Topological superfluids with time reversal symmetry

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It is shown that superfluids in two and three dimensions which have time reversal invariant ground states have phases which are distinguished by a topological invariant. Further, it is shown that the B-phase of $^3$He is a superfluid in the non-trivial topological class. Superfluids in the non-trivial topological class are shown to have gapless edge states and support various kinds of vortices with zero energy modes localized in their cores. Some of these vortices have non-abelian statistics.

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There has been a considerable amount of interest in phases characterized by a topological invariant (topological phases). The most well known examples of such phases are the quantum Hall states [1, 2, 3]. Chiral superconductors and superfluids are another class of systems which are characterized by a topological invariant [4]. A droplet of $^3$He in the A-phase [2] is an example of a 2D chiral superfluid while strontium ruthenate is thought to be a chiral superconductor [5]. Chiral superconductors and superfluids have a number of interesting properties. A neutral chiral superfluid has a macroscopic current which runs along the edge of a bounded sample. In a superconductor, these currents are screened but still substantial [2, 8]. Chiral superconductors and superfluids have gapless edge states. They also support exotic defects called half quantum vortices which carry half a quantum of flux. Each half quantum vortex carries a Majorana fermion and a pair of Majorana fermions is equivalent to be a chiral superconductor [6]. Chiral superconductors and superfluids have non-trivial topological class. They also support exotic defects which have zero-energy Majorana fermions and can lead to non-abelian statistics.

There has been considerable recent work on materials and systems with time reversal symmetry which have a non-trivial topological $Z_2$ invariant and some of these are believed to have been experimentally detected [12, 13]. These systems are generically called quantum spin Hall systems [14]. We show that the B phase of superfluid Helium-3 can be identified as a “topological superfluid” in the non-trivial topological class.

For the rest of this work, unless explicitly stated, by superfluids, we mean superfluids with unbroken translation symmetry. Lattice superconductors which are not invariant under the full translational group have previously been studied in an earlier work [11]. In Sec. I, we study and obtain a topological classification of superfluids in two and three dimensions. In Sec. II, we study the edge states in these superfluids. In Sec. III, we present examples of superfluids in the non-trivial topological class. In Sec. IV, we consider exotic vortices and their statistics in these systems.

I. TOPOLOGICAL SUPERFLUIDS WITH TIME REVERSAL SYMMETRY

The mean field BdG Hamiltonian which characterizes a superfluid may be written in the form:

$$H_{BdG} = \int d^d k \left[ \psi^\dagger(k) \hat{H} \psi(k) \right],$$

where

$$\hat{H} = \left( \hat{h}(k) \Delta(k) \right) \Delta^\dagger(k) - \hat{h}^T(-k).$$

Here $\hat{h}$ represents the single particle Hamiltonian, $\Delta$ the order parameter that characterizes superconductivity and $\psi, \psi^\dagger$ are the fermionic operators in the two component Nambu formalism. The BCS ground state wavefunction of the superfluid is annihilated by operators of the form $\gamma(k) = \sum_{\alpha=1,1} u_{k,\alpha} \psi_{k,\alpha} + v_{k,\alpha} \psi_{-k,\alpha}^\dagger$ where $(u_{k,\alpha}, v_{k,\alpha})^T$ is a negative energy eigenvector of the matrix $\hat{H}$.

As long as the system has a bulk gap, at each point in momentum space, there are two eigenvectors of this
matrix, say $e_1(k)$ and $e_2(k)$ with negative eigenvalues. As $k \to \infty$, the two dimensional vector space spanned by these states, which we denote by $V(k)$ goes to a fixed two dimensional space $[21]$. As far as the topology of the ground state wavefunction is concerned, the base space which is momentum space is thus a sphere, $S^n$ obtained by the one point compactification of $R^n$, where $n$ is equal to 2 or 3 in the cases of our interest.

When the ground state wavefunctions has time reversal symmetry, the eigenvectors of $H$ at $k,-k$ are not independent. Time reversal symmetry requires that if $u$ is an eigenstate at $k$, then $\Theta u$ is also an eigenstate with the same energy at $-k$. While the two dimensional vector space consisting of the negative energy eigenvectors can be written in terms of a continuous basis locally in momentum space, finding a global continuous basis is not always possible. The ground state thus defines a twisted two dimensional vector bundle on $S^2$ or $S^3$ as the case may be.

**Superfluids in 2D**

Let us first study the topology of the ground state wavefunction of a two dimensional superfluid following [10]. We divide $S^2$ into two patches, E and E' where $E = \{k \ni k < 2\}$ and $E' = \{k \ni k > 0.5\}$. On each of these patches we define a basis of two continuous orthogonal vector functions, $|v_{1,i}(k)|, |v_{2,i}(k)| \in V(k)$ where the index $i$ is a label for the patch E or E'. We represent the vector $v_{1}(k) = c_{1}v_{1,i}(k) + c_{2}v_{2,i}(k)$ as the spinor $(c_1,c_2)^T$. The operation of time reversal in this basis then corresponds to the operator $\sigma_2K_0\Phi$ where $K_0$ is the complex conjugation operator and $\Phi$ is the involution operator that takes vectors at $k$ to vectors at $-k$. If a vector $v$ is represented by the spinors $v_E$ and $v_{E'}$ in the two different bases defined above, then $v_E$ and $v_{E'}$ can be related by means of a transformation matrix $U$ such that $v_E = U v_{E'}$ where $U$ is an element of the unitary group, $U(2)$. The transition function, $U(k)$, which glues the vector representations $v_E(k), v_{E'}(k)$ on the two patches, on the circle $T = \{k \ni k = 1\} \subset E \cap E'$ can be written as $e^{i\alpha(k)}e^{in(\phi)\sigma \omega(k)}$. Then time reversal invariance leads to the condition:

$$U(-k) = e^{-i\alpha(k)}e^{in(\phi)\sigma \omega(k)}. \quad (3)$$

Then, as shown in previous work [13], $U_{2D}$, the set of all transition functions for these ground states in two dimensions consists of two topologically distinct classes of transition functions. The trivial class corresponds to the case when it is possible to find a globally continuous basis for the vector space spanned by $e_1(k), e_2(k)$. The non trivial $Z_2$ phase corresponds to the case when the transition function cannot continuously be deformed to unity. A simple formula for the $Z_2$ invariant can be written down when the wavefunctions $e_1(k), e_2(k)$ have well defined Chern numbers. When this is not the case, one can still always find a set of two orthogonal “wavefunctions” $e_1'(k), e_2'(k) \in V(k)$ for which well defined Chern numbers exist. The $Z_2$ invariant can then be expressed in a particularly simple form in terms of these Chern numbers. It is given by $|c_n| \mod 2$ where $c_n$ is the Chern number of either of these functions.

**Superfluids in 3D**

Now consider a superfluid in three dimensions. We again divide momentum space, which is now topologically equivalent to the three sphere $S^3$ for our purposes, into two patches $E = \{k \ni k < 2\}$ and $E' = \{k \ni k > 0.5\}$. As before, on each of these patches we define a basis of two continuous orthogonal vector functions, $|v_{1,i}(k)|, |v_{2,i}(k)| \in V(k)$ where the index $i$ is a label for the patch E or E'. The transition function, $U(k)$ which glues the vectors on the two patches, $v_E(k), v_{E'}(k)$ on the two sphere $T = \{k \ni k = 1\} \subset E \cap E'$ is again an element of $U(2)$ and can be written as $e^{i\alpha(k)}e^{in(\phi)\sigma \omega(k)}$. Time reversal symmetry results again in the constraint given by Eq. (3). Let C be any great circle on T. Then the function obtained by restricting the domain of $U(k)$ to T is a function which belongs to the class of two dimensional transition functions $U_{2D}$. This thus defines a projection from the class of transition functions on $S^2$ for the three dimensional case, which we call $U_{3D}$, to the class of transition functions for the two dimensional case, $U_{2D}$. As previously discussed, there are two topologically inequivalent classes in $U_{2D}$.

Further, it is well known that $\pi_2(U(2)) = \pi_2(SU(2)) = 0$. This implies that the class of transition functions in 3D is determined by the topological class of the 2D transition function that it projects to.

Thus the topological invariant for the 3D superfluid may also be evaluated using an extension of the simple Chern number formula from the 2D case. Let P be any two dimensional plane which maps onto itself under time reversal and contains the origin and let $e_1'(k), e_2'(k)$ be a set of orthogonal wavefunctions which span $V(k)$ at each point $k$ in the plane, which map onto each other under time reversal symmetry and for which well defined Chern numbers exist. Then, the 3D $Z_2$ invariant is simply $|c_n| \mod 2$, where $c_n$ is the Chern number of one of the wavefunctions.

We have shown above that there are thus two distinct topological phases of superfluids at the mean field level. This implies that if the full Hamiltonian of the system is adiabatically perturbed, as long as the ground state does not spontaneously break time reversal symmetry, the ground state remains in one of two distinct phases, unless there is a phase transition.

We note that superfluids in 2D have a single $Z_2$ invar-
ant as do lattice superconductors. However in 3D, lattice superconductors have four $Z_2$ invariants, while superfluids with unbroken translational symmetry have a single $Z_2$ invariant. This invariant corresponds to the fourth $Z_2$ invariant which is intrinsically three dimensional in nature and corresponds to the invariant in insulators and lattice superconductors which determines whether the system is in the strong or weak topological class.

II. EDGE STATES IN TOPOLOGICAL SUPERFLUIDS

Superfluids in 2D

A chiral 2D superconductor and a quantum Hall insulator are both characterized by the same topological number, namely the first Chern number $\mathbb{Z}_2$. The connection between the Chern number and the presence of robust edge states is well known in the context of the integer quantum Hall effect. Further, it is also known that these edge states exist in chiral superconductors and superfluids. This follows both from the Chern number - edge state connection and also from explicit calculations using the BdG Hamiltonian. The gapless edge excitations in the case of a chiral superfluid/superconductor are Majorana fermions or equal linear combinations of particles and holes. In the case of insulators with time reversal symmetry, the $Z_2$ invariant determines whether robust edge states exist or not. A superconductor whose ground state respects time reversal symmetry can not have net charge currents along the edge. Nevertheless since the ground state of a non-trivial 2D $Z_2$ superconductor may be adiabatically continued to a product state of two wavefunctions each of which has the same form in position space as the ground state wavefunction of a chiral superconductor with an odd Chern number, it follows that the TR invariant superconductor of the non trivial $Z_2$ class has a robust pair of edge states. When the Chern number is one, there is precisely one set of edge states and a single pair of Majorana fermions at zero energy at the edge. This pair of Majorana edge states is stable as long as time reversal symmetry is preserved because no operator that is invariant under time reversal symmetry can have a non-zero matrix element between the two states.

Superfluids in 3D

We now study the 3D case. Consider a semi infinite three dimensional superfluid with periodic boundary conditions in any two directions, say in the x and y directions and which exists in the region $z < 0$. The momentum variables $k_x, k_y$ are then good quantum numbers. The Hamiltonian $H(\alpha k_x + \beta k_y = 0, z)$ represents a 2D superfluid Hamiltonian which from our previous discussion is a 2D TR invariant Hamiltonian in the non-trivial $Z_2$ class. It therefore follows that the eigenstates of this Hamiltonian has a set of robust edge states with a 1D Dirac spectrum. Since this is true for arbitrary $\alpha, \beta$, it follows that the eigenstates of the Hamiltonian in general have a 2D Dirac energy spectrum at low energies. While the spectrum is Dirac-like, the eigenstates are Majorana rather than Dirac fermions. There is a single pair of zero energy Majorana fermions at the edge corresponding to the $k_x = k_y = 0$ state. More generally when there are open boundary conditions, there are two surfaces. From the above argument, it is expected that each one will have a single pair of zero energy Majorana fermions.

III. EXAMPLES OF TOPOLOGICAL SUPERFLUIDS

In this section, we shall present some examples of states in two and three dimensions that are in the non-trivial $Z_2$ class. Since these states are triplet states, it is useful to introduce the $d$-vector notation. The order parameter of a general triplet superconductor/superfluid can be written in the form

$$\Delta = \Delta_0 \left( \begin{array}{c} idy - dx \\ dz \\ idy + dx \end{array} \right)$$

where in the absence of textures, defects and edges, $d$ is a function only of $k$.

The state whose $d$-vector is given by

$$d(k) = i(k_x + ik_y)\hat{y}$$

is an example of a chiral $p+ip$ superconductor/superfluid. On the other hand, the state with

$$d(k) = k_x \hat{x} + k_y \hat{y}$$

is an example of a superfluid with a non-trivial $Z_2$ invariant. Since the spins decouple for such a state, the up and the down spins may be considered separately. The wavefunction for the up/down-spins are identical to those of a chiral $p_x - ip_y/p_x + ip_y$ superfluid. Thus, the form of the wavefunctions and their spectrum for chiral superfluids in rectangular and circular geometries with an edge may be used to deduce the wavefunction for the ground state of the above system, but shall not be presented explicitly here. It follows from the analogy with chiral superfluids, that there is a single pair of stable zero energy Majorana edge states as previously stated in Sec. II.

The Balian Werthamer state which is believed to exist in the B-phase of $^3$He has the description:

$$d(k) = k$$
This is a gapped 3D superfluid which is time reversal invariant. To determine its topological invariant, we consider the plane \( k_z = 0 \) in momentum space. The order parameter then corresponds to the 2D superfluid given by Eq. (5) which is in the non-trivial \( Z_2 \) class of 2D superfluids. From the results of Sec. I, it follows therefore that \(^3\text{He} - \text{B}\) is a superfluid in the non-trivial \( Z_2 \) class.

Indeed, as anticipated in Sec. II, the superfluid state when confined to a boundary has gapless edge states. Since the BdG Hamiltonian for this state is invariant under simultaneous rotations in spin and orbital space, one may consider without loss of generality a three dimensional sample confined with an infinite wall at \( y = 0 \). We further consider the set of states with \( k_z = 0 \). The resulting system can be regarded as the sum of two two dimensional chiral superfluids with opposite spin and chirality [24]. The edge states of a chiral 2D superfluid in a rectangular geometry were analyzed in Ref. [7]. It was found that the spectrum was linear in the momentum component parallel to the edge and that there was precisely one zero energy Majorana mode. It follows that the confined 3D superfluid considered here has a spectrum \( E \propto \sqrt{k_x^2 + k_y^2} \) which is 2D Dirac-like.

### IV. EXOTIC VORTICES AND NON-ABELIAN STATISTICS

The 2D topological superfluid state whose order parameter is given by Eq. (5) supports many kinds of exotic vortices. In the vortices that we shall be interested in, the order parameter takes the form:

\[
\Delta(r, \theta) = \Delta_0(r) \begin{pmatrix} e^{i\phi_-(\theta)}(id_y - d_x) & 0 \\ 0 & e^{i\phi_+(\theta)}(id_y + d_x) \end{pmatrix}.
\]

(7)

Here \( r, \theta \) are the polar coordinates in the vortex and \( \phi_{\pm} \) are the phase windings of the different components of the order parameter. When \( \phi_+ = \theta, \phi_- = 0 \) or vice versa, the defect is a half quantum vortex. In a superconductor such a defect would carry half a quantum of flux. Only one of the spin components are involved in the low energy physics. We may therefore consider a spinless chiral superconductor with a full quantum vortex, which as pointed out in Ref. [2] has a single zero energy Majorana core state at the center of a vortex. Further the statistics of these states is non-abelian.

When the system has both spin and orbital rotational symmetry the state given by Eq. (5) is degenerate with the set of states

\[
d(k) = R[k]
\]

where \( R[k] \) represents the vector obtained by a rotation, \( R \), of the k-vector in the two dimensional momentum space plane. One can therefore consider defects which are represented by

\[
\phi_+ = \pm \theta, \quad \phi_- = \mp \theta
\]

(9)

These are spin-vortices which may be seen as arising from the slow rotation of the \( d(k) \) in position space. The overall phase of the order parameter does not vary and thus these defects preserve time reversal symmetry. The spins are still decoupled and each spin component can be regarded as a chiral superfluid which now has a full quantum vortex but of opposite relative chirality. Further, it follows that there is a single pair of zero energy Majorana fermions which are stable to local perturbations which do not break time reversal symmetry. The statistics of these fermions for each individual spin component is non-abelian [24] and this could possibly be harnessed for quantum computation if the spins are kept decoupled.

The 3D superfluid previously discussed, which is described by the equation: \( d(k) = k \) is also degenerate with the set of states \( d(k) = R[k] \) where \( R \) is any rotation in real space when the system has both spin and orbital rotational symmetry. Since \( R \) is an element of \( SO(3) \) and \( \sigma_1(SO(3)) = Z_2 \), the superfluid then supports line defects which correspond to non-contractible paths in \( SO(3) \). An example of such a vortex is

\[
d(k, \theta) = R(n, \theta)[k]
\]

(10)

where \( \theta \) is the polar coordinate in the vortex and \( R(n, \theta)[k] \) is the vector obtained by rotating \( k \) through the angle \( \theta \) about the axis \( n \). To determine the low energy states, we apply periodic boundary conditions in the \( z \) direction and set \( n = z \). Then for the set of states \( k_z = 0 \), we see that the order parameter at the vortex has the same form as the defect in the 2D superfluid considered in Eqs. (7) and (9). There is thus a stable pair of zero energy Majorana states at the core of these vortices. The 3D superfluid state also hosts analogs of the half quantum vortex. The discussion from the 2D case can be easily carried over to the defects in the 3D superfluid states.

The above discussion pertained to superfluids with unbroken translational symmetry. Lattice superconductors in two dimensions are described by a single \( Z_2 \) invariant, while in 3D, they are described by four such invariants. The fourth \( Z_2 \) invariant for superconductors is analogous to the single \( Z_2 \) invariant for superfluids discussed above. The discussions pertaining to the edge states and the exotic defects presented above is also valid for superconductors with a non-trivial \( Z_2 \) invariant in 2D and a non trivial fourth \( Z_2 \) invariant in three dimensions.

In conclusion, we have shown that superfluids which do not break translational symmetry have two phases determined by topology. We also showed that the non-trivial topological class of superfluids have a number of interesting properties such as edge states and exotic vortices which obey non-abelian statistics. We also provided examples of such superfluids and showed that the B-phase of \(^3\text{He}\) is an example of a non-trivial topological super-
fluid. It is hoped that the present work will spur further interest and work in these systems.

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[21] This is based on the assumptions that $h(k)$ becomes a function only of the magnitude of $k$ and that $\Delta(k)$ vanishes as $k \to \infty$.
[22] The energy splitting induced by the finite distance between the edges falls of exponentially with the distance.
[23] This is a two dimensional version of the Balian-Werthamer state.
[24] More precisely, the Hamiltonian for these states is a direct sum of up and down spin components which are analogous to the Hamiltonians of chiral spinless superfluids.
[25] When the statistics of the vortices as a whole are considered however, the statistics is abelian.