The Curvature Property of a Linear Dynamical System

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Abstract
In this work a two-dimensional smooth autonomous dynamical system is regarded as a three-dimensional Riemannian manifold and it is shown that the scalar curvature of a linear dynamical system $\frac{dx}{dt} = ax + by$, $\frac{dy}{dt} = cx + dy$ is non-positive. The manifold is scalar-flat iff $b = -c$ and $a = d = 0$.

Keywords: Linear dynamical systems, Riemann curvature tensor, scalar curvature

Bir Lineer Dinamik Sistemin Eğrilik özelliği

Öz
Bu çalışmada iki-boyutlu, düzgün, otonom bir dinamik sistem üç-boyutlu bir Riemann manifoldu olarak değerlendirilmiş ve bir $\frac{dx}{dt} = ax + by$, $\frac{dy}{dt} = cx + dy$ lineer dinamik sisteminin skaler eğrilğinin pozitif olmadığı gösterilmiştir. Manifold skaler-düzdür ancak ve ancak $b = -c$ ve $a = d = 0$.

Anahtar Kelimeler: Lineer dinamik sistemler, Riemann eğrilik tensörü, skaler eğrilik
1. Introduction

A smooth autonomous dynamical system (SADS) on a two-dimensional manifold \( N = (D; x, y) \), where \( D \) is a connected open set in \( \mathbb{R}^2 \) endowed with coordinates \( (x, y) \), is given by a system of first order ordinary differential equations

\[
\frac{dx}{dt} = f(x, y) \\
\frac{dy}{dt} = g(x, y)
\]

(1)

such that \( f \) and \( g \) are smooth functions on \( N \). The system (1) defines a smooth vector field

\[ \xi = f(x, y) \partial_x + g(x, y) \partial_y \]

(2)

on \( N \) which is a smooth section of the tangent bundle \( TN \), i.e. is a mapping \( \xi : N \to TN \) defined to be

\[ \xi(x, y) = (x, y, \dot{x} = f(x, y), \dot{y} = g(x, y)). \]

Since the rank of \( d\xi \), the differential of \( \xi \), equals 2 on \( N \), a SADS may be regarded as a surface in \( TN \).

In this work, to capture all the information about the dynamics, we identify a SADS as a three-dimensional manifold \( M = I \times N \) with local coordinates \( (t, x, y) \). Here \( t \) is a coordinate function on an open interval \( I \subset \mathbb{R} \) and stands for the time coordinate. This consideration gives rise to define a SADS as a submanifold of the first-order jet bundle \( J^1(\mathbb{R}, N) \) of maps \( \mathbb{R} \to N \) [1, 2].

This paper is devoted to the investigation of the curvature property of a linear system

\[
\frac{dx}{dt} = ax + by \\
\frac{dy}{dt} = cx + dy,
\]

(3)

where \( a, b, c, d \in \mathbb{R} \), within the context of Riemannian geometry. By means of the method of moving frames we evaluate the connection 1-form and the curvature 2-form of the Levi-Civita connection in \( TM \) compatible with the Riemannian metric which is defined by the sum of squares of the one-forms

\[
\omega^1 = dt \\
\omega^2 = dx - (ax + by)dt \\
\omega^3 = dy - (cx + dy)dt,
\]

and show that the scalar curvature of the connection is non-positive and the curvature vanishes if and only if \( b = -c \) and \( a = d = 0 \).

2. Riemannian structure and the curvature

The solutions of (3) are identified with the solutions of the Pfaffian system

\[
\omega^2 = 0, \quad \omega^3 = 0
\]

such that on a solution curve we have \( \omega^1 \neq 0 \). The column vector \( \omega = (\omega^1, \omega^2, \omega^3)^t \), where \( ^t \) denotes the transposition, defines a coframe on \( M \) which is dual to the frame of the vector fields

\[
e_1 = \frac{\partial}{\partial t} + f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y}, \quad e_2 = \frac{\partial}{\partial x}, \quad e_3 = \frac{\partial}{\partial y}.
\]

If we introduce the Riemannian metric on \( M \)

\[ ds^2 = \sum_i \omega^i \otimes \omega^i, \]

(4)

then \( M \) becomes a Riemannian manifold and the frame \( e = (e_1, e_2, e_3) \) forms an orthonormal frame. Let \( \nabla \) be the connection compatible with the Riemannian metric (4). The structure equations for the coframe \( \omega \) are given by

\[
\begin{align*}
d\omega^1 &= 0 \\
d\omega^2 &= a \omega^1 \wedge \omega^2 + b \omega^1 \wedge \omega^3 \\
d\omega^3 &= c \omega^1 \wedge \omega^2 + d \omega^1 \wedge \omega^3.
\end{align*}
\]

As it is considered in [4], the \( \mathfrak{so}(3, \mathbb{R}) \)-valued connection form is obtained by solving the system of equations

\[
d\omega^i = -\theta^i \wedge \omega^1, \quad \theta^i = -\theta^i.
\]

The unique 1-form

\[
\theta = \begin{pmatrix}
0 & \theta^1_1 & \theta^1_2 \\
-\theta^2_1 & 0 & \theta^2_3 \\
-\theta^3_1 & -\theta^3_2 & 0
\end{pmatrix}
\]

satisfying the structure equations is obtained by the Cartan's Lemma, where

\[
\begin{align*}
\theta^1_1 &= -a \omega^2 - \frac{1}{2} (b + c) \omega^3 \\
\theta^1_3 &= -\frac{1}{2} (b - c) \omega^2 - d \omega^3 \\
\theta^2_3 &= -\frac{1}{2} (b - c) \omega^1.
\end{align*}
\]

The 1-form \( \theta \) is called \( \mathfrak{so}(3, \mathbb{R}) \)-valued connection form of the Levi-Civita connection.

The curvature 2-form of the Levi-Civita connection is defined by the anti-symmetric matrix of two-forms:

\[
\Omega^i_j = d\theta^i_j + \sum_k \theta^i_k \wedge \theta^j_k, \quad \Omega^i_j = -\Omega^j_i.
\]

In terms of \( \omega^i \)'s, the components of the Riemannian curvature tensor are determined by

\[
\Omega^i_j = \sum_{k<l} R^i_{jkl} \omega^k \wedge \omega^l,
\]

For the details on a connection in an arbitrary vector bundle we refer to [3]. By a direct calculation we obtain

\[
\begin{align*}
\Omega^1_2 &= -\frac{1}{2} \left( 2a^2 + \frac{3}{2} c^2 + bc - \frac{1}{2} b^2 \right) \omega^1 \wedge \omega^2 \\
\Omega^1_3 &= -\frac{1}{2} \left( 2a^2 + bc + 2d^2 - \frac{1}{2} c^2 \right) \omega^1 \wedge \omega^3.
\end{align*}
\]
\[ \Omega_3^2 = \frac{1}{2} \left( \frac{1}{2} (b + c)^2 - 2ad \right) \omega^2 \wedge \omega^3. \]

The independent nonzero components of the Riemann curvature tensor are

\[ R_{122}^1 = -\frac{1}{2} \left( 2a^2 + c(b + c) - \frac{1}{2} (b^2 - c^2) \right), \]
\[ R_{131}^1 = -(ab + cd), \]
\[ R_{313}^1 = -\frac{1}{2} \left( b(b + c) + 2d^2 + \frac{1}{2} (b^2 - c^2) \right), \]
\[ R_{232}^2 = \frac{1}{2} \left( 1 \frac{1}{2} (b + c)^2 - 2ad \right). \]

At a point, the sectional curvatures of the two dimensional subspaces spanned by \((e_1, e_2), (e_1, e_3)\) and \((e_2, e_3)\) are given respectively by \( R_{122}^1, R_{313}^1, \) and \( R_{232}^2. \) The scalar curvature is defined by the trace \( R = \sum_{ij} R_{ij} \) and is found to be \( R = 2(R_{122}^1 + R_{313}^1 + R_{232}^2). \) Note that the scalar curvature is an invariant, that is, it does not depend on the choice of an orthonormal frame. It follows that

\[ R = -\left( 2a^2 + \frac{3}{2} c^2 + bc - \frac{1}{2} b^2 + \frac{3}{2} b^2 + bc + 2d^2 - \frac{1}{2} c^2 \right) \]
\[ -\frac{1}{2} b^2 - \frac{1}{2} c^2 - bc + 2ad \right). \]

By arranging the terms we obtain

\[ 2R = -((b + c)^2 + (2a + d)^2 + 3d^2). \]

As a result we have the following:

**Theorem 1** The Riemannian manifold corresponding to a linear system

\[
\begin{align*}
\frac{dx}{dt} &= ax + by \\
\frac{dy}{dt} &= cx + dy,
\end{align*}
\]

has non-positive scalar curvature. The scalar curvature vanishes if and only if \( b = -c \) and \( a = d = 0. \)

We say that the Riemannian manifold is flat iff the curvature tensor identically vanishes. Substituting \( b = -c \) and \( a = d = 0 \) into the components of the Riemannian curvature tensor yields the following:

**Corollary 2** The Riemannian manifold corresponding to a linear system

\[
\begin{align*}
\frac{dx}{dt} &= \lambda y \\
\frac{dy}{dt} &= -\lambda x,
\end{align*}
\]

where \( \lambda \in \mathbb{R} \), is flat.

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