A FROBENIUS-SCHREIER-SIMS ALGORITHM TO TENSOR DECOMPOSE ALGEBRAS

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Abstract. We introduce a decomposition of associative algebras into a tensor product of cyclic modules. This produces a means to encode a basis with logarithmic information and thus extends the reach of calculation with large algebras. Our technique is an analogue to the Schreier-Sims algorithm for permutation groups and is a by-product of Frobenius reciprocity.

1. Introduction

In 1967 by Charles C. Sims [18] introduced an algorithm that given a group $G$ generated by permutations $S$ on a finite set $\Omega$ produced a data structure that amongst other things could efficiently compute $|G|$, decide if an arbitrary permutation $\sigma$ was in $G$, and if so write it as a word in the original generators. The algorithm was put to immediate and effective use in the Classification of Finite Simple Groups. In the years to follow this algorithm would be improved by several measures. Some improved worst-case complexity, others made faster implementations for computer algebra systems such as GAP and Magma, and randomized nearly-linear time alternatives were created [4, 8, 9, 11, 14, 16]. Independently the concept of a base (one of the outputs of the algorithm) became a powerful device to explore subgroup lattices of large groups [2, 5, 6]. Today this family of techniques we collectively known as Schreier-Sims algorithms.

Here we introduce a Schreier-Sims type algorithm for computing bounds on the dimension of large algebras, e.g. of group, Hecke, Hopf, and finitely presented algebras.

Notation. We prefer here the notation $g\omega$ for the action of an element $g \in G$ on a term $\omega$ in a set $\Omega$ as this will accord well with our use of left $A$-modules $M$. We write $S_n$ for the group of permutations on $\Omega = \{1, \ldots, n\}$. A group generated by a set $S$ is denoted $\langle S \rangle$.

The free $K$-algebra on a set of indeterminants $X$ is denoted $K\langle X \rangle$ and consists of all $K$-linear combinations of words in $X$. Also a $K$-algebra generated by a set $S$ is denoted $K\langle S \rangle$. If the elements of $S$ are known to commute we may also write $K[S]$, e.g. $K[X]$ denotes the usual polynomial ring in $X$. Note that in our notation a group algebra is denoted $K\langle G \rangle$, not as $K[G]$. Let $\text{Ann}_A(M) = \{a \in A \mid aM = 0\}$. Call $M$ faithful if $\text{Ann}_A(M) = 0$.

An unfaithful use of Schreier-Sims. Consider a case where an unfaithful representation $\rho : A \to M_n(\mathbb{C})$ of a $\mathbb{C}$-algebra $A$ could be used to compute $\dim A$. Assume first that $G$ is a group with a faithful permutation representation $\rho : G \to S_n$ into

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the symmetric group $S_n$ on $\{1, \ldots, n\}$. Take $A = \mathbb{C} \langle G \rangle$ to be the group algebra and $\hat{\rho} : A \to M_n(\mathbb{C})$ to be the linear extension of $G$’s action to permute coordinates of vectors. Notice $\hat{\rho}$ is usually far from injective. E.g. if $n > 3$, 
\[
\dim \mathbb{C} \langle S_n \rangle = n! \gg n^2 = \dim M_n(\mathbb{C}).
\]
However, to compute $\dim A$ from these data, we simply apply the above mentioned Schreier-Sims type algorithms to compute $|G|$ as a permutation group.

Our simple question is whether something like the above example can be done without a priori knowledge of a group, or other baked-in structure. For instance, if each monomial term is bounded in length buy $\ell$
\[
A
\]
Also given the means to operate in $A$ however, to compute $\dim A$? On the one hand it seems $M$ is unlikely to be helpful as it is not faithful. Yet on the other hand the above example shows that with the right representation theory perhaps $M$ is more informative than expected. Replicating the success of Schreier-Sims for general algebras turns up a happy coincidence. Schreier-Sims is a special case of Frobenius reciprocity.

Call an algebra $A$ semiprimary if its Jacobson radical $J(A)$ is nilpotent and $A/J(A)$ is semisimple.

**Theorem 1.1.** Fix a field $K$. Given a semiprimary $K$-algebra $A$ and an $A$-module $M$, there exists $x_1, \ldots, x_\ell \in M$, a chain of subalgebras $A = A_0 > A_1 > \cdots > A_\ell$, where all irreducible representations of $A_\ell$ on $M$ are trivial, and a $K$-linear epimorphism
\[
A_0 x_1 \otimes \cdots \otimes A_{\ell-1} x_\ell \otimes A_\ell \twoheadrightarrow A.
\]
Also given the means to operate in $A$ and the image of a generating set of $A$ acting on $M$, there is a polynomial-time algorithm to construct the above data.

Amongst the implications of Theorem 1.1 is a way to parameterize a spanning set for $A$ by constructing bases $B_i$ for each cyclic module $M_i = A_{i-1} x_i$, and $A_\ell$. The image of $B_1 \otimes \cdots \otimes B_\ell \otimes A_\ell$ spans $A$ and provides a generating set for which each monomial term is bounded in length buy $\ell + 1$. (Note that monomials in the original generators $S$ of $G$ are words of arbitrary length in $K\langle S \rangle$ and thus have numerous unspecified relations.) Furthermore we obtain a bound
\[
\dim A \leq \dim M_1 \cdots \dim M_\ell \dim A_\ell.
\]
Each $\dim M_i$ is known, the lingering ambiguity lies with $A_\ell$. While we have not discovered a property of $A$ and $M$ that will force $\dim A_\ell = 1$, we have found in examples it is quite common that $A_\ell = K$ or an algebra for which the dimension is self-evident.

Continuing with our illustration. Suppose our group $G$ represented on $\Omega = \{1, \ldots, n\}$ has a subset $\beta = \{\beta_1, \ldots, \beta_\ell\}$ such that the following stabilizer subgroups
\[
G[i] = \{g \in G \mid 1 \leq j \leq i, \beta_j^g = \beta_j\}
\]
end in 1, i.e. $G[\ell] = 1$. In the usual Schreier-Sims parlance, $\beta$ is a base for $G$. Use $A_i := \mathbb{C} \langle G[i] \rangle$ as subalgebras of $A = \mathbb{C} \langle G \rangle$. Let $x_i = e_\beta_i$ be the vector with zero’s in all positions except $\beta_i$ and write $\Delta_i$ for the $G[i-1]$-orbit of $\beta_i$. Then for each $i$:
\[
A_{i-1} x_i = \text{Span}_\mathbb{C} \{ e_\delta \mid \delta \in \Delta_i \}.
\]
Hence, the dimension of $A_{i-1}x_i$ is $|\Delta_i|$. Finally note that $A_{i+1} = C$ in this case. So we obtain a surjection

$$C^{\Delta_1} \otimes \cdots \otimes C^{\Delta_\ell} \rightarrow \mathbb{C} \langle G \rangle.$$ 

Indeed, in this case of we get a bijection. Observe that all the ingredients now are in terms of algebras and modules and thus achieve our goal of removing the permutation interpretation.

**Related work.** Ours is not the first attempt at a generalization of Schreier-Sims for linear representations. A notable related work by Białynicki [1] considered groups $G$ of matrices acting on $K^n$ for finite fields $K$ or order $q$. There the strategy was to induce from vectors in $K^n$ a permutation representation of $G$ and then proceed with the conventional Schreier-Sims methods. The limitations of such an approach are that in general linear groups $G$ may fail to have any low-index subgroups. For instance the proper subgroups of $\text{SL}_d(q)$ have index at least $q^{cd}$ for some $c$.

Also, while our interest is in an algorithm for so-called “black-box algebra”, where we know nothing of the algebra when we begin, it is worth specific mention that many far superior algorithms for algebras exist when something is known about $A$. Several authors have contributed over decades to computer algebra systems including GAP [9] and Magma [4], and developed special purpose systems like CHEVIE [15] that calculate with either arbitrary but small algebras, or large but prescriptive algebras. For instance, an arbitrary algebra can be provided by a basis and structure constants or by a faithful representation. Algorithms for such algebras where investigated by Ronyai and later several others. These are small in our sense because we can provide them by a basis. Large algebras in computation include group algebra, quantum groups (Hopf algebras), Hecke algebras, and finitely presented algebras. Algorithms for these algebras can be efficient if they are first told of additional features, such as the group of a group algebra or appropriate Lie or Chevalley data. See the above references for details.

## 2. From Schreier-Sims to Frobenius

In the coarsest explanation of our proof, we point out that having functions from tensors of submodules into other modules is what one expects when considering induction-restriction functors – a natural tool in representation theories both linear and permutation based. In retrospect it seems obvious that when working with stabilizer subgroups we could make arguments using induction-restriction with appeals to Frobenius reciprocity. However, the original Schreier-Sims algorithm was so elegantly explained by a rewriting formula known as Schreier’s lemma that the Frobenius interpretation never appeared. Its linear analog therefore had not surfaced either. We only discovered this relationship by a coincidence observing that the tedious calculations one is loathed to write when proving Schreier’s lemma and Frobenius reciprocity turn up identical formulas. This seems the right place to begin our proof.

Throughout the many improvements to the original Schreier-Sims algorithm one key aspect survives intact which is the idea to build generators for the stabilizer of a point by computing representatives of the cosets of the stabilizer. The reason this works, and the reason to attach Schreier’s name to the algorithm, is because of the following observation.
Lemma 2.1 (Schreier). Given $G = \langle S \rangle$, $H \leq G$, and a function $\tau : G \to G$, where $\tau(g)H = gH$, it follows that:

$$H = \langle \tau(stH)^{-1}st \mid s \in S, t \in \tau(G) \rangle.$$ 

The set $\{ \tau(stH)^{-1}st \mid s \in S, t \in \tau(G) \}$ is known as a set of Schreier generators for $H$. Schreier proved this lemma in the context of free groups and the proof is an early example of rewriting in groups. Its relevance to permutation groups is as follows. Fix a group $G$ acting on a set $\Omega$. Note that for $\omega \in G$, the function $G \to G\omega$ given by $g \mapsto g\omega$ is constant on the cosets of the stabilizer $G\omega = \{ g \in G \mid g\omega = \omega \}$.

Such a function $f$ is said to hide the subgroup $G\omega$. Schreier’s lemma says that to discover the hidden subgroup we need only enumerate the orbit $G\omega = \{ g_1\omega, \ldots, g_m\omega \}$.

So we define $\tau(g) := g_i$ where $g\omega = g_i\omega$ and apply Schreier’s lemma to produce a set of generators for $G\omega$ of size $|S| \cdot |G\omega|$. By recursion (together with a careful reduction of the number of generators as we go) we end up with the following data.

**Base:** a subset $\{ \beta_1, \ldots, \beta_\ell \}$ of $\Omega$ whose stabilizer chain from $[1,2]$ ends in 1.

**Strong Generators:** set $X$ of generators for $G$ with the property $G^{[i]} = \langle X_i \rangle$ where $X_i := X \cap G^{[i]}$.

This allows us to treat the $G^{[i-1]}$ orbit $\Delta_i = \{ g\beta_i \mid g \in G^{[i]} \}$ as a connected Cayley graph $\text{Cay}(X_i, \Delta_i) = \{ (\delta, x\delta) \mid \delta \in \Delta_i, x \in X_i \}$.

**Schreier tree:** a spanning tree for $\text{Cay}(X_i, \Delta_i)$.

These data are considered the output of the Schreier-Sims algorithm. For a thorough account we refer the reader to [16]. Our own algorithm will produce a similar output though regrettably it is far less understood than the output of Schreier-Sims.

**A coincidence with Frobenius reciprocity.** Now let us consider the effect of studying $G\omega$ and $\Delta = \omega^G$ as problem of induction and restriction of permutation representations.

Consider an algebra $B$ contained in an algebra $A$. To every $A$-module $M$ there is an associated $B$-module $\text{Res}_B^A(M) = \text{hom}_B(B, M)$ which simply restricts the action of $A$ to $B$. Likewise, every $B$-module $N$ induces an $A$-module $\text{Ind}_B^A(N) = A \otimes_B N$.

**Theorem 2.2 (Frobenius Reciprocity).** There is a natural isomorphism

$$\text{hom}_A(\text{Ind}_B^A(N), M) \cong \text{hom}_B(N, \text{Res}_B^A(M)).$$

As is standard with adjoint pairs we compose them to get endofunctor which can be applied to modules in a single category. The one we consider is the composition $\text{Ind}_B^A \circ \text{Res}_B^A$. The natural isomorphism in Frobenius reciprocity then asserts a natural transformation from this endofunctor to the identity, a so-called counit $\epsilon : \text{Ind}_B^A \circ \text{Res}_B^A \to 1$, i.e.:

$$\text{Ind}_B^A(\text{Res}_B^A(M)) = A \otimes_B \text{hom}_B(B, M) \to M,$$

Usually it is beneficial to operate with such statements at the level of objects in the category, after all these are functors. However, if one takes care to express this...
relationship specifically we discover our connection to Schreier-Sims. First observe that (2.3) is realized by the following map: for $a \in A$ and $\phi \in \text{hom}_B(B, M)$,

$$a \otimes \phi \mapsto a \cdot \phi(1).$$

Now consider this in the following special case. Fix a faithful representation $\rho : G \to S_n$. Set $A = K \langle G \rangle$ and take $M = K^n$ to be the $A$-module induced by $\rho$. Next define $B = K \langle H \rangle$, where $H = G_\omega$. Unpacking the formulas above we observe that

$$\text{Ind}_B^A(\text{Res}_B^A(M)) \cong K^{G/H} \otimes_K K^n$$

and the action by $A = K \langle G \rangle$ is described as follows. Fix a transversal $\tau : G/H \to G$. For $s \in S$, $tH \in G/H$, and basis vector $e_i \in K^d$,

$$s(tH \otimes e_i) = stH \otimes e_{\tau(stH)^{-1}st \cdot i}.$$  

Notice that this expression includes precisely the data in Schreier’s Lemma 2.1. For example, if we consider induction-restriction of $G$-sets $\Omega$ we find the following formula. We need to choose a transversal (which will not alter the result up to isomorphism) and set:

$$\text{Ind}_H^G(\text{Res}_H^G(g\Omega)) = cG/H_H \times \tau_H \Omega$$

where

$$s(tH, \delta) = (stH, \tau(stH)^{-1}st\delta).$$

We now see that the role of Schreier’s Lemma in Sims’ algorithm was to realize the Frobenius counit of the pair $(\text{Ind}_B^A, \text{Res}_B^A)$. Fortunately for us, Frobenius reciprocity holds for much more than permutation modules.

3. Proof of Theorem 1.1

Our proof is devised in the following way. We use a point $x \in M$ to describe cyclic module $Ax \leq M$. This replace the concept of an orbit. Next we devise subalgebras $B \leq Ax = K + \text{Ann}_{A}(x)$ to replace the role of point stabilizers in the original Schreier-Sims algorithm. We then want to apply the induction-restriction process to $Ax$ to obtain a surjection $A \otimes_B \text{hom}_B(B, M) \to M$. Recovering relations to reduce the size of the vector space on the left to $Ax \otimes B$ we produce a process similar to Schreier’s lemma and so create generators for $B$. Hence, we can efficiently write down a surjection $Ax \otimes B \to A$. The final stage is to recursively apply the strategy to $B$.

Notice we have not elected to use $B = A_x$. Doing so would create a decomposition as well but one that is vastly larger than $A$. For example in the case of a group algebra $A = C\langle G \rangle$, the point stabilizer $H = G_\omega$ forms a subalgebra $B = C\langle H \rangle$ of dimension $|H|$ where as the stabilizer subalgebra $A_{x\omega}$ has dimension $|G| - |H| = |H|(|G : H| - 1)$. Instead what we require of $B$ is simply that we be able to produce a surjection of $Ax \otimes B \to A$. In fact the ability to choose many algebras for $B$ has made it unclear whether one can expect to build a surjection with is also a bijection and thus obtain a precise dimension for $A$. This would be a helpful question to resolve.

Our proof is split into three parts. First we introduce a substitute for the concept of Schreier generators (Proposition 3.1). We shall call these Frobenius-Schreier-Sims (FSS) generators in part to acknowledge the critical role of each person and to distinguish our generators from those used in permutation group algorithms.
Second we prove that FFS generators exists for semi-local algebras (Proposition 3.1). Along with that proof we discern a reasonable algorithm to construct FSS generators for an algebra (Section 3.3). We use that to confirm it is possible to compute a decomposition:

\[ A_0 \otimes \cdots \otimes A_{t-1} x_t \otimes A_t \to A. \]

3.1. Frobenius-Schreier-Sims generators. Fix a \(K\)-algebra \(A = K\langle S \rangle\) generated by \(S\) and a cyclic (left) \(A\)-module \(M = Ax\). By a transversal for \(M\) we mean a function \(T: x \in M \mapsto 1\), such that \(T(0) = 0\) and \(T(ax) = T(x)\) for \(a \in K\). In particular, \(K\langle T \rangle \cap M = Ax\), and there is a linear epimorphism \(Ax \otimes_K K\langle T \rangle = \text{Ann}(x)\).

From these data we define the Frobenius-Schreier-Sims (FSS) generators as the following set.

\[
U(S, T, \sigma, \tau) = \{1\} \cup \{\sigma(st)st : s \in ST\} \cup \{\sigma(st)^{-1} - \tau(stx) : s \in ST\} \cup \{st : \tau(stx) = 0\}.
\]

**Proposition 3.1.** Under the notation above, for each \(a \in A\), there exists \(\lambda_i \in K\), \(t_i \in T\), and \(u_{i1}, \ldots, u_{i\ell(i)} \in U(S, T, \sigma, \tau)\) such that

\[ a = \sum_i \lambda_i t_i u_{i1} \cdots u_{i\ell(i)}. \]

In particular, \(K\langle U \rangle \leq K + \text{Ann}(x)\) and there is a linear epimorphism \(Ax \otimes_K K\langle U \rangle \to A\) given by \(ax \otimes b \mapsto \tau(ax)b\). So each \(s \in S\) is has tensor rank at most 2, specifically

\[ sx \otimes \sigma(s)s + x \otimes (s - \tau(sx))\sigma(s)s \mapsto s. \]

**Proof.** Let \(s \in S\) and \(t \in T\). If \(\tau(stx) = 0\) then \(st \in \text{Ann}_A(x)\). Likewise, for \(st \in ST\), by definition \(\sigma(st)^{-1} - \tau(stx) \in \text{Ann}_A(x)\). Finally, as \(stx = \tau(stx)x = \tau(stx)x + \alpha(st)x = \sigma(st)^{-1}x\), it follows that \(\sigma(st)x = x\). Therefore \(\sigma(st)st - 1 \in \text{Ann}(x)\). So in all cases \(U \subset K + \text{Ann}(x)\).

Assuming \(a \in A = K\langle S \rangle\), it follows that there are \(\alpha_i \in K\) and a sequence of sequences \(s_{i1}, \ldots, s_{i\ell(i)} \in S\) such that

\[ a = \sum_i \alpha_i s_{i\ell(i)} \cdots s_{i1}. \]

As \(1 \in T\) and \(1 \in U(S, T, \sigma, \tau)\), use \(u_{i1} = t_i = 1\) so that:

\[ a = \sum_i \alpha_i s_{i\ell(i)} \cdots s_{i1} t_i u_{i1}. \]
Now suppose for induction that for some \( j \), there exists \( \beta_j \in K \), \( t_i \in T \), and \( u_{i(j-1)}, \ldots, u_{i1} \in U \) such that
\[
(3.2) \quad a = \sum_i \beta_i s_{id(i)} \cdots s_{ij} t_i u_{i(j-1)} \cdots u_{i1}.
\]

Now we proceed to rewrite each summand. If \( \tau(s_{ij} t_i) = 0 \) then replace that term with \( t_j u_{ij} \) where \( t_{i+1} = 1 \) and \( u_{ij} = s_{ij} t_i \in U \). Otherwise, set \( u_{i(j+1)} = \sigma(s_{ij} t_i) s_{ij} t_i \in U(S, T, \sigma) \). Note that \( s_{ij} t_i = \tilde{\tau}(s_{ij} t_i) \sigma(s_{ij} t_i) s_{ij} t_i \). So also set \( \sum_k \lambda_k t_{ik} = \tau(xs_{ij} t_i) \).

By re-writing we prove:
\[
\begin{align*}
\beta_i s_{id(i)} \cdots s_{ij} t_i u_{i(j-1)} \cdots u_{i1} &= \beta_i s_{id(i)} \cdots s_{i(j+1)} \tilde{\tau}(s_{ij} t_i x) \sigma(s_{ij} t_i) s_{ij} t_i u_{i(j-1)} \cdots u_{i1} \\
&= \sum_k \beta_i \lambda_k s_{id(i)} \cdots s_{i(j+1)} t_{ik} u_{ij} u_{i(j-1)} \cdots u_{i1} \\
&\quad + \beta_i s_{id(i)} \cdots s_{i(j+1)} \gamma(s_{ij} u_{i(j-1)} \cdots u_{i1}) \tag{3.2}
\end{align*}
\]

In particular, every summand now has been converted into a sum of possibly several summands each with one fewer \( S \) terms, followed by a \( T \) term, and \( U \) terms (recalling that \( \alpha(st) \in U \) as well). Therefore re-indexing if necessary
\[
a = \sum_m \gamma_1 s_{m(t(m))} \cdots s_{m(j+1)} t_{m} u_{mj} u_{m(j-1)} \cdots u_{m1}.
\]

Carrying out the recursion we arrive at
\[
a = \sum_i \lambda_i t_i u_{id(i)} \cdots u_{i1}.
\]

Finally let \( \Gamma : M \otimes K[U] \to A \) be defined on pure-tensors as
\[
m \otimes b \mapsto \tau(m)b.
\]

Here we have used the assumption that \( \tau \) is linear. From the decomposition above, given \( a \in A \),
\[
\sum_i \lambda_i (t_i x) \otimes (u_{id(i)} \cdots u_{i1}) \mapsto \sum_i \lambda_i t_i u_{id(i)} \cdots u_{i1} = a.
\]

Therefore \( \Gamma \) is surjective. \( \square \)

3.2. Existence of FSS generators. To prove the existence of FSS generators we want to reduce to the case of central simple rings, i.e. matrices \( M_n(\Delta) \) over division rings \( E \) extending \( K \). As we are afforded a module \( M \) for \( A \) it is possible to begin with a simple submodule. However, as we cannot assume that \( A \) is faithfully represented in on \( M \) we need a device to lift our results to \( A \) no matter the presence of a nontrivial annihilator. The tool we choose is to lift the Pierce decomposition by the lifting of idempotents.

So recall that in an associative algebra \( A \), a set \( e_1, \ldots, e_m \) of elements in \( A \) is a set of pairwise orthogonal idempotents if \( e_i e_j = \delta_{ij} e_i \). These idempotents are supplemental if \( 1 = e_1 + \cdots + e_m \). Idempotents other than 0 and 1 are called proper nontrivial. An idempotent \( e \) is primitive if it not the sum of proper nontrivial idempotents. Finally by a frame for \( A \) we mean a set of pairwise orthogonal primitive idempotents that sum to 1. For instance, in \( M_n(\Delta) \), the usual matrix units \( E_{ij} = [\delta_{ij}] \) give a natural frame: \( \{E_{11}, \ldots, E_{nn}\} \). Finally we need the following classic lemma on the lifting of idempotents.
Lemma 3.3. Let $A$ be an algebra an $N$ a nilpotent ideal. Suppose $e \in A$ such that $e^2 \equiv e \mod N$ and $(e^2 - e)^n = 0$. Define

$$\hat{e} = e^n \sum_{i=1}^{n-1} \binom{2n-1}{j} e^{n-j-1}(1-e)^j.$$ 

It follows that $\hat{e}^2 = \hat{e}$, $e \equiv \hat{e} \mod N$, and $1 - \hat{e} = 1 - e$. In particular in a semiprimary algebra we can lift a frame for $A/J(A)$.

Proposition 3.4. If $A$ is semiprimary and $M$ is a simple $A$-module, then there exists a transversal with a Frobenius-Schreier-Sims section.

Proof. Let $\rho : A \to \text{End}(M)$ be the induced representation. As $M$ is simple, by Jacobson’s density theorem the image of $A$ is dense, and in particular $A/\ker \rho$ is primitive. By assumption $A$ is semiprimary so $A/J(A)$ is semisimple Artinian, and so $A/\ker \rho$ is simple. Therefore we have an epimorphism $A \to M_n(\Delta)$ for some division ring $\Delta = \text{End}_A(M)$. By lifting the idempotents $E_{ij}$ to idempotents $e_{ij} \in A$ we can construct explicit elements in $A$ whose image is a prescribed matrix, i.e. $\sum_{ij} x_{ij} e_{ij} \mapsto \begin{bmatrix} x_{ij} \end{bmatrix}$, where $x_{ij} \in e_1 Ae_1$ (here we are using the assumption that the radical has finite length).

Now up to a choice of basis of $M$ as a $\Delta$-vector space, each $ax = \sum_{i=1}^{n} x_i e_i$. Choose

$$\tau(ax) = \sum_{i=1}^{n} x_i e_{i1}.$$ 

For $\sigma$ we proceed as follows. Since we may assume $\tau(stx) \neq 0$, there is some $x_i \neq 0$. If $x_1 \neq 0$, choose

$$\sigma(st) = 1 - e_{11} + \tau(stx).$$ 

This is invertible with inverse

$$\sigma(st)^{-1} = 1 - e_{11} - x_{11}^{-1} \tau(stx) + x_{11}^{-1} e_{11}.$$ 

Furthermore, $(\sigma(st)^{-1} - \tau(stx)) e_1 = 0$, which satisfies our desired hypothesis.

If $x_1 = 0$ then let $x_i$ be the first non-zero. Apply a permutation by $(1i)$ to the rows and columns and apply the construction above, then conjugate back. This is the value for $\sigma(st)$. □

3.3. Repeating decompositions. We should now like to consider a recursive application of the above decomposition. Evidently $B = K\langle U \rangle$ is a subalgebra so we can treat $M$ as a $B$-module. Thus we can select a new $y \in M$ and proceed with $By$ in the role of $M$ and $B$ in the role of $A$. The result would be to decompose

$$M_1 \otimes \cdots \otimes M_t \otimes A_t \to A$$

of cyclic modules $M$. The only complication is if $B = A$, as then we may find ourselves in an infinite corecursion.

So when do we get $A = B$? Well observe that $B \leq K + \text{Ann}_A(x)$ and so $A = B$ would imply $A = K + \text{Ann}_A(x)$. Thus we cease our corecursion when $Ax$ is the trivial module $Kx$. We are free to choose a cyclic submodule from $M$. Which means we bottom out once $M$ itself is a product of trivial $A$-modules. I.e. $M = K^n$ with the action by $A$ simply as scalars.
3.4. **An algorithm.** Now we discuss how to realize this decomposition. We suppose we have an algebra of black-box type \( A = K(S) \) and a representation \( \rho : A \to \text{End}(M) \). What we mean by black-box in this context is that we have algorithms that perform the operations of the algebras, and the module, and to test equality of elements. It is typically the last assumption that cause some concern. For example operating in a quotient of \( K[x_1, \ldots, x_n] \) requires testing when one polynomial is member in an ideal. That problem is known to be NP-hard and it is solved in general by often difficult Gröbner bases methods. Fortunately once this has been done the results can be recycled for every subsequent comparison of elements. To if this is necessary cost then at least it is a one time cost.

**Remark 3.5.** As a technical matter this input model is not yet usable in the decision problems such in the study of P vs. NP since as stated we cannot prove the the operations satisfy the axioms of an algebra. Since our algorithms performance can only be guaranteed under that assumption we should also demand that such algebras be input along with proofs of the axioms. That can be done but requires a form of computation based on type theory and that is a subject for another context; see [7]. Even so, with our assumptions so far our algorithms below should be considered as black-box algorithms in the *promise hierarchy*.

We assume also several now standard algorithms for computing with small rings and modules, that is ones for which a basis is small enough to produce. Detailed accounts of the many methods can be found in [10, 19].

Here are the steps of our algorithm: given an algebra \( A = K(S) \) and the images of \( S \) in \( \text{End}(M) \), do as follows.

1. If \( M \) is a trivial \( A \)-module return \( A \to A \).
2. Otherwise, use the MeatAxe (or Ronayi’s deterministic algorithm) to compute an basis for an simple submodule \( N \leq M \), and fix \( 0 \neq x \in N \); so, \( Ax = N \), and also compute \( \Delta = \text{End}_A(N) \) producing a representation \( A \to \mathbb{M}_n(\Delta) \).
3. Choose a set of supplementary pairwise orthogonal primitive idempotents \( E_{11}, \ldots, E_{nn} \) for \( \mathbb{M}_n(\Delta) \) and write them as polynomials in the image of \( S \) in \( \mathbb{M}_n(\Delta) \), for instance by expanding \( S \) into a basis of the image.
4. Apply the idempotent lifting formula to produce pairwise orthogonal primitive idempotents \( e_1, \ldots, e_n \) in \( A \). Add also \( e_0 := 1 - \sum e_i \).
5. Implement the choice of \( \tau \) and \( \sigma \) above from the Pierce decomposition given by the idempotents just calculated.
6. Compute the set \( U \). Repeat the process with \( A_1 := K(U) \). and return \( Ax \otimes K(U) \). Return \( Ax_1 \otimes \cdots \otimes A_\ell x_\ell \otimes A_{\ell+1} \to A \).

4. **Examples**

In the master’s thesis of the first author [12] an implementation of parts of our generalized Schreier-Sims algorithm were developed. A particular technological adaptation was to explore the algorithm in a parallel functional programming paradigm. A full implementation of our algorithm has not been attempted but the following examples are included as a demonstration.

**Example: the dihedral group** \( D_8 \). To give the reader a sense of how the FSS algorithm operates in a classical setting, we apply it to the toy example where \( A \) is the group algebra of the dihedral group \( D_8 \) over the complex numbers, \( \mathbb{C}(D_8) \).
Recall that $D_8$ can be described by

$$D_8 = \{ r, s \mid r^4 = s^2 = 1, srs = r^{-1} \}.$$ 

$D_8$ permutes the points of the square $\{(1,0),(0,1),(-1,0),(0,-1)\} \subset \mathbb{R}^2$ in the usual way. We can associate each of these points to a basis vector $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{C}^4$. Denote this representation by $W$. Then a 2-dimensional irreducible submodule of $V$ is generated by the element $e_1 - e_3$. Setting $v_1 = e_1 - e_3$ and $v_2 = e_2 - e_4$, the action of $D_8$ is defined by

$$r \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad s \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (4.1)$$

We choose $\tau : V \to D_8$ to be given by

$$\tau(v_1) = 1 \quad \text{and} \quad \tau(v_2) = r$$

and $T = \{1, r\}$. Then

$$ST = \{r, s, r^2, sr\}.$$ 

We observe that

$$\tau(rv_1) = r, \quad \tau(sv_1) = 1, \quad \tau(r^2v_1) = -1, \quad \tau(srv_1) = -r.$$ 

As the above elements are all invertible in $\mathbb{C}\langle D_8 \rangle$, we define $\sigma : ST \to \mathbb{C}\langle D_8 \rangle^\times$ so that $\sigma(st) = \tau(stv_1)^{-1}$:

$$\sigma(r) = r^3, \quad \sigma(s) = 1, \quad \sigma(r^2) = -1, \quad \sigma(sr) = -r^3.$$ 

Then since

$$\{\sigma(st)^{-1} - \tau(stv_1) \mid st \in ST\} = \{0\}$$

and

$$\{st \mid \tau(st) = 0\} = \emptyset,$$

we have that

$$U(S, T, \sigma, \tau) = \mathbb{C}\langle r^2, s \rangle \subseteq \mathbb{C} + \text{Ann}(v_1).$$

Of course $\mathbb{C}\langle r^2, s \rangle \cong C_2 \times C_2$ (the product of two order 2 cyclic groups). Thus there is an epimorphism from

$$D_8v_1 \otimes \mathbb{C}(C_2 \times C_2) \to D_8.$$ 

But due to dimension considerations, this is in fact an isomorphism.

It can be checked that

$$W \cong L \oplus L' \oplus V$$

where $L$ and $L'$ are both 1-dimensional ($L$ is the trivial representation and $L'$ is the representation where $r$ acts as $-1$ and $s$ as $1$). Thus as described in Section 3.3 the algorithm stops at this point.
4.2. Example: degenerate cyclotomic Hecke algebras. As a demonstration of the generality of our method, we will apply it to a representation of a level three degenerate cyclotomic Hecke algebra $H_n^\lambda$. We begin by describing this algebra and justifying its general interest.

The degenerate affine Hecke algebra $H_n$ is a generalization of the symmetric group $S_n$. For simplicity, in this example we will take $H_n$ to be an algebra over $\mathbb{C}$. $H_n$ is generated by elements $s_1, s_2, \ldots, s_{n-1}$ and $x_1, x_2, \ldots, x_n$, such that $s_1, \ldots, s_{n-1}$ satisfy the usual Coxeter generator relations for the symmetric group:

\begin{align}
(4.2) & \quad s_i^2 = 1, \quad 1 \leq i \leq n-1, \\
(4.3) & \quad s_is_j = s_js_i, \quad |i - j| > 1, \quad 1 \leq i, j \leq n-1, \\
(4.4) & \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \quad 1 \leq i \leq n-2,
\end{align}

the elements $x_1, \ldots, x_n$ commute, and:

\begin{align}
(4.5) & \quad s_jx_i = x_is_j, \quad i \neq j, j + 1, \\
(4.6) & \quad s_ix_i = x_{i+1}s_i - 1, \quad 1 \leq i \leq n-1.
\end{align}

As a $\mathbb{C}$-vector space $H_n \cong \mathbb{C}(S_n) \otimes_\mathbb{C} \mathbb{C}[x_1, \ldots, x_n]$.

Choose some $d$-tuple $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d$. Let $I^\lambda$ be the two-sided ideal of $H_n^\lambda$ generated by the element

\begin{equation}
(4.7) \quad \prod_{i=1}^d (x_1 - \lambda_i).
\end{equation}

The quotient algebra $H_n^\lambda = H_n/I^\lambda$ is called the degenerate cyclotomic Hecke algebra associated to $\lambda$. $H_n^\lambda$ is said to be of level $d$. By abuse of notation we write $x_i, s_i \in H_n^\lambda$ for the images of $x_i, s_i \in H_n$ in this quotient. $H_n^\lambda$ has dimension $\dim(H_n^\lambda) = d^n n!$ [13, Theorem 3.2.2]. In particular, as $\mathbb{C}$-vector spaces,

\begin{equation}
(4.8) \quad H_n^\lambda \cong \mathbb{C}(S_n) \otimes_\mathbb{C} \mathbb{C}[x_1, x_2, x_3]
\end{equation}

with $\dim(\mathbb{C}[x_1, x_2, x_3]) = d^n$.

The claim that degenerate cyclotomic Hecke algebras are generalizations of symmetric groups is justified by the fact that when $\lambda = (0)$, $H_n^\lambda \cong \mathbb{C}(S_n)$. These algebras have deep connections to Lie theory, for example their representation theory is intimately connected to crystals for quantum groups [13, Part I] and their centers are related to parabolic orbits $O$ for $\mathfrak{gl}_n(\mathbb{C})$. Yet despite this, many aspects of their representation theory are still not fully understood. For example, when $d > 2$, $H_n^\lambda$ is generally not semisimple and the dimensions of simple $H_n^\lambda$-modules are not known.

Consider a simple 6-dimensional $H_n^\lambda$-module for $\lambda = (2, 2, 4)$, which we denote by $V_{2,2,4}$. Explicitly, this representation can be described by:

$$
s_1 \mapsto \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}, \quad s_2 \mapsto \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}.
$$
Since \( V \) be defined so that to the group algebra of \( T \) will generate \( V \), the elements of \( \tau \) will take one of two forms. Either \( x \sigma \) or \( x \tau \). Some care must be taken when choosing \( T \). Our goal is that the element obtained after commuting \( x_1, x_2, \) or \( x_3 \) past \( t \in T \) will be invertible in the algebra. In this case, the easiest choice is

\[
T = \{ 1, s_1, s_2, s_1 s_2, s_1 s_2 s_1 \} \subset \mathbb{C}(S_3).
\]

Then \( \tau(V_{2,2,4}) \) is exactly the subalgebra generated by \( s_1 \) and \( s_2 \) which is isomorphic to the group algebra of \( S_3, \mathbb{C}(S_3) \subset H_3^\lambda \). Some care must be taken when choosing the elements of \( T \). Our goal is that the element obtained after commuting \( x_1, x_2, \) or \( x_3 \) past \( t \in T \) will be invertible in the algebra. In this case, the easiest choice is

\[
\{ v_1, v_2, v_3, v_4, v_5, v_6 \}.
\]

Since \( V_{2,2,4} \) is already assumed to be simple, any non-zero element of \( V_{2,2,4} \) will generate \( V_{2,2,4} \). We pick \( V_{2,2,4} = H_3^\lambda v_1 \). It can be checked that \( \tau : V_{2,2,4} \to H_3^\lambda \) can be defined so that

\[
\tau(v_1) = 1, \quad \tau(v_2) = s_1, \\
\tau(v_3) = s_2, \quad \tau(v_4) = s_2 s_1, \\
\tau(v_5) = s_1 s_2, \quad \tau(v_6) = s_1 s_2 s_1.
\]

Denote the basis of \( V_{2,2,4} \) chosen via the matrix representation above by:

\[
\{ v_1, v_2, v_3, v_4, v_5, v_6 \}.
\]

Then \( \tau(V_{2,2,4}) \) is exactly the subalgebra generated by \( s_1 \) and \( s_2 \) which is isomorphic to the group algebra of \( S_3, \mathbb{C}(S_3) \subset H_3^\lambda \). Some care must be taken when choosing the elements of \( T \). Our goal is that the element obtained after commuting \( x_1, x_2, \) or \( x_3 \) past \( t \in T \) will be invertible in the algebra. In this case, the easiest choice is

\[
T = \{ 1, s_1, s_2, s_1 s_2, s_1 s_2 s_1 \} \subset \mathbb{C}(S_3).
\]

In order to define \( \sigma : ST \to (H_3^\lambda)^\times \) we first observe that any element of \( st \in ST \) will take one of two forms. Either \( s = s_1, s_2 \) in which case \( st \) is an element of \( S_3 \) and we define \( \sigma(st) = (st)^{-1} \). Otherwise \( s = x_1, x_2, x_3 \) and in this case we observe that since \( t \in T \subset \mathbb{C}(S_3) \), using relations (1.5) we can rewrite

\[
x_i t = a_{t,i} x_j + b_{t,i}
\]

where \( a_{t,i}, b_{t,i} \in \mathbb{C}(S_3) \) and \( j \in \{ 1, 2, 3 \} \). Then

\[
\tau(x_i v_1) = \tau(a_{t,i} x_j v_1 + b_{t,i} v_1) = k_j \tau(a_{t,i} v_1) + \tau(b_{t,i} v_1) = k_j a_{t,i} + b_{t,i}
\]

where \( k_1, k_2 = 2 \) and \( k_3 = 4 \). Thus, provided that \( k_j a_{t,i} + b_{t,i} \) is invertible, we can define

\[
\sigma(x_i t) = (k_j a_{t,i} + b_{t,i})^{-1}.
\]

For the choices of \( T \) made above, \( k_j a_{t,i} + b_{t,i} \) does happen to be invertible for all \( i \in \{ 1, 2, 3 \} \).

For such \( \tau \) and \( \sigma \),

\[
\{ \sigma(st)^{-1} - \tau(st v_1) \mid s \in S, T \} = \{ 0 \},
\]
and

\[ \{ st : \tau(st) = 0 \} = \emptyset. \]

On the other hand

\[ \{1, x_1, x_2, x_3\} \subseteq \{ \sigma(st)st \mid s \in S, t \in T \} \]

as we would hope, but the span of \( \{ \sigma(st)st \mid s \in S, t \in T \} \) also contains additional elements that act by 0 on \( v_1 \). For example, both \( s_1x_1 - 2s_1 \), and \( s_1x_2 - 2s_1 \) are in this set.

Theorem 1.1 then says that there is a linear epimorphism

\[ (4.9) \quad H_3^\lambda v_1 \otimes \mathbb{C} \langle U \rangle \rightarrow H_3^\lambda. \]

It is clear that \( H_3^\lambda v_1 \cong \mathbb{C}\langle S_3 \rangle \) as vector spaces and we suspect that \((4.9)\) is an alternative decomposition to \((4.8)\).

What aspects of the structure of \( V_{2,2,4} \) does the FSS algorithm recognize? \( V_{2,2,4} \) can be realized as follows: let \( L(2) \boxtimes L(2) \boxtimes L(4) \) be the 1-dimensional representation of \( \mathbb{C}[x_1, x_2, x_3] \subset H_3 \) where \( x_1 \) and \( x_2 \) act by multiplication by 2 and \( x_3 \) acts by multiplication by 4. Since \( \mathbb{C}[x_1, x_2, x_3] \) is a subalgebra of \( H_3 \), then we can construct an induced representation of \( H_3 \) from \( L(2) \boxtimes L(2) \boxtimes L(4) \),

\[ \text{Ind}_{H_3^\lambda}^{H_3} \mathbb{C}[x_1, x_2, x_3] \otimes L(2) \boxtimes L(2) \boxtimes L(4). \]

It turns out that this representation factors through the quotient \( H_3 \twoheadrightarrow H_3^\lambda \) and that

\[ V_{2,2,4} \cong p\left( \text{Ind}_{H_3^\lambda}^{H_3} \mathbb{C}[x_1, x_2, x_3] \otimes L(2) \boxtimes L(2) \boxtimes L(4) \right). \]

The FSS algorithm cannot see this quotient map, but it does seem to see shadows of it. \( H_3^\lambda \) is free as a right \( \mathbb{C}[x_1, x_2, x_3] \)-module, with basis \( S_3 \subset H_3^\lambda \). The FSS algorithm sees this via the first tensor term on the left hand side of \((4.9)\). The right tensor term of \((4.9)\) does not appear to be equal to \( \mathbb{C}[x_1, x_2, x_3] \), but as suggested above, probably corresponds to an alternative basis decomposition of \( H_3^\lambda \).

Within the theory of degenerate cyclotomic Hecke algebras and for many related generalizations of symmetric groups, the entire representation theory of the tower of algebras can be built up from induced representations from the appropriate subalgebra. In this example at least, the FSS algorithm was able to identify such structure.

5. Closing remarks

Our interest here has been in the discovery and proof of the decomposition of Theorem 1.1. We have therefore not sought the most efficient solution or the broadest applications. We do however encourage such future development. There are several questions left unanswered, we offer some we can see ourselves.

Our first and most pressing concern is to capture the behavior of the final term \( A_\ell \). We have found often \( A_\ell = K \) or context makes its value immaterial. But we should like to learn qualities of rings and modules that would predict when \( A_\ell = K \).

A second concern is that we do not appear to be able to “sift” most elements in \( A \) over the FSS generators. What we mean by this is that in a traditional Schreier-Sims algorithm and arbitrary permutation can be written as an product of the strong generating set by having it act on the individual orbits and removing the corresponding transversal term. If at the end the element is nontrivial we have proved that the permutation was not in the group. Now if we apply the same
reasoning to our FSS generators we have a clear problem. Elements $a$ of $A$ may be in the annihilator of $M$ and so no amount of work acting with $a$ on the points $x_1, \ldots, x_\ell \in M$ will tell us more than that $a$ annihilates $M$. That being said, as we observed in the introduction for many elements of $A$ not in the annihilator this process allows us to write the element with short monomials.

Finally, a different rewriting option. If we have an element expressed as a polynomial in the non-commuting generators $S$ of $A$ we can use the rewriting in our proof of Proposition 3.1 to produce a new expression with fixed monomial lengths. This step does not depend on the faithfulness of our modules. However, because we deal with linear combinations and not simply single terms as in Schreier’s lemma, the number of terms in our some grows perhaps exponentially. It would be nice to find a rewriting algorithm that can express a word in FSS form in time polynomial in the final number of terms of the word.

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