|$ corresponding to $u$. We have natural maps:

$$H_n \otimes u \to H_{2n} \otimes u \to H_{3n},$$

$$|nK| \overset{+U}{\to} |2nK| \overset{+U}{\to} |3nK|,$$

and $P_n' \overset{l_1}{\to} P_{2n}' \overset{l_2}{\to} P_{3n}'$.

If we take $[nU]$ to be the origin of $P_n'$, then $l_i'(i = 1, 2)$ are embeddings of linear spaces. We will identify $P_n'$ and $P_{2n}'$ as subspaces of $P_{3n}'$ in this way in the future.

Similarly, let $v_1, v_2$ be two semi-invariants in $H_{2n}$, and $p, q$ the corresponding points in $\Sigma_{2n}$. Let $D$ be the divisor in $|4nK|$ corresponding to the vector $v_1 \otimes v_2$. Then $l_2([D])$ corresponds to a point in $P_{3n}$, which is just the mid-point $\frac{1}{2}(l_2(p) + q)$.

Lemma 3.1. Let $\Sigma_n$ be a basic set in $P_n$. Then

(i) The set $\Sigma_n$ is integrally convex with respect to the lattice $L$ consisting of points corresponding to divisors linearly equivalent to $nK$, and $\Sigma_{ni}$ is relatively convex in $\Sigma_{n(i+1)}$ for $i \geq 1$, as we consider $\Sigma_{ni}$ as a subset of $\Sigma_{n(i+1)}$ in the above way.

(ii) Suppose $P_n$ is uniquely decomposable. If the dimension of $\Sigma_n$ is less then 3, then $\dim \Phi_n(X) \leq 2$.

(iii) Suppose $P_n$ is uniquely decomposable. If $\Phi_n$ is birational, then the dimension of $\Sigma_n$ is at least 4.

Proof. (i) See [X1, Lemma 3].

(ii) Suppose the dimension of $\Sigma_n$ is 2. By hypothesis, we have

$$\Sigma_n = \{[D_1], [D_2], \ldots, [D_{\delta_n}]\}.$$
(a) \([D_2]\) and \([D_3]\) generate the additive group \(\mathcal{L}\), and
(b) for each \([D_j]\) \((j > 3)\), we have \([D_j] = a_j[D_2] + b_j[D_3]\) with \(a_j, b_j \in \mathbb{Z}^+\).

By (b) we have
\[
D_j + (a_j + b_j - 1)D_1 = a_jD_2 + b_jD_3
\]
for \(j \geq 4\), that is,

\[
(3.1.1) \quad v_j \otimes v_1^{(a_j + b_j - 1)} = \xi_j v_2^{a_j} \otimes v_3^{b_j}, \quad \text{for some } \xi_j \in \mathbb{C}, \ j \geq 4.
\]

Let \(\phi: X \to \mathbb{P}^2\) be the rational map defined by \((v_1 : v_2 : v_3)\). Now for general points \(p, q \in X\) (here by general we mean that \(v_1(p) \neq 0\) and \(v_1(q) \neq 0\)), such that \(\pi(p) = \pi(q)\) \((i.e., v_1(p) = \lambda v_1(q)\) for some \(\lambda \in \mathbb{C}, i = 1, 2, 3\)), by (3.1.1) we have \(v_j(p) = \lambda v_j(q)\) for \(j = 4, \ldots, \delta_n\). Hence \(\Phi_n = (v_1 : v_2 : v_3 : \cdots : v_{\delta_n})\) is factor through \(\phi\).

(iii) Otherwise, by (ii) we can suppose that the dimension of \(\Sigma_n\) is 3. Then \(\Phi_n\) is factor through \(\phi: X \to \mathbb{P}^3\), where \(\phi\) is similarly defined as in (ii). This is a contradiction. \(\square\)

**Lemma 3.2.** Let \(\Sigma_n\) be a basic set in \(H_n\). Then the length of the longest chain in \(\Sigma_n\) is less than \(\frac{12}{(2n-1)(n-2)}\#\Sigma_n\).

**Proof.** Let \(k\) be the length of the longest chain in \(\Sigma_n\). Then there is a pencil \(\Lambda\) of surfaces on \(X\) such that there is a divisor in \(|nK_X|\) containing \(k\) times a fiber \(F\) of \(\Lambda\). Hence \(nK_X \equiv kF + D\) for some divisor \(D \geq 0\); we have \(kK_X^2F \leq nK_X^3\) since \(K_X\) is nef.

We claim \(K_X^2F \geq 1\). Indeed, let \(\pi: X' \to X\) be a succession of blowing-ups with nonsingular centers such that \(\phi \circ \pi\) is a morphism, where \(\phi\) is a rational map induced by \(\Lambda\). We set \(\pi^*F \equiv F' + E\), where \(F'\) consists of some fibers of \(\phi \circ \pi\), and \(E\) is an exceptional divisor for \(\pi\). We have
\[
K_X^2F = \pi^*(K_X)^2 \pi^*F = \pi^*(K_X)^2F' + \pi^*(K_X)^2E
\]
Since \(\pi^*K_X\) is nef, we have \(\pi^*(K_X)^2E \geq 0\), and Since \(\pi^*K_X\) is nef and big, \(F'\) is nef, and \(F' \neq 0\), we have \(\pi^*(K_X)^2F' \geq 1\).

Hence we have \(k \leq nK_X^3 < \frac{12}{(2n-1)(n-2)}p_n\) by Corollary 1.3. \(\square\)

**Proposition 3.3.** Suppose \(K_X\) is nef. Then there exists an integer \(N\) such that \(H_{4N}\) is not uniquely decomposable.

**Proof.** By Lemma 3.1 and Lemma 3.2, we can choose \(N_0 \gg 0\) such that for \(n \geq N_0\), the condition of Lemma 2.6 is satisfied for \(\Sigma_n \subset \Sigma_{2n} \subset \Sigma_{3n}\), and we can suppose \(H_i\) is uniquely decomposable for \(i < 4n\), for otherwise the corollary is trivially true. By Corollary 1.3, we have
#(Σₙ.Σ₃ₙ ∪ Σ₂ₙ.Σ₂ₙ) − dim H₄ₙ
≥ (1 − ϵ)#Σ₃ₙ + \frac{14(1 − 4ϵ)}{3}#Σ₂ₙ − 57 − dim H₄ₙ
= (1 − ϵ)dimH₃ₙ + \frac{14(1 − 4ϵ)}{3}dimH₂ₙ − 57 − dim H₄ₙ
= \left(1 − \frac{529}{18}n^3 + O(n^2)\right)K_X^3
> 0 \quad \text{for } n ≥ N₀ \text{ and } n ≫ 0.

Hence there exists an integer N independent of the choice of X such that #(Σₙ.Σ₃ₙ ∪ Σ₂ₙ.Σ₂ₙ) > dim H₄ₙ, i.e., there are more than dimH₄ₙ- semi-invariants in H₄ₙ. □

Proof of theorem 3.0. Let b = 4N. The Proposition 3.3 guarantees that there is a pencil Λ in |bK| each of whose elements is a fixed divisor by G. We consider the following commutative diagram:

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & C \\
\downarrow \pi & & \downarrow g \\
\downarrow h & & \\
X \xrightarrow{φ} \cdots & \mapsto & \mathbb{P}^1
\end{array}
\]

where φ is the rational map induced by Λ, π is a succession of blowing-ups with nonsingular centers such that g := φ ∘ π is a morphism, and g = h ∘ f is the stein factorization. Let d = deg h, and F a general fiber of f. Clearly, d ≤ bK_X^3. By Fact 1.4 F is a surface of general type. Let F₀ be the minimal model of F.

The automorphism group G can be lifted to be an automorphism group G' of X'. (Clearly #G' = #G; for any element σ in G, denote by σ' the corresponding element in G'.) Let H be the stabilizer of F, then the index H in G' is at most d. We have a natural group homomorphism ρ: H → Aut(F₀). We claim that Ker ρ = 1. Indeed, by the definition of π, we can identify X' − π⁻¹(Y) with X − Y, where Y is the base locus of the moving part of Λ. Let σ' ∈ Ker ρ. Since F is general, σ' equals identity on X' − Exceptional divisors of π. Since X' − π⁻¹(Y) = X − Y, we have σ equals identity on X − Y. Hence σ equals identity on X since the codimension of Y in X is ≥ 2.

Since H is an abelian automorphism group of F₀, we have (cf. [X1], [Ca].)

\[
#H \leq \begin{cases} 
36K_{F₀}^2 + 24, & \text{if } \chi(O_{F₀}) ≥ 8; \\
270K_{F₀}^2, & \text{otherwise.}
\end{cases}
\]

Consequently, we have

\[
#G ≤ d \cdot #H \leq \begin{cases} 
335p_g(F₀)d, & \text{if } p_g(F₀) ≥ 34; \\
270 - 34k - 878 - 34k & \text{otherwise.}
\end{cases}
\]
Since $p_g(F_0) = p_g(F)$, Theorem A follows from the following claim and Proposition 1.6.

**Claim.** $d p_g(F) \leq p_{b+4}(X)$.

**Proof of the claim.** We have

\[(2.6) \quad \pi^*(bK_X) \equiv f^*h^*(p) + Z,\]

where $p$ is a closed point of $\mathbb{P}^1$, and $Z$ is the fixed part of $\pi^*(bK_X)$. For general $p \in \mathbb{P}^1$, we have $D := f^*h^*(p) = F_1 + \cdots + F_d$ such that $F_i$ are general fibers of $f$ and $F_i \neq F_j$ when $i \neq j$.

Since $p_3(X) > 0$ by Corollary 1.3, we have

\[(K_X' + D)|_D < (K_X' + D + \pi^*3K_X)|_D,\]

so

\[dh^0(K_F) = h^0(K_X' + D)|_D \leq h^0((K_X' + D + \pi^*3K_X)|_D).\]

Consider the following exact sequence

\[0 \to \mathcal{O}_{X'}(K_{X'} + \pi^*(3K_X)) \to \mathcal{O}_{X'}(K_{X'} + D + \pi^*(3K_X)) \to \mathcal{O}_D(K_{X'} + D + \pi^*(3K_X)) \to 0.\]

Since $\pi^*(3K_X)$ is nef and big, by Fact 1.2 we have

\[h^1(\mathcal{O}_{X'}(K_{X'} + \pi^*(3K_X))) = 0,\]

so

\[h^0(\mathcal{O}_{X'}(K_{X'} + D + \pi^*(3K_X))) \leq h^0(\mathcal{O}_{X'}(K_{X'} + D + \pi^*(3K_X))).\]

since $\pi_*\mathcal{O}_{X'}(K_{X'}) = \mathcal{O}_{X}(K_{X})$ (cf. [Mo, p. 280] ) and $D < \pi^*(bK_X)$ (by (2.6)), we have

\[h^0(\mathcal{O}_{X'}(K_{X'} + D + \pi^*(3K_X))) \leq h^0(\mathcal{O}_{X'}(K_{X'} + \pi^*((b+3)K_X))) = p_{b+4}(X).\]

Summing up the above inequalities, we get $d p_g(F) \leq p_{b+4}(X)$.

---

**Chapter II** Abelian automorphism groups of surfaces: an improvement on Xiao’s results
§4. Automorphisms of non-hyperelliptic curves

It is well-known that for a complex curve of genus \( g \geq 2 \), its abelian automorphism group is of order \( \leq 4g + 4 \) [N]. In this section we give a further analysis to abelian automorphism groups for non-hyperelliptic curves. Our main results are Theorems 4.4 and 4.10 which shall be used in the sequel.

We begin by establishing notations.

**Notation 4.1.** Let \( C \) be a smooth projective curve of genus \( g \geq 2 \), \( G \) an abelian subgroup of \( \text{Aut}(C) \). We have a finite abelian covering

\[ \pi: C \to X := C/G. \]

Let \( q_1, ..., q_k \) be the points over which \( \pi \) is ramified, and \( r_i \) the ramification number of \( \pi \) over \( q_i \). We assume that \( r_1 \geq r_2 \geq ... \geq r_k \). Choose a point \( Q_i \in C \) lying above \( q_i \), and put

\[ G(q_i) = \{ \sigma \in G \mid \sigma Q_i = Q_i \}. \]

Since \( G \) is assumed to be abelian, \( G(q_i) \) is well defined; we have \( |G(q_i)| = r_i \). Using Hurwitz formula to the morphism \( \pi \), we get

\[
\frac{(2g - 2)}{|G|} = 2g(X) - 2 + \sum_{i=1}^{k} \left( 1 - \frac{1}{r_i} \right).
\]

For convenience, we denote \( \frac{(2g - 2)}{|G|} \) by \( \tau \).

**Lemma 4.3.** Notations being as above. Let \( |G| = p_1^{t_1} ... p_l^{t_l} \), where \( p_i \)s are prime numbers, and let \( m_j = \prod_{1 \leq i \leq k, i \neq j} r_i \) for \( j = 1, ..., k \). Suppose \( g(X) = 0 \). Then we have

(i) \( m_j \) is a multiple of \( |G| \) for any \( j \);

(ii) For each \( i \), there are at least two points \( q_j, q'_j \) such that \( p_i | r_j, p_i | r'_j \);

(iii) \( |G| \) is a multiple of the least common multiple of \( r_i \)’s.

(iv) If \( G \) is cyclic, then \( |G| \) equals the least common multiple of \( r_i \)’s, and for each \( i \), there are at least two points \( q_j, q'_j \) such that \( p_i^{t_i} | r_j, p_i^{t_i} | r'_j \);

**Proof.** The Galois group \( G \) of \( \pi \) is an abelian quotient \( \pi_1(X - \{q_1, ..., q_k\}) \) which is generated by \( \gamma_1, ..., \gamma_k \), where \( \gamma_i \) is a small loop around \( p_i \). Let \( \bar{\gamma}_i \) be the image of \( \gamma_i \) in \( G \). \( r_i \) being the order of \( \bar{\gamma}_i \). Since \( \bar{\gamma}_1 ... \bar{\gamma}_k = 1 \), we have \( G \) is generated by, say, \( \bar{\gamma}_2, ..., \bar{\gamma}_k \). Hence (i) follows. Since \( r_i | |G| \) for each \( i \), we have (iii). Now (ii) follows from (i) and (iii).

If \( G \) is cyclic, then the order of the generator of \( G \) divides the least common multiple of \( r_i \)’s. Hence by (iii) we have that \( |G| \) equals the least common multiple of \( r_i \)’s. Since say, \( r_1 \) is the order of \( \bar{\gamma}_1^{-1} = \gamma_2 ... \gamma_k \), we have \( r_1 \) divides the least common multiple of \( r_i \)’s. Hence if \( p_i^{t_i} | r \), then there is a \( i \geq 2 \) such that \( p_i^{t_i} | r' \). □
**Remark** The assumption of being cyclic is indispensable in Lemma 4.3 (iv). For example, let $C$ is the plane Fermat curve $x^5 + y^5 + z^5 = 0$, and $G$ consists of the automorphisms $(x, y, z) \rightarrow (\mu x, \nu y, z)$, where $\mu, \nu$ are $\exp(\frac{2\pi i}{5})$. Then $X/G \simeq \mathbb{P}^1$, $|G| = 25$, and $r_i = 5$ for $i = 1, 2, 3$.

**Theorem 4.4.** Notations being as in (4.1). Suppose that $C$ is nonhyperelliptic. Then

$$|G| \leq 3g + 6$$

with one exception below. The exceptional $C$ is the plane Fermat curve $x^d + y^d + z^d = 0$ of degree $d = 4$ and 5, and $G$ consists of the automorphisms $(x, y, z) \rightarrow (\mu x, \nu y, z)$, where $\mu, \nu$ are $\exp(\frac{2\pi i}{d})$. (In this case, $g = \frac{(d-1)(d-2)}{2}$, $|G| = d^2$.)

**Proof.** If $g(X) \geq 1$, by (4.2) we get the result.

If $g(X) = 1$, then the commutativity of $G$ implies $k \geq 2$. By (4.2) we get $\tau \geq 1$, that is, $|G| \leq 2g - 2$.

Now we can assume $g(X) = 0$. If $k \geq 6$, by (4.2) we get $\tau \geq 1$, that is, $|G| \leq 2g - 2$. Hence we can assume $k \leq 5$. We prove the case $k = 3$; for the case $k = 4$ and 5, the proof is similar, and left to the reader.

Clearly, by (4.2) we can assume $r_3 \leq 8$. On the other hand, we claim $r_3 \geq 3$. Otherwise, let $Y = C/G(q_3)$. Using Hurwitz formula to $Y \rightarrow X$, which is unramified except $q_1, q_2 \in X$, we have $Y \simeq \mathbb{P}^1$, and $C$ is a hyperelliptic curve. A contradiction.

Let

$$h_i = \frac{r_i r_3}{|G|}, \quad \tau = 1, 2.$$ 

By Lemma 4.3, we have $h_i \in \mathbb{N}$, and $h_1 \geq h_2$ by our assumption. By (4.2) we have

$$\tau = 1 - \frac{r_3}{h_1 |G|} - \frac{r_3}{h_2 |G|} - \frac{1}{r_3}.$$ 

If $r_3 = 3$, by (4.5) we have $\tau \geq \frac{2}{3} - \frac{6}{|G|}$; consequently, $|G| \leq 3g + 6$.

Now we can assume $4 \leq r_3 \leq 8$. By (4.2), we have $\tau \geq 1 - \frac{3}{r_3}$; consequently,

$$|G| \leq \frac{r_3}{(r_3 - 3)} (2g - 2).$$ 

**Case 1.** $h_2 \geq 2$. By (4.5) we have $\tau \geq 1 - \frac{1}{r_3} - \frac{r_2}{|G|}$; consequently,

$$|G| \leq \frac{r_3}{(r_3 - 1)} (2g - 2 + r_3) \leq \frac{4}{3} (2g - 2 + r_3) \leq 3g + 6$$ 

provided $g \geq 4r_3 - 26$. 


Hence if either $r_3 \leq 7$ or $r_3 = 8$ and $g \geq 6$, we get the result. If $r_3 = 8$ and $g < 6$, by (4.6) we have

$$|G| \leq \frac{16}{5}(g-1) < 3g + 6.$$  

Case 2. $h_2 = 1$, $h_1 \geq 2$. Since $|G| = r_2r_3 \geq r_3^2$ in this case, we get

$$g \geq \frac{1}{2}(r_3^2 - 3r_3 + 2).$$  

By (4.5) we have

$$|G| \leq \frac{r_3}{(r_3 - 1)}(2g - 2 + \frac{3r_3}{2}) < 3g + 6.$$  

Case 3. $h_1 = h_2 = 1$. By (4.5) we have

$$|G| = \frac{r_3}{(r_3 - 1)}(2g - 2 + 2r_3) \leq 3g + 6$$

provided $g \geq 2(r_3 - 1)$.

By (4.7) we get the result when $r_3 \geq 6$.

When $r_3 = 5$, by (4.7) we have $g \geq 6$, and by (4.8) we have $g \equiv 0 \pmod{2}$.

Hence we have either $|G| \leq 3g + 6$ or

$$(g, |G|, r_1, r_2, r_3) = (6, 25, 5, 5, 5).$$

In the latter case, since $g(X) = 0$, we can choose an isomorphism between $X$ and $\mathbb{P}^1$, which maps $q_1, q_2$ and $q_3$ to the points $0, 1$ and $\infty$ of $\mathbb{P}^1$. Now it is easy to verify that the above $C$ is isomorphic to the Fermat curve $x^5 + y^5 + z^5 = 0$, and $G$ is the same as given in Theorem 4.4.

When $r_3 = 4$, by (4.7) we have $g \geq 3$, and by (4.8) we have $g \equiv 0 \pmod{3}$.

Hence we have either $|G| \leq 3g + 6$ or

$$(g, |G|, r_1, r_2, r_3) = (3, 16, 4, 4, 4).$$

In the latter case, $C$ is isomorphic to the Fermat curve of degree 4.

**Examples 4.9.** The estimates of Theorem 4.4 is best possible. Let $m \geq 2$. Let $C$ be given by the complete nonsingular model of affine equation $y^3 = x^{3m} - 1$, and $G$ consist of $x \to \mu x; y \to \nu y$, where $\mu^{3m} = 1$ and $\nu^3 = 1$. Clearly, $C$ is nonhyperelliptic. We have $g(C) = 3m - 2$, and $|G| = 9m = 3g(G) + 6$.

We say a smooth curve $C$ is *bi-elliptic*, if $C$ can be represented as a ramified double covering of an elliptic curve.
Theorem 4.10. Let $f: S \to B$ be a fibration of variable moduli. Let $G$ be an abelian automorphism group of $S$, inducing trivial action on $B$. Then

(i) $|G| \leq 4g - 4$;

(ii) If the general fiber is neither hyperelliptic nor bi-elliptic, then $|G| \leq 3g - 3$;

(iii) Suppose that $G$ is cyclic. Then $|G| \leq 2g + 2$.

Proof. Let $C$ be a general fiber of $f$. We have a finite abelian covering

$$\pi: C \to X := C/G.$$ 

Let $k, q_i, r_i, \tau$ and $G(q_i)$ be as in Notation 4.1.

If $g(X) \geq 1$, by (4.2) we get $\tau \geq 1$, that is, $|G| \leq 2g - 2$.

We can assume $g(X) = 0$. In this case we have $k \geq 4$ by the hypothesis of variable moduli.

When $k \geq 6$, by (4.2) we get $\tau \geq 1$, that is, $|G| \leq 2g - 2$.

When $k = 5$, we get either $\tau \geq 2/3$, that is, $|G| \leq 3g - 3$ or $\tau = 1/2$, that is, $|G| = 4g - 4$. In the latter case, $r_1 = \ldots = r_5 = 2$.

Now we assume $k = 4$. Using Lemma 4.3 and (4.2), it is easy to see that if $r_3 \geq 3$, then $\tau \geq 2/3$, that is, $|G| \leq 3g - 3$.

Hence we assume $r_3 = r_4 = 2$. If $r_2 \geq 6$, then $\tau \geq 2/3$, that is, $|G| \leq 3g - 3$. If $r_2 = 5$, by (4.2) and Lemma 4.3, we have either $\tau \geq 2/3$, that is, $|G| \leq 3g - 3$ or $(r_1, r_2, r_3, r_4) = (5, 5, 2, 2)$. In the latter case, we have $|G| \mid 10$, and $g = 4$ when $|G| = 10$. Hence, $|G| < 4g - 4$. If $r_2 = 4$, by (4.2) and Lemma 4.3, we have either $\tau \geq 2/3$, that is, $|G| \leq 3g - 3$ or

$$(r_1, r_2, r_3, r_4) \in \{(8, 4, 2, 2), (4, 4, 2, 2)\}.$$ 

In the latter case, we have $\tau \geq 1/2$, that is, $|G| \leq 4g - 4$. If $r_2 = 3$, by Lemma 4.3 (i), we have $|G| \mid 12$. In this case, we can assume $|G| = 12$. Then we have either $|G| \leq 3g - 3$ (when $g \geq 5$) or $(g, r_1) = (4, 6)$. (Note that $(g, r_1) = (3, 3)$ is impossible by Lemma 4.3.) If $r_2 = 2$, by Lemma 4.3 (i), we have $|G| \mid 8$. Hence we get $|G| \leq 3g - 3$. (Note that In this case, we have $g > 3$ by (4.2).)

Summing up, we have $|G| \leq 4g - 4$; moreover, $|G| \leq 3g - 3$ with the exceptional \{g, |G|, k, (r_1, \ldots, r_k)\} below:

$$\{5, 16, 5, (2, 2, 2, 2, 2)\}, \{4, 10, 4, (5, 5, 2, 2)\};$$

$$\{6, 16, 4, (8, 4, 2, 2)\}; \{4, 12, 4, (6, 3, 2, 2)\};$$

$$\{3, 8, 4, (4, 4, 2, 2)\}.$$ 

Now we finish the proof of this theorem by proving the following claim.

Claim. If $\pi: C \to C/G$ is one of the above exceptional cases, then $C$ is either hyperelliptic or bi-elliptic.
Proof of the claim. Let $Y = C/G(q_k)$. Then $|G(q_k)| = 2$. Using Hurwitz formula to $Y \to X$, we have $g(Y) \leq 1$. □

(iii) follows from Lemma 4.3 (iv). See [X3, Appendix A, Lemma A3] for a proof.

Remark. It is likely that (i) and (ii) do not give the best bound when $g \gg 0$, but here we only need the result for small $g$.

§5. ABELIAN AUTOMORPHISM GROUPS OF SMALL GENUS FIBRATIONS

In [Ch1], Chen has shown that under the condition $K^2 \geq 5$ and $S$ is not a product of two curves of genus 2, the order of an abelian subgroup of Aut$(S)$ is at most $12.5K^2 + 100$. The purpose of this section is to give a similar estimate for surfaces with small genus 3, 4, and 5. As a consequence, we give an estimate of the order of an abelian subgroup of Aut$(S)$ for the surface $S$ whose 1-canonical map is composed with a pencil.

Let me start by giving a convenient lemma, which plays an important role in this section. For a proof, see [X3, Appendix A].

Lemma 5.1 ([X3, Lemma A1]). Let $f: S \to C$ be a fibration of constant moduli, of which the general fibers are of genus $g \geq 2$. Let $F'$ be a singular fiber of $f$. Denote by $\chi_{\text{top}}(F')$ the Euler topological character of $F'$. Then

$$\chi_{\text{top}}(F') + 2g - 2 \geq \begin{cases} 
  g - 1, & \text{if } g \text{ is odd, and } F' \text{ is a double curve of genus } \frac{g+1}{2}; \\
  g + 2, & \text{otherwise.}
\end{cases}$$

In particular, we have $\chi_{\text{top}}(F') + 2g - 2 \geq 4$ except if $g = 3$, and $F'$ is a double curve of genus 2.

Let $S$ be a surface of general type with a pencil $\Lambda$, and $G$ an abelian group of automorphisms of $S$ whose elements map $\Lambda$ onto itself. Suppose $S$ has no $(-1)$-curves contained in a fiber of $\Lambda$, and let $g$ be the genus of a general element of $\Lambda$, with $g \geq 2$. Blowing up the base points (if necessary) of $\Lambda$, $\Lambda$ corresponds to a relatively minimal fibration $f: S \to C$. By hypothesis, we have a homomorphism of $G$ into Aut$(C)$. Let $H$ be its kernel. There is an exact sequence

$$0 \to H \to G \to I \to 0,$$

where $I$ is the image of $G$ in Aut$(C)$. Let $F$ be a general fiber of $f$. Then $H$ is an abelian subgroup of Aut$(F)$, and $H$ induces trivial action on $C$. We have $\#G = \#H\#I$, $\#H \leq 4g + 4$, and

$$(5.2) \quad \#I \leq \begin{cases} 
  4b + 4, & \text{if } b \geq 2; \\
  \max\{6, n\}, & \text{if } b = 1; \\
  \max\{4, n\}, & \text{if } b = 0.
\end{cases}$$
where \( b = g(C) \), and \( n \) is the number of singular fibers of \( f \).

Let \( \tau := \min \{ \chi_{\text{top}}(F') + 2g - 2 \} \). Then \( \tau \geq 1 \), and we have

\[
\begin{align*}
(5.3) \quad n \leq & \frac{1}{\tau} \sum (\chi_{\text{top}}(F') + 2g - 2) \\
& \leq \frac{1}{\tau} \left[ c_2(S) - 4(g - 1)(b - 1) \right] \\
& \leq \frac{1}{\tau} \left( \frac{2g + 1}{g - 1} K_S^2 - 16(g - 1)(b - 1) - 24(b - 1) \right)
\end{align*}
\]

according to \([X4]\), where the sum is taken over singular fibers \( F' \) of \( f \).

**Proposition 5.4.** Let \( S \) be a minimal surface of general type having two pencils \( \Lambda_i \) \((i = 1, 2)\) of genus \( g \geq 2 \). Let \( G \) be an abelian subgroup of \( \text{Aut}(S) \). Suppose that \( K_S^2 > 4(g - 1)^2 \). Then \( \#G \leq 16K_S^2 \).

**Proof.** By assumption, we have that \( S \cong C_1 \times C_2 \) for some smooth curve \( C_i \in \Lambda_i, i = 1, 2 \) (cf. \([X5, \text{Proposition 6.4}]\)). Since the order of an abelian group of automorphisms of genus \( g \) is at most \( 4g + 4 \), we have \( \#G \leq 32(g + 1)^2 \) (cf. \([X1, \text{Example 2}]\)). Since \( K_S^2 = 8(g - 1)^2 \) in this case, we get the result. \( \square \)

**Proposition 5.5.** Let \( S, G \) be as above, and that \( S \) has a pencil \( \Lambda \) of curves of genus 3. Suppose that \( K_S^2 > 16 \), and \( K_S^2 \geq 3\chi(O_S) \). Then

\[
\#G \leq 24K^2 + 64.
\]

**Proof.** We note that \( \Lambda \) has no base point: we have \((mK - nC)C = 0\) for \( C \in \Lambda \), and some positive integral numbers \( m, n \). Then the Hodge’s index theorem implies \( K^2 \leq 9 \), a contradiction.

Let \( f: S \to C \) be the fibration of genus 3 associated with \( \Lambda \), and \( b = g(C) \). Since \( K^2 > 16 \), by Proposition 5.4, we can assume that \( S \) has only one fibration of genus 3. Hence each element of \( G \) maps \( \Lambda \) onto itself.

If \( f \) is of variable moduli, then by lemma 2 we have \( \#H \leq 8 \). By \((5.2), (5.3)\) and the assumption we have \( \#I \leq 3K_S^2 + 8 \), and consequently

\[
\#G \leq 24K^2 + 64.
\]

If \( f \) is of constant moduli, Lemma 5.1 allows us to conclude \( \#I \leq \frac{1}{5}(c_2(S) - 4(g - 1)(b - 1)) \); consequently,

\[
\#G \leq \frac{56}{5}(K^2 + 16)
\]

except when there are many double curves of genus 2. In the latter case let \( D \) be a general divisor in the moving part of \( |K| \). Then \( 0 < DE < 4 \).
Now we consider two cases separately.

Case 1. $DF > 0$. Consider the projection $\psi: D \to C$ induced by $f$. $\psi$ must be ramified along each double fiber. Let $m$ be the number of double fibers of $f$. By Hurwitz formula, we have

$$2K^2 \geq 2p_a(D) - 2 \geq (DF)(2b - 2 + \frac{m}{2});$$

so $m \leq 2K^2 + 4$, and by Lemma 5.1, (5.2) and (5.3) we have

$$\#G \leq (4g + 4)\frac{m}{2} \leq 16K^2 + 32.$$

Case 2. $DF = 0$. In this case, $|K|$ is composed with a pencil of genus 3, and $f = \varphi_K$, the 1-canonical map of $S$. Let $|K| = |M| + Z$, where $Z$ is a fixed part of $|K|$, and $|M|$ is the moving part of $|K|$. Then $M \sim aF$, where $a \geq p_g - 1$ (see [Be, proposition 2.1]), and

$$K^2 \geq aKF \geq (p_g - 1)KF \geq 4\chi - 8.$$

Hence we have

$$c_2(S) - 4(g - 1)(b - 1) = 12\chi - K^2 - 4(g - 1)(b - 1) \leq 2K^2 + 32;$$

consequently, by Lemma 5.1 and (5.3), $\#I \leq K^2 + 16$, and we get $\#G \leq 16(K^2 + 16)$. □

**Proposition 5.6.** Notations being as in proposition 5.5. Suppose $K^2 > 36$, and that $S$ has a pencil $\Lambda$ of curves of genus 4. Suppose that $K_S^2 > 36$, and $K_S^2 \geq 4\chi(O_S)$. Then

$$\#G \leq 24K^2 + 144.$$

**Proof.** It is easy to see that $\Lambda$ has no base point as in the proof of proposition 5.5.

Let $f: S \to C$ be the fibration of genus 4 associated with $\Lambda$, and $b = g(C)$. Since $K^2 > 36$, by Proposition 5.4, we can assume that $S$ has only one fibration of genus 4. Hence each element of $G$ maps $\Lambda$ onto itself.

If $f$ is of constant moduli, then by Lemma 5.1 we have

$$\#G \leq 10K^2 + 240.$$

If $f$ is of variable moduli, Lemma 4.10 allows us to conclude $\#H \leq 12$; consequently, $\#G \leq 24K^2 + 144$ as in the proof of Proposition 5.5. □
Proposition 5.7. Notations being as in proposition 5.5. Suppose that $K_S^2 > 64$, $K_S^2 \geq \frac{24}{5} \chi(\mathcal{O}_S)$, and that $S$ has a pencil $\Lambda$ of curves of genus 5. Then

$$\#G \leq 24K^2 + 256.$$ 

Proof. The proof is similar to that of Proposition 5.6, and left to the reader. \qed

Proposition 5.8. Let $S$ be a minimal smooth surface of general type, and $K$ the canonical divisor of $S$. Let $G$ be an abelian group of automorphisms of $S$ (i.e. $G \subset \text{Aut}(S)$). And let $\varphi_1$ be the 1-canonical map of $S$. Suppose that $\chi(\mathcal{O}_S) \geq 21$, and that $\dim \varphi_1(S) = 1$. Then

$$\#G \leq 12.5K^2 + 469.$$ 

Proof. By assumption, $\varphi$ is composed with a pencil of curves of genus $g$ with $2 \leq g \leq 5$, and the pencil has no base points according to [Be]. Let $f: S \to C$ be a fibration of genus $g$ induced by $\varphi_1$. Let $F$ be a general fiber of $f$. If $g = 2$, then $\#G \leq 12.5K_S^2 + 100$ [Ch]. Hence we can assume that $g \geq 3$.

Let $|K_S| = |M| + Z$, where $Z$ is the fixed part of $|K_S|$, and $|M|$ is the moving part of $|K_S|$. Then $M \sim aF$, where $a \geq p_g - 1$ (see [Be, Proposition 2.1]), and

$$K_S^2 \geq aK_S.F \geq (2g - 2)(p_g - 1) \geq (2g - 2)(\chi - 2).$$

Consequently, $\# I \leq \frac{1}{7}(\frac{6}{g-1} - 1)K_S^2 + 24 + 4(g - 1))$, where $\tau$ is as in (5.3). Now it is easy to see that

$$\#G \leq \begin{cases} 8K^2 + 640 \leq 12K^2 + 496, & \text{if } g = 4; \\ 12K^2 + 432, & \text{if } g = 5. \end{cases}$$

When $g = 3$, one has $K_S^2 \geq \frac{21}{4}p_g - \frac{71}{6}$ [Su]. Consequently, $\# I \leq \frac{1}{7}(\frac{9}{7}K_S^2 + 47 + \frac{1}{21})$, and $\#G \leq \frac{21}{7}K^2 + 376 + \frac{8}{21}$. \qed

§6. Abelian subgroups for surfaces with large $K_S^2$

Our main result in this section is the following.

Theorem 6.1. Let $S$ be a minimal smooth surface of general type over the complex number field, and $K$ the canonical divisor of $S$. Let $G$ be an abelian group of automorphisms of $S$ (i.e. $G \subset \text{Aut}(S)$). Suppose that the dimension of the 1-canonical image of $S$ is 2, $\chi \geq 14$, $K^2 \geq 82$, and that $S$ has no pencil of curves of genus $\leq 5$. Then

$$\#G \leq 24K^2 + 16.$$
Moreover, if the 1-canonical map of $S$ is birational, then

$$\#G \leq 18K^2 + 18.$$  

The arguments here are inspired by the work of Xiao [X1]. We consider the natural action of the abelian group $G$ on the space $H_n = H^0(S, nK_S)$, for a fixed positive integer $n$. Because $G$ is finite abelian, such an action is diagonalisable, in other words $H_n$ has a basis composed of semi-invariant vectors. Consider two such semi-invariants $v_1, v_2$ in $H_n$, with

$$\sigma(v_i) = \alpha_i(\sigma)v_i \text{ for } \sigma \in G,$$

where $\alpha_i$ are the corresponding characters of $G$. Suppose that the two semi-invariants $v_1, v_2$ corresponding to a same character of $G$ (i.e., $\alpha_1 = \alpha_2$), and let $D_1$ and $D_2$ be the corresponding divisors in $|nK_S|$. Then $D_1$ and $D_2$ generate a pencil $\Lambda$ whose general fiber $F$ is fixed by $G$. Therefore $\#G$ is limited by the order of the group of automorphisms of the normalization $\tilde{F}$ of $F$ as a smooth curve. But $\#\text{Aut}(\tilde{F})$ increases proportionally with $K_S^2$, as $g(\tilde{F})$ so does.

Instead of considering the natural map $H_n \otimes H_n \rightarrow H_{2n}$,

We may consider the natural map

$$H_{n-t} \otimes H_{n+t} \oplus H_n \otimes H_n \rightarrow H_{2n},$$

which is compatible with the above actions of $G$: i.e., if $v_1 \in H_{n-t}$, $v_2 \in H_{n+t}$ (resp. $w_i \in H_n$ $i = 1, 2$) are two semi-invariants, then $v_1 \otimes v_2$ (resp. $w_1 \otimes w_2$) is semi-invariant in $H_{2n}$. If there are more than dim $H_i$ semi-invariants in $H_i$ for some $i \leq n + t$, then there are semi-invariants in $H_i$ (therefore in $H_{2n}$) with the same character, and we are done. So we may assume that there are exactly $\delta_i = \text{dim } H_i$ ($i \leq n + t$) semi-invariants $v^i_j$ ($j = 1, ..., \delta_i$) in $H_i$, corresponding to mutually different characters. Each vector $v^i_j$ corresponds to a unique divisor $D^i_j$ in $|iK|$. The relation $v^i_j \otimes v^k_l = cv^r_m \otimes v^n_s$ (where $c$ is a constant, and $i + l = r + s = 2n$) in $H_{2n}$ translates to a relation

$$(*) \quad D^i_j + D^l_k = D^r_m + D^n_s$$

between these divisors.

Fix an semi-invariant $u \in H_t$ when $H_t \neq 0$. $u$ corresponds to a unique divisor $U$ in $|tK|$ which is fixed by $G$. Then we can consider the finite set $\Sigma$ of points...
corresponding to $D_j^i$ in a certain divisorial space $P_i$ defined in [X1, §1] (see also §3), and there are natural embeddings:

$$\Sigma_{n-t} \to \Sigma_n \to \Sigma_{n+t}, \quad P_{n-t} \xrightarrow{l_2} P_n \xrightarrow{l_3} P_{n+t}$$

defined by $U$. In such a setting, an semi-invariant in $H_{2n}$ of the form $v_j^i \otimes v_k^l$ corresponds naturally to the mid-point of two points in $\Sigma_{n+t}$ corresponding to $D_j^i$ and $D_l^k$, and a relation of the form (*) means that the corresponding mid-points coincide.

Denote by $S_1$ the set of mid-points of two points $p, q$ in $\Sigma_{n+t}$ such that either $p$ is in $\Sigma_{n-t}$ and $q$ is in $\Sigma_{n+t}$ or $p$ and $q$ are in $\Sigma_n$. We reduce the problem to that of comparing the number of points in $S_1$ and the dimension of $H_{2n}$. In this way we show that for $n = 2$ and $t = 1$ the number of points in $S_1$ is larger than the dimension of $H_{4}$, provided that the dimension of the 1-canonical image of $S$ is 2, $\chi \geq 14$, $K^2 \geq 82$, and that $S$ has no pencil of curves of genus $\leq 5$.

Assume $H_1 \neq 0$. Fix an semi-invariant $u \in H_1$ for the natural action of $G$ on $H_1$. Let $U$ be the divisor in $|K|$ corresponding to $u$. We have natural maps:

$$H_1 \xrightarrow{\otimes u} H_2 \xrightarrow{\otimes u} H_3;$$

$$|K| \xrightarrow{+U} |2K| \xrightarrow{+U} |3K|;$$

$$P_1' \xrightarrow{l_i'} P_2' \xrightarrow{l_2} P_3'.$$

If we take $[nU]$ be the origin of $P_n'$, then $l_i'(i = 1, 2)$ are embeddings of linear spaces. Fix $P_1'$ and $P_2'$ as subspaces of $P_3'$ in this way in the future. Since $U$ is fixed by $G$, $l_i'(i = 1, 2)$ induce natural embeddings:

$$\Sigma_1 \to \Sigma_2 \to \Sigma_3,$$

and natural embeddings:

$$P_1 \xrightarrow{l_2} P_2 \xrightarrow{l_3} P_3.$$

Let $v$ (resp.$w$) be an semi-invariant in $H_1$ (resp. $H_3$), and $p$ (resp. $q$) the corresponding point in $\Sigma_1$ (resp. $\Sigma_3$). Let $D$ be the divisor in $|4K|$ corresponding to the vector $v \otimes w$. Then $l_2([\frac{1}{2}D])$ corresponds to a point in $P_2$, which is just the mid-point $\frac{1}{2}(l_2l_1(p) + q)$: in fact, Let $D'$ (resp. $D''$) be the divisor in $|K|$ (resp. $|3K|$) corresponding to the vector $v$ (resp. $w$). Then $D = D' + D''$, and

$$l_2([\frac{1}{2}D]) = [\frac{1}{2}(D' + U)]$$

$$= [\frac{1}{2}(D' + 2U)] = [\frac{1}{2}(D' + 2U) + D'' - D_1]$$

$$= \frac{1}{2}([D' + 2U] + [D'']) = \frac{1}{2}(l_2l_1(p) + q).$$
Similarly, let \( v_1, v_2 \) be two semi-invariants in \( H_2 \), and \( p, q \) the corresponding points in \( \Sigma_2 \). Let \( D \) be the divisor in \( |4K| \) corresponding to the vector \( v_1 \otimes v_2 \). Then \( l_2(\frac{1}{2}D) \) corresponds to a point in \( P_2 \), which is just the mid-point \( \frac{1}{2}(l_2(p) + l_2(q)) \).

**Remark.** \( \Sigma_n \) is integrally convex with respect to the lattice \( L \) consisting of points corresponding to divisors linearly equivalent to \( nK \) [X1, Lemma 3]. Clearly it is easy to verify that \( \Sigma_i \) is relatively convex in \( \Sigma_{i+1} \) for all \( i \), as we consider \( \Sigma_i \) as a subset of \( \Sigma_{i+1} \) in the above way.

**Lemma 6.2.** Suppose \( K^2 \geq 82 \), and let \( \Sigma_i \) be a basic set in \( H_i \) for \( i = 2, 3 \). Assume that either there is a chain in \( \Sigma_3 \) of length \( \geq \frac{1}{10} \# \Sigma_3 \) or there is a chain in \( \Sigma_2 \) of length \( \geq \frac{1}{5} \# \Sigma_2 \). Then \( S \) has a pencil of curves of genus \( \leq 5 \).

**Proof.** The proof is a modification of [X1, lemma 4]. Suppose there is a chain in \( \Sigma_3 \) of length \( \geq \frac{1}{10} \# \Sigma_3 \). (For the latter case, the proof is similar.) Let \( k \) be the length of the longest chain in \( \Sigma_3 \). Then there is a pencil \( \Lambda \) of curves on \( S \) such that there is a divisor in \( |3K| \) containing \( k \) times a fiber \( F \) of \( \Lambda \). We have \( KF \leq 9 \). Since \( K^2 \geq 82 \), the Hodge index theorem implies \( F^2 = 0 \). Now the evenness of \( F^2 + KF \) forces \( KF \leq 8 \), which means that \( \Lambda \) is a pencil of curves of genus \( \leq 5 \). □

**Proposition 6.3.** Let \( S \) be a minimal surface whose 1-canonical image is of dimensional 2. Suppose \( \chi \geq 14 \) and that \( S \) has no pencils of curves of genus \( \leq 5 \). Then \( H_4 \) is not uniquely decomposable.

**Proof.** We can suppose \( H_n \) is uniquely decomposable for \( n \leq 3 \), for otherwise the corollary is trivially true. Let \( \Sigma_i \) be a basic set in \( P_i \) (\( i = 1, 2, 3 \)). By [X1, lemma 3] and by the hypothesis that 1-canonical image is of dimensional 2, we have the dimension of \( \Sigma_1 \) is \( \geq 2 \); also by [X1, lemma 3], we have the dimension of \( \Sigma_2 \) is \( \geq 3 \); By lemma 6.2, the condition of lemma 2.7 is satisfied for

\[
\Sigma_1 \subset \Sigma_2 \subset \Sigma_3.
\]

Then this corollary results from the relations

\[
\# \Sigma_i = \dim H_i = \frac{i(i-1)}{2} K^2 + \chi \quad (i = 2, 3),
\]

\[
K^2 \leq 9 \chi \quad \text{(Bogomolov-Miyaoka-Yau’s inequality)},
\]

that \( \# (\Sigma_1, \Sigma_3 \cup \Sigma_2, \Sigma_2) > \dim H_4 \). In particular, there are more than \( \dim H_4 \) semi-invariants in \( H_4 \). □

**Lemma 6.4.** Suppose \( K^2 \geq 10 \) and with no pencils of curves of genus 2. Let \( G \) be an abelian group of automorphisms of \( S \), such that \( |4K| \) contains a pencil \( \Lambda \) whose general numbers are fixed under the action of \( G \). Then

\[
\# G \leq 24K^2 + 16.
\]
Moreover, if the 1-canonical map of $S$ is birational, then

$$\#G \leq 18K^2 + 18.$$ 

**Proof.** Blowing up the base points of $\Lambda$, we get a surface $S'$ such that $\Lambda$ is associated to a fibration $f: S' \to C$. Let $F$ be a general fiber of $f$, and let $k$ be the number of fibers of $f$ contained in a general member of the moving part of $\Lambda$. We have

$$g(F) - 1 \leq \frac{2k + 8}{k^2}K^2.$$ 

Let $H$ be the stabilizer of $F$. Then the index of $H$ in $G$ is at most $k$. Because $H$ is an abelian group of automorphisms of curve $F$, we have

$$\#H \leq 4g(F) + 4.$$ 

Therefore if $k \geq 2$, we get

$$\#G \leq \begin{cases} 
36K^2 + 24; \\
18K^2 + 18, \text{ if } f \text{ is non-hyperelliptic.}
\end{cases}$$

This allows us to assume $k \leq 1$, consequently there is no divisor in $|2K|$ whose pull-back on $S'$ contains $F$.

Let $\pi: F \to B := F/H$ be the projection map. Let $O_1, \ldots, O_l$ be the orbits of the action of $H$ on $F$ contained in pull-backs of fixed divisors in $|2K|$, $n_i$ the number of points in $O_i$. If $l < 4$, then the image of the bicanonical map of $S$ is a rational surface (see e.g. [X1, p 624] for a proof); therefore $S$ has a pencil of curves of genus 2 [1?], contradiction to the hypothesis. So we can assume $l \geq 4$. Using Hurwitz formula to $\pi$, we get

$$(l - 2)\#H \leq 2g(F) - 2 + \sum_{i=1}^{l} n_i.$$ 

Because $n_i \leq 2KF \leq 8K^2/k$, we see that $\#G \leq 17K^2$, once $l \geq 6$. When $4 \leq l \leq 5$, remark that the pull-back $E$ of each fixed divisor in $|2K|$ has a unique representation

$$E = a_1O_1 + \cdots + a_lO_l,$$

with $0 \leq a_in_i \leq 8K^2$, $i = 1, \ldots, l$. Let $b_i$ ($i = 1, \ldots, l$) be the maximum of the $i$-th coefficient in all such representations, we must have

$$b_1 \cdots b_l \geq \dim H_2 = K^2 + \chi, \quad b_in_i \leq 8K^2.$$ 

Now if $l = 5$, then $\sum \frac{1}{b_i} < 4 + 1/K^2$; consequently,

$$\sum n_i \leq 8(4 + \frac{1}{K^2})K^2, \quad \#G \leq \frac{52}{K^2} + 8.$$
When \( l = 4 \), taking also into account the fact that no such pull-back divisor can contain another, it is easy to see that the minimum for \( \sum_{i=1}^{4} n_i \) is achieved when, say

\[
\begin{align*}
b_1 = b_2 &= \frac{1}{2} \dim H_2 - 1, \quad b_3 = b_4 = 1, \\
n_1 = n_2 &= 16, \quad n_3 = n_4 = 8K^2,
\end{align*}
\]

and consequently \( \#G \leq 18K^2 + 16. \) \( \square \)

**Proof of theorem 6.1.** Proposition 6.3 guarantees that there is a pencil \( \Lambda \) in \( |4K| \) whose every element is a fixed divisor by \( G \). Hence by Lemma 6.4, we get the result. \( \square \)

**Theorem 6.5.** Let \( S \) be a minimal smooth surface of general type over the complex number field, and \( K \) the canonical divisor of \( S \). Let \( G \) be an abelian group of automorphisms of \( S \) (i.e. \( G \subset \text{Aut}(S) \)). Suppose that \( K^2 \geq 181 \), and that either \( S \) has no pencil of curves of genus \( g \), \( 3 \leq g \leq 5 \) or \( K_S^2 \geq \frac{12(g-1)}{g+5} \chi(O_S) \) when \( S \) has a pencil of curves of genus \( g \), \( 3 \leq g \leq 5 \). Then

\[
\#G \leq 24K^2 + 256.
\]

**Proof.** By a theorem of Beauville (cf. [Be], [BPV]), we have that either \( \varphi_1 \) is composed with a pencil of curves of genus \( g \), \( 2 \leq g \leq 5 \), and the pencil has no base points, or \( \dim \varphi_1(S) = 2 \). Hence the result follows from Theorem 6.1 and Propositions 5.5, 5.6, and 5.7. \( \square \)

§7. Abelian subgroups for small numerical invariants

**Theorem 7.1.** Let \( S \) be a minimal smooth surface of general type over the complex number field, \( K \) the canonical divisor of \( S \), and \( \chi(O_S) \) the Euler characteristic of the structure sheaf of \( S \). Let \( G \) be an abelian group of automorphisms of \( S \) (i.e. \( G \subset \text{Aut}(S) \)). Then

\[
\#G \leq 36K^2 + 24,
\]

provided \( \chi(O_S) \geq 8 \).

**Lemma 7.2.** Suppose \( \chi \geq 5 \), and that \( P_6 \) is uniquely decomposable. Then the dimension of \( \Sigma_3 \) is 3.

**Proof.** Let \( d \) be the dimension of \( \Sigma_3 \), Then

\[
\#(\Sigma_3, \Sigma_2) > (d + 1)(\#\Sigma_2 - d) \quad \text{(cf. [5, Lemma 1])}.
\]
Now assume \( d \geq 4 \), then taking into account the inequality \( \# \Sigma_3 \geq d + 1 \), we have

\[
\#(\Sigma_3, \Sigma_3) \geq 5\# \Sigma_3 - 10.
\]

On the other hand, we have

\[
d \leq 3.\]

So we get \( \#(\Sigma_3, \Sigma_3) \geq 5\dim H_3 - 10 > \dim H_6 \), when \( \chi \geq 3 \), a contradiction. Hence we have \( d \leq 3 \). Now \( d \leq 2 \) is impossible: otherwise, the image of the 3-canonical map of \( S \) is either a rational curve or a rational surface (cf. [5, Lemma 3]). □

we may copy the proof of [X1, Lemma 4] to get the following.

**Lemma 7.3.** Suppose \( K^2 \geq 10 \), and let \( \Sigma_i \) be a basic set in \( H_i \) for \( i = 3, 4 \). Assume that either there is a chain in \( \Sigma_4 \) of length \( \geq \frac{1}{6} \# \Sigma_4 \) or there is a chain in \( \Sigma_3 \) of length \( \geq \frac{1}{4} \# \Sigma_3 \). Then \( S \) has a pencil of curves of genus 2. □

**Lemma 7.4.** Suppose \( \chi \geq 8 \) and that \( S \) has no pencils of curves of genus 2. Then \( H_6 \) is not uniquely decomposable.

*Proof.* We can suppose \( H_n \) is uniquely decomposable for \( n \leq 4 \), for otherwise the corollary is trivially true. Let \( \Sigma_i \) be a basic set in \( P_i (i = 2, 3, 4) \). By Lemma 7.2 and Lemma 7.3, the condition of Lemma 2.75 is satisfied for

\[
\Sigma_2 \subset \Sigma_3 \subset \Sigma_4.
\]

Then this corollary results from the relations

\[
\# \Sigma_i = \dim H_i = \frac{i(i-1)}{2} K^2 + \chi \quad (i = 2, 3, 4),
\]

\[
K^2 \leq 9 \chi \quad \text{(Bogomolov- Miyaoka-Yau’s inequality)},
\]

that \( \#(\Sigma_2, \Sigma_4 \cup \Sigma_3, \Sigma_3) > \dim H_6 \). In particular, there are more than \( \dim H_6 \) semi-invariants in \( H_6 \). □

**Lemma 7.5.** Suppose \( K^2 \geq 10 \) and with no pencils of curves of genus 2. Let \( G \) be an abelian group of automorphisms of \( S \), such that \( |6K| \) contains a pencil \( \Lambda \) whose general numbers are fixed under the action of \( G \). Then

\[
\# G \leq 36K^2 + 24.
\]

*Proof.* We modify the proof of [X1, lemma 7] to get our results: we can assume that there is no divisor in \( |2K| \) whose pull-back contains a general fiber \( F \) of the fibration \( f: S' \rightarrow C \) induced by a pencil \( \Lambda \) of \( G \)-invariant divisors in \( |4K| \). Let \( O \).
..., \( O_l \) be the orbits on \( F \) contained in pull-backs of fixed divisors in the moving parts of the pull-backs of \( |2K| \). As \( K^2 \geq 10 \), the 2-canonical map is birational by Bombieri’s theorem, and so \( l \geq 4 \). Using Hurwitz formula to \( \pi \), we get

\[
(l - 2)\#H \leq 2g(F) - 2 + \sum_{i=1}^{l} n_i.
\]

Because \( n_i \leq 2KF \leq 12K^2/k \), we see that \( \#G \leq 34K^2 \), once \( l \geq 5 \), and \( \#G \leq 33K^2 + 24 \) when \( l = 4 \) as in [X1, Lemma 7]. □

Proof of theorem 1. If \( S \) has a relatively minimal genus 2 fibration, then \( \#G \leq 12.5K^2 + 100 \) (cf. [Ch1, Theorem 0.2]). Hence we can suppose that \( S \) has no pencil of curves of genus 2, then Lemma 7.4 guarantees that there is a pencil \( \Lambda \) in \( |6K| \) each of whose elements is a fixed divisor by \( G \). Hence by Lemma 7.5, we get the result. □

As a consequence of Theorem 7.1, we have that the order of an Abelian subgroup of Aut(\( S \)) is at most \( \#G \leq 36K^2 + 24 \), provided \( K^2 \geq 64 \). Here we give a similar estimates for surfaces with \( K^2 < 64 \).

**Lemma 7.6.** Suppose \( K^2 \geq 4 \) (resp. \( K^2 \geq 2 \)). Then \( H_{12} \) (resp. \( H_{16} \)) is not uniquely decomposable.

**Proof.** We may consider the natural map

\[
H_4 \otimes H_8 \oplus H_6 \otimes H_6 \to H_{12}
\]

(resp. \( H_5 \otimes H_{11} \oplus H_8 \otimes H_8 \to H_{16} \))

instead of

\[
H_2 \otimes H_4 \oplus H_3 \otimes H_3 \to H_6.
\]

Note that if there is a chain in \( \Sigma_6 \) (resp. \( \Sigma_8, \Sigma_{11} \)) of length \( \geq \frac{1}{4}\#\Sigma_6 \) (resp. \( \geq \frac{1}{4}\#\Sigma_8, \geq \frac{1}{6}\#\Sigma_{11} \)) then \( S \) has a pencil of curves of genus 1, contradicting that \( S \) is of general type (cf. [X1, Lemma 4] for a proof). We choose and fix an semi-invariant \( u' \) in \( H_2 \) (resp. \( u'' \) in \( H_3 \)) instead of \( u \) in \( H_1 \), as in §1. Then one checks immediately that the first half of §6 goes for the new pair. □

**Theorem 7.7.** Let \( S \) be a minimal smooth surface of general type over the complex number field, and \( K \) the canonical divisor of \( S \). Let \( G \) be an abelian group of automorphisms of \( S \) (i.e. \( G \subset \text{Aut}(S) \)). Then

\[
\#G \leq \begin{cases} 
114K^2 + 24, & \text{provided } 4 \leq K^2 \leq 63; \\
200K^2 + 22, & \text{provided } 2 \leq K^2 \leq 3; \\
270, & \text{provided } K^2 = 1.
\end{cases}
\]
Proof. If $K^2 \geq 4$ (resp. $K^2 \geq 2$), by Lemma 7.6, $H_{12}$ (resp $H_{16}$) is not uniquely decomposable. We modify the proof of [X1, Lemma 7] to get our results: we can assume that there is no divisor in $|3K|$ (resp. $|4K|$) whose pull-back contains a general fiber $F$ of the fibration $f : S' \to C$ induced by a pencil $\Lambda$ of $G$-invariant divisors in $|12K|$ (resp. $|16K|$). Let $O_1, \ldots, O_l$ be the orbits on $F$ contained in pull-backs of fixed divisors in the moving parts of the pull-backs of $|3K|$ (resp. $|4K|$). As $K^2 \geq 4$ (resp. $K^2 \geq 2$), the tricanonical map (resp. 4-canonical map) is birational by Bombieri’s theorem, and so $l \geq 4$. And as $g(F) - 1 \leq \frac{6k + 72}{k^2} K^2$ (resp. $g(F) - 1 \leq \frac{8k + 128}{k^2} K^2$), $n_i \leq 36K^2/k$ (resp. $\leq 64K^2/k$), we get $\#G \leq 112K^2$ (resp. $\leq 198K^2$), once $l \geq 5$, and $\#G \leq 114K^2 + 24$ (resp. $\leq 200K^2 + 22$) when $l = 4$ as in the proof of Lemma 7.5.

If $K^2 = 1$, the proof of [X2, Appendix A, Theorem A1] shows that $\#G \leq 270$. □

§8. Miscellaneous results

Theorem 8.1. Let $S$ be a simply connected even smooth surface of general type over the complex number field, i.e., the intersection form on $H^2(S, \mathbb{Z})$ is even, or equivalently, $K_S = 2L$ for some integral divisor $L \in \text{Div}(S)$. Let $G$ be an abelian group of automorphisms of $S$ (i.e. $G \subset \text{Aut}(S)$). Suppose that

(i) $\phi_L$ is generally finite, and $\phi_{2L}$ is birational;
(ii) $S$ has no fibrations of genus $3 \leq g \leq 8$.
(iii) $K_S^2 > 196$. Then

$$\#G \leq 12K^2 + 24.$$ 

Proof. Since $S$ is simply connected, for any $\sigma \in G$, we have $\sigma^* L \equiv L$. Therefore there is a natural action of the abelian group $G$ on $H_n(L) := H^0(S, nL)$, for a positive integer $n$. Hence we can use $nL$ (instead of $nK$) to define the space $P_n(L)$ and the basic set $\Sigma_n(L)$ as in §3.

Then we can modify the proof of Theorem 6.1 to get the result. We leave the detail of the proof to the reader. □

Corollary 8.2. Let $d_i \in \mathbb{Z}$, $d_i \geq 2$, for $i = 1, \ldots, N - 2$, let $S$ be a smooth complete intersection of type $(d_1, \ldots, d_{N-2})$ in $\mathbb{CP}^N$, and let $G$ be an abelian group of automorphisms of $S$ (i.e. $G \subset \text{Aut}(S)$). Suppose that $\sum_{i=1}^{N-2} d_i - N$ is odd, and $K_S^2 > 196$. Then

$$\#G \leq 12K^2 + 24.$$ 

Proof. By Theorem 8.1, it is enough to show that $S$ has no fibrations of genus $\leq 8$. Otherwise, suppose that $S$ has a fibration of genus $g \leq 8$. Let $F$ be a general fiber of such a fibration. Since $\deg F \geq N$ (cf. [Mu, p 77]), we have

$$14 \geq 2g - 2 \geq N(\sum d_i - N - 1) \geq N(N - 5).$$

Hence $N \leq 7$, and $\sum d_i \leq N + 1 + [\frac{14}{N}]$. Now we can verify case-by-case that this is impossible since $K^2_S > 196$. □

For surfaces in $\mathbb{CP}^3$, we have a better estimation for the order of abelian automorphism groups.

**Theorem 8.3.** Let $G$ be an abelian group of automorphisms of a surface $S$ in $\mathbb{CP}^3$ of degree $d \geq 5$. Then $\#G \leq 3d^2(d - 2) + 9$.

**Proof.** Changing coordinates, we can assume that each element $\sigma \in G$ has the form $\text{diag}(\ast, \ast, \ast, \ast)$ since the finite abelian group $G$ can be diagonalisable. Let $a_1m_1(x) + \cdots + a_tm_t(x)$ be the defining equation of $S$, where $m_i(x)$ are monomials of degree $d$ in $\mathbb{C}[x_0, \ldots, x_3]$, and $a_i \in \mathbb{C}$. Since $S$ is smooth, there exists a surface $Y$, the defining equation given by a general linear combination of $m_i(x)$, such that $Y$ and the complete intersection $C := X \cap Y$ are both smooth. We have a natural group map $\beta: G \to \text{Aut}(C)$. $\text{Ker} \beta = 1$ since $C$ is not contained in a hyperplane of $\mathbb{P}^3$. Since $C$ is non-hyperelliptic, we have the order of an abelian automorphism group is $\leq 3g(C) + 6$ (Theorem 4.9), where $g(C) = d^2(d - 2) + 1$. Consequently, we have $\#G \leq 3d^2(d - 2) + 9$. □

It is an interesting problem whether a generic nonhyperelliptic surface in its moduli space has no automorphisms except the identity. For surfaces in $\mathbb{P}^3$, the situation is simple; we can modify the proof of [Ha2, Ex. 5.7(c)] to prove the following.

**Proposition 8.4.** If $d \geq 5$, then smooth surfaces in $\mathbb{CP}^3$ of degree $d$ having general moduli has no automorphisms except the identity. Here of general moduli means that there is a countable union $V$ of subvarieties of the space $\mathbb{P}^N$ of surfaces of degree $d$ in $\mathbb{P}^3$, such that the statement $\text{Aut}(S) = 1$ holds for $S \in \mathbb{P}^N - V$. □

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