Scaling of dynamical susceptibility at the onset of rigidity for disordered viscoelastic matter

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The onset of rigidity in interacting liquids, as they undergo a transition to a disordered solid, is associated with a dramatic rearrangement of the low-frequency vibrational spectrum. In this letter, we derive scaling forms for the singular dynamical response of disordered viscoelastic networks near both jamming and rigidity percolation. Using effective-medium theory, we extract critical exponents, invariant scaling combinations and analytical formulas for universal scaling functions near these transitions. Our scaling forms describe the behavior in space and time near the various onsets of rigidity, for rigid and floppy phases and the crossover region, including diverging length and time scales at the transitions. We expect that these behaviors can be measured in systems ranging from colloidal suspensions to anomalous charge-density fluctuations of “strange” metals.

Jamming [1] and Rigorous Percolation (RP) [2] provide suitable frameworks to characterize the fascinating invariant scaling behavior exhibited by several classes of disordered viscoelastic materials near the onset of rigidity [3]. Both are often described by elastic networks near the Maxwell limit of mechanical stability [4], and represent transitions from a rigid phase to a floppy one when the average coordination number z falls below the isostatic value z_{c}. RP appears in network glasses [5], fiber networks [6, 7] and soft colloidal gels [8], and is described in terms of networks in which bonds are randomly removed; the bulk modulus vanishes [9] at the transition [10–12]. Jamming is also a ubiquitous phenomenon arising in systems ranging from amorphous solids and glasses [13] to cell tissues [14] and deep learning [15]. Jamming is commonly described in terms of sphere packings that possess a finite bulk modulus B > 0 at the transition. Recently, it was shown that jamming can be described as a multi-critical point that terminates a line of continuous transitions associated with rigidity percolation and that there is a deep connection between the universal scaling forms for both transitions [16]. Determining explicit formulas for the susceptibilities and space-time correlations has been challenging, however, since there is a scarcity both of comprehensive numerical data and of analytic models for these transitions (with the exception of jamming in high dimensions [17–19]). Here, we leverage the analytically-tractable effective-medium theory (EMT) of Ref. [16] to fill this gap and extract explicit equations for these universal forms.

At jamming [1], two-dimensional disk packings form a disordered contact network [blue lines in Fig. 1(a)] that supports compression but not shear. Mimicking compression by randomly adding next-nearest neighbor bonds between disks [red N-bonds in Fig. 1(a)] and/or randomly removing B-bonds can lead to either jamming or RP depending on the population for each type of bond [16]. A simpler model that yields the same scaling behavior consists of randomly placing ‘B’ and ‘N’-bonds between nearest and next-nearest neighbor pairs of sites [blue and red solid lines in Fig. 1(b)] of a periodic honeycomb lattice. This network describes a diluted version of a 3-sub-lattice system consisting of a honeycomb lattice [shaded blue in Fig. 1(b)] and two triangular lattices (shaded red; here we show only the bonds of one triangular lattice). Detailed knowledge of the mechanical behavior of periodic lattices allowed the development of an EMT at finite dimension [20] for jamming [16] and for the crossover from jamming to RP, valid in both rigid and floppy states. We will employ these results to derive explicit solutions for the critical scaling of the susceptibilities of disordered viscoelastic matter near jamming and RP. Our analysis not only allows for quick assessment of scale-invariant behavior of quantities such as viscosities and correlations (without the need for computationally-expensive simulations); it also serves as an example of how one may analyze rigidity transitions for which the universality class has not been determined.

Figure 1(c) shows the phase diagram of the honeycomb-triangular lattice (HTL) model in terms of occupation probability of nearest neighbor B bonds and next-nearest neighbor N bonds. Rigid (yellow) and floppy regions are separated by an RP line that terminates in a multicritical jamming point J (red disk). Arrows denote paths approaching jamming and RP from the rigid and floppy states. From Fig. 1(c), one can also extract definitions for the scaling variables δ_{p_b} and δ_{p_{_{RP}}}, chosen so that δ_{p_{_{RP}}} = 0 at RP, and δ_{p_b} is also zero at jamming.

RP should generically be codimension one, because only one constraint (isostaticity) needs to be satisfied. In the HTL model of Fig. 1(b), jamming is codimension two. But the jump in bulk modulus characteristic of jamming here demands a complete honeycomb lattice; one can see that if the three orientations of hexagon bonds were independently populated, the jamming transition would be codimension four (their three probabilities set to one plus isostaticity). This special tuning of the system to favor the bulk modulus is echoed in the jamming of frictionless spheres, where the first state of self stress [21] leads
to a jump in the bulk modulus because the conjugate degree of freedom (a uniform compression) was used to tune the system to the rigidity transition. As evidence for this, shear jamming of frictionless spheres has a jump in a single anisotropic modulus [22].

We conjecture that there is a class of disordered elastic systems for which a renormalization-group scheme leads to the typical crossover flow diagram [23] (projected in \( \delta p_{\text{R}} \times \delta p_{\text{B}} \) space) illustrated in Fig. 1(d). The scaling variable \( \delta p_{\text{R}} \propto \Delta z \equiv z - z_c \) must be relevant for both jamming and RP, but the depletion probability of the B-lattice \( \delta p_{\text{B}} \) is relevant only for jamming. This behavior is captured by the direction of the arrows coming in and out of the putative jamming and RP fixed points (red and black disks, respectively) in Fig. 1(d).

A system near the \( J \) fixed point (\( \delta p_{\text{B}}, |\delta p_{\text{R}}| \ll 1 \)) will be controlled either by \( J \) if a crossover variable \( \delta p_{\text{B}}/|\delta p_{\text{R}}|^{\varphi} \ll 1 \) for some exponent \( \varphi \), or by \( \text{RP} \) if \( \delta p_{\text{B}}/|\delta p_{\text{R}}|^{\varphi} \gg 1 \), i.e., for trajectories such as the black dashed line passing sufficiently close to the critical line (black solid line.) Though \( \delta p_{\text{B}} \) does not have a direct interpretation in the jamming of sphere packings [except for the network model of Fig. 1(a)], there might be variables that play a similar role, such as attractive interactions in soft gels [24].

We now introduce a scaling ansatz for the longitudinal response function [25] near jamming:

\[
\chi_L \approx |\delta p_{\text{R}}|^{-\gamma} L \left( \frac{q}{|\delta p_{\text{R}}|^{\nu}}, \frac{\omega}{|\delta p_{\text{R}}|^{\gamma}}, \frac{\delta p_{\text{B}}}{|\delta p_{\text{R}}|^{\varphi}} \right),
\]

where \( q \) is the wavevector, \( \omega \) the frequency, \( \gamma, \nu, z \) and \( \varphi \) are critical exponents for the susceptibility, correlation length, correlation time, and crossover behavior, respectively [23, 26], and \( L \) is a universal scaling function. Many other properties can be derived from \( L \) (Table II).

Such space-time susceptibilities, and the corresponding structure and correlation functions, are the fundamental linear response quantities for materials. They have been well studied in glassy systems, but have hitherto not been a focus in the study of jamming or RP. Baumgarten et al. [27] and Hexner et al. [28] have studied the static response of frictionless jammed spheres to a sinusoidal perturbation; they find diverging length scales that are different from the ones presented here. Because our system is on a regular lattice, and particularly because our analysis replaces the disordered lattice with a uniform one, it is natural for us to fill this gap.

Our approach goes beyond previous work [29] in two aspects. First, rather than starting with an ansatz for the free energy in terms of the excess contact number \( \Delta z \), excess packing fraction \( \Delta \phi \), shear stress \( \epsilon \) and system size \( N \), we consider the longitudinal response in terms of \( \delta p_{\text{R}}, q, \nu, \omega \) and \( \delta p_{\text{B}} \). Our variable \( \delta p_{\text{R}} \) is proportional to \( \Delta z \). Though we do not consider an explicit dependence of \( \chi_L \) on \( \epsilon \) or \( \Delta \phi \) [30], we can extract equivalent expressions for moduli and correlations from the dependence of \( \chi_L \) on \( q \). Importantly, the inclusion of \( \omega \) in our analysis allows us to predict dynamical properties such as viscosities.

Second, we use EMT [16] to derive and validate both the universal exponents and the universal scaling functions (\( L \), for both jamming and RP. This form of EMT is based on the coherent-potential approximation [10, 31] (CPA), and is known to reproduce well results obtained from simulations of randomly-diluted lattices with two-body [32] harmonic interactions [33, 34], even for undamped [11, 16] and overdamped dynamics [35, 36]. Although the CPA involves mean-field-like uncontrolled approximations, it preserves the topology of the original lattices — an essential ingredient that ultimately allows one to describe jamming. Here we focus on the longitudinal response, since the full response of isotropic elastic systems can be decomposed into longitudinal and transverse components, and the latter has the same scaling form near both jamming and RP as the longitudinal response near RP; see Supplemental Material (SM) [37].
We use the long wavelength limit of the longitudinal response $\chi_l$ along with EMT results from Ref. [16] to derive critical exponents (see Table I) and the universal scaling function $L$ in Eq. (1) (see SM [37]),

$$L(u, v, w) = \left[ \frac{a u^2}{1 + w/\mathcal{M}_\pm(v)} - \delta(v) \right]^{-1}. \quad (2)$$

Here $\mathcal{M}_\pm(v) = b \sqrt{1 - c \delta(v) \pm 1}$ and $a$, $b$ and $c$ are constants. The plus and minus signs in $\mathcal{M}_\pm$ correspond to solutions in the elastic and floppy states, respectively. Finally, $\delta(v) = \rho v^2$ and $i \gamma v$ for undamped and overdamped dynamics, respectively, where $\rho$ is a mass density and $\gamma$ is a drag coefficient. Equation (2) embodies the central results of this paper. From Eqs. (1) and (2), we will extract the universal behavior of the elastic moduli, viscosities as well as the density response and correlation functions (dynamic structure factor). Though it is not certain that these functions are as universal as critical exponents, recent simulations of compressed hyperspheres [38] indicate that critical amplitudes calculated using mean-field models at infinite dimension are preserved for low-dimensional jammed packings.

|               | $\gamma$ | $z$ | $\nu$ | $\varphi$ | $\beta_B$ | $\gamma_B$ |
|---------------|----------|-----|-------|----------|-----------|-----------|
| Jamming       | 2        | 1 (2)| 1     | 0        | 1 (2)     |           |
| Rigidity Percolation | 2        | 2 (4)| 1/2   | -        | 1 (1)     | 0 (1)     |

TABLE I. Critical exponents for the longitudinal susceptibility ($\gamma$), correlation length ($\nu$), correlation time ($z$) and crossover behavior ($\varphi$) near jamming and RP for undamped and overdamped (between parentheses if different from undamped) dynamics. The exponents $\beta_B$ and $\gamma_B$ can be derived from $\gamma$, $\nu$ and $z$ (see Table II), and describe power-law singularities for the bulk modulus and viscosity, respectively.

For $|\delta p_{\text{cr}}| \ll |\delta p|$ ($w \gg 1$ in Eq. (2)), our model exhibits RP criticality: $\delta p_{\text{cr}}$ becomes an irrelevant variable, and $L(u, v, w) \rightarrow \tilde{L}(u, v)$, with

$$\tilde{L}(u, v) = \left[ a' u^2 \mathcal{M}_\pm(v) - \tilde{\delta}(v) \right]^{-1}. \quad (3)$$

Here $a' = a/|\delta p_{\text{cr}}|$ is a constant, and the change in $L$ is accompanied by a change in the critical exponents $\nu$ and $z$ (see Table I). Note that the invariant scaling combinations $v/|\delta p_{\text{cr}}|^\nu$ and $\omega/|\delta p_{\text{cr}}|^{z\nu}$ lead to definitions for diverging length and time scales, $l_* \sim 1/|\delta p_{\text{cr}}|^\nu$ and $\tau^* \sim 1/|\delta p_{\text{cr}}|^{z\nu}$, respectively. The exponent $z\nu$ depends only on the type of dynamics, but the exponent $\nu$ (equals 1 and 1/2 for jamming and RP, respectively) emerges naturally in our approach and is consistent with more elaborate definitions involving cutting boundaries [13].

To validate Eqs. (2) and (3), we show in Fig. 2 the scaling collapse plots of the longitudinal response as a function of the frequency scaling combination $v = \omega/|\delta p_{\text{cr}}|^{z\nu}$, near jamming and RP, for overdamped dynamics, in the rigid and floppy phases. The solid and dashed curves are the asymptotic universal scaling predictions at two different values of the wavevector scaling variable $u = q/|\delta p_{\text{cr}}|^\nu$. We approach the jamming point along paths of constant $\delta p_{\text{cr}}/|\delta p_{\text{cr}}|^\nu \equiv w$. Real parts are in blue; imaginary (dissipative) parts in red. Symbols show the convergence of the full solutions of the EMT equations to our universal scaling forms (Eqs. (2) and (3)).

![FIG. 2. Scaling collapse of the longitudinal response near jamming and RP for overdamped dynamics following the paths shown in Fig. 1(c). Blue disks and red triangles correspond to full solutions of the EMT equations for the real and imaginary parts of $|\delta p_{\text{cr}}|^{z\nu} \chi_l$, respectively. Solid and dashed curves correspond to the universal scaling predictions of Eqs. (2) and (3). Dashed [solid] lines correspond to $q/|\delta p_{\text{cr}}|^\nu = u = 1$ [0.1]. We approach the jamming point along $\delta p_{\text{cr}}/|\delta p_{\text{cr}}|^\nu \equiv w$ equal to $\sqrt{5}/4$ from the rigid side (a), and equal to 2 from the floppy side (b). Full solutions run at $|\delta p_{\text{cr}}| = 10^{-2}, 10^{-3}$, and $10^{-4}$ for RP and a range $|\delta p_{\text{cr}}| \in [5 \times 10^{-2}, 5 \times 10^{-4}]$ for jamming show convergence to our universal asymptotic predictions.

The universal function $L$ (and $\tilde{L}$) describes the scale-invariant behavior of the longitudinal response. On the overdamped solid side [Fig. 2(a) and (c)], the real part $L'(v)$ plateaus and the imaginary part $L''(v)$ (the dissipation) vanishes at low frequency $v$. At high frequency, both $L'$ and $L''$ decay to zero, but $L'$ decays faster than $L''$ [39]. This leads to a frequency scale above which $L'' > L'$ in the “diffusive limit” $\omega/|\delta p_{\text{cr}}|^\nu \sim D^*$, where $D^* \sim |\delta p_{\text{cr}}|^\nu$ is a characteristic diffusion constant, with the exponent $\nu \equiv (z - 2)/\nu$ equal to zero and one for jamming and RP, respectively. The precise form of the decay of $L'$ and $L''$ depends on the wavevector $u$ (see SM [37]). On the liquid side, $L'$ behaves as in the solid side, but $L''$ diverges rather than vanishing at low $v$ due to the...
predominant viscous response of the liquid state.

Equations (1) and (2) determine the scaling behavior of several quantities characterized by the general form,

\[
Y = |\delta p_{\text{bulk}}|^y \mathcal{Y} \left( \frac{q}{|\delta p_{\text{bulk}}|^z}, \frac{\omega}{|\delta p_{\text{bulk}}|^z}, \frac{\delta p_{\text{bulk}}}{|\delta p_{\text{bulk}}|^\nu} \right),
\]

where in Table II we present explicit expressions for the exponent \( y \) and universal function \( \mathcal{Y} \) describing the bulk modulus \( B \), viscosity \( \zeta \), density response \( \Pi \) and correlation function \( S \). The behavior near RP is obtained by replacing \( \mathcal{Y} \) and \( \mathcal{L} \) (now functions of \( u \) and \( v \) only), respectively, along with appropriate changes for the exponents (see Table I). The scaling behavior of the shear modulus and viscosity near jamming and RP is the same as that of \( B \) and \( \zeta \), respectively, near RP.

![Table II. Critical exponent \( y \) and universal scaling function \( \mathcal{Y} \) describing the singular behavior of the bulk modulus \( B \) and viscosity \( \zeta \), density response \( \Pi \) and correlation function \( S \), according to Eq. (4).](image)

We now discuss an intriguing application of our universal function \( \mathcal{S}(u, v) \) (see Table II) for the density-density correlation (the structure function for isotropic fluids at \( q \neq 0 \)), shown in Fig. 3(a) for undamped fluids near RP. At fixed \( u \), \( \mathcal{S}(u, v) \) has a maximum (blue dashed line) at \( v = v^* \approx O(1) \) [i.e. \( \omega^* \propto |\delta p_{\text{bulk}}| \) (see SM [37]), which coincides with the crossover from Debye to isotropic behavior, interpreted as the paradigmatic boson peak [40–42] of glasses [43]. Near jamming or RP, this point marks the onset of the enhancement of the population of low-energy modes [44] leading to a flat density of states at low frequency [16, 44]. We analyze the surprising connections between these featureless low-energy modes and the unconventional particle-hole continuum measured recently using momentum and energy-resolved spectroscopic probes in certain strange metals [45, 46] in Ref. [47]. The damping of the plasmon into the broad, featureless continuum in the cuprate strange metal [45] is possibly related to the behavior of the longitudinal response analyzed in this paper; we leave a detailed discussion of this analogy for elsewhere [47]. At fixed \( v \), \( \mathcal{S} \) plateaus at a value of \( u \) of \( O(1) \) (i.e. at \( q \propto |\delta p_{\text{bulk}}|^{1/2} \)). Figure 3(b) shows a diagram in terms of \( u \) and \( v \) marking the boson peak (blue-dashed line) and regions where \( \mathcal{S}(u, v) \) exhibits power-law behavior. The blue region indicates the neighborhood of the boson peak, in which \( \mathcal{S}(u, v) > \mathcal{S}(u, v^*)/2 \), and the red and yellow regions show power-law regimes in the rescaled frequency \( v \) and wavevector \( u \).

![Figure 3. (a) 3D plot of the universal scaling function for the correlation function \( \mathcal{S}(u, v) \), for undamped fluids near RP. The blue dashed line corresponds to the rescaled frequency \( v^* \) (the boson peak) at which \( \mathcal{S}(u, v) \) is maximum for fixed rescaled wavevector \( u \). (b) \( u \times v \) diagram showing the boson peak (blue dashed line) and power law regions for which \( \mathcal{S}(u, v) \propto u^\alpha v^\beta \), with \( \alpha, \beta \) within 10% of their asymptotic values (0, –2) (red) and (4, –4) (yellow). In the blue region the condition \( \mathcal{S}(u, v) > \mathcal{S}(u, v^*)/2 \) is satisfied.](image)

In this letter, we have combined scaling theory and the EMT of Ref. [16] to produce analytical formulas for universal scaling functions for the longitudinal dynamical response near both jamming and RP. Our equations can be used to determine the space-time dependence of universal functions for several quantities (such as moduli, viscosities and correlations) near the onset of rigidity in both the solid and liquid phases. A direct approach to experimentally validate our predictions consists of using 3D printers to fabricate and perform experiments on the disordered elastic networks illustrated in Figs. 1(a) and (b). We also expect these scaling forms to apply to more traditional glass forming systems such as colloidal suspensions. Here, in addition to more standard scattering measurements, new techniques for measuring 3D particle positions and even stresses with high precision may make it feasible to measure these functional forms and test our predictions [48–51]. In such suspensions, we expect that the scaling functions will capture the behavior in the elastic regime. However, our theory is built on a fixed network topology and lacks some features of the liquid phase. Annealed rather than quenched disorder [23] (or even intermediate disorder [52]) could be needed to describe viscoelastic fluids. Extensions of our analysis could include the incorporation of anisotropic bond occupation [53], which plays a major role in the crossover scaling of thickening suspensions near frictional jamming [54] and that can lead to simpler models for both shear jamming [55] and thickening [56], as well as the incorporation of random stress fields, which can elucidate the unjamming of colloidal suspensions (such as titanium dioxide) due to activity [57].
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Supplemental Material for “Scaling of dynamic susceptibility at the onset of rigidity for disordered viscoelastic matter”

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Here we present a brief review of the main results of Ref. (Liarte et al., 2019) (Sec. I), details for the derivation of the critical exponents and universal scaling functions for the longitudinal response (Sec. II), as well as scaling collapse plots near both jamming and rigidity percolation for undamped and overdamped dynamics in the rigid and floppy phases (Sec. III), and derivations for the universal scaling behavior of the transverse dynamic response, moduli and viscosities, density response and correlation functions (Sec. IV.)

I. BRIEF REVIEW OF “JAMMING AS A MULTICRITICAL POINT”

Here we discuss key results of “Jamming as a multicritical point” (Liarte et al., 2019), which we use in the derivation of critical exponents and universal scaling functions for the dynamic response presented in Secs. II and IV.

We use the honeycomb-triangular lattice (HTL) model to describe both static and dynamical properties of jamming and rigidity percolation (RP) near the threshold of mechanical stability. The HTL model combines suitable mechanical properties of two periodic lattices: The honeycomb lattice (solid lines in Fig. 1) with finite bulk modulus $B > 0$ and zero shear modulus $G = 0$, and two triangular lattices (dashed and dotted lines) with finite $B, G > 0$. In simulations, bonds (harmonic elastic interactions) of the honeycomb lattice and the triangular lattices have unit spring constant and are populated with probability $p_b$ and $p_n$, respectively. In the effective-medium theory, the honeycomb and triangular lattices are fully populated with bonds with a complex frequency-dependent effective spring constant $k_b(\omega)$ and $k_n(\omega)$, respectively.

We use the Coherent Potential Approximation (CPA) to derive the set of self-consistent equations for $k_b(\omega)$ and $k_n(\omega)$, and then describe elastic and phonon properties of the HTL model near jamming and RP. The derivation of the self-consistent equations is standard and is explained in many references [see e.g. (Elliott et al., 1974; Feng et al., 1985; Mao and Lubensky, 2011)]. In the CPA, the randomly-diluted lattice is modeled by a homogeneous lattice with effective spring constants that satisfy a set of prescribed equations. The self-consistent equations for the jamming model of Ref. (Liarte et al., 2019) are given by

$$k_\alpha = \frac{p_\alpha - h_\alpha}{1 - h_\alpha},$$

where $p_\alpha$ and $k_\alpha$ are the occupancy probability of each bond and the effective spring constant for sub-lattice $\alpha \in \{B, N\}$, respectively. The functions $h_\alpha = h_\alpha(p_b, p_n, \omega)$

FIG. 1 Illustration of the honeycomb-triangular lattice model. In simulations, bonds of the honeycomb lattice (solid lines) and the two triangular lattices (dashed and dotted lines) are populated with probability $p_b$ and $p_n$, respectively. In the effective medium theory, the fully-occupied honeycomb and triangular sub-lattices have frequency-dependent effective spring constants $k_b(\omega)$ and $k_n(\omega)$, respectively, satisfying a set of self-consistent equations.

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1 This is a special case of the more general model in which the bond occupation probability can be different for the two triangular lattices.
are defined by
\[ h_\alpha = \frac{1}{\bar{z}_\alpha N_c} \sum_q \text{Tr} [D_\alpha(q) \cdot G(q,\omega)], \quad (2) \]

where \( N_c \) is the total number of cells, \( \omega \) is the frequency, \( q \) is the wavevector, and \( D_\alpha(q) \) and \( \bar{z}_\alpha \) are the dynamical matrix and the number of bonds per unit cell for sublattice \( \alpha \), respectively. The trace is taken over an \( mD \)-dimensional space, where \( m \) is the number of sites per unit cell and \( D \) is the spatial dimension. The Green’s function \( G \) is defined by
\[ G(q,\omega) = \left[ \sum_\alpha D_\alpha(q) - \omega^2 \mathbb{I} \right]^{-1}, \quad (3) \]

where \( \mathbb{I} \) is an identity matrix. Note that \( \sum_\alpha D_\alpha \) depends on all effective spring constants \( k_\alpha \), so that Eq. 1 self-consistently determines the values of all \( k_\alpha \) for given \( p_n, p_s \) and \( \omega \).

Elastic moduli can be expressed in terms of the effective springs constants by taking the long-wavelength limit of the dynamical matrix (with the caveat that internal degrees of freedom must be relaxed before applying the limit of small wavevector). The HTL has isotropic elasticity, with
\[ B = \frac{3}{4} k_n + \frac{9}{2} k_s, \quad G = \frac{9}{4} k_s, \quad (4) \]

for the bulk and shear moduli, respectively. These definitions, along with solutions of the CPA equations allow us to draw the zero-frequency phase diagram shown in Fig. 2.

Using perturbation analysis, one can write the asymptotic equations for the low-frequency behavior of \( k_n \) and \( k_s \) near the jamming point \( J \):
\[ k_n \approx \frac{k_s}{k_n + \delta p_n/b_1}, \quad (5) \]
\[ k_s \approx b_2 |\delta p_{\nu\nu}| \left( \sqrt{1 - c |\bar{\omega}(\omega)/|\delta p_{\nu\nu}|^2| \pm 1} \right), \quad (6) \]

where \( b_1, b_2 \) and \( c \) are constants, the plus and minus signs on the second equation corresponds to the rigid and floppy phases, respectively, and
\[ \bar{\omega}(\omega) = \begin{cases} \rho \omega^2, & \text{for undamped dynamics,} \\ i \gamma \omega, & \text{for overdamped dynamics,} \end{cases} \quad (7) \]

where \( \rho \) and \( \gamma \) represent the mass density and a drag coefficient, respectively. From Eqs. (5) and (6), we can write,
\[ k_n \approx \frac{\delta p_n}{b |\delta p_{\nu\nu}| \left( \sqrt{1 - c |\bar{\omega}(\omega)/|\delta p_{\nu\nu}|^2| \pm 1} \right)}^{-1}, \quad (8) \]

where \( b \equiv b_1 \cdot b_2 \) is constant. From Eq. (8), we can derive the scaling behavior of the frequency-dependent bulk modulus:
\[ B(\omega) \approx a \left[ 1 + \frac{|\delta p_{\nu\nu}| |\delta p_{\nu\nu}|^\varphi}{M(\omega/|\delta p_{\nu\nu}|^\nu)} \right]^{-1}, \quad (9) \]

where \( a \) is a constant corresponding to the value of the bulk modulus of the fully-populated honeycomb lattice in our model. The exponent \( \varphi = 1 \), and the product \( \nu = 1 \) and 2 for undamped and overdamped dynamics, respectively. The universal scaling function \( M \) is given by
\[ M(\nu) = b \left[ \sqrt{1 - c \hat{\nu}(\nu) \pm 1} \right], \quad (10) \]

where \( b \) and \( c \) are constants, the plus and minus correspond to solutions in the elastic and floppy states, respectively, and
\[ \hat{\nu}(\nu) = \begin{cases} \rho \nu^2, & \text{for the undamped case,} \\ i \gamma \nu, & \text{for the overdamped case.} \end{cases} \quad (11) \]

We use Eq. (9) to derive the scaling behavior of the longitudinal response in Sec. II. The scaling behavior for the shear modulus \( G \) follows directly from the asymptotic behavior of \( k_s \) in Eq. (6): \[ G(\omega) \approx g |\delta p_{\nu\nu}|^{\beta_G} M(\omega/|\delta p_{\nu\nu}|^\nu), \quad (12) \]

where \( \beta_G = 1 \) and \( g \) is a constant.

Note that these scaling forms were obtained using an approximation that is valid near the multicritical point \( J \).
Some of the nonuniversal constants change as one moves away from the J point towards larger values of $\delta p_n$ [see e.g. the slope of the shear modulus in Ref. (Pliarte et al., 2019)]. Though these constants depend on the model, we expect the functional forms be universal.

II. EXPONENTS AND UNIVERSAL SCALING FUNCTION FOR THE LONGITUDINAL RESPONSE

Here we use the results presented in Sec. I to derive the critical exponents and universal scaling functions for the longitudinal response near both jamming and RP, for undamped and overdamped dynamics in the solid and fluid phases.

In the long wavelength limit, the longitudinal component of the dynamic response function $\chi_l$ of an isotropic viscoelastic material is given by (Bland, 2016; Chaikin and Lubensky, 1995; Kubo et al., 2012)

$$\chi_l = \left\{ -\rho \omega^2 - i \gamma \omega + q^2 \left[ B(\omega) + 2 \frac{D - 1}{D} \frac{1}{G(\omega)} \right] \right\}^{-1}.$$ (13)

The complex moduli $B(\omega) = B' + iB''$ and $G(\omega)$ can be decomposed into storage ($B'$ and $G'$) and loss ($B''$ and $G''$) components. Interestingly, a nonzero loss modulus develops even if there is no dissipation term in the dynamics (i.e. if $\gamma = 0$). This happens because the dynamical CPA involves an average over disorder that transforms simple springs into Kelvin-Voigt elements [combinations of springs and dashpots (Bland, 2016)], which lead to nonzero imaginary parts if the frequency is sufficiently high. Physically, this is the way CPA incorporates scattering of long-wavelength phonons off the disordered lattice.

For $G/B \ll 1$ (near jamming) we can write

$$\chi_l \approx \left\{ -\rho \omega^2 - i \gamma \omega + q^2 B(\omega) \right\}^{-1} \approx \left\{ -\rho \omega^2 - i \gamma \omega + a q^2 \right\}^{-1} \times \left[ 1 + \frac{\delta p_n/|\delta p_n|^\varphi}{\mathcal{M}_\pm(\omega/|\delta p_n|^{2\varphi})} \right]^{-1},$$ (14)

where $\varphi = 1, z \nu = 1$ and 2 for undamped and overdamped dynamics, respectively, and we have used the asymptotic form for the bulk modulus [Eq. (27)]. Now we multiply both sides of (14) by $|\delta p_n|^\varphi$ to write

$$|\delta p_n|^2 \chi_l \approx \left\{ -\rho \left( \frac{\omega}{|\delta p_n|} \right)^2 - i \gamma \frac{\omega}{|\delta p_n|^2} + a \right\} \times \left( \frac{q}{|\delta p_n|} \right)^2 \left[ 1 + \frac{\delta p_n/|\delta p_n|^\varphi}{\mathcal{M}_\pm(\omega/|\delta p_n|^{2\varphi})} \right]^{-1},$$ (15)

which leads to the scaling form:

$$\chi_l \approx |\delta p_n|^{-\gamma} \mathcal{L} \left( \frac{q}{|\delta p_n|^\nu}, \frac{\omega}{|\delta p_n|^{\nu}}, \frac{\delta p_n}{|\delta p_n|^\varphi} \right),$$ (16)

where

$$\mathcal{L}(u, v, w) = \left[ \frac{a u^2}{1 + w/\mathcal{M}_\pm(v)} - \tilde{\nu}(v) \right]^{-1},$$ (17)

with the exponents $\gamma = 2$ and $\nu = 1$ for jamming.

The crossover to rigidity percolation occurs when the invariant scaling combination $\delta p_n/|\delta p_n|^\varphi \gg 1$, so that, from Eq. (15):

$$|\delta p_n|^2 \chi_l \approx \left\{ -\rho \left( \frac{\omega}{|\delta p_n|} \right)^2 - i \gamma \frac{\omega}{|\delta p_n|^2} + a \right\} \times \left( \frac{q}{|\delta p_n|} \right)^2 \frac{|\delta p_n|^\varphi}{|\delta p_n|^{2\varphi}} \mathcal{M}_\pm \left( \frac{\omega}{|\delta p_n|^{2\varphi}} \right) \right] \right\}^{-1}.$$ (18)

Since $\delta p_n$ is an irrelevant variable for rigidity percolation, the term $|\delta p_n|^\varphi$ has to be incorporated into the invariant scaling combination for $q$, i.e.

$$a q/(|\delta p_n|^\varphi/|\delta p_n|) = a'(q/|\delta p_n|^\nu)^2,$$ (18)

where $\nu = 1/2$ for RP, and $a' = a/\delta p_n$. Since the product $z \nu$ depends only on the type of dynamics (undamped or overdamped,) the exponent $z$ also change for rigidity percolation. The longitudinal response then behaves as:

$$\chi_l \approx |\delta p_n|^{-\gamma} \mathcal{L} \left( \frac{q}{|\delta p_n|^\nu}, \frac{\omega}{|\delta p_n|^{\nu}}, \frac{\delta p_n}{|\delta p_n|^\varphi} \right),$$ (19)

where

$$\mathcal{L}(u, v, w) = \left[ a' u^2 \mathcal{M}_\pm(v) - \tilde{\nu}(v) \right]^{-1}.$$ (20)

Table I lists the values of the critical exponents $\gamma$, $z$, $\nu$, $\varphi$, $\beta_B$, and $\gamma_B$ (the last two will be defined in Sec. IV) for both jamming and RP, for both undamped and overdamped (between parentheses, if different from undamped) dynamics.

|                | $\gamma$ | $z$ | $\nu$ | $\varphi$ | $\beta_B$ | $\gamma_B$ |
|----------------|----------|-----|------|-----------|------------|------------|
| Jamming        | 2        | 1   | 2    | 1         | 0          | 1 (2)      |
| Rigidity Percolation | 2      | 1/2 | -    | 1         | 0          | 0 (1)      |

Table I Critical exponents extracted from the longitudinal response function near jamming and rigidity percolation for undamped and overdamped (between parentheses) dynamics.

III. SCALING COLLAPSES AND GLOBAL BEHAVIOR

To validate the universal scaling functions derived in Sec. II, here we show scaling collapse plots for the rescaled longitudinal response as a function of rescaled frequency at fixed values of $u$ (rescaled wavevector) and $w$ (rescaled crossover variable; only for plots near jamming). We
consider both overdamped [(a), (b), (c) and (f)] and undamped [(c), (d), (g) and (h)] dynamics, and paths approaching jamming (first row) and rigidity percolation (second row) from both the rigid and floppy phases (see Fig. 2). Although there are model-specific predictions for some of the nonuniversal constants, here we choose the constants $a$, $a'$, $b$ and $c$ to best fit the collapsed data.

Note that the asymptotic solutions derived in (Liarte et al., 2019) do not capture the small but nonzero imaginary parts of the effective spring constants at frequencies smaller than $\sim \omega^*$ (the characteristic crossover to isostaticity) when there is no damping. This feature has important consequences for energy dissipation in systems believed to exhibit behavior related to RP, such as electrons in certain strange metals (Thornton et al., 2021). The corrections to scaling appear as singular perturbations to the self-consistency equations and vanish as powers of $|\delta \rho_p|$ in dimensions larger than three. Moreover, the scaling variables contain logarithms in two dimensions. This analysis is beyond the scope of the present work, and will be presented in a separate manuscript.

Equations (17) and (20) imply that our universal functions $\mathcal{L}(u, v, w)$ and $\tilde{\mathcal{L}}(u, v)$ generally behave as $u^\alpha v^\beta$ with the exponents $\alpha$ and $\beta$ depending on the region in the $u$ (rescaled wavevector) $\times v$ (rescaled frequency) plane. To illustrate this global behavior and map it in the different regions, we present diagrams showing power-law regions for which $\mathcal{L}(u, v, w) \propto u^\alpha v^\beta$ (Fig. 4) and $\tilde{\mathcal{L}}(u, v) \propto u^\alpha v^\beta$ (Fig. 5), with $(\alpha, \beta)$ within 10% of their asymptotic values (except when either $\alpha$ or $\beta$ is zero, in which case we plot the region in which its absolute value is less than 0.1.) In both Fig. 4 and Fig. 5, the first and second rows correspond the real part and the imaginary part of the universal function, respectively. To generate each panel, we numerically calculate the exponents using $f_\alpha \equiv \partial \log \mathcal{L} / \partial \log u$ and $f_\beta \equiv \partial \log \mathcal{L} / \partial \log v$, and then we plot the regions in which $f_\alpha - \alpha < 0.1 \alpha$ and $f_\beta - \beta < 0.1 \beta$, for several values of $\alpha$ and $\beta$. We have set the constants $a$, $a'$, $b$ and $c$ equal to the numerical values used to collapse the data in Fig. 3.

IV. DERIVATION OF THE SCALING BEHAVIOR OF OTHER QUANTITIES

Here we present a brief derivation of the universal scaling functions and critical exponents for the transverse dynamic response, elastic moduli, viscosities, density response and correlation functions.

A. Transverse dynamic response

To extract the universal scaling functions and critical exponents associated with the transverse dynamic response, we follow the same steps that we used in Sec. II. We start with the long-wavelength limit of the transverse component of the dynamic response function $\chi_T$ of an isotropic viscoelastic material (Bland, 2016; Chaikin and Lubensky, 1995; Kubo et al., 2012; Sethna, 2006):

$$\chi_T = \left\{ -\rho \omega^2 - i \gamma \omega + q^2 G(\omega) \right\}^{-1}. \quad (21)$$

Near jamming, the complex shear modulus $G(\omega)$ satisfies Eq. (12), so that

$$\chi_T \approx \left\{ -\rho \omega^2 - i \gamma \omega + q^2 |\delta \rho_p| |\delta \rho_p|^2 \mathcal{M}_\pm \left( \frac{\omega}{|\delta \rho_p|^z} \right) \right\}^{-1}, \quad (22)$$

where $z = 1$ and 2 for undamped and overdamped dynamics, respectively. Multiplying both sides by $|\delta \rho_p|^2$, we obtain

$$|\delta \rho_p|^2 \chi_T \approx \left\{ -\rho \left( \frac{\omega}{|\delta \rho_p|} \right)^2 - i \gamma \frac{\omega}{|\delta \rho_p|^2} + \frac{q}{|\delta \rho_p|^2} \mathcal{M}_\pm \left( \frac{\omega}{|\delta \rho_p|^z} \right) \right\}^{-1}, \quad (23)$$

which leads to

$$\chi_T \approx |\delta \rho_p|^{-\gamma_T} \tilde{\mathcal{L}} \left( \frac{q}{|\delta \rho_p|^z}, \frac{\omega}{|\delta \rho_p|^z} \right), \quad (24)$$

where $\tilde{\mathcal{L}}$ is given by Eq. (20), and we have relaxed the original definitions of the nonuniversal constants $a$, $a'$, $b$, and $c$. The exponent $\gamma_T = 2$ and the exponents $z$ and $\nu$ are the same as the ones for the longitudinal response near RP (see Table I). Note that the critical behavior of $\chi_T$ does not change if one approaches RP instead of jamming.

B. Elastic moduli and viscosities

We have already presented a short derivation of the scaling behavior of the elastic moduli in Sec. I [see Eqs. (9) and (12).] Here we explore the connection between the dynamic response and the moduli to discuss the critical exponents and universal scaling function for $B$ and $G$.

Near jamming, Eq. (13) leads to

$$B \approx \frac{1}{2 q} \frac{\partial \chi_T^{-1}}{\partial q}. \quad (25)$$

Using Eq. (16), we then find,

$$B \approx \frac{1}{2 q} |\delta \rho_p|^\gamma \frac{\partial \mathcal{L}^{-1}}{\partial q}$$

$$= \frac{1}{2 q} |\delta \rho_p|^\gamma |\delta \rho_p|^{-\nu} \frac{\partial \mathcal{L}^{-1}}{\partial (q/|\delta \rho_p|^z)}$$

$$= \frac{1}{2 q} (q/|\delta \rho_p|^z) |\delta \rho_p|^\gamma |\delta \rho_p|^{-2\nu} \frac{\partial \mathcal{L}^{-1}}{\partial (q/|\delta \rho_p|^z)}, \quad (26)$$
so that,
\[
B \approx |\delta p_r|^{|\beta_B|} B \left( \frac{q}{|\delta p_r|^w}, \frac{\omega}{|\delta p_r|^v}, \frac{\delta p_n}{|\delta p_r|^w} \right), \tag{27}
\]
where \(\beta_B = \gamma - 2 z \nu\), and
\[
B(u, v, w) = \frac{1}{2u} \frac{\partial}{\partial u} \left[ \frac{1}{\mathcal{L}(u, v, w)} \right]. \tag{28}
\]
Near rigidity percolation, the universal function $B(u,v,w) \rightarrow \overline{B}(u,v)$, which is given by Eq. (28) with $L$ replaced by $\overline{L}$.

To derive the scaling for the shear modulus, we follow the same steps described in the last paragraph. Now we explore the connection between $G$ and the transverse response $\chi_T$. Equation (21) leads to

$$G = \frac{1}{2q} \varnothing \overline{L}^{-1},$$

which is valid near both jamming and RP. Using Eq. (24), we then find,

$$G \approx \frac{1}{2q} |\delta_{\text{hp}}|^\gamma \varnothing \overline{L}^{-1}$$

$$= \frac{1}{2q} |\delta_{\text{hp}}|^\gamma |\delta_{\text{hp}}|^{-\nu} \varnothing \overline{L}^{-1}$$

$$= \frac{1}{2q/|\delta_{\text{hp}}|^\nu} |\delta_{\text{hp}}|^\gamma |\delta_{\text{hp}}|^{-2\nu} \varnothing \overline{L}^{-1},$$

so that,

$$G \approx |\delta_{\text{hp}}|^\beta G \left( \frac{q}{|\delta_{\text{hp}}|^\nu}, \frac{\omega}{|\delta_{\text{hp}}|^\nu}, \frac{\delta_{\text{hp}}}{|\delta_{\text{hp}}|^\nu} \right),$$

where $\beta_G = \gamma_T - 2z\nu$, with $\gamma_T = 2$ and $z$ and $\nu$ given in Table I for RP. The universal scaling function

$$G(u,v) = \frac{1}{2u} \varnothing \overline{L} \left[ \frac{1}{G(u,v)} \right].$$

As expected, the scaling behavior of $G$ near both RP and jamming is the same as the scaling behavior of $B$ near RP.

To extract the scaling behavior for the bulk and shear viscosities, we use the definitions $\zeta = B''(\omega)/\omega$ and $\eta = G''(\omega)/\omega$, so that,

$$\zeta = |\delta_{\text{hp}}|^{-\gamma_B} \mathcal{Z} \left( \frac{q}{|\delta_{\text{hp}}|^\nu}, \frac{\omega}{|\delta_{\text{hp}}|^\nu}, \frac{\delta_{\text{hp}}}{|\delta_{\text{hp}}|^\nu} \right),$$

$$\eta = |\delta_{\text{hp}}|^{-\gamma_G} \mathcal{E} \left( \frac{q}{|\delta_{\text{hp}}|^\nu}, \frac{\omega}{|\delta_{\text{hp}}|^\nu}, \frac{\delta_{\text{hp}}}{|\delta_{\text{hp}}|^\nu} \right),$$

with the exponents on the r.h.s. of Eq. (34) corresponding to the ones listed in Table I for RP, and

$$\mathcal{Z}(u,v) = \frac{1}{v} \Im \left[ B(u,v,w) \right],$$

$$\mathcal{E}(u,v) = \frac{1}{v} \Im \left[ G(u,v) \right],$$

are the universal scaling functions. Near RP, $\mathcal{Z}(u,v,w) \rightarrow \overline{\mathcal{Z}}(u,v)$, which is given by Eq. (37) with $B$ replaced by $\overline{B}$. As expected, $\mathcal{E}$ does not change near rigidity percolation.
C. Density Response

The derivation of the density response $\Pi$ proceeds from the equations of motion, in a way that is similar to the derivation of $\chi$ (Chaikin and Lubensky, 1995). Whereas $\chi \equiv u_L/f_L$ is defined in Fourier space as the ratio of the longitudinal part of the displacement field $u_L$ to its conjugate external field $f_L$, the density response can be defined as $\Pi \equiv n/h$, where $n$ is the density and $h$ is the density conjugate field. For small displacements,

$$n \equiv n_0 (1 - i q u_L) \quad (39)$$

where $n_0$ is a constant given by the average background density. The appropriate conjugate field in Fourier space that linearly couples to the density in the Hamiltonian is $h \equiv f_L/(i q n_0)$. Recasting the equation of motion for $u$ as an equation of motion for $n$ leads to a factor of $q^2$ originating from the divergence operator in Eq. (39) and another factor of $q$ that is present in the definition of $h$. Thus,

$$\Pi = d' q^2 \chi_L, \quad (40)$$

where $d'$ is a constant.

Equations (40) and (16) lead to

$$\Pi \approx d' \left( \frac{q}{|\delta p_{\rho^n}|} \right)^2 |\delta p_{\rho^n}|^{-1} \gamma L, \quad (41)$$

resulting in the scaling form:

$$\Pi \approx |\delta p_{\rho^n}|^{-1} \gamma L \left( \frac{q}{|\delta p_{\rho^n}|} \right)^2, \quad (42)$$

which reduces to

$$\Pi \approx |\delta p_{\rho^n}|^{-1} \gamma L \left( \frac{q}{|\delta p_{\rho^n}|} \right)^2, \quad (43)$$

near RP, where

$$\mathcal{P}(u,v,w) = d' u^2 L(u,v,w), \quad (44)$$

$$\mathcal{P}(u,v) = d' u^2 L(u,v). \quad (45)$$

D. Correlation functions

We end this section with derivations of the scaling behavior of the Ursell function $S_{nn}(q,\omega)$ (the structure factor for isotropic fluids at nonzero $q$) and the scaling behavior of the density-density correlation function in real space: $S_{nn}(r,r',t-t') = \langle n(r,t)n(r',t') \rangle - \langle n(r,t) \rangle \langle n(r',t') \rangle$.

Using the fluctuation-dissipation theorem,

$$\omega S_{nn}(q,\omega) = 2 T \Im [\Pi(q,\omega)], \quad (46)$$

where $T$ is the temperature, and Eq. (42), we obtain

$$S_{nn}(q,\omega) \approx \frac{2 T}{\omega} \Im \left[ \frac{|\delta p_{\rho^n}|^{-2} |\mathcal{P}|}{|\delta p_{\rho^n}|^{-1} \gamma} \right], \quad (47)$$

so that

$$S_{nn}(q,\omega) \approx |\delta p_{\rho^n}|^{(2-D)\nu - \gamma} \mathcal{S} \left( \frac{q}{|\delta p_{\rho^n}|}, \frac{\omega}{|\delta p_{\rho^n}|^{\nu}} \right), \quad (48)$$

which reduces to:

$$S_{nn}(q,\omega) \approx |\delta p_{\rho^n}|^{(2-D)\nu - \gamma} \mathcal{S} \left( \frac{q}{|\delta p_{\rho^n}|}, \frac{\omega}{|\delta p_{\rho^n}|^{\nu}} \right), \quad (49)$$

near RP, where

$$\mathcal{S}(u,v,w) = \frac{2 T}{v} \Im \left[ \mathcal{P}(u,v,w) \right], \quad (50)$$

$$\mathcal{S}(u,v) = \frac{2 T}{v} \Im \left[ \mathcal{P}(u,v) \right]. \quad (51)$$

The two-time density-density correlation function $S_{nn}(r-r',t-t')$ is given by,

$$S_{nn}(r-r',t-t') = \int d\omega \int dq e^{i \omega (t-t') + iq (r-r')} \times S_{nn}(q,\omega)$$

$$= \int d\omega \left( \frac{\omega}{|\delta p_{\rho^n}|^{\nu}} \right) \int dq \left( \frac{q}{|\delta p_{\rho^n}|} \right) |\delta p_{\rho^n}|^{(2-D)\nu} \times \exp \left\{ i \left[ \frac{q}{|\delta p_{\rho^n}|} (r-r') - \frac{\omega}{|\delta p_{\rho^n}|^{\nu}} (t-t') \right] \right\} \times S_{nn}(q,\omega). \quad (52)$$

Using Eq. (48), we then obtain

$$S_{nn}(r-r',t-t') \approx |\delta p_{\rho^n}|^{(2-D)\nu - \gamma} \mathcal{S} \left( \frac{r-r'}{\ell}, \frac{t-t'}{\tau}, |\delta p_{\rho^n}|^{\nu} \right), \quad (53)$$

which reduces to

$$S_{nn}(r-r',t-t') \approx |\delta p_{\rho^n}|^{(2-D)\nu - \gamma} \mathcal{S} \left( \frac{r-r'}{\ell}, \frac{t-t'}{\tau}, |\delta p_{\rho^n}|^{-\nu} \right), \quad (54)$$

near RP, where $\ell = |\delta p_{\rho^n}|^{-\nu}$ and $\tau = |\delta p_{\rho^n}|^{-\nu}$ are diverging length and times scales, respectively, and

$$S(p,s,w) = 2 T \int du dv e^{i(w-p-u)} \frac{\Im \mathcal{P}(u,v,w)}{v}, \quad (55)$$

$$\bar{S}(p,s) = 2 T \int du dv e^{i(w-p-u)} \frac{\Im \mathcal{P}(u,v)}{v}. \quad (56)$$
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