Nonvanishing boundary condition for the mKdV hierarchy and the Gardner equation

J F Gomes, Guilherme S França and A H Zimerman

Instituto de Física Teórica - IFT/UNESP, Rua Dr. Bento Teobaldo Ferraz, 271, Bloco II, 01140-070, São Paulo, SP, Brazil

E-mail: jfg@ift.unesp.br, guisf@ift.unesp.br and zimerman@ift.unesp.br

Received 2 August 2011, in final form 14 October 2011
Published 6 December 2011
Online at stacks.iop.org/JPhysA/45/015207

Abstract

A Kac–Moody algebra construction for the integrable hierarchy containing the Gardner equation is proposed. Solutions are systematically constructed by employing the dressing method and deformed vertex operators, which take into account the nonvanishing boundary value problem for the modified Korteweg–de Vries (mKdV) hierarchy. Explicit examples are given and besides the usual KdV-like solitons, our solutions contemplate the large amplitude table-top solitons, kinks, dark solitons, breathers and wobbles.

PACS numbers: 05.45.Yv, 02.10.Ud

1. Introduction

The Gardner equation appeared a long time ago when Miura [1, 2] introduced the remarkable transformation

\[ u = v^2 + v_x, \quad (1) \]

connecting the Korteweg–de Vries (KdV) equation, 4u_t = u_{3x} - 6uu_x, to the modified KdV (mKdV) equation, 4v_t = v_{3x} - 6v^2v_x. Both equations are ideal prototypes for integrable models. The transformation (1) is highly nontrivial and relates the solutions of the two nonlinear equations.

Miura [1] pointed out that the KdV equation is invariant under Galilean transformation. If we change the coordinates according to

\[ x \rightarrow \xi + at, \quad t \rightarrow \tau, \quad a = \text{const}, \quad (2) \]

then \( \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \tau} = \frac{\partial}{\partial \xi} - a \frac{\partial}{\partial \xi} \) and the KdV equation becomes \( 4u_\tau - 4au_\xi = u_{3\xi} - 6uu_\xi \). The undesirable \(-4au_\xi\) term on the left-hand side is cancelled by the nonlinear term \(-6uu_\xi\) on the right-hand side by a constant shift in the field variable. Therefore, the KdV equation is invariant under the transformation (2) with

\[ u(x, t) \rightarrow u(\xi, \tau) + \frac{2}{9}a. \quad (3) \]
Following the same idea for the mKdV equation, we obtain
\[ 4v_\tau - 4av_\xi = v_{3\xi} - 6v^2 v_\xi. \]
Because of the quadratic term in the nonlinearity, a constant shift in the field can still remove the undesirable \(-4a v_\xi\) term, but at the cost of an extra nonlinear one. Then, the mKdV equation is not invariant under the transformation (2), but it is transformed into the Gardner equation
\[ 4v_\tau = v_{3\xi} - 6v^2 v_\xi - 12\mu vv_\xi, \quad \mu \equiv \left(\frac{2}{3}a\right)^{1/2}, \]
if we shift the field
\[ v(x, t) \rightarrow v(\xi, \tau) + \left(\frac{2}{3}a\right)^{1/2}. \]
Equation (4) is evidently interesting because it may be considered as a combination of the KdV and mKdV equations or as an integrable deformation of the KdV or mKdV equations. In a sequence of three papers, Wadati [3–5] obtained the \(n\)-soliton solution using the inverse scattering transform and Bäcklund transformation, and showed that (4) is indeed a completely integrable Hamiltonian system.

It is known that integrable soliton equations appear as members of a more general structure called integrable hierarchies. In [6] the results previously obtained for the KdV equation individually were generalized to the whole KdV hierarchy and its deformation related to the Gardener transformation was discussed.

The properties of the Gardner equation are fairly well known and can be obtained through the properties of the mKdV equation, and are the subject of current research [7, 8]. Besides the mathematical interest, equation (4) has a wide application in atmospheric and ocean waves and was extensively studied by Grimshaw et al [9] where they obtained the large-amplitude table-top solitons, often observed in ocean coastal zones, and breathers [10]. Recently, the solutions of the Gardner equation were used to construct static solutions of the Gross–Pitaevskii equation, known to describe the dynamics of Bose–Einstein condensates [11].

A general algebraic approach for the construction of integrable hierarchies and its solitonic solutions can be formulated in terms of representation of affine Lie algebras. The soliton solutions can be systematically constructed by the dressing method, which connects a trivial (vanishing) vacuum to a nontrivial soliton solution employing vertex operators. See for instance [12–16] and references therein.

The transformation (5), connecting solutions of the Gardner and mKdV equations, clearly shows that both solutions have different boundary values due to a constant shift in the field variable. This implies that both equations have different vacuum solutions. In particular, those solutions of the Gardner hierarchy with trivial vacuum are related to nonvanishing vacuum solutions of mKdV equation and vice versa.

In [17] we have extended the mKdV hierarchy for negative flows by introducing negative even-graded Lax pairs. The nonlinear equations within that class do not present solutions with a vanishing boundary condition and, in order to circumvent this problem, a deformed vertex operator was introduced. This paper is a natural continuation of [17] in which the same algebraic formulation is employed to construct solutions of the Gardner equation. Interesting solutions arise where a table-top soliton or kink can coexist with normal solitons. We also obtain dark solitons, breathers and the wobble [18, 19] solutions.

In sections 2 and 3, we briefly introduce the KdV and mKdV hierarchies, respectively, and show that the Miura transformation represents a map between both hierarchies and not only between two single equations. Our main results are contained in section 4 where we propose a deformed hierarchy grounded on the \(\hat{sL}_2\) Kac–Moody algebra, containing the Gardner equation as one of its members. In section 5, we introduce a new vertex operator, and in section 6 construct explicit solutions for the whole hierarchy. Concluding remarks are in section 7.
2. The KdV hierarchy

Following [20], the mathematical construction of the KdV hierarchy can be introduced by the Lax equation in terms of pseudo-differential operators

\[ L_n = \left[ t_n^{3/2}, L \right], \quad n = 1, 3, 5, \ldots, \]  

where \( L \equiv \partial^2 - u(x, t) \) is the Schrödinger operator. The subscript \( L_n \) denotes the differential part of the operator. We then calculate

\[ L_n^{3/2} = \partial^3 - \frac{1}{2} u \partial - \frac{3}{4} u_t, \quad L_n^{5/2} = \partial^5 - \frac{5}{2} u \partial^3 - \frac{15}{4} u_t \partial^2 + \left( \frac{15}{16} u^2 - \frac{25}{8} u_{2t} \right) \partial - \frac{15}{16} u_{3t} \partial + \frac{15}{8} u_{4t}. \]  

Substituting (7) in (6), we obtain the KdV equation

\[ 4u_t = u_{3t} - 6uu_{t}. \]  

Repeating the same calculation with (8), we have the Sawada–Kotera equation\(^1\)

\[ 16u_{tt} = u_{5t} - 10uu_{3t} - 20u_{tt}u_{2t} + 30u^2 u_{t}. \]  

Higher order equations are obtained from (6) and their soliton solutions are well known [20]. It is important to note that while (9) is invariant under Galilean transformation, (10) and higher order KdV equations are not.

3. The mKdV hierarchy

The mKdV hierarchy can be constructed from a zero curvature condition based on the affine Kac–Moody algebra \( \hat{s}_2 \) [12–15, 17], generated by

\[ \hat{s}_2 \equiv \{ E_a^{(n)} \cdot E_{-a}^{(n)} \cdot H^{(n)} \} \]  

(11)

together with the spectral derivative operator \( \hat{d} \) and the central term \( \hat{c} \). The \( \hat{s}_2 \) commutation relations are

\[ [H^{(n)}, H^{(m)}] = 2n \delta_{0+n+m,0} \hat{c}, \quad [H^{(n)}, E_{\pm a}^{(m)}] = \pm 2E_{\pm a}^{(n+m)}, \quad [E_{-a}^{(n)}, E_{a}^{(m)}] = 0, \]  

\[ [E_a^{(n)}, E_{-a}^{(m)}] = H^{(n+m)} + n \delta_{n+m,0} \hat{c}, \quad [\hat{c}, T^{(n)}] = 0, \quad [\hat{d}, T^{(n)}] = nT^{(n)}, \]  

(12)

where \( T^{(n)} \in \hat{s}_2 \). The operator \( \hat{Q} = \frac{1}{2} H^{(0)} + 2\hat{d} \) defines the grading operation \([\hat{Q}, T^{(n)}] = nT^{(n)}\) that decomposes the algebra into even- and odd-graded subspaces:

\[ \hat{s}_2^{(2n)} = \{ H^{(n)} \}, \quad \hat{s}_2^{(2n+1)} = \{ E_a^{(n)}, E_{-a}^{(n)} \}, \quad \hat{s}_2^{(0)} = \{ H^{(0)} \}. \]  

(13)

Considering this grading structure, it follows from the Jacobi identity that for \( T^{(i)} \in \hat{s}_2^{(i)} \) and \( T^{(j)} \in \hat{s}_2^{(j)} \), we have \([T^{(i)}, T^{(j)}] \in \hat{s}_2^{(i+j)} \).

The mKdV hierarchy is then defined by the zero curvature equation

\[ \partial_t + E_a^{(0)} + E_{-a}^{(1)} + E_{\pm a}^{(0)} + vH^{(0)}, \partial_t + D^{(n)} + D^{(n-1)} + \cdots + D^{(0)} = 0, \]  

(14)

where \( n = 1, 3, 5, \ldots, v = v(x, t) \) and \( D^{(i)} \in \hat{s}_2^{(i)} \). In the construction of integrable models from the zero curvature equation, only the loop algebra that corresponds to set \( \hat{c} = 0 \) in the commutation relations (12) is employed. Equation (14) is solved grade by grade, starting from the highest one, until the zero grade projection leads to the nonlinear time-evolution equation. This procedure works in the following way. Let \( n = 3 \) for example; then from the grading

\footnote{The coefficients of this fifth-order KdV are different from the original Sawada–Kotera, but the nonlinear terms are the same.}
structure (13), the operators involved in the construction (14) must be a linear combination of the algebra generators in the form

\[ D^{(3)} = a_3 E_a^{(3)} + b_3 E_a^{(2)}, \quad D^{(2)} = c_2 H^{(1)}, \]  

\[ D^{(1)} = a_1 E_a^{(0)} + b_1 E_a^{(1)}, \quad D^{(0)} = c_0 H^{(0)}. \]  

(15)

(16)

The coefficients \((a_i, b_i, c_i)\) will be determined as functions of the field \(v(x,t)\) by projecting the zero curvature equation (14) into each graded subspace. In this way, we obtain the nonlinear partial differential equation, from the zero grade projection, and also its Lax pair. Note that only the commutation relations (12) (with \(\hat{c} = 0\)) are used in this calculation and no matrix representation is needed. So, for \(n = 3\) we obtain the well-known mKdV equation

\[ 4v_t = v_{3x} - 6uv_x, \]  

(17)

while for \(n = 5\) we find

\[ 16v_t = v_{5x} - 10u^2v_{3x} - 40vv_xv_{2x} - 10v_x^3 + 30u^4v_x. \]  

(18)

Without loss of generality in the above calculations, all integration constants were chosen to vanish.

Using the Miura transformation (1) in (9) and (10), we can relate them with the corresponding equations in the mKdV hierarchy, (17) and (18), respectively. This operation can be expressed for both equations as

\[ \text{KdV}(u \rightarrow v^2 + v_x) = (2v + \partial_x) \text{mKdV}(v). \]  

(19)

After verifying this expression for these two particular cases, we now show that, in fact, the Miura transformation holds for all higher orders, i.e. the Miura transformation is a map between solutions of the entire mKdV hierarchy into solutions of the KdV hierarchy. Consecutive time evolutions of the KdV hierarchy are given by the recursion formula

\[ u_{n+2} = Ru_n, \quad u_1 = u_x, \]  

(20)

where \(R \equiv \frac{1}{4} \partial_t^2 - \frac{1}{2} u_x \partial_x^{-1}\). Substituting \(u = v^2 + v_x\) on both sides of (20), we can write it in a factorized form as

\[ u_{n+2} = (2v + \partial_x)u_{n+1} = (2v + \partial_x)(\frac{1}{4} \partial_t^2 - v_x - v^2)u_n. \]  

(21)

Define now \(R' \equiv \frac{1}{4} \partial_t^2 - v^2 - v_x \partial_x^{-1}v\); thus

\[ u_{n+2} = R'v_n, \quad v_t = v_x, \]  

(22)

which shows that entire mKdV hierarchy is recursively generated by \(R'\) and (19) holds for \(n = 1, 3, 5, \ldots\).

4. A hierarchy containing the Gardner equation

Motivated by the transformation (5), we now propose a deformation of the mKdV hierarchy (14),

\[ [\partial_t + E_a^{(0)} + E_a^{(1)} + (\mu + v)H^{(0)}, \partial_x + D^{(n)} + D^{(n-1)} + \cdots + D^{(0)}] = 0, \]  

(23)

where \(\mu = \text{const}, v = v(x,t)\) and \(n = 1, 3, 5, \ldots\). In the usual algebraic construction [14], as used in (14), the semi-simple element \(E = E_a^{(0)} + E_a^{(1)}\) is responsible for the algebra decomposition \(sl_2 = \mathcal{K} \oplus \mathcal{M}\), its kernel \(\mathcal{K}\) and image \(\mathcal{M}\) subspaces. In (23) the deformed element \(E_a^{(0)} + E_a^{(1)} + \mu H^{(0)}\) contains both \(\mathcal{K}\) and \(\mathcal{M}\) components. Obviously, the modification from (14) to (23) corresponds to a simple translation in the field \(v \rightarrow v + \mu\); however, it introduces important changes in constructing its solutions through the dressing method.
Reconsidering (14), but now with a nonvanishing constant boundary condition, \( v \to v_0 \), the situation is quite the same as in (23) with \( v \to 0 \). Therefore, solutions with the vanishing boundary condition of (23) are related to solutions with the nonvanishing boundary condition of (14).

Solving (23) for \( n = 3 \), using the same procedure described in the previous section for the mKdV hierarchy, we obtain

\[
v_i = \frac{1}{4} v_{3i} - \left( \mu^2 - \frac{\alpha}{2} \right) v_i - 3\mu v v_i - \frac{3}{2} v^2 v_i,
\]

where \( \alpha \) is an arbitrary integration constant. Choosing \( \alpha = 2\mu^2 \), we have the well-known Gardner equation

\[
4v_i = v_{3i} - 12\mu v v_i - 6v^2 v_i
\]

and its Lax pair\(^2\)

\[
A = E_a^{(0)} + E_{-a}^{(1)} + (\mu + v)H^{(0)},
\]

\[
B = E_a^{(1)} + E_{-a}^{(2)} + (\mu + v)H^{(1)} + \frac{1}{2}(v_i - v^2 - 2\mu v + 2\mu^2)E_a^{(0)} - \frac{1}{2}(v_{2i} + 2\mu v - 2\mu^2)E_a^{(1)} + \frac{1}{2}(v_{2i} - 2v^3 - 6\mu v^2 + 4\mu^3)H^{(0)}.
\]

Solving now (23) for \( n = 5 \), we obtain

\[
16v_i = v_{3i} - 10v^2 v_{3i} - 40v v_{v3i} - 10v_5^{3i} + 30v^4 v_i - 20\mu vv_{v5i} - 40\mu u v_{2v5i} + 12\mu^2 v^3 v_i + 12\mu v^3 v_i.
\]

The two arbitrary integration constants that show up are conveniently chosen: \( \alpha = 4\mu^2 \) and \( \beta = 6\mu^4 \). Equation (27) is a combination of (18) and (10). Note that when \( \mu \to 0 \), (27) becomes (18) in the same way that (25) becomes (17). We point out that (27) is not obtained from (18) by the Galilean transformation (5). The Lax pair for (27) is

\[
A = E_a^{(0)} + E_{-a}^{(1)} + (\mu + v)H^{(0)},
\]

\[
B = E_a^{(2)} + E_{-a}^{(3)} + (\mu + v)H^{(2)} + \frac{1}{2}(v_i - v^2 - 2\mu v + 4\mu^2)E_a^{(1)} - \frac{1}{2}(v_{2i} + 2\mu v - 4\mu^2)E_a^{(2)} - \frac{1}{2}(v_{2i} - 2v^3 - 6\mu v^2 + 4\mu^3)H^{(1)} + \frac{1}{8}(v_{3i} - 6v^2 v_i - 2v v_{v2i} + v_{v3i} + 3v^4 - 12\mu vv_{v3i} - 2v_{3i} + 12\mu v v_{v2i} + 2\mu v_{v3i})
\]

\[
+ 4\mu^2 v_{v2} - 12\mu v^3 + 8\mu^2 v^2 - 8\mu^3 v + 8\mu^4)E_a^{(0)} - \frac{1}{8}(v_{3i} - 6v^2 v_i + 2v v_{v2i} - v_{v3i} - 3v^4 - 12\mu vv_{v3i} + 2\mu v_{v3i})
\]

\[
+ 4\mu^2 v_{v2} - 12\mu v^3 + 8\mu^2 v^2 - 8\mu^3 v + 8\mu^4)E_{-a}^{(1)} + \frac{1}{16}(v_{4i} - 10v^2 v_{3i} - 10v_5^{3i} + 6v^5 - 20\mu vv_{2v5i} - 10\mu v^2
\]

\[
+ 30\mu v^4 + 40\mu^2 v^3 + 16\mu^3)H^{(0)}.
\]

The hierarchy defined in (23) can also be recursively generated through the following pseudo-differential operator:

\[
v_{n+2} = Rv_n, \quad v_i = v_i,
\]

\[
R \equiv \frac{1}{4} \partial_x^2 - (v + 2\mu) v - v_i (\mu \partial_x^{-1} + \partial_x^{-1} v).
\]

\(^2 L = \partial + A, L_i = [L, B].\)
5. Vertex operator

The usual $\hat{s}\ell_2$ vertex operator with principal gradation

$$V(\kappa) = \sum_{n=-\infty}^{\infty} \kappa^{-2n} \left[ H^{(n)} - \frac{1}{2} \delta_{00} \hat{c} + \kappa^{-1} E^{(n)}_a - \kappa^{-1} E^{(n+1)}_{-a} \right],$$

(31)

solves the mKdV and sinh-Gordon equations with a vanishing boundary condition. Here, we introduce a modified vertex operator that generalizes and takes into account the nonvanishing boundary value problem for the mKdV hierarchy and, henceforth, the Gardner hierarchy with a vanishing boundary condition.

Define the following vertex operator depending on the parameters $(\kappa, \mu)$:

$$V_i \equiv \sum_{n=-\infty}^{\infty} (\kappa_i^2 - \mu^2)^n \left[ H^{(n)} + \frac{\mu - \kappa_i}{2\kappa_i} \delta_{00} \hat{c} + \frac{1}{\kappa_i + \mu} E^{(n)}_a - \frac{1}{\kappa_i - \mu} E^{(n+1)}_{-a} \right].$$

(32)

This vertex was proposed in [17] to solve the negative even-graded part of the mKdV hierarchy. Note that when $\mu \to 0$, the vertex (31) is recovered. The parameter $\mu$, also present in equation (25), is related to the nonvanishing boundary condition of the mKdV hierarchy through the identification $\mu \leftrightarrow v_0$, where $v \to v_0$ is the constant value of the field in $|x| \to \infty$. Consider the operator

$$\Omega_m \equiv E^{(m)}_a + E^{(m+1)}_{-a} + \mu H^{(m)},$$

(33)

which corresponds to the vacuum $v = 0$ configuration of the Lax component $A_{\text{vac}} = A_0$ in (26a), for $m = 0$. Taking the commutator of (33) with the vertex (32), we verify that

$$[\Omega_m, V_i] = -2\kappa_i (\kappa_i^2 - \mu^2)^n V_i.$$  

(34)

As will be clear in the following section, (34) determines the dispersion relation for all nonlinear evolution equations in the hierarchy. Consider the highest weight states for $\hat{s}\ell_2$—$|\lambda_0\rangle$, $|\lambda_1\rangle$—which obey the following actions: $E^{(0)}_a|\lambda_j\rangle = 0$, $E^{(0)}_{-a}|\lambda_j\rangle = 0$, and $H^{(n)}|\lambda_j\rangle = 0$ for $n > 0$, and $H^{(0)}|\lambda_j\rangle = \delta_{j1}|\lambda_j\rangle$ and $\hat{c}|\lambda_j\rangle = |\lambda_j\rangle$ for $j = 0, 1$. The adjoint relations are $H^{(n)\dagger} = H^{(-n)}$, $E^{(n)\dagger} = E^{(-n)}$, and $\hat{c}^\dagger = \hat{c}$. Taking (32) between the highest weight states,

$$\langle \lambda_j | V_i | \lambda_j \rangle = \frac{\mu + \sigma_i \kappa_i}{2\kappa_i}, \quad \text{where} \quad \sigma_0 = -1, \quad \sigma_1 = 1.$$  

(35)

A more involved calculation of two vertices yields

$$\langle \lambda_j | V_i V_j | \lambda_j \rangle = \left( \frac{\mu + \sigma_i \kappa_i}{2\kappa_i} \right) a_{ij}, \quad \text{where} \quad a_{ij} \equiv \frac{(\kappa_j - \kappa_i)^2}{(\kappa_j + \kappa_i)^2}.$$  

(36)

It is possible to prove that the expectation value of a product of $n$ vertices decomposes into

$$\langle \lambda_j | \prod_{i=1}^{n} V_i | \lambda_j \rangle = \prod_{i=1}^{n} \langle \lambda_j | V_i | \lambda_j \rangle \prod_{i<k}^{n} a_{ik}.$$  

(37)

The matrix elements (36) and (37) determine the nonlinear interaction between solitons. Note that the nilpotency property of the vertex representation is a direct consequence of (36) when $\kappa_i = \kappa_j$. This corresponds to the physical interpretation that when $\kappa_i \to \kappa_j$, two solitons degenerate into a single soliton (no interaction). Equation (37) also corresponds to the physical property that $n$-solitons interact in pairs.
6. Dressing the vacuum

The two dressing group elements \( \Theta_\pm \) correspond to gauge transformations mapping trivial vacuum potentials, \( A_0 \) and \( B_0 \), into nontrivial ones involving field-dependent potentials, \( A \) and \( B \) [15, 13],

\[
A = \Theta_+ A_0 \Theta_+^{-1} - \partial_\nu \Theta_+ \Theta_+^{-1}, \quad \tag{38a}
\]

\[
B = \Theta_- B_0 \Theta_-^{-1} - \partial_\nu \Theta_- \Theta_-^{-1}. \quad \tag{38b}
\]

They are factorized into positive and negative grade generators

\[
\Theta_+ = e^{\lambda_0} e^{\lambda_1} \ldots, \quad \Theta_- = e^{\nu_0} e^{\nu_1} \ldots \tag{39}
\]

which together with the zero curvature condition yield

\[
\Theta_-^{-1} \Theta_+ = \Psi_0 G \Psi_0^{-1}, \tag{40}
\]

where \( G \) is an arbitrary constant group element and \( \Psi_0 \) is such that \( A_0 = -\partial_\nu \Psi_0 \Psi_0^{-1} \) and \( B_0 = -\partial_\nu \Psi_0 \Psi_0^{-1} \). For the Gardner equation (25), taking in (26a) and (26b) we end up with the vacuum potentials \( A_0 = \Omega_0 \) and \( B_0 = \Omega_1 + \mu^2 \Omega_0 \); therefore,

\[
\Psi_0 = \exp[-\Omega_0 x - (\Omega_1 + \mu^2 \Omega_0) t], \tag{41}
\]

where \( \Omega_m \) is defined in (33). The vacuum takes into account information about the boundary condition at \( |x| \to \infty \) \( (v \to 0) \).

The dressing method requires the use of highest weight states representation so it is necessary to include the central term \( \hat{c} \) in the calculations. In the construction of the models, because \( \hat{c} \) commutes with every other operator, the zero curvature equation is invariant under the addition of a central term. Therefore, we change \( A \to A - \nu_1 \hat{c} \), where \( A = E^{(0)} + E^{(1)} + \nu \tau \) is one of the Lax operators and \( \nu \) is a function that will be determined. Solving for the zero grade projection of (38a) with \( \Theta_+ \), we find

\[
\exp[X^{(0)}] = \exp[\phi H^{(0)} + \nu \hat{c}], \quad \nu = -\phi_1. \tag{42}
\]

Using this result when taking the left-hand side of (40) between highest weight states,

\[
e^\nu = \langle \lambda_0 | \Theta_-^{-1} \Theta_+ | \lambda_0 \rangle \equiv \tau_0, \quad e^{\nu + \nu} = \langle \lambda_1 | \Theta_-^{-1} \Theta_+ | \lambda_1 \rangle \equiv \tau_1, \quad \nu = \partial_\nu \ln \frac{\tau_0}{\tau_1}. \tag{43}
\]

Choosing now \( G = \prod_{i=1}^n e^{V_i} \) and projecting the right-hand side of (40) between these states

\[
\tau_j = \langle \lambda_j | \Psi_0 \left[ \prod_{i=1}^n \exp[V_i] \right] \Psi_0^{-1} | \lambda_j \rangle \\
= \langle \lambda_j | \prod_{i=1}^n \exp \left\{ \Psi_0 V_i \Psi_0^{-1} \right\} | \lambda_j \rangle \\
= \langle \lambda_j | \prod_{i=1}^{n-1} \exp[e^{\xi_i} V_i] | \lambda_j \rangle \\
= \langle \lambda_j | \prod_{i=1}^{n-1} (1 + e^{\xi_i} V_i) | \lambda_j \rangle, \tag{44a}
\]

where in (44a) we have used the fact that \( V_i = V(\kappa_i) \) are eigenstates of \( \Omega_m \) with eigenvalues given by (34). The dispersion relation therefore follows

\[
\xi_i = 2 \nu_1 x + \left( 2 \nu_1 (\nu_1^2 - \mu^2) + 2 \nu_1 \mu^2 \right) t = 2 \nu_1 x + 2 \nu_1^2 t, \tag{45}
\]
where in (44b) the nilpotency property of the vertex V was enforced. The general formula (44b) is valid for all equations within the hierarchy and can be solved explicitly by using the factorization property of the vertices (37) and (36). The explicit spacetime dependence, however, is specified according to the choice of vacuum potentials $A_0$ and $B_0$ for each individual model. Let us consider for instance the vacuum for potentials (28a) and (28b), $A_0 = \Omega_0$ and $B_0 = \Omega_2 + 2\mu^2\Omega_1 + \mu^4\Omega_0$. From (34) we therefore have
\[
\xi_{j} = 2\kappa_{j}x + \left[2\kappa_{j}(\kappa_{j}^2 - \mu^2) - \mu^2\kappa_{j}(\kappa_{j}^2 - \mu^2) + 2\kappa_{j}^2\right]t.
\]

### 6.1. Nonvanishing boundary condition for the mKdV hierarchy

We have constructed solutions to the hierarchy associated with the Gardner equation, where $v \to 0$ when $|x| \to \infty$. We now address the nonvanishing boundary value problem for the mKdV hierarchy.

Recall that in (23) if we take $\mu = 0$, we recover the mKdV hierarchy (14). When taking a constant vacuum $v \to v_0$ in (14), the term $v_0$ plays the role of $\mu$, and hence throughout this subsection we shall consider all expressions after (32) with $\mu$ replaced by $v_0$.

Reconsider the zero grade projection of (38a) for $\Theta_+, but now with $A_0 = E_0^{(0)} + E_0^{(1)} + v_0H^{(0)}$ and $A = E_a^{(0)} + E^a_{-a} + v(x, t)H^{(0)} - v_0\hat{c}$. The analogue of (42) becomes
\[
\exp[X^{(0)}] = \exp[(v_0x + \phi)H^{(0)} + v\hat{c}], \quad v = -\phi_x,
\]
and (43) implies the following nonvanishing boundary solution:
\[
v = v_0 + \partial_x \ln \frac{t_0}{t_1}.
\]
Equation (44b) does not change except for the dispersion relation. The vacuum potentials are obtained by setting $v \to v_0$ in (14). For the mKdV equation (17), we have $A_0 = \Omega_0$ and $B_0 = \Omega_1 - \frac{1}{2}v_0\Omega_0$, and for (18) $A_0 = \Omega_0$ and $B_0 = \Omega_2 - \frac{1}{2}v_0^2\Omega_1 + \frac{1}{2}v_0^4\Omega_0$. Then, using (34), we have the respective dispersion relations
\[
\xi_1 = 2\kappa_1x + (2\kappa_1^3 - 3v_0^2\kappa_1)t, \\
\xi_1 = 2\kappa_1x + (2\kappa_1^3 - 5v_0^2\kappa_1^3 + \frac{5}{4}v_0^4\kappa_1)t.
\]
Note that the dispersion relations depend on the boundary condition, so the velocities of these solitons depend on the vacuum field $v_0$.

### 6.2. Examples

We now present some explicit examples of the exact solutions we have obtained. The tau functions for a general n-soliton solution are given by (44b). The solution is then obtained from (43) or (48), with spacetime dependence specified by the dispersion relations (45) and (46) or (49) and (50), according to the Gardner or mKdV hierarchies, respectively.

The general expression for one-soliton solution is
\[
\tau_j = 1 + \langle V_j \rangle e^{\xi_1},
\]
where $j = 0, 1$ and $\langle \bullet \rangle_j \equiv \langle \lambda_j \mid \bullet \mid \lambda_j \rangle$. The vertex element is given by (35) with $\mu \to v_0$ for the mKdV hierarchy. The two- and three-soliton solutions are respectively given by
\[
\tau_j = 1 + \langle V_1 \rangle e^{\xi_1} + \langle V_2 \rangle e^{\xi_1} + \langle V_1V_2 \rangle e^{\xi_1 + \xi_2}. 
\]
Figure 1. Soliton for the Gardner equation when $\kappa_1 \geq \mu$ and $\mu > 0$. We choose $\mu = 2$ and $\kappa_1 = (3, 2.0001, 2)$ in figures (a), (b) and (c), respectively. When $\kappa_1 \to \mu^+$, the solution shown in (a) deforms, as shown in (b), until it becomes a kink when $\kappa_1 = \mu$ as shown in (c).

Figure 2. Soliton for the Gardner equation when $\kappa_1 < 0$ and $|\kappa_1| \leq \mu$. $\mu = 2$ and $\kappa_1 = (-1.5, -1.9999, -2)$ in figures (a), (b) and (c), respectively.

and

$$\tau_j = 1 + (V_1)_j e^{\xi_1} + (V_2)_j e^{\xi_2} + (V_3)_j e^{\xi_3} + (V_1 V_2)_j e^{\xi_1 + \xi_2} + (V_1 V_3)_j e^{\xi_1 + \xi_3} + (V_2 V_3)_j e^{\xi_2 + \xi_3} + (V_1 V_2 V_3)_j e^{\xi_1 + \xi_2 + \xi_3}. \tag{53}$$

The one-soliton solution of the Gardner equation, with dispersion (45), has different behaviour according to the values of $\kappa_1$ and $\mu$. When $\kappa_1 \geq \mu$ and $\mu > 0$, the solutions are plotted as shown in figure 1. Note that in figure 1(c) we have a kink (anti-kink if you are used to sine-Gordon terminology). When $\kappa_1 < 0$ but $|\kappa_1| \leq \mu$, we have figure 2. For $|\kappa_1| > \mu$ we recover the behaviour of figure 1(a). The more interesting solution occurs when $0 < \kappa_1 < \mu$ has the usual KdV solitons and also a maximum amplitude table-top soliton, as shown in figure 3. The same graphs apply to (27); the only difference is in the dispersion relation (46). If we change sign $\mu \to -\mu$, $\kappa_1 \to -\kappa_1$, then the graphs are reflected through $y$- and $x$-axes, so we have an elevation instead of a depression wave, figure 3(b). In figure 3(c) we have the mKdV nonvanishing boundary solutions with dispersion (49). The mKdV equation can also have a kink, like in figure 1(a), but with the asymptotes in $+2$ and $-2$ corresponding to the constant background $v_0 = 2$. The same solutions also apply to (18) with dispersion (50). These
Figure 3. Soliton for Gardner ((a), (b)) and mKdV (c) when $0 < \kappa_1 < \mu$. The usual KdV-like soliton increases its amplitude and becomes narrower as $\kappa_1$ increases, but when $\kappa_1 \to \mu^-$, it becomes the maximum amplitude table-top soliton. $\mu = 2$ and $\kappa_1 = (1.5, 1.9, 1.999999999)$ in (a). In (b) we changed sign $\mu \to -\mu$, $\kappa_1 \to -\kappa_1$ with the same numerical values. In (c) we have mKdV dark solitons and table-top soliton, with the same numerical values as in (a). Note the background corresponding to $v_0 = 2$.

Figure 4. (a) Breather for the Gardner equation, (b) breather for the mKdV equation: $\mu = 2$ and $\alpha = 0.5, \beta = 5$ and (c) breather for the KdV equation: $\mu = 2$, $\alpha = 1.5$ and $\beta = 4$.

solutions with a constant background are known as dark solitons [21, 22]. Dark solitons of the NLS equation have a wide application in nonlinear optics.

Substituting the previous solutions of the mKdV hierarchy in the Miura transformation (1), we obtain dark solitons of the KdV hierarchy. Unlike the mKdV, the KdV hierarchy does not have kinks or table-top solitons; therefore, the interesting solutions of the Gardner equation are inherited from the nonvanishing boundary solutions of the mKdV equation.

For the two-soliton solution (52), we can have a combination of two usual solitons, one soliton with a kink and one soliton with a table-top soliton. Moreover, we can have the interesting breather, a spatially localized but oscillating solution, by choosing complex conjugate wave numbers $\kappa_1 = \alpha + i\beta$ and $\kappa_2 = \alpha - i\beta$ (the explicit expression for the breather is given by (A.4) in the appendix). Figure 4 shows the breather for the Gardner, mKdV and KdV equations. The nonvanishing boundary breather of the KdV is obtained through Miura transformation.

In figure 5 we have a specific situation of the three-soliton solution (53). Note that the final profile is different from the initial one. The waves did not keep their initial form after
The waves travel to the left so one should read this figure from (d) to (a). We are plotting a three-soliton solution of the Gardner equation with parameters $\mu = -2$, $\kappa_1 = -1.9999$, $\kappa_2 = -1.5$ and $\kappa_3 = -1.2$.

The mKdV equation can also have the same kind of three-soliton solution with a background $v_0$.

In [18], Kölbermann proposed the wobble solution for the sine-Gordon equation; then Ferreira et al [19] showed that the wobble is a three-soliton solution, where two solitons combine to form a breather and the third one is a kink. We follow this line of thought and take our three-soliton solution (53) with $\kappa_1 = \alpha + i\beta$, $\kappa_2 = \alpha - i\beta$ and $\kappa_3 = \mu$ to obtain the wobble solution for the Gardner equation (the explicit expression is (A.9) in the appendix). We can also have the wobble for the mKdV equation with $\kappa_3 = v_0$. These solutions are shown in figure 6. Despite the fact that KdV equation can have a breather, it cannot have a wobble because it does not have a kink. We could also combine the breather with a table-top soliton and with usual solitons as Grimshaw et al considered in [10].

7. Concluding remarks

We have constructed an integrable hierarchy (23) that contains the Gardner equation (25) as one of its members. This construction is based on the Kac–Moody algebra $\tilde{\mathfrak{s}}\mathfrak{e}_2$ with principal gradation. Besides the Gardner equation, another fifth-order PDE (27) that is a combination of Sawada–Kotera and fifth-order mKdV (18) was explicitly considered.

We have introduced a new vertex operator (32) that enabled us to obtain explicit $n$-soliton solutions of the Gardner and mKdV hierarchies, this last one with a nonvanishing...
boundary condition. Besides the usual KdV-like solitons, our solutions contemplate table-top solitons and kinks. The two-soliton solution can be used to form a breather, and using the Miura transformation we obtained a breather for the KdV equation, also with a nonvanishing boundary condition. The three-soliton solution showed that the interaction of individual waves when a kink or table-top soliton is present changes its initial profile such that energy is always conserved. Combining a breather with a kink, we also obtained the wobble solution for the Gardner and mKdV hierarchies. We stress that the our solutions are valid for the whole hierarchy of nonlinear equations, the only modification relying on the dispersion relations. Further exploration of our results can be made by considering the nonvanishing boundary value problem for the non-Abelian AKNS hierarchy that contains the NLS equation which has practical applications in nonlinear optics and water waves.

Acknowledgments

We thank CNPq and Fapesp for support. We also thank the anonymous referees for valuable suggestions.

Appendix. Breather and wobble solutions

The general breather expression is obtained from (52) by setting complex conjugate wave numbers, $\kappa_1 = \kappa_2^* = \alpha + i\beta$, which implies that the dispersion relations will be in the form $\xi_1 = \xi_2^* = \eta + i\zeta$, for some real functions $\eta = \eta(x, t)$ and $\zeta = \zeta(x, t)$ to be determined later.
From (35) we have \( \langle V_1 \rangle_j = \langle V_2 \rangle_j^* = a_j - ib \), where
\[
a_j = \frac{\mu \alpha}{2(\alpha^2 + \beta^2)} + \frac{\sigma_j}{2}, \quad (\sigma_0 = -1, \ \sigma_1 = 1),
\]
(\text{A.1})
\[
b = \frac{\mu \beta}{2(\alpha^2 + \beta^2)}.
\]
(\text{A.2})
and from (36)
\[
\langle V_1 V_2 \rangle_j = -\frac{\beta^2}{\alpha^2}(a_j^2 + b^2).
\]
(\text{A.3})
Therefore, the general tau functions for the breather are
\[
\tau_j = 1 + 2e^\eta \left[ a_j \cos \xi - b \sin \xi - \frac{\beta^2}{2\alpha^2}(a_j^2 + b^2)e^\eta \right].
\]
(\text{A.4})
The wobble is obtained from (53) with \( \kappa_1 = \kappa_2 = \alpha + i\beta \) and \( \kappa_3 = \mu \), so \( \xi_1 = \xi_2^* = \eta + i\zeta \) and \( \xi_3 = \eta \mu \) is a real function depending on \( \mu \). From (35) we obtain \( \langle V_3 \rangle_j = \delta_j \), and from (36) we calculate \( \langle V_1 V_3 \rangle_j = \langle V_2 V_3 \rangle_j^* = \delta_j(c_j + id_j) \), where
\[
\frac{\kappa_1 - \kappa_3}{\kappa_1 + \kappa_3} = \left( \frac{\kappa_2 - \kappa_3}{\kappa_2 + \kappa_3} \right)^* = \gamma + iv,
\]
(\text{A.5})
\[
\gamma = \frac{\alpha^2 + \beta^2 - \mu^2}{(\alpha + \mu)^2 + \beta^2}, \quad v = \frac{2\beta \mu}{(\alpha + \mu)^2 + \beta^2},
\]
(\text{A.6})
and
\[
c_j = a_j(\gamma^2 - v^2) + 2b\eta v,
\]
(\text{A.7})
\[
d_j = 2a_j \eta v - b(\gamma^2 - v^2).
\]
(\text{A.8})
The general wobble tau functions are then given by
\[
\tau_j = 1 + 2e^\eta \left[ a_j \cos \xi - b \sin \xi - \frac{\beta^2}{2\alpha^2}(a_j^2 + b^2)e^\eta \right]
+ 2\delta_j e^{\eta \pm \eta_\mu} \left[ c_j \cos \xi - d_j \sin \xi - \frac{\beta^2}{2\alpha^2}(c_j^2 + d_j^2)e^\eta + \frac{1}{2} e^{-\eta} \right].
\]
(\text{A.9})

\text{A.1. Gardner hierarchy solutions}

The breather or wobble of the Gardner equation (25) is given by
\[
v = \partial_t \ln \frac{\tau_0}{\tau_1}
\]
(\text{A.10})
replacing (\text{A.4}) or (\text{A.9}), respectively. From (45) we have
\[
\eta = 2\alpha x + 2\alpha(\alpha^2 - 3\beta^2)t,
\]
(\text{A.11})
\[
\xi = 2\beta x - 2\beta(\beta^2 - 3\alpha^2)t,
\]
(\text{A.12})
\[
\eta_\mu = 2\mu x + 2\mu^3 t.
\]
(\text{A.13})
For (27), the only modification comes from (46) that yields
\[
\eta = 2\alpha x + 2\alpha(\alpha^4 - 10\alpha^2 \beta^2 + 5\beta^4)t,
\]
(\text{A.14})
\[
\xi = 2\beta x + 2\beta(\beta^4 - 10\alpha^2 \beta^2 + 5\alpha^4)t,
\]
(\text{A.15})
\[
\eta_\mu = 2\mu x + 2\mu^5 t.
\]
(\text{A.16})
A.2. mKdV hierarchy solutions

For the mKdV equation (17), the breather or wobble is given by

\[ v = v_0 + \partial_x \ln \frac{\tau_0}{\tau_1} \]  

(A.17)

using (A.4) or (A.9), respectively. In the coefficients (A.1), (A.2), (A.7) and (A.8) we should replace \( \mu \rightarrow v_0 \).

Taking into account the dispersion (49) we have

\[ \eta = 2\alpha x + 2\alpha (\alpha^2 - 3\beta^2 - \frac{5}{2} v_0^2) t, \]  

(A.18)

\[ \zeta = 2\beta x - 2\beta (\beta^2 - 3\alpha^2 + \frac{3}{2} v_0^2) t, \]  

(A.19)

\[ \eta_{v_0} = 2v_0 x - \frac{3}{4} v_0^3 t. \]  

(A.20)

For (18) the dispersion (50) implies

\[ \eta = 2\alpha x + 2\alpha (\alpha^4 - 10\alpha^2 \beta^2 + 5\beta^4 - \frac{5}{2} v_0^2 \alpha^2 + \frac{15}{2} v_0^2 \beta^2 + \frac{15}{4} v_0^4) t, \]  

(A.21)

\[ \zeta = 2\beta x + 2\beta (\beta^4 - 10\alpha^2 \beta^2 + 5\alpha^4 + \frac{5}{2} v_0^2 \alpha^2 - \frac{15}{2} v_0^2 \beta^2 + \frac{15}{4} v_0^4) t, \]  

(A.22)

\[ \eta_{v_0} = 2v_0 x + \frac{3}{4} v_0^3 t. \]  

(A.23)

References

[1] Miura R M 1968 J. Math. Phys. 9 1202
[2] Miura R M, Gardner C S and Kruskal M D 1968 J. Math. Phys. 9 1204
[3] Wadati M 1975 J. Phys. Soc. Japan 38 681
[4] Wadati M 1975 J. Phys. Soc. Japan 38 673
[5] Wadati M 1976 J. Phys. Soc. Japan 41 1499
[6] Kupershmidt B A 1981 J. Math. Phys. 22 449
[7] Kiselev A V 2007 Theor. Math. Phys. 152 963
[8] Muñoz C 2011 arXiv:1106.0648v2 [math.AP]
[9] Grimshaw R, Pelinovsky D, Pelinovsky E and Slunyaev A 2002 Chaos 12 1070
[10] Grimshaw R, Slunyaev A and Pelinovsky E 2010 Chaos 20 013102
[11] Malomed B A and Stepanyants Y A 2010 Chaos 20 013130
[12] Babelon O and Bernard D 1993 Int. J. Mod. Phys. A 8 507
[13] Ferreira L A, Miramontes J L and Guillén J S 1997 J. Math. Phys. 38 882
[14] Aratyn H, Gomes J F and Zimerman A H 2004 Algebraic construction of integrable and super integrable hierarchies SYMPHYS-11: Proc. 11th Int. Conf. on Symmetry Methods in Physics (Prague, Czech Republic) (arXiv:hep-th/0408231v1)
[15] Aratyn H, Gomes J F, Nissimov E, Pacheva S and Zimerman A H 2000 Proc. NATO Advanced Research Workshop on Integrable Hierarchies and Modern Physical Theories (NATO ARW-UIC 2000) (Chicago, IL) (arXiv:nlin/0012042v1 [nlin.SI])
[16] Olive D I, Turok N and Underwood J W R 1993 Nucl. Phys. B 409 509
[17] Gomes J F, França G S, de Melo G R and Zimerman A H 2009 J. Phys. A: Math. Theor. 42 445204 (arXiv:0906.5579)
[18] Kalbermann G 2004 J. Phys. A: Math. Gen. 37 11603
[19] Ferreira L A, Piette B and Zakrzewski W J 2008 Phys. Rev. E 77 036613
[20] Miwa T, Jimbo M and Date E 2000 Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras (Cambridge: Cambridge University Press)
[21] Chen Z-Y, Huang N-N, Liu Z-Z and Xiao Y 1993 J. Phys. A: Math. Gen. 26 1365
[22] Huang N-N, Chen Z-Y and Yue H 1996 Phys. Lett. A 221 167