ON THE PICTURE DEPENDENCE OF RAMOND-RAMOND COHOMOLOGY

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ABSTRACT

Closed string physical states are BRST cohomology classes computed on the space of states annihilated by $b^{-0}$. Since $b^{-0}$ does not commute with the operations of picture changing, BRST cohomologies at different pictures need not agree. We show explicitly that Ramond-Ramond (RR) zero-momentum physical states are inequivalent at different pictures, and prove that non-zero momentum physical states are equivalent in all pictures. We find that D-brane states represent BRST classes that are nonpolynomial on the superghost zero modes, while RR gauge fields appear as polynomial BRST classes. We also prove that in $x$-cohomology, the cohomology where the zero mode of the spatial coordinates is included, there is a unique ghost-number one BRST class responsible for the Green-Schwarz anomaly, and a unique ghost number minus one BRST class associated with RR charge.

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1. Introduction and Summary

Ramond-Ramond states of the closed superstring play a crucial role in strong-weak duality conjectures. At zero momentum, these states decouple from perturbative amplitudes, but couple non-perturbatively to D-brane solitons [1]. Unlike all other bosonic states of the string or superstring, the kinetic terms of Ramond-Ramond (RR) fields appear without a dilaton dependent $e^{-2\phi}$ factor in the spacetime effective action in string gauge.

Despite the importance of these states, their role in superstring theory is still poorly understood. Sigma models in Ramond-Ramond backgrounds have been constructed with worldsheet fields of the Type II superstring only in the manifestly spacetime-supersymmetric formalism, using either the Green-Schwarz [2] or the modified Green-Schwarz description [3]. The Green-Schwarz sigma model is probably incomplete since it lacks a Fradkin-Tseytlin term coupling the spacetime dilaton to worldsheet curvature [5]. Although the modified Green-Schwarz sigma model includes such a Fradkin-Tseytlin term [6,3,7], it is unsuitable for the uncompactified superstring since it lacks manifest SO(9,1) covariance. Superstring field theory actions for RR fields have been constructed only at the free level [8], and require extra non-minimal worldsheet variables to be added to the usual Ramond-Neveu-Schwarz (RNS) variables [9,10]. It is not clear whether or not a consistent free closed string field theory can be formulated using only RNS variables, as would have been naturally expected to be the case.

The lack of a free RR kinetic term for closed superstrings using the standard RNS variables implies that, precisely speaking, we do not really have a definition of physical states in the RR sector. For bosonic strings, physical closed string states are simply defined as BRST cohomology classes computed on the semirelative complex, i.e., on the subvector space of states that are annihilated by the antighost zero mode ($\bar{b}_0 - \bar{\bar{b}}_0$). The necessity to restrict the set of states of the conformal field theory to the semirelative complex is well understood and can be seen in a variety of ways. For similar reasons, and a new one (see below), it seems clear that the restriction to the semirelative complex is also necessary in closed superstring field theory. On the other hand, BRST cohomology in superstring theory requires a choice of picture for the superghost vacua [17]. While it was known that absolute BRST cohomology, i.e., the cohomology computed on the unrestricted complex, is independent of the choice of picture [18,17], it was not known before this paper if semirelative cohomology would be independent of such choice. Indeed, the literature deals with the chiral superstring complex only [19,20,21], and the closed superstring semirelative cohomology appears not to have been computed.

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‡ Sigma models in Ramond-Ramond backgrounds have also been constructed in [4] using worldsheet fields of the heterotic superstring.

§ These include the need to accommodate topological constraints on sections of bundles over moduli spaces of Riemann surfaces [11], the requirement of having an action principle for the cohomology problem [12,13,14], and the required existence of a zero-momentum physical (ghost) dilaton state [15,16].
One could ask if any additional constraints should be imposed on the closed superstring fields, e.g. perhaps string fields in the RR sector need to be annihilated by \((\beta_0 - \bar{\beta}_0)\) where \(\beta_0\) and \(\bar{\beta}_0\) are the zero modes of bosonic ghosts. If spacetime supersymmetry is to be an off-shell symmetry, however, powerful constraints on such possibilities arise. In the NS-NS sector, the constraints defining the BRST complex (the off-shell string field) are \((b_0 - \bar{b}_0)\Phi = 0\), and \((L_0 - \bar{L}_0)\Phi = 0\) where \(L_0 - \bar{L}_0 = 0\) is the usual level-matching condition. Since the spacetime-supersymmetry generators anti-commute with \(b_0 - \bar{b}_0\), and commute with \((L_0 - \bar{L}_0)\), it is natural to impose the same constraints on the R-NS, NS-R, and RR BRST complexes. In this way, supersymmetry manifestly maps the complexes into each other, as should be the case for generators of off-shell symmetries.

Since the superghost fields do not have simple commutation relations with the supersymmetry operators, further constraints based on the superghosts in the R-NS, NS-R or RR sectors are likely to imply further conditions in the NS-NS sector. Moreover, all such conditions should not affect BRST cohomology at non-zero momentum and at the physical ghost number, for this cohomology matches the light-cone physical states. While all this argumentation does not prove that further conditions cannot be imposed, it does not look easy to do. In this paper, we only impose the \((b_0 - \bar{b}_0) = 0\) and \((L_0 - \bar{L}_0) = 0\) conditions, we show that the non-zero momentum cohomology is the expected one, and we find that the subtle zero-momentum cohomology appears to be physically sensible.

Using the standard RNS description, we perform a careful analysis of BRST semirelative cohomology in all sectors of closed superstrings, focusing in particular on the Ramond-Ramond sector. We show that the Ramond-Ramond BRST semirelative classes are indeed picture-dependent. The analysis of semi-relative cohomology is performed at all ghost numbers and in the pictures \((-\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, -3/2), (-3/2, -1/2),\) and \((-3/2, -3/2)\), which are the four Ramond-Ramond pictures in which all positive oscillator modes of the left and right-moving \((\beta, \gamma)\) ghosts annihilate the vacuum.

In section 2 we prove that for every sector of the closed superstring, the BRST semirelative classes at non-zero momentum are the same in all possible pictures. This is expected since, at non-zero momentum, one can compute the physical spectrum in light-cone gauge. At a more technical level, this happens because at non-zero momentum one can define both picture-raising and picture-lowering operators that commute with \((b_0 - \bar{b}_0)\).

In section 3, we show that the RR semirelative cohomology at zero momentum is indeed inequivalent in the four different pictures mentioned above. This inequivalence is caused by the fact that \(b_0 - \bar{b}_0\) does not commute with any picture-lowering operator containing a finite

\footnote{This is easily seen from the fact that the supersymmetry charge is independent of the reparametrization ghost \(c(z)\) in both the \(-\frac{1}{2}\) and \(+\frac{1}{2}\) pictures \cite{17}.}

\footnote{Note that in the closed superstring field theory action constructed using non-minimal worldsheet variables \cite{8}, all sectors of the superstring satisfy these off-shell constraints.}

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number of terms.** On the other hand, both for the NS-R (or R-NS) sector of closed super-strings and for the R-sector of heterotic strings there are two obvious pictures to consider and we find that zero-momentum semirelative states are picture independent.

In our analysis of zero-momentum RR states, we found it is useful to distinguish between the conventional “finite” cohomology and a novel “infinite” cohomology [24]. These are just BRST cohomologies computed in two complexes that are defined in a slightly different ways. Taking basis vectors to be monomials built as the product of oscillators acting on the vacuum, states in the finite complex are restricted to be finite linear combinations of the basis vectors, while we impose no such restriction for the infinite complex. The infinite complex is just an enlarged version of the finite complex, and thus the usual phenomena can occur; finite BRST classes might become trivial in the infinite complex, and there may exist infinite BRST classes that have no representatives in the finite complex. These latter states will be called strictly infinite BRST classes.

We find that the finite and infinite cohomologies at zero momentum are equivalent both in the \((-1/2, -1/2)\) picture and in the \((-3/2, -3/2)\) picture. This happens because upon restriction to a fixed ghost number and to \(L_0 = \overline{L}_0 = 0\), the state space in these pictures becomes finite dimensional and the finite and infinite complexes coincide. On the other hand, for the mixed \((-3/2, -1/2)\) picture (or the \((-1/2, -3/2)\) picture), restriction to a fixed ghost number and to \(L_0 = \overline{L}_0 = 0\) leaves an infinite dimensional state space. As a result, the finite and infinite cohomologies are different in the mixed \((-3/2, -1/2)\) picture; the same is true also for the \((-1/2, -3/2)\) picture. Furthermore, we find in these mixed pictures that the finite cohomology classes are all trivial in the infinite complex, and that there exist strictly infinite BRST classes. Therefore the infinite cohomology in these mixed pictures is all strictly infinite, and it is natural to think of the finite and infinite cohomologies as disjoint vector spaces.

While it is thus clear that (zero-momentum) semirelative cohomologies are not the same for different pictures, we prove some relations that show that the \((-1/2, -1/2)\) and the \((-3/2, -3/2)\) pictures together contain the information that is encoded in the mixed picture \((-3/2, -1/2)\) (or the picture \((-1/2, -3/2)\)). For zero momentum states, we prove that the finite cohomology at picture \((-1/2, -3/2)\) or \((-3/2, -1/2)\) is equivalent to the cohomology at picture \((-3/2, -3/2)\), while the infinite cohomology at picture \((-1/2, -3/2)\) or \((-3/2, -1/2)\) is equivalent to the cohomology at picture \((-1/2, -1/2)\). In summary, for Ramond-Ramond

** It is interesting to note that there are similar problems with defining a picture-lowering operator for the open \(N=2\) string [22]. As shown in [23], a BRST-invariant inverse to the \(N=2\) picture-raising operators can be constructed only if one allows an infinite number of terms. This suggests that some of the techniques developed in this paper may also be applicable for open \(N=2\) strings.
zero-momentum semirelative cohomology we have the following isomorphisms

\[ H^{-\frac{3}{2}, -\frac{3}{2}} \leftrightarrow H^{fin}_{-\frac{3}{2}, -\frac{1}{2}} \]  

(1.1) 

\[ H_{-\frac{3}{2}, -\frac{1}{2}} \leftrightarrow H_{-\frac{1}{2}, -\frac{1}{2}} \]  

In addition to being of mathematical interest, the distinction between finite and infinite cohomology is also of physical interest. Boundary states for superstring D-branes are shown to be representatives for strictly infinite BRST classes (classes in \( H^{-\frac{3}{2}, -\frac{1}{2}} \)). This is a superstring phenomenon. Although D-brane states in bosonic string theory are also nonpolynomial functions of the oscillators (and thus define an infinite cohomology class), all except a finite set of basis vectors entering into the expansion of the state are BRST exact [25]. Therefore, the class of the bosonic D-brane state has a representative in the finite complex. As defined above, this means that the BRST class associated to a bosonic D-brane state, in contrast to that of superstring D-brane states, is not strictly infinite. The difference comes from the fact that superstring D-brane states have non-polynomial dependence on bosonic ghost zero modes. We also show that the finite cohomology contains RR central charges and zero-momentum RR gauge fields [26].

In section 4, we refine our analysis of zero-momentum RR cohomology to \( x_0 \)-cohomology, where the states in the BRST complex can explicitly depend on the zero mode of the non-compact bosonic coordinates \( X^\mu(z, \bar{z}) \) [27,28]. We explain why this is the cohomology problem that must be used to discuss anomalies, and the relation to the Fischler-Susskind mechanism. In \( x_0 \)-cohomology, ghost-number one states are associated with anomalies, ghost-number zero states are associated with zero-momentum physical fields that cannot be gauged away even with gauge parameters that diverge at infinity, and ghost-number minus one states are associated with global symmetries; for superstrings, super-Poincaré generators.

In the Type-I and Type IIB Ramond-Ramond sector with infinite cohomology, we find the expected \( D_9 \)-brane anomaly at ghost-number plus one. In the Type IIB Ramond-Ramond sector with finite cohomology, we find the zero-momentum axion at ghost-number zero. The zero-momentum axion has properties quite analogous to those of the zero-momentum ghost dilaton; it would be BRST trivial had we not imposed the semirelative condition, and, it is also present in the relative closed string cohomology (the cohomology on the complex where states satisfy \( b_0 = \bar{b}_0 = 0 \)). Finally, in the Type IIA Ramond-Ramond sector with finite cohomology, we find a zero-brane charge at ghost-number minus one. The presence of this zero-brane charge suggests that an eleventh coordinate may be present in superstring theory without having to introduce membranes [29].

An important question that remains open is the construction of a superstring field theory using only RNS variables. Our work here should help understand the role of the various pictures. Further geometrical insight into picture changing [30] could also be of use. We
believe our work gives evidence that the intricate structure of pictures in superstring theory is not a technical nuisance but rather a key element in the future understanding of duality transformations, D-branes, and the non-perturbative physics of superstring theory.

2. The cohomology at nonzero momentum

The main purpose of the present section is to show that at non-zero momentum the closed superstring cohomology, defined as the BRST cohomology evaluated on the subcomplex where all states are annihilated by $b_0 - \bar{b}_0 \equiv \bar{b}_0^-$, is the same in all pictures. The result will hold for each sector of the closed superstring, that is, for the NS-NS sector, for the NS-R sector, and for the RR sector.

This result follows from a similar result for the chiral complex, the complex associated to the left moving part of the conformal theory. Let $H_P$ denote the relative chiral complex, i.e. the states of the chiral sector that have picture number $P$ and are annihilated by $b_0$. These states may be in the NS or R sectors. Moreover, let $H_P(Q, b_0)$ denote the cohomology of $Q$ on $H_P$, i.e. the relative cohomology at picture number $P$. We now claim that the following result is true:

Theorem: For any fixed non-zero momentum, the chiral relative cohomologies $H_P(Q, b_0)$ and $H_{P-1}(Q, b_0)$ are isomorphic vector spaces. This holds both for the NS and for the R sectors.

Proof. Our strategy to establish this result is similar to that used in [18], namely, we identify an invertible picture changing operator acting on the relative cohomologies. As we will show, the required operator is the momentum operator in the $-1$ picture,

$$\tilde{p}^\mu = \oint \frac{dz}{2\pi i} e^{-\phi(z)} \psi^\mu(z),$$

(2.1)

which was used in [31] to discuss supersymmetry charges. This operator carries picture number minus one, and thus can be used as an inverse picture changing operator. Since the integrand does not involve the conformal ghost $c$, we have that $[b_0, \tilde{p}^\mu] = 0$. Therefore, $\tilde{p}^\mu : H_P \to H_{P-1}$, that is, $\tilde{p}^\mu$ maps the relative chiral complexes at different pictures. Furthermore, given that

$$[Q, e^{-\phi} \psi^\mu(z)] = \partial(ce^{-\phi} \psi^\mu)(z),$$

(2.2)

one finds that $[Q, \tilde{p}^\mu] = 0$. As a consequence, $\tilde{p}^\mu : H_P(Q, b_0) \to H_{P-1}(Q, b_0)$. It remains to show that $\tilde{p}^\mu$ has a well defined inverse. To this end, consider now the picture changing

Note that in defining the chiral relative complex, we only impose a $b_0 = 0$ condition. This differs from the usual approach where the relative complex is defined as one where the computations of BRST cohomology do not involve ghost zero modes [21].
operator $X_0$ defined as \cite{17}

$$X_0 = \oint \frac{dz}{2\pi i z} X(z),$$

(2.3)

where the local operator $X(z)$ is given by\footnote{Our conventions can be found in section 3.1.}

$$X(z) = \{Q, \xi(z)\} = c\partial \xi - \frac{1}{2} e^\phi \psi \cdot \partial + (\partial \eta) e^{2\phi} b + \partial (\eta e^{2\phi} b).$$

(2.4)

This operator is BRST nontrivial despite appearances, and it commutes with $Q$. Furthermore, as mentioned in \cite{32}, $X_0$ commutes with the zero mode of the antighost field

$$[b_0, X_0] = (\partial \xi)_0 = 0.$$  

(2.5)

It follows that $X_0 : H_{P-1}(Q, b_0) \to H_P(Q, b_0)$ and is therefore a candidate for an inverse to $\tilde{p}^\mu$.

We now claim that the following operator relations hold

$$X_0 \tilde{p}^\mu = -\frac{1}{2} p^\mu + \{Q, m^\mu\},$$

$$\tilde{p}^\mu X_0 = -\frac{1}{2} p^\mu + \{Q, n^\mu\},$$

(2.6)

where $p^\mu$ is the momentum operator

$$p^\mu = \oint \frac{dz}{2\pi i} \partial X^\mu,$$

(2.7)

and the operators $m^\mu$ and $n^\nu$ satisfy

$$[b_0, m^\mu] = 0, \quad [b_0, n^\mu] = 0.$$  

(2.8)

Before establishing these relations let us prove that they imply the desired result, namely, that at any fixed non-zero momentum the relative cohomologies $H_P(Q, b_0)$ and $H_{P-1}(Q, b_0)$ are isomorphic. For this, it is enough to show that the map $\tilde{p}^\mu : H_P(Q, b_0) \to H_{P-1}(Q, b_0)$ is (i) one to one and (ii) onto. One to one: consider two states $|a\rangle$ and $|b\rangle$ of the same momentum, and both belonging to $H_P(Q, b_0)$. Moreover, assume that $\tilde{p}^\mu |a\rangle = \tilde{p}^\mu |b\rangle$ on $H_{P-1}(Q, b_0)$.  

\[ [b_0, m^\mu] = 0, \quad [b_0, n^\mu] = 0. \]
Explicitly, this means that
\[ \tilde{p}^\mu (|a\rangle - |b\rangle) = Q |\epsilon^\mu\rangle, \] (2.9)
where \( b_0 |\epsilon^\mu\rangle = 0 \). Then multiplying by \( X_0 \) one finds
\[ X_0 \tilde{p}^\mu (|a\rangle - |b\rangle) = QX_0 |\epsilon^\mu\rangle, \] (2.10)
and using the first equation above
\[ -\frac{1}{2} p^\mu (|a\rangle - |b\rangle) = Q \left(-m^\mu (|a\rangle - |b\rangle) - X_0 |\epsilon^\mu\rangle\right). \] (2.11)
Since \( p^\mu \) has a definite non-zero eigenvalue in the left hand side, and the gauge parameter on the right hand side is annihilated by \( b_0 \), this equation shows that \( |a\rangle = |b\rangle \) on \( H^P(Q,b_0) \). This shows that the map is one to one.

To show that the map is onto, consider any nontrivial state \( |d\rangle \in H_{P-1}(Q,b_0) \) of fixed non-vanishing momentum. We claim it is the image under \( \tilde{p}^\mu \) of the state \( |d'\rangle = -2X_0 \frac{1}{p^\mu} |d\rangle \) where \( \mu \) is chosen in a direction for which \( p^\mu |d\rangle \neq 0 \). Indeed, it is clear that \( |d'\rangle \in H_P(Q,b_0) \), and moreover, using the second relation in (2.6) we find
\[ \tilde{p}^\mu |d'\rangle = |d\rangle - 2Q \frac{1}{p^\mu} m^\mu |d\rangle \] (2.12)
where it is understood that there is no sum over \( \mu \) in the second term. Since the second term in the right hand side is an allowed gauge parameter, this verifies that the map is onto. This completes the verification that equations (2.6) imply isomorphic non-zero momentum cohomologies in the various pictures.

We must now establish equations (2.6). We begin by evaluating \( X_0 \tilde{p}^\mu \),
\[ X_0 \tilde{p}^\mu = \oint \frac{dz_1}{2\pi iz_1} X(z_1) \oint \frac{dz_2}{2\pi i} (e^{-\phi \psi^\mu}) (z_2). \] (2.13)
Using the relation
\[ X(z_1) - X(z'_2) = \left\{ Q, \int_{z'_2}^{z_1} du \partial \xi(u) \right\}, \] (2.14)
we find
\[ X_0 \tilde{p}^\mu = \oint \frac{dz_1}{2\pi iz_1} \oint \frac{dz_2}{2\pi i} X(z'_2) (e^{-\phi \psi^\mu}) (z_2) \]
\[ + \oint \frac{dz_1}{2\pi iz_1} \left\{ Q, \int_{z'_2}^{z_1} du \partial \xi(u) \right\} \oint \frac{dz_2}{2\pi i} (e^{-\phi \psi^\mu}) (z_2), \] (2.15)
where we define \( z'_2 = z_2 + \epsilon \), and we are taking the limit as \( \epsilon \to 0 \). In the last term of the right hand side, the integral to the right can be placed inside the BRST commutator, and in
the first term we can evaluate explicitly the operator product to find

\[ X_0 \tilde{p}^\mu = -\frac{1}{2} p^\mu + \left\{ Q, \oint \frac{dz_1}{2\pi i z_1} \oint \frac{dz_2}{2\pi i} \int_{z_2}^{z_1} du \, \partial \xi(u) \left( e^{-\phi} \psi^\mu \right) (z_2) \right\} \tag{2.16} \]

This equation has the form anticipated in (2.6) where we can now see the explicit form of the operator \( m^\mu \), and confirm that it commutes with \( b_0 \). The second equation in (2.6) is proven in an exactly identical fashion, the operator \( n^\mu \) differs from \( m^\mu \) only in the order of the operator product \( \partial \xi \) and \( e^{-\phi} \psi^\mu \) in the above expression. This concludes the proof of equations (2.6), and therefore completes the proof of the theorem.

We note that the proof presented above does not depend on the sector of the superstring we are considering. For different sectors the picture numbers \( P \) are of different type (integer for NS and half integer for R), but no separate treatment is required, as picture changing operators satisfy the same identities as they act on the two different sectors.

We also note that the above proof does more than establish an isomorphism of vector spaces. It produces a “canonical” identification in the sense that there are natural picture changing operators implementing the isomorphism. This means that we expect states related by the isomorphism to be physically equivalent.

**Corollary.** At fixed nonzero momentum, the closed superstring semirelative cohomologies \( H_{P,\bar{P}}(Q,b_0^-) \) of each sector (NS, NS-R, and RR) are isomorphic for all pictures \( P, \bar{P} \).

**Proof.** The corollary follows by simple observations. First, once we guarantee that \( \tilde{p}^\mu, X_0, m^\mu, \) and \( n^\mu \) commute with \( b_0 \), being holomorphic, they will commute with \( b_0^- = b_0 - \bar{b}_0 \). Moreover equations (2.6) hold for \( Q \) the total BRST operator. It thus follows that \( p^\mu \) and \( X_0 \) guarantee the isomorphism (at fixed momentum) between \( H_{P,\bar{P}}(Q,b_0^-) \) and \( H_{P-1,\bar{P}}(Q,b_0^-) \). Completely analogous remarks hold for the antiholomorphic sector. This completes the verification of the corollary.
3. Cohomology at Zero-Momentum

In the previous section, we proved that the semi-relative cohomology at non-zero momentum of the closed RNS superstring is isomorphic in all pictures. This proof holds for all sectors of the superstring, i.e. the NS-NS, NS-R, and R-R sectors. At zero momentum, however, there is a more complicated relationship between the semi-relative cohomologies in different pictures. In this section we will explore the picture dependence of the zero-momentum cohomology.

We will not attempt here a full analysis of all possible pictures. We will only relate the cohomologies for superstring pictures in which all positive modes of the worldsheet fields annihilate the vacuum. This includes the $-1$ picture in the NS sector (where $\beta_r$ and $\gamma_r$ annihilate the vacuum for all $r \geq \frac{1}{2}$), the $-3/2$ picture in the R sector (where $\beta_{n+1}$ and $\gamma_n$ annihilate the vacuum for $n \geq 0$) and the $-1/2$ picture in the R sector (where $\beta_n$ and $\gamma_{n+1}$ annihilate the vacuum for $n \geq 0$).

This section is organized as follows. In subsection 3.1 we give our conventions. In subsection 3.2 we show for the chiral Ramond complex that the zero-momentum relative cohomology, i.e., the cohomology in the chiral subcomplex of zero-momentum states annihilated by $b_0$, is not the same in pictures $-1/2$ and $-3/2$. This shows that the theorem proven in section 2 does not extend to zero momentum.

In subsection 3.3, we extend the analysis to the semirelative RR complex of zero momentum states. We consider the four cases defined by pictures $(P, \bar{P})$, where $P$ and $\bar{P}$ are either $(-1/2)$ or $(-3/2)$. We note that when $P \neq \bar{P}$, the nontrivial physical states would be BRST trivial if we allowed gauge parameters that are infinite linear combinations of Fock space states. This suggests refined definitions of BRST complexes, a finite complex where states must be finite linear combinations of Fock space states, and an infinite complex where there is no such condition. We then compute explicitly these two types of cohomology for the four pictures in question.

In subsection 3.4, we relate the various cohomologies obtaining the relations indicated in equation (1.1) of the introduction. To this end, we build a suitable picture-raising and picture-lowering operator, $X_0$ and $Y_0$, that map the cohomologies into each other. We observe that the maps are one-to-one and surjective maps, and thus define canonical isomorphisms of the cohomologies. This should imply that these cohomologies represent the same physical content.

Finally, in subsection 3.5 we show that the two possible pictures that can be used to describe the R-sector of heterotic strings yield equivalent zero momentum semirelative cohomologies. Similarly, the two possible pictures that can be used to describe the R-NS sector of closed

* All other pictures contain states with arbitrarily negative energy, e.g. $(\gamma_{1/2})^n|0\rangle$ in the 0 picture of the NS sector. This is not truly pathological since at any given ghost number the energy is bounded below. One can certainly work with these pictures, but they have been used less than the canonical pictures. A superstring field theory with NS picture 0 has been considered in [33,34].
superstrings yield equivalent zero-momentum semirelative cohomologies. For both cases, we build suitable picture changing operators that establish the isomorphisms.

3.1. Notation and conventions for vacua

The chiral Ramond vacuum in the $P$ picture will be denoted by $|P\rangle^\alpha$ where $\alpha$ is a sixteen-component Majorana-Weyl spinor index. Its vertex operator in terms of the SL(2)-invariant vacuum $|1\rangle$ is $cS^\alpha e^{P\phi}(z)$ where $c(z)$ is the reparametrization ghost, $S^\alpha(z)$ is the spin field of conformal weight $5/8$, and $\phi(z)$ is the chiral boson coming from fermionization of the $(\beta, \gamma)$ system as $\beta = e^{-\phi} \partial \xi$ and $\gamma = e^\phi \eta$. We will write

$$|P\rangle^\alpha = cS^\alpha e^{P\phi(0)} |1\rangle, \quad (3.1)$$

and, by construction this state is annihilated by $b_0$ (but not by $c_0$) and by all positively moded oscillators of the reparametrization ghosts $(b, c)$ and the matter fields.

The inequivalent Majorana-Weyl spinor will be denoted as $|P\rangle^{\alpha'}$. The zero modes of the worldsheet fermions act on the vacua as

$$\psi_0^\mu |P\rangle^\alpha = \Gamma^\mu_{\alpha \beta'} |P\rangle^{\beta'},$$

$$\psi_0^\mu |P\rangle^{\beta'} = \Gamma^\mu_{\beta' \alpha} |P\rangle^\alpha, \quad (3.2)$$

and therefore the two vacua have opposite GSO (chirality) eigenvalue. If we conventionally declare $| -\frac{1}{2} \rangle^\alpha$ to have GSO eigenvalue $+1$, the GSO-projection implies that states must be built on vacua $|P\rangle$ with spinor index $\alpha/\alpha'$ when $(P + 1/2 + N)$ is even/odd, where $N$ is the number of $\beta_0$’s plus the number of $\gamma_0$’s. For notational convenience, we will leave the spinor index out of many equations. The spinor type can be deduced from the above rule. Comments made for vacua with the spinor index left out hold for both kind of spinor indices.

For convenience, we will set the ghost numbers of the $|P\rangle$ vacua to be zero for all $P$. This follows from defining the ghost-number current as [35]

$$j_{ghost} = -bc + \eta \xi, \quad (3.3)$$

rather than as $j_{ghost} = -bc - \partial \phi$, which is the conventional definition. Note that the ghost numbers of $Q, b, c, \beta$, and $\gamma$ are unchanged. In this convention, $\xi$ carries ghost number $-1$, and $\eta$ carries ghost number $+1$. The picture changing operators $X(z)$ and $Y(z)$ now carry ghost number zero. Both for $P = -1/2$ and for $P = -3/2$, the $L_0$ eigenvalue of $|P\rangle$ is zero. In these two pictures, as mentioned earlier, all positively moded superghost oscillators kill the vacuum. Moreover, $\beta_0 | -\frac{1}{2} \rangle = 0$ and $\gamma_0 | -\frac{3}{2} \rangle = 0$. 

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The BRST operator in the chiral complex is defined as

$$Q = \oint \frac{dz}{2\pi i} \left\{ c(T_M + \frac{1}{2}T_{gh}) + \gamma(G_M + \frac{1}{2}G_{gh}) \right\},$$  \hspace{1cm} (3.4)

and it is strictly conserved, squares to zero, and satisfies

$$\{Q, b(z)\} = T_M + T_{gh},$$

$$[Q, \beta(z)] = G_M(z) + G_{gh}(z),$$ \hspace{1cm} (3.5)

where*

$$T_M(z) = -\frac{1}{2} \partial X \cdot \partial X - \frac{1}{2} (\partial \psi) \cdot \psi,$$

$$T_{gh}(z) = c\partial b + 2\partial cb - \frac{1}{2} \gamma\partial \beta - \frac{3}{2} \partial \gamma\beta,$$

$$G_M(z) = -\frac{1}{2} \psi \cdot \partial X,$$

$$G_{gh}(z) = c\partial \beta + \frac{3}{2} \partial c\beta - 2b\gamma.$$ \hspace{1cm} (3.6)

In the R-sector, the BRST operator can be expanded as

$$Q = L_0c_0 + (-\gamma_0^2 + M)b_0 + DR\gamma_0 + N\beta_0 + \tilde{Q}$$ \hspace{1cm} (3.7)

where $M$, $N$ and $\tilde{Q}$ are independent of ghost and matter zero modes. Here $D_R = \frac{1}{2} \psi_\mu p_\mu + \cdots$, and $L_0 = \frac{1}{2} p^2 + \cdots$ where the dots indicate terms independent of ghost and matter zero modes.

In the R-R sector, states are built on vacua of the type $(|P\rangle \otimes |\bar{P}\rangle)$ where the spinor index on each vacuum is determined by the GSO projection. The left moving GSO projection is defined as above, and the right moving GSO-projection implies that the second spinor index is $(\alpha/\alpha')$ when ($\bar{P} + 1/2 + N + \gamma$) is even/odd, where $N$ is the total number of $\bar{\beta}_0$’s and $\gamma_0$’s, $y = 0$ for the Type IIB superstring, and $y = 1$ for the Type IIA superstring.

### 3.2. Relative chiral cohomology in the R-sector

We are guaranteed by the result of [18] that absolute cohomologies are isomorphic in all pictures, in particular, this applies to zero-momentum cohomology in the R sector. Moreover, at non-zero momentum, we have shown that relative chiral cohomologies are the same in all pictures (Theorem, section 2). There is no guarantee, however, that at zero momentum the relative cohomologies will be the same. They are not, as we show now by explicit consideration of the R sector.

* We use the standard ope’s $c(z)b(w) \sim \gamma(z)\beta(w) \sim (z - w)^{-1}$. 
We will calculate the relative cohomology classes for the $-1/2$ and $-3/2$ pictures. At zero momentum and with $L_0 = 0$, we can only build candidate states using zero modes. Therefore, the chiral BRST operator in (3.7) reduces to $Q = -\gamma^2_0 b_0$. It follows that in the relative complex (where all states are annihilated by $b_0$) all states are $Q$-closed and no state can be $Q$-exact. Thus every (non-zero) zero-momentum state annihilated by $b_0$ and $L_0$ represents a cohomology class. The matter zero modes have been used (the index $\alpha$ on the vacuum), $c_0$ cannot be used, so we are only left with $\beta_0$ and $\gamma_0$. For the $-1/2$ picture only $\gamma_0$ can be used and we therefore find,

$$H_{-1/2}^{n<0}(Q, b_0) = 0,$$
$$H_{-1/2}^{0}(Q, b_0) = |-\frac{1}{2}\rangle^\alpha,$$
$$H_{-1/2}^{n>0}(Q, b_0) = \gamma^n_0 | -\frac{1}{2}\rangle.$$

Here $H^n$ denotes the cohomology class at ghost number $n$. There are no classes for negative ghost numbers, one spinor state at ghost number zero, and one spinor state for every other positive ghost number. The spinor index in the last equation is $\alpha$ for $n$ even, and $\alpha'$ for $n$ odd. On the other hand, for picture $-3/2$ we find

$$H_{-3/2}^{n<0}(Q, b_0) = \beta^n_0 | -\frac{3}{2}\rangle,$$
$$H_{-3/2}^{0}(Q, b_0) = |-\frac{3}{2}\rangle^\alpha',$$
$$H_{-3/2}^{n>0}(Q, b_0) = 0.$$

The spinor index on the right hand side of the first equation is $\alpha$ for $n$ odd, and $\alpha'$ for $n$ even. We note that, given the structure of these cohomologies, no picture changing operator of definite ghost number could map one cohomology to the other. These cohomologies, however, are dual with respect to the linear inner product defined by computing the correlation function of the corresponding operators on a sphere having a $c_0$ insertion.

### 3.3. Zero-momentum RR semirelative cohomology computations

In the NS-NS sector of the superstring, the only picture where the vacuum satisfies the condition of highest weight state for the positively moded superghost oscillators is the $(-1, -1)$ picture. There is therefore nothing to relate. In the NS-R sector there are two available pictures; the $(-1, -3/2)$ picture and the $(-1, -1/2)$ picture. The zero momentum semirelative cohomologies will be proven to be equivalent in section 3.5. In the R-R sector, however, there

† Rather than describing the cohomology classes $H^n(Q, b_0)$ by the corresponding vector spaces as is customary in the mathematics literature, we simply list representatives.
is a surprising difference between the cohomologies in the four available pictures. In the \((-1/2, -3/2)\) picture (or \((-3/2, -1/2)\) picture), there exists two inequivalent ways of defining cohomology. One of the them is called ‘finite cohomology’ and will be proven equivalent to the cohomology in the \((-3/2, -3/2)\) picture. The other one is called ‘infinite cohomology’ and will be proven equivalent to the cohomology in the \((-1/2, -1/2)\) picture.

When computing BRST cohomology for a complex \(\mathcal{H}\), one typically computes “finite cohomology”, in the sense that the complex is defined to include only vectors that are finite linear superpositions of Fock state vectors. A cohomology class in this complex must have a representative built as a finite linear superposition of Fock states, and cannot be written as \([Q, \Lambda]\) where \(\Lambda\) also contains a finite number of terms. Infinite cohomology is defined as BRST cohomology in a complex where general states include infinite linear combinations of Fock space states. A cohomology class in this complex may be represented by a state which is an infinite superposition of Fock space states, and cannot be written as \([\hat{Q}, \hat{\Lambda}]\) where \(\hat{\Lambda}\) can contain an infinite number of terms.

In practice, one computes cohomology by restricting oneself to subspaces of states with fixed momentum, fixed ghost number, and \(L_0\) eigenvalue zero. In the bosonic string theory complex, these subspaces are always finite dimensional, and therefore finite and infinite cohomologies are identical as the finite and infinite complexes are the same. In superstring theory we have the superghosts zero modes \(\beta_0\) and \(\gamma_0\), and there exists a double infinity of objects, \((\beta_0 \bar{\gamma}_0)^n\), and \((\gamma_0 \bar{\beta}_0)^n\), with \(n \geq 0\), having total ghost number zero and \(L_0\) eigenvalue zero. These objects vanish on the \((-1/2, -1/2)\) and \((-3/2, -3/2)\) pictures, and thus the relevant subspaces for cohomology computations in these pictures are always finite dimensional. This is not the case for the mixed \((-3/2, -1/2)\) and \((-1/2, -3/2)\) pictures. The objects \((\beta_0 \bar{\gamma}_0)^n\) do not annihilate the \((-3/2, -1/2)\) vacuum, and similarly the objects \((\gamma_0 \bar{\beta}_0)^n\) do not annihilate the \((-1/2, -3/2)\) vacuum. Therefore, in the case of mixed pictures we can distinguish between finite and infinite cohomology.

Let us now consider the explicit computation of semirelative cohomologies at zero momentum in the RR sector of the superstring. No separate discussion will be necessary for IIA and IIB superstrings; we will leave the spinor indices of the vacua unspecified, and they can be reconstructed from the GSO condition, as explained at the end of section 3.1. At zero momentum, and with \(L_0 = \bar{L}_0 = 0\) and \(b_0^\pm = 0\), all states are built by acting with \(\gamma_0, \bar{\gamma}_0, \beta_0, \bar{\beta}_0\) and \(c_0^+\) on the vacua. The BRST operator can then be read from (3.7), and up to an irrelevant overall factor reads

\[
Q = -b_0^+(\gamma_0^2 + \bar{\gamma}_0^2).
\]

The results of the computations of BRST cohomology are summarized in the table below (\(\gamma_{\pm}\) and \(\beta_{\pm}\) are defined in (3.11)), and explanations on the computations follow.
Table. List of semirelative cohomology classes in the Ramond-Ramond sector of the superstring at zero momentum. Shown are the standard choices of picture.

Diagonal Pictures. We begin with the computation in the $(-1/2, -1/2)$ picture. It follows from the form of $Q$ that a state is $Q$-closed if and only if it involves no $c_0^+$. Since the $\beta_0$ and $\bar{\beta}_0$ modes kill the vacuum in this picture, there are no candidate states at negative ghost numbers, and as a consequence there is no cohomology at negative ghost numbers. At ghost number zero, the vacuum states are all the candidate states, they are all $Q$-closed, and given the absence of ghost minus one states, they represent BRST classes. We indicate this as an entry “one” in the table, standing for the fact that there are no oscillators acting on the vacuum. At $G = 1$, there are no trivial states, since all candidates at $G = 0$ were $Q$-closed. Therefore, the BRST classes are represented by $\gamma_0$ acting on the vacuum and $\bar{\gamma}_0$ acting on the vacuum. The states where $c_0$ only acts on the vacuum are unphysical $G = 1$ states, so starting at $G = 2$ there exist trivial states. The general pattern at $G \geq 2$ is readily elucidated. For this purpose it is convenient to introduce new zero modes.*

* The vector spaces are naturally complex, so this change of variables is acceptable. When writing states in terms of $\pm$ zero modes, the properly projected GSO states are found by rewriting the states in terms of the original holomorphic and antiholomorphic zero modes, and applying the rules explained earlier. For example, the bispinor state $\gamma_\pm | -\frac{1}{2}, -\frac{1}{2} \rangle$ in the IIB superstring is $\gamma_0 | -\frac{1}{2})^\alpha \otimes | -\frac{1}{2})^\beta \pm i\gamma_0 | -\frac{1}{2})^\beta \otimes | -\frac{1}{2})^\alpha \rangle$. 

| $k$ | $H^{(k)}_{-\frac{1}{2}, -\frac{3}{2}}$ | $H^{(k)fin}_{-\frac{1}{2}, -\frac{1}{2}}$ | $H^{(k)\infty}_{-\frac{1}{2}, -\frac{1}{2}}$ | $H^{(k)}_{-\frac{1}{2}, -\frac{1}{2}}$ |
|-----|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| $-n - 1$ | $c_0^+ \beta^{n+2}$ | $\pm i\gamma_\mp \beta_0^{n+2}$ | $\emptyset$ | $\emptyset$ |
| $-1$ | $c_0^+ \beta^2$ | $\pm i\gamma_\mp \beta_0^2$ | $\emptyset$ | $\emptyset$ |
| $0$ | $c_0^+ \pm \beta_0$ | $\pm i\gamma_\mp \beta_0$ | $\frac{c_0^+}{\gamma_0^2} \sin(\beta_0 \gamma_0)$ | $1$ |
| $+1$ | $c_0^+$ | $\gamma_0$ | $c_0^+ \exp(\pm i\beta_0 \gamma_0)$ | $\gamma_\pm$ |
| $n + 1$ | $\emptyset$ | $\emptyset$ | $c_0^+ \gamma_0^n \exp(\pm i\beta_0 \gamma_0)$ | $\gamma_\pm^{n+1}$ |
\[ \gamma_\pm = \frac{1}{2}(\gamma_0 \pm i\bar{\gamma}_0), \]
\[ \beta_\pm = \beta_0 \pm i\bar{\beta}_0, \]  
(3.11)  

satisfying commutation relations
\[ [\gamma_\pm, \beta_\mp] = 1, \quad [\gamma_\pm, \beta_\pm] = 0. \]  
(3.12)  

The BRST operator then reads \( Q \sim b_0^+ \gamma_+ \gamma_- \). The BRST closed states at \( G = n + 1 \) are represented by homogeneous polynomials \( p_{n+1}(\gamma_+, \gamma_-) \) of degree \( n + 1 \). This is a vector space of dimension \( n + 2 \). Consider now representatives for unphysical states at \( G = n \). Any state built by acting on the vacuum with \( c_0^+ \) and with a monomial \( m_{n-1}(\gamma_+, \gamma_-) \) of degree \( n - 1 \) is unphysical. Since \( Q \) simply deletes the \( c_0^+ \) and multiplies by \( \gamma_+ \gamma_- \), it is clear that on the vector space spanned by these monomials, the kernel of \( Q \) is the zero vector. Thus the unphysical states are represented (non-canonically) by the vector space of homogeneous polynomials \( p_{n-1}(\gamma_+, \gamma_-) \) of degree \( n - 1 \), a vector space of dimension \( n \). It now follows that the dimension of the physical space at ghost number \( (n + 1) \) is given by \( (n + 2) - n = 2 \). Two representatives are readily chosen, they are \( \gamma_\pm^{n+1} \). It is clear they cannot be trivial as any trivial state is the sum of monomials each of which contains at least one \( \gamma_+ \) and one \( \gamma_- \). This completes the computation of the semirelative cohomology in the \((-1, 2)\) picture.

The computations in the \((-3/2, -3/2)\) picture are not all that different. Candidate states are built acting with \( \beta_+, \beta_- \), and \( c_0^+ \). We first claim that a state is \( Q \)-exact if and only if it has no \( c_0^+ \). On the one hand, an arbitrary state \( \beta_+^m \beta_-^m |0\rangle \) without \( c_0^+ \) can be written as \( \sim Qc_0^+ \beta_+^{m+1} \beta_-^{m+1} |0\rangle \). On the other hand, any exact state is obtained by \( Q \) action, which includes multiplicative action with \( b_0^+ \). Since \( b_0^+ \) annihilates the vacuum, in order to get a nonzero result the unphysical state must include a \( c_0^+ \), and the trivial state will not include it. All physical states at ghost number \(-n\) are therefore of the form \( c_0^+ p_{n+1}(\beta_+, \beta_-) |0\rangle \), where \( p_{n+1}(\beta_+, \beta_-) \) is an homogeneous polynomial of degree \( n + 1 \). The physical states are now those annihilated by \( \gamma_+ \gamma_- \), these are simply the states build with either \( \beta_+ \)'s or \( \beta_- \)'s but not both. Therefore, at ghost number \(-n\) we simply have the states \( c_0^+ \beta_\pm^{n+1} \).

It is readily seen from the above table that the number of elements in the \((-3/2, -3/2)\) cohomology at ghost-number \( G \) is equal to the number of elements in the \((-1/2, -1/2)\) cohomology at ghost-number \( 1 - G \). This is reasonable, as it implies that the non-degenerate bilinear form
\[ \langle \Psi_{-3/2, -3/2} | c_0^- | \Phi_{-1/2, -1/2} \rangle \]  
(3.13)  

pairing the semirelative complexes \( \mathcal{H}_{-1/2, -1/2} \) and \( \mathcal{H}_{-3/2, -3/2} \) induces a nondegenerate bilinear form on the zero-momentum semirelative cohomologies.

**Mixed Pictures.** We now focus on the \((-3/2, -1/2)\) picture (the results for the \((-1/2, -3/2)\) picture are obtained simply by exchanging holomorphic and antiholomorphic sectors). We
begin our analysis by calculating cohomologies for \( G = n + 1 \geq 2 \), which requires consideration of \( G = n \) states that are not annihilated by \( Q \). These are of the form

\[
a_p \equiv c_0^+ \tilde{\gamma}_0^{n-1} (\beta_0 \tilde{\gamma}_0)^p |0\rangle, \quad p = 0, 1, \ldots \infty. \tag{3.14}
\]

Note that there are an infinite number of states, due to the fact that \((\beta_0 \tilde{\gamma}_0)\) does not kill the vacuum, is Grassmann even, has ghost number zero, and \( L_0 = \bar{L}_0 = 0 \). Since \( \gamma_0 \) kills the vacuum and \([\gamma_0, \beta_0] = 1\), we can write the BRST operator as

\[
Q \sim b_0^+ \left( \frac{\partial^2}{\partial \beta_0^+} + \tilde{\gamma}_0^2 \right). \tag{3.15}
\]

It is clear that \( Qa_p \neq 0 \) for any fixed \( p \), and given that \( Q \) includes multiplication by \( \tilde{\gamma}_0^2 \), \( Q \) cannot annihilate any finite linear combination of \( a_p \)'s. The candidate states at \( G = n + 1 \geq 2 \) are of the form

\[
b_p \equiv \tilde{\gamma}_0^{n+1} (\beta_0 \tilde{\gamma}_0)^p \left| -\frac{3}{2}, -\frac{1}{2} \right\rangle, \quad c_p \equiv c_0^+ \tilde{\gamma}_0^n (\beta_0 \tilde{\gamma}_0)^p \left| -\frac{3}{2}, -\frac{1}{2} \right\rangle, \quad p = 0, 1, \ldots \infty. \tag{3.16}
\]

It follows from (3.15) and (3.14) that

\[
Qa_0 = b_0, \\
Qa_1 = b_1, \\
Qa_p = b_p + p(p - 1)b_{p-2}, \quad p \geq 2,
\]

showing that all \( b_p \) are trivial as they can all be written as \( Q \) acting on a finite linear combination of \( a_p \)'s with \( p' \leq p \). On the other hand none of the \( c_p \)'s can be exact, since exact states cannot have a \( c_0^+ \). As remarked above, no finite linear combination of \( c_p \)'s can be \( Q \) closed, but it is clear that a state of the form \( c_0^+ \tilde{\gamma}_0^n h(\beta_0 \tilde{\gamma}_0) \), where \( h(x) \) is simply a function of a single variable, will be \( Q \)-closed if \( h'' + h = 0 \). We therefore find two solutions

\[
c_0^+ \tilde{\gamma}_0^n \sin(\beta_0 \tilde{\gamma}_0), \quad c_0^+ \tilde{\gamma}_0^n \cos(\beta_0 \tilde{\gamma}_0) \quad n \geq 1. \tag{3.18}
\]

In summary, for \( G = n + 1 \geq 2 \) we have found no finite cohomology and two states in the infinite cohomology.

The case of \( G = 1 \) is slightly different with regards to the finite cohomology. The \( G = 0 \) states (in the finite complex) that are not annihilated by \( Q \) are of the form

\[
d_p \equiv c_0^+ \beta_0^+ (\beta_0 \tilde{\gamma}_0)^p \left| -\frac{3}{2}, -\frac{1}{2} \right\rangle, \quad p = 0, 1, \ldots \infty, \tag{3.19}
\]

and the states at \( G = 1 \) that could be in finite cohomology are

\[
e_p \equiv \tilde{\gamma}_0 (\beta_0 \tilde{\gamma}_0)^p \left| -\frac{3}{2}, -\frac{1}{2} \right\rangle, \quad p = 0, 1, \ldots \infty. \tag{3.20}
\]
It is then simple to find

\[
Qd_0 = e_1 , \\
Qd_1 = e_2 + 2e_0 , \\
Qd_p = e_{p+1} + p(p+1)e_{p-1} , \quad p \geq 2 .
\] (3.21)

It is clear that all \( e_k \) with \( k \) odd are trivial. On the other hand \( e_0 = \bar{\gamma}_0 | -\frac{3}{2}, -\frac{1}{2} \rangle \) is not trivial in the finite complex. It is trivial on the infinite complex since, using the above equations for \( p \) odd, we can write \( e_0 = \frac{1}{2} Q(d_1 - \frac{1}{12} d_3 + \cdots) \). This is an example of a phenomenon we find here; every cohomology class in the finite complex is trivial on the infinite complex.

The computations for other ghost numbers follow the above lines quite closely so they will not be discussed explicitly. The results are indicated on the table. We mention that for \( G = -n - 1 \), we find two states

\[
\beta_0^{n+1} | -\frac{3}{2}, -\frac{1}{2} \rangle , \quad \text{and} \quad \beta_0^{n+2} \bar{\gamma}_0 | -\frac{3}{2}, -\frac{1}{2} \rangle ,
\] (3.22)

but for later purposes it is convenient to express them as the linear combinations

\[
[\pm i(n + 2) \beta_0^{n+1} + \beta_0^{n+2} \bar{\gamma}_0 ] | -\frac{3}{2}, -\frac{1}{2} \rangle = \pm i \gamma_\mp \beta_0^{n+2} | -\frac{3}{2}, -\frac{1}{2} \rangle ,
\] (3.23)

as indicated on the table. Note that the pattern actually holds for \( G = 1 \) (corresponding to \( n = -2 \)) since \( \pm i \gamma_\mp | -\frac{3}{2}, -\frac{1}{2} \rangle = \bar{\gamma}_0 | -\frac{3}{2}, -\frac{1}{2} \rangle \).

Comments. In our conventions the string field has ghost number zero, thus a D-brane state, which is naturally a bra that couples to the string field, can be thought as a ket of ghost number plus one. If the string field is represented in the \((P, \bar{P})\) picture, the D-brane state must be in the \((-2 - P, -2 - \bar{P})\) picture. The superghost dependence of D-brane states has been discussed in [36] where it was shown that

\[
|B, \pm \rangle = \exp(\mp i \beta_0 \bar{\gamma}_0) \exp \left( \pm i \sum_{r \geq 1} (\gamma_{-r} \bar{\gamma}_{-r} - \beta_{-r} \bar{\beta}_{-r}) \right) | -\frac{3}{2}, -\frac{1}{2} \rangle .
\] (3.24)

These states are \( Q \)-closed, and we note that the \( L_0 = \bar{L}_0 = 0 \) part are exactly the infinite cohomology classes we have found (and the rest of the states is \( Q \)-exact). They are appropriate electric and magnetic D-branes for a string field in the \((-1/2, -3/2)\) picture. The isomorphisms of cohomology stated in (1.1), to be established in the next section, imply that \( G = 1 \) cohomology in the \((-1/2, -1/2)\) picture describes the same cohomology classes that are associated to the D-brane states.

* This is different from the observation of [36] that the above D-brane states can be rewritten in the form \( |B, \pm \rangle = \epsilon^\pm_0 \delta(\gamma_\pm) | -\frac{1}{2}, -\frac{1}{2} \rangle \). This rewriting does not imply that the states \( |B, \pm \rangle \) belong to the \((-1/2, -1/2)\) picture, since delta functions of superghosts are indeed picture changing operators.
Similarly, electric and magnetic gauge fields at zero-momentum are represented by \( G = 0 \) elements of the finite \((-3/2, -1/2)\) cohomology (or \((-3/2, -3/2)\) cohomology, via (1.1)). This is consistent with the non-degeneracy of (3.13) since zero-momentum electric and magnetic gauge fields have non-zero coupling to electric and magnetic D-branes [37]. Note that the \( G = 0 \) elements of \((-1/2, -1/2)\) cohomology (or \((-3/2, -1/2)\) infinite cohomology) are field-strengths rather than gauge fields since they are invariant under \( x \)-dependent gauge-transformations.

### 3.4. Isomorphisms of semirelative cohomology

The explicit results of the previous section, as summarized in the table, suggest natural isomorphisms of the zero momentum semi-relative cohomologies at the various pictures. Since the relevant vector spaces defined by the cohomology classes are finite dimensional, naive counting shows that the vector spaces obey the isomorphism indicated in (1.1). It is the purpose of the present section to construct a picture-raising operator and a picture-lowering operator that induce a “canonical” isomorphism between the various cohomologies. This means that the matching of physical states it is not just a counting coincidence, but that the states related by the canonical isomorphism are really physically equivalent.

The picture-raising operator \( X_0 \) is defined as in (2.4), and we can can read out the zero mode piece by acting on zero-momentum states in the \(-\frac{3}{2}\) picture. One can readily show that \( X_0 c_0 \left| -\frac{3}{2} \right\rangle^\alpha = \gamma_0 \left| -\frac{1}{2} \right\rangle^\alpha \), \( X_0 c_0 \beta_0 \left| -\frac{3}{2} \right\rangle^\alpha = \left| -\frac{1}{2} \right\rangle^\alpha \), and \( X_0 \) annihilates all other states. Making use of the familiar relation \( \delta(\beta_0) \left| -\frac{3}{2} \right\rangle^\alpha = \left| -\frac{1}{2} \right\rangle^\alpha \), we find that

\[
X_0 c_0 \left| -\frac{3}{2} \right\rangle^\alpha = b_0 \{\gamma_0, \delta(\beta_0)\} c_0 \left| -\frac{3}{2} \right\rangle^\alpha, \quad X_0 c_0 \beta_0 \left| -\frac{3}{2} \right\rangle^\alpha = b_0 \{\gamma_0, \delta(\beta_0)\} c_0 \beta_0 \left| -\frac{3}{2} \right\rangle^\alpha, \quad (3.25)
\]

where \( \{\cdot, \cdot\} \) denotes the anticommutator. Note that \( \delta(\beta_0) \left| -\frac{3}{2} \right\rangle^{\alpha'} = \left| -\frac{1}{2} \right\rangle^{\alpha'} \) since \( \delta(\beta_0) \) is fermionic and therefore anti-commutes with \( \psi_0^\mu \).

From (3.25), we are led to consider the picture-raising operator

\[
X_0 \equiv b_0^+ \{\gamma_0, \delta(\beta_0)\}, \quad (3.26)
\]

which is built exclusively of zero modes. In the zero-momentum \( L_0 = 0 \) sector, \( Q \sim b_0 \gamma_0^2 + b_0 \bar{\gamma}_0^2 \), and therefore \( X_0 Q = Q X_0 = 0 \). Also, \( X_0 \) clearly commutes with \( b_0^0 \) and therefore defines a map of semirelative closed string cohomologies.

For the picture-lowering operator, consider the usual operator \( Y = c \partial \xi e^{-2\phi} \), and its zero mode piece \( Y_0 = \oint \frac{dz}{2\pi i z} Y(z) \). The operator \( Y_0 \) involves both zero modes and non-zero modes, and we can read out the zero mode piece by acting on zero-momentum states in the \(-\frac{1}{2}\) picture. One can readily show that \( Y_0 \left| -\frac{1}{2} \right\rangle^\alpha = c_0 \beta_0 \left| -\frac{3}{2} \right\rangle^\alpha \), \( Y_0 \gamma_0 \left| -\frac{1}{2} \right\rangle^\alpha = c_0 \left| -\frac{3}{2} \right\rangle^\alpha \), and \( Y_0 \)
annihilates all other states. Making use of the relation \( \delta(\gamma_0) \left| -\frac{1}{2}\right\rangle^\alpha = \left| -\frac{3}{2}\right\rangle^\alpha \) (which implies that \( \delta(\gamma_0) \left| -\frac{1}{2}\right\rangle^\alpha' = -\left| -\frac{3}{2}\right\rangle^\alpha' \)), we find that

\[
Y_0 \left| -\frac{1}{2}\right\rangle^\alpha = c_0 [\beta_0, \delta(\gamma_0)] \left| -\frac{1}{2}\right\rangle^\alpha, \quad Y_0 \gamma_0 \left| -\frac{1}{2}\right\rangle^\alpha = c_0 [\beta_0, \delta(\gamma_0)] \gamma_0 \left| -\frac{1}{2}\right\rangle^\alpha.
\] (3.27)

From this, we are led to consider the picture-lowering operator

\[
Y_0 \equiv c_0 [\beta_0, \delta(\gamma_0)],
\] (3.28)

which is built exclusively of zero modes. One readily verifies that \( Y_0 Q = Q Y_0 = 0 \). Indeed, since \( Q \sim b_0 \gamma_0^2 \), and given that \( Y_0 \) has a single \( \beta_0 \), the \( \gamma_0^2 \) factor of \( Q \) can be made to hit the delta function \( \delta(\gamma_0) \). The operator \( Y_0 \), however, does not commute with \( b_0 \). This had to be the case, for otherwise we could prove an isormorphism of the zero-momentum chiral relative cohomologies, in direct contradiction with the explicit computation discussed in section 3.2.

For the closed string case, we can attempt to set

\[
Y_0 \sim c_0^+ [\beta_0, \delta(\gamma_0)]
\] (3.29)

and while this operator commutes with \( b_0^- \), it does not anymore commute with the complete BRST operator which reads \( Q \sim b_0^+ (\gamma_0^2 + \bar{\gamma}_0^2) \). In order to obtain an operator that commutes with \( Q \), we begin by introducing some notation. Let

\[
B^1 \equiv [\beta_0, \delta(\gamma_0)], \quad B^2 \equiv [\beta_0, B^1], \quad \cdots \quad B^{n+1} = [\beta_0, B^n].
\] (3.30)

It is straightforward to verify that

\[
[B^n, \gamma_0] = 0, \quad [B^n, \bar{\gamma}_0] = 0,
\] (3.31)

where the second equation is trivially satisfied. It is also simple to verify by induction that

\[
\gamma_0 B^n = n B^{n-1},
\] (3.32)

which indicates that formally, thinking of the superscript of \( B \) as an exponent, \( \gamma_0 \sim \partial / \partial B \). Finally, acting on the vacua we have

\[
B^n \left| -\frac{1}{2}\right\rangle = \pm \beta_0^n \left| -\frac{3}{2}\right\rangle,
\] (3.33)

where the top sign applies to an \( \alpha \) spinor index and the bottom sign applies to an \( \alpha' \) spinor index.
We now claim that
\[ Y_0 \equiv c_0^+ \cdot \frac{1}{\gamma_0} \cdot \sin(\gamma_0 B), \] (3.34)
is the desired picture-lowering operator which commutes with \( Q \). In the power series expansion of \( \sin \) we take \((B)^n \equiv B^n\), as suggested above. The first term in the series expansion is the operator indicated in (3.29) and the other terms are the corrections necessary to make the operator commute with \( Q \). Indeed, \([Y_0, b^-] = 0\) is manifest, and \([Q, Y_0] = 0\) simply requires

\[(\gamma_0^2 + \bar{\gamma}_0^2) \sin(\gamma_0 B) = 0, \] (3.35)

where we made use of relations (3.31). But this equation is manifestly satisfied by virtue of the identification of \( \gamma_0 \) with \( \partial/\partial B \). This confirms that the inverse picture changing operator \( Y_0 \) in (3.34) defines a map of semirelative cohomologies at zero momentum. Is is also clear from the explicit expressions that the operators \( Y_0 \) and \( X_0 \) commute with both left and right GSO projections.

In order to prove isomorphisms of cohomology we will verify explicitly that \( Y_0 \) defines a one to one surjective map of cohomologies. In fact, we will also see that \( X_0 \) defines an explicit operator inverse for \( Y_0 \). Let us first show that

\[ Y_0 : H^{(k)}_{-\frac{1}{2}, -\frac{1}{2}} \rightarrow H^{(k)\infty}_{-\frac{3}{2}, -\frac{1}{2}} \] (3.36)
is a one to one surjective map. By construction, (3.34) does indeed map correctly the cohomology at \( G = 0 \) (see table). At \( G = 1 \) we have

\[ Y_0 \gamma_\pm \left| -\frac{1}{2}, -\frac{1}{2} \right\rangle = Y_0 \left( \gamma_\pm \pm i \bar{\gamma}_0 \right) \left| -\frac{3}{2}, -\frac{1}{2} \right\rangle \]
\[ = (\gamma_\pm \pm i \bar{\gamma}_0) Y_0 \left| -\frac{1}{2}, -\frac{1}{2} \right\rangle \]
\[ = \pm c_0^+ \left( \frac{1}{\gamma_0} \right) \left( \gamma_\pm \pm i \bar{\gamma}_0 \right) \sin(\beta_0 \gamma_0) \left| -\frac{3}{2}, -\frac{1}{2} \right\rangle \]
\[ = \pm c_0^+ \exp(\pm i \beta_0 \bar{\gamma}_0) \left| -\frac{3}{2}, -\frac{1}{2} \right\rangle, \] (3.37)

where use was made of (3.31) and (3.33), and the \( \pm \) must be chosen depending on the spinor type of the holomorphic vacuum in the ket appearing in the left hand side. In the last step, we

\* Note that this equation, as written, requires a specific ket and therefore does not incorporate the GSO condition. This is not a problem since the proper \( \gamma_\pm \left| -\frac{1}{2}, -\frac{1}{2} \right\rangle \) states in any theory are the GSO projection of \( \gamma_\pm \left[ | -\frac{1}{2}, -\frac{1}{2}\rangle, \alpha\beta + | -\frac{1}{2}, -\frac{1}{2}\rangle, \alpha'\beta \right] \). The action of \( Y_0 \) on this state follows immediately from (3.37), and GSO projection can be applied then.
recognized that \( \gamma_0 \) acts as \( \partial/\partial \beta_0 \). Comparing with the table, we see that \( Y_0 \) does act correctly at \( G = 2 \). For higher ghost numbers the verification is now very simple

\[
Y_0 \gamma^{n+1} \left| \pm \frac{1}{2}, -\frac{1}{2} \right\rangle = \pm c_0^{+} (\gamma_0 \pm i\gamma_0)^n \exp(\pm i\beta_0 \gamma_0) \left| \pm \frac{3}{2}, -\frac{1}{2} \right\rangle ,
\]

\[
\sim \pm c_0^{+} \gamma_0^n \exp(\pm i\beta_0 \gamma_0) \left| \pm \frac{3}{2}, -\frac{1}{2} \right\rangle ,
\]

which is the expected result. This confirms our claims about (3.36).

Let us now show that the map

\[
X_0 : H^{(k)}(-\frac{3}{2}, -\frac{1}{2}) \rightarrow H^{(k)}(-\frac{1}{2}, -\frac{1}{2})
\]

provides an inverse for (3.36). At \( G = 0 \), only the first term in the Taylor expansion of \( b^{-}_0 \) survives when hit by \( X_0 \), and it is easy to check that the image of this term is \( \pm \left| \pm \frac{1}{2}, -\frac{1}{2} \right\rangle \), where once more, the \( \pm \) choice depends on the spinor type of the holomorphic vacuum. At \( G = 1 \), the first two terms in the Taylor expansion of the state \( c_0^{+} \exp(\pm i\beta_0 \gamma_0) \left| \pm \frac{3}{2}, -\frac{1}{2} \right\rangle \) survive when hit by \( X_0 \), and these two terms combine to give \( \pm \gamma_0 \left| \pm \frac{1}{2}, -\frac{1}{2} \right\rangle \). At \( G > 1 \), the first two terms in the Taylor expansion of the states survive to give \( \pm \gamma_0^n \left| \pm \frac{1}{2}, -\frac{1}{2} \right\rangle \), which are in the same cohomology class as \( \pm \gamma_0^{n+1} \left| \pm \frac{1}{2}, -\frac{1}{2} \right\rangle \). This confirms our claims about (3.39).

We now want to verify that

\[
\overline{Y}_0 : H^{(k)\text{fin}}(-\frac{3}{2}, -\frac{1}{2}) \rightarrow H^{(k)}(-\frac{3}{2}, -\frac{3}{2})
\]

is a one to one surjective map. Here, as we must change the picture of the right-movers, we consider the operator

\[
\overline{Y}_0 \equiv c_0^{+} \cdot \frac{1}{\gamma_0} \cdot \sin(\gamma_0 \bar{B}) ,
\]

which also commutes with \( b^{-}_0 \) and with the BRST operator. The first nontrivial check is at \( G = 1 \) where we find

\[
\overline{Y}_0 \gamma_0 \left| \mp \frac{3}{2}, -\frac{1}{2} \right\rangle = c_0^{+} \bar{B} \gamma_0 \left| \mp \frac{3}{2}, -\frac{1}{2} \right\rangle
\]

\[
= c_0^{+} \gamma_0 \bar{B} \left| \mp \frac{3}{2}, -\frac{1}{2} \right\rangle
\]

\[
= \pm c_0^{+} \gamma_0 \beta_0 \left| \mp \frac{3}{2}, -\frac{3}{2} \right\rangle
\]

\[
= \pm c_0^{+} \left| \mp \frac{3}{2}, -\frac{3}{2} \right\rangle ,
\]

in agreement with the result of the table. Note that the power expansion of \( \overline{Y}_0 \) collapsed to the first term since \( \gamma_0 \) annihilates the vacuum, and there are no \( \beta_0 \)’s in the state upon
consideration. We now consider the general term at \( G = -n - 1 \), this time we must evaluate

\[
c_0^+ \frac{1}{\gamma_0} \sin(\gamma_0 B) \left[ \pm i (n + 2) \beta_0^{n+1} + \beta_0^{n+2} \bar{\gamma}_0 \right] \left| -\frac{3}{2}, -\frac{1}{2} \right) \quad (3.43)
\]

Consider now the case when \( n = 2l \) is even (the odd case can be treated similarly). Using the series expansion for \( \sin \) we find

\[
\pm c_0^+ \sum_{k=0}^{\infty} \frac{(\pm i)^{2k}}{(2k+1)!} \gamma_0^{2k} \left[ \pm i (2l + 2) \beta_0^{2l+1} + \beta_0^{2l+2} \bar{\gamma}_0 \right] \beta_0^{2k+1} \left| -\frac{3}{2}, -\frac{1}{2} \right) \quad (3.44)
\]

where we used \([\bar{B}^n, \bar{\gamma}_0] = 0\). Here both \( \gamma_0 \) and \( \bar{\gamma}_0 \), since they kill the vacuum, act as derivatives and the infinite sum is truncated to

\[
\pm c_0^+ \left[ \sum_{k=0}^{l} \left( \frac{2l + 2}{2k+1} \right) \beta_0^{2l+1-2k} (\pm i \bar{\beta}_0)^{2k+1} + \sum_{k=0}^{l+1} \left( \frac{2l + 2}{2k} \right) \beta_0^{2l+2-2k} (\pm i \bar{\beta}_0)^{2k} \right] \left| -\frac{3}{2}, -\frac{3}{2} \right) \quad (3.44)
\]

The two sums can now be combined into a single sum

\[
\pm c_0^+ \sum_{k=0}^{2l+2} \left( \frac{2l + 2}{k'} \right) \beta_0^{2l+2-k'} (\pm i \bar{\beta}_0)^{k'} \left| -\frac{3}{2}, -\frac{3}{2} \right) = \pm c_0^+ (\beta_0 \pm i \bar{\beta}_0)^{2l+2} \left| -\frac{3}{2}, -\frac{3}{2} \right) ,
\]

and this is recognized as the state of \( G = -n - 1 \) in the \((-3/2, -3/2)\) semirelative cohomology. This verifies that (3.40) induces the claimed isomorphism.

Similarly, it is straightforward to verify that

\[
\overline{\mathcal{X}}_0 : H^{(k)}_{-\frac{3}{2}, -\frac{3}{2}} \rightarrow H^{(k) fin}_{-\frac{3}{2}, -\frac{1}{2}} \quad (3.45)
\]

acts as the inverse of \( \overline{\mathcal{Y}}_0 \).

3.5. Semirelative Classes for Heterotic R-sector, and R-NS Superstrings

In this section, we consider first the R-sector of heterotic strings. While the left movers here define a bosonic string, the right movers can be in either the \(-1/2\) or \(-3/2\) pictures. We will compute zero momentum cohomologies in both pictures and show that the cohomologies are equivalent by constructing a picture changing operator that gives an explicit isomorphism.

The antiholomorphic sector here is that of bosonic strings and will be considered only at zero momentum, and with \( \bar{L}_0 = 0 \). The vacuum \( |\Omega\rangle = \bar{c}_1 |1\rangle \), has \( \bar{L}_0 = -1 \) and is annihilated by all positively moded oscillators. The constraint \( \bar{L}_0 = 0 \) implies that states must have
an oscillator of mode number $-1$ acting on $|\Omega\rangle$, and oscillators with mode number less than $-1$ cannot be used. Under those circumstances, the relevant antiholomorphic part of BRST operator reads $\tilde{Q} \sim \tilde{b}_0 \tilde{c}_1$. Now consider using the $(-1/2)$ picture for the holomorphic sector. The complete vacuum will be denoted as $|-\frac{1}{2}\rangle_H \equiv |\Omega\rangle \otimes |\Omega\rangle$ where the $H$ stands for heterotic. Its ghost number is defined to be zero. The semirelative complex here is built by action with the set of oscillators $\{\gamma_0, \alpha_0^+, \tilde{b}_-1, \tilde{c}_-1, \tilde{\alpha}_-^2\}$ on the vacuum. The BRST operator, using a specific (but conventional) relative normalization between left and right sectors reads

$$Q = b_0^+ (\tilde{c}_1 \tilde{c}_1 + \gamma_0^2).$$

(3.46)

A straightforward computation gives that the only nonvanishing cohomology classes are

$$H^{-1}(Q, b_0^-) = \tilde{b}_-1 | -\frac{1}{2}\rangle_H ,$$

$$H^0(Q, b_0^-) = \tilde{\alpha}_-^2 | -\frac{1}{2}\rangle_H , \quad \tilde{b}_-1 \gamma_0 | -\frac{1}{2}\rangle_H ,$$

$$H^1(Q, b_0^-) = \tilde{\alpha}_-^2 \gamma_0 | -\frac{1}{2}\rangle_H , \quad (\tilde{c}_-1 - \tilde{b}_-1 \gamma_0^2) | -\frac{1}{2}\rangle_H ,$$

$$H^2(Q, b_0^-) = \gamma_0 (\tilde{c}_-1 - \gamma_0^2 \tilde{b}_-1) | -\frac{1}{2}\rangle_H .$$

(3.47)

The computations for the $(-3/2)$ picture uses the complex built on $|-\frac{3}{2}\rangle_H \equiv |\Omega\rangle \otimes |\Omega\rangle$ by the action of the same set of oscillators, except that $\gamma_0$ is replaced by $\beta_0$. The answer this time is

$$H^{-1}(Q, b_0^-) = c_0^+ (\tilde{b}_-1 \beta_0 - \frac{1}{6} \tilde{c}_-1 \beta_0^3) | -\frac{3}{2}\rangle_H ,$$

$$H^0(Q, b_0^-) = c_0^+ \tilde{\alpha}_-^2 \beta_0 | -\frac{3}{2}\rangle_H , \quad c_0^+ (\tilde{b}_-1 - \frac{1}{2} \tilde{c}_-1 \beta_0^2) | -\frac{3}{2}\rangle_H ,$$

$$H^1(Q, b_0^-) = c_0^+ \tilde{\alpha}_-^2 \gamma_0 | -\frac{3}{2}\rangle_H , \quad c_0^+ \tilde{c}_-1 \beta_0 | -\frac{3}{2}\rangle_H ,$$

$$H^2(Q, b_0^-) = c_0^+ | -\frac{3}{2}\rangle_H .$$

(3.48)

Comparing the last two lists, we see that the dimensionalities of the cohomologies agree. More importantly, using the methods of the previous subsection we can construct picture-raising and picture-lowering operators $\mathcal{X}_0^H$ and $\mathcal{Y}_0^H$. The operator $\mathcal{X}_0^H$ is defined as in (3.26) while

$$\mathcal{Y}_0^H = c_0^+ \left( B - \frac{1}{6} \tilde{c}_1 B^3 \right) ,$$

(3.49)

where this operator manifestly commutes with $b_0^-$ and a short computation shows that it commutes with the BRST operator indicated in (3.46). It is straightforward to show that $\mathcal{Y}_0^H$ acting on the list (3.47) gives us precisely the list in (3.48). This confirms that the two pictures of the heterotic string contain the same physical zero momentum states. Since that is also the case for non-zero momentum, there is no ambiguity in the R sector of the heterotic string.

Consider now the R-NS sector of closed superstrings. The antiholomorphic sector here is NS and will be considered only at zero momentum, and with $\tilde{L}_0 = 0$. The vacuum $|\Omega\rangle_{NS} = \tilde{c}_1 | -1\rangle$,
is based on the $-1$ picture, has $\bar{L}_0 = -1/2$, and is annihilated by all positively moded oscillators. The constraint $\bar{L}_0 = 0$ implies that states must have an oscillator of mode number $-1/2$ acting on $|\Omega\rangle_{NS}$, and oscillators with mode number less than $-1/2$ cannot be used. Under those circumstances the relevant antiholomorphic part of BRST operator reads $\bar{Q} \sim \bar{b}_0 \bar{\gamma}_{-1/2} \bar{\gamma}_{1/2}$. Now consider using the ($-1/2$) picture for the holomorphic sector. The complete vacuum will be denoted as $|\bar{-1/2}, -1\rangle \equiv |\bar{-1/2}\rangle \otimes |\Omega\rangle_{NS}$. Its ghost number is defined to be zero. The semirelative complex here is built by action with the set of oscillators $\{\bar{\gamma}_0, \bar{c}_0^+, \bar{\psi}_{-1/2}^\mu, \bar{\beta}_{-1/2}, \bar{\gamma}_{-1/2}\}$ on the vacuum. The BRST operator, using a specific (but conventional) relative normalization between left and right sectors reads

$$Q = \bar{b}_0^+ (\bar{\gamma}_{-1/2} \bar{\gamma}_{1/2} + \bar{\gamma}_0^2). \quad (3.50)$$

A straightforward computation gives that the only nonvanishing cohomology classes are

$$H^{-1}(Q, \bar{b}_0) = \bar{\beta}_{-1/2} |\bar{-1/2}, -1\rangle,$$

$$H^0(Q, \bar{b}_0) = \bar{\psi}_{-1/2}^\mu |\bar{-1/2}, -1\rangle, \quad \bar{\beta}_{-1/2} \bar{\gamma}_0 |\bar{-1/2}, -1\rangle,$$

$$H^{+1}(Q, \bar{b}_0) = \bar{\psi}_{-1/2}^\mu \bar{\gamma}_0 |\bar{-1/2}, -1\rangle, \quad (\bar{\gamma}_{-1/2} - \bar{\beta}_{-1/2} \bar{\gamma}_0^2) |\bar{-1/2}, -1\rangle,$$

$$H^{+2}(Q, \bar{b}_0) = \gamma_0 (\bar{\gamma}_{-1/2} - \gamma_0^2 \bar{\beta}_{-1/2}) |\bar{-1/2}, -1\rangle. \quad (3.51)$$

The computations for the ($-3/2$) R-picture uses the complex built on $|\bar{-3/2}, -1\rangle \equiv |\bar{-3/2}\rangle \otimes |\Omega\rangle_{NS}$ by the action of the same set of oscillators, except that $\gamma_0$ is replaced by $\beta_0$. The answer this time is

$$H^{-1}(Q, \bar{b}_0) = c_0^+ (\bar{\beta}_{-1/2} \beta_0 - \frac{1}{6} \bar{\gamma}_{-1/2} \beta_0^3) |\bar{-3/2}, -1\rangle,$$

$$H^0(Q, \bar{b}_0) = c_0^+ \bar{\psi}_{-1/2}^\mu \beta_0 |\bar{-3/2}, -1\rangle, \quad c_0^+ (\bar{\beta}_{-1/2} - \frac{1}{2} \bar{\gamma}_{-1/2} \beta_0^2) |\bar{-3/2}, -1\rangle,$$

$$H^{+1}(Q, \bar{b}_0) = c_0^+ \bar{\psi}_{-1/2}^\mu |\bar{-3/2}, -1\rangle, \quad c_0^+ \bar{\gamma}_{-1/2} \beta_0 |\bar{-3/2}, -1\rangle,$$

$$H^{+2}(Q, \bar{b}_0) = c_0^+ \bar{\gamma}_{-1/2} |\bar{-3/2}, -1\rangle. \quad (3.52)$$

Once again the last two lists have the same number of cohomology classes at each ghost number. The picture-raising operator is the same as before while the picture-lowering operator reads

$$\mathcal{Y}_0^{NS} = c_0^+ \left( B - \frac{1}{6} \bar{\gamma}_{-1/2} \bar{\gamma}_{1/2} B^3 \right), \quad (3.53)$$

where this operator manifestly commutes with $\bar{b}_0^-$ and a short computation shows that it commutes with the BRST operator indicated in (3.50). This last computation requires noting that $\bar{\gamma}_{1/2}^2$ annihilates all relevant states (i.e., $\bar{L}_0 = 0$, zero-momentum states). It is straightforward to show that $\mathcal{Y}_0^{NS}$ acting on the list (3.51) gives us precisely the list in (3.52). This confirms that
the two pictures of the R-NS sector of the superstring contain the same physical zero momentum states. Since that is also the case for non-zero momentum, there is no ambiguity in the R-NS sector of the superstrings. All in all, this section showed that inequivalent cohomologies only occur in the RR sector of superstrings.

4. Zero momentum $x_0$-cohomology

In principle, a computation of $x_0$-cohomology requires a complete reconsideration of the earlier computations. Both the Fock spaces of fields and gauge parameters must be extended to include multiplication by arbitrary finite order polynomials on the zero modes $x_0^\mu$ of the non-compact bosonic coordinates. As usual, comparing with the previous calculation where $x_0$ was not included, we can both lose physical states, due to new gauge parameters, or gain states, due to new fields. Given the results of [27], we expect that the physically relevant $x_0$-cohomology at zero momentum is a subset of the zero-momentum cohomology in the original semirelative complex. There is one predictable exception to this; at ghost number minus one, $x_0$-cohomology includes new states linear in $x_0$, these give rise to Lorentz symmetries. The present analysis will only consider the zero momentum physical states found in the previous section and we will ask which ones can be gauged away in $x_0$-cohomology. The states that cannot be gauged away are definitely $x_0$-cohomology classes, but there could be additional classes represented by states that contain factors of $x_0$.

For the purposes of the present section, the relevant BRST operator reads

$$Q = c_0^+ p^2 - b_0^+ (\gamma_0^2 + \bar{\gamma}_0^2) + (\gamma_0 \psi_0^\mu + \bar{\gamma}_0 \bar{\psi}_0^\mu) p_\mu ,$$

where the zero modes $\psi_0^\mu$ act on the vacua as indicated in (3.2). This BRST operator can be related to that of (3.4) by rescaling $c \rightarrow \frac{1}{2} c$, $b \rightarrow 2 b$, $\gamma \rightarrow \frac{1}{2} \gamma$ and $\beta \rightarrow 2 \beta$.

Since the picture changing operators discussed in the previous section commute with the above BRST operator $Q$, the isomorphisms of cohomology will hold for $x_0$-cohomology; the $(-3/2, -3/2)$ $x_0$-cohomology will coincide with the finite $(-3/2, -1/2)$ $x_0$-cohomology, and the infinite $(-3/2, -1/2)$ $x_0$-cohomology will coincide with the $(-1/2, -1/2)$ $x_0$-cohomology. We will therefore only compute the finite $(-3/2, -1/2)$ cohomology and the $(-1/2, -1/2)$ cohomology; in both cases considering Types I, IIA, and IIB superstrings. Our results are summarized in the table at the end of this section.
4.1. Anomalies, Fischler-Susskind and $x_0$-cohomology

Here we wish to make some comments that relate to the appearance of anomalies in string theory. These remarks give the appropriate BRST cohomological framework for the observations first made in Ref. [38]. We claim that a string theory has candidate anomalies if the $x_0$-cohomology at ghost number plus one is non-vanishing. Here, as in the rest of this paper, we are using the convention that the physical string field is at ghost number zero. In a nutshell, the anomaly arises because we are required to solve an equation of the type $Q|\Phi_0\rangle = -|D\rangle$, where $|\Phi_0\rangle$ is of ghost number zero. Consistency of this equation implies that $|D\rangle$ is exact. If the appropriate cohomology at ghost number one vanishes, $|D\rangle$ is necessarily exact and thus the above equation has a solution. But if there is nontrivial cohomology at $G = 1$, there is a potential anomaly. We have an anomaly if $|D\rangle$ is a representative of a nontrivial class since then the above equation does not have a solution. Let us first explain how the equation arises, and second, why one should use $x_0$-cohomology.

A consistent quantum background in string field theory is a background where the effective string action has vanishing one point functions. As usual, the effective action is obtained by computing (using the Batalin-Vilkovisky quantum master action) all one-particle (string!) irreducible graphs. This effective action takes the form

$$\Gamma(\Psi) \sim \frac{1}{2} \langle \Psi, Q\Psi \rangle + \langle \Psi, D \rangle + \cdots,$$

(4.2)

where we have shown the kinetic term, and a possible one-point interaction that could arise in the computation of the effective action. In order to have a consistent background, the linear term must vanish. To achieve this, one must shift the background, the essence of the Fischler-Susskind mechanism [39], and one tries $\Psi \rightarrow \Psi + \Psi_0$, where $\Psi_0$ represents a set of vacuum expectation values to be determined. It follows immediately from the above equation that the condition that the linear term vanish after the shift is simply $Q|\Psi_0\rangle = -|D\rangle$. Note that in a momentum conserving theory, $|D\rangle$ must be concentrated at zero-momentum.

The choice of $x_0$-cohomology is the obvious choice, as this is the choice that allows giving vacuum expectation values that are polynomials in the spacetime coordinates. Indeed while zero-momentum standard cohomology in bosonic closed string theories does not vanish at ghost number one, the $x_0$-cohomology does vanish [27]. This is rather natural, as we expect no candidate anomalies in bosonic string theory. In superstring theory, we will see that while there are many zero-momentum semi-relative classes at $G = 1$, only one state survives, and only for type IIB or type I string theory.

In an open-closed string theory, to lowest possible order in the topological genus expansion, the term $\langle \Psi, D \rangle$ arises from D-branes and orientifolds (once-punctured disks or crosscaps). If there is a candidate $x_0$-cohomology class, explicit computation is necessary to see that it does not appear in $|D\rangle$. This is precisely what happens for type I string theory; the candidate class exists and its disappearance from $|D\rangle$ fixes the gauge group to be $SO(32)$. In a purely closed
string theory, one-point functions would arise should one-point amplitudes for closed string states fail to vanish. This is not supposed to happen for type IIB string theory, thus despite the existence of a candidate class for an anomaly, the anomaly is not present.

### 4.2. Prototype calculations

In this section, we will discuss the two prototype calculations that are useful for the computation of $x_0$-cohomology. For these two cases, we will write the gauge transformations and find the gauge invariant states. For this purpose, we need the description of bispinors in terms of differential forms. The well-known expansions are

\[ a_{\alpha\beta} = a^{(1)}_{\mu} (C\Gamma^\mu)_{\alpha\beta} + a^{(3)}_{\mu_1\mu_2\mu_3} (C\Gamma^{\mu_1\mu_2\mu_3})_{\alpha\beta} + a^{(5+)}_{\mu_1\cdots\mu_5} (C\Gamma^{\mu_1\cdots\mu_5})_{\alpha\beta}, \]

\[ b_{\alpha'\beta'} = b^{(1)}_{\mu} (C\Gamma^\mu)_{\alpha'\beta'} + b^{(3)}_{\mu_1\mu_2\mu_3} (C\Gamma^{\mu_1\mu_2\mu_3})_{\alpha'\beta'} + a^{(5-)}_{\mu_1\cdots\mu_5} (C\Gamma^{\mu_1\cdots\mu_5})_{\alpha'\beta'}, \]  

\[ c_{\alpha\beta'} = c^{(0)} (C)_{\alpha\beta'} + c^{(2)}_{\mu_1\mu_2} (C\Gamma^{\mu_1\mu_2})_{\alpha\beta'} + c^{(4)}_{\mu_1\cdots\mu_4} (C\Gamma^{\mu_1\cdots\mu_4})_{\alpha\beta'}. \]  

The spinor indices $\alpha$ and $\alpha'$ are identified with the irreducible spinor representations 16 and 16' respectively. The degree of the differential form is indicated by the superscript, with the plus/minus in the five-forms indicating self dual and anti-self dual pieces. In the first two equations, the one and five forms appear in the symmetric part of the product of spinor indices, while the three form appears in the antisymmetric part of the product. The matrices $C_{\alpha\beta'}$ and $C_{\alpha'\beta'}$ are the nonvanishing pieces of the $32 \times 32$ charge conjugation matrix. As such, they satisfy $\Gamma^\mu T C = -C\Gamma^\mu$. The totally antisymmetric products of $\Gamma$ matrices satisfy a basic relation

\[ \Gamma^{\mu_1\cdots\mu_n} \Gamma^{\nu} = \Gamma^{\mu_1\cdots\mu_{n-1}\nu} + \{ \delta^{\nu\mu_1} \Gamma^{\mu_2\cdots\mu_n-1} \pm \cdots \}. \]  

The gauge parameters that appear in the calculations of $x_0$-cohomology are of the form $\Lambda_{\mu\alpha\beta}$, $\Lambda_{\mu\alpha'\beta'}$ or $\Lambda_{\mu\alpha\beta'}$; bi-spinors with an additional vector index. The spinor indices can be treated as in (4.3). The presence of the additional vector index allows for two operations. In the first case, that index can be antisymmetrized with respect to the other antisymmetric indices. For example, doing this to the two-forms $\Lambda^{(2)}_{\mu,\nu_1,\nu_2}$ yields a three-form $\Lambda^{(2)}_{\mu,\nu_1,\nu_2}$. We will denote the resulting three-form as $a\Lambda^{(2)}$, where the prefix $a$ indicates taking the antisymmetric part. In the second operation, we can contract the vector index against one of the form indices. For example $\Lambda^{(2)}_{\mu,\nu_1,\nu_2}$ would this time yield the one form $\Lambda^{(2)}_{\mu,\nu_1}$, which is the prefix $t$ indicates taking trace. Although it is also possible to combine the indices to form a partially-symmetric three-form tensor, such a tensor will not appear in the cohomology calculations below.
**Gauge System I** We now begin with the first prototype problem. This is the gauge system described by a gauge parameter with two spinor indices of the same type. We have

\[
\delta a_{\alpha\beta'} = \Lambda_{\mu\alpha\gamma} \Gamma^{\mu\gamma}_{\beta'}, \\
\delta b_{\beta'\alpha} = \Lambda_{\mu\gamma\alpha} \Gamma^{\mu\gamma}_{\beta'}. \tag{4.5}
\]

It is convenient to form linear combinations

\[
\delta A_{\pm\alpha\beta'} \equiv \delta (a_{\alpha\beta'} \pm b_{\beta'\alpha}) = \Lambda_{\pm\mu\alpha\gamma} \Gamma^{\mu\gamma}_{\beta'}, \tag{4.6}
\]

where \(\Lambda_{+\mu}\) and \(\Lambda_{-\mu}\) denote the symmetric and antisymmetric parts (with respect to the spinor indices) of the gauge parameter \(\Lambda_{\mu}\). The fields \(A_{\pm\alpha\beta'}\) define zero-, two- and four-forms \(A^{(0)}_{\pm}, A^{(2)}_{\pm}, A^{(4)}_{\pm}\). The symmetric gauge parameters define one-forms \(\Lambda^{(1)}_{\mu}\) and self-dual five forms \(\Lambda^{(5+)}_{\mu}\), while the antisymmetric gauge parameters define three forms \(\Lambda^{(3)}_{\mu}\). Using the expansions (4.3), substituting into (4.6), and using (4.4) we find

\[
\delta A^{(0)}_{+} \sim t\Lambda^{(1)}, \quad \delta A^{(2)}_{+} \sim a\Lambda^{(1)}, \quad \delta A^{(4)}_{+} \sim t\Lambda^{(5+)}, \tag{4.7}
\]

implying that all the \(A_{+}\) fields can be gauged away. For the \(A_{-}\) fields we find

\[
\delta A^{(0)}_{-} \sim 0, \quad \delta A^{(2)}_{-} \sim t\Lambda^{(3)}, \quad \delta A^{(4)}_{-} \sim a\Lambda^{(3)}. \tag{4.8}
\]

We thus see that the scalar \(A^{(0)}_{-}\) cannot be gauged away. It corresponds to \(A_{-\alpha\beta'} = C_{\alpha\beta'}\), \(A_{+\alpha\beta'} = 0\), or, up to an overall irrelevant factor, to

\[
a_{\alpha\beta'} = -b_{\beta'\alpha} = C_{\alpha\beta'}. \tag{4.9}
\]

In a nutshell, the gauge system contained two scalars and a single one-form \(\Lambda^{(1)}_{\mu}\) gauge parameter whose trace could only be used to gauge away one of the scalars.

**Gauge System II** We now have a gauge parameter of mixed spinor type, and gauge transformations reading

\[
\delta a_{\alpha\beta} = \Lambda_{\mu\alpha\gamma} \Gamma^{\mu\gamma}_{\beta'} = (\Lambda_{\mu} \Gamma^{\mu})_{\alpha\beta}, \\
\delta b_{\alpha'\beta'} = \Lambda_{\mu\gamma\beta'} \Gamma^{\mu\gamma}_{\alpha'} = (\Gamma^{\mu} T\Lambda_{\mu})_{\alpha'\beta'}. \tag{4.10}
\]

We again expand the gauge parameters and the fields in terms of differential forms and then find the following gauge transformations

\[
\delta a^{(1)} \sim a\Lambda^{(0)} - t\Lambda^{(2)}, \quad \delta a^{(3)} \sim a\Lambda^{(2)} - t\Lambda^{(4)}, \quad \delta a^{(5+)} \sim a_{+}\Lambda^{(4)}, \tag{4.11}
\]

\[
\delta b^{(1)} \sim -a\Lambda^{(0)} - t\Lambda^{(2)}, \quad \delta b^{(3)} \sim -a\Lambda^{(2)} - t\Lambda^{(4)}, \quad \delta b^{(5-)} \sim -a_{-}\Lambda^{(4)},
\]

where the subscripts in \(a_{\pm}\) indicate taking the self-dual or anti-self-dual combinations. Considering the gauge transformations of sums and differences of the \(a\) and \(b\) one forms, and of the three forms, we see that they can all be gauged away. In addition, the five-forms can be gauged away separately. All of \(a_{\alpha\beta}\) and \(b_{\alpha'\beta'}\) is pure gauge.
4.3. \(x_0\)-COHOMOLOGY IN THE \((-1/2,-1/2)\) PICTURE

We discuss in turn the IIB, type-I and IIA superstrings.

**IIB superstring.** There is no cohomology at \(G = -1\), in fact, there are no zero-momentum \(L_0 = \bar{L}_0 = 0\) states at this ghost number because the candidate oscillators \(\beta_0, \bar{\beta}_0, b_0, \bar{b}_0\) vanish on the vacuum. This fact does not change when we include polynomials in \(x_0\). At \(G = 0\) the set of physical states is described by

\[
\Psi = -a_{\alpha\beta} \gamma_0 \left| -\frac{1}{2} \right\rangle^\alpha \otimes \left| -\frac{1}{2} \right\rangle^\beta + b_{\beta\alpha} \gamma_0 \left| -\frac{1}{2} \right\rangle^\beta \otimes \left| -\frac{1}{2} \right\rangle^\alpha.
\]

(4.12)

The gauge parameters are of the form

\[
\Lambda = x_0^\mu \Lambda_{\mu\alpha\beta} \left| -\frac{1}{2} \right\rangle^\alpha \otimes \left| -\frac{1}{2} \right\rangle^\beta.
\]

(4.13)

We do not include a term quadratic in \(x_0\) since, upon action by \(Q\), it would give rise to a term containing \(c_0^+\) and there are no such terms in (4.12). Since \(Q\) reduces the powers of \(x_0\) by one or two units, higher polynomials in \(x_0\) are not necessary. Recalling that \(|-\frac{1}{2}\rangle^\alpha\) and \(|-\frac{1}{2}\rangle^\beta\) are respectively Grassmann odd and Grassmann even, a short computation gives gauge transformations identical to those in (4.5). As discussed there, we have a gauge invariant state in the cohomology. Making use of (4.9) and (4.12), the state is found to be

\[
\left| \mathcal{A}^0 \right\rangle = C_{\alpha\beta'} (\gamma_0 \left| -\frac{1}{2} \right\rangle^\alpha \otimes \left| -\frac{1}{2} \right\rangle^\beta + \gamma_0 \left| -\frac{1}{2} \right\rangle^\beta \otimes \left| -\frac{1}{2} \right\rangle^\alpha) .
\]

(4.14)

This semirelative \(x_0\)-cohomology class at ghost number one represents a candidate anomaly, as discussed earlier. This completes our computation of \(x_0\)-cohomology in the \((-1/2,-1/2)\) picture for the IIB string.

**Type I superstring.** We now consider the Type I superstring, whose BRST complex is defined as the subcomplex of the IIB superstring complex spanned by states that are preserved by an exchange of left and right-movers. More precisely, when the left and right movers are in the same picture, there is an obvious way to map the left moving and right moving state spaces into each other, namely, we exchange holomorphic and antiholomorphic labels on all oscillators.
and vacua. If we denote this map by $I$, type-I states $|\Psi\rangle$ must satisfy
\begin{equation}
T(|\Psi\rangle) = |\Psi\rangle ,
\end{equation}
where the exchange map $T$ is defined as
\begin{equation}
T\left(a_{ij}|i\rangle \otimes |j\rangle\right) = (-)^{ij} a_{ij} I(|j\rangle) \otimes I(|i\rangle) ,
\end{equation}
and the sign prefactor takes into account the grassmanality of the states that have been exchanged. As in the IIB case, there is no $G = -1$ cohomology since there are no candidate states. At $G = 0$, the cohomology consists of the states $f_{\alpha\beta} |\frac{-1}{2}\rangle^\alpha \otimes |\frac{-1}{2}\rangle^\beta$, where $f_{\alpha\beta} = -f_{\beta\alpha}$ is a constant anti-symmetric bispinor field. Note that $f_{\alpha\beta}$ is anti-symmetric on account of (4.16) given that $|\frac{-1}{2}\rangle^\alpha$ and $|\frac{-1}{2}\rangle^\beta$ are both Grassman odd.

At $G = 1$, the candidate states are those of (4.12) where $a_{\alpha\beta'} = -b_{\beta'\alpha}$ (since $|\frac{-1}{2}\rangle^\alpha$ and $|\frac{-1}{2}\rangle^\beta$ commute). One can compute the cohomology using Gauge System I remembering that the gauge parameter $\Lambda$ must be anti-symmetric in its spinor indices and that $a_{\alpha\beta'} = -b_{\beta'\alpha}$. Therefore, equation (4.7) is unnecessary while equation (4.8) is the same as in the Type IIB case. So the $G = 1$ cohomology of the Type I superstring contains the same scalar as in (4.14).

**IIA superstring.** There is no $x_0$-cohomology for $G = -1$, and at $G = 0$ the semirelative cohomology classes $f_{\alpha\beta'} |\frac{-1}{2}\rangle^\alpha \otimes |\frac{-1}{2}\rangle^\beta$ remain nontrivial in $x_0$-cohomology. For $G = 1$, the candidate states are the semirelative classes
\begin{equation}
\Psi = -a_{\alpha\beta} \gamma_0 |\frac{-1}{2}\rangle^\alpha \otimes |\frac{-1}{2}\rangle^\beta + b_{\alpha'\beta'} \gamma_0 |\frac{-1}{2}\rangle^{\alpha'} \otimes |\frac{-1}{2}\rangle^{\beta'} .
\end{equation}
The gauge parameters are taken to be of the form
\begin{equation}
\Lambda = x_0^\mu A_{\mu \alpha' \beta'} |\frac{-1}{2}\rangle^{\alpha'} \otimes |\frac{-1}{2}\rangle^{\beta'} .
\end{equation}
A little calculation gives the gauge transformations considered in (4.10). It follows from the analysis of these equations that the IIA $x_0$-cohomology at $G = 1$ is absent.
4.4. Finite $x_0$-cohomology in the $(-3/2,-1/2)$ picture

**IIB superstring** Consider IIB theory at $G = -1$. The semirelative classes found earlier are

$$\Psi = a_{\alpha\beta} \beta_0 \ket{-\frac{3}{2}}^\alpha \otimes \ket{-\frac{1}{2}}^\beta - \frac{1}{2} b_{\alpha'\beta'} \beta_0^2 \gamma_0 \ket{-\frac{3}{2}}^{\alpha'} \otimes \ket{-\frac{1}{2}}^{\beta'}.$$  (4.19)

Here the ket $\ket{-\frac{3}{2}}^\alpha \otimes \ket{-\frac{1}{2}}^\beta$ is Grassmann odd, with $\ket{-\frac{3}{2}}^{\alpha'}$ even and $\ket{-\frac{1}{2}}^{\beta'}$ odd. It is now sufficient to consider the $x_0$-dependent $G = -2$ gauge parameter

$$\Lambda = \frac{1}{2} x_0^\mu \Lambda_{\mu\alpha'\beta'} \beta_0^2 \ket{-\frac{3}{2}}^{\alpha'} \otimes \ket{-\frac{1}{2}}^{\beta'},$$  (4.20)

and the resulting gauge transformations read

$$\delta a_{\alpha\beta} = \Lambda_{\mu\gamma'\beta} \Gamma_{\gamma'\alpha} = \Gamma_{\mu\beta}^T \Lambda_{\mu\alpha},$$

$$\delta b_{\alpha'\beta'} = \Lambda_{\mu\alpha'\gamma} \Gamma_{\gamma'\beta'} = \Lambda_{\mu\gamma} \Gamma_{\beta'\alpha'},$$  (4.21)

which after the exchange of primed and unprimed indices and the exchange of $a$ and $b$, coincide precisely with those considered in (4.10). We therefore conclude that there is no $x_0$-cohomology at $G = -1$ for IIB string theory.

Let us now consider cohomology at $G = 0$. The candidate states are

$$\Psi = a_{\alpha\beta} \ket{-\frac{3}{2}}^\alpha \otimes \ket{-\frac{1}{2}}^\beta + b_{0\alpha'\beta'} \beta_0 \gamma_0 \ket{-\frac{3}{2}}^{\alpha'} \otimes \ket{-\frac{1}{2}}^{\beta'}.$$  (4.22)

Here the ket $\ket{-\frac{3}{2}}^\alpha \otimes \ket{-\frac{1}{2}}^\beta$ is Grassmann even, with both $\ket{-\frac{3}{2}}^{\alpha'}$ and $\ket{-\frac{1}{2}}^{\beta'}$ odd. As we will claim that there is a gauge invariant scalar, it is necessary to consider the most general $x_0$-dependent $G = -2$ gauge parameter. We take

$$\Lambda = x_0^\mu \left( \Lambda_{\mu\alpha\beta} \beta_0 \ket{-\frac{3}{2}}^\alpha \otimes \ket{-\frac{1}{2}}^\beta + \frac{1}{2} \tilde{\Lambda}_{\mu\alpha'\beta'} \beta_0^2 \gamma_0 \ket{-\frac{3}{2}}^{\alpha'} \otimes \ket{-\frac{1}{2}}^{\beta'} \right) + \frac{1}{2} \Omega_{\alpha'\beta} c_0^+ \beta_0^2 \ket{-\frac{3}{2}}^{\alpha'} \otimes \ket{-\frac{1}{2}}^{\beta'},$$  (4.23)

where the last term, necessary in order to cancel terms containing a $\beta_0^2 \gamma_0^2$ factor in the gauge variations, fixes $\Omega_{\alpha'\beta} = -\tilde{\Lambda}_{\mu\alpha'\gamma} \Gamma_{\gamma'\beta}$. The gauge transformations then read

$$\delta a_{\alpha'\beta} = \Lambda_{\mu\gamma'\beta} \Gamma_{\gamma'\alpha'} + \tilde{\Lambda}_{\mu\alpha'\gamma'} \Gamma_{\gamma'\beta},$$

$$\delta b_{\alpha'\beta'} = \Lambda_{\mu\gamma'\beta'} + \tilde{\Lambda}_{\mu\gamma'\beta'} \Gamma_{\gamma'\alpha}.$$  (4.24)

This is essentially gauge system I ((4.5)) with two gauge parameters $\Lambda$ and $\tilde{\Lambda}$ playing similar roles. We now confirm that there is a gauge invariant scalar. To show this, it is convenient to
rewrite the above equations as

\[ \delta a_{\alpha'\beta} = (\Gamma^\mu T \Lambda_\mu + \tilde{\Lambda}_\mu \Gamma^\mu)_{\alpha'\beta} \]
\[ \delta b_{\beta\alpha'} = (\Gamma^\mu T \Lambda^T_\mu + \tilde{\Lambda}^T_\mu \Gamma^\mu)_{\alpha'\beta}. \] (4.25)

Defining linear combinations, we now find

\[ \delta A_{\pm\alpha'\beta} \equiv \delta(a_{\alpha'\beta} \pm b_{\beta\alpha'}) = (\Gamma^\mu T \Lambda_{\pm\mu} + \tilde{\Lambda}_{\pm\mu} \Gamma^\mu)_{\alpha'\beta} \] (4.26)

where the ± subscripts in the gauge parameters indicate the exchange property for spinor indices. Since the gauge transformations of $A_-$ involve the antisymmetric gauge parameters, and they only contain three forms, it is clear that the scalar part of $A_-$ cannot be gauged away. We thus have $A_{-\alpha'\beta} = C_{\alpha'\beta}$, $A_{+\alpha'\beta} = 0$ and the state representing the cohomology class is given by

\[ |A_-\rangle = C_{\alpha'\beta} \left( |-\frac{3}{2}\rangle \otimes |\frac{1}{2}\rangle^{\alpha'} - \beta_0 \gamma_0 |\frac{3}{2}\rangle \otimes |\frac{1}{2}\rangle^{\alpha'} \right). \] (4.27)

This state is the zero momentum axion. Just as the zero momentum ghost-dilaton, it would be trivial in absolute cohomology.

We now consider $G = 1$ cohomology. The candidate states and gauge parameters read

\[ \Psi = f_{\alpha'\beta'} \gamma_0 \left| -\frac{3}{2}\right>^{\alpha'} \otimes \left| -\frac{1}{2}\right>^{\beta'}, \]
\[ \Lambda = x_0^\mu \Lambda_{\alpha'\beta} \left| -\frac{3}{2}\right>^{\alpha'} \otimes \left| -\frac{1}{2}\right>^{\beta}. \] (4.28)

The resulting gauge transformations $\delta f_{\alpha'\beta'} = -\Lambda_{\alpha'\beta} \Gamma^\mu \gamma_\mu$, are a subset of those in gauge system II. It then follows that there is no IIB $x_0$-cohomology at $G = 1$.

**Type I superstring.** Since the left-moving and right-moving pictures are different, the Type I string fields $|\Psi\rangle$ will be defined to satisfy the condition

\[ T(\nabla_0 |\Psi\rangle) = \nabla_0 |\Psi\rangle, \] (4.29)

where $T$ is defined in (4.16) and $\nabla_0$ is defined in (3.41). Note that the state $\nabla_0 |\Psi\rangle$ is in the diagonal $(-3/2, -3/2)$ picture, and thus the action of $T$ is well defined.

First consider the $G = -1$ cohomology. If $\Psi$ is defined as in (4.19), condition (4.29) implies that $a_{\alpha\beta} = a_{\beta\alpha}$ and $b_{\alpha'\beta'} = b_{\beta'\alpha'}$. Defining the gauge parameter

\[ \Lambda = \frac{1}{2} x_0^\mu \Lambda_{\mu\alpha'\beta} \left( \beta_0^2 \left| -\frac{3}{2}\right>^{\alpha'} \otimes \left| -\frac{1}{2}\right>^\beta - \frac{1}{3} \beta_0^3 \gamma_0 \left| -\frac{3}{2}\right>^\beta \otimes \left| -\frac{1}{2}\right>^{\alpha'} \right) \]
\[ - \frac{1}{6} c_0^+ \beta_0^3 \Lambda_{\mu\alpha'\beta} \Gamma^\mu \alpha' \left| -\frac{3}{2}\right>^\beta \otimes \left| -\frac{1}{2}\right>^{\alpha}, \] (4.30)

which can be verified to satisfy (4.29), one finds the gauge transformations as (4.21) for the symmetric parts of the fields. It follows that there is no cohomology.

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At $G = 0$, the string field $\Psi$ of (4.22) satisfies (4.29) if $a_{\alpha'\beta} = b_{\beta'\alpha}$. Similarly, the gauge parameter of (4.23) satisfies (4.29) if $\Lambda_{\mu\alpha\beta} = \Lambda_{\mu\beta\alpha}$, and $\tilde{\Lambda}_{\mu\alpha'\beta'} = \tilde{\Lambda}_{\mu\beta'\alpha'}$. So the gauge transformation of $A_{+\alpha'\beta}$ is the same as in (4.26), while the field $A_{-\alpha'\beta}$ vanishes. Therefore, there is no cohomology at $G = 0$.

At $G = 1$ the candidate state is the same as in (4.28), but with $f_{\alpha'\beta'} = -f_{\beta'\alpha'}$. The appropriate gauge parameter in the type I complex is

$$\Lambda = x_0^\mu \Lambda_{\mu\alpha'\beta} \left( | -\frac{3}{2} \rangle^{\alpha'} \otimes | -\frac{1}{2} \rangle^\beta + \beta_0 \gamma_0 | -\frac{3}{2} \rangle^\beta \otimes | -\frac{1}{2} \rangle^{\alpha'} \right) + c_0^+_\mu \Lambda_{\mu\alpha'\beta} \beta_0 | -\frac{3}{2} \rangle^\beta \otimes | -\frac{1}{2} \rangle^{\alpha'}.$$

(4.31)

The resulting gauge transformation is $\delta f_{\alpha'\beta'} = -\Lambda_{\mu\alpha'\gamma} \Gamma_{\beta'\gamma} + \Lambda_{\mu\beta'\gamma} \Gamma_{\alpha'\gamma}$, and therefore, using the same argument as in the Type IIB case, we conclude that there is no cohomology at $G = 1$.

IIA superstring. We begin with the case $G = -1$. Here the candidate states are

$$\Psi = a_{\alpha'\beta} \beta_0 | -\frac{3}{2} \rangle^{\alpha'} \otimes | -\frac{1}{2} \rangle^\beta - \frac{1}{2} b_{\alpha'\beta'} \beta_0^2 \gamma_0 | -\frac{3}{2} \rangle^\alpha \otimes | -\frac{1}{2} \rangle^{\beta'}.$$

(4.32)

After consideration of the appropriate gauge parameters, one finds gauge transformations identical to those in (4.24). We therefore conclude that one scalar survives in $x_0$-cohomology. This is the state

$$|G\rangle = C_{\alpha'\beta} \left( \beta_0 | -\frac{3}{2} \rangle^{\alpha'} \otimes | -\frac{1}{2} \rangle^\beta + \frac{1}{2} \beta_0^2 \gamma_0 | -\frac{3}{2} \rangle^\beta \otimes | -\frac{1}{2} \rangle^{\alpha'} \right),$$

(4.33)

which, being at $G = -1$ represents a symmetry generator. The associated scalar charge can be identified with RR-charge, or as the momentum generator along an extra dimension curled up into a circle. This extra dimension is the eleventh direction of M-theory. The fact that no extra states were found for IIB superstrings is consistent with the viewpoint that the extra dimensions in F-theory seem to be nondynamical.

At $G = 0$, the candidate states read

$$\Psi = a_{\alpha\beta} | -\frac{3}{2} \rangle^{\alpha} \otimes | -\frac{1}{2} \rangle^\beta + b_{\alpha'\beta'} \beta_0 \gamma_0 | -\frac{3}{2} \rangle^{\alpha'} \otimes | -\frac{1}{2} \rangle^{\beta'}.$$

(4.34)

The gauge transformations for this case are found to be those in (4.21). It thus follows that the IIA $x_0$-cohomology at $G = 0$ vanishes. Finally, for $G = 1$ we find that all the candidate states $f_{\alpha'\beta'\gamma_0} | -\frac{3}{2} \rangle^{\alpha'} \otimes | -\frac{1}{2} \rangle^{\beta'}$ can be gauged away in $x_0$-cohomology. This completes our calculations of $x_0$-cohomology, the results of which are summarized in the table below.
Table. Summary of $x_0$-semirelative cohomology classes for IIA, IIB and type I superstrings.

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