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ABSTRACT
Proof mass can adjust the natural frequency of a cantilevered energy harvester to fit the vibration source frequency and, hence, improve energy efficiency. In this paper, a cantilevered energy harvesting model including a proof mass is presented based on the flexoelectric theory. The electromechanical coupling responses at steady state are obtained for harmonic excitations and then reduced to single-mode expressions for modal excitations. The flexoelectric coupling coefficient, which represents conversion of energy, is investigated. The numerical results reveal that the flexoelectric coupling coefficient can be improved by adjusting the proof mass to make the vibration frequency of the microbeam adapt to that of the ambient vibration source. The adjusting strategies have also been formulated. In addition, the flexoelectric coupling coefficient increases with the decrease in the thickness of the microbeam. As expected, the flexoelectric coupling coefficient can further be enhanced when the beam thickness reaches nanometer scale. For the beam thickness $h = 0.3 \mu m$, the current output decreases and the voltage output increases with the increase in the electrical load resistance. When the electrical load resistance is around 100 M$\Omega$, the power output arrives at its maximum. The resonance frequency shifts from 34 693 Hz to 35 350 Hz with the increase in the load resistance from short- to open-circuit conditions, and the flexoelectric coupling coefficient for this thickness lever is $k_r \approx 0.19$.

I. INTRODUCTION
Since the development of micro-electro-mechanical-system, harvesting ambient mechanical energy into electrical energy holds great promise for powering the low-powered electronic devices, replacing the batteries, and achieving self-powered electronic devices in a variety of applications, such as personal electronics, wireless sensing, implantable medical devices, and so on.1–6 Until now, many energy harvesters have been developed based on piezoelectricity.6–8 However, piezoelectricity is inherent only in non-centrosymmetric materials. The materials with the largest piezoelectric coefficient are ferroelectric, and hence, piezoelectric devices are subjected to hysteretic and nonlinear behavior, and they can only work below the Curie temperature.7–9 Actually, in centrosymmetric dielectrics, a polarization can also be induced by the strain gradient or inhomogeneous strain field, known as flexoelectricity,10 which can broaden the choice of materials that can be used for electromechanical coupling.

In general, as a new electromechanical coupling property, flexoelectricity exists in all dielectrics. The direct flexoelectricity is the coupling between the strain gradient and its induced electric polarization, while the inverse flexoelectricity refers to the coupling of the polarization gradient to the elastic stress.11 Since reduced dimensions imply larger gradients, flexoelectricity will be more competitive at a smaller scale.12 By measuring the transverse flexoelectric coefficients of barium strontium titanate microcantilevers with thickness ranging from 1.4 mm to 30 $\mu m$, Huang et al.13 demonstrated that the polarization induced by flexoelectricity can be increased significantly with structures scaling down. Recently, the flexoelectric coefficient components of polyvinylidene fluoride were investigated by Zhang et al.14–16 Shu et al.17 measured the transverse flexoelectric coefficient of the 0.5 wt. % Al$_2$O$_3$-doped BTS ceramics to be...
40.5 μC/m. Zhang and Chu found that the flexoelectric coefficient of reduced (Na0.5Bi0.5)0.95Sr0.05TiO3 ceramics is dramatically increased to more than 1.5 mC/m.14 which is approximately 10 times higher than that of insulating dielectric materials with the highest flexoelectric coefficient of (Ba,Sr)TiO3 ceramics.15 In addition, Huang et al. investigated the thermal dependence of flexoelectricity in BT-8BZT and suggested that BT-8BZT can be used in a broad range of temperatures.16

Many flexoelectric theories have been proposed to capture the flexoelectricity. In addition to the traditional strain and polarization, the strain gradient and polarization gradient are also introduced into the extended linear theory for dielectrics by Majdoub et al.17 Then, a variational principle was established by Shen and Hu for nanosized dielectrics considering the flexoelectricity, surface effect, and electrostatic force.18 For centro-symmetric dielectrics, a flexoelectric theory has been presented by Maranganti et al. in the assumption of an extended linear theory19 by excluding the piezoelectricity. Furthermore, by decomposing the strain gradient tensor into mutually independent parts, a reformulated flexoelectric theory has been presented by Maranganti et al.20 In addition, an isogeometric approach was proposed to account for Maxwell stresses by Thai et al.21 For topology optimization of flexoelectric composites, Ghasemi et al. presented design methodologies.22,23 Nguyen et al. developed a model to capture the Maxwell-Wagner polarization effect in a bilayer structure24 and established the equations of motion to investigate the role of dynamic flexoelectricity in piezoelectric nanobeams.25

The energy harvesting by utilizing the flexoelectric effect has been examined by Deng et al.26 Yan further introduced the surface effect and proposed a flexoelectric energy harvester model.27 Hamdia et al. identified the key input parameters influencing the flexoelectric energy conversion factor by applying sensitivity analysis.28 Considering the strain gradient and flexoelectric effects, Managheb et al. studied the energy harvesting from a Timoshenko beam.29 In addition, Moura and Erturk presented an electroelasticodynamic framework for flexoelectric energy harvesting.30 However, it should be noted that the cantilever flexoelectric energy harvester is used to harvest ambient vibrational energy. The energy harvester should be designed to fit the vibration source. In fact, the vibration frequency of ambient vibration source is usually random. The natural frequency of the cantilevered energy harvester can be adjusted by a proof mass to fit that of vibration source effectively.

In this paper, based on the flexoelectric theory of Li et al., a cantilever beam model with a proof mass is proposed for energy harvesting. The flexoelectric theory is reviewed in Sec. II. In Sec. III, the flexoelectric Bernoulli-Euler beam model is established by considering a proof mass. The electromechanical coupling responses at steady state are obtained for harmonic excitations and then reduced to single-mode expressions for modal excitations in Sec. IV. In Sec. V, the electromechanical coupling of a flexoelectric cantilever beam subjected to translation base excitation is analyzed. Finally, Sec. VI summarizes conclusions.

II. A REVIEW OF FLEXOELECTRIC THEORY

Flexoelectric theory can describe the size-dependent electromechanical coupling phenomena in dielectrics. By using the hydrostatic/deviatoric and symmetric/antisymmetric splitting of strain gradient tensor, a reformulated flexoelectric theory has been proposed by Li et al. for isotropic centro-symmetric dielectrics.26 Constitute relations have further been shown according to its orthogonal strain gradient components.21 However, by comparing the two different decomposition schemes of strain gradient tensor ηijk, three of the same orthogonal components can be obtained as22,23

\[ \eta_{ijk} = \eta^V_{ijk} + \eta^{(1)}_{ijk} + \eta^{\alpha\beta}_{ijk}, \]  

in which η^V_{ijk} is the strain gradient spherical tensor and η^{(1)}_{ijk} and η^{\alpha\beta}_{ijk} constitute the strain gradient deviatoric tensor. Thus, the internal energy density is expressed as

\[ U = \frac{1}{2} k e_i e_j + \mu \epsilon_i^{f} \epsilon_j^{f} + 3 \mu_1^2 \eta_{ijk} \eta^V_{ijk} + \mu_2^2 \eta^{(1)}_{ijk} \eta^{(1)}_{ijk} + 3 \mu_3^2 \eta^{\alpha\beta}_{ijk} \eta^{\alpha\beta}_{ijk} \]

\[ + \frac{1}{2} \alpha P_i P_j + f_1 P_i \eta_{ijk} + 2 f_2 P_j \eta^V_{ijk} - f_3 e_i Q_j - 2 f_4 e_j Q_i, \]

where \( k \) and \( \mu \) are the bulk and shear modulus, \( l_0, l_1, \) and \( l_2 \) are the independent material length scale parameters, \( a \) is the reciprocal dielectric susceptibility, \( f_1 \) and \( f_2 \) are the flexoelectric coefficients, respectively, \( e_i \) is the strain tensor, \( \eta_{ijk} \) is the strain gradient tensor, \( P_i \) is the polarization vector, and \( Q_i = P_i \) is the polarization gradient tensor with a comma denoting the differentiation with respect to coordinates. The tensors of strain \( e_i \) and its gradient \( \eta_{ijk} \) are obtained, given as

\[ e_i = \frac{1}{2} (u_i + u_j), \eta_{ijk} = e_i e_j, \]

with \( u_i \) denoting the displacement vector, \( e_i = e_i - \frac{1}{2} \delta_i m \) is the deviatoric strain tensor with \( \delta_i \) denoting the Kronecker delta. The three orthogonal strain gradient components are defined, respectively, as

\[ \eta^V_{ijk} = \frac{1}{10} \delta_{ij} (3 \eta^\alpha_{mmn} - \eta^\alpha_{mnm}) + \frac{1}{3} \delta_{jk} (3 \eta^\alpha_{nmm} - \eta^\alpha_{mmn}) + \frac{1}{5} \delta_{ik} (2 \eta^\alpha_{nmm} - \eta^\alpha_{mmn}), \]

\[ \eta^{(1)}_{ijk} = \frac{1}{3} (\eta_{ijk} + \eta_{jik} + \eta_{kji}) - \frac{1}{15} \delta_{ij} (2 \eta^{\alpha\beta}_{mmn} + \eta^{\alpha\beta}_{nmm}) + \delta_{jk} (2 \eta^{\alpha\beta}_{nmn} + \eta^{\alpha\beta}_{mmn}) + \delta_{ik} (2 \eta^{\alpha\beta}_{nmm} + \eta^{\alpha\beta}_{mmn}), \]

\[ \eta^{\alpha\beta}_{ijk} = \frac{1}{6} (4 \eta_{ijk} - 2 \eta_{kji} - 2 \eta_{jki}) + \delta_{ij} (\eta_{mnm} - \eta_{mmn}) + \delta_{ji} (\eta_{mmn} - \eta_{mnm}) + 2 \delta_{ki} (\eta_{nmm} - \eta_{nmn}). \]

Accordingly, the stress tensor \( \sigma_{ij} \), the higher-order stress tensor \( \tau^V_{ijk}, \tau^{(1)}_{ijk}, \) and \( \tau^{\alpha\beta}_{ijk} \), the effective local electric field \( E_i \), and higher-order electric field \( V_i \), which are work-conjugate to the strain tensor \( e_i \), the strain gradient tensor \( \eta_{ijk} \), \( \eta_{ijkl} \), and \( \eta^{\alpha\beta}_{ijk} \), the polarization vector \( P_i \), and the polarization gradient tensor \( Q_i \), respectively, can be
obtained from the internal energy density in Eq. (2). The constitutive equations are written as

\[ \sigma_{ij} = k \delta_{ij} \varepsilon_{nn} + 2 \mu \varepsilon_{ij} - f_i \delta_{ij} Q_{ik} - f_x (Q_{ij} + Q_{ki}), \]

\[ \tau_{ijk} = 6 \mu \delta_{ij} \eta_{ijk} + f_i \delta_{jk} P_i + f_x (\delta_{ij} P_k + \delta_{ik} P_j), \]

\[ \tau_{ijk}^{(1)} = 2 \mu \delta_{ij} \eta_{ijk}^{(1)}, \]

\[ \tau_{ijk}^{(2)} = 6 \mu \delta_{ij} \eta_{ijk}^{(2)}, \]

\[ E_i = \alpha P_i + f_i \eta_{ij} + 2f_x \eta_{ij}, \]

\[ V_0 = -f_i \delta_{ij} \xi_{kk} - 2f_x \xi_{ij}. \]

Moreover, the higher-order stress components \( \tau_{ijk}^{V}, \tau_{ijk}^{(1)}, \) and \( \tau_{ijk}^{(2)} \) constitute the total higher-order stress tensor \( \tau_{ijk} \), which is conjugate to the strain gradient tensor \( \eta_{ijk} \), given as

\[ \tau_{ijk} = \tau_{ijk}^{V} + \tau_{ijk}^{(1)} + \tau_{ijk}^{(2)} \]

\[ = 6 \mu \delta_{ij} \eta_{ijk}^{(2)} + 2 \mu \delta_{ij} \eta_{ijk}^{(1)} + 6 \mu \delta_{ij} \eta_{ijk}^{(2)} + f_i \delta_{jk} P_i + f_x (\delta_{ij} P_k + \delta_{ik} P_j). \]

From Eqs. (2)–(13), it can be known that only the strain gradient spherical tensor can induce polarization, while the flexoelectric effect of the strain gradient deviatoric tensor diminishes in isotropic centro-symmetric dielectrics.

### III. FLEXOELECTRIC ENERGY HARVESTING MODEL OF BERNOULLI-EULER MICROBEAM

#### A. Mechanical equation including flexoelectricity

Consider a flexoelectric harvester, as shown in Fig. 1, which is a flexoelectric cantilever microbeam with width \( b \), thickness \( h \), and length \( L \) subjected to a proof mass \( M_1 \) at \( x = a \). Consider its base undergoes an arbitrary translation \( g(t) \) and a small rotation \( h(t) \). Due to the bending vibration of microbeam under the base excitation, the generated dynamic strain gradient will induce potential difference, via the flexoelectric effect, across the thickness of the beam. A resistive electrical load \( R \) is connected to the electrodes covering the upper and lower surfaces of microbeam to qualify the voltage output \( V(t) \).

For a Bernoulli-Euler microbeam under base excitation, its partial differential equation of forced vibration (with a proof mass) can be written as

\[ \frac{\partial^3 M^b(x, t)}{\partial x^3} = \frac{\partial^2 M(x, t)}{\partial x^2} + \xi_x \frac{\partial^2 w(x, t)}{\partial t^2} \]

\[ + [\rho_1 + M_1 \delta(x - a)] \frac{\partial^2 w(x, t)}{\partial t^2} + \xi_x \frac{\partial w(x, t)}{\partial t} \]

\[ = -[\rho_1 + M_1 \delta(x - a)] \frac{\partial^2 w(x, t)}{\partial t^2} - \xi_x \frac{\partial w(x, t)}{\partial t} \]

where \( M \) and \( M^b \) are the redefined bending moment and higher-order bending moment on a Bernoulli-Euler beam by considering the higher-order stresses, given as

\[ M = b \int_{y_2}^{y_1} (z \sigma_{11} + \tau_{311}) \text{d}z, \]

\[ M^b = b \int_{y_2}^{y_1} z \tau_{111} \text{d}z. \]

\( w(x, t) \) is the transverse displacement relative to its base of microbeam, \( \xi_x \) and \( \xi_z \) are the strain-rate and viscous air damping coefficients, respectively, \( I \) is the moment of inertia, \( \rho_1 \) is the mass per unit length of the beam, and \( w_0 \) is the base excitation given in the form of displacement which can be represented by the translation and the small rotation.

\[ w_0(x, t) = g(t) + x h(t). \]

For the Bernoulli–Euler microbeam, the displacement components are taken as

\[ u = -z \frac{\partial w}{\partial x}, \quad v = 0, \quad w = w(x, t), \]

in which \( u, v, \) and \( w \) are the \( x, y, \) and \( z \)-components of the displacement vector relative to its base, respectively. According to the flexoelectric theory, Eqs. (3)–(12), the nonzero strain, strain gradient, stress, higher-order stress, electric field, and higher-order electric field components are shown in the Appendix.

Substituting the nonzero stress and higher-order stress components (A4)–(A7) into Eq. (15), together with Eq. (13), the redefined higher-order bending moment \( M^b \) equals zero and the bending moment \( M \) is written as

\[ M = b \int_{y_2}^{y_1} \left( E_z \frac{\partial^2 w}{\partial x^2} - f_i \frac{\partial P_i}{\partial z} - \frac{12}{5} \mu \varepsilon_{ij} \frac{\partial^2 w}{\partial x^2} - \frac{8}{15} \mu \varepsilon_{ij} \frac{\partial^2 w}{\partial x^2} \right. \]

\[ - \frac{2}{5} \mu \varepsilon_{ij} \frac{\partial^2 w}{\partial x^2} \left. + f_i P_i \right) \text{d}z. \]

Based on the blocking boundary condition, the first flexoelectric term in Eq. (18), by using integration by parts, can be written as

\[ \int_{y_2}^{y_1} \left( -f_i \frac{\partial P_i}{\partial z} \right) \text{d}z = \int_{y_2}^{y_1} f_i P_i \text{d}z. \]

Substituting Eqs. (A10) and (19) into Eq. (18) and considering the electric field \( E_0 = -V(t)/h \), the bending moment \( M \) is further expressed as

\[ M = -(E_0 I + \mu A \varepsilon_z) \frac{\partial^2 w}{\partial x^2} - \frac{2b f_i V}{\alpha}, \]

where \( A \) denotes the cross-sectional area and \( I \) denotes the equivalent length-scale parameter which satisfies

\[ I = \frac{12}{5} \frac{b}{h} \frac{8}{15} \frac{r^2}{l^2} + 2 \frac{f_i^2}{\mu a}. \]
Obviously, the last flexoelectric term in Eq. (20) is independent of x-coordinate and its differentiation with respect to x-coordinate will be zero when the bending moment is substituted into Eq. (14). Hence, the Heaviside function H(x) is introduced to ensure the survival of this term. Hence, Eq. (20) is rewritten as

\[
M = -(E^I I + \mu A^I) \frac{\partial^2 w}{\partial x^2} - \frac{2bf_1 V}{\alpha} [H(x) - H(x - L)].
\]  

(22)

Substituting Eq. (22) into (14), together with \(M^h = 0\), the flexoelectrically coupled mechanical equation for transverse vibrations is given as

\[
\left( E^I I + \mu A^I \right) \frac{d^4 w}{dx^4} + \xi d^2 w(x, t) \frac{d^2 w(x, t)}{dt^2} + \xi_1 \frac{\partial w(x, t)}{\partial t} + \frac{2bf_1 V}{\alpha} \left[ \frac{d\delta(x)}{dx} - \frac{d\delta(x - L)}{dx} \right] \frac{d^2 w}{dx^2} - \frac{\xi_2}{\alpha} \frac{\partial w(x, t)}{\partial t} = [\rho_1 + M_1 \delta(x - a)] \frac{\partial^2 w(x, t)}{\partial t^2} - \xi_2 \frac{\partial w_0(x, t)}{\partial t},
\]

(23)

in which, as the Dirac delta function, \(\delta(x)\) satisfies

\[
\int_{-\infty}^{+\infty} \frac{d\delta(x - p)}{dx^4} f(x) dx = (-1)^n \left[ \frac{d^n f(x)}{dx^n} \right]_{x = p}.
\]

(24)

The vibration response of microbeam in Fig. 1 can be expressed by considering proportional damping assumption as

\[
w = \sum_{r=1}^{\infty} \phi_r(x) \eta_r(t),
\]

(25)

in which \(\phi_r(x)\) is the normalized eigenfunction and \(\eta_r(t)\) is the modal mechanical coordinate of the \(r\)th vibration mode. By using Laplace transform, eigenfunction \(\phi_r(x)\) is given as

\[
\phi_r(x) = \frac{L^2}{2\lambda_r^2} \left[ \phi_r''(0) ch(\frac{\lambda_r}{L} x) + \frac{1}{k} \phi_r'''(0) sh(\frac{\lambda_r}{L} x) - \phi_r''(0) \right] \times \left[ \cos(\frac{\lambda_r}{L} x) - \frac{1}{k} \phi_r'''(0) \sin(\frac{\lambda_r}{L} x) \right] + \frac{M_1 \omega_r^2 \phi_r(a)L^3}{2(E^I I + \mu A^I) \lambda_r^4} \times \left[ sh(\frac{\lambda_r}{L} (x - a)) - \sin(\frac{\lambda_r}{L} (x - a)) \right] H(x - a),
\]

(26)

where \(\omega_r\) denotes the \(r\)th undamped natural frequency, in short circuit conditions, of the \(r\)th vibration mode, given by

\[
\omega_r = \lambda_r^2 \sqrt{\frac{E^I I + \mu A^I}{\rho_1 L^4}},
\]

(27)

and the eigenvalues \(\lambda_r > 0, \ r = 1, 2, \ldots \) can be obtained from the following characteristic equation:

\[
1 + \chi \lambda \cos \lambda + \frac{M_1}{2\rho_1 L} \left[ sh(\frac{a}{L}) \cos(\frac{a}{L}) \cos(\frac{\lambda}{L}) + \sin(\frac{a}{L}) \sin(\frac{\lambda}{L}) \right] - \sin \lambda ch(\frac{a}{L}) \sin(\frac{a}{L}) \cos(\frac{\lambda}{L}) - \sin \lambda ch(\frac{a}{L}) \cos(\frac{a}{L}) \cos(\frac{\lambda}{L}) - \sin(\frac{a}{L}) \sin(\frac{a}{L}) \cos(\frac{\lambda}{L}) = 0.
\]

(28)

By substituting Eq. (25) into Eq. (23), the mechanical equation can be derived in modal coordinates according to the orthogonality conditions of vibration mode, given by

\[
\frac{d^2 \eta_r(t)}{dt^2} + 2\xi_0 \omega_r \frac{d\eta_r(t)}{dt} + \omega_r^2 \eta_r(t) - \bar{g} V(t) = f_r(t),
\]

(29)

in which \(\xi_0\) is the modal mechanical damping ratio, satisfying

\[
2\xi_0 = \frac{\xi_1 E^I I + \mu A^I \omega_r^2 + \xi_2}{\rho_1},
\]

(30)

the modal flexoelectric coupling term can be obtained in terms of Eq. (24) as

\[
\bar{g}_r = \int_0^L \frac{2bf_1}{\alpha} \left[ \frac{d\delta(x)}{dx} - \frac{d\delta(x - L)}{dx} \right] \phi_r(x) dx = -\frac{2bf_1}{\alpha} \frac{d\phi_r(L)}{dx},
\]

(31)

and the modal mechanical forcing function is

\[
f_r(t) = -\rho_1 \left[ \frac{d^2 (\bar{g}_r \phi_r(x))}{dt^2} \int_0^L \phi_r(x) dx + \frac{d^2 h(t)}{dt^2} \int_0^L x \phi_r(x) dx \right] - \xi_2 \left[ \frac{d\bar{g}_r(t)}{dt} \int_0^L \phi_r(x) dx + \frac{d h(t)}{dt} \int_0^L x \phi_r(x) dx \right] - M_1 \phi_r(a) \left[ \frac{d^2 \bar{g}_r(t)}{dt^2} + a \frac{d^2 h(t)}{dt^2} \right].
\]

(32)

B. Electrical circuit equation including flexoelectricity

For the flexoelectric microbeam, the electric field equilibrium equation is given as

\[
E_3 - V_{33,3} + \frac{\partial \phi}{\partial z} = 0,
\]

(33)

which can be rewritten according to the Appendix as

\[
a P_3 - \frac{2f_1}{\alpha} \frac{\partial^2 w}{\partial x^2} + \frac{\partial \phi}{\partial z} = 0.
\]

(34)

Moreover, the circuit equation is expressed as

\[
\frac{d}{dt} \int_0^L \vec{D} \cdot \vec{n} dA' = \frac{V(t)}{R},
\]

(35)

in which \(A'\) is the electrode area and \(\vec{n}\) denotes the unit outward normal of the electrodes. \(\vec{D}\) is the electric displacement vector, and its nonzero component is \(D_3\) only, which can be obtained by using Eq. (34), given by

\[
\frac{D_3}{\alpha} = P_3 - \frac{\partial \phi}{\partial z} = -(\frac{1}{\alpha} + \epsilon_0) \frac{\partial \phi}{\partial z} + \frac{2f_1}{\alpha} \frac{\partial^2 w}{\partial x^2}.
\]

(36)

Substituting Eq. (36) into Eq. (35), the circuit equation is further written as

\[
C \frac{dV(t)}{dt} + \frac{V(t)}{R} + \sum_{r=1}^{\infty} \bar{g}_r \frac{d\eta_r(t)}{dt} = 0,
\]

(37)
in which the capacitance \(C\) is given as

\[
C = \frac{bL}{\rho} \left(\varepsilon_0 + \frac{1}{\alpha}\right). \tag{38}
\]

### C. Steady state response of voltage and vibration

For the flexoelectric microbeam harvester, the flexoelectrically coupled mechanical equation and circuit equation are given as Eqs. (29) and (37), respectively. For harmonic base excitation, the translational and rotational components of the base displacement are written as

\[
g(t) = W_0 e^{j\omega t}, \quad h(t) = \theta_0 e^{j\omega t}, \tag{39}
\]

in which \(W_0\) and \(\theta_0\) are the amplitudes corresponding to its translation and rotation, \(j\) is the unit imaginary number, and \(\omega\) is the frequency. Then, by substituting Eq. (39) into (32), the modal mechanical forcing function is written by

\[
f_r(t) = F_r e^{j\omega t}, \tag{40}
\]

in which the amplitude \(F_r\) is

\[
F_r = \omega^2 \left[ \rho I_0 \int_0^L \phi_r(x) dx + \theta_0 \int_0^L x \phi_r(x) dx \right] + M_r \phi_r(a)(W_0 + \theta_0 a) - \xi_0 a \int_0^L \phi_r(x) dx + \theta_0 \int_0^L x \phi_r(x) dx. \tag{41}
\]

Correspondingly, the voltage and modal mechanical response at steady state are assumed to be harmonic at the same frequency \(\omega\) as

\[
\eta_r(t) = H_r e^{j\omega t}, \quad V(t) = V_0 e^{j\omega t}, \tag{42}
\]

in which the amplitudes \(H_r\) and \(V_0\) are complex valued. Thus, substituting Eqs. (40) and (42) into (29) and (37) yields

\[
(\omega^2 - \omega^2 + 2\xi_0 j\omega) H_r - \theta_0 V_0 = F_r, \tag{43}
\]

\[
(C j\omega + \frac{1}{R}) V_0 + j\omega \sum_{r=1}^\infty \theta_r H_r = 0. \tag{44}
\]

The amplitude \(H_r\) and voltage amplitude \(V_0\) can be obtained from Eq. (44). Therefore, the voltage response at steady state is written as

\[
V(t) = \sum_{r=1}^\infty \frac{-j\omega \theta_r F_r}{\omega_r^2 - \omega^2 + 2\xi_0 j\omega} e^{j\omega t}, \tag{45}
\]

and the modal mechanical response at steady state is expressed by

\[
\eta_r(t) = \frac{e^{j\omega t}}{\omega_r^2 - \omega^2 + 2\xi_0 j\omega} \cdot \left( F_r - \theta_r \left( \sum_{r=1}^\infty \frac{j\omega \theta_r F_r}{\omega_r^2 - \omega^2 + 2\xi_0 j\omega} \right) C j\omega + \frac{1}{R} \sum_{r=1}^\infty \frac{j\omega \theta_r^2}{\omega_r^2 - \omega^2 + 2\xi_0 j\omega} \right). \tag{46}
\]

The transverse displacement response can be obtained by substituting Eq. (46) into (25),

\[
w = \sum_{r=1}^\infty \frac{\phi_r(x) e^{j\omega t}}{\omega_r^2 - \omega^2 + 2\xi_0 j\omega} \left( F_r - \theta_r \left( \sum_{r=1}^\infty \frac{j\omega \theta_r F_r}{\omega_r^2 - \omega^2 + 2\xi_0 j\omega} \right) C j\omega + \frac{1}{R} \sum_{r=1}^\infty \frac{j\omega \theta_r^2}{\omega_r^2 - \omega^2 + 2\xi_0 j\omega} \right). \tag{47}
\]

### IV. ELECTROMECHANICAL FRFs FOR MULTI- AND SINGLE-MODE

In the flexoelectric microbeam model for energy harvesting, the outputs are the responses of vibration and voltage under the inputs of translation and rotation excitation. Thus, the four electromechanical FRFs of vibration or voltage output to translational or rational base acceleration are defined, respectively.

#### A. Electromechanical FRFs for multimode

Because \(g(t) = W_0 e^{j\omega t}, h(t) = \theta_0 e^{j\omega t}\), the corresponding translational and rotational base accelerations are \(d^2 g(t)/dt^2 = -\omega^2 W_0 e^{j\omega t}\), \(d^2 h(t)/dt^2 = -\omega^2 \theta_0 e^{j\omega t}\), respectively. From Eq. (41), the modal forcing amplitude is rewritten as

\[
F_r = -\alpha_r \omega^2 W_0 - \beta_r \omega^2 \theta_0, \tag{48}
\]

with

\[
\alpha_r = -\rho I_0 \int_0^L \phi_r(x) dx - M_r \phi_r(a) + \frac{\xi_0}{\omega} \int_0^L \phi_r(x) dx, \tag{49}
\]

\[
\beta_r = -\rho \int_0^L x \phi_r(x) dx - M_r a \phi_r(a) + \frac{\xi_0}{\omega} \int_0^L x \phi_r(x) dx.
\]

According to the base accelerations of translation and rotation, together with Eq. (48), the voltage response, Eq. (45), at steady state is rewritten as

\[
V(t) = s(\omega) (-\omega^2 W_0 e^{j\omega t}) + y(\omega) (-\omega^2 \theta_0 e^{j\omega t}), \tag{50}
\]

in which the FRF of the voltage output to base acceleration for translation is

\[
k(\omega) = -\sum_{r=1}^\infty \frac{j\omega \theta_r \alpha_r}{\omega_r^2 - \omega^2 + 2\xi_0 j\omega} C j\omega + \frac{1}{R} \sum_{r=1}^\infty \frac{j\omega \theta_r^2}{\omega_r^2 - \omega^2 + 2\xi_0 j\omega}, \tag{51}
\]

and the voltage output is related to rotational base acceleration by

\[
y(\omega) = -\sum_{r=1}^\infty \frac{j\omega \theta_r \beta_r}{\omega_r^2 - \omega^2 + 2\xi_0 j\omega} C j\omega + \frac{1}{R} \sum_{r=1}^\infty \frac{j\omega \theta_r^2}{\omega_r^2 - \omega^2 + 2\xi_0 j\omega}. \tag{52}
\]

Similarly, the vibration response in Eq. (46) at steady state is rewritten as

\[
\eta_r(t) = \frac{e^{j\omega t}}{\omega_r^2 - \omega^2 + 2\xi_0 j\omega} \cdot \left( F_r - \theta_r \left( \sum_{r=1}^\infty \frac{j\omega \theta_r F_r}{\omega_r^2 - \omega^2 + 2\xi_0 j\omega} \right) C j\omega + \frac{1}{R} \sum_{r=1}^\infty \frac{j\omega \theta_r^2}{\omega_r^2 - \omega^2 + 2\xi_0 j\omega} \right). \tag{46}
\]
The single-mode vibration response at steady state is

\[ w = \chi(\omega, x)(-\omega^2 W_0 e^{j\omega t}) + m(\omega, x)(-\omega^2 \theta_0 e^{j\omega t}), \]  

(53)

where the FRFs that relate the response of transverse displacement to translational and rotational base accelerations, respectively, are

\[ \chi(\omega, x) = \sum_{r=1}^{\infty} \phi_r(x) \left( \frac{\omega_r^2 - \omega^2 + 2\xi\omega, j\omega}{\omega_r^2 - \omega^2 + 2\xi\omega, j\omega} \right), \]  

(54)

\[ m(\omega, x) = \sum_{r=1}^{\infty} \phi_r(x) \left( \frac{\omega_r^2 - \omega^2 + 2\xi\omega, j\omega}{\omega_r^2 - \omega^2 + 2\xi\omega, j\omega} \right), \]  

(55)

B. Electromechanical FRFs for single-mode

If the flexoelectric microbeam is excited at its resonance frequency \( \omega = \omega_r \), the maximum electrical response can be obtained. In this condition, only the contribution of the \( r \)th vibration mode is necessary to be considered in the summation terms and other modes can be ignored. Then, the voltage and transverse displacement responses given by Eqs. (45) and (47) at steady state can be reduced, respectively, to

\[ \dot{V}(t) = \frac{-j\omega R_0 E_0 e^{j\omega t}}{(C_j \omega + 1)(\omega_r^2 - \omega^2 + 2\xi\omega, j\omega) + j\omega R_0^2}, \]  

(56)

\[ \dot{w} = \frac{(C_j \omega + 1) \phi_r(x) e^{j\omega t}}{(C_j \omega + 1)(\omega_r^2 - \omega^2 + 2\xi\omega, j\omega) + j\omega R_0^2}. \]  

(57)

In parallel, the single-mode FRFs can be obtained from Eqs. (50)–(55) by reserving the contribution of the \( r \)th mode only in the summation terms. The single-mode voltage response at steady state is

\[ \dot{V}(t) = \dot{\chi}(\omega)(-\omega^2 W_0 e^{j\omega t}) + \dot{\chi}(\omega)(-\omega^2 \theta_0 e^{j\omega t}), \]  

(58)

in which

\[ \dot{\chi}(\omega) = \frac{-j\omega R_0 \alpha_r}{(C_j \omega + 1)(\omega_r^2 - \omega^2 + 2\xi\omega, j\omega) + j\omega R_0^2}. \]  

(59)

\[ \dot{\chi}(\omega) = \frac{-j\omega R_0 \beta_r}{(C_j \omega + 1)(\omega_r^2 - \omega^2 + 2\xi\omega, j\omega) + j\omega R_0^2}. \]  

(60)

The single-mode vibration response at steady state is

\[ \dot{w} = \dot{\chi}(\omega, x)(-\omega^2 W_0 e^{j\omega t}) + \dot{m}(\omega, x)(-\omega^2 \theta_0 e^{j\omega t}), \]  

(61)

V. CASE STUDIES AND RESULTS

A flexoelectric energy harvester of the cantilever microbeam subjected to a harmonic translational base excitation only is considered in this section. Note that no rotation is considered \((\theta_0 = 0)\). For the numerical simulation, one chose PVDF (polyvinylidene difluoride) which has the following properties: \( E = 3.7 \) GPa, \( \alpha = 1.38 \times 10^{-10} \) N m\(^2\)/C\(^2\), \( \rho = 1.78 \times 10^3 \) kg/m\(^3\), \( \varepsilon_0 = 8.854 \times 10^{-12} \) C\(^2\)/(N m\(^2\)), and \( f = -179 \) N m/C. The geometry of the cantilever microbeam is set as \( h = 0.3 \) \( \mu \text{m} \), \( b = 3 \) \( \mu \text{m} \), and \( L = 30 \) \( \mu \text{m} \), and the proof mass is \( M_f = 1.5 \text{p} \mu \text{b} \).

The modal flexoelectric coupling coefficient and the frequency response are investigated in the following. It should be noted here that the frequency response functions (FRFs) are normalized by base acceleration quantified in terms of the gravitational acceleration, \( g = 9.81 \) m/s\(^2\), since the mechanical excitation is the translational acceleration, \( d^2 w_0/dt^2 = -\omega^2 W_0 e^{j\omega t} \).

A. Modal flexoelectric coupling coefficient

The flexoelectric coupling coefficient \( "k_e" \), which is a measure of energy conversion for the \( r \)th vibration mode, is defined as

\[ k_e^2 = \frac{(\omega_e^2)^2 - (\omega_0^2)^2}{(\omega_0^2)^2}, \]  

(64)

where \( \omega_e^2 \) and \( \omega_0^2 \) denote the natural frequencies in the short-circuit and open-circuit, respectively. The undamped \( r \)th natural frequency in short circuit conditions can be written as Eq. (27),

\[ \omega_r^2 = \omega_r^2 = \lambda^2 \sqrt{\frac{E_1 + \mu \varepsilon_0^2}{\rho_0 L^4}}. \]  

(65)

**FIG. 2.** The size effect on the flexoelectric coupling coefficient.
FIG. 3. The influence of normalized proof mass on the flexoelectric coupling coefficient.

In open-circuit condition \( R \to \infty \), for \( r \)th modal vibrations, Eq. (44) can be reduced to

\[
V(t) = \frac{\theta_r h_r(t)}{C}, \quad R \to \infty.
\]  

(66)

Substituting Eq. (66) into Eq. (29), the undamped \( r \)th natural frequency \( \omega_{rc} \) is given by

\[
\omega_{rc} = \omega_r \sqrt{1 + \frac{\theta_r^2}{\omega_r^2 C}}.
\]  

(67)

Substituting Eqs. (65) and (67) into Eq. (64) yields

\[
k_r^2 = \frac{1}{\left(\frac{q_r}{\omega_r}\right)^2 C + 1}.
\]  

(68)

From Eqs. (27), (31), and (38), together with Eq. (26), Eq. (68) can further be written as

\[
k_r^2 = \frac{1}{\left(\varepsilon_0 + \frac{1}{\alpha}\right) \left(\frac{E}{2\mu^2} + \frac{2f^2}{\alpha} \right) + 1},
\]  

(69)

with

\[
q_r = -\sin(\lambda_r) - \sinh(\lambda_r) + \zeta_r \left[\cos(\lambda_r) - \cosh(\lambda_r)\right].
\]  

(70)

The influence of thickness and proof mass of microbeam on the flexoelectric coupling coefficient when the proof mass is located

FIG. 4. Voltage FRFs changing with frequency at different positions of proof mass.

FIG. 5. Voltage FRFs changing with frequency at different proof mass.

FIG. 6. Voltage FRFs changing with frequency at different load resistances.
TABLE I. The resonance frequency and flexoelectric coupling coefficient at different proof mass.

| Normalized proof mass $M_l/p_0/L$ | 3        | 1.5      | 1       | 0.5     | 0.2     | 0       |
|----------------------------------|----------|----------|---------|---------|---------|---------|
| Resonance frequency in short-circuit condition (Hz) | 25 414   | 34 693   | 41 106  | 53 222  | 68 966  | 92 809  |
| Resonance frequency in open-circuit condition (Hz) | 25 901   | 35 350   | 41 878  | 54 198  | 70 176  | 94 281  |
| Flexoelectric coupling $k_r$ | 0.193 1  | 0.191 8  | 0.191 0 | 0.188 9 | 0.184 9 | 0.176 0 |

at the tip of the cantilever beam is shown in Figs. 2 and 3, respectively. Figure 2 shows that the flexoelectric coupling coefficient monotonously increases as the microbeam thickness decreases. As expected, when the beam thickness reaches nanometer scale, the flexoelectric coupling coefficient can further be enhanced. From the curve in Fig. 3, it can be seen that the flexoelectric coupling coefficient monotonously increases as the proof mass increases although the enhancement effect is not strong enough. The main role of a proof mass is to adjust the resonance frequency of cantilever microbeam to fit that of the vibration source and, hence, make the cantilevered energy harvester work at the resonant frequency to improve energy efficiency. Both of the two conclusions shown in Figs. 2 and 3 are consistent with that shown in Eq. (68) since the natural frequency decreases as the microbeam thickness decreases and proof mass increases, respectively. From the aspect of energy, the flexoelectric effect associated with strain gradient is size-dependent. As the beam thickness decreases and proof mass increases, strain gradients increase obviously. Therefore, the conversion efficiency of mechanical to electrical energy advances with the enhanced flexoelectric effect.

B. Electromechanical frequency response

1. Voltage FRFs

The voltage output FRFs of the flexoelectric cantilever microbeam harvester are shown in Figs. 4–6. The vibration frequency of the ambient vibration source is usually variable. Figure 4 shows that the resonance frequency decreases and the corresponding voltage output increases monotonously as the distance ($a$) from the proof mass to the fixed end increases. The natural frequency of the cantilever microbeam can be adjusted to fit that of the ambient vibration source by changing the position of the proof mass. For the case that the proof mass is located at the tip of the cantilever beam, the resonance frequency decreases and the corresponding voltage output increases monotonously as the proof mass increases, shown in Fig. 5. A proof mass can also change the natural frequency of the cantilever microbeam to that of the ambient vibration source. The curve of $M_l = 0$ in Fig. 5 represents the voltage FRFs of the cantilever microbeam without a proof mass. In this case, the present energy harvester is similar with that of Deng et al., in which the difference in the natural frequency is due to different material length-scale parameters.

The resonance frequency in short- and open-circuit conditions and the flexoelectric coupling coefficient at different proof mass are shown in Table I. Table I shows that the flexoelectric coupling coefficient decreases with the decrease in proof mass, which is consistent with that in Fig. 3. However, the difference in the flexoelectric coupling coefficient at different proof mass is negligible.

Figure 6 shows the voltage output increases as the load resistance $R$ increases from zero (short-circuit) through infinite (open-circuit). There is a shift from 34 693 Hz to 35 350 Hz in the resonance frequency with the increase in the load resistance from $R → 0$ to $R → ∞$. For this thickness lever, the flexoelectric coupling coefficient can be obtained in terms of Eq. (64) as $k_r ≈ 0.19$, which is consistent with that shown in Fig. 3. The flexoelectric coupling is size-dependent, which will be comparable with piezoelectric coupling when the thickness of the beam is much smaller.

2. Current FRFs

According to Ohm’s law, $i(t) = V(t)/R$, the electric current can be calculated. The electric current output FRFs of the flexoelectric cantilever microbeam harvester are shown in Fig. 7. Figure 7 shows that the electric current decreases as the load resistance increases from $R → 0$ to $R → ∞$, which is an opposite trend in comparison with the voltage output. The resonance frequency shift (from 34 693 Hz to 35 350 Hz) which is the same with that discussed for the voltage output in Fig. 6 reveals a strong flexoelectric coupling.

3. Transverse displacement FRFs

The transverse displacement FRFs of the flexoelectric cantilever microbeam harvester are shown in Fig. 8. The resonance frequency shifts from 34 693 Hz to 35 350 Hz when the load resistance increases.
from zero to infinite, which is in agreement with that in Fig. 6 in the same thickness and proof mass. In addition, for certain load resistances, the proof deflection is attenuated because of the shunt damping of resistor, which is analogous to that of the piezoelectric energy harvesting case.

4. Power density FRFs

The electrical power can be calculated as $|V(t)|^2/R$. Normalizing by base acceleration squared, the power density FRF is shown in Fig. 9. From these curves, it can be found that with the increase in load resistance, the power shows an opposite trend, which reveals that there must be a load resistance making the power arrive its maximum. For the current case, a maximum power needs the electrical load around 100 MΩ and the corresponding resonance frequency is 348.231 Hz. In addition, the same shift of the resonance frequency is shown again as the load resistance changes.

VI. CONCLUSIONS

In this paper, the electromechanical coupling behaviors of a flexoelectric cantilever beam with a proof mass are investigated under base excitations based on the flexoelectric theory of Li et al. The electromechanical coupling responses at steady state are obtained for harmonic excitations. To illustrate the current model, the electromechanical coupling of a flexoelectric cantilever beam subjected to translation base excitation is analyzed. Numerical results show that the voltage output increases and the current output decreases with the increase in electrical load resistance at any frequency. The power output arrives at its maximum when the electrical load resistance is around 100 MΩ. Moreover, as the load resistance increases from zero to infinity, the resonance frequency shifts from 34693 Hz to 35350 Hz and the flexoelectric coupling coefficient for this thickness lever is $k_r \approx 0.19$. Actually, the flexoelectric coupling coefficient is size-dependent, which increases with the decrease in the thickness of the microbeam. The flexoelectric coupling coefficient can further be enhanced when the beam thickness reaches nanometer scale. In addition, proof mass can reduce the vibration frequency of the cantilever microbeam to adapt the vibration frequency of the ambient vibration source and, hence, improve the electrical energy output under the resonant condition.

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APPENDIX: CONSTITUTIVE EQUATIONS FOR FLEXOELECTRIC BEAM

For the Bernoulli–Euler microbeam, the only nonzero strain component can be obtained from Eq. (3) as

$$\varepsilon_{11} = -\frac{1}{h} \frac{\partial^2 w}{\partial x^2}. \tag{A1}$$

Note that $h \ll L$ is assumed for the Bernoulli-Euler beam, the strain gradient $\varepsilon_{11,1}$ can be neglected in comparison with $\varepsilon_{11,3}$. Thus, in the following analysis, only the strain gradient along the thickness direction is considered, which is given as

$$\eta_{311} = -\frac{1}{h} \frac{\partial^2 w}{\partial x^2}. \tag{A2}$$

Substituting Eq. (A2) into Eqs. (4)–(6), the nonzero strain gradient components are defined as
According to the constitutive relations in Eqs. (7)–(11), the nonzero components of stress tensor, higher-order stress tensor, and local electric field can be obtained, given as

$$\sigma_{11} = -E^\nu \frac{\partial^2 w}{\partial x^2} - f_1 \frac{\partial P_3}{\partial z}, \hspace{1cm} (A4)$$

$$\tau_{311}^\nu = \frac{12}{5} \mu E^\nu \frac{\partial^2 w}{\partial x^2} + f_1 P_3, \hspace{1cm} (A5)$$

$$\tau_{311}^{(1)} = \frac{8}{15} \mu E^\nu \frac{\partial^2 w}{\partial x^2}, \hspace{1cm} (A6)$$

$$\tau_{311}^{(2)} = -2\mu E^\nu \frac{\partial^2 w}{\partial x^2}, \hspace{1cm} (A7)$$

$$E_3 = \alpha P_3 - f_1 \frac{\partial^2 w}{\partial x^2}, \hspace{1cm} (A8)$$

$$V_{33} = f_1 z \frac{\partial^2 w}{\partial x^2}, \hspace{1cm} (A9)$$

with $E^\nu$ denoting Young's modulus. From Eq. (A8), the polarization component $P_3$ is given as

$$P_3 = \frac{1}{\alpha} E_3 + f_1 \frac{\partial^2 w}{\partial x^2}. \hspace{1cm} (A10)$$

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