Heisenberg Limit Superradiant Super-resolving Metrology

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We propose a superradiant metrology scheme to achieve Heisenberg limit super-resolving displacement measurement by encoding multiple light momenta into a three-level atomic ensemble. We use 2N coherent pulses to prepare a single excitation superradiant state in a superposition of two timed Dicke states that are 4N light momenta apart in the momentum space. The phase difference between these two states induced by a uniform displacement of the atomic ensemble has 1/4N sensitivity. Experiments based on Ramsey interferometry are proposed in crystal and in ultracold atoms.

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Measurements play a key role in physics, not only in the direct sense of defining standards [1,2], but also in verifying predictions of theories such as gravitational wave [3,4]. The quantum noise sets a standard quantum limit in the sensitivity of phase measurement [5] which is equal superposition means that we are totally unable to figure out where the photons are. The uncertainty of the photon number in the direct sense of defining standards [1−3], but also in verifying predictions of theories such as gravitational wave [4−5]. The quantum noise sets a standard quantum limit in the sensitivity of phase measurement [5].

The metrology with Schrödinger cat states indicates that the above statement can be used inversely, i.e., to improve the sensitivity of one observable by preparing a quantum state with large uncertainty in its conjugate observable. Therefore, the sensitivity of the displacement x can be improved by preparing a Schrödinger cat state of the momentum p. The N00N state composed by two optical modes with opposite momenta seems to be a good candidate, but the fragile high photon number N00N state is very difficult to prepare, to preserve, and to manipulate [21−27]. On the other hand, single photon can be easily prepared in an entangled state of two opposite momenta, 1/√N (|1k0−k⟩ + |0k1−k⟩) [28]. To manipulate this entangled state, we can guide it to an ensemble of N atoms and the momentum can be translated into the phase of the collective excitation, 1/√2 (|bk⟩ + |b−k⟩), where |bk⟩ ≡ 1/√N0 ∑Nj=1 exp (ikxj) |c1, c2, ..., bj, ...cN⟩

is the timed Dicke state [29]. Here we prepare the pencil-like atomic ensemble along x direction and make the wave vector k = k̂x. |cj⟩ and |bj⟩ are the excited and ground states of the atom at position xj. A uniform displacement of the atomic ensemble, xj → xj + r0, will attach opposite phases to the two timed Dicke state, 1/√2 (eikr0|bk⟩ + e−ikr0|b−k⟩). The phases can be retrieved by the interference pattern of photon signal emitted in the two opposite directions, cos² (k(r0)). If we prepare the ensemble in an atomic N00N state with entangled N-fold collective excitations [30], the interference pattern can have N times higher resolution [31]. However, the efficiency in preparing these N00N states drops exponentially with N [31,32].

In this Letter, we show that the “momentum
We propose an experimental scheme based on ground ensemble with coherent pulses. The scheme is similar to the pulse pair scheme used to prepare the atomic ensemble in the superradiant state with N pulses. The N00N state with high atomic efficiency can be prepared in the timed Dicke state. Instead of atomic N00N state with high atomic efficiency, we only need a single excitation superradiant state. Instead of atomic N00N state with high atomic efficiency, we only need a single excitation superradiant state. Instead of atomic N00N state with high atomic efficiency, we only need a single excitation superradiant state. Instead of atomic N00N state with high atomic efficiency, we only need a single excitation superradiant state. Instead of atomic N00N state with high atomic efficiency, we only need a single excitation superradiant state.

\[
U_l = \exp \left[ i \frac{\pi}{2} \sum_{j=1}^{N_a} \left( e^{i k_1 x_j + i \phi_l} \sigma_j^+ + e^{-i k_1 x_j - i \phi_l} \sigma_j^- \right) \right],
\]

for \( \frac{1}{2} \Omega(t) e^{-i \omega t + i k_1 x + i \phi_l} \) and each Rabi frequency area \( S = \int \Omega(t') dt' = \pi (l = 1, 2) \). \( \sigma_j^+ = |b_j\rangle\langle a_j| \) and \( \sigma_j^- = |a_j\rangle\langle b_j| \) are the raising and lowering operators for the jth atom. \( \phi_l \) is the phase of field \( \Omega_l \).

In the following, we set \( \phi_1 = \phi_2 = 0 \). At time \( t_1 \), we apply \( U_l \) to \( |b_0\rangle \),

\[
U_1 |b_0\rangle = \sum_{j=1}^{N_a} \left( e^{i k_1 x_j} \sigma_j^+ + e^{-i k_1 x_j} \sigma_j^- \right) |a_j\rangle.
\]

where we have defined \( |a_j\rangle \) in the same way as Eq. \( 4 \) by replacing \( b \) with \( a \). The atomic ensemble transit to \( |a_{-k_1}\rangle \) by collectively emitting a photon in mode \( k_1 \), and therefore acquire momentum \( -\hbar k_1 \) based on momentum conservation. In calculating Eq. \( 4 \), we should note that the terms \( \sigma_j^+, \sigma_j^- \) and the higher order ones in the expansion of \( U_l \) applied to the single photon Dicke state lead to zero.

At time \( t_2 \), we send the \( \pi \)-pulse of \( k_2 \) and the state evolves to

\[
U_2 |a_{-k_1}\rangle = - |b_{-2k_1}\rangle.
\]

The atomic ensemble get another momentum \( \hbar k_2 = - \hbar k_1 \) by collectively absorbing a photon from mode \( k_2 \). The pulse pair \( U_2 U_1 \) encode a total momentum \( -2\hbar k_1 \) in the atomic ensemble. The above process can be repeated for another \( N - 1 \) times and the final state becomes \((-1)^N \langle b_{-2Nk_1}\rangle \) with a large momentum \(-2N\hbar k_1\).

By combining the above mechanism and the technique of Ramsey interferometry, we can measure a displacement to the Heisenberg limit. To prepare a “momentum Schrödinger cat state”, we should first prepare a superposition state of \( |b_0\rangle \) and \( |a_0\rangle \). Then the \( \pi \)-pulses will drive these two states in two opposite directions in the momentum space. The whole scheme is shown in Fig. 2.

In the following, we show the explicit procedure based on a tripod Raman configuration which has been proved to have decoherence time as long as 1 minute \( \Delta t \). The atom has three degenerate ground states which can be lifted by a Zeeman magnetic field, as shown in Fig. 3 (a). We first pump all the atoms to state \( |c\rangle \). An off-resonant coherent field induces a Raman transition via intermediate state \( |d\rangle \) to prepare the atomic ensemble in the state \( |b_0\rangle \), accompanied by the emission of a single Stokes photon \( |S\rangle \). A zero-phase Raman \( \frac{\pi}{2} \)-pulse
Now we let the atomic ensemble experience a collective displacement \( x_j \rightarrow x_j + r_0 \) by a uniform optical force or gravity. The state becomes
\[
\left(-1\right)^N \left( \frac{1}{\sqrt{2}} b_{-2Nk_1} + \frac{i}{\sqrt{2}} a_{2Nk_1} \right).
\]

The relative phase between the two states is \( 4Nk_1r_0 \) and the sensitivity is enhanced by \( 4N \) times compared with a single Raman transition. To retrieve this phase, we apply the inverse \( \pi \)-pulse sequences \( U_1U_2 \) to the state in Eq. (10). Because the unitary transform is reversible, we will finally get
\[
|\Psi\rangle = \frac{1}{\sqrt{2}} e^{-i2Nk_1r_0} |b_0\rangle + \frac{i}{\sqrt{2}} e^{i2Nk_1r_0} |a_0\rangle.
\]

A zero-phase \( \frac{\pi}{2} \)-pulse will transform the state in Eq. (11) to
\[
U_{ba} \left( \frac{\pi}{2} \right) |\Psi\rangle = -i \sin (2Nk_1r_0) |b_0\rangle + i \cos (2Nk_1r_0) |a_0\rangle.
\]
Now the probability of the state $|b\rangle$, $P_b = \sin^2(2Nk_1r_0)$ can be got by observing the retrieved photon via the forward Raman transition $|b\rangle \rightarrow |d\rangle \rightarrow |c\rangle$ after a pumping pulse coupling $|b\rangle$ to $|d\rangle$ is applied. Here the Raman transition $|b\rangle \rightarrow |d\rangle \rightarrow |c\rangle$ will happen rather than $|b\rangle \rightarrow |d\rangle \rightarrow |a\rangle$ due to a superradiant enhancement of the vacuum interaction for the former one [23]. The probability of $|a_0\rangle$, $P_a = \cos^2(2Nk_1r_0)$ can be simultaneously measured in a different direction. Their difference
\[
P = P_b - P_a = -\cos(4Nk_1r_0)
\] (13)
is the signal from which the displacement $r_0$ can be measured. The noise is $\Delta P = |\sin(4Nk_1r_0)|$ and the phase sensitivity
\[
\Delta(k_1r_0) = \frac{\Delta P}{|\partial P/\partial(k_1r_0)|} = \frac{1}{4N}
\] (14)
scales at the Heisenberg limit.

One obstacle that can kill the interference pattern is the random overall phase of each operation $U_1$ and $U_2$, if the control pulses do not have a fixed relative phase. However, we can group $U_1$ and $U_2$ in pairs, as shown in Fig. 3(c). If the fields $k_{ad}$ and $k_{da}$ carry the phases $\phi_{bd}$ and $\phi_{da}$, and then the phases of $U_1$ and $U_2$ are $\phi_1 = \phi_2 = \phi_{bd} - \phi_{da} \equiv \phi_r$, the pair operation of $U_1$ and $U_2$ is therefore $U_2U_1 = -\sum_{j}^{A_{1}} \cos(2k_jx)/|b_j\rangle\langle b_j| + e^{2ik_jx}|a_j\rangle\langle a_j|$ where the additional phases in the delay line paths are tuned to be a multiple of $2\tau$. The operation $(U_2U_1)^N$ is therefore free of the random phase $\phi_r$.

The imperfection of the $\pi$-pulses in amplitude and in phase due to the environmental noises, such as the oscillation and the rotation of the optical devices, can reduce the phase sensitivity. We suppose the area $S$ and the phase $\phi$ of the $\pi$-pulses have Gaussian distribution with variations $\Delta S, \Delta \phi \ll 1/\sqrt{N}$. The phase sensitivity is then $\Delta(k_1r_0) = [4N (1 - N\Delta S^2/2)]^{-1} + \Delta \phi/\sqrt{4N}$. We simulate the interference patterns in Fig. 4 for $\Delta S = 0.1$ and $\Delta \phi = 0.01$. Although $\Delta S$ reduces the visibility and $\Delta \phi$ blurs the interference pattern, the phase sensitivities for $N = 16$ and $32$ are still enhanced to $1/55$ and $1/98$, marginally lower than the Heisenberg limit $1/64$ and $1/128$ whereas much higher than the shot-noise limit $1/8$ and $1/11$.

The pure dephasing between the ground states due to the environmental noise field does not make the “momentum Schrödinger cat state” more fragile as the number $N$ increases. Although there are $2N$ light momenta encoded in the media, the ensemble only contain a single excitation whose dephasing is independent of $N$. This robustness against the noise field is a big advantage of our scheme over the |N00N⟩ state and the spin Schrödinger cat state |CAT⟩. However, the relative motion between the atoms will reduce the visibility by a factor $e^{-cN^2}$ as we will discuss later. Therefore, to propose an experimental implementation, a solid system where the relative distance between atoms is fixed is preferred, especially the earth-ion-doped crystal, such as Pr$^{3+}$:Y$_2$SiO$_5$ whose quantum memory storage time reaches to 1 minute [37, 38]. The three states $|a\rangle, |b\rangle$ and $|c\rangle$ can be chosen from the ground state $^3H_g$. The intermediate state $|d\rangle$ is a sublevel of the excited state $^1D_2$. The Zeeman splitting can be $\sim$ MHz. The control pulses of $U_{1,2}$ can have duration of $10\mu s$ to avoid transition to unwanted levels. The prepare and read stage of the measurement can cost time $10\mu s$ for $N = 10$. Then for the transition wavelength $\sim 600$nm, the resolution is $7.5\mu m$. To move the crystal for $100$nm can be achieved in several ms by gravitation. The whole measurement can be completed well within the decoherence time of 1 minute.

For a proof of principle verification of the mechanism, the ultra cold atoms can also work. The obstacle is that the thermal motions of the atoms will randomize the relative phase between the atoms, and reduce the visibility $P = e^{-(2Nk_1v_m\tau)^2} \cos(4Nk_1r_0)$ where $v_m = \sqrt{2k_BT/m}$ is the most probable velocity and $\tau$ is the overall time of measurement. The phase sensitivity is thus $\Delta(k_1r_0) = \epsilon^2 \Delta N/4N$, where $\epsilon = 2k_1v_m\tau$, roughly the number of wavelengths the atom travelled. The minimum phase sensitivity is $\sqrt{2\epsilon\epsilon}/4$ when $N = 1/\sqrt{2\epsilon}$. We use a copropagating configuration in the Raman pulses to achieve a small $k_j$ and consequently small $\epsilon$. Take $^{87}$Rb as an example [31, 36], the three levels are chosen to be the three Zeeman sublevels of $5^2S_{1/2}, F = 1$. The intermediate state is $5^2S_{1/2}, F = 2, m_F = 0$. If we use nanosecond Raman transition pulses with GHz positive detuning, the $\pi$-pulses require only a moderate average power $10$W/cm$^2$. At temperature $T \sim \mu K$, $v_m \sim 1$cm/s. We suppose the whole measurement costs time $\tau = 100\mu s$ during which a displacement of $10\mu m$ can be achieved by optical force [31]. Then the random displacement is $v_m\tau \sim 1\mu m$. If the effective wavelength in the Raman transition is $\lambda_1 = 2\pi/k_1 \sim 200\mu m$. For resolution of $5\mu m$, we need $N = 10$. The interference pattern becomes
$P \approx -0.67 \cos(40k_1r_0)$ which still has large visibility. The phase sensitivity 1/27 still exceeds the shot-noise limit $1/\sqrt{40} \sim 1/6$.

In conclusion, we propose a scheme of subwavelength metrology with an atomic ensemble in a “momentum Schrödinger cat state” prepared by encoding multiple photon momenta into the phase of a timed Dicke state. In stead of multiple excitations in photon and spin N00N state, we only manipulate the single excitation superradiant state which can be prepared with high efficiency. By sending $2N$ coherent pulses into the atomic ensemble, a superposition of two timed Dicke state are separated by 4N light momenta in the momentum space, which can yield a phase sensitivity of 1/4N. We analysed the feasibility of a proof of principle experiment in Pr$^{3+}$:Y$_2$SiO$_5$ crystal and in ultra cold $^{87}$Rb atoms.

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