Distributed support-vector-machine over dynamic balanced directed networks

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Abstract—In this paper, we consider the binary classification problem via distributed Support-Vector-Machines (SVM), where the idea is to train a network of agents, with limited share of data, to cooperatively learn the SVM classifier for the global database. Agents only share processed information regarding the classifier parameters and the gradient of the local loss functions instead of their raw data. In contrast to the existing work, we propose a continuous-time algorithm that incorporates network topology changes in discrete jumps. This hybrid nature allows us to remove chattering that arises because of the discretization of the underlying CT process. We show that the proposed algorithm converges to the SVM classifier over time-varying weight balanced directed graphs by using arguments from the matrix perturbation theory.

Index Terms—Support Vector Machine, constrained distributed convex optimization, matrix perturbation theory.

I. INTRODUCTION

Machine learning has been an area of significant research in recent signal processing and control literature [1]–[4]. Among the topics of interest, Support Vector Machines (SVMs) are supervised-learning methods with several applications ranging from image/video processing to bioinformatics. Motivated by the recent progress in computing hardware and wireless communication, we are interested in developing distributed solutions for SVM classification. The basic idea is to process the raw data at each node in order to train a local classifier and then fuse these classifiers among the neighboring nodes. D-SVM (distributed SVM) finds applications where a subset of the data is acquired by different nodes/servers/agents possibly at different geographic locations, privacy is of concern, and communicating data to a fusion center (FC) is practically infeasible.

In binary classification, SVM defines the maximum-margin hyperplane (as the classifier) determined by the closest data samples known as the Support-Vectors (SVs). The preliminary work on D-SVM (referred as Distributed Parallel SVM (DP-SVM) [5] and Parallel SVM (P-SVM) [6]) is focused on local calculations and sharing of the SVs, as the representatives of discriminant information of the dataset [5]–[9]. In [6]–[8], these local SVs are updated via a FC to improve the D-SVM performance. Ref. [5] implements D-SDV on multi-agent networks and requires a Hamiltonian cycle that visits every agent exactly once. A FC-free approach is considered in [9], where every agent locally solves a (coupled) convex optimization sub-problem via alternating direction method of multipliers (ADMM), which is not computationally efficient. A main drawback is that these approaches require sharing raw data over the communication network, raising data privacy and information security issues. More recently, consensus-based distributed optimization methods are proposed in [10]–[18], where instead of raw data, agents share processed information, which in case of leakage to unauthorized parties reveals little information about the original data. Among these, the solution in [16] requires distributed computation of the Hessian inverse, which is not practical since even for a sparse Hessian matrix, its inverse is not necessarily sparse. Penalty-based approaches are proposed in [17], [18], where the constrained convex cost function is reformulated by adding a new penalty term on consensus constraint violation. It is shown that there is a gap of $O(\kappa)$ (with $\kappa$ as the penalty constant) between the optimal penalty-based solution and the original constrained one [19].

Other methods include finite/fixed-time algorithms [11]–[15] that are prone to steady-state oscillations (known as chattering) due to non-Lipschitz dynamics.

In this paper, a D-SVM method is proposed that overcomes the challenges of (semi-centralized) FC-based solution and the chattering phenomena. Moreover, in contrast to Refs. [10]–[18], where either continuous-time (CT) or discrete-time (DT) protocols are considered, we propose a hybrid algorithm to address the topology switching of the multi-agent network in DT incorporated in a CT gradient-descent update [20]. Our hybrid approach enables more flexibility in considering mixed-dynamics [20], [21], which allows solving D-SVM via CT protocols over general dynamic digraphs in DT domain. To analyze the proposed hybrid model, we use matrix perturbation theory [22] to characterize the eignespectrum of the proposed dynamics. The proposed D-SVM is fully distributed, as opposed to FC-based approaches, and does not require solving convex sub-problems unlike [5]–[9]. Due to Lipschitz-continuity of the proposed CT approach, it’s DT approximation is free of the aforementioned chattering inherent to the non-Lipschitz dynamics [11]–[15]. Note that we directly solve the original constrained optimization free of penalty-based approximation inaccuracies in [17]–[19].
We now describe the rest of the paper. Section II recap
some preliminaries on algebraic graph theory while Sec-
tion III formulates the D-SVM problem. Section IV states
our CT gradient descent method to address D-SVM whereas the
convergence analysis over dynamic WB-digraphs is available
in Section V. Section VI provides an illustrative example, and
finally, Section VII concludes the paper with some future
research directions.

II. PRELIMINARIES ON ALGEBRAIC GRAPH THEORY

We represent the multi-agent network by a strongly-
connected directed graph (SC digraph) \( G \). Assuming a posi-
tive weight \( w_{ij} \) for every link (from node \( j \) to node \( i \))
and zero otherwise, the irreducible adjacency matrix of \( G \)
is \( W = \{ w_{ij} \} \), and the Laplacian matrix \( \bar{W} = \{ \bar{w}_{ij} \} \) is
defined as,

\[
\bar{w}_{ij} = \begin{cases} w_{ij}, & i \neq j; \\ -\sum_{j=1}^{n} w_{ij}, & i = j. \end{cases}
\]

(1)

The SC property of the graph is directly related to the rank
of its Laplacian matrix as given in the next lemma.

Lemma 1: [23] The given Laplacian \( \bar{W} \) in (1) for a SC
digraph has eigenvalues whose real-parts are non-positive
with one eigenvalue at zero.

Next, we define a WB-digraph as an SC digraph with equal
weight-sum of incoming and outgoing links at every node \( i \),
i.e., \( \sum_{j=1}^{n} w_{ij} = \sum_{j=1}^{n} w_{ji} \), implying the following lemma.

Lemma 2: [23] For the Laplacian \( \bar{W} \) of a WB-digraph,
the vectors \( 1_n^T \) and \( 1_n \) are respectively the left and right
eigenvector associated with the zero eigenvalue, i.e., \( 1_n^T \bar{W} = 0_n \) and \( \bar{W} 1_n = 0_n \), where \( 1_n \) and \( 0_n \) are the column vectors
of \( 1\)'s and \( 0\)'s of size \( n \), respectively.

In the rest of the paper, \( \| A \|_\infty \) denotes the infinity norm
of a matrix, i.e., \( \| A \|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| \).

III. PROBLEM STATEMENT

Consider a binary classification problem for a given set
of \( N \) data points \( \chi_i \in \mathbb{R}^{m-1}, \ i = 1, \ldots, N \), each belonging
to one of two classes labeled by \( l_i \in \{-1, 1\} \). Using the
entire training set, the SVM problem is to find a hyper-
plane \( \omega^T \chi - \nu = 0 \), for \( \chi \in \mathbb{R}^{m-1} \), based on the maximum
margin linear classification to partition the data into two
classes. Subsequently, a new test data point \( \hat{\chi} \) belongs
to the class labeled as \( g(\hat{\chi}) = \text{sgn}(\omega^T \hat{\chi} - \nu) \). In case the
data points are not linearly separable, the input data is first
projected into a high-dimensional space \( \mathcal{F} \) via a nonlinear
mapping \( \phi(\cdot) \). This mapping is such that the inner products
of two projected data points can be computed via a kernel
function \( K(\cdot, \cdot) \), i.e., \( K(\chi_i, \chi_j) = \phi(\chi_i)^\top \phi(\chi_j) \). By proper
selection of \( \phi(\cdot) \), a linear optimal hyperplane defined by \( \omega \\
and \( \nu \) can be found in \( \mathcal{F} \) such that \( g(\hat{\chi}) = \text{sgn}(\omega^T \phi(\hat{\chi}) - \nu) \)
determines the class of \( \hat{\chi} \). In centralized SVM, all the data
points are sent to a central computation entity (the FC) that
finds the optimal \( \omega \) and \( \nu \) by minimizing the following
convex loss function [24]:

\[
\min_{\omega, \nu} \omega^T \omega + C \sum_{j=1}^{N} \max\{1 - l_j(\omega^T \phi(\chi_j) + \nu), 0\}^q
\]

(2)

where \( q = \{1, 2, \ldots\} \) defines smoothness of the loss function
and its derivatives, and the positive constant \( C \) determines
the trade-off between increasing the margin size and ensuring
that the projected data \( \phi(\chi_j) \) lies on correct side of the
hyperplane. We note that the SVM loss function (2) is not
continuously twice-differentiable for \( q = \{1, 2\} \). Therefore,
it is common to approximate \( \max\{z, 0\}^q \) for \( q = 1 \) by the
smooth function \( L(z, \mu) = \frac{\mu}{q} \log(1 + \exp(\mu z)) \), which is
the integral of the well-known sigmoid function [12]. It can
be shown that by setting \( \mu \) large enough \( L(z, \mu) \) becomes
arbitrarily close to \( \max\{z, 0\}; \) see [25] for more smooth
loss functions, e.g., for logistic regression with cross-entropy
loss.

In distributed SVM (D-SVM), the data points are available
over a network of \( n \) agents and each agent \( i \) possesses a local
dataset with \( N_i \) data points denoted by \( \chi^i_j, j = 1, \ldots, N_i \).
Since each agent has access to partial data, the locally found
values \( \omega^i \) and \( \nu^i \), obtained by solving (2) over the local
dataset \( \chi^i_j, j = 1, \ldots, N_i \), may differ for each agent \( i \). The
idea behind D-SVM is thus to develop a distributed mech-
anism to learn the global classifier parameters by making sure
that no agent reveals its local data to any other agent.
The corresponding distributed optimization problem is given by:

\[
\min_{\omega_i, \nu_i, \ldots, \omega_n, \nu_n} \sum_{i=1}^{n} f_i(\omega_i, \nu_i)
\]

subject to \( \omega_1 = \cdots = \omega_n \), \( \nu_1 = \cdots = \nu_n \),

where each local cost \( f_i : \mathbb{R}^m \rightarrow \mathbb{R} \) is approximated as

\[
f_i(\omega_i, \nu_i) = \omega_i^T \omega_i + C \sum_{j=1}^{N_i} \frac{1}{\mu} \log(1 + \exp(\mu \nu_i)).
\]

(3)

Let \( \chi_i = [\omega_i^T; \nu_i] \in \mathbb{R}^m \) and let \( \chi \in \mathbb{R}^{mn} \) be the global vec-
tor concatenating all \( \chi_i \)'s, i.e., \( \chi = [\chi_1; \chi_2; \ldots; \chi_n] \), where
the symbol ‘;’ denotes the column concatenation of the \( \chi_i \)
 vectors. Then, Problem (3) takes the following form:

\[
\min_{x \in \mathbb{R}^{mn}} F(x), \\
F(x) = \sum_{i=1}^{n} f_i(x_i)
\]

subject to \( x_1 = x_2 = \cdots = x_n \).

(4)

We next provide the following lemma on the local costs.

Lemma 3: [12] Each local cost \( f_i \) is twice differentiable
and strictly convex, i.e., the \( m \times m \) Hessian matrix \( \nabla^2 f_i(x_i) \)
is positive definite, for all non-zero \( x_i \in \mathbb{R}^m \).

Clearly, any solution \( x_i^*, i = 1, \ldots, n \), of (4) must sat-
sify \( \sum_{i=1}^{n} \nabla f_i(x_i^*) = 0_m \), such that \( x_1^* = \ldots = x_n^* = x^* \),
for some \( x^* \in \mathbb{R}^m \). In other words, the optimality condi-
tion \( \nabla F(x^*) = 0_m \) must hold for some \( x^* \in \mathbb{R}^{mn} \) such
that \( x^* = 1_n \otimes x^* \), where \( \nabla F : \mathbb{R}^{mn} \rightarrow \mathbb{R}^{mn} \) is the gradient
of \( F : \mathbb{R}^{mn} \rightarrow \mathbb{R} \).
IV. PROPOSED ALGORITHM: DYNAMICS AND AUXILIARY RESULTS

We now provide a distributed solution to the D-SVM problem. Let \( x_i(t) \in \mathbb{R}^m \) be the state of agent \( i \) at time \( t \), where \( t \geq 0 \) is the continuous-time variable. To solve problem (4), we consider the following continuous-time linear dynamics for all \( x_i(t) \in \mathbb{R}^m \), \( i \in \{1, \ldots, n\} \),

\[
\dot{x}_i = - \sum_{j=1}^{n} w_{ij} (x_i - x_j) - \alpha y_i,
\]

where \( x_i = \frac{dx_i}{dt} \), \( W = \{w_{ij}\} \) is the weighted adjacency matrix associated with \( G \), and \( \alpha > 0 \) is the stepsize. We note that instead of the standard descend direction of \( \nabla f_i(x_i) \), the \( x_i \)-update in (5) descends in the direction of an auxiliary variable \( y_i(t) \in \mathbb{R}^m \). The variable \( y_i(t) \) in fact tracks the sum of local gradients, asymptotically, and is updated via the following dynamics (see [1]–[3] for similar DT methods):

\[
\dot{y}_i = - \sum_{j=1}^{n} a_{ij} (y_i - y_j) + \frac{d}{dt} \nabla f_i(x_i),
\]

where \( \dot{y}_i = \frac{dy_i}{dt} \) and the matrix \( A = \{a_{ij}\} \) is the weighted adjacency matrix with the same structure as \( W \). In (6),

\[
\frac{d}{dt} \nabla f_i(x_i) = \nabla^2 f_i(x_i) \dot{x}_i.
\]

Note that the proposed algorithm, (5) and (6), is in continuous-time. However, the structure of the underlying graph \( G \) may change in time instances, that we consider as time steps in a discrete-time framework. This makes the proposed dynamics hybrid where the state variables, \( x \) and \( y \), evolve in CT over DT switching of the network topology. For the ease of notation, we define an auxiliary global variable \( y = [y_1; y_2; \ldots; y_n] \in \mathbb{R}^{mn} \) that concatenates the local \( y_i(t) \)'s. We make the following assumption on the weight matrices \( W \) and \( A \).

**Assumption 1:** The weights \( W = \{w_{ij}\} \) and \( A = \{a_{ij}\} \) are associated to a WB-digraph with \( w_{ij} \geq 0 \) and \( a_{ij} \geq 0 \), respectively. Further, \( \sum_{j=1}^{n} w_{ij} < 1 \) and \( \sum_{j=1}^{n} a_{ij} < 1 \).

Following Assumption [1] we obtain from (5) and (6):

\[
\sum_{i=1}^{n} \dot{y}_i = \sum_{i=1}^{n} \frac{d}{dt} \nabla f_i(x_i),
\]

\[
\sum_{i=1}^{n} \dot{x}_i = -\alpha \sum_{i=1}^{n} y_i.
\]

Integrating (8) with respect to \( t \) and initializing the auxiliary variable \( y(0) = 0_{nm} \), we have

\[
\sum_{i=1}^{n} \dot{x}_i = -\alpha \sum_{i=1}^{n} y_i = -\alpha \sum_{i=1}^{n} \nabla f_i(x_i),
\]

which shows that the time-derivative of the sum of states \( x_i \)'s is towards sum gradient, and therefore, the equilibrium \( (x_i = 0_m) \) of the dynamics (5)–(6) is \( x^* \) satisfying \((1_n^T \otimes I_m) \nabla F(x^*) = 0_m \) (\( I_m \) as the identity matrix of size \( m \)), which is the optimal state of problem (4) [10].

**Lemma 4:** Initializing from any \( x(0) \neq 1_n \otimes x_0 \), for some non-zero \( x_0 \in \mathbb{R}^m \), and \( y(0) = 0_{nm} \), the state \( [x^*; 0_{nm}] \) with \((1_n^T \otimes I_m) \nabla F(x^*) = 0_m \) is an invariant equilibrium point of the dynamics (5)–(6).

**Proof:** From (10), the following uniquely holds at \( x = x^* = 1_n \otimes x^* \),

\[
\sum_{i=1}^{n} \dot{x}_i = -\alpha (1_n^T \otimes I_m) \nabla F(x^*) = 0_m.
\]

Further, from (5) we have \( \dot{x}_i = 0_m \) and from (6) and (7),

\[
\dot{y}_i = \frac{d}{dt} \nabla f_i(x^*) = \nabla^2 f_i(x^*) \dot{x}_i = 0_m,
\]

which shows that \([x^*; 0_{nm}]\) is an invariant equilibrium point of the dynamics (5)–(6).

The above lemma only shows that the state \([x^*; 0_{nm}]\), with \( x^* \) as the optimal point of problem (4), is the equilibrium of the proposed networked dynamics (5)–(6). Note that the first term in Eq. (5) drives the agents to reach consensus on \( x_i \)'s while the second term along with the dynamics (6) implements the gradient correction.

V. PROOF OF CONVERGENCE

In this section, we show that dynamics (5)–(6) converge to the equilibrium state described in Lemma 4. As it is the case in Section [7] in which the Laplacian matrix for \( W \), the Laplacian matrix for \( A \) is denoted by \( \bar{A} \). Define the \( nm \text{-by-} nm \) Hessian matrix \( H \) as \( \text{blockdiag}[\nabla^2 f_i(x_i)] \).

The dynamics (5)–(6) can be written in compact form as

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = M(t, \alpha)
\begin{pmatrix}
x \\
y
\end{pmatrix},
\]

where \( M(t, \alpha) = \begin{pmatrix} W & 0 \\ H(W \otimes I_m) & -\alpha I_{nm} \end{pmatrix} \).

[11]–[12] Networked dynamics (11)–(12) represents a hybrid dynamical system because: (i) the matrix \( H \) varies in CT; and (ii) the structure of \( W \) and \( \bar{A} \) may change in DT in case of dynamic network topology. In this direction, towards convergence analysis, we first evaluate the stability properties of the matrix \( M \) at every time-instant, and then generalize the convergence to the entire time horizon. In the rest of this paper for notation simplicity, we drop the dependence of \( M \) on \((t, \alpha)\), unless where needed, despite the fact that it is a function of both time \( t \) and stepsize \( \alpha \).

**Lemma 5:**[26], [27] Let \( P(\alpha) \) be an \( n \text{-by-} n \) matrix depending smoothly on a real parameter \( \alpha \geq 0 \). Assume \( P(0) \) has \( l \) \( n \) equal eigenvalues, denoted by \( \lambda_1 = \ldots = \lambda_l \), associated with right eigenvectors \( v_1, \ldots, v_l \) and left eigenvectors \( u_1, \ldots, u_l \) which are linearly independent. Let \( \lambda_1(\alpha) \) denote the eigenvalues of \( P(\alpha) \), as a function of \( \alpha \), corresponding to \( \lambda_1, i \in \{1, \ldots, l\} \), and \( P^2 = \frac{dP(\alpha)}{d\alpha} |_{\alpha=0} \). Then, \( \frac{dP(\alpha)}{d\alpha} |_{\alpha=0} \) are the eigenvalues of the following \( l \text{-by-} l \) matrix,

\[
\begin{pmatrix}
u_1^T P' v_1 & \ldots & \nu_1^T P' v_l \\
\vdots & \ddots & \vdots \\
\nu_l^T P' v_1 & \ldots & \nu_l^T P' v_l
\end{pmatrix},
\]
Theorem 1: Let Assumption \[1\] hold. For a sufficiently small \(\alpha\), all eigenvalues of \(M\) have non-positive real-parts and the algebraic multiplicity of the zero eigenvalue is \(m\).

Proof: Let \(M = M_0 + \alpha M_1\) with

\[
M_0 = \begin{pmatrix}
\mathbf{W} \otimes I_m & 0_{mn \times mn} \\
H(\mathbf{W} \otimes I_m) & \mathbf{A} \otimes I_m
\end{pmatrix},
\]

\[
M_1 = \begin{pmatrix}
0_{mn \times mn} & -I_{mn} \\
0_{mn \times mn} & -H
\end{pmatrix},
\]

where \(0_{mn \times mn}\) is the zero matrix of size \(mn\). Since matrix \(M_0\) is block (lower) triangular we have,

\[
\sigma(M_0) = \sigma(\mathbf{W} \otimes I_m) \cup \sigma(\mathbf{A} \otimes I_m),
\]

(13)

where \(\sigma(\cdot)\) represents the eigenspectrum of the matrix. From Lemma \[1\] both matrices \(\mathbf{W}\) and \(\mathbf{A}\) have \(n-1\) eigenvalues in the LHP (left-half plane) and one isolated eigenvalue at zero. Therefore, matrix \(M_0\) has \(m\) sets of eigenvalues associated with \(m\) dimensions of vector states \(x_i\) i.e.,

\[
\text{Re}\{\lambda_{2n,j}\} \leq \ldots \leq \text{Re}\{\lambda_{3,j}\} < \lambda_{2,j} = \lambda_{1,j} = 0,
\]

where \(j = \{1, \ldots, m\}\). Using Lemma \[5\] we analyze the spectrum of \(M\) by considering it as the perturbed version of \(M_0\) via the term \(\alpha M_1\). We check the variation of the zero eigenvalues \(\lambda_{1,j}\) and \(\lambda_{2,j}\) by adding the (small) perturbation \(\alpha M_1\). Denote these perturbed eigenvalues by \(\lambda_{1,j}(\alpha)\) and \(\lambda_{2,j}(\alpha)\). To apply Lemma \[5\] define the right eigenvectors corresponding to \(\lambda_{1,j}\) and \(\lambda_{2,j}\) as,

\[
V = [V_1 \ V_2] = \begin{pmatrix}
1_n & 0_n \\
0_n & 1_n
\end{pmatrix} \odot I_m,
\]

(14)

Similarly, the left eigenvectors are \(V^\top\). Note that these eigenvectors are defined using Lemma \[2\] and satisfy \(V^\top V = I_{2mn}\). Recall that,

\[
\left.\frac{dM}{d\alpha}\right|_{\alpha=0} = M_1 \text{ and following Lemma \[5\],}
\]

\[
V^\top M_1 V = \begin{pmatrix}
0_{m \times m} & 0_{m \times m} \\
-nI_m & -(1_n \otimes I_m)^\top H(1_n \otimes I_m)
\end{pmatrix}.
\]

(15)

Following the definition of the Hessian matrix \(H\),

\[
-(1_n \otimes I_m)^\top H(1_n \otimes I_m) = -\sum_{i=1}^n \nabla^2 f_i(x_i) < 0,
\]

where the last inequality follows the strict convexity of the local functions \(f_i(x_i)\) (Lemma \[3\]). Recall that from Lemma \[3\] the derivatives \(\frac{d\lambda_{1,j}}{d\alpha}|_{\alpha=0}\) and \(\frac{d\lambda_{2,j}}{d\alpha}|_{\alpha=0}\) depend on the eigenvalues of \(\mathbf{M}\) which clearly form a lower triangular matrix with \(m\) zero eigenvalues and \(m\) negative eigenvalues (following \[16\]). Therefore, \(\frac{d\lambda_{1,j}}{d\alpha}|_{\alpha=0} = 0\) and \(\frac{d\lambda_{2,j}}{d\alpha}|_{\alpha=0} < 0\), which implies that considering \(\alpha M_1\) as a perturbation, the \(m\) zero eigenvalues \(\lambda_{2,j}(\alpha)\) of \(M\) move toward the LHP while \(\lambda_{1,j}(\alpha)\)'s remain zero. We recall that the eigenvalues are a continuous functions of the matrix elements [22], and therefore, for sufficiently small \(\alpha\) we have,

\[
\text{Re}\{\lambda_{2n,j}(\alpha)\} \leq \ldots \leq \text{Re}\{\lambda_{3,j}(\alpha)\} \leq \lambda_{2,j}(\alpha) = \lambda_{1,j}(\alpha) = 0.
\]

(17)

The proof is completed. \[\square\]
Following (18), for \( \gamma < 1 \),
\[
4(2 + \gamma) + \max \{2 + \gamma(2 + \alpha), 2 + \alpha\}^{1 - \frac{1}{\alpha n}} \lessgtr \lambda_{\min},
\]
and for \( \gamma \geq 1 \),
\[
4(4 + \gamma(4 + \alpha))^{1 - \frac{1}{\alpha n}} < \lambda_{\min}.
\]
Since the functions on the left-hand-side of the above inequalities are monotonically increasing for \( \alpha > 0 \), the largest \( \alpha \) satisfying the above inequalities is given by (19)-(20).

Lemma 7 gives a conservative upper-bound on \( \alpha \) which guarantees the rest of the eigenvalues, other than \( \lambda_{1,1}(\alpha) = 0 \) and \( \lambda_{2,j}(\alpha) < 0 \), remain in the LHP and Theorem 1 is valid. However, the eigenvalues of \( M \) may still remain in the LHP for a possible less-conservative choice of \( \alpha > \pi \). In general, for a proper \( \alpha \), matrix \( M \) has \( m \) zero eigenvalues associated with the eigenvectors \( \hat{V}_i \) given in (14), and the null space of the time-varying matrix \( M \) is,
\[
\mathcal{N}(M) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0_n \end{pmatrix} \otimes I_m \right\},
\]
which is independent of time.

Theorem 2: Let the conditions in Lemma 4, Lemma 7, and Theorem 1 hold. The proposed dynamics (5)-(9) converges to \( [x^*; 0_{nm}] \) with \( x^* \) as the optimal value of problem (4).

Proof: Define the following proper positive-definite Lyapunov function proposed in [29],
\[
V(\delta) = \frac{1}{2} \delta^T \delta = \frac{1}{2} \|\delta\|^2_2
\]
with \( \delta \in \mathbb{R}^{2mn} \) defined as the difference of state system and the optimal state,
\[
\delta = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x^* \\ 0_{nm} \end{pmatrix}.
\]
Since from Lemma 4, \([x^*; 0_{nm}]\) is an invariant state of the networked dynamics (11)-(12), we have \( \delta = M(\delta) \). Then, the time-derivative of the proposed Lyapunov function is as follows,
\[
\dot{V} = \delta^T \dot{\delta} = \delta^T M \delta.
\]
Following Theorem 1, let \( \lambda_{1,j}(\alpha) = 0, \ Re\{\lambda_{2,j}(\alpha)\} < 0 \) for \( 2 \leq i \leq 2n, 1 \leq j \leq m \) represent the real-parts of the eigenspectrum of \( M \). It is known that (23),
\[
\delta^T M \delta \leq \max_{1 \leq j \leq m} \ Re\{\lambda_{2,j}(\alpha)\} \delta^T \delta.
\]
Since \( M \) varies in time, \( \max_{1 \leq j \leq m} \ Re\{\lambda_{2,j}(\alpha)\} \) also changes in time. However, from Theorem 1, it is always negative, implying that \( \dot{V} < 0 \) for \( \delta \neq 0_{2mn} \). We have,
\[
\dot{V} = 0 \iff \delta = 0_{2mn},
\]
and, from LaSalle’s invariance principle, convergence to the invariant set \( \{\delta = 0_{2mn}\} \) follows (see [29] Section 4.1).

Remark 1: Following (24), the convergence rate of the dynamics (11)-(12) depends on \( \Re\{\lambda_{2,j}(\alpha)\} \). Note that \( \Re\{\lambda_{2,j}(\alpha)\} \) is tightly related to the parameter \( \alpha \) and therefore, to improve the convergence rate parameter \( \alpha \) needs not to be very small.

VI. SIMULATION: NONLINEAR SVM EXAMPLE

For simulation consider the academic example given in [30] (page 747). Consider \( N = 60 \) uniformly distributed sample data points, shown in Fig. 1 (Left), represented in two classes: blue ‘+’s and red ‘o’s. Clearly, these points \( x_i = [x_i(1); x_i(2)] \) are not linearly separable in \( \mathbb{R}^2 \). The nonlinear mapping \( \phi(x_i) = [x_i(1)^2; x_i(2)^2; \sqrt{2} x_i(1) x_i(2)] \), proposed by [30], properly maps the data points to \( \mathbb{R}^3 \) such that the projected data points are linearly separable via a hyperplane as shown in Fig. 4 (Right). It can be shown that the associated kernel function is \( K(x_i, x_j) = (\phi(x_i) \phi(x_j))^2 \). We evaluate the proposed dynamics (11)-(12) (with \( \alpha = 10 \)) for D-SVM over a network of \( n = 5 \) agents each having access to 50% random selection of the data points. The loss function \( f_i(x_i) \) follows the smooth approximation discussed in Section III with \( \mu = 3 \) and \( C = 1.5 \). Every agent finds the optimal separating hyperplane defined by \( x_i = [\omega_i^T; \nu_i] \) (\( \omega_i \in \mathbb{R}^3 \)) and shares this value along with the auxiliary variable \( y_i \) with its direct neighbors in \( \mathcal{G} \). The agents’ network \( \mathcal{G} \) is considered as the union of a directed cycle and a 2-hop digraph (see examples in [2]). To satisfy the weight-balanced condition the link weights in each network are equal and randomly chosen in the range \((0, 0.5)\). Using MATLAB’s \texttt{randperm} function, we randomly change the permutation of the nodes in the network and the link weights every 0.05 seconds to simulate a dynamic network in DT domain. The time-evolution of \( x_i = [\omega_i^T; \nu_i] \in \mathbb{R}^4 \), loss function \( F(x_i) \), and sum of the gradients \( \sum_{i=1}^n \nabla f_i(x_i) \in \mathbb{R}^4 \) are shown in Fig. 2. The agents reach consensus on the optimal value \( x^* = [\omega^*(1), \omega^*(2), \omega^*(3), \nu]^T \) as the parameters of the separating hyperplane in \( \mathbb{R}^3 \). Via the inverse mapping, the hyperplane in \( \mathbb{R}^3 \) represents an ellipse formulated as \( \{(x_1^2 + x_2^2 - \tau = 0) \} \) in \( \mathbb{R}^2 \), which separates the original data points \( x_i \)’s in \( \mathbb{R}^2 \). The calculated separating ellipses by all 5 agents are shown in Fig. 3 at two different time-instants.

VII. CONCLUSION AND FUTURE RESEARCH

In this work, a Lipschitz dynamics is proposed to solve D-SVM over a dynamic WB-digraph in a hybrid setting. We adopt matrix perturbation analysis to prove convergence of the CT dynamics (11)-(12) whose parameters vary due to switching network topology in DT domain. In particular, our proposed distributed optimization in D-SVM setup enables
the agents to cooperatively learn the classifier over a dynamic network via local information, improving classical D-SVM methods in terms of data privacy [6]–[8] and computational complexity [9].

As future research direction, one can extend the results to the DT counterpart. For example, for Euler-Forward method, the DT version of matrix $M$ in (12) is $M_t = I + TM$ with $T$ as the sampling period. Then, the explicit upper bound on $T$ such that a stable CT dynamics from Theorem 1-2 remains stable after discretization can be defined. Additionally, extensions to time-delayed networks, online D-SVM, and sparse digraphs are directions of interest.

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