THE $\infty$-EIGENVALUE PROBLEM WITH A SIGN-CHANGING WEIGHT

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $m \in C(\overline{\Omega})$ be a sign-changing weight function. For $1 < p < \infty$, consider the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda m(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

where $\Delta_p u$ is the usual $p$-Laplacian. Our purpose in this article is to study the limit as $p \to \infty$ for the eigenvalues $\lambda_{k,p}(m)$ of the aforementioned problem. In addition, we describe the limit of some normalized associated eigenfunctions when $k = 1$.

1. Introduction

Our main goal in this paper is to study the limit as $p \to \infty$ in the eigenvalue problem for the $p$-Laplacian with a sign-changing weight.

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $1 < p < \infty$, and consider

$$\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$$
the usual $p$-Laplacian operator. Let $m \in C(\overline{\Omega})$ be a function (the weight) that changes sign in $\Omega$. We set

$$\Omega_+ := \{m > 0\}, \quad \Omega_- := \{m < 0\}, \quad \Omega_0 := \{m = 0\}.$$

Since we assume that $m$ changes sign we have that $\Omega_+ \neq \emptyset$ and $\Omega_- \neq \emptyset$.

The eigenvalue problem associated with the $p$-Laplacian with a weight function $m$ is given by

$$\begin{cases} -\Delta_p u(x) = \lambda m(x)|u|^{p-2}u(x) & x \in \Omega, \\ u(x) = 0 & x \in \partial \Omega. \end{cases}$$

It is a well-known fact in the literature (cf. [1, 10, 11] and references therein) that the first (positive) eigenvalue can be characterized variationally as follows:

$$\lambda_{1,p} := \lambda_{1,p}(m) = \inf_{\mathcal{A}^+(m)} \int_{\Omega} |\nabla u|^p > 0,$$

where $\mathcal{A}^+(m) := \{u \in W^{1,p}_0(\Omega) : \int_{\Omega} m|u|^p = 1\}.$

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In a similar way the first negative eigenvalue is given by
\[ \mu_{1,p} := \mu_{1,p}(m) = -\lambda_{1,p}(-m) = -\inf_{A^{-}(m)} \int_{\Omega} |\nabla u|^p < 0, \]
where \( A^{-}(m) := \{ u \in W^{1,p}_0(\Omega) : \int_{\Omega} m|u|^p = -1 \} \).

Regarding higher eigenvalues, it is also known that a sequence of positive eigenvalues \( \lambda_{k,p}(m) \) can be obtained by the Ljusternik-Schnirelman theory. In fact, it holds that
\[ 0 < \lambda_{1,p}(m) < \lambda_{2,p}(m) \leq \lambda_{3,p}(m) \leq \ldots \leq \lambda_{k,p}(m) \to \infty \quad \text{as} \quad k \to \infty, \]
see e.g. [1, 15] and references therein. Of course, the same ideas also give the existence of a sequence of negative eigenvalues \( \mu_{k,p}(m) \),
\[ 0 > \mu_{1,p}(m) > \mu_{2,p}(m) \geq \mu_{3,p}(m) \geq \ldots \geq \mu_{k,p}(m) \to -\infty \quad \text{as} \quad k \to \infty. \]

Eigenvalue problems have received an increasing amount of attention along the last decades by many authors, being studied mainly via variational methods. We quote, among many others, [2, 3, 4, 5, 6, 10, 11, 13, 14, 15, 17, 18, 20, 21, 22, 23, 25, 27, 28]. In some of these references the limit as \( p \to \infty \) of the eigenvalue problem associated to the classical case, \( m = 1 \), was considered. In particular, this limit as \( p \to \infty \) was studied in detail in [18] (for the first eigenvalue) and [17] (for higher eigenvalues), see also [4] for an anisotropic version. In those papers it is proved that
\[ \lambda_{1,\infty}(1) := \lim_{p \to \infty} (\lambda_{1,p}(1))^{1/p} = \inf \left\{ \frac{\|\nabla v\|_{L^\infty(\Omega)}}{\|v\|_{L^\infty(\Omega)}} : v \in W^{1,\infty}_0(\Omega), v \neq 0 \right\} = \frac{1}{R}, \]
where \( R \) is the largest possible radius of a ball contained in \( \Omega \). In addition, they take the limit as \( p \to \infty \) in the eigenfunctions of the \( p \)-Laplacian eigenvalue problems (see [18]) and are viscosity solutions of the following eigenvalue problem (called the infinity eigenvalue problem in the literature and studied in [7, 9, 16, 18, 29])
\[
\begin{cases}
\min \{ |\nabla u| - \lambda_{1,\infty}(1)u, \Delta_{\infty} u \} = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]
The operator \( \Delta_{\infty} \) that appears here is called the \( \infty \)-Laplacian and is given by \( \Delta_{\infty} u := -\langle D^2 u Du, Du \rangle \).

Our main first result for the weighted case gives a geometric characterization of the first \( \infty \)-eigenvalue and establishes that it is associated to an eigenfunction that satisfies a limiting variational problem, as well as a partial differential equation, the later being satisfied in the viscosity sense. These results generalize classical results for the \( p \)-Laplace eigenvalue problem without the weight. It is interesting to emphasize that positive \( \infty \)-eigenvalues only take into account the geometry of the set where the weight \( m \) is positive.

**Theorem 1.1.** The limit as \( p \to \infty \) in the minimization problem [12] is given by
\[ \lambda_{1,\infty}(m) := \lim_{p \to \infty} \sqrt[p]{\lambda_{1,p}(m)} = \inf_{u \in W^{1,\infty}_0(\Omega)} \|\nabla u\|_{L^\infty(\Omega)}/\|u\|_{L^\infty(\Omega^+)} . \]
Moreover, this value \( \lambda_{1,\infty}(m) \) has a geometric characterization:
\[ \lambda_{1,\infty}(m) = \frac{1}{R_+}, \quad \text{where} \quad R_+ := \max_{x \in \Omega^+} d(x, \partial \Omega), \]
i.e., $R_+$ is the radius of the largest ball in $\Omega$ centered at a point in $\Omega_+$. Let $u_p$ be an eigenfunction associated with $\lambda_{1,p}(m)$, that is, a minimizer to (1.2), normalized by $\int_\Omega m|u_p|^p = 1$. Then, up to a subsequence,

$$u_p \to u_\infty,$$

uniformly in $\Omega$ and weakly in $W^{1,q}_0(\Omega)$ for every $1 < q < \infty$. Also, $u_\infty \in W^{1,\infty}_0(\Omega)$, it is a minimizer of (1.3) and a viscosity solution to

$$\begin{aligned}
-\Delta_\infty v &= 0 & \text{in } & \{mv = 0\}, \\
\min\{-\Delta_\infty v, |\nabla v| - \lambda_{1,\infty} v\} &= 0 & \text{in } & \{mv > 0\}, \\
\max\{-\Delta_\infty v, -|\nabla v| - \lambda_{1,\infty} v\} &= 0 & \text{in } & \{mv < 0\}, \\
v &= 0 & \text{on } & \partial\Omega.
\end{aligned}
$$

Concerning higher eigenvalues, which will be properly defined in section 4, we have been able to establish an upper bound. This bound is analogous to the one obtained in [17] for the unweighted case, but again the balls need to be centered in the set $\Omega_+$. We have the following result:

**Theorem 1.2.** Let $\lambda_{k,p}$ be the $k$-th eigenvalue of the $p$-Laplacian problem, as defined in (4.1). Then we have that

$$\lim_{p \to \infty} \left( \lambda_{k,p} \right)^{1/p} \leq \frac{1}{R_{k,+}},$$

where

$$R_{k,+} := \sup_{r > 0} \left\{ \text{there are } k \text{ disjoint balls of radius } r \text{ in } \Omega \text{ centered at } x_1, \ldots, x_k \in \Omega_+ \right\}.$$

For the case of the second $\infty$-eigenvalue, $k = 2$, we are also able to completely determine $\lambda_{2,\infty}$ and give a geometric characterization similar to the classical one by [17], once again depending only on the set where $m$ is positive.

**Theorem 1.3.** Let $\lambda_{2,p}$ be the second eigenvalue of the $p$-Laplacian problem, as defined in (4.1). We have that

$$\lambda_{2,\infty} := \lim_{p \to \infty} \left( \lambda_{2,p} \right)^{1/p} = \frac{1}{R_{2,+}},$$

where

$$R_{2,+} := \sup_{r > 0} \left\{ \text{there are two disjoint balls } B_r(x_1), B_r(x_2) \subset \Omega \text{ with } x_1, x_2 \in \Omega_+ \right\}.$$

**Remark 1.4.** Although in the above theorems we focus on the first positive eigenvalue, we can obtain analogous results for the first negative eigenvalue. It holds that

$$\mu_{1,\infty}(m) := \lim_{p \to \infty} -\sqrt[p]{-\mu_{1,p}(m)} = -\inf_{u \in W^{1,\infty}_0(\Omega)} \frac{\| \nabla u \|_{L^\infty(\Omega)}}{\| u \|_{L^\infty(\Omega_+)}},$$

where

$$\mu_{1,\infty}(m) = -\inf_{u \in W^{1,\infty}_0(\Omega)} \frac{\| \nabla u \|_{L^\infty(\Omega)}}{\| u \|_{L^\infty(\Omega_-)}} = -\frac{1}{R_-},$$

where $R_-$ the radius of the largest ball included in $\Omega$ centered at a point in $\Omega_-$, i.e., $R_- := \max_{x \in \Omega_-} d(x, \partial\Omega)$. Also, the limit of the associated eigenfunctions satisfies an eigenvalue problem analogous to (1.4).

A similar result concerning higher eigenvalues also holds for the negative ones.
Finally, let us observe that with the same ideas we can analyze a slightly different operator. Namely, we now add a term $C(x)|u|^{p-2}u$ to the $p$-Laplacian and obtain the following eigenvalue problem:

$$\begin{cases}
-\Delta_p u(x) + C(x)|u(x)|^{p-2}u(x) = \lambda m(x)|u|^{p-2}u(x) & x \in \Omega, \\
u(x) = 0 & x \in \partial \Omega,
\end{cases}$$

where $C$ is continuous and positive in $\Omega$ and $m$ changes sign and satisfies the previous conditions. For this problem, it is known (see [11]) that there exists a principal eigenvalue given by

$$\lambda_{1,p}(C, m) = \min_{f_\Omega \leq 1} \int_{\Omega} |\nabla u|^p + C|\nabla u|^p.$$

Concerning the limit as $p \to \infty$ we have the following result.

**Theorem 1.5.** The limit as $p \to \infty$ in the minimization problem (1.7) is given by

$$\lambda_{1,\infty}(C, m) := \lim_{p \to \infty} \sqrt[p]{\lambda_{1,p}(C, m)} = \max \left\{ \frac{1}{R_+}, 1 \right\},$$

where, as before, $R_+ := \max_{x \in \Omega_+} d(x, \partial \Omega)$.

The rest of the paper is organized as follows: In Section 2 we collect previous necessary results, namely we recall the definition of viscosity solution and the equivalence between viscosity and weak solution in the $p$-Laplacian setting. Next, in Section 3 we concentrate on the first $\infty$-eigenvalue and prove Theorem 1.1. In Section 4 we deal with higher eigenvalues and include some simple examples to see how the eigenvalues depend on the set $\Omega_+$. Finally, in Section 5 we deal with Theorem 1.5.

## 2. Preliminary results

In this section we collect some results that will be used along this paper.

First, we observe that we can rewrite the first equation in (1.1) as

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = \lambda m(x)|u|^{p-2}u$$

or, expanding the divergence operator, as

$$-|\nabla u_p|^{p-4} (|\nabla u_p|^2 \Delta u_p + (p-2) \Delta_\infty u_p) = \lambda m(x)|u|^{p-2}u.$$  

This equation is in divergence form and is nonlinear. Nevertheless it is elliptic (degenerate) and there are multiple ways in which we can define solution to this problem. The first one is the concept of weak solution (that is closely related to the variational nature of this problem).

**Definition 2.1.** A function $u \in W_0^{1,p}(\Omega)$ is a weak solution of (2.1) if

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi = \int_{\Omega} \lambda m(x)|u|^{p-2}u \varphi$$

for every $\varphi \in W_0^{1,p}(\Omega)$.

Since our goal is to consider the limit as $p \to \infty$, we need to choose an appropriate concept of solution such that it is somehow “stable” under the limit, in order to identify the limiting problem. The right notion of solution to this problem is the
viscosity one (see e.g. [19]). Notice that the limit equation that appears in (1.4) is not in divergence form.

For the reader’s convenience we briefly include the basics of the notion of viscosity solution, that will be used in the next section to establish the equation satisfied by the limiting function. Let \( x, y \in \mathbb{R} \), \( z \in \mathbb{R}^N \), and \( S \) be a real symmetric matrix. We define the following continuous function

\[
H_p(x, y, z, S) := -|z|^{p-4} \left( |z|^2 \text{trace}(S) + (p-2) \langle S \cdot z, z \rangle \right) - \lambda_{1,p} m(x)|u_p|^{p-2} u_p.
\]

Observe that \( H_p \) is elliptic in the sense that \( H_p(x, y, z, S) \geq H_p(x, y, z, S') \) if \( S \leq S' \) in the sense of bilinear forms, and also that (2.2) can then be written as \( H_p(x, u_p, \nabla u_p, D^2 u_p) = 0 \). We are thus interested in viscosity sub and supersolutions of the partial differential equation

\[
\left\{ \begin{array}{l}
H_p(x, u, \nabla u, D^2 u) = 0 \quad \text{in } \Omega, \\
\quad u = 0 \quad \text{on } \partial \Omega.
\end{array} \right.
\]

**Definition 2.2.** An upper semicontinuous function \( u \) defined in \( \Omega \) is a viscosity subsolution of (2.3) if \( u|_{\partial \Omega} \leq 0 \) and, whenever \( x_0 \in \Omega \) and \( \phi \in C^2(\Omega) \) are such that

i) \( u(x_0) = \phi(x_0) \),

ii) \( u(x) < \phi(x) \) if \( x \neq x_0 \),

then

\[
H_p(x_0, \phi(x_0), \nabla \phi(x_0), D^2 \phi(x_0)) \leq 0.
\]

**Definition 2.3.** A lower semicontinuous function \( u \) defined in \( \Omega \) is a viscosity supersolution of (2.3) if \( u|_{\partial \Omega} \geq 0 \) and, whenever \( x_0 \in \Omega \) and \( \phi \in C^2(\Omega) \) are such that

i) \( u(x_0) = \phi(x_0) \),

ii) \( u(x) > \phi(x) \) if \( x \neq x_0 \),

then

\[
H_p(x_0, \phi(x_0), \nabla \phi(x_0), D^2 \phi(x_0)) \geq 0.
\]

We observe that in both of the above definitions the second condition is required just in a neighbourhood of \( x_0 \) and the strict inequality can be relaxed. We refer to [8] for more details about the general theory of viscosity solutions, and to [19] for viscosity solutions related to the \( \infty \)-Laplacian and the \( p \)-Laplacian operators. The following result can be shown as in [24, Proposition 2.4] (recall that \( \lambda_{k,p} \) are as in (4.1) below).

**Lemma 2.4.** A continuous weak solution to the eigenvalue problem

\[
\left\{ \begin{array}{l}
-\Delta_p u(x) = \lambda_{k,p} m(x)|u_p|^{p-2} u_p(x) \quad x \in \Omega, \\
u(x) = 0 \quad x \in \partial \Omega,
\end{array} \right.
\]

is also a viscosity solution in the sense of the previous definition.

Note that, from the results in [10], variational eigenvalues in the sequences of positive/negative eigenvalues to our problem have associated eigenfunctions that are weak solutions (and hence viscosity solutions) to (2.4).
3. The first eigenvalue.

Proof of Theorem 1.1. Let \( u_p \) be a solution to (1.1) and \( v \) be any test function. We have that, due to the variational characterization,
\[
\lambda_{1,p}^1 = \frac{\int_\Omega |\nabla u_p|^p}{\int_\Omega m|u_p|^p} \leq \frac{\int \Omega |\nabla v|^p}{\int_\Omega m|v|^p}.
\]

Let \( B_r(c) := \{ x \in \mathbb{R}^n : |x - c| < r \} \) be a ball contained in \( \Omega \) and centered at a point \( c \in \{ m > 0 \} \), and define the following function:
\[
w := \begin{cases} 
  d(x, \partial B_r(c)) & \text{if } x \in B_r(c), \\
  0 & \text{if } x \notin B_r(c).
\end{cases}
\]

Using \( w \) as a test function above we have
\[
\lambda_{1,p}^1 \leq \frac{|\Omega|}{\int_{B_r(c)} m|w|^p},
\]
which is equivalent to
\[
\lambda_{1,p}^{1/p} \leq \frac{|\Omega|^{1/p}}{\|w\|_{L^p(B_r(c), m)}}.
\]

Now, choosing \( \delta > 0 \) such that \( B_{\delta/2}(c) \subset \{ m > 0 \} \), we observe that
\[
\|w\|_{L^p(B_r(c), m)} = \left( \int_{B_{\delta/2}(c)} m|w|^p + \int_{B_r(c) \setminus B_{\delta/2}(c)} m|w|^p \right)^{1/p} \\
\geq \left( \int_{B_{\delta/2}(c)} m|w|^p + \int_{B_r(c) \setminus B_{\delta/2}(c)} m|w|^p \right)^{1/p} \\
\geq (r - \delta/2) \left( C - \|m\|_{L^\infty} \int_{B_r(c) \setminus B_{\delta/2}(c)} \left( \frac{r - \delta}{r - \delta/2} \right)^p \right)^{1/p} \\
\rightarrow r - \delta/2 \quad \text{as } p \rightarrow \infty.
\]

On the other hand,
\[
\|w\|_{L^p(B_r(c), m)} = \left( \int_{B_{\delta/2}(c)} m|w|^p + \int_{B_r(c) \setminus B_{\delta/2}(c)} m|w|^p \right)^{1/p} \\
\leq \left( \int_{B_{\delta/2}(c)} m|w|^{p \prime} + \int_{B_r(c) \setminus B_{\delta/2}(c)} m|w|^p \right)^{1/p} \\
\rightarrow r \quad \text{as } p \rightarrow \infty.
\]

Since \( \delta \) can be chosen arbitrarily small we conclude that \( \lim_{p \rightarrow \infty} \|w\|_{L^p(B_r(c), m)} = r \). Now, taking limits in \( p \) in (3.1) we deduce that
\[
(3.2) \quad \limsup_{p \rightarrow \infty} \lambda_{1,p}^{1/p} \leq \frac{1}{r}.
\]
Therefore, as this inequality holds being \( r \) the radius of any ball contained in \( \Omega \) and centered at \( c \in \{m > 0\} \), we get

\[
(\text{3.3}) \quad \limsup_{p \to \infty} \lambda_{1,p}^{1/p} \leq \inf_{\{r > 0; B_r(c) \subset \Omega, e \in \{m > 0\}\}} \frac{1}{r} = \inf_{\{r > 0; B_r(c) \subset \Omega, e \in \{m > 0\}\}} \frac{1}{r} = \frac{1}{R+}.
\]

On the other hand, using Hölder's inequality we have, for \( q < p \),

\[
(\text{3.4}) \quad \|\nabla u_p\|_q \leq \|\nabla u_p\|_p |\Omega|^{1/q-1/p} = \lambda_{1,p}^{1/p} |\Omega|^{1/q-1/p} \leq C.
\]

Hence, \( \{u_p\} \) is a bounded sequence in \( W_0^{1,q}(\Omega) \) and therefore there is a subsequence (that we still call \( u_p \)) that converges weakly in \( W_0^{1,q}(\Omega) \) and uniformly in \( \overline{\Omega} \) to a limit \( u_\infty \) (we are using here that \( W_0^{1,q}(\Omega) \to C(\overline{\Omega}) \) when \( q > N \)). By a diagonal procedure we can obtain a subsequence \( u_p \) that converges weakly in \( W_0^{1,q}(\Omega) \) for every \( 1 < q < \infty \) and uniformly in \( \overline{\Omega} \) to \( u_\infty \).

Now, recalling (3.3) and letting \( p \to \infty \) in (3.3) we derive that

\[
\|\nabla u_\infty\|_q \leq \limsup_{p \to \infty} \lambda_{1,p}^{1/p} |\Omega|^{1/q-1/p} \leq \frac{1}{R+} |\Omega|^{1/q},
\]

and now taking \( q \to \infty \) we finally get

\[
\|\nabla u_\infty\|_\infty \leq \frac{1}{R+}.
\]

Hence \( u_\infty \) belongs to \( W_0^{1,\infty}(\Omega) \). Moreover, since we normalized the eigenfunctions by \( \int_\Omega m|u_p|^p = 1 \),

\[
1 = \left( \int_\Omega m|u_p|^p \right)^{1/p} \leq \left( \int_\Omega m^+|u_p|^p \right)^{1/p} \to \|u_\infty\|_{L^\infty(\Omega_+)} \quad \text{as} \quad p \to \infty.
\]

Therefore, \( \|u_\infty\|_{L^\infty(\Omega_+)} \geq 1 \). Next we notice that (since \( u_\infty \) is Lipschitz continuous in \( \overline{\Omega} \)) there exists \( x_0 \in \Omega_+ \) with

\[
u_\infty(x_0) = \|u_\infty\|_{L^\infty(\Omega_+)} \geq 1.
\]

Now we observe that, if we take \( y \in \partial\Omega \) such that \( |x_0 - y| = \text{dist}(x_0, \partial\Omega) \), we have

\[
1 \leq u_\infty(x_0) = u_\infty(x_0) - u_\infty(y) \leq \|\nabla u_\infty\|_\infty |x_0 - y| \leq \frac{1}{R+} |x_0 - y|.
\]

Hence, as \( R_+ := \max_{x \in \Omega_+} d(x, \partial \Omega) \) we get that all the previous inequalities must be equalities and so

\[
u_\infty(x_0) = 1, \quad \|\nabla u_\infty\|_\infty = \frac{1}{R+}, \quad \text{and} \quad d(x_0, \partial \Omega) = R+.
\]

Notice that this implies that \( u_\infty \) is a minimizer for the limit variational problem, that is,

\[
\|\nabla u_\infty\|_{L^\infty(\Omega)} = \inf_{\nu \in W^{1,\infty}_0(\Omega)} \|\nabla \nu\|_{L^\infty(\Omega_+)}.
\]

On the other hand, again employing (3.4) we infer that

\[
\|\nabla u_\infty\|_q \leq \liminf_{p \to \infty} \|\nabla u_p\|_q \leq \left( \liminf_{p \to \infty} \lambda_{1,p}^{1/p} \right) |\Omega|^{1/q},
\]
and letting \( q \to \infty \) we conclude that
\[
\frac{1}{R_+} = \| \nabla u_\infty \|_\infty \leq \liminf_{p \to \infty} \lambda_{1,p}^{1/p}.
\]

Taking into account (3.3) we derive that there exists the limit as \( p \to \infty \) of \((\lambda_{1,p})^{1/p} (:= \lambda_{1,\infty})\) and that is given by
\[
\lambda_{1,\infty} = \frac{1}{R_+} \frac{\| \nabla u_\infty \|_{L^\infty(\Omega)}}{\| u_\infty \|_{L^\infty(\Omega^+)}} = \inf_{\nu \in W_0^{1,\infty}(\Omega)} \frac{\| \nabla \nu \|_{L^\infty(\Omega)}}{\| \nu \|_{L^\infty(\Omega^+)}}.
\]

This ends the proof of the first assertion of the theorem.

The next and final step in this proof is to find the equation satisfied by \( u_\infty \). We start by addressing the set \( \{ m = 0 \}^\circ \) and prove that
\[-\Delta_\infty u_\infty = 0 \quad \text{in} \quad \{ m = 0 \}^\circ \text{ in the viscosity sense.}\]

Following the definition of viscosity solution as stated in the previous section, let \( x_0 \in \{ m = 0 \}^\circ \) and \( \phi \in C_0^{\infty}(\Omega) \) be such that \( u_\infty(x_0) = \phi(x_0) \) and \( u_\infty(x) < \phi(x) \), for all \( x \in B \) where \( B \) is an open ball containing \( x_0 \). We need to show that
\[-\Delta_\infty \phi(x_0) \leq 0.\]

Since \( u_p \to u_\infty \) uniformly, the function \( u_p - \phi \) reaches a maximum over \( B \) at an interior point, say \( x_p \). First we see that \( x_0 \) is the only limit point of \( \{ x_p \} \). In fact, if there existed another cluster point \( x^* \neq x_0 \), then \( x_{p'} \to x^* \) for \( x_{p'} \) maximum point of \( u_{p'} - \phi \) in \( B \). In particular, we would have
\[u_{p'}(x_{p'}) - \phi(x_{p'}) \geq u_{p'}(x_0) - \phi(x_0).\]

Letting \( p' \) tend to infinity and recalling that \( u_p \) tends to \( u_\infty \) in \( C(\Omega) \) due to classical compactness theorems,
\n\[u_\infty(x^*) - \phi(x^*) \geq u_\infty(x_0) - \phi(x_0) = 0,\]

which is a contradiction with the definition of \( x_0 \) and \( \phi \). Therefore, \( x_{p'} \to x_0 \).

Since \( x_{p'} \) is a maximum point of \( u_{p'} - \phi \) in \( B \) from the equation satisfied by \( u_p \) at \( x_p \in B \) we obtain
\[-|\nabla \phi(x_p)|^{p-2} (|\nabla \phi(x_p)|^2 \Delta \phi(x_p) + (p - 2) \Delta_\infty \phi(x_p)) \leq 0.
\]

Assuming \( \phi \) is such that \( \nabla \phi(x_0) \neq 0 \) (otherwise we trivially obtain \(-\Delta_\infty \phi(x_0) = 0\)) we have that \( \nabla \phi(x_0) \neq 0 \), and hence we may divide by \((p-2)|\nabla \phi(x_p)|^{p-4} \) and obtain
\[-\frac{|\nabla \phi(x_p)|^2 \Delta \phi(x_p)}{p - 2} - \Delta_\infty \phi(x_p) \leq 0.
\]

Now, letting \( p \to \infty \) we obtain
\[-\Delta_\infty \phi(x_0) \leq 0,\]
that is, \( u_\infty \) is a viscosity subsolution to \(-\Delta_\infty v = 0\).

Similarly one can establish that \( u_\infty \) is a viscosity supersolution to \(-\Delta_\infty v = 0\), and hence we conclude that \( u_\infty \) is a viscosity solution to \(-\Delta_\infty v = 0 \) in \( \Omega_0 \).

Now, we deal with the other cases. We start by looking at points where \( u_\infty \) is positive.
We consider \( x_0 \in \{ m > 0 \} \) and \( \phi \in C^2_{\text{loc}} \) be such that \( u_\infty(x_0) = \phi(x_0) \) and \( u_\infty(x) < \phi(x) \), for all \( x \in B \) where \( B \) is an open ball containing \( x_0 \). Following the steps used before we now arrive to

\[
-\frac{\|\nabla \phi(x_p)\|^2 \Delta \phi(x_p)}{p-2} - \Delta_\infty \phi(x_p) \geq \frac{\lambda_{1,p} m(x_p) |u_p(x_p)|^{p-1}}{(p-2)|\nabla \phi(x_p)|^{p-4}}.
\]

Again we may assume that \( \phi \) is such that \( \nabla \phi(x_p) \neq 0 \) (since the right hand side is positive) and then we may divide by \((p-2)|\nabla \phi(x_p)|^{p-4}\). Again due to the fact that the right hand side is positive we may rewrite it as

\[
\frac{\lambda_{1,p} m(x_p) |u_p(x_p)|^{p-1}}{(p-2)|\nabla \phi(x_p)|^{p-4}} = \frac{1}{p-2} \left( \frac{\lambda_{1,p}^{1/p} m^{1/p}(x_p) |u_p(x_p)|^{\frac{p-1}{p}}}{|\nabla \phi(x_p)|^{\frac{p-2}{p}}} \right)^p.
\]

As \( p \to \infty \) we have

\[
-\Delta_\infty \phi(x_0) \geq \lim_{p \to \infty} \frac{1}{p-2} \left( \frac{\lambda_{1,p}^{1/p} m^{1/p}(x_p) |u_p(x_p)|^{\frac{p-1}{p}}}{|\nabla \phi(x_p)|^{\frac{p-2}{p}}} \right)^p.
\]

Since \( \phi \) is in \( C^2 \) the left hand side is well defined and that implies that the right hand side must be finite. This in turn leads to

\[
\lambda_{1,\infty} \phi(x_0) \leq |\nabla \phi(x_0)|.
\]

Therefore, we have obtained

\[
\min \{ -\Delta_\infty \phi(x_0), |\nabla \phi(x_0)| - \lambda_{1,\infty} \phi(x_0) \} \geq 0.
\]

That is, \( u_\infty \) is a viscosity subsolution.

To show that \( u_\infty \) is a viscosity supersolution we consider \( x_0 \in \{ m > 0 \} \) and \( \phi \in C^2_{\text{loc}} \) be such that \( u_\infty(x_0) = \phi(x_0) \) and \( u_\infty(x) > \phi(x) \), for all \( x \in B \) where \( B \) is an open ball containing \( x_0 \). In this case we arrive to

\[
-\frac{\|\nabla \phi(x_p)\|^2 \Delta \phi(x_p)}{p-2} - \Delta_\infty \phi(x_p) \leq \frac{\lambda_{1,p} m(x_p) |u_p(x_p)|^{p-1}}{(p-2)|\nabla \phi(x_p)|^{p-4}}.
\]

Again we may assume that \( \phi \) is such that \( \nabla \phi(x_p) \neq 0 \) and then we may divide by \((p-2)|\nabla \phi(x_p)|^{p-4}\). Since the right hand side is positive we may rewrite it as

\[
\frac{\lambda_{1,p} m(x_p) |u_p(x_p)|^{p-1}}{(p-2)|\nabla \phi(x_p)|^{p-4}} = \frac{1}{p-2} \left( \frac{\lambda_{1,p}^{1/p} m^{1/p}(x_p) |u_p(x_p)|^{\frac{p-1}{p}}}{|\nabla \phi(x_p)|^{\frac{p-2}{p}}} \right)^p.
\]

As \( p \to \infty \) we get

\[
-\Delta_\infty \phi(x_0) \leq \lim_{p \to \infty} \frac{1}{p-2} \left( \frac{\lambda_{1,p}^{1/p} m^{1/p}(x_p) |u_p(x_p)|^{\frac{p-1}{p}}}{|\nabla \phi(x_p)|^{\frac{p-2}{p}}} \right)^p.
\]

Now, if

\[
|\nabla \phi(x_0)| - \lambda_{1,\infty} \phi(x_0) = |\nabla \phi(x_0)| - \lambda_{1,\infty} u(x_0) > 0
\]

then the right hand side goes to 0 as \( p \to \infty \) and we get that

\[
\lambda_{1,\infty} \phi(x_0) < |\nabla \phi(x_0)| \implies -\Delta_\infty \phi(x_0) \leq 0
\]

Therefore, we have obtained

\[
\min \{ -\Delta_\infty \phi(x_0), |\nabla \phi(x_0)| - \lambda_{1,\infty} \phi(x_0) \} \leq 0.
\]

That is, \( u_\infty \) is a viscosity supersolution.
The equation in the set \( \{ m < 0 \} \) when \( u_\infty \) is positive can be obtained with analogous computations. When \( u_\infty \) is negative we argue in the same way noticing that the inequalities are reversed. \( \square \)

### 4. Higher eigenvalues.

In this section we analyse the case of higher eigenvalues. In order to do so, we first recall that there exists a sequence of positive eigenvalues that can be constructed by variational methods. Since \( m^+ := \max \{ m, 0 \} \neq 0 \), \( m \in C(\Omega) \) and \( \Omega \) is a bounded domain, we are in the setting described in [10]. If we want to allow the domain \( \Omega \) to be unbounded we would need other restrictions on \( m \) to assure our variational problem is set on a manifold (see [27] and also [28] for further details) and similar results hold.

In fact, positive eigenvalues to our problem correspond (via Lagrange multipliers type arguments) to positive critical values of the functional \( \Phi : W^{1,p}_0(\Omega) \to \mathbb{R} \),

\[
\Phi(u) := \int_\Omega |\nabla u|^p,
\]

restricted to the \( C^1 \) manifold \( A^+(m) := \{ u \in W^{1,p}_0(\Omega) : \int_\Omega m|u|^p = 1 \} \). Such critical values can be characterized by being the image through \( \Phi \) of a function \( u \in A^+(m) \) such that \( \Phi'(u) \) is orthogonal to the tangent space of \( A^+(m) \) at \( u \), \( T_u A^+(m) \). We emphasize that, since we choose \( m^+ \neq 0 \), then \( A^+(m) \neq \emptyset \) is a \( C^1 \) manifold.

Therefore, we now focus on the analysis of such critical values. Since this is a nonlinear setting and we seek a min-max type principle, we need an appropriate notion of measure, such as the genus of Krasnoselskii. For the sake of completeness we include it here (see Juutinen-Lindqvist [17]):

**Definition 4.1.** Let \( E \) be a real Banach space and let \( A \subset E \) be any closed symmetric set (that is, \( v \in A \Rightarrow -v \in A \)). The genus \( \gamma(A) \) of \( A \) is the smallest integer \( m \) such that there exists a continuous odd mapping \( \phi : A \to \mathbb{R}^m \setminus \{0\} \). If no such integer exists we write \( \gamma(A) = \infty \).

If \( 0 \in A \) then immediately \( \gamma(A) = \infty \). On the other hand if \( \gamma(A) = 1 \) then \( A \) is non-connected.

If we restrict ourselves to \( \Sigma_k \), \( k = 1, 2, \ldots \) the collection of all symmetric compact subsets \( A \subset A^+(m) \) such that \( \gamma(A) \geq k \) then, such as for the \( p \)-Laplacian case (see [15]), for the problem with weights it is known that (see [28]) there exists an increasing sequence of positive eigenvalues of \( (4.1) \), converging to \( \infty \), characterized by

\[
\lambda_{k,p} = \inf_{A \in \Sigma_k} \sup_{u \in A} \int_\Omega |\nabla u|^p.
\]

Observe that, since \( \gamma(\{u,-u\}) = 1 \) we recover the usual definition for \( \lambda_{1,p} \). We also recall the following lemma (see [26]) that provides a way to compute the genus of some specific subsets of \( W^{1,p}_0 \).

**Lemma 4.2.** Let \( A \subset W^{1,p}_0(\Omega) \) and \( U \subset \mathbb{R}^k \) be a bounded neighborhood of \( 0 \). If there exists an odd homeomorphism \( h : A \to \partial U \) then \( \gamma(A) = k \).
Using this characterization we now proceed to prove the second theorem stated in the introduction. Namely we establish an upper bound for the sequence of eigenvalues.

**Proof of Theorem 1.2** For simplicity we present the proof for $k = 2$, that is, for the second eigenvalue. The proof for $k > 2$ follows by the same ideas. Let $r_2 > 0$ be such that there exist disjoint open balls $B_1 = B(c_1, r_2) \subset \Omega$ and $B_2 = B(c_2, r_2) \subset \Omega$ with $c_1, c_2 \in \Omega_+$. Using $r_2$ we define the truncated cone functions $C_1, C_2$ by

$$C_1(x) := (r_2 - |x - c_1|)^+, \quad C_2(x) := (r_2 - |x - c_2|)^+.$$ 

Set $A := \langle C_1, C_2 \rangle \cap \{v \in W_0^{1,\infty} : \|v\|_{\infty, \Omega} = 1\}$. We have that $A$ is closed and, by the previous lemma, has genus 2. Therefore

$$\lambda_{2,p}^{1/p} \leq \sup_{v \in A} \left( \frac{\int_\Omega |\nabla v|^p}{\int_\Omega m|v|^p} \right)^{1/p}.$$ 

Now let $v := \alpha C_1 + \beta C_2$. Since $C_1$ and $C_2$ have disjoint support we can write,

$$\int_\Omega |\nabla v|^p = (|\alpha|^p + |\beta|^p) \int_{B(r_2)}.$$ 

On the other hand, after a change of variables, we obtain,

$$\int_\Omega m(x)|v|^p = \int_{B_{r_2}(0)} (|\alpha|^p m(x + c_1) + |\beta|^p m(x + c_2)) |r_2 - |x||^p.$$ 

By assumption we have that $m(c_1), m(c_2) > 0$ and thus there exists $\delta > 0$ such that $m(x + c_1), m(x + c_2) > 0$ for $x \in B_\delta(0)$. Therefore,

$$\|v\|_{L^p(\Omega, m)} = \left( \int_{B_{\delta/2}(0)} (|\alpha|^p m(x + c_1) + |\beta|^p m(x + c_2)) |r_2 - |x||^p + \int_{B_{r_2}(0) \setminus B_{\delta/2}(0)} (|\alpha|^p m(x + c_1) + |\beta|^p m(x + c_2)) |r_2 - |x||^p \right)^{1/p} \geq \left( \int_{B_{\delta/2}(0)} (|\alpha|^p m(x + c_1) + |\beta|^p m(x + c_2))^{1/p} (r_2 - \delta/2)^p - (|\alpha|^p + |\beta|^p) \|m\|_{L^\infty} \int_{B_{\delta/2}(0) \setminus B_{\delta}(0)} |r_2 - \delta/2|^p \right)^{1/p} \to r_2 - \delta/2 \quad \text{as } p \to \infty.$$
Similarly,
\[ \|v\|_{L^p(\Omega,m)} = \left( \int_{B_k(0)} (|\alpha|^p m(x + c_1) + |\beta|^p m(x + c_2)) r_2 - |x|^p + \int_{B_{k/2}(0) \setminus B_k(0)} (|\alpha|^p m(x + c_1) + |\beta|^p m(x + c_2)) r_2 - |x|^p \right)^{1/p} \]
\[ \leq \left( \int_{B_{k/2}(0) \setminus B_k(0)} (|\alpha|^p m(x + c_1) + |\beta|^p m(x + c_2)) r_2^{1/p} + \int_{B_{k/2}(0) \setminus B_k(0)} (|\alpha|^p + |\beta|^p) m_{\infty} \int_{B_{k/2}(0) \setminus B_k(0)} |r_2 - \delta|^p \right)^{1/p} \]
\[ \to r_2 \quad \text{as } p \to \infty. \]

Since \( \delta \) can be chosen arbitrarily small we conclude that
\[ \lim_{p \to \infty} \|v\|_{L^p(B_{r_2}(0),m)} = r_2. \]

Now, taking limits in \( p \) in the inequality for the eigenvalue we have that
\[ \limsup_{p \to \infty} \lambda_{1,1/p}^{1/p} \leq \frac{1}{r_2}. \]

Therefore, as this inequality holds for any \( r_2 \) as above, we get
\[ \limsup_{p \to \infty} \lambda_{1,1/p}^{1/p} \leq \inf_{r_2} \frac{1}{r_2} = \frac{1}{R_{2,+}}. \]

The proof is completed. \( \square \)

The upper bound established above is actually attained in the case \( k = 2 \), that is, we can completely characterize \( \lambda_{2,\infty} \) by means of \( R_{2,+} \), the maximum possible radius of two disjoint balls in \( \Omega \) centered at \( \Omega_+ \). Given the result of Theorem 1.2 we only need to prove that the reverse inequality holds, when \( k = 2 \).

Arguing as in the proof of Theorem 1.1 we can easily deduce that at least a subsequence of the sequence of eigenfunctions \( \{u_{2,p}\} \) converges uniformly in \( \Omega+ \) to \( u_{2,\infty} \). Moreover, this function \( u_{2,\infty} \) is a viscosity solution of a problem such as (1.4) with \( \lambda_{1,\infty} \) replaced by some \( \Lambda \) satisfying \( \Lambda \leq \frac{1}{R_{2,+}} \).

**Proof of Theorem 1.3** From the condition imposed on all eigenfunctions we can deduce that
\[ 1 = \left( \int_{\Omega} m|u_{2,p}|^p \right)^{1/p} \]
\[ \leq \left( \int_{\Omega_+} m|u_{2,p}|^p \right)^{1/p} \]
\[ = \left( \int_{\Omega_+} m|u_{2,+}|^p + \int_{\Omega_+} m|u_{2,-}|^p \right)^{1/p} \]
\[ \to \max \left\{ \|u_{2,+}\|_{L^\infty(\Omega_+)}, \|u_{2,-}\|_{L^\infty(\Omega_+)} \right\} \quad \text{as } p \to \infty, \]
where \( u_{2,+} \) denote the positive and negative parts of \( u_{2,\infty} \).

Since \( u_{2,+} \) are Lipschitz continuous in \( \Omega+ \) there exist \( x_1, x_2 \in \Omega_+ \) such that
\[ \|u_{2,+}\|_{L^\infty(\Omega_+)} = u_{2,+}(x_1) \quad \text{and} \quad \|u_{2,-}\|_{L^\infty(\Omega_+)} = -u_{2,-}(x_2). \]
Let now $\mathcal{N}^\pm \subset \Omega^\pm$ be nodal sets of $u_{2,\infty}^\pm$ respectively and such that $x_1 \in \mathcal{N}^+$ and $x_2 \in \mathcal{N}^-$. By Theorem 3.2 in [10] (see also Theorem 8.1 in [17] for the classical setting), since $u_{2,\infty}^+ > 0$ is a viscosity solution to $\min\{|\nabla v| - \Lambda v, -\Delta v\} = 0$ in $\mathcal{N}^+$ and $u_{2,\infty}^+ = 0$ on $\partial \mathcal{N}^+$, then

$$\Lambda = \lambda_1,\infty(\mathcal{N}^+) = \|\nabla u_{2,\infty}^+\|_{\infty,\mathcal{N}^+} = \frac{1}{R_{1,+}(\mathcal{N}^+)}.$$ 

Moreover $u_{2,\infty}^+(x_1) = 1$. We obtain a similar result for $u_{2,\infty}^-$. As a first conclusion we see that

$$\lambda_2,\infty(\mathcal{N}^+) = 1.$$

We obtain a similar result for $u_{2,\infty}^-$. As a first conclusion we see that

$$\lambda_2,\infty(\mathcal{N}^+) = 1.$$

Here we are using that the only (positive) eigenvalue of (1.1) that has an associated eigenfunction of constant sign is the first eigenvalue, see e.g. [11, Section 1].

On the other hand, we also conclude that there exist two balls $B_+ \subset \mathcal{N}^+$ and $B_- \subset \mathcal{N}^-$ with radius $R_{1,+}(\mathcal{N}^\pm)$ respectively. Since $B_+$ and $B_-$ are disjoint and both contained in $\Omega^+$, by the definition of $R_{2,+}$ we have that

$$R_{2,+} \geq \max\{R_{1,+}(\mathcal{N}^+), R_{1,+}(\mathcal{N}^-)\}.$$

Finally,

$$\frac{1}{R_{2,+}} \leq \frac{1}{R_{1,+}} = \|\nabla u_{2,\infty}^+\|_{\infty,\mathcal{N}^+} \leq \|\nabla u_{2,\infty}^-\|_{\infty,\mathcal{N}^-} \leq \liminf_{p \to \infty} \lambda_{2,p}^{1/p} \leq \frac{1}{R_{2,+}},$$

where we have used the result of the previous theorem in the last inequality. Hence,

$$\lambda_{2,\infty} = \frac{1}{R_{2,+}}.$$

This ends the proof. \(\square\)

4.1. Examples. Now let us present some simple examples to see how the geometry of $\Omega^+$ affects the eigenvalues $\lambda_{1,\infty}(m)$ and $\lambda_{2,\infty}(m)$. Notice that the size of the weight is not relevant for the limit eigenvalue problem, what matters here is the set $\Omega^+ = \{m > 0\}$.

In what follows we will fix $\Omega$ as being the unit ball in $\mathbb{R}^2$ (for simplicity). In this case we have

$$\lambda_{1,\infty}(1) = 1, \quad \text{and} \quad \lambda_{2,\infty}(1) = 2,$$

see [17, 18].

**Example 1.** Let $\Omega^+ = B_\delta(0)$ with $\delta$ small be a ball centered at the origin. From our results we obtain

$$\lambda_{1,\infty}(m) = 1, \quad \text{and} \quad \lambda_{2,\infty}(m) = \frac{1}{\delta}.$$

**Example 2.** Let $\Omega^+ = \{x \in B_1(0) : \text{dist}(x, \partial B_1(0)) \leq \delta\}$ be a small strip around the boundary $\partial B_1(0)$ of width $\delta$. Now, we have

$$\lambda_{1,\infty}(m) = \frac{1}{\delta}, \quad \text{and} \quad \lambda_{2,\infty}(m) = \frac{1}{\delta}.$$

Notice that in this case we have $\lambda_{1,\infty}(m) = \lambda_{2,\infty}(m)$. 

Example 3. Let $\Omega_+ = B_3((1/2,0)) \cup B_3((-1/2,0))$ the union of two small balls. In this case we get
\[
\lambda_{1,\infty}(m) = \frac{2}{1+2\delta}, \quad \text{and} \quad \lambda_{2,\infty}(m) = 2.
\]

5. The First Eigenvalue for a Slightly Different Operator

In this section we analyze a slightly different operator, namely we now investigate the following eigenvalue problem
\[
\begin{align*}
\{-\Delta_p u(x) + C(x)|u(x)|^{p-2}u(x) &= \lambda m(x)|u(x)|^{p-2}u(x) & x \in \Omega, \\
u(x) &= 0 & x \in \partial \Omega,
\end{align*}
\]
where $C$ is continuous and positive in $\overline{\Omega}$ and $m$ changes sign and satisfies the conditions imposed in the previous sections.

It is known (see [11]) that there exists a principal eigenvalue
\[
\lambda_{1,p}(C, m) = \min_{f_{\Omega}} \frac{\int_{\Omega} m|u|^{p} + \int_{\Omega} |\nabla u|^{p} + C|u|^{p}}{\int_{\Omega} |u|^{p}}.
\]

Our aim is to compute the limit
\[
\lim_{p \to \infty} \left( \lambda_{1,p}(C, m) \right)^{1/p}.
\]

Proof of Theorem 5.3. Following the ideas of Theorem 4.1 we search for an upper bound for $\limsup_p \lambda_{1,p}^{1/p}$. To this end, let $c \in \Omega_+$. Associated with this $c \in \Omega_+$, let $R = R(c) > 0$ be the radius of the biggest ball $B_R(c)$ such that $B = B_R(c) \subset \Omega$. Once again we consider the function
\[
w := \begin{cases} 
d(x, \partial B_R(c)) & \text{if } x \in B_R(c), \\
0 & \text{if } x \notin B_R(c).
\end{cases}
\]

Using the definition of $\lambda_{1,p}$ we see that
\[
\lambda_{1,p}^{1/p}(C, m) \leq \left( \frac{\int_{\Omega} m|w|^{p}}{\int_{\Omega} |w|^{p}} \right)^{1/p}.
\]

We already know that
\[
\lim_{p \to \infty} \left( \int_{\Omega} m|w|^{p} \right)^{1/p} = R.
\]

On the other hand, as $C$ is positive, we obtain
\[
\left( \int_{B} C|w|^{p} \right)^{1/p} = \left( \int_{B} C|d(x, c)|^{p} \right)^{1/p} \to R,
\]
as $p \to \infty$. Therefore, letting $p$ to infinity, we obtain
\[
\limsup_{p \to \infty} (\lambda_{1,p}(C, m))^{1/p} \leq \max \left\{ \frac{1}{R}, 1 \right\}.
\]

We have that if $R \leq 1$ then $1/R \geq 1$ so that the maximum is achieved for $1/R$. On the other hand, if $R > 1$ then the maximum is 1.
Now, taking into account that \( c \in \Omega_+ \) we obtain
\[
\limsup_{p \to \infty} (\lambda_{1,p}(C, m))^{1/p} \leq \inf_{c \in \Omega_+} \max \left\{ \frac{1}{R(c)}, 1 \right\} = \max \left\{ \frac{1}{R_+}, 1 \right\},
\]
where, as before, \( R_+ := \max_{x \in \Omega_+} d(x, \partial \Omega) \).

From this bound we can argue as before to obtain that \( \{u_p\} \) is a bounded sequence in \( W_0^{1,q}(\Omega) \) and then there is a subsequence (that we still call \( u_p \) that converges weakly in \( W_0^{1,q}(\Omega) \) and uniformly in \( \Omega \) to a limit \( u_\infty \). Moreover, it holds that
\[
\|\nabla u_\infty\|_\infty \leq \limsup_{p \to \infty} (\lambda_{1,p}(C, m))^{1/p} \leq \max \left\{ \frac{1}{R_+}, 1 \right\}.
\]
Hence \( u_\infty \) belongs to \( W_0^{1,\infty}(\Omega) \). Moreover,
\[
1 = \left( \int_\Omega m^+ |u_p|^p \right)^{1/p} \leq \left( \int_\Omega m^+ |u_p|^p \right)^{1/p} \to \|u_\infty\|_{L^\infty(\Omega_+)} \quad \text{as } p \to \infty.
\]
Therefore, \( \|u_\infty\|_{L^\infty(\Omega_+)} \geq 1 \). Next we notice that (since \( u_\infty \) is Lipschitz continuous in \( \Omega \)) there exists \( x_0 \in \Omega_+ \) with
\[
u_\infty(x_0) = \|u_\infty\|_{L^\infty(\Omega_+)} \geq 1.
\]
Now we observe that, if we take \( y \in \partial \Omega \) such that \( |x_0 - y| = \text{dist}(x_0, \partial \Omega) \), we have
\[
1 \leq u_\infty(x_0) - u_\infty(y) \leq \|\nabla u_\infty\|_{\infty} |x_0 - y| \leq \max \left\{ \frac{1}{R_+}, 1 \right\} |x_0 - y| \leq \max \left\{ 1, R_+ \right\}.
\]
Hence, if \( R_+ := \max_{x \in \Omega_+} d(x, \partial \Omega) \leq 1 \) we get that all the previous inequalities must be equalities and so
\[
u_\infty(x_0) = 1, \quad \|\nabla u_\infty\|_\infty = \frac{1}{R_+}, \quad \text{and} \quad d(x_0, \partial \Omega) = R_+.
\]
Notice that this implies that \( u_\infty \) is a minimizer for the limit variational problem, that is,
\[
\max \left\{ \|\nabla u_\infty\|_{L^\infty(\Omega)} : \|u_\infty\|_{L^\infty(\Omega)} \right\} = \inf_{v \in W_0^{1,\infty}(\Omega)} \max \left\{ \|\nabla v\|_{L^\infty(\Omega)} : \|v\|_{L^\infty(\Omega)} \right\} = \frac{1}{R_+}.
\]
Moreover, we have
\[
\lim_{p \to \infty} (\lambda_{1,p}(C, m))^{1/p} = \frac{1}{R_+} = \max \left\{ \frac{1}{R_+}, 1 \right\}.
\]
On the other hand, if \( R_+ := \max_{x \in \Omega_+} d(x, \partial \Omega) \geq 1 \) we have
\[
\liminf_{p \to \infty} \left( \int_\Omega C|u_p|^p \right)^{1/p} \geq u_\infty(x_0) = 1.
\]
Therefore,
\[
\liminf_{p \to \infty} (\lambda_{1,p}(C, m))^{1/p} = \liminf_{p \to \infty} \left( \min_{I_{\Omega}} m^+ |u_p|^p \int_\Omega |\nabla u|^p + C|u|^p \right)^{1/p} \geq \liminf_{p \to \infty} \left( \int_\Omega C|u_p|^p \right)^{1/p} \geq 1.
\]
We conclude that also in this case
\[
\lim_{p \to \infty} (\lambda_{1,p}(C, m))^{1/p} = 1 = \max \left\{ \frac{1}{R_+}, 1 \right\}.
\]
This ends the proof. \(\square\)

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