MALLIAVIN CALCULUS APPROACH TO STATISTICAL INFEERENCE FOR LÉVY DRIVEN SDE’S

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ABSTRACT. By means of the Malliavin calculus, integral representations for the likelihood function and for the derivative of the log-likelihood function are given for a model based on discrete time observations of the solution to equation $dX_t = a_\theta(X_t)dt + dZ_t$ with a Lévy process $Z$. Using these representations, regularity of the statistical experiment and the Cramer-Rao inequality are proved. Malliavin calculus and Likelihood function and Lévy driven SDE and Regular statistical experiment and Cramer-Rao inequality

1. Introduction

Consider stochastic equation of the form

$$dX_t = a_\theta(X_t)dt + dZ_t,$$

where $Z$ is a one-dimensional Lévy process, $a: \Theta \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function, $\Theta \subset \mathbb{R}$ is a parametric set. The main objective of our study is the statistical inference of the unknown parameter $\theta$ given the observations of the solution to this equation at a discrete time set.

The likelihood function in the above model is highly implicit. In this paper, we develop an approach which makes it possible to control the properties of the likelihood and log-likelihood functions only in the terms of the objects involved in the model: the function $a_\theta(x)$, its derivatives, and the Lévy measure of the Lévy process $Z$. This approach is based on an appropriate version of the Malliavin calculus for a Poisson point measure.

The Malliavin calculus, developed first as a tool for proving existence and smoothness of distribution densities, appears to be very efficient in a study of sensitivities w.r.t. parameters. This field of applications, motivated by the analysis of volatilities in the models of financial mathematics, comes back to [8] and was studied intensively during the last years. This technique has natural extensions to statistical problems. In [10], [11], for discretely observed diffusion models, a Malliavin-type integral representation of the derivative of the log-likelihood ratio w.r.t. parameter is given, and then is used as a key tool in the proof of the LAN (LAMN) property of the model. In the recent paper [7], several versions of Malliavin calculus-based sensitivity analysis on the Wiener-Poisson probability space was developed, with applications to evaluation of the Cramer-Rao inequality and to a study of asymptotic properties of MLE for discretely observed diffusion processes.

We are mainly concentrated on the study of equation (1), where $Z$ is a Lévy process without a diffusion component. We develop a particular version of the Malliavin calculus for Poisson point measures from [3], [2] (see also more recent
papers [1, 4] and references therein), which is convenient for the purposes of the further sensitivity analysis. We give integral representations for the likelihood function and for the derivative of the log-likelihood function w.r.t. parameter. These representations are used then as the key ingredient in the proof of the regularity of a statistical experiment generated by discrete time set observations of the solution to (1), and consequent evaluation of the Cramer-Rao inequality. These representations also give a tool for the further asymptotical analysis of the properties on the model when the size of the sample tends to $\infty$; see forthcoming papers [13] and [14], addressed to the LAN property of the model and to the asymptotic efficiency of the MLE, respectively.

For simplicity reasons, here we restrict ourselves by the case of both observations $X_t$ and parameter $\theta$ being one-dimensional, and postpone the study of the multidimensional case for a further research.

2. Notation, assumptions, and main results

2.1. Notation and assumptions. Let $Z$ be a Lévy process without a diffusion component; that is,

$$Z_t = ct + \int_0^t \int_{|u|>1} u\nu(ds, du) + \int_0^t \int_{|u|\leq1} u\tilde{\nu}(ds, du),$$

where $\nu$ is a Poisson point measure with the intensity measure $ds \mu(du)$, and $\tilde{\nu}(ds, du) = \nu(ds, du) - ds \mu(du)$ is respective compensated Poisson measure. In the sequel, we assume the Lévy measure $\mu$ to satisfy the following:

**H.**
(i) for some $\kappa > 0$,

$$\int_{|u|\geq1} u^{2+\kappa} \mu(du) < \infty;$$

(ii) for some $u_0 > 0$, the restriction of $\mu$ on $[-u_0, u_0]$ has a positive density $\sigma \in C^2([-u_0, 0) \cup (0, u_0])$;

(iii) there exists $C_0$ such that

$$|\sigma'(u)| \leq C_0|u|^{-1}\sigma(u), \quad |\sigma''(u)| \leq C_0u^{-2}\sigma(u), \quad |u| \in (0, u_0];$$

(iv)

$$\left(\log \frac{1}{\varepsilon}\right)^{-1} \mu\left(\{u : |u| \geq \varepsilon\}\right) \to \infty, \quad \varepsilon \to 0.$$  

One particularly important class of Lévy processes satisfying **H** consists of tempered $\alpha$-stable processes (see [17]), which arise naturally in models of turbulence [15], economical models of stochastic volatility [5], etc.

Without loss of generality, $\Theta$ is assumed to be a finite open interval on $\mathbb{R}$. For a given $\theta \in \Theta$, assuming that the drift term $a_\theta$ satisfies the standard local Lipschitz and linear growth conditions, Eq. (1) uniquely defines a Markov process $X$. We denote by $P^\theta_x$ the distribution of this process in $\mathcal{D}([0, \infty))$ with $X_0 = x$, and by $E^\theta_x$ the expectation w.r.t. this distribution. Respective finite-dimensional distribution for given time moments $t_1 < \cdots < t_n$ is denoted by $P^\theta_x(t_{1n})$. On the other hand, solution $X$ to Eq. (1) is a random function defined on the same probability space $(\Omega, \mathcal{F}, P)$ with the process $Z$, which depends additionally on the parameter $\theta$ and the initial value $x = X(0)$. We do not indicate this dependence in the notation, i.e. write $X_t$ instead of e.g. $X^\theta_{x,t}$, but it will be important in the sequel that,
under certain conditions, \( X_t \) is \( L_2 \)-differentiable w.r.t. \( \theta \) and is \( L_2 \)-continuous w.r.t \((t, x, \theta)\).

In the sequel we will show that, under appropriate conditions, Markov process \( X \) admits a transition probability density \( p_t^\theta(x, y) \) w.r.t. Lebesgue measure, which is continuous w.r.t. \((t, x, y) \in (0, \infty) \times \mathbb{R} \times \mathbb{R}\). Then (see [6]), for every \( t > 0, x, y \in \mathbb{R} \) such that

\[
(2) \quad p_t^\theta(x, y) > 0,
\]

there exists a weak limit in \( \mathbb{D}([0, t]) \)

\[
P_{t, \theta}^{x, y} = \lim_{\varepsilon \to 0} P_{x}^{\delta} \left( \cdot \mid |X_t - y| \leq \varepsilon \right),
\]

which can be interpreted naturally as a *bridge* of the process \( X \) started at \( x \) and conditioned to arrive to \( y \) at time \( t \). We denote by \( E_{x, y}^{t, \theta} \) the expectation w.r.t. \( P_{x, y}^{t, \theta} \).

In what follows, \( C \) denotes a constant which is not specified explicitly and may vary from place to place. By \( C^{k, m}(\mathbb{R} \times \Theta) \), \( k, m \geq 0 \) we denote the class of functions \( f : \mathbb{R} \times \Theta \to \mathbb{R} \) which has continuous derivatives

\[
\partial_x^{i+j} f := \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial \theta^j} f, \quad i \leq k, \quad j \leq m.
\]

### 2.2. Main results: formulation.

Here we formulate two main theorems of this paper. The first one concerns the local properties of the transition probabilities of the Markov process \( X \). The functionals \( \Xi_t, \Xi_1 \), involved in its formulation, will be introduced explicitly in the proof below; see formulae (37) and (42).

**Theorem 1.** I. Let \( a \in C^{2,0}(\mathbb{R} \times \Theta) \) with bounded derivatives \( \partial_x a, \partial_\theta a \).

Then the Markov process \( X \) defined by (1) has a transition probability density \( p_t^\theta \) w.r.t. the Lebesgue measure, which has an integral representation

\[
(3) \quad p_t^\theta(x, y) = E_{x}^{\delta} \left[ \Xi_t \mathbb{1}_{Y} \right], \quad t > 0, \quad x, y \in \mathbb{R}.
\]

The function \( p_t^\theta(x, y) \) is continuous w.r.t. \((t, x, y, \theta) \in (0, \infty) \times \mathbb{R} \times \mathbb{R} \times \Theta\).

II. Let \( a \in C^{3,1}(\mathbb{R} \times \Theta) \) have bounded derivatives \( \partial_x a, \partial_\theta a, \partial_x^2 a, \partial_\theta^2 a, \partial_x \partial_\theta a \), \( \partial_x^3 a, \partial_\theta^3 a \) and

\[
(4) \quad |a(x)| + |\partial_\theta a(x)| \leq C(1 + |x|), \quad \theta \in \Theta, \quad x \in \mathbb{R}.
\]

Then the transition probability density has a derivative \( \partial_\theta p_t^\theta(x, y) \), which is continuous w.r.t. \((t, x, y, \theta) \in (0, \infty) \times \mathbb{R} \times \mathbb{R} \times \Theta\).

III. Under the conditions of statement II, one has

\[
(5) \quad \partial_\theta p_t^\theta(x, y) = g_t^\theta(x, y) p_t^\theta(x, y)
\]

with

\[
(6) \quad g_t^\theta(x, y) = \begin{cases} E_{x, y}^{t, \theta} \mathbb{1}, & p_t^\theta(x, y) > 0, \\ 0, & \text{otherwise}. \end{cases}
\]

**Remark 1.** By statements II and III, the logarithm of the transition probability density has a continuous derivative w.r.t. \( \theta \) on the open subset of \((0, \infty) \times \mathbb{R} \times \mathbb{R} \times \Theta\) defined by inequality (2) and, on this subset, admits the integral representation

\[
(7) \quad \partial_\theta \log p_t^\theta(x, y) = E_{x, y}^{t, \theta} \mathbb{1}.
\]
The second theorem concerns the basic properties of the statistical experiment
\[(\mathbb{R}^n, \mathcal{B} (\mathbb{R}^n), P^\theta_{x, (t_k)} \mid \theta \in \Theta)\],
generated by observations of the Markov process \(X\) with fixed \(X_0 = x\) at time
moments \(t_1 < \cdots < t_n\); we refer to [12] for the notation and terminology. Recall
that a statistical experiment \((\mathcal{X}, \mathcal{U}, P^\theta, \theta \in \Theta)\) is called regular, if \(dP^\theta = p^\theta d\lambda\) for
some \(\sigma\)-finite measure \(\lambda\), and
(a) the function \(\theta \mapsto p^\theta (x)\) is continuous for \(\lambda\)-a.a. \(x \in \mathcal{X}\);
(b) the function \(\theta \mapsto \sqrt{p^\theta} \in L_2 (\mathcal{X}, \lambda)\) is differentiable;
(c) the function \(\theta \mapsto \partial_\theta \sqrt{p^\theta} \in L_2 (\mathcal{X}, \lambda)\) is continuous (the derivative is under-
stood in the \(L_2 (\mathcal{X}, \lambda)\) sense).

For a regular statistical experiment with \(\Theta \subset \mathbb{R}^1\), respective
Fisher information is defined as
\[I (\theta) = 4 \int_{\mathcal{X}} \left( \partial_\theta \sqrt{p^\theta} \right)^2 d\lambda.\]

**Theorem 2.** Let conditions of statement II of Theorem 1 hold true and \(x \in \mathbb{R}, n \in \mathbb{N}\),
\(0 < t_1 < \cdots < t_n\) be fixed.
Then the statistical experiment (8) is regular. Respective Fisher information
equals
\[I (\theta) = \sum_{k=1}^n E^\theta_x \left( \frac{g^\theta_{t_k-t_{k-1}} (X_{t_{k-1}}, X_{t_k})}{g^\theta (x)} \right)^2,
\]
where \(t_0 := 0\).

**Remark 2.** For the statistical experiment (8) one has \(\mathcal{X} = \mathbb{R}^n\), and the natural
choice of \(\lambda\) is the Lebesgue measure. Then by Theorem 4
\[p^\theta (x) = \prod_{k=1}^n p^\theta_{t_k-t_{k-1}} (x_{k-1}, x_k), \quad x = (x_1, \ldots, x_n),\]
where \(x_0 := x\), and there exists a point-wise derivative
\[\partial_\theta p^\theta (x) = p^\theta (x) g^\theta (x), \quad g^\theta (x) := \sum_{k=1}^n g^\theta_{t_k-t_{k-1}} (x_{k-1}, x_k), \quad x = (x_1, \ldots, x_n).\]
In particular, one can interpret (4) and (10) as integral representations for the
log-likelihood function and the derivative of the likelihood function in a one-point ob-
servation model.

Combining Theorem 2 and Theorem I.7.3 in [12], we obtain the following version
of the Cramer-Rao inequality.

**Corollary 1.** Let conditions of statement II of Theorem 1 hold true and \(x \in \mathbb{R}, n \in \mathbb{N}\),
\(0 < t_1 < \cdots < t_n\) be fixed. Assume that
\[I (\theta) > 0\]
and \(T : \mathbb{R}^n \to \mathbb{R}\) is a Borel measurable function such that the function
\[\theta \mapsto E^\theta_x T^2 (X_{t_1}, \ldots, X_{t_n})\]
is locally bounded.
Then the bias
\[d (\theta) = E^\theta_x T (X_{t_1}, \ldots, X_{t_n}) - \theta\]
is differentiable, and
\[ \mathbb{E}_x^\theta \left(T(X_{t_1}, \ldots, X_{t_n}) - \theta \right)^2 \geq \frac{(1 + \partial_\theta d(\theta))^2}{I(\theta)} + d^2(\theta). \]

2.3. Main results: discussion. Let us emphasize one particularly important property of our model, which makes a substantial difference with those studied in [10], [11], 7. In the diffusive models studied in [10], [11], the likelihood function is positive, hence the log-likelihood function is a \( C^1 \) function w.r.t. \( \theta \). The approach of [7] requires, among others, the following structural assumption:

\[ (11) \text{ the support of the density } p^\theta \text{ does not depend on } \theta, \]
which also makes it possible to obtain the \( C^1 \) log-likelihood function by considering respective support set as a state space \( X \). However, as one can see from the example below, in the context of Eq. (1), the assumption (11) would restrict the model substantially.

Example 1. Let \( Z \) be a tempered \( \alpha \)-stable with \( \alpha \in (0, 1) \), which has positive jumps, only; that is,
\[ \mu(du) = r(u)u^{-\alpha-1}1_{u>0}du \]
with some (smooth enough) \( r \) such that \( r(u) = \text{const} > 0 \) in a neighborhood of \( u = 0 \), and \( r(u) \to 0 \) (rapidly enough) as \( u \to \infty \). Then by the support theorem from [19], the topological support of \( P^\theta_t(x, dy) \) equals \( [y^\theta_t(x), \infty) \), where \( y^\theta_t(x) \) is the value at time moment \( s = t \) of the solution to the Cauchy problem
\[ y'(s) = a_\theta(y(s)), \ y(0) = x. \]

Generically, \( y^\theta_t(x) \) depends on \( \theta \); for instance, for \( a_\theta(x) = \theta x \) one has \( y^\theta_t(x) = xe^{\theta t} \).

Because the topological support of \( P^\theta_t(x, dy) \) is the closure of the support of the transition probability density \( p^\theta_t(x, y) \), this indicates that in this case (11) fails.

This observation mainly motivates the particular form of our approach: because we would like to exclude the assumption (11) completely, we do not rely on the path-wise regularity of the log-likelihood function. Instead of that, we prove that our model is regular. Regularity of the experiment yields the Cramer-Rao inequality, and, which is maybe even more important, is a natural pre-requisite for the Ibragimov-Khasminskii’s version of the Hayek-Le Cam approach to the study of the asymptotic properties of a model ([12], Chapter II.3 and Chapter III.1). In our forthcoming papers [13] and [14], we use both the integral representations (3), (6) and the regularity of the model to prove the LAN property of the model and the asymptotic efficiency of the MLE.

3. Malliavin calculus for Poisson random measures

Typically, a Malliavin calculus-based sensitivity analysis requires a pair of a derivation operator \( D \) and an adjoint operator \( \delta = D^* \) to be defined on the probability space under the consideration. Below we outline such a construction, based on perturbations of “jump amplitudes”, which is well known in the field, goes back to [3], [2], and has various modifications; see e.g. [1], [4], and references therein. To keep the exposition self-sufficient and transparent, we explain the main components of this construction; in addition, we specially modify it in order to provide the integral representations, involved into the further statistical applications, in as explicit form as it is possible.
3.1. Perturbations of Poisson random measures and associated differential operators. Let \( q : \mathbb{R} \to \mathbb{R}^+ \) be a \( C^2 \)-function with bounded derivative. Denote by \( Q_c(x) \), \( c \in \mathbb{R} \) the value at the time moment \( s = c \) of the solution to Cauchy problem

\[ q'(s) = \phi(q(s)), \quad q(0) = x. \]

Then \( \{Q_c, c \in \mathbb{R}\} \) is a group of transformations of \( \mathbb{R} \), and \( \partial_t Q_c(x)|_{c=0} = \phi(x) \).

Denote by \( \mathcal{O} \) the space of locally finite configurations in \( \mathbb{R}^+ \times (\mathbb{R} \setminus \{0\}) \); that is, the family of all sets \( \varpi \subset \mathbb{R}^+ \times (\mathbb{R} \setminus \{0\}) \) such that for any \( \varepsilon > 0, R > 0, T > 0 \) the set

\[ \varpi \cap \left( [0, T] \times \{ u : \varepsilon < |u| < R \} \right) \]

is finite. This space is naturally endowed by the vague topology; that is, the minimal topology w.r.t. which any map of the form

\[ \varpi \mapsto \sum_{(\tau, u) \in \varpi} f(\tau, u) \]

with a continuous \( f : \mathbb{R}^+ \times (\mathbb{R} \setminus \{0\}) \), which is supported by some set of the form \( [0, T] \times \{ u : \varepsilon < |u| < R \} \), is continuous. This is the natural state space when the random point measure \( \nu \) is considered as a random element; denote by \( \nu_0 \) the distribution of \( \nu \) in \( (\mathcal{O}, \mathcal{B}(\mathcal{O})) \). In what follows, we identify the initial probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with \( (\mathcal{O}, \mathcal{B}(\mathcal{O}), \mathbb{P}_\nu) \), and assume \( \nu(\omega) = \omega \). Under this convention, which does not restrict generality, every \( \omega \in \mathcal{O} \) is a locally finite collection of points \( (\tau, u) \), where \( \tau \in \mathbb{R}^+ \) is the “jump time”, and \( u \in \mathbb{R} \setminus \{0\} \) is the “jump amplitude”.

For given \( T > 0 \) and \( \varrho \), define the group \( \{Q_c, c \in \mathbb{R}\} \) of transformations of the configuration space \( \mathcal{O} \) in the following way. Transformation \( Q_c \) maps a configuration \( \omega \) into the collection of points of the form

\[ \begin{cases} (\tau, Q_c(u)), & (\tau, u) \in \omega \text{ is such that } \tau < T; \\ (\tau, u), & (\tau, u) \in \omega \text{ is such that } \tau > T. \end{cases} \]

Then \( Q_c \) transforms \( \nu \) into the Poisson point measure \( \nu_c \),

\[ \nu_c(A \times B) = \nu \left( (A \cap [0, T]) \times Q_c^{-1}(B) \right) + \nu \left( (A \cap [0, T]) \times B \right), \quad A \in \mathcal{B}([0, T]), B \in \mathcal{B}(\mathbb{R}), \]

and the intensity measure for \( \nu_c \) has the form

\begin{equation}
I_{s \leq T} ds \left[ \mu \circ Q_c^{-1} \right](du) + I_{s > T} ds \mu(du).
\end{equation}

Fix \( u_1 \in (0, u_0) \), where \( u_0 \) comes from \( H \) (ii). In what follows, we choose the function \( \varrho \) involved in the definition of \( Q_c, c \in \mathbb{R} \) in such a way that

\[ \varrho(u) = \begin{cases} u^2, & |u| \leq u_1; \\ 0, & |u| \geq u_0. \end{cases} \]

Then the intensity measure (12) has the density w.r.t. \( ds \mu(du) \) equal to

\begin{equation}
m_{c,T}(s, u) = I_{s \leq T} m_c(u) + I_{s > T},
\end{equation}

where

\[ m_c(u) = \frac{d[\mu \circ Q_c^{-1}](u)}{d\mu}(u) = \frac{1}{R_c(Q_c^{-1}(u))} \sigma(Q_c^{-1}(u)) \sigma(u) \]

and
with

\[ R_c(x) := \partial_x Q_c(x) = \exp \left( \int_0^x \dot{g}(Q_s(x)) \, ds \right). \]

By the construction, \( Q_c(u) \equiv u, c \in \mathbb{R} \) if \( |u| \geq u_0 \). On the other hand, for any given \( c \in \mathbb{R} \) there exists \( u(c) > 0 \) s.t.

\[ Q_s(u) = \left( \frac{1}{u} - s \right)^{-1} = \frac{u}{1 - us}, \quad |s| \leq |c|, \quad |u| \leq u(c). \]

Therefore there exists \( \hat{u}(c) \in (0, u(c)) \) s.t.

\[ \frac{Q_s(u)}{u} \in \left[ \frac{1}{2}, 2 \right], \quad |s| \leq |c|, \quad |u| \leq \hat{u}(c). \]

Because \( Q_c^{-1} = Q_{-c} \), this yields immediately that

\[ R_c(Q_c^{-1}(u)) = O(|u|), \quad u \to 0. \]

Using (14) and H (iii), we get for \( |u| \leq \hat{u}(c) \)

\[ |\sigma(Q_c^{-1}(u)) - \sigma(u)| = \left| \int_0^c \sigma'(Q_s^{-1}(u)) Q_s^{-1}(u)^2 \, ds \right| \leq 2|u| \int_0^{|c|} \sigma(Q_s^{-1}(u)) \, ds. \]

It is straightforward to deduce from H(iii) that, for some \( u_2 > 0 \) and \( K > 1 \),

\[ \sigma(\gamma u) \leq K \sigma(u), \quad \gamma \in [1/2, 2], \quad |u| \leq u_2. \]

Summarizing all the above, we conclude that for a given \( c \) the function \( \log m_c \) is continuous, vanishes when \( |u| \geq u_0 \), and satisfies

\[ \log m_c(u) = O(|u|), \quad u \to 0. \]

Therefore, one has

\[ \int_{|\log m_c| \geq \log 2} |1 - m_c(s, u)| \, ds \mu(du) + \int_{\mathbb{R}^+ \times \mathbb{R}} \log^2 m_c(s, u) \, ds \mu(du) < \infty. \]

Applying Skorokhod’s criterion for absolute continuity of the laws of Poisson point measures [20], we arrive at following.

**Proposition 1.** The distribution \( P_{\nu_c} \) of \( \nu_c \) in \((\mathcal{O}, \mathcal{B}(\mathcal{O}))\) is absolutely continuous w.r.t. \( P_{\nu} \), and

\[ \frac{dP_{\mu_c}}{dP_{\nu}}(\nu) = \exp \left\{ \int_0^T \int_{\mathbb{R}} \log m_c(u) \tilde{\nu}(ds, du) \right\}. \]

Consequently, the map \( Q_c : \Omega \to \Omega \) generates the map of \( L_0(\Omega, \mathcal{F}, P) \) into itself in the following way (we keep the same symbol \( Q_c \) for this map):

\[ Q_c F(\omega) = F(Q_c \omega), \quad F \in L_0(\Omega, \mathcal{F}, P). \]

Straightforward computation shows that for every \( u \neq 0 \)

\[ \partial_x m_c(u)|_{x=0} = -\frac{(\sigma(u)\dot{g}(u))'}{\sigma(u)} =: \chi(u). \]
In addition,

\[(16) \quad \int_{\mathbb{R}} \left( \frac{m_c(u) - 1}{c} - \chi(u) \right)^2 \mu(du) \to 0, \quad c \to 0.\]

Because \(m_c(u) = 1, |u| \geq u_0\), the latter relation and \((15)\) yield

\[(17) \quad \kappa_c(u) - 1 \quad \to \quad \int_0^T \int_{\mathbb{R}} \chi(u) \tilde{\nu}(ds, du) \text{ in } L^2(\Omega, \mathcal{F}, P), \quad c \to 0.\]

The proofs of \((16)\) and \((17)\) are straightforward but cumbersome, and therefore are omitted.

**Definition 1.** A functional \(F \in L^2(\Omega, \mathcal{F}, P)\) is called stochastically differentiable, if there exists an \(L^2(\Omega, \mathcal{F}, P)\)-limit

\[(18) \quad \hat{D}F = \lim_{c \to 0} \frac{1}{c} \left( Q_c F - F \right).\]

The closure \(D\) of the operator \(\hat{D}\) defined by \((18)\) is called the stochastic derivative. The adjoint operator \(\delta = D^*\) is called the divergence operator or the extended stochastic integral.

**Remark 3.** By \((19)\) and \((24)\) below, \(\text{dom}(D)\) is dense in \(L^2(\Omega, \mathcal{F}, P)\), hence \(\delta\) is well defined. In addition, by statement 3 of Proposition 2 below \(\text{dom}(\delta)\) is dense in \(L^2(\Omega, \mathcal{F}, P)\), hence \(\hat{D}\) is closable. The operator \(\delta\) itself is closed as an adjoint one; e.g. Theorem VIII.1 in \([18]\).

The following proposition collects the main properties of the operators \(D, \delta\).

**Proposition 2.** 1. Let \(\varphi \in C^1(\mathbb{R}^d, \mathbb{R})\) have bounded derivatives and \(F_k \in \text{dom}(D), \quad k = 1, \ldots, d.\)

Then \(\varphi(F_1, \ldots, F_d) \in \text{dom}(D)\) and

\[(19) \quad D \left[ \varphi(F_1, \ldots, F_d) \right] = \sum_{k=1}^d [\partial_x \varphi](F_1, \ldots, F_d) D F_k.\]

2. The constant function 1 belongs to \(\text{dom}(\delta)\) and

\[(20) \quad \delta(1) = \int_0^T \int_{\mathbb{R}} \chi(u) \tilde{\nu}(ds, du).\]

3. Let \(G \in \text{dom}(D)\) and

\[(21) \quad \mathbb{E} (\delta(1) G)^2 < \infty.\]

Then \(G \in \text{dom}(\delta)\) and \(\delta(G) = \delta(1) G - DG.\)

**Proof.** 1. It is sufficient to consider \(F_k, k = 1, \ldots, d\) which satisfy \((18)\). Then the fraction

\[(22) \quad \frac{1}{c} \left( Q_c \varphi(F_1, \ldots, F_d) - \varphi(F_1, \ldots, F_d) \right)\]

converges in probability to the right hand side of \((19)\). Its square is dominated by

\[\sup_x \|\nabla \varphi(x)\|^2 \sum_{k=1}^d \left( \frac{Q_c F_k - F_k}{c} \right)^2,\]

and hence is uniformly integrable. Consequently, \((22)\) converges in \(L^2(\Omega, \mathcal{F}, P).\)
2. For any \( F \) satisfying (18) we have by (17) 
\[
EF\left( \int_0^T \int_{\mathbb{R}} \chi(u)\tilde{\nu}(ds, du) \right) = \lim_{c \to 0} \frac{1}{c} EF(\kappa_c - 1) = \lim_{c \to 0} \frac{1}{c} E(Q_cF - F) = E(DF),
\]
which gives by the definition of \( \delta = D^* \) that \( \delta(1) = \int_0^T \int_{\mathbb{R}} \chi(u)\tilde{\nu}(ds, du) \).

3. For bounded \( F, G \in \text{dom}(D) \) one has by (19) that \( FG \in \text{dom}(D) \) and \( D(FG) = FDG + GDF. \) Then by statement 2 we have
\[
(23) \quad EGDF = EF\left( \delta(1)G - DG \right).
\]

For arbitrary \( F \in \text{dom}(D) \), using (19), one can choose a sequence of bounded \( F_n \in \text{dom}(D) \) such that \( F_n \to F \) and \( DF_n \to DF \) in \( L_2(\Omega, \mathcal{F}, P) \). This proves (23) for arbitrary \( F \in \text{dom}(D) \), and yields the required statement under the additional assumption that \( G \) is bounded. Approximating \( G \) by bounded \( G_n \in \text{dom}(D) \) and using that \( \delta \) is a closed operator completes the proof.

3.2. Differential properties of the solution to (1). Denote \( Z_t^c = Q_cX_t \). It can be seen straightforwardly that
\[
(24) \quad \frac{1}{c} (Z_t^c - Z_t) \to DZ_t := \int_0^T \int_{\mathbb{R}} \vartheta(u)\nu(ds, du) \text{ in } L_2(\Omega, \mathcal{F}, P)
\]
uniformly by \( t \in [0, T] \) for every \( T \). Then one can consider \( X_t^c = Q_cX_t \) as the solution to the following perturbed SDE:
\[
(25) \quad dX_t^c = a_\theta(X_t^c)dt + dZ_t^c.
\]
Applying Theorem II.2.8.5 [9], under conditions of statement I, Theorem II.2.8.5 we get that for any fixed \( \theta \in \Theta \) and initial value \( x \in \mathbb{R} \) the family \( X_t^c \) is differentiable in \( L_2 \); that is,
\[
(26) \quad \frac{1}{c} (X_t^c - X_t) \to DX_t \text{ in } L_2(\Omega, \mathcal{F}, P), \quad c \to 0.
\]
Clearly, the derivative in the right hand side of (26) is just the stochastic derivative of \( X_t \); see Definition II.2.8.5. Moreover, convergence (25) holds true uniformly by \( t \in [0, T] \) for every \( T \). The process \( Y_t := DX_t \) satisfies the linear SDE
\[
dY_t = \partial_\theta a_\theta(X_t)Y_t dt + dDZ_t, \quad Y_0 = 0,
\]
and hence can be written explicitly:
\[
(27) \quad DX_t = E_t \int_0^T \int_{\mathbb{R}} \mathcal{E}_s^{-1}(\vartheta(u)\nu(s, du), \quad \mathcal{E}_t := \exp \left\{ \int_0^T \partial_\theta a_\theta(X_t)dt \right\}.
\]
The same argument gives (we omit the detailed exposition):
\[
(28) \quad D\delta(1) = \int_0^T \int_{\mathbb{R}} \chi'(u)\vartheta(u)\nu(ds, du),
\]
\[
(29) \quad D\mathcal{E}_t = \mathcal{E}_t \int_0^T \partial_{xx}^2 a_\theta(X_t)DX_t dt,
\]
\[
D^2 X_t := D(DX_t) = D\mathcal{E}_t \left( \int_0^T \int_{\mathbb{R}} \mathcal{E}_s^{-1}(\vartheta(u)\nu(s, du) \right.
\]
\[
+ \mathcal{E}_t \int_0^T \int_{\mathbb{R}} (\mathcal{E}_s^{-1}(\vartheta(u)\vartheta'(u) - \mathcal{E}_s^{-2}(\partial_{xx}^2 \vartheta(u))) \nu(ds, du).
\]
The second derivative $D^2 X_t$ is stochastically differentiable, as well; a cumbersome explicit formula for $D^3 X_t$, analogous to (30), is omitted. Note that the assumption on $\sigma''$ from $H$ (iii) is required to bound $\chi'$ and prove (25), and the assumption on the derivatives $\partial_x a, \partial_x^2 a, \partial_x^3 a$ is used to get the existence of the derivatives $DX_t, D^2 X_t, D^3 X_t$.

Recall that, although this is not given explicitly in the notation, the solution $X_t$ to Eq. (1) is a function which depends on the parameter $\theta$. Applying Theorem II.2.8.5 [9] once more, we get that, under $H$ (i) and conditions of statement II of Theorem II $X_t$ is $L_2$-differentiable w.r.t. $\theta$ (in the sense similar to (26)), and respective derivative equals

$$\partial_\theta X_t = \mathcal{E}_t \int_0^t \mathcal{E}_s^{-1} |\partial_\theta a_\theta|(X_s)ds$$

This derivative is $L_2$-continuous w.r.t. $\theta$ and stochastically differentiable with

$$D(\partial_\theta X_t) = D\mathcal{E}_t \int_0^t \mathcal{E}_s^{-1} |\partial_\theta a_\theta|(X_s)ds$$

and the assumption $\partial_x a$ is bounded, there exist positive $C_1, C_2$ such that

$$C_1 \leq \mathcal{E}_t \leq C_2, \quad t \in [0, T].$$

Then (33) would follow from the same bound for the Itô integral of the deterministic function $\rho$ w.r.t. $\nu$. To prove that bound, we apply Lemma 5.1 [2] with $L \equiv 1, \eta = \bar{\rho}$ and estimate separately the integral over the compensator $d\mu(du)$. Because $\partial_x^2 a$ is bounded, the analogue of (33) holds true for $D\mathcal{E}_t$ instead of $DX_t$; see (29). Repeating (with minor modifications) the same argument, we prove that the analogue of (33) holds true for $D^2 X_t, D^3 X_t$. For instance, to get the analogue of (33) for $D^2 X_t$ given by (30), we use (34), the analogue of (33) for $D\mathcal{E}_t$, the Hölder inequality, and Lemma 5.1 [2] with $L_s = D\mathcal{E}_s, \eta = \bar{\rho}$.

Finally, by assumption $H$ (iii) and the choice of $\rho$, the function

$$\chi = -(\sigma'/\sigma)\bar{\rho} - \bar{\rho}$$

is bounded by $C|u|$ and vanishes for $|u| \geq u_0$. Hence by the same argument the analogue of (33) holds true for $\delta(1)$, given by (20).

Next, note that by the condition $H$ (i) we have $E|Z_t|^{2+\kappa} \leq C, t \leq T$. Recall that $a_\theta(x)$ has a linear growth bound w.r.t. $x$. Then the standard argument based on the Gronwall lemma shows that

$$E^\theta|X_t|^p \leq C(1 + |x|^p), \quad p \in [1, 2 + \kappa).$$
Similar bounds with \( p < 2 + \kappa \) hold true for \( \partial \varphi X_t \) and \( D(\partial \varphi X_t) \). This follows from formulae (31), (32), the Hölder inequality, and \( L_p \)-bounds (with arbitrarily large \( p \)) for \( DX_t, D^2X_t \). Similarly, \( DX_t, D^2X_t, D^3X_t \) are \( L_2 \)-differentiable w.r.t. parameter \( \theta \) (we omit the details, but note that this is the place where we require the assumption on \( \partial^2 \varphi \delta \), \( \partial^3_x \theta \delta \), \( \partial^4_{xx} \theta \delta \)). Respective derivatives \( \partial \varphi DX_t, \partial \varphi D^2X_t, \partial \varphi D^3X_t \) can be written explicitly, and satisfy bounds analogous to (35).

Finally, we show that for every \( p \geq 1, t \in (0, T] \) there exists \( C_{p,t} \) s.t.
\[
E_t^\theta((DX_t)^{-p} \leq C_{p,t}, \quad x \in \mathbb{R}, \theta \in \Theta
\]
(recall that \( \varphi \geq 0 \) and therefore \( DX_t \) is non-negative). By (33) and non-negativity of \( \varphi \),
\[
DX_t \geq (C_2/C_1) \int_0^t \int_{|u| \leq u_0} \varphi(u)\nu(ds, du) \geq (C_2/C_1) \int_0^t \int_{|u| \leq u_0} u^2\nu(ds, du).
\]
Hence by assumption \( H \) (iv)
\[
P_t^\theta(DX_t < \varepsilon) \leq P_t^\theta \left( \int_0^t \int_{(C_1\varepsilon/C_2)^{1/2} \leq |u| \leq u_0} u^2\nu(ds, du) = 0 \right) = \exp \left[ -t \mu \left( \{ (C_1\varepsilon/C_2)^{1/2} \leq |u| \leq u_0 \} \right) \right] = o(\varepsilon^p), \quad \varepsilon \to 0
\]
for every \( p \geq 1 \), which implies the required statement.

4. Proof of Theorem II

Our first step is to prove that \( (DX_t)^{-1} \in \text{dom}(\delta) \) and
\[
\Xi_t := \delta \left( \frac{1}{DX_t} \right) = \frac{\delta(1)}{DX_t} + \frac{D^2X_t}{(DX_t)^2}.
\]
To do that, for arbitrary \( \varepsilon > 0 \) we consider the variable \( G_{t,\varepsilon} = (DX_t + \varepsilon)^{-1} \). By statement 1 of Proposition 2 applied to \( F = DX_t \) and \( \phi \in C^1_b(\mathbb{R}) \) such that \( \phi(x) = 1/x, x \geq \varepsilon \), the variable \( G_{t,\varepsilon} \) has the stochastic derivative \( -D^2X_t/(DX_t + \varepsilon)^2 \). Then by the statement 3 of Proposition 2
\[
\Xi_{t,\varepsilon} := \delta \left( G_{t,\varepsilon} \right) = \frac{\delta(1)}{DX_t + \varepsilon} + \frac{D^2X_t}{(DX_t + \varepsilon)^2}.
\]
By (36) and (39), \( G_{t,\varepsilon} \to (DX_t)^{-1} \) in \( L_2 \). Using, in addition, analogues of (35) for \( \delta(1) \) and \( D^2X_t \), we see that the right hand side term in (38) converges to that in (37) in \( L_2 \) as \( \varepsilon \to 0 \). Hence (37) holds true because \( \delta \) is a closed operator.

Now we can finalize the proof of statement I; the argument here is quite analogous to the one from the proof of Proposition 3.1.2 in [16], hence we omit details. By Proposition 2 the definition of \( \delta = D^* \), and (37), for every \( \varphi \in C^1_b(\mathbb{R}) \) we have
\[
E_t^\theta \varphi(X_t) = E_t^\theta D(\varphi(X_t)) \left( \frac{1}{DX_t} \right) = E_t^\theta \varphi(X_t) \delta \left( \frac{1}{DX_t} \right) = E_t^\theta \varphi(X_t) \Xi_t.
\]
Approximating \( \varphi_y := I_{[0,\infty)}(\cdot - y) \) by a sequence of \( \varphi_n \in C^1_b(\mathbb{R}) \), we get the representation (33).

By Theorem II.2.8.3 [9] applied to Eq. (1), \( X_t \) depend continuously (in \( L_2 \)) w.r.t. parameters \( x, t, \theta \). The same argument gives the continuity (in \( L_2 \)) of \( DX_t \) and \( D^2X_t \) w.r.t. parameters \( x, t, \theta \); note that both these derivatives can be defined as solutions to certain SDE’s, hence one can apply Theorem II.2.8.3 [9] iteratively.
Then it is easy to show that $\Xi_t$ depend continuously (in $L_2$) on the parameters $x, t, \theta$, as well. Indeed, by the continuity of $DX_t$, $D^2X_t$ and non-negativity of $DX_t$, for every $\varepsilon > 0$ the functional $\Xi_{t, \varepsilon}$, defined by (35), is continuous. Using the moment bounds (33) and (36) it is easy to show that $\Xi_{t, \varepsilon}$ converges to $\Xi_t$ in $L_2$ as $\varepsilon \to \infty$ uniformly in some neighborhood of any given point $(x, t, \theta) \in \mathbb{R} \times (0, \infty) \times \Theta$, hence the limiting functional $\Xi_t$ depend continuously on $x, t, \theta$.

By representation (39), we have
\[
P^\varepsilon_x(X_t = y) = 0, \quad x, y \in \mathbb{R}, \quad t > 0, \quad \theta \in \Theta.
\]

Then $L_2$-continuity of $\Xi_t$ and representation (39) provide that $p^\varepsilon_t(x, y)$ is continuous w.r.t. $(t, x, y, \theta)$, where

To prove statement II, we make one more integration by parts in the right hand side in (41), which gives

By (3), we have
\[
p^\varepsilon_t(x, y) = \mathbb{E}_x^\theta \psi_y(X_t) \delta \left( \frac{\Xi_t}{DX_t} \right),
\]

where $\psi_y = (\cdot - y) \vee 0$ is an absolutely continuous function with the derivative equal to $\varphi_y$. Recall that $X_t$ is $L_2$-differentiable w.r.t. parameter $\theta$, see (31) for its derivative. In addition, $DX_t$, $D^2X_t$, and $D^3X_t$ are $L_2$-differentiable w.r.t. $\theta$, and all these derivatives satisfy moment bounds similar to (35). Now it is easy to prove that $\delta(\Xi_t/(DX_t))$ is $L_2$-differentiable w.r.t. $\theta$ (the explicit formula of the derivative is omitted). One can just replace $DX_t$ in the denominator in the formula (40) by $DX_t + \varepsilon$, prove that this new functional is $L_2$-differentiable w.r.t. $\theta$ using the chain rule, and then show using (36) that both this functional and its derivative w.r.t. $\theta$ converge (locally uniformly) in $L_2$ as $\varepsilon \to 0$, respectively, to $\delta(\Xi_t/(DX_t))$ and to the functional $\partial_\theta \delta(\Xi_t/(DX_t))$ which comes from the formal differentiation of (40). This argument also shows that $\delta(\Xi_t/(DX_t))$ and $\partial_\theta \delta(\Xi_t/(DX_t))$ depend continuously (in $L_2$) on $x, t, \theta$. Therefore, we can take a derivative at the right hand side in (41), which gives

This function is continuous w.r.t. $(t, x, y, \theta)$ because $X_t, \partial_\theta X_t, \delta(\Xi_t/(DX_t))$, and $\partial_\theta \delta(\Xi_t/(DX_t))$ depend continuously (in $L_2$) on $x, t, \theta$, and (39) holds true.

To prove statement III, we use moment bounds for $\partial_\theta X_t$, $D(\partial_\theta X_t)$, $DX_t$, and $DX^2_t$ to get, similarly to the proof of (37), that $\partial_\theta X_t/(DX_t)$ belongs to dom($\delta$) and

Then for any test function $f \in C^1(\mathbb{R})$ with a bounded derivative we have
\[
\partial_\theta \mathbb{E}_x^\theta f(X_t) = \mathbb{E}_x^\theta f'(X_t)(\partial_\theta X_t) = \mathbb{E}_x^\theta Df(X_t) \frac{\partial_\theta X_t}{DX_t} = \mathbb{E}_x^\theta f(X_t) \Xi^\theta_t = \mathbb{E}_x^\theta f(X_t) g^\theta_2(x, X_t);
\]
see (6) for the definition of $g^\theta_t(x,y)$. Because the test function $f$ is arbitrary, the integral identity (43) proves (5).

5. Proof of Theorem 2

First, we formulate some properties of $p^\theta_t$ and $g^\theta_t = \partial_\theta \log p^\theta_t$, which follows from the integral representations for these functions and moment bounds obtained above.

Lemma 1. For every $p < 2 + \kappa$ there exists constant $C$ which depends on $t$ and $p$ only, such that

\begin{align}
(44) & \quad p^\theta_t(x,y) \leq C(1 + |x - y|)^{-p} \\
(45) & \quad \mathbb{E}_x^\theta |g^\theta_t(x, X_t)|^p \leq C(1 + |x|)^p.
\end{align}

Proof. By the moment bounds from Section 3.3 and formulae (37), (42), we have

\begin{align}
(46) & \quad \mathbb{E}_x^\theta |\Xi^1_t|^{p'} \leq C
\end{align}

for every $p' \geq 1$, and

\begin{align}
(47) & \quad \mathbb{E}_x^\theta |\Xi^1_t|^p \leq C(1 + |x|^p)
\end{align}

for every $p \in [1, 2 + \kappa)$, with the constants $C$ depending on $t, p'$ (resp. $p$), only.

Because

$$g^\theta_t(x, X_t) = \mathbb{E}_x^\theta [\Xi^1_t | X_t],$$

inequality (45) follows directly from (47) and Jensen’s inequality. To get (44), we use the standard argument, e.g. [16], Lemma 3.1.3 and Example afterwards. By the representation (3) and the Hölder inequality,

$$p^\theta_t(x,y) \leq C \left( \mathbb{P}_x^\theta(X_t > y) \right)^{(p'-1)/p'}. $$

Recall that $Z_t$ has finite moment of the order $2 + \kappa$ and $a^\theta$ has bounded derivative in $x$. Then by the Gronwall lemma

$$\mathbb{E}_x^\theta |X_t - x|^{2+\kappa} \leq C.$$ 

Then for $y > x$

$$\mathbb{P}_x^\theta(X_t > y) \leq \min\{1, C|x - y|^{-(2+\kappa)}\},$$

which gives (44) with $p = (p'-1)(2+\kappa)/p'$; the latter value can be made arbitrarily close to $2 + \kappa$ by the choice of $p'$. For $y < x$ one should use instead of (3) the representation

$$p^\theta_t(x,y) = -\mathbb{E}_x^\theta |\Xi^1_{\mathbb{I}_{X_t \leq y}}|,$$

which is equivalent to (4), because $\Xi^1$ is a stochastic integral and therefore has zero expectation.

\[ \square \]

Let us proceed with the proof of the Theorem. Formula (9) and statement I of Theorem 1 immediately provide the continuity property (a) from the definition of
a regular statistical experiment. To prove the $L_2$-differentiability property (b), and $L_2$-continuity of the derivative (c), we put for $\varepsilon > 0$

$$\psi_\varepsilon(z) = \begin{cases} 0, & z < \varepsilon/2, \\ \frac{(z-\varepsilon/2)^2}{\varepsilon^2}, & z \in [\varepsilon/2, \varepsilon], \\ \sqrt{\varepsilon - \frac{z^2}{8}}, & z \geq \varepsilon. \end{cases}$$

By the construction, $\psi_\varepsilon \in C^1$ and $\psi_\varepsilon(z) = 0$ for $z \leq \varepsilon/2$. Then, by statement II of Theorem 1 and statement 2 of Lemma 1 the mapping $\theta \mapsto \eta^\theta := \psi_\varepsilon(p^\theta) \in L_2(\mathbb{R}^n, \lambda)$ is continuously differentiable with the derivative equal

$$\zeta^\theta := \frac{d}{d\theta} \eta^\theta = \psi'_\varepsilon(p^\theta) \partial \theta p^\theta.$$ 

see (11) for the formula for $\partial \theta p^\theta$. By the construction,

$$\psi_\varepsilon(z) \to \psi_0(z) := \sqrt{\varepsilon}, \quad \psi'_\varepsilon(z) \to \psi'_0(z) = \frac{1}{2\sqrt{\varepsilon}}, \quad \varepsilon \to 0.$$ 

Hence to prove properties (b), (c) it is enough to show that

$$(48) \quad \eta^\theta \to \eta^0 := \psi_0(p^\theta), \quad \zeta^\theta \to \zeta^0 := \psi'_0(p^\theta) \partial \theta p^\theta \quad \text{in} \quad L_2(\mathbb{R}^n, \lambda)$$

uniformly by $\theta$. We show the second convergence in (48), the proof of the first one is similar and simpler. By the explicit form of $\psi'_\varepsilon$ and the Hölder inequality,

$$\int_{\mathbb{R}^n} (\zeta^\theta - \zeta^0)^2 d\lambda \leq \frac{1}{4} \int_{p^\theta \leq \varepsilon} (g^\theta)^2 p^\theta d\lambda \leq \frac{1}{4} \left( \int_{\mathbb{R}^n} |g^\theta|^p p^\theta d\lambda \right)^{\frac{2}{p}} \left( \int_{p^\theta \leq \varepsilon} p^\theta d\lambda \right)^{\frac{p-2}{p}}$$

for $p \geq 2$. Take $p \in (2, 2 + \kappa)$, then by Jensen’s inequality, representation (10), and (35),

$$\int_{\mathbb{R}^n} |g^\theta|^p p^\theta d\lambda = \mathbb{E}_x^\theta \left| \sum_{k=1}^n g^\theta_{t_k-t_{k-1}}(X_{t_{k-1}}, X_{t_k}) \right|^p \leq n^{p-1} \sum_{k=1}^n \mathbb{E}_x^\theta \left| g^\theta_{t_k-t_{k-1}}(X_{t_{k-1}}, X_{t_k}) \right|^p \leq n^{p-1} C \sum_{k=1}^n \mathbb{E}_x^\theta \left| 1 + |X_{t_{k-1}}| \right|^p \leq \hat{C},$$

where constant $\hat{C}$ does not depend on $\theta$. On the other hand,

$$\int_{p^\theta \leq \varepsilon} p^\theta d\lambda \leq \sqrt{\varepsilon} \int_{\mathbb{R}^n} \sqrt{p^\theta} d\lambda,$$

and by (44) with $p \in (2, 2 + \kappa)$

$$\int_{\mathbb{R}^n} \sqrt{p^\theta} d\lambda \leq C \int_{\mathbb{R}^n} \prod_{k=1}^n \left( 1 + |x_{k-1} - x_k| \right)^{-p/2} dx_1 \ldots dx_n \leq \hat{C}$$

with constant $\hat{C}$ which does not depend on $\theta$. Summarizing all the above, we get the second convergence in (48), uniform by $\theta$. This completes the proof of the regularity. The formula for the Fisher information follows from the identity

$$\int_{\mathbb{R}^n} \left( \partial_{\theta} \sqrt{p^\theta} \right)^2 d\lambda = \frac{1}{4} \int_{\mathbb{R}^n} (g^\theta)^2 p^\theta d\lambda = \frac{1}{4} \mathbb{E}_x^\theta \left( \sum_{k=1}^n g^\theta_{t_k-t_{k-1}}(X_{t_{k-1}}, X_{t_k}) \right)^2,$$

Markov property, and the observation that

$$\mathbb{E}_x^\theta y^\theta_x(x, X_t) = 0.$$
for every $x \in \mathbb{R}, \theta \in \Theta, t > 0$, which follows from (13) with $f \equiv 1$.

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