Representation Theory of Lattice Current Algebras

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Abstract

Lattice current algebras were introduced as a regularization of the left- and right moving degrees of freedom in the WZNW model. They provide examples of lattice theories with a local quantum symmetry $U_q(\mathfrak{g})$. Their representation theory is studied in detail. In particular, we construct all irreducible representations along with a lattice analogue of the fusion product for representations of the lattice current algebra. It is shown that for an arbitrary number of lattice sites, the representation categories of the lattice current algebras agree with their continuum counterparts.

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1 Introduction

Lattice current algebras were introduced and first studied several years ago (see [8, 13] and references therein). They were designed to provide a lattice regularization of the left- and right-moving degrees of freedom of the WZNW model [30] and gave a new appealing view on the quantum group structure of the model. In spite of many similarities between lattice and continuum theory, fundamental relations between them remain to be understood. In this paper we prove the conjecture of [4] that the representation categories of the lattice and continuum model agree.

1.1 Lattice current algebras. Lattice current algebras are defined over a discretized circle, i.e., their fundamental degrees of freedom are assigned to $N$ vertices and $N$ edges of a 1-dimensional periodic lattice. We enumerate vertices by integers $n (mod N)$. Edges are oriented such that the $n^{th}$ edge point from the $(n − 1)^{st}$ to the $n^{th}$ vertex. Being defined over lattices of size $N$, the lattice current algebras come in families $\mathcal{K}_N, N$ a positive integer. A precise definition of these (associative $\ast$)-algebras $\mathcal{K}_N$ is given in the next section. We shall see that elements of $\mathcal{K}_N$ can be assembled into $(s \times s)$-matrices, $J_n$ and $N_n, n \in \mathbb{Z} mod N$, with $\mathcal{K}_N$-valued matrix elements such that

\[ J_n R J_{n+1} = J_{n+1} J_n J_n  \quad , \quad R' J_n J_n = J_n J_n R , \]
\[ R' \hat{N}_n R \hat{N}_n = \hat{N}_n R' \hat{N}_n R , \]
\[ \hat{N}_n J_n = J_n R \hat{N}_n R' , \quad J_n \hat{N}_{n-1} = R \hat{N}_{n-1} R' J_n . \]

(1.1)

To explain notations we view the matrices $J_n, N_n$ as elements in $End(V) \otimes \mathcal{K}_N$, where $V$ is an $s$-dimensional vector space. In this way $J_n = \sum m_{n, \xi} \otimes l_{n, \xi}$ determines elements $m_{n, \xi} \in End(V)$ and $l_{n, \xi} \in \mathcal{K}_N$ which are used to define

\[ \frac{1}{J_n} = \sum m_{n, \xi} \otimes e \otimes l_{n, \xi} , \quad \frac{1}{J_n} = \sum e \otimes m_{n, \xi} \otimes l_{n, \xi} \]

where $e$ is the unit in $End(V)$. Similar definitions apply to $N_n$. Throughout the paper we will use the symbol $\sigma$ for the permutation map $\sigma : End(V) \otimes End(V) \rightarrow End(V) \otimes End(V)$, and application of $\sigma$ to an object $X \in End(V) \otimes End(V)$ is usually abbreviated by putting a prime, i.e. $X' \equiv \sigma(X)$. The matrix $R = R(h) \in End(V) \otimes End(V)$ which appears in eqs. (1.1) is a one-parameter solution of the Yang Baxter Equation (YBE). Such solutions can be obtained from arbitrary simple Lie algebras.

The lattice Kac-Moody algebra $\mathcal{K}_N$ depends on a number of parameters, including the “Planck constant” $h$ in the solution $R(h)$ of the YBE and the “lattice spacing” $\Delta = 1/N$. A first, nontrivial test for the algebraic relations \cite{30} comes from the classical continuum limit, i.e., from the limit in which $h$ and $\Delta = 1/N$ are sent to zero. Using the rules

\[ N_n \sim 1 - \Delta \eta(x) , \quad J_n \sim 1 - \Delta j(x) , \quad R \sim 1 + i \gamma hr \]
with \( x = n/N \), \( \gamma \) being a deformation parameter and the standard prescription
\[
\{\ldots\} = \lim_{\hbar \to 0} \frac{\ldots}{\hbar}
\]
to recover the Poisson brackets from the commutators, one finds that
\[
\{ j_\ell (x), j_\ell (y) \} = \frac{\gamma}{2} [C, j_\ell (x) - j_\ell (y)] \delta(x - y) + \gamma C \delta'(x - y) ,
\]
\[
\{ \eta_\ell (x), \eta_\ell (y) \} = \frac{\gamma}{2} [C, \eta_\ell (x) - \eta_\ell (y)] \delta(x - y) ,
\]
\[
\{ \eta_\ell (x), j_\ell (y) \} = \frac{\gamma}{2} [C, j_\ell (x) - j_\ell (y)] \delta(x - y) + \gamma C \delta'(x - y) .
\]
Here \( C \) is the Casimir element \( C = r + r' = r + \sigma(r) \). For clarity, let us rewrite these relations in terms of components. When we express \( C = t^a \otimes t^a \) and \( j(x) = j_a(x)t^a \) in terms of generators \( t^a \) of the classical Lie algebra, the relations become
\[
\{ j_a(x), j_b(y) \} = \gamma f^c_{ab} j_c(x) \delta(x - y) + \gamma \delta_{ab} \delta'(x - y) ,
\]
\[
\{ \eta_a(x), \eta_b(y) \} = \gamma f^c_{ab} j_c(x) \delta(x - y) ,
\]
\[
\{ \eta_a(x), j_b(y) \} = \gamma f^c_{ab} j_c(x) \delta(x - y) + \gamma \delta_{ab} \delta'(x - y) .
\]
The \( f^c_{ab} \)'s are the structure constants of the Lie algebra, i.e., \([t^a, t^b] = f^c_{ab} t^c\). We easily recognize the first equation as the classical Poisson bracket of the left currents in the WZNW model. Furthermore, the quantity \( j^R(x) \equiv j^R(x) - \eta(x) \) Poisson commutes with \( j^L(x) \equiv j(x) \) and satisfies the Poisson commutation relations of the right currents, i.e.,
\[
\{ j^R(x), j^R(y) \} = -\frac{\gamma}{2} [C, j^R(x) - j^R(y)] \delta(x - y) - \gamma C \delta'(x - y) ,
\]
\[
\{ \eta^R(x), j^R(y) \} = 0 .
\]
Hence we conclude that the lattice current algebra as described in eqs. \([1.1]\) is the quantum lattice counterpart of the classical left and right currents.

One would like to establish a close relationship between the lattice current algebra and its counterpart in the continuum model. A first step in this direction is described in this paper. We find that the representation categories of the lattice and the continuum theory coincide. For this to work, it is rather crucial to combine left- and right-moving degrees of freedom. For instance, the center of the lattice current algebra with only one chiral sector changes dramatically depending on whether the number of lattice sites is odd or even. The only \(*\)-operation known for such algebras \([26]\) is constructed in the case of \( U_q(sl(2)) \) and does not admit straightforward generalizations. However, no such difficulties appear in the full theory. It therefore appears to be rather unnatural to constrain the discrete models to one chiral sector.

1.2 Main results. In the next section we use the methods developed in \([3]\) to provide a precise definition of lattice current algebras. In contrast to the
heuristic definition we use in this introduction, our precise formulation is applicable to general modular Hopf algebras $G$, in particular to $U_q(g)$, for an arbitrary semisimple Lie algebra $g$. The main result of Section 3 provides a complete list of irreducible representations for the lattice current algebras $K_N$.

**Theorem A** (Representations of $K_N$) For every semisimple modular Hopf algebra $G$ and every integer $N \geq 1$, there exists a lattice current algebra $K_N$ which admits a family of irreducible $*$-representations $D_{IJ}^N$ on Hilbert spaces $W_{IJ}^N$. Here the labels $I, J$ run through classes of finite-dimensional, irreducible representations of the algebra $G$.

The two labels $I, J$ that are needed to specify a representation of $K_N$ correspond to the two chiralities in the theory of current algebras. In fact, the algebra $K_1$ is isomorphic to the quantum double of the algebra $G$ [21] and the pairs $I, J$ label its representations. These results are in agreement with the investigation of related models in [22].

Next, we introduce an inductive limit $K_\infty$ of the family of finite-dimensional algebras $K_N$. It can be done using the block-spin transformation [13] $K_N \to K_{N+1}$.

Under this embedding, every irreducible representation of $K_{N+1}$ splits into a direct sum of irreducible representations of $K_N$. It appears that the representation $D_{IJ}^{N+1}$ always splits into several copies of the representation $D_{IJ}^N$. Thus, representations of the inductive limit $K_\infty$ are in one to one correspondence with representations of $K_1$ (or $K_N$ for arbitrary finite $N$).

In order to be able to take tensor products of representations of the lattice current algebras, we introduce a family of homomorphisms $\Lambda_{N,M} : K_{N+M-1} \to K_N \otimes K_M$ which satisfy the co-associativity condition

$$(id \otimes \Lambda_{M,L}) \circ \Lambda_{N,L+M-1} = (\Lambda_{N,M} \otimes id) \circ \Lambda_{M+N-1,L}.$$ 

Let us notice that the co-product $\Lambda_{N,M}$ is supposed to provide a lattice counterpart of the co-product defined by the structure of superselection sectors in algebraic field theory [1], [10], [27], [28].

By combining the co-product $\Lambda_{N,M}$ with the block-spin transformation we construct a new co-product $\Delta_N : K_N \to K_N \otimes K_N$ which preserves the number of sites in the lattice (see subsection 4.3). This co-product is compatible with the block-spin transformation and, hence, it defines a co-product for the inductive limit $K_\infty$:

$$\Delta_\infty : K_\infty \to K_\infty \otimes K_\infty.$$ 

Our second result concerns tensor products of representations of the lattice current algebra $K_\infty$. 
Theorem B (Representation category of the lattice current algebra) The braided tensor categories of representations of the lattice current algebra $\mathcal{K}_\infty$ with the co-product $\Delta_\infty$ and of the Hopf algebra $\mathcal{K}_1$ with the co-product $\Delta_1$ coincide.

In principle, our theory must be modified to apply to $U_q(G), q^p = 1$. It is well known that $U_q(G)$ at roots of unity is not semi-simple. This can be cured by a process of truncation which retains only the “physical” part of the representation theory of quantized universal enveloping algebras. The algebraic implementation of this idea has been explained in [4] and can be transferred easily to the present situation. We plan to propose an alternative treatment in a forthcoming publication.

2 Definition of the Lattice Current Algebra

Our goal is to assign a family of lattice current algebras (parametrized by the number $N$ of lattice sites) to every modular Hopf-*-algebra $G$. Before we describe the details, we briefly recall some fundamental ingredients from the theory of Hopf algebras.

2.1 Semisimple, modular Hopf-algebras. By definition, a Hopf algebra is a quadruple $(G, \epsilon, \Delta, S)$ of an associative algebra $G$ (the “symmetry algebra”) with unit $e \in G$, a one-dimensional representation $\epsilon : G \to \mathbb{C}$ (the “co-unit”), a homomorphism $\Delta : G \to G \otimes G$ (the “co-product”) and an anti-automorphism $S : G \to G$ (the “antipode”). These objects obey a set of basic axioms which can be found e.g. in [1]. The Hopf algebra $(G, \epsilon, \Delta, S)$ is called quasitriangular if there is an invertible element $R \in G \otimes G$ such that

$$ R \Delta(\xi) = \Delta(\xi) R \text{ for all } \xi \in G,$$

$$(id \otimes \Delta)(R) = R_{13} R_{12}, \quad (\Delta \otimes id)(R) = R_{13} R_{23}. $$

Here $\Delta' = \sigma \circ \Delta$, with $\sigma : G \otimes G \to G \otimes G$ the permutation map, and we are using the standard notation for the elements $R_{ij} \in G \otimes G \otimes G$.

For a ribbon Hopf-algebra one postulates, in addition, the existence of a certain invertible central element $v \in \mathcal{G}$ (the “ribbon element”) which factorizes $R' R \in \mathcal{G} \otimes \mathcal{G}$ (here $R' = \sigma(R)$), in the sense that

$$ R' R = (v \otimes v) \Delta(v^{-1}) $$

(see [24] for details). The ribbon element $v$ and the element $R$ allow us to construct a distinguished grouplike element $g \in \mathcal{G}$ by the formula

$$ g^{-1} = v^{-1} \sum S(r^2_\xi) r^1_\xi, $$

where the elements $r^i_\xi$ come from the expansion $R = \sum r^1_\xi \otimes r^2_\xi$ of $R$. The element $g$ is important in the definition of q-traces below.
We want this structure to be consistent with a *-operation on \( \mathcal{G} \). To be more precise, we require that
\[
R^* = (R^{-1})' = \sigma(R^{-1}) , \quad \Delta'(\xi)^* = \Delta(\xi^*) ,
\]
and that \( v, g \) are unitary \footnote{Here we have fixed \(*\) on \( \mathcal{G} \otimes \mathcal{G} \) by \((\xi \otimes \eta)^* = \xi^* \otimes \eta^*\). Following \cite{18}, we could define an alternative involution \( \dagger \) on \( \mathcal{G} \otimes \mathcal{G} \) which incorporates a permutation of components, i.e., \((\xi \otimes \eta)^\dagger = \eta^\dagger \otimes \xi^\dagger\) and \(\xi^\dagger = \xi^*\) for all \(\xi, \eta \in \mathcal{G}\). With respect to \(\dagger\), \(\Delta\) becomes an ordinary \(*\)-homomorphism and \(R\) is unitary.\}. This structure is of particular interest, since it appears in the theory of the quantized universal enveloping algebras \( U_q(\mathfrak{g}) \) when the complex parameter \( q \) has values on the unit circle \cite{18}.

At this point we assume that \( \mathcal{G} \) is semisimple, so that every representation of \( \mathcal{G} \) can be decomposed into a direct sum of finite-dimensional, irreducible representations. From every equivalence class \([I]\) of irreducible representations of \( \mathcal{G} \), we may pick a representative \( \tau_I \), i.e., an irreducible representation of \( \mathcal{G} \) on a \( \delta_I \)-dimensional Hilbert space \( V_I \). The quantum trace \( \text{tr}_q^I \) is a linear functional acting on elements \( X \in \text{End}(V_I) \) by
\[
\text{tr}_q^I(X) = \text{Tr}_I(X\tau_I(g)) .
\]
Here \( \text{Tr}_I \) denotes the standard trace on \( \text{End}(V_I) \) with \( \text{Tr}_I(e_I) = \delta_I \) and \( g \in \mathcal{G} \) has been defined above. Evaluation of the unit element \( e_I \in \text{End}(V_I) \) with \( \text{tr}_q^I \) gives the quantum dimension \( d_I \equiv \text{tr}_q^I(e_I) \).

Furthermore, we assign a number \( S_{IJ} \) to every pair of representations \( \tau_I, \tau_J \),
\[
S_{IJ} = \mathcal{N}(\text{tr}_q^I \otimes \text{tr}_q^J)(R'R')^{IJ} \quad \text{with} \quad (R'R)^{IJ} = (\tau^I \otimes \tau^J)(R'R) ,
\]
for a suitable, real normalization factor \( \mathcal{N} \). The numbers \( S_{IJ} \) form the so-called \( S\)-matrix \( S \). Modular Hopf algebras are ribbon Hopf algebras with an invertible \( S\)-matrix \cite{2}.

Let us finally recall that the tensor product, \( \tau \otimes \tau' \), of two representations \( \tau, \tau' \) of a Hopf algebra is defined by
\[
(\tau \otimes \tau')(\xi) = (\tau \otimes \tau')\Delta(\xi) \quad \text{for all} \quad \xi \in \mathcal{G} .
\]
In particular, one may construct the tensor product \( \tau^I \otimes \tau^J \) of two irreducible representations. According to our assumption that \( \mathcal{G} \) is semisimple, such tensor products of representations can be decomposed into a direct sum of irreducible representations, \( \tau^K \). The multiplicities \( N_{K}^{IJ} \) in this Clebsch-Gordan decomposition of \( \tau^I \otimes \tau^J \) are called fusion rules.\footnote{If a diagonal matrix \( T \) is introduced according to \( T_{IJ} = \pi \delta_{I,J} d_I^2 \tau^I(v) \) (with an appropriate complex factor \( \pi \)), then \( S \) and \( T \) furnish a projective representation of the modular group \( SL(2, \mathbb{Z}) \).}
Among all our assumptions on the structure of the Hopf-algebra \((\mathcal{G}, \epsilon, \Delta, S)\) (quasi-triangularity, existence of a ribbon element \(v\), semisimplicity of \(\mathcal{G}\) and invertibility of \(S\)), semisimplicity of \(\mathcal{G}\) is the most problematic one. In fact it is violated by the algebras \(U_q(\mathcal{g})\) when \(q\) is a root of unity. It is sketched in [4] how “truncation” can cure this problem, once the theory has been extended to weak quasi-Hopf algebras [18].

Example: (Hopf-algebra \(\mathbb{Z}_q\)) We wish to give one fairly trivial example for the algebraic structure discussed so far. Our example comes from the group \(\mathbb{Z}_p\). To be more precise, we consider the associative algebra \(G\) generated by one element \(g\) subject to the relation \(g^p = 1\). On this algebra, a co-product, co-unit and an antipode can be defined by

\[
\Delta(g) = g \otimes g, \quad S(g) = g^{-1}, \quad \epsilon(g) = 1.
\]

We observe that \(\mathcal{G}\) is a commutative semisimple algebra. It has \(p\) one-dimensional representations \(\tau^r(g) = q^r, r = 0, \ldots, p - 1\), where \(q\) is a root of unity, \(q = e^{2\pi i/p}\). We may construct characteristic projectors \(P^r \in \mathcal{G}\) for these representations according to

\[
P^r = \frac{1}{p} \sum_{s=0}^{p-1} q^{-rs} g^s \quad \text{for} \quad r = 0, \ldots, p - 1.
\]

One can easily check that \(\tau^r(P^s) = \delta_{r,s}\). The elements \(P^r\) are employed to obtain a nontrivial \(R\)-matrix,

\[
R = \sum_{r,s} q^{rs} P^r \otimes P^s.
\]

When evaluated with a pair of representations \(\tau^r, \tau^s\) we find that \((\tau^r \otimes \tau^s)(R) = q^{rs}\). The \(R\)-matrix satisfies all the axioms stated above and thus turns \(\mathcal{G}\) into a quasitriangular Hopf-algebra. Moreover, a ribbon element is provided by \(v = \sum q^{-r^2} P_r\). We can finally introduce a \(*\)-operation on \(\mathcal{G}\) such that \(g^* = g^{-1}\). The consistency relations follow from the co-commutativity of \(\Delta\), i.e. \(\Delta' = \Delta\), and the property \(R = R'\). A direct computation shows that the \(S\)-matrix is invertible only for odd integers \(p\). Summarizing all this, we have constructed a family of semisimple ribbon Hopf-\(*\)-algebras \(\mathbb{Z}_q, q = \exp(2\pi i/p)\). They are modular Hopf-algebras for all odd integers \(p\).

2.2 \(R\)-matrix formalism. Before we propose a definition of lattice current algebras, we mention that Hopf algebras \(\mathcal{G}\) are intimately related to the objects \(N_n, n \in \mathbb{Z}_{\text{mod}N}\), introduced in eq. (1.1). To understand this relation, let us introduce another (auxiliary) copy, \(\mathcal{G}_a\), of \(\mathcal{G}\) and let us consider the \(R\)-matrix as an object in \(\mathcal{G}_a \otimes \mathcal{G}\). To distinguish the latter clearly from the usual \(R\), we denote it by \(N_{\pm}\),

\[
N_- \equiv R^{-1} \in \mathcal{G}_a \otimes \mathcal{G}, \quad N_+ \equiv R' \in \mathcal{G}_a \otimes \mathcal{G}.
\]
Quasi-triangularity of the $R$-matrix furnishes the relations

$$
\Delta_a(N_{\pm}) = \tilde{N}_{\pm} \tilde{N}_{\pm}, \quad R \tilde{N}_{+} \tilde{N}_{-} = \tilde{N}_{-} \tilde{N}_{+} R, \quad (2.2)
$$

Here we use the same notations as in the introduction, and $\Delta_a(N_{\pm}) = (\Delta \otimes id)(N_{\pm}) \in G_a \otimes G_a \otimes G$. The subscript $a$ reminds us that $\Delta_a$ acts on the auxiliary (i.e. first) component of $N_{\pm}$. To be perfectly consistent, the objects $R$ in the preceding equations should all be equipped with a lower index $a$ to show that $R \in G_a \otimes G_a$ etc.. We hope that no confusion will arise from omitting this subscript on $R$. The equations (2.2) are somewhat redundant: in fact, the exchange relations on the second line follow from the first equation in the first line. This underlines that the formula for $\Delta_a(N_{\pm})$ encodes information about the product in $G$ rather than the co-product $\Delta$. More explanations of this point follow in Subsection 2.3.

Next, we combine $N_+$ and $N_-$ into one element

$$
N \equiv N_+(N_-)^{-1} = R'R \in G_a \otimes G.
$$

From the properties of $N_{\pm}$ we obtain an expression for the action of $\Delta_a$ on $N$,

$$
\Delta_a(N) = \tilde{N}_{+} \tilde{N}_{+}(N_-)^{-1}(\tilde{N}_-)^{-1} = R^{-1} \tilde{N}_{+} \tilde{N}_{+} R \tilde{N}_-(N_-)^{-1}(\tilde{N}_-)^{-1} = R^{-1} \tilde{N}_-(N_-)^{-1} R \tilde{N}_{+}(\tilde{N}_-)^{-1} = R^{-1} \tilde{N}_R \tilde{N}_R.
$$

As seen above, the formula for $\Delta_a(N)$ encodes relations in the algebra $G$ and implies, in particular, the following exchange relations for $N$:

$$
R' \tilde{N} R \tilde{N} = R'R\Delta_a(N) = R'\Delta'_a(N)R = \tilde{N} R' \tilde{N} R.
$$

This kind of relations first appeared in [23]. We used them in the introduction when describing the objects $N_n$ assigned to the sites of the lattice. Thus we have shown that, for any modular Hopf-algebra $G$, one may construct objects $N$ obeying the desired quadratic relations.

The other direction, namely the problem of how to construct a modular Hopf-algebra $G$ from an object $N$ satisfying the exchange relations described above, is addressed in the next section.

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3The co-product $\Delta$ of $G$ acts on $N_{\pm}$ according to $\Delta(N_{\pm}) = (id \otimes \Delta)(N_{\pm}) = N_{\pm} \tilde{N}_{\pm} \in G_a \otimes G \otimes G$. Here $N_{\pm}$ $[\tilde{N}_{\pm}]$ on the right hand side of the equation have the unit element $e \in G$ in the third [second] tensor factor.
above, is more subtle. To begin with, one has to choose linear maps \( \pi : G_a \to C \) in the dual \( G'_a \) of \( G_a \). When such linear forms \( \pi \in G'_a \) act on the first tensor factor of \( N \in G_a \otimes G \) they produce elements in \( G_a \):

\[
\pi(N) \equiv (\pi \otimes \text{id})(N) \in G \quad \text{for all} \quad \pi \in G'_a .
\]

\( \pi(N) \in G \) will be called the \( \pi \)-component of \( N \) or just component of \( N \). Under certain technical assumptions it has been shown in \cite{6} that the components of \( N \) generate the algebra \( G \). In this sense one can reconstruct the modular Hopf-algebra \( G \) from the object \( N \).

**Lemma 1** \cite{6} Let \( G_a \) be a finite-dimensional, semisimple modular Hopf algebra and \( N \) be the algebra generated by components of \( N \in G_a \otimes N \) subject to the relations

\[
R \frac{1}{a} N R \frac{2}{a} = R \Delta_a(N) ,
\]

where we use the same notations as above. Then \( N \) can be decomposed into a product of elements \( N \pm \in G_a \otimes N \),

\[
N = N_+ N_-^{-1} \quad \text{such that}
\]

\[
\Delta(N) \equiv N_+ \tilde{N} N_-^{-1} \in G_a \otimes N \otimes N ,
\]

\[
\epsilon(N) \equiv e \in G_a , \quad S(N_\pm) \equiv N_\mp^{-1} \in G_a \otimes N
\]

define a Hopf-algebra structure on \( N \). Here, the action of \( \Delta, \epsilon, S \) on the second tensor component of \( N, N_\pm \) is understood. In the equation for \( \Delta(N) \), \( N_+, N_-^{-1} \) are supposed to have a trivial entry in the third component while \( \tilde{N} = \sum n_\varsigma \otimes e \otimes N_\varsigma \) with \( e \) being the unit in \( N \) and \( N = \sum n_\varsigma \otimes N_\varsigma \in G_a \otimes N \). As a Hopf algebra, \( N \) is isomorphic to \( G_a \).

Let us remark that the \( * \)-operation in \( G \) induces a \( * \)-operation in \( N \) which looks as follows:

\[
N_+^* = N_- . \quad (2.3)
\]

In our definition of the lattice current algebras below, we shall describe the degrees of freedom at the lattice sites directly in terms of elements \( \xi \in G \), instead of working with \( N \) (as in the introduction).

The \( \delta_I \)-dimensional representations \( \tau^I \) of \( G_a \cong G \) furnish a \( \delta_I \times \delta_I \)-matrix of linear forms on \( G_a \). When these forms act on the first tensor factor of \( N \), we obtain a matrix \( N^I \in \text{End}(V^I) \otimes G \) of elements in \( G \),

\[
N^I \equiv \tau^I(N) = (\tau^I \otimes \text{id})(N) .
\]

These matrices will turn out to be useful.

**Example:** \((R\text{-matrix formalism for } \mathbb{Z}_q)\) Let us illustrate all these remarks on the example of \( G = \mathbb{Z}_p \). Recall that \( R = \sum q^{rs} P^r \otimes P^s \) and that \( \mathbb{Z}_p \) has only one-dimensional representations given by \( \tau^r(g) = q^r \). Evaluation of the
objects \( N_\pm \) in representations \( \tau^r \) produces elements \( N'_\pm = (\tau^r \otimes id)(N_\pm) \in C \otimes \mathbb{Z}_q \cong \mathbb{Z}_q \). Explicitly, they are given by

\[
N'_\pm = \sum q^{rs} P_s = g^r \quad \text{and} \quad N^r = \sum q^{-rs} P_s = g^{-r} .
\]

Together with the property \( (\tau^r \otimes \tau^s) \Delta \cong \tau^{r+s} \) the relations (2.2) become

\[
g^{\pm(r+s)} = g^{\pm s} g^{\pm r} , \quad q^{rs} g^s g^{-r} = g^{-r} g^s q^{rs} .
\]

For \( N \in \mathbb{Z}_q \otimes \mathbb{Z}_q \) we find

\[
N = \sum q^{2rs} P_r \otimes P_s \quad \text{and} \quad N^r = g^{2r} .
\]

As predicted by the general theory, the elements \( N^r \in \mathbb{Z}_q \) generate the algebra \( \mathbb{Z}_q \) when \( p \) is odd.

2.3 **Definition of \( K_N \).** Next, we turn to the definition of the lattice current algebras \( K_N \) associated to a fixed modular Hopf algebra. Before entering the abstract formalism, it is useful to analyse the classical geometry of the discrete model. Our classical continuum theory contains two Lie-algebra valued fields, namely \( \eta(x) \) and \( j(x) \). To describe a configuration of \( \eta \), for instance, we have to place a copy of the Lie-algebra at every point \( x \) on the circle. On the lattice, there are only \( N \) discrete points left and hence configurations of the lattice field \( \eta \) involve only \( N \) copies of the Lie algebra. When passing from the continuum to the lattice, we encode the information about the field \( j(x) \) in the holonomies along links,

\[
j_n = P \exp\left( \int_n j(x) dx \right) .
\]

Here \( \int_n \) denotes integration along the \( n^{th} \) link that connects the \( (n-1)^{st} \) with the \( n^{th} \) site. The classical lattice field \( j_n \) has values in the Lie group. Let us remark that, even at the level of Poisson brackets, the variables \( j_n \) can not be easily included into the Poisson algebra. The reason is that \( j_n \)'s fail to be continuous functions of the currents. Therefore, we should regularize the Poisson brackets (or commutation relations) of the lattice currents. This regularization is done in the most elegant way with the help of \( R \)-matrices. This consideration explains an immediate appearance of the quantum groups in the description of the lattice current algebras.

In analogy to the classical description of the lattice field \( \eta \), the lattice current algebras contain \( N \) commuting copies of the algebra \( \mathcal{G} \) or, more precisely, \( K_N \) contains an \( N \)-fold tensor product \( \mathcal{G} \otimes N \) of \( \mathcal{G} \) as a subalgebra. We denote by \( \mathcal{G}_n \) the subalgebra

\[
\mathcal{G}_n = e \otimes \ldots \otimes \mathcal{G} \otimes \ldots \otimes e \subset \mathcal{G} \otimes N
\]

where \( \mathcal{G} \) appears in the \( n^{th} \) position and all other entries in the tensor product are trivial. The canonical isomorphism of \( \mathcal{G} \) and \( \mathcal{G}_n \subset \mathcal{G} \otimes N \) furnishes the homomorphisms

\[
\iota_n : \mathcal{G} \to \mathcal{G} \otimes N \quad \text{for all} \quad n = 1, \ldots , N.
\]
We think of the copies $G_n$ of $G$ as being placed at the $N$ sites of a periodic lattice, with $G_n$ assigned to the $n^{th}$ site. In addition, the definition of $K_N$ will involve generators $J_n, n = 1, \ldots, N$. The generator $J_n$ sits on the link connecting the $(n-1)^{st}$ with the $n^{th}$ site.

**Definition 1** The lattice current algebra $K_N$ is generated by components of $J_n \in G_a \otimes K_N, n = 1, \ldots, N$, along with elements in $G \otimes a \otimes K_N$. These generators are subject to three different types of relations.

1. **Covariance properties** express that the $J_n$ are tensor operators transforming under the action of elements $\xi_m \in G_m$ like holonomies in a gauge theory, i.e.,

$$
\iota_n(\xi)J_n = J_n \Delta_n(\xi) \quad \text{for all } \xi \in G \\
\Delta_{n-1}(\xi)J_n = J_n \iota_{n-1}(\xi) \quad \text{for all } \xi \in G \\
\iota_m(\xi)J_n = J_n \iota_m(\xi) \quad \text{for all } \xi \in G, m \neq n, n-1 \mod N.
$$

The covariance relations (2.4) make sense as relations in $G_a \otimes K_N$, if $\iota_n(\xi) \in G_n \subset K_N$ is regarded as an element $\iota_n(\xi) \in G_a \otimes K_N$ with trivial entry in the first tensor factor and $\Delta_n(\xi) \equiv (id \otimes \iota_n)\Delta(\xi) \in G_a \otimes G_n \subset G_a \otimes K_N$.

2. **Functoriality** for elements $J_n$ on a fixed link means that

$$
\frac{1}{J_n} \frac{2}{J_n} = R \Delta_a(J_n)
$$

This is to be understood as a relation in $G_a \otimes G_a \otimes K_N$ where $\Delta_a : G_a \otimes K_N \to G_a \otimes G_a \otimes K_N$ acts trivially on the second tensor factor $K_N$ and $R = R \otimes e \in G_a \otimes G_a \otimes K_N$. The other notations were explained in the introduction. We also require that the elements $J_n$ possess an inverse $J_n^{-1} \in G_a \otimes K_N$ such that

$$
J_n J_n^{-1} = e, \quad J_n^{-1} J_n = e.
$$

3. **Braid relations** between elements $J_n, J_m$ assigned to different links have to respect the gauge symmetry and locality of the model. These principles require

$$
\frac{1}{J_n} \frac{2}{J_m} = \frac{2}{J_m} \frac{1}{J_n} \quad \text{if } n \neq m, m \pm 1 \mod N, \\
\frac{1}{J_n} R \frac{2}{J_{n+1}} = \frac{2}{J_{n+1}} \frac{1}{J_n}
$$

$R$ denotes the element $R \otimes e \in G_a \otimes G_a \otimes K_N$ as before.

---

4Recall that a component of $J_n$ is an element $\pi(J_n) \equiv (\pi \otimes id)(J_n)$ in the algebra $K_N$. Here $\pi$ runs through the dual $G'_a$ of $G_a$. 
The lattice current algebra $\mathcal{K}_N$ contains a subalgebra $\mathcal{J}_N$ generated by components of the $J_n$ only. They are subject to functoriality (2.) and braid relations (3.). The subalgebra $\mathcal{J}_N$ admits an action of $\mathcal{G}^\otimes N$ (by generalized derivations) such that the full lattice current algebra $\mathcal{K}_N$ can be regarded as a semi-direct product of $\mathcal{J}_N$ and $\mathcal{G}^\otimes N$ with respect to this action. Our covariance relations (1.) give a precise definition of the semi-direct product.

Let us briefly explain how Definition 1 is related to the description we used in the introduction. The relation of the Hopf algebras $\mathcal{G}_n$ and the objects $N_n$ has already been discussed. Our covariance relations in eq. (2.4) correspond to the exchange relations between $N$ and $J$ in the third line of eq. (1.1). They can be related explicitly with the help of the quasi-triangularity of $\mathcal{R}$, using the formula $N = R' R_n$. We have, for instance,

$$N_n^2 J_n^1 = (e \otimes (R' R)_n)^1 J_n^1 = (id \otimes \Delta_n)(R' R)_{213} = 1 J_n^1 R(e \otimes (R' R)_n) R' = 1 J_n^1 R \ N_n R' ,$$

where we use the notation $(R' R)_n = (id \otimes \iota_n)(R' R) \in \mathcal{G}_a \otimes \mathcal{G}_a$ and $R = (R \otimes e) \in \mathcal{G}_a \otimes \mathcal{G}_a \otimes \mathcal{K}_N$ as usual. $[\cdot]_{213}$ means that the first and the second tensor factors of the expression inside the brackets are exchanged. For finite-dimensional semisimple modular Hopf algebras $\mathcal{G}$, Lemma 1 implies that one could define $\mathcal{K}_N$ using the objects $N_n \in \mathcal{G}_a \otimes \mathcal{G}_a \subset \mathcal{G}_a \otimes \mathcal{K}_N$ instead of elements $\eta \in \mathcal{G}^\otimes N$. The generators $N_n$ would have to obey the exchange relations stated in eq. (1.1) and

$$N_n^2 R \ N_n^1 = R \Delta_a(N_n) .$$

The functoriality relation for $J_n$ did not appear in the introduction. But we can use it now to derive quadratic relations for the $J_n$ in much the same way as has been done for $N$, earlier in this section. Indeed we find

$$R' J_n^1 J_n^2 = R' R \Delta_a(J_n) = R' \Delta_a(J_n) R$$

$$= J_n^1 J_n^2 R .$$

This exchange relation is the one used in the introduction to describe the lattice currents $J_n$. When the first two tensor factors in this equation are evaluated with representations of $\mathcal{G}_a \cong \mathcal{G}$ one derives quadratic relations for the $\mathcal{K}_N$-valued matrices

$$J_n^I = (\tau^I \otimes id)(J_n) \in End(V^I) \otimes \mathcal{K}_N .$$

As we discussed earlier in this section, elements in the algebra $\mathcal{K}_N$ can be obtained from $J_n$ with the help of linear forms $\pi \in \mathcal{G}_a'$. To understand functoriality properly one must realize that it describes the “multiplication table” for elements $\pi(J_n) \in \mathcal{K}_N$. If we pick two linear forms $\pi_1, \pi_2 \in \mathcal{G}_a'$ on $\mathcal{G}_a$, the corresponding elements in $\mathcal{K}_N$ satisfy

$$\pi_1(J_n)\pi_2(J_n) = (\pi_1 \otimes \pi_2)(R \Delta(J_n)) .$$
We can rewrite this equation by means of the \((twisted)\) associative product \(*\) for elements \(\pi_i \in \mathcal{G}'_a\),

\[ (\pi_1 * \pi_2)(\xi) \equiv (\pi_1 \otimes \pi_2)(R\Delta(\xi)) \ . \]

It allows us to express the product \(\pi_1(J_n)\pi_2(J_n)\) in terms of the element \(\pi_1 * \pi_2 \in \mathcal{G}'_a\),

\[ \pi_1(J_n)\pi_2(J_n) = (\pi_1 * \pi_2)(J_n) \quad \text{for all} \quad \pi_1, \pi_2 \in \mathcal{G}'_a \ . \]

**Remark:** It may help here to invoke the analogy with a more familiar situation. In fact, the Hopf algebra structure of \(\mathcal{G}\) induces the standard (non-twisted) product \(\cdot\) on its dual \(\mathcal{G}'\),

\[ (\pi_1 \cdot \pi_2)(\xi) \equiv (\pi_1 \otimes \pi_2)(\Delta(\xi)) \quad \text{for all} \quad \xi \in \mathcal{G} \ . \]

Let us identify \(\pi \in \mathcal{G}'_a\) with the image \(\pi(T) = (\pi \otimes id)(T)\) of some universal object \(T \in \mathcal{G}_a \otimes \mathcal{G}'\) and insert \(T\) into the definition of the product \(\cdot\),

\[ \pi_1(T)\pi_2(T) \equiv (\pi_1 \otimes \pi_2)(\Delta_\alpha(T)) = (\pi_1 \cdot \pi_2)(T) \ . \]

Here and in the following we omit the \(\cdot\) when multiplying elements \(\pi(T)\). The derived multiplication rules for components of \(T \in \mathcal{G}_a \otimes \mathcal{G}'\) are equivalent to the following functoriality

\[ T \begin{array}{c} 1 \end{array} T \begin{array}{c} 2 \end{array} = \Delta_\alpha(T) \]

and imply \(RTT\)-relations:

\[ R T \begin{array}{c} 1 \end{array} T \begin{array}{c} 2 \end{array} = \begin{array}{c} 2 \end{array} T \begin{array}{c} 1 \end{array} R \ . \]

Such relations are known to define quantum groups, and hence our variables \(J_n\) describe some sort of twisted quantum groups. This fits nicely with the nature of the classical lattice field \(j_n\). As we have noted earlier, the latter takes values in a Lie group.

We saw above that the definition of a product for components of \(J_n\) implies the desired quadratic relations. The converse is not true in general, i.e. functoriality is a stronger requirement than the quadratic relations. In the familiar case of \(U_q(sl_2)\) for example, functoriality furnishes also the standard determinant relations which are usually “added by hand” when algebras are defined in terms of quadratic relations. Due to functoriality we are thus able to develop a universal theory which does not explicitly depend on the specific properties of the Hopf algebra \(\mathcal{G}\).

We have shown that the mathematical definition of \(\mathcal{K}_N\) represented in this section agrees with the description used in the introduction. The algebra \(\mathcal{K}_N\) now appears as a special example of the “graph algebras” defined and studied in [4] to quantize Chern-Simons theories. This observation will enable us to use some of the general properties established there.

The most important one among such general properties is the existence of a \(*\)-operation. On the copies \(\mathcal{G}_n\), a \(*\)-operation comes from the structure of
the modular Hopf-*-algebra $G$. Its action can be extended to the algebra $K_N$
by the formula
\[ J_n^* = S_n^{-1} J_n^{-1} S_{n-1} \quad (2.8) \]
where $S_n = (id \otimes \iota_n) [\Delta(\kappa^{-1})(\kappa \otimes \kappa)R^{-1}] \in G_a \otimes G_n \subset G_a \otimes K_N$. Here $\iota_n : G \to G_n \subset K_N$ is the canonical embedding, and the central element $\kappa \in G$ is a certain square root of the ribbon element $v$, i.e. $\kappa^2 = v$ (cp. [4] for details). Let us note that the formula (2.8) can be rewritten using the elements $N_{n,\pm} \in G_a \otimes G_n$ constructed from the $R$-element according to
\[ N_{n,+} \equiv (id \otimes \iota_n)(R^t) \quad , \quad N_{n,-} \equiv (id \otimes \iota_n)(R)^{-1} \quad . \]
The conjugate current is expressed as
\[ J_n^* = (\kappa_{n-1} \kappa_n)^{-1} N_{n-1}^{-1} J_n^{-1} N_{n-1,-} (\kappa_{n-1} \kappa_n) \quad . \quad (2.9) \]
Here $\kappa_{n-1} = \iota_{n-1}(\kappa)$, $\kappa_n = \iota_n(\kappa)$. In order to verify the property $(J_n^*)^* = J_n$, one uses the following identities:
\[ v_n^{-1} J_n v_{n-1} = v_n N_{n-1}^{-1} J_n \quad , \quad v_n^{-1} J_n v_n = v_n^{-1} J_n N_n , \]
where $v_n$ is the ribbon element in the auxiliary Hopf algebra $G_a$. The ribbon elements $v$ at different lattice sites generate automorphisms of the lattice current algebra which resemble the evolution automorphism in the quantum top [3].

**Example:** *(The $U(1)$-current algebra on the lattice)* It is quite instructive to apply the general definition of lattice current algebras to the case $G = Z_q$. Recall that $Z_q$ is generated by one unitary element $g$ which satisfies $g^p = 1$. Representations $\tau^s$ of $G$ were labeled by an integer $s = 1, \ldots , p-1$, and $\tau^s(g) = q^s$ with $q = \exp(2\pi i/p)$. We can apply the one-dimensional representations $\tau^s$ to the current $J_n \in G_a \otimes K_N$ to obtain elements $J_n^s = \tau^s(J_n) = (\tau^s \otimes id)(J_n) \in K_N$. Functoriality becomes
\[ J_n^s J_n^t = q^{st} J_n^{s+t} , \]
where we have used that $(\tau^s \otimes \tau^t)\Delta_a(\xi) = \tau^{s+t}(\xi)$ and $(\tau^s \otimes \tau^t)(R^{-1}) = q^{-st}$ ($s + t$ is to be understood modulo $p$). The relation allows to generate the elements $J_n^s$ from $J_n^1 \in K_N$. Observe that $J_n^0$ is the unit element $e$ in the algebra $K_N$. From the previous equation we deduce that the $p^{th}$ power of the generator $J_n^1$ is proportional to $e$,
\[ (J_n^1)^p = q^{p(p-1)/2} J_n^p = q^{p(p-1)/2} e . \]
This motivates us to introduce the renormalized generators $w_n = q^{(1-p)/2} J_n^1 \in K_n$ which obey $w_n^p = e$. In the following it suffices to specify relations for the generators $w_n$ and $g_n = \iota_n(g) \in K_N$. The covariance equations (2.4) read
\[ g_n w_n = q w_n g_n \quad , \quad w_n g_{n-1} = q g_{n-1} w_n . \]
since \((\tau^1 \otimes \text{id}) \Delta(g) = qg\). The exchange relations for currents become

\[ w_n w_{n+1} = q^{-1} w_{n+1} w_n . \]

The identity \(\Delta(\kappa^{-1})(\kappa \otimes \kappa) R^{-1} = e \otimes e\) with \(\kappa = \sum q^{-r^2/2} P_r\) finally furnishes

\[ w_n^* = w_n^{-1} . \]

At this point one can easily recognize the algebra of \(w_n\)'s as the lattice \(U(1)\)-current algebra \([12]\).

### 2.4 The right currents

Let us stress that there is a major ideological difference between our discussion of lattice current algebras and the work in \([4]\). In the context of Chern Simons theories, the graph algebras were introduced as auxiliary objects, and physical variables of the theory were to be constructed from objects assigned to the links, i.e. from the variables \(J_n\). Here the \(J_n\)'s represent only the left-currents, and we expect also right-currents to be present in the theory. They are constructed from the variables \(J_n\) and the elements \(\xi_n \in G_n\) and hence give a physical meaning to the graph algebras. We define a family of new variables \(J_R^R \in G_a \otimes K_N\) on the lattice by setting

\[ J_R^R_n = N_{n-1}^{-1} J_R^R_{n-1} N_{n-1} . \]

The \(J_R^R\) turn out to provide the right currents in our theory. For the rest of this section we will use the symbol \(J_L^L\) to denote the original left currents \(J_n\).

**Proposition 2** (Right-currents on the lattice) With \(J_R^R \in G_a \otimes K_N\) defined as above, one finds that

1. the elements \(J_R^R\) and \(J_L^L\) commute for arbitrary \(n, m\),

\[ J_R^R_n J_L^L_m = J_L^L_m J_R^R_n ; \]

2. the elements \(J_R^R\) satisfy the following exchange and functoriality relations

\[ J_R^R_{n+1} R J_R^R_n = J_R^R_n J_R^R_{n+1} , \quad \text{(2.10)} \]

\[ J_R^R_n J_R^R_n = R \Delta_a(J_R^R_n) . \quad \text{(2.11)} \]

(Here we are using the same notations as in the definition of the lattice current algebra above.)

If we denote the subalgebra in \(K_N\) generated by the components of left currents \(J_L^L\) by \(J_L^L_N\) and, similarly, use \(J_R^R_N\) to denote the subalgebra generated by components of \(J_R^R\), the result of this proposition can be summarized in the following statement: \(J_R^R_N\) and \(J_L^L_N\) form commuting subalgebras in \(K_N\), and \(J_R^R_N\) is isomorphic to \((J_L^L_N)^{op}\). Here the subscript \(\text{op}\) means opposite multiplication.
Both statements have their obvious counterparts in the continuum theory (cp. Eq. (1.2)).

**Example:** (*The right U(1)-currents*) We continue the discussion of the U(1)-current algebra on the lattice by constructing the right currents. Our general theory teaches us to consider

\[ w_R^n = g_n w^{-1}_n g_{n-1} \]

Here \( g_n = N_{n,+}^1 = \iota_n(g) \) and \( g_{n-1} = N_{n-1,-}^1 = \iota_{n-1}(g) \). The reader is invited to check that these elements commute with \( w_L^n \).

2.5 **Monodromies.** In the continuum theory one is particularly interested in the behavior of the chiral fields \( g^C(x) = P \exp(\int_0^x j^C(x) dx) \) under rotations by \( 2\pi \), i.e. the monodromy of \( g^C \). Here and in the following, \( C \) stands for either \( L \) or \( R \). The monodromy of \( g^C \) is determined by the expression

\[ m^C = P \exp(\oint j^C(x) dx). \]

Due to the regularizing effect of the lattice, left and right monodromies, \( M^L, M^R \), are relatively easy to construct and control for our discrete current algebra. They are obtained as an ordered product of the chiral lattice holonomies \( J^L_n \) or \( J^R_n \) along the whole circle, i.e.

\[
M^L = v_{a}^1 \cdots J^L_2 \cdots J^L_N \quad \text{and} \quad \quad \quad (2.12)
\]

\[
M^R = v_{a}^1 \cdots J^R_N \cdots J^R_2 J^R_1. \quad \quad (2.13)
\]

When we derive relations for the monodromies, it is convenient to include the factors involving \( v_a = v \otimes e \in G_a \otimes K_N \). The definition in terms of left and right currents produces elements \( M^L \) and \( M^R \) in \( G_a \otimes K_N \). Their algebraic structure differs drastically from the properties of the currents \( J_n \). We encourage the reader to verify the following list of equations:

\[
\tilde{M}^L R \tilde{M}^L = R \Delta_a(M^L) , \quad \tilde{M}^R R \tilde{M}^R = R \Delta_a(M^R) ,
\]

\[
\Delta_0(\xi) M^L = M^L \Delta_0(\xi) , \quad \Delta_0(\xi) M^R = M^R \Delta_0(\xi)
\]

for all \( \xi \in \mathcal{G} \) and with \( \Delta_0(\xi) = (id \otimes \iota_0)(\Delta(\xi)) = (id \otimes \iota_0)(\sigma \circ \Delta(\xi)) \). The functoriality relations are familiar from Subsection 2.2 and imply that the algebra generated by components of the monodromy \( M^L \) or \( M^R \) is isomorphic to \( \mathcal{G} \) or \( \mathcal{G}_{op} \). There are several places throughout the paper where this observation becomes relevant for a better understanding of our results.

As usual, we may evaluate the elements \( M^C \) in irreducible representations of \( \mathcal{G}_a \). This results in a set of \( K_N \) valued matrices \( M^{C,I} = (\tau^I \otimes id)(M^C) \). Their quantum traces

\[
c^I_C \equiv tr^I(M^{C,I})
\]
are elements in the algebra $\mathcal{K}_N$ which have a number of remarkable properties. First, they are central elements in the lattice current algebra $\mathcal{K}_N$, i.e. the $c^I_C$ commute with all elements $A \in \mathcal{K}_N$. Even more important is that $c^I_R, c^I_L \in \mathcal{K}_N$ generate two commuting copies of the Verlinde algebra \[29\]. Explicitly this means that
\[
c^I_C c^J_C = \sum N^I_{KJ} c^K_C \quad \text{and} \quad (c^I_C)^* = c^\bar{K}_C
\]
for $C = R, L$. Here and in the following $\bar{K}$ denotes the unique label such that $N^K_0 = 1$, and 0 stands for the trivial representation $\tau^0 = \epsilon$. If the $S$-matrix $S_{IJ} = N(\text{tr}_q^I \otimes \text{tr}_q^J)(R'R)$ is invertible, and $N$ is suitably chosen, the linear combinations
\[
\chi^I_C = \sum_J N d^I_J S^I_C \text{c}^J_C
\]
provide a set of orthogonal central projectors in $\mathcal{K}_N$, for each chirality $C = R, L$, i.e.,
\[
\chi^I_C \chi^J_C = \delta_{IJ} \chi^I_C \quad \text{and} \quad (\chi^I_C)^* = \chi^I_C
\]
Proofs of all these statements can be found in \[4\]. We will see in the next section that products $\chi^I_L \chi^J_R$ provide a complete set of minimal central projectors in the lattice current algebra or, in other words, they furnish a complete set of characteristic projectors for the irreducible representations of $\mathcal{K}_N$.

Example: (The center of the lattice $U(1)$ current algebra) In terms of the variable $w_n = q^{(1-p)/2}J^1_n = -q^{1/2}J^1_n$ (cf. Subsection 2.3 for notations), the definition of the monodromy $M_{L,1} = (\tau^1 \otimes \text{id})(M^L)$ becomes
\[
M_{L,1} = (-1)^N q^{3N/2 - 1} w_1 w_2 \cdots w_N \in \mathcal{K}_N.
\]
In this particular example, the quantum trace is trivial so that $c^1_L = M_{L,1}$. It is easily checked that $c^1_L$ commutes with all the generators $w_n, g_n \in \mathcal{K}_N$ and that it satisfies
\[
(c^1_L)^p = c^0_L = e.
\]
Of course, this relation follows also from the formula $c^r_L c^s_L = N r^s L c^r_L = c^{r+s}_L$.

Similar considerations apply to the right currents.

2.6 The inductive limit $\mathcal{K}_\infty$. So far, the lattice current algebras $\mathcal{K}_N$ depend on the number $N$ of lattice sites, and one may ask what happens when $N$ tends to infinity. A mathematically precise meaning to this question is provided by the notion of inductive limit. The latter requires an explicit choice of embeddings of lattice current algebras for different numbers of lattice sites. They will come from some kind of inverse block-spin transformation \[13\].

Suppose we are given two lattice current algebras $\mathcal{K}_N$ and $\mathcal{K}_{N+1}$ with generators $J_n, n = 1, \ldots, N$ and $\tilde{J}_m, m = 1, \ldots, N + 1$ respectively. The embeddings of $\mathcal{G}$ into $\mathcal{K}_N$ or $\mathcal{K}_{N+1}$ will be denoted by $\iota_n$ or $\iota_m$. An embedding
\[ \gamma_N : \mathcal{K}_N \rightarrow \mathcal{K}_{N+1} \]

is furnished by

\[
\begin{align*}
\gamma_N(J_n) &= \hat{J}_n \text{ for all } n < N, \\
\gamma_N(J_N) &= v_a^{-1}\hat{J}_N\hat{J}_{N+1} \quad \text{and} \\
\gamma_N(\iota_n(\xi)) &= \hat{\iota}_n(\xi) \quad \text{for all } n < N, \\
\gamma_N(\iota_N(\xi)) &= \hat{\iota}_{N+1}(\xi).
\end{align*}
\]

The intuitive idea behind \( \gamma_N \) is to pass from \( \mathcal{K}_N \) to \( \mathcal{K}_{N+1} \) by dividing the \( N \)th link on the lattice of length \( N \) into two new links, so that we end up with a lattice of length \( N + 1 \). Observe that \( \gamma_N \) maps the monodromies \( M \in G_a \otimes \mathcal{K}_N \) to \( \hat{M} \in G_a \otimes \mathcal{K}_{N+1} \), and, consequently, the same holds for our projectors \( \chi_L^I, \chi_R^J \in \mathcal{K}_N \) and \( \hat{\chi}_L^I, \hat{\chi}_R^J \in \mathcal{K}_{N+1} \),

\[
\gamma_N(\chi_L^I) = \hat{\chi}_L^I, \quad \gamma_N(\chi_R^J) = \hat{\chi}_R^J.
\] (2.14)

Since the set of numbers \( N \) is directed, the collection of \( \mathcal{K}_N \), together with the maps \( \gamma_N \), forms a directed system, and we can define the inductive limit

\[ \mathcal{K}_\infty \equiv \lim_{N \to \infty} \mathcal{K}_N. \]

By definition, \( \mathcal{K}_\infty = \bigcup_N \mathcal{K}_N/\sim \) where two elements \( A_N \in \mathcal{K}_N \) and \( A_{N'} \in \mathcal{K}_{N'} \) are equivalent, iff \( A_N \) is mapped to \( A_{N'} \) by a string of embeddings \( \gamma_M \), i.e.,

\[ A_{N'} = \gamma_{N'-1} \circ \ldots \circ \gamma_{N+1} \circ \gamma_N(A_N). \]

For the lattice \( U(1) \)-current algebra, a detailed investigation of this inductive limit was performed in [3].

We have chosen to define the block-spin transformation by dividing the \( N \)th link of the lattice. Now we introduce another block spin operation by dividing the 1st link of the lattice:

\[
\begin{align*}
\tilde{\gamma}_N(J_n) &= \hat{J}_{n+1} \text{ for all } n > 1, \\
\tilde{\gamma}_N(J_1) &= v_a^{-1}\hat{J}_1\hat{J}_2 \quad \text{and} \\
\tilde{\gamma}_N(\iota_n(\xi)) &= \hat{\iota}_{n+1}(\xi) \quad \text{for all } n.
\end{align*}
\]

Notice that the two block-spin operations ‘commute’ with each other: \( \tilde{\gamma}_{N+1} \circ \gamma_N = \gamma_{N+1} \circ \tilde{\gamma}_N \). While the map \( \gamma_N \) is used in the definition of the inductive limit, we reserve \( \tilde{\gamma}_N \) for the definition of the co-product for lattice current algebras (see Section 4).

### 3 Representations of the Lattice Current Algebra

The stage is now set to describe our main result on the representation theory of the lattice current algebras. We will begin with a much simpler problem of representing two important subalgebras of \( \mathcal{K}_N \). Their representations will
serve as building blocks for the representation theory of the lattice current algebra $K_N$.

3.1 The algebra $U$. The lattice current algebras $K_N$ contain $N - 1$ non-commuting holonomies $U_\nu, \nu = 1, \ldots, N - 1$,

$$U_\nu \equiv v_1^{1-\nu}J_1 \cdots J_\nu \in G_a \otimes K_N.$$ 

The objects $U_\nu$ commute with all elements in the symmetry algebras $G_n$, except the ones for $n = 0$ and $n = \nu$. These properties of $U_\nu$ remind us of holonomies in a gauge theory, which transform nontrivially only under gauge transformations acting at the endpoints of the paths. Together, the elements in $G_0 \otimes G_\nu \subset K_N$ and the components of $U_\nu$ generate a subalgebra, $U_\nu$, of the lattice current algebra $K_N$. These subalgebras $U_\nu \subset K_N$ are all isomorphic to the algebra $U$ which we investigate in this subsection. We begin with a definition of $U$.

**Definition 3** The algebra $U$ is the *-algebra generated by components of elements $U, U^{-1} \in G_a \otimes U$ together with elements in $G_0 \otimes G_1$ such that

$$U^\dagger U = R\Delta_a(U),$$

$$\iota_1(\xi)U = U\Delta_1(\xi), \quad \Delta_0(\xi)U = U\iota_0(\xi)$$

and $U^{-1}$ is the inverse of $U$. Here we use the same notations as in Subsection 2.3. In particular, $\iota_{0,1}$ denote the canonical embeddings of $G$ into $G_0 \otimes G_1$. The *-operation on $U$ extends the *-operation on $G_0 \otimes G_1 \subset U$, so that

$$U^* = S_1^{-1}U^{-1}S_0.$$ 

Here $S_i = (id \otimes \iota_i)(\Delta(\kappa^{-1})(\kappa \otimes \kappa)R^{-1}) \in G_a \otimes U$ for $i = 0, 1$, and $\kappa$ is a certain central square root of the ribbon element, as before.

The algebra $U$ admits a very nice irreducible representation, $D$, which is constructed by acting with elements in $U$ on a “ground state” $|0\rangle$. The state $|0\rangle$ may be characterized by the following (invariance-) properties

$$\iota_i(\xi)|0\rangle = |0\rangle \epsilon(\xi) \quad \text{for all} \quad \xi \in G \quad i = 0, 1$$

where $\epsilon$ is the trivial representation (co-unit) of $G$. Here and in the following we neglect to write the letter $D$ when elements in $U$ act on vectors. Since we are dealing with a unique representation of $U$, ambiguities are excluded. While the preceding formula means that $|0\rangle$ is invariant under the action of $G_0 \otimes G_1$, the components of $U \in G_a \otimes U$ create new states in the carrier space, $\Re$, of the representation $D$,

$$|\pi\rangle \equiv \pi(U)|0\rangle = (\pi \otimes id)(U)|0\rangle \in \Re.$$
for all \( \pi \in \mathcal{G}' \). In particular, one identifies \( |e\rangle = |0\rangle \) because \( \epsilon(U) = U^0 = e \).

We can think of the vectors \( |\pi\rangle \) as coming from a universal object \( u \in \mathcal{G}_a \otimes \mathbb{R} \), i.e.,

\[
|\pi\rangle \equiv \pi(u) = (\pi \otimes \text{id})(u)
\]

In other words, \( u = U|0\rangle \). A complete description of the representation \( D \) on \( \mathbb{R} \) is given in the following proposition.

**Proposition 4** (Representation of \( \mathcal{U} \)) There exists an irreducible \(*\)-representation \( D \) of the algebra \( \mathcal{U} \) on a carrier space \( \mathbb{R} \) such that

\[

U^{1} U^{2} \equiv R\Delta_{a}(u) \quad , \quad (e \otimes \iota_{1}(\xi))u = u(\xi \otimes e) \quad , \quad (e \otimes \iota_{0}(\xi))u = (\mathcal{S}(\xi) \otimes e)u

\]

Here \( u \in \mathcal{G}_a \otimes \mathbb{R} \) and \( \Delta_{a}(u) = (\Delta \otimes \text{id})(u) \). The representation space \( \mathbb{R} \) contains a unique invariant vector \( |0\rangle \in \mathbb{R} \).

**Proof:** The formulas for the action of \( \mathcal{U} \) on \( \mathbb{R} \) follow from \( u = U|0\rangle \in \mathcal{G}_a \otimes \mathbb{R} \) by using the invariance of \( |0\rangle \) under the action of \( \iota_{1}(\xi) \) and \( \iota_{0}(\xi) \). In particular, we have that

\[

U^{1} U^{2} \equiv U^{1} U^{2} |0\rangle = R\Delta_{a}(U)|0\rangle = R\Delta_{a}(u) \quad \text{and} \quad (e \otimes \iota_{1}(\xi))u = U\Delta_{1}(\xi)|0\rangle = U|0\rangle (\text{id} \otimes e)(\Delta(\xi)) = u(\xi \otimes e) \quad .
\]

To derive the action of \( \mathcal{G}_0 \) on \( \mathbb{R} \) we rewrite the equation \( U\iota_{0}(\xi) = \Delta_{0}(\xi)U \) according to

\[

\iota_{0}(\xi)U = (\mathcal{S}(\xi_{1}) \otimes e)U(e \otimes \iota_{0}(\xi_{2})) \quad .
\]

Here we have inserted the expansion \( \Delta(\xi) = \xi_{1} \otimes \xi_{2} \) and used some standard Hopf-algebra properties of the antipode \( \mathcal{S} \). Then one proceeds as in the computation of \( \iota_{1}(\xi)u \) to obtain the last formula claimed in Proposition 4.

Observe that the formulas in the preceding Proposition define an action – not a co-action – of the algebra \( \mathcal{U} \) on the representations space \( \mathbb{R} \). Again, it is important to keep in mind that the co-product \( \Delta \) in the first formula of Proposition 4 acts on the first tensor factor \( \mathcal{G}_a \) of \( u \in \mathcal{G}_a \otimes \mathbb{R} \). In terms of the multiplication \( * \) in \( \mathcal{G}' \) (cf. Subsection 2.3) one has that

\[

\pi_{1}(U)|\pi_{2}\rangle = |\pi_{1} \ast \pi_{2}\rangle
\]

for all \( \pi_{1}, \pi_{2} \in \mathcal{G}' \). Consequently, the components \( \pi(U) \) act on \( \mathbb{R} \) as some kind of (twisted) multiplication operators. Furthermore, the matrix elements of \( u^I = \tau^I(u) \in \text{End}(V^I) \otimes \mathbb{R} \) span a \( \delta^2 \)-dimensional subspace of \( \mathbb{R} \) which is invariant under the action of \( \mathcal{G}_0 \otimes \mathcal{G}_1 \subset \mathcal{U} \). With the help of Proposition 4 we deduce

\[

\iota_{1}(\xi)u^I = u^I \tau^I(\xi) \quad , \quad \iota_{0}(\xi)u^I = \tau^I(\mathcal{S}(\xi))u^I
\]
for all $\xi \in G$ and with $\iota_i(\xi) = (e^I \otimes \iota_i(\xi))$, where $e^I$ is the unit element in $End(V^I)$. The two formulas furnish the following decomposition of $\mathcal{R}$ into a direct sum of $G_0 \otimes G_1$-modules,

$$\mathcal{R} \cong \bigoplus_I \hat{V}^I \otimes V^I ,$$

(3.1)

where $\hat{V}^I$ is dual to $V^I$ and the sum extends over the classes of irreducible representations of $G$. $\hat{V}^0 \otimes V^0 \subset \mathcal{R}$ coincides with the one-dimensional subspace spanned by $|0\rangle$. All these features of the representation space $\mathcal{R}$ resemble those of the algebra of square-integrable functions on a Lie-group $G$ with its characteristic action of left and right invariant vector fields. This similarity is not too surprising and can be traced back to the analogy between the objects $U$ and $T$. In Subsection 2.3, $T$ was found to satisfy “RTT- relations” which are the key ingredient in the deformation theory of groups $G$.

Mainly for technical reasons we finally look at a certain subalgebra $D$ of $U$ and its action on $\mathcal{R}$.

**Lemma 2** Let $D$ be the subalgebra of $U$ which is generated by elements $\xi \in G_1$ and components of $U$. When the representation $D$ of $U$ is restricted to $D \subset U$ it furnishes an irreducible representation $D$ of $D$ on $\mathcal{R}$.

**Proof:** To prove this Lemma we show that every vector $|\pi\rangle$ in the representation space $\mathcal{R}$ is cyclic under the action of $D$. Since $\mathcal{R}$ contains the cyclic vector $|0\rangle$, our task simplifies to the following problem: show that for every $|\pi\rangle \in \mathcal{R}$ there is a representation operator $A_\pi \in D(D)$ such that $A_\pi |\pi\rangle = |0\rangle$. For the proof it is crucial to find the projector on $|0\rangle$ in $D(D)$. It is constructed from the minimal central projector $P^0 \in G$ that corresponds to the trivial representation $\epsilon = \tau^0$. By definition, $P^0$ satisfies $\tau^I(P^0) = \delta_{I,0}$, so that

$$\iota_1(P^0)|\pi'\rangle = |0\rangle \pi'$$

holds for all $\pi' \in G'$. The other ingredient we need below is a distinguished element $\mu \in G'$ – called the right integral of $G$ – with the properties

$$(\mu \otimes id)\Delta(\xi) = \mu(\xi) , \quad \mu(P^0) = 1 .$$

Now we choose $\xi_\pi \in G$ such that $\pi(\xi_\pi) = 1$. A short technical computation shows that

$$(e \otimes \xi_0)\Delta(P^0) = (S(u^{-1}\xi_0) \otimes e)\Delta(P^0) ,$$

where $u = g^{-1}v \in G$ and $g$ was introduced in Subsection 2.1. We abbreviate $\eta_0 \equiv S(u^{-1}\xi_0)$ and regard $\eta_0$ as a map from $G$ to $G$ acting by left multiplication so that $\mu \circ \eta_0$ makes sense as an element in $G'$. Let us define

$$A_\pi \equiv \iota_1(P^0)(\mu \circ \eta_0)(U) .$$
The following calculation proves that $A_\pi |\pi\rangle = |0\rangle$ and hence completes the proof of the lemma.

\[
A_\pi |\pi\rangle = \iota_1(P_0)(\mu \circ \eta_0)(U)|\pi\rangle
= \iota_1(P_0)((\mu \circ \eta_0) \ast \pi)
= |0\rangle((\mu \circ \eta_0) \ast \pi)(P_0)
= |0\rangle(\mu \circ \pi)((\eta_0 \circ \epsilon)R\Delta(P_0))
= |0\rangle(\pi(\xi_0)\mu(P^0)) = |0\rangle .
\]

The algebra $\mathcal{D}$ is a semidirect product of $\mathcal{G}$ and its dual $\mathcal{G}'$, the latter being supplied with the twisted product $\ast$ that we discussed in Subsection 2.3. In this light, $\mathcal{D}$ appears as a close relative of the deformed cotangent bundle $T_q^*\mathcal{G}$ over a group $\mathcal{G}$ which differs from the structure of $\mathcal{D}$ only through the use of the standard product $\cdot$ in $\mathcal{G}'$.

**Example:** Let us continue our tradition and illustrate the theory with the example $\mathcal{G} = \mathbb{Z}_q$. The algebra $\mathcal{U}$ is then generated by the unitary elements $g_0, g_1$ and $w$ satisfying

\[
w^0 = 1 , \quad g_1 w = qwg_1 , \quad wg_0 = qg_0 w
\]

and $g_0$ commutes with $g_1$ (cf. Subsection 2.3 for further details). States in $\mathcal{R}$ are created from a ground state $|0\rangle$ with invariance properties

\[
g_1|0\rangle = |0\rangle \quad \text{and} \quad g_0|0\rangle = |0\rangle.
\]

Through iterated application of $w$ on $|0\rangle$ we may produce $p$ linearly independent vectors

\[
|r\rangle \equiv w^r|0\rangle \in \mathcal{R} \quad \text{for} \quad r = 0, \ldots, p - 1 .
\]

Specializing the proof of Proposition 4 to our example, we obtain

\[
g_1|r\rangle = q^r w^r g_1|0\rangle = q^r|r\rangle
\]

and similarly for $g_0$. This shows that the one-dimensional subspace spanned by $|r\rangle$ corresponds to the summand $\hat{V}^r \otimes V^r$ in the decomposition (3.1) of $\mathcal{R}$. The subalgebra $\mathcal{D}$ of $\mathcal{U}$ is generated by $g = g_1$ and $w$, with Weyl commutation relations

\[
w g = q gw
\]

and $g^p = e = w^p$. It acts irreducibly on the $p$-dimensional space $\mathcal{R}$.

3.2 The algebra $\mathcal{K}$. Before we deal with the general situation, it is helpful to study the simplest example of a lattice current algebra for which the lattice consists of only one (closed) link and one site, i.e., the case $N = 1$. Strictly speaking, $\mathcal{K}_1$ has not been defined above. So we must first give a definition.
Definition 5 (The algebra $\mathcal{K}$) The *-algebra $\mathcal{K} \equiv \mathcal{K}_1$ is generated by components of $M, M^{-1} \in \mathcal{G}_a \otimes \mathcal{K}$ and elements $\xi \in \mathcal{G}$ with the following relations

$$M^1 R M^2 = R \Delta_a(M) ,$$

$$\Delta(\xi)M = M \Delta(\xi) \text{ for all } \xi \in \mathcal{G}$$

$M^{-1}$ is the inverse of $M$, so that $M^{-1}M = e = MM^{-1}$. The action of * is extended from $\mathcal{G}$ to $\mathcal{K}$ by the formula

$$M^* = S^{-1} M^{-1} S ,$$

where $S = \Delta(\kappa^{-1})(\kappa \otimes \kappa)R^{-1}$, as before.

Components $\pi(M)$ of the monodromy $M$ can be represented on the carrier spaces $V^I$ of the representations $\tau^I$ of $\mathcal{G}$. This is accomplished by the formula

$$D^I(\pi(M)) = (\pi \otimes \tau^I)(R'R) \in \text{End}(V^I)$$

for all linear forms $\pi \in \mathcal{G}'_a$ on $\mathcal{G}_a$. An equivalent universal formulation without reference to linear forms $\pi$ is

$$D^I(M) = (id \otimes \tau^I)(R'R) \in \mathcal{G}_a \otimes \text{End}(V^I) .$$

Indeed, one may check that such an action on $V^I$ is consistent with the functoriality of $M$, i.e., with the first relation in Definition 5,

$$D^I(M R M^2) = (id \otimes id \otimes \tau^I)[R'_{13} R_{12} R'_{13} R_{23} R'_{23}]$$

$$= (id \otimes id \otimes \tau^I)[R'_{13} R_{12} R_{13} R_{23}]$$

$$= (id \otimes id \otimes \tau^I)[R_{12} R'_{23} R'_{13} R_{23}]$$

$$= (id \otimes id \otimes \tau^I)[R_{12}(\Delta \otimes id)(R'R)]$$

$$= R(\Delta_a \otimes id)(D^I(M)) .$$

From our discussion in Subsections 2.4, 2.5 we know already that $\mathcal{G}_a \otimes \mathcal{K}$ contains not only the left monodromy $M^L = M$ but also the right monodromy

$$M^R = N^{-1}_- M^L N_+ \in \mathcal{G}_a \otimes \mathcal{K} \text{ with }$$

$$N_+ = R', \quad N_- = R^{-1} .$$

Here $N_{\pm}$ are regarded as elements in $\mathcal{G}_a \otimes \mathcal{G} \subset \mathcal{G}_a \otimes \mathcal{K}$, as before. Since the components of $M^L$ and $M^R$ commute, they can be represented on spaces $V^I \otimes V^J$ such that the right/left-monodromies act trivially on the first/second tensor factor, respectively. This action of left and right monodromies on

$$W^{IJ} \equiv V^I \otimes V^J$$

can be extended to an action of the entire algebra $\mathcal{K}$. Actually, the algebra $\mathcal{K}$ is isomorphic to the Drinfeld double of $\mathcal{G}$ [23, 21].
Proposition 6 (Representations of $K$) The algebra $K$ has a series of irreducible *-representations, $D^{IJ}$, defined on the spaces $W^{IJ}$. Explicitly, the action is given by

\[
D^{IJ}(M) = (id \otimes \tau^J)(R'R) ,
\]

\[
D^{IJ}(\xi) = (\tau^I \boxtimes \tau^J)(\xi) = (\tau^I \otimes \tau^J)(\Delta(\xi)) .
\]

Here $D^{IJ}(M) \in G_a \otimes \text{End}(V^I)$ is regarded as an element of $G_a \otimes \text{End}(W^{IJ})$ with trivial action on the second tensor factor $V^J$ in $W^{IJ}$.

**Proof:** To check the representation property is left as an exercise to the reader. It may be helpful to consult Theorem 12 of [6]. Irreducibility follows from the fact that the action of the monodromies on the spaces $V^I$ is irreducible (cf. Lemma 1 of [6]).

**Example:** For $G = \mathbb{Z}_q$, the definition of $K$ furnishes an algebra with generators $c = c_L^1$ and $g$ such that

\[
c^p = 1 , \quad g^p = 1 \quad \text{and} \quad cg = gc .
\]

Some explanation can be found at the end of Subsection 2.5. The abelian algebra $K$ has one-dimensional representations, $D^{st}$, on spaces $W^{st}$ labeled by two integers $s,t = 0, \ldots, p-1$,

\[
D^{st}(c) = q^s , \quad D^{st}(g) = q^{s+t} ,
\]

where $q = \exp(2\pi i/p)$. For comparison with the general formulas in Proposition 3, one should keep in mind that $g$ is a generator of the algebra $G$ while $c$ coincides with the element $\tau^1(M) \in K$, up to a scalar factor.

3.3 Representations of $K_N$. In representing the full lattice current algebras $K_N$ it is convenient to pass to a new set of generators: Let us define elements $U_\nu \in G_a \otimes K_N, \nu = 1, \ldots, N-1$, as above by

\[
U_\nu \equiv v_1^{1-\nu} J_1 \cdots J_\nu .
\]

Then the $U_\nu$, together with $M \equiv M^L$ and the elements $\xi \in G_n, n = 1, \ldots, N$, generate $K_N$. They obey the following relations

\[
\begin{align*}
U^1_\nu \ U^2_\nu & = R \Delta_a(U_\nu) , \\
M^1 R^2 M & = R \Delta_a(M) , \\
R' \ U^1_\nu \ U^2_\mu & = U^1_\mu \ U^1_\nu \quad \text{for} \quad 1 \leq \nu < \mu \leq N-1 , \\
R' \ U^1_\nu \ M^2 & = M^1 R^1 \ U^1_\nu , \\
\Delta^0(\xi) M & = M \Delta^0(\xi) \quad \text{for} \quad \xi \in G , \\
\iota^0(\xi) U_\nu & = U_\nu \Delta^0(\xi) \quad \text{for} \quad \xi \in G , \\
\Delta^0(\xi) U_\nu & = U_\nu \iota^0(\xi) \quad \text{for} \quad \xi \in G .
\end{align*}
\]
We infer that $\mathcal{K}_N$ contains $(N-1)$ copies of the algebra $\mathcal{U}$ generated by $U_\nu, \mathcal{G}_\nu, \mathcal{G}_0$ and one copy of the algebra $\mathcal{K}$ generated by $M \equiv M^L$ and $\mathcal{G}_0$. These subalgebras do not commute, but the non-commutativity is felt only by $U_\nu, M$ and $\mathcal{G}_0$. In any case, the exchange relations motivate us to look for representations of the lattice current algebra $\mathcal{K}_N$ on spaces

$$W_N^{IJ} \equiv \mathbb{R}^{\otimes N-1} \otimes W^{IJ}.$$  

(3.3)

To state the formulas we introduce the notation $D_\nu, \nu = 1, \ldots, N-1$, which stands for the representation $D$ of the algebra $\mathcal{U}_\nu$ on the $\nu^{th}$ factor of the tensor product (3.3), i.e., for every $X \in \mathcal{U}_\nu$,

$$D_\nu(X) = id_1 \otimes \ldots \otimes id_{\nu-1} \otimes D(X) \otimes id_{\nu+1} \otimes \ldots \otimes id_{N-1} \otimes Id,$$

where $id_\mu$ acts as the identity on the $\mu^{th}$ factor $\mathbb{R}$ in $W_N^{IJ}$, and $Id$ is the identity on $W^{IJ}$. Similarly, $D^{IJ}$ denotes the action of the subalgebra $\mathcal{K} \subset \mathcal{K}_N$ on the last factor $W^{IJ}$ in $W_N^{IJ}$.

The map $\iota_0$ embeds elements of $\mathcal{G}$ into all the subalgebras $\mathcal{U}_\nu$ of the lattice current algebra $\mathcal{K}_N$. Hence $\mathcal{G}$ acts on each tensor factor $\mathbb{R}$ in $W_N^{IJ}$ independently with the help of representations $D_\nu$. We employ the co-product $\Delta$ on $\mathcal{G}$ to obtain a family $\vartheta_\nu, \nu = 1, \ldots, N$, of $\mathcal{G}$-actions on $W_N^{IJ}$. In the following, $\vartheta_1$ is the trivial representation, $\vartheta_1 = \epsilon$, and

$$\vartheta_\nu(\xi) = (D_1 \otimes \ldots \otimes D_{\nu-1})(\iota_0(\xi)) \otimes id_{\nu} \otimes \ldots \otimes id_{N-1} \otimes Id$$

for all $\xi \in \mathcal{G}$. The symbol $\otimes$ denotes the tensor product of representations.

With these conventions we are prepared to define representations of $\mathcal{K}_N$ on $W_N^{IJ}$. The essential idea is borrowed from the well known Jordan-Wigner transformation. In fact, in writing the actions of our generators on $W_N^{IJ}$ we have to relate the $U_\nu$’s and $M$ (which obey non-trivial exchange relations among each other) to the operators $D_\nu(U_\nu)$ and $D^{IJ}(M)$. The latter act on different tensor factors in $W_N^{IJ}$ and hence commute. In analogy with the ‘tail’-factors $\prod_{i=0}^{n} \sigma_i^3$ of the Jordan-Wigner transformation, we will employ tails of $R$-matrices to express the $U_\nu$ in terms of $D_\nu(U_\nu)$ and $M$ in terms of $D^{IJ}(M)$.

Related constructions appear in Majid’s ‘transmutation theory’ (see e.g. [19]).

**Theorem 7** (Representations of $\mathcal{K}_N$) The algebra $\mathcal{K}_N$ has a series of irreducible *-representations $D_N^{IJ}$ realized on the spaces $W_N^{IJ}$ given in equation (3.3). In the representation $D_N^{IJ}$, the generators of $\mathcal{K}_N$ act as

$$D_N^{IJ}(U_\nu) = (id \otimes \vartheta_\nu)(R^{-1})D_\nu(U_\nu),$$

$$D_N^{IJ}(M) = (id \otimes \vartheta_N)(R^{-1})D^{IJ}(M)(id \otimes \vartheta_N)(R),$$

$$D_N^{IJ}(\iota_\nu(\xi)) = D_\nu(\iota_\nu(\xi)) \text{ for } \nu = 1, \ldots, N-1, \xi \in \mathcal{G},$$

$$D_N^{IJ}(\iota_0(\xi)) = (\vartheta_N \otimes D^{IJ})(\Delta_0(\xi)) \text{ for all } \xi \in \mathcal{G}.$$ 

Every *-representation of $\mathcal{K}_N$ can be decomposed into a direct sum of the irreducible representations $D_N^{IJ}$. 


Proof: The proof is similar to the one of Theorem 15 in [3]. Although we do not present the details, we stress that the factors \((id \otimes \vartheta_\nu)(R^{\pm 1})\) produce “tails of \(R\)-elements” which are responsible for the correct exchange relations of \(U_\nu, M\) in the representations \(D^I_J\). From the definition of \(\vartheta_\nu\) and quasitriangularity of \(R\) we infer (for \(\nu > 1\))

\[(id \otimes \vartheta_\nu)(R^{-1}) = R_{12}^{-1} R_{13}^{-1} \ldots R_{1\nu}^{-1}.\]

Here it is understood that the second tensor factor of \(R_1\mu\) is represented on the \(\mu\)th factor \(\Re\) in \(W^I_J\) with the help of \(D_\nu \circ \iota_0\). A simple example suffices to illustrate how nontrivial exchange relations between \(U_\nu, M\) arise in our representations

\[D^I_J(R_1^{12} U_1^{12} U_2^{2}) = R_1^{12} D_1(U_1^{12}) R_2^{23} D_2(U_2^{2}) = R_1^{12} (R_2^{12})^{-1} R_2^{23} D_1(U_1^{12}) D_2(U_2^{2}) = R_2^{23} D_2(U_2^{2}) D_1(U_1^{12}) = D^I_J(U_2^{2} U_1^{1}).\]

These equations are to be understood as equations in \(G_a \otimes G_a \otimes \text{End}(W^I_J)\). To reach the second line, we insert the exchange relation of \(U_1\) with elements \(\iota_0(\xi)\) and quasi-triangularity of \(R^{-1}\). Then we use that the images of \(D_1\) and \(D_2\) commute.

Irreducibility of the representations \(D^I_J\) follows from the case \(N = 1\) which we have treated in Subsection 3.2, together with Lemma 2 of Section 3.1.

It is quite instructive to evaluate the projectors \(\chi^K_L \chi^K_L\) (defined at the end of Subsection 2.5) in the representations \(D^I_J\). The answer is given by (cf. [3])

\[D^I_J(\chi^K_L \chi^K_L) = \delta_{I,K} \delta_{J,L}.\]

As we have promised above, the elements \(\chi^K_L \chi^K_L\) are characteristic projectors for the irreducible representations of the lattice current algebra \(K_N\).

The representation theory described here survives the limit in which \(N\) tends to infinity. With the help of our embeddings \(\gamma_N : K_N \rightarrow K_{N+1}\) (see Section 2.6) we can define an action \(D^I_J(\gamma_N)\) of the lattice current algebra \(K_N\) on \(W^I_J\). It is, of course, no longer irreducible, so that \(W^I_J\) decomposes into a direct sum of irreducible representations of the algebra \(K_N\). The observations made in the preceding paragraph combined with formula (2.14) furnish an isomorphism

\[W^I_J \cong W^I_J \otimes \Re\]

of \(K_N\) modules, with \(\Re\) being a multiplicity space of the reduction. In particular, all the irreducible subrepresentations of the \(K_N\)-action on \(W^I_J\) are isomorphic to \(D^I_J\). This implies that the inductive limit for the directed system \((K_N, \gamma_N)\) splits into independent contributions coming from the simple summands of \(K_N\). Since inductive limits of simple algebras are simple, we
conclude that $\mathcal{K}_\infty$ possesses irreducible representations $D^{IJ}_\infty$ on $W^{IJ}_\infty$. As for the algebras $\mathcal{K}_N$, the labels $I, J$ run through the classes of irreducible representations of $\mathcal{G}$.

**Example:** (Representations of the $U(1)$-current algebra) To discuss the representation theory of the $U(1)$-current algebra, we introduce the generators

$$v_\nu = q^{(\nu-1)/2} w_1 \cdots w_\nu \in \mathcal{K}_N$$

which agree with $U^1_\nu = (\tau^1 \otimes id)(U_\nu)$, up to a scalar factor. The whole algebra $\mathcal{K}_N$ is generated from the elements $g_\nu \in \mathcal{G}_\nu \subset \mathcal{K}_N$, the monodromy $c \in \mathcal{K}_N$ and the holonomies $v_\nu \in \mathcal{K}_N$ such that

$$g_\nu v_\nu = q v_\nu g_\nu, \quad v_\nu g_0 = q g_0 v_\nu, \quad v_\nu v_\mu = q^{-1} v_\mu v_\nu \quad \text{for} \quad \nu < \mu, \quad g_\nu^p = g_\nu^q = c^p = v_\nu^p = 1 \quad \text{for all} \quad \nu = 1, \ldots, N - 1$$

and $c$ commutes with every other element. Vectors in the carrier space of $D^I_N$ are denoted by

$$|r_1, r_2, \ldots, r_{N-1})_{s,t},$$

where $r, s, t = 0, \ldots, p - 1$. We can easily define an action of our generators on such states,

$$v_\nu |r_1, \ldots, r_\nu, \ldots, r_{N-1})_{s,t} = q^{r_1 + \cdots + r_\nu - \nu} |r_1, \ldots, r_\nu + 1, \ldots, r_{N-1})_{s,t},$$

$$g_\nu |r_1, \ldots, r_\nu, \ldots, r_{N-1})_{s,t} = q^\nu |r_1, \ldots, r_\nu, \ldots, r_{N-1})_{s,t},$$

$$g_0 |r_1, \ldots, r_{N-1})_{s,t} = q^{r_1 + \cdots + r_{N-1} + s + t} |r_1, \ldots, r_{N-1})_{s,t},$$

$$c |r_1, \ldots, r_{N-1})_{s,t} = q^s |r_1, \ldots, r_{N-1})_{s,t}.$$

The numerical factor on the right hand side of the first line is an example of the “tail of $R$-elements” discussed above. A similar term usually appears in the action of monodromies $M$ but is absent here. It can be seen that the contributions from $(id \otimes \vartheta_N)(R^{-1})$ and $(id \otimes 
\vartheta_N)(R)$, which occur in the general expression of $D^{IJ}_N(M)$, cancel for $D^I_N(c)$, since all irreducible representations $\tau^r$ of $\mathcal{Z}_q$ are one-dimensional.

From the experience with conformal field theories we expect that the diagonal representation $\bigoplus D^{I1}_N$ on the Hilbert space $\mathcal{H} \equiv \bigoplus I \ W^{I1}_N$ is particularly relevant. Here we just wish to remark that this representation can be realized by a very natural construction. Indeed, it was observed (cf. [4]) that algebras such as $\mathcal{K}_N$ admit a distinguished invariant linear functional $\omega: \mathcal{K}_N \rightarrow \mathbb{C}$,

$$\omega(j_1^I \ldots J^N_{I_N} \xi) = \epsilon(\xi) \delta_{I_1,0} \cdots \delta_{I_N,0}$$

for all $\xi \in \mathcal{G}_n$, $n \in \mathbb{Z} mod N$ and $J^I_n = (\tau^I \otimes id)(J_n)$, as usual. When the quantum dimensions $d_I$ are positive, this functional is positive and hence
furnishes – by the GNS construction – a Hilbert space $\mathcal{H}_\omega$ together with a representation $\pi$ of $K_N$ on $\mathcal{H}_\omega$. If $|0\rangle_\omega$ denotes the GNS vacuum, states in $\mathcal{H}_\omega$ are obtained from 

$$j_1^I \cdots j_N^I |0\rangle_\omega.$$ 

The formula shows that $\mathcal{H}_\omega$ is isomorphic to the diagonal sum $\mathcal{H} = \bigoplus_I W^I_N \cong \mathbb{R}^\otimes N$. An explicit evaluation of $c^I_{R,L}$ on $\mathcal{H}_\omega$ establishes an isomorphism of the two spaces as $K_N$ modules.

**Proposition 8** The GNS-representation arising from the state $\omega : K_N \to \mathbb{C}$ is unitarily equivalent to the diagonal representation $\bigoplus_I W^I_N$ of the lattice current algebra.

The quantum lattice analog of the group-valued local fields of the WZNW model act in this diagonal representation [2, 8].

4 Product of Representations

All continuous current algebras are equipped with a trivial co-product which can be written for Fourier modes of currents $j(n)$ as

$$\Delta(j(n)) = j(n) \otimes 1 + 1 \otimes j(n). \quad (4.4)$$

From the point of view of CFT this co-product is not satisfactory, because it changes the central charge of representations. In the framework of CFT, the central charge is characteristic of the model, and one must define a new co-product which preserves it. Such a co-product is provided by the structure of CFT [15, 20]:

$$\Delta_{\text{CFT}}(j(n)) = j(n) \otimes 1 + 1 \otimes \sum_{k \leq n} C_k^n z^{n-k} j(k). \quad (4.5)$$

Here $C_k^n \equiv n! / k!(n-k)!$ are binomial coefficients. Observe that the co-product $\Delta_{\text{CFT}}$ is not symmetric and explicitly depends on the parameter $z$. The aim of this section is to introduce a lattice counterpart of $\Delta_{\text{CFT}}$.

4.1 Co-product for lattice current algebras. Because lattice current algebras are labelled by the number of lattice sites, it is not necessary that both current algebras on the right hand side of the co-product correspond to chains of the same length. We shall define a family of embeddings

$$\Lambda_{M,N} : K_{N+M-1} \to K_M \otimes K_N \quad (4.6)$$

for any $N$ and $M$. The homomorphisms $\Lambda_{M,N}$ determine an action of $K_{N+M-1}$ on the tensor products $W^I_M \otimes W^L_N$ of representation spaces for $K_M$ and $K_N$.

Pictorially, $\Lambda_{M,N}$ corresponds to gluing two closed chains of length $M$ and $N$ by identifying some site of the first chain with some site of the second
chain. In this way, the co-product $\Lambda_{M,N}$ explicitly depends on the positions of the identified points. This property is similar to the $z$-dependence of $\Delta^z_{\text{CFT}}$.

Below, we always assume that the enumeration starts from the gluing points, so that this extra parameter does not show up in our formulas; in the same fashion one can put e.g. $z = 1$ in the continuum theory.

After gluing we cut the resulting eight-like loop at the middle point and get one connected chain of length $N + M - 1$. Similarly to $\Delta^z_{\text{CFT}}$, the co-product $\Lambda_{M,N}$ depends on the order in which $M$ and $N$ appear. Next, we present the constructive description of $\Lambda_{M,N}$.

Let us denote the left currents of $K_N$ by $J^\beta_n$, $n = 1,\ldots,N$, and similarly by $J^\alpha_m$, $m = 1,\ldots,M$, the left currents of $K_M$. $J^\beta_n$ and $J^\alpha_m$ are regarded as elements in $G \otimes K_M \otimes K_N$ with the property

$$J^\alpha_m \cdot J^\beta_n = J^\beta_n \cdot J^\alpha_m$$

for all $n, m$. In addition to the left currents, we need $N + M$ commuting copies of the symmetry algebra $G$ to generate $K_M \otimes K_N$.

With these notations we can define the announced embedding

$$\Lambda_{M,N} : K_{N+M-1} \to K_M \otimes K_N .$$

It maps the generators $J_\rho, \rho = 1,\ldots,N + M - 1$, of $K_{N+M-1}$ in the following way to generators of $K_M \otimes K_N$:

$$\Lambda_{M,N}(J_\rho) = \begin{cases} J_\rho^\alpha & \text{for } \rho = 1,\ldots,M - 1 \\ J_1^\alpha N_1^\alpha J_1^\beta & \text{for } \rho = M \\ J_\rho^\beta J_\rho^{-M+1} & \text{for } \rho = M + 1,\ldots,N + M - 1 \\ J_N^\beta (N_\rho^\alpha)^{-1} & \text{for } \rho = N + M - 1 \end{cases} \quad (4.7)$$

where $N_\rho^\alpha = N_{0,\rho}^\alpha$. For elements $\xi \in G$, we define

$$\Lambda_{N,M}(\iota_\rho(\xi)) = \begin{cases} \iota_\rho^\alpha(\xi) \otimes e & \text{for } \rho = 1,\ldots,M - 1 \\ e \otimes \iota_\rho^{-M+1}(\xi) & \text{for } \rho = M,\ldots,N + M - 1 \\ (\iota_0^\alpha \otimes \iota_0^\beta)(\Delta(\xi)) & \text{for } \rho = N + M - 1 \end{cases} \quad (4.8)$$

where $e$ means the unit element in $K_M$ or $K_N$.

**Proposition 9** The map $\Lambda_{M,N} : K_{N+M-1} \to K_M \otimes K_N$ defined through eqs. (4.7) and (4.8), is an algebra homomorphism.

One can prove this proposition by directly verifying the defining relations for $K_{N+M-1}$. In order to justify calling $\Lambda_{M,N}$ a co-product, we need some kind of co-associativity. For lattice current algebras this holds in the following form.

**Proposition 10** The family of homomorphisms $\Lambda_{M,N}$ satisfies the following property

$$(id \otimes \Lambda_{N,L}) \circ \Lambda_{M,L+N-1} = (\Lambda_{M,N} \otimes id) \circ \Lambda_{M+N-1,L} .$$

(4.9)
Applying our formulas for $\Lambda_{M,N}$ twice to the generators of $K_{L+M+N-2}$ one easily sees that the homomorphisms on the left and on the right hand sides of (4.9) coincide.

Let us notice that along with the co-product $\Lambda$ one can introduce a co-product $\tilde{\Lambda}$ defined by

$$
\tilde{\Lambda}_{M,N}(J_\rho) = \begin{cases} 
N_+^\beta J_\rho^\alpha & \text{for } \rho = 1 \\
J_\rho^\alpha & \text{for } \rho = 2, \ldots, M-1 \\
J_1^\beta (N_+^\beta)^{-1} J_1^\alpha & \text{for } \rho = M \\
J_{\rho-M+1}^\beta & \text{for } \rho = M+1, \ldots, N+M-1 
\end{cases}
$$

and $\tilde{\Lambda}_{M,N}$ acts on elements $\xi \in G$ according to eq. (4.8) with the $\Delta$ in the last line being replaced by $\Delta'$. These two co-products $\Lambda$ and $\tilde{\Lambda}$ are ‘intertwined’ by the $\ast$-operation. To make a precise statement we introduce an element $K \in K_M \otimes K_N$ by the expression $K = \tilde{\Lambda}_{M,N}(\iota_0(\kappa)) (\iota_0^\ast(\kappa) \otimes \iota_0^\ast(\kappa))^{-1}$. With this notation we have

$$
\Lambda_{M,N}(x^\ast) = K^{-1} \tilde{\Lambda}_{M,N}(x^\ast) K
$$

(4.11)

for all $x \in K_{N+M-1}$. This property is similar to (2.1) where the $\ast$-operation intertwines $\Delta$ and $\Delta'$ and, in fact, it reduces to the latter on elements $\xi \in G_n \subset K_{N+M-1}$.

4.2 Special cases. There are important special cases of the maps $\Lambda_{M,N}$ which we would like to consider in more detail. First, observe that the map

$$
\Lambda_{1,1} : K_1 \rightarrow K_1 \otimes K_1
$$

(4.12)

satisfies the co-associativity condition

$$
(id \otimes \Lambda_{1,1}) \circ \Lambda_{1,1} = (\Lambda_{1,1} \otimes id) \circ \Lambda_{1,1}.
$$

(4.13)

This is a specification of equation (4.9) for the case of $L = M = N = 1$. We conclude that the map $\Delta_1 = \Lambda_{1,1}$ furnishes a co-product for the algebra $K_1$. It has been recently shown [21] that as a Hopf algebra $K_1$ is isomorphic to the Drinfeld double of $G$. In particular, this implies existence of an $R$-matrix for the Hopf-algebra $K_1$.

As we know (see Section 3), the algebra $K_1$ is generated by the elements $\xi \in G$ and by the universal element $M \in G \otimes K_1$. Irreducible representations of $K_1$ are labelled by the pairs $(I, J)$ of irreducible representations of $G$. This implies that as an algebra $K_1$ is isomorphic to $G \otimes G$ (see also [23] where an isomorphism of quasi-triangular Hopf algebras is described).

There is another interesting special choice of chain lengths $N$ and $M$:

$$
\Lambda_{1,N} : K_N \rightarrow K_1 \otimes K_N
$$

(4.14)
The co-associativity condition (4.9) adapted to this case reads

$$\left((\text{id} \otimes \Lambda_{1,N}) \circ \Lambda_{1,N} = (\Lambda_{1,1} \otimes \text{id}) \circ \Lambda_{1,N}\right) . \quad (4.15)$$

This shows that $\Lambda_{1,N}$ also provides a co-action of the Hopf algebra $K_1$ on $K_N$. Such a structure has been noticed already in [2]. We shall see that it permits us to establish a one-to-one correspondence between representations of $K_N$ and $K_1$.

Our last remark on the properties of $\Lambda_{M,N}$ concerns the inductive limit $K_\infty$. Using the block-spin embeddings $K_N \rightarrow K_{N+1}$, one can construct a commutative diagram:

$$\Lambda_{M,N} : K_{N+M-1} \rightarrow K_M \otimes K_N$$

$$\Lambda_{M,N+1} : K_{N+M} \rightarrow K_M \otimes K_{N+1} . \quad (4.16)$$

Commutativity (4.16) ensures that the sequence of homomorphisms $\Lambda_{M,N}$ defines homomorphisms

$$\Lambda_{M,\infty} : K_\infty \rightarrow K_M \otimes K_\infty . \quad (4.17)$$

Then equation (4.9) implies the co-associativity for $\Lambda_{M,\infty}$:

$$\left((\text{id} \otimes \Lambda_{N,\infty}) \circ \Lambda_{M,\infty} = (\Lambda_{M,N} \otimes \text{id}) \circ \Lambda_{N+M-1,\infty}\right) . \quad (4.18)$$

Thus, $\Lambda_{M,\infty}$ provides a co-module structure for $K_\infty$ with respect to the family $K_M$. In particular, $K_\infty$ is a co-module over $K_1$.

4.3 Implications for representation theory. The co-product $\Lambda_{M,N}$ yields a notion of tensor product for representations of the algebras $K_M$ and $K_N$.

**Definition 11** The representation $D$ of the algebra $K_{N+M-1}$ is called a tensor product of the representations $D_M$ of $K_M$ and $D_N$ of $K_N$ if it acts on the tensor product of the corresponding vector spaces $W_M \otimes W_N$ according to the following formula:

$$D(x) = (D_M \otimes D_N)\Lambda_{M,N}(x) \quad (4.19)$$

for all elements $x$ of the lattice current algebra $K_{N+M-1}$. The resulting representation will be denoted by $D_M \boxtimes D_N$.

We would like to analyse the structure of this new tensor product. We denote by 0 the trivial representation of the symmetry Hopf algebra.

**Proposition 12** For any $M$ and $N$ and for arbitrary labels $I$ and $J$ of the representations of the symmetry algebra the following representations of $K_{M+N-1}$ are isomorphic:

$$D_{M+N-1}^{IJ} \simeq D_M^{IJ} \boxtimes D_N^{00} \simeq D_N^{00} \boxtimes D_M^{IJ} \quad . \quad (4.20)$$

In this sense, tensoring with the vacuum representation $D_N^{00}$ is trivial.
To prove this proposition one first checks that on the spaces $W_{M}^{I} \otimes W_{N}^{0}$ and $W_{N}^{0} \otimes W_{M}^{I}$ the central elements of $\mathcal{K}_{M+N-1}$ have eigenvalues corresponding to the representation $D_{M+N-1}^{I}$. Then one checks that the dimensions of all these spaces coincide with the dimension of $W_{M+N-1}^{I}$. This completes the proof.

Observe that the tensor product of representations that we have introduced, relates representations of different algebras. For instance, if we take $N = M$, we obtain a representation of the algebra $\mathcal{K}_{2N-1}$ on the tensor product of representation spaces of $\mathcal{K}_{N}$. The idea now is to embedd the algebra $\mathcal{K}_{N}$ into $\mathcal{K}_{2N-1}$ with the help of the block spin maps $\tilde{\gamma}_{M}$,

$$\tilde{\gamma}_{2N-1,N} := \tilde{\gamma}_{2N-2} \circ \cdots \circ \tilde{\gamma}_{N+1} \circ \tilde{\gamma}_{N} : \mathcal{K}_{N} \rightarrow \mathcal{K}_{2N-1} .$$

In this way we may represent the algebra $\mathcal{K}_{N}$ on tensor products of its own representation spaces. The resulting representation certainly has a huge commutant and is not appropriate to describe the representation theory of $\mathcal{K}_{N}$. We shall define a certain projection operator $\mathcal{P}_{N}^{0} \in \mathcal{K}_{N} \otimes \mathcal{K}_{N}$ that projects to more interesting subrepresentation.

The construction of $\mathcal{P}_{N}^{0}$ proceeds as follows. Notice that the lattice current algebra $\mathcal{K}_{N}$ contains $N - 1$ local projectors $p_{i} \in \mathcal{G}_{\alpha} \subset \mathcal{K}_{N}$ where $\alpha$ runs from 1 to $N - 1$. They are uniquely determined by the property $\iota_{i}(\xi)p_{i} = \epsilon(\xi)p_{i}$ for all $\xi \in \mathcal{G}$. An explicit formula for $p_{i}$ in terms of the objects $N_{i}$ can be obtained along the lines of Subsection 2.5. When $N_{i}$ replaces the monodromy $M$ then we obtain $p_{i}$ instead of $\chi^{0}$. These projectors $p_{i}$ commute with each other, i.e., $p_{i}p_{j} = p_{j}p_{i}$ for all $i, j$ so that their product defines again a projector $\mathcal{P}_{N}$,

$$\mathcal{P}_{N} := \prod_{i=1}^{N-1} p_{i} \in \mathcal{K}_{N} .$$

From the defining relations of $\mathcal{K}_{N}$ it is fairly obvious that $\mathcal{P}_{N}$ commutes with $N_{0}$ and the monodromy $M$. In the following, $\mathcal{P}_{N}^{\alpha}$ will denote the projector $\mathcal{P}_{N} \otimes e \in \mathcal{K}_{N} \otimes \mathcal{K}_{N}$ and similarly $\mathcal{P}_{N}^{\beta} = e \otimes \mathcal{P}_{N} \in \mathcal{K}_{N} \otimes \mathcal{K}_{N}$.

With these objects at hand, we are now able to define a new co-product $\Delta_{N}$ for the algebra $\mathcal{K}_{N},$

$$\Delta_{N}(x) := \mathcal{P}_{N}^{0} \Lambda_{N,N}(\tilde{\gamma}_{2N-1,N}(x)) \quad \text{for all} \quad x \in \mathcal{K}_{N} . \quad (4.21)$$

It is easy to check that $\Delta_{N}$ defines a homomorphism because $\tilde{\gamma}_{2N-1,N}$ and $\Lambda_{N,N}$ are homomorphisms and the projector $\mathcal{P}_{N}^{0}$ commutes with the image of $\Lambda_{N,N} \circ \tilde{\gamma}_{2N-1,N} : \mathcal{K}_{N} \rightarrow \mathcal{K}_{N} \otimes \mathcal{K}_{N}$. Let us also mention that $\Delta_{N}(\mathcal{P}_{N} x) = \mathcal{P}_{N} \Delta_{N}(x)$ so that the co-associativity of $\Delta_{N}$ follows from that of $\Lambda_{N,M}$,

$$(\Delta_{N} \otimes id)\Delta_{N}(x) = (id \otimes \Delta_{N})\Delta_{N}(x) \quad \text{for all} \quad \xi \in \mathcal{K}_{N} .$$

In deviation from the standard properties of co-products, $\Delta_{N}$ is not unit preserving, i.e., $\Delta_{N}(e) \neq e \otimes e \in \mathcal{K}_{N} \otimes \mathcal{K}_{N}$ ($e \in \mathcal{K}_{N}$ denotes the unit element),
and there is no one-dimensional trivial representation of $\mathcal{K}_N$. The role of the co-unit is actually played by the vacuum representation $D_{N}^{00}$ of $\mathcal{K}_N$. Such algebraic properties are characteristic for *weak Hopf-algebras* [7] and the closely related *weak quasi-Hopf algebras* of [8].

Notice, that the co-product $\Delta_N$ is compatible with the block spin operation: $(\gamma_N \otimes \gamma_N)\Delta_N = \Delta_{N+1}$, $\gamma_N$. This property is ensured by the fact that two block spin operations $\gamma_N$ and $\tilde{\gamma}_N$ commute with each other (see Section 2). Thus, one can define an operation $\Delta_\infty : \mathcal{K}_\infty \rightarrow \mathcal{K}_\infty \otimes \mathcal{K}_\infty$ which provides a co-product for the inductive limit of lattice current algebras.

We would finally like to compare the representation category of $\mathcal{K}_N$ with that of $\mathcal{K}_1$. To this end notice that the formula

$$D_{N}^{IJ} \simeq D_{1}^{IJ} \otimes D_{N}^{00}$$

provides a one-to-one correspondence between representations of $\mathcal{K}_1$ and $\mathcal{K}_N$ for arbitrary $N$. In fact, this implies the same kind of correspondence for representations of $\mathcal{K}_1$ and $\mathcal{K}_\infty$. Because $\mathcal{K}_N$ is semisimple, for all $N$, the isomorphism (4.22) induces a map

$$F_N(D_1) = D_N$$

which assigns to each representation $D_1$ of the algebra $\mathcal{K}_1$ a representation $D_N$ of the algebra $\mathcal{K}_N$.

To describe the properties of the map $F$, it is convenient to use the language of the theory of categories (see e.g. [7]). It is clear that the map $F_N$ is invertible and that it defines a co-variant tensor functor mapping the category of representations of the Hopf-algebra $\mathcal{K}_1$ into the category of representations of the lattice current algebras $\mathcal{K}_N$. Actually, on the image of $D_{N}^{00}$, the tensor product of representations of $\mathcal{K}_N$ defined through $\Delta_N$ is isomorphic to the representation obtained with the help of the co-action $\Lambda_{1,N}$. Since tensor operators for the latter may be trivially identified with tensor operators of the quasitriangular Hopf-algebra $\mathcal{K}_1$, the functor $F_N$ provides an equivalence of braided tensor categories. In the limit $N \rightarrow \infty$ we arrive at the following conclusion.

**Theorem 13** *The functor $F$ establishes an isomorphism between representations the Hopf algebra $\mathcal{K}_1$ and the lattice current algebra $\mathcal{K}_\infty$ which is compatible with a co-products of $\mathcal{K}_1$ and $\mathcal{K}_\infty$ and establishes an equivalence of braided monoidal categories.*

We can view this fact as the lattice analogue of a theorem in [16], [14] on the equivalence of tensor categories corresponding to quantum groups and current algebras. Here the algebra $\mathcal{K}_\infty$ replaces the current algebra, and $\mathcal{K}_1 = \mathcal{G} \otimes \mathcal{G}$ is the direct product of two quantum groups corresponding to two chiral sectors. From this point of view, finding an exact relationship between lattice and continuum current algebras emerges as a challenging problem.
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