Nonparametric Estimation of Linear Multiplier in Stochastic Differential Equations Driven by \( \alpha \)-Stable Noise

B.L.S. Prakasa Rao
CR Rao Advanced Institute of Mathematics, Statistics and Computer Science, Hyderabad, India

Abstract: We discuss nonparametric estimation of linear multiplier in a trend coefficient in models governed by a stochastic differential equation driven by an \( \alpha \)-stable small noise.

Keywords and phrases: Stochastic differential equation; Trend coefficient; Linear multiplier; Nonparametric estimation; Kernel method; Levy process.

AMS Subject classification (2010): Primary 62M09; Secondary 60G52.

1 Introduction

Let \((\Omega, \mathcal{F}, P)\) be a probability space equipped with a right continuous and increasing family of \(\sigma\)-algebras \(\{\mathcal{F}_t, t \geq 0\}\). Let \(\{Z_t, t \geq 0\}\) be a standard \(\alpha\)-stable Levy process with \(Z_1\) distributed as \(S_{\alpha}(1, \beta, 0)\). A random variable \(Z\) is said to have a stable distribution \(S_{\alpha}(\sigma, \beta, \mu)\) with index of stability \(\alpha \in (0, 2]\), scale parameter \(\sigma \in (0, \infty)\), skewness parameter \(\beta \in [-1, 1]\), and location parameter \(\mu \in (-\infty, \infty)\) if it has the characteristic function (c.f) of the following form:

\[
\phi_Z(u) = E[\exp(iuZ)] = \exp\{-\sigma^\alpha |u|^\alpha (1 - i\beta \text{sgn}(u) \tan(\frac{\alpha \pi}{2})) + i\mu u\} \quad \text{if} \quad \alpha \neq 1
\]

\[
= \exp\{-\sigma |u|(1 + i\beta \frac{2}{\pi} \text{sgn}(u) \log |u|) + i\mu u\} \quad \text{if} \quad \alpha = 1.
\]

If \(\mu = 0\), then we say that the random variable \(Z\) is strictly stable. If, in addition \(\beta = 0\), we say that the random variable \(Z\) is symmetric \(\alpha\)-stable (cf. Samorodnitsky and Taqcu (1994), Sato (1999)). Here after we assume that \(1 < \alpha < 2\).

Suppose that \(X = \{X_t, 0 \leq t \leq T\}\) is a stochastic process satisfying the stochastic differential equation (SDE)

\[
dX_t = \theta(t)X_t dt + \epsilon dZ_t, 0 \leq t \leq T, X_0 = x_0
\]

where \(Z = \{Z_t, 0 \leq t \leq T\}\) is a standard \(\alpha\)-stable Levy process with \(Z_1\) distributed as \(S_{\alpha}(1, \beta, 0)\). Suppose that \(\alpha\) and \(\beta\) are known with \(1 < \alpha < 2\) but the linear multiplier \(\theta(.)\) is unknown. The problem is to estimate the function \(\theta(.)\) and the trend function based on
the observations of the process $X$ over the interval $[0, T]$. Parameter estimation for Ornstein-Uhlenbeck process driven by $\alpha$-stable Levy motion is investigated in Hu and Long (2007, 2009). Least squares estimation for discretely observed Ornstein-Uhlenbeck processes with small Levy noises is studied in Long (2009) and Ma (2010). Parametric estimation for a class of SDE driven by small stable noises from discrete observations is studied in Long (2010). Shen and Xu (2014) studied estimation of the drift parameter for SDEs driven by small noises which is more than pure jump $\alpha$-stable noises. Shen et al. (2018) discussed parameter estimation for Ornstein-Uhlenbeck processes driven by a fractional Levy process. Nonparametric estimation of the trend function for stochastic differential equations driven by a mixed fractional Brownian motion or a sub-fractional Brownian motion are investigated in Prakasa Rao (2019, 2020).

2 Levy processes

An $\mathcal{F}_t$-adapted stochastic process $\{Z_t, t \geq 0\}$ is called $\alpha$-stable Levy process if (i) $Z_0 = 0$ a.s., (ii) $Z_t - Z_s$ has the distribution $S_\alpha((t - s)^{1/\alpha}, \beta, 0)$ for $0 \leq s < t < \infty$, and (iii) for any $0 \leq t_0 < t_1 < \ldots < t_m < \infty$, the random variables $Z_{t_0}, Z_{t_1} - Z_{t_0}, \ldots, Z_{t_m} - Z_{t_{m-1}}$ are independent. Ito-type stochastic integrals with respect to $\alpha$-stable Levy processes were investigated in Kallenberg (1975, 1992) and Rosinski and Woyczynski (1986). Let $L^\alpha$ be the family of all real-valued $\mathcal{F}_t$-predictable processes on $\Omega \times [0, \infty)$ such that for every $T > 0$, $\int_0^T |\phi(t, \omega)|^\alpha \, dt < \infty$ a.s. It is known that a predictable process $\phi$ is integrable with respect to a strictly $\alpha$-stable Levy process $Z$, that is, $\int_0^T \phi(t) \, dZ_t$ exists for every $T > 0$, if and only if $\phi \in L^\alpha$. The following moment inequality is due to Long (2010) improving the result in Theorem 3.2 in Rosinski and Woyczynski (1985).

**Theorem 2.1:** Suppose $Z$ is a $\alpha$-stable Levy process and $\phi$ is a predictable process such that $\phi \in L^\alpha$. Then there exists a positive constant $c$ depending only on $\alpha$ and $\beta$ independent of $T$ such that

\begin{equation}
E[ \sup_{0 \leq t \leq T} |\int_0^t \phi(s) \, dZ_s|] \leq cE[|\int_0^T |\phi(t)|^\alpha \, dt|^{1/\alpha}].
\end{equation}

In particular, by choosing $\phi(s) \equiv 1$, it follows that

\begin{equation}
E[ \sup_{0 \leq t \leq T} |Z_t|] \leq cT^{1/\alpha}
\end{equation}

where $c$ is a positive constant depending on $\alpha$.

The following result, due to Rosinski and Woyczynski (1985) and Kallenberg (1992, Theorem 4.1), gives a method for finding the distribution of a stochastic integral with respect to
an $\alpha$-stable Levy process. Let $\phi_+$ and $\phi_-$ denote the positive and negative parts of a function $\phi$.

**Theorem 2.2:** Suppose $\phi \in L^\alpha$ and $Z$ is a strictly $\alpha$-stable Levy process. Then (i) there exists independent processes $Z'$ and $Z''$ with the same distribution as $Z$ such that

$$
\int_0^t \phi(s)dZ_s = Z'(\int_0^t [\phi_+(s)]^\alpha ds) - Z''(\int_0^t [\phi_-(s)]^\alpha ds)
$$

almost surely. (ii) If $Z$ is symmetric, that is $\beta = 0$, then there exists a $\alpha$-stable Levy process $Z'$ such that $Z'$ and $Z$ have the same distribution and

$$
\int_0^t \phi(s)dZ_s = Z'(\int_0^t |\phi(s)|^\alpha ds)
$$

almost surely.

## 3 Preliminaries

Let us consider the stochastic differential equation

$$
dX_t = \theta(t)X_t dt + \epsilon dZ_t, X_0 = x_0, 0 \leq t \leq T
$$

where the trend function $\theta(.)$ is unknown. We would like to estimate the function $\theta(.)$ and the drift $\theta(t)X_t$ based on the observation $X = \{X_t, 0 \leq t \leq T\}$. Let $x = \{x_t, 0 \leq t \leq T\}$ be the solution of the ordinary differential equation

$$
\frac{dx_t}{dt} = \theta(t)x_t, x_0, 0 \leq t \leq T.
$$

Observe that

$$x_t = x_0 \exp(\int_0^t \theta(s)ds).
$$

We assume that (A1) the function $\theta(t)$ is bounded over the interval $[0, T]$ by a constant $L$.

**Lemma 3.1:** Let $X_t$ and $x_t$ be the solutions of the equation (3.1) and (3.2) respectively. Then, with probability one,

$$
(a) |X_t - x_t| \leq e^{Lt} \epsilon \sup_{0 \leq s \leq t} |Z_s|
$$

and

$$
(b) \sup_{0 \leq t \leq T} E|X_t - x_t| \leq e^{LT} \epsilon T^{1/\alpha}.
$$

**Proof of (a):** Let $u_t = |X_t - x_t|$. Then

$$
u_t \leq \int_0^t |\theta(v)(X_v - x_v)| dv + \epsilon |Z_t|
\leq L \int_0^t u_v dv + \epsilon \sup_{0 \leq s \leq t} |Z_s|.
$$
Applying the Gronwall’s lemma (cf. Lemma 1.12, Kutoyants (1994), p.26), it follows that

\[ u_t \leq \epsilon \sup_{0 \leq s \leq t} |Z_s| e^{Lt}. \]

(3. 6)

**Proof of (b):** From (3.3), we have,

\[ E|X_t - x_t| \leq e^{Lt} \epsilon E(\sup_{0 \leq s \leq t} |Z_s|) \]

\[ = e^{Lt} \epsilon t^{1/\alpha} \]

(3. 7)

by the moment inequality in Theorem 2.1 applied to the function \( \phi \equiv 1 \). Hence

\[ \sup_{0 \leq t \leq T} E|X_t - x_t| \leq e^{LT} \epsilon T^{1/\alpha}. \]

(3. 8)

4 Estimation of the Drift function

Let \( \Theta_0(L) \) denote the class of all functions \( \theta(\cdot) \) with the same bound \( L \). Let \( \Theta_k(L) \) denote the class of all functions \( \theta(\cdot) \) which are uniformly bounded by the same constant \( L \) and which are \( k \)-times differentiable satisfying the condition

\[ |\theta^{(k)}(x) - \theta^{(k)}(y)| \leq L'|x - y|, x, y \in \mathbb{R} \]

for some constant \( L' > 0 \). Here \( g^{(k)}(x) \) denotes the \( k \)-th derivative of \( g(\cdot) \) at \( x \) for \( k \geq 0 \). If \( k = 0 \), we interpret the function \( g^{(0)} \) as the function \( g(\cdot) \).

Let \( G(u) \) be a bounded function with finite support \([A, B]\) with \( A < 0 < B \) satisfying the condition

\( (A_2) \) \( G(u) = 0 \) for \( u < A \) and \( u > B \); and \( \int_A^B G(u) du = 1 \).

Boundedness of the function \( G(\cdot) \) with finite support \([A, B]\) implies that

\[ \int_{-\infty}^{\infty} |G(u)|^\alpha du < \infty; \int_{-\infty}^{\infty} |u^j G(u)| du < \infty, j \geq 0. \]

We define a kernel type estimator \( \hat{\theta}_t \) of the function \( \theta(t) \) by the relation

\[ \hat{\theta}_t X_t = \frac{1}{\varphi_\epsilon} \int_0^T G \left( \frac{\tau - t}{\varphi_\epsilon} \right) dX_\tau \]

(4. 1)

where the normalizing function \( \varphi_\epsilon \to 0 \) as \( \epsilon \to 0 \). Let \( E\theta(\cdot) \) denote the expectation when the function \( \theta(\cdot) \) is the linear multiplier.
Theorem 4.1: Suppose that the linear multiplier \( \theta(.) \in \Theta_0(L) \) and the function \( \varphi_\epsilon \to 0 \) such that \( \epsilon \varphi_\epsilon^{-1} \to 0 \) as \( \epsilon \to 0 \). Further suppose that the conditions \((A_1),(A_2)\) hold. Then, for any \( 0 < c \leq d < T \), the estimator \( \hat{\theta}_t \) satisfies the property

\[
\lim_{\epsilon \to 0} \sup_{\theta(.) \in \Theta_0(L)} \sup_{c \leq t \leq d} E_\theta(|\hat{\theta}_t X_t - \theta(t)x_t|) = 0.
\]

In addition to the conditions \((A_1),(A_2)\), suppose the following condition holds.

\((A_3)\int_{-\infty}^{\infty} u^j G(u)du = 0 \) for \( j = 1, 2, \ldots k \).

Theorem 4.2: Suppose that the function \( \theta(.) \in \Theta_{k+1}(L) \) and \( \varphi_\epsilon = \epsilon^{k-1/2} \). Suppose the conditions \((A_1)-(A_3)\) hold. Then

\[
\lim_{\epsilon \to 0} \sup_{\theta(.) \in \Theta_{k+1}(L)} \sup_{c \leq t \leq d} E_\theta(|\hat{\theta}_t X_t - \theta(t)x_t|) \epsilon^{k-1/2} < \infty.
\]

Theorem 4.3: Suppose that the function \( \theta(.) \in \Theta_{k+1}(L) \) and \( \varphi_\epsilon = \epsilon^\gamma \) where \( \gamma = \frac{1}{(k+2)-\alpha} \). Suppose the conditions \((A_1)-(A_3)\) hold. Then the asymptotic distribution of

\[
\varphi_\epsilon^{-(k+1)}(\hat{\theta}_t X_t - \theta(t)x_t),
\]

as \( \epsilon \to 0 \) is the distribution of the random variable

\[
\left( \int_0^T G(u)^{\alpha_+}du \right)^{1/\alpha} U_1 - \left( \int_0^T G(u)^{\alpha_-}du \right)^{1/\alpha} U_2
\]

where \( U_1 \) and \( U_2 \) are independent random variables with \( \alpha \)-stable distribution \( S_\alpha(1, \beta, 0) \) shifted by the constant \( m \) defined by

\[
m = \frac{J^{(k+1)}(x_t)}{(k+1)!} \int_{-\infty}^{\infty} G(u)u^{k+1}du
\]

and \( J(t) = \theta(t)x_t \).

5 Proofs of Theorems

Proof of Theorem 4.1: From the equation (3.1), we have

\[
E_\theta(|\hat{\theta}_t X_t - \theta(t)x_t|) = E_\theta\left[ \left| \frac{1}{\varphi_\epsilon} \int_0^T G \left( \frac{\tau - t}{\varphi_\epsilon} \right) (\theta(\tau)X_\tau - \theta(\tau)x_\tau) d\tau \right| + \frac{1}{\varphi_\epsilon} \int_0^T G \left( \frac{\tau - t}{\varphi_\epsilon} \right) \theta(\tau)x_\tau d\tau - \theta(t)x_t + \frac{\epsilon}{\varphi_\epsilon} \int_0^T G \left( \frac{\tau - t}{\varphi_\epsilon} \right) dZ_\tau \right]
\]

\[
\leq E_\theta\left[ \left| \frac{1}{\varphi_\epsilon} \int_0^T G \left( \frac{\tau - t}{\varphi_\epsilon} \right) (\theta(\tau)X_\tau - \theta(\tau)x_\tau) d\tau \right| \right]
\]

\[
+E_{\theta} \left[ \frac{1}{\varphi_\epsilon} \int_{0}^{T} G \left( \frac{\tau - t}{\varphi_\epsilon} \right) \theta(\tau)x_\tau d\tau - \theta(t)x_t \right] \\
+\frac{\epsilon}{\varphi_\epsilon} E_{\theta} \left[ \left| \int_{0}^{T} G \left( \frac{\tau - t}{\varphi_\epsilon} \right) dZ_\tau \right| \right] \\
= I_1 + I_2 + I_3 \text{ (say).}
\]

Apply the change of variables \( u = (t - \tau)\varphi_\epsilon^{-1} \) and denote \( \epsilon_1 = \min(\epsilon', \epsilon'') \), where \( \epsilon' = \sup \{ \epsilon : \varphi_\epsilon \leq -\frac{T}{A} \} \) and \( \epsilon'' = \sup \{ \epsilon : \varphi_\epsilon \leq -\frac{T - d}{B} \} \). Then, for \( \epsilon < \epsilon_1 \),

\[
(5.2) \quad I_3 = \frac{\epsilon}{\varphi_\epsilon} E_{\theta} \left( \int_{0}^{T} G \left( \frac{\tau - t}{\varphi_\epsilon} \right) dZ_\tau \right) \\
\leq \frac{\epsilon}{\varphi_\epsilon} C_1 \left[ \int_{0}^{T} \left\{ \left| G \left( \frac{\tau - t}{\varphi_\epsilon} \right) \right| \right\}^\alpha d\tau \right]^{1/\alpha} \\
\leq \frac{\epsilon}{\varphi_\epsilon} \int_{-\infty}^{\infty} |G(u)|^{1/\alpha} |\varphi_\epsilon|^{1/\alpha} \\
\leq \frac{C_2 \epsilon}{\varphi_\epsilon} |\varphi_\epsilon|^{1/\alpha} \text{ (by using (A2))}
\]

for some positive constant \( C_2 \) depending on \( T, \alpha, L, A \) and \( B \). Since \( \epsilon \varphi_\epsilon^{-1} \to 0 \) and \( \varphi_\epsilon \to 0 \) as \( \epsilon \to 0 \), it follows that \( I_1 \) tends to zero as \( \epsilon \to 0 \).

Furthermore

\[
(5.3) \quad I_2 = E_{\theta} \left[ \frac{1}{\varphi_\epsilon} \int_{0}^{T} G \left( \frac{\tau - t}{\varphi_\epsilon} \right) \theta(\tau)x_\tau d\tau - \theta(t)x_t \right] \\
\leq E_{\theta} \left[ \int_{-\infty}^{\infty} |G(u) (\theta(t + \varphi_\epsilon u) x_{t+\varphi_\epsilon u} - \theta(t)x_t) | du \right] \\
\leq L \int_{-\infty}^{\infty} |G(u)| \varphi_\epsilon du \\
\leq C_3 \varphi_\epsilon
\]

for some positive constant \( C_3 \) depending on \( T, \alpha, L, L_1, A \) and \( B \). Hence \( I_2 \) tends to zero as \( \epsilon \to 0 \). Furthermore note that

\[
(5.4) \quad I_1 = E_{\theta} \left[ \frac{1}{\varphi_\epsilon} \int_{0}^{T} G \left( \frac{\tau - t}{\varphi_\epsilon} \right) (\theta(\tau)x_\tau - \theta(t)x_t) d\tau \right]
\]
\[
E_\theta \left[ \int_{-\infty}^{\infty} G(u) \left( \theta(t + \varphi_\varepsilon u)X_{t+\varphi_\varepsilon u} - \theta(t + \varphi_\varepsilon u)x_{t+\varphi_\varepsilon u} \right) \, du \right]
\]

\[
\leq \left( \int_{-\infty}^{\infty} |G(u)| L E_\theta \left( |X_{t+\varphi_\varepsilon u} - x_{t+\varphi_\varepsilon u}| \right) \, du \right) \quad (\text{by using the condition } (A_1))
\]

\[
\leq \int_{-\infty}^{\infty} |G(u)| L \sup_{0 \leq t + \varphi_\varepsilon u \leq T} E_\theta \left( |X_{t+\varphi_\varepsilon u} - x_{t+\varphi_\varepsilon u}| \right) \, du
\]

\[
\leq C_4 \varepsilon \quad (\text{by using (3.4)})
\]

for some positive constant \( C_4 \) depending on \( T, \alpha \) and \( L \). Hence \( I_3 \) tends to zero as \( \varepsilon \to 0 \). Theorem 4.1 is now proved by using the equations (5.1) to (5.4).

**Remarks:** From the proof presented above, it is possible to choose the functions \( c_\varepsilon \) and \( d_\varepsilon \) such that \( c_\varepsilon \to 0, d_\varepsilon \to T \) and satisfy the conditions

\[
\frac{c_\varepsilon}{\varphi_\varepsilon} \geq -A, \quad \frac{T - d_\varepsilon}{\varphi_\varepsilon} \geq B
\]

(for instance, choose \( c_\varepsilon = -A\varphi_\varepsilon \) and \( d_\varepsilon = T - B\varphi_\varepsilon \). Then the estimator \( \hat{\theta}_t \) satisfies the property that

\[
\lim_{\varepsilon \to 0} \sup_{\theta(t) \in \Theta_0(L)} \sup_{c_\varepsilon \leq t \leq d_\varepsilon} E_\theta(\left| \hat{\theta}_t X_t - \theta(t)x_t \right|) = 0.
\]

**Proof of Theorem 4.2:** Let \( J(t) = \theta(t)x_t \). By the Taylor’s formula, for any \( u \in R \),

\[
J(y) = J(u) + \sum_{r=1}^{k} J^{(r)}(u) \frac{(y-u)^r}{r!} + [J^{(k)}(z) - J^{(k)}(u)] \frac{(y-u)^k}{k!}
\]

for some \( z \) such that \( |z - u| \leq |y - u| \). Using this expansion, the equation (3.2) and the conditions in the expression \( I_2 \) defined in the proof of Theorem 4.1, it follows that

\[
I_2 \leq \left[ \int_{-\infty}^{\infty} G(u) \left( J(t + \varphi_\varepsilon u) - J(t) \right) \, du \right]
\]

\[
= \sum_{j=1}^{k} J^{(j)}(t) \left[ \int_{-\infty}^{\infty} G(u)u^j \, du \right] \varphi_\varepsilon^j(j!)^{-1}
\]

\[
+ \left( \int_{-\infty}^{\infty} G(u)u^k \left( J^{(k)}(z_u) - J^{(k)}(x_t) \right) \, du \right) \varphi_\varepsilon^k(k!)^{-1}
\]

for some \( z_u \) such that \( |x_t - z_u| \leq |x_{t+\varphi_\varepsilon u} - x_t| \leq C|\varphi_\varepsilon u| \) for some positive constant \( C \). Hence

\[
I_2 \leq C_5 L \left[ \int_{-\infty}^{\infty} |G(u)u^{k+1}| \varphi_\varepsilon^{k+1}(k!)^{-1} \, du \right]
\]

\[
\leq C_6 (k!)^{-1} \varphi_\varepsilon^{k+1} \int_{-\infty}^{\infty} |G(u)u^{k+1}| \, du
\]
for some positive constant $C_7$ depending on $A, B, \alpha, T$ and $L$. Combining the relations (5.2), (5.4) and (5.6), we get that there exists a positive constant $C_8$ depending on $\alpha, T, L, A, B$ such that

$$\sup_{\theta \in \Theta} E_{\theta} |\hat{\theta}_t x_t - \theta(t)x_t| \leq C_8 (\epsilon \varphi^{-1} \varphi^1/\alpha + \varphi^{k+1} + \epsilon).$$

Choosing $\varphi = \epsilon^{k/2}$, we get that

$$\limsup_{\epsilon \to 0} \sup_{\theta(.) \in \Theta_{k+1}(L)} \sup_{c \leq t \leq d} E_{\theta} |\hat{\theta}_t X_t - \theta(t)x_t| \epsilon^{-\frac{(k+1)}{2}} < \infty.$$

This completes the proof of Theorem 4.2.

**Proof of Theorem 4.3:**

From the equation (3.1), we obtain that

$$\hat{\theta}_t X_t - \theta(t)x_t = \left[ \frac{1}{\varphi \epsilon} \int_0^T G \left( \frac{\tau - t}{\varphi \epsilon} \right) (\theta(\tau)X_{\tau} - \theta(\tau)x_{\tau}) \tau \right]
+ \frac{1}{\varphi \epsilon} \int_0^T G \left( \frac{\tau - t}{\varphi \epsilon} \right) \theta(\tau)x_{\tau} d\tau - \theta(t)x_t + \frac{\epsilon}{\varphi \epsilon} \int_0^T G \left( \frac{\tau - t}{\varphi \epsilon} \right) dZ_{\tau}$$

$$= \left[ \int_{-\infty}^{\infty} G(u)(\theta(t + \varphi \epsilon u)x_{t+\varphi \epsilon u} - \theta(t + \varphi \epsilon u)x_t + \varphi \epsilon \int_0^T G \left( \frac{\tau - t}{\varphi \epsilon} \right) dZ_{\tau} \right]
+ \int_{-\infty}^{\infty} G(u)(\theta(t + \varphi \epsilon u)x_{t+\varphi \epsilon u} - \theta(t)x_t) du
+ \frac{\epsilon}{\varphi \epsilon} \int_0^T G \left( \frac{\tau - t}{\varphi \epsilon} \right) dZ_{\tau}.$$

Let $J(t) = \theta(t)x_t$. By the Taylor’s formula, for any $u \in R$,

$$J(y) = J(u) + \sum_{r=1}^{k+1} \frac{J^{(r)}(u)(y-u)^j}{j!} + \frac{[J^{(k+1)}(z) - J^{(k+1)}(x)](y-u)^{k+1}}{(k+1)!}$$

for some $z$ such that $|z - u| \leq |y - u|$. Let

$$m = \frac{J^{(k+1)}(x)}{(k+1)!} \int_{-\infty}^{\infty} G(u)u^{k+1} du$$

and

$$R_1(t) = \varphi^{-k+1} \int_0^T G \left( \frac{\tau - t}{\varphi \epsilon} \right) (\theta(\tau)X_{\tau} - \theta(\tau)x_{\tau}) d\tau.$$
By arguments similar to those given in (5.2) for obtaining upper bounds, it follows that

\[ E|R_1(t)| \leq C \varphi^{-k} \epsilon. \]

Let

\[ R_2(t) = \varphi_{\epsilon}^{-(k+1)} \int_0^T G \left( \frac{\tau - t}{\varphi_{\epsilon}} \right) \theta(\tau)x_{\tau} - \theta(t)x_t d\tau. \]

Observe that

\[ R_2(t) = m + o(1) \]

by an application of the Taylor’s expansion under the condition \((A_3)\). Furthermore

\[ \varphi_{\epsilon}^{-(k+1)}(\hat{\theta}_t x_t - \theta(t)x_t) = \epsilon \varphi_{\epsilon}^{-(k+2)} \int_0^T G \left( \frac{\tau - t}{\varphi_{\epsilon}} \right) dZ_{\tau} + R_2(t) + R_1(t) \]

Let \( \varphi_{\epsilon} \) be chosen so that \( \varphi_{\epsilon}^{-1/\alpha} = \epsilon \varphi_{\epsilon}^{-(k+2)} \). One such choice is \( \varphi_{\epsilon} = \epsilon^v \) where \( v = (k + 2 - \frac{1}{\alpha})^{-1} \). We will now study the asymptotic behaviour of the random variable

\[ W_{\epsilon} = (\varphi_{\epsilon})^{-1/\alpha} \int_0^T G \left( \frac{\tau - t}{\varphi_{\epsilon}} \right) dZ_{\tau} \]

as \( \epsilon \to 0 \). Note that

\[ \varphi_{\epsilon}^{-(k+1)}(\hat{\theta}_t x_t - \theta(t)x_t) = W_{\epsilon} + m + o_p(1). \]

By Theorem 2.2, there exist two independent processes \( Z' \) and \( Z'' \) with the same distribution as \( Z \) such that

\[ \int_0^T G \left( \frac{\tau - t}{\varphi_{\epsilon}} \right) dZ_{\tau} \]

is the distribution of the random variable

\[ \varphi_{\epsilon}^{-(k+1)}(\hat{\theta}_t x_t - \theta(t)x_t) \]

is the distribution of the random variable

\[ (\int_0^T G(u)^{\alpha} du)^{1/\alpha} U_1 - (\int_0^T G(u)^{\alpha} du)^{1/\alpha} U_2 \]

where \( U_1 \) and \( U_2 \) are two independent random variables with \( \alpha \)-stable distribution \( S_{\alpha}(1, \beta, 0) \). Hence the asymptotic distribution of the random variable

\[ \varphi_{\epsilon}^{-(k+1)}(\hat{\theta}_t x_t - \theta(t)x_t) \]
where $U_1$ and $U_2$ are independent random variables with $\alpha$-stable distribution $S_\alpha(1, \beta, 0)$ shifted by the constant $m$ defined by

$$m = J^{(k+1)}(x_t) \int_{-\infty}^{\infty} G(u)u^{k+1}du$$

where $J(t) = \theta(t)x_t$.

Note that the results given above deal with asymptotic properties of the estimator for the function

$$J(t) = \theta(t)x_t = \theta(t)x_0 \exp(\int_0^t \theta(s) \, ds).$$

We will now present another method for the estimation of the linear multiplier $\theta(t)$.

### 6 Estimation of the Multiplier $\theta(.)$

Let $\Theta_p(L_\gamma)$ be a class of functions uniformly bounded and $k$-times continuously differentiable for some integer $k \geq 1$ with the $k$-th derivative satisfying the Holder condition of the order $\gamma \in (0, 1)$:

$$|\theta^{(k)}(t) - \theta^{(k)}(s)| \leq L_\gamma |t - s|^\gamma, \rho = k + \gamma.$$

From the Lemma 3.1, it follows that

$$|X_t - x_t| \leq e^{Lt} \sup_{0 \leq s \leq T} |Z_s|.$$

Let

$$A_t = \{\omega : \inf_{0 \leq s \leq t} X_s(\omega) \geq \frac{1}{2} x_0 e^{-Lt}\}$$

and let $A = A_T$. Define the process $Y$ with the differential

$$dY_t = \theta(t) I(A_t) dt + \epsilon X_t^{-1} I(A_t) \, dZ_t, 0 \leq t \leq T.$$

We will now construct an estimator of the function $\theta(.)$ based on the observation of the process $Y$ over the interval $[0, T]$. Define the estimator

$$\tilde{\theta}(t) = I(A) \frac{1}{\varphi_\epsilon} \int_0^T G\left(\frac{t-s}{\varphi_\epsilon}\right) dY_s$$

where the kernel function $G(.)$ satisfies the conditions $(A1) - (A3)$. Observe that

$$E|\tilde{\theta}(t) - \theta(t)| = E[I(A) \frac{1}{\varphi_\epsilon} \int_0^T G\left(\frac{t-s}{\varphi_\epsilon}\right)(\theta(s) - \theta(t)) \, ds$$

$$+ I(A^c)\theta(t) + I(A) \frac{\epsilon}{\varphi_\epsilon} \int_0^T G\left(\frac{t-s}{\varphi_\epsilon}\right) X_s^{-1} \, dZ_s|$$

$$\leq E[I(A) \int_R G(u)|\theta(t + u\varphi_\epsilon) - \theta(t)| \, du] + |\theta(t)|P(A^c)$$
\[ I_1 = \varphi_\varepsilon \mathbb{E}[I(A) \int_0^T G\left( \frac{t-s}{\varphi_\varepsilon} \right) X_s^{-1} dZ_s] \]
\[ = I_1 + I_2 + I_3. \text{ (say).} \]

Applying the Taylor’s theorem and using the fact that the function \( \theta(t) \in \Theta_\rho(L_\gamma) \), it follows that

\[ I_1 \leq \frac{L_\gamma}{(k+1)!} \varphi_\varepsilon \int_R |G(u) u^\rho| du. \]

Note that, by Lemma 3.1,

\[ P(A_c) = P\left( \inf_{0 \leq t \leq T} X_t < \frac{1}{2} x_0 e^{-LT} \right) \]
\[ \leq P\left( \inf_{0 \leq t \leq T} |X_t - x_t| + \inf_{0 \leq t \leq T} x_t < \frac{1}{2} x_0 e^{-LT} \right) \]
\[ \leq P\left( \inf_{0 \leq t \leq T} |X_t - x_t| < \frac{1}{2} x_0 e^{-LT} \right) \]
\[ \leq P\left( \sup_{0 \leq t \leq T} |X_t| > \frac{1}{2} x_0 e^{-LT} \right) \]
\[ \leq P\left( \epsilon e^{LT} \sup_{0 \leq t \leq T} |Z_t| > \frac{1}{2} x_0 e^{-LT} \right) \]
\[ = P\left( \sup_{0 \leq t \leq T} |Z_t| > \frac{x_0}{2\epsilon} e^{-2LT} \right) \]
\[ \leq \frac{D}{\alpha(2 - \alpha)} T \left( \frac{x_0}{2\epsilon} e^{-2LT} \right)^{-\alpha} \]

for some positive constant \( D \) by a maximal inequality for stable stochastic integrals in Gine and Marcus (1983) (cf. Joulin (2006)). The upper bound obtained above and the fact that \( |\theta(s)| \leq L, 0 \leq s \leq T \) leads to an upper bound for the term \( I_2 \). We have used the inequality

\[ x_t = x_0 \exp\left( \int_0^t \theta(s) ds \right) \geq x_0 e^{-Lt} \]

in the computations given above. Applying Theorem 2.1, it follows that

\[ |E[I(A) \int_0^T G\left( \frac{t-s}{\varphi_\varepsilon} \right) X_s^{-1} dZ_s]| \]
\[ = |E[\int_0^T G\left( \frac{t-s}{\varphi_\varepsilon} \right) X_s^{-1} I(A_s) dZ_s]| \]
\[ \leq CE\left[ \int_0^T |G\left( \frac{t-s}{\varphi_\varepsilon} \right) X_s^{-1} I(A_s)|^{1/\alpha} ds \right] \]
\[ \leq Ce^{LT} \left( \int_0^T \left| G\left( \frac{t-s}{\varphi_\varepsilon} \right) \right|^{1/\alpha} ds \right)^{1/\alpha} \]
\[ = Ce^{LT} (\varphi_\varepsilon)^{1/\alpha} \left( \int_R |G(u)|^{1/\alpha} du \right)^{1/\alpha} \]
for some positive constant $C$ depending on $\alpha$ which leads to an upper bound on the term $I_3$. Combining the above estimates, it follows that

$$E|\tilde{\theta}(t) - \theta(t)| \leq C_1 \varphi^\rho + C_2 \epsilon^\alpha + C_3 \epsilon^{(1/\alpha)-1}$$

for some positive constants $C_i$, $i = 1, 2, 3$ depending on $\alpha$ and $L$. Choosing $\varphi_\epsilon = \epsilon^{\alpha/\rho}$, we obtain that

$$E|\tilde{\theta}(t) - \theta(t)| \leq C_4 \epsilon^{(\rho-\alpha+1)/\rho} + C_5 \epsilon^\alpha$$

for some positive constants $C_4$ and $C_5$. It easy to see that $0 < \frac{\rho-\alpha+1}{\rho} < \alpha$ for all $\rho > \alpha - 1$ which holds since $1 < \alpha < 2$ and $\rho > 1$. Hence we obtain the following result implying the uniform consistency of the estimator $\tilde{\theta}(t)$ as an estimator of $\theta(t)$ as $\epsilon \to 0$.

**Theorem 6.1:** Let $\theta \in \Theta_\rho(L)$ and $\varphi_\epsilon = \epsilon^{\alpha/\rho}$. Suppose the conditions $(A_1) - (A_3)$ hold. Then, for any interval $[c, d] \subset [0, T]$,

$$\limsup_{\epsilon \to 0} \sup_{\theta(.) \in \Theta_\rho(L)} \sup_{c \leq t \leq d} E|\tilde{\theta}(t) - \theta(t)| \epsilon^{-\frac{\rho-\alpha+1}{\rho}} < \infty.$$ 

**Funding:** This work was supported under the scheme “INSA Senior Scientist” by the Indian National Science Academy while the author was at the CR Rao Advanced Institute for Mathematics, Statistics and Computer Science, Hyderabad 500046, India.

**References:**

Gine, E. and Marcus, M. (1983) The central limit theorem for stochastic integrals with respect to Levy processes, *Ann. Probability*, 11, 58-77.

Hu, Y. and Long, H. (2007) Parameter estimation for Ornstein-Uhlenbeck processes driven by $\alpha$-stable Levy motions, *Communications on Stochastic Analysis*, 1, 175-192.

Hu, Y. and Long, H. (2009) Least squares estimator for Ornstein-Uhlenbeck processes driven by $\alpha$-stable Levy motions, *Stochastic Process Appl.*, 119, 2465-2480.

Joulin, A. (2006) On maximal inequalities for stable stochastic integrals, Prepint, Department of Mathematics, Universite de La Rochelle.

Kallenberg, O. (1975) On the existence and path properties of stochastic integrals, *Ann. Probab.*, 3, 262-280.

Kallenberg, O. (1992) Some time change representations of stable integrals via predictable transformations of local martingale, *Stochastic Process appl.*, 40, 199-223.
Kutoyants, Y.A.(1994) Identification of Dynamical Systems with Small Noise. Kluwer: Dordrecht.

Long, H. (2009) Least squares estimator for discretely observed Ornstein-Uhlenbeck processes with small Levy noises, Statist. Probab. Lett., 79, 2076-2085.

Long, H. (2010) Parameter estimation for a class of stochastic differential equations driven by small stable noises from discrete observations, Acta Mathematica Scientia, 30B, 645-663.

Ma, C. (2010) A note on “Least squares estimator for discretely observed Ornstein-Uhlenbeck processes with small Levy noises”, Statist. Probab. Lett., 80, 1528-1531.

Prakasa Rao, B.L.S. (2019) Nonparametric estimation of trend for stochastic differential equations driven by mixed fractional Brownian motion, Stoch. Anal. and Appl., 37, 271-280.

Prakasa Rao, B.L.S. (2020) Nonparametric estimation of trend for stochastic differential equations driven by sub-fractional Brownian motion, Random Oper. Stoch. Equ. (to appear).

Rosinski, J. and Woyczynski, W.A. (1985) Moment inequalities for real and vector p-stable stochastic integrals, Probability Theory in Banach Spaces V, Lecture Notes in Math. Vol. 1153: 369-386, Springer.

Rosinski, J. and Woyczynski, W.A. (1986) On Ito stochastic integration with respect to p-stable motion: inner clock, integrability of sample paths, double and multiple integrals, Ann. Probab., 14, 271-286.

Samorodnitsky, G. and Taqqu, M.S. (1994) Stable non-Gaussian Random Processes: stochastic Models with Infinite Variance, Chapman and Hall, New York.

Sato, K.I. ( 1999) Levy Processes and Infinitely Divisible Distributions, Cambridge University Press, Cambridge.

Shen, L. and Xu, Q. (2014) Statistical inference for stochastic differential equations with small noises, Abstract and Applied analysis, Volume 2014, Article ID 473681, 6 pages.

Shen, G., Li, Y. and Gao, Z. (2018) Parameter estimation for Ornstein-Uhlenbeck process driven by fractional Levy process, Journal of Inequalities and Applications, Volume 2018, 2018: 356.