Freely Falling Finite Frames Near a Black Hole

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It is well-known that the Riemann curvature tensor has no discontinuity at the black hole horizon. It is also well-known that a freely falling observer takes finite time to reach the horizon from an outside point. However, the usual assumption is that such an observer resides in a frame of reference (spaceship) of infinitesimal size. This assumption is justified as long as the coordinates are continuous enough to assume that the observer’s frame is small compared to the variations of the metric from a local flat metric. Such an assumption may be invalid when the coordinate system has not only a discontinuity but a singularity like the one at the horizon. Hence, here, the characteristics of a finite frame (a spaceship) near a black hole horizon is discussed. It is shown that clocks placed at the front and rear ends have different time scales even in the limit when they reach the horizon at the same time. This renders such a frame physically meaningless. It is also argued that the forces that are expected to keep a realistic frame (like a spaceship) in one piece tend to zero near the horizon. So, a physical spaceship is expected to fall apart near the horizon.

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I. INTRODUCTION

It is commonly understood that an observer falling freely into a black hole does not notice anything unusual while passing through the event horizon. This understanding is based on the assumption that the observer’s frame of reference (say a spaceship) is infinitesimal in size. However, real frames can never be truly infinitesimal. So, one needs to investigate the validity of the approximation of a finite frame by an infinitesimal one. The approximation should be valid as long as the finite frame is small enough to keep variations of its metric from a local flat metric negligible. This approximation can always be made if there are no discontinuities in coordinates. In the presence of coordinate discontinuities, and in particular singularities, more careful consideration is needed to see if the approximation is still valid. In particular, the coordinate singularity at the event horizon deserves such careful consideration. Hence, in the following, some features of a finite freely falling frame of reference will be considered. To keep the discussion simple, the frame will be assumed to fall radially towards the black hole.

II. DIFFERENCE IN TIME MEASUREMENTS BETWEEN FRONT AND REAR ENDS OF FALLING FRAME

The Schwarzschild line element in standard spherical polar coordinates $(t, r, \theta, \phi)$ is given as

$$d\tau^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\Omega^2,$$  

where

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2,$$

the speed of light $c = 1$, the Schwarzschild radius $r_s = 2GM/c^2$, $G$ is the universal gravitational constant and $M$ is the mass of the source. Using $\tau$ as the affine parameter the two equations of motion of a radially freely falling point particle are\[1–3,\]

$$\left(1 - \frac{r_s}{r}\right)^{-1} \frac{d^2r}{d\tau^2} + \frac{r_s}{2r^2} \left(\frac{dt}{d\tau}\right)^2 = 0,$$

$$\left(1 - \frac{r_s}{r}\right)^{-2} \frac{r_s}{2r^2} \left(\frac{dr}{d\tau}\right)^2 = 0,$$

On integration, these give,

$$\left(\frac{dr}{d\tau}\right)^2 = k^2 - (1 - r_s/r),$$

$$\left(1 - \frac{r_s}{r}\right) \frac{dt}{d\tau} = k,$$

where $k$ is a constant that depends on initial conditions. If the falling particle starts at $r = r_0$ at zero velocity, then equation [3] gives,

$$k = \left(1 - \frac{r_s}{r_0}\right)^{1/2}.$$  

A more compact way of writing equations [5] and [6] is as follows.

$$dr = -\sqrt{k^2 - A} \frac{dt}{k},$$

$$d\tau = \frac{A}{k} dt,$$

where $A$ is a constant that depends on initial conditions.
where,

\[ A = 1 - r_s/r. \]  

(10)

The negative sign for the square root is chosen as \( dr/dt \) is negative for a falling particle.

Now, consider a finite sized frame of reference (spaceship) falling freely along a radial direction towards the event horizon at \( r = r_s \). Let the front end of the spaceship be given by \( r = r_1 \) and the rear end be given by \( r = r_2 \) at any time \( t \). Also, let the frame start from rest at \( r_1 = r_{01} \) and \( r_2 = r_{02} \). So the initial length of the spaceship is,

\[ L = r_{02} - r_{01}. \]  

(11)

The non-gravitational forces keeping the spaceship together are expected to be small compared to the tidal forces. Hence, each of the two ends can be thought of as falling freely. So, the equations 5 and 4 can be written for each of the two ends as follows (using the subindices ‘1’ and ‘2’ for the front and rear ends respectively).

\[
\begin{align*}
    dr_1 &= -\sqrt{k_1^2 - A_1/k_1} \, dt, \\
    d\tau_1 &= A_1/k_1 \, dt, \\
    dr_2 &= -\sqrt{k_2^2 - A_2/k_2} \, dt, \\
    d\tau_2 &= A_2/k_2 \, dt,
\end{align*}
\]

(12) \hspace{1cm} (13) \hspace{1cm} (14) \hspace{1cm} (15)

where,

\[
\begin{align*}
    k_1 &= (1 - r_s/r_{01})^{1/2}, & A_1 &= 1 - r_s/r_1, \\
    k_2 &= (1 - r_s/r_{02})^{1/2}, & A_2 &= 1 - r_s/r_2.
\end{align*}
\]

(16) \hspace{1cm} (17)

If a clock placed at the front end measures the proper time interval \( d\tau_1 \), it is the invariant space-time separation between two events given by \((t, r_1)\) and \((t + dt, r_1 + dr_1)\) where \( dr_1 \) is the change in the position of the falling front end in time \( dt \). Let us now compute the time interval of the same two events as measured by a clock placed at the rear end. First, we transform \( dr_1 \) and \( dt \) to the stationary, locally flat coordinate system at the instantaneous position of the rear end. This requires the metric scale factors as shown in equation 11. So, the space and time intervals in this frame are,

\[
\begin{align*}
    d\tau' &= A_2^{-1/2} \, dr_1, \\
    dt' &= A_2^{1/2} \, dt.
\end{align*}
\]

(18) \hspace{1cm} (19)

Next, this needs to be transformed to the moving frame of the rear end using a Lorentz boost due to the velocity \( v_2 \) of the rear end with respect to the local stationary frame. This gives,

\[
\begin{align*}
    dR &= \frac{dr' - v_2 dt'}{\sqrt{1 - v_2^2}}, \\
    dT &= \frac{dt' - v_2 dr'}{\sqrt{1 - v_2^2}}.
\end{align*}
\]

(20) \hspace{1cm} (21)

where \( dR \) is the radial component and \( dT \) the time component. The velocity of a falling particle with respect to the stationary, locally flat frame can be found (using the metric scaling of equation 11) to be,

\[
v = A^{-1/2} dr/dt = \frac{1}{A} \frac{dr}{dt}.
\]

(22)

Using equation 3, this gives,

\[
v = -\sqrt{k^2 - A/k}.
\]

(23)

Specifically for the rear end this gives,

\[
v_2 = -\sqrt{k_2^2 - A_2/k_2}.
\]

(24)

Using this in equation 21 and making substitutions from equations 18, 19 and 12 gives,

\[
\begin{align*}
    dT &= \left( k_2 - \frac{A_1}{k_1} \sqrt{(k_1^2 - A_1)(k_2^2 - A_2)} \right) \, dr_1. \\
    &= \left( \frac{k_1 k_2}{A_1} - \sqrt{(k_1^2 - A_1)(k_2^2 - A_2)} \right) \, A_2 \, dt.
\end{align*}
\]

(25) \hspace{1cm} (26)

So the time interval in consideration is measured at the rear end to be different from the front end measurement by a scale factor \( S \).

\[
\begin{align*}
    dT &= S \, d\tau_1, \\
    S &= \left( \frac{k_1 k_2}{A_1} - \sqrt{(k_1^2 - A_1)(k_2^2 - A_2)} \right) / A_2.
\end{align*}
\]

(27) \hspace{1cm} (28)

To test the integrity of the finite frame at the event horizon, we need to find \( S \) in the limit of the spaceship reaching the horizon. This is the limit of \( t \to \infty \), and hence, \( r_1 \to r_s \) and \( r_2 \to r_s \). So, in this limit \( A_1 \to 0 \) and \( A_2 \to 0 \). As a result, we see that the limiting value of the scale factor,

\[
S_0 = \lim_{t \to \infty} S,
\]

(29)

has an indeterminate form. So, the limit has to be computed carefully. This has been done in appendix A and the result is as follows,

\[
S_0 = \frac{k_1^2 + k_2^2}{2k_1 k_2}.
\]

(30)

Note that \( S_0 = 1 \), if \( k_1 = k_2 \). This is expected, as \( k_1 = k_2 \) means both front and rear ends start at the same place making them the same frame. However, if \( k_1 \neq k_2 \), \( S_0 > 1 \). This means that, at the horizon, the front and
rear end time measurements for the same space-time interval will be different by a finite factor. Note that the two ends coincide on reaching the horizon. Also, using equation 28 it can be seen that both ends travel at the speed of light at the horizon. Hence, their time measurements being different by a finite factor is meaningless. So, finite frames of reference are meaningless at the event horizon.

One may also consider the extreme limit of the front end starting at the horizon \((r_{01}=r_s)\). In this case, as long as the rear end starts at a finite distance from the front, the scale factor \(S_0\) becomes infinity \([6]\). This is definitely not acceptable for any frame of reference!

### III. RELATIVE VELOCITY OF FRONT AND REAR ENDS

Now, let us compute the relative velocity of the front end with respect to the rear end. This would be \(dR/dT\), \(dR\) is computed using equations \([18\ 19\ 20\ 21\ 22]\) and also \([12\ 13]\). The result is,

\[
dR = S_r d\tau_1,
\]

where,

\[
S_r = \frac{k_1 \sqrt{k_2^2 - A_2} - k_2 \sqrt{k_1^2 - A_1}}{A_1}.
\]

Then the relative velocity is,

\[
V = \frac{dR}{dT} = \frac{S_r}{S}.
\]

In the limit of the frame reaching the horizon, \(S_r\) is found to be (see appendix \([A]\)),

\[
S_{r0} = \lim_{t \to \infty} S_r = \frac{k_2^2 - k_1^2}{2k_1k_2}.
\]

Hence, the limiting case of the relative velocity of the front end with respect to the rear end is,

\[
V_0 = \lim_{t \to \infty} V = \frac{k_2^2 - k_1^2}{k_1^2 + k_2^2}.
\]

For \(k_1 \neq k_2\), it can be seen that \(V_0 > 0\) as \(k_2 > k_1\). A positive relative velocity using the radial coordinate means the front end is moving outwards and hence, towards the rear end. So, tidal forces near the horizon will tend to compress the finite frame. This contradicts the common understanding derived from the weak gravity limit (Newtonian gravity).

In the extreme limit of the front end starting at the horizon \((r_{01}=r_s)\), \(V_0\) becomes unity which is the speed of light \([6]\).

### IV. FORCES NEAR THE EVENT HORIZON

A physical spaceship, in order to stay in one piece, relies on non-gravitational forces. A non-gravitational force can communicate between the ends of the spaceship at most at the speed of light. The above results show that there can be a finite time delay in information transmitted from the front to the rear even when the two ends are only infinitesimally apart. This means the effective speed of light tends to zero and communication of forces between the ends fails. Hence, at the horizon, the spaceship is expected to come apart.

### V. CONCLUSION

Here it has been shown that a finite frame of reference, like a spaceship, becomes meaningless when it reaches the event horizon of a black hole. Its front and rear ends reach the horizon together and at the same speed. But their time measurements of a space-time interval are different by a finite factor. Also, it is argued that forces that keep a spaceship together fail to transmit between the ends. As a result, the spaceship comes apart at the horizon. This, and other unphysical aspects of the black hole event horizon \([4]\), make the existence of such an event horizon suspect. A possible resolution is presented in an earlier publication \([5]\).

**Appendix A: Limiting Values of Scale Factors**

Equation \([28]\) shows that \(S\) takes an indeterminate form as \(t \to \infty\). As \(A_1\) and \(A_2\) tend to zero in this limit, one can expand the square roots in powers of \(1/k_1^2\) and \(A_2/k_2^2\) and keep up to first order terms. This gives,

\[
S \simeq k_1k_2 \left(\frac{1}{A_1} - \frac{1}{A_2}\right) + \frac{k_2A_1}{2k_1A_2} + \frac{k_1}{2k_2},
\]

(A1)

This still contains indeterminate forms. So, we need to find the relationship of \(A_1\) and \(A_2\) near the horizon. This can, of course, be found by integrating equations \([12\ 14]\) but a simpler approach is to realize that in the limit of \(A_1 \to 0\) and \(A_2 \to 0\), they can be related as follows.

\[
A_2 = \alpha A_1^p,
\]

(A2)

where \(\alpha\) and \(p\) are positive constants. Now, noting that \(A_2 \geq A_1\), it can be seen that \(S \to \infty\) as \(A_1 \to 0\) unless \(\alpha = 1\) and \(p = 1\). This is true even for \(k_1 = k_2\). But, we must have \(S = 1\) for \(k_1 = k_2\) as that is the case of the front and rear ends being the same throughout their trajectories. Hence, we conclude that \(\alpha = p = 1\). This gives,

\[
A_2 = A_1,
\]

(A3)
in the limit $A_1 \to 0$. Hence, in the same limit, we get (using equation [A1]),

$$S_0 = \lim_{t\to\infty} S = \frac{k_2^2 + k_2^2}{2k_1k_2}. \quad \text{(A4)}$$

Similarly, $S_{r_0}$ of equation [A4] can be computed from equation [22] The result is,

$$S_{r_0} = \frac{k_2^2 - k_1^2}{2k_1k_2}. \quad \text{(A5)}$$

[1] J. Foster and J. D. Nightingale, *A Short Course in General Relativity*, (Springer-Verlag, 2006).
[2] P. G. Bergmann, *Introduction to the Theory of Relativity*, (Dover, 1976).
[3] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, (W. H. Freeman and Company, 1973).
[4] T. Biswas, arXiv:1006.4185 [gr-qc] (2010).
[5] T. Biswas, arXiv:0809.1452 [gr-qc] (2008).
[6] This limit has to be interpreted only as a limit. If $r_{01} = r_s$ exactly, the procedure of appendix [A] fails.