A hybrid approach for modelling multidimensional degradation processes

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Abstract. Degradation processes are often multidimensional. Modeling such degradation processes needs to address two key issues: indicator fusion and degradation model selection; and they have been separately addressed in the literature. This paper proposes a hybrid approach to jointly address these two issues. The proposed approach first fuses multiple degradation indicators into a composite degradation indicator. This fusion step involves data normalization, aggregation model selection and determination of indicator weights. After the fusion step, the problem becomes one-dimensional, and the existing method to select the degradation model for a one-dimensional degradation process can be applied. The resulting model obtained from the proposed approach can be a two-phase model; and the model for the second phase has a closed-form expression. This considerably facilitates residual life prediction. A real-world example is included to illustrate the proposed approach and its appropriateness.

1. Introduction
Degradation processes are often multidimensional [1-3], and a fusion technique is usually applied to combine all the degradation indicators into a composite degradation indicator [4]. The indicator fusion needs to address several issues such as data normalization, aggregation model selection and determination of indicator weights. To reduce the subjectivity, the weights are usually derived using a mathematical method [5]; and this may require estimating the time to the failure threshold of a degradation path using interpolation or extrapolation.

The stochastic process models for modeling one-dimensional degradation processes can be age-, state- and age-state-dependent. The traditional methods to build such models are usually to directly fit a set of degradation data to an assumed model. Since the assumption on the degradation model can be unrealistic, the resulting model can be unreliable. To improve, Jiang presents an innovative approach to aid model selection and the basic idea is as follows [6]. A one-dimensional degradation process can be graphically displayed in a two-dimensional plane, where the horizontal axis (or simply h-axis) can be age or state variable and the vertical axis (or simply v-axis) can be degradation rate or its reciprocal. Thus, there are four coordinate systems for selection. For each coordinate system, the sample correlation coefficient (CC) and the average of five-point moving coefficients of variation (CV) of the ordered data can be calculated. The best coordinate system has the largest |CC| (i.e., good linearity) and the smallest CV (i.e., small variability). In such a coordinate system, the degradation rate or its reciprocal can be approximately represented by a simple function of age or state variable so that the
mean degradation function has a closed-form expression. This will considerably facilitate residual life prediction and ensure good prediction accuracy.

According to the above discussion, the indicator fusion and coordinate system selection have been separately addressed in [5-6]. The purpose of this paper is to propose a hybrid approach to jointly address these two problems. The proposed approach consists of three main steps as shown in figure 1.

![Figure 1. Proposed three-step modeling framework.](image)

In the first step, a preliminary data analysis will be carried out to examine the monotonicity of degradation indicators and their dependence. The second step combines all the degradation indicators into a composite degradation indicator. This will involve several issues such as data normalization, aggregation model selection, determination of indicator weights and failure threshold modeling. The weights are mathematically derived and the dependence is handled according to a competing risk approach. Finally, the coordinate system for displaying the degradation process of composite indicator is selected based on the value of |CC/CV| in the third step. The resulting model obtained from the proposed approach can be a two-phase model, which is typically a mixture of the delay time model and stochastic model [7], and analytically tractable. A real-world example is included to illustrate the proposed approach and its appropriateness. Thus, the main contribution of the paper is the three-step modelling framework shown in figure 1, which is generally applicable to multidimensional degradation processes. In addition, a weighted linear method will be developed to estimate pseudo-failure time, which is applicable for the situation where the extrapolation is needed; and a method to estimate the phase boundary will be developed, which is applicable for the situations where the degradation rate curve in the best coordinate system is S-shaped or inverse S-shaped.

The paper is organized as follows. The proposed approach is presented in Section 2 and illustrated in Section 3. The paper is concluded in Section 4.

2. PROPOSED APPROACH

2.1. Preliminary data analysis

Let
denote the observed value of degradation indicator \( Y_i \) of the \( k \)-th unit at time point \( t \). Without loss of generality, assume that \( Y_0(0) = 0 \) and \( Y_i(t) \) has an increasing trend though it may locally decrease. Let \( L_i \) denote the failure threshold of \( Y_i \).

A preliminary data analysis can be carried out based on the information in equation (1). For any unit and \( t_k \), if \( \hat{y}_i^{(t_k)} - \hat{y}_i^{(t_{k-1})} \geq 0 \), \( Y_i \) can be considered to be monotonic.

Let \( r_d \) denote the CC between \( Y_i \) and \( Y_k \). Its critical value is given by [8]

\[
r_c = t_r / \sqrt{1 + t_r^2}
\]

where \( t_r \) is the Student’s \( t \)-value associated with the 95% level, one tail, and the degrees of freedom \( nK - 1 \). If \( r_d > r_c \), \( Y_i \) can be thought to be linearly correlated with \( Y_k \); otherwise, they can be thought to be mutually independent.

2.2. Indicator fusion

According to [4-5], the indicator fusion deals with specifying the two relations in equation (3):

\[
z = \phi(y_i^*, w_i), \quad L(z) = \phi(z; \mu, \sigma)
\]

where \( z \) is the composite indicator, \( \phi(.) \) is an aggregation model to be specified, \( y_i^* \) is the normalized value of \( y_i \), and \( w_i \) is the weight of \( Y_i \), which meets

\[
w_i \in (0, 1), \sum_{i=1}^{n} w_i = 1
\]

\( L(.) \) is the normal random failure threshold of composite indicator \( Z \). \( \phi(.) \) is the normal density function with parameters \( \mu \) and \( \sigma \). The indicator fusion involves the following four issues

- Normalization of the data;
- Determination of indicators’ weights;
- Selection of the aggregation model;
- Determination of parameters of the random failure threshold model.

Since the method to normalize the data may change the weights in an implicit way [9-10], it is necessary to ensure their compatibility.

2.2.1. Equal-failure-threshold transformation. For the above setting, the data is normalized by:

\[
y_i^* = y_i / L_i
\]

This transformation has the following features:

- \( y_i^* \) is dimensionless;
- all the normalized indicators have a common failure threshold 1;
- the time for \( Y_i \) to reach \( L_i \) (denoted as \( T_i \)) is equal to the time for \( Y_i^* \) to reach 1, implying that the transformation does not impact the distribution of \( T_i \), and hence does not impact the weights derived from the probability method of [9].

2.2.2. Determination of indicators’ weights. Let \( F_i(t) \) and \( f_i(t) \) denote the cumulative distribution function and density function of \( T_i \), respectively. The probability of event \( (T_i < T_i) \) is given by

\[
p_d = \int_0^t [1 - F_i(t)] f_i(t) dt
\]

Specially, when \( i = I \), \( p_d = 0.5 \). When \( T_i \) is small, \( Y_i \) is an important indicator. Therefore, a small \( T_i \) corresponds to a large \( p_d \) and a large \( w_i \). For a fixed \( I \) value (\( I = 1, 2, \ldots, m \)), \( w_i \) is calculated by

\[
w_i = p_d / \sum_{i=1}^{m} p_d
\]

The problem is now to determine \( F_i(t) \). This needs to estimate the pseudo-failure time of each degradation path [11-12]. Reference [12] uses a weighted least squares method (LSM) to fit a
degradation path to a power-law model and calls the model as the local mean degradation model. The weight function is given by:

\[ w(t) = \exp\left[-\frac{(t - \mu)^2}{2\sigma^2}\right] \]

where \( \mu = t_a \) and \( \sigma = \mu/3 \). The pseudo-failure time is the intersection between the fitted power-law model and the degradation limit. Since the estimation for the pseudo-failure time is generally based on extrapolation, a large weight should be assigned to the recent observations. Equation (8) can meet this requirement well. Jiang and Huang use a cross-validation approach to optimize the value of \( \sigma \) [13].

To make the estimated pseudo-failure time robust, the local mean degradation model is taken as a linear model, given by

\[ y_p(t) = a + bt \]

(9)

To make the estimated pseudo-failure time accurate, the fitted local mean degradation model should be based on the last three observations. Thus, the value of \( \sigma \) is taken as

\[ \sigma = (t_a - t_{a-3})/3 \]

(10)
The values of \( a \) and \( b \) are determined through minimizing the following

\[ SSE = \sum_{j=1}^{n} w(t_j)[y_j - y_p(t_j)]^2 \]

(11)

and the pseudo-failure time is estimated as

\[ t_p = (L - a)/b \]

(12)
Thus, \( F(t) \) can be obtained through fitting \( m \) pseudo-failure times to a proper distribution; and the weight of indicator can be obtained from equation (6) and equation (7).

2.2.3. Aggregation model. Reference [4] presents four categories of aggregation models; and the most widely used model is the generalized weighted mean model, given by [14]

\[ z_p = \left( \sum_{j=1}^{n} w_i y_j^p \right)^{1/p}, \quad p \in (-\infty, \infty) \]

(13)
When \( p = 0, 1 \) and 2, the model reduces into the weighted geometric mean, weighted arithmetic mean and weighted quadratic mean, respectively.

Aggregation model selection needs to consider the compensation among the indicators [4]. If the indicators are highly correlated, the weighted arithmetic mean model with a full compensability is proper; otherwise, an aggregation model with a relatively weak compensability is desired, and the aggregated value should be closer to the value of the poorer indicator. For example, for a non-decreasing degradation path, the aggregated value should be larger than the weighted arithmetic mean value. This requires \( p > 1 \).

2.2.4. Determination of random failure threshold. The time to failure for a unit can be defined as \( t_f \), which meets

\[ t_f = \min_t (t_i') \]

(14)
where \( t_i' \) corresponds to \( y_i^* = 1 \). Equation (14) is actually a competing risk model and applicable for both independent and dependent cases. The value of \( Z \) at \( t = t_f \) corresponds to the failure threshold of the composite indicator, which is generally not a constant. This implies that the failure threshold of the composite indicator is a random variable. Without loss of generality, it can be assumed to follow a normal distribution \( \Phi(Z; \mu, \sigma) \), where \( \mu \) and \( \sigma \) are distribution parameters.

To estimate \( \mu \) and \( \sigma \), the data of \( Z(t) \), \( (z, s) \), are extracted from equation (1). If \( \max(y_i^*) > 1 \) [ \( \max(y_i^*) < 1 \)], then \( s = 1 \) [ \( s = 0 \)] and the corresponding \( z \) is a left [right] censored observation of \( Z(t) \).

The log-likelihood function is given by

\[ \ln(L) = \sum_{z=1} \ln[\Phi(z)] + \sum_{z=0} \ln[1 - \Phi(z)] \]

(15)
The parameters are determined through maximizing ln(L).

2.3. Coordinate system selection for composite indicator

The data of composite indicator can be written as

\[ z_j^{(k)}; j = 1, 2, \ldots, n, k = 1, 2, \ldots, K \]  

The data are transformed into degradation rate data given by [6]

\[ r_j^{(k)} = [z_j^{(k)} - z_{j-1}^{(k)}]/(t_j^{(k)} - t_{j-1}^{(k)}), z_j = [z_j^{(k)} + z_{j+1}^{(k)}]/2, t_j = (t_{j-1}^{(k)} + t_j^{(k)})/2 \]  

Here, \( r_j^{(k)} \) is the degradation rate at age \( t_j \) and state \( z_j \) and the sample size of the dataset given by equation (17) is \( nK \). The data can be displayed in a two-dimensional plane with the h-axis being \( t_j \) or \( z_j \) and the v-axis being \( r_j \) or \( 1/r_j \). For each coordinate system, calculate the values of CC and CV. The best coordinate system corresponds to the largest value of |CC/CV|. The degradation model is built based on the information of the data plot in the best coordinate system. This is illustrated in the next section.

3. Illustration

3.1. a Data and earlier model

Xu et al. deal with the reliability evaluation of a certain product. The degradation indicators are capacitance \( (Y_1) \) and wear amount \( (Y_2) \), increase with time and have zero initial values [15]. The failure thresholds are \( L_1 = 14 \) and \( L_2 = 100 \), respectively. Table 1 shows the observed degradation paths.

| Time | Unit  | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|------|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1    | Y_1   | 0.45| 0.76| 0.64| 0.40| 0.68| 0.83| 0.34| 0.27| 0.68| 0.40|
|      | Y_2   | 0.55| 0.80| 0.46| 0.20| 0.10| 2.96| 0.42| 0.74| 1.53| 3.21|
| 2    | Y_1   | 2.19| 0.79| 0.56| 0.56| 1.49| 1.53| 0.95| 1.51| 1.33| 1.41|
|      | Y_2   | 1.60| 3.42| 7.92| 16.85|11.88| 8.93| 8.91| 10.97|12.21| 5.29|
| 3    | Y_1   | 4.14| 2.45| 1.21| 2.87| 3.03| 2.62| 1.24| 3.50| 2.21| 2.28|
|      | Y_2   | 7.11| 9.43| 24.63|18.3|14.94|12.35|14.92|11.97|18.65|21.98|
| 4    | Y_1   | 5.28| 3.72| 3.81| 4.90| 4.10| 4.81| 2.12| 3.91| 3.03| 3.00|
|      | Y_2   | 21.70|19.09|38.08|20.81|19.85|18.44|36.10|16.40|20.12|26.29|
| 5    | Y_1   | 6.46| 4.95| 5.68| 5.67| 6.09| 5.68| 3.76| 6.49| 4.40| 4.63|
|      | Y_2   | 29.66|22.53|44.60|28.56|52.27|26.41|43.97|19.63|34.71|33.80|
| 6    | Y_1   | 7.35| 6.53| 6.49| 6.95| 6.96| 7.34| 4.51| 8.54| 6.00| 5.07|
|      | Y_2   | 38.71|32.09|59.35|34.24|64.06|30.18|55.57|24.26|41.65|41.57|
| 7    | Y_1   | 7.35| 6.88| 7.29| 8.47| 8.79| 7.92| 4.63| 9.68| 6.19| 6.42|
|      | Y_2   | 49.04|38.16|65.21|45.86|76.69|45.30|61.80|26.53|45.37|48.30|
| 8    | Y_1   | 8.05| 8.88| 8.41|10.16|9.65|10.97|6.35|11.10|6.76| 8.33|
|      | Y_2   | 54.07|47.54|67.70|51.80|87.62|52.07|80.72|36.77|51.89|52.53|
| 9    | Y_1   | 8.88| 9.65| 9.57|11.36|10.88|14.11|8.64|13.98|6.27| 9.85|
|      | Y_2   | 59.83|61.57|91.90|66.10|100.51|62.16|84.55|42.86|82.09|60.53|
| 10   | Y_1   | 10.60|11.23|12.84|12.76|12.03|16.23|9.45|17.23|8.99|13.71|
|      | Y_2   | 65.14|73.18|114.72|79.39|112.76|93.54|96.23|48.47|92.74|68.34|

TABLE 1. Degradation paths of \( Y_1 \) and \( Y_2 \).

Under the assumption that \( Y_1 \) and \( Y_2 \) are dependent, Xu et al. use the \( t \)-Copula function to model their dependence [15]. The estimated reliability functions under both the independent and dependent assumptions are displayed in figure 2. As seen, the different assumptions result in different estimates.
3.2. Preliminary data analysis
A preliminary data analysis shows:

- $Y_1$ is non-monotonic and $Y_2$ is monotonic.
- The CC between $Y_1$ and $Y_2$ is 0.7922 and the critical value of CC associated with the 95% level, one tail, and the degrees of freedom $nK - 1 = 99$ is 0.8566, implying that $Y_1$ and $Y_2$ can be thought to be mutually independent.

3.3. Aggregation of indicators
The data are normalized by equation (5), and the weighted LSM is used to find the pseudo-failure time of each degradation path. The parameters of the weight function are $\mu = 10$ and $\sigma = 1$ time unit for the current example. The results are shown in the second and third columns of table 2, where $T_1$ and $T_2$ are times for $Y_1$ and $Y_2$ to reach their failure thresholds, respectively. The 4th column of table 2 shows the values of $T = \min(T_1, T_2)$, which is the pseudo-failure time of a unit.

The pseudo-failure time data are fitted to the Weibull, inverse Weibull, gamma, normal and lognormal distributions, respectively, and the results from the best model are shown in table 3. It is noted that the inverse Weibull distribution (IWD) is the best model for $T_2$ and $T$.

### Table 2. Pseudo-failure times, $X_1$ and $X_2$.

| Unit | $T_1$ | $T_2$ | $T$ | $X_1$ | $X_2$ |
|------|------|------|-----|------|------|
| 1    | 12.45| 16.41| 12.45| 5.07 | 7.38 |
| 2    | 12.09| 12.18| 12.09| 5.87 | 6.22 |
| 3    | 10.52| 9.35 | 9.35 | 4.75 | 4.60 |
| 4    | 10.93| 11.55| 10.93| 5.61 | 5.32 |
| 5    | 11.69| 8.97 | 8.97 | 4.42 | 4.55 |
| 6    | 9.06 | 10.34| 9.06 | 5.54 | 3.51 |
| 7    | 13.46| 10.44| 10.44| 5.28 | 5.16 |
| 8    | 8.96 | 18.70| 8.96 | 5.19 | 3.77 |
| 9    | 13.11| 10.37| 10.37| 5.33 | 5.03 |
| 10   | 10.16| 14.10| 10.16| 6.16 | 4.00 |

### Table 3. Distributions of $T_1$, $T_2$, $T$ and $X_1$.

| $\beta$ or $\mu$ | $\eta$ or $\sigma$ |
|------------------|-------------------|
| Weibull          | IWD               |
| IWD              | Normal            |
| 8.720            | 11.90             |
| 5.769            | 10.71             |
| 10.96            | 9.673             |
| 5.322            | 0.4854            |
From equation (6) yields \( \Pr(T_1 < T_2) = 0.5426 \). This yields \((w_1, w_2) = (0.5426, 0.4574)\). Since \( Y_1 \) and \( Y_2 \) are independent, the generalized weighted mean with \( p = 2 \) is selected as the aggregation model. Figure 3 shows the degradation paths of composite indicator in the coordinate system of \( z(t) \) vs. \( t \). The mean degradation function can be approximated by the power-law model with parameters \((\beta, \eta) = (1.353, 10.95)\). Since \( \beta > 1 \), \( z(t) \) vs. \( t \) is non-linear.

Using the approach outlined in Section II.B.4 yields the normal random failure threshold with parameters \( \mu_t = 0.8627 \) and \( \sigma_z = 0.01851 \). It is noted that \( \mu_t + 3\sigma_z = 0.9182 < 1 \), implying that the failure limit of the composite indicator is smaller than the failure limit of individual indicators.

**Figure 3.** Degradation paths of composite indicator.

**3.4. Degradation model of composite degradation indicator**

**3.4.1. Coordinate system selection.** Table 4 shows the values of CC, CV and \(|CC/CV|\) under 4 coordinate systems. Clearly, the best coordinate system is the one that the \( h \)-axis is \( z \) and the \( v \)-axis is the degradation rate \( r \). Figure 4 shows the plot of data under the best coordinate system. As seen, the plot has an inverse S-shaped trend, which implies that the plot is concave [convex] when \( z \) is small [large]. In other words, the process consists of two phases and their boundary is called the change point (denoted as \( z_0 \)).

**Table 4.** Values of CC, CV and \(|CC/CV|\) under different coordinate systems.

| \( h \)-axis | \( r \) | \(|CC| \) | \( CV \) | \(|CC/CV| \) | \( r \) | \(|CC| \) | \( CV \) | \(|CC/CV| \) |
|---|---|---|---|---|---|---|---|---|
| \( h \)-axis | \( v \)-axis | \( r \) | \( |CC| \) | \( CV \) | \( |CC/CV| \) | \( r \) | \( |CC| \) | \( CV \) | \( |CC/CV| \) |
| \( z \) | \( z_0 \) | 0.5415 | 0.3626 | 1.493 | 0.4227 | 0.3959 | 1.068 |
| 0.5829 | 0.3600 | 1.619 | 0.4215 | 0.3986 | 1.057 |

**Figure 4:** Plot of \( r \) vs. \( z \).
3.4.2. Determination of change point. The left tail of the plot can be approximated by a power-law model given by

\[ R_1(z) = az^b \]  

(18)

Fitting the data in \((0, z_{max}/3)\) to equation (18) yields \((a, b) = (0.1684, 0.4249)\).

The right tail of the plot can be approximated by

\[ R_2(z) = a + bz \]  

(19)

Fitting the data in \((2z_{max}/3, z_{max})\) to equation (19) yields \((a, b) = (-0.0273, 0.2133)\). The asymptotical relations are displayed in figure 5, and look reasonable.

![Figure 5. Plots of asymptotical relations.](image)

Absolute errors between the data and asymptotical relations are given by

\[ \varepsilon_1(z) = |R_1(z) - r(z)|, \varepsilon_2(z) = |R_2(z) - r(z)| \]  

(20)

Figure 6 shows the error curves. As seen, \(\varepsilon_1(z)\) [\(\varepsilon_2(z)\)] has an increasing [decreasing] trend so that there exists a point where \(\varepsilon_1(z) = \varepsilon_2(z)\). This point is defined as the change point of \(r(z)\). For the current example, \(z_0 = 0.4026 \approx 0.40\), which is close to the inflection point of the cubic function shown in figure 4.

![Figure 6. Plots of \(\varepsilon_1(z)\) and \(\varepsilon_2(z)\).](image)

3.4.3. Degradation model in the first phase. The first phase is the interval of \(z < z_0\). Generally, the unit in this phase is safe and there is not a need to make a failure prediction. Therefore, we model the degradation in this phase using a distribution of time to change point (denoted as \(X_1\)) where \(z(t) = z_0\). For each unit, the value of \(X_1\) can be obtained using interpolation and the results are shown in the 5th column of table 2. For the data of \(X_1\), the normal distribution provides the best fitting and the parameters are shown in the last column of table 3.

Figure 6 shows the error curves. As seen, \(\varepsilon_1(z)\) [\(\varepsilon_2(z)\)] has an increasing [decreasing] trend so that there exists a point where \(\varepsilon_1(z) = \varepsilon_2(z)\). This point is defined as the change point of \(r(z)\). For the current example, \(z_0 = 0.4026 \approx 0.40\), which is close to the inflection point of the cubic function shown in figure 4.

The second phase is the interval of \(z > z_0\). Let \(X_2\) denote the residual life at \(X_1\), given by

\[ X_2 = T - X_1 \]  

(21)
The values of $X_2$ are shown in the last column in table 2.

3.4.4. Degradation model in the second phase. Since $z(t)$ is monotonic, we model the degradation in the second phase using a gamma process model with shape parameter $u(z)$ and scale parameter $v$. The mean function is given by equation (19) with $R_v(z)$ being replaced by $dz/dt$. For a given initial condition $(t_0, z_0)$, solving equation (19) yields

$$z(t) = [(a + b z_0)e^{b(t-t_0)} - a] / b$$

Equation (22) is actually the mean degradation function. Therefore, we replace “$z(t)$” with $\mu(t)$. Thus, the shape parameter at $t$ is given by

$$u(t) = [\mu(t) - \mu(t_0)] / v$$

Model parameter set, $(a, b, v)$, can be obtained using the maximum likelihood method. For the current example, $(a, b, v) = (0.0000, 0.1688, 0.01478)$. Since $a \approx 0$, we have

$$\mu(t) = z_0 e^{b(t-t_0)} \cdot \mu(t) - \mu(t_0) = z_0 [e^{b(t-t_0)} - 1]$$

This implies that the mean increment of $Z$ is proportional to $z_0$ and exponentially increases with time.

3.4.5. Distribution of residual lifetime. Suppose that the current observation of a path is $(t^*, z^*)$ and $z^* > z_0$. The distribution of residual life, $X = T - t^*$, is given by

$$F(x) = 1 - G[1 - z^* (e^{b(t^*-t_0)} - 1)] / v, v$$

where $G(.)$ is the gamma distribution function. Clearly, the residual life depends on both age and state.

To illustrate, consider the last observation point of degradation path of Unit 1 as the current observation, which is $(t^*, z^*) = (10, 0.7109)$. Figure 7 shows the degradation path, failure threshold and residual life distribution. The residual life distribution can be approximated by a normal distribution with parameters $(\mu, \sigma) = (1.181, 0.3441)$. That is, the mean residual life at $t = 10$ is 1.181 or the mean life of Unit 1 is 11.18. The pseudo-failure time obtained from the weighted LSM is 12.45. The relative error between the two estimates is 11.4%, implying that the two estimates are fairly close to each other.

![Degradation path and residual life distribution of Unit 1](image)

3.4.6. Distribution of time to failure for unit population. From equation (21), the life of unit population is given by

$$T = X_1 + X_2$$

Here, $X_3$ is the residual life at $z_0 = 0.40$. The distribution of $X_2$ can be approximated by the normal distribution with parameters $(\mu, \sigma) = (4.581, 0.5778)$.

From table 2 yields CC$(X_1, X_2) = 0.0392$. This implies that $X_1$ and $X_2$ can be thought to be mutually independent. Using numerical integration yields the distribution of $T$, which also can be approximated by a normal distribution with parameters $(\mu, \sigma) = (9.903, 0.7546)$. The tradeoff Bx life of $T$ is 8.573, which corresponds to a reliability of 96.1%.
Finally, the reliability functions estimated from the pseudo-failure time approach and two-phase degradation modeling approach are also displayed in figure 2. From the figure, the following remarks can be made:

- The reliability function obtained from the pseudo-failure time approach is close to the reliability function of [15] under the independent assumption. This confirm the fact that \( Y_1 \) and \( Y_2 \) are independent.
- For \( t < 10 \), the reliability functions obtained from the pseudo-failure time approach and two-phase degradation model are close to each other. This confirm the appropriateness of the pseudo-failure time approach.
- The reliability function obtained from the two-phase degradation model has the smallest life dispersion while a small dispersion implies a good failure prediction ability. This validates the appropriateness of the proposed modeling approach.

4. CONCLUSIONS
This paper has dealt with modeling of multidimensional degradation processes. A hybrid modeling approach has been proposed, which is generally applicable to multidimensional degradation processes. This is the main contribution of the paper. Another contribution of the paper is a weighted LSM that combines with a linear model to estimate pseudo-failure time, which is applicable for the situation where extrapolation is needed.

A real-world example that involves two degradation indicators has been analyzed in detail, and a two-phase degradation model has been developed to fit the data. The model in the first phase is similar to the first part of the delay time model, and the model in the second phase is an age-state-dependent stochastic process model. The contribution and finding obtained from this example have been:

- An asymptotic-relation-based method is developed to estimate the change point, which is applicable for the situations where the degradation rate curve in the best coordinate system is S-shaped or inverse S-shaped.
- The distribution of time to the failure threshold generally has a light left tail so that the normal and inverse Weibull distributions can be appropriate for approximating this distribution.

A topic for future research is to examine the influence of parameter \( p \) of the generalized weighted mean model on the resulting model.

ACKNOWLEDGMENT
The research was supported by the National Natural Science Foundation of China (No. 71771029).

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