DERIVATION OF EFFECTIVE TRANSMISSION CONDITIONS FOR DOMAINS SEPARATED BY A MEMBRANE FOR DIFFERENT SCALING OF MEMBRANE DIFFUSIVITY

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Abstract. We consider a system of non-linear reaction–diffusion equations in a domain consisting of two bulk regions separated by a thin layer with periodic structure. The thickness of the layer is of order $\epsilon$, and the equations inside the layer depend on the parameter $\epsilon$ and an additional parameter $\gamma \in [-1, 1)$, which describes the size of the diffusion in the layer. We derive effective models for the limit $\epsilon \to 0$, when the layer reduces to an interface $\Sigma$ between the two bulk domains. The effective solution is continuous across $\Sigma$ for all $\gamma \in [-1, 1)$. For $\gamma \in (-1, 1)$, the jump in the normal flux is given by a non-linear ordinary differential equation on $\Sigma$. In the critical case $\gamma = -1$, a dynamic transmission condition of Wentzell-type arises at the interface $\Sigma$.

1. Introduction. In this paper, we consider a system of reaction-diffusion equations in a domain $\Omega$ consisting of two bulk regions $\Omega^+_\epsilon$ and $\Omega^-_\epsilon$, which are separated by a thin layer $\Omega^{M\epsilon}$ with periodic heterogeneous structure. The thickness of the layer between the bulk domains is of order $\epsilon > 0$, where the parameter $\epsilon$ is much smaller than the length scale of the whole domain. The equations in the membrane depend on $\epsilon$, the periodicity of the heterogeneities being of order $\epsilon$, and the diffusion coefficients having the size $\epsilon^\gamma$ with $\gamma \in [-1, 1)$. Our aim is to derive effective models in the limit $\epsilon \to 0$. In the limit, the thin layer reduces to an interface $\Sigma$ between the bulk domains, and the processes in the bulk are again described by a system of non-linear reaction-diffusion equations. The crucial part of the paper is to derive the effective transmission conditions across $\Sigma$. It turns out that different results are obtained for different values of the parameter $\gamma$. For $\gamma \in (-1, 1)$, the solution is continuous across the interface $\Sigma$, and the jump in the normal flux is given by a non-linear ordinary differential equation on $\Sigma$. In the critical case $\gamma = -1$, we have again continuity of the effective solution, however, the jump in the normal flux is described by a non-linear reaction-diffusion equation on the interface $\Sigma$. Such a condition is also called a dynamic Wentzell-transmission condition. These kind of boundary conditions were first considered in a general setting in [19] from a probabilistic point of view. Applications of these conditions can be found e.g., in [4] for fractured medium, and [18] for water waves. In [11] a Wentzell-boundary condition was derived in the limit $\epsilon \to 0$ of a Allen-Cahn-type equation in a homogeneous boundary layer of thickness $\epsilon$. The effective models in this paper differ from those

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obtained in [15] for the case $\gamma = 1$, where the transmission conditions involved jumps in the solution and in the normal flux, the jumps in the normal flux being computed by means of solutions to local problems in the layer. The effective model obtained in [15] was analyzed and numerical approaches for its computation were developed in [16]. The situation of curved layers with locally periodic microstructure is considered in [14]. A further contribution to the investigation of processes through thin layers with periodic microstructure, described by periodically varying constitutive properties is given e. g., in [13]. In this paper, the asymptotic behavior of an elastic body containing a thin layer with periodically varying rigidity modulus is considered, and the $\Gamma$-convergence framework is used to find the approximating energy functional. We emphasize that in our paper, we perform not only the homogenization of the heterogeneous structure, but also a dimension reduction of the thin layer to an hyperplane. Further problems with this feature can be found in the framework of elasticity, where elastic bodies containing material inclusions periodically distributed along a hyper-surface are considered, see e. g., [9], and also in fluid flow through thin filters, where the two bulk regions are connected by thin channels, see e. g., [2]. Finally, we mention the paper [6], where two bodies joined along a rapidly oscillating periodic surface are considered, the interface being situated in a $\varepsilon$ neighborhood of an hyperplane. Here, an effective model for the stationary heat diffusion is derived.

For the derivation of effective transmission conditions across $\Sigma$, we use two-scale convergence for thin layers, introduced in [12] for homogeneous layers, and extended to heterogeneous layers with periodic structure in [15]. Since the scaling in our paper is different from that in [15], we first prove some general compactness results concerning weak two-scale convergence for gradients in thin layers. To pass to the limit in the non-linear terms in the membrane equation, we need strong two-scale convergence of the microscopic solutions in the thin domain. To establish this strong convergence, we use its characterization via the unfolding operator for thin domains, and apply a Kolmogorov-type compactness result for Banach-space valued functions in $L^2(\Sigma, L^2((0,T) \times Z))$. This is based on estimates for the shifts of the microscopic solutions in the thin layer, with respect to the variable in $\Sigma$. A similar idea was used in [15] for the case $\gamma = 1$. There, the classic Kolmogorov compactness criterion was employed. However, the proof cannot be transferred to the case $\gamma \in [-1,1)$. Moreover, by using the compactness result from Lemma 4.6, the $L^\infty$-estimates for the microscopic solutions can be avoided.

For the critical case $\gamma = -1$, additional difficulties arise due to the fact that the diffusion term does not vanish in the limit. We obtain a surface diffusion with an effective diffusion coefficient computed with help of the solution of a cell-problem in the membrane. For the derivation of this cell-problem, we prove $H^2$-regularity for the microscopic solutions in $\Omega^\pm_\varepsilon$ and $\Omega^M_\varepsilon$, together with trace estimates for the first derivatives, in which the $\varepsilon$-dependence is given explicitly.

Membranes occur in many applications and we can find a whole range of diffusion-coefficients, depending on the role of the membrane for the particular application. In biology, the cell membrane for example, acts as a selectively permeable barrier between the intracellular and extracellular environment, thus having large diffusion coefficients for chemical species required for cellular function and small ones e. g., for chemicals that can be toxic to a cell. On the other hand, there are also isolating cell layers like e. g., mucous membranes or the epidermis with its upper layer stratum corneum, which act as barriers to protect underlying tissue from infection,
dehydration and chemicals. In engineering disciplines cooling devices (thin layers of highly conducting material) are employed to transmit heat smoothly between two media, while ceramic coatings act as barriers to diffusion in combustion engines. Since the content of membranes defines their physical and biological properties, in the microscopic model the structural details have to be resolved.

When considering a special application, the starting point in the multi-scale analysis is a dimensional analysis by which, starting from characteristic values of the physical parameters, relevant parameter combinations are identified and are set in relation with the scale parameter $\epsilon^\gamma$, with $\gamma$ being a real parameter. In this paper, the range of $\gamma$ is restricted to the interval $[−1,1)$, due to the fact that our compactness results used in the derivation of the effective model can be applied in these cases. As already mentioned the case $\gamma = 1$ is treated in [15].

For other choices of $\gamma$, we expect that for large coefficients (of order $\epsilon^{\gamma'}, \gamma' < −1$) the membrane has no contribution to the effective model, whereas for very small coefficients (of order $\epsilon^{\gamma'}, \gamma' > 1$) the membrane acts as an impermeable layer. These are however expectations which have to be validated by a rigorous analysis in future investigations. Further, open problems arise when more than one small parameter have to be considered, e.g., in the case when the thickness of the membrane and the period of the membrane heterogeneities are of different orders of magnitude. Furthermore, different modeling and multi-scale techniques could be applied, taking into account the stochastic aspects of the membrane structure.

Our paper is organized as follows. In Section 2, we present the microscopic problem and show a priori estimates and regularity results. In Section 3, we shortly recapitulate the notions of two-scale convergence for thin heterogeneous domains, and unfolding operator for thin domains, together with related results. Further, we establish some general compactness results concerning two-scale convergence of gradients in thin layers. In Section 4, we prove convergence results for the sequence of microscopic solutions, especially the strong two-scale convergence in the membrane. In the last Section 5, the effective models for the different values of $\gamma$ are derived.

2. The microscopic model. Let $\Omega := (0,1)^{n−1} \times (-H,H) \subset \mathbb{R}^n$ with fixed $H > 0$ and $n \geq 2$. Further, let $\epsilon > 0$ be a sequence with $\epsilon^{-1} \in \mathbb{N}$, and define the subdomains

$$\Omega^+_\epsilon := (0,1)^{n−1} \times (\epsilon, H),$$
$$\Omega^M_\epsilon := (0,1)^{n−1} \times (-\epsilon, \epsilon),$$
$$\Omega^-_\epsilon := (0,1)^{n−1} \times (-H, -\epsilon),$$

see Figure 1. The domains $\Omega^\pm_\epsilon$ and $\Omega^M_\epsilon$ are separated by an interface $S^\pm_\epsilon$, i.e.,

$$S^+_\epsilon := (0,1)^{n−1} \times \{\epsilon\} \quad \text{and} \quad S^-_\epsilon := (0,1)^{n−1} \times \{-\epsilon\},$$

hence, we have $\Omega = \Omega^+_\epsilon \cup \Omega^-_\epsilon \cup \Omega^M_\epsilon \cup S^+_\epsilon \cup S^-_\epsilon$. For $\epsilon \to 0$, the membrane $\Omega^M_\epsilon$ reduces to the interface $\Sigma := (0,1)^{n−1} \times \{0\}$, separating the domains

$$\Omega^+ := (0,1)^{n−1} \times (0, H) \quad \text{and} \quad \Omega^- := (0,1)^{n−1} \times (-H, 0).$$

To describe the microscopic structure of $\Omega^M_\epsilon$, we introduce the standard cell $Z$ given by $Z := Y \times (-1,1)$ with $Y := (0,1)^{n−1}$, and denote the upper and lower boundary
Figure 1. The microscopic domain containing the thin layer $\Omega^M_\epsilon$ with periodic structure. The heterogeneous structure of the membrane is modeled by the diffusion-coefficient $D^M_\epsilon$. In biology such a layer is e.g., the stratum corneum which consists of flattened cells (corneocytes) surrounded by lipid components.

of $Z$ by

$$S^+ := Y \times \{1\} = (0, 1)^{n-1} \times \{1\} \quad \text{and} \quad S^- := Y \times \{-1\} = (0, 1)^{n-1} \times \{-1\}.$$  

In $\Omega$, we consider a reaction-diffusion system for the unknown vector-valued function $u_\epsilon := (u_{1,\epsilon}, \ldots, u_{m,\epsilon}) : (0, T) \times \Omega \to \mathbb{R}^m$ with $m \in \mathbb{N}$, and we denote the restriction of $u_\epsilon$ to the subdomains $\Omega^\pm_\epsilon$ and $\Omega^M_\epsilon$ by $u^\pm_\epsilon$ and $u^M_\epsilon$, and the restrictions of the components for $i = 1, \ldots, m$ by $u^\pm_{i,\epsilon}$ and $u^M_{i,\epsilon}$. Thus, $u_\epsilon$ is a solution of the problem

$$\begin{align*}
\partial_t u^\pm_{i,\epsilon} - D^\pm_1 \Delta u^\pm_{i,\epsilon} &= f^\pm_i(t, x, u^\pm_\epsilon) \quad \text{in } (0, T) \times \Omega^\pm_\epsilon, \\
\frac{1}{\epsilon} \partial_t u^M_{i,\epsilon} - \gamma \nabla \cdot \left(D^M_1 \left(\frac{x}{\epsilon}\right) \nabla u^M_{i,\epsilon}\right) &= \frac{1}{\epsilon} g_i \left(\frac{x}{\epsilon}, u^M_\epsilon\right) \quad \text{in } (0, T) \times \Omega^M_\epsilon,  
\end{align*}$$

with $\gamma \in [-1, 1)$. On the outer boundary $\partial \Omega$, with outer unit normal $\nu$, we assume a zero Neumann-boundary condition, i.e.,

$$-\nabla u_{i,\epsilon} \cdot \nu = 0 \quad \text{on } (0, T) \times \partial \Omega.$$  

The initial condition is given by

$$u_\epsilon(0, x) = \begin{cases} U^\pm_0 \left(\frac{x}{\epsilon}, \frac{x_n}{H-\epsilon} H\right) & \text{for } x \in \Omega^\pm_\epsilon, \\
U^M_0 \left(\frac{x}{\epsilon}\right) & \text{for } x \in \Omega^M_\epsilon. \end{cases}$$

For the coupling condition between the bulk regions $\Omega^\pm_\epsilon$ and the membrane $\Omega^M_\epsilon$, we assume natural transmission conditions on the interfaces $S^+_\epsilon$ and $S^-_\epsilon$, i.e., continuity of the traces and the normal fluxes:

$$\begin{align*}
u^\pm_\epsilon &= u^M_\epsilon \quad \text{on } S^+_\epsilon, \\
-D^\pm_1 \nabla u^\pm_{i,\epsilon} \cdot \nu &= -\gamma D^M_1 \left(\frac{x}{\epsilon}\right) \nabla u^M_{i,\epsilon} \cdot \nu \quad \text{on } S^-_\epsilon.  
\end{align*}$$

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\partial_t u^\pm_{i,\epsilon} - D^\pm_1 \Delta u^\pm_{i,\epsilon} &= f^\pm_i(t, x, u^\pm_\epsilon) \quad \text{in } (0, T) \times \Omega^\pm_\epsilon, \\
\frac{1}{\epsilon} \partial_t u^M_{i,\epsilon} - \gamma \nabla \cdot \left(D^M_1 \left(\frac{x}{\epsilon}\right) \nabla u^M_{i,\epsilon}\right) &= \frac{1}{\epsilon} g_i \left(\frac{x}{\epsilon}, u^M_\epsilon\right) \quad \text{in } (0, T) \times \Omega^M_\epsilon,  
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\end{align*}$$
Here $\nu$ is the outer unit normal on $S^\pm_x$ with respect to $\Omega^M_x$.

**Assumptions on the data:**

**A1**) For $i = 1, \ldots, m$ it holds $D^+_i > 0$ and $D^+_i \in C_{\text{per}}^{0,1}(\overline{\mathbb{Y}}, C^{0,1}([1,1]))$. Further, we set $D^+_i(x) := D^+_i \left( \frac{x}{\epsilon} \right)$ and $D^+_i$ is strictly positive.

**A2**) The function $f^\pm = (f_1^\pm, \ldots, f_m^\pm) : [0,T] \times \overline{\Omega^\pm} \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous, uniformly continuous with respect to the second variable (independent of the third variable) and uniformly Lipschitz continuous with respect to the third variable. The uniform Lipschitz condition ensures the estimate

$$|f^\pm_i(t,x,z)| \leq C(1 + |z|) \quad \text{for all} \quad (t,x,z) \in [0,T] \times \overline{\Omega^\pm} \times \mathbb{R}^m.$$

**A3**) The function $g = (g_1, \ldots, g_m) : [0,T] \times \overline{\mathbb{Y}} \times [-1,1] \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous, uniformly Lipschitz continuous with respect to the last variable and $Y$-periodic with respect to the second variable. As above, we have

$$|g_i(t,y,z)| \leq C(1 + |z|) \quad \text{for all} \quad (t,y,z) \in [0,T] \times \overline{\mathbb{Y}} \times [-1,1] \times \mathbb{R}^m.$$

**A4**) For the initial functions, we assume $U^+_0 \in H^2(\Omega^+)^m$ and $U^+_0 \in H^2(\Sigma)^m$, i.e., $U^+_0 \ast \epsilon$ is independent of the last variable and it holds

$$\frac{1}{\sqrt{\epsilon}} \left\| U^+_0 \right\|_{L^2(\Omega^+_0)} + \epsilon^2 \left\| \nabla U^+_0 \right\|_{L^2(\Omega^+_0)} \leq C$$

for all $\gamma \in [-1,1]$.

**Remark 2.1.** (i) In [15], problem (1) was treated assuming the scaling $\gamma = 1$ for the diffusion coefficients in the layer $\Omega^+_x$, and taking Dirichlet-boundary condition on the upper and lower boundary $(0,1)^{n-1} \times \{ \pm H \}$ of $\Omega$. Replacing the zero Neumann-boundary condition on the upper and lower boundary of $\Omega$ by a Dirichlet-boundary condition, would not influence the result of our paper.

(ii) The results of our paper remain also valid for more general diffusion-coefficients, like e.g., $D^+_i \in C_{\text{per}}^{0,1}(\overline{\mathbb{Y}}, C^{0,1}([1,1])^{n \times n}$ satisfying the ellipticity condition

$$(D^+_i(y)\xi,\xi) \geq \alpha\|\xi\|^2 \quad \text{for all} \quad y \in \overline{\mathbb{Z}},$$

with a positive constant $\alpha$, independent of $y$. A similar generalization can be made for the diffusion-coefficients $D^+$ and $D^-$. The simplified structure of the coefficients is considered in order to keep the notation clear.

**Definition 2.2.** A function $u_\epsilon \in L^2((0,T), H^1(\Omega))^m \cap H^1((0,T), L^2(\Omega))^m$ is called a weak solution of Problem (1), if for all $\phi \in H^1(\Omega)$, $i = 1, \ldots, m$, and a.e. $t \in (0,T)$ it holds that

$$\int_{\Omega^+} \partial_t u^\pm_{i,\epsilon} \phi dx + \int_{\Omega^-} \partial_t u^-_{i,\epsilon} \phi dx + \frac{1}{\epsilon} \int_{\Omega^M} \partial_t u^M_{i,\epsilon} \phi dx$$

$$+ D^+_i \int_{\Omega^+} \nabla u^+_{i,\epsilon} \nabla \phi dx + D^-_i \int_{\Omega^-} \nabla u^-_{i,\epsilon} \nabla \phi dx + \epsilon^2 \int_{\Omega^M} D^+_i \left( \frac{x}{\epsilon} \right) \nabla u^M_{i,\epsilon} \nabla \phi dx,$$

$$= \int_{\Omega^+} f^+_{i,\epsilon}(t,x,u^+_{i,\epsilon}) \phi dx + \int_{\Omega^-} f^-_{i,\epsilon}(t,x,u^-_{i,\epsilon}) \phi dx + \frac{1}{\epsilon} \int_{\Omega^M} g_i \left( \frac{x}{\epsilon}, u^M_{i,\epsilon} \right) \phi dx$$

(2)

together with the initial condition (1b).


The existence of a weak solution can be established by the Galerkin-method. The uniqueness of a solution follows by standard energy estimates.

Our aim is to derive effective approximations for the microscopic solutions \( u_\epsilon \) of Problem (1) by passing to the limit \( \epsilon \to 0 \). For this purpose, we use compactness results based on estimates for \( u_\epsilon \). In a first step, we obtain the following a priori estimates:

**Lemma 2.3.** For the solution \( u_\epsilon \) of Problem (1) it holds that

\[
\| u_{\epsilon,\ell}^\pm \|_{L^\infty((0,T),H^1(\Omega_\epsilon^\pm))} \leq C,
\]

\[
\frac{1}{\sqrt{\epsilon}} \| u_{\epsilon,\ell}^M \|_{L^\infty((0,T),L^2(\Omega_\epsilon^M))} + \epsilon^2 \| \nabla u_{\epsilon,\ell}^M \|_{L^\infty((0,T),L^2(\Omega_\epsilon^M))} \leq C,
\]

\[
\| \partial_t u_{\epsilon,\ell}^\pm \|_{L^2((0,T),L^2(\Omega_\epsilon^\pm))} + \frac{1}{\sqrt{\epsilon}} \| \partial_t u_{\epsilon,\ell}^M \|_{L^2((0,T),L^2(\Omega_\epsilon^M))} \leq C,
\]

with a constant \( C > 0 \) independent of \( \epsilon \).

**Proof.** The proof is standard and follows similar ideas as in [15, Lemma 3.1]. \( \square \)

Additionally to the a priori estimates from Lemma 2.3, we prove \( H^2 \) regularity of the functions \( u_\epsilon^\pm \) and \( u_\epsilon^M \) up to the interfaces \( S_\epsilon^\pm \), together with trace estimates for the first derivatives of the microscopic solution \( u_\epsilon \) on \( S_\epsilon^\pm \). These results will be used later in the critical case \( \gamma = -1 \), see Proposition 5.2.

**Lemma 2.4.** Let \( u_\epsilon \) be the solution of Problem (1), then we have

\( u_\epsilon^\pm \in L^2((0,T),H^2(\Omega_\epsilon^\pm)^m) \quad \text{and} \quad u_\epsilon^M \in L^2((0,T),H^2(\Omega_\epsilon^M)^m) \),

and the following estimate is valid

\[
\epsilon \sum_{k,l=1}^n \| \partial_{kl}u_{\epsilon,\ell}^\pm \|_{L^2((0,T)\times\Omega_\epsilon^\pm)} + \epsilon^{2+1} \sum_{k,l=1}^n \| \partial_{kl}u_{\epsilon,\ell}^M \|_{L^2((0,T)\times\Omega_\epsilon^M)} \leq C,
\]

for a constant \( C > 0 \) independent of \( \epsilon \), and \( i = 1, \ldots, m \).

**Proof.** For the proof, we use ideas from [10]. We define for \( V \subset \Omega \) with \( \nabla \subset \Omega \) and \( u: \Omega \to \mathbb{R}^m \) for \( |h| < \text{dist}(V, \partial \Omega) \) and \( l \in \{1, \ldots, n\} \) the difference quotient

\[
D^h l u(x) := \frac{u(x + he_l) - u(x)}{h},
\]

where \( e_l \) is the \( l \)-th canonical basis vector of \( \mathbb{R}^n \) and \( u^h_l(x) := u(x + he_l) \). If it is clear from the context, we suppress the index \( l \), i.e., we write \( D^h u \) and \( u^h \). More details about estimates and properties of difference quotients can be found e.g., in [10]. In the following, we use the short notation \( \| \cdot \|_U := \| \cdot \|_{L^2(U)} \) for \( U \subset \mathbb{R}^n \) measurable.

We cover \( \Omega \) with balls \( \{B_\alpha\}_{\alpha=1}^N \), with \( N \in \mathbb{N} \). For \( \overline{B_\alpha} \subset \Omega^\pm \) the proof is standard and follows by similar estimates as below. Due to the Neumann-zero boundary condition on \( \partial \Omega \), we assume without loss of generality that \( B_\alpha \subset \Omega \) for all \( \alpha = 1, \ldots, m \), otherwise we use a mirror argument for balls intersecting the boundary \( \partial \Omega \). Hence, we assume that \( B \in \{B_\alpha\}_{\alpha=1}^N \), with center \( x_B \) and radius \( r \), fulfills \( \overline{B} \subset \Omega \) and \( B \cap S_\epsilon^- \neq \emptyset \). Without loss of generality we assume \( \overline{B} \cap S_\epsilon^+ = \emptyset \), since the arguments remain the same for the general case, where in the following equations we have to deal with additional terms depending on \( u_\epsilon^- \). For \( \delta > 0 \) small, the ball \( \tilde{B} \) with center \( x_B \) and radius \( (r + \delta) \) is relatively compact in \( \Omega \) (and again we assume that it does not intersect \( S_\epsilon^- \)). Then we can choose a cut-off
function $\xi \in C_0^\infty (\hat{B})$ with $0 \leq \xi \leq 1$, $\xi = 1$ in $B$, and $\|\nabla \xi\| \leq \frac{C}{h}$, for a constant $C > 0$ independent of $h$. Now, fix $l \in \{1, \ldots, n - 1\}$ and choose $h > 0$ small. For $i \in \{1, \ldots, m\}$, we test the variational equation (2) with $\phi = -D^{-h}(\xi^2 D^h u_{i,\epsilon})$ and obtain for a.e. $t \in (0, T)$:

$$
\begin{align*}
- \int_{\Omega^*_l} f_i^+(u_{\epsilon}^+) D^{-h}(\xi^2 D^h u_{i,\epsilon}^+) dx - \frac{1}{\epsilon} \int_{\Omega^*_l} g_i \left(\frac{x}{\epsilon} u_{\epsilon}^M\right) D^{-h}(\xi^2 D^h u_{i,\epsilon}^M) dx & \\
= \int_{\Omega^*_l} \partial_t (D^h u_{i,\epsilon}^+) \xi^2 D^h u_{i,\epsilon} - \sum_{j=1}^{n} D^h_j \partial_j u_{i,\epsilon} \partial_j (D^{-h}(\xi^2 D^h u_{i,\epsilon}^+)) dx & \\
+ \int_{\Omega^*_M} \frac{1}{\epsilon} \partial_t (D^h u_{i,\epsilon}^M) \xi^2 D^h u_{i,\epsilon}^M - \epsilon \sum_{j=1}^{n} D^M_i \left(\frac{x}{\epsilon}\right) \partial_j u_{i,\epsilon} \partial_j (D^{-h}(\xi^2 D^h u_{i,\epsilon}^M)) dx,
\end{align*}
$$

where we used the integration by parts formula for difference quotients and the fact that the time derivative commutes with the operator $D^h$. Using again integration by parts for difference quotients and the commutativity of $\partial_j$ and $D^{-h}$, we get that the right-hand side in the equality above is equal to

$$
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left(\|\xi D^h u_{i,\epsilon}^+\|_{\Omega^*_l}^2 + \frac{1}{\epsilon} \|\xi D^h u_{i,\epsilon}^M\|_{\Omega^*_l}^2\right) + D^h_i \sum_{j=1}^{n} \int_{\Omega^*_l} D^h \partial_j u_{i,\epsilon} \partial_j (\xi^2 D^h u_{i,\epsilon}^+) dx & \\
+ \epsilon \sum_{j=1}^{n} \int_{\Omega^*_M} D^h \left(D^M_i \left(\frac{x}{\epsilon}\right) \partial_j u_{i,\epsilon}^M\right) \partial_j (\xi^2 D^h u_{i,\epsilon}^M) dx =: I_{i,h} + I_{i,h}^2 + I_{i,h}^3.
\end{align*}
$$

Since the operator $D^h$ commutes with $\partial_j$, we obtain for the second term with the product rule

$$
I_{i,h}^2 = D^h_i \sum_{j=1}^{n} \int_{\Omega^*_l} 2\xi \partial_j \xi D^h \partial_j u_{i,\epsilon}^+ D^h u_{i,\epsilon}^+ + \xi^2 |D^h \partial_j u_{i,\epsilon}^+|^2 dx.
$$

Further, for arbitrary functions $v$ and $w$ it holds $D^h(vw) = v^h D^h w + w D^h v$ with $v^h(x) = v(x + he_l)$, and with the definition $D^M_i \left(x + he_l\right) := D^M_i \left(\frac{x + he_l}{\epsilon}\right)$ and similar arguments as above it follows that

$$
I_{i,h}^3 = \epsilon \sum_{j=1}^{n} \left[ \int_{\Omega^*_M} 2D^M_i D^h \partial_j u_{i,\epsilon}^M \xi \partial_j \xi D^h u_{i,\epsilon}^M + D^M_i \left(x + he_l\right) \xi^2 |D^h \partial_j u_{i,\epsilon}^M|^2 \right] & \\
+ 2D^h \left(D^M_i \left(\frac{x}{\epsilon}\right)\right) \partial_j u_{i,\epsilon}^M \xi \partial_j \xi D^h u_{i,\epsilon}^M + D^h \left(D^M_i \left(\frac{x}{\epsilon}\right)\right) \partial_j u_{i,\epsilon}^M \xi^2 D^h \partial_j u_{i,\epsilon}^M dx.
$$

Further, we have for arbitrary $\theta > 0$
\[- \int_{\Omega^+} f_i(t, x, u^{+}_i) D^{-h}(\xi^2 D^h u^{+}_{i,e}) \, dx - \frac{1}{\epsilon} \int_{\Omega^+} g_i \left( \frac{x}{\epsilon}, u^M_i \right) D^{-h}(\xi^2 D^h u^{+}_{i,e}) \, dx \]
\[\leq \int_{\Omega^+} \frac{C}{\theta} \left| f_i(t, x, u^{+}_i) \right|^2 + \theta \left| D^{-h}(\xi^2 D^h u^{+}_{i,e}) \right|^2 \, dx \]
\[+ \int_{\Omega^+} \frac{C \epsilon^{-\gamma-2}}{\theta} \left| g_i \left( \frac{x}{\epsilon}, u^M_i \right) \right|^2 + \epsilon^\gamma \theta \left| D^{-h}(\xi^2 D^h u^{+}_{i,e}) \right|^2 \, dx \]
\[\leq C \int_{\Omega^+} \frac{1}{\theta} (1 + \| u^{+}_i \|^2) + \theta \left( \left| \partial_t u^{+}_{i,e} \right|^2 + \xi^2 |D^h(\partial_t u^{+}_{i,e})|^2 \right) \, dx \]
\[+ C \int_{\Omega^+} \frac{\epsilon^{-\gamma-2}}{\theta} (1 + \| u^M_i \|^2) + \epsilon^\gamma \theta \left( \left| \partial_t u^M_{i,e} \right|^2 + \xi^2 |D^h(\partial_t u^M_{i,e})|^2 \right) \, dx,\]

where we used the inequality
\[\| D^{-h}(\xi^2 D^h u^{+}_{i,e}) \|^2_{L^2(\Omega^+)} \leq C \left( \| \partial_t u^{+}_{i,e} \|^2_{\Omega^+} + \| \xi D^h(\partial_t u^{+}_{i,e}) \|^2_{\Omega^+} \right),\]
which is an easy consequence of the properties of \(\xi\) and [10, Lemma 7.23]. Nevertheless, we should keep in mind that the constant \(C\) depends on \(\delta\), and \(C = C(\delta) \to \infty\) for \(\delta \to 0\). But since \(\Omega\) is compact, the covering of \(\Omega\) with balls \(B_\alpha\) is finite and we can choose a minimal strictly positive \(\delta\). A similar result holds for the term \(D^{-h}(\xi^2 D^h u^M_{i,e})\). Altogether we showed
\[I_{1,h}^1 + I_{2,h}^1 + I_{3,h}^1 \leq C \int_{\Omega^+} \frac{1}{\theta} (1 + \| u^{+}_i \|^2) + \theta \left( \left| \partial_t u^{+}_{i,e} \right|^2 + \xi^2 |D^h(\partial_t u^{+}_{i,e})|^2 \right) \, dx \]
\[+ C \int_{\Omega^+} \frac{\epsilon^{-\gamma-2}}{\theta} (1 + \| u^M_i \|^2) + \epsilon^\gamma \theta \left( \left| \partial_t u^M_{i,e} \right|^2 + \xi^2 |D^h(\partial_t u^M_{i,e})|^2 \right) \, dx.\]

Using the a priori estimates from Lemma 2.3, the positivity of the diffusion-coefficients \(D_1^E \) and \(D_1^M \), and choosing \(\theta\) small we get
\[\frac{d}{dt} \left( \| \xi D^h u^{+}_{i,e} \|^2_{\Omega^+} + \frac{1}{\epsilon} \| \xi D^h(\nabla u^{+}_{i,e}) \|^2_{\Omega^+} \right) + \| \xi D^h(\nabla u^{+}_{i,e}) \|^2_{\Omega^+} + \epsilon^\gamma \| \xi D^h(\nabla u^{+}_{i,e}) \|^2_{\Omega^+} M \]
\[\leq -D_1^E \int_{\Omega^+} 2 \xi D^h \nabla u^{+}_{i,e} \cdot \nabla \xi D^h u^{+}_{i,e} \, dx - \epsilon^\gamma \int_{\Omega^+} 2 D_1^E \xi D^h \nabla u^{+}_{i,e} \cdot \nabla \xi D^h u^{+}_{i,e} \, dx + C \epsilon^{-\gamma-1} \]
\[+ D^h \left( D_1^M \left( \frac{X}{\epsilon} \right) \right) \left( 2 \xi D^h u^{+}_{i,e} \cdot \nabla \xi D^h u^{+}_{i,e} + \xi^2 |\nabla u^{+}_{i,e} | \cdot |D^h(\nabla u^{+}_{i,e})| \right) \, dx + C \epsilon^{-\gamma-1} \]
\[\leq C \left( \int_{\Omega^+} \xi \| D^h \nabla u^{+}_{i,e} \| \| D^h u^{+}_{i,e} \| \, dx + \epsilon^\gamma \int_{\Omega^+} \xi \| D_1^M \| \| D^h \nabla u^{+}_{i,e} \| \| D^h u^{+}_{i,e} \| \, dx + \epsilon^{-2} \right).\]
Integration with respect to time and using the properties of the initial values leads to

$$
\|D^h u_{i,e}^+(t)\|^2_{\Omega_e^2} + \frac{1}{\epsilon} \|D^h u_{i,e}^M(t)\|^2_{\Omega_e^2} + \int_0^t \|D^h \nabla u_{i,e}^+\|^2_{\Omega_e^2} + \epsilon \|D^h \nabla u_{i,e}^M\|^2_{\Omega_e^2} d\tau \\
\leq C \left( \int_0^t \int_{\Omega_e^2} \xi \|D^h (\nabla u_{i,e}^M)\| \|D^h u_{i,e}^+\| d\tau + \epsilon \gamma \left( \int_0^t \int_{\Omega_e^2} \xi |D^h u_{i,e}^M| \|D^h u_{i,e}^M\| d\tau \right) \\
+ \left[ D^h D^h \left( \frac{\gamma}{\epsilon} \right) \right] \left( \xi \|D^h u_{i,e}^M\| \|D^h u_{i,e}^M\| \right) \right) \\
+ \|D^h u_{i,e}^+(0)\|^2_{L^2(\Omega_e^2)} + \frac{1}{\epsilon} \|D^h u_{i,e}^M(0)\|^2_{L^2(\Omega_e^2)} + C \epsilon^{-2} \\
\leq C \left( I_{\epsilon,h}^5 + I_{\epsilon,h}^5 + I_{\epsilon,h}^6 + I_{\epsilon,h}^7 \right) + C \epsilon^{-2}.
$$

First of all, we have for arbitrary $\theta > 0$

$$
I_{\epsilon,h}^5 \leq \theta \int_0^t \int_{\Omega_e^2} \xi \|D^h \nabla u_{i,e}^+\|^2 d\tau + \frac{C}{\theta} \int_0^t \int_{\Omega_e^2 \cap \bar{B}} \|D^h u_{i,e}^+\|^2 d\tau \\
\leq \theta \int_0^t \int_{\Omega_e^2} \xi \|D^h \nabla u_{i,e}^+\|^2 d\tau + \frac{C}{\theta},
$$

where we used the a priori estimate (3). For $\theta$ small enough, the first term can be absorbed by the left-hand side. For the term $I_{\epsilon,h}^5$, we use $|D^h(x)| \leq C$ for all $x \in \Omega_e^M$ to obtain

$$
I_{\epsilon,h}^5 \leq \theta \gamma \int_0^t \int_{\Omega_e^2} \xi \|D^h (\nabla u_{i,e}^M)\|^2 d\tau + \frac{CC}{\theta} \int_0^t \int_{\Omega_e^2 \cap \bar{B}} \|D^h u_{i,e}^M\|^2 d\tau \\
\leq \theta \gamma \int_0^t \int_{\Omega_e^2} \xi \|D^h (\nabla u_{i,e}^M)\|^2 d\tau + C,
$$

for an arbitrary $\theta > 0$, where we used similar arguments as above. Again, for small $\theta$, the first term can be absorbed by the left-hand side. For the terms $I_{\epsilon,h}^6$ and $I_{\epsilon,h}^7$ we use the inequality

$$
\left| D^h D^h \left( \frac{\gamma}{\epsilon} \right) \right| = \left| D^h D^h \left( \frac{x+h}{\epsilon} \right) - D^h D^h \left( \frac{x}{\epsilon} \right) \right| \leq \frac{C}{\epsilon} \quad \text{for all } x \in \Omega_e^M,
$$

which holds due to the Lipschitz continuity of $D^h$. It follows

$$
I_{\epsilon,h}^6 \leq C \epsilon^{-1} \int_0^t \int_{\Omega_e^2} \xi \|D^h \nabla u_{i,e}^M\|^2 + \|D^h u_{i,e}^M\|^2 d\tau \leq \frac{C}{\epsilon},
$$

and for the last term (for $\theta > 0$)

$$
I_{\epsilon,h}^7 \leq C \epsilon \int_0^t \int_{\Omega_e^2} \xi \|D^h \nabla u_{i,e}^M\|^2 \|D^h (\nabla u_{i,e}^M)\| d\tau \\
\leq \frac{C}{\theta \epsilon} + C \theta \epsilon \int_0^t \int_{\Omega_e^2} \xi \|D^h (\nabla u_{i,e}^M)\|^2 d\tau.
$$
Using again the absorption argument we finally obtain
\[
\|\xi D^h u_{t,\epsilon}^+ (t)\|_{\Omega^\epsilon_t}^2 + \frac{1}{\epsilon} \|\xi D^h u_{t,\epsilon}^- (t)\|_{\Omega^\epsilon_t}^2 \\
+ \int_0^t \|\xi D^h \nabla u_{t,\epsilon}^+ \|_{\Omega^\epsilon_t}^2 + \epsilon \|\xi D^h \nabla u_{t,\epsilon}^- \|_{\Omega^\epsilon_t}^2 \, dt \leq \frac{C}{\epsilon^2},
\]
for all \( t \in [0, T] \) and especially for \( t = T \). Hence, we have
\[
\|D^h (\partial_j u_{t,\epsilon}^+)\| \leq \frac{C}{\epsilon}, \\
\|D^h (\partial_j u_{t,\epsilon}^-)\| \leq Ce^{-\frac{1}{2}},
\]
for all \( j \in \{1, \ldots, n\} \). Now, [10, Lemma 7.24] implies \( \partial_j u_{t,\epsilon}^+ \in L^2((0, T) \times (B \cap \Omega_t^+)) \) and \( \partial_j u_{t,\epsilon}^- \in L^2((0, T) \times (B \cap \Omega_t^-)) \), for \( j \in \{1, \ldots, n\} \) and \( l \in \{1, \ldots, n-1\} \), together with the inequalities
\[
\|\partial_j u_{t,\epsilon}^+\|_{(0, T) \times (B \cap \Omega_t^+)} \leq \frac{C}{\epsilon}, \\
\|\partial_j u_{t,\epsilon}^-\|_{(0, T) \times (B \cap \Omega_t^-)} \leq Ce^{-1-\frac{1}{2}}.
\]
It remains to establish the existence of \( \partial_{nn} u_{t,\epsilon}^+ \) and \( \partial_{nn} u_{t,\epsilon}^- \). Keep in mind that \( D_i^M \in C^{0,1}(\mathbb{Z}) \) with \( D_i^M > 0 \), especially \( D_i^M \in W^{1,\infty}(\mathbb{Z}) \), and the same holds for \( \frac{1}{D_i^M} \). Hence, from the variational equation (2) (rigorously this can be done by testing with \( \phi(x) = \frac{\phi}{D^M((\frac{x}{\epsilon}))} \) for \( \phi \in D(B \cap \Omega_M^\epsilon) \)), we obtain
\[
\partial_{nn} u_{t,\epsilon}^+ = \frac{1}{D_i^M((\frac{x}{\epsilon}))} \left( -\epsilon^{-1} \epsilon^{-1} \partial_i u_{t,\epsilon}^+ - \epsilon^{-1} \epsilon^{-1} g_i \left( \frac{x}{\epsilon} , u_{t,\epsilon}^+ \right) \right) \\
- \epsilon^{-1} \nabla_y D_i^M \left( \frac{x}{\epsilon} \right) \nabla u_{t,\epsilon}^+ - D_i^M \left( \frac{x}{\epsilon} \right) \Delta_x u_{t,\epsilon}^+,
\]
with \( \Delta_x = \sum_{j=1}^{n} \partial_j^2 \). Hence, \( \partial_{nn} u_{t,\epsilon}^+ \in L^2((0, T) \times (B \cap \Omega_M^\epsilon)) \), and the a priori estimates in Lemma 2.3, the growth condition for \( g_i \), and the estimate in (7) imply
\[
\|\partial_{nn} u_{t,\epsilon}^+\|_{(0, T) \times (B \cap \Omega_M^\epsilon)} \leq Ce^{-\frac{1}{2}}.
\]
The same arguments imply \( \partial_{nn} u_{t,\epsilon}^- \in L^2((0, T) \times \Omega_M^\epsilon) \) with
\[
\|\partial_{nn} u_{t,\epsilon}^-\|_{(0, T) \times (B \cap \Omega_M^\epsilon)} \leq Ce^{-1}.
\]
The assertion of the theorem is proved, since this result holds for all \( B \in \{B_a\}_{a=1}^N \) and this is a finite covering of \( \Omega \).

Lemma 2.5. Let \( u_{\epsilon} \) be the solution of Problem (1) for \( \gamma = -1 \). Then the derivatives of \( u_{\epsilon}^\pm \) satisfy the following trace estimates on \( S_{\epsilon}^\pm \) for \( j = 1, \ldots, n \)
\[
\|\partial_j u_{t,\epsilon}^\pm\|_{L^2((0, T) \times S_{\epsilon}^\pm)} \leq \frac{C}{\sqrt{\epsilon}}.
\]

Proof. We use the following trace-inequality: For all \( v_{\epsilon}^\pm \in H^1(\Omega_{\epsilon}^\pm) \), we have
\[
\|v_{\epsilon}^\pm\|_{L^2(\Omega_{\epsilon}^\pm)} \leq C \left( \|v_{\epsilon}^\pm\|_{L^2(\Omega_{\epsilon}^\pm)} + \|v_{\epsilon}^\pm\|_{L^2(\Omega_{\epsilon}^\pm)}^{\frac{1}{2}} \|\nabla v_{\epsilon}^\pm\|_{L^2(\Omega_{\epsilon}^\pm)}^{\frac{1}{2}} \right),
\]
with a constant \( C > 0 \) independent of \( \epsilon \). This result is a consequence of [8, Theorem II.4.1] and it is easy to show that that constant \( C > 0 \) is independent of \( \epsilon \) by transforming the set \( \Omega_{\epsilon}^\pm \) onto \( \Omega_{\epsilon}^\pm \). Applying this trace estimate to the solutions \( u_{\epsilon}^\pm \) and using Lemma 2.3 and Lemma 2.4 yields the estimates (8).
3. Two-scale convergence and the unfolding operator for thin domains.

To pass to the limit \( \epsilon \to 0 \) in the thin layer \( \Omega^M \), we will use the concepts of two-scale convergence for thin heterogeneous domains and the unfolding operator for thin heterogeneous domains, which were introduced and investigated in [15]. However, to treat the situation of our paper, i.e., with the scaling exponent \( \gamma \in [-1, 1) \), we have to derive additional compactness results regarding mainly the gradients.

3.1. Two-scale convergence for thin heterogeneous domains. For \( n \in \mathbb{N} \) with \( n > 1 \) we consider a bounded, open, and connected set \( G \subset \mathbb{R}^{n-1} \) and for \( \epsilon > 0 \) we define \( G_\epsilon := G \times (-\epsilon, \epsilon) \subset \mathbb{R}^n \), \( \Sigma := G \times \{0\} \), \( Y := (0,1)^{n-1} \) and \( Z := Y \times (-1,1) \). Let \( T > 0 \) be given. In the following, for a given sequence \( \{u_\epsilon\} \), we always denote a convergent subsequence again by \( \{u_\epsilon\} \) to simplify the notation. In [15] the following definitions were given (see also [12] for the case of a thin layer without microscopic structure).

**Definition 3.1.** We say that a sequence \( u_\epsilon \in L^2((0,T) \times G_\epsilon) \), converges (weakly) in the two-scale sense to the limit function \( u_0 \in L^2((0,T) \times \Sigma \times Z) \), if for all \( \phi \in C^0([0,T] \times \Sigma, C^0_{per}([0,1]^{n-1}, C^0([-1,1])) \) it holds

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^T \int_{G_\epsilon} u_\epsilon(t,x)\phi\left(t,\bar{x},\frac{x}{\epsilon}\right) \, dx \, dt = \int_0^T \int_Z u_0(t,\bar{x},y)\phi(t,\bar{x},y) \, dy \, d\bar{x} \, dt.
\]

Further, we say that a (weakly) two-scale convergent sequence \( u_\epsilon \) converges strongly in the two-scale sense to \( u_0 \), if we additionally have

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \|u_\epsilon\|_{L^2((0,T) \times G_\epsilon)} = \|u_0\|_{L^2((0,T) \times \Sigma \times Z)}.
\]

The following lemma, see [15, Proposition 4.2], gives the fundamental result for the compactness theory of the two-scale convergence in thin domains.

**Lemma 3.2.** Let \( u_\epsilon \) be a sequence in \( L^2((0,T) \times G_\epsilon) \), such that \( \|u_\epsilon\|_{L^2((0,T) \times G_\epsilon)} \leq C\sqrt{\epsilon} \) with \( C > 0 \) independent of \( \epsilon \). Then, there exists a subsequence and a function \( u_0 \in L^2((0,T) \times \Sigma \times Z) \), such that \( u_\epsilon \) converges in the two-scale sense to \( u_0 \).

In the following we consider functions \( u_\epsilon \) in the space \( L^2((0,T), H^1(G_\epsilon)) \), such that

\[
\frac{1}{\sqrt{\epsilon}} \|u_\epsilon\|_{L^2((0,T) \times G_\epsilon)} + \epsilon^2 \|\nabla u_\epsilon\|_{L^2((0,T) \times G_\epsilon)} \leq C,
\]

with \( \gamma \in [-1,1) \), and prove convergence properties for the gradients. Thereby, we use the following notation:

\[
\mathcal{H}^0_{\text{per}} := \left\{ u \in H^1(Z) : u \text{ is } Y\text{-periodic and } \int_Z u \, dy = 0 \right\}.
\]

**Theorem 3.3.** Let \( \gamma = -1 \). For a given sequence \( u_\epsilon \in L^2((0,T), H^1(G_\epsilon)) \) with \( \|u_\epsilon\|_{L^2((0,T), H^1(G_\epsilon))} \leq C\sqrt{\epsilon} \), there exists a subsequence and limit functions \( u_0 \in L^2((0,T), H^1(\Sigma)) \), \( u_1 \in L^2((0,T) \times \Sigma, \mathcal{H}^0_{\text{per}}) \), such that

\[
\begin{align*}
u_\epsilon & \to u_0 & \text{in the two-scale sense,} \\
\nabla u_\epsilon & \to \nabla_x u_0 + \nabla_y u_1 & \text{in the two-scale sense,}
\end{align*}
\]

where \( \nabla_x u_0 := (\partial_1 u_0, \ldots, \partial_{n-1} u_0, 0) \).
Proof. The result of this theorem was already stated in [15, Proposition 4.4(i)], however without proof. Since the proof is not straightforward, we elaborate it here.

For the proof, we transform the functions $u_\varepsilon$ to functions defined on the fixed domain $G_1$. A similar idea can be found in [12].

Due to Theorem 3.2 and the assumptions on $u_\varepsilon$, we can find a subsequence and functions $u_\varepsilon \in L^2((0, T) \times \Sigma \times Z)$, $\xi_0 \in L^2((0, T) \times \Sigma \times Z)^n$, such that

$$u_\varepsilon \to u_0 \quad \text{in the two-scale sense},$$

$$\nabla u_\varepsilon \to \xi_0 \quad \text{in the two-scale sense}.$$

For $\Phi \in \mathcal{D}((0, T) \times \Sigma, C^\infty_{\text{per}}(\overline{V}, C^\infty_{\text{per}}(-1, 1)))^n$, we obtain with integration by parts

$$0 = \lim_{\varepsilon \to 0} \int_0^T \int_{G_\varepsilon} \nabla u_\varepsilon(t, x, \varepsilon) \cdot \Phi(t, \tilde{x}, \varepsilon) \, dxdt$$

$$= \int_0^T \int_{\Sigma} \int_{Z} u_0(t, \tilde{x}, y) \nabla_y \cdot \Phi(t, \tilde{x}, y) dyd\tilde{x}dt,$$

i.e., $u_0$ is independent of $y$.

By a change of variables, we define the function $\tilde{u}_\varepsilon(x) := u_\varepsilon(x, \varepsilon x_n)$ with $x$ in the fixed domain $G_1$. Of course, we have $\tilde{u}_\varepsilon \in L^2((0, T), H^1(G_1))$, and from the assumption on $u_\varepsilon$ it follows $\|\tilde{u}_\varepsilon\|_{L^2((0, T), H^1(G_1))} \leq C$. Hence, we find a subsequence and a function $\tilde{u}_0 \in L^2((0, T), H^1(G_1))$ with $\tilde{u}_\varepsilon \to \tilde{u}_0$ weakly in $L^2((0, T), H^1(G_1))$. Further, it holds

$$\|\partial_n \tilde{u}_\varepsilon\|^2_{L^2((0, T) \times G_1)} = \int_0^T \int_{G_1} c^2 |\partial_n u_\varepsilon(t, \tilde{x}, \varepsilon x_n)|^2 \, dxdt = \varepsilon \|\partial_n u_\varepsilon\|^2_{L^2((0, T) \times G_\varepsilon)} \leq C\varepsilon^2.$$

Passing to the limit $\varepsilon \to 0$, we immediately obtain $\partial_n \tilde{u}_0 = 0$, and therefore $\tilde{u}_0$ is independent of the $n$-th variable and we can write $\tilde{u}_0(t, x) = \tilde{u}_0(t, \tilde{x})$, and $\tilde{u}_0 \in L^2((0, T), H^1(\Sigma))$. Now, choose $\phi \in \mathcal{D}((0, T) \times \Sigma)$ and use the two-scale convergence of $u_\varepsilon$, the weak convergence of $\tilde{u}_\varepsilon$, and Fubini’s theorem to obtain

$$\int_{\Sigma} \int_{Z} u_0(t, \tilde{x}, y) \phi(t, \tilde{x}, \tilde{y}) \, dyd\tilde{x}dt$$

$$= \lim_{\varepsilon \to 0} \int_{\Sigma} \int_{Z} \tilde{u}_0(t, \tilde{x}) \phi(t, \tilde{x}, \varepsilon x_n) \, dyd\tilde{x}dt$$

$$= \int_{\Sigma} \int_{G_1} \tilde{u}_0(t, \tilde{x}) \phi(t, \tilde{x}) \, dxdt = 2 \int_{\Sigma} \int_{\Sigma} \tilde{u}_0(t, \tilde{x}) \phi(t, \tilde{x}) \, dxdt.$$

Since $|Z| = 2$ we get $u_0 = \tilde{u}_0$ and especially $u_0 \in L^2((0, T), H^1(\Sigma))$.

It remains to identify the limit of the gradients. Therefore, we multiply $\nabla u_\varepsilon$ with test-functions $\Phi_\tau(t, x) := \frac{1}{\tau} \Phi(t, \xi, \frac{x}{\tau})$ with $\Phi \in \mathcal{D}((0, T) \times \Sigma, C^\infty_{\text{per}}(\overline{V}, C^\infty_{\text{per}}(-1, 1)))^n$ and $\nabla_y \cdot \Phi = 0$. Integrating over $G_\varepsilon$, the integration by parts formula, and the convergence results for $u_\varepsilon$ and $\nabla u_\varepsilon$ imply

$$\int_0^T \int_{\Sigma} \int_{Z} \xi_0(t, \tilde{x}, y) \cdot \Phi(t, \tilde{x}, \tilde{y}) \, dyd\tilde{x}dt = \int_0^T \int_{\Sigma} \int_{Z} \nabla_x u_0(t, \tilde{x}) \cdot \Phi(t, \tilde{x}, \tilde{y}) \, dyd\tilde{x}dt.$$

Hence, there exists $u_1 \in L^2((0, T) \times \Sigma, \mathcal{H}_{\text{per}}^0)$, such that for almost every $(t, \tilde{x}, y) \in (0, T) \times \Sigma \times Z$ it holds that $\xi_0(t, \tilde{x}, y) = \nabla_x u_0(t, \tilde{x}) + \nabla_y u_1(t, \tilde{x}, y)$.

Next, we establish a convergence result for the case $\gamma \in (-1, 1)$. Similar results for the usual two-scale convergence in fixed domains can be found in [17].
Theorem 3.4. Let $\gamma \in (-1, 1)$ and $u_\epsilon \in L^2((0, T), H^1(G_\epsilon))$ with
\[
\frac{1}{\sqrt{\epsilon}} \|u_\epsilon\|_{L^2((0, T) \times G_\epsilon)} + \epsilon^2 \|\nabla u_\epsilon\|_{L^2((0, T) \times G_\epsilon)} \leq C.
\]
Then, there exist $u_0 \in L^2((0, T) \times \Sigma)$ and $u_1 \in L^2((0, T) \times \Sigma, \mathcal{H}^0_{\text{per}})$, such that up to a subsequence
\[
\begin{align*}
\epsilon^{-1/2} \nabla u_\epsilon &\to u_0 & \text{in the two-scale sense,} \\
\epsilon^{-1/2} \nabla u_\epsilon &\to \nabla_y u_1 & \text{in the two-scale sense.}
\end{align*}
\]
Proof. From our assumption follows
\[
\frac{1}{\sqrt{\epsilon}} \|u_\epsilon\|_{L^2((0, T) \times G_\epsilon)} + \sqrt{\epsilon} \|\nabla u_\epsilon\|_{L^2((0, T) \times G_\epsilon)} \leq C,
\]
and due to Lemma 3.2 and Theorem 3.3, we find a subsequence, and limit functions $u_0 \in L^2((0, T) \times \Sigma, \mathcal{H}^0_{\text{per}})$, and $\xi_0 \in L^2((0, T) \times \Sigma \times Z)^n$, with
\[
\begin{align*}
u_\epsilon &\to u_0 & \text{in the two-scale sense,} \\
\epsilon \nabla u_\epsilon &\to \nabla_y u_0 & \text{in the two-scale sense,} \\
\epsilon^{-1/2} \nabla u_\epsilon &\to \xi_0 & \text{in the two-scale sense.}
\end{align*}
\]
Since $2^{-1/2} < 1$, the last two convergences imply $\nabla_y u_0 = 0$ and therefore $u_0(t, \bar{x}, y) = u_0(t, \bar{x})$. Now, choose a test function $\Phi \in D\left((0, T) \times \Sigma, C^\infty\left(\nabla, C^\infty_0(-1, 1)\right)\right)$ with $\nabla_y \cdot \Phi = 0$. Again by integration by parts we get
\[
\int_0^T \int_\Sigma \int_Z \xi_0(t, \bar{x}, y) \cdot \Phi(t, \bar{x}, y) dy d\bar{x} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^T \int_{G_\epsilon} \epsilon^{-1/2} \nabla u_\epsilon(t, x) \cdot \Phi(t, \bar{x}, \frac{x}{\epsilon}) \, dx
\]
\[
= \lim_{\epsilon \to 0} \left(-\frac{1}{\epsilon} \int_{G_\epsilon} \epsilon^{2^{-1/2}} u_\epsilon(t, x) \nabla_x \cdot \Phi(t, \bar{x}, \frac{x}{\epsilon}) \, dx\right) = 0,
\]
since $2^{-1/2} > 0$. Hence, using the Helmholtz-decomposition we get the existence of $u_1 \in L^2((0, T) \times \Sigma, \mathcal{H}^0_{\text{per}})$ with $\xi_0 = \nabla_y u_1$. \qed

3.2. The Unfolding Operator $\mathcal{T}_M^\epsilon$ for thin domains. The unfolding operator for thin domains maps functions defined on domains with varying thickness of order $\epsilon$ to functions defined on fixed domains. The method of unfolding was first introduced in [1] and further developed in [5]. The unfolding operator for thin domains was defined in [15]. In case of the domain $\Omega^M_\epsilon$ from Section 2, the unfolding operator $\mathcal{T}_M^\epsilon : L^2((0, T) \times \Omega^M_\epsilon) \to L^2((0, T) \times \Sigma \times Z)$ is defined by
\[
\mathcal{T}_M^\epsilon u_\epsilon(t, \bar{x}, y) = u_\epsilon\left(t, \epsilon \left[\frac{\bar{x}}{\epsilon}\right], 0 \right) + cy,
\]
where $[\cdot]$ denotes the Gauß-bracket. Here, $\Sigma$ is identified with the unit cube $(0, 1)^{n-1}$ in $\mathbb{R}^{n-1}$. This definition makes sense, due to the assumptions on our domain, namely that $\Sigma = (0, 1)^{n-1} \setminus \{0\}$ and $\frac{1}{\epsilon} \in \mathbb{N}$. These imply that $\Omega^M_\epsilon$ is the union of translations of the scaled cell $\epsilon Z$ by vectors with integer components.

The following basic properties of the unfolding operator $\mathcal{T}_M^\epsilon$ are used in the present paper:

Lemma 3.5. For $u_\epsilon, v_\epsilon \in L^2((0, T) \times \Omega^M_\epsilon)$ we have
\[
\left(\mathcal{T}_M^\epsilon u_\epsilon, v_\epsilon\right)_{L^2((0, T) \times \Sigma \times Z)} = \frac{1}{\epsilon}(u_\epsilon, v_\epsilon)_{L^2((0, T) \times \Omega^M_\epsilon)}.
\]
If we additionally have \( u_\epsilon \in L^2((0, T), H^1(\Omega^M_\epsilon)) \), then \( T^M_\epsilon u_\epsilon \in L^2((0, T) \times \Sigma, H^1(Z)) \) and almost everywhere in \((0, T) \times \Sigma \times Z\) it holds

\[
\nabla_y T^M_\epsilon u_\epsilon = \epsilon T^M_\epsilon (\nabla u_\epsilon).
\]

For a function \( u_\epsilon \in H^1((0, T), L^2(\Omega^M_\epsilon)) \), we have \( T^M_\epsilon u_\epsilon \in H^1((0, T), L^2(\Sigma \times Z)) \) with

\[
\partial_t T^M_\epsilon u_\epsilon = T^M_\epsilon \partial_t u_\epsilon.
\]

Furthermore, the relation between the unfolding operator and the two-scale convergence from [15, Proposition 4.7] will be employed.

4. Convergence results. The equations in the bulk regions \( \Omega^\pm_\epsilon \) do not depend on \( \gamma \). Therefore the convergence results for the solutions \( u^\pm_\epsilon \) given in [15, Proposition 2.1] in the case \( \gamma = 1 \) remain valid for arbitrary \( \gamma \in [-1, 1] \). It holds:

**Proposition 4.1.** Let \( u_\epsilon \) be the sequence of solutions of Problem (1). Then, there exists a subsequence and \( u^\pm_i \in L^2((0, T), H^1(\Omega^\pm)) \cap H^1((0, T), L^2(\Omega^\pm)) \), such that

\[
\begin{align*}
\chi_{\Omega^+} u_{i,\epsilon} &\to u_{i,0}^+ \quad \text{strongly in } L^2((0, T) \times \Omega^+), \\
\chi_{\Omega^+} \nabla u_{i,\epsilon} &\to \nabla u_{i,0}^+ \quad \text{weakly in } L^2((0, T) \times \Omega^+)\,, \\
\chi_{\Omega^-} \partial_t u_{i,\epsilon} &\to \partial_t u_{i,0}^- \quad \text{weakly in } L^2((0, T) \times \Omega^-)\,.
\end{align*}
\]

The convergence result for the traces of \( u^\pm_i \) on \( S^\pm_\epsilon \) also remains valid for all \( \gamma \in [-1, 1] \). In fact, we have, see [15, Proposition 2.2]:

**Proposition 4.2.** For a subsequence of \( u^\pm_\epsilon \) it holds that

\[
\lim_{\epsilon \to 0} \int_0^T \int_{S^\pm_\epsilon} u^\pm_\epsilon(t, x) \cdot \Phi(t, \bar{x}, \frac{x}{\epsilon}) \, dx \, dt = \int_0^T \int_{\Sigma} \int_Y u^\pm_0(t, \bar{x}, 0) \cdot \Phi(t, \bar{x}, \bar{y}, \pm 1) d\bar{y} \, dt,
\]

for all \( \Phi \in C^\infty([0, T] \times \Sigma, C^\infty_{\text{per}}(\overline{\Sigma}, C^\infty([-1, 1])) \)\(^m\).

From the a priori estimates in Lemma 2.3 and the developed compactness results for the two-scale convergence in thin domains, see Section 3.1, we now derive the basic convergence results for the sequence \( u^M_\epsilon \) in the membrane \( \Omega^M_\epsilon \). We have to distinguish between the cases \( \gamma \in (-1, 1) \) and \( \gamma = -1 \). Theorem 3.3 and Theorem 3.4 together with Lemma 2.3 immediately imply:

**Proposition 4.3.** (i) Let \( \gamma \in (-1, 1) \) and \( u_\epsilon \) be the solution of Problem (1). Then, there exists a subsequence and limit functions \( u^M_{i,0} \in L^2((0, T), L^2(\Sigma)) \cap H^1((0, T), L^2(\Sigma)) \) and \( u^M_{i,1} \in L^2((0, T) \times \Sigma, H^0_{\text{per}}) \) for \( i = 1, \ldots, m \), such that

\[
\begin{align*}
\chi_{\Omega^+} u_{i,\epsilon} &\to u^M_{i,0} \quad \text{in the two-scale sense}, \\
\epsilon^{\frac{1}{2}+\frac{2}{p}} \nabla u_{i,\epsilon} &\to \nabla_y u^M_{i,1} \quad \text{in the two-scale sense,} \\
\partial_t u_{i,\epsilon} &\to \partial_t u^M_{i,0} \quad \text{in the two-scale sense.}
\end{align*}
\]

(ii) Let \( \gamma = -1 \) and \( u_\epsilon \) the solution of Problem (1). Then, there exists a subsequence and limit functions \( u^M_{i,0} \in L^2((0, T), H^1(\Sigma)) \cap H^1((0, T), L^2(\Sigma)) \) and
\[ u_{e,1}^M \in L^2((0, T) \times \Sigma, \mathcal{H}_{\perp}^0), \text{ for } i = 1, \ldots, m, \text{ such that} \]
\[ u_{e,1}^M \to u_{e,0}^M \tag{10a} \]
\[ \nabla u_{e,1}^M \to \nabla_x u_{e,0}^M + \nabla_y u_{e,1}^M \tag{10b} \]
\[ \partial_t u_{e,1}^M \to \partial_t u_{e,0}^M \tag{10c} \]

Next, we derive a relation between the limit functions \( u_{i,0}^\pm \) and \( u_{i,0}^M \). We emphasize that for \( \gamma \in [-1, 1] \) the results are different from those obtained in the case \( \gamma = 1 \). In fact, we obtain continuity of the traces of \( u_{i,0}^+ \) and \( u_{i,0}^- \) at the interface \( \Sigma \), whereas in [15] the solution can be discontinuous across \( \Sigma \).

**Proposition 4.4.** Let \( \gamma \in [-1, 1], u_{i,0}^\pm \) and \( u_{i,0}^M \) be the limit functions from Proposition 4.1 and 4.3, respectively. Then, for \( i = 1, \ldots, m \) it holds that
\[ u_{i,0}^\pm(t, \bar{x}, 0) = u_{i,0}^M(t, \bar{x}) \quad \text{a.e. on } (0, T) \times \Sigma, \]
especially we have \( u_{i,0}^+ = u_{i,0}^- \) a.e. on \( (0, T) \times \Sigma \).

**Proof.** For \( \Phi \in \mathcal{D}(((0, T) \times \Sigma, C_\infty^\perp(\overline{Y}, C_\infty([-1, 1])))^n \), it holds due to (9b) and (10b) that
\[ \lim_{\epsilon \to 0} \frac{e^{1-\gamma}}{\epsilon} \int_0^T \int_{\Omega_{\epsilon}^M} \epsilon^{1-\gamma} \nabla u_{i,e}^M(t, x) \cdot \Phi(t, \bar{x}, \frac{y}{\epsilon}) \, dx \, dt = 0, \]

since \( \frac{1-\gamma}{2} > 0 \). Further, integration by parts on the left-hand side, the continuity of the solutions at \( S_\epsilon^\pm \), see (1c), Proposition 4.2, and (9a) respectively (10a) imply
\[ 0 = -\int_0^T \int_{\Sigma} \int_{Y} u_{i,0}^M(t, \bar{x}) \nabla_y \cdot \Phi(t, \bar{x}, y) dy \, dx \, dt \]
\[ + \int_0^T \int_{\Sigma} \int_{Y} u_{i,0}^+(t, \bar{x}, 0) \Phi_n(t, \bar{x}, \bar{y}, 1) - u_{i,0}^-(t, \bar{x}, 0) \Phi_n(t, \bar{x}, \bar{y}, -1) dy \, d\bar{x}, \]
\[ \square \]

### 4.1. Strong convergence in the thin domain.
To pass to the limit \( \epsilon \to 0 \) in the non-linear terms of the microscopic problem (2), we need strong two-scale convergence for the sequence \( u_{\epsilon}^M \) in \( \Omega_{\epsilon}^M \). As was shown in [15, Proposition 4.7], this is equivalent to the strong convergence of the unfolded sequence \( \mathcal{T}_{\epsilon}^M u_{\epsilon}^M \) in \( L^2((0, T) \times \Sigma \times Z) \). To show the latter, we use the compactness result from Lemma 4.6 below. The basic idea consists in estimating the shifts of \( u_{\epsilon}^M \). A similar idea was used in [15] for the case \( \gamma = 1 \). There the classic Kolmogorov compactness criterion, see e.g., [3, 20], was employed. However, the proof of the estimate (2.22) in [15, Theorem 2.3], does not hold for \( \gamma \in [-1, 1] \). An improvement compared to [15] is that we can avoid the use of \( L^\infty \)-estimates for the solution \( u_{\epsilon}^M \). In the previous paper, such estimates were used to control solutions on a part of the domain \( \Omega_{\epsilon}^M \) which initially was cut off, see [15, page 707-708]. We avoid this cut off argument by using the special domain decomposition (ii)\( ^\prime \) in the compactness result from Lemma 4.6.

To work out the strong compactness results, we need the following notations.

**Notation 4.5.**
1. For an arbitrary set \( U \subset \mathbb{R}^k, k \in \mathbb{N} \), and \( \xi \in \mathbb{R}^k \) we define \( U_\xi := U \cap (U - \xi) \). Especially, for \( \xi \in \mathbb{R}^n \) we write \( \Omega_{\epsilon, \xi}^\pm := (\Omega_{\epsilon}^\pm)_\xi \) and \( \Omega_{\epsilon, \xi}^M := (\Omega_{\epsilon}^M)_\xi \).
2. For $\sigma \in \{-1,1\}^k$ and $\xi \in \mathbb{R}^k$ define $\xi_\sigma := (\sigma_1\xi_1, \ldots, \sigma_k\xi_k)$, and for $\xi \in \mathbb{R}^{k+1}$ we write $\xi_\sigma := (\sigma_1\xi_1, \ldots, \sigma_k\xi_k, \xi_{k+1})$.

3. Let $\xi \in \mathbb{R}^k$, then $(0, \xi) := \prod_{i=1}^k (0, \xi_i)$, where for $a, b \in \mathbb{R}$, $(a, b) := (b, a)$ if $b < a$ and $(0, 0) := \{0\}$.

4. For every $\xi \in \mathbb{R}^k$ and $v : (0, T) \times \mathbb{R}^k \to \mathbb{R}^l$, $k, l \in \mathbb{N}$, we can define the shifts

$$\delta_\xi v(t, x) := v(t, x + \xi) - v(t, x) \quad \text{for } (t, x) \in (0, T) \times \mathbb{R}^k.$$

Especially for $\xi = le$ with $l \in \mathbb{Z}^{n-1} \times \{0\}$ we shortly write

$$\delta v(t, x) := v(t, x + le) - v(t, x) \quad \text{for } (t, x) \in (0, T) \times \mathbb{R}^n,$$

where we suppress the index $le$ for the shifts, since it should be clear from the context which shifts are considered. Further, we shortly write for the translated function $v_l(t, x) := v(t, x + le)$.

5. Since $\Omega$ is a Lipschitz domain, we can extend the function $u_\epsilon$ to a function $u_\epsilon \in L^2((0, T), H^1(\Omega)) \cap H^1((0, T), L^2(\Omega^n))$ which we will again denote by $u_\epsilon$. Hence, the shifts of $u_\epsilon$ are well-defined and for the translated function we shortly write $u_{\epsilon,l}(t, x) := u_\epsilon(t, x+le)$, $u_{\epsilon,l}^M(t, x) := u_{\epsilon}^M(t, x+le)$, and $u_{\epsilon,l}^M(t, x) := v_{\epsilon}^M(t, x+le)$, with $l \in \mathbb{Z}^n$, and in a similar way, we define the translated components $u_{\epsilon,l,t}$, $u_{\epsilon,l}^M$, and $u_{\epsilon,l}^M$.

**Lemma 4.6.** Let $p \in [1, \infty)$, $U \subset \mathbb{R}^n$ an open rectangle, $B$ a Banach-space and $F \subset L^p(U, B)$. $F$ is relatively compact iff

(i) for every cube $C \subset U$ the set $\left\{ \int_C f dx : f \in F \right\}$ is relatively compact in $B$,

(ii) for $z \in \mathbb{R}^n$ with $z_i \geq 0$ holds

$$\sup_{f \in F} \| \delta z f \|_{L^p(U, B)} \to 0 \quad \text{for } z \to 0.$$

The condition (ii) is equivalent to the following one:

(ii)” for $z \in \mathbb{R}^n$ with $z_i \geq 0$ holds

$$\sup_{f \in F} \| \delta z f \|_{L^p(\Omega_{z_\sigma}, B)} \to 0 \quad \text{for } z \to 0$$

for all $\sigma \in \{-1,1\}^n$, where $\xi = \frac{1}{2}(1, \ldots, 1) \in \mathbb{R}^n$.

**Proof.** See [7].

**Lemma 4.7.** Let $u_\epsilon$ be the solution of the Problem (1), $\xi := \frac{1}{2}(1, \ldots, 1, 0) \in \mathbb{R}^n$ and $\kappa \in (0, 1)$. Then for every $\rho > 0$ there exists $\delta_0 > 0$, such that for all $\epsilon > 0$, $\sigma \in \{-1,1\}^{n-1}$ and $l \in \mathbb{Z}^n$ with $|l| \in (0, \xi_\sigma)$ and $|l| < \delta_0$ holds for a.e. $t \in (0, T)$

$$\frac{1}{\epsilon} \| \delta u_\epsilon^M(t) \|_{L^2(\Omega_{\epsilon,\sigma}, \mathbb{R}^n)}^2 + \epsilon^\gamma \| \nabla \delta u_\epsilon^M(t) \|_{L^2((0, T) \times \Omega_{\epsilon,\sigma}, \mathbb{R}^n)}^2 \leq C \left( \frac{1}{\epsilon} \| \delta u_\epsilon^M(0) \|_{L^2(\Omega_{\epsilon,\sigma}, \mathbb{R}^n)}^2 + \epsilon^{\frac{2\gamma + 1}{4}} + \sum_{\sigma \in \{\pm\}} \left( \| \delta u_\epsilon^M(0) \|_{L^2((0, T) \times \Omega_{\epsilon,\sigma}, \mathbb{R}^n)}^2 + \| \delta u_\epsilon^M(0) \|_{L^2(\Omega_{\epsilon,\sigma}, \mathbb{R}^n)}^2 \right) + \rho. \right)$$

**Proof.** First of all, we construct a cut-off function in the following way. For every corner of $\Sigma$, i.e., for every element $\lambda \in \{0,1\}^{n-1} \times \{0\}$ there exists a (unique) $\sigma_\lambda \in \{-1,1\}^{n-1}$, such that $\lambda + \sigma_\lambda \in \Sigma$ for all $\xi \in \Sigma$. Now, let $\xi \in (0, 1)^{n-1}$ be fixed and $l = (\lambda, 0) \in \{0,1\}^{n-1} \times \{0\}$, i.e., a corner of $\Sigma$. Then for $l \in (0, 1)$ there exists a cut-off function $\eta_{\xi,\kappa} \in C_0^\infty(\mathbb{R}^{n-1})$ with

1. $0 \leq \eta_{\xi,\kappa} \leq 1$ in $\mathbb{R}^{n-1}$.
2. $\eta = 1$ in $\Sigma_{\epsilon_{z\sigma}}$.
3. \( \eta_{\xi,\lambda} = 0 \) in \( \mathbb{R}^{n-1} \setminus \Sigma_{n,\xi,\lambda} \).

Now, we consider \( \eta_{\xi,\lambda} \) as a function defined on \( \mathbb{R}^n \), which is independent of the last variable, i.e., \( \eta_{\xi,\lambda}(x) := \eta_{\xi,\lambda}(\bar{x}) \) for all \( x \in \mathbb{R}^n \).

In the following, we only consider the case \( \sigma = (1, \ldots, 1) \), i.e., \( \xi_{\sigma} = \xi \). The other cases can be treated in the same way. First of all, we can choose \( \delta_0 \) so small, that for every \( \epsilon > 0 \) and \( l \in \mathbb{Z}^n \) with \( l\epsilon \in (0, \xi) \) and \( |l\epsilon| < \delta_0 \) the translation \( x + l\epsilon \) lies in \( \Omega \) for \( x \in \Omega_{\xi,\lambda} \). Now, we consider the test function \( \eta_{\xi,\lambda} \) constructed above with \( \lambda = 0 \in \mathbb{R}^n \), and we shortly write \( \eta := \eta_{\xi,\lambda} \). An easy transformations argument implies that for all \( \phi \in H^1(\Omega) \) and a.e. \( t \in (0, T) \) the following variational equation holds

\[
\int_{\Omega^+_t} \partial_t \delta u_{i+1}^+ \eta^2 \phi dx + \int_{\Omega^-_t} \partial_t \delta u_{i-1}^- \eta^2 \phi dx + \frac{1}{\epsilon} \int_{\Omega^+_t} \partial_t \delta u_{i+1}^M \eta^2 \phi dx + \frac{1}{\epsilon} \int_{\Omega^-_t} \partial_t \delta u_{i-1}^M \eta^2 \phi dx
\]

\[+ \sum_{\alpha \in \{\pm\}} D^\alpha \int_{\Omega^+_t} \nabla \delta u_{i,\epsilon}^+ \cdot \nabla (\eta^2 \phi) dx + \epsilon^j \int_{\Omega^M} D^M \left( \frac{x}{\epsilon} \right) \nabla \delta u_{i,\epsilon}^M \cdot \nabla (\eta^2 \phi) dx \]

\[= \int_{\Omega^+_t} \delta f_{i+1}^+(x, u_{i+1}^+) \eta^2 \phi dx + \int_{\Omega^-_t} \delta f_{i-1}^-(x, u_{i-1}^-) \eta^2 \phi dx + \frac{1}{\epsilon} \int_{\Omega^M} \delta g_i \left( \frac{x}{\epsilon}, u_{i,\epsilon}^M \right) \eta^2 \phi dx,
\]

with \( \delta f_{i+1}^+(x, u_{i+1}^+) := f_{i+1}^+(x + l\epsilon, u_{i+1}^+) - f_{i+1}^+(x, u_{i+1}^+) \) and \( \delta g_i \left( \frac{x}{\epsilon}, u_{i,\epsilon}^M \right) := g_i \left( \frac{x}{\epsilon}, u_{i,\epsilon}^M \right) - g_i \left( \frac{x}{\epsilon}, u_{i,\epsilon}^M \right) \). Choose \( \phi := \delta u_{i,\epsilon} \) and use the product rule to get

\[
\frac{1}{2} \frac{d}{dt} \left( \|\eta \delta u_{i,\epsilon}^+\|_{L^2(\Omega^+_t)}^2 + \|\eta \delta u_{i,\epsilon}^-\|_{L^2(\Omega^-_t)}^2 + \frac{1}{\epsilon} \|\eta \delta u_{i,\epsilon}^M\|_{L^2(\Omega^M)}^2 \right)
\]

\[+ \sum_{\alpha \in \{\pm\}} D^\alpha \|\eta \nabla \delta u_{i,\epsilon}^+\|_{L^2(\Omega^+_t)}^2 + \epsilon^j \int_{\Omega^M} \eta^2 D^M \left( \frac{x}{\epsilon} \right) \nabla \delta u_{i,\epsilon}^M \cdot \nabla \delta u_{i,\epsilon}^M dx \]

\[= \sum_{\alpha \in \{\pm\}} \int_{\Omega^+_t} \delta f_{i+1}^+(x, u_{i+1}^+) \eta^2 \delta u_{i,\epsilon}^+ dx + \frac{1}{\epsilon} \int_{\Omega^M} \delta g_i \left( \frac{x}{\epsilon}, u_{i,\epsilon}^M \right) \eta^2 \delta u_{i,\epsilon}^M dx
\]

\[- 2 \sum_{\alpha \in \{\pm\}} D^\alpha \int_{\Omega^+_t} \eta \delta u_{i,\epsilon}^+ \nabla \delta u_{i,\epsilon}^+ \cdot \nabla \eta dx - 2 \epsilon \int_{\Omega^M} \eta \delta u_{i,\epsilon}^M D^M \left( \frac{x}{\epsilon} \right) \nabla \delta u_{i,\epsilon}^M \cdot \nabla \eta dx,
\]

and we denote the terms on the right-hand side by \( I^l_{\epsilon} \) with \( l = 1, \ldots, 6 \). The terms \( I^1_{\epsilon} \) and \( I^2_{\epsilon} \) can be treated in the same way. Since \( f_{i+1}^+ \) and \( g_i \) are Lipschitz continuous with respect to the last variable, and \( f_{i+1}^+ \) uniformly continuous with respect to the \( x \)-variable it holds for small \( l\epsilon \)

\[I^1_{\epsilon} \leq \rho + C \|\eta \delta u_{i,\epsilon}^+\|_{L^2(\Omega^+_t)}^2 \quad \text{and} \quad I^2_{\epsilon} \leq \frac{C}{\epsilon} \|\eta \delta u_{i,\epsilon}^M\|_{L^2(\Omega^M)}^2.
\]

For the fourth term, we have for arbitrary \( \theta > 0 \), due to the properties of \( \eta \):

\[I^4_{\epsilon} \leq \theta \|\eta \nabla \delta u_{i,\epsilon}^+\|_{L^2(\Omega^+_t)}^2 + \frac{C}{\theta} \|\delta u_{i,\epsilon}^+\|_{L^2(\Omega^+_t)}^2.
\]

For \( \theta \) small enough, the first term can be absorbed by the left-hand side. In the same way, we can estimate the term \( I^5_{\epsilon} \). For the last term, we use the H"{o}lder-inequality and the a priori estimate (4) to obtain

\[I^6_{\epsilon} \leq C \epsilon \int_{\Omega^M_{\epsilon,\xi}} \|\delta u_{i,\epsilon}^M\| \|\nabla \delta u_{i,\epsilon}^M\| dx \leq C \epsilon \frac{\eta^2}{\epsilon^4}.
\]
Integration with respect to time and using Gronwall’s inequality gives us the desired result for shifts of the functions \( u^M_{i,\epsilon} \). Now, using again (11), we obtain the shifts for the gradients.

To prove the strong two-scale convergence of \( u^M_{i,\epsilon} \), we have to choose \( \epsilon \) and \( l \) in Lemma 4.7 in such a way, that the right-hand side of the estimate becomes arbitrary small, but this is not possible for \( \gamma = -1 \). Nevertheless, in this case the shifts can be estimated in a much more simpler way by using the mean-value theorem and the a priori estimates from Lemma 2.3:

**Lemma 4.8.** Let \( \gamma = -1 \) and \( u^\gamma \) the solution of Problem (1) and \( \xi := \frac{1}{2}(1, \ldots, 1, 0) \in \mathbb{R}^n \). Then, for all \( \sigma \in \{-1, 1\}^{n-1}, l \in \mathbb{Z}^{n-1} \times \{0\} \) with \( \|\xi\| < (0, \xi_\sigma) \) it holds for almost every \( t \in (0, T) \)

\[
\frac{1}{\epsilon} \|\delta u^M_{\epsilon}(t)\|_{L^2(\Omega^M_{\epsilon, \xi_\sigma})}^2 \leq C|t|^2\epsilon^2.
\]

With Lemma 4.7 and Lemma 4.8, and the compactness result form Lemma 4.6, we have the necessary tools to prove the strong two-scale convergence for the sequence \( u^M_{i,\epsilon} \).

**Theorem 4.9.** Let \( u^\gamma \) be the solution of Problem (1) for \( \gamma \in [-1, 1] \). Then, for \( i = 1, \ldots, m \), it holds that

\[
u^M_{i,\epsilon} \rightarrow u^M_{i,0} \text{ strongly in the two-scale sense.}
\]

This is equivalent to \( T^M_{\epsilon} u^M_{i,\epsilon} \rightarrow u^M_{i,0} \) (strongly) in \( L^2((0, T) \times \Sigma \times Z) \).

**Proof.** The proof follows almost the same lines as in [15, pp.709-711]. The only differences occur, since our shift-estimates in Lemma 4.7 and Lemma 4.8 differ from the shift estimates in [15], see Theorem 2.3, and since we use a decomposition of the domain \( \Omega^M_{\epsilon} \). We consider the unfolded functions \( T^M_{\epsilon} u^M_{i,\epsilon} \) as elements of the Banach-valued functions space \( L^2(\Sigma, L^2((0, T) \times Z)) \), since we want to apply Lemma 4.6, where the crucial step is to show the condition (ii), respectively (ii)’. We show \( T^M_{\epsilon} u^M_{i,\epsilon} \rightarrow u^M_{i,0} \) in \( L^2((0, T) \times \Sigma \times Z) \) for \( i = 1, \ldots, m \), hence [15, Proposition 4.7] gives the desired result.

First of all, we show condition (ii)’ in Lemma 4.6, i.e., for \( \overline{\xi} := \frac{1}{2}(1, \ldots, 1, 1) \in \mathbb{R}^{n-1} \), we show that for all \( \rho > 0 \) there exists a \( \delta > 0 \), such that for all \( \epsilon > 0 \) and all \( \xi \in \mathbb{R}^{n-1} \) with \( 0 \leq \xi_j < \delta \) for \( j = 1, \ldots, n-1 \) it holds

\[
\|T^M_{\epsilon} u^M_{i,\epsilon}(\cdot + \overline{\xi}) - T^M_{\epsilon} u^M_{i,\epsilon}\|_{L^2(\Sigma_{\epsilon,\xi}, L^2((0, T) \times Z))} < \rho.
\]

(12)

Obviously all the shifts are well-defined for \( \overline{\xi} \) small enough. We give the proof only for the case \( \sigma = (1, \ldots, 1) \in \mathbb{R}^{n-1} \) and suppress the index \( \sigma \). The other cases follow in the same way. Following the same lines as in the proof of [15, Theorem 2.3, page 709], we obtain

\[
\|T^M_{\epsilon} u^M_{i,\epsilon}(\cdot + \overline{\xi}) - T^M_{\epsilon} u^M_{i,\epsilon}\|_{L^2(\Sigma_{\epsilon,\xi}, L^2((0, T) \times Z))} \leq \sum_{k \in (0,1)^{n-1}} \frac{1}{\epsilon} \int_0^T \int_{\Omega^M_{\epsilon,\xi}} \left| u^M_{i,\epsilon}(t, x + \epsilon k + \left[\frac{\xi}{\epsilon}\right], 0) - u^M_{i,\epsilon}(t, x) \right|^2 \, dx \, dt
\]

We have to distinguish between the cases \( \gamma \in (-1, 1) \) and \( \gamma = -1 \). For the first case we can apply Lemma 4.7 with \( l = (k + \left[\frac{\xi}{\epsilon}\right], 0) =: l_\epsilon \) (we suppress the index \( k \))
and \( \kappa \in (0, 1) \), and obtain
\[
\| T^M u_i^M(\cdot, + \bar{z}, \cdot) - T^M u_i^M \|_{L^2(\Sigma \times L^2((0,T) \times Z))}^2
\leq \tilde{p} + C \sum_{k \in \{0,1\}^{n-1}} \left( \frac{1}{\epsilon} \| u^M_\epsilon(0, \cdot, + t\epsilon) - u_\epsilon^M(0) \|_{L^2(\Omega^\alpha_{\epsilon, \kappa_\epsilon})}^2 + \epsilon^{\frac{n+1}{2}} \right)
+ \sum_{\alpha \in \{\pm\}} \left( \| u^\alpha_\epsilon(\cdot, + t\epsilon) - u^\alpha_\epsilon(0) \|_{L^2((0,T) \times \Omega^\alpha_{\epsilon, \kappa_\epsilon})}^2 + \| u^\alpha_\epsilon(0, \cdot, + t\epsilon) - u^\alpha_\epsilon(0) \|_{L^2(\Omega^\alpha_{\epsilon, 0})}^2 \right),
\]
where we can choose \( \tilde{p} \) arbitrary small for small \( \epsilon \) and \( \bar{z} \), since \( |t\epsilon| \leq \| \bar{z} \| + \sqrt{n-1}\epsilon \).
Further, for \( \gamma \in (-1, 1) \) we have \( \frac{2\gamma+1}{2\gamma+2} > 0 \) and therefore \( C\epsilon^{\frac{n+1}{2}} \to 0 \) for \( \epsilon \to 0 \). The terms \( \| u^\pm_\epsilon(\cdot, + t\epsilon) - u^\pm_\epsilon \|_{L^2((0,T) \times \Omega^\pm_{\epsilon, \kappa_\epsilon})} \) tend to zero due the mean-value theorem, in fact with the a priori estimates in Lemma 2.3 it follows
\[
\| u^\pm_\epsilon(\cdot, + t\epsilon) - u^\pm_\epsilon \|_{L^2((0,T) \times \Omega^\pm_{\epsilon, \kappa_\epsilon})}^2 \leq \| u^\pm_\epsilon \|_{L^2((0,T) \times \Omega^\pm_{\epsilon, \kappa_\epsilon})} \leq C \| u \|_{L^2} \to 0
\]
for \( \epsilon \to 0 \) and \( \bar{z} \to 0 \). Due to the Assumption A4), we can handle the terms including the initial values of \( u^\pm_\epsilon \) in the same way. Further, it holds that
\[
\frac{1}{\epsilon} \| u^M_\epsilon(0, \cdot, + t\epsilon) - u^M_\epsilon(0) \|^2_{L^2(\Omega^\alpha_{\epsilon, \kappa_\epsilon})} = \| U^M_0(\cdot, + t\epsilon) - U^M_0 \|^2_{L^2(\Sigma \times (-1,1))},
\]
and this term converges to zero for \( \epsilon \to 0 \) and \( \bar{z} \to 0 \) Hence, we showed for \( \gamma \in (-1, 1) \) there exist \( \epsilon_0 > 0 \) and \( \delta_0 > 0 \), such that for all \( \epsilon \leq \epsilon_0 \) and \( \| z \| < \delta_0 \) inequality (12) is fullfilled. For \( \gamma = -1 \) we can directly apply Lemma 4.8 and obtain the same result as for \( \gamma \in (-1, 1) \). Since inequality (12) has to be true for all \( \epsilon \), we still have to establish the cases \( \epsilon > \epsilon_0 \). But due to the choice of the sequence \( \epsilon \), there are only finitely many, say \( \{\epsilon_\alpha\}_{\alpha=1}^N \) with \( N \in \mathbb{N} \), and for every \( \alpha \in \{1, \ldots, N\} \), we can find a \( \delta_\alpha > 0 \), such that (12) holds. Then, we can choose \( \delta := \min_{\alpha \in \{0,\ldots,N\}} \{\delta_\alpha\} \) and obtain the condition \((ii)'\) of Lemma 4.6.

We still have to establish the condition \((i)\) in Lemma 4.6. We define the Banach-space \( X := L^2((0,T), H^1(Z)) \cap H^1((0,T), L^2(Z)) \) together with norm \( \| u \|_X := \| u \|_{L^2((0,T), H^1(Z))} + \| \partial_t u \|_{L^2((0,T) \times Z)} \). The a priori estimates in Lemma 2.3 and the properties of the unfolding operator \( T^M, \) see Lemma 3.5, immediately imply that for every cube \( C \subset \Sigma \) the set \( \{ \int_C T^M u_i^M(\bar{x})d\bar{x} \} \) is bounded in \( X \). Further, due the Aubin-Lions compactness Theorem, the embedding \( X \subset L^2((0,T) \times Z) \) is compact and therefore the set \( \{ \int_C T^M u_i^M(\bar{x})d\bar{x} \} \) is relatively compact in \( L^2((0,T) \times Z) \). From Lemma 4.6 and [15, Proposition 4.7] follows the desired result. \( \square \)

**Remark 4.10.** To use classical Kolmogorov-compactness results, see [3, 20], one has to extend functions defined on a bounded domain \( U \subset \mathbb{R}^n \) to the whole space, and has to derive estimates in a neighbourhood of the boundary of \( U \). With the compactness result in Lemma 4.6 we avoid this problem by using the special domain decomposition in \((ii)''\). Further, in contrast to classical results, in Lemma 4.6, we consider Banach-valued functions. In the proof of Theorem 4.9 this result is applied to the unfolded sequence \( T^M u_i^M \), where we consider these functions as elements of the space \( L^2(\Sigma, L^2((0,T) \times Z)) \). This has the advantage that we only have to consider shifts with respect to the \( \bar{x} \)-variable and not shifts with respect to \( t \) and \( y \), what makes the proof less technical, but we have to check condition \((i)\) in Lemma 4.6 instead.
The strong two-scale convergence of \( u^M \) implies the two-scale convergence of the non-linear function \( g 
\):

**Proposition 4.11.** For \( \gamma \in [-1, 1) \), let \( u_\epsilon \) be the solution of the Problem (1). Then, up to a subsequence, it holds

\[
g\left(t, \frac{\cdot}{\epsilon}, u^M_\epsilon(t, \cdot)\right) \rightarrow g\left(t, \cdot, u^M(t, \cdot)\right)
\]

in the two-scale sense, where \( t \) denotes the time-variable, \( \cdot, \cdot \) denotes the variable with respect to \( \Omega^M \), \( \cdot, \cdot \) the variable with respect to \( \Gamma \).

**Proof.** The proof can be found in [15, Section 6]. \( \square \)

5. **The macroscopic problems.** We now take the limit \( \epsilon \rightarrow 0 \) in the variational formulation (2) to obtain the macroscopic problems for \( \gamma \in [-1, 1) \).

**Theorem 5.1 (The case \( \gamma \in (-1, 1) \)).** Let \( \gamma \in (-1, 1) \) and \( u_0 := (u_0^+, u_0^M, u_0^-) \), where the components of \( u_0 \) are the limit functions from Proposition 4.1 and Proposition 4.3. Then \( u_0 \) is the unique weak solution of \( i = 1, \ldots, m \)

\[
\partial_t u^\pm_i - D^\pm_i \Delta u^\pm_i = f^\pm_i(t, x, u^\pm_0) \quad \text{in } (0, T) \times \Omega^\pm, \\
-D^\pm_i \nabla u^\pm_i \cdot \nu^\pm = 0 \quad \text{on } (0, T) \times \partial \Omega^\pm \setminus \Sigma, \\
u^+_0 = u_0^- = u_0^M \quad \text{on } (0, T) \times \Sigma, \\
\left[D^\pm_i \nabla u^\pm_i \cdot \nu\right]_{\Sigma} = -|Z| \partial_t u^M_{i,0} + \int_Z g_i(y, u^M_0)dy \quad \text{on } (0, T) \times \Sigma, \\
u^+_0(0) = U^\pm_0 \quad \text{in } \Omega^\pm, \\
u^M_0(0) = |Z| U^M_0(\bar{x}) \quad \text{in } \Sigma,
\]

with \( \left[D^\pm_i \nabla u^\pm_i \cdot \nu\right]_{\Sigma} := D^+_i \nabla u^+_i \cdot \nu^+ - D^-_i \nabla u^-_i \cdot \nu^+ \). Thereby, we call \( u_0 \) a weak solution, if the function \( \bar{u}_0 \) defined by \( \bar{u}_0 := u^\pm_0 \) in \( \Omega^\pm \) is an element of \( L^2((0, T), H^1(\Omega)) \cap H^1((0, T), L^2(\Omega)) \) with \( \bar{u}_0|_{\Sigma} = u^M_{0, \Sigma} \), and for all \( \phi \in L^2((0, T), H^1(\Omega)) \) the variational equation (13) below holds.

**Proof.** We test the microscopic equation (2) with \( \phi \in \mathcal{D}((0, T) \times \bar{\Omega}) \) and obtain for \( \epsilon \rightarrow 0 \) from Proposition 4.1, 4.3, and 4.11

\[
\int_0^T \int_{\Omega^+} \partial_t u^+_i \phi dt dx + \int_0^T \int_{\Omega^-} \partial_t u^-_i \phi dt dx + \int_0^T \int_{\Sigma} \partial_t u^M_i \phi dt dy dx dt \\
+ D^+_i \int_0^T \int_{\Omega^+} \nabla u^+_i \nabla \phi dt dx + D^-_i \int_0^T \int_{\Omega^-} \nabla u^-_i \nabla \phi dt dx
\]

(13)

By density the equation holds for all \( \phi \in L^2((0, T), H^1(\Omega)) \). The uniqueness follows by a standard energy estimate. \( \square \)

To derive the effective model for the limit function \( (u_0^+, u_0^M, u_0^-) \) in the case \( \gamma = -1 \), we have to find a cell-problem to express the function \( u^M_\epsilon \) from Proposition 4.3 in terms of the function \( u^M_0 \). This can be done in a natural way by using the two-scale convergence of the function \( u^M_\epsilon \). Nevertheless, we have to deal with additional boundary terms on \( S^\pm_\epsilon \). These terms can be controlled by the trace estimate from Lemma 2.5.
Proposition 5.2. For \( i = 1, \ldots, m \), let \( u_{i,0}^M \in L^2((0,T), H^1(\Sigma)) \cap H^1((0,T), L^2(\Sigma)) \) and \( u_{i,1}^M \in L^2((0,T) \times \Sigma, \nu_{\text{per}}^0) \) be the limit functions from Proposition 4.3. Then, for \( u_{i,1}^M \) we have the representation

\[
u_{i,1}^M(t,\bar{x},y) = \sum_{j=1}^{n-1} \partial_j \nu_{i,0}^M(t,\bar{x}) w_{i,j}(y),
\]

for almost every \((t,\bar{x},y) \in (0,T) \times \Sigma \times Z\), and \( w_{i,j} \in H^1(Z), \ j = 1, \ldots, n-1\), is the unique solution of the cell-problem

\[-\nabla \cdot (D_i^M (\nabla w_{i,j} + e_j)) = 0 \text{ in } Z, \quad -D_i^M (\nabla w_{i,j} + e_j) \cdot \nu = 0 \text{ on } S^+ \cup S^-,
\]

for almost every \((t,\bar{x},y) \in (0,T) \times \Sigma \times Z\). (14)

Proof. The \( H^2 \)-regularity results from Lemma 2.4 imply \( D_i^M (\nabla u_{i,\epsilon}^M) \cdot \nu^M = -\epsilon D_i^{\epsilon} u_{i,\epsilon}^M \cdot \nu^M \) in \( L^2((0,T) \times S^+ \cup S^-) \), with the outer unit normal \( \nu^M \) and \( \nu^\epsilon \) with respect to \( \Omega^M \) and \( \Omega^\epsilon \), respectively. Now, the divergence-formula implies that for all \( \phi \in \mathcal{D}((0,T) \times \Sigma, C^\infty_{\text{per}}(\Sigma, C^\infty([-1,1]))) \) it holds that

\[
\int_0^T \int_{\Omega^M} \partial_t u_{i,\epsilon}^M(t,x) \phi \left(t,\bar{x},\frac{x}{\epsilon}\right) dx dt - \int_0^T \int_{\Omega^M} g(t,\frac{x}{\epsilon}, u_{i,\epsilon}^M(t,x)) \phi \left(t,\bar{x},\frac{x}{\epsilon}\right) dx dt \\
+ \int_0^T \int_{\Omega^M} D_i^{\epsilon} \left(\frac{x}{\epsilon}\right) \nabla u_{i,\epsilon}^M(t,x) \cdot \left[\nabla_x \phi \left(t,\bar{x},\frac{x}{\epsilon}\right) + \frac{1}{\epsilon} \nabla_y \phi \left(t,\bar{x},\frac{x}{\epsilon}\right)\right] dx dt \\
= - \sum_{\alpha \in \{\pm\}} \epsilon D_i^{\epsilon} \int_0^T \int_{S^\alpha} \nabla u_{i,\epsilon}^\alpha(t,x) \cdot \nu^\alpha(x) \phi \left(t,\bar{x},\frac{x}{\epsilon}\right) |d\sigma_x| dt \\
\leq C\epsilon \left(\|\partial_x u_{i,\epsilon}^+\|_{L^2((0,T) \times S^+)} + \|\partial_x u_{i,\epsilon}^-\|_{L^2((0,T) \times S^-)}\right) \leq C\sqrt{\epsilon},
\]

where in the last inequality we used Lemma 2.5. Now, for \( \epsilon \to 0 \), we obtain with Proposition 4.3 and Proposition 4.11 that

\[
\int_0^T \int_Z D_i^M(y) \left(\nabla_x u_{i,0}^M(t,\bar{x}) + \nabla_y u_{i,1}^M(t,\bar{x},y)\right) \cdot \nabla_y \psi(t,x,y) dy dx dt = 0,
\]

for all \( \psi \in \mathcal{D}((0,T) \times \Sigma, C^\infty_{\text{per}}(\Sigma, C^\infty([-1,1]))) \). An elementary calculation yields the desired result. \( \square \)

The following density result will be used in the proof of Theorem 5.4 below, for the derivation of the variational formulation of the effective model.

Lemma 5.3. Let \( H^\Sigma := \{u \in H^1(\Omega) : u|_{\Sigma} \in H^1(\Sigma)\} \) together with the inner product \( (u,v)_{H^\Sigma} := (u,v)_{H^1(\Omega)} + \|u,v\| \) and the associated norm \( \|\cdot\|_{H^\Sigma} \). Then the set \( C^\infty(\Omega) \) is dense in \( H^\Sigma \).

Proof. We first show the following result: Let \( B \) be a ball in \( \mathbb{R}^n \) with center 0 and radius \( R > 0 \), \( \Sigma_B := \{x \in B : x_0 = 0\} \), and \( H^\Sigma_0(B) := \{u \in H^1_0(\Sigma_B) : u|_{\Sigma_B} \in H^1_0(\Sigma_B), \supp(u|_{\Sigma_B}) \subset \Sigma_B\} \) together with the inner product \( \langle \cdot, \cdot \rangle_{H^\Sigma_0} \) and norm \( \|\cdot\|_{H^\Sigma_0} \) defined in the same way as for \( H^\Sigma \). Then the set \( C^\infty_0(B) \) is dense in \( H^\Sigma_0(B) \).
Let $Y$ be the closure of $C_0^\infty(B)$ in $H_0^{\Sigma}(B)$, then we have $H_0^{\Sigma}(B) = Y \oplus Y^\perp$. We show $Y^\perp = \{0\}$. Let $u_0 \in Y^\perp$, i.e.,
\[
0 = (u_0, \phi)_{H_0^{\Sigma}} = (u, \phi)_{H_1(B)} + (u, \phi)_{H^1(\Sigma_B)} \quad \text{for all } \phi \in C_0^\infty(B).
\]

Now, we shortly write $u_0^\pm := u_0|_{B^\pm}$, where $B^\pm := \{x \in B : \pm x_n < 0\}$, and for all $\phi^\pm \in C_0^\infty(B^\pm)$ it holds that
\[
0 = (u_0, \phi)_{H^1(B^\pm)},
\]
i.e., $\Delta u_0^\pm = u_0^\pm$ in $B^\pm$, especially the generalized normal-trace of $\nabla u_0$ exists with $\nabla u_0^\pm \cdot \nu \in H^{-\frac{1}{2}}(\partial B^\pm)$. Now, for all $\phi \in C_0^\infty(B)$ we have (we denote the duality pairing of $H^{-\frac{1}{2}}(\partial B^\pm)$ and $H^\frac{1}{2}(\partial B^\pm)$ by $\langle \cdot, \cdot \rangle_{\partial B^\pm}$)
\[
0 = (u_0, \phi)_{H^1(\Sigma_B)} + (u_0, \phi)_{L^2(\Sigma_B)} + \int_{B^\pm} \nabla u_0 \nabla \phi \, dx
\]
\[
= (u_0, \phi)_{H^1(\Sigma_B)} + (u_0, \phi)_{L^2(\Sigma_B)} + \sum_{\alpha \in \{\pm\}} \int_{B^\pm} \Delta u_0^\alpha \phi \, dx + \sum_{\alpha \in \{\pm\}} \langle \nabla u_0^\alpha \cdot \nu, \phi \rangle_{\partial B^\pm}
\]
\[
= (u_0, \phi)_{H^1(\Sigma_B)} + \langle \nabla u_0^+ \cdot \nu, \phi \rangle_{\partial B^+} + \langle \nabla u_0^- \cdot \nu, \phi \rangle_{\partial B^-}.
\]

Since $C_0^\infty(\Sigma_B)$ is dense in $H_0^1(\Sigma_B)$, we can find a sequence $\phi_k \in C_0^\infty(B)$, such that
\[
\phi_k|_{\Sigma_B} \to u_0|_{\Sigma_B} \quad \text{in } H^1(\Sigma_B),
\]
\[
\phi_k|_{\partial B^\pm} \to u_0|_{\partial B^\pm} \quad \text{in } H^\frac{1}{2}(\partial B^\pm),
\]
where the last convergence follows from the compact support of $u_0|_{\Sigma_B}$ in $\Sigma_B$ and the convergence of $\phi_k$ to $u_0|_{\Sigma_B}$ in $H^1(\Sigma_B)$, thus especially in $H^\frac{1}{2}(\Sigma_B)$. Hence, the equation above also holds for $\phi = u_0$ in the trace-sense, and since
\[
\langle \nabla u_0^\pm \cdot \nu, u_0^\pm \rangle_{\partial B^\pm} = \int_{B^\pm} \Delta u_0^\pm u_0 + \|\nabla u_0\|^2 \, dx = \|u_0\|^2_{H^{\frac{3}{2}}(B^\pm)},
\]
we obtain $\|u_0\|^2_{H_0^{\Sigma}(B)} = 0$, i.e., $u_0 = 0$. This gives us the density of $C_0^\infty(B)$ in $H_0^{\Sigma}(B)$.

Let $u \in H^{\Sigma}$. By mirroring $u$ at the faces of $\Omega$ and multiplying this function by a smooth cut-off function which is 1 on $\Omega$, we can construct an extension $\tilde{u}$ of $u$, such that $\tilde{u} \in H_0^{\Sigma}(B)$, where the radius $R$ of $B$ has to be chosen large enough. Now, we can apply the result above to find a sequence $\phi_k$ in $C_0^\infty(B)$ converging to $\tilde{u}$ in $H_0^{\Sigma}(B)$, and of course this convergence also holds in the space $H^{\Sigma}$, i.e., $\phi_k|_{\Omega} \to u$ in $H^{\Sigma}$.

**Theorem 5.4 (The case $\gamma = -1$).** Let $\gamma = -1$ and $u_0 := (u_0^+, u_0^M, u_0^-)$, where the components of $u_0$ are the limit functions from Proposition 4.1 and Proposition
4.9. Then, \( u_0 \) is the unique weak solution of
\[
\begin{align*}
\partial_t u_{i,0}^\pm - D_i^\pm \Delta u_{i,0}^\pm &= f_i^\pm(t, x, u_0^\pm) \quad \text{in } (0, T) \times \Omega^\pm, \\
-D_i^\pm \nabla u_{i,0}^\pm \cdot \nu^\pm &= 0 \quad \text{on } (0, T) \times \partial \Omega^\pm \setminus \Sigma, \\
\hat u_0^+ = u_0^- = u_0^M \quad &\text{on } (0, T) \times \Sigma,
\end{align*}
\]
\[
|Z| \partial_t u_{i,0}^M - \nabla \cdot \left( D_i^{M,*} \nabla u_{i,0}^M \right) = -[D_i \nabla u_{i,0} \cdot \nu]_{\Sigma}
\]
\[
+ \int_Z g_i(y, u_0^M) dy \quad \text{on } (0, T) \times \Sigma,
\]
\[
-D_i^{M,*} \nabla \hat z u_{i,0}^M = 0 \quad \text{on } (0, T) \times \partial \Sigma,
\]
\[
u_0^+(0) = U_0^+ \quad \text{in } \Omega^+,
\]
\[
u_0^M(0) = U_0^M(\bar x) \quad \text{in } \Sigma,
\]
for \( i = 1, \ldots, m \), where we used the same notations as in Theorem 5.4. The homogenized diffusion-coefficient \( D_i^{M,*} \in \mathbb{R}^{(n-1) \times (n-1)} \) is given by
\[
(D_i^{M,*})_{k\ell} = \int_Z D_i^M(y) \left( \nabla w_{i,k} + e_k \right) \cdot \left( \nabla w_{i,l} + e_l \right) dy,
\]
and the \( w_{i,k} \) are the solutions of the cell-problems (14). Thereby, we call \( u_0 \) a weak solution, if the function \( \hat u_0 \) defined by \( \hat u_0 := u_0^\pm \) in \( \Omega^\pm \) is an element of \( L^2((0, T), H^1) \cap H^1((0, T), L^2(\Omega)) \) with \( \hat u_0|_{\Sigma} = u_0^M \), and for all \( \phi \in L^2((0, T), H^2(\Sigma)) \) the variational equation (15) below hold.

Proof. We test the microscopic equation (2) with \( \phi \in C^\infty([0, T] \times \overline{\Omega}) \) and use Proposition 4.1, 4.3, 4.11, and 5.2. Since the arguments are similar to those in the proof of Theorem 5.1, we only consider the diffusion-term in \( \Omega^M \) in more detail:
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^T \int_{\Omega^M} \partial_t \left( \frac{\Sigma}{\epsilon} \right) \nabla u_{i,\epsilon}^M(t, x) \nabla \phi(x) dx dt
\]
\[
= \int_0^T \int_{\Sigma} \int_Z D_i^M(y) \left( \nabla \hat z u_{i,0}^M(t, \bar x) + \nabla y w_{i,1}^M(t, x, y) \right) \cdot \nabla \phi(\bar x, 0) dy d\bar x dt,
\]
where \( \nabla \hat z \phi(\bar x, 0) \) is the projection of \( \nabla \phi(\bar x, 0) \) on \( \Sigma \). Now, from Proposition 5.2 it follows that
\[
\int_{\Sigma} \int_Z D_i^M(y) \left( \nabla y w_{i,1}^M(t, \bar x, y) + \nabla \hat z u_{i,0}^M(t, \bar x) \right) \cdot \nabla \phi(\bar x, 0) dy d\bar x
\]
\[
= \sum_{k,l=1}^{n-1} \int_{\Sigma} \int_Z D_i^M(y) \left( \nabla y w_{i,k}(y) + e_k \right) \cdot \left( \nabla y w_{i,l}(y) + e_l \right) \partial_{\bar x} u_{i,0}^M(t, \bar x) \partial_{\bar x} \phi(\bar x, 0) dy d\bar x
\]
\[
+ \sum_{k=1}^{n-1} \int_{\Sigma} \int_Z D_i^M(y) \left( \nabla y w_{i,k}(y) + e_k \right) \cdot e_{n} \partial_{\bar x} u_{i,0}^M(t, \bar x) \partial_{\bar x} \phi(\bar x, 0) dy d\bar x
\]
\[
= \int_{\Sigma} D_i^{M,*} \nabla \hat z u_{i,0}^M(t, \bar x) \cdot \nabla \phi(\bar x, 0) d\bar x,
\]
where (•) follows from the cell problem of $w_{i,k}$, since $c_n = \nabla y_n$ and $y_n \in \mathcal{H}^0_{\text{per}}$. Hence, almost everywhere in $(0, T)$ and for all $\phi \in \mathcal{C}^\infty(\overline{\Omega})$ it holds

\[
\int_0^T \int_{\Omega^+} \partial_t u_{i,0}^+ \phi \, dx \, dt + \int_0^T \int_{\Omega^-} \partial_t u_{i,0}^- \phi \, dx \, dt + \int_0^T \int_{\Sigma} \int_{\Omega^+} \partial_t u_{i,0}^+ \phi \, dy \, dx \, dt + \\
+ \int_0^T \int_{\Sigma} D_i^M \nabla_x u_{i,0}^+ \nabla_x \phi \, dx \, dt + \sum_{\alpha \in \{\pm\}} D_i^\alpha \int_0^T \int_{\Omega^\alpha} \nabla u_{i,0}^\alpha \nabla \phi \, dx \, dt
\]

\[= \int_0^T \int_{\Omega^+} f_i^+(u_{i,0}^+) \phi \, dx \, dt + \int_0^T \int_{\Omega^-} f_i^-(u_{i,0}^-) \phi \, dx \, dt + \int_0^T \int_{\Sigma} \int_{\Omega} g_i(u_{i,0}^+) \phi \, dy \, dx \, dt. \tag{15}\]

By Lemma 5.3 the equation holds for all $\phi \in L^2((0, T), H^2\Sigma)$. The uniqueness follows from standard energy estimates by plugging in the solution itself which is an admissible test-function.

For the sake of completeness we repeat the result for $\gamma = 1$ from [15]. The limit function $u_0 := (u_0^+, u_0^M, u_0^-)$ fulfills again the equations in the bulk domains $\Omega^\pm$, however, the solutions are not continuous across the interface $\Sigma$. The effective transmission conditions on the interface $\Sigma$ are given by

\[
D_j^+ \nabla u_{j,0}^+(t, \bar{x}, 0) \cdot \nu^+ = \int_{S^+} D_j^M(y) \nabla u_{j,0}^M(t, \bar{x}, y) \cdot \nu^+ \, dy,
\]

\[
D_j^- \nabla u_{j,0}^-(t, \bar{x}, 0) \cdot \nu^- = \int_{S^-} D_j^M(y) \nabla u_{j,0}^M(t, \bar{x}, y) \cdot \nu^- \, dy,
\]

which hold in distributional sense with respect to $t$ and $\bar{x}$. The limit function $u_{0}^M$ is the weak solution of the following local problem

\[
\partial_t u_{0}^M(t, \bar{x}, y) - \nabla_y \cdot (D_j^M(y) \nabla_y u_{j,0}^M(t, \bar{x}, y)) = g_j(y, u_{0}^M(t, \bar{x}, y)) \quad \text{in } (0, T) \times Z,
\]

\[
u_j^M(t, \bar{x}, y) = u_{0}^M(t, \bar{x}, 0), \quad \text{on } (0, T) \times S^\pm,
\]

\[
u_j^M(0, \bar{x}, y) = U_0^M(\bar{x}, y), \quad \text{in } Z,
\]

$u_{0}^M$ is periodic in $Y$

for almost every $\bar{x} \in \Sigma$.

In conclusion, we see that for all $\gamma \in (-1, 1)$ and $\gamma = -1$, the effective model consists of reaction-diffusion equations in the bulk domains $\Omega^+$ and $\Omega^-$, the solutions being continuous across the interface $\Sigma$, in contrast to the case $\gamma = 1$ where jumps across $\Sigma$ are possible. The different behavior for the cases $\gamma \in (-1, 1)$ and $\gamma = -1$ appears in the jump condition for the normal flux across $\Sigma$. In fact, for $\gamma \in (-1, 1)$, the jump of the normal flux is given by an ordinary differential equation with a non-linear reaction term, whereas for $\gamma = -1$, we additionally obtain surface-diffusion on $\Sigma$, i.e., the jump of the normal flux is described by the Wentzell-type transmission condition.

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