From PDEs to Pfaffian bundles

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Abstract

We explain how to encode the essential data of a PDE on jet bundle into a more intrinsic object called Pfaffian bundle. We provide motivations to study this new notion and show how prolongations, integrability and linearisations of PDEs generalise to this setting.

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1 Introduction

The history and the importance of theory of Partial Differential Equations (PDEs) are themselves subjects of entire monographs. Very briefly, one of the central questions is that of integrability—i.e., that of the existence of local solutions of a PDE passing through each point. There are various techniques to handle this problem, each one with its own advantages. For instance, the Cartan-Kähler theorem can be applied in many instances but it is bound to the analytic setting. Another standard approach starts with the attempt to solve the PDE formally—and then one talks about formal integrability. One also discovers the notion of prolongations, which allows one to replace a given PDE with a new, "larger" one, but which may be easier to handle and, of course, has the same solutions as the original one. Another standard technique is that of linearising a PDE—the outcome is a PDE that is much easier to handle and which, although it usually has different solutions than the original one, often carries important informations about the behaviour of the solutions one is looking for.

While the role of jets is clear already in the local study of PDEs, formalising it was important for a more geometric approach to PDEs; this was carried out by Charles Ehresmann [5] in the 50’s, leading to the notion of jet bundle as the standard formalism to study PDEs on manifolds. Solutions of a PDE were then becoming sections of a bundle \( R \to M \) over a manifold \( M \), the PDEs themselves were becoming subspaces \( P \subseteq J^k R \) of the bundles of jets of sections of \( R \), and the condition for a section \( s \) of \( R \) to be a solution of \( P \) was that \( j^k_x s \in P \) for all \( x \in M \). Many of the notions and techniques known in the local study (e.g. prolongations, linearisations, etc) were then recast in this formalism; that process quickly revealed the notion of Cartan distribution(s), or Cartan form(s), on the jet bundles \( J^k R \) and its central role to the entire geometric theory. The various ways of understanding these objects gave rise to different schools/approaches to the subject, e.g. depending on whether (and how) one works with vector fields or differential forms; see, among others, the monographs [1,12,14,16,19,22]. For instance, the Cartan-Kähler theorem mentioned above is now part of the standard material on Exterior Differential Systems [2].

The aim of this paper is to emphasise and (hopefully) to clarify the importance of the Cartan distribution/form even further. The main message is that what is needed for the theory to work is not the jet bundles \( J^k R \) but just the bundle \( P \) together with the induced Cartan distribution; or, in our language, a Pfaffian bundle. Of course, there are points at which the jet bundles are still important, but often they are just “noise” in the background, giving rise to unnecessarily complicated formulas. Also, we are aware that this point may be, in principle, rather obvious to the specialists (and there are similar theories carried out at the level of infinite jet bundles), but we find it useful to spell it out in detail, taking care of the subtleties that arise along the way. We hope that, in this way, various techniques and notions that are often presented in a rather pragmatic way, via "down to earth" (but complicated) local formulas, become more transparent to people with a more geometric background/interests.

On the other hand, our main motivation for carrying this out comes from the study of Lie pseudogroups and of geometric structures: the theory is now ready to be used right away to understand the main structures underlying the theory of Lie pseudogroups \( \Gamma \) and, furthermore, of \( \Gamma \)-structures on manifolds. E.g. one may say that the Pfaffian groupoids of [15] are just the multiplicative version of the Pfaffian bundles discussed in this paper. Again, while this may still seem rather abstract for someone whose interest on Lie pseudogroups comes from the study of symmetries of concrete PDEs, it reveals the theory from a more geometric perspective,
pinpointing the actual structure that makes everything work, and uncovers rather unexpected bridges with other parts of Differential Geometry. For instance, the abstract (Pfaffian) groupoids arising from pseudogroups behave surprisingly similar to the symplectic groupoids of Poisson Geometry- and this similarity can really be exploited: e.g. the analogues of the Hamiltonian spaces and of Morita equivalences of Poisson Geometry turn out to be precisely what is needed to study general geometric structures and their integrability- as carried out in [3]. In all of these, the notion of Pfaffian bundle that is being discussed in this paper has the role of building block.

A few words on the structure of this paper. In section 2 we review the basics on PDEs: this include the notion of jet bundle and Cartan form, as well as its linear counterpart, the classical Spencer operator. Moreover, we recall the concepts of prolongation and of integrability of a PDE, and various important theorems in this area, together with the necessary technical tools, i.e. tableaux and Spencer cohomology.

In section 3 we introduce the definition of Pfaffian bundle in a double way, using either a differential form or a distribution. We define as well a number of objects naturally inspired from the theory of PDEs, such as symbol spaces and curvatures, and then we focus on the particular case of linear Pfaffian bundles and the process of linearising Pfaffian bundle along a solution.

Section 4 is the core of the paper: we use the definitions and the ideas from the previous section to develop a theory of prolongation in the context of Pfaffian bundles. In particular, we present first the general notions of morphism and prolongation in the Pfaffian category, followed by the concrete construction of a universal prolongation. Since this process is not always possible, we show concrete criteria for the prolongability of a Pfaffian bundle, and then see how these results translate to the linear picture.

Last, in section 5 we apply the theorems from section 4 in order to tackle integrability of Pfaffian bundles up to a finite order, as well as formal integrability. Borrowing ideas and terminology from the theory of G-structures, we associate inductively to a Pfaffian bundle certain obstructions called intrinsic torsions. In this setting, we can prove fundamental result such as the Goldschmidt criterion for formal integrability, the integrability criterion for Pfaffian bundles of finite type and the fact that analytic formally integrable Pfaffian bundles are integrable.

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2 PDEs on jet bundles

The different notions developed for Pfaffian bundles arise as a way to geometrically encapsulate the fundamental properties of the theory of PDEs. In this section we review the various geometrical notions that motivated and inspired the analogous ones for Pfaffian bundles. In particular, we will restrict our attention to jets of sections, which are easier to deal with, more widely studied in the literature and powerful enough for many applications. We will therefore not consider jets of submanifolds, even if we think that a suitable generalisation of Pfaffian bundles could be introduced also in that more general setting.
2.1 Jets, PDEs, and the Cartan form

A PDE of order $k$ in the function $u = u(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}^m$ is an equation of the form

$$F \left( x_i, u, \frac{\partial^{\vert \alpha \vert} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \right) = 0$$

for all $m$-multi-indices $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\vert \alpha \vert = \alpha_1 + \cdots + \alpha_m \leq k$. However, in order to describe a conceptual theory of PDEs on manifolds, the language of jets will be very well suited, since it sees the PDE as a submanifold of the $k$-jet bundle given by the zero locus of $F$ (see [12, 16] as references for jets).

More precisely, the $k$-jet of $u$ at $x \in \mathbb{R}^n$ is encoded in all the partial derivatives of $u$ up to order $k$: this means that two such functions $u$ and $v$ have the same $k$-jet at $x$ if they have the same Taylor polynomial of degree $k$ at $x$. This defines an equivalence relation $\sim_k$ on the space of smooth maps $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$, whose induced equivalence class (the $k$-jet of $u$ at $x$) is denoted by $j^k_x u$. Such an element of this quotient has coordinates $u^\alpha = \frac{\partial^{\vert \alpha \vert} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$, with $\alpha$ as above.

More generally, given a fibration (by which we mean a surjective submersion) $\pi : \mathbb{R} \to M$, we denote by $\Gamma(\mathbb{R})$ the set of sections of $\pi$, and by $\Gamma_{loc}(\mathbb{R})$ the local ones. For any integer $k \geq 0$, the space of $k$-jets of sections of $\pi$ is defined as

$$J^k \mathbb{R} := \{ j^k_x \beta \mid \beta \in \Gamma_{loc}(\mathbb{R}), \ x \in \text{Dom}(\beta) \}.$$ 

This set has a canonical manifold structure which fibres over $M$: indeed, the collection of $k$-jets of functions $u : \mathbb{R}^n \to \mathbb{R}^m$ coincides with $J^k \mathbb{R}$, when $R = \mathbb{R}^n \times \mathbb{R}^m$ is the trivial bundle over $\mathbb{R}^n$ with fibre $\mathbb{R}^m$, hence the coordinates described above can be taken as local coordinates for $J^k \mathbb{R}$ when $\dim(M) = n$ and $\text{rk}(R) = m$. The various jet bundles are related to each other by the obvious projection maps

$$\cdots \to J^2 \mathbb{R} \to J^1 \mathbb{R} \to J^0 \mathbb{R} = \mathbb{R},$$

and each projection $J^k \mathbb{R} \to J^{k-1} \mathbb{R}$ is an affine bundle modelled on the pullback of $S^k(T^* M) \otimes T^\pi R$. To simplify notation we denote all the projections by $\text{pr}$, and the fibration of $J^k \mathbb{R}$ over $M$ by $\pi$. Having at hand the language of jets, we can naturally formalise the following definition (see [7]): a PDE of order $k$ on $\pi$ is a (connected) fibred submanifold

$$P \subset J^k \mathbb{R}.$$ 

A (local) solution of $P$ is any (local) section $\beta$ of $\mathbb{R}$ with the property that

$$j^k_x \beta \in P \ \forall x \in \text{Dom}(\beta);$$

this means that the (local) section $j^k \beta$ of $J^k \mathbb{R}$ must be a (local) section of $P$. In other words, the set of solutions of $P$, denoted by $\text{Sol}(P)$, is made up by all the sections $\alpha$ of $P$ which are holonomic, i.e. of the form $\alpha = j^k \beta$ for $\beta$ a section of $R$. Accordingly, in order to detect which sections are holonomic, we introduce the Cartan 1-form

$$\omega \in \Omega^1(J^k \mathbb{R}; \text{pr}^* (T^\pi (J^{k-1} \mathbb{R})))$$.
A linear PDE on $E$ only if addition defined by $J$ kernel of the Cartan form, called the Cartan distribution $\in X^k$. In the general case, at level $k$, solutions of $F$ where $i$ plays the role of the Cartan form (1), i.e. detecting holonomic classical Spencer operator. In other words, for the study of PDEs, the only relevant data is a fibration $P \to M$ endowed with an appropriate 1-form (or, equivalently, with its kernel): this will be our starting point for the definition of Pfaffian bundles (which forget the ambient jet space).

2.2 Linear PDEs and Spencer operators

If $R = E$ is a vector bundle over $M$, $J^kE$ is canonically a vector bundle over $M$ with fibrewise addition defined by

$$j^k_x\beta + j^k_x\eta = j^k_x(\beta + \eta)$$

A linear PDE on $E$ of order $k$ is a vector subbundle $F \subset J^kE$ over $M$. As in the general case, solutions of $F$ are sections of $F$ that are holonomic; however, in this linear setting the classical Spencer operator plays the role of the Cartan form (1), i.e. detecting holonomic sections. As for the Cartan form, we will define explicitly this operator when $k = 1$, using a very convenient way to describe sections of $J^1E$, known as the Spencer decomposition: it is the canonical isomorphism of vector spaces

$$\Gamma(J^1E) = \Gamma(E) \oplus \Omega^1(M; E).$$

This decomposition comes from the short exact sequence of vector bundles over $M$

$$0 \to \text{Hom}(TM; E) \to J^1E \xrightarrow{pr} E \to 0$$

where $i$, at the level of sections, is defined as $i(df \otimes s) = fj^1s - j^1(fs)$. Although this sequence does not have a canonical right splitting, at the level of sections it does: $s \mapsto j^1s$. This gives the decomposition (2), so that the Spencer operator $D^{\text{class}}$ is by definition the projection to the second component:

$$D^{\text{class}} : \Gamma(J^1E) \to \Omega^1(M; E)$$

This operator has been extensively studied, see for example [9, 11, 17, 18, 20, 21].

Moreover, it is clear from its description that holonomic sections of $F \subset J^1E$ are precisely the sections $\alpha$ with the property that $D^{\text{class}}(\alpha) = 0$. The same story can be also repeated for higher jets, obtaining classical Spencer operators of the form $D^{\text{class}} : \Gamma(J^kE) \to \Omega^1(M; J^{k-1}E)$, which
vanish on the solutions of (higher order) linear PDEs $F$. Hence, in analogy with the Cartan form, we can characterise the solutions of $F$ only in terms of $F$ viewed as a vector bundle (and not as a subbundle of $J^k E$), together with the restriction of $D = D^{\text{clas}}$ to $F$:

$$\text{Sol}(F) \simeq \Gamma(F, D) := \{ \alpha \in \Gamma(F) \mid D(\alpha) = 0 \}.$$ 

After defining Pfaffian bundles as generalisation of PDEs with their Cartan forms, their linear counterpart (the linear Pfaffian bundles) will be in turn a generalisation of linear PDEs with their classical Spencer operators.

**Remark 2.2.** We will also show that the classical Spencer operator can be seen as the linearisation of the Cartan form. Actually the whole picture relating the two objects can be more clearly seen in the world of Lie groupoids endowed with multiplicative forms and Lie algebroids endowed with (non classical) Spencer operators: the linearisation of a Lie groupoid is its Lie algebroid, and the linearisation of a multiplicative forms is a Spencer operator. See [4] as a reference for this topic.

## 2.3 Prolongations of PDEs

The theory of prolongations of a PDE is a powerful tool to find solutions for a PDE; the literature on this topic is very rich and dates back several decades: we mention [9, 10, 14, 1, 22, 19] and we will briefly recall here some of these notions.

A prolongation of a PDE $P$ of order $k$ on $\pi : R \rightarrow M$ can be thought as the $(k+1)$-order PDE on $\pi$ of first order differential consequences of $P$, with the fundamental property of having the same space of solutions. The first naive guess to define the prolongation of $P$ would be simply $J^1 P = \{ j^1 \sigma \mid \sigma \in \Gamma(P) \}$. However, one immediately sees that $J^1 P$ fails to be a PDE of order $(k+1)$ on $\pi$, since $J^1 P$ is by construction a subset of $J^1 (J^k R)$, not of $J^{k+1} R \subset J^1 (J^k R)$. The way to solve this (set-theoretical) problem is to define the **prolongation** $P^{(1)}$ as

$$P^{(1)} = J^1 P \cap J^{k+1} R$$

(5)

However, $P^{(1)}$ may fail to be a subbundle of $J^{k+1} R$; even more, $P^{(1)}$ may fail to be smooth. If $P^{(1)}$ happens to be “nice enough” (e.g. it is indeed a new PDE, and the projection $P^{(1)} \rightarrow P$ is a surjective submersion) then $P$ is said to be **integrable up to order $k+1$**; if it is integrability up to any order it is said to be **formally integrable**. In this case we obtain a tower of bundles over $M$

$$\ldots \rightarrow P^{(2)} \rightarrow P^{(1)} \rightarrow P$$

(6)

each of them endowed with the restriction of the Cartan form at every order, and all the maps being surjective submersions.

The study of formal integrability of a PDE is a very useful tool in order to find its solutions. This can be best seen in the analytic case, where formal integrability becomes a sufficient condition for **integrability**, i.e. finding local solutions at every point.

**Theorem 2.3** (Theorem 9.1 of [7]). If $P \subset J^k R$ is an analytic formally integrable PDE, then for every $p \in P^{(l)} \subset J^{k+l} R$ over $x \in M$ there is an analytic local solution $\beta$ of $P$ such that $j^x_{k+l} \beta = p$ on a neighbourhood of $x \in \text{dom}(\beta)$.

In particular, for every $p \in P$ there passes a local (analytic) solution.
However, in the smooth category this is not always true, since there are formally integrable PDEs admitting no solution: see the famous Lewy counterexample [13].

To understand better the structure of the prolongations and the notion of formal integrability, one arrives at the notion of tableau (see [2, 6] and the next section). The tableaux are linear spaces that provide the framework to handle the intricate linear algebra behind PDEs; they also provide (Spencer) cohomological criteria for integrability of PDEs.

In particular, the symbol space \( g \) of the PDE \( P \subseteq J^k R \) is the tableau
\[
\mathfrak{g} = \ker(dpr : T^\pi P \to T^\pi J^{k-1} R) \subset \ker(dpr : T^\pi J^k R \to T^\pi J^{k-1} R) \simeq S^k(T^* M) \otimes T^\pi R. \tag{7}
\]
This last isomorphism comes from the standard exact sequence over \( J^k R \):
\[
0 \to S^k T^* M \otimes T^\pi R \to T^\pi J^k R \xrightarrow{dpr} T^\pi J^{k-1} R \to 0 \tag{8}
\]
where we understand that all the vector bundles sit on top \( J^k R \) as pullback of the obvious maps.

Equivalently, using the definition of the Cartan form, one checks that
\[
\mathfrak{g} = \{ v \in T^\pi P | dpr(v) = 0 \} = \{ v \in T^\pi P | \omega(v) = 0 \} = T^\pi P \cap \ker(\omega) \simeq T^\pi P \cap (S^k(T^* M) \otimes T^\pi R). \tag{9}
\]

We can use the symbol space to provide a sufficient criterion for formal integrability of PDEs, in terms of the Spencer cohomology of \( \mathfrak{g} \), which we recall in the next section (see [7] for the original result and [23] for a more careful and modern proof):

**Theorem 2.4 (Goldschmidt formal integrability criterion).** Let \( P \) be a PDE whose symbol space \( \mathfrak{g} \) is 2-acyclic, i.e. its Spencer cohomology \( H^{k,2}(\mathfrak{g}) \) vanishes \( \forall k \geq 0 \). If, moreover \( P^{(1)} \to P \) is surjective and the prolongation \( \mathfrak{g}^{(1)} = \{ \eta \in S^{k+1}(T^* M) \otimes T^\pi R \mid \forall X \in \mathfrak{g}, \forall Y \in X(\mathcal{M}) \} \) is of constant rank, then \( P \) is formally integrable.

**Remark 2.5.** In the same way that the theory of (linear) Pfaffian bundles was inspired by that of (linear) PDEs presented above, the notion of prolongation of a Pfaffian bundle (developed in section 4) comes as a geometrical way to describe the prolongation of a PDE only in terms of \( P \) and the Cartan form, i.e. it isolates the properties that each map of (6) has in terms of that form, forgetting the ambient jet space where \( P \) lived. 

\( \diamond \)

### 2.4 Tableaux and Spencer cohomology

As stated in Theorem 2.4, Goldschmidt provides in [7] a cohomological criterion for formal integrability of a PDE in terms of its tableau. In this section we recall the general notions of tableau and Spencer cohomology, and state some facts relevant to the theory of PDEs. Also we describe a small variant of the Spencer cohomology which will appear in the theory of Pfaffian bundle, when dealing with a slightly more general notion of tableau.

**Definition 2.6.** Let \( E, F \) be vector spaces. A **tableau** on \( (E, F) \) is a linear subspace
\[
\mathfrak{g} \subset \text{Hom}(E; F).
\]
We define the 1\(^{st}\) **prolongation** of \( \mathfrak{g} \) as
\[
\mathfrak{g}^{(1)} := \{ \eta \in \text{Hom}(E, \mathfrak{g}) : \eta(X)(Y) = \eta(Y)(X) \ \forall \ X, Y \in E \} = \text{Hom}(E, \mathfrak{g}) \cap S^2 E^* \otimes F,
\]
and we define inductively the 1\(^{st}\) **prolongation** of \( \mathfrak{g} \) by
\[
\mathfrak{g}^{(i)} := (\mathfrak{g}^{(i-1)})^{(1)} = \text{Hom}(E, \mathfrak{g}^{(i-1)}) \cap S^{i+1} E^* \otimes F.
\]
Next, we recall that the De Rham operator on $E$

$$\delta : S^k E^* \to E^* \otimes S^{k-1} E^*,$$

$$\delta(\eta)(v) = \iota v \eta \in S^{k-1} E^*$$

extends to a linear map

$$\delta : \wedge^j E^* \otimes S^k E^* \to \wedge^{j+1} E^* \otimes S^{k-1} E^*,$$

$$\delta(\omega \otimes \eta) = (-1)^j \omega \wedge \delta(\eta).$$

The resulting sequence of complexes (i.e. $\delta \circ \delta = 0$) is of the form

$$0 \to S^k E^* \xrightarrow{\delta} E^* \otimes S^{k-1} E^* \to \cdots \to \wedge^k E^* \otimes S^{k-n} E^* \to 0$$

for each $k$ (we set $S^l E^* = 0$ for $l < 0$). We tensor then the sequence of complexes above by $F$, and the operator $\delta$ by $Id_F$, keeping still the same notation $\delta$. Note that for a tableau $g \subset \text{Hom}(E; F)$, each prolongation $g^{(i)}$ can be described as the kernel of the restriction of the appropriate $\delta$ to $\text{Hom}(E, g^{(i-1)})$:

$$\delta = \delta_i : \text{Hom}(E, g^{(i-1)}) \to \text{Hom}(\Lambda^2 E, g^{(i-2)}),$$

$$\delta(\eta)(X, Y) = \eta(X)(Y) - \eta(Y)(X),$$

hence, it is not difficult to see that the sequence of complex (10) tensored with $F$ contains the subsequence of complexes

$$0 \to g^{(i)} \xrightarrow{\delta} E^* \otimes g^{(i-1)} \xrightarrow{\delta} \wedge^2 E^* \otimes g^{(i-2)} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \wedge^i E^* \otimes g \xrightarrow{\delta} \wedge^{i+1} E^* \otimes F,$$

for each $i$. At $\wedge^m E^* \otimes g^{(l)}$ the cocycles are denoted by

$$Z^{l,m}(g) := \ker(\delta : \wedge^m E^* \otimes g^{(l)} \to \wedge^{m+1} E^* \otimes g^{(l-1)}),$$

and the coboundaries by

$$B^{l,m}(g) := \text{Im}(\delta : \wedge^{m-1} E^* \otimes g^{(l+1)} \to \wedge^m E^* \otimes g^{(l)});$$

the cohomology groups are denoted by

$$H^{l,m}(g) := Z^{l,m}(g)/B^{l,m}(g).$$

Note that by construction $H^{l,1}(g) = 0$ for all $l \geq 0$. The resulting cohomology is called the **Spencer cohomology of the tableau $g$.**

**Definition 2.7.** Let $r \geq 1$ be an integer. A tableau $g$ is said to be $r$-acyclic if

$$H^{l,m}(g) = 0, \quad \forall 1 \leq m \leq r, \ l \geq 0.$$  

and is involutive if it is $r$-acyclic for all $r \geq 1$, i.e.

$$H^{l,m}(g) = 0, \quad \forall m \geq 1, \ l \geq 0.$$  

Later on, in the theory of Pfaffian bundles, we will need small variant of the Spencer complex of a tableau $g \subset \text{Hom}(E, F)$ in which the inclusion $g \hookrightarrow \text{Hom}(E, F)$ is replaced by a linear map

$$\partial : g \to \text{Hom}(E; F).$$
In this case we define the 1st prolongation of \(g\) (with respect to \(\partial\)) by

\[
g(1)(\partial) := \{ \eta \in \text{Hom}(E; g) \mid \partial(\eta(X))(Y) = \partial(\eta(Y))(X), \ \forall X, Y \in E \} \tag{13}
\]

We can regard \(g(1)(\partial)\) as a (classical) tableau on \((E, g)\) and prolong it repeatedly, giving rise to the higher prolongations

\[
g(i)(\partial) = S^i E^* \otimes g \cap \text{Hom}(E; g^{(i-1)}),
\]

\(i > 1\). The Spencer sequence for \(g(1)(\partial)\) can be extended in the following way: we extend \(\partial\) to the linear map

\[
\delta \partial : \wedge^j E^* \otimes g \rightarrow \wedge^{j+1} E^* \otimes F, \quad \delta \partial(\omega \otimes v) = (-1)^j \omega \wedge \partial(v).
\]

A simple computation shows that the sequence of Spencer complexes of \(g(1)(\partial)\) extends to the sequence of complexes

\[
0 \rightarrow g^{(i)} \xrightarrow{\delta} E^* \otimes g^{(i-1)} \xrightarrow{\delta} \ldots \xrightarrow{\delta} \wedge^{i-1} E^* \otimes g \xrightarrow{\delta} \wedge^i E^* \otimes g \xrightarrow{\delta} \wedge^{i+1} E^* \otimes F, \tag{14}
\]

for each \(i\). We call the \(\partial\)-Spencer cohomology of \(g\) the cohomology of the sequence (14).

Now, when dealing with vector bundles \(E, F\) over \(M\) instead of vector spaces, all the notions discussed above extend naturally. In particular, a tableau bundle on \((E, F)\) is a bundle \(g \subset \text{Hom}(E, F)\) of linear subspaces \(\{g_x \subset \text{Hom}(E_x; F_x)\}_{x \in M}\), whose rank may vary; \(g\) is therefore a (smooth) vector subbundle over \(M\) only when it is of constant rank. However, let us point out that the prolongations \(g^{(i)}\) may fail to be smooth even if we start with a smooth tableau bundle \(g\); at certain points the rank of some prolongations may not be constant anymore. One of the roles of the acyclicity condition is to ensure the smoothness of the prolongations (see [7, 23] for the proof):

**Lemma 2.8.** Let \(g \subset \text{Hom}(E; F)\) be a tableau bundle over a connected manifold \(M\). If \(g\) is 2-acyclic and \(g^{(1)} \subset \text{Hom}(E; g)\) is a vector bundle of constant rank, then \(g^{(i)} \subset \text{Hom}(E; g^{(i-1)})\) is also a vector bundle of constant rank for all \(i \geq 0\).

**Remark 2.9.** Lemma 2.8 above also holds when dealing with a tableau bundle defined by a vector bundle map \(\partial : g \rightarrow \text{Hom}(E; F)\) over \(M\); in that case we are considering of course the 1st prolongation \(g^{(1)}(\partial)\) w.r.t. \(\partial\). The proof follows the same lines as the proof of Lemma 2.8.  

A fundamental result in the theory of prolongations of PDEs states that, even if a tableau bundle is not involutive, it becomes so after a finite number of prolongations (see [8, Lemma 2]):

**Theorem 2.10.** Let \(g\) be a tableau bundle. There exists an integer \(l_0\) such that \(g^{(l)}\) is involutive for all \(l \geq l_0\).

## 3 Pfaffian bundles and their geometry

We present now the central object of this paper, which we obtain by replacing the jet bundles with their “PDE structure”; furthermore, we explain how to recover many concepts from the theory of PDEs. As anticipated in the introduction, we stress that the leading idea in this picture is not to give an abstract generalisation of the notion of PDE, but to shed light on its geometry.
3.1 Pfaffian bundles

Definition 3.1. A Pfaffian bundle \((P, \theta)\) over \(M\) is a fibration \(\pi : P \to M\) together with a pointwise surjective form \(\theta \in \Omega^1(P, N)\) with coefficients in some vector bundle \(N \to P\) such that

- \(\theta\) is \(\pi\)-regular, i.e. the restriction of \(d\pi\) to \(\ker(\theta)\) is pointwise surjective, or equivalently, \(\ker(\theta)\) is transversal to the \(\pi\)-fibres:
  \[TP + \ker(\theta) = TP\]

- \(\theta\) is \(\pi\)-involutive, i.e. the following distribution is involutive
  \[g(\theta) := T^\pi P \cap \ker \theta\]

The form \(\theta\) satisfying the properties above is called a Pfaffian form, the vector bundle \(N\) the coefficient bundle, and the distribution \(g(\theta)\) the symbol space of \(\theta\).

From the \(\pi\)-regularity of the Pfaffian form \(\theta\) it follows that it has constant rank, hence it defines a vector subbundle \(g(\theta) \subset TP\) over \(P\), i.e. a regular distribution (therefore it makes sense to ask it to be involutive).

As anticipated, the \(k\)-jet space \(\pi : J^k R \to M\) of a fibration \(R \to M\), together with the canonical Cartan form \(\omega\), turns out to be our main example of Pfaffian bundle. Moreover, given any PDE \(P \subset J^k R\) of order \(k\), the restriction of \(\omega\) to \(P\) is still \(\pi\)-regular and \(\pi\)-involutive, thus \((P, \omega |_P)\) is a Pfaffian bundle as well (under some regularity conditions on \(P\)).

Remark 3.2. (Pfaffian distributions) We can look at \(\pi\)-regular 1-forms from the equivalent point of view of distributions transversal to the \(\pi\)-fibres (or \(\pi\)-transversal distributions).

In particular, starting with a \(\pi\)-transversal distribution \(H \subset TP\), one defines the normal bundle \[N_H := TP/H = T^\pi P/H^\pi\]
and the symbol space of \(H\)
\[g(H) = T^\pi P \cap H\]
If, moreover, \(H\) has involutive symbol space we call it Pfaffian distribution. We can then produce the 1-form \(\theta_H\) (and say that \(\theta_H\) is induced by \(H\)) given by the projection \(TP \to N_H\): by construction \(\theta_H\) satisfies \(\ker(\theta_H) = H\), is \(\pi\)-regular, and its symbol space coincides with that of \(H\).

Viceversa, if \(H_\theta\) is already the kernel of a \(\pi\)-regular 1-form \(\theta \in \Omega^1(P, N)\), then its normal bundle becomes isomorphic to the coefficient bundle \(N\) via the map \(N_H \ni [u] \mapsto \theta(u) \in N\), under this isomorphism \(\theta\) can be trivially written as the projection map \(TP \to N_H\). Clearly, \(H_\theta\) is \(\pi\)-transversal and its symbol space coincides with that of \(\theta\).

Lemma 3.3. The previous construction gives a 1-1 correspondence:

\[
\begin{align*}
\left\{ \text{Pfaffian distributions} \right\} & \leftrightarrow \left\{ \text{(equivalence classes) of Pfaffian forms} \right\} \\
H \subset TP & \quad \theta \in \Omega^1(P; N)
\end{align*}
\]

where two forms \(\theta_1, \theta_2\) are equivalent if there exists a vector bundle isomorphism \(\phi : N_1 \to N_2\) such that \(\phi(\theta_1(v)) = \theta_2(v)\) \(\forall v \in TP\).

This correspondence, in the case of PDEs, is precisely the one between the Cartan form and the Cartan distribution.
Accordingly, we have the equivalent notion of a Pfaffian bundle \((P, H)\) over \(M\) when dealing with a Pfaffian distribution; in the following, we will switch freely between these two definitions (with forms or with distributions).

**Remark 3.4. (Pfaffian systems)** Pfaffian bundles are related to another way of studying differential equations, namely exterior differential systems (EDSs): every Pfaffian bundle induces a special kind of EDS.

We refer to [2] for an introduction on EDSs, which are differential ideals of the exterior algebra of a manifold. In particular, a Pfaffian system is an EDS \(J \subset \Omega^*(P)\), generated as an exterior differential ideal in degree one, together with some transversal (or independence) condition. It can be proved then that a \(\pi\)-transversal distribution \(H \subset TP\) induces such kind of Pfaffian system, and moreover, if \(H\) is also \(\pi\)-involutive, the Pfaffian system turns out to be linear (another notion from the theory of EDSs, which is completely unrelated with that of linear Pfaffian bundle in section 3.2).

**Remark 3.5.** The framework of Pfaffian bundles fits nicely in between two classical ways of studying differential equations:

- The formalism of jet bundles becomes a particular case (we give up the jets and retain the main structure given by the Cartan form)
- The formalism of exterior differential systems is a more general case (we concentrate only on Pfaffian systems which have a transversal condition and are linear)

In both cases, a (local) solution of a PDE (i.e. a holonomic section in the jet bundle language, an “integral manifold” in the EDS language) corresponds to a (local) section of the Pfaffian bundle which pullbacks the Pfaffian form to zero:

**Definition 3.6.** Given a Pfaffian bundle \((P, \theta)\), a holonomic (local) section of \((P, \theta)\) is any (local) section \(\beta\) of \(P\) with the property that \(\beta^*\theta = 0\). The set of holonomic sections is denoted by \(\Gamma(P, \theta)\) and the local ones by \(\Gamma_{loc}(P, \theta)\).

Analogously, a holonomic section of a Pfaffian bundle \((P, H)\) is any section \(\beta\) tangent to \(H\) (i.e. \(d\beta : TM \to TP\) takes values in \(H\)). We denote by \(\Gamma(P, H)\) the set of holonomic sections, and by \(\Gamma_{loc}(P, H)\) the local ones.

Alternatively, one can define the (infinite-dimensional) vector bundle \(E\) over the (infinite-dimensional, assuming it is not empty) manifold \(\Gamma(P)\) by setting the fibres

\[ E_{\beta} := \Omega^1(M; \beta^*N) \]

and consider its global section

\[ \Theta : \Gamma(P) \to E, \quad \beta \mapsto \beta^*\theta \] (16)

The holonomic sections of \((P, \theta)\) are precisely the ones that are mapped by \(\Theta\) to the zero section of \(M\), hence \(\Theta\) can be called holonomator (when it vanishes, all sections are holonomic). This point of view will be very useful in section 3.3, when looking at the linearisation of a Pfaffian bundle along a holonomic section. Alternatively, the holonomator can be recovered from the section of the finite dimensional vector bundle

\[ e : J^1P \to \text{Hom}(\pi^*TM, N), j^1_x\beta \mapsto \theta \circ d_x\beta \]
as $\Theta(\beta) = e \circ j^1 \beta$. In section 4.3 we will use $e$ to define the partial prolongation of a Pfaffian bundle.

One of the main questions for Pfaffian bundles is the integrability from the PDE point of view:

**Definition 3.7.** A Pfaffian bundle $(P, \theta)$ (or $(P, H)$) is said to be **PDE-integrable** if for each point $p \in P$ passes a local holonomic section $\beta \in \Gamma_{\text{loc}}(P, \theta)$ (or $\beta \in \Gamma_{\text{loc}}(P, H)$), i.e. $\beta(\pi(p)) = p$.

**Remark 3.8.** Of course the notion of holonomic section makes sense for any 1-form on a fibration $P \to M$, without any a priori relation with $T^\pi P$; however, PDE-integrability implies $\pi$-regularity of $\theta$, which is therefore a posteriori a meaningful condition to ask in the definition. This can be more easily seen using $H = \ker \theta$: if for any $p$ there is a local section $\beta : M \to P$ passing through $p$ which is tangent to $H$, then

$$T_x M = d(\pi \circ \beta)(T_x M) = d\pi(d\beta(T_x M)) \subset d\pi(H_p),$$

where $x = \pi(p)$. This means that $d\pi$ is surjective when restricted to $H$, i.e. $H$ is $\pi$-transversal (or $\theta$ is $\pi$-regular).

A natural notion that comes into play when studying PDE-integrability is that of **integral element** (see [2] for the analogous notion for an EDS). Intuitively, an integral element of $(P, H)$ is a linear subspace $V \subset T_p P$, $p \in P$, which is a “good” candidate to be the tangent space of a holonomic (local) section $\beta$ that passes through $p$. Suppose that $V$ is indeed tangent to $\beta$ i.e. $V = d\beta(T_x M)$, $x = \pi(p)$: this immediately implies that the dimension of $V$ is the dimension of $M$ and that $T_p P$ can be written as the direct sum $V \oplus T^\pi_p P$. Due to the holonomicity of $\beta$, one further obtains that

$$V \subset H_p, \quad \text{and} \quad [u, v]_p \in V, \quad (17)$$

for any $u = d\beta(X), v = d\beta(Y)$ with $X, Y \in \mathfrak{X}(M)$.

In order to rewrite this last condition (independently of the extensions of $u_p, v_p$) we need to introduce the **curvature map** of $H$

$$c_H : H \times H \to \mathcal{N}_H$$

which is the $C^\infty(P)$-bilinear map defined at the level of sections by $(U, V) \mapsto [U, V] \mod H$; the Leibniz identity of the Lie bracket of vector fields implies that $c_H$ is indeed well defined. Alternatively, if $H = \ker \theta$, such a map is denoted by $c_\theta : H \times H \to \mathcal{N}$ and can be described by $(U, V) \mapsto \theta([U, V])$; therefore, it coincides with the restriction of $d_\nabla \theta$ to $\ker(\theta)$, where $d_\nabla$ is the (De Rham-type) differential of any linear connection $\nabla$ on $P$.

**Definition 3.9.** Given a Pfaffian bundle $(P, H)$ (or $(P, \theta)$), a linear subspace $V \subset T_p P$ of dimension equal to the dimension of $M$ is called a **partial integral element** if

$$V \subset H_p, \quad \text{and} \quad T_p P = V \oplus T^\pi_p P.$$

If, moreover, the restriction of the curvature map $(c_H)_p$ to $V \times V$ is zero, then $V$ is called an integral element.
3.2 Linear Pfaffian bundles and relative connections

Let $\pi : E \to M$ be a vector bundle with fibrewise addition $a(v, w) = v + w$, and zero section $0(x) = (x, 0)$; its tangent vector bundle $TE \to TM$ has as structure maps the differential of the structure maps of $E$; in particular the fibrewise addition is given by the differential $da$.

- A differential form $\theta \in \Omega^1(E; \pi^*F)$ with values in the (pullback of the) coefficient bundle $F \to M$ is called linear if $a^*\theta = pr_1^*\theta + pr_2^*\theta$, where $pr_1, pr_2 : E \times_M E \to E$ denote the canonical projections.

- A distribution $H \subset TE$ is called linear if it is a vector subbundle of $TE$ over the same base $TM$.

**Remark 3.10.** Many of the notions for general distributions are over $E$, but when we deal with linear distribution these objects descend on $M$. For instance, we show now that the normal bundle $TE/H$ can be recovered from the $\pi$-pullback of the following vector bundle (over $M$):

$$F_H = (TE/H)|_M$$

and that the same happens for the symbol space $g(H)$:

$$g(H) \simeq \pi^*(g(H)|_M)$$

First, as soon as we deal with vectors tangent to the fibres we can right translate them to the zero section. Any vector $V$ at $u \in E$ tangent to the fibre $E_x$, $x = \pi(u)$, moves to a vector based at $0(x) = (x, 0)$ by taking the differential of the right translation $a_{-u}$ by $-u$:

$$da_u : T_u(E_x) \to T_x(E_x), V \mapsto da(V, 0_{-u})$$

The advantage of this is that $da_u$ takes $g(H)_u$ to $g(H)_x$ because $H$ is linear, hence we obtain (20).

Second, as $H$ is linear, $TM = d0(TM) \subset H|_M$ and this shows that $H$ is $\pi$-transversal on $M$. This, together with the identification (20), implies the $\pi$-transversality:

$$TE = H + T^\pi E$$

Indeed, it is enough to compute $\text{rk}(H_u + T^\pi_u E) = \text{rk}(H_u) + \text{rk}(T^\pi_u E) - \text{rk}(g(H)_u)$ and compare it with the ranks at $x = \pi(u)$.

Now, condition (21) implies in turn that the normal bundle can be rewritten as

$$TE/H = T^\pi E/g(H).$$

Passing to the normal bundle and using (20) we obtain the isomorphism we wanted:

$$\pi^*F_H \simeq TE/H.$$  

**Remark 3.11. (Equivalence between linear forms and distributions)** Any pointwise surjective linear form $\theta$ induces a distribution $H = \ker(\theta)$ which is clearly linear too; conversely, any linear distribution arises by taking $F = F_H$ as in (19) and $\theta_H$ the canonical projection $TE \to TE/H = T^\pi E/H^\pi$ followed by the isomorphism $T^\pi E/H^\pi \simeq \pi^*F_H$ given by right translation to the units: $V_u \mapsto da(V_u, 0_{-u})$ (see Remark 3.10 and the lemma below).

Analogously to observation 3.2, this defines a correspondence between linear distributions and pointwise surjective linear forms (up to isomorphisms of the coefficient bundle $F$).
Lemma 3.12. \( \theta_H \in \Omega^1(E; \pi^*F) \) is linear.

Proof. Due to the transversality of \( H \) one writes \( \theta_u(V) = \theta_u(V - \bar{V}) \), with \( \bar{V} \in H_u = \ker(\theta_u) \) so that \( d\pi(V) = d\pi(\bar{V}) \); hence, for any other vectors \( W \in T_wE \) with \( d\pi(V) = d\pi(W) \), and \( \bar{W} \in H_w \) with \( d\pi(W) = d\pi(\bar{W}) \), we have

\[
\theta_u(V) + \theta_w(W) = \theta_0(da(da(V - \bar{V}, 0_w), da(W - \bar{W}, 0_w))) = \theta_0(da(da(V, W) - da(\bar{V}, W), 0_w)) = \theta_u+w(da(V, W) - da(\bar{V}, \bar{W})) = \theta_u+w(da(V, W))
\]

where in the last line we used that \( da \) takes \( H_u \times_{T\pi} H_w \) to \( H_{u+w} = \ker(\theta_{u+w}) \) because \( H \) is linear.

This implies that the following definition is well given:

Definition 3.13. A linear Pfaffian bundle is a vector bundle \( E \to M \), together with a pointwise surjective linear form \( \theta \) or a linear distribution \( H \subset TE \).

It can be easily seen that vertical vector fields constant along the fibre of \( \pi \) commute. Writing any vector field tangent to \( g(H) \) (\( H \) linear) as a linear combination of such vectors tangent to \( g(H) \) and constant along the fibres, follows that \( g(H) \) is involutive. This, together with Remarks 3.10 and 3.11, implies:

Lemma 3.14. If \( (E, \theta) \) is a linear Pfaffian bundle then it is a Pfaffian bundle. Analogously for a linear Pfaffian bundle \( (E, H) \).

In this setting, linear forms and linear distributions can be encoded by a generalised version of linear connections, called relative connections.

Definition 3.15. A connection on the vector bundle \( \pi : E \to M \), relative to the pointwise surjective vector bundle map \( \sigma : E \to F \) over \( M \), is an \( \mathbb{R} \)-linear map

\[
D : \Gamma(E) \to \Omega^1(M; F),
\]

satisfying, for any section \( s \) and function \( f \in C^\infty(M) \), the Leibniz-type identity

\[
D(fs)(X) = fD(s)(X) + L_X(f)\sigma(s), \quad \forall X \in \mathfrak{X}(M)
\]  

(22)

We also say that \( (D, \sigma) \) is a relative connection and \( \sigma \) is its symbol map.

In particular, any linear form \( \theta \in \Omega^1(M, F) \) is fully encoded by the operator

\[
D : \Gamma(E) \to \Omega^1(M; F), \quad s \mapsto s^*\theta.
\]  

(23)

together with the vector bundle map \( \sigma : E \to F \), \( \sigma(v) = \theta(v) \). Indeed, we have the following:

Proposition 3.16. The above procedure induces a 1-1 correspondence between pointwise surjective linear 1-forms on the vector bundle \( \pi : E \to M \) and relative connections on \( \pi \).

Proof. The linearity of \( \theta \) is translated into the fact that \( D \) as in (23) is \( \mathbb{R} \)-linear and satisfies the Leibniz-type identity (22), where \( \sigma : E \to F \) is the vector bundle map over \( M \) defined by

\[
\sigma_x(u) = \theta_f(u)
\]

under the canonical identification \( T^*_F E = E_x, f \in E, x = \pi(f) \in M \). Conversely, if \( D \) is a connection relative to \( \sigma \), then \( s^*\theta = D(s) \) (for any \( s \in \Gamma(E) \)) and \( \theta(v) = \sigma(v) \) (for any \( v \in E = T^*_F E|_M \)) determines uniquely a well defined linear form. Q.E.D.
When there is no confusion, we denote a **linear Pfaffian bundle** by \((E, D)\). Of course, all definitions and properties can be translated from the point of view of linear forms to the one of relative connections and viceversa. Accordingly, we call

\[ g(D) := \ker(\sigma) \]

the **symbol space of** \( D \), we say that a section \( s \) is holonomic if \( D(s) = 0 \), and we denote by \( \Gamma(E, D) \) the set of **holonomic sections**. As in the case of linear distributions, the linearity of the form \( \theta \) associated to \( D \) implies that the natural identification between \( T^\pi E \) and the pullback \( \pi^*(E) \) restricts to the symbol spaces:

\[ g(\theta) \simeq \pi^*(g(D)). \tag{24} \]

**Remark 3.17. (Relative connections induced by linear distributions)** One can also describe directly the correspondence between linear distributions \( H \subset TE \) and relative connections (instead of getting it for free from the previous correspondence and Remark 3.10). Out of such an \( H \), one produces a connection

\[ D : \Gamma(E) \to \Omega^1(M; E/\mathfrak{g}), \]

relative to the projection \( \text{pr} : E \to E/\mathfrak{g} \), where \( \mathfrak{g} \subset E \) is the subbundle defined by

\[ \mathfrak{g} := (g(H))_M \subset (T^\pi E)_M \simeq E, \]

where we are identifying canonically \( T^\pi E \) with \( \pi^*E \).

This procedure gives a correspondence between linear distributions and connections relative to projections \( \text{pr} : E \to E/\mathfrak{g} \), with \( \mathfrak{g} \subset E \) any subbundle. Of course this can be thought as a generalisation of the well-known correspondence between linear connections \( \nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E) \), and transversal linear distributions, given by the horizontal distribution of \( \nabla \).

**Remark 3.18.** With the notation of the previous Remark 3.17, one can alternatively write the relative connection (23) associated to \( \theta \) as

\[ D_X(s)(x) = [\bar{s}, \bar{X}](x) \mod H \]

for \( X \in \mathfrak{X}(M) \), where \( \bar{X} \in \mathfrak{X}(E) \) is any \( \pi \)-projectable extension of \( X \), tangent to \( H \) and \( \bar{s} \) is the vertical vector field constant along the fibres induced by \( s \). Of course the above formula coincides with (23) when \( \theta_H \) is the canonical projection \( TE \to \pi^*F_H \).

**Remark 3.19 (Relative connections as Spencer operators).** Any vector bundle \( E \) can be thought as a Lie algebroid with zero bracket and zero anchor. The appropriate generalisation of relative connection in the world of algebroids is the notion of Spencer operators: these are relative connections compatible with the Lie bracket and the anchor; they play the infinitesimal counterpart of multiplicative distributions (see [4]). These compatibility conditions are trivially satisfied in the vector bundle case, hence they coincide in this case with our notion of relative connection.
In the linear case, the $k$-jet vector bundle $\pi : J^kE \to M$ together with the Cartan form $\omega$ is a linear Pfaffian bundle. The coefficient bundle of $\omega$ is $J^{k-1}E$ because we have the canonical identification $\text{pr}^*T^\pi (J^{k-1}E) \simeq \pi^*J^{k-1}E$, with $\text{pr} : J^kE \to J^{k-1}E$ the projection. This explains the reason why the Cartan form and the classical Spencer operator play the same role in the theory of linear PDEs: the classical Spencer operator $D : \Gamma (J^kE) \to \Omega^1 (M; J^{k-1}E)$ is just the connection relative to the projection $J^kE \to J^{k-1}E$, and defined by equation (23) via $\omega$:

$$D(s) = s^*\omega$$

In other words, the Cartan form on a linear jet space is fully encoded by the classical Spencer operator (see also sections 2.1 and 2.2).

### 3.3 Linearisation of Pfaffian bundles along holonomic sections

Also the process of linearisation fits nicely into the context of Pfaffian bundles. We talk about the linearisation of a Pfaffian bundle $\pi : (P, \theta) \to M$ along any holonomic section $\beta$. The outcome is a linear Pfaffian bundle for which the underlying vector bundle (over $M$) is $\text{Lin}_\beta (P, \theta) := \beta^*T^\pi P$ and the corresponding 1-form is with values in $\beta^*F$.

Geometrically, the situation is rather simple: while the holonomic sections of $(P, \theta)$ correspond to sections $\beta \in \Gamma (P)$ with $\beta^*\theta = 0$, we are looking at the zeroes of the holonomator (16) map: $\sigma \mapsto \sigma^*\theta$. Allowing ourselves to pass to infinite dimensional manifolds, recall that this map is itself a section

$$\Theta \in \Gamma (P, \mathcal{F})$$

where $\mathcal{P} = \Gamma (P)$ and $\mathcal{F}$ is the vector bundle over $\Gamma (P)$ whose fibre above $\beta$ is

$$\mathcal{F}_\beta := \Omega^1 (M, \beta^*N).$$

Solutions of $(P, \theta)$ are now realised as zeroes of $\Phi$, and the linearisation procedure becomes the usual linearisation of a section $\Theta$ at a zero $\beta$- which will be a linear map

$$d_\beta \Theta : T^\beta \mathcal{P} \to \mathcal{F}_\beta.$$

While, intuitively, $T^\beta \mathcal{P} = \Gamma (\beta^*T^\pi P)$, the linearisation becomes an operator

$$D^\beta := d_\beta \Theta : \Gamma (\beta^*T^\pi P) \to \Omega^1 (M, \beta^*F);$$

together with $l^\beta$ given by $\theta$ restricted to $T^\pi P$, we obtain a relative connection $(D^\beta, l^\beta)$ on $\beta^*T^\pi P$ with coefficients in $\beta^*F$. This is precisely the linearisation of $(P, \theta)$ along $\beta$, described in terms of relative connections.

Of course, the previous description is a bit problematic since it involves infinite dimensional manifolds. However, with the resulting intuition at hand, we can just copy the usual formulas from the finite dimensional case and obtain the following explicit description of $D^\beta$. Starting with $s \in \Gamma (\beta^*T^\pi P)$, choose a family $\beta_t$ (varying smoothly w.r.t. $t \in (-\epsilon, \epsilon)$) such that

$$\beta_0 = \beta, \quad \frac{d}{dt} \bigg|_{t=0} \beta_t (x) = s(x).$$
For $X_x \in T_xP$ we now consider the resulting curve

$$(-\epsilon, \epsilon) \ni t \mapsto \beta^*_t(\theta)(X_x) \in F_{\beta_t(x)}$$

which vanishes at 0. Hence we can linearise it at 0

$$\left. \frac{d}{dt} \right|_{t=0} \beta^*_t(\theta)(X_x) \in T_0 F \cong T_x M \oplus F_x$$

and then define

$$D^\beta_X(s)(x) = \text{pr}_{F_x} \left( \left. \frac{d}{dt} \right|_{t=0} (\beta_t)^*(\theta)(X_x) \right) \in F_x.$$ 

**Definition 3.20.** Let $(P, \theta)$ be a Pfaffian bundle over $M$ and $\beta \in \Gamma(P, \theta)$ a holonomic section, i.e. $\beta^* \theta = 0$. The **linearisation of $(P, \theta)$ along $\beta$** is the linear Pfaffian bundle

$$(\text{Lin}_\beta(P, \theta), D^3)$$

It is straightforward to check now that the intrinsic objects associated to Pfaffian bundles (tableaux, prolongations, etc) are compatible with the linearisation procedure. For instance, the symbol bundle of $\text{Lin}_\beta(P, \theta)$ coincides with the pull-back via $\beta$ of the symbol bundle of $(P, \theta)$,

$$g(\text{Lin}_\beta(P, \theta)) = \beta^* g.$$ 

**Remark 3.21 (Linearisation of a linear Pfaffian bundle).** When a Pfaffian bundle is already linear, the previous procedure becomes the identity when linearising along the zero section (of course, the zero section is always holonomic for a linear $\theta$).

Indeed, the linearisation of $(E, \theta)$ along the holonomic section $0$ recovers the vector bundle $E = E^0 = 0^*(T^n E)$ and the associated relative connection $D$ of $\theta$ as in (23): a section $s$ of $E$ can by written as

$$s = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} (0 + \epsilon s)$$

hence,

$$D^\theta(s) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} (0 + \epsilon s)^*(\theta) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \epsilon(s^*(\theta)) = s^*(\theta) = D(s)$$

where in the second equality we used again the linearity of $\theta$ to write $(0 + \epsilon s)^*(\theta) = 0^* \theta + \epsilon(s^* \theta) = \epsilon(s^* \theta)$. As $\theta$ and $D$ encode the same Pfaffian bundle (see observation 3.19), we see that linearising a linear Pfaffian bundle along the zero section does not do anything; we end up recovering the same linear Pfaffian bundle. ♦

**Remark 3.22 (Linearisation of a Pfaffian groupoid).** Intuitively, a Pfaffian groupoid is a Pfaffian bundle together with a multiplicative (group-like) structure; such multiplicativity translates into a richer geometrical content and simpler objects. Passing to the infinitesimal counterpart, we found Lie algebroids endowed with Spencer operators (see observation 3.19); the linearisation of a Pfaffian groupoid along its unit map coincides precisely with the Spencer operator associated to a multiplicative form as in [4]. ♦
4 Prolongations

The purpose of this section is to understand geometrically and intrinsically the notion of prolongation of a Pfaffian bundle and its fundamental properties. We start by exploring the type of morphisms between Pfaffian bundles which induce maps on the set of holonomic sections, and then move forward to study morphisms with more specific requirements. These extra conditions extract, in a sense, all the fundamental properties of the prolongations of a PDE (see section 2.3), in the same way that the conditions of a Pfaffian bundle extract the fundamental properties of the solutions of a PDE.

4.1 Morphisms and fibrations of Pfaffian bundles

Given two Pfaffian bundles over the same manifolds, the most natural notion of morphism between them should impose a relation between the two Pfaffian forms.

Definition 4.1. A Pfaffian morphism between two Pfaffian bundles $(P', \theta')$, $(P, \theta)$ over $M$, is a smooth fibre bundle map $\phi : P' \to P$ with the property that

$$\phi^*\theta = \Phi \circ \theta'$$

for some vector bundle map $\Phi : N' \to \phi^*N$ between the coefficient bundles; we note immediately that, since $\theta'$ and $\theta$ are surjective, $\Phi$ will be unique.

An example is given by a PDE $P \hookrightarrow J^kR$: in this case, the Cartan form $\theta_P = i^*\theta$ on $P$ is just the pullback of the Cartan form $\theta$ on $P$ by the injection $i$.

Similarly, if the PDE $P \subseteq J^kR$ is $(k + 1)$-integrable (see section 2), the projection

$$\text{pr} : (P^{(1)}, \theta^{(1)}) \to (P, \theta)$$

is a Pfaffian morphism, where $\theta^{(1)}$ is the restriction of the Cartan form of $J^{k+1}R$, and $\theta$ the restriction of the Cartan form of $J^kR$. In both these cases, the map $\Phi$ is the identity.

Remark 4.2. From the general definition above, it follows also that the morphism $\phi$ induced on the sections preserves the holonomic ones:

$$\phi : \Gamma_{loc}(P', \theta') \to \Gamma_{loc}(P, \theta)$$

Moreover, since $\pi' = \pi \circ \phi$, the differential $d\phi$ maps the symbol space $g(\theta')$ to $g(\theta)$. 

However, there are a number of reasons to add some constraints to the above definition of Pfaffian morphism. First, such notion does not behave well with respect to important objects associated to Pfaffian bundles, such as curvature or integral elements. Second, given $\phi : P' \to (P, \theta)$, we cannot always produce Pfaffian morphisms by endowing $P'$ with the form $\phi^*\theta$ (as we did before), since it might not be $\pi$-involutive, $\pi$-regular, or even surjective. Last, the definition revealed to be too weak for our further study of prolongations of Pfaffian bundles.

A first extra condition one could impose on $\phi$ is surjectivity. Indeed, under such assumption, the PDE-integrability of $P'$ implies the PDE-integrability of $P$: given any $p \in P$ we can consider a point $p' \in \phi^{-1}(p) \subseteq P'$, take a holonomic section $\sigma'$ around it and obtain the holonomic section $\sigma = \phi \circ \sigma'$ around $p$. 

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A second natural condition to ask is that $\phi$ is a submersion. This will make sure that the pullback $\phi^*\theta$ is pointwise surjective and becomes a Pfaffian form on $P'$: indeed, $\phi^*\theta$ is $\pi'$-regular because the maps $d_{\phi(p)}\phi : T_pP' \to T_{\phi(p)}P$ and $d_{\phi(p)}\pi : T_{\pi(p)}P \to T_{\pi(\phi(p))}M = T_{\pi'(p)}M$ are surjective and the diagram

$$
\begin{array}{c}
\text{ker}(\phi^*\theta)_p \\
\downarrow d\pi'
\end{array}
\begin{array}{c}
d\phi \\
\downarrow d\pi
\end{array}
\begin{array}{c}
\text{ker}(\theta)_{\phi(p)} \\
T_{\pi'(p)}M
\end{array}
$$

commutes, hence $d\pi'$ is surjective as well (we do not need to ask it at the beginning). To prove the involutivity of $g(\phi^*\theta)$, we recall that $d\phi$ maps $g(\phi^*\theta)$ to $g(\theta)$, and then see that the curvature maps (18) are related by the equation

$$c_{\phi^*\theta} = \Phi \circ c_{\theta}$$

The last part can be checked using $\phi$-projectable vector fields (see Proposition 4.4); hence, for any two vector fields $X, Y$ tangent to $g(\phi^*\theta)$, we have

$$c_{\phi^*\theta}(X, Y) = c_{\theta}(d\phi(X), d\phi(Y)) = 0$$

because $g(\theta)$ is involutive. This says on one hand that the bracket $[X, Y]$ belongs to ker($\phi^*\theta$); on the other hand, since $T\pi'P'$ is involutive, that the bracket $[X, Y]$ is also tangent to $T\pi'P'$, hence to $g(\phi^*\theta)$, proving that $\phi^*\theta$ is $\pi'$-involutive and Pfaffian.

In conclusion, we can define

**Definition 4.3.** A Pfaffian fibration between two Pfaffian bundles $(P', \theta')$, $(P, \theta)$ over $M$, is a surjective submersion $\pi : P' \to P$ which is also a Pfaffian morphism.

The trivial example of a Pfaffian fibration is simply given by a surjective submersive bundle map $\phi$ between the fibration $\pi : P \to M$ and the map $\pi' : P' \to M$, where we endow $P$ with a Pfaffian form $\theta$ and $P'$ with the pullback Pfaffian structure $\theta' = \phi^*\theta$ described above; in this case, $\Phi$ is just the identity.

The PDE $P \rightarrow J^kR$ is not a Pfaffian fibration, since $i$ is not a surjective submersion, whereas the prolongation $pr : (P^{(1)}, \theta^{(1)}) \to (P, \theta)$ is. In fact, this projection is not only a Pfaffian fibration but it has a richer geometrical structure, which are manifested in the properties of a normalised prolongation (see Definition 4.6 and Proposition 4.22).

Let us recap all the properties of Pfaffian fibrations.

**Proposition 4.4.** Given the Pfaffian fibration $\phi : (P', \theta') \to (P, \theta)$,

- $\phi$ sends holonomic sections of $(P', \theta')$ to holonomic sections of $(P, \theta)$
- $d\phi$ sends the symbol space $g(\theta')$ to the symbol space $g(\theta)$
- If $(P, \theta')$ is PDE-integrable, $(P, \theta)$ is PDE-integrable
- The curvature maps are related by the equation

$$\phi^*c_{\theta} = \Phi \circ c_{\theta'}$$

(28)
• \( \phi \) sends (partial) integral elements of \((P', \theta')\) to (partial) integral elements \((P, \theta)\)

**Proof.** We have only to prove the last two statements. First we use equations (27) and (25) to conclude that \( \phi^*c_\theta = c_{\Phi(\theta')} \); then we use two linear connections \( \nabla' \) and \( \nabla \), respectively on the coefficient bundles \( N' \) and \( N \), to show that \( d\phi_*(\Phi(\theta')) = \Phi(d\gamma'(\theta')) \); last we argue that the restrictions of \( c_{\Phi(\theta')} \) and \( \Phi(c_{\theta'}) \) to \( \ker(\theta') \) coincide (see the discussion after equation (18)).

The relations (25) and (28) imply also that \( \phi \) preserves (partial) integral elements (definition 3.9). Q.E.D.

**Remark 4.5. (Pfaffian morphisms between Pfaffian distributions)** Paraphrasing the above discussion in the language of Pfaffian distributions \( H' \subset TP' \), \( H \subset TP \), one obtains the corresponding conditions of Pfaffian morphisms only in terms of the distributions, when applied to the associated forms \( \theta = \theta_H \) and \( \theta' = \theta_{H'} \). First of all, (25) corresponds to

\[
d\phi(H') \subset H.
\]

The map \( \Phi : TP'/H' \to \phi^*TP/H \) is forced to be \([u] \mapsto [d\phi(u)]\) (which is well defined by (29)); in this case we denote \( \Phi \) by \([d\phi]\). Hence, in this setting, a Pfaffian morphism is a fibre bundle map \( \phi : P' \to P \) satisfying (29); as in (26), \( \phi \) preserves holonomic sections.

A Pfaffian morphism \( \phi \) is called a Pfaffian fibration when it is also a surjective submersion. Again, such condition will imply an equation on the curvatures analogous to (28):

\[
\phi^*c_H = [d\phi] \circ c_{H'}.
\]

Moreover, as in proposition 4.4, it sends (partial) integral elements to (partial) integral elements and PDE-integrability of \((P', H')\) would imply integrability of \((P, H)\).

\[
\Diamond
\]

### 4.2 Abstract prolongations

Going back to the definition of prolongation of a PDE (seen as a Pfaffian fibration) one finds that, for a PDE \( P \subset J^kR \) integrable up to order \( k + 1 \), the projection \( \text{pr} \) from the prolongation \( P^{(1)} \) to \( P \) maps the kernel of \( \theta^{(1)} \) at a given point \( p \) to a single integral element of \((P, \theta)\) at \( \text{pr}(p) \). The reason is the following: first, the image of \( \ker(\theta^{(1)}_p) \) is horizontal because \( \theta^{(1)} \) is \( \pi \)-transversal, \( \text{pr} \) is a bundle map over \( M \), and most importantly,

\[
\ker(d\text{pr}) = g(\theta^{(1)}_p).
\]

So, for every \( u, u' \in \theta^{(1)}_p \) projecting via \( d\pi \) to \( X \in T_{\pi(p)}M \), the difference \( u - u' \) lies in \( g(\theta^{(1)}_p) \), hence \( d\text{pr}(u) = d\text{pr}(u') \). Second, \( d\text{pr}(\ker(\theta^{(1)}_p)) \) is actually an integral element because for any \( u, v \in \ker(\theta^{(1)}), \)

\[
c_\theta(d\text{pr}(u), d\text{pr}(v)) = 0.
\]

Equation (30) can be proven directly by the definition of the symbol space of \( \theta^{(1)}_p \), while equation (31) is part of Proposition 4.22. In fact, the Definition of prolongation is so that this happens: partial integral elements based at a given point are mapped to a single integral element.

**Definition 4.6.** An (abstract) prolongation \( \phi : (P', \theta') \to (P, \theta) \) of a Pfaffian bundle \((P, \theta)\) consists of a Pfaffian bundle \((P', \theta')\) together with a Pfaffian fibration \( \phi : P' \to P \), such that

\[
g(\theta') \subset \ker(d\phi)
\]
and such that for any \( u, v \in \ker(\theta') \),

\[
c_{\theta}(d\phi(u), d\phi(v)) = 0. \tag{33}
\]

We say that \( \phi : P' \to P \) is a **normalised prolongation** if \( g(\theta') = \ker(d\phi) \).

Again, as in Remark 4.5, Definition 4.6 can be reformulated using distributions instead of forms: we say that \( \phi : (P', H') \to (P, H) \) is a Pfaffian prolongation if it is a Pfaffian fibration (i.e. \( d\phi(H') \subset H \)) and

\[
g(H') \subset \ker(d\phi), \text{ and } c_H(d\phi(u), d\phi(v)) = 0 \tag{34}
\]

for all \( u, v \in H' \).

The prolongation \( \phi \) is normalised when

\[
g(H') = \ker(d\phi). \tag{35}
\]

**Remark 4.7 (Cartan-Ehresmann connections [23])**. Continuing with the discussion previous to Definition 4.6, we see that Pfaffian bundles \((P, \theta)\) that have Pfaffian prolongation \( \phi : P' \to P \), with sections \( \sigma : P \to P' \) (of \( \phi \)), admit \( \sigma \)-dependent horizontal (w.r.t. \( \pi \)) subdistributions \( H_\sigma \subset \ker(\theta) \), made of integral elements of \( \theta \). We call such a distribution a **Cartan-Ehresmann connection** of \((P, H)\). At \( p \in P \),

\[
H_{\sigma, p} = d_{\sigma(p)} \phi(\ker(\theta')).
\]

That \( H_{\sigma, p} \) is a horizontal vector space inside \( \ker(\theta) \) is because equation \( (32) \) holds, and that the curvature on \( H_{\sigma, p} \) is zero is of course implied by equation \( (33) \) ♦

**Remark 4.8. (Alternative definition of prolongation)** Because \( \phi \) is a Pfaffian fibration, the equation \( (28) \) involving the curvatures \( \phi^* c_{\theta} = \Phi \circ c_{\theta'} \) holds, hence we can replace condition \( (33) \), for the equivalent one:

\[
\Phi(c_{\theta'}(u, v)) = 0, \quad \forall u, v \in \ker(\theta^{(1)}).
\]

In terms of distributions it is instead

\[
[d\phi](c_{H'}(u, v)) = 0
\]

where \([d\phi] : TP'/H' \to \phi^*(TP/H)\) is the induced map on the quotient. ♦

In the picture using distributions, the name **normalised** has a natural explanation:

**Lemma 4.9.** The Pfaffian prolongation \( \phi : (P', H') \to (P, H) \) is normalised if and only if its differential \( d\phi \) descends to an isomorphism between \( TP'/H' \) and the pullback via \( \phi \) of \( T\pi P' \):

\[
T_pP'/H'_p \simeq T_{\phi(p)}^\pi P, \quad [u] \mapsto d\phi(u - v), \tag{36}
\]

where \( v \in H'_p \) is any vector with the property that \( d\pi'(u) = d\pi'(v) \).
The map (36) comes from composing the maps (38) and (37) below: the transversality of \( H \) implies that its normal bundle is isomorphic to \( T\pi' P'/H^{\pi'} \) (we denote \( H^{\pi'} = (H')^{\pi'} \)):

\[
TP'/H' \simeq T\pi' P'/H^{\pi'}, \quad [u] \mapsto [u - v],
\]

(37)

where \( v \in H \) is as in the Lemma 4.9. On the other hand, \( d\phi(g(H')) = 0 \) implies that map \( d\phi \) induces a well defined surjective map

\[
T\pi' P'/H^{\pi'} \to T\pi P, \quad [w] \mapsto d\phi(w).
\]

(38)

Hence, only the conditions for the prolongation imply that the map (38) is well defined and surjective; the injectivity is precisely condition (35). This also suggests that if \( \phi \) is not normalised but we want to make it normalised, we could “fatten” \( H' \) by \( \ker(d\phi) \subset T\pi' P' \) to a new distribution

\[
\bar{H'} := H' + \ker(d\phi),
\]

(39)

so that (38) becomes injective when we replace \( H^{\pi'} \) by \( \bar{H'}^{\pi'} = \ker(d\phi) \).

**Proposition 4.10.** For a prolongation \( \phi : (P', H') \to (P, H) \) the new Pfaffian bundle \( (P', \bar{H'}) \) makes the surjective submersive bundle map \( \phi : P' \to P \), a normalised prolongation.

In the previous lemma we called \( \bar{H'} \) as in (39) the **canonical normalised prolongation**. That \( \bar{H'} \) indeed has constant rank follows from dimension counting:

\[
\text{rk}(\bar{H'}) = \text{rk}(H') + \text{rk}(\ker(d\phi)) - \text{rk}(H' \cap \ker(d\phi)) = \text{rk}(H') + \text{rk}(\ker(d\phi)) - \text{rk}(g(H')),
\]

were we used that \( g(H') = H' \cap \ker(d\pi') = H' \cap \ker(d\phi) \), because on the one hand as \( \phi \) is a bundle morphism, \( \ker(d\phi) \subset \ker(d\pi') \), and on the other hand the first condition for the prolongation \( \phi \) is the inclusion \( g(H') \subset H' \cap \ker(d\phi) \). That \( \bar{H'} \) is transversal follows from the transversality of \( H' \subset \bar{H'} \), and the \( \pi' \)-involutivity is just the involutivity of \( \ker(d\phi) \).

**Remark 4.11.** If we look at normalised prolongations in terms of the 1-forms, we have various identifications that put us in the following specific case. Lemma 4.9 identifies the quotient \( TP'/\ker(\theta') \) with the pullback of \( T\pi P \) via \( \phi \) on the one hand, and \( \theta' \) identifies this quotient with its coefficient bundle \( N' \); hence, we can think that the coefficient bundle is \( T\pi P \):

\[
N' = \phi^* (T\pi P).
\]

Moreover, under this identification, the maps on the vertical bundles \( d\phi : T\pi' P' \to T\pi P \) and \( \theta' : T\pi' P' \to N' \) are the same, so we are left in the situation where a prolongation \( \phi : (P', \theta') \to (P, \theta) \) is normalised if \( \theta' \) takes values on \( T\pi P \):

\[
\theta' \in \Omega^1(P'; \phi^* (T\pi P)),
\]

and the differential \( d\phi \) coincides with \( \theta' \) on vertical tangent vectors \( T\pi' P' \). The remaining conditions for a prolongations of course remain the same, namely

\[
\phi^* \theta = \theta \circ \theta', \quad \text{and} \quad \theta(c_{\theta'}(u, v)) = c_\theta(d\phi(u), d\phi(v)) = 0
\]

for all \( u, v \in \ker(\theta') \) \( \diamond \)
4.3 The partial prolongation

To simplify the exposition, we will tell the whole story using distributions; at the end, we will make the appropriate comments about how this picture is adapted using 1-forms.

The classical prolongation of a Pfaffian bundle \((P, H)\) may be thought as the space of first order consequences of the Pfaffian bundle, in analogy with the notion of prolongation of a PDE (section 2.3). As such, the classical prolongation consists of integral elements of \((P, H)\) (see equation (17)), i.e. linear subspaces \(V \subset T_p P\) which projects isomorphically to \(T_{\pi(p)} M\) via \(d\pi\), and satisfying the two conditions (17). Identifying \(V\) with the image of a linear splitting \(\zeta : T_{\pi(p)} M \to H_p \subset T_p P\) (think of \(\zeta\) as the differential \(d_x \beta\) for some \(\beta \in \Gamma_{\text{loc}}(P)\)), the two conditions are

\[
\text{Im}(\zeta) \subset H_p, \quad \zeta^*(e_H) = 0.
\]

The partial prolongation of \((P, H)\) takes care of the first condition, and identifying jets \(j^1_x \beta\), \(\beta \in \Gamma_{\text{loc}}(P)\), with the differential of \(\beta\) at \(x\), the partial prolongation can be thought as the biggest submanifold of \(J^1 P\) with the property that when endowed with the restriction of the Cartan distribution, the projection is a Pfaffian fibration (Theorem 4.13). With this we have:

**Definition 4.12.** The partial prolongation of \((P, H)\), denoted by \(J^1_H P\), is the subset of \(J^1 P\) defined by

\[
J^1_H P := \{(p, \zeta) \mid p \in P, \quad \zeta : T_{\pi(p)} M \to H_p \subset T_p P \text{ linear, } d\pi \circ \zeta = \text{id}\}
\]

Of course the classical prolongation sits inside \(J^1_H P\), hence, many of its properties are inherited from \(J^1_H P\). Both the partial and the classical prolongation can be seen as universal, the first in the world of Pfaffian fibration (Proposition 4.20), and the second in the world of Pfaffian prolongations (Proposition 4.16). In this section we study the structure of \(J^1_H P\), as well as its main properties.

Let us start by explaining the underlying smooth structure of \(J^1_H P\). First of all, recall that, in general, \(pr : J^1 R \to R\) is an affine bundle over \(R\); this is immediately clear if one represents, as explained above, the points of \(J^1 P\) as pairs \((p, \xi)\) with \(p \in P\) and \(\xi : T_x M \to T_p P\) splitting of \((d\pi)\), where \(x = \pi(p)\). We see that each two points in the same fibre above \(p \in P\), \((p, \xi)\) and \((p, \xi')\), differ by

\[
\xi' - \xi : T_x M \to T_{\pi(p)} P,
\]

which can be arbitrary, hence \(pr : J^1 P \to P\) is an affine bundle with underlying vector bundle \(\text{Hom}(\pi^* T M, T^\pi P)\). We remark that for \(J^1_H P\) is the kernel of the horizontalisation map

\[
e : J^1 P \to \text{Hom}(\pi^* T M; N_H), \quad e(j^1_x \beta) : v \mapsto d_x \beta(v) \mod H_{\beta(x)}
\]

and that \(e\) is an affine map with the underlying vector bundle map

\[
\overline{\nu} : \text{Hom}(\pi^* T M, T^\pi P) \to \text{Hom}(\pi^* T M, N_H), \quad \xi \mapsto \xi \mod H.
\]

Since \(H\) is \(\pi\)-transversal and therefore \(pr : T^\pi P \to T \pi P/H^\pi = N_H, v \mapsto v \mod H\) is surjective, it follows that \(pr : J^1_H P \to P\) is an affine bundle with underlying vector bundle

\[
\ker(\overline{\nu}) = \text{Hom}(\pi^* T M, g(H)).
\]
Theorem 4.13. The partial prolongation $J^1_H P$ of a Pfaffian bundle $(P, H)$ is the largest subbundle of $J^1 R$ such that, when endowed with the restriction of the Cartan distribution

$$H^{(1)} := \mathcal{C} \cap T J^1_H P,$$

the restriction of the projection $pr : J^1_H P \to P$ becomes a Pfaffian fibration. Moreover, $pr : J^1_H P \to P$ is an affine bundle modelled on $\text{Hom}(\pi^* TM, g(H))$.

Proof. To see that $H^{(1)}$ is $\pi$-transversal, we compute its vertical part $H^{(1)} \cap T^\pi J^1_H P$, which is the same as the kernel of the Cartan form $\omega$ when restricted to $T^\pi J^1_H P$. From the explicit definition (1) of $\omega$, we see that the Cartan form when restricted to $T^\pi J^1_H P$ is precisely $dpr : T^\pi J^1_H P \to T^\pi P$; but the kernel of $dpr$ is the first term of the exact sequence over $J^1_H P$,

$$0 \to g(H^{(1)}) = \text{Hom}(\pi^* TM; pr^*(g(H))) \to T^\pi J^1_H P \xrightarrow{dpr} pr^*(T^\pi P) \to 0,$$  \hspace{1cm} (42)

where this sequence comes from restricting

$$0 \to \text{Hom}(\pi^* TM; pr^*(T^\pi P)) \to T^\pi J^1_H P \xrightarrow{dpr} pr^*(T^\pi P) \to 0.$$  \hspace{1cm} (43)

to $T J^1_H P$. This also shows that as $dpr : T^\pi J^1_H P \to T^\pi P$ is pointwise surjective, $\omega$ on $T J^1_H P \subset T^\pi J^1_H P$ is surjective as well; hence $H^{(1)} = \ker(\omega|_{T J^1_H P})$ is a distribution and

$$\text{rk}(H^{(1)}) = \text{rk}(T J^1_H P) - \text{rk}(T^\pi P).$$  \hspace{1cm} (44)

The $\pi$-transversality follows from dimension counting using (42) and (44):

$$\text{rk}(H^{(1)} + T^\pi J^1_H P) = \text{rk}(H^{(1)}) + \text{rk}(T^\pi J^1_H P) - \text{rk}(\text{Hom}(\pi^* TM; g(H))) = \text{rk}(T J^1_H P).$$

The involutivity of the vertical part of $H^{(1)}$ is immediate as it is the intersection of the tangent space of a submanifold with the involutive distribution $\mathcal{C}^\pi$.

We now see that $J^1_H P$ is the biggest submanifold of $J^1 P$ so that $pr$ becomes a Pfaffian fibration. Indeed, a vector $v \in T_{j^{1, \beta}} J^1 P$ belongs to the Cartan distribution if

$$0 = \omega(v) = dpr(v) - d_x \beta(d\pi(v)).$$

Hence, $dpr(\mathcal{C}^{j^{1, \beta}})$ sits inside $H_{\beta(x)}$ if $d_x \beta(T_x M) \subset H_{\beta(x)}$, i.e. $j^{1, \beta}$ belongs to the partial prolongation $J^1_H P$.

Q.E.D.

Remark 4.14. From the proof we see that the symbol space $g(H^{(1)})$ of the partial prolongation is precisely the kernel of the differential of the projection $pr : J^1_H P \to P$,

$$g(H^{(1)}) = \ker(dpr) = \text{Hom}(\pi^* TM; pr^*(g(H))).$$

This condition is shared with normalised prolongations and means that we have an isomorphism for each $p \in J^1_H P$,

$$T_p J^1_H P / H^{(1)}_p \simeq T^\pi_{pr(p)} P, \quad [u] \mapsto dpr(u - v),$$

where $v \in H^{(1)}_p$ is any vector with $d\pi(u) = d\pi(v)$; see Lemma 4.9.  \hspace{1cm} \Diamond
Remark 4.15. Being a Pfaffian fibration, the projection $\text{pr} : J^1_H P \to P$ induces a map between holonomic sections
\[ \Gamma(J^1_H P, H^{(1)}) \to \Gamma(P, H), \quad \xi \mapsto \text{pr}^*(\xi). \]

In fact, this map defines a 1-1 correspondence with inverse given by $\Gamma(P, H) \ni \beta \mapsto j^1 \beta$. In principle, the first jet of $\beta$ is a section of $J^1 P$ tangent to the Cartan distribution $\mathcal{C}$ (see lemma 2.1), but because $\beta$ is holonomic, $d_\beta(T_x M)$ is a subset of $H_{\beta(x)}$ for all $x \in \text{dom}(\beta)$, i.e. $j^1 \beta$ is actually a section of the partial prolongation $J^1_H P$, tangent to $H^{(1)}$ (equation (41)).

Another possible characterisation of the partial prolongation is that it is “universal” among the world of Pfaffian fibration with target $(P, H)$. More precisely,

**Proposition 4.16.** Any Pfaffian fibration $\phi : (P', H') \to (P, H)$ with the property that $\mathfrak{g}(H') \subset \ker(d\phi)$ factors through a unique bundle morphism $\varphi : P' \to J^1_H P$ over $P$ so that
\[ d\varphi(H') \subset H^{(1)} \quad \text{and} \quad [\text{dpr}] \circ \varphi^* c_{H^{(1)}} = [d\phi] \circ c_{H'}, \]

where $[d\text{pr}] : N_{H^{(1)}} \to \varphi^* N_H, \ [u] \mapsto [\text{dpr}(u)]$ and $[d\phi] : N_{H'} \to \phi^* N_H, \ [u] \mapsto [d\phi(u)]$ are the induced maps on the normal bundles.

**Proof.** The condition $d\varphi(H') \subset H^{(1)}$ forces the definition of $\varphi$ to be as follows: for $u \in H'_p$, $d\varphi(v)$ is an element of $H^{(1)}_{\varphi(p)}$. This means that for $j^1_{\pi'(p)} \beta = \varphi(p)$,
\[ 0 = \text{dpr}(d\varphi(v)) - d_{\pi'(p)} \beta(d\pi(d\varphi(v))) = d\phi(v) - d_{\pi'(p)} \beta(d\pi'(v)), \]

where in the second equality we are using that $\varphi$ is a bundle map over $P$ (and hence, over $M$), thus $\text{pr} \circ \varphi = \phi$ and $\pi \circ \varphi = \pi'$. This defines uniquely $\varphi(p)$ as the linear splitting $\varphi(p) : T_{\pi'(p)} H' \to H_{\phi(p)}$ of $d\pi$ given by $X \mapsto d\phi(v)$, where $v$ is any vector tangent to $H'_p$ with the property that $d\pi'(v) = X$. Of course, we still need to check that $\varphi$ is indeed well-defined, but this is a direct consequence of $\mathfrak{g}(H') \subset \ker(d\phi)$, as one can see easily.

Showing the equality involving the curvatures is a direct consequence of the relations between the curvatures, respectively, of the Pfaffian fibrations $\phi$ and $\text{pr}$ (Remark 4.5). We already have that
\[ \phi^* c_H = [d\phi] \circ c_{H'}, \quad \text{and} \quad \text{pr}^* c_H = [d\text{pr}] \circ c_{H^{(1)}}, \]

and we apply then $\varphi^*$ to the second equation and use $\text{pr} \circ \varphi = \phi$ to substitute in the first equation. Q.E.D.

### 4.4 The classical prolongation

Recall that the classical prolongation of a Pfaffian bundle $(P, H)$ may be thought as the space of first order consequences of the Pfaffian bundle, in analogy with the notion of prolongation of a PDE. It is defined as the set of integral elements of $(P, H)$ (see equation (17)), and hence it sits inside the partial prolongation $J^1_H P \subset J^1 P$ as the subset of the partial prolongation where the condition on the curvature (34) for prolongations
\[ \text{pr}^* c_H = 0 \]
holds. Indeed, if $j^1_{\beta}$ is an element of $J^1_H P$ satisfying the condition that for any $u, v \in H^{(1)}_{j^1 \beta}$, $c_H(d\text{pr}(u), \text{pr}(v)) = 0$, then this says that
\[ c_H(d_x \beta(d\pi(u)), d_x \beta(d\pi(v))) = 0 \]
because \( dpr(u) - d\pi_{\beta}(d\pi(u)) = 0 \) (i.e. \( u \in H^{(1)}_J \)), and analogously for \( v \). This is exactly saying that \( J^1_{\beta} \) is an integral element.

**Definition 4.17.** The classical prolongation of \((P, H)\), denoted by \( \text{Prol}(P, H) \), is the subset of \( J^1_H P \) defined by

\[
\text{Prol}(P, H) = \{(p, \zeta) \mid p \in P, \zeta : T_{\pi(p)}M \rightarrow H_p \subset T_pP \text{ linear; } d\pi \circ \zeta = \text{id}, \zeta^*(c_H) = 0\} \quad (45)
\]

Whenever smooth, and the projection \( pr : \text{Prol}(P, H) \rightarrow P \) surjective, the classical prolongation is characterised by the property that it is the biggest submanifold of the first jet bundle \( J^1_P \) with the property that when endowed with the restriction of the Cartan distribution \( \mathcal{C} \), the projection \( pr \) becomes a Pfaffian normalised prolongation (see Theorem 4.18).

However, studying its underlying structure is a bit more subtle. Similarly as in the case of the partial prolongation, the classical prolongation is the zero-set of the 1\(^{\text{st}}\) curvature map

\[
c_1 : J^1_H P \rightarrow \text{Hom}(\pi^* \wedge^2 TM, N_H), \quad (p, \zeta) \mapsto \zeta^* c_H \quad (46)
\]

Hence, the smoothness of \( \text{Prol}(P, H) \) can be study by understanding \( c_1 \). First, \( c_1 \) is an affine map and a simple computation reveals that the underlying vector bundle morphism is precisely the map

\[
\delta_H : \text{Hom}(\pi^* TM; g(H)) \rightarrow \text{Hom}(\pi^*(\wedge^2 TM); N_H) \quad (47)
\]

\[
\delta_H(\eta_p)(X, Y) = \partial_H(\eta_p)(X)(Y) - \partial_H(\eta_p)(Y)(X),
\]

where \( \partial_H \), called the symbol map, is given by

\[
\partial_H : g(H) \rightarrow \text{Hom}(\pi^* (TM); N_H), \quad \partial_H(v)(Y) = c_H(v, \hat{Y}) \quad (48)
\]

with \( \hat{Y} \) any vector tangent to \( H_p \) that projects to \( Y \): \( d\pi(\hat{Y}) = Y \); as one can check, it is well-defined because \( g(H) \) is involutive. We deduce that \( \text{Prol}(P, H) \) is a smooth affine sub-bundle of \( J^1_P \) if and only if:

- \( \delta_H \) has constant rank.
- \( pr : \text{Prol}(P, H) \rightarrow P \) is surjective.

Related to the first point, we see that the kernel of \( \delta \), called the prolongation of the symbol space (w.r.t. \( \partial \))

\[
g(H)^{(1)} := \{ \eta : T_{\pi(p)}M \rightarrow g(H) \mid p \in P, \partial_H(\eta)(X)(Y) = \partial_H(\eta)(Y)(X) \forall X, Y \in T_{\pi(p)}M \} \quad (49)
\]

is a bundle of vector spaces \( g(H)^{(1)} \subset \text{Hom}(\pi^*(TM); g(H)) \) whose rank may vary, and it is just the 1st prolongation of the tableau \( \partial_H : g(H) \rightarrow \text{Hom}(\pi^*(TM); N_H) \) in the sense of equation (13). Of course, \( \delta \) has constant rank if and only if \( g(H)^{(1)} \) is of constant rank. Now, related to the last point, the previous discussion also implies that \( c_1 \) descends to a map, called the higher curvature map of \((P, H)\):

\[
\kappa : P \rightarrow \text{Hom}(\pi^*(\wedge^2 TM); N_H)/\text{Im}(\delta_H), \quad p \mapsto [\zeta^*(c_H) = c_1(p, \zeta)], \quad (50)
\]

It is now a simple exercise to check that the zero-set of \( \kappa \) is precisely the image of \( pr : \text{Prol}(P, H) \rightarrow P \). In particular:
Theorem 4.18. For any Pfaffian bundle \( \pi : (P, H) \to M \), the following are equivalent:

1. The prolongation \( \text{Prol}(P, H) \) is a smooth affine sub-bundle of \( J^1 R \).

2. The prolongation \( g(H)^{(1)} \) of \( g(H) \) is of constant rank, and \( \kappa = 0 \) (or, equivalently, \( \text{pr} : \text{Prol}(P, H) \to P \) is surjective).

Moreover, in this case:

- the vector bundle underlying the affine bundle \( \text{Prol}(P, H) \) is precisely \( g(H)^{(1)} \).

- the restriction of the Cartan distribution \( \mathcal{C} \) of \( J^1 P \) to \( \text{Prol}(P, H) \)

\[
H^{(1)} := \mathcal{C} \cap T \text{Prol}(P, H),
\]

(51)

and, \( (\text{Prol}(P, H), H^{(1)}) \) becomes a Pfaffian bundle over \( M \) with symbol space \( \text{pr}^* g(H)^{(1)} \subset \text{Hom}(\pi^* TM, \text{pr}^* g) \).

- the projection from \( (\text{Prol}(P, H), H^{(1)}) \) to \( (P, H) \) is a normalised prolongation.

Proof. From the discussion previous to the theorem we know that the first three items are equivalent.

Checking that \( H^{(1)} \) as in (51) is a Pfaffian distribution is completely analogous to the proof given for the partial prolongation. Also, \( \ker(dpr) \) when restricted to the vertical tangent of the classical prolongation \( T^\pi \text{Prol}(P, H) \) can be computed to be \( g(H)^{(1)} \): we know that \( g(H)^{(1)} \subset \text{Hom}(\pi^* TM; g(H)) \) is the vector bundle that models the affine bundle \( \text{pr} : \text{Prol}(P, H) \to P \), and hence it can be computed as the kernel of

\[
dpr : T^\pi \text{Prol}(P, H) \to T^\pi P
\]

(see sequences (42) and (43)). On the other hand,

\[
g(H^{(1)}) = \ker(dpr : T^\pi \text{Prol}(P, H) \to T^\pi P)
\]

because by the very definition of \( H^{(1)} \) as the kernel of the Cartan form \( \omega \) when restricted to \( \text{Prol}(P, H) \). Hence, \( g(H^{(1)}) = \text{pr}^* g(H)^{(1)} \).

To conclude that \( \text{pr} : (\text{Prol}(P, H), H^{(1)}) \to (P, H) \) is a normalised prolongation, we note that as the projection from \( (J^1_H P, H^{(1)}) \) to \( (P, H) \) is a Pfaffian fibration and \( \text{Prol}(P, H) \) is a subbundle of \( J^1_H P \), the only thing left to see is that \( \text{pr}^* c_H = 0 \), which holds by construction of \( \text{Prol}(P, H) \) (see the discussion previous to the Definition 4.17) Q.E.D.

Remark 4.19. The same observation as 4.15 goes here. In fact, one of the main properties of the classical prolongation is that whenever \( \text{pr} : \text{Prol}(P, H) \to P \) is a prolongation (i.e. it is a bundle map), then there is 1-1 correspondence between holonomic sections

\[
\Gamma(\text{Prol}(P, H), H^{(1)}) \to \Gamma(P, H), \quad \xi \mapsto \text{pr}^* (\xi),
\]

with inverse \( \Gamma(P, H) \ni \beta \mapsto j^1 \beta \). In remark 4.15 we saw that \( j^1 \beta \) takes values on the partial prolongation; here we observe that, as \( [d\beta(X), d\beta(Y)]_{\beta(x)} \) coincides with \( d\beta([X, Y]_x) \) for any \( X, Y \in \mathfrak{x}(M) \), and \( x \in \text{dom}(\beta) \), then \( j^1 \beta \) lies in \( \text{Prol}(P, H) \). ♦

Again, the classical prolongation can be thought as “universal” among prolongations. Let us assume that \( \text{pr} : \text{Prol}(P, H) \to P \) is a (smooth) bundle map.
Proposition 4.20. Any Pfaffian prolongation \( \phi : (P', H') \to (P, H) \) factors through a unique bundle map \( \varphi : P' \to \text{Prol}(P, H) \) over \( P \), so that

\[
d\varphi(H') \subset H^{(1)}, \quad \text{and} \quad [d\varphi] \circ \varphi^*c_{H^{(1)}} - [d\phi] \circ c_{H^1} = 0,
\]

where \([d\varphi] : \mathcal{N}_{H^{(1)}} \to \varphi^*\mathcal{N}_H, [u] \to [d\varphi(u)]\), and \([d\phi] : \mathcal{N}_{H^1} \to \phi^*\mathcal{N}_H, [u] \to [d\phi(u)]\), are the induced maps on the normal bundles.

Remark 4.21. Actually the above proposition can be stated in a slightly greater generality. Even if \( \text{Prol}(P, H) \) is not smooth, any prolongation factors through the map \( \varphi : P' \to J^1_H P \) given in Proposition 4.16. We can slightly modify the above statement by saying that this map takes values in the subset \( \text{Prol}(P, H) \), and that the relations with the distributions, and the curvatures hold when we take \( H^{(1)} \) as the Pfaffian distribution (41) of \( J^1_H P \).

As a consequence we obtain that when \( (P, H) \) admits a prolongation then the projection \( \text{pr} : \text{Prol}(P, H) \to P \) is surjective. We will give the proof of the above proposition without the smoothness assumption.

Proof. We let \( \varphi : P' \to J^1_H P \) defined as in the proof of Proposition 4.16, and we show that it takes values in \( \text{Prol}(P, H) \). A closer look to \( \varphi(p) : T^x\pi(p)P \to H_p \), shows that its image \( \varphi(p)(T^x\pi(p)P) \) coincides with \( d\phi(H_p) \), because \( d\phi(g(H')) = 0 \). As we remarked in the discussion above the definition 4.6 of prolongation, the conditions that \( \phi \) satisfy imply that \( d\phi(H_p) \) is an integral element, hence \( \varphi(p) \) belongs to \( \text{Prol}(P, H) \).

The left hand side condition (52) for the distributions is immediately implied by the same condition in Proposition 4.16 for the partial prolongation, and the right hand side condition (52) also follows from the commutativity of the curvatures in the same proposition taking into account that on \( J^1_H P \), \( \text{pr}^*c_H = [d\varphi] \circ c_{H^{(1)}} \) is zero at points of \( \text{Prol}(P, H) \), and that \( \phi \) satisfies \( \varphi^*c_H = [d\phi] \circ c_{H^1} = 0 \).

Q.E.D.

Again, the motivating and inspiring example comes from the classical definition (5) of prolongation of a PDE \( P \subset J^kR \); the next result states that it coincides with our definition of classical prolongation.

Proposition 4.22. Let \( P \subset J^kR \) be a PDE with the property that the restriction of the Cartan distribution \( H = \mathcal{C} \cap TP \) is of constant rank, so that \( (P, H) \) is a Pfaffian bundle. Then,

\[
\text{Prol}(P, H) = P^{(1)} := J^1P \cap J^{k+1}R, \quad g(H)^{(1)} \simeq g^{(1)}
\]

where \( g^{(1)} \) is as in Theorem 2.4. Moreover, if \( P \) is integrable up to order \( k + 1 \), then \( \text{pr} : (P^{(1)}, H^{(1)}) \to (P, H) \) is a normalised prolongation with \( H^{(1)} = \mathcal{C} \cap TP^{(1)} \), and \( \text{pr} : P^{(1)} \to P \) is an affine subbundle modelled on \( g^{(1)} \).

Proof. We first recall that \( J^{k+1}R \) sits inside \( J^1(J^kR) \) as the splitting \( \sigma : T_xM \to T_qJ^kR \) of \( d\pi \) tangent to the Cartan distribution \( \mathcal{C} \subset T(J^kR) \), and so that

\[
c_c(\sigma(X), \sigma(Y)) = 0, \quad \text{for all } X, Y \in T_xM
\]

(this can be checked in local coordinates). Since \( P^{(1)} \) is the intersection of \( J^1(J^kR) \) with \( J^1P \), then the \( \sigma \) that belong to \( P^{(1)} \) are the ones satisfying the previous conditions plus the fact that its image \( \sigma(T_xM) \) lies in \( T_qP \). Putting all these conditions together, we see that \( \sigma \) is an element of \( P^{(1)} \) if and only if it also belongs to the classical prolongation \( \text{Prol}(P, H) \).
To finish, we observe that the definition integrable up to order $k+1$ is saying precisely that $p_r : P^{(1)} \to P$ is a bundle map, hence, by Theorem 4.18, it is a normalised prolongation. Moreover, in this case we have the inclusion $\mathfrak{g}(H) \subset \ker(d\text{pr} : T^\pi J^k R \to J^{k-1} R) \simeq S^k T^\ast M \otimes T^\pi R$ (see the exact sequence (8)), and $\partial_\mathcal{H}$ is precisely the restriction of $\partial_\mathcal{C} : S^k T^\ast M \otimes T^\pi R \to \text{Hom}(TM; S^k T^\ast M \otimes T^\pi R)$, $\eta \mapsto \partial_\mathcal{C}(\eta)(X) = \iota_X \eta$; hence $\mathfrak{g}(H)^{(1)} = \mathfrak{g}^{(1)}$, and the rest follows from Theorem 4.18. Q.E.D.

Coming back to Pfaffian bundles using the language of form we have the following remark:

**Remark 4.23. (Classical prolongation for forms)** Let us go back to the picture of Pfaffian bundles $(P, \theta)$ in terms of 1-forms. All the definitions related to the partial and classical prolongation go through in this picture, only that instead of writing them in terms of the distributions we can choose to write them directly in terms of the 1-forms. For example, instead of considering the distribution $H^{(1)}$ as in (41) and (51), we look at the dual 1-form denoted by $\theta^{(1)}$, given by the restriction of the Cartan form $\omega$ on $J^1 P$ to the partial or classical prolongation. All the results go through in this setting with the appropriate modifications: for Theorems 4.13 and 4.18, since the projection $\text{pr}$ in each case is a Pfaffian morphism, then the forms $\theta^{(1)}$ and $\theta$ are related by $\text{pr}^* \theta = \tilde{\theta} \circ \theta^{(1)}$, where $\tilde{\theta} : pr^*(T^\pi P) \to pr^* N$, $v \mapsto \theta(v)$ is the vector bundle map between the coefficient bundle of $\theta^{(1)}$, and $\theta$. For propositions 4.16 and 4.20, the condition for the distributions translate into $\varphi^* \theta^{(1)} = [d\phi] \circ \theta'$, where $[d\phi] : N' \to \varphi^* T^\pi P$, is the composition of the identification $T^\pi P'/H^\pi$ with $N'$ via $\theta'$, with the map $T^\pi P'/H^\pi \to T^\pi P$, $[v] \mapsto [d\phi(v)]$, and the relation between the curvatures become $\tilde{\theta} \circ \varphi^* c_{\theta^{(1)}} = \Phi \circ c_{\theta'}$, where $\Phi : N' \to \phi^* N$ is the vector bundle map between the coefficient bundles, associated to the Pfaffian fibration $\phi$. Of course in Proposition 4.20 this last expression is equal to zero. ♦

**Other results about prolongations**

There are some other nice consequences about the Pfaffian distributions and the prolongations involving the curvature and the prolongation of the symbol space; we list some of them.

**Corollary 4.24.** Assume that $\mathfrak{g}(H)^{(1)}$ has constant rank; then $(P, H)$ admits a Pfaffian prolongation if and only if the higher curvature $\kappa$ vanishes.

**Proof.** If $(P, H)$ admits a prolongation, then Remark 4.21 says that the projection $pr : \text{Prol}(P, H) \to P$ is surjective, hence $\kappa = 0$ by part of Theorem 4.18. The converse is Theorem 4.18. Q.E.D.

**Corollary 4.25.** The Pfaffian distribution $H \subset TP$ is involutive if and only if $\text{Prol}(P, H)$ coincides with $J^1_H P$ and the symbol map $\partial$ (48) vanishes.

**Proof.** If $H$ is involutive then all partial integral elements are integral elements, hence $\text{Prol}(P, H) = J^1_H P$; moreover, $\partial$ vanishes trivially.
Conversely, if we let \( p \in P \), we can split \( H_p \) as a direct sum \( V \oplus g(H)_p \), where \( V \) is a partial integral element. Because \( \text{Prol}(P,H) = J^1_HP \), \( V \) is actually an integral element. Now, we compute the bracket modulo \( H \) using the direct sum: for \( v + u, v' + u' \in V \oplus g(H)_p \),

\[
\begin{align*}
&c_H(v + u, v' + u') = c_H(v, v') + c_H(v, u') + c_H(u, v') + c_H(u, u') \\
&= -\partial(u')(d\pi(v)) - \partial(u)(d\pi(v')) = 0
\end{align*}
\]

where we used the involutivity of \( g(H) \).

**Corollary 4.26.** Let \( H \subset TP \) be a Pfaffian distribution whose curvature \( \kappa \) vanishes; then if two of the following three conditions hold, the third holds as well:

1. \( \text{pr} : \text{Prol}(P,H) \to P \) is a bijection;
2. \( g(H) \) is zero;
3. \( H \) is involutive.

**Proof.** Items (1) and (2) imply (3) follows from a computation similar to that of Corollary 4.25. From (1) and (3) to (2), we have that (3) implies that \( \text{Prol}(P,H) = J^1_HP \) by Corollary 4.25, and by (1) we have that for the fibre bundle \( \text{pr} : J^1_HP \to P \), \( \ker(d\pi) = \text{Hom}(\pi^*TM; \text{pr}^*g(H)) \) (Remark 4.14), which is zero because \( \text{pr} \) is a bijection, hence (2). From (2) and (3) to (1), we see that \( H \) is a horizontal distribution if and only if \( g(H) \) is zero; in this case \( \text{pr} : J^1_HP \to P \) is a bijection. If, moreover, \( H \) is involutive \( J^1_HP = \text{Prol}(P,H) \) by Corollary 4.25. Q.E.D.

### 4.5 Prolongations in the linear case

Let \((E', D')\), \((E, D)\) be linear Pfaffian bundles over \(M\), with \((D', \sigma')\) a relative connection taking values in \(E\), and \((D, \sigma)\) a relative connection taking values in \(F\):

\[
D' : \Gamma(E') \to \Omega^1(M; E), \quad D : \Gamma(E) \to \Omega^1(M; F),
\]

**Definition 4.27.** The relative connections \((D', \sigma')\) and \((D, \sigma)\) as in (53) are **compatible** if

1. \( D \circ \sigma' = \sigma \circ D' \);
2. \( D_X \circ D'_Y - D_Y \circ D'_X = 0 \) for all \( X, Y \in \mathcal{X}(M) \).

The two condition of Definition 4.27 above have a clear cohomological interpretation, which appeared already in [9,20]: for a relative connection \((D, \sigma)\) there exists a linear operator, denoted by the same letter \( D \)

\[
D : \Omega^*(M; E) \to \Omega^{*+1}(M, F)
\]

uniquely defined by the property that on \( \Gamma(E) = \Omega^0(M; E) \) it coincides with the connection \( D \), and that it satisfies the Leibniz identity relative to \( \sigma \):

\[
D(\omega \otimes s) = d\omega \otimes \sigma(s) + (-1)^k \omega \wedge D(s),
\]

for any \( k \)-form \( \omega \in \Omega^k(M) \), and any section \( s \in \Gamma(E) \). This operator can be given explicitly by the Koszul formula

\[
D\eta(X_0, \ldots, X_k) = \sum_i (-1)^i D_{X_i}(\eta(X_0, \ldots, \hat{X}_i, \ldots, X_k))
\]

\[
+ \sum_{i<j} (-1)^{i+j} \sigma(\eta([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k))
\]

for any \( \eta \in \Omega^k(M; E) \). A direct check shows the following lemma:
Lemma 4.28. Let \((D', \sigma')\), and \((D, \sigma)\) be relative connections as in (53). If \(\dim M > 0\), then the relative connections are compatible if and only if the composition
\[
\Omega^*(M; E') \xrightarrow{D'} \Omega^{*+1}(M; E) \xrightarrow{D} \Omega^{*+2}(M; F)
\]
is zero.

For compatible relative connections \((D', \sigma')\) and \((D, \sigma)\) as above, the condition (1) of Definition 4.27 implies that \(\sigma'\) preserves holonomic sections. In general, the resulting map
\[
\Gamma(E', D') \to \Gamma(E, D), \quad s \mapsto \sigma'(s)
\]
is not necessarily surjective; its surjectivity is measured in the sense of Proposition (4.30) below, by the map \(k\) which we now present.

Denote by \(\partial' : \mathfrak{g}' \to \text{Hom}(TM; E)\), the linear map given by the restriction of \(D'\) to its symbol space \(\mathfrak{g}' = \ker(\sigma')\). Linearity is a consequence of equation (22). Condition (1) of Definition 4.27 implies that the image of \(\partial'\) lies inside \(\text{Hom}(TM; \mathfrak{g})\), \(\mathfrak{g} = \ker(\sigma)\), hence \(\partial'\) takes the form
\[
\partial' : \mathfrak{g}' \to \text{Hom}(TM; \mathfrak{g}), \quad \partial' = D'|_{\mathfrak{g}'}.
\]
By the very definition of the operators (54) we get that at higher order \(\partial' (\omega \otimes s) = (-1)^k \omega \wedge \partial'(s)\), for \(\omega \in \Omega^k(M)\), and any section \(s \in \Gamma(\mathfrak{g}')\); hence, together with Lemma 4.28 above, this implies that the composition
\[
\wedge^k T^* M \otimes \mathfrak{g}' \xrightarrow{\partial'} \wedge^{k+1} T^* M \otimes \mathfrak{g} \xrightarrow{\partial} \wedge^{k+2} T^* M \otimes F
\]
of vector bundles over \(M\) is zero. Denote by \(H^{0,1}(\mathfrak{g})\) the quotient
\[
H^{0,1}(\mathfrak{g}) := \frac{\ker\{\partial : T^* M \otimes \mathfrak{g} \to \wedge^2 T^* M \otimes F\}}{\text{Im}\{\partial' : \mathfrak{g}' \to T^* M \otimes \mathfrak{g}\}},
\]
and define
\[
k : \Gamma(E, D) \to H^{0,1}(\mathfrak{g}), \quad s \mapsto [D'(\bar{s})],
\]
where \(\bar{s}\) is a section of \(E'\) so that \(\sigma'(\bar{s}) = s\).

Lemma 4.29. \(k\) as above is well defined.

Proof. If \(s' \in \Gamma(E')\) is any other such a section, then \(\alpha = \bar{s} - s'\) belongs to \(\mathfrak{g}'\), and \(\partial'(\alpha) = D'(\bar{s}) - D'(s')\), which means that \(D'(\bar{s}), D'(s')\), a priori sections of \(\text{Hom}(TM; \mathfrak{g})\) as \(\sigma(D'(\bar{s})) = D(\sigma'(\bar{s})) = D(s) = 0\) (and the same for \(s')\), represent the same class on the quotient by \(\ker(\partial')\). Moreover, for vector fields \(X, Y \in \mathfrak{X}(M)\),
\[
\partial(D'(\bar{s}))(X, Y) = DX D'_Y (\bar{s}) - DY D'_X (\bar{s}) - \sigma D'_{[X,Y]} (\bar{s})
\]
\(D'\) is zero by condition (2) of Definition 4.27. Hence, \(k\) is indeed well defined.

Q.E.D.

Proposition 4.30. For compatible connections as in (53), the sequence
\[
\Gamma(E', D') \xrightarrow{\sigma'} \Gamma(E, D) \xrightarrow{k} H^{0,1}(\mathfrak{g})
\]
is exact.
and we conclude that \( \sigma \) is holonomic. Conversely, if \( k(s) = [D'(s)] = 0 \), then there is a section \( \beta \) of \( g' \) so that \( D'(s) = \partial'(\beta) = D'((\beta)) \). Then the section \( s' = s - \beta \) of \( E' \) is such that \( \sigma'(s') = \sigma'(s) = s \), and \( D'(s') = 0 \), i.e. \( s' \) is a holonomic section of \( D' \).

Q.E.D.

When looking at linear Pfaffian bundles in terms of the linear Pfaffian forms, we realise that the definition of compatible connections coincides with the linear counterpart of normalised prolongations (see Remark 4.11). Let \( \theta' \) be a linear form, and let \( D' \) associated to \( \theta' \) as in (23):

\[
D' : \Gamma(E') \to \Omega^1(M; E), \quad s \mapsto s^*(\theta'),
\]

and \( D \) associated to \( \theta \) in the same way: \( D(u) = u^*\theta, u \in \Gamma(E) \).

**Lemma 4.31.** \((D', \sigma')\) is compatible with \((D, \sigma)\) if and only if \( \sigma' : (E', \theta') \to (E, \theta) \) is a normalised prolongation. Moreover, any normalised prolongation \( \phi : (E', \theta') \to (E, \theta) \) with \( \phi \) linear is, up to automorphisms of \( E \), of the form \( \phi = \sigma' = \theta'|_{g'} \), with \((D', \sigma')\) compatible with \((D, \sigma)\).

**Proof.** First of all, as \( \sigma' \) is by definition the restriction of \( \theta' \) to \( g(\theta') = \ker(\theta')^{\pi} \), and as \( \sigma' \) is linear its differential coincides with \( \sigma' \), when restricted to \( T^\pi_v E' = E'_\pi(V) \), for any \( v \in E' \) (of course we are using the canonical identification of these vector spaces). From this we get for the condition that

\[
g(\theta') = \pi^{\pi}(g(D)) = \pi^{\pi}(\ker(\sigma')) = \ker d\sigma',
\]

and hence, the coefficient bundle of \( \theta' \) is precisely \( \pi^*E \) (see also Remark 4.11). Now, let us investigate what are the other conditions of prolongation of Pfaffian bundles in terms of the relative connections: clearly from the correspondence (23) the relation \( \sigma^*\theta = \sigma \circ \theta' \) between the forms is translated into the equivalent condition (1) \( D \circ \sigma' = \sigma \circ D' \).

To see that the condition on the curvatures of \( \theta' \) and \( \theta \) is the same as condition (2) for compatible connections, we write \( \sigma \circ c_{g'} \) as the restriction to \( \ker(\theta') \), of the skew-symmetric bilinear map

\[
T^\pi E \times T^\pi E \to \pi^{\pi}E, \quad (u, v) \mapsto -d_D\theta'(u, v),
\]

where \( d_D \in \Omega^2(E'; \pi^{\pi}(F)) \) at \( U, V \in X(E') \) is defined by the De-Rham-type formula

\[
d_D\theta(U, V) = D_{\pi'}(\theta'(V)) - D_{\pi'}(\theta'(U)) - \sigma(\theta'[U, V])
\]

with \( D_{\pi'} : X(E') \times \Gamma(\pi^{\pi}E) \to \Gamma(\pi^{\pi}F) \) the pullback of \( D \) via \( \pi' : E' \to M \); of course, when \( U, V \) belong to \( \ker(\theta') \), \( -d_D\theta(U, V) \) coincides with \( \sigma(c_{g'}(U, V)) \). As \( \sigma \circ c_{g'} = \sigma^{\pi}c_{g'} \), and \( d\sigma' \) is zero on the vertical part \( \ker(\theta')^{\pi} \) because it coincides with \( \sigma' \) on \( g(D') = \ker(\theta')^{\pi} \), then a straightforward check shows that \( \sigma \circ c_{g'} \) is zero if and only if \( \sigma^*(\sigma \circ c_{g'})_x = 0 \) for any \( x \in M \), and any section \( s \) is \( \in \Gamma(E') \) such that \( s^*(\sigma \circ c_{g'})_x = 0 \). But

\[
s^*(\sigma \circ c_{g'})_x(X, Y) = s^*(d_D\theta')(x, Y) = D_X \circ D_Y(s)(x) - D_Y \circ D_X(s)(x) - \sigma D_{[X,Y]}(s)(x),
\]

and we conclude that \( \sigma \circ c_{g'} \) is zero if and only if condition (2) of Definition 4.27 holds.

Now, if \( \phi : (E', \theta') \to (E, \theta) \) is a normalised prolongation with \( \phi \) linear, in view of Remark 4.11, we assume that \( \theta' \) takes values on \( \phi^*T^\pi E \), which, in turn, is isomorphic to \( \phi^*\pi^*(E) = (\pi^*)^*(E) \) (again we use the canonical isomorphism of \( T^\pi E \) with \( \pi^*(E) \)). Also, we assume that under
these isomorphisms, \( d\phi \) coincides with \( \theta' \) on \( T^n V' \). Again, as \( \phi \) is linear its differential \( d\phi \) when restricted to the vertical vector bundle \( T^n V' = \pi^n E' \) coincides with \( \phi \); hence, on \( E' = T^n V'|_M \)

\[
\phi = d\phi = \theta' = \sigma'
\]

Q.E.D.

Passing to the classical prolongation of linear Pfaffian bundles we find again that many objects, which were in general over \( E \), become linear objects over \( M \) described in terms of relative connections. The linear version of the **partial prolongation** of \( (E, D) \) (with associated linear form \( \theta \)) is described in terms of \( D \) by

\[
J^1_B E := J^1_B E = \{ j^1_x s \mid s \in \Gamma_{loc}(E), \ D(s)(x) = 0 \}
\]

(recall that the linear form associated to \( D \) is defined by \( s^*\theta = D(s) \) and \( \theta|_E = \sigma \)). Similarly to Theorem 4.13 (together with the fact that the \( J^1 E \) is a linear Pfaffian bundle), we can characterise \( J^1_B E \) as the largest vector subbundle of \( J^1 E \) over \( M \), with the property that the projection \( \text{pr} : J^1_B E \to E \) is a Pfaffian fibration. In this language this means that \( J^1_B E \) is the largest subbundle so that condition (1) of Definition 4.27

\[
\sigma \circ D^{(1)} = D \circ \text{pr}
\]

holds for the restriction \( D^{(1)} : \Gamma(J^1_B E) \to \Omega^1(M; E) \) of the classical Spencer operator (4). At the level of sections, the partial prolongation is described using the decomposition (2) by

\[
\Gamma(J^1_B E) = \{ (\alpha, \omega) \in \Gamma(E) \oplus \Omega^1(M; E) \mid D(\alpha) = \sigma \circ \omega \}. \tag{56}
\]

This is seen by recalling that using the decomposition (2), a section \( (\alpha, \omega) \) of \( J^1 E \) at \( x \) is precisely the splitting

\[
d_x \alpha - \omega_x : T_x M \to T_{\alpha(x)} E
\]

where \( \omega_x \) is viewed as a map from \( T_x M \) to \( T^n_{\alpha(x)} E \) when canonically identifying \( T^n_{\alpha(x)} E \) with \( E_x \); hence, the image of \( (\alpha, \omega)_x \) belongs to \( \text{ker}(\theta) \) if and only if for all \( X \in T_x M \)

\[
0 = \theta(d_x \alpha(X) - \omega(X)) = \theta(d_x \alpha(X)) - \theta(\omega(X)) = \alpha^* \theta_x(X) - \sigma(\omega(X)) = D_X(\alpha) - \sigma(\omega(X)).
\]

The **classical prolongation** comes as the kernel of the vector bundle map \( C \),

\[
C : J^1_B E \to \text{Hom}(\wedge^2 TM; F)
\]

defined at the level of sections by

\[
C(\alpha, \omega)(X, Y) = D_X(\omega(Y)) - D_Y(\omega(X)) - \sigma(\omega[X, Y]),
\]

for any \( X, Y \in \mathfrak{X}(M) \), i.e.

\[
\text{Prol}(E, D) := \text{Prol}(E, \theta) = \ker(C).
\]

This last equality is consequence of the Lemma (4.32) below. As the relative connection \( D^{(1)} \) of \( J^1_B E \) is the projection to the second component of \( \Gamma(J^1_B E) \subset \Gamma(E) \oplus \Omega^1(M; E) \), the classical prolongation can be alternatively written as

\[
\text{Prol}(E, D) = \{ j^1_x s \mid D(s)(x) = 0, \ D_X \circ D^{(1)}_Y(s)(x) - D_Y \circ D^{(1)}_X(s)(x) - \sigma \circ D^{(1)}_{[X, Y]}(s)(x) \},
\]

i.e. \( \text{Prol}(E, D) \) is the largest bundle of vector spaces of \( J^1_B E \), where the condition (2) for compatible connections holds.
Lemma 4.32. Let \((E, \theta) \in \Omega^1(E; \pi^*F)\) be a linear Pfaffian bundle, and let \((D, \sigma)\) be the associated relative connections. The curvature map \(c_1: J^1_D E \to \text{Hom}(\pi^* \wedge^2 TM; \pi^* F)\) of equation \((46)\) is precisely \(-\pi^* C\), with \(C\) as in \((58)\).

Proof. Using the Spencer decomposition \((56)\), let \((\alpha, \omega) \in \Gamma(J^1_D E)\). In terms of the form \(\theta\), this means that \(\alpha^* \theta = \theta \circ \omega\). Following \((57)\), for \(X, Y \in \mathfrak{X}(M)\) we regard \(d\alpha(X) - \omega(X)\) as a \(\pi\)-projectable vector field on \(\ker(\theta)\) in such a way that \(\omega(X)\) is the vector field constant along the fibres of \(E\), extending \(\omega(X)\) (strictly speaking, we choose a \(\pi\)-projectable extension inside \(\ker(\theta)\) so that it coincides with \(d\alpha(X) - \omega(X)\) along \(\alpha(M) \subset TE\), and the same we do for \(d\alpha(Y) - \omega(Y)\). With this

\[
(\alpha, \omega)^* c_0(X, Y) = \theta([d\alpha(X) - \omega(X), d\alpha(Y) - \omega(Y)])
\]

\[
= \theta([d\alpha(X), d\alpha(Y)] - \theta([d\alpha(X), \omega(Y)]) - \theta([\omega(X), d\alpha(Y) - \omega(Y)])
\]

\[
= \alpha^* \theta([X, Y]) - \theta([d\alpha(X), \omega(Y)]) + D_Y(\omega(X))
\]

\[
= D_{[X, Y]}(\alpha) - \theta([d\alpha(X), \omega(Y)]) + D_Y(\omega(X))
\]

where in third line we use Remark 3.18 saying that \(D_Y(\omega(X))\) is precisely \(\theta([\omega(X), d\alpha(Y) - \omega(Y)])\) (recall that \((\alpha, \omega)\) belongs to \(J^1_D E = J^1_0 E\) means precisely that \(d\alpha(X) - \omega(X) \in \ker(\theta)\) for all \(X \in \mathfrak{X}(M)\)). Now, using that vector fields constant along the fibres of \(E\) commute, we get that \([\omega(X), \omega(Y)]\) = 0, and therefore \(\theta([d\alpha(X), \omega(Y)])\) can be computed as

\[
\theta([d\alpha(X), \omega(Y)]) = \theta([d\alpha(X), \omega(Y)]) - \theta([\omega(X), \omega(Y)]) + \theta([\omega(X), \omega(Y)])
\]

\[
= \theta([d\alpha(X) - \omega(X), \omega(Y)]) = -D_X(\omega)
\]

Putting the two equations above together and using that \(D(\alpha) = \sigma(\omega)\), we conclude the proof.

Q.E.D.

As pointed out in the general discussion, \(\text{Prol}(E, D)\) might fail to be a (smooth) fibre bundle over \(E\), the reasons being the lack of surjectivity of the projection \(\text{pr} : \text{Prol}(E, D) \to E\), and that the rank over \(M\) might vary. In fact, in this linear picture things simplify and what is going on is that the exact sequence \((3)\) for \(J^1 E\) restricts to the exact sequence of vector bundles over \(M\),

\[
0 \to g(D)^{(1)} \to \text{Prol}(E, D) \overset{\text{pr}}{\to} E,
\]

with \(g(D)^{(1)}\) the bundle of vector spaces over \(M\), called the \textbf{prolongation of the symbol space w.r.t.} \(\partial_D\) (see Section 2.4)

\[
g(D)^{(1)} := \{\eta : T_x M \to g(D)_x \mid x \in M, \partial_D(\eta(X))(Y) = \partial_D(\eta(Y))(X), \forall X, Y \in T_x M\},
\]

and where

\[
\partial_D : g(D) \to \text{Hom}(TM; F), \quad \partial_D (v)(X) \mapsto D_X(v)
\]

(because \(g(D) = \ker(\sigma)\), and the Leibniz identity of \(D\) w.r.t. \(\sigma\) one can easily verify that \(\partial_D\) is a well-defined linear map). Notice that \(g(D)^{(1)}\) is just the 1st prolongation of the tableau \(\partial_D : g(D) \to \text{Hom}(TM; F)\) in the sense of equation \((13)\). One checks that the sequence is exact by considering a section of \(J^1_D E\) that belongs to \(\text{Prol}(E, D)\), which lives in the kernel of \(\text{pr}\), i.e. its second component in the decomposition \((56)\) is zero.

Now, the surjectivity of \(\text{pr} : \text{Prol}(E, D) \to E\) is of course related to the map \(C\) of equation \((58)\). Letting

\[
\delta_D : \text{Hom}(TM; g(D)) \to \text{Hom}(\wedge^2 TM; F),
\]
defined by $\delta_D(\eta)(X,Y) = \partial D(\eta(X))(Y) - \partial D(\eta(Y))(X)$, we see that $C$ descends to a vector bundle map

$$K : E \to \text{Hom}(\wedge^2 TM; F)/\text{Im}(\delta_D), \ p \mapsto [C(\xi)]$$

where $\xi \in J^1_E$ is any element that projects to $p$ (it is a straightforward computation using the decomposition (56) that $K$ is well defined). It is now a simple exercise to check that the zero-set of $K$ is precisely the image of $\text{pr} : \text{Prol}(E,D) \to E$. Thus,

**Proposition 4.33.** The classical prolongation $\text{Prol}(E,D)$ is a (smooth) subbundle of $J^1_E \to E$ if and only if $K = 0$, and the partial prolongation space $g(D)^{(1)}$ has constant rank. In this case, the restriction of the Spencer operator

$$D^{(1)} : \Gamma(\text{Prol}(E,D)) \to \Omega^1(M;E),$$

is compatible with $D$.

One of the main importance of the classical prolongation is that, even not assuming any smoothness condition on $\text{Prol}(E,D)$, the map

$$\Gamma(\text{Prol}(E,D), D^{(1)}) \to \Gamma(E, D), \ \xi \mapsto \text{pr} \circ \xi$$

defines a bijection, with inverse $s \in \Gamma(E, D) \mapsto j^1s$. Moreover, it is universal among the connections relative to $D$, in the following sense:

**Proposition 4.34.** If $(E', D')$ is compatible with $(E, D)$, then there exists a unique vector bundle map $j : E' \to \text{Prol}(E,D)$ so that

$$D' = D^{(1)} \circ j.$$ 

Of course the above proposition is consequence of Proposition 4.20, for general prolongations. We only remark that in this case, $j = \varphi$ is defined in terms of $D'$, and at the level of sections is given by

$$j(s) = (\sigma'(s), D'(s)) \in \Gamma(E) \oplus \Omega^1(M;E).$$

The conditions for compatible connections mean that $j(s)$ actually lands in $\text{Prol}(E,D)$.

**Remark 4.35.** As we had remarked on 3.17, in the linear case many of the objects sit on top of $M$. Of course, the symbol map $\partial_\theta$ of equation (48), prolongation space $g^{(1)}(\theta)$ of equation (49), and the higher curvature map $\kappa$ of equation (50) of the linear form $\theta$, are just pullbacks of the analogous objects for the associated relative connection $D$. In fact, from Remark 3.18 we know that $g(\theta) \simeq \pi^*g(D)$ and this isomorphism comes from the canonical identification of $T^*E$ with $\pi^*E$ by translating vertical vectors to the zero section; hence, using the description of $D$ in terms of $\theta$ as in Remark 3.18) we have

$$\partial_\theta = \pi^*\partial_D, \quad g^{(1)}(\theta) \simeq \pi^*g^{(1)}(D), \quad \kappa = \pi^*K.$$

$\diamond$

**Remark 4.36 (Linearisation of Pfaffian prolongations along holonomic sections).** As we did for Pfaffian bundles (3.3), we can linearise Pfaffian normalised prolongations

$$\phi : (P', \theta') \to (P, \theta)$$
along a holonomic section $\xi \in \Gamma(P', \theta')$, and obtain compatible connections

$$\text{Lin}_\xi (P', \theta') \xrightarrow{D^\xi} \text{Lin}_\phi(\xi)(P, \theta) \xrightarrow{D^\phi(\xi)} \phi(\xi)^*N,$$

where $(\text{Lin}_\xi(P', \theta'), D^\xi)$ is the linearisation of $(P', \theta')$ along $\xi$, and $(\text{Lin}_\phi(\xi)(P, \theta), D^\phi(\xi))$ the linearisation of $(P, \theta)$ along $\phi(\xi) \in \Gamma(P, \theta)$. In particular, the classical (and the partial) prolongation of $(P, \theta)$ can be linearised along a holonomic section $\beta$ of $\theta$ to a connection compatible with $D$. The functoriality of linearisation implies that

$$\text{Prol}(\text{Lin}_\beta(P, \theta), D^\beta) = \text{Lin}_j \beta(\text{Prol}(P, \theta)), \quad D^{(1)} = D^{j^1 \beta},$$

Similarly, objects such as the symbol space, the prolongation space, etc., are linearised along $\beta$ to the analogous objects on $(P^j, D^\beta)$. Of course all the properties of $(P, \theta)$ pass to its linearisation.

This Lie functor becomes particularly nice when linearising Pfaffian groupoids along the unit section, where the multiplicativity allows us to translate properties of the linearisation to the analogous properties of the Pfaffian groupoid. \hfill $\diamondsuit$

## 5 Integrability of Pfaffian bundles

One can think that when we prolong a Pfaffian bundle $(P, H)$ over $M$, we are trying to determine if an element of $(P, H)$ comes from a section which is “holonomic up to order 1”; if we prolong again then we are looking for sections which are “holonomic up to order 2”, etc. If we can repeat this process indefinitely, we find a formal holonomic section of the Pfaffian bundle i.e. a Taylor series of a potential holonomic section of $(P, H)$

Let us be more specific. To set and simplify notation denote by $(P^{(1)} = \text{Prol}(P, H), H^{(1)})$ the classical prolongation of $(P, H)$ from definition 4.17. Under the conditions of Theorem 4.18, such prolongation is in turn a smooth Pfaffian bundle over $P$. We could therefore build the classical prolongation of $P^{(1)}$ and denote it by $(P^{(2)}, H^{(2)})$; this sits inside a jet bundle, as $P^{(2)} \subseteq J^1_{H^{(1)}} P^{(1)} \subseteq J^1 P^{(1)}$, but may not be a smooth submanifold, and the projection over $P^{(1)}$ may not be a surjective submersion. However, if we apply again Theorem 4.18, we find conditions under which also $P^{(2)}$ is a Pfaffian bundle over $P^{(1)}$. When this process can be carried out up to “infinity” we say that $(P, H)$ is formally integrable. The goal of this section is to formalise this procedure and describing precisely the obstructions to formal integrability.

### 5.1 Integrability up to finite order

We begin with the precise definition of the properties we want to consider:

**Definition 5.1.** A Pfaffian bundle $(P, H) = (P^{(0)}, H^{(0)})$ is called **integrable up to order** $k \geq 1$ when, for all $i = 1, \ldots, k$, the universal prolongations

$$P^{(i)} = \text{Prol}(P^{(i-1)}, H^{(i-1)}) \subseteq J^1_{H^{(i-1)}} P^{(i-1)},$$

are smooth submanifolds, and the projections $P^{(i)} \to P^{(i-1)}$ are surjective submersions.

In particular, if $(P, H)$ is integrable up to order $k$, it follows that each $P^{(i)}$ is a Pfaffian bundle over $M$, when endowed with the distribution $H^{(i)} = (H^{(i-1)})^{(1)}$, and pr : $(P^{(i)}, H^{(i)}) \to (P^{(i-1)}, H^{(i-1)})$ is precisely the classical prolongation of the Pfaffian bundle $(P^{(i-1)}, H^{(i-1)})$. We call

$$(P^{(i)}, H^{(i)})$$

the $i^{th}$ **classical prolongation** of the Pfaffian bundle $(P, H)$, $i = 1 \ldots k$. 

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Remark 5.2. This definition has some immediate consequences:

- If \( P \subseteq J^k R \) is a PDE, the notion of integrability up to order \( k \) in the sense of Pfaffian bundles coincides with the notion of integrability up to order \( k \) in the sense of PDEs (see Section 2.3).

- Let \((P, H)\) be a Pfaffian bundle integrable up to order \( k \). Then, for every integers \( i, l \leq k \) with \( i + l \leq k \),
  1. \((P, H)\) is also integrable up to order \( i \).
  2. The Pfaffian bundle \((P^{(i)}, H^{(i)})\) is integrable up to order \( l \), and its \( l^{th}\)-prolongation \((P^{(i)})(l)\) coincide with the \((i + l)^{th}\)-prolongation \(P^{(i+l)}\) of \((P, H)\).
  3. The holonomic sections of \((P, H)\) are in bijections with the holonomic sections of \((P^{(i)}, H^{(i)})\).

The equivalence with the definition of integrability for PDEs follows directly from Proposition 4.22. Properties 1 and 3 are also immediate from the definition and from remark 4.19. For the second property, note that \( P^{(i)} \subseteq J^i P \) is a PDE and prolongations of Pfaffian bundles and PDEs coincide. Our statement become then precisely [7, Theorem 7.2].

We describe now the main obstructions for integrability. The first step, which takes care of \( \text{pr} : P^{(1)} \to P \), was discussed in Theorem 4.18, and one needs:

- c1) the projection \( \text{pr} : P^{(1)} \to P \) to be surjective which, in turn, was shown to be equivalent to the higher curvature map \((50)\) to vanish.

- c2) the prolongation \( g^{(1)} = g(H)^{(1)} \) of the symbol space \( g = g(H) \) to be of constant rank (hence \( g^{(1)} \) is given by \((13)\), applied to \( \partial_H : g = g(H) \to \text{Hom}(\pi^*TM; N_H) \)).

Under these conditions \( P^{(1)} \) becomes an affine bundle over \( P \) modelled on \( g^{(1)} \) and itself a smooth Pfaffian bundle (over \( M \)). Moving one step upwards, we would now like to unravel these conditions c1 and c2) when applied to the prolongation of \( P^{(1)} \), \( \text{pr} : P^{(2)} \to P^{(1)} \); and then to continue this analysis inductively. First of all, the (higher) prolongations that are relevant in condition c2) will be precisely the ones from Section 2.4:

\[
 g^{(i)} = \pi^*(S^i T^* M) \otimes g \cap \text{Hom}(\pi^*TM; g^{(i-1)}) = \ker(\delta_i), \quad \text{(for } i > 1) \]

with \( \delta_i \) as in \((11)\). One can inductively check that \( (g^{(j)})^{(r)} = g^{(j+r)} \) for all \( j + r > 1 \) and then, indeed:

Lemma 5.3. If \((P, H)\) is integrable up to order \( k \geq 1 \), then we have the following canonical isomorphisms of bundles of vector spaces over \( P^{(i)} \), \( 1 \leq i \leq k \)

\[
 \text{pr}^* g^{(i+1)} \simeq \text{pr}^* g(H^{(1)})^{(i)} \simeq ... \simeq \text{pr}^* g(H^{(i-1)})^{(2)} \simeq g(H^{(i)})^{(1)}. \quad (60)\]

Moreover \( \text{pr} : P^{(i)} \to P^{(i-1)} \) is an affine bundle modelled on the vector bundle \( \text{pr}^* g^{(i)} \) over \( P^{(i-1)} \) (where \( P^0 := P \)).

Proof. First of all, we regard \( g^{(i)} \) sitting inside of \( \pi^*(S^i T^* M) \otimes g \subset \pi^*(S^i T^* M) \otimes T^* P \). Having in mind the exact sequence \((8)\) of vector bundles over \( J^i P \), and that the symbol space of \((J^i P, \xi)\) is precisely \( \ker(\text{dpr} : TM^k R \to T(J^{k-1} R) \simeq \pi^* S^{i-1} T^* M \otimes \text{pr} T^* P \), one can check that \( \delta_i \) coincides with the restriction of the symbol map \( \delta_c : \text{Hom}(\pi^* TM, \pi^* S^{i-2} T^* M \otimes \text{pr} T^* P) \to \text{Hom}(\pi^* \Lambda^2 TM, \pi^* S^{i-2} T^* M \otimes \text{pr} T^* P) \).
(see also the proof of Proposition 4.22, where we look at this $\partial c$). Also, we can regard $(P^{(i)}, H^{(i)})$, for $i = 1, \ldots, k$, as a PDE endowed with $H^{(i)}$-the restriction of the Cartan distribution $\mathcal{C} \subset T J^i P$. Having all these in mind, and the equality of the prolongation spaces of Proposition 4.22 we can see inductively the canonical isomorphisms (60). Moreover, $pr : P^{(i)} \rightarrow P^{(i-1)}$ is an affine bundle modelled on the vector bundle $pr^* g^{(i)} = (g^{(i-1)})^{(1)}$ (we set $g^{(0)} = g$).

Q.E.D.

We now move to the condition c1). For a Pfaffian bundle $(P, H)$ integrable up to order $k$ the discussion following the Definition 4.17 tells us that the classical prolongation of $(P^{(k)}, H^{(k)})$ appears as the kernel of the 1st curvature map (46)

$$
c_{1, H^{(k)}} : J^1_{H^{(k)}} P^{(k)} \rightarrow \text{Hom}(\pi^* \wedge^2 TM, pr^* T^\pi P^{(k-1)}), \quad j^1_{\pi} \alpha \mapsto (\alpha^* c_{H^{(k)}})_x = (c_{H^{(k)}})_x (d_x \sigma(\cdot), d_x \sigma(\cdot)).
$$

In the last Hom-space we have used the identification of the normal bundle $N_{H^{(k)}}$ with $pr^* T^\pi P^{(k-1)}$ (via the differential $dpr$), as it follows from the fact that $pr : (P^{(k)}, H^{(k)}) \rightarrow (P^{(k-1)}, H^{(k-1)})$ is a normalised prolongation. Also, $c_{1, H^{(k)}}$ is an affine map of affine bundles over $P^{(k)}$, where $J^1_{H^{(k)}} P^{(k)} \rightarrow P^{(k)}$ is modelled on $\text{Hom}(\pi^* TM; g(H^{(k)}))$, with

$$g(H^{(k)}) = pr^* g(H^{(k-1)})^{(1)} \simeq pr^* g^{(k)}$$

where the first equality is by (part of) Theorem 4.18, and the second by Lemma 5.3. Thus, the underlying vector bundle morphism of $c_{1, H^{(k)}}$ is of the form

$$\overline{c_{1, H^{(k)}}} : \text{Hom}(\pi^* TM; pr^* g^{(k)}) \rightarrow \text{Hom}(\pi^* \wedge^2 TM, pr^* T^\pi P^{(k-1)}),$$

and a computation reveals that it is precisely the pullback via $pr$ of the map $\delta_k$ of equation (11) (see the proofs of Lemma 5.3 and Proposition 4.22). Thus, $P^{(k+1)} := \text{Prol}(P^{(k)}, H^{(k)})$ is a smooth affine subbundle of $J^1_{H^{(k)}} P^{(k)} \rightarrow P^{(k)}$ if and only if

- $\delta_k$ has constant rank, i.e. $\ker(\delta_k) = g^{(k+1)}$ has constant rank;
- $P^{(k+1)} \rightarrow P^{(k)}$ is surjective.

Related to the last point, this discussion also implies that $c_{1, H^{(k)}}$ descends to the following map:

**Definition 5.4.** Let $(P, H)$ be a Pfaffian bundle integrable up to order $k \geq 1$. The **intrinsic torsion of order** $k$ of $(P, H)$ is defined to be the higher curvature (50) of $(P^{(k)}, H^{(k)})$, i.e. the map

$$\tau_k := \kappa_{P^{(k)}} : P^{(k)} \rightarrow \frac{\text{Hom}(\pi^* \wedge^2 TM, pr^* T^\pi P^{(k-1)})}{\delta(\text{Hom}(\pi^* TM, pr^* g^{(k)}))}, \quad p \mapsto [\sigma^*(c_{H^{(k)}})_x] = [c_{1, H^{(k)}}(j^1_{\pi} \sigma)] \quad (62)$$

where $j^1_{\pi} \sigma$ is an element of the partial prolongation $J^1_{H^{(k)}} P^{(k)}$ s.t. $\sigma(x) = p$. We set $\tau_0 = \kappa$ to be the higher curvature of $(P, H)$.

From the general discussion of the classical prolongation, we know already that the zero-set of $\tau_k$ is precisely the image of $P^{(k+1)} \rightarrow P^{(k)}$. Hence, form Theorem 4.18 we obtain:

**Proposition 5.5.** Let $(P, H)$ be a Pfaffian bundle integrable up to order $k$. If

- the intrinsic torsion $\tau_k$ vanishes

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• the prolongation $g^{(k+1)}$ is smooth

then $(P,H)$ is integrable up to order $k + 1$, and the converse also holds. Moreover, the classical prolongation

$$\text{pr} : (P^{(k+1)}, H^{(k+1)}) \rightarrow (P^{(k)}, H^{(k)})$$

has symbol $g(H^{(k+1)}) = \text{pr}^* g^{(k+1)}$, and it is an affine bundle over $P^{(k)}$ modelled on $\text{pr}^* g^{(k+1)}$.

To understand better $\tau_k$ we look at its image; at the end of the section we will prove the following:

**Proposition 5.6.** Let $(P,H)$ be a Pfaffian bundle integrable up to order $k \geq 1$. Then its intrinsic torsions $\tau_k$ take values in the Spencer cohomology groups (12) of the tableau bundle $g = g^{(0)} = g(H)$

$$H^{k-1,2}(g) = \frac{\ker(\delta : \text{Hom}(\pi^* \wedge^2 TM, g^{(k-1)}) \rightarrow \text{Hom}(\pi^* \wedge^3 TM, g^{(k-2)}))}{\text{Im}(\delta : \text{Hom}(\pi^* TM, g^{(k)}) \rightarrow \text{Hom}(\pi^* \wedge^2 TM, g^{(k-1)}))}$$

where we set $g^{(-1)} = N_H$, and we regard the prolongations $g^{(i)}$ sitting on top of $P^{(k)}$ via the pullback by $\text{pr}$.

If we assume that some prolongation $g^{(i)}$ of the symbol space has rank 0, then the Spencer cohomology group $H^{1,2}(g)$ vanishes. In particular, by Proposition 5.6, the torsion $\tau_{i+1}$ is zero; this suggests that for certain types of Pfaffian bundles, Proposition 5.5 becomes simpler.

This leads us to the following definition:

**Definition 5.7.** A Pfaffian bundle $(P,H)$ is of **finite type** $l$ if $l$ is the smallest integer $l \geq 0$ such that $g^{(l)} = 0$. We say that $(P,H)$ is of **infinite type** if $g^{(l)} \neq 0 \forall l$.

With this,

**Proposition 5.8.** Let $(P,H)$ be a Pfaffian bundle of finite type $l$. If $(P,H)$ is integrable up to order $k$ and $l < k$, then it is integrable up to order $k + i$, $i \geq 0$. Moreover, $\text{pr} : P^{(j)} \rightarrow P^{(j-1)}$ is a bijection for all $j \geq l$.

**Proof.** Because $i \geq 0$, then the finite type condition says that $g^{(k+i-1)} = 0$ (as $k + i - 1 \geq l$), and therefore $\tau_{k+i}$ vanishes (see the discussion before Definition 5.7). Also $g^{(k+i+1)}$ has obviously constant rank equal to 0, and we can apply Proposition 5.5 inductively on $i$ to conclude that $(P,H)$ is integrable up to order $k + i$. Now, Lemma 5.3 tells us that $P^{(j)} \rightarrow P^{(j-1)}$ is an affine bundle modelled on $\text{pr}^* g^{(j)}$, so if $j \geq l$, then $g^{(j)} = 0$, and therefore $P^{(j)} \rightarrow P^{(j-1)}$ is a bijection.

**Q.E.D.**

**Remark 5.9 (Pfaffian bundles and geometric structures).** The name intrinsic torsion comes from the theory of $G$-structures, i.e. reductions of the structure group of the frame bundle of a manifold $M$: examples include Riemannian metrics, almost symplectic structures or almost complex structures.

In that setting, intrinsic torsions are tensors defined recursively and with values in the Spencer cohomology of the Lie algebra of $G \subseteq GL(n, \mathbb{R})$; each one is an obstruction for the prolongation of the $G$-structure to an higher order.

In a similar way, one can define the notion of formally integrable $G$-structure (for which all these torsions vanish): this turns out to be a necessary but not sufficient condition for the
integrability of a $G$-structure, i.e. the existence of an atlas on $M$ whose charts are "adapted" to the structure. The concepts of structures of finite and infinite type arise also from this field.

Generalising these ideas, one can use the theory of Pfaffian bundles to study (formal) integrability of geometric structures described by a Lie pseudogroup (not necessarily transitive or arising from a Lie group): see the forthcoming [3].

**Proof of Proposition 5.6.** We check the case $k = 1$, using the Pfaffian form $\theta$ associated to $H \subset TP$, and the Pfaffian form $\theta^{(1)}$ associated to $H^{(1)} \subset TP^{(1)}$. The general case $k \geq 1$ follows similarly.

First of all, we check that the map $c_{1,H^{(1)}} = c_{1,\theta^{(1)}}$ of equation (61) takes values in

$$\text{Hom}(\pi^* \wedge^2 TM, g) \subseteq \text{Hom}(\pi^* \wedge^2 TM, pr^* T^\pi P).$$

Indeed, an element $j^1_x \sigma$ belongs to $J^1_{H^{(1)}} P^{(1)}$ if $d_x \sigma(T_x M) \subset H^{(1)}_{\sigma(x)}$; thus, as $pr : (P^{(1)}, H^{(1)}) \rightarrow (P, H)$ is the classical prolongation, it is normalised, i.e.

$$\theta(c_{\theta^{(1)}}(d_x \sigma(X), d_x \sigma(Y))) = 0,$$

for any $X, Y \in T_x M$ (see Remark 4.11). In conclusion, $c_{1,H^{(1)}}(j^1_x \sigma)(X,Y) \in \ker(\theta)$, therefore it is in $g = \ker(\theta) \cap T^\pi P$.

Now, we check that $c_{1,\theta^{(1)}}$ takes values in the kernel of

$$\delta_\theta = \delta_H : \text{Hom}(\pi^* \wedge^2 TM, g = g^{(0)}) \rightarrow \text{Hom}(\pi^* \wedge^2 TM, N_H = g^{(-1)}).$$

In order to do that, let $j^1_x \sigma \in J^1_{H^{(1)}} P^{(1)}$ and $X, Y, Z$ vector field on $M$; we need to compute

$$\partial_H(c_{H^{(1)}}(j^1_x \sigma)(X,Y))(Z) = \partial_H(c_{H^{(1)}}(d_x \sigma(X), d_x \sigma(Y)))(Z) = c_{H^{(1)}}(c_{H^{(1)}}(d_x \sigma(X), d_x \sigma(Y)), \sigma(X)(Z)) \quad (63)$$

First, extend locally $d \sigma(X), d \sigma(Y), d \sigma(Z) \in TP^{(1)}$ to local vector fields $\vec{X}, \vec{Y}, \vec{Z}$ of $P^{(1)}$ which are simultaneously $\pi$- and pr-projectable; in particular, this means $d \sigma(\vec{X}) = \vec{X}$, and similarly for $\vec{Y}$ and $\vec{Z}$. These extensions are always possible as pr is a submersion and a fibre bundle map over $M$, hence one can simultaneously trivialise $P^{(1)}$ around $\sigma(x)$ as $\mathbb{R}^{k+n+m}$, $P$ around $pr(\sigma(x))$ as $\mathbb{R}^{n+m}$, and $M$ around $x$ as $\mathbb{R}^n$, so that pr and the two maps to $M$ become standard projections.

Moreover, consider the pullback via $pr : P^{(1)} \rightarrow P$ of some torsion-free linear connection $\nabla : \mathfrak{X}(P) \times \mathfrak{X}(P) \rightarrow \mathfrak{X}(P)$ (e.g. the Levi-Civita connection of some fixed Riemannian metric on $P$); in the following we will use the same notation $\nabla$ also for the pullback connection on $pr^* TP$. We can now compute the term $c_{H^{(1)}}(d_x \sigma(X), d_x \sigma(Y))$ in equation (63) using $\nabla$ (see the discussion after equation (18)):

$$c_{H^{(1)}}(d_x \sigma(X), d_x \sigma(Y)) = d_\sigma \theta^{(1)}(d_x \sigma(X), d_x \sigma(Y)) = (\nabla_{\vec{X}} \theta^{(1)}(\vec{Y}))_{\sigma(x)} - (\nabla_{\vec{Y}} \theta^{(1)}(\vec{X}))_{\sigma(x)} - \theta^{(1)}([\vec{X}, \vec{Y}]_{\sigma(x)}) \quad (64)$$

From the the definition of $\theta^{(1)}$ as Cartan form, we see that the last addendum vanishes

$$\theta^{(1)}_{\sigma(x)}([\vec{X}, \vec{Y}]_{\sigma(x)}) = \theta^{(1)}_{\sigma(x)}(d_\sigma([X,Y])_{\sigma(x)}) = 0$$

Note that we use $\sigma(x)$ also to denote the splitting $\sigma(x) : T_x M \rightarrow T_{pr(\sigma(x))} P$. In the second equality we also used that $[\vec{X}, \vec{Y}]_{\sigma(x)} = d \sigma([X,Y]_{\sigma(x)})$ because $X, Y$ are $\pi$-projectable and $\sigma$ is

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a section of \( \pi \). In the second equality we used the fact that \( j_1^1 \sigma \) is an element of \( J_{H^{(1)}}P^{(1)} \); therefore \( \theta^{(1)} \circ d_x \sigma = 0 \).

On the other side, to rewrite the other two terms of (64) we use

\[
\theta^{(1)}_p(\bar{X}) = d_p\text{pr}(\bar{X}) - p(X) = d_p\text{pr}(\bar{X}) - \bullet(X)(p), \quad \forall p \in P^{(1)}
\]

where we write again \( p \) for the induced splitting \( p : T_{\pi(p)}M \to T_{\text{pr}(p)}P \), and we denote by \( \bullet(\bar{X}) \) the section of \( \text{pr}^*(TP) \to P^{(1)} \) defined by \( \bullet(X)(p) = p(X) \). We have therefore written \( \theta^{(1)}(\bar{X}) \) as the sum of two sections of \( \text{pr}^*(TP) \); doing the same also for \( Y \) we get

\[
\nabla_X(\theta^{(1)}(\bar{Y})) - \nabla_Y(\theta^{(1)}(\bar{X})) = \nabla_{\text{pr}(X)}(d\text{pr}(\bar{Y})) - \nabla_{\text{pr}(Y)}(d\text{pr}(\bar{X})) - \nabla_X(\bullet(Y)) + \nabla_Y(\bullet(X)) =
\]

\[
= [d\text{pr}(\bar{X}), d\text{pr}(\bar{Y})] - \nabla_X(\bullet(Y)) + \nabla_Y(\bullet(X)) = d\text{pr}[\bar{X}, \bar{Y}] - \nabla_X(\bullet(Y)) + \nabla_Y(\bullet(X))
\]

(65)

Here we used in the first line the definition of pullback connection via \( \text{pr} \), i.e. \( \nabla_X(\text{pr}(\bar{Y})) = \nabla_{\text{pr}(X)}(d\text{pr}(\bar{Y})) \), because the section \( d\text{pr}(\bar{Y}) \in \Gamma(\text{pr}^*(TP)) \) is already the pullback of the section \( \text{pr}^*(d\text{pr}(\bar{Y})) \in \mathfrak{X}(P) \) (recall that they are \( \text{pr} \)-projectable vector fields). The first equality of the second line follows from the fact that \( \nabla \) is torsion-free. For the last equality, as \( d_x \sigma \) takes values in \( H^{(1)}_{\sigma(x)} \), we have \( d\text{pr}(X_{\sigma(x)}) = \sigma(x)(X) \); in particular, \( d\text{pr}[\bar{X}, \bar{Y}]_{\sigma(x)} = d\text{pr}d_x \sigma[X, Y] = \sigma(x)[X, Y] \).

We compute the last two terms of (65) at \( \sigma(x) \in P^{(1)} \): since \( \bullet(X)_{\sigma(x)} = \sigma(x)(X) = \bar{X}_{\sigma(x)} \), and similarly for \( Y \), we have

\[
-(\nabla_X(\bullet(Y))\sigma(x)) + (\nabla_Y(\bullet(X))\sigma(x)) = -\nabla_{\bullet(x)_{\sigma(x)}}(\bullet(Y)) + \nabla_{\bullet(Y)_{\sigma(x)}}(\bullet(X)).
\]

(66)

Now, choose a local Cartan-Ehresmann connection \( C \subset H \) extending \( \sigma(x)(T_x M) = d\text{pr}(H^{(1)}_{\sigma(x)}) \subset H_{\text{pr}(\sigma(x))} \) (see Remark 4.7). As \( p : T_{\pi(p)}M \to T_pP \) denotes an integral element of \( (P, H) \) for \( p \in P^{(1)} \), then locally \( p(X) = C_p(X) + \eta_p(X) \), \( \forall X \in X(M) \) for some \( \eta_p \in \mathfrak{g}^{(1)}_{\text{pr}(p)} \). That is, locally

\[
\bullet(X) = \text{pr}^*C(X) + S
\]

where \( S \) is a finite sum of terms of the form \( f \text{pr}^*(\eta)(X) \), for \( \eta \in \Gamma(\mathfrak{g}^{(1)}) \), \( f \in C^\infty(P^{(1)}) \) and with \( f(\sigma(x)) = 0 \) (as \( C_{\sigma(x)} = d_x \sigma \), and \( \sigma(x)(X) = d_x \sigma(X) \)). To simplify notation we assume that locally \( S \) is given by a single term, i.e.

\[
\bullet(X) = \text{pr}^*C(X) + f \text{pr}^*(\eta)(X), \quad \eta \in \Gamma(\mathfrak{g}^{(1)}), f \in C^\infty(P^{(1)}), f(\sigma(x)) = 0, \forall X \in \mathfrak{X}(M).
\]

A direct calculation shows that the right-hand side of (66) is (up to pullbacks and coefficients) a \( C^\infty(P^{(1)}) \)-linear combinations of five kinds of terms (the first three come from the torsion-free property of \( \nabla \) and the last two from the Leibniz one):

(i) \([C(X), C(Y)]\), (ii) \([\eta(X), \eta(Y)]\), (iii) \([C(X) + \eta(X), C(Y) + \eta(Y)]\), (iv) \(\eta(X)\), (v) \(\eta(Y)\)

(67)

In conclusion, we plug our results in equation (63) to get

\[
\partial_0(c_{1,H^{(1)}}(j_1^1 \sigma)(X, Y))(Z) = c_0(d_x \theta^{(1)}(d_x \sigma(X), d_x \sigma(Y)), \sigma(x)(Z))
\]

\[
= c_0(\sigma(x)[X, Y], \sigma(x)(Z)) + c_0(t_1(iv) + t_2(v), \sigma(x)(Z)) + c_{\theta, \sigma(x)}(r_1(i) + r_2(ii) + r_3(iii), C(Z) + f\eta(Z))
\]

(68)
where the enumeration indicates terms as in (67), \( t_1, t_2 \in \mathbb{R} \), and \( r_1, r_2, r_3 \in C^\infty(P^{(1)}) \). Now, the theorem is proved once we show that

\[
\delta_\theta(c_{1,H^{(1)}}(j^1_x\sigma))(X,Y,Z) = \delta_\theta(c_{1,H^{(1)}}(j^1_x\sigma)(X,Y),Z) + \text{cyclic permutations of } (X,Y,Z) = 0.
\]

Indeed, terms like the first one in the second line of (68) are zero because \( \sigma(x) \) is an integral element, i.e. \( \sigma(x)^*c_\theta = 0 \). Terms involving \( \eta(\cdot) \) and \( \sigma(x)(\cdot) \), such as the second one in the second line of (68), vanish as well, since \( \eta \in g^{(1)} \) (equation (49)).

Last, all the terms inside \( c_\theta \) in the third line of (68) are vector fields taking values in \( H: [C(X),C(Y)], [C(X) + \eta(X), C(Y) + \eta(Y)] \subset H \) because \( C \) is a Cartan-Ehresmann connection, the same holds for \( C + \eta \), since \( \eta \in g^{(1)} \), and \( \eta(X), \eta(Y) \in g \subset H \). Therefore, \( c_\theta \) evaluated in these terms can be computed as \( \theta([\cdot, \cdot]) \); we can use the Jacobi identity to show that the part of \( \delta_\theta \) involving these terms vanishes.

Q.E.D.

### 5.2 Formal integrability

**Definition 5.10.** A Pfaffian bundle is called formally integrable when it is integrable up to any order.

It follows from Corollary 5.2 that, when \( (P,H) \) is a PDE, this definition coincides with the homonymous one, introduced in Section 2.3.

In particular, as in the theory of PDEs, formal integrability is not always a sufficient condition for PDE-integrability, but it becomes so in the analytic setting:

**Theorem 5.11 (Existence of analytic local holonomic sections).** If \( (P,H) \) is an analytic formally integrable Pfaffian bundle, then for every \( p \in P^{(k)} \subseteq J^kP \) over \( x \in M \) there is an analytic local holonomic section \( \beta \) of \( (P,H) \) such that \( j^k_x\beta = p \) on a neighbourhood of \( x \in \text{dom}(\beta) \).

In particular, \( (P,H) \) is PDE-integrable.

**Proof.** If \( (P,H) \) is formally integrable, its universal prolongation \( P^{(1)} \subseteq J^1P \) is a formally integrable PDE. Moreover, since \( P \) is an analytic manifold, \( J^1_P \) is analytic as well, being the kernel of the analytic bundle map \( e \) of equation (40). Similarly, \( P^{(1)} \subseteq J^1_P \) is analytic because it is the kernel the 1st curvature map \( c_1 \), which is also an analytic bundle map. We conclude that \( P^{(1)} \) is an analytic formally integrable PDE; we can apply Theorem 2.3, which gives precisely our statement.

In particular, for every \( p \in P^{(k)} = (P^{(1)})^{(k-1)} \) over \( x \) there exists a solution \( \beta \) of \( P^{(1)} \) such that \( j^k_x\beta = p \), and \( \beta \) being the solution of the PDE \( P^{(1)} \) means that \( \alpha = j^1_x\beta \) sits inside \( P^{(1)} \), i.e. \( \alpha \) is an holonomic section of \( (P^{(1)},H^{(1)}) \) and therefore \( \text{pr}(\alpha) = \beta \) is a holonomic section of \( (P,H) \).

The PDE-integrability of \( (P,H) \) follows from the PDE-integrability of \( P^{(1)} \) and the fact that \( \text{pr}: P^{(1)} \to P \) is surjective. Q.E.D.

We look now for sufficient conditions for formal integrability. An immediate one follows from Proposition 5.8

**Proposition 5.12.** Let \( (P,H) \) be a Pfaffian bundle of finite type \( k \). If \( P \) is integrable up to order \( k > l \), then it is formally integrable.
However, this Proposition follows also as a corollary from a straightforward generalisation of the cohomological integrability criterion 2.4 of Goldschmidt, which holds also for Pfaffian bundle of infinite type.

**Theorem 5.13.** Let \((P, H)\) be a Pfaffian bundle such that

- The symbol space \(g\) is 2-acyclic, i.e. \(H^{l,2}(g) = 0 \forall l \geq 0\)
- \(g^{(1)}\) is smooth and \(P^{(1)} \to P\) is surjective

Then \(P\) is formally integrable.

**Proof.** From the fact that \(g\) is 2-acyclic and \(g^{(1)}\) is smooth, it follows from Lemma 2.8 and Remark 2.9 that \(g^{(l)}\) is smooth also for \(l \geq 1\). Moreover, thanks to our hypotheses, \(P\) is already integrable up to order 1 by Theorem 4.18. Assume now that \(P\) is integrable up to order \(l \geq 1\): then the intrinsic torsion \(\tau_l : P^{(l)} \to H^{2,l-1}(g) = 0\) must vanish, hence \(P\) is integrable up to order \(l + 1\) by Proposition 5.5. By induction we find that \(P\) is formally integrable. Q.E.D.

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