Geometry of conjunction set of smooth stationary Gaussian fields

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Abstract

We investigate the conjunction probability that at a same point the values of the Gaussian fields exceed the given threshold. This problem was studied by Double-sum method or Euler characteristic method. In this paper, using a recent result of Azaïs and Wschebor describing the shape of the excursion set, we give the explicit values of the generalized Pickands constants and compare them with the predictions by Euler characteristic method. Our results give a partial validity of Euler characteristic method.

Key words: Conjunction problem, Gaussian fields, Double-sum method, Pickands constant, Euler characteristic method.

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1 Introduction

Let $X_i(t), 1 \leq i \leq n$, be the independent copies of a centered stationary Gaussian field with unit variance and both define on a compact set $S \subset \mathbb{R}^d$. The following probability, so called conjunction probability, has drawn much of interest,

$$P\left(\sup_{t \in S} \min_{1 \leq i \leq n} X_i(t) \geq u\right),$$

where $u$ is a fixed threshold, or equivalently the probability that the conjunction set (excursion set)

$$C_u = \{t \in S : X_i(t) \geq u, \forall 1 \leq i \leq n\}$$

is non-empty.

When $n = 1$, then it becomes the tail distribution of the maximum of stationary Gaussian field. Finding the exact value of the tail distribution is very challenging. Therefore, ones would like to consider the asymptotic formula as $u$ tends to infinity. This problem has been studied extensively in literature. One could mention three main techniques to deal with it: double-sum method (see \cite{13, 15}), Euler characteristic method (see \cite{17, 16}) and Rice method (see \cite{4, 7, 8}).

The first method was introduced by Pickands \cite{13} for stationary Gaussian process and later was extended to non-stationary Gaussian one and also non-Gaussian one by Piterbarg \cite{15}. However, the Pickands constant is difficult to estimate and therefore is known explicitly for only 2 cases.
The second one was provided by Adler and Taylor [1], it concerns differentiable processes. It is an important tool to study the geometry of random surfaces. By an heuristic argument, the main idea of this method is as follows. When the level \( u \) is large enough, if the excursion set \( C_u \) is non-empty, then it would be a simply-connected domain. Therefore its Euler characteristic, denoted by \( \mu_0(C_u) \), is equal to 1. Since \( \mu_0(C_u) \) only takes values 0 and 1, its expectation could be used as an approximation to the excursion probability. Adler and Taylor gave that

\[
E(\mu_0(C_u)) = \sum_{i=0}^{d} \rho_i \mu_i(S),
\]

where \( \rho_i \)'s are the Euler characteristic densities defined as

\[
\rho_0 = \Phi(u) = \int_{\mathbb{R}} e^{-x^2/2} dx,
\]

\[
\rho_i = (2\pi)^{-i+1/2} H_{i-1}(u) e^{-u^2/2} = (2\pi)^{-i/2} H_{i-1}(u) \varphi(u), \forall i > 0,
\]

with \( H_j(x) = (-1)^j e^{x^2/2} \frac{d^j}{dx^j} e^{-x^2} \) is the Hermite polynomial of degree \( j \); and \( \mu_i(S) \)'s are the Minkowski functionals (or the Killing-Lipschitz curvatures) of \( S \) (see [1]). Note that \( \mu_0(S) \) is the Euler characteristic of \( S \), for example, it is equal to number of connected components minus the number of holes inside when \( d = 2 \); and \( \mu_d(S) \) is equal to \( \lambda_d(S) \), the volume of \( S \). The Rice method based on local maxima leads to the same approximation. It gives also an upper bound. The first proof of validity is due to Piterbarg [14]. The expectation given in (2) is proved to be a very accurate approximation when the domain \( S \) is "nice" in the sense that it is a tamed and locally convex subset of \( \mathbb{R}^d \) (see [4, Theorem 14.3.3]). Note that in the case both apply, the Euler Characteristic method gives an extra term with respect to the double sum method is thus more accurate, see Azais and Mourareau [3].

In this paper, we are interested in the case \( n \geq 2 \). The motivation of this problem comes from the statistical applications in neurology, for example, to determine whether the functional organization of the brain for language differs according to sex (see [19]). In this application, \( X_i(t) \) is the value of image \( i \) at the location \( t \in \mathbb{R}^d \) representing the intensity with respect to some actions. Here both the double-sum method and Euler characteristic method are still useful.

By the double-sum method, Debicki et al. [10, 11] considered the one-dimensional processes and proved that

\[
P \left( \sup_{t \in [0,T]} \min_{1 \leq i \leq n} X_i(t) > u \right) = H_{n,2}Tu\Phi(u)(1 + o(1)),
\]

where \( H_{n,2} \) is so-called the generalized Pickands constant defined as

\[
H_{n,2} = \lim_{n \to \infty} \frac{1}{n} \max_{k \geq 1} \mathbb{P}(Z(ak) \leq 0),
\]

with

\[
Z(t) = \min_{1 \leq i \leq n} \left( \sqrt{2} Y_i(t) - t^2 + E_i \right),
\]

here \( Y_i \)'s are independent copies of a centered Gaussian process \( Y(t) \) with covariance function \( \text{Cov}(Y(t), Y(s)) = |ts|, \forall t, s \geq 0 \), and \( E_i \)'s are mutually independent unit mean exponential random variables being further independent of \( Y_i \)'s.

They also extended to non-stationary processes and mentioned that the result can be extended to the Gaussian fields but it requires more heavy notations. Note that their works are applied for a wide class of processes but here we just state the result for smooth stationary ones.
By Euler characteristic method, Worsley and Friston [19] considered the upper-triangular Toeplitz matrix \( R \) defined as

\[
R = \begin{pmatrix}
\rho_0/b_0 & \rho_1/b_1 & \cdots & \rho_d/b_d \\
0 & \rho_0/b_0 & \cdots & \rho_{d-1}/b_{d-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \rho_0/b_0
\end{pmatrix},
\]

(3)

where \( b_i = \Gamma((i + 1)/2)/\Gamma(1/2) \) with gamma function \( \Gamma(.) \), and \( \rho_i \)'s are the Euler characteristic densities as defined above, and gave a heuristic argument that

\[
P(C_u \neq \emptyset) = P \left( \sup \min_{1 \leq i \leq t \leq n} X_i(t) \geq u \right) \approx E(\mu_0(C_u)) = (1, 0, \ldots, 0) R^n \mu(S),
\]

(4)

where \( \mu(S) = (\mu_0(S)b_0, \mu_1(S)b_1, \ldots, \mu_d(S)b_d) \) is the column vector of the scale of the Minkowski functionals of \( S \). However, to prove that \( E(\mu_0(C_u)) \) is a good approximation is still an open question.

For further discussion, see also [2, 3, 18].

Let us consider the simplest case that is conjunction probability of the processes (i.e. \( d=1 \)). Then the matrix \( R \) defined in (3) becomes

\[
R = \begin{pmatrix}
\rho_0 & \rho_1/b_1 \\
0 & \rho_0
\end{pmatrix},
\]

and it is clear that

\[
R^n = \begin{pmatrix}
\rho_0^n & n\rho_0^{n-1}\rho_1/b_1 \\
0 & \rho_0^n
\end{pmatrix}.
\]

Also note that in this case the domain \( S \) is the interval \([0, T]\) with \( \mu_0([0, T]) = 1 \) and \( \mu_1([0, T]) = T \). Therefore, if one can prove the validity of the Euler characteristic, then from (4),

\[
P \left( \sup \min_{1 \leq i \leq t \leq n} X_i(t) \geq u \right) \approx \rho_0^n + n\rho_0^{n-1}\rho_1 T = \Phi(u) + n\Phi^{n-1}(u)\varphi(u)T/\sqrt{2\pi}.
\]

So Euler characteristic method could give an asymptotic with more terms than the one given by the double-sum method. And since

\[
\Phi(u) = \varphi(u) \left( \frac{1}{u} + o \left( \frac{1}{u} \right) \right),
\]

it could imply that

\[
H_{n,2} = \frac{n}{\sqrt{2\pi}}.
\]

In a recent paper [12], the equality in (5) has been proved to be true. Here, the author exploited the one-dimensional structure of the processes, and used Rice formula to calculate the expected number of "up-crossing" that is the number of time \( t \) such that at that time the value of a process goes up through the threshold value \( u \) while the values of the other ones are both greater than \( u \). However, this idea seems hard to extend in higher dimensions.

In this paper, we consider the conjunction problem from another point of view. Our approach relies on a recent result of Azais and Wschebor [4] describing the size of the excursion set and also the connection between the volume and the capacity of the index set with the tail distribution of the maximum. In [4], this idea has been used to provide the asymptotic formula of the tail of the maximum corresponding to the coefficients of the volume of the neighborhood of non-locally convex index set. With the same spirit, our general result, that is Theorem [11] gives an asymptotic formula for the conjunction probability with one term where the coefficient comes from the local geometry (or local volume) of the conjunction set.
The statement and the proof of the main theorem is presented in Section 2. In Section 3, we examine the main theorem in three special cases: case $n = 2$, case $d = 1$, and case $d = 2$, $n = 3$, that provides the explicit value of the leading coefficient. In case $d = 1$, we give a new proof to the equality (5). In two other cases, our results agree with the heuristic leading coefficients given by Euler characteristic method. Therefore, in some sense, we partially confirm the validity of Euler characteristic method. However, to derive a full expansion as described in (4) is still an open question.

Throughout this paper, we will use the following assumption and notations.

**Assumption A**: Assume $X$ be a random field defined on a ball $B \subset \mathbb{R}^d$ containing the domain $S$ such that $X$ satisfies:

i. The index set $S$ is compact and is the closure of its interior.

ii. $X$ is a stationary centered Gaussian field.

iii. Almost surely the paths of $X(t)$ are of class $C^3$.

iv. For all $s \neq t \in B$, the distribution of $(X(s), X(t), X'(s), X'(t))$ does not degenerate.

v. For all $t \in B$ and $\gamma$ in the unit sphere $S^{d-1}$, the distribution of $(X(t), X'(t), X''(t)\gamma)$ does not degenerate.

**Notations:**

- $\lambda_k(.)$ stands for the usual $k$-dimensional Lebesgue measure.
- $B(t, r)$ stands for the ball of radius $r$ at center $t$.
- For a $n$-dimensional vector $m = (m_1, \ldots, m_n)$, the $\ell_1$ norm of $m$, denoted by $\|m\|$, is defined as $\|m\| = |m_1| + |m_2| + \ldots + |m_n|$. 
- For a $n$-dimensional vector $m = (m_1, \ldots, m_n)$ and a $n$-tuple of non-negative integers $r = (r_1, \ldots, r_n)$, the notations $m^r$ stands for $M^r = m_1^{r_1}m_2^{r_2}\ldots m_n^{r_n}$.
- For a given set $S \subset \mathbb{R}^d$ and a positive constant $\epsilon$, the $\epsilon$- neighborhood of $S$, denoted by $S^{+\epsilon}$, is defined as $S^{+\epsilon} = \{t \in \mathbb{R}^d : \operatorname{dist}(t, S) \leq \epsilon\}$.
- For a given set $S \subset \mathbb{R}^d$ and a small enough positive constant $\epsilon$, the set $S^{-\epsilon}$, is defined as $S^{-\epsilon} = \{t \in \mathbb{R}^d : B(t, \epsilon) \subset S\}$.

## 2 Main theorem

The general result in this paper is stated as follows.

**Theorem 1.** Let $X_i(t), 1 \leq i \leq n$, be the independent copies of a Gaussian field $X$ satisfying Assumption (A). Assume that there exist the constants $k > 0$ and $C_m$ such that for $r_1, \ldots, r_n$ small enough,

$$
\lambda_{nd} \left( \{t_1, \ldots, t_n\} : t_i \in S^{+r_i}, \forall i \text{ and } \bigcap_{1 \leq i \leq n} B(t_i, r_i) \neq \emptyset \right) = \lambda_{d}(S) \sum_{\|m\|=k} C_{m}r_{m}^{n} + O(\sum_{\|m\|=k+1} r_{m}),
$$

(6)
and
\[
\lambda_n \left( \{t_1, \ldots, t_n \} : t_i \in S^{-(r_1 + \ldots + r_n)}, \forall i \text{ and } \bigcap_{1 \leq i \leq n} B(t_i, r_i) \neq \emptyset \right) = \lambda_n(S) \sum_{\|m\|=k} C_m r^m + o(\sum_{\|m\|=k+1} r^m).
\]

Then as \( u \) tends to infinity,
\[
P \left( \max_{i \leq n} \min_{t \in S} X_i(t) \geq u \right) = \left( \frac{\gamma u^\alpha}{(2\pi)^{n/2}} \sum_{\|m\|=k} C_m \prod_{i=1}^n \Gamma(1 + m_i/2) + o(1) \right). \tag{7}
\]

In the proof of the main theorem, we need the following lemma due to Azais and Wschebor [9].

**Lemma 1.** Let \( X \) be a random field satisfying Assumption (A) and \( \alpha \) be a given real number such that \( 0 < \alpha < 1 \). Then there exist two constants \( C, c > 1 \) such that the probabilities of the following events are at most equal to \( Ce^{-cu^{2/2}} \):

\[
A_1 = \{ \exists \text{ a local maximum in } B \text{ with value } \geq u + 1 \},
\]

\[
A_2 = \left\{ \exists \text{ two or more local maxima in } \hat{B} \text{ with value } \geq u \right\},
\]

\[
A_3 = \left\{ \exists \text{ a local maximum } t \in \hat{B} \text{ such that } u < X(t) < u + 1 \text{ and } \right.

\[
\min \left\{ \gamma^T X''(s) : s \in B(t, u^{-\beta}), \gamma = \frac{s - t}{\|s - t\|} \right\} \leq -X(t) - u^\alpha \},
\]

\[
A_4 = \left\{ \exists \text{ a local maximum } t \in \hat{B} \text{ such that } u < X(t) < u + 1 \text{ and } \right.

\[
\max \left\{ \gamma^T X''(s) : s \in B(t, u^{-\beta}), \gamma = \frac{s - t}{\|s - t\|} \right\} \geq -X(t) + u^\alpha \},
\]

where \( \beta \) is a positive constant in \((0, 1)\) such that \( \beta > (1 - \alpha)/2 \).

Now we are able to prove the main theorem.

**Proof of Theorem 1.** Thanks to Lemma 1 as in [9], one knows that for each \( i = 1, \ldots, n \), with high probability, there exists only one local maximum of \( X_i(t) \) at the location \( t_i \in \hat{B} = \hat{B}_i \) with value at least \( u \), then the corresponding excursion set

\[
K_{n,i} := \{ s \in B_i : X_i(s) \geq u \}
\]

consists of only one connected component, and moreover,

\[
B(t, r_i) \subset K_{n,i} \subset B(t, r_i),
\]

where \( r_i = \sqrt{\frac{2}{\hat{X}_i(t_i) + u^\alpha}} \) and \( r_i = \sqrt{\frac{2}{\hat{X}_i(t_i) - u^\alpha}} \).

Moreover, if for some \( k \in \{1, \ldots, n\} \), at least one event of \( A_{1k}, A_{2k}, A_{3k}, \text{ and } A_{4k} \) (in Lemma 1) occurs, then

\[
P \left( A_{1k} \cup A_{2k} \cup A_{3k} \cup A_{4k} \cap \left\{ \sup_{t \in S} \min_{1 \leq i \leq n} X_i(t) \geq u \right\} \right) \leq P \left( A_{1k} \cup A_{2k} \cup A_{3k} \cup A_{4k} \right) P \left( \sup_{t \in S} \min_{1 \leq i \leq n, i \neq k} X_i(t) \geq u \right) \leq P \left( A_{1k} \cup A_{2k} \cup A_{3k} \cup A_{4k} \right) \prod_{1 \leq i \leq n, i \neq k} P(\max_{t \in S} X_i(t) \geq u) \leq (\text{const}) e^{-cu^{2/2}} \left( \lambda_n(S) u^{d-1} \varphi(u) \right)^{n-1} = o(u^{nd-n-k} \varphi^u(u)).
\]
Here we use the fact that (see [13])
\[ P \left( \max_{t \in S} X_i(t) \geq u \right) \leq (\text{const}) \lambda_d(S) u^{d-1} \varphi(u). \]

Therefore, from the fact that
\[ P \left( \sup_{t \in S^1 \leq t \leq n} X_i(t) \geq u \right) = P(\exists t \in S : t \in K_u, \forall i = 1, \ldots, n) \]
\[ = P(S \cap K_{u,1} \cap \ldots \cap K_{u,n} \neq \emptyset), \]
we obtain the upper bound
\[ P \left( \sup_{t \in S^1 \leq t \leq n} X_i(t) \geq u \right) \leq P(S \cap B(t_1, \overline{r}) \cap \ldots \cap B(t_n, \overline{r}) \neq \emptyset) + o(u^{d-n-k} \varphi^n(u)), \]
and the lower bound
\[ P \left( \sup_{t \in S^1 \leq t \leq n} X_i(t) \geq u \right) \geq P(S \cap B(t_1, \overline{r}) \cap \ldots \cap B(t_n, \overline{r}) \neq \emptyset) + o(u^{d-n-k} \varphi^n(u)). \]

- At first, deal with the upper bound. By Markov inequality, it is at most equal to
\[ P(\exists t = (t_1, \ldots, t_n) \in B^\otimes n : \forall i = 1, \ldots, n, X_i(t) \text{ has a local maximum at } t_i, \]
\[ X_i(t_i) \in [u, u + 1], t_i \in S^{+\overline{r}} \text{ and } \bigcap_{1 \leq i \leq n} B(t_i, \overline{r}) \neq \emptyset) + o(u^{d-n-k} \varphi^n(u)) \]
\[ \leq E(\text{card} \{ t = (t_1, \ldots, t_n) \in B^\otimes n : \forall i = 1, \ldots, n, X_i(t) \text{ has a local maximum at } t_i, \]
\[ X_i(t_i) \in [u, u + 1], t_i \in S^{+\overline{r}} \text{ and } \bigcap_{1 \leq i \leq n} B(t_i, \overline{r}) \neq \emptyset) + o(u^{d-n-k} \varphi^n(u)), \]
where \( B^\otimes n \) stands for the Cartesian product set \( B \times \ldots \times B. \)

By Rice formula applied to the vector-valued Gaussian field \( Z(t) = (X'_1(t_1), \ldots, X'_n(t_n)) \) with \( t = (t_1, \ldots, t_n) \in \mathbb{R}^n, \) the above expectation is equal to
\[ E = \int_{[u, u+1]^n} du_1 \ldots du_n \int_{B^\otimes n} dt \, p_{X_1(t_1), \ldots, X_n(t_n), X'_1(t_1), \ldots, X'_n(t_n)}(u_1, \ldots, u_n, 0, \ldots, 0) \]
\[ \times \mathbb{E} \left( \prod_{i=1}^n \det \left( X''_i(t_i) \right) \mathbb{I}_{X''_i(t_i) \leq 0} \mathbb{I}_{X_i(t_i) = u_i, X'_i(t_i) = 0} \bigg| X_i(t_i) = u_i \right), \]
where \( p_{X_1(t_1), \ldots, X_n(t_n), X'_1(t_1), \ldots, X'_n(t_n)}(\cdot) \) is the joint density function of the random vector \( (X_1(t_1), \ldots, X_n(t_n), X'_1(t_1), \ldots, X'_n(t_n)). \)

Using (4) and the fact that the fields \( X_i's \) are independent and \( X'_i(t_i) \) is independent to \( X_i(t_i) \) and \( X''_i(t_i), \) we have
\[ E = \frac{\lambda_d(S)}{(2\pi)^{nd/2}} \int_{[u, u+1]^n} \prod_{i=1}^n \mathbb{E} \left( \left| \det \left( X''_i(t_i) \right) \mathbb{I}_{X''_i(t_i) \leq 0} \right| \bigg| X_i(t_i) = u_i \right) \varphi(u_i) \]
\[ \times \left( \sum_{\|m\|=k} C_m \mathbb{I}_{m} + O \left( \sum_{\|m\|=k+1} \mathbb{I}_{m} \right) \right) du_1 \ldots du_n. \]  

Note that under the condition \( X_i(t_i) = u_i \) then \( \overline{r} \) is no more random and is equal to
\[ \overline{r} = \sqrt{\frac{u_i u_i - \overline{r}}{u_i - u_i^d}}. \]

Using the fact that (see [4])
\[ E \left( \left| \det(X'_i(t)) \mathbb{I}_{X''_i(t) \leq 0} \right| \bigg| X_i(t) = u_i, X'_i(t) = 0 \right) = u_i^d + O \left( u_i^{d-2} \right) \text{ as } u_i \to \infty, \]
Then
\[ \int_u^{u+1} \mathcal{P}^{m_2} E \left( \left| \det \left( X'_i(t_i) \right) \right| \big| X_i(t_i) = u, X'_i(t_i) = 0 \right) \varphi(u) du \]
\[ \simeq \int_u^{u+1} u^d \left( 2 \frac{u_i - u}{u_i - u^n} \right)^{m_2/2} \varphi(u) du. \]
By the change of variable \( u_i = u + x/u \), the above integral is equal to
\[ \int_u^{u+1} (u + x/u)^d \left( \frac{2x/u}{u + x/u - u^n} \right)^{m_2/2} \varphi(u + x/u) dx/u \]
\[ \simeq 2^{m_2/2} u^d \varphi(u) \int_0^m x^{m_2/2} e^{-x} dx \simeq 2^{m_2/2} u^d (\varphi(u) \Gamma(1 + m_2/2)). \]
Therefore, for each vector \( m = (m_1, \ldots, m_n) \) with norm \( k \),
\[ \int_{[u,u+1]^{\otimes n}} \prod_{i=1}^n E \left( \left| \det \left( X'_i(t_i) \right) \right| \big| X_i(t_i) = u_i \right) \varphi(u_i) \mathcal{P}^m du_1 \ldots du_n \]
\[ = \int_{[u,u+1]^{\otimes n}} \prod_{i=1}^n E \left( \left| \det \left( X'_i(t_i) \right) \right| \big| X_i(t_i) = u_i \right) \varphi(u_i) du_i \]
\[ \simeq \prod_{i=1}^n 2^{m_i/2} u_i^{d - (m_i + 1)} \varphi(u) \Gamma(1 + m_i/2) \]
\[ = 2^{k/2} u^{d - n - k} \varphi^*(u) \prod_{i=1}^n \Gamma(1 + j_i/2). \]
Hence, substituting into \( S \),
\[ E = u^{n - d - n - k} \varphi^*(u) \left( \frac{2^{k/2} \lambda_d(S)}{(2\pi)^{d/2}} \sum_{|m| = k} C_m \prod_{i=1}^n \Gamma(1 + m_i/2) + o(1) \right). \]
- For the lower bound, let us denote
\[ S^{-r_1 + \ldots - r_n} = \{ t \in S : B(t, r_1 + \ldots + r_n) \subset S \}. \]
Then the lower bound is at least equal to
\[ P \left( \exists t \in B^{\otimes n} : t_i \in S^{-r_1 + \ldots + r_n} \forall i, X_i(t) \text{ has a local maximum at } t_i, X_i(t_i) \in [u, u + 1], \right. \]
\[ \text{and } \bigcap_{1 \leq i \leq n} B(t_i, r_i) \neq \emptyset \bigg) + o(u^{n-d-n-k} \varphi^*(u)) \]
\[ = P(M_n \geq 1) + o(u^{n-d-n-k} \varphi^*(u)) \]
\[ \geq E(M_n) - E(M_n(M_n - 1))/2 + o(u^{n-d-n-k} \varphi^*(u)) \]
where
\[ M_n = \text{card} \left\{ t \in B^{\otimes n} : t_i \in S^{-r_1 + \ldots + r_n}, \forall i, \bigcap_{1 \leq i \leq n} B(t_i, r_i) \neq \emptyset, \text{ and } X_i(t) \text{ has a local maximum at } t_i, X_i(t_i) \in [u, u + 1] \right\}. \]
It is clear that
\[ M_n \leq M = \text{card} \{ t = (t_1, \ldots, t_n) \in B^{\otimes n} : X_i(t) \text{ has a local maximum at } t_i, X_i(t_i) \geq u, \forall i \}. \]
Then applying Rice formula and using the independent property of the given fields, we have
\[
E(M_\bullet(M, M-1)) \leq E(M(M-1))
\]
\[
= \int_{[u, \infty)^n} dydz \int_{B^n \times B^n} dt ds \times E \left( \prod_{i=1}^{n} \det \left( X_i''(t_i) \right) I_{\{X''_{i}(t_i) \geq 0\}} \mid X_i(t_i) = y_i, X_i'(t_i) = 0, X_i(s_i) = z_i, X_i'(s_i) = 0, \forall t \right) \times pX_1(t_1), \ldots, X_n(t_n), X_1'(t_1), \ldots, X_n'(t_n), X_1(s_1), \ldots, X_n(s_n), X_1'(s_1), \ldots, X_n'(s_n) \{y_1, \ldots, y_n, 0, \ldots, 0, z_1, \ldots, z_n, 0, \ldots, 0\} = \prod_{i=1}^{n} E(M_i(M_i-1)),
\]
where
\[
M_i = \text{card} \left\{ t_i \in B_i : X_i(.) \text{ has a local maximum at } t_i, X(t) \geq u \right\}.
\]
In [3], it is true that there exist two constants \( C, c > 1 \) such that
\[
E(M_i(M_i-1)) \leq C e^{-cu^2/2}.
\]
Hence we have
\[
E(M(M-1)) = o(u^{nd-n-k} \varphi^n(u)).
\]
The calculation of the expectation \( E(M_\bullet) \) is similar as in the upper bound part and we obtain the same asymptotic formula. Then the result follows. \( \square \)

3 Applications

In this section, we provide some interesting examples where we can verify the conditions [9] and [7], to illustrate the main theorem. Since the structure of the \( n \)-tuple \((t_1, \ldots, t_n)\) satisfying the condition
\[
\bigcap_{1 \leq i \leq n} B(t_i, r_i) \neq \emptyset
\]
is local as the \( r_i \)'s are small enough and we observe that
\[
\lambda_d(t_1 : t_1 \in S^{+r_1}) = \lambda_d(S) + O(r_1),
\]
\[
\lambda_d(t_1 : t_1 \in S^{-(r_1+\ldots+r_n)}) = \lambda_d(S) + O(r_1 + \ldots + r_n),
\]
the conditions [9] and [7] are met as soon as we can check that for a fixed point \( t_1 \) and a small enough fixed radius \( r_1 \), there exist the constants \( k > 0 \) and \( C_m \) such that
\[
\lambda_{(n-1)d} \left( (t_2, \ldots, t_n) : \bigcap_{1 \leq i \leq n} B(t_i, r_i) \neq \emptyset \right) = \sum_{\|m\|=k} C_m r^m. \tag{9}
\]

3.1 Case \( n = 2 \)

The very first and most intuitive example is devoted to the case \( n = 2 \) corresponding to the practical application mentioned in the introduction. It is clear that
\[
\{ t_2 : B(t_1, r_1) \cap B(t_2, r_2) \neq \emptyset \} = B(t_1, r_1 + r_2).
\]

Therefore
\[
\lambda_d (t_2 : B(t_1, r_1) \cap B(t_2, r_2) \neq \emptyset) = \lambda_d (B(t_1, r_1 + r_2))
\]
\[
= \frac{\pi^{d/2}}{\Gamma(1 + d/2)} (r_1 + r_2)^d = \frac{\pi^{d/2}}{\Gamma(1 + d/2)} \sum_{j=0}^{d} \binom{d}{j} r_1^{d-j} r_2^j,
\]
here we use the fact that the volume of a \(d\)-dimensional unit ball is \(\pi^{d/2}/\Gamma(1 + d/2)\).

Then we have an immediate consequence of the main theorem as follows.

**Corollary 1.** Let \(S\) be a compact set satisfying Assumption(A). Consider \(X_i(t), 1 \leq i \leq 2\), being two independent copies of a Gaussian field \(X\) satisfying Assumption (A). Then as \(u\) tends to infinity,

\[
P \left( \max_{t \in S} \min(X_1(t), X_2(t)) \right) \geq u \right) = \frac{u^{d-2} \varphi^2(u) \lambda_d(S)}{(2\pi)^{d/2} \Gamma(1 + d/2)} \left( \sum_{j=0}^{d} \binom{d}{j} \frac{\Gamma(1 + j/2) \Gamma(1 + (d - j)/2)}{\Gamma(1 + j/2) \Gamma(1 + (d - j)/2)} + o(1) \right).
\]

(10)

**Proof.** We substitute the following parameters in the statement of the main theorem

\[
k = d, \quad m = (j, d - j), \quad C_m = \frac{\pi^{d/2} (d)}{\Gamma(1 + d/2)}.
\]

\(\square\)

**Remark.** Let us now consider the estimation given by Euler characteristic method. It is clear that (1) becomes

\[
(1, 0, \ldots, 0) R^2 \mu(S).
\]

Here the term corresponding to \(\mu_d(S)\) (or \(\lambda_d(S)\)) is

\[
\lambda_d(S) b_d \sum_{i=0}^{d} \frac{\rho_i}{b_i} \frac{\rho_{d-i}}{b_{d-i}}.
\]

From the definition of the Euler characteristic densities \(\rho_i\)'s, this term is equivalent to

\[
\frac{\lambda_d(S) u^{d-2} \varphi^2(u)}{(2\pi)^{d/2} \Gamma(1 + d/2)} \sum_{i=0}^{d} \frac{1}{\Gamma((i + 1)/2) \Gamma((d - i + 1)/2)} \Gamma((d + 1)/2) \Gamma(1/2) \sum_{i=0}^{d} \frac{\Gamma(1 + i/2) \Gamma(1 + (d - i)/2)}{\Gamma(1 + d/2)}.
\]

Comparing with the asymptotic formula given in (10), it is surprising to see that

\[
\frac{\Gamma((d + 1)/2) \Gamma(1/2)}{\Gamma((i + 1)/2) \Gamma((d - i + 1)/2)} = \frac{1}{\Gamma(1 + d/2)} \sum_{i=0}^{d} \frac{\Gamma(1 + i/2) \Gamma(1 + (d - i)/2)}{\Gamma(1 + d/2)}.
\]

(11)

Indeed, we will prove that for every \(i = 0, \ldots, d,\)

\[
\frac{\Gamma((d + 1)/2) \Gamma(1/2)}{\Gamma((i + 1)/2) \Gamma((d - i + 1)/2)} = \frac{1}{\Gamma(1 + d/2)} \binom{d}{i} \Gamma(1 + i/2) \Gamma(1 + (d - i)/2).
\]

We consider the case when \(d = 2p + 1\) and \(i = 2k\), the other cases are similar. In this case, the equality (11) is equivalent to

\[
\frac{\Gamma(p + 1) \Gamma(1/2)}{\Gamma(k + 1/2) \Gamma(p + 1 - k)} = \frac{1}{\Gamma(p + 1 + 1/2)} \binom{2p + 1}{2k} \Gamma(1 + k) \Gamma(p - k + 1 + 1/2).
\]

From the fact that for every positive integer \(k\)

\[
\Gamma(k) = (k-1)! \quad \text{and} \quad \Gamma \left( k + \frac{1}{2} \right) = \frac{(2k)! \sqrt{\pi}}{4^k k!},
\]

it is easy to check that (11) holds true.
3.2 Case $d = 1$

In this subsection, we would like to revisit the conjunction probability of stationary centered Gaussian processes. Although that the corresponding result given in [12] is more powerful and more informative than an asymptotic formula as in the statement of the main theorem, it would be nice to see whether our proposing method is enough to prove that

$$H_{n,2} = \frac{n}{\sqrt{2\pi}}$$

The affirmative answer is deduced by the following lemma.

**Lemma 2.** For a fixed point $t_1$ on the real axis and a small enough fixed radius $r_1$, we have

$$\lambda_{n-1} \left( (t_2, \ldots, t_n) \in \mathbb{R}^{n-1} : \bigcap_{1 \leq i \leq n} B(t_i, r_i) \neq \emptyset \right) = 2^{n-1} \sum_{i=1}^{n} \left( \prod_{j \neq i} r_j \right).$$

(12)

**Proof.** We will prove by induction on $n$.

- For $n = 2$, it is obvious as in the above subsection.
- Assume that the statement is true from 2 to $n - 1$.
- For $n$-tuple $(t_1, t_2, \ldots, t_n)$, we would like to calculate the volume as the following integral

$$\int_{\mathbb{R}^{n-1}} I_1(\bigcap_{1 \leq i \leq n} B(t_i, r_i) \neq \emptyset) dt_2 \ldots dt_n = \int_{\mathbb{R}^{n-2}} \int_1 \int_{\bigcap_{1 \leq i \leq n-1} B(t_i, r_i) \neq \emptyset} dt_2 \ldots dt_{n-1}.$$ (13)

Again by induction, it is clear that if the intersection $\bigcap_{1 \leq i \leq n-1} B(t_i, r_i)$ is non-empty, it is an interval. Therefore

$$\int_{\mathbb{R}^{n-1}} \int_1 \int_{\bigcap_{1 \leq i \leq n-1} B(t_i, r_i) \neq \emptyset} dt_2 \ldots dt_{n-1} = \lambda_1 \left( \bigcap_{1 \leq i \leq n-1} B(t_i, r_i) \right) + 2r_n,$$

and substitute in (13), the considering volume is equal to

$$\int_{\mathbb{R}^{n-2}} \int_1 \int_{\bigcap_{1 \leq i \leq n-1} B(t_i, r_i) \neq \emptyset} \left( \lambda_1 \left( \bigcap_{1 \leq i \leq n-1} B(t_i, r_i) \right) + 2r_n \right) dt_2 \ldots dt_{n-1}. (14)$$

By inductive hypothesis,

$$2r_n \int_{\mathbb{R}^{n-2}} \int_1 \int_{\bigcap_{1 \leq i \leq n-1} B(t_i, r_i) \neq \emptyset} dt_2 \ldots dt_{n-1} = 2r_n 2^{n-2} \sum_{i=1}^{n-1} \left( \prod_{j \neq i} r_j \right).$$

For the rest term in the integral (14), let us introduce a new variable $y$ corresponding to the point in the intersection, and we have

$$\int_{\mathbb{R}^{n-2}} \int_1 \int_{\bigcap_{1 \leq i \leq n-1} B(t_i, r_i) \neq \emptyset} \lambda_1 \left( \bigcap_{1 \leq i \leq n-1} B(t_i, r_i) \right) dt_2 \ldots dt_{n-1}$$

$$= \int_{\mathbb{R}^{n-2}} \int_1 \int_{\bigcap_{1 \leq i \leq n-1} B(t_i, r_i) \neq \emptyset} \lambda_1(y) dt_2 \ldots dt_{n-1} dy$$

$$= \int_{B(t_1, r_1)} \int_{\mathbb{R}^{n-2}} \int_{\bigcap_{1 \leq i \leq n-1} B(t_i, r_i) \neq \emptyset} dt_2 \ldots dt_{n-1} dy$$

$$= \int_{B(t_1, r_1)} \int_{B(y, r_2)} \int_{\mathbb{R}^{n-2}} \int_{B(y, r_{n-1})} dt_{n-1}$$

$$= \int_{B(t_1, r_1)} \left( \prod_{i=2}^{n-1} (2r_i) \right) dy = 2^{n-1} \prod_{i=1}^{n-1} r_i$$

where the equality in the third line follows from Fubini theorem.

Then the proof follows easily. \(\square\)
Applying the main theorem in this case with respect to \( k = n - 1 \), \( m \) is \( n \)-dimensional vector with \( n - 1 \) unit entries and only one zero entry and \( C_m = 2^{n-1} \), we have

**Corollary 2.** Let \( X_i(t), 1 \leq i \leq n \), be the independent copies of a Gaussian process \( X \) satisfying Assumption (A). Then as \( u \) tends to infinity,

\[
P \left( \max_{t \in [0,T]} \min_{1 \leq i \leq n} X_i(t) \geq u \right) = u^{-(n-1)} \varphi^n \left( \frac{nT}{\sqrt{2\pi}} + o(1) \right).
\]

### 3.3 Case \( d = 2, n = 3 \)

**Lemma 3.** For a fixed point \( t_1 \) in the plane and a small enough fixed radius \( r_1 \), we have

\[
\lambda_4 \left( (t_2, t_3) \in \mathbb{R}^3 : \cap_{1 \leq i \leq 3} B(t_i, r_i) \neq \emptyset \right) = \pi r_1^2 + r_2 \text{peri}(B(t_2, r_2) \cap B(t_1, r_1)) + \lambda_2 (B(t_2, r_2) \cap B(t_1, r_1)).
\]

**Proof.** We have

\[
\lambda_4 \left( (t_2, t_3) \in \mathbb{R}^3 : \cap_{1 \leq i \leq 3} B(t_i, r_i) \neq \emptyset \right) = \int_{\mathbb{R}^2} dt_2 \left[ \int_{B(t_2, r_2) \cap B(t_1, r_1) \neq \emptyset} \int_{B(t_3, r_3) \cap B(t_1, r_1) \neq \emptyset} dt_3 \right] = \int_{\mathbb{R}^2} dt_2 \left[ \int_{(B(t_2, r_2) \cap B(t_1, r_1) \neq \emptyset)} \lambda_2 \left( (B(t_2, r_2) \cap B(t_1, r_1) \neq \emptyset) \right) \right].
\]

Since the intersection \( (B(t_2, r_2) \cap B(t_1, r_1)) \) is a convex set then from the Steiner formula,

\[
\lambda_2 ((B(t_2, r_2) \cap B(t_1, r_1)) \neq \emptyset) = \pi r_3^2 + r_3 \text{peri}(B(t_2, r_2) \cap B(t_1, r_1)) + \lambda_2 (B(t_2, r_2) \cap B(t_1, r_1)),
\]

where \( \text{peri}(\cdot) \) stands for the perimeter of the set.

Therefore, the considering volume is equal to

\[
\int_{B(t_1, r_1 + r_2)} \left[ \pi r_2^2 + r_3 \text{peri}(B(t_2, r_2) \cap B(t_1, r_1)) + \lambda_2 (B(t_2, r_2) \cap B(t_1, r_1)) \right] dt_2. \tag{16}
\]

- For the first term in (16), it is clear that

\[
\pi r_3^2 \int_{B(t_1, r_1 + r_2)} dt_2 = r_3^2 \pi (r_1 + r_2)^2 = \pi^2 (r_1^2 r_2^2 + r_2^2 r_3^2) + 2\pi^2 r_1 r_2 r_3^2.
\]

- For the third term in (16), we introduce a new variable \( y \) corresponding to the point in the intersection, and we use the Fubini theorem to obtain that

\[
\int_{B(t_1, r_1 + r_2)} \lambda_2 (B(t_2, r_2) \cap B(t_1, r_1)) dt_2 = \int_{B(t_1, r_1 + r_2)} \left[ \int_{B(t_2, r_2) \cap B(t_1, r_1)} dy \right] dt_2 = \int_{B(t_1, r_1)} \left[ \int_{B(t_2, r_2)} \right. dy = \pi r_2^2 \int_{B(t_1, r_1)} dy = \pi r_2^2 r_1^2.
\]

- For the second in (16), let us denote \( S(t, r) \) the circle with radius \( r \) at center point \( t \), i.e. the boundary of the disk \( B(t_1, r_1) \). It is clear that the perimeter of the intersection of two disks is the sum of the length of the arc on the circle \( S(t_1, r_1) \) and the length of the one on \( S(t_2, r_2) \). For the first kind with respect to the arc on \( S(t_1, r_1) \), we have

\[
\int_{S(t_1, r_1)} \left[ \int_{B(t_1, r_1 + r_2)} \right. dt_2 \left. \left[ \int_{S(t, r)} \right] \right. dy = \int_{S(t_1, r_1)} \left. \int_{B(t, r_2)} dt_2 \right] dy = \pi r_2^2 \text{int}_{S(t_1, r_1)} = 2\pi^2 r_1 r_2^2.
\]
For the second kind with respect to the arc on $S(t_2, r_2)$, since the role of two points $t_1$ and $t_2$ are the same, we can fix $t_2$ and let $t_1$ move around $t_2$ in the integral. By the same argument as above, we obtain that

$$\int_{B(t_1, r_1 + r_2)} dt_2 \left[ \int_{S(t_2, r_2)} ||y \in B(t_1, r_1)|| dy \right] = 2\pi^2 r_1^2 r_2.$$ 

Then the result follows by summing up three terms in (16).

From the above lemma, we can apply the main theorem with respect to $k = 4$ and six 3-dimensional vectors $m$ divided into two group

$$\{(2, 2, 0), (2, 0, 2), (0, 2, 2)\} \text{ with } C_m = \pi^2,$$

$$\{(1, 2, 1), (2, 1, 1), (1, 1, 2)\} \text{ with } C_m = 2\pi^2.$$ 

**Corollary 3.** Consider $X_i(t), 1 \leq i \leq 3$, being three independent copies of a Gaussian field $X$ satisfying Assumption (A). Then as $u$ tends to infinity,

$$P \left( \max_{t \in S} \min_{1 \leq i \leq n} X_i(t) \geq u \right) = u^{-1} \varphi^2(u) \left[ \frac{3\lambda_2(S)}{2\pi} \left( 1 + \frac{\pi}{2} \right) + o(1) \right].$$

(17)

It is easy to check that the leading coefficient \( \frac{3\lambda_2(S)}{2\pi} \left( 1 + \frac{\pi}{2} \right) \) coincides with the one given by Euler characteristic method.

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