Tadpoles and vacuum bubbles in light-front quantization

S.S. Chabyshova

Department of Physics, University of Idaho, Moscow ID 83844 USA

J.R. Hiller

Department of Physics, University of Idaho, Moscow ID 83844 USA and

Department of Physics and Astronomy,
University of Minnesota-Duluth, Duluth, Minnesota 55812 USA
(Dated: January 4, 2022)
Abstract

We develop a method by which vacuum transitions may be included in light-front calculations. This allows tadpole contributions which are important for symmetry-breaking effects and yet are missing from standard light-front calculations. These transitions also dictate a nontrivial vacuum and contributions from vacuum bubbles to physical states. In nonperturbative calculations these separate classes of contributions (tadpoles and bubbles) cannot be filtered; instead, we regulate the bubbles and subtract the vacuum energy from the eigenenergy of physical states. The key is replacement of momentum-conserving delta functions with model functions of finite width; the width becomes the regulator and is removed after subtractions. The approach is illustrated in free scalar theory, quenched scalar Yukawa theory, and in a limited Fock-space truncation of φ⁴ theory.

I. INTRODUCTION

Recent calculations of the critical coupling in two-dimensional φ⁴ theory [1–13] have shown that there is a discrepancy in the nonperturbative equivalence of equal-time and light-front quantization. Although this discrepancy can be explained with a computed shift in the renormalized mass [3, 6, 8, 11], this explanation is a correction to the light-front calculation, rather than a direct calculation. In addition, calculations with coordinates that interpolate between equal-time and light-front quantizations [15, 16], indicate that the light-front limit should obtain the same critical coupling as obtained in equal time quantization [17]. The key is the inclusion of tadpole contributions, which on the light front requires zero modes, modes with zero longitudinal momentum, to represent transitions to and from the vacuum, as illustrated in Fig. 1(a). In a nonperturbative calculation, where one cannot pick and choose classes of diagrams, the presence of vacuum transitions necessarily imports (divergent) vacuum bubbles, of a sort shown in Fig. 1(b), as well as tadpoles.

Zero-mode contributions to physical states and the corresponding nontriviality of the light-front vacuum [19, 20, 22, 23] are of broader interest than just the critical coupling

![Fig. 1. Tadpole graph (a) and vacuum bubble (b) in φ⁴ theory. Note the momentum-conserving transitions to and from the vacuum that imply light-front zero-mode contributions.](image)

1 These figures were drawn with JaxoDraw [18].
2 This is a separate question from perturbative equivalence, which has been generally established. For recent discussions, see [19, 21].
in $\phi^4$ theory. They enter into any discussion of symmetry breaking, such as the Higgs mechanism, and of vacuum condensates [24]. More recently they have been identified as possible contributions to higher-twist distribution functions [25]. For these reasons, we explore a possible method for inclusion, within the context of two-dimensional scalar theories; extension to three and four dimensions should be straightforward.

Contributions such as tadpoles and vacuum bubbles that involve transitions to and from the vacuum must rely on terms normally excluded from light-front Hamiltonians. These are terms with only creation operators or only annihilation operators. With light-front longitudinal momenta constrained to be nonnegative, momentum conservation requires that the operators create or annihilate zero momentum. On this basis they are always dropped. However, depending on the zero-momentum behavior of the Fock-space wave functions, matrix elements of such terms need not be zero.

For example, consider light-front quantization [28–34] of a two-dimensional scalar theory. We define light-front coordinates [28] and momenta as

\[
\begin{align*}
\text{matrix element of a vacuum transition in } \phi \text{ theory reduces to } \phi^4 \\
\text{with } n \text{ the number of constituents. The symmetry for bosons then requires that the small momentum behavior of this wave function is } \\
\psi_n \sim \frac{1}{\sqrt{\prod_i^n p_i^+ \sum_i^n \frac{1}{p_i^+}}}.
\end{align*}
\]

The matrix element of a vacuum transition in $\phi^4$ theory reduces to

\[
\begin{align*}
\langle \psi_{n+4} | & \int \frac{\prod_i^4 dp_i^+}{\sqrt{\prod_i^4 p_i^+}} \delta(\sum_i^n p_i^+) \prod_i^4 a_i(p_i^+) | \psi_n \rangle \\
\sim & \int \frac{\prod_i^{n+4} dp_i^+}{\prod_i^{n+4} p_i^+} \frac{\delta(\sum_i^{n+4} p_i^+)}{\left(\sum_i^{n+4} \frac{1}{p_i^+}\right)} \delta(P^+ - \sum_i^n p_i^+).
\end{align*}
\]

With $Q \equiv \sum_i^n p_i$, $P_n^{+} + x_i = Q$, and $\prod_i^{n+4} dp_i^+ = Q^3 dQ \prod_i^4 dx_i \delta(1 - \sum_i^4 x_i)$, this becomes

\[
\begin{align*}
\langle \psi_{n+4} | & \int \frac{\prod_i^4 dp_i^+}{\sqrt{\prod_i^4 p_i^+}} \delta(\sum_i^4 p_i^+) \prod_i^4 a_i(p_i^+) | \psi_n \rangle \\
\sim & \int dQ \delta(Q) \int \frac{\prod_i^4 dx_i \delta(1 - \sum_i^4 x_i)}{\left(\prod_i^4 x_i\right) \left(\prod_i^4 \frac{1}{x_i}\right)} \int \frac{\prod_i^n dp_i^+ \delta(P^+ - \sum_i^n p_i^+)}{\left(\prod_i^n p_i^+\right) \left(\prod_i^n \frac{1}{p_i^+}\right)}.
\end{align*}
\]

For alternative methods, see [26] and [27].
which is finite and nonzero. Thus, such vacuum-transition terms cannot be ignored automatically.

Vacuum transitions also generate vacuum bubbles which make contributions proportional to the size $L$ of the spatial dimension as expressed through $4\pi \delta(0) = \int dx^- \equiv L$. A non-perturbative calculation requires a cutoff, to regulate this infinity, and a subtraction of the vacuum energy from any eigenenergy of a physical state. We regulate by replacing delta functions of momentum with model functions $\delta_\epsilon$ that have a width parameter $\epsilon$ and take the limit of $\epsilon \to 0$ at the end of a calculation. For the (nontrivial) vacuum state, we compute a finite energy density, with the model parameter $\epsilon$ related to the spatial volume $L$ in a model-dependent way: $4\pi \delta_\epsilon(0) = L$.

For any finite width $\epsilon$, there will be additional modes present in any calculation. We call these ephemeral modes, since they are not zero modes but instead disappear in the limit of zero width. The remaining imprint is essentially a zero-mode contribution, but obtained as a limit. Contributions to massive states, beyond the vacuum-energy shift and tadpoles, are generally negligible for weak coupling; however, for strong coupling, the Fock-state momentum wave functions can be become broad enough that they overlap with ephemeral modes. Depending on the zero-momentum behavior of these wave functions, there can be additional contributions from vacuum transitions.

Such contributions cannot be readily captured with the DLCQ formalism \cite{35}, and DLCQ calculations with constrained zero modes \cite{36,38} are incomplete. For good resolution of the ephemeral modes, the DLCQ resolution $K$ must satisfy $1/K \ll \epsilon/P^+$. Also, the integrals that must be represented by the rectangular DLCQ grid are highly singular, for which the grid is ill suited. Calculations would be best undertaken with a basis function expansion, for which matrix elements can be computed once and for all with an adaptive Monte Carlo integration, such as is available in the VEGAS package \cite{39}.

Even with antiperiodic boundary conditions, the DLCQ approach cannot neglect zero modes. With such boundary conditions one can avoid the constraint equation, but the approximation to the integral operators in $P^-$ is the midpoint rule with an error no better than the $1/K^2$ for periodic boundary conditions, where the integrals are approximated by the trapezoidal rule\footnote{Without a solution to the constraint equation, the error in DLCQ with periodic boundary conditions is of order $1/K$, unless the endpoints (the zero modes) make no contribution to the integrals.}. The trouble is that the coefficient of the $1/K^2$ correction is small only if the integrand is slowly varying. If instead there is rapid variation, such as can happen near zero momentum, the approximation becomes quite poor except at very high resolution. In other words, if zero-mode contributions are important, antiperiodic boundary conditions do not provide an approximation any better than periodic boundary conditions.

To explore the inclusion of vacuum transitions, we first consider $\phi^4$ theory in more detail: in Sec. II we consider the leading tadpole and vacuum-bubble contributions. Next, in Sec. III we develop an analytic solution for a free scalar as a generalized coherent state of ephemeral modes. The vacuum bubble contributions replicate the one-loop calculation emphasized by Collins \cite{22}. We also consider the solution for a shifted scalar with nonzero vacuum expectation value. This is done in the continuum, without interpolation from equal-time quantization and without discretization. Finally, we consider quenched scalar Yukawa theory in lowest-order Fock truncation in Sec. IV to see the subtraction of the vacuum energy of the neutral scalar in the charge-zero sector from the dressed scalar energy in the charge-one sector.
sector. Numerical calculations are postponed to future work.

II. LOWEST-ORDER $\phi^4$ THEORY

The Lagrangian for two-dimensional $\phi^4$ theory is

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} \mu^2 \phi^2 - \frac{\lambda}{4!} \phi^4,$$

where $\mu$ is the mass of the boson and $\lambda$ is the coupling constant. The light-front Hamiltonian density is

$$\mathcal{H} = \frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{4!} \phi^4.$$  \hfill (2.2)

The mode expansion for the field is

$$\phi(x^+ = 0, x^-) = \int \frac{dp}{\sqrt{4\pi p}} \{ a(p)e^{-ipx^-/2} + a^\dagger(p)e^{ipx^-/2} \}. \hfill (2.3)$$

The nonzero commutation relation is

$$[a(p), a^\dagger(p')] = \delta(p - p'). \hfill (2.4)$$

The light-front Hamiltonian is $\mathcal{P}^- = \mathcal{P}_{0}^- + \mathcal{P}_{\text{int}}^-$, with

$$\mathcal{P}_{0}^- = \int \frac{dp}{p} \mu^2 a^\dagger(p)a(p) + \frac{\mu^2}{2} \int \frac{dp_1 dp_2}{\sqrt{p_1 p_2}} \delta(p_1 + p_2) \left[ a(p_1)a(p_2) + a^\dagger(p_1)a(p_2) \right], \hfill (2.5)$$

$$\mathcal{P}_{\text{int}}^- = \mathcal{P}_{04}^- + \mathcal{P}_{40}^- + \mathcal{P}_{22}^- + \mathcal{P}_{13}^- + \mathcal{P}_{31}^-; \hfill (2.6)$$

where

$$\mathcal{P}_{04}^- = \frac{\lambda}{24} \int \frac{dp_1 dp_2 dp_3 dp_4}{4\pi \sqrt{p_1 p_2 p_3 p_4}} \delta^4 \left( \sum_i p_i \right) a(p_1)a(p_2)a(p_3)a(p_4), \hfill (2.7)$$

$$\mathcal{P}_{40}^- = \frac{\lambda}{24} \int \frac{dp_1 dp_2 dp_3 dp_4}{4\pi \sqrt{p_1 p_2 p_3 p_4}} \delta^4 \left( \sum_i p_i \right) a^\dagger(p_1)a^\dagger(p_2)a(p_3)a^\dagger(p_4), \hfill (2.8)$$

$$\mathcal{P}_{22}^- = \frac{\lambda}{4} \int \frac{dp_1 dp_2}{4\pi \sqrt{p_1 p_2}} \int \frac{dp'_1 dp'_2}{\sqrt{p'_1 p'_2}} \delta(p_1 + p_2 - p'_1 - p'_2) \hfill (2.9)$$

$$\times a^\dagger(p_1)a^\dagger(p_2)a(p'_1)a(p'_2),$$

$$\mathcal{P}_{13}^- = \frac{\lambda}{6} \int \frac{dp_1 dp_2 dp_3}{4\pi \sqrt{p_1 p_2 p_3 (p_1 + p_2 + p_3)}} a^\dagger(p_1 + p_2 + p_3)a(p_1)a(p_2)a(p_3), \hfill (2.10)$$

$$\mathcal{P}_{31}^- = \frac{\lambda}{6} \int \frac{dp_1 dp_2 dp_3}{4\pi \sqrt{p_1 p_2 p_3 (p_1 + p_2 + p_3)}} a^\dagger(p_1)a^\dagger(p_2)a(p_3)a(p_1 + p_2 + p_3). \hfill (2.11)$$

The subscripts indicate the number of creation and annihilation operators in each term.

\footnote{For convenience we drop the $+$ superscript and will from here on write light-front momenta such as $p^+$ as just $p.$}
To isolate the contribution from the tadpole and vacuum bubble in Fig. 1, we consider only two terms in the Fock-state expansion of the eigenstate

\[ |\psi(P)\rangle = \psi_1 a^\dagger(P)|0\rangle + \cdots + \int \prod_i^5 dp_i \delta(P - \sum_i^5 p_i) \psi_5(p_1, \ldots, p_5) \frac{1}{\sqrt{5!}} \prod_i^5 a^\dagger(p_i)|0\rangle + \cdots, \quad (2.12) \]

in order to represent the five constituents in the intermediate states, and we keep only the first term of \( P_0^- \) and the first three terms of \( P_{\text{int}}^- \), as the only terms that connect the two Fock sectors. We then consider the eigenvalue problem

\[ (P_0^- + P_{\text{int}}^-)|\psi(P)\rangle = \left( \frac{M^2}{P} + P_{\text{vac}}^- \right)|\psi(P)\rangle, \quad (2.13) \]

where we include the shift of vacuum energy \( P_{\text{vac}}^- \), to be obtained from solving the corresponding vacuum eigenvalue problem

\[ (P_0^- + P_{\text{int}}^-)|\text{vac}\rangle = P_{\text{vac}}^-|\text{vac}\rangle, \quad (2.14) \]

with \( |\text{vac}\rangle \) the lowest eigenstate. Projection of the eigenvalue problem for the lowest massive state onto Fock sectors yields a system of equations for the Fock-state wave functions

\[ \mu^2 \psi_1 + \frac{\lambda}{\sqrt{24}} \int \frac{\prod_i^4 dp_i}{4\pi \sqrt{\prod_i^4 p_i}} \delta(\sum_i^4 p_i) \psi_5(p_1, \ldots, p_5) = \left( \frac{M^2}{P} + P_{\text{vac}}^- \right) \psi_1, \quad (2.15) \]

\[ \left( \sum_i^5 \frac{\mu^2}{p_i} \right) \psi_5 + \frac{\lambda}{24} \left[ \frac{\delta(\sum_i^4 p_i)}{4\pi \sqrt{\prod_i^4 p_i}} + (p_5 \leftrightarrow p_1, p_2, p_3, p_4) \right] \psi_1 + 20 \frac{\lambda}{4} \int \frac{dp'_1 dp'_2}{4\pi \sqrt{p_1 p_2 p'_1 p'_2}} \delta(p_1 + p_2 - p'_1 - p'_2) \psi_5(p'_1, p'_2, p_3, p_4, p_5) = \frac{M^2}{P} \psi_5, \quad (2.16) \]

where we have invoked a sector-dependent energy shift, with no \( P_{\text{vac}}^- \) in the top Fock sector\(^7\).

The second equation can be solved iteratively with respect to the self-coupling of the five-constituent Fock state in the third term of (2.16); this corresponds to a diagrammatic expansion. The leading term in the expansion generates the vacuum bubble in Fig. 1(b) that contributes \( P_{\text{vac}}^- \) in (2.15). The second term, where the self interaction acts once, produces the tadpole in Fig. 1(a). Both are written explicitly in (2.17) and (2.18) below. Subtraction of \( P_{\text{vac}}^- \) from both sides of (2.15) eliminates the divergent bubble.

From (2.15) and (2.16), the contributions take the forms

\[ \text{bubble} \rightarrow \int \frac{\prod_i^5 dp_i \delta(P - \sum_i^5 p_i) \delta(\sum_i^4 p_i)^2}{\prod_i^4 p_i} \sim - \int \delta_\epsilon(Q)^2 \frac{dQ}{\mu^2} \int \frac{\prod_i^4 dx_i}{\prod_i^4 x_i} \delta(1 - \sum_i^4 x_i), \quad (2.17) \]

\[ \text{tadpole} \rightarrow \int \frac{\prod_i^5 dp_i \delta(\sum_i^4 p_i) \delta(P - \sum_i^5 p_i)}{\sqrt{\prod_i^4 p_i}} \int \frac{dp'_1 dp'_2}{\sqrt{p_4 p_5 p'_1 p'_2}} \frac{\delta(p_4 + p_5 - p'_1 - p'_2)}{\sqrt{p_4 p_5 p'_1 p'_2}} \delta(\sum_i^3 p_i + p'_i) \delta_\epsilon(p_4 + p_5 - p'_1 - p'_2) \delta_\epsilon(\sum_i^3 p_i + p'_i), \quad (2.18) \]

The expression for the bubble diverges as \( \epsilon \to 0 \) and is proportional to \( \delta(0) = L/4\pi \); however, the same expression is obtained for the nontrivial vacuum energy \( P_{\text{vac}}^- \) and is subtracted.

---

\(^7\) In the top sector, there are, of course, no vacuum corrections from higher Fock sectors.
The expression for the tadpole contribution can be simplified by noting that $\delta(P - \sum_i p_i)$ reduces to $\delta(P - p_3)$, which can be used to do the $p_3$ integral and $\delta(p_4 + p_5 - p_1' - p_2')$ becomes $\delta(p_4 + P - p_1' - p_2')$, which can be used to do the $p_2'$ integral. Finally, $\delta(P - \sum_i p_i + p_1')$ can be written $\delta(p_4 - p_1')$ and used to do the $p_1'$ integral. These leave

$$\text{tadpole} \sim \int \frac{1}{\prod_i p_i} \frac{1}{p_4 P} \left[ \frac{\delta_i(\sum_i p_i)}{M^2 - \sum_i \frac{\mu_i^2}{p_i} - \frac{\mu_i^2}{P}} \right]^2 \sim \frac{1}{P} \int \frac{\delta_i(Q) dQ}{\prod_i x_i} \left( \sum_i \frac{\mu_i^2}{x_i} \right)^2,$$

which is finite and inversely proportional to $P$, the mark of a light-front self-energy correction.

Of course, in a nonperturbative calculation, these contributions cannot be separated. However, with the bubbles regulated, one can solve the eigenproblems for the vacuum and the massive states and then carry out the necessary $P_{\text{vac}}^-$ subtraction prior to taking the width parameter $\epsilon$ to zero.

**III. FREE SCALAR**

**A. Free vacuum**

The free vacuum $|\text{vac}\rangle$ is an eigenstate of the free scalar Hamiltonian $\mathcal{P}_0^-$ in (2.5):

$$\mathcal{P}_0^- |\text{vac}\rangle = P_{\text{vac}}^- |\text{vac}\rangle. \tag{3.1}$$

We will show that the vacuum is a generalized coherent state,

$$|\text{vac}\rangle = \sqrt{Z} e^{A^\dagger} |0\rangle, \tag{3.2}$$

where

$$A^\dagger = \int_0^\infty dp_1 dp_2 \frac{f(p_1, p_2)}{\sqrt{p_1 P_1 + \frac{1}{p_1}}} a^{\dagger}(p_1) a^{\dagger}(p_2). \tag{3.3}$$

For such a state, we have

$$a(p)|\text{vac}\rangle = 2 \int \frac{dp'}{\sqrt{pp'}} \frac{f(p, p')}{\frac{1}{p} + \frac{1}{p'}} a^{\dagger}(p')|\text{vac}\rangle \tag{3.4}$$

and

$$a(p_1) a(p_2)|\text{vac}\rangle = \frac{2}{\sqrt{p_1 P_2}} \frac{f(p_1, p_2)}{\frac{1}{p_1} + \frac{1}{p_2}} |\text{vac}\rangle \tag{3.5}$$

$$+ 4 \int \frac{dp' dp'_2}{\sqrt{p_1 P_2 p'_1 p'_2}} \frac{f(p_1, p'_1) f(p_2, p'_2)}{\frac{1}{p_1} + \frac{1}{p'_1} + \frac{1}{p_2} + \frac{1}{p'_2}} a^{\dagger}(p'_1) a^{\dagger}(p'_2)|\text{vac}\rangle.$$

With these we can apply $\mathcal{P}_0^-$ to obtain

$$\mathcal{P}_0^- |\text{vac}\rangle = \frac{\mu^2}{2} \int \frac{dp_1 dp_2}{\sqrt{p_1 P_2}} \delta_i(p_1 + p_2) \left[ a^{\dagger}(p_1) a^{\dagger}(p_2) + \frac{2}{\sqrt{p_1 P_2}} \frac{f(p_1, p_2)}{\frac{1}{p_1} + \frac{1}{p_2}} \right]|\text{vac}\rangle \tag{3.6}$$

$$+ 4 \int \frac{dp' dp'_2}{\sqrt{p_1 P_2 p'_1 p'_2}} \frac{f(p_1, p'_1) f(p_2, p'_2)}{\frac{1}{p_1} + \frac{1}{p'_1} + \frac{1}{p_2} + \frac{1}{p'_2}} a^{\dagger}(p'_1) a^{\dagger}(p'_2)|\text{vac}\rangle,$$

$$+ \int dp \frac{\mu^2}{p} a^{\dagger}(p) \int dp' \frac{2}{\sqrt{pp'}} \frac{f(p, p')}{\frac{1}{p} + \frac{1}{p'}} a^{\dagger}(p')|\text{vac}\rangle.$$
The solution to $\mathcal{P}_0^-|\text{vac}\rangle = P_{\text{vac}}^-|\text{vac}\rangle$ is then possible if

$$P_{\text{vac}}^- = \frac{\mu^2}{2} \int \frac{dp_1 dp_2}{\sqrt{p_1 p_2}} \frac{\delta_\epsilon(p_1 + p_2)}{\sqrt{p_1 + p_2}} \frac{f(p_1, p_2)}{p_1 + \frac{1}{p_1} + \frac{1}{p_2}}$$

(3.7)

and the symmetrized coefficients of $a^\dagger(p_1) a^\dagger(p_2)$ sum to zero:

$$0 = \frac{\mu^2}{2} \frac{\delta_\epsilon(p_1 + p_2)}{\sqrt{p_1 p_2}} + 2\mu^2 \int \frac{dp'_1 dp'_2}{p'_1 p'_2 \sqrt{p_1 p_2}} \delta_\epsilon(p'_1 + p'_2) \frac{f(p_1, p'_1)}{p_1 + \frac{1}{p_1} + \frac{1}{p_2}} \frac{f(p_2, p'_2)}{p_2 + 1 + \frac{1}{p_2}} \frac{1}{\sqrt{p_1 p_2}} \frac{1}{p_1} + \frac{1}{p_2}$$

(3.8)

In the second term of (3.8), we can compute

$$\int \frac{dp'_1 dp'_2}{p'_1 p'_2} \delta_\epsilon(p'_1 + p'_2) \frac{f(p_1, p'_1)}{p_1 + \frac{1}{p_1} + \frac{1}{p_2}} \frac{f(p_2, p'_2)}{p_2 + 1 + \frac{1}{p_2}} \frac{1}{\sqrt{p_1 p_2}} \frac{1}{p_1} + \frac{1}{p_2}$$

(3.9)

$$= p_1 p_2 \int Q dQ \delta_\epsilon(Q) \int dx \frac{f(p_1, xQ) f(p_2, (1 - x)Q)}{(p_1 + xQ)(p_2 + (1 - x)Q)} = f(p_1) f(p_2) \int Q dQ \delta_\epsilon(Q) = 0.$$

The sum of coefficients in (3.8) is then zero if

$$f(p_1, p_2) = -\frac{1}{2} \delta_\epsilon(p_1 + p_2).$$

(3.10)

This determines the vacuum state.

With this solution for the coherent-state wave function, the energy of the vacuum is

$$P_{\text{vac}}^- = -\frac{\mu^2}{2} \int \frac{dp_1 dp_2}{p_1 p_2} \frac{\delta_\epsilon(p_1 + p_2)}{\sqrt{p_1 + p_2}} = -\frac{\mu^2}{2} \int \frac{Q dQ dx}{Q^2 x(1 - x)} \frac{\delta_\epsilon(Q)}{\sqrt{Q x(1 - x)}}$$

$$= -\frac{\mu^2}{2} \int dQ \delta_\epsilon(Q)^2 = -\frac{\mu^2}{2} \delta_\epsilon(0) \int_0^\infty dQ \delta_\epsilon(Q) = -\frac{\mu^2}{2} \frac{L}{4\pi} \frac{1}{2} = -\frac{\mu^2 L}{16\pi}.$$

(3.11)

Here $L$ is the (infinite) volume of light-front space; however, $\delta_\epsilon$ at finite $\epsilon$ regulates $P_{\text{vac}}^-$ when it is embedded in a nonperturbative calculation.
This result is proportional to the one-loop vacuum bubble computed by Collins [22]. The equivalent perturbative calculation, corresponding to the loop in Fig. 2, is

$$2\mu^2 \int {dq_1 dq_2 \over \sqrt{q_1 q_2}} \delta_\epsilon(q_1 + q_2) \left( M^2 - p^2 - q_1^2 - q_2^2 \right)^2 \sqrt{q_1 q_2} = 2\mu^4 \int {dq_1 dq_2 \over q_1 q_2} \delta_\epsilon(q_1 + q_2)^2, \quad (3.12)$$

which matches (3.11). The leading factor of 2 comes from the two possible contractions of the double scalar creation and annihilation operators.

Any massive state in the free theory has a Fock-state wave function that is a product of delta functions of the individual particle momenta $p_i$. This part of the state will not mix with the ephemeral modes, provided the width parameter $\epsilon$ is chosen such that it is much less than all the $p_i$. For example, the single-particle state with mass $\mu$ and momentum $P$ is just $a\dagger(P)| vac \rangle$. It is an eigenstate of $P^- - P^-_{\text{vac}}$ with eigenvalue $\mu^2/P$. Weak couplings that broaden the momentum-space wave functions only slightly will produce effectively unmixed contributions, but calculations with strong couplings require more care.

**B. Shifted scalar**

Next we consider the shifted free scalar where $\phi \to \phi + v$. The new Lagrangian is

$$\mathcal{L} = \mathcal{L}_{v=0} - \mu^2 v \phi - {1 \over 2} \mu^2 v^2, \quad (3.13)$$

and the Hamiltonian is $\mathcal{H}^- = \mathcal{H}^-_0 + \mathcal{H}^-_{\text{int}}$ with the interaction part

$$\mathcal{H}^-_{\text{int}} = \int dx \mu^2 v \phi = \sqrt{4\pi} \mu^2 v \int {dp \over \sqrt{p}} \delta_\epsilon(p)[a(p) + a\dagger(p)] + {1 \over 2} \mu^2 v^2 L. \quad (3.14)$$

The constant term represents the shift in the energy of the vacuum and is therefore proportional to the spatial size $L$.

The vacuum state $| v \rangle$ is now an eigenstate of $\mathcal{H}^-$

$$\left( \mathcal{H}^-_0 + \mathcal{H}^-_{\text{int}} \right) | v \rangle = \mathcal{H}^-_{\text{vac}} | v \rangle. \quad (3.15)$$

It can be constructed from the free vacuum as $e^B| v \rangle$ with

$$B \equiv v \int {dp \over \sqrt{4\pi p}} \delta_\epsilon(p)[a\dagger(p) - a(p)]. \quad (3.16)$$

This works because

$$e^B \phi(x^-) e^{-B} = \phi(x^-) + v \quad (3.17)$$

and

$$e^B \mathcal{H}^-_0 e^{-B} = \mathcal{H}^- - \mathcal{H}^-_{\text{int}}. \quad (3.18)$$

This then permits

$$\left( \mathcal{H}^-_0 + \mathcal{H}^-_{\text{int}} \right) | v \rangle = e^B \mathcal{H}^-_0 e^{-B} e^B | v \rangle = e^B \mathcal{H}^-_0 | v \rangle = \mathcal{H}^-_{\text{vac}} e^B | v \rangle = \mathcal{H}^-_{\text{vac}} | v \rangle. \quad (3.19)$$

Thus, in both the free and shifted cases, the vacuum is a generalized coherent state of ephemeral modes.
The state is also correctly normalized, because
\[
\langle \text{vac} | \text{vac} \rangle_v = \langle \text{vac} | e^{B^\dagger} e^B | \text{vac} \rangle = \langle \text{vac} | e^{-B} e^B | \text{vac} \rangle = \langle \text{vac} | \text{vac} \rangle = 1. 
\]
(3.20)
The vacuum expectation value of the field can also be computed:
\[
\langle \text{vac} | \phi(x^-) | \text{vac} \rangle_v = \langle \text{vac} | e^{B^\dagger} \phi(x^-) e^B | \text{vac} \rangle = \langle \text{vac} | e^{-B} \phi(x^-) e^B | \text{vac} \rangle = \langle \text{vac} | (\phi(x^-) - v) | \text{vac} \rangle = -v.
\]
(3.21)
This restores the shift.

IV. QUENCHED SCALAR YUKAWA THEORY

In order to look at a case with more structure, we consider scalar Yukawa theory \[40\], for which the Lagrangian is
\[
\mathcal{L} = |\partial_\mu \chi|^2 - m^2 |\chi|^2 + \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} \mu^2 \phi^2 - g \phi |\chi|^2,
\]
(4.1)
where \(\chi\) is a complex scalar field with mass \(m\) and \(\phi\) is a real scalar field with mass \(\mu\). The two fields are coupled by a Yukawa term with strength \(g\). In two dimensions, the light-front Hamiltonian density is
\[
\mathcal{H} = m^2 |\chi|^2 + \frac{1}{2} \mu^2 \phi^2 + g \phi |\chi|^2.
\]
(4.2)
The mode expansions for the fields are (2.3) for \(\phi\) and
\[
\chi = \int \frac{dp}{\sqrt{4\pi p}} \left[ c_+^\dagger(p) e^{-ipx^-/2} + c_-^\dagger(p) e^{ipx^-/2} \right].
\]
(4.3)
The nonzero commutation relations of the creation and annihilation operators are (2.4) and
\[
[c_\pm(p), c_\pm^\dagger(p')] = \delta(p - p').
\]
(4.4)
In terms of these operators, the quenched light-front Hamiltonian \(\mathcal{P}^- = \int dx^- \mathcal{H} = \mathcal{P}_{0^-} + \mathcal{P}_{\text{int}}\) is specified by
\[
\mathcal{P}_{0^-} = \int dp \frac{m^2}{p} \left[ c_+^\dagger(p)c_+(p) + c_-^\dagger(p)c_-(p) \right] + \int dq \frac{\mu^2}{q} a_\dagger(q)a(q)
\]
\[+ \frac{\mu^2}{2} \int \frac{dq_1 dq_2}{\sqrt{q_1 q_2}} \delta_\varepsilon(q_1 + q_2) \left[ a(q_1)a(q_2) + a_\dagger(q_1)a_\dagger(q_2) \right],
\]
(4.5)
and
\[
\mathcal{P}_{\text{int}} = g \int \frac{dp dq}{\sqrt{4\pi pq(p + q)}} \left\{ \left[ c_+^\dagger(p + q)c_+(p) + c_-^\dagger(p + q)c_-(p) \right] a(q) + \text{h.c.} \right\}.
\]
(4.6)
Pair creation and annihilation terms are suppressed for the complex scalar; without this quenching, the theory is unstable \[41\]. This also suppresses ephemeral modes for the complex scalar, which would need to appear in pairs to conserve charge, leaving only those of the
neutral scalar. The vacuum in the charge-zero sector is that of the free scalar, as given in
the previous section; this provides the value of $P_\text{vac}^-$ for subtraction in the charge-one sector.

We seek eigenstates of $\mathcal{P}^-$, for which the two-dimensional light-front mass eigenvalue problem is

$$
\mathcal{P}^-|\psi(P)\rangle = \left( \frac{M^2}{P} + P_\text{vac}^- \right) |\psi(P)\rangle. \tag{4.7}
$$

We limit this to the charge-one sector. This sector is characterized as a single complex
scalar dressed by a cloud of neutrals. For the present purposes we will consider only a severe
Fock-space truncation that keeps no more than two neutrals. The Fock-state expansion for
the eigenstate is then

$$
|\psi(P)\rangle = \psi_0 c_+^\dagger(P)|0\rangle + \int dq dp \delta(P - q - p) \psi_1(q)a_+^\dagger(q)c_+^\dagger(p)|0\rangle 
+ \int dq_1 dq_2 dp \delta(P - q_1 - q_2 - p) \psi_2(q_1, q_2) \frac{1}{\sqrt{2}} a_+^\dagger(q_1)a_+^\dagger(q_2)c_+^\dagger(p)|0\rangle. \tag{4.8}
$$

The normalization condition $\langle \psi(P')|\psi(P)\rangle = \delta(P' - P)$ becomes

$$
1 = |\psi_0|^2 + \int dq|\psi_1|^2 + \int dq_1 dq_2|\psi_2|^2. \tag{4.9}
$$

To construct the eigenvalue problem for the wave functions, we act with $\mathcal{P}_0^-$ and $\mathcal{P}_{\text{int}}^-$
on the eigenstate and then project onto the three Fock sectors included in the truncation. Terms that generate higher Fock sectors are dropped. For $\mathcal{P}_0^-$ we have

$$
\mathcal{P}_0^-|\psi(P)\rangle = \frac{m^2}{P} \psi_0 c_+^\dagger(P)|0\rangle + \int dq dp \delta(P - q - p) \left( \frac{\mu^2}{q} + \frac{m^2}{p} \right) \psi_1(q)a_+^\dagger(q)c_+^\dagger(p)|0\rangle 
+ \int dq_1 dq_2 dp \delta(P - q_1 - q_2 - p) \left( \frac{\mu^2}{q_1} + \frac{\mu^2}{q_2} + \frac{m^2}{p} \right) \psi_2(q_1, q_2) \frac{1}{\sqrt{2}} a_+^\dagger(q_1)a_+^\dagger(q_2)c_+^\dagger(p)|0\rangle 
+ \frac{\mu^2}{2} \int dq_1 dq_2 \delta_\epsilon(q_1 + q_2) \psi_1(q_1)a_+^\dagger(q_2)c_+^\dagger(P)|0\rangle 
+ \frac{\mu^2}{2} \int dq_1 dq_2 \delta_\epsilon(q_1 + q_2) \psi_2(q_1, q_2)c_+^\dagger(P)|0\rangle. \tag{4.10}
$$

The last two terms violate momentum conservation but only by amounts of order $\epsilon$, the
width of $\delta_\epsilon$. For $\mathcal{P}_{\text{int}}^-$ we find

$$
\mathcal{P}_{\text{int}}^-|\psi(P)\rangle = \frac{g}{\sqrt{4\pi}} \int \frac{dq dp}{\sqrt{q p(p + q)}} \delta(P - q - p) \left[ \psi_1(q)c_+^\dagger(P) + \psi_0 a_+^\dagger(q)c_+^\dagger(p) \right]|0\rangle \tag{4.11}
+ \frac{\sqrt{2}}{\sqrt{4\pi}} \frac{g}{\sqrt{q p(p + q)}} \delta(P - q - p) \left[ \psi_2(q_1, q_2)a_+^\dagger(q_1)c_+^\dagger(p + q_2) 
+ \psi_1(q_1)a_+^\dagger(q_2)c_+^\dagger(q_2)|0\rangle. \right.
$$

Projection of $\mathcal{P}^-|\psi(P)\rangle = \left( \frac{M^2}{P} + P_\text{vac}^- \right) |\psi(P)\rangle$ onto each of the three Fock sectors yields the
following three equations:

$$
\frac{m^2}{P} \psi_0 + \frac{g}{\sqrt{4\pi}} \int_0^P \frac{dq \psi_1(q)}{\sqrt{q P(P - q)}} + \frac{\mu^2}{\sqrt{2}} \int \frac{dq_1 dq_2}{\sqrt{q_1 q_2}} \delta_\epsilon(q_1 + q_2) \psi_2(q_1, q_2) = \left( \frac{M^2}{P} + P_\text{vac}^- \right) \psi_0. \tag{4.12}
$$
\[
\left(\frac{\mu^2}{q} + \frac{m^2}{P-q}\right)\psi_1(q) + \frac{g}{\sqrt{4\pi}} \frac{\psi_0}{\sqrt{qP(P-q)}} + \sqrt{2} \frac{g}{\sqrt{4\pi}} \int_0^{P-q} \frac{dq' \psi_2(q, q')}{{q}'(P-q)(P-q-q')} = \frac{M^2}{P} \psi_1(q),
\]
and
\[
\left(\frac{\mu^2}{q_1} + \frac{\mu^2}{q_2} + \frac{m^2}{P-q_1-q_2}\right)\psi_2(q_1, q_2) + \frac{\mu^2}{\sqrt{2}} \delta_{q_1 + q_2} \frac{\psi_0}{\sqrt{q_1q_2}} \left[\psi_1(q_1) + \psi_1(q_2)\right] = \frac{M^2}{P} \psi_2(q_1, q_2).
\]

The vacuum energy \(P_{\text{vac}}\) appears only in the first equation, because the Fock-space truncation prevents any such correction in all but the lowest Fock sector.

We build a matrix representation for these equations by introducing basis-function expansions
\[
\psi_1(q) = \frac{1}{\sqrt{P}} \sum_n a_n f_n(q), \quad \psi_2(q_1, q_2) = \frac{1}{\sqrt{P}} \sum_{n_1} b_{n_1} g_{n_1}(q_1, q_2),
\]
with \(p = P - q\) and \(p = P - q_1 - q_2\), respectively, \(\tilde{m} \equiv m/\mu\), and
\[
f_{-1}(q) = \frac{C_{-1}}{\sqrt{q(P-q)}} \frac{P \delta_q}{P_{-q}} = C_{-1} \sqrt{qP} \delta_q(q),
\]
\[
f_n(q) = \frac{C_n}{\sqrt{q(P-q)}} \frac{(q/P)^n}{\tilde{m}^n}, \quad n \geq 0,
\]
\[
g_{-10}(q_1, q_2) = \frac{D_{-10} \sqrt{P}}{\sqrt{q_1q_2(P-q_1-q_2)}} \frac{P \delta_{q_1 = q_2}}{q_1 + \frac{1}{q_2} + \frac{\tilde{m}^2}{P-q_1-q_2}} = D_{-10} \sqrt{q_1q_2} \delta_{q_1 + q_2},
\]
\[
g_{n_1}(q_1, q_2) = \frac{D_{n_1} \sqrt{P}}{\sqrt{q_1q_2(P-q_1-q_2)}} \frac{(q_1^{n-j}q_2^{n-j})/(q_1^{n-j}q_2^{n-j})}{q_1^{n-j}q_2^{n-j}}, \quad n \geq 0, j = 0, \ldots, n/2.
\]
These have the desired small-momentum behavior shown in (1.3). The negative index \(n = -1\) is reserved for the ephemeral-mode contributions. The normalization condition (4.19) reduces to
\[
1 = |\psi_0|^2 + \sum_{nm} B^{(1)}_{nm} a_n a_m + \sum_{n_1, n_2} B^{(2)}_{n_1, n_2} b_{n_1} b_{n_2},
\]
where the overlaps between the nonorthogonal basis functions are the symmetric matrices
\[
B^{(1)}_{nm} = \frac{1}{P} \int f_n(q) f_m(q) dq, \quad B^{(2)}_{n_1, n_2} = \frac{1}{P^2} \int g_{n_1}(q_1, q_2) g_{n_2}(q_1, q_2) dq_1 dq_2.
\]
The normalization coefficients \(C_n\) and \(D_{n_1}\) are fixed by requiring \(B^{(1)}_{nn} = 1\) and \(B^{(2)}_{n_1,n_2} = 1\), and one can show that the \(n = -1\) basis functions are orthogonal to the others, making \(B^{(1)}_{-1,n} = 0\) and \(B^{(2)}_{-10,n_2} = 0\) for \(n \geq 0\).

The system of equations (4.12, 4.14) becomes, with \(\lambda \equiv g/\sqrt{4\pi} \mu^2\) and \(\tilde{M} \equiv M/\mu\),
\[
\tilde{m}^2 \psi_0 - \frac{P}{\mu^2} P_{\text{vac}} \psi_0 + \lambda \sum_n V^{(0)}_n a_n + \sum_{n_1} U^{(0)}_{n_1} b_{n_1} = \tilde{M}^2 \psi_0,
\]
\[
\sum_m T^{(1)}_{nm} a_m + \lambda V^{(0)}_n \psi_0 + \lambda \sum_{ml} V^{(1)}_{n,ml} b_{ml} = \tilde{M}^2 \sum_m B^{(1)}_{nm} a_m, \tag{4.23}
\]
\[
\sum_{ml} T^{(2)}_{n,j,ml} b_{ml} + U^{(0)}_n \psi_0 + \lambda \sum_m V^{(1)}_{n,mj} a_m = \tilde{M}^2 \sum_m B^{(2)}_{nj,ml} b_{ml}, \tag{4.24}
\]

The various matrix elements are defined by
\[
T^{(1)}_{nm} = \int dq f_n(q) \left( \frac{1}{q} + \frac{\bar{m}^2}{P - q} \right) f_m(q), \tag{4.25}
\]
\[
T^{(2)}_{n,j,ml} = \frac{1}{P} \int dq_1 dq_2 g_{nj}(q_1, q_2) \left( \frac{1}{q_1} + 1 + \frac{\bar{m}^2}{P - q_1 - q_2} \right) g_{ml}(q_1, q_2), \tag{4.26}
\]
\[
U^{(0)}_n = \frac{1}{\sqrt{2}} \int dq_1 dq_2 \delta_\epsilon(q_1 + q_2) g_{nj}(q_1, q_2), \tag{4.27}
\]
\[
V^{(0)}_n = \sqrt{P} \int \frac{dq f_n(q)}{\sqrt{q P(P - q)}}, \quad V^{(1)}_{n,ml} = \sqrt{\frac{2}{P}} \int \frac{dq dq' f_n(q) g_{ml}(q, q')}{\sqrt{q' P(P - q - q')}}, \tag{4.28}
\]

Details of matrix element computations are left to Appendix A. These include the definition of a factor \( \beta \equiv 1/[\int_0^\infty dq \delta_\epsilon(q^2)] \) which enters the normalization for ephemeral modes\(^8\). With these matrix elements, the system of equations can be written as

\[
\bar{m}^2 \psi_0 = \frac{P}{\mu^2} P^{-a_{-1}} + \lambda \sum_{n \geq 0} V^{(0)}_n a_n - \frac{P \sqrt{2} \beta}{\mu^2} a_{-10} + \frac{D_{00}}{2\sqrt{2}} b_{00} = \tilde{M}^2 \psi_0 \tag{4.29}
\]

\( n = -1: \) \[-\frac{2P}{\mu^2} \beta P^{-a_{-1}} + \frac{1}{2} \sqrt{\beta} C_0 a_0 + \frac{1}{2} P \sqrt{2 \beta} \lambda \psi_0 + \lambda \sqrt{2} g_{-10} = \tilde{M}^2 a_{-1} \tag{4.30}\]

\( n = 0: \) \[\frac{1}{2} \sqrt{\beta} C_0 a_{-1} + \sum_{m \geq 0} T^{(1)}_{0m} a_m + \lambda V^{(0)}_0 \psi_0 + \lambda \sum_{m \geq 0} V^{(1)}_{0,ml} b_{ml} = \tilde{M}^2 \sum_{m \geq 0} B^{(1)}_{0m} a_m, \tag{4.31}\]

\( n > 0: \) \[\sum_{m \geq 0} T^{(1)}_{nm} a_m + \lambda V^{(0)}_n \psi_0 + \lambda \sum_{m \geq 0} V^{(1)}_{n,ml} b_{ml} = \tilde{M}^2 \sum_{m \geq 0} B^{(1)}_{nm} a_m, \tag{4.32}\]

\( n = -1: \) \[-\frac{12P}{\mu^2} \beta P^{-a_{-10}} + 1/2 D_{-10} D_{00} b_{00} - \frac{\sqrt{2} \beta P^{-a_{-10}}}{\mu^2} \sqrt{2 \beta} P^{-a_{-10}} \psi_0 + \lambda \sqrt{2} a_{-1} = \tilde{M}^2 b_{-10} \tag{4.33}\]

\( n = 0: \) \[\frac{1}{2} D_{-10} D_{00} b_{-10} + \sum_{m \geq 0} T^{(2)}_{00,ml} b_{ml} + \frac{D_{00}}{2\sqrt{2}} \psi_0 + \lambda \sum_{m \geq 0} V^{(1)}_{m,00} a_m = \tilde{M}^2 \sum_{m \geq 0} B^{(2)}_{00,ml}, \tag{4.34}\]

\( n > 0: \) \[\sum_{m \geq 0} T^{(2)}_{nj,ml} b_{ml} + \lambda \sum_{m \geq 0} V^{(1)}_{n,mj} a_m = \tilde{M}^2 \sum_{m \geq 0} B^{(2)}_{nj,ml} b_{ml}. \tag{4.35}\]

Cancellation of the infinite vacuum energy \( P_{\text{vac}} \) in \((4.29)\), \((4.30)\), and \((4.33)\) is achieved if \( a_{-1} = 0 \) and \( b_{-10} = -\psi_0 / \sqrt{2 \beta} \). These values correspond to the structure of the vacuum; in other words, as a part of solving the dressed particle state, we have reconstituted the vacuum as the foundation of the physical eigenstate and thereby cancelled the (infinite) vacuum energy. The projection onto \( f_{-1} \), which is equation \((4.30)\), is no longer needed or used. The

\(^8\) The value of \( \beta \) depends on the model used for \( \delta_\epsilon \); it is not zero because the integral is over only half the real line. Physical results are independent of \( \beta \).
factor $\beta$ disappears in the eigenstate by cancelling in the product $b_{-10} f_{-10} \propto b_{-10} D_{-10}$ with $D_{-10} = \sqrt{6} \beta$.

This leaves a finite matrix problem with finite corrections due to nonzero matrix elements of vacuum transitions. In particular, there is a finite matrix element $(D_{00}/2\sqrt{2})$ coupling the three-particle sector $(b_{00})$ to the one-particle sector $(\psi_0)$ between (129) and (134). The two extra particles are ephemeral modes.

In this severe Fock-space truncation, the matrix elements are simple enough to invoke $\epsilon \to 0$ explicitly. A more general calculation would require a model for $\delta_\epsilon$ and extrapolation of the limit $\epsilon \to 0$ numerically, in addition to consideration of several $\delta_\epsilon$ models to confirm model independence.

V. SUMMARY

We have developed a formalism by which vacuum transitions can be included in light-front calculations and have argued that they must be included to have full equivalence with equal-time quantization and to be consistent with the perturbative equivalence of the two quantizations. The latter equivalence follows, at least on a formal level, as a choice of coordinates for evaluation of Feynman diagrams, with proper care as emphasized in [19, 20]. In that context, contributions such as nonzero vacuum bubbles and tadpoles are recovered. These have been missing from nonperturbative calculations due to the neglect of vacuum transitions in light-front Hamiltonians. The inclusion of such transitions means that the light-front vacuum is not trivial and instead can be characterized as a generalized coherent state of ephemeral modes, even for a free theory.

Our approach is based on the realization that vacuum transition matrix elements are nonzero with respect to Fock-state wave functions with the correct small-momentum behavior. These matrix elements lead to tadpole contributions as well as disconnected vacuum bubbles. The vacuum bubbles are regulated by the introduction of a finite width $\epsilon$ in momentum-conserving delta functions, so that a bubble’s proportionality to $\delta(0)$ is replaced by $\delta_\epsilon(0) = L/4\pi$, where $L$ is the light-front spatial volume. The width $\epsilon$ is taken to zero (and $L$ to infinity) after the (infinite) vacuum energy is subtracted. The modes with momentum of order $\epsilon$ that are removed in this limit are the ephemeral modes. They represent the accumulation of contributions at zero momentum.

The use of proper basis functions is critical. A standard DLCQ approximation [29, 35] cannot capture these effects, partly because the zero-mode contributions form sets of measure zero and partly because the DLCQ grid provides a poor approximation to integral operators with modes of order $\epsilon \ll P^+$, for either periodic or antiperiodic boundary conditions.

We have considered several applications of these ideas. The most basic was to show that the vacuum bubbles and tadpoles expected in $\phi^4$ theory are in fact reproduced. We next considered the free scalar case in detail, constructing the vacuum state as a generalized coherent state of ephemeral modes and extending this to include the shifted scalar, with recovery of the correct vacuum expectation value. The shifted case can, of course, be handled in DLCQ by inclusion of the constraint equation for the spatial average of the field [36, 38]. Here, however, we have an exact analytic solution with no discretization. Also, the analytic solution contains the one-loop vacuum bubble discussed by Collins [22] as a prime example of light-front vacuum structure in perturbation theory.

To illustrate how the approach functions in an interacting theory, we considered the charge-one sector of quenched scalar Yukawa theory. There we have shown how the vacuum
subtraction can be implemented and how strong coupling can result in residual effects from ephemeral modes, which in the limit translate to zero-mode effects.

This work was done in two dimensions. The extension to three and four dimensions should be straightforward. The transverse momenta have the full range of $-\infty$ to $\infty$ and therefore can be balanced without being individually zero. The coherent state for the free scalar vacuum would be built from an operator such as

$$A^\dagger = \int \frac{dp_1^+ dp_2^+ dp_\perp}{\sqrt{p_1^+ p_2^+}} f(p_1^+, p_2^+, p_\perp) \delta_\epsilon(p_1^+ + p_2^+, p_\perp) a^\dagger(p_1^+, p_\perp) a^\dagger(p_2^+, -p_\perp)$$

(5.1)

that creates two ephemeral modes with opposite transverse momenta.

The ideal demonstration that our approach is useful would be to compute the critical coupling in $\phi^4$ theory. The tadpole contributions that were absent previously [5] would now be included. Such a calculation is a natural next step.

**ACKNOWLEDGMENTS**

This work was supported in part by the Minnesota Supercomputing Institute and the Research Computing and Data Services at the University of Idaho through grants of computing time.

**Appendix A: Matrix elements for scalar Yukawa theory**

We compute the matrix elements needed to resolve the system of Fock-space equations for scalar Yukawa theory. The matrices are defined in (4.21) through (4.28). With the definition of the model-dependent factor $\beta$

$$\frac{1}{\beta} = \int_0^\infty dq q \delta_\epsilon(q)^2,$$

(A1)

the basis function overlaps (4.21) are, for $n, m \geq 0$,

$$B^{(1)}_{-1-1} = \frac{(C_{-1})^2}{P} \int dq q^{\alpha} \delta_\epsilon(q)^2 = \frac{(C_{-1})^2}{\beta},$$

(A2)

$$B^{(1)}_{-m} = \frac{C_{-1} C_m}{P} \int dq q^{\alpha} \delta_\epsilon(q)^2 \left(\frac{q/P}{P-q}\right)^n \frac{1}{q + \frac{m^2}{q}} = C_{-1} C_m \frac{1}{P} \int dq q (q/P)^n \delta_\epsilon(q) \rightarrow 0,$$

(A3)

$$B^{(1)}_{nn} = \frac{C_n C_m}{P} \int_0^P dq q^{\alpha} \left(\frac{q/P}{P-q}\right)^{n+m} \frac{1}{q + \frac{m^2}{q}}^2 = C_n C_m \int_0^1 \frac{x^{n+m+1} (1-x) dx}{(1-x + m^2 x)^2},$$

(A4)

$$B^{(2)}_{-10-10} = (D_{-10})^2 \int dq_1 dq_2 q_1 q_2 \frac{\delta_\epsilon(q_1 + q_2)^2}{(q_1 + q_2)^2} = (D_{-10})^2 \int dx x (1-x) \int Q dQ \delta_\epsilon(Q)^2 = \frac{(D_{-10})^2}{6 \beta},$$

(A5)

For any $\delta_\epsilon$ model that scales properly with $\epsilon$, $\beta$ is independent of $\epsilon$. 

---

For any $\delta_\epsilon$ model that scales properly with $\epsilon$, $\beta$ is independent of $\epsilon$. 

---

15
\[ B_{-10, nj}^{(2)} = \frac{D_{-10} D_{nj}}{P} \int dq_1 dq_2 \frac{\delta_\epsilon(q_1 + q_2)}{q_1 + q_2} \frac{q_1 q_2}{P} \frac{q_1^{n-j} q_2^{j}}{P^n} \rightarrow 0, \quad (A6) \]

\[ B_{nj, ml}^{(2)} = D_{nj} D_{ml} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{(x_1^j x_2^{n-j} + x_1^{n-j} x_2^j) (x_1^j x_2^{m-l} + x_1^{m-l} x_2^j)}{x_1 x_2 (1 - x_1 - x_2) \left( \frac{1}{x_1} + \frac{1}{x_2} + \frac{\bar{m}^2}{x_1 x_2} \right)^2}, \quad (A7) \]

The normalization conditions, that diagonal elements of \( B^{(1)} \) and \( B^{(2)} \) be unity, yield \( C_{-1} = \sqrt{3} \) and \( D_{-10} = \sqrt{6} \beta \).

The matrix elements for kinetic energy are, with \( P_{\text{vac}} \) defined in (3.11),

\[ T_{-11}^{(1)} = (C_{-1})^2 \int dq \frac{q P \delta_\epsilon(q)^2}{q} = (C_{-1})^2 P \int \delta_\epsilon(q)^2 = -(C_{-1})^2 \frac{2P}{\mu^2} P_{\text{vac}}, \]

\[ T_{-11n}^{(1)} = C_{-1} C_n \int dq \frac{\delta_\epsilon(q) (q/P)^n}{q} = \frac{1}{2} C_{-1} C_n \delta_{n0}, \]

\[ T_{nm}^{(1)} = C_n C_m \int \frac{dx x^{n+m}}{1 - x + \bar{m}^2 x}, \]

\[ T_{-10,-10}^{(2)} = \frac{(D_{-10})^2}{P} \int dq_1 dq_2 P^2 \frac{q_1 q_2}{q_1 + q_2} \frac{\delta_\epsilon(q_1 + q_2)^2}{q_1 + q_2} \left( \frac{1}{q_1} + \frac{1}{q_2} + \frac{\bar{m}^2}{P - q_1 - q_2} \right) = -(D_{-10})^2 \frac{2P}{\mu^2} P_{\text{vac}}, \]

\[ T_{-10, nj}^{(2)} = D_{-10} D_{nj} \int dq_1 dq_2 \frac{\delta_\epsilon(q_1 + q_2)}{q_1 + q_2} \frac{q_1^{n-j} q_2^{j}}{P} \rightarrow \frac{1}{2} D_{-10} D_{00} \delta_{n0}, \]

\[ T_{nj, ml}^{(2)} = 2 D_{nj} D_{ml} \int dx_1 dx_2 \frac{x_1^j x_2^{n-j} (x_1^j x_2^{m-l} + x_1^{m-l} x_2^j)}{(x_1 + x_2) (1 - x_1 - x_2) + \bar{m}^2 x_1 x_2}. \]

In \( T_{-11}^{(1)} \) and \( T_{-10, nj}^{(2)} \) we have used \( \int dq \delta_\epsilon(q) = \frac{1}{2} \), which follows from integrating only over positive \( q \).

The potential terms have the following matrix elements:

\[ U_{-10}^{(0)} = \frac{D_{-10}}{\sqrt{2}} \int \frac{dq_1 dq_2}{\sqrt{q_1 q_2}} \frac{\delta_\epsilon(q_1 + q_2)}{q_1 + q_2} P \frac{\sqrt{q_1 q_2}}{q_1 + q_2} = -\frac{D_{-10} 2P}{\sqrt{2} \mu^2} P_{\text{vac}}, \]

\[ U_{nj}^{(0)} = \frac{1}{\sqrt{2}} \int \frac{dq_1 dq_2}{\sqrt{q_1 q_2}} \frac{\delta_\epsilon(q_1 + q_2)}{q_1 + q_2} \frac{D_{nj} \sqrt{P}}{\sqrt{q_1 q_2 (P - q_1 - q_2)}} \frac{q_1^{n-j} q_2^{j} / P^n}{\frac{1}{q_1} + \frac{1}{q_2} + \frac{\bar{m}^2}{P - q_1 - q_2}} = D_{00} \frac{2\sqrt{2}}{\mu^2} \delta_{n0}, \]

\[ V_{-1}^{(0)} = C_{-1} \sqrt{P} \int \frac{dq}{\sqrt{q P (P - q)}} \sqrt{q P} \delta_\epsilon(q) = \frac{1}{2} C_{-1}, \]

\[ V_{n}^{(0)} = C_n \int \frac{dx x^n}{1 - x + \bar{m}^2 x}, \]

\[ V_{-1,-10}^{(1)} = \sqrt{2} C_{-1} D_{-10} \int dq q \delta_\epsilon(q)^2 = \frac{\sqrt{2}}{\beta} C_{-1} D_{-10} = \sqrt{12}, \]

\[ V_{n,-10}^{(1)} = \sqrt{2} C_n D_{-10} \int dq dq' \left( \frac{q}{P} \right)^{n+1} \frac{\delta_\epsilon(q + q')}{q + q'} = \sqrt{2} C_n D_{-10} \int dq \left( \frac{q}{P} \right)^{n+1} \delta_\epsilon(q) \rightarrow 0, \]

\[ V_{-1, nj}^{(1)} = \sqrt{2} C_{-1} D_{nj} \int dq dq' \frac{\delta_\epsilon(q)}{q'} \frac{q'^{n-j} + q^{n-j} q'^j}{P^n} \rightarrow 0, \]

\[ V_{n, -10}^{(1)} = \sqrt{2} C_{-1} D_{-10} \int dq dq' \frac{q^{n-j} + q'^{n-j} q'^j}{P^n} \rightarrow 0, \]

\[ V_{-1, nj}^{(1)} = \sqrt{2} C_{-1} D_{nj} \int dq dq' \frac{\delta_\epsilon(q)}{q'} \frac{q'^{n-j} + q^{n-j} q'^j}{P^n} \rightarrow 0, \]
\[ V_{n,ml}^{(1)} = \sqrt{2}C_nD_{ml} \int \frac{dx_1dx_2x_1^{n+1}}{1-x_1+m^2x_1(x_1+x_2)(1-x_1-x_2)+m^2x_1x_2}. \]  

(A21)

In \( V_{n,-10}^{(1)} \) we have used a representation of the Dirac delta function

\[ \delta(q) = \int dq' \frac{\delta(q+q')}{q+q'} \]  

(A22)

which follows from

\[ \int dqf(q)dq' \frac{\delta(q+q')}{q+q'} = \int QdxQf(xQ)\frac{\delta(Q)}{Q} = \frac{1}{2}f(0). \]  

(A23)

1. S. Rychkov and L.G. Vitale, Hamiltonian truncation study of the \( \phi^4 \) theory in two dimensions, Phys. Rev. D 91, 085011 (2015); Hamiltonian truncation study of the \( \phi^2 \) theory in two dimensions. II. The \( Z_2 \)-broken phase and the Chang duality, Phys. Rev. D 93, 065014 (2016).
2. J. Elias-Miro, M. Montull and M. Riembau, The renormalized Hamiltonian truncation method in the large \( E_T \) expansion, JHEP 1604, 144 (2016); J. Elias-Miro, S. Rychkov and L. G. Vitale, NLO Renormalization in the Hamiltonian Truncation, Phys. Rev. D 96, 065024 (2017); High-precision calculations in strongly coupled quantum field theory with next-to-leading-order renormalized Hamiltonian truncation, JHEP 1710, 213 (2017).
3. A. Pelissetto and E. Vicari, Critical mass renormalization in renormalized \( \phi^4 \) theories in two and three dimensions, Phys. Lett. B 751, 532 (2015).
4. P. Bosetti, B. De Palma and M. Guagnelli, Monte Carlo determination of the critical coupling in \( \phi_2^4 \) theory, Phys. Rev. D 92, 034509 (2015); B. De Palma and M. Guagnelli, Monte Carlo simulation of \( \phi_2^4 \) and \( O(N)\phi_3^4 \) theories,” PoS LATTICE 2016, 277 (2016); S. Bronzin, B. De Palma and M. Guagnelli, New Monte Carlo determination of the critical coupling in \( \phi_2^4 \) theory, Phys. Rev. D 99, 034508 (2019).
5. M. Burkardt, S.S. Chabysheva, and J.R. Hiller, Two-dimensional light-front \( \phi^4 \) theory in a symmetric polynomial basis, Phys. Rev. D 94, 065006 (2016); S.S. Chabysheva and J.R. Hiller, Light-front \( \phi_2^4 \) theory with sector-dependent mass, Phys. Rev. D 95, 096016 (2017).
6. N. Anand, V. X. Genest, E. Katz, Z. U. Khandker and M. T. Walters, RG flow from \( \phi^4 \) theory to the 2D Ising model, JHEP 1708, 056 (2017).
7. A. L. Fitzpatrick, J. Kaplan, E. Katz, L. G. Vitale and M. T. Walters, Lightcone effective Hamiltonians and RG flows, JHEP 1808, 120 (2018).
8. A. L. Fitzpatrick, E. Katz and M. T. Walters, Nonperturbative Matching Between Equal-Time and Lightcone Quantization, JHEP 2010, 092 (2020).
9. N. Anand, A. L. Fitzpatrick, E. Katz, Z. U. Khandker, M. T. Walters and Y. Xin, Introduction to Lightcone Conformal Truncation: QFT Dynamics from CFT Data, arXiv:2005.13511 [hep-th].
10. M. Serone, G. Spada, and G. Villadoro, \( \lambda \phi^4 \) Theory I: The Symmetric Phase Beyond NNNNNNLO, JHEP 1808, 148 (2018); \( \lambda \phi_2^4 \) theory — Part II. the broken phase beyond NNNN(NNNN)LO, JHEP 1905, 047 (2019); G. Sberveglieri, M. Serone and G. Spada, Renormalization scheme dependence, RG flow, and Borel summability in \( \phi^4 \) Theories in \( d < 4 \), Phys. Rev. D 100, 045008 (2019).
[11] D. Kadoh, Y. Kuramashi, Y. Nakamura, R. Sakai, S. Takeda, and Y. Yoshimura, Tensor network analysis of critical coupling in two dimensional $\phi^4$ theory, JHEP 1905, 184 (2019).
[12] G. O. Heymans and M. B. Pinto, Critical behavior of the 2d scalar theory: resumming the N$^8$LO perturbative mass gap, JHEP 07, 163 (2021).
[13] J. P. Vary, M. Huang, S. Jawadekar, M. Sharaf, A. Harindranath and D. Chakrabarti, Critical Coupling for Two-dimensional $\phi^4$ Theory in Discretized Light-Cone Quantization, arXiv:2109.13372 [hep-th].
[14] M. Burkardt, Light-front quantization of the sine-Gordon model, Phys. Rev. D 47, 4628 (1993).
[15] K. Hornbostel, Nontrivial vacua from equal time to the light cone, Phys. Rev. D 45, 3781 (1992).
[16] C. R. Ji and C. Mitchell, Poincare invariant algebra from instant to light front quantization, Phys. Rev. D 64, 085013 (2001); C. R. Ji and A. T. Suzuki, Interpolating scattering amplitudes between the instant form and the front form of relativistic dynamics, Phys. Rev. D 87, 065015 (2013); C. R. Ji, Z. Li, and A. T. Suzuki, Electromagnetic gauge field interpolation between the instant form and the front form of the Hamiltonian dynamics, Phys. Rev. D 91, 065020 (2015) Z. Li, M. An, and C. R. Ji, Interpolating Helicity Spinors Between the Instant Form and the Light-front Form, Phys. Rev. D 92, 105014 (2015); C. R. Ji, Z. Li, B. Ma, and A. T. Suzuki, Interpolating quantum electrodynamics between instant and front forms, Phys. Rev. D 98, 036017 (2018); B. Ma and C. R. Ji, Interpolating 't Hooft model between instant and front fields, Phys. Rev. D 104, 036004 (2021).
[17] S.S. Chabysheva and J.R. Hiller, Transitioning from equal-time to light-front quantization in $\phi^4_2$ theory, Phys. Rev. D 102, 116010 (2020).
[18] D. Binosi and L. Theußl, JaxoDraw: A graphical user interface for drawing Feynman diagrams, Comp. Phys. Comm. 161, 76 (2004).
[19] P.D. Mannheim, P. Lowdon and S.J. Brodsky, Structure of light front vacuum sector diagrams, Phys. Lett. B 797, 134916 (2019);
[20] P. D. Mannheim, Equivalence of light-front quantization and instant-time quantization, Phys. Rev. D 102, 025020 (2020); P. D. Mannheim, P. Lowdon and S. J. Brodsky, Comparing light-front quantization with instant-time quantization, Phys. Rept. 891, 1 (2021).
[21] W.N. Polyzou, Relation between instant and light-front formulations of quantum field theory, Phys. Rev. D 103, 105017 (2021).
[22] J. Collins, The non-triviality of the vacuum in light-front quantization: An elementary treatment, arXiv:1801.03960 [hep-ph].
[23] L. Martinovic and A. Dorokhov, Vacuum loops in light-front field theory, Phys. Lett. B 811, 135925 (2020).
[24] M. Burkardt, F. Lenz, and M. Thies, Chiral condensate and short time evolution of QCD(1+1) on the light cone, Phys. Rev. D 65, 125002 (2002) F. Lenz, K. Ohta, M. Thies and K. Yazaki, Chiral symmetry in light cone field theory, Phys. Rev. D 70, 025015 (2004) S. R. Beane, Broken chiral symmetry on a null plane, Ann. Phys. 337, 111-142 (2013)
[25] F.P. Aslan and M. Burkardt Singularities in twist-3 quark distributions, Phys. Rev. D 101, 016010 (2020); X. Ji, Fundamental Properties of the Proton in Light-Front Zero Modes, Nucl. Phys. B 960, 115181 (2020).
[26] G. McCartor, Light cone quantization for massless fields, Z. Phys. C 41, 271 (1988); T. Heinzl, A. Ilderton, and D. Seipt, Mode truncations and scattering in strong fields, Phys. Rev. D 98, 016002 (2018).
[27] M. Herrmann and W. N. Polyzou, Light-front vacuum, Phys. Rev. D 91, 085043 (2015).
[28] P.A.M. Dirac, Forms of relativistic dynamics, Rev. Mod. Phys. 21, 392 (1949).
[29] S.J. Brodsky, H.-C. Pauli, and S.S. Pinsky, Quantum chromodynamics and other field theories on the light cone, Phys. Rep. 301, 299 (1998).
[30] J. Carbonell, B. Desplanques, V.A. Karmanov, and J.F. Mathiot, Explicitly covariant light front dynamics and relativistic few body systems, Phys. Rep. 300, 215 (1998).
[31] G.A. Miller, Light front quantization: A Technique for relativistic and realistic nuclear physics, Prog. Part. Nucl. Phys. 45, 83 (2000).
[32] T. Heinzel, Light cone quantization: Foundations and applications, Lect. Notes Phys. 572, 55 (2001).
[33] M. Burkardt, Light front quantization, Adv. Nucl. Phys. 23, 1 (2002).
[34] J.R. Hiller, Nonperturbative light-front Hamiltonian methods, Prog. Part. Nucl. Phys. 90, 75 (2016).
[35] H.-C. Pauli and S.J. Brodsky, Solving field theory in one space one time dimension, Phys. Rev. D 32, 1993 (1985); Discretized light cone quantization: Solution to a field theory in one space one time dimension, Phys. Rev. D 32, 2001 (1985).
[36] D. G. Robertson, On spontaneous symmetry breaking in discretized light cone field theory, Phys. Rev. D 47, 2549 (1993).
[37] S. S. Pinsky, B. van de Sande, and J. R. Hiller, Spontaneous symmetry breaking of (1 + 1)-dimensional φ^4 theory in light front field theory. 3, Phys. Rev. D 51, 726 (1995).
[38] S. S. Chabysheva and J. R. Hiller, Zero momentum modes in discrete light-cone quantization, Phys. Rev. D 79, 096012 (2009).
[39] G. P. Lepage, Adaptive multidimensional integration: VEGAS enhanced, J. Comput. Phys. 439, 110386 (2021); A New Algorithm for Adaptive Multidimensional Integration, J. Comput. Phys. 27, 192 (1978).
[40] G.C. Wick, Properties of Bethe-Salpeter wave functions, Phys. Rev. 96, 1124 (1954); R.E. Cutkosky, Solutions of a Bethe-Salpeter equation, Phys. Rev. 96, 1135 (1954); E. Zur Linden and H. Mitter, Bound-state solutions of the bethe-salpeter equation in momentum space, Nuovo Cim. B 61, 389 (1969); D. Bernard, Th. Cousin, V.A. Karmanov, and J.-F. Mathiot, Nonperturbative renormalization in a scalar model within light front dynamics, Phys. Rev. D 65, 025016 (2001); Y. Li, V.A. Karmanov, P. Maris, and J.P. Vary, Ab initio approach to the non-perturbative scalar Yukawa model, Phys. Lett. B 748, 278 (2015).
[41] G. Baym, Phys. Rev. 117, 886 (1960); Inconsistency of cubic boson-boson interactions, F. Gross, C. Savkli, and J. Tjon, Stability of the scalar χ^2φ interaction, Phys. Rev. D 64, 076008 (2001).