SOME RESULTS ON A CROSS-SECTION 
IN THE TENSOR BUNDLE 

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Abstract. The present paper is devoted to some results concerning with 
the complete lifts of an almost complex structure and a connection in a manifold 
to its (0,q)-tensor bundle along the corresponding cross-section.

1. Introduction 

The behaviour of the lifts of tensor fields and connections on a manifold to its 
different bundles along the corresponding cross-sections are studied by several 
authors. For the case tangent and cotangent bundles, see [13, 14, 15] and also tangent 
bundles of order 2 and order r, see [11, 3]. In [2], the first author and his collaborator 
studied the complete lift of an almost complex structure in a manifold on the 
so-called pure cross-section of its (p,q)-tensor bundle by means of the Tachibana 
operator (for diagonal lift to the (p,q)-tensor bundle see [1] and for the (0,q)-tensor 
bundle see [5]). Moreover they proved that if a manifold admits an almost complex 
structure, then so does on the pure cross-section of its (p,q)-tensor bundle provided 
that the almost complex structure is integrable. In [6], the authors give detailed 
description of geodesics of the (p,q)-tensor bundle with respect to the complete 

2. Preliminaries 

Let $M$ be a differentiable manifold of class $C^\infty$ and finite dimension $n$. Then 
the set $T_q^0(M) = \bigcup_{P \in M} T_q^0(P)$, $q > 0$, is the tensor bundle of type $(0,q)$ over $M$, 
where $\bigcup$ denotes the disjoint union of the tensor spaces $T_q^0(P)$ for all $P \in M$. 

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For any point \( \hat{P} \) of \( T^0_q(M) \) such that \( \hat{P} \in T^0_q(M) \), the surjective correspondence \( \hat{P} \to P \) determines the natural projection \( \pi : T^0_q(M) \to M \). The projection \( \pi \) defines the natural differentiable manifold structure of \( T^0_q(M) \), that is, \( T^0_q(M) \) is a \( \mathcal{C}^\infty \)-manifold of dimension \( n + n^q \). If \( x^j \) are local coordinates in a neighborhood \( U \) of \( P \in M \), then a tensor \( t \) at \( P \) which is an element of \( T^0_q(M) \) is expressible in the form \((x^j, t_{j_1...j_q})\), where \( t_{j_1...j_q} \) are components of \( t \) with respect to natural base.

We may consider \((x^j, t_{j_1...j_q})\) to the fibre is locally expressed by 
\[ x^k = x^k, \]
\[ x^k = \xi_{k_1...k_q}(x^k) \]
with respect to the coordinates \((x^k, x^k)\) in \( T^0_q(M) \). Differentiating (2.3) by \( x^j \), we see that \( n \) tangent vector fields \( B_j \) to \( \sigma_\xi(M) \) have components
\[ (B^K_j) = \frac{\partial x^K}{\partial x^j} = \left( \begin{array}{c} \delta^K_k \\ \delta^K_{j_1...j_q} \end{array} \right) \]
with respect to the natural frame \( \{\partial_k, \partial_k\} \) in \( T^0_q(M) \).

On the other hand, the fibre is locally expressed by
\[ t_{k_1...k_q} \]
taking \( t_{k_1...k_q} \) being considered as parameters. Thus, on differentiating with respect to \( x^j = t_{j_1...j_q} \), we see that \( n^q \) tangent vector fields \( C^k_j \) to the fibre have components
\[ (C^K_j) = \frac{\partial x^K}{\partial x^j} = \left( \begin{array}{c} 0 \\ \delta^K_{j_1...j_q} \end{array} \right) \]
with respect to the natural frame \( \{\partial_k, \partial_k\} \) in \( T^0_q(M) \).
We consider in $\pi^{-1}(U) \subset T^0_q(M)$, $n + n^q$ local vector fields $B_j$ and $C_j$ along $\sigma_\xi(M)$. They form a local family of frames $[B_j, C_j]$ along $\sigma_\xi(M)$, which is called the adapted $(B, C)$-frame of $\sigma_\xi(M)$ in $\pi^{-1}(U)$. Taking account of (2.2) on the cross-section $\sigma_\xi(M)$, and also (2.4) and (2.5), we can easily prove that, the complete lift $C V$ has along $\sigma_\xi(M)$ components of the form

$$C V = \begin{pmatrix} V^j \\
-L_V \xi_{j_1 \cdots j_q} \end{pmatrix}$$

with respect to the adapted $(B, C)$-frame. From (2.1), (2.4) and (2.5), the vertical lift $V A$ also has components of the form

$$V A = \begin{pmatrix} 0 \\
A_{j_1 \cdots j_q} \end{pmatrix}$$

with respect to the adapted $(B, C)$-frame.

3. Almost complex structures on a pure cross-section in the $(0, q)$-tensor bundle

A tensor field $\xi \in \mathfrak{S}^0_q(M)$ is called pure with respect to $\varphi \in \mathfrak{S}^1_q(M)$, if \[3.1\]

$$\varphi^T_{j_1 \cdots j_q} = \cdots = \varphi^T_{j_1} \xi_{j_1} = \xi_{j_1 \cdots j_q}.$$  

In particular, vector and covector fields will be considered to be pure.

Let $\mathfrak{S}^0_q(M)$ denotes a module of all the tensor fields $\xi \in \mathfrak{S}^0_q(M)$ which are pure with respect to $\varphi$. Now, we consider a pure cross-section $\sigma_\xi^q(M)$ determined by $\xi \in \mathfrak{S}^0_q(M)$. The complete lift $C \varphi$ of $\varphi$ along the pure cross-section $\sigma_\xi^q(M)$ to $T^0_q(M)$ has local components of the form

$$C \varphi = \begin{pmatrix} \varphi^k_l \\
-(\Phi_\varphi \xi)_{l_1 \cdots l_q} \\
\varphi^T_{j_1} \delta^j_{k_2} \cdots \delta^q_{k_q} \end{pmatrix}$$

with respect to the adapted $(B, C)$-frame of $\sigma_\xi^q(M)$, where $(\Phi_\varphi \xi)_{l_1 \cdots l_q} = \varphi^m_l \partial_m \xi_{l_1 \cdots l_q} - \delta^*_{l_1} \delta^*_{l_2} \cdots \delta^*_{l_q} \delta^*_{k_1 \cdots k_q} + \sum_{a=1}^q (\partial_{k_a} \varphi^a) \xi_{l_1 \cdots l_q} \Delta_{k_a}$ is the Tachibana operator.

We consider that the local vector fields

$$C X(i) = C \left( \frac{\partial}{\partial x^i} \right) = \left( \delta^h_{i_1} \frac{\partial}{\partial x^{i_1}} \right)$$

and

$$V X(i) = V \left( dx^{i_1} \otimes \cdots \otimes dx^{i_q} \right) = V \left( \delta^i_{h_1} \cdots \delta^i_{h_q} dx^{h_1} \otimes \cdots \otimes dx^{h_q} \right) = \begin{pmatrix} 0 \\
\delta^i_{h_1} \cdots \delta^i_{h_q} \end{pmatrix}$$

$i = 1, ..., n, \overline{i} = n + 1, ..., n + n^q$ span the module of vector fields in $\pi^{-1}(U)$. Hence, any tensor fields is determined in $\pi^{-1}(U)$ by their actions on $C V$ and $V A$ for any
$V \in \mathfrak{X}_q^0(M)$ and $A \in \mathfrak{X}_q^0(M)$. The complete lift $C\varphi$ along the pure cross-section $\sigma^\varphi(M)$ has the properties

\begin{equation}
\begin{cases}
C\varphi(CV)^C = (\varphi(V))^C + V ((L_\varphi) \circ \xi), \forall V \in \mathfrak{X}_q^0(M), (i)
C\varphi(VA)^C = (\varphi(A)), \forall A \in \mathfrak{X}_q^0(M), \ (ii)
\end{cases}
\end{equation}

which characterize $C\varphi$, where $\varphi(A) \in \mathfrak{X}_q^0(M)$. Remark that $V ((L_\varphi) \circ \xi)$ is a vector field on $T_q^0(M)$ and locally expressed by

$$V ((L_\varphi) \circ \xi) = \begin{pmatrix} 0 \\
(L_\varphi)^C_{ij} \xi_{j_{i_2\ldots i_q}} \end{pmatrix}$$

with respect to the adapted $(B,C)$-frame, where $\xi_{i_1\ldots i_q}$ are local components of $\xi$ in $M$ [5].

**Theorem 1.** Let $M$ be an almost complex manifold with an almost complex structure $\varphi$. Then, the complete lift $C\varphi \in \mathfrak{X}^1_q(T_q^0(M))$, when restricted to the pure cross-section determined by an almost analytic tensor $\xi$ on $M$, is an almost complex structure.

**Proof.** If $V \in \mathfrak{X}_q^0(M)$ and $A \in \mathfrak{X}_q^0(M)$, in view of the equations (i) and (ii) of (3.2), we have

\begin{equation}
(C\varphi)^2(CV) = (\varphi^2)(CV) + V (N_\varphi \circ \xi)(CV)
\end{equation}

and

\begin{equation}
(C\varphi)^2(VA) = (\varphi^2)(VA),
\end{equation}

where $N_\varphi(X)(Y) = (L_\varphi X \varphi - \varphi(L_X \varphi))(Y) = [\varphi X, \varphi Y] - \varphi [X, \varphi Y] - \varphi [\varphi X, Y] + \varphi^2 [X, Y] = N_\varphi(X,Y)$ is nothing but the Nijenhuis tensor constructed by $\varphi$.

Let $\varphi \in \mathfrak{X}_q^1(M)$ be an almost complex structure and $\xi \in \mathfrak{X}_q^0(M)$ be a pure tensor with respect to $\varphi$. If $(\Phi_\varphi \xi) = 0$, the pure tensor $\xi$ is called an almost analytic $(0,q)$-tensor. In [9] [7] [4], it is proved that $\xi \circ \varphi \in \mathfrak{X}_q^0(M)$ is an almost analytic tensor if and only if $\xi \in \mathfrak{X}_q^0(M)$ is an almost analytic tensor. Moreover if $\xi \in \mathfrak{X}_q^0(M)$ is an almost analytic tensor, then $N_\varphi \circ \xi = 0$. When restricted to the pure cross-section determined by an almost analytic tensor $\xi$ on $M$, from (3.3), (3.4) and linearity of the complete lift, we have

$$\langle C\varphi \rangle^2 = C(\varphi^2) = C(-I_M) = -I_{T_q^0(M)}.$$ 

This completes the proof. $\square$

**4. Complete lift of a symmetric affine connection on a cross-section in the $(0,q)$-tensor bundle**

We now assume that $\nabla$ is an affine connection (with zero torsion) on $M$. Let $\Gamma^h_{ij}$ be components of $\nabla$. The complete lift $C\nabla$ of $\nabla$ to $T_q^0(M)$ has components $C\Gamma^I_{MS}$.
such that

\begin{align}
C^i_{m,s} &= \Gamma^i_{m,s}, \\
C^s_{m,s} &= C^i_{m,s} = C^i_{m,s} = C^i_{m,s} = C^i_{m,s} = 0,
\end{align}

\begin{align}
C^r_{m,s} &= -\sum_{c=1}^q (\Gamma^{s}_{mc} \delta^s_{i_1} ... \delta^s_{i_{c-1}} \delta^s_{i_{c+1}} ... \delta^s_{i_q}), \\
C^r_{m,s} &= -\sum_{c=1}^q (\Gamma^{m}_{sc} \delta^m_{i_1} ... \delta^m_{i_{c-1}} \delta^m_{i_{c+1}} ... \delta^m_{i_q}), \\
C^r_{m,s} &= \sum_{c=1}^q (\partial^m \Gamma_s^{i_c} + \Gamma^{r}_{mi_c} \Gamma_s^{i_r} + \Gamma^{r}_{ms_r} \Gamma_s^{i_r} t_{i_1 ... i_{c-1} a i_{c+1} ... i_q}) \\
&\quad + \frac{1}{2} \sum_{b=1}^q \sum_{c=1}^q (\Gamma^{l}_{mi_b} \Gamma_s^{i_l} + \Gamma^{l}_{mi_c} \Gamma_s^{i_l} t_{i_1 ... i_{b-1} a i_{b+1} ... i_{c-1} a i_{c+1} ... i_q}) \\
&\quad + \sum_{d=1}^q l_{i_1 ... i_q} R_{i_d km l}
\end{align}

with respect to the natural frame in $T^0_q(M)$, where $\delta^i_j$ is the Kronecker delta and $R_{ikm l}$ is components of the curvature tensor $R$ of $\nabla$ \cite{6}.

We now study the affine connection induced from $C^i_j$ on the cross-section $\sigma_\xi(M)$ determined by the $(0, q)$-tensor field $\xi$ in $M$ with respect to the adapted $(B, C)$-frame of $\sigma_\xi(M)$. The vector fields $\tilde{\Gamma}^r_{m,s}$ given by (2.5) are linearly independent and not tangent to $\sigma_\xi(M)$. We take the vector fields $\tilde{\Gamma}^r_{m,s}$ as normals to the cross-section $\sigma_\xi(M)$ and define an affine connection $\tilde{\nabla}$ induced on the cross-section. The affine connection $\tilde{\nabla}$ induced $\sigma_\xi(M)$ from the complete lift $C^i_j$ of a symmetric affine connection $\nabla$ in $M$ has components of the form

\begin{align}
\tilde{\Gamma}^h_{ji} = (\partial_j B^h_i A^C + C^h_{CB} B^C_i B^A_j) B^h_A,
\end{align}

where $B^h_A$ are defined by

\begin{align}
(B^h_A, C^h_A) = (B^h_A, C^h_A)^{-1}
\end{align}

and thus

\begin{align}
B^h_A = (\delta^h_j 0, C^h_A) = (-\partial_j \xi_{k_1 ... k_q}, \delta^h_{k_1} ... \delta^h_{k_q}).
\end{align}

Substituting (4.1), (2.4), (2.5) and (4.3) in (4.2), we get

\begin{align}
\tilde{\Gamma}^h_{ji} = \Gamma^h_{ji},
\end{align}

where $\Gamma^h_{ji}$ are components of $\nabla$ in $M$.

From (4.2), we see that the quantity

\begin{align}
\partial_j B^A_i + C^A_{CB} B^C_i B^A_j - \Gamma^h_{ji} B^h_A
\end{align}

is a linear combination of the vectors $C^h_{ji} A$. To find the coefficients, we put $A = \tilde{h}$ in (4.4) and find

\begin{align}
\nabla_j \nabla_i \xi_{h_1 ... h_q} + \sum_{\lambda=1}^q \xi_{h_1 ... l_{h_q}} R_{l_{h_q} i_j l}.
\end{align}
Hence, representing (4.4) by \( \tilde{\nabla}_j B_i^A \), we obtain

\[
(4.5) \quad \tilde{\nabla}_j B_i^A = (\nabla_j \nabla_i \xi_{h_1 \ldots h_q} + \sum_{\lambda=1}^q \xi_{h_1 \ldots h_q} R^A_{h\lambda ij} l) C^A_{h}.
\]

The last equation is nothing but the equation of Gauss for the cross-section \( \sigma_\xi(M) \) determined by \( \xi_{h_1 \ldots h_q} \). Hence, we have the following proposition.

**Proposition 1.** The cross-section \( \sigma_\xi(M) \) in \( T^0_q(M) \) determined by a \((0, q)\) tensor \( \xi \) in \( M \) with symmetric affine connection \( \nabla \) is totally geodesic if and only if \( \xi \) satisfies

\[
\nabla_j \nabla_i \xi_{h_1 \ldots h_q} + \sum_{\lambda=1}^q \xi_{h_1 \ldots h_q} R^A_{h\lambda ij} l = 0.
\]

Now, let us apply the operator \( \tilde{\nabla}_k \) to (4.5), we have

\[
(4.6) \quad \tilde{\nabla}_k \tilde{\nabla}_j B_i^A = \nabla_k (\nabla_j \nabla_i \xi_{h_1 \ldots h_q} + \sum_{\lambda=1}^q \xi_{h_1 \ldots h_q} R^A_{h\lambda ij} l) C^A_{h}.
\]

Recalling that

\[
\nabla_k \nabla_j B_i^A - \nabla_j \nabla_k B_i^A = \tilde{R}_{DCB}^A B_k^D B_j^C B_i^B - R^A_{kji} B_h^A,
\]

and using the Ricci identity for a tensor field of type \((0, q)\), from (4.6) we get

\[
\tilde{R}_{DCB}^A B_k^D B_j^C B_i^B - R^A_{kji} B_h^A
\]

\[
= \sum_{\lambda=1}^q (\nabla_k R^A_{h\lambda ij} l - \nabla_j R^A_{h\lambda ik} l) \xi_{h_1 \ldots h_q} - R^A_{kji} \xi_{l} \nabla_i \xi_{h_1 \ldots h_q}
\]

\[
- \sum_{\lambda=1}^q R^A_{kji} \nabla_k \xi_{h_1 \ldots h_q} - \sum_{\lambda=1}^q R^A_{h\lambda ij} \xi_{l} \nabla_i \xi_{h_1 \ldots h_q} - \sum_{\lambda=1}^q R^A_{h\lambda ij} \xi \nabla_i \xi_{h_1 \ldots h_q} - \sum_{\lambda=1}^q R^A_{h\lambda ij} \xi \nabla_i \xi_{h_1 \ldots h_q}
\]

Thus we have the result below.

**Proposition 2.** \( \tilde{R}_{DCB}^A B_k^D B_j^C B_i^B \) is tangent to the cross-section \( \sigma_\xi(M) \) if and only if

\[
\sum_{\lambda=1}^q (\nabla_k R^A_{h\lambda ij} l - \nabla_j R^A_{h\lambda ik} l) \xi_{h_1 \ldots h_q} = R^A_{kji} \xi \nabla_i \xi_{h_1 \ldots h_q} - \sum_{\lambda=1}^q R^A_{h\lambda ij} \xi \nabla_i \xi_{h_1 \ldots h_q}
\]

\[
+ \sum_{\lambda=1}^q R^A_{h\lambda ij} \xi \nabla_i \xi_{h_1 \ldots h_q}.
\]

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