Focalization and phase models for classical extensions of non-associative Lambek calculus

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Abstract

Lambek’s non-associative syntactic calculus (NL, 18) excels in its resource consciousness: the usual structural rules for weakening, contraction, exchange and even associativity are all dropped. Recently, there have been proposals for conservative extensions dispensing with NL’s intuitionistic bias towards sequents with single conclusions: De Groote and Lamarche’s classical NL (CNL, 10) and Moortgat’s Lambek-Grishin calculus (LG, 21). We demonstrate Andreoli’s focalization property (2) for said proposals: a normalization result for Cut-free sequent derivations identifying to a large extent those differing only by trivial rule permutations. In doing so, we proceed from a ‘uniform’ sequent presentation, deriving CNL from LG through the addition of structural rules. The normalization proof proceeds by the construction of syntactic phase models wherein every ‘truth’ has a focused proof, similar to 23.

1 Introduction

Logics without structural rules were first proposed by Lambek in the late fifties and early sixties. His syntactic calculus from 17 was a logic of strings, making no appeal to weakening, contraction or exchange. Associativity was subsequently dropped in 18, allowing for reasoning with (binary-branching) trees. These days, said calculi are referred to by the associative and non-associative Lambek calculus respectively (L/NL).

Attempts at lifting (N)L’s intuitionistic bias towards sequents with single conclusions first culminated in bilinear logic and cyclic linear logic (11), both conservative extensions of L (hence associative). More recently, two proposals were put forward within the non-associative setting: De Groote and Lamarche’s classical NL (CNL, 10) and Moortgat’s Lambek-Grishin calculus (LG, 21). While the latter extends NL’s logical vocabulary by a coresiduated family of connectives, consisting of a par and coimplications, the former instead takes as primitives the tensor and par, augmented by classical linear negation, with the (co)implications reduced to defined operations.

Proof-theoretic investigations into CNL and LG have thus far concentrated on Cut-free sequent calculi and proof nets (22). The current work contributes a proof of the focalization property. First observed by Andreoli within full linear logic (2), the latter is a normalization result for Cut-free sequent derivations, identifying (to a large extent) those that differ only by trivial rule permutations.
Roughly, a focused derivation is one where invertible (logical) inferences are always applied as soon as possible, and, for every application of a non-invertible inference, the active formulas appearing in its premises are principal.

Our method of proof avoids the (local) rewriting of derivations found in the usual syntactic demonstrations of Cut elimination \([19]\), proceeding instead by model-theoretic means similarly to \([23]\) and \([14]\). More specifically, we propose a phase semantics \([23]\) for \(\text{CNL}\) and \(\text{LG}\), seeking to define syntactic models wherein every ‘truth’ has a focused derivation. The desired result then follows by composition with soundness of unfocused provability. Like in \([23]\), our treatment is uniform in the sense that we demonstrate focalization of both \(\text{CNL}\) and \(\text{LG}\) by a single proof, applying as well to any further extensions by structural rules.

We proceed as follows. \(\S\) 2 recapitulates material on \(\text{CNL}\) and \(\text{LG}\), its modest contribution being a uniform one-sided sequent presentation, expressing their differences by a number of structural rules. The definition of focused derivations is taken up in \(\S\) 3. As an intermediate step, we introduce polarized adaptations of \(\text{CNL}\) and \(\text{LG}\), recording alternations between chains of invertible and non-invertible inferences within the logical vocabulary through Girard’s shift connectives \([12]\). Phase spaces are defined in \(\S\) 4, together with proofs of soundness and completeness. The former result is stated for polarized sequent derivations with Cut, whereas the latter concerns focused provability, thus obtaining normalization by composition. Figure 1 summarizes our results.

![Figure 1: Summary of results: composition of arrows yields focalization.](image)
Definition 2.1. Connectives come in dual pairs: a non-commutative, non-associative multiplicative disjunction (par) accompanies a similarly resource-sensitive multiplicative conjunction (tensor), while direction-sensitive divisions (implications) are complemented by left- and right subtractions (coimplications). Expanding upon [21] and [10], we also take additives into consideration:

\[ A, B ::= p | \bar{p} \]  
(Positive vs. negative atoms)

\[ (A \otimes B) | (B \oplus A) \]  
(Tensor vs. par)

\[ (A / B) | (B \triangleright A) \]  
(Right implication vs. left coimplication)

\[ (B \setminus A) | (A \otimes B) \]  
(Left implication vs. right coimplication)

\[ (A \land B) | (A \lor B) \]  
(Additive conjunction and disjunction)

Note that, for each atomic formula \( p \), we also assume to have at our disposal its negation \( \bar{p} \). While conflicting with Moortgat’s account of LG, we will find the choice of (positive/negative) polarity for atoms to influence the shape of focused proofs found in §3 (cf. Example [13]).

Definition 2.2. Made explicit, the duality present in the above discussion is realized as a classical linear negation (\( ^\bot \)):

\[ p^\bot = \text{def} \bar{p} \]
\[ (A \otimes B)^\bot = \text{def} B^\bot \triangleright A^\bot \]
\[ (A / B)^\bot = \text{def} B^\bot \otimes A^\bot \]
\[ (B \setminus A)^\bot = \text{def} A^\bot \otimes B^\bot \]
\[ (A \land B)^\bot = \text{def} B^\bot \lor A^\bot \]

Indeed, involutivity (\( A^{\bot \bot} = A \)) is easily established. Note that we have not identified \( A/B \) with \( A\otimes B^\bot \), and similarly for the other (co)implications. Instead, said formulas will turn out interderivable in CNL, though not in LG.

2.2 LG sequentialized

Our presentation of LG proceeds in two steps. First, in §2.2.1, we briefly recapitulate the algebraic account, adapted to the extended logical vocabulary. §2.2.2 introduces our sequent calculus and justifies its rules by dual translations into algebraic derivations.

2.2.1 The minimal logic of (co)residuation

Defined algebraically, derivability in LG characterizes inequalities \( A \leq B \). Figure 2 presents an axiomatization: \( \leq \) satisfies the preorder laws (identity) and (transitivity), \( /\lor \triangleright \setminus \) are realized as meets/joins, \( \bot \) act as residuals to \( \otimes \) (r), while, finally, the connectives \( \otimes, \triangleright \) and \( \otimes \) listen to the dual coresiduation laws (cr), obtained by reversing \( \leq \). Note the complete absence of axioms licensing the structural rules of sequent calculi: the tensor and par fail to satisfy weakening (e.g., \( A \otimes B \leq A \) or \( A \leq A \otimes B \)) and contraction (\( A \leq A \otimes A \) or \( A \otimes A \leq A \)), nor are they associative or commutative.

Lemma 2.3. Derived rules of inference include the following monotonicity laws:

\[
\begin{align*}
\frac{A \leq B \quad C \leq D}{A \otimes C \leq B \otimes D} & \quad \text{m} & \frac{A \leq B \quad C \leq D}{A \otimes C \leq B \otimes D} & \quad \text{m}
\end{align*}
\]

\[
\begin{align*}
\frac{A \setminus D \leq B / C}{D \setminus A \leq C / B} & \quad \text{m} & \frac{A \otimes D \leq B \triangleleft C}{D \otimes A \leq C \triangleleft B} & \quad \text{m}
\end{align*}
\]

3
Preorder laws

\[
\frac{A \leq A}{I} \quad \frac{A \leq B \quad B \leq C}{A \leq C \quad T}
\]

(Co)residuation

\[
\begin{array}{cccc}
A \otimes B \leq C & B \leq A \backslash C & C \leq A \oplus B & C \ominus B \leq A \\
A \leq C \backslash B & B \leq A \backslash C & A \leq C \backslash B & C \leq A \oplus B \\
 & B \leq A \backslash C & A \leq C \backslash B & C \leq A \oplus B
\end{array}
\]

Meets/Joins

\[
\begin{array}{cc}
\frac{A \leq B \quad A \leq C}{A \leq B \land C} & \frac{B \leq A \lor C}{B \leq A \lor C}
\end{array}
\]

Figure 2: LG characterized algebraically. Double horizontal inference lines indicate interchangeability of premises and conclusion, where multiple conclusions (in the rules for meets/joins) are to be interpreted conjunctively.

Proof. As a typical case, we take the rule for \( \otimes \), derived thus:

\[
\begin{array}{c}
C \leq D \\
B \otimes D \leq B \otimes D \\
B \leq (B \otimes D)/(B \otimes D)
\end{array}
\]

Grishin \((13)\) has considered possible extensions of LG by groups of axioms establishing interaction between \( \{\otimes, /, \backslash\} \) and \( \{\oplus, \otimes, \ominus\} \), while remaining conservative over the two families separately. To illustrate the applicability of our method to Grishin’s studies, we single out the laws for linear distributivity of \( \otimes \) over \( \oplus \) \((S)\), presented in rule format in Figure 5.

**Lemma 2.4.** The following inequalities are derivable in the presence of the rules of Figure 5.

\[
\begin{align*}
A \otimes (B \oplus C) & \leq (A \otimes B) \oplus C \\
A \otimes (B \oplus C) & \leq B \oplus (A \otimes C)
\end{align*}
\]

\[
\begin{align*}
(A \oplus B) \otimes C & \leq A \oplus (B \otimes C) \\
(A \oplus B) \otimes C & \leq (A \otimes C) \oplus B
\end{align*}
\]
Proof. As a typical case, we check \( A \otimes (B \oplus C) \leq (A \otimes B) \oplus C \).

\[
\begin{align*}
B \oplus C & \leq B \oplus C \quad I \\
B \oplus C & \leq B \\
(B \oplus C) \otimes C & \leq B \quad c.r. \\
B & \leq A \backslash (A \otimes B) \quad r \quad T \\
(B \oplus C) \otimes C & \leq A \backslash (A \otimes B) \\
A \otimes (B \oplus C) & \leq (A \otimes B) \oplus C
\end{align*}
\]

As stressed by Moortgat, one may argue there to be different conceptions of \( \text{LG} \), depending on which of Grishin’s groups are adopted. Therefore, we will henceforth refer by (the algebraic presentation of) \( \text{LG} \) to the calculus defined by the inference rules of Figure 2 only, while \( \text{LG}_I \) denotes the extension by linear distributivity (following Moortgat’s notation). On those occasions where the difference is inessential, we keep using \( \text{LG} \). For a more thorough exploration of the wider landscape of Lambek-Grishin calculi, the reader is referred to [21].

We note that the methods used in this article are general enough so as to be applicable to said alternatives. We conclude with the realization of \( \cdot \) at the level of derivations, again demonstrable through a straightforward induction:

Lemma 2.5. For any \( A, B \), \( A \leq B \) iff \( B^I \leq A^I \).

2.2.2 One-sided sequents

As shown by Moortgat ([21]), \( \text{LG} \) has a Cut-free display calculus. Like in ordinary two-sided calculi, connectives are introduced as hypotheses or conclusions. To guarantee Cut-admissibility, however, structural commas no longer associate exclusively to (multiplicative) conjunctions and disjunctions, but may also appear as counterparts for the (co)implications. The current section presents a one-sided retelling of Moortgat’s display calculus, in the sense that, for any \( A, B \), the inequalities \( A \leq B \) and \( B^I \leq A^I \) will have the same sequent counterpart.

Definition 2.6. Proofs establish presentations, being pairs of structures \( \Gamma, \Delta \):

\[
\Gamma, \Delta, \Theta \quad ::= 
A \mid (\Gamma \bullet \Delta) \mid (\Gamma \leftarrow \Delta) \mid (\Delta \rightarrow \Gamma)
\]

Structures

\[
\omega \quad ::= 
\Gamma, \Delta
\]

Presentations

Terminology is adapted from Andreoli ([4]), who distinguished between presentations and varieties. The intuition, further pursued below, is that presentations are closed under reversible structural rules allowing any of its formulas to be displayed as the whole of one of its components. The equivalence classes of presentations generated by said rules are the (freely generated) varieties, presentations thus ‘presenting’ a variety from the point of view of one of its substructures (particularly formulas). We refrain from explicating the latter concept, however, as the focused derivations defined in §3.3 already compile away the reversible structural rules. We note Lamarche ([16]) uses ‘terms’ and ‘reversible terms’ for denoting syntactic objects similar to our structures and Andreoli’s varieties.

Definition 2.7. Structures \( \Gamma \) interpret by pairs of dual formulas \( \Gamma^+, \Gamma^- \):

\[
\begin{align*}
A^+ & \stackrel{=def}{=} A \\
(\Gamma \bullet \Delta)^+ & \stackrel{=def}{=} \Gamma^+ \otimes \Delta^+ \\
(\Delta \rightarrow \Gamma)^+ & \stackrel{=def}{=} \Delta^+ \otimes \Gamma^+ \\
(\Gamma \leftarrow \Delta)^+ & \stackrel{=def}{=} \Gamma^+ \odot \Delta^+
\end{align*}
\]

\[
\begin{align*}
A^- & \stackrel{=def}{=} A^I \\
(\Gamma \bullet \Delta)^- & \stackrel{=def}{=} \Gamma^- \otimes \Delta^- \\
(\Delta \rightarrow \Gamma)^- & \stackrel{=def}{=} \Delta^- \odot \Gamma^- \\
(\Gamma \leftarrow \Delta)^- & \stackrel{=def}{=} \Gamma^- \odot \Delta^-
\end{align*}
\]

\[\]

5
\[ \Gamma, \Delta \vdash A, A \quad Ax \quad \Delta, A \vdash \Gamma, A \vdash \Gamma, \Delta \vdash \text{Cut} \]
\[ \Gamma, \Delta \vdash \Gamma, \Delta \quad dp \quad \Gamma \cdot \Delta, \Theta \vdash \Gamma, \Delta \vdash \Theta \quad dp \]
\[ \Gamma, A \vdash B \quad \Gamma, A \vdash B \quad \Gamma \vdash B \quad \Gamma \vdash B \quad \Gamma \vdash \Gamma \]
\[ \Gamma, A \vdash \Delta, B \quad \Delta \cdot \Gamma, A \vdash B \quad \Delta \cdot \Gamma, A \vdash B \quad \Delta \vdash \Gamma, A \vdash B \]
\[ \Gamma, A \vdash \Delta, B \quad \Delta \cdot \Gamma, A \vdash B \quad \Delta \vdash \Gamma, A \vdash B \]
\[ \Gamma, A \vdash \Delta, B \quad \Delta \cdot \Gamma, A \vdash B \quad \Delta \vdash \Gamma, A \vdash B \]

Figure 4: A left-sided sequent calculus for \( \text{LG} \): base logic

\[ \Gamma_2 \Gamma_1 \cdot \Gamma_2, \Delta_1 \Delta_2 \Delta_1 \vdash A \]
\[ \Delta_1 \rightarrow \Gamma_1, \Gamma_2 \rightarrow \Delta_2 \rightarrow \Delta_1 \vdash A \]
\[ \Delta_2 \rightarrow \Gamma_1, \Delta_1 \rightarrow \Gamma_2 \rightarrow \Delta_2 \rightarrow \Delta_1 \vdash C_1 \]

Figure 5: Structural rules: Linear Distributivity

One easily shows \( \Gamma^+ = \Gamma^{-1} \) and \( \Gamma^- = \Gamma^{+1} \). The ambiguity extends to the level of derivability, where we will find a witness for the provability of \( \Gamma, \Delta \) to be realizable into algebraic derivations of both \( \Gamma^+ \leq \Delta^- \) and \( \Delta^+ \leq \Gamma^- \). Conversely, inequalities \( A \leq B \) and \( B^+ \leq A^+ \) are both presented by \( A, B \).

**Definition 2.8.** Figure 4 defines derivability judgements \( \omega \vdash \) for \( \text{LG} \), written as left-sided sequents[^1]. Next to the familiar axioms and Cut, we have the display postulates \( (dp) \), ensuring, for any presentation \( \omega \) and an occurrence therein of a formula \( A \), the (unique) existence of \( \Delta \) s.t. \( \omega \) may be rewritten into \( \omega' = \Delta, A \).

We say \( A \) is displayed in \( \omega' \). Presentations \( \omega \) and \( \omega' \) interderivable through display postulates are said to be display equivalent, a situation often abbreviated

\[ \frac{\omega}{\omega'} \quad \text{Dp} \]

**Definition 2.9.** We fix terminology for referring to occurrences of formulas in (instances of) logical rules. Given one of the form

\[ \frac{\Gamma_1, A_1 \vdash \ldots \Gamma_n, A_n \vdash R}{\Gamma, A \vdash} \]

we call \( A \) the main or principal formula of \( R \), and the subformulas \( A_1, \ldots, A_n \) of \( A \) occurring in the premises the active formulas of \( R \). We also say \( A_i \) (\( 1 \leq i \leq n \)) is principal if main in the rule deriving the corresponding premise.

[^1]: Our (non-conventional) preference for left-sided sequents over right-sided ones is motivated by the former’s transparent correspondence with intuitionistic sequents, constituting the target of double negation translations ([21]).
being similar. By induction hypothesis, we know

$$\text{Lemma 2.11.}$$

In general, any two derivations of the same sequent employing different axiom matchings will likewise remain distinct under focusing, although the converse need not always hold. In particular, the other two derivations employ different axiom matchings, in the precise sense that they generalize to distinct presentations $a/(d/b) \cdot (s/(d/c))\backslash b, a$ and $a/(c/b) \cdot (s/(c/d))\backslash d, d$, and therefore remain distinct under focalization.

We proceed to show soundness and completeness w.r.t. algebraic derivability.

$$\text{Lemma 2.11.}$$ If $A \leq B$, then $A, B^i \vdash$.

$$\text{Proof.}$$ By an induction on the derivation witnessing $A \leq B$. The preorder laws trivially translate to Axioms and Cut, while the rules for meets and joins are immediate by $(\vee)$ and $(\wedge^i)$. This leaves us with the (co)residuation laws and the Grishin interactions as the only nontrivial cases.

$$\text{(Co)residuation.}$$ We explicitly check $A \leq C/B$ if $A \otimes B \leq C$, the other cases being similar. By induction hypothesis, we know $A \otimes B, C^i \vdash$. Hence,

$$\begin{align*}
A \cdot B, B^i \vdash & I \\
\Delta & A, A^i \vdash I \oplus \Delta \\
A \otimes B, C^i \vdash & T \\
A \cdot B, C^i \vdash & Dp \\
A, B \vdash & C^i \vdash \\
A, B^i \otimes C^i \vdash & \otimes
\end{align*}$$

$\text{In general, any two derivations of the same sequent employing different axiom matchings will likewise remain distinct under focusing, although the converse need not always hold. In particular, focalization still identifies less derivations than proof nets do.}$

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Grishin interactions. Again, we consider only one case. Take $C \otimes A \leq D \otimes B$ if $A \otimes B \leq C \setminus D$. By induction hypothesis, $A \otimes B, D^i \otimes C^i \vdash$. Hence,

$$
\begin{array}{c}
C, C^i \vdash I \\
D^i, D \vdash I \\
D^i \leftarrow C, A \otimes B \vdash \\
D^i \leftarrow C, A \leftarrow B^i \vdash \otimes \\
B^i \bullet D^i, C \bullet A \vdash \otimes \\
B^i \bullet D^i, C \otimes A \vdash \\
C \otimes A, B^i \bullet D^i \vdash \otimes \\
\end{array}
$$

Lemma 2.12. If $\Gamma, \Delta \vdash$, then $\Gamma^+ \leq \Delta^-$ and $\Delta^+ \leq \Gamma^-$.

Proof. We proceed by induction on $\Gamma, \Delta \vdash$. The cases $(Ax)$ and $(Cut)$ are trivial, while $(\otimes)$, $(\otimes)$ and $(\otimes)$ are immediate from the induction hypotheses.

Case $(dp)$. In general, the display postulates are justified by (co)residuation. As a typical case, we take $\Gamma \bullet \Delta, \Theta \vdash$ implies $\Gamma, \Delta \rightarrow \Theta \vdash$. By the induction hypothesis, $\Gamma^+ \otimes \Delta^+ \leq \Theta^-$ and $\Theta^+ \leq \Delta^- \otimes \Gamma^-$. Hence,

$$
\frac{\Gamma^+ \otimes \Delta^+ \leq \Theta^-}{\Gamma^+ \leq \Theta^-/\Delta^+} \quad \frac{\Theta^+ \leq \Delta^+ \oplus \Gamma^-}{\Theta^+ \otimes \Delta^- \leq \Gamma^-}
$$

Cases $(\otimes)$, $(/) and (/)$. In general, said cases all depend on Lemma 2.3. Consider $(/).$ Assuming $\Delta, B^i \vdash$ and $\Gamma, A \vdash$, we have as induction hypotheses $B^i \leq \Delta^-$, $\Delta^+ \leq B$, $A \leq \Gamma^-$ and $\Gamma^+ \leq A^i$. Hence,

$$
\frac{\Gamma^+ \leq A^i}{\Delta^- \otimes \Gamma^+ \leq B^i \otimes A^i} \quad \frac{\Delta^+ \leq B}{A/B \leq \Gamma^+} \quad \frac{A \leq \Gamma^-}{A \otimes \Delta^- \leq \Gamma^-}
$$

Cases $(\wedge)$ and $(\vee)$. We consider $(\wedge)$, $(\vee)$ being similar. Assuming $\Gamma, A \vdash$, we have induction hypotheses $\Gamma^+ \leq A^i$ and $A \leq \Gamma^-$. Hence,

$$
\frac{\Gamma^* \leq A^i}{\Gamma^* \leq B^i \vee A^i} \quad \frac{A \leq \Gamma^-}{A \wedge \Gamma^- \leq \Gamma^-}
$$

Case $(v)$. Assuming $\Gamma, A \vdash$ and $\Gamma, B \vdash$, we have as induction hypotheses $\Gamma^* \leq A^i$, $A \leq \Gamma^-$, $\Gamma^+ \leq B^i$ and $B \leq \Gamma^-$. Hence,

$$
\frac{\Gamma^* \leq A^i}{\Gamma^* \leq B^i \wedge A^i} \quad \frac{A \leq \Gamma^-}{B \leq \Gamma^-} \quad \frac{B \leq \Gamma^-}{A \vee B \leq \Gamma^-}
$$

Case $(A_{1/2}^j/C_1)$. We consider $\Delta_2 \leftarrow \Gamma_1, \Delta_1 \rightarrow \Gamma_2 \vdash$ if $\Gamma_1 \bullet \Delta_2, \Delta_2 \bullet \Delta_1 \vdash$ as a typical case. By induction hypothesis, $\Delta_1 \otimes \Gamma_2^* \leq \Gamma_1^+ \setminus \Delta_2^*$ and $\Delta_2 \otimes \Gamma_1^* \leq \Gamma_2^+ / \Delta_1^*$:

$$
\frac{\Delta_1 \otimes \Gamma_2^* \leq \Gamma_1^+ \setminus \Delta_2^*}{\Gamma_1^+ \otimes \Delta_2^* \leq \Delta_1^* \oplus \Delta_2^*} \quad \frac{\Delta_2 \otimes \Gamma_1^* \leq \Gamma_2^+ / \Delta_1^*}{\Delta_2^* \otimes \Delta_1^* \leq \Gamma_2^+ \oplus \Gamma_1^*}
$$
\[ \Gamma, \Delta \rightarrow \Theta \vdash \quad \Gamma, \Delta \rightarrow \Theta \vdash \]

Figure 7: Structural rules: Classical non-associative Lambek calculus

\[
A, B ::= p \mid \bar{p} \mid (A \otimes B) \mid (A \oplus B) \mid (A \land B) \mid (A \lor B)
\]

\[
\Gamma, \Delta ::= A \mid (\Gamma \bullet \Delta)
\]

\[
\frac{\Gamma, A \vdash \Delta, B \vdash}{\Gamma, A \otimes B \vdash} \quad \frac{\Gamma, A \vdash \Delta, B \vdash}{\Gamma, A \oplus B \vdash} \quad \frac{\Gamma, A \vdash \Delta, B \vdash}{\Gamma, A \land B \vdash} \quad \frac{\Gamma, A \vdash \Delta, B \vdash}{\Gamma, A \lor B \vdash}
\]

\[
\frac{\Gamma, \Delta \vdash}{\Gamma, \Delta \bullet \Theta \vdash} \quad \frac{\Gamma, \Delta \vdash}{\Gamma, \Delta \oplus \Theta \vdash}
\]

Figure 8: A left-sided retelling of De Groote and Lamarche’s original right-sided sequent calculus for \textit{CNL}, augmented by rules for the additives.

2.3 CNL sequentialized

\textit{CNL} derives from \textit{LG}_{\emptyset} by identifying \(\bullet\), \(\multimap\) and \(\rightarrow\), i.e., by adding the structural rules of Figure 7.

\textbf{Lemma 2.13.} In \textit{CNL}, we have \(B \mid A, A^\downarrow \ssim B \vdash, B \oplus A^\downarrow, A \otimes B \vdash, A / B, B \otimes A^\downarrow\) and \(A^\downarrow \ssim B, B \otimes A^\downarrow\).

\textbf{Proof.} We demonstrate the first two claims, the latter two being similar.

\[
\frac{B, B^\downarrow \vdash}{A, B^\downarrow \vdash} \quad \frac{A^\downarrow \ssim B, A \vdash}{B \mid A, A^\downarrow \ssim B \vdash} \quad \frac{A^\downarrow \ssim B, B \otimes A^\downarrow \vdash}{B \otimes A^\downarrow, A \ssim B^\downarrow \vdash}
\]

When mapped into inequalities \(A \leq B\) via \(\cdot +\) and \(\cdot -\), the previous lemma suggests the following identifications between formulas:

\[
A / B = A \oplus B^\downarrow \quad B \otimes A = B^\downarrow \ominus A
\]

In particular, we have the more economic axiomatization for \textit{CNL} of Figure 8, as originally employed in [10], save for a few notational differences. We here stick to the presentation of \textit{CNL} as derived from \textit{LG}_{\emptyset} with structural postulates, as it allows for a uniform proof of the focalization property.
3  Focusing proofs

Sequent calculi have seen widespread application in backward-chaining proof search: an attempt at proving a goal sequent proceeds by matching it against the conclusion of an inference rule, and replacing it by the latter’s premises. The process terminates successfully once all goals have been replaced by axioms, and fails when there remain goals to prove while the applicable inferences have been exhausted. Cut elimination guarantees a reasonable bound on the search space: the only formulas found in the premises of the remaining inference rules already occur as subformulas of the conclusion.

While satisfying the subformula property, Cut-free proof search still suffers from inessential non-determinism: neighboring logical inference steps involving different main and active formulas freely permute, making the choice of their relative ordering meaningless for settling the question of provability. The problem seeming inherent to the sequentialization of rule applications, Girard proposed a parallel representation of proofs, called proof nets. In contrast, Andreoli stuck with sequent calculus, seeking instead a method for obtaining canonical representatives of derivations differing only by trivial rule permutations. Thus was born focused proof search: greedily apply invertible inferences (preserving provability of the conclusion in the premises), while the active formulas appearing in the premises of non-invertible inferences always are to be principal. In other words, once chosen as main, a formula is ‘focused upon’ in the sense of fixing the choice for subsequent rule applications to those targeting its subformulas.

The current section treats a succession of formalisms, each further realizing Andreoli’s focusing strategy for LG and CNL. A brief review in §3.1 of the causes for inessential nondeterminism in proof search reveals a partitioning of formulas into those of positive or negative polarity, depending on whether their inferences are always invertible. This leads in §3.2 to an extension of the logical vocabulary by connectives for explicitly recording polarity shifts, with sequent derivations being adapted accordingly. The latter’s normal (i.e., Cut-free) forms turn out to already satisfy weak focalization, tackling permutations between non-invertible inferences. Full, or strong focalization is obtained in §3.3 through a sequent calculus of synthetic inferences ([3]), collapsing multiple inference steps that freely permute. The latter are furthermore compiled from the formulas appearing in the goal sequent(s), thus explicating the subformula property. Soundness and completeness w.r.t. unfocused provability, as discussed in §2, are dealt with to the extent that all will be left to check is the normalization of polarized derivations into those considered strongly focalized.

3.1  Polarities and rule permutations

In §2.1, algebraic considerations led us to group the multiplicatives into the families \{\otimes, \slash.left, \slash.right\} and \{\oplus, \oslash, \obslash\}, finding support in [1]. Inspection of the logical rules in Figure 4 however, reveals another natural classification:

\[
\begin{align*}
P, Q & \quad ::= \quad p \mid (A \otimes B) \mid (A \oslash B) \mid (B \oslash A) \mid (A \lor B) \quad \text{Positive formulas} \\
N, M & \quad ::= \quad p \mid (A \oplus B) \mid (B \oslash A) \mid (A/B) \mid (A \land B) \quad \text{Negative formulas}
\end{align*}
\]

\[\text{The widespread appearance of } \bot \text{ in the rules of Figure 4 necessitates a slightly nonconventional definition of the notion of subformula, as explicated in §3.3.}\]

\[\text{The notions of strong and weak focalization as used here were, to the author’s knowledge, first used in [19].}\]

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Again, the dual of a positive under \( \cdot \) is negative and vice versa, motivating the choice of terminology. Call a logical inference with a positive (negative) main formula positive (negative). We observe that positive inferences are always invertible, meaning premises and conclusion may be interchanged. For example, invertibility of (\( \otimes \)) and (one half of) (\( \land \)) is witnessed by the following Cuts:

\[
\begin{align*}
B^+, B & \vdash I & A, A^+ & \vdash I \\
B^+ \rightarrow A, A^+/B^+ & \vdash & \Gamma, B \otimes A & \vdash T \\
\end{align*}
\]

The positive/negative distinction provides a neat classification of rule permutations, always involving inference steps with disjoint active and main formulas:

1. Positive/negative. E.g., noting \( B^+, C \cdot D \vdash \) iff \( C \cdot D, B^+ \vdash \) by (\( Dp \)):

\[
\begin{align*}
B^+, C \cdot D & \vdash & B^+, C \otimes D & \vdash \otimes \\
C \otimes D, B^+ & \vdash & \Gamma, A & \vdash / \\
\end{align*}
\]

2. Negative/negative. E.g., noting \( C^+, A \vdash \) iff \( A, C^+ \vdash \) by (\( Dp \)):

\[
\begin{align*}
\Delta, B^+ & \vdash C^+, A & \Delta & \vdash C^+, A/B \vdash / \\
A/B \cdot \Delta, C^+ & \vdash & \Gamma, D & \vdash / \\
\end{align*}
\]

3. Positive/positive. E.g., noting \( C \leftarrow D, A \cdot B \vdash \) iff \( A \cdot B, C \leftarrow D \vdash \) by (\( Dp \)):

\[
\begin{align*}
C & \leftarrow D, A \cdot B & \leftarrow & \otimes \\
C & \leftarrow D, A \otimes B & \leftarrow & \otimes \\
A \otimes B, C & \leftarrow D & \leftarrow & \otimes \\
\end{align*}
\]

Besides the reorderings caused by the logical rules, the current sequent calculus is home to an additional form of redundancy, courtesy of the display postulates. Recall the latter’s purpose is to isolate the main formula of a logical inference from within a presentation. While always possible in a canonical fashion, nothing prevents us from taking detours, e.g., displaying formulas without applying the corresponding logical inference, or even introducing cycles (revisiting the same sequent multiple times throughout a derivation). Thus, we wish to constrain their applicability, ideally doing away with them altogether.

In the sequel, we consider variations of \( \text{CNL} \) and \( \text{LG} \) that internalizes polarity shifts within the syntax of formulas. We define the corresponding sequent derivations in \( \S 3.2 \), these being two-sided in the sense that all invertible inferences apply on the left-hand side, while all non-invertible, formerly negative
derivability judgements

\[ \omega \]

Definition 3.5. The sequent calculus for the stub

Definition 3.4. As before, structures \( \Gamma \) interpret by dual formulas \( \Gamma^+ \) and \( \Gamma^- \):

\[
\begin{align*}
\Gamma \Delta & \equiv (\Gamma \bullet \Delta) | (\Gamma \leftarrow \Delta) | (\Delta \rightarrow \Gamma) \\
\omega & \equiv \Gamma, \Delta
\end{align*}
\]

In particular, antecedent negative formulas appear only as \( \downarrow N \).

Definition 3.4. As before, structures \( \Gamma \) interpret by dual formulas \( \Gamma^+ \) and \( \Gamma^- \):

\[
\begin{align*}
\Gamma^+ & \equiv (\Gamma \bullet \Delta)^+ \equiv (\Delta \rightarrow \Gamma)^+ \equiv (\Gamma \leftarrow \Delta)^+ \\
\Gamma^- & \equiv (\Gamma \bullet \Delta)^- \equiv (\Delta \rightarrow \Gamma)^- \equiv (\Gamma \leftarrow \Delta)^-
\end{align*}
\]

Definition 3.5. The sequent calculus for \( \text{LG}^{pol}_\omega \) is provided in Figure 6 involving derivability judgements \( \omega^+ \) and \( \Gamma = P \) defined by mutual induction. We refer by the stub to the right-hand side of the turnstile, adapting terminology of (11). The extension by structural rules remains unchanged from Figures 5 and 7.

Note that Figure 5 makes no explicit mention of rules for deriving negative formulas. Instead, these are hidden inside the right introductions, as is clear when the latter precede an application of (\( \downarrow L \)).

\[ \text{Remark 3.2.} \]
\[ \frac{\Delta \vdash P \quad \Gamma, P \vdash}{\Gamma, \Delta \vdash} T \]

\[ \frac{\neg P \vdash P}{I} \]

\[ \frac{\Gamma, P \vdash Q}{\Gamma \vdash P \otimes Q \vdash \otimes L} \]

\[ \frac{\Gamma, P \vdash \Delta \vdash Q}{\Gamma, \Delta \vdash P \otimes Q \vdash \otimes R} \]

\[ \frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma, P \lor Q \vdash \lor L} \]

\[ \frac{\Delta \vdash N \vdash}{\Delta, \Delta \vdash dp} \]

\[ \Delta \vdash N \vdash, \Delta, \Delta \vdash \downarrow L \]

Thus, a right introduction of a positive formula may be understood as a left introduction of its negative dual. This intuition is further pursued through a completeness proof w.r.t. the sequent derivations of Section 2, demonstrated using the following decoration of unpolarized formulae with shifts.

**Definition 3.6.** For a (n unpolarized) formula, let \( \epsilon(A) = + \) if \( A \) is of positive polarity, i.e., of the form \( p, B \otimes C, B \otimes C \) or \( C \otimes B \), and \( \epsilon(A) = - \) otherwise. We translate \( A \) into a formula \( \ddagger(A) \) of \( CNL^{pol}/LG^{pol} \) avoiding ‘vacuous’ polarity shifts by excluding subformulas of the form \( \vartriangleright P \) or \( \vartriangleright N \). For the base cases, we define \( \ddagger(p) = p \) and \( \ddagger(\vartriangleright p) = \vartriangleright p \). For complex \( A \) with \( \epsilon(A) = + \), we set

\[
\begin{array}{c|c|c|c|c}
\epsilon(A) & \epsilon(B) & \ddagger(A \otimes B) & \ddagger(B \otimes A) & A \lor B \\
\hline
+ & + & \ddagger(A) \otimes \ddagger(B) & \ddagger(B) \otimes \ddagger(A) & \ddagger(A) \lor \ddagger(B) \\
+ & - & \ddagger(A) \otimes \ddagger(B) & \ddagger(B) \otimes \ddagger(A) & \ddagger(A) \lor \ddagger(B) \\
- & + & \vartriangleright \ddagger(A) \otimes \ddagger(B) & \vartriangleright \ddagger(B) \otimes \ddagger(A) & \ddagger(A) \lor \ddagger(B) \\
- & - & \vartriangleright \ddagger(A) \otimes \ddagger(B) & \vartriangleright \ddagger(B) \otimes \ddagger(A) & \ddagger(A) \lor \ddagger(B) \\
\end{array}
\]

Finally, for complex \( A \) with \( \epsilon(A) = - \),

\[
\begin{array}{c|c|c|c|c}
\epsilon(A) & \epsilon(B) & \ddagger(\vartriangleright A \otimes \vartriangleright B) & \ddagger(\vartriangleright B \otimes \vartriangleright A) & \vartriangleright A \land \vartriangleright B \\
\hline
+ & + & \ddagger(\vartriangleright A) \vartriangleright \ddagger(\vartriangleright B) & \ddagger(\vartriangleright B) \vartriangleright \ddagger(\vartriangleright A) & \ddagger(\vartriangleright A) \land \ddagger(\vartriangleright B) \\
+ & - & \ddagger(\vartriangleright A) \vartriangleright \ddagger(\vartriangleright B) & \ddagger(\vartriangleright B) \vartriangleright \ddagger(\vartriangleright A) & \ddagger(\vartriangleright A) \land \ddagger(\vartriangleright B) \\
- & + & \ddagger(\vartriangleright A) \vartriangleright \ddagger(\vartriangleright B) & \ddagger(\vartriangleright B) \vartriangleright \ddagger(\vartriangleright A) & \ddagger(\vartriangleright A) \land \ddagger(\vartriangleright B) \\
- & - & \ddagger(\vartriangleright A) \vartriangleright \ddagger(\vartriangleright B) & \ddagger(\vartriangleright B) \vartriangleright \ddagger(\vartriangleright A) & \ddagger(\vartriangleright A) \land \ddagger(\vartriangleright B) \\
\end{array}
\]

An easy induction establishes

**Lemma 3.7.** The map \( \ddagger(\cdot) \) commutes with linear negation. I.e., for any \( A \), \( \ddagger(A)^\vartriangleleft = \ddagger(A^\vartriangleleft) \).

13
Below, we show such shifts through subformulas to cross the turnstile only upon the encounter of polarity shifts, and ‘vacuous’ at most a single formula. Since, in the absence of Cut, formulas are allowed indeed, all non-invertible inferences take place entirely within the stoop, housing negative/negative permutations, though leaving the others unaddressed. In-latter’s admissibility suffices for showing a weak form of focalization, restricting making repeated use of Cut. As a consequence, we claim that a proof of the must consider all possible values for including the display postulates, are immediate. In each remaining case, we

**Definition 3.8.** We extend \( \vdash \) to the level of structures \( \Gamma \) as follows:

\[
A \mapsto \begin{cases} 
\vdash (A) & \text{if } \epsilon (A) = + \\
\vdash (A) & \text{if } \epsilon (A) = - \\
\vdash (\Gamma \bullet \Delta) & =_{\text{def}} \vdash (\Gamma) \bullet \vdash (\Delta)
\end{cases}
\]

Below, we show \( \Gamma, \Delta \vdash \) in \( \text{LG} (\text{CNL}) \) implies \( \vdash (\Gamma), \vdash (\Delta) \vdash \) in \( \text{LG}^{\text{pol}} (\text{CNL}^{\text{pol}}) \), making repeated use of Cut. As a consequence, we claim that a proof of the latter’s admissibility suffices for showing a weak form of focalization, restricting negative/negative permutations, though leaving the others unaddressed. Indeed, all non-invertible inferences take place entirely within the stoop, housing at most a single formula. Since, in the absence of Cut, formulas are allowed to cross the turnstile only upon the encounter of polarity shifts, and ‘vacuous’ such shifts through subformulas \( \vdash P \) and \( \vdash N \) have been avoided, maximal chains of non-invertible inferences are enforced, traversing the formula tree of the particular \( N \) designated main through \( (\downarrow L) \).

**Theorem 3.9.** If \( \Gamma, \Delta \vdash \) in \( \text{LG} / \text{CNL} \), then \( \vdash (\Gamma), \vdash (\Delta) \vdash \) in \( \text{LG}^{\text{pol}} / \text{CNL}^{\text{pol}} \).

**Proof.** By induction on \( \Gamma, \Delta \vdash \), making free use of L3.7. The structural rules, including the display postulates, are immediate. In each remaining case, we must consider all possible values for \( \epsilon (A_1), \ldots, \epsilon (A_n) \) of the active and main formulas \( A_1, \ldots, A_n \) involved. For axioms and Cut, we have

\[
\frac{\Delta, \vdash (A) \vdash \downarrow L \quad \vdash (\Gamma), \vdash (A) \vdash \downarrow I \quad \vdash (\Gamma), \vdash (A) \vdash \downarrow I \quad \vdash (\Gamma), \vdash (A) \vdash \downarrow I}{\vdash (\Gamma), \vdash (\Delta) \vdash}
\]

The remaining cases (\( \otimes \)), (\( \odot \)) and (\( \ominus \)) trivially translate to left introductions. \( \square \)
3.3 Strong focalization

Our efforts so far have left all but negative/negative permutations unaddressed. Our interest, however, is in full, or strong focalization, providing further normalization of Cut-free polarized derivations. In particular, we structure top-down proof-search into alternating invertible and non-invertible phases. The latter proceeds as in the previous section, taking place entirely within the stoup, and ends in each branch with an application of \( \downarrow R \). All applicable invertible inferences are subsequently to be exhausted before \( \downarrow L \) is made available, ushering in a new non-invertible phase. By furthermore collapsing both phases into single inference steps, the problem posed by permutations between the inferences within a single invertible phase is remedied.

Despite offering increased control over rule ordering, focused derivations remain sequent derivations at heart, satisfying in particular a local subformula property. From the above discussion, however, it should be clear that subformulas are identified only up to polarity shifts, marking the boundaries between the invertible and non-invertible phases.

**Definition 3.10.** For any \( P \) or \( N \), the maps \( \sigma \) and \( \tau \), defined by mutual induction, pick out the subformulas relevant for focused proof search:

\[
\begin{align*}
\sigma(\uparrow P) &= \text{def} \ (P^\downarrow) \cup \tau(P) \\
\sigma(\downarrow p) &= \text{def} \ \{ p \} \\
\sigma(M \land N) &= \text{def} \ \sigma(M) \cup \sigma(N) \\
\sigma(M \lor N) &= \text{def} \ \sigma(M) \cup \sigma(N) \\
\sigma(M/Q) &= \text{def} \ \sigma(M) \cup \tau(Q) \\
\sigma(Q\setminus M) &= \text{def} \ \sigma(M) \cup \tau(Q)
\end{align*}
\]

\[
\begin{align*}
\tau(\downarrow N) &= \text{def} \ \{ N \} \cup \sigma(N) \\
\tau(\downarrow p) &= \text{def} \ \{ p \} \\
\tau(P \lor Q) &= \text{def} \ \tau(P) \cup \tau(Q) \\
\tau(P \land Q) &= \text{def} \ \tau(P) \cup \tau(Q) \\
\tau(N \Rightarrow P) &= \text{def} \ \tau(P) \cup \sigma(N) \\
\tau(N \Leftarrow P) &= \text{def} \ \tau(P) \cup \sigma(N)
\end{align*}
\]

Note carefully the definition of \( \sigma(\uparrow P) \) as \( (P^\downarrow) \cup \tau(P) \) instead of \( P \cup \tau(P) \). Explained in terms of the sequent derivations of §3.2, this is due to the formulation of the rules for \( \uparrow \): since in the premise of \( \downarrow L \) the stoup contains the negation of the main formula \( \downarrow N \), the latter’s subformulas of the form \( \uparrow P \) also appear as \( P^\downarrow \) when main in an instance of \( \downarrow R \). Thus, it is \( P^\downarrow \), and not \( P \), that we wish to remember at this particular polarity shift. As a further indication of the harmony of this definition, a straightforward induction proves

**Lemma 3.11.** For any \( N, P, \sigma(N) = \tau(N^\downarrow) \) and (dually) \( \tau(P) = \sigma(P^\downarrow) \).

In what is to follow, fix a set \( X \) of negative formulas. Concepts are defined relative to \( X \), its intended instantiation being as follows. The aim of this section being the normalization of polarized derivations, we fix \( X \) by \( \{ \Gamma^-, \Delta^- \} \) for an initial goal sequent \( \Gamma, \Delta \vdash \) according to D[3.3]. This determines a set of goal sequents, as further elaborated upon below, s.t. the latter’s strongly focused provability guarantees the provability of the initial \( \Gamma, \Delta \vdash \) (cf. C[4.14]). While not a necessary ingredient for the definitions to follow, being easily ignorable, we find the explicit parameterization over such sets better emphasizes the goal-driven nature of the current take on focalization, while furthermore serving as an explicit check on the satisfaction of the subformula property. In particular, inference rules are defined only for the members of the closure \( X^\tau = \text{def} \ \{ \tau(\downarrow N) \mid N \in X \} \) of \( X \) under \( \tau(\cdot) \).

\[\text{The following is a simple case analysis.}\]

}\text{Bottom-up variations on focused proof search have also been considered by Chaudhuri and Plenning (12) in the context of linear logic.}
Definition 3.13. Structures are revised so as to prohibit positive formulas other than the form $\downarrow N$ or $p$. In the latter case, we can leave the shift $\downarrow$ implicit, arriving at the following definition, where $p, N \in X^\tau$ in the base cases:

$$
\Pi, \Sigma, \Upsilon \quad ::= \quad p \mid N \mid (\Pi \bullet \Sigma) \mid (\Pi \leftarrow \Sigma) \mid (\Sigma \rightarrow \Pi)
$$

$$
\omega \quad ::= \quad \Pi, \Sigma
$$

The interpretation of structures $\Pi$ by dual formulas $\Pi^*$ and $\Pi^-$ is a straightforward adaption of $D[\Xi]$ where in particular $N^* = N$ and $N^- = N^\downarrow$. Note we do not require $\Pi^*, \Pi^- \in X^\tau$.

Since the logical vocabulary remains unchanged from §3.2, we have chosen not to overload notation any further and use different metavariables for denoting structures to prevent confusion.

Definition 3.14. To absorb the display postulates, we resort to the use of contexts $\omega[\Sigma]$, representing presentations with a distinguished occurrence of $\Delta$.

$$
\Pi[], \Sigma[] \quad ::= \quad [] \mid (\Pi[] \bullet \Sigma) \mid (\Pi[] \Sigma[]) \quad \omega[] \quad ::= \quad \Pi[], \Sigma \mid \Pi, \Sigma[]
$$

Let $\Pi[\Sigma] (\omega[\Sigma])$ denote the result of substituting $\Sigma$ for the unique occurrence of $[]$ in $\Pi[] (\omega[])$. For $\Pi[]$ and $\Sigma[]$ contexts, we denote by $\Pi[\Sigma][]$ their composition, with insertion of $\Upsilon$ understood as the insertion of $\Sigma[][\Upsilon]$ in $\Pi[]$.

Definition 3.15. The map $\vdash$ takes pairs of contexts and structures into structures, being defined by induction over its first argument:

$$
[\vdash \Upsilon \quad ::= \quad \Upsilon \mid (\Pi[] \bullet \Sigma) \mid (\Pi[] \Sigma[]) \quad (\Pi[] \Sigma[]) \vdash \Upsilon \quad ::= \quad \Pi[], \Sigma \mid \Pi, \Sigma[]
$$

The intuition we pursue is that $\Pi[\Sigma], \Upsilon \vdash$ if $\Pi[] \vdash \Upsilon, \Sigma \vdash$ through the display postulates. In particular, defining $\omega^*[]$ by $\Pi[] \vdash \omega$ for any $\omega[] = \Pi[], \Sigma$ or $\omega[] = \Sigma, \Pi[]$, $\omega[\Pi] \vdash$ if $\omega^*[]$, $\Pi \vdash$.

Definition 3.16. For each positive $P$, the set $\|P\|$ decomposes $P$ into its structural counterparts:

$$
\|P \otimes Q\| \quad ::= \quad \{\Pi \bullet \Sigma \mid \Pi \in \|P\|, \Sigma \in \|Q\|\} \quad \|P \vee Q\| \quad ::= \quad \|P\| \cup \|Q\|
$$

$$
\|P \otimes N\| \quad ::= \quad \{\Pi \leftarrow \Sigma \mid \Pi \in \|P\|, \Sigma \in \|N\|\} \quad \|p\| \quad ::= \quad \{p\}
$$

$$
\|N \otimes P\| \quad ::= \quad \{\Sigma \rightarrow \Pi \mid \Pi \in \|P\|, \Sigma \in \|N\|\} \quad \|\downarrow N\| \quad ::= \quad \{N\}
$$

One easily shows that if $N \in X^\tau$, then $\|N\|$ is well-defined relative to $X^\tau$.

Figure [10] provides a first approximation of strong focalization. Compared to Figure [9] invertible inferences are compiled away into the right introduction of $\downarrow$, ensuring their greedy application. Roughly, the inference of $\downarrow N$ requires a premise for each element of $\|N\|$, calling to attention the fact that the only branching left introductions of Figure [9] are those introducing additives, and
\[
\begin{align*}
\omega^*[[N]] & \vdash N^i \quad D \quad \{\Pi, \Sigma \vdash | \Sigma \in \|N^i\|\} \downarrow \\
\omega[[N]] & \vdash \quad \Pi \vdash \bot \\
\vdash & \quad \Pi \vdash P \\
\Pi & \vdash \bot \quad \Pi \vdash P \quad \forall^l \\
\Pi & \vdash Q \\
\Pi \cdot \Sigma & \vdash P \otimes Q \\
\Pi & \vdash P \quad \Pi \vdash P \\
\Pi \cdot \Sigma & \vdash P \otimes Q \\
\Pi & \vdash \bot \quad \Pi \vdash P \\
\Pi \cdot \Sigma & \vdash P \otimes Q \\
\Pi & \vdash \bot \\
\Pi \cdot \Sigma & \vdash P \otimes Q \\
\end{align*}
\]

Figure 10: A first approximation of strongly focalized derivations.

\[
\begin{align*}
\omega^*[[N]] & \vdash \Sigma \quad \Pi \vdash N^i \\
\omega[[N]] & \vdash \quad \Pi \vdash N^i \\
\vdash & \quad \Pi \vdash P \\
\Pi & \vdash P \quad \Pi \vdash P \quad \forall^r \\
\Pi & \vdash Q \\
\Pi \cdot \Sigma & \vdash P \otimes Q \\
\Pi & \vdash P \quad \Pi \vdash P \\
\Pi \cdot \Sigma & \vdash P \otimes Q \\
\Pi & \vdash \bot \\
\Pi \cdot \Sigma & \vdash P \otimes Q \\
\Pi & \vdash \bot \\
\Pi \cdot \Sigma & \vdash P \otimes Q \\
\end{align*}
\]

Figure 11: Strongly focalized derivations: base logic.

Similarly, \(\|N^i\|\) is a singleton if no additives are encountered in \(N^i\) up to the first immediate polarity switches. For the above revision of \((\downarrow R)\) and the renaming of \((\downarrow L)\) into decisions \((D)\), right introductions remain unaltered from \(F[9]\) violating our faithfulness to \(X^\tau\). What is needed is a reformulation of \((D)\) so as to take the sets \(\|P\|\) into account, just like we did for the invertible phase.

**Definition 3.17.** Figure \([1]\) defines strong normalization for \(LG_{\sigma}\), involving judgements \(\Pi, \Sigma \vdash \Pi \vdash \Sigma\). The latter addresses context splitting during the non-invertible phase (i.e., the distribution of a structure appearing in the conclusion over the premises), and replaces the previous judgement form \(\Pi \vdash P\). In particular, right introductions of \(\downarrow\) have been renamed \(\text{Reactions} (R)\), while \(\bullet\) and \(\rightarrow\) resemble \((\otimes R), (\odot R)\) and \((\ominus R)\) respectively. The remaining \((\lor R^l)\) and \((\lor R^r)\) are compiled away into the revised Decisions \((D)\). Optional structural extensions are listed in Figure \([12]\).

The current treatment of structural rules emphasizes their contribution to context splitting. In particular, whereas for the base logic the latter process is easily seen to be deterministic, the same cannot be said of the structural extensions. Compare this situation to those of logics less resource sensitive, where the non-determinism of context splitting is left implicit in the representation of sequents using lists or (multi)sets of formulas.

**Remark 3.18.** By restricting to structures containing no non-atomic positive formulas other than \(\downarrow N\), focused proof search may proceed from a non-singleton set of initial goal presentations. In particular, we will prove in §4 (cf. C[14]) that if \(\Gamma, \Delta \vdash\) according to \(F[9]\) then also \(\Pi, \Sigma \vdash\) for all \(\Pi \in \|\Gamma^+\|\) and \(\Sigma \in \|\Delta^+\|\), and vice versa.
Proof. As a typical case, we check (b). Evidently, any derivation of $\Pi$, Revisit the derivations of F.6 from the point of view of focalization, applying the decorations of D.3.6. Note that only a

Figure 13: Deriving $\Pi \vdash \top$.

Figure 12: Structural rules for strong focalization, applied during context-splitting. Compared to Figure 5, each mixed associativity principle is split into two rules, while each mixed commutativity principle splits into four. (REVISE)

Example 3.19. Figures 13 and 14 revisit the derivations of F.6 from the point of view of focalization, applying the decorations of D.3.6. Note that only a single focused counterpart remains for $(p/q \cdot q) \cdot p \vdash r, r \vdash$. We ensure closure under the display postulates and Linear Distributivity (F.5).

Lemma 3.20. For any $\Pi, \Sigma, \Upsilon$, we have the following implications:

(a) $\Sigma, \Pi \vdash$ implies $\Pi, \Sigma \vdash$
(b) $\Pi \cdot \Sigma, \Upsilon \vdash$ implies $\Pi, \Sigma \vdash \Upsilon \vdash$
(c) $\Pi, \Sigma \cdot \Upsilon \vdash$ implies $\Pi \vdash \Sigma, \Upsilon \vdash$

Proof. As a typical case, we check (b). Evidently, any derivation of $\Pi \cdot \Sigma, \Upsilon \vdash$
whether the main formula \(N\). Again, \(\Pi\)

We demonstrate (a), the same technique applying for proving (b)-(f).

Proof. Ending with an application of \((D)\). The main formula \(N\) must occur in either \(\Pi, \Sigma\) or \(\Upsilon\). Without loss of generality, assume \(N\) is in \(\Pi\), i.e., \(\Pi = \Pi'[N]\):

\[
\frac{(\Pi'[\cdot \Sigma]) \Downarrow \Upsilon \vdash \Upsilon'}{
\Pi'[N] \cdot \Sigma, \Upsilon \vdash D}
\]

for some \(\Upsilon' \in \|N\|.\) Since, by definition, \((\Pi'[\cdot \Sigma]) \Downarrow \Upsilon = \Pi'[\cdot (\Sigma \rightarrow \Upsilon)]\), we can also derive \(\Pi[N], \Sigma \rightarrow \Upsilon \vdash\):

\[
\frac{\Pi'[\cdot (\Sigma \rightarrow \Upsilon)] \Downarrow \Upsilon'}{
\Pi[N], \Sigma \rightarrow \Upsilon \vdash D}
\]

Lemma 3.21. We have the following admissible rules (compare with Figure 5):

(a) In the presence of \(A_{1}^{a,b}\), \(\Gamma_1 \bullet \Gamma_2, \Delta_2 \bullet \Delta_1 \vdash\) if \(\Gamma_2 \rightarrow \Delta_2, \Delta_1 \rightarrow \Gamma_1 \vdash\)

(b) In the presence of \(A_{2}^{a,b}\), \(\Gamma_1 \bullet \Gamma_2, \Delta_2 \bullet \Delta_1 \vdash\) if \(\Delta_1 \rightarrow \Gamma_1, \Gamma_2 \rightarrow \Delta_2 \vdash\)

(c) In the presence of \(C_{a}^{d}\), \(\Gamma_1 \bullet \Gamma_2, \Delta_2 \bullet \Delta_1 \vdash\) if \(\Delta_2 \rightarrow \Delta_1, \Delta_1 \rightarrow \Gamma_2 \vdash\)

Proof. We demonstrate (a), the same technique applying for proving (b)-(f). Again, \(\Pi_2 \leftarrow \Sigma_2, \Sigma_1 \leftarrow \Pi_1 \vdash\) can have been witnessed by a derivation ending with an application of \((D)\), so we consider four subcases, depending on whether the main formula \(N\) is in \(\Pi_1, \Pi_2, \Sigma_1\) or \(\Sigma_2:

\[
\frac{(\Pi_2 \leftarrow \Sigma_2, \Sigma_1 \leftarrow \Pi_1'[N] \Downarrow)}{(N \text{ in } \Pi_1)}
\]

\[
\frac{(\Pi_2 \leftarrow \Sigma_2, \Sigma_1'[N] \leftarrow \Pi_1 \vdash)}{(N \text{ in } \Pi_2)}
\]

\[
\frac{(\Sigma_2'[\cdot (\Sigma_2 \leftarrow \Pi_1) \downarrow \Upsilon)}{(N \text{ in } \Sigma_1)}
\]

\[
\frac{(\Sigma_2'[\cdot (\Sigma_1 \leftarrow \Pi_1) \downarrow \Upsilon)}{(N \text{ in } \Sigma_2)}
\]

for some \(\Upsilon \in \|N\|.\) We receive the desired results by applications of \((A_{1}^{a/b})\):
At the level of structures,

\[ \Pi_1^a \vdash (\Pi_2 \rightsquigarrow \Sigma_2) \rightsquigarrow \Sigma_1 \vdash \Upsilon \]

\[ \Pi_1^b \vdash (\Pi_2 \rightsquigarrow \Sigma_2) \rightsquigarrow \Sigma_1 \vdash \Upsilon \]

\[ \Pi_1 \cdot \Pi_2, \Sigma_2 \cdot \Sigma_1 \vdash \]

\[ \Sigma_1^b \vdash \left( (\Pi_1 \cdot \Pi_2) \cdot (\Pi_2 \rightsquigarrow \Sigma_2) \right) \vdash \Upsilon \]

\[ \Pi_1 \cdot \Pi_2, \Sigma_2 \cdot \Sigma_1 \vdash \]

\[ \Sigma_2^b \vdash (\Sigma_1 \rightsquigarrow \Pi_2) \vdash \Upsilon \]

\[ \Pi_1 \cdot \Pi_2, \Sigma_2 \cdot \Sigma_1 \vdash \]

\[ \Sigma_2^b \vdash (\Sigma_1 \cdot \Pi_2) \vdash \Upsilon \]

\[ \Pi_1 \cdot \Pi_2, \Sigma_2 \cdot \Sigma_1 \vdash \]

We proceed to demonstrate soundness of strong focalization w.r.t. derivability in \( \text{LG/CNL} \). Combined with T 3.29 all that will be left to explicate in order to close the square of F is the correspondence between weak and strong focalization, to which we will dedicate the entirety of §4.

**Definition 3.22.** We define the forgetful maps taking a positive \( P \) or negative \( N \) of \( \text{LG}^\text{pol} (\text{CNL}^\text{pol}) \) into shift-free formulas \( (P)^4 \) and \( (N)^4 \) of \( \text{LG} (\text{CNL}) \):

\[
\begin{align*}
(P \otimes Q)^4 &= \text{def} \ (P)^4 \otimes (Q)^4 \\
(N \otimes P)^4 &= \text{def} \ (N)^4 \otimes (P)^4 \\
(P \otimes N)^4 &= \text{def} \ (P)^4 \otimes (N)^4 \\
(P \lor Q)^4 &= \text{def} \ (P)^4 \lor (Q)^4 \\
(\downarrow N)^4 &= \text{def} \ (N)^4 \\
\end{align*}
\]

At the level of structures, \((\Gamma)^4\) denotes the result of substituting occurrences of \( N \) by \((N)^4\), while leaving positive atoms intact. Finally, for presentations, \((\Pi, \Sigma)^4 = \text{def} \ (\Pi)^4, (\Sigma)^4\).

Our goal is demonstrate \( \Pi, \Sigma \vdash \) implies \( (\Pi)^4, (\Sigma)^4 \vdash \).

**Lemma 3.23.** For any \( \Pi^a, \Sigma \) and \( \Upsilon \), \((\Pi[\Upsilon^b])^4, (\Sigma)^4 \vdash \) iff \((\Pi^a \cdot \Sigma)^4, (\Upsilon)^4 \vdash \).

*Proof.* By induction on \( \Pi^a \). The base case is immediate from \((dp)\). For the inductive cases, we check \( \Pi^a = \Pi_1^a \cdot \Pi_2^a \) and \( \Pi^a = \Pi_1 \cdot \Pi_2^a \):

\[
\begin{align*}
(\Pi_1^a + (\Pi_2 \rightarrow \Pi_2))^4, (\Upsilon)^4 \vdash & \quad 1H \\
(\Pi_1^a \cdot (\Pi_2 \rightarrow \Pi_2))^4, (\Sigma)^4 \vdash & \quad Dp
\end{align*}
\]

the desired result being immediate from \( (\Pi_1^a \cdot \Pi_2^a \cdot \Pi_1^a \cdot (\Pi_2 \rightarrow \Pi_2)) \) and \( (\Pi_1^a \cdot (\Pi_2 \rightarrow \Pi_2)) \cdot (\Sigma)^4 \). In applying the induction hypothesis, we implicitly assumed \( (\Pi_1^a \cdot (\Sigma)^4)^4 = (\Pi_1^a \cdot \Sigma)^4 \), which is easy to check.

**Corollary 3.24.** For any \( \omega^b \) and \( \Pi, (\omega[\Pi]^b) \vdash \) iff \((\omega[\Pi]^b)^4, (\Pi)^4 \vdash \).

**Lemma 3.25.** For any \( P, \Gamma \), if \( \Gamma, (\Sigma)^4 \vdash \) for all \( \Sigma \in \|P\| \), then \( \Gamma, (P)^4 \vdash \). I.e., the following inference is admissible for (unpolarized) \( \text{LG}_5 \), and hence \( \text{LG}_i / \text{CNL} \):

\[
\frac{\Gamma, (\Sigma)^4 \vdash \Sigma \in \|P\|}{\Gamma, (P)^4 \vdash}
\]

*Proof.* By induction on \( P \). If \( P = p \) or \( P = \downarrow N \), \( \|P\| = P \) and the desired result is immediate. For the remaining inductive cases, we consider explicitly \( P = P_1 \lor P_2 \) and \( P = P_1 \cdot P_2 \). The former is demonstrated thus:
Thus, applying the induction hypotheses, we have
\[ \text{(1)} \]
\[
\begin{array}{c}
\{\Gamma, \Pi \vdash_{\Sigma} P_1\} \\
\Gamma_i (P_1) \vdash_{\Pi, \Sigma} \text{IH} \\
\Gamma_i (P_1) \vdash_{\Pi, \Sigma} \text{IH}
\end{array}
\]
noting \( \| P_1 \vee P_2 \| = \| P_1 \| \cup \| P_2 \| \). In case \( P = P_1 \otimes P_2 \), we have
\[
\begin{array}{c}
\Pi \in \| P_1 \| \\
\Sigma \Rightarrow \Gamma, \Pi \otimes \Sigma \vdash_{\Pi, \Sigma} Dp \text{ IH} \\
\Sigma \in \| P_2 \| \\
\Gamma \Rightarrow (P_1)^{\otimes}, (\Sigma)^{\oplus} \vdash_{\Pi, \Sigma} Dp \\
\Gamma, (P_1)^{\otimes} \otimes (P_2)^{\otimes} \vdash_{\Pi, \Sigma} \otimes
\end{array}
\]
noting \( \| P_1 \otimes P_2 \| = \{ \Pi \otimes \Sigma \mid \Pi \in \| P_1 \|, \Sigma \in \| P_2 \| \} \).

**Lemma 3.26.** For any \( N \) and \( \Sigma \in \| N^t \|, (N)^t, (\Sigma)^t \vdash_{\Pi, \Sigma} \).

**Proof.** By induction on \( N \). If \( N = t \) \( P \) or \( N = \overline{p} \), the desired result is immediate by applying (1). For the remaining inductive cases, we check \( N = N_1 \setminus N_2 \) and \( N = N_1 \oplus N_2 \). In the former case, note \( \Sigma \in \| N_1^t \setminus N_2^t \| \) iff \( \Sigma \in \| N_1^t \| \) or \( \Sigma \in \| N_1^t \| \) for the remaining inductive cases, we check
\[
\begin{array}{c}
\frac{\text{IH}}{(N_1)^t, (\Sigma)^t \vdash_{\Pi, \Sigma} dp} \\
\frac{\text{IH}}{(N_2)^t, (\Sigma)^t \vdash_{\Pi, \Sigma} dp}
\end{array}
\]
In case \( N = N_1 \oplus N_2 \), note \( \Sigma \in \| N_1^t \oplus N_2^t \| \) iff \( \Sigma = \Sigma_2 \otimes \Sigma_1 \) for \( \Sigma_1 \in \| N_1^t \| \) and \( \Sigma_2 \in \| N_2^t \| \). Hence, by the induction hypotheses,
\[
\begin{array}{c}
\frac{\text{IH}}{(N_1)^t, (\Sigma_1)^t \vdash_{\Pi, \Sigma} dp} \\
\frac{\text{IH}}{(N_2)^t, (\Sigma_2)^t \vdash_{\Pi, \Sigma} dp}
\end{array}
\]

**Lemma 3.27.** We have the following implications:
\[
\Pi, \Sigma \vdash \implies (\Pi)^h, (\Sigma)^h \vdash \\
\Pi \vdash \Sigma \implies (\Pi)^h, (\Sigma)^h \vdash
\]

**Proof.** By a mutual induction. The case (I) is immediate, so we are left to check

**Case (II), (\( \rightarrow \), (\( \leftarrow \)).** We check (\( \bullet \)), the others being similar. By induction hypothesis, \((\Pi)^h, (\Pi^t)^t \vdash \) and \((\Sigma)^h, (\Sigma^t)^t \vdash \), so we apply (\( \oplus \)): 
C.3.24, but this necessitates writing $\Pi = \text{inductive cases, consider } \Theta$

Suffice it to show the admissibility of (*). In the base case, $\Theta = \text{Lemma, for arbitrary } \Gamma$ and proceeding by induction on $\Pi$.

Next, we check $\Pi = \text{Lemma 3.25 with } \text{Assume (*) } \Theta$

We might try using C.3.24 but this necessitates writing $\Pi[] + \Gamma'$, which need not be defined w.r.t. $X^\prime$, seeing as we cannot assume $\Gamma' \in X^\prime$. Thus, we prove as an additional Lemma, for arbitrary $\Gamma$ and proceeding by induction on $\Pi[]$, admissibility of

instituting $\Gamma$ with $(\Upsilon^-)^{}$ for the desired result. In the base case $\Pi[] = []$, $\Theta$

Next, we check $\Pi[] = \Pi''$, the other inductive cases being handled similarly.

Case (R). Immediate by L.3.25

Case (D). By induction hypothesis, $(\omega^+[])^+$, $(\Sigma^-)^+ \vdash$, while $N, (\Sigma^+)^+ \vdash$ by L.3.26. An easy induction will conform $(\Sigma^-)^+$ and $(\Sigma^+)^+$ are dual, and hence we can apply (T), invoking C.3.24 afterwards:

\[ (N)^+, (\Sigma^+)^+ \vdash L.3.26 \]
\[ (\omega^+[])^+, (\Sigma^-)^+ \vdash IH \]
\[ (\omega[N])^+ \vdash C.3.24 \]

Theorem 3.28. $\Gamma, \Delta \vdash$ in LG/CNL only if $\Pi, \Sigma \vdash$ for all $\Pi \in \|\Gamma\|$, $\Sigma \in \|\Delta\|$.

Proof. Assume (*) $\Theta, \Theta' \vdash$ if $(\forall \Pi \in \|\Theta\|)((\Pi)^+, \Theta' \vdash)$ in LG (CNL). Then

\[ (\forall \Pi \in \|\Theta\|)(\Pi)^+, (\Sigma)^+ \vdash Dp \]
\[ (\forall \Pi \in \|\Gamma\|)(\Pi)^+, (\Delta)^+ \vdash Dp \]

Suffice it to show the admissibility of (*). In the base case, $\Theta = A$ and we apply L.3.25 with $\Upsilon(A)$ if $\Upsilon(A)$ is positive, and with $\Upsilon(A)$ otherwise. For the inductive cases, consider $\Theta = \Theta_1 \bullet \Theta_2$, handled thus:
4 Normalization as Completeness

The current section demonstrates provability in \( \mathbf{LG}_{\text{pol}}(\mathbf{CNL}_{\text{pol}}) \) \((F.9)\) implies focused provability \((F.11)\). Note the converse direction already obtains by composing Theorems 3.28 and 3.9. The standard approach proceeds via Cut elimination, as explained in [19]. Here, instead, we provide a model-theoretic argument along the lines of [23] and [14]. That is, we define phase models for \( \mathbf{LG} \) and \( \mathbf{CNL} \) and construct a syntactic model for which we show ‘truth’ to imply focused provability. Composed with soundness for the derivations of \( F.9 \), the desired result immediately follows. We define our phase models and establish soundness in §4.1, while §4.2 will be dedicated to showing completeness.

4.1 Phase models

**Definition 4.1.** A phase space is a 5-tuple \( \langle P, \cdot, \leftarrow, \rightarrow, \bot \rangle \) where:

1. \( P \) is a non-empty set of phases with operations \( \cdot, \leftarrow, \rightarrow : P \times P \to P \). We use metavariables \( x, y, z \) for denoting elements of \( P \) and \( A, B, C \) for denoting subsets of \( P \).
2. \( \bot \subseteq P \times P \) s.t.

   \[
   \left\{ \begin{array}{c}
   \langle x, y \rangle \in \bot \Rightarrow \langle y, x \rangle \in \bot \\
   \langle x \cdot y, z \rangle \in \bot \leftrightarrow \langle x, y \rightarrow z \rangle \in \bot \\
   \langle x, y \cdot z \rangle \in \bot \leftrightarrow \langle x \leftarrow y, z \rangle \in \bot
   \end{array} \right.
   \]

3. A phase space may be required to satisfy further conditions depending on which structural rules are added to the base logic, as detailed in Table 4.

As usual, we often identify a phase space by its carrier set \( P \). Given a phase space, the operation \( \cdot^\bot : \mathcal{P}(P) \to \mathcal{P}(P) \) is defined by mapping \( A \subseteq P \) to \( \{ x \mid (\forall y \in A)((x, y) \in \bot) \} \).

**Remark 4.2.** If we were to restrict our attention to \( \mathbf{CNL} \), a more parsimonious definition for phase spaces seems naturally available: take any 3-tuple \( \langle P, \cdot, \bot \rangle \), where \( \cdot : P \times P \to P \) and \( \bot \subseteq P \times P \) s.t., for all \( x, y \in P \),

\[
\left\{ \begin{array}{c}
\langle x, y \rangle \in \bot \Rightarrow \langle y, x \rangle \in \bot \\
\langle x \cdot y, z \rangle \in \bot \leftrightarrow \langle x, y \cdot z \rangle \in \bot
\end{array} \right.
\]

The following are some easy observations on the operation \( \cdot^\bot \) on phase spaces.
Lemma 4.3. Given $P$, we have $A \subseteq B \downarrow$ iff $B \subseteq A \uparrow$ ($A, B \in \mathcal{P}(P)$). Equivalently, $A \subseteq A^{\downarrow}$, $A \subseteq B$ implies $B^{\downarrow} \subseteq A^{\downarrow}$ and $(A_{\downarrow})^{\downarrow} \subseteq A^{\downarrow}$. In other words, $\downarrow$ is a Galois connection, and hence $\downarrow$ a closure operator, meaning (at the cost of some redundancy), $A \subseteq A^{\downarrow}$, $A \subseteq B$ implies $A^{\downarrow} \subseteq B^{\downarrow}$, $(A^{\downarrow})^{\downarrow} \subseteq A^{\downarrow}$.

Formulas will be interpreted by facts: subsets $A \subseteq P$ s.t. $A = A^{\downarrow}$. The following is a consequence of the well-known property of closure operators being closed under intersection:

Lemma 4.4. Facts are closed under finite intersections.

Definition 4.5. A model consists of a phase space $P$ and a valuation $v$ taking positive atoms $p$ into facts. $v$ extends to maps $v^{+}(\cdot)$ and $v^{-}(\cdot)$, defined by mutual induction and acting on arbitrary positive and negative formulas respectively:

\[
\begin{align*}
  v^{+}(p) & \quad \overset{\text{def}}{=} v(p) \\
  v^{+}(P \otimes Q) & \quad \overset{\text{def}}{=} v^{+}(P) \times v^{+}(Q) \\
  v^{+}(N \odot P) & \quad \overset{\text{def}}{=} v^{+}(N) \rightarrow v^{+}(P) \\
  v^{+}(P \vee Q) & \quad \overset{\text{def}}{=} v^{+}(P) \cap v^{+}(Q) \\
  v^{+}(\downarrow N) & \quad \overset{\text{def}}{=} v^{-}(N)^{\uparrow} \\
  v^{-}(p) & \quad \overset{\text{def}}{=} v(p) \\
  v^{-}(M \odot N) & \quad \overset{\text{def}}{=} v^{-}(M) \times v^{-}(N) \\
  v^{-}(Q \vee M) & \quad \overset{\text{def}}{=} v^{-}(Q) \rightarrow v^{-}(M) \\
  v^{-}(\uparrow P) & \quad \overset{\text{def}}{=} v^{-}(P)^{\downarrow}
\end{align*}
\]

Here, we have employed the following operations, evidently facts by Lemma 4.3:

\[
\begin{align*}
  \times & : \mathcal{P}(P) \times \mathcal{P}(P) \rightarrow \mathcal{P}(P) & (A, B) \mapsto \{x \cdot y \mid x \in A^{\downarrow}, y \in B^{\downarrow}\}^{\downarrow} \\
  \leftarrow & : \mathcal{P}(P) \times \mathcal{P}(P) \rightarrow \mathcal{P}(P) & (A, B) \mapsto \{x \leftarrow y \mid x \in A^{\downarrow}, y \in B^{\downarrow}\}^{\downarrow} \\
  \rightarrow & : \mathcal{P}(P) \times \mathcal{P}(P) \rightarrow \mathcal{P}(P) & (A, B) \mapsto \{x \rightarrow y \mid x \in A^{\downarrow}, y \in B^{\downarrow}\}^{\downarrow}
\end{align*}
\]

Lemma 4.6. $v^{+}(P) = v^{-}(\uparrow P)^{\downarrow}$ and $v^{-}(N) = v^{+}(\downarrow N)^{\uparrow}$ for any $N, P$.

Proof. Immediate, since the sets involved are facts. \qed

Lemma 4.7. For any $N, P$, $v^{+}(P) = v^{-}(\downarrow P)^{\uparrow}$ and (dually) $v^{-}(N) = v^{+}(N^{\downarrow})$.

Proof. By a straightforward inductive argument. \qed

Lemma 4.8. We have the following equivalences:

\[
\begin{align*}
  v^{+}(\downarrow \Gamma^{\uparrow}) & \subseteq v^{+}(\downarrow \Delta^{\uparrow}) & \iff v^{+}(\uparrow \Gamma^{\downarrow}) \subseteq v^{+}(\uparrow \Delta^{\downarrow}) \\
  v^{+}(\downarrow \Gamma^{\downarrow}) & \subseteq v^{+}(\downarrow \Delta^{\downarrow}) & \iff v^{+}(\uparrow \Gamma^{\uparrow}) \subseteq v^{+}(\uparrow \Delta^{\uparrow})
\end{align*}
\]

Proof. Recalling $\Theta^{+ \downarrow} = \Theta^{\downarrow}$ and $\Theta^{\downarrow +} = \Theta^{\uparrow}$ for arbitrary $\Theta$, we have

\[
\begin{align*}
  v^{+}(\downarrow \Delta^{\downarrow}) & \subseteq v^{+}(\Gamma^{\downarrow}) & \iff v^{+}(\Gamma^{\uparrow}) \subseteq v^{+}(\downarrow \Delta^{\uparrow}) & \quad \text{(Lemma 4.3)} \\
  & \iff v^{+}(\uparrow \Gamma^{\downarrow}) \subseteq v^{+}(\Delta^{\uparrow}) & \quad \text{(Lemma 4.6)}
\end{align*}
\]
and

\[ v^*(\downarrow \Delta^-) \subseteq v^*(\Gamma^+) \quad \text{iff} \quad v^*(\Gamma^+) \subseteq v^*(\downarrow \Delta^-) \quad \text{(Lemma 4.3)} \]
\[ v^*(\Gamma^-) \subseteq v^*(\downarrow \Delta^+) \quad \text{(Lemma 4.4)} \]
\[ v^*(\downarrow \Gamma^+) \subseteq v^*(\Delta^-) \quad \text{(Lemma 4.6)} \]

and similarly \( v^-((\uparrow \Gamma^+)) \subseteq v^-((\Delta^-)) \) iff \( v^-((\uparrow \Delta^+)) \subseteq v^-((\Gamma^-)) \).

We state soundness for sequent derivability in \( \text{LG}^{pol} \) and \( \text{CNL}^{pol}(\text{F}_{\Theta}) \).

**Theorem 4.9.** All phase models satisfy the following implications:

\[ \Gamma, \Delta \vdash \implies v^*(\downarrow \Gamma^-) \subseteq v^*(\Delta^+) \]
\[ \Gamma \vdash P \implies v^*(P) \subseteq v^*(\Gamma^+) \]

**Proof.** By induction, freely applying \( \text{L.4.3} \). Note axioms \( \mathcal{I} \) and \( \mathcal{T} \) trivially reduce to reflexivity and transitivity of set inclusion, while \( \downarrow \) \( \mathcal{L} \) and \( \downarrow \mathcal{R} \) are immediate by \( \text{L.4.3} \). The cases \( \forall \mathcal{L} \) and \( \forall \mathcal{R} \) are equally trivial, reducing to the defining properties of greatest lower bounds. This leaves us to check

**Case** \( \forall \mathcal{P} \). As a typical instance, we check \( \Gamma \bullet \Delta, \Theta \vdash \) if \( \Gamma, \Delta \rightarrow \Theta \vdash \). The following hypotheses will be used:

\[ \text{(IH)} \quad v^*(\downarrow \Gamma^-) \subseteq v^*((\Delta \rightarrow \Theta)^+), \text{iff} \quad v^-((\Gamma^-)^+) \subseteq v^-((\Delta^-)^+) \rightarrow v^*(\Theta^+) \]
\[ (a) \quad x \in v^*((\Theta^-)^+) = v^*(\Theta^+) \]
\[ (b) \quad y \in v^*(\Gamma^+) = v^*(\Gamma^-)^+ \]
\[ (c) \quad z \in v^*(\Delta^+) = v^*(\Delta^-)^+ \]

(IH) being the induction hypothesis. We desire \( v^*((\downarrow (\Gamma \bullet \Delta)^+)) \subseteq v^*(\Theta^+) \), iff \( v^-((\Gamma \bullet \Delta)^+) \subseteq v^*((\Gamma^+ \times v^*(\Delta^+)) \) after unfolding. So assume \( (a)-(c) \). We show \( (x, y, z) \in \downarrow \), iff \( (z \rightarrow x, y) \in \downarrow \). By \( (b) \) and \( (IH) \), \( y \in \{ z \rightarrow x \mid z \in v^*(\Delta^+) \}, x \in v^*(\Theta^+) \) \), so we apply \( (a) \) and \( (c) \).

**Cases** \( \forall \mathcal{L} \), \( \forall \mathcal{L} \), \( \forall \mathcal{L} \). Immediate, upon the realization that \( v^*((P \bullet Q)^+) = v^*(P \otimes Q), v^*((N^+ \rightarrow P)^+) = v^*(N \otimes P) \) and \( v^*((P \rightarrow N^+)^+) = v^*(P \otimes N) \).

**Cases** \( \forall \mathcal{R} \), \( \forall \mathcal{R} \), \( \forall \mathcal{R} \). We explicitly check \( \forall \mathcal{R} \). By the induction hypothesis, \( v^*(N^+) = v^*(N \otimes P) \subseteq v^*(\Delta^+) \) and \( v^*(P) \subseteq v^*(\Gamma^+) \). Thus, by \( \text{L.4.3} \)

\[ \{ x \rightarrow y \mid x \in v^*(\Gamma^+)^+, y \in v^*(\Delta^+)^+ \} \subseteq \{ x \rightarrow y \mid x \in v^*(P)^+, y \in v^*(N)^+ \} \]

with another application of \( \text{L.4.3} \) deriving the desired \( v^*(P \otimes N) \subseteq v^*((\Gamma \leftarrow \Delta)^+) \), noting \( v^*(\Gamma^+ \otimes \Delta^-) = v^*(\Gamma^+) \prec v^*(\Delta^-) = v^*(\Gamma^+) \prec v^*(\Delta^+) \).

**Cases** \( \mathcal{A}_{\mathcal{T}/\mathcal{IV}}^{\mathcal{A}_{\mathcal{T}/\mathcal{IV}}} \), \( \mathcal{A}_{\mathcal{I}/\mathcal{IV}}, \{ \bullet, \cdot \}, \{ \cdot, \bullet \} \). As a typical instance, we check \( \mathcal{A}_{\mathcal{I}}^{\mathcal{I}} \), i.e., \( \Gamma_1 \bullet \Gamma_2 \Delta \bullet \Delta_1 \vdash \) if \( \Gamma_2 \rightarrow \Delta_2, \Delta_1 \rightarrow \Gamma_1 \vdash \). We use the following hypotheses:

\[ \mathcal{F} \quad \{ z \rightarrow u, v \rightarrow y \} \in \downarrow \Rightarrow \{ y \circ z, u \circ v \} \in \downarrow \]
\[ \text{(IH)} \quad v^*((\Gamma_2 \rightarrow \Delta_2)^+) \subseteq v^*((\Delta_1 \leftarrow \Gamma_1)^+) \]
\[ (a) \quad x \in (v^-((\Gamma_1^+)^+) \times v^-((\Delta_2^+)^+)) \]
\[ (b) \quad y \in v^*((\Delta_2^+) \quad v^*(\Delta_2^+) \]
\[ (c) \quad z \in v^*((\Delta_1^+) \]
\[ (d) \quad u \in v^-((\Gamma_1^+)^+) \]
\[ (e) \quad v \in v^*((\Delta_2^+) \]

\[ = v^*(\Gamma_2^+) \]

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recalling (F) to be the frame condition associated with \(A_1^1\). We must show 

\[
v^*(i(\Gamma_1, \Gamma_2)) \subseteq v^*((\Delta_2 \Delta_1))\]

iff \((v^*(\Gamma_1', \Gamma_2')) \subseteq v^*(\Delta_2' \Delta_1')\) by definition unfolding. Thus, we establish \((x, y, z) \in I\) on the assumptions (a)-(c).

By (a), it suffices to show \((y \circ z, u \circ v) \in \perp\) given (d), (e), reducing to \((z \leftarrow u, v \leftarrow y) \in \perp\) by (F), if \((v \leftarrow y, z \leftarrow u) \in \perp\). By (III), (c) and (d), the desired result follows from \(v \leftarrow y \in (v^*(\Gamma_2') \leftarrow v^*(\Delta_2'))^{i}\). But this is a consequence of (b), (e) and the fact that \(v \leftarrow y \in (v^*(\Gamma_2') \leftarrow v^*(\Delta_2'))^{i}\)

4.2 Completeness

The purpose of this section is to demonstrate the completeness of strong focalization w.r.t. the phase models. Like in the previous section, we fix a set \(X\) of formulas relative to which the relevant concepts are defined.

Definition 4.10. Completeness will be established w.r.t. the syntactic (phase) model, defined by taking the structures \(\Pi\) (relative to \(X\)) as phases, setting \((\Pi, \Sigma) \in I\) if \(\Pi, \Sigma \vdash\) and letting \(v(p) = \{p\}^{i} = \{\Pi \mid \Pi, p \vdash\}\). The well-definedness of the syntactic model is a consequence of Lemmas 3.20 and 3.21 the frame conditions for \(\text{CNL}\) being easily checked. The following is the central lemma of this section.

Lemma 4.11. For arbitrary \(N, P, \Pi, \Sigma, \) we have

(i) \(\Pi \in v^*(N)\) implies \(\Pi, \Sigma \vdash\) for all \(\Sigma \in N^+\)

(ii) \((\forall \Pi)((\exists \Sigma \in N^+)((\Pi \vdash \Sigma \Rightarrow \Pi, \Sigma \vdash)\text{ implies } \Sigma \in v^*(N))\)

(iii) \(\Pi \in v^*(P)\) implies \(\Pi, \Sigma \vdash\) for all \(\Sigma \in P^+\)

(iv) \((\forall \Pi)((\exists \Sigma \in P^+)((\Pi \vdash \Sigma \Rightarrow \Pi, \Sigma \vdash)\text{ implies } \Sigma \in v^*(P))\)

Proof. First, note that if for some \(\Sigma \in N^+\) \((\Sigma \in P^+)\) \(\Pi, \Sigma \vdash\), then also \(\Pi, N \vdash (\Pi, P^+) \vdash\) by applying (D). Consequently, (ii) and (iv) imply, respectively, \(N \in v^*(N)\) and \(P^+ \in v^*(P)\). In practice, when invoking the induction hypothesis for (ii) or (iv), we often immediately instantiate them by the latter consequences. To prove (i)-(iv), we proceed by a simultaneous induction on \(P, N\). As typical cases, we check \(p, \perp, N, P \circ N\) and \(P \lor Q\).

Case \(p\). Since \(\{p\} = \{p\}\), it suffices to show \(\Pi \in v^*(p)\) implies \(\Pi, p \vdash\) for (iii), and if \(\Pi, p \vdash\) implies \(\Pi, \Sigma \vdash\) then also \(\Sigma \in v^*(p)\), iff \(\Sigma, p \vdash\) for (iv).

(iii) By definition, as \(v^*(p) = \{p\}^{i}\).

(iv) Immediate from the observation that \(\Pi, p \vdash\) if \(\Pi = p\), as a simple case analysis on (D) will show.

Case \(\perp\). Since \(\{\perp\} = \{\perp\}\), it suffices to show \(\Pi \in v^*(\perp)\) implies \(\Pi, N \vdash\) for (iii), and if \(\Pi \vdash N\) implies \(\Pi, \Sigma \vdash\) then also \(\Sigma \in \perp, N\) for (iv).

(iii) Suppose \(\Pi \in v^*(\perp)\). By (IV)(i), \(N \in v^*(N)\), so that \(\Pi, N \vdash\).

(iv) We show \(\Sigma \in v^*(\perp)\) \(= v^*(N)^{i}\), assuming (a) \(\Pi \vdash N\) implies \(\Pi, \Sigma \vdash\) for any \(\Pi\). Letting (b) \(\Theta \in v^*(N)\), it suffices to ensure \(\Sigma, \Theta \vdash\). (IH(i) and (b) imply \(\Theta, \Theta'\) for all \(\Theta' \in N^{i}\|\), hence \(\Theta \vdash N\) by (R). Thus, \(\Theta, \Sigma \vdash\) by (a), and we apply (dP).
Case $P \otimes N$. We show (iii) and (iv).

(iii) Let (a) $\Pi \in v^*(P \otimes N)$ and (b) $\Sigma \in \|P \otimes N\|$, iff $\Sigma = \Sigma_1 \leftarrow \Sigma_2$ for some $\Sigma_1 \in \|P\|$ and $\Sigma_2 \in \|N\|$. We show $\Pi, \Sigma \vdash$. By (a), it suffices to show $\Sigma_1 \in v^*(P)$ and $\Sigma_2 \in v^*(N)$. I.e., we must ascertain $\Sigma_1, \Upsilon_1 \vdash$ and $\Sigma_2, \Upsilon_2 \vdash$ on the assumptions $\Upsilon_1 \in v^*(P)$ and $\Upsilon_2 \in v^*(N)$. The desired result follows from IH(i), IH(iii), (dp) and (b).

(iv) The following hypotheses will be used:

- (a) $\Pi \vdash \Upsilon$ for some $\Upsilon \in \|P \otimes N\|$ implies $\Pi, \Sigma \vdash$ for all $\Pi$
- (b) $\Upsilon_1 \in v^*(P)$
- (c) $\Upsilon_2 \in v^*(N)$
- (d) $(\forall \Pi_1)((\exists \Upsilon_1 \in \|P\|)(\Pi_1 \vdash \Upsilon_1) \Rightarrow \Pi_1, \Upsilon_2 \vdash)$ implies $\Upsilon_2 \cdot \Sigma \in v^*(P)$
- (e) $\Pi_1 \vdash \Upsilon_1$ for some $\Upsilon_1 \in \|P\|$ implies $\Sigma \Rightarrow \Pi_1 \in v^*(P)$
- (f) $(\forall \Pi_2)((\exists \Upsilon_2 \in \|N\|)(\Pi_2 \vdash \Upsilon_2) \Rightarrow \Pi_1, \Sigma \Rightarrow \Pi_1 \vdash)$ implies $\Sigma \Rightarrow \Pi_1 \in v^*(P)$
- (g) $\Pi_2 \vdash \Upsilon_2$ for some $\Upsilon_2 \in \|N\|

Assuming (a), we show $\Sigma \in v^*(P \otimes N)$. So let (b), (c). Since $\Sigma, \Upsilon_1 \leftarrow \Upsilon_2 \vdash$ iff $\Theta_1, \Upsilon_2 \cdot \Sigma \vdash$ by (dp), it suffices by (b) to prove $\Upsilon_2 \cdot \Sigma \in v^*(P)$. By (d), i.e., IH(iv), we need only prove $\Pi_1, \Upsilon_2 \cdot \Sigma \vdash$ on the assumption (e), iff $\Upsilon_2, \Sigma \Rightarrow \Pi_1 \vdash$ by (dp). Applying (c), we must show $\Sigma \Rightarrow \Pi_1 \in v^*(N)$, which follows from (f), i.e., IH(ii), if we can prove $\Pi_2, \Sigma \Rightarrow \Pi_1 \vdash$ on the assumption (g). By (dp) and (a), taking $\Upsilon = \Upsilon_1 \leftarrow \Upsilon_2$ in the latter case, this follows from $\Pi_1 \leftarrow \Pi_2 \vdash \Upsilon_1 \leftarrow \Upsilon_2$, witnessed by (e), (e) and (f).

Case $P \lor Q$. We show (iii) and (iv).

(iii) List of hypotheses:

- (a) $\Pi \in v^*(P \lor Q)$, iff $\Pi \in v^*(P)$ and $\Pi \in v^*(Q)$
- (b) $\Pi \in v^*(P)$ implies $\Pi, \Sigma \vdash$ for all $\Sigma \in \|P\|$
- (c) $\Pi \in v^*(P)$ implies $\Pi, \Upsilon \vdash$ for all $\Upsilon \in \|Q\|$

Assume (a). The induction hypotheses (b) and (c) immediately imply $\Pi, \Pi' \vdash$ for all $\Pi' \in \|P \lor Q\| = \|P\| \cup \|Q\|$. 

(iv) List of hypotheses:

- (a) $(\exists \Upsilon \in \|P \lor Q\|)(\Pi \vdash \Upsilon$ implies $\Pi, \Sigma \vdash$ for all $\Pi$
- (b) $(\forall \Pi)((\exists \Upsilon' \in \|P\|)(\Pi \vdash \Upsilon' \Rightarrow \Pi, \Sigma \vdash$) implies $\Sigma \in v^*(P)$
- (c) $\Pi \vdash \Upsilon'$ for some $\Upsilon' \in \|P\|

We show $\Sigma \in v^*(P \lor Q) = v^*(P) \lor v^*(Q)$ on the assumption (a). We only prove $\Sigma \in v^*(P)$, with $\Sigma \in v^*(Q)$ following similarly. By the induction hypothesis (b), it suffices to show, for any given $\Pi$, that $\Pi, \Sigma \vdash$ on the assumption (c). Applying (a), we may suffice by demonstrating there exists $\Upsilon$ s.t. $\Pi \vdash \Upsilon$. By (c), we may take $\Upsilon = \Upsilon'$. 

$\square$

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Lemma 4.12. For arbitrary $N, P$, $\| N \| \subseteq v^-(N)^k$, $\| P \| \subseteq v^+(P)^k$

Proof. We show the former statement, the second being dual. Suppose $\Sigma \in \| N \|$. We apply L.4.11(iv) to show $\Sigma \in \| \downarrow N \| (= v^-(N)^\ast)$. So let $\Pi$ be such that for some $\Upsilon \in \| \downarrow N \|$, $\Pi \vdash \Upsilon$. Note, however, $\| \downarrow N \| = \{ N \}$, so that $\Pi \vdash N$. Since the latter is derivable only by $(R)$, we have $\Pi, \Sigma' \vdash$ for all $\Sigma' \in \| N \|$, hence also $\Pi, \Sigma \vdash$.

We state completeness w.r.t. the syntactic model, implying in particular completeness w.r.t. all phase models.

Theorem 4.13. For arbitrary $N, P$, if $v^+(\downarrow N) \subseteq v^+(P)$, then $\Pi, \Sigma \vdash$ for every $\Pi \in \| N \|$ and $\Sigma \in \| P \|$ with $X$ instantiated by $\{ N, P \}$.

Proof. If $X = \{ N, P \}$, then $\| N \|$ and $\| P \|$ are well-defined relative to $X^\ast$ as argued in D.3.16 so that it makes sense to speak of presentations $\Pi, \Sigma$ for every $\Pi \in \| N \|$ and $\Sigma \in \| P \|$. Now, by L.4.12 $\| N \| \subseteq v^-(N)^k = v^+(\downarrow N)$, hence $\| N \| \subseteq v^+(P)$. The desired result follows immediately from L.4.11(iii).

Corollary 4.14. We have $\Gamma, \Delta \vdash$ in LGpol or CNLpol iff $\Pi, \Sigma \vdash$ for all $\Pi \in \| \Gamma^+ \|$ and $\Sigma \in \| \Delta^+ \|$, instantiating $X$ by $\{ \Gamma^-, \Delta^- \}$.

Proof. The direction from left to right follows from composing Theorems 4.9 and 4.13 while the composition of Theorems 3.28 and 3.9 takes care of the other direction.

5 Related Topics

We consider some related topics and directions for future research.

5.1 Synthetic inference rules

Since the works of Girard ([12]) and Andreoli ([3]), the literature on focused proof search has become home to various implementations of synthetic inference rules, mostly concerning classical (linear) logic. The current account borrows a bit from everything, but is perhaps most similar to that of Zeilberger ([24]) in its depiction of the non-invertible phase, while more strongly resembling [3] for the invertible rules. It should be noted, however, that Zeilberger’s work stresses a higher order interpretation of focused proofs through the use of Martin-Löf’s generalized inductive definitions, and proves normalization accordingly.

5.2 Focusing as a semantics of proofs

Following Andreoli, we have explained focusing as a method of streamlining Cut-free backward chaining proof search. Around the same time as Andreoli’s initial [2], however, Girard ([11]) independently published on a similar sequent calculus (weakly focalized, by current terminology) for classical logic, with the aim of restoring the Church-Rosser property for Cut elimination, bypassing Lafont’s critical pairs. In particular, Girard’s results inspired a novel translation into intuitionistic logic, achieving parsimony by making the introduction of double negations contingent upon the polarity of the formula being translated. Focused
derivations thus seem particularly suited to serve as a constructive theory of (classical) proofs, a theme further pursued by Zeilberger ([24]).

The original intended application of LG and CNL being the study of natural language syntax (argued to be similarly resource sensitive), an intuitionistic translation for focused derivations would be expected to similarly benefit investigations of natural language semantics along the line of Montague’s work ([20]). Double negation translations for LG have been previously studied by Bernardi and Moortgat ([6]) for precisely this purpose, although their work does not yet benefit of the structure of focused derivations.

5.3 Normalization by evaluation

Save for the naïve use of the set theoretic language, none of our proofs resort to classical reasoning. In particular, the completeness result of §4.2 proceeds not via the usual proof by contraposition through the construction of counter-models, but rather shows directly that any ‘truth’ in the syntactic model has a focused proof. Thus, through a formalization in a constructive meta language like Martin-Löf type theory or the calculus of constructions, we might hope to explicate the underlying algorithmic content underlying our work, being a mapping of sequent derivations in LG or CNL into focused derivations. Already for intuitionistic logic, such formalizations of constructive completeness proofs have been studied by Coquand ([9]) and Herbelin and Lee ([14]), while Ilik ([15]) additionally considers classical logic. Each of the works cited further stress the connection to normalization by evaluation, first appearing in [5], seeking normalization proofs for the λ-calculus (and later, arbitrary term rewriting systems) making no recourse to the usual reduction relations. We leave the study of such connections for LG and CNL as future research.

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